Existence and uniqueness of positive solutions for a class of logistic type elliptic equations in \( \mathbb{R}^N \) involving fractional Laplacian

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Abstract

In this paper, we study the existence and uniqueness of positive solutions for the following nonlinear fractional elliptic equation:

\[
(-\Delta)^{\alpha} u = \lambda a(x) u - b(x) u^p \quad \text{in} \quad \mathbb{R}^N,
\]

where \( \alpha \in (0,1), N \geq 2, \lambda > 0, a \) and \( b \) are positive smooth function in \( \mathbb{R}^N \) satisfying

\[
a(x) \to a^\infty > 0 \quad \text{and} \quad b(x) \to b^\infty > 0 \quad \text{as} \quad |x| \to \infty.
\]

Our proof is based on a comparison principle and existence, uniqueness and asymptotic behaviors of various boundary blow-up solutions for a class of elliptic equations involving the fractional Laplacian.

1 Introduction

A celebrated result of Du and Ma [10] asserts that the uniqueness positive solution of

\[
-\Delta u = \lambda u - u^p \quad \text{in} \quad \mathbb{R}^N
\]

for \( N \geq 1, \lambda > 0 \) and \( p > 1 \), is \( u \equiv \lambda^{-\frac{1}{p-1}} \). Moreover, in [10], the authors also consider the following logistic type equation:

\[
-\Delta u = \lambda a(x) u - b(x) u^p \quad \text{in} \quad \mathbb{R}^N, \tag{1.1}
\]

where \( p > 1, a \) and \( b \) are positive smooth function in \( \mathbb{R}^N \) satisfying

\[
a(x) \to a^\infty > 0 \quad \text{and} \quad b(x) \to b^\infty > 0 \quad \text{as} \quad |x| \to \infty.
\]

AMS Subject Classifications 2010: 35J60, 47G20.

Key words: Fractional Laplacian, comparison principle, blow-up solution, uniqueness.
Then they proved that problem (1.1) has a unique positive solution for each \( \lambda > 0 \). A similar problem for quasi-linear operator has been studied by Du and Guo [9].

In the present work, we are interested in understanding whether similar results hold for equations involving a nonlocal diffusion operator, the simplest of which is perhaps the fractional Laplacian. For \( \alpha \in (0, 1) \), we study the following fractional elliptic problem:

\[
(-\Delta)^{\alpha} u = \lambda u - u^p \quad \text{in } \mathbb{R}^N,
\]

where \( N \geq 2 \). The fractional Laplacian is defined, up to a normalization constant, by

\[
(-\Delta)^{\alpha} u(x) = \int_{\mathbb{R}^N} \frac{2u(x) - u(x + y) - u(x - y)}{|y|^{N+2\alpha}} \, dy, \quad \forall \, x \in \mathbb{R}^N.
\]

Our first main result is

**Theorem 1.1** Let \( \lambda > 0 \). Suppose \( u \in C^{2\alpha + \beta}_{loc}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, \omega) \) for some \( \beta > 0 \) and \( \omega = 1/(1 + |y|^{N+2\alpha}) \) is a nonnegative solution of (1.2). Then \( u \) must be a constant if \( p \) verifies

\[
1 + 2\alpha < p < \frac{1 + \alpha}{1 - \alpha}.
\]

**Remark 1.1** We notice that

\[
\frac{N + 2\alpha}{N - 2\alpha} \leq \frac{1 + \alpha}{1 - \alpha},
\]

if \( N \geq 2 \).

As in [10] and [9], our proof of this result based on a comparison principle for concave sublinear problems (see Lemma 2.1) and involves boundary blow-up solutions. We use a rather intuitive squeezing method to proof Theorem 1.1 as follows. Denote \( B_R \) as a ball centered at the origin with radius \( R \). Then problem

\[
\begin{cases}
(-\Delta)^\alpha v = \lambda v - v^p & \text{in } B_R, \\
v = 0 & \text{in } \mathbb{R}^N \setminus B_R,
\end{cases}
\]

has a unique positive solution \( v_R \) if \( R \) is large enough for any fixed \( \lambda > 0 \). On the other hand, the following boundary blow-up problem

\[
\begin{cases}
(-\Delta)^\alpha w = \lambda w - w^p & \text{in } B_R, \\
\lim_{x \in B_R, x \to \partial B_R} w(x) = +\infty, \\
w = g & \text{in } \mathbb{R}^N \setminus \bar{B}_R,
\end{cases}
\]
for some $g \in L^1(\mathbb{R}^N \setminus \bar{B}_R, \omega)$ and $\lambda > 0$, has a positive solution $w_R$ for any $R > 0$. The comparison principle implies that any entire positive solution of (1.2) satisfies $v_R \leq u \leq w_R$ in $B_R$. Moreover, one can show (see Lemmas 2.2 and 2.3 in Section 2) that both $v_R$ and $w_R$ converge locally uniformly to $\frac{\lambda}{p-1}$ as $R \to +\infty$. Therefore, $u \equiv \frac{\lambda}{p-1}$ in $\mathbb{R}^N$.

Next, we make use of Theorem 1.1 to study logistic type fractional elliptic problems with variable coefficients that are asymptotically positive constants. More precisely, we study the following problem

$$(-\Delta)^{\alpha} u = \lambda a(x) u - b(x) u^p \quad \text{in} \ \mathbb{R}^N,$$

(1.5)

where $a$ and $b$ are positive smooth function in $\mathbb{R}^N$. Moreover, we suppose that

$$a(x) \to a^\infty > 0 \quad \text{and} \quad b(x) \to b^\infty > 0 \quad \text{as} \ |x| \to \infty.$$  

(1.6)

We can prove that

**Theorem 1.2** Let $\lambda > 0$. Suppose $a$ and $b$ are positive smooth function in $\mathbb{R}^N$ and satisfying (1.6). Then equation (1.3) has a unique positive solution if $p$ verifies (1.3).

We prove Theorem 1.2 by a similar argument as in the proof of Theorem 1.1, we consider the Dirichlet problem and the boundary blow-up problem in a ball $B_R$. When $R$ is large, these problems have positive solutions $v_R$ and $w_R$ respectively. By comparison principle, as $R \to \infty$, $v_R$ increase to a minimal positive solution of (1.5) and $w_R$ decrease to a maximal positive solution of (1.5). Therefore, when (1.5) has a unique positive solution, $v_R$ and $w_R$ approximate this unique solution from below and above, respectively.

We mentioned that, in [10] and [9], the existence and uniqueness results hold provided $p > 1$, but in our Theorems 1.1 and 1.2 we require $p$ satisfying (1.3). This is because we will use Perron’s method (we refer the reader to User’s guide [6] for the presentation of Perron’s method which extends to the case of nonlocal equations, see for example [3, 4, 11]) to construct solution of problem 1.4 by applying Proposition 2.2 and choosing

$$\tau = -\frac{2\alpha}{p-1} \in (-1, \tau_0(\alpha))$$

in $V_\tau(x)$ (see (2.13)). This implies

$$p < 1 - \frac{2\alpha}{\tau_0(\alpha)}.$$
Moreover, in [5], the authors proved that \( \tau_0(\alpha) \) has a simplicity formula, that is, \( \tau_0(\alpha) = \alpha - 1 \). Thus, we have
\[
p < 1 - \frac{2\alpha}{\tau_0(\alpha)} = \frac{1 + \alpha}{1 - \alpha}.
\]

This article is organized as follows. In Section 2 we present some preliminary lemmas to prove a comparison principle involving the fractional Laplacian, existence and asymptotic behaviors of boundary blow-up solutions. Section 3 is devoted to prove the existence and uniqueness results of problems (1.2) and (1.5), i.e., Theorems 1.1 and 1.2.

## 2 Preliminary lemmas

In this section, we introduce some lemmas which are useful in the proof of our main results. The first important ingredient is the comparison principle involving the fractional Laplacian which is useful in dealing with boundary blow-up problems.

**Lemma 2.1 (Comparison principle)** Suppose that \( \Omega \) is a bounded domain in \( \mathbb{R}^N \), \( a(x) \) and \( b(x) \) are continuous functions in \( \Omega \) with \( \|a\|_{L^\infty(\Omega)} < \infty \) and \( b(x) \) nonnegative and not identity zero. Suppose \( u_1, u_2 \in C^{2\alpha+\beta}(\Omega) \) for some \( \beta > 0 \) are positive in \( \Omega \) and satisfy
\[
(-\Delta)\alpha u_1 - a(x)u_1 + b(x)u_1^p \geq 0 \geq (-\Delta)\alpha u_2 - a(x)u_2 + b(x)u_2^p \quad \text{in } \Omega \quad (2.1)
\]
and \( \lim \sup_{x \to \partial \Omega} (u_2 - u_1) \leq 0 \) with \( u_2 - u_1 \leq 0 \) in \( \mathbb{R}^N \setminus \overline{\Omega} \), where \( p > 1 \). Then \( u_2 \leq u_1 \) in \( \Omega \).

In order to prove Lemma 2.1, we need the following proposition.

**Proposition 2.1** For \( u \geq 0 \) and \( v > 0 \), we have
\[
L(u, v) \geq 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}^N,
\]
where
\[
L(u, v)(x, y) = (u(x) - u(y))^2 - (v(y) - v(x)) \left( \frac{u(y)^2}{v(y)} - \frac{u(x)^2}{v(x)} \right).
\]
Moreover, the equality holds if and only if \( u = kv \) a.e. for some constant \( k \).
We note that Proposition 2.1 is a special case \((p = 2)\) of Lemma 4.6 in [13] and we omit the proof here.

**Proof of Lemma 2.1** Let \(\phi_1\) and \(\phi_2\) be nonnegative functions in \(C^\infty_0(\Omega)\). By (2.1), we obtain that
\[
\int_{\mathbb{R}^{2N}} \frac{(u_1(x) - u_1(y))(\phi_1(x) - \phi_1(y))}{|x - y|^{N+2\alpha}} - \frac{(u_2(x) - u_2(y))(\phi_2(x) - \phi_2(y))}{|x - y|^{N+2\alpha}} \, dx \, dy \geq 2 \int_\Omega b(x)[u_2^\alpha \phi_2 - u_1^\alpha \phi_1] \, dx + \int_\Omega a(x)(u_1 \phi_1 - u_2 \phi_2) \, dx \tag{2.2}
\]

For \(\varepsilon > 0\), we denote \(\varepsilon_1 = \varepsilon\) and \(\varepsilon_2 = \varepsilon/2\) and let
\[
v_i = \frac{[(u_2 + \varepsilon_2)^2 - (u_1 + \varepsilon_1)^2]^+}{u_i + \varepsilon_i}, \quad i = 1, 2.
\]

By our our assumption, \(v_i\) is zero near \(\partial\Omega\) and in \(\mathbb{R}^N \setminus \bar{\Omega}\). Hence \(v_i \in X_0^a(D_0)\), where \(D_0 \subset \subset \Omega\) and \(X_0^a(D_0) = \{w \in H^a(\mathbb{R}^N) : w = 0 \ a.e\ in \ \mathbb{R}^N \setminus D_0\}\). In fact, it is clear that \(\|v_1\|_{L^2(\mathbb{R}^N)} = \|v_1\|_{L^2(D_0)} \leq C\) and thus it remains to verify that the Gagliardo norm of \(v_1\) in \(\mathbb{R}^N\) is bounded by a constant. Using the symmetry of the integral in the Gagliardo norm with respect to \(x\) and \(y\) and the fact that \(v_1 = 0\) in \(\mathbb{R}^N \setminus \Omega\), we can split as follows
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_1(x) - v_1(y)|^2}{|x - y|^{N+2\alpha}} \, dx \, dy = \int_\Omega \int_{\mathbb{R}^N} \frac{|v_1(x) - v_1(y)|^2}{|x - y|^{N+2\alpha}} \, dx \, dy + 2 \int_\Omega \left( \int_{\mathbb{R}^N \setminus \Omega} \frac{|v_1(x)|^2}{|x - y|^{N+2\alpha}} \, dy \right) \, dx \tag{2.3}
\]

Next, we estimate both integrals in the right hand side of (2.3) is finite. We first notice that, for any \(y \in \mathbb{R}^N \setminus D_0\),
\[
\frac{|v_1(x)|^2}{|x - y|^{N+2\alpha}} = \chi_{D_0}(x)|v_1(x)|^2 \leq \chi_{D_0}(x)|v_1(x)|^2 \sup_{x \in D_0} \frac{1}{|x - y|^{N+2\alpha}}.
\]

This implies that
\[
\int_\Omega \left( \int_{\mathbb{R}^N \setminus \Omega} \frac{|v_1(x)|^2}{|x - y|^{N+2\alpha}} \, dy \right) \, dx \leq \left( \int_{\mathbb{R}^N \setminus D_0} \frac{1}{\text{dist}(y, \partial D_0)^{N+2\alpha}} \, dy \right) \|v_1\|^2_{L^2(D_0)} < +\infty
\]
since \(\text{dist}(\partial\Omega, \partial D_0) \geq \gamma > 0\) and \(N + 2\alpha > N\). Hence, the second term in the right hand side of (2.3) is finite by the above inequality. In order to show the
first term in the right hand side of (2.3) is also finite, we need the following estimates

\[
\frac{(u_2(x) + \varepsilon_2)^2 - (u_1(x) + \varepsilon_1)^2}{u_1(x) + \varepsilon_1} - \frac{(u_2(y) + \varepsilon_2)^2 - (u_1(y) + \varepsilon_1)^2}{u_1(y) + \varepsilon_1} = \left| \frac{(u_2(x) + \varepsilon_2)^2}{u_1(x) + \varepsilon_1} - \frac{(u_2(y) + \varepsilon_2)^2}{u_1(y) + \varepsilon_1} \right| + \left| \frac{u_1(y) - u_1(x)}{u_1(y) + \varepsilon_1} \right| \]  

\leq \left| \frac{(u_2(x) + \varepsilon_2)^2}{u_1(x) + \varepsilon_1} - \frac{(u_2(y) + \varepsilon_2)^2}{u_1(y) + \varepsilon_1} \right| + |u_1(y) - u_1(x)| \right| 

(2.4)

and

\[
\frac{(u_2(x) + \varepsilon_2)^2 - (u_1(x) + \varepsilon_1)^2}{u_1(x) + \varepsilon_1} - \frac{(u_2(y) + \varepsilon_2)^2 - (u_1(y) + \varepsilon_1)^2}{u_1(y) + \varepsilon_1} = \left| \frac{(u_2(x) + \varepsilon_2)^2}{u_1(x) + \varepsilon_1} - \frac{(u_2(y) + \varepsilon_2)^2}{u_1(y) + \varepsilon_1} \right| + \left| \frac{u_1(y) - u_1(x)}{u_1(y) + \varepsilon_1} \right| \]  

\leq \frac{u_2(x) + u_2(y) + 2\varepsilon_2}{u_1(x) + \varepsilon_1} |u_2(x) - u_2(y)| + \frac{(u_2(x) + \varepsilon_2)^2}{(u_1(x) + \varepsilon_1)(u_1(y) + \varepsilon_1)} |u_1(y) - u_1(x)| \right| 

(2.5)

Combining (2.4) and (2.5), we have

\[
\frac{(u_2(x) + \varepsilon_2)^2}{u_1(x) + \varepsilon_1} - \frac{(u_2(y) + \varepsilon_2)^2}{u_1(y) + \varepsilon_1} \leq C(\varepsilon_1, \varepsilon_2, ||u_1||_{L^\infty(\Omega)}, ||u_2||_{L^\infty(\Omega)}) (|u_1(y) - u_1(x)| - |u_2(y) - u_2(x)|) \]  

\leq \tilde{C}|x - y|^{2\alpha + \beta}.

In the last inequality of above estimate, we have used the fact \(u_1, u_2 \in C^{2\alpha + \beta}(\Omega)\). This implies

\[
\int_\Omega \int_\Omega \frac{|v_1(x) - v_1(y)|^2}{|x - y|^{N+2\alpha}} \, dx \, dy < +\infty
\]

since the following inequality

\[
|w + (x) - w^+(y)| \leq |w(x) - w(y)|
\]

for all \((x, y) \in \mathbb{R}^N \times \mathbb{R}^N\) and function \(w : \mathbb{R}^N \to \mathbb{R}\). Therefore, \(v_1 \in X_0^\alpha(D_0)\).

Similarly, we can show \(v_2 \in X_0^\alpha(D_0)\). On the other hand, by Theorem 6 in \cite{12}, we know that \(v_i\) can be approximate arbitrarily closely in the \(X_0^\alpha(D_0)\) norm by \(C_0^\infty(D_0)\) functions. Hence, we see that (2.2) holds when \(\phi_i\) is replaced by \(v_i\) for \(i = 1, 2\).

Denote

\[
D(\varepsilon) = \{ x \in \Omega : u_2(x) + \varepsilon_2 > u_1(x) + \varepsilon_1 \}.
\]
Hence, $0 \leq x, y$.

Finally, if $(x, y) \in \mathbb{R}^N \setminus D(\varepsilon)$, we know that $v_i(x) = v_i(y) = 0$, $i = 1, 2$. Therefore,

$$A_1 := \int_{\mathbb{R}^N \setminus D(\varepsilon)} \int_{\mathbb{R}^N \setminus D(\varepsilon)} \frac{\mathcal{L}(u_1, u_2)}{|x - y|^{N+2\alpha}} dxdy = 0,$$

where

$$\mathcal{L}(u_1, u_2) = (u_1(x) - u_1(y))\left(v_1(x) - v_1(y)\right) - (u_2(x) - u_2(y))\left(v_2(x) - v_2(y)\right).$$

For $(x, y) \in D(\varepsilon) \times \mathbb{R}^N \setminus D(\varepsilon)$, we notice that $v_1(y) = v_2(y) = 0$ and, by the definition of $D(\varepsilon)$,

$$u_2(x) + \varepsilon_2 > u_1(x) + \varepsilon_1 \quad \text{and} \quad u_2(y) + \varepsilon_2 \leq u_1(y) + \varepsilon_1. \quad (2.6)$$

It follows that

$$\mathcal{L}(u_1, u_2) = \left[u_1(x) - u_1(y)\right]v_1(x) - \left[u_2(x) - u_2(y)\right]v_2(x)$$

$$= \left[(u_1(x) + \varepsilon_1) - (u_1(y) + \varepsilon_1)\right]v_1(x) - \left[(u_2(x) + \varepsilon_2) - (u_2(y) + \varepsilon_2)\right]v_2(x)$$

$$= \frac{\varepsilon_1 u_1(x) + \varepsilon_2 u_2(x)}{u_1(x) + \varepsilon_1 (u_2(x) + \varepsilon_2)} \cdot \left[(u_1(x) + \varepsilon_1)(u_2(y) + \varepsilon_2) - (u_1(y) + \varepsilon_1)(u_2(x) + \varepsilon_2)\right]$$

$$\leq 0.$$

Hence,

$$A_2 = \int_{D(\varepsilon)} \int_{\mathbb{R}^N \setminus D(\varepsilon)} \frac{\mathcal{L}(u_1, u_2)}{|x - y|^{N+2\alpha}} dydx \leq 0.$$

A similar argument implies that

$$A_3 = \int_{\mathbb{R}^N \setminus D(\varepsilon)} \int_{D(\varepsilon)} \frac{\mathcal{L}(u_1, u_2)}{|x - y|^{N+2\alpha}} dydx \leq 0.$$

Finally, if $(x, y) \in D(\varepsilon) \times D(\varepsilon)$, it is easy to check that

$$\mathcal{L}(u_1, u_2) = (u_1(x) - u_1(y))\left(v_1(x) - v_1(y)\right) - (u_2(x) - u_2(y))\left(v_2(x) - v_2(y)\right)$$

$$= -(u_1(x) - u_1(y))^2 + (u_1(y) - u_1(x))\left(\frac{(u_2(y) + \varepsilon_2)^2}{u_1(y) + \varepsilon_1} - \frac{(u_2(x) + \varepsilon_2)^2}{u_1(x) + \varepsilon_1}\right)$$

$$- (u_2(x) - u_2(y))^2 + (u_2(y) - u_2(x))\left(\frac{(u_1(y) + \varepsilon_1)^2}{u_2(y) + \varepsilon_2} - \frac{(u_1(x) + \varepsilon_1)^2}{u_2(x) + \varepsilon_2}\right).$$
By Proposition 2.1, we know that $L(u_1, u_2)(x, y) \leq 0$ in $D(\varepsilon) \times D(\varepsilon)$. Therefore,

$$A_4 = \int_{D(\varepsilon)} \int_{D(\varepsilon)} \frac{L(u_1, u_2)}{|x - y|^{N+2\alpha}} dxdy \leq 0.$$ 

Summing up these estimates from $A_1$ to $A_4$, we know that the left hand side of (2.2) is nonpositive.

On the other hand, as $\varepsilon \to 0$, the first term in the right hand side of (2.2) converges to

$$\int_{D(0)} b(x) \left( u_2^{p-1} - u_1^{p-1} \right) (u_2^2 - u_1^2) dx,$$

while the last term in the right side of (2.2) converges to 0.

Next, we show that $D(0) = \emptyset$. Suppose to the contrary that $D(0) \neq \emptyset$. Since the left side of (2.2) is nonpositive by the estimates from $A_1$ to $A_4$ and right hand side of (2.2) tends to 0 as $\varepsilon \to 0$, we easy deduce

$$\int_{\mathbb{R}^N} L(u_1, u_2) \frac{dxdy}{|x - y|^{N+2\alpha}} = 0,$$

where $L(u_1, u_2) = \lim_{\varepsilon \to 0} L(u_1, u_2)$ and

$$\int_{D(0)} b(x) \left( u_2^{p-1} - u_1^{p-1} \right) (u_2^2 - u_1^2) dx = 0.$$

This imply that

$$b \equiv 0 \quad \text{in } D(0)$$

and

$$L(u_1, u_2) \equiv 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}^N.$$

Hence, by Proposition 2.1, we know $u_1 = ku_2$ in $D(0)$ for some constant $k$. Since $b \neq 0$ in $\Omega$, it follows from the above that $D(0) \neq \emptyset$. Thus, $D(0) \subset \Omega$, $\partial D(0) \cap \Omega \neq \emptyset$. It follows that the open set $D(0)$ has connected component $G$ such that $\partial G \cap \Omega \neq \emptyset$. Now on $G$, $u_1 = ku_2$. On the other hand, we have $u_1|_{\partial G \cap \Omega} = u_2|_{\partial G \cap \Omega} > 0$. Thus, $k = 1$. So we have $u_1 = u_2$ in $G$, which contradicts $G \subset D(0)$. Therefore, we must have $D(0) = \emptyset$ and thus $u_1 \geq u_2$ in $\Omega$. We complete the proof of Lemma 2.1. □

By applying this comparison principle together with the Perron’s method for the nonlocal equation, we can obtain the following two lemmas.

**Lemma 2.2** Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with smooth boundary and $p > 1$. Suppose $a$ and $b$ are smooth positive functions in $\overline{\Omega}$, and let $\mu_1$ denote
the first eigenvalue of \((-\Delta)^\alpha u = \mu a(x)u\) in \(\Omega\) with \(u = 0\) in \(\mathbb{R}^N \setminus \Omega\). Then the equation
\[
\begin{cases}
(-\Delta)^\alpha u = \mu u[a(x) - b(x)w^{p-1}] & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega
\end{cases}
\]
has a unique positive solution for every \(\mu > \mu_1\). Furthermore, the unique solution \(u_\mu\) satisfies \(u_\mu \to [a(x)/b(x)]^{1/(p-1)}\) uniformly in any compact subset of \(\Omega\) as \(\mu \to +\infty\).

**Proof.** (Existence) The existence follows from a simple sub- and super-solution argument. In fact, any constant greater than or equal to \(M = \max_{\bar{\Omega}}[a(x)/b(x)]^{1/(p-1)}\) is a super-solution. Let \(\phi\) be a positive eigenfunction corresponding to \(\mu_1\) (for the existence of the first eigenvalue and corresponding eigenfunction has been obtained in [13] and [15]), then for each fixed \(\mu > \mu_1\) and small positive \(\varepsilon\), \(\varepsilon \phi < M\) and is a sub-solution. Therefore, by the sub- and super-solution method (see [14]), there exist at least one positive solution.

(Uniqueness) If \(u_1\) and \(u_2\) are two positive solutions, by Lemma 2.1 we have \(u_1 \leq u_2\) and \(u_2 \leq u_1\) both hold in \(\Omega\). Hence, \(u_1 = u_2\). This proves the uniqueness.

(Asymptotic behaviour) Given any compact subset \(K\) of \(\Omega\) and any small \(\varepsilon > 0\) such that \(\varepsilon < v_0(x) = [a(x)/b(x)]^{1/(p-1)}\) in \(\Omega\). Let
\[
v_\varepsilon(x) = \begin{cases} v_0(x) + \varepsilon & \text{in } K, \\
l(x) & \text{in } \Omega \setminus K, \\
0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]
where \(l(x)\) is nonnegative function such that \(v_\varepsilon\) is smooth in \(\Omega\) and satisfying \(D_0 := \text{supp}(v_\varepsilon) \subset \subset \Omega\). Thus, for any \(x \in \Omega\),
\[
|(-\Delta)^\alpha v_\varepsilon(x)| \leq \int_{\mathbb{R}^N} \frac{|v_\varepsilon(x) - v_\varepsilon(y)|}{|x - y|^{N+2\alpha}} dy
= \int_{\Omega} \frac{|v_\varepsilon(x) - v_\varepsilon(y)|}{|x - y|^{N+2\alpha}} dy + \int_{\mathbb{R}^N \setminus \Omega} \frac{|v_\varepsilon(x)|}{|x - y|^{N+2\alpha}} dy
\leq \int_{\Omega} \frac{|v_\varepsilon(x) - v_\varepsilon(y)|}{|x - y|^{N+2\alpha}} dy + \left( \int_{\mathbb{R}^N \setminus \Omega} \frac{1}{\text{dist}(y, \partial D_0)^{N+2\alpha}} dy \right) \|v_\varepsilon\|_{L^\infty(\mathbb{R}^N)}
\leq C
\]
for some positive constant \(C = C(\varepsilon)\) since \(v_\varepsilon\) is smooth and \(\text{dist}(\partial \Omega, \partial D_0) \geq \gamma > 0\). On the other hand, we notice that \(v_\varepsilon(a(x) - b(x)v_\varepsilon^{p-1}) \leq -\delta\) in \(\Omega\) for
some positive constant $\delta = \delta(\varepsilon)$. Hence, for all large $\mu$, $v_\varepsilon$ is a super-solution of our problem.

On the other hand, let $\phi$ be a positive eigenfunction corresponding to $\mu_1$. Then we can find a small neighborhood of $\partial \Omega$ in $\Omega$, say $U$, such that $\phi$ is very small in $U$. Therefore, for all $\mu > \mu_1 + 1$, we have

$$(-\Delta)^\alpha \phi = \mu_1 a(x) \phi - \mu \phi[a(x) - b(x)\phi^{p-1}] \quad \text{in } U. \quad (2.7)$$

By shrinking $U$ further if necessary, we can assume that $\bar{U} \cap K = \emptyset$ and $\phi < v_0 - \varepsilon$ in $U$. Next, we choose smooth function $w_\varepsilon$ as

$$w_\varepsilon(x) = \begin{cases} v_0(x) - \varepsilon & \text{in } K, \\ \phi(x) & \text{in } U, \\ l(x) & \text{in the rest of } \Omega, \\ 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $l$ is a positive function such that $w_\varepsilon$ is smooth in $\Omega$ and satisfying $l \leq v_0 - \varepsilon/2$. Moreover, we let

$$\phi(x) \leq w_\varepsilon(x) \quad \text{in } \Omega \quad (2.8)$$

otherwise we choose $\tilde{\phi} = \phi/C$ for some constant $C > 0$ large replace $\phi$. Then we can see that, for $x \in \Omega \setminus U$,

$$\begin{align*}
\|(-\Delta)^\alpha w_\varepsilon(x)\| & \leq \int_{\mathbb{R}^N} \frac{|w_\varepsilon(x) - w_\varepsilon(y)|}{|x-y|^{N+2\alpha}} dy \\
& = \int_{\Omega} \frac{|w_\varepsilon(x) - w_\varepsilon(y)|}{|x-y|^{N+2\alpha}} dy + \int_{\mathbb{R}^N \setminus \Omega} \frac{|w_\varepsilon(x) - w_\varepsilon(y)|}{|x-y|^{N+2\alpha}} dy \\
& = \int_{\Omega} \frac{|w_\varepsilon(x) - w_\varepsilon(y)|}{|x-y|^{N+2\alpha}} dy + \int_{\mathbb{R}^N \setminus \Omega} \frac{|w_\varepsilon(x)|}{|x-y|^{N+2\alpha}} dy \\
& = \int_{\Omega} \frac{|w_\varepsilon(x) - w_\varepsilon(y)|}{|x-y|^{N+2\alpha}} dy + \left( \int_{\mathbb{R}^N \setminus \Omega} \frac{1}{\text{dist}(y, \partial U \cap \Omega)^{N+2\alpha}} dy \right) \|w_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \\
& \leq C,
\end{align*}$$

for some positive constant $C = C(\varepsilon)$ since $\text{dist}(\partial \Omega, \partial U \cap \Omega) \geq \gamma > 0$. Moreover, we know $w_\varepsilon(a(x) - b(x)w_\varepsilon^{p-1}) \geq \delta$ in $\Omega \setminus U$ for some positive constant $\delta = \delta(\varepsilon)$. Therefore,

$$(-\Delta)^\alpha w_\varepsilon \leq \mu w_\varepsilon(a(x) - b(x)w_\varepsilon^{p-1}) \quad \text{in } \Omega \setminus U \quad (2.9)$$

for all large $\mu$. For $x \in U$, by (2.7) and (3.7), we have

$$\begin{align*}
(-\Delta)^\alpha w_\varepsilon(x) & = \int_{\Omega} \frac{w_\varepsilon(x) - w_\varepsilon(y)}{|x-y|^{N+2\alpha}} dy + \int_{\mathbb{R}^N \setminus \Omega} \frac{w_\varepsilon(x)}{|x-y|^{N+2\alpha}} dy
\end{align*}$$
\[
\begin{align*}
&\int_{\Omega} \phi(x) - w_\varepsilon(y)\,dy + \int_{\mathbb{R}^N \setminus \Omega} \frac{\phi(x)}{|x-y|^{N+2\alpha}}\,dy \\
\leq &\int_{\Omega} \phi(x) - \phi(y)\,dy + \int_{\mathbb{R}^N \setminus \Omega} \frac{\phi(x)}{|x-y|^{N+2\alpha}}\,dy \\
= &\ (-\Delta)^\alpha \phi(x) \\
\leq &\ \mu \phi[a(x) - b(x)\phi^{p-1}] \\
= &\ \mu w_\varepsilon[a(x) - b(x)w_\varepsilon^{p-1}], \\
\end{align*}
\]

for \( \mu > \mu_1 + 1 \). Finally, combining (2.9) and (2.10), we know \( w_\varepsilon \) is a sub-solution of our problem for all large \( \mu \).

Since \( w_\varepsilon < v_\varepsilon \), we deduce that \( w_\varepsilon \leq u_\mu < v_\varepsilon \) in \( \Omega \). In particular,

\[
[a(x)/b(x)]^{1/(p-1)} - \varepsilon \leq u_\mu \leq [a(x)/b(x)]^{1/(p-1)} + \varepsilon
\]

in \( K \) for all large \( \mu \). Hence, \( u_\mu \to [a(x)/b(x)]^{1/(p-1)} \) as \( \mu \to +\infty \) in \( K \), as required. \( \square \)

**Lemma 2.3** Let \( \Omega, a \) and \( b \) be as in Lemma 2.2. Suppose \( p \) verifies (1.3), then equation

\[
\begin{cases}
(-\Delta)^\alpha u = \mu u[a(x) - b(x)u^{p-1}] & \text{in } \Omega, \\
\lim_{x \to \partial \Omega} u = +\infty, & \text{in } \mathbb{R}^N \setminus \Omega
\end{cases}
\]

has at least one positive solution for each \( \mu > 0 \) if the measurable function \( g_\mu \) satisfying

\[
\int_{\mathbb{R}^N \setminus \Omega} \frac{g_\mu(y)}{1 + |y|^{N+2\alpha}}\,dy \leq C,
\]

where positive constant \( C \) is independent of \( \mu \). Furthermore, suppose \( u_\mu \) is a positive solution of (2.11), then \( u_\mu \) satisfies \( u_\mu \to [a(x)/b(x)]^{1/(p-1)} \) uniformly in any compact subset of \( \Omega \) as \( \mu \to +\infty \).

We first recall the following result in [4]. Assume that \( \delta > 0 \) such that the distance function \( d(x) = \text{dist}(x, \partial \Omega) \) is of \( C^2 \) in \( A_\delta = \{x \in \Omega: d(x) < \delta \} \) and define

\[
V_\tau(x) = \begin{cases}
l(x), & x \in \Omega \setminus A_\delta, \\
d(x)^\tau, & x \in A_\delta, \\
0, & x \in \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

where \( \tau \) is a parameter in \((-1, 0)\) and the function \( l \) is positive such that \( V_\tau \) is \( C^2 \) in \( \Omega \).
Proposition 2.2 ([4], Proposition 3.2) Assume that \( \Omega \) is a bounded, open subset of \( \mathbb{R}^N \) with a \( C^2 \) boundary. Then there exists \( \delta_1 \in (0, \delta) \) and a constant \( C > 1 \) such that if \( \tau \in (-1, \tau_0(\alpha)) \) where \( \tau_0(\alpha) \) is the unique solution of

\[
C(\tau) = \int_{0}^{+\infty} \chi_{(0,1)} |1-t|^{\alpha} + (1+t)^{\alpha} - 2dt
\]

for \( \tau \in (-1, 0) \) and \( \chi_{(0,1)} \) is the characteristic function of the interval \((0,1)\), then

\[
\frac{1}{C} d(x) r^{2\alpha} \leq -(-\Delta)^{\alpha} V_r(x) \leq C d(x) r^{-2\alpha}, \quad \text{for all } x \in A_{\delta_1}.
\]

Next, we will the existence result in Lemma 2.3 by applying Perrod’s method and thus we need to find ordered sub and super-solution of (2.11). As in [4], we begin with a simple lemma that reduce the problem to find them only in \( A_{\delta_1} \).

Lemma 2.4 Let \( \Omega, a \) and \( b \) be as in Lemma 2.2. Suppose \( U \) and \( W \) are order super and sub-solution of (2.11) in the sub-domain \( A_{\delta_1} \). Then there exists \( \lambda \) large such that \( U_{\lambda} = U + \lambda \eta \) and \( W_{\lambda} = W - \lambda \eta \) are ordered super and sub-solution of (2.11), where \( \eta \in C^\infty_0(\mathbb{R}^N) \) satisfying \( 0 \leq \eta \leq 1 \) and \( \text{supp}(\eta) \subset \Omega \ \backslash \ A_{\delta_1} \).

Proof. The proof is similar as Lemma 4.1 in [4] and we just need replace \( \bar{V} \) in Lemma 4.1 in [4] to \( \eta \) for our lemma. So we omit the proof here. \( \square \)

Now we in position to prove Lemma 2.3

Proof of Lemma 2.3 (Existence) We define

\[
G_{\mu}(x) = \frac{1}{2} \int_{\mathbb{R}^N} \frac{\hat{g}_\mu(x+y)}{|y|^{N+2\alpha}} dy \quad \text{for } x \in \Omega,
\]

where

\[
\hat{g}_\mu(x) = \begin{cases} 
0, & x \in \Omega, \\
g_\mu(x), & x \in \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

We observe that

\[
G(x) = -(-\Delta)^{\alpha} \hat{g}_\mu(x) \quad \text{for } x \in \Omega.
\]

Moreover, we know that \( G_{\mu} \) is continuous (see Lemma 2.1 in [4]) and nonnegative in \( \Omega \). Therefore, if \( u \) is a solution of (2.11), then \( u - \hat{g}_\mu \) is the solution of

\[
\begin{cases} 
(-\Delta)^{\alpha} u = \mu u [a(x) - b(x)u^{p-1}] + G_{\mu}(x) & \text{in } \Omega, \\
\lim_{x \to \Omega, x \to \partial \Omega} u = +\infty, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases} \quad (2.14)
\]
and vice versa, if \( u \) is a solution of \((2.14)\), then \( u + \tilde{g}_\mu \) is a solution of \((2.11)\).

Next, we will look for solution of \((2.14)\) instead of \((2.11)\).

Define
\[
U_\lambda(x) = \lambda V_\epsilon(x) \quad \text{and} \quad W_\lambda(x) = \lambda W_\epsilon(x),
\]
where \( \tau = -2\alpha/(p - 1) \). Notice that \( \tau = -2\alpha/(p - 1) \in (-1, \alpha - 1) \), \( \tau p = \tau - 2\alpha \) and \( \tau p < \tau < 0 \).

By Proposition 2.2 we find that for \( x \in A_\delta \) and \( \delta > 0 \), small
\[
(-\Delta)^\alpha U_\lambda + \mu b(x)U_\lambda^p - \mu a(x)U_\lambda - G_\mu(x)
\geq -C\lambda d(x)^{\tau - 2\alpha} + \mu b(x)\lambda d(x)^{\tau p} - \mu a(x)\lambda d(x)^\tau - G_\mu(x)
\geq -C\lambda d(x)^{\tau - 2\alpha} + \mu b(x)\lambda d(x)^{\tau p} - \mu a(x)\lambda d(x)^\tau - G_\mu(x),
\]
for some \( C > 0 \). Then there exists a large \( \lambda > 0 \) such that \( U_\lambda \) is a supersolution of \((2.14)\) with the first equation in \( A_\delta \) since \( G_\mu \) is continuous in \( \Omega \). Similarly, by Proposition 2.2 we have that for \( x \in A_\delta \) and \( \delta > 0 \), small
\[
(-\Delta)^\alpha W_\lambda + \mu b(x)W_\lambda^p - \mu a(x)W_\lambda - G_\mu(x)
\leq -\frac{\lambda}{C} d(x)^{\tau - 2\alpha} + \mu b(x)\lambda d(x)^{\tau p} - \mu a(x)\lambda d(x)^\tau - G_\mu(x)
\leq -\frac{\lambda}{C} d(x)^{\tau - 2\alpha} + \mu b(x)\lambda d(x)^{\tau p}
\leq 0,
\]
if \( \lambda > 0 \) small. Here we have used the fact \( G_\mu \) is nonnegative.

Finally, by using Lemma 2.4 there exists a solution \( \tilde{u}_\mu \) of problem \((2.14)\) and thus a solution \( u_\mu = \tilde{u}_\mu + \tilde{g}_\mu \) is a solution of \((2.11)\). Moreover, \( u_\mu > 0 \) in \( \Omega \). Indeed, since 0 is a sub-solution of \((2.11)\), by Lemma 2.1 we have \( u_\mu \geq 0 \) in \( \Omega \). If \( u_\mu(x') = 0 \) for some points \( x' \in \Omega \) and \( u_\mu \neq 0 \) in \( \mathbb{R}^N \), then by the definition of fractional Laplacian \(-\Delta)^\alpha u_\mu(x') < 0\) which is a contradiction. Therefore, \( u_\mu > 0 \) in \( \Omega \).

(Asymptotic behaviour) Let \( K \) be an arbitrary compact subset of \( \Omega \), \( v_0(x) = [a(x)/b(x)]^{1/(p - 1)} \) in \( \Omega \) and \( \varepsilon > 0 \) any small positive number satisfies \( v_0 > \varepsilon \) in \( \bar{\Omega} \). Define
\[
\tilde{w}_\varepsilon(x) = \begin{cases} 
 v_0(x) - \varepsilon + \lambda \eta(x) & \text{in } K, \\
 \mu^{-1} d(x)^\tau & \text{in } A_\delta, \\
 l(x) & \text{in the rest of } \Omega, \\
 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]
where \( \tau \) is a parameter in \((-1, 0)\), \( \lambda \) and \( \eta \) defined as in Lemma 2.4 and the function \( l \) is positive such that \( w_\varepsilon \) is \( C^2 \) in \( \Omega \). By a similar argument as
Proposition 3.2 in [4], there exists \( \delta \in (0, \delta) \) and constants \( c > 0 \) and \( C > 0 \) such that
\[
c(1 + \mu^{-1}d(x)^{\tau-2\alpha}) \leq -(-\Delta)^{\alpha} \tilde{w}_\varepsilon(x) \leq C(1 + \mu^{-1}d(x)^{\tau-2\alpha})
\]
for all \( x \in A_\delta \) and \( \tau \in (-1, \alpha - 1) \). Hence, for \( x \in A_\delta \) and \( \delta > 0 \) small,
\[
(-\Delta)^{\alpha} \tilde{w}_\varepsilon + \mu b(x)\tilde{w}_\varepsilon^\frac{\mu}{p} - \mu a(x)\tilde{w}_\varepsilon - G_\mu(x)
\leq -c\mu^{-1}d(x)^{\tau-2\alpha} + \mu^{1-p}\beta(x)d(x)\tau_p - a(x)\lambda d(x)^\tau - G_\mu(x) - c
\leq -c\mu^{-1}d(x)^{\tau-2\alpha} + \mu^{1-p}\beta(x)d(x)\tau_p
\leq 0,
\]
if \( \mu \) is large enough. Hence, \( \tilde{w}_\varepsilon \) is sub-solution in \( A_\delta \). By applying Lemma 2.4, we know that \( w_\varepsilon = \tilde{w}_\varepsilon - \lambda \eta \tilde{g}_\mu \) is a sub-solution of problem (2.11) for all large \( \mu > 0 \).

On the other hand, we define choose a function
\[
\tilde{v}_\varepsilon(x) = \begin{cases} 
  v_0(x) + \varepsilon - \lambda \eta & \text{in } K, \\
  \mu d(x)^\tau & \text{in } A_\delta, \\
  l(x) & \text{in the rest of } \Omega, \\
  0, & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]
where \( \tau \) is a parameter in \((-1,0)\), \( \lambda \) and \( \eta \) defined as in Lemma 2.4 and the function \( l \) is positive such that \( v_\varepsilon \) is \( C^2 \) in \( \Omega \). By a similar argument as Proposition 3.2 in [4], there exists \( \delta_1 \in (0, \delta) \) and constants \( c > 0 \) and \( C > 0 \) such that
\[
c(1 + \mu d(x)^{\tau-2\alpha}) \leq -(-\Delta)^{\alpha} \tilde{v}_\varepsilon(x) \leq C(1 + \mu d(x)^{\tau-2\alpha})
\]
for all \( x \in A_\delta \) and \( \tau \in (-1, \alpha - 1) \). Hence, for \( x \in A_\delta \) and \( \delta > 0 \) small,
\[
(-\Delta)^{\alpha} \tilde{v}_\varepsilon + \mu b(x)\tilde{v}_\varepsilon^\frac{\mu}{p} - \mu a(x)\tilde{v}_\varepsilon - G_\mu(x)
\geq -c\mu d(x)^{\tau-2\alpha} + \mu^{p+1}\beta(x)d(x)\tau_p - \mu^2 a(x)\lambda d(x)^\tau - G_\mu(x) - C
\geq -c\mu d(x)^{\tau-2\alpha} + \mu^{p+1}\beta(x)d(x)\tau_p - \mu^2 a(x)\lambda d(x)^\tau - G_\mu(x) - C
\geq 0,
\]
if \( \mu \) is large enough since \( \|G_\mu\|_{L^\infty(\Omega)} \leq C \) for some constant \( C > 0 \) independent of \( \mu \) by (2.14). Hence, \( \tilde{v}_\varepsilon \) is sub-solution in \( A_\delta \). By applying Lemma 2.4, we know that \( v_\varepsilon = \tilde{v}_\varepsilon + \lambda \eta + \tilde{g}_\mu \) is a super-solution of problem (2.11) for all large \( \mu > 0 \).

As \( w_\varepsilon < v_\varepsilon \) in \( \Omega \), we must have \( w_\varepsilon \leq u_\mu \leq v_\varepsilon \) in \( \Omega \). This implies that \( u_\mu \rightarrow v_0 \) in \( K \) as \( \mu \rightarrow \infty \), as required. We complete the proof of this Lemma. \( \square \)
3 Proofs

The main purpose of this section is to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let us first observe that a nonnegative entire solution of (1.2) is either identically zero or positive everywhere. Indeed, if \( u(x') = 0 \) for some points \( x' \in \mathbb{R}^N \) and \( u \not\equiv 0 \) in \( \mathbb{R}^N \), then by the definition of fractional Laplacian \( (-\Delta)^{\alpha}u(x') < 0 \) which is a contradiction. Therefore, we only consider positive solution.

Suppose \( \lambda > 0 \) and let \( u \) be an arbitrary positive entire solution of (1.2). We will show that \( u(x_0) = \lambda^{1/(p-1)} \) for any point \( x_0 \in \mathbb{R}^N \) by using pointwise convergence of Lemmas 2.2 and 2.3.

For any \( t > 0 \), define \( u_t(x) = u[x_0 + t(x - x_0)] \).

Then \( u_t \) satisfies
\[
(-\Delta)^{\alpha}u = t^{2\alpha}(\lambda u - u^p) \quad \text{in} \quad \mathbb{R}^N.
\]

Let \( B \) denote the unit ball with center \( x_0 \). By Lemma 2.2, for all large \( t \), the problem
\[
\begin{cases}
(-\Delta)^{\alpha}v = t^{2\alpha}u(\lambda - u^{p-1}) & \text{in } B, \\
u = 0 & \text{in } \mathbb{R}^N \setminus B
\end{cases}
\]
has a unique positive solution \( v_t \) and \( v_t \to \lambda^{1/(p-1)} \) as \( t \to \infty \) at \( x = x_0 \in B \).

By Lemma 2.1, we have that \( u_t \geq v_t \) in \( B \) and thus
\[
u(x_0) = u_t(x_0) \geq v_t(x_0).
\]

Letting \( t \to \infty \) in the above inequality we conclude that \( u(x_0) \geq \lambda^{1/(p-1)} \).

Let \( w_t \) be a positive solution of
\[
\begin{cases}
(-\Delta)^{\alpha}w = t^{2\alpha}w(\lambda - w^{p-1}) & \text{in } B, \\
\lim_{x \in B, x \to \partial B} w = +\infty, \\
w = u_t & \text{in } \mathbb{R}^N \setminus \bar{B}.
\end{cases}
\]

By our assumption, we know that
\[
\int_{\mathbb{R}^N} \frac{u_t(x)}{1 + |x|^{N+2\alpha}} dx \leq C,
\]
where constant \( C > 0 \) independent of \( t \) for \( t \) large enough. In fact
\[
\int_{\mathbb{R}^N} \frac{u_t(x)}{1 + |x|^{N+2\alpha}} dx = \int_{\mathbb{R}^N} \frac{u(x_0 + t(x - x_0))}{1 + |x|^{N+2\alpha}} dx \\
= \int_{\mathbb{R}^N} \frac{u(x)}{t^N \left(1 + \frac{|x+(t-1)x_0|}{t}^{N+2\alpha}\right)} dx.
\]
Define function
\[ f(t) = t^N \left( 1 + \left| \frac{x + (t - 1)x_0}{t} \right|^{N+2\alpha} \right), \]

we know \( f(1) = 1 + |x|^{N+2\alpha} \) and \( f(t) \to +\infty \) as \( t \to +\infty \). Then, we can choose \( t \) large enough such that \( f(t) \geq f(1) \). So by (3.2), for \( t \) large enough, we have
\[ \int_{\mathbb{R}^N} \frac{u_t(x)}{1 + |x|^{N+2\alpha}} dx \leq \int_{\mathbb{R}^N} \frac{u(x)}{1 + |x|^{N+2\alpha}} dx \leq C, \]
since \( u \in L^1(\mathbb{R}^N, \omega) \).

Then, applying Lemma 2.3, we see that \( w_t \to \lambda^{1/(p-1)} \) as \( t \to \infty \) at \( x = x_0 \in B \). Applying Lemma 2.1, we have that \( u_t \leq w_t \) in \( B \) and thus
\[ u(x_0) = u_t(x_0) \leq w_t(x_0). \]
Letting \( t \to \infty \) in the above inequality we conclude that \( u(x_0) \leq \lambda^{1/(p-1)} \).
Therefore, \( u(x_0) = \lambda^{1/(p-1)} \). Since \( x_0 \) is arbitrary, we conclude that \( u \equiv \lambda^{1/(p-1)} \) in \( \mathbb{R}^N \) for \( \lambda > 0 \), the unique constant solution of (1.2). □

Next, we will extend Theorem 1.1 to similar problem with variable coefficients, that is, Theorem 1.2. We first consider the following equation which is more general than (1.5):
\[ (-\Delta)^\alpha u = a(x)u - b(x)u^p, \quad x \in \mathbb{R}^N, \quad (3.3) \]
where \( a(x) \) and \( b(x) \) are continuous functions in \( \mathbb{R}^N \) and satisfying
\[ \lim_{|x| \to \infty} a(x) = a^\infty > 0, \quad \lim_{|x| \to \infty} b(x) = b^\infty > 0. \quad (3.4) \]
Here we allow \( a \) and \( b \) can be change sign which is more general than (1.5).

**Theorem 3.1** Under the above assumptions, if \( u \in C_{\text{loc}}^{2\alpha+\beta}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, \omega) \) for some \( \beta > 0 \) is a positive solution of (3.3) with \( p \) verifies (1.3), then
\[ \lim_{|x| \to \infty} u(x) = \left( \frac{a^\infty}{b^\infty} \right)^{\frac{1}{p-1}}. \]

We postpone the proof of Theorem 3.1 and first we use it to prove the following result.

**Corollary 3.1** Under the assumptions in Theorem 3.1, if we further assume that \( b \) is a nonnegative, then problem (3.3) has at most one positive solution.
**Proof.** Suppose $u_1$ and $u_2$ are two positive solutions of (3.3). By Theorem 3.1, we have

$$\lim_{|x| \to \infty} [(1 + \varepsilon)u_1 - u_2] = \varepsilon (a^\infty / b^\infty)^{1/(p-1)} > 0$$

for any positive constant $\varepsilon$.

Since $b$ is nonnegative, then $(1 + \varepsilon)u_1$ is a super solution of (3.3). Therefore, applying Lemma 2.1 in a large ball to conclude that $(1 + \varepsilon)u_1 \geq u_2$ in a large ball. It follows that this is true in all of $\mathbb{R}^N$. Hence, $u_1 \geq u_2$ in $\mathbb{R}^N$ since $\varepsilon$ is arbitrary. Similarly, we also can deduce $u_2 \geq u_1$ in $\mathbb{R}^N$. Finally, we must have $u_1 = u_2$ in $\mathbb{R}^N$, that is, (3.3) has at most one positive solution. □

Now we are in the position to prove Theorem 3.1.

**Proof of Theorem 3.1.** We prove it by a contradiction argument. Assume that there exists a sequence points $x_n \in \mathbb{R}^N$ satisfying $|x_n| \to \infty$ such that $|u(x_n) - (a^\infty / b^\infty)^{1/(p-1)}| \geq \varepsilon_0$ for some constant $\varepsilon_0 > 0$.

We define

$$a_n(x) = a(x_n + x), \quad b_n(x) = b(x_n + x) \quad \text{and} \quad u_n(x) = u(x_n + x).$$

Then $u_n$ satisfies

$$(-\Delta)^\alpha u_n = a_n(x)u_n - b_n(x)u_n^p \quad \text{in} \quad \mathbb{R}^N. \quad (3.5)$$

If we let

$$L^n_\alpha u = -(-\Delta)^\alpha u - (a^\infty - a_n)u,$$

then we can rewrite (3.5) as

$$-L^n_\alpha u_n = a^\infty u_n - b_n(x)u_n^p \quad \text{in} \quad \mathbb{R}^N.$$

Next, we fix a ball $B_r = \{ x \in \mathbb{R}^N \mid |x| < r \}$ and consider the following problem

$$\begin{cases}
-L^n_\alpha w = a^\infty w - b_n w^p & \text{in} \ B_r, \\
w = 0 & \text{in} \ \mathbb{R}^N \setminus B_r. \quad (3.6)
\end{cases}$$

By using the variational characterization of the first eigenvalue and (3.4), we see that $\lambda_1(-L^n_\alpha, B_r) \to \lambda_1((-\Delta)^\alpha, B_r)$ as $n \to \infty$, where $\lambda_1(-L^n_\alpha, B_r)$ and $\lambda_1((-\Delta)^\alpha, B_r)$ denote the first eigenvalues of $-L^n_\alpha$ and $(-\Delta)^\alpha$ in $B_r$ with Dirichlet boundary conditions in $\mathbb{R}^N \setminus B_r$, respectively. Since we can choose $r$ large enough such that $\lambda_1((-\Delta)^\alpha, B_r) < a^\infty$, then we may assume that $\lambda_1(-L^n_\alpha, B_r) < a^\infty$ for all $n$. On the other hand, we know that $b_n \to b^\infty$.
uniformly in $B_r$ and thus we may also assume that $b_n \geq b^\infty/2$ in $B_r$ for all $n$.

Let $\phi_n \in X^\beta_0(B_r)$ be the first eigenfunction corresponding to $\lambda_1(-L^n_\alpha, B_r)$, that is,

$$\begin{cases} (-\Delta)^\alpha \phi_n + (a^\infty - a_n)\phi_n = \lambda_1(-L^n_\alpha, B_r)\phi_n & \text{in } B_r, \\ \phi_n = 0 & \text{in } \mathbb{R}^N \setminus B_r, \end{cases} \quad (3.7)$$

with $\|\phi_n\|_{L^\infty(B_r)} = 1$. By Theorems 1 and 2 in [16] and using (3.4), we know $\phi_n$ is also a viscosity solution of (3.7). Then by Theorem 2.6 in [8], we have $\phi_n \in C^{\beta}_{\text{loc}}(B_r)$. Then, by Corollary 4.6 in [7], $\phi_n$ converges uniformly to a $\phi_\infty$ and $\phi_\infty$ satisfies

$$\begin{cases} (-\Delta)^\alpha \phi_\infty = \lambda_1((-\Delta)^\alpha, B_r)\phi_\infty & \text{in } B_r, \\ \phi_\infty = 0 & \text{in } \mathbb{R}^N \setminus B_r. \end{cases}$$

in viscosity sense. Next, by a similar argument as Theorem 2.1 in [4], we know $\phi_\infty \in C^{2\alpha+\beta}_{\text{loc}}(B_r)$ and is a classical solution. Then $\phi_\infty$ is the normalized positive eigenfunction corresponding to $\lambda_1((-\Delta)^\alpha, B_r)$.

It is easily to check that $\varepsilon \phi_n$ is a subsolution of (3.6) for every $n$ if we choose $\varepsilon$ small enough. Furthermore, $(2a^\infty/b^\infty)^{1/(p-1)}$ is a supersolution of (3.6) for all $n$. Then (3.6) has a positive solution $w_n$ satisfies $\varepsilon \phi_n \leq w_n \leq (2a^\infty/b^\infty)^{1/(p-1)}$. Then, using the regularity results again, we know $w_n$ converges in $C^{2\alpha+\beta}_{\text{loc}}(B_r)$ to some function $w$ satisfying $\varepsilon \phi_\infty \leq w \leq (2a^\infty/b^\infty)^{1/(p-1)}$ and

$$\begin{cases} (-\Delta)^\alpha w = a^\infty w - b^\infty w^p & \text{in } B_r, \\ w = 0 & \text{in } \mathbb{R}^N \setminus B_r. \end{cases}$$

Applying Lemma 2.2, we know the above problem has a unique positive solution. Therefore, $w = w_r$ is uniquely determined and the whole sequence $w_n$ converges to $w_r$.

By the comparison principle (see Lemma 2.1), we know that

$$u_n \geq w_n \rightarrow w_r \text{ in } B_r. \quad (3.8)$$

Next, we show $u_n$ has a uniformly bounded in $\mathbb{R}^N$ for all $n$ large enough, that is, there exists a positive constant $C$ independent of $n$ such that $u_n(x_0) \leq C$ for any $x_0 \in \mathbb{R}^N$. We define, for any $t > 0$,

$$u_{t,n}(x) = u_n(x_0 + t(x - x_0)).$$

Then $u_{t,n}$ satisfying

$$(-\Delta)^\alpha u = t^{2\alpha}(\tilde{a}_n u - \tilde{b}_n u^p) \text{ in } \mathbb{R}^N,$$
where \( \tilde{a}_n(x) = a_n(x_0 + t(x - x_0)) \) and \( \tilde{b}_n(x) = b_n(x_0 + t(x - x_0)) \). On the other hand, since \( \tilde{a}_n \to a^\infty \) and \( \tilde{b}_n \to b^\infty \) uniformly in \( B \) where \( B \) denote the unit ball with center \( x_0 \), we may assume \( \tilde{a}_n \leq 2a^\infty \) and \( \tilde{b}_n \geq b^\infty / 2 \) in \( B \) for all \( n \).

We consider the following problem

\[
\begin{cases}
(-\Delta)^{\alpha} v = t^{2\alpha}(2a^\infty v - (b^\infty / 2)v^p) & \text{in } B, \\
\lim_{x \in B, x \to \partial B} v = +\infty, \\
v = u_{t,n} & \text{in } \mathbb{R}^N \setminus B.
\end{cases}
\] (3.9)

As a argument before, we know \( u_{t,n} \in L^1(\mathbb{R}^N, \omega) \) for \( t \) and \( n \) large enough. Thus, by applying Lemma 2.3, we know this problem has at least one positive solution. Let \( v_t \) be a solution of (3.9), then \( v_t \to (4a^\infty / b^\infty)^{1/(p-1)} \) as \( t \to \infty \) at \( x = x_0 \in B \). Then the comparison principle deduce that \( u_{t,n} \leq v_t \) in \( B \) and thus

\[ u_n(x_0) = u_{t,n}(x_0) \leq v_t(x_0). \]

Letting \( t \to \infty \) in the above inequality we conclude that \( u_n(x_0) \leq (4a^\infty / b^\infty)^{1/(p-1)} \) as we required.

Hence, \( \|u_n\|_{L^\infty(\mathbb{R}^N)} \leq C \) for all \( n \) large enough, where constant \( C > 0 \) independent of \( n \). On the other hand, \( u_n \in C_{loc}^{2\alpha + \beta}(\mathbb{R}^N) \) implies that \( u_n \) converges uniformly to some function \( u^\infty \) and

\[ (-\Delta)^{\alpha} u_n \to (-\Delta)^{\alpha} u^\infty \quad \text{in } B_r \]

is strongly as \( n \to +\infty \). Hence, \( u^\infty \) is nonnegative and satisfies

\[ (-\Delta)^{\alpha} u = a^\infty u - b^\infty u^p \quad \text{in } B_r. \]

Furthermore, \( u^\infty \geq w_r > 0 \). Thus \( u^\infty \) is a positive solution and \( |u^\infty(0) - (a^\infty / b^\infty)^{1/(p-1)}| \geq \varepsilon_0 \) due to the choice of \( x_0 \).

Choose a sequence \( r = r_1 \leq r_2 \leq \cdots \leq r_m \to \infty \) as \( m \to \infty \). We can apply the above argument to each \( r_m \) and then use a diagonal process to obtain a positive solution \( U \) of

\[ (-\Delta)^{\alpha} u = a^\infty u - b^\infty u^p \quad \text{in } \mathbb{R}^N, \] (3.10)

which satisfies \( U(0) \geq w_{r_m}(0) \) and \( |U(0) - (a^\infty / b^\infty)^{1/(p-1)}| \geq \varepsilon_0 \). By changing of variables of the form \( x = \theta y, \theta \in \mathbb{R}, \) then (3.10) can write as

\[ (-\Delta)^{\alpha} v = (a^\infty / b^\infty)v - v^p \quad \text{in } \mathbb{R}^N, \]

where \( v(y) = u(x) = u(\theta y) \). In fact, we can choose \( \theta = (b^\infty)^{-1/(2\alpha)} \). Then applying Theorem L.1 to the above equation, we have \( v \equiv (a^\infty / b^\infty)^{1/(p-1)}. \)
Hence, \( u \equiv (a^{\infty}/b^{\infty})^{1/(p-1)} \). This a contradiction. We complete the proof. \( \square \)

**Proof of Theorem 1.2** First, we let \( \lambda > 0 \). We consider the following eigenvalue problem with weight function:

\[
\begin{cases}
(-\Delta)^{\alpha} u = \lambda a(x) u & \text{in } B_r, \\
u = 0 & \text{in } \mathbb{R}^N \setminus B_r.
\end{cases}
\]

We denote \( \mu_1 \) be the first eigenvalue of this problem. Since \( \mu_1 \to 0 \) as \( r \to \infty \), we can choose \( r_1 > 0 \) large enough such that \( \mu_1 \leq \lambda \) when \( r \geq r_1 \). So we can choose an increasing sequence \( r_1 < r_2 < \cdots < r_n \to \infty \) and consider the following problem

\[
\begin{cases}
(-\Delta)^{\alpha} u = \lambda a(x) u - b(x) u^p & \text{in } B_{r_n}, \\
u = 0 & \text{in } \mathbb{R}^N \setminus B_{r_n},
\end{cases}
\]

(3.11)

where \( B_{r_n} = B_{r_{n}} \). By Lemma 2.2, problem (3.11) has a unique positive solution \( u_n \) for each \( n \). Furthermore, by the comparison principle (see Lemma 2.1), we know \( u_n \leq u_{n+1} \). On the other hand, any positive constant \( M \) satisfying \( M^{p-1} \geq M_0^{p-1} = \lambda \sup_{\mathbb{R}^N} a(x)/\inf_{\mathbb{R}^N} b(x) \) is a supersolution of (3.11). It follows that \( u_n \leq M_0 \) for all \( n \). Therefore, \( u_n \) is increasing in \( n \) and \( u_{\infty}(x) = \lim_{n \to \infty} u_n(x) \) is well defined in \( \mathbb{R}^N \). Then, \( u_{\infty} \) satisfying (1.5). Since \( u_{\infty} \geq u_n > 0 \) in \( B_{r_n} \) for each \( n \), we know that \( u_{\infty} \) is a positive solution of (1.5). Moreover, by Corollary 3.1, \( u_{\infty} \) is the unique solution of (1.5). We complete the proof. \( \square \)

4 Acknowledgements

A. Quaas was partially supported by Fondecyt Grant No. 1151180 Programa Basal, CMM. U. de Chile and Millennium Nucleus Center for Analysis of PDE NC130017.

References

[1] N. Abatangelo, On a fractional keller osserman condition. arXiv:1412.6298 [math.AP].

[2] N. Abatangelo, Large s-harmonic functions and boundary blow-up solutions for the fractional Laplacian. To appear in Discrete Contin. Dyn. Syst. DOI: 10.3934/dcds.2015.35.5555
[3] G. Barles, E. Chasseigne and C. Imbert, On the Dirichlet problem for second-order elliptic integro-differential equations. Indiana Univ. Math. J. 57 (2008), no. 1, 213-246.

[4] H. Chen, P. Felmer and A. Quaas, Large solutions to elliptic equations involving fractional Laplacian. Annales de l’Institut Henri Poincare (C) Non Linear Analysis, 2013.

[5] H. Chen, H. Hajaiej and Y. Wang, Boundary blow-up solutions to fractional elliptic equations in a measure framework. Discrete Contin. Dyn. Syst. 36 (2016), no. 4, 1881-1903.

[6] M.G Crandall, H.Ishii and P.L Lions, Users guide to viscosity solutions of second order partial differential equations. Bull. Amer. Soc. 27 (1992), pp 1-67.

[7] L. Caffarelli and L. Silvestre, Regularity theory for fully nonlinear integro-differential equations. Comm. Pure Appl. Math. 62 (2009), no. 5, 597-638.

[8] L. Caffarelli and L. Silvestre, Regularity results for nonlocal equations by approximation. Arch. Ration. Mech. Anal. 200 (2011), no. 1, 59-88.

[9] Y. Du and Z. Guo, Boundary blow-up solutions and their applications in quasilinear elliptic equations. J. Anal. Math. 89 (2003), 277-302.

[10] Y. Du and L. Ma, Logistic type equations on $\mathbb{R}^N$ by a squeezing method involving boundary blow up solutions. J. London Math. Soc. (2) 64 (2001), 107-124.

[11] P. Felmer and A. Quaas, Boundary blow up solutions for fractional elliptic equations. Asymptot. Anal. 78 (2012), no. 3, 123-144.

[12] A. Fiscella, R. Servadei, and E. Valdinoci, Density properties for fractional Sobolev spaces. Ann. Acad. Sci. Fenn. Math. 40 (2015), no. 1, 235-253.

[13] L. del Pezzo and A. Quaas, Global bifurcation for fractional $p$-Laplacian and application. arXiv:1412.3722 [math.AP]

[14] M. Struwe, Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems. Springer-Verlag, Berlin, 1990. xiv+244 pp.
[15] R. Servadei and E. Valdinoci, Variational methods for non-local operators of elliptic type. Discrete Contin. Dyn. Syst. 33 (2013), no. 5, 2105-2137.

[16] R. Servadei and E. Valdinoci, Weak and viscosity solutions of the fractional Laplace equation. Publ. Mat. 58 (2014), no. 1, 133-154.