On global dynamics of 2D convective Cahn–Hilliard equation

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Abstract

In this paper, we study the long time behavior of solution for the initial-boundary value problem of convective Cahn–Hilliard equation in a 2D case. We show that the equation has a global attractor in $H^4(\Omega)$ when the initial value belongs to $H^1(\Omega)$.

Keywords: Global attractor; Convective Cahn–Hilliard equation; Absorbing set

1 Introduction

The dynamic properties of diffusion equations ensure the stability of diffusion phenomena and provide the mathematical foundation for the study of diffusion dynamics. There are many studies on the existence of global attractors for diffusion equations. For the classical results, we refer the reader to [1–9].

The convective Cahn–Hilliard equation [10–16], which arises naturally as a continuous model for the formation of facets and corners in crystal growth, is a typical fourth order nonlinear parabolic equation. Let $\Omega = [0, L] \times [0, L]$, where $L > 0$, $\gamma$ is a positive constant, $\vec{\beta}$ is a vector. We consider the convective Cahn–Hilliard equation in the 2D case:

$$u_t + \gamma \Delta^2 u = \Delta \phi(u) - \vec{\beta} \cdot \nabla \psi(u), \quad x = (x_1, x_2) \in \mathbb{R}^2, t \geq 0. \quad (1)$$

Equation (1) is supplemented by the following boundary conditions:

$$u(x_1 + L, x_2, t) = u(x_1, x_2 + L, t) = u(x_1, x_2, t), \quad x \in \mathbb{R}^2, t \geq 0, \quad (2)$$

and the initial condition

$$u(x, 0) = u_0(x). \quad (3)$$

In this paper, we denote by $H = L^2(\Omega)$, $(\cdot, \cdot)$ the $H$-inner product and by $\| \cdot \|$ the corresponding $H$-norm, denote $A = -\Delta$, where $\Delta$ is the Laplace operator. Assume that the initial function has zero mean, i.e., $\int_{\Omega} u_0(x) \, dx = 0$, then it follows that $\int_{\Omega} u(x, t) \, dx = 0$ for $t > 0$. Here, as [3], we set

$$H^k_{\text{per}} = \left\{ u \mid u \in H^k_{\text{per}}(\Omega), \int_{\Omega} u(x, t) \, dx = 0, \right\}, \quad k = 1, 2, \ldots.$$
Using the same method as [13], we obtain the lemma on the existence of global weak solution to problem (1)–(3).

**Lemma 1.1** Suppose that $u_0 \in \dot{H}^1_{\text{per}}(\Omega)$ and the functions $\varphi(r) \in C^2(\mathbb{R})$, $\psi(r) \in C^1(\mathbb{R})$ satisfy

$$
\varphi'(r) > 0, \quad \varphi^{(i)} \leq c_0 r^{k-i} + c_1, \quad \psi'(r) \leq c_0 r \sqrt{\varphi(r)} + c_1,
$$

where $k \leq 3$ is a positive constant and $i = 0, 1, 2$. Then there exists a unique solution $u$ for problem (1)–(3) such that

$$
u \in C(\mathbb{R}_+; \dot{H}^1_{\text{per}}(\Omega)) \cap L^2_{\text{loc}}(\mathbb{R}_+; \dot{H}^2_{\text{per}}(\Omega)).$$

By Lemma 1.1, we can define the operator semigroup $S(t)u_0 : \dot{H}^1_{\text{per}}(\Omega) \times \mathbb{R}_+ \rightarrow \dot{H}^1_{\text{per}}(\Omega)$, which is $(\dot{H}^1_{\text{per}}, \dot{H}^1_{\text{per}})$-continuous. In what follows, we always assume that $\{S(t)\}_{t \geq 0}$ is the semigroup generated by the weak solutions of problem (1). It is sufficient to see that the restriction of $\{S(t)\}$ on the affined space $\dot{H}^1_{\text{per}}(\Omega)$ is a well-defined semigroup.

**Proposition 1.2** ([17–19]) Suppose that $\mathcal{A}$ is an $(H^1, H^1)$-global attractor for $\{S(t)\}_{t \geq 0}$. Suppose further that $(S(t))_{t \geq 0}$ has a bounded $(H^1, H^1)$-absorbing set and $\{S(t)\}_{t \geq 0}$ is $(H^1, H^4)$-asymptotically compact. Then $\mathcal{A}$ is also an $(H^1, H^4)$-global attractor.

The main result of this paper will be stated in the following.

**Theorem 1.3** Suppose that $u_0 \in \dot{H}^1_{\text{per}}(\Omega)$ and the functions $\varphi(r) \in C^3(\mathbb{R})$, $\psi(r) \in C^2(\mathbb{R})$ satisfy

$$
\varphi'(r) > 0, \quad \varphi^{(i)} \leq c_0' r^{k-i} + c'_1, \quad \psi'(r) \leq c_0' r \sqrt{\varphi(r)} + c_1,
$$

where $k \leq 3$ is a positive constant and $i = 0, 1, 2$. Then there exists an $(H^1, H^4)$-global attractor for the solution $u(x, t)$ of problem (1)–(3), which is invariant and compact in $H^1(\Omega)$ and attracts every bounded subset of $H^1(\Omega)$ with respect to the norm topology of $H^4(\Omega)$.

**Remark 1.4** In the previous papers [18, 20, 21], my cooperators and I also studied the existence of global attractor for a 2D convective Cahn–Hilliard equation. There are two main differences between the previous results and Theorem 1.3. First, in [18, 20], we assumed that there exists double-well potential for the convective Cahn–Hilliard equation, which was replaced by the higher order polynomial in [21]. But, in this paper, this assumption is changed by (4), which seems more abroad than double-well potential and polynomial. Second, in [18], the existence of $(H^2, H^2)$-global attractor was obtained, and in [20, 21], the existence of $(H^4, H^4)$-global attractor was proved. In this paper, we only assume that the initial data belongs to $H^1(\Omega)$ and obtain the $(H^1, H^4)$-global attractor for the 2D convective Cahn–Hilliard equation.

The remaining parts are organized as follows. We begin by giving some uniform estimates of solutions for the 2D convective Cahn–Hilliard equation in Sect. 2. Then, in Sect. 3, we prove the main results on the existence of global attractor.
2 Uniform estimates of solutions

First of all, we establish the uniform estimates of solutions of problem (1) as \( t \to \infty \). These estimates are necessary to prove the existence of global attractors.

**Lemma 2.1** Suppose that \( u_0 \in L^2(\Omega) \) and the functions \( \varphi(r) \in C^1(\mathbb{R}) \), \( \psi(r) \in C^1(\mathbb{R}) \) satisfy

\[
\varphi'(r) > 0, \quad \psi'(r) \leq c_0 r \sqrt{\varphi'(r)} + c_1.
\]

Then, for problem (1)–(3), we have

\[
\|u(t)\| \leq M_0, \quad \forall t \geq T_0,
\]

and

\[
\int_t^{t+1} \|Au(t)\|^2 \, d\tau \leq M_0, \quad t \geq T_0.
\]

Here, \( M_0 \) is a positive constant depending on \( \gamma \) and \( c_i \) \( (i = 0, 1) \). \( T_0 \) depends on \( \gamma \), \( c_i \) \( (i = 0, 1) \) and \( R \), where \( \|u_0\|^2 \leq R^2 \).

**Proof** Multiplying equation (1) by \( u \) and integrating the resulting relation over \( \Omega \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 + \gamma \|\Delta u\|^2 + \int_{\Omega} \varphi'(u) \|\nabla u\|^2 \, dx = \beta \cdot \int_{\Omega} \psi'(u) u \nabla u \, dx.
\]  

(5)

Note that

\[
\beta \cdot \int_{\Omega} \psi'(u) u \nabla u \, dx = \beta \cdot \int_{\Omega} \psi'(u) u \nabla u \, dx \\
\leq c_2 |\beta| \int_{\Omega} |u \nabla u \sqrt{\varphi'(u)}| \, dx + c_3 |\beta| \int_{\Omega} |u| \, dx \\
\leq \frac{1}{2} \int_{\Omega} \varphi'(u) \|\nabla u\|^2 \, dx + \frac{c_2}{2} \|u\|^2 + \frac{c_3}{2}.
\]

Hence

\[
\frac{d}{dt} \|u\|^2 + 2\gamma \|\Delta u\|^2 + \int_{\Omega} \varphi'(u) \|\nabla u\|^2 \, dx \leq c_2 \|u\|^2 + c_3.
\]  

(6)

Applying Poincaré’s inequality, we arrive at

\[
\|u\|^2 \leq c' \|\nabla u\|^2.
\]

Moreover,

\[
c' \|\nabla u\|^2 = -c' \int_{\Omega} u \Delta u \, dx \leq \frac{1}{2} \|u\|^2 + \frac{(c')^2}{2} \|\Delta u\|^2.
\]

Therefore, the following inequality holds:

\[
\|u\|^2 \leq (c')^2 \|\Delta u\|^2.
\]
Summing up, we get
\[
\frac{d}{dt}\|u\|^2 + \left(\frac{2\gamma}{(c')^2} - c_4\right)\|u\|^2 \leq c_5, \tag{7}
\]
where \(\gamma\) satisfies \(\frac{2\gamma}{(c')^2} - c_4 > 0\). Using Gronwall’s inequality, we deduce that
\[
\|u\|^2 \leq e^{-\frac{(2\gamma)(c')^2}{2\gamma - c_4(c')^2}}\|u_0\|^2 + \frac{c_3(c')^2}{2\gamma - c_4(c')^2} \leq \frac{2c_3(c')^2}{2\gamma - c_4(c')^2} \tag{8}
\]
for all \(t \geq T^* = \frac{(c')^2}{2\gamma - c_4(c')^2} \ln \left[\frac{2\gamma - c_4(c')^2}{c_3(c')^2}\right] R^2\).

Integrating (6) over \((t, t + 1)\) with \(t \geq T^*\) yields
\[
\int_{t}^{t+1} \|\Delta u\|^2 d\tau \leq c_4. \tag{9}
\]

By using a mean value theorem for integrals, we obtain the existence of a time \(t'_0 \in (T^*, T^* + 1)\) such that
\[
\|\Delta u(t'_0)\|^2 \leq c_5
\]
holds uniformly, the proof is complete. □

**Lemma 2.2** Suppose that \(u_0 \in H^1_{per}(\Omega)\) and the functions \(\psi(r) \in C^2(\mathbb{R})\), \(\psi(r) \in C^1(\mathbb{R})\) satisfy
\[
\psi'(r) > 0, \quad \psi^{(i)} \leq c_0 r^{k-i} + c_1, \quad \psi'(r) \leq c_0 r \sqrt{\psi(r)} + c_1,
\]
where \(k \leq 3\) is a positive constant and \(i = 0, 1, 2\). Then, for problem (1)–(3), we have
\[
\|\nabla u(t)\| \leq M_1, \quad \forall t \geq T_1,
\]
and
\[
\int_{t}^{t+1} \|\nabla u(t)\|^2 d\tau \leq M_1, \quad t \geq T_1.
\]

Here, \(M_1\) is a positive constant depending on \(\gamma\) and \(c_i, c'_i\) (\(i = 0, 1\)). \(T_1\) depends on \(\gamma, c_i, c'_i\) (\(i = 0, 1\)) and \(R\), where \(\|u_0\|^2_{H^1_{per}} \leq R^2\).

**Proof** Multiplying equation (1) by \(-\Delta u\) and integrating the resulting relation over \(\Omega\) yields
\[
\frac{1}{2} \frac{d}{dt}\|\nabla u\|^2 + \gamma \|\nabla u\|^2 = -\int_{\Omega} \Delta \psi(u)\Delta u \, dx - \beta \cdot \int_{\Omega} \nabla \psi(u)\nabla u \, dx
\]
\[
= -\int_{\Omega} \psi'(u)\Delta u^2 \, dx - \int_{\Omega} \psi''(u)|\nabla u|^2 \Delta u \, dx
\]
\[
- \beta \cdot \int_{\Omega} \psi'(u)|\nabla u|^2 \Delta u \, dx.
\]
Hence
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + \gamma \|\nabla \Delta u\|^2 + \int_\Omega \varphi'(u) \Delta u^2 dx \\
= -\int_\Omega \varphi''(u)\Delta u^2 dx - \beta \int_\Omega \varphi'(u) \Delta u dx \\
\leq c \int_\Omega |\Delta u| |\nabla u|^2 dx + c|\beta| \int_\Omega |u^2\sqrt{\varphi(u)} \nabla \Delta u| dx + c\|\nabla u\|^2 \\
\leq \frac{c}{2} \int_\Omega |\nabla u|^4 dx + \frac{c}{2} \int_\Omega |\Delta u|^2 dx + \int_\Omega \varphi'(u) |\Delta u|^2 dx + c^2|\beta|^2 \int_\Omega u^4 |\nabla u|^2 dx \\
+ \frac{c_6}{2} \|\nabla u\|^2 .
\]

By Nirenberg's inequality, we obtain
\[
\|u\|_4 \leq c_1 \|\nabla u\|_2 \|u\|_\frac{5}{2} + c_2 \|u\| , \quad \|\nabla u\|_4 \leq c_1' \|\nabla \Delta u\|_\frac{1}{2} \|u\|_2 + c_2' \|u\| , \\
\|\Delta u\|_8 \leq c_1' \|\nabla \Delta u\|_2 \|u\|_\frac{3}{2} + c_2' \|u\| , \quad \|\Delta u\|_4 \leq c_1' \|\nabla \Delta u\|_\frac{5}{2} \|u\|_1 + c_2' \|u\| .
\]

Thus, by Hölder's inequality and the above inequalities, we deduce that
\[
\frac{c}{2} \int_\Omega |\nabla u|^4 dx + \frac{c}{2} \int_\Omega |\Delta u|^2 dx + \frac{c^2|\beta|^2}{4} \int_\Omega u^4 |\nabla u|^2 dx \leq \frac{\gamma}{2} \|\nabla \Delta u\|^2 + \frac{c_7}{2} .
\]

Summing up, we obtain
\[
\frac{d}{dt} \|\nabla u\|^2 + \gamma \|\nabla \Delta u\|^2 \leq c_6 \|\nabla u\|^2 + c_7 . \quad (10)
\]

On the other hand,
\[
\|\nabla u\|^2 = -\int_\Omega u \Delta u dx \leq \|u\| \|\Delta u\| \leq \sqrt{\frac{2c_3(c')^2}{2\gamma - c_2(c')^2}} \|\Delta u\| 
\]

and
\[
\|\Delta u\|^2 = -\int_\Omega \nabla u \cdot \nabla \Delta u dx \leq \|\nabla u\| \|\nabla \Delta u\| .
\]

Adding the above two inequalities together gives
\[
c_6 \|\nabla u\|^2 \leq c \|\nabla \Delta u\|_\frac{3}{2} \|\nabla u\| \leq \frac{\gamma}{2} \|\nabla \Delta u\|^2 + c_8 . \quad (11)
\]

It then follows from (10) and (11) that
\[
\frac{d}{dt} \|\nabla u\|^2 + \frac{\gamma}{2} \|\nabla \Delta u\|^2 \leq c_7 + c_8 .
\]

Applying Gronwall's inequality yields
\[
\|\nabla u\|^2 \leq e^{-\frac{\gamma}{2} t} \|\nabla u_0\|^2 + \frac{2(c_7 + c_8)}{\gamma} \leq \frac{4(c_7 + c_8)}{\gamma} \|\nabla u_0\|^2 . \quad (12)
\]
for all $t \geq T' = \max\{T^*, \frac{2}{\gamma} \ln \frac{\gamma R^2}{2(c_7 + c_8)}\}$. Integrating (10) over $(t, t + 1)$ with $t \geq T'$ gives

$$\int_t^{t+1} \|\nabla u\|^2 \, d\tau \leq c_9.$$ 

Using a mean value theorem for integrals, we obtain the existence of a time $t_0 \in (T', T' + 1)$ such that

$$\|\nabla u(t_0)\|^2 \leq c_{10}$$

holds uniformly. Since we consider problem (1)–(3) in the 2D case, based on Sobolev’s embedding theorem, we can get

$$\|u\|_p = \left(\int_\Omega u^p \, dx\right)^{\frac{1}{p}} \leq c_{11}, \quad 1 \leq p < \infty.$$ 

Set $T_1 = T'$, we complete the proof. □

**Lemma 2.3** Suppose that $u_0 \in H^1_{\text{per}}(\Omega)$ and the functions $\varphi(r) \in C^2(\mathbb{R})$, $\psi(r) \in C^1(\mathbb{R})$ satisfy

$$\varphi'(r) > 0, \quad \varphi^{(i)}(r) \leq c_0^i |r|^{k-i} + c_1, \quad \varphi'(r) \leq c_0 \sqrt{\varphi'(r)} + c_1,$$

where $k \leq 3$ is a positive constant and $i = 0, 1, 2$. Then, for problem (1)–(3), we have

$$\|Au(t)\| \leq M_2, \quad \forall t \geq T_2,$$

and

$$\int_t^{t+1} \|u_t\|^2 \, d\tau \leq M_2, \quad t \geq T_2.$$ 

Here, $M_2$ is a positive constant depending on $\gamma$ and $c_i, c_i'$ ($i = 0, 1, 2$), $T_2$ depends on $\gamma, c_i, c_i'$ ($i = 0, 1$) and $R$, where $\|u_0\|^2_{H^1_{\text{per}}} \leq R^2$.

**Proof** Multiplying equation (1) by $\Delta^2 u$ and integrating the resulting relation over $\Omega$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\Delta u\|^2 + \gamma \|\Delta^2 u\|^2 = (\Delta \varphi(u), \Delta^2 u) + (\beta \cdot \nabla \varphi(u), \Delta^2 u)$$

$$= (\varphi'(u) \Delta u + \varphi''(u) |\nabla u|^2, \Delta^2 u) + \beta \cdot (\varphi(u) \nabla u, \Delta^2 u)$$

$$\leq \frac{\gamma}{2} \|\Delta^2 u\|^2 + \frac{2}{\gamma} \|\varphi(u) \Delta u\|^2 + \frac{2}{\gamma} \|\varphi''(u) |\nabla u|^2\|^2 + \frac{1}{\gamma} \|\varphi'(u) \nabla u\|^2.$$
Simple calculation shows that
\[
\frac{d}{dt} \| \Delta u \|^2 + \gamma \| \Delta^2 u \|^2 \\
\leq \frac{4}{\gamma} \int_{\Omega} \left| \frac{\partial^2 u}{\partial t^2} \right|^2 \, dx + \frac{4}{\gamma} \int_{\Omega} \left| \frac{\partial^3 u}{\partial t^3} \right|^2 \, dx + c \int_{\Omega} u^2 \left| \nabla \frac{\partial u}{\partial t} \right|^2 \, dx + c \| \nabla u \|^2 \\
\leq c \left( \int_{\Omega} \left| \frac{\partial^2 u}{\partial t^2} \right|^2 \, dx + \int_{\Omega} u^2 \left| \nabla u \right|^2 \, dx + \int_{\Omega} u^2 \left| \nabla u \right|^2 \, dx \right) \\
\quad + c \| \Delta u \|^2 + c \| \nabla u \|^4 + c \| \nabla u \|^2 \\
\leq c \left( \| u \|^2 \| \Delta u \|^2 + \| u \|^3 \| \nabla u \|^4 + \| u \|^6 \| \nabla u \|^2 \right) + c \| \Delta u \|^2 + c \| \nabla u \|^4 + c \\
\leq c \left( \| \Delta u \|^2 + \| \nabla u \|^4 + \| \nabla u \|^2 + \| \nabla u \|^4 \right) + c \| \Delta u \|^2 + c.
\]

By Sobolev’s embedding theorem, we deduce that
\[
\| \Delta u \|^2 \leq \left( c_1' \left\| \Delta^2 u \right\|^\frac{5}{2} \| u \|^\frac{5}{2} + c_2' \| u \| \right)^2 \leq \frac{\varepsilon}{c} \| \Delta^2 u \|^2 + c_e, \\
\| \nabla u \|^2 \leq \left( c_1' \left\| \Delta^2 u \right\|^\frac{5}{2} \| u \|^\frac{5}{2} + c_2' \| u \| \right)^2 \leq \frac{\varepsilon}{c} \| \Delta^2 u \|^2 + c_e, \\
\| \nabla u \|^4 \leq \left( c_1' \left\| \Delta^2 u \right\|^\frac{5}{4} \| u \|^\frac{5}{4} + c_2' \| u \| \right)^4 \leq \frac{\varepsilon}{c} \| \Delta^2 u \|^2 + c_e, \\
\| \nabla u \|^8 \leq \left( c_1' \left\| \Delta^2 u \right\|^\frac{5}{8} \| u \|^\frac{5}{8} + c_2' \| u \| \right)^8 \leq \frac{\varepsilon}{c} \| \Delta^2 u \|^2 + c_e.
\]

Moreover,
\[
c \| \Delta u \|^2 = -c \int_{\Omega} \nabla u \cdot \nabla \Delta u \, dx = c \int_{\Omega} u \Delta^2 u \, dx \leq \| u \| \| \Delta^2 u \| \leq \varepsilon \| \Delta^2 u \|^2 + c_e.
\]

Summing up and setting \( \varepsilon = \frac{\gamma}{10} \) gives
\[
\frac{d}{dt} \| \Delta u \|^2 + \frac{\gamma}{2} \| \Delta^2 u \|^2 \leq c_{12}. \tag{13}
\]

By a Calderón–Zygmund type estimate, the following inequality holds:
\[
\frac{d}{dt} \| \Delta u \|^2 + \frac{\gamma' \varepsilon}{2} \left( \| \Delta u \|^2 + \| \nabla \Delta u \|^2 \right) \leq c_{12}.
\]

Then, using Gronwall’s inequality, we obtain
\[
\| \Delta u \|^2 \leq e^{-\frac{\gamma' \varepsilon}{2} (t-t_0)} \| \Delta u (t_0) \|^2 + \frac{2c_{12}}{\gamma' \varepsilon} \leq \frac{4c_{12}}{\gamma' \varepsilon} \tag{14}
\]

for all \( t \geq t_0' = \max \{ T_0, t_0' + \frac{2}{\gamma' \varepsilon} \ln \frac{\gamma' \varepsilon \| \Delta u \|^2}{4c_{12}} \} \). Setting \( t \geq t_0' \), taking \( s \in (t, t+1) \), integrating (14) over \( (s, t+1) \), we derive that
\[
\| \Delta u (t+1) \|^2 \leq c_{13} + \| \Delta u (s) \|^2. \tag{15}
\]
Integrating (15) with respect to $s$ in $(t, t + 1)$, we can obtain
\[
\| \Delta u(t + 1) \|^2 \leq c_{13} + \int_t^{t+1} \| \Delta u(s) \|^2 \, dx \leq c_{14}, \quad \forall t \geq T_0'.
\] (16)

By (14), (12), (7), and Sobolev’s embedding theorem, we conclude
\[
\| u \|_\infty \leq c_{15}, \quad \| \nabla u \|_p \leq c_{16}, \quad 1 \leq p < \infty.
\] (17)

Multiplying equation (1) by $u_t$, integrating the resulting relation over $\Omega$ yields
\[
\| u_t \|^2 + \frac{\gamma}{2} \frac{d}{dt} \| \Delta u \|^2
= \int_{\Omega} \Delta \varphi(u) u_t \, dx + \beta \int_{\Omega} \nabla \psi(u) u_t \, dx
= \int_{\Omega} \varphi'(u) \Delta u u_t \, dx + \int_{\Omega} \varphi''(u) |\nabla u|^2 u_t \, dx + \beta \int_{\Omega} \psi'(u) \Delta u u_t \, dx
\leq \| \varphi'(u) \|_\infty \| \Delta u \| \| u_t \| + \| \varphi''(u) \|_\infty \| \nabla u \|^2 \| u_t \| + |\beta| \| \psi'(u) \|_\infty \| \Delta u \| \| u_t \|
\leq \frac{1}{2} \| u_t \|^2 + c \left( \| \varphi'(u) \|_\infty \| \Delta u \|^2 + \| \varphi''(u) \|_\infty \| \nabla u \|^4 + |\beta|^2 \| \psi'(u) \|_\infty \| \nabla u \|^2 \right)
\leq \frac{1}{2} \| u_t \|^2 + \frac{c_{17}}{2},
\]

that is,
\[
\| u_t \|^2 + \frac{d}{dt} \| \Delta u \|^2 \leq c_{17}.
\] (18)

Integrating (18) over $(t + 1, t + 2)$, using (14), we derive that
\[
\int_t^{t+2} \| u_t \|^2 \, dx \leq c_{18}, \quad \forall t \geq T_0'.
\]

Using a mean value theorem for integrals, we obtain the existence of a time $t_1 \in (T_0'' + 1, T_0'' + 2)$ such that the following estimate holds uniformly:
\[
\| u_t(t_1) \|^2 \leq c_{19}.
\]

Then the proof is complete. \qed

**Lemma 2.4** Suppose that $u_0 \in H^1_{\text{per}}(\Omega)$ and the functions $\varphi(r) \in C^2(\mathbb{R})$, $\psi(r) \in C^2(\mathbb{R})$ satisfy
\[
\varphi'(r) > 0, \quad \varphi^{(i)}(0) \leq c_0 r^{k-i} + c_1, \quad \varphi'(r) \leq c_0 r^{\sqrt{\varphi'(r)} + c_1},
\]
where $k \leq 3$ is a positive constant and $i = 0, 1, 2$. Then, for problem (1)–(3), we have
\[
\| \nabla u(t) \| \leq M_3, \quad \forall t \geq T_3,
\]
and
\[ \int_t^{t+1} \left\| \Delta^2 u(t) \right\|^2 dt \leq M_3, \quad \forall t \geq T_3. \]

Here, \( M_3 \) is a positive constant depending on \( \gamma, c_i, c'_i \) (\( i = 0, 1 \)). \( T_3 \) depends on \( \gamma, c_i, c'_i \) (\( i = 0, 1 \)) and \( R \), where \( \|u_0\|_{\Omega_1} \leq R^2 \).

Proof: Multiplying (1) by \( \Delta^3 u \) and integrating the resulting relation over \( \Omega \), we obtain
\[
\frac{1}{2} \frac{d}{dt} \left\| \nabla \Delta u \right\|^2 + \gamma \left\| \nabla \Delta^2 u \right\|^2 \\
= \int_\Omega \nabla \Delta \phi(u) \nabla \Delta^2 u \; dx + \beta \cdot \int_\Omega \Delta \phi(u) \nabla \Delta^2 u \; dx \\
= \int_\Omega \phi'(u) \nabla \Delta u \nabla \Delta^2 u \; dx + 3 \int_\Omega \phi''(u) \nabla u \Delta u \nabla \Delta^2 u \; dx \\
+ \int_\Omega \phi'''(u) |\nabla u|^2 \nabla \Delta^2 u \; dx \\
+ \beta \cdot \int_\Omega \phi'(u) \Delta u \nabla \Delta^2 u \; dx + \beta \cdot \int_\Omega \phi''(u) |\nabla u|^2 \nabla \Delta^2 u \; dx \\
\leq \frac{\gamma}{2} \left( \int_\Omega |\nabla \Delta u|^2 + c \left( \int_\Omega \phi''(u)|\nabla u| \Delta u \right)^2 + 3 \int_\Omega \phi'(u)|\nabla u|^2 \| \Delta u \|^2 \right) \\
+ \int_\Omega \phi''(u) |\nabla u|^2 |\phi'(u)| \Delta u \| \Delta u \|^2 + |\beta|^2 |\phi''(u)| \| \Delta u \|^2 \| \nabla \Delta^2 u \|^4 \right). \tag{19}
\]

It follows form (17) that
\[
\left\| \phi'(u) \right\|_{\Omega_1}^2 \left\| \nabla \Delta u \right\|^2 \leq \frac{c}{2} \left\| \nabla \Delta u \right\|^2, \\
3 \left\| \phi''(u) \right\|_{\Omega_1}^2 \| \Delta u \|^2 \leq \frac{c}{2} \| \Delta u \|^2,
\]
and
\[
\left\| \phi''(u) \right\|_{\Omega_1}^2 \left\| \nabla u \right\|^6 + |\beta|^2 \left\| \phi'(u) \right\|_{\Omega_1}^2 \| \Delta u \|^2 + |\beta|^2 \left\| \phi''(u) \right\|_{\Omega_1}^2 \left\| \nabla \Delta^2 u \right\|^4 \leq \frac{c_{19}}{2}.
\]

Summing up, we find that
\[
\frac{d}{dt} \left\| \nabla \Delta u \right\|^2 + \gamma \left\| \nabla \Delta^2 u \right\|^2 \leq c \left( \left\| \nabla \Delta u \right\|^2 + \| \Delta u \|^2 + c_{19} \right). \tag{20}
\]

Using Nirenberg’s inequality, we obtain
\[
c' \| \Delta u \|^2 \leq c' \left( c'_1 \left\| \nabla \Delta^2 u \right\|^\frac{2}{5} \| \Delta u \|^\frac{8}{5} + c'_2 \| \Delta u \| \right)^2 \leq \frac{\gamma}{4} \left\| \nabla \Delta^2 u \right\|^2 + c_{20}.
\]

On the other hand,
\[
c' \left\| \nabla \Delta u \right\|^2 = c' \int_\Omega \nabla u \cdot \nabla \Delta^2 u \; dx \leq c' \left\| \nabla u \right\| \left\| \nabla \Delta^2 u \right\| \leq \frac{\gamma}{4} \left\| \nabla \Delta^2 u \right\|^2 + c_{21}.
\]
Hence
\[
\frac{d}{dt} \| \nabla \Delta u \|^2 + \gamma \frac{\beta}{2} \| \nabla \Delta^2 u \|^2 \leq c_{20} + c_{21} + c_{19}. \tag{21}
\]

A simple calculation shows that
\[
\frac{d}{dt} \| \nabla \Delta u \|^2 + c_{22} \| \nabla \Delta u \|^2 \leq c_{23}. \tag{22}
\]

By Gronwall’s inequality, we immediately obtain
\[
\| \nabla \Delta u(t) \|^2 \leq e^{-c_{22}(t-t_0)} \| \nabla \Delta u(t_0) \|^2 + \frac{c_{23}}{c_{22}} \leq \frac{2c_{23}}{c_{22}} \tag{23}
\]
for all \( t \geq T^*_1 = \max \{ T_1, t_0 + \frac{1}{c_{22}} \ln \frac{c_{22}}{2c_{23}} \} \). Combining (23), (14), (12), and (7) together gives
\[
\| \nabla u \|_\infty \leq c_{24}, \quad \| \Delta u \|_q \leq c_{25}, \quad 1 \leq q < \infty, \forall t \geq T^*_1. \tag{24}
\]

Multiplying equation (1) by \( A \alpha \), integrating the resulting relation over \( \Omega \), we obtain
\[
\begin{align*}
\| \nabla u_t \|^2 + \gamma \frac{d}{dt} \| \nabla \Delta u \|^2 &= \\
&= \int_\Omega \nabla \Delta \varphi(u) \nabla u_t \, dx + \gamma \int_\Omega \Delta \varphi(u) \nabla u_t \, dx \\
&= \int_\Omega [\varphi'(u) \nabla \Delta u + 3 \varphi''(u) \nabla u \Delta u + \varphi''(u) \nabla u \nabla u \nabla u_t \, dx \\
&\quad + \beta \cdot \int_\Omega [\psi'(u) \nabla \Delta u + \psi''(u) \nabla u \nabla u_t \, dx \\
&\leq \| \varphi'(u) \|_\infty \| \nabla \Delta u \| \| \nabla u_t \| + 3 \| \varphi''(u) \|_\infty \| \nabla u \|_\infty \| \Delta u \| \| \nabla u_t \| \\
&\quad + \| \varphi''(u) \|_\infty \| \nabla u \|_\infty \| \nabla \Delta u \| \| \nabla u_t \| \\
&\quad + \| \beta \| \| \psi'(u) \|_\infty \| \Delta u \| \| \nabla u_t \| + \| \beta \| \| \psi''(u) \|_\infty \| \nabla u \|_\infty \| \nabla u \| \| \nabla \Delta u \| \\
&\leq c \| \nabla u_t \| \leq \frac{1}{2} \| \nabla u_t \|^2 + \frac{c_{26}}{2},
\end{align*}
\]

Summing up, using the result of (23) gives
\[
\| \nabla u_t \|^2 + \gamma \frac{d}{dt} \| \nabla \Delta u \|^2 \leq c_{26}. \tag{25}
\]

Then
\[
\gamma \frac{d}{dt} \| \nabla \Delta u \|^2 \leq c_{26}.
\]

Setting \( t \geq T^*_1 \), taking \( s \in (t, t+1) \), integrating the above inequality over \( (s, t+1) \), we obtain
\[
\| \nabla \Delta u(t + 1) \|^2 \leq \frac{1}{\gamma} (c_{26} + \| \nabla \Delta u(s) \|^2).
\]
Integrating the above inequality with respect to \(s\) in \((t, t+1)\), we have
\[
\| \nabla \Delta u(t+1) \|^2 \leq \frac{1}{\gamma} \left( c_{26} + \int_t^{t+1} \| \nabla \Delta u(s) \|^2 \, ds \right) \leq c_{27}, \quad \forall t \geq T^*_1. \tag{26}
\]

Integrating (25) over \((t+1, t+2)\), using (26) yields
\[
\int_{t+1}^{t+2} \left\| A^{1/2} u_t \right\|^2 \, dt \leq c_{28}, \quad \forall t \geq T^*_1.
\]

Using a mean value theorem for integrals, we obtain the existence of a time \(t_2 \in (T^*_1 + 1, T^*_1 + 2)\) such that the following estimate holds uniformly:
\[
\left\| A^{1/2} u_t(t_2) \right\|^2 \leq c_{29}.
\]

Then we complete the proof. \(\square\)

**Lemma 2.5** Suppose that \(u_0 \in H^1_{\text{per}}(\Omega)\) and the functions \(\psi(r) \in C^3(\mathbb{R})\), \(\varphi(r) \in C^2(\mathbb{R})\) satisfy
\[
\psi'(r) > 0, \quad \varphi^{(i)} \leq c_0 |r|^{k-i} + c_1, \quad \psi'(r) \leq c_0 \sqrt{\varphi(r)} + c_1,
\]
where \(k \leq 3\) is a positive constant and \(i = 0, 1, 2\). Then, for problem (1)–(3), we have
\[
\| u_t \| \leq M_4, \quad \forall t \geq T_4.
\]

Here, \(M_4\) is a positive constant depending on \(\gamma, c_i, c'_i\) \((i = 0, 1)\). \(T_4\) depends on \(\gamma, c_i, c'_i\) \((i = 0, 1)\) and \(R\), where \(\| u_0 \|^2_{H^1_{\text{per}}} \leq R^2\).

**Proof** Setting \(v = u_t\), differentiating (1) with respect to the time \(t\), we deduce that
\[
v_t + \gamma \Delta^2 v - \left[ \Delta \varphi(u) \right]_t - \beta : \left[ \nabla \psi(u) \right]_t = 0.
\tag{27}
\]

Multiplying (27) by \(v\), integrating the resulting relation over \(\Omega\) yields
\[
\frac{1}{2} \frac{d}{dt} \| v \|^2 + \gamma \| \Delta v \|^2 - \int_{\Omega} \left[ \Delta \varphi(u) \right] v \, dx - \int_{\Omega} \beta : \left[ \nabla \psi(u) \right] v \, dx = 0. \tag{28}
\]

Using Sobolev’s embedding theorem, we get
\[
\int_{\Omega} \left[ \Delta \varphi(u) \right] v \, dx + \beta : \int_{\Omega} \left[ \varphi(u) \nabla u \right] v \, dx
= \int_{\Omega} \varphi'(u) v \Delta v \, dx + \int_{\Omega} \varphi''(u) v^2 \Delta u \, dx + \int_{\Omega} \varphi'''(u) |\nabla u|^2 v^2 \, dx
+ 2 \int_{\Omega} \psi'(u) v \nabla u \nabla v \, dx + \beta : \int_{\Omega} \psi''(u) v \nabla \psi \, dx + \beta : \int_{\Omega} \psi'''(u) |\nabla u|^2 \, dx
\leq \| \varphi'(u) \|_{\infty} \| \Delta v \| \| v \| + \| \varphi''(u) \|_{\infty} \| \Delta u \| \| \nabla v \|^2 + \| \varphi'''(u) \|_{\infty} \| \nabla u \| \| v \|^2
+ 2 \| \varphi'(u) \nabla u \|_{\infty} \| v \| \| \nabla v \| + |\beta| \| \psi'(u) \|_{\infty} \| \nabla \psi \| \| v \| + |\beta| \| \psi''(u) \|_{\infty} \| \nabla u \| \| \nabla v \| ^2
\]
\[
\leq c\left(\|\Delta v\|\|v\| + \|\nabla v\|^2 + \|\nabla v\|\|v\|\right)
\leq \frac{\gamma}{2}\|\Delta v\|^2 + \frac{c_{30}}{2}\|v\|^2 + \frac{c_{31}}{2}.
\]

Hence,
\[
\frac{d}{dt}\|v\|^2 + \frac{\gamma}{2}\|\Delta v\|^2 \leq c_{30}\|v\|^2 + c_{31}.
\]

A simple calculation shows that
\[
\|v\|^2 \leq \frac{1}{c'}\|\Delta v\|^2.
\]

It then follows from (29) and the above inequality that
\[
\frac{d}{dt}\|v\|^2 + (c'\gamma - c_{30})\|v\|^2 \leq c_{31},
\]
where \(\gamma\) is sufficiently large, it satisfies \(c'\gamma - c_{30} > 0\). Using Gronwall’s inequality, we derive that
\[
\|v\|^2 \leq e^{-c'\gamma(t - t_1)}\|v(t_1)\|^2 + \frac{c_{31}}{c'\gamma - c_{30}}\leq c_{19}e^{-c'\gamma(t - t_1)} + \frac{c_{31}}{c'\gamma - c_{30}}
\]
for all \(t \geq t_1 + \frac{1}{c'\gamma - c_{30}}\ln\frac{c_{19}(c'\gamma - c_{30})}{c_{31}}\). Then the proof is complete. \(\square\)

**Lemma 2.6** Suppose that \(u_0 \in H^1_{per}(\Omega)\) and the functions \(\varphi(r) \in C^3(\mathbb{R})\), \(\psi(r) \in C^2(\mathbb{R})\) satisfy
\[
\varphi'(r) > 0, \quad \varphi^{(i)} \leq c'_0 r^{k-i} + c'_i, \quad \psi'(r) \leq c_0 r^{\sqrt{\varphi'(r)}} + c_1,
\]
where \(k \leq 3\) is a positive constant and \(i = 0, 1, 2\). Then, for problem (1)–(3), we have
\[
\|A^{\frac{1}{2}}v(t)\| \leq M_5, \quad \forall t \geq T_5.
\]

Here, \(M_5\) is a positive constant depending on \(\gamma\), \(c_i\), \(c'_{i}\) \((i = 0, 1)\). \(T_5\) depends on \(\gamma\), \(c_i\), \(c'_{i}\) \((i = 0, 1)\) and \(R\), where \(\|u_0\|^2_{H^1_{per}} \leq R^2\).

**Proof** Multiplying (27) by \(\Lambda v\), integrating the resulting relation over \(\Omega\), we obtain
\[
\frac{1}{2} \frac{d}{dt}\|
\nabla v\|^2 + \gamma\|\nabla \Delta v\|^2 = -\int_{\Omega} [\Delta \varphi(u)]_t \Delta v \, dx - \beta \cdot \int_{\Omega} [\nabla \psi(u)]_t \Delta v \, dx
\]
By Sobolev’s embedding theorem, we get
\[
- \int_\Omega [\Delta \varphi(u)]_t \Delta v \, dx - \beta \cdot \int_\Omega [\nabla \psi(u)]_t \Delta v \, dx
= - \int_\Omega \varphi(u) \Delta v^2 \, dx - \int_\Omega \varphi''(u) v \Delta u \Delta v \, dx - \int_\Omega \varphi''(u) v \nabla u \nabla v \, dx
+ 2 \int_\Omega \varphi''(u) \nabla v \nabla u \nabla v \, dx + \beta \cdot \int_\Omega \psi'(u) \nabla v \, dx + \beta \cdot \int_\Omega \psi'(u) v \nabla u \nabla v \, dx
\leq \|\varphi(u)\|_\infty \|\Delta v\|^2 + \|\varphi''(u)\|_\infty \|\Delta u\| \|\Delta v\| \|v\|_\infty + \|\varphi''(u)\|_\infty \|\nabla u\|_\infty \|\nabla v\| \|v\|
+ 2 \|\varphi''(u)\|_\infty \|\nabla u\|_\infty \|\nabla v\| \|v\|_\infty + |\beta| \|\psi'(u)\|_\infty \|\nabla v\| \|\Delta v\|
+ |\beta| \|\psi'(u) \nabla v\|_\infty \|\nabla u\|_\infty \|\Delta v\| \|v\|
\leq c(\|\Delta v\|^2 + \|\Delta u\| \|\nabla v\| + \|\nabla v\| \|\Delta v\|) \leq \frac{\gamma}{2} \|\nabla \Delta v\|^2 + \frac{c_{32}}{2} \|\nabla v\|^2.
\]

Summing up gives
\[
\frac{d}{dt} \|\nabla v\|^2 + \gamma \|\Delta v\|^2 \leq c_{32} \|\nabla v\|^2.
\]

Using Nirenberg’s inequality, we obtain
\[
c_{32} \|\nabla v\|^2 \leq c_{32} (c'_1 \|\nabla v\|^{\frac{1}{3}} \|v\|^{\frac{2}{3}} + c'_2 \|v\|) \leq \frac{\gamma}{2} \|\nabla \Delta v\|^2 + c_{33}.
\]

Adding the above two inequalities together gives
\[
\frac{d}{dt} \|\nabla v\|^2 + c_{32} \|\nabla v\|^2 \leq 2c_{33}.
\]

By Gronwall’s inequality, we can obtain
\[
\|\nabla v\|^2 \leq e^{-c_{32}(t-t_2)} \|\nabla v(t_2)\|^2 + \frac{2c_{33}}{c_{32}}
\leq c_{29} e^{-c_{32}(t-t_2)} + \frac{2c_{33}}{c_{32}} \leq \frac{4c_{33}}{c_{32}}
\]
for all \( t \geq t_2 + \frac{1}{c_{32}} \ln \frac{c_{33}}{2c_{32}} \). Then the proof is complete. \( \square \)

**Lemma 2.7** Suppose that \( u_0 \in H^1_{\text{per}}(\Omega) \) and the functions \( \varphi(r) \in C^2(\mathbb{R}) \), \( \psi(r) \in C^2(\mathbb{R}) \) satisfy
\[
\varphi'(r) > 0, \quad \varphi^{(i)}(r) \leq c_i |r|^{k-i} + c'_i, \quad \psi'(r) \leq c_0 r \sqrt{\varphi'(r)} + c_1,
\]
where \( k \leq 3 \) is a positive constant and \( i = 0, 1, 2 \). Then, for problem (1)–(3), we have
\[
\|A^2 u(t)\| \leq M_6, \quad \forall t \geq T_6.
\]

Here, \( M_6 \) is a positive constant depending on \( \gamma, c_i, c'_i \) (i = 0, 1). \( T_6 \) depends on \( \gamma, c_i, c'_i \) (i = 0, 1) and \( R \), where \( \|u_0\|_{H^1_{\text{per}}} \leq R^2 \).
Lemma 3.1  Using Lemmas 2.2 and 2.7, we see that
\[ \dot{\text{the embedding is satisfied}} \]
and the functions
\[ u(t) \text{ for } t \in [0, \infty) \] and \[ \text{the dynamical system } S(t), t \in [0, \infty) \] satisfy
\[ k \leq 3 \text{ is a positive constant and } i = 0, 1, 2. \]
where \( k \leq 3 \) is a positive constant and \( i = 0, 1, 2 \).
Then, for the solution \( u(x, t) \) of problem (1)–(3), the dynamical system \( S(t), t \in [0, \infty) \) is a global attractor for \( \dot{H}^1_\text{per} \)-asymptotically compact.

Proof For (1), we have
\[ \gamma A^2 u = -u_t + \Delta \psi(u) + \beta \cdot \nabla \psi(u). \]
By Lemmas 2.6 and 2.7, there exists $T > 0$ such that

$$\|v_n\|_{D(A^\frac{1}{2})} \leq M_5, \quad \|u_n\|_{D(A^\frac{1}{2})} \leq M_6, \quad \forall t \geq T, n = 1, 2, \ldots.$$  (36)

Since $t_n \to \infty$, there exists $N > 0$ such that $t_n \geq T$ for all $n \geq N$. Therefore, by (36), we get

$$\|v_n(t_n)\|_{D(A^\frac{1}{2})} \leq M_5, \quad \|u_n(t_n)\|_{D(A^\frac{1}{2})} \leq M_6, \quad \forall n \geq N.$$  (37)

Note that the embedding $D(A^\frac{1}{2}) \hookrightarrow H$ and $D(A^1) \hookrightarrow D(A)$ are compacted. Hence, by (36), there exist $v \in D(A^\frac{1}{2})$, $\Delta u \in D(A)$, $\nabla u \in H^3_{\text{per}}$, and $u \in H^3_{\text{per}}$ such that, up to a subsequence,

$$\begin{aligned}
v_n(t_n) & \to v \quad \text{strongly in } H, \\
\Delta u_n(t_n) & \to \Delta u \quad \text{strongly in } D(A^\frac{1}{2}), \\
\nabla u_n(t_n) & \to \nabla u \quad \text{strongly in } D(A), \\
u_n(t_n) & \to u \quad \text{strongly in } H^3_{\text{per}}.
\end{aligned}$$  (38)

By (37) and Sobolev’s embedding theorem, we obtain

$$\|u_n(t_n)\|_{W^{2,\infty}} \leq C, \quad \forall n \geq N.$$  

It then follows from (36) and (38) that

$$\|u_n(t_n) - u\| \to 0, \quad \|v_n(t_n) - v\|^2 \to 0, \quad \|\Delta u_n(t_n) - \Delta u\|^2 \to 0,$$

and

$$\|\Delta \phi(u_n(t_n)) - \Delta \phi(u)\|
\leq c\|\psi'(u_n(t_n))\|_\infty \|\Delta u_n(t_n) - \Delta u\| + c\|\Delta u\|_\infty \|\psi'(u_n(t_n))\|_\infty \|\nabla u_n(t_n) - \nabla u\|
+ c\|\nabla u\|_\infty \|\psi''(u_n(t_n))\|_\infty \|\nabla u_n(t_n) - \nabla u\|
\leq c\|\Delta u_n(t_n) - \Delta u\| + c\|\nabla u\|_\infty \|\psi'(u_n(t_n))\|_\infty \|\nabla u_n(t_n) - \nabla u\|
+ c\|\nabla u\|_\infty \|\psi''(u_n(t_n))\|_\infty \|\nabla u_n(t_n) - \nabla u\|
\leq c\|\Delta u_n(t_n) - \Delta u\|
+ c\|\nabla u\|_\infty \|\psi'(u_n(t_n))\|_\infty \|\nabla u_n(t_n) - \nabla u\| + 1 - \theta_2)u\|_\infty \|u_n(t_n) - u_n(t_n)\|
\to 0, \quad \text{(39)}
where $\theta_1, \theta_2 \in (0, 1)$. Using the same method as above, we also have
\[
\left\| \nabla \psi(u_n(t)) - \nabla \psi(u) \right\| \to 0.
\]

Therefore
\[
\gamma A^2 u_n(t) \to -u_t + \Delta \psi(u) + \beta \cdot \nabla \psi(u), \quad \text{strongly in } H,
\]
that is, $(u_n(t))_{n=1}^{\infty}$ converges to $A^{-2}(-v + \Delta \psi(u) + \beta \cdot \nabla \psi(u))$ in $H^4_{\text{per}}(\Omega)$. Then we complete the proof. \qed

Now we give the proof of the main result.

**Proof of Theorem 1.3** Note that $(S(t))_{t \geq 0}$ has an $(\dot{H}^1_{\text{per}}, \dot{H}^4_{\text{per}})$-global attractor $\mathcal{A}$. By Lemma 2.7, $B_2$ is a bounded $(\dot{H}^1_{\text{per}}, \dot{H}^4_{\text{per}})$-absorbing set for $(S(t))_{t \geq 0}$. On the other hand, by Lemma 3.1, we can obtain $(S(t))_{t \geq 0}$ is $(\dot{H}^1_{\text{per}}, \dot{H}^4_{\text{per}})$-asymptotically compact. Then, by Proposition 1.2, $\mathcal{A}$ is actually an $(\dot{H}^1_{\text{per}}, \dot{H}^4_{\text{per}})$-global attractor for $(S(t))_{t \geq 0}$. The proof of Theorem 1.3 is complete. \qed

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**Authors’ contributions**
The main idea of this paper was proposed by XZ. XZ prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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