On the Kendall Correlation Coefficient

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Abstract

In the present paper, we first discuss the Kendall rank correlation coefficient $\tau_n$. In continuous case, we define $\tau_n$ in terms of the concomitants of order statistics, find the expected value of $\tau_n$ and show that the later is free of $n$. We also prove that in continuous case the Kendall correlation coefficient converges in probability to its expected value $\tau = E\tau_n$. We then propose to consider $\tau$ as a new theoretical correlation coefficient which can be an alternative to the classical Pearson product-moment correlation coefficient. At the end of this work we analyze illustrative examples.

Keywords and Phrases: bivariate distributions; concomitants of order statistics; Pearson product-moment correlation coefficient; sample correlation coefficient; Kendall rank correlation coefficient.

1 Introduction

Let $(X, Y), (X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be independent and identically distributed random vectors with bivariate distribution $F(x, y) = P(X < x, Y < y)$ and corresponding marginal distributions $H(x) = P(X < x)$ and $G(y) = P(Y < y)$. The purpose of slight modifications in the definitions of $F, H$ and $G$ is that in Section 4 we define a new correlation coefficient and we wish to have the same form of this correlation coefficient for different types of distributions. If $F$ is an absolutely continuous distribution then the corresponding densities will be denoted as $f(x, y), h(x)$ and $g(y)$, respectively. Let $X_{1,n} \leq X_{2,n} \leq \ldots \leq X_{n,n}$ be the order statistics obtained from the sample $X_1, X_2, \ldots, X_n$. For these order statistics, let us define their concomitants $Y_{[1,n]}, Y_{[2,n]}, \ldots, Y_{[n,n]}$. Let $X_i = X_{j,n}$. Then $Y_{[j,n]} = Y_i$ is the concomitant of the order statistic $X_{j,n}$. Concomitants of order statistics were proposed by David (1973) and Bhattacharya (1974). Concomitants of order statistics were further discussed in David and Galambos (1974), Bhattacharya (1984), Egorov and Nevzorov (1984), David (1994), Goel and Hall (1994), Chu et al. (1999), David and Nagaraja (2003), Bairamov and Stepanov (2010) and others.

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It is known that the rate of dependence between random variables $X$ and $Y$ can be measured in terms of the Pearson product-moment correlation coefficient
$$\rho = \frac{E(X - EX)(Y - EY)}{\sigma_X \sigma_Y}.$$ 

The sample correlation coefficient
$$\rho_n = \frac{\sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^{n} (X_i - \bar{X})^2 \sum_{i=1}^{n} (Y_i - \bar{Y})^2}}$$
is a good approximation for $\rho$ for large values of $n$ since $\rho_n \xrightarrow{p} \rho$; see, for example, Fisher (1921).

Let $Y_{[1,n]}, \ldots, Y_{[n,n]}$ be the concomitants of order statistics $X_{1,n} \leq \ldots \leq X_{n,n}$. Let
$$r_{j,n} = \sum_{i=1}^{j-1} I_{i,j,n} = \sum_{i=1}^{j-1} I_{Y_{[i,n]} \leq Y_{[j,n]}} \quad (1 \leq i < j \leq n)$$
be the rank of the concomitant $Y_{[j,n]}$ among the concomitants $Y_{[1,n]}, \ldots, Y_{[j,n]}$. The following value
$$\tau_n = \frac{4 \sum_{j=2}^{n} r_{j,n}}{n(n-1)} - 1 = \frac{4}{n(n-1)} \sum_{j=2}^{n} \sum_{i=1}^{j-1} I_{i,j,n} - 1$$
is known as Kendall’s rank correlation coefficient (see, for example, Kendall (1970)). This rank correlation coefficient $\tau_n$ can be used along with $\rho_n$. Sometimes samples $X_i$ and $Y_i$ are not known but the ranks $r_{j,n}$ are known. In this case one can use $\tau_n$ instead of $\rho_n$. It should be noted that there is another rank correlation coefficient – Spearman’s rank correlation coefficient, which we do not discuss in the present work.

Basic properties of $\tau_n$ are as follows. If the agreement/disagreement between the sequence $X_{j,n}$ and the rankings of $Y_{[j,n]}$ is perfect, the coefficient value $\tau_n$ is near $1/-1$. Further, if variables $X$ and $Y$ are independent, then $Y_{[j,n]} \ i = 1, \ldots, n$ are independent and identically distributed. Obviously, $P(Y_{[i,n]} \leq Y_{[j,n]}) = 1/2 \forall i, j, i \neq j$. Then $\tau_n \xrightarrow{p} 0$. Different bounds for $\tau_n$ (basically in the normal case) were obtained in Daniels (1950), Durbin and Stuart (1951), Kendall (1970) and Xu et al. (2009); see also references in these works.

Further in the paper, we discuss moment and asymptotic properties of $\tau_n$. In Section 2, we find $E\tau_n$ in general continuous case and show that it is free of $n$. In Section 3, we prove that in general continuous case the Kendall correlation coefficient converges in probability to its expected value. The last observation motivates us to introduce in Section 4 a new correlation coefficient: $\tau = E\tau_n$. This correlation coefficient $\tau$ is a possible alternative to the classical Pearson product-moment correlation coefficient $\rho$. It should be noted that no moment assumption is needed for the existence of $\tau$. We illustrate our theoretical results in Section 5 by examples and simulation results.
2 The expected value of $\tau_n$

We assume in this section that $F$ is a continuous distribution.

**Lemma 2.1.** For $n \geq 2$, we have

$$E\tau_n = 4 \int_{x \leq u, y \leq v} F(dx, dy)F(du, dv) - 1$$

(2.1)

$$= 4 \int_{\mathbb{R}^2} (1 - H(x) - G(y) + F(x, y))F(dx, dy) - 1$$

(2.2)

$$= 4 \int_{\mathbb{R}^2} F(x, y)F(dx, dy) - 1 = 4EF(X, Y) - 1.$$  

(2.3)

**Proof** It is easily seen that (2.1) implies both (2.2) and (2.3). We present the proof (2.1) for the case when $F$ is an absolutely continuous distribution. Let

$$Z_{[i,n]} = (X_{1,n}, Y_{[i,n]}), \ldots, Z_{[n,n]} = (X_{n,n}, Y_{[n,n]}).$$

One can find that

$$f_{Z_{[i,n]},\ldots,Z_{[n,n]}}(x_1, y_1, \ldots, x_n, y_n) = n! f(x_1, y_1) \ldots f(x_n, y_n) \quad (x_1 \leq \ldots \leq x_n, y_i \in \mathbb{R})$$

(2.4)

and

$$f_{Y_{[i,n]},Y_{[j,n]},\ldots,Y_{[n,n]}}(y_1, y_2, \ldots, y_n) = n! \int_{x_1 \leq \ldots \leq x_n} f(x_1, y_1) \ldots f(x_n, y_n)dx_1 \ldots dx_n.$$  

(2.5)

By integration in (2.5), one can find that for $y, v \in \mathbb{R}$

$$f_{Y_{[i,n]},Y_{[j,n]}}(y, v) = \frac{n!}{(i - 1)!(j - i - 1)!(n - j)!} \times \int_{x \leq u} H^{i-1}(x)f(x, y)(H(u) - H(x))^{j-i-1}f(u, v)(1 - H(u))^{n-j}dxdu.$$  

Then

$$Er_{j,n} = \sum_{i=1}^{j-1} P(Y_{[i,n]} \leq Y_{[j,n]}) = \frac{n!}{(j - 2)!(n - j)!} \times \int_{x \leq u, y \leq v} f(x, y)f(u, v)H^{j-2}(u)(1 - H(u))^{n-j}dxdudydv.$$  

We finish at the following identity

$$E\tau_n = 4 \sum_{j=2}^{n-1} Er_{j,n} \frac{n!}{n(n - 1)} - 1 = 4 \int_{x \leq u, y \leq v} f(x, y)f(u, v)dxdudydv - 1.$$  

(2.6)

The result readily follows. When $F$ is continuous the result can be proved in the same manner. □

**Remark 2.1.** Since $E\tau_n$ is free of $n$, let $\tau = E\tau_n$. 


3 Asymptotic properties of $\tau_n$

We assume in this section that $F$ is a continuous distribution.

**Theorem 3.1.** The following asymptotic property holds true

$$\tau_n \rightarrow \tau.$$

**Proof** We present the proof of Theorem 3.1 for the case when $F$ is an absolutely continuous distribution. It follows from Chebyshev's inequality that for any $\varepsilon > 0$

$$P(|\tau_n - \tau| > \varepsilon) \leq \frac{\text{Var}\tau_n}{\varepsilon^2} = \frac{16}{\varepsilon^2 n^2 (n-1)^2} \left[ \sum_{j=2}^{n} \sum_{i=1}^{j-1} \sum_{k=2}^{n} \sum_{l=1}^{k-1} E(I_{i,j,n}I_{l,k,n}) - \left( E \left( \sum_{j=2}^{n} \sum_{i=1}^{j-1} I_{i,j,n} \right) \right)^2 \right]. \quad (3.1)$$

By (2.6), we have

$$\left( E \left( \sum_{j=2}^{n} \sum_{i=1}^{j-1} I_{i,j,n} \right) \right)^2 = \left( n(n-1) \int_{x \leq u, y \leq v} f(x,y)f(u,v)dx dy du dv \right)^2.$$

Let

$$\sum = \sum_{j=2}^{n} \sum_{i=1}^{j-1} \sum_{k=2}^{n} \sum_{l=1}^{k-1} E(I_{i,j,n}I_{l,k,n}) = \sum_{j=2}^{n} \sum_{i=1}^{j-1} \sum_{k=2}^{n} \sum_{l=1}^{k-1} P(Y[i,n] \leq Y[j,n], Y[l,n] \leq Y[k,n]).$$

One can write $\Sigma = \Sigma_1 + \Sigma_2$, where

$$\Sigma_1 = \sum_{1 \leq i < j < k \leq n} + \sum_{1 \leq i < l < j < k \leq n} + \sum_{1 \leq l < i < j < k \leq n} + \sum_{1 \leq l < i < j < k \leq n},$$

$$\Sigma_2 = 2 \sum_{1 \leq i < j < k \leq n} + 2 \sum_{1 \leq i < l < j < k \leq n} + 2 \sum_{1 \leq l < i < j < k \leq n} + 2 \sum_{1 \leq l < i < j < k \leq n} + 2 \sum_{1 \leq i < l < j < k \leq n}.$$

Here

$$\sum_{1 \leq i < j < k \leq n} = \sum_{k=4}^{n} \sum_{l=3}^{k-1} \sum_{j=2}^{l-1} \sum_{i=1}^{j-1} P(Y[i,n] \leq Y[j,n], Y[l,n] \leq Y[k,n]).$$
\[
\sum_{1 \leq i < j < l < k \leq n} = \sum_{k=3}^{n} \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} P(Y[i,n] \leq Y[j,n], Y[j,n] \leq Y[k,n])
\]

and the other terms (sums) in \(\Sigma_1\) and \(\Sigma_2\) are designated in the same fashion. We show in this section that the terms in \(\Sigma_1\) behave like \(O(n^4)\) when the terms in \(\Sigma_2\) behave like \(o(n^4)\). Let us start with \(\Sigma_2\). Let us, for example, show that \(\sum_{1 \leq i < j < k < l \leq n} \rightarrow 0\) as \(n \rightarrow \infty\). It follows from (2.5) that

\[
\sum_{k=3}^{n} \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} P(Y[i,n] \leq Y[j,n] \leq Y[k,n]) = \sum_{k=3}^{n} \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} \frac{n!}{(i-1)!(j-i-1)!(k-j-1)!(n-k)!} \\
\times \int_{x_i \leq x_j \leq x_k, y_i \leq y_j \leq y_k} H^i(x_i)f(x_i, y_i)(H(x_j) - H(x_i))^j-i-1 f(x_j, y_j)(H(x_k) - H(x_j))^k-j-1 \\
\times f(x_k, y_k)(1 - H(x_k))^{n-k}dx_idy_idx_jdy_jdx_kdy_k \\
= n(n-1)(n-2) \int_{x_i \leq x_j \leq x_k, y_i \leq y_j \leq y_k} f(x_i, y_i)f(x_j, y_j)f(x_k, y_k)dx_idy_idx_jdy_jdx_kdy_k \\
\leq n(n-1)(n-2) = o(n^4).
\]

One can show that the other terms of \(\Sigma_2\) behave like \(o(n^4)\) too. Let us now estimate the terms in \(\Sigma_1\). Let us take, for example, \(\sum_{1 \leq i < j < l < k \leq n}\). It follows from (2.5) that

\[
\sum_{1 \leq i < j < l < k \leq n} = \sum_{k=4}^{n} \sum_{l=3}^{k-1} \sum_{j=2}^{l-1} \sum_{i=1}^{j-1} P(Y[i,n] \leq Y[j,n], Y[l,n] \leq Y[k,n]) \\
= \sum_{k=4}^{n} \sum_{l=3}^{k-1} \sum_{j=2}^{l-1} \sum_{i=1}^{j-1} \frac{n!}{(i-1)!(j-i-1)!(l-j-1)!(k-l-1)!(n-k)!} \\
\times \int_{x_i \leq x_j \leq x_l, y_i \leq y_j \leq y_l} H^i(x_i)f(x_i, y_i)(H(x_j) - H(x_i))^j-i-1 f(x_j, y_j)(H(x_l) - H(x_j))^l-j-1 \\
\times f(x_l, y_l)(H(x_k) - H(x_l))^{k-l-1} f(x_k, y_k)(1 - H(x_k))^{n-k}dx_idy_idx_jdy_jdx_ldy_ldx_kdy_k \\
= n(n-1)(n-2)(n-3) \int_{x_i \leq x_j \leq x_l, y_i \leq y_j \leq y_l} f(x_i, y_i)f(x_j, y_j)f(x_k, y_k)dx_idy_idx_jdy_jdx_ldy_ldx_kdy_k \\
\leq T,
\]

where

\[
T = f(x_i, y_i)f(x_j, y_j)f(x_l, y_l)f(x_k, y_k)dx_idy_idx_jdy_jdx_ldy_ldx_kdy_k.
\]
In the same way one can estimate the other terms in $\Sigma_1$. Then

$$
\sum = n(n-1)(n-2)(n-3) 
\times \left( \int_{x_i \leq x_j \leq x_k, y_i \leq y_j, y_i \leq y_k} T + \int_{x_i \leq x_j \leq x_k, y_i \leq y_j, y_i \leq y_k} T + \int_{x_i \leq x_j \leq x_k, y_i \leq y_j, y_i \leq y_k} T 
+ \int_{x_i \leq x_k \leq x_j, y_i \leq y_j, y_i \leq y_k} T + \int_{x_i \leq x_k \leq x_j, y_i \leq y_j, y_i \leq y_k} T + \int_{x_i \leq x_k \leq x_j, y_i \leq y_j, y_i \leq y_k} T \right) + o(n^4)
$$

Observe that

$$
\int_{x_i \leq x,j, x_k \leq x_j, y_i \leq y_j, y_i \leq y_k} T
= \int_{x_i \leq x_j, x_k \leq x_j, y_i \leq y_j, y_i \leq y_k} f(x_i, y_i) f(x_j, y_j) f(x_k, y_k) dx_i dy_i x_j dy_j dx_k dy_k
= \left( \int_{x_i \leq x_j, y_i \leq y_j} f(x_i, y_i) f(x_j, y_j) dx_i dy_i x_j dy_j dx_k dy_k \right)^2.
$$

It follows from (3.1) that

$$P(|\tau_n - \tau| > \varepsilon) \to 0 \quad (n \to \infty).$$

The result readily follows. When $F$ is continuous the result can be proved in the same manner. \(\square\)

### 4 New correlation coefficient

In continuous case, we propose to consider the value

$$
\tau = 4EF(X,Y) - 1 = 4 \int_{x \leq a, y \leq v} F(dx, dy) F(du, dv) - 1 = 4 \int_{\mathbb{R}^2} F(x, y) F(dx, dy) - 1
$$

as a new theoretical correlation coefficient, which measures the rate of dependence between the random variables $X$ and $Y$. Basic properties of $\tau$ are as follows.

**Property 4.1.** It is obvious that $-1 \leq \tau \leq 1$.

**Property 4.2.** It is easily seen that if $X$ and $Y$ are independent, then $\tau = 0$.

**Property 4.3.** Let $Y = \varphi(X)$, where $\varphi$ is a nondecreasing function. Then $\tau = 1$. Let $Y = \psi(X)$, where $\psi$ is a non increasing function. Then $\tau = -1$. 
Proof We prove only the first statement. The second statement can be proved in the same manner. We again assume that $F$ is an absolutely continuous distribution. Let us consider the integral $\int_{\mathbb{R}^2} F_{X,Y}(x,y) f_{X,Y}(x,y) dxdy$, where the random variables $X$ and $Y$ are now attached to the corresponding bivariate distribution and density. It is obvious that

$$P(x \leq X < x + dx, y \leq Y < y + dy) \approx f_{X,Y}(x,y) dxdy.$$ 

Let $Y = X$. Then for all small enough $dx$ and $dy$

$$P(x \leq X < x + dx, y \leq X < y + dy) = \begin{cases} h_X(x) \min\{dx, dy\}, & \text{if } y = x \\ 0, & \text{otherwise.} \end{cases}$$

It follows from the last identity that

$$\int_{\mathbb{R}^2, Y=X} F_{X,Y}(x,y) f_{X,Y}(x,y) dxdy = \int_{\mathbb{R}} H_X(x) h_X(x) dx = 1/2.$$ 

That way, if $Y = X$, then $\tau = 4EF(X,Y) - 1 = 1$. The result readily follows since $EF(X, \varphi(X)) \geq EF(X, X)$.\] 

Observe that the corresponding Property 4.3 for $\rho$ is as follows: $\rho = \pm1$ iff $Y = aX + b$.

In discrete case, one can define $\tau$ by the identity

$$\tau = 4EF(X,Y) - 1 = 4 \sum_j \sum_k f(j,k) F(j,k) - 1,$$

where $f(j,k) = P(X = j, Y = k)$ and $F(j,k) = P(X < j, Y < k)$. One can check here the validity of Properties 4.1-4.3. To check Property 4.1 one can apply the inequality $\sum_j h(j) H(j) \leq 1/2$, where $h(j) = P(X = j)$ and $H(j) = P(X < j)$.

In general case, when $F$ has discrete and continuous components, one can define $\tau$ as follows: $\tau = 4EF(X,Y) - 1$.

The proposed correlation coefficient $\tau$ has some advantages and disadvantages in comparison with the Pearson product-moment correlation coefficient $\rho$.

Advantages.

(1) We think that $\tau$ reflects the rate of dependence between $X$ and $Y$ better than $\rho$ since $\tau$ is based on the whole information about a distribution. Observe that $\rho$ is based only on the first and second moments.

(2) No moment assumption is needed for the existence of $\tau$.

(3) In continuous case, we have proved that $\tau_n \xrightarrow{P} \tau$. It is also true that $\rho_n \xrightarrow{P} \rho$. That way, both $\tau$ and $\rho$ are approximated for large values of $n$ by $\tau_n$ and $\rho_n$, respectfully. However, $E\tau_n = \tau$.\]
Disadvantages.

(1) The Pearson product-moment correlation coefficient $\rho$ is simpler than $\tau$. To compute $\rho$ one should only know the first and the second moments. To compute $\tau$ one should know the distribution $F$. In this respect, $\rho$ has an advantage over $\tau$, because in many real experiments (say, in the time-series analysis) one often works only with first and second moments.

(2) The Pearson product-moment correlation coefficient $\rho$ is visibly presented in many important statistical models and formulas such as linear regression models, bivariate normals densities and so on.

5 Examples

Example 5.1. Let

$$F(x, y) = 1 - e^{-x} - \frac{1 - e^{-x(y+1)t}}{(y+1)^t} \quad (x > 0, y > 0, t > 0)$$

be a bivariate distribution with marginal distributions $H(x) = 1 - e^{-x} (x > 0)$ and $G(y) = 1 - \frac{1}{(y+1)^t} (y > 0)$. It follows from (2.2) that for any $t > 0$

$$\tau = E\tau_n = -0.5.$$  
We made a corresponding simulation experiment, i.e. $\tau_n$ was computed many times for ”large” values of $n$ by simulation in Matlab. The code for Kendall’s correlation coefficient is

```
corrcoef(x,y,’type’,Kendall).
```

The experiment gave us the same value $-0.5$.

Observe that here $\rho = -\frac{\sqrt{t(t-2)}}{2t-1} \quad (t > 2)$.

Example 5.2. Let

$$F(x, y) = 1 - \frac{1}{(y+1)^t} - \frac{1}{(x+1)^t} + \frac{1}{(x+y+1)^t} \quad (x > 0, y > 0, t > 0)$$

be a bivariate distribution with marginal distributions $H(x) = 1 - \frac{1}{(x+1)^t} (x > 0)$ and $G(y) = 1 - \frac{1}{(y+1)^t} (y > 0)$. It follows from (2.2) that for any $t > 0$

$$\tau = \frac{1}{2t+1}.$$  
A corresponding simulation experiment made for different values of $t > 0$ confirmed this result.

Observe that here $\rho = \frac{1}{t} \quad (t > 2)$. 
Example 5.3. Let

\[ F(x, y) = xy(1 + \alpha(1 - x)(1 - y)) \quad (0 < x, y < 1, -1 \leq \alpha \leq 1) \]

be a bivariate distribution with marginal distributions \( H(x) = x (0 < x < 1) \) and \( G(y) = y (0 < y < 1) \). It follows from (2.2) that for any \( \alpha \)

\[ \tau = \frac{2\alpha}{9}. \]

A corresponding simulation experiment made for different values of \( \alpha \) confirmed this result.

Observe that here \( \rho = \frac{\alpha}{3} \) \((-1 \leq \alpha \leq 1)\).

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