Mean Convergence of Vector–valued Walsh Series

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Abstract

Given any Banach space $X$, let $L_2^X$ denote the Banach space of all measurable functions $f : [0, 1] \to X$ for which

$$
\|f\|_2 := \left( \int_0^1 \|f(t)\|^2 \, dt \right)^{1/2}
$$

is finite. We show that $X$ is a UMD–space (see [1]) if and only if

$$
\lim_n \|f - S_n(f)\|_2 = 0 \quad \text{for all } f \in L_2^X,
$$

where

$$
S_n(f) := \sum_{i=0}^{n-1} \langle f, w_i \rangle w_i
$$

is the $n$–th partial sum associated with the Walsh system $(w_i)$.

1 Introduction

There are several enumerations of the system of Walsh functions. Therefore we first give the appropriate definition.

For $i \geq 1$, the Rademacher functions $(r_i)$ are defined as follows

$$
r_1(t) := \begin{cases} +1 & \text{for } t \in [0, \frac{1}{2}) + \mathbb{Z} \\ -1 & \text{for } t \in [\frac{1}{2}, 1) + \mathbb{Z} \end{cases} \quad \text{and} \quad r_{i+1}(t) := r_1(2^i t).
$$

Let $n \in \mathbb{N}$. Then $n$ has a unique representation of the form

$$
n = \sum_{i=0}^\infty n_i 2^i,
$$

with $n_i \in \{0, 1\}$. Note that in fact only a finite number of the $n_i$ are different from zero. We let

$$
w_n(t) := \prod_{i=0}^\infty r_{i+1}(t)^{n_i}.
$$
Again the formally infinite product on the righthand side is finite, hence \( w_n \) is well defined.

For \( f \in L^2_X \), we denote by \( S_n(f) \) the \( n \)–th partial sum of the Walsh series of \( f \),

\[
S_n(f) := \sum_{i=0}^{n-1} \langle f, w_i \rangle w_i,
\]

where

\[
\langle f, w_i \rangle := \int_0^1 f(t) w_i(t) \, dt.
\]

Let \( X \) and \( Y \) be Banach spaces. For \( T : X \to Y \), the ideal norm \( \delta(T|\mathcal{W}_n, \mathcal{W}_n) \) is defined to be the least constant \( c \geq 0 \) such that for all \( f \in L^2_X \)

\[
\|TS_n(f)\|_2 \leq c\|f\|_2.
\]

Note that

\[
\delta(T|\mathcal{W}_{2p}, \mathcal{W}_{2p}) = \|T\| (1)
\]

for all operators \( T \) and \( p \in \mathbb{N} \); see e. g. [2]. In order to get a non–decreasing sequence of ideal norms, we let

\[
\delta_{\text{max}}(T|\mathcal{W}_n, \mathcal{W}_n) := \max_{1 \leq k \leq n} \delta(T|\mathcal{W}_k, \mathcal{W}_k).
\]

For a more general treatment of ideal norms associated with orthogonal systems we refer to [4], from where the above notation is adopted.

For \( k = 1, \ldots, 2^n \), let

\[
\Delta_k^{(n)} := \left[ \frac{k - 1}{2^n}, \frac{k}{2^n} \right]
\]

be the \( k \)–th dyadic intervall of order \( n \).

A dyadic martingale is a martingale \( (M_0, M_1, \ldots) \) relative to the dyadic filtration \( \mathcal{F} = (\mathcal{F}_n) \), where \( \mathcal{F}_n \) is generated by \( \{\Delta_k^{(n)} : k = 1, \ldots, 2^n\} \). If \( (M_0, M_1, \ldots) \) is an \( X \)–valued dyadic martingale, then there exist elements \( x_j \in X \) such that

\[
M_i = \sum_{j=0}^{2^i-1} x_j h_j,
\]

where \( h_j \) denotes the \( j \)–th Haar function

\[
h_0 \equiv 1
\]

\[
h_j(t) := \begin{cases} +2^{(p-1)/2} & \text{for } t \in \Delta_{2m+1}^{(p)}, \\
-2^{(p-1)/2} & \text{for } t \in \Delta_{2m+2}^{(p)}, \\
0 & \text{otherwise},
\end{cases}
\]

and \( j = 2^{p-1} + m, m = 0, \ldots, 2^{p-1} - 1 \).
As usual, we let
\[ \text{d}M_i := M_{i+1} - M_i. \]

Given \( p \in \{1, 2, \ldots \} \), let \( \mu_p(T) \) denote the least constant \( c \geq 0 \) such that for all \( X \)-valued dyadic martingales \((M_0, M_1, \ldots, M_p)\) and for all sequences \( \varepsilon_0, \ldots, \varepsilon_{p-1} \) of signs \( \pm 1 \) we have
\[
\left\| \sum_{i=0}^{p-1} \varepsilon_i T dM_i \right\|_2 \leq c \|M_p\|_2.
\]

We write \( \mu_p(X) \) instead of \( \mu_p(I_X) \), where \( I_X \) denotes the identity map of the Banach space \( X \).

Note that for all \( T : X \to Y \)
\[ \mu_{p-1}(T) \leq \mu_p(T). \]  
(3)

Choosing \( M_p := M_{p-1} \) in the defining inequality of \( \mu_p(T) \), we get \( dM_{p-1} = 0 \) and hence
\[
\left\| \sum_{i=0}^{p-2} \varepsilon_i T dM_i \right\|_2 \leq \mu_p(T) \|M_{p-1}\|_2,
\]
which proves the desired inequality.

With the above notation we can prove the following result.

**Theorem**  For all operators \( T : X \to Y \) and \( p \in \mathbb{N} \), we have
\[ \delta_{\max}(T|W_{2p}, W_{2p}) \leq \mu_p(T) \leq 2\delta_{\max}(T|W_{2p}, W_{2p}). \]

By definition a Banach space \( X \) has the UMD–property if there exists a constant \( c \geq 0 \) such that
\[
\left\| \sum_{i=0}^{n} \varepsilon_i dM_i \right\|_2 \leq c \left\| \sum_{i=0}^{n} dM_i \right\|_2
\]
for all martingales \((M_0, M_1, \ldots)\) with values in \( X \) and all \( n \in \mathbb{N} \). This is equivalent to the boundedness of the sequence \( \mu_p(X) \); see [1].

Thus the theorem gives a characterization of UMD–spaces by the mean convergence of \( X \)-valued Walsh series.

## 2 Preliminaries

Let \( s, t \in [0, 1] \). Then \( s \) and \( t \) have unique representations \( s = \sum_{j=0}^{\infty} s_j 2^{-j-1} \) and \( t = \sum_{j=0}^{\infty} t_j 2^{-j-1} \), respectively, supposed we choose them to be finite if possible. By \( s \oplus t \) we denote the dyadic sum of \( s \) and \( t \),
\[ s \oplus t := \sum_{j=0}^{\infty} |t_j - s_j|2^{-j-1}. \]
Then
\[ \int_0^1 f(s) \, ds = \int_0^1 f(s \oplus t) \, ds \] (4)
for all \( f \in L_1 \) and \( t \in [0, 1] \). Moreover
\[ w_n(s \oplus t) = w_n(s) w_n(t). \] (5)

For \( n \geq 1 \), let
\[ D_n(t) := \sum_{i=0}^{n-1} w_i(t) \]
be the \( n \)-th Dirichlet kernel associated with the Walsh functions.

We have
\[ S_n(f)(t) = \int_0^1 f(s) D_n(s \oplus t) \, ds. \] (6)

For \( n \geq 1 \), let \( 0 \leq k_1 < k_2 < \ldots < k_s \) be defined by
\[ n = \sum_{l=1}^{s} 2^{k_l}. \] (7)

We will use the following result from [3, Theorem 8, p. 28].

Lemma 1
\[ D_n = w_n \sum_{i \in \{k_1, \ldots, k_s\}} (D_{2^{i+1}} - D_{2^i}). \]

3 Proof of the theorem

For \( n \) as in (7), we have by (4), (6) and lemma 1
\[ \|TS_n(f)\|_2 = \left\| \sum_{i \in \{k_1, \ldots, k_s\}} (TS_{2^{i+1}}(fw_n) - TS_{2^i}(fw_n)) \right\|_2 \]
\[ \leq \frac{1}{2} \sum_{i=0}^{k_s} \left\| TS_{2^{i+1}}(fw_n) - TS_{2^i}(fw_n) \right\|_2 + \]
\[ + \frac{1}{2} \sum_{i=0}^{k_s} \varepsilon_i (TS_{2^{i+1}}(fw_n) - TS_{2^i}(fw_n)) \right\|_2, \] (8)
where \( \varepsilon_i \) is defined by
\[ \varepsilon_i := \begin{cases} +1 & \text{if } i \in \{k_1, \ldots, k_s\} \\ -1 & \text{if } i \notin \{k_1, \ldots, k_s\} \end{cases}. \]
Note that $M_i := S_2(fw_n)$ form a dyadic martingale of the form (2), since the linear span of the Walsh functions $w_0, \ldots, w_{2^{p}-1}$ coincides with the linear span of the Haar functions $h_0, \ldots, h_{2^{p}-1}$; see [4]. Hence we have

$$\left\| \sum_{i=0}^{k_s} \varepsilon_i(TS_{2^{i+1}}(fw_n) - TS_2(Tfw_n)) \right\|_2 = \left\| \sum_{i=0}^{k_s} \varepsilon_i TdM_i \right\|_2 \leq \mu_{k_s+1}(T) \|S_{2^{k_s+1}}(fw_n)\|_2.$$ 

The same argument applied with $\varepsilon_i = +1$ for all $i = 0, \ldots, k_s$ yields

$$\left\| \sum_{i=0}^{k_s} (TS_{2^{i+1}}(fw_n) - TS_2(fw_n)) \right\|_2 = \left\| \sum_{i=0}^{k_s} TdM_i \right\|_2 \leq \mu_{k_s+1}(T) \|S_{2^{k_s+1}}(fw_n)\|_2.$$ 

Therefore we obtain from (8) that

$$\|TS_n(f)\|_2 \leq \mu_{k_s+1}(T) \|S_{2^{k_s+1}}(fw_n)\|_2.$$ 

If $n < 2^p$, then $k_s + 1 \leq p$ and it follows from (1) and (2) that

$$\|TS_n(f)\|_2 \leq \mu_{k_s+1}(T) \|fw_n\|_2 \leq \mu_p(T) \|f\|_2.$$ 

If $n = 2^p$, then again by (1)

$$\|TS_n(f)\|_2 \leq \|T\| \|S_n(f)\|_2 \leq \mu_p(T) \|f\|_2.$$ 

Consequently

$$\|TS_n(f)\|_2 \leq \mu_p(T) \|f\|_2$$

for all $1 \leq n \leq 2^p$ and hence

$$\delta_{\max}(T|\mathcal{W}_{2^p}, \mathcal{W}_{2^p}) \leq \mu_p(T).$$

This proves the lefthand inequality of the theorem.

To check the righthand inequality we use the following fact.

**Lemma 2** Let $I \subseteq \{0, \ldots, p-1\}$ and let $n$ be defined by $n := \sum_{i \in I} 2^i < 2^p$. Then we have

$$\left\| \sum_{i \in I} TdM_i \right\|_2 \leq \delta(T|\mathcal{W}_n, \mathcal{W}_n) \|M_p\|_2$$

for all martingales $(M_0, \ldots, M_p)$ of the form (3).

**Proof:** We write $M_i$ in the form

$$M_i = \sum_{j=0}^{2^i-1} x_j w_j,$$
where

\[ x_j := \int_0^1 M_p(t) w_j(t) \, dt \in X. \]

Then, by lemma [I],

\[
\left\| \sum_{i \in I} T dM_i \right\|_2 \leq \left\| \sum_{i \in I} \left( \sum_{j=2^i}^{2^{i+1}-1} T x_j w_j \right) \right\|_2
\]

\[
= \left\| \sum_{i \in I} (TS_{2^{i+1}}(M_p) - TS_2(M_p)) \right\|_2
\]

\[
= \left\| \sum_{i \in I} \left( \int_0^1 T M_p(s) \left( D_{2^{i+1}}(s \oplus t) - D_{2^i}(s \oplus t) \right) \, ds \right) \right\|_2
\]

\[
= \left\| \int_0^1 T M_p(s) w_n(s \oplus t) D_n(s \oplus t) \, ds \right\|_2
\]

\[
= \left\| TS_n(M_p w_n) \right\|_2
\]

\[
\leq \delta(T|W_n, W_n) \| M_p w_n \|_2
\]

\[
= \delta(T|W_n, W_n) \| M_p \|_2.
\]

\[\square\]

We are now able to complete the proof of the theorem. Let a sequence \( \varepsilon_0, \ldots, \varepsilon_{p-1} \) of signs \( \pm 1 \) be given. Define \( n \) and \( m \) by

\[ n := \sum_{\{i : \varepsilon_i = +1\}} 2^i \quad \text{and} \quad m := \sum_{\{i : \varepsilon_i = -1\}} 2^i. \]

Then we get from lemma [I] that

\[
\left\| \sum_{i=0}^{p-1} \varepsilon_i T dM_i \right\|_2 \leq \left\| \sum_{\{i : \varepsilon_i = +1\}} T dM_i \right\|_2 + \left\| \sum_{\{i : \varepsilon_i = -1\}} T dM_i \right\|_2
\]

\[
\leq \delta(T|W_n, W_n) \| M_p \|_2 + \delta(T|W_m, W_m) \| M_p \|_2
\]

\[
\leq 2\delta^{\max}(T|W_{2p}, W_{2p}) \| M_p \|_2.
\]

Since this holds for all sequences \( (\varepsilon_i) \), we have

\[ \mu_p(T) \leq 2\delta^{\max}(T|W_{2p}, W_{2p}), \]

which is the desired righthand inequality.

### 4 Some consequences

**Corollary 1** The following conditions are equivalent.
(i) \( \| f - S_n(f) \|_2 \to 0 \) for all \( f \in L^X_2 \).

(ii) \( X \) has the UMD–property.

**Proof:** If \( X \) has the UMD–property, then by the theorem
\[
\delta_{\max}(X|W_n, W_n) \leq c
\]
for all \( n \in \mathbb{N} \). Since the Walsh functions form a complete orthonormal system in \( L^2_1[0, 1] \), we can find a linear combination \( \sum_{k=0}^{N} x_k w_k \in L^X_2 \) with
\[
\left\| f - \sum_{k=0}^{N} x_k w_k \right\|_2 \leq \varepsilon.
\]
Then, for \( n \geq N \),
\[
\| f - S_n(f) \|_2 \leq \left\| f - \sum_{k=0}^{N} x_k w_k \right\|_2 + \left\| S_n(\sum_{k=0}^{N} x_k w_k - f) \right\|_2 \\
\leq \varepsilon + \delta_{\max}(X|W_n, W_n) \varepsilon \leq (1 + c)\varepsilon.
\]
If on the other hand
\[
\| f - S_n(f) \|_2 \to 0 \quad \text{for all } f \in L^X_2,
\]
then, by the uniform boundedness theorem, we get
\[
\| S_n(f) \|_2 \leq c \| f \|_2
\]
and hence \( X \) is a UMD–space. \( \square \)

As a further easy application of the theorem we get the order of growth of \( \mu_p(X) \) for \( X = L_1[0, 1] \).

To this end, let
\[
L_n := \int_{0}^{1} |D_n(t)| \, dt \tag{9}
\]
be the Lebesgue constants associated with the Walsh system. We also consider
\[
L^\max_n := \max_{k \leq n} L_k.
\]

**Corollary 2**
\[
\frac{p}{8} \leq \mu_p(L_1[0, 1]) \leq 2p.
\]
Proof: For $X = L_1[0, 1]$ and $f \in L_1[0, 1]$, we define $F \in L_2^X$ by

$$F(t) := f(t \oplus \cdot).$$

Then, by (4) and (5), we have

$$\langle F, w_j \rangle = \frac{1}{0} \int F(t)w_j(t) \, dt = \frac{1}{0} \int f(t \oplus \cdot)w_j(t) \, dt = \int f(t)w_j(t)w_j(\cdot) \, dt = \langle f, w_j \rangle w_j.$$ 

Hence

$$\frac{1}{n-1} \sum_{j=0}^{n-1} \langle F, w_j \rangle w_j(t) = \frac{1}{n-1} \sum_{j=0}^{n-1} \langle f, w_j \rangle w_j(t \oplus \cdot)$$

Furthermore

$$\|F\|_2^2 = \frac{1}{0} \int \|F(t)\|^2_2 \, dt = \frac{1}{0} \left( \int \|f(t \oplus s)\|_1 \, ds \right)^2 \, dt = \frac{1}{0} \left( \int |f(s)| \, ds \right)^2 \, dt = \|f\|_1^2.$$ 

Similarly

$$\left\| \sum_{j=0}^{n-1} \langle F, w_j \rangle w_j \right\|_2 = \left\| \sum_{j=0}^{n-1} \langle f, w_j \rangle w_j \right\|_1.$$ 

If we now choose $f$ to be the characteristic function of the interval $[0, 2^{-p}]$, then for $k \leq n < 2^p$

$$\langle f, w_k \rangle = 2^{-p} \quad \text{and} \quad \|f\|_1 = 2^{-p}.$$ 

Consequently

$$\delta(L_1|W_n, W_n) \geq \frac{\left\| \sum_{j=0}^{n-1} \langle F, w_j \rangle w_j \right\|_2}{\|F\|_2} = L_n,$$

where $L_n$ denotes the Lebesgue constant as defined in (9).

Since

$$\delta(X|W_n, W_n) \leq L_n$$

for all Banach spaces $X$, we have

$$\delta(L_1|W_n, W_n) = L_n.$$ 

This proves corollary 2 by taking into account our theorem and the following result from [6, Theorem 9, p. 34].
Lemma 3
\[ \frac{p}{8} \leq L^\max_{2p} \leq p. \]

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