A NOTE ON THE SCHMID-WITT SYMBOL AND HIGHER LOCAL FIELDS

MATTHEW SCHMIDT

Abstract. For a local field of characteristic $p > 0$, $K$, the combination of local class field theory and Artin-Schreier-Witt theory yield what is known as the Schmid-Witt symbol. The symbol encodes interesting data about the ramification theory of $p$-extensions of $K$ and we can, for example, use it to compute the higher ramification groups of such extensions. In 1936, Schmid discovered an explicit formula for the Schmid-Witt symbol of Artin-Schreier extensions of local fields. Later, his formula was generalized to Artin-Schreier-Witt extensions, but still over a local field. In this paper we generalize Schmid’s formula to compute the Artin-Schreier-Witt-Parshin symbol for Artin-Schreier-Witt extensions of two-dimensional local fields.

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1. Introduction

Let $k$ be a finite field of characteristic $p$ and $K = k((T))$. Denote by $W(K)$ the Witt vectors of $K$ and by $W_n(K)$ the truncated Witt vectors of $K$ of length $n$. Define a map on $W(K)$, $\phi(x_0, x_1, \cdots) = (x_0^p - x_0, x_1^p - x_1, \cdots)$. Combining Local class field theory and Artin-Schreier-Witt theory, the Schmid-Witt symbol

\[ [\cdot, \cdot)_n : W_n(K)/\phi W_n(K) \times K^\times / (K^\times)^{p^n} \to W_n(F_p), \]

was first studied in the $n = 1$ case by [11] who discovered the explicit formula

\[ [x, y]_1 = \text{Tr}_{k/F_p}(\text{res}(x \cdot \log y)), \]

where $\text{res} : K \to k$ is the residue map and $d\log$ is logarithmic differentiation. For $n > 1$, [13], [13] and more recently [6] have produced explicit formulas generalizing

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In this paper, we generalize (2) to two-dimensional local fields $K = k((S))((T))$ for $n \geq 1$, following the work of (4).

More precisely, a two-dimensional local field $K$ is a complete discrete valuation field which has as its residue field a local field (i.e. a “one-dimensional” local field). For example, if $K = k((S))((T))$, then the residue field of $K$ is $F = \bar{K} = k((S))$, which is a classical local field. For such fields (and higher local fields in general) we have the Artin-Schreier-Witt-Parshin symbol (2):

$$[\cdot, \cdot] : W_n(K)/\wp W_n(K) \times K_2^{top}(K)/(K_2^{top}(K))^{p^n} \to W_n(\mathbb{F}_p),$$

where $K_2^{top}(K)$ is the topological Milnor $K$-group (2). We will study this symbol after taking $n \to \infty$,

$$[\cdot, \cdot] : W(K)/\wp W(K) \times K_2^{top}(K) \to W_n(\mathbb{F}_p),$$

where $K_2^{top}(K)$ is the $p$-adic completion of the topological Milnor $K$-group.

Much like (13) and (6), we rely on a pair of very explicit liftings in order to express the Artin-Schreier-Witt-Parshin symbol as the trace of a residue. Namely, let $\hat{K} = \text{Fr}(W(k)((S))((T)))$ and define lifts $\hat{\cdot} : W(K) \to W(\hat{K})$ and $\hat{\cdot} : K_2^{top}(K) \to K_2^{top}(\hat{K})$. (See Section 4 for more details.) Our main result is then:

**Theorem 1.** For $x \in W(K)$ and $y \in K_2^{top}(\hat{K})$,

$$[x, y] = \text{Tr}_{W(k)/W(\mathbb{F}_p)}(\text{res}(\hat{x} \cdot d \log(\hat{y}))).$$

As an application, we use this formula to explicitly compute the upper ramification groups as defined in (7) and (3).

**Theorem 2.** For two vectors $\vec{r} = (r_1, r_2), \vec{m} = (m_1, m_2) \in \mathbb{Z}_{\geq 0}^2$, let $\ell(\vec{r}, \vec{m})$ be the minimum integer $\ell$ such that $(p^\ell m_1, p^\ell m_2) \notin \vec{r}$. Then

$$G^{\vec{r}} = \left(\prod_{p^\ell m_1, p^\ell m_2 \in \mathbb{Z}_{\geq 0}^2} W(k) \right)^{(m_1, m_2) \in \mathbb{Z}_{\geq 0}^2}.$$ 

Theorem 2 generalizes results of (13) which describe the ramification groups of pro-$p$ abelian extensions for a one-dimensional local field $K$.

**Remark.** While we prove our results only for two-dimensional local fields, similar formulas and proofs easily generalize to higher dimensions.

### 2. Preliminaries

#### 2.1. Witt Vectors.

Let $R$ be a commutative ring with unity, and $A$ an $R$-algebra. Denote by $W(R) = \varprojlim_m W_m(R)$ the ring of Witt vectors over $R$, with $W_m(R)$ the truncated Witt vectors. Denote the $m$th ghost vector component of a Witt vector $x = (x_i) \in W(A)$ by

$$g^{(m)} = \sum_{i=0}^m p^i x_i^{p^m-1},$$

and let $g := (g^{(0)}, g^{(1)}, \cdots) : W(R) \to \mathbb{R}_{\geq 0}$ be the ghost vector map.

Finally, for an $r \in R$, define the Teichmüller lift of $a$ in $W(R)$ to be $[a] = (a, 0, \cdots)$.

For a detailed exposition of Witt vectors see (14).
2.2. Higher Local Fields. For a discrete valuation field \( K \), let \( O_K \) be the valuation ring of \( K \) and \( m_K \) the maximal ideal of \( O_K \). The residue field of \( K \) is defined as \( \bar{K} = O_K/m_K \). A complete discrete valuation field \( K \) is said to be an \( n \)-dimensional local field if there is a sequence of complete discrete valuation fields \( K_0, \ldots, K_n = K \) such that:

1. \( K_0 \) is a finite field.
2. \( K_i = K_{i-1} \) for \( 1 \leq i \leq n \).

In this paper, we will only consider 2-dimensional local fields of equal characteristic. That is, fix some finite field of characteristic \( p > 0 \), \( k \), and let \( K = k((S))((T)) \). \( K \) is then an equal characteristic 2-dimensional local field because \( K \) and all of its residue fields have characteristic \( p \).

Place an order on \( \mathbb{Z}^2 \) as follows: for \( \vec{i} = (i_1, i_2), \vec{j} = (j_1, j_2) \in \mathbb{Z}^2 \),

\[
\vec{i} < \vec{j} \iff \begin{cases} i_1 < j_1, i_2 = j_2 & \text{or} \\ i_1 = j_1, i_2 < j_2. \end{cases}
\]

\( K \) can then be given a valuation \( v_K : K \to \mathbb{Z}^2 \) by \( \sum_{(j,i) \in \Omega} a_{ji} S^j T^i \mapsto (j_0, i_0), \ i_0 \) chosen minimally (\( f \) is a Laurent series in \( T \)) and \( j_0 \) chosen minimally among the terms \( S^j T^{i_0} \). (The coefficient of \( T^{i_0} \) is a laurent series in \( S \) and so a minimum exists.)

We will close this section with a couple structure theorems regarding \( K \) and \( W(K) \). Let \( B \) be an \( \mathbb{F}_p \)-basis for \( k/\wp k \) and \( C \) an \( \mathbb{F}_p \)-basis over \( k \). Fix some \( \beta = [\alpha] \in W(k) \) where \( \alpha \in \bar{k} \) such that \( \text{Tr}_{k/\mathbb{F}_p}(\alpha) \neq 0 \).

**Proposition 3.** The set

\[
\mathcal{D} = B \cup \{ cS^{-i}T^{-j} | c \in C, i \geq 0 \text{ or } j \geq 0, p \nmid (i, j) \}
\]

is a \( \mathbb{F}_p \)-basis for \( K/\wp K \).

**Proof.** See Lemma 2, Section 1 of [9]. \( \square \)

**Corollary 1.** Every \( x \in W(K) \) has a unique representative in \( W(K)/\wp W(K) \) of the form

\[
c\beta + \sum_{(i,j) \in \mathbb{Z}^2, i \text{ or } j < 0, (i,p) = 1 \text{ or } (j,p) = 1} c_{ij}[S^i][T^j],
\]

where \( c \in W(\mathbb{F}_p) \) and \( c_{ij} \in W(k) \) with \( c_{ij} \to 0 \) as \( j \to \infty \) for any fixed \( i \).

**Proof.** The unique representation follows from Proposition 3.10 in [6] and Proposition 3. \( \square \)

2.3. Milnor K-groups. Denote the Milnor two \( K \)-group of \( K \) by \( K_2(K) \). We write \( K_2(K) \) as a multiplicative abelian group on symbols \( \{a, b\} \), with \( a, b \in K^\times \). Define the topological Milnor \( K \)-group, \( K_2^{\text{top}}(K) \), to be the quotient of \( K_2(K) \) by the intersection of all its neighborhoods of zero. (See [2] or [8].)

Let \( \vec{i} \in \mathbb{Z}^2_{\geq 0} \). Define the \( \vec{i} \)th unit group of \( K_2^{\text{top}}(K) \) to be:

\[
U^{\vec{i}}K_2^{\text{top}}(K) = \{ \{u, x\} | u, x \in K^\times, v_K(u - 1) \geq \vec{i} \}.
\]

Similarly, let \( V_K = 1 + TO_K \) and denote by \( VK_2^{\text{top}}(K) \) the subgroup of \( K_2^{\text{top}}(K) \) generated by elements of the form \( \{u, x\}, u \in V_K \) and \( x \in K \). \( VK_2^{\text{top}}(K) \) has the following decomposition:
Proposition 4. Every $y \in V K_2^{top}(K)$ can be written uniquely in the form:

$$
\prod_{i \geq 1, j \in \mathbb{Z}} \left( \prod_{k=0}^{\infty} \{1 + a_{ijk}S^iT_i, S\}^{c_{ijk}p^k} \right) \prod_{i \geq 1, j \in \mathbb{Z}} \left( \prod_{k=0}^{\infty} \{1 + b_{ijk}S^iT_i, T\}^{d_{ijk}p^k} \right),
$$

for some $a_{ijk}, b_{ijk} \in k$ and $c_{ijk}, d_{ijk} \in [0, p - 1]$.

Proof. Proposition 2.4 in [1]. (Or Proposition 0.4.5 in [16] for a slightly stronger version on $U^1 K_2^{top}(K)$.)

2.4. Higher Local Class Field Theory. Let $G_K = \text{Gal}(K^{ab}/K)$ be the Galois group of the maximal abelian extension of $K$, $G_{p^n} = G_K/(G_K)^{p^n}$ be the Galois group of the maximal abelian extension of exponent $p^n$ of $K$, and $G_{p^\infty} = \varprojlim_n G_{p^n}$ be the Galois group of the maximal abelian pro-$p$ extension of $K$.

Let $\omega_F : F^x \to G_F$ be the regular Artin reciprocity map from one dimensional local class field theory and let $\omega_K : K_2(K) \to G_K$ be Kato’s reciprocity map for higher local fields ([9], [4]).

3. Computation of $G_{p^\infty}$

The goal of this section will be to get an explicit description of $G_{p^\infty}$ via Artin-Schreier-Witt theory and higher local class field theory.

Proposition 5. $K_2^{top}(K)/(K_2^{top}(K))^{p^n} \cong G_{p^n}$.

Proof. By Theorem 3.4 in [6], $W_n(K)/\wp W_n(K)$ and $G_{p^n}$ are dual to each other. However, by Proposition 1 in [10], $K_2^{top}(K)/(K_2^{top}(K))^{p^n} \cong \text{Hom}_{cont}(W_n(K)/\wp W_n(K), \mathbb{Q}/\mathbb{Z})$, and so Pontryagin duality yields the isomorphism.

Corollary 2. $\hat{K}_2^{top}(K) = \varprojlim_n K_2^{top}(K)/p^n \cong G_{p^\infty}$.

Lemma 6.

$VK_2^{top}(K) \cong \varprojlim_n VK_2^{top}(K)/(VK_2^{top}(K))^{p^n}$.

Proof. By Remark 1 in Section 4 in [2] (p.11), the natural map

$$VK_2^{top}(K) \to \varprojlim_n VK_2^{top}(K)/(VK_2^{top}(K))^{p^n}$$

is surjective. Moreover, by an earlier remark following 4.2 in [1] (p.8),

$$\cap_n (VK_2^{top}(K))^{p^n} = \{1\},$$

so the natural map must also be injective.

Proposition 6.

$G_{p^\infty} \cong \hat{K}_2^{top}(K) \cong \mathbb{Z}_p \times VK_2^{top}(K)$.

Proof. It is known that $K_2^{top}(K) \cong \langle \{S, T\} \rangle \times (k^\times)^2 \times VK_2^{top}(K)$ (See [1], p.8). So let $\{S, T\}^i \in \langle \{S, T\} \rangle$. Then for any $i > 0$, $\{S, T\}^i = \{S^i, T\}$, and we see the map $\langle \{S, T\} \rangle \to \mathbb{Z}$ given by $\{S^i, T\} \mapsto i$ is an isomorphism. This implies $\langle \{S, T\} \rangle/(\{S, T\})^{p^n} \cong \mathbb{Z}/p^n\mathbb{Z}$.
Because $k^p = k$, $k^\times/(k^\times)^p$ = 1 and $\lim_{n \to \infty} k^\times/(k^\times)^p^n = 1$. Then, by Lemma 6 we see that:

$$K_2^{\text{top}}(K) \cong \frac{\lim_n \langle \{S, T\} \rangle/(\{S, T\} \rangle)^p \times V K_2^{\text{top}}(K)/(V K_2^{\text{top}}(K))^p}{\cong \mathbb{Z}_p \times \lim_n V K_2^{\text{top}}(K)/(V K_2^{\text{top}}(K))^p}$$

Then, by Lemma 6, we see that:

$$\hat{K} \cong \mathbb{Z}_p \times \lim_n V K_2^{\text{top}}(K)/(V K_2^{\text{top}}(K))^p$$

\[\square\]

4. THE EXPLICIT LOCAL SYMBOL

The formula in Theorem 1 depends on several liftings and maps on the Khler differential $\Omega^2_{K/k}$. We define those now.

Let $\tilde{K} = \text{Fr}(W(k))((S))((T))$. For $x \in W(K)$, define $\hat{x}$ to be a lift of $x$ to $W(\tilde{K})$ (by taking Teichmuller liftings coefficient-wise). Similarly, for $y \in K_2^{\text{top}}(K)$, define $\tilde{y}$ to be the lift to $K_2^{\text{top}}(\tilde{K})$.

Explicitly, we can define $\hat{x}$ in the following manner. Write $x = \left(\sum_{(i,j)} a_{i,j,k} S^i T^j\right)_k \in W(K) = W(k)((S)((T)))$, where $(i,0,k,j,0,k)$ is with minimal $j_0,k$ so that $a_{i,0,k,j_0,k} \neq 0$. Then

$$\hat{x} = \left(\sum_{(i,j)} A_{i,j,k} S^i T^j\right)_k \in W(\text{Fr}(W(k))((S)((T))))$$

where $A_{i,j,k} = (a_{i,j,k}, \cdots) \in W(k)$.

**Definition 8.** Let $\omega = \sum_{i,j} a_{i,j,k} S^i T^j dS \wedge dT \in \Omega^2_{K/k}$. Define the map

$$\text{res} : \Omega^2_{K/k} \to W(k)$$

$$\text{res}_K(\omega) = a_{-1,-1}.$$

**Definition 9.** For $\{a_1, a_2\} \in K_2(\tilde{K})$ let

$$d \log : K_2(\tilde{K}) \to \Omega^2_{K/k}$$

$$d \log(\{a_1, a_2\}) \mapsto \frac{da_1}{a_1} \wedge \frac{da_2}{a_2}$$

We are now ready to define the non-degenerate higher dimensional Schmid-Witt symbol as defined in (9):

**Definition 10.** (Parshin’s Formula) The explicit Artin-Schreier-Witt-Parshin symbol is given by:

$$[\cdot, \cdot : W_m(K)/\wp W_m(K) \times K_2^{\text{top}}(K)/p^m \to W_m(F_p) = \mathbb{Z}/p^m \mathbb{Z}$$

$$(x, y) \mapsto \text{Tr}_{W_m(K)/W_m(F_p)}(\tilde{y}^{-1}(\text{res}(g^{(i)} x \cdot d \log \tilde{y}))_{i=0}^{m-1} \mod p),$$

where the mod $p$ is taken component-wise.

In order to prove Theorem 2, we will compare our formula to Parshin’s formula and show they are equal (just as [6] compares their formula to [13]).
4.1. A new 2-dimensional formula for $[\ , \ )_K$. We define two liftings.

The first is from $W(K) / ϕW(K)$ to $\hat{K}$ uniquely by

$$x = cβ + \sum_{(i,j) \in \mathbb{Z}^2, i < 0, (i,p) = 1 \text{ or } (j,p) = 1} c_{ij} [S^i][T^j] \mod ϕW(K) \mapsto cβ + \sum_{(i,j) \in \mathbb{Z}^2, i < 0, (i,p) = 1 \text{ or } (j,p) = 1} c_{ij} S^i T^j = \hat{x},$$

where $c \in W(\mathbb{F}_p)$ and $c_{ij} \in W(k)$. (The existence and uniqueness of the representation follows from Corollary 1)

Second, we have a (not yet necessarily unique) lifting of $K_2^{top}(K)$ to $K_2^{top}(\hat{K})$.

That is, by Proposition 1, every $y \in K_2^{top}(K)$ has an explicit lifting to $K_2^{top}(\hat{K})$:

$$y = \{S^e, T\} \prod_{i} \{1 + a_{ij} S^i T^j, S\} \cdot \prod_{i} \{1 + b_{ij} S^i T^j, T\} \mapsto \{S^e, T\} \prod_{i} \{1 + [a_{ij}] S^i T^j, S\} \cdot \prod_{i} \{1 + [b_{ij}] S^i T^j, T\} = \hat{y}.$$

**Lemma 11.** Suppose that $i, j, ℓ_1, ℓ_2 \in \mathbb{Z}_{\geq 0}$ with $ij \neq 0$ such that $p \nmid (i, j)$ and $p \nmid (ℓ_1, ℓ_2)$. Then for all $h \in \mathbb{Z}_{\geq 0}$,

$$\frac{ℓ_1 p^h}{i} = \frac{ℓ_2 p^h}{j} \in \mathbb{Z}_{\geq 0} \iff \frac{ℓ_1}{i} = \frac{ℓ_2}{j} \in \mathbb{Z}_{\geq 0}.$$

**Proof.** The reverse direction is trivial. So suppose $\frac{ℓ_1 p^h}{i} = \frac{ℓ_2 p^h}{j} \in \mathbb{Z}_{\geq 0}$ and $i \mid ℓ_1 p^h$ but $i \nmid ℓ_1$. (That is, $v_p(i) > 0$.) Taking the $p$-adic valuation of $\frac{ℓ_1 p^h}{i} = \frac{ℓ_2 p^h}{j}$ yields the equation

$$v_p(ℓ_1) + h - v_p(i) = v_p(ℓ_2) + h - v_p(j).$$

Since $p \mid i$, we must have $p \nmid j$, so

$$v_p(ℓ_1) - v_p(i) = v_p(ℓ_2).$$

But the condition $p \nmid (ℓ_1, ℓ_2)$ implies $v_p(ℓ_1) = 0$ or $v_p(ℓ_2) = 0$. In the first case, $v_p(ℓ_1) = -v_p(ℓ_2)$, which is a contradiction (assuming $v_p(ℓ_2) \neq 0$) since $i \in \mathbb{Z}_{\geq 0}$. So $v_p(ℓ_2) = 0$. But then $v_p(ℓ_1) = v_p(i)$, so because we assumed $i \mid ℓ_1 p^h$, $i \mid ℓ_1$, a contradiction. Finally, if $v_p(ℓ_1) = 0 = v_p(ℓ_2)$, then $v_p(i) = 0$, a contradiction. \[\square\]

**Proof of Theorem 4.7.** We prove our result following the proof of Theorem 4.7 in [3].

Namely, via the structure of $W(K) / ϕW(K)$ and $K_2^{top}(\hat{K})$, and the $\mathbb{Z}_p$-bilinearity of $[\ , \ )$, it suffices to prove the claim for the following cases:

1. $[c[S^{ℓ_1} T^{ℓ_2}], \{S, T\}], c \in W(k), ℓ_1, ℓ_2 \in \mathbb{Z}_{\geq 0}, p \nmid (ℓ_1, ℓ_2),$
2. $[b[S^{ℓ_1} T^{ℓ_2}], \{1 + aS T^j, S\}, [b[S^{ℓ_1} T^{ℓ_2}], \{1 + aS T^j, T\}], a \in k, b \in W(k), ℓ_1, ℓ_2, i, j \in \mathbb{Z}_{\geq 0}, p \nmid (ℓ_1, ℓ_2) \text{ and } p \nmid (i, j).$

**Proof of (1):** Here $d \log(\{S, T\}) = \frac{dS}{S} \wedge \frac{dT}{T} = S^{-1}T^{-1}dS \wedge dT$ and $\hat{x} = cS^{ℓ_1} T^{ℓ_2}$. Then,

$$\text{Tr}_{W(k)/W(\mathbb{F}_p)}(\text{res}(\hat{x} \cdot d \log(\hat{y}))) = \text{Tr}_{W(k)/W(\mathbb{F}_p)}(\text{res}(cS^{ℓ_1} T^{ℓ_2} \cdot S^{-1}T^{-1}dS \wedge dT)) = \text{Tr}_{W(k)/W(\mathbb{F}_p)}(\text{res}(cS^{ℓ_1} T^{ℓ_2-1}dS \wedge dT)).$$

Therefore if $ℓ_1, ℓ_2 \neq 0, [c[S^{ℓ_1} T^{ℓ_2}], \{S, T\}] = 0$. If $ℓ_1 = ℓ_2 = 0$,

$$\text{Tr}_{W(k)/W(\mathbb{F}_p)}(\text{res}(cS^{ℓ_1-1} T^{ℓ_2-1}dS \wedge dT)) = \text{Tr}_{W(k)/W(\mathbb{F}_p)}(c).$$
On the other hand, by Proposition 10 in [14],

\[ c[S^{f_1} T^{f_2}] = (c_k S^{f_1} p^k T^{f_2} p^k)_{k=0}^{\infty}, \]

where \( c = (c_k)_0^{\infty} \). Then \( (c_k S^{f_1} p^k T^{f_2} p^k)_{k=0}^{\infty} = ([c_k] S^{f_1} p^k T^{f_2} p^k)_{k=0}^{\infty} \), and so

\[ g^{(h)}([c_k] S^{f_1} p^k T^{f_2} p^k)_{k=0}^{\infty} = \sum_{k=0}^{h} p^k ([c_k] S^{f_1} p^k T^{f_2} p^k)^{p^h-k} = \sum_{k=0}^{h} p^k [c_k]^{p^h-k} S^{f_1} p^k T^{f_2} p^k \]

\[ = (\sum_{k=0}^{h} p^k [c_k]^{p^h-k}) S^{f_1} p^k T^{f_2} p^k. \]

Thus:

\[ \text{(3)} \quad g^{(h)} \hat{x} \cdot S^{-1} T^{-1} dS \wedge dT = (\sum_{k=0}^{h} p^k [c_k]^{p^h-k}) S^{f_1} p^h-1 T^{f_2} p^h-1 dS \wedge dT, \]

and consequently (3) has nonzero residue if and only if \( \ell_1 = \ell_2 = 0 \). If \( \ell_1 = \ell_2 = 0 \), we get

\[ g^{-1}((\sum_{k=0}^{h} p^k [c_k]^{p^h-k})_{h=0}^{\infty}) = g^{-1}((g^h(c'))_{h=0}^{\infty}) = c', \]

where \( c' = ([c_k])_{k=0}^{\infty} \in W(W(k)) \). Because \( c' \equiv c \mod p \), Parshin’s formula finally gives:

\[ \text{Tr}_{W(k)/W(F_p)}(g^{-1}((\text{res}(g^{(i)} \hat{x} \cdot S^{-1} T^{-1} dS \wedge dT))_{i=0}^{m-1}) \mod p) = \text{Tr}_{W(k)/W(F_p)}(c). \]

**Proof of (2):** For simplicity, we will assume \( ij \neq 0 \). The other cases are easier and can be proven similarly. We first compute

\[ d \log(\{1 + a S^i T^j, S\}) = \frac{d(1 + a S^i T^j)}{1 + a S^i T^j} \wedge \frac{dS}{S} \]

\[ = -a_j \sum_{k=0}^{\infty} (-a)^k S^{(k+1)i-1} T^{(k+1)j-1} dS \wedge dT. \]

Likewise, \( d \log(\{1 + a S^i T^j, T\}) = a_i(1 + a S^i T^j)^{-1} S^{-1} T^{-1} dS \wedge dT \). Then as before, \( \hat{x} = c S^{f_1} T^{f_2} \), so

\[ \hat{x} \cdot d \log(\{1 + a S^i T^j, S\}) = -ajc \sum_{k=0}^{\infty} (-a)^k S^{(k+1)i+\ell_1-1} T^{(k+1)j+\ell_2-1}. \]

This has nonzero residue if and only if \( k + 1 = -\frac{\ell_1}{i} = -\frac{\ell_2}{j} \in \mathbb{Z}_{\geq 1} \), that is, \( k = \frac{-i-\ell_1}{i} = \frac{-j-\ell_2}{j} \in \mathbb{Z}_{\geq 0} \). (If \( \ell_1 = \ell_2 = 0 \), then \( (k+1)i-1 = (k+1)j-1 = -1 \), so that \( i = j = 0 \). The following implies that the resulting residue is zero regardless.) In this case, we get

\[ \text{(4)} \quad \text{Tr}_{W(k)/W(F_p)}(\text{res}(\hat{x} \cdot d \log(\{1 + a S^i T^j, S\})) = \text{Tr}_{W(k)/W(F_p)}(-jc(-a)^{\frac{\ell_1}{i}}). \]
Now, we compute the symbol using Parshin’s formula. As before, $$g^{(h)}(\bar{x}) = g^{(h)}(c'T)S^\ell T^{\ell \frac{p^h}{i}}.$$ Then

$$g^{(h)}(\bar{x}) \cdot d \log \{ 1 + aS^i T^j, S \} = -aj g^{(h)}(c') \sum_{k=0}^{\infty} (-a)^k S^{(k+1)i + \ell \frac{p^h}{j}} T^{(k+1)j + \ell \frac{p^h}{j}} - dz \wedge dT.$$ 

This term has nonzero residue if and only if $$k = -\ell \frac{p^h}{i} - 1 = -\ell \frac{p^h}{j} - 1 \in \mathbb{Z}_{\geq 0}.$$ (By Lemma 11, this occurs exactly when $$cS^i \in \ell \mathbb{Z}_{\geq 0}.$$ For this $$k,$$ we get a residue $$-aj g^{(h)}(c')(-a)^{-\ell \frac{p^h}{i}} - dz \wedge dT.$$ Then

$$g^{-1}((-aj g^{(h)}(c')(-a)^{-\ell \frac{p^h}{i}})_{\infty} = -j \log g^{-1}((-aj)^{-\ell \frac{p^h}{i}})$$

$$= -jc' g^{-1}(g((-aj)^{-\ell \frac{p^h}{i}}))$$

$$= -jc' ((-aj)^{-\frac{\ell}{i}}) \equiv -jc(-\frac{\ell}{i}) \mod p.$$ 

Taking the trace and comparing this with (9), the claim follows. □

Remark. Computations like those in the proof of Theorem 1 can show that for $$\ell, i \in \mathbb{Z}_{\geq 0},$$

$$[cS^\ell, \{ 1 + aS^i, T \}] = [cS^\ell, 1 + aS^i]$$

$$[cT^\ell, \{ 1 + aT^i, S \}] = [cT^\ell, 1 + aT^i],$$

where the symbol $$[,]$$ on the right hand side is the one-dimensional Schmid-Witt symbol from [6] and [13].

Some further explicit computations using Theorem 1 yield:

Corollary 3. Let $$x = c\beta + \sum_{(i,j) \in \Omega} c_{ij}[S^{-i} T^{-j}]$$ and $$y = \{ S^\epsilon, T \} \Pi_{(i,j) \in \Omega_1} \{ 1 + a_{ij} S^\ell T^j, S \} \cdot \Pi_{(i,j) \in \Omega_2} \{ 1 + b_{ij} S^\ell T^j, T \}.$$ Then

$$[x, y] = \sum_{(m,n) \in \Omega} \sum_{\substack{(i,j) \in \Omega_1, \ 0 \leq n \leq 2 \mathbb{Z}_{\geq 0} \leq \ell \mathbb{Z}_{\geq 0}}} \sum_{k=0}^{\infty} c_{ik} p^k \text{Tr}_{W(k)/W(\mathbb{F}_p)}(-jc_{m,n}([-a_{ijk}]^{m/i})$$

$$\sum_{(i,j) \in \Omega_2, \ 0 \leq n \leq 2 \mathbb{Z}_{\geq 0}} \sum_{k=0}^{\infty} d_{ik} p^k \text{Tr}_{W(k)/W(\mathbb{F}_p)}(ic_{m,n}([-b_{ijk}]^{m/i}))$$

5. Computation of $$G^\tilde{i}$$

For any $$n$$-dimensional local field, we can define upper ramification groups as studied in [7] and [3].

Definition 12. For $$\tilde{i} \in \mathbb{Z}^2,$$ define:

$$G^\tilde{i} = \omega_K(U^\tilde{i} K^{\text{top}}(K)).$$
Let $H = \text{Hom}(W(K)/\wp W(K), W(F_p))$. By Theorem 3.4 in [6] (and more generally just Artin-Schreier-Witt theory), we have the isomorphism:

$$
\tau : G_{p^\infty} \to H \\
g \mapsto (ga \text{ mod } \wp W(K) \mapsto ga - a).
$$

Let $H^0$ be the image of the inertia subgroup of $G_{p^\infty}$ in $H$.

For $\chi \in H$, observe that for any $x = c\beta + \sum_{(i,j) \in \Omega} c_{ij}[S]^{-i}[T]^{-j} \in W(K)/\wp W(K)$,

$$
\chi(c\beta + \sum_{(i,j) \in \Omega} c_{ij}[S]^{-i}[T]^{-j}) = \chi(c\beta) + \sum_{(i,j) \in \Omega} \chi(c_{ij}[S]^{-i}[T]^{-j}),
$$

so an element of $H$ is determined entirely by its values on the elements $c\beta$ and $c_{ij}[S]^{-i}[T]^{-j}$, and

$$
H \cong \text{Hom}(W(F_p), W(F_p)) \times \prod_{(i,j) \in \mathbb{Z}_{\geq 0}^2 \atop p|(i,j)} \text{Hom}(W(k), W(F_p)).
$$

But since $W(k) \cong \text{Hom}(W(k), W(F_p))$ (See Remark 4.10 in [6]),

$$
H \cong W(F_p) \times \prod_{(i,j) \in \mathbb{Z}_{\geq 0}^2 \atop p|(i,j)} W(k).
$$

Combining this with Theorem 3.4 in [6] and Proposition 7 yields:

$$
\mathbb{Z}_p \times V K^{\text{top}}_2(K) \cong G_{p^\infty} \cong H \cong W(F_p) \times \prod_{(i,j) \in \mathbb{Z}_{\geq 0}^2 \atop p|(i,j)} W(k).
$$

As

$$
\prod_{(i,j) \in \mathbb{Z}_{\geq 0}^2 \atop p|(i,j)} W(k) \cong \prod_{(i,j) \in \mathbb{Z}_{\geq 0}^2 \atop (j,p)=1} W(k) \prod_{(i,j) \in \mathbb{Z}_{\geq 0}^2 \atop (i,p)=1, (j,p) \neq 1} W(k),
$$

this map induces an isomorphism

$$
V K^{\text{top}}_2(K) \cong \prod_{(i,j) \in \mathbb{Z}_{\geq 0}^2 \atop (j,p)=1} W(k) \prod_{(i,j) \in \mathbb{Z}_{\geq 0}^2 \atop (i,p)=1, (j,p) \neq 1} W(k).
$$

Using Corollary 3 and the Proposition we will explicitly compute this map.
Define maps
\[
\phi_S : \left\{ \prod_{i,j \geq 1, j \in \mathbb{Z}} \prod_{k=0}^{\infty} \{1 + a_{ijk} S^i T^j, S\}^{c_{ijk} p^k} \right\} \to \prod_{(i,j) \in \Omega_1, (j,p) = 1} W(k)
\]
\[
\prod_{(i,j) \in \Omega_1, (j,p) = 1} \prod_{k=0}^{\infty} \{1 + a_{ijk} S^i T^j, S\}^{c_{ijk} p^k} \mapsto \left( \sum_{k=0}^{\infty} c_{ijk} p^k \sum_{m/n \in \mathbb{Z}_{\geq 0}} \left(-j\right)[-a_{ijk}]^{m/n} \right)
\]
\[
\phi_T : \left\{ \prod_{i,j \in \mathbb{Z}} \prod_{k=0}^{\infty} \{1 + a_{ijk} S^i T^j, T\}^{d_{ijk} p^k} \right\} \to \prod_{(i,j) \in \Omega_2, (i,p) = 1, (j,p) \neq 1} W(k)
\]
\[
\prod_{(i,j) \in \Omega_2, (i,p) = 1, (j,p) \neq 1} \prod_{k=0}^{\infty} \{1 + b_{ijk} S^i T^j, T\}^{d_{ijk} p^k} \mapsto \left( \sum_{k=0}^{\infty} d_{ijk} p^k \sum_{m/n \in \mathbb{Z}_{\geq 0}} i[-b_{ijk}]^{m/n} \right)
\]

The isomorphism in (5) is therefore given by the map \( \phi := \phi_S \times \phi_T \).

**Remark.** To see how the above maps compare with the map in (6) (Remark 4.10), we shall compute a couple of examples. Observe that (assuming \( ij \neq 0 \)),

\[
\{1 + a_{110} S^1 T^1, S\} \xrightarrow{\phi_S} \begin{cases} 0 & \text{if } m \neq n \\ \frac{a_{110}}{m/n} & \text{if } m = n \end{cases}, \quad \left( m, n \right) \in \mathbb{Z}_{\geq 0}, \quad (m,n) = 1
\]

\[
\{1 + a_{110} S^1 T^p, T\} \xrightarrow{\phi_T} \begin{cases} 0 & \text{otherwise} \\ \frac{a_{110}}{n} & \text{if } n = pm \end{cases}, \quad \left( m, n \right) \in \mathbb{Z}_{\geq 0}, \quad (m,p) = 1, (n,p) \neq 1
\]

To compare, the map in (6) (and more generally Lemma 2.1 in [5]) yields:

\[
(1 - a_{10} T^1)^{p^0} \mapsto \left( [a_{10}]^{1} \right)_{(i,p) = 1}.
\]

We now have all the tools to compute \( G^T \).

**Proof of Theorem** (2). The proof will follow from computing \( \phi(UF_t K^\text{top}_{2}(K)) \) via the maps \( \phi_S \) and \( \phi_T \). For notational convenience, let

\[
P = \left( p^{(\ell, (m_1, m_2))} W(k) \right)_{(m_1, m_2) \in \mathbb{Z}_{\geq 0}, p/(m_1, m_2)}.
\]

Let \( y \in V K^\text{top}_{2}(K) \), and write \( y \) as in Proposition (4). Then \( y \in UFK^\text{top}_{2}(K) \) if and only if for all \( (p^{k_i}, p^{k_j}) < \ell, a_{ij} = 0 \) and for all \( (p^{k_i}, p^{k_j}) \geq \ell, b_{ij} = 0 \). Hence \( \phi(UFK^\text{top}_{2}(K)) < P \).

The other inclusion is clear from the bijective nature of the maps \( \phi_S \) and \( \phi_T \). \( \Box \)
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