Fields, trees, and forests

Jesper Lykke Jacobsen
LPTMS, Université Paris-Sud, Bâtiment 100, 91405 Orsay, France
Service de Physique Théorique, CEA Saclay, 91191 Gif-sur-Yvette, France

Abstract. A large class of combinatorial objects related to trees and forests can be represented by non-Gaussian Grassmann integrals, generalizing the Kirchhoff matrix-tree theorem. In particular, unrooted spanning forests with weight $a$ per tree, which also arise as particular $Q \to 0$ limits of the Potts model, can be represented by a Grassmann theory involving a Gaussian term and a particular bilocal four-fermion term. This latter model can be mapped, to all orders in perturbation theory, onto the $N$-vector model at $N = -1$ or, equivalently, onto the sigma-model taking values in the unit hemi-supersphere in $\mathbb{R}^{1|2}$. In two dimensions, the sigma model generically is either massive or flows to a critical symplectic fermion theory. However, the antiferromagnetic critical point of the Potts model corresponds to another critical point of the forest model (at negative $a$). The corresponding conformal theory on the square lattice (with $a = -4$) has a non-linearly realized $OSP(2|2) = SL(1|2)$ symmetry, and involves non-compact degrees of freedom, with a continuous spectrum of critical exponents.

1. Introduction

The $Q$-state Potts model comprises many interesting models of statistical physics. For example, the $Q \to 1$ limit gives access to the bond percolation problem. It is also widely known that the $Q \to 0$ limit yields the combinatorial problem of covering a graph with spanning trees, but it is slightly less obvious that if one takes simultaneously the infinite-temperature limit in a particular way (see below) the spanning tree problem generalizes to that of spanning forests, i.e., many-component spanning subgraphs with no cycles.

Indeed, let the Potts model be defined on a finite undirected graph $G = (V,E)$ with vertex set $V$ and edge set $E$. Each vertex sustains a spin variable $\sigma_i = 1, 2, \ldots, Q$ where the number of states $Q$ is initially considered an integer. The hamiltonian is given by

$$-\beta H = \sum_{\langle ij \rangle} J_{ij} \delta(\sigma_i, \sigma_j)$$

(1.1)

where $J_{ij}$ is the coupling constant of the edge $\langle ij \rangle$, and the collection of all edges is denoted $\langle ij \rangle$. A high-temperature expansion can be obtained in a standard way by using the identity

$$e^{J_{ij} \delta(\sigma_i, \sigma_j)} = 1 + v_{ij} \delta(\sigma_i, \sigma_j)$$

(1.2)

where we have set $v_{ij} = e^{J_{ij}} - 1$. Indeed, compute now the partition function

$$Z = \sum_{\{\sigma\}} \prod_{\langle ij \rangle} (1 + v_{ij} \delta(\sigma_i, \sigma_j))$$

(1.3)
by expanding out the product $\prod_{ij}$. To each term in the expansion we associate a subset $E' \subseteq E$ so that $(ij) \in E'$ iff the term contains the factor $v_{ij} \delta(\sigma_i, \sigma_j)$. Summing now over $\{\sigma_i\}$ gives [1]

$$Z = \sum_{E' \subseteq E} Q^{k(E')} \prod_{(ij) \in E'} v_{ij},$$

(1.4)

where $k(E')$ denotes the number of connected components in $E'$. Note that in this relation, $Q$ can be promoted to a real variable. Let us therefore finally set $v_{ij} = w_{ij}Q$ and take the limit $Q \to 0$ with $w_{ij}$ finite. This yields

$$\lim_{Q \to 0} \frac{Z}{Q^V} = \sum_{E' \subseteq E \text{ forests}} \prod_{(ij) \in E'} w_{ij},$$

(1.5)

where the sum is only over subsets $E' \subseteq E$ without cycles. Thus, those $E'$ are spanning forests, i.e., a collection of trees that jointly span $V$. A review of other possible ways of taking the $Q \to 0$ limit of the Potts model is given in Ref. [2].

**Figure 1.** Unrooted spanning forest of six trees.

Figure 1 shows a spanning forest consisting of six component trees. Note in particular that a tree may be an isolated vertex.

In this note we review parts of our recent work [2–4] on spanning forests, done in collaboration with S. Caracciolo, J. Salas, H. Saleur, A.D. Sokal, and A. Sportiello.

Apart from the connection to the Potts model, spanning forests are closely related to Kirchhoff’s matrix-tree theorem [6] and its generalizations [7], which express the generating polynomials of spanning trees and rooted spanning forests in a graph as determinants associated to the graph’s Laplacian matrix. The matrix-tree theorem plays a central role in electrical circuit theory [8] and in certain exactly-soluble models in statistical mechanics [9, 10]. Like all determinants, those arising in Kirchhoff’s theorem can of course be rewritten as Gaussian integrals over fermionic (Grassmann) variables, a fact that we shall exploit shortly.

We begin by proving (following largely Ref. [3]) a generalization of Kirchhoff’s theorem in which a large class of combinatorial objects are represented by suitable non-Gaussian Grassmann integrals. Although these integrals can no longer be calculated in closed form, the resulting identities allow the use of field-theoretic methods to shed new light on the critical behavior of the underlying geometrical models.

As a special case, we show that the unrooted spanning forests appearing in Eq. (1.5) can be retrieved (with the correct weighting) by combining particular rooted objects which are in turn linked to Grassmann integrals. This culminates in Eq. (2.15) which recasts the partition function as a Grassmann theory involving a Gaussian term and a particular bilocal four-fermion term.

Furthermore, this latter model can be mapped, to all orders in perturbation theory, onto the $N$-vector model [$O(N)$-invariant $\sigma$-model] at $N = -1$ or, equivalently, onto the $\sigma$-model taking values in the unit hemi-supersphere in $\mathbb{R}^{1|2}$ [$OSP(1|2)$-invariant $\sigma$-model].
It should be stressed that all of this is valid for any graph $G = (V, E)$. Further results however follow in two dimensions. First, the fermionic model is perturbatively asymptotically free, in close analogy to (large classes of) two-dimensional $\sigma$-models and four-dimensional nonabelian gauge theories. Second, the existing knowledge of the phase diagram of the square-lattice Potts model gives insight into the phase diagram of the spanning forest model and the $O(N = -1) \sigma$-model alike. In particular, giving a weight $a = -4$ to each component tree yields a novel critical point whose critical behavior is different from that of the well-known theory of a single spanning tree. Indeed, the corresponding conformal field theory has a non-linearly realized $OSP(2|2) = SL(1|2)$ symmetry, and involves non-compact degrees of freedom, with a continuous spectrum of critical exponents. To our knowledge, this is the second time only that a non-compact boson is encountered in a statistical mechanics model with a finite number of degrees of freedom per vertex (the first time being the super Goldstone phases of Ref. [5]).

2. Combinatorial identities

Let $G = (V, E)$ be a finite undirected graph with vertex set $V$ and edge set $E$. Associate to each edge $e$ a weight $w_e$, which can be a real or complex number or, more generally, a formal algebraic variable. For $i \neq j$, let $w_{ij} = w_{ji}$ be the sum of $w_e$ over all edges $e$ that connect $i$ to $j$. The (weighted) Laplacian matrix $L$ for the graph $G$ is then defined by $L_{ij} = -w_{ij}$ for $i \neq j$, and $L_{ii} = \sum_{k \neq i} w_{ik}$. This is a symmetric matrix with all row and column sums equal to zero.

Since $L$ annihilates the vector with all entries 1, its determinant is zero. Kirchhoff’s matrix-tree theorem [6] and its generalizations [7] express determinants of square submatrices of $L$ as generating polynomials of spanning trees or rooted spanning forests in $G$. For any vertex $i \in V$, let $L(i)$ be the matrix obtained from $L$ by deleting the $i$th row and column. Then Kirchhoff’s theorem states that $\det L(i)$ is independent of $i$ and equals

$$\det L(i) = \sum_{T \in \mathcal{T}} \prod_{e \in T} w_e, \quad (2.1)$$

where the sum runs over all spanning trees $T$ in $G$. (We recall that a subgraph of $G$ is called a tree if it is connected and contains no cycles, and is called spanning if its vertex set is exactly $V$.) The $i$-independence of $\det L(i)$ expresses, in electrical-circuit language, that it is physically irrelevant which vertex $i$ is chosen to be “ground”. Another interpretation (which will become clearer below) is that $\det L(i)$ gives spanning trees which are rooted in the vertex $i$; exactly because the trees span $V$ the location of the root vertex does not matter at this stage.

There are many different proofs of Kirchhoff’s formula (2.1); one simple proof is based on the Cauchy–Binet theorem in matrix theory (see e.g. [11]).

More generally, for any sets of vertices $I, J \subseteq V$, let $L(I|J)$ be the matrix obtained from $L$ by deleting the columns $I$ and the rows $J$; when $I = J$, we write simply $L(I)$. The “principal-minors matrix-tree theorem” reads

$$\det L(i_1, \ldots, i_r) = \sum_{F \in \mathcal{F}(i_1, \ldots, i_r)} \prod_{e \in F} w_e, \quad (2.2)$$

where the sum runs over all spanning forests $F$ in $G$ composed of $r$ disjoint trees, each of which contains exactly one of the “root” vertices $i_1, \ldots, i_r$. This theorem can easily be derived by applying Kirchhoff’s theorem (2.1) to the graph in which the vertices $i_1, \ldots, i_r$ are contracted to a single vertex. Namely, since the contributing configurations on the contracted graph have no cycles, undoing the contraction will make the trees fall into $r$-component forests.

Finally, the “all-minors matrix-tree theorem” (whose proof is more difficult, see [7]) states that for any subsets $I, J$ of the same cardinality $r$,

$$\det L(I|J) = \sum_{F \in \mathcal{F}(I|J)} \epsilon(F, I, J) \prod_{e \in F} w_e, \quad (2.3)$$
where the sum runs over all spanning forests \( F \) in \( G \) composed of \( r \) disjoint trees, each of which contains exactly one vertex from \( I \) and exactly one vertex (possibly the same one) from \( J \); here \( \epsilon(F, I, J) = \pm 1 \) are signs whose precise definition is not needed here.

Let us now introduce, at each vertex \( i \in V \), a pair of Grassmann variables \( \psi_i, \bar{\psi}_i \). All of these variables are nilpotent (\( \psi_i^2 = \bar{\psi}_i^2 = 0 \)), anticommute, and obey the usual rules for Grassmann integration [12]. For example, \( \int d\psi_i = 0 \) and \( \int d\psi_i \, d\bar{\psi}_j = \delta_{ij} \). Writing \( D(\psi, \bar{\psi}) = \prod_{i \in V} d\psi_i \, d\bar{\psi}_i \) for the integration measure, we have, for any matrix \( A \),

\[
\int D(\psi, \bar{\psi}) \, e^{\bar{\psi}A\psi} = \det A . \tag{2.4}
\]

To see this, simply expand out the exponential (using the nilpotent property), and remark that the only terms surviving the integration are products containing each \( \psi_i, \bar{\psi}_i \) exactly once. Bringing this product into the proper order (using the anticommuting property) yields precisely the sign required to produce the determinant.

More generally

\[
\int D(\psi, \bar{\psi}) \, \bar{\psi}_{i_1} \psi_{j_1} \cdots \bar{\psi}_{i_r} \psi_{j_r} \, e^{\bar{\psi}A\psi} = \epsilon(i_1, \ldots, i_r | j_1, \ldots, j_r) \det A(i_1, \ldots, i_r | j_1, \ldots, j_r) \tag{2.5}
\]

where the sign \( \epsilon(i_1, \ldots, i_r | j_1, \ldots, j_r) = \pm 1 \) depends on how the vertices are ordered but is always +1 when \( (i_1, \ldots, i_r) = (j_1, \ldots, j_r) \). These formulae allow us to rewrite the matrix-tree theorems in Grassmann form; for instance, (2.2) becomes

\[
\int D(\psi, \bar{\psi}) \left( \prod_{\alpha = 1}^{r} \bar{\psi}_{i_{\alpha}} \psi_{i_\alpha} \right) e^{\bar{\psi}L\psi} = \sum_{F \in \mathcal{F}(i_1, \ldots, i_r)} \prod_{e \in F} w_e . \tag{2.6}
\]

In Eq. (2.6) each tree in the forest is rooted in a vertex. Our goal is however to obtain a Grassmann integral form for the generating function of unrooted forests, cf. Eq. (1.5). As we shall see, this can be obtained by combining formulae where trees are rooted in vertices and in edges. Before tackling this, we now derive a general combinatorial formula [Eq. (2.13)] in which trees are rooted in arbitrary subgraphs of \( G \).

To this end, we introduce, for each connected (not necessarily spanning) subgraph \( \Gamma = (V_\Gamma, E_\Gamma) \) of \( G \), the operator

\[
Q_\Gamma = \left( \prod_{e \in E_\Gamma} w_e \right) \left( \prod_{i \in V_\Gamma} \bar{\psi}_i \psi_i \right) . \tag{2.7}
\]

(Note that each \( Q_\Gamma \) is even and hence commutes with the entire Grassmann algebra.) Now consider an unordered family \( \Gamma = \{ \Gamma_1, \ldots, \Gamma_l \} \) with \( l \geq 0 \), and let us try to evaluate an expression of the form

\[
\int D(\psi, \bar{\psi}) \, Q_{\Gamma_1} \cdots Q_{\Gamma_l} \, e^{\bar{\psi}L\psi} . \tag{2.8}
\]

If the subgraphs \( \Gamma_1, \ldots, \Gamma_l \) have one or more vertices in common, then this integral vanishes on account of the nilpotency of the Grassmann variables. If, by contrast, the \( \Gamma_1, \ldots, \Gamma_l \) are vertex-disjoint, then (2.6) expresses

\[
\int D(\psi, \bar{\psi}) \left( \prod_{k=1}^{l} \prod_{i \in V_k} \bar{\psi}_i \psi_i \right) e^{\bar{\psi}L\psi} \tag{2.9}
\]

as a sum over forests \( F \) rooted at the vertices of \( V_\Gamma = \bigcup_{k=1}^{l} V_{\Gamma_k} \). In particular, all the edges of \( E_\Gamma = \bigcup_{k=1}^{l} E_{\Gamma_k} \) must be absent from these forests, since otherwise two or more of the root vertices
would lie in the same component (or one of the root vertices would be connected to itself by a loop edge). On the other hand, by adjoining the edges of $E_\Sigma$, these forests can be put into one-to-one correspondence with what we shall call $\Sigma$-forests, namely, spanning subgraphs $H$ in $G$ whose edge set contains $E_\Sigma$ and which, after deletion of the edges in $E_\Sigma$, leaves a forest in which each tree component contains exactly one vertex from $V_\Sigma$.

An example of a $\Sigma$-forest is shown in Fig. 2. An equivalent definition is the following: A $\Sigma$-forest is a subgraph $H$ with $l$ connected components in which each component contains exactly one $\Sigma_i$, and which does not contain any cycles other than those lying entirely within the $\Sigma_i$. Note, in particular, that a $\Sigma$-forest is a forest if and only if all the $\Sigma_i$ are trees.

Observing that adjoining the edges of $E_\Sigma$ provides precisely the factor $\prod_{e \in E_\Sigma} w_e$, we have

$$\int D(\psi, \bar{\psi}) Q_{\Sigma_1} \cdots Q_{\Sigma_l} e^{\bar{\psi} L \psi} = \sum_{H \in \mathcal{F}_\Sigma} \prod_{e \in H} w_e$$

(2.10)

where the sum runs over all $\Sigma$-forests $H$.

We can now combine all the formulae (2.10) into a single generating function, by introducing a coupling constant $t_\Sigma$ for each connected subgraph $\Sigma$ of $G$. Since $1 + t_\Sigma Q_\Sigma = e^{t_\Sigma Q_\Sigma}$, we have

$$\int D(\psi, \bar{\psi}) e^{\bar{\psi} L \psi + t_\Sigma Q_\Sigma} = \sum_{\Gamma \text{ vertex-disjoint}} \left( \prod_{\Sigma \in \mathcal{G}} t_\Sigma \right) \sum_{H \in \mathcal{F}_\Sigma} \prod_{e \in H} w_e.$$ 

(2.11)

We can express this in another way by interchanging the summations over $\mathcal{G}$ and $H$. Consider an arbitrary spanning subgraph $H$ with connected components $H_1, \ldots, H_l$; let us say that $\Sigma$ marks $H_i$ (denoted $\Sigma \prec H_i$) in case $H_i$ contains $\Sigma$ and contains no cycles other than those lying entirely within $\Sigma$. Define the weight

$$W(H_i) = \sum_{\Sigma \prec H_i} t_\Sigma.$$ 

(2.12)

Then saying that $H$ is a $\Sigma$-forest is equivalent to saying that each of its components is marked by exactly one of the $\Sigma_i$; summing over the possible families $\mathcal{G}$, we obtain

$$\int D(\psi, \bar{\psi}) e^{\bar{\psi} L \psi + t_\Sigma Q_\Sigma} = \sum_{H \text{ spanning } \subseteq G \atop H = (H_1, \ldots, H_l)} \left( \prod_{i=1}^l W(H_i) \right) \prod_{e \in H} w_e.$$ 

(2.13)

This is our general combinatorial formula. Extensions allowing prefactors $\bar{\psi}_{i_1} \psi_{j_1} \cdots \bar{\psi}_{i_r} \psi_{j_r}$ are also easily derived.
Eq. (2.13) has potentially many applications, but for the purpose of obtaining the generating function of unrooted forests we restrict attention here to the special case in which \( t_{Γ} = t \) whenever \( Γ \) consists of a single vertex with no edges, \( t_{Γ} = u \) whenever \( Γ \) consists of two vertices linked by a single edge, and \( t_{Γ} = 0 \) otherwise. We have

\[
\int D(\psi, \bar{\psi}) \exp \left[ \bar{\psi}L\psi + t \sum_i \bar{\psi}_i \psi_i + u \sum_{\langle ij \rangle} w_{ij} \bar{\psi}_i \psi_i \bar{\psi}_j \psi_j \right] = \sum_{\mathcal{F} \in \mathcal{F}} \left( \prod_{i=1}^l \left( t|V_{F_i} | + u|E_{F_i} | \right) \right) \prod_{e \in \mathcal{F}} w_e \quad (2.14)
\]

where the sum runs over spanning forests \( \mathcal{F} \) in \( G \) with components \( F_1, \ldots, F_l \); here \( |V_{F_i} | \) and \( |E_{F_i} | \) are, respectively, the numbers of vertices and edges in the tree \( F_i \). [We remark that the four-fermion term \( u \sum_{\langle ij \rangle} w_{ij} \bar{\psi}_i \psi_i \bar{\psi}_j \psi_j \) can equivalently be written, using nilpotency of the Grassmann variables, as \( -(u/2) \sum_{i,j} \bar{\psi}_i \psi_i \bar{\psi}_j \psi_j \).] If \( u = 0 \), this formula represents vertex-weighted spanning forests as a massive fermionic free field \([9, 13]\). More interestingly, since \( |V_{F_i} | - |E_{F_i} | = 1 \) for each tree \( F_i \), we can take \( u = -t \) and obtain the generating function of \emph{unrooted} spanning forests with a weight \( t \) for each component:

\[
\int D(\psi, \bar{\psi}) \exp \left[ \bar{\psi}L\psi + t \sum_i \bar{\psi}_i \psi_i - t \sum_{\langle ij \rangle} w_{ij} \bar{\psi}_i \psi_i \bar{\psi}_j \psi_j \right] = \sum_{\mathcal{F} \in \mathcal{F}} t^l \prod_{e \in \mathcal{F}} w_e \quad (2.15)
\]

This is furthermore equivalent to giving each edge \( e \) a weight \( w_e/t \), and then multiplying by an overall prefactor \( t^{|V|} \). This fermionic representation of unrooted spanning forests is the translation to generating functions and Grassmann variables of a little-known but important paper by Liu and Chow \([14]\).

3. Mapping onto lattice \( \sigma \)-models

We now claim that the model \( (2.14) \) with \( u = -t \) can be mapped, to all orders in perturbation theory, onto the \( N \)-vector model at \( N = -1 \). Recall that the \( N \)-vector model consists of spins \( \sigma_i \in \mathbb{R}^N \), \( |\sigma_i | = 1 \), located at the sites \( i \in V \), with Boltzmann weight \( e^{-H} \) where \( H = -T^{-1} \sum_{\langle ij \rangle} w_{ij} (\sigma_i \cdot \sigma_j - 1) \) and \( T = \text{temperature} \).

Low-temperature perturbation theory is obtained by decomposing \( \sigma_i \) into longitudinal and transverse fluctuations with respect to the spontaneous magnetization, i.e., by writing \( \sigma_i = (\sqrt{1 - T \pi_i^2}, T^{1/2} \pi_i) \) with \( \pi_i \in \mathbb{R}^{N-1} \) and expanding in powers of \( \pi \). Taking into account the Jacobian, the Boltzmann weight is \( e^{-H'} \) where

\[
H' = H + \frac{1}{2} \sum_i \log(1 - T \pi_i^2) \quad (3.1)
\]

\[
= \frac{1}{2} \sum_{i,j} L_{ij} \pi_i \cdot \pi_j - T \sum_i \pi_i^2 + \frac{T}{4} \sum_{\langle ij \rangle} w_{ij} \pi_i^2 \pi_j^2 + O(\pi_i^4, \pi_j^4) .
\]

When \( N = -1 \), the bosonic field \( \pi \) has \(-2\) components, and so can be replaced by a fermion pair \( \psi, \bar{\psi} \) if we make the substitution \( \pi_i \cdot \pi_j \rightarrow \psi_i \bar{\psi}_j - \bar{\psi}_i \psi_j \). Higher powers of \( \pi_i^2 \) vanish due to the nilpotence
of the Grassmann fields, and we obtain the model (2.14) if we identify \( t = -T, u = T \). Note the reversed sign of the coupling: the spanning-forest model with positive weights \( (t > 0) \) corresponds to the antiferromagnetic \( N \)-vector model \((T < 0)\).

An alternate mapping can be obtained by introducing at each site, in addition to the Grassmann fields \( \psi_i, \bar{\psi}_i, \) an auxiliary one-component bosonic field \( \varphi_i \), satisfying the constraint \( \varphi_i^2 + 2t\bar{\psi}_i\psi_i = 1 \). Solving this constraint yields \( \varphi_i = 1 - t\bar{\psi}_i\psi_i = e^{-t\bar{\psi}_i\psi_i} \) and

\[
\delta(\varphi_i^2 + 2t\bar{\psi}_i\psi_i - 1) = \frac{1}{2\varphi_i} \delta(\varphi_i - (1 - t\bar{\psi}_i\psi_i)) = \frac{e^{t\bar{\psi}_i\psi_i}}{2} \delta(\varphi_i - (1 - t\bar{\psi}_i\psi_i)).
\]

If we define the superfield \( \hat{\sigma}_i = (\varphi_i, \psi_i, \bar{\psi}_i) \) with inner product \( \hat{\sigma}_i \cdot \hat{\sigma}_j = \varphi_i \varphi_j + t(\bar{\psi}_i\psi_j - \psi_i\bar{\psi}_j) \), then the \( \sigma \)-model with Hamiltonian \( \mathcal{H} = -T^{-1} \sum_{(ij)} \omega_{ij}(\hat{\sigma}_i \cdot \hat{\sigma}_j - 1) \) and constraint \( \hat{\sigma}_i \cdot \hat{\sigma}_i = 1 \) corresponds to the fermionic model (2.14) if we again make the identification \( t = -T, u = T \). This \( \sigma \)-model, which is invariant under the supergroup \( OSP(1|2) \), has been studied previously in Ref. [15]. It is presumably nonperturbatively equivalent to the \( N \)-vector model at \( N = -1 \), on the grounds that each fermion equals \(-1\) boson.

It is worth mentioning that the correspondence between the spanning-forest model and these two \( \sigma \)-models, while valid at all orders of perturbation theory, does not hold nonperturbatively. (This can be checked explicitly in the exact solution for the two-site model.) The error arises from neglecting the second square root when solving the constraints; we did not, in fact, parametrize a (super)sphere but rather a (super)hemisphere. Indeed, since \( t > 0 \) corresponds to an antiferromagnetic \( \sigma \)-model, the terms we have neglected are actually dominant! But no matter: the perturbative correspondence is still correct, and has the renormalization-group consequences discussed below.

In Ref. [4] the above mappings have been generalized so as to make the target space the full supersphere. The choice between the two square roots then defines an Ising variable which couples to the fermions. The end result is a forest model in which trees getting near or across an Ising domain wall get their weights modified. Intuitively, this modified forest model can only exhibit distinct critical behavior if the coupling \( t \) that renders the forest model critical is simultaneously exactly the one required for making the Ising variable critical. In the absence of a more detailed study, we judge this possibility unlikely, and we conclude that the critical behavior of the unmodified forest model [i.e., the hemi-supersphere \( \sigma \)-model] will coincide with that of the modified forest model [i.e., the full supersphere \( \sigma \)-model].

### 4. Continuum limit in two dimensions

Suppose now that the graph \( G \) is a regular two-dimensional lattice, with weight \( w_{ij} = w > 0 \) for each nearest-neighbor pair. We can then read off, from known results on the \( N \)-vector model [16], the RG flow for the spanning-forest model: it is

\[
\frac{d\bar{\ell}}{d\ell} = \frac{3}{2\pi} \bar{\ell}^2 - \frac{3}{(2\pi)^2} \bar{\ell}^3 + \frac{2.34278457}{(2\pi)^3} \bar{\ell}^4 + \frac{1.43677}{(2\pi)^4} \bar{\ell}^5 + \ldots
\]

where \( \bar{\ell} = \frac{t}{w} \) and \( \ell \) is the logarithm of the length rescaling factor; here the first two coefficients are universal (after suitable normalization of the kinetic term), while the remaining coefficients are for the square lattice only. The positive coefficient of the \( \bar{\ell}^2 \) term indicates that for \( t > 0 \) the model is perturbatively asymptotically free. Indeed, two-dimensional \( N \)-vector models are asymptotically free for the usual sign of the coupling \((T > 0)\) when \( N > 2 \), but for the reversed sign of the coupling \((T < 0)\) when \( N < 2 \). Assuming that the asymptotic freedom holds also nonperturbatively, we conclude that for \( t > 0 \) the model is attracted to the infinite-temperature fixed point at \( t = +\infty \), hence is massive and \( OSP(1|2) \)-symmetric. For \( t_{\text{crit}} < t < 0 \), by contrast, the model is attracted to the symplectic fermion
fixed point at $t = 0$, and hence is massless with central charge $c = -2$, with the $OSP(1|2)$ symmetry spontaneously broken. Finally, for $t < t_{\text{crit}}$ we expect that the model will again be massive, with the $OSP(1|2)$ symmetry restored.

More specifically, for $t > 0$ it is predicted that the correlation length diverges for $t \downarrow 0$ (or $w \uparrow +\infty$) as

$$\xi = C_\xi e^{(2\pi/3)(w/t)} \left(\frac{2\pi w}{3 t}\right)^{1/3} \left[1 - 0.0116221204 \frac{t}{w} + 0.00446142 \frac{t^2}{w^2} + \ldots\right]$$

(4.2)

where $C_\xi$ is a nonperturbative constant (the terms in brackets are for the square lattice only). The numerical results of [2], based on transfer matrices and finite-size scaling, are consistent with the nonperturbative validity of the asymptotic-freedom predictions (4.1)/(4.2), but are inconclusive because the strip widths are small ($L \leq 10$).

The numerics of [2] are also consistent with the central charge $c = -2$ in the massless phase $t_{\text{crit}} < t < 0$, but are not definitive because of the strong $(1/\log)$ corrections to scaling induced by the marginally irrelevant operator.

Finally, the critical point $t_{\text{crit}}$ presumably corresponds to the $Q \to 0$ limit of the antiferromagnetic critical curve in the $Q$-state Potts model, under the identification $w/t = (e^{\beta J} - 1)/Q$. Known exact results for the square lattice [17, 18] yield $(w/t)_{\text{crit}} = -1/4$. There is much numerical and analytical evidence for this identification. In particular, it is known [18] that the antiferromagnetic critical properties for $Q = 4 \cos(\frac{\pi k}{4})$ and $k$ integer are essentially those of $Z_k$-symmetric parafermions $SU(2)_k/U(1)$. Taking the $k \to 0$ limit of this, the analytical predictions for the central charge $c = -1$ and the dimension of the twist operator $h_{tw} = -1/16$ follow. Both of these are in excellent agreement with the numerics [2], provided that the twist operator is identified with the parity effect occurring when defining the forest model on cylinders of odd circumference.

The full analysis of the critical theory proves rather difficult [19], as the bosonic degree of freedom is non-compact and provides a continuous component to the spectrum of critical exponents. In fact, it turns out that the Bethe Ansatz equations of the $Q$-state antiferromagnet become, in the $Q \to 0$ limit, exactly those of an integrable spin chain with $OSP(2|2) = SL(2|1)$ symmetry [4].

One particularity of the critical point is that the derivative of the free energy is discontinuous at $t_{\text{crit}}$ [2]. Thus, the phase transition is simultaneously first order and second order: this is commonly referred to as a first-order critical point.

Let us also remark that there exists a much-studied variant of the $N$-vector model in which the high-temperature expansion on the lattice has been truncated so as to forbid loop crossings [20]. For $-2 < N < 2$ this model possesses several critical points; in particular, the dilute-loop critical point is expected to be generic in the sense that adding loop crossings acts as an irrelevant perturbation. For $N = -1$ this yields a $c = -3/5$ theory [21]; the relation to the $c = -1$ theory discussed above is mysterious and deserves further study.

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