Existence of Asymptotic Expansions in Noncommutative Quantum Field Theories

C.A. Linhares\(^{(1)}\), A.P.C. Malbouisson\(^{(2)}\) and I. Roditi\(^{(2)}\)

\(^{(1)}\)Instituto de Física, Universidade do Estado do Rio de Janeiro, Rua São Francisco Xavier, 524, 20559-900 Rio de Janeiro, RJ, Brazil
\(^{(2)}\)Centro Brasileiro de Pesquisas Físicas, Rua Dr. Xavier Sigaud, 150, 22290-180 Rio de Janeiro, RJ, Brazil

February 3, 2008

Abstract

Starting from the complete Mellin representation of Feynman amplitudes for noncommutative vulcanized scalar quantum field theory, introduced in a previous publication, we generalize to this theory the study of asymptotic behaviours under scaling of arbitrary subsets of external invariants of any Feynman amplitude. This is accomplished for both convergent and renormalized amplitudes.

1 Introduction

The possibility of studying both the ultraviolet and infrared behaviours of Feynman amplitudes in quantum field theories, obtained directly without the need of first calculating explicitly the complete expressions for them, is a subject that is still finding new applications. In the present paper, we seek such asymptotic expansions for noncommutative theories known in the literature as of the ‘vulcanized’ type, that is, those which incorporate suitable modifications in order to avoid the occurrence of the ultraviolet–infrared divergence mixing, and thus become renormalizable \([1, 2, 3, 4, 5, 6]\).

The direct approach to asymptotic behaviours was formulated for commuting theories in the 1970’s in the papers \([7, 8, 9]\) within the Bogoliubov–Parasiuk–Hepp–Zimmermann renormalization scheme. It is based on the Feynman–Schwinger parametric representation of amplitudes, expressed in terms of Symanzik polynomials in the Schwinger parameters \([10, 11]\).

However, in vulcanized noncommutative theories, propagators are based on the Mehler kernel, instead of the heat kernel of commutative theories. This leads to propagators that are quadratic in the position space, so that the noncommutative parametric representation involves integration over position and moment-
tum variables, which can be performed. It results that one obtains hyperbolic polynomials in the Schwinger parameters, not just the Symanzik polynomials of the commutative case \[12, 13\]. See also the reviews \[14, 15, 16\].

In \[7, 8, 17\], the Mellin transform technique was applied in order to prove theorems implying the existence of asymptotic expansions of the amplitudes and in \[18\] the concept of ‘FINE’ polynomials was introduced, that is, those having the property of being factorizable in each Hepp sector \[19\] of the variables (a complete ordering of the Schwinger parameters). Under scaling by a parameter \(\lambda\) of (at least a few of) external invariants associated to a diagram, the Mellin transform with respect to this scaling parameter leads, as \(\lambda\) is taken to infinity, to an asymptotic series in powers of \(\lambda\) and powers of logarithms of \(\lambda\). This was possible because for amplitudes having the FINE property the Mellin transform may be ‘desingularized’, which means that the integrand of the inverse Mellin transform, which gives back the Feynman amplitude as a function of \(\lambda\), has a meromorphic structure, so that the residues of its various poles generate the asymptotic expansion. However, this is not the case under arbitrary scaling, as the FINE property simply does not occur in many diagrams.

For those non-FINE diagrams, it was introduced in \[18\] the so-called ‘multiple Mellin’ representation, which consists in splitting the Symanzik polynomials in a certain number of pieces, each one of which having the FINE property. Then, after scaling by the parameter \(\lambda\), an asymptotic expansion can be obtained as a sum over all Hepp sectors. This is always possible to do if one adopts, as in \[20, 21, 22\], the extreme point of view of splitting the Symanzik polynomials in all its monomials, which leads to the so-called ‘complete Mellin’ (CM) representation. The CM representation provides a general framework to the study of asymptotic expansions of Feynman amplitudes. Moreover, the integrations over the Schwinger parameters can be explicitly performed without any division of the integral into Hepp sectors, and we are left with the pure geometrical study of convex polyhedra in the Mellin variables \[20\]. Also, the CM representation allows a unified treatment of the asymptotic behaviour of both ultraviolet convergent and divergent amplitudes. This happens because, as shown in \[20, 21\], the renormalization procedure does not alter the algebraic structure of integrands in the CM representation. It only changes the set of relevant integration domains in the Mellin variables. The method allows the study of dimensional regularization \[21, 22\] and of the infrared behaviour of amplitudes relevant to critical phenomena \[23\]. With the CM representation one is also able to prove the existence of asymptotic expansions for most useful commutative field theories, including gauge theories in an arbitrary gauge \[24\].

In what regards noncommutative field theories, one expects that an adaptation of the general results of all these references could be developed. In fact, recently \[25\], the CM representation has been extended to the ‘vulcanized’ noncommutative \(\phi^4\) massless theory and a proof of dimensional meromorphy of its Feynman amplitudes has been presented. Our choice of a massless theory is due to the fact that the CM representation becomes less explicit and less appealing in the massive model. In any case, masses are not essential for vulcanized noncommutative field theories which have no ‘infrared divergences’ and
only ‘half-a-direction’ for their renormalization group. Based on Ref. [25], in the present paper we intend to show that asymptotic expansions exist for this noncommutative theory, in a similar way as the analogous result for the respective commutative theory. We also study explicitly the case of divergent noncommutative amplitudes in the CM representation, by adapting to this context the renormalization procedure of subtraction of suitably truncated Taylor expansions of amplitude integrand functions along the lines of Refs. [1, 20, 21, 26].

We find that the renormalization procedure in the CM representation, as already mentioned for commutative theories, also does not alter the algebraic structure of integrands for the noncommutative Feynman amplitudes, only the set of relevant integration domains in the Mellin variables changes. This allows to transpose to divergent Feynman integrals the machinery used in the convergent case and prove the existence of asymptotic expansions for renormalized amplitudes.

The paper is organized as follows. In section 2, we very briefly recall the main features of the complete Mellin representation for commutative scalar theories. Next, in section 3, we review the CM representation for the vulcanized $\phi^4$ theory. In sections 4 and 5 we present the generalizations to the noncommutative theory of the respective theorems on the existence of the asymptotic expansions for the convergent and renormalized amplitudes. In the last section we summarize our conclusions.

2 Complete Mellin representation in the commutative scalar case

Let us first consider the simpler case of a Feynman amplitude in a commutative massive scalar theory. The amplitude related to an arbitrary diagram $G$, with $I$ internal lines, $V$ vertices, and $L$ loops, in $d$ spacetime dimensions, reads

$$A_G = C_G \int_0^\infty \frac{\prod_{\ell=1}^I d\alpha_\ell}{(4\pi)^{dL/2} U^{d/2}(\alpha)} e^{-\sum_\ell \alpha_\ell m_\ell^2} e^{-N(s;\alpha)/U(\alpha)},$$

where $C_G$ is a constant, $U$ and $N$ are homogeneous polynomials in the $\alpha_\ell$ variables, known in the literature as the Symanzik polynomials, which are written as

$$U(\alpha) = \sum_j \prod_{\ell=1}^I \alpha_\ell^{u_{ij}} \equiv \sum_j U_j, \quad N(\alpha) = \sum_k s_k \left( \prod_{\ell=1}^I \alpha_\ell^{n_{ik}} \right) \equiv \sum_k N_k,$$

where $j$ runs over the set of 1-trees and $k$ over the set of 2-trees of the diagram $G$; $s_k$ are $O(d)$-invariants given by the square of the sum of all external momenta at one of the components of the 2-tree $k$; also,

$$u_{ij} = \begin{cases} 0 & \text{if the line } \ell \text{ belongs to the 1-tree } j \\ 1 & \text{otherwise} \end{cases}$$
and
\[ n_{\ell k} = \begin{cases} 
0 & \text{if the line } \ell \text{ belongs to the 2-tree } k \\
1 & \text{otherwise}.
\end{cases} \quad (4) \]

The complete Mellin representation for \( A_G \), following the steps shown in \cite{20, 21, 24} is given by
\[
A_G(s_k, m^2_\ell) = \int_\delta \frac{\prod_j \Gamma(-x_j)}{\Gamma(-\sum_j x_j)} \prod_k x_k^{s_k} \Gamma(-y_k) \prod_\ell (m^2_\ell)^{-\phi_\ell} \Gamma(\phi_\ell),
\] (5)

where
\[
\phi_\ell = \sum_j u_{\ell j} x_j + \sum_k n_{\ell k} y_k + 1. \quad (6)
\]

The symbol \( \int_\delta \) means integration over the independent variables \( \text{Im } x_j, \text{Im } y_k \) in the convex domain \( \delta \) defined by (\( \sigma \) and \( \tau \) standing respectively for \( \text{Re } x_j \) and \( \text{Re } y_k \))
\[
\delta = \left\{ \sigma, \tau \mid \sigma_j < 0; \tau_k < 0; \sum_j x_j + \sum_k y_k = -\frac{d}{2}; \forall i, \text{Re } \phi_i = \sum_j u_{ij} \sigma_j + \sum_k n_{ik} \sigma_k + 1 > 0 \right\}. \quad (7)
\]

This domain \( \delta \) is nonempty as long as \( d \) is positive and small enough so that every subdiagram of \( G \) has convergent power counting \cite{20}; hence in particular for the \( \phi^4 \) theory it is always nonempty for any diagram for \( 0 < d < 2 \).

Let us denote collectively by \( \zeta_\mu \) the arguments of the \( \Gamma \)-functions: \( -x_j, -y_k, \phi_\ell \). Also we call collectively \( t_\mu \) the set of invariants \( s_k \) and the squared masses \( m^2_\ell \). A general asymptotic regime is then defined as the scaling \( t_\mu \to \lambda^{b_\mu} t_\mu \), in such a way that the amplitude (5) becomes a function of \( \lambda \) written in the convenient form \cite{20}
\[
A_G(\lambda) = \int_\delta \lambda^\xi \frac{t_\mu^{-\zeta_\mu} \Gamma(\zeta_\mu)}{\Gamma(-\sum_j x_j)}, \quad (8)
\]
with \( \zeta = \sum_\mu b_\mu \zeta_\mu \). This representation can be extended to complex values of \( d \).

For instance, for a massive \( \phi^4 \) diagram, it is analytic in \( d \) for \( \text{Re } d < 2 \) and meromorphic in \( d \) in the whole complex plane with singularities at rational values; furthermore, its dimensional analytic continuation has the same unchanged CM integrand but translated integration contours. Also, it is valid without change in the form of the integrand for renormalized amplitudes \cite{20, 21}. Using the meromorphic properties of the integrand of eq. (8), an asymptotic expansion in powers of \( \lambda \) and powers of logarithms of \( \lambda \) is obtained for \( A_G(\lambda) \) in Ref \cite{20}.

### 3 Complete Mellin representation for noncommutative scalar theories

In order to establish notation, we review in this section the results of \cite{24}, which we take as the starting point of the study of asymptotic behaviours and
renormalization, to be developed in the following sections, and which constitutes
the main subject of the present paper.

According to the analysis exposed in [12], the amplitude related to a ribbon
diagram $G$ with $L$ internal lines, by choosing a particular root vertex $\bar{V}$, has a
parametric representation in terms of the variable $t_\ell = \tanh \alpha_\ell/2$, where $\alpha_\ell$ are
the former Schwinger parameters as

$$A_G(\{x_e\}, p_{\bar{V}}) = K_G \int_0^1 \prod_\ell dt_\ell \left(1 - t_\ell^2\right)^{d/2-1} \int dx dp \exp \left[\frac{-\Omega}{2} X \bar{G} X^t\right], \quad (9)$$

where $K_G$ is a constant, $d$ is the spacetime dimension, $\Omega$ is the Grosse–Wulkenhaar
vulcanization coefficient, $X$ summarizes all positions and hypermomenta, and $\bar{G}$ is a certain quadratic form. Calling $x_e$ and $p_{\bar{V}}$ the external variables and $x_i, p_i$ the internal ones, we decompose $\bar{G}$ into an internal quadratic form $Q$, an
external one $M$ and a coupling part $P$, so that

$$X = \begin{pmatrix} x_e & p_{\bar{V}} & x_i & p_i \end{pmatrix}, \quad \bar{G} = \begin{pmatrix} M & P \\ P^t & Q \end{pmatrix}. \quad (10)$$

Performing the Gaussian integration over all internal variables, one gets the
noncommutative parametric representation given by

$$A_G(\{x_e\}, p_{\bar{V}}) = K_G \int_0^1 \prod_\ell dt_\ell \left(1 - t_\ell^2\right)^{d/2-1} e^{-\Omega H_G, \bar{V} / H_G, \bar{V}} \prod_\ell t_\ell \prod_{\ell \in J} t_\ell, \quad (11)$$

where new polynomials, in the $t_\ell$ variables ($\ell = 1, \ldots, L$), $H_G, \bar{V}$ and $H_V, \bar{V}$,
have been introduced, which are the analogs of the Symanzik polynomials $U$ and $N$ of the commutative case. It has been shown in [12] that for the Grosse–Wulkenhaar $\phi^4$ model we have

$$H_G, \bar{V} = \sum_{K_U = I \cup J; n + |K_U| even} \frac{s^{2g - k_{K_U}} n_{K_U}^2 \prod_\ell t_\ell \prod_{\ell \in I} t_\ell}{[H_{G, \bar{V}}(t)]^{d/2}},$$

$$= \sum_{K_U} a_{K_U} \prod_\ell t_\ell^{n_{K_U}} \equiv \sum_{K_U} H_{K_U}, \quad (12)$$

where $I$ is a subset of the first $L$ indices, with $|I|$ elements, and $J$ a subset of the next $L$ indices, with $|J|$ elements; $s = 1/4 \Theta \Omega$ is a constant containing the noncommutative parameter $\Theta$ and the vulcanization coefficient $\Omega$; $g$ is the genus of the diagram, $a_{K_U} = s^{2g - k_{K_U}} n_{K_U}^2$, $k_{K_U} = |K_U| - L - F - 1$, $F$ being the number of faces of the diagram; $n_{K_U} = \text{Pf} \left( B_{K_U} \right)$, where $B$ is the antisymmetric part of the quadratic form $Q$ restricted by omitting hypermomenta, so $n_{K_U}$ is the Pfaffian of the antisymmetric matrix obtained from $B$ by deleting the lines and columns in the set $K_U = I \cup J$; finally,

$$u_{\ell K_U} = \begin{cases} 0 & \text{if } \ell \in I \text{ and } \ell \notin J \\ 1 & \text{if } \ell \notin I \text{ and } \ell \notin J \\ 2 & \text{if } \ell \notin I \text{ and } \ell \in J. \end{cases} \quad (13)$$
The second polynomial $HV$ has both a real part $HV^R$ and an imaginary part $HV^I$. We need to introduce beyond $I$ and $J$ as above a particular line $\tau \notin I$ which is the analog of a 2-tree cut. Then it is shown in [12] that

$$HV^R_{G, \bar{V}} = \sum_{K_V = I \cup J} \prod_{\tau \notin I} t_\tau \prod_{\tau' \in J} t_{\tau'} \left[ \sum_{e_1} \sum_{\tau \notin K_V} P_{e_1, \tau} \epsilon_{K_V, \tau} \text{Pf}(B_{K_V, \tau}) \right]^2 = \sum_{K_V} s^R_{K_V} \left( \prod_{\ell = 1}^L t_\ell \right) \equiv \sum_{K_V} HV^R_{K_V}, \quad (14)$$

where

$$s^R_{K_V} = \left[ \sum_{e} x_e \sum_{\tau \notin K_V} P_{e, \tau} \epsilon_{K_V, \tau} \text{Pf}(B_{K_V, \tau}) \right]^2 \quad (15)$$

and $v_{lK_V}$ is given by the same formula as $u_{lK_U}$. The imaginary part involves pairs of lines $\tau, \tau'$ and corresponding signatures [12, 13]:

$$HV^I_{G, \bar{V}} = \sum_{K_V = I \cup J} \prod_{\tau \notin I} t_\tau \prod_{\tau' \in J} t_{\tau'} \epsilon_{K_V, \tau} \text{Pf}(B_{K_V, \tau}) \times \left[ \sum_{w, \tau} x_w \sum_{\tau' \notin K_V} P_{w, \tau'} \epsilon_{K_V, \tau'} \text{Pf}(B_{K_V, \tau'}) \right] x_{w''} \sigma x_{w''} = \sum_{K_V} HV^I_{K_V}, \quad (16)$$

where

$$s^I_{K_V} = \epsilon_{K_V, \tau} \text{Pf}(B_{K_V}) \left[ \sum_{w, \tau} \sum_{\tau' \notin K_V} P_{w, \tau'} \epsilon_{K_V, \tau'} \text{Pf}(B_{K_V, \tau'}) \right] x_{w''} \sigma x_{w''} = \sum_{K_V} HV^I_{K_V}, \quad (17)$$

where $\sigma = \left( \begin{array}{cc} \sigma_2 & 0 \\ 0 & \sigma_2 \end{array} \right)$ and $\sigma_2$ is the second Pauli matrix.

The main differences of the noncommutative parametric representation with respect to the commutative case are the presence of the constants $a_{K_U}$ in $HU$ (which contains the noncommutative quantity $s = 1/4\Theta\Omega$), the presence of the imaginary part $iHV^I$ in $HV$, and the fact that the parameters $u_{\ell j}$ and $v_{\ell k}$ in the formulas above can have also the value 2 (and not only 0 and 1).

In order to proceed, we now introduce the Mellin parameters. For the real part $HV^R$ of $HV$, we use the identity [25]

$$e^{-HV^R_{K_V} / HU_{K_U}} = \int_{[\tau_{K_V}]^R} \Gamma \left( -y_{K_V} \right) \left( \frac{HV^R_{K_V}}{HU_{K_U}} \right)^{y_{K_V}} u_{K_V}^{y_{K_V}}, \quad (18)$$
where \( \int_{K_V} \) is a short notation for \( \int_{-\infty}^{\infty} \frac{d \text{Im} y_{K_V}^R}{2\pi} \), with \( \text{Re} y_{K_V}^R \) fixed at \( \tau_{K_V}^R < 0 \). However, for the imaginary part one cannot apply anymore the same identity. It nevertheless remains true in the sense of distributions. More precisely, we have for \( HV_{K_V}^R / HU_{K_U} > 0 \) and \(-1 < \tau_{K_V}^I < 0\) (see [25])

\[
e^{-HV_{K_V}^I / HU_{K_U}} = \int_{\tau_{K_V}^I} \Gamma (-y_{K_V}^I) \left( \frac{i HU_{K_U}^I}{HV_{K_V}^I} \right)^{y_{K_V}^I}, \quad (19)
\]

which introduces another set of Mellin parameters. The distributional sense of the formula above is a major difference with respect to the commutative case.

For the polynomial \( HU \) one can use the formula [25]

\[
\Gamma \left( \sum y_{K_V} + \frac{d}{2} \right) (HU_{K_U})^{-\sum y_{K_V}} (y_{K_V}^R + y_{K_V}^I)^{-d/2} = \int \prod \Gamma (-x_{K_U}) HU_{x_{K_U}}^{y_{K_V}^I}.
\]

As in the commutative case, we now insert the distribution formulas (18), (19) and (20) into the general form of the Feynman amplitude. This gives

\[
\mathcal{A} G = K_G \int_{\Delta} \prod a_{K_U}^{x_{K_U}} \Gamma (-x_{K_U}) \left( \prod s_{K_U}^{y_{K_V}} \right)^{y_{K_V}^R} \Gamma (-y_{K_V}^R) \\
\times \left( \prod s_{K_V}^{y_{K_V}^I} \right)^{y_{K_V}^I} \Gamma (-y_{K_V}^I) \int_0^1 \prod \ell d_{\ell} \left( 1 - t_{\ell}^2 \right)^{d/2-1} t_{\ell}^{\phi_{\ell} - 1}, \quad (21)
\]

where

\[
\phi_{\ell} = \sum_{K_U} u_{\ell K_U} x_{K_U} + \sum_{K_U} \left( t_{\ell K_U} y_{K_V}^R + v_{\ell K_U} y_{K_V}^I \right) + 1. \quad (22)
\]

Here \( \int_{\Delta} \) means integration over the variables \( \frac{\text{Im} x_{K_U}}{2\pi i}, \frac{\text{Im} y_{K_V}^R}{2\pi i} \) and \( \frac{\text{Im} y_{K_V}^I}{2\pi i} \), where \( \Delta \) is the convex domain

\[
\Delta = \left\{ \sigma, \tau^R, \tau^I \left| \begin{array}{l}
\sigma_{K_U} < 0; \quad \tau_{K_V}^R < 0; \quad -1 < \tau_{K_V}^I < 0; \\
\sum_{K_U} x_{K_U} + \sum_{K_V} (y_{K_V}^R + y_{K_V}^I) = -d/2; \\
\forall \ell, \text{Re} \phi_{\ell} = \sum_{K_U} u_{\ell K_U} x_{K_U} + \sum_{K_V} (v_{\ell K_V} y_{K_V}^R + v_{\ell K_V} y_{K_V}^I) + 1 > 0
\end{array} \right. \right\} \quad (23)
\]

and \( \sigma, \tau^R \) and \( \tau^I \) stand for Re \( x_{K_U} \), Re \( y_{K_V}^R \), and Re \( y_{K_V}^I \). The \( t_{\ell} \) integrations in (21) may be performed using the representation for the beta function

\[
\int_0^1 dt_{\ell} \left( 1 - t_{\ell}^2 \right)^{d/2-1} t_{\ell}^{\phi_{\ell} - 1} = \frac{1}{2} B \left( \frac{\phi_{\ell}}{2}, \frac{d}{2} \right) = \frac{\Gamma \left( \frac{\phi_{\ell}}{2} \right) \Gamma \left( \frac{d}{2} \right)}{2 \Gamma \left( \frac{\phi_{\ell} + d}{2} \right)}. \quad (24)
\]

The representation is convergent for \( 0 < \text{Re} \ d < 2 \). Therefore, we can claim that any Feynman amplitude of a \( \phi^4 \) diagram is analytic at least in the strip
0 < \text{Re } d < 2, \text{ where it admits the following CM representation}\ [25]

\[ A_G = K_G \int_{\Delta} \frac{\prod_{K_U} a_{K_U} x_{K_U}}{\Gamma(-\sum_{K_U} x_{K_U})} \left( \prod_{K_V} (s_{K_V}^R)^{y_{K_V}^R} \Gamma(-y_{K_V}^R) \right) \]

\[ \times \left( \prod_{\lambda_{K_V}} (s_{K_V}^I)^{y_{K_V}^I} \Gamma(-y_{K_V}^I) \right) \left( \prod_{\ell=1}^L \Gamma \left( \frac{\phi_{\ell}}{2} \right) \Gamma \left( \frac{d}{2} \right) \right) \lambda^\psi, \quad (25) \]

which holds as a tempered distribution of the external invariants.

We have thus obtained the complete Mellin representation of Feynman amplitudes for a noncommutative quantum field theory. The beta functions, which result from the \( t_\ell \)-integrations, lead to the appearance of gamma functions that were not present in the commutative case. We will comment about this in the next section.

4 Asymptotic expansions for convergent amplitudes

A general asymptotic regime is defined by scaling the invariants \( s_{K_V}^R, s_{K_V}^I \) and \( a_{K_U} \),

\[ s_{K_V}^R \rightarrow \lambda^{b_{K_V}} s_{K_V}^R \]

\[ s_{K_V}^I \rightarrow \lambda^{c_{K_V}} s_{K_V}^I \]

\[ a_{K_U} \rightarrow \lambda^{d_{K_U}} a_{K_U}, \quad (26) \]

where \( b_{K_V}, c_{K_V}, \) and \( d_{K_U} \) may have positive, negative or null values, and letting \( \lambda \) go to infinity. We then obtain under these scalings

\[ A_G(\lambda) = K_G \int_{\Delta} \frac{\prod_{K_U} a_{K_U} x_{K_U}}{\Gamma(-\sum_{K_U} x_{K_U})} \left( \prod_{K_V} (s_{K_V}^R)^{y_{K_V}^R} \Gamma(-y_{K_V}^R) \right) \]

\[ \times \left( \prod_{\lambda_{K_V}} (s_{K_V}^I)^{y_{K_V}^I} \Gamma(-y_{K_V}^I) \right) \left( \prod_{\ell=1}^L \Gamma \left( \frac{\phi_{\ell}}{2} \right) \Gamma \left( \frac{d}{2} \right) \right) \lambda^\psi, \quad (27) \]

where the exponent of \( \lambda \) is a linear function of the Mellin variables:

\[ \psi = \sum_{K_V, K_U} \left( b_{K_V} y_{K_V}^R + c_{K_V} y_{K_V}^I + d_{K_U} x_{K_U} \right). \quad (28) \]

Notice that the factor \( \left[ \prod_{\ell=1}^L \Gamma \left( \frac{d}{2} \right) 2\Gamma \left( \frac{\phi_{\ell}+d}{2} \right) \right]^{-1} \) in the integrand of eq. \( (27) \) does not affect the meromorphic structure of the amplitude \( (27) \). Moreover, for strictly positive dimensions \( d > 0 \) and \( \phi_{\ell} \in \Delta \), this factor also does not introduce \textit{zeroes} in the integrand.
From the above expressions, we can show that the proof of the theorem given in [20] can be extended for the noncommutative case. To do this, let us rewrite the above expression for $A_G(\lambda)$ in a convenient way. Let us denote collectively the variables $\{x_{KU}, y_{KV}\}$ as $\{z_K\}$, whereas the arguments of the gamma functions leading to singularities, $-x_{KU}, -y^R_{KV}, -y^I_{KV}$, and $\frac{\phi_\ell}{2}$ will be renamed $\psi_\nu(z_K)$. The convex domain $\Delta$ can then be rewritten simply as

$$\Delta = \{z_K \text{ such that } \text{Re } \psi_\nu(z_K) > 0, \text{ for all } \nu\}.$$  \hspace{1cm} (29)

Let us define the set of quantities $\{s_\nu\}$ such that it includes the quantity $a_{KU}$, which are functions of the objects $\Theta$ and $\Omega$ having no correspondents in ordinary commutative field theory,

$$s_\nu = \begin{cases} a_{KU} & \text{if } z_K = x_{KU} \\ s^R_{KV} & \text{if } z_K = y^R_{KV} \\ s^I_{KV} & \text{if } z_K = y^I_{KV} \\ 1 & \text{if } z_K = \frac{\phi_\ell}{2}. \end{cases}$$  \hspace{1cm} (30)

Also, we introduce the factors $f_\nu$ (in general functions of the variables $x_{KU}$ and $y_{KV}$) such that

$$f_\nu = \begin{cases} \Gamma\left(\frac{d}{2}\right) \left[\prod_{\ell=1}^{L} 2\Gamma\left(\frac{\phi_\ell + d}{2}\right)\right]^{-1} & \text{if } \psi_\nu = \phi_\ell/2 \\ 1 & \text{otherwise}. \end{cases}$$  \hspace{1cm} (31)

Therefore the expression for $A_G(\lambda)$ in (27) can be simplified to

$$A_G(\lambda) = K_G \int_\Delta \lambda^\psi \prod_\nu f_\nu s_\nu^{-\psi_\nu} \Gamma(\psi_\nu) \Gamma(-\sum_{KU} x_{KU}) \Gamma^{-1}.$$  \hspace{1cm} (32)

Eq. (32) has exactly the same singularity structure as (8), the factors $f_\nu$ only modify the residues at the poles. Thus we can translate to the present situation all the steps of the proof of the asymptotics theorem of Ref. [20], since it relies entirely on displacements of the integration contours crossing the singularities of the gamma functions $\Gamma(\psi_\nu) \Gamma(\zeta_\nu)$, in the commutative counterpart of eq. (8). For completeness, this demonstration is given in the Appendix. Thus the result of Ref. [20] remains valid mutatis mutandis for the the vulcanized $\phi^{*4}$ theory and we are allowed to state the following theorem:

**Theorem 4.1.** Let us consider a ribbon diagram $G$ of the vulcanized $\phi^{*4}$ theory, and its related amplitude $A_G(\lambda)$ under the general scaling of its invariants,

$$s^R_{KV} \rightarrow \lambda^{b_{KV}} s^R_{KV}$$

$$s^I_{KV} \rightarrow \lambda^{c_{KV}} s^I_{KV}$$

$$a_{KU} \rightarrow \lambda^{d_{KU}} a_{KU},$$

where $b_{KV}, c_{KV}$ and $d_{KU}$ may have positive, negative or null values, and as $\lambda \rightarrow \infty$. Then there exists an asymptotic expansion of $A_G(\lambda)$ of the form

$$A_G(\lambda) = \sum_{p=p_{\text{max}}}^{-\infty} \sum_{q=0}^{q_{\text{max}(p)}} A_{pq}(s^R_{KV}, s^I_{KV}, a_{KU}) \lambda^p \ln^q \lambda,$$  \hspace{1cm} (34)
where $p$ runs over the rational values of a decreasing arithmetic progression, with $p_{\text{max}}$ as a ‘leading power’, and $q$, for a given $p$, runs over a finite number of nonnegative integer values.

The coefficients $A_{pq}(s_{Kv}^R,s_{Kv}^I,a_{Ku})$ of the expansion in (34) are functions only of the invariants associated to the hyperbolic polynomials. Notice, in particular, that the invariants $a_{Ku}$ contain the noncommutative entities $\Theta$ and $\Omega$ encoded.

5 Renormalized amplitudes

Let us now consider the complete Mellin representation of divergent amplitudes. The analysis follows the steps taken in [20, 21] for the commutative case, and the recent results of Ref. [26] for noncommutative theories. We have to go back to the amplitude given in eq. (11), for the occurrence of ultraviolet divergences in an expression such as this one prevents the interchange of the integral over the $t_\ell$ variables with the one over the domain $\Delta$. It means that the $t_\ell$-integral cannot be performed and a renormalization prescription is therefore required. For this, we use the method of subtracting the first terms of a generalized Taylor expansion corresponding to the infinities of the divergent subdiagrams [7, 8], as adapted to the $t_\ell$ integrations [26]. Each $t_\ell$ variable belonging to a divergent subdiagram $S$ is scaled by a parameter $\rho^2$, $t_\ell \in S \rightarrow \rho^2 t_\ell \in S$, and the integral in eq. (11) becomes a function of $\rho$, which we call $g(\rho)$. Next, following the steps of [7], we define the generalized Taylor operator of order $n$,

$$\tau^n [\rho^\nu g(\rho)] = \rho^\nu T^n - E[\nu] [g(\rho)], \quad (35)$$

with $E[\nu]$ being the smallest integer greater than $\nu$, and $T^n [g(\rho)]$ being the (truncated) usual Taylor operator over a function $g(\rho)$,

$$T^n [g(\rho)] = \sum_{k=0}^{q} \frac{\rho^k}{k!} g^{(k)}(0), \quad (36)$$

which makes sense only for $q \geq 0$. The generalized Taylor operator acts on the $t_\ell$ integrand, so that for each primitively divergent subdiagram $S$ of $G$ one associates a subtraction operator $\tau_{S}^{-2l_S}$, where $l_S$ is the number of lines in the subdiagram $S$. The $\tau_S$ operator is equivalent to the introduction of counterterms in the theory, in that it is defined in order to suppress the ultraviolet divergent terms from the integrand: the $t_\ell$ variables associated to the subdiagram $S$ are first scaled by the parameter $\rho$, and the first few terms of the generalized Taylor expansion in $\rho$ are kept in $\tau_S$. In fact, this corresponds to the Taylor operator in eq. (36), truncated at the order $q$, which is the superficial degree of convergence (the negative of the superficial degree of divergence) of the subdiagram $S$. $q = dL_S - 2l_S$, $L_S$ being the number of loops of $S$. At the end of the that computation, one takes $\rho = 1$.

Now, a crucial point, argued in Ref. [26], is that since one is interested in the region of ultraviolet divergences, the factor $\prod\ell dt_\ell (1 - t_\ell^2)^{d/2-1}$ in eq. (11)
can be bounded in such a way that it cannot contribute to divergences and so it is included in the integration measure. This factor plays exactly the same rôle of the integration measure \( \prod_\ell d\alpha_\ell \exp (-\sum_\ell m_\ell \alpha_\ell) \) in the massive commutative case. Thus the action of the generalized Taylor operator on the integrand is

\[
\tau_S^{-2l_S} \left( \frac{e^{-HV_G/HU_G}}{HU_G^{d/2}} \right) = \left[ \tau_S^{-2l_S} \left( \frac{e^{-HV_G/HU_G}}{HU_G^{d/2}} \bigg|_{t_s=-\rho^2 t_s} \right) \right]_{\rho=1}.
\]

The renormalized amplitude is defined by introducing the operator \( R \) \[7\],

\[
R = \prod_S \left( 1 - \tau_S^{-2l_S} \right),
\]

which satisfies the identity \[26\]

\[
R = 1 + \sum_F \prod_{S \in F} (-\tau_S^{-2l_S}) = \prod_S \left( 1 - \tau_S^{-2l_S} \right),
\]

where \( F \) is the set of all nonempty forests of primitively divergent subdiagrams. Then the renormalized amplitude is

\[
A_G^{\text{ren}}(\{x_\ell\},p_\bar{V}) = K_G \int_0^1 \prod_\ell dt_\ell (1 - t_\ell^2)^{d/2-1} R \left\{ \frac{e^{-HV_G,\bar{V}(t,x_\ell,p_\bar{V})}/HU_G,\bar{V}(t)}{[HU_G,\bar{V}(t)]^{d/2}} \right\}.
\]

Now, within the context of the complete Mellin representation, we have, from \[21\],

\[
A_G^{\text{ren}} = K_G \int_\Delta \prod_{K_U} a_{K_U}^{x_{K_U}} \frac{\Gamma(-x_{K_U})}{\Gamma(-\sum_{K_U} x_{K_U})} \left( \prod_{K_V} (s_{K_V}^R)^{y_{K_V}} \Gamma(-y_{K_V}) \right)
\]

\[
\times \left( \prod_{K_V} (s_{K_V}^I)^{y_{K_V}^I} \Gamma(-y_{K_V}^I) \right) \int_0^1 \prod_\ell dt_\ell (1 - t_\ell^2)^{d/2-1} R \left[ s_{\phi_\ell}^{-1} \right].
\]

This is the analogous of the starting point of the analysis of \[21\] on renormalized amplitudes in the complete Mellin representation. We see that the renormalization operator \( R \) acts on the \( t_\ell \)-variables, \textit{exactly in the same way} as it acts on the \( \alpha_\ell \)-variables in the commutative situation of Refs. \[20 \[21\], the only difference (which does not affect the validity of the theorems in \[21\]) being in the integration measure over the \( t_\ell \)-variables.

Therefore, the theorems stated in \[21\] remain valid in the vulcanized non-commutative case. Then we can follow the same steps as in Ref. \[21\], that is, we define cells \( C \), such that

\[
\sup_C \text{Re}(\phi_\ell) > 0, \forall \ell; \quad \inf_S \left( \inf_C \sum_{\ell \in C} \text{Re}(\phi_\ell) \right) \leq 0.
\]
The effect of the $R$ operator in eq. (42) is to split the factor $R \left( t_\ell^{\phi_\ell - 1} \right)$ into a set of terms $\left\{ \mu_C t_\ell^{\phi_\ell}, \phi_\ell \in C \right\}$, where $\mu_C$ are numerical coefficients. This allows the $t_\ell$ integral to be evaluated just as in the convergent case. The renormalized amplitude in the CM representation is then given by

$$A_{G}^{\text{ren}} = K_G \sum_C \mu_C \int_{\Delta C} I_C(x_{K_U}, y_{K_V}^R, y_{K_V}^I),$$  \hspace{1cm} (44)$$

where we have defined the integrands

$$I_C = \prod_{K_U} \frac{a_{x_{K_U}}}{\Gamma(-x_{K_U})} \left( \prod_{K_V} \frac{(s_{K_V}^R)^{y_{K_V}^R}}{\Gamma(-y_{K_V}^R)} \right) \times \left( \prod_{K_V} (s_{K_V}^I)^{y_{K_V}^I} \Gamma(-y_{K_V}^I) \right) \left( \prod_{l=1}^{L} \Gamma \left( \frac{\phi_l}{2} \right) \right).$$ \hspace{1cm} (45)$$

We now have a set of integration domains given by

$$\Delta_C = \left\{ \sigma, \tau_R, \tau^I \in C \mid \begin{array}{l}
\sigma_{K_U} < 0; \tau_{K_V}^R < 0; -1 < \tau_{K_V}^I < 0; \\
\sum_{K_U} x_{K_U} + \sum_{K_V} (y_{K_V}^R + y_{K_V}^I) = -d/2; \\
\forall \ell, \Re \phi_\ell = \sum_{K_U} u_{\ell K_U} x_{K_U} \\
+ \sum_{K_V} (v_{\ell K_V} y_{K_V}^R + v_{\ell K_V} y_{K_V}^I) + 1 > 0
\end{array} \right\},$$ \hspace{1cm} (46)$$

instead of the single one ($\Delta$) of the convergent amplitude. As in the commutative case, we see that the renormalization procedure only changes the relevant integration domains in the Mellin variables. The structure of the integrands $I_C$ in the cells $C$ remains exactly of the same form as for convergent amplitudes. This then implies that we can apply in each cell the machinery used in the previous section and we can state the following theorem:

**Theorem 5.1.** Under the scaling of eq. (33), each integral $\int_{\Delta C} I_C(x_{K_U}, y_{K_V}^R, y_{K_V}^I)$ has an asymptotic expansion of the same form of the the one of eq. (34); therefore the amplitude $A_{G}^{\text{ren}}(\lambda)$ has an asymptotic expansion of the form

$$A_G^{\text{ren}}(\lambda) = \sum_C \mu_C I_C(\lambda),$$ \hspace{1cm} (47)$$

with

$$I_C(\lambda) = \sum_{p=-\mu_{C\text{max}}}^{\infty} \phi_{\mu_{C\text{max}}(p)} \sum_{q=0}^{\mu_{C\text{max}}} A_{C(p,q)}(s_{K_V}^R, s_{K_V}^I, a_{K_U}) \lambda^p \ln^{q} \lambda$$ \hspace{1cm} (48)$$

and where in each cell $C$, $p$ runs over the rational values of a decreasing arithmetic progression, with $\mu_{C\text{max}}$ as a 'leading power', and $q$, for a given $p$, runs over a finite number of nonnegative integer values.

As in the convergent case, the coefficients $A_{C(p,q)}(s_{K_V}^R, s_{K_V}^I, a_{K_U})$ of the expansion in $I_C(\lambda)$ are functions only of the invariants associated to the hyperbolic polynomials.
6 Conclusions

We have shown in this paper that all the steps in the proofs of the theorems in Refs. [20, 21] may be reproduced in the context of the vulcanized noncommutative scalar $\phi^{*4}$ model. In particular, the proof of the existence of asymptotic expansions for Feynman amplitudes in commutative field theories done in [20] may be transposed to the present situation, for both convergent and renormalized amplitudes. The resulting theorems take into account the influence of the specificities of the noncommutative generalization of the theory in the details of the proof. In particular, it was crucial to observe that the parameters $a_{K_U}$, within which the noncommutative entities $\Theta$ and $\Omega$ are encoded, and are of course inexistent in the commutative case, may be defined as part of the `invariants' $s_{\nu}$, and therefore are related to the meromorphic structure of the amplitude and its asymptotic behaviour can be studied. Another difference with respect to the commutative case is the fact that the next set of invariants $s_{\nu}$, $s_{K_U}$, have real and imaginary parts ($s_{R_{K_U}}$ and $s_{I_{K_U}}$), and they contribute separately. Also, as the field we are considering is massless, the $s_{\nu}$ related to the functions $\phi_{\ell}$ are trivial. In principle, the explicit calculation of the coefficients of the expansions, in both theorems 4.1 and 5.1, is possible but, for a general amplitude, is an extremely hard task. Nevertheless, those corresponding to the leading terms can be evaluated (see Appendix) along the same lines as in the commutative case in Ref. [20].

7 Appendix: Proof of the theorem 4.1

In this Appendix we perform a “translation” for the noncommutative theory, of the proof of the asymptotics theorem of Ref. [20]. In eq. (32), when $\psi_{\nu} \in \Delta$, the integral is absolutely convergent, so we have a first bound:

$$\mathcal{A}_\mathcal{G}(\lambda) < \text{const.} \lambda^{p_{\text{max}}+\epsilon}; \quad p_{\text{max}} = \inf_{\Delta} (\Re \psi(z_K)),$$

(49)

where $\epsilon$ is an arbitrary small number. Therefore, the function $\psi(z_K) - p_{\text{max}}$ is positive in $\Delta$, and reaches zero on its boundary. It then ensues that there exist nonnegative coefficients $d_{\nu}$ such that $\psi(z_K) - p_{\text{max}} = \sum_{\nu} d_{\nu} \psi_{\nu}$, which implies

$$\frac{1}{\prod_{\nu} \psi_{\nu}} \equiv \frac{1}{\psi(z_K) - p_{\text{max}}} \sum_{\nu} \frac{d_{\nu}}{\prod_{\nu' \neq \nu} \psi_{\nu'}}.$$

(50)

For a given $\nu$, if the subset $\{\psi_{\nu'}, \nu' \neq \nu\}$ still generates $\psi(z_K) - p_{\text{max}}$, we can repeat the procedure, which is iterated until we obtain

$$\frac{1}{\prod_{\nu} \psi_{\nu}} \equiv \sum_{E} \frac{d_{E}}{(\psi(z_K) - p_{\text{max}})^{q_{E}+1}} \prod_{\nu \in E} \frac{1}{\psi_{\nu}}.$$

(51)
For each $E \subset \{ \nu \}$, $\psi(z_K) - p_{\text{max}}$ does not belong to the convex domain defined by the subset $\{ \psi_\nu \geq 0, \ \nu \in E \}$ and it becomes negative somewhere in

$$\Delta_E = \{ z_K \text{ such that } \psi_\nu + \theta_\nu E > 0, \text{ for all } \nu \}; \ \ \ \theta_\nu E = \begin{cases} 0 & \text{if } \nu \in E \\ 1 & \text{otherwise.} \end{cases}$$

(52)

Therefore, the amplitude $A_G(\lambda)$ in eq (32) becomes,

$$A_G(\lambda) = \sum E d_E \int_{\Delta_E; \text{Re } (\psi(z_K) - p_{\text{max}}) > 0} \lambda^\psi M_E(z_K) \left( \psi(z_K) - p_{\text{max}} \right)^{\theta_\nu E + 1},$$

(53)

where we have defined the function

$$M_E(z) = \frac{\prod_\nu f_\nu s_\nu^{-\psi_\nu} \Gamma(\psi_\nu + \theta_\nu E)}{\Gamma(-\sum K_U x_K U)},$$

(54)

which is analytical in $\Delta_E$.

The integration path can be moved up to a point where $\psi(z_K) - p_{\text{max}} < 0$, and applying Cauchy’s integral formula we obtain

$$A_G(\lambda) = \sum E d_E \left\{ \lambda^p_{\text{max}} \sum_{q=0}^{\theta_\nu E} A_{p_{\text{max}}}^E q \ln^{q+1} \lambda + \int_{\Delta'_E}; \text{Re } (\psi(z_K) - p_{\text{max}}) < 0 \psi(z_K) - p_{\text{max}} \right\},$$

(55)

where in $\Delta'_E$: $\text{Re } (\psi(z_K) - p_{\text{max}}) < 0$ and

$$A_{p_{\text{max}}}^E q = \frac{1}{q! (q E - q)!} \int_{\Delta_E; \psi(z_K) - p_{\text{max}} = 0} \nabla^{q E - q} M_E(z_K),$$

(56)

with $\nabla$ being the differential operator along any direction crossing the plane $\psi = p_{\text{max}}$. The integral in the second term of (55) is bounded, being less than a constant times $\lambda^{p_{\text{max}} - b_E + 1}$, where $p_{\text{max}} - b_E = \inf_{\Delta_E} \text{Re } \psi(z_K)$, in which $b_E$ is a strictly positive rational number.

The remaining gamma-function singularities are treated in a similar fashion, by applying the identity

$$\Gamma(\psi_\nu + \theta_\nu E) = \frac{1}{\psi_\nu + \theta_\nu E} \Gamma(\psi_\nu + \theta_\nu E + 1)$$

(57)

in the second term of (55), leading to an analogous term with $\lambda^{p_{\text{max}} - b_E}$, and another integral in the next analyticity strip, and so forth. In this way, a complete asymptotic expansion is produced.

References

[1] Grosse, H., Wulkenhaar, R.: Power-counting theorem for non-local matrix models and renormalisation. Commun. Math. Phys. 254, 91-27 (2005).
[2] Grosse, H., Wulkenhaar, R.: Renormalization of $\phi^4$-theory on noncommutative $\mathbb{R}^2$ in the matrix base. JHEP 12, 019 (2003).

[3] Grosse, H., Wulkenhaar, R.: Renormalization of $\phi^4$-theory on noncommutative $\mathbb{R}^4$ in the matrix base. Commun. Math. Phys. 256, 305-374 (2005).

[4] Rivasseau, V., Vignes-Tourneret, F., Wulkenhaar, R.: Renormalization of noncommutative $\phi^4$-theory by multi-scale analysis. Commun. Math. Phys. 262, 565-594 (2006).

[5] Gurau, R., Magen, J., Rivasseau, V., Vignes-Tourneret, F.: Renormalization of non-commutative $\phi_4^4$ field theory in x space. Commun. Math. Phys. 267, 515-542 (2006).

[6] Vignes-Tourneret, F.: Renormalization of the orientable non-commutative Gross–Neveu model. Ann. H. Poincaré 8, 427-474 (2007).

[7] Bergère, M.C., Zuber, J.-B.: Renormalization of Feynman amplitudes and parametric integral representation. Commun. Math. Phys. 35, 113-140 (1974).

[8] Bergère, M.C., Lam, Y.-M.P.: Asymptotic expansion of Feynman amplitudes. Part I. The convergent case. Commun. Math. Phys. 39, 1-32 (1974).

[9] Bergère, M.C., Lam, Y.-M.P.: Bogolubov–Parasiuk theorem in the alpha-parametric representation. J. Math. Phys. 17, 1546-1557 (1976).

[10] Nakanishi, N.: Graph Theory and Feynman Integrals. Gordon and Breach, New York (1971).

[11] Itzykson, C., Zuber, J.-B.: Quantum Field Theory. McGraw-Hill, New York (1980).

[12] Gurau, R., Rivasseau, V.: Parametric representation of noncommutative field theory. Commun. Math. Phys. 272, 811-835 (2007).

[13] Rivasseau, V., Tanasă, A.: Parametric representation of “critical” noncommutative QFT models, arXiv: math-ph/0701034.

[14] Rivasseau, V., Vignes-Tourneret, F.: Renormalisation of non-commutative field theories, arXiv: hep-th/0702068.

[15] Rivasseau, V., Vignes-Tourneret, F.: Non-commutative renormalization. In: de Monvel, A.B., Buchholz, D., Iagolnitzer, D., Moschella, U. (eds.) Rigorous Quantum Field Theory: A Festschrift for Jacques Bros, Birkhäuser, Basel (2006); arXiv: hep-th/0409312.

[16] Rivasseau, V.: Non-commutative renormalization, Séminaire Poincaré X (2007) 1-81; arXiv: 0705.0705 [hep-th].
[17] Bergère, M.C., Lam, Y.-M.P.: preprint, Freie Universität, Berlin, HEP May 1979/9 (unpublished).

[18] Bergère, M.C., de Calan, C., Malbouisson, A.P.C.: A theorem on asymptotic expansion of Feynman amplitudes. Commun. Math. Phys. 62, 137-158 (1978).

[19] Hepp, K.: Proof of the Bogoliubov–Parasiuk theorem on renormalization. Commun. Math. Phys. 2, 301-326 (1966).

[20] de Calan, C., Malbouisson, A.P.C.: Complete Mellin representation and asymptotic behaviours of Feynman amplitudes. Ann. Inst. Henri Poincaré 32, 91-107 (1980).

[21] de Calan, C., David, F. Rivasseau, V.: Renormalization in the complete Mellin representation of Feynman amplitudes. Commun. Math. Phys. 78, 531-544 (1981).

[22] de Calan, C., Malbouisson, A.P.C.: Infrared and ultraviolet dimensional meromorphy of Feynman amplitudes. Commun. Math. Phys. 90, 413-416 (1983).

[23] Malbouisson, A.P.C.: A convergence theorem for asymptotic expansions of Feynman amplitudes. J. Phys. A: Math. Gen. 33, 3587-3595 (2000); Critical behaviour of correlation functions and asymptotic expansions of Feynman amplitudes. In: Janke, W., Pelster, A., Schmidt, H.-J., Bachmann, M. (eds.) Fluctuating Paths and Fields: Festschrift Dedicated to Hagen Kleinert, World Scientific, Singapore (2001).

[24] Linhares, C.A., Malbouisson, A.P.C., Roditi, I.: Asymptotic expansions of Feynman amplitudes in a generic covariant gauge, arXiv: hep-th/0612010.

[25] Gurau, R., Malbouisson, A.P.C., Rivasseau, V., Tanasă, A.: Non-commutative Complete Mellin Representation for Feynman Amplitudes. Lett. Math. Phys. 81, 161-175 (2007).

[26] Gurau, R., Tanasă, A.: Dimensional regularization and renormalization of non-commutative QFT, arXiv:0706.1147 [math-ph].