AN ANALOGUE OF THE KOSTANT CRITERION FOR QUADRATIC LIE SUPERALGEBRAS

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Abstract. Assume that \( r \) be a finite dimensional complex Lie superalgebra with a non-degenerate super-symmetric invariant bilinear form, \( p \) is a finite dimensional complex super vector space with a non-degenerate super-symmetric bilinear form, and \( \nu : r \to \text{osp}(p) \) is a homomorphism of Lie superalgebras. In this paper, we give a necessary and sufficient condition for \( r \oplus p \) to be a quadratic Lie superalgebra. The criterion obtained is an analogue of a constancy condition given by Kostant in Lie algebra setting. As an application, we prove an analogue of the Parthasarathy’s formula for the square of the Dirac operator attached to a pair of quadratic Lie superalgebras.

1. Introduction

A quadratic Lie superalgebra is a Lie superalgebra \( g = g_0 \oplus g_1 \) with a non-degenerate invariant super-symmetric bilinear form \( (\cdot, \cdot) \). We always assume that \( (\cdot, \cdot) \) is consistent, that is, \( (x, y) = 0 \) for any \( x \in g_0 \) and \( y \in g_1 \). Let \( r \) be a subalgebra of a finite dimensional complex quadratic Lie superalgebra \( g \) such that the restriction of \( (\cdot, \cdot) \) on \( r \) is non-degenerate. Denote by \( p \) the orthogonal complement of \( r \) in \( g \) with respect to \( (\cdot, \cdot) \). Then we have an orthogonal decomposition \( g = r \oplus p \), where the restriction of \( (\cdot, \cdot) \) on \( p \) is also non-degenerate and \( p \) is an \( r \)-module.

Conversely, let \( r \) be a finite dimensional complex quadratic Lie superalgebra with respect to a bilinear form \( (\cdot, \cdot)_r \), let \( p \) be a finite dimensional complex super vector space with a non-degenerate super-symmetric bilinear form \( (\cdot, \cdot)_p \), and let \( \nu : r \to \text{osp}(p) \) be a \( (\cdot, \cdot)_p \)-invariant representation of \( r \) on \( p \). Define

\[ g = r \oplus p \]

and define a non-degenerate super-symmetric bilinear form \( (\cdot, \cdot)_g \) on \( g \) by

\[ (\cdot, \cdot)_g|_r = (\cdot, \cdot)_r, \quad (\cdot, \cdot)_g|_p = (\cdot, \cdot)_p, \quad (p, r)_g = 0. \]

The pair \( (\nu, (\cdot, \cdot)_g) \) is of Lie super type if there exists a Lie superalgebraic structure \([\cdot, \cdot]\) on \( g \) satisfying the following conditions:

(a) \( g \) is a quadratic Lie superalgebra with respect to \( (\cdot, \cdot)_g \), and
(b) \( r \) is a subalgebra of \( g \) and \( [x, y] = \nu(x)y \) for any \( x \in r, y \in p \).
In [8, 9], Kostant studied the above problem in Lie algebra setting and obtained a constancy condition involving the Casimir element of \( \mathfrak{g} \) and a cubic element in \( (\Lambda^3(\mathfrak{p}))^2 \) which is used to construct the cubic Dirac operator. In this paper, we obtain an analogue of a constancy condition for the case of Lie superalgebras based on the study in [1].

We begin with the case \( \mathfrak{r} = 0 \). For this case, it is to find all quadratic Lie superalgebraic structures on a complex super vector space with a non-degenerate super-symmetric bilinear form. Clearly, for any quadratic Lie superalgebra \( \mathfrak{g} \), there exists a unique \( \phi \in \Lambda^3_0(\mathfrak{g}) \) such that

\[
(\phi, z_1 \wedge z_2 \wedge z_3) = \frac{1}{2}([z_1, z_2], z_3), \quad [z_1, z_2] = 2\iota(z_1)\iota(z_2)\phi.
\]

Motivated by the above fact, for any \( \phi \in \Lambda^3_0(\mathfrak{g}) \), define a bracket \( [\cdot, \cdot]^\phi \) on \( \mathfrak{g} \) by

\[
[z_1, z_2]^{\phi} = 2\iota(z_1)\iota(z_2)\phi.
\]

We prove that the bracket \( [\cdot, \cdot]^\phi \) defines a Lie superalgebraic structure on \( \mathfrak{g} \) if and only if the Clifford square \( \phi^2 \) is a constant.

In the general case, let \( \phi_\mathfrak{r} \in \Lambda^3_0(\mathfrak{r}) \) be the cubic element corresponding to the quadratic Lie superalgebraic structure on \( \mathfrak{r} \), and let \( \phi_\mathfrak{p} \in \Lambda^3_0(\mathfrak{p}) \) be the cubic element given as the projection of \( \phi \) relative to the decomposition \( \mathfrak{g} = \mathfrak{r} \oplus \mathfrak{p} \). If \( (\nu, \langle \cdot, \cdot \rangle_\mathfrak{g}) \) is of Lie super type, then the cubic element \( \phi \) is decomposed as

\[
(1.1) \quad \phi = \phi_\mathfrak{r} + \phi_\mathfrak{p} + \sum_{1 \leq i < r} \nu_i(x_i) \wedge x^i,
\]

where \( \{x_1, \ldots, x_r\} \) is a basis of \( \mathfrak{r} \), \( \{x^1, \ldots, x^r\} \) is the \( (\cdot, \cdot)_\mathfrak{r} \)-dual basis to \( \{x_1, \ldots, x_r\} \), and

\[
\nu_\mathfrak{r} : \mathfrak{r} \rightarrow \Lambda^2(\mathfrak{p})
\]

is the unique Lie superalgebraic homomorphism induced by \( \nu \). Moreover, \( \phi_\mathfrak{p} \in (\Lambda^3_0(\mathfrak{p}))^2 \). Conversely, for any \( \phi_\mathfrak{p} \in (\Lambda^3_0(\mathfrak{p}))^2 \), define the cubic element \( \phi \) by (1.1). We prove that \( \phi^2 \) is a scalar if and only if \( \nu_\mathfrak{r}(\text{Cas}_\mathfrak{r}) + \phi_\mathfrak{p}^2 \) is a constant, thus \( (\nu, \langle \cdot, \cdot \rangle_\mathfrak{g}) \) is of Lie super type if and only if \( \nu_\mathfrak{r}(\text{Cas}_\mathfrak{r}) + \phi_\mathfrak{p}^2 \) is a constant.

This paper is organized as follows. In Section 2, we recall some basic facts about Clifford algebras and exterior algebras over super vector spaces. Sections 3 and 4 are to study the case \( \mathfrak{r} = 0 \) and the general case, respectively. As an application, we prove an analogue of the Parthasarathy’s formula for the square of the Dirac operator attached to a pair of quadratic Lie superalgebras in Section 5.

2. Preliminaries

2.1. Super vector spaces. A \( \mathbb{Z}_2 \)-graded space \( V = V_0 + V_1 \) is called a super vector space, where the elements of \( V_0 \) are even and those of \( V_1 \) are odd. Denote by \( |x| \in \{0, 1\} \) the parity of a homogeneous element \( x \in V \). (Whenever this notation is used, it implies that \( x \) is homogeneous.) We say that a bilinear form \( \langle \cdot, \cdot \rangle \) on \( V \) is super-symmetric if \( \langle x, y \rangle = (-1)^{|x||y|}\langle y, x \rangle \) for any \( x, y \in V \); consistent if \( (V_0, V_1) = 0 \). Throughout this paper, we always assume that \( \langle \cdot, \cdot \rangle \) is consistent, that is,

\[
(2.1) \quad \langle x, y \rangle = 0, \quad \text{if } |x| \neq |y|.
\]
For a finite dimensional super vector space $V$, let $\{e_1, \ldots, e_m\}$ of $V_0$ be a basis of $V_0$ and $\{e_{m+1}, \ldots, e_{m+n}\}$ a basis of $V_1$. Corresponding to the homogeneous basis $\{e_1, \ldots, e_{m+n}\}$ of $V$, the matrix of an endomorphism $T$ on $V$ is the form $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, where $\alpha$ is an $(m \times m)$-, $\beta$ an $(m \times n)$-, $\gamma$ an $(n \times m)$-, and $\delta$ an $(n \times n)$-matrix. Define the supertrace $\text{str}(T)$ of $T$ by

$$\text{str}(T) = \text{tr}(\alpha) - \text{tr}(\delta).$$

It is clear that $\text{str}(T)$ is independent of the choice of a homogeneous basis.

**Lemma 2.1.** Let $V$ be a finite dimensional super vector space with a non-degenerate super-symmetric bilinear form $(\cdot, \cdot)$, let $A$ be an associative algebra, and let $f, g : V \to A$ be two linear mappings. Assume that $\{x_1, \ldots, x_n\}$ is a homogeneous basis of $V$ and $\{x^1, \ldots, x^n\}$ is the $(\cdot, \cdot)$-dual basis of $\{x_1, \ldots, x_n\}$. Then $\sum_{i=1}^n f(x_i)g(x^i)$ is independent of the choice of basis. In particular,

$$\sum_{i=1}^n f(x_i)g(x^i) = \sum_{i=1}^n (-1)^{|x_i||x^i|}f(x_i)g(x_i).$$

**Proof.** Let $\{y_1, \ldots, y_n\}$ be another basis of $V$ and let $\{y^1, \ldots, y^n\}$ be the $(\cdot, \cdot)$-dual basis of $\{y_1, \ldots, y_n\}$. Let $S = (s_{ij})$ and $T = (t_{ij})$ be $n \times n$ matrices satisfying

$$(y_1, \ldots, y_n) = (x_1, \ldots, x_n)S, \quad (y^1, \ldots, y^n) = (x^1, \ldots, x^n)T,$$

that is, $y_i = \sum_{j=1}^n s_{ji}x_j$ and $y^i = \sum_{j=1}^n t_{ji}x^j$ for any $i = 1, \ldots, n$. Since

$$\delta_{ij} = (y^i, y_j) = \frac{1}{n} \sum_{k=1}^n t_{ki}x^k = \frac{1}{n} \sum_{k=1}^n s_{kj}x_k = \sum_{k=1}^n s_{kj}t_{ki},$$

we have that $S^T = E_n$, which implies that $TS^T = E_n$, that is, $\sum_{i=1}^n t_{ki}s_{li} = \delta_{kl}$. Here $E_n$ is the $n \times n$ identity matrix. Now,

$$\sum_{i=1}^n f(y_i)g(y^i) = \sum_{i=1}^n \sum_{k=1}^n s_{li}t_{ki}f(x_i)g(x^k) = \sum_{k=1}^n \sum_{i=1}^n \delta_{kl}f(x_i)g(x^k) = \sum_{i=1}^n f(x_i)g(x^i),$$

which implies that $\sum_{i=1}^n f(x_i)g(x^i)$ is independent of the choice of basis. The last statement follows from the fact that $\{(-1)^{|x_i||x^i|}x_i\}$ is the $(\cdot, \cdot)$-dual basis of $\{x_1, \ldots, x_n\}$. 

A superalgebra is a super vector space $A = A_0 + A_1$ with a multiplication satisfying $A_0A_1 \subset A_{i+j}$ for any $i, j \in \mathbb{Z}_2$. For superalgebras $A$ and $B$, $A \otimes B$ is a superalgebra with the multiplication defined by

$$(x \otimes y)(x' \otimes y') = (-1)^{|y||x'|}xx' \otimes yy'.$$

A Lie superalgebra $\mathfrak{g}$ is a superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with a bracket $[\cdot, \cdot]$ satisfying

$$[x, y] = -(1)^{|y||x|}y[x],$$

$$[x, [y, z]] = [[x, y], z] + (1)^{|y||z|}[y, [x, z]].$$
Moreover, we may identify \( \mathcal{A} \) algebra generated by \( \mathcal{A} \) with the relation
\[ xy + (-1)^{|x||y|}yx = 2x, y \]
\( \text{resp. } x \wedge y + (-1)^{|x||y|}y \wedge x = 0 \).
where \( xy \) (resp. \( x \wedge y \)) is the Clifford multiplication of \( C(V) \) (resp. the exterior multiplication of \( \Lambda(V) \)) for any \( x, y \in V \). The pair \((C(V), \zeta_C)\) (resp. \((\Lambda(V), \zeta_C)\)) has the following standard universal mapping property.

**Proposition 2.2.** Assume that \( \mathcal{A} \) is an associative algebra with the unity element \( 1_\mathcal{A} \) and \( \phi : V \to \mathcal{A} \) is a linear mapping such that

\[
\phi(x)\phi(y) + (-1)^{|x||y|}\phi(y)\phi(x) = 2\phi(x, y)1_\mathcal{A} \quad (\text{resp. } \phi(x)\phi(y) + (-1)^{|x||y|}\phi(y)\phi(x) = 0)
\]

for any \( x, y \in V \). Then \( \phi \) extends uniquely to an algebra homomorphism \( \phi_C : C(V) \to \mathcal{A} \) (resp. \( \phi_\Lambda : \Lambda(V) \to \mathcal{A} \)).

It is well-known that \( T(V) \) has a natural \( \mathbb{Z} \times \mathbb{Z}_2 \)-gradation. The degree of \( x_1 \otimes \cdots \otimes x_n \) is equal to \((n, |x_1| + \cdots + |x_n|)\). Since \( C(V) \) and \( \Lambda(V) \) inherit the \( \mathbb{Z}_2 \)-gradation from \( T(V) \), we still denote by \(|u|\) the parity of a homogeneous element \( u \) in \( T(V) \) (resp. \( C(V), \Lambda(V) \)). The \( \mathbb{Z} \)-gradation of \( T(V) \) induces a \( \mathbb{Z}_2 \)-gradation of \( \Lambda(V) \), but only induces a \( \mathbb{Z}_2 \)-gradation of \( C(V) \).

Denote by \( T^n_a(V) \) the subspace spanned by the elements of degree \((n, a)\) in \( T(V) \). Then

\[
T(V) = \bigoplus_{n \in \mathbb{Z}, a \in \mathbb{Z}_2} T^n_a(V).
\]

Set \( T^n(V) = \sum_{n \in \mathbb{Z}} T^n_a(V) \) and \( T_a(V) = \sum_{n \in \mathbb{Z}} T^n_a(V) \).

**Definition 2.3.** A linear mapping \( D : T(V) \to T(V) \) is called a derivation of degree \((k, d)\) if

(i) \( D(T^n(V)) \subset T^{k+n}(V) \) and \( D(T_a(V)) \subset T_{n+d}(V) \),

(ii) \( D(u \otimes v) = D(u) \otimes v + (-1)^{kn}(-1)^{da}u \otimes D(v) \), \( \forall u \in T^n_a(V), v \in T(V) \).

Similarly, one can define the derivation of \( C(V) \) and \( \Lambda(V) \). If \( D_T \) is a derivation of \( T(V) \) which stabilizes both \( I_C(V) \) and \( I_\Lambda(V) \), then \( D_T \) descends a derivation \( D_C \) of \( C(V) \) and a derivation \( D_\Lambda \) of \( \Lambda(V) \).

For any homogeneous element \( x \in V \), there is a unique derivation \( \iota_T(x) \) of \( T(V) \) such that \( \iota_T(x)(y) = (x, y) \) for any \( y \in V \). Explicitly,

\[
\iota_T(x)(x_1 \otimes \cdots \otimes x_n) = \sum_{k=1}^n (-1)^{k-1}(-1)^{|x|(|x_0|+\cdots+|x_{k-1}|)}(x, x_k)x_1 \otimes \cdots \otimes \hat{x}_k \otimes \cdots \otimes x_n,
\]

where \( x_1, \ldots, x_n \in V \) and \(|x_0| = 0\). Clearly, \( \iota_T(x) \) is a derivation of degree \((-1, |x|)\) by the identity (2.1). By Proposition 4.5 in \([1]\), \( \iota_T(x) \) stabilizes both \( I_C(V) \) and \( I_\Lambda(V) \). Then \( \iota_T(x) \) descends to derivations \( \iota_C(x) \) and \( \iota_\Lambda(x) \) of \( C(V) \) and \( \Lambda(V) \), respectively.

For any \( x \in V \), let \( \epsilon_\Lambda(x) \) be the left exterior multiplication operator by \( x \) on \( \Lambda(V) \). By Proposition 4.6 in \([1]\), we have

\[
\epsilon_\Lambda(x)\epsilon_\Lambda(y) + (-1)^{|x||y|}\epsilon_\Lambda(y)\epsilon_\Lambda(x) = 0,
\]

\[
\iota_\Lambda(x)\iota_\Lambda(y) + (-1)^{|x||y|}\iota_\Lambda(y)\iota_\Lambda(x) = 0,
\]

\[
\iota_\Lambda(x)\epsilon_\Lambda(y) + (-1)^{|x||y|}\epsilon_\Lambda(y)\iota_\Lambda(x) = (x, y).
\]

Set \( \gamma(x) = \epsilon_\Lambda(x) + \iota_\Lambda(x) \). Then

\[
\gamma(x)\gamma(y) + (-1)^{|x||y|}\gamma(y)\gamma(x) = 2(x, y).
\]
The linear map $V \to \text{End}(\Lambda(V))$ defined by $x \mapsto \gamma(x)$ naturally extends to a homomorphism $T(V) \to \text{End}(\Lambda(V))$, which, by the identity (2.8), descends to a homomorphism

$$\gamma : C(V) \to \text{End}(\Lambda(V)).$$

The homomorphism $\gamma$ defines a $(C(V))$-module structure on $\Lambda(V)$. Let $\eta : C(V) \to \Lambda(V)$ be the linear map defined by

$$\eta(u) = \gamma(u)1_{\Lambda(V)},$$

where $1_{\Lambda(V)}$ is the unity element of $\Lambda(V)$.

Define the skew super symmetrization map $s : \Lambda(V) \to T(V)$ by

$$s(x_1 \wedge x_2 \wedge \cdots \wedge x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{N_\sigma(x_1,\ldots,x_n)} \text{sgn}(\sigma)x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(n)}$$

for homogeneous elements $x_1, \ldots, x_n \in V$. Here $\text{sgn}(\sigma)$ denotes the signature of the permutation $\sigma$, and $N_\sigma(x_1,\ldots,x_n)$ is the number of pairs $i < j$ such that $x_i, x_j$ are odd elements and $\sigma^{-1}(i) > \sigma^{-1}(j)$. Let $\Sigma(V) = s(\Lambda(V))$ be the space of skew super-symmetric tensors. Set $\hat{\pi}_C = \pi_C|_{\Sigma(V)}$ and $\hat{\pi}_\Lambda = \pi_{\Lambda|_{\Sigma(V)}}$.

**Lemma 2.4** (1, Theorem 4.8). The map $\eta$ is bijective. Moreover, one has

$$\eta \circ \hat{\pi}_C = \hat{\pi}_\Lambda,$$

that is, the following diagram commutes.

\[
\begin{array}{ccc}
\Sigma(v) & \xrightarrow{\hat{\pi}_C} & \Lambda(V) \\
\downarrow{\eta} & & \downarrow{\hat{\pi}_\Lambda} \\
C(V) & \xrightarrow{\eta} & \Lambda(V)
\end{array}
\]

Using $\eta$, we may identify $C(V)$ with $\Lambda(V)$. There exist two multiplications on $\Lambda(V)$, that is, the exterior multiplication $u \wedge v$ and the Clifford multiplication $uv$.

Set $\Lambda^n(V) = \pi_\Lambda(T^n(V))$ and denote the component of $u$ in $\Lambda^n(V)$ by $(u)^n$, for any $u \in \Lambda(V)$. One can extends the bilinear form $(\cdot, \cdot)$ on $V$ to a non-degenerate bilinear form on $\Lambda(V)$, which is still denote by $(\cdot, \cdot)$,

$$(u, v) = \begin{cases} \frac{n(n-1)}{2} (uv)_0, & m = n, \\ 0, & m \neq n, \end{cases}$$

for any $u \in \Lambda^m(V)$ and $v \in \Lambda^n(V)$, where we identify $\Lambda^0(V)$ with $\mathbb{C}$. If $u = x_1 \wedge \cdots \wedge x_n$ is an element in $\Lambda^n(V)$, then

$$(u, v) = \frac{n(n-1)}{2} \iota_\Lambda(x_1) \cdots \iota_\Lambda(x_n) v$$

for any $v \in \Lambda^n(V)$. Moreover, we have

**Lemma 2.5** (1, Theorem 5.4). Let $x \in V$ and $u, v \in \Lambda(V)$. Then

(i) $(u, v) = (-1)^{|u||v|}(v, u)$,
(ii) $(\epsilon_\Lambda(x) u, v) = (-1)^{|x||u|}(u, \iota_\Lambda(x) v)$,
(iii) $(\iota_\Lambda(x) u, v) = (-1)^{|x||u|}(u, \epsilon_\Lambda(x) v)$. 
Every \( u \in \Lambda^2(V) \) defines an operator \( \text{ad} \ u \) on \( \Lambda(V) \):
\[
\text{ad} \ u(v) = [u, v]_C = uv - (-1)^{|u||v|}vu, \quad \forall v \in \Lambda(V).
\]
It is proved in [1] that \( \text{ad} \ u \) is a derivation of degree \( (0, |u|) \) of \( \Lambda(V) \). Moreover, we have
\[
(2.10) \quad \text{ad} \ u(z) = -2(-1)^{|u||z|}i_{\Lambda}(z)u
\]
for any \( z \in V \) and
\[
(2.11) \quad (\text{ad} \ u(v_1), v_2) + (-1)^{|u||v_1|}(v_1, \text{ad} \ u(v_2)) = 0
\]
for any \( v_1, v_2 \in \Lambda(V) \). Now define a map
\[
A : \Lambda^2(V) \to \mathfrak{osp}(V, \varepsilon)
\]
by \( A(u) = (\text{ad} \ u)|_V \).

**Lemma 2.6** ([1], Theorem 6.3). \( \Lambda^2(V) \) is a Lie superalgebra under the commutator
\[
[u, v]_C = uv - (-1)^{|u||v|}vu,
\]
and the map \( A : \Lambda^2(V) \to \mathfrak{osp}(V, \varepsilon) \) is an isomorphism.

Let \( \xi_T \in \text{End}(T(V)) \). Suppose that both \( I_A(V) \) and \( I_C(V) \) are stable under \( \xi_T \). Then \( \xi_T \) descends to a map \( \xi_A \) (resp. \( \xi_C \)) of \( \Lambda(V) \) (resp. \( C(V) \)).

**Lemma 2.7** ([1], Lemma 4.10). If \( \Sigma(V) \) is stable under \( \xi_T \), then \( \eta \circ \xi_C = \xi_A \circ \eta \) on \( C(V) \).

If \( I_A(V) \), \( I_C(V) \) and \( \Sigma(V) \) are stable under \( \xi_T \), then by Lemma 2.6 we may identify \( \xi_C \) with \( \xi_A \) on \( \Lambda(V) \) by means of \( \eta \).

**Lemma 2.8.** For any homogeneous element \( x \in V \), \( \nu_T(x) \) stabilizes \( \Sigma(V) \).

*Proof.* Let \( \sigma \in S_n \) be a permutation of the set \( \{1, \ldots, n\} \). For any integer \( l \) between 1 and \( n \), there exists a unique integer \( k \) such that \( \sigma(k) = l \). Denote the permutation group of the set of \( \{1, \ldots, n\} \) by \( S_{n,l} \). Let \( \tau(\sigma; k, l) \in S_{n,l} \) be the permutation such that
\[
\tau(\sigma; k, l)(1, \ldots, \hat{l}, \ldots, n) = (\sigma(1), \ldots, \sigma(k), \ldots, \sigma(n)).
\]
Recall that \( N_{\sigma}(x_1, \ldots, x_n) \) is the number of pairs \( i < j \) such that \( x_i, x_j \) are odd elements and \( \sigma^{-1}(i) > \sigma^{-1}(j) \). Then
\[
x_1 \wedge \cdots \wedge x_n = (-1)^{N_{\sigma}(x_1, \ldots, x_n)} \text{sgn}(\sigma)x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)}
\]
\[
= (-1)^{k-1+N_{\sigma}(x_1, \ldots, x_n)}(-1)^{|x_{\sigma(k)}|(\langle x_{\sigma(1)} \rangle + \cdots + \langle x_{\sigma(k-1)} \rangle)} \text{sgn}(\sigma)
\]
\[
x_{\sigma(k)} \wedge x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(k)} \wedge \cdots \wedge x_{\sigma(n)}.
\]
On the other hand, we have
\[
x_1 \wedge \cdots \wedge x_n = (-1)^{l-1}(-1)^{|x_1| + \cdots + |x_{l-1}|}x_1 \wedge x_1 \wedge \cdots \wedge x_{\hat{l}} \wedge \cdots \wedge x_n
\]
\[
= (-1)^{l-1+N_{\tau(\sigma; k, l)}(x_1, \ldots, \hat{x}_l, \ldots, x_n)}(-1)^{|x_1| + \cdots + |x_{l-1}|} \text{sgn}(\tau(\sigma; k, l))
\]
\[
x_1 \wedge x_{\tau(\sigma; k, l)(1)} \wedge \cdots \wedge x_{\tau(\sigma; k, l)(l)} \wedge \cdots \wedge x_{\tau(\sigma; k, l)(n)}
\]
\[
= (-1)^{l-1+N_{\tau(\sigma; k, l)}(x_1, \ldots, \hat{x}_l, \ldots, x_n)}(-1)^{|x_1| + \cdots + |x_{l-1}|} \text{sgn}(\tau(\sigma; k, l))
\]
\[
x_{\sigma(k)} \wedge x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(k)} \wedge \cdots \wedge x_{\sigma(n)}.
\]
which implies that
\begin{equation}
(-1)^{k-1+N_\sigma(x_1,\ldots,x_n)}(-1)^{|x_\sigma(k)|(|x_{\sigma(1)}|+\cdots+|x_{\sigma(k-1)}|)} \text{sgn}(\sigma)
= (-1)^{l-1+N_{\tau(\sigma';k,l)}(x_1,\ldots,\widehat{x_i},\ldots,x_n)}(-1)^{|x_l|(|x_{1}|+\cdots+|x_{l-1}|)} \text{sgn}(\tau(\sigma; k, l)).
\end{equation}

Conversely, for any \(\tau \in S_{n,l}\), there exist permutations \(\sigma_i (1 \leq i \leq n)\) of \(\{1, \ldots, n\}\) such that
\begin{equation}
\tau(\sigma_i; i, l) = \tau.
\end{equation}

In fact, \(\sigma_i\) is defined by \(\sigma_i(i) = l\) and
\[
\sigma_i(1, \ldots, \hat{i}, \ldots, n) = (\tau(1), \ldots, \widehat{\tau(l)}, \ldots, \tau(n)).
\]

Let \(S(k; x_1, x_2, \ldots, x_k)\) denote
\[
\sum_{\sigma \in S_k} (-1)^{N_\sigma(x_1,\ldots,x_k)} \text{sgn}(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}.
\]

Identities (2.12) and (2.13) imply that
\[
\iota_T(x)(S(n; x_1, x_2, \ldots, x_n)) = \iota_T(x)\left( \sum_{\sigma \in S_n} (-1)^{N_\sigma(x_1,\ldots,x_n)} \text{sgn}(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \right)
= \sum_{\sigma \in S_n} \sum_{k=1}^n (-1)^{k-1+N_\sigma(x_1,\ldots,x_n)}(-1)^{|x_\sigma(k)|(|x_{\sigma(1)}|+\cdots+|x_{\sigma(k-1)}|)} \text{sgn}(\sigma)
\]
\[
(x, x_{\sigma(k)}) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)} \otimes \cdots \otimes x_{\sigma(n)}
= \sum_{\sigma \in S_n} \sum_{l=1}^n (-1)^{l-1+N_{\tau(\sigma';\sigma^{-1}(l),l)}(x_1,\ldots,\widehat{x_i},\ldots,x_n)}(-1)^{|x_l|(|x_{1}|+\cdots+|x_{l-1}|)} \text{sgn}(\tau(\sigma; \sigma^{-1}(l), l))
\]
\[
(x, x_l) x_{\tau(\sigma';\sigma^{-1}(l),l)} x_{\sigma(1)} \otimes \cdots \otimes x_{\tau(\sigma';\sigma^{-1}(l),l)} \otimes \cdots \otimes x_{\tau(\sigma;\sigma^{-1}(l),l)} \otimes \cdots \otimes x_{\tau(\sigma;\sigma^{-1}(l),l)}
= \sum_{l=1}^n \sum_{\tau \in S_{n,l}} (-1)^{l-1+N_{\tau(\sigma';\sigma^{-1}(l),l)}(x_1,\ldots,\widehat{x_i},\ldots,x_n)}(-1)^{|x_l|(|x_{1}|+\cdots+|x_{l-1}|)} \text{sgn}(\tau)(x, x_l)
\]
\[
x_{\tau(1)} \otimes \cdots \otimes x_{\tau(l)} \otimes \cdots \otimes x_{\tau(n)}
= \sum_{l=1}^n (-1)^{l-1+|x_l|(|x_{1}|+\cdots+|x_{l-1}|)}(x, x_l) S(n-1; x_1, \ldots, \widehat{x_l}, \ldots, x_n).
\]

The lemma follows. \(\square\)

Thus, we may identify \(\iota_A\) with \(\iota_C\) on \(\Lambda(V)\) by means of \(\eta\), and we denote \(\iota_A (= \iota_C)\) by \(\iota\).

Let \(\alpha_T\) be the linear map of \(T(V)\) defined by
\[
\alpha_T(x_1 \otimes \cdots \otimes x_n) = (-1)^{n(n+1)/2+N_{\sigma_0(x_1,\ldots,x_n)} \text{sgn}(\sigma)}(x_n \otimes \cdots \otimes x_1),
\]
where \(x_1, x_2, \ldots, x_k\) are homogeneous elements in \(V\) and
\[
\sigma_0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}.
\]
It is proved in [1] that $\alpha_T$ stabilizes $I_A(V)$, $I_C(V)$ and $\Sigma(V)$. Hence, $\alpha_T$ descends to the linear map $\alpha_\Lambda$ (resp. $\alpha_C$) of $\Lambda(V)$ (resp. $C(V)$); we may identify $\alpha_\Lambda$ with $\alpha_C$ on $\Lambda(V)$ by means of $\eta$, and we denote $\alpha_\Lambda(=\alpha_C)$ by $\alpha$.

**Lemma 2.9 (I).** The linear map $\alpha$ has the following properties.

(i) $\alpha^2 = 1$.

(ii) $\alpha(u) = (-1)^{n(u-1)}u$ for any $u \in \Lambda^n(V)$.

(iii) $\alpha(u \wedge v) = (-1)^{|u||v|}\alpha(v) \wedge \alpha(u)$ and $\alpha(uv) = (-1)^{|u||v|}\alpha(v)\alpha(u)$ for any $u, v \in \Lambda(V)$.

**Lemma 2.10.** Let $x, u$ be homogeneous elements in $V$ and $\Lambda^k(V)$ respectively. Then

$$xu + (-1)^{k-1}(-1)^{|x||u|}ux = 2\epsilon(x)u.$$

*Proof.* Recall that $\gamma(x) = \epsilon(x) + \epsilon(x)$ is the operator of left Clifford multiplication in $\Lambda(V)$ by $x$. Let $\gamma_R(x) \in \text{End}(\Lambda(V))$ be the operator of right Clifford multiplication in $\Lambda(V)$ by $x$. Then

$$xu + (-1)^{k-1}(-1)^{|x||u|}ux = \gamma(x)(u) + (-1)^{k-1}(-1)^{|x||u|}\gamma_R(x)(u).$$

Let

$$\text{End}^j(\Lambda(V)) = \{\xi \in \text{End}(\Lambda(V))|\xi(\Lambda^j(V)) \subset \Lambda^{i+j}(V)\}.$$

Then

$$\text{End}(\Lambda(V)) = \bigoplus_{j \in \mathbb{Z}} \text{End}^j(\Lambda(V)).$$

Write $\gamma(x) = \sum_{j \in \mathbb{Z}} a_j$ and $\gamma_R(x) = \sum_{j \in \mathbb{Z}} b_j$, where $a_j, b_j \in \text{End}^j(\Lambda(V))$. Then $a_j = 0$ unless $j \in \{-1, 1\}$, and $a_{-1} = \epsilon(x), a_1 = \epsilon(x)$. By Lemma 2.11

$$\gamma_R(x)(u) = ux = \alpha(\alpha(ux)) = (-1)^{k(k-1)/2}(-1)^{|x||u|}\alpha(xu) = (-1)^{k(k-1)/2}(-1)^{|x||u|}\alpha(\gamma(x)u).$$

It follows that

$$b_j(u) = (-1)^{k(k-1)/2}(-1)^{|x||u|}\alpha(a_j(u)) = (-1)^{k(k-1)/2}(-1)^{|x||u|}a_j(u).$$

Hence we have $b_j = 0$ unless $j \in \{-1, 1\}$, and

$$b_{-1}(u) = (-1)^{k-1}(-1)^{|x||u|}a_{-1}(u), \quad b_1 = (-1)^k(-1)^{|x||u|}a_1(u).$$

Therefore, we have

$$xu + (-1)^{k-1}(-1)^{|x||u|}ux = 2a_{-1}(u) = 2\epsilon(x)u.$$

The lemma follows. \hfill \square

### 3. Quadratic Lie superalgebraic structures on super vector spaces

Let $\mathfrak{g}$ be a finite dimensional complex super vector space with a non-degenerate supersymmetric bilinear form $(\cdot, \cdot)$.

If there exists a Lie superalgebraic structure $[\cdot, \cdot]$ on $\mathfrak{g}$ such that $\mathfrak{g}$ is quadratic with respect to $(\cdot, \cdot)$, then there exists a unique $\phi \in \Lambda^3_{0}\mathfrak{g}$ such that

$$(\phi, z_1 \wedge z_2 \wedge z_3) = -\frac{1}{2}([z_1, z_2], z_3)$$

for any $z_1, z_2, z_3 \in \mathfrak{g}$. 
Lemma 3.1. If \( u \phi \) that is, \( z \) for homogeneous elements and for any \( 1 \), by the identity (2.6), 
\[
\lbrack z_1, z_2 \rbrack = 2 \iota(z_1)\iota(z_2)\phi,
\]
First, by the identity (2.6), \([z_1, z_2] = -2 (\phi, z_1 \wedge z_2 \wedge z_3)\)
for any \( z_1, z_2 \in g \). Next, by Lemma 2.5, we have
\[
\lbrack [z_1, z_2], z_3 \rbrack = 2 \iota(z_1)\iota(z_2)\phi = -2 (\phi, z_1 \wedge z_2 \wedge z_3).
\]
for homogeneous elements \( z_1, z_2, z_3 \in g \). It follows that
\[
\lbrack [z_1, z_2], z_3 \rbrack = (z_1, [z_2, z_3] \phi),
\]
that is, \((\cdot, \cdot)\) is invariant with respect to the bracket \([\cdot, \cdot]\phi\). Finally, we will give the condition for the bracket \([\cdot, \cdot]\phi\) satisfying the super Jacobi identity. Denote the Clifford square of \( u \) by \( u^2 \) for any \( u \in \Lambda g \). By Lemma 2.9, we have \( \alpha(\phi^2) = -1 \phi|\phi| \alpha(\phi) \alpha(\phi) = \phi^2 \). Then
\[
\phi^2 = (\phi^2)^4 + (\phi^2)^0.
\]

Lemma 3.1. If \( \phi \in \Lambda^3 g \), then
\[
\iota(z_1)\iota(z_2)\iota(z_3)\phi^2 = \frac{1}{2} \lbrack [z_1, [z_2, z_3] \phi] - [[z_1, z_2] \phi, z_3] \phi - (-1)^{|z_1||z_2|} [z_2, [z_1, z_3] \phi] \phi \rbrack
\]
for homogeneous elements \( z_1, z_2, z_3 \in g \).

Proof. By a direct calculation, we have
\[
\iota(z_1)\iota(z_2)\iota(z_3)\phi^2 = \frac{1}{2} [(\iota(z_1)\phi)(\iota(z_2)\iota(z_3)\phi) - (-1)^{|z_1||z_2|+|z_1||z_3|} (\iota(z_1)\iota(z_3)\phi)(\iota(z_2)\phi))
\]
\[- (-1)^{|z_1||z_2|} (\iota(z_2)\phi)(\iota(z_1)\iota(z_3)\phi) - (-1)^{|z_1||z_2|+|z_2||z_3|} (\iota(z_1)\iota(z_3)\phi)(\iota(z_2)\phi))
\]
\[- (-1)^{|z_1||z_3|+|z_2||z_3|} (\iota(z_3)\phi)(\iota(z_1)\iota(z_2)\phi) - (-1)^{|z_1||z_3|+|z_2||z_3|} (\iota(z_1)\iota(z_2)\phi)(\iota(z_3)\phi)).\]
Note that \( \iota(z_2)\iota(z_3)\phi = \frac{1}{2} [z_2, [z_3] \phi] \), by the identity (2.10), we have
\[
\iota(z_1)\phi[\iota(z_2)\iota(z_3)\phi) = -(-1)^{|z_1||z_2|+|z_1||z_3|} \iota(z_1)\iota(z_3)\phi(\iota(z_2)\phi) = \frac{1}{2} [z_1, [z_2, z_3] \phi] \).
\]
Similarly, we have
\[
\iota(z_2)\phi(\iota(z_1)\iota(z_3)\phi) = -(-1)^{|z_1||z_2|+|z_2||z_3|} \iota(z_1)\iota(z_3)\phi(\iota(z_2)\phi) = \frac{1}{2} [z_2, [z_1, z_3] \phi] \).
\]
and
\[
\iota(z_3)\phi(\iota(z_1)\iota(z_2)\phi) = -(-1)^{|z_1||z_3|+|z_2||z_3|} \iota(z_1)\iota(z_3)\phi(\iota(z_2)\phi) = \frac{1}{2} [z_3, [z_1, z_2] \phi] \).
\]
Hence
\[
\iota(z_1)\iota(z_2)\iota(z_3)\phi^2 = \frac{1}{2} \lbrack [z_1, [z_2, z_3] \phi] - [[z_1, z_2] \phi, z_3] \phi - (-1)^{|z_1||z_2|} [z_2, [z_1, z_3] \phi] \phi \rbrack.
\]
If \( i, j, k \) are different from each other and \( |e_i| + |e_j| + |e_k| = 0 \), then
\[
\phi^2(e^i \wedge e^j \wedge e^k, e^i \wedge e^j \wedge e^k) = (-1)^{|e^i||e^j||e^k|} = (-1)^{|e_i|+|e_j|+|e_k||e_i|}.
\]

If \( j = k, i \neq j, |e_i| + |e_j| + |e_k| = 0 \) and \( e_i \wedge e_j \wedge e_k \neq 0 \), then \( |e_i| = 0 \) and \( |e_j| = \bar{1} \), thus
\[
\phi^2(e^i \wedge e^j \wedge e^k, e^i \wedge e^j \wedge e^k) = -2 = 2(-1)^{|e_i||e_j|+|e_k||e_k|}
\]
and
\[
\phi^2(e^i \wedge e^k \wedge e^i, e^j \wedge e_k \wedge e_i) = -2 = 2(-1)^{|e_j||e_k|+|e_i||e_i|}.
\]

If \( i = j = k \) and \( |e_i| + |e_j| + |e_k| = 0 \), then
\[
|e_i| = 0, \quad e_i \wedge e_j \wedge e_k = 0.
\]
By the above discussion, we have

\[(3.2) \quad \phi = \frac{1}{6} \sum_{1 \leq i,j,k \leq n} ( -1)^{\|e_i\| + \|e_j\| + \|e_k\|} (\phi, e_i \wedge e_j \wedge e_k) e_i \wedge e_j \wedge e_k. \]

Thus,

\[
\phi^2 = (\phi^2)_0 = - (\phi, \phi)
\]

\[
= - \frac{1}{6} \sum_{1 \leq i,j,k \leq n} ( -1)^{\|e_i\| + \|e_j\| + \|e_k\|} (\phi, e_i \wedge e_j \wedge e_k)(e_i \wedge e_j \wedge e_k, \phi)
\]

\[
= - \frac{1}{24} \sum_{1 \leq i,j \leq n} ( -1)^{\|e_i\| + \|e_j\|} ([e_i, e_j]^\phi, [e_i, e_j]^\phi)([e_i, e_j]^\phi, e_i \wedge e_j)
\]

\[
= \frac{1}{24} \sum_{1 \leq i,j \leq n} ( -1)^{\|e_i\| + \|e_j\|} ([e_i, e_j]^\phi, [e_i, e_j]^\phi)([e_i, e_j]^\phi, e_i \wedge e_j),
\]

which implies that

\[(3.3) \quad (\phi^2) = (\phi^2)_0 = \frac{1}{24} \text{str} \sum_{i=1}^{n} \text{ad} e_i \text{ad} e_i. \]

Since the map \( \text{ad} : \mathfrak{g} \to \text{End}(\mathfrak{g}) \) naturally extends to a homomorphism of associative algebras \( \text{ad} : T(\mathfrak{g}) \to \text{End}(\mathfrak{g}) \), we have that

\[(\phi^2) = (\phi^2)_0 = \frac{1}{24} \text{str ad}(\text{Cas}_\mathfrak{g}). \]

**Theorem 3.4.** For any \( \phi \in \mathcal{V} \), let \( \mathfrak{g} \) be the corresponding quadratic Lie superalgebra. Then the constant

\[\phi^2 = \frac{1}{24} \text{str ad}(\text{Cas}_\mathfrak{g}).\]

4. **The criterion for \((\nu, (\cdot, \cdot)_\mathfrak{g})\) to be of Lie super type**

Let \( \mathfrak{r} \) be a finite dimensional complex Lie superalgebra with a non-degenerate invariant super-symmetric bilinear form \((\cdot, \cdot)_\mathfrak{r}\), let \( \mathfrak{p} \) be a finite dimensional complex super vector space with a non-degenerate super-symmetric bilinear form \((\cdot, \cdot)_\mathfrak{p}\), and let

\[\nu : \mathfrak{r} \to \mathfrak{osp}(\mathfrak{p})\]

be a \((\cdot, \cdot)_\mathfrak{p}\)-invariant representation of \( \mathfrak{r} \) on \( \mathfrak{p} \). Define

\[\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{p},\]

and define a non-degenerate super-symmetric bilinear form \((\cdot, \cdot)_\mathfrak{g}\) on \( \mathfrak{g} \) by

\[(\cdot, \cdot)_\mathfrak{g}|_{\mathfrak{r}} = (\cdot, \cdot)_{\mathfrak{r}}, \quad (\cdot, \cdot)_\mathfrak{g}|_{\mathfrak{p}} = (\cdot, \cdot)_{\mathfrak{p}}, \quad (\mathfrak{r}, \mathfrak{r})_\mathfrak{g} = 0.\]

The goal of this section is to give a necessary and sufficient condition for \((\nu, (\cdot, \cdot)_\mathfrak{g})\) to be of Lie super type. The case when \( \mathfrak{r} = 0 \) has been studied in Section 3; this section is to study the case for \( \mathfrak{r} \neq 0. \)
Let \( r = \dim \mathfrak{r} \), \( p = \dim \mathfrak{p} \), and \( n = r + p \). Take a homogeneous basis \( \{ e_1, \ldots, e_n \} \) of \( \mathfrak{g} \), given by a basis \( \{ x_1, \ldots, x_r \} \) of \( \mathfrak{r} \) followed by a basis \( \{ y_1, \ldots, y_p \} \) of \( \mathfrak{p} \). Since \( \mathfrak{r} \) is \((\cdot,\cdot)\)\(\mathfrak{g}\)-orthogonal to \( \mathfrak{p} \), we have the \((\cdot,\cdot)\)\(\mathfrak{g}\)-dual basis \( \{ e^1, \ldots, e^n \} = \{ x^1, \ldots, x^r, y^1, \ldots, y^p \} \), where \( \{ x^1, \ldots, x^r \} \) is the \((\cdot,\cdot)\)\(\mathfrak{r}\)-dual basis of \( \{ x_1, \ldots, x_r \} \) and \( \{ y^1, \ldots, y^p \} \) is the \((\cdot,\cdot)\)\(\mathfrak{p}\)-dual basis of \( \{ y_1, \ldots, y_p \} \).

If \( (\nu, (\cdot,\cdot)\mathfrak{g}) \) is of Lie super type, by Theorem 3.2, the quadratic Lie superalgebraic structure on \( \mathfrak{g} \) determines a cubic element \( \phi \in \Lambda_2^3 \mathfrak{g} \). The corresponding bracket \([\cdot,\cdot]\phi\) is defined by

\[
[\phi(x), \phi(y)] = 2\epsilon(x)y - \epsilon(y)x + \epsilon(x)\epsilon(y)[x, y] - \epsilon([x, y])\epsilon(x)y + \epsilon([x, y])\epsilon(y)x.
\]

By the identity (4.3) and Lemma 2.5, we have

\[
for any \ z_1, z_2 \in \mathfrak{g}.
\]

The condition (b) in the definition of a Lie super type says that

\[
[\phi(x), \phi(y)] = [x, y] + \nu(x)y - \nu(y)x.
\]

Let \( \phi_\mathfrak{r} \in \Lambda_2^3 \mathfrak{r} \) be the element corresponding to the quadratic Lie superalgebraic structure on \( \mathfrak{r} \). Then

\[
[x_i, x_j] = 2\epsilon(x_i)x_j - \epsilon(x_j)x_i + \epsilon(x_i)\epsilon(x_j)[x_i, x_j] - \epsilon([x_i, x_j])\epsilon(x_i)x_j + \epsilon([x_i, x_j])\epsilon(x_j)x_i.
\]

Define a cubic element \( \phi_\mathfrak{p} \in \Lambda_2^3 \mathfrak{p} \) by

\[
\epsilon(\phi_\mathfrak{p}) = \frac{1}{12} \sum_{1 \leq i,j,k \leq p} (-1)^{|x_i||x_j|+|x_k|} ([x_i, x_j], x_k)x^i \wedge x^j \wedge x^k.
\]

By Lemma 2.6 there exists a unique Lie superalgebra homomorphism

\[
\nu_\mathfrak{r} : \mathfrak{r} \rightarrow \Lambda^2 \mathfrak{p}
\]

such that, for any \( x \in \mathfrak{r} \) and \( y \in \mathfrak{p} \),

\[
[\nu_\mathfrak{r}(x), y] = \nu_\mathfrak{r}(x)y - (-1)^{|\nu_\mathfrak{r}(x)||y|} y\nu_\mathfrak{r}(x) = \nu(x)y = [x, y] \phi.
\]

Note that \( \nu_\mathfrak{r} \) preserves the grading, that is, \( |\nu_\mathfrak{r}(x)| = |x| \) for any \( x \in \mathfrak{r} \). Clearly,

\[
\{ y_i \wedge y_j | 1 \leq i < j \leq p, y_i \wedge y_j \neq 0 \}
\]

is a basis of \( \Lambda^2 \mathfrak{p} \) with the dual basis

\[
\left\{ \frac{1}{(y^i \wedge y^j, y_i \wedge y_j)} y^i \wedge y^j | 1 \leq i \leq j \leq p, y^i \wedge y^j \neq 0 \right\}.
\]

Note that

\[
(y^i \wedge y^j, y_i \wedge y_j) = (-1)^{|y^i||y^j|}
\]

if \( i \neq j \), and

\[
(y^i \wedge y^j, y_i \wedge y_j) = -2 = 2(-1)^{|y^i||y^j|}
\]

if \( i = j \).
for any \( y_i \in p_1 \). Then for any \( u \in \Lambda^2 p \),

\[
(4.7) \quad u = \sum_{1 \leq i \leq j \leq p, \ y_i \wedge y_j \neq 0} \frac{(u, y_i \wedge y_j)}{(y_i \wedge y_j, y_i \wedge y_j)} y^i \wedge y^j = \frac{1}{2} \sum_{1 \leq i, j \leq p} (-1)^{|y_i^i||y_j^j|}(u, y_i \wedge y_j)y^i \wedge y^j.
\]

It follows that

\[
u_\ast(x) = \frac{1}{2} \sum_{1 \leq i, j \leq p} (-1)^{|y_i^i||y_j^j|}(\nu_\ast(x), y_i \wedge y_j)y^i \wedge y^j
\]

\[
= \frac{1}{2} \sum_{1 \leq i, j \leq p} (-1)^{|y_j^j| + |\nu_\ast(x)|}(\nu_\ast(x), y_j)y^i \wedge y^j
\]

\[
= -\frac{1}{4} \sum_{1 \leq i, j \leq p} (-1)^{|y_i^i||y_j^j|}([x, y_i], y_j)y^i \wedge y^j,
\]

since

\[
[x, y_i]^0 = [\nu_\ast(x), y] = -2(-1)^{|\nu_\ast(x)| |y_i|} \nu_\ast(x).
\]

Let

\[
(4.8) \quad \phi_\nu = \sum_{1 \leq i \leq r} \nu_\ast(x_i) \wedge x^i.
\]

By a direct calculation, we have

\[
\phi = \sum_{1 \leq i \leq j \leq k \leq n, \ e_i \wedge e_j \wedge e_k \neq 0} \frac{(\phi, e_i \wedge e_j \wedge e_k)}{(e_i \wedge e_j \wedge e_k, e_i \wedge e_j \wedge e_k)} e^i \wedge e^j \wedge e^k
\]

\[
= -\frac{1}{2} \sum_{1 \leq i, j, k \leq n, \ e_i \wedge e_j \wedge e_k \neq 0} \frac{([e_i, e_j]^0, e_k)}{(e_i \wedge e_j \wedge e_k, e_i \wedge e_j \wedge e_k)} e^i \wedge e^j \wedge e^k
\]

\[
= -\frac{1}{12} \sum_{1 \leq i, j, k \leq r} (-1)^{|x_i||x_j| + |x_k||x_k|}([x_i, x_j], x_k)x^i \wedge x^j \wedge x^k
\]

\[-\frac{1}{4} \sum_{1 \leq i, j, l \leq p} (-1)^{|y_i||y_j| + |y_l||y_l|}([y_i, y_j]^0, y_k)y^i \wedge y^j \wedge y^k
\]

\[-\frac{1}{12} \sum_{1 \leq l, j, k \leq p} (-1)^{|y_l||y_j| + |y_k||y_k|}([y_l, y_j]^0, y_k)y^i \wedge y^j \wedge y^k
\]

\[
= \phi_\nu + \sum_{1 \leq i \leq r} (-1)^{|x_i||x_i|} x^i \wedge \nu_\ast(x_i) + \phi_p
\]

\[
= \phi_\nu + \phi_\nu + \phi_p.
\]
Furthermore, by the identity (2.11), for any homogeneous element \( x \in r \),
\[
([\nu_*(x), \phi_p]_C, y_i \wedge y_j \wedge y_k)
= (-\phi_{p}, [\nu_*(x), y_i \wedge y_j \wedge y_k]_C)
= (-\phi_{p}, [\nu_*(x), y_i \wedge y_j \wedge y_k]_C)
- (-1)^{|y_i||y_j|}([\phi_{p}, y_i \wedge [\nu_*(x), y_j]_C \wedge y_k])
- (-1)^{|y_i||y_j|}([\phi_{p}, y_i \wedge y_j \wedge [\nu_*(x), y_k]_C])
= -([x, y_i]_\phi \wedge y_j \wedge y_k) - (-1)^{|y_i||y_j|}([\phi_{p}, y_i \wedge [x, y_j]_\phi \wedge y_k])
- (-1)^{|y_i||y_j|}([\phi_{p}, y_i \wedge y_j \wedge [x, y_k]_\phi])
\]
\[
= -([x, y_i]_\phi \wedge y_j \wedge y_k) - (-1)^{|y_i||y_j|}([y_i, [x, y_j]_\phi] y_k) - (-1)^{|y_i||y_j|}([y_i, y_j]_\phi, [x, y_k]_\phi)
\]
\[
= -([x, y_i]_\phi \wedge y_j \wedge y_k) - (-1)^{|y_i||y_j|}([y_i, [x, y_j]_\phi] y_k) + ([x, [y_i, y_j]_\phi]_\phi, y_k).
\]
It follows that \([\nu_*(x), \phi_p]_C = 0\) by the super Jacobi identity, that is,
\[
\phi_p \in (\Lambda^3_0(p))^r.
\]

**Remark 4.1.** If \((\nu, (\cdot, \cdot)_g)\) is of Lie super type, \(\phi_\nu\) and \(\phi_p\) are determined completely by the quadratic Lie superalgebraic structure of \(r\) and the \(r\)-module structure on \(p\). In order to give a Lie superalgebraic structure satisfying the conditions of a Lie super type, we only need to determine the cubic element \(\phi_p\).

Conversely, for any \(\phi_p \in (\Lambda^3_0(p))^r\), define the cubic element \(\phi \in \Lambda^3_0\mathfrak{g}\) by
\[
\phi = \phi_\nu + \phi_p,
\]
where \(\phi_\nu\) and \(\phi_p\) are defined by identities (4.4) and (4.8), respectively. Let \([\cdot, \cdot]_\phi^\circ\) be the bracket on \(\mathfrak{g}\) determined by the identity (4.11). By Section 3, we know that \([\cdot, \cdot]_\phi^\circ\) has skew super-symmetry and \((\cdot, \cdot)_g\) is invariant with respect to the bracket \([\cdot, \cdot]_\phi^\circ\). Moreover,
\[
[x_i, x_j]_\phi = 2\nu(x_i)\iota(x_j)\phi = 2\nu(x_i)\iota(x_j)\phi_\nu = [x_i, x_j]
\]
for any \(1 \leq i, j \leq r\) and
\[
[x_i, y_j]_\phi = 2\nu(x_i)\iota(y_j)_\phi = 2\nu(x_i)\iota(y_j)\phi_\nu = 2(-1)^{|x_i||y_j|}\iota(y_j)\nu_*(x_i) = [\nu_*(x_i), y_j]_C = \nu(x_i)y_j
\]
for any \(1 \leq i \leq r\) and \(1 \leq j \leq p\).

Clearly, the super Jacobi identity holds for \(x_i, x_j, y_k\) since \(r\) is a Lie superalgebra and for \(x_i, x_j, y_k\) since \(p\) is a \(r\)-module.

We claim that the super Jacobi identity holds for \(x_i, y_j, y_k\). In fact, let \(P_r : \mathfrak{g} \to r\) and \(P_p : \mathfrak{g} \to p\) be the projections with respect to the decomposition \(\mathfrak{g} = r \oplus p\). Since \([\cdot, \cdot]_\phi^\circ\) has skew super-symmetry and \((\cdot, \cdot)_g\) is invariant with respect to the bracket \([\cdot, \cdot]_\phi^\circ\), we have
\[
([x_i, [y_j, y_k]_\phi^\circ]_\phi - [[x_i, y_j]_\phi^\circ, y_k]_\phi - (-1)^{|x_i||y_j|}[y_j, [x_i, y_k]_\phi^\circ]_\phi, x_i)
= -(-1)^{|x_i||y_j|+|y_k|}([y_j, y_k]_\phi^\circ, [x_i, x_i]_\phi) - ([x_i, y_j]_\phi^\circ, [y_k, x_i]_\phi) - (-1)^{|x_i||y_j|}[y_j, [x_i, y_k]_\phi^\circ, x_i]_\phi
= -(-1)^{|x_i||y_j|+|y_k|}([y_j, y_k]_\phi^\circ, [x_i, x_i]_\phi) - (-1)^{|x_i||y_k|}[x_i, [y_k, x_i]_\phi^\circ]_\phi - ([y_k, x_i]_\phi^\circ, x_i]_\phi
= 0
\]
for any \(1 \leq l \leq r\). It follows that
\[
P_r([x_i, [y_j, y_k]_\phi^\circ]_\phi - [[x_i, y_j]_\phi^\circ, y_k]_\phi - (-1)^{|x_i||y_j|}[y_j, [x_i, y_k]_\phi^\circ]_\phi) = 0.
\]
Since $\phi_p \in (\Lambda^2_\nu(p))^r$, for any $1 \leq l \leq p$, we have
\[
([x_i, [y_j, y_k]]_\phi^\nu - [[x_i, y_j]_\phi^\nu, y_k^\nu] - (-1)^{|x_i||y_j|}[y_j, [x_i, y_k]_\phi^\nu]_\phi^\nu, y_l])
= ([\nu_*(x), \phi_p^\nu]_{C, y_j \wedge y_k \wedge y_l}) = 0,
\]
that is,
\[
(P_p([x_i, [y_j, y_k]]_\phi^\nu - [[x_i, y_j]_\phi^\nu, y_k^\nu] - (-1)^{|x_i||y_j|}[y_j, [x_i, y_k]_\phi^\nu]_\phi^\nu, y_l)) = 0.
\]
Then the claim holds by identities (4.9) and (4.10).

At last, we consider the super Jacobi identity for $y_i, y_j, y_k$. For any $1 \leq l \leq r$, we have
\[
([y_i, [y_j, y_k]]_\phi^\nu - [[y_i, y_j]_\phi^\nu, y_k^\nu] - (-1)^{|y_i||y_j|}[y_j, [y_i, y_k]_\phi^\nu]_\phi^\nu, x_i)
= ([y_i, [y_j, y_k]]_\phi^\nu, x_i^\nu) - ([y_i, y_j]_\phi^\nu, [y_k, x_i]_\phi^\nu) + (-1)^{|y_j||y_k|}([y_i, y_k]_\phi^\nu, [y_j, x_i]_\phi^\nu)
= ([y_i, [y_j, y_k]]_\phi^\nu, x_i^\nu) - ([y_i, y_k]_\phi^\nu, [y_j, x_i]_\phi^\nu) + (-1)^{|y_j||y_k|}[y_j, [y_i, y_k]_\phi^\nu]_\phi^\nu,
\]
and
\[
= 0.
\]
It follows that
\[
P_t([y_i, [y_j, y_k]]_\phi^\nu - [[y_i, y_j]_\phi^\nu, y_k^\nu] - (-1)^{|y_i||y_j|}[y_j, [y_i, y_k]_\phi^\nu]_\phi^\nu, y_l)) = 0.
\]
Thus we only need to consider
\[
P_p([y_i, [y_j, y_k]]_\phi^\nu - [[y_i, y_j]_\phi^\nu, y_k^\nu] - (-1)^{|y_i||y_j|}[y_j, [y_i, y_k]_\phi^\nu]_\phi^\nu).
\]
By Lemma 3.4 we have
\[
([y_i, [y_j, y_k]]_\phi^\nu - [[y_i, y_j]_\phi^\nu, y_k^\nu] - (-1)^{|y_i||y_j|}[y_j, [y_i, y_k]_\phi^\nu]_\phi^\nu) = 2\nu_t(y_i)\nu_t(y_j)\nu_t(y_k)\phi^\nu.
\]
Note that, for any $1 \leq l \leq p$,
\[
(\nu_t(y_i)\nu_t(y_j)\nu_t(y_k)\phi^2, y_l) = (\nu_t(y_i)\nu_t(y_j)\nu_t(y_k)\phi^2, y_l).
\]
By Lemma 2.1 $\phi_\nu$ is independent of the choice of basis of $r$. It follows that
\[
\phi_\nu = \sum_{1 \leq s \leq r} \nu_*(x_i) \wedge x^s = \sum_{1 \leq s \leq r} (-1)^{|x_j||x^s|} \nu_*(x^s) \wedge x_j.
\]
Thus,
\[
\phi_\nu^2 = \left( \sum_{1 \leq s \leq r} \nu_*(x_i) \wedge x^s \right) \left( \sum_{1 \leq s \leq r} (-1)^{|x_j||x^s|} \nu_*(x^s) \wedge x_j \right).
\]
Since $\nu_*(x_i) \in \Lambda^2(p)$, we have $\nu_t(x_i)\nu_*(x_i) = 0$. By Lemma 2.4
\[
\nu_*(x_i)x^l = \alpha_2(\nu_*(x_i)x^l) = (-1)^{|x_i||x^l|} \alpha(x^l)\alpha(\nu_*(x_i)) = -(-1)^{|x_i||x^l|} \alpha(x^l)\nu_*(x_i))
= (-1)^{|x_i||x^l|} \alpha(x^l) \wedge \nu_*(x_i) + \nu_t(x_i) \nu_*(x_i) = -(-1)^{|x_i||x^l|} \alpha(x^l) \wedge \nu_*(x_i))
= -\alpha(\nu_*(x_i)) \wedge \alpha(x^l) = \nu_*(x_i) \wedge x^l,
\]
which implies that
\[
\phi_\nu^2 = \left( \sum_{1 \leq s \leq r} \nu_*(x_i)x^s \right) \left( \sum_{1 \leq s \leq r} (-1)^{|x_j||x^s|} \nu_*(x^s) \wedge x_j \right)
= \left( \sum_{1 \leq s \leq r} \nu_*(x_i) \right) \left( \sum_{1 \leq s \leq r} (-1)^{|x_j||x^s|} \nu_*(x^s) \wedge x_j \right) + \sum_{1 \leq s \leq r} \nu_*(x_i) \nu_*(x^s).
\]
Note that
\[(\iota(y_i)\iota(y_j)\iota(y_k))((\sum_{1 \leq s \leq r} \nu_s(x_s))((\sum_{1 \leq t \leq r} (-1)^{|x_t||x^t|}x^t \wedge \nu_s(x^t) \wedge x_t), y_i)) = 0.\]

It follows that
\[(4.15) \quad (\iota(y_i)\iota(y_j)\iota(y_k)\phi^2, y_l) = (\iota(y_i)\iota(y_j)\iota(y_k)(\sum_{1 \leq s \leq r} \nu_s(x_s)\nu_s(x^s)), y_l).\]

Let $\Lambda^{\text{even}}(p) = \sum_{i=0}^{\infty} \Lambda^{2i}(p)$. Then $\Lambda^{\text{even}}(p)$ is a subalgebra of $\Lambda(p)$ with respect to the Clifford multiplication. We can extend $\nu_\ast : \tau \to \Lambda^2(p)$ to a homomorphism of associative algebras

\[\nu_\ast : T(\tau) \to \Lambda^{\text{even}}(p).\]

By identities (4.13) and (4.15), we have
\[(4.16) \quad (\iota(y_i)\iota(y_j)\iota(y_k)\phi^2, y_l) = (\iota(y_i)\iota(y_j)\iota(y_k)(\nu_\ast(Cas_\ast) + \phi^2), y_l)\]

It is clear that $(\nu_\ast(Cas_\ast))_k = 0$ if $k \notin \{0, 2, 4\}$. By Lemma 2.9,

\[\nu_\ast(Cas_\ast) = \sum_{i=1}^{r} \nu_s(x_i)\nu_s(x^i) = \sum_{i=1}^{r} (-1)^{|x_i||x^i|}\nu_s(x^i)\nu_s(x_i).\]

Then, by Lemma 2.9

\[\alpha(\nu_\ast(Cas_\ast)) = \sum_{i=1}^{r} \alpha(\nu_s(x_i)\nu_s(x^i)) = \sum_{i=1}^{r} (-1)^{|x_i||x^i|}\nu_s(x^i)\nu_s(x_i) = \nu_\ast(Cas_\ast),\]

which implies that

\[\nu_\ast(Cas_\ast) = (\nu_\ast(Cas_\ast))_4 + (\nu_\ast(Cas_\ast))_0.\]

Clearly, $(\phi^2)_k = 0$ if $k \notin \{0, 2, 4, 6\}$. Note that

\[\alpha(\phi^2) = (-1)^{|\phi^2||\phi^2|}\alpha(\phi_\ast)\alpha(\phi_\ast) = \phi^2,\]

which implies that

\[\phi^2 = (\phi^2)_4 + (\phi^2)_0.\]

Hence

\[\nu_\ast(Cas_\ast) + \phi^2 = (\nu_\ast(Cas_\ast) + \phi^2)_4 + (\nu_\ast(Cas_\ast) + \phi^2)_0.\]

Furthermore, by identities (4.12) and (4.14),

\[P_\phi([y_i, [y_j, y_k]]\phi\phi - [[y_i, y_j]\phi, y_k]\phi - (-1)^{|y_i||y_j|}[y_j, [y_i, y_k]]\phi) = 0\]

if and only if

\[(\nu_\ast(Cas_\ast) + \phi^2)_4 = 0,\]

that is, $(\nu_\ast(Cas_\ast) + \phi^2)$ is a constant.

In summary, we have the following theorem.
Then by Theorem 3.2, 3.4 and 4.2, we have

\[ [z_1, z_2]^\phi = 2\iota(z_1)\iota(z_2) \phi, \quad \forall z_1, z_2 \in \mathfrak{g}. \]

For any \( \phi \in \mathcal{V} \), set \( \phi = \phi_t + \phi_\nu + \phi_p \). Let \( \mathfrak{g} \) be the corresponding quadratic Lie superalgebra with the bracket \( [z_1, z_2]^\phi = 2\iota(z_1)\iota(z_2) \phi \). We denote by \( \text{ad}_g \) (resp. \( \text{ad}_r \)) the adjoint representation of \( \mathfrak{g} \) on itself (resp. \( \mathfrak{r} \) on itself) and that extended to \( U(\mathfrak{g}) \) (resp. \( U(\mathfrak{r}) \)).

Recall that \( \phi_t \in \Lambda^3(\mathfrak{r}) \), \( \phi_\nu \in \pi_\Lambda(T^2(\mathfrak{p}) \otimes T(\mathfrak{r})) \) and \( \phi_p \in \Lambda^3(\mathfrak{p}) \). Since \( \mathfrak{r} \) is \( (\cdot, \cdot)^\mathfrak{g} \)-orthogonal to \( \mathfrak{p} \), we have

\[ (\phi^2)_0 = (\phi_t^2)_0 + (\phi_\nu^2)_0 + (\phi_p^2)_0 \]

By the identity (4.14), we have

\[ (\phi_p^2)_0 = (\sum_{1 \leq i \leq r} \nu_\iota(x_i)\nu_\iota(x^i))_0 = (\nu_\iota(\text{Cas}_\mathfrak{r}))_0. \]

Then by Theorem 3.2, 3.4 and 4.2, we have

\[ \nu_\iota(\text{Cas}_\mathfrak{r}) + \phi_p^2 = (\nu_\iota(\text{Cas}_\mathfrak{r}) + \phi_p^2)_0 = (\phi_t^2)_0 + (\phi_p^2)_0 = (\phi^2)_0 - (\phi_t^2)_0 = \frac{1}{24}(\text{str ad}_g(\text{Cas}_\mathfrak{g}) - \text{str ad}_r(\text{Cas}_\mathfrak{r})). \]

**Theorem 4.3.** Assume that \( (\nu, (\cdot, \cdot)^\mathfrak{g}) \) is of Lie super type corresponding to \( \phi_\mathfrak{p} \in \mathcal{V} \), i.e. the bracket of \( \mathfrak{g} \) is defined by \( [z_1, z_2]^\phi = 2\iota(z_1)\iota(z_2) \phi \), where \( \phi = \phi_t + \phi_\nu + \phi_p \). Then the constant

\[ \nu_\iota(\text{Cas}_\mathfrak{r}) + \phi_p^2 = \frac{1}{24}(\text{str ad}_g(\text{Cas}_\mathfrak{g}) - \text{str ad}_r(\text{Cas}_\mathfrak{r})). \]

5. **Dirac operator for quadratic Lie superalgebras**

Let \( \mathfrak{g} \) be a finite dimensional complex quadratic Lie superalgebra with respect to \( (\cdot, \cdot) \), let \( \mathfrak{r} \) be a subalgebra of \( \mathfrak{g} \) such that the restriction of \( (\cdot, \cdot) \) to \( \mathfrak{r} \) is non-degenerate, let \( \mathfrak{g} = \mathfrak{r} \oplus \mathfrak{p} \) be the orthogonal decomposition with respect to \( (\cdot, \cdot) \), and let \( \nu \) be the adjoint representation of \( \mathfrak{r} \) on \( \mathfrak{p} \). Then \( (\nu, (\cdot, \cdot)) \) is of Lie super type.

Analogous to the cubic Dirac operator of quadratic Lie algebra introduced by Kostant in [8], we will define the cubic Dirac operator \( D(\mathfrak{g}, \mathfrak{r}) \) of \( \mathfrak{g} \) corresponding to the above decomposition \( \mathfrak{g} = \mathfrak{r} \oplus \mathfrak{p} \). Denote by \( \xi \) the injection map \( \mathfrak{g} \to U(\mathfrak{g}) \) and its extension \( U(\mathfrak{g}) \to U(\mathfrak{g}) \). Define the cubic Dirac operator \( D(\mathfrak{g}, \mathfrak{r}) \in U(\mathfrak{g}) \otimes C(\mathfrak{p}) \) by

\[ D(\mathfrak{g}, \mathfrak{r}) = \sum_{i=1}^{\mathfrak{p}} \xi(y_i) \otimes y^i + 1 \otimes \phi_\mathfrak{p}, \]

where \( \phi_\mathfrak{p} = -\frac{1}{12} \sum_{1 \leq i, j, k \leq \mathfrak{p}} (-1)^{|y_i||y_j|+|y_i||y_k|} (\{y_i, y_j\}, y_k) y^i \wedge y^j \wedge y^k \). It is clear that \( D(\mathfrak{g}, \mathfrak{r}) \) is independent of the choice of basis. Note that

\[ (\phi_\mathfrak{p}, y_i \wedge y_j \wedge y_k) = (\phi_\mathfrak{p}, y_i \wedge y_j \wedge y_k) = -\frac{1}{2} (\{y_i, y_j\}, y_k) \]
and \(|y_i| = |y'_i|\) for 1 \(\leq i \leq p\). Set \(\Box_1 = \sum_{i=1}^{p} \xi(y_i) \otimes y^i\) and \(\Box_2 = 1 \otimes \varphi_p\). Then
\[
D(\mathfrak{g}, \mathfrak{r}) = \Box_1 + \Box_2.
\]
Recall that \(U(\mathfrak{g})\) and \(C(\mathfrak{r})\) are both superalgebras and the multiplication on \(U(\mathfrak{g}) \otimes C(\mathfrak{r})\) is defined by the identity (22). Since \(y^j y^i + (-1)^{|y_j||y_i|} y^i y^j = 2(y^j, y^i)\), \(y^j y^i = y^i \wedge y^j + (y^j, y^i)\), and \(\xi(y_i) \xi(y_j) - (-1)^{|y_i||y_j|} \xi(y_j) \xi(y_i) = \xi([y_i, y_j])\) we have
\[
(\Box_1)^2 = \sum_{i=1}^{p} \xi(y_i) \otimes y^i \left( \sum_{j=1}^{p} \xi(y_j) \otimes y^j \right) = \sum_{1 \leq i, j \leq p} (-1)^{|y_i||y_j|} \xi(y_i) \xi(y_j) \otimes y^i y^j
\]
\[
= \frac{1}{2} \sum_{1 \leq i, j \leq p} \left( (-1)^{|y_i||y_j|} \xi(y_j) \xi(y_i) \otimes y^i y^j + (-1)^{|y_j||y_i|} \xi(y_j) \xi(y_i) \otimes y^j y^i \right)
\]
\[
= \sum_{1 \leq i, j \leq p} (-1)^{|y_i||y_j|} (y^j, y^i) \xi(y_j) \xi(y_i) \otimes 1 + \frac{1}{2} \sum_{1 \leq i, j \leq p} (-1)^{|y_j||y_i|} \xi([y_i, y_j]) \otimes y^i y^j
\]
\[
+ \frac{1}{2} \sum_{1 \leq i, j \leq p} (-1)^{|y_i||y_j|} (y^i, y^j) \xi([y_i, y_j]) \otimes 1.
\]
Since \(y^j = \sum_{k=1}^{p} (y^k, y^j) y_k\), we have
\[
\sum_{1 \leq i, j \leq p} (y^i, y^j) \xi(y_j) \xi(y_i) \otimes 1 = \sum_{1 \leq i, j, k \leq p} (y^k, y^j) (y^i, y_k) \xi(y_j) \xi(y_i) \otimes 1.
\]
Note that \(|y_j| = |y_k|\) if \((y^k, y^j) \neq 0\). Then \(\sum_{j=1}^{p} (y^k, y^j) \xi(y_j) = (-1)^{|y_k||y_j|} \xi(y^k)\). Hence
\[
\sum_{1 \leq i, j \leq p} (y^i, y^j) \xi(y_j) \xi(y_i) \otimes 1 = \sum_{1 \leq i, k \leq p} (-1)^{|y_k||y_i|} (y^i, y_k) \xi(y^k) \xi(y_i) \otimes 1
\]
\[
= \sum_{i=1}^{p} (-1)^{|y_i||y_i|} \xi(y^i) \xi(y_i) \otimes 1.
\]
By Lemma 2.1, we have
\[
\sum_{1 \leq i, j \leq p} (y^i, y^j) \xi(y_j) \xi(y_i) \otimes 1 = \sum_{i=1}^{p} \xi(y_i) \xi(y^i) \otimes 1.
\]
Since
\[
\sum_{1 \leq i, j \leq p} (-1)^{|y_i||y_j|} (y^i, y^j) \xi([y_i, y_j]) \otimes 1
\]
\[
= \frac{1}{2} \sum_{1 \leq i, j \leq p} \left( (-1)^{|y_i||y_j|} (y^j, y^i) \xi([y_i, y_j]) \otimes 1 + (-1)^{|y_j||y_i|} (y^j, y^i) \xi([y_j, y_i]) \otimes 1 \right)
\]
\[
= \frac{1}{2} \sum_{1 \leq i, j \leq p} (y^j, y^i) \xi([y_i, y_j]) + (-1)^{|y_j||y_i|} ([y_j, y_i]) \otimes 1
\]
\[
= 0,
\]
we have

\[(\Box_1)^2 = \sum_{i=1}^{p} \xi(y_i)\xi(y^i) \otimes 1 + \frac{1}{2} \sum_{1 \leq i, j \leq p} (-1)^{|y_i||y_j|} \xi([y_i, y_j]) \otimes y^i \wedge y^j\]

\[= \sum_{i=1}^{p} \xi(y_i)\xi(y^i) \otimes 1 + \frac{1}{2} \sum_{1 \leq i, j \leq p} (-1)^{|y_i||y_j|} \xi([y_i, y_j]_r) \otimes y^i \wedge y^j\]

\[+ \frac{1}{2} \sum_{1 \leq i, j \leq p} (-1)^{|y_i||y_j|} \xi([y_i, y_j]_p) \otimes y^i \wedge y_j.\]

Denote by \(I, II\) and \(III\) the three summands in the right side of the above equation respectively, that is,

\[(\Box_1)^2 = I + II + III.\]

By identities (2.8), (2.10), (4.6) and (4.7)

\[II = \frac{1}{2} \sum_{1 \leq i, j \leq p} \sum_{k=1}^{r} (-1)^{|y_i||y_j|} (x^k, [y_i, y_j]) \xi(x_k) \otimes y^i \wedge y^j\]

\[= \frac{1}{2} \sum_{1 \leq i, j \leq p} \sum_{k=1}^{r} (-1)^{|y_i||y_j|} ([x^k, y_i], y_j) \xi(x_k) \otimes y^i \wedge y^j\]

\[= - \sum_{1 \leq i, j \leq p} \sum_{k=1}^{r} (-1)^{|y_i||y_j|} (\nu_s(x^k), y_i \wedge y_j) \xi(x_k) \otimes y^i \wedge y^j\]

\[= -2 \sum_{k=1}^{r} \xi(x_k) \otimes \nu_s(x^k).\]

Define a diagonal embedding \(\zeta : \mathfrak{r} \to U(\mathfrak{g}) \otimes C(\mathfrak{p})\) by

\[\zeta(x) = \xi(x) \otimes 1 + 1 \otimes \nu_s(x), \quad \forall x \in \mathfrak{r}.\]

Extending \(\zeta\) to a homomorphism \(\zeta : U(\mathfrak{r}) \to U(\mathfrak{g}) \otimes C(\mathfrak{p})\), by Lemma 2.1 we have

\[\zeta(\text{Cas}_x) = \sum_{i=1}^{r} (\xi(x_i) \otimes 1 + 1 \otimes \nu_s(x_i)) (\xi(x^i) \otimes 1 + 1 \otimes \nu_s(x^i))\]

\[= \sum_{i=1}^{r} (\xi(x_i)\xi(x^i) \otimes 1 + \xi(x_i) \otimes \nu_s(x^i) + (-1)^{|x_i||x^i|} \xi(x^i) \otimes \nu_s(x_i) + 1 \otimes \nu_s(x_i)\nu_s(x^i))\]

\[= \sum_{i=1}^{r} (\xi(x_i)\xi(x^i) \otimes 1 + 2\xi(x_i) \otimes \nu_s(x^i) + 1 \otimes \nu_s(x_i)\nu_s(x^i)).\]
It follows from identities (5.3) and (5.4) that
\[ I + II + \zeta(\text{Cas}_g) = \xi(\text{Cas}_g) \otimes 1 + 1 \otimes \nu_*(\text{Cas}_t). \]

By Lemma 2.10 and the identity (4.7), we have
\[ \square_1 \square_2 + \square_2 \square_1 = \sum_{k=1}^p \xi(y_k) \otimes (y^k \phi_p + \phi_p y^k) = 2 \sum_{k=1}^p \xi(y_k) \otimes \nu(y^k) \phi_p \]
\[ = \sum_{k=1}^p \sum_{1 \leq i,j \leq p} (-1)^{|y_i| |y_j|} (\nu(y^k) \phi_p, y_i \wedge y_j) \xi(y_k) \otimes y^i \wedge y^j \]
\[ = \sum_{1 \leq i,j,k \leq p} (-1)^{|y_i| |y_j|} (\phi_p, y^k \wedge y_i \wedge y_j) \xi(y_k) \otimes y^i \wedge y^j \]
\[ = -\frac{1}{2} \sum_{1 \leq i,j,k \leq p} (-1)^{|y_i| |y_j|} [y^k, [y_i, y_j]] \xi(y_k) \otimes y^i \wedge y^j \]
\[ = -\frac{1}{2} \sum_{1 \leq i,j \leq p} (-1)^{|y_i| |y_j|} \xi([y_i, y_j]) \otimes y^i \wedge y^j \]
\[ = -III. \]

By identities (5.3), (5.5) and (5.6), we have
\[ (D(\mathfrak{g}, \mathfrak{r}))^2 = \xi(\text{Cas}_g) \otimes 1 - \zeta(\text{Cas}_t) + 1 \otimes (\nu_*(\text{Cas}_t) + \phi_p^2). \]

By Theorem 4.3, we have an analogue of the Parthasarathy’s formula.

**Theorem 5.1.** Let \( \mathfrak{g} \) be a finite dimensional complex quadratic Lie superalgebra with respect to \( (\cdot, \cdot) \) and let \( \mathfrak{r} \) be a subalgebra of \( \mathfrak{g} \) such that the restriction of \( (\cdot, \cdot) \) to \( \mathfrak{r} \) is non-degenerate. Define \( D(\mathfrak{g}, \mathfrak{r}) \in U(\mathfrak{g}) \otimes \mathbb{C}(\mathfrak{p}) \) by the identity (5.1). Then
\[ (D(\mathfrak{g}, \mathfrak{r}))^2 = \xi(\text{Cas}_g) \otimes 1 - \zeta(\text{Cas}_t) + \frac{1}{24} (\text{str} \ ad_\mathfrak{g}(\text{Cas}_g) - \text{str} \ ad_\mathfrak{r}(\text{Cas}_t))(1 \otimes 1). \]

**Remark 5.2.** In [8], Kostant proved the identity (5.7) when \( \mathfrak{g} \) is a quadratic Lie algebra and \( \mathfrak{r} \) is a subalgebra of \( \mathfrak{g} \). In fact, for the Lie algebraic case, the formula of \( (D(\mathfrak{g}, \mathfrak{r}))^2 \) in terms of Casimir elements goes back to Parthasarathy. In [12], he obtained the formula under the assumption that \( \mathfrak{g} = \mathfrak{r} \oplus \mathfrak{p} \) is a Cartan decomposition and \( \text{rank}(\mathfrak{g}) = \text{rank}(\mathfrak{r}) \). For this case, \( \phi_p = 0 \). The Dirac operators for Lie superalgebras have been studied by several groups of researchers ([2, 4, 5, 10, 13]). In [2], Huang and Pandžić proved the identity (5.7) for the case \( \mathfrak{r} = \mathfrak{g}_0 \). In [13], Pengpan constructed the cubic Dirac operator for both full Lie superalgebra \( \mathfrak{g} \) and its equal rank embeddings, where \( \mathfrak{g} \) is a basic Lie superalgebra. The author also derived a formula for the square of Dirac operators (see the formula (40) in [13]). In the infinite dimensional Lie superalgebras case, Landweber studied an affine analogue of the cubic Dirac operator ([10]) for loop algebras, which was introduced much earlier by Kac and Todorov in [6] on unitary representations of Neveu-Schwarz and Ramond superalgebras, and was studied further by Kac, Möseneder Frajria and Papi in [4].
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