AN EXPONENTIAL DIOPHANTINE EQUATION RELATED TO ODD PERFECT NUMBERS

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Abstract. We shall show that, for any given primes \( \ell \geq 17 \) and \( p, q \equiv 1 \) \( \pmod{\ell} \), the diophantine equation \( \frac{x^\ell - 1}{x - 1} = p^m q \) has at most four positive integral solutions \((x, m)\) and give its application to odd perfect number problem.

1. Introduction

The purpose of this paper is to bound the number of integral solutions of the diophantine equation

\[ \frac{x^\ell - 1}{x - 1} = p^m q, m \geq 0. \]

This equation arises from our study of odd perfect numbers of a certain form. \( N \) is called perfect if the sum of divisors of \( N \) except \( N \) itself is equal to \( N \). It is one of the oldest problem in mathematics whether or not an odd perfect number exists. Euler has shown that an odd perfect number must be of the form \( N = p^\alpha q_1^{2\beta_1} \cdots q_k^{2\beta_k} \) for distinct odd primes \( p, q_1, \ldots, q_k \) and positive integers \( \alpha, \beta_1, \ldots, \beta_k \) with \( p \equiv \alpha \equiv 1 \) \( \pmod{4} \).

However, we do not know a proof of the nonexistence of odd perfect numbers even of the special form \( N = p^\alpha (q_1 q_2 \cdots q_k)^{2\beta} \), although [15] conjectures that there exists no such one. Gathering various results such as [4], [8], [9], [10], [14], [15] and [18], we know that \( \beta \geq 9, \beta \not\equiv 1 \pmod{3}, \beta \not\equiv 2 \pmod{5} \) and \( \beta \) cannot take some other values such as 11, 14, 18, 24.

We have shown that, if \( N = p^\alpha (q_1 q_2 \cdots q_k)^{2\beta} \) is an odd perfect number, then \( k \leq 4\beta^2 + 2\beta + 2 \) in [19]. Recently, we have improved this upper bound by \( 2\beta^2 + 8\beta + 3 \) in [21], where the coefficient 8 of \( \beta \) can be replaced by 7 if \( 2\beta + 1 \) is not a prime or \( \beta \geq 29 \). Since it is known that \( N < 2^{4k+1} \) from [17], we have

\[ N < 2^{4\beta^2 + 8\beta + 4}. \]

The key point for this result is the diophantine lemma that, if \( \ell, p, q \) are given primes such that \( \ell \geq 19 \) and \( p \equiv q \equiv 1 \pmod{\ell} \), then [1] has at most six integral

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solutions \((x, m)\) such that \(x\) is a prime below \(2^{4^2}\) if \(\ell\) is a prime \(\geq 59\) and at most five such solutions if \(\ell\) is a prime \(\geq 59\) (we note that, by Theorems 94 and 95 in Nagell [10], any prime factor of \((x^\ell - 1)/(x - 1)\) with \(\ell\) prime must be \(\equiv 1\) (mod \(\ell\)) or equal to \(\ell\)). Combining this result with an older upper bound in [19], we obtain the above upper bound for \(N\).

Now we return to the equation (1), which is a special type of Thue-Mahler equations. Evertse gave an explicit upper bound for the numbers of solutions of such equations. Theorem 3 of [6] gives that a slightly generalized equation \((x^\ell - y^\ell)/(x - y) = p^m q^n\) has at most \(7(\ell - 1)^3\) integral solutions for \(\ell \geq 4\). In this paper, we would like to obtain a more strong upper bound for the numbers of solutions of (1).

**Theorem 1.1.** If \(\ell, p, q\) are given primes such that \(\ell \geq 19\) and \(p \equiv q \equiv 1\) (mod \(\ell\)), then (1) has at most four positive integral solutions \((x, m)\). Moreover, if (1) has five integral solutions \((x_i, m_i)\) with \(m_5 > m_4 > \cdots > m_1 \geq 0\), then \(m_1 = 0\) and \(x_2 = x_1^r\) for some integer \(r \geq 1\).

Combining this result with an argument in [21], we obtain the following new upper bound for odd perfect numbers of a special form.

**Corollary 1.2.** If \(N = p^e(q_1 q_2 \cdots q_k)^{2\beta}\) is an odd perfect number with \(p, q_1, q_2,\ldots, q_k\) distinct primes and \(p \equiv e \equiv 1\) (mod 4), then \(k \leq 2\beta^2 + 6\beta + 3\) and \(N < 2^{42\beta^2 + 6\beta + 4}\).

Our method is similar to the approach used in [21]. In this paper, we use upper bounds for sizes of solutions of (1) derived from a Baker-type estimate for linear forms of logarithms by Matveev [13], which may be interesting itself, while [21] used an older upper bound for odd perfect numbers of the form given above. We note that Padé approximations using hypergeometric functions given by Beukers [2, 3] does not work in our situation since our situation will give much weaker approximation to \(\sqrt{D}\), although Beukers’ gap argument is still useful (see Lemma 2.4 below).

In the next section, we introduce some preliminary results from [21] and Matveev’s lower bounds for linear forms of logarithms. In Section 3, using Matveev’s lower bounds, upper bounds for the sizes of solutions of (1) is given. In Section 4, using these results, we prove Theorem 1.1. For large \(\ell\), this can be done combining results in Sections 2 and 3 with a general estimates for class numbers and regulators of quadratic fields. For small \(\ell\), we settle the case \(x_1\) is large and then check the remaining \(x_1\)’s.

A more generalized equation of (1) is

\[
(2) \quad \frac{x^m - 1}{x - 1} = y^n z^l, \quad x \geq 2, y \geq 2, m \geq 3, n \geq 2.
\]

Assuming the abc-conjecture, the author [20] proved that any integral solution of (2) with \(\ell \geq 3, m \geq 1, n \geq 2, 1 \leq y < z\) and \(x^\ell\) sufficiently large must satisfy \((\ell, m, n) = (4, 1, 2), (3, 1, 3)\) or \((\ell, n) = (3, 2)\).
2. A preliminary lemmas

In this section, we shall introduce some notations and lemmas.

We begin by introducing a well-known result concerning prime factors of values of the \( n \)-th cyclotomic polynomial, which we denote by \( \Phi_n(X) \). This result has been proved by Bang [1] and rediscovered by many authors such as Zsigmondy [22], Dickson [5] and Kanold [10, 11].

**Lemma 2.1.** If \( a \) is an integer greater than 1, then \( \Phi_n(a) \) has a prime factor which does not divide \( a^m - 1 \) for any \( m < n \), unless \( (a, n) = (2, 1), (2, 6) \) or \( n = 2 \) and \( a + 1 \) is a power of 2.

In order to introduce further results on values of cyclotomic polynomials, we need some notations and results from the arithmetic of a quadratic field. Let \( \ell \geq 17 \) be a prime and \( D = \left( -1 \right) \frac{\ell + 1}{6} \). Let \( K \) and \( \mathcal{O} \) denote \( \mathbb{Q}(\sqrt{D}) \) and its ring of integers \( \mathbb{Z}[\frac{1 + \sqrt{D}}{2}] \) respectively. We use the overline symbol to express the conjugate in \( K \). In the case \( D > 0 \), \( \epsilon \) and \( R = \log \epsilon \) shall denote the fundamental unit and the regulator in \( K \) respectively. In the case \( D < -4 \), we set \( \epsilon = -1 \) and \( R = \pi i \). We note that neither \( D = -3 \) nor \( D = -4 \) occurs since we have assumed that \( \ell \geq 17 \).

Moreover, we define the absolute logarithmic height \( h(\alpha) \) of an algebraic number \( \alpha \) in \( K \). For an algebraic number \( \alpha \) in \( K \) and a prime ideal \( p \) over \( K \) such that \( \alpha = \left( \zeta_1/\zeta_2 \right) \xi \) with \( \xi \in \mathcal{O} \) and \( \zeta_1, \zeta_2 \) in \( \mathcal{O} \), we define the absolute value \( |\alpha|_p \) by

\[
|\alpha|_p = Np^{-k}
\]

as usual, where \( Np \) denotes the norm of \( p \), i.e., the rational prime lying over \( p \). Now the absolute logarithmic height \( h(\alpha) \) is defined by

\[
h(\alpha) = \frac{1}{2} \left( \log^+ |\alpha| + \log^+ |\bar{\alpha}| + \sum_p \log^+ |\alpha|_p \right),
\]

where \( \log^+ t = \max\{0, \log t\} \) and \( p \) in the sum runs over all prime ideals over \( K \).

The following three lemmas on the value of the cyclotomic polynomial \( \Phi_\ell(x) \) are quoted from [21], except the latter part of Lemma 2.3.

**Lemma 2.2.** If \( x \) is an integer \( > 3^{(\ell+1)/6} \), then \( \Phi_\ell(x) \) can be written in the form \( X^2 - DY^2 \) for some coprime integers \( X \) and \( Y \) with \( 0.3791/x < \left| Y/(X - Y\sqrt{D}) \right| < 0.6296/x \). Moreover, if \( p, q \) are primes \( \equiv 1 \pmod{\ell} \) and \( \Phi_\ell(x) = p^m q \) for some integer \( m \), then

\[
\left[ \frac{X + Y\sqrt{D}}{X - Y\sqrt{D}} \right] = \left( \frac{p}{p} \right)^{\pm m} \left( \frac{q}{q} \right)^{\pm 1},
\]

where \( [p] = pp \) and \( [q] = qq \) are prime ideal factorizations in \( \mathcal{O} \).

**Lemma 2.3.** Assume that \( \ell \) is a prime \( \geq 17 \). If \( x_2 > x_1 > 0 \) are two multiplicatively independent integers and \( \Phi_\ell(x_1) = p^{m_1} q_j \) and \( \Phi_\ell(x_2) = p^{m_2} q_j \), then
Let \( x_2 > x_1 > 0 \) are multiplicatively dependent integers and \( \Phi_{\ell}(x_i) = p^{m_i}q \) for \( i = 1, 2 \), then \( m_1 = 0 \) and \( x_2 = x_1^{\ell} \) for some prime \( r \).

**Lemma 2.4.** If \( \Phi_{\ell}(x_i) = p^{m_i}q_j \) for three integers \( x_3 > x_2 > x_1 > 0 \) with \( x_2 > x_1^{1/6+1/6} \), then \( m_3 > 0.397 |R|x_1 \).

**Proof of lemmas.** Lemma 2.2, the former statement of Lemma 2.3 and Lemma 2.4 are Lemmas 2.3, 4.1 and 4.2 of [21] respectively. Hence, what we should prove here is only the latter statement of Lemma 2.3.

The assumption implies that \( x_1 = y_1 \) and \( x_2 = y_2 \) for some positive integers \( y, r_1, r_2 \) with \( r_2 > r_1 \). Assume that \( r_1 > 1 \). Then, for each \( i, p^{m_i}q = \Phi_{\ell}(x_i) \) must be divisible by \( \Phi_{r_i\ell}(y)\Phi_{\ell}(y) \). Since we have assumed that \( \ell \geq 17 \), Lemma 2.1 yields that each of \( \Phi_{\ell}(y), \Phi_{r_1\ell}(y), \Phi_{r_2\ell}(y) \) must have a primitive prime factor. Hence, the product \( \Phi_{\ell}(y)\Phi_{r_1\ell}(y)\Phi_{r_2\ell}(y) \) must have at least three distinct prime factors, which contradicts to the assumption.

Thus we must have \( r_1 = 1 \) and \( x_2 = x_1^{\ell} \). We see that \( p^{m_2}q = (x_1^{r_\ell} - 1)/(x_1^{r_\ell} - 1) = \prod_{d | \ell} \Phi_{d\ell}(x_1) \), while each \( \Phi_{d\ell}(x_1) \) has a primitive prime factor. Hence, \( r \) must be prime and, since \( \Phi_{\ell}(x_1) \) must be divisible by \( q \), we conclude that \( \Phi_{r\ell}(x_1) = p^{m_2} \) and \( \Phi_{\ell}(x_1) = q \), proving the latter statement of Lemma 2.3. \( \square \)

In order to obtain an upper bound for the size of solutions, we use an lower bound for linear forms of logarithms due to Matveev [13] Theorem 2.2.

**Lemma 2.5.** Let \( \alpha_1, \alpha_2, \ldots, \alpha_n \) be algebraic integers in \( \mathcal{O} \) which are multiplicatively independent and \( b_1, b_2, \ldots, b_n \) be arbitrary integers. Let \( A(\alpha) = \max\{2h(\alpha), |\log \alpha|\} \) and \( A_j = A(\alpha_j) \). Moreover, we put \( \kappa = 1 \) if \( D > 0 \) and \( \kappa = 2 \) if \( D < 0 \).

Put

\[
B = \max\{1, |b_1|A_1/A_n, |b_2|A_2/A_n, \ldots, |b_n|\},
\]

\[
\Omega = A_1A_2\ldots A_n,
\]

\[
C(n) = \frac{16}{n!\kappa}e^n(2n + 1 + 2\kappa)(n + 2)(4(n + 1))^{n+1}\left(\frac{1}{2\kappa n}\right)^{\kappa} (4.4n + 5.5 \log n + 7)
\]

and

\[
\Lambda = b_1 \log a_1 + \ldots + b_n \log a_n.
\]

Then we have \( \Lambda = 0 \) or

\[
\log |\Lambda| > -C(n)(1 + \log 3 - \log 2 + \log B) \max\left\{1, \frac{n}{\kappa}\right\} \Omega.
\]

3. Upper bounds for the sizes of solutions

In this section, we shall give upper bounds for the sizes of solutions of (1), which itself may be of interest. As in the previous sections, for a prime \( \ell \geq 17 \), we let \( D = (-1)\ell \), \( \kappa \) and \( \mathcal{O} \) denote the quadratic field \( \mathbb{Q}(\sqrt{\ell}) \) and its ring of
integers $\mathbb{Z}[(1 + \sqrt{D})/2]$ respectively and $h$ be the class number of $K$. In the case $D > 0$, $\epsilon$ and $R = \log \epsilon$ shall denote the fundamental unit and the regulator in $K$. In the case $D < -4$, we set $\epsilon = -1$ and $R = \pi i$.

We let $p, q$ be primes $\equiv 1 \pmod{\ell}$. Then we can factor $[p] = \overline{p}p$ and $[q] = \overline{q}q$ in $O$ and we see that $p^h = [\pi]$ and $q^h = [\eta]$ for some $\pi, \eta \in O$. Moreover, we can take such $\pi, \eta$ in $O$ such that $\left[ \pi \right] = p^h/2\epsilon^{-1/2}, q^h/2\epsilon^{-1/2} \leq |\pi| \leq q^h/2\epsilon^{-1/2}$ if $D > 0$ and $|\arg \pi|, |\arg \eta| < \pi/4$.

**Theorem 3.1.** Assume that $\Phi_\ell(x) = p^m q$. Then we have the following upper bounds for $m$:

i) If $h \log q > h \log p \geq R$, then

$$m < 4.56C(3) \ell h^2 R (\log q)(\log(8C(3)\ell^2 R) + \log \log p).$$

ii) If $h \log q \geq R \geq h \log p$, then

$$m < 4.56C(3) \ell \frac{h R^2}{2} (\log q) \log \left( \frac{8C(3)\ell R^3}{2\ell} \right).$$

iii) If $h \log p > h \log q \geq R$, then

$$m < 4.56C(3) \ell h R^2 \log(4C(3)\ell R^2).$$

iv) If $h \log p \geq R \geq h \log q$, then

$$m < 4.56C(3) \ell h R^2 \log(4C(3)\ell R^2).$$

v) If $R \geq h \log \max\{p, q\}$, then

$$m < 4.56C(3) \ell R^3 \log(8C(3)\ell R^3) \log \ell.$$ 

**Proof.** If $\Phi_\ell(x) = p^m q$, then Lemma 2.2 yields that there exist two integers $X, Y$ such that

$$\left[ \frac{X + Y\sqrt{D}}{X - Y\sqrt{D}} \right] = \left( \frac{\overline{p}p}{p} \right)^{\pm m} \left( \frac{\overline{q}q}{q} \right)^{\pm 1},$$

with $0 < \left| Y/(X - Y\sqrt{D}) \right| < 0.6296/x$. Taking the $h$-th powers, we have

$$\left\langle \frac{X + Y\sqrt{D}}{X - Y\sqrt{D}} \right\rangle^h = e^u \left( \frac{\overline{\pi}}{\pi} \right)^{\pm m} \left( \frac{\overline{\eta}}{\eta} \right)^{\pm 1} \neq 1$$

for some integer $u$. Now let

$$\Lambda = u \log \epsilon \pm m \log \left( \frac{\overline{\pi}}{\pi} \right) \pm \left( \frac{\overline{\eta}}{\eta} \right) = h \log \left( \frac{X + Y\sqrt{D}}{X - Y\sqrt{D}} \right).$$

Then (13) immediately gives that

$$0 < |\Lambda| < \frac{2hY\sqrt{D}}{X - Y\sqrt{D}} < \frac{1.2588h}{x}. $$
Before applying Lemma 2.5, we can easily see that $h \log p \leq A(\bar{\pi}/\pi) \leq h \log p + R$, $h \log q \leq A(\bar{\eta}/\eta) \leq h \log q + R$ and $A(\epsilon) \leq |R|$.

We begin by treating the first case $h \log q > h \log p > |R|$. In particular, we have $q > p$ and therefore

$$
\frac{uA(\epsilon)}{A(\bar{\eta}/\eta)} = \frac{|u \log \epsilon|}{2A(\bar{\eta}/\eta)} \leq \frac{m |\log(\bar{\pi}/\pi)| + |\log(\bar{\eta}/\eta)| + |\Lambda|}{h \log q} < \frac{(m + 1)R + |\Lambda|}{\log q} < \frac{2mR}{\log q}
$$

and

$$
\frac{mA(\bar{\pi}/\pi)}{A(\bar{\eta}/\eta)} = \frac{m |\log p + |\log(\bar{\pi}/\pi)||}{\log q + |\bar{\eta}/\eta|} \leq \frac{m(\log p + R)}{\log q}.
$$

Since $h \log p > |R|$, we see that $A(\bar{\pi}/\pi) < h \log p + R < 2h \log p$, $A(\bar{\eta}/\eta) < h \log q + R < 2h \log q$ and $B \leq 2m \log p / \log q$. Hence, Matveev’s theorem gives

$$
\log x - \log(1.2588h) < -\log |\Lambda| < C(3)(2h)^2 \log \left(\frac{2m \log p}{\log q}\right) R(\log p)(\log q)
$$

and therefore

$$
\frac{m \log p}{\log q} < \ell \log x < \ell \left(\frac{\log(1.2588h)}{\log q} + 4C(3)h^2 R \log \left(\frac{2m \log p}{\log q}\right) (\log p)\right).
$$

Taking it into account that $C(3) > 10^{10}$, we may assume that $(2m \log p) / \log q > 10^{10}$. Now, using $h < \ell^{1/2} \log(4\ell)$ from p. 199 in [7], we obtain

$$
\frac{2m \log p}{\log q} < 4(2C(3) + 1)\ell h^2 R \log \left(\frac{2m \log p}{\log q}\right) (\log p)
$$

$$
=: U \log \left(\frac{2m \log p}{\log q}\right).
$$

Since $2C(3) + 1 > 3.6 \times 10^{10}$, we have

$$
\frac{m \log p}{\log q} < 0.569U \log U
$$

$$
< 4.56C(3)\ell h^2 R(\log p)(\log(\ell h^2 R) + \log \log p + \log(8C(3)))
$$

proving i).

Nextly, if $h \log q > |R| > h \log p$, then $A(\bar{\pi}/\pi) < 2R$, $A(\bar{\eta}/\eta) < 2h \log q$ and $B \leq 2mR/h \log q$. Moreover, (16) and (17) hold as in the previous case. Hence, an argument similar to above yields that

$$
\frac{m \log p}{\log q} < \ell \left(\frac{\log(1.2588h)}{\log q} + 4C(3)h R^2 \log \left(\frac{2m R}{h \log q}\right)\right)
$$

(22)
Table 1. Estimates when $\ell \leq 41$ and $x_1$ is large

| $\ell$ | $h$ | $R$ | $x_1 \geq$ | $x_2 >$ | $x_3 >$ |
|--------|-----|-----|-------------|---------|---------|
| 17     | 1   | $\log(4 + \sqrt{17})$ | 63 | $x_1^4$ | $\max\{q^{9/17}, 63^{1/17}\}$ |
| 19     | 1   | $\pi i$ | 68 | $x_1^3$ | $\max\{q^{9/19}, 68^{1/19}\}$ |
| 23     | 3   | $\pi i$ | 13 | $x_1^4$ | $\max\{q^{14/23}, 13^{23/14}\}$ |
| 29     | 1   | $\log((5 + \sqrt{29})/2)$ | 5 | $x_1^5$ | $\max\{q^{25/29}, 6^{29/25}\}$ |
| 31     | 3   | $\pi i$ | 5 | $x_1^6$ | $\max\{q^{25/31}, 5^{31/25}\}$ |
| 37     | 1   | $\log(6 + \sqrt{37})$ | 3 | $x_1^7$ | $\max\{q^{36/37}, 3^{36/37}\}$ |
| 41     | 1   | $\log(32 + 5\sqrt{41})$ | 3 | $x_1^8$ | $\max\{q^{49/41}, 3^{49/41}\}$ |

and, observing that $p > 2\ell$,

\begin{equation}
\frac{mR}{h \log q} < \ell \left( \frac{R \log(1.2588h)}{h(\log(2\ell))(\log q)} + 4C(3) \frac{R^3}{\log(2\ell)} \log \left( \frac{2mR}{h \log q} \right) \right).
\end{equation}

Proceeding as above, we obtain

\begin{equation}
\frac{mR}{h \log q} < 4.56C(3) \frac{\ell}{\log(2\ell)} R^3 \left( \frac{8C(3)\ell R^3}{2\ell} \right),
\end{equation}

which proves ii).

In the remaining cases, similar arguments give iii), iv) and v).

4. PROOF OF THE MAIN THEOREM

In this section, we shall prove the main theorem.

Assume that $\Phi_\ell(x_i) = p^m q$ has five solutions $0 < m_1 < m_2 < m_3 < m_4 < m_5$.

It is clear that $x_1 \geq \max\{q^{1/\ell}, 2\}$. Since we have assumed that $m_1 > 0$, Lemma 2.3 yields that $x_3 \geq \max\{q, 2\ell\}^{1/(\ell+1)/6}/\ell$. Now it follows from Lemma 2.4 that

\begin{equation}
m_5 > 0.397\pi x_3 > 0.397\pi \max\{q^{(\ell+1)/6}/\ell, 2^{(\ell+1)/6}/\ell\} := M.
\end{equation}

We begin by the case $\ell \geq 47$. With the aid of the upper bound $|R| < \ell^{1/2} \log(4\ell)$ from p. 199 of [7], Theorem 3.1 implies that $m_5 < M$, which contradicts to (25). Hence, if $\ell \geq 47$, then $\Phi_\ell(x) = p^m q$ with $m > 0$ can have at most four solutions.

Next, assume that $\ell = 43$. We must have $x_1 \geq 3$ since $2^{43} - 1 = 431 \times 9719 \times 2099863$ has three distinct prime factors. Thus we must have $m_5 > 0.397\pi \max\{q^{49/43}, 3^{49}\}$, which exceeds the upper bounds given in Theorem 3.1 with $h = 1, R = \pi i$. Indeed, Theorem 3.1 would yield that, if $q < 3^{43}$, then $m_5 < 4.7 \times 10^{16} < 0.397\pi \times 3^{49} < m_5$ and, if $q > 3^{43}$, then $m_5 < 2.8 \times 10^{15} \log q (\log \log q + 32) < 0.397\pi q^{49/43} < m_5$. In both cases, we are led to a contradiction. Hence, $\Phi_\ell(x) = p^m q$ with $m > 0$ can never have five solutions.
If $\ell \leq 41$ and $x_1$ is at least the corresponding value given in Table 4, then $x_2$ and $x_3$ exceeds the value given in this table. Now we see that $m_5 > 0.397\pi x_3$ exceeds our upper bound $M$, which leads to contradiction.

We have examined the remaining cases. Then $x_1$ must be one of the values given in Table 4 and $p, q$ must be in the range given in this table. Hence, in any case, Theorem 3.1 gives $m < 1.3 \times 10^{17}$. But we have confirmed that $x_2 > p^4 > 10^6$ for these cases. Hence, we must have $x_3 > x_2 > 10^{24}$ and $m_5 > x_3 > 10^{24}$ for all cases given in Table 4, which is a contradiction again.

For example, in the case $\ell = 23$ (in this case, we have $h = 3$ and $R = \pi i$), if $x_1 \geq 13$, then we must have $m_5 > 0.397\pi \max\{q^{16/23}, 13^{16}\}$, which exceeds the upper bounds given in Theorem 3.1.

If $x_1 < 13$, then we must have $x_1 = 2, 3, 5$; $(10^{23} - 1)/9$ is prime and $(x^{23} - 1)/(x - 1)$ with $x = 4, 6, 7, 8, 9, 11$ or 12 has more than two distinct prime factors.

If $x_1 = 2, 3$ or 5, then $p, q \leq 332207361361$ and $m < 1.3 \times 10^{17}$. But, in any case, we have confirmed that $x_2 > p^4 > 10^6$. Hence, we must have $x_3 > x_2 > 10^{24}$ and $m_5 > x_3 > 10^{24}$, which is a contradiction.

Thus, we have proved that $\Phi_\ell(x_i) = p^m q$ can never have five solutions $0 < m_1 < m_2 < m_3 < m_4 < m_5$. Combining the latter part of Lemma 2.3, we have Theorem 1.1.

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