On the algebraic Gordian distance

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Abstract

To find restrictions that two $S$-equivalence classes should bear when their algebraic Gordian distance is one, we construct the Blanchfield pairings of two Seifert matrices mutually convertible by an algebraic unknotting operation. We improve a theorem of Kawauchi. Our results show that two Alexander polynomials cannot be realized by a pair of matrices with Gordian distance one if a corresponding quadratic equation does not have an integer solution. We also give examples of how our results help in calculating the Gordian distances, algebraic Gordian distances and polynomial distances.

1 Introduction

A knot is defined to be an oriented circle embedded in the three sphere $S^3$. A crossing change on a knot is called an unknotting operation. The unknotting operation is originally a geometrical operation. In this paper, we consider an analogous operation. The object of this operation is a Seifert matrix, which is a square unimodular matrix, say $V$, satisfying $\det(V - V^T) = 1$.

A Seifert matrix $W$ is said to be congruent to $V$ if $W = PVP^T$ for a unimodular matrix $P$. A Seifert matrix $W$ is called an enlargement of $V$ if

$$W = \begin{pmatrix} 0 & 0 & 0 \\ 1 & x & M \\ 0 & N^T & V \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & x & M \\ 0 & N^T & V \end{pmatrix},$$

where $M$ and $N$ are row vectors. Then $V$ is a reduction of $W$. The $S$-equivalence is an equivalence relation generated by congruences, enlargements and reductions. The set of $S$-equivalence classes consists of all Seifert
matrices modulo $S$-equivalence \cite{19,20}. Let $[V]$ denote the $S$-equivalence class of a Seifert matrix $V$, and $[K]$ denote the $S$-equivalence class of a knot $K$.

Motivated by the unknotting operation, the *algebraic unknotting operation* assigns a Seifert matrix $W$ to

$$
\begin{pmatrix}
\varepsilon & 0 & 0 \\
1 & x & M \\
0 & N^T & W
\end{pmatrix}
$$

for $\varepsilon = \pm 1$ and $x \in \mathbb{Z}$, where $M$ and $N$ are row vectors \cite{14}.

There is a natural connection between the algebraic unknotting operation and the unknotting operation. Now we focus on the local part of a knot where the crossing change takes place. The unknotting operation can be seen as adding a twist to a knot, turning Figure 1a into Figure 1b or 1c. The twist may fall into two types, corresponding to two types of the algebraic unknotting operations. To distinguish them, set $\varepsilon = 1$ for Figure 1b and $\varepsilon = -1$ for Figure 1c. We call the corresponding operation an $\varepsilon$-unknotting operation.

It is easy to see that Figure 1a and Figure 1d are of the same knot and the latter one bounds a Seifert surface shown in Figure 1e. Then we add a twist to Figure 1e, so that Figure 1f corresponds to Figure 1b for $\varepsilon = 1$, and Figure 1g corresponds to Figure 1c for $\varepsilon = -1$. By choosing the direction of $\alpha$ such that $lk(\beta, \alpha^+) = 1$, we have $lk(\alpha, \alpha^+) = \varepsilon$ and $lk(\beta, \beta^+) = x$. The Seifert matrix of Figure 1f coincides with the result of the algebraic unknotting operation.

Figure 1: Unknotting operation

The Gordian distance \cite{13} between $K$ and $K'$, denoted by $d_G(K, K')$, is
the minimum number of crossing changes needed to turn $K$ into $K'$. The *unknotting number* of $K$, denoted by $u(K)$, is defined by $u(K) = d_G(K, O)$, where $O$ is the trivial knot. Let $\mathcal{V}$ and $\mathcal{V}'$ be two $S$-equivalence classes. For any two representative matrices for $\mathcal{V}$ and $\mathcal{V}'$, there exists a sequence of algebraic unknotting operations and $S$-equivalences transforming one matrix to the other one. The *algebraic Gordian distance* between $\mathcal{V}$ and $\mathcal{V}'$, denoted by $d^a_G(\mathcal{V}, \mathcal{V}')$, is the minimum number of algebraic unknotting operations in such a sequence \[4,14\].

**Remark.** Note that if a Seifert matrix $V$ can be turned into $V'$ by an algebraic unknotting operation, then there exist knots $K$ and $K'$ corresponding to $V$ and $V'$, respectively, such that $K$ and $K'$ are mutually convertible by an unknotting operation. Since Seifert matrices of the same knot are all $S$-equivalent, $V'$ can be turned into a Seifert matrix that is $S$-equivalent to $V$. Consequently, $d^a_G$ is a well-defined distance function and the set of $S$-equivalence classes equipped with $d^a_G$ is a metric space.

It is obvious that $d_G(K_1, K_2) \geq d^a_G([K_1], [K_2])$. The *algebraic unknotting number* $u^a(V)$ is defined to be $d^a_G(V, O)$, where $O$ is the $S$-equivalence class of the $0 \times 0$ matrix \[14\].

The *Alexander polynomial* of a Seifert matrix $V$, denoted by $\Delta_V$, can be calculated by $\Delta_V = |t^{\frac{1}{2}}V - t^{-\frac{1}{2}}V^T|$. The Alexander polynomial is a knot invariant, which means that any two Seifert matrices of a given knot have the same Alexander polynomial. If $V$ is a Seifert matrix of $K$, we write $\Delta_K = \Delta_V$. Saeki proved $u_a([K]) = \min_{K_0} d_G(K, K_0)$, where $K_0$ is a knot with $\Delta_{K_0} = 1$; see \[18\]. We write $u_a(K)$ instead of $u_a([K])$ for convenience.

Analogously, the *Alexander polynomial distance* between two Alexander polynomials $\Delta$ and $\Delta'$, denoted by $\rho(\Delta, \Delta')$, is defined by $\rho(\Delta, \Delta') = \min_{K_\Delta, K_{\Delta'}} d_G(K_\Delta, K_{\Delta'})$, where $K_\Delta$ and $K_{\Delta'}$ are knots with Alexander polynomials $\Delta$ and $\Delta'$, respectively. Remark that $1 \leq \rho(\Delta, \Delta') \leq 2$ for any distinct $\Delta$ and $\Delta'$; see \[9\].

Kawauchi called the following question Jong’s Problem \[9, p.954\] as it is mentioned in Jong’s papers \[6–8\].

**Question.** Determine two Alexander polynomials $\Delta$ and $\Delta'$ such that $\rho(\Delta, \Delta') = 2$.

Equivalently, this question asks when two Alexander polynomials cannot be realized by a pair of knots with Gordian distance one. Kawauchi gave a criterion for the Alexander polynomials of degree two \[9\]. In Section 3, we give some criteria in Corollary 3.3 and Corollary 3.4 for two Alexander polynomials with only one Alexander polynomial of degree two.
Note that a question given by Nakanishi asks if \( \rho(\Delta_3, \Delta_4) = 2 \); see [16, p.334]. It is answered positively by Kawauchi [9]. Our result gives another method to answer it.

The study of the unknotting number and the Gordian distance are often related to pairing relations of covering spaces. Let \( V \) be a Seifert matrix. The **Alexander module**, denoted by \( A_V \), is defined by

\[
A_V = \Lambda^{2n}/(tV - V^T)\Lambda^{2n},
\]

where \( \Lambda \) is the Laurent polynomial ring \( \mathbb{Z}[t^\pm 1] \) and \( 2n \) is the size of \( V \). If \( V \) is a Seifert matrix of \( K \), we have \( A_V \cong H_1(\tilde{X}(K); \mathbb{Z}) \), where \( \tilde{X}(K) \) is the infinite cyclic cover of the complement of \( K \). If two Seifert matrices, say \( V \) and \( V' \), belong to the same \( S \)-equivalence class \( V \), then their Alexander modules are isomorphic. Denote the Alexander module of \( V \) by \( A_V \). The **Blanchfield pairing** of \( V \) is a map \( \beta : A_V \times A_V \to \mathbb{Q}(\Lambda)/\Lambda \), which is a sesquilinear form, meaning \( \beta(ax, by) = a\bar{b}\beta(x, y) \), where \( \bar{b} = b|_{t=t-1} \); see [1]. The Blanchfield pairing of \( V \) is given by the matrix \((t-1)(tV - V^T)^{-1}\); see [20]. Note that two matrices have the same Blanchfield pairing structure, if and only if they are \( S \)-equivalent [20].

There are a lot of papers on how to calculate the Gordian distance for two given knots. A large table of the Gordian distances is given by Moon [12]. However, the algebraic Gordian distance of knots is rarely studied. We are interested in the restrictions that two \( S \)-equivalence classes should bear when their algebraic Gordian distance is one. Such restrictions provide us detections on the lower bounds for various distances in knot theory.

We now list some existing results for future use. To detect the Gordian distance, some lower bound criteria are proven. The signature criterion [13] is \( d_G(K, K') \geq \frac{1}{2}|\sigma(K) - \sigma(K')| \), where \( \sigma(K) \) is the signature of \( K \).

Murakami generalized Lickorish’s result [11] on the double branched cover and showed that if \( u(K) = d_G(K, K') = 1 \), then there exists an integer \( d \) such that

\[
\frac{2d^2}{D(K)} \equiv \pm \frac{D(K) - D(K')}{2D(K)} \pmod{1},
\]

where \( D(K) \) and \( D(K') \) denote the determinants of \( K \) and \( K' \), respectively [13]. We call it Murakami’s obstruction.

As to the algebraic unknotting number, we refer to the following lemma of Murakami, which we will use later in Section 3.

**Lemma 1.1** ([14, Theorem 5, p.288]). If \( u_a(K) = 1 \), then there exists a generator \( \alpha \) for the Alexander module of \( K \) such that the Blanchfield pairing \( \beta(\alpha, \alpha) = \pm \frac{1}{\Delta_K} \). Moreover, the Blanchfield pairing is given by a \( 1 \times 1 \)-matrix \((\pm \frac{1}{\Delta_K})\).
Kawauchi used residue modules and determinant rings and gave the following theorem, concluding an obstruction on the Alexander polynomials realizable by two knots with Gordian distance one.

**Theorem 1.2** ([9, Theorem 1.2, p.949]). If $u(K) = d_G(K, K') = 1$, then there exists $c \in \Lambda$ such that $\pm \Delta_{K'} \equiv c^\ast c \pmod{\Delta_K}$.

We improve this result in corollaries to our main theorem in Section 3 so that Theorem 1.2 is shown to be a special case of our result.

**Theorem 1.3** (Main Theorem). Let $V$ and $V'$ be two Seifert matrices. If the algebraic Gordian distance $d^a_G([V], [V'])=1$, then there exist $a \in A_V$ and $a' \in A_{V'}$ such that $\beta(a, a) \equiv \pm \frac{\Delta_{V'}}{\Delta_V} (\pmod{\Lambda})$ and $\beta(a', a') \equiv \pm \frac{\Delta_V}{\Delta_{V'}} (\pmod{\Lambda})$.

**Corollary 1.4.** If $u_a([V]) = d^a_G([V], [V']) = 1$, then there exists $c \in \Lambda$ such that $\pm \Delta_{V'} \equiv c^\ast c \pmod{\Delta_V}$.

The remainder of this paper is organized as follows. We give some auxiliary results in Section 2. In Section 3, we prove our main theorem. We show how our result improves Kawauchi’s and give new answers to Jong’s Problem. In addition, our result provides a new method to answer Nakanishi’s question. The final section consists of some examples of how our result helps in calculating various distances in knot theory.

## 2 The Seifert matrix

In this section, we recall the definition of the algebraic unknotted operation.

We notice that the matrix $\begin{pmatrix} \varepsilon & 0 & 0 \\ 1 & x & M \\ 0 & N^T & W \end{pmatrix}$ is not the only possible result of adding a twist.

**Lemma 2.1.** Let $W$ be a Seifert matrix of $K$. If an $\varepsilon$-unknotting operation assigns $K$ to $K'$, then both $\begin{pmatrix} \pm 1 & x & M \\ 0 & N^T & W \end{pmatrix}$ and $\begin{pmatrix} \varepsilon & 0 & 0 \\ \varepsilon & \pm 1 & 0 \\ 0 & x & M \end{pmatrix}$ are Seifert matrices of $K'$.

**Proof.** The Seifert surface of $K'$ can be constructed as in Figure 2a or Figure 2b, corresponding to $\begin{pmatrix} \varepsilon & 0 & 0 \\ \varsigma & x & M \\ 0 & N^T & W \end{pmatrix}$ and $\begin{pmatrix} \varepsilon & \varsigma & 0 \\ 0 & x & M \end{pmatrix}$, respectively. The direction of $\alpha$ determines $\varsigma = 1$ or $\varsigma = -1$. \qed
It is often hard to tell whether two matrices are $S$-equivalent or not, especially for matrices of size larger than $2 \times 2$. As a consequence to Lemma 2.1, we have the following equivalence.

**Lemma 2.2.** $\begin{pmatrix} \varepsilon & 0 & 0 \\ \pm 1 & x & M \\ 0 & N^T & W \end{pmatrix}$ is $S$-equivalent to $\begin{pmatrix} \varepsilon & \pm 1 & 0 \\ 0 & x & M \\ 0 & N^T & W \end{pmatrix}$ for $\varepsilon = \pm 1$.

Now we show that some Alexander polynomials are only realizable by Seifert matrices with algebraic unknotting number one.

**Lemma 2.3.** If a $2 \times 2$ Seifert matrix $V$ has $\det V = D \in \{1, 2, 3, 5\}$, then $u_a(V) = 1$.

**Proof.** Since $\det V > 0$ and the matrix size is $2 \times 2$, either $V$ or $-V$ is positive definite. Every $2 \times 2$ positive definite Seifert matrix is congruent to a matrix $\begin{pmatrix} a & b+1 \\ b & c \end{pmatrix}$, where $0 < 2b + 1 \leq \min(a, c)$; see [20, p.204]. Since $b = 0$ is the only solution to $ac - b(b+1) = D$, we obtain $ac = D$. Therefore, we have either $a = D$ and $c = 1$, or $a = 1$ and $c = D$. By Lemma 2.2, $\begin{pmatrix} 1 & 1 \\ 0 & D \end{pmatrix}$ is $S$-equivalent to $\begin{pmatrix} 1 & 0 \\ 1 & D \end{pmatrix}$, which is $\begin{pmatrix} D & 1 \\ 0 & 1 \end{pmatrix}$ by switching the basis’s elements.

By Lemma 2.1, both $\begin{pmatrix} 1 & 1 \\ 0 & D \end{pmatrix}$ and $\begin{pmatrix} -1 & -1 \\ 0 & -D \end{pmatrix}$ have algebraic unknotting number one, so the proof is complete. \(\Box\)

**Lemma 2.4.** For a Seifert matrix $V$, if $\Delta_V = ht + ht^{-1} + 1 - 2h$ with $h \in \{1, 2, 3, 5\}$, then $u_a(V) = 1$.

**Proof.** Because $\Delta_V = ht + ht^{-1} + 1 - 2h$, $V$ is $S$-equivalent to a $2 \times 2$ Seifert matrix $V'$ with $\det V' = h$ [20]. By Lemma 2.3, we have $u_a(V) = 1$. \(\Box\)

The next lemma relates the distance between matrices with the distance between polynomials.
Lemma 2.5. We have $d_G^0([K_1], [K_2]) \geq \rho(\Delta_{K_1}, \Delta_{K_2})$.

Proof. If $d_G^0([K_1], [K_2]) = 0$, $K_1$ is $S$-equivalent to $K_2$ and hence $\Delta_{K_1} = \Delta_{K_2}$, which gives $\rho(\Delta_{K_1}, \Delta_{K_2}) = 0$. If $d_G^0([K_1], [K_2]) = 1$, by definition, there exist $K'_1 \in [K_1]$ and $K'_2 \in [K_2]$ such that $d_G(K'_1, K'_2) = 1$. Since $K'_1 \in [K_1]$ and $K'_2 \in [K_2]$, we obtain $\Delta_{K'_1} = \Delta_{K_1}$, $\Delta_{K'_2} = \Delta_{K_2}$ and consequently $\rho(\Delta_{K_1}, \Delta_{K_2}) = 1$. If $d_G^0([K_1], [K_2]) \geq 2$, the inequality holds because $\rho(a, b) \leq 2$.

Consequently, we have $d_G(K_1, K_2) \geq d_G^0([K_1], [K_2]) \geq \rho(\Delta_{K_1}, \Delta_{K_2})$.

3 Proof of the Theorem 1.3 and its consequence

In this section, we first deduce the structure of the Blanchfield pairing realized by a pair of $S$-equivalence classes with algebraic Gordian distance one. It yields the obstructions expressed in terms of the Alexander polynomials for two $S$-equivalence classes with algebraic Gordian distance one. In corollaries to the main theorem, we conclude another answer to Jong’s problem that two Alexander polynomials cannot be realized by a pair of knots with Gordian distance one if a corresponding quadratic equation does not have an integer solution.

Proof of Theorem 1.3. If $[V]$ and $[V']$ have algebraic Gordian distance one, there exist $W \in [V]$ and $W' \in [V']$ such that $W$ can be obtained from $W'$ by an algebraic unknotting operation. By definition, the algebraic unknotting operation assigns $W'$ to

$$
\begin{pmatrix}
\varepsilon & 0 & 0 \\
1 & x & M \\
0 & N^T & W'
\end{pmatrix}
$$

for $\varepsilon = \pm 1$.

Let $a$ be the first element of the basis for $A_V$ so that the Blanchfield pairing $\beta(a, a)$ is the $(1, 1)$-entry of matrix $(t - 1)(tW - W^T)^{-1}$. It is given by

$$
\beta(a, a) \equiv (t - 1) \frac{|x(t - 1) - tM - N|}{|tW - W^T|}.
$$

The Alexander polynomials are given by $\Delta_V = \Delta_W = t^{-g}|tW - W^T|$ and $\Delta_{V'} = \Delta_{W'} = t^{1-g}|tW' - W'^T|$, where $2g$ is the size of $W$. Because $|tW - W^T| = \varepsilon(t - 1) \left| x(t - 1) - tM - N - tW' + W'^T \right|$, we have

$$
\beta(a, a) \equiv \frac{\varepsilon(\Delta_K - \Delta_{K'})}{\Delta_K} \equiv -\varepsilon \frac{\Delta_{K'}}{\Delta_K} \pmod{\Lambda}.
$$
The equation of $\beta(a',a')$ can be proved in a similar way.

Theorem 1.3 states the Blanchfield pairing obstruction when the algebraic Gordian distance is one. By the same method, we conclude the following corollary, which gives the same obstruction as above for a pair of knots with Gordian distance one.

**Corollary 3.1.** If $K$ and $K'$ are two knots with $d_G(K,K') = 1$, then there exist $a \in H_1(\tilde{X}(K))$ and $a' \in H_1(\tilde{X}(K'))$ such that $\beta(a,a) \equiv \pm \frac{\Delta_{K'}}{\Delta_K} \pmod{\Lambda}$ and $\beta(a',a') \equiv \pm \frac{\Delta_K}{\Delta_{K'}} \pmod{\Lambda}$.

As the Blanchfield pairing is a complicated form, we now focus on the case where the Alexander module is cyclic. We prove the following corollary and show that it improves existing results.

**Proof of Corollary 1.4.** Since $u_a([V]) = d^a_G([V],[V']) = 1$, by Theorem 1.3 there exist $a \in A_V$ and $g \in A_{V'}$ such that $\beta(a,a) \equiv \pm \frac{\Delta_{V'}}{\Delta_V} \pmod{\Lambda}$ and $\beta(g,g) \equiv \pm \frac{1}{\Delta_V} \pmod{\Lambda}$.

By Lemma 1.1, the Blanchfield pairing on $A_V$ is cyclic and generated by $a$. Therefore, there exists $c \in \Lambda$ such that $a = cg$. Hence we have

$$\pm \frac{\Delta_{V'}}{\Delta_V} \equiv \beta(cg,cg) \equiv c\bar{c}\beta(g,g) \equiv \frac{c\bar{c}}{\Delta_V} \pmod{\Lambda},$$

which gives $\pm \Delta_{V'} = c\bar{c} \pmod{\Delta_V}$ and completes the proof.

Then the following corollary comes naturally.

**Corollary 3.2.** If $u_a(K) = d_G(K,K') = 1$, then there exists $c \in \Lambda$ such that $\pm \Delta_{K'} \equiv c\bar{c} \pmod{\Delta_K}$.

**Remark.** It is worth mentioning that Corollary 1.4 (or Corollary 3.2) implies Theorem 1.2, which is a result of Kawauchi [9, Theorem 1.2, p.949]. Note that there are infinitely many knots with trivial Alexander polynomial. Since $u(K) = 1$ is a special case of $u_a(K) = 1$, Corollary 3.2 implies Theorem 1.2. Moreover, Corollary 1.4 concerns the algebraic Gordian distance. By the inequality $d_G(K_1,K_2) \geq d^a_G([K_1],[K_2])$, the algebraic Gordian distance subsumes the lower bound of the Gordian distance. Therefore, Corollary 1.4 also implies Theorem 1.2. We give an example in Section 4 to show
this by letting $K_1$ and $K_2$ be a pair of knots with same Alexander polynomials as $9_{25}$ and $3_1$, respectively. From Lemma 2.4, we know $u_a(K_2) = 1$, so we can deduce $d_{G}(K_1, K_2) \geq d_{G}^c([K_1], [K_2]) \geq 2$ from a consequence of Corollary 1.4. On the contrary, Theorem 1.2 cannot work in this case, because we do not know if $u_a(K_2) = 1$.

Our next aim is to give new answers to Jong’s problem, which asks how to detemine two Alexander polynomials $\Delta$ and $\Delta'$ such that $\rho(\Delta, \Delta') = 2$.

**Corollary 3.3.** Let $\Delta_K$ and $\Delta_{K'}$ be the Alexander polynomials for $K$ and $K'$, respectively, with $\Delta_K = h(t + t^{-1}) + 1 - 2h$, $|h|$ being a prime or 1 and $\Delta_{K'} \equiv d \ (\text{mod } \Delta_K)$, where $d \in \Z$. If $u_a(K) = 1$ and if $h^2 x^2 + y^2 + (2h - 1)xy = \pm d$ does not have an integer solution, then the algebraic Gordian distance $d_{G}^c([K], [K']) > 1$.

**Proof.** Seeking a contradiction, suppose $d_{G}^c([K], [K']) = 1$. By Corollary 3.2, there exists $c \in \Lambda$ such that $c \bar{c} \equiv \pm \Delta_{K'} \equiv \pm d \ (\text{mod } \Delta_K)$. Let

$$c = \sum_{-n \leq i \leq m} a_i t^i \quad \text{and} \quad \bar{c} = \sum_{-m \leq i \leq n} a_{-i} t^i,$$

which gives

$$c \bar{c} = a_{-n}a_m t^{m+n} + \cdots + a_m a_{-n} t^{-(m+n)}.$$

If $c$ can be expressed as $c = pt^{j+1} + qt^j$, where $p$ and $q$ are integers, we have $(p^2 + q^2) + pq(t + t^{-1}) \equiv \pm d \ (\text{mod } \Delta_K)$. Because $|h|$ is prime or 1, we have either $h|p$ or $h|q$. Without loss of generality, we may assume $p = hx$. By substituting $p$, we obtain that $h^2 x^2 + q^2 + (2h - 1)xq = \pm d$. If it does not have an integer solution, the algebraic Gordian distance cannot be one.

If $c$ has more than two terms, we have $h|a_{-n}a_m$ follows from $c \bar{c} \equiv \pm d \ (\text{mod } \Delta_K)$, so either $h|a_{-n}$ or $h|a_m$. Hence we obtain

$$c \equiv \sum_{-n \leq i \leq m} a'_i t^i \quad \text{or} \quad \sum_{-m \leq i \leq n} a'_i t^i \ (\text{mod } \Delta_K),$$

where $\{a'_i\}$ are integer coefficients. Repeat this step until we deduce $c \equiv pt^{j+1} + qt^j \ (\text{mod } \Delta_K)$, where $p$ and $q$ are integers. The rest of the proof follows in the same manner.

Then the following corollary comes instantly.

**Corollary 3.4.** The Alexander polynomial distance $\rho(t - 1 + t^{-1}, \Delta) = 2$ if $\Delta \equiv 2 + 4m \ (\text{mod } t - 1 + t^{-1})$ for some $m \in \Z$. 

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Proof. Lemma 2.4 concludes that any knot with Alexander polynomial $t - 1 + t^{-1}$ has algebraic unknotting number one. By Corollary 3.3, it suffices to show that $x^2 + y^2 + xy = 2 + 4m$ does not have an integer solution.

Now we check the parities of $x$ and $y$. If $x$ and $y$ are both odd or one even one odd, then $x^2 + y^2 + xy$ is odd, which is a contradiction. If $x$ and $y$ are both even, then $x^2 + y^2 + xy \equiv 0 \pmod{4}$, which is also a contradiction. Hence the proof is complete.

Remark. We have other applications of Corollary 3.3. By Lemma 2.4, Corollary 3.3 gives the same result for $\rho(\Delta, ht + ht^{-1} + 1 - 2h) = 2$ with $h \in \{1, 2, 3, 5\}$ if polynomial $h^2 x^2 + y^2 + (2h - 1)xy = \pm d$ does not have an integer solution.

Remark. Corollary 3.4 offers another route to answer Nakanishi’s question [16, p.334], which asks if $\rho(\Delta_{31}, \Delta_{41}) = 2$. Moreover, it implies two Seifert matrices with Alexander polynomials same as $3_1$ and $4_1$, respectively, cannot be turned into each other by one algebraic unknotting operation.

4 Examples

Corollary 3.4 could be applied for cases where Murakami’s obstruction and the signature criterion fail to work. By our method, we can update some entries in the Gordian distance table given by Moon [12]. Take $3_1$ and $9_{25}$ for example, the diagrams of which are shown in Figure 3; see [3]. By Moon’s table, the lower bound of $d_G(9_{25}, 3_1)$ is one and we now improve it to two as follows.

\[
\Delta_{31} = t + t^{-1} - 1 \quad \Delta_{9_{25}} = -3t^2 - 3t^{-2} + 12t + 12t^{-1} - 17
\]
\[
\sigma(3_1) = -2 \quad \sigma(9_{25}) = -2
\]
\[
D(3_1) = 3 \quad D(9_{25}) = 47
\]

Figure 3: $3_1$ and $9_{25}$

From [3] we have $\Delta_{31} = t + t^{-1} - 1$ and $\Delta_{9_{25}} = -3t^2 - 3t^{-2} + 12t + 12t^{-1} - 17$, so $\Delta_{9_{25}} = (-3t + 9 - 3t^{-1})\Delta_{31} - 2$. By Corollary 3.4, we obtain
\( \rho(\Delta_{925}, \Delta_{31}) = 2 \). From Lemma \[2.5\], we have \( d_G(K_1, K_2) \geq d^*_G([K_1], [K_2]) \geq \rho(\Delta_{925}, \Delta_{31}) = 2 \) for any pair of knots \( K_1 \) and \( K_2 \) with \( \Delta_{K_1} = \Delta_{925} \) and \( \Delta_{K_2} = \Delta_{31} \). Then we obtain \( d_G(9_{25}, 3_{1}) \geq d^*_G([9_{25}], [3_{1}]) \geq \rho(\Delta_{925}, \Delta_{31}) = 2 \).

Moreover, this example demonstrates how our result helps in calculating the algebraic Gordian distance of two given \( S \)-equivalent classes. We know \( u_a(9_{25}) = u_a(3_{1}) = 1 \); see \[2\]. It gives \( d^*_G([9_{25}], [3_{1}]) \leq u_a(9_{25}) + u_a(3_{1}) = 2 \). Therefore, we have \( d^*_G([9_{25}], [3_{1}]) = 2 \).

It is worth mentioning that Murakami’s method does not work in this example. Following Murakami’s method, we have to prove there does not exist an integer \( d \) such that

\[
\frac{2d^2}{D(3_{1})} \equiv \pm \frac{D(3_{1}) - D(9_{25})}{2D(3_{1})} \quad (\text{mod } 1).
\]

In fact, any integer \( d \), such that \( d \not\equiv 0 \pmod{3} \), satisfies the requirement, which means Murakami’s method does not work in this case.

Meanwhile, the knot signature criterion fails in this case. Since \( \sigma(3_{1}) = \sigma(9_{25}) = -2 \), the signature criterion cannot tell whether \( d_G(K, K') \) is one or not.

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