A superconducting quantum interference device (SQUID) comprising 0- and Ψ-Josephson junctions (JJs), called 0-π-SQUID, is studied by the resistively shunted junction model. The π-SQUID shows half-integer Shapiro-steps (SS) under microwave irradiation at the voltage V = (Δ/2e)Ωn/2, with angular frequency Ω and half-integer n/2 in addition to integer n. We show that the π-SQUID can be a π-qubit with spontaneous loop currents by which the half-integer SS are induced. Making the 0- and Ψ-JJs equivalent is a key for the half-integer SS and realizing the π-qubit.

**In this letter, we show that the** π-JS** Quinnuq juction, Eq. (2), the equation of motion for phase differences φa and φb are given by,**

\[
\frac{dφ_a}{dτ} + \sin φ_a + \frac{1}{β}(φ_a - φ_b) = \frac{1}{2} \left[ ib - \frac{4π φ_π - φ_α}{β φ_0}. \right]
\]

\[
\frac{dφ_b}{dτ} + \sin φ_b - \frac{1}{αβ}(φ_a - φ_b) = \frac{1}{2α} \left[ ib + \frac{4π φ_π - φ_α}{β φ_0}. \right]
\]

where R′ = R0/β, RL ≡ RL/φ0, τ ≡ ω0t, β ≡ 2π/Ω/φ0, i ≡ IJ/φ0, iL ≡ IJ/Λ, and I′ ≡ IJ/φ0. As we will discuss later, one of the key parameters is α ≡ Jα/β, which indicates the asymmetry of two JJs.

By numerically solving Eqs. (6) and (7) for φ0 = 0, I−V curves with step structures are obtained as shown in Figs. 2.
Fig. 1. (Color online) Schematic of π-SQUID (upper) and the RSJ (lower) models.

Fig. 2. (Color online) (a) $I$–$V$ curves with $\Omega/\omega_0 = 0.1$, $\beta = 1.0$, and $i_{ac} = 0.5$, for $\alpha = 1.0$ (red upper triangles), $\alpha = 0.8$ (blue lower triangles), and $\alpha = 0.6$ (purple circles). For clarity, the latter two curves are vertically shifted by $i_{ac} = 0.2$ and 0.4, respectively. The half-integer SS are suppressed by decreasing $\alpha$, which is controlled by changing the ratio of junction areas, $W_a$ and $W_c$. Making both Josephson coupling and resistance of two JJs equivalent is a key to observe the half-integer SS. In Fig. 2 (b) for $\alpha = 1$, $I$–$V$ curves are plotted for $i_{ac} = 0.2$ and $\beta = 1.0$ (green circles), $i_{ac} = 0.5$ and $\beta = 1.0$ (red upper triangles), and $i_{ac} = 0.5$ and $\beta = 0.2$ (black crosses). The height of half-integer SS is enhanced by increasing $i_{ac}$, whereas it is suppressed by decreasing $\beta$. When $\beta$ is small by decreasing the SQUID loop, the half-integer Shapiro steps can be observed by increasing the $i_{ac}$. $\beta$ is estimated as $\beta \sim 1$ for $2.5 \times 2.5 \mu m^2$ loop and $J_0 \sim 70 \mu A$, meaning $L \sim 4.7$ pH. It satisfies the criteria to overcome the thermal noise, i.e. $L$ must be less than about 20 nH at 4.2 K.

The half-integer SS can be understood using the following approximation. In the first order of $\beta$, $i_B$ is given by

$$i_b \sim 4 \frac{\alpha}{1+\alpha} \left[ \cos \left( \frac{\Phi_q - \Phi_0}{\Phi_0} \right) \sin \phi \right. + \left. \frac{\beta}{2} \frac{\alpha}{1+\alpha} \sin \left( \frac{\Phi_q - \Phi_0}{\Phi_0} \right)^2 \sin 2\phi \right],$$

where $\phi \equiv (\phi_a + \phi_b)/2$. Notably, the second term in Eq. (8) including “$\sin 2\phi$” is the origin of the half-integer SS. By applying a voltage $V(t) = V + \Omega \cos \Omega t$, $\phi = \phi(0) + at + b \sin \Omega t$ with $a = 2\pi V/\Phi_0$ and $b = 2\pi V/(\Phi_0 \Omega)$, $i_B$ becomes

$$i_B \sim 4 \frac{\alpha}{1+\alpha} \text{Im} \left[ \sum_k A e^{i\beta k} J_k(b) e^{i \left( a + k + 1 \right) \Omega t} \right. + \left. \frac{\beta}{2} \frac{\alpha}{1+\alpha} B^2 e^{2i\phi(0)} \sum_k J_k(2b) e^{i \left( 2a + k + 1 \right) \Omega t} \right],$$

where $A \equiv \cos \left( \frac{\Phi_q - \Phi_0}{\Phi_0} \right)$, $B \equiv \sin \left( \frac{\Phi_q - \Phi_0}{\Phi_0} \right)$, and the $k$th order Bessel function $J_k(b)$. For the $\pi$-SQUID with $\Phi_c = 0$, the first term in Eq. (9) is zero because $\Phi_q/\Phi_0 = 1/2$. When $V$ satisfies $V/(\Omega \Phi_0/2\pi) = k'/2$, the SS with a half-integer $k'/2$ and an integer $q(k' = 2q)$ appear with a DC-component, $2\beta [\alpha/(1+\alpha)]^2 J_2(b)$, for $\phi(0) = \pi/4$. Meanwhile, in the conventional SQUID with $\Phi_c = 0$, the second term in Eq. (9) is zero because $\Phi_q/\Phi_0 = 0$. Only integer SS appear at voltages of integer multiples of $V/(\Omega \Phi_0/2\pi) = k$ with a DC-component, $4[\alpha/(1+\alpha)] J_2(b)$, for $\phi(0) = \pi/2$.

The half-integer SS are explained by the onset of spontaneous current, leading to a flip-flop between two fluid states of the $\pi$-SQUID synchronized to the alternating field. It suggests that the present system can become a qubit. Equation (8) shows that the two potentials with $\phi$ and $2\phi$ are convoluted similar to the $ads$-wave JJ. The half-integer SS and $\pi$-qubit are the two sides of the same coin. The potential energy of the $\pi$-SQUID $U(\phi_a, \phi_b)$ with Eq. (3) is given by
where $x \equiv \phi$, and $y \equiv (\pi/\Phi_0)\Phi$. The ground state is obtained by minimizing $f(x, y)$ with respect to $y$ for a fixed $x$.

\[
\frac{\partial f(x, y)}{\partial y} \bigg|_{y=y_0} = -\cos(x + y_0) - \alpha \cos(x - y_0) + \frac{4}{\beta}y_0 = 0,
\]

by which $y_0$ is determined as a function of $x$, i.e., $y_0 = y_0(x)$. Equation (13) means that $f(x, y)$ is minimized with respect to $\Phi_{ex}$, since we study the $\pi$-SQUID with two JJs shown in Fig. 1. To avoid $\Phi_{ex}$, the minimization with respect to $\Phi_{ex}$ can be substituted by another Josephson phase including one more JJ in the $\pi$-SQUID, as discussed in the previous studies.\(^{28,31}\) Numerically solving Eq. (13), we find that $f(x, y_0(x))$ has the double minimum with respect to $x$ as shown in Fig. 3. The right minimum corresponds to the current circulating state, whereas the left one has no circulating current. Similar to the previous case,\(^{30}\) the barrier height is suppressed by decreasing $\alpha$, which coincides with the suppression of the half-integer SS. When the barrier height was zero, the stable state is $x = \pi/2$, which may be realized by setting the phase-lock to $\phi_b = \phi_0$ using Eqs. (3) and (4). In this case, the spontaneous loop current $I_a - I_b = 0$. Then, the SS appear at voltages with integer multiples. The two minima correspond to the clockwise and anticlockwise loop currents in addition to the $\pi$-shift because of the $\pi$-JJ, meaning that the spontaneous loop current is induced. Then, $\phi_b$ and $\phi_0$ are not synchronized as discussed in Refs. 6 and 14 because the loop current means the time-evolution of the phase difference in each junction.

It is useful to compare the present model with the previous one, in which a metallic transport is assumed in the $\pi$-JJ.\(^{30}\) The Josephson current of the metallic junction assigned to the junction-$b$ is given by

\[
J_b = \frac{\sin \phi_b}{\sqrt{1 - T \sin^2(\phi_b/2)}},
\]

with transmittance $T$ ($0 \leq T \leq 1$).\(^{33-35}\) This case also shows the half-integer SS as shown in Fig. 4, although its magnitude becomes small when $T$ approaches 1, where the magnitude of the metallic junction becomes large and dominates the current. Even in such a metallic case, the half-integer SS can be observed by increasing the $I_{bc}$.

The potential energy corresponding to Eq. (12) is given by

\[
g(x, y) = -\sin(x + y) + \frac{4\alpha}{T} \times \sqrt{1 - \frac{T}{2} [1 + \sin(x - y)]} + \frac{2}{\beta}y_0^2.
\]

The ground state is obtained by minimizing $g(x, y)$ with respect to $y$ for a fixed $x$.

\[
\frac{\partial g(x, y)}{\partial y} \bigg|_{y=y_0} = -\cos(x + y_0) - \frac{\alpha \cos(x - y_0)}{\sqrt{1 + \frac{2}{\beta} [1 + \sin(x - y_0)]}} + \frac{4}{\beta}y_0 = 0.
\]

Numerically solving Eq. (16), we find that $g(x, y_0(x))$ exhibits the double minimum with respect to $x$ as shown in Fig. 5. Because the magnitude of potential highly depends on $T$, $g(x, y_0(x)) \times T$ is plotted instead of $g(x, y_0(x))$ for clarity. The rather insulating case with $\alpha = 0.6$, which corresponds to the black circles in Fig. 3, shows the shallow minimum, whereas the double minimum becomes clear by increasing $T$; this would contradict to the suppression of the half-integer SS in Fig. 4, i.e., red upper triangles and blue lower triangles. However, it is caused by a large magnitude of potential in the

Fig. 3. (Color online) The potential energy for $\beta = 1.0$ with $\alpha = 1.0$ (red upper triangles), $\alpha = 0.8$ (blue lower triangles), and $\alpha = 0.6$ (black circles).
metallic junction. In fact, the half-integer SS are revived by increasing $i_{ac}$.

In this paper, we assumed that the Josephson critical current densities $j_i \ (i = a, b)$ defined by $j_i = J_i / A_i$ with junction area $A_i$ are common between junctions-$a$ and $b$, i.e. $j_a / j_b = 1$. Even in the case of $j_a / j_b \neq 1$, the half-integer SSs can remain for $0.7 \leq j_a / j_b \leq 1.3$. We can consider another type of SQUID including two 0-JJs and one $\pi$-JJ, which is experimentally realized. In this case, the $\pi$-JJ is used as a $\pi$-phase shifter. The condition is quite similar to the SQUID with an external flux studied by Vanneste et al. We can expect the half-integer SSs in such a geometry as well.

So far, we have focused on the overdamped JJs to clarify the close relation between the half-integer SSs and the $\pi$-qubit. From a viewpoint of qubit application and/or operation, on the other hand, we need to include the capacitance in Eq. (1) and should estimate the coherence time of qubit. In fact, Kato et al., discussed a long coherence time in a $\pi$-qubit. Although the coherence time of qubit is a crucial factor, it goes beyond our aim in this paper. We will examine the coherence time of $\pi$-SQUID by considering a capacitance and thermal fluctuations in the near future.

In summary, the half-integer SS in $\pi$-SQUID comprising 0- and $\pi$-JJs have been studied using the RSJ model. We have shown that the $\pi$-SQUID can be a $\pi$-qubit with spontaneous loop currents, by which the half-integer SSs are induced, meaning that the half-integer SS and the $\pi$-qubit are the two sides of the same coin. Making the 0- and $\pi$-JJs equivalent is a key for the half-integer Shapiro steps and realizing the $\pi$-qubit.

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Fig. 5. (Color online) Potential energy for $\alpha = 0.6$ and $\beta = 1.0$ with $T = 0.8$ (black circles), $T = 0.5$ (blue lower triangles), and $T = 0.2$ (red upper triangles).

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