ABSTRACT. We use Gale duality for complete intersections and adapt the proof of the
fewnomial bound for positive solutions to obtain the bound
\[ \frac{e^4 + 3}{4} 2^{\binom{n}{2}} n^k \]
for the number of non-zero real solutions to a system of \( n \) polynomials in \( n \) variables
having \( n+k+1 \) monomials whose exponent vectors generate a subgroup of \( \mathbb{Z}^n \) of odd
index. This bound only exceeds the bound for positive solutions by the constant factor
\( (e^4 + 3)/(e^2 + 3) \) and it is asymptotically sharp for \( k \) fixed and \( n \) large.

INTRODUCTION

In [3], the sharp bound of \( 2n+1 \) was obtained for the number of non-zero real solutions to
a system of \( n \) polynomial equations in \( n \) variables having \( n+2 \) monomials whose exponents
affinely span the lattice \( \mathbb{Z}^n \). In [4], the sharp bound of \( n+1 \) was given for the positive
solutions to such a system of equations. This last bound was generalized in [7], which
showed that the number of positive solutions to a system of \( n \) polynomial equations in \( n \)
variables having \( n+k+1 \) monomials was less than
\[ \frac{e^2 + 3}{4} 2^{\binom{n}{2}} n^k, \]
which is asymptotically sharp for \( k \) fixed and \( n \) large [3]. This dramatically improved
Khovanskii’s fewnomial bound [8] of \( 2^{(n+k)/2} (n+1)^{n+k} \).

We give a bound for all non-zero real solutions. Under the assumption that the exponent
vectors \( W \) span a subgroup of \( \mathbb{Z}^n \) of odd index, we show that the number of non-degenerate
non-zero real solutions to a system of polynomials with support \( W \) is less than
\[ \frac{e^4 + 3}{4} 2^{\binom{n}{2}} n^k. \]

The novelty is that this bound exceeds the bound for solutions in the positive orthant by
a fixed constant factor \( (e^4 + 3)/(e^2 + 3) \), rather than by a factor of \( 2^n \), which is the number
of orthants. By the construction in [5], it is asymptotically sharp for \( k \) fixed and \( n \) large.

We follow the outline of [7]—we use Gale duality for real complete intersections [6]
and then bound the number of solutions to the dual system of master functions. The
key idea is that including solutions in all chambers in a complement of an arrangement of hyperplanes in $\mathbb{RP}^k$, rather than in just one chamber as in [7], does not increase our estimate on the number of solutions very much. This was discovered while implementing a numerical continuation algorithm for computing the positive solutions to a system of polynomials [1]. That algorithm was improved by this discovery to one which finds all real solutions. It does so without computing complex solutions and is based on [7] and the results of this paper. Its complexity depends on [11], and not on the number of complex solutions.

We state our main theorem in Section 1 and then use Gale duality to reduce it to a statement about systems of master functions, which we prove in Section 2.

1. Gale duality for systems of sparse polynomials

Let $W = \{w_0 = 0, w_1, \ldots, w_{n+k}\} \subset \mathbb{Z}^n$ be a collection of $n+k+1$ integer vectors ($|W| = n+k+1$), which correspond to monomials in variables $x_1, \ldots, x_n$. A (Laurent) polynomial $f$ with support $W$ is a real linear combination of monomials with exponents from $W$,

$$f(x_1, \ldots, x_n) = \sum_{i=0}^{n+k} c_i x^{w_i} \text{ with } c_i \in \mathbb{R}.$$

A system with support $W$ is a system of polynomial equations

$$f_1(x_1, \ldots, x_n) = f_2(x_1, \ldots, x_n) = \cdots = f_n(x_1, \ldots, x_n) = 0,$$

where each polynomial $f_i$ has support $W$. Since multiplying every polynomial in (3) by a monomial $x^\alpha$ does not change the set of non-zero solutions but translates $W$ by the vector $\alpha$, we see that it was no loss of generality to assume that $0 \in W$.

The system (3) has infinitely many solutions if $W$ does not span $\mathbb{Z}^n$. We say that $W$ spans $\mathbb{Z}^n \mod 2$ if the $\mathbb{Z}$-linear span of $W$ is a subgroup of $\mathbb{Z}^n$ of odd index.

Theorem 1. Suppose that $W$ spans $\mathbb{Z}^n \mod 2$ and $|W| = n+k+1$. Then there are fewer than $\frac{n+n!}{2}$ non-degenerate non-zero real solutions to a sparse system (3) with support $W$.

The importance of this bound for the number of real solutions is that it has a completely different character than Kouchnirenko’s bound for the number of complex solutions.

Proposition 2 (Kouchnirenko [2]). The number of non-degenerate solutions in $(\mathbb{C}^\times)^n$ to a system (3) with support $W$ is no more than $n! \text{vol}(\text{conv}(W))$.

Here, $\text{vol}(\text{conv}(W))$ is the Euclidean volume of the convex hull of $W$.

Perturbing coefficients of the polynomials in (3) so that they define a complete intersection in $(\mathbb{C}^\times)^n$ can only increase the number of non-degenerate solutions. Thus it suffices to prove Theorem 1 under this assumption. Such a complete intersection is equivalent to a complete intersection of master functions in a hyperplane complement [6].

Let $\mathbb{R}^{n+k}$ have coordinates $z_1, \ldots, z_{n+k}$. A polynomial (2) with support $W$ is the pullback $\Phi_W^*(\Lambda)$ of the degree 1 polynomial $\Lambda := c_0 + c_1 z_1 + \cdots + c_{n+k} z_{n+k}$ along the map

$$\Phi_W : (\mathbb{R}^\times)^n \ni x \mapsto (x^{w_i} \mid i = 1, \ldots, n+k) \in \mathbb{R}^{n+k}.$$

If we let $\Lambda_1, \ldots, \Lambda_n$ be the degree 1 polynomials which pull back to the polynomials in the system (3), then they cut out an affine subspace $L$ of $\mathbb{R}^{n+k}$ of dimension $k$. 
Let $\{p_i \mid i = 1, \ldots, n+k\}$ be degree 1 polynomials on $\mathbb{R}^k$ which induce an isomorphism between $\mathbb{R}^k$ and $L$,
$$
\Psi_p : \mathbb{R}^k \ni y \mapsto (p_1(y), \ldots, p_{n+k}(y)) \in L \subset \mathbb{R}^{n+k}.
$$
Let $\mathcal{A} \subset \mathbb{R}^k$ be the arrangement of hyperplanes defined by the vanishing of the $p_i(y)$. This is the pullback along $\Psi_p$ of the coordinate hyperplanes of $\mathbb{R}^{n+k}$.

The image $\Phi_W((\mathbb{R}^\times)^n)$ inside of the torus $(\mathbb{R}^\times)^{n+k}$ has equations
$$
z^{\beta_1} = z^{\beta_2} = \cdots = z^{\beta_k} = 1,$$
where the weights $\{\beta_1, \ldots, \beta_k\}$ form a basis for the $\mathbb{Z}$-submodule of $\mathbb{Z}^{n+k}$ of linear relations among the vectors $W$. To these data, we associate a system of master functions on the complement $M_A$ of the arrangement $A$ of $\mathbb{R}^k$,
$$
(4) \quad p(y)^{\beta_1} = p(y)^{\beta_2} = \cdots = p(y)^{\beta_k} = 1.
$$
Here, if $\beta = (b_1, \ldots, b_{n+k})$ then $p^\beta := p_1(y)^{b_1} \cdots p_{n+k}(y)^{b_{n+k}}$.

A basic result of [6] is that if $W$ spans $\mathbb{Z}^n$ modulo 2 and either of the systems (3) or (4) defines a complete intersection, then the other defines a complete intersection and the maps $\Phi_W$ and $\Psi_p$ induce isomorphisms between the two solution sets, as analytic subschemes of $(\mathbb{R}^\times)^n$ and $M_A$. Since we assumed that the system (3) is general, these hypotheses hold and the arrangement is essential in that the polynomials $p_i$ span the space of all degree 1 polynomials on $\mathbb{R}^k$.

**Theorem 3.** A system (4) of master functions in the complement of an essential arrangement of $n+k$ hyperplanes in $\mathbb{R}^k$ has at most (1) non-degenerate real solutions.

We actually prove a bound for a more general system than (4), namely for
$$
p(z)^{2\beta_1} = p(z)^{2\beta_2} = \cdots = p(z)^{2\beta_k} = 1.
$$
We write this more general system as
$$
(5) \quad |p(z)|^{\beta_1} = |p(z)|^{\beta_2} = \cdots = |p(z)|^{\beta_k} = 1.
$$
In a system of this form we may have real number weights $\beta_i \in \mathbb{R}^{n+k}$. We give the strongest form of our theorem.

**Theorem 4.** A system of the form (5) with real weights $\beta_i$ in the complement of an essential arrangement of $n+k$ hyperplanes in $\mathbb{R}^k$ has at most (1) non-degenerate real solutions.

**2. Proof of Theorem 4**

We follow [7] with minor, but important, modifications. Perturbing the polynomials $p_i(y)$ and the weights $\beta_j$ will not decrease the number of non-degenerate real solutions in $M_A$. This enables us to make the following assumptions.

The arrangement $\mathcal{A}^+ \subset \mathbb{RP}^k$, where we add the hyperplane at infinity, is general in that every $j$ hyperplanes of $\mathcal{A}^+$ meet in a $(k-j)$ dimensional linear subspace, called a codimension $j$ face of $\mathcal{A}$. If $B$ is the matrix whose columns are the weights $\beta_1, \ldots, \beta_k$, then the entries of $B$ are rational numbers and no minor of $B$ vanishes. This last technical
condition as well as the freedom to further perturb the $\beta_j$ and the $p_i$ are necessary for the results in [7, Section 3] upon which we rely.

For functions $f_1, \ldots, f_j$ on $M_A$, let $V(f_1, \ldots, f_j)$ be the subvariety they define. Suppose that $\beta_j = (b_{1,j}, \ldots, b_{n+k,j})$. For each $j = 1, \ldots, k$, define

$$\psi_j(y) := \sum_{i=1}^{n+k} b_{i,j} \log |p_i(y)|.$$  

Then (5) is equivalent to $\psi_1(y) = \cdots = \psi_k(y) = 0$. Inductively define $\Gamma_k, \Gamma_{k-1}, \ldots, \Gamma_1$ by

$$\Gamma_j := \text{Jac}(\psi_1, \ldots, \psi_j, \Gamma_{j+1}, \ldots, \Gamma_k),$$  

the Jacobian determinant of $\psi_1, \ldots, \psi_j, \Gamma_{j+1}, \ldots, \Gamma_k$. Set

$$C_j := V(\psi_1, \ldots, \psi_{j-1}, \Gamma_{j+1}, \ldots, \Gamma_k),$$  

which is a curve in $M_A$.

Let $\♭(C)$ be the number of unbounded components of a curve $C \subset M_A$. We have the estimate from [7], which is a consequence of the Khovanskii-Rolle Theorem,

$$|V(\psi_1, \ldots, \psi_k)| \leq b(C_k) + \cdots + b(C_1) + |V(\Gamma_1, \ldots, \Gamma_k)|.$$  

Here, $|S|$ is the cardinality of the set $S$. We estimate these quantities.

**Lemma 5.**

1. $|V(\Gamma_1, \ldots, \Gamma_k)| \leq 2^{(\ell)(k)} n^k$.
2. $C_j$ is a smooth curve and

$$b(C_j) \leq \frac{1}{2} 2^{(\ell)(k-j)} n^{k-j} \binom{n+k+1}{j} \cdot 2^j \leq \frac{1}{2} 2^{(\ell)(k)} n^k \cdot \frac{2^{2j-1}}{j!}.$$  

**Proof of Theorem 4.** By (6) and Lemma 5, we have

$$|V(\psi_1, \ldots, \psi_k)| \leq 2^{(\ell)(k)} n^k \left(1 + \frac{1}{4} \sum_{j=1}^{k} \frac{4^j}{j!}\right) < 2^{(\ell)(k)} n^k \cdot \frac{e^4 + 3}{4}. \quad \square$$  

**Proof of Lemma 2** The bound (1) is from Lemma 3.4 of [7]. Statements analogous to (2) for $\tilde{C}_j$, the restriction of $C_j$ to a single chamber (connected component) of $M_A$, were established in Lemma 3.4 and the proof of Lemma 3.5 in [7]:

$$b(\tilde{C}_j) \leq \frac{1}{2} 2^{(\ell)(k-j)} n^{k-j} \binom{n+k+1}{j} \leq \frac{1}{2} 2^{(\ell)(k)} n^k \cdot \frac{2^{2j-1}}{j!}.$$  

The bound we claim for $b(C_j)$ has an extra factor of $2^j$. *A priori* we would expect to multiply this bound (7) by the number of chambers of $M_A$ to obtain a bound for $b(C_j)$, but the correct factor is only $2^j$.

We work in $\mathbb{RP}^k$ and use the extended hyperplane arrangement $A^+$, as we will need points in the closure of $C_j$ in $\mathbb{RP}^k$. The first inequality in (7) for $b(\tilde{C}_j)$ arises as each
unbounded component of \( \tilde{C}_j \) meets \( \mathcal{A}^+ \) in two distinct points (this accounts for the factor \( \frac{1}{2} \)) which are points of codimension \( j \) faces where the polynomials

\[
F_i(y) := \Gamma_k(y) \cdot \left( \prod_{i=1}^{n+k} p_i(y) \right)^{2^i}
\]

for \( i = 0, \ldots, k - j - 1 \) vanish. (By Lemma 3.4(1) of [7], \( F_i \) is a polynomial of degree \( 2^i n \).)

The genericity of the weights and the linear polynomials \( p_i(y) \) imply that these points will lie on faces of codimension \( j \) but not of higher codimension. The factor \( 2^{(k-j)j} n^{k-j} \) is the Bézout number of the system \( F_0 = \cdots = F_{k-j-1} \) on a given codimension \( j \) plane, and there are exactly \( \binom{n+k+1}{j} \) codimension \( j \) faces of \( \mathcal{A}^+ \).

At each of these points, \( C_j \) will have one branch in each chamber of \( M_\mathcal{A} \) incident on that point. Since the hyperplane arrangement \( \mathcal{A}^+ \) is general there will be exactly \( 2^j \) such chambers. \( \square \)

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