GROWTH CONDITIONS AND REGULARITY, AN OPTIMAL LOCAL BOUNDEDNESS RESULT

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Abstract. We prove local boundedness of local minimizers of scalar integral functionals
\[ \int_{\Omega} f(x, \nabla u(x)) \, dx, \]
where the integrand satisfies \((p, q)\)-growth of the form
\[ |z|^p \lesssim f(x, z) \lesssim |z|^q + 1 \]
under the optimal relation \( \frac{1}{p} - \frac{1}{q} \leq \frac{1}{n-1} \).

1. Introduction and main result

In this note, we establish a sharp local boundedness result for local minimizers of integral functionals
\[ F[u, \Omega] := \int_{\Omega} f(x, \nabla u) \, dx, \]
where \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \), is a bounded domain and the integrand \( f(x, \nabla u) \) satisfies \((p, q)\)-growth of the form
\[ |z|^p \lesssim f(x, z) \lesssim |z|^q + 1, \]
see Assumption 1 below. Local boundedness and Hölder continuity of local minimizers of (1) in the case
\( 1 < p = q \) are classical, see the original reference [18] or the textbook [20]. Giaquinta [16] provided
an example of an autonomous convex integrand satisfying (2) with \( p = 2 \) and \( q = 4 \) that admits
unbounded minimizer in dimension \( n \geq 6 \). Similar examples can be found in [25, 21], in particular it
follows from [25, Section 6] that if
\[ q > \frac{(n-1)p}{n-1-p} =: p_n^* \quad \text{and} \quad 1 < p < n-1, \]
then one cannot expect local boundedness for minimizers of (1) in general. In this paper we show that
condition (3) is sharp. Before we state our main result, we recall a standard notion of local minimizer
and quasi-minimizer in the context of functionals with \((p, q)\)-growth

Definition 1. Given \( Q \geq 1 \), we call \( u \in W^{1,1}_{\text{loc}}(\Omega) \) a \( Q \)-minimizer of (1) if and only if
\[ F(u, \text{supp} \varphi) < \infty \quad \text{and} \quad F(u, \text{supp} \varphi) \leq Q F(u + \varphi, \text{supp} \varphi) \]
for any \( \varphi \in W^{1,1}(\Omega) \) satisfying \( \text{supp} \varphi \subset \Omega \). If \( Q = 1 \), then \( u \) is a local minimizer of (1). Moreover,
we call \( u \) a quasi-minimizer if and only if there exists \( Q \geq 1 \) such that \( u \) is a \( Q \)-minimizer.

Assumption 1. Let \( f : \Omega \times \mathbb{R}^n \to \mathbb{R} \) be a Carathéodory function and suppose that \( z \mapsto f(x, z) \) is
convex for almost every \( x \in \Omega \). Moreover, there exist \( 1 \leq L < \infty \), \( \mu \geq 0 \) such that for all \( z \in \mathbb{R}^n \) and
almost every \( x \in \Omega \)
\[ |z|^p \leq f(x, z) \leq L |z|^q + 1, \]
\[ f(x, 2z) \leq \mu + L f(x, z), \]
Now we are in position to state the main result of the present paper
**Theorem 1.** Let \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \) and suppose Assumption \( \mathcal{A} \) is satisfied with \( 1 \leq p \leq q < \infty \) such that
\[
\frac{1}{q} \geq \frac{1}{p} - \frac{1}{n-1}.
\]
Let \( u \in W^{1,1}_{\text{loc}}(\Omega) \) be a quasi-minimizer of the functional \( \mathcal{F} \) given in \( \mathcal{A} \). Then, \( u \in L^\infty_{\text{loc}}(\Omega) \).

**Remark 1.** We provide the proof of Theorem 1 in Section 3. We establish a slightly more general results in which the growth condition \( \mathcal{A} \) is replaced by
\[
|z|^p - g(x) \frac{1}{r-1} \leq f(x, z) \leq L(|z|^q + g(x) \frac{1}{r-1}),
\]
and optimal assumptions (in the Lorentz-scale) on \( g \) are imposed.

Let us now relate Theorem 1 to previous results in the literature. To the best of our knowledge, the best previously known relation between \( p \) and \( q \) that ensures local boundedness under Assumption \( \mathcal{A} \) can be found in the paper by Fusco & Sbordone [14] and reads
\[
\frac{1}{q} \geq \frac{1}{p} - \frac{1}{n},
\]
see [14] Theorem 2 (see also the more recent result [11, Theorem 2.3]) which also implies local boundedness with condition (7). Obviously, relation (6) is less restrictive than (7) and in view of the discussion above optimal for local boundedness (compare (6) and (3)). However, we want to emphasize that [11, 14] (and similarly [6, 15]) contain sharp local boundedness results under additional structural assumptions on the growth of \( f \), namely anisotropic growth of the form
\[
\sum_{i=1}^{n} |z_i|^{p_i} \lesssim f(x, z) \lesssim \sum_{i=1}^{n} M (1 + |z_i|^{p_i}).
\]
In this case, local boundedness is proven under the condition \( q \leq p^* \), where \( \frac{1}{p^*} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i} \) and \( p^* = \frac{n}{n-1} \). This condition is optimal for local boundedness in view of the above mentioned counterexamples (the integrands in [16, 25, 21] satisfy growth of the form (8)).

The systematic study of higher regularity of minimizers of functionals with \((p, q)\)-growth was initiated by Marcellini [24, 25]. By now there is a large and quickly growing literature on regularity results for minimizers of functionals with \((p, q)\)-growth, and more general non-standard growth [23, 26]. We refer to [27] for an overview. A currently quite active field of research is the regularity theory for so-called double phase problems where the model functional is given by
\[
\int_{\Omega} |\nabla u(x)|^p + a(x)|\nabla u(x)|^q \, dx,
\]
where \( 0 \leq a \in C^{0,\alpha} \) with \( \alpha \in (0, 1] \) see e.g. [11, 9, 10, 12, 13] and [31, 22] for some motivation for functionals of the form (9). For this kind of functionals rather sharp conditions for higher \((C^{1,\beta})\) regularity are known, where \( \alpha \) has to be balanced with \( p, q, \) and \( n \). In [10] it was observed that the conditions on the data can be relaxed if one a priori knows that the minimizer is bounded. Obviously by Theorem 1 the results of [10] can be applied without any a priori assumption whenever \( \frac{1}{q} \geq \frac{1}{p} - \frac{1}{n} \) and in particular can be used to improve [10, Theorem 5.3]. Similarly, Theorem 1 improves the applicability of some results in [7, 8] were also higher regularity results are proven assuming a priori boundedness of the minimizer.

Let us very briefly explain the strategy of the proof of Theorem 1 and the origin of our improvement. In principle, we use a variation of the De-Giorgi type iteration similar to e.g. [14, 15, 11]. Recall that De-Giorgi iteration is based on a Caccioppoli inequality (which yield a reverse Poincaré inequality) and Sobolev inequality. The main new ingredient here is to use in the Caccioppoli inequality cut-off functions that are optimized with respect to the minimizer \( u \) (instead of using affine cut-offs). This enables us to use Sobolev inequality on \((n-1)\)-dimensional spheres instead of \( n \) dimensional balls and thus get the desired improvement. This idea, combined with a variation of Moser-iteration, was recently used by the second author and Bella in the analysis of linear non-uniformly elliptic equations [3]
Lemma 1. which is a variation of [5, Lemma 3]

2.1. Preliminary lemmata. A key ingredient in the proof of Theorem 1 is the following lemma which is a variation of [5, Lemma 3]

Lemma 1. Fix $n \geq 2$. For given $0 < \rho < \sigma < \infty$, $v \in L^1(B_\sigma)$ and $s > 1$, we consider

$$J(\rho, \sigma, v) := \inf \left\{ \int_{B_\rho} |v| |\nabla \eta|^s \, dx \mid \eta \in C_0^1(B_\sigma), \eta \geq 0, \eta = 1 \text{ in } B_\rho \right\}.$$  

Then for every $\delta \in (0, 1]$  

$$J(\rho, \sigma, v) \leq (\sigma - \rho)^{-(s-1+\frac{1}{s})} \left( \int_\rho^\sigma \left( \int_{S_r} |v| \right)^{\delta} \, dr \right)^{\frac{1}{\delta}}.$$  

Proof of Lemma 1. Estimate (10) follows directly by minimizing among radial symmetric cut-off functions. Indeed, we obviously have for every $\varepsilon \geq 0$

$$J(\rho, \sigma, v) \leq \inf \left\{ \int_\rho^\sigma |\eta'(r)| s \int_{S_r} |v| + \varepsilon \, dr \mid \eta \in C^1(\rho, \sigma), \eta(\rho) = 1, \eta(\sigma) = 0 \right\} =: J_{\text{id}, \varepsilon}.$$  

For $\varepsilon > 0$, the one-dimensional minimization problem $J_{\text{id}, \varepsilon}$ can be solved explicitly and we obtain  

$$J_{\text{id}, \varepsilon} = \left( \int_\rho^\sigma \left( \int_{S_r} |v| + \varepsilon \right)^{\frac{1}{s-1}} \, dr \right)^{-(s-1)}.$$  

Let us give an argument for (11). First we observe that using the assumption $v \in L^1(B_\sigma)$ and a simple approximation argument we can replace $\eta \in C^1(\rho, \sigma)$ with $\eta \in W^{1, \infty}(\rho, \sigma)$ in the definition of $J_{\text{id}, \varepsilon}$. Let $\tilde{\eta} : [\rho, \sigma] \to [0, \infty)$ be given by

$$\tilde{\eta}(r) := 1 - \left( \int_\rho^\sigma b(r)^{-\frac{s-1}{s}} \, dr \right)^{-1} \int_\rho^r b(r)^{-\frac{s-1}{s}} \, dr,$$

where $b(r) := \int_{S_r} |v| + \varepsilon$.

Clearly, $\tilde{\eta} \in W^{1, \infty}(\rho, \sigma)$ (since $b \geq \varepsilon > 0$), $\tilde{\eta}(\rho) = 1$, $\tilde{\eta}(\sigma) = 0$, and thus

$$J_{\text{id}, \varepsilon} \leq \int_\rho^\sigma |\tilde{\eta}'(r)| s b(r) \, dr = \left( \int_\rho^\sigma b(r)^{-\frac{s-1}{s}} \, dr \right)^{-(s-1)}.$$  

The reverse inequality follows by Hölder’s inequality: For every $\eta \in W^{1, \infty}(\rho, \sigma)$ satisfying $\eta(\rho) = 1$ and $\eta(\sigma) = 0$, we have

$$1 = \left( \int_\rho^\sigma \eta'(r) \, dr \right)^s \leq \int_\rho^\sigma |\eta'(r)| s b(r) \, dr \left( \int_\rho^\sigma b(r)^{-\frac{s-1}{s}} \, dr \right)^{s-1}.$$  

Clearly, the last two displayed formulas imply (11).

Due to the monotonicity of $(-\infty, \infty) \ni m \mapsto (\int_\rho^\sigma v^m(r) \, dr)^{\frac{1}{m}}$, we deduce from (11) for every $\delta > 0$

$$J_{\text{id}, \varepsilon} \leq (\sigma - \rho)^{-(s-1+\frac{1}{s})} \left( \int_\rho^\sigma \left( \int_{S_r} |v| + \varepsilon \right)^{\delta} \, dr \right)^{\frac{1}{\delta}}.$$  

Sending $\varepsilon$ to zero, we obtain (10). \qed

In order to derive a suitable Caccioppoli type inequality in the proof of Theorem 1, we make use of the so-called 'hole-filling' trick combined with the following useful (and well-kown) lemma.
Lemma 2 (Lemma 6.1, [29]). Let $Z(t)$ be a bounded non-negative function in the interval $[\rho, \sigma]$. Assume that for every $p \leq s < t \leq \sigma$ it holds

$$Z(s) \leq \theta Z(t) + (t - s)^{-\alpha} A + B,$$

with $A, B \geq 0$, $\alpha > 0$ and $\theta \in [0, 1)$. Then, there exists $c = c(\alpha, \theta) \in [1, \infty)$ such that

$$Z(s) \leq c((t - s)^{-\alpha} A + B).$$

2.2. Non-increasing rearrangement and Lorentz-spaces. We recall the definition and useful properties of the non-increasing rearrangement $f^*$ of a measurable function $f$ and Lorentz spaces, see e.g. [29] Section 22. For a measurable function $f : \mathbb{R}^n \to \mathbb{R}$, the non-increasing rearrangement is defined by

$$f^*(t) := \inf\{\sigma \in (0, \infty) : |\{x \in \mathbb{R}^n : |f(x)| > \sigma\}| \leq t\}.$$ 

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function with $\text{supp} f \subset \Omega$, then it holds for all $p \in [1, \infty)$

$$\int_{\Omega} |f(x)|^p \, dx = \int_{0}^{[\Omega]} (f^*(t))^p \, dt.$$ 

A simple consequence of (12) and the fact $f \leq g$ implies $f^* \leq g^*$ is the following inequality

$$\sup_{|A| \leq t} \int_{A} |f(x)|^p \leq \int_{0}^{t} (f^*_{\Omega}(t))^p \, dt,$$

where $f^*_{\Omega}$ denotes the non-increasing rearrangement of $f\chi_{\Omega}$ (inequality (13) is in fact an equality but for our purpose the upper bound suffices).

The Lorentz space $L^{n,1}(\mathbb{R}^d)$ can be defined as the space of measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ satisfying

$$\|f\|_{L^{n,1}(\mathbb{R}^d)} := \int_{0}^{\infty} t^{\frac{n}{d}} f^*(t) \frac{dt}{t} < \infty.$$ 

Moreover, for $\Omega \subset \mathbb{R}^d$ and a measurable function $f : \mathbb{R}^d \to \mathbb{R}$, we set

$$\|f\|_{L^{n,1}(\Omega)} := \int_{0}^{[\Omega]} t^{\frac{n}{d}} f^*_{\Omega}(t) \frac{dt}{t} < \infty,$$

where $f^*_{\Omega}$ defined as above. Let us recall that $L^{n+\varepsilon}(\Omega) \subset L^{n-1}(\Omega) \subset L^{n}(\Omega)$ for every $\varepsilon > 0$, where $L^{n-1}(\Omega)$ is the space of all measurable functions $f : \Omega \to \mathbb{R}$ satisfying $\|f\|_{L^{n-1}(\Omega)} < \infty$ (here we identify $f$ with its extension by zero to $\mathbb{R}^n \setminus \Omega$). Following [29] Section 9], we define for given $\alpha > 0$ the Lorentz-Zygmund space $L^{n,1}(\log L)^{\alpha}(\mathbb{R}^d)$ as the space of all measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ satisfying

$$\|f\|_{L^{n,1}(\log L)^{\alpha}(\mathbb{R}^d)} := \int_{0}^{\infty} t^{\frac{n}{d}} (1 + |\log(t)|)^{\alpha} f^*(t) \frac{dt}{t} < \infty.$$ 

As above, for $\Omega \subset \mathbb{R}^d$ and a measurable function $f : \mathbb{R}^d \to \mathbb{R}$, we set

$$\|f\|_{L^{n,1}(\log L)^{\alpha}(\Omega)} := \int_{0}^{[\Omega]} t^{\frac{n}{d}} (1 + |\log(t)|)^{\alpha} f^*_{\Omega}(t) \frac{dt}{t} < \infty,$$

and denote by $L^{n,1}(\log L)^{\alpha}(\Omega)$ the space of all measurable functions $f : \Omega \to \mathbb{R}$ satisfying $\|f\|_{L^{n,1}(\log L)^{\alpha}(\Omega)} < \infty$. Obviously, we have for every bounded domain $\Omega$ that $L^{n+\varepsilon}(\Omega) \subset L^{n,1}(\log L)^{\alpha}(\Omega) \subset L^{n,1}(\Omega)$ for every $\varepsilon > 0$.

3. Proof of Theorem 1

In this section, we provide a proof of Theorem 1. As mentioned in the introduction, we establish a slightly stronger statement where the growth condition (11) is relaxed in order to introduce a right-hand side (see Remark 2 below).
Assumption 2. Let \( f : \Omega \times \mathbb{R}^n \to \mathbb{R} \) be a Carathéodory function and suppose that \( z \mapsto f(x,z) \) is convex for almost every \( x \in \Omega \). Moreover, there exist \( 1 \leq L < \infty, \mu \geq 0 \) such that for all \( z \in \mathbb{R}^n \) and almost every \( x \in \Omega \)
\[
(14) \quad |z|^p - g(x) \frac{|z|^p}{p} \leq f(x,z) \leq L|z|^p + g(x) \frac{|z|^p}{p},
\]
\[
(15) \quad f(x,2z) \leq \mu + Lf(x,z),
\]
where \( g \) is a non-negative function satisfying \( g \in L^{\frac{p}{p-1}}(\Omega) \).

In order to state an a priori estimate it is convenient to introduce suitable scale invariant versions of Sobolev and \( L^p \) norms. For any bounded domain \( \Omega \subset \mathbb{R}^n \), we set
\[
\|v\|_{W^{1,p}(\Omega)} := |\Omega|^{-\frac{1}{p}} \|v\|_{L^p(\Omega)} + \|\nabla v\|_{L^p(\Omega)},
\]
where
\[
\|v\|_{L^p(\Omega)} := |\Omega|^{-\frac{1}{p}} \|v\|_{L^p(\Omega)}.
\]

Note that by definition of \( \| \cdot \|_{W^{1,p}(\Omega)} \), it holds
\[
(16) \quad \forall v \in W^{1,p}(B_R), R > 0 : \quad \|v\|_{W^{1,p}(B_R)} = \|v_R\|_{W^{1,p}(B_1)} \quad \text{where} \quad v_R := \frac{1}{R}v(R) \in W^{1,p}(B_1).
\]

Theorem 2. Let \( \Omega \subset \mathbb{R}^n, n \geq 2 \) and suppose Assumption 3 is satisfied with \( 1 < p < q < \infty \) satisfying
\[
(17) \quad \varepsilon := \varepsilon(n,p,q) := \min \left\{ \frac{1}{q} + \frac{1}{n-1}, 1 \right\} - \frac{1}{p} > 0,
\]
and suppose that
\[
g^{\frac{1}{p-1}} \in L^{n,1}(\Omega) \quad \text{if} \quad p < n \quad \text{and} \quad g^{\frac{1}{p-1}} \in L^{n,1}(\log L)^{\frac{n-1}{p}}(\Omega) \quad \text{if} \quad p = n.
\]

Let \( u \in W^{1,1}_{\text{loc}}(\Omega) \) be a quasi-minimizer of the functional \( \mathcal{F} \) given in (1). Then, \( u \in L^{\infty}_{\text{loc}}(\Omega) \). Moreover, if
\[
(18) \quad \varepsilon(n,p,q) > 0 \quad \text{and} \quad 1 < p < n,
\]
there exists \( c = c(L,n,p,q,Q) \in [1,\infty) \) such that every \( Q \)-minimizer of (1) satisfies for every \( x_0 \in \Omega \) with \( B_R := B_R(x_0) \subset \Omega \) the estimate
\[
(19) \quad \|u\|_{L^{\infty}(B_{R\frac{1}{2}})} \leq c\|u\|_{W^{1,p}(B_{R})} + R\|u\|_{W^{1,q}(B_{R})}^{1+\frac{1}{q}(1-\frac{1}{p})} + \|g^{\frac{1}{p-1}}\|_{L^{n,1}(B_{R})}.
\]

Remark 2. As mentioned above, Theorem 2 is optimal with respect to the relation between the exponents \( p \) and \( q \). Moreover, it is also optimal with respect to the assumption on \( g \) (at least for \( p < n \)). Indeed, for \( p > 1 \) consider
\[
(20) \quad f(x,z) := \frac{1}{p-1} |z|^p + G \cdot z,
\]
where \( G \in L^{\frac{p}{p-1}}(\Omega,\mathbb{R}^n) \). Clearly \( f \) satisfies Assumption 4 with \( 1 < p = q, \quad g = \frac{1}{p-1}|G| \) and \( L = \frac{p+2}{p} \).

Note that \( u \in W^{1,1}_{\text{loc}}(\Omega) \) is a local minimizer of the functional \( \mathcal{F} \) given in (1) and \( f \) given as in (20) if and only if it solves locally
\[
(21) \quad -\Delta_p u := -\text{div}(|\nabla u|^{p-2} \nabla u) = \frac{1}{p+1} \text{div} G.
\]

Hence, Theorem 2 yields local boundedness for every weak solution of (21) provided \( |G|^{\frac{1}{p-1}} \in L^{n,1}(\Omega) \). On the contrary, the (unbounded) function \( u(x) = \log(-\log(|x|)) \) solves (trivially) (21) on \( B_{\frac{1}{2}} \) with right-hand side \( G = -(p+1)|\nabla u|^{p-2} \nabla u \) satisfying \( |G|^{\frac{1}{p-1}} = c(p) |\nabla u| \) and thus \( |G|^{\frac{1}{p-1}} \in L^{n,1}(B_{\frac{1}{2}}) \) for every \( \delta > 0 \) (in particular \( |G|^{\frac{1}{p-1}} \in L^{n,n}(B_{1}) = L^{n}(B_{1}) \)) but \( |G|^{\frac{1}{p-1}} \notin L^{n,1}(B_{\frac{1}{2}}) \).

In the interesting recent paper [2], a related result is proven on the Lipschitz-scale. More precisely, it is proven that local minimizer of \( \int_{\Omega} f(|\nabla u|) - gu \, dx \) are locally Lipschitz if \( f \) satisfies (controlled) \( (p,q) \)-growth i.e.
\[
(22) \quad (1 + |z|^2)^{\frac{p-2}{2}} |\lambda|^2 \lesssim \langle D^2 f(z)\lambda, \lambda \rangle \lesssim (1 + |z|^2)^{\frac{p-2}{2}} |\lambda|^2
\]
with $\frac{2}{p} < 1 + \frac{2}{n}$ and $g$ is in the optimal Lorentz space $L^{n,1}(\Omega)$ (provided $n \geq 3$). Very recently, Lipschitz-regularity of minimizers for integrands satisfying (22) is proven in [5] under the less restrictive relation $\frac{2}{p} < 1 + \frac{2}{n-1}$ in the case $g \equiv 0$. It would interesting if the methods of [5] and [2] can be combined to obtain Lipschitz estimates under the assumption $\frac{2}{p} < 1 + \frac{2}{n-1}$ and $g \in L^{n,1}$ provided $n \geq 3$.

Proof of Theorem [4] By standard scaling and translation arguments it suffices to suppose that $B_1 \Subset \Omega$ and prove that $u$ is locally bounded in $B_\frac{1}{2}$. Hence, we suppose from now on that $B_1 \Subset \Omega$. In Step 1–Step 3 below, we consider the case $p \in [1, n)$. We first derive a suitable Caccioppoli-type inequality (Step 1) and perform a De Giorgi-type iteration (Step 2 and 3) to prove boundedness from above for a $Q$-minimiser. In Step 4, we discuss how this implies the claim of the theorem in the case $p \in [1, n)$. In Step 5, we sketch the adjustments for the remaining non-trivial case $p = \infty$.

Step 1. Basic energy estimate.

We claim that there exists $c = c(n, p, q, Q) \in [1, \infty)$ such that for every $k \geq 0$ and every $\frac{1}{q} \leq \rho < \sigma \leq 1$ it holds

$$ (23) \qquad \|\nabla (u - k)\|_{L^p(B_\rho)}^p \leq c\left( \omega(|A_{k,\sigma}|) + \frac{L |A_{k,\sigma}|}{(\sigma - \rho)^\gamma} \|\nabla (u - k)\|_{W^{1,p}(B_\sigma)}^q \right), $$

where $\gamma = \gamma(q) := q - 1 + q \min\left\{ \frac{1}{q} + \frac{1}{n-1}, 1 \right\}$, $\epsilon$ as in (17),

$$ (24) \quad A_{l,r} := B_r \cap \{ x \in \Omega : u(x) > l \} \quad \text{for all } r > 0 \text{ and } l > 0, $$

and $\omega : [0, |B_1|] \rightarrow [0, \infty)$ is defined by

$$ (25) \quad \omega(t) := \int_0^t ((g \frac{1}{r^{p-1}} \chi_{B_1})^+(t))^p \, dt. $$

Fix $M > k$ and let $\eta \in C^1(B_1, [0, 1])$ be such that $\eta = 1$ in $B_\rho$ and supp $\eta \subset B_\sigma$. Define $w := \max\{u_M - k, 0\}$ where $u_M := \min\{u, M\}$ and set $\varphi := -\eta^q w$. Since $u$ is a quasi-minimiser, we obtain with help of convexity of $f$ that

$$ \int_{A_{k,\sigma}} f(x, \nabla u(x)) \, dx \leq \int_{A_{k,\sigma}} f(x, \nabla (u + \varphi)(x)) \, dx $$

$$ = \int_{A_{k,\sigma} \cap \{u \leq M\}} f(x, (1 - \eta^q)\nabla u - q\eta^q - \nabla \eta(u_M - k)_+ ) \, dx $$

$$ + \int_{A_{k,\sigma} \cap \{u > M\}} f(x, \nabla u + q\eta^q - \nabla \eta(-u_M - k)_+ ) \, dx $$

$$ \leq \int_{A_{k,\sigma}} (1 - \eta^q) f(x, \nabla u) + \eta^q f(x, -q\nabla \eta(u_M - k)_+ ) \, dx $$

$$ + \frac{Q}{2} \int_{A_{k,\sigma} \cap \{u > M\}} f(x, 2\nabla u) + f(x, -2q\eta^q - \nabla \eta(u_M - k)_+ ) \, dx $$

and thus, using (14), (15) and $|\eta| \leq 1$,

$$ \int_{A_{k,\sigma}} f(x, \nabla u(x)) \, dx \leq \int_{A_{k,\sigma} \setminus B_\rho} f(x, \nabla u) \, dx + \frac{Q}{2} \int_{A_{k,\sigma} \cap \{u > M\}} \mu + Lf(x, \nabla u) \, dx $$

$$ + Q \int_{A_{k,\sigma}} g \frac{1}{r^{p-1}} + L\eta^q (1 + 2q)|\nabla q^q| (u_M - k)_+ |q^q| \, dx. $$

We claim that there exists $c = c(n, q) \in [1, \infty)$ such that

$$ (27) \quad \inf_{\eta \in A(\rho, \sigma)} \int_{A_{k,\sigma}} |\nabla \eta|^q (u_M - k)_+ |q^q| \leq c(\sigma - \rho)^{-\gamma} \|(u - k)_+ + |q^q|_{W^{1,p}(B_\sigma \setminus B_\rho)} |A_{k,\sigma}| q^q, $$

and thus, using (14), (15) and $|\eta| \leq 1$,
where $A(\rho, \sigma) := \{ \eta \in C^1(B_r), \eta \equiv 1 \text{ on } B_r \}$. To show (27), we use the Sobolev inequality on spheres, i.e. there exists $c = c(n, q) \in [1, \infty)$ such that for every $r > 0$

\[
(28) \quad \left( \int_{S_r} |\varphi|^q \right)^{\frac{1}{q}} \leq c \left( \left( \int_{S_r} |\nabla \varphi|^{q^*} \right)^{\frac{1}{q^*}} + \frac{1}{r} \left( \int_{S_r} |\varphi|^{q^*} \right)^{\frac{1}{q^*}} \right),
\]

where $q_\ast \geq 1$ is given by $\frac{1}{q_\ast} = \min\{\frac{1}{q} + \frac{1}{n-1}, 1\}$. Combining (28) applied to $\varphi = (u-k)_+$ and Lemma 1 with $\delta := \frac{4}{q_\ast} > 0$ yield

\[
\begin{align*}
\inf_{\eta \in A(\rho, \sigma)} \int_{A_{k, r}} |\nabla \eta|^q (u_M - k)_+^q &\leq (\sigma - \rho)^{-q} \left( \int_{\rho}^\sigma \left( \int_{S_r} |(u-k)_+^q \right) \right) \frac{d\sigma}{\sigma} \\
&\leq c(\sigma - \rho)^{-q} \left( \int_{\rho}^\sigma \left( \int_{S_r} |(u-k)_+^q \right) + \left( \int_{S_r} |(u-k)_+^q \right) \right) \frac{d\sigma}{\sigma},
\end{align*}
\]

(note that we ignored the factor $\frac{1}{r}$ in (28) in view of $\frac{1}{r} \leq \rho < \sigma \leq 1$). Finally, we observe that $\varepsilon \equiv 0$ implies that $q_\ast \leq \rho$ and we obtain with help of Hölder inequality

\[
\begin{align*}
\inf_{\eta \in A(\rho, \sigma)} \int_{A_{k, r}} |\nabla \eta|^q (u_M - k)_+^q &\leq c(\sigma - \rho)^{-q} \left( \int_{\rho}^\sigma \left( \int_{S_r} |(u-k)_+^q \right) \right) \frac{d\sigma}{\sigma} \\
&\leq c(\sigma - \rho)^{-q} \left( \int_{\rho}^\sigma \left( \int_{S_r} |(u-k)_+^q \right) \right) \frac{d\sigma}{\sigma},
\end{align*}
\]

and (27) is proven.

Since (28) is valid for all $\eta \in A(\rho, \sigma)$, we deduce from (27), (13) and $f(x, z) \geq -g(x)\frac{\mu}{\mu^*}$

\[
\int_{A_{k, r}} f(x, \nabla u) dx \leq Q \int_{A_{k, r}} f(x, \nabla u) dx + \frac{Q}{2} \int_{A_{k, r} \cap \{u > M\}} \mu + Lf(x, \nabla u) dx
\]

(29)

\[
\int_{A_{k, r}} f(x, \nabla u(x)) dx \leq \theta \int_{A_{k, r}} f(x, \nabla u(x)) dx + c \left( \omega(\{A_{k, r}\}) + \frac{cLQ}{(\sigma - \rho)^\gamma} \right) \left( \int_{A_{k, r} \cap \{u > M\}} \mu + Lf(x, \nabla u) dx \right),
\]

with $\theta = \frac{Q}{Q+1} \in [0, 1)$ and $c = c(n, p, q) \in [1, \infty)$. Estimate (29) follows by Lemma 1 and (14).

**Step 2.** One-step improvement.

We claim that there exist $c_1 = c_1(n, p, q, Q) \in [1, \infty)$ and $c_2 = c_2(n, p) \in [1, \infty)$ such that for every $k > h \geq 0$ and every $\frac{1}{2} \leq \rho < \sigma < 1$ it holds

\[
(30) \quad J(k, \rho) \leq c_1 \left( \omega \left( \frac{c_2 J(h, \sigma)^{\frac{\mu}{\mu}}}{(k-h)^{p_\ast}} \right) + L \left( \frac{J(h, \sigma)^{\frac{\mu}{\mu}}}{(k-h)^{p_\ast}} \right) \right) \left( \frac{p_\ast}{(k-h)^{p_\ast}} \right) + \left( \frac{J(h, \sigma)^{\frac{\mu}{\mu}}}{(k-h)^{p_\ast}} \right) \left( \frac{p_\ast}{(k-h)^{p_\ast}} \right),
\]

where $p_\ast := \frac{pn}{n-p}$ and for any $l \geq 0$ and $r > 0$

\[
J(l, r) := \|u - l\|_{L^{p_\ast}(B_r)}.
\]

Note that $k-h < u-h$ on $A_{k,r}$ for every $r > 0$ and thus with help of Sobolev inequality

\[
\begin{align*}
|A_{k, r}| \leq &\int_{A_{k, r}} \left( \frac{u(x) - h}{k-h} \right)^{p_\ast} \leq \frac{\|u - h\|_{L^{p_\ast}(B_r)}}{(k-h)^{p_\ast}} \leq c \frac{J(h, \sigma)^{\frac{\mu}{\mu}}}{(k-h)^{p_\ast}},
\end{align*}
\]
where \( c = c(n, p) \in [1, \infty) \). Combining the above estimate with (23), we obtain

\[
\|\nabla (u - k)\|_{L^p(B_{\rho})} \leq c_1 \left( \omega \left( \frac{c_2 J(h, \sigma)^{1/p}}{(k-h)^{p_n}} \right) + L \left( \frac{J(h, \sigma)^{1/p}}{(k-h)^{p_n}} \right) \right)^{\nu_{p,q}^\tau} J(h, \sigma)^{\tilde{\nu}_{p,q}^\tau},
\]

where \( c_1 = c_1(n, p, q, Q) \in [1, \infty) \) and \( c_2 = c_2(n, p) \in [1, \infty) \). It is left to estimate \( \| (u - k)^+ \|_{L^p(B_{\rho})} \). A combination of Hölder inequality, Sobolev inequality and estimate (31) yield

\[
\| (u - k)^+ \|_{L^p(B_{\rho})} \leq \| (u - h)^+ \|_{L^p(B_{\rho})} \| A_k \|_{L^p(B_{\rho})} \leq c \left( \frac{J(h, \sigma)^{1/p}}{(k-h)^{p_n}} \right)^{\nu_{p,q}^\tau} J(h, \sigma)^{\tilde{\nu}_{p,q}^\tau},
\]

Combining (32) and (33), we obtain (30).

**Step 3.** Iteration.

For \( k_0 \geq 0 \) and a sequence \( (\Delta_\ell)_{\ell \in \mathbb{N}} \subset [0, \infty) \) specified below, we set

\[
k_\ell := k_0 + \Delta_\ell, \quad \sigma_\ell = \frac{1}{2} + \frac{1}{2^{\ell+1}}.
\]

For every \( \ell \in \mathbb{N} \cup \{0\} \), we set \( J_\ell := J(k_\ell, \sigma_\ell) \). From (30), we deduce for every \( \ell \in \mathbb{N} \)

\[
J_\ell \leq c_1 \left( \omega \left( \frac{c_2 J_{\ell-1}^{(1/q)}(k-h)^{p_n}}{\Delta_\ell^{p_n}} \right) + L 2^{(\ell+1)\gamma} \left( \frac{J_{\ell-1}^{1/q}}{\Delta_\ell} \right)^{\nu_{p,q}^\tau} J_{\ell-1}^{\nu_{p,q}^\tau} + \left( \frac{J_{\ell-1}^{1/q}}{\Delta_\ell} \right)^{\tilde{\nu}_{p,q}^\tau} J_{\ell-1}^{\tilde{\nu}_{p,q}^\tau} \right),
\]

where \( c_1 \) and \( c_2 \) are as in Step 2. Fix \( \tau = \tau(n, p, q) \in (0, 1) \) such that

\[
2^{\gamma \tau^{p}(1+p_n)\epsilon} = \frac{1}{2}.
\]

We claim that we can choose \( \{\Delta_\ell\}_{\ell \in \mathbb{N}} \) satisfying

\[
\sum_{\ell \in \mathbb{N}} \Delta_\ell < \infty
\]

and \( k_0 \) (in the borderline case \( \epsilon = 0 \)) in such a way that

\[
J_\ell \leq \tau^{\ell} J_0 \quad \text{for all } \ell \in \mathbb{N} \cup \{0\}.
\]

**Substep 3.1.** Suppose that \( \epsilon > 0 \). Set \( k_0 = 0 \) and choose \( \Delta_\ell \) to be the smallest number such that

\[
c_1 \omega \left( \frac{c_2 (\tau^{\ell-1} J_0)^{1/q}}{(\Delta_\ell)^{p_n}} \right)^{\frac{1}{q}} \leq \frac{1}{3} \tau^{\ell} J_0, \quad c_1 \tau^{-(p_n + 1)} J_0^{p_n + \nu_{p,q}^\tau} \leq \frac{1}{3} \Delta_\ell^{\nu_{p,q}^\tau}
\]

and

\[
c_1 L 2^{\gamma \tau^{p}(1+p_n)\epsilon} J_0^{\frac{1}{\tau^{p}(1+p_n)\epsilon}} 2^{-\ell} \leq \frac{1}{3} \Delta_\ell^{\nu_{p,q}^\tau}
\]

is valid. The choice of \( \tau \) (see (36)), \( \Delta_\ell \) and estimate (35) combined with a straightforward induction argument yield (38). Using \( \sum_{\ell \in \mathbb{N}} (2^{-\alpha + \tau^{\beta}}) < \infty \) for any \( \alpha, \beta > 0 \), we deduce from (39) and (40)

\[
\sum_{\ell \in \mathbb{N}} \Delta_\ell \leq \sum_{\ell \in \mathbb{N}} \frac{c_1^{-1}}{\omega^{-1} \left( \frac{J_0^{1/q}}{\Delta_\ell^{p_n}} \right)^{1/q}} + c (J_0^{\frac{1}{\tau^{p}(1+p_n)\epsilon}} + J_0^{\frac{1}{\tau^{p}(1+p_n)\epsilon}} + \frac{1}{2^{\ell+1}}).
\]
where \( c = c(L, n, p, q, Q) \in [1, \infty) \). Next, we show that \( g^{\frac{1}{p}} \in L^{n,1}(B_1) \) ensures that the first term on the right-hand side of (41) is bounded and thus (37) is valid. Indeed,

\[
\sum_{\ell \in \mathbb{N}} \frac{(\tau, J_0)^{\frac{1}{p'}}}{(\omega^{-1}(\frac{\tau - J_0}{3c_1}))^{\frac{p'}{p}}} \leq \int_{1}^{\infty} \frac{(\tau, J_0)^{\frac{1}{p'}}}{(\omega^{-1}(\frac{\tau - J_0}{3c_1}))^{\frac{p'}{p}}} \, dt = \frac{1}{\log \tau} \int_{0}^{\tau} \frac{(t, J_0)^{\frac{1}{p'}}}{(\omega^{-1}(\frac{t - J_0}{3c_1}))^{\frac{p'}{p}}} \, dt \\
\leq (3c_1)^{\frac{1}{p'}} \int_{0}^{\infty} \frac{\omega' \omega}{s^{\frac{1}{p'}} \omega(s)} \, ds.
\]

(42)

Recall \( \omega(t) = \int_{0}^{t} ((g^{\frac{1}{p}} \chi_{B_1})^*(s))^p \, ds \) and \( (g^{\frac{1}{p}} \chi_{B_1})^* \) is non-increasing, thus \( \omega(t) \geq t ((g^{\frac{1}{p}} \chi_{B_1})^*(t))^p \) and

\[
\int_{0}^{\infty} \frac{(\omega(s))^{\frac{1}{p'}} \omega'(s)}{s^{\frac{1}{p'}} \omega(s)} \, ds \leq \int_{0}^{\infty} \frac{s^{\frac{1}{p'}} (s ((g^{\frac{1}{p}} \chi_{B_1})^*(s))^p)^{-1 - \frac{1}{p'}} ((g^{\frac{1}{p}} \chi_{B_1})^*(s))^p \, ds}{s^{\frac{1}{p'}} (g^{\frac{1}{p}} \chi_{B_1})^*(s) ds} = \| g^{\frac{1}{p}} \|_{L^{n,1}(B_1)}.
\]

(43)

Notice that (38) and \( k_0 = 0 \) implies

\[
\|(u - \sum_{\ell \in \mathbb{N}} \Delta \ell)_+\|_{L^p(B_1)} = 0 \Rightarrow \sup_{B_1 \setminus B_2} u \leq \sum_{\ell \in \mathbb{N}} \Delta \ell
\]

and thus

\[
\sup_{B_1 \setminus B_2} u \leq \sum_{\ell \in \mathbb{N}} \Delta \ell.
\]

Hence, appealing to (41)-(43), we find \( c = c(L, n, p, q, Q) \in [1, \infty) \) such that

\[
\sup_{B_1 \setminus B_2} u \leq c(\|(u)_+\|_{W^{1,p}(B_1)} + \|(u)_+\|_{W^{1,p}(B_1)}^{1 + (\frac{1}{p} - \frac{1}{p'}) (1 - \frac{1}{p}) \frac{1}{2}} + \|g\|_{L^{n,1}(B_1)})
\]

(44)

Substep 3.1. Suppose that \( \varepsilon = 0 \). We claim that

\[
\lim_{k_0 \to \infty} J_0 = 0.
\]

Before, we give the argument for (45) we explain how (45) implies the desired claim (38) in the case \( \varepsilon = 0 \). Choose \( \Delta \ell \) to be the smallest number such that (39) is satisfied and choose \( k_0 \) sufficiently large such that

\[
c_1 L^{2 \gamma} \tau^{-\frac{1}{p'}} J_0 \frac{\eta - 1}{\tau} \leq \frac{1}{3}.
\]

It is now easy to see that the choice of \( \tau, \Delta \ell, k_0 \) and estimate (35) yield (38). In view of Substep 3.1 we also have \( \sum_{\ell \in \mathbb{N}} \Delta \ell < \infty \) and we have

\[
\sup_{B_1 \setminus B_2} u \leq k_0 + c(\|(u)_+\|_{W^{1,p}(B_1)} + \|g\|_{L^{n,1}(B_1)}) < \infty,
\]

where \( c = c(L, n, p, q, Q) \in [1, \infty) \).

Let us now show (45). For \( k \geq \frac{2 \tau}{|B_1|^{\frac{1}{p'}} |u|_{L^p(B_1)} } \) we have \( |A_{k,1}| \leq \frac{1}{2} |B_1| \) and thus a suitable version of Poincare inequality (see e.g. [17, Proposition 1.3]) yields

\[
\int_{B_1} |(u - k)_+|^p \, dx \leq c \int_{B_1} |\nabla (u - k)|_p^p \, dx,
\]

(46)
where $c = c(n, p) \in [1, \infty)$. Hence, it suffices to show $\lim_{k \to \infty} \|
abla(u - k)\|_{L^p(B_1)} = 0$. By (14), we have for every $k \geq 0$

$$\int_{B_1} |\nabla(u - k)|^p = \int_{A_{k, \epsilon}} |\nabla u|^p \leq \int_{A_{k, \epsilon}} f(x, \nabla u) + g^{\frac{p}{p-1}}(x) \, dx.$$

Since $B_1 \subseteq \Omega$ and $f(x, \nabla u), g^{\frac{p}{p-1}} \in L^1_{\text{loc}}(\Omega)$, the right-hand side in (16) tends to zero as $k$ tends to infinity and thus (15) is proven.

**Step 4.** Conclusion in the case $p < n$.

In view of Step 1–Step 3, we have that if $B_1 \subseteq \Omega$ than $u$ is locally bounded from above in $B_\frac{1}{2}$ and in the case $\varepsilon > 0$, we have the estimate (16). Moreover, if $u$ is a $Q$-minimizer of $\mathcal{F}$, then $-u$ is a $Q$-minimizer of the functional $\hat{\mathcal{F}}(u) := \int_{\Omega} \hat{f}(x, \nabla v(x)) \, dx$ with $\hat{f}(x, z) := f(x, -z)$. Clearly, $\hat{f}$ is convex in the second component and satisfies the same growth conditions as $f$. Hence, we obtain that $u$ is locally bounded in $B_\frac{1}{2}$. Moreover, if $\varepsilon > 0$ there exists $c = c(L, n, p, q, Q) \in [1, \infty)$ such that

$$\|u\|_{L^\infty(B_\frac{1}{2})} \leq c(\|u\|_{W^{1,p}(B_1)} + \|v\|_{W^{1,p}(B_1)}^{1 + \frac{(\frac{1}{p} - \frac{1}{q})(1 - \frac{1}{p})}{2}} + \|g\|_{L^{n,1}(B_1)}).$$

The conclusion of the theorem in the case $p \in [1, n)$ now follows by standard scaling, translation and covering arguments (here we use (16)).

**Step 5.** The case $p = n$.

We use the same notation as in the previous steps and sketch the necessary adjustments. Note that for $p = n$ we cannot use Sobolev inequality in the form (31). In the parts not involving $\omega$ it suffices to replace $p^*_n$ by any $\tilde{p} \in [q, \infty)$ (recall $q > p = n$) and we leave the details to the reader. Using this replacement for the estimates related to $\omega$, we obtain local boundedness under slightly stronger assumptions on $g$, namely $g^{\frac{1}{p-1}} \in L^{n+\delta}(\Omega)$ for some $\delta > 0$ (in fact this statement is already contained in [19]). Thus we may appeal to the Moser-Trudinger inequality, which gives for some dimensional constant $c > 0, 0 \leq h < k, \frac{1}{2} < \sigma < 1$

$$\|A_{k, \sigma} \| \leq c \exp\left( -\frac{1}{c} \left( k - \frac{h}{J(h, \sigma)^{\frac{1}{p}}(n)} \right)^{\frac{1}{p-1}} \right).$$

Let us first conclude and present the the derivation of the above inequality below.

In view of Step 3 and (17) it suffices to show that the sequence $\{\Delta_t\}_{t \in \mathbb{N}}$ defined by the identity

$$\omega\left( c \exp\left( -\frac{1}{c} \left( \frac{\Delta_t}{\epsilon} \right)^{\frac{1}{p-1}} \right) \right) = \epsilon \Delta_t J_0,$$

for some $\epsilon > 0$ and $\tau \in (0, 1)$ is summable. Indeed, we have

$$\sum_{t \in \mathbb{N}} \Delta_t \lesssim \sum_{t \in \mathbb{N}} (\epsilon^{\frac{1}{p-1}} J_0)^{\frac{1}{p}} \left( 1 + |\log(\omega^{-1}(\epsilon \Delta_t J_0))| \right)^{\frac{n+1}{n}}$$

$$\lesssim \frac{1}{\tau^{\frac{1}{p}} |\log(\tau)|} \int_0^\tau (t J_0)^{\frac{1}{p}} \left( 1 + |\log(\omega^{-1}(\epsilon \Delta_t J_0))| \right)^{\frac{n+1}{n}} \frac{dt}{t}$$

$$= \frac{1}{(\epsilon \Delta_t)^{\frac{1}{p}} |\log(\tau)|} \int_0^{\omega^{-1}(\epsilon \Delta_t J_0)} (\omega(s))^{\frac{1}{p}} \left( 1 + \frac{\log(s)}{\omega(s)} \right)^{\frac{n+1}{n}} \frac{ds}{s}.$$

Now we can continue as before, i.e. using $\omega(s) \geq s (g^{\frac{1}{n-1}} \chi_{B_1})^n(s)$ and $\omega'(s) = (g^{\frac{1}{n-1}} \chi_{B_1})^n(s)$, we obtain

$$\int_0^{\omega^{-1}(\epsilon \Delta_t J_0)} (\omega(s))^{\frac{1}{p}} \left( 1 + \frac{\log(s)}{\omega(s)} \right)^{\frac{n+1}{n}} \frac{\omega'(s)}{\omega(s)} \, ds \leq \int_0^\infty s^{\frac{1}{p}} \left( 1 + \frac{\log(s)}{\omega(s)} \right)^{\frac{n+1}{n}} ((g^{\frac{1}{n-1}} \chi_{B_1})^n(s)) \, ds$$

$$= \|g^{\frac{1}{n-1}}\|_{L^{n,1}(\log L)^{\frac{n+1}{n}}(B_1)} < \infty.$$
Finally, we present the argument for \([\ell]\). For this we recall the Moser-Trudinger inequality in the following form: there exists \(c_i = c_i(u) > 0, i = 1, 2\) such that for every ball \(B \subset \mathbb{R}^d\) and every \(v \in W^{1,n}(B)\)

\[
\int_B \exp \left( \left( \frac{|v - f_B|}{c_1 \|v\|_{L^n(B)}} \right)^{\frac{n}{n-1}} \right) \leq c_2
\]

(see e.g. [19] Chapter 7). Since \(A_{k,\sigma} \subset B_{\alpha} \cap \{x: (u - h)_+ \geq k - h\} =: E_{h,k,\sigma}\) Chebychev’s inequality combined with Moser-Trudinger inequality gives \([\ell]\):

\[
|E_{h,k,\sigma}| \lesssim \exp \left(- \left( \frac{k - h}{2^{n-1} c_1 J(h, \sigma)^{\frac{1}{n}}} \right)^{\frac{n}{n-1}} \right) \int_{B_{\sigma}} \exp \left( \left( \frac{(u - h)_+}{2^{n-1} c_1 J(h, \sigma)^{\frac{1}{n}}} \right)^{\frac{n}{n-1}} \right) \, dx
\]

\[
\leq \exp \left(- \left( \frac{k - h}{2^{n-1} c_1 J(h, \sigma)^{\frac{1}{n}}} \right)^{\frac{n}{n-1}} \right) \exp \left( \left( \frac{\int_{B_{\sigma}} (u - h)_+}{c_1 J(h, \sigma)^{\frac{1}{n}}} \right)^{\frac{n}{n-1}} \right) c_2 |B_\sigma|
\]

\[
\lesssim \exp \left(- \left( \frac{k - h}{2^{n-1} c_1 J(h, \sigma)^{\frac{1}{n}}} \right)^{\frac{n}{n-1}} \right),
\]

where we use in the last estimate the Poincaré inequality the assumption \(\sigma \leq 1\).

\[\square\]

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