BASED MODULES OVER THE $\mathfrak{q}$QUANTUM GROUP OF TYPE AI

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Abstract. This paper studies classical weight modules over the $\mathfrak{q}$quantum group $U^q$ of type AI. We introduce the notion of based $U^q$-modules by generalizing the notion of based modules over quantum groups (quantized enveloping algebras). We prove that each finite-dimensional irreducible classical weight $U^q$-module with integer highest weight is a based $U^q$-module. As a byproduct, a new combinatorial formula for the branching rule from $\mathfrak{sl}_n$ to $\mathfrak{so}_n$ is obtained.

1. Introduction

Based modules over quantum groups. In representation theory of quantum groups (quantized enveloping algebras), there is a notion of based modules. Let $U = U_q(\mathfrak{g})$ denote the quantum group over the field $K := \mathbb{C}(q)$ of rational functions in one variable $q$ associated with a complex semisimple Lie algebra $\mathfrak{g}$. A based $U$-module is a $U$-module equipped with two distinguished structures: a crystal basis and a canonical basis, which is also known as a global crystal basis.

Crystal basis theory, which was discovered by Kashiwara [16] and Lusztig [22], is a powerful tool in combinatorial representation theory. The crystal basis $B_M$ of a based $U$-module $M$ is a local basis in the sense that it forms a basis of $M$ at $q = \infty$ (in the literature, the variable $q$ in this paper corresponds to $q^{-1}$). To be more specific, $B_M$ is a $\mathbb{C}$-basis of $L_M := L_M/q^{-1}L_M$, where $L_M$ is a free submodule of $M$ over the ring $K_\infty := \mathbb{C}[q^{-1}][q^{-1}]$ of rational functions regular at $q = \infty$, called the crystal lattice. Although $B_M$ is not a genuine $K$-basis of $M$, it has a lot of information about the $U$-module structure of $M$.

Canonical basis theory was initiated by Lusztig [22], and then Kashiwara [17] found a different approach. The canonical basis of a based $U$-module $M$ is a globalization of the crystal basis in the sense that it tends to the crystal basis as $q$ goes to $\infty$.

One of the most distinguished properties of based $U$-modules is cellularity ([23, Chapter 27]), which gives a canonical stratification of based $U$-modules. This generalizes a well-known property of the Hecke algebra of type $A$ with the Kazhdan-Lusztig basis [18].

$\mathfrak{q}$Quantum groups. Recall that the quantum group $U$ is a $q$-deformation of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Namely, $U$ tends to $U(\mathfrak{g})$ as $q$ goes to 1. There is another family of $q$-deformations of the universal enveloping algebras of complex Lie algebras, called the quantum groups.

The quantum groups appear in the theory of quantum symmetric pairs initiated by Letzter [21] unifying earlier constructions by Koornwinder [20], Gavriliuk and Klimyk [11], Noumi [25], and others. Kolb [19] generalized Letzter’s theory to symmetrizable Kac-Moody algebras. In the theory of quantum symmetric pairs, we consider a complex semisimple Lie algebra $\mathfrak{g}$ and an involutive Lie algebra automorphism $\theta$ on $\mathfrak{g}$. Set $\mathfrak{g}^\theta := \{x \in \mathfrak{g} \mid \theta(x) = x\}$ to be the fixed point subalgebra. A quantum symmetric pair...
associated with \((g, \mathfrak{t})\) is a pair \((U, U')\) consisting of the quantum group \(U = U_q(g)\) and its coideal subalgebra \(U'\) which tends to \(U(\mathfrak{t})\) as \(q\) goes to 1. The \(U'\) itself is referred to as an \(\text{qum}\) quantum group.

The quantum groups \(U'\) (with the embeddings \(U' \hookrightarrow U\)) are thought of as generalizations of the quantum groups in the sense that a quantum group \(U\) (with the comultiplication map \(\Delta : U \rightarrow U \otimes U \simeq U_q(g \oplus g)\)) is an \(\text{qum}\) quantum group associated with \((g \oplus g, g)\). Such an \(\text{qum}\) quantum group is said to be of diagonal type.

Based on this viewpoint, many results in the theory of quantum groups have been generalized to the \(\text{qum}\) quantum groups setting, e.g., the bar-involution \([2, 5]\), the quasi-\(K\)-matrix \([4, 5]\), the universal \(K\)-matrix \([1, 3, 8, 26]\), quantum Schur and Howe dualities \([4, 10, 27]\), and the canonical basis \([4, 5]\). However, the theory of based \(U\)-modules has not been generalized to the \(\text{qum}\) quantum groups setting.

**Results.** The aim of this paper is to define the notion of based \(U'\)-modules, and study their properties for the quantum group of type AI, i.e., the \(\text{qum}\) quantum group associated with \((g, \mathfrak{t}) \simeq (sl_n, so_n)\) (see \([29, 30]\) for related works for the \(\text{qum}\) quantum group of quasi-split type AI(III)). The symmetric pair \((sl_n, so_n)\) is of split type, and hence, its structure, such as defining relations, is relatively simple. Moreover, it is one of the generalized Onsager algebras, which have background in integrable systems (see \([28]\) and references therein for detail).

In \([31]\), the notion of classical weight \(
U'\)-modules was introduced, and the finite-dimensional irreducible classical weight \(U'\)-modules were classified in terms of highest weight theory for several types including AI. The isoclasses of such irreducible modules are parametrized by the set \(X^+_{\mathfrak{t}}\) of dominant integral weights for \(\mathfrak{t}\). For each \(\nu \in X^+_{\mathfrak{t}}\), let \(V(\nu)\) denote the corresponding irreducible \(U'\)-module. Recall that the integral weights for \(\mathfrak{t} \simeq so_n\) are divided into two groups, integer weights and half integer weights. Let \(X^+_{\mathfrak{t}, \text{int}}\) denote the set of integer weights, and set \(X^+_{\mathfrak{t}, \text{int}} := X^+ \cap X^+_{\mathfrak{t}, \text{int}}\). The first main result in this paper is the following.

**Theorem A** (Theorem 8.1.2). Let \(\nu \in X^+_{\mathfrak{t}}\) be an integer weight. Then, \(V(\nu)\) has a based \(U'\)-module structure.

As a byproduct, we obtain a purely combinatorial formula for the branching rule from \(U\) to \(U'\). Before stating the formula, let us prepare some notation. First of all, let us identify \(X^+_{\mathfrak{t}, \text{int}}\) with \(\mathbb{Z}^m\), where \(m\) denotes the rank of \(\mathfrak{t}(\simeq so_n)\):

\[
m = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even}, \\ \frac{n-1}{2}, & \text{if } n \text{ is odd}. \end{cases}
\]

Let \(\tilde{E}_i, \tilde{F}_i\) denote the Kashiwara operators acting on crystal bases \([16]\). Given a finite-dimensional \(U\)-module and its crystal basis \(\mathcal{B}\), for each \(b \in \mathcal{B}\), set

\[
\varphi_i(b) := \max\{k \mid \tilde{F}_i^k b \neq 0\}, \\
\epsilon_i(b) := \max\{k \mid \tilde{E}_i^k b \neq 0\}, \\
\deg_i(b) := \begin{cases} \epsilon_i(b), & \text{if } \varphi_i(b) \text{ is even}, \\ \epsilon_i(b) + 1, & \text{if } \varphi_i(b) \text{ is odd}, \end{cases} \\
\tilde{\mathcal{B}}_i(b) := \begin{cases} \tilde{E}_i b, & \text{if } \varphi_i(b) \text{ is even}, \\ \tilde{F}_i b, & \text{if } \varphi_i(b) \text{ is odd}. \end{cases}
\]


**Theorem B** (Theorem 8.2.6). Let $\lambda \in X^+$ and $\nu = (\nu_1, \nu_3, \ldots, \nu_{2m-1}) \in X^+_{\text{int}} = \mathbb{Z}^m$. Let $[\lambda : \nu]$ denote the multiplicity of the irreducible $U^\prime$-module $V(\nu)$ of highest weight $\nu$ in the irreducible $U$-module $V(\lambda)$ of highest weight $\lambda$.

1. Suppose $n$ is even and $\nu_{2m-1} \neq 0$. Then, we have
   \[
   [\lambda : \nu] = \frac{1}{2} \sharp \{ b \in \mathcal{B}(\lambda) \mid \deg_{2i-1}(b) = |\nu_{2i-1}| \text{ for all } i \in \{1, \ldots, m\}, \quad \deg_{2i}(b) = 0 \text{ for all } i \in \{1, \ldots, m-1\}, \quad \deg_{2i+1}(\tilde{B}_2 \tilde{B}_1^{-1})^{\nu_{2i+1}}b = 0 \text{ for all } i \in \{1, \ldots, m-1\}\}.
   \]

2. Suppose $n$ is even and $\nu_{2m-1} = 0$. Then, we have
   \[
   [\lambda : \nu] = \sharp \{ b \in \mathcal{B}(\lambda) \mid \deg_{2i-1}(b) = \nu_{2i-1} \text{ for all } i \in \{1, \ldots, m\}, \quad \deg_{2i}(b) = 0 \text{ for all } i \in \{1, \ldots, m-1\}, \quad \deg_{2i+1}(\tilde{B}_2 \tilde{B}_1^{-1})^{\nu_{2i+1}}b = 0 \text{ for all } i \in \{1, \ldots, m-1\}\}.
   \]

3. Suppose $n$ is odd. Then, we have
   \[
   [\lambda : \nu] = \sharp \{ b \in \mathcal{B}(\lambda) \mid \deg_{2i-1}(b) = \nu_{2i-1} \text{ for all } i \in \{1, \ldots, m\}, \quad \deg_{2i}(b) = 0 \text{ for all } i \in \{1, \ldots, m\}, \quad \deg_{2i+1}(\tilde{B}_2 \tilde{B}_1^{-1})^{\nu_{2i+1}}b = 0 \text{ for all } i \in \{1, \ldots, m-1\}\}.
   \]

Although $\mathfrak{k}$ is not set-theoretically the same as $\mathfrak{so}_n$, the Lie algebra of $n$ by $n$ skew-symmetric matrices, this formula also tells us the branching rule from $\mathfrak{sl}_n$ to $\mathfrak{so}_n$ by a similar argument to [24, 2.4]. This classical branching problem has been studied for a long time by many people, and several (partial) answers have been provided. Among them, Jang and Kwon [14] recently described this branching rule in terms of Littlewood-Richardson coefficients. Their formula identifies the multiplicities with the numbers of certain combinatorial objects; Littlewood-Richardson tableaux satisfying certain conditions. On the other hand, since $\mathcal{B}(\lambda)$ is realized as the set of semistandard tableaux of shape $\lambda$, our formula identifies the multiplicities with the numbers of certain semistandard tableaux. Hence, it would be interesting to construct an explicit bijection between such Littlewood-Richardson tableaux and semistandard tableaux.

Recall that the crystal basis $\mathcal{B}_M$ of a finite-dimensional based $U$-module $M$ tells us how $M$ decomposes into irreducible ones. In particular, the crystal basis theory provides us a branching rule from $U \otimes U$ to $U$. Recall also that $(U \otimes U, U)$ is an quantum group of diagonal type. Then, our formula can be seen as a generalization of the branching rule for the quantum groups of diagonal type to that of type $AI$.

**Organization.** This paper is organized as follows. In Section 2, we briefly review the theory of based modules over the quantum groups. In Section 3, we formulate the notion of based modules over the quantum groups by generalizing that of based modules over the quantum groups. From Section 4 on, we restrict our attention to the quantum group of type $AI$. For such an quantum group, we introduce the notion of standard $X^\prime$-weight modules, which plays a key role when proving that the finite-dimensional irreducible highest weight modules of integer highest weights are based modules. In Sections 5–7, we analyze low-rank cases. Based on results obtained there, we finally prove our main theorems in Section 8.

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Notation. Throughout this paper, we use the following notation:

- $\mathbb{Z}_{\geq 0}$: the set of nonnegative integers.
- $\mathbb{Z}_{\text{ev}}$: the set of even integers.
- $\mathbb{Z}_{\text{odd}}$: the set of odd integers.
- $\mathbb{Z}_{0,\text{ev}} := \mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{\text{ev}}$.
- $\mathbb{Z}_{0,\text{odd}} := \mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{\text{odd}}$.
- For $a \in \mathbb{Z}$, $p(a) := \begin{cases} \text{ev} & \text{if } a \in \mathbb{Z}_{\text{ev}}, \\ \text{odd} & \text{if } a \in \mathbb{Z}_{\text{odd}}. \end{cases}$
- For $a \in \mathbb{Z}$, $q(a) := p(a - 1) = \begin{cases} \text{odd} & \text{if } a \in \mathbb{Z}_{\text{ev}}, \\ \text{ev} & \text{if } a \in \mathbb{Z}_{\text{odd}}. \end{cases}$
- For $a, b \in \mathbb{Z}$ and $p \in \{\text{ev, odd}\}$, $[a, b]_p := [a, b] \cap \mathbb{Z}_p$.
- $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$: the abelian group of order 2.
- $\tilde{\tau} : \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$: the quotient map.
- For $a \in \mathbb{Z}/2\mathbb{Z}$, $p(a) := \begin{cases} \text{ev} & \text{if } a = 0, \\ \text{odd} & \text{if } a = 1. \end{cases}$

2. QUANTUM GROUPS

In this section, we briefly review basic results concerning finite-dimensional based modules over a quantum group. We refer the reader to [23].

2.1. Hermitian inner product. Let $\mathbb{K}$ denote the field $\mathbb{C}(q)$ of complex rational functions. In this subsection, we formulate the notion of $\mathbb{K}$-valued Hermitian inner product and generalize basic results about complex metric spaces. The proofs of statements in this subsection are straightforward, and hence, omitted.

Set $\mathbb{K}_1 := \left\{ \frac{f}{g} \in \mathbb{K} \mid f, g \in \mathbb{C}[q], g|_{q=1} \neq 0 \right\}$.

$$\mathbb{K}_\infty := \left\{ \frac{f}{g} \in \mathbb{K} \mid f, g \in \mathbb{C}[q^{-1}], g|_{q^{-1}=0} \neq 0 \right\}.$$ Let $\text{ev}_1 : \mathbb{K}_1 \rightarrow \mathbb{K}_1/(q - 1)\mathbb{K}_1 \simeq \mathbb{C}$ (resp., $\text{ev}_\infty : \mathbb{K}_\infty \rightarrow \mathbb{K}_\infty/q^{-1}\mathbb{K}_\infty \simeq \mathbb{C}$) denote the evaluation map at $q = 1$ (resp., $q^{-1} = 0$).

**Definition 2.1.1.** Let $c \in \mathbb{K}^\times$, and write $c = \frac{f}{g}$ with $f = \sum_m a_m q^m \in \mathbb{C}[q, q^{-1}]$, $g \in 1 + q^{-1}\mathbb{C}[q^{-1}]$.

1. The degree $\text{deg}(c) \in \mathbb{Z}$ of $c$ is defined to be $\max\{m \mid a_m \neq 0\}$.
2. The leading coefficient $\text{lc}(c) \in \mathbb{C}$ of $c$ is defined to be $a_{\text{deg}(c)}$.
3. The leading term $\text{lt}(c) \in \mathbb{C}q^\mathbb{Z}$ of $c$ is defined to be $\text{lc}(c)q^{\text{deg}(c)}$.

Furthermore, we set $\text{deg}(0) := -\infty$, $\text{lc}(0) := 0$, $\text{lt}(0) := 0$.

Let $z^* \in \mathbb{C}$ denote the complex conjugate of $z \in \mathbb{C}$. Extend the notion of complex conjugate to a ring automorphism on $\mathbb{K}$ by setting $q^* = q$.

**Definition 2.1.2.** Let $V$ be a $\mathbb{K}$-vector space. A ($\mathbb{K}$-valued) Hermitian inner product on $V$ is a map $(\cdot, \cdot) : V \times V \rightarrow \mathbb{K}$ satisfying the following:
(1) \((au + bv, w) = a(u, w) + b(v, w)\) for all \(a, b \in K, u, v, w \in V\).
(2) \((v, u) = (u, v)^*\) for all \(u, v \in V\).
(3) \(lc((v, v)) \geq 0\) for all \(v \in V\).
(4) \(lc((v, v)) = 0\) implies \(v = 0\).
(5) \(\deg((v, v)) \in \mathbb{Z}_{ev} \cup \{-\infty\}\) for all \(v \in V\).

**Definition 2.1.3.** Let \(V\) be a \(K\)-vector space equipped with a Hermitian inner product. For \(v \in V\), we set

\[
\deg(v) := \frac{1}{2} \deg((v, v)), \quad lc(v) := \sqrt{lc((v, v))}, \quad lt(v) := lc(v)q^{\deg(v)}.
\]

Here, we understand that \(\frac{1}{2}(-\infty) = -\infty\) and \(q^{-\infty} = 0\).

**Definition 2.1.4.** Let \(V\) be a \(K\)-vector space equipped with a Hermitian inner product.

(1) \(v \in V\) is said to be almost normal if \(lt(v) = 1\).
(2) \(u, v \in V\) are said to be orthogonal (resp., almost orthogonal) if \((u, v) = 0\) (resp., \((u, v) \in q^{-1}K_{\infty}\)).
(3) A basis of \(V\) is said to be orthogonal if it consists of vectors which are pairwise orthogonal.
(4) A basis of \(V\) is said to be almost orthonormal if it consists of almost normal vectors which are pairwise almost orthogonal.

**Remark 2.1.5.** Because of conditions (3)–(5) in Definition 2.1.2, every nonzero vector can be normalized to an almost normal vector proportional to it.

The following is an easy analogue of an elementary result about complex metric spaces.

**Proposition 2.1.6.** Let \(V\) be a finite-dimensional \(K\)-vector space equipped with a Hermitian inner product \((\cdot, \cdot)\), and \(W \subseteq V\) a subspace.

(1) \(V\) possesses an orthogonal and almost orthonormal basis.
(2) The restriction \((\cdot, \cdot)|_W := (\cdot, \cdot)|_{W \times W}\) is a Hermitian inner product on \(W\).
(3) Let \(B_W\) be an almost orthonormal basis of \(W\). Then, there exists an almost orthonormal basis \(B_V\) of \(V\) which extends \(B_W\).

Let \(V\) and \(W\) be as in Proposition 2.1.6. In the sequel, unless otherwise stated, whenever we consider a Hermitian inner product of \(W\), we assume that it is \((\cdot, \cdot)|_W\) in 2.1.6 (2).

For each \(K\)-vector space \(V\) equipped with a Hermitian inner product, we use the following notation throughout this paper:

- \(\mathcal{L}_V := \{v \in V \mid (v, v) \in K_{\infty}\}\).
- \(\overline{\mathcal{L}}_V := \mathcal{L}_V/q^{-1}\)\(\mathcal{L}_V\).
- \(ev_{\infty} : \mathcal{L}_V \to \overline{\mathcal{L}}_V\) the quotient map.

**Proposition 2.1.7.** Let \(V\) be a finite-dimensional \(K\)-vector space equipped with a Hermitian inner product \((\cdot, \cdot)\) and an almost orthonormal basis \(B\).

(1) We have \(\mathcal{L}_V = K_{\infty}B\).
(2) We have \(\mathcal{L}_V = \{v \in V \mid (u, v) \in K_{\infty}\ for all \ u \in \mathcal{L}_V\}\).
(3) \((\cdot, \cdot)\) induces a \(\mathbb{C}\)-valued Hermitian inner product (denoted by the same symbol) on \(\overline{\mathcal{L}}_V\) given by

\[(ev_{\infty}(u), ev_{\infty}(v)) := ev_{\infty}((u, v)).\]
(4) \(ev_{\infty}(B)\) is an orthonormal basis of \(\overline{\mathcal{L}}_V\).
Proposition 2.1.8. Let $V$ be a finite-dimensional $\mathbb{K}$-vector space equipped with a Hermitian inner product. Suppose that $V$ admits an orthogonal decomposition $V = \bigoplus_{\omega \in \Omega} V_{\omega}$. Then, we have orthogonal decompositions

$$L_V = \bigoplus_{\omega \in \Omega} L_{V_{\omega}}, \quad L_V = \bigoplus_{\omega \in \Omega} L_{V_{\omega}},$$

here, we identify $L_{V_{\omega}}/q^{-1}L_{V_{\omega}}$ with $L_{V_{\omega}}/(q^{-1}L_{V_{\omega}} \cap L_{V_{\omega}})$, which equals $L_{V_{\omega}}.

Definition 2.1.9. Let $V, W$ be $\mathbb{K}$-vector spaces equipped with Hermitian inner products $(\cdot, \cdot)_V$, $(\cdot, \cdot)_W$, respectively.

1. We say that a linear map $f \in \text{Hom}_K(V, W)$ almost preserves the metrics if for each $u, v \in L_V$, we have $(f(u), f(v))_W - (u, v)_V \in q^{-1}K_\infty$.

2. We say that $V$ and $W$ are almost isometric if there exists an isomorphism $V \to W$ of vector spaces which almost preserves the metrics. Such an isomorphism is called an almost isometry.

The following is an easy analog of an elementary result about complex metric spaces.

Proposition 2.1.10. Let $V, W$ be $\mathbb{K}$-vector spaces equipped with Hermitian inner products $(\cdot, \cdot)_V$, $(\cdot, \cdot)_W$, respectively. Let $f \in \text{Hom}_K(V, W)$. Then, the following are equivalent.

1. $f$ almost preserves the metrics.

2. For each almost orthonormal basis $B$ of $V$, its image $f(B)$ forms an almost orthonormal basis of $f(V)$.

3. There exists an almost orthonormal basis $B$ of $V$ such that $f(B)$ forms an almost orthonormal basis of $f(V)$.

Let $V$ be a $\mathbb{K}$-vector space equipped with a Hermitian inner product, and $W \subseteq V$ a subspace. Throughout this paper, we always equip the quotient space $V/W$ with a Hermitian inner product $(\cdot, \cdot)_{V/W}$ as follows. Let $B_W \subseteq B_V$ be almost orthonormal bases of $W \subseteq V$. Set $W' := K(B_V \setminus B_W)$. For each $v \in V = W \oplus W'$, let $v_1 \in W$ and $v_2 \in W'$ be such that $v = v_1 + v_2$. Also, let $[v]$ denote the image of $v$ in $V/W$. Then, define

$$([u], [v])_{V/W} := (a_2, v_2).$$

This is a Hermitian inner product on $V/W$. Clearly, $[B_V] := \{[b] \mid b \in B_V \setminus B_W\}$ forms an almost orthonormal basis of $V/W$. Then, by Proposition 2.1.7 (1), we have

$$L_{V/W} = K_\infty[B_V] = K_\infty B_V/(W \cap K_\infty B_V) = L_V/L_W.$$

This implies that although the Hermitian inner product thus constructed depends on $B_W$ and $B_V$, the $K_\infty$-submodule $L_{V/W}$ does not.

2.2. Quantum groups. For $n \in \mathbb{Z}$ and $a \in \mathbb{Z}_{>0}$, set

$$[n]_{q^a} := \frac{q^{an} - q^{-an}}{q^a - q^{-a}}.$$

Also, for $m \geq n \geq 0$, set

$$[n]_{q^a}! := [n]_{q^a} \cdots [2]_{q^a}[1]_{q^a}, \quad \left[\begin{array}{c} m \\ n \end{array}\right]_{q^a} := \frac{[m]_{q^a}!}{[m-n]_{q^a}! [n]_{q^a}!},$$

where we understand that $[0]_{q^a}! = 1$. When $a = 1$, we often omit the subscript $q^a$.

Let $A$ be an associative algebra over $\mathbb{K}$. For $x, y \in A$, $z \in A^\times$, $a \in \mathbb{Z}_{>0}$ and $b \in \mathbb{Z}$, set

$$[z; n]_{q^a} := \frac{q^{an}z - q^{-an}z^{-1}}{q^a - q^{-a}}, \quad \{z; n\}_{q^a} := q^{an}z + q^{-an}z^{-1}, \quad [x, y]_{q^a} := xy - q^b yx.$$
When $a = 1$ or $b = 0$, we often omit the subscripts $q^a$ or $q^b$, respectively.

Let $(a_{i,j})_{i,j \in I}$ be a Cartan matrix of finite type. We identify the index set $I$ with the set of vertices of the Dynkin diagram of the Cartan matrix, or with the Dynkin diagram itself. Let $\{\alpha_i\}_{i \in I}$ denote the set of simple roots, $\{\varpi_i\}_{i \in I}$ the set of simple coroots, $\{\alpha^*\}_{i \in I}$ the set of fundamental weights, $X := \bigoplus_{i \in I} \mathbb{Z} \varpi_i$ the weight lattice, $Y := \bigoplus_{i \in I} \mathbb{Z} h_i$ the coroot lattice, $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{Z}$ the perfect pairing given by $\langle h_i, \varpi_j \rangle = \delta_{i,j}$. Let $d_i \in \mathbb{Z}_{>0}$ be such that $d_i a_{i,j} = d_j a_{j,i}$ for all $i, j \in I$.

Let $g = g(I)$ denote the finite-dimensional semisimple Lie algebra over $\mathbb{C}$ associated with the Dynkin diagram $I$. Let $e_i, f_i, h_i, i \in I$ denote the Chevalley generators.

The quantum group $U = U_q(\mathfrak{g})$ is a unital associative algebra over $K$ generated by $E_i, F_i, K_i, h, i \in I, h \in Y$ subject to the following relations; for each $i, j \in I$ and $h, h' \in Y,$

\[
K_0 = 1, \quad K_h K_{h'} = K_{h + h'},
\]

\[
K_h E_i = q^{h, \alpha_i} E_i K_h, \quad K_h F_i = q^{-(h, \alpha_i)} F_i K_h,
\]

\[
[E_i, F_j] = \delta_{i,j} [K_i; 0]_q, \quad \sum_{k=0}^{1-a_{i,j}} (-1)^k \left[ \frac{1 - a_{i,j}}{k} \right] E_i^{1-a_{i,j}} F_j^{k} E_i^k = 0 \quad \text{if } i \neq j,
\]

\[
\sum_{k=0}^{1-a_{i,j}} (-1)^k \left[ \frac{1 - a_{i,j}}{k} \right] F_i^{1-a_{i,j}} F_j^{k} F_i^k = 0 \quad \text{if } i \neq j,
\]

where $K_i := K_{d_i h_i}$ and $q_i := q^{d_i}.$

The quantum group $U$ admits a Hopf algebra structure with comultiplication $\Delta$ defined by

\[
\Delta(E_i) := E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) := 1 \otimes F_i + F_i \otimes K_i^{-1}, \quad \Delta(K_h) = K_h \otimes K_h,
\]

Let $\varphi$ denote the algebra anti-involution on $U$ defined by

\[
\varphi(E_i) := q_i^{-1} F_i K_i, \quad \varphi(F_i) := q_i^{-1} F_i K_i^{-1}, \quad \varphi(K_h) := K_h.
\]

By the definitions of $\Delta$ and $\varphi$, one can verify that

\[
\Delta \circ \varphi = (\varphi \otimes \varphi) \circ \Delta.
\]

Let $(\cdot)^* : U \to U$ denote the ring automorphism which extends the complex conjugate on $K$ in a way such that

\[
E_i^* = E_i, \quad F_i^* = F_i, \quad K_i^* = K_i, \quad \text{for all } i \in I, h \in Y.
\]

Then, we have $\varphi \circ * = * \circ \varphi$, and $\Delta \circ * = (* \otimes *) \circ \Delta$. Therefore, $\varphi^* := \varphi \circ *$ is an algebra anti-involution, and satisfies

\[
\Delta \circ \varphi^* = (\varphi^* \otimes \varphi^*) \circ \Delta.
\]

Let $Br = Br(I)$ denote the braid group associated with the Dynkin diagram $I$ with generators $T_i, i \in I$. The braid group $Br$ acts on $U$ via Lusztig’s automorphisms $T_i^p,$
Proposition 2.4.2. · there exists a unique contragredient Hermitian inner product

\( (2) \)

\[ T_i(E_j) := \begin{cases} -F_i K_i & \text{if } j = i, \\ \sum_{r+s=-a_{i,j}} (-1)^r q_i^{-r} E_i^{(s)} E_j^{(r)} & \text{if } j \neq i, \end{cases} \]

\[ T_i(F_j) := \begin{cases} -K_i^{-1} E_i & \text{if } j = i, \\ \sum_{r+s=-a_{i,j}} (-1)^r q_i^{r} F_i^{(r)} F_j^{(s)} & \text{if } j \neq i, \end{cases} \]

\[ T_i K_h := K_{h-(h,a_i)} h_i, \]

where \( E_i^{(n)} := \frac{1}{|i|^{n}} E_i^n \) and \( F_i^{(n)} := \frac{1}{|i|^{n}} F_i^n \) are divided powers.

2.3. **Weight U-modules.** Given a \( U \)-module \( M \) and a weight \( \lambda \in X \), the subspace

\[ M_\lambda := \{ m \in M \mid K_h m = q^{(h,\lambda)} m \ \text{for all } h \in Y \}, \]

is called the weight space of \( M \) of weight \( \lambda \), and each element of \( M_\lambda \) is said to be a weight vector of weight \( \lambda \). A \( U \)-module is said to be a weight module if it admits a weight space decomposition \( M = \bigoplus_{\lambda \in X} M_\lambda \).

For each \( \lambda, \mu \in X \), we write \( \mu \leq \lambda \) to indicate that \( \lambda - \mu = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i \). This defines a partial order on \( X \), called the dominance order.

Let \( X^+ := \{ \lambda \in X \mid \langle h_i, \lambda \rangle \geq 0 \ \text{for all } i \in I \} \) denote the set of dominant weights.

**Theorem 2.3.1** (see e.g. [23, Part I]).

1. For each weight \( U \)-module \( M \), \( \lambda \in X \), and \( i \in I \), we have

\[ E_i M_\lambda \subseteq M_{\lambda + \alpha_i}, \quad F_i M_\lambda \subseteq M_{\lambda - \alpha_i}. \]

2. Each finite-dimensional \( U \)-module is completely reducible.

3. The isoclasses of finite-dimensional irreducible \( U \)-modules are parametrized by \( X^+ \).

   Let \( V(\lambda) \) denote the finite-dimensional irreducible \( U \)-module of highest weight \( \lambda \in X^+ \), and \( v_\lambda \in V(\lambda) \) the highest weight vector.

4. For each \( \lambda \in X^+ \), we have

\[ V(\lambda) = \text{Span}_K \{ F_{i_1} \cdots F_{i_r} v_\lambda \mid r \geq 0, \ i_1, \ldots, i_r \in I \}. \]

5. For each \( \lambda \in X^+ \) and \( \mu \in X \), we have \( \dim V(\lambda)_{\lambda} = 1 \) and \( \dim V(\lambda)_{\mu} = 0 \) unless \( \mu \leq \lambda \).

6. For each \( \lambda \in X^+ \) and \( \mu \in X \) such that \( \mu < \lambda \), we have

\[ V(\lambda)_{\mu} = \sum_{i \in I} F_i V(\lambda)_{\mu + \alpha_i}. \]

2.4. **U-modules with contragredient Hermitian inner products.**

**Definition 2.4.1.** Let \( M \) be a \( U \)-module equipped with a Hermitian inner product \( (\cdot, \cdot) \). We say that \( (\cdot, \cdot) \) is contragredient if it satisfies

\[ (xu, v) = (u, \varphi^*(x)v) \quad \text{for all } x \in U, \ u, v \in M. \]

**Proposition 2.4.2.** Let \( \lambda \in X^+ \), and consider the irreducible \( U \)-module \( V(\lambda) \). Then, there exists a unique contragredient Hermitian inner product \( (\cdot, \cdot)_\lambda \) on \( V(\lambda) \) such that \( (v_\lambda, v_\lambda)_\lambda = 1 \).
Proof. Let $U_Q$ denote the $\mathbb{Q}(q)$-subalgebra of $U$ generated by $E_i, F_i, K_h, i \in I, h \in Y$, and set $V(\lambda)_Q := U_Qv_\lambda$. Then, we have

$$U = U_Q \otimes_{\mathbb{Q}(q)} K, \quad V(\lambda) = V(\lambda)_Q \otimes_{\mathbb{Q}(q)} K.$$ 

By [23, Proposition 19.1.2], there exists a unique symmetric bilinear form $(\cdot, \cdot)_\lambda : V(\lambda)_Q \times V(\lambda)_Q \to \mathbb{Q}(q)$ such that $(v_\lambda, v_\lambda)' = 1$, and $(xu, v)' = (u, \varphi(x)v)'_\lambda$ for all $x \in U_Q$ and $u, v \in V(\lambda)_Q$. Define a map $(\cdot, \cdot)_\lambda : V(\lambda) \times V(\lambda) \to K$ by

$$(au, bv)_\lambda := ab^*(u, v)'_\lambda, \quad a, b \in K, \quad u, v \in V(\lambda)_Q.$$ 

Then, $(\cdot, \cdot)_\lambda$ satisfies conditions (1) and (2) in Definition 2.1.2, condition (2) in Definition 2.4.1, and $(v_\lambda, v_\lambda)_\lambda = 1$.

Moreover, by [23, Lemma 19.1.4], there exists a basis $B$ of $V(\lambda)_Q$ such that

$$(b, b')_\lambda \in \delta_{b,b'}q^{-1}K_\infty.$$ 

This shows that $(\cdot, \cdot)_\lambda$ satisfies conditions (3)–(5) in Definition 2.1.2. Thus the proof completes.

**Lemma 2.4.3.** Let $M$ and $N$ be $U$-modules with contragredient Hermitian inner products $(\cdot, \cdot)_M$ and $(\cdot, \cdot)_N$. Then, the form $(\cdot, \cdot)_{M,N}$ on $M \otimes N$ given by

$$(m_1 \otimes n_1, m_2 \otimes n_2)_{M,N} := (m_1, m_2)_{M}(n_1, n_2)_N$$

is a contragredient Hermitian inner product.

**Proof.** The assertion follows from equation (1) on page 7.

**Lemma 2.4.4.** Let $M$ be a weight $U$-module equipped with a contragredient Hermitian inner product. Then, we have $(M_\lambda, M_\mu) = 0$ for all $\lambda, \mu \in X$ such that $\lambda \neq \mu$.

**Proof.** The assertion follows from the fact that $\varphi^*(K_h) = K_h$ for all $h \in Y$.

**Proposition 2.4.5.** Let $M$ be a finite-dimensional $U$-module equipped with a contragredient Hermitian inner product. Then, there exists an irreducible decomposition $M = \bigoplus_{k=1}^r M_k$ satisfying the following:

1. $(M_\lambda, M_\mu) = 0$ unless $k = l$.
2. For each $k \in [1, r]$, there exists a unique $\lambda_k \in X^+$ and an isomorphism $\phi_k : M_k \to (\lambda_k)$ of $U$-modules which is an almost isometry.

**Proof.** For each $\lambda \in X^+$, set $H_\lambda := \{m \in M_\lambda \mid E_i m = 0 \text{ for all } i \in I\}$. Then, by Proposition 2.1.6 (1), $H_\lambda$ has an orthogonal basis. By Lemma 2.4.4, we have $(H_\lambda, H_\mu) = 0$ if $\lambda \neq \mu$. Then one can take an orthogonal basis $m_1, \ldots, m_r$ of $\bigoplus_{\lambda \in X^+} H_\lambda$ consisting of weight vectors. Setting $M_k := Um_k$, we obtain an irreducible decomposition $M = \bigoplus_{k=1}^r M_k$ of $M$.

For each $k \in [1, r]$, let $\lambda_k \in X^+$ be such that $m_k \in H_{\lambda_k}$. Then, we have $M_k \simeq V(\lambda_k)$. Each element of $M_k$ is of the form $xm_k$ for some $x \in U$. Let $k \neq l$ and $x, y \in U$. Without loss of generality, we may assume that $\lambda_k \geq \lambda_l$. Then, we have

$$(xm_k, ym_l) = (m_k, \varphi^*(x)ym_l).$$

Since $\varphi^*(x)ym_l \in M_l \simeq V(\lambda_l)$, by Theorem 2.3.1 (5), it is a linear combination of weight vectors of weight less than or equal to $\lambda_l$, while $m_k$ is of weight $\lambda_k(\geq \lambda_l)$. Hence, if $\lambda_l \neq \lambda_k$, then we have $(m_k, \varphi^*(x)ym_l) = 0$ by Lemma 2.4.4. Otherwise, one can write as $\varphi^*(x)ym_l = cm_l + m'$,
where \( c \in K \), and \( m' \) is a sum of weight vectors of weights less than \( \lambda_i(= \lambda_k) \). Then, we have
\[
(m_k, \varphi^*(x)ym_l) = c(m_k, m_l) = 0
\]
since \( m_k \) and \( m_l \) are orthogonal. Thus, we obtain that \( (M_k, M_l) = 0 \), as desired.

For a proof of the second assertion, replace \( m_k \) with \( \frac{1}{\lambda(m_k)}m_k \) in the argument above. Then, we have \( \lambda(t(m_k)) = 1 \). Since \( m_k \) is a highest weight vector of weight \( \lambda_k \), there exists an isomorphism \( \phi_k : M_k \to V(\lambda_k) \) of \( U \)-modules such that \( \phi_k(m_k) = v_{\lambda_k} \). Then, for each \( u, v \in M_k \), we have
\[
(u, v) = (m_k, m_k)(\phi_k(u), \phi_k(v))_{\lambda_k}.
\]
In particular, for each \( u, v \in \mathcal{L}_{M_k} \), we have
\[
(\phi_k(u), \phi_k(v))_{\lambda_k} - (u, v) = ((m_k, m_k)^{-1} - 1)(u, v) \in q^{-1}K_{\infty}.
\]
This implies that \( \phi_k \) is an almost isometry. Thus, the proof completes. \( \square \)

2.5. Crystal bases of \( U \)-modules.

**Definition 2.5.1.** Let \( V \) be a \( K \)-vector space and \( A \) a subring of \( K \). An \( A \)-lattice of \( V \), or a lattice of \( V \) over \( A \), is a free \( A \)-submodule \( L \) of \( V \) such that \( L \otimes_A K = V \).

**Example 2.5.2.** Let \( V \) be a finite-dimensional \( K \)-vector space equipped with a Hermitian inner product. Then, \( \mathcal{L}_V \) is a \( K_{\infty} \)-lattice of \( V \).

Let \( M \) be a finite-dimensional \( U \)-module equipped with a contragredient Hermitian inner product. Let \( \widetilde{E}_i, \widetilde{F}_i, i \in I \) denote the Kashiwara operators. They preserve \( \mathcal{L}_M \), and hence, induce \( \mathbb{C} \)-linear operators on \( \mathcal{L}_M \). A \( \mathbb{C} \)-basis of \( \mathcal{L}_M \) satisfying certain conditions is called a crystal basis of \( M \) (see e.g. [16] for detail). For each \( b \in \mathcal{B}_M \) and \( i \in I \), set
\[
\varphi_i(b) := \max\{k \mid \widetilde{E}_i^kb \neq 0\}, \quad \varepsilon_i(b) := \max\{k \mid \widetilde{F}_i^kb \neq 0\}.
\]

Let \( M \) and \( N \) be \( U \)-modules with crystal bases \( \mathcal{B}_M \) and \( \mathcal{B}_N \). Then, \( \mathcal{B}_M \otimes \mathcal{B}_N \) becomes a crystal basis of \( M \otimes N \) on which the Kashiwara operators and \( \varphi_i, \varepsilon_i \) act as
\[
\widetilde{E}_i(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \widetilde{E}_i(b_2) & \text{if } \varepsilon_i(b_1) < \varphi_i(b_2), \\ \widetilde{F}_i(b_1) \otimes b_2 & \text{if } \varepsilon_i(b_1) \geq \varphi_i(b_2), \end{cases}
\]
\[
\widetilde{F}_i(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \widetilde{F}_i(b_2) & \text{if } \varepsilon_i(b_1) \leq \varphi_i(b_2), \\ \widetilde{E}_i(b_1) \otimes b_2 & \text{if } \varepsilon_i(b_1) > \varphi_i(b_2), \end{cases}
\]
\[
\varphi_i(b_1 \otimes b_2) = \varphi_i(b_1) + \max(0, \varphi_i(b_2) - \varepsilon_i(b_1)),
\]
\[
\varepsilon_i(b_1 \otimes b_2) = \varepsilon_i(b_2) + \max(0, \varepsilon_i(b_1) - \varphi_i(b_2)).
\]

These formulas can be found, for example, in [12, Theorem 4.1]; one should note that \( \widetilde{E}_i, \widetilde{F}_i, \varphi_i, \varepsilon_i \) in this paper are \( \widetilde{E}_i, \widetilde{F}_i, \varphi_i, \varepsilon_i \) there.

When we consider the irreducible \( U \)-module \( V(\lambda) \), \( \lambda \in X^+ \) equipped with the contragredient Hermitian inner product \((\cdot, \cdot)_\lambda \) constructed in Proposition 2.4.2, we set
\[
\mathcal{L}(\lambda) := \mathcal{L}_{V(\lambda)}, \quad \mathcal{L}(\lambda) := \mathcal{L}_{V(\lambda)}, \quad b_\lambda := \text{ev}_{\infty}(v_\lambda).
\]
By [17, Theorem 2], \( V(\lambda) \) possesses a unique crystal basis \( \mathcal{B}(\lambda) \subseteq \mathcal{L}(\lambda) \) containing \( b_\lambda \).
Now, let us turn back to a general $U$-module $M$. By Proposition 2.4.5, we have an orthogonal irreducible decomposition $M = \bigoplus_{k=1}^{r} M_k$. Then, by Proposition 2.1.8, we have

$$L_M = \bigoplus_{k=1}^{r} L_k, \quad \overline{L}_M = \bigoplus_{k=1}^{r} \overline{L}_k,$$

where $L_k := L_M \cap M_k$ and $\overline{L}_k := L_k / q^{-1} L_k$. Moreover, the almost isometries $\phi_k$ in Proposition 2.4.5 induce isomorphisms

$$L_k \simeq L(\lambda_k), \quad \overline{L}_k \simeq \overline{L}(\lambda_k).$$

Let $B_k$ denote the basis of $\overline{L}_k$ corresponding to $B(\lambda_k)$ under the isomorphism $\overline{L}_k \simeq \overline{L}(\lambda_k)$ above. Then, we obtain a crystal basis $B_M := \bigcup_{k=1}^{r} B_k$ of $M$. Furthermore, every crystal basis of $M$ is of this form.

For each $\lambda \in X^+$, set $M[\lambda] := \bigoplus_{\lambda_k = \lambda} M_k$ to be the sum of all submodules of $M$ isomorphic to $V(\lambda)$. Set

$$L_M[\lambda] := L_M \cap M[\lambda] = \bigoplus_{\lambda_k = \lambda} L_k,$n$$

$$\overline{L}_M[\lambda] := L_M[\lambda] / q^{-1} L_M[\lambda] = \bigoplus_{\lambda_k = \lambda} \overline{L}_k,$n$$

$$B_M[\lambda] := B_M \cap \overline{L}_M[\lambda] = \bigcup_{\lambda_k = \lambda} B_k.$n$$

Also, set

$$M[\geq \lambda] := \bigoplus_{\mu \geq \lambda} M[\mu], \quad M[> \lambda] := \bigoplus_{\mu > \lambda} M[\mu],$$

and define $L_M[\geq \lambda], L_M[> \lambda], \overline{L}_M[\geq \lambda], \overline{L}_M[> \lambda], B_M[\geq \lambda], B_M[> \lambda]$ in the obvious way. Then, $M[\geq \lambda], M[> \lambda], \text{ and } M[> \lambda]/M[> \lambda]$ are $U$-modules possessing crystal bases $B_M[\geq \lambda], B_M[> \lambda], \text{ and } \{[b] \mid b \in B_M[\lambda]\}$, respectively, where $[b]$ denotes the image of $b \in L[\geq \lambda]$ in $\overline{L}[\geq \lambda]/\overline{L}[> \lambda]$.

2.6. Based U-modules. Let $\hat{U} = \bigoplus_{\lambda, \mu \in X} \lambda U_\mu = \bigoplus_{\lambda \in X} \hat{U}_1 \lambda$ denote the modified quantum group, where the $1_\lambda$ are the associated orthogonal idempotents (see [23, Chapter 23] for a precise definition).

**Definition 2.6.1** ([23, 23.1.4]). A unital $\hat{U}$-module is a $\hat{U}$-module $M$ such that for each $m \in M$, we have the following:

1. $1_\lambda m = 0$ for all but finitely many $\lambda \in X$.
2. $\sum_{\lambda \in X} 1_\lambda m = m$.

As explained in [23, 23.1.4], a weight $U$-module has a unital $\hat{U}$-module structure, and vice versa.

Set

$$A := \mathbb{C}[q, q^{-1}] \subseteq K.$$
Let $\hat{U}_A$ denote the $A$-form of $U$. It is an $A$-subalgebra of $\hat{U}$ generated by the divided powers $E_i^{(n)}1_{\lambda}, F_i^{(n)}1_{\lambda}, i \in I, \lambda \in X, n \in \mathbb{Z}_{\geq 0}$.

**Definition 2.6.2.** Let $M$ be a weight $U$-module. An $A$-form of $M$ is an $A$-lattice $M_A$ of $M$ which is simultaneously an $\hat{U}_A$-submodule.

**Definition 2.6.3.** Let $M$ be a $U$-module. A bar-involution $\psi_M$ on $M$ is an involutive $C$-linear automorphism such that
\[
\psi_M(xm) = \psi(x)\psi_M(m) \quad \text{for all } x \in U, \ m \in M.
\]

**Definition 2.6.4.** Let $M$ be a weight $U$-module equipped with a contragredient Hermitian inner product $(\cdot, \cdot)_M$, an $A$-form $M_A$, a bar-involution $\psi_M$, and a crystal basis $B_M$. We say that $(M, (\cdot, \cdot)_M, M_A, \psi_M, B_M)$, or simply $M$, is a based $U$-module if the following two conditions are satisfied:

1. The quotient map $ev_\infty : L_M \to \overline{L}_M$ restricts to an isomorphism
\[
L_M \cap M_A \cap \psi_M(L_M) \to \overline{L}_M
\]

of $C$-vector spaces; let $G : \overline{L}_M \to L_M \cap M_A \cap \psi_M(L_M)$ denote its inverse.

2. For each $b \in B_M$, we have $\psi_M(G(b)) = G(b)$.

**Remark 2.6.5.** This definition is equivalent to [23, 27.1.2]. We choose this definition so that we can straightforwardly generalize it to the quantum groups setting.

**Example 2.6.6.** Let $\lambda \in X^+$. Then, the irreducible $U$-module $V(\lambda)$ possesses a unique bar-involution $\psi_\lambda$ fixing the highest weight vector $v_\lambda$. Set $V(\lambda)_A := \hat{U}_A v_\lambda$. Then, $(V(\lambda), (\cdot, \cdot)_\lambda, V(\lambda)_A, \psi_\lambda, B(\lambda))$ is a based $U$-module. The set $G(\lambda) := G(B(\lambda))$ is called the canonical basis of $V(\lambda)$.

A finite-dimensional based $U$-module is cellular in the following sense.

**Proposition 2.6.7** ([23, Propositions 27.1.7 and 27.1.8]). Let $(M, (\cdot, \cdot)_M, M_A, \psi_M)$ be a finite-dimensional based $U$-module, and $B_M$ a crystal basis. Then, for each $\lambda \in X^+$, the $U$-modules $M[\geq \lambda], M[> \lambda]$, and $M[\geq \lambda]/M[> \lambda]$ are based $U$-modules with the same (or induced) contragredient Hermitian inner products and bar-involutions as $M$, and the $A$-forms spanned by $G(B_M[\geq \lambda]), G(B_M[> \lambda]), [G(B_M[\lambda])] := \{G(b) + M[> \lambda] \mid b \in B_M[\lambda]\}$, respectively. Furthermore, there exists an isomorphism
\[
M[\geq \lambda]/M[> \lambda] \to V(\lambda)^{\otimes m_\lambda}
\]
of $U$-modules which restricts to a bijection
\[
[G(B_M[\lambda])] \to G(\lambda)^{m_\lambda}
\]

between the canonical bases, where $m_\lambda := \dim_k \text{Hom}_U(V(\lambda), M)$ denotes the multiplicity of $V(\lambda)$ in $M$.

3. Quantum symmetric pairs

In this section, we formulate the quantum groups following Kolb [19]. Then, we introduce the notion of based modules over the quantum groups generalizing that over the quantum groups.
3.1. Quantum groups. Let \((I, I_\bullet, \tau)\) be a Satake diagram, and \(\mathfrak{k}\) the associated symmetric pair subalgebra (see e.g., [5, Table 4] for the list of Satake diagrams). Namely, \(I_\bullet\) is a subdiagram of \(I\), and \(\tau\) is a diagram involution on \(I\) satisfying certain conditions (see [19, Definition 2.3]), and \(\mathfrak{k}\) is the Lie subalgebra of \(\mathfrak{g}\) generated by the following elements:

- \(e_i, i \in I_\bullet\).
- \(h_i^\tau := \begin{cases} h_i & \text{if } i \in I_\bullet, \\ h_i - h_{\tau(i)} & \text{if } i \in I_0. \end{cases}\)
- \(b_i := \begin{cases} f_i & \text{if } i \in I_\bullet, \\ f_i + \zeta_i T_{w_\bullet}(e_{\tau(i)}) & \text{if } i \in I_0. \end{cases}\)

Here, \(I_0 := I \setminus I_\bullet\), \(w_\bullet\) is the longest element of the Weyl group of \(I_\bullet\), \(\bar{T}_{w_\bullet}\) is the braid group action on \(\mathfrak{g}\) in terms of triple exponentials, and \(\zeta_i \in \mathbb{C}\) are certain scalars. There exists a Lie algebra involution \(\theta\) on \(\mathfrak{g}\) such that

\[
\mathfrak{k} = \mathfrak{g}_\theta := \{ x \in \mathfrak{g} \mid \theta(x) = x \}.
\]

It is known that \(\mathfrak{k}\) is a reductive Lie algebra ([9, Proposition 1.13.3]).

Let \(U^\flat = U^\flat_{\xi, \kappa}\) denote the quantum group associated with the Satake diagram \((I, I_\bullet, \tau)\) and parameters \(\xi = (\xi_i)_{i \in I_\bullet} \in (K_1^\times)_I\), \(\kappa = (\kappa_i)_{i \in I_\bullet} \in K_1^\times\) obeying constraints [5, Equations (3.4)–(3.8) and the assumption in Proposition 4.6]. Namely, \(U^\flat\) is the subalgebra of \(U\) generated by \(E_i, i \in I_\bullet, B_i, k_i, i \in I\) where

\[
k_i^{\pm 1} := \begin{cases} K_i^{\pm 1} & \text{if } i \in I_\bullet, \\ (K_i K_{\tau(i)})^{\pm 1} & \text{if } i \in I_0, \end{cases}
\]

\[
B_i := \begin{cases} F_i & \text{if } i \in I_\bullet, \\ F_i + \zeta_i T_{w_\bullet}(E_{\tau(i)}) K_i^{-1} + \kappa_i K_i^{-1} & \text{if } i \in I_0. \end{cases}
\]

One of the most distinguished properties of \(U^\flat\) is that it is a right coideal of \(U\), i.e., \(\Delta(U^\flat) \subseteq U^\flat \otimes U\). Therefore, the tensor product of a \(U^\flat\)-module and a \(U\)-module becomes a \(U^\flat\)-module via \(\Delta\).

**Proposition 3.1.1** ([5, Proposition 4.6]). For each \(i \in I_\bullet\), we have \(\varphi(B_i) \in U^\flat\), and

\[
\varphi(B_i) = q_i^{-1} \zeta_{\tau(i)}^{-1} T_{w_\bullet}^{-1}(B_{\tau(i)}) K_{w_\bullet(h_{\tau(i)})} K_i^{-1}.
\]

Consequently, the involution \(\varphi^*\) on \(U^\flat\) preserves \(U^\flat\).

**Proposition 3.1.2** ([2, Theorem 3.11]). There exists a unique \(\mathbb{C}\)-algebra involution \(\psi^\flat\), called the \(\flat\)-involution on \(U^\flat\) such that

\[
\psi^\flat(E_i) = E_i, \quad \psi^\flat(B_i) = B_i, \quad \psi^\flat(k_i) = k_i^{-1}, \quad \psi^\flat(q) = q^{-1}.
\]

3.2. Canonical bases of based \(U\)-modules.

**Definition 3.2.1.** Let \(M\) be a \(U^\flat\)-module. An \(\flat\)-involution on \(M\) is a \(\mathbb{C}\)-linear involution \(\psi^\flat_M\) on \(M\) such that

\[
\psi^\flat_M(xm) = \psi^\flat(x)\psi^\flat_M(m) \quad \text{for all } x \in U^\flat \text{ and } m \in M.
\]

Let \(M\) be a finite-dimensional \(U\)-module equipped with a \(\bar{\flat}\)-involution \(\bar{\psi}_M\). Let \(T\) denote the quasi-\(K\)-matrix (see [5] for a precise definition). Then, \(\psi^\flat_M := T \circ \psi_M\) becomes an \(\bar{\flat}\)-involution of the \(U^\flat\)-module \(M\). We call it the \(\bar{\flat}\)-involution associated with \(\psi_M\).
Theorem 3.2.2 ([5, Theorem 5.7]). Let $M$ be a finite-dimensional based $U$-module. Then, for each $b \in B_M$, there exists a unique $G^i(b) \in \mathcal{L}_M \cap M$ satisfying the following:

1. $\psi^i_M(G^i(b)) = G^i(b)$.
2. $\text{ev}_\infty(G^i(b)) = b$.

We call $G^i(B_M)$ the canonical basis associated with the canonical basis $G(B_M)$. When $B_M = B(\lambda)$ for some $\lambda \in X^+$, we denote by $G^i(\lambda)$ the canonical basis associated with $G(\lambda)$.

Lemma 3.2.3. Let $M$ be a finite-dimensional based $U$-module. Let $M = \bigoplus_{k=1}^r M_k$ be the irreducible decomposition such that $B_M = \bigsqcup_{k=1}^r B_k$, where $B_k := B_M \cap \mathcal{L}_M$. Let $\lambda_k \in X^+$ be such that $M_k \cong V(\lambda_k)$. Then, for each $b \in B_k$, we have

$$G^i(b) \in M_k \oplus \bigoplus_{l \in \mathbb{Z} \setminus \mathbb{N}, \lambda_l > \lambda_k} M_l.$$

Proof. From Proposition 2.6.7, we see that

$$G(b') \in M_l \oplus \bigoplus_{l' \in \mathbb{Z} \setminus \mathbb{N}, \lambda_l > \lambda_l} M_{l'}$$

for all $l \in [1, r]$, $b' \in B_l$. Then, by the construction of $G^i(b)$ in [5, Theorem 5.7], we have

$$G^i(b) \in M_k + \bigoplus_{l \in \mathbb{Z} \setminus \mathbb{N}, \lambda_l > \lambda_k} KG(b').$$

Combining these two facts, we obtain the assertion. \hfill \Box

3.3. Modified quantum groups. Following [5, 3.1], we set

$$X^\ast := X/\{x + w \circ \tau(x) \mid x \in X\}, \quad Y^\ast := \{h \in Y \mid h + w \circ \tau(h) = 0\}.$$

Then, the perfect pairing on $Y \times X$ induces a bilinear map $Y^\ast \times X^\ast \rightarrow \mathbb{Z}$ given by

$$\langle h, \lambda \rangle := \langle h, \lambda \rangle, \quad h \in Y^\ast, \lambda \in X,$$

where $\lambda$ denotes the image of $\lambda$ in $X^\ast$.

Recall that $Q = \sum_{i \in I} \mathbb{Z} \alpha_i$ is the root lattice. From the definition of quantum groups, we see that $U$ is a $Q$-graded algebra such that $E_i, F_i, K_i^{\pm1}$ are homogeneous of degree $\alpha_i, -\alpha_i, 0$, respectively. Set

$$Q' := \{\lambda \in Q \mid \lambda \in X^\ast\} \subseteq X^\ast.$$

Then, $U$ is $Q'$-graded, and $E_i, F_i, K_i^{\pm1}$ are homogeneous of degree $\overline{\alpha}_i, -\overline{\alpha}_i, \overline{\alpha}_i$, respectively. Hence, $U'$ is a $Q'$-graded algebra. Let $U'(\overline{\lambda})$ denote the homogeneous part of degree $\overline{\lambda} \in Q'$. For $\zeta \in X^\ast \setminus Q'$, set $U'(\zeta) := 0$. Then, we have

$$U' = \bigoplus_{\overline{\lambda} \in Q'} U'(\overline{\lambda}) = \bigoplus_{\zeta \in X^\ast} U'(\zeta).$$

For each $\zeta, \eta \in X^\ast$, let $\pi_{\zeta, \eta}$ denote the composition

$$\pi_{\zeta, \eta} : U'(\zeta - \eta) \hookrightarrow U' \twoheadrightarrow U'/\left(\sum_{h \in Y^\ast} (K_h - q(h, \zeta))U' + \sum_{h \in Y^\ast} U'(K_h - q(h, \eta))\right),$$

and set $\zeta U_{\eta}$ to be the image of $\pi_{\zeta, \eta}$. Clearly, we have $\zeta U_{\eta} = 0$ if $\zeta - \eta \notin Q'$. \hfill \Box
The modified $u$ quantum group $\hat{U}^u$ is defined by

$$\hat{U}^u := \bigoplus_{\zeta, \eta \in X^\prime} \zeta \hat{U}^u \eta = \bigoplus_{\zeta, \eta \in X^\prime} \zeta U^u_\eta.$$  

**Remark 3.3.1.** This definition slightly differs from the one in [5]. However, it is what the authors of [5] actually considered. The author would like to thank Weiqiang Wang for a fruitful discussion on this matter.

The modified $u$ quantum group $\hat{U}^u$ admits an associative algebra structure just like $\hat{U}$ does (see [23, Chapter 23]). For each $\zeta, \zeta_1, \zeta_2, \eta_1, \eta_2 \in X^\prime$ and $x_1, x_2 \in U^u(\zeta_i - \eta_i)$, set

$$\pi_{\zeta_1, \eta_1}(x_1) \pi_{\zeta_2, \eta_2}(x_2) := \delta_{\eta_1, \zeta_2} \pi_{\zeta_1, \eta_2}(x_1x_2).$$

For each $\zeta \in X^\prime$, set

$$1_\zeta := \pi_{\zeta, \zeta}(1).$$

These $1_\zeta$'s form a family of orthogonal idempotents:

$$1_\zeta 1_\eta = \delta_{\zeta, \eta} 1_\zeta.$$

Also, we have

$$\zeta U^u_\eta = 1_\zeta \hat{U}^u 1_\eta.$$

The modified $u$ quantum group $\hat{U}^u$ is equipped with a $U^u$-bimodule structure defined as follows: For each $x, z \in U^u$ and $y \in \hat{U}^u$, set

$$xyz := \sum_{\zeta, \eta \in X^\prime} (x1_\zeta)y(1_\eta z).$$

Especially, we have for each $i \in I$ and $j \in I^*$,

$$E_j 1_\zeta = 1_{\zeta + \eta} E_j,$$

$$B_i 1_\zeta = 1_{\zeta - \eta} B_i,$$

$$k_i^{\pm 1} 1_\zeta = 1_{\zeta + k_i^{\pm 1} = q^{\pm \langle h^*_i, \zeta \rangle} 1_\zeta}.$$  

**Definition 3.3.2.** A $U^u$-module $M$ is said to be an $X^\prime$-weight module if it admits an $X^\prime$-gradation

$$M = \bigoplus_{\zeta \in X^\prime} M_\zeta$$

satisfying the following:

1. $E_j M_\zeta \subseteq M_{\zeta + \eta}$ for all $j \in I^*$, $\zeta \in X^\prime$.
2. $B_i M_\zeta \subseteq M_{\zeta - \eta}$ for all $i \in I$, $\zeta \in X^\prime$.
3. $k_i^{\pm 1} m = q^{\pm \langle h^*_i, \zeta \rangle} m$ for all $i \in I$, $\zeta \in X^\prime$, $m \in M_\zeta$.

The subspace $M_\zeta$ (resp., a vector in $M_\zeta$) is referred to as the $X^\prime$-weight space (resp., an $X^\prime$-weight vector) of weight $\zeta$.

**Example 3.3.3.** Let $M = \bigoplus_{\lambda \in X} M_\lambda$ be a weight $U$-module. Then, it admits an $X^\prime$-weight module structure such that

$$M_\zeta = \bigoplus_{\lambda \in X} M_\lambda$$

for all $\zeta \in X^\prime$. We call it the canonical $X^\prime$-weight module structure of $M$. 
Remark 3.3.4. Since the pairing \( Y^* \times X^* \to \mathbb{Z} \) is not perfect, there may exist \( \zeta, \eta \in X^* \) such that \( \langle h, \zeta \rangle = \langle h, \eta \rangle \) for all \( h \in Y^* \), but \( \zeta \neq \eta \). Hence, the condition \( \sum_{i}^{k} m_i = q^{\pm(h, \zeta)} m \) for all \( i \in I \) cannot ensure that \( m \in M_\zeta \).

Definition 3.3.5. A unital \( \hat{U}^\prime \)-module is a \( \hat{U}^\prime \)-module \( M \) such that for each \( m \in M \), we have

1. \( 1 \zeta m = 0 \) for all but finitely many \( \zeta \in X^* \).
2. \( \sum_{\zeta \in X^*} 1 \zeta m = m \).

As in the quantum groups case, each unital \( \hat{U}^\prime \)-module is equipped with an \( X^* \)-weight module structure, and vice versa.

Let \( M \) be a unital \( \hat{U}^\prime \)-module. Then, it admits a \( \hat{U}^\prime \)-module structure. This action is compatible with the \( U^\prime \)-bimodule structure of \( \hat{U}^\prime \).

Definition 3.3.6. Let \( M \) be an \( X^* \)-weight module. A \( U^\prime \)-submodule \( N \subseteq M \) is said to be an \( X^* \)-weight submodule if we have \( N = \bigoplus_{\zeta \in X^*} (N \cap M_\zeta) \).

The following is immediate from the definition.

Proposition 3.3.7. Let \( M \) be an \( X^* \)-weight module, and \( N \subseteq M \) an \( X^* \)-weight submodule.

1. \( N \) itself is an \( X^* \)-weight module such that \( N_\zeta = N \cap M_\zeta \) for all \( \zeta \in X^* \).
2. The quotient \( \hat{U}^\prime \)-module \( M/N \) admits an \( X^* \)-weight module structure such that \( (M/N)_\zeta = M_\zeta/N = M_\zeta \cap N_\zeta \) for all \( \zeta \in X^* \).

Proposition 3.3.8. Let \( M \) be an \( X^* \)-weight module, and \( N \subseteq M \) a \( U^\prime \)-submodule. Suppose that \( N \) is generated by \( X^* \)-weight vectors of \( M \). Then, \( N \) is an \( X^* \)-weight submodule.

Proof. As one can easily see, a sum of \( X^* \)-weight submodules of \( M \) is an \( X^* \)-weight submodule of \( M \). Hence, it suffices to show the assertion for the case when \( N \) is generated by a single \( X^* \)-weight vector \( m \in M_\eta \) for some \( \eta \in X^* \). Let \( \zeta' \in Q^0 \) and \( x \in U^\prime(\zeta') \). Then, by equation (5), we have

\[
1_\zeta x = x 1_{\zeta - \zeta'}.
\]

Hence, it follows that

\[
1_\zeta (xm) = x 1_{\zeta - \zeta'} m = \delta_{\zeta - \zeta', \eta} xm \in N.
\]

Therefore, \( N \) is a \( \hat{U}^\prime \)-submodule of \( M \). Hence, we have

\[
1_\zeta N = N \cap M_\zeta.
\]

On the other hand, since \( 1_\zeta \)'s are orthogonal idempotents and \( M \) is a unital \( \hat{U}^\prime \)-module, we obtain

\[
N = \bigoplus_{\zeta \in X^*} 1_\zeta N.
\]

This proves the assertion.

The modified quantum group \( \hat{U} \) admits a \( \hat{U}^\prime \)-bimodule structure given as follows: For each \( x_1, x_2 \in U^\prime \), \( \zeta \in X^* \), and \( y \in \hat{U} \), set

\[
(x_1 1_{\zeta_1}) y (1_{\zeta_2} x_2) := \sum_{\lambda \in X \atop \lambda_i = \zeta_i} (x_1 1_{\lambda_1}) y (1_{\lambda_2} x_2).
\]

In the right-hand side, we regard \( x_i \) as an element of \( U \) via the inclusion \( U^\prime \hookrightarrow U \).

Let \( M \) be a unital \( \hat{U} \)-module. Then, it admits a weight module structure \( M = \bigoplus_{\lambda \in X} M_\lambda \), and hence, the canonical \( X^* \)-weight module structure \( M = \bigoplus_{\zeta \in X^*} M_\zeta \) (see
Example 3.3.3. Therefore, it admits a unital $\mathcal{U}$-module structure. This structure is compatible with the $\mathcal{U}$-bimodule structure of $\mathcal{U}$.

More generally, let us consider a Satake subdiagram.

**Definition 3.3.9.** A Satake subdiagram of a Satake diagram $(I, I_\bullet, \tau)$ is a triple $(J, J_\bullet, \sigma)$ satisfying the following:

- $(J, J_\bullet, \sigma)$ itself is a Satake diagram.
- $J \subseteq I$, $J_\bullet \subseteq I_\bullet$, $\sigma(j) \in \{j, \tau(j)\}$ for all $j \in J$.
- $w_\bullet(\alpha_{\tau(j)}) = w_\bullet(J)(\alpha_{\sigma(j)})$ for all $j \in J \cap I_\circ$.

Here and after, $A(J)$ denotes an object $A$ constructed from $(J, J_\bullet, \sigma)$ instead of $(I, I_\bullet, \tau)$.

Let $(J, J_\bullet, \sigma)$ be a Satake subdiagram of our Satake diagram $(I, I_\bullet, \tau)$, and $\varsigma(J), \kappa(J)$ parameters for $U_\bullet(J)$ such that $\varsigma_j = \varsigma(J)_j, \kappa_j = \kappa(J)_j$ for all $j \in J \cap I_\circ$. We claim that $U_\bullet(J) \subseteq U$. In fact, for each $j \in J \cap I_\circ$, we have

$$T_{w_\bullet}(E_{\tau(j)}) = T_{w_\bullet(J)}T_{w_\bullet(J)^{-1}}(E_{\tau(j)}) = T_{w_\bullet(J)}(E_{\sigma(j)}).$$

The last equality follows from the identity $w_\bullet(J)^{-1}w_\bullet(\alpha_{\tau(j)}) = \alpha_{\sigma(j)}$ and [15, Proposition 8.20]. Hence, we obtain

$$B_j(J) = F_j - \varsigma(J)_j T_{w_\bullet(J)}(E_{\sigma(j)})K_j^{-1} = F_j - \varsigma J T_{w_\bullet(J)}(E_{\sigma(j)})K_j^{-1} = B_j \in U_\bullet.$$

Similar statements for the other generators are verified more easily.

**Example 3.3.10.** Let $I$ be a Dynkin diagram, and consider two Satake diagrams $(I, \emptyset, \text{id})$ and $(I, I, -w_0)$, where $-w_0$ denotes the diagram automorphism induced by the longest element of the Weyl group of $I$. Namely, the first diagram consists of only black nodes, while the second one only white nodes with the same underlying Dynkin diagram $I$. Then, $(I, I, \text{id})$ is a Satake subdiagram of $(I, I, -w_0)$.

For each $\lambda \in X$, define $\lambda' \in X(J)$ by

$$\langle h_j, \lambda' \rangle = \begin{cases} \frac{1}{2}\langle w_\bullet(J)(h_j) - h_j, \lambda \rangle & \text{if } j \in J_\circ \cap I_\bullet, \\ \langle h_j, \lambda \rangle & \text{otherwise.} \end{cases}$$

By [19, (5.6)], for each $j \in J_\circ$, we have

$$w_\bullet(J)(h_j) - h_j = w_\bullet(J)(h_{\sigma(j)}) - h_{\sigma(j)} \in \sum_{k \in J_\bullet} \mathbb{Z}h_k.$$

Then, it follows that

$$\langle h_j, \lambda + w_\bullet \circ \tau(\lambda) \rangle = \langle h_j, \lambda' + w_\bullet(J) \circ \sigma(\lambda') \rangle \quad \text{for all } j \in J.$$

This shows that there exists a well-defined $\mathbb{Z}$-linear map $\cdot|_{h(J)} : X^1 \to X^1(J)$ making the following diagram commute

$$\begin{array}{ccc}
X & \longrightarrow & X^1(J) \\
\downarrow \cdot|_{h(J)} & & \downarrow \cdot|_{h(J)} \\
X(J) & \longrightarrow & X^1(J),
\end{array}$$

where the left vertical arrow $\cdot|_{h(J)} : X \to X(J)$ denotes the ordinary restriction map from the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ to the one $\mathfrak{h}(J)$ of $\mathfrak{g}(J)$. Then, as before, $\mathcal{U}^1$ admits a
\( \mathring{U}^t(J) \)-bimodule structure: For each \( x_1, x_2 \in \mathring{U}^t(J), \eta_i \in X^t(J), \) and \( y \in \mathring{U}, \) set
\[
(x_1 1_{\eta_i})y(1_{\eta_j}x_2) = \sum_{\zeta_i \in X^t, \zeta_i|_{h(J)} = \eta_i} (x_1 1_{\zeta_i})y(1_{\zeta_i}x_2).
\]

Let \( M \) be a unital \( \mathring{U}^t \)-module. Then, it admits an \( X^t \)-weight module structure \( M = \bigoplus_{\zeta \in X^t} M_\zeta \), where \( M_\zeta = 1_{\zeta} M \). By restriction, we obtain a \( \mathring{U}^t(J) \)-module structure on \( M \). Furthermore, it becomes an \( X^t(J) \)-weight module such that \( M = \bigoplus_{\eta \in X^t(J)} M_\eta \), where
\[
M_\eta := \bigoplus_{\zeta \in X^t, \zeta|_{h(J)} = \eta} M_\zeta.
\]

Therefore, \( M \) admits a unital \( \mathring{U}^t(J) \)-module structure. Again, this structure is compatible with the \( \mathring{U}^t(J) \)-bimodule structure of \( \mathring{U}^t \).

### 3.4. Based \( \mathring{U}^t \)-modules.

Recall that \( \mathring{U}^t \) acts on each weight \( \mathring{U} \)-module via the canonical \( X^t \)-weight module structure. Set
\[
\mathring{U}^t_\mathbf{A} := \{ x \in \mathring{U}^t \mid xV(\lambda)_\mathbf{A} \subseteq V(\lambda)_\mathbf{A} \text{ for all } \lambda \in X^+ \}.
\]

**Definition 3.4.1.** Let \( M \) be an \( \mathring{X}^t \)-weight module. An \( \mathbf{A} \)-form of \( M \) is an \( \mathbf{A} \)-lattice \( M_\mathbf{A} \) of \( M \) which is simultaneously a \( \mathring{U}^t_\mathbf{A} \)-submodule.

**Definition 3.4.2.** Let \( M \) be a \( \mathring{U}^t \)-module equipped with a Hermitian inner product \( (\cdot, \cdot)_M \). We say that \( (\cdot, \cdot)_M \) is contragredient if it satisfies
\[
(6) \quad (xu, v)_M = (u, \psi^*(x)v)_M \quad \text{for all } x \in \mathring{U}^t, u, v \in M.
\]

**Remark 3.4.3.** Each contragredient Hermitian inner product on a \( \mathring{U} \)-module \( M \) is a Hermitian inner product of the \( \mathring{U}^t \)-module \( M \).

**Definition 3.4.4.** Let \( M \) be an \( \mathring{X}^t \)-weight module equipped with a contragredient Hermitian inner product \( (\cdot, \cdot)_M \), an \( \mathbf{A} \)-form \( M_\mathbf{A} \), an \( \mathbf{A} \)-linear involution \( \psi^*_M \), and a basis \( \mathbf{B}_M \) of \( \mathcal{L}_M \). We say that \((M, (\cdot, \cdot)_M, M_\mathbf{A}, \psi^*_M, \mathbf{B}_M)\), or simply \( M \), is a based \( \mathring{U}^t \)-module if the following two conditions are satisfied:

1. The quotient map \( \text{ev}_\infty : \mathcal{L}_M \to \mathcal{L}_M \) restricts to an isomorphism
\[
\mathcal{L}_M \cap M_\mathbf{A} \cap \psi^*_M(\mathcal{L}_M) \to \mathcal{L}_M
\]
   of \( \mathbb{C} \)-vector spaces; let \( G^t : \mathcal{L}_M \to \mathcal{L}_M \cap M_\mathbf{A} \cap \psi^*_M(\mathcal{L}_M) \) denote its inverse.

2. For each \( b \in \mathbf{B}_M \), we have \( \psi^*_M(G^t(b)) = G^t(b) \).

**Example 3.4.5.** Let \( \lambda \in X^+ \), and consider the irreducible \( \mathring{U} \)-module \( V(\lambda) \). Then, by Theorem 3.2.2, the quintuple \((V(\lambda), (\cdot, \cdot)_\lambda, V(\lambda)_\mathbf{A}, \psi^*_\lambda, \mathbf{B}(\lambda))\) is a based \( \mathring{U}^t \)-module.

**Lemma 3.4.6.** Let \( M \) be a based \( \mathring{U}^t \)-module and \( v \in \mathcal{L}_M \cap M_\mathbf{A} \). Suppose that \( \psi^*_M(v) = v \).

Then, we have
\[
v = G^t(\text{ev}_\infty(v)).
\]

In particular, if \( v \in q^{-1} \mathcal{L}_M \), then \( \text{ev}_\infty(v) = 0 \).

**Proof.** By our assumption that \( v = \psi^*_M(v) \in \psi^*_M(\mathcal{L}_M) \), we have
\[
v \in \mathcal{L}_M \cap M_\mathbf{A} \cap \psi^*_M(\mathcal{L}_M).
\]
Since \( G^t \) is the inverse of \( \text{ev}_\infty \), the assertion follows. \( \square \)
Proposition 3.4.7. Let $M$ be a finite-dimensional based $\mathcal{U}$-module with $\mathcal{B}_M$ being an orthonormal basis of $\overline{\mathcal{E}}_M$. Then, $G^i(\mathcal{B}_M)$ forms an almost orthonormal $K$-basis of $M$, a free $K_{\infty}$-basis of $\mathcal{L}_M$, and a free $A$-basis of $M_A$.

**Proof.** Since $\dim \mathcal{E}_M = \dim K M$, $G^i(\mathcal{B}_M)$ forms a $K$-basis of $M$. For each $b, b' \in \mathcal{B}_M$, we have

$$ev_\infty((G^i(b), G^i(b'))) = (ev_\infty(G^i(b)), ev_\infty(G^i(b'))) = (b, b')_M = \delta_{b, b'}.$$ 

This implies that $G^i(\mathcal{B}_M)$ is almost orthonormal. \qed

**Definition 3.4.8.** Let $M, N$ be based $\mathcal{U}$-modules and $f \in \text{Hom}_{\mathcal{U}}(M, N)$.

1. We say that $f$ is a homomorphism of based $(\mathcal{U})$-modules if

$$f(G^i(b)) \in G^i(\mathcal{B}_N) \cup \{0\}$$

for all $b \in \mathcal{B}_M$.

2. We say that $M$ is a based submodule of $N$ if $f$ is the inclusion map and a homomorphism of based modules.

3. We say that $N$ is a based quotient module of $M$ if $f$ is the quotient map and a homomorphism of based modules.

**Proposition 3.4.9.** Let $M$ be a finite-dimensional based $\mathcal{U}$-module with $\mathcal{B}_M$ being an orthonormal basis, and $N \subseteq M$ a $\mathcal{U}$-submodule. Suppose that $N = K G^i(\mathcal{B}_N)$, where $\mathcal{B}_N := \mathcal{B}_M \cap \overline{\mathcal{E}}_N$.

1. Set $N_A := N \cap M_A$, and $\psi_M^i := \psi_M^i|_N$. Then, $(N, (\cdot, \cdot)_N, N_A, \psi_M^i, \mathcal{B}_N)$ is a based submodule of $M$.

2. Let $\psi_{M/N}^i$ denote the $\mathcal{U}$-bar-involution on $M/N$ induced from $\psi_M^i$, and $\mathcal{B}_{M/N} := \{b + \overline{\mathcal{E}}_N \mid b \in \mathcal{B}_M \setminus \mathcal{B}_N\}$. Then, $(M/N, (\cdot, \cdot)_{M/N}, M_A/N_A, \psi_{M/N}^i, \mathcal{B}_{M/N})$ is a based quotient module of $M$.

**Proof.** By Proposition 3.4.7, $G^i(\mathcal{B}_N)$ and $G^i(\mathcal{B}_M)$ form $\psi_M^i$-invariant almost orthonormal free $A$-bases of $N_A$ and $M_A$, respectively. Furthermore, $\{[G^i(b)] \mid b \in \mathcal{B}_M \setminus \mathcal{B}_N\}$ forms a $\psi_{M/N}^i$-invariant almost orthonormal free $A$-basis of $M_A/N_A$. Then, the assertion follows if we prove that $N_A$ is a $\mathcal{U}^i_A$-submodule.

Let $b \in \mathcal{B}_N$ and $x \in \mathcal{U}^i_A$. Then, we have $x G^i(b) \in N \cap M_A = N_A$. Thus, the proof completes. \qed

4. *Quantum group of type AI*

In the remainder, we restrict our attention to the $\mathfrak{k}$-quantum group of type AI with parameters $q_i = q^{-1}$ and $\kappa_i = 0$ for all $i \in I_q$. It is a quantum group associated with Satake diagram $(I, 0, \text{id})$, where $I = [1, n - 1]$ is the Dynkin diagram of type $A_{n-1}$. We assume that $n \geq 3$ unless otherwise specified. The case when $n = 2$ will be studied separately because $\mathfrak{k}$ is not a simple Lie algebra in this case.

Let $\mathfrak{g} = \mathfrak{g}(I)$ denote the complex simple Lie algebra associated with $I$. Recall that $Y = \bigoplus_{i=1}^{n-1} \mathbb{Z}h_i$ and $X = \bigoplus_{i=1}^{n-1} \mathbb{Z}w_i$ are the coroot lattice and the weight lattice for $\mathfrak{g}$, respectively. We often identify $\lambda = \sum_{i=1}^{n} \lambda_i w_i \in X$ with $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in \mathbb{Z}^{n-1}$. For each $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in X$, we have $\lambda \in X^+$ if and only if $\lambda_i \geq 0$ for all $i \in I$. 
4.1. Symmetric pair of type $A_l$. Let $\mathfrak{g}$ be the subalgebra of $\mathfrak{g}$ generated by $b_i := f_i + e_i$, $i \in I$. The defining relations of $\mathfrak{g}$ for the generators $b_1, \ldots, b_{n-1}$ are as follows (see [28, Definition 2.5 and Theorem 2.7]):

$$
\begin{align*}
[b_i, b_j] &= 0 \quad \text{if } |i - j| > 1, \\
[b_i, [b_i, b_j]] &= b_j \quad \text{if } |i - j| = 1.
\end{align*}
$$

(7)

Let us identify $\mathfrak{g}$ with the special linear Lie algebra $\mathfrak{sl}_n := \{X \in \text{Mat}_n(\mathbb{C}) \mid \text{tr} X = 0\}$ in the usual way. Consider the special orthogonal Lie algebra $\mathfrak{so}_n := \{X \in \mathfrak{g} \mid X + {}^t X = 0\} \subseteq \mathfrak{g}$. It is generated by $b'_i := f_i - e_i$, $i \in I$. Let $m$ denote the rank of $\mathfrak{so}_n$:

$$
m = \begin{cases} 
\frac{n}{2} & \text{if } n \in \mathbb{Z}_{ev}, \\
\frac{n-1}{2} & \text{if } n \in \mathbb{Z}_{odd}.
\end{cases}
$$

Then, $\bigoplus_{i=1}^m \mathfrak{ch}_{2i-1}$ forms a Cartan subalgebra of $\mathfrak{so}_n$ since it forms an abelian subalgebra of dimension $m$.

Set $I_\mathfrak{t} := [1, m]$, and identify it with the Dynkin diagram of type $D_m$ if $n \in \mathbb{Z}_{ev}$ or $B_m$ if $n \in \mathbb{Z}_{odd}$ whose vertices are labeled by $I_\mathfrak{t}$ in the same manner as [13, Section 11.4]. Let $e'_i, f'_i, h'_i$, $i \in I_\mathfrak{t}$ be the Chevalley generators of $\mathfrak{g}(I_\mathfrak{t})$.

Let $I_\otimes := I \cap \mathbb{Z}_{odd} = \{2i - 1 \mid i \in I_\mathfrak{t}\}$, and set $\mathfrak{h}_\mathfrak{t}$ to be the subalgebra of $\mathfrak{t}$ generated by $b_i$, $i \in I_\otimes$. As before, we see that $\mathfrak{h}_\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{t}$. Let $\{b'_i \mid i \in I_\otimes\} \subseteq \mathfrak{h}_\mathfrak{t}$ denote the dual basis of $\{b_i \mid i \in I_\otimes\}$. For each $i \in I_\mathfrak{t}$, set

$$
\gamma_i := \begin{cases} 
\frac{b^{2i-1} - b^{2i+1}}{2} & \text{if } i \neq m, \\
\frac{b^{2m-3} + b^{2m-1}}{2} & \text{if } i = m \text{ and } n \in \mathbb{Z}_{ev}, \\
\frac{b^{2m-1}}{2} & \text{if } i = m \text{ and } n \in \mathbb{Z}_{odd}.
\end{cases}
$$

For each $i \in I_\mathfrak{t}$ such that $2i < n$, set

$$
b_{2i, \pm} := \frac{1}{2}(b_{2i} \pm [b_{2i-1}, b_{2i}]).
$$

Also, for each $i \in I_\mathfrak{t}$ such that $2i < n - 1$ and for each $e \in \{+, -\}$, set

$$
b_{2i, e, \pm} := \frac{1}{2}(b_{2i, e} \pm [b_{2i+1}, b_{2i, e}]).$$

Now, for each $i \in I_\mathfrak{t}$, set

$$
x_i := \begin{cases} 
2b_{2i, +} & \text{if } i \neq m, \\
2b_{2m-2, +} & \text{if } i = m \text{ and } n \in \mathbb{Z}_{ev}, \\
2b_{2m, +} & \text{if } i = m \text{ and } n \in \mathbb{Z}_{odd},
\end{cases}
$$

$$
y_i := \begin{cases} 
2b_{2i, -} & \text{if } i \neq m, \\
2b_{2m-2, -} & \text{if } i = m \text{ and } n \in \mathbb{Z}_{ev}, \\
2b_{2m, -} & \text{if } i = m \text{ and } n \in \mathbb{Z}_{odd},
\end{cases}
$$

$$
w_i := \begin{cases} 
\frac{b_{2i-1} - b_{2i+1}}{2} & \text{if } i \neq m, \\
\frac{b_{2m-3} + b_{2m-1}}{2} & \text{if } i = m \text{ and } n \in \mathbb{Z}_{ev}, \\
2b_{2m-1} & \text{if } i = m \text{ and } n \in \mathbb{Z}_{odd}.
\end{cases}
$$

Using relations (7), one can verify that there exists an isomorphism $\mathfrak{g}(I_\mathfrak{t}) \rightarrow \mathfrak{t}$ of Lie algebras sending $e'_i, f'_i, h'_i$ to $x_i, y_i, w_i$, respectively (see also [31, Sections 4.1–4.2]).
Then, there exists an isomorphism 
\[ X \] 
respectively. However, one should note that 
\[ k \] 
Here, we extend \( \nu \) we have 
\[ \mathbb{I} \] 
4.2. Let \( \mathcal{I} \) denote the irreducible \( \mathfrak{I} \)-module of highest weight \( \nu \in X_+ \).

We often identify \( \nu \in X_\mathfrak{I} \) with \( (\nu_1, \nu_3, \ldots, \nu_{2m-1}) \in (\frac{1}{2}\mathbb{Z})^{\mathfrak{I}} \), where
\[ \nu_i := \langle b_i, \nu \rangle . \]
Here, we extend \( \nu \) to a linear form on \( \mathfrak{h}_\mathfrak{I} = Y_\mathfrak{I} \otimes \mathbb{C} \). With this notation, we have
\[ \langle w_i, \nu \rangle = \begin{cases} \nu_{2i-1} - \nu_{2i+1} & \text{if } i \neq m, \\ \nu_{2m-3} + \nu_{2m-1} & \text{if } i = m \text{ and } n \in \mathbb{Z}_{\text{ev}}, \\ 2\nu_{2m-1} & \text{if } i = m \text{ and } n \in \mathbb{Z}_{\text{odd}}. \end{cases} \]

Note that we have either \( \nu \in \mathbb{Z}_{\mathfrak{I}}^{\mathfrak{I}} \) or \( \nu \in (\frac{1}{2} + \mathbb{Z})^{\mathfrak{I}} \). We call \( \nu \in \mathbb{Z}_{\mathfrak{I}}^{\mathfrak{I}} \) an integer weight. Set \( X_{\mathfrak{I}, \text{int}} \) to be the set of integer weights. Let \( \nu = (\nu_1, \nu_3, \ldots, \nu_{2m-1}) \in X_\mathfrak{I} \). When \( n \in \mathbb{Z}_{\text{ev}} \), we have \( \nu \in X^+_\mathfrak{I} \) if and only if \( \nu_1 \geq \cdots \geq \nu_{2m-3} \geq |\nu_{2m-1}| \), while when \( n \in \mathbb{Z}_{\text{odd}} \), we have \( \nu \in X^+_\mathfrak{I} \) if and only if \( \nu_1 \geq \cdots \geq \nu_{2m-1} \geq 0 \).

4.2. Type AI. Let \( \mathbf{U}' \) denote the quantum group of type AI with parameters \( q_i = q^{-1}, \kappa_i = 0 \) for all \( i \in I \). It is a subalgebra of \( \mathbf{U} \) generated by \( B_i = F_i + q^{-1}E_iK_i^{-1}, i \in I \). The defining relations of \( \mathbf{U}' \) for \( B_1, \ldots, B_{n-1} \) are as follows (see e.g. [19, Section 7]):
\[ [B_i, B_j] = 0 \quad \text{if } |i - j| > 1, \]
\[ [B_i, [B_i, B_j]q]q^{-1} = B_j \quad \text{if } |i - j| = 1. \] 

Let us recall the notion of classical weight \( \mathbf{U}' \)-modules, which was introduced in [31].

**Definition 4.2.1.** Let \( M \) be a \( \mathbf{U}' \)-module. \( M \) is said to be a classical weight module if for each \( i \in I_{\mathfrak{I}}, B_i \) acts on \( M \) diagonally with eigenvalues belonging to \( K_1 \).

Let \( M \) be a classical weight module. For each \( \nu \in \mathfrak{h}_\mathfrak{I}' \), set \( M_\nu \) to be the subspace of \( M \) consisting of \( v \in M \) satisfying the following: For each \( i \in I_{\mathfrak{I}}, B_i v = av \) for some \( a \in K_1 \) such that \( \text{ev}_1(a) = \langle b_i, \nu \rangle \). Then, we have by definition
\[ M = \bigoplus_{\nu \in \mathfrak{h}_\mathfrak{I}'} M_\nu. \]

The subspace \( M_\nu \) (resp., an element of \( M_\nu \)) is referred to as the \( X_\mathfrak{I} \)-weight space of \( M \) (resp., \( X_\mathfrak{I} \)-weight vector) of weight \( \nu \).

**Proposition 4.2.2** ([31, Proposition 3.1.4 and Remark 3.1.5]). Let \( M \) be a \( \mathbf{U} \)-module of finite dimension. Then, as a \( \mathbf{U}' \)-module, it is a classical weight module. Moreover, for each \( i \in I_{\mathfrak{I}}, \) the eigenvalues of \( B_i \) are of the form \( [a], a \in \mathbb{Z} \).

**Proposition 4.2.3** ([31, Proposition 3.1.6]). Let \( M \) be a classical weight module, and \( N \) a finite-dimensional \( \mathbf{U} \)-module. Then, \( M \otimes N \) is a classical weight module.

**Remark 4.2.4.** A weight vector of a finite-dimensional \( \mathbf{U} \)-module \( M \) is not necessarily an \( X_\mathfrak{I} \)-weight vector of \( M \). Similarly, the tensor product of an \( X_\mathfrak{I} \)-weight vector of a classical weight module \( M \) and a weight vector of a finite-dimensional \( \mathbf{U} \)-module \( N \) is not necessarily an \( X_\mathfrak{I} \)-weight vector of \( M \otimes N \).
In [31], we defined linear operators $l_j^{±1}$ for $j \in I_\oplus$, and $B_{i,e}, B_{i,e_1,e_2}$ for $i \in I \setminus I_\oplus$, $e,e_1,e_2 \in \{+,-\}$ acting on each classical weight module $M$. By definition, for each $v \in M$ and $e \in \{+, -, \}$, we have

\[
l_j^{±1}v = q^{±a}v \quad \text{if } B_jv = [a]v,
\]

\[
B_{2i,±}(l_{2i-1}^{±1} ± [B_{2i-1}, B_{2i}]q) \frac{1}{\{l_{2i-1}; 0\}},
\]

\[
B_{2i,e±}(l_{2i+1}^{±1} ± [B_{2i+1}, B_{2i,e}]q) \frac{1}{\{l_{2i+1}; 0\}}.
\]

Set

\[
\tilde{I}_t := \begin{cases} 
\{(2i, ±) | i \in [1, m - 1]\} & \text{if } n \in \mathbb{Z}_{ev}, \\
\{(2i, ±) | i \in [1, m - 1]\} \cup \{2m\} & \text{if } n \in \mathbb{Z}_{odd}.
\end{cases}
\]

For each $j \in \tilde{I}_t$, set

\[
X_j := \begin{cases} 
B_{2i,±\{l_{2i-1}; 0\}} & \text{if } j = (2i, ±), \\
B_{2m,\{l_{2m-1}\}} & \text{if } j = 2m,
\end{cases}
\]

\[
Y_j := \begin{cases} 
B_{2i,−\{l_{2i+1}; 0\}} & \text{if } j = (2i, ±), \\
B_{2m,−\{l_{2m-1}\}} & \text{if } j = 2m,
\end{cases}
\]

\[
\gamma_j := \begin{cases} 
l_{2i-1}^{2i-1} ± l_{2i+1}^{2i+1} & \text{if } j = (2i, ±), \\
l_{2m-1}^{2m-1} & \text{if } j = 2m.
\end{cases}
\]

For each $\nu, \xi \in X_t$, we write $\xi \leq \nu$ to indicate that $\nu - \xi \in \sum_{i \in I_t} \mathbb{Z}_{\geq 0} \gamma_i = \sum_{j \in \tilde{I}_t} \mathbb{Z}_{\geq 0} \gamma_j$. This defines a partial order on $X_t$, called the dominance order.

**Theorem 4.2.5 ([31, Corollaries 4.1.4, 4.1.7, 4.2.2, and 4.2.4])**

1. For each $i \in I_\oplus$, $l_i$ (resp., $B_i$) acts on each finite-dimensional classical weight module diagonally with eigenvalues of the form $q^a$ (resp., $[a]$), $a \in \mathbb{Z}$.
2. For each classical weight module $M, \nu \in X_t$, and $j \in \tilde{I}_t$, we have

\[
X_jM_\nu \subseteq M_{\nu+\gamma_j}, \quad Y_jM_\nu \subseteq M_{\nu-\gamma_j}.
\]
3. Each finite-dimensional classical weight module is completely reducible.
4. The isoclasses of finite-dimensional irreducible classical weight modules are parametrized by the set $X^+_{t,\text{int}}$. The finite-dimensional irreducible classical weight module $V(\nu)$ corresponding to $\nu \in X^+_{t,\text{int}}$ is characterized by the condition that there exists a unique (up to nonzero scalar multiple) $v_\nu \in V(\nu) \setminus \{0\}$ such that $X_jv_\nu = 0$ for all $j \in \tilde{I}_t$; the vector $v_\nu$ is referred to as a highest weight vector.
5. For each $\nu \in X^+_{t,\text{int}}$, we have

\[
V(\nu) = \text{Span}_K\{Y_{j_1} \cdots Y_{j_r}, v_\nu | r \geq 0, \ j_1, \ldots, j_r \in \tilde{I}_t\}.
\]
6. For each $\nu \in X^+_{t,\text{int}}$ and $\xi \in X_t$, we have $\dim V(\nu)_{\nu} = 1$, and $\dim V(\nu)_{\xi} = 0$ unless $\xi \leq \nu$.
7. For each $\nu \in X^+_{t,\text{int}}$ and $\xi \in X_t$ such that $\xi < \nu$, we have

\[
V(\nu)|_\xi = \sum_{j \in I_t} Y_jV(\nu)|_{\xi+\gamma_j}.
\]
Proposition 4.2.6. Let \( \nu \in X^+_{t,\text{int}} \). Then, we have
\[
\chi(U,M) = \sum_{\nu \in X_{t,\text{int}}} (\dim_K M_\nu) e^\nu.
\]
where the right-hand side denotes the character of the irreducible \( k \)-module \( V_\nu \) of highest weight \( \nu \).

Proof. The assertion essentially follows from [31, arguments before Proposition 3.3.9]. \( \square \)

Let us recall from [31] some formulas which will be frequently used below.

(9) \( B_{2i-1} = [l_{2i-1}; 0] \),
(10) \( B_{2i,\pm} = (B_{2i}l_{2i-1}^{\pm1} \pm [B_{2i-1}, B_{2i}]) \frac{1}{\{l_{2i-1}; 0\}} \),
(11) \( B_{2i} = B_{2i,+} + B_{2i,-} \),
(12) \( l_{2j-1}B_{2i,\pm} = q^{\pm \delta_{ij}} B_{2i,\pm} l_{2j-1} \) if \( j \neq i + 1 \),
(13) \( [B_{2i,\pm}\{l_{2i-1}; 0\}, B_{2i,-}\{l_{2i-1}; 0\}] = [l_{2i}^2; 0] \),
(14) \( B_{2i,e,\pm} = (B_{2i,e}l_{2i+1}^{\pm1} \pm [B_{2i+1}, B_{2i,e}]) \frac{1}{\{l_{2i+1}; 0\}} \),
(15) \( B_{2i,\pm} = B_{2i,\pm,+} + B_{2i,\pm,-} \),
(16) \( l_{2j-1}B_{2i,e,\pm e_2} = q^{e_1 \delta_{ij} + e_2 \delta_{ij-1}} B_{2i,e,\pm e_2} l_{2j-1} \),
(17) \( [B_{2i,\pm}\{l_{2i-1}; 0\}, B_{2i,-\pm}\{l_{2i+1}; 0\}] = (1 + (q - q^{-1})^2 B_{2i,-\pm} B_{2i,\pm})[l_{2i-1}l_{2i+1}; 0] \),
(18) \( [B_{2i,\pm}\{l_{2i-1}; 0\}, B_{2i,-\pm}\{l_{2i-1}; 0\}] = 0 \),
(19) \( [B_{2i,\pm}\{l_{2i+1}; 0\}, B_{2i,-\pm}\{l_{2i+1}; 0\}] = 0 \).

Lemma 4.2.7. Let \( M \) be a finite-dimensional classical weight module equipped with a contragredient Hermitian inner product. Then, for each \( u, v \in M, i \in I_t \), we have (if \( B_{2i,\pm}, \text{ etc. is defined} \)
\[
(l_{2i-1}u, v) = (u, l_{2i-1}v),
(B_{2i,\pm}u, v) = (u, B_{2i,\pm}v),
(B_{2i,\pm}u, v) = (u, B_{2i,-\pm}v),
(B_{2i,-\pm}u, v) = (u, B_{2i,\pm}v).
\]

Proof. The first assertion is reduced to the case when both \( u \) and \( v \) are \( X_t \)-weight vectors. The latter is clear from the definitions.

Applying Proposition 3.1.1 to our case, we have
\[
\varphi^*(B_i) = B_i
\]
for all \( i \in I \). Then, the remaining assertions are directly verified by using the first assertion and equations (9)–(16). \( \square \)
Lemma 4.2.8. Let $M$ be a finite-dimensional classical weight module equipped with an $\bar{\imath}$-bar-involution $\psi_M^\iota$. Then, for each $v \in M$, $i \in I_\iota$, we have (if $B_{2i,\pm}$, etc. is defined)

\begin{align*}
\psi_M^\iota(l_{2i-1}v) &= l_{2i-1}^{-1}\psi_M^\iota(v), \\
\psi_M^\iota(B_{2i,\pm}v) &= B_{2i,\pm}\psi_M^\iota(v), \\
\psi_M^\iota(B_{2i,\pm}v) &= B_{2i,\pm}\psi_M^\iota(v), \\
\psi_M^\iota(B_{2i,-\pm}v) &= B_{2i,-\pm}\psi_M^\iota(v).
\end{align*}

Proof. Similar to the proof of Lemma 4.2.7. \hfill \Box

4.3. Contragredient Hermitian inner products and $\bar{\imath}$-bar-involutions on irreducible modules. In this subsection, we construct a contragredient Hermitian inner product and an $\bar{\imath}$-bar-involution on $V(\nu)$, $\nu \in X_{t,\text{int}}$.

Let $V_\iota$ denote the natural representation of $U$. Namely, it possesses a basis $\{u_1, \ldots, u_n\}$ such that

\begin{align*}
E_iu_j &= \delta_{i,j-l}u_i, \quad F_iu_j := \delta_{i,j-l+1}u_i, \quad K_iu_j := q^{\delta_{i,j-l}-\delta_{i,j-1}}u_j.
\end{align*}

The natural representation $V_\iota$ is equipped with a contragredient Hermitian inner product $(\cdot, \cdot)_\iota$ given by

\[(u, v)_\iota = \delta_{i,j}.
\]

Set

\[\mathcal{L}_\iota := \mathcal{L}_{V_\iota}, \quad \overline{\mathcal{L}}_\iota := \overline{\mathcal{L}}_{V_\iota}, \quad \mathcal{B}_\iota := \{\overline{\imath}_1, \ldots, \overline{\imath}_n\},
\]

where $\overline{\imath}_i := \text{ev}_{\infty}(u_i)$. Note that $\mathcal{B}_\iota$ forms a crystal basis of $V_\iota$ on which the Kashiwara operators act as

\[\overline{E}_i\overline{\imath}_j = \delta_{i,j-1}{\overline{\imath}_i}, \quad \overline{F}_i\overline{\imath}_j = \delta_{i,j+1}{\overline{\imath}_i}.
\]

The following follows from a direct calculation.

Lemma 4.3.1. Let $M$ be a $U'$-module, $v \in M$. Suppose that $B_iv = [a_i]v$ for some $a_i \in \mathbb{Z}$ for all $i \in I_\iota$. Let $i \in I_\iota$, and set $v'_\pm := v \otimes (u_i \pm q^{\pm n}u_{i+1})$. Also, set $v_0 := v \otimes u_n$ if $n \in \mathbb{Z}_{\text{odd}}$. Then, for each $j \in I_\iota$, we have

\[B_jv'_\pm = [a_j \pm \delta_{i,j}][v'_\pm], \quad B_jv_0 = [a_j]v_0.
\]

Proposition 4.3.2. Let $d \geq 0$ and $\nu \in X_{t,\text{int}}$. Set $M := V_\iota^{\otimes d}$.

1. If $n \in \mathbb{Z}_{\text{ev}}$, then, $\dim M_\nu$ equals the number of sequences

\[(e_1, \ldots, e_d) \in \{(i, +), (i, -) \mid i \in I_\iota\}^d
\]

such that

\[\sharp\{k \mid e_k = (i, +)\} - \sharp\{k \mid e_k = (i, -)\} = \nu_i \text{ for all } i \in I_\iota.
\]

2. If $n \in \mathbb{Z}_{\text{odd}}$, then, $\dim M_\nu$ equals the number of sequences

\[(e_1, \ldots, e_d) \in \{(i, +), (i, -), 0 \mid i \in I_\iota\}^d
\]

such that

\[\sharp\{k \mid e_k = (i, +)\} - \sharp\{k \mid e_k = (i, -)\} = \nu_i \text{ for all } i \in I_\iota.
\]

Proof. Using Lemma 4.3.1, the assertions follow by induction on $d$. \hfill \Box

Let $d \geq 0$ and set $M := V_\iota^{\otimes d}$. Let $(\cdot, \cdot)_\iota$ denote the contragredient Hermitian inner product on $M$ obtained from $(\cdot, \cdot)_\iota$ by means of Lemma 2.4.3. Set

\[M_{A_1} := \text{Span}_{A_1}\{u_{i_1} \otimes \cdots \otimes u_{i_d} \mid i_1, \ldots, i_d \in [1, n]\}.
\]
Then, \( M_1 := M_{A_1}/(q-1)M_{A_1} \) is a \( g(= sl_n) \)-module isomorphic to \( (\mathbb{C}^n)^{\otimes d} \), where \( \mathbb{C}^n \) is equipped with the natural \( sl_n \)-module structure. As a \( \mathfrak{t} \)-module, we have an irreducible decomposition

\[
M_1 \simeq \bigoplus_{\nu \in X_{t,\text{int}}^+} V_{\mathfrak{t}}(\nu)^{\oplus m_{\nu}}
\]

for some \( m_{\nu} \geq 0 \).

On the other hand, by induction on \( d \) and Lemma 4.3.1, we see that each \( X_{t^d} \)-weight space \( M_{\nu}, \nu \in X_{t^d} \) is spanned by vectors in \( M_{A_1} \). Therefore, we obtain

\[
\text{ch}_{U^}\ M = \text{ch}_{t}M_1 = \sum_{\nu} m_{\nu}\text{ch}_{t}V_{t}(\nu).
\]

Furthermore, we have an irreducible decomposition

\[
M \simeq \bigoplus_{\nu \in X_{t,\text{int}}^+} V(\nu)^{\oplus m'_{\nu}}
\]

as a \( U^\cdot \)-module for some \( m'_{\nu} \geq 0 \). Then, we have

\[
\sum_{\nu} m_{\nu}\text{ch}_{t}V_{t}(\nu) = \text{ch}_{U^\cdot}M = \sum_{\nu} m'_{\nu}\text{ch}_{U^\cdot}V(\nu) = \sum_{\nu} m'_{\nu}\text{ch}_{U^\cdot}V_{t}(\nu).
\]

Since the irreducible characters of \( \mathfrak{t} \) are linearly independent, we conclude that \( m_{\nu} = m'_{\nu} \) for all \( \nu \in X_{t,\text{int}}^+ \).

**Proposition 4.3.3.** Let \( \nu \in X_{t,\text{int}}^+ \). Then, there exists \( d \geq 0 \) such that \( V_{\mathfrak{t}}^{\otimes d} \) possesses a highest weight vector of weight \( \nu \).

**Proof.** It is a classical result that the irreducible \( \mathfrak{t} \)-module \( V_{t}(\nu) \) of highest weight \( \nu \in X_{t,\text{int}}^+ \) appears as a submodule of \( (\mathbb{C}^n)^{\otimes d} \) for some \( d \geq 0 \). This implies, by the argument above, that the irreducible \( U^\cdot \)-module \( V(\nu) \) appears as a submodule of \( V_{\mathfrak{t}}^{\otimes d} \). \( \square \)

**Corollary 4.3.4.** Let \( \nu \in X_{t,\text{int}}^+ \).

1. There exists a unique contragredient Hermitian inner product \( \langle \cdot, \cdot \rangle_\nu \) on \( V(\nu) \) such that \( (v_\nu, v_\nu)_\nu = 1 \).
2. There exists a unique \( \mathfrak{g} \)-ubar-involution \( \psi_\nu^\dagger \) on \( V(\nu) \) such that \( \psi_\nu^\dagger(v_\nu) = v_\nu \).

**Proof.** Let \( d \geq 0 \) be such that \( M := V_{\mathfrak{t}}^{\otimes d} \) possesses a highest weight vector \( v \) of weight \( \nu \) (see Proposition 4.3.3). By Lemma 4.2.8, we see that \( \psi_\nu^M(v) \) is also a highest weight vector of weight \( \nu \). Replacing \( v \) with \( v + \psi_\nu^M(v) \), we may assume that \( \psi_\nu^M(v) = v \). Furthermore, replacing \( v \) with \( \frac{1}{\text{ord}(v)}v \), we may assume that \( v \) is almost normal. Let \( \phi : V(\nu) \to M \) be an \( U^\cdot \)-isomorphism such that \( \phi(v_\nu) = v \). Define maps \( (\cdot, \cdot)_\nu : V(\nu) \times V(\nu) \to K \) and \( \psi_\nu^\dagger : V(\nu) \to V(\nu) \) by

\[
(u_1, u_2)_\nu := (v, v)^{-1}M \cdot (\phi(u_1), \phi(u_2))M,
\]

\[
\psi_\nu^\dagger(u) := \phi^{-1}(\psi_\nu^M(\phi(u))).
\]

Then, it is straightforwardly verified that these are a desired Hermitian inner product and an \( \mathfrak{g} \)-ubar-involution.

The uniqueness is easily verified. Hence, the proof completes. \( \square \)

Let \( \nu \in X_{t,\text{int}}^+ \). As in the quantum groups setting, with respect to the contragredient Hermitian inner product \( (\cdot, \cdot)_\nu \), we write

\[
\mathcal{L}(\nu) := L V(\nu), \quad \mathcal{L}(\nu) := L V(\nu), \quad b_\nu := e v_\infty(v_\nu).
\]
4.4. **Divided powers.** Following [7], for each \( i \in I \) and \( k \in \mathbb{Z}_{\geq 0} \), set
\[
B_{i,\text{ev}}^{(k)} := \begin{cases} 
\frac{1}{[2a]!} B_{i} \prod_{b=a+1}^{n} (B_{i} - [2b]) & \text{if } k = 2a \in \mathbb{Z}_{\text{ev}}, \\
\frac{1}{2a+1} \prod_{b=a+1}^{n} (B_{i} - [2b]) & \text{if } k = 2a + 1 \in \mathbb{Z}_{\text{odd}}, 
\end{cases}
\]
\[
B_{i,\text{odd}}^{(k)} := \begin{cases} 
\frac{1}{[2a]!} \prod_{b=a+1}^{n} (B_{i} - [2b - 1]) & \text{if } k = 2a \in \mathbb{Z}_{\text{ev}}, \\
\frac{1}{2a+1} B_{i} \prod_{b=a+1}^{n} (B_{i} - [2b - 1]) & \text{if } k = 2a + 1 \in \mathbb{Z}_{\text{odd}}.
\end{cases}
\]

**Proposition 4.4.1** ([7, equations (2.5) and (3.2)]). For each \( p \in \{\text{ev, odd}\} \) and \( k \geq 0 \), we have
\[
B_{i,p}^{(k)} B_{i,p}^{(k+1)} = [k + 1] B_{i,p}^{(k+1)} + \delta_{p,p(k)} [k] B_{i,p}^{(k-1)},
\]
where we understand \( B_{i,p}^{(-1)} = 0 \).

Recall \( X^s \) from equation (4) in page 14. In our setting, we have
\[
X^s = X/2X = (\mathbb{Z}/2\mathbb{Z})^{n-1}.
\]
For each \( \zeta = (\zeta_1, \ldots, \zeta_{n-1}) \in X^s \), set
\[
B_{i,\zeta}^{(k)} := B_{i,p(\zeta)}^{(k)} 1_{\zeta}.
\]
These elements are called divided powers.

**Theorem 4.4.2** ([6, Corollary 7.5]). \( \check{U}_\Lambda \) is generated by \( B_{i,\zeta}^{(k)} \), \( i \in I \), \( \zeta \in X^s \), \( k \in \mathbb{Z}_{\geq 0} \).

**Definition 4.4.3.** Let \( M = \bigoplus_{\zeta \in X^s} M_\zeta \) be an \( X^s \)-weight module. We say that \( M \) is standard if for each \( \zeta = (\zeta_1, \ldots, \zeta_{n-1}) \in X^s \), we have
\[
M_\zeta = \{ v \in M \mid B_{i,p(\zeta)}^{(k)} v = 0 \text{ for some } k_i > 0 \text{ such that } \overline{k_i} \neq \zeta_i, \text{ for all } i \in I \}.
\]

The following two propositions are immediate from the definition.

**Proposition 4.4.4.** Let \( M \) be a standard \( X^s \)-weight module, and \( N \subseteq M \) an \( X^s \)-weight submodule. Then, \( N \) is standard.

**Proposition 4.4.5.** Let \( M \) and \( N \) be standard \( X^s \)-weight modules. Then, each \( U^s \)-module homomorphism \( f : M \to N \) preserves the \( X^s \)-weight spaces. In particular, it lifts to a \( U \)-module homomorphism.

The rest of this subsection is devoted to proving that each finite-dimensional \( U \)-module with the canonical \( X^s \)-weight module structure is standard.

**Lemma 4.4.6.** Let \( M \) be a \( U^s \)-module. Let \( v \in M \), \( i \in I \), and \( p \in \{\text{ev, odd}\} \). Then, the following are equivalent:
\begin{enumerate}
\item \( B_{i,p}^{(k)} v = 0 \) for some \( k > 0 \) such that \( p(k) \neq p \).
\item \( v \) is a sum of \( B_i \)-eigenvectors of eigenvalues of the form \([a] \) with \( a \in [-k+1, k-1] \).
\end{enumerate}

**Proof.** By the definition of divided powers, if \( p(k) \neq p \), then we have
\[
B_{i,p}^{(k)} = \prod_{a \in [-k+1, k-1]} (B_i - [a]).
\]
This implies the assertion. \( \square \)

**Proposition 4.4.7.** Let \( M \) be a weight \( U \)-module. Then, its canonical \( X^s \)-weight module structure is standard.
Proof. Let \( i \in I \) and \( l \in \mathbb{Z}_{\geq 0} \), and consider the \((l+1)\)-dimensional irreducible \( U_i \)-module \( V(l) \), where \( U_i \) denotes the subalgebra of \( U \) generated by \( E_i, F_i, K_i^{\pm 1} \). Let \( v \in V(l) \) be a highest weight vector. By [7, Theorems 2.10 and 3.6], \( V(l) \) has a basis \( \{ B_{i,p(l)}^{(k)} v \mid k \in [0, l] \} \), and, we have \( B_{i,p(l)}^{(l+1)} v = 0 \). Since \( B_{i,p(l)}^{(k)} \)'s commute with each other, we obtain

\[
B_{i,p(l)}^{(l+1)} u = 0 \quad \text{for all } u \in V(l).
\]

By Lemma 4.4.6, we see that each vector in \( V(l) \) is a sum of \( B_i \)-eigenvectors of eigenvalues of the form \([a], a \in [-l, l]_{p(l)}\).

Let \( \lambda \in X_i, v \in M_\lambda \), and \( i \in I \). We show that \( B_{i,p(\lambda_i)}^{(k_i)} v = 0 \) for some \( k_i > 0 \) such that \( \overline{k_i} \neq \overline{\lambda_i} \). As a \( U_i \)-module, \( M \) decomposes as

\[
M = \bigoplus_{l \geq 0} M[l],
\]

where \( M[l] \) denotes the isotypic component of \( M \) of type \( V(l) \). By weight consideration, we have \( v \in \bigoplus_{l \in \mathbb{Z}_{p(\lambda_i)}} M[l] \). By argument above, \( v \) is a sum of \( B_i \)-eigenvectors of eigenvalues of the form \([a], a \in \mathbb{Z}_{p(\lambda_i)}\). By Lemma 4.4.6, we have \( B_{i,p(\lambda_i)}^{(k_i)} v = 0 \) for some \( k_i > 0 \) such that \( \overline{k_i} \neq \overline{\lambda_i} \).

For each \( \zeta \in X^I_i \), set

\[
M'_\zeta := \{ v \in M \mid B_{i,p(\zeta_i)}^{(k_i)} v = 0 \text{ for some } k_i > 0 \text{ such that } \overline{k_i} \neq \overline{\zeta_i} \text{ for all } i \in I \}.
\]

By above, we have

\[
M_\zeta = \bigoplus_{\overline{\lambda} = \overline{\zeta}} M_\lambda \subseteq M'_\zeta \subseteq M.
\]

Summing up through \( \zeta \in X^I_i \), we obtain

\[
M = \bigoplus_{\zeta \in X^I_i} M_\zeta = \sum_{\zeta \in X^I_i} M'_\zeta.
\]

Since \( M'_\zeta \)'s intersects trivially with each other, we conclude that

\[
M_\zeta = M'_\zeta.
\]

This proves the assertion. \( \square \)

4.5. Classical weight modules with contragredient Hermitian inner products.

Let \( M \) be a finite-dimensional classical weight module equipped with a contragredient Hermitian inner product. The following are easy analogs of Lemma 2.4.4 and Proposition 2.4.5.

Lemma 4.5.1. Let \( \nu \neq \xi \in X_{t,\mathrm{int}} \). Then, we have \((M_\nu, M_\xi) = 0\). Consequently, we have \( \mathcal{L}_M = \bigoplus_{\nu \in X_{t,\mathrm{int}}} \mathcal{L}_M,\nu \) and \( \overline{\mathcal{L}_M} = \bigoplus_{\nu \in X_{t,\mathrm{int}}} \overline{\mathcal{L}_M,\nu} \), where \( \mathcal{L}_M,\nu := \mathcal{L}_M \cap M_\nu \) and \( \overline{\mathcal{L}_M,\nu} := \mathcal{L}_M,\nu/q^{-1} \mathcal{L}_M,\nu \).

Proposition 4.5.2. There exists an orthogonal irreducible decomposition \( M = \bigoplus_{k=1}^r M_k \) of \( M \).

Because of Proposition 4.5.2, we can define \( M[\nu], M[\geq \nu], M[> \nu] \), etc. in the same way as in the quantum groups setting.

Suppose further that \( M \) is a standard \( X^I_i \)-weight module. Then, its distinct \( X^I_i \)-weight spaces are orthogonal to each other:
Lemma 4.5.3. For each \( \xi \neq \eta \in \mathcal{X}^* \), we have \((M_\xi, M_\eta) = 0\).

Proof. Suppose that \( \xi \neq \eta \). Then, there exists \( i \in I \) such that \( \xi_i \neq \eta_i \). Let \( u \in M_\xi \) and \( v \in M_\eta \). By Lemma 4.4.6, we can write \( u = \sum_k u_k \) (resp., \( v = \sum_l v_l \)) with \( B_i u_k = [a_k] u_k \), \( \overline{a_k} = \xi_i \) (resp., \( B_i v_l = [b_l] v_l \), \( \overline{b_l} = \eta_l \)). Then, for each \( k, l \), we have

\[
[a_k](u_k, v_l) = (B_i u_k, v_l) = (u_k, B_i v_l) = [b_l](u_k, v_l).
\]

Since \( a_k \neq b_l \), it follows that \( (u_k, v_l) = 0 \). Hence, we obtain \( (u, v) = \sum_{k, l} (u_k, v_l) = 0 \), as required.

From this lemma together with Proposition 2.1.8, we obtain orthogonal decompositions

\[
\mathcal{L}_M = \bigoplus_{\zeta \in \mathcal{X}^*} \mathcal{L}_{M, \zeta}, \quad \overline{\mathcal{L}}_M = \bigoplus_{\zeta \in \mathcal{X}^*} \mathcal{L}_{M, \zeta},
\]

where \( \mathcal{L}_{M, \zeta} := \mathcal{L}_M \cap M_\zeta \) and \( \overline{\mathcal{L}}_{M, \zeta} := \mathcal{L}_M / q^{-1} \mathcal{L}_{M, \zeta} \). Hence, for each \( \zeta \in \mathcal{X}^* \), the idempotent \( 1_\zeta \) defines a projection

\[
1_\zeta : \overline{\mathcal{L}}_M \rightarrow \overline{\mathcal{L}}_{M, \zeta}.
\]

5. \( n = 2 \) Case

Sections 5–7 are devoted to investigating low rank cases; \( n = 2, 3, 4 \). The results obtained in these sections will be used to prove our main theorems.

In this section, we consider the \( n = 2 \) case. In this case, we can identify \( \mathcal{X} = \mathbb{Z} \), \( \mathcal{X}^* = \mathbb{Z}_{\geq 0} \), \( \mathcal{X}^r = \mathbb{Z} / 2\mathbb{Z} \). Set \( \mathcal{X}_{t, \text{int}} := \mathbb{Z} \). For each \( \nu \in \mathcal{X}_{t, \text{int}} \), let \( \mathbf{K}_\nu = \mathbf{K}_{\nu} \) denote the 1-dimensional \( \mathbf{U}^* \)-module such that \( B_1 v_\nu = [\nu] v_\nu \). Clearly, \( \mathbf{K}_\nu \) possesses a unique contragredient Hermitian inner product \( (\cdot, \cdot)_\nu \) such that \( (v_\nu, v_\nu)_\nu = 1 \). Set \( \mathcal{L}(\nu) := \mathbf{K}_\nu v_\nu \), \( \overline{\mathcal{L}}(\nu) := \mathcal{L}(\nu) / q^{-1} \mathcal{L}(\nu) \), and \( b_\nu := \text{ev}_\infty(\nu) \).

In the \( n = 2 \) case, we define a classical weight module to be a \( \mathbf{U}^* \)-module isomorphic to \( \bigoplus_{\nu \in \mathcal{X}_{t, \text{int}}} \mathbf{K}_{\nu}^{\otimes m_\nu} \) for some \( m_\nu \geq 0 \). Then, each classical weight module \( M \) admits an \( \mathcal{X}_r \)-weight space decomposition \( M = \bigoplus_{\nu \in \mathcal{X}_{t, \text{int}}} M_\nu \) as in the \( n \geq 3 \) case.

5.1. Modified action of \( B_1 \). Let \( M = \bigoplus_{\nu \in \mathcal{X}_{t, \text{int}}} M_\nu \) be a finite-dimensional classical weight module equipped with a contragredient Hermitian inner product.

Let \( \widetilde{B}_1 \) be a linear operator on \( M \) defined by

\[
\widetilde{B}_1 v = \begin{cases} v & \text{if } v \in M_\nu \text{ for some } \nu > 0, \\ 0 & \text{if } v \in M_0, \\ -v & \text{if } v \in M_\nu \text{ for some } \nu < 0. \end{cases}
\]

Then, \( \widetilde{B}_1 \) preserves \( \mathcal{L}_M \). Hence, it induces a \( \mathbb{C} \)-linear operator \( \overline{\widetilde{B}}_1 \) on \( \overline{\mathcal{L}}_M \).

Let \( \nu \in \mathcal{X}_{t, \text{int}} \) and consider the irreducible \( \mathbf{U}^* \)-module \( \mathbf{K}_\nu = \mathbf{K}_{\nu} \). Let \( V_1 = \mathbf{K}_1 \oplus \mathbf{K}_2 \) denote the natural representation of \( \mathbf{U} = U_q(\mathfrak{sl}_2) \). Recall from Proposition 4.2.3 that \( \mathbf{K}_\nu \otimes V_1 \) is a classical weight module. Set

\[
v_{\nu, \pm} := v_\nu \otimes (u_1 \pm q^{\nu} u_2),
\]

(20)

\[
v_{\nu, \pm} := \frac{1}{\text{tr}(v_{\nu, \pm})} v'_{\nu, \pm} = \begin{cases} v_\nu \otimes (q^\nu u_1 \pm u_2) & \text{if } \pm \nu > 0, \\ v_\nu \otimes (u_1 \pm q^\nu u_2) & \text{if } \nu = 0, \\ v_\nu \otimes (u_1 \pm q^{\nu} u_2) & \text{if } \pm \nu < 0. \end{cases}
\]
By Lemma 4.3.1, we have

\begin{equation}
B_1 v_{\nu, \pm} = [\nu \pm 1] v_{\nu, \pm}.
\end{equation}

Since \(\text{dim} K_\nu \otimes V_\pm = 2\), we obtain the irreducible decomposition

\[ K_\nu \otimes V_\pm = K v_{\nu, +} \oplus K v_{\nu, -} \simeq K_{\nu+1} \oplus K_{\nu-1}. \]

By the definition (20) of \(v_{\nu, \pm}\), we see that

\begin{equation}
ev_\infty(v_{\nu, \pm}) = \begin{cases}
\pm b_\nu \otimes \pi_2 & \text{if } \pm \nu > 0, \\
\frac{1}{\sqrt{2}} b_\nu \otimes (\pi_1 \pm \pi_2) & \text{if } \nu = 0, \\
b_\nu \otimes \pi_1 & \text{if } \pm \nu < 0.
\end{cases}
\end{equation}

**Proposition 5.1.1.** Let \(\nu \in X_{t, \text{int}}\). Then, we have

\[ \tilde{B}_1 (b_\nu \otimes \pi_1) = \begin{cases}
\pm b_\nu \otimes \pi_2 & \text{if } \pm \nu > 1, \\
0 & \text{if } \nu = \pm 1, \\
b_\nu \otimes \pi_2 & \text{if } \nu = 0,
\end{cases} \]

\[ = \begin{cases}
\tilde{B}_1 b_\nu \otimes \pi_1 & \text{if } |\nu| > 1, \\
0 & \text{if } |\nu| = 1, \\
b_\nu \otimes \pi_2 & \text{if } \nu = 0,
\end{cases} \]

\[ \tilde{B}_1 (b_\nu \otimes \pi_2) = \begin{cases}
\pm b_\nu \otimes \pi_2 & \text{if } \pm \nu > 0, \\
b_\nu \otimes \pi_1 & \text{if } \nu = 0,
\end{cases} \]

\[ = \begin{cases}
\tilde{B}_1 b_\nu \otimes \pi_2 & \text{if } |\nu| > 0, \\
b_\nu \otimes \pi_1 & \text{if } \nu = 0.
\end{cases} \]

**Proof.** The assertion is a consequence of formulas (21) and (22). \(\square\)

Let \(M\) be a finite-dimensional classical weight module equipped with a contragredient Hermitian inner product.

**Definition 5.1.2.** A vector \(b \in \overline{L}_M\) is said to be \(B_1\)-homogeneous of degree \(\nu\) if it belongs to \(\overline{L}_{M, \nu} + \overline{L}_{M, -\nu}\) for some \(\nu \in X_{t, \text{int}}\) such that \(\nu \geq 0\). If \(b \in \overline{L}_M\) is \(B_1\)-homogeneous of degree \(\nu\), we write \(\deg_1(b) = \nu\).

The following lemmas are clear from the definitions and previous results.

**Lemma 5.1.3.** Let \(b \in \overline{L}_M \setminus \{0\}\) be \(B_1\)-homogeneous. Then, we have exactly one of either \(\tilde{B}_1 b = 0\) or \(\tilde{B}_1^2 b = b\). Furthermore, \((1 \pm \tilde{B}_1) b\) is an \(X_t\)-weight vector of weight \(\pm \deg_1(b)\).
Lemma 5.1.4. Let $b \in \mathcal{L}_M$ be $B_1$-homogeneous. Then, we have
\[
\deg_1(b \otimes \overline{u}_1) = \begin{cases} 
\deg_1(b) - 1 & \text{if } \deg_1(b) > 0, \\
1 & \text{if } \deg_1(b) = 0,
\end{cases}
\]
\[
\deg_1(b \otimes \overline{u}_2) = \deg_1(b) + 1,
\]
\[
\tilde{B}_1(b \otimes \overline{u}_1) = \begin{cases} 
\tilde{B}_1 b \otimes \overline{u}_1 & \text{if } \deg_1(b) > 1, \\
0 & \text{if } \deg_1(b) = 1, \\
b \otimes \overline{u}_2 & \text{if } \deg_1(b) = 0.
\end{cases}
\]
\[
\tilde{B}_1(b \otimes \overline{u}_2) = \begin{cases} 
\tilde{B}_1 b \otimes \overline{u}_2 & \text{if } \deg_1(b) > 0, \\
b \otimes \overline{u}_1 & \text{if } \deg_1(b) = 0.
\end{cases}
\]

We want to describe the action of $\tilde{B}_1$ on $\mathcal{L}_M \otimes \mathcal{L}_\odot^d$ for an arbitrary $d \geq 0$. The space $\mathcal{L}_\odot^d$ possesses a crystal basis $\mathcal{B}_\odot^d := \{\overline{u}_{i_1} \otimes \cdots \otimes \overline{u}_{i_d} \mid i_1, \ldots, i_d \in \{1, 2\}\}$. For the reader’s convenience, we write down the crystal structure of $\mathcal{B}_\odot^d \otimes \mathcal{B}_\odot^d$, which can be derived from equations (3) in page 10: For each $\overline{u} = \overline{u}_{i_1} \otimes \cdots \otimes \overline{u}_{i_d}, i_1, \ldots, i_d \in \{1, 2\}$, we have
\[
\tilde{F}_1(\overline{u} \otimes \overline{u}_1) = \begin{cases} 
\overline{u} \otimes \overline{u}_2 & \text{if } \varepsilon_1(\overline{u}) < 1, \\
\tilde{F}_1 \overline{u} \otimes \overline{u}_1 & \text{if } \varepsilon_1(\overline{u}) \geq 1,
\end{cases}
\]
\[
\tilde{E}_1(\overline{u} \otimes \overline{u}_1) = \begin{cases} 
0 & \text{if } \varepsilon_1(\overline{u}) \leq 1, \\
\tilde{E}_1 \overline{u} \otimes \overline{u}_1 & \text{if } \varepsilon_1(\overline{u}) > 1,
\end{cases}
\]
\[
\varphi_1(\overline{u} \otimes \overline{u}_1) = \begin{cases} 
\varphi_1(\overline{u}) + 1 & \text{if } \varepsilon_1(\overline{u}) < 1, \\
\varphi_1(\overline{u}) & \text{if } \varepsilon_1(\overline{u}) \geq 1,
\end{cases}
\]
(23)
\[
\varepsilon_1(\overline{u} \otimes \overline{u}_1) = \begin{cases} 
0 & \text{if } \varepsilon_1(\overline{u}) \leq 1, \\
\varepsilon_1(\overline{u}) - 1 & \text{if } \varepsilon_1(\overline{u}) > 1,
\end{cases}
\]
\[
\tilde{F}_1(\overline{u} \otimes \overline{u}_2) = \tilde{F}_1 \overline{u} \otimes \overline{u}_2,
\]
\[
\tilde{E}_1(\overline{u} \otimes \overline{u}_2) = \begin{cases} 
\overline{u} \otimes \overline{u}_1 & \text{if } \varepsilon_1(\overline{u}) = 0, \\
\tilde{E}_1 \overline{u} \otimes \overline{u}_2 & \text{if } \varepsilon_1(\overline{u}) > 0,
\end{cases}
\]
\[
\varphi_1(\overline{u} \otimes \overline{u}_2) = \varphi_1(\overline{u}),
\]
\[
\varepsilon_1(\overline{u} \otimes \overline{u}_2) = \varepsilon_1(\overline{u}) + 1.
\]

Proposition 5.1.5. Let $M$ be a finite-dimensional classical weight module equipped with a contragredient Hermitian inner product. Let $b \in \mathcal{L}_M$ be $B_1$-homogeneous and $\overline{u} = \overline{u}_{i_1} \otimes \cdots \otimes \overline{u}_{i_d} \in \mathcal{B}_\odot^d$. Then, $b \otimes \overline{u}$ is $B_1$-homogeneous. Moreover, we have
\[
\deg_1(b \otimes \overline{u}) = \begin{cases} 
\deg_1(b) - \varphi_1(\overline{u}) + \varepsilon_1(\overline{u}) & \text{if } \deg_1(b) > \varphi_1(\overline{u}), \\
\varepsilon_1(\overline{u}) & \text{if } \varphi_1(\overline{u}) - \deg_1(b) \in \mathbb{Z}_{\geq 0, \text{ev}}, \\
\varepsilon_1(\overline{u}) + 1 & \text{if } \varphi_1(\overline{u}) - \deg_1(b) \in \mathbb{Z}_{\geq 0, \text{odd}},
\end{cases}
\]
\[
\tilde{B}_1(b \otimes \overline{u}) = \begin{cases} 
\tilde{B}_1 b \otimes \overline{u} & \text{if } \deg_1(b) > \varphi_1(\overline{u}), \\
b \otimes \tilde{E}_1 \overline{u} & \text{if } \varphi_1(\overline{u}) - \deg_1(b) \in \mathbb{Z}_{\geq 0, \text{ev}}, \\
b \otimes \tilde{F}_1 \overline{u} & \text{if } \varphi_1(\overline{u}) - \deg_1(b) \in \mathbb{Z}_{\geq 0, \text{odd}}.
\end{cases}
\]
Proof. We proceed by induction on \(d\); the \(d = 0\) case is clear. Let \(d \geq 0\) and assume that the proposition is true, and let us prove the \(d + 1\) case. During the proof below, we often use Lemma 5.1.4 and equations (23) without mentioning one by one. First, we compute \(\text{deg}_1\) and \(\tilde{B}_1\) for \(b \otimes \overline{u} \otimes \overline{u}_1\).

Suppose that \(\varepsilon_1(\overline{u}) > 0\). Then, we have \(\varphi_1(\overline{u} \otimes \overline{u}_1) = \varphi_1(\overline{u})\) and \(\varepsilon_1(\overline{u} \otimes \overline{u}_1) = \varepsilon(\overline{u}) - 1\).

- When \(\varphi_1(\overline{u} \otimes \overline{u}_1) > \deg_1(b)\). In this case, we have
  \[
  \deg_1(b \otimes \overline{u}_1) = \deg_1(b) - \varphi_1(\overline{u}_1) + \varepsilon_1(\overline{u}_1) \geq \varepsilon_1(\overline{u}) + 1 \geq 2.
  \]
  Hence, we obtain
  \[
  \deg_1(b \otimes \overline{u} \otimes \overline{u}_1) = \deg_1(b \otimes \overline{u}_1) - 1 = \deg_1(b) - \varphi_1(\overline{u}_1) + \varepsilon_1(\overline{u}_1),
  \]
  and
  \[
  \tilde{B}_1(b \otimes \overline{u} \otimes \overline{u}_1) = \tilde{B}_1(b \otimes \overline{u}) \otimes \overline{u}_1 = b \otimes \tilde{E}_1(\overline{u} \otimes \overline{u}_1).
  \]

- When \(\varphi_1(\overline{u} \otimes \overline{u}_1) - \deg_1(b) \in \mathbb{Z}_{\geq 0, \text{ev}}\). In this case, we have
  \[
  \deg_1(b \otimes \overline{u}) = \varepsilon_1(\overline{u}) \geq 1.
  \]
  Hence, we obtain
  \[
  \deg_1(b \otimes \overline{u} \otimes \overline{u}_1) = \deg_1(b \otimes \overline{u}_1) - 1 = \varepsilon_1(\overline{u}) - 1 = \varepsilon_1(\overline{u} \otimes \overline{u}_1),
  \]
  and
  \[
  \tilde{B}_1(b \otimes \overline{u} \otimes \overline{u}_1) = \begin{cases} \tilde{B}_1(b \otimes \overline{u}) \otimes \overline{u}_1 & \text{if } \varepsilon_1(\overline{u}) > 1, \\ 0 & \text{if } \varepsilon_1(\overline{u}) = 1 \end{cases} = b \otimes \tilde{E}_1(\overline{u} \otimes \overline{u}_1).
  \]

- When \(\varphi_1(\overline{u} \otimes \overline{u}_1) - \deg_1(b) \in \mathbb{Z}_{>0, \text{odd}}\). In this case, we have
  \[
  \deg_1(b \otimes \overline{u}) = \varepsilon_1(\overline{u}) + 1 \geq 2.
  \]
  Hence, we obtain
  \[
  \deg_1(b \otimes \overline{u} \otimes \overline{u}_1) = \deg_1(b \otimes \overline{u}_1) - 1 = \varepsilon_1(\overline{u}) = \varepsilon_1(\overline{u} \otimes \overline{u}_1) + 1,
  \]
  and
  \[
  \tilde{B}_1(b \otimes \overline{u} \otimes \overline{u}_1) = \tilde{B}_1(b \otimes \overline{u}) \otimes \overline{u}_1 = b \otimes \tilde{F}_1(\overline{u} \otimes \overline{u}_1).
  \]
This case-by-case analysis proves the assertion for the $\varepsilon_1(\overline{u}) > 0$ case.

Next, suppose that $\varepsilon_1(\overline{u}) = 0$. Then, we have $\varphi_1(\overline{u} \otimes \overline{u}_1) = \varphi_1(\overline{u}) + 1$ and $\varepsilon_1(\overline{u} \otimes \overline{u}_1) = 0$.

- When $\deg_1(b) > \varphi_1(\overline{u} \otimes \overline{u}_1)$. In this case, we have
  \[ \deg_1(b \otimes \overline{u}) = \deg_1(b) - \varphi_1(\overline{u}) > 1. \]

  Therefore, we obtain
  \[ \deg_1(b \otimes \overline{u} \otimes \overline{u}_1) = \deg_1(b \otimes \overline{u}) - 1 = \deg_1(b) - \varphi_1(\overline{u}) - 1 = \deg_1(b) - \varphi_1(\overline{u} \otimes \overline{u}_1) + \varepsilon_1(\overline{u} \otimes \overline{u}_1), \]

  and
  \[ \widetilde{B}_1(b \otimes \overline{u} \otimes \overline{u}_1) = \widetilde{B}_1(b \otimes \overline{u}) \otimes \overline{u}_1 = \widetilde{B}_1 b \otimes \overline{u} \otimes \overline{u}_1. \]

- When $\varphi_1(\overline{u} \otimes \overline{u}_1) - \deg_1(\overline{u}) \in \mathbb{Z}_{\geq 0, \text{odd}}$. If $\deg_1(b) > \varphi_1(\overline{u})$, then we must have $\deg_1(b) = \varphi_1(\overline{u}) + 1$. Hence, we have
  \[ \deg_1(b \otimes \overline{u}) = \deg_1(b) - \varphi_1(\overline{u}) = 1. \]

  Therefore, we obtain
  \[ \deg_1(b \otimes \overline{u} \otimes \overline{u}_1) = \deg_1(b \otimes \overline{u}_1) - 1 = 0 = \varepsilon_1(\overline{u} \otimes \overline{u}_1), \]

  and
  \[ \widetilde{B}_1(b \otimes \overline{u} \otimes \overline{u}_1) = 0 = b \otimes \widetilde{E}_1(\overline{u} \otimes \overline{u}_1). \]

  On the other hand, if $\deg_1(b) \leq \varphi_1(\overline{u})$, then we have $\varphi_1(\overline{u}) - \deg_1(b) \in \mathbb{Z}_{\geq 0, \text{odd}}$, and hence
  \[ \deg_1(b \otimes \overline{u}) = \varepsilon_1(\overline{u}) + 1 = 1. \]

  Therefore, $\deg_1(b \otimes \overline{u} \otimes \overline{u}_1)$ and $\widetilde{B}_1(b \otimes \overline{u} \otimes \overline{u}_1)$ are the same as before.

- When $\varphi_1(\overline{u} \otimes \overline{u}_1) - \deg_1(b) \in \mathbb{Z}_{\geq 0, \text{odd}}$. In this case, we have $\varphi_1(\overline{u}) - \deg_1(b) \in \mathbb{Z}_{\geq 0, \text{ev}}$. Hence, we have
  \[ \deg_1(b \otimes \overline{u}) = \varepsilon_1(\overline{u}) = 0. \]

  Therefore, we obtain
  \[ \deg_1(b \otimes \overline{u} \otimes \overline{u}_1) = 1 = \varepsilon_1(\overline{u} \otimes \overline{u}_1) + 1, \]

  and
  \[ \widetilde{B}_1(b \otimes \overline{u} \otimes \overline{u}_1) = b \otimes \overline{u} \otimes \overline{u}_2 = b \otimes \widetilde{F}_1(\overline{u} \otimes \overline{u}_1). \]

This case-by-case analysis proves the assertion for the $\varepsilon_1(\overline{u}) = 0$ case.

Now, let us compute $\deg_1$ and $\widetilde{B}_1$ for $b \otimes \overline{u} \otimes \overline{u}_2$.

- When $\deg_1(b) > \varphi_1(\overline{u} \otimes \overline{u}_2)$. In this case, we have
  \[ \deg_1(b \otimes \overline{u}) = \deg_1(b) - \varphi_1(\overline{u}) + \varepsilon_1(\overline{u}) \geq \varepsilon_1(\overline{u}) + 1 \geq 1. \]

  Hence, we obtain
  \[ \deg_1(b \otimes \overline{u} \otimes \overline{u}_2) = \deg_1(b \otimes \overline{u}) + 1 = \deg_1(b) - \varphi_1(\overline{u}) + \varepsilon_1(\overline{u}) + 1 = \deg_1(b) - \varphi_1(\overline{u} \otimes \overline{u}_2) + \varepsilon_1(\overline{u} \otimes \overline{u}_1), \]
\[ \tilde{B}_1(b \otimes \bar{\pi} \otimes \bar{\pi}_2) = \tilde{B}_1(b \otimes \bar{\pi}) \otimes \bar{\pi}_2 = \tilde{B}_1 b \otimes \bar{\pi} \otimes \bar{\pi}_2. \]

- When \( \varphi_1(\bar{\pi} \otimes \bar{\pi}_2) - \text{deg}_1(b) \in \mathbb{Z}_{\geq 0, \text{ev}} \). In this case, we have
  \[ \text{deg}_1(b \otimes \bar{\pi}) = \epsilon_1(\bar{\pi}) \geq 0. \]
  Hence, we obtain
  \[ \text{deg}_1(b \otimes \bar{\pi} \otimes \bar{\pi}_2) = \text{deg}_1(b \otimes \bar{\pi}) + 1 = \epsilon_1(\bar{\pi}) + 1 = \epsilon_1(\bar{\pi} \otimes \bar{\pi}_2), \]

  and
  \[ \tilde{B}_1(b \otimes \bar{\pi} \otimes \bar{\pi}_2) = \begin{cases} 
  \tilde{B}_1(b \otimes \bar{\pi}) \otimes \bar{\pi}_2 & \text{if } \epsilon_1(\bar{\pi}) > 0, \\
  b \otimes \bar{\pi} \otimes \bar{\pi}_1 & \text{if } \epsilon_1(\bar{\pi}) = 0
  \end{cases} = \begin{cases} 
  b \otimes \tilde{E}_1 \bar{\pi} \otimes \bar{\pi}_2 & \text{if } \epsilon_1(\bar{\pi}) > 0, \\
  b \otimes \bar{\pi} \otimes \bar{\pi}_1 & \text{if } \epsilon_1(\bar{\pi}) = 0
  \end{cases} = b \otimes \tilde{E}_1(\bar{\pi} \otimes \bar{\pi}_2). \]

- When \( \varphi_1(\bar{\pi} \otimes \bar{\pi}_2) - \text{deg}_1(b) \in \mathbb{Z}_{\geq 0, \text{odd}} \). In this case, we have
  \[ \text{deg}_1(b \otimes \bar{\pi}) = \epsilon_1(\bar{\pi}) + 1 \geq 1. \]
  Hence, we obtain
  \[ \text{deg}_1(b \otimes \bar{\pi} \otimes \bar{\pi}_2) = \text{deg}_1(b \otimes \bar{\pi}) + 1 = \epsilon_1(\bar{\pi}) + 2 = \epsilon_1(\bar{\pi} \otimes \bar{\pi}_2) + 1, \]

  and
  \[ \tilde{B}_1(b \otimes \bar{\pi} \otimes \bar{\pi}_2) = \tilde{B}_1(b \otimes \bar{\pi}) \otimes \bar{\pi}_2 = b \otimes \tilde{F}_1 \bar{\pi} \otimes \bar{\pi}_2 = b \otimes \tilde{F}_1(\bar{\pi} \otimes \bar{\pi}_2). \]

Thus, the proof completes. \( \square \)

**Corollary 5.1.6.** Let \( M \) be a finite-dimensional \( U \)-module with a crystal basis \( \mathcal{B}_M \). For each \( b \in \mathcal{B}_M \), the following hold:

1. \( b \) is \( B_1 \)-homogeneous;

   \[ \text{deg}_1(b) = \begin{cases} 
  \epsilon_1(b) & \text{if } \varphi_1(b) \in \mathbb{Z}_{\text{ev}}, \\
  \epsilon_1(b) + 1 & \text{if } \varphi_1(b) \in \mathbb{Z}_{\text{odd}}.
  \end{cases} \]

2. \( \tilde{B}_1 b \in \mathcal{B}_M \sqcup \{0\}; \)

   \[ \tilde{B}_1 b = \begin{cases} 
  \tilde{E}_1 b & \text{if } \varphi_1(b) \in \mathbb{Z}_{\text{ev}}, \\
  \tilde{F}_1 b & \text{if } \varphi_1(b) \in \mathbb{Z}_{\text{odd}}.
  \end{cases} \]
Proof. Since $\mathcal{B}_M$ can be embedded into $\mathcal{B}_\otimes^d$ for some $d \geq 0$, it suffices to prove the assertions for $M = V^{\otimes d}_2$. The latter follows from Proposition 5.1.5 by considering $K_0 \otimes V^{\otimes d}_2 \simeq V^{\otimes d}_2$.

□

Corollary 5.1.7. Let $M$ be a finite-dimensional $\mathbf{U}$-module with a crystal basis $\mathcal{B}_M$. Then, the following forms an orthonormal basis of $\overline{\mathcal{L}}_M$ consisting of $X_\xi$-weight vectors:

$$\{ b \in \mathcal{B}_M \mid \varphi_1(b) \in \mathbb{Z}_{ev} \text{ and } \varepsilon_1(b) = 0 \} \sqcup \{ \frac{1}{\sqrt{2}} (b \pm \tilde{F}_1(b)) \mid b \in \mathcal{B}_M \text{ and } \varphi_1(b) \in \mathbb{Z}_{odd} \}.$$ 

Proof. The assertion follows from Lemma 5.1.3 and Corollary 5.1.6. □

Remark 5.1.8. In the general $n \geq 3$ case, we define $\tilde{B}_i$ and $\deg_i$ for all $i \in I$ in the obvious way.

5.2. Based $\mathbf{U}$-module structures of irreducible $\mathbf{U}$-modules. Let $\lambda \in X^+ = \mathbb{Z}_{\geq 0}$, and consider the irreducible $\mathbf{U}$-module $V(\lambda)$. Since the weights of $V(\lambda)$ belong to the interval $[-\lambda, \lambda]_{p(\lambda)}$, each vector in $V(\lambda)$ is an $X_\xi$-weight vector of weight $\overline{\lambda}$.

Recall that the set $\{ F^{(k)}_1 v_\lambda \mid k \in [0, \lambda] \}$ forms the canonical basis $G(\lambda)$, and the set $\{ B^{(k)}_{1,p(\lambda)} v_\lambda \mid k \in [0, \lambda] \}$ forms the canonical basis $G^s(\lambda)$ of $V(\lambda)$. Viewing $V(\lambda)$ as a based $\mathbf{U}^s$-module in the sense of Example 3.4.5, we have $G^s(\tilde{F}_1^k b_\lambda) = B^{(k)}_{1,p(\lambda)} v_\lambda$.

For each $m \in \mathbb{Z}$, set

$$V(\lambda) \{ \geq m \} := K \{ B^{(k)}_{1,p(\lambda)} v_\lambda \mid k \geq m \}.$$ 

Then, we have a filtration

$$0 = V(\lambda) \{ \geq \lambda + 1 \} \subseteq V(\lambda) \{ \geq \lambda - 1 \} \subseteq \cdots \subseteq V(\lambda) \{ \geq \lambda - 2 \lfloor \frac{1}{2} \rfloor - 1 \} = V(\lambda)$$

of $X_\xi$-weight modules, where $\lfloor r \rfloor$ denotes the greatest integer not greater than $r \in \mathbb{Q}$. In fact, by Proposition 4.4.1, we have

$$B_1 B^{(\lambda-2m-1)}_{1,p(\lambda)} v_\lambda \equiv [\lambda - 2m] B^{(\lambda-2m)}_{1,p(\lambda)} v_\lambda, \quad B_1 B^{(\lambda-2m)}_{1,p(\lambda)} v_\lambda \equiv [\lambda - 2m] B^{(\lambda-2m-1)}_{1,p(\lambda)} v_\lambda$$

modulo $V(\lambda) \{ \geq \lambda - 2m + 1 \}$.

For each $\nu \in [-\lambda, \lambda]_{p(\lambda)}$, set

$$b_{\lambda,\nu} := \begin{cases} \frac{b_\lambda}{\sqrt{2}} (\tilde{F}_1^{[\nu]-1} b_\lambda \pm \tilde{F}_1^{[\nu]} b_\lambda) & \text{if } \nu = 0, \\ \frac{1}{\sqrt{2}} (B^{([\nu]-1)}_{1,p(\lambda)} \pm B^{([\nu])}_{1,p(\lambda)}) & \text{if } \pm \nu > 0. \end{cases}$$

Then, by Corollary 5.1.7, the vector $b_{\lambda,\nu} \in \overline{\mathcal{L}}(\lambda)$ is an $X_\xi$-weight vector of weight $\nu$, and the set $\{ b_{\lambda,\nu} \mid \nu \in [-\lambda, \lambda]_{p(\lambda)} \}$ forms an orthonormal basis of $\overline{\mathcal{L}}(\lambda)$.

Proposition 5.2.1. Let $\lambda \in X^+$, and $\nu \in [-\lambda, \lambda]_{p(\lambda)}$. Then, $G^s(b_{\lambda,\nu})$ is a sum of $X_\xi$-weight vectors of weights in $\{ \nu, \pm (|\nu| + 2), \pm (|\nu| + 4), \ldots, \pm \lambda \}$.

Proof. For each $\nu \in [-\lambda, \lambda]_{p(\lambda)}$, define $B_{1,\nu} \in \mathbf{U}^s$ by

$$B_{1,\nu} := \begin{cases} 1 & \text{if } \nu = 0, \\ \frac{1}{\sqrt{2}} (B^{([\nu]-1)}_{1,p(\lambda)} \pm B^{([\nu])}_{1,p(\lambda)}) & \text{if } \pm \nu > 0. \end{cases}$$

Then, by above, we have

$$B_{1,\nu} v_\lambda = G^s(b_{\lambda,\nu}),$$

and

$$B_1 B_{1,\nu} v_\lambda \equiv [\nu] B_{1,\nu} v_\lambda \pmod{V(\lambda) \{ \geq |\nu| + 1 \}}.$$ 

Then, the assertion follows by descending induction on $|\nu|$.

□
6. **n = 3 case**

In this section, we consider the $n = 3$ case. In this case, we can identify $X = \mathbb{Z}^2$, $X^+ = \mathbb{Z}_{\geq 0}^2$, $X_{t, \text{int}} = \mathbb{Z}$, $X^+_{t, \text{int}} = \mathbb{Z}_{\geq 0}$, and $X^* = (\mathbb{Z}/2\mathbb{Z})^2$.

6.1. **Lowering and raising operators.** Set

$$X_2 = B_{2,+}\{l_1; 0\}, \quad Y_2 = B_{2,-}\{l_1; 0\}, \quad l := l_1.$$ 

In this section, when there is no ambiguity, we abbreviate $X_2$ and $Y_2$ as $X$ and $Y$, respectively.

By Lemma 4.2.7 and equation (12), we have

$$(Xu, v) = (u, Y\{l; -1\}\{l_1; 0\}v), \quad (Yu, v) = (u, X\{l_1; 1\}\{l; 0\}v).$$

for all $u, v \in M$ with $M$ being a $U_\mathbb{I}$-module equipped with a contragredient Hermitian inner product.

Let $\nu \in X^+_t$, and consider the corresponding irreducible $U_\mathbb{I}$-module $V(\nu)$.

**Lemma 6.1.1.** Let $k \geq 0$. Then, we have

$$XY^{(k)}v_\nu = [2\nu - k + 1]Y^{(k-1)}v_\nu,$$

where $Y^{(k)} := \frac{1}{[k]!}Y^k$.

**Proof.** By equation (13), we have

$$[X, Y] = [l^2; 0].$$

Then, the assertion is verified by induction on $k$. \hfill \Box

From this lemma (with a standard argument), we see that $\{Y^{(k)}v_\nu \mid k \in [0, 2\nu]\}$ forms a basis of $V(\nu)$. Note that $Y^{(k)}v_\nu \in V(\nu)_{\nu-k}$. For each $k \in [0, 2\nu]$, we can choose $c_{k, \nu} \in \mathbb{K}^\times$ in a way such that $\text{lt}(c_{k, \nu}Y^{(k)}v_\nu) = 1$. For each $k \in \mathbb{Z}$, set

$$\tilde{Y}^kv_\nu := \begin{cases} c_{k, \nu}Y^{(k)}v_\nu & \text{if } k \in [0, 2\nu], \\ 0 & \text{otherwise}. \end{cases}$$

Then, $\{\tilde{Y}^kv_\nu \mid k \in [0, 2\nu]\}$ forms an almost orthonormal basis of $V(\nu)$.

Using Lemma 6.1.1, we see by induction on $k$ that

$$(Y^{(k)}v_\nu, Y^{(k)}v_\nu)_\nu = \left[\frac{2\nu}{k}\right]\frac{\{\nu\}}{\{\nu-k\}}.$$

This shows that for each $k \in [0, 2\nu]$, we have

$$\deg(Y^{(k)}v_\nu) = \frac{1}{2}((2\nu-k)k + \nu - |\nu-k|) = \begin{cases} \frac{1}{2}(2\nu-k+1)k & \text{if } k \leq \nu, \\ \frac{1}{2}(2\nu-k)(k+1) & \text{if } k \geq \nu, \end{cases}$$

$$(24)$$

$$\text{lc}(Y^{(k)}v_\nu) = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } k = \nu > 0, \\ 1 & \text{otherwise}. \end{cases}$$
Consequently, we obtain
\[
\text{lt}(\mathcal{Y}^{(k)} v_\nu) = \begin{cases} 
1 & \text{if } \nu = 0, \\
\frac{q^k}{2(2\nu-k+1)n} & \text{if } 0 \leq k < \nu, \\
\frac{1}{\sqrt{2}} q^{\nu(\nu+1)} & \text{if } k = \nu > 0, \\
q^{k(2\nu-k)(k+1)} & \text{if } \nu < k \leq 2\nu.
\end{cases}
\]

Remark 6.1.2. For each \( k \in [0, \nu] \), we have
\[
\begin{align*}
\deg(\prod_{j=0}^{k-1} (\nu - j)) &= \sum_{j=0}^{k-1} (\nu - j) = \frac{1}{2}(2\nu - k + 1)k, \\
\text{lc}(\prod_{j=0}^{k-1} (\nu - j)) &= 1.
\end{align*}
\]
Hence, we may choose
\[
c_{k,\nu} := \begin{dcases} 
1 & \text{if } \nu = 0, \\
\prod_{j=0}^{k-1} \frac{1}{\nu - j} & \text{if } 0 \leq k < \nu, \\
\sqrt{2}\prod_{j=0}^{\nu-1} \frac{1}{\nu - j} & \text{if } k = \nu > 0, \\
\prod_{j=0}^{2\nu-k-1} \frac{1}{\nu - j} & \text{if } \nu < k \leq 2\nu.
\end{dcases}
\]
In the sequel, \( c_{k,\nu} \) always means this value. Note that we have
\[
\psi(\mathcal{Y}^{k} v_\nu) = \mathcal{Y}^{k} v_\nu
\]
for all \( k \in [0, 2\nu] \), and
\[
c_{0,\nu} = c_{2\nu,\nu} = 1.
\]

Definition 6.1.3. The lowering operator \( \mathcal{Y} \) and the raising operator \( \mathcal{X} \) are \( K \)-linear endomorphisms on \( V(\nu) \) defined by
\[
\mathcal{Y}(\mathcal{Y}^{k} v_\nu) := \begin{dcases} 
\mathcal{Y}^{k+1} v_\nu & \text{if } k \in [0, 2\nu], \\
0 & \text{otherwise},
\end{dcases}
\]
\[
\mathcal{X}(\mathcal{Y}^{k} v_\nu) := \begin{dcases} 
\mathcal{Y}^{k-1} v_\nu & \text{if } k \in [0, 2\nu], \\
0 & \text{otherwise}.
\end{dcases}
\]
Clearly, the operators \( \mathcal{Y} \) and \( \mathcal{X} \) preserve \( \mathcal{L}(\nu) \). Hence, they induce linear operators \( \mathcal{Y} \) and \( \mathcal{X} \) on \( \mathcal{L}(\nu) \).

Since each finite-dimensional classical weight module is completely reducible, we can extend the definitions of \( \mathcal{Y}, \mathcal{X} \) on the irreducible modules to the finite-dimensional classical weight modules. The following are counterparts of well-known results about the Kashiwara operators. The proofs are similar, so we omit them.

Proposition 6.1.4. Let \( M \) be a finite-dimensional classical weight module. Let \( \xi \in X_{t,\text{int}} \). Then, we have \( \mathcal{Y} M_\xi = \mathcal{Y} M_\xi \).

Proposition 6.1.5. Let \( M \) be a finite-dimensional classical weight module equipped with a contragredient Hermitian inner product. Then, for each \( u, v \in \mathcal{L}_M \), we have
\[
(\mathcal{Y} u, v) - (u, \mathcal{X} v) \in q^{-1}K_\infty.
\]
6.2. Tensor product rule. In this subsection, we investigate the behavior of $\tilde{Y}, \tilde{X}$ on tensor product modules. Let $V_2 = Ku_1 \oplus Ku_2 \oplus Ku_3$ denote the natural representation of $U = U_q(sl_3)$.

Let $M$ be a finite-dimensional classical weight module equipped with a contra gredient Hermitian inner product, and $\nu \in X_{\text{int}}$. For $v \in M_\nu$, set

$$v'_\pm := v \otimes (u_1 \pm q^{\pm \nu} u_2), \quad v_0 := v \otimes u_3,$$

and

$$v_\pm := \frac{\text{lt}(v)}{\text{lt}(v_\pm)} v'_\pm = \begin{cases} q^{\pm \nu} v'_\pm & \text{if } \pm \nu > 0, \\ \frac{1}{\sqrt{2}} v'_\pm & \text{if } \nu = 0, \\ v'_\pm & \text{if } \pm \nu < 0. \end{cases}$$

Then, we have

$$\text{deg}(v'_\pm) = \text{deg}(v) + \max\{\pm \nu, 0\}, \quad \text{deg}(v_0) = \text{deg}(v),$$

$$(26) \quad \text{lc}(v'_\pm) = \begin{cases} \text{lc}(v) & \text{if } \nu \neq 0, \\ \sqrt{2} \text{lc}(v) & \text{if } \nu = 0. \end{cases}$$

Lemma 6.2.1. We have $v'_\pm \in (M \otimes V_2)_{\nu \pm 1}$ and $v_0 \in (M \otimes V_2)_{\nu}$.

Proof. The assertion follows from Lemma 4.3.1. \hfill \square

For convenience, we write down easy formulas:

$$(27) \quad v \otimes u_1 = \frac{1}{\{\nu\}} (q^{-\nu} v'_+ + q^\nu v'_-),$$

$$(28) \quad v \otimes u_2 = \frac{1}{\{\nu\}} (v'_+ - v'_-).$$

Proposition 6.2.2. We have

$$X v'_+ = q^{-1} (Xv)'_+, \quad Y v'_+ = \frac{\nu + 1}{\{\nu\}} (Yv)'_+ + q^\nu \{\nu + 1\} v_0 + \frac{q^\nu (q - q^{-1})}{\{\nu\}} (Xv)'_-, \quad X v'_- = q^{-\nu} (q - q^{-1}) \{\nu\} (Yv)'_+ - q^{-\nu} \{\nu - 1\} v_0 + \frac{\nu - 1}{\{\nu\}} (Xv)'_-,$$

$$Y v'_- = q^{-1} (Yv)'_-, \quad X v_0 = v'_+ + q (Xv)_0, \quad Y v_0 = q (Yv)_0 - v'_-.$$

Proof. Since $\Delta(B_2) = B_2 \otimes K_2^{-1} + 1 \otimes B_2$, we have

$$B_2 v'_\pm = B_2 v \otimes (u_1 \pm q^{\pm \nu - 1} u_2) \pm q^{\pm \nu} v \otimes u_3$$

$$= (B_{2,+} + B_{2,-} v \otimes u_1 \pm q^{\pm \nu - 1} (B_{2,+} - B_{2,-}) v \otimes u_2) \pm q^{\pm \nu} v_0$$

$$= \frac{1}{\{\nu + 1\}} (q^{-\nu - 1} (B_{2,+} v)'_+ + q^{\nu + 1} (B_{2,+} v)'_-) + \frac{1}{\{\nu - 1\}} (q^{-\nu + 1} (B_{2,-} v)'_+ + q^{\nu - 1} (B_{2,-} v)'_-)$$

$$\pm q^{\pm \nu - 1} \left( \frac{1}{\{\nu + 1\}} ((B_{2,+} v)'_+ - (B_{2,+} v)'_-) + \frac{1}{\{\nu - 1\}} ((B_{2,-} v)'_+ - (B_{2,-} v)'_-) \right) \pm q^{\pm \nu} v_0.$$
For the last equality, we used equations (27) and (28). Rearranging this, we obtain

\[ B_2 v'_+ = q^{-1} \left\{ \frac{1}{\nu} \right\} (B_2 v')'_+ + q'' q^{-1} \left\{ \frac{1}{\nu + 1} \right\} (B_2 v')'_- + (B_2 v')'_+ + q' v_0, \]

\[ B_2 v'_- = (B_2 v')'_+ + q'' q^{-1} \left\{ \frac{1}{\nu - 1} \right\} (B_2 v')'_- + q^{-1} \left\{ \frac{1}{\nu} \right\} (B_2 v')'_- - q^{-\nu} v_0. \]

By weight consideration, we conclude

\[ B_{2,+} v'_+ = q^{-1} \left\{ \frac{1}{\nu} \right\} (B_2 v')'_+ , \]

\[ B_{2,-} v'_+ = q'' q^{-1} \left\{ \frac{1}{\nu + 1} \right\} (B_2 v')'_+ + (B_2 v')'_+ + q' v_0, \]

\[ B_{2,+} v'_- = (B_2 v')'_+ + q'' q^{-1} \left\{ \frac{1}{\nu - 1} \right\} (B_2 v')'_- - q^{-\nu} v_0, \]

\[ B_{2,-} v'_- = q^{-1} \left\{ \frac{1}{\nu} \right\} (B_2 v')'_-. \]

These imply the first four assertions.

Also, we have

\[ B_2 v_0 = v \otimes u_2 + q B_2 v \otimes u_3 \]

\[ = \frac{1}{\nu} (v'_+ - v'_-) + q (B_2 v)_0 + q (B_2 v)_0. \]

Again, we used equation (28) for the last equality. By weight consideration, we obtain

\[ B_{2,+} v_0 = \frac{1}{\nu} v'_+ + q (B_2 v)_0, \]

\[ B_{2,-} v_0 = - \frac{1}{\nu} v'_- + q (B_2 v)_0. \]

These imply the remaining assertions. Thus, we complete the proof. \( \square \)

**Lemma 6.2.3.** Let \( M \) be a finite-dimensional classical weight module, \( \nu \in X_{t,\text{int}} \), and \( v \in M_\nu \). Then, for each \( k \geq 0 \), we have

\[ Y^{(k)} v'_+ = \frac{\nu + 1}{\nu - k + 1} (Y^{(k)} v)_+ + q'' \left\{ \frac{\nu + 1}{\nu - k + 1} \right\} (Y^{(k-1)} v)_0 \]

\[ - \frac{\nu + 1}{\nu - k + 1} (Y^{(k-1)} v)_- + q^{\nu - k + 1} (q - q^{-1}) \left\{ \frac{\nu - k + 1}{\nu - k + 1} \right\} (Y^{(k-1)} X v)_-, \]

\[ Y^{(k)} v_0 = q^k (Y^{(k)} v)_0 - (Y^{(k-1)} v)_-, \]

\[ Y^{(k)} v'_- = q^{-k} (Y^{(k)} v)_-. \]

**Proof.** Using Proposition 6.2.2, the assertion is verified by induction on \( k \). \( \square \)
Proposition 6.2.4. Let $\nu \in X^+_{t,\text{int}}$. Then, for each $k \in [0, 2\nu + 2]$, we have

$$X(v_\nu)'_+ = 0,$$

$$Y^{(k)}(v_\nu)'_+ = \frac{\nu + 1}{\nu - k + 1} (Y^{(k)}v_\nu)'_+ + q'\nu + 1 (Y^{(k-1)}v_\nu) - \frac{\nu + 1}{\nu - k + 1} (Y^{(k-2)}v_\nu)'_-, $$

$$\text{ev}_\infty(\widetilde{Y}^k(v_\nu)_+) = \begin{cases} \text{ev}_\infty((v_\nu)_+) & \text{if } k = 0, \\
\text{ev}_\infty((\widetilde{Y}^{k-1}v_\nu)_0) & \text{if } 1 \leq k \leq 2\nu + 1, \\
-\text{ev}_\infty((\widetilde{Y}^{2\nu}v_\nu)_-^{-1}) & \text{if } k = 2\nu + 2. 
\end{cases}$$

Proof. The first and second assertions are verified by using Proposition 6.2.2 and Lemma 6.2.3. Let us prove the third assertion. By the second equality and the definition of $\widetilde{Y}$, we can write

$$d_0\widetilde{Y}^k(v_\nu)_+ = d_1(\widetilde{Y}^k v_\nu)_+ + d_2(\widetilde{Y}^{k-1}v_\nu)_0 - d_3(\widetilde{Y}^{k-2}v_\nu)_-, $$

where $d_0, d_1, d_2, d_3 \in \mathbb{K}$ such that $\text{lc}(d_j) \in \mathbb{R}_{\geq 0}$ and

$$\text{deg}(d_0) = \text{deg}(Y^{(k)}v_\nu)'_+, \quad \text{deg}(d_1) = \text{deg}(\frac{\nu + 1}{\nu - k + 1} (Y^{(k)}v_\nu)'_+),$$

$$\text{deg}(d_2) = \text{deg}(q'\nu + 1 (Y^{(k-1)}v_\nu)_0), \quad \text{deg}(d_3) = \text{deg}(\frac{\nu + 1}{\nu - k + 1} (Y^{(k-2)}v_\nu)'_-).$$

More explicitly, by equations (24) and (26), we have

$$\text{deg}(d_1) = \nu + 1 - |\nu - k + 1| + \frac{1}{2}(n(2\nu - k) + \nu - |\nu - k|) + \max\{\nu - k, 0\}$$

$$= \nu + 1 + \frac{1}{2}(k + 1)(2\nu - k) - |\nu - k + 1|,$$

$$\text{deg}(d_2) = 2\nu + 1 + \frac{1}{2}((k - 1)(2\nu - k + 1) + \nu - |\nu - k + 1|),$$

$$\text{deg}(d_3) = \nu + 1 - |\nu - k + 1| + \frac{1}{2}((k - 2)(2\nu - k + 2) + \nu - |\nu - k + 2|)$$

$$+ \max\{-\nu + k - 2, 0\}$$

$$= \nu + 1 + \frac{1}{2}(k - 2)(2\nu - k + 3) - |\nu - k + 1|.$$

Here, we used an easy formula

$$\max\{x, 0\} = \frac{1}{2}(x + |x|).$$

Then, we compute as

$$\text{deg}(d_2) - \text{deg}(d_1) = \nu + \frac{1}{2}(-3\nu + 3k - 1 + |\nu - k + 1|)$$

$$= \begin{cases} k & \text{if } 0 \leq k \leq \nu + 1, \\
-\nu + 2k - 1 & \text{if } \nu + 1 \leq k \leq 2\nu + 2, 
\end{cases}$$

$$\text{deg}(d_2) - \text{deg}(d_3) = \nu + \frac{1}{2}(3\nu - 3k + 5 + |\nu - k + 1|)$$

$$= \begin{cases} 3\nu - 2k + 3 & \text{if } k \leq \nu + 1, \\
2\nu - k + 2 & \text{if } \nu + 1 \leq k \leq 2\nu + 2. 
\end{cases}$$
From these, we obtain
\begin{align}
\deg(d_3) & \leq \frac{\deg(d_1)}{3(\nu - k + 1)} \quad \text{if } 0 \leq k \leq \nu + 1, \\
\deg(d_1) & \leq \frac{\deg(d_3)}{2\nu - k + 2} \quad \text{if } \nu + 1 \leq k \leq 2\nu + 2.
\end{align}

(30)

Here, \( a \leq b \) means \( c = b - a \geq 0 \).

On the other hand, since the terms in the right-hand side of (29) are orthogonal to each other, we have
\[
\deg(d_0) = \begin{cases} 
\deg(d_1) & \text{if } k = 0, \\
\max\{\deg(d_1), \deg(d_2)\} & \text{if } k = 1, \\
\max\{\deg(d_1), \deg(d_2), \deg(d_3)\} & \text{if } 2 \leq k \leq 2\nu, \\
\max\{\deg(d_2), \deg(d_3)\} & \text{if } k = 2\nu + 1, \\
\deg(d_3) & \text{if } k = 2\nu + 2.
\end{cases}
\]

Combining this and inequalities (30), we obtain
\[
ev_{\infty}(\tilde{Y}^n(v_\nu)_+) = \begin{cases} 
\frac{\deg(d_1)}{\deg(d_0)}ev_{\infty}((v_\nu)_+) & \text{if } k = 0, \\
\frac{\deg(d_1)}{\deg(d_0)}ev_{\infty}((\tilde{Y}^{k-1}v_\nu)_0) & \text{if } 1 \leq k \leq 2\nu + 1, \\
-k\frac{\deg(d_1)}{\deg(d_0)}ev_{\infty}((\tilde{Y}^{2\nu}v_\nu)_-) & \text{if } k = 2\nu + 2.
\end{cases}
\]

Noting that \( \deg(d_j) \in \mathbb{R}_{\geq 0} \) for all \( j = 0, 1, 2, 3 \), and that
\[
\deg(\tilde{Y}^k(v_\nu)_+) = \deg((v_\nu)_+) = \deg((\tilde{Y}^{k-1}v_\nu)_0) = \deg((\tilde{Y}^{2\nu}v_\nu)_-) = 1,
\]
we finally obtain the required equation. \( \square \)

The following two results can be proved in a similar way to Proposition 6.2.4.

**Proposition 6.2.5.** Let \( \nu \in X_{t,\text{int}}^+ \) be such that \( \nu > 0 \). Set
\[
v' := (Yv_\nu)_+ - q^{-1}[2\nu](v_\nu)_0, \\
v := \frac{1}{\deg(v')}(v').
\]

Then, for each \( k \in [0, 2\nu] \), we have
\[
Xv' = 0,
\]
\[
Y^{(k)}v' = \frac{\{\nu\}}{\{\nu - k\}}(k + 1)(Y^{(k+1)}v_\nu)_+ - q^{-1}[2\nu - 2k](Y^{(k)}v_\nu)_0 + [2\nu - k + 1](Y^{(k-1)}v_\nu)_-,
\]
\[
ev_{\infty}(\tilde{Y}^k v) = \begin{cases} 
ev_{\infty}((\tilde{Y}^{k+1}v_\nu)_+) & \text{if } 0 \leq k < \nu, \\
\frac{1}{\nu^2}ev_{\infty}((\tilde{Y}^{\nu+1}v_\nu)_+ + (\tilde{Y}^{\nu-1}v_\nu)_-) & \text{if } k = \nu, \\
ev_{\infty}((\tilde{Y}^{k-1}v_\nu)_-) & \text{if } \nu < k \leq 2\nu.
\end{cases}
\]

**Proposition 6.2.6.** Let \( \nu \in X_{t,\text{int}}^+ \) be such that \( \nu > 0 \). Set
\[
v' := -[2](Y^{(2)}v_\nu)_+ + q^{-\nu-1}[2\nu - 1](v_\nu)_0 + [2\nu][2\nu - 1](v_\nu)_-,
\]
\[
v := \frac{1}{\deg(v')}(v').
\]
Then, for each $k \in [0, 2\nu - 2]$, we have

\[ Xv' = 0, \]

\[ Y^{(k)}v' = -\frac{[k + 2][k + 1](\nu - 1)}{\nu - k - 1}(Y^{(k+2)}v)_+ + q^{-\nu-1}[k + 1][2\nu - k - 1](\nu - 1)(Y^{(k+1)}v)_0 + \frac{[2\nu - k][2\nu - k - 1](\nu - 1)}{\nu - k - 1}(Y^{(k)}v)'_-, \]

\[ \text{ev}_\infty(\tilde{\gamma}^k v) = \begin{cases} \text{ev}_\infty((\tilde{\gamma}^k v)_-) + \frac{1}{\sqrt{2}}\text{ev}_\infty((\tilde{\gamma}^{k+1} v)_+) - (\tilde{\gamma}^{k-1} v)_-) & \text{if } 0 \leq k < \nu - 1, \\ -\text{ev}_\infty((\tilde{\gamma}^{k+2} v)_+) & \text{if } k = \nu - 1, \\ \text{ev}_\infty((\tilde{\gamma}^{k+2} v)_+) & \text{if } \nu - 1 < k \leq 2\nu - 2. \end{cases} \]

6.3. Based module structures of irreducible $U$-modules. In this subsection, we study $V(\nu)$, $\nu \in X^+_\text{int}$, and show that it admits a based $U$-module structure.

**Proposition 6.3.1.** Let $\nu \in X^+_\text{int}$. Then, $V(\nu)$ is a standard $X^*$-weight module. Furthermore, the highest weight vector $v_\nu$ has the following $X^*$-weight vector decomposition

\[ v_\nu = \frac{1}{2}(v_\nu + (-1)^\nu Y^{(2\nu)}v_\nu) + \frac{1}{2}(v_\nu - (-1)^\nu Y^{(2\nu)}v_\nu) \in V(\nu)_{(\nu,0)} \oplus V(\nu)_{(\nu,1)} \]

\[ = \frac{1}{2}(v_\nu + Y^{(2\nu)}v_\nu) + \frac{1}{2}(v_\nu - Y^{(2\nu)}v_\nu) \in V(\nu)_{(\nu,\nu)} \oplus V(\nu)_{(\nu,\nu-1)} \]

**Proof.** Let $M := V_2^{\otimes \nu}$. Then, its canonical $X^*$-weight module structure is standard by Proposition 4.4.7. Define $u^{\pm \nu} \in M$ inductively by

\[ u^{\pm 1} := u_1 \pm u_2, \quad u^{\pm k} := (u^{\pm (k-1)})_{\pm} = u^{\pm (k-1)} \otimes (q^{-k+1}u_1 \pm u_2). \]

Explicitly, we have

\[ u^{\pm \nu} = \sum_{i_1, ..., i_\nu \in \{1, 2\}} (\pm 1)^{\# \{i_k = 2\}} \prod_{i_k = 1}^k q^{-k+1}u_{i_1} \otimes \cdots \otimes u_{i_\nu}. \]

Hence, we obtain

\[ u^\nu - u^{-\nu} = 2 \sum_{i_1, ..., i_\nu \in \{1, 2\}} \prod_{i_k = 1}^k q^{-k+1}u_{i_1} \otimes \cdots \otimes u_{i_\nu} \in M_{(\nu,0)}, \]

and

\[ u^\nu + u^{-\nu} = 2 \sum_{i_1, ..., i_\nu \in \{1, 2\}} \prod_{i_k = 1}^k q^{k-1}u_{i_1} \otimes \cdots \otimes u_{i_\nu} \in M_{(\nu,\nu)}. \]

These imply that

\[ u^\nu = \frac{1}{2}(u^\nu + u^{-\nu}) + \frac{1}{2}(u^\nu - u^{-\nu}) \in M_{(\nu,0)} \oplus M_{(\nu,\nu)} \]

is the $X^*$-weight vector decomposition of $u^\nu$.

On the other hand, by the definition of $u^{\pm \nu}$ and Proposition 6.2.4, we have

\[ Xu^\nu = 0, \quad lu^\nu = q^\nu u^\nu, \quad Y^{(2\nu)}u^\nu = (-1)^\nu u^\nu. \]

Therefore, the $U$-submodule $U^\nu u^\nu$ generated by $u^\nu$ is isomorphic to $V(\nu)$, and it is also generated by two $X^*$-weight vectors $u^\nu + u^{-\nu}$ and $u^\nu - u^{-\nu}$. Hence, $U^\nu u^\nu$ is a standard $X^*$-weight module by Propositions 3.3.8, and 4.4.4. This completes the proof. \[ \square \]
Now, we set
\[ V(\nu)A := \hat{U}_A v_\nu, \]
and call it the A-form of \( V(\nu) \). That it is actually an A-form of \( V(\nu) \) in the sense of Definition 3.4.1 will be proved later.

**Lemma 6.3.2.** Let \( \nu \in X^+_t, \ M := V^s_t. \) Let \( u^\nu \in M \) be as before. Then, we have
\[ \psi^1_M(u^\nu) = u^\nu. \]

**Proof.** By Proposition 4.3.2, we see that \( M \) has no \( X_t \)-weight vectors of weight greater than \( \nu \) or less than \( -\nu \).

The vector \( u^\nu \), expanded by the basis \( \{ u_i \otimes \cdots \otimes u_{i_v} \mid i_1, \ldots, i_v \in \{1, 2, 3\} \} \), contains \( u_2^{\otimes \nu} \) with coefficient 1. Note that we have
\[
\widetilde{E}_1(\pi_1^{\otimes \nu}) = 0,
\widetilde{F}_1^{\nu-1}(\pi_1^{\otimes \nu}) = \pi_1 \otimes \pi_2^{\otimes \nu-1},
\widetilde{F}_1(\pi_1^{\otimes \nu}) = \pi_2^{\otimes \nu}.
\]

This together with Proposition 5.2.1, implies that \( G^i(\pi_1 \otimes \pi_2^{\otimes \nu-1} + \pi_2^{\otimes \nu}) \) is an \( X_t \)-weight vector of weight \( \nu \). Furthermore, the vector \( G^i(\pi_1 \otimes \pi_2^{\otimes \nu-1} + \pi_2^{\otimes \nu}) \), expanded by the basis \( \{ u_i \otimes \cdots \otimes u_{i_v} \mid i_1, \ldots, i_v \in \{1, 2, 3\} \} \), contains \( u_2^{\otimes \nu} \) with coefficient 1. Since \( \dim M_\nu = 1 \) by Proposition 4.3.2, we conclude that
\[ u^\nu = G^i(\pi_1 \otimes \pi_2^{\otimes \nu-1} + \pi_2^{\otimes \nu}), \]
and hence, it is \( \psi^1_M \)-invariant. This completes the proof \( \square \)

**Proposition 6.3.3.** Let \( \nu \in X^+_t, \int. \)

1. For each \( k \geq 0 \), we have
\[ B^{(k)}_{2,p(\nu)}(v_\nu + Y^{(2\nu)}v_\nu), \ B^{(k)}_{2,q(\nu)}(v_\nu - Y^{(2\nu)}v_\nu) \in V(\nu)_A. \]

2. We have \( B^{(k)}_{2,p(\nu)}(v_\nu + Y^{(2\nu)}v_\nu) = 0 \) for all \( k > \nu \).

3. We have \( B^{(k)}_{2,q(\nu)}(v_\nu - Y^{(2\nu)}v_\nu) = 0 \) for all \( k > \nu - 1 \).

4. \( \{ B^{(k)}_{2,p(\nu)}(v_\nu + Y^{(2\nu)}v_\nu) \mid k \in [0, \nu] \} \cup \{ B^{(k-1)}_{2,q(\nu)}(v_\nu - Y^{(2\nu)}v_\nu) \mid k \in [0, \nu - 1] \} \)
forms a free \( \mathbb{K}_\infty \)-basis of \( \mathcal{L}(\nu) \).

5. \( \{ \text{ev}_\infty(B^{(k)}_{2,p(\nu)}(v_\nu + Y^{(2\nu)}v_\nu)) \mid k \in [0, \nu] \} \cup \{ \text{ev}_\infty(B^{(k-1)}_{2,q(\nu)}(v_\nu - Y^{(2\nu)}v_\nu)) \mid k \in [0, \nu - 1] \} \)
forms a \( \mathbb{C} \)-basis of \( \tilde{\mathcal{L}}(\nu) \).

6. We have \( \text{ev}_\infty(B^{(k)}_{2,p(\nu)}(v_\nu + Y^{(2\nu)}v_\nu)) = \bar{Y}^kB_\nu + \bar{Y}^{2\nu-k}b_\nu \) for all \( k \in [0, \nu] \).

7. We have \( \text{ev}_\infty(B^{(k)}_{2,q(\nu)}(v_\nu - Y^{(2\nu)}v_\nu)) = \bar{Y}^kB_\nu - \bar{Y}^{2\nu-k}b_\nu \) for all \( k \in [0, \nu - 1] \).

**Proof.** Let \( M = V^s_t, \) and \( u^\nu \in M \) be as before. Then, there exists a \( \mathbb{U} \)-module homomorphism \( \phi : V(\nu) \rightarrow M \) such that \( \phi(v_\nu) = \frac{1}{\sqrt{2}} u^\nu \). We identify \( V(\nu) \) with \( \phi(V(\nu)) \). By Proposition 6.3.1, we have
\[
B^{(k)}_{2,p(\nu)}(v_\nu + Y^{(2\nu)}v_\nu) = B^{(k)}_{2,p(\nu)}(v_\nu + Y^{(2\nu)}v_\nu) = 2B^{(k)}_{2,p(\nu)}v_\nu \in V(\nu)_A,
B^{(k)}_{2,q(\nu)}(v_\nu - Y^{(2\nu)}v_\nu) = B^{(k)}_{2,q(\nu)}(v_\nu - Y^{(2\nu)}v_\nu) = 2B^{(k)}_{2,q(\nu)}v_\nu \in V(\nu)_A.
\]
This proves assertion (1).
Similarly, for each $k$ our claim follows from the recursive formula for the $M_k$ for each $d_k$ for the same $d_k$.

Let $
u > 1$. These prove assertions (2) and (3).

For each $p \in \{ev, odd\}$ and $k \in [0, \nu)$, we show by induction on $k$ that there exist $d_{m,k,p} \in K\ast$ such that $d_{0,k,p} = 1$ if $k < \nu$, $d_{0,\nu,p} = \frac{1}{\sqrt{2\nu}}$, and $d_{m,k,p} \in q^{-1}K\ast$ if $m > 0$, and

$$B^{(k)}_{2,p} (v_{\nu} - Y^{(2\nu)} v_{\nu}) = 0 \quad \text{if } k > \nu - 1.$$

These prove assertions (2) and (3).

For each $p \in \{ev, odd\}$ and $k \in [0, \nu)$, we show by induction on $k$ that there exist $d_{m,k,p} \in K\ast$ such that $d_{0,k,p} = 1$ if $k < \nu$, $d_{0,\nu,p} = \frac{1}{\sqrt{2\nu}}$, and $d_{m,k,p} \in q^{-1}K\ast$ if $m > 0$, and

$$B^{(k)}_{2,p} v_{\nu} = \sum_{m=0}^{\left\lfloor \frac{\nu}{2} \right\rfloor} d_{m,k,p} \tilde{Y}^{k-2m} v_{\nu}.$$

Let $k \in [0, \nu - 1]$. Then, inductively, we have

$$B_2 B^{(k)}_{2,p} v_{\nu} = (Y + X) \frac{1}{\{l; 0\}} \sum_{m=0}^{\left\lfloor \frac{\nu}{2} \right\rfloor} d_{m,k,p} c_{k-2m} \tilde{Y}^{(k-2m)} v_{\nu}$$

$$= \sum_{m=0}^{\left\lfloor \frac{\nu}{2} \right\rfloor} d_{m,k,p} c_{k-2m} \tilde{Y}^{(k-2m)} v_{\nu}$$

By degree consideration, the right-hand side is a sum of $\tilde{Y}^{k-2m+1}$ with $m \in [0, \left\lfloor \frac{k+1}{2} \right\rfloor]$ whose coefficient equals $\frac{[k+1]}{\sqrt{2^{k+1}}} \nu$ if $m = 0$, and belongs to $q^{-1}K\ast$ if $m \neq 0$. Then, our claim follows from the recursive formula for the divided powers in Proposition 4.4.1. Similarly, for each $k \in [0, \nu]$ we have

$$B^{(k)}_{2,p} Y^{(2\nu)} v_{\nu} = \sum_{m=0}^{\left\lfloor \frac{\nu}{2} \right\rfloor} d_{m,k,p} \tilde{Y}^{2\nu-k+2m} v_{\nu}$$

for the same $d_{m,k,p} \in K\ast$ as before. Therefore, we obtain

$$ev_{\infty}(B^{(k)}_{2,p} (v_{\nu} + Y^{(2\nu)} v_{\nu})) = \tilde{Y}^{k} b_{\nu} + \tilde{Y}^{2\nu-k} b_{\nu}$$

for each $k \in [0, \nu]$, and

$$ev_{\infty}(B^{(k)}_{2,q} (v_{\nu} - Y^{(2\nu)} v_{\nu})) = \tilde{Y}^{k} b_{\nu} - \tilde{Y}^{2\nu-k} b_{\nu}$$

for each $k \in [0, \nu - 1]$. These imply the remaining assertions. \hfill \-box

**Theorem 6.3.4.** Let $\nu \in X^+_{ext}$. Then, $V(\nu)$ is a based $U^+$-module. Furthermore, for each $k \in [0, \nu]$, we have

$$G''(\tilde{Y}^{k} b_{\nu} + \tilde{Y}^{2\nu-k} b_{\nu}) = B^{(k)}_{2,p} (v_{\nu} + Y^{(2\nu)} v_{\nu}),$$

$$G''(\tilde{Y}^{k} b_{\nu} - \tilde{Y}^{2\nu-k} b_{\nu}) = B^{(k)}_{2,q} (v_{\nu} - Y^{(2\nu)} v_{\nu}).$$
Proof. Let \( u^\nu \in M := V_s^{\otimes \nu} \) be as before. Then, there exists an almost isometry

\[ \phi : V(\nu) \to \hat{U}^\nu u^\nu (= U^\nu u^\nu) \]

of \( U^\nu \)-modules such that \( \phi(v_\nu) = \frac{1}{\sqrt{2}} u^\nu \). Note that \( u^\nu \) belongs to the \( A \)-form \( M_A := \bigoplus_{\nu=1}^\infty A u_\nu \). Also, by Lemma 6.3.2, it is \( \psi_{M^\nu} \)-invariant. Then, by Propositions 4.4.5 and 6.3.3 (1), we have

\[ \phi(B_{2,2(\nu)}^{(k)}(v_\nu + Y^{(2\nu)}v_\nu)), \phi(B_{2,2(\nu)}^{(k)}(v_\nu - Y^{(2\nu)}v_\nu)) \in \hat{U}_A^\nu u^\nu \subseteq M_A, \]

for all \( k \geq 0 \), and they are \( \psi_{M^\nu} \)-invariant. Furthermore, by Proposition 6.3.3 (6) and (7), we obtain

\[ \text{ev}_\infty(\phi(B_{2,2(\nu)}^{(k)}(v_\nu + Y^{(2\nu)}v_\nu))) = \tilde{Y}^k b + \tilde{Y}^{2\nu - k} b \quad \text{if } k \in [0, \nu], \]
\[ \text{ev}_\infty(\phi(B_{2,2(\nu)}^{(k)}(v_\nu - Y^{(2\nu)}v_\nu))) = \tilde{Y}^k b - \tilde{Y}^{2\nu - k} b \quad \text{if } k \in [0, \nu - 1], \]

where \( b := \text{ev}_\infty(\phi(v_\nu)) \). Therefore, by Lemma 3.4.6, we conclude that

\[ \phi(B_{2,2(\nu)}^{(k)}(v_\nu + Y^{(2\nu)}v_\nu)) = G^\nu(\tilde{Y}^k b + \tilde{Y}^{2\nu - k} b) \quad \text{if } k \in [0, \nu], \]
\[ \phi(B_{2,2(\nu)}^{(k)}(v_\nu - Y^{(2\nu)}v_\nu)) = G^\nu(\tilde{Y}^k b - \tilde{Y}^{2\nu - k} b) \quad \text{if } k \in [0, \nu - 1]. \]

These show that \( U^\nu u^\nu \) is spanned by \( G^\nu(\text{ev}_\infty(\phi(L(\nu)))) \). Hence, by Proposition 3.4.9 (1), \( U^\nu u^\nu \) is a based submodule of \( M \). This shows that \( V(\nu) = \phi^{-1}(U^\nu u^\nu) \) is a based \( U^\nu \)-module. Thus, the proof completes. \( \square \)

Corollary 6.3.5. Let \( \nu \in X^+_{t_{\text{int}}} \).

1. For each \( k \in [0, \nu] \), we have

\[ \hat{B}_1(\tilde{Y}^k + \tilde{Y}^{2\nu - k})b_\nu = (\tilde{Y}^k - \tilde{Y}^{2\nu - k})b_\nu, \]

and for each \( k \in [0, \nu - 1] \),

\[ \hat{B}_1(\tilde{Y}^k - \tilde{Y}^{2\nu - k})b_\nu = (\tilde{Y}^k + \tilde{Y}^{2\nu - k})b_\nu. \]

2. For each \( k \in [0, \nu] \), we have

\[ \hat{B}_2(\tilde{Y}^k + \tilde{Y}^{2\nu - k})b_\nu = \begin{cases} (\tilde{Y}^{k - 1} + \tilde{Y}^{2\nu - k + 1})b_\nu & \text{if } \nu - k \in \mathbb{Z}_{\text{ev}} \setminus \{0\}, \\ \sqrt{2}(\tilde{Y}^{\nu - 1} + \tilde{Y}^{\nu + 1})b_\nu & \text{if } \nu - k = 0, \\ (\tilde{Y}^{k + 1} + \tilde{Y}^{2\nu - k - 1})b_\nu & \text{if } \nu - k \in \mathbb{Z}_{\text{odd}} \setminus \{1\}, \\ \sqrt{2}\tilde{Y}^\nu b_\nu & \text{if } \nu - k = 1, \end{cases} \]

and for each \( k \in [0, \nu - 1] \),

\[ \hat{B}_2(\tilde{Y}^k - \tilde{Y}^{2\nu - k})b_\nu = \begin{cases} (\tilde{Y}^{k + 1} - \tilde{Y}^{2\nu - k - 1})b_\nu & \text{if } \nu - k \in \mathbb{Z}_{\text{ev}}, \\ (\tilde{Y}^{k - 1} - \tilde{Y}^{2\nu - k + 1})b_\nu & \text{if } \nu - k \in \mathbb{Z}_{\text{odd}}. \end{cases} \]
Example 6.3.6. We give graphical descriptions of $\tilde{B}_1$ and $\tilde{B}_2$ on $\mathcal{L}(\nu)$ for $\nu = 2$ (left) and $\nu = 3$ (right).

Proposition 6.3.7. Let $M$ be a finite-dimensional classical weight standard $X^\ast$-weight module. Let $v \in M$ be a highest weight vector of weight $\nu \in X^+_t \text{int}$. Then, we have

$$U^\ast v = \check{U}^\ast v.$$  

Furthermore, for each $\zeta = (\zeta_1, \zeta_2) \in X^\ast$, we have

$$1_\zeta v = \begin{cases} \frac{1}{2}(v + X(2\nu)v) & \text{if } \zeta = (\overline{\nu}, \overline{\nu}), \\ \frac{1}{2}(v - X(2\nu)v) & \text{if } \zeta = (\overline{\nu}, \overline{\nu} - 1), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The assertions follow from Propositions 4.4.5 and 6.3.1. \qed

Proposition 6.3.8. Let $M$ be a finite-dimensional classical weight module equipped with a contragredient Hermitian inner product. Set

$$L_1 := \{ b \in \mathcal{L}_M \mid \tilde{X}b = 0 \},$$

$$L_2 := \{ b \in \mathcal{L}_M \mid \tilde{B}_2b = 0 \}.$$

Then, the linear map

$$L_2 \rightarrow L_1; \quad b \mapsto (1 + \tilde{B}_1)b$$

is an isomorphism of $\mathbb{C}$-vector spaces, with inverse

$$L_1 \rightarrow L_2; \quad b \mapsto \sum_{\zeta \in X^\ast} 1_\zeta b.$$

Proof. Write an orthogonal irreducible decomposition of $M$ as

$$M = \bigoplus_{k=1}^r M_k, \quad M_k \simeq V(\nu_k), \quad \nu_k \in X^+_t \text{int}.$$  

This induces an orthogonal decomposition

$$\mathcal{L}_M = \bigoplus_{k=1}^r \mathcal{L}_k, \quad \mathcal{L}_k := \mathcal{L}_{M_k} \simeq \mathcal{L}(\nu_k).$$

Since both $\tilde{X}$ and $\tilde{B}_2$ preserve $\mathcal{L}_k$ for all $k \in [1, r]$, it suffices to prove the assertion for the $M = V(\nu), \nu \in X^+_t \text{int}$ case.
When $M = V(\nu)$ for some $\nu \in X_{\text{int}}^+$, we have
$$L_1 = \mathbb{C}b_\nu.$$ 
Furthermore, by Corollary 6.3.5 (2), we see that
$$L_2 = \mathbb{C}(1 + (-1)^{2\nu})b_\nu.$$ 
Using Proposition 6.3.7, we compute as
$$(1 + \tilde{B}_1)(\sum_{\zeta \in \mathbb{U}} 1_\zeta b_\nu) = \frac{1}{2}(1 + \tilde{B}_1)(1 + (-1)^{2\nu})b_\nu$$
$$= \frac{1}{2}(1 + (-1)^{2\nu})b_\nu + \frac{1}{2}(1 - (-1)^{2\nu})b_\nu = b_\nu.$$ 
This, together with dimension consideration, proves the assertion.

6.4. Applications to the general $n \geq 3$ case. In this subsection, we consider the general $n \geq 3$ case; results obtained here are necessary for the $n = 4$ case. For a subset $J = \{j_1, \ldots, j_r\} \subseteq I$, let $U_J = U_{j_1, \ldots, j_r}$ (resp., $U_J^\dagger = U_{j_1, \ldots, j_r}^\dagger$) denote the subalgebra generated by $E_j, F_j, K_j^{\pm 1}, j \in J$ (resp., $B_j, j \in J$). Then, $(J, \emptyset, \text{id})$ is a Satake subdiagram of $(I, \emptyset, \text{id})$ of type $A_l$. Let $U' = U_{j_1, \ldots, j_r}^\dagger := U'(J)$ denote the corresponding modified quantum group.

For each $\eta = (\eta_j)_{j \in J} \in (\mathbb{Z}/2\mathbb{Z})^J = X'(J)$, let $1_{J, \eta} = 1_{j_1, \ldots, j_r, \eta}$ denote the corresponding idempotent in $U_J$. It acts on each $X'$-weight module by
$$1_{J, \eta} = \sum_{\zeta \in X': \zeta_j = \eta_j \text{ for all } j \in J} 1_\zeta.$$ 
In particular, for each $\zeta = (\zeta_1, \ldots, \zeta_{n-1}) \in X'$, we have
$$1_{1, \zeta_1} \cdots 1_{n-1, \zeta_{n-1}} = 1_\zeta$$
on each $X'$-weight module.

Lemma 6.4.1. Let $M$ be a finite-dimensional classical weight standard $X'$-weight module. Let $v \in M$ be a highest weight vector of weight $\nu \in X_{\text{int}}^+$. Then, $U'v$ is a standard $X'$-weight module.

Proof. For each $\zeta \in X'$, by equation (31), we have
$$1_\zeta v = 1_{\zeta_1} \cdots 1_{n-1, \zeta_{n-1}} v.$$ 
For a proof, by Propositions 3.3.8 and 4.4.4, it suffices to show that $1_\zeta v \in U'v$ for all $\zeta \in X'$.

Let $\zeta \in X'$ be such that $1_\zeta v \neq 0$. We show that $\nu_{2i-1} = \zeta_{2i-1}$ for all $i \in I$ and
$$1_\zeta v = \frac{1}{2^{m'}}(1 - (-1)^{\delta_{\zeta_1, 2}} Y_2^{(2\nu_1)})(1 - (-1)^{\delta_{\zeta_3, 4}} Y_4^{(2\nu_3)})(1 - (-1)^{\delta_{\zeta_2m'-1, 2m'}} Y_{2m'}^{(2\nu_{2m'-1})}) v,$$
where
$$m' := \begin{cases} \frac{n}{2} - 1 & \text{if } n \in \mathbb{Z}_{\text{ev}}, \\ \frac{n}{2} & \text{if } n \in \mathbb{Z}_{\text{odd}}. \end{cases}$$ 

We proceed by induction on $n$. The case when $n = 3$ has been already proved in Proposition 6.3.7.
First, consider the case when \( n \in \mathbb{Z}_{\text{ev}} \). Since \( B_{n-1}v = [\nu_{n-1}]v \), we obtain
\[
1_{n-1,\zeta_{n-1}}v = \delta_{\nu_{n-1},\zeta_{n-1}}v.
\]
This, together with our hypothesis that \( 1_{\zeta} v \neq 0 \), implies that \( \zeta_{n-1} = \zeta_{n-1} \). As a \( \mathbf{U}_{1,2,\ldots,n-2} \)-module vector, \( 1_{\nu_{n-1},\zeta_{n-1}}v = v \) is a highest weight vector of weight \( (\nu_1, \nu_3, \ldots, \nu_{n-3}) \).

Therefore, our claim follows from the induction hypothesis.

Next, consider the case when \( n \in \mathbb{Z}_{\text{odd}} \). By the \( n = 3 \) case, we have
\[
1_{n-2,\zeta_{n-2}}1_{n-1,\zeta_{n-1}}v = \delta_{\nu_{n-2},\zeta_{n-2}} \frac{1}{2} (1 - (-1)^{\delta_{\nu_{n-2},\zeta_{n-1}}} Y^{(2\nu_{n-2})})v.
\]
As before, this implies that \( \zeta_{n-2} = \zeta_{n-2} \). By weight consideration, as a \( \mathbf{U}_{1,2,\ldots,n-3} \)-module vector, both \( v \) and \( Y^{(2\nu_{n-2})}v \) are highest weight vectors of weight \( (\nu_1, \nu_3, \ldots, \nu_{n-4}) \). Then, our claim follows from the induction hypothesis. Thus, the proof completes.

**Proposition 6.4.2.** Let \( \nu \in X_{\text{t.int}} \). Then, \( V(\nu) \) is a standard \( X^1 \)-weight module.

**Proof.** The assertion follows from Propositions 4.3.3, 4.4.7, and Lemma 6.4.1.

For each \( \nu \in X_{\text{t.int}} \), set \( V(\nu)_A := \tilde{U}_A v_{\nu} \), and call it the \( A \)-form of \( V(\nu) \).

For a based classical weight module \( M \), consider the following condition:
\[
G^{i}(b) \in M_{\nu} \oplus \bigoplus_{\xi \in X_{\text{t.int}} \setminus \{\nu\}, |\xi| \geq |\nu|} M_{\xi} \quad \text{for all } \nu \in X_{\text{t.int}}, b \in \overline{\mathbf{L}}_{M,\nu}.
\]

When \( M \) satisfies this condition, for each \( \nu \in X_{\text{t.int}} \) and \( b \in \overline{\mathbf{L}}_{M,\nu} \), set \( G_0^{i}(b) \) to be the image of \( G^{i}(b) \) under the projection onto the weight space \( M_{\nu} \). This defines a \( \mathbb{C} \)-linear map
\[
G_0^{i} : \overline{\mathbf{L}}_{M} \rightarrow M
\]
which preserves the \( X_\nu \)-weight spaces. Note that we have
\[
e_{V,\infty}(G_0^{i}(b)) = b
\]
for all \( b \in \overline{\mathbf{L}}_{M} \).

**Proposition 6.4.3.** Let \( (M, (\cdot, \cdot)_M, M_A, \psi_M, \mathcal{B}_M) \) be a finite-dimensional based \( \mathbf{U} \)-module such that \( \psi_M := \mathcal{Y} \circ \psi_M \) is defined. Then, \( (M, (\cdot, \cdot)_M, M_A, \psi_M, \mathcal{B}_M) \) is a based classical weight standard \( X^1 \)-weight module satisfying condition (32).

**Proof.** Let \( \mathcal{B} \) be a crystal basis of \( M \). Let \( i \in I \), and set
\[
\mathcal{B}_i := \{ b \in \mathcal{B} \mid \tilde{E}_ib = 0 \}.
\]
Then, we have
\[
\mathcal{B} = \{ \tilde{F}_i^k b \mid b \in \mathcal{B}_i, k \in [0, \varphi_i(b)] \}.
\]
Let \( \preceq \) be a total ordering on \( \mathcal{B}_i \) such that for each \( b_1, b_2 \in \mathcal{B}_i \), \( \text{wt}(b_1) < \text{wt}(b_2) \) implies \( b_1 \prec b_2 \). For each \( b \in \mathcal{B}_i \), set
\[
\mathcal{B}_{\geq b} := \{ \tilde{F}_i^k b' \in \mathcal{B} \mid b' \in \mathcal{B}_i, b' \succeq b, k \in [0, \varphi_i(b')] \},
\]
\[
\mathcal{B}_{> b} := \{ \tilde{F}_i^k b' \in \mathcal{B} \mid b' \in \mathcal{B}_i, b' \succ b, k \in [0, \varphi_i(b')] \},
\]
\[
\mathcal{B}_b := \mathcal{B}_{\geq b} \setminus \mathcal{B}_{> b} = \{ \tilde{F}_i^k b \mid k \in [0, \varphi_i(b)] \}.
\]
Also, for each \( b \in B \) and \( \nu \in [-\varphi_i(b), \varphi_i(b)]_{[\varphi_i(b)]} \), set
\[
\bar{b}_\nu := \begin{cases} 
\frac{1}{\sqrt{2}}(P^{-|\nu|-1}b \pm P^{\nu}b) & \text{if } \nu = 0, \\
\frac{1}{\sqrt{2}}(P^{-|\nu|}b \pm P^{\nu}b) & \text{if } \nu > 0.
\end{cases}
\]

By Proposition 2.6.7, for each \( b \in B \), the \( U_1 \)-submodule \( M_{\geq b} \) (resp., \( M_{\leq b} \)) spanned by \( \{G(b') \mid b' \in B_{\geq b}\} \) (resp., \( \{G(b') \mid b' \in B_{\leq b}\} \)) is a based \( U_1 \)-submodule with a crystal basis \( B_{\geq b} \) (resp., \( B_{\leq b} \)). Furthermore, as a \( U_1 \)-submodule, it is a based \( U_1 \)-submodule by Lemma 6.4.4 and Proposition 3.4.9 (1).

Now, we prove by descending induction on \( b \in B \) that \( M_{\geq b} \) satisfies condition (32). Let \( b \in B \), and assume that our claim is true for all \( b' \in B \) such that \( b' \prec b \). Then, the quotient \( U_1 \)-module \( M_b := M_{\geq b}/M_{\leq b} \) is a based \( U_1 \)-module (by Proposition 3.4.9 (2)) isomorphic to the \([l + 1]\)-dimensional irreducible \( U \)-module \( V(l) \), where \( l := \varphi_i(b) \). Then, for each \( \nu \in [-l, l]_{[l]} \), we have
\[
B_{\nu, l}[G^i(b)] = [G^i(b)_{\nu}],
\]
where \([v] \) denotes the image of \( v \in M_{\geq b} \) in \( M_b \), and \( B_{\nu, l} \) is as in the proof of Proposition 5.2.1. Let us write
\[
B_{\nu, l}G^i(b) = G^i(b) + \sum_{b' \in B, \xi \in [-\varphi_i(b), \varphi_i(b)]} \sum_{\xi' \in \xi} c_{\nu, \xi}G^i(b'_\xi),
\]
where \( c_{\nu, \xi} \in A_{\text{inv}} := \{f \in A \mid f(q^{-1}) = f(q)\} \). We show that \( c_{\nu, \xi} = 0 \) for all \( \xi \notin \{\nu, \pm(|\nu| + 2), \pm(|\nu| + 4), \ldots\} \). Assume contrary and take \((b', \xi)\) such that \( \deg(c_{\nu, \xi}) \) is maximal. Let us write
\[
G^i(b'_\xi) = \sum_{\omega \in X_f} m_{\omega}, \quad m_{\omega} \in M_{\omega}.
\]
Then, we have \( m_{\omega} \in L_M, ev_{\infty}(m_{\omega}) = \delta_{\omega, \xi}b'_\xi \). Consider
\[
(B_{\nu, l}G^i(b), m_{\xi})_M.
\]
Since the first factor is a sum of \( B_{\nu, l}\)-eigenvector of eigenvalues in \( \{\nu, \pm(|\nu| + 2), \pm(|\nu| + 4), \ldots\} \), this value equals 0. On the other hand, by the maximality of \( \deg(c_{\nu, \xi}) \), this value belongs to \( c_{\nu, \xi}q^{\deg(c_{\nu, \xi})-1}K_{\infty} \). This contradicts our assumption that \( c_{\nu, \xi} \neq 0 \).

By above, we obtain that
\[
G^i(b)_\nu = B_{\nu, l}G^i(b) - \sum_{b' \in B, \xi \in \nu, \pm(|\nu| + 2), \ldots} \sum_{\xi' \in \xi} c_{\nu, \xi}G^i(b'_\xi).
\]
By the definition of \( B_{\nu, l} \) and our induction hypothesis, the right-hand side is a linear combination of \( B_{\nu, l}\)-eigenvectors of eigenvalues in \( \{\nu, \pm(|\nu| + 2), \ldots\} \). Therefore, we conclude that \( M_{\geq b} \) satisfies condition (32), as desired. This completes the proof.

**Lemma 6.4.4.** Let \( M \) be a finite-dimensional based classical weight standard \( X^\ast \)-weight module satisfying condition (32). Let \( \nu \in X_{\text{int}} \) be such that \( M[\geq \nu] = 0 \). Assume that \( V(\nu) \) is a based \( U \)-module. Then, there exists a based \( U \)-submodule \( N \) of \( M \) isomorphic to \( V(\nu) \).

**Proof.** Let \( b \in \overline{L}_M[\nu] \cap L_{M, \nu} \). Then, by condition (32) and our hypothesis that \( M[\geq \nu] = 0 \), the vector \( G^i(b) \) is a highest weight vector of weight \( \nu \). Set \( N := U^\ast G^i(b) \). Then, there exists an isomorphism \( \phi : V(\nu) \to N \) of \( U \)-modules which sends \( v_\nu \) to \( G^i(b) \). It almost preserves the metrics. In particular, it induces an isomorphism \( \phi : \overline{L}(\nu) \to \overline{L}_N \subseteq \overline{L}_M \) of \( \mathbb{C} \)-vector spaces.
Furthermore, since \( \phi \) almost preserves the metrics, we obtain
\[
\text{ev}_\infty(\phi(G'(b'))) = \phi(\text{ev}_\infty(G'(b'))) = \phi(b').
\]
Then, by Lemma 3.4.6, we conclude that
\[
\phi(G'(b')) = G'(\phi(b')).
\]
This far, we have shown that \( N \) is spanned by \( G'(\phi(\Omega(\nu))) \). Since \( \phi(\Omega(\nu)) = \Omega_N \), by Proposition 3.4.9 (1), we finally see that \( N \) is a based \( \mathbf{U}^* \)-submoule of \( M \).

\[\square\]

7. \( n = 4 \) case

In this section, we consider the \( n = 4 \) case. In this case, we can identify \( X = \mathbb{Z}^3 \), \( X^+ = \mathbb{Z}_{\geq 0}^3 \), \( X_{t, \text{int}} = \mathbb{Z}^2 \), \( X_{t, \text{ext}} = \{(\nu_1, \nu_3) \in X_{t, \text{int}} \mid \nu_1 \geq |\nu_3|\} \), and \( X^t = (\mathbb{Z}/2\mathbb{Z})^3 \).

7.1. Lowering and raising operators. Set
\[
X_\pm := X_{2, \pm} = B_{2, \pm \{l_1; 0\}}, \quad Y_\pm := Y_{2, \pm} = B_{2, \pm \{l_3; 0\}}.
\]

Let \( \nu = (\nu_1, \nu_3) \in X_{t, \text{int}}^+ \), and consider the irreducible \( \mathbf{U}^* \)-module \( V(\nu) \).

**Lemma 7.1.1.** For each \( l, m \in \mathbb{Z}_{\geq 0} \), we have
\[
Y_+^{(m)} Y_-^{(l)} v_\nu = Y_-^{(l)} Y_+^{(m)} v_\nu,
\]
\[
X_- Y_-^{(l)} Y_+^{(m)} v_\nu = \frac{[\nu_1 - \nu_3 - l + 1] \{\nu_1 - l + 1\} \{\nu_3 + l\} Y_-^{(l-1)} Y_+^{(m)} v_\nu}{\{\nu_1 - l - m + 1\} \{\nu_3 + l - m\}},
\]
\[
X_+ Y_-^{(l)} Y_+^{(m)} v_\nu = \frac{[\nu_1 + \nu_3 - m + 1] \{\nu_1 - m + 1\} \{\nu_3 - m\} Y_-^{(l)} Y_+^{(m-1)} v_\nu}{\{\nu_1 - l - m + 1\} \{\nu_3 + l - m\}},
\]
where \( Y_\pm^{(k)} := \frac{1}{k!} Y_\pm^k \).

**Proof.** By equations (16)–(19), we have
\[
[Y_+, Y_-] = 0,
\]
\[
[X_\pm, Y_\mp \{l_1; 0\}] = 0,
\]
\[
[X_+, Y_+] = [l_1 l_3; 0] + (q - q^{-1})^2 Y_- X_- \frac{[l_1 l_3; 0]}{\{l_1; 0\} \{l_3; -1\}},
\]
\[
[X_-, Y_-] = [l_1 l_3^{-1}; 0] + (q - q^{-1})^2 Y_+ X_+ \frac{[l_1 l_3^{-1}; 0]}{\{l_1; 0\} \{l_3; 1\}}.
\]

For each \( m \geq 0 \),
\[
X_+ Y_+^{(m)} = Y_+^{(m)} X_+ \frac{\{l_1; 1\} \{l_3; 0\}}{\{l_1; -m + 1\} \{l_3; \pm m\}},
\]
\[
X_\pm Y_\pm^{(m)} = Y_\pm^{(m-1)} [l_1 l_3^{-1}; -m + 1] + Y_\pm^{(m)} X_\pm + Y_\pm^{(m-1)} Y_\pm X_\mp P_\pm(m),
\]
for some rational function \( P_\pm(m) \in \mathbb{Q}(l_1, l_3) \) in variables \( l_1, l_3 \). Now, it is straightforward to obtain the desired equations by induction on \( l, m \). \( \square \)
From this lemma (with a standard argument), we see that the set
\[ \{ Y_-^{(l)} Y_+^{(m)} v_\nu \mid l \in [0, \nu_1 - \nu_3], \ m \in [0, \nu_1 + \nu_3] \} \]
forms a basis of \( V(\nu) \). Note that \( Y_-^{(l)} Y_+^{(m)} v_\nu \in V(\nu)(\nu_1 - l - m, \nu_3 + l - m) \). Then, for each \( l \in [0, \nu_1 - \nu_3], \ m \in [0, \nu_1 + \nu_3] \), we can choose \( c_{l,m,\nu} \in K^V \) in a way such that
\[ \text{lt}(c_{l,m,\nu} Y_-^{(l)} Y_+^{(m)} v_\nu) = 1. \]
For each \( l, m \in \mathbb{Z} \), set
\[ \hat{Y}_-^{l} \hat{Y}_+^{m} v_\nu := \begin{cases} c_{l,m,\nu} Y_-^{(l)} Y_+^{(m)} v_\nu & \text{if } l \in [0, \nu_1 - \nu_3] \text{ and } m \in [0, \nu_1 + \nu_3], \\ 0 & \text{otherwise.} \end{cases} \]
Then, \( \{ \hat{Y}_-^{l} \hat{Y}_+^{m} v_\nu \mid l \in [0, \nu_1 - \nu_3], \ m \in [0, \nu_1 + \nu_3] \} \) forms an almost orthonormal basis of \( V(\nu) \).

Using Lemma 7.1.1, we see by induction on \( l \) and \( m \) that
\[ (Y_-^{(l)} Y_+^{(m)} v_\nu, Y_-^{(l)} Y_+^{(m)} v_\nu)_\nu = \begin{bmatrix} \nu_1 - \nu_3 \\ l \\ m \end{bmatrix} \begin{bmatrix} \nu_3 - m \\ \nu_3 + l - m \end{bmatrix} \prod_{i=1}^{l} \frac{\{\nu_1 - i + 1\} \{\nu_3 + i\}}{\{\nu_1 - i - m + 1\} \{\nu_1 - i - m\}} \prod_{j=1}^{m} \frac{\{\nu_3 - j + 1\}}{\{\nu_1 - j\}}. \]

**Lemma 7.1.2.** For each \( k \geq 0 \), we have
\[ Y^{(k)} = \sum_{l + m = k} Y_-^{(l)} Y_+^{(m)} \frac{\{l; l - m\} \prod_{j=0}^{k} \{l; -j + 1\}}{\prod_{j=0}^{k} \{l; -j\}}. \]

**Proof.** By equation (15), we have
\[ Y = (Y_- + Y_+) \{l; 0\} \]
Then, the assertion follows by induction on \( k \). \( \square \)

Set
\[ Y^{(l, m)} v_\nu := \begin{bmatrix} \nu_3 + l - m \\ \nu_1 - l - m \end{bmatrix} \prod_{j=0}^{l+m} \frac{\{\nu_1 - j\}}{\{\nu_3 + l - j\}} Y_-^{(l)} Y_+^{(m)} v_\nu. \]
By Lemma 7.1.2, we see that
\[ Y^{(k)} v_\nu = \sum_{l + m = k} Y^{(l, m)} v_\nu. \]
Also, we have
\[ (Y_-^{(l, m)} v_\nu, Y_-^{(l, m)} v_\nu)_\nu = \begin{bmatrix} \nu_1 - \nu_3 \\ l \\ m \end{bmatrix} \begin{bmatrix} \nu_1 + \nu_3 \\ \nu_3 + l - m \end{bmatrix} \prod_{i=1}^{l} \frac{\{\nu_1 + i\} \{\nu_3 + l - m\}}{\{\nu_3 + l - i + 1\} \{\nu_3 - j\}}. \]
This implies that for each \( l \in [0, \nu_1 - \nu_3] \) and \( m \in [0, \nu_1 + \nu_3] \), we have
\[ \text{deg}(Y_-^{(l, m)} v_\nu) \]
\[ = \frac{1}{2} \left( (\nu_1 - \nu_3 - l) l + (\nu_1 + \nu_3 - m) m + \nu_1 - |\nu_3| - |\nu_1 - l - m| + |\nu_3 + l - m| \right. \]
\[ + \sum_{i=1}^{l} (|\nu_1 - i + 1| - |\nu_3 + l - i + 1|) + \sum_{j=1}^{m} (|\nu_1 + j + 1| - |\nu_3 - j|). \]

(34)
**Definition 7.1.3.** The lowering operator $\tilde{Y}_±$ and the raising operator $\tilde{X}_±$ are $K$-linear endomorphisms on $V(\nu)$ defined by

\[
\tilde{Y}_+(\tilde{Y}_+^m v_\nu) := \begin{cases} 
\tilde{Y}_+^{l+1} v_\nu & \text{if } l \in [0, \nu_1 - \nu_3] \text{ and } m \in [0, \nu_1 + \nu_3], \\
0 & \text{otherwise},
\end{cases}
\]

\[
\tilde{Y}_-(\tilde{Y}_+^m v_\nu) := \begin{cases} 
\tilde{Y}_+^{-l+1} v_\nu & \text{if } l \in [0, \nu_1 - \nu_3] \text{ and } m \in [0, \nu_1 + \nu_3], \\
0 & \text{otherwise},
\end{cases}
\]

\[
\tilde{X}_+(\tilde{Y}_+^m v_\nu) := \begin{cases} 
\tilde{Y}_+^{-l} v_\nu & \text{if } l \in [0, \nu_1 - \nu_3] \text{ and } m \in [0, \nu_1 + \nu_3], \\
0 & \text{otherwise},
\end{cases}
\]

\[
\tilde{X}_-(\tilde{Y}_+^m v_\nu) := \begin{cases} 
\tilde{Y}_+^{-l-1} v_\nu & \text{if } l \in [0, \nu_1 - \nu_3] \text{ and } m \in [0, \nu_1 + \nu_3], \\
0 & \text{otherwise}.
\end{cases}
\]

Clearly, the operators $\tilde{Y}_±$ and $\tilde{X}_±$ preserve $L(\nu)$. Hence, they induce linear operators $\tilde{Y}_±$ and $\tilde{X}_±$ on $\tilde{L}(\nu)$.

Since each finite-dimensional classical weight module is completely reducible, we can extend the definitions of $\tilde{Y}_±, \tilde{X}_±$ on the irreducible modules to each finite-dimensional classical weight modules. The following can be proved in a similar way to the $n = 3$ case.

**Proposition 7.1.4.** Let $M$ be a finite-dimensional classical weight $U'$-module. Let $\xi \in X_{t,\text{int}}$. Then, we have $Y_± M_\xi = \tilde{Y}_± M_\xi$.

**Proposition 7.1.5.** Let $M$ be a finite-dimensional classical weight $U'$-module equipped with a contragredient Hermitian inner product. Then, for each $u, v \in L_M$, we have

\[
(\tilde{Y}_+ u, v) - (u, \tilde{X}_+ v) \in q^{-1} K_\infty.
\]

### 7.2. Based module structures of irreducible modules

For each $l, m \in \mathbb{Z}$, set

\[
\tilde{V}^{l,m} := \tilde{V}_+^l \tilde{V}_+^m.
\]

Let $\nu \in X^+_{t,\text{int}}$, and consider the irreducible module $V(\nu)$. Note that for each $l \in [0, \nu_1 - \nu_3]$ and $m \in [0, \nu_1 + \nu_3]$, we have

\[
ev_\infty \left( \frac{1}{\text{ht}(\tilde{V}^{l,m} v_\nu)} \tilde{V}^{l,m} v_\nu \right) = \tilde{V}^{l,m} b_\nu.
\]

**Lemma 7.2.1.** Let $l \in [0, \nu_1 - \nu_3]$, $m \in [0, \nu_1 + \nu_3]$ be such that $l + m \leq \nu_1$. Then, we have

\[
\deg(\tilde{V}^{l,m} v_\nu) = \deg(Y^{(\nu_1 - l, \nu_1 + m)} v_\nu).
\]

**Proof.** By equation (34), we have

\[
\deg(Y^{(\nu_1 - l, \nu_1 + m)} v_\nu)
\]

\[
= \frac{1}{2}((\nu_1 - \nu_3 - l) + (\nu_1 + \nu_3 - m)) + \nu_3 + l - m
\]

\[
+ \sum_{i=1}^{\nu_1 - \nu_3} (|\nu_1 - i + 1| - |\nu_1 - i - 1|) + \sum_{j=1}^{\nu_1 + \nu_3} (|\nu_1 - j + 1| - |\nu_3 - j|).
\]
Here, we have
\[
\sum_{i=1}^{v_1-v_3-l} (|\nu_1 - i + 1| - |\nu_1 - l - i + 1|)
\]
\[
= \begin{cases}
\sum_{s=1}^{v_1-v_3-l} |\nu_1 - s + 1| - \sum_{t=1}^{v_1-v_3-l} |\nu_1 - t + 1| & \text{if } \nu_1 - l \geq \nu_3 + l, \\
(\sum_{s=1}^l |\nu_3 + l - s + 1| - \sum_{t=1}^l |\nu_3 + l - t + 1|) & \text{if } \nu_1 - l \leq \nu_3 + l
\end{cases}
\]
\[
= \sum_{s=1}^l |\nu_1 - s + 1| - \sum_{s=1}^l |\nu_3 + l - s + 1|
\]
\[
= \sum_{i=1}^l (|\nu_1 - i + 1| - |\nu_3 + l - i + 1|).
\]

Similarly, we obtain
\[
\sum_{j=1}^{v_1+v_3-m} (|\nu_1 - j + 1| - |\nu_3 - j|) = \sum_{j=1}^m (|\nu_1 - j + 1| - |\nu_3 - j|).
\]

Thus, the assertion follows. \qed

Set \(d^l_m := \deg(Y^{(l,m)}_{V_\nu})\). By direct calculation, we obtain the following.

**Lemma 7.2.2.** Let \(l \in [0, \nu_1 - \nu_3]\), \(m \in [0, \nu_1 + \nu_3]\) be such that \(l + m \leq \nu_1\).

1. If \(\nu_3 + l - m \geq 0\), then we have
   \[
   d^l_m - d^{l+1,m-1}_m = \begin{cases}
   2(\nu_3 + l - m) + 1 > 0 & \text{if } \nu_3 - m \leq 0, \\
   (\nu_3 + l - m) + l + 1 > 0 & \text{if } \nu_3 - m \geq 0.
   \end{cases}
   \]

2. If \(\nu_3 + l - m \leq 0\), then we have
   \[
   d^l_m - d^{l-1,m+1}_m = \begin{cases}
   -2(\nu_3 + l - m) + 1 > 0 & \text{if } \nu_3 + l \geq 0, \\
   -(\nu_3 + l - m) + m + 1 > 0 & \text{if } \nu_3 + l \leq 0.
   \end{cases}
   \]

3. If \(\nu_3 + l - m = -1\), then we have
   \[
   d^l_m = d^{l+1,m-1}_m.
   \]

**Proposition 7.2.3.** Let \(k \in [0, \nu_1]\).

1. Suppose that \(\nu_3 \geq 0\) and set
   \[
   (l_0, m_0) := \begin{cases}
   (0, k) & \text{if } 0 \leq k \leq \nu_3, \\
   \left(\frac{k-\nu_3}{2}, \frac{k+\nu_3}{2}\right) & \text{if } \nu_3 \leq k \leq 2\nu_1 - \nu_3 \text{ and } k - \nu_3 \in \mathbb{Z}_{\text{ev}}, \\
   \left(\frac{k-\nu_3+1}{2}, \frac{k+\nu_3-1}{2}\right) & \text{if } \nu_3 \leq k \leq 2\nu_1 - \nu_3 \text{ and } k - \nu_3 \in \mathbb{Z}_{\text{odd}}, \\
   (\nu_1 - \nu_3, k - \nu_1 + \nu_3) & \text{if } 2\nu_1 - \nu_3 \leq k \leq 2\nu_1.
   \end{cases}
   \]
   Then, we have
   \[
   \widetilde{Y}_k b_{V_\nu} = \begin{cases}
   \frac{1}{x_2} \frac{1}{Y^{(l_0,m_0)}_{V_\nu}} + \frac{1}{x_2} \frac{1}{Y^{(l_0-1,m_0+1)}_{V_\nu}} b_{V_\nu} & \text{if } \nu_3 \leq k \leq 2\nu_1 - \nu_3 \text{ and } k - \nu_3 \in \mathbb{Z}_{\text{odd}}, \\
   \text{otherwise.}
   \end{cases}
   \]
(2) Suppose that \( \nu_3 \leq 0 \) and set

\[
(l_0, m_0) := \begin{cases} (k, 0) & \text{if } 0 \leq k \leq -\nu_3, \\ \left( \frac{k - \nu_3}{2}, \frac{k + \nu_3}{2} \right) & \text{if } -\nu_3 \leq k \leq 2\nu_1 + \nu_3 \text{ and } k + \nu_3 \in \mathbb{Z}_{ev}, \\ \left( \frac{k - \nu_3 + 1}{2}, \frac{k + \nu_3}{2} \right) & \text{if } -\nu_3 \leq k \leq 2\nu_1 + \nu_3 \text{ and } k + \nu_3 \in \mathbb{Z}_{odd}, \\ (k - \nu_1 - \nu_3, \nu_1 + \nu_3) & \text{if } 2\nu_1 + \nu_3 \leq k \leq 2\nu_1. 
\end{cases}
\]

Then, we have

\[
\tilde{Y}^k b_\nu = \begin{cases} \frac{1}{\sqrt{2}}(\tilde{Y}^{l_0,m_0} + \tilde{Y}^{l_0-1,m_0+1})b_\nu & \text{if } -\nu_3 \leq k \leq 2\nu_1 + \nu_3 \text{ and } k + \nu_3 \in \mathbb{Z}_{odd}, \\ \tilde{Y}^{l_0, m_0} b_\nu & \text{otherwise}. 
\end{cases}
\]

**Proof.** The assertion follows from the definition of \( \tilde{Y}^{l,m} \) and Lemmas 7.2.1 and 7.2.2. \( \square \)

**Lemma 7.2.4.** Let \( \nu \in X_{\text{int}}^+, M \) a based classical weight standard \( X^1 \)-weight module satisfying condition (32). Suppose that \( M[\nu] = 0 \). Then, \( M \) contains a \( U^\ast \)-submodule isomorphic to \( V(\nu) \) which is also a based submodule of \( M \).

**Proof.** We prove the statement under the assumption that \( \nu_3 \geq 0 \). The \( \nu_3 \leq 0 \) case can be proved similarly.

Let \( b \in \mathcal{L}_M \) and set \( v := G^i(b) \). By our hypothesis, \( v \) is a highest weight vector of weight \( \nu \). Hence, in order to prove the assertion, it suffices to show that \( G^i(\tilde{Y}^{l,m}b) \in U^\ast v \) for all \( l \in [0, \nu_1 - \nu_3] \) and \( m \in [0, \nu_1 + \nu_3] \). Below, we concentrate on the case when \( l + m \leq \nu_1 \). The case when \( l + m \geq \nu_1 \) can be proved similarly by exchanging the roles of \( v \) and \( Y^{(2\nu)}(= Y^{(\nu_1 - \nu_3, \nu_1 + \nu_3)}v) \).

Set

\[
B := \{ \tilde{Y}^{l,m} v \mid l \in [0, \nu_1 - \nu_3], m \in [0, \nu_1 + \nu_3] \}.
\]

Then, it forms a basis of \( U^\ast v \) consisting of vectors in \( \mathcal{L}_M \). Let \( B \subseteq \mathcal{L}_M \) be a basis of \( M \) which extends \( B \).

We show that \( G^i(\tilde{Y}^{l,m}b) \in U^\ast v \) inductively. Assume that we have \( G^i(\tilde{Y}^{l',m'}b) \in U^\ast v \) for the following cases:

- \(|\nu_1 - (l' + m')| > \nu_1 - (l + m)\), or
- \(l' + m' = l + m\) and \(|\nu_3 + l' - m'| < |\nu_3 + l - m|\).

For such \( l', m' \), we can write

\[
G^i(\tilde{Y}^{l',m'}b) = \sum_{l'' \in [0, \nu_1 - \nu_3]} \sum_{m'' \in [0, \nu_1 + \nu_3]} p_{l', m'}^{l'', m''} \tilde{Y}^{l'', m''} v
\]

for some \( p_{l', m'}^{l'', m''} \in K_\infty \). By condition (32), we see that \( p_{l', m'}^{l'', m''} = 0 \) if

- \(|\nu_1 - (l'' + m'')| < |\nu_1 - (l' + m')|\), or
- \(|\nu_3 + l'' - m''| < |\nu_3 + l' - m'|\).

Assume further that for each \( l'', m'' \) such that \( l'' + m'' = l' + m' \) and \(|\nu_3 + l'' - m''| \geq |\nu_3 + l - m|\), we have

\[
\deg(p_{l', m'}^{l'', m''}) \leq d_{l'}^{l'', m''} - d_{l'}^{l', m'} + \sum_{j=1}^{k_{l'', m''}} (|\nu_3 + l'' - m''| - j),
\]

where \( k_{l'', m''} \) is the smallest integer such that

\[
d_{l'}^{l'', m''} - d_{l'}^{l', m'} + \sum_{j=1}^{k_{l'', m''}} (|\nu_3 + l'' - m''| - j) \geq 0.
\]
where \((l_0', m_0')\) is as in Lemma 7.2.3 with \(l\) being replaced with \(l' + m'\), and

\[
\tilde{k}_{l', m'} := \begin{cases} \\
|l' - l_0'| & \text{if } l' + m' \leq \nu_3 \text{ or } l' + m' \geq 2 \nu_1 - \nu_3, \\
|\nu_3 + l' - m'| & \text{if } \nu_3 \leq l' + m' \leq 2 \nu_1 - \nu_3 \text{ and } |\nu_3 + m' - n'| \leq l' + m' - \nu_3, \\
l' & \text{if } \nu_3 \leq l' + m' \leq 2 \nu_1 - \nu_3 \text{ and } |\nu_3 + l' - m'| \geq l' + m' - \nu_3.
\end{cases}
\]

Under the induction hypothesis above, we show that \(G^r(\tilde{Y}^{l',mb}) \in \mathbb{U}^v\). Before doing so, let us prepare some notation. Set \(k := l + m\). Then, we can write as

\[
\tilde{Y}^k v = \sum_{l'+m'=k} r_{l',m'} \tilde{Y}^{l',m'} v
\]

for some \(r_{l',m'} \in \mathbb{K}_\infty\) such that

\[
\text{lt}(r_{l',m'}) = q^{d_{l',m'} - d_{l_0',m_0'}}.
\]

Also, for each \(j \geq 0\), we have

\[
B_3^{(j)}\tilde{Y}^k v = \sum_{l'+m'=k} r_{l',m'} j \tilde{Y}^{l',m'} v,
\]

where \(r_{l',m',j} := r_{l',m'} (\nu_3 + l' - m'|j) / |j|!\). Hence, we have

\[
\text{lt}(r_{l',m',j}) = q^{d_{l',m'} - d_{l_0',m_0'} + \sum_{i=1}^j (\nu_3 + l' - m'|i)}.
\]

Note that we have

\[
\deg(r_{l',m',0}) < \deg(r_{l',m',1}) < \ldots < \deg(r_{l',m',|\nu_3 + l' - m'| - 1}).
\]

Let us prove that \(G^r(\tilde{Y}^{l,mb}) \in \mathbb{U}^v\). First, suppose that \(k \leq \nu_3\). In this case, we have \((l_0, m_0) = (0, k)\), and \(\tilde{k}_{l,m} = l\).

Using Lemma 7.2.2, we see that for each \(l', m'\) such that \(l' + m' = k\) and for \(j \geq 0\), we have

\[
\deg(r_{l',m',j}) = \frac{1}{2} l' (2(\nu_3 - m') + l' - 1) + \frac{1}{2} j (2(\nu_3 - m') + 2l' - j - 1)
\]

\[
= \frac{1}{2} (l' - j) (2(\nu_3 - m') + l' - j - 1).
\]

This implies that

\[
\deg(r_{l',m',j}) = 0, \quad \text{deg}(r_{l',m',j}) < 0 \quad \text{if } j < l'.
\]

By Theorem 6.3.4, we have

\[
G^r(\tilde{Y}^{0,k}b) = G^r(\tilde{Y}^k b) \in \mathbb{U}^v.
\]

By the proof of Proposition 6.3.3, we see that

\[
G^r(\tilde{Y}^k b) \in \tilde{Y}^k v + \bigoplus_{j < k \text{ or } j > 2 \nu_1 - k} q^{-1} \mathbb{K}_\infty \tilde{Y}^j v.
\]

This shows that \(G^r(\tilde{Y}^k b)\) is a sum of \(B_3\)-eigenvectors of eigenvalues \([a], a \in \mathbb{Z}_{p(\nu_3-k)}\). Since \(M\) is a standard \(X^\nu\)-weight module, we see that

\[
1_\zeta G^r(\tilde{Y}^k b) = 0
\]
for all $\zeta \in X$ such that $\zeta_3 \neq \nu_3 - k$. Therefore, we obtain

$$\sum_{\zeta \in X, \zeta_3 = \nu_3 - k} 1_\zeta G^r(\tilde{Y}^kB) = \sum_{\zeta \in X} 1_\zeta G^r(\tilde{Y}^kB) = G^r(\tilde{Y}^kB).$$

Hence, we have

$$B_{3,p(\nu_3-k)}(\tilde{Y}^kB) = \sum_{\zeta_3 = \nu_3 - k} B_{3,p}^{(l)}(\tilde{Y}^kB) \notin \mathcal{U}_A \nu.$$

By the definition of divided powers, we can write as

$$B_{3,p(\nu_3-k)}^{(l)}(\tilde{Y}^kB) = \sum_{j=0}^{l} c_j B_3^{(j)}(\tilde{Y}^kB)$$

$$\in \sum_{j=0}^{l} c_j B_3^{(j)} \tilde{Y}^k v + \sum_{l'+m' < k \text{ or } l'+m' > 2\nu_3 - k} K \tilde{Y}^{l',m'} v$$

for some $c_j \in \delta_{j,l} + q^{-1}K_\infty$. For each $j \in [0,l]$, set

$$u_j := c_j B_3^{(j)} \tilde{Y}^k v = \sum_{l'+m' = k} c_j r^{l',m',j} \tilde{Y}^{l',m'} v,$$

and

$$l' := \min\{l'' \neq l \mid \deg(c_j r^{l'',m'',j}) \geq 0\}.$$

By (36), we must have $l' < j$. Set $m' := k - l'$. Then, there exists $a_{l',j} \in A_{\text{inv}}$ such that $u_j - a_{l',j} G^r(\tilde{Y}^{l',m'} b)$, expanded by the basis $B$, contains $\tilde{Y}^{l',m'} v$ with coefficient in $q^{-1}K_\infty$. Note that we have

$$\deg(a_{l',j}) = \deg(c_j r^{l'',m'',j}).$$

Then, for each $l'' > l'$, setting $m'' := k - l''$, we compute as

$$\deg(c_j r^{l'',m'',j}) - \deg(a_{l',j} r^{l'',m''})$$

$$= \deg(c_j) + \deg(r^{l'',m'',j}) - (\deg(c_j) + \deg(r^{l',m',j}) + \deg(p_{l',m'}))$$

$$= \deg(r^{l'',m'',j}) - \deg(r^{l',m',j}) - \deg(p_{l',m'})$$

$$\geq \deg(r^{l'',m'',j}) - \deg(r^{l',m',j})$$

$$= d_{l''}^{m''} - d_{l'}^{m',m''} + j(\nu_3 + l'' - m'' - 1) - \frac{1}{2} j(j + 1)$$

$$- (d_{l'}^{m',m''} - d_{l'}^{m',m''} + j(\nu_3 + l' - m' - 1) - \frac{1}{2} j(j + 1))$$

$$- (d_{l''}^{m'',m''} - d_{l''}^{m'',m''} + l'(\nu_3 + l'' - m'' - 1) - \frac{1}{2} l'(l' + 1))$$

$$= (j - l')(\nu_3 + l'' - m'' - 1) - \deg(r^{l',m',j}) - (j - l')(\nu_3 + l' - m' - 1)$$

$$= (j - l')((l'' - m'') - (l' - m')) > 0.$$

Therefore, both $u'$ and $u' - a_{l',j} G^r(\tilde{Y}^{l',m'} b)$ contain $\tilde{Y}^{l'',m''} v$, $l'' > l'$, $l'' + m'' = k$ with coefficient in $q^{\deg(c_j r^{l'',m'',j})}K_\infty$. 


Replacing \( u' \) with \( u' - a_{l',j} G^i(\tilde{Y}^{l',m'} b) \), and repeating the procedure above, we finally obtain that

\[
u_j := c_j B^{(j)}_3(\bar{Y}^k v - \sum_{l'=0}^{j-1} a_{l',j} G^i(\tilde{Y}^{l',m'} b))
\]

contains \( \tilde{Y}^{l''} m'' v, l'' < j, l'' + m'' = k \) with coefficient in \( q^{-1} K_\infty \), and \( \tilde{Y}^{l''} m'' v, l'' \geq j, l'' + m'' = k \) with coefficient in \( q^{\deg(c_{r''} m'' j)} K_\infty \), where \( a_{l',j} \in A_{\text{inv}} \).

By above, we see that

\[
\begin{align*}
u := B^{(l)}_{3,p(l_3+l-1-m)} G^i(\tilde{Y}^{k} b) - \sum_{j=0}^{l} \sum_{l'=0}^{j-1} a_{l',j} G^i(\tilde{Y}^{l',k-l'} b) \in \sum_{j=0}^{l} u_j + \sum_{l'+m'<k \text{ or } l'+m'>2 \nu_1-k} K \tilde{Y}^{l''} m'' v
\end{align*}
\]

contains \( \tilde{Y}^{l''} m'' v, l'' < l, l'' + m'' = k \) with coefficient in \( q^{-1} K_\infty \), and \( \tilde{Y}^{l''} m'' v, l'' \geq l, l'' + m'' = k \) with the degree of coefficient being

\[
\max_{0 \leq j \leq l} \{\deg(c_{r''} m'' j)\}.
\]

Since \( m'' \leq k \leq \nu_3 \), we have \( \nu_3 + l'' - m'' \geq l'' \geq l \). Then, by equation (35), we see that

\[
\max_{0 \leq j \leq l} \{\deg(c_{r''} m'' j)\} = \deg(r'' m'' l) \leq 0;
\]

the equality holds if and only if \( l'' = l \) by equation (36).

By the construction above, \( u - G^i(\tilde{Y}^{l,m} b) \) is contained in \( M_A \), and is invariant under \( \psi_M \). Furthermore, the vector \( u \), expanded by \( \bar{B} \), contains \( w \in \bar{B} \) with coefficient in \( q^{-1} K_\infty \) unless \( w = \tilde{Y}^{l''} m'' v \) with \( l' + m' < k \) or \( l' + m' > 2 \nu_1 - k \). From these observations and condition (32), we can find \( a_{l'',m'} \in A_{\text{inv}} \) such that

\[
u - G^i(\tilde{Y}^{l'',m''} b) - \sum_{l'+m'<k \text{ or } l'+m'>2 \nu_1-k} a_{l',m'} G^i(\tilde{Y}^{l',m'} b) \in q^{-1} L_M \cap M_A.
\]

Then, by Lemma 3.4.6, we see that

\[
u - G^i(\tilde{Y}^{l,m} b) - \sum_{l'+m'<k \text{ or } l'+m'>2 \nu_1-k} a_{l',m'} G^i(\tilde{Y}^{l',m'} b) = 0.
\]

This shows that

\[
G^i(\tilde{Y}^{l,m} b) \in U^v,
\]

as desired. By the construction, the degree of \( \tilde{Y}^{l',m'} v, l' > l, l' + m' = k \) in \( G^i(\tilde{Y}^{l,m} b) \) is at most \( \deg(r'' m'' l) \). Hence, we can proceed our induction.

Next, suppose that \( \nu_3 \leq k \leq \nu_1 \) and \( k - \nu_3 \in \mathbb{Z}_{\text{ev}} \) (the case when \( k - \nu_3 \in \mathbb{Z}_{\text{odd}} \) is similar). By the same argument as above, we obtain

\[
G^i(\tilde{Y}^{l,m} b) = G^i(\tilde{Y}^{k} b) \in U^v.
\]

Computing \( B^{(k,m)}_{3,\mathbb{Z}_{\text{ev}}} G^i(\tilde{Y}^{k} b) \), one can construct \( G^i((\tilde{Y}^{l,m} b) + \tilde{Y}^{l,-m} b) \) if \( |\nu_3 + l - m| \leq k - \nu_3 \), and \( G^i(\tilde{Y}^{l,m} b) \) if \( \nu_3 + l - m > k - \nu_3 \) in the same way as above, where \( (l_\pm, m_\pm) \) is such that \( l_\pm + m_\pm = k \) and

\[
\nu_3 + l_- - m_+ = -(\nu_3 + l_+ - m_-) = |\nu_3 + l - m|.
\]

This completes the proof. \(\square\)
Theorem 7.2.5. Let \( \nu \in X^+_{t,\text{int}} \). Then, \( V(\nu) \) is a based \( U^t \)-module.

Proof. Let \( M \) be a finite-dimensional based classical weight standard \( X^t \)-weight module satisfying condition (32) such that \( M[\nu] \neq 0 \). Such \( M \) surely exists by Propositions 4.3.3 and 4.4.7. Let us write an irreducible decomposition of \( M \)

\[
M = \bigoplus_{k=1}^r M_k, \quad M_k \simeq V(\nu_k), \quad \nu_k \in X^+_{t,\text{int}}.
\]

Without loss of generality, we may assume that \( k \leq l \) implies \( \nu_k \leq \nu_l \). By Lemma 7.2.4, we see that \( M_\nu \) is a base \( U^t \)-submodule of \( M \), and hence, \( V(\nu) \) is a based \( U^t \)-module. Then, \( M/M_\nu \) is a finite-dimensional based classical weight standard \( X^t \)-weight module satisfying condition (32) by Propositions 3.3.7 (2) and 3.4.9 (2). Replacing \( M \) with \( M/M_\nu \), we see that \( V(\nu_{r-1}) \) is a based \( U^t \)-module. Proceeding in this way, we conclude that \( V(\nu_k) \) is a based \( U^t \)-module for all \( k \in [1, r] \). This completes the proof. \( \square \)

Proposition 7.2.6. Let \( M \) be a finite-dimensional classical weight \( U^t \)-module equipped with a contragredient Hermitian inner product. Set

\[
L_1 := \{ b \in \mathcal{Z}_M \mid \tilde{X}b = \tilde{X}_b = 0 \},
\]

\[
L_2 := \mathbb{C}\{ b \in \mathcal{Z}_M \mid b \text{ is } B_1\text{-homogeneous, } \tilde{B}_2b = 0, \text{ and } \deg_3((\tilde{B}_2\tilde{B}_1)^{\deg_3(b)}b) = 0 \}.
\]

Then, the linear map

\[
L_2 \to L_1; \quad b \mapsto (1 + \tilde{B}_1)b.
\]

is an isomorphism of \( \mathbb{C} \)-vector spaces with inverse

\[
L_1 \to L_2; \quad b \mapsto \sum_{\xi \in X^t \atop \xi_2 = \sigma} 1_{\xi}b.
\]

Proof. As the proof of Proposition 6.3.8, it suffices to consider the case when \( M = V(\nu) \) for some \( \nu \in X^+_{t,\text{int}} \). Clearly, we have

\[
L_1 = \mathbb{C}b_\nu.
\]

Let \( b \in \mathcal{Z}(\nu) \) be \( \tilde{B}_1 \)-homogeneous such that \( \tilde{B}_2b = 0 \) and \( \deg_3((\tilde{B}_2\tilde{B}_1)^{\deg_3(b)}b) = 0 \). Set \( l_1 := \deg_1(b) \) and \( l_3 := \deg_3(b) \). By Proposition 6.3.8, there exists \( b' \in \mathcal{Z}(\nu) \) such that \( b' \) is a sum of \( X_t \)-weight vectors of weights \( \xi \in X_{t,\text{int}} \) with \( \xi_1 = l_1 \), and such that \( \tilde{X}b' = 0 \) and \( b = (1 + (-1)^{l_1}\tilde{Y}^{2l_1})b' \). Then, from Corollary 6.3.5, we see that

\[
(\tilde{B}_2\tilde{B}_1)^{l_3}b \in \mathbb{C}(\tilde{Y}^{l_3} + (-1)^{l_1+l_3}\tilde{Y}^{2l_1-l_3})b'.
\]

On the other hand, by Proposition 7.2.3, we see that for each \( b'' \in \mathcal{Z}(\nu) \), we have \( \deg_3(b'') = 0 \) only if \( b'' \in \bigoplus_{n=0}^{2\nu_1} \mathbb{C}\tilde{Y}^n b_\nu \). Therefore, we obtain \( l_1 = \nu_1 \), \( b' \in \mathbb{C}b_\nu \), and hence,

\[
L_2 \subseteq \mathbb{C}(1 + (-1)^{\nu_1}\tilde{Y}^{2\nu_1})b_\nu.
\]

Conversely, by Proposition 7.2.3 again, we have

\[
\deg_3((\tilde{B}_2\tilde{B}_1)^{\nu_1}(1 + (-1)^{\nu_1}\tilde{Y}^{2\nu_1})b_\nu) = \deg_3((\tilde{Y}^{\nu_1} + (-1)^{\nu_1+\nu_3}\tilde{Y}^{2\nu_1-\nu_3})b_\nu) = 0.
\]

Thus, we see that

\[
L_2 = \mathbb{C}(1 + (-1)^{\nu_1}\tilde{Y}^{2\nu_1})b_\nu.
\]

Now, by the same calculation as in the proof of Proposition 6.3.8, the assertion follows. \( \square \)
8. General case

In this section, we consider the general $n \geq 3$ case, and set

$$m := \begin{cases} 
\frac{n}{2} & \text{if } n \in \mathbb{Z}_{ev}, \\
\frac{n-1}{2} & \text{if } n \in \mathbb{Z}_{odd}.
\end{cases}$$

Let $j \in \tilde{I}$, and set

$$J = \begin{cases} 
\{2i - 1, 2i, 2i + 1\} & \text{if } j = (2i, \pm), \\
\{2m - 1, 2m\} & \text{if } j = 2m.
\end{cases}$$

Then, $U'_j$ is canonically isomorphic to the quantum group of type $\text{AI}$ with $n = 2$ or 3.

Hence, we can define linear operators $\tilde{X}_j, \tilde{Y}_j$ to correspond $\tilde{X}, \tilde{Y}$ if $j = 2m$ or $\tilde{X}_\pm, \tilde{Y}_\pm$ if $j = (2i, \pm)$.

8.1. Based module structure of irreducible modules. In this subsection, we prove that $V(\nu), \nu \in X^+_t$ is a based $U^t$-module.

**Lemma 8.1.1.** Let $M$ be a finite-dimensional based classical weight standard $X^*$-weight module satisfying condition (32). Let $N \subseteq M$ be a $U^t$-submodule, $\nu \in X^+_t$, and $j \in \tilde{I}$. Suppose that for each $\xi \geq \nu$ and $b \in L_{N,\xi}$ we have $G^t(b) \in N$. Then, for each $b \in L_{N,\nu}$, we have $G^t(\tilde{Y}_j b) \in N$.

**Proof.** Define $J \subseteq I$ as above. Let us view $M$ and $N$ as $U'_j$-modules. Then, we have an orthogonal decomposition

$$L_N = \bigoplus_{\xi \in X^+_t(J)} \bigcap_{\nu \geq \xi} L_N[\xi].$$

Let $\{b_1, \ldots, b_r\}$ be an orthonormal basis of $\bigoplus_{\xi \geq \nu}(L_N[\xi] \cap L_{N,\xi})$ consisting of $X_t$-weight vectors. Let $\xi_k \in X^+_t(J)$ be such that $b_k \in L_{N,\xi_k}$. We may assume that $k \leq l$ implies $\xi_k \leq \xi_l$. Note that we have

$$L_{N,\nu} \subseteq \text{Span}_{\mathbb{C}}\{\tilde{Y}_{j_1} \cdots \tilde{Y}_{j_s} b_k \mid s \geq 0, k \in [1, r], j_1, \ldots, j_s \in \tilde{I}(J)\}.$$ 

Hence, to prove the assertion, it suffices to show that $G^t(\tilde{Y}_{j_1} \cdots \tilde{Y}_{j_s} b_k) \in N$ for all $s, k, j_1, \ldots, j_s$.

By weight consideration, $G^t(b_r) \in N$ is a highest weight vector of weight $\xi_r$. Then, by Theorems 6.3.4 and 7.2.5, $U'_j G^t(b_r)$ is a based $U'_j$-submodule of $M$, and $G^t(\tilde{Y}_{j_1} \cdots \tilde{Y}_{j_s} b_r) \in U'_j G^t(b_r) \subseteq N$. Replacing $M$ with $M/U'_j G^t(b_r)$ and repeating the same argument as above, we see that $G^t(\tilde{Y}_{j_1} \cdots \tilde{Y}_{j_s} b_{r-1}) \in N$ for all $j_1, \ldots, j_s \in J$. Proceeding in this way, we conclude that $G^t(\tilde{Y}_{j_1} \cdots \tilde{Y}_{j_s} b_k) \in N$ for all $s \geq 0, k \in [1, r], j_1, \ldots, j_s \in \tilde{I}(J)$, as desired. \hfill $\square$

**Theorem 8.1.2.** Let $\nu \in X^+_t$. Then, $V(\nu)$ is a based $U^t$-module. Moreover, we have the following:

1. $\mathcal{L}(\nu) = \text{Span}_{\mathbb{K}}\{\tilde{Y}_{j_1} \cdots \tilde{Y}_{j_r} v_{\nu} \mid r \geq 0, j_1, \ldots, j_r \in \tilde{I}\}$.
2. $\mathcal{L}(\nu) = \text{Span}_{\mathbb{C}}\{\tilde{Y}_{j_1} \cdots \tilde{Y}_{j_r} b_{\nu} \mid r \geq 0, j_1, \ldots, j_r \in \tilde{I}\}$.
3. For each $b \in \mathcal{L}(\nu)$, if $X_j b = 0$ for all $j \in \tilde{I}$, then we have $b \in \mathcal{C}b_{\nu}$. 

Proof. Let $M$ be a finite-dimensional based classical weight standard $X^+$-weight module satisfying condition (32) such that $M[\nu] \neq 0$. We proceed by descending induction on \{ $\nu' \in X^+_{\text{int}} \mid M[\nu'] \neq 0$ \} that $V(\nu')$ is a based $U^+$-module. Assuming that our claim is true for all $\nu' > \nu$ and replacing $M$ with $M/M[>\nu]$, we may assume that $M[>\nu] = 0$. Let $b_0 \in \overline{L}_{\nu}$. Then, $G^*(b_0)$ is a highest weight vector of weight $\nu$. Hence, we can identify $V(\nu)$ with an $X^+$-weight submodule $U^\nu G^*(b_0)$ of $M$. Under this identification, we have $b_0 = b_\nu$.

Let $\xi \in X_{\text{int}}$ be such that $\xi < \nu$. Assume that for all $\xi' > \xi$ and $b \in \overline{L}(\nu)_{\xi'}$, we have $G^*(b) \in V(\nu)$. Then, $V(\nu)_{\xi'}$ is spanned by $G^0(\overline{L}(\nu)_{\xi'})$. Actually, for a basis $B$ of $\overline{L}(\nu)_{\xi'}$, $G^0_0(B)$ forms a basis of $V(\nu)_{\xi'}$.

Let $v \in L(\nu)_\xi$, and set $b := \text{ev}_\infty(v) \in \overline{L}(\nu)_\xi$. Since we have

$$V(\nu)_\xi = \sum_{j \in \tilde{I}_t} Y_j V(\nu)_{\xi+\gamma_j} = \sum_{j \in \tilde{I}_t} Y_j V(\nu)_{\xi+\gamma_j}$$

by Theorem 4.2.5 (7) and Propositions 6.1.4 and 7.1.4, we can write as

$$v = \sum_{j \in \tilde{I}_t} v_j, \quad v_j \in \overline{Y}_j V(\nu)_{\xi+\gamma_j}.$$ 

We show that we can take $v_j \in L(\nu)$ for all $j \in \tilde{I}_t$.

For each $j \in \tilde{I}_t$, choose an orthonormal basis $B_j$ of $\overline{Y}_j \overline{L}(\nu)_{\xi+\gamma_j}$. By Lemma 8.1.1, $G^*(B_j) \subseteq V(\nu)$. Consequently, $G^0_0(B_j)$ forms an almost orthonormal basis of $\overline{Y}_j V(\nu)_{\xi+\gamma_j}$. Let us write

$$v_j = \sum_{b \in B_j} c_b G^0_0(b)$$

with $c_b \in K$. Set $d := \max\{ \deg(c_b) \mid b \in \bigsqcup_j B_j \}$. Assume that $d > 0$. Then, we have

$$v_j \in \bigcup_{b \in B_j} \text{lt}(c_b) G^0_0(b) + q^{d-1} \mathcal{L}_M.$$ 

This shows that

$$\sum_{b \in \bigsqcup_j B_j} \text{lt}(c_b) G^0_0(b) \equiv q^{-d} v \equiv 0$$

modulo $q^{-1} \mathcal{L}_M$. Taking $\text{ev}_\infty$, we obtain

$$\sum_{b \in \bigsqcup_j B_j} \text{lt}(c_b) b = 0,$$

and hence,

$$\sum_{b \in \bigsqcup_j B_j} \text{lt}(c_b) G^0_0(b) = q^d G^0_0(\sum_{b \in \bigsqcup_j B_j} \text{lt}(c_b) b) = 0.$$ 

Replacing $v_j$ with

$$v_j - \sum_{b \in B_j} \text{lt}(c_b) G^0_0(b) \in \overline{Y}_j V(\nu)_{\xi+\gamma_j},$$
we still have \( v = \sum_j v_j \). Furthermore, if we write \( v_j = \sum_{b \in B_j} c_b G_0^j(b) \) again, then, we have
\[
\max\{\deg(c_b) \mid b \in \bigcup_j B_j\} < d.
\]
Repeating this procedure, we conclude that \( v_j \in \mathcal{L}(\nu) \) for all \( j \), as desired.

Now, we have \( v = \sum_j v_j \) with \( v_j \in \tilde{Y}_j \mathcal{L}(\nu)_{\xi + \gamma_j} \). Taking \( \text{ev}_\infty \), we see that \( b \in \sum_j \tilde{Y}_j \mathcal{L}(\nu)_{\xi + \gamma_j} \). This shows that
\[
\mathcal{L}(\nu)_\xi = \sum_{j \in \tilde{I}_j} \tilde{Y}_j \mathcal{L}(\nu)_{\xi + \gamma_j}.
\]
Then, by Lemma 8.1.1, we obtain
\[
G''(\mathcal{L}(\nu)_\xi) \subseteq V(\nu)
\]
Now, the assertions, except (3), follow.

Let us prove (3). Let \( b \in \mathcal{L}(\nu) \) be such that \( \tilde{X}_j b = 0 \) for all \( j \in \tilde{I}_j \). Then, by Propositions 6.1.5 and 7.1.5, for each \( b' \in \mathcal{L}(\nu) \) we have
\[
(b, \tilde{Y}_j b')_\nu = (\tilde{X}_j b, b')_\nu = 0.
\]
This shows that
\[
(b, \bigoplus_{\xi < \nu} \mathcal{L}(\nu)_\xi)_\nu = 0.
\]
By weight consideration, we obtain \( b \in \mathcal{L}(\nu)_\nu = Cb_\nu \), as desired.

\[\square\]

**Corollary 8.1.3.** Let \( M \) be a finite-dimensional based classical weight standard \( X^* \)-weight module satisfying condition (32). Then, for each \( \nu \in X^*_\text{int} \), the following hold.

1. \( M[\geq \nu] \) is a based \( U^* \)-submodule of \( M \).
2. \( M[\rangle \nu] \) is a based \( U^* \)-submodule of \( M \) and \( M[\geq \nu] \).
3. \( M[\geq \nu]/M[\rangle \nu] \) is a based \( U^* \)-module isomorphic to \( V(\nu)^{m_\nu} \), where \( m_\nu \) denotes the multiplicity \( \dim_{\mathbb{K}} \text{Hom}_{U^*}(V(\nu), M) \) of \( V(\nu) \) in \( M \).

**Proof.** The assertion follows from Lemma 6.4.4 and Theorem 8.1.2. \( \square \)

### 8.2 Branching rule

In this subsection, we prove a combinatorial formula describing the branching rule from \( U^* \) to \( U \), which coincides with that from \( g \) to \( \mathfrak{t} \). Before doing so, let us explain that our formula also describes the branching rule from \( \mathfrak{sl}_n \) to \( \mathfrak{so}_n \). Recall that \( g(I_t) \) denote the complex simple Lie algebra of type \( D_m \) if \( n \in \mathbb{Z}_{ov} \) or \( B_m \) if \( n \in \mathbb{Z}_{od} \). Let \( e_i, f_i, h_i \in g(I_t) \), \( i \in I_t \) denote the Chevalley generators. Then, we have seen that there exist isomorphisms \( g(I_t) \to \mathfrak{t} \) and \( g(I_t) \to \mathfrak{so}_n \) which send \( h'_i \) to \( w_i \) and \( w'_i \), respectively (see Subsection 4.1 for the definitions of \( w_i \) and \( w'_i \)). Namely, we have two realizations of \( g(I_t) \) inside \( g = \mathfrak{sl}_n \). By [24, 2.4] and the following lemma, we see that the branching rules from \( g \) to \( \mathfrak{t} \) and from \( g \) to \( \mathfrak{so}_n \) coincide.

**Lemma 8.2.1.** Let \( V := \mathbb{C}^n \) denote the natural representation of \( g = \mathfrak{sl}_n \). Then, as \( g(I_t) \)-modules, we have \( V|_{\mathfrak{t}} \cong V|_{\mathfrak{so}_n} \).

**Proof.** It suffices to show that \( V|_{\mathfrak{t}} \) and \( V|_{\mathfrak{so}_n} \) have the same character. Recall that the subspaces \( \mathbb{C}\{b_{2i-1} \mid i \in [1, m]\} \) and \( \mathbb{C}\{b_{2i-1} \mid i \in [1, m]\} \) form Cartan subalgebras of \( \mathfrak{t} \) and \( \mathfrak{so}_n \), respectively. Under the isomorphisms \( g(I_t) \to \mathfrak{t} \) and \( g(I_t) \to \mathfrak{so}_n \), for each \( i \in [1, m] \), \( b_{2i-1} \) and \( \sqrt{-1}b_{2i-1} \) correspond to the same vector in \( g(I_t) \). Hence, in order to prove the
assertion, it suffices to show that for each $i \in [1, m]$, $b_{2i-1}$ and $\sqrt{-1}b_{2i-1}$ have the same spectra on $V$.

Let $\{v_1, \ldots, v_n\}$ denote the standard basis of $V = \mathbb{C}^n$. Then, for each $i \in [1, m]$ such that $2i \leq n$, the vector $v_{2i-1} \pm v_{2i}$ (resp., $v_{2i-1} \pm \sqrt{-1}v_{2i}$) is an eigenvector of $b_{2i-1}$ (resp., $\sqrt{-1}b_{2i-1}$) of eigenvalue $\pm \delta_{ij}$. Also, if $n \in \mathbb{Z}_{\text{odd}}$, the vector $v_n$ is an eigenvector of both $b_{2i-1}$ and $\sqrt{-1}b_{2i-1}$ of eigenvalue 0. Therefore, our claim follows. Thus, the proof completes. □

**Remark 8.2.2.** Another often used realization of $\mathfrak{g}(I_t)$ is

$$\mathfrak{t}' := \{X = (x_{i,j})_{1 \leq i, j \leq n} \in \mathfrak{g} \mid x_{i,j} = -x_{n-j+1, n-i+1}\},$$

with $h'_i \in \mathfrak{g}(I_t)$ corresponding to $h_i + h_{n-i}$ if $i \neq m$, $h_{m-1} + 2h_m + h_{m+1}$ if $i = m$ and $n \in \mathbb{Z}_{\text{ev}}$, or $2(h_m + h_{m+1})$ if $i = m$ and $n \in \mathbb{Z}_{\text{odd}}$. By the same way as above, we see that the branching rule from $\mathfrak{g}$ to $\mathfrak{t}'$ is the same as that from $\mathfrak{g}$ to $\mathfrak{t}$.

Let $\lambda \in X^+$ and consider the irreducible $\mathfrak{g}$-module $V_{\mathfrak{g}}(\lambda)$ of highest weight $\lambda$. As a $\mathfrak{t}$-module, it decomposes as

$$V_{\mathfrak{g}}(\lambda) \simeq \bigoplus_{\nu \in X^+_{\mathfrak{t}, \text{int}}} V_{\mathfrak{t}}(\nu)^{\oplus [\lambda : \nu]}$$

for some $[\lambda : \nu] \geq 0$.

Also, consider the irreducible $\mathfrak{U}$-module $V(\lambda)$, and set $V(\lambda)_{\mathfrak{K}_1} := \mathfrak{U}_{\mathfrak{K}_1}v_{\lambda}$. We have

$$V(\lambda)_1 := V(\lambda)_{\mathfrak{K}_1} / (q - 1)V(\lambda)_{\mathfrak{K}_1} \simeq V_{\mathfrak{g}}(\lambda).$$

Let $v \in V(\lambda)_{\mathfrak{K}_1}$ and write its $X_{\mathfrak{t}}$-weight vector decomposition as $v = \sum_{\xi \in X_{\mathfrak{t}, \text{int}}} v_{\xi}$ with $v_{\xi} \in V(\lambda)_{\xi}$. Then, for each $\xi \in X_{\mathfrak{t}, \text{int}}$, we have for sufficiently large $N \geq 0$,

$$\prod_{i \in I} \prod_{a \in [-N,N] \setminus \{\xi\}} (B_i - [a])v = \prod_{i \in I} \prod_{a \in [-N,N] \setminus \{\xi\}} ([\xi] - [a])v_{\xi}.$$

Hence, we obtain

$$v_{\xi} = \prod_{i \in I} \prod_{a \in [-N,N] \setminus \{\xi\}} \frac{1}{[\xi] - [a]}(B_i - [a])v \in \mathfrak{U}_{\mathfrak{K}_1}v_{\lambda} \subseteq V(\lambda)_{\mathfrak{K}_1}.$$

This shows that

$$V(\lambda)_{\mathfrak{K}_1} = \bigoplus_{\xi \in X_{\mathfrak{t}, \text{int}}} (V(\lambda)_{\xi} \cap V(\lambda)_{\mathfrak{K}_1}),$$

and hence,

$$\text{ch}_{\mathfrak{U}, V(\lambda)} = \text{ch}_{\mathfrak{t}, V_{\mathfrak{g}}(\lambda)}.$$

By character consideration, we have

$$V(\lambda) \simeq \bigoplus_{\nu \in X_{\mathfrak{t}, \text{int}}^+} V(\nu)^{\oplus [\lambda : \nu]},$$

and hence,

$$\overline{Z}(\lambda) \simeq \bigoplus_{\nu \in X_{\mathfrak{t}, \text{int}}^+} \overline{Z}(\nu)^{\oplus [\lambda : \nu]}.$$

This, together with Theorem 8.1.2 (3), shows that

$$[\lambda : \nu] = \dim_{\mathbb{C}} \{b \in \overline{Z}(\lambda)_{\nu} \mid X_jb = 0 \text{ for all } j \in \tilde{I}_t\}.$$
However, we do not know how \( \tilde{X}_j \) acts on \( \mathcal{Z}(\lambda) \) explicitly. In the sequel, we aim to describe this multiplicity \([\lambda; \nu]\) in terms of \( \deg_i \) and \( \tilde{B}_i \), which in turn, can be described in terms of the crystal structure of \( \mathcal{B}(\lambda) \) by Corollary 5.1.6.

**Lemma 8.2.3.** Let \( M \) be a finite-dimensional \( U \)-module with a crystal basis \( \mathcal{B} \). Let \( b \in \mathcal{B} \), and write its \( X_t \)-weight vector decomposition as \( b = \sum_{\nu \in X_t} b_\nu \) with \( b_\nu \in \mathcal{Z}_{M,\nu} \). Then, we have \( b_\nu \neq 0 \) if and only if \( |\nu_{2i-1}| = \deg_{2i-1}(b) \) for all \( i \in [1, m] \). Furthermore, if \( b_\nu \neq 0 \), then we have

\[
b_\nu = \frac{1}{2^s} \prod_{i \in [1, m], \nu_{2i-1} > 0} (1 + \tilde{B}_{2i-1}) \prod_{i \in [1, m], \nu_{2i-1} < 0} (1 - \tilde{B}_{2i-1})b,
\]

where \( s := \# \{ i \mid \nu_{2i-1} \neq 0 \} \).

**Proof.** Let \( \nu \in X_{t,\text{int}} \) be such that \( |\nu_{2i-1}| = \deg_{2i-1}(b) \) for all \( i \in [1, m] \), and set \( s \) to be the number of \( i \) such that \( \nu_{2i-1} \neq 0 \). By Lemma 5.1.3 and Corollary 5.1.6, we see that

\[
\frac{1}{2^s} \prod_{i \in [1, m], \nu_{2i-1} > 0} (1 + \tilde{B}_{2i-1}) \prod_{i \in [1, m], \nu_{2i-1} < 0} (1 - \tilde{B}_{2i-1})b
\]

is a nonzero \( X_t \)-weight vector of weight \( \nu \). Also, by an elementary calculation, we have

\[
b = \sum_{\nu \in X_{t,\text{int}}} \frac{1}{2^s} \prod_{i \in [1, m], \nu_{2i-1} > 0} (1 + \tilde{B}_{2i-1}) \prod_{i \in [1, m], \nu_{2i-1} < 0} (1 - \tilde{B}_{2i-1})b.
\]

Therefore, this is the \( X_t \)-weight decomposition of \( b \). Thus, the proof completes. \( \square \)

For a finite-dimensional classical weight module \( M \) equipped with a contragredient Hermitian inner product, and \( b \in \mathcal{Z}_M \), consider the following condition:

\[
(37) \quad \text{deg}_{2i}(b) = 0 \quad \text{for all } i \in [1, m] \text{ such that } 2i < n, \quad \text{deg}_{2i+1}(\tilde{B}_i \tilde{B}_{2i-1} \text{deg}_{2i+1}(b)) = 0 \quad \text{for all } i \in [1, m] \text{ such that } 2i + 1 < n.
\]

**Proposition 8.2.4.** Let \( M \) be a finite-dimensional classical weight module equipped with a contragredient Hermitian inner product. Set

\[
L_1 := \{ b \in \mathcal{Z}_M \mid \tilde{X}_j b = 0 \text{ for all } j \in \tilde{I}_t \}, \quad L_2 := \mathbb{C}\{ b \in \mathcal{Z}_M \mid b \text{ satisfies condition } (37) \}.
\]

Then, the linear map

\[
L_2 \to L_1; \quad b \mapsto (1 + \tilde{B}_1)(1 + \tilde{B}_3) \cdots (1 + \tilde{B}_{2m'-1})b,
\]

where

\[
m' := \begin{cases} m - 1 & \text{if } n \in \mathbb{Z}_{\text{ev}}, \\ m & \text{if } n \in \mathbb{Z}_{\text{odd}} \end{cases}
\]

is an isomorphism of \( \mathbb{C} \)-vector spaces with inverse

\[
L_1 \to L_2; \quad b \mapsto \sum_{\zeta \in X_t, \zeta_{2i} = 0 \text{ for all } i \in [1, m']} 1_{\zeta} b.
\]
Proof. We prove by induction on \( m' \). The \( m' = 1 \) case follows from Propositions 6.3.8 and 7.2.6.

As the proof of Proposition 6.3.8, it suffices to consider the case when \( M = V(\nu) \) for some \( \nu \in X_{\text{int}}^+ \). Clearly, we have

\[
L_1 = \mathbb{C}b_\nu.
\]

As the proof of Lemma 6.4.1, we have

\[
e_{\nu} := \sum_{\zeta \in X^+} \chi_{\nu} b_\nu
\]

\[
= \text{ev}_\infty \left( \sum_{\zeta \in X^+} \chi_{\nu} v_\nu \right)
\]

\[
= \text{ev}_\infty \left( \frac{1}{2m'} (1 + (-1)^{\nu_1} Y_2^{(2\nu_1)}) (1 + (-1)^{\nu_2} Y_4^{(2\nu_2)}) \cdots (1 + (-1)^{\nu_{2m' - 1}} Y_{2m'}^{(2\nu_{2m' - 1})}) v_\nu \right)
\]

\[
= \frac{1}{2m'} (1 + (-1)^{\nu_1} Y_2^{(2\nu_1)}) (1 + (-1)^{\nu_2} Y_4^{(2\nu_2)}) \cdots (1 + (-1)^{\nu_{2m' - 1}} Y_{2m'}^{(2\nu_{2m' - 1})}) b_\nu \neq 0.
\]

We show that \( c'' \in L_2 \). As a \( U_{1,2,\ldots,2m'}^{-1} \)-module vector, both \( v_\nu \) and \( Y_{2m'}^{(2\nu_{2m' - 1})} v_\nu \) are highest weight vectors of weights \( (\nu_1, \nu_3, \ldots, \nu_{2m' - 3}, \nu_{2m' - 1}) \) and \( (\nu_1, \nu_3, \ldots, \nu_{2m' - 3}, -\nu_{2m' - 1}) \), respectively. By the induction hypothesis, we see that \( c'' \) satisfies condition (37) as a \( U_{1,2,\ldots,2m'}^{-1} \)-module vector.

On the other hand, by the definition of \( 1_{i,p} \)’s \( (i \in I, p \in \{\text{ev, odd}\}) \), they pairwise commute. Hence, we have

\[
1_{1,2,3,\ldots} 1_{2m' - 1,2m'} \cdots 1_{2m' - 1,2m'} = 1_{2m' - 1,2m'} 1_{1,2,3,\ldots} \cdots 1_{2m' - 1,2m'}
\]

\[
= \frac{1}{2m'} (1 + (-1)^{\nu_1} Y_2^{(2\nu_1)}) (1 + (-1)^{\nu_2} Y_4^{(2\nu_2)}) \cdots (1 + (-1)^{\nu_{2m' - 1}} Y_{2m'}^{(2\nu_{2m' - 1})}) v_\nu,
\]

and consequently,

\[
c'' = \frac{1}{2m'} (1 + (-1)^{\nu_1} Y_2^{(2\nu_1)}) (1 + (-1)^{\nu_2} Y_4^{(2\nu_2)}) \cdots (1 + (-1)^{\nu_{2m' - 1}} Y_{2m'}^{(2\nu_{2m' - 1})}) b_\nu.
\]

Now, set

\[
J := \begin{cases} 
\{2m' - 1, 2m', 2m' + 1\} & \text{if } n \in \mathbb{Z}_{\text{ev}}, \\
\{2m' - 1, 2m'\} & \text{if } n \in \mathbb{Z}_{\text{odd}}.
\end{cases}
\]

By weight consideration, as a \( U_J \)-module vector,

\[
(1 + (-1)^{\nu_1} Y_2^{(2\nu_1)}) (1 + (-1)^{\nu_2} Y_4^{(2\nu_2)}) \cdots (1 + (-1)^{\nu_{2m' - 1}} Y_{2m'}^{(2\nu_{2m' - 1})}) v_\nu
\]

is a sum of highest weight vectors of weight \( (\nu_{2m' - 1}, \nu_{2m' + 1}) \) and a lowest weight vector of weight \( (-\nu_{2m' - 1}, \nu_{2m' + 1}) \) if \( n \in \mathbb{Z}_{\text{ev}} \), or a sum of highest weight vectors of weight \( \nu_{2m' - 1} \) and lowest weight vectors of weight \( -\nu_{2m' - 1} \) if \( n \in \mathbb{Z}_{\text{odd}} \). Here, lowest weight vector means
a vector of the form \(Y^{(2\nu_{2m'-1})}v\) for some highest weight vector \(v\). Then, by Proposition 6.3.7, we see that \(c'\) is a linear combination of vectors of the form
\[
(1 + (-1)^{2m'-1}Y)b'
\]
with \(b'\) being a highest weight vector of weight \((\nu_{2m'-1}, \pm \nu_{2m'+1})\) if \(n \in \mathbb{Z}_{\text{ev}}\), or \(\nu_{2m' - 1}\) if \(n \in \mathbb{Z}_{\text{odd}}\). Therefore, by the \(n = 3, 4\) cases, \(c'\) satisfies condition (37) as a \(U'_{\delta}\)-module vector. Thus, we conclude that
\[
c' \in L_2,
\]
as desired.

Next, let \(c \in \mathcal{L}(\nu)\) satisfy condition (37). As the proof of Lemma 8.2.3, we can write the \(X_t\)-weight vector decomposition of \(c\) as
\[
c = \sum_{\xi \in X_{t, \text{int}}} c_{\xi}, \quad c_{\xi} \in \mathcal{L}(\nu)\xi.
\]
Let \(i \in [1, m']\). Then, we have
\[
(1 + \tilde{B}_{2i-1})c = \sum_{|\xi_{2i-1}| = \deg_{2i-1}(c)\text{ for all } k \in [1, m]} 2^{1 - \delta_{2i-1}(c)k} c_{\xi}.
\]
Define \(J_i \subseteq I\) by
\[
J_i := \begin{cases} 
{2i - 1, 2i, 2i + 1} & \text{if } 2i + 1 < n, \\
{2i - 1, 2i} & \text{if } 2i + 1 = n.
\end{cases}
\]
Then, by the \(n = 3, 4\) cases, we have
\[
\tilde{X}_j(1 + \tilde{B}_{2i-1})c = 0 \quad \text{for all } j \in \tilde{I}_t(J_i).
\]
By weight consideration, we see that
\[
(38) \quad \tilde{X}_j c_{\xi} = 0 \quad \text{for all } j \in \tilde{I}_t(J_i) \text{ and } \xi \in X_{t, \text{int}} \text{ such that } |\xi_{2i-1}| = \deg_{2i-1}(c).
\]
Now, remark that
\[
c' := (1 + \tilde{B}_1)(1 + \tilde{B}_3) \cdots (1 + \tilde{B}_{2m'-1})c = \sum_{|\xi_{2k-1}| = \deg_{2k-1}(c)\text{ for all } k \in [1, m]} 2^s c_{\xi},
\]
where \(s = \#\{i \in [1, m'] \mid \deg_{2i-1}(c) \neq 0\}\). By (38), we conclude that
\[
\tilde{X}_j c' = 0 \quad \text{for all } j \in \tilde{I}_t,
\]
and hence,
\[
c' \in \mathbb{C}b_\nu = L_1
\]
by Theorem 8.1.2 (3).

This far, we have verified that the linear maps in consideration are well-defined, and that \(\dim L_2 \geq \dim L_1 = 1\). Hence, in order to complete the proof, it suffices to show that
\[
\sum_{\xi \in X^s} 1_{\xi}(1 + \tilde{B}_1)(1 + \tilde{B}_3) \cdots (1 + \tilde{B}_{2m'-1})c = c.
\]
Set
\[
J' := \{1, 2, 3\}, \quad J'' := \{3, 4, \ldots, n - 1\}.
\]
It is easily verified that \((1 + \tilde{B}_1)c\) satisfies condition (37) as a \(U_q\)-module vector. Then, by the induction hypothesis, we have
\[
\sum_{\zeta'' \in X'(J'')} 1_{j''} \zeta''(1 + \tilde{B}_3)(1 + \tilde{B}_5) \cdots (1 + \tilde{B}_{2m'-1})(1 + \tilde{B}_1)c = (1 + \tilde{B}_1)c.
\]
Hence, we compute as
\[
\sum_{\zeta_i = 0 \text{ for all } i \in [1, m']} 1_{j'}(1 + \tilde{B}_1)(1 + \tilde{B}_3) \cdots (1 + \tilde{B}_{2m'-1})c
= \sum_{\zeta_i = 0 \text{ for all } i \in [1, m']} 1_{j'} \sum_{\zeta'' \in X'(J'')} 1_{j''} \zeta''(1 + \tilde{B}_3)(1 + \tilde{B}_5) \cdots (1 + \tilde{B}_{2m'-1})(1 + \tilde{B}_1)c
= \sum_{\zeta_i = 0 \text{ for all } i \in [1, m']} 1_{j'}(1 + \tilde{B}_1)c.
\]
By the \(n = 4\) case, the last term equals \(c\), as desired. Thus, the proof completes.

**Lemma 8.2.5.** Let \(\lambda \in X^+\). Let \(b \in \mathcal{Z}(\lambda)\) and write \(b = \sum_{k=1}^r a_k b_k\) with \(a_k \in \mathbb{C}^x\), \(b_k \in \mathcal{B}(\lambda)\). If \(b\) satisfies condition (37), then so does \(b_k\) for all \(k \in [1, r]\).

**Proof.** Since each element in \(\mathcal{B}(\lambda)\) is homogeneous, we must have \(\deg_i(b_k) = \deg_i(b)\) for all \(i \in [1, n - 1]\). In particular, we obtain
\[
\deg_{2i}(b_k) = 0 \text{ for all } i \in [1, m'].
\]
Next, let \(i \in I_\mathcal{B}\) be such that \(2i + 1 < n\), and set \(\{k_1, \ldots, k_l\} \subseteq [1, r]\) to be the subset such that \((\tilde{B}_{2i} \tilde{B}_{2i-1})^{\deg_{2i+1}(b_k)}b_k \neq 0\) and \(\deg_{2i+1}((\tilde{B}_{2i} \tilde{B}_{2i-1})^{\deg_{2i+1}(b_k)}b_k) \neq 0\). Since \(\deg_{2i+1}((\tilde{B}_{2i} \tilde{B}_{2i-1})^{\deg_{2i+1}(b_k)}b_k) = 0\), we must have
\[
\sum_{j=1}^l a_{k_j}(\tilde{B}_{2i} \tilde{B}_{2i-1})^{\deg_{2i+1}(b_k)}b_k_j = 0.
\]
Since \((\tilde{B}_{2i} \tilde{B}_{2i-1})^{\deg_{2i+1}(b_k)}b_k_j\) are distinct vectors in \(\mathcal{B}(\lambda)\), we see that \((\tilde{B}_{2i} \tilde{B}_{2i-1})^{\deg_{2i+1}(b_k)}b_k = 0\) for all \(j \in [1, l]\). Therefore, we conclude that \(\deg_{2i+1}((\tilde{B}_{2i} \tilde{B}_{2i-1})^{\deg_{2i+1}(b_k)}b_k) = 0\) for all \(k \in [1, r]\). Thus, the proof completes.

**Theorem 8.2.6.** Let \(\lambda \in X^+\) and \(\nu \in X_i^\text{int}\).

1. Suppose \(n \in \mathbb{Z}_{\text{ev}}\) and \(\nu_{2m-1} \neq 0\). Then, we have
   \[
   [\lambda : \nu] = \frac{1}{2} \sharp\{b \in \mathcal{B}(\lambda) \mid \deg_{2i-1}(b) = |\nu_{2i-1}| \text{ for all } i \in [1, m],
   \deg_{2i}(b) = 0 \text{ for all } i \in [1, m - 1],
   \deg_{2i+1}((\tilde{B}_{2i} \tilde{B}_{2i-1})^{\nu_{2i+1}}b) = 0 \text{ for all } i \in [1, m - 1]\}.
   \]

2. Suppose \(n \in \mathbb{Z}_{\text{ev}}\) and \(\nu_{2m-1} = 0\). Then, we have
   \[
   [\lambda : \nu] = \sharp\{b \in \mathcal{B}(\lambda) \mid \deg_{2i-1}(b) = |\nu_{2i-1}| \text{ for all } i \in [1, m],
   \deg_{2i}(b) = 0 \text{ for all } i \in [1, m - 1],
   \deg_{2i+1}((\tilde{B}_{2i} \tilde{B}_{2i-1})^{\nu_{2i+1}}b) = 0 \text{ for all } i \in [1, m - 1]\}.
   \]
(3) Suppose \( n \in \mathbb{Z}_{\text{odd}} \). Then, we have
\[
[\lambda : \nu] = \# \{ b \in \mathcal{B}(\lambda) \mid \deg_{\xi_{2i-1}}(b) = |\nu_{2i-1}| \text{ for all } i \in [1, m], \\
\deg_{\xi_i}(b) = 0 \text{ for all } i \in [1, m], \\
\deg_{\xi_{2i+1}}((\tilde{B}_2, \tilde{B}_{2i-1})^{\nu_{2i+1}}b) = 0 \text{ for all } i \in [1, m-1] \}.
\]

Proof. Let \( b \in \mathcal{B}(\lambda) \) be such that
\[
|\deg_1(b)| \geq |\deg_3(b)| \geq \cdots \geq |\deg_{2m-1}(b)|.
\]
By Lemma 8.2.3, we can write
\[
b = \sum_{\xi \in \chi_{\text{t, int}}^{\pm}} b_\xi, \quad b_\xi \in \mathcal{L}(\lambda) \setminus \{0\}.
\]
Note that when \( n \in \mathbb{Z}_{\text{ev}} \) and \( \deg_{\xi_{2m-1}}(b) \neq 0 \), there are exactly two dominant weights \( \xi^+, \xi^- \in X_{\text{t, int}}^+ \) such that \( \xi_{2m-1}^+ > 0 \), \( \xi_{2m-1}^- < 0 \), and \( b_{\xi^+}, b_{\xi^-} \neq 0 \). Otherwise, there is exactly one dominant weight \( \xi \in X_{\text{t, int}}^+ \) such that \( b_\xi \neq 0 \).

From now on, we concentrate on the case when \( n \in \mathbb{Z}_{\text{ev}} \) and \( \nu_{2m-1} \neq 0 \). The other cases can be proved in a similar way more easily.

Set
\[
L' := \{ b \in \mathcal{L}(\lambda)_{\nu^+} \oplus \mathcal{L}(\lambda)_{\nu^-} \mid \tilde{X}_j b = 0 \text{ for all } j \in \tilde{I}_t \},
\]
and
\[
B := \{ b \in \mathcal{B}(\lambda) \mid \deg_{\xi_{2i-1}}(b) = |\nu_{2i-1}| \text{ for all } i \in [1, m], \\
\deg_{\xi_i}(b) = 0 \text{ for all } i \in [1, m-1], \\
\deg_{\xi_{2i+1}}((\tilde{B}_2, \tilde{B}_{2i-1})^{\nu_{2i+1}}b) = 0 \text{ for all } i \in [1, m-1] \}.
\]
By Theorem 8.2.4, we have an isomorphism
\[
\mathbb{C}B \rightarrow L'
\]
which sends
\[
b \mapsto (1 + \tilde{B}_1)(1 + \tilde{B}_3) \cdots (1 + \tilde{B}_{2m-3})b,
\]
with inverse given by
\[
b \mapsto \sum_{\xi \in X^+} 1_\xi b.
\]
Let \( b \in B \) and write its \( X_t \)-weight vector decomposition as
\[
b = \sum_{\xi \in \chi_{\text{t, int}}^{\pm}} b_\xi, \quad b_\xi \in \mathcal{L}(\lambda) \setminus \{0\}.
\]
Set
\[
b^+ := \sum_{\xi_1, \xi_3, \cdots, \xi_{2m-1} > 0} b_\xi, \quad b^- := \sum_{\xi_1, \xi_3, \cdots, \xi_{2m-1} < 0} b_\xi.
\]
Let us show that \( b^\pm \in \mathbb{C}B \).

By the definitions, we have
\[
b = b^+ + b^-,
\]
and
\[
(1 + \tilde{B}_1)(1 + \tilde{B}_3) \cdots (1 + \tilde{B}_{2m-3})b^+ = 2^{m-1}b_{\nu^\pm}.
\]
Since we have $\bar{X}_j(b_{\nu^+} + b_{\nu^-}) = 0$ for all $j \in \tilde{I}_t$, by weight consideration, we obtain
$$\bar{X}_j b_{\nu^\pm} = 0 \text{ for all } j \in \tilde{I}_t.$$  
In particular, we have
$$b_{\nu^\pm} \in L'.$$

On the other hand, we have
$$\sum_{\zeta \in X'} 1 \zeta 2^{m-1} (b_{\nu^+} + b_{\nu^-}) = b = b^+ + b^-.$$  
Also, since $b_{\nu^\pm}$ is a highest weight vector of weight $\nu^\pm$, we see that
$$\sum_{\zeta \in X'} 1 \zeta 2^{m-1} b_{\nu^\pm}$$
$$= (1 + (-1)^{\nu_1} \tilde{Y}_2^{2\nu_1}) (1 + (-1)^{\nu_2} \tilde{Y}_4^{2\nu_2}) \cdots (1 + (-1)^{\nu_{2m-2}} \tilde{Y}_{2m-2}^{2\nu_{2m-2}}) b_{\nu^\pm}$$
is a sum of $X_{\nu}$-weight vectors of weights $\zeta' \in X_{\nu, \text{int}}$ with $\pm \zeta'_1 \cdot \zeta'_2 \cdots \zeta'_{2m-1} > 0$. Therefore, we see that
$$\sum_{\zeta \in X'} 1 \zeta 2^{m-1} b_{\nu^\pm} = b^\pm.$$  
This shows that
$$b^\pm \in \mathbb{C}B,$$
as desired.

Now, we show that $\bar{B}_1 \bar{B}_3 \cdots \bar{B}_{2m-1}$ restricts to an involution on $B$. In fact, for each $b = b^+ + b^- \in B$, we have
$$\bar{B}_1 \bar{B}_3 \cdots \bar{B}_{2m-1} b = b^+ - b^- \in \mathbb{C}B.$$  
Since $\bar{B}_1 \bar{B}_3 \cdots \bar{B}_{2m-1} b \in B(\lambda)$, we conclude that $\bar{B}_1 \bar{B}_3 \cdots \bar{B}_{2m-1} b \in B$.

Also, we have $\bar{B}_1 \bar{B}_3 \cdots \bar{B}_{2m-1} b = b$ if and only if $b^- = 0$, which cannot occur by Lemma 8.2.3. Thus, we obtain an isomorphism
$$\mathbb{C} \{ (1 \pm \bar{B}_1 \bar{B}_3 \cdots \bar{B}_{2m-1}) b \mid b \in B \} \rightarrow L'_\pm := \{ b \in \mathbb{Z}(\lambda)_{\nu^\pm} \mid \bar{X}_j b = 0 \text{ for all } j \in \tilde{I}_t \}.$$  
This implies
$$[\lambda : \nu^\pm] = \dim L'_\pm = \frac{1}{2} |B|.$$  
Since we have either $\nu = \nu^+$ or $\nu = \nu^-$, the assertion follows.  

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