Front propagation in random media:
From extremal to activated dynamics

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Abstract

Front propagation in a random environment is studied close to the depinning thresh-
old. At zero temperature we show that the depinning force distribution exhibits a
universal behavior. This property is used to estimate the velocity of the front at very
low temperature. We obtain a Arrhenius behavior with a prefactor depending on the
temperature as a power law. These results are supported by numerical simulations.

1 Introduction

Front propagation in a random environment has in recent years become a generic problem
describing the motion of interfaces as different as magnetic domain walls[1], wetting[2] or
fracture fronts[3]. In all cases the balance between a roughening effect due to the quenched
random environment and the smoothing effect of the elastic interactions leads to a rich phe-
nomenology. At zero temperature one observes a depinning transition: a threshold force is
required for unlimited motion. When the front is driven at a force close to the threshold,
one observes spatio-temporal organization, the front roughness exhibits self-affine correla-
tions and the critical exponents characterising the propagation of the front can be shown
to depend on the nature of the elastic interactions[4]. At finite temperature the dynamics
is controlled by thermal activation and a creep regime characterized by a stretched expo-
nential dependence of the velocity on the driving force has been predicted by scaling[5] or
renormalisation [6] arguments. This creep regime has been experimentally observed in weak
pinning conditions for the propagation of a magnetic domain wall [1].

In the following we consider the case of strong pinning, we first give results obtained in
the framework of an extremal dynamics focusing on the depinning force distribution in the
vicinity of the macroscopic depinning threshold. We then consider the thermally activated
propagation of a front driven at a constant force and we show that at low temperature the
knowledge of the depinning force distribution leads to a simple Arrhenius regime with a
prefactor that depends on the temperature as a power law. To illustrate these results we
present in the following numerical simulations performed with two different elastic interactions: Laplacian (corresponding to the motion of magnetics walls or fluid invasion in a Hele-Shaw cell) and long range elasticity (which corresponds to the propagation of a wetting or a fracture front).

2 Force distribution in the extremal model

We consider the propagation of an elastic 1D front through a 2D random environment. The position of the front at abscissa $x_i$ is given by $h_i = h(x_i)$. The random environment consists in traps of depth $\gamma_i$ which are randomly distributed in the direction of propagation. In the following we restrict ourselves to the case of an overdamped dynamics in a strong pinning situation. When submitted to an external driving force $f_{ext}$ the depinning criterion for the site $i$ can be written as:

$$f_{ext} + f_{el}[x_i, h_i] > \gamma_i$$

where the elastic component can be Laplacian or long range. In the particular case of an interfacial fracture front, the long range elastic kernel comes from a first order expansion of the stress intensity factor and with periodic boundary conditions it can be written:

$$f_{el}[x_i, h_i] = \sum_{j \neq i} \frac{h_j - h_i}{\sin \left[ \frac{\pi (x_i - x_j)}{L} \right]^2}$$

where $L$ is the size of the system. In the Laplacian case, we use a discrete form of the Laplace operator:

$$f_{Lap}[x_i, h_i] = \frac{1}{2} (h_{i+1} - 2h_i + h_{i-1})$$

In the extremal dynamics the external force is adapted at every step to the value $f_{ext} = f_c(t)$ such that only one site can depin. This corresponds to selecting the weakest site $i_0$ of the front:

$$f_c = \gamma_{i_0} - f_{el}[x_{i_0}, h_{i_0}] = \min_i (\gamma_i - f_{el}[x_i, h_i])$$

The weakest site is then advanced up to the next trap, the elastic forces are updated and the process is iterated on the weakest site of the new configuration. The maximum of these depinning forces $f_c$ over time corresponds to the macroscopic depinning threshold $f^*$.

The front then presents a multiscale roughness: it remains statistically invariant under the anisotropic scaling transformations $x \rightarrow \lambda x$, $y \rightarrow \lambda^\zeta y$. The front is said to be self-affine and the exponent $\zeta$ is called the roughness exponent. This scaling invariance is responsible for long range correlations of the height differences on the front. In particular when measured over a size $d$, the height standard deviation grows as a power law: $\sqrt{\langle \Delta y^2 \rangle(d)} \propto d^\zeta$. A direct consequence of this self-affinity property is that the height power spectrum is a power law of exponent $-1 - 2\zeta$ where $\zeta \simeq 0.35$ and $\zeta \simeq 1.25$ in the fracture front case and in the Laplacian case respectively.

It can be shown that the behavior of the distribution $Q(f_c)$ in the vicinity of the threshold $f^*$ exhibits universal features. In order to illustrate this point let us focus on a sequence of successive depinning events on the front. Such sequences are known to be time and space
correlated and present an avalanche-like behavior[4]. The distance $d$ between two successive events is distributed according to $p(d) \propto d^{-a}$ where $a = 3$ in the Laplacian case and $a = 2$, the exponent of the elastic kernel, in the case of long range elastic interactions. Large separations between two successive events correspond to the jump from one avalanche to another one, the larger the jump, the closer the depinning force to the threshold $f^*$. Let us consider the force distribution $q_d(f)$ conditioned to the size $d$ of the jump. From the knowledge of the elastic interactions and of the front roughness statistics we can give estimates of the moments of these distributions. The typical height fluctuations on a distance $d$ scaling as $d^\zeta$, the force fluctuations scale as $d^{-b}$ where $b = 1 - \zeta \simeq 0.65$ in the case of long range interactions or $b = 2 - \zeta \simeq 0.75$ in the Laplacian case. As $d$ increases, $q_d(f)$ becomes narrower and closer to the threshold $f^*$. In particular we get the linear relationship:

$$\langle f_c \rangle_d = f^* - A \sigma_d(f_c),$$  \hspace{1cm} (5)$$

where $\langle f_c \rangle_d$ and $\sigma_d(f_c)$ are the mean and the standard deviation of the depinning forces conditioned to a jump of size $d$. This relationship allows a precise determination of the threshold $f^*$. Moreover all distributions $q_d$ are identical up to a rescaling transformation:

$$q_d(f^* - f_c) = d^b \psi[(f^* - f_c) d^b].$$  \hspace{1cm} (6)$$

The knowledge of the distribution $p(d) \propto d^{-a}$ of the distances between successive active sites allows to express the depinning force distribution close to the threshold:

$$Q(f^* - f_c) = \int x^{b-a} \psi[(f^* - f_c)x^b] dx \propto (f^* - f_c)^\nu, \hspace{0.5cm} \nu = \frac{a - b - 1}{b},$$  \hspace{1cm} (7)$$

with the numerical values $\nu \simeq 0.5$ in the fracture front case and $\nu \simeq 1.7$ in the Laplacian case.

In Fig. 1 we present numerical results obtained in the case of long range elastic interactions, we see that the low tail contribution corresponds to activity jumps of size 0 or 1. For contributions corresponding to larger jumps, the details of the random trap distribution are washed out and we can check that the force distributions obey the scaling proposed in Eq. (6). Note in addition that the universal behavior proposed in Eq. (6) only applies in a very close vicinity of the threshold $f^*$. The size of the system is $L = 1024$, the trap depth and the distance between two traps in the propagation direction are uniformly distributed in the range $[0 - 1]$. The results are averaged over 5 millions iteration steps. A linear fit using Eq. (5) gives $f^* = 0.373 \pm 0.001$ and we get for the roughness exponent $\zeta = 0.34 \pm 0.01$.

### 3 From extremal to activated dynamics

Instead of adapting the driving force at every step so that only one site can depin at a time we consider now the front propagation under a constant external driving force $f_{ext}$. At zero temperature the front moves freely if $f_{ext} > f^*$, while it will only advance a finite distance before being pinned if $f_{ext} < f^*$. This travelling distance diverges when $f_{ext}$ approaches the threshold $f^*$. At finite temperature we now allow a pinned site $i$ to depin at any time with an Arrhenius probability $p_i$: 
\begin{equation}
\rho_i = e^{\frac{\Delta_i}{T}}, \quad \Delta_i = f_{\text{ext}} + f_i^{\text{el}} - \gamma_i < 0, \quad (8)
\end{equation}
and (assuming all other sites to be frozen) the probability that the site has remained pinned
during the time \( \tau \) and advances in the interval \([\tau, \tau + d\tau]\) can be written
\begin{equation}
P_i(\tau)d\tau = \frac{1}{\tau_i}e^{-\frac{\tau}{\tau_i}}d\tau, \quad \tau_i = \frac{1}{p_i} = e^{-\frac{\Delta_i}{T}}. \quad (9)
\end{equation}
If we consider now all sites of the front, the typical waiting time for the first depinning event
on the front is \( \tau^* \) such that:
\begin{equation}
\frac{1}{\tau^*} = \sum_i \frac{1}{\tau_i} = \sum_i e^{\frac{\Delta_i}{T}}, \quad (10)
\end{equation}
the probability that the depinning takes place at site \( i \) being \( p_i / \sum p_i = \tau^*/\tau_i \). At very low
temperature the waiting time \( \tau^* \) is dominated by the waiting time of the weakest site \( i_0 \)
\begin{equation}
\frac{1}{\tau^*} = \frac{1}{\tau_{i_0}} \left( 1 + \sum_{i \neq i_0} e^{\frac{\Delta_i - \Delta_{\text{max}}}{T}} \right). \quad (11)
\end{equation}

In the following we focus on this situation of very low temperatures. We consider the
case of an elastic front driven at a constant force \( f_{\text{ext}} \). The (very low) temperature allows
to introduce a physical time in the extremal model which is adapted as follows. When
the driving force \( f_{\text{ext}} \) exceeds the local threshold \( f_c \), all sites which fulfill the depinning
Figure 2: Velocity of the front after rescaling by the Arrhenius term \(\exp\left(\frac{(f^* - f_{ext})}{T}\right)\) for different driving forces in the case of long range elastic interactions (left) in Laplacian elasticity (right). The symbols correspond to the numerical simulations and the lines to the expected results from Eq. (12) with an exponent \(\nu = 0.25\) (long range) and \(\nu = 1.66\) (Laplacian)

criterion are allowed to advance at the same time. Then the elastic forces are updated to take into account the change of front conformation. If however the site reaches a blocked configuration where no site is able to depin at zero temperature, the weakest site is allowed to advance by thermal activation. The waiting time \(t_0\) is chosen according to the distribution \((1/\tau_{00}) \exp(-t_0/\tau_{00})\) corresponding to the extremal site and the elastic forces are updated before the next iteration.

The distribution of the depinning forces being known in the vicinity of the macroscopic threshold \(f^*\), we can estimate the creep velocity at very low temperature via the average waiting time of an activated step:

\[
\langle \tau \rangle \approx T^{1+\nu} \exp \left( \frac{f^* - f_{ext}}{T} \right) \int_0^{f^* - f_{ext}} u^\nu e^{-u} du ,
\]

so that we expect at low temperatures:

\[
v(f_{ext}, T) = T^{-(1+\nu)} \exp \left( -\frac{f^* - f_{ext}}{T} \right). \tag{13}
\]

and a simple Arrhenius behavior when \(T \gg f^* - f_{ext}\).

In Fig. 2 we present the numerical results obtained for a front driven at different forces below the threshold. The symbols correspond to the velocity results after correction by the Arrhenius term and the lines to the expected behavior following Eq. (12). For the latter expressions we used the values \(\nu = 0.25\) in the long range case and \(\nu = 1.66\) and Laplacian case to compare with the expected values \(\nu = 0.5\) and \(\nu = 1.7\) respectively. These differences
may be due to the fact that the universal behavior of the depinning force distribution is only valid in a region close to the threshold ($\Delta f < 0.1$ in our case, see Fig. 1) and for lower driving forces the system becomes more sensitive to the details of the distribution.

This behavior is different from the stretched exponential described in [5, 6, 7, 8]; the main reason is probably that although we consider very low temperatures, our study is restricted to the very close vicinity of the threshold while the usual scaling obtained for creep motion corresponds to driving forces far below the threshold. Note in addition that at very low temperature, the time scales of depinning become very large and that it may be necessary to take other mechanisms into account. This is for example the case for crack propagation in glass for which the subcritical motion of the front competes with a stress assisted ion interdiffusion phenomenon which tends to locally reinforce the material [10]. Within these restrictions, our knowledge of the depinning force distribution gives us the evolution of the front velocity with temperature by way of the front roughness, thus relating a dynamical behavior to a geometrical property.

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