On functionals involving the torsional rigidity related to some classes of nonlinear operators

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March 12, 2018

ABSTRACT: In this paper we study optimal estimates for two functionals involving the anisotropic $p$-torsional rigidity $T_p(\Omega)$, $1 < p < +\infty$. More precisely, we study $\Phi(\Omega) = T_p(\Omega) |\Omega|^{-1} M(\Omega)$ and $\Psi(\Omega) = T_p(\Omega) |\Omega|^{-1} R_F(\Omega)^{p-1}$, where $M(\Omega)$ is the maximum of the torsion function $u_\Omega$ and $R_F(\Omega)$ is the anisotropic inradius of $\Omega$.

KEYWORDS: torsional rigidity, anisotropic operators, optimal estimates

MSC 2010: 49Q10, 35J25

1 INTRODUCTION

Let $F : \mathbb{R}^N \to [0, +\infty[, N \geq 2$, be a convex, even, 1-homogeneous and $C^3(\mathbb{R}^N \setminus \{0\})$ function such that $[F^p]_{L^p}$ is positive definite in $\mathbb{R}^N \setminus \{0\}$, $1 < p < +\infty$. The anisotropic $p$-laplacian is the operator defined by

$$\Omega_p u := \sum_{i=1}^N \frac{\partial}{\partial x_i} (F(\nabla u)^{p-1} F_{x_i}(\nabla u)).$$

For $p = 2$, $Q_2$ is the so-called Finsler Laplacian, while when $F(\xi) = |\xi|$ is the Euclidean norm, $Q_p$ reduces to the well known $p$-Laplace operator.

Given a bounded domain $\Omega$ in $\mathbb{R}^N$, let us consider the torsion problem for $\Omega_p$:

$$\begin{cases} -\Omega_p u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega. \end{cases}$$

(1)

The anisotropic $p$-torsional rigidity of $\Omega$ is the number $T_p(\Omega) > 0$ defined by

$$T_p(\Omega) = \int_\Omega F(\nabla u_\Omega)^p \, dx = \int_\Omega u_\Omega \, dx,$$

where $u_\Omega \in W^{1,p}_0(\Omega)$ is the torsion function, that is the unique solution of (1).
The main aim of the paper is the study of optimal estimates for the following two functionals involving \( T_p(\Omega) \):

\[
\Phi(\Omega) = \frac{T_p(\Omega)}{|\Omega| M(\Omega)}, \quad \Psi(\Omega) = \frac{T_p(\Omega)}{|\Omega| R_F(\Omega)^q}.
\]

Here and after we will denote by \( q \) the Hölder conjugate of \( p, q = \frac{p}{p-1} \), by \( M(\Omega) \) the maximum of the torsion function \( u_\Omega \) and by \( R_F(\Omega) \) the anisotropic inradius of \( \Omega \) (see Section 2 for the precise definitions). Observe that the functionals \( \Phi \) and \( \Psi \) are scaling invariant with respect to the domain. Indeed:

\[
T_p(t\Omega) = t^{N+q}T_p(\Omega), \quad |t\Omega| = t^N|\Omega|, \quad M(t\Omega) = t^QM(\Omega), \quad R_F(t\Omega) = tR_F(\Omega).
\]

Our main result is the following.

**Theorem 1.1.** Let \( \Omega \) be a convex bounded domain in \( \mathbb{R}^N \). It holds that

i) \[
\frac{q}{Nq-1(N+q)} \leq \Phi(\Omega) \leq \frac{q}{q+1}.
\]

The right-hand side inequality is optimal for a suitable sequence of thinning rectangles.

ii) \[
\frac{1}{Nq-1(N+q)} \leq \Psi(\Omega) \leq \frac{1}{q+1}.
\]

The left-hand side inequality holds as an equality if and only if \( \Omega \) is a Wulff shape, that is a ball in the dual norm \( F^* \); the right-hand side inequality is optimal for a suitable sequence of thinning rectangles.

When \( F = E \) is the Euclidean norm, there is a wide literature on sharp estimates for \( T_p(\Omega) \) related to several geometrical quantities depending on \( \Omega \). For example, in the classical case of the torsional rigidity for the Laplace operator (\( p = 2 \)), with \( N = 2 \), it is known that

\[
\frac{1}{8} \leq \Psi(\Omega) = \frac{T_2(\Omega)}{R_E^2|\Omega|} \leq \frac{1}{3},
\]

where \( R_E(\Omega) \) is the standard Euclidean inradius of \( \Omega \). The left-hand side inequality is due to Pólya and Szegő (see [PZ]), while the right-hand side inequality was proved by Makai in [M].

As regards the case \( p \neq 2 \), in [FGL], among other results, estimates for \( \Psi(\Omega) \) are given in the planar case, obtaining an upper bound and a sharp lower bound. In the anisotropic case, in [BGM] the estimates in ii) are proved for \( p = 2 \).

As regards the functional \( \Phi(\Omega) \), up to our knowledge, it seems that the only known result is in the Euclidean case for \( p = 2 \). Indeed, in [HLP] the authors prove the following estimates:

\[
\frac{1}{(N+1)^2} \leq \Phi(\Omega) = \frac{T_2(\Omega)}{|\Omega| M(\Omega)} \leq \frac{2}{3}.
\]

Moreover, they show the optimality of the upper bound, while they conjecture that the lower bound is not optimal, and that the sharp constant in the plane is \( \frac{1}{3} \), achieved on a sequence of thinning isosceles triangles. In our result, we improve the constant \( (N+1)^{-2} \), replacing it with \( [N(N+2)]^{-1} \). Anyway, we believe that \( [N(N+2)]^{-1} \) is not optimal, and for \( N = 2 \) we show that there is a sequence of thinning isosceles triangles \( \tau_n \) such that \( \Phi(\tau_n) \to \frac{1}{3} \).

In order to prove our main result, among the main tools involved, the following estimate for the maximum \( M(\Omega) \) of the torsion function \( u_\Omega \) plays a key role.
Theorem 1.2. Let $\Omega$ be a bounded convex domain in $\mathbb{R}^N$, $N \geq 2$, and let $R_F(\Omega)$ the anisotropic inradius of $\Omega$. Let $u_\Omega$ be the solution of (1). For $1 < p < +\infty$ it holds that

$$\frac{R_F^q(\Omega)}{qN} \leq M(\Omega) \leq \frac{R_F^q(\Omega)}{q}. \tag{2}$$

The right-hand side inequality is optimal for a suitable sequence of thinning $N$-rectangular domains. The other inequality holds as an equality if and only if $\Omega$ is the Wulff shape $W_R(x_0)$. 

The upper bound in (2) has been proved in [P] in the Euclidean case for $p = 2$, $N = 2$ (see also [S]), by using a $P$-function computation and a maximum principle. Anyway, many other estimates for the torsion function are known; the interested reader can refer, for example, to [vB, BFr, HLP] and the reference therein contained. We prove inequality (2) generalizing the $P$-function technique to the case $1 < p < +\infty$, and in the anisotropic case.

Finally, we recall that in the Euclidean case, several other estimates for the $p$-torsional rigidity, involving different geometrical quantities, are known (for the Euclidean case, see for instance [vBBV, vBFNT, S] $(p = 2)$, [FGL] $(1 < p < +\infty)$, and [DG1] for the anisotropic case $(1 < p < +\infty)$).

The paper is organized as follows. In Section 2 we fix some notation and recall preliminary results about Finsler metrics and the anisotropic $p$-torsional rigidity. In Section 3 we prove Theorem 1.2 by using the $P$-function method. Finally, in Section 4 we give the proof of the main Theorem 1.1. We will split it in several partial results.

2 NOTATION AND PRELIMINARIES

Throughout the paper we will consider a convex even $1$-homogeneous function

$$\xi \in \mathbb{R}^N \mapsto F(\xi) \in [0, +\infty[,$$

that is a convex function such that

$$F(t\xi) = |t|F(\xi), \quad t \in \mathbb{R}, \xi \in \mathbb{R}^N, \tag{3}$$

and such that

$$a|\xi| \leq F(\xi), \quad \xi \in \mathbb{R}^N, \tag{4}$$

for some constant $a > 0$. The hypotheses on $F$ imply there exists $b \geq a$ such that

$$F(\xi) \leq b|\xi|, \quad \xi \in \mathbb{R}^N. \tag{5}$$

Moreover, throughout the paper we will assume that $F \in C^{3,\beta}(\mathbb{R}^N \setminus \{0\})$, and

$$[F^p]_{\xi \xi}(\xi)$$

is positive definite in $\mathbb{R}^N \setminus \{0\}, \tag{6}$$

with $1 < p < +\infty$.

The hypothesis (6) on $F$ ensures that the operator

$$\Omega_p u := \text{div} \left( \frac{1}{p} \nabla \xi [F^p](\nabla u) \right)$$

is elliptic, hence there exists a positive constant $\gamma$ such that

$$\frac{1}{p} \sum_{i,j=1}^n \nabla_{\xi_i \xi_j}^2 [F^p](\eta)\xi_i \xi_j \geq \gamma|\eta|^{p-2}|\xi|^2,$$

for some positive constant $\gamma$, for any $\eta \in \mathbb{R}^n \setminus \{0\}$ and for any $\xi \in \mathbb{R}^n$. 


\textbf{Remark 2.1.} We stress that for \( p \geq 2 \) the condition

\[ \nabla^2_\xi[F^2](\xi) \text{ is positive definite in } \mathbb{R}^N \setminus \{0\}, \]

implies (6).

The polar function \( F^\circ : \mathbb{R}^N \to [0, +\infty] \) of \( F \) is defined as

\[ F^\circ(v) = \sup_{\xi \neq 0} \frac{\langle \xi, v \rangle}{F(\xi)}. \]

It is easy to verify that also \( F^\circ \) is a convex function which satisfies properties (3) and (4). Furthermore,

\[ F(v) = \sup_{\xi \neq 0} \frac{\langle \xi, v \rangle}{F^\circ(\xi)}. \]

From the above property it holds that

\[ |\langle \xi, \eta \rangle| \leq F(\xi)F^\circ(\eta), \quad \forall \xi, \eta \in \mathbb{R}^N. \quad (7) \]

The set

\[ \mathcal{W} = \{ \xi \in \mathbb{R}^N : F^\circ(\xi) < 1 \} \]

is the so-called Wulff shape centered at the origin. We put \( \kappa_n = |\mathcal{W}| \), where \( |\mathcal{W}| \) denotes the Lebesgue measure of \( \mathcal{W} \). More generally, we denote with \( \mathcal{W}_r(x_0) \) the set \( r\mathcal{W} + x_0 \), that is the Wulff shape centered at \( x_0 \) with measure \( \kappa_n r^n \), and \( \mathcal{W}_r(0) = \mathcal{W}_r \).

We observe that \( F \) is the support function of \( \mathcal{W} \). In general for a nonempty closed convex set \( K \subset \mathbb{R}^N \), the support function \( h_K \) is defined by

\[ h_K(x) := \sup\{\langle x, \xi \rangle, \xi \in K\}, \quad \text{for } x \in \mathbb{R}^N. \quad (8) \]

The following properties of \( F \) and \( F^\circ \) hold true:

\[ \langle F^\circ(\xi), \xi \rangle = F(\xi), \quad \langle F^\circ(\xi), \xi \rangle = F^\circ(\xi), \quad \forall \xi \in \mathbb{R}^N \setminus \{0\} \]

(9)

\[ F(F^2(\xi)) = F^\circ(F^2(\xi)) = 1, \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}, \]

(10)

\[ F^\circ(\xi)F(\xi) = F(\xi)F^\circ(\xi) = \xi, \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}. \]

(11)

\subsection{2.1 Anisotropic mean curvature}

Let \( \Omega \) be a \( C^2 \) bounded domain, and \( \nu(x) \) be the unit outer normal at \( x \in \partial \Omega \), and let \( \mathbf{u} \in C^2(\overline{\Omega}) \) such that \( \Omega_t = \{ u > t \} \), \( \partial \Omega_t = \{ u = t \} \) and \( \nabla u \neq 0 \) on \( \partial \Omega_t \). The anisotropic outer normal \( n_F \) to \( \partial \Omega_t \) is given by

\[ n_F(x) = F^\circ(\nu(x)) = F^\circ(-\nabla u), \quad x \in \partial \Omega. \]

It holds

\[ F^\circ(n_F) = 1. \]

The anisotropic mean curvature of \( \partial \Omega_t \) is defined as

\[ \mathcal{H}_F(x) = \text{div}(n_F(x)) = \text{div}[\nabla \xi F(-\nabla u(x))], \quad x \in \partial \Omega_t. \quad (12) \]
It holds that
\[
\frac{\partial u}{\partial n_F} = \nabla u \cdot F_{\xi}(-\nabla u) = -F(\nabla u). \tag{13}
\]
In [X] it has been proved that for a smooth function \(u\), on its level sets \(\{u = t\}\) it holds
\[
\Omega_2 u = \frac{\partial u}{\partial n_F} J_F + \frac{\partial^2 u}{\partial n_F^2}, \tag{14}
\]
where \(\frac{\partial u}{\partial n_F} = \nabla u \cdot n_F\). In the next result we generalize (14) for \(\Omega_p u\).

**Proposition 2.2.** Let \(u\) be a \(C^2(\overline{\Omega})\) function with a regular level set \(\partial \Omega_t\). Then we have
\[
\Omega_p u = F^{p-2}(\nabla u) \left( \frac{\partial u}{\partial n_F} J_F + (p - 1) \frac{\partial^2 u}{\partial n_F^2} \right), \tag{15}
\]
where \(J_F\) is the anisotropic mean curvature of \(\partial \Omega_t\) as defined in (12).

**Proof.** By definition of \(\Omega_p\), (14) and (13), we have
\[
\Omega_p u = \text{div} \left( F^{p-2}(\nabla u) F(\nabla u) F_{\xi}(\nabla u) \right) \\
= F^{p-2}(\nabla u) \left( \Omega_2 u + (p - 2) F_{\xi_i}(\nabla u) F_{\xi_j}(\nabla u) u_{x_i x_j} \right) \\
= F^{p-2}(\nabla u) \left( \frac{\partial u}{\partial n_F} J_F + \frac{\partial^2 u}{\partial n_F^2} + (p - 2) F_{\xi_i}(\nabla u) F_{\xi_j}(\nabla u) u_{x_i x_j} \right) \\
= F^{p-2}(\nabla u) \left( \frac{\partial u}{\partial n_F} J_F + (p - 1) \frac{\partial^2 u}{\partial n_F^2} \right),
\]
that is the thesis. \(\square\)

Finally we recall the definition of the anisotropic distance from the boundary and the anisotropic inradius.

Let us consider a domain \(\Omega\), that is a connected open set of \(\mathbb{R}^N\), with non-empty boundary. The anisotropic distance of \(x \in \Omega\) to the boundary of \(\partial \Omega\) is the function
\[
d_F(x) = \inf_{y \in \partial \Omega} F^0(x - y), \quad x \in \overline{\Omega}.
\]
We stress that when \(F = |\cdot|\) then \(d_F = d_E\), the Euclidean distance function from the boundary.

It is not difficult to prove that \(d_F\) is a uniform Lipschitz function in \(\overline{\Omega}\) and, using the property of \(F\) we have
\[
F(\nabla d_F(x)) = 1 \quad \text{a.e. in } \Omega.
\]
Obviously, assuming \(\sup_{\Omega} d_F < +\infty\), \(d_F \in W^{1,\infty}_0(\Omega)\) and the quantity
\[
R_F(\Omega) = \sup \{d_F(x), \ x \in \Omega\}, \tag{16}
\]
is called anisotropic inradius of \(\Omega\).

For further properties of the anisotropic distance function we refer the reader to [CM].
2.2 Anisotropic p-torsional rigidity

In this subsection we summarize some properties of the anisotropic p-torsional rigidity. We refer the reader to [DG1] for further details.

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, and $1 < p < +\infty$. Throughout the paper we will denote by $q$ the Hölder conjugate of $p$,

$$q := \frac{p}{p-1}.$$  

Let us consider the torsion problem for the anisotropic p-Laplacian

$$\begin{cases}
-\Omega_p u := -\text{div} \left( F^{p-1}(\nabla u)F_\xi(\nabla u) \right) = 1 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases} \quad (17)$$

By classical result there exists a unique solution of (17), that we will always denote by $u_\Omega$, which is positive in $\Omega$. Moreover, by (6) and being $F \in C^3(\mathbb{R}^n \setminus \{0\})$, then $u_\Omega \in C^{1,\alpha}(\Omega) \cap C^3(\{\nabla u_\Omega \neq 0\})$ (see [LU, To]).

In view of the above considerations, we define the p-torsional anisotropic rigidity of $\Omega$ the number $T_p(\Omega) > 0$ such that

$$T_p(\Omega) = \int_\Omega F(\nabla u_\Omega)^p \, dx = \int_\Omega u_\Omega \, dx. \quad (18)$$

A characterization of $T_p$ is provided by the equality $T_p(\Omega) = \sigma(\Omega)^{\frac{1}{p-1}}$, where $\sigma(\Omega)$ is the best constant in the Sobolev inequality

$$\|\varphi\|_{L^p(\Omega)}^p \leq \sigma(\Omega) \|F(\nabla \varphi)\|_{L^p(\Omega)}^p,$$

that is

$$T_p(\Omega)^{p-1} = \sigma(\Omega) = \max_{\psi \in W^{1,p}_0(\Omega) \setminus \{0\}} \left( \frac{\int_\Omega |\psi| \, dx}{\int_\Omega F(\nabla \psi)^p \, dx} \right)^p \quad (19)$$

and the solution $u_\Omega$ of (17) realizes the maximum in (19).

It is immediate to see that if $\Omega \subset \tilde{\Omega}$, then

$$T_p(\Omega) \leq T_p(\tilde{\Omega}). \quad (20)$$

Moreover, by the maximum principle it holds that

$$M(\Omega) \leq M(\Omega), \quad \text{(21)}$$

where $M(\Omega)$ is the maximum of the torsion function in $\Omega$.

A consequence of the anisotropic Pólya-Szegő inequality (see [AFLT]) is the following upper bound for $T_p(\Omega)$ in terms of the measure of $\Omega$.

**Theorem 2.3.** Let $\Omega$ be a bounded open set of $\mathbb{R}^N$. Then,

$$T_p(\Omega) \leq T_p(W_R), \quad (22)$$

where $W_R$ is the Wulff shape centered at the origin with the same Lebesgue measure as $\Omega$. 

Remark 2.4. If $\Omega = W_R$, by the symmetry of the problem, $T_p(W_R)$ and the solution $u$ of (17) can be explicitly calculated. We have:

$$u_W(x) = \frac{R^q - F^p(x)^q}{qNq-1} \quad \text{and} \quad T_p(W_R) = \frac{1}{Nq-1} |W_R| \frac{|W_R|}{N + qR^q}. \quad (23)$$

Remark 2.5. We point out that the lower bound in statement ii) of Theorem 1.1 gives a stability type inequality for (22). Indeed we have

$$0 \leq T_p(W_R) - T_p(\Omega) \leq \frac{1}{8} (R^2 - R_F(\Omega)), \quad \text{where } |W_R| = |\Omega|.$$

3 AN ESTIMATE OF THE MAXIMUM OF THE TORSION FUNCTION

In order to give a sharp upper bound for the maximum $M(\Omega)$ of the torsion function $u_\Omega$, we will take into account the following $P$-function:

$$P(x) = \frac{p-1}{p} F^p(\nabla u_\Omega) + u_\Omega - M(\Omega),$$

where $M(\Omega) = \max_\Omega u_\Omega$. The following result is proved in [CFV].

Proposition 3.1. Let $\Omega$ be a domain in $\mathbb{R}^N$, $N \geq 2$, and $u_\Omega \in W^{1,p}_0(\Omega)$ be a solution of (17). Set

$$d_{ij} := \frac{1}{F(\nabla u_\Omega)} \frac{\partial^2}{\partial x_i \partial x_j} \left[ \frac{F^p}{p} \right] (\nabla u_\Omega),$$

Then it holds that

$$(d_{ij}P_i)_{ij} - b_k P_k \geq 0 \quad \text{in } \{\nabla u_\Omega \neq 0\}$$

where

$$b_k = \frac{p-2}{F^3(\nabla u_\Omega)} F_x^p(\nabla u_\Omega) P_x F_x^k(\nabla u_\Omega) + \frac{2p-3}{F^2(\nabla u_\Omega)} \left( \frac{F_{x_k}^p(\nabla u_\Omega) P_{x_k} F_{x_k}^p(\nabla u_\Omega)}{p-1} - F_{x_k} F_x(\nabla u_\Omega) \right)$$

As a consequence of the previous result we get the following maximum principle for $P$.

Theorem 3.2. Let $\Omega$ be a bounded $C^2$ domain in $\mathbb{R}^N$, $N \geq 2$, with nonnegative anisotropic mean curvature $H_F \geq 0$ on $\partial \Omega$, and $u_\Omega$ the torsion function. Then

$$\frac{p-1}{p} F^p(\nabla u_\Omega) + u_\Omega \leq M(\Omega) \quad \text{in } \Omega,$$

that is the function $P$ achieves its maximum at the points $x_M \in \Omega$ such that $u_\Omega(x_M) = M(\Omega)$.

Proof. Let us denote by $E_{u_\Omega}$ the set of the critical points of $u_\Omega$, that is $E_{u_\Omega} = \{x \in \Omega : \nabla u_\Omega(x) = 0\}$. Being $\partial \Omega$ $C^2$, by the Hopf Lemma (see for example [CT]), $E_{u_\Omega} \cap \partial \Omega = \emptyset$.

Applying Proposition 3.1, the function $P$ verifies a maximum principle in the open set $\Omega \setminus E_{u_\Omega}$. Then we have

$$\max_{\Omega \setminus E_{u_\Omega}} P = \max_{\partial(\Omega \setminus E_{u_\Omega})} P.$$ 

Hence one of the following three cases occur
1. the maximum point of $\mathcal{P}$ is on $\partial \Omega$;
2. the maximum point of $\mathcal{P}$ is on $E_u$;
3. the function $\mathcal{P}$ is constant in $\overline{\Omega}$.

In order to prove the theorem we have to show that statement 1 cannot happen. Let us compute the derivative of $\mathcal{P}$ in the direction of the anisotropic normal $n_F$, in the sense of (13). Hence we get

$$\frac{\partial \mathcal{P}}{\partial n_F} = \frac{p-1}{p} \frac{\partial}{\partial n_F} \left( -\frac{\partial u_\Omega}{\partial n_F} \right)^p + \frac{\partial u_\Omega}{\partial n_F} = -(p-1) \left( -\frac{\partial u}{\partial n_F} \right)^p \frac{\partial^2 u_\Omega}{\partial n_F^2} + \frac{\partial u_\Omega}{\partial n_F} =$$

$$= -F(\nabla u_\Omega)Q[u] - F^{p-1}(\nabla u_\Omega)J_F - F(\nabla u_\Omega) = -F^{p-1}(\nabla u_\Omega)J_F, \quad (24)$$

where last identity follows by (15). On the other hand, if a maximum point $\bar{x}$ of $\mathcal{P}$ is on $\partial \Omega$, by Hopf Lemma either $\mathcal{P}$ is constant in $\overline{\Omega}$, or $\frac{\partial \mathcal{P}}{\partial n_F}(\bar{x}) > 0$. Hence being $J_F \geq 0$ we have a contradiction. \hfill $\square$

As a consequence of the previous result we get the following optimal estimate for the maximum of $u_\Omega$.

**Theorem 3.3.** Let $\Omega$ be a bounded convex domain in $\mathbb{R}^N$, $N \geq 2$, and $1 < p < +\infty$. It holds that

$$\frac{R_F^p(\Omega)}{qN^{q-1}} \leq M(\Omega) \leq \frac{R_F^p(\Omega)}{q}. \quad (25)$$

**Remark 3.4.** In the next section we will show that the right-hand side inequality in (25) is optimal on a suitable sequence of thinning rectangles (see Proposition 4.4 and (36)). We stress that, in general, the quotient $\frac{R_F^p(\Omega)}{qM(\Omega)}$ approaches the value 1 also for different sequences of sets (see the example 4.7).

**Proof.** The left-hand side inequality of (25) follows by (21) and (23). Hence, let us prove the other inequality.

First of all, suppose that $\Omega$ is a $C^2$, strictly convex domain. Let $v$ be a direction in $\mathbb{R}^N$. By Theorem 3.2 and property (7) we have

$$\frac{d\partial u_\Omega}{dv} = (\nabla u, v) \leq F(\nabla u_\Omega)F^0(v) \leq \left[ \frac{p}{p-1} (M(\Omega) - u_\Omega) \right]^\frac{1}{p} F^0(v), \quad (26)$$

where $M(\Omega)$ is the maximum of $u_\Omega$ in $\overline{\Omega}$. Let us denote by $x_M$ the point of $\Omega$ such that $M(\Omega) = u_\Omega(x_M)$, by $\bar{x} \in \partial \Omega$ such that $F^0(x_M - \bar{x}) = d_F(x_M)$ and by $v$ the direction of the straight line joining the points $x_m$ and $\bar{x}$. Then by (26) we get

$$\int_{u(\bar{x})}^{M(\Omega)} \frac{1}{(M(\Omega) - u_\Omega)^\frac{1}{p}} du \leq \left( \frac{p}{p-1} \right)^\frac{1}{p} F^0(v)|\bar{x} - x_M| = \left( \frac{p}{p-1} \right)^\frac{1}{p} F^0(\bar{x} - x_M).$$

Being $F^0(\bar{x} - x_M) \leq R_F(\Omega)$, we get

$$\left( \frac{p}{p-1} \right)^\frac{p-1}{p} \leq \left( \frac{p}{p-1} \right)^\frac{1}{p} R_F(\Omega),$$

which gives the estimate (25) for smooth convex domains. To prove the estimate in the case of a general convex body $\Omega$, we proceed by approximation. It is well-known (see for example [BF]) that a convex body $\Omega$ can be approximated in the Hausdorff distance by an increasing sequence of smooth strictly convex bodies $\Omega_n \subseteq \Omega$. Clearly, $R_F(\Omega_n) / R_F(\Omega)$. 


Let \( u_n \geq 0 \) be the torsion function in \( \Omega_n \). In order to conclude the proof we have to show that \( M(\Omega_n) \to M(\Omega) \) as \( n \to \infty \). We first observe that by (25),

\[
M(\Omega_n) \leq \frac{R_F^q(\Omega_n)}{q} \leq \frac{R_F^q(\Omega)}{q},
\]

hence \( u_n \) are bounded in \( L^\infty(\Omega_n) \). Furthermore, applying Theorem 3.2 in \( \Omega_n \) we have

\[
\frac{p-1}{p}F^p(\nabla u_n) + u_n \leq M(\Omega_n) \quad \text{in } \overline{\Omega}_n.
\]

Then by property (5)

\[
|\nabla u_n| \leq C \quad \text{in } \overline{\Omega}_n.
\]

Hence by (27) and (28), using Ascoli-Arzelà theorem we get that \( u_n \to u_\Omega \) uniformly in \( \Omega \) and this allows to pass to the limit in (27) and the proof is completed.

\[\square\]

**Remark 3.5.** We point out that if we take \( \Omega \) smooth, the thesis of Theorem 3.3 holds if we assume only that the anisotropic mean curvature of \( \Omega \) is nonnegative.

## 4 Proof of Theorem 1.1

We split the proof in various theorems. We first prove the lower bound for \( \Psi(\Omega) \) in \( \ii \).

**Theorem 4.1.** If \( \Omega \subset \mathbb{R}^N \) is a convex bounded domain, \( N \geq 2 \), and \( 1 < p < +\infty \), then

\[
\frac{T_p(\Omega)}{|\Omega|} \geq \frac{1}{N^{q-1}} \frac{1}{N+q} R_F(\Omega)^q,
\]

where \( R_F(\Omega) \) is the anisotropic inradius of \( \Omega \) defined in (16). Moreover the equality holds when \( \Omega \) is a Wulff shape.

**Proof.** Let us assume first that \( \Omega \) is a strictly convex domain and then we remove this assumption with a proof that follows by approximation as in Theorem 3.3. Let us consider as test function into (19) the following

\[
\varphi(x) = \frac{1 - \mathcal{K}^o(x)^q}{qN^{q-1}}
\]

where \( \mathcal{K}^o \) is the support function of the polar set of \( \Omega \), defined in (8). Then \( \Omega = \{ \mathcal{K}^o < 1 \} \). By (23), we observe that when \( \Omega = W \) then \( \varphi \) is exactly the torsion function of the Wulff shape. We start computing

\[
\int_\Omega \mathcal{K}^o(x)^q \ dx = \int_0^\infty \int_{\mathcal{K}^o=t} \frac{t^q}{|\nabla \mathcal{K}^o(x)|} \ d\mathcal{H}^{N-1} \ dt = \int_0^\infty t^q \int_{\mathcal{K}^o=t} \frac{\mathcal{K}(\nabla \mathcal{K}^o(x))}{|\nabla \mathcal{K}^o|} \ d\mathcal{H}^{N-1} \ dt,
\]

\[
= \frac{1}{N+q} \int_{\partial \Omega} \frac{\mathcal{K}(\nabla \mathcal{K}^o(x))}{N+q} \ d\mathcal{H}^{N-1} \ dt = \frac{1}{N+q} \int_{\partial \Omega} \mathcal{K}(\nabla \mathcal{K}^o(x)) \ d\mathcal{H}^{N-1} \ dt
\]

where \( \mathcal{X} = (\mathcal{K}^o)^o \). Then we have

\[
\left( \int_\Omega \varphi \ dx \right)^p = \frac{|\Omega|^p}{N^q(N+q)^p} = \left( \frac{|\Omega|}{N^q(N+q)^p} \right)^p.
\]
Let us now compute
\[ \int_\Omega F^p(\nabla \varphi) \, dx = \frac{1}{q^p N^q} \int_\Omega F^p(\nabla K^0(x))^q \, dx = \frac{1}{N^q} \int_0^1 t^p(q-1) \int_{K^0=t} \frac{F^p(\nabla K^0)}{|\nabla K^0|} \, d\mathcal{C}^{N-1} \, dt \]
\[ = \frac{1}{N^q} \int_0^1 t^{q+N-1} \int_{\partial \Omega} |\nabla K^0|^{p-1} F^p(y_\Omega) \, d\mathcal{C}^{N-1} \, dt \]
\[ = \frac{1}{N^q} \int_0^1 t^{q+N-1} \int_{\partial \Omega} F^p(y_\Omega) \, d\mathcal{C}^{N-1} \, dt, \]
where last equality follows by the identity \( K(\nabla K^0(x)) = 1 \). Being \( W_{R_F(\Omega)} \subseteq \Omega \), it follows that \( K(x) \geq R_F(\Omega)F(x) \), so we have
\[ \int_\Omega F^p(\nabla \varphi) \, dx \leq \frac{1}{(N + q) N^q} \cdot \frac{1}{R_F(\Omega)^p} \int_{\partial \Omega} K(y_\Omega(x)) \, d\mathcal{C}^{N-1} = \]
\[ = \frac{1}{(N + q) N^q - 1} \cdot \frac{|\Omega|}{R_F(\Omega)^p}. \quad (31) \]

Joining together (30) and (31), we have the thesis.

Now we prove the validity of (29) without the assumption on the strict convexity of the domain \( \Omega \). As in the proof of Theorem 3.3, let \( \Omega_n \) be a sequence of smooth strictly convex bodies such that \( \Omega_n \to \Omega \). Such a convergence ensures that, as \( n \to \infty \),
\[ |\Omega_n| \to |\Omega| \quad \text{and} \quad R_F(\Omega_n) \to R_F(\Omega). \quad (32) \]
By (20), it follows that
\[ T_p(\Omega) \geq T_p(\Omega_n), \]
and by applying (29) to each \( \Omega_n \), we find
\[ T_p(\Omega) \geq \frac{|\Omega_n|}{N^q - 1(N + q)} R_F(\Omega_n)^q, \]
which, combined with (32), gives the desired result. Finally we stress that if \( \Omega \) is a Wulff shape, the equality case follows from Remark 2.4. On the other hand, if the equality holds in (29), then equality must hold in (31), and then \( K(x) = R_F(\Omega)F(x) \), which implies \( \Omega = W_{R_F} \). \( \square \)

Let us consider the functional
\[ \Phi(\Omega) = \frac{T_p(\Omega)}{|\Omega|M(\Omega)}. \quad (33) \]
As consequence of theorems 4.1 and 3.3, we can prove the following estimates for (33) which is statement i) of Theorem 1.1.

**Theorem 4.2.** For any bounded convex domain \( \Omega \subset \mathbb{R}^N \), \( N \geq 2 \), \( 1 < p < +\infty \) it holds that
\[ \frac{q}{N^q - 1(N + q)} \leq \Phi(\Omega) \leq \frac{q}{q + 1}. \quad (34) \]
**Proof.** We first prove the lower bound for the functional \( \Phi \). By (29) and (25) we have
\[ \Phi(\Omega) = \frac{T_p(\Omega)}{|\Omega|M(\Omega)} \geq \frac{q}{N^q - 1(N + q)}, \]
which gives the lower bound in (34).
In order to prove the inequality in the right-hand side in (34), by Theorem 3.2 we have
\[
\frac{p-1}{p} F^p(\nabla u_\Omega) + u_\Omega \leq M(\Omega) \quad \text{in } \Omega.
\]
Integrating in both sides and recalling (18), we get
\[
\left( \frac{p-1}{p} + 1 \right) T_p(\Omega) \leq M(\Omega) |\Omega|,
\]
which implies the upper bound in (34).

\[\square\]

In the following last result we prove the upper bound in statement ii) of Theorem 1.1, which follows immediately by the preceding results. We stress that in the anisotropic setting, the case \( p = 2 \) was previously considered in [BGM] with a completely different proof.

**Theorem 4.3.** Let \( \Omega \in \mathbb{R}^N \) be a bounded convex domain, \( N \geq 2, 1 < p < +\infty \). It holds that
\[
\frac{T_p(\Omega)}{|\Omega|} \leq \frac{R_F(\Omega)^q}{q + 1}.
\]

**Proof.** By the right-hand side inequality in (34), and (25), we have
\[
\frac{T_p(\Omega)}{|\Omega|} \leq \frac{q}{q + 1} M(\Omega) \leq \frac{R_F(\Omega)^q}{q + 1}.
\]

\[\square\]

The final part of the section is devoted to prove the optimality of (35). As a consequence, by (36) this will give the optimality of the right-hand side inequality of (34), and of (25).

**Proposition 4.4.** Let \( \Omega_\varepsilon \) be the N-rectangle \( ] - \varepsilon, \varepsilon [ \times ] - a_2, a_2 [ \times \ldots \times ] - a_N, a_N [ \), and suppose that \( R_F(\Omega) = \varepsilon F^0(e_1) \). Then
\[
\lim_{\varepsilon \to 0^+} \frac{T_p(\Omega_\varepsilon)}{(R_F(\Omega_\varepsilon))^q |\Omega_\varepsilon|} = \frac{1}{q + 1}.
\]

The hypothesis \( R_F(\Omega) = \varepsilon F^0(e_1) \) is not restrictive, in the sense that if it is not true we can choose a rotated N-rectangle where \( R_F(\Omega) = \varepsilon F^0(\nu) \) for some direction \( \nu \), and use the remark below.

**Remark 4.5.** If \( A \in SO(N) \) is a rotation matrix, then, denoting by \( F_A(\xi) = F(A\xi) \), it holds that
\[
(F_A)^0(\xi) = (F^0)_A(\xi), \quad \text{and then} \quad R_{F_A}(A\Omega) = R_F(\Omega)
\]
(see [DGP] for the details). Hence, emphasizing the dependence on \( F \) by denoting \( T_p(\Omega) = T_{p,F}(\Omega) \), we have
\[
T_{p,F}(\Omega) = \max_{\varphi \in W_{0,p}^0(\Omega)} \left( \int_\Omega |\varphi(x)|dx \right)^p = \max_{\varphi \in W_{0,p}^0(\Omega)} \left( \int_{A\Omega} |\varphi(A^T x)|dx \right)^p = T_{p,F_{A^T}}(A\Omega) \geq \frac{|\Omega|}{q + 1} R_{F_{A^T}}(A\Omega)^q = \frac{|\Omega|}{q + 1} R_F(\Omega)^q.
\]
We notice that both

\[ F^0(e_1) = \frac{1}{F(e_1)}. \]

Indeed, being \( R_F(\Omega) = \varepsilon F^0(e_1) \), it holds that

\[ v_\Omega(\varepsilon e_1) = e_1 = \frac{F^0(e_1)}{|F^0(e_1)|}, \]

where \( v_\Omega(\varepsilon e_1) \) is the Euclidean outer normal vector to \( \partial \Omega \). Hence by (10) and (9), we have

\[ F(e_1) = \frac{1}{|F^0(e_1)|} = \frac{1}{F^0(e_1)}, \]

where last equality follows by \( F^0(e_1) = F^0(e_1) \cdot e_1 = |F^0(e_1)| \).

Let \( \Omega_\varepsilon = C_\varepsilon \cup D_\varepsilon \), where \( C_\varepsilon = [\varepsilon^2, \varepsilon^2+\varepsilon, \varepsilon^2-\varepsilon \times \ldots \times -\varepsilon] + \varepsilon N, a_\varepsilon = \varepsilon N, a_\varepsilon - \varepsilon \], and \( D_\varepsilon = \Omega_\varepsilon \setminus C_\varepsilon \). Setting \( x = (x_1, z) \) with \( z \in \mathbb{R}^{N-1} \) and \( a = (a_2, \ldots, a_N) \), we consider the function \( \varphi_\varepsilon \) defined by

\[
\begin{align*}
\varphi_\varepsilon(x_1, z) &= \frac{\varepsilon^q - x_1^q}{q} \quad \text{in } C_\varepsilon \\
\varphi_\varepsilon(x_1, z) &= \min \{|a-z|, |a-z|\} \frac{\varepsilon^q - x_1^q}{q} \quad \text{in } D_\varepsilon.
\end{align*}
\]

We can estimate the anisotropic p-torsional rigidity by using \( \varphi_\varepsilon \) as test function. We have:

\[
T_p(\Omega_\varepsilon)^{q-1} \geq \left( \int_{\Omega_\varepsilon} \varphi_\varepsilon \right)^p \left( \int_{C_\varepsilon} \varphi_\varepsilon \right) \left( \int_{D_\varepsilon} \varphi_\varepsilon \right) = \left( \int_{C_\varepsilon} F^p(\nabla \varphi_\varepsilon) \right) \left( \int_{D_\varepsilon} F^p(\nabla \varphi_\varepsilon) \right) + \left( \int_{D_\varepsilon} F^p(\nabla \varphi_\varepsilon) \right)
\]

We now compute

\[
\int_{C_\varepsilon} \varphi_\varepsilon \ dx = \int_{C_\varepsilon} \frac{\varepsilon^q - x_1^q}{q} \ dx = \frac{|C_\varepsilon|\varepsilon^q}{q+1}
\]

and

\[
\int_{C_\varepsilon} F^p(\nabla \varphi_\varepsilon) \ dx = F^p(e_1) \int_{C_\varepsilon} x_1^p \ dx = F^p(e_1) \frac{|C_\varepsilon|\varepsilon^q}{q+1}.
\]

We notice that both \( \int_{D_\varepsilon} \varphi_\varepsilon \ dx \) and \( \int_{D_\varepsilon} F^p(\nabla \varphi_\varepsilon) \ dx \) are negligible, since they go to zero as \( \varepsilon^{N+q-1} \). By recalling that

\[
(R_F(\Omega))^q = \varepsilon^q F^0(e_1)^q = \frac{\varepsilon^q}{F(e_1)^q},
\]

we have

\[
\frac{1}{q+1} \geq \lim_{\varepsilon \to 0} \frac{T_p(\Omega_\varepsilon)^q}{|\Omega_\varepsilon|} \geq \frac{1}{q+1}
\]

which concludes the proof. \(\square\)
Remark 4.6. We believe that the lower bound of $\Phi(\Omega)$ in (34) is not optimal. Actually, in the Euclidean setting, with $p = 2$ our bound improves the analogous result of [HLP]:

$$\Phi(\Omega) \geq \frac{2}{N(N+2)} > \frac{1}{(N+1)^2}$$

Moreover in [HLP] the authors conjecture that for $F = \mathcal{E}$, $p = 2$ and $N = 2$ it holds

$$\Phi(\Omega) \geq \frac{1}{3}$$

and

$$\Phi(\Omega_n) \to \frac{1}{3}, \quad \text{as } n \to \infty,$$

where $\Omega_n$ is a sequence of isosceles triangles degenerating to a segment.

In the following example, for $F = \mathcal{E}$, $N = 2$ and $p = 2$, we find a sequence of degenerating triangles $\Omega_n$ such that (37) holds.

Example 4.7. Let

$$N = 2, \quad F(\xi) = \sqrt{\xi_1^2 + \xi_2^2}, \quad p = 2.$$  

We want to show that there exists a sequence of thinning isosceles triangles $\tau_a$ of the plane such that

$$\Phi(\tau_a) = \frac{T_2(\tau_a)}{|\tau_a|M(\tau_a)} \to \frac{1}{3} \quad \text{as } a \to 0,$$

where $T_2(\tau_a)$ is the torsional rigidity of $\tau_a$, $M(\tau_a)$ is the maximum of the torsion function in $\tau_a$ and $|\tau_a|$ is the area of the triangle.

First of all, we recall that by a result contained in [FGL], for any sequence of isosceles triangles $\tau_n$ such that the ratio $R(\tau_n)/w(\tau_n) \to 0$, where $w(\tau_n)$ is the width of $\tau_n$, then

$$\lim_{n \to \infty} \frac{T_2(\tau_n)}{|\tau_n|} \frac{P^2(\tau_n)}{|\tau_n|^2} = \frac{2}{3}.$$  

Hence, recalling that in a triangle it holds that $R(\tau_n) = \frac{2|\tau_n|}{P(\tau_n)}$, then

$$\Phi(\tau_n) = \frac{T_2(\tau_n)}{|\tau_n|} \frac{P^2(\tau_n)}{|\tau_n|^2} \frac{R^2(\tau_n)}{4M(\tau_n)},$$

the result is proved if we find a sequence of triangles with vanishing ratio $R(\tau_n)/w(\tau_n)$ and such that $\frac{R^2(\tau_n)}{4M(\tau_n)}$ tends to 1.

![Figure 1](image-url)

To this aim, let

$$\mathcal{E}_a = \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{1-a^2} + \frac{(y-a)^2}{a^2} = 1 \right\},$$
and consider a point \((a, y_a)\), with \(y_a = a + a\sqrt{\frac{1 - 2a^2}{1 - a^2}}\). Let \(\tau_a\) be the isosceles triangle constructed with one side on the x-axis and with each side tangent to the ellipse at the points \((0, 0), (a, y_a), (-a, y_a)\), as in Figure 1.

The vertices of the triangle are:

\[
V_1 = \left(0, y_a + \frac{a^4}{(1 - a^2)(y_a - a)}\right), \quad V_2 = \left(a + \frac{y_a(y_a - a)}{a^3}(1 - a^2), 0\right), \quad V_3 = -V_2.
\]

Let us observe that \(V_1 \to (0, 0)\) as \(a \to 0\), while the first coordinate of \(V_2\) diverges.

Then, denoting by \(A(\tau_a)\) and \(P(\tau_a)\) respectively the area and the perimeter of \(\tau_a\), and by

\[
h = y_a + \frac{a^4}{(1 - a^2)(y_a - a)}, \quad \frac{b}{2} = a + \frac{y_a(y_a - a)}{a^3}(1 - a^2),
\]

we have:

\[
R(\tau_a) = \frac{2|\tau_a|}{P(\tau_a)} = \frac{bh}{b + \sqrt{2h^2 + b^2}}.
\]

Now, being \(\mathcal{E}_a \subset \tau_a\), by the comparison principle and (25) it holds that

\[
M(\mathcal{E}_a) \leq M(\tau_a) \leq \frac{R^2(\tau_a)}{2}, \quad M(\mathcal{E}_a) = \frac{a^2(1 - a^2)}{2},
\]

where the maximum of the torsion function on \(\mathcal{E}_a\) follows by a direct computation. Then, being \(h = 2a + o(a^2)\) and \(b \to +\infty\) as \(a \to 0\), we have

\[
1 \leq \frac{R(\tau_a)^2}{2M(\mathcal{E}_a)} = \left(\frac{b}{b + \sqrt{2h^2 + b^2}}\right)^2 \frac{h^2}{a^2(1 - a^2)} = \left(\frac{1}{1 + \sqrt{\frac{2h^2}{b^2} + 1}}\right)^2 \frac{h^2}{a^2(1 - a^2)} \to 1
\]

as \(a \to 0\)

and (38) is proved.

We explicitly observe that, from the above computations, it holds

\[
\frac{R^2(\mathcal{E}_a)}{2M(\mathcal{E}_a)} \to 1 \quad \text{as} \quad a \to 0.
\]

**Acknowledgements**

This work has been partially supported by the FIRB 2013 project “Geometrical and qualitative aspects of PDE’s” and by GNAMPA of INdAM.

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