Coarse structure of ultrametric spaces with applications

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Abstract
We show how to decompose all separable ultrametric spaces into a “Lego” combinations of scaled versions of full simplices. To do this we introduce metric resolutions of metric spaces, which describe how a space can be broken up into roughly independent pieces. We use these metric resolutions to define the coarse disjoint union of metric spaces, which provides a way of attaching large scale metric spaces to each other in a “coarsely independent way”. We use these notions to construct universal spaces in the categories of separable and proper metric spaces of asymptotic dimension 0, respectively. In doing so we generalize a similar result of Dranishnikov and Zarichnyi as well as Nagórko and Bell. However, the new application is a universal space for proper metric spaces of asymptotic dimension 0, something that eluded those authors. We finish with a description of some countable groups that can serve as such universal spaces.

Keywords
Asymptotic dimension · Coarse geometry · Ultrametric spaces · Universal spaces

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1 Introduction

Coarse geometry considers metric spaces and abstract spaces from a “large scale” perspective, wherein one pays most attention to the properties that persist through “zooming out” from the object of study while ignoring whatever small scale structure that may exist. Such ideas have existed since the early 20th century but were first explicitly laid out by Gromov [14] where several properties relevant to the coarse geometry of countable groups were clearly defined. Among these there was the asymptotic dimension of a metric space, which was intended to serve as a large scale analog of the classical covering dimension. This particular property would come to some prominence when Yu proved in [16] that the Novikov conjecture is satisfied by groups of finite asymptotic dimension. Such progress and the intuitive appeal of the subject has motivated much of the study into developing the methods of coarse geometry.

In this paper we completely characterize the coarse geometry of a large class of spaces. Specifically separable ultrametric spaces. The breadth of this class of spaces was realized when Brodskiy et al. proved in [9] that every separable metric space of asymptotic dimension 0 is coarsely equivalent to a countable ultrametric space whose metric takes values in the nonnegative integers. Such spaces include all countable locally finite groups (i.e. those where every finitely generated subgroup is finite). We characterize these spaces by showing how to decompose them into several scaled full simplices on certain sets of vertices. This in itself is via two concepts. The first of which is metric resolution (introduced in Sect. 3) which provides a way of distinguishing particular “pieces” of a metric space. The second concept is that of a coarse disjoint union (introduced in Sect. 5), which describes how to put families of metric spaces together into one space such that each metric space used in the construction is “coarsely independent” from the other spaces used in the construction. Our characterization of ultrametric spaces is then performed by distinguishing certain pieces, each of which is a scaled full simplex, of the spaces via metric resolutions, then attaching them together via a coarse disjoint union in such a way that the resulting space is coarsely equivalent to the original. In doing this we find that there are only countably many such scaled full simplices that can be employed which allows us in Sect. 7 to construct universal spaces for the categories of separable and proper metric spaces of asymptotic dimension 0, respectively. In the case of separable spaces of bounded geometry such a universal space has been constructed previously by Dranishnikov and Zarichnyi [10]. In the broader case of separable spaces, this has been done previously by Nagórko and Bell [7]. Banakh and Zarichnyi constructed universal spaces for coarsely homogeneous spaces of asymptotic dimension 0 in [2] (that paper remains unpublished and its content is included in [3–6] combined). The methods employed in these papers differ from the methods we employ. The universal space for proper metric spaces of asymptotic dimension 0 is the first that appears in the literature. The question of whether or not there are universal spaces for proper metric spaces of asymptotic dimension greater than 0 is open and seems to be an interesting question. We finish with some discussion of infinite groups that can be realized as such universal spaces.
2 Preliminaries

We begin with the few basic preliminary definitions needed in subsequent sections.

**Definition 2.1** A metric space \((X, d)\) is called an *ultrametric space* if in place of the usual triangle inequality for metric spaces the metric \(d\) satisfies the stronger *ultrametric triangle inequality* which says that for all \(x, y, z \in X\)

\[
d(x, z) \leq \max \{d(x, y), d(y, z)\}.
\]

**Definition 2.2** Given a set \(D\) of non-negative integers, a *\(D\)-ultrametric space* is an ultrametric space with all distances belonging to \(D\). If \(D\) is the set of all non-negative integers, then a \(D\)-ultrametric space will be called an *integral ultrametric space*.

**Definition 2.3** A metric space \((X, d)\) is said to be of *asymptotic dimension 0* if for every uniformly bounded cover \(\mathcal{U}\) of \(X\) (i.e. one in which the diameters of its elements have a finite upper bound), there is a uniformly bounded cover \(\mathcal{V}\) that is refined by \(\mathcal{U}\) and whose elements are disjoint.

Recall that an *\(r\)-chain* joining \(a, b \in Y\) is a sequence \(x_0 = a, \ldots, x_k = b\) such that \(d(x_i, x_{i+1}) < r\) for each \(i < k\).

Alternatively, one could define for each \(r > 0\) the relation \(\sim_r\) on \(X\) by setting \(x \sim_r y\) if \(d(x, y) < r\). Then say that \(x, y \in X\) are *\(r\)-connected* if there is an \(r\)-chain joining them and define the *\(r\)-component* of \(X\) to be a maximal \(r\)-connected subset of \(X\). A metric space \((X, d)\) is of asymptotic dimension 0 if and only if the collection of \(r\)-components of \(X\) is uniformly bounded for every \(r > 0\). For a more in depth discussion of asymptotic dimension the reader is referred to [15].

**Definition 2.4** A function \(f : (X, d_1) \to (Y, d_2)\) of metric spaces is called:

- **uniformly bornologous** (or *large scale continuous*) if for all \(R > 0\) there is an \(S > 0\) such that if \(d_1(x, y) \leq R\) then \(d_2(f(x), f(y)) \leq S\).
- **proper** if for every bounded \(B \subseteq Y\), \(f^{-1}(B)\) is bounded in \(X\).
- **uniformly proper** if for every \(R > 0\) there is an \(S > 0\) such that if the diameter of \(B \subseteq Y\) is bounded by \(R\), then the diameter of \(f^{-1}(B)\) is bounded by \(S\).
- **coarsely surjective** if there is an \(R > 0\) such that for every \(y \in Y\) there is an \(x \in X\) such that \(d_2(f(x), y) \leq R\).
- **a coarse equivalence** if it is uniformly bornologous, uniformly proper, and coarsely surjective.
- **a coarse embedding** if it is uniformly bornologous and uniformly proper.

The following construction provides a universal construction of ultrametrics for functions \(f : X \to (Y, d_Y)\) from sets to metric spaces:

**Definition 2.5** Given a function \(f : X \to (Y, d)\) from a set to a metric space, the *induced integral ultrametric* \(d^\text{ul}_f(x, y)\) by \(f\) and \(d\) on \(X\) is defined, for \(x \neq y\), as the minimum of all integers \(r \geq 1\) such that there is an \(r\)-chain in \(Y\) joining \(f(x)\) and \(f(y)\). Obviously, \(d^\text{ul}_f(x, y) = 0\) if \(x = y\).
Proposition 2.6 If \( f : (X, d_X) \to (Y, d) \) is surjective large scale continuous and \((Y, d)\) is of asymptotic dimension 0, then \( f : (X, d^{ul}_X) \to (Y, d) \) is a coarse equivalence.

Proof Choose a selection \( s : Y \to X \) for \( f \). Since \( d^{ul}_{id}(x, y) \leq d(f(x), f(y)) + 1 \) for all \( x, y \in X \), the function \( s : (Y, d) \to (X, d^{ul}_X) \) is large scale continuous. If \((Y, d)\) is of asymptotic dimension 0 and there is \( S > 0 \) such that every two points in \( Y \) that can be connected by an \( r \)-chain have to be of distance at most \( S \). That shows \( f : (X, d^{ul}_X) \to (Y, d) \) is a large scale continuous map as well. \( \square \)

Corollary 2.7 Suppose \((X, d)\) is a metric space. The following conditions are equivalent:

1. \((X, d)\) is of asymptotic dimension 0.
2. \( id : (X, d^{ul}_{id}) \to (X, d) \) is large scale continuous.
3. \( id : (X, d) \to (X, d^{ul}_{id}) \) is a coarse equivalence.

The following is a generalization of ultrametric spaces.

Definition 2.8 A metric space \((X, d)\) is called an isosceles space if \( d \) is an isosceles metric, i.e. every triangle in \( X \) is isosceles.

Proposition 2.9 Every isosceles space \((X, d)\) is of asymptotic dimension at most 0.

Proof In view of Corollary 2.7 it suffices to show \( id : (X, d^{ul}_{id}) \to (X, d) \) is large scale continuous. Suppose \( x_0, \ldots, x_k \) is an \( m \)-chain joining \( x \) to \( y \) in \( X \). If \( k \leq 2 \), then obviously \( d(x, y) < 2m \). If \( k > 2 \), then \( d(x_0, x_3) < 2m \) and continuing by induction, we conclude \( d(x, y) < 2m \). Thus, \( id : (X, d^{ul}_{id}) \to (X, d) \) is 2-Lipschitz, hence large scale continuous. \( \square \)

The following result from [9] is the critical observation needed to construct the universal spaces in Sect. 7. Our version is slightly more general and it follows from Corollary 2.7.

Theorem 2.10 If \((X, d_X)\) is a separable metric space of asymptotic dimension zero, then there is a countable integral ultrametric space \((Y, d)\) coarsely equivalent to \((X, d_X)\). Moreover, if \( X \) is proper (in the sense that its bounded subsets have compact closure), then \( Y \) can be chosen to have finite bounded subsets only.

Proof Choose a maximal subset \( Y \) of \( X \) such that \( d_X(x, y) > 1 \) if \( x \neq y \) in \( Y \). Use the integral ultrametric \( d \) on \( Y \) from Corollary 2.7. \( \square \)

3 Metric resolutions

In this section we introduce a formal definition of “Lego” combinations of metric spaces.

Definition 3.1 A metric resolution is a surjective function \( f : X \to S \) of metric spaces so that \( d_X(x, y) = d_S(f(x), f(y)) \) if \( f(x) \neq f(y) \).
Example 3.2 $d^\text{ul}_f$ from Definition 2.5, provided $f : X \to Y$ is surjective, is a metric resolution if $Y$ is given the integral ultrametric induced by $d_Y$.

In applications, metric resolutions will be constructed as indexing functions $i : \bigsqcup_{t \in S} X_t \to S$ (that means $i(x) = t$ if $x \in X_t$) from a disjoint union $\bigsqcup_{t \in S} X_t$ of metric spaces $(X_t, d_t)$ so that there are a metric $d_S$ on $S$ and the metric $d$ on $\bigsqcup_{t \in S} X_t$ having the following properties:

- $d$ restricted to each $X_t$ equals $d_t$,
- $d(x, y) = d_S(u, t)$ if $x \in X_u$, $y \in X_t$, and $u \neq t$.

Proposition 3.3 Suppose $f : X \to S$ is a surjective function, $d_S$ is a metric on $S$, and $d_t$ is a metric on $X_t := f^{-1}(t)$ for each $t \in S$. Define $d : X \times X \to [0, \infty)$ as extending each $d_t$, $t \in S$, and satisfying $d(x, y) = d_S(f(x), f(y))$ if $f(x) \neq f(y)$. $d$ is a metric on $X$ making $f$ a metric resolution if $\text{diam}(X_t) \leq 2 \cdot \inf_{u \neq t} d_S(u, t)$ for each $t \in S$. Moreover, if $(X_t, d_t)$, $t \in S$, are isosceles and $(S, d_S)$ is isosceles, then the metric of $(X, d)$ is isosceles as well. Additionally, if the diameter of each $X_t := f^{-1}(t)$ is at most $\inf_{u \neq t} d_S(u, t)$, and $(X_t, d | X_t)$, $t \in S$, are ultrametrics, then $d$ is an ultrametric as well.

Proof Obviously, there is at most one $d$ that can be considered as the total metric on $X$. Given $x, y, z \in X$, we need to show the Triangle Inequality $d(x, z) \leq d(x, y) + d(y, z)$. It is so if they belong to three different fibers of $f$. Also, the inequality holds if $x, z$ belong to the same fiber of $f$ due to the assumption that $\text{diam}(X_t) \leq 2 \cdot \inf_{u \neq t} d_S(u, t)$ for each $t \in S$. Assume $x \in X_a$, $z \in X_b$, and $a \neq b$. The Triangle Inequality is obvious if $y \in X_a$ or $y \in X_b$ as either $d(x, z) = d(y, z)$ or $d(x, z) = d(x, y)$ in those cases.

It is clear that $d$ is an isosceles metric from the above analysis given the initial metrics $d_S$, $d_t$, $t \in S$, are isosceles as well. If the initial metrics $d_S$, $d_t$, $t \in S$, are ultrametrics and $d$ is not an ultrametric, then there are three points $x, y, z \in X$ such that $d(x, y) = d(y, z) < d(x, z)$. Two of those points must belong to the same fiber of $f$ and $x, z$ must belong to different fibers of $f$ as the diameter of each $X_t := f^{-1}(t)$ is at most $\inf_{u \neq t} d_S(u, t)$. Assume $x \in X_a$, $z \in X_b$, and $a \neq b$. Now, either $d(x, z) = d(y, z)$ or $d(x, z) = d(x, y)$, a contradiction. \qed

Proposition 3.4 Suppose $(X, d)$ is an ultrametric space, $x_0 \in X$, and $S$ is the set of all possible values of $d(x_0, x)$, $x \in X$. If $d_S$ is defined by $d(u, t) = \max(u, t)$ for $u \neq t$, then $d_S$ is an ultrametric on $S$ and $i : X \to S$ defined by $i(x) = d(x_0, x)$ is a metric resolution.

Proof $d_S$ is clearly an ultrametric. Suppose $i(x) = u < t = i(y)$. By the Ultrametric Triangle Inequality we get $d(x, y) = t = \max(u, t) = d_S(i(x), i(y))$. \qed

Proposition 3.5 Suppose $D$ is a finite subset of non-negative integers and $(X, d)$ is a $D$-ultrametric space. If $m = \max(D)$, then there is a metric resolution $i : X \to m \cdot C$ for some full simplex $C$, i.e. a finite set with the metric on it admitting only values 0 and 1.
Definition 4.1 Given a surjective function \( f : X \to S \) in Definition 2.5 and metric resolutions in Sect. 3. The idea is to splice metrics on the fibers of a function \( f : X \to S \) with a metric \( d_S \) on \( S \). The construction in this section is on the other side of spectrum than metric resolutions. There, fibers are basically parallel, in the new concept different fibers diverge at infinity.

Proposition 4.2 Consider the equivalence relation \( \sim \) on \( X \) defined by \( x \sim y \) if \( d(x, y) < m \). Let \( C \) be the set of all equivalence classes. Notice that \( d(x, y) = m \) if \( x \) and \( y \) belong to different equivalence classes. Define \( i(x) \) as the equivalence class of \( [x] \) of \( x \).

Proof Consider the equivalence relation \( \sim \) on \( X \) defined by \( x \sim y \) if \( d(x, y) < m \). Let \( C \) be the set of all equivalence classes. Notice that \( d(x, y) = m \) if \( x \) and \( y \) belong to different equivalence classes. Define \( i(x) \) as the equivalence class of \( [x] \) of \( x \).

Proposition 3.6 Suppose \((X, d)\) is an ultrametric space. The function \( \rho(A, B) = \sup\{d(x, y) | x \in A, y \in B\} \) for \( A \neq B \) defines an ultrametric on the space of non-empty bounded subsets of \( X \).

Proof If \( \rho(A, C) > \max(\rho(A, B), \rho(B, C)) \), then there exist points \( a \in A \) and \( c \in C \) such that \( d(a, c) > \max(\rho(A, B), \rho(B, C)) \). Choose \( b \in B \). Now, \( d(a, c) \leq \max(d(a, b), d(b, c)) \leq \max(\rho(A, B), \rho(B, C)) \), a contradiction.

Proposition 3.7 If \( i : X \to S \) is a metric resolution and \( X \) is an ultrametric space, then the distance between \( u \) and \( t \) in \( S \), \( u \neq t \), is the same as the distance between \( i^{-1}(u) \) and \( i^{-1}(t) \) in \( X \).

Proof If \( x \in i^{-1}(u) \) and \( y \in i^{-1}(t) \), then \( d_X(x, y) = d_S(u, t) \). Therefore the distance between \( u \) and \( t \) in \( S \), \( u \neq t \), is the same as the distance between \( i^{-1}(u) \) and \( i^{-1}(t) \) in \( X \).

4 Splicing of metrics

In this section we will discuss a more general concept than both \( d^\text{ul} \) in Definition 2.5 and metric resolutions in Sect. 3. The idea is to splice metrics on the fibers of a function \( f : X \to S \) with a metric \( d_S \) on \( S \). The construction in this section is on the other side of spectrum than metric resolutions. There, fibers are basically parallel, in the new concept different fibers diverge at infinity.

Definition 4.1 Given a surjective function \( f : X \to S \) from a set to a metric space \((S, d_S)\), given metrics \( d_t \) on fibers \( f^{-1}(t), t \in S \), and given a section \( g : S \to X \), we define the splicing \( d \) of metrics along \( g \) on \( X \) as follows:

- \( d \) restricted to each fiber \( f^{-1}(t) \) equals \( d_t \).
- If \( x \in f^{-1}(u), y \in f^{-1}(t) \) and \( u \neq t \), then
  \[
  d(x, y) = \max\{d_S(u, t), d_u(x, g(u)), d_t(y, g(t))\}.
  \]

Example 4.2 A metric resolution \( f : X \to S \) is a splicing of metrics on fibers with a metric on \( S \) along any section \( g : S \to X \), i.e. all sections give the same metrics, if the diameters of fibers satisfy the following condition: \( \text{diam}(f^{-1}(u)) \leq \text{dist}(u, S \setminus \{u\}) \) for all \( u \in S \).

Proof In this case the formula \( d(x, y) = \max\{d_S(u, t), d_u(x, g(u)), d_t(y, g(t))\} \) yields \( d_S(u, t) \) if \( u \neq t \).

Proposition 4.3 A splicing \( d \) of metrics is a metric. Moreover, if the metric \( d_S \) on \( S \) and all the metrics on the fibers of \( f \) are ultrametrics, then so is \( d \).
Proof Suppose $d(x, z) > d(x, y) + d(y, z)$ for some points $x, y, z \in X$. Obviously, all three points $f(x), f(y), f(z)$ cannot be equal to the same point of $S$. The other extreme is all three points $u := f(x), t := f(y), v := f(z)$ being pairwise different. Now, $d(x, z) \neq d_S(u, v) \leq d_S(u, t) + d_S(t, v) \leq d(x, y) + d(y, z)$, so $d(x, z) = d_u(x, g(u)) \leq d(x, y)$ or $d(x, z) = d_v(z, g(v)) \leq d(y, z)$, a contradiction in both cases.

Therefore, it remains to consider the following two cases:

- **Case A**: $u = f(x) = f(z) \neq t = f(y)$. Notice $d(x, y) \geq d_u(x, g(u))$ and $d(z, y) \geq d_u(z, g(u))$, so $d(x, y) + d(y, z) \geq d_u(x, z) = d(x, z)$, a contradiction.

- **Case B**: $u = f(x) \neq t = f(z), v = f(y)$, and $v = u$ or $v = t$.

  If $v = u$, then $d(x, z)$ cannot be $d_f(z, g(z)) \leq d(y, z)$ and it cannot be $d_u(x, g(u)) \leq d_u(x, y) + d_a(y, g(u)) \leq d(x, y) + d(y, z)$, so $d(x, z) = d_S(u, t) = d_S(v, t) \leq d(y, z)$, a contradiction.

  If $v = t$, $d(x, z)$ cannot be $d_f(z, g(z)) \leq d(x, y)$ and it cannot be $d_u(x, g(u)) \leq d(x, y)$, so $d(x, z) = d_S(u, t) \leq d(y, z)$, a contradiction.

That proves $d$ is a metric.

Assume the metric $d_S$ on $S$ and all the metrics on the fibers of $f$ are ultrametrics. Suppose $d(x, z) > \max(d(x, y), d(y, z))$ for some points $x, y, z \in X$. Obviously, all three points $f(x), f(y), f(z)$ cannot be equal to the same point of $S$. The other extreme is all three points $u := f(x), t := f(y), v := f(z)$ being pairwise different. Now, $d(x, z) \neq d_S(u, v) \leq \max(d_S(u, t), d_S(t, v)) \leq \max(d(x, y), d(y, z))$, so $d(x, z) = d_u(x, g(u)) \leq d(x, y)$ or $d(x, z) = d_v(z, g(v)) \leq d(y, z)$, a contradiction in both cases.

Therefore, it remains to consider the following two cases:

- **Case A**: $u = f(x) = f(z) \neq t = f(y)$. Notice $d(x, y) \geq d_u(x, g(u))$ and $d(z, y) \geq d_u(z, g(u))$, so $\max(d(x, y), d(y, z)) \geq d_u(x, z) = d(x, z)$, a contradiction.

- **Case B**: $u = f(x) \neq t = f(z), v = f(y)$, and $v = u$ or $v = t$.

  If $v = u$, then $d(x, z)$ cannot be $d_f(z, g(z)) \leq d(y, z)$ and it cannot be $d_u(x, g(u)) \leq \max(d_u(x, y), d(y, g(u)) \leq \max(d(x, y), d(y, z))$, so $d(x, z) = d_S(u, t) = d_S(v, t) \leq d(y, z)$, a contradiction.

  If $v = t$, $d(x, z)$ cannot be $d_f(z, g(z)) \leq d(x, y)$ and it cannot be $d_u(x, g(u)) \leq d(x, y)$, so $d(x, z) = d_S(u, t) \leq d(y, z)$, a contradiction.

That proves $d$ is an ultrametric. 

\end{proof}

5 Coarse disjoint unions

In this section we define the concept of a coarse disjoint union, which is a means of joining a family of metric spaces together in such a way that they are “coarsely independent” of one another.

**Definition 5.1** Given a family $\{(X_s, d_s)_{s \in S}\}$ of metric spaces, a **coarse disjoint union** of that family is a disjoint union $\bigsqcup_{s \in S} X_s$ equipped with a metric $d$ satisfying the following properties:

1. $d$ restricted to each $X_s$ equals $d_s$. 
2. Given \( M > 0 \) there are bounded subsets \( B_t \) of \( X_s \), all but finitely many of them empty, such that if \( x \in X_s \setminus B_t \) and \( y \in X_t \setminus B_t \) for some \( s \neq t \), then \( d(x, y) > M \).

**Observation 5.2** Notice that Condition 2 in Definition 5.1 can be split into the following two conditions:

2a. Every bounded subset \( B \) of \( \bigsqcup_{s \in S} X_s \) is contained in \( \bigsqcup_{s \in F} X_s \) for some finite \( F \subset S \).

2b. Given \( M > 0 \), there is a bounded subset \( B \) of \( \bigsqcup_{s \in S} X_s \) such that if \( x \in X_s \setminus B \) and \( y \in X_t \setminus B \) for some \( s \neq t \), then \( d(x, y) > M \).

**Observation 5.3** If a coarse disjoint union \( X \) exists and each \( X_s \) is non-empty, then \( S \) is countable. Indeed, \( X \) is the union of its balls \( B(x_0, n), n \geq 1 \), and each ball intersects finitely many terms \( X_s \).

**Proposition 5.4** A disjoint union \( \bigsqcup_{s \in S} X_s \) is a coarse disjoint union of a family \( \{(X_s, d_s)\}_{s \in S} \) of metric spaces if and only if it can be equipped with a metric \( d \) satisfying the following properties:

1. \( d \) restricted to each \( X_s \) equals \( d_s \).
2. Given a sequence \( \{x_n\}_{n \geq 1} \) of points in \( \bigsqcup_{s \in S} X_s \) belonging to different parts \( X_s \), one has \( x_n \to \infty \) (that means \( d(a, x_n) \to \infty \) for some, hence for all, \( a \in \bigsqcup_{s \in S} X_s \)).
3. Given \( M > 0 \) and a sequence of pairs \( (x_n, y_n), n \geq 1 \), of points in \( \bigsqcup_{s \in S} X_s \) such that \( d(x_n, y_n) < M \) for all \( n \geq 1 \) and \( x_n \to \infty \), there is \( k \geq 1 \) such that for each \( n \geq k \) there is an index \( s \in S \) so that \( x_n, y_n \in X_s \).

**Proof** Suppose \( \bigsqcup_{s \in S} X_s \) is a coarse disjoint union in the sense of Definition 5.1. Given a sequence \( \{x_n\}_{n \geq 1} \) of points in \( \bigsqcup_{s \in S} X_s \) belonging to different parts \( X_s \) such that \( d(a, x_n) \) is not divergent to infinity for some \( a \in \bigsqcup_{s \in S} X_s \), we may reduce this case to the one where there is \( M > 0 \) satisfying \( d(a, x_n) < M \) for all \( n \geq 1 \). There are bounded subsets \( B_s \) of \( X_s \), all but finitely many of them empty, such that if \( x \in X_s \setminus B_s \) and \( y \in X_t \setminus B_t \) for some \( s \neq t \), then \( d(x, y) > 2M \). There are \( t \neq s \) in \( S \) such that \( B_t = B_s = \emptyset \) and \( x_k \in X_t, x_m \in X_s \) for some \( k, m \), a contradiction as \( d(x_k, x_m) < 2M \). Given \( M > 0 \) and a sequence of pairs \( (x_n, y_n), n \geq 1 \), of points in \( \bigsqcup_{s \in S} X_s \) such that \( d(x_n, y_n) < M \) for all \( n \geq 1 \) and \( x_n \to \infty \), assume there is no \( k \geq 1 \) such that for each \( n \geq k \) there is an index \( s \in S \) so that \( x_n, y_n \in X_s \). We may reduce this case to the one where \( x_n \) and \( y_n \) do not belong to the same \( X_s \) for all \( n \geq 1 \). There are bounded subsets \( B_s \) of \( X_s \), all but finitely many of them empty, such that if \( x \in X_s \setminus B_s \) and \( y \in X_t \setminus B_t \) for some \( s \neq t \), then \( d(x, y) > M \). There are \( t \neq s \) in \( S \) such that \( B_t = B_s = \emptyset \) and \( x_k, y_k \in X_s \) for some \( k, m \), a contradiction as \( d(x_k, y_k) < M \).

The proof in the reverse direction is similar. \( \square \)

**Corollary 5.5** A metric space \( (S, d_S) \) is a coarse disjoint union of its points if and only if it is proper (its bounded subsets are finite) and has the property that \( d_S(x_n, y_n) < M < \infty \) for all \( n \) and \( d_S(x_n, x_1) \to \infty \) imply existence of \( k \) such that \( x_n = y_n \) for all \( n > k \).
Proposition 5.6 Suppose \( f : X \to S \) is a surjective function and \( X \) is given the metric equal to splicing of metrics on fibers along a section \( g : S \to X \). If \( S \) is a coarse disjoint union of its points, then \( X \) is a coarse disjoint union of fibers of \( f \).

\textbf{Proof} Apply Corollary 5.5:

If \( B \) is a bounded subset of \( X \), then \( f(B) \) is bounded as \( f \) is 1-Lipschitz. Hence \( f(B) \) is finite.

If \( d(x_n, y_n) < M, x_n \to \infty \), and \( f(x_n) \neq f(y_n) \) for all \( n \geq 1 \), then \( d_S(f(x_n), f(y_n)) < M \) and \( f(x_n) \to \infty \). Hence \( f(x_n) = f(y_n) \) for almost all \( n \), a contradiction. \( \square \)

Corollary 5.7 Every integral ultrametric space \( (X, d) \) is a coarse disjoint union of countably many of its bounded subsets.

\textbf{Proof} Pick \( x_0 \in X \) and let \( S \) be the set of all possible distances \( d(x_0, x), x \in X \). The metric \( d_S \) of \( S \) is defined by \( d_S(u, t) = \max(u, t) \) if \( u \neq t \). By Proposition 3.4 the function \( f : X \to S \) given by \( f(x) = d(x_0, x) \) is a metric resolution. Notice \( S \) is a coarse disjoint union of its points and apply Proposition 5.6. \( \square \)

Corollary 5.8 If \( S \) is countable, then any family \( \{(X_s, d_s)\}_{s \in S} \) of non-empty metric spaces has a coarse disjoint union. Moreover, if each \( d_s \) is an (integral) ultrametric, then there is a coarse disjoint union equipped with an (integral) ultrametric.

\textbf{Proof} We may assume \( S = \mathbb{N} \) and we equip \( S \) with the max ultrametric \( d(m, n) = \max(m, n) \) if \( m \neq n \). Let \( X \) be a disjoint union of all \( X_s, s \in S \), equipped with a splicing metric for some section \( g : S \to X \). By Corollary 5.5 and Proposition 5.6, \( X \) is a coarse disjoint union of \( \{(X_s, d_s)\}_{s \in S} \). \( \square \)

Proposition 5.9 Given two coarse disjoint unions \( (\bigsqcup_{s \in S} X_s, d) \) and \( (\bigsqcup_{s \in S} Y_s, \rho) \),

\begin{enumerate}
\item isometric embeddings \( i_s : X_s \to Y_s, s \in S \), induce a coarse embedding \( i \) from \( (\bigsqcup_{s \in S} X_s, d) \) to \( (\bigsqcup_{s \in S} Y_s, \rho) \),
\item identity functions \( i_s : X_s \to Y_s, s \in S \), induce a coarse equivalence \( i \) from \( (\bigsqcup_{s \in S} X_s, d) \) to \( (\bigsqcup_{s \in S} Y_s, \rho) \).
\end{enumerate}

\textbf{Proof} Notice (1) implies (2), so only (1) needs to be proved. Suppose, on the contrary, that there is \( M > 0 \) and a sequence of pairs \( (x_n, y_n), n \geq 1 \), of points in \( \bigsqcup_{s \in S} X_s \) such that \( d(x_n, y_n) < M \) for all \( n \geq 1 \) but \( \rho(i(x_n), i(y_n)) \to \infty \). There is \( k \geq 1 \) such that for each \( n \geq k \) there is an index \( s \in S \) so that \( i(x_n), i(y_n) \in Y_s \). Therefore \( \rho(i(x_n), i(y_n)) = d(x_n, y_n) \) for all \( n > k \), a contradiction. \( \square \)

Proposition 5.10 Suppose \( S \) is countable and \( \{(X_s, d_s)\}_{s \in S} \) is a family of metric spaces. A coarse disjoint union of \( \{(X_s, d_s)\}_{s \in S} \) is separable (proper) if and only if each \( X_s \) is separable (proper).

\textbf{Proof} It follows from the fact that any bounded subset \( B \) of the coarse disjoint union is a union of finitely many bounded subsets of some among spaces \( X_s \). \( \square \)
6 Special ultrametric spaces

In this section we construct universal spaces (with respect to isometric embeddings) for specific classes of bounded ultrametric spaces. More specifically, for a finite subset $D \subseteq \mathbb{N}$ that contains 0 we construct universal spaces for the class of countable $D$-ultrametric spaces, and for each $m \geq 1$ we construct a universal space for the class of $D$-ultrametric spaces with at most $m$ points. The spaces constructed in this section serve as the building blocks for the universal spaces constructed in Sect. 7.

Proposition 6.1 Suppose $p_i: E_i \to B_i, i = 1, 2,$ are metric resolutions, $f: B_1 \to B_2$ is an isometric embedding and for each $s \in B_1$ there is an isometric embedding $f_s: p_1^{-1}(s) \to p_2^{-1}(f(s))$. There is an isometric embedding $F: E_1 \to E_2$ such that $p_2 \circ F = f \circ p_1$.

Proof $F$ is the union of all $f_s, s \in B_1$. Clearly, it is an isometric embedding. $\square$

The following can be seen as an inductive step in splitting bounded ultrametric spaces into Lego pieces.

Proposition 6.2 Suppose $D$ is a finite set of non-negative integers and $k = \max(D) > 0$. Every $D$-ultrametric space $X$ of diameter $k$ admits a surjective metric resolution $f: X \to \{0, k\} \subset \mathbb{N}$ such that $\text{diam}(f^{-1}(0)) < k$.

Proof Pick two points $a, b \in X$ at distance $k$. $f$ maps all points $x \in X$ such that $d(x, a) < k$ to 0 and $f$ maps all points $x \in X$ such that $d(a, x) = k$ to $k$. Notice $\text{diam}(f^{-1}(0)) < k$, $\text{diam}(f^{-1}(k)) \leq k$, so $f$ is indeed a metric resolution. $\square$

Proposition 6.3 Given a finite set $D$ of non-negative integers and given $m \geq 1$, there is a finite $D$-ultrametric space $FU(m, D)$ such that any $D$-ultrametric space $X$ containing at most $m$ points isometrically embeds in $FU(m, D)$.

Proof We will construct $FU(m, D)$ by induction on $m + |D|$, $|D|$ being the cardinality of $D$. Let $FU(1, D)$ be a one-point metric space for any $D$. Given $D$ with $k = \max(D) > 0$, $FU(m, D)$ is the disjoint union of $FU(m, D \setminus \{k\})$ and $FU(m - 1, D)$ so that any distance between points in different parts is $k$. Thus, as in Proposition 6.2, $FU(m, D)$ admits a metric resolution onto $\{0, k\}$ with fibers $FU(m, D \setminus \{k\})$ and $FU(m - 1, D)$.

Now, by induction on $m + |D|$, we can show the space $FU(m, D)$ has the needed property. Indeed, it is so for $m + |D| = 2$. Once we assume the spaces $FU(k, C)$, $k + |C| < n$ and $C$ a finite set of non-negative integers, have the needed property, we use Proposition 6.2 jointly with Proposition 6.1 to conclude $FU(m, D)$ has the property that any $D$-ultrametric space $X$ containing at most $m$ points isometrically embeds in $FU(m, D)$ provided $m + |D| \leq n$. Indeed, by Proposition 6.2, if $X$ is of diameter $k$, it admits a surjective metric resolution $g: X \to \{0, k\} \subset \mathbb{N}$ such that $\text{diam}(g^{-1}(0)) < k$. By inductive assumption $g^{-1}(0)$ isometrically embeds in $F(m, D \setminus \{k\})$ and $g^{-1}(k)$ isometrically embeds into $F(m - 1, D)$. By Proposition 6.1 applied to $B_1 = B_2 = \{0, k\}, f: B_1 \to B_2$ being the identity map, $p_1: E_1 := X \to B_1$ being equal to $g$, and $p_2: E_2 := FU(m, D) \to B_2$ being the metric resolution from the inductive definition of $FU(m, D)$, $X$ isometrically embeds into $FU(D, m)$. $\square$
Proposition 6.4  Given a finite set $D$ of non-negative integers there is a countable $D$-ultrametric space $CU(D)$ such that any countable $D$-ultrametric space $X$ isometrically embeds in $CU(D)$.

Proof  Let $CU(\{0\})$ be a one-point metric space. Suppose spaces $CU(D)$ are known for all $D$ containing at most $n$ integers and $G$ contains $(n+1)$ integers with $k = \max(G)$. Define $CU(G)$ as the total space in a metric resolution $f : CU(G) \to k \cdot \Delta$, where $k \cdot \Delta$ is an infinite countable space with all non-zero distances equal $k$ (i.e. $k$ times the infinite simplex), so that $f^{-1}(x)$ is a copy of $CU(G \setminus \{k\})$ for each $x \in k \cdot \Delta$. By Proposition 3.3 such a resolution exists and $CU(G)$ is an ultrametric space.

Given a countable $G$-ultrametric space $X$ if $\text{diam}(X) < k$, $X$ clearly embeds into $CU(G)$ isometrically due to the inductive step, so assume $\text{diam}(X) = k$ and choose a maximal subset $S$ of $X$ such that all non-zero distances in $S$ are equal to $k$. For each $s \in S$ consider the set $X_s$ of all $x \in X$ such that $d(s, x) < k$. Notice the following:

(a) $\text{diam}(X_s) < k$ for each $s \in S$,
(b) if $x \in X_s$, $y \in X_t$ and $s \neq t$, then $d(x, y) = k$,
(c) $X = \bigcup_{s \in S} X_s$.

Choose an injective function $g : S \to k \cdot \Delta$ and choose isometric embeddings $f_s : X_s \to f^{-1}(g(s))$ for each $s \in S$. Apply Proposition 6.1.

Corollary 6.5  There is a countable integral ultrametric space $CU$ such that any countable integral ultrametric space $X$ isometrically embeds in $CU$.

Proof  Consider $\mathbb{N}$ equipped with the ultrametric $d(k, m) = \max(k, m)$ for $k \neq m$ and use spaces $CU(\mathbb{N} \cap [0, i])$ from Proposition 6.4. Define $CU$ as the total space in a metric resolution $f : CU \to \mathbb{N}$ so that $f^{-1}(i) = CU(\mathbb{N} \cap [0, i])$ for each $i \in \mathbb{N}$. By Proposition 3.3 such a resolution exists and by Proposition 6.1 together with Proposition 3.4 the space $CU$ has the needed property.

7 Universal spaces

In this section we prove our main applications. That is, we give a detailed construction of universal spaces (with respect to coarse embeddings) in the classes of separable metric spaces of asymptotic dimension 0 and the class of proper metric spaces of asymptotic dimension 0.

Theorem 7.1  There is a countable integral ultrametric space $CU$ such that any separable metric space $X$ of asymptotic dimension 0 coarsely embeds in $CU$.

Proof  Consider $CU$ from Corollary 6.5. We claim that $CU$ is the desired universal space. In light of Theorem 2.10 it will suffice to show that if $(X, d)$ is a countable integral ultrametric space, then $X$ coarsely embeds into $CU$. Then let $(X, d)$ be such a space. Apply Corollary 6.5.

Theorem 7.2  There is a countable and proper integral ultrametric space $PU$ such that any proper metric space $X$ of asymptotic dimension 0 coarsely embeds in $PU$. 
Proof We again use the enumeration $D_1, D_2, \ldots$ of the finite subsets of $\mathbb{N}$ that contain 0. The set of all $\text{FU}(m, D_n)$ (where defined) is countable. Enumerate these spaces and denote this sequence by $\{Y_1, Y_2, \ldots\}$. Let $r_i = i + \sum_{j=1}^i \text{diam}(Y_j)$ for $i \geq 1$.

We then define $\text{PU}$ as the total space of a metric resolution $f: \text{PU} \rightarrow S$, where $S = \{r_i\}_{i \geq 1}$ is equipped with the max ultrametric and $f^{-1}(r_i)$ is the corresponding $\text{FU}(m, D_n)$ for each $i \geq 1$.

It is proper by Proposition 5.10. Let $(X, d)$ be a countable proper metric space of asymptotic dimension 0. By Theorems 2.10 and 5.7 we may assume without loss of generality that that $X$ can be written as a coarse disjoint union of finite $G_k$-ultrametric spaces $X_k$. There is a strictly increasing sequence $(n_k)_{k \in \mathbb{N}}$ such that $G_k \subset D_{n_k}$ for each $k \geq 1$ and $D_{n_k} \subseteq D_{n_{k+1}}$ for each $k \geq 1$. Then, by Proposition 5.9 we have that $X$ coarsely embeds into $\text{PU}$.

The following result means that, in dimension 0, our Theorem 7.2 is not implied by the work of Bell and Nagórko [7]:

**Proposition 7.3** There is a separable space $X$ of asymptotic dimension 0 that is not coarsely embeddable into a proper metric space.

**Proof** Consider a sequence of bounded $\mathbb{N}$-ultrametric spaces $X_i$ such that for each $n \geq 1$ there is infinitely many $i$ such that $X_i$ contains infinitely countably many points and the distance between any two different points is $n + 1$. Let $X$ be a coarse disjoint union of the sequence $X_i$. If $X$ coarsely embeds into a proper metric space, then it contains $Y$ with the property that $B(Y, n) = X$ for some $n \geq 2$ and every bounded subset of $Y$ is finite. There is $i$ such that $X_i$ contains infinitely countably many points, the distance between any two different points is $n + 1$, and $\text{dist}(X_i, X \setminus X_i) > n$. Now, $Y \cap X_i$ must be finite, $B(Y, n) \cap X_i = B(Y \cap X_i, n) = Y \cap X_i$, a contradiction. □

**Remark 7.4** Proposition 7.3 can be easily generalized to any dimension $n$.

In the first version of the paper the authors posed the following question: Given $n \geq 1$, is there a universal space in the class of proper metric spaces of asymptotic dimension at most $n$? Since then it was negatively solved for $n = 1$ by Mykhailo Zarichnyi [17].

8 Ultrametric groups as universal spaces

In this section we show that certain unbounded ultrametric groups are universal in respective categories of spaces of asymptotic dimension 0.

**Definition 8.1** An (integral) ultrametric group is a group equipped with a left-invariant (integral) ultrametric $d$.

**Proposition 8.2** Suppose $G$ is a group and $D$ is a discrete subset of non-negative reals containing 0. Assigning $G$ a left-invariant $D$-ultrametric $d$ is equivalent to picking subgroups $G_a$, $a \in D$, of $G$ satisfying the following conditions:

1. $G_0 = \{1_G\}$.  

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(2) $G_a$ is a subgroup of $G_b$ if $a < b$ belong to $D$.

(3) $\bigcup_{a \in D} G_a = G$.

**Proof** Given a left-invariant $D$-ultrametric $d$ on $G$ and given $a \in D$, define $G_a$ as all $g \in G$ satisfying $d(g, 1_G) \leq a$. Notice $g \in G_a$ implies $g^{-1} \in G_a$ as $d(g^{-1}, 1_G) = d(g \cdot g^{-1}, 1_G) = d(1_G, g)$. Also, if $g, h \in G_a$, then $d(g \cdot h, 1_G) \leq \max(d(g, h), d(g, 1_G)) = \max(d(h, 1_G), d(g, 1_G)) \leq a$. It is obvious that $\{G_a\}_{a \in D}$ satisfy Conditions (1)–(3).

Given $\{G_a\}_{a \in D}$ satisfying Conditions (1)–(3), define $d(g, h)$ as the infimum of $a \in D$ satisfying $g^{-1} \cdot h \in G_a$. If $d(g, h), d(h, k) \leq a$, then $g^{-1} \cdot h \in G_a$ and $h^{-1} \cdot k \in G_a$, so their product $g^{-1} \cdot k$ belongs to $G_a$ and $d(g, k) \leq a$. That means $d$ is an ultrametric, indeed. 

**Definition 8.3** Given a discrete subset $D$ of non-negative reals containing 0 and given subgroups $G_a, a \in D$, of $G$ satisfying the following conditions:

- $G_0 = \{1_G\}$,
- $G_a$ is a subgroup of $G_b$ if $a < b$ belong to $D$,
- $\bigcup_{a \in D} G_a = G$,

the ultrametric $d$ in Proposition 8.2 is said to be induced by $\{G_a\}_{a \in D}$.

**Proposition 8.4** Suppose $G$ is a group and $d_i, i = 1, 2$, are two ultrametrics induced by families $\{G^i_a\}_{a \in D_i}$ of subgroups of $G$. $(G, d_1)$ is coarsely equivalent to $(G, d_2)$ if and only if for each $a \in D_i$ and $j \neq i$ there is $b \in D_j$ such that $G^i_a \subset G^j_b$.

**Proof** Assume the identity $(G, d_i) \rightarrow (G, d_2)$ is large scale continuous (aka bornological) and $a \in D_i$. There is $b \in D_2$ such that $d_1(g, h) \leq a$ implies $d_2(g, h) \leq b$, so $g \in G_a^1$ implies $g \in G_b^1$ as $d_1(g, 1_G) \leq a$ implies $d_2(g, 1_G) \leq b$ and $g \in G_b^2$.

The reverse implication is similar. 

**Proposition 8.5** Suppose $(X, d_X)$ is an integral ultrametric space and $(G, d_G)$ is an integral ultrametric group. If every bounded subset $B$ of $X$ isometrically embeds in $G$, then $(X, d_X)$ coarsely embeds in $(G, d_G)$.

**Proof** Of interest is only the case of $X$ being unbounded, so $G$ is also unbounded. Pick $x_0 \in X$ and a sequence $\{x_n\}_{n \geq 1}$ of points in $X$ such that $d(x_{n+1}, x_0) > d(x_n, x_0) + 1$ for each $n \geq 1$. Put $r_n = d(x_n, x_0)$ for $n \geq 1$ and pick an isometric embedding $i_n: B_n \rightarrow G$, where $B_n = \{x \in X \mid r_{n-1} < d(x, x_0) \leq r_n\}$ for $n \geq 2$ and $B_1 = \{x \in X \mid d(x, x_0) \leq r_1\}$. We may assume $i_n(x_0) = 1_G$ for each $n \geq 1$. Now pick a sequence $\{g_n\}_{n \geq 1}$ of elements of $G$ such that $d_G(g_1, 1_G) > r_1$ and $s_n \equiv d_G(g_n, 1_G) > d_G(g_{n-1}, 1_G) + \text{diam}(B_n)$. Replacing $i_n$ by $j_n := g_n \cdot i_n$ we obtain a sequence of isometric embeddings of $B_n$ into $C_n = \{g \in G \mid s_{n-1} < d(g, 1_G) \leq s_n\}$. By Proposition 6.1, $X$ coarsely embeds in $G$.

**Corollary 8.6** Suppose $(X, d_X)$ is an integral ultrametric space such that for each $n$ there is a cardinal number $c(n)$ with the property that each open $(n + 2)$-ball $B(x, n + 2)$, $x \in X$, has cardinality at most $c(n)$. If $(G, d_G)$ is an integral ultrametric group induced by a sequence of subgroups $\{G_n\}_{n \geq 1}$ with the property that the cardinality of cosets of $G_n^1$ in $G_{n+1}^1$ is at least $c(n)$ for each $n \geq 1$, then $(X, d_X)$ coarsely embeds in $(G, d_G)$.
Proof Suppose each bounded subset of $X$ of diameter at most $n$ isometrically embeds in $G$. Therefore, for each $x \in X$, there is an isometric embedding $i_x : B(x, n+1) \to G_n$ such that $i_x(x) = 1_G$. Suppose $x_0 \in X$. Consider the equivalence relation $x \sim y$ on $B(x_0, n+2)$ defined by $d_X(x, y) < n+1$. For each equivalence class $c$ not containing $x_0$ choose $x(c) \in B(x_0, n+2) \setminus B(x_0, n+1)$ and $g_c \in G_{n+1}$ such that if $c \neq k$, then $g_c^{-1} \cdot g_k \notin G_n$. Extend $i_{x_0}$ over $B(x_0, n+2)$ to a function $j$ by sending $x(c)$ to $g_c$ and by sending any $x$ equivalent to $x_c$ to $g_c^{-1} \cdot i_x(c)(x)$. Notice $j$ is an isometric embedding when restricted to each equivalence class, the images of different equivalence classes are disjoint, and if $d_X(x, y) = n+1$, then $d_G(j(x), j(y)) = n+1$. That means $j$ is an isometric embedding.

Corollary 8.7 Suppose $G$ is a countable group that is the union of an increasing sequence of its subgroups $\{G_i\}_{i \geq 1}^\infty$ with the property that the index of $G_i$ in $G_{i+1}$ is infinite for each $i \geq 1$. There is an integral ultrametric $d_G$ on $G$ such that $(G, d_G)$ is a universal space in the category of separable metric spaces of asymptotic dimension $0$.

Corollary 8.8 Let $G$ be a countable vector space over the rationals $\mathbb{Q}$ that is of infinite algebraic dimension. There is an integral ultrametric $d_G$ on $G$ such that $(G, d_G)$ is a universal space in the category of separable metric spaces of asymptotic dimension $0$.

Theorem 8.9 Suppose $G$ is a countable group that is the union of a strictly increasing sequence of its finite subgroups $\{G_i\}_{i \geq 1}^\infty$. There is a proper integral ultrametric $d_G$ on $G$ such that $(G, d_G)$ is a universal space in the category of metric spaces of bounded geometry that have asymptotic dimension $0$.

Proof In the category of proper discrete metric spaces, $X$ being of bounded geometry means that for each natural number $n$ there is a natural number $c(n)$ such that every $n$-ball $B(x, n)$ contains at most $c(n)$ elements.

Consider a proper integral ultrametric space $(X, d_X)$ of bounded geometry and choose natural numbers $c(n)$ with the property that each ball $B(x, n+2)$, $x \in X$, contains at most $c(n)$ elements. Replace $\{G_n\}$ by its subsequence $\{H_n\}$ such that the index of $H_n$ in $H_{n+1}$ is larger than $c(n + 1)$ for each $n \geq 1$. By Corollary 8.6 and Proposition 8.4, $(X, d_X)$ coarsely embeds into $(G, d_G)$.

Corollary 8.10 Let $G$ be a countable vector space over the $\mathbb{Z}/2\mathbb{Z}$ that is of infinite algebraic dimension. There is a proper integral ultrametric $d_G$ on $G$ such that $(G, d_G)$ is a universal space in the category of metric spaces of bounded geometry that have asymptotic dimension $0$.

Remark 8.11 See [1, 2] for coarse classifications of groups of asymptotic dimension $0$.

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