Kernels of perturbed Toeplitz operators in vector-valued Hardy spaces

Arup Chattopadhyay¹ · Soma Das¹ · Chandan Pradhan¹

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Abstract
Recently, Liang and Partington (Integr Equ Oper Theory 92(4): 35, 2020) show that kernels of finite-rank perturbations of Toeplitz operators are nearly invariant with finite defect under the backward shift operator acting on the scalar-valued Hardy space. In this article, we provide a vectorial generalization of a result of Liang and Partington. As an immediate application, we identify the kernel of perturbed Toeplitz operator in terms of backward shift-invariant subspaces in various important cases by applying the recent theorem (see Chattopadhyay et al. in Integr Equ Oper Theory 92(6): 52, 2020, Theorem 3.5 and O’Loughlin in Complex Anal Oper Theory 14(8): 86, 2020, Theorem 3.4) in connection with nearly invariant subspaces of finite defect for the backward shift operator acting on the vector-valued Hardy space.

Keywords Vector-valued Hardy space · Nearly invariant subspaces · Toeplitz operator · Shift operator · Beurling’s theorem · Multiplier operator

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Arup Chattopadhyay
arupchatt@iitg.ac.in; 2003arupchattopadhyay@gmail.com
Soma Das
soma18@iitg.ac.in; dsoma994@gmail.com
Chandan Pradhan
chandan.math@iitg.ac.in; chandan.pradhan2108@gmail.com

¹ Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati 781039, India
1 Introduction

It is well known that the kernel of a Toeplitz operator is nearly invariant under the backward shift operator acting on the scalar-valued Hardy space, and the concept of nearly backward shift-invariant subspaces was first introduced by Hitt [10] as a generalization to Hayashi’s results concerning Toeplitz kernels in [9]. Later Sarason [18] further investigated these spaces and modified Hitt’s algorithm for scalar-valued Hardy space to study the kernels of Toeplitz operators. In 2010, Chalendar–Chevrot–Partington (C–C–P) [2] gives a complete characterization of nearly invariant subspaces under the backward shift operator acting on the vector-valued Hardy space, providing a vectorial generalization of a result of Hitt. In [3], Chalendar–Gallardo–Partington (C–G–P) introduces the notion of the nearly invariant subspace of finite defect for the backward shift operator acting on the scalar-valued Hardy space as a generalization of nearly invariant subspaces and provides a complete characterization of these spaces in terms of backward shift-invariant subspaces. Recently in [4], the authors of this article characterize nearly invariant subspace of finite defect for the backward shift operator acting on the vector-valued Hardy space, providing a vectorial generalization of a result of Chalendar–Gallardo–Partington (C–G–P). In this connection, it is also noted that a similar type of connection was also obtained independently by O’Loughlin [16]. Furthermore, in this context, Liang and Partington [12] recently provide a connection between kernels of finite-rank perturbations of Toeplitz operators and nearly invariant subspaces with finite defect under the backward shift operator acting on the scalar-valued Hardy space. In other words, they give an affirmative answer to the following question in several important cases, which is closely related to the invariant subspace problem:

Given a Toeplitz operator $T$ acting on the scalar-valued Hardy space,

is the kernel of a finite-rank perturbation of $T$ is nearly backward shift invariant with finite defect?  \( \text{(1)} \)

Moreover, they also identify the kernel of perturbed Toeplitz operator in terms of backward shift-invariant subspaces in several important cases by applying a recent theorem by Chalendar–Gallardo–Partington (C–G–P).

The purpose of this paper is to study the kernels of finite-rank perturbations of Toeplitz operators and their connection with nearly invariant subspaces with finite defect under the backward shift operator acting on the vector-valued Hardy space. In other words, we give an affirmative answer to the above question (1) in several important cases in the vector-valued Hardy space, providing a vectorial generalization of a result of Liang and Partington. Furthermore, we also identify the kernel of perturbed Toeplitz operator in terms of backward shift-invariant subspaces by applying our recent theorem (see [4, Theorem 3.5]) in connection with nearly invariant subspaces of finite defect for the backward shift operator acting on the vector-valued Hardy space in several important cases as mentioned by Liang and Partington [12]. For more information on this direction of research, we refer the
reader to [5, 9, 19] and the references therein. To state the precise contribution of this paper, we need to introduce first some definitions and notations.

The \( \mathbb{C}^m \)-valued Hardy space \([17]\) over the unit disc \( \mathbb{D} \) is denoted by \( H^2_{\mathbb{C}^m}(\mathbb{D}) \) and defined by

\[
H^2_{\mathbb{C}^m}(\mathbb{D}) := \left\{ F(z) = \sum_{n \geq 0} A_n z^n : \|F\|^2 = \sum_{n \geq 0} \|A_n\|^2_{\mathbb{C}^m} < \infty, A_n \in \mathbb{C}^m \right\}.
\]

We can also view the above Hilbert space as the direct sum of \( m \)-copies of \( H^2_{\mathbb{C}^m}(\mathbb{D}) \) or sometimes it is useful to see the above space as a tensor product of two Hilbert spaces \( H^2_{\mathbb{C}^m}(\mathbb{D}) \) and \( \mathbb{C}^m \), that is,

\[
H^2_{\mathbb{C}^m}(\mathbb{D}) \equiv H^2_{\mathbb{C}^m}(\mathbb{D}) \oplus \cdots \oplus H^2_{\mathbb{C}^m}(\mathbb{D}) \equiv H^2_{\mathbb{D}} \otimes \mathbb{C}^m.
\]

On the other hand, the space \( H^2_{\mathbb{C}^m}(\mathbb{D}) \) can also be defined as the collection of all \( \mathbb{C}^m \)-valued analytic functions \( F \) on \( \mathbb{D} \) such that

\[
\|F\| = \left[ \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} \|F(re^{i\theta})\|^2 \, d\theta \right]^{\frac{1}{2}} < \infty.
\]

Moreover, the nontangential boundary limit (or radial limit) \( F(e^{i\theta}) := \lim_{r \to 1^-} F(re^{i\theta}) \) exists almost everywhere on the unit circle \( \mathbb{T} \) (for more details, see [15], I.3.11). Therefore, \( H^2_{\mathbb{C}^m}(\mathbb{D}) \) can be embedded isometrically as a closed subspace of \( L^2(\mathbb{T}, \mathbb{C}^m) \) by identifying \( H^2_{\mathbb{C}^m}(\mathbb{D}) \) through the nontangential boundary limits of the \( H^2_{\mathbb{C}^m}(\mathbb{D}) \) functions. Furthermore, \( L^2(\mathbb{T}, \mathbb{C}^m) \) can be decomposed in the following way

\[
L^2(\mathbb{T}, \mathbb{C}^m) = H^2_{\mathbb{C}^m}(\mathbb{D}) \oplus \overline{H^2_0},
\]

where

\[
\overline{H^2_0} = \{ F \in L^2(\mathbb{T}, \mathbb{C}^m) : F \in H^2_{\mathbb{C}^m}(\mathbb{D}) \text{ and } F(0) = 0 \}.
\]

In other words, in the above decomposition \( H^2_{\mathbb{C}^m}(\mathbb{D}) \) is identified with the subspace spanned by \( \{ e^{int} \otimes e_i : n \geq 0, 1 \leq i \leq m \} \) and \( \overline{H^2_0} \) is the subspace spanned by \( \{ e^{int} \otimes e_i : n < 0, 1 \leq i \leq m \} \), respectively, where \( \{ e_i : 1 \leq i \leq m \} \) is the standard basis for \( \mathbb{C}^m \). Let \( S \) denote the forward shift operator (multiplication by the independent variable) acting on \( H^2_{\mathbb{C}^m}(\mathbb{D}) \), that is, \( SF(z) = zF(z), \ z \in \mathbb{D} \). The adjoint of \( S \) is denoted by \( S^* \) and defined in \( H^2_{\mathbb{C}^m}(\mathbb{D}) \) as the operator

\[
S^*(F)(z) = \frac{F(z) - F(0)}{z}, \ F \in H^2_{\mathbb{C}^m}(\mathbb{D})
\]

which is known as backward shift operator. The Banach space of all \( \mathcal{L}(\mathbb{C}^r, \mathbb{C}^m) \) (set of all bounded linear operators from \( \mathbb{C}^r \) to \( \mathbb{C}^m \))-valued bounded analytic functions on \( \mathbb{D} \) is denoted by \( H^\infty_{\mathcal{L}(\mathbb{C}^r, \mathbb{C}^m)}(\mathbb{D}) \) and the associated norm is

\[
\|F\|_\infty = \sup_{z \in \mathbb{D}} \|F(z)\|.
\]

Moreover, the space \( H^\infty_{\mathcal{L}(\mathbb{C}^r, \mathbb{C}^m)}(\mathbb{D}) \) can be embedded isometrically as a closed
subspace of $L^\infty(\mathbb{T}, \mathcal{L}(\mathbb{C}^r, \mathbb{C}^m))$. Note that each $\Theta \in H^\infty_{\mathcal{L}}(\mathbb{C}^r, \mathbb{C}^m)(\mathbb{D})$ induces a bounded linear map $T_\Theta \in \mathcal{L}(H^2_{\mathcal{L}}(\mathbb{D}), H^2_{\mathcal{L}}(\mathbb{D}))$ defined by

$$T_\Theta F(z) = \Theta(z)F(z), \quad (F \in H^2_{\mathcal{L}}(\mathbb{D})).$$

The elements of $H^\infty_{\mathcal{L}}(\mathbb{C}^r, \mathbb{C}^m)(\mathbb{D})$ are called the multipliers and are determined by

$$\Theta \in H^\infty_{\mathcal{L}}(\mathbb{C}^r, \mathbb{C}^m)(\mathbb{D}) \text{ if and only if } ST_\Theta = T_\Theta S,$$

where the shift $S$ on the left-hand side and the right-hand side act on $H^2_{\mathcal{L}}(\mathbb{D})$ and $H^2_{\mathcal{L}}(\mathbb{D})$, respectively. A multiplier $\Theta \in H^\infty_{\mathcal{L}}(\mathbb{C}^r, \mathbb{C}^m)(\mathbb{D})$ is said to be inner if $T_\Theta$ is an isometry, or equivalently, $\Theta(e^{it}) \in \mathcal{L}(\mathbb{C}^r, \mathbb{C}^m)$ is an isometry almost everywhere with respect to the Lebesgue measure on $\mathbb{T}$. Inner multipliers are among the most important tools for classifying invariant subspaces of reproducing kernel Hilbert spaces. For instance:

**Theorem 1.1** (*Beurling–Lax–Halmos [14]*) A non-zero closed subspace $\mathcal{M} \subseteq H^2_{\mathcal{L}}(\mathbb{D})$ is shift invariant if and only if there exists an inner multiplier $\Theta \in H^\infty_{\mathcal{L}}(\mathbb{C}^r, \mathbb{C}^m)(\mathbb{D})$ such that $\mathcal{M} = \Theta H^2_{\mathcal{L}}(\mathbb{D})$, for some $r \ (1 \leq r \leq m)$.

Consequently, the space $\mathcal{M}_\perp$ of $H^2_{\mathcal{L}}(\mathbb{D})$ which is invariant under $S^*$ (backward shift) can be represented as $\mathcal{K}_\Theta := \mathcal{M}_\perp = H^2_{\mathcal{L}}(\mathbb{D}) \ominus \Theta H^2_{\mathcal{L}}(\mathbb{D})$, which also known as model spaces [7, 8, 13, 15]. Let $P_m : L^2(\mathbb{T}, \mathbb{C}^m) \rightarrow H^2_{\mathcal{L}}(\mathbb{D})$ be an orthogonal projection onto $H^2_{\mathcal{L}}(\mathbb{D})$ defined by

$$P_m(F) = \sum_{n=-\infty}^{\infty} A_n e^{int} \rightarrow \sum_{n=0}^{\infty} A_n e^{int}.$$  

Therefore, $P_m(F) = (Pf_1, Pf_2, \ldots, Pf_m)$, where $P$ is the Riesz projection on $H^2_{\mathcal{L}}(\mathbb{D})$ [7] and $F = (f_1, f_2, \ldots, f_m) \in L^2(\mathbb{T}, \mathbb{C}^m)$. Also note that for any $\Phi \in L^\infty(\mathbb{T}, \mathcal{L}(\mathbb{C}^m, \mathbb{C}^m))$, the Toeplitz operator $T_\Phi : H^2_{\mathcal{L}}(\mathbb{D}) \rightarrow H^2_{\mathcal{L}}(\mathbb{D})$ is defined by

$$T_\Phi(F) = P_m(\Phi F)$$

for any $F \in H^2_{\mathcal{L}}(\mathbb{D})$. Since $H^2_{\mathcal{L}}(\mathbb{D})$ can be written as direct sum of $m$-copies of $H^2_{\mathcal{L}}(\mathbb{D})$, then we have the following matrix-representation of $T_\Phi$:

$$T_\Phi = \begin{bmatrix}
T_{\phi_11} & T_{\phi_12} & \cdots & T_{\phi_1m} \\
T_{\phi_21} & T_{\phi_22} & \cdots & T_{\phi_2m} \\
\vdots & \vdots & \ddots & \vdots \\
T_{\phi_m1} & T_{\phi_m2} & \cdots & T_{\phi_mm}
\end{bmatrix}_{m \times m},$$

where $\Phi = (\phi_{ij})_{m \times m}$ is an element of $L^\infty(\mathbb{T}, \mathcal{L}(\mathbb{C}^m, \mathbb{C}^m))$ and each $T_{\phi_{ij}}$ is a Toeplitz operator in $H^2_{\mathcal{L}}(\mathbb{D})$ [6]. Furthermore, it is well known that $T_{\phi}^* = T_{\Phi^*}$, where $\Phi^* = \overline{\Phi}$.
Next we introduce the definition of nearly invariant subspaces for $S^*$ in vector-valued Hardy space.

**Definition 1.2** (See [4, Definition 3.1]) A closed subspace $M$ of $H^2_{C^m} (\mathbb{D})$ is said to be nearly invariant for $S^*$ if every element $F \in M$ with $F(0) = 0$ satisfies $S^*F \in M$. Moreover, a closed subspace $M \subset H^2_{C^m} (\mathbb{D})$ is said to be nearly $S^*$-invariant with defect $p$ if and only if there is a $p$-dimensional subspace $\mathcal{F} \subset H^2_{C^m} (\mathbb{D})$ (which may be taken to be orthogonal to $M$) such that if $F \in M, F(0) = 0$ then $S^* F \in M \oplus \mathcal{F}$, this subspace $\mathcal{F}$ is called the defect space.

In our earlier work [4], we have shown a connection between nearly $S^*$ invariant subspaces and $S^*$ invariant subspaces in vector-valued Hardy space. It can be easily seen that the kernel of a Toeplitz operator is nearly $S^*$ invariant in vector-valued Hardy space. Next we consider a finite rank perturbation (say rank $n$) of a Toeplitz operator $T_U : H^2_{C^m} (\mathbb{D}) \rightarrow H^2_{C^m} (\mathbb{D})$, denoted by $T_n$ and defined by

$$T_n(F) = T_U(F) + \sum_{i=1}^{n} \langle F, G_i \rangle H_i, \quad \forall F \in H^2_{C^m} (\mathbb{D}),$$

where $\{G_i\}_{i=1}^{n}$ and $\{H_i\}_{i=1}^{n}$ are orthonormal sets in $H^2_{C^m} (\mathbb{D})$. Therefore, it is natural to ask whether the kernel of $T_n$ is nearly $S^*$ invariant subspace with a finite defect or not. In this article, we provide an affirmative answer to this question in several important cases mentioned by Liang and Partington [12]. In other words, we solve the above-mentioned problem (1) in various important cases in vector-valued Hardy spaces, providing a vectorial generalization of a result of Liang and Partington [12].

For simplicity, we first discuss the problem for rank two perturbation, that is for $T_2$, and then we state our main theorem for rank $n$ perturbation of $T_\Phi$, that is for $T_n$.

The rest of the paper is organized as follows: in Sect. 2, we study the kernel of $T_n$ whenever $\Phi = 0$ almost everywhere on the circle and provide some applications of our earlier theorem [4, Theorem 3.5]. Sections 3, 4 and 5 deal with the study of the kernel of $T_n$ in several important cases as mentioned by Liang and Partington [12] whenever $\Phi$ is non zero almost everywhere on the circle along with few applications of our earlier theorem [4, Theorem 3.5].

### 2 Kernel of finite rank perturbation of Toeplitz operator having symbol zero almost everywhere on the circle

In this section, we study the kernel of $T_n = T_\Phi + \sum_{i=1}^{n} \langle \cdot, G_i \rangle H_i$, where $\Phi = 0$ almost everywhere on $\mathbb{T}$. As we have discussed earlier first we study the kernel of $T_2$ and later we will state the main theorem corresponding to $T_n$. Note that if $\Phi = 0$ almost everywhere on $\mathbb{T}$, then the kernel of $T_2$ is given by

$$\text{Ker}T_2 = H^2_{C^m} (\mathbb{D}) \ominus \{G_1, G_2\}.$$  

It is easy to check that the kernel of $T_2$ is nearly $S^*$ invariant with defect 2 because if we consider any element $F \in \text{Ker}T_2$ with $F(0) = 0$, then

$$\langle \cdot, \overline{\phi}_j \rangle_{m \times m} [6].$$
\[ S'(F) \in \text{Ker}T_2 \cup \left( \bigvee \{G_1, G_2\} \right) = H^2_{\mathbb{C}^m}(\mathbb{D}), \]

and the defect space is \( \mathcal{F} = \bigvee \{G_1, G_2\} \). Now for the general case we have the following result:

**Theorem 2.1** Suppose \( \Phi = 0 \) almost everywhere on \( \mathbb{T} \). Then the subspace \( \text{Ker}T_n \) is nearly \( S^* \) invariant with defect \( n \) and the defect space is \( \mathcal{F} = \bigvee \{G_1, G_2, \ldots, G_n\} \).

Next, we provide a nice application of the following theorem obtained by us (see Theorem 3.5, [4]) as well as obtained independently by Ryan O’Loughlin (see Theorem 3.4, [16]) to understand the kernel of perturbed Toeplitz operator in a better way in terms of backward shift-invariant subspaces. Before that, let us recall that theorem concerning nearly \( S^* \)-invariant subspaces with defect \( p \) on vector-valued Hardy spaces \( H^2_{\mathbb{C}^m}(\mathbb{D}) \).

**Theorem 2.2** Let \( \mathcal{M} \) be a closed subspace that is nearly \( S^* \)-invariant with defect \( p \) in \( H^2_{\mathbb{C}^m}(\mathbb{D}) \) and let \( \{E_1, E_2, \ldots, E_p\} \) be any orthonormal basis for the \( p \)-dimensional defect space \( \mathcal{F} \). Let \( \{W_1, W_2, \ldots, W_r\} \) be an orthonormal basis of \( \mathcal{W} := \mathcal{M} \oplus (\mathcal{M} \cap zH^2_{\mathbb{C}^m}(\mathbb{D})) \) and let \( F_0 \) be the \( m \times r \) matrix whose columns are \( W_1, W_2, \ldots, W_r \). Then

(i) in the case where there are functions in \( \mathcal{M} \) that do not vanish at \( 0 \),

\[
\mathcal{M} = \left\{ F : F(z) = F_0(z)K_0(z) + \sum_{j=1}^{p} zk_j(z)E_j(z) : (K_0, k_1, \ldots, k_p) \in \mathcal{K} \right\},
\]

where \( \mathcal{K} \subset H^2_{\mathbb{C}^m}(\mathbb{D}) \times H^2_{\mathbb{C}^m}(\mathbb{D}) \times \cdots \times H^2_{\mathbb{C}^m}(\mathbb{D}) \) is a closed \( S^* \oplus \cdots \oplus S^* \)-invariant subspace of the vector-valued Hardy space \( H^2_{\mathbb{C}^m}(\mathbb{D}) \) and

\[
\|F\|^2 = \|K_0\|^2 + \sum_{j=1}^{p} \|k_j\|^2,
\]

(ii) in the case where all the functions in \( \mathcal{M} \) vanish at \( 0 \),

\[
\mathcal{M} = \left\{ F : F(z) = \sum_{j=1}^{p} zk_j(z)E_j(z) : (k_1, \ldots, k_p) \in \mathcal{K} \right\},
\]

with the same notation as in (i) except that \( \mathcal{K} \) is now a closed \( S^* \oplus \cdots \oplus S^* \)-invariant subspace of the vector-valued Hardy space \( H^2_{\mathbb{C}^m}(\mathbb{D}) \) and

\[
\|F\|^2 = \sum_{j=1}^{p} \|k_j\|^2. \]

Conversely, if a closed subspace \( \mathcal{M} \) of the vector-valued Hardy space \( H^2_{\mathbb{C}^m}(\mathbb{D}) \) has a representation like (i) or (ii) as above, then it is a nearly \( S^* \)-invariant subspace of defect \( p \).

For simplicity we will deal with a rank one perturbation of Toeplitz operator and let it be denoted by \( T \), that is \( T = T_\Phi + \langle G, \cdot \rangle H \) with \( \|G\|_2 = 1 \) and \( S^*H \neq 0 \). Now as an application of the above Theorem 2.2, our aim is to represent the kernel of the
operator $T$ in some special cases. It should also be observed that we can find $K$ as the $S'$ invariant subspace in $H^2_{\mathbb{C}^m}(\mathbb{D})$ like the scalar case such as

$$F_0(z)S^nK_0(z) + z \sum_{j=1}^{p} S^n k_j(z)E_j(z) \in \mathcal{M}$$

for all $n \in \mathbb{N} \cup \{0\}$. In the case $\Phi = 0$, we have $\mathcal{M} = \text{Ker}T = H^2_{\mathbb{C}^m}(\mathbb{D}) \ominus \langle G \rangle$ which is nearly $S'$ invariant with defect space $\mathcal{F} = \langle G \rangle$ by Theorem 2.1. Assume furthermore that the function $G \in H^\infty(\mathbb{D}, \mathcal{L}(\mathbb{C}, \mathbb{C}^m)) = H^\infty_{\mathbb{C}^m}(\mathbb{D})$. Now consider $F_i = P\mathcal{M}(k_0 \otimes e_i)$, where $k_0$ is the reproducing kernel at 0 and $\{e_i : 1 \leq i \leq m\}$ is a standard orthonormal basis of $\mathbb{C}^m$, generate the subspace $\mathcal{V}$ in Theorem 2.2. Without loss of generality we assume that $\{F_1, F_2, \ldots, F_r\}$ (where $r \leq m$) is a basis of $\mathcal{V}$. Using Gram–Schmidt orthonormalization, we find an orthonormal basis of $\mathcal{V}$ as follows:

$$W_1 = C_{11}F_1, W_2 = C_{21}F_1 + C_{22}F_2, \ldots, W_r = C_{r1}F_1 + C_{r2}F_2 + \cdots + C_{rr}F_r,$$

where the constant $C_{ij}$ can be determined via the process of orthonormalization. Now if we consider $G = (g_1, g_2, \ldots, g_m) \in H^\infty_{\mathbb{C}^m}(\mathbb{D})$ with $\|G\|_2 = 1$ and $\mathcal{M} = H^2_{\mathbb{C}^m}(\mathbb{D}) \ominus \langle G \rangle$, then for any $i \in \{1, 2, \ldots, m\}$ we have

$$F_i = P\mathcal{M}(k_0 \otimes e_i) = (-g_i(0)g_1, -g_i(0)g_2, \ldots, 1 - g_i(0)g_i, \ldots, -g_i(0)g_m).$$

Therefore, by Theorem 2.2, the $m \times r$ matrix $F_0$ whose columns are $\{W_1, W_2, \ldots, W_r\}$ has the following representation:

$$F_0 = \begin{bmatrix}
C_{11}(1 - \overline{g_1(0)}g_1) & C_{12}(1 - \overline{g_1(0)}g_1) + C_{22}(\overline{g_2(0)}g_1) & \cdots & C_{1r}(1 - \overline{g_1(0)}g_1) + \cdots + C_{rr}(\overline{g_r(0)}g_1) \\
C_{11}(\overline{g_1(0)}g_2) & C_{21}(\overline{g_1(0)}g_2) + C_{22}(1 - \overline{g_2(0)}g_2) & \cdots & C_{r1}(\overline{g_1(0)}g_2) + \cdots + C_{rr}(\overline{g_r(0)}g_2) \\
\vdots & \vdots & \ddots & \vdots \\
C_{11}(\overline{g_1(0)}g_m) & C_{21}(\overline{g_1(0)}g_m) + C_{22}(\overline{g_2(0)}g_m) & \cdots & C_{r1}(\overline{g_1(0)}g_m) + \cdots + C_{rr}(\overline{g_r(0)}g_m)
\end{bmatrix}.$$  

(6)

According to Theorem 2.2, $\mathcal{M}$ has the following representation:

1. In the case when $\mathcal{V} \neq \{0\}$, $\mathcal{M} = \left\{ F : F(z) = F_0(z)K_0(z) + zk_1(z)G(z) : (K_0, k_1) \in \mathcal{K} \subseteq H^2_{\mathbb{C}^r}(\mathbb{D}) \times H^2_{\mathbb{C}^r}(\mathbb{D}) \right\}$. Suppose $|G|^2 = |g_1|^2 + |g_2|^2 + \cdots + |g_m|^2$ and if we consider

$$G_0 = \begin{bmatrix}
C_{11}P(g_1 - g_1(0)|G|^2) \\
C_{21}P(g_1 - g_1(0)|G|^2) + C_{22}P(g_2 - g_2(0)|G|^2) \\
\vdots \\
C_{r1}P(g_1 - g_1(0)|G|^2) + \cdots + C_{rr}P(g_r - g_r(0)|G|^2)
\end{bmatrix} \in H^2_{\mathbb{C}^r}(\mathbb{D}).$$  

(7)
and \( g = P(\bar{z}|G|^2) \in H^2_C(\mathbb{D}) \), then the \( S^* \oplus S^* \) invariant subspace corresponding to \( \mathcal{M} \) is
\[
\mathcal{K} = \left\{ (K_0, k_1) \in H^2_C(\mathbb{D}) \times H^2_C(\mathbb{D}) : \langle K_0, \bar{z}^n G_0 \rangle_{H^2_C(\mathbb{D})} + \langle k_1, \bar{z}^n g \rangle_{H^2_C(\mathbb{D})} = 0 \right\}.
\]

for \( n \in \mathbb{N} \cup \{0\} \).

(2) In case \( \mathcal{W} = \{0\} \), \( \mathcal{M} = \left\{ F : F(z) = zk_1(z)G(z) : k_1 \in \mathcal{K} \right\} \) with \( S^* \) invariant subspace is
\[
\mathcal{K} = \left\{ k_1 \in H^2_C(\mathbb{D}) : \langle k_1, \bar{z}^n g \rangle_{H^2_C(\mathbb{D})} = 0 \text{ for } n \in \mathbb{N} \cup \{0\} \right\}.
\]

Remark 2.3 If \( G = (g_1, g_2, \ldots, g_m) \) is an arbitrary element of \( H^2_C(\mathbb{D}) \), then \( \mathcal{M} = H^2_C(\mathbb{D}) \otimes \langle G \rangle \) will be of the form:

1. In the case when \( \mathcal{W} \neq \{0\} \),
\[
\mathcal{M} = \left\{ F : F(z) = F_0(z)K_0(z) + zk_1(z)G(z) : (K_0, k_1) \in \mathcal{K} \subseteq H^2_C(\mathbb{D}) \times H^2_C(\mathbb{D}) \right\},
\]
where the \( S^* \oplus S^* \) invariant subspace corresponding to \( \mathcal{M} \) is
\[
\mathcal{K} = \left\{ (K_0, k_1) \in H^2_C(\mathbb{D}) \times H^2_C(\mathbb{D}) : \langle F_0 S^n K_0 + z S^n k_1 G, G \rangle = 0 \right\}
\]
for \( n \in \mathbb{N} \cup \{0\} \).

2. In case \( \mathcal{W} = \{0\} \), \( \mathcal{M} = \left\{ F : F(z) = zk_1(z)G(z) : k_1 \in \mathcal{K} \right\} \), where the \( S^* \) invariant subspace corresponding to \( \mathcal{M} \) is
\[
\mathcal{K} = \left\{ k_1 \in H^2_C(\mathbb{D}) : \langle z S^n k_1 G, G \rangle = 0 \text{ for } n \in \mathbb{N} \cup \{0\} \right\}.
\]

Next, we give some concrete examples without any detail which are based on the examples given [12] (see Sect. 3) with minor modifications through which we calculate the space \( \mathcal{K} \) explicitly. From now onward we denote by \( \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) \) the \( m \times m \) diagonal matrix with diagonal entries \( \{\lambda_1, \lambda_2, \ldots, \lambda_m\} \), that is
\[
\text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) = \begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
0 & 0 & \cdots & \lambda_m \\
\end{bmatrix}_{m \times m}.
\]

To proceed further, we need the following useful result in the vector-valued Hardy space \( H^2_C(\mathbb{D}) \) setting.
Proposition 2.4 Let $\Phi \in H_{\mathcal{L}(\mathcal{C}_m; \mathcal{C}_m)}^{\infty}(\mathbb{D}) \setminus \{0\}$ be of the form $\Phi = \text{diag}(\phi_1, \phi_2, \ldots, \phi_m)$ and let $\Theta$ be the inner part of $\Phi$. Then $\text{Ker} T_{\Phi^*} = \mathcal{K}_\Theta = H_{\mathcal{C}_m}^2(\mathbb{D}) \oplus \Theta H_{\mathcal{C}_m}^2(\mathbb{D})$. In particular if, $\Psi = \text{diag}(\psi_1, \psi_2, \ldots, \psi_m)$ with each $\psi_i$ is an outer function in $H_{\mathcal{C}}^2(\mathbb{D})$, then $T_{\Psi^*}$ is an injective Toeplitz operator.

Proof Let $\Psi$ be of the above form with each $\psi_i$ is an outer function and consider the element $F = (f_1, f_2, \ldots, f_m) \in \text{Ker} T_{\Psi^*}$. Then for all $i \in \{1, 2, \ldots, m\}$, $P(\overline{\psi}_i f_i) = 0$ which implies that $\overline{\psi}_i f_i \in \overline{H}_0^2$. Since each $\psi_i$ is outer, then $f_i \in \overline{H}_0^2$ and hence each $f_i = 0$. Therefore, $F = 0$ and hence $T_{\Psi^*}$ is injective. Now using the canonical factorization of each $\phi_i = \psi_i \theta_i$, where $\psi_i$ is outer and $\theta_i$ is inner, we have the following decomposition of $\Phi$:

$$\Phi = \text{diag}(\phi_1, \phi_2, \ldots, \phi_m) = \text{diag}(\phi_1, \phi_2, \ldots, \phi_m) \times \text{diag}(\theta_1, \theta_2, \ldots, \theta_m) = \Psi \cdot \Theta \quad \text{(say)}.$$

Therefore, $\Theta$ is an inner multiplier, and $\Psi$ is an outer function in the sense of V. I. Smirnov [11]. We call $\Theta$ as the inner part of $\Phi$ and $\Psi$ as the outer part of $\Phi$. Note that

$$\text{Ker} T_{\Phi^*} = \left\{ F \in H_{\mathcal{C}_m}^2(\mathbb{D}) : T_{\Psi^*} T_{\Theta^*}(F) = 0 \right\}$$

$$= \left\{ F \in H_{\mathcal{C}_m}^2(\mathbb{D}) : T_{\Theta^*}(F) = 0 \right\} = \mathcal{K}_\Theta,$$

where at the second quality we have used the fact that $T_{\Psi^*}$ is injective. This completes the proof. \(\square\)

Example

(i) Let $G = (1, 0, \ldots, 0) = k_0 \otimes e_1$, then

$$\mathcal{M} = \text{span}\{e_2, e_3, \ldots, e_m\} \oplus z H_{\mathcal{C}_m}^2(\mathbb{D}).$$

In this case, $\mathcal{W} = \text{span}\{e_2, e_3, \ldots, e_m\}$ and consider $F_0 = [e_2, e_3, \ldots, e_m]_{m \times (m-1)}$. Therefore,

$$\mathcal{M} = \left\{ F : F(z) = F_0(z)K_0(z) + zk_1(z) \otimes e_1 : (K_0, k_1) \in \mathcal{K} \subset H_{\mathcal{C}_m}^2(\mathbb{D}) \times H_{\mathcal{C}}^2(\mathbb{D}) \right\},$$

where $\mathcal{K} = H_{\mathcal{C}_m}^2(\mathbb{D})$ is a trivial $S^*$ invariant subspace.

(ii) Let $G = \frac{1}{\sqrt{m}}(\theta_1, \theta_2, \ldots, \theta_m)$ with each $\theta_i$ a non constant inner function in $H_{\mathcal{C}}^2(\mathbb{D})$ and each $\theta_i(0) = 0$. If $\Theta = \text{diag}(\theta_1, \theta_2, \ldots, \theta_m)$, then

$$\mathcal{M} = \mathcal{K}_\Theta \oplus z \mathcal{M} H_{\mathcal{C}_m}^2(\mathbb{D}) \oplus \Theta(\mathbb{C}^m \ominus \{(1, 1, \ldots, 1)\}).$$
In this case, \( G_0 = \left[ \frac{1}{\sqrt{m}} \theta_1, \frac{1}{\sqrt{m}} \theta_2, \ldots, \frac{1}{\sqrt{m}} \theta_m \right]^t \in H_{C^m}(\mathbb{D}), \ g = 0 \) and consider \( F_0 = I_{m \times m} \). Therefore, using Case(i), \( \mathcal{M} \) has the following representation:

\[
\mathcal{M} = \left\{ F : F(z) = F_0(z)K_0(z) + zk_1(z)G(z) : (K_0, k_1) \in \mathcal{K} \subset H_{C^m}(\mathbb{D}) \times H_C^2(\mathbb{D}) \right\}
\]

with the \( S^* \) invariant subspace \( \mathcal{K} = \text{Ker} T_\xi \times H_C^2(\mathbb{D}), \) where \( \xi \in H_{C(\mathbb{C},C^\infty)}^2(\mathbb{D}) \) is given by \( \xi = [\theta_1, \theta_2, \ldots, \theta_m]^t \). Using the fact 2 one can easily check that \( \mathcal{K} \) is \( S^* \) invariant subspace of \( H_{C^m+1}(\mathbb{D}) \).

(iii) If we consider \( G(z) = \left( 1 + \frac{z^k}{\sqrt{2}}, 0, \ldots, 0 \right) \) for \( k \geq 1 \), then

\[
\mathcal{M} = \text{span}\left\{ 1 - \frac{z^k}{\sqrt{2}}, z, \ldots, z^{k-1}, \frac{z^{k+1}}{\sqrt{2}}, \ldots \right\} \oplus H_C^2(\mathbb{D}) \oplus H_C^2(\mathbb{D}) \oplus \cdots \oplus H_C^2(\mathbb{D})
\]

In this case, \( G_0 = \left[ \frac{1}{\sqrt{2}}, 0, \ldots, 0 \right]^t \in H_{C^m}(\mathbb{D}), \ g(z) = \frac{z^k}{\sqrt{2}} \in H_C^2(\mathbb{D}) \) and consider \( F_0 = \text{diag}\left( \frac{1 - z^k}{\sqrt{2}}, 1, \ldots, 1 \right) \). Thus, using Case(i),

\[
\mathcal{M} = \left\{ F : F(z) = F_0(z)K_0(z) + zk_1(z)G(z) : (K_0, k_1) \in \mathcal{K} \subset H_{C^m}(\mathbb{D}) \times H_C^2(\mathbb{D}) \right\}
\]

with the \( S^* \) invariant subspace

\[
\mathcal{K} = \left\{ (K_0, k_1) \in H_{C^m}(\mathbb{D}) \times H_C^2(\mathbb{D}) : \langle (S^*)^{k-1}k_1(z) \rangle = -\langle (S^*)^kK_0, k_z \otimes e_1 \rangle \right\}.
\]

(iv) Now consider \( G(z) = \sqrt{1 - \frac{|x|^2}{m}}(k_x(z), k_x(z), \ldots, k_x(z)) \), where \( k_x \) is a reproducing kernel at \( x \in \mathbb{D} \setminus \{ 0 \} \) in \( H_C^2(\mathbb{D}) \). Then \( \mathcal{M} = \left\{ F = (f_1, f_2, \ldots, f_m) \in H_{C^m}(\mathbb{D}) : \sum_{i=1}^m f_i(z) = 0 \right\} \). Since \( P_{\mathcal{M}}(k_0 \otimes e_1) \) is non zero, then \( \mathcal{W} \) is non trivial and hence \( F_0 \) has the same form as in (6) with each \( g_i(z) = k_x(z) \). In this case \( G_0 = 0, g(z) = m\overline{x}k_x(z) \) which is an outer function. Moreover, using Case(i), \( \mathcal{M} = \left\{ F : F(z) = F_0(z)K_0(z) + zk_1(z)G(z) : (K_0, k_1) \in \mathcal{K} \subset H_C^2(\mathbb{D}) \times H_C^2(\mathbb{D}) \right\} \) with the \( S^* \) invariant subspace \( \mathcal{K} = H_C^2(\mathbb{D}) \times \{ 0 \} \).

For the case \( n = m \), we have the following interesting example in this context.

**Example** If we consider \( T_m = T_\Phi + \sum_{i=1}^m \langle G_i \rangle H_l \) with \( \Phi = 0 \) almost everywhere on \( \mathbb{T} \). Then the defect space \( \mathcal{F} = \text{span}\{ G_1, G_2, \ldots, G_m \} \). Consider \( G_1 = \theta \otimes
\[ e_1, G_2 = \theta_2 \otimes e_2, \ldots, G_m = \theta_m \otimes e_m, \] where each \( \theta_i \) is a non constant inner function in \( H^2_{\mathcal{L}}(\mathbb{D}) \). Let \( \Theta = \text{diag}(\theta_1, \theta_2, \ldots, \theta_m) \). Therefore, the kernels of perturbed Toeplitz operators are given by

\[ K = \text{Ker} T_m = H^2_{\mathcal{L}}(\mathbb{D}) \ominus \mathcal{F} = K_{\Theta} \oplus \mathcal{Z} H^2_{\mathcal{L}}(\mathbb{D}) \].

Thus, \( \mathcal{W} \neq \{0\} \) and consider

\[ F_0 = \text{diag} \left( \frac{1 - \bar{\theta}_1(0)\theta_1}{1 - |\theta_1(0)|^2}, \frac{1 - \bar{\theta}_2(0)\theta_2}{1 - |\theta_2(0)|^2}, \ldots, \frac{1 - \bar{\theta}_m(0)\theta_m}{1 - |\theta_m(0)|^2} \right) \].

Therefore, using case (i) of Theorem 2.2 we have

\[ \mathcal{M} = \left\{ F : F(z) = F_0(z)K_0(z) + \sum_{i=1}^{m} z^{k_i}G_i(z) \right\} \cap \mathcal{K} \subseteq H^2_{\mathcal{L}}(\mathbb{D}) \times H^2_{\mathcal{L}}(\mathbb{D}) \),

where the \( S^* \oplus S^* \oplus \cdots \oplus S^* \) invariant subspace \( \mathcal{K} \) corresponding to \( \mathcal{M} \) is \( \mathcal{K} = K_{\zeta} \times H^2_{\mathcal{L}}(\mathbb{D}) \) and \( \zeta = \text{diag}(\theta_1 - \theta_1(0), \theta_2 - \theta_2(0), \ldots, \theta_m - \theta_m(0)) \). One can check that \( \mathcal{K} \) is the invariant subspace of \( H^2_{\mathcal{L}}(\mathbb{D}) \) using Proposition (2.4).

Next we consider the case when \( \Phi \in L^\infty(\mathbb{T}, \mathcal{L}(\mathbb{C}^m, \mathbb{C}^m)) \) is non-zero almost everywhere on \( \mathbb{T} \) and with this assumption we consider three important subcases in the next three sections. To proceed further we need the following useful results:

**Theorem 2.5** For \( \Phi, \Psi \in L^\infty(\mathbb{T}, \mathcal{L}(\mathbb{C}^m, \mathbb{C}^m)) \); if either \( \Psi^* \in H^\infty_{\mathcal{L}}(\mathbb{C}^m, \mathbb{C}^m) \) or \( \Phi \in H^\infty_{\mathcal{L}}(\mathbb{C}^m, \mathbb{C}^m) \), then \( T_{\Psi} T_{\Phi} \) is a Toeplitz operator; in both cases \( T_{\Psi} T_{\Phi} = T_{\Psi \Phi} \).

The proof of the above theorem follows similarly as in the scalar-valued case and hence left it to the reader. On the other hand, it is important to observe that the converse of the theorem is not true in general. For example, consider \( \Psi = \text{diag}(e^{i0}, 0, 0, \ldots, 0) \), and \( \Phi = \text{diag}(0, e^{-i0}, 0, \ldots, 0) \). Then it is easy to observe that \( \Psi \in H^\infty_{\mathcal{L}}(\mathbb{C}^m, \mathbb{C}^m) \) and \( \Phi^* \in H^\infty_{\mathcal{L}}(\mathbb{C}^n, \mathbb{C}^n) \) but \( T_{\Psi} T_{\Phi} = 0 \) is a Toeplitz operator. Now we denote \( Z = \text{diag}(z, z, \ldots, z) \in H^\infty_{\mathcal{L}}(\mathbb{C}^m, \mathbb{C}^m) \) and hence \( T_{Z}^* = T_{Z^*} \). Thus,

\[ T_{Z^*} T_{\Phi} = T_{Z^* \Phi} = T_{\Phi Z^*}, \forall \Phi \in L^\infty(\mathbb{T}, \mathcal{L}(\mathbb{C}^m, \mathbb{C}^m)). \]

Next we provide an equivalent condition on the element \( F \) to be in the kernel of \( T_2 = T_{\Phi} + \sum_{i=1}^{2} \langle \cdot, G_i \rangle H_i \). To do that suppose \( F \in \text{Ker} T_2 \) with \( F(0) = 0 \). Then the following equivalent conditions hold.

\[ T_2(F) = 0 \iff T_{\Phi}(F) + \sum_{i=1}^{2} \langle F, G_i \rangle H_i = 0 \iff \Phi F + \sum_{i=1}^{2} \langle F, G_i \rangle H_i \in \overline{H}_0^2. \]  

(9)

Applying \( T_{Z^*} \) on both sides of (9) and using Theorem (2.5), we have the following equivalent conditions.
\[ T_{\phi \Sigma}(F) + \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i = 0 \Leftrightarrow P_m \left( \phi \frac{F}{z} + \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i \right) = 0 \] 
(10)

\[ \Leftrightarrow T_{\phi \Sigma}(\frac{F}{z}) + \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i = 0 \Leftrightarrow \phi \frac{F}{z} + \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i \in \mathcal{H}^2_0. \]

Now if we recapitulate our problem, actually we have to show the kernel of \( T_2 \) is nearly \( S^* \) invariant with finite defect and to do so we have to find a vector \( V \) in some appropriate finite dimensional subspace \( \mathcal{F} \) such that

\[ S^* F + V \in \text{Ker} T_2 \quad \text{with} \quad F \in \text{Ker} T_2, \, F(0) = 0, \]

which is equivalent to the following equations:

\[ T_2(S^* F + V) = 0 \Leftrightarrow T_{\phi \Sigma}(\frac{F}{z} + V) + \sum_{i=1}^{2} \left( \frac{F}{z} + V, G_i \right) H_i = 0 \] 
(11)

\[ \Leftrightarrow P_m \left( \phi V - \sum_{i=1}^{2} \left( F, G_i \right) S^* H_i + \sum_{i=1}^{2} \left( \frac{F}{z} + V, G_i \right) H_i \right) = 0, \text{(using (2.7))} \] 
(12)

\[ \Leftrightarrow \phi V - \sum_{i=1}^{2} \left( F, G_i \right) S^* H_i + \sum_{i=1}^{2} \left( \frac{F}{z} + V, G_i \right) H_i \in \mathcal{H}^2_0. \] 
(13)

In the following three sections, we are going to show that kernel of \( T_2 \) is nearly \( S^* \) invariant with a finite defect in various important cases mentioned by Liang and Partington [12]. Furthermore, we also calculate the defect space \( \mathcal{F} \) explicitly in those mentioned cases.

### 3 Kernel of finite rank perturbation of Toeplitz operator having symbol an inner multiplier

In this section, we consider \( \Phi = \Theta \) is an inner function. Our main aim is to show that the kernel of \( T_n = T_{\phi} + \sum_{i=1}^{n} \langle ., G_i \rangle H_i \) is nearly \( S^* \) invariant with finite defect and to calculate the defect space explicitly. To avoid complicacies in the calculations, we restrict our self in the case \( n = 2 \) and finally we state our result in general setting. For this purpose, let us consider \( F \in \text{Ker} T_2 \) with \( F(0) = 0 \). In this context the Eq. (10) becomes

\[ \Theta \frac{F}{z} + \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i = 0. \]
(14)

Next by acting \( T_{\Theta}^* \) on both sides of (14), we have
\[
T_\Theta^* \left( \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i \right) = \Theta^* \left( \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i \right) = - \frac{F}{z} \in H^2_\mathbb{C}^n(\mathbb{D}).
\]

Therefore, (11) becomes \( \Theta \left( \frac{F}{z} + V \right) + \sum_{i=1}^{2} \left( \frac{F}{z} + V, G_i \right) H_i = 0 \). Using (14), the above equation is equivalent to

\[
\Theta V - \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i + \sum_{i=1}^{2} \left( \frac{F}{z} + V, G_i \right) H_i = 0. \tag{15}
\]

Let \( V \in H^2_\mathbb{C}^n(\mathbb{D}) \) be such that \( V = \sum_{i=1}^{2} \langle F, G_i \rangle T_\Theta^* (S^* H_i) = \sum_{i=1}^{2} \langle F, G_i \rangle T_\Theta^* (S^* (T_\Theta^* (H_i))) \),

because \( T_\Theta^* S^* = T_\Theta^* T_z = T_z T_\Theta^* = S^* T_\Theta^* \) and, therefore, using (14) we conclude that \( V \) satisfies the above Eq. (15).

Using the above construction of \( V \) in (16) we define the defect space

\[
\mathcal{F} = \bigvee_{i=1}^{2} \left\{ T_\Theta^* (S^* H_i) \right\} = \bigvee_{i=1}^{2} \left\{ S^* (T_\Theta^* H_i) \right\},
\]

with dimension at most 2. Hence, \( \text{Ker} T_2 \) is nearly \( S^* \) invariant with defect at most 2. Therefore, by repeating the above calculations again we have the following theorem regarding the kernel of \( T_n \).

**Theorem 3.1** If \( \Phi = \Theta \) is an inner multiplier, then the subspace \( \text{Ker} T_n \) is nearly \( S^* \) invariant subspace with defect at most \( n \) and the defect space is

\[
\mathcal{F} = \bigvee_{i=1}^{n} \left\{ T_\Theta^* (S^* H_i) \right\} = \bigvee_{i=1}^{n} \left\{ S^* (T_\Theta^* H_i) \right\}.
\]

The following example is similar to the example given in [12] (see Sect. 2) for the scalar-valued case.

**Example** If \( \Phi(z) = \text{diag}(z^p, z^p, \ldots, z^p) \), \( p \in \mathbb{N} \) and \( n = 1 \). Then \( \text{Ker} T_1 \) is nearly \( S^* \) invariant subspace with defect 1 and the defect space is \( \mathcal{F} = \langle (S^*)_1 H_1 \rangle \).

Now coming back to the application part of Theorem 2.2, as discussed earlier we deal with rank one perturbation of Toeplitz operator. Thus, \( T = T_\Theta + \langle ., G \rangle H \), with \( \|G\|_2 = 1 \) and \( S^* H \neq 0 \). Therefore, \( \mathcal{M} = \text{Ker} T \subseteq \langle \Theta^* H \rangle \) and \( \mathcal{F} = \langle S^* (T_\Theta^* H) \rangle = \langle S^* (\Theta^* H) \rangle \). Next we consider \( F = \mu \Theta^* H \in \mathcal{M} \) satisfying \( T(F) = 0 \) which is equivalent to

\[
\mu (1 + \langle \Theta^* H, G \rangle) = 0.
\]

Therefore, we have the following two cases to consider:

Case 1. If \( 1 + \langle \Theta^* H, G \rangle \neq 0 \), then \( \mathcal{M} = \{0\} \) is a trivial \( S^* \) invariant subspace.
Case 2. If $1 + \langle \Theta^* H, G \rangle = 0$, then it yields $\mathcal{M} = \langle \Theta^* H \rangle$. Let $\{A_k\}_{k=0}^{\infty}$ be the coefficients of the Taylor series expansion of $\Theta^* H$ (since $\Theta^* H \in H^2_{\mathbb{C}}(\mathbb{D})$) and we calculate the subspace $\mathcal{K}$ in two sub-cases.

(i) If $A_0 \neq 0$, then there exists at least one function in $\mathcal{M}$ that do not vanish at 0. Now $\mathcal{W} = \mathcal{M} \ominus (\mathcal{M} \cap zH^2_{\mathbb{C}}(\mathbb{D}))$ has dimension 1 (since $\dim \mathcal{M} = 1$). Therefore, exactly one $F_i \neq 0$ for some $i (1 \leq i \leq m)$ which generates $\mathcal{W}$. On the other hand,

\[
F_i = P_{\mathcal{M}}(k_0 \otimes e_i) = \frac{\langle k_0 \otimes e_i, \Theta^* H \rangle \Theta^* H}{\| \Theta^* H \|_2^2},
\]

and hence $F_0 = [C_{11}F_i]_{m \times 1}$. Thus,

\[
\mathcal{M} = \left\{ F : F = K_0 C_{11} \frac{\langle k_0 \otimes e_i, \Theta^* H \rangle \Theta^* H}{\| \Theta^* H \|_2^2} + \frac{k_1 (\Theta^* H - A_0)}{\| S^* (\Theta^* H) \|} : (K_0, k_1) \in \mathcal{K} \subset H^2_{\mathbb{C}}(\mathbb{D}) \times H^2_{\mathbb{C}}(\mathbb{D}) \right\}.
\]

Note that the structure of $\mathcal{M}$ forces to conclude that $K_0 \in \mathbb{C}$ and $k_1 (\Theta^* H - A_0) = \xi \Theta^* H$ with $\xi \in H^2_{\mathbb{C}}(\mathbb{D})$ which will be valid if and only if $K_0 \in \mathbb{C}$ and $k_1 = 0$. Therefore, the required nearly $S^*$ invariant subspace of finite defect is

\[
\mathcal{M} = \left\{ F : F = K_0 C_{11} \frac{\langle k_0 \otimes e_i, \Theta^* H \rangle \Theta^* H}{\| \Theta^* H \|_2^2} : (K_0, 0) \in \mathcal{K} \subset H^2_{\mathbb{C}}(\mathbb{D}) \times H^2_{\mathbb{C}}(\mathbb{D}) \right\}
\]

and the corresponding $S^* \oplus S^*$ invariant subspace is $\mathcal{K} = \mathbb{C} \times \{0\}$ of $H^2_{\mathbb{C}}(\mathbb{D})$.

(ii) If $A_0 = 0$, then $\mathcal{W} = \{0\}$ and hence $F_0 = 0$. In that case the nearly $S^*$ invariant subspace of finite defect is

\[
\mathcal{M} = \langle \Theta^* H \rangle = \left\{ F : F = k_1 \frac{\Theta^* H}{\| S^* (\Theta^* H) \|} : k_1 \in \mathcal{K} \subset H^2_{\mathbb{C}}(\mathbb{D}) \right\}
\]

with the corresponding $S^*$ invariant subspace is $\mathcal{K} = \mathbb{C}$ of $H^2_{\mathbb{C}}(\mathbb{D})$.

4 Kernel of finite rank perturbation of Toeplitz operator with symbol having factorization in $\mathcal{G}H^\infty (\mathcal{L}(\mathbb{C}^m))$

In this section, we deal with very special type of $\Phi \in L^\infty (T, \mathcal{L}(\mathbb{C}^m, \mathbb{C}^m))$ such that $\Phi = F_1^* F_2$, with $F_j \in \mathcal{G}H^\infty (\mathcal{L}(\mathbb{C}^m, \mathbb{C}^m))$ for $j = 1, 2$. Here $\mathcal{G}H^\infty (\mathcal{L}(\mathbb{C}^m, \mathbb{C}^m))$ denotes the set of all invertible elements in $H^\infty (\mathcal{L}(\mathbb{C}^m, \mathbb{C}^m))$. For more details in the literature, we refer to [1], and the references cited therein. Note that the commutativity property makes a major difference while dealing with the vector-valued case in comparison to the scalar-valued case. Indeed, in our setting, we do
not have the commutativity between $F_1$ and $F_2$ whereas in the scalar-valued case, they do so (see Section 2.3). Therefore, it is essential to give little more details of the analysis in our setting for a better understanding of the reader. Note that for any vector $F \in \text{Ker}T_2$ with $F(0) = 0$, (9) can be rewritten as

$$T_{F_1}T_{F_2}(F) + \sum_{i=1}^{2} \langle F, G_i \rangle H_i = 0. \quad (17)$$

Since $F(0) = 0$ and $T_zT_{F_1} = T_{F_1}T_z$, then using these fact along with the action of $T_z$ on both sides of (17), we get

$$T_{F_1}\left( F_2 \frac{F}{z} \right) + \sum_{i=1}^{2} \langle F, G_i \rangle S^*H_i = 0. \quad (18)$$

Now by applying $T_{F_2}, T_{F_1^{-1}}$ on both sides of (18), we have

$$\frac{F}{z} + \sum_{i=1}^{2} \langle F, G_i \rangle T_{F_2}T_{F_1^{-1}}(S^*H_i) = 0. \quad (19)$$

Thus, using (18) our desired result (11) is equivalent to

$$T_{F_1}T_{F_2}(V) - \sum_{i=1}^{2} \langle F, G_i \rangle S^*H_i + \sum_{i=1}^{2} \left\langle \frac{F}{z} + V, G_i \right\rangle H_i = 0. \quad (20)$$

Now we choose $V$ in such a way so that

$$T_{F_1}(F_2V) = \sum_{i=1}^{2} \langle F, G_i \rangle S^*H_i \Leftrightarrow V = \sum_{i=1}^{2} \langle F, G_i \rangle T_{F_2}T_{F_1^{-1}}(S^*H_i).$$

Next if we consider the above $V$, then using (19) the left-hand side of (20) becomes

$$\sum_{i=1}^{2} \langle F, G_i \rangle S^*H_i - \sum_{i=1}^{2} \langle F, G_i \rangle S^*H_i$$

$$+ \sum_{i=1}^{2} \left\langle \frac{F}{z} + \sum_{i=1}^{2} \langle F, G_i \rangle T_{F_2}T_{F_1^{-1}}(S^*H_i), G_i \right\rangle H_i = 0.$$

Moreover, from the choice of $V$ it is clear that the defect space should be $\mathcal{F} = \bigvee_{i=1}^{2} \left\{ F_2^{-1}S^*(T_{F_1^{-1}}H_i) \right\}$ with dimension at most 2. Therefore, in general setting, we have the following theorem regarding the kernel of $T_n$:

**Theorem 4.1** Let $\Phi = F_1F_2$ with $F_1, F_2 \in \mathcal{C}^{m}(\mathcal{L}(\mathbb{C}^m))$. Then the subspace $\text{Ker}T_n$ is nearly $S^*$ invariant with defect at most $n$ and the defect space is
\[
\mathcal{F} = \bigvee_{i=1}^{n} \left\{ F_{2}^{-1}T_{F_{1}^{-1}}(S^{*}H_{i}) \right\} = \bigvee_{i=1}^{n} \left\{ F_{2}^{-1}S^{*}(T_{F_{1}^{-1}}H_{i}) \right\}.
\]

**Remark 4.2**

1. If we suppose \( \Phi^{*} \in \mathcal{G}H^{\infty}(\mathcal{L}(\mathbb{C}^{m})) \), then the kernel of \( T_{n} \) is nearly \( S^{*} \) invariant with defect at most \( n \) and the defect space is \( \mathcal{F} = \bigvee_{i=1}^{n} \left\{ S^{*}(T_{\Phi^{*}H_{i}}) \right\} \).
2. If we consider \( \Phi \in \mathcal{G}H^{\infty}(\mathcal{L}(\mathbb{C}^{m})) \), then the kernel of \( T_{n} \) is also nearly \( S^{*} \) invariant with defect at most \( n \) and the defect space will be \( \mathcal{F} = \bigvee_{i=1}^{n} \left\{ T_{\Phi^{*}}S^{*}(H_{i}) \right\} \).

For simplicity, if we assume \( n = 1 \) and \( \Phi \in \mathcal{G}H^{\infty}(\mathcal{L}(\mathbb{C}^{m})) \), then we have the following corollary.

**Corollary 4.3** *For* \( n = 1 \) *and* \( \Phi \in \mathcal{G}H^{\infty}(\mathcal{L}(\mathbb{C}^{m})) \), *the subspace* \( \text{Ker}T_{1} \) (i.e., \( \text{Ker}T \)) *is nearly* \( S^{*} \) *invariant with defect at most* \( 1 \) *and the defect space is* \( \mathcal{F} = \left( \frac{S^{*}H}{\Phi} \right) \).

We now discuss about the application part of Theorem 2.2. Since in this section we consider that \( U \in \mathcal{G}H^{\infty}(\mathcal{L}(\mathcal{C}^{m})) \) almost everywhere on \( \mathbb{T} \), then \( \mathcal{M} = \text{Ker}T \subseteq \langle F_{2}^{-1}(T_{F_{1}^{-1}}H) \rangle \) and the defect space is

\[
\mathcal{F} = \langle F_{2}^{-1}T_{F_{1}^{-1}}(S^{*}H) \rangle = \langle F_{2}^{-1}S^{*}(T_{F_{1}^{-1}}H) \rangle.
\]

Let us consider any vector \( F = \lambda F_{2}^{-1}(T_{F_{1}^{-1}}H) \in \mathcal{M} \) satisfying \( T(F) = 0 \). Then it is equivalent to the following

\[
\lambda(1 + \langle F_{2}^{-1}(T_{F_{1}^{-1}}H), G \rangle) = 0. \tag{21}
\]

Therefore, according to (21), we have the following two cases.

**Case I:** If \( 1 + \langle F_{2}^{-1}(T_{F_{1}^{-1}}H), G \rangle \neq 0 \), then it yields that \( \mathcal{M} = \{0\} \) which is a trivial \( S^{*} \) invariant subspace.

**Case II:** If \( 1 + \langle F_{2}^{-1}(T_{F_{1}^{-1}}H), G \rangle = 0 \), then \( \mathcal{M} = \langle F_{2}^{-1}(T_{F_{1}^{-1}}H) \rangle \).

Let \( \{A_{k}\}_{k=0}^{\infty} \) be the Taylor coefficients of \( T_{F_{1}^{-1}}H \) and let \( \{\Phi_{k}\}_{k=0}^{\infty} \) be the Taylor coefficients of \( F_{2}^{-1} \). Therefore, it follows that

\[
[F_{2}^{-1}(T_{F_{1}^{-1}}H)](z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \Phi_{k}A_{n-k} \right) z^{n}, \quad \text{and} \quad zS^{*}(T_{F_{1}^{-1}}H)(z) = (T_{F_{1}^{-1}}H)(z) - A_{0}.
\]

Depending upon the context, we again divide our analysis into two sub-cases to calculate \( \mathcal{K} \).
In this section, we consider kernels of finite rank perturbation of Toeplitz operator having
and hence kernels of perturbed Toeplitz operators Page 17 of 28

Following the above identities, we have

$$K = \left\{ F : F = F_0 K_0 + z k_1 E_1 \mid (K_0, k_1) \in \mathcal{K} \subset H^2_{\mathbb{C}}(\mathbb{D}) \times H^2_{\mathbb{C}}(\mathbb{D}) \right\}.$$ 

Sub-case I: If $F^{-1}_0(T_{F^{-1}_1H})(0) \neq 0$, then $\mathcal{W}$ has dimension exactly 1 and,

$$F_0 = [C_1 F^{-1}_0(T_{F^{-1}_1H})]_{m \times 1}, \text{ and } E(z) = \frac{F^{-1}_2 S^*(T_{F^{-1}_1H})}{\|F^{-1}_2 S^*(T_{F^{-1}_1H})\|^2}.$$ 

Therefore, from the first case of Theorem 2.2 it follows that

$$\mathcal{M} = \{F^{-1}_0(T_{F^{-1}_1H})\} = \{F : F = F_0 K_0 + z k_1 E_1 : (K_0, k_1) \in \mathcal{K} \subset H^2_{\mathbb{C}}(\mathbb{D}) \times H^2_{\mathbb{C}}(\mathbb{D})\}.$$ 

Thus, the subspace $\mathcal{M}$ has the following representation:

$$\mathcal{M} = \{F : F = C_1 K_0 \frac{F^{-1}_2(T_{F^{-1}_1H})}{\|F^{-1}_2(T_{F^{-1}_1H})\|^2} : (K_0, 0) \in \mathcal{K}\}.$$ 

with $\mathcal{K} = \mathbb{C} \times \{0\}$ is a $S^* \oplus S^*$ invariant subspace of $H^2(\mathbb{D}, \mathbb{C}^2)$.

Sub-case II: If $F^{-1}_0(T_{F^{-1}_1H})(0) = \Phi_0 A_0 = 0$, then $\mathcal{W} = \{0\}$ and hence from case(ii) of Theorem 2.2, we have

$$\mathcal{M} = \{F^{-1}_0(T_{F^{-1}_1H})\} = \{F : F = k_1 \frac{F^{-1}_2(T_{F^{-1}_1H} - A_0)}{\|F^{-1}_2 S^*(T_{F^{-1}_1H})\|} : k_1 \in \mathcal{K} \subset H^2_{\mathbb{C}}(\mathbb{D})\}.$$ 

which will be true if and only if $A_0 = 0$ and $k_1 \in \mathbb{C}$. Thus, we have the subspace

$$\mathcal{M} = \{F : F = k_1 \frac{F^{-1}_2(T_{F^{-1}_1H})}{\|F^{-1}_2 S^*(T_{F^{-1}_1H})\|} : k_1 \in \mathcal{K}\}$$ 

with $\mathcal{K} = \mathbb{C}$ an $S^*$ invariant subspace of $H^2_{\mathbb{C}}(\mathbb{D})$.

5 Kernel of finite rank perturbation of Toeplitz operator having symbol adjoint of an inner multiplier

In this section, we consider $\Phi(z) = \Theta^*(z)$, where $\Theta$ is a non constant inner multiplier. It is well known that the kernel of the Toeplitz operator $T_{\Theta^*}$ is the model space $\mathcal{K}_\Theta$. In other words $\text{Ker} T_{\Theta^*} = \mathcal{K}_\Theta = H^2_{\mathbb{C}}(\mathbb{D}) \ominus \Theta H^2_{\mathbb{C}}(\mathbb{D})$. Compare to
scalar valued case our analysis mainly differ in two places. First, in Sect. 5.1 to show that the subspace $K$ is backward shift invariant and second, in Sect. 5.2 conditions (38) and (39) are not exactly similar with conditions (3.8) and (3.9) given in [12] (see Sect. 3) to get the explicit expression of $K$. Perhaps it is better for the reader if we provide little details of the analysis in this section. To proceed further, like in the previous section first we find an equivalent condition (10) for any vector $F \in \text{Ker} T_2$ with $F(0) = 0$. Note that

$$\Theta^* F + \sum_{i=1}^2 \langle F, G_i \rangle S^* H_i \in \overline{H}_0^2 \iff \frac{F}{z} + \sum_{i=1}^2 \langle F, G_i \rangle \Theta S^* H_i \in \Theta \overline{H}_0^2. \quad (22)$$

Therefore, in this case, Eqs. (12) and (13) changed into

$$P_m \left( \Theta^* V - \sum_{i=1}^2 \langle F, G_i \rangle S^* H_i + \sum_{i=1}^2 \left( \frac{F}{z} + V, G_i \right) H_i \right) = 0. \quad (23)$$

Next, we consider three cases for the construction of the defect space $\mathcal{F}$ in this case.

Case 1: Suppose $G_1 \in \Theta H_{C^n}^2(\mathbb{D})$ and $G_2 \in \Theta H_{C^n}^2(\mathbb{D})$. Due to condition (22), it follows that

$$\left( \frac{F}{z} + \sum_{i=1}^2 \langle F, G_i \rangle \Theta S^* H_i, G_1 \right) = 0. \quad (24)$$

and

$$\left( \frac{F}{z} + \sum_{i=1}^2 \langle F, G_i \rangle \Theta S^* H_i, G_2 \right) = 0. \quad (25)$$

Now we consider $V = \sum_{i=1}^2 \langle F, G_i \rangle \Theta S^* H_i$. Then substituting this $V$ and using the above two identities (24) and (25), the left-hand side of (23) becomes

$$P_m \left( \Theta^* \left( \sum_{i=1}^2 \langle F, G_i \rangle \Theta S^* H_i \right) - \sum_{i=1}^2 \langle F, G_i \rangle S^* H_i \right.$$  
$$+ \sum_{i=1}^2 \left( \frac{F}{z} + \sum_{j=1}^2 \langle F, G_j \rangle \Theta S^* H_j, G_i \right) H_i \right) = 0.$$

Thus, we define the defect space as

$$\mathcal{F} = \bigvee_{i=1}^2 \{ \Theta S^* H_i \} \quad (26)$$

and hence $\text{Ker} T_2$ is nearly $S^*$ invariant with defect at most 2.
Case 2: In this case, we consider $G_1 \in \Theta H^2_\mathbb{C}_m(\mathbb{D})$ and $G_2 \notin \Theta H^2_\mathbb{C}_m(\mathbb{D})$ (On a similar note, one can assume that $G_1 \notin \Theta H^2_\mathbb{C}_m(\mathbb{D})$ and $G_2 \notin \Theta H^2_\mathbb{C}_m(\mathbb{D})$, then the corresponding analysis will be of similar kind). First, we note that (24) holds in this case. Now if the identity (25) also hold in this case, then the defect space will be exactly same as in Case 1. Therefore, we assume that the identity (25) does not hold in this case, then the vector $G_2$ can be decomposed in the following way:

$$G_2 = U + W \quad \text{with} \quad U \neq 0 \in \mathcal{K}_\Theta = \text{Ker} T_{\Theta^*} \quad \text{and} \quad W \in \Theta H^2_\mathbb{C}_m(\mathbb{D}).$$

Thus,

$$T_{\Theta^*}(U) = P_m(\Theta^* U) = 0.$$  \quad (27)

Now we choose $V = \sum_{j=1}^2 \langle F, G_j \rangle \Theta S^* H_j + \xi_F U$, where $\xi_F$ is a constant which satisfies the following identity

$$\xi_F \|U\|^2 = -\left\langle \frac{F}{z} + \sum_{j=1}^2 \langle F, G_j \rangle \Theta S^* H_j, G_2 \right\rangle \neq 0.$$  \quad (28)

If we substitute $V$ in the left-hand side of (23), and then using (27) we have

$$\sum_{i=1}^2 \left\langle \frac{F}{z} + \left( \sum_{j=1}^2 \langle F, G_j \rangle \Theta S^* H_j + \xi_F U \right), G_i \right\rangle H_i$$

$$= \left\langle \frac{F}{z} + \left( \sum_{j=1}^2 \langle F, G_j \rangle \Theta S^* H_j + \xi_F U \right), G_2 \right\rangle H_2 \quad \text{(using (5.3) and } \langle U, G_1 \rangle = 0)$$

$$= \left( \left\langle \frac{F}{z} + \sum_{j=1}^2 \langle F, G_j \rangle \Theta S^* H_j, G_2 \right\rangle + \xi_F \|U\|^2 \right) H_2 = 0. \quad \text{(using (5.7))}$$

Therefore, using the construction of $V$, we define the defect space as

$$\mathcal{F} = \sqrt{\{ \Theta S^* H_1, \Theta S^* H_2, P_{\mathcal{K}_\Theta} G_2 \}},$$  \quad (29)

where $P_{\mathcal{K}_\Theta} : L^2 \to \mathcal{K}_\Theta$ is an orthogonal projection onto $\mathcal{K}_\Theta$. Therefore, Ker$T_2$ is nearly $S^*$-invariant with defect at most 3. The defect space is either given by (29) or given by (26) depending upon the condition (25).

Case 3: In this case, we consider that none of $G_1, G_2$ is in $\Theta H^2_\mathbb{C}_m(\mathbb{D})$, that is, $G_1 \notin \Theta H^2_\mathbb{C}_m(\mathbb{D})$ and $G_2 \notin \Theta H^2_\mathbb{C}_m(\mathbb{D})$. Therefore, we decompose the vectors $G_1, G_2$ in the following way:

$$G_1 = X_1 + \Theta Y_1 \quad \text{and} \quad G_2 = X_2 + \Theta Y_2,$$

where $X_i(\neq 0) \in \mathcal{K}_\Theta$ and $Y_i \in H^2_\mathbb{C}_m(\mathbb{D})$ for $i = 1, 2$. From the condition (22) we have the following:
\[ F \left( \frac{F}{z} + \sum_{i=1}^{2} \langle F, G_i \rangle \Theta S^* H_i \right) = \Theta F^0, \text{ for some } F^0 \in \mathbb{H}^2_0. \]  

(30)

In this context, we will find such a \( V \) so that the equivalent condition (11) holds, i.e.,

\[ \Theta^* \left( \frac{F}{z} + V \right) + \sum_{i=1}^{2} \left( \frac{F}{z} + V, G_i \right) H_i \in \mathbb{H}^2_0 \]  

(31)

which by (30) is equivalent to

\[ F^0 - \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i + \Theta^* V \]

\[ + \sum_{i=1}^{2} \left( F^0 - \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i + \Theta^* V, \Theta^* G_i \right) H_i \in \mathbb{H}^2_0. \]  

(32)

Now we can decompose \( F^0 \) as \( F^0 = F^0_1 + F^0_2 \), where

\[ F^0_1 \in \mathcal{N} := \bigvee \{ \Theta^* X_1, \Theta^* X_2 \} \subseteq \mathbb{H}^2_0 \text{ and } F^0_2 \in \mathbb{H}^2_0 \cap \mathcal{N}. \]

Then it implies that

\[ \langle F^0_2, \Theta^* X_i \rangle = 0 \text{ and } \langle F^0_2, Y_i \rangle = 0 \text{ for } i = 1, 2. \]

Hence, using these facts, (32) is equivalent to

\[ F^0_1 - \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i + \Theta^* V \]

\[ + \sum_{i=1}^{2} \left( F^0_1 - \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i + \Theta^* V, \Theta^* G_i \right) H_i \in \mathbb{H}^2_0 \]

\[ \Leftrightarrow F^0_1 - \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i + \Theta^* V \]

\[ + \sum_{i=1}^{2} \left( F^0_1 - \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i + \Theta^* V, \Theta^* X_i + Y_i \right) H_i \in \mathbb{H}^2_0. \]

Let us choose \( V = \sum_{i=1}^{2} \langle F, G_i \rangle \Theta S^* H_i - \Theta F^0_1 \). Thus, our chosen vector \( V \) fulfill all the requirements and from the construction of \( V \), we can define the defect space as follows:

\[ \mathcal{F} = \bigvee \left\{ \Theta S^* H_1, \Theta S^* H_2, P_{K_0} G_1, P_{K_0} G_2 \right\} \]

with dimension at most 4. Consequently, the kernel of \( T_2 \) is nearly \( S^* \)-invariant with defect at most 4. Therefore, by repeating the above argument once again we have the following theorem in general context regarding the kernel of \( T_n \).
Theorem 5.1 Suppose \( \Phi(z) = \Theta^*(z) \), where \( \Theta \in H^\infty_{\mathbb{C}^m} (\mathbb{D}) \) is an inner multiplier, then the following statements hold.

(i) If \( G_j \in \Theta H^2_{\mathbb{C}^m} (\mathbb{D}) \), \( j \in \{1, 2, \ldots, n\} \), then the subspace \( \text{Ker} T_n \) is nearly \( S^* \) invariant with defect at most \( n \) and the defect space is \( \mathcal{F} = \sqrt{\Theta S^* H_j : j = 1, 2, \ldots, n} \).

(ii) If \( G_j \not\in \Theta H^2_{\mathbb{C}^m} (\mathbb{D}) \) for \( j \in \Lambda_i \subset \{1, 2, \ldots, n\} \), then the kernel of \( T_n \) is nearly \( S^* \) invariant subspace of \( H^2_{\mathbb{C}^m} (\mathbb{D}) \) with defect at most \( n + 1 \) and the defect space is

\[
\mathcal{F} = \sqrt{\Theta S^* H_i, P_{K_\Theta} G_j : i = 1, 2, \ldots, n \text{ and } j \in \Lambda_i}. 
\]

Like other sections, we now discuss the application part of Theorem 2.2 in this context. For the operator \( T \), Eq. (9) is equivalent to

\[
T\Theta^*F + \langle F, G \rangle H = 0 \iff \Theta^*F + \langle F, G \rangle H \in \overline{H_0^2} \iff F + \langle F, G \rangle \Theta H \in \overline{\Theta H_0^2}.
\]

Observing the above equivalent criteria, we say that the kernel of the operator \( T \) satisfies \( \mathcal{M} = \text{Ker}T \subset (H^2 \cap \Theta H_0^2) \oplus \langle \Theta H \rangle = K_\Theta \oplus \langle \Theta H \rangle \). Now consider the vector \( F \in \mathcal{M} = \text{Ker}T \) which is of the form \( F = F_\zeta + \mu \Theta H \), where \( F_\zeta \in K_\Theta \) and \( \mu \in \mathbb{C} \). Then the above equivalent condition reduces to

\[
\mu(1 + \langle \Theta H, G \rangle) = -\langle F_\zeta, G \rangle. \quad (33)
\]

Therefore, from the first part of this section, we conclude the defect space of the nearly \( S^* \) invariant subspace \( \text{Ker} T \) and which is as follows:

\[
\mathcal{F} = \left\{ \begin{array}{ll}
\langle \Theta S^* H \rangle, & \text{for } G \in \Theta H^2_{\mathbb{C}^m} (\mathbb{D}), \\
\sqrt{\Theta S^* H, P_{K_\Theta} G}, & \text{for } G \not\in \Theta H^2_{\mathbb{C}^m} (\mathbb{D}).
\end{array} \right.
\]

Accordingly, we analyze the kernel of \( T \) in two different subsections due to the above two different representation of defect spaces. Before we proceed, we would like to mention the following: We know that the subspace \( \mathcal{W} = \mathcal{M} \ominus (\mathcal{M} \cap zH^2_{\mathbb{C}^m} (\mathbb{D})) \) (if non trivial) is generated by the vectors \( \{F_1, F_2, \ldots, F_m\} \), where \( F_i = P_M(k_0 \otimes e_i) \). If \( \dim \mathcal{W} = r(1 \leq r \leq m) \) and \( \{W_1, W_2, \ldots, W_r\} \) is an orthonormal basis for \( \mathcal{W} \), then \( F_0 \) is the \( m \times r \) matrix whose columns are \( W_1, W_2, \ldots, W_r \). Using Gram–Schmidt orthonormalization, we can form an orthonormal basis for \( \mathcal{W} \) from the generator set \( \{F_1, F_2, \ldots, F_m\} \) and the resulting vectors are nothing but the linear combination of \( \{F_1, F_2, \ldots, F_m\} \). Therefore, it is sufficient to prove that everything holds good for the generating set of vectors. For simplicity of calculations, we do the whole analysis for single \( F_i \), that means from now onward we assume that \( \dim \mathcal{W} = 1 \) and \( F_0 = [x_i F_i]_{m \times 1} \) in each cases and subsections.
5.1 \( G \in \Theta H^2_{C} (D) \)

In this subsection, we assume that \( G \in \Theta H^2_{C} (D) \). Then the corresponding defect space for the nearly \( S^* \) invariant subspace \( \mathcal{M} = \text{Ker} T \) is \( \mathcal{F} = \langle \Theta S^* H \rangle \).

Therefore, Eq. (33) reduces to

\[
\mu(1 + \langle \Theta H, G \rangle) = 0.
\]

Let us denote \( \Theta_i(0) = \left[ \theta_{i1}(0), \theta_{i2}(0), \ldots, \theta_{im}(0) \right]^t \). Now considering the defect space as \( \mathcal{F} = \langle \Theta S^* H \rangle \), we have the following two cases.

Case 1. If \( 1 + \langle \Theta H, G \rangle = 0 \), then \( \mathcal{M} = K_0 + \langle \Theta H \rangle \).

Therefore, case (i) of Theorem 2.2 implies that the nearly invariant subspace \( \mathcal{M} \) for \( S^* \) with the finite defect can be written in the following form.

\[
\mathcal{M} = \left\{ F : F = \alpha_i \left( k_0 \otimes e_i - \Theta \cdot \Theta_i(0) \right) K_0 + k_1 \frac{\Theta H(0)}{||S^* H||} \\
+ \left( \alpha_i \frac{\langle k_0 \otimes e_i, \Theta H \rangle}{||\Theta H||^2} K_0 + \frac{k_1}{||S^* H||} \right) \Theta H : (K_0, k_1) \in \mathcal{K} \right\},
\]

where \( \mathcal{K} = \{(K_0, k_1) \in H^2_{C} (D) : K_0 \) and \( k_1 \) satisfies the following (34) \}

\[
\begin{cases}
\alpha_i \left( k_0 \otimes e_i - \Theta \cdot \Theta_i(0) \right) K_0 + k_1 \frac{\Theta H(0)}{||S^* H||} \in K_\Theta, \quad \text{and} \\
\alpha_i \frac{\langle k_0 \otimes e_i, \Theta H \rangle}{||\Theta H||^2} K_0 + \frac{k_1}{||S^* H||} \in \mathbb{C}.
\end{cases}
\]

Our next aim is to show the subspace \( \mathcal{K} \) of \( H^2_{C} (D) \) is a \( S^* \oplus S^* \) invariant subspace which is not exactly same as in the scalar valued case. First, we observe that if we replace \( K_0, k_1 \) by \( S^* K_0, S^* k_1 \), then the second condition of (34) is trivially true. Thus, we only have to check that the first condition also true if we replace \( K_0, k_1 \) by \( S^* K_0, S^* k_1 \). Since \( \mathcal{K}_\Theta \) is an \( S^* \) invariant subspace of \( H^2_{C} (D) \), then

\[
H_\Theta := S^* \left( \alpha_i \left( k_0 \otimes e_i - \Theta \cdot \Theta_i(0) \right) K_0 + k_1 \frac{\Theta H(0)}{||S^* H||} \right) \in \mathcal{K}_\Theta.
\]

On the other hand,
\[
 z_i \left( k_0 \otimes e_i - \Theta \cdot \Theta_i(0) \right) S^* K_0 + S^* k_1 \frac{\Theta H(0)}{\|S^* H\|} = H_\Theta + S^* \left( z_i \Theta \cdot \Theta_i(0) K_0 - k_1 \frac{\Theta H(0)}{\|S^* H\|} \right) \\
 - z_i \Theta \cdot \Theta_i(0) S^* K_0 + S^* k_1 \frac{\Theta H(0)}{\|S^* H\|} = H_\Theta + z_i S^* \Theta \cdot \Theta_i(0) K_0(0) - k_1(0) \frac{\Theta H(0)}{\|S^* H\|} \\
 S^* \Theta(H(0)) \in K_\Theta.
\]

Since \( H_\Theta \in K_\Theta \), \( S^* \Theta \cdot \Theta_i(0) \in K_\Theta \) and \( S^* \Theta(H(0)) \in K_\Theta \), then from the above equation we conclude that \( K \) is an \( S^* \oplus S^* \) invariant subspace of \( H^2_{C^2}(\mathbb{D}) \).

Case 2. If \( 1 + \langle \Theta H, G \rangle \neq 0 \), then \( \mathcal{M} = K_\Theta \). Therefore,
\[
 F_i = P_{\mathcal{M}}(k_0 \otimes e_i) = P_{K_\Theta}(k_0 \otimes e_i) \\
 = k_0 \otimes e_i - \sum_{j=1}^{m} \langle k_0 \otimes e_i, \Theta e_j \rangle \Theta e_j = k_0 \otimes e_i - \Theta \cdot \Theta_i(0).
\]

Using case(i) of Theorem 2.2, we have
\[
 \mathcal{M} = \left\{ F : F = z_i \left( k_0 \otimes e_i - \Theta \cdot \Theta_i(0) \right) K_0 + k_1 \frac{\Theta(H - H(0))}{\|S^* H\|} : (K_0, k_1) \in K \right\} \\
 = \left\{ F : F = z_i \left( k_0 \otimes e_i - \Theta \cdot \Theta_i(0) \right) K_0 : (K_0, 0) \in K \right\}.
\]

Therefore, the corresponding \( S^* \oplus S^* \) invariant subspace is
\[
 K = \left\{ (K_0, 0) \in H^2_{C^2}(\mathbb{D}) : K_0 \text{ satisfies the following } (5.14) \right\}
\]
such that \( z_i \left( k_0 \otimes e_i - \Theta \cdot \Theta_i(0) \right) K_0 \in K_\Theta \), which is equivalent to the fact that
\[
 K_0 \in \bigcap_{j=1}^{m} \text{Ker} T_{\theta_i - \theta_i(0)} = \bigcap_{j=1}^{m} K_{\zeta_i}, \text{ where } \zeta_i \text{ is an inner factor of } \theta_i - \theta_i(0). \tag{35}
\]

Furthermore, it is easy to check that \( K \) is an \( S^* \oplus S^* \) invariant subspace of \( H^2_{C^2}(\mathbb{D}) \) using the scalar version of Proposition 2.4.

**5.2 G \notin \Theta H^2_{C^m}(\mathbb{D})**

In this subsection, we consider the case that \( G \notin \Theta H^2_{C^m}(\mathbb{D}) \), then the kernel of \( T \) is a nearly \( S^* \) invariant subspace with defect at most 2 and the defect space is \( \mathcal{F} = \sqrt{\Theta S^* H, P_{K_\Theta} G} \). Since \( G \notin \Theta H^2_{C^m}(\mathbb{D}) \), then we find a nonzero \( G_\zeta \in K_\Theta \) and \( G_\Theta \in \Theta H^2_{C^m}(\mathbb{D}) \) such that \( G = G_\zeta + G_\Theta \). Therefore, the identity (33) reduces to
\[ \mu(1 + \langle \Theta H, G_\Theta \rangle) = -\langle F_\gamma, G_\gamma \rangle. \] (36)

As we have already mentioned in the beginning of Sect. 5 that the obtained conditions (38) and (39) in our setting are not exactly similar to (3.8) and (3.9) in the scalar case [12] (see Sect. 3). In other words, we can not break the inner product mentioned in conditions (38) and (39) into smaller pieces because we do not know whether the individual term lies in respective \( L^2 \)-spaces or not. To proceed further, we need the following remark concerning the projection \( P_M(k_0 \otimes e_i) \).

**Remark 5.2** If \( \mathcal{M} = \ker T \subset \mathcal{N} := K_\Theta \oplus \langle \Theta H \rangle \) satisfies \( \mathcal{N} = \mathcal{M} \oplus \langle R \rangle \), where \( R = U + \rho \Theta H \) with \( U \in K_\Theta \) and \( \rho \in \mathbb{C} \). Then

\[
P_M(k_0 \otimes e_i) = k_0 \otimes e_i - \Theta \cdot \Theta_i(0) + \frac{\langle k_0 \otimes e_i, \Theta H \rangle \Theta H}{\| \Theta H \|^2} - \frac{\langle k_0 \otimes e_i, U + \rho \Theta H \rangle}{\| U + \rho \Theta H \|^2} (U + \rho \Theta H).
\] (37)

Now we need to analyze two cases according to whether \( v_\Theta := 1 + \langle \Theta H, G_\Theta \rangle \) is zero or not along with the defect space \( F = \sqrt{\{ \Theta S^* H, P_{k_0} G \}} \).

Case 1. If \( v_\Theta = 0 \), then (36) holds if and only if \( F_\gamma \in \langle P_{k_0} G \rangle \) and, therefore, substituting \( U = G_\gamma \) and \( \rho = 0 \) in (37), we have

\[
P_M(k_0 \otimes e_i) = k_0 \otimes e_i - \Theta \cdot \Theta_i(0) + \frac{\langle k_0 \otimes e_i, \Theta H \rangle \Theta H}{\| \Theta H \|^2} - \frac{\langle k_0 \otimes e_i, G_\gamma \rangle}{\| G_\gamma \|^2} G_\gamma.
\]

Therefore, \( F_0 = \left[ x_i P_M(k_0 \otimes e_i) \right]_{m \times 1} \) and hence using case (i) of Theorem 2.2, we have the following representation of \( \mathcal{M} \):

\[
\mathcal{M} = \left\{ F : F = x_i \left( k_0 \otimes e_i - \Theta \cdot \Theta_i(0) \right) K_0 - \frac{k_1}{\| S^* H \|} \Theta H(0) + \left( \frac{k_1}{\| S^* H \|} + \frac{\langle k_0 \otimes e_i, \Theta H \rangle}{\| \Theta H \|^2} K_0 \right) \Theta H - \left( \frac{\langle k_0 \otimes e_i, G_\gamma \rangle x_i K_0}{\| G_\gamma \|^2} - \frac{zk_2}{\| G_\gamma \|^2} \right) G_\gamma : (K_0, k_1, k_2) \in K \right\}
\]

and the corresponding \( S^* \oplus S^* \oplus S^* \) invariant subspace is

\[
K = \left\{ (K_0, k_1, k_2) \in H^2_{\mathbb{D}} : K_0, k_1, k_2 \text{ satisfies the following (5.17)} \right\},
\]

where
where we have the following representation of $M$

$$
\begin{aligned}
\mathcal{K}erns \ of \ perturbed \ Toeplitz \ operators
\end{aligned}
$$

and $G_\tilde{z} = (G_1, G_2, \ldots, G_m)$. In a similar fashion like (34), we prove that the first two conditions of (38) also hold for $S^*K_0, S^*k_1$. Moreover, the last condition also holds for $S^*K_0$ and $S^*k_2$ trivially. Thus $\mathcal{K}$ is an $S^* \oplus S^* \oplus S^*$ invariant subspace of $H^2_{C_3}(\mathbb{D})$.

Case 2. Next we consider $\nu_\Theta \neq 0$, then Eq. (36) gives $\mu = -\nu_\Theta^{-1}\langle F_\tilde{z}, G_\tilde{z} \rangle$ and thus we have,

$$
\mathcal{M} = \text{Ker}T = \{F : F = K - \nu_\Theta^{-1}\langle K, G_\tilde{z} \rangle \Theta H, K \in K_\Theta\}.
$$

Therefore, by simple calculations we conclude that $\mathcal{N} = \mathcal{M} \oplus \langle G_\tilde{z} + \frac{\nu_\Theta}{\|H\|^2} \Theta H \rangle$.

Thus, by letting $U = G_\tilde{z}$ and $\rho = \frac{\nu_\Theta}{\|H\|^2}$ in (5.20), we get

$$
P_{\mathcal{M}}(k_0 \otimes e_i)
= k_0 \otimes e_i - \Theta \cdot \Theta_i(0) + \langle k_0 \otimes e_i, \Theta H \rangle \Theta H - \omega_\Theta \left( G_\tilde{z} + \frac{\nu_\Theta}{\|H\|^2} \Theta H \right),
$$

where $\omega_\Theta = \frac{\langle k_0 \otimes e_i, G_\tilde{z} + \frac{\nu_\Theta}{\|H\|^2} \Theta H \rangle}{\|G_\tilde{z} + \frac{\nu_\Theta}{\|H\|^2} \Theta H\|^2}$. Therefore, from Case (i) of Theorem 2.2,

we have the following representation of $\mathcal{M}$:

$$
\mathcal{M} = \left\{ F : F = \alpha_i(k_0 \otimes e_i)K_0 - \left( \alpha_i k_0 \Theta \cdot \Theta_i(0) + k_1 \frac{\Theta H(0)}{\|S^*H\|} \right)
+ \left( \alpha_i \frac{\langle k_0 \otimes e_i, \Theta H \rangle}{\|H\|^2} K_0 + \frac{k_1}{\|S^*H\|} - \frac{z k_2}{\|G_\tilde{z}\| \|H\|^2} \right) \Theta H
+ \left( - \alpha_i k_0 \omega_\Theta + \frac{z k_2}{\|G_\tilde{z}\|} \right) \left( G_\tilde{z} + \frac{\nu_\Theta}{\|H\|^2} \Theta H \right) : (K_0, k_1, k_2) \in \mathcal{K} \right\}
$$

and the corresponding $S^* \oplus S^* \oplus S^*$ invariant subspace is

$$
\mathcal{K} = \left\{ (K_0, k_1, k_2) : K_0, k_1, k_2 \text{ satisfies the following (5.18)} \right\},
$$

where
\[
\begin{align*}
\{z_i(k_0 \otimes e_i)K_0 - \left( z_iK_0 \Theta \cdot \Theta_i(0) + k_1 \frac{\Theta H(0)}{\|S^*H\|} \right) \in \mathcal{K}_\Theta, \\
\{z_i(k_0 \otimes e_i, \Theta H) \frac{1}{\|H\|^2} K_0 + \frac{k_1}{\|S^*H\|} - \frac{zk_2}{\|G_\zeta\| \|H\|^2} \in \mathbb{C}, \\
\langle K_0, z^n \bar{z}G_i + L(n) \rangle + \sum_{i} -z_i S^n k_0 \omega G_0 + \frac{z^n k_2}{\|G_\zeta\|} \left( G_\zeta + \frac{\Theta G_\zeta}{\|H\|^2} \right), G_\zeta + \frac{\Theta G_\zeta}{\|H\|^2} = 0, \\
& \text{form } \in \mathbb{N} \cup \{0\},
\end{align*}
\]

and \(G_\zeta = (G_1, G_2, \ldots, G_m)\), where

\[
L(n) = \begin{cases} 
\frac{(S^n k_2)(0)}{\|G_\zeta\| \|H\|^2} |\frac{\Theta G_\zeta}{\|H\|^2}| & \text{if } n \in \mathbb{N} \\
\left( z_i(k_0 \otimes e_i, \Theta H) \frac{1}{\|H\|^2} K_0 + \frac{k_1}{\|S^*H\|} - \frac{zk_2}{\|G_\zeta\| \|H\|^2} \right) \frac{\Theta G_\zeta}{\|H\|^2} & \text{if } n = 0.
\end{cases}
\]

By repeating the similar explanations as in (34), the conditions of (39) also hold for \(S^* K_0, S^* k_1, S^* k_2\) and hence we conclude that, \(\mathcal{K}\) is an \(S^* \oplus S^* \oplus S^*\) invariant subspace of \(H^2_\mathcal{C}(\mathbb{D})\). Now we give an example motivated from [12] with minor modifications to understand the case \(\Phi = \Theta^*\).

**Example** Suppose \(\Theta(z) = \text{diag}(z^1, z^2, \ldots, z^m) \in H^\infty_{\mathcal{C}(\mathbb{D}, m)}(\mathbb{D}), s \geq 1, \quad G = G_\zeta + G_\theta\) with \(G_\zeta = (z^{-1}, 1, \ldots, 1), G_\theta = \Theta H^2_{\mathcal{C}}(\mathbb{D}) \oplus \cdots \oplus \Theta H^2_{\mathcal{C}}(\mathbb{D})\) and \(H = \frac{1}{\sqrt{2}}(1-z, 0, \ldots, 0)\). Then it follows that, \(\mathcal{M}\) is a nearly \(S^*\) invariant subspace with defect space \(\mathcal{F} = \sqrt{\{z^j H, G_\zeta\}}\). Case 1. If \(v_\Theta = 1 + \langle \Theta H, G_\theta \rangle = 0\), then it follows that

\[
\mathcal{M} = \sqrt{\{1 \otimes e_1, \{z \otimes e_i\}_{i=1}^m, \{z^2 \otimes e_i\}_{i=1}^m, \ldots, \{z^{s-2} \otimes e_i\}_{i=1}^m, \{z^{s-1} \otimes e_j\}_{j=2}^m\}} \oplus \{z^j H\}
\]

\[
= \left\{ F : F = K_0 \otimes e_1 - k_1 \frac{z^j H(0)}{\|S^*H\|} + k_1 \frac{z^j H}{\|S^*H\|} + zk_2 \frac{G_\zeta}{\|G_\zeta\|} : (K_0, k_1, k_2) \in \mathcal{K} \right\}
\]

with an \(S^* \oplus S^* \oplus S^*\) invariant subspace \(\mathcal{K} = \{(K_0, k_1, k_2) : K_0, k_1\) satisfies the following (40) and \(k_2 \in H^2_\mathcal{C}(\mathbb{D})\)\} such that
\[
\begin{cases}
K_0 \otimes e_1 - k_1 \frac{z^s H(0)}{\|S^s H\|} \in \mathcal{K}_\Theta, \\
\frac{k_1}{\|S^s H\|} \in \mathbb{C}, \text{ and} \\
\langle K_0, z^{n+s-1} \rangle = 0 \text{ for } n \in \mathbb{N} \cup \{0\}.
\end{cases}
\] (40)

Case 2. If \(v_\Theta = 1 + \langle \Theta H, G_\Theta \rangle \neq 0\), then it follows that
\[
\mathcal{M} = \sqrt{1} \otimes e_1, \{z \otimes e_i \}_{i=1}^m, \{z^2 \otimes e_i \}_{i=1}^m, \ldots, \{z^{s-2} \otimes e_i \}_{i=1}^m, \{z^{s-1} \otimes e_j \}_{j=1}^m
\]
\[
\oplus \langle z^s H \rangle \otimes \left( G_\zeta + \frac{\bar{v}_\Theta}{\|H\|} z^s H \right)
\]
\[
= \left\{ F : F = K_0 \otimes e_1 - k_1 \frac{z^s H(0)}{\|S^s H\|} + \left( \frac{k_1}{\|S^s H\|} - \frac{z k_2}{\|G_\zeta\|} \right) z^s H \right. \\
+ \frac{z k_2}{\|G_\zeta\|} \left( G_\zeta + \bar{v}_\Theta z^s H \right) : (K_0, k_1, k_2) \in \mathcal{K} \right\}
\]
with an \(S^s \oplus S^s \oplus S^s\) invariant subspace
\[
\mathcal{K} = \left\{ (K_0, k_1, k_2) : K_0, k_1, k_2 \text{ satisfies the following (5.20)} \right\}
\]
such that
\[
\begin{cases}
K_0 \otimes e_1 - k_1 \frac{z^s H(0)}{\|S^s H\|} \in \mathcal{K}_\Theta, \\
\frac{k_1}{\|S^s H\|} - \frac{z k_2}{\|G_\zeta\|} \bar{v}_\Theta \in \mathbb{C}, \text{ and} \\
\langle K_0, z^n \left( z^{s-1} + \frac{\bar{v}_\Theta z^s}{\sqrt{2}} (1 - z) \right) \rangle + \langle k_1, \frac{\bar{v}_\Theta}{\sqrt{2}} z^n \rangle \\
+ \langle k_2, \frac{z^n \bar{v}_\Theta}{\sqrt{2}} (1 - z) \rangle = 0 \text{ for } n \in \mathbb{N} \cup \{0\}.
\end{cases}
\] (41)

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