On the information to work conversion: a view from ancient fluctuation-dissipation relations

Yu. E. Kuzovlev
Donetsk Free Statistical Physics Laboratory

The “generalized fluctuation-dissipation relations”, which had anticipated the “fluctuation theorems” fifteen years before, are applied to presently popular “information to useful work conversion” to demonstrate that its success and bounds are determined by a specific “insider” information rather than formal one.

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1 Introduction. The subject to be under our attention in general was interestingly reviewed e.g. in [1] while a curious example of its individualizing can be found e.g. in [2]. The means for our consideration will be so called “generalized fluctuation-dissipation relations” (FDR) which originally appeared in [3] - [13] and for the first time presented the so called “fluctuation theorems” (FT), as it was explained in [14] (see also [15], especially Appendix therein, notes in [16, 17], and articles [18] - [28] for various applications and modifications of FDR).

In all reasonings below we take in mind statistical thermodynamics grounded on Hamiltonian mechanics (see introductory words in [16]), not a “stochastic thermodynamics” mixing principles with uncontrolled conjectures (like e.g. “Marcovianity”). Our assumptions concern construction of a system’s Hamiltonian only but not its evolution (in this respect see [29, 30]).

2 Hamiltonian. Let us consider a system with Hamiltonian

$$H_f(X,\Gamma,Q) = H_0(X,\Gamma,Q) - f \cdot Q,$$
(1)

where \{X, \Gamma, Q\}, with \(X = \{X_1, X_2 \ldots X_n\}\), is full set of system’s variables (including that of an inner “reservoir” as well as any inner “Maxwell demons”) and \(f = f(t)\) is an external force.

We assume that, first, the system is thermodynamically indifferent in respect to variables \(X\) and \(Q\) in the sense that in absence of the force they make no influence on “partition sum” over the rest of variables:

$$q \equiv \int \exp\{-\beta H_0(X,\Gamma,Q)\} \, d\Gamma = \text{const} \ldots$$
(2)

Second, system’s phase space is finite in respect to \(X\),

$$0 \leq X_j \leq \Delta X_j \ldots$$
(3)

although \(Q\) can take arbitrary values. For instance, all \(X\), and \(Q\) along with them, may take meaning of some “angles” like similar variables in the model of [2]. So, for certainty, let \(Q\) and \(X\) have positive parities (i.e. do not change signs) under time reversal.

3 Initial state and processes under interest. Because of [2] the system has no stationary equilibrium probability distribution even if \(f = 0\), not speaking about \(f \neq 0\). Therefore it is reasonable to consider distributions

$$\rho(\cdot \mid Q_0) = \frac{\delta(Q - Q_0)}{\Delta X_n^q} e^{-\beta H_0(\cdot)}$$
(4)

with \(\cdot\) - replacing \(X,\Gamma,Q\) and factor \(\Delta X_n^q\) in denominator for normalization to unit. They describe quasi-equilibrium states (ensembles of states) characterized by fixed instant, e.g. at time \(t = 0\), \(Q\)’s value in free unperturbed system (at \(f = 0\)).

Then, at \(t > 0\), evolution starting from such initial condition’s ensemble but obeying the full perturbed Hamiltonian [1] with \(f \neq 0\) will reveal possible connections between start values \(X(t = 0) = X_0\) and further work

$$W(t) = f \cdot (Q(t) - Q(0)) = f \cdot (Q(t) - Q_0)$$
(5)

of the external force as switched on at \(t = 0\). In particular, possibilities to make quantity [3] artificially negative, - and thus convert some amount of reservoir’s heat (represented by \(\Gamma\)) into useful work against the force, - due to a “successful” special choice of initial \(X(t = 0) = X_0\).

Such the ladder set \(X_0\) may be thought as a “program” for the inner “demons” (also represented by \(\Gamma\)) to achieve the success. In other words, program of converting initial information into useful work. Thanks to property [2] establishing of this program in itself meets no energy cost and destroys informational, i.e. statistical, equilibrium (uncertainty) only, but not thermodynamical equilibrium.

4 Backward processes an FDR. To examine this question, consider statistical ensemble average

$$I \equiv \int \delta(Q(t) - Q_1) e^{-\beta W(t)} \times \delta(X(0) - X_0) \rho(\cdot \mid Q_0)$$
(6)

with \(\int\) denoting full phase space integration. Clearly, in view of [14], it may be written also as

$$I = \frac{1}{\Delta X_n^q} e^{-\beta f \Delta Q} P(\Delta Q, t \mid X_0, Q_0, f)$$
(7)

with

$$\Delta Q = Q(t) - Q(0) = Q_1 - Q_0 \ldots$$

and

$$p = \int P(\Delta Q, t \mid X_0, Q_0, f)$$
and \( P(\Delta Q, t \mid X_0, Q_0, f) \) being \( \Delta Q \)’s probability density distribution (PDD) after time \( t \) under the given initial information about \( X \).

On the other hand, reverting system’s phase trajectories back in time from a given moment \( t > 0 \) to \( t = 0 \), recalling that

\[
W(t) = H_0(X(t), \Gamma(t), Q(t)) - H_0(X(0), \Gamma(0), Q(0))
\]

and using general recipes of [3], one can express the same quantity in the form

\[
I = \int_0^\Delta \delta(X(t) - X_0) \delta(Q(t) - Q_0) \rho(\cdot|Q_1). \tag{8}
\]

It is nothing but

\[
I = P(X_0, -\Delta Q, t \mid Q_1, f), \tag{9}
\]

where \( -\Delta Q = Q_0 - Q_1 \), and function \( P(X, \Delta Q, t \mid Q_0, f) \) is joint PDD of \( X \) and \( \Delta Q \) at time \( t \) in absence of any initial information about \( X \).

Thus, we obtain FDR

\[
e^{-\beta f \Delta Q} P(\Delta Q, t \mid X, Q_0, f) \frac{1}{\Delta X^n} = \tag{10}
\]

\[
= P(X, -\Delta Q, t \mid Q_0 + \Delta Q, f),
\]

which connects \( \ast \) influence of initially stated “program” \( X \) onto later \( \Delta Q \)’s statistics and \( \ast \ast \) mutual statistical correlations between simultaneous current \( X \)’s and \( \Delta Q \)’s values in the course of “non-programmed” evolution.

5 Natural simplifications and FDR for conditional probabilities.

As far as the variable \( Q \) conjugated with the external force \( f \) meets no restrictions, its absolute value may be of no importance for an interaction between its increments \( \Delta Q \) and the “programming variables” \( X \). Then (10) simplifies to

\[
e^{-\beta f \Delta Q} P(\Delta Q, t \mid X, f) \frac{1}{\Delta X^n} = \tag{11}
\]

\[
= P(X, -\Delta Q, t \mid f).
\]

Next notice, in view of the properties [1-3], that the fraction \( 1/\Delta X^n \) in (11) plays role of eigen marginal PDD of variables \( X \), and integration of this FDR over \( X \) reduces it to the native FDR for the work in itself [3, 4],

\[
e^{-\beta f \Delta Q} P(\Delta Q, t \mid f) = P(-\Delta Q, t \mid f). \tag{12}
\]

Division of (11) by (12) yields

\[
P(\Delta Q, t \mid X, f) P(\Delta Q, t \mid f) = \Delta X^n P(X, t - \Delta Q, f) \tag{13}
\]

with function \( P(X, t \mid \Delta Q, f) \) meaning conditional PDD of \( X(t) \) under condition of given increment \( \Delta Q \) during previous time interval \((0, t)\).

This FDR shows that \( X \)-program-induced relative increase (decrease) in probability of the work to be \( W = f\Delta Q \) equals to relative increase (decrease) in probability of finding just the programming values \( X(t) = X \) after spontaneous occurrence of the opposite-sign work \( -W = -f\Delta Q \). Hence, in order to provide (enlarge probability of negative work, \( W < 0 \), that is “information to work conversion”, one has to start from those “good” \( X \Rightarrow X(0) \) whose appearance, \( X(t) \Rightarrow X \), is stimulated by opposite-sign positive work (as if the latter was spent on writing useful information).

6 On statistical averages, inequalities and estimates.

Now let us integrate FDR (11) over \( \Delta Q \) to obtain

\[
\left\langle e^{-\beta W(t)} \mid X, f \right\rangle = \Delta X^n P(X, t \mid f), \tag{14}
\]

where angle brackets on the left designate conditional averaging under fixed \( X(0) = X \), while on the right-hand side we see ratio of actual a posteriori marginal PDD of \( X = X(t) \) to its a priori uniform (“quasi-equilibrium”) PDD \( 1/\Delta X^n \). This FDR, in turn, standardly implies inequality

\[
-\beta \left\langle W(t) \mid X, f \right\rangle \leq \ln (\Delta X^n P(X, t \mid f)). \tag{15}
\]

In principle, of course, the ratio \( P(X, t \mid f)/(1/\Delta X^n) \) may be non-constant (\( X \)-dependent), - and thus inevitably different from unit, - since generally \( P(X, t \mid f) \) represents essentially non-equilibrium processes and non-equilibrium system’s states. Hence, for some good \( X \) choices definitely the logarithm’s argument in (15) is greater than unit. Consequently, for such \( X \) this inequality allows negative mean value of the work, that is success, at least on average, in production of useful work against the force for the expense of reservoir’s heat energy.

At that, (15) understandably estimates upper bound of the success and prompts that it can be made arbitrary high, up to values growing \( \propto t \), at sufficiently large \( n \) (under obvious presumptions about system’s dynamics and “kinematics”).

7 Information and phase volume exchange.

Leaving more complex statistical averages for separate speculations, let us pay attention to one more probabilistic characterization of our “key” quantity (14). Namely, merely reformulate (11) as

\[
\exp (-\beta f \Delta Q) P(\Delta Q, t \mid X(0) = X, f) = \Delta X^n P(X, t \mid f),
\]

where denominator is conditional PDD with condition stated at final point of observation time interval instead of its initial point in the nominator.

Naturally, if \( X(0) = X \) favors, say, greater \( \Delta Q \), then \( X(t) = X \) favors smaller \( -\Delta Q \). So, the whole left-side fraction indicates distortion of generic time symmetry or
asymmetry of $\Delta Q$'s statistics under influence of the conditions. If the latter make no real effect, then the fraction must be unit because of phase volume conservation ("Liouville theorem") in dynamics responsible for $\Delta Q$. Correspondingly, factual effectiveness of the conditions (usefulness of "programming" information) requires significant phase volume exchange between different microscopic dynamic channels of transport and fluctuations. In our context here, channels related to $\Delta Q$ or to $X$.

At that, the Liouville theorem keeps on the whole but not for separate channels (subsystems) \[8, 13\]. This is subject of "generalized fluctuation-dissipation reciprocity relations" \[4, 8, 13\].

8 Concluding remarks.

To resume, we illustrated that the ancient classical "generalized fluctuation-dissipation relations" help to recognize time-distributed statistical correlations valuable for transforming a part of system's thermal energy into useful work against external forces.

A quantum formulation of this subject (in particular, quantum analogues of above FDR) depends on one or another rule of mutual ordering of non-commutative quantum variables (operators) and thus a rule of symmetrization of operator exponentials (and other functions). Most general quantum FDR not pinned to special orderings were exhaustively written out already in \[3\]. One interesting variant of the ordering was under our consideration in \[23, 27, 28\]. Investigations of other variants and approaches to "quantum FT" are reflected e.g. in \[31, 32\].

Quantum translation of the aforesaid joke reasonings may be placed elsewhere. I hope they bring a food for thinking and will not overflow today's sea of "fluctuation theorems".

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* kuzovlev@kinetic.ac.donetsk.ua, yuk-1370@yandex.ru, kuzovlev@ddnu.ru

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