ON GENERALIZED SYMMETRIC POWERS AND A GENERALIZATION OF KOLMOGOROV–GELFAND–BUCHSTABER–REES THEORY

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In 1939, Gelfand and Kolmogorov published a paper [4], where they showed that for a (compact Hausdorff) topological space $X$, homomorphisms of the algebra of continuous functions $C(X)$ to the field of real numbers are in a one-to-one correspondence with points of $X$. The algebra $C(X)$ is considered purely algebraically, without a topology. This result is less known than its analog that gave birth to the theory of normed rings. The Gelfand–Kolmogorov theorem may be viewed as a description of the image of the canonical embedding of $X$ into the infinite-dimensional linear space $V = A^*$, where $A = C(X)$, by a system of quadratic equations $f(1) = 1$, $f(a)^2 - f(a^2) = 0$, indexed by elements of $A$. This aspect was recently emphasized by Buchstaber and Rees (see [1] and references therein), who showed that there is a natural embedding into $V$ not only for $X$, but also for all its symmetric powers $\text{Sym}^n X$. To this end, algebra homomorphisms should be replaced by the so-called `$n$-homomorphisms', and quadratic equations describing the image, by certain algebraic equations of higher degree. Buchstaber and Rees’ theory was motivated by their earlier study of Hopf objects for multi-valued groups. (In the hindsight, the notion of an $n$-homomorphism of algebras was present, implicitly, in Frobenius’s notion of higher group characters.)

In the present note we suggest a further natural generalization of Buchstaber–Rees’s theory. Namely, for a set or topological space $X$ there is a functorial object $\text{Sym}^{p\mid q} X$, $p, q \geq 0$, and for a commutative algebra with unit $A$, a corresponding algebra $\text{S}^{p\mid q} A$ (see definitions below). We call them ‘generalized symmetric powers’. There is a canonical map from $\text{Sym}^{p\mid q} X$ to $V = A^*$; we introduce certain algebraic equations (see below) that should describe its image, thus extending the statements of Gelfand–Kolmogorov and Buchstaber–Rees. On the level of algebras, this corresponds to a description of algebra homomorphisms $\text{S}^{p\mid q} A \to B$ in terms of the new notion of a '$p\mid q$-homomorphism'. Our work was motivated by the results in supergeometry in [3], from which comes our main tool, the ‘characteristic function’ of a linear map of algebras. The methods that we propose yield, in particular, a simple direct proof of the main theorem of Buchstaber and Rees.

Let $A$ and $B$ be commutative associative algebras with unit. Consider an arbitrary linear map $f: A \to B$. Its characteristic function is defined to be $R(f, a, z) = e^{\ln(1 + az)}$, where $a \in A$ and $z$ is a formal parameter. Example: if $f$ is an algebra homomorphism, then $R(f, a, z) = 1 + f(a)z$, i.e., a linear polynomial. Algebraic properties of the map $f$ are reflected in functional properties of $R(f, a, z)$ w.r.t. the variable $z$. Another example: let $R(f, a, z)$ be a polynomial of degree $n$. This corresponds to the Buchstaber–Rees theory, as we show below.
We call a linear map \( f \), a \( p|q \)-homomorphism if \( R(f, a, z) \) is a rational function that can be represented by the ratio of polynomials of degrees \( p \) and \( q \). Properties of \( p|q \)-homomorphisms follow from general properties of \( R(f, a, z) \). For an arbitrary map \( f \), \( R(f, a, z) \) has the power expansion at zero \( R(f, a, z) = 1 + \psi_1(f, a)z + \psi_2(f, a)z^2 + \ldots \) where \( \psi_k(f, a) = P_k(s_1, \ldots, s_k) \). Here \( s_k = s_k(f, a) = f(a^k) \), and \( P_k \) are the classical Newton polynomials giving expression of elementary symmetric functions via sums of powers.

There is an important exponential property \( R(f + g, a, z) = R(f, a, z)R(g, a, z) \). Suppose \( R(f, a, z) \) extends to a genuine function of \( z \) regarded, say, as a complex variable. Consider its behaviour at infinity. By a formal transformation one can see that \( R(f, a, z) = e^{f(a^k)}e^{f(1)z}e^{f(1)z^2} \ldots \). In particular, for \( a = 1 \) we have \( R(f, 1, z) = (1 + z)^{f(1)} \). The assumption that \( R(f, a, z) \) has no essential singularity implies that \( f(1) = \chi \in \mathbb{Z} \) and the integer \( \chi \) is the order of the pole at infinity. Hence we have the expansion \( R(f, a, z) = \sum_{k \leq \chi} \psi_k(f, a)z^k \) near infinity where \( \psi_k(f, a) := e^{f(a^k)}\psi_{\chi-k}(f, a^{-1}) \). We denote the leading term of the expansion \( e^{f(a^k)} =: \ber(f, a) \) and call it, the \( f \)-Berezinian of \( a \in A \). Our formal transformations and the definition of \( f \)-Berezinian make sense at least for a rational characteristic function or for elements in the neighborhood of identity, if the algebras are over \( \mathbb{C} \) or \( \mathbb{R} \). One can immediately see that \( f \)-Berezinian is multiplicative: \( \ber(f, a_1a_2) = \ber(f, a_1)\ber(f, a_2) \). Note that \( a \mapsto \ber(f, a) \) is, in general, a partially defined map \( A \to B \). In the rational case, \( \ber(f, a) \) is the ratio of polynomials in the elements \( f(a^k) \) with values in \( B \).

Here are the examples to be kept in mind. If \( \rho: A \to \Mat(n, B) \) is a matrix representation by \( n \times n \) matrices and \( f(a) = \Tr \rho(a) \), then \( R(f, a, z) = \det(1 + \rho(a)z) \), and the \( f \)-Berezinian of \( a \) is \( \det \rho(a) \). If \( \rho : A \to \Mat(p|q, B) \) is a matrix representation by matrices \( p|q \times p|q \) and \( f(a) = \Str \rho(a) \) is the supertrace, then \( R(f, a, z) = \Ber(1 + \rho(a)z) \). In this case \( \ber(f, a) = \Ber \rho(a) \) is the ordinary Berezinian.

Multilinear symmetric functions \( \Phi_k(f, a_1, \ldots, a_k) \) of elements \( a_i \in A \) such that \( \Phi_k(f, a, \ldots, a) = k!\psi_k(f, a) \) satisfy relations identical with the Frobenius recursion for higher group characters (see [1]). The examples above make this connection apparent. For the case of a matrix representation, \( s_k(f, a) = \Tr \rho(a)^k \), \( \psi_k(f, a) = \Tr \Lambda^k \rho(a) \), and \( \Phi_k(f, a_1, \ldots, a_k) = k! \Tr(\rho(a_1) \wedge \ldots \wedge \rho(a_k)) \). The ‘abstract’ case simply reproduces the relations between these traces.

Let us return to the case when \( R(f, a, z) \) is polynomial in \( z \). The Buchstaber–Rees theory can be recovered as follows. Here \( f(1) = \chi = n \geq 0 \); it gives the degree of \( R(f, a, z) \), so \( \psi_k(f, a) = 0 \) for all \( k \geq n + 1 \) and all \( a \in A \). This is equivalent to the equations \( f(1) = n \in \mathbb{N} \) and \( \Phi_{n+1}(f, a_1, \ldots, a_{n+1}) = 0 \) for all \( a_i \), which is precisely the definition of an \( n \)-homomorphism according to Buchstaber and Rees [1]. Since here \( \ber(f, a) = \psi_n(f, a) \) (in particular, is a polynomial function of \( a \)), the latter function is multiplicative in \( a \), and therefore its polarization \( \Phi_A(f, a_1, \ldots, a_n) \) yields an algebra homomorphism \( S^nA \to B \). This gives a one-to-one correspondence between \( n \)-homomorphisms \( A \to B \) and algebra homomorphisms \( S^nA \to B \).

Consider a topological space \( X \). Its \( p|q \)-th symmetric power \( \Sym^{p|q}X \) is defined as the identification space of \( X^{p+q} \) with respect to the action of \( S_p \times S_q \) and the relations

\[ (x_1, \ldots, x_{p-1}, y, x_{p+1} \ldots, x_{p+q-1}, y) \sim (x_1, \ldots, x_{p-1}, z, x_{p+1} \ldots, x_{p+q-1}, z) \].
The algebraic analog of it is the $p/q$-th symmetric power $S^{p/q}A$ of a commutative associative algebra with unit $A$. We define $S^{p/q}A$ as the subalgebra $\mu^{-1}(S^{p-1}A \otimes S^{q-1}A)$ in $S^p A \otimes S^q A$ where $\mu: S^p A \otimes S^q A \rightarrow S^{p-1}A \otimes S^{q-1}A \otimes A$ is the multiplication of the last arguments. Example: for $A = \mathbb{C}[x]$, the algebra $S^{p/q}A$ is the algebra of all polynomial invariants of $p/q$ by $p/q$ matrices. There is a relation between algebra homomorphisms $S^{pq}A \rightarrow B$ and $p/q$-homomorphisms $A \rightarrow B$.

To each homomorphism $S^{pq}A \rightarrow B$ canonically corresponds a $p/q$-homomorphism $A \rightarrow B$, and we have managed to establish the inverse in special cases. Example. An element $[x_1, \ldots, x_{p+q}] \in \text{Sym}^{pq}X$ defines a $p/q$-homomorphism on $A = C(X)$: $a \mapsto a(x_1) + \ldots + a(x_p) - \ldots - a(x_{p+q})$. In general, a linear combination of algebra homomorphisms of the form $\sum n_\alpha f_\alpha$ where $n_\alpha \in \mathbb{Z}$ is a $p/q$-homomorphism with $\chi = \sum n_\alpha$, $p = \sum n_\alpha$, and $q = -\sum n_\alpha$. This follows from the exponential property of characteristic function.

By using formulas from [3], the condition that $f: A \rightarrow B$ is a $p/q$-homomorphism can be expressed as $f(1) = p - q$ and $|\psi_k(f,a), \ldots, \psi_{k+q}(f,a)|_{q+1}$ for $k \geq p - q + 1$, where $|\psi_k(f,a), \ldots, \psi_{k+q}(f,a)|_{q+1}$ is a Hankel determinant. It is a system of polynomial equations for ‘coordinates’ of the linear map $f$. (In particular, it should describe the image of $\text{Sym}^{pq}X$.)

The notions of the characteristic function and $f$-Berezinian, introduced in this note, are powerful tools for studying algebraic properties of linear maps generalizing ring homomorphisms. Applications of our results may, in particular, include topological ramified coverings with branching more general than that considered by L. Smith and Dold. The possibility of such an application was pointed out to us by V. M. Buchstaber (compare with [2]). We thank him for fruitful discussions.

References

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Appendix. Some comments and a short proof of the Buchstaber–Rees main theorem

In this Appendix we show how our approach allows to give a quick direct proof of the main theorem of Buchstaber and Rees, namely, that algebra homomorphisms $S^n A \rightarrow B$ are in one-to-one correspondence with $n$-homomorphisms $A \rightarrow B$. For this reason, we are concerned mainly with the case of a polynomial characteristic function, which corresponds to the Buchstaber–Rees theory. We also elucidate some general constructions, trying not to duplicate the main text. The questions of the anonymous referee of the first version of this note prompted us to write this Appendix, and we would like to thank him for this.
1.

Let $A$ and $B$ be two associative, commutative and unital algebras over $\mathbb{C}$ or $\mathbb{R}$. Consider a linear map $f: A \to B$. We say that an arbitrary function $\phi: A \to B$ is polynomial (or $f$-polynomial) if it is given by a polynomial expression in $f(a)$, $f(a^2)$, etc. It is a useful notion.

The characteristic function of a linear map $f$ introduced in the main text is defined as

$$R(f,a,z) = \exp (f(\ln (1+az))) = 1 + \psi_1(a)z + \psi_2(a)z^2 + \psi_3(a)z^3 + \ldots$$

as a formal power series. For brevity denote $R(a,z) = R(f,a,z)$. It is a function of both $z$ and $a$. The coefficients $\psi_k(a)$ are polynomial functions of $a$ of degree $k$, $\psi_k(\lambda a) = \lambda^k \psi_k(a)$. Indeed, by differentiating the definition w.r.t. $z$ we can see that $\psi_k(a)$ can be obtained by standard Newton-like recurrent formulae:

$$\psi_1(a) = f(a), \quad \psi_{k+1}(a) = \frac{1}{k+1} (f(a)\psi_k(a) - f(a^2)\psi_{k-1}(a) + f(a^3)\psi_{k-2}(a) - \ldots).$$

Note that the characteristic function according to its definition obeys the relation

$$R(a,z)R(a',z') = R(az + a'z' + aa'zz',1)$$

or

$$R(a,1)R(b,1) = R(c,1) \quad \text{if} \quad 1 + c = (1 + a)(1 + b)$$

(makes sense as formal power series). We shall use it in what follows.

2.

One can make the following formal transformation of the characteristic function aimed at obtaining its expansion near infinity. Note that initially $R(a,z)$ is a formal power series, which can be seen as the Taylor expansion at zero of some function of $z$ if such a function exists. Assume that it exists and denote it by the same $R(a,z)$. Then we have

$$R(a,z) = e^{\int \ln(1+az)} = e^{\int \ln(az(1+a^{-1}z^{-1}))} = e^{\int \ln(az) + f(1)\int \ln(1+a^{-1}z^{-1})} = e^{\int \ln z + f(1)\int \ln(1+a^{-1}z^{-1})} = z^{f(1)} e^{\ln a} \sum_{k \geq 0} \frac{e^{\ln a} \psi_k(a^{-1})}{k!} z^{-k} = e^{\ln a} z^\chi + e^{\ln a} \psi_1(a^{-1}) z^{\chi - 1} + e^{\ln a} \psi_2(a^{-1}) z^{\chi - 2} + \ldots$$

where we denoted $\chi = f(1)$. Initially $f(1) \in B$, but the assumption that there is a Laurent expansion at infinity forces us to conclude that $\chi$ must be a number in $\mathbb{Z}$. Here we assume whatever we need, e.g., that $a^{-1}$ exists, and so on. Instead of describing how this calculation can be justified, we show below how one can go around it.

3.

Suppose now that the formal power series $R(a,z)$ terminates, i.e., it is a polynomial function in $z$ for all $a \in A$. Under a mild technical assumption that the degree of $R(a,z)$ is uniformly bounded by some $N \in \mathbb{N}$, we deduce that $f(1) = n \in \mathbb{N}$ and that $R(a,z)$ is a polynomial of degree $n$, i.e., the degree is at most $n$ for all $a$ and is exactly $n$ for some $a$. The arguments below may be used to replace the formal calculation above.
Indeed, consider $R(1, z) = \exp[f(\ln(1 + z))] = \exp[\chi \ln(1 + z)]$, where $\chi = f(1) \in B$. We shall show first that the element $\chi \in B$ is an integer. We have $\exp[\chi \ln(1 + z)] = (1 + z)^\chi$ where $(1 + z)^\chi$ is considered is a formal power series:

$$(1 + z)^\chi = 1 + \chi z + \frac{\chi(\chi - 1)}{2} z^2 + \cdots = \sum_{k=0}^{\infty} \frac{\chi(\chi - 1)\cdots(\chi - k + 1)}{k!} z^k.$$

But $R(a, z)$ is a polynomial of degree at most $N$ for all $a$. Hence $\chi(\chi - 1)\cdots(\chi - k + 1) = 0$ for all $k > N$. Under another technical condition (introduced by Buchstaber and Rees), that the algebra $B$ is “connected” \footnote{That means by the definition that the equation $b(b - 1)(b - 2)\cdots(b - k) = 0$ in $B$ implies that $b = j$ for some $j = 0, 1, \ldots, k$.}, we conclude that $\chi = n$ for some integer $n$ between 1 and $N$.

For an arbitrary $a$, we have, from the above identity \footnote{From the Frobenius recursion formula, it is easy to prove by induction that the multilinear maps $\Phi_k$ are symmetric, therefore are defined by the restrictions to the diagonal, and then, again using induction, deduce that the functions $\varphi_k(a) = \Phi_k(a, \ldots, a)$ obey Newton type recurrence relations and thus can be identified with $k!\psi_k(a)$.},

$$R(a, z) = R(za, 1) = R(z - 1, 1)R\left(\frac{1}{z} + a - 1, 1\right) = z^n R\left(\frac{1}{z} + a - 1, 1\right).$$

More explicitly, the expansion at the RHS has the form

$$z^n \left[1 + f\left(\frac{1}{z} + a - 1\right) + \psi_1\left(\frac{1}{z} + a - 1\right) + \cdots + \psi_N\left(\frac{1}{z} + a - 1\right)\right]$$

for some $N \in \mathbb{N}$. The coefficients $\psi_k(a)$ are polynomial functions of $a$ of degree $k$, in particular $\psi_k(\lambda a) = \lambda^k \psi_k(a)$. By comparing the expansions at the LHS and RHS, we conclude that the degree of $R(a, z)$ in $z$ can be at most $n$:

$$R(a, z) = 1 + \psi_1(a)z + \psi_2(a)z^2 + \psi_3(a)z^3 + \cdots + \psi_n(a)z^n,$$

for any $a$.

Since $\psi_k(a)$ are in one-to-one correspondence with the functions $\Phi_k(a_1, \ldots, a_k)$ appearing in the “Frobenius recursion” of Buchstaber and Rees (up to the factor of $k!$), which can be recovered from $\psi_k(a)$ by polarization \footnote{From the Frobenius recursion formula, it is easy to prove by induction that the multilinear maps $\Phi_k$ are symmetric, therefore are defined by the restrictions to the diagonal, and then, again using induction, deduce that the functions $\varphi_k(a) = \Phi_k(a, \ldots, a)$ obey Newton type recurrence relations and thus can be identified with $k!\psi_k(a)$}, this reproduces the Buchstaber–Rees definition of $n$-homomorphisms.

**Remark.** Here is a formula for the polarization of a homogeneous polynomial of degree $k$ (the restriction of a symmetric $k$-linear function to the diagonal):

$$\Phi_k(a_1, a_2, \ldots, a_k) = \sum_{r=1}^{k} \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq k} (-1)^{k+r} \psi_k(a_{i_1} + a_{i_2} + \cdots + a_{i_r}). \tag{4}$$

Here $\Phi_k(a, \ldots, a) = k!\psi_k(a)$. For example,

$$\Phi_2(a_1, a_2) = \psi_2(a_1 + a_2) - \psi_2(a_1) - \psi_2(a_2)$$

and

$$\Phi_3(a_1, a_2, a_3) = \psi_3(a_1 + a_2 + a_3) - \psi_3(a_1 + a_2) - \psi_3(a_1 + a_3) - \psi_3(a_2 + a_3) + \psi_3(a_1) + \psi_3(a_2) + \psi_3(a_3).$$

This should be a “textbook formula”, however it is difficult to find a reference for it.
4.

We have defined the \( f\)-Berezinian as a map \( \text{ber}_f : A \to B \) by the formula \( \text{ber}_f(a) = \exp f(\ln a) = R(a - 1, 1) \) when it makes sense. (In the main text, the notation is \( \text{ber}(f, a) \).) It is clearly multiplicative, for

\[
\exp f(\ln(ab)) = \exp f(\ln a + \ln b) = \exp (f\ln a + f\ln b) = \exp f(\ln a) \exp f(\ln b).
\]

This holds even if \( B \) is non-commutative, but \( f \) is a trace.

Let \( f \) be an \( n \)-homomorphism. Then the formally defined \( \text{ber}_f(a) = \exp f(\ln a) \) is a polynomial function of \( a \) with values in \( B \):

\[
\text{ber}_f(a) = \exp f(\ln(1 + (a - 1))) = R(a - 1, 1) = 1 + f(a - 1) + \psi_2(a - 1) + \cdots + \psi_n(a - 1),
\]

well-defined for all \( a \).

**Proposition.** For an \( n \)-homomorphism \( f \),

\[
\text{ber}_f(a) = \psi_n(a).
\]

**Proof.** Consider the equality

\[
1 + \psi_1(a)z + \psi_2(a)z^2 + \psi_3(a)z^3 + \cdots + \psi_n(a)z^n = z^n \left[ 1 + f \left( \frac{1}{z} + a - 1 \right) + \psi_2 \left( \frac{1}{z} + a - 1 \right) + \cdots + \psi_n \left( \frac{1}{z} + a - 1 \right) \right]
\]

(same as (3); we have legitimately set \( N = n \)). Collecting all terms of degree \( n \) in \( z \), we arrive at

\[
\psi_n(a) = 1 + f(a - 1) + \psi_2(a - 1) + \cdots + \psi_n(a - 1) = \text{ber}_f(a).
\]

\( \square \)

**Corollary.** For an \( n \)-homomorphism \( f \), the function \( \psi_n(a) \) is multiplicative in \( a \).

(The original proof of this fact by Buchstaber and Rees was based on rather hard combinatorial arguments involving at one instance hypergeometric polynomials.)

**Remark.** The apparatus of characteristic functions allows to obtain easily many facts. For example, if \( f \) and \( g \) are \( n \)- and \( m \)-homomorphisms \( A \to B \), respectively, then the exponential property of characteristic functions immediately implies that \( f + g \) is an \((n + m)\)-homomorphism, since its characteristic function is the product of polynomials of degrees \( \leq n \) and \( \leq m \). If \( g \) is an \( m \)-homomorphism \( A \to B \) and \( f \) is an \( n \)-homomorphism \( B \to C \), then \( R(f \circ g, a, z) = e^{fg\ln(1 + az)} = e^{fgR(g, a, z)} = \text{ber}_f R(g, a, z) \). Since we know that \( R(g, a, z) \) is a polynomial in \( z \) of degree at most \( m \) and the \( f \)-Berezinian \( \text{ber}_f b \) is a polynomial in \( b \in B \) of degree \( n \), we conclude that \( R(f \circ g, a, z) \) has degree at most \( nm \) in \( z \), therefore \( f \circ g \) is an \( nm \)-homomorphism.

5.

Buchstaber and Rees proved that there is a one-to-one correspondence between \( n \)-homomorphisms of \( A \to B \) and algebra homomorphisms \( S^n A \to B \).

One can obtain this result in our framework as follows.

Let \( \mathcal{H}^n(A, B) \) be the set of all \( n \)-homomorphisms from the algebra \( A \) to the algebra \( B \). We shall construct two mutually inverse maps between the spaces \( \mathcal{H}^n(A, B) \) and \( \mathcal{H}^1(S^n A, B) \),
thus establishing their one-to-one correspondence. It is convenient to introduce an \( n \times n \) matrix \( \mathcal{L}(a) \) with entries in \( A^{\otimes n} = A \otimes \ldots \otimes A \), where
\[
\mathcal{L}(a) = \text{diag } [a \otimes 1 \otimes \cdots \otimes 1, 1 \otimes a \otimes 1 \otimes \cdots \otimes 1, \ldots, 1 \otimes 1 \otimes \cdots \otimes 1 \otimes a].
\]
The map \( a \mapsto \mathcal{L}(a) \) is a matrix representation.

To every algebra homomorphism \( F \in \mathcal{H}(S^n A, B) \) we assign an \( n \)-homomorphism \( f_F \in \mathcal{H}(A, B) \), by setting
\[
f_F(a) = F(\text{Tr} \mathcal{L}(a)).
\]
(Note that \( \text{Tr} \mathcal{L}(a) = \Delta(a) \) in the Buchstaber–Rees notations.) Conversely, to every \( n \)-homo morphism \( f \in \mathcal{H}(A, B) \) we assign an algebra homomorphism \( F_f \in \mathcal{H}(S^n A, B) \) defined by the condition that for all \( z \)
\[
F_f(\det(1 + \mathcal{L}(a)z)) = R(f, a, z). \tag{5}
\]
We shall prove that these maps are well-defined and that
\[
f_{F_f} = f, \quad F_{f_F} = F.
\]

Let \( F \) be a homomorphism from \( S^n A \) to \( B \). Then evidently \( f_F \) is a linear map. A calculation shows that the characteristic function of \( f_F \) is the following polynomial:
\[
R(f_F, a, z) = F(\det(1 + \mathcal{L}(a)z)). \tag{6}
\]
It is polynomial of degree \( n \). Hence \( f \) is an \( n \)-homomorphism. To obtain (6), we note that \( f_F \ln(1 + az) = F(\text{Tr} \mathcal{L}(\ln(1 + az))) = F(\text{Tr} \ln(1 + \mathcal{L}(a)z)) \), hence by exponentiating we arrive at \( e^{f_F \ln(1 + az)} = e^{F(\text{Tr} \ln(1 + \mathcal{L}(a)z))} = F e^{\text{Tr} \ln(1 + \mathcal{L}(a)z)} = F \det(1 + \mathcal{L}(a)z) \).

The formula (5) requires a little bit more work. It defines a map \( F_f \) on elements of the form \( \text{Tr} \mathcal{L}(a) \in S^n A \), including \( \det \mathcal{L}(a) \) (which is a polynomial in traces). In particular,
\[
F_f(\det \mathcal{L}(a)) = \text{ber} \Phi_n(a) = \psi_n(a).
\]
It is the restriction of the linear map \( \frac{1}{n!} \Phi_n : S^n A \to B \) to the elements of the form \( \det \mathcal{L}(a) = a \otimes \cdots \otimes a \), where \( \Phi_n \) corresponds to the symmetric multilinear map \( \Phi_n \). Recall that \( \frac{1}{n!} \Phi_n \) is the polarization of \( \psi_n \), as given by (2). The elements \( \det \mathcal{L}(a) \) linearly span \( S^n A \) and we see that the linear map \( F_f \) is multilinear on them. Hence it is a homomorphism.

We proved that the maps \( f \mapsto F_f, F \mapsto f_F \) are well-defined. It remains to prove that their composition is identity. We have
\[
F_{f_F}(\text{Tr} \mathcal{L}(a)) = f_F(a) = F(\text{Tr} \mathcal{L}(a)).
\]
We see that the maps \( F_{f_F} \) and \( F \) coincide on the elements \( \text{Tr} \mathcal{L}(a) \). Hence they coincide on the elements \( \det \mathcal{L}(a) \). This implies that they coincide on all elements of \( S^n A \). To compare \( f_{F_f} \) and \( f \), look again at the formula
\[
R(f, a, z) = F(\det(1 + \mathcal{L}(a)z)).
\]
Expanding it in \( z \) and comparing the first terms we see that \( F(\text{Tr} \mathcal{L}(a)) = f(a) \), i.e., \( f_{F_f} = f \).
This concludes the proof.

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