Solution of the problem of rotation of a homogeneous disk by the variational method, the apparatus of vector analysis and the finite element method

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Abstract. This work considers two global approaches to obtaining solutions to problems in mechanics: analytical and numerical, their division into various methods, allowing a more flexible approach to the choice of optimal methods for obtaining solutions to problems of continuum mechanics. Analyzed the effectiveness of various analytical methods for obtaining resolving continuum mechanics. In particular, the derivation of differential equations of equilibrium from the condition of minimality of the potential energy of elastic deformation of a solid, as well as the mathematical apparatus of vector analysis of classical field theory. In addition to the analytical solution, a numerical solution of this problem was presented by the finite element method, performed in the ANSYS Mechanical APDL software package, followed by verification of the obtained simulation results with the results of the analytical solution.

1. Introduction
Consideration of two global approaches to obtaining solutions to problems in mechanics: analytical and numerical, their division into different methods, allows a more flexible approach to the choice of optimal methods for obtaining solutions to problems of continuum mechanics. For example, approaches to the analytical solution of problems in mechanics are the derivation of differential equations of equilibrium from the condition of minimality of the potential energy of elastic deformation of a solid as a whole or the recording of differential operators of vector analysis in curvilinear coordinates, which most fully reflect the symmetry of the problem. It should be noted that the first method, due to the static nature of the problem, is a simplified case of applying the Lagrangian [1] and Hamiltonian formalisms of the classical field theory, and due to the absence of thermal effects in the problem – the method of thermodynamic potentials. Variational methods are the most effective methods of modern theoretical physics, analytical and numerical mathematics, economics and other mathematical sciences. In particular, Hamilton's variational principle of least action makes it possible to obtain equations in just a few steps, the derivation of which by conventional methods requires the use of long and cumbersome calculations. In addition, analytical methods allow one to obtain the necessary equations without trying to imagine a visual representation of the considered mechanical system, but only formally describing mathematical transformations.

The numerical implementation of problems in continuum mechanics has both classical solution methods, such as, for example, the Finite Difference Methods (FDM), the Boundary Element Method
(BEM), the Finite Element Method (FEM), and more narrowly focused (specialized) methods with special capabilities for some sections of continuum mechanics. These include methods such as the Discrete Element Method (DEM), the Finite Volume Method (FVM), as well as meshfree methods ones, such as Smoothed Particle Galerkin (SPG) and others \[2\].

The advantage of numerical methods lies in their versatility with respect to the original topology of structures studied in boundary value of problems. The advantage of the analytical solution lies in the ability to express the desired functions in a formulaic form, as well as in the fact that the resulting equations are reversible. Due to this, such equations can be differentiated, integrated, and also solved with respect to any parameter to study the dependence of the desired function on various variables.

In this article, the derivation of the analytical resolving equation was presented for the problem of the rotation of a homogeneous solid plane disk. Rotating discs and shafts are widely used in various fields of technology, such as in the creation of steam and gas turbines, compressors, fans, disintegrators, as well as in machines of the chemical industry, in the rotary dispersion of melts. Discs and shafts are subjected to loads because of the inertial component (centrifugal force) that leads to the stress-strain state of the structure at a high level of intensity. The stresses caused by the inertial component are distributed symmetrically about the axis of rotation of the disk (Figure 1).

![Design model for the problem of rotation of a homogeneous solid plane disk.](image)

**Figure 1.** Design model for the problem of rotation of a homogeneous solid plane disk.

### 2. Analytical solution

#### 2.1. Variational method

The total potential energy of the disk is the sum of the potential energy of elastic deformation and the potential energy in the centrifugal force field:

\[
U = \int_V \left\{ \frac{\sigma_{\text{xx}} E}{2} \right\} - \rho \Omega^2 ru_r = \int_0^R \int_0^{2\pi} \int_0^L \frac{\sigma_{\text{xx}} E}{2} - \rho \Omega^2 ru_r \ dV = 2\pi H \int_0^R \int_0^{2\pi} \int_0^L \frac{\sigma_{\text{xx}} E}{2} - \rho \Omega^2 ru_r \ dr d\theta \ dr =
\]

\[
= 2\pi H \int_0^R \int_0^{2\pi} \int_0^L r \sigma_{\text{xx}} u_r' + \frac{\sigma_{\text{xx}}}{2} - \rho \Omega^2 r u_r \ dr d\theta \ dr.
\]

When varying the function \(\delta u_r\), which describes the displacement, variation of potential energy (considering the linear dependence of stresses on deformations according to Hooke’s law):

\[
\Delta U = 2\pi H \int_0^R \int_0^{2\pi} \int_0^L r \sigma_{\text{xx}} \delta u_r' + \frac{\sigma_{\text{xx}}}{2} - \rho \Omega^2 r \delta u_r \ dr d\theta \ dr.
\]

Integrating the equation by parts considering \(\sigma_{\text{xx}} (R) = 0\):
\[ \delta U = 2\pi H \left( \int_0^R dr \left\{ -r \sigma'' \right\} \delta u_r + \sigma_{\varphi \varphi} \delta u_\varphi - \rho \Omega^2 r^2 \delta u_r \right) + 2\pi H r \sigma'' \delta u_r \bigg|_0^R = 2\pi H \int_0^R dr \left\{ -r \sigma'' \right\} + \sigma_{\varphi \varphi} - \rho \Omega^2 r^2 \delta u_r. \] (3)

Equating, according to the variational principle, this variation to zero (the energy should be minimal) and due to the arbitrariness \( \delta u_r \) inside the disk, we get:

\[-(r \sigma'')' + \sigma_{\varphi \varphi} - \rho \Omega^2 r^2 = 0. \] (4)

Using Hooke’s law [3]:

\[ \sigma_{ij} = \frac{E}{1+\nu} \left( u_{ij} + \nu u_{kkl} \right), \quad u_{ij} = u_{xx} + u_{yy} + u_{zz}. \] (5)

Due to \( \sigma_{zz} = 0 \) we’ll get:

\[ u_{zz} = -\frac{\nu}{1-\nu} (u_{xx} + u_{yy}). \] (6)

Consequently:

\[ \sigma_{xx} = \frac{E}{1-\nu^2} (u_{xx} + \nu u_{yy}); \]
\[ \sigma_{yy} = \frac{E}{1-\nu^2} (u_{yy} + \nu u_{xx}). \] (7)

Since these equations do not consist of derivatives, they have a tensor form, so in polar coordinates they look the same:

\[ \sigma_{rr} = \frac{E}{1-\nu^2} (u_{rr} + \nu u_{\varphi \varphi}); \]
\[ \sigma_{\varphi \varphi} = \frac{E}{1-\nu^2} (u_{\varphi \varphi} + \nu u_{rr}). \] (8)

Strain tensor \( u_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \) in the polar coordinate system has the form:

\[ u_{rr} = \frac{\partial u_r}{\partial r}; \]
\[ u_{\varphi \varphi} = \frac{1}{r} \frac{\partial u_\varphi}{\partial \varphi} + \frac{u_\varphi}{r} = u_\varphi. \] (9)

Substituting the obtained equations (9-10) into (4), we have:

\[ r^2 u'' + ru' - u_r = -\frac{(1-\nu^2)\rho \Omega^2 r^3}{E}. \] (10)

This equation is a linearly inhomogeneous differential equation one-dimensional in the variable \( r \). General solution of a homogeneous differential equation:

\[ u_r = C_1 r + \frac{C_2}{r}. \] (11)

Obviously the constant \( C_2 = 0 \) due to the absence of a singularity at zero. A particular solution (11) can be found by substituting \( u_r = Ar^3 \):
\[ A(6+3-1)r^3 = -\frac{(1-v^2)\rho\Omega^2 r^3}{E}, \]  
(12)

From this we get that \( A = -\frac{(1-v^2)\rho\Omega^2}{8E} \). Finally:

\[ u_r = C_1 r - \frac{(1-v^2)\rho\Omega^2}{8E} r^3. \]  
(13)

Constant \( C_1 \) we find by substituting into the boundary conditions:

\[ \sigma_{rr}(R) = 0. \]  
(14)

\[ C_1 = \frac{(1-v)(v+3)\rho\Omega^2 R^2}{8E}. \]  
Finally:

\[ u_r = \frac{(1-v^2)\rho\Omega^2}{8E} \left[ \frac{v+3}{v+1} \right] \left[ r^3 - \frac{R^3}{3(v+1)} \right]. \]  
(15)

\( u_r \) has a maximum \( \left(1-v^2\right)\rho\Omega^2 R^2 \left[ \frac{v+3}{v+1} \right] \), where: \( r = R \sqrt{\frac{v+3}{3(v+1)}} \).

The non-zero components of the stress tensor will be:

\[ \sigma_{rr} = \frac{(v+3)\rho\Omega^2}{8} \left[ R^2 - r^2 \right]; \]

\[ \sigma_{\phi\phi} = \frac{\rho\Omega^2}{8} \left[ (v+3)R^2 - (3v+1)r^2 \right]. \]  
(16)

In particular:

\[ \sigma_{rr}(0) = \sigma_{\phi\phi}(0) = \frac{(v+3)\rho\Omega^2 R^2}{8}; \]

\[ \sigma_{\phi\phi}(R) = \frac{(1-v)\rho\Omega^2 R^2}{4}. \]  
(17)

2.2. The apparatus of vector analysis of the classical field theory.

Another way to obtain (4) is to use vector analysis from the equilibrium equation [3]. The main advantages of vector methods over traditional coordinate methods:

1. Compactness. One vector equation unites several coordinate equations, and its study can most often be carried out directly, without replacing vectors with their coordinate notation.

2. Invariance. The vector equation does not depend on the choice of the coordinate system and can be easily translated into a coordinate notation in any convenient curvilinear coordinate system.

3. Visibility. Differential operators of vector analysis and the relations connecting them usually have a simple and clear physical interpretation.

The equations of vector analysis for the problem under consideration, considering the equilibrium condition, Hooke's law and plane-stressed state, we obtain [3]:

\[ \text{grad} \left( \text{div} \bar{u} \right) - \frac{1-v}{2} \text{rot} \left( \text{rot} \bar{u} \right) = -\tilde{f} = \frac{p\Omega^2 r\tilde{e}_r}{E}, \]  
(18)

where: \( \tilde{f} = p\Omega^2 r\tilde{e}_r \) – body force component.

Let’s describe in detail in terms of polar coordinates:

\[ \text{rot} \bar{u} = \bar{0}. \]  
(19)
\[
\text{div} \vec{u} = \frac{1}{r} \frac{d}{dr} \left( ru_r \right),
\]

(20)

\[
\text{grad} (\text{div} \vec{u}) = \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} \left( ru_r \right) \right] \vec{e}_r.
\]

(21)

\[
\frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} \left( ru_r \right) \right] \vec{e}_r = -\frac{1-v^2}{E} \rho \Omega^2 r \vec{e}_r.
\]

(22)

Expanding (22), we obtain (11).

Thus, the final resolving differential equations of equilibrium were obtained for the problem of rotation of a homogeneous solid plane disk using the variational principle of minimality of the potential energy of elastic deformation of a solid, as well as the apparatus of vector analysis of classical field theory.

3. Numerical simulation

Let’s consider the solution of the problem by the finite element method using the ANSYS Mechanical APDL software package. The advantages of numerical methods, in particular, the finite element method is its versatility regarding the initial topology of structures studied in boundary value problems, thus making it possible to calculate the stress and strain fields in real parts, considering all their design features.

Numerical modeling of static, temperature and dynamic stress-strain state in the ANSYS Mechanical APDL software package is based on the implementation of the finite element method (FEM) in the matrix form of displacements [4-6]. To solve the main system of equations are formed by global matrices of stiffness \([K]\), damping \([C]\) and mass matrices \([M]\), as well as the vector of external nodal load \([F]\).

The following system of linear algebraic equations (SLAE) is solved for problems of linear statics:

\[
[K] \cdot \{u\} = \{F\},
\]

(23)

where: \([K]\) – global stiffness matrix, which is defined as:

\[
[K] = \sum_{n=1}^{N} [K_n],
\]

(24)

where: \([u]\) – vector of nodal displacements, \(N\) – number of elements, \([K_n]\) – stiffness matrix element, \([F]\) – global external load vector.

To solve the problem under consideration, we describe the main stages of computational modeling:

1. Calculations are carried out in a formulation that allows linear two-dimensional modeling to assess the stress-strain state of the disk under inertial action, which is caused by the rotation of the disk with a constant angular velocity.

2. Considered the axisymmetric for this problem calculations are performed for a quarter of the disk.

3. As a finite element discretizing the computational domain, we used PLANE183, which simulates the plane mechanical behavior (plane stressed, plane strained and generalized plane stress states) in a two-dimensional formulation with a quadratic approximation by the element.

4. The model of the material is assumed to be linear-elastic with constant values of the elastic modulus \(E\), Poisson's ratio \(v\) and density \(\rho\).

5. The size of the finite element was varied to study the mesh convergence.

6. Boundary conditions are accepted as for a symmetric disk with fixing of all nodes of the model along the X axis from displacements along UY, and all nodes along the Y axis from displacement along UX. The inertial component of its own weight is taken as the load caused by the rotation of the disk at a constant angular velocity.
Using the radius \( r = 0.35 \text{m} \), the elastic modulus \( E = 200,000 \text{ MPa} \), Poisson's ratio \( \nu = 0.3 \), density \( \rho = 7850 \text{ kg/m}^3 \) and angular velocity \( \Omega = 314 \text{ Hz} \) as variables, we obtain the results of the numerical solution presented in Figure 2.

![Figure 2. The Isofields of radial \( \sigma_r \) (a) and tangential \( \sigma_\phi \) (b) stresses, MPa.](image)

4. Verification

The results of the analytical solution can be interesting from the point of view of verification of software systems that implement numerical methods, in particular, the finite element method [6]. For verification, we will consider both qualitative and quantitative characteristics of the functions under consideration. Let's show graphs of changes in functions \( \sigma_r \) and \( \sigma_\phi \) along the radius of the disk, as well as the maximum values of the obtained values. The results of comparing the graphs of functions obtained by the finite element implementation in the ANSYS Mechanical APDL software package and analytically are presented in Figure 3 and 4, a comparison of quantitative characteristics is presented in table 1.

![Figure 3. Distribution of radial stresses \( \sigma_r \) along the disk radius (left – the results of the analytical solution, right – the results of ANSYS Mechanical APDL), MPa.](image)
Figure 4. Distribution of tangential stresses $\sigma_{\phi\phi}$ along the disk radius (left – the results of the analytical solution, right – the results of ANSYS Mechanical APDL), MPa.

The results of comparison of numerical values are presented in table 1.

Table 1. Comparison of numerical values.

| Verifiable parameter | Analytical solution | ANSYS Mechanical APDL | Relative error $\delta$ (%) |
|----------------------|---------------------|------------------------|-----------------------------|
| Maximum stress $\sigma_{rr}$ (MPa) | 39.11 | 39.11 | 0 |
| Maximum stress $\sigma_{\phi\phi}$ (MPa) | 39.11 | 39.11 | 0 |

5. Conclusions

1. The application of various approaches to obtaining the resolving equation was presented for the problem of the rotation of a homogeneous solid plane disk, in particular, the variational principle of minimality of the potential energy of elastic deformation of a solid, as well as through the application of the apparatus of vector analysis of classical field theory.

2. The efficiency of various analytical methods was analyzed for obtaining the resolving equations of continuum mechanics. In particular, using the variational principle, which makes it possible to obtain equations in just a few steps, the derivation of which by conventional methods requires the use of long and cumbersome calculations, as well as using the apparatus of vector analysis of the classical field theory, which allows, due to its invariance with respect to the choice of the coordinate system, write down the equations directly in coordinates that most fully represent the symmetry of the problem.

3. In the ANSYS Mechanical APDL software package was developed a finite element model of a rotating homogeneous solid plane disk. The main stages of computational modeling were considered of the considered problem.

4. The verification results confirmed the correctness of this finite element model, which makes it possible to use this finite element model in the future for the numerical implementation of more complex problems.

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