Classical emulation of quantum states with coherent mixtures

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We propose a classical emulation methodology to emulate quantum phenomena arising from any non-classical quantum state using only a finite set of coherent states or their statistical mixtures. This allows us to successfully reproduce well-known quantum effects using resources that can be much more feasibly generated in the laboratory. We present a simple procedure to experimentally carry out quantum-state emulation with coherent states that also applies to any general set of classical states that are easier to generate, and demonstrate its capabilities in observing the Hong-Ou-Mandel effect, violating Bell inequalities and witnessing quantum non-classicality.

Introduction.—Understanding the extent to which classical elements can be used to still reveal nontrivial quantum effects in experiments can offer a deeper perspective on the interplay between classical and quantum resources. On the one hand, important cornerstone results were already established much earlier in quantum information and quantum optics, such as antibunching of photons and suppression of the field amplitude noise below the classical level [1, 2], exceeding the standard quantum limit in measurement precision [3], violating Bell inequalities [4], and the exhibition of non-classicality signatures [5–7]. On the other hand, interesting new research on how these quantum effects can still be observed using classical resources have emerged. For example, recently it was shown that Bell inequalities violation and other quantum-like signatures may be brought to light by “classical entanglement”, that is, by local classical correlations of different degrees of freedom [8–13].

In this work, going by a more direct route with classical resources, we investigate the potential of emulating all quantum phenomena arising from any non-classical quantum state using only a finite set of coherent states or their statistical mixtures. This allows us to successfully reproduce well-known quantum effects using resources that can be much more feasibly generated in the laboratory. We present a simple procedure to experimentally carry out quantum-state emulation with coherent states that also applies to any general set of classical states that are easier to generate, and demonstrate its capabilities in observing the Hong-Ou-Mandel effect, violating Bell inequalities and witnessing quantum non-classicality.

Data-pattern representation.—It is already well-known that it is possible to represent an arbitrary quantum state as a mixture of coherent states projectors, either by continuous Glauber-Sudarshan P representation [6, 21], or its "coarse-grained" discrete version [22, 23]. Here we are exploiting another possibility. Namely, we use non-orthogonality of coherent states to approximate a given signal state with a finite number of coherent state projectors on some predefined lattice in a finite-dimensional subspace. This representation might be very different from the coarse-grained P-representation [15, 16]. Let us consider a state described by $\rho_{\text{true}}$ and assume that for an arbitrarily small $\epsilon$ we can choose a finite-dimensional subspace $S$ containing $\rho_{\text{true}}$. $1 - \{\text{Tr}_S[\rho_{\text{true}}]\} \leq \epsilon$. We approximate $\rho_{\text{true}}$ by the operator $\rho$ represented through a set of probe states $\{\rho_j\}$ in the following way:

$$\rho_{\text{true}} \approx \rho = \sum_j c_j \rho_j, \quad \sum_j c_j = 1,$$

where the coefficients $c_j$ can be negative for non-orthogonal $\rho_j$. It has already been shown that using coherent probe states, it is possible to achieve an arbitrarily small distance between $\rho$ and $\rho_{\text{true}}$ in the subspace $S$, and semipositivity of the projection of the matrix $\rho$ on the subspace $S$ [15, 16, 18, 19].
Notice that for few-photon and few-mode non-classical states one can achieve higher than 0.99 fidelity of the approximation (1) and reproduction of the experiment results with just few tens of the probe coherent states [15, 16, 19]. For some states, it is possible to develop quite economical representations in terms of phase-averaged coherent states: $\rho_j = \sum_{n=0}^{\infty} c_i^n \exp(-|\alpha_i|^2) |n\rangle \langle n|$. Fock states are examples of such states. Figure 1 illustrates an example of the single-photon Fock state representation. As shown in Fig. 1a, an appropriate set of $\rho_j$s may contain only five phase-averaged coherent states. The fidelity of the representation exceeds 0.9996, as confirmed in Fig. 1a. Just seven phase-averaged coherent states allow to represent two-photon Fock states with the representation fidelity exceeding 0.999. Non-classically of the single-photon Fock state representation can be witnessed experimentally, as discussed in Sec. B of Supplemental material, with the experimental scheme involving beam-splitters and on-off detectors. Implementing linear transformations for the Fock states (in particular, beam-splitting and phase-shifting), one is able to emulate entangled states of several modes with the practically the same resources that are necessary for emulating Fock states in these modes. Indeed, for example, by 50/50 beam-splitting one turns two single-photon Fock states into the two-photon NOON state, $|1\rangle_a |1\rangle_b \rightarrow \frac{1}{\sqrt{2}} (|2\rangle_a |0\rangle_b + |0\rangle_a |2\rangle_b)$. More elaboration on this is found in Sec. A.C.D of Supplemental Material. In particular, in Sec. D we provide a recipe for producing NOON states for an arbitrary number of photons from two-mode Fock states, and discuss matters involving two-, three- and four-photon NOON states.

**Classical emulation of quantum states.**—Now let us demonstrate how it is possible to emulate an arbitrary measurement over an arbitrary quantum state approximated according to Eq. (1) using info about the state preparation. We use this info to build the following combined state of our signal and the two-state ancilla labeling the prepared states:

$$\rho_c = \frac{\zeta_+}{\zeta_+ + \zeta_-} \rho_+ \otimes |+\rangle \langle +| + \frac{\zeta_-}{\zeta_+ + \zeta_-} \rho_- \otimes |–\rangle \langle –|,$$

where $\zeta_+(-) = \sum_{c_j > 0 (<0)} |c_j|; \rho_+(-) = \sum_{c_j > 0 (<0)} |c_j| \rho_j$.

And the two mutually orthonormal ancilla states $|\pm\rangle$ encode classical information about the sign of the coefficient $c_j$ before the sampled signal state $\rho_1$. Notice that all the weights in the mixture of combined probe states (2) are positive. To utilize the knowledge about the state preparation for measuring the observable $A$, we suggest measuring the combined observable $A \otimes B$, where the ancilla observable allowing to infer the info about the state preparation is $B = (\zeta_+ + \zeta_-) (|+\rangle \langle +| - |–\rangle \langle –|)$. Thus, up to the accuracy of the representation, $\langle A \rangle = \text{Tr}(B \rho_c)$. Notice that for the classical states representable as positive-weighted mixtures of coherent-state projectors, the state (2) becomes trivial, and the measurement procedure is the same as for a usual, preparation-indifferent measurement. The scheme for measuring $\rho_c$ defined in Eq. (2) for classical and emulated states are shown in Fig. 2.

As follows from Eq.(2), the described measurement can be realized in the following simple way. One samples the probes $\rho_j$ according to the probability distribution $p_j = |c_j|^2 / (\zeta_+ + \zeta_-)$, labels each probe state by the ancilla state $|+\rangle$ or $|–\rangle$ depending on $\text{sgn}(c_j)$, and performs the measurement of the observable $A$ on the signal state and $B$ on the ancilla. If the $k$-th sample is the probe $\rho_{j_k}$, let us denote the particular measurement result of the observable $A$ as $A_{k}$ and the classical weight (measurement result for $B$) as $B_{k}$. For $N$ samples,
we calculate the following combination of the measurement results:

\[
\langle A \rangle_N = \frac{1}{N} \sum_k A_k B_k = \frac{\zeta_+ + \zeta_-}{N} \sum_k \text{sgn}(c_{jk}) A_k, \tag{3}
\]

with \(\langle A \rangle = \langle A \rangle_N + O(1/\sqrt{N})\) (see Sec. A of the Supplemental Material).

Eqs. (1)–(3) show that using just coherent states, it is possible to emulate results of any measurements on the quantum state, \(\rho_{\text{true}}\), with arbitrary precision. However, one needs to pay for it by the necessity of additional measurements of the ancilla resulting in positive and negative weights \(B_k\), and that leads to increase in statistical errors. Indeed, from Eqs. (1) and (2) one can get the following difference between the variance, \(\Delta_{AB}\), of \(A \otimes B\) evaluated with the mixed state \(\rho_c\), and the variance, \(\Delta_A\), of the observable \(A\) evaluated with \(\rho\):

\[
\Delta_{AB} - \Delta_A = 2\zeta_+ \zeta_- \left( \text{Tr}\{\rho_+ A^2\} + \text{Tr}\{\rho_- A^2\} \right) \geq 0. \tag{4}
\]

The price for the ability to model quantum states by classical ones is a larger number of the measurement runs for getting the same statistical error.

**Hong-Ou-Mandel effect.**—Let us illustrate a possibility of emulating measurement of quantum states by an archetypal quantumness demonstrator: the Hong-Ou-Mandel single-photon interference. If one has a single photon per each entry port of the 50/50 beam-splitter (BS) (Fig. 3a), in case of the ideal interference of both fields \(a\) and \(b\), the probability \(p_{12}\) of having the detectors \(D_1\) and \(D_2\) clicking simultaneously is zero. If the interference is no more ideal (for example, due to imperfect overlapping of the pulses or misaligned polarization), a probability of simultaneous clicks is arising and increasing with worsening of interference (which is usually illustrated with the Hong-Ou-Mandel “dip” in the dependence of \(p_{12}\) on the parameter describing interference quality, i.e. overlap of the fields on the BS).\cite{24, 25} If the detectors have the efficiency \(\eta\) and do not distinguish modes in the impinging fields, a registration of a click on \(j\)-th detector is described by the expression \cite{25}

\[
\Pi_j = 1 - \exp\left\{ -\frac{\eta}{2}(a^†a + b^†b + fa^†b + f^*b^†a) \right\},
\]

where the signs “+” and “-” correspond to the \(j = 1\) and 2 respectively, the operators \(x^\dagger\), \(x\) are creation and annihilation operators for \(x\)-th mode, \(x = a, b; :\) denotes the normal ordering, and the parameter \(f\) describes the degree of the overlap. Upon considering, for simplicity, a real and positive \(f\), the probability of both detectors clicking is \(p_{12} = \langle 1_a, 1_b | \Pi_1 \Pi_2 | 1_a, 1_b \rangle = (1 - f^2)\eta^2/2\), where the \(\langle 1_a, 1_b \rangle\) describes the single-photon Fock states in the modes \(a\) and \(b\).

The following scheme reproduces the Hong-Ou-Mandel effect (Fig. 3b): the randomly chosen phase-averaged coherent states are produced by appropriate splitting of an input coherent state, while the additional random phase shift \(\varphi\) introduces the effect of the phase averaging. In this manner the representation of \(\rho^\prime\) for the input single-photon states can be built in terms of the phase-averaged coherent states,

\[
\rho^\prime = \sum_{k,l} c_k c_l \rho_{k,l}^a \rho_{k,l}^b. \tag{5}
\]

For the probe state \(\rho_0^a \rho_0^b\) with a non-ideal overlap, the two-detector click probability now reads

\[
p_{12}^k = \frac{1}{2\pi} \int d\varphi \left[ 1 - p_{12}^k(\varphi) \right] \left[ 1 - p_{12}^l(\varphi) \right], \tag{6}
\]

where

\[
p_{12}^k(\varphi) = \exp\left\{ -\frac{\eta}{2} \left( |\alpha_k|^2 + |\alpha_l|^2 \pm 2f|\alpha_k||\alpha_l|\cos\varphi \right) \right\}.
\]

Let us now estimate to which extent the classical representation (5) is more expensive in terms of the necessary number of state copies. From Eq. (5), for \(\eta = 0.8\) and the overlap \(f^2 = 0.95\), one gets \(p_{12} = 0.017\). For the representation of the single-photon state shown in Fig. 1, the probability \(p_{21}\) is the same as for the quantum single-photon input. However, the variance of photocounts statistics is hugely different. For the input states being true single photons, a single-trial variance is less than unity. However, the single-trial variance, estimated according to Eq. (4), is \(\text{Var}(p_{12}) = 1.5 \times 10^4\). So, one needs the number of measurement runs (samples of probe states), \(N\), of about \(10^6\) for reliable demonstration of Hong-Ou-Mandel effect.

**Bell inequalities violation.**—Another famous manifestation of quantumness is Bell-type inequalities violation for a distinguishable (for example, spatially separated) quantum systems. Let us show here how it is possible to emulate the state of two entangled modes \(a\) and \(b\) sharing a single photon state \(|\Psi_{ab}\rangle = |\Psi_{ab}\rangle = (|1\rangle_a |0\rangle_b - |0\rangle_a |1\rangle_b)/\sqrt{2}\), and to demonstrate violation of the Clauser-Horn inequality\cite{26} using a modification of the scheme discussed in Ref.\cite{27}. In Ref.\cite{27} the coherently displaced signals are measured with simple on-off detectors and the following inequality is considered:

\[
-1 \leq \delta = q(\alpha, \gamma) - q(\alpha, \delta) + q(\beta, \gamma) + q(\beta, \delta) - q(\beta) - q(\gamma) \leq 0, \tag{7}
\]

FIG. 3. Scheme of observing Hong-Ou-Mandel effect with single photons (a) and the setup for its classical emulation (b). Gray dashed lines show how the phase-averaged coherent states can be generated by splitting a reference coherent state with subsequent application of a random phase shift \(\varphi\) in one of the arms.
where the no-click probabilities for the coherently displaced input state are defined as
\[ q_x(\mu) = \langle Q_x(\mu) \rangle = \langle D_x(\mu) \Pi^{(x)}_{\text{off}} D_x(\mu) \rangle, \]
and \( q(\mu, \nu) = \langle Q_{ab}(\mu) Q_{ab}(\nu) \rangle \), where the operator \( D_x(\mu) = \exp \{ \mu x^d - \mu^* x \} \) describes coherent displacement of \( x \)-th mode by the amplitude \( \mu, x = a, b \). The operator \( \Pi^{(x)} \) describes appearance of count absence on the detector measuring the \( x \)-th mode. For the detectors with the efficiency \( \eta \) this operator can be expressed as \( \Pi^{(x)} = (1 - \eta)^{n_x} \) in terms of the number operator \( n_x = x^d x \). The value of \( j_0 \) can be minimized for all possible shifts \( \alpha, \beta, \gamma, \) and \( \delta \). To have the minimal \( j_0 \), one needs setting \( \alpha = -\delta = \mu_1 \) and \( \gamma = -\beta = \mu_2 \). Particular values of \( \mu_1,2 \) depend on the efficiency. For example, for the minimal value of \( j_0 \) and the optimal amplitudes \( \mu_1 \) and \( \mu_2 \) for ideal detectors (\( \eta = 1 \)) are equal to \((-1.118, 0.563, 0.165) \); for \( \eta = 0.95 \) the optimal values are \((-1.118, 0.587, 0.177) \), for \( \eta = 0.90 \) they are \((-1.066, 0.615, 0.191) \). Taking into account the relations between the optimal amplitudes and inequality (7), one can introduce the observable \( J_0 = Q_a(-\mu_2)Q_b(\mu_2) - Q_a(\mu_1)Q_b(-\mu_1) - Q_a(-\mu_2)[1 - Q_b(-\mu_1)] - [1 - Q_a(\mu_1)]Q_b(\mu_2) \), corresponding to the quantity \( j_0 = \langle J_0 \rangle \). To classically emulate \( |\Psi_{ab}\rangle \langle \Psi_{ab}| \), it is convenient to represent this state as a result of beamsplitting of a single-mode one-photon state. Emulating this single-photon state with a mixture of phase-averaged coherent states, \( |\Psi_{ab}\rangle \langle \Psi_{ab}| \) can be rewritten in the form of Eq. (1) with \( \rho_j = \frac{1}{2\pi} \int_0^{2\pi} d\varphi f_j(\varphi)|f_j(\varphi)\rangle \langle f_j(\varphi)| \), where the states \( |f_j(\varphi)\rangle \) are products of the coherent states of the modes \( a \) and \( b \): \( |f_j(\varphi)\rangle = \alpha_j e^{i\varphi}/\sqrt{2} \) if \( \alpha_j e^{i\varphi}/\sqrt{2} \) and \(-\alpha_j e^{i\varphi}/\sqrt{2} \) otherwise.

A scheme, suitable for implementation of the discussed classical emulation of the state \( |\Psi_{ab}\rangle \langle \Psi_{ab}| \) is shown in Fig. 4. For each trial, the two parties choose the same random index \( j \) of the probe state and the same random phase shift \( \varphi \) and prepare the coherent states \( |\alpha_j e^{i\varphi}/\sqrt{2} \rangle \) and \(-\alpha_j e^{i\varphi}/\sqrt{2} \rangle \) of the modes \( a \) and \( b \) respectively. Notice that for emulation of \( |\Psi_{ab}\rangle \langle \Psi_{ab}| \) with our approach and the state representation used in Fig. 1, one needs only five phase-averaged probe states. The expectation values of \( J_0 \) over such probe states can be found using the rules

\[ \text{Tr} \{ \rho_j Q_a(\mu)Q_b(\nu) \} = \frac{1}{2\pi} \int d\varphi q_j(\mu, \nu) q_j(-\nu, \varphi), \]

\[ \text{Tr} \{ \rho_j Q_{ab}(\mu) \} = \frac{1}{2\pi} \int d\varphi q_j(\pm \mu, \varphi), \]

where \( q_j(\nu, \varphi) = \exp(-\eta |\alpha_j e^{i\varphi}/\sqrt{2} - \nu|^2) \).

As it is to be expected, the mean value of the measured observable remain approximately the same as for the true state, \( \langle J_0 \rangle = -1.118, -1.077 \) for \( \eta = 0.95, 0.9 \), while the single-trial variances are expectedly large: \( \text{Var}(J_0) = 5.1 \times 10^3 \) and \( 5.0 \times 10^3 \) respectively. The numbers of the measurement repetitions, required for reliable non-classicality demonstration, are \( N \gtrsim 1.5 \times 10^6 \) for \( \eta = 0.95 \) and \( N \gtrsim 3.4 \times 10^6 \) for \( \eta = 0.90 \).

Conclusions.—We have shown that non-classical effects (such as the observation of two-photon interference, and violation of Bell-type inequalities) can be emulated in a purely classical way. For that one just needs to know how the signal state is prepared by labeling each probe and using that classical information during the measurement. However, the price for the ability to use classical light sources is the rapid growth of the measurement repetition number, required for getting reliable results. The proposed approach to modeling a quantum state by the combination of classical ones is likely to be a handy toolbox for proof-of-principle experiments with classical tests of quantum effects. It is clear that possible applications and measurements with emulated non-classical states are not limited to just the two experiments discussed above. In the Supplemental Material, Sec. B describes the experiment on non-classicality witnessing for the emulated single-photon state. In Secs. C and D respectively, additional technical details on how to emulate phase estimation with NOON states, and arbitrary multi-photon NOON states are presented. By experimenting with such NOON states, one can confirm the expected quantum effects without the troubles related to the generation and preservation of complex non-classical quantum states (it is also useful to mention that decoherence very quickly deteriorates metrological advantage expected from the true NOON states [28]). Very generally in metrology, the presented emulation procedure provides a systematic tool to translate quantum measurement protocols into classical ones, and to study their respective efficiencies.

Our results shed new light on an alternative operational meaning to non-classicality. The number of sampling copies required to faithfully emulate a non-classical state is generally larger when classical states are used for this task than non-classical ones. In this dual sense, the degree of non-classicality is closely related to the complexity of classical emulation. In practical terms, if one attempts to emulate multi-mode entangled states with classical states, the number of probe-state components needed to achieve a faithful emulation increases with the number of modes and photon numbers. Nevertheless, classical emulation can still present a
more practical solution in place of direct generation of highly-entangled non-classical states, since classical emulation resource states, such as coherent states, are far easier to implement.

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SUPPLEMENTARY MATERIAL

A. DEVELOPING CLASSICAL REPRESENTATIONS

Rate of convergence of $\langle A \rangle_N$

Suppose that the set of classical component probe states $\{\rho_j\}$ are used to emulate the target true state $\rho_{true} \approx \sum_j c_j \rho_j$. In the perspective of statistical inference, one can rewrite $\langle A \rangle_N$ defined in Eq. (3) of the main text as

$$\langle A \rangle_N = \frac{\zeta}{N} \sum_{j,l} \text{sgn}(c_j) a_l n_{jl}, \quad (10)$$

where $\zeta = \sum_j |c_j| = \zeta_+ + \zeta_-$, and now $\langle A \rangle_N$ is a double sum over all probe states employed and eigenvalues $a_l$ of the observable $A$ measured. The frequencies $n_{jl}$ of having $l$-th result of the measurement with $j$-th probe are summed to the total number of trials, $N$: $\sum_{j,l} n_{jl} = N$. Thus the relative frequencies $\nu_{jl} = n_{jl}/N$ inherently follow a multinomial distribution with respect to the indices $j$ and $l$, with the statistical average

$$\nu_{jl} = \left\{ \begin{array}{ll} [\delta_{j,l'} \tilde{p}_{jl} + (N-1)\tilde{p}_{j'l}]/N & \text{when } j = j', \\ (N-1)\tilde{p}_{j'l}/N & \text{otherwise}. \end{array} \right. \quad (11)$$

determined by the observation $\nu_{jl} = \tilde{p}_{jl} = p_j p_{j'l}$, $p_j = |c_j|/\zeta$ and $p_{j'} = |a_l|/\zeta$. These immediately give

$$\frac{\langle (A \rangle_N - \langle A \rangle)^2}{\frac{\zeta^2}{N} \left[ \sum_j p_j \text{Tr}\{\rho_j A^2\} - \left( \sum_j p_j \text{sgn}(c_j)\text{Tr}\{\rho_j A\} \right)^2 \right] = \frac{\Delta_{AB}}{N}, \quad (12)$$

Alternatively, one arrives at this result by considering the extended model $\langle A \rangle = \text{Tr}\{\langle A \otimes B \rangle \rho_c\}$ discussed in the text and recognizing the fact that each independently sampled eigenvalue incurs a quantum variance of $\Delta_{AB}$, such that scaling $\Delta_{AB}$ with $N$ gives the right answer. Hence, in the limit of large $N$, we indeed expect that $\langle A \rangle - \langle A \rangle_N = O(1/\sqrt{N})$.

On the other hand, for the same number of copies $N$, if one can generate $\rho_{true}$ directly, then the naive linear estimator $\langle A \rangle_{\text{LIN}} = \sum_j a_l \nu_l$ for $\langle A \rangle$, where $\nu_l = n_l/N \rightarrow p_l = \langle a_l | \rho_{true} | a_l \rangle$, leads to

$$\left( \langle A \rangle_{\text{LIN}} - \langle A \rangle \right)^2 = \frac{\Delta_A}{N} \quad (13)$$

owing to the basic multinomial rule $\langle \nu_{jl} - p_j \rangle (\nu_{k'l} - p_{k'}) = (p_j \delta_{j,k'} - p_{j'k'})/N$. As argued in the text, the fact that $\Delta_{AB} > \Delta_A$ simply reiterates that non-classical state emulation using classical component states reduces the complexity of quantum-state generation at the expense of a larger $N$ to estimate $\langle A \rangle$ up to some fixed accuracy.

Sampling accuracy in non-classical quantum-state emulation

Let us show how well sampling of the probe states from the set with the help of $\rho_c$ (equivalent to using a classical random-number generator) can approximate the state $\rho$ from Eq. (1) of the main text.

After a multinomial sampling of $N_s$ copies of $\rho$ (not to be confused with $N$, the total number of copies used to estimate $\langle A \rangle_N$), one obtains the estimator $\hat{\rho} = \zeta \sum_j \nu_j \text{sgn}(c_j) \rho_j$, where the relative frequencies $\nu_j$ of the probe states $p_j$ tend to the probabilities $p_j = |c_j|/\zeta$ for $N_s \gg 1$. We may consider the mean squared-error (MSE), $\text{MSE} = \overline{\text{Tr}\{\langle \rho - \hat{\rho} \rangle^2\}}$ with $\overline{\cdot}$ denoting the statistical mean (expectation value), as the figure of merit for determining the accuracy of such a sampling with a given value of $N_s$ with respect to the actual state $\rho = \sum_j c_j \rho_j$ being classically emulated. For a multinomial distribution, as $\overline{\nu_j} = p_j$ and $(\nu_j - p_j)(\nu_k - p_k) = (p_j \delta_{j,k} - p_{j'k'})/N_s$, the MSE can be easily computed to be

$$\text{MSE} = \frac{1}{N_s} \sum_{j,j'} \left\{ |c_j| |c_{j'}| \text{Tr}\{\rho_j^2\} - c_j c_{j'} \text{Tr}\{\rho_j \rho_{j'}\} \right\} \quad (14).$$

An important special case corresponds to pure probe states $\langle \rho_j^2 \rangle = 1$. They allow to see more clearly into the essence of statistical noise introduced by classical emulation of quantum states. For mixed probe states $\rho_j$, their intrinsic classical noise is masked by the assumption about their noise-less sampling. For pure probes, one can represent the MSE (14) as

$$\text{MSE} = \frac{1}{N_s} \left\{ (1 - \text{Tr}\{\rho^2\}) + (\zeta^2 - 1) \right\} \quad (15).$$

The first term corresponds to the internal classical noise of the mixed state $\rho$, while the latter one describes the additional sampling noise introduced by classical representation.
of a non-classical state. If the state $\rho$ is classical, one can find its representation with positive weights $c_j > 0$. Therefore, according to standard normalization of the density matrices $\rho$ and $\rho_j$, we have $\zeta = \sum_j |c_j| = \sum_j c_j = 1$ and the second term in Eq. (15) vanishes. Notice that this term also vanishes when one samples the actual physical state (2) of the main text used for reproducing measurement results instead of $\rho$ having MSE $= \left(1 - \text{Tr}(\rho^2)\right) / N$. 

It is also worth noting that if $\rho_{\text{true}}$ is also pure, the MSE in (15) is defined by accuracy and purity of the representation, and goes to zero with the fidelity going to unity.

Optimizing the representation

Our task is to simulate the result of quantum-state measurement. So, the optimization task for developing the representation would consist of choosing the minimal possible number of classical probes providing for the least error is estimating a specified observable. Also, these probes themselves should be specified. The choice of the probes for the quantum state reconstruction was discussed in a number of works [16, 17, 19]. However, measurement-oriented optimization as discussed in the current contribution, was not carried on. We leave this discussion for the future works. Here we consider a particular case of optimization which allows shedding some light on the best choice of the probe states for the experiments.

First, let us search for the optimal decomposition of the state $\rho_{\text{true}}$, which simultaneously minimizes the additional sampling noise $\left(\zeta^2 - 1\right) / N_s$ (see Eq. (15)) and the decomposition error $\text{Tr}\{(\rho - \rho_{\text{true}})^2\}$:

$$
\min_{\{\rho_j\}, \{c_j\}} D, \quad D = \left(\frac{\zeta^2 - 1}{N_s} + \text{Tr}\{(\rho - \rho_{\text{true}})^2\}\right). \quad (16)
$$

A variation in $\rho_j$ and $c_j$ gives

$$
\delta D = 2 \sum_j \left(\frac{\zeta \text{sgn}(c_j)}{N_s} + \text{Tr}\{\rho_j(\rho - \rho_{\text{true}})\}\right) \delta c_j + 2 \sum_j c_j \text{Tr}\{\delta \rho_j(\rho - \rho_{\text{true}})\}. \quad (17)
$$

The normalization of density operators imposes the constraints $\sum_j \delta c_j = 0$ and $\text{Tr}\{\delta \rho_j\} = 0$ on the variations. Strictly speaking, one should also impose certain constraints ensuring semi-positivity of $\rho$ during the considered variation. However, if the state $\rho$ is mixed with all its eigenvalues being strictly positive, the semi-positivity condition is not violated for any infinitesimally small variation.

First, let us assume that we do not require the states $\rho_j$ to be classical, i.e. we do not impose any additional constraints on $\delta \rho_j$. In that case, the variation in $\rho_j$ (the lower line in Eq. (17)) implies that the optimal representation should be accurate: $\rho = \rho_{\text{true}}$. Then, variation in $c_j$ leads to the equation $\sum_j \text{sgn}(c_j) \delta c_j = 0$ and, finally, to positivity of all the weights: $c_j > 0$ for all $j$. Therefore, if we are not limited in terms of the structure of $\rho_j$, the optimal choice is either $\rho_{\text{true}}$ itself or any of its representations in the form of a positive-weight mixture (if $\rho_{\text{true}}$ is mixed).

The latter conclusion about the positivity of optimal $c_j$ as soon as the representation is accurate ($\rho = \rho_{\text{true}}$) stems from the variation in $c_j$ and remains valid regardless of any constraints imposed on $\delta \rho_j$. If, for the available set of probe states $\rho_j$, an accurate decomposition of $\rho_{\text{true}}$ requires negative weights, such a decomposition is not optimal. For usage of a simpler representation with a smaller number of components $\rho_j$, the advantage of having smaller sampling noise will exceedingly compensate for the loss in representation accuracy.

Let us now assume that the set of probe states is limited by coherent states only: $\rho_j = |\alpha_j\rangle\langle\alpha_j|$. Their variations can be written as

$$
\delta \rho_j = (a^\dagger - a_j^\dagger)|\alpha_j\rangle\langle\alpha_j| + |\alpha_j\rangle\langle\alpha_j|(a - a_j)\delta \alpha_j^\dagger. \quad (18)
$$

Independence of the variations of the coherent states amplitudes implies that the following condition should be satisfied for the optimal representation:

$$
\langle \alpha_j| (a, \rho - \rho_{\text{true}})|\alpha_j\rangle = 0. \quad (19)
$$

From this equation it follows, for example, that for the set of coherent probes not limited to just a single state, difference between the optimally represented $\rho$ and $\rho_{\text{true}}$ cannot be proportional to a coherent or a number state. Generally, the operator $[a, \rho - \rho_{\text{true}}]$ should be expressible as a sum of linearly independent operators. The number of these operators should be at least that of the probe states.

Now let us consider a measurement-oriented version of the procedure described above. We look for the optimal component set that minimizes sampling noise (14) for a given $N_s$ and under the condition that a certain measurement result is to be obtained. As the properties of the optimal components $\rho_j$ are of interest here, we shall focus only on their variation. Furthermore, we suppose that there is some desirable physical property about the decomposition $\rho$ that needs to be fixed in the form of an expectation-value constraint, namely $\text{Tr}\{\rho O\} = \mu$ for some observable $O$. The corresponding distance to be minimized, $D’ = N_s \text{MSE} - \lambda(\text{Tr}\{\rho O\} - \mu)$. This distance is parametrized by a Lagrange scalar $\lambda$. A variation in $\rho_j$ therefore gives

$$
\delta D’ = 2\zeta \sum_j |c_j|\text{Tr}\{\rho_j \delta \rho_j\} - 2 \sum_j c_j \text{Tr}\{\rho \delta \rho_j\} - \lambda \sum_j c_j \text{Tr}(O \delta \rho_j). \quad (20)
$$

Since the $\rho_j$’s are quantum states, they take the form $\rho_j = W^\dagger_j W_j / \text{Tr}\{W^\dagger_j W_j\}$. This implies the variation

$$
\delta \rho_j = \frac{\delta W^\dagger_j W_j + W^\dagger_j \delta W_j}{\text{Tr}\{W^\dagger_j W_j\}} - \rho_j \frac{\text{Tr}\{\delta W^\dagger_j W_j + W^\dagger_j \delta W_j\}}{\text{Tr}\{W^\dagger_j W_j\}}. \quad (21)
$$
that leads to the extremal equation

$$c_j \rho_j M = 2c_j (\rho_j^2 - \rho_j \text{Tr}(\rho_j^2)) + c_j \rho_j \text{Tr}(\rho_j M),$$

$$M = 2 \rho + \lambda O,$$

(22)

when $\delta D'$ is set to zero.

In order to satisfy (22), we now need $\rho_j$ to commute with $2 \rho + \lambda O$. This practically means that $\rho_j$ shares common eigenstates with $\rho$ and $O$. A specific situation is when $O = a^\dagger a$ and $\mu$ is the mean photon number of the system. Then, an extremal set of $\rho_j$’s is some set of Fock-state mixture ($\rho_j = \sum_n |n\rangle w_{jn} \langle n|$ with $\sum_n w_{jn} = 1$), which is compatible with a $\rho$ that is also a Fock state.

**Single-photon state emulation**

A general non-classical quantum state $\rho_{\text{true}}$ can be emulated by a linear combination of classical states inasmuch as

$$\rho_{\text{true}} \approx \rho = \sum_j c_j \rho_j, \quad \sum_j c_j = 1.$$  

(23)

In general, given a fixed set of $\rho_j$’s, the coefficients $c_j$, which approximate $\rho_{\text{true}}$ in the best way, can be determined by solving the following numerical problem:

$$\max_{\{c_j\}} F(\rho_{\text{true}}, \rho)$$

subject to: $\sum_j c_j = 1, \ \rho \geq 0,$

(24)

where the fidelity $F(\rho_{\text{true}}, \rho) = \langle \text{Tr} \{ \sqrt{\rho_{\text{true}} \rho \sqrt{\rho_{\text{true}}}} \} \rangle^2$ is maximized over the coefficients $c_j$ conditioned on the positive semidefiniteness of $\rho$ [19]. The solution to this problem can be found using semidefinite programming.

When $\rho_{\text{true}} = |1\rangle \langle 1|$ is the single-photon state, the set of classical probe states $\rho_j$ can be chosen in the form of 5 phase-averaged coherent states with amplitudes $\alpha_i = 0, 0.25, 0.5, 0.75, 1$ (Fig. 5a):

$$\rho_j = |\alpha_j\rangle \langle \alpha_j| = \frac{1}{2\pi} \int d\varphi |\alpha_j e^{i\varphi}\rangle \langle \alpha_i e^{i\varphi}|$$

$$= \sum_n \frac{|\alpha_j|^{2n}}{n!} e^{-|\alpha_j|^2} |n\rangle \langle n|.$$  

(25)

The semidefinite program in (24) produces the resulting optimal coefficients $c_j = -21.8, 25.6, -3.1, 0.33, -0.0028$. The fidelity of the constructed representation for the single photon state exceeds 0.9996 (Fig. 5b).

**B. NON-CLASSICALITY WITNESS: CLASSICAL EMULATION OF NON-CLASSICALITY**

To prove non-classicality of a given state (or a class of states), one can construct a witness operator $W$, find the classical limit

$$W_0 = \max_{\rho' \text{ is classical}} \text{Tr}(W \rho'),$$

(26)

and check that the investigated state $\rho$ yields

$$\text{Tr}(W \rho) > W_0.$$  

(27)

For example, to prove non-classicality of the single-photon state, one can build the following witness operator:

$$W = 2|1\rangle \langle 1| - |0\rangle \langle 0| - |2\rangle \langle 2|.$$  

(28)

Using Glauber representation of classical states $\rho'$ and taking into account diagonality of the operator $W$ in Fock state

![FIG. 5. Representation of the single-photon state in terms of phase-averaged coherent states: decomposition coefficients (a), diagonal elements of the optimal linear combination of the coherent states, shown in different scales in the main plot and the inset (b), and the values of non-classicality witness operator $W$ (Eq. (28)) for the coherent states (c). The dashed horizontal line shows the expectation value of the non-classicality witness operator for the single-photon state.](image)
basis, one can show that

\[ W_0 = \max_{\rho \text{ classical}} \text{Tr}\{W\rho\} = \max_{P : P(\alpha) \geq 0} \int d^2 \alpha P(\alpha) W(\alpha), \]

(29)

where

\[ W(\alpha) = \langle \alpha | W | \alpha \rangle = \left( 2|\alpha|^2 - 1 - \frac{|\alpha|^4}{2} \right) e^{-|\alpha|^2}, \]

(30)

and the normalization condition holds:

\[ \int d^2 \alpha P(\alpha) = 1. \]

(31)

Non-negativity of \( P(\alpha) \) for classical states implies that the maximal classical value \( W_0 \) corresponds to the maximum of the function \( W(|\alpha|) \), which equals

\[ W_0 = \max_\alpha W(|\alpha|) = 0.206 \]

(32)

and is reached for the coherent state with the amplitude \( |\alpha^{(0)}| = 1.134 \).

For the single-photon state, the expectation value of the witness operator equals \( \langle 1 | W | 1 \rangle = 2 > W_0 \).

The mean value \( \langle W \rangle \) for the emulated state (25) equals 1.9992, which clearly exceeds the classical limit \( W_0 = 0.206 \).

On the other hand, Fig. 5c shows that the values \( \text{Tr}\{W\rho_j\} \) fit into the classical region \([-1, W_0]\) for all \( j \). The two reasons for the final result exceeding the classical limit are:

- minus sign for certain \( \rho_j \): classical maximum of \(-W\) is \( 1 > 0.206 \) (but still less than 2);
- the measurement results are multiplied by the factor \( \zeta_+ + \zeta_- = 50.8 \).

The excess variance of a single-trial measurement (given by Eq. (5) of the main text) is \( 2.0 \times 10^4 \). Therefore, to demonstrate the non-classicality reliably, one needs of about \( 10^8 \) copies of the state.

Non-classicality witnessing under realistic measurement conditions

The non-classicality witness \( W \), described by Eq. (28), requires a photon number resolving measurement. To stay more realistic, it is worth constructing a witness, which can be measured with usual single-photon detectors, possessing final efficiency and dark count rate.

Let us consider the measurement setup, shown in Fig. 6a and consisting of 4 single-photon detectors with the detection efficiency \( \eta \) and the dark count rate \( \varepsilon \). The five possible outcomes of the measurement correspond to detection of \( m = 0, 1, 2, 3, \) and 4 counts respectively and can be described by the positive operator-valued measure (POVM) \( \{\Pi_0, \ldots, \Pi_4\} \).

The POVM elements have the following Fock-state basis representation:

\[ \Pi_m = \sum_{n=0}^{\infty} |n\rangle \langle n| p(m|n), \]

(33)

where

\[ p(m|n) = \sum_{k=0}^{m} \frac{4!(1-m-k)(1-\varepsilon)^{4-k}}{(4-m)!k!(m-k)!} \times \left( 1 - \frac{4-k}{4} \eta \right)^n \]

(34)

is the probability of detecting \( m \) counts if the input state of the measurements scheme in Fig. 6a is the Fock state \( n \) (Fig. 6b).

Following the ideas from Eq. (23), one can try to approximate the witness operator \( W \), introduced by Eq. (28), in terms of the available POVM elements:

\[ W \approx W_4 \equiv \sum_{m=0}^{4} z_m \Pi_m, \]

(35)

where the coefficients \( z_m \) (see the inset in Fig. 6c) can be found, for example, by minimization of the quadratic distance between \( W \) and \( W_4 \) (the fidelity \( F \) cannot be used here because neither \( W \) nor \( W_4 \) are positive semi-definite operators):

\[ \min_{\{z_m\}} \sum_{n=0}^{\infty} \left( \langle n | W | n \rangle - \langle n | W_4 | n \rangle \right)^2. \]

(36)

Fig. 6c shows the resulting witness operator \( W_4 \). While being different from the ideal witness \( W \) because of detectors’ non-ideality, it is still suitable for detection of non-classicality. During the performed numerical calculations, we assumed that the detection efficiency equals \( \eta = 0.8 \) and the dark count probability is \( \varepsilon = 0.001 \).

Similarly to Eq. (32), the maximal classical value \( W_{40} \) of the constructed witness corresponds to the coherent state \( |\alpha\rangle \) with \( \alpha = 1.176 \), maximizing \( W_4(|\alpha\rangle) = \langle \alpha | W_4 | \alpha \rangle \):

\[ W_{40} = \max_\alpha W_4(|\alpha\rangle) = 0.248. \]

(37)

The witness value, reached for the single-photon state, equals \( \langle 1 | W_4 | 1 \rangle = 1.538 > W_{40} \). Unlike the ideal witness \( W \) yielding zero variance for the state \( |1\rangle \), the variance of the observable \( W_4 \) for the single-photon state is 1.59. Therefore, one needs to perform at least several repetitions of the measurement to be sure that the results are incompatible with the assumption of a classical input state if the state \( |1\rangle \) is supplied.

The classically emulated single-photon state, discussed in the previous sections, yields the mean value \( \langle W_4 \rangle = 1.537 \), which still noticeably exceeds the classical limit. The excess variance of the observable \( W_4 \) is \( 1.8 \times 10^3 \). Therefore, the number of the measurement repetitions, required for reliable proof of the single-photon state non-classicality, remains approximately the same as for the measurement of the ideal witness \( W \).
FIG. 6. Approximation of the single-photon non-classicality witness by the four-detector measurement setup: detection scheme (a), Fock-basis decomposition coefficients of the POVM elements $\Pi_m$ (b), and comparison of the ideal witness $W$ (gray bars) and the constructed witness $W_4$ (blue line) - plot (c). The inset in plot (c) shows the found decomposition coefficients $z_m$ in Eq. (35). The detection efficiency $\eta = 0.8$ and the dark count rate $\varepsilon = 0.001$ were used for the calculations.

C. PHASE ESTIMATION WITH 2-PHOTON NOON-STATE

Here we shown how one can emulate the phase estimation with NOON states using our emulation scheme.

The state obtained after the interference of 2 photons at a beamsplitter, is the 2-photon NOON-state $(|2\rangle_a|2\rangle_b)/\sqrt{2}$ and, therefore, can be used for sensitivity enhancement in phase estimation. For the scheme, shown in Fig. 7, the probability of encountering both photons in one arm (coupled to either $D_1$ or $D_2$) equals

$$p_2^{(0)}(\theta) = \frac{1 + f}{4} \sin^2 \theta,$$

(38)

while the probability of having one photon in each arm is

$$p_1^{(0)}(\theta) = \frac{1 - f}{2} + \frac{1 + f}{2} \cos^2 \theta,$$

(39)

where $\theta$ is the phase shift to be measured.

Similarly to the previous section, one can calculate the probability of coincidence count $p_{11}$ and the unconditional probabilities of single-photon detection $p_1$, and $p_1$ and introduce the normalized coincidence rate:

$$g_2(\theta) = \frac{p_{11}}{p_1 \cdot p_1} = \frac{16 \left( \eta(1 - \epsilon)(z + \eta(3 - f - 4\epsilon) + 8\epsilon) + 4\epsilon^2 \right)}{(\eta(1 - \epsilon)z + \eta(1 - \epsilon)(8 - f\eta - \eta) + 8\epsilon)^2},$$

(40)

where $z = (1 + f)\eta \cos 2\theta$. The solid line in Fig. 8a shows the dependence of $g_2(\theta)$ on the phase shift $\theta$.

For the 2-mode probe state $\rho_i \otimes \rho_j$, the probabilities of clicks on both detectors are

$$p_{11}(i, j; \theta) = \frac{1}{2\pi} \int d\varphi \left[ 1 - p_-(|\alpha_i|, |\alpha_j|, \theta, \varphi) \right]$$

$$\times \left[ 1 - p_+(|\alpha_i|, |\alpha_j|, \theta, \varphi) \right],$$

(41)

$$p_1(i, j) = \frac{1}{2\pi} \int d\varphi \left[ 1 - p_-(|\alpha_i|, |\alpha_j|, \theta, \varphi) \right],$$

(42)

and

$$p_1(i, j) = \frac{1}{2\pi} \int d\varphi \left[ 1 - p_+(|\alpha_i|, |\alpha_j|, \theta, \varphi) \right],$$

(43)

where

$$p_{\pm}(x, y, \theta, \varphi) = (1 - \varepsilon) \exp \left[ -\frac{\eta}{2} \left( x^2 (1 \pm \cos \theta) + y^2 (1 \mp \cos \theta) \pm 2\sqrt{f} x y \sin \theta \sin \varphi \right) \right].$$

(44)
The results of the calculation of the normalized second-order functions for the true and emulated states are shown in Fig. 8a. The values \( \eta = 0.8, \varepsilon = 0.001, f = 0.95 \) and \( N = 10^8 \) repetitions were assumed.

**D. CLASSICAL EMULATION OF AN ARBITRARY NOON-STATE**

The technique, described in the previous section for emulation of a NOON-state with \( N = 1 \), can be generalized to emulation of a state with an arbitrary \( N \):

\[
|\Psi^{(N)}_{ab}\rangle = \frac{1}{\sqrt{2}} (|N\rangle_a |0\rangle_b - |0\rangle_a |N\rangle_b). \tag{45}
\]

Fock states can be decomposed in terms of phase-averaged coherent states and require affordable resources. Linear optical transformation of the Fock states correspond to trivial arithmetic operations with the amplitudes of the coherent states, used for their representation. Therefore, finding a way to represent a NOON state as a result of some linear operations applied to Fock states will be sufficient for construction of its efficient decomposition.

First, let us consider the linear optical transformation

\[
a^\dagger \rightarrow (a^\dagger + e^{i\theta}b^\dagger)/\sqrt{2}, \quad b^\dagger \rightarrow (a^\dagger - e^{i\theta}b^\dagger)/\sqrt{2} \tag{46}
\]

applied to the 2-mode Fock state \( |n\rangle_a |m\rangle_b \) with \( n + m = N \). The density matrix of the resulting state is

\[
\rho(n, m, \theta) = \sum_{j=0}^{N} \sum_{l=0}^{N} A_{nm}^{(j)} A_{nm}^{(l)} e^{i\theta(\ell-j)} \times |j\rangle_a \langle\ell| \otimes |N-j\rangle_b \langle N-\ell|, \tag{47}
\]

where

\[
A_{nm}^{(j)} = \sum_{k=\max(0, j-m)}^{\min(n, j)} \frac{\sqrt{n!m!j!(N-j)!(-1)^{m-j+k}}}{k!(n-k)!(j-k)!(m-j+k)!} 2^{-N/2}. \tag{48}
\]

In the symmetric case \( n = m = N/2 \), only even indices \( j = 2s \) yield non-zero coefficients (it can be considered as a generalization of Hong-Ou-Mandel effect):

\[
A_{nn}^{(2s)} = \sqrt{2s}! |(N-2s)!(-1)^{n-s}|s!(n-s)! 2^{-N/2}. \tag{49}
\]
To generate the target NOON state, we need to remove all the terms from Eq. (47), except for those with \( j \) and \( l \) equal to 0 or \( N \). Here, we can use the equality

\[
\frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi ik/N} = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}.
\]  

(50)

Therefore,

\[
\frac{1}{N} \sum_{k=0}^{N-1} \rho(n, m, \theta_0 + 2\pi k/N) = \frac{N!}{n!\sqrt{m!}} 2^{-N+1} |\Psi_N(\theta_0, m)\rangle_a \langle \Psi_N(\theta_0, m)| + \sum_{j=1}^{N-1} \left( A_{nm}^{(j)} \right)^2 |j\rangle_a \langle j| \otimes |N - j\rangle_b \langle N - j|,
\]  

(51)

where

\[
|\Psi_N(\theta_0, m)\rangle_{ab} = \frac{1}{\sqrt{2}} \left( |\Psi_N(0)\rangle_0 + (-1)^m e^{iN\theta_0} |\Psi_N(0)\rangle_N \right).
\]  

(52)

By the choice \( \theta_0 = \pi/N \) for even \( m \) and \( \theta_0 = 0 \) for odd \( m \), one can ensure that the first term of Eq. (51) corresponds to the target state (45): \( |\Psi_N(\theta_0, m)\rangle_{ab} = |\Psi_N\rangle_{ab} \).

Finally, the target state can be expressed from Eq. (51):

\[
|\Psi_N\rangle_{ab} |\Psi_N\rangle = \frac{N!\sqrt{2}^{N-1}}{N!} \left[ \frac{1}{N} \sum_{k=0}^{N-1} \rho(n, m, \theta_0 + 2\pi k/N) \right] - \sum_{j=1}^{N-1} \left( A_{nm}^{(j)} \right)^2 |j\rangle_a \langle j| \otimes |N - j\rangle_b \langle N - j|.
\]  

(53)

As discussed above, the derived expression implies that emulating the NOON-state is not much more complex than emulation of the Fock state with \( N - 1 \) photons.

When \( N \) is even, the procedure can be simplified. Eq. (49) implies that only even multipliers \((l - j)\) of the parameter \( \theta \) are present in Eq. (47) if \( n = m = N/2 \). Therefore, \( \rho(n, n, \theta + \pi) = \rho(n, n, \theta) \), and the summation over \( j \) can be limited by \( N/2 - 1 \) instead of \( N - 1 \):

\[
|\Psi_N\rangle_{ab} |\Psi_N\rangle = \frac{(n!)\sqrt{2}^{N-1}}{N!} \left[ \frac{2^{N/2-1}}{N} \sum_{k=0}^{N/2-1} \rho(n, m, \theta_0 + 2\pi k/N) \right] - \sum_{k=1}^{N/2-1} \left( A_{nm}^{(2k)} \right)^2 |2k\rangle_a \langle 2k| \otimes |N - 2k\rangle_b \langle N - 2k|.
\]  

(54)

In comparison with Eq. (53), the derived expression contains almost twice smaller number of terms and requires emulation of Fock states with up to \( \max(N/2, N - 2) \) photons only.

The expressions (53) and (54) have exactly the same form as required by Eq. (1). Therefore, the emulation of the target NOON-state can be performed as discussed above, but applied in two steps. First, one randomly chooses one of the states from the right-hand side of Eq. (53) or (54) with their probabilities being proportional to the decomposition coefficients. Then, the selected state is emulated classically according to its decomposition in terms of coherent states. Let us consider the emulation of the states from the right-hand side of Eqs. (53) and (54) in more details.

Suppose that the Fock states \( |n\rangle \) can be approximated by linear combinations of phase-averaged coherent states in the following way:

\[
|n\rangle \langle n| \approx \sum_i c_{ni} |\alpha_{ni}\rangle \langle \alpha_{ni}|
\]  

(55)

Therefore, 2-mode Fock states can be emulated as

\[
|n\rangle_a \langle n| \otimes |m\rangle_b \langle m| \approx \sum_{i,j} c_{ni} c_{mj} |\alpha_{ni}\rangle \langle \alpha_{ni}||\alpha_{mj}\rangle \langle \alpha_{mj}|
\]  

(56)

Finally, the states \( \rho(n, m, \theta) \) can be decomposed as

\[
\rho(n, m, \theta) = \sum_{i,j} c_{ni} c_{mj} \rho_{ij}(n, m, \theta),
\]  

(57)

where

\[
\rho_{ij}(n, m, \theta) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2 \left( |\alpha_{ni}|e^{i\varphi_1} + |\alpha_{mj}|e^{i\varphi_2} \right) \langle a_{ij} | \left( |\alpha_{ni}|e^{i\varphi_1} + |\alpha_{mj}|e^{i\varphi_2} \right) \otimes \left( e^{i\theta} |\alpha_{ni}|e^{i\varphi_1} - |\alpha_{mj}|e^{i\varphi_2} \right) \langle b_{ij} |.
\]  

(58)

Technically, generation of a phase-averaged coherent state corresponds to generation of a coherent state with the given amplitude and addition of a random uniformly distributed phase shift. I.e. to emulate the state \( \rho(n, m, \theta) \), one chooses the pair of indices \((i, j)\) with the probabilities proportional to \( c_{ni} c_{mj} \), then choose two random phases \( \varphi_1, \varphi_2 \in [0, 2\pi) \), and finally generates the 2-mode coherent state according to the integrand of Eq. (57).

For \( N = 1, n = 1, \) and \( m = 0 \), Eq. (53) yields

\[
|\Psi_{-}\rangle_{ab} |\Psi_{-}\rangle = \rho(1, 0, \pi),
\]  

(59)

which completely agrees with the previously obtained results.

For \( N = 2, n = 1, \) and \( m = 1 \), one can use Eq. (54) to obtain the representation

\[
|\Psi_2\rangle_{ab} |\Psi_2\rangle = \rho(1, 1, 0),
\]  

(60)

known from Hong-Ou-Mandel effect.

For \( N = 3, n = 2, m = 1 \) and \( N = 4, n = 2, m = 2 \) the results are

\[
|\Psi_3\rangle_{ab} |\Psi_3\rangle = \frac{4}{5} [\rho(2, 1, 0) + \rho(2, 1, 2\pi/3) + \rho(2, 1, 4\pi/3)] - \frac{1}{6} (|1\rangle_a (1 | \otimes |2\rangle_b (2 | \otimes |1\rangle_b (1))
\]  

(61)
| $N = 1$ | $N = 2$ | $N = 3$ | $N = 4$ |
|--------|--------|--------|--------|
| Fidelity | $0.9996$ | $0.9992$ | $0.99$  | $0.982$ |
| $\zeta_+ + \zeta_-$ | $2.6 \times 10^4$ | $2.8 \times 10^4$ | $1.8 \times 10^3$ |

TABLE I. Results for classical emulation of NOON-states

and
\[
|\Psi_4\rangle_a\langle\Psi_4| = \frac{2}{3} \left[ \rho(2, 2, \pi/4) + \rho(2, 2, 3\pi/4) \right]
\]
\[ - \frac{1}{3} |2\rangle_a^\ast (2) \otimes |2\rangle_b(2) \] (62)

respectively.

For emulation of NOON-states with $N = 1$ and 2 the already constructed decomposition of the single-photon state in term of phase-averaged coherent states is sufficient. To emulate the NOON-states with $N = 3$ and 4, one also need to represent the 2-photon state. The state $|2\rangle$ can be approximated by the phase-averaged coherent states with amplitudes $\alpha_2 = 0, 1/3, 2/3, \ldots, 2$ according to Eq. (25). The resulting decomposition (Fig. 9) has the fidelity $F = 0.99$. The final fidelity of NOON-states decomposition are listed in Table I.

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