Stability and nonlinear secondary modes of double-periodic flows with pumping

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Abstract. We investigate, both qualitatively and numerically, the family of forced doubly periodic plane flows of viscous fluids. These flows generalize the seminal Kolmogorov flow to the case of two-dimensional periodicity of the driving force and to the presence of pumping in two spatial directions. The dimensionless parameters of the problem are the flow rates in two perpendicular directions, the forcing intensity, and two spatial periods of the force. The Kolmogorov flow itself corresponds to the particular case when the force depends on a single coordinate and the mean drift is absent. When the forcing amplitude is increased, the basic stationary flow pattern, described by the explicit solution of the Navier-Stokes equations, displays structural rearrangements: isolated vortices appear on the background of the global flow. In the present study, we consider destabilization of the basic flow pattern: there, onset of time dependence can influence the previously reported unusual spectral and transport properties of Lagrangian dynamics. Analysis of possible stationary states, of their stability and forms of the secondary oscillatory and stationary modes is performed. Equations of fluid dynamics are solved numerically through the spectral and finite-difference methods. Stability of explicit stationary solutions with respect to small hydrodynamical perturbations is investigated.

1. Introduction

Spatially periodic flows became a subject of research in turbulence studies as models for mechanisms of cascade energy transfer. The flow, imposed by the force that sinusoidally depended on a spatial coordinate, was proposed for mimicking the transition to turbulence by A.N. Kolmogorov in 1959 [1]; below we refer to it as K59. Analysis of stability and nonlinear regimes of that flow was expected to provide insights into the nonlinear evolution and the onset of spectral characteristics of turbulence, previously deduced by Kolmogorov from dimensional considerations [2].

These ideas were realized in a number of subsequent theoretical works. In particular, the stability of the Kolmogorov flow was analyzed in [3–5], where the longwave nature of the instability was established, the boundaries of stability in the space of parameters were identified, and the secondary spatially periodic motions near the threshold were investigated. Experimental realizations of these
flows [6] were based on the "MHD drive" - the electromagnetic forces invoked in a weakly conducting liquid by the regular spatial arrangement of electrodes and/or permanent magnets, with the ability to control the flow parameters by changing the electric current and the magnetic field strength. Experiments confirmed the instability of the basic flow and the transition to secondary regimes with periodicity in two coordinates. Experiments using electromagnetic excitation of fluid motion were extended to flows initially periodic in two directions [6,7]. For these flows, raising the intensity of the driving force distorts the initial flow pattern and causes transitions to ever more complex flow regimes, up to the onset of turbulence. For practical purposes such flows were used e.g., as elements of the MHD pumps [8].

Several variants of the generalizations of the Kolmogorov flow to the case of two-dimensional periodicity, imposed by doubly periodic forces, were suggested. For certain configurations of the force, explicit flow patterns with steady vortices were obtained, including the case of pumping in two directions [9], and the “cat’s eyes” pattern [10].

2. Formulation of the problem. Equations and boundary conditions

We consider a two-dimensional flow of a viscous incompressible fluid in the rectangular cell of a plane layer with periodicity conditions along coordinates x and y (or, equivalently, on a 2D torus), caused by the external force, constant in time and doubly-periodic in space, \( F(x, y) = (\lambda_1 \sin y, \lambda_2 \sin x) \).

The plane velocity field \( \vec{v} = [v_x, v_y] \) obeys the Navier-Stokes equations with the periodicity conditions and establishes flow rates in the perpendicular directions (mean drift). Within the framework of the two-field method, in terms of the stream function \( \Psi(x, y) \) and vorticity \( \Omega(x, y) \), the governing equations and conditions in dimensionless form are:

\[
\frac{\partial \Omega}{\partial t} + \frac{\partial \Psi}{\partial y} \frac{\partial \Omega}{\partial x} - \frac{\partial \Psi}{\partial x} \frac{\partial \Omega}{\partial y} = \Phi(x, y); \Omega = \frac{\partial^2 \Psi}{\partial y^2} - \frac{\partial^2 \Psi}{\partial x^2}; V_x = \frac{\partial \Psi}{\partial y}, V_y = -\frac{\partial \Psi}{\partial x}
\]

\[
\Phi(x, y) = -\left( \frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} \right) = -\lambda_1 \cos y + \lambda_2 \cos x;
\]

\[
\Psi(x, y) = Re_x y - Re_x x + \Psi(x, y); \Psi(x + l_x, y) = \Psi(x, y + l_y) = \Psi(x, y);
\]

\[
I_x = \int_0^{l_x} dx \left( \Psi(x, l_y) - \Psi(x, 0) \right) = l_x \cdot Re_x; I_y = -\int_0^{l_y} dy \left( \Psi(l_x, y) - \Psi(0, y) \right) = l_y \cdot Re_y
\]

\[
0 \leq x \leq l_x, 0 \leq y \leq l_y.
\]

The problem (2.1) is characterized by the following dimensionless parameters: the flow rates (Reynolds numbers) \( Re_x = \alpha / \nu; Re_y = \beta / \nu \); external force: \( \lambda_1 = f_1 L / \nu; \lambda_2 = f_2 L / \nu \); cell sizes: \( l_x = L_x / L; l_y = L_y / L \). Here \( \nu \) is the kinematic viscosity coefficient; \( L_x, L_y \) are, respectively, the dimensional length and width of the cell; \( f_1, f_2 \) are the dimensional amplitudes of the external force, and, finally, \( \alpha, \beta \) are the dimensional flow rates in the x and y directions.

3. Known flow patterns and their stability

3.1. Explicit solutions

The explicit solution of the problem (2.1) is written as follows:

\[
\Psi_0(x, y) = Re_x y - Re_x x + \lambda_2 \frac{\sin(x + \phi)}{\sqrt{Re_x^2 + 1}} - \lambda_1 \frac{\sin(y + \phi)}{\sqrt{Re_y^2 + 1}}.
\]

\[
\Omega_0(x, y) = \lambda_2 \frac{\sin(x + \phi)}{\sqrt{Re_x^2 + 1}} - \lambda_1 \frac{\sin(y + \phi)}{\sqrt{Re_y^2 + 1}}; \phi_x = \arctan \left( 1 / Re_x \right); \phi_y = \arctan \left( 1 / Re_y \right)
\]
The special cases of (3.1) are the known explicit solution K59 for the Kolmogorov flow and the flow on the surface of a two-dimensional torus [9] (hereinafter ZPK96).

In the first case, the solution K59 corresponding to $\lambda_2 = \text{Re}_x = \text{Re}_y = 0$, is $\Psi(y) = -\lambda_1 \cos(y)$; $\Omega(y) = -\lambda_1 \cos(y)$, and in the more general form for the case of nonzero pumping $\text{Re}_y \neq 0$ (K59a) it has the form:

$$\Psi(y) = -\frac{\lambda_1}{\sqrt{1 + \text{Re}_y^2}} \sin \left[ y + \arctan \left( \frac{1}{\text{Re}_y} \right) \right]; \quad \Omega(y) = -\frac{\lambda_1}{\sqrt{1 + \text{Re}_y^2}} \sin \left[ y + \arctan \left( \frac{1}{\text{Re}_y} \right) \right]$$

(3.2)

Another known solution ZPK96 [9] has the doubly periodic structure and corresponds to $\lambda_1 = \lambda_2 = \lambda$, $l_x = l_y = 2\pi$. In the absence of an external force ($\lambda = 0$) it is a trivial flow realized on a 2D torus with a rotation number $\rho = \text{Re}_x / \text{Re}_y$, with straight streamlines and uniform velocity field. When $\lambda$ is increased, the streamlines become curved, and at $\lambda = \lambda_{cr} = \sqrt{(\text{Re}_x)^2 + \text{max}(\text{Re}_x^2, (\text{Re}_y)^2)}$, two stagnation points of the flow appear. At higher values of $\lambda$, the pattern of streamlines combines the “global” component with the two “local” vortices. In the global part, the non-trivial Lagrangian dynamics with anomalous diffusion of particle tracers takes place. Fig. 1 illustrates evolution of the streamline pattern, caused by the increase of the driving force. For convenience, we show 2x2 elementary cells. On the left panel (weak forcing) the velocity field has no singularities, and the transport of tracers is unbounded. On the central panel (moderate forcing) two small-scale vortices are present within each cell, and the velocity field is decomposed into the local component (closed blue streamlines inside the vortices) and the unbounded global component (shown in red), responsible for the transport. The right panel shows that further increase of the forcing strength results in growth of the vortices and squeezing of the global component.

![Figure 1](image_url)

**Figure 1.** Transformation of the flow pattern for ZPK 96 flow in the course of the increase of the forcing amplitude: a) $\lambda/\lambda_{cr} = 0.85$; b) $\lambda/\lambda_{cr} = 1.19$; c) $\lambda/\lambda_{cr} = 8.5$ ($\text{Re}_x = 1, \text{Re}_y = (\sqrt{5} - 1)/2$).

3.2. Numerical methods for stability analysis

To analyze the linear stability of (3.1) we impose small perturbations $\Psi'(x,y) \cdot \exp(\sigma t), \Omega'(x,y) \cdot \exp(\sigma t)$, in general, with doubly periodic $\Psi'(x,y), \Omega'(x,y)$. Representing the fields in (2.1) and (3.1) as Fourier expansions,

$$\Psi'(x,y) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} \Psi_{m,n} e^{imx + iny}, \quad I_x' = \frac{l_x}{2\pi}, \quad I_y' = \frac{l_y}{2\pi}$$

substituting this form into (2.1), excluding $\Omega'(x,y)$, and using the standard Galerkin-Kantorovich procedure yields the set of equations for Fourier amplitudes:
Without forcing (at $\lambda_1 = \lambda_2 = 0$) Eq. (3.4) becomes $\sigma = -\frac{m^2}{l_x^2} - \frac{n^2}{l_y^2} - \frac{im}{l_x} \Re_e - \frac{in}{l_y} \Re_y$, which implies stability of the trivial solution. At finite $\lambda_1$, $\lambda_2$, the increments $\sigma$ are found by representing (3.3) in the form $(A - \sigma E) X = 0$, where $A$ is the matrix of the system, and $X$ is the column vector of the coefficients. The main calculations were performed with up to 65x65 terms in the Fourier series. Another numerical approach, the finite difference method, has been used for checking the results and nonlinear calculations, with the number of nodes up to 200x100. In numerics, the multidimensional Newton method has recovered both stable and unstable patterns [11,12].

3.3. Study of stability of K59, K59a

As shown in [3-5], the solution of K59 is stable for $l < 2\pi$. The longwave instability ($l \rightarrow \infty$) was obtained for $\lambda_{1r} = \sqrt{2}$, for cells of finite length $l > 2\pi$, the oscillatory instability was found in [13], and is confirmed in the present study for the more general case of K59a at a nonzero flow rate $\Re_y$.

Numerical results indicate that for a finite periodicity cell, the stability of the solution increases. Instability takes place for the wavelength exceeding the critical value: $\lambda_{1r} > \sqrt{2}$, and as a result of the pitchfork bifurcation, several single-vortex (SV) and two-vortex (TV) secondary patterns branch from the K59 solution (Fig. 2).

![Figure 2](image)

**Figure 2.** Dependence of increments $\sigma_i$ on $\lambda_1$ (a), on $\Re_y$ for $\lambda_1=10$ (b) and pattern of one of instability modes for K59 ($l'_x = 2$).

3.4. Study of stability of ZPK96

Surprisingly, we failed to destabilize the pattern (3.1) in the most challenging geometry where this flow features fractal power spectra of Lagrangian variables and anomalous transport [9], namely, in a square cell, corresponding to the spatial periods of the driving force: there, the pattern with $\lambda_1=\lambda_2=\lambda$ stays stable at all tested values of the force amplitude $\lambda$ and in the wide range of the flow rates $\Re_x, \Re_y$ (in intervals $[0, 100 \lambda_{cr}]$ for $\lambda$, and $[0, 10]$ for $\Re_x, \Re_y$). Real parts of the eigenvalues (see black curves on the left panel of Fig. 3) display no visible tendency to growth under the increase of the driving force.
Figure 3. Dependence of largest real parts of decrements on the force amplitude at $\text{Re}_x = \text{Re}_y = 0$ at $l_x' = 1$ and various $l_y'$ (a); forms of instability modes for $l_x' = 2, l_y' = 1$ (b) and $l_x' = 3, l_y' = 1$ (c).

To enable instability, the region should be elongated, with the length of one side being at least twice the size of the other (see color curves in Fig. 3,a and isolines of unstable modes in Fig. 3,b,c). The possibility of stationary and self-oscillating flow regimes after destabilization of the main solution (3.1) is shown in Fig. 4,5. Fig. 4 illustrates the dependence of maximum value of the real part and the corresponding imaginary part (frequency of neutral oscillations) of increments $\sigma = \sigma_r + i\sigma_i$ on the force amplitude $\lambda$, and Fig. 5 is a bifurcation diagram on planes $(\lambda, \text{Re}_x),(\lambda, \text{Re}_y)$.

Figure 4. Dependences of real (a) and imaginary (b) parts of decrements on the force amplitude at $l_x' = 2, l_y' = 1$.

Figure 5. Bifurcation diagram in $(\text{Re}_x, \lambda)$ (a) and $(\text{Re}_y, \lambda)$ (b) planes, $(l_x' = 2, l_y' = 1)$. 

Conclusions
The calculations disclose that in square-shaped cells with periodic boundary conditions the family of spatially periodic stationary flow patterns with steady vortices features stability in a sufficiently wide (potentially unbounded?) range of parameter values. The instability is encountered only in elongated rectangles when several periods of force fit into a periodic cell of the velocity field. In the instability parameter region, the growth of perturbations of the stationary flow is, as a rule, oscillatory. This can significantly affect the Lagrangian dynamics of the flow, since after the onset of time-dependence the effective phase space becomes three-dimensional, opening possibilities for chaotic streamlines.

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