On correlation functions in the perturbed minimal models $\mathcal{M}_{2,2n+1}$

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Abstract

Two-point correlation functions of spin operators in the minimal models $\mathcal{M}_{p,p'}$ perturbed by the field $\Phi_{13}$ are studied in the framework of conformal perturbation theory [1]. The first-order corrections for the structure functions are derived analytically in terms of gamma functions. Together with the exact vacuum expectation values of local operators, this gives the short-distance expansion of the correlation functions.

The long-distance behaviors of these correlation functions in the case $\mathcal{M}_{2,2n+1}$ have been worked out using a form-factor bootstrap approach.

The results of numerical calculations demonstrate that the short- and long-distance expansions match at the intermediate distances. Including the descendent operators in the OPE drastically improves the convergency region. The combination of the two methods thus describes the correlation functions at all length scales with good precision.
1 Introduction

A complete set of correlation functions is the main object completely characterizing field theories. The well-known nontrivial examples where exact correlation functions have been found are the two-dimensional scaling Ising model in the zero magnetic field and the conformal field theories \[2, 3\]. Unfortunately, away from the free fermion/conformal points, the specific methods that have been applied for those models are difficult to generalize. In particular, the problem of finding exact analytic expressions for correlation functions of integrable two dimensional massive models is still open. In approaching this problem, we follow \[1\] and study general behaviors of correlation functions in the perturbed conformal field theories at all scales by applying a combination of conformal perturbation theory and the form-factor bootstrap approach.

In what follows, we concentrate on the minimal models of CFT \(\mathcal{M}_{p,p'}\) (with coprime integers \(1 < p < p'\)) perturbed by the field \(\Phi_{13}\). These massive field theories having an infinite number of conservation laws are integrable \[4\]. From the statistical mechanics standpoint, the given massive models describe a universality class of the integrable RSOS-type models \[5\] in the corresponding antiferromagnetic regime.

The conformal field theories \(\mathcal{M}_{p,p'}\) underlying the massive models under consideration are characterized by the central charge of the Virasoro algebra

\[
c = 1 - 6\frac{(p' - p)^2}{pp'}. \tag{1}
\]

There are \((p - 1) \times (p' - 1)/2\) primary fields \(\Phi_{l,k}\) \((l = 1, \ldots, p - 1\) and \(k = 1, \ldots, p' - 1\)) in the model. The conformal dimensions \((\Delta_{l,k}, \Delta_{l,k})\) of these fields are determined by the Kac formula:

\[
\Delta_{l,k} = \frac{(p'l - pk)^2 - (p' - p)^2}{4pp'}. \tag{2}
\]

In what follows, we impose the standard normalization for the primary fields adopted in CFT,

\[
\langle \Phi_{l,k}(x)\Phi_{l,k}(0) \rangle_{\text{CFT}} = |x|^{-4\Delta_{l,k}}. \tag{3}
\]

We use the symbol \(I = \Phi_{11}\) for the unity operator throughout this work. The operator \(\Psi = \Phi_{12}\) is identified with the spin operator, and \(\Phi = \Phi_{13}\), with the energy operator. It is convenient to use the positive rational parameter

\[
\xi = \frac{p}{p' - p}.
\]
instead of \( p \) and \( p' \) because, for example, the operators \( I, \Psi, \Phi, \) and \( \Phi_{15} \), which are important for us, then have the conformal dimensions

\[
\Delta_I = 0, \quad \Delta_{\Psi} = \frac{\xi - 2}{4(\xi + 1)}, \quad \Delta_{\Phi} = \frac{\xi - 1}{\xi + 1}, \quad \Delta_{\Phi_{15}} = \frac{4\xi - 2}{\xi + 1}.
\]

The general procedure developed in [2, 9] allows computing structure constants in the operator algebra and determining all correlation functions. This knowledge can be used to study correlation functions of the \( \Phi_{13} \) perturbation of the model \( M_{p,p'} \) as follows.

The scaling model we study can be formally defined through the action

\[
A_{S_{p,p'}} = A_{M_{p,p'}} + g \int \Phi \, d^2x, \tag{4}
\]

where \( A_{M_{p,p'}} \) denotes the “action” of the critical model and \( g \) is a coupling constant having the dimension \( g \sim m_1^{2-2\Delta_{\Phi}} \), where \( m_1 \) is a mass gap (the mass of lightest particle). We note that the operator \( \Phi \) has the conformal dimension \( \Delta_{\Phi} < 1 \) and the perturbation is relevant. More explicitly, the dimensional parameters \( g \) and \( m_1 \) are connected via the exact “mass–coupling-constant relation” found in [8]:

\[
\pi g = \frac{(\xi + 1)^2}{(\xi - 1)(2\xi - 1)} \left( \frac{3\xi}{\xi + 1} \right) \left( \frac{\xi}{\xi + 1} \right)^{\frac{1}{2}} m_1^{\frac{2-2\Delta_{\Phi}}{2}}. \tag{5}
\]

Here and hereafter, the massive parameter \( m \) is related to the mass of the lightest particle \( m_1 \) by

\[
m = m_1 \left( \frac{\pi \gamma(1 - \frac{\xi}{2}) \gamma(\frac{1+\xi}{2})}{8 \sin \pi \xi} \right)^{\frac{1}{2}}
\]

and we set

\[
\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}.
\]

Formal definition (4) is understood in the sense of the perturbation series. For example, for the correlation function of the spin fields, we have

\[
\langle \Psi(x)\Psi(0) \rangle = \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \int \langle \Psi(x)\Psi(0)\Phi(y_1)\cdots\Phi(y_n) \rangle_{\text{CFT}} d^2y_1 \cdots d^2y_n.
\]

Here the symbols \( \Psi \) in the l.h.s. denote the scaling fields in the perturbed model; in the UV limit, these fields become the conformal spin fields \( \Phi_{12} \). Throughout this work, we
generally assume that the renormalized fields in $\Phi$ have definite scaling dimensions and are denoted by the same letter as in the conformal $g \to 0$ limit. We also require that the normalization of scaling fields at the limit be fixed as in (3).

The UV and IR regularization schemes for the integrals over the plane were discussed extensively in [1]. We only briefly recall the final prescription here. Since the perturbing field is relevant, the UV renormalization can be achieved by adding a finite number of counterterms. To handle the IR divergences and to work out the infrared-safe perturbation theory, the so-called conformal perturbation theory is developed [1].

The conformal perturbation theory is based on the following hypothesis. Let \{\(A_a, a = 0, 1, \ldots\)\} be a complete set of local fields in the scaling theory. It is assumed that if the renormalized fields that are eigenvectors of the dilatation operator, i.e., the fields with a definite scaling dimensions, are chosen as basis elements, then the structure functions in the corresponding operator-product expansion

\[
A_m(x) A_a(0) = \sum_b C_{A_m A_a}^{A_b}(r) A_b(0),
\]  

are analytic functions of the coupling constant \(g\). This is a rather natural conjecture since the functions \(C_{A_m A_a}^{A_b}(r)\) as local quantities are assumed to not develop any nonanalyticity. We propose that under this choice, the renormalized fields \(A_b(r)\) turn out to be perturbations of the corresponding basis fields (primaries and descendants) of the conformal model. Then the functions \(C_{A_m A_a}^{A_b}(r)\) are given by regular expansions

\[
C_{A_m A_a}^{A_b}(r) = r^{2(\Delta_b - \Delta_a - \Delta_m)} \left( C_{A_m A_a}^{A_b} + gr^{2-2\Delta} Q^{(1)} + (gr^{2-2\Delta})^2 Q^{(2)} + \cdots \right). 
\]  

The zeroth-order terms here are determined by the structure constants \(C_{A_m A_a}^{A_b}\) from the critical model, and the problem of perturbation theory is to define \(g\)-independent corrections \(Q^{(i)}\).

To define the correlation functions following [10], we must also find vacuum expectation values (VEVs) of the local fields \(A_a\), which might be nonzero in the off-critical case. These important quantities are nonlocal and, generally, nonanalytic in the coupling constant \(g\). From counting dimensions, we have \(\langle A_b \rangle \sim m^{2\Delta_b}\). The set of VEVs contain all nonperturbative information on the theory. The VEVs for perturbed primaries were proposed in [18] (also see [19]). The Lukyanov and Zamolodchikov formula for the VEVs in the first ground state [5] can be written as

\[
\langle \Phi_{1k} \rangle = (-1)^{k-1} m^{2\Delta_{1k}} Q(1 - \xi(k - 1)).
\]  

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1. See [13] for the conformal perturbation theory in all orders.

2. We note that because of fusion rules, many of the \(C_{A_m A_a}^{A_b}\) in the conformal case are zeroes. But some of the corresponding \(C_{A_m A_a}^{A_b}(r)\) may become nonzero because of the correction terms.
The function $Q(\eta)$ is defined in terms of the following exponent of an integral that is understood in the sense of analytic continuation from the region where it converges:

$$Q(\eta) = \exp \int_0^\infty \frac{dt}{t} \left( \cosh(2t) \sinh((\eta - 1)t) \sinh((\eta + 1)t) - \frac{\eta^2 - 1}{2\xi(\xi + 1)} e^{-2(\xi + 1)t} \right).$$  \tag{9}

We note that if $\xi > 1$, for instance, as in the models $\mathcal{M}_{p,p+1}$ of principal series perturbed by the energy operator, then there are no bound states. In that case, we use the kink mass in the expressions like (9).

The short distance expansion for the two-point correlation function of spin fields is determined by the VEVs of fields $\Phi_{1k}$ (and their descendants) with odd $k = 2s + 1$ (see Eq. (13) below). After some computations we found that for these fields the expression (9) simplifies and can be written explicitly in terms of gamma functions as following

$$Q(1 - 2s\xi) = \prod_{i=1}^{s} Q_i(\xi),$$  \tag{10}

where

$$Q_i(\xi) = \frac{(1 - (2l - 1)\xi)(1 - 2l\xi)}{2(\xi + 1)} \times \gamma \left( \frac{(2l - 1)\xi}{2} \right) \gamma \left( \frac{1 - (2l - 1)\xi}{2} \right) \left[ \gamma \left( \frac{1 - 2(l - 1)\xi}{\xi + 1} \right) \gamma \left( 1 - \frac{(2l + 1)\xi}{\xi + 1} \right) \right]^2.\tag{11}$$

The one-point VEVs for the first nontrivial descendent operators are also known in the analytic form [20],

$$\langle L_{-2} \bar{L}_{-2} \Phi_{1k} \rangle = -(1 + \xi)^4 W(1 - \xi(k - 1)) m^4 \langle \Phi_{1k} \rangle,$$  \tag{12}

where the function $W(\eta)$ is

$$W(\eta) := \frac{1}{\xi^2(\xi + 1)^2} \gamma \left( \frac{1 + \eta + \xi}{2} \right) \gamma \left( \frac{\eta - \xi}{2} \right) \gamma \left( \frac{1 - \eta + \xi}{2} \right) \gamma \left( -\frac{\eta + \xi}{2} \right).$$

Although the problem of determining exact VEVs of all descendants is still open, the knowledge achieved up to now can already be used to analyze correlation functions. For example, VEVs (11)–(12) determine the leading terms in the short-distance expansion of the two-point correlation function of spin operators,

$$\langle \Psi(x) \Psi(0) \rangle = C_{\eta \psi}^I(r) \langle I \rangle + C_{\eta \psi}^\Phi(r) \langle \Phi(0) \rangle + C_{\eta \psi}^{\Phi_{15}}(r) \langle \Phi_{15} \rangle$$

$$+ C_{\eta \psi}^{L_{-2} \bar{L}_{-2} I}(r) \langle L_{-2} \bar{L}_{-2} I \rangle + C_{\eta \psi}^{L_{-2} \bar{L}_{-2} \Phi}(r) \langle L_{-2} \bar{L}_{-2} \Phi(0) \rangle + \cdots,$$  \tag{13}
which is the main object of study in the first part of this work. The conformal perturbation theory based on an exact knowledge of VEVs of local operators is thus applicable for an effective study of the short-distance \((mr \ll 1)\) behaviors of correlation functions.

On the other hand, theories \((\ref{14})\) are massive integrable models with factorized scattering. The scattering matrices in those theories coincide with those in the restricted sine-Gordon model \([10, 11, 12]\). We can apply the spectral expansion \([6, 7]\) (see \([17]\) and the references therein for details)

\[
\langle \Psi(x)|\Psi(0)\rangle = \sum_{n=0}^{\infty} \sum_{\{a_n\}} \frac{1}{n!} \int \frac{d\beta_1}{2\pi} \cdots \frac{d\beta_n}{2\pi} \langle 0|\Psi(0)|\beta_1, \ldots, \beta_n\rangle_{a_1 \cdots a_n} \times \\
\times a_1 \cdots a_n \langle \beta_1, \ldots, \beta_n|\Psi(0)|0\rangle e^{-r \sum m_j \cosh \beta_j}
\]

for the correlation functions and study the same correlation functions at long distances \(mr \gg 1\). Here, \(a_i\) are the particle types, and the quantities

\[
\langle 0|\Psi|\beta_1, \ldots, \beta_n\rangle_{a_1 \cdots a_n}, \quad a_1 \cdots a_n \langle \beta_1, \ldots, \beta_n|\Psi|0\rangle
\]

are matrix elements of the local operators in the basis of asymptotic states, the form factors. The latter are assumed to be meromorphic functions of the rapidity variables \(\beta\) satisfying a set of locality axioms \([7]\). For the case of primary fields, the exact solutions for the form-factor equations have already been found \([1, 7, 10, 27, 28, 29]\).

The difficulty not easily handled in the form-factor approach is that the axioms for scalar primary fields do not fix an overall normalization of the form factors or, in other words, the normalization of local operators. To impose conformal normalization \((\ref{3})\), we determine the overall rapidity-independent factor as in \([1, 29]\) by the VEVs of local operators \((\ref{8})\) (also see \([30]\)). This prescription seems quite natural from the standpoint of the quantum group procedure \([8, 10, 19]\).

In the integrable cases, there are thus two different expansions \((\ref{13})\) and \((\ref{14})\) of the same correlation functions of the local operators. A natural idea would be to compare these formulae for short and long distances. As soon as the expansions are matched somewhere at the intermediate distances, we would obtain a description of correlation functions at all scales. The practical question, of course, is how easily that can be achieved.

We mention that this program of studying correlation functions, initiated in \([1]\) (also see \([20]\)) for the case of the Lee–Yang model, has already been applied to several more complicated cases \([32, 34, 33]\).

We have found that for the models \(M_{p,p'}\) perturbed by the field \(\Phi_{13}\) in the resonance-free cases, an elegant analytic form for the first-order perturbation theory for the structure
functions can be easily provided. With answers (8)–(12) taken into account, this leads to expression (35) controlling the UV behaviors of the spin correlation functions.

For the case \( M_{2,2n+1} \), this together with the form-factor expansion up to three particles already gives a broad region where the two expansions match.\(^3\) For example, one of the plots illustrating this phenomenon for the model \( M_{2,7} \) perturbed by the energy operator is shown in Fig. 1. The large region of convergence between 1 and 3 makes it clear that the given method provides a good approximation for correlation functions at all scales.

This matching between different expansions at the very first steps demonstrates that the combination of the conformal-perturbation and form-factor approaches is a very useful tool for checking the consistency of different proposals used in both constructions. For instance, in our case, we obtain an additional confirmation for statements that the form factors for the primary fields are given by the minimal solutions with the prescribed analytic properties \([27, 29]\), that the normalization of the form factors is fixed correctly, and so on. Moreover, that the contribution from the descendent operators drastically

\(^3\)We choose the model \( M_{2,2n+1} \) just for simplicity. The form factors of primaries in this case do not contain contour integrals. Another perturbed CFT with that property is a parafermionic model perturbed by the first energy field \([31]\).
improves the convergence of the two expansions (see Fig. 1) can be treated as a good check for the correctness of (12). In addition, this method can also be treated as an alternative way for studying universal scaling behaviors of the correlation functions in the corresponding lattice RSOS models [3, 35].

Although many technical components that we used are already known, we collect all necessary facts for the reader’s convenience and to make the text self-consistent. This work is organized as follows. In Sec. 2, we consider the procedure for finding first-order corrections to the structure functions. We demonstrate that the double integrals over a complex plane in the physically interesting situations can be taken analytically leading to an elegant answer in terms of gamma functions. We thus obtain the UV expansion for the correlation function of spin operators. We provide some details of the numerical computation for $\mathcal{M}_{2,7}$ as an example. In Sec. 3, we recall the scattering data and form-factor axioms applicable to our case with many particles. We restrict our attention to the three-particle case, which already gives good IR expansion formulae. In Sec. 4, we give some results of the numerical computations and demonstrate the consistency of both expansions. In Sec. 5, we draw final conclusions. The technical details are collected in the appendix.

2 Short-distance expansion scaling $\mathcal{M}_{p,p'}$ model

Our main aim is to study two-point correlation functions of the spin operators in the massive $\Phi_{13}$ perturbation of $\mathcal{M}_{p,p'}$ model [4]. In this section, we analyze the short-distance expansion based on the operator algebra

$$
\langle \Psi(x)\Phi(0) \rangle = C^I_{\Psi\Psi}(r)\langle I \rangle + C^{\Phi}_{\Psi\Psi}(r)\langle \Phi(0) \rangle + C^{\Phi_{15}}_{\Psi\Psi}(r)\langle \Phi_{15} \rangle + C^{L_{-2}L_{-2}}_{\Psi\Psi}(r)\langle L_{-2}L_{-2} \rangle + \cdots .
$$

(15)

In the r.h.s., we keep only the leading terms in the small $r$ that have nonzero VEVs. We thus omit the derivative operators as well as operators with nonzero spins. Taking the known VEVs of local operators (8)–(12), we study correlation functions (15) following the program initiated in [1].

2.1 Perturbation theory for structure functions

According to [1], the structure functions in (15) have the expansion with the zeroth order given by the structure constants $C^K_{\Psi\Psi}$ from the underlying $\mathcal{M}_{p,p'}$ (7),

$$
C^K_{\Psi\Psi}(r) = C^K_{\Psi\Psi} r^{2\Delta_K - 4\Delta_{\Psi}} + C^{(1)}_{\Psi\Psi}(r) + \cdots .
$$

(16)
The first-order corrections \( C_{\Psi \Psi}^{(1)}(r) \) can be represented in the regularized form with \( R \) being the parameter of the infrared cutoff,

\[
C_{\Psi \Psi}^{(1)}(r) := \lim_{R \to \infty} \left[ -g \int_{|y|<R} \langle \tilde{A}_K(\infty) \Phi(y) \Psi(x) \Psi(0) \rangle_{\text{CFT}} d^2y + \pi g \sum_A \frac{C_{\Psi \Psi}^A C_{\Phi}^{K}}{\Delta_K - \Delta_A - \Delta_\Phi + 1} R^{2(\Delta_K - \Delta_A - \Delta_\Phi + 1)} R^{2(\Delta_K - \Delta_A - \Delta_\Phi - 4\Delta_\Psi)} \right].
\] (17)

As in the Lee–Yang case [1], it is convenient for practical purposes to apply an analytic regularization procedure for the integral. In our case of two operators \( \Psi \equiv \Phi_{12} \), we analyze the regularized integrals

\[
C_{\Psi \Psi}^{I(1)}(r) = -g \int' \langle \Phi(y) \Psi(x) \Psi(0) \rangle_{\text{CFT}} d^2y,
\]

\[
C_{\Psi \Psi}^{\Phi(1)}(r) = -g \int' \langle \Phi(\infty) \Phi(y) \Psi(x) \Psi(0) \rangle_{\text{CFT}} d^2y,
\]

\[
C_{\Psi \Psi}^{\Phi_{15}(1)}(r) = -g \int' \langle \Phi_{15}(\infty) \Phi(1) \Psi(z) \Psi(0) \rangle_{\text{CFT}} d^2y.
\] (18)

We recall the exact form of the conformal correlators appearing here. The correlation functions in the integral in the first line above are defined via the structure constants of the operator algebra

\[
\langle \Phi(y) \Psi(x) \Psi(0) \rangle_{\text{CFT}} = C_{\Psi \Psi}^\Phi |y|^{-2\Delta_\Psi} |x|^{-2(2\Delta_\Psi - \Delta_\Phi)} |y-x|^{-2\Delta_\Phi}.
\] (19)

The correlation function including the field \( \Phi_{15} \) has a very similar form. Indeed, the correction term in the third line can be rewritten as

\[
C_{\Psi \Psi}^{\Phi_{15}(1)}(r) = -gr^{2(1-\Delta_\Phi - 2\Delta_\Psi + \Delta_{15})} \times \int' |z|^{2(\Delta_\Phi + 2\Delta_\Psi - 2 - \Delta_{15})} \langle \Phi_{15}(\infty) \Phi(1) \Psi(z) \Psi(0) \rangle_{\text{CFT}} d^2z,
\]

where

\[
\langle \Phi_{15}(\infty) \Phi(1) \Psi(z) \Psi(0) \rangle_{\text{CFT}} = C_{\Psi \Psi}^\Phi C_{\Phi \Phi}^{\Phi_{15}} |z|^{\frac{r}{\xi + 1}} |1-z|^{\frac{2r}{\xi + 1}}.
\] (20)

In general, the four-point correlation functions of the primary fields in the conformal models can be found, for example, using the free-field construction of Dotsenko and Fateev [9] or the differential equations method. For example, the integral in the second
line of Eq. (18) is expressed through the correlation function
\[
\langle \Psi(0)\Psi(z)\Phi(1)\Phi(\infty) \rangle_{\text{CFT}}
= C_{\Psi\Psi}^{\Phi}C_{\Phi\Phi}^{\Phi} \left| z^{\frac{4}{\xi+1}} (1-z)^{\frac{4}{2(\xi+1)}} \right|_{\text{CFT}} 2F_1 \left( \frac{\xi}{\xi+1}, \frac{3\xi-1}{\xi+1}, \frac{2\xi}{\xi+1} | z \right)^2
+ \left| z^{\frac{4}{\xi+1}} (1-z)^{\frac{4}{2(\xi+1)}} \right|_{\text{CFT}} 2F_1 \left( \frac{2}{\xi+1}, \frac{2\xi}{\xi+1}, \frac{2}{\xi+1} | z \right)^2.
\]

(21)

With Eq. (75), it has the form
\[
C_{\Psi\Phi}^{\Phi(1)}(r) = -g r^{2-4\Delta_\Psi} \int |z|^{2\Delta_\Psi-2} \langle \Psi(0)\Psi(z)\Phi(1)\Phi(\infty) \rangle_{\text{CFT}} d^2 z
= -g r^{\xi+4} \frac{\gamma\left(\frac{2}{\xi+1}\right)}{\gamma^2\left(\frac{1}{\xi+1}\right)} \lim_{\epsilon \to 0} \left( \int d^2 x \int d^2 y |x|^{\frac{2\epsilon}{\xi+1}} |1-x|^{\frac{4\epsilon}{\xi+1}} \right.
\left. \times |y|^{-\frac{2\epsilon+3}{\xi+1}+4\epsilon} |1-y|^{\frac{2\epsilon}{\xi+1}} |x-y|^{\frac{2\epsilon}{\xi+1}} \right).
\]

(22)

Here, we fixed the way of an analytic regularization in the second line that allows computing the integral explicitly before taking the limit. To do this, we use formula (26), which is derived in the next section.

### 2.2 The integrals

From explicit formulae (19)–(21), we find that the properly regularized corrections (18) are reduced to integrals of the forms

\[
J_1(p, q) = \int d^2 y |y|^{2p} |y-1|^{2q},
\]

(23)

\[
J_2(a, b, d, e, c) = \int d^2 x \int d^2 y |x|^{2a} |1-x|^{2b} |y|^{2d} |1-y|^{2e} |x-y|^{2c}.
\]

(24)

The first integral can be easily computed by reducing it to a product of beta-integrals, giving
\[
J_1(p, q) = \frac{\pi \gamma(p+1) \gamma(q+1)}{\gamma(p+q+2)}.
\]

(25)

Integral (24) is more complicated. It can be rewritten in terms of a product of two contour integrals admitting a representation via the higher hypergeometric functions \( _3F_2 \) at unity (also see 34). For completeness, we collect the details of computations in Appendix B.
In our cases \((17)\), the integral \(J_2(a, b, d, e, c)\) appears at special values of the parameters and can be expressed in terms of gamma functions due to the following formula

\[
J_2\left(\frac{\beta - \alpha - 1}{2}, \delta - \beta - 1, \alpha - 1, -\frac{\delta}{2}, \frac{\beta - \alpha - 1}{2}\right)
= 2^{2\alpha - 2} \pi^2 \gamma\left(\frac{\alpha}{2}\right) \gamma\left(\frac{\beta}{2}\right) \gamma\left(\frac{1 - \delta}{2}\right) \gamma\left(\frac{\delta - \beta}{2}\right) \gamma\left(\frac{\alpha - \delta + 1}{2}\right) \\
\times \gamma\left(\frac{\beta - \alpha + 1}{2}\right) \gamma\left(\frac{\delta - \alpha - \beta + 1}{2}\right).
\]  

Using \((25)\)–\((26)\), we can analytically derive first corrections for the structure functions for arbitrary \(\xi > 0\).

### 2.3 First-order corrections to the structure functions

Applying the formulae for integrals \((25)\)–\((26)\) and the exact values of the CFT structure constants from Appendix A, we now give exact expressions for the structure functions up to the first order in \(g\).

The functions \(C_{\Psi\Psi}^{I}(r)\) and \(C_{\Psi\Psi}^{\Phi_{15}}(r)\) are the simplest. (We recall that the conformal field theory structure constants were \(C_{\Psi\Psi}^{I} = 1\) and \(C_{\Psi\Psi}^{\Phi_{15}} = 0\).) Using \((25)\), we find that they have the forms

\[
C_{\Psi\Psi}^{I}(r) = r^{\frac{2+\xi}{4+\xi}} + \pi g r^{\frac{\xi}{4+\xi}} \left(\frac{\gamma\left(\frac{\xi}{4+\xi}\right)\gamma\left(\frac{2}{4+\xi}\right)}{\gamma\left(\frac{2}{4+\xi}\right)\gamma\left(\frac{4}{4+\xi}\right)}\right)^{\frac{1}{2}},
\]

\[
C_{\Psi\Psi}^{\Phi_{15}}(r) = -\pi g r^{\frac{2+7\xi}{4+\xi}} C_{\Psi\Psi}^{\Phi_{15}} \frac{\gamma\left(\frac{1+2\xi}{4+\xi}\right)}{\gamma\left(\frac{2+4\xi}{4+\xi}\right)}.
\]  

The first-order correction to the function \(C_{\Psi\Psi}^{\Phi}(r)\) \((22)\)

\[
C_{\Psi\Psi}^{\Phi(1)}(r) = -gr^{\frac{\xi+4}{4+\xi}} \frac{\gamma\left(\frac{2}{4+\xi}\right)}{\gamma\left(\frac{1}{4+\xi}\right)} \lim_{\epsilon \to 0} J(\epsilon),
\]

\[
J(\epsilon) = J_2\left(-\frac{\xi}{\xi + 1}, -\frac{2\xi}{\xi + 1}, -\frac{\xi + 3}{\xi + 1} + 2\epsilon, \frac{\xi}{\xi + 1} - \epsilon, -\frac{\xi}{\xi + 1}\right)
\]

is computed by applying Eq. \((26)\). Taking the limit, we obtain the expression

\[
\lim_{\epsilon \to 0} J(\epsilon) = \left(\frac{\pi\xi(1-\xi)^2}{2(\xi + 1)^2}\right) 2 \gamma^3\left(\frac{1-\xi}{4+\xi}\right) \gamma^2\left(\frac{\xi}{4+\xi}\right) \gamma\left(\frac{2-2\xi}{4+\xi}\right).
\]
Finally, the analytic expression for the structure function $C_{\Psi\Psi}^\Phi(r)$ in the first-order perturbation theory is

$$C_{\Psi\Psi}^\Phi(r) = -r^{\frac{\xi}{2}+1} \left( \frac{\gamma(\frac{\xi}{2}+1)}{\gamma(\frac{2\xi}{2+1})} \right)^{1/2} + \pi g r^{\frac{\xi+1}{2}} \left( \frac{(1-\xi)^2}{2(\xi+1)^2} \right)^2 \gamma^4(\frac{1-\xi}{\xi+1}) \gamma^4(\frac{\xi}{\xi+1}) \gamma^2(\frac{2-2\xi}{\xi+1}).$$

(28)

It is convenient to rewrite structure functions (27)–(28) in terms of the lightest particle mass. Applying the mass-coupling-constant relation, we have

$$C_{\Psi\Psi}^I(r) = r^{\frac{\xi}{2}+1} \left\{ 1 - (mr)^{\frac{\xi}{2}+1} \left( \frac{(\xi+1)^2}{(\xi-1)(2\xi+1)} \right)^{\frac{1}{2}} \right\},$$

(29)

$$C_{\Psi\Psi}^\Phi(r) = -r^{\frac{\xi}{2}+1} \left\{ \left( \frac{\gamma(\frac{\xi}{\xi+1})}{\gamma(\frac{2\xi}{\xi+1})}\right)^{\frac{1}{2}} \right. + (mr)^{\frac{\xi}{2}+1} \left( \frac{\xi^2(1-\xi)^3}{4(\xi+1)^2(2\xi+1)} \right)^{\frac{1}{2}} \left( \frac{\gamma^8(\frac{1-\xi}{\xi+1}) \gamma(\frac{3\xi}{\xi+1})}{\gamma^2(\frac{2-2\xi}{\xi+1})} \right) \right\},$$

(30)

$$C_{\Psi\Psi}^{I^2}(r) = -r^{\frac{2\xi}{2}+2} (mr)^{\frac{\xi}{2}+1} \left( \frac{\xi^2(1-\xi)^3}{4(1-2\xi)(3\xi+1)^2} \right)^{\frac{1}{2}} \left( \frac{\gamma^7(\frac{\xi}{\xi+1}) \gamma^4(\frac{1-\xi}{\xi+1})}{\gamma^2(\frac{2-2\xi}{\xi+1}) \gamma^2(\frac{2-3\xi}{\xi+1})} \right) \right\}. \right\}$$

(31)

We also need the structure functions for the descendent fields in the main order. These are expressed using the corresponding CFT structure constants of the primary fields just as

$$C^{L_{-2}L_{-2}}_{\Psi\Psi} = \frac{\xi^2}{4(\xi+3)^2} (\xi+1)^3 C_{\Psi\Psi}^I,$$

(32)

$$C^{L_{-2}L_{-2}}_{\Psi\Psi}^\Phi = \frac{\xi^2}{4(3\xi+1)^2} (\xi+1)^3 C_{\Psi\Psi}^\Phi.$$  

(33)

### 2.4 UV expansion for the correlation function of spin operators

Taking these formulae for structure functions, VEVs (34) as well as the expressions

$$\langle L_{-2}L_{-2}\rangle = -m^4 \left( \frac{\gamma(\frac{\xi}{2})}{\gamma(\frac{\xi+1}{2})} \right)^2,$$

$$\langle L_{-2}L_{-2}\Phi \rangle = m^4 \frac{4(\xi+1)^2 \gamma(\frac{3\xi}{2}) \gamma(\frac{1-3\xi}{2})}{\xi^2(1-\xi)^2 \gamma(\frac{1}{2}) \gamma(\frac{1-\xi}{2})} \langle \Phi \rangle$$

(34)
for VEVs (12) into account, we obtain the explicit first-order UV expansion in $g$ for the correlation function

$$\langle \Psi(r) \Psi(0) \rangle = r^{\frac{2-\xi}{\xi+1}} \left[ A(r) - B(r)Q(1-2\xi) + D(r)Q(1-4\xi) \right].$$  \hspace{1cm} (35)$$

We recall that the function $Q(\eta)$ (9) for the fields $\Phi_{1,2s+1}$ is given by the Eqn. (10) - (11). For the fields $\Phi_{13}$ and $\Phi_{15}$ the respective expressions in (35) turn out to be

\begin{align*}
Q(1-2\xi) &= \frac{(1-\xi)(1-2\xi)}{2(\xi+1)} \gamma \left( \frac{\xi}{2} \right) \gamma \left( \frac{1-\xi}{2} \right) \left[ \gamma \left( \frac{1}{\xi+1} \right) \gamma \left( \frac{1-2\xi}{\xi+1} \right) \right]^{\frac{1}{2}}, \hspace{1cm} (36)
\end{align*}

\begin{align*}
Q(1-4\xi) &= \frac{(1-\xi)(1-2\xi)(1-3\xi)(1-4\xi)}{4(\xi+1)^2} \gamma \left( \frac{\xi}{2} \right) \gamma \left( \frac{3\xi}{2} \right) \times \gamma \left( \frac{1-\xi}{2} \right) \gamma \left( \frac{1-2\xi}{2} \right) \gamma \left( \frac{1-3\xi}{2} \right) \left[ \gamma \left( \frac{1}{\xi+1} \right) \gamma \left( \frac{1-4\xi}{\xi+1} \right) \right]^{\frac{1}{2}}. \hspace{1cm} (37)
\end{align*}

The $r$-dependent functions $A(r), B(r),$ and $D(r)$ in (35) have the forms

\begin{align*}
A(r) &= 1 - \frac{\xi^2}{4(\xi+3)^2} \frac{\gamma^2 \left( \frac{\xi}{2} \right)}{\gamma^2 \left( \frac{1+\xi}{2} \right)} (mr)^4 \\
&\quad - \frac{(\xi+1)^2}{(1-\xi)(1-2\xi)} \frac{\gamma^2 \left( \frac{\xi}{\xi+1} \right) \gamma^5 \left( \frac{2}{\xi+1} \right) \gamma \left( \frac{3\xi}{\xi+1} \right)}{\gamma \left( \frac{2\xi}{\xi+1} \right) \gamma \left( \frac{2-\xi}{\xi+1} \right) \gamma^2 \left( \frac{1}{\xi+1} \right)} \left( mr \right)^{\frac{1}{\xi+1}}, \hspace{1cm} (38)
\end{align*}

\begin{align*}
B(r) &= \left\{ \left( \frac{\gamma \left( \frac{\xi}{\xi+1} \right) \gamma \left( \frac{2}{\xi+1} \right)}{\gamma \left( \frac{2\xi}{\xi+1} \right) \gamma \left( \frac{2-\xi}{\xi+1} \right)} \right)^{\frac{1}{2}} \left[ 1 + \frac{(\xi+1)^2}{(\xi-1)^2(3\xi+1)^2} \frac{\gamma \left( \frac{3\xi}{2} \right) \gamma \left( \frac{1-3\xi}{2} \right)}{\gamma \left( \xi \right) \gamma \left( \frac{1-\xi}{2} \right)} \right] \right\} (mr)^{\frac{1}{\xi+1}} \hspace{1cm} (39)
\end{align*}

\begin{align*}
D(r) &= -\frac{\xi^2(1-\xi)^3}{4(\xi+1)^2(1-2\xi)} \left( \frac{\gamma^5 \left( \frac{1-\xi}{\xi+1} \right) \gamma \left( \frac{3\xi}{\xi+1} \right) \gamma^2 \left( \frac{2-2\xi}{\xi+1} \right)}{\gamma \left( \frac{4\xi}{\xi+1} \right) \gamma \left( \frac{2-2\xi}{\xi+1} \right) \gamma \left( \frac{2-3\xi}{\xi+1} \right)} \right)^{\frac{1}{2}} (mr)^{\frac{2(\xi-1)}{\xi+1}}. \hspace{1cm} (40)
\end{align*}

2.5 Model $M_{2,7}$ as an example

The first model in the series $M_{2,5}$ is identified with the Lee-Yang model [14] with the central charge of Virasoro algebra $c = -\frac{22}{5}$. There are only two primary fields in the
theory: the unit field \( I := \Phi_{11} = \Phi_{14} \) with the conformal dimension zero and the spin field \( \Phi_{12} = \Phi_{13} \) with the dimension \(-\frac{1}{5}\). This example was analyzed in detail in [1, 20].

The next example is the model \( \mathcal{M}_{2,7} \) [15] with the central charge
\[
c = -\frac{68}{7}. \tag{41}
\]
In this case, there are three primaries: the unit field \( I := \Phi_{11} = \Phi_{16} \), the spin field \( \Psi := \Phi_{12} = \Phi_{15} \), and the energy field \( \Phi := \Phi_{13} = \Phi_{14} \) with the respective conformal dimensions
\[
\Delta_I = 0, \quad \Delta_\Psi = -\frac{2}{7}, \quad \Delta_\Phi = -\frac{3}{7}. \tag{42}
\]

The first-order corrections in \( g \) to the structure functions are given by expressions [29–31] with \( \xi = \frac{2}{5} \).

Providing an analytic continuation of the Lukyanov–Zamolodchikov solution, we have the following data for the VEVs of operators giving the leading contribution:
\[
m_1^I \langle \Phi_{13} \rangle = 2.269550689,
\]
\[
m_1^I \langle \Phi_{15} \rangle = -2.325136i
\]
for primary fields and
\[
m_1^\Psi C_{\Psi^2 \Phi}^{L_2 \bar{L}_2 I}(r) \langle L_2 \bar{L}_2 I \rangle =
\]
\[
= -0.0005899419474(m_1 r)^{\frac{36}{7}},
\]
\[
m_1^\Phi C_{\Phi^2 \Phi}^{L_2 \bar{L}_2 \Phi}(r) \langle L_2 \bar{L}_2 \Phi \rangle =
\]
\[
= 0.01123584810(m_1 r)^{\frac{30}{7}}
\]
for the first nontrivial descendants. Taking into account that the mass coupling constant for \( \xi = 2/5 \) is
\[
g = -0.04053795542378225m_1^{20/7} \tag{43}
\]
and collecting all data together, we obtain the short-distance expansion up to \( r^6 \):
\[
m_1^\Psi \langle \Psi(x) \Psi(0) \rangle \sim -5.83(m_1 r)^2 + (m_1 r)^\frac{3}{7} - 0.265(m_1 r)^{22/7} + 0.233(m_1 r)^{24/7}
\]
\[-0.045(m_1 r)^4 + 0.011(m_1 r)^{30/7}
\]
\[-0.0006(m_1 r)^{\frac{36}{7}} + O(r^6). \tag{44}
\]
The plots for this function are shown in Figs. [11, 12]. It can be clearly seen that the contributions from the descendent fields \( \langle L_2 \bar{L}_2 \Phi \rangle \) and \( \langle L_2 \bar{L}_2 I \rangle \) are essential.
3 Long-distance expansion

Massive models allow an alternative description as scattering theories with factorized S matrices [10, 11]. In this section, we use the form-factor approach to study the long-distance behaviors of the spin correlation functions. The structure of ground states of the model as well as the spectrum of particles depend on the numbers \((p, p')\) [5, 10, 12]. There are two kinks interpolating between \(p - 1\) different vacuums and \([(p' - p)/p]\) bound states.

For simplicity, we restrict our attention to the case of nonunitary models \(M_{2,2n+1}\) having only \(n\) primary fields \(\Phi_{1,k} = \Phi_{1,2n-k+1}, (k = 1, \ldots, n)\). The corresponding perturbed models are the simplest integrable two-dimensional models in the sense that the scattering matrices are diagonal. The multiparticle form factors of the fields \(\Psi, \Phi, \ldots\) in these theories do not contain complicated contour integrals originating from solitonic form factors [7]. Hereafter in this section, we use the symbol \(\xi\) only for \(\xi = 2/2n - 1\).

We note that for studying a general case, one must also take kink contributions into account.

3.1 Perturbed \(M_{2,2n+1}\) models as scattering theories.

We briefly recall the basic facts about the scattering theories under consideration (see, e.g., [27] for a detailed review). In \(M_{2,2n+1}\) models perturbed by the field \(\Phi\), there are \(j = 1, \ldots, n - 1\) scalar self-conjugate particles with the masses

\[
m_j := m_1 \frac{\sin(\frac{\pi \xi_j}{2})}{\sin(\frac{\pi \xi}{2})}.
\]  

(45)

Here, \(m_1\) is a mass of the lightest particle “1”. The standard momentum parameterization is given in terms of rapidities \(\beta\) as

\[
p_a^0 = m_a \cosh \beta,
\]

\[
p_a^1 = m_a \sinh \beta.
\]

(46)

The S matrix for these models originates from the breathers S matrix of the corresponding sine-Gordon model [10, 26]:

\[
S_{ab}(\beta) = f_{\frac{|a-b|}{2}}(\beta)f_{\frac{a+b}{2}}(\beta) \prod_{s=1}^{\min(ab)-1} \left(f_{\frac{|a-b| + 2s}{2}}(\beta)\right)^2.
\]

(47)
Here, the basic building block \( f_a(\beta) \) is defined through the meromorphic functions
\[
f_a(\beta) = \frac{\tanh \frac{1}{2}(\beta + i\pi a\xi)}{\tanh \frac{1}{2}(\beta - i\pi a\xi)}.
\] (48)

In the physical strip \( 0 < \text{Im} \beta < \pi \), the scattering matrices satisfy the unitarity and crossing symmetry conditions given respectively as
\[
S_{ab}(\beta)S_{ba}(-\beta) = 1, \quad S_{ab}(\beta) = S_{ab}(i\pi - \beta),
\]
as well as the additional bound-state condition described as follows. If \( S_{ab}(\beta) \) has a simple pole \( \beta_{ab} = iu_{ab} \) in the direct channel, then the particle “c” of mass
\[
m_c^2 = m_a^2 + m_b^2 + 2m_a m_b \cos(u_{ab}^c)
\]
is a bound state of “a” and “b”. The bound state \( |B_c(\beta)\rangle \) with \( \beta = \beta_a + \beta_b \) is defined as the projection on the pole of the two-particle state \( |B_a(\beta_a)B_b(\beta_b)\rangle \) at the relative rapidity \( \beta_{ab} = iu_{ab}^c \). Then the S matrices for the composite particle can be found via the bootstrap equations
\[
S_{cd}(\beta) = S_{ad}(\beta + i(\pi - u_{ac}^b))S_{bd}(\beta - i(\pi - u_{bc}^a)).
\]
In particular, the total scattering matrix \( S_{ab} \) can be reconstructed starting from the S matrix \( S_{11}(\beta) = f_1(\beta) \) for the lightest particle assuming that the bootstrap tree for our massive models is closed under the fusions
\[
a_i \times a_j \longrightarrow a_{i+j} \quad \text{or} \quad a_i \times a_j \longrightarrow a_{2n-1-i-j},
a_i \times a_j \longrightarrow a_{i-j}.
\]
The rule in the first line above is to choose the variant where a final particle \( i + j \) or \( 2n - 1 - i - j \) is in the range \( (1, \ldots, n - 1) \).

### 3.2 Form-factor approach

As we already mentioned, the correlation functions admit a form-factor expansion that is very useful for studying the infrared behaviors. For example, for spin fields, the corre-
sponding spectral decomposition is
\[
\langle \Psi(x)\Psi(0) \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d\beta_1}{2\pi} \cdots \frac{d\beta_n}{2\pi} \langle 0|\Psi(x)|\beta_1, \ldots, \beta_n \rangle_{a_1 \cdots a_n} \times
\]
\[
\times a_1 \cdots a_n \langle \beta_1, \ldots, \beta_n |\Psi(0)|0 \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\{a_j\}} \int \frac{d\beta_1}{2\pi} \cdots \frac{d\beta_n}{2\pi} e^{-r \sum_j m_{a_j} \cosh \beta_j}
\]
\[
\times F_{a_1 \cdots a_n}(\beta_1, \ldots, \beta_1) F_{a_1 \cdots a_n}(\beta_1, \ldots, \beta_n) \ , \quad (49)
\]
where we introduce the matrix elements of the local operators in the basis of the asymptotic states
\[
F_{a_1 \cdots a_n}(\beta_1, \ldots, \beta_n) = \langle 0|\Psi(0)|\beta_1, \ldots, \beta_n \rangle_{a_1 \cdots a_n} \ . \quad (50)
\]
Form factors (50) are defined in this approach as a set of functions satisfying Smirnov’s axioms [7] (also see [6]). We recall them for completeness. The first two axioms are the Watson equations
\[
F_{a_1 \cdots a_j a_{j+1} \cdots a_n}(\beta_1, \ldots, \beta_1, \beta_{j+1}, \cdots, \beta_n) = S_{a_j a_{j+1}}(\beta_j - \beta_{j+1}) F_{a_1 \cdots a_j a_{j+1} a_{j+2} \cdots a_n}(\beta_1, \ldots, \beta_{j+1}, \beta_j, \ldots, \beta_n) ,
\]
\[
F_{a_1 a_2 \cdots a_n}(\beta_1 + 2\pi i, \beta_2, \cdots, \beta_n) = F_{a_2 \cdots a_n a_1}(\beta_2, \cdots, \beta_n, \beta_1) . \quad (51)
\]
The requirement for relativistic invariance implies that for the local operator with spin \( s \), one has
\[
F_{a_1 \cdots a_n}(\beta_1 + \Lambda, \cdots, \beta_n + \Lambda) = e^{s\Lambda} F_{a_1 \cdots a_n}(\beta_1, \cdots, \beta_n) . \quad (52)
\]
The ultraviolet bound condition requires that there exist a finite constant \( t(j, n) < \infty \) such that
\[
F_{a_1 \cdots a_n}(\beta_1 + \Lambda, \cdots, \beta_j + \Lambda, \beta_{j+1}, \ldots, \beta_n) = O(\exp e^{t(j,n)|\Lambda|}) , \quad \Lambda \rightarrow \infty . \quad (53)
\]
In addition, there are two pole conditions. The first is a kinematic pole condition. In our case for self-conjugate particles, it looks like
\[
-i \lim_{\beta' \rightarrow \beta} (\beta' - \beta) F_{a a_1 \cdots a_n}(\beta' + i\pi, \beta, \beta_1, \ldots, \beta_n) =
\]
\[
= (1 - \prod_{j=2}^{n} S_{a_j}(\beta - \beta_j)) F_{a_1 \cdots a_n}(\beta_1, \ldots, \beta_1) . \quad (54)
\]

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The second pole condition is the equation for bound states. Let $a_i \times a_j \rightarrow a_k$. Then

$$-i \lim_{\beta' \to \beta} (\beta' - \beta) F_{a_1 \cdots a_i a_j \cdots a_n}(\beta_1, \ldots, \beta' + i(\pi - u^{a_j}_{a_ia_k}), \beta - i(\pi - u^{a_i}_{a_ia_k}), \ldots) =$$

$$= \Gamma_{a_ia_j}^{a_k} F_{a_1 \cdots a_i \cdots a_n}(\beta_1, \ldots, \beta, \ldots, \beta_n),$$

where the three particle on-shell vertex is defined as\(^4\)

$$-i \text{Res}_{\beta} S_{ab}(\beta)\big|_{\beta = i u^c_{ab}} = (\Gamma^c_{ab})^2.$$

Smirnov \(^7\) proved that the set of functions (50) satisfying conditions (51)–(55) defines a matrix element of a local operator in the scattering theory with the given S matrix.

### 3.3 First form factors

The solutions of locality axioms (51)–(55) for the primaries in perturbed $\mathcal{M}_{2,2n+1}$ models have already been discussed in the literature. The form factors of the fields $\Psi$ and $\Phi$ were found in \(^10\) via the quantum group reduction procedure from the sine-Gordon model. The case of the scaling Lee–Yang model $\mathcal{M}_{2,5}$ was directly elaborated in \(^1\). The general expressions for the primaries $\Phi_{1k}$ appeared in \(^27\) based on the form factors for exponential operators in the sinh-Gordon model \(^28\). But for our aim of studying the correlation functions that are in agreement with (15), we must complete these prescriptions by choosing a proper overall factor determining normalization \(^3\) of local operators. This was essentially provided in \(^29\).\(^5\) Indeed, after the quantum group reduction \(^8\) \(^19\), the expressions for the first breather form factors of the exponential operators in the sine-Gordon model can be essentially treated as the following particle “1” form factors

\(^4\)We note that the perturbed minimal models $\mathcal{M}_{2,2n+1}$ violate the one-particle unitarity since some of its couplings are purely imaginary.

\(^5\)Another advantage of Lukyanov’s bosonization rules is that they can be easily extended to include the kink sector for a perturbed minimal CFT with general $(p,p')$.  

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of the primaries in the reduced model:\(^6\)

\[
\langle 0|\Phi_{1k}|0 \rangle = \langle \Phi_{1k} \rangle, \tag{56}
\]

\[
\langle 0|\Phi_{1k}|\beta \rangle = iC\frac{\sin \left(\frac{(k-1)/2}{2}\right)}{\sin(\pi \xi)} \langle \Phi_{1k} \rangle, \tag{57}
\]

\[
\langle 0|\Phi_{1k}|\beta_2, \beta_1 \rangle = i^2C^2\frac{\sin^2 \left(\frac{(k-1)/2}{2}\right)}{\sin^2(\pi \xi)} R(\beta_1 - \beta_2) \langle \Phi_{1k} \rangle, \tag{58}
\]

\[
\langle 0|\Phi_{1k}|\beta_3, \beta_2, \beta_1 \rangle = i^3C^3\frac{\sin \left(\frac{(k-1)/2}{2}\right)}{\sin(\pi \xi)} \prod_{s<j} R(\beta_s - \beta_j)
\]

\[
\times \left\{\frac{\sin^2 \left(\frac{(k-1)/2}{2}\right)}{\sin^2(\pi \xi)} + \frac{1}{\prod_{s<j} 2 \cosh \left(\frac{\beta_s - \beta_j}{2}\right)}\right\} \langle \Phi_{1k} \rangle, \tag{59}
\]

and so on. Here, the function \(R(\beta)\) determining the rapidity-dependent part of the two-particle form factor is given explicitly as

\[
R(\beta) = \exp \left\{ 4 \int_0^\infty \frac{dt \sinh t \sinh \xi t \sinh(\xi + 1)t}{\sinh^2 2t} \cosh 2(1 - \frac{i}{\pi} \beta)t \right\}. \tag{60}
\]

We note that it has a property that is very useful for checking the locality axioms at low particle levels (while one of the easiest ways to prove the validity of the general formulae is to use the free-field realization):

\[
R(\beta)R(\beta \pm i\pi) = \frac{\sinh(\beta)}{\sinh(\beta) \mp i \sin(\pi \xi)},
\]

\[
\prod_{k=0}^{n-2} R \left( \beta + \frac{i\pi \xi}{2}(2k + 1) \right) R \left( \beta - \frac{i\pi \xi}{2}(2k + 1) \right) = \frac{\cosh(\beta) - \cos(\pi \xi)}{\cosh(\beta) + 1} R(\beta). \tag{61}
\]

The \(\beta\)-independent constant \(C\) fixing the normalization of the first particles is

\[
C^2 = 8 \cos^2 \left(\frac{\pi \xi}{2}\right) \sin \left(\frac{\pi \xi}{2}\right) \exp \left( - \int_0^{\pi \xi} \frac{dt}{\pi \sin t} \right). \tag{62}
\]

\(^6\)We note that the same result can be obtained using the exact realization of the form factors in the RSOS model and taking the scaling limit \(\xi \to 0\).
3.4 Correlation functions at long distances

One can see that the leading terms for the correlation function already come from the zero- and one-particle form factors. For example, even the first two-term expression

$$\langle \Phi_1 k(x) \Phi_1 k(0) \rangle = \langle \Phi_1 k \rangle^2 \left\{ 1 - \frac{2 \sin^2 \left( \frac{\pi}{2} (k - 1) \xi \right)}{\sin \left( \frac{\pi}{2} \xi \right)} \exp \left( - \int_0^{\pi \xi} \frac{dt}{\pi \sin t} \right) K_0(m_1 r) \right\} + \cdots, \quad (63)$$

where $K_0(x)$ is a Macdonald function, turns out to be a good approximation for the correlation functions at long distances.

Of course, for more accurate calculations, one must take higher form factors with more particles into account, including those with particles $a_j$ with $j > 1$. The latter can be easily derived directly through the bound-state axiom \[55\]. It is clear from \[49\] that at long distances the contribution of an $n$-particle form factor with particles $a_1, \ldots, a_n$ and masses $m_{a_1}, \ldots, m_{a_n}$ to the correlation function is essentially approximated by the quantity $\exp(-r \sum_j m_{a_j})$. In practice, the form factors with a higher number of particles $a_1$ and particles of high mass become important only at small distances. It is useful to order the form factors according to the value of this quantity. For example, for the perturbed $M_{2,7}$ model that is considered further, the leading terms thus come from $F_1, F_2, F_11, F_{21}, F_{111}, \text{etc.}$, in agreement with mass relations \[15\] $m_1 < m_2 < 2m_1 < m_1 + m_2 < 3m_1$, etc.

4 Numerical results

The form-factor and conformal perturbation expansions work well at long and short distances respectively. In this section, we collect the numerical data demonstrating matching between these two at $mr \sim 1$ for the case of scaling $M_{2,7}$ models as an example. Substituting $\xi = \frac{2}{5}$ for the $M_{2,7}$ case, we obtain the expression

$$\langle \Phi_{12} \lvert B_1(\beta) \rangle \langle B_1(\beta) \rvert \Phi_{12} \rangle = -\langle \Phi_{12} \rangle^2 \cdot 0.7574894945.$$

Taking into account the numerical value for the VEV

$$m_1^4 \langle \Phi_{12} \rangle = -2.325598436i, \quad (64)$$

7The additional requirements for the consistency of formulae \[56\] – \[59\], etc., appearing from the bound-state condition are automatically satisfied for the operators $\Phi_{1,k}$ as was argued in \[27\]. For form factors with a small number of particles, this can be verified straightforwardly.
we find that the leading term in the long-distance expansion is given by the formula

\[
m_1^8 \langle \Phi(x) \Phi(0) \rangle \sim -5.408408086 \left( 1 - 0.2411163946 \cdot K_0(rm_1) \right). \tag{65}
\]

Comparison of the short- and long-distance expansions for the two-point correlation function \( \langle \Psi(x) \Psi(0) \rangle \) is shown in Figs. 1–3.

Figure 2: Form factor contributions at small distances

The numerically calculated data is presented in Table I. The difference between the short- and long-distance expansions indicates the self-consistency of the zeroth-order perturbation theory and the form-factor expansion up to two particles.

5 Conclusion

We have demonstrated that there is a region in the mass scale where the long- and short-distance expansions for the correlation functions of the spin operators match each other. We can therefore conclude that the combination of the two approaches allows reconstructing the correlation functions at all scales.
| $mr$ | up to $r^{8/7}$ | up to $r^{28/7}$ | up to $r^{36/7}$ | up to $F_{11}$ | up to $F_{12}$ | up to $F_{111}$ |
|------|----------------|-----------------|-----------------|----------------|----------------|-----------------|
| 0.02 | -1.89536       | -1.89536        | -1.89536        | -1.78911       | -2.14822       | -1.97677        |
| 0.04 | -2.29916       | -2.29917        | -2.29917        | -2.20386       | -2.41955       | -2.34353        |
| 0.06 | -2.56977       | -2.56979        | -2.56979        | -2.48822       | -2.63930       | -2.59599        |
| 0.08 | -2.77773       | -2.77778        | -2.77778        | -2.70767       | -2.82117       | -2.79351        |
| 0.10 | -2.94806       | -2.94817        | -2.94817        | -2.88736       | -2.97622       | -2.95732        |
| 0.12 | -3.09288       | -3.09306        | -3.09306        | -3.03986       | -3.11138       | -3.09787        |
| 0.14 | -3.21905       | -3.21935        | -3.21934        | -3.17245       | -3.23117       | -3.22120        |
| 0.16 | -3.33092       | -3.33135        | -3.33135        | -3.28975       | -3.33870       | -3.33115        |
| 0.18 | -3.43139       | -3.43200        | -3.43199        | -3.39489       | -3.43619       | -3.43036        |
| 0.20 | -3.52254       | -3.52336        | -3.52335        | -3.49009       | -3.52528       | -3.52071        |
| 0.22 | -3.60588       | -3.60696        | -3.60694        | -3.57702       | -3.60725       | -3.60361        |
| 0.24 | -3.68258       | -3.68397        | -3.68395        | -3.65692       | -3.68306       | -3.68014        |
| 0.26 | -3.75354       | -3.75530        | -3.75526        | -3.73077       | -3.75352       | -3.75114        |
| 0.28 | -3.81950       | -3.82167        | -3.82162        | -3.79937       | -3.81926       | -3.81732        |
| 0.30 | -3.88103       | -3.88367        | -3.88361        | -3.86334       | -3.88081       | -3.87921        |
| 0.35 | -4.01850       | -4.02249        | -4.02247        | -4.00626       | -4.01911       | -4.01810        |
| 0.40 | -4.13682       | -4.14280        | -4.14259        | -4.12947       | -4.13911       | -4.13845        |
| 0.50 | -4.33032       | -4.34153        | -4.34097        | -4.33216       | -4.33782       | -4.33752        |
| 0.70 | -4.60061       | -4.62933        | -4.62699        | -4.62274       | -4.62492       | -4.62484        |
| 0.90 | -4.77130       | -4.82907        | -4.82226        | -4.82018       | -4.82110       | -4.82108        |
| 1.00 | -4.83076       | -4.90821        | -4.89757        | -4.89617       | -4.89679       | -4.89677        |
| 1.20 | -4.91088       | -5.03975        | -5.01671        | -5.01633       | -5.01661       | -5.01661        |
| 1.40 | -4.95019       | -5.14911        | -5.10491        | -5.10540       | -5.10554       | -5.10554        |
| 1.60 | -4.95765       | -5.24871        | -5.17110        | -5.17252       | -5.17258       | -5.17258        |
| 1.80 | -4.93933       | -5.34862        | -5.22123        | -5.22367       | -5.22371       | -5.22371        |
| 2.00 | -4.89960       | -5.45779        | -5.25949        | -5.26302       | -5.26304       | -5.26304        |
| 2.20 | -4.84170       | -5.58470        | -5.28902        | -5.29349       | -5.29349       | -5.29349        |
| 2.40 | -4.76811       | -5.73780        | -5.31231        | -5.31721       | -5.31721       | -5.31721        |
| 2.60 | -4.68080       | -5.92578        | -5.33150        | -5.33577       | -5.33577       | -5.33577        |
| 2.80 | -4.58133       | -6.15775        | -5.34855        | -5.35034       | -5.35034       | -5.35034        |
| 3.00 | -4.47099       | -6.44339        | -5.36541        | -5.36181       | -5.36181       | -5.36181        |
| 3.20 | -4.35086       | -6.79303        | -5.38416        | -5.37087       | -5.37087       | -5.37087        |
| 3.40 | -4.22182       | -7.21775        | -5.40703        | -5.37805       | -5.37805       | -5.37805        |
| 3.60 | -4.08465       | -7.72941        | -5.43656        | -5.38374       | -5.38374       | -5.38374        |

Table 1:
This also serves as a good consistency check for the correctness of both methods as well as the VEVs of local operators.

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When the paper was completed we learned from V. Fateev that the quantities \( J_2(a, b, d, e, c) \) in Eqn. (24) at the special values of parameters as in the l.h.s. of Eq. (26) can
be transformed to the class of integrals studied in [38], where analytical expressions in terms of gamma functions had already been proposed. We are grateful to V. Fateev for informing us about this.

Appendix

A Structure constants and conformal blocks

For completeness, we here give a derivation of structure constants for the subalgebra of conformal fields $\Phi_{ik}$ and relevant conformal blocks that are necessary for building the corresponding correlation functions in conformal field theory. To reproduce these known results [9], we use the method of functional equations [36].

We consider the four-point conformal block with the chiral $\Phi_{12}$ operator

$$G(z) = \langle \Phi_{12}(z)\Phi_{\Delta_1}(0)\Phi_{\Delta_2}(1)\Phi_{\Delta_3}(\infty) \rangle,$$  

where $\Phi_{\Delta_j}(z)$ denotes chiral parts of the primary field $\Phi_{1n_j}(z,\bar{z})$ ($k = 1, \ldots, \rho' - 1$) with the conformal dimensions $\Delta_{1,n_j}$. This holomorphic function satisfies the null vector equation [2]

$$\mu \frac{d^2}{dz^2} G(z) + \left(\frac{1}{z} + \frac{1}{z-1}\right) \frac{d}{dz} G(z) + \left(\frac{\Delta_1 + \Delta_2 + \delta - \Delta_3}{z(z-1)} - \frac{\Delta_1}{z^2} - \frac{\Delta_2}{(z-1)^2}\right) G(z) = 0$$  

with $\mu = \frac{3}{2(2\Delta_{12}+1)}$. The solutions of the equation and their monodromy properties can be easily found by noting that it turns out to be the Riemann equation

$$\frac{d^2 u}{dz^2} + \left(\frac{\alpha + \alpha' + \beta + \beta' + \gamma + \gamma'}{z} + \frac{\gamma'\gamma}{(z-1)^2} + \frac{\beta\beta' - \alpha\alpha' - \gamma\gamma'}{z(z-1)}\right) u = 0,$$  

with the parameters

$$\alpha = \frac{n_1 - 1}{2} \rho, \quad \alpha' = 1 - \frac{n_1 + 1}{2} \rho,$$

$$\beta = \frac{2 - n_3}{2} \rho, \quad \beta' = \frac{n_3 + 2}{2} \rho - 1,$$

$$\gamma = \frac{n_2 - 1}{2} \rho, \quad \gamma' = 1 - \frac{n_2 + 1}{2} \rho.$$
Here, the parameter $\rho$ is related to the labels $(p, p')$ of the minimal model as $\rho = p/p'$. The equation (68) has two linearly independent solutions having power-law behaviors at $z = 0$. These correspond to two conformal blocks in the S-channel,

\[ S_1(z) = z^\alpha (1 - z)^\gamma \, _2F_1 \left( \begin{array}{c} \alpha + \beta + \gamma, \\ 1 + \alpha - \alpha' \end{array} \left| z \right| \right), \]
\[ S_2(z) = z^{\alpha'} (1 - z)^\gamma \, _2F_1 \left( \begin{array}{c} \alpha' + \beta + \gamma, \\ 1 + \alpha' - \alpha \end{array} \left| z \right| \right). \]

(69)

Another set of linear independent solutions having power law form at $z = 1$ corresponds to T-channel conformal blocks:

\[ T_1(z) = z^\alpha (1 - z)^\gamma \, _2F_1 \left( \begin{array}{c} \alpha + \beta + \gamma, \\ 1 + \gamma - \gamma' \end{array} \left| 1 - z \right| \right), \]
\[ T_2(z) = z^{\alpha'} (1 - z)^\gamma \, _2F_1 \left( \begin{array}{c} \alpha' + \beta + \gamma', \\ 1 + \gamma' - \gamma \end{array} \left| 1 - z \right| \right). \]

(70)

Since second-order differential equation (68) has two linear independent solutions, $S_1$ and $S_2$ are expressed in terms of $T_1$ and $T_2$ as

\[ S_1(z) = AT_1(z) + BT_2(z), \]
\[ S_2(z) = CT_1(z) + DT_2(z). \]

(71)

Using the relations between solutions of Riemann equations (or just using the formulae for the analytic continuation for the hypergeometric functions), we obtain the expressions for the $z$-independent coefficients

\[
A = \frac{\Gamma(1 + \alpha - \alpha') \Gamma(\gamma' - \gamma)}{\Gamma(\alpha + \beta' + \gamma') \Gamma(\alpha + \beta + \gamma)}, \\
B = \frac{\Gamma(1 + \alpha - \alpha') \Gamma(\gamma - \gamma')}{\Gamma(\alpha + \beta + \gamma) \Gamma(\alpha + \beta' + \gamma)}, \\
C = \frac{\Gamma(1 + \alpha' - \alpha) \Gamma(\gamma' - \gamma)}{\Gamma(\alpha' + \beta + \gamma') \Gamma(\alpha' + \beta + \gamma)}, \\
D = \frac{\Gamma(1 + \alpha' - \alpha) \Gamma(\gamma - \gamma')}{\Gamma(\alpha' + \beta + \gamma') \Gamma(\alpha' + \beta + \gamma)}.
\]

In what follows, we need the ratio $AB/CD$. It can be expressed in terms of the $\gamma(x)$ function as

\[
\frac{AB}{CD} = \frac{\gamma(1 + \alpha - \alpha') \gamma(\alpha' + \beta + \gamma') \gamma(\alpha' + \beta + \gamma) \gamma(\alpha' + \beta' + \gamma)}{\gamma(1 + \alpha' - \alpha) \gamma(\alpha + \beta + \gamma)}.
\]
For the conformal blocks in model $M_{p,p'}$, the following expression holds for the operators $\Phi_{1,n}^j$:

\[
\frac{AB}{CD} = -\frac{\gamma(n_1\rho)}{\gamma(2-n_1\rho)} \frac{\gamma(2-(n_1+n_2+n_3)\frac{\rho}{2})}{\gamma((n_1-n_2+n_3)\frac{\rho}{2})} \frac{\gamma((-n_1+n_2+n_3)\frac{\rho}{2})}{\gamma((n_1+n_2-n_3)\frac{\rho}{2})}.
\]

The crossing symmetry condition [2] (see fig. 4) leads to the equation for the correlation functions

\[
C(2, n_1, n_1 + 1)C(n_1 + 1, n_2, n_3)|S_1|^2 + C(2, n_1, n_1 - 1)C(n_1 - 1, n_2, n_3)|S_2|^2 = C(2, n_2, n_2 + 1)C(n_2 + 1, n_1, n_3)|T_1|^2 + C(2, n_2, n_2 - 1)C(n_2 - 1, n_1, n_3)|T_2|^2,
\]

where $C(n_1, n_2, n_3)$ denotes the structure constants $C_{n_1n_2}^{n_3}$ of the operator algebra of fields $\Phi_{1,n}^j$. In particular, this gives

\[
C(2, n_1, n_1 + 1)C(n_1 + 1, n_2, n_3)AB + C(2, n_1, n_1 - 1)C(n_1 - 1, n_2, n_3)CD = 0.
\]

Figure 4: Conformal blocks.
We consider the case $n_3 = n_1 = n$ and $n_2 = 2$. If $A_0, B_0, \ldots$ are the corresponding elements of the monodromy matrix, then we find

$$C^2(2, n, n + 1)A_0 B_0 + C^2(2, n, n - 1)C_0 D_0 = 0.$$ 

Therefore,

$$\left[ \frac{C(2, n, n + 1)}{C(2, n, n - 1)} \right]^2 = - \frac{C_0 D_0}{A_0 B_0}. $$

We can now write the general difference equation for the structure constants in the form

$$\frac{C(n_1 + 1, n_2, n_3)}{C(n_1 - 1, n_2, n_3)} = - \left[ - \frac{A_0 B_0}{C_0 D_0} \right]^\frac{1}{2} \times \frac{CD}{AB}. $$

Rewriting everything via the variables $n_i$, we obtain the functional equation for structure constants in the $\Phi_{1,n}$ subalgebra of $\mathcal{M}_{p,p'}$

$$\frac{C(n_1 + 2, n_2, n_3)}{C(n_1, n_2, n_3)} = \left[ \frac{\gamma((2 - (n_1 + 1)\rho) \gamma(2 - (n_1 + 2)\rho)}{\gamma(n_1\rho) \gamma((n_1 + 1)\rho)} \right]^\frac{1}{2} \times $$

$$\times \frac{\gamma((n_1 - n_2 + n_3 + 1)\frac{\rho}{2}) \gamma((n_1 + n_2 - n_3 + 1)\frac{\rho}{2})}{\gamma((2 - n_1 + n_2 + n_3 + 1)\frac{\rho}{2}) \gamma((-n_1 + n_2 + n_3 - 1)\frac{\rho}{2})}. (72)$$

We seek the solution in the ansatz

$$C(n_1, n_2, n_3) = N(n_1) \prod_{k=1}^{n} \frac{\gamma(k\rho)}{\gamma((n_1 - k)\rho) \gamma((n_2 - k)\rho) \gamma(2 - (n_3 + k)\rho)},$$

where

$$n = (n_1 + n_2 - n_3 - 1)/2. $$

Substituting this expression in (72), we obtain the constraint for $N(n_1)$

$$\frac{N(n_1 + 2)}{N(n_1)} = \left( \frac{\gamma(n_1\rho) \gamma((n_1 + 1)\rho) \gamma(2 - (n_1 + 1)\rho) \gamma(2 - (n_1 + 2)\rho)}{\gamma((n_1 - 1)\rho) \gamma((n_2 - 1)\rho) \gamma(2 - (n_3 + 1)\rho)} \right)^\frac{1}{2},$$

which can be easily resolved as

$$N(n) = \left( \prod_{k=1}^{n-1} \gamma(k\rho) \gamma(2 - (k + 1)\rho) \right)^\frac{1}{2}. (73)$$

27
Taking the symmetry $\Delta_i \leftrightarrow \Delta_j$ into account, we find the expression for the structure constants

$$C(n_1, n_2, n_3) = \frac{N(n_1)N(n_2)}{N(n_3)} \prod_{k=1}^{n} \frac{\gamma(k \rho)}{\gamma((n_1 - k) \rho) \gamma((n_2 - k) \rho) \gamma(2 - (n_3 + k) \rho)}.$$  \hspace{1cm} (74)

We briefly comment on the admissible triples $(n_1, n_2, n_3)$. The procedure used in the derivation assumes that the field $\Phi_{1,n_3}$ must be related to $\Phi_{1,n_1}$ and $\Phi_{1,n_2}$ by fusion rules [2]. Otherwise, the proposed structure constants should be reconstructed by symmetry.

### B Integrals for the first corrections

The integrals from the correlation functions that appear at the first order correction can be represented in the form of double integrals over a plane because of the equation [9, 25, 13]

$$\frac{\pi \gamma(a + 1) \gamma(c + 1)}{\gamma(a + c + 2)} \int d^2x |x|^{2(a+d+c+1)} |1 - x|^{2e} \left| \text{}_2F_1 \left( \begin{array}{c} a + 1, -b \\ a + c + 2 \end{array} \right| x \right|^2$$

$$+ \pi \frac{\gamma(b + 1) \gamma(a + c + 1)}{\gamma(a + b + c + 2)} \int d^2x |x|^{2d} |1 - x|^{2e} \left| \text{}_2F_1 \left( \begin{array}{c} -a - b - c - 1, -c \\ -a - c \end{array} \right| x \right|^2$$

$$= \int d^2x \int d^2y |x|^{2a} |1 - x|^{2b} |y|^{2d} |1 - y|^{2e} |x - y|^{2c}. $$

For practical computations it is convenient to decompose this integral into a sum of holomorphic and antiholomorphic parts using the way proposed in works [16, 9] (see [24, 13] for alternative methods). Starting with the integral

$$J = \int \int d^2x d^2y \left( |x|^{2a} |1 - x|^{2b} |y|^{2d} |1 - y|^{2e} |x - y|^{2c} \right), $$

performing Wick rotation

$$x_2 = ix_0(1 - 2i\epsilon), \hspace{1cm} y_2 = iy_0(1 - 2i\epsilon), $$

and introducing the new variables

$$x = x_1 + x_0, \hspace{1cm} \bar{x} = x_1 - x_0, $$

$$y = y_1 + y_0, \hspace{1cm} \bar{y} = y_1 - y_0,$$
we can easily obtain
\[ J = -\frac{1}{4} \int \int dx dy \ x^a(x-1)^b\ y^d(y-1)^e(x-y)^c \]
\[ \times \int \int d\bar{x} d\bar{y} \ (\bar{x} + i\epsilon(x-\bar{x}))^a(\bar{x} - 1 + i\epsilon(x-\bar{x}))^b(\bar{y} + i\epsilon(y-\bar{y}))^d \]
\[ \times (\bar{y} - 1 + i\epsilon(y-\bar{y}))^e(\bar{x} - \bar{y} + i\epsilon((x-\bar{x}) - (y-\bar{y})))^c. \] (76)

Integral (76) has the branch points
\[ \bar{x} = -i\epsilon x, \quad \bar{y} = -i\epsilon y, \]
\[ \bar{x} = 1 - i\epsilon(x-1), \quad \bar{y} = 1 - i\epsilon(y-1), \] (77)

By deforming integration contours is easy to demonstrate that non-trivial contributions to (76) come only from two domains of integration, namely,
\[ \{0 < x < 1, \quad 0 < y < x\} \quad \text{and} \quad \{0 < x < 1, \quad x < y < 1\}. \] (78)

One observes that the integral \( J \) is represented as sum
\[ J = I(a, b, d, e, c) + I(d, e, a, b, c), \] (79)

where
\[ I(a, b, d, e, c) = -\frac{1}{4} \int_0^1 dx \int_0^x dy \ x^a(x-1)^b\ y^d(y-1)^e(x-y)^c \times \]
\[ \times \int_{\mathcal{C}_1} d\bar{x} \int_{\mathcal{C}_2} d\bar{y} \ (\bar{x} - 1)^b\ (\bar{y} - 1)^e(\bar{x} - \bar{y})^c, \]

\[ I(d, e, a, b, c) = -\frac{1}{4} \int_0^1 dx \int_0^1 dy x^a(x-1)^b\ y^d(y-1)^e(x-y)^c \times \]
\[ \times \int_{\mathcal{C}_4} d\bar{y} \int_{\mathcal{C}_3} d\bar{x} \ (\bar{x} - 1)^b\ (\bar{y} - 1)^e(\bar{x} - \bar{y})^c, \]
We introduce notations $I(a, b, d, e, c)$, $I(d, e, a, b, c)$ for the integrals above since that two integrals are related by transform $a \rightarrow d$, $b \rightarrow e$ as it can be easily checked by change of variables. The contours of integrations are shown in Fig. 5. Let us now transform the contour integrals $I(a, b, d, e, c)$ into standard type integrals. This is easily provided by applying the technique of analytical continuations and manipulation with the contours described in [9]. We obtain the following expression

$$I(a, b, d, e, c) = \sin \pi b K(a, b, d, e, c)[\sin \pi e L(a, b, d, e, c) + \sin \pi (e + c) L(d, e, a, b, c)], \quad (80)$$

where

$$K = \int_{0}^{1} dx \int_{0}^{1} dy \ x^{a+d+c+1} (1-x)^{y_d}(1-y)^c(1-xy)^e,$$

$$L = \int_{0}^{1} dy \int_{0}^{1} dx \ x^{-a-b-c-2}(1-x)^c y^{-a-b-d-e-c-3}(1-y)^e (1-xy)^b. \quad (81)$$

Now one uses standard formulae for the integral representations of the hypergeometric
functions
\[
\int_0^1 dy y^p(1-y)^q(1-xy)^r = B(p+1,q+1) \, _2F_1\left( -r, \frac{p+1}{p+q+2} \right| x),
\]
\[
\int_0^1 x^s(1-x)^t \, _2F_1\left( \frac{a}{c}, \frac{b}{c} | x \right) dx = B(s+1,t+1) \, _3F_2\left( \frac{a,b}{s+t+2}, \frac{s+1}{c} | 1 \right),
\]
to find that the integrals \( K \) and \( L \) admit another representation in terms of the higher hypergeometric functions. Here \( B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y) \) is a beta function. Namely,

\[
K = B(d+1,c+1)B(a+d+c+2,b+1) \times _3F_2\left( \frac{-e}{a+d+c+3+b}, \frac{a+d+c+2}{d+2+c} | 1 \right),
\]

\[
L = B(-a-b-d-e-c-2,e+1)B(-a-b-c-1,c+1) \times _3F_2\left( \frac{-b}{-a-b-d-c-1}, \frac{-a-b-d-e-c-2}{-a-b} | 1 \right).
\]

Equations (79), (80) and (82) give a form convenient for numerical studies. We note that the functions \( _3F_2 \) at unity satisfy specific identities [37]. This simplifies the answer and leads to (26).

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