ON RIEMANNIAN MANIFOLDS WITH POSITIVE WEIGHTED RICCI CURVATURE OF NEGATIVE EFFECTIVE DIMENSION

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Abstract. In this paper, we investigate complete Riemannian manifolds satisfying the lower weighted Ricci curvature bound $\text{Ric}_N \geq K$ with $K > 0$ for the negative effective dimension $N < 0$. We analyze two one-dimensional examples of constant curvature $\text{Ric}_N \equiv K$ with finite and infinite total volumes. We also discuss when the first non-zero eigenvalue of the Laplacian takes its minimum under the same condition $\text{Ric}_N \geq K > 0$, as a counterpart to the classical Obata rigidity theorem. Our main theorem shows that, if $N < -1$ and the minimum is attained, then the manifold splits off the real line as a warped product of hyperbolic nature.

1. Introduction

Riemannian manifolds of Ricci curvature bounded below are classical research subjects in comparison geometry and geometric analysis. Recently, diverse developments on the curvature-dimension condition in the sense of Lott, Sturm and Villani have shed new light on this theory. The curvature-dimension condition $\text{CD}(K, N)$ is a synthetic notion of lower Ricci curvature bounds for metric measure spaces. The parameters $K$ and $N$ are usually regarded as ‘a lower bound of the Ricci curvature’ and ‘an upper bound of the dimension’, respectively. Thus $N$ is sometimes called the effective dimension. The roles of $K$ and $N$ are better understood when we consider a weighted Riemannian manifold $(M, g, m)$, a Riemannian manifold of dimension $n$ equipped with an arbitrary smooth measure $m$. Then the Ricci curvature is modified into the weighted Ricci curvature $\text{Ric}_N$ involving a parameter $N$, and $\text{CD}(K, N)$ is equivalent to $\text{Ric}_N \geq K$ (see [vRS, St1, St2, LV] as well as [Vi] for $N \in [n, \infty)$, [Oh2, Oh3] for $N \leq 0$, and also [Oh1] for the Finsler analogue).

For $N \in [n, \infty]$, the weighted Ricci curvature $\text{Ric}_N$ (also called the Bakry–Émery–Ricci curvature) has been intensively studied by, for instance, Bakry and his collaborators in the framework of the $\Gamma$-calculus (see [BGL]). Recently it turned out that there is a rich theory also for $N \in (-\infty, 1]$, though this range seems strange due to the above interpretation of $N$ as an upper dimension bound. Among others, various Poincaré-type inequalities [KM], the curvature-dimension condition [Oh2, Oh3], isoperimetric inequalities [Mi, Kl, Oh3], and the splitting theorem [Wy] were studied for $N < 0$ or $N \leq 1$.

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The aim of this article is to contribute to the study of the structure of Riemannian manifolds with $\text{Ric}_N \geq K$ for $K > 0$ and $N < 0$. In Section 3 we analyze two one-dimensional examples $M_1, M_2$ of $\text{Ric}_N \equiv K$ (appearing in [KM, Mi]) having different natures. The first example $M_1$ has finite volume and enjoys only the exponential concentration; in particular, $M_1$ does not satisfy the logarithmic Sobolev inequality. For $N < -1$, this space attains the minimum of the first non-zero eigenvalue of the (weighted) Laplacian (derived from the Bochner inequality, see Proposition 2.5). The second example $M_2$ has infinite volume and reveals the difficulty in obtaining a volume comparison under $\text{Ric}_N \geq K$.

In Section 4, we investigate when the minimum of the first non-zero eigenvalue is attained under the condition $\text{Ric}_N \geq K > 0$ with $N < -1$, as a counterpart to the Obata rigidity theorem [Ob]. Our main theorem (Theorem 4.5) asserts that, when $\dim M \geq 2$, we have a warped product splitting $M \cong \mathbb{R} \times \cosh(\sqrt{K/(1 - N)} \Sigma^{n - 1}$ of hyperbolic nature. Moreover, $\Sigma$ enjoys $\text{Ric}_{N - 1} \geq K(2 - N)/(1 - N)$. This warped product splitting should be compared with Cheng–Zhou’s theorem in the case of $N = \infty$, where the sharp spectral gap forces the space to isometrically split off a one-dimensional Gaussian space (see [CZ] and Theorem 4.3 for details). Our proof considers the equality case in the Bochner inequality, which is related to the dimension-free version in [CZ] but requires a further discussion due to the fact that the Hessian of the eigenfunction does not vanish (Lemma 4.4). Therefore we have only the warped product rather than the isometric product in [CZ]. We stress that in our setting of $N < 0$, it is interesting to have a hyperbolic structure even when the curvature is positive.

2. Preliminaries

2.1. Weighted Riemannian manifolds

A weighted Riemannian manifold $(M, g, m)$ will be a pair of a complete, connected, boundaryless manifold $M$ equipped with a Riemannian metric $g$ and a measure $m = e^{-\psi} \text{vol}_g$, where $\psi \in C_\infty(M)$ and $\text{vol}_g$ is the standard volume measure on $(M, g)$. On $(M, g, m)$, we define the weighted Ricci curvature as follows.

**Definition 2.1.** (Weighted Ricci curvature) Given a unit vector $v \in U_x M$ and $N \in (-\infty, 0) \cup [n, \infty]$, the weighted **Ricci curvature** $\text{Ric}_N(v)$ is defined by:

1. $\text{Ric}_N(v) := \text{Ric}_g(v) + \text{Hess}_g(\psi(v,v) - \langle \nabla \psi(x), v \rangle^2/(N - n)$ for $N \in (-\infty, 0) \cup (n, \infty)$;
2. $\text{Ric}_n(v) := \text{Ric}_g(v) + \text{Hess}_g(\psi(v,v)$ if $\langle \nabla \psi(x), v \rangle = 0$, and $\text{Ric}_n(v) := -\infty$ otherwise;
3. $\text{Ric}_\infty(v) := \text{Ric}_g(v) + \text{Hess}_g(\psi(v,v)$,

where $n = \dim M$ and $\text{Ric}_g$ denotes the Ricci curvature of $(M, g)$. The parameter $N$ is sometimes called the **effective dimension**. We also define $\text{Ric}_N(cv) := c^2 \text{Ric}_N(v)$ for $c \geq 0$.

Note that if $\psi$ is constant then the weighted Ricci curvature coincides with $\text{Ric}_g(v)$ for all $N$. When $\text{Ric}_N(v) \geq K$ holds for some $K \in \mathbb{R}$ and all unit vectors $v \in TM$, we will write $\text{Ric}_N \geq K$. By definition, $\text{Ric}_n(v) \leq \text{Ric}_N(v) \leq \text{Ric}_\infty(v) \leq \text{Ric}_{N'}(v)$.
holds for $n \leq N < \infty$ and $-\infty < N' < 0$, and $\text{Ric}_N(v)$ is non-decreasing in $N$ in the ranges $(-\infty, 0)$ and $[n, \infty]$.

Remark 2.2. The weighted Ricci curvature for $N \in [n, \infty]$ has been intensively and extensively investigated, see [Qi, BGL] for instance. The study for the negative effective dimension $N < 0$ is more recent. One can find in [Mi, Kl, Oh3] the isoperimetric inequality, in [Wy] the Cheeger–Gromoll type splitting theorem, and in [Oh2, Oh3] the curvature-dimension condition in this context.

We also define the weighted Laplacian with respect to $m$.

Definition 2.3. (Weighted Laplacian) The weighted Laplacian (also called the Witten Laplacian) of $u \in C^\infty(M)$ is defined as follows:

$$\Delta_m u := \Delta u - \langle \nabla u, \nabla \psi \rangle.$$  

Notice that the Green formula (the integration by parts formula)

$$\int_M u \Delta_m v \, dm = -\int_M \langle \nabla u, \nabla v \rangle \, dm = \int_M v \Delta_m u \, dm$$

holds provided $u$ or $v$ belongs to $C^\infty_c(M)$ (smooth functions with compact supports) or $H^1_0(M)$.

2.2. Bochner inequality and eigenvalues of the Laplacian

Recall that the Bochner–Weitzenböck formula associated with the weighted Ricci curvature $\text{Ric}_\infty$ and the weighted Laplacian $\Delta_m$ holds as follows. For $u \in C^\infty(M)$, we have

$$\Delta_m \left( \frac{\|\nabla u\|^2}{2} \right) - \langle \nabla \Delta_m u, \nabla u \rangle = \text{Ric}_\infty (\nabla u) + \|\text{Hess} u\|_{HS}^2,$$  

where $\|\cdot\|_{HS}$ stands for the Hilbert–Schmidt norm (with respect to $g$). As a corollary we obtain the Bochner inequality for $N \in (-\infty, 0) \cup [n, \infty]$ (the case of negative effective dimension was studied independently in [KM, Oh2]). We give an outline of the proof for later use.

Theorem 2.4. (Bochner inequality) For $N \in (-\infty, 0) \cup [n, \infty]$ and any $u \in C^\infty(M)$, we have

$$\Delta_m \left( \frac{\|\nabla u\|^2}{2} \right) - \langle \nabla \Delta_m u, \nabla u \rangle \geq \text{Ric}_N (\nabla u) + \frac{(\Delta_m u)^2}{N}.$$  

Proof. Fix $x \in M$ and let $B$ be the matrix representation of $\text{Hess} u(x)$ for some orthonormal basis. Then we find

$$\|\text{Hess} u\|^2_{HS} = \text{tr}(B^2) \geq \frac{(\text{tr} B)^2}{n} = \frac{(\Delta u)^2}{n}.$$  

Putting $p = \Delta_m u$ and $q = \langle \nabla u, \nabla \psi \rangle$, we have

$$\frac{(\Delta u)^2}{n} = \frac{(p + q)^2}{n} = \frac{p^2}{N} - \frac{q^2}{N - n} + \frac{N(N - n)}{n} \left( \frac{p}{N} + \frac{q}{N - n} \right)^2 \geq \frac{p^2}{N} - \frac{q^2}{N - n} = \frac{(\Delta_m u)^2}{N} - \frac{\langle \nabla u, \nabla \psi \rangle^2}{N - n}.$$  

Combining these with the Bochner–Weitzenböck formula (1) we obtain the Bochner inequality. □
We remark that \( N(N - n) \geq 0 \) used in the proof fails for \( N \in (0, n) \). The Bochner inequality has quite rich applications in geometric analysis. In this article we are interested in the first non-zero eigenvalue of the weighted Laplacian, a generalization of the Lichnerowicz inequality. The case of the negative effective dimension \( N < 0 \) was studied in \([KM, Oh2]\). Here we give a proof for completeness.

**Proposition 2.5.** (First non-zero eigenvalue) Let \((M, g, m)\) be a complete weighted Riemannian manifold satisfying \( \text{Ric}_N \geq K \) for some \( K > 0 \) and \( N < 0 \), and assume \( m(M) < \infty \). Then the first non-zero eigenvalue of the non-negative operator \(-\Delta_m\) is bounded from below by \( K \frac{N}{N - 1} \).

**Proof.** Let \( u \in H^1_0 \cap C^\infty(M) \) be a non-constant eigenfunction of \(-\Delta_m\) with \( \Delta_m u = -\lambda u \), \( \lambda > 0 \). We deduce from the Bochner inequality (Theorem 2.4) and the condition \( \text{Ric}_N \geq K \) that

\[
\Delta_m \left( \frac{|\nabla u|^2}{2} \right) \geq \langle \nabla \Delta_m u, \nabla u \rangle + K |\nabla u|^2 + \frac{(\Delta_m u)^2}{N}.
\]

Since constant functions belong to \( H^1_0(M) \) thanks to the hypothesis \( m(M) < \infty \), the integration of the above inequality yields

\[
0 \geq \left( \frac{1}{N} - 1 \right) \int_M (\Delta_m u)^2 \, dm + K \int_M |\nabla u|^2 \, dm.
\]

We again use the integration by parts to see

\[
\int_M (\Delta_m u)^2 \, dm = -\lambda \int_M u \Delta_m u \, dm = \lambda \int_M |\nabla u|^2 \, dm.
\]

Therefore we have

\[
\frac{1 - N}{N} \lambda + K \leq 0,
\]

which shows the claim \( \lambda \geq K \frac{N}{N - 1} \).

It was observed in \([KM, §3.2]\) and \([Mi, Theorem 6.1]\) that the estimate \( \lambda \geq K \frac{N}{(N - 1)} \) is sharp for \( N \in (-\infty, -1] \) and, somewhat surprisingly, not sharp for \( N < 0 \) close to 0. See Section 4 for a further discussion, where we discuss the rigidity for the constant \( K \frac{N}{(N - 1)} \) in the case \( N < -1 \). The first non-zero eigenvalue of the weighted Laplacian is related to the concentration of measures. When \( m(M) = 1 \), we define the concentration function of \((M, g, m)\) by

\[
\alpha(r) := \sup \{ 1 - m(B(A, r)) \mid A \subset M : \text{Borel}, \ m(A) \geq 1/2 \}.
\]

See \([Le, Theorem 3.1]\) for the following corollary.

**Corollary 2.6.** (Exponential concentration) Let \((M, g, m)\) be a compact weighted Riemannian manifold satisfying \( \text{Ric}_N \geq K \) for some \( K > 0 \) and \( N < 0 \), and assume \( m(M) = 1 \). Then \((M, g, m)\) satisfies the exponential concentration \( \alpha(r) \leq e^{-\sqrt{KN/(N-1)r}/3} \).
3. Two examples of one-dimensional spaces of constant curvature

In this section we analyze two examples of one-dimensional spaces such that \( \text{Ric}_N \equiv K \) for some \( K > 0 \) and \( N < 0 \). These examples will be helpful to understand the general picture of the spaces satisfying \( \text{Ric}_N \geq K > 0 \). In particular, we shall give negative answers to some naive guesses.

The first example is the following.

**Example 3.1.** For \( K > 0 \) and \( N < 0 \), the space

\[ M_1 := \left( \mathbb{R}, |\cdot|, \cosh\left(\sqrt{\frac{K}{1-N}}x \right)^{N-1} dx \right), \]

where \(|\cdot|\) denotes the canonical distance structure, satisfies \( \text{Ric}_N \equiv K \).

This is a model space in Milman’s isoperimetric inequality (see Case 1 of [Mi, Corollary 1.4]). Notice that the total volume is finite since

\[
\int_{\mathbb{R}} \cosh\left(\sqrt{\frac{K}{1-N}}x \right)^{N-1} dx \leq 2 \int_0^\infty \left( e^{\sqrt{K/(1-N)}x} \right)^{N-1} dx = 2 \int_0^\infty e^{-\sqrt{K/(1-N)}x} dx < \infty.
\]

In order to see the curvature identity, we observe that the weight function is given by

\[
\psi(x) = (1 - N) \log\left( \cosh\left(\sqrt{\frac{K}{1-N}}x \right) \right),
\]

and

\[
\psi'(x) = \sqrt{K(1-N)} \tanh\left(\sqrt{\frac{K}{1-N}}x \right),
\]

\[
\psi''(x) = K - K \tanh\left(\sqrt{\frac{K}{1-N}}x \right)^2.
\]

Since the Ricci curvature vanishes in the one-dimensional case, we have the curvature identity

\[
\text{Ric}_N \left( \frac{\partial}{\partial x} \bigg|_x \right) = \psi''(x) + \frac{\psi'(x)^2}{1-N}
\]

\[
= K - K \tanh\left(\sqrt{\frac{K}{1-N}}x \right)^2 + \frac{K(1-N)}{1-N} \tanh\left(\sqrt{\frac{K}{1-N}}x \right)^2
\]

\[
= K.
\]

In [KM] it was observed that this space attains the minimum value of the first non-zero eigenvalue in respect of Proposition 2.5.

**Lemma 3.2.** (Minimal first non-zero eigenvalue) For \( N < -1 \), the first non-zero eigenvalue of \(-\Delta_m\) in the space \( M_1 \) in Example 3.1 coincides with \( KN/(N-1) \).
Proof. We shall show that the function
\[ u(x) = \sinh \left( \sqrt{\frac{K}{1-N}} x \right) \]
gives the minimal first non-zero eigenvalue. Notice that \( u \in L^2(M_1) \) by the assumption \( N < -1 \), and
\[ u'(x) = \sqrt{\frac{K}{1-N}} \cosh \left( \sqrt{\frac{K}{1-N}} x \right), \quad u''(x) = \frac{K}{1-N} u. \]
Thus the weighted Laplacian of \( u \) is calculated as
\[
\Delta_m u(x) = u''(x) - u'(x)\psi'(x) = \frac{K}{1-N} u(x) - K \sinh \left( \sqrt{\frac{K}{1-N}} x \right)
\]
\[
= \frac{K}{1-N} u(x) - K u(x) = -\frac{KN}{N-1} u(x). \quad \Box
\]

Remark 3.3. (Failure of log-Sobolev inequality) For \( N \in [n, \infty) \), the bound \( \text{Ric}_N \geq K > 0 \) is known to imply the logarithmic Sobolev inequality (see, for example, [BGL] for proofs based on the \( \Gamma \)-calculus):
\[
\int_{\{f > 0\}} f \log f \, dm \leq \frac{N-1}{2KN} \int_{\{f > 0\}} \frac{|\nabla f|^2}{f} \, dm \tag{2}
\]
for non-negative functions \( f \in H^1(M) \) with \( \int_M f \, dm = 1 \). It is well known that the logarithmic Sobolev inequality implies the normal concentration, see [Le, Theorem 5.3]. One might expect that the logarithmic Sobolev inequality (2) has a counterpart for \( N < 0 \) similarly to the spectral gap. This is, however, not the case since \( M_1 \) (normalized as \( m(M_1) = 1 \)) given in Example 3.1 enjoys only the exponential concentration (consider \( A = [0, \infty) \)). See [Mi, Proposition 6.4] for a detailed estimate of the concentration function.

Our second example is as follows.

Example 3.4. For \( K > 0 \) and \( N < 0 \), the space
\[
M_2 := (\mathbb{R}, | \cdot |, e^{-\sqrt{K(1-N)}x} \, dx)
\]
satisfies \( \text{Ric}_N \equiv K \).

This space also appeared in Case 2 of [Mi, Corollary 1.4]. The weight function is the linear function \( \psi(x) = \sqrt{K(1-N)}x \) and the curvature identity is straightforward:
\[
\text{Ric}_N \left( \frac{\partial}{\partial x} \bigg|_x \right) = \frac{\psi'(x)^2}{1-N} = K.
\]

We stress that \( M_2 \) has very different natures from \( M_1 \) in Example 3.1. Recall that, when \( N = \infty \), \( \text{Ric}_\infty \geq K > 0 \) implies that the measure has Gaussian decay and the total volume is finite [St1, Theorem 4.26]. From Example 3.1 one may expect that \( \text{Ric}_N \geq K > 0 \) for \( N < 0 \) still implies the exponential decay; however, it is not the case by the second example \( M_2 \) above. We refer to [Sa2] for a related work on weighted Riemannian manifolds.
with boundaries, where a volume comparison for regions spreading from the boundary was established for \( N \leq 1 \).

We also remark that, different from the case of \( N = \infty \), taking products will destroy the curvature bound for \( N < 0 \) due to the nonlinearity of the term \( \langle \nabla \psi, v \rangle^2 \) in \( \psi \) in the definition of \( \text{Ric}_N(v) \). For instance,

\[
(\mathbb{R}^n, | \cdot |, e^{-x_1^2-x_2^2-\cdots-x_n^2} \, dx_1 \, dx_2 \cdots \, dx_n)
\]
satisfies only \( \text{Ric}_N \geq 0 \) (consider \( v \) in the kernel of \( d\psi \)).

### 4. First non-zero eigenvalue of the Laplacian

As we saw in Proposition 2.5, the first non-zero eigenvalue of \( -\Delta_m \) on a weighted Riemannian manifold satisfying \( \text{Ric}_N \geq K > 0 \) is bounded from below by \( KN/(N-1) \) (or \( K \) if \( N = \infty \)), which is sharp for \( N \leq -1 \). In this section we consider the rigidity problem when the minimum value is attained. In the unweighted (\( N = n \)) case, the classical Obata theorem \([Ob]\) asserts the following.

**Theorem 4.1. (Obata rigidity theorem)** Let \((M, g)\) be an \( n \)-dimensional Riemannian manifold with \( \text{Ric}_g \geq K > 0 \). Then the first non-zero eigenvalue of \( -\Delta \) coincides with \( K n/(n-1) \) if and only if \( M \) is isometric to the \( n \)-dimensional sphere \( \mathbb{S}^n \) with radius \( \sqrt{(n-1)/K} \).

The case of \( N \in (n, \infty) \) turns out void as follows.

**Theorem 4.2. (Ketterer, Kuwada)** Let \((M, g, m)\) be a weighted Riemannian manifold of dimension \( n \) with \( \text{Ric}_N \geq K > 0 \) for some \( N \in [n, \infty) \). Then the first non-zero eigenvalue of \( -\Delta_m \) coincides with \( KN/(N-1) \) if and only if \( N = n, m = c \text{ vol}_g \) for some constant \( c > 0 \) and \( M \) is isometric to the \( n \)-dimensional sphere \( \mathbb{S}^n \) with radius \( \sqrt{(n-1)/K} \).

Precisely, it is shown in \([Ke]\) in the general framework of RCD-spaces (metric measure spaces satisfying the Riemannian curvature-dimension condition) that the existence of an eigenfunction for the eigenvalue \( KN/(N-1) \) implies the maximal diameter \( \text{diam}(M) = \pi \sqrt{(n-1)/K} \) in the Bonnet–Myers type theorem. This is, however, achieved only when \( N = n \) by Kuwada’s result in \([Ku]\). If we admit singularities, then the sharp spectral gap is attained for spherical suspensions as discussed in \([Ke]\) for RCD-spaces (see also \([KM, \S 3.2]\)).

Next, in the case of \( N = \infty \), an interesting rigidity theorem was established in \([CZ]\).

**Theorem 4.3. (Cheng–Zhou)** Let \((M, g, m)\) be an \( n \)-dimensional weighted Riemannian manifold with \( \text{Ric}_\infty \geq K > 0 \). If the first non-zero eigenvalue of \( -\Delta_m \) coincides with \( K \), then \( M \) is isometric to the product space \( \Sigma^{n-1} \times \mathbb{R} \) as weighted Riemannian manifolds, where \( \Sigma^{n-1} \) is an \((n-1)\)-dimensional manifold with \( \text{Ric}_\infty \geq K \) and \( \mathbb{R} \) is equipped with the Gaussian measure \( e^{-Kx^2/2} \, dx \).

The proof of Theorem 4.3 has a certain similarity with the Cheeger–Gromoll splitting theorem. The role of the Busemann function in \([CG]\) is replaced with the eigenfunction in \([CZ]\). Theorem 4.3 was recently generalized to RCD-spaces in \([GKKO]\).

Now we consider the case of \( N < 0 \). We begin with an important equation for an eigenfunction.
LEMMA 4.4. Let \((M, g, m)\) be a complete weighted Riemannian manifold satisfying \(\text{Ric}_N \geq K\) for some \(N < 0\) and \(K > 0\), and \(m(M) < \infty\). Suppose that the first non-zero eigenvalue of \(-\Delta_m\) coincides with \(KN/(N - 1)\). Then the eigenfunction \(u\) of \(KN/(N - 1)\) necessarily satisfies
\[
\text{Hess } u = -\frac{Ku}{N - 1} g
\]
as operators \(TM \times TM \rightarrow \mathbb{R}\).

Proof. Let \(u\) be an eigenfunction of the eigenvalue \(KN/(N - 1)\), namely
\[
\Delta_m u = -\frac{KN}{N - 1} u.
\]
Tracing back to the proof of the lower bound of the first non-zero eigenvalue, the Bochner inequality for \(u\) necessarily becomes an equality. Thus we find, from the proof of Theorem 2.4, that
\[
\text{Hess } u = f \cdot g
\]
for some function \(f : M \rightarrow \mathbb{R}\); \(n\)
\[
\frac{\Delta_m u}{N} + \frac{\langle \nabla u, \nabla \psi \rangle}{N - n} \equiv 0.
\]
On the one hand, we observe from the former equation (4) that \(\Delta u = nf\). On the other hand, the latter equation (5) yields
\[
\Delta u = \Delta_m u + \langle \nabla u, \nabla \psi \rangle = \frac{n}{N} \Delta_m u = -\frac{Kn}{N - 1} u.
\]
Therefore we have \(f = -Ku/(N - 1)\) and complete the proof.

THEOREM 4.5. Let \((M, g, m)\) be a complete weighted Riemannian manifold of dimension \(n\) satisfying \(\text{Ric}_N \geq K\) for some \(N < -1\) and \(K > 0\), and \(m(M) < \infty\). Assume that the first non-zero eigenvalue of \(-\Delta_m\) coincides with \(KN/(N - 1)\).

(i) If \(n \geq 2\), then \(M\) is isometric to the warped product
\[
\mathbb{R} \times_{\cosh(\sqrt{K/(1-N)t})} \Sigma = \left( \mathbb{R} \times \Sigma, dt^2 + \cosh^2 \left( \frac{K}{1 - N} t \right) \cdot g_\Sigma \right)
\]
and the measure \(m\) is written through the isometry as
\[
m(dt \, dx) = \cosh^{N-1} \left( \frac{K}{1 - N} t \right) dt \, m_\Sigma(dx),
\]
where \((\Sigma, g_\Sigma, m_\Sigma)\) is an \((n - 1)\)-dimensional weighted Riemannian manifold satisfying \(\text{Ric}_{N-1} \geq K(2 - N)/(1 - N)\).

(ii) If \(n = 1\), then \((M, g, m)\) is isometric to the model space \(M_1\) in Example 3.1 up to a constant multiplication of the measure.

Proof. (i) By rescaling the Riemannian metric, we can assume that \(K = 1 - N\). Let \(u \in H^1_0 \cap C^\infty(M)\) be an eigenfunction satisfying \(\Delta_m u = Nu\). Along any geodesic \(\gamma : I \rightarrow M\) for an interval \(I \subset \mathbb{R}\), it follows from Lemma 4.4 that \((u \circ \gamma)^\prime = (u \circ \gamma)\). This implies that
\[
u \circ \gamma(t) = u \circ \gamma(0) \cdot \cosh t + (u \circ \gamma)'(0) \cdot \sinh t.
\]
Put $\Sigma := u^{-1}(0)$ (which is non-empty since $\int_M u \, dm = 0$). We deduce from (6) that, for $x, y \in \Sigma$, every geodesic connecting them is included in $\Sigma$. Hence $\Sigma$ is totally geodesic and the mean curvature of $\Sigma$ is identically zero. Lemma 4.4 also shows that, for any smooth vector field $V$ along a geodesic $\gamma$ included in $\Sigma$,

$$\langle \nabla_{\dot{\gamma}} (\nabla u \circ \gamma), V \rangle = \text{Hess} \, u(\dot{\gamma}, V) = u \cdot \langle \dot{\gamma}, V \rangle \equiv 0.$$ 

Hence $\nabla u \circ \gamma$ is a parallel vector field and $|\nabla u|$ is constant on $\Sigma$. We will normalize $u$ so as to satisfy $|\nabla u| \equiv 1$ on $\Sigma$.

Fix $x \in \Sigma$ and consider the geodesic $\gamma_x : [0, \infty) \rightarrow M$ such that $\dot{\gamma}_x(0) = \nabla u(x)$. Then (6) and our normalization $|\nabla u|(x) = 1$ yield

$$(u \circ \gamma_x)(t) = \sinh t, \quad t \in \mathbb{R}. \quad (7)$$

For any unit speed geodesic $\eta : [0, s) \rightarrow M$ from $x$ to $\gamma_x(t)$, (7) and (6) imply $\sinh t = u(\eta(s)) \leq \sinh s$. Since $\sinh t$ is monotone increasing, we have $s \geq t$ and $\gamma_x$ is globally minimizing. Denoting the distance function from $\Sigma$ by $\rho_\Sigma$, we obtain from (7) that $\rho_\Sigma = |\arcsinh(u)|$ and, on $u^{-1}((0, \infty))$,

$$\Delta u = \Delta(\sinh \rho_\Sigma) = \cosh \rho_\Sigma \cdot \Delta \rho_\Sigma + \sinh \rho_\Sigma \cdot |\nabla \rho_\Sigma|^2.$$ 

Since $|\nabla \rho_\Sigma| = 1$ and $\Delta u = nu = n \sinh \rho_\Sigma$, we have for $t \geq 0$

$$\Delta \rho_\Sigma(\gamma_x(t)) = (n - 1) \tanh \rho_\Sigma(\gamma_x(t)) = (n - 1) \tanh t. \quad (8)$$

We shall compare the Laplacian comparison inequality (8) with the (unweighted) Ricci curvature along $\gamma_x$. Using (7) and (5), we have

$$\langle \psi \circ \gamma_x, \dot{\gamma}_x \rangle(t) = \langle \nabla \psi \circ \gamma_x, \dot{\gamma}_x \rangle(t) = \frac{1}{\cosh t} \langle \nabla \psi, \nabla u \rangle(\gamma_x(t))$$

$$= \frac{n - N}{\cosh t} u(\gamma_x(t)) = (n - N) \tanh t.$$

Hence

$$\psi \circ \gamma_x(t) = (n - N) \log(\cosh t) + \psi(x) \quad (9)$$

and

$$\text{Ric}(\dot{\gamma}_x(t)) = \text{Ric}_N(\dot{\gamma}_x(t)) - (\psi \circ \gamma_x)''(t) - \frac{\langle \psi \circ \gamma_x \rangle'(t)^2}{n - N}$$

$$= \text{Ric}_N(\dot{\gamma}_x(t)) - \frac{n - N}{\cosh^2 t} - (n - N) \tanh^2 t$$

$$= \text{Ric}_N(\dot{\gamma}_x(t)) - (n - N) \geq 1 - n.$$ 

This curvature bound together with the minimality of $\Sigma$ implies that

$$\Delta \rho_\Sigma(\gamma_x(t)) \leq (n - 1) \tanh t \quad (10)$$

by [Ka, Corollary 2.44] (see also [HK] and [Sa1, Theorem 4.3]). Then, since we have equality (8), the rigidity theorem for the Laplacian comparison shows that $u^{-1}((0, \infty))$ is
isometric to the warped product
\[ [0, \infty) \times_{\text{cosh}} \Sigma := ([0, \infty) \times \Sigma, dt^2 + \cosh^2 t \cdot g_{\Sigma}), \]
where \( g_{\Sigma} \) is the Riemannian metric of \( \Sigma \) induced from \( g \) of \( M \) (see the proof of [Sa1, Theorem 1.8] for details). Here we give a sketch of the proof for the isometry along [Sa1].

For a regular point \( x \in \Sigma \) of \( u \), let \( T_x^u \Sigma \) be the orthogonal complement of \( T_x \Sigma \). Choose an orthonormal basis \( \{ e_{x,i} \}_{i=1}^{n-1} \) of \( T_x \Sigma \) and for \( i = 1, \ldots, n-1 \), let \( Y_{x,i} \) be the \( \Sigma \)-Jacobi field along \( \gamma_x \) such that \( Y(0) = e_{x,i} \) and \( Y_{x,i}'(0) = -A_v x_{e_{x,i}} \), where \( A_v : T_x \Sigma \to T_x \Sigma \) is the shape operator of the tangent vector \( v \). Since the mean curvature on \( \Sigma \) is 0 and \( \text{Ric}_N \left( \gamma_x(t) \right) \geq 1 - n \), we have \( \Delta \rho_{\Sigma}(\gamma_x(t)) \leq (n - 1) \tanh t \) as in (10) and equality holds if and only if \( Y_{x,i}(t) = \cosh(t) E_{x,i}(t) \) where \( E_{x,i} \) are the parallel vector fields along \( \gamma_x \) with \( E_{x,i}(0) = e_{x,i} \). Then the map \( \phi : [0, \infty) \times \Sigma \to M \) defined by \( \phi(t, x) := \gamma_x(t) \) provides the desired isometry between \( [0, \infty) \times_{\text{cosh}} \Sigma \) and \( M \).

By a similar discussion on \( u^{-1}((-\infty, 0]) \) we can extend the isometry to
\[ M \cong \mathbb{R} \times_{\text{cosh}} \Sigma = (\mathbb{R} \times \Sigma, dt^2 + \cosh^2 t \cdot g_{\Sigma}). \]

Concerning the splitting of the measure, let \( \Sigma \) be equipped with the measure \( m_{\Sigma} := e^{-\psi} \text{vol}_{g_{\Sigma}} \). By (9) and the structure of the warped product, we see that the measure \( m \) is written through the isometry as
\[ m(dt \, dx) = \cosh^{N-n} t \cdot e^{-\psi(x)} \cdot (\cosh^{n-1} t \, dt \, \text{vol}_{g_{\Sigma}}(dx)) \]
\[ = \cosh^{n-1} t \, dt \, m_{\Sigma}(dx). \]

With this expression one can see that the eigenfunction \( u \) is in \( L^2(M) \) when \( N < -1 \) (but not in \( L^2(M) \) for \( N \in [-1, 0) \)).

Finally, for any unit vector \( v \in T_x \Sigma \) at \( x \in \Sigma \), the sectional curvature of the plane \( \gamma_x(0) \wedge v \) coincides with \(-1\) (see [ON, Proposition 42(2) in §7]). Therefore we have, on the weighted Riemannian manifold \( (\Sigma, g_{\Sigma}, m_{\Sigma}) \),
\[ \text{Ric}_{N-1}^\Sigma(v) = \text{Ric}^E(v) + \text{Hess}^\Sigma \psi(v, v) - \frac{\langle \nabla^\Sigma \psi, v \rangle^2}{(N-1) - (n-1)} \]
\[ = \text{Ric}^M(v) + 1 + \text{Hess}^M \psi(v, v) - \frac{\langle \nabla^M \psi, v \rangle^2}{N-n} \]
\[ = \text{Ric}^M_N(v) + 1 \geq 2 - N. \]

(ii) Fix an eigenfunction \( u \) of the eigenvalue \( KN/(N-1) \), take \( x \in M \) with \( u(x) = 0 \) and choose a unit speed geodesic \( \gamma : \mathbb{R} \to M \) with \( \gamma(0) = x \). It follows from Lemma 4.4 that
\[ u \circ \gamma(t) = (u \circ \gamma)'(0) \cdot \sinh \left( \sqrt{\frac{K}{1-N}} t \right). \]
Since \( u \) is not constant, we have \( (u \circ \gamma)'(0) \neq 0 \) and hence \( u \) is injective. Therefore \( M \) is isometric to \( \mathbb{R} \) and we will identify them via \( \gamma \) as \( \gamma(t) = t \).

Denote by \( \psi : \mathbb{R} \to \mathbb{R} \) the weight function, namely \( m = e^{-\psi} \, dt \). Then it follows from
\[ -\frac{KN}{N-1} u = \Delta_m u = u'' - u' \psi' \]
that

\[ \psi'(t) = \frac{1}{u'(t)} \left( u''(t) + \frac{KN}{N-1} u(t) \right) \]

\[ = \sqrt{\frac{1-N}{K}} \left( \frac{K}{1-N} + \frac{KN}{N-1} \right) \tanh \left( \sqrt{\frac{K}{1-N}} t \right) \]

\[ = \sqrt{K(1-N)} \tanh \left( \sqrt{\frac{K}{1-N}} t \right). \]

Integrating this we have

\[ \psi(t) = (1-N) \log \left( \cosh \left( \sqrt{\frac{K}{1-N}} t \right) \right) + \psi(0), \]

which implies \( m = e^{-\psi(0)} \cosh(\sqrt{K/(1-N)} t)^N \) as desired. \( \square \)

The following corollary is a byproduct of the proof above (see [KM, Mi] for the one-dimensional case).

**Corollary 4.6.** Let \((M, g, m)\) be a complete weighted Riemannian manifold of dimension \( n \) satisfying \( \text{Ric}_N \geq K \) for some \( N \in [-1, 0) \) and \( K > 0 \), and \( m(M) < \infty \). Then \( KN/(N-1) \) cannot be an eigenvalue of \( -\Delta_m \).

**Remark 4.7.** The curvature bound of \( \Sigma \) implies the estimate of the first non-zero eigenvalue \( \lambda_1(\Sigma) \) as

\[ \lambda_1(\Sigma) \geq \frac{K(2-N)}{1-N} \frac{N-1}{N-2} = K. \]

This is better than the estimate \( \lambda_1(M) \geq KN/(N-1) \) for \( M \).

We close the article with a concrete example of the splitting phenomenon described in Theorem 4.5(i) (with \( K = 1-N \)).

**Example 4.8.** Let \((\mathbb{H}^2, g)\) be the hyperbolic plane of constant sectional curvature \(-1\). There are smooth functions \( u \) and \( \psi \) satisfying

\[ \Delta_m u = Nu, \quad \text{Ric}_N = 1-N, \]

where \( m = e^{-\psi} \text{vol}_g \).

Let us give the precise expressions of \( u \) and \( \psi \) in the upper half-plane model:

\[ (\mathbb{H}^2, g) = \left( \mathbb{R} \times (0, \infty), \frac{dx^2 + dy^2}{y^2} \right). \]

By the consideration in the proof of Theorem 4.5(i), we can expect that the function \( u \) is written by using the distance function from the \( y \)-axis (\( \Sigma = u^{-1}(0) \) coincides with the
One can explicitly calculate it as

\[
u(x, y) = \pm \sinh \left( d \left( x, y, \left( 0, \sqrt{x^2 + y^2} \right) \right) \right) = \pm \sinh \left( \arccosh \left( 1 + \frac{x^2 + (y - \sqrt{x^2 + y^2})^2}{2y\sqrt{x^2 + y^2}} \right) \right) = \pm \sqrt{\frac{x^2 + y^2}{y^2} - 1} = \frac{x}{y},
\]

where we choose the sign + for \( x > 0 \) and – for \( x < 0 \), and we used the equation \( \sinh(\arccosh t) = \sqrt{t^2 - 1} \) for \( t \geq 0 \). The Christoffel symbols are readily calculated as

\[
\Gamma_{11}^1 = \Gamma_{12}^2 = \Gamma_{12}^1 = 0, \quad \Gamma_{12}^1(x, y) = \Gamma_{22}^2(x, y) = -\frac{1}{y}, \quad \Gamma_{21}^2(x, y) = \frac{1}{y}.
\]

Using these we find

\[
\text{Hess } u(x, y) = \begin{pmatrix}
0 & -y^2 \frac{1}{2xy} \\
-y^2 \frac{1}{2xy} & -\frac{1}{y} \end{pmatrix} - \frac{1}{y} \begin{pmatrix}
0 & -y^{-1} \\
-y^{-1} & 0 \end{pmatrix} + \frac{x}{y^2} \begin{pmatrix}
y^{-1} & 0 \\
0 & -y^{-1} \end{pmatrix} = \frac{x}{y} \cdot \frac{1}{y^2} \begin{pmatrix}
1 & 0 \\
0 & 1 \end{pmatrix} = u(x, y) \cdot g(x, y).
\]

Next we consider the weight function. Let

\[
\psi_1(x, y) := (2 - N) \log \left( \cosh \left( d \left( x, y, \left( 0, \sqrt{x^2 + y^2} \right) \right) \right) \right) = (2 - N) \log \left( \frac{\sqrt{x^2 + y^2}}{y} \right),
\]

and

\[
\psi_2(x, y) := -(2 - N) \log \left( \sqrt{x^2 + y^2} \right).
\]

Notice that \( \psi_1 \) naturally appears in respect of (5), and \( \psi_2 \) is employed to improve the convexity in the directions perpendicular to \( \nabla u \). Indeed, we have \( \langle \nabla u, \nabla \psi_2 \rangle = 0 \) and

\[
\langle \nabla u, \nabla \psi_1 \rangle(x, y) = (2 - N) y^2 \left( \frac{1}{y} \cdot \frac{x}{x^2 + y^2} - \frac{x}{y^2} \cdot \left( \frac{y}{x^2 + y^2} - \frac{1}{y} \right) \right) = (2 - N) \frac{x}{y} = (2 - N) u(x, y).
\]

Our weighted function will be \( \psi = \psi_1 + \psi_2 \), namely

\[
\psi(x, y) = -(2 - N) \log y.
\]

The above calculations yield

\[
\Delta_m u = \Delta u - \langle \nabla u, \nabla \psi \rangle = 2u - (2 - N)u = Nu.
\]

We finally calculate \( \text{Ric}_N \) of this example. To this end, we observe that

\[
\frac{\text{Hess } \psi}{2 - N} = \begin{pmatrix}
y^{-2} & 0 \\
0 & 0 \end{pmatrix}, \quad \frac{d\psi \otimes d\psi}{(2 - N)^2} = \begin{pmatrix}
0 & 0 \\
0 & y^{-2} \end{pmatrix}.
\]
Therefore we obtain
\[
(Ric_N)_{(x,y)} = -\frac{1}{y^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (\text{Hess } \psi)_{(x,y)} + \frac{(d\psi \otimes d\psi)_{(x,y)}}{2 - N}.
\]

\[
= \frac{1 - N}{y^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

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