On explicit free field realizations of current algebras

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Abstract

We construct the explicit free field representations of the current algebras $so(2n)_k$, $so(2n+1)_k$ and $sp(2n)_k$ for a generic positive integer $n$ and an arbitrary level $k$. The corresponding energy-momentum tensors and screening currents of the first kind are also given in terms of free fields.

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1 Introduction

Conformal field theories (CFTs) [1, 2] have played a fundamental role in the framework of string theory and the theories of modern condensed matter physics and statistical physics at critical point. The Wess-Zumino-Novikov-Witten (WZNW) models [2], whose symmetry algebras are current algebras [3], stand out as an important class of CFTs because these models are “building blocks” of all rational CFTs through the so-called GKO coset construction [4]. The Wakimoto free field realizations of current algebras [5, 2] have been proven to be powerful in the study of the WZNW models [6, 7, 8, 9, 10] due to the fact that an explicit free field representation enables one to construct integral representations of correlators of the CFT.

Free field realizations of current algebras have been extensively investigated by many authors [11, 12, 13, 14, 15, 16, 17, 18]. However, it is very complicated to apply the general procedure proposed in these references to derive explicit free field expressions of the affine currents for higher-rank algebras [14, 18, 19, 20, 21]. To our knowledge, explicit expressions have so far been known only for some isolated cases: those associated with Lie algebras $su(n)$ [11], $B_2$ (or $so(5)$) [12], $G_2$ [14], and Lie superalgebra $gl(m|n)$ [22]. In particular, explicit free field expressions for affine currents associated with Lie algebras $so(2n)$, $so(2n + 1)$ and $sp(2n)$ for generic $n$ are still lacking.

In this paper, we find a way to overcome the complication in the above-mentioned general method. In our approach, the construction of the differential operator realization becomes much simpler (c.f. [14, 18, 19]). We shall work out the explicit forms of the differential realizations of $so(2n)$, $so(2n + 1)$ and $sp(2n)$, and apply them to construct explicit free field representations of the corresponding current algebras. These representations provide the Verma modules of the algebras.

This paper is organized as follows. In section 2, we briefly review the definitions of finite-dimensional Lie algebras and the associated current algebras, which also introduces our notation and some basic ingredients. In section 3, after constructing explicitly the differential operator realization of $so(2n)$, we construct the explicit free field representations of the $so(2n)$ currents, the corresponding energy-momentum tensor and the associated screening currents at a generic level $k$. In sections 4 and 5, we present the corresponding results for the $so(2n+1)$

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1The authors in [23] proposed certain explicit free field expressions for the $so(2n)$, $so(2n + 1)$ and $sp(2n)$ current algebras, but one can check that their results are incorrect, as was also pointed out in [14].
2 Notation and preliminaries

Let \( g \) be a simple Lie algebra with a finite dimension \( \dim(g) = d < \infty \) and \( \{E_i | i = 1, \ldots, d\} \) be a basis of \( g \). The generators \( \{E_i\} \) satisfy commutation relations,

\[
[E_i, E_j] = \sum_{m=1}^{d} f_{ij}^m E_m,
\]

(2.1)

where \( f_{ij}^m \) are the structure constants of \( g \). Alternatively, one can use the associated root system \([24]\) to label the generators of \( g \) as follows. Let us assume the rank of \( g \) to be \( \text{rank}(g) = n \) with a generic positive integer \( n \geq 1 \), and \( h \) be a Cartan subalgebra of \( g \). The set of (positive) roots is denoted by \( (\Delta_+)^{\Delta} \), and we write \( \alpha > 0 \) if \( \alpha \in \Delta_+ \). Among the positive roots, the simple roots are \( \{\alpha_i | i = 1, \ldots, n\} \). Associated with each positive root \( \alpha \), there are a raising operator \( E_\alpha \), a lowering operator \( F_\alpha \) and a Cartan generator \( H_\alpha \). Then one has the Cartan-Weyl decomposition of \( g \)

\[
g = g_- \oplus h \oplus g_+.
\]

(2.2)

One can introduce a nondegenerate and invariant symmetric metric or bilinear form \((E_i, E_j)\) for \( g \). For \( so(2n) \), \( so(2n + 1) \) and \( sp(2n) \), which we consider in this paper, the corresponding bilinear forms are given in \((A.7)\), \((B.10)\) and \((C.9)\) respectively. Then the current algebra \( g_k \) is generated by the currents \( E_i(z) \) associated with the generators \( E_i \) of \( g \). The current algebra at a general level \( k \) obeys the following OPEs \([2]\),

\[
E_i(z) E_j(w) = \frac{k (E_i, E_j)}{(z - w)^2} + \sum_{m=1}^{d} f_{ij}^m E_m(w) \left( \frac{1}{z - w} + \frac{1}{w - w} \right), \quad i, j = 1, \ldots, d,
\]

(2.3)

where \( f_{ij}^m \) are the structure constants \((2.1)\). The aim of this paper is to construct explicit free field realizations of the current algebras associated with \( so(2n) \), \( so(2n + 1) \) and \( sp(2n) \) at a generic level \( k \).

3 Free field realization of the \( so(2n) \) currents

As mentioned in the introduction, practically it would be very involved (if not impossible) to obtain explicit free field realizations of current algebras associated with higher-rank algebras.
by the general method outlined in [11, 12, 14, 16, 18]. We have found a way to overcome the complication. In our approach, the construction of differential operator realizations becomes much simpler, giving rise to explicit expressions of differential operators (see (3.11)-(3.15) for \(so(2n)\), (4.9)-(4.13) for \(so(2n+1)\) and (5.8)-(5.12) for \(sp(2n)\) below). In this section, we consider the \(so(2n)\) current algebra for generic \(n\) and arbitrary level \(k\).

### 3.1 Differential operator realization of \(so(2n)\)

The root system of \(D_n\) (or \(so(2n)\)) are:

\[\pm \epsilon_i \pm \epsilon_j, \text{ for } i \neq j \text{ and } i, j = 1, \ldots, n.\]

Among them, the positive roots \(\Delta^+\) can be chosen as:

\[\epsilon_i \pm \epsilon_j \text{ for } 1 \leq i < j \leq n.\]

The simple roots are

\[
\alpha_1 = \epsilon_1 - \epsilon_2, \quad \alpha_2 = \epsilon_2 - \epsilon_3, \ldots, \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n, \quad \alpha_n = \epsilon_{n-1} + \epsilon_n. \tag{3.1}
\]

Hereafter, we adopt the convention that

\[
E_i \equiv E_{\alpha_i}, \quad F_i \equiv F_{\alpha_i}, \quad i = 1, \ldots, n. \tag{3.2}
\]

The matrix realization of the generators associated with all roots of \(so(2n)\) is given in Appendix A, from which one may derive the structure constants for the particular choice of the basis.

Let us introduce a coordinate \(x_{i,j}\) associated with each positive root \(\epsilon_i - \epsilon_j\) (\(i < j\)) and a coordinate \(\bar{x}_{i,j}\) associated with each positive root \(\epsilon_i + \epsilon_j\) (\(i < j\)) respectively. These \(n \times (n-1)\) coordinates satisfy the following commutation relations:

\[
[x_{i,j}, x_{m,l}] = 0, \quad [\partial x_{i,j}, \partial x_{m,l}] = 0, \quad [\partial x_{i,j}, x_{m,l}] = \delta_{im} \delta_{jl}, \tag{3.3}
\]

\[
[\bar{x}_{i,j}, \bar{x}_{m,l}] = 0, \quad [\partial \bar{x}_{i,j}, \partial \bar{x}_{m,l}] = 0, \quad [\partial \bar{x}_{i,j}, \bar{x}_{m,l}] = \delta_{im} \delta_{jl}, \tag{3.4}
\]

and the other commutation relations are vanishing. Let \(\langle \Lambda \rangle\) be the highest weight vector of the representation of \(so(2n)\) with highest weights \(\{\lambda_i\}\), satisfying the following conditions:

\[
\langle \Lambda \rangle | F_i = 0, \quad 1 \leq i \leq n, \tag{3.5}
\]

\[
\langle \Lambda \rangle | H_i = \lambda_i \langle \Lambda \rangle, \quad 1 \leq i \leq n. \tag{3.6}
\]

Here the generators \(H_i\) are expressed in terms of some linear combinations of \(H_{\alpha}\) \((A.5)\).

An arbitrary vector in the corresponding Verma module \(^2\) is parametrized by \(\langle \Lambda \rangle\) and the

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\(^2\)The irreducible highest weight representation can be obtained from the Verma module through the cohomology procedure \([12]\) with the help of screening operators (e.g. (3.28) and (3.29) below).
coordinates \((x, \bar{x})\) as
\[
\langle \Lambda, x, \bar{x} \rangle = \langle \Lambda | G_+(x, \bar{x}) \rangle, \tag{3.7}
\]
where \(G_+(x, \bar{x})\) is given by (c.f. [14, 18])
\[
G_+(x, \bar{x}) = (\bar{G}_{n-1,n} G_{n-1,n}) (\bar{G}_{n-2,n-1} G_{n-2,n} G_{n-2,n} G_{n-2,n-1}) \ldots
\times (G_{1,2} \ldots G_{1,n} G_{1,n} \ldots G_{1,2}). \tag{3.8}
\]
Here, for \(i < j\), \(G_{i,j}\) and \(\bar{G}_{i,j}\) are given by
\[
G_{i,j} = e^{x_{i,j} E_{\lambda_i - \lambda_j}}, \quad \bar{G}_{i,j} = e^{\bar{x}_{i,j} E_{\bar{\lambda}_{i+1}}}. \tag{3.9}
\]
One can define a differential operator realization \(\rho^{(d)}\) of the generators of \(so(2n)\) by
\[
\rho^{(d)}(g) \langle \Lambda, x, \bar{x} \rangle \equiv \langle \Lambda, x, \bar{x} | g \rangle, \quad \forall g \in so(2n). \tag{3.10}
\]
Here \(\rho^{(d)}(g)\) is a differential operator of the coordinates \(\{x_{i,j}, \bar{x}_{i,j}\}\) associated with the generator \(g\), which can be obtained from the defining relation (3.10). The defining relation also assures that the differential operator realization is actually a representation of \(so(2n)\). Therefore it is sufficient to give the differential operators related to the simple roots, as the others can be constructed through the simple ones by the commutation relations. Using the relation (3.10) and the Baker-Campbell-Hausdorff formula, after some algebraic manipulations, we obtain the following differential operator representation of the simple generators:
\[
\rho^{(d)}(E_i) = \sum_{m=1}^{i-1} (x_{m,i} \partial_{x_m,i+1} - \bar{x}_{m,i+1} \partial_{\bar{x}_m,i}) + \partial_{x_{i,i+1}}, \quad 1 \leq i \leq n - 1, \tag{3.11}
\]
\[
\rho^{(d)}(E_n) = \sum_{m=1}^{n-2} (x_{m,n-1} \partial_{x_m,n} - x_{m,n} \partial_{x_{m,n-1}}) + \partial_{x_{n-1,n}}, \tag{3.12}
\]
\[
\rho^{(d)}(F_i) = \sum_{m=1}^{i-1} (x_{m,i+1} \partial_{x_m,i} - \bar{x}_{m,i} \partial_{\bar{x}_m,i+1})
- \sum_{m=i+2}^{n} (x_{i,m} \partial_{x_{i+1,m}} - x_{i,m} \partial_{\bar{x}_{i+1,m}} + \bar{x}_{i,m} \partial_{\bar{x}_{i+1,m}} - \bar{x}_{i,m} \partial_{\bar{x}_{i+1,m}} - x_{i+1}^{2} \partial_{x_{i+1,i+1}}
- x_{i,i+1} \left[ \sum_{m=i+2}^{n} (x_{i,m} \partial_{x_{i,m}} + \bar{x}_{i,m} \partial_{\bar{x}_{i,m}} - x_{i+1,m} \partial_{x_{i+1,m}} - \bar{x}_{i+1,m} \partial_{\bar{x}_{i+1,m}}) \right]
+ x_{i,i+1}(\lambda_i - \lambda_{i+1}), \quad 1 \leq i \leq n - 1, \tag{3.13}
\]
A direct computation shows that these differential operators (3.11)-(3.15) satisfy the $so(2n)$ commutation relations corresponding to the simple roots and the associated Serre relations. This implies that the differential representation of non-simple generators can be consistently constructed from the simple ones. Hence, we have obtained an explicit differential realization of $so(2n)$.

### 3.2 Free field realization of $so(2n)_k$

With the help of the differential realization given by (3.11)-(3.15) we can construct the explicit free field representation of the $so(2n)$ current algebra at arbitrary level $k$ in terms of $n \times (n - 1)$ bosonic $\beta$-$\gamma$ pairs $\{(\beta_{i,j}, \gamma_{i,j}), (\bar{\beta}_{i,j}, \bar{\gamma}_{i,j}), 1 \leq i < j \leq n\}$ and $n$ free scalar fields $\phi_{i}, i = 1, \ldots, n$. These free fields obey the following OPEs:

$$
\beta_{i,j}(z) \gamma_{m,l}(w) = -\gamma_{m,l}(z) \beta_{i,j}(w) = \frac{\delta_{im} \delta_{jl}}{(z - w)}, \quad 1 \leq i < j \leq n, \quad 1 \leq m < l \leq n,
$$

(3.16)

$$
\bar{\beta}_{i,j}(z) \bar{\gamma}_{m,l}(w) = -\bar{\gamma}_{m,l}(z) \bar{\beta}_{i,j}(w) = \frac{\delta_{im} \delta_{jl}}{(z - w)}, \quad 1 \leq i < j \leq n, \quad 1 \leq m < l \leq n,
$$

(3.17)

$$
\phi_{i}(z) \phi_{j}(w) = \delta_{ij} \ln(z - w), \quad 1 \leq i, j \leq n,
$$

(3.18)

and the other OPEs are trivial.

The free field realization of the $so(2n)$ current algebra is obtained by the substitution in the differential realization (3.11)-(3.15) of $so(2n)$,

$$
x_{i,j} \rightarrow \gamma_{i,j}(z), \quad \partial_{x_{i,j}} \rightarrow \beta_{i,j}(z), \quad 1 \leq i < j \leq n,
$$

(3.19)

$$
\bar{x}_{i,j} \rightarrow \bar{\gamma}_{i,j}(z), \quad \partial_{\bar{x}_{i,j}} \rightarrow \bar{\beta}_{i,j}(z), \quad 1 \leq i < j \leq n,
$$

(3.20)

$$
\lambda_{j} \rightarrow \sqrt{k + 2(n - 1)} \partial \phi_{j}(z), \quad 1 \leq j \leq n.
$$

(3.21)

Moreover, in order that the resulting free field realization satisfy the desirable OPE for $so(2n)$ currents, one needs to add certain extra (anomalous) terms which are linear in $\partial \gamma(z)$ and $\partial \bar{\gamma}(z)$ in the expressions of the currents associated with negative roots (e.g. the last
term in the expressions of $F_i(z)$, see (3.22)-(3.23) below. Here we present the results for the currents associated with the simple roots,

$$
E_i(z) = \sum_{m=1}^{i-1} \left( \gamma_{m,i}(z) \beta_{m,i+1}(z) - \bar{\gamma}_{m,i+1}(z) \bar{\beta}_{m,i}(z) \right) + \beta_{i,i+1}(z), \quad 1 \leq i \leq n - 1, \quad (3.22)
$$

$$
E_n(z) = \sum_{i=m=1}^{n-2} \left( \gamma_{m,n-1}(z) \bar{\beta}_{m,n}(z) - \gamma_{m,n}(z) \bar{\beta}_{m,n-1}(z) \right) + \bar{\beta}_{n-1,n}(z);
$$

$$
F_i(z) = \sum_{m=1}^{i-1} \left( \gamma_{m,i}(z) \beta_{m,i+1}(z) - \bar{\gamma}_{m,i+1}(z) \bar{\beta}_{m,i}(z) \right) - \gamma_{i,i+1}^2(z) \beta_{i,i+1}(z)
- \sum_{m=i+2}^{n} \left( \gamma_{i,m}(z) \beta_{i+1,m}(z) - \bar{\gamma}_{i,m}(z) \bar{\beta}_{i+1,m}(z) \right)
- \gamma_{i,i+1}(z) \left[ \sum_{m=i+2}^{n} \gamma_{i,m}(z) \beta_{i,m}(z) + \bar{\gamma}_{i,m}(z) \bar{\beta}_{i,m}(z) \right]
+ \gamma_{i,i+1}(z) \left[ \sum_{m=i+2}^{n} \gamma_{i+1,m}(z) \beta_{i+1,m}(z) + \bar{\gamma}_{i+1,m}(z) \bar{\beta}_{i+1,m}(z) \right]
+ \sqrt{k + 2(n-1)} \gamma_{i,i+1}(z) (\partial \phi_i(z) - \partial \phi_{i+1}(z))
+ (k + 2(i-1)) \partial \gamma_{i,i+1}(z), \quad 1 \leq i \leq n - 1,
$$

$$
F_n(z) = \sum_{m=1}^{n-2} \left( \gamma_{m,n}(z) \beta_{m,n-1}(z) - \bar{\gamma}_{m,n-1}(z) \beta_{m,n}(z) \right) - \gamma_{n-1,n}^2(z) \bar{\beta}_{n-1,n}(z)
+ \sqrt{k + 2(n-1)} \gamma_{n-1,n}(z) (\partial \phi_{n-1}(z) + \partial \phi_n(z)) + (k + 2(n-2)) \partial \gamma_{n-1,n}(z),
$$

$$
H_i(z) = \sum_{m=1}^{i-1} \left( \gamma_{m,i}(z) \beta_{m,i}(z) - \bar{\gamma}_{m,i}(z) \bar{\beta}_{m,i}(z) \right) - \sum_{m=i+1}^{n} \left( \gamma_{i,m}(z) \beta_{i,m}(z) + \bar{\gamma}_{i,m}(z) \bar{\beta}_{i,m}(z) \right)
+ \sqrt{k + 2(n-1)} \partial \phi_i(z), \quad 1 \leq i \leq n. \quad (3.23)
$$

Here and throughout normal ordering of free fields is implied whenever necessary. The free field realization of the currents associated with the non-simple roots can be obtained from the OPEs of the simple ones. We can straightforwardly check that the above free field realization of the currents satisfies the OPEs of the $so(2n)$ current algebra: Direct calculation shows that there are at most second order singularities (e.g. the coefficients of $\frac{1}{(z-w)^2}$) in the OPEs of the currents. Comparing with the definition of the current algebra (2.3), terms with first order singularity (e.g. the coefficients of $\frac{1}{(z-w)}$) are fulfilled due to the very substitution (3.19)-(3.21) and the fact that the differential operator realizations (3.11)-(3.15) are a representation of the corresponding finite-dimensional Lie algebra $so(2n)$; terms with second order singularity
also match those in the definition (2.3) after the suitable choice we made for the anomalous terms in the expressions of the currents associated with negative roots.

The free field realization of the $so(2n)$ current algebra (3.22)-(3.23) gives rise to the Fock representations of the current algebra in terms of the free fields (3.16)-(3.18). These representations are in general not irreducible for the current algebra. In order to obtain irreducible ones, one needs certain screening charges, which are the integrals of screening currents (see (3.30)-(3.31) below), and performs the cohomology procedure as in [7, 11, 12, 13]. We shall construct the associated screening currents in subsection 4.

### 3.3 Energy-momentum tensor

In this subsection we construct the free field realization of the Sugawara energy-momentum tensor $T(z)$ of the $so(2n)$ current algebra. After a tedious calculation, we find

$$
T(z) = \frac{1}{2 (k + 2(n - 1))} \left\{ \sum_{i<j} \left( E_{\epsilon_i-\epsilon_j}(z) F_{\epsilon_i-\epsilon_j}(z) + F_{\epsilon_i-\epsilon_j}(z) E_{\epsilon_i-\epsilon_j}(z) \right) \\
+ \sum_{i<j} \left( E_{\epsilon_i+\epsilon_j}(z) F_{\epsilon_i+\epsilon_j}(z) + F_{\epsilon_i+\epsilon_j}(z) E_{\epsilon_i+\epsilon_j}(z) \right) + \sum_{i=1}^{n} H_i(z) H_i(z) \right\} \\
= \sum_{i=1}^{n} \left( \frac{1}{2} \partial \phi_i(z) \partial \phi_i(z) - \frac{n - i}{\sqrt{k + 2(n - 1)}} \partial^2 \phi_i(z) \right) \\
+ \sum_{i<j} \left( \beta_{i,j}(z) \partial \gamma_{i,j}(z) + \bar{\beta}_{i,j}(z) \partial \bar{\gamma}_{i,j}(z) \right). \tag{3.24}
$$

It is straightforward to check that $T(z)$ satisfy the following OPE,

$$
T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}. \tag{3.25}
$$

The corresponding central charge $c$ is

$$
c = \frac{kn(2n - 1)}{k + 2(n - 1)} \equiv \frac{k \dim(so(2n))}{k + 2(n - 1)}. \tag{3.26}
$$

Moreover, we find that with regard to the energy-momentum tensor $T(z)$ defined by (3.24) the $so(2n)$ currents associated with the simple roots (3.22)-(3.23) are indeed primary fields with conformal dimensional one, namely,

$$
T(z)E_i(w) = \frac{E_i(w)}{(z-w)^2} + \frac{\partial E_i(w)}{(z-w)}, \quad 1 \leq i \leq n,
$$
\[
T(z)F_i(w) = \frac{F_i(w)}{(z-w)^2} + \frac{\partial F_i(w)}{(z-w)}, \quad 1 \leq i \leq n,
\]
\[
T(z)H_i(w) = \frac{H_i(w)}{(z-w)^2} + \frac{\partial H_i(w)}{(z-w)}, \quad 1 \leq i \leq n.
\]

It is expected that the \(so(2n)\) currents associated with non-simple roots, which can be constructed through the simple ones, are also primary fields with conformal dimensional one. Therefore, \(T(z)\) is the very energy-momentum tensor of the \(so(2n)\) current algebra.

### 3.4 Screening currents

Important objects in the application of free field realizations to the computation of correlation functions of the CFTs are screening currents. A screening current is a primary field with conformal dimension one and has the property that the singular part of its OPE with the affine currents is a total derivative. These properties ensure that the integrated screening currents (screening charges) may be inserted into correlators while the conformal or affine Ward identities remain intact \([6, 8]\).

Free field realization of the screening currents may be constructed from certain differential operators \([12, 18]\) which can be defined by the relation,

\[
\rho^{(d)}(s_\alpha) \langle \Lambda, x, \bar{x} | \equiv \langle \Lambda | E_\alpha G_+(x, \bar{x}), \quad \text{for} \ \alpha \in \Delta_+.
\]  \hspace{1cm} (3.27)

The operators \(\rho^{(d)}(s_\alpha) \ (\alpha \in \Delta_+)\) give a differential operator realization of a subalgebra of \(so(2n)\), which is spanned by \(\{E_\alpha, \alpha \in \Delta_+\}\). Again it is sufficient to construct \(s_i \equiv \rho^{(d)}(s_{\alpha_i})\) related to the simple roots. Using (3.27) and the Baker-Campbell-Hausdorff formula, after some algebraic manipulations, we obtain the following explicit expressions for \(s_i\):

\[
s_i = \sum_{m=i+2}^n (\bar{x}_{i+1,m} \partial_{\bar{x}_{i,m}} - \bar{x}_{i+1,m} x_{i+1,m} \partial_{x_{i,i+1}} + x_{i+1,m} \partial_{x_{i,m}}) + \partial_{x_{i,i+1}},
\]  \hspace{1cm} 1 \leq i \leq n - 1,

\[
s_n = \partial_{x_{n-1,n}}.
\]  \hspace{1cm} (3.29)

One may obtain the differential operators \(s_\alpha\) associated with the non-simple generators from the above simple ones. Following the procedure similar to \([12, 18]\), we find that the free field realization of the screening currents \(S_i(z)\) corresponding to the differential operators \(s_i\) is given by

\[
S_i(z) = \sum_{m=i+2}^n \left( \bar{\gamma}_{i+1,m}(z)\bar{\beta}_{i,m}(z) - \bar{\gamma}_{i+1,m}(z)\gamma_{i+1,m}(z)\beta_{i,i+1}(z) + \gamma_{i+1,m}(z)\beta_{i,m}(z) \right)
\]
\[ + \beta_{i,i+1}(z) e^{-\frac{\alpha_i \phi(z)}{\sqrt{k+2(n-1)}}}, \quad 1 \leq i \leq n-1, \]  
(3.30)

\[ S_n(z) = \bar{\beta}_{n-1,n}(z) e^{-\frac{\alpha_{n-1} \phi(z)}{\sqrt{k+2(n-1)}}}. \]  
(3.31)

Here \( \phi(z) \) is

\[ \bar{\phi}(z) = \sum_{i=1}^{n} \phi_i(z) \epsilon_i. \]  
(3.32)

The OPEs of the screening currents with the energy-momentum tensor and the \( so(2n) \) currents (3.22)-(3.23) are

\[ T(z)S_j(w) = \frac{S_j(w)}{(z-w)^2} + \partial S_j(w) = \partial_w \left\{ \frac{S_j(w)}{(z-w)} \right\}, \quad j = 1, \ldots, n, \]  
(3.33)

\[ E_i(z)S_j(w) = 0, \quad i, j = 1 \ldots, n, \]  
(3.34)

\[ H_i(z)S_j(w) = 0, \quad i, j = 1 \ldots, n, \]  
(3.35)

\[ F_i(z)S_j(w) = \delta_{ij} \partial_w \left\{ \frac{(k + 2(n-1)) e^{-\frac{\alpha_i \phi(z)}{\sqrt{k+2(n-1)}}}}{(z-w)} \right\}, \quad i, j = 1, \ldots, n. \]  
(3.36)

The screening currents obtained this way are called screening currents of the first kind \[25\].

## 4 Results for \( so(2n+1) \)

### 4.1 Differential operator realization of \( so(2n+1) \)

The root system of \( B_n \) (or \( so(2n+1) \)) are: \( \{ \pm \epsilon_i \pm \epsilon_j | i \neq j, i, j = 1, \ldots, n \} \) and \( \{ \pm \epsilon_i | i = 1, \ldots, n \} \).

Among them, the positive roots \( \Delta_+ \) can be chosen as:

\[ \epsilon_i \pm \epsilon_j, \text{ for } 1 \leq i < j \leq n, \quad \text{and} \quad \epsilon_i, \text{ for } i = 1, \ldots, n. \]

The simple roots are

\[ \alpha_1 = \epsilon_1 - \epsilon_2, \quad \alpha_2 = \epsilon_2 - \epsilon_3, \ldots, \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n, \quad \alpha_n = \epsilon_n. \]  
(4.1)

Associated with each positive root \( \alpha \), there are a raising operator \( E_{\alpha} \), a lowering operator \( F_{\alpha} \) and a Cartan generator \( H_{\alpha} \). The matrix realization of the generators associated with all roots of \( so(2n+1) \) is given in Appendix B, from which one may derive the structure constants for the particular choice of the basis. Similar to the \( so(2n) \) case, we adopt the convention (3.2) for the raising/lowering generators associated with the simple roots.
In addition to the coordinates \( \{ x_{i,j}, \bar{x}_{i,j} \mid 1 \leq i < j \leq n \} \), which are associated with the positive roots \( \{ \epsilon_i \pm \epsilon_j \mid i < j \} \), we also need to introduce extra \( n \) coordinates \( \{ x_i \mid i = 1, \ldots, n \} \) associated with the positive roots \( \{ \epsilon_i \mid i = 1, \ldots, n \} \). The coordinates \( \{ x_{i,j}, \bar{x}_{i,j} \} \) and their differentials satisfy the same commutation relations as (3.3)-(3.4). The other non-trivial commutation relations are

\[
[x_i, x_j] = [\partial_{x_i}, \partial_{x_j}] = 0, \quad [\partial_{x_i}, x_j] = \delta_{ij}.
\] (4.2)

Let \( \langle \Lambda \mid \) be the highest weight vector of the highest weight representation of \( so(2n+1) \) satisfying the following conditions:

\[
\langle \Lambda \mid F_i = 0, \quad 1 \leq i \leq n, \tag{4.3}
\]

\[
\langle \Lambda \mid H_i = \lambda_i \langle \Lambda \mid, \quad 1 \leq i \leq n. \tag{4.4}
\]

Here the generators \( H_i \) are some linear combinations of \( H_\alpha \) (B.8). An arbitrary vector in the corresponding Verma module is parametrized by \( \langle \Lambda \mid \) and the coordinates \( (x, \bar{x}) \) as

\[
\langle \Lambda, x, \bar{x} \mid = \langle \Lambda | G_+ (x, \bar{x}), \tag{4.5}
\]

where \( G_+ (x, \bar{x}) \) is given by (c.f. [14, 18])

\[
G_+ (x, \bar{x}) = (G_n) (\bar{G}_{n-1,n} G_{n-1} G_{n-1,n}) (\bar{G}_{n-2,n-1} G_{n-2,n} G_{n-2,n} G_{n-2,n-1}) \ldots \times (\bar{G}_{1,2} \ldots G_{1,n} G_1 G_{1,n} \ldots G_{1,2}). \tag{4.6}
\]

Here \( G_{i,j} \) and \( \bar{G}_{i,j} \) for \( i < j \), and \( G_i \) are given by

\[
G_{i,j} = e^{x_{i,j} E_{\epsilon_i - \epsilon_j}}, \quad \bar{G}_{i,j} = e^{\bar{x}_{i,j} E_{\epsilon_i + \epsilon_j}}, \quad G_i = e^{x_i E_{\epsilon_i}}. \tag{4.7}
\]

Then one can define a differential operator realization \( \rho^{(d)} \) of the generators of \( so(2n+1) \) by

\[
\rho^{(d)} (g) \langle \Lambda, x, \bar{x} \mid = \langle \Lambda, x, \bar{x} \mid g, \quad \forall g \in so(2n+1). \tag{4.8}
\]

After tedious calculations analogous to those in the \( so(2n) \) case, we have found the differential realization of \( so(2n+1) \). Here we give the results for the generators associated with the simple roots,

\[
\rho^{(d)} (E_i) = \sum_{m=1}^{i-1} (x_{m,i} \partial_{x_{m,i+1}} - \bar{x}_{m,i+1} \partial_{\bar{x}_{m,i}}) + \partial_{x_{i,i+1}}, \quad 1 \leq i \leq n - 1, \tag{4.9}
\]
The free field realization of the $so(2n + 1)_k$ current algebra is obtained by the substitution in the differential realization (4.9)-(4.13) of $so(2n + 1)$,

\[
\begin{align*}
   x_{i,j} &\rightarrow \gamma_{i,j}(z), & \partial_{x_{i,j}} &\rightarrow \beta_{i,j}(z), & 1 \leq i < j \leq n, \\
   \bar{x}_{i,j} &\rightarrow \bar{\gamma}_{i,j}(z), & \partial_{\bar{x}_{i,j}} &\rightarrow \bar{\beta}_{i,j}(z), & 1 \leq i < j \leq n, \\
   x_i &\rightarrow \gamma_i(z), & \partial_{x_i} &\rightarrow \beta_i(z), & i = 1, \ldots, n, \\
   \lambda_i &\rightarrow \sqrt{k + 2n - 1}\partial\phi_i(z), & i = 1, \ldots, n,
\end{align*}
\]

followed by an addition of anomalous terms linear in $\partial\gamma(z)$ and $\partial\bar{\gamma}(z)$ in the expressions of

\[
\begin{align*}
   \rho^{(d)}(E_n) &= \sum_{m=1}^{n-1} (x_{m,n}\partial_{x_m} - x_m\partial_{x_{m,n}}) + \partial_{x_n}, \\
   \rho^{(d)}(F_i) &= \sum_{m=1}^{i-1} (x_{m,i+1}\partial_{x_{m,i}} - \bar{x}_{m,i}\partial_{\bar{x}_{m,i+1}}) - x_i\partial_{x_{i+1}} + \frac{x_i^2}{2}\partial_{\bar{x}_{i,i+1}} \\
   &\quad - \sum_{m=i+2}^{n} (x_{m,i}\partial_{x_{i,m}} - x_{m,i}\bar{x}_{m,i}\partial_{\bar{x}_{i,m}} + \bar{x}_{i,m}\partial_{\bar{x}_{i+1,m}}) - x_{i,i+1}\partial_{x_{i,i+1}} \\
   &\quad - x_{i,i+1} \left[ \sum_{m=i+2}^{n} (x_{m,i}\partial_{x_{i,m}} + \bar{x}_{m,i}\partial_{\bar{x}_{i,m}} - x_{i+1,m}\partial_{x_{i+1,m}} - \bar{x}_{i+1,m}\partial_{\bar{x}_{i+1,m}}) \right] \\
   &\quad + x_{i,i+1} (x_{i+1}\partial_{x_{i+1}} - x_i\partial_{x_i} + \lambda_i - \lambda_{i+1}), & 1 \leq i \leq n - 1, \\
   \rho^{(d)}(F_n) &= \sum_{m=1}^{n-1} (x_{m,n}\partial_{x_{m,n}} - \bar{x}_{m,n}\partial_{\bar{x}_{m,n}}) - \frac{x_n^2}{2}\partial_{x_n} + x_n\lambda_n, \\
   \rho^{(d)}(H_i) &= \sum_{m=1}^{i-1} (x_{m,i}\partial_{x_{m,i}} - \bar{x}_{m,i}\partial_{\bar{x}_{m,i}}) - \sum_{m=i+1}^{n} (x_{m,i}\partial_{x_{i,m}} + \bar{x}_{m,i}\partial_{\bar{x}_{i,m}}) \\
   &\quad - x_i\partial_{x_i} + \lambda_i, & i = 1, \ldots, n.
\end{align*}
\]
the currents. Here we present the results for the currents associated with the simple roots,

\[ E_i(z) = \sum_{m=1}^{i-1} \left( \gamma_{m,i}(z) \tilde{\beta}_{m,i+1}(z) - \bar{\gamma}_{m,i+1}(z) \tilde{\beta}_{m,i}(z) \right) + \beta_{i,i+1}(z), \quad 1 \leq i \leq n - 1, \quad (4.15) \]

\[ E_n(z) = \sum_{m=1}^{n-1} \left( \gamma_{m,n}(z) \beta_{m}(z) - \gamma_{m}(z) \tilde{\beta}_{m,n}(z) \right) + \beta_{n}(z), \]

\[ F_i(z) = \sum_{m=1}^{i-1} \left( \gamma_{m,i+1}(z) \beta_{m,i}(z) - \bar{\gamma}_{m,i}(z) \tilde{\beta}_{m,i+1}(z) \right) - \gamma_i(z) \beta_{i+1}(z) + \frac{1}{2} \gamma_i^2(z) \tilde{\beta}_{i,i+1}(z) \]

\[ - \sum_{m=i+2}^{n} \left( \gamma_{i,m}(z) \beta_{i+1,m}(z) - \gamma_{i,m}(z) \gamma_{i,m}(z) \tilde{\beta}_{i,i+1}(z) + \bar{\gamma}_{i,m}(z) \tilde{\beta}_{i+1,m}(z) \right) \]

\[ - \gamma_i(z) \beta_{i+1}(z) \left[ \sum_{m=i+2}^{n} \gamma_{i,m}(z) \beta_{i,m}(z) + \bar{\gamma}_{i,m}(z) \tilde{\beta}_{i,m}(z) \right] \]

\[ + \gamma_i(z) \beta_{i+1}(z) \left[ \sum_{m=i+2}^{n} \gamma_{i+1,m}(z) \beta_{i+1,m}(z) + \bar{\gamma}_{i+1,m}(z) \tilde{\beta}_{i+1,m}(z) \right] \]

\[ - \gamma_{i,i+1}^2(z) \beta_{i,i+1}(z) + \gamma_{i,i+1}(z) \gamma_{i,i+1}(z) \beta_{i,i+1}(z) - \gamma_{i,i+1}(z) \gamma_i(z) \beta_i(z) \]

\[ + \sqrt{k + 2n - 1} \gamma_{i,i+1}(z) \left( \partial \phi_i(z) - \partial \phi_{i+1}(z) \right) \]

\[ + (k + 2(i - 1)) \partial \gamma_{i,i+1}(z), \quad 1 \leq i \leq n - 1, \]

\[ F_n(z) = \sum_{m=1}^{n-1} \left( \gamma_{m,n}(z) \beta_{m,n}(z) - \bar{\gamma}_{m,n}(z) \beta_{m,n}(z) \right) - \frac{1}{2} \gamma_n^2(z) \beta_n(z) \]

\[ + \sqrt{k + 2n - 1} \gamma_n(z) \partial \phi_n(z) + (k + 2(n - 1)) \partial \gamma_n(z), \]

\[ H_i(z) = \sum_{m=1}^{i-1} \left( \gamma_{m,i}(z) \beta_{m,i}(z) - \bar{\gamma}_{m,i}(z) \tilde{\beta}_{m,i}(z) \right) - \sum_{m=i+1}^{n} \left( \gamma_{i,m}(z) \beta_{i,m}(z) + \bar{\gamma}_{i,m}(z) \tilde{\beta}_{i,m}(z) \right) \]

\[ - \gamma_i(z) \beta_i(z) + \sqrt{k + 2n - 1} \partial \phi_i(z), \quad 1 \leq i \leq n. \quad (4.16) \]

### 4.3 Energy-momentum tensor

After a tedious calculation, we find that the Sugawara tensor corresponding to the quadratic Casimir of \( so(2n+1) \) is given by

\[ T(z) = \frac{1}{2(k + 2n - 1)} \left\{ \sum_{i<j} \left( E_{\epsilon_i-\epsilon_j}(z) F_{\epsilon_i-\epsilon_j}(z) + F_{\epsilon_i-\epsilon_j}(z) E_{\epsilon_i-\epsilon_j}(z) \right) \right. \]

\[ + \sum_{i<j} \left( E_{\epsilon_i+\epsilon_j}(z) F_{\epsilon_i+\epsilon_j}(z) + F_{\epsilon_i+\epsilon_j}(z) E_{\epsilon_i+\epsilon_j}(z) \right) \]

\[ + \sum_{i=1}^{n} \left( E_{\epsilon_i}(z) F_{\epsilon_i}(z) + F_{\epsilon_i}(z) E_{\epsilon_i}(z) \right) + \sum_{i=1}^{n} H_i(z) H_i(z) \left\} \right. \]
\[\sum_{i=1}^{n} \left( \frac{1}{2} \partial \phi_i(z) \partial \phi_i(z) - \frac{2n - 2i + 1}{2\sqrt{k + 2n - 1}} \partial^2 \phi_i(z) \right) + \sum_{i<j} \left( \beta_{i,j}(z) \partial \gamma_{i,j}(z) + \bar{\beta}_{i,j}(z) \partial \bar{\gamma}_{i,j}(z) \right) + \sum_{i=1}^{n} \beta_i(z) \partial \gamma_i(z). \tag{4.17}\]

It is straightforward to check that \(T(z)\) satisfy the following OPE,

\[T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}. \tag{4.18}\]

The corresponding central charge \(c\) is

\[c = \frac{kn(2n+1)}{k+2n-1} = \frac{k \dim(\so(2n+1))}{k+2n-1}. \tag{4.19}\]

Moreover, we find that with regard to the energy-momentum tensor \(T(z)\) defined by (4.17) the \(so(2n+1)\) currents associated with the simple roots (4.15)-(4.16) are indeed primary fields with conformal dimensional one, namely,

\[T(z)E_i(w) = \frac{E_i(w)}{(z-w)^2} + \frac{\partial E_i(w)}{(z-w)}, \quad 1 \leq i \leq n,\]

\[T(z)F_i(w) = \frac{F_i(w)}{(z-w)^2} + \frac{\partial F_i(w)}{(z-w)}, \quad 1 \leq i \leq n,\]

\[T(z)H_i(w) = \frac{H_i(w)}{(z-w)^2} + \frac{\partial H_i(w)}{(z-w)}, \quad 1 \leq i \leq n.\]

4.4 Screening currents

Free field realization of the screening currents of \(so(2n+1)_k\) can be constructed from the differential operators similar to those of the \(so(2n)_k\) case, which are defined by the relation

\[\rho^{(d)}(s_\alpha) \langle \Lambda, x, \bar{x} \rangle \equiv \langle \Lambda | E_\alpha G_+(x, \bar{x}) \rangle, \quad \text{for } \alpha \in \Delta_+. \tag{4.20}\]

After some algebraic manipulations, we obtain the following explicit expressions of \(s_i\) associated with the simple roots of \(so(2n+1)\):

\[s_i = \sum_{m=i+2}^{n} \left( \bar{x}_{i+1,m} \partial \bar{x}_{i,m} - \bar{x}_{i+1,m} x_{i+1,m} \partial \bar{x}_{i,i+1} + x_{i+1,m} \partial x_{i,m} \right) + x_{i+1} \partial x_i \]

\[-\frac{1}{2} x_{i+1,i+1}^2 \partial \bar{x}_{i,i+1} + \partial \bar{x}_{i,i+1}, \quad 1 \leq i \leq n - 1, \tag{4.21}\]

\[s_n = \partial x_n. \tag{4.22}\]
The free field realization of the screening currents $S_i(z)$ of the $so(2n+1)$ current algebra corresponding to the above differential operators $s_i$ is given by

$$S_i(z) = \left\{ \sum_{m=1}^{n} \left( \bar{\gamma}_{i+1,m}(z) \bar{\beta}_{i,m}(z) - \bar{\gamma}_{i+1,m}(z) \gamma_{i+1,m}(z) \bar{\beta}_{i,i+1}(z) + \gamma_{i+1,m}(z) \beta_{i,m}(z) \right) 
+ \gamma_{i+1}(z) \beta_i(z) - \frac{1}{2} \gamma_{i+1}^2 \bar{\beta}_{i,i+1}(z) + \beta_{i,i+1}(z) \right\} e^{-\frac{\alpha_i \cdot \vec{\phi}(z)}{\sqrt{k+2n-1}}}, \quad 1 \leq i \leq n-1,$$

$$S_n(z) = \beta_n(z) e^{-\frac{\alpha_n \cdot \vec{\phi}(z)}{\sqrt{k+2n-1}}}, \quad (4.24)$$

where $\vec{\phi}(z)$ is defined in (3.32). From direct calculation we find that the screening currents satisfy the required OPEs with the energy-momentum tensor (4.17) and the $so(2n+1)$ currents (4.15)-(4.16), namely,

$$T(z)S_j(w) = \frac{S_j(w)}{(z-w)^2} + \frac{\partial S_j(w)}{(z-w)} = \partial w \left\{ \frac{S_j(w)}{(z-w)} \right\}, \quad j = 1, \ldots, n, \quad (4.25)$$

$$E_i(z)S_j(w) = 0, \quad i, j = 1 \ldots, n, \quad (4.26)$$

$$H_i(z)S_j(w) = 0, \quad i, j = 1 \ldots, n, \quad (4.27)$$

$$F_i(z)S_j(w) = \delta_{ij} \partial w \left\{ \frac{(k+2n-1) e^{-\frac{\alpha_i \cdot \vec{\phi}(z)}{\sqrt{k+2n-1}}}}{(z-w)} \right\}, \quad i, j = 1, \ldots, n. \quad (4.28)$$

These screening currents, (4.23) and (4.24), are screening currents of the first kind [25].

5 Results for $sp(2n)_k$

5.1 Differential operator realization of $sp(2n)$

The root system of $C_n$ (or $sp(2n)$) are: $\{ \pm \epsilon_i \pm \epsilon_j | i \neq j, i, j = 1, \ldots, n \}$ and $\{ \pm 2\epsilon_i | i = 1, \ldots, n \}$. Among them, the positive roots $\Delta_+$ can be chosen as:

$$\epsilon_i \pm \epsilon_j, \text{ for } 1 \leq i < j \leq n, \quad \text{and } 2\epsilon_i, \text{ for } i = 1, \ldots, n.$$

The simple roots are

$$\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \ldots, \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n, \alpha_n = 2\epsilon_n. \quad (5.1)$$

Associated with each positive root $\alpha$, there are a raising operator $E_\alpha$, a lowering operator $F_\alpha$ and a Cartan generator $H_\alpha$. The matrix realization of the generators associated with all roots of $sp(2n)$ is given in Appendix C, from which one may derive the structure constants
for the particular choice of the basis. Like the $so(2n)$ and $so(2n + 1)$ cases, we adopt the
convention (3.2) for the raising/lowering generators associated with the simple roots.

Similar to the $so(2n + 1)$ case, besides the coordinates $\{x_{i,j}, \bar{x}_{i,j} | 1 \leq i < j \leq n\}$, which
are associated with the positive roots $\{\epsilon_i \pm \epsilon_j | i < j\}$, we also need to introduce extra $n$
coordinates $\{x_i | i = 1, \ldots, n\}$ associated with the positive roots $\{2\epsilon_i | i = 1, \ldots, n\}$. These
coordinates $\{x_{i,j}, \bar{x}_{i,j}\}$ and $\{x_i\}$ satisfy the same commutation relations as (3.3)-(3.4) and (4.2).

Let $\langle \Lambda |$ be the highest weight vector of the highest weight representation of $sp(2n)$ sat-
ifying the following conditions:

$$\langle \Lambda | F_i = 0, \quad 1 \leq i \leq n,$$
$$\langle \Lambda | H_i = \lambda_i \langle \Lambda |, \quad 1 \leq i \leq n. \tag{5.2}$$

Here the generators $H_i$ are some linear combinations of $H_\alpha$ (C.7). An arbitrary vector in
the corresponding Verma module is parametrized by $\langle \Lambda |$ and the coordinates ($x$ and $\bar{x}$) as

$$\langle \Lambda, x, \bar{x} | = \langle \Lambda | G_+(x, \bar{x}), \quad (5.4)$$

where $G_+(x, \bar{x})$ is given by (c.f. [13, 18])

$$G_+(x, \bar{x}) = (G_n) (\bar{G}_{n-1,n} G_{n-1,1} G_{n-1,n}) (\bar{G}_{n-2,n-1} G_{n-2,1} G_{n-2,n} G_{n-2,n-1}) \cdots \times (\bar{G}_{1,n} G_{1,1} G_{1,n} \cdots G_{1,2}). \tag{5.5}$$

Here $G_{i,j}$ and $\bar{G}_{i,j}$ for $i < j$, and $G_i$ are given by

$$G_{i,j} = e^{x_{i,j} E_{i,j}}, \quad \bar{G}_{i,j} = e^{\bar{x}_{i,j} E_{i,j}}, \quad G_i = e^{x_i E_{2i}}. \tag{5.6}$$

Then one can define a differential operator realization $\rho^{(d)}$ of the generators of $sp(2n)$ by

$$\rho^{(d)} (g) \langle \Lambda, x, \bar{x} | \equiv \langle \Lambda, x, \bar{x} | g, \quad \forall g \in sp(2n). \tag{5.7}$$

After tedious calculations analogous to those in the previous cases, we have found the differ-
ential realization of $sp(2n)$. Here we give the results for the generators associated with the simple roots,

$$\rho^{(d)} (E_i) = \sum_{m=1}^{i-1} (x_{m,i} \partial_{x_{m,i+1}} - \bar{x}_{m,i+1} \partial_{\bar{x}_{m,i}}) + \partial_{x_{i,i+1}}, \quad 1 \leq i \leq n - 1. \tag{5.8}$$
With the help of the differential realization given by (5.8)-(5.12) we can construct the free field representation of the differential realization (5.8)-(5.12) of $sp$ fields.

5.2 Free field realization of $sp(2n)_k$

With the help of the differential realization given by (5.8)-(5.12) we can construct the free field representation of the $sp(2n)$ current algebra with arbitrary level $k$ in terms of $n^2$ bosonic $\beta$-$\gamma$ pairs $\{(\beta_{i,j}, \gamma_{i,j}), (\bar{\beta}_{i,j}, \bar{\gamma}_{i,j})\}, 1 \leq i < j \leq n$ and $\{(\beta_i, \gamma_i) | i = 1, \ldots, n\}$, and $n$ free scalar fields $\phi_i, i = 1, \ldots, n$. These free fields $\{(\beta_{i,j}, \gamma_{i,j}), (\bar{\beta}_{i,j}, \bar{\gamma}_{i,j}), (\beta_i, \gamma_i)\}$ and $\{\phi_i\}$ obey the same OPEs as (3.16)-(3.18) and (4.14).

The free field realization of the $sp(2n)$ current algebra is obtained by the substitution in the differential realization (5.8)-(5.12) of $sp(2n)$,

$$
\rho^{(d)}(E_n) = \sum_{m=1}^{n-1} (x_{m,n} \partial x_{m,n} + x_{m,n}^2 \partial x_m) + \partial x_n, \quad (5.9)
$$

$$
\rho^{(d)}(F_i) = \sum_{m=1}^{i-1} (x_{m,i+1} \partial x_{m,i} - \bar{x}_{m,i} \partial \bar{x}_{m,i+1}) - x_i \partial x_{i+1} - 2 \bar{x}_{i,i+1} \partial x_{i+1}
- \sum_{m=i+2}^{n} (x_{i,m} \partial x_{i+1,m} - \bar{x}_{i,m} \partial \bar{x}_{i+1,m} + \bar{x}_{i,m} \partial \bar{x}_{i+1,m} + 2 \bar{x}_{i,m} \partial x_{i+1,m})
- x_{i,i+1} \left[ \sum_{m=i+2}^{n} (x_{i,m} \partial x_{i,m} + \bar{x}_{i,m} \partial \bar{x}_{i,m} - x_{i+1,m} \partial x_{i+1,m} - \bar{x}_{i+1,m} \partial \bar{x}_{i+1,m}) \right]
+ x_{i,i+1} (x_{i+1,j+1} - 2 x_{i+1} \partial x_{i+1} - 2 x_i \partial x_i)
+ x_{i,i+1} (\lambda_i - \lambda_{i+1}), \quad 1 \leq i \leq n - 1, \quad (5.10)
$$

$$
\rho^{(d)}(F_n) = \sum_{m=1}^{n-1} (x_{n,m} \partial x_m + \bar{x}_{n,m} \partial \bar{x}_m) - x_n^2 \partial x_n + x_n \lambda_n, \quad (5.11)
$$

$$
\rho^{(d)}(H_i) = \sum_{m=1}^{i-1} (x_{m,i} \partial x_{m,i} - \bar{x}_{m,i} \partial \bar{x}_{m,i}) - \sum_{m=i+1}^{n} (x_{i,m} \partial x_{i,m} + \bar{x}_{i,m} \partial \bar{x}_{i,m})
- 2 x_i \partial x_i + \lambda_i, \quad i = 1, \ldots, n. \quad (5.12)
$$

followed by an addition of anomalous terms linear in $\partial \gamma(z)$ and $\partial \bar{\gamma}(z)$ in the expressions of $\beta_i(z)$. 

$$
\lambda_i \rightarrow \sqrt{k + 2(n + 1)} \partial \phi_i(z), \quad i = 1, \ldots, n,
$$

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the currents. Here we present the results for the currents associated with the simple roots,

\[
E_i(z) = \sum_{m=1}^{i-1} \left( \gamma_{m,i}(z) \beta_{m,i+1}(z) - \bar{\gamma}_{m,i+1}(z) \bar{\beta}_{m,i}(z) \right) + \beta_{i,i+1}(z), \quad 1 \leq i \leq n - 1, \tag{5.13}
\]

\[
E_n(z) = \sum_{m=1}^{n} \left( \gamma_{m,n}(z) \bar{\beta}_{m,n}(z) + \gamma_{m,n}^2(z) \beta_{m,n}(z) \right) + \beta_n(z),
\]

\[
F_i(z) = \sum_{m=1}^{i-1} \left( \gamma_{m,i+1}(z) \beta_{m,i}(z) - \bar{\gamma}_{m,i+1}(z) \bar{\beta}_{m,i+1}(z) \right) - \gamma_i(z) \bar{\beta}_{i,i+1}(z) - 2\bar{\gamma}_{i,i+1}(z) \beta_{i+1}(z)
- \sum_{m=i+2}^{n} \left( \gamma_{i,m}(z) \bar{\beta}_{i+1,m}(z) - \gamma_{i,m}(z) \bar{\beta}_{i+1,m}(z) \right)
- \sum_{m=i+2}^{n} \left( \bar{\gamma}_{i,m}(z) \beta_{i+1,m}(z) + 2\bar{\gamma}_{i,m}(z) \gamma_{i+1,m}(z) \beta_{i+1}(z) \right)
- \gamma_{i,i+1}(z) \sum_{m=i+2}^{n} \left( \gamma_{i,m}(z) \beta_{i,m}(z) + \bar{\gamma}_{i,m}(z) \bar{\beta}_{i,m}(z) \right)
+ \gamma_{i,i+1}(z) \sum_{m=i+2}^{n} \left( \gamma_{i+1,m}(z) \beta_{i+1,m}(z) + \bar{\gamma}_{i+1,m}(z) \bar{\beta}_{i+1,m}(z) \right)
- \gamma_{i,i+1}(z)^2 \beta_{i,i+1}(z) + 2\gamma_{i,i+1}(z) \gamma_{i+1}(z) \beta_{i+1}(z) - 2\gamma_{i,i+1}(z) \gamma_i(z) \beta_i(z)
+ \sqrt{k + 2(n + 1) \gamma_{i,i+1}(z) (\partial \phi_{i}(z) - \partial \phi_{i+1}(z))}
+ (k + 2(i - 1)) \partial \gamma_{i,i+1}(z), \quad 1 \leq i \leq n - 1,
\]

\[
F_n(z) = \sum_{m=1}^{n-1} \left( \gamma_{m,n}(z) \beta_{m,n}(z) + \bar{\gamma}_{m,n}(z) \bar{\beta}_{m,n}(z) \right) - \gamma_n^2(z) \beta_n(z)
+ \sqrt{k + 2(n + 1) \gamma_n(z) \partial \phi_n(z) + \left( \frac{k}{2} + (n - 1) \right) \partial \gamma_n(z),
\]

\[
H_i(z) = \sum_{m=1}^{i-1} \left( \gamma_{m,i}(z) \beta_{m,i}(z) - \bar{\gamma}_{m,i}(z) \bar{\beta}_{m,i}(z) \right) - \sum_{m=i+1}^{n} \left( \gamma_{i,m}(z) \beta_{i,m}(z) + \bar{\gamma}_{i,m}(z) \bar{\beta}_{i,m}(z) \right)
- 2\gamma_i(z) \beta_i(z) + \sqrt{k + 2(n + 1) \partial \phi_i(z)}, \quad 1 \leq i \leq n. \tag{5.14}
\]

### 5.3 Energy-momentum tensor

After a tedious calculation, we find that the Sugawara tensor corresponding to the quadratic Casimir of \( sp(2n) \) is given by

\[
T(z) = \frac{1}{2(k + 2(n + 1))} \left\{ \sum_{i<j} \left( E_{\epsilon_i-\epsilon_j}(z) F_{\epsilon_i-\epsilon_j}(z) + F_{\epsilon_i-\epsilon_j}(z) E_{\epsilon_i-\epsilon_j}(z) \right) + \sum_{i<j} \left( E_{\epsilon_i+\epsilon_j}(z) F_{\epsilon_i+\epsilon_j}(z) + F_{\epsilon_i+\epsilon_j}(z) E_{\epsilon_i+\epsilon_j}(z) \right) \right\}
\]
After some algebraic manipulations, we obtain the following explicit expressions of the associated with the simple roots of \( sp \) defined by the relation

\[
+2 \sum_{i=1}^{n} \left( E_{2i}(z)F_{2i}(z) + F_{2i}(z)E_{2i}(z) \right) + \sum_{i=1}^{n} H_{i}(z)H_{i}(z) \right) 
\]

\[
= \sum_{i=1}^{n} \left( \frac{1}{2} \partial \phi_{i}(z) \partial \phi_{i}(z) - \frac{n - i + 1}{\sqrt{k + 2(n + 1)}} \partial^{2} \phi_{i}(z) \right) 
\]

\[
+ \sum_{i<j} \left( \beta_{i,j}(z) \partial \gamma_{i,j}(z) + \tilde{\beta}_{i,j}(z) \partial \tilde{\gamma}_{i,j}(z) \right) + \sum_{i=1}^{n} \beta_{i}(z) \partial \gamma_{i}(z). \tag{5.15} \]

It is straightforward to check that \( T(z) \) satisfy the following OPE,

\[
T(z)T(w) = \frac{c/2}{(z - w)^4} + \frac{2T(w)}{(z - w)^2} + \partial T(w). \tag{5.16} \]

The corresponding central charge \( c \) is

\[
c = \frac{kn(2n + 1)}{k + 2(n + 1)} \equiv \frac{k \dim(sp(2n))}{k + 2(n + 1)}. \tag{5.17} \]

Moreover, we find that with regard to the energy-momentum tensor \( T(z) \) defined by (5.15), the \( sp(2n) \) currents associated with the simple roots (5.13)-(5.14) are indeed primary fields with conformal dimensional one, namely,

\[
T(z)E_{i}(w) = \frac{E_{i}(w)}{(z - w)^2} + \frac{\partial E_{i}(w)}{(z - w)}, \quad 1 \leq i \leq n, 
\]

\[
T(z)F_{i}(w) = \frac{F_{i}(w)}{(z - w)^2} + \frac{\partial F_{i}(w)}{(z - w)}, \quad 1 \leq i \leq n, 
\]

\[
T(z)H_{i}(w) = \frac{H_{i}(w)}{(z - w)^2} + \frac{\partial H_{i}(w)}{(z - w)}, \quad 1 \leq i \leq n. 
\]

### 5.4 Screening currents

Free field realization of the screening currents of \( sp(2n) \) can be constructed from the differential operators similar to the \( so(2n) \) and \( so(2n + 1) \) cases in previous sections, which are defined by the relation

\[
\rho^{(d)}(s_{\alpha}) \langle \Lambda, x, \bar{x} \rangle \equiv \langle \Lambda | E_{\alpha} G_{+}(x, \bar{x}), \quad \text{for} \ \alpha \in \Delta_{+}. \tag{5.18} \]

After some algebraic manipulations, we obtain the following explicit expressions of \( s_{i} \) associated with the simple roots of \( sp(2n) \):

\[
s_{i} = \sum_{m=i+2}^{n} \left( \bar{x}_{i+1,m} \partial x_{i,m} - \bar{x}_{i+1,m} x_{i+1,m} \partial x_{i,i+1} + x_{i+1,m} \partial x_{i,m} + 2x_{i+1,m} \bar{x}_{i,m} \partial x_{i} \right) + x_{i+1} \partial x_{i,i+1} + 2\bar{x}_{i,i+1} \partial x_{i} + \partial x_{i,i+1}, \quad 1 \leq i \leq n - 1, \tag{5.19} \]

\[
s_{n} = \partial x_{n}. \tag{5.20} \]
The free field realization of the screening currents $S_i(z)$ of the $sp(2n)$ current algebra corresponding to the above differential operators $s_i$ is given by

$$S_i(z) = \left\{ \sum_{m=i+2}^{n} (\bar{\gamma}_{i+1,m}(z)\beta_{i,m}(z) - \bar{\gamma}_{i+1,m}(z)\gamma_{i+1,m}(z)\bar{\beta}_{i,i+1}(z)) + \sum_{m=i+2}^{n} (\gamma_{i+1,m}(z)\beta_{i,m}(z) + 2\gamma_{i+1,m}(z)\bar{\gamma}_{i,m}(z)\beta_{i}(z)) + \gamma_{i+1}(z)\bar{\beta}_{i,i+1}(z) + 2\bar{\gamma}_{i,i+1}(z) + \beta_{i,i+1}(z)\right\} e^{-\frac{\alpha_i \vec{\phi}(z)}{\sqrt{k+2(n+1)}}}, \quad 1 \leq i \leq n-1,$$

$$S_n(z) = \beta_n(z)e^{-\frac{\alpha_n \vec{\phi}(z)}{\sqrt{k+2(n+1)}}},$$

where $\vec{\phi}(z)$ is defined in (3.32). From direct calculations we find that the screening currents satisfy the required OPEs with the energy-momentum tensor (5.15) and the $sp(2n)$ currents (5.13)-(5.14), namely,

$$T(z)S_j(w) = \frac{S_j(w)}{(z-w)^2} + \partial S_j(w) = \partial_w \left\{ \frac{S_j(w)}{(z-w)} \right\}, \quad j = 1, \ldots, n, \quad (5.23)$$

$$E_i(z)S_j(w) = 0, \quad i, j = 1, \ldots, n, \quad (5.24)$$

$$H_i(z)S_j(w) = 0, \quad i, j = 1, \ldots, n, \quad (5.25)$$

$$F_i(z)S_j(w) = \frac{\delta_{ij}}{1 + \delta_{im}} \partial_w \left\{ (k + 2(n+1)) e^{-\frac{\alpha_i \vec{\phi}(z)}{\sqrt{k+2(n+1)}}} \right\}, \quad i, j = 1, \ldots, n. \quad (5.26)$$

These screening currents, given in (5.21) and (5.22), are screening currents of the first kind.

### 6 Discussions

We have constructed the explicit expressions of the free field representations for the $so(2n)$, $so(2n+1)$ and $sp(2n)$ current algebras at an arbitrary level $k$, and the corresponding energy-momentum tensors. We have also found the free field representation of $n$ screening currents of the first kind for each current algebra. Our results reduce to those in [12] for the $so(5)$ case.

The free field realizations (3.22)-(3.23), (4.15)-(4.16) and (5.13)-(5.14) of the current algebras $so(2n)_k$, $so(2n+1)_k$ and $sp(2n)_k$ respectively give rise to the Fock representations of the corresponding current algebras in terms of the free fields (3.16), (3.18) and (4.14). They provide explicit realizations of the vertex operator constructions of representations for affine
Lie algebras [26, 27]. Moreover, these representations are in general not irreducible for the
current algebras. To obtain irreducible representations, one needs the associated screening
charges, which are the integrals of the corresponding screening currents ((3.30)-(3.31), (4.23)-
(4.24) and (5.21)-(5.22)), and performs the cohomology procedures as in [7, 11, 12, 13].

Our explicit expressions of the affine currents, energy-momentum tensor and screening
currents in terms of free fields should allow one to construct the primary fields and correlation
functions of the associated WZNW models using the method developed in [18]. The approach
presented in this paper can be generalized to construct the explicit free field realizations of the
current superalgebra \(osp(m|2n)\) with generic \(m\) and \(n\). Results will be reported elsewhere
[28].

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Appendix A: Fundamental representation of \(so(2n)\)

Let \(e_{ij}, i, j = 1, \ldots, n\), be an \(n \times n\) matrix with entry 1 at the \(i\)th row and the \(j\)th column
and zero elsewhere. The \(2n\)-dimensional fundamental representation of \(so(2n)\), denoted by
\(\rho_0\), is given by the following \(2n \times 2n\) matrices,

\[
\rho_0 \left( E_{e_i - e_j} \right) = \begin{pmatrix} e_{ij} & 0 \\ 0 & -e_{ji} \end{pmatrix}, \quad \rho_0 \left( F_{e_i - e_j} \right) = \begin{pmatrix} e_{ji} & 0 \\ 0 & -e_{ij} \end{pmatrix}, \quad 1 \leq i < j \leq n, \quad (A.1)
\]

\[
\rho_0 \left( E_{e_i + e_j} \right) = \begin{pmatrix} 0 & e_{ij} - e_{ji} \\ 0 & 0 \end{pmatrix}, \quad \rho_0 \left( F_{e_i + e_j} \right) = \begin{pmatrix} 0 & 0 \\ -e_{ij} + e_{ji} & 0 \end{pmatrix}, \quad 1 \leq i < j \leq n, \quad (A.2)
\]

\[
\rho_0 \left( H_{e_i - e_j} \right) = \begin{pmatrix} e_{ii} - e_{jj} & 0 \\ 0 & -e_{ii} + e_{jj} \end{pmatrix}, \quad 1 \leq i < j \leq n, \quad (A.3)
\]

\[
\rho_0 \left( H_{e_i + e_j} \right) = \begin{pmatrix} e_{ii} + e_{jj} & 0 \\ 0 & -e_{ii} - e_{jj} \end{pmatrix}, \quad 1 \leq i < j \leq n. \quad (A.4)
\]

We introduce \(n\) linear-independent generators \(H_i (i = 1, \ldots n)\),

\[
H_i = \frac{1}{2} (H_{e_i - e_j} + H_{e_i + e_j}). \quad (A.5)
\]
Actually, the above generators \( \{H_i\} \) span the Cartan subalgebra of \( so(2n) \). In the fundamental representation, these generators can be realized by

\[
\rho_0(H_i) = \begin{pmatrix} e_{ii} & 0 \\ 0 & -e_{ii} \end{pmatrix}, \quad i = 1, \ldots, n.
\] (A.6)

The corresponding nondegenerate invariant bilinear symmetric form of \( so(2n) \) is given by

\[
(x, y) = \frac{1}{2} tr(\rho_0(x)\rho_0(y)), \quad \forall x, y \in so(2n).
\] (A.7)

**Appendix B: Fundamental representation of \( so(2n + 1) \)**

Let \( e_i \) \( (i = 1, \ldots, n) \) be an \( n \)-dimensional row vector with the \( i \)th component being 1 and all others being zero, and \( e_i^T \) be the transport of \( e_i \), namely,

\[
e_i = (0, \ldots, 0, 1, 0, \ldots, 0), \quad e_i^T = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
\] (B.1)

Then, the \( (2n + 1) \)-dimensional fundamental representation of \( so(2n + 1) \), denoted by \( \rho_0 \), is given by the following \( (2n + 1) \times (2n + 1) \) matrices,

\[
\rho_0(e_i - e_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_{ij} & 0 \\ 0 & 0 & -e_{ji} \end{pmatrix}, \quad \rho_0(F_{e_i - e_j}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_{ji} & 0 \\ 0 & 0 & -e_{ij} \end{pmatrix}, \quad 1 \leq i < j \leq n,
\] (B.2)

\[
\rho_0(e_i + e_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_{ij} - e_{ji} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho_0(F_{e_i + e_j}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -e_{ij} + e_{ji} & 0 \end{pmatrix}, \quad 1 \leq i < j \leq n,
\] (B.3)

\[
\rho_0(e_i) = \begin{pmatrix} 0 & 0 & e_i \\ -e_i^T & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho_0(F_{e_i}) = \begin{pmatrix} 0 & -e_i & 0 \\ 0 & 0 & 0 \\ e_i^T & 0 & 0 \end{pmatrix}, \quad i = 1, \ldots, n,
\] (B.4)

\[
\rho_0(H_{e_i - e_j}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_{ii} - e_{jj} & 0 \\ 0 & 0 & -e_{ii} + e_{jj} \end{pmatrix}, \quad 1 \leq i < j \leq n,
\] (B.5)

\[
\rho_0(H_{e_i + e_j}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_{ii} + e_{jj} & 0 \\ 0 & 0 & -e_{ii} - e_{jj} \end{pmatrix}, \quad 1 \leq i < j \leq n,
\] (B.6)
\[
\rho_0 (H_{\epsilon_i}) = \begin{pmatrix}
0 & 0 & 0 \\
0 & e_{ii} & 0 \\
0 & 0 & -e_{ii}
\end{pmatrix}, \quad i = 1, \ldots, n. \tag{B.7}
\]

We introduce \( n \) linear-independent generators \( H_i \) (\( i = 1, \ldots n \)),
\[
H_i = \frac{1}{2} (H_{\epsilon_i-\epsilon_j} + H_{\epsilon_i+\epsilon_j}) = H_\epsilon_i. \tag{B.8}
\]

Actually, the above generators \( \{H_i\} \) span the Cartan subalgebra of \( so(2n + 1) \). Moreover, the matrix realization of \( \{H_i\} \) in the fundamental representation is given by
\[
\rho_0 (H_i) = \begin{pmatrix}
0 & 0 & 0 \\
0 & e_{ii} & 0 \\
0 & 0 & -e_{ii}
\end{pmatrix}, \quad i = 1, \ldots, n. \tag{B.9}
\]

The corresponding nondegenerate invariant bilinear symmetric form of \( so(2n + 1) \) is given by
\[
(x, y) = \frac{1}{2} tr (\rho_0(x)\rho_0(y)), \quad \forall x, y \in so(2n + 1). \tag{B.10}
\]

**Appendix C: Fundamental representation of \( sp(2n) \)**

The \( 2n \)-dimensional fundamental representation of \( sp(2n) \), denoted by \( \rho_0 \), is given by the following \( 2n \times 2n \) matrices,
\[
\rho_0 (E_{\epsilon_i-\epsilon_j}) = \begin{pmatrix}
e_{ij} & 0 \\
0 & -e_{ji}
\end{pmatrix}, \quad \rho_0 (F_{\epsilon_i-\epsilon_j}) = \begin{pmatrix}
e_{ji} & 0 \\
0 & -e_{ij}
\end{pmatrix}, \quad 1 \leq i < j \leq n, \tag{C.1}
\]
\[
\rho_0 (E_{\epsilon_i+\epsilon_j}) = \begin{pmatrix}
0 & e_{ij} + e_{ji} \\
0 & 0
\end{pmatrix}, \quad \rho_0 (F_{\epsilon_i+\epsilon_j}) = \begin{pmatrix}
0 & 0 \\
e_{ij} + e_{ji} & 0
\end{pmatrix}, \quad 1 \leq i < j \leq n, \tag{C.2}
\]
\[
\rho_0 (E_{2\epsilon_i}) = \begin{pmatrix}
e_{ii} \\
0 & 0
\end{pmatrix}, \quad \rho_0 (F_{2\epsilon_i}) = \begin{pmatrix}
0 & 0 \\
e_{ii} & 0
\end{pmatrix}, \quad i = 1, \ldots, n, \tag{C.3}
\]
\[
\rho_0 (H_{\epsilon_i-\epsilon_j}) = \begin{pmatrix}
e_{ii} - e_{jj} & 0 \\
0 & -e_{ii} + e_{jj}
\end{pmatrix}, \quad 1 \leq i < j \leq n, \tag{C.4}
\]
\[
\rho_0 (H_{\epsilon_i+\epsilon_j}) = \begin{pmatrix}
e_{ii} + e_{jj} & 0 \\
0 & -e_{ii} - e_{jj}
\end{pmatrix}, \quad 1 \leq i < j \leq n, \tag{C.5}
\]
\[
\rho_0 (H_{2\epsilon_i}) = \begin{pmatrix}
e_{ii} \\
0 & -e_{ii}
\end{pmatrix}, \quad i = 1, \ldots, n. \tag{C.6}
\]

We introduce \( n \) linear-independent generators \( H_i \) (\( i = 1, \ldots n \)),
\[
H_i = \frac{1}{2} (H_{\epsilon_i-\epsilon_j} + H_{\epsilon_i+\epsilon_j}) = H_{2\epsilon_i}. \tag{C.7}
\]
Actually, the above generators \( \{H_i\} \) span the Cartan subalgebra of \( sp(2n) \). Moreover, the matrix realization of \( \{H_i\} \) in the fundamental representation is given by

\[
\rho_0 (H_i) = \begin{pmatrix} e_{ii} & 0 \\ 0 & -e_{ii} \end{pmatrix}, \quad i = 1, \ldots, n. \quad (C.8)
\]

The corresponding nondegenerate invariant bilinear symmetric form of \( sp(2n) \) is given by

\[
(x, y) = \frac{1}{2} tr(\rho_0 (x) \rho_0 (y)), \quad \forall x, y \in sp(2n).
\quad (C.9)
\]

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