The Effective Resistance of the $N$-Cycle Graph with Four Nearest Neighbors

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Abstract The exact expression for the effective resistance between any two vertices of the $N$-cycle graph with four nearest neighbors $C_N(1, 2)$, is given. It turns out that this expression is written in terms of the effective resistance of the $N$-cycle graph $C_N$, the square of the Fibonacci numbers, and the bisected Fibonacci numbers. As a consequence closed form formulas for the total effective resistance, the first passage time, and the mean first passage time for the simple random walk on the the $N$-cycle graph with four nearest neighbors are obtained. Finally, a closed form formula for the effective resistance of $C_N(1, 2)$ with all first neighbors removed is obtained.

Keywords Effective resistance · Kirchhoff index · Resistor electrical network · Simple random walk · First and mean first passage time

1 Introduction

A classical problem in electric circuit is the computation of the resistance between two nodes in a resistor network (some relevant references may be found in [1]). Calculating the resistance using the traditional analysis methods, such as the Kirchhoff's laws, can be done in principle for any given network, however, with the growing number of nodes, the number of equations to solve grows very rapidly. Most of resistance computations in the past are carried out for infinite networks using the Greens function technique [1,2].

In 2004, Wu [3] proposed a new formulation of finite resistor networks that gives a formula for the two-point resistance (effective resistance) in terms of the eigenvalues and the eigenvectors of the Laplacian matrix. This formula may be used to compute the effective resistance of the one-dimensional lattice, two-dimensional network and higher-dimensional lattices. Known results for infinite networks may be obtained by taking the infinite-size limit [1].
It has been known for sometime now, that electrical networks and random walks are closely related. They are connected through the effective resistance of an electrical network. One of the interesting parameter that characterizes and give information about random walks is the first passage time (FPT), or the hitting time, this is the expected time to hit a target node for the first time for a walker starting from a source node. Also, other quantities of interests to know are the mean first passage time (MFPT) and the commute time (the random round trip between two nodes) [4–6]. The close link between the effective resistance and the commute time of a random walker on the graph was established in [6]. Therefore, the effective resistance provides an alternative way to compute the FPT.

In this paper, the resistor electrical network is a connected, undirected graph $G = (V, E)$, with vertex set $V = \{1, 2, \ldots, N\}$ and edge set $E$ in which each edge $ij$ has a resistance $r_{xy} = 1$. An interesting invariant quantity related to the effective resistance of a graph $G$, is the total effective resistance of $G$, denoted by $R(G)$ [7], it is called the Kirchhoff’s index this is the analogue of the Wiener index $W(G)$ of a graph $G$ [8]. This quantity turns out to be connected to the MFPT.

For a given graph $G$, it is not always possible to get closed formula for the effective resistance of graphs, however, for certain graphs with symmetries like the undirected circulant graphs [9], this may be possible. By using spectral graph theory [10], Chau and Basu [11] obtained a formula to compute the FPT of the random walk on the $N$-cycle graph with $2p$ neighbors. However, We find that in this case, it is easier to use Wu’s formula to obtain the effective resistance between any two nodes. This follows from the fact that the eigenvalues of the Laplacian matrix may be obtained without any difficulty in this case. Then it is not difficult to show that this formula when multiplied by the number of edges $|E|$ is identical to the (FPT) given in [11]. The undirected circulant graphs enjoy rotational symmetry, i.e., each vertex of these graphs looks the same from any vertex. Therefore, we may as well consider that the random walk has started at vertex 0, and after some steps reaches a given vertex say $l$.

Using the commute time formula given by Chandra et al. [6], $C_{ij} = 2|E|R_{ij}$, then the first passage time $H_{0l}$ may be written as $H_{0l} = |E|R_{0l}$. For example, the effective resistance between the vertex 0 and any other vertex $l$ of the complete graph is $R_{0l} = 2/N$, and since the number of edges is $|E| = \frac{N(N-1)}{2}$, then the expression for the FPT of the random walk on the complete graph gives $H_{0l} = N - 1$. This result was derived previously using probabilistic techniques on graphs [10,12].

It was pointed out by Wu [3], that little attention has been paid to finite electrical network, even though these are the ones occurring in real life. Therefore, it interesting to apply such a formalism to other finite electrical networks beyond the cycle and the complete graphs [3]. We have already done so for the complete graph $K_N$ in which the number of vertices $N$ is odd, minus the $N$ edges that connect the opposite vertices [13]. Here, we give an exact expression for the effective resistance between any two vertices of the $N$-cycle graph with four nearest neighbors, i.e., every vertex is connected to its two neighbors and neighbor’s neighbors, this graph is denoted by $C_N(1, 2)$.

The formalism developed in this paper and in [13], enables us to obtain closed formulas for the effective resistance of some finite two-dimensional electrical networks, like The circular ladder and the $2 \times N$ electrical network with free boundaries [14]. It turns out that the effective resistance of the graph $C_N(1, 2)$ with its first neighbors being removed contributes to the effective resistance of the circular ladder.

In general, the effective resistance of the circulant graphs are given by trigonometrical sums, in particular, when considering the graph $C_N(1, 2)$, one has to deal with trigonometrical sums.
power sums. This may be evaluated using a formula by Schwatt [15] on trigonometrical power sums, however, this formula does not give the right answer when the powers are congruent to \( N \). Therefore, one has to modify the formula slightly to get the right answer.

Here, the exact computations of the effective resistance and the total effective resistance are explicit, this is done in Sect. 2 In Sect. 3, the the FPT, and the MFPT of the simple random walks on the graph \( C_N(1, 2) \) are also given by closed formulas. Computation of the effective resistance of the graph \( C_N(1, 2) \) minus \( 2N \) vertices of the first neighbors is considered in Sect. 4. Finally, in Sect. 5, we give our discussion and check our formulas using the Foster [16] first identity and the recursion formula satisfied by the hitting time.

### 2 The Exact Effective Resistance and the Total Effective Resistance of the Graph \( C_N(1, 2) \)

Here, we give the exact formula for the effective resistance of the \( N \)-cycle graph with four nearest neighbors, see Fig. 1. According to Wu’s formula [3], given a resistor network with unit resistors, the effective resistance between any two nodes \( \alpha \) and \( \beta \) is given by

\[
R_{\alpha, \beta} = \sum_{n=1}^{N-1} \frac{|\psi_{n \alpha} - \psi_{n \beta}|^2}{\lambda_n},
\]

where \( 1 \leq \alpha, \beta \leq N \) and \( \lambda_n, \psi_n \) are the eigenvalues and the eigenvectors of the Laplacian \( L \) of the resistor network having unit resistors (edges), that is, graphs.

The Laplacian matrix \( L \) of the graph \( G = (V, E) \) is \( L = D - A \), here, \( D \) is the diagonal matrix of degrees whose \( i \)th element is the number of resistors connected to the \( i \)th node. And \( A \) is the adjacency matrix representing the edge set \( E \).

The matrix elements of the Laplacian of the graph \( C_N(1, 2) \) are;

\[
L_{mn} = \begin{cases} 
4 & \text{if } m = n, \\
-1 & \text{if } m = n \pm 1, n \pm 2 \mod (N), \\
0 & \text{otherwise,}
\end{cases}
\]

equivalently, we may writes \( L_{mn} = 4\delta_{m,n} - (\delta_{m,n+1} + \delta_{m,n-1}) - (\delta_{m,n+2} + \delta_{m,n-2}) \).

The laplacian matrix \( L \) may be diagonalized using the basis \( \psi_{n,k} := \frac{1}{\sqrt{N}} \exp(\frac{2\pi i nk}{N}) \).

To that end, we make use of the fact that the matrix elements \( \delta_{m,n+k} \) may be considered as matrix elements of the \( k \)th power of the rotation matrix \( R \) of finite closed lattice [17], i.e., \( (R^k)_{mn} = \delta_{m,n+k} \). Using the expression of the matrix \( \psi \), then \( \delta_{m,n+k} \) may be written as

![Octahedral graph](https://example.com/figure1.png)
Similarly one shows \((R^{-k})_{mn} = \delta_{m,n-k}\) in the diagonalized form reads;

\[
(R^{-k})_{mn} = \delta_{m,n} \exp[2\pi i km/N].
\]  

As a consequence, the eigenvalues of the laplacian of the \(N\)-cycle graph with four nearest neighbors, are \(\lambda_n = 4 \sin^2(n\pi/N) + 4 \sin^2(2n\pi/N)\). Thus, the effective resistance of the graph \(C_N(1, 2)\) is

\[
R(l) = \frac{1}{N} \sum_{n=1}^{N-1} \frac{4 \sin^2(nl\pi/N)}{4 \sin^2(n\pi/N) + 4 \sin^2(2n\pi/N)}
\]

\[
= \frac{1}{5N} \sum_{n=1}^{N-1} \frac{\sin^2(nl\pi/N)}{\sin^2(n\pi/N)} + \frac{4}{5N} \sum_{n=1}^{N-1} \frac{\sin^2(nl\pi/N)}{5(1 - 4/5 \sin^2(n\pi/N))},
\]  

up to a constant factor, the first term in the above expression is nothing but the effective resistance of the \(N\)-cycle graph [3],

\[
\frac{1}{N} \sum_{n=1}^{N-1} \frac{\sin^2(nl\pi/N)}{\sin^2(n\pi/N)} = l(1 - l/N),
\]  

while the second trigonometrical sum is a non-trivial sum that we want to evaluate.

This term may be written explicitly as;

\[
\sum_{n=1}^{N-1} \frac{\sin^2(nl\pi/N)}{1 - 4/5 \sin^2(n\pi/N)}
\]

\[
= \frac{1}{2} \sum_{j=0}^{\infty} (4/5)^j \left[ \sum_{s=1}^{l} (-1)^{s+1} \frac{l}{l+s} \left( \frac{l+s}{l-s} \right) 2^{2s} \sum_{n=1}^{N-1} \sin^2(s+j)(n\pi/N) \right],
\]  

in obtaining Eq. (6), we used the identity \(\cos 2(ln\pi/N) = \sum_{s=0}^{l} (-1)^s \frac{l}{l+s} \left( \frac{l+s}{l-s} \right) 2^{2s} \sin^2(n\pi/N)\). The sum over \(n\) in Eq. (6), may be found in Schwatt’s classic book [15], however, it is not suitable for \(j\) a multiple of \(N\), so his formula must be modified. We may show that the right formula that takes into consideration this fact is

\[
\sum_{n=1}^{N-1} \sin^2(s+j)(n\pi/N) = \frac{N}{2^{2(s+j)}} \binom{2(j+s)}{j+s} + \frac{N}{2^{2(j+s)-1}} \sum_{p=1}^{[(j+s)/N]} (-1)^p N \binom{2(j+s)}{j-pN}.
\]  

Fortunately, we do not need to have a closed form formula for the binomial sum to do our computations. This problem is by-passed through the residue representation of binomials, this will be discussed shortly. Let \(R_1(l)\) be the above second term of Eq. (4), then our computation is equivalent to evaluate the following terms
Thus, where

\[
R_1(l)' = \frac{1}{25} \sum_{s=1}^{l} (-1)^{s+1} 5^s \frac{2l}{l+s} \left( \frac{l+s}{l-s} \right) \left[ \sum_{j=0}^{\infty} \frac{(1/5)^j}{j!} \left( \frac{2J}{J} \right) - \sum_{j=0}^{s-1} \frac{(1/5)^j}{j!} \left( \frac{2J}{J} \right) \right], \tag{8}
\]

and

\[
R_1(l)'' = \frac{4}{25} \sum_{j=0}^{\infty} \frac{(1/5)^{J-5}}{j!} \sum_{s=1}^{l} (-1)^{s+1} \frac{l}{l+s} \left( \frac{l+s}{l-s} \right) \sum_{p=1}^{[J/N]} (-1)^{pN} \left( \frac{2J}{J-pN} \right), \tag{9}
\]

where \( J = j + s \), note that in obtaining the above second sum over \( J \), we added the terms \( J = 0, \ldots, J = s - 1 \), since they do not contribute to the binomial \( \binom{2J}{J-pN} \), and \( l \leq N - 1 \).

Now, we will evaluate the sum over \( J \), is done through the representation of binomial coefficients by residue \([18]\). First, recall if \( G(w) \) is the generating function of the binomial coefficient sequence \( \binom{n}{k} \) for a fixed \( n \) given by \( G(w) = \sum_{k=0}^{n} \binom{n}{k} w^k = (1 + w)^n \), then \( \binom{n}{k} = \text{res}_w (1 + w)^n w^{-k-1} \). The following linearity property of the residue operator \( \text{res} \) is very crucial in evaluating the effective resistance. This may be defined as follows; given some constants \( \alpha \) and \( \beta \), then \( \alpha \text{res}_w G_1(w) w^{-k-1} + \beta \text{res}_w G_2(w) w^{-k-1} = \text{res}_w (\alpha G_1(w) + \beta G_2(w)) w^{-k-1} \). Using the residue representation of the binomials and the linearity property, then the sum over \( J \) in Eq. (8) gives

\[
R_1(l)' = \frac{1}{25} \sum_{s=1}^{l} (-1)^{s+1} 5^s \frac{2l}{l+s} \left( \frac{l+s}{l-s} \right) \left[ \sqrt{5} - 5^{1-s} \text{res}_w \left( \frac{(1 + w)2^s}{(w)^s((1 + w)^2 - 5w)} \right) \right]. \tag{10}
\]

The first sum in Eq. (10) may be written in terms of the Lucas numbers through the normalized Chebyshev polynomial of the first kind \([19]\). Let this sum be denoted by \( L_1(l) \), then

\[
L_1(l) = (-1)^{l+1} \frac{\sqrt{5}}{25} \sum_{k=0}^{l-1} (-1)^k 5^{l-k} \frac{2l}{2l-k} \binom{2l-k}{k} = (-1)^{l+1} \frac{\sqrt{5}}{25} \left[ C_{2l}(\sqrt{5}) - (-1)^l 2 \right], \tag{11}
\]

where

\[
C_{2l}(x) = 2T_{2l}(x/2) = \sum_{k=0}^{l} (-1)^k \frac{2l}{2l-k} \binom{2l-k}{k} x^{2l-2k}
\]

is the normalized Chebyshev polynomial of the first kind and

\[
T_{2l}(x/2) = \frac{1}{2} \left[ \left( \frac{x}{2} + \frac{\sqrt{(x/2)^2 - 1}}{2l} \right)^{2l} + \left( \frac{x}{2} - \frac{\sqrt{(x/2)^2 - 1}}{2l} \right)^{2l} \right].
\]

Thus,

\[
L_1(l) = (-1)^{l+1} \frac{\sqrt{5}}{25} \left( L_{2l} - 2(-1)^l \right) = (-1)^{l+1} \frac{\sqrt{5}}{5} F_l^2,
\]

where

\[
L_{2l} = \left( \frac{1 + \sqrt{5}}{2} \right)^{2l} + \left( \frac{1 - \sqrt{5}}{2} \right)^{2l},
\]

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are the bisection of the Lucas numbers, and

\[ F_l = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^l - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^l, \]

are the well-known Fibonacci numbers.

The second sum in Eq. (10) denoted by \( F_1(l) \), may be computed to give

\[ F_1(l) = (-1)^l \frac{1}{5} \sum_{w=0}^{\infty} \frac{1}{(w-(3-\sqrt{5})/2)(w-(3+\sqrt{5})/2)} \left( C_2l \left( \frac{1}{\sqrt{w}} + \sqrt{w} \right) - (-1)^l/2 \right) \]

\[ = (-1)^l+1 \frac{1}{5\sqrt{5}} \sum_{w=0}^{\infty} \frac{1}{w^l} \left( \frac{1}{(w-(3-\sqrt{5})/2))} - \frac{1}{(w-(3+\sqrt{5})/2))} \right) \]

\[ = (-1)^l \frac{1}{5} F_{2l}. \]  

(12)

Thus, the sum \( R_1(l)' \) is given by the following closed form formula

\[ R_1(l)' = (-1)^l+1 \frac{\sqrt{5}}{5} F_{l}^2 + (-1)^l \frac{1}{5} F_{2l} \]  

(13)

For \( N \) even, the sum \( R_1(l)'' \) given in Eq. (9) is

\[ R_1(l)'' = \frac{4}{25} \sum_{J=0}^{\infty} (1/5)^{J-5} \sum_{l=1}^{l} (-1)^{l+1} \frac{l}{l+s} \left( \sum_{p=1}^{[J/N]} \left( \frac{2J}{J - pN} \right) \right) \]

(14)

where

\[ \sum_{p=1}^{[J/N]} \left( \frac{2J}{J - pN} \right) = \text{res}_w (1+w)^{2J} w^{-J-1} \left( \frac{w^N}{1-w^N} \right). \]  

(15)

Then the second term \( R_1(l)'' \), becomes

\[ R_1(l)'' = \frac{2}{5} \sum_{s=1}^{l} (-1)^{s+1} 5^s \frac{2l}{l+s} \left( \frac{l+s}{l-s} \right) \text{res}_w \left( \frac{w^N}{1-w^N} \right) \frac{1}{5w - (1+w)^2}. \]  

(16)

the acceptable residue is \( w = (3 - \sqrt{5})/2 \), and so the expression for \( R_1(l)'' \) is

\[ R_1(l)'' = \frac{2}{5\sqrt{5}} \sum_{s=1}^{l} (-1)^{s+1} 5^s \frac{2l}{l+s} \left( \frac{l+s}{l-s} \right) \left( \frac{\left( \frac{3-\sqrt{5}}{2} \right)^N}{1 - \left( \frac{3-\sqrt{5}}{2} \right)^N} \right). \]  

(17)

The sum over \( s \), was done previously, see Eq. (11), therefore,

\[ R_1(l)'' = (-1)^l+1 \frac{2}{\sqrt{5}} F_{l}^2 \left( \frac{\left( \frac{3-\sqrt{5}}{2} \right)^N}{1 - \left( \frac{3-\sqrt{5}}{2} \right)^N} \right). \]  

(18)
Finally, the exact formula for the effective resistance between any two vertices of the graph $C_N(1, 2)$, reads

$$R(l) = \frac{1}{5} l(1 - l/N) + (-1)^{l+1} \frac{\sqrt{5}}{5} F_l^2 + (-1)^l \frac{1}{5} F_{2l}$$

$$+ (-1)^{l+1} \frac{2}{\sqrt{5}} F_l^2 \left( \frac{\left(\frac{3-\sqrt{5}}{2}\right)^N}{1 - \left(\frac{3-\sqrt{5}}{2}\right)^N} \right)$$

$$= \frac{1}{5} l(1 - l/N) + (-1)^{l+1} \frac{F_l^2}{\sqrt{5}} \left( \frac{1 + \left(\frac{3-\sqrt{5}}{2}\right)^N}{1 - \left(\frac{3-\sqrt{5}}{2}\right)^N} \right) + (-1)^l \frac{F_{2l}}{5}. \quad (19)$$

Our formula is consistent with the recursion relation, $R(l + 1) = l(1 - l/N) + 2R(1) - 3R(l) - R(l-1)$. The effective resistance $R(l)$ given by Eq. (4) enjoys the following symmetry $R(l) = R(N - l)$, this is obvious from the first term of our formula given in Eq. (19), however, from the second term, this symmetry is not obvious. If our formula were to be correct, then the symmetry enjoyed by the effective resistance $R(l)$ would give the following identity,

$$\frac{1}{5} \left( F_{2(N-l)} - F_{2l} \right) = \frac{1}{\sqrt{5}} \left( F_{2(N-l)} - F_l^2 \right) \left( \frac{1 + \left(\frac{3-\sqrt{5}}{2}\right)^N}{1 - \left(\frac{3-\sqrt{5}}{2}\right)^N} \right). \quad (20)$$

This turns out to be correct and follows simply by using the different expressions of the Fibonacci numbers.

The effective resistance considered so far, assumes that the number of vertices is even, the odd case, however, is not so different as far as the above computations are concerned, the only difference between the two cases is that the sum over $p$ of the binomial given by Eq. (15), should be replaced by

$$\sum_{p=1}^{[l/N]} (-1)^p \binom{2J}{J - pN} = -\text{res}_w (1 + w)^{2J} w^{J-1} \left( \frac{w^N}{1 + w^N} \right). \quad (21)$$

Therefore, the exact effective resistance of the graph $C_N(1, 2)$, for $N$ odd reads

$$R(l) = \frac{1}{5} \frac{l}{N} (N - l) + (-1)^{l+1} \frac{F_l^2}{\sqrt{5}} \left( \frac{1 - \left(\frac{3-\sqrt{5}}{2}\right)^N}{1 + \left(\frac{3-\sqrt{5}}{2}\right)^N} \right) + (-1)^l \frac{F_{2l}}{5}. \quad (22)$$

The symmetry of the effective resistance in this case implies the following identity

$$\frac{1}{5} \left( F_{2(N-l)} + F_{2l} \right) = \frac{1}{\sqrt{5}} \left( F_{2(N-l)}^2 + F_l^2 \right) \left( \frac{1 - \left(\frac{3-\sqrt{5}}{2}\right)^N}{1 + \left(\frac{3-\sqrt{5}}{2}\right)^N} \right). \quad (23)$$

Therefore, the computed effective resistance $R(l)$ is correct and satisfies the hidden symmetry property $R(l) = R(N - l)$.

Having computed the effective resistance of the graph $C_N(1, 2)$, we now compute the total effective resistance of $C_N(1, 2)$, this is also known as the Kirchhoff index [7]. The total effective resistance of $C_N(1, 2)$ in terms of the effective resistance $R(l)$, is
\[ R(C_N(1, 2)) = \sum_{i<j} R_{ij} = \frac{N}{2} \sum_{l=1}^{N-1} R(l). \]

In order to sum over all the effective resistances given by the formula (19) for \( N \) even, one needs to evaluate the alternating sums \( \sum_{l=1}^{N-1} (-1)^l F_l^2 \) and \( \sum_{l=1}^{N-1} (-1)^l F_{2l}^2 \). The first alternating sum may be obtained using the identity due to Lucas \( L_{2l} = F_{2l+2} - F_{2l-2} \), to give

\[ \sum_{l=1}^{N-1} (-1)^l F_l^2 = -\frac{1}{5} (F_{2N-1} - 1 + 2N). \]

Similarly, the second alternating sum may be evaluated from another Lucas formula \( F_{2l}^2 = F_{2l+1}^2 - F_{2l-1}^2 \), to give

\[ \sum_{l=1}^{N-1} (-1)^l F_{2l} = F_{N-1}^2 - F_N^2 - 1. \]

Therefore the exact formula for the total effective resistance reads

\[
R(C_N(1, 2)) = \frac{1}{5} N^3 - N \frac{1}{12} + \frac{N}{2} \frac{1}{5} (F_{2N-1} + 2N - 1) \left( \frac{1 + \left( \frac{3-\sqrt{5}}{2} \right)^N}{1 - \left( \frac{3-\sqrt{5}}{2} \right)^N} \right) + \frac{N}{2} \frac{1}{5} (F_{N-1}^2 - F_N^2 - 1).
\]

For \( N \) odd, the total effective resistance is

\[
R(C_N(1, 2)) = \frac{1}{5} N^3 - N \frac{1}{12} - \frac{N}{2} \frac{1}{5} (F_{2N-1} - 2N + 1) \left( \frac{1 - \left( \frac{3-\sqrt{5}}{2} \right)^N}{1 + \left( \frac{3-\sqrt{5}}{2} \right)^N} \right) + \frac{N}{2} \frac{1}{5} (F_N^2 - F_{N-1}^2 - 1).
\]

3 Simple Random Walk on the Graph \( C_N(1, 2) \)

The connection between simple random walks and electrical networks via the effective resistances in electrical networks is well established [4–6]. In particular, the covering and commute times of random walks in graphs can be determined by the effective resistance [5,6]. The simple random walk on a graph \( G \) is defined by the jumping probability \( p_{ij} \) between nearest neighbor vertices \( i \) and \( j \):

\[
p_{ij} = \begin{cases} 
\frac{1}{d(i)} & \text{if } i, j \text{ are adjacent vertices,} \\
0 & \text{otherwise}
\end{cases}
\]

here, \( d(i) \) is the degree of the vertex \( i \).

This may be interpreted as the probability the walk move from vertex \( i \) to vertex \( j \), given that we are at vertex \( i \). The simple random walk on a graph \( G \), may be described by the first passage time, or the hitting time FPT, \( H_{ij} \), this is the expected number of jumps that the walk takes until it lands on \( j \), assuming the walk starts at \( i \).
The other related parameter is the commute time $C_{ij}$, this is the expected number of jumps in a random walk starting at $i$ before vertex $j$ and then vertex $i$ is reached again, that is, $C_{ij} = H_{ij} + H_{ji}$. The discussion of hitting times in terms of the effective resistance seems to be due to Chandra et al. [6]. In particular, it has been proved by Chandra et al., that the commute time $C_{ij}$ is equal to $2|E|R_{ij}$.

It has been shown in [20], that the total effective resistance of a graph $G$, $R(G) = \sum_{i<j} R_{ij}$, may be written as

$$R(G) = N \sum_{n=1}^{N-1} \frac{1}{\lambda_n},$$

where $\lambda_n$ are the eigenvalues of the Laplacian of a graph $G$. Thus, the total effective resistance is an invariant quantity of a graph $G$, for symmetric graphs like the circulant graphs with the property $H_{ij} = H_{ji}$, this quantity reads,

$$R(G) = \frac{1}{2|E|} \sum_{i<j} (H_{ij} + H_{ji}) = \frac{1}{|E|} \sum_{i<j} H_{ij}$$

(27)

that is, the total effective resistance of the symmetric graph $G$, is the hitting time averaged over all pairs of vertices.

If the expression for the effective resistance is known in a closed form, then so are the total effective resistance, the FPT and MFPT. Since the graph $C_N(1, 2)$ has rotational symmetry, that is, the FPT may be written as $H_{0,l} = |E|R(l)$, and with the total number of edges $|E| = 2N$, then, the exact expression for the FPT of the simple random walk on the $N$-cycle graph with 4-nearest neighbors is

$$H_{0,l} = \begin{cases} \frac{2}{5}l(N - l) + (-1)^{l+1} \frac{2N}{\sqrt{5}} F_{2l} \left( \frac{1 + (\frac{1-\sqrt{5}}{2})^N}{1 - (\frac{1+\sqrt{5}}{2})^N} \right) + (-1)^{l} \frac{2N}{\sqrt{5}} F_{2l} & \text{for } N \text{ even}, \\ \frac{2}{5}l(N - l) + (-1)^{l+1} \frac{2N}{\sqrt{5}} F_{2l} \left( \frac{1 - (\frac{1-\sqrt{5}}{2})^N}{1 + (\frac{1+\sqrt{5}}{2})^N} \right) + (-1)^{l} \frac{2N}{\sqrt{5}} F_{2l} & \text{for } N \text{ odd} \end{cases}$$

(28)

One may check that our expressions for the effective resistance of the $N$-cycle with $2p$ neighbors, and hence that of the first passage time agree with those in [11]. This may be checked as follows; the eigenvalues of the laplacian of the $N$-cycle graph with $2p$ neighbors are; $\lambda_n = 4 \sum_{m=1}^{p} \sin^2(mn\pi/N)$. Thus, the two point resistance formula may be written as

$$R(l) = \frac{1}{N} \sum_{n=1}^{N-1} \sin^2(nl\pi/N) \left( \sum_{m=1}^{p} \sin^2 mn\pi/N \right)^{-1}.$$ 

The first passage time $H_{0,l}$ is $|E|R(l)$, where the number of edges for the $N$-cycle with $2p$ neighbors is $|E| = pN$, and so

$$H_{0,l} = p \sum_{n=1}^{N-1} \sin^2(nl\pi/N) \left( \sum_{m=1}^{p} \sin^2 mn\pi/N \right)^{-1}.$$ 

This is exactly equivalent to the formula given in [11].

The MFPT, may be obtained from the following simple formula that holds for all regular graphs, i.e., all vertices that have the same degree $d$. This formula may be derived as follows;

$$H_{0,l} = \frac{1}{N} \sum_{l=1}^{N-1} H_{0,l} = \frac{2|E|}{N^2} \sum_{l=1}^{N-1} \frac{N-1}{2} R(l) = d \frac{N}{R(G)}.$$ 

(29)
where we have used the formula $|E| = Nd$ for regular graphs. Since $d = 4$ for $C_N(1, 2)$, the closed formula for the MFPT is

$$
H_{0,l} = \left\{ \begin{array}{ll}
\frac{4}{5} N^2 - \frac{1}{12} & + \frac{2}{3} (F_{2N-1} + 2N - 1) \left( \frac{1 + \left( \frac{3-\sqrt{5}}{2} \right)^N}{1 - \left( \frac{3-\sqrt{5}}{2} \right)^N} \right) \text{ for } N \text{ even} \\
\frac{4}{5} N^2 - \frac{1}{12} & - \frac{2}{3} (F_{2N-1} - 2N + 1) \left( \frac{1 - \left( \frac{3+\sqrt{5}}{2} \right)^N}{1 + \left( \frac{3+\sqrt{5}}{2} \right)^N} \right) \text{ for } N \text{ odd} 
\end{array} \right.
$$

By using the formula $R(G) = N \sum_{n=1}^{N-1} \frac{1}{\lambda_n}$, then the equivalent formula for the MFPT is

$$
H_{0,l} = \sum_{n=1}^{N-1} \frac{1}{\sin^2(n\pi/N) + \sin^2(2n\pi/N)}.
$$

This is the eigentime identity relating the mean first passage time to the eigenvalues of the Laplacian of the regular graph $C_N(1, 2)$ of degree 4.

In the literature, the eigentime identity is given in terms of the eigenvalues of the normalized Laplacian $\mathcal{L}$ [10], for regular graphs one has, $\mathcal{L} = D^{-1}L$, i.e., the eigentime identity is independent of the Laplacian used. It is not difficult to show that the formula to compute the MFPT between vertex 0 and vertex $l$, on the $N$-cycle graph with $2p$ neighbors, is

$$
H_{0,l} = p \sum_{n=1}^{N-1} \left( 2 \sum_{m=1}^{p} \sin^2 mn\pi/N \right)^{-1},
$$

for $p = 1$, gives the well known eigentime identity on the $N$-cycle [12], and for $p = 2$ gives Eq. (31).

### 4 The Graph $C_N(1, 2)$ minus 2$N$ vertices

Having obtained a closed formula for the effective resistance of the graph $C_N(1, 2)$, next we compute the two-point resistance of the graph $C_N(1, 2)$ minus 2$N$ vertices, the total number of adjacent vertices in the $N$-cycle graph. Let us denote this graph by $C_N^{-2N}(1, 2)$, the eigenvalues of this graph are given by $\lambda_n = \sin^2 2n\pi/N$, then the effective resistance may be written as

$$
R(l) = \frac{1}{N} \sum_{n=1}^{N-1} \frac{\sin^2 nl\pi/N}{\sin^2 2n\pi/N}.
$$

Before computing the effective resistance, we make the following observations, for $N$ even, the graph $C_N^{-2N}(1, 2)$ splits into two $N/2$-cycles, see Fig. 2. Therefore, the effective resistance for $N$ even reeds;

$$
R(l) = \frac{2}{N} \sum_{n=1}^{N/2-1} \frac{\sin^2 2nl\pi/N}{\sin^2 2n\pi/N} = l(1 - 2l/N),
$$

in obtaining the above equation we used Wu’s formula Eq. (5).

Now, if $N$ is odd, then the graph $C_N^{-2N}(1, 2)$ is connected, see Fig. 3. Thus, the effective resistance is given by Eq. (32). This is a variant of the Wu’s [3] two-point resistance of the cycle graph. Using the trigonometrical identity
we may show that the effective resistance satisfy the following recursion

\[ R(l + 1) = l(1 - l/N) + 1/2(N - 1/N) - 2R(l) - R(l - 1), \]  

(35)

where the first term in the above equation is nothing but the effective resistance of the $N$-cycle graph. From the above recursion, one may propose the following formula for the effective resistance
\[
R(l) = \begin{cases} 
\frac{l}{2} \left(1 - \frac{l}{2N}\right) & \text{for } l \text{ even,} \\
\frac{1}{4} \left(N - \frac{l^2}{N}\right) & \text{for } l \text{ odd.}
\end{cases}
\] (36)

It is not difficult to check that our formula given by Eq. (36) is correct, i.e., satisfy the above recursion formula given by Eq. (35). To do so, one has to take into account that for \(l\) even, both \(l + 1\) and \(l - 1\) are odd, similarly for \(l\) odd, both \(l + 1\) and \(l - 1\) are even. As a result, the effective resistance of the connected graph \(C_{N}^{-2N}(1, 2)\) is

\[
R(l) = \frac{1}{N} \sum_{n=1}^{N-1} \frac{\sin^2 n l \pi / N}{\sin^2 2n \pi / N} = \begin{cases} 
\frac{l}{2} \left(1 - \frac{l}{2N}\right) & \text{for } l \text{ even,} \\
\frac{1}{4} \left(N - \frac{l^2}{N}\right) & \text{for } l \text{ odd.}
\end{cases}
\] (37)

By using the above formula, one can easily check the symmetry enjoyed by the effective resistance \(R(l) = R(N - l)\). This formula will play a crucial role in obtaining a closed formula for the effective resistance of the circular ladder.

5 Discussion

To conclude, in this work, we were able to obtain exact formulas for the effective resistance, the total effective resistance, the FPT and the MFPT of the simple random walk on the \(N\)-cycle with 4-nearest neighbors, all these quantities are written in terms of the Fibonacci numbers. The techniques used in this paper are related to those given by Chair [13], in which we computed the effective resistance of the complete graph minus \(N\) edges, where \(N\) is odd.

The effective resistance was written in terms of certain numbers that we called the Bejaia and the Pisa numbers. These numbers generalize the bisected Fibonacci and Lucas numbers. It was shown recently that the effective resistance of a \(2 \times N\) resistor network, is written in terms of the Pisa numbers.\(^1\)

We may check that our formula for the effective resistance is correct using the Foster first identity [16], namely,

\[
\sum_{i \sim j} R_{ij} = N - 1,
\]

where the sum is taken over all pairs of adjacent vertices. Due to the symmetry that we have for the graph \(C_N(1, 2)\), the sum over all effective resistances that connect adjacent vertices may be written as

\[
\sum_{i \sim j} R_{ij} = NR(1) + NR(2).
\]

From our expression for the effective resistance given by Eq. (19), and substituting \(F_1 = F_2 = 1\), then a simple computation gives indeed the Foster identity \(\sum_{i \sim j} R_{ij} = N - 1\). One might as well check that the hitting time satisfies the set of equations [10] given by

\[
H_{i, j} = 1 + \frac{1}{d(i)} \sum_{k \sim i} H_{k, j},
\]

\(^1\) Work in progress.
or equivalently,

\[ \sum_{k \sim i} (H_{i,j} - H_{k,j}) = d(i). \]

For the \( N \)-cycle with 4-nearest neighbors, set \( i = 0, j = 1 \) then the above equation reduces to \( 3H_{0,1} - H_{0,2} - H_{0,3} = 4 \), that is, the degree for the graph \( C_N(1, 2) \), in obtaining this result we have used our equation for the hitting time, (28) and the fact that \( F_1 = F_2 = 1, F_3 = 2 \).

It is interesting to note that by identifying the total effective resistance \( R(C_N(1, 2)) \) given in the Eqs. (24), (25) with the invariant quantity

\[ N \sum_{n=1}^{N-1} \frac{1}{4 \sin^2(n\pi/N) + 4 \sin^2(2n\pi/N)}, \]

then, we deduce the following identities,

\[ \sum_{n=1}^{N-1} \frac{1}{\sin^2 n\pi/N} = \frac{N^2 - 1}{3}, \quad (38) \]

\[ \sum_{n=1}^{N-1} \frac{1}{(1 + 4 \cos^2 n\pi/N)} = \frac{1}{2} (F_{2N-1} + 2N - 1) \frac{1}{\sqrt{5}} \left( \frac{1 + \left(3 - \sqrt{5} \right)^N}{1 - \left(3 - \sqrt{5} \right)^N} \right) + \frac{1}{2} (F_{N-1}^2 - F_N^2 - 1) \quad \text{for } N \text{ even,} \]

\[ \sum_{n=1}^{N-1} \frac{1}{(1 + 4 \cos^2 n\pi/N)} = -\frac{1}{2} (F_{2N-1} - 2N + 1) \frac{1}{\sqrt{5}} \left( \frac{1 - \left(3 - \sqrt{5} \right)^N}{1 + \left(3 - \sqrt{5} \right)^N} \right) + \frac{1}{2} (F_N^2 - F_{N-1}^2 - 1) \quad \text{for } N \text{ odd.} \]

The first identity is well known, however, we have not seen the last two identities in the literature.

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