Chiral anomalies
in noncommutative YM theories

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Abstract. Using cohomological methods we discuss several issues related to chiral anomalies in noncommutative $U(N)$ YM theories in any even dimension. We show that for each dimension there is only one solution of the WZ consistency condition and that there cannot be any reducible anomaly, nor any mixed anomaly when the gauge group is a product group. We also clarify some puzzling aspects of the issue of the anomaly when chiral fermions are in the adjoint representation.

1 Introduction

The subject of chiral anomalies in noncommutative $U(N)$ gauge field theories has been addressed by several authors [1, 2, 3, 4, 5, 6, 7]. The generally accepted conclusion is that in order for noncommutative gauge theories to be anomaly–free they must be nonchiral (which includes also formally chiral theories with adjoint matter in D=4) and that mixed anomalies are absent. In this paper we would like to add some further evidence to these conclusions and extend them to dimensions other than 4.

The method we employ is based on the WZ consistency conditions [8] and relies on the concept of \textit{nc-locality} (almost an oxymoron), which means that the space of cochains
we consider is the same as in ordinary local field theories with the ordinary product replaced by the Weyl–Moyal product. This principle of nc–locality is suggested by one–loop renormalization of noncommutative field theories, where counterterms are precisely of the above type, and, in the cases in which the noncommutative field theories can be embedded in string theory in the presence of a B field, can be traced back to the properties of (tree and planar one–loop) string amplitudes, precisely to the fact that such string amplitudes factorize into noncommutative factors and ordinary string amplitudes. However we do not know whether nc-locality is compatible with IR–UR and with higher loops renormalizations. Therefore, for the time being we take it as a working hypothesis.

The advantage of using this method is that, once the formalism is established, many conclusions are evident without resorting to explicit Feynman diagram calculations.

This paper is an elaboration upon [4], where descent equations for anomalies in noncommutative theories were introduced. It is organized as follows. In the next section we introduce our notation. In section 3 we discuss the problem of deriving anomalies from the descent equations and concentrate in particular on the uniqueness of the solutions. Section 4 is devoted to the anomaly problem in the adjoint representation. We show that in $D = 4k$ this anomaly vanishes and in $D = 4k + 2$ is equal to $2N$ times the anomaly in the fundamental representation. In section 5 we summarize our results.

2 Notation and conventions

In the following we will consider $U(N)$ gauge theories in a noncommutative $\mathbb{R}^D$, with Moyal deformation parameters $\theta^{\mu\nu}$. The gauge potential will be denoted by $A^i_{\mu}$ with $i,j = 1,\ldots,N$ being the indices of the fundamental and antifundamental representation of $U(N)$. Next we introduce a basis of hermitean matrices $t^A = (t^A)^i_j$, (capital letters $A, B, \ldots = 0, \ldots N^2 - 1$ will denote indices in the Lie algebra $u(N)$), with the normalization

$$\text{tr}(t^A t^B) = \frac{1}{2} \delta^{AB}.$$  \hspace{1cm} (1)

This can be done, for example, by using a basis of hermitean matrices for the Lie algebra of $SU(N)$, $t^a$, (whenever necessary, lower case letters $a, b, \ldots = 1, \ldots N^2 - 1$ will denote indices in the adjoint of $su(N)$), and adjoining $t^0 = \frac{1}{\sqrt{2N}} 1_N$. The basis $t^A$ satisfies

$$[t^A, t^B] = i f_{ABC} t^C, \quad \{t^A, t^B\} = d_{ABC} t^C$$  \hspace{1cm} (2)

where $f_{ABC}$ is completely antisymmetric, $f_{abc}$ is the same as for $su(N)$ and $f_{0BC} = 0$; while $d_{ABC}$ is completely symmetric; $d_{abc}$ is the same as for $su(N)$, $d_{0BC} = \sqrt{\frac{2}{N}} \delta_{BC}$, $d_{00c} = 0$ and $d_{000} = \sqrt{\frac{2}{N}}$, see [4]. Here and henceforth summation over repeated indices is understood and upper or lower indices are used interchangeably since the metric is $\delta_{AB}$.

Using this basis we write

$$A^i_{\mu} \equiv A^B_{\mu} (t^B)^j_i, \quad A_{\mu} = A^B_{\mu} t^B, \quad A^B_{\mu} = 2 \text{tr}(t^B A_{\mu})$$
and $\text{tr}$ denotes the trace in the fundamental representation.

With this notation, the action of the chiral fermions in the fundamental representation interacting with an external gluon is

$$ S = \int d^D x \bar{\psi} \gamma^\mu (i \partial_\mu \psi + A_\mu P_+ \psi) $$

where $P_\pm = \frac{1}{2} (1 \pm \gamma^5)$ and $\gamma = i^{1-n} \gamma_0 \gamma_1 \cdots \gamma_{D-1}$, with $D = 2n$.

For later use we introduce also the $N^2 \times N^2$ matrices $F^A, D^A$

$$(F^A)^{BC} = f^{ABC}, \quad (D^A)^{BC} = d^{ABC}$$

They satisfy the commutation rules

$$[F^A, F^B] = f^{ABC} F^C, \quad [F^A, D^B] = f^{ABC} D^C, \quad [D^A, D^B] = -f^{ABC} F^C. $$

Now let us form the combination $G^A = \frac{1}{2} (D^A + i F^A)$. It is easy to prove that they are hermitean and satisfy the relations (use the identities in [9])

$$[G^A, G^B] = i f^{ABC} G^C, \quad \{G^A, G^B\} = d^{ABC} G^C$$

isomorphic to (2). We notice that, if $T$ denotes transposition, then

$$[G^A, (G^B)^T] = 0. $$

We also have

$$\hat{\text{tr}}(G^A G^B) = \frac{N}{2} \delta^{AB} $$

$\hat{\text{tr}}$ is the trace in the representation of $u(N)$ spanned by the $G^A$’s, while the symbol $\text{Tr}$ will be used in a generic sense without regard to a particular representation. In the following we will need to consider the combinations

$$\hat{A}_\mu = A_\mu^B G^B.$$

One may wonder what is the representation of $u(N)$ spanned by the generators $G^A$. This representation is equivalent to the direct sum of $N$ copies of the fundamental representation. A way to see this is by computing traces of generators. By repeatedly using (2) on one side and (3) on the other side, and, finally, utilizing (1) and (8) one can easily show that

$$\hat{\text{tr}}(G^{A_1} \cdots G^{A_n}) = N \text{tr}(t^{A_1} \cdots t^{A_n}).$$
3 Anomalies from cocycles

It is well-known by now that only the (anti)fundamental and the adjoint representations of \( u(N) \) extend to representations of the Lie algebra of noncommutative \( U(N) \) gauge transformations, [10]. So we can build noncommutative gauge theories only with the latter representations or direct sums of them.

Taking this into account, let us give a more detailed reformulation of the approach in [4]. To this end we consider a one-form gauge potential \( A = A_\mu dx^\mu = A^B X^B \), with gauge field strength two-form \( F = dA + iA \star A = F^B X^B \) and gauge transformation parameter \( C = C^B X^B \) (which we take to be a Grassmann–odd Faddeev–Popov ghost with ghost number 1). All these quantities are valued in the Lie algebra generated by \( X^A \) which stand either for \( t^A \) or \( G^A \)'s or by direct sums of them. They are therefore hermitean matrices.

The gauge (BRST) transformations are:

\[
\begin{align*}
  sA &= dC - iA \star C + iC \star A, \\
  sC &= -C \star C
\end{align*}
\]

(11)

\( d \) and \( s \) are assumed to commute. As a consequence the transformations (11) are nilpotent as in the ordinary case.

Now, as in ordinary theories, we would like to write down the descent equations [8, 11, 12, 13, 14] relevant to \( D = 2n \) dimensions, starting from a closed and BRST invariant \((2n + 2)\)-form \( \Omega_{2n+2} \), constructed as a polynomial of \( F \) and referred to as the top form:

\[
\begin{align*}
  \Omega_{2n+2} &= d\Omega_{2n+1}^0 \\
  s\Omega_{2n+1}^0 &= d\Omega_{2n}^1 \\
  s\Omega_{2n}^1 &= d\Omega_{2n-1}^2
\end{align*}
\]

(12)

where the upper index is the ghost number and the lower index is the form order. \( \Omega_{2n}^1 \) is the (unintegrated) anomaly. The virtue of the descent equations formalism is that it provides explicit expressions for anomalies and one is spared the details of the complicated verification that \( \Omega_{2n}^1 \) does satisfy the Wess–Zumino consistency conditions. The latter is an automatic consequence of the top form \( \Omega_{2n+2} \) being closed and invariant (and, of course, nontrivial, i.e. non–exact, otherwise the corresponding anomaly would be trivial).

However in noncommutative gauge theories there is a complication. The above method does not work straightforwardly, because there exists no closed invariant polynomial that can be built with the noncommutative curvature \( F \). But there is a way out that was pointed out in [4]: the differential space of cochains must be constituted by forms that are defined up to an overall cyclic permutations of the Moyal product factors involved. This will be spelt out in more detail in a moment (see the definition of cyclic equivalence below).

But, before, let us pause to make a comment on this method. At first sight it may look artificial, but it is a very effective method to derive the expression of the anomaly. What is relevant is that the last equation in (12) can now be rewritten as

\[
  s\Omega_{2n}^1 = d\Omega_{2n-1}^2 + \ldots
\]
where dots denote terms that can be cast in the form of graded \(*\)-commutators. It is well-known that these terms are total derivatives of the form \(\theta^{\mu\nu}\partial_\mu\ldots\). So upon integration, this equation gives

\[
s \left( \int d^Dx \Omega^1_{2n} \right) = 0
\]

which precisely says that \(\int d^Dx \Omega^1_{2n}\) satisfies the Wess-Zumino consistency conditions.

From now on the descent equations (12) have to be understood in the framework of the new definition of the BRST cohomology.

What the above discussion boils down to is that, in order to know what anomalies we have in a given theory, we can simply concentrate on the possible closed and invariant forms \(\Omega^2_{D+2}\) we can construct out of the curvature \(\mathcal{F}\). In [4] the forms considered were simply traces of \(*\)-products of \(\mathcal{F}\) in the fundamental representation. Here we wish to be more general. The Feynman rules tell us that the anomaly will contain traces of the matrices that appear in the fermion–gluon vertex, i.e. traces of \(t^A\) or \(G^A\) or direct sums of them (let us denote them collectively by \(X^A\)). However one cannot exclude a priori that these traces may have symmetry properties in some of the indices (similarly to what happens in ordinary theories). Therefore we will limit ourselves to writing the polynomials that are involved in the descent equations (12) as

\[
h^{A_1A_2\ldots A_k} E_1^{A_1} \ast E_2^{A_2} \ast \ldots \ast E_k^{A_k}
\]

where \(E_i^{A_i}\) is any form of the type \(C^{A_i}, A^{A_i}\) or exterior differentials of them, and \(h\) is a tensor obtained as a combination of traces of the appropriate generators. All the polynomials appearing in the descent equations are considered the same if they differ by a cyclic ordering of the factors \(E_1, \ldots, E_k\) (with the correct grading). We refer to the latter identification as cyclic equivalence.

As for the forms \(\Omega^2_{D+2}\), we will write them as

\[
h^{A_1A_2\ldots A_{n+1}} F^{A_1} \ast F^{A_2} \ast \ldots \ast F^{A_{n+1}}.
\]

Due to the cyclic equivalence, we can assume that the \(h^{A_1\ldots A_{n+1}}\) is cyclically symmetric.

As pointed out above the forms (13) must be closed and BRST invariant. Due to the Bianchi identity, these two requirements amount to the same property. Using

\[
\Gamma \ast \Lambda = G^{ABC} \Gamma^A \ast \Lambda^B X^C
\]

for any two forms \(\Gamma = \Gamma^A X^A\) and \(\Lambda = \Lambda^A X^A\) and cyclicity of \(h\), one can see that the latter must satisfy

\[
h^{A_1\ldots A_{n+1}}X^C G^{XBD} - h^{A_1\ldots A_{n+1}}X^C G^{XDC} = 0
\]

for any couple of contiguous indices \(B, C\). This set of constraints together with the cyclic symmetry in the indices characterize the tensors \(h\).
It is perhaps useful to remark that in ordinary theories the condition (16) is replaced by a weaker one, which therefore allows in general for more solutions.

Now, let us examine the consequences of (16). The simplest case, n=1 (D=2), is trivial; we can only have \( h_{AB} \sim \delta_{AB} \), which correspond to the trace of the product of two generators, and is easily seen to satisfy (16). The next case, n=2 (D=4), has two possibilities: either \( h_{ABC} = f_{ABC} \) or \( h_{ABC} = d_{ABC} \). Using the identities for the \( f,d \) tensors, see [9], one can see that (16) is not satisfied for either of these possibilities separately, while it is satisfied for the combination \( d_{ABC} + if_{ABC} \). But this precisely means that \( h_{ABC} \sim Tr(X^AX^BX^C) \). One can similarly proceed to higher dimensions and convince oneself that the only solution is in any case \( h_{A_1...A_n} \sim Tr(X^{A_1}...X^{A_n}) \).

Therefore we end up with the ansatz made in [4] for the top form of \( \Omega_{D+2} \), with the additional specification that \( F = F^AX^A \), where \( X^A \) can be \( t^A, G^A \) or direct sums of them. Now, one has simply to apply the formulas of [4] to get the anomalies in any even \( D = 2n \) dimension. It is the one determined by the top form \( \Omega_{D+2} = tr(F \star ... \star F) \) with \( n+1 \) entries. The corresponding (unintegrated) anomaly is given by the formula, see [4],

\[
\Omega_{2n}^1 = n \int_0^1 dt (t-1) Tr(dC \star A \star F \star ... \star F + dC \star F \star A \star ... \star F + \ldots + dC \star F \star F \star \ldots \star A) 
\]  

(17)

where the sum under the trace symbol includes \( n-1 \) terms. In [17] we have introduced a parameter \( t, 0 \leq t \leq 1 \), and the traditional notation \( F_t = tdA + it^2A \star A \).

For example the anomaly of the action (3) is obtained from the above formula by replacing \( A,C \) and \( F \) with the corresponding fields in the fundamental representation \( A,C \) and \( F \), respectively, by integrating the expression (17) over the space-time and multiplying it by the factor

\[
\frac{2^n}{(n+1)(4\pi)^n \Gamma(n+1)}.
\]

Let us now draw some conclusions. The first is that in noncommutative gauge theories, as opposed to ordinary ones, there cannot be reducible anomalies, that is anomalies derived from a top form made of product of traces such as \( Tr(F \star ... \star F) Tr(F \star ... \star F) \), the reason being that such forms are not closed nor invariant as one easily sees by applying cyclic equivalence [4].

A similar argument shows that in noncommutative gauge theories there cannot be mixed anomalies [3, 7]. For suppose we have a bifundamental gauge theory with gauge group \( U(N_1) \times U(N_2) \). In this theory we can have \( U(N_1) \) anomalies and \( U(N_2) \) anomalies, but not mixed \( U(N_1) \times U(N_2) \) anomalies. The reason is that the latter kind of anomalies should come for instance from a top form like \( Tr_1(F_1 \star ... \star F_1) Tr_2(F_2 \star ... \star F_2) \), where the

\footnote{One may think to get around this obstacle by allowing for cyclic equivalence in each trace separately, which amounts to changing the cohomology. In this case one would get a closed and invariant top form but such cyclic equivalence would clash with the integration rule in D dimensions, so that the resulting integral \( \int d^Dx \Omega_D \) would not satisfy the Wess-Zumino consistency conditions.}
first trace refers to $U(N_1)$ and the second to $U(N_2)$. For the same reason as before such top form is neither closed nor invariant.

Finally we would like to make a comment on the $U(1)$ anomaly inside a $U(N)$ theory. From what has been just said it is apparent that there is no room for a separate $U(1)$ anomaly (differently from ordinary $U(N)$ gauge theories). Anyhow one can verify it directly. For example, in the simplest case, $n = 1 (D = 2)$, we can try to split $h^{AB} = \frac{1}{2}\delta^{AB}$ into a $U(1)$ part $\delta^{00}$ and an $SU(N)$ part $\delta^{ab}$ and build two corresponding separate top forms. However it is easy to see that $\delta^{00}$ and $\delta^{ab}$ do not satisfy separately (16). This can be extended to higher dimensional cases and is of course in keeping with the impossibility to disentangle the $U(1)$ factor in a noncommutative $U(N)$ gauge theory.

4 Anomalies in the adjoint representation

In ordinary gauge theories with chiral fermions in the adjoint representation the chiral anomaly identically vanishes in $D = 4$ while it is nonvanishing in $D = 4k + 2$ dimensions. In ordinary gauge theories in $D = 4k$ one can verify this by a direct Feynman diagram computation. Or else one can get the explicit form of the adjoint anomaly via descent equations starting from the top form with $\mathcal{F}$ valued in the adjoint representation, i.e. expanded over the set of antisymmetric matrices $F^A$ ([3]). In this way one sees that, in $D = 4k$ dimensions, the top form (and consequently the anomaly) is determined by the trace of the symmetric product of $2k + 1$ antisymmetric $F^A$ matrices. Therefore it identically vanishes.

In noncommutative gauge theories the question of chiral anomaly in the adjoint representation looks at first a bit puzzling. Let us see why. In [3] the adjoint chiral anomaly has been shown to vanish in $D = 4$ by writing the action in terms of Majorana fermions and showing the vector nature of the vertex, and by a direct Feynman diagram calculation. On the other hand if we try to apply to this case the formula obtained from the descent equations we see immediately that in $D = 4$ we will never get zero, the reason being that $Tr(X^AX^BX^C)$ cannot vanish for any of the representations considered in the previous section, nor do we know of other representations of the group of gauge transformations whose generators, when inserted in the above trace, can give zero. So it is evident that for the adjoint representation the formulas obtained via the descent equations must be applied with a grain of salt.

To clarify the situation let us start from the relevant noncommutative action

$$S = \int d^Dx \bar{\psi}^j i \star \gamma^\mu (i \partial_\mu \psi^j + A^i_{jk} \star P_+ \psi^k_j - P_+ \psi^i_k \star A_{ij}^k).$$

(18)

We find it useful to rewrite this action in terms of the basis introduced in section 2: $\psi^j_i = \psi_A(t^a)^j_i$ and so on. The action (18) takes the form

$$S = \int d^Dx \bar{\psi} \star \gamma^\mu (i \partial_\mu \psi + \tilde{A}_\mu \star P_+ \psi - P_+ \psi \star \tilde{A}_\mu),$$

(19)
where we use a vector notation for $\psi = \{\psi^A\}$ and a matrix notation for the gauge potential, $\hat{A}_\mu = A^B_\mu G^B$. In particular the last term in (19) means

$$\bar{\psi} \ast \gamma^\mu P_+ \psi \ast \hat{A}_\mu = \bar{\psi}^B \ast \gamma^\mu P_+ \psi^C \ast (\hat{A}_\mu)_{CB}$$

This action is invariant under

$$\delta \hat{A}_\mu = \partial_\mu \lambda - i\hat{A}_\mu \ast \lambda + i\lambda \ast \hat{A}_\mu,$$

$$\delta \psi = i\lambda \ast \psi - i\psi \ast \lambda,$$

$$\delta \bar{\psi} = -i\lambda \ast \bar{\psi} + i\bar{\psi} \ast \hat{\lambda}$$

and, again, $\hat{\lambda} = \lambda^B G^B$.

Now let us introduce the charge conjugate field $\psi^c = C^\dagger \bar{\psi}^T$. The charge conjugation operator $C$ is defined to have the following properties:

$$C^\dagger C = 1, \quad C\gamma_\mu C^\dagger = -\gamma^T_\mu \quad (20)$$

Moreover we assume a metric $g_{\mu\nu}$ with signature $(+,-,\ldots,-)$ and $\gamma_0^\dagger = -\gamma_0$. As a consequence, in dimension $D = 2n$ we have

$$C\hat{\gamma} C^\dagger = (-1)^n \hat{\gamma}. \quad (21)$$

where $\hat{\gamma}$ has been defined in section 2. If we express the action (19) in terms of the charge–conjugate fields we get

$$S = \int d^D x \bar{\psi}^c \ast \gamma^\mu (i\partial_\mu \psi^c + \hat{A}_\mu \ast P_+^c \psi^c - P_+^c \psi^c \ast \hat{A}_\mu) \quad (22)$$

where $P_+^c = P_-$ in dimension $D = 4k$ and $P_+^c = P_+$ in dimension $D = 4k+2$. Since integrating over $\psi$ or $\psi^c$ in the path integral does not entail any difference, we see that, when computing the anomaly by Feynman diagrams techniques, (i) in dimension $D = 4k$, the action (19) and the action (22) give opposite contributions to the anomaly because they contain fermions with opposite chirality, therefore the anomaly must vanish, (ii) in dimension $D = 4k+2$, the action (19) and the action (22) give the same contribution, therefore the anomaly presumably will not vanish.

This clarifies the problem in $D = 4k$ dimensions. What remains for us to do is to compute the anomaly in $D = 4k+2$. The solution is actually very simple: from section 3 we learned that there is only one nontrivial cocycle in any even $D = 2n$ dimension, it is the one determined by the top form $\hat{\mathrm{tr}}(\hat{F} \ast \ldots \ast \hat{F})$ with $n+1$ entries. The corresponding (unintegrated) anomaly is given by formula (17).

Of course what remains to be determined is the coefficient in front of the anomaly for the present case. To determine it we must resort to other methods. Here we follow [6] and express the gauge current in (19) as the sum of two pieces

$$j_\mu^B = j_\mu^+_B + j_\mu^-_B \quad (23)$$

$$j_\mu^+ B = \bar{\psi} \ast G^B \gamma_\mu P_+ \psi,$$

$$j_\mu^- B = -\bar{\psi} \ast \gamma_\mu P_+ \psi G^B = \bar{\psi}^c \ast G^B \gamma_\mu P_+ \psi^c$$
Since the two pieces represent the same vertex (replacing $\psi$ by $\psi^c$) they must contribute the same amount to the anomaly, so, in fact, it is enough to compute the contribution from one, $j^+$ for instance.

Let us solve now an auxiliary problem. We remark that $j^+$ specifies the fermion-gluon interaction corresponding to the following action

$$S = \int d^D x \bar{\psi} \star \gamma^\mu (i\partial_\mu \psi + \hat{A}_\mu \star P_+ \psi)$$

(24)

which looks like (3), except that instead of the generators $t^A$ we have here the generators $G^A$. Therefore this action is invariant under

$$\delta \hat{A}_\mu = \partial_\mu \hat{\lambda} - i\hat{A}_\mu \star \hat{\lambda} + i\hat{\lambda} \star \hat{A}_\mu$$

$$\delta \psi = i\hat{\lambda} \star \psi, \quad \delta \bar{\psi} = -i\bar{\psi} \star \hat{\lambda}.$$  

We know how to compute the anomaly of (24) in any even dimension. Simply we apply the descent equations method to the closed invariant form $\Omega_{n+2} = \hat{\text{tr}}(\hat{F} \star \hat{F} \cdots \star \hat{F})$ with $n+1$ $\hat{F}$ entries, where $\hat{F}$ is the curvature of $\hat{A}$. From (5) we know that $\hat{\text{tr}}(\hat{F} \star \hat{F} \cdots \star \hat{F}) = N \text{tr}(F \star F \cdots \star F)$ where the RHS refers to the fundamental representation. Therefore the corresponding anomaly is $N$ times the anomaly in the fundamental representation.

As far as the adjoint anomaly is concerned, we would be therefore led to conclude that this anomaly cancels against the corresponding negative contribution from $j^-$ in $D = 4k$, while it gets doubled in $D = 4k+2$. Therefore our conclusion would be that in $D = 4k+2$ dimensions the chiral adjoint anomaly is $2N$ times the chiral anomaly in the fundamental representation.

However we are not quite finished yet. One may rightly object that above we did not really compute the anomaly of (24) or (24), but only twice the anomaly of (24). We have still to prove that there are no interference terms between $j^+$ and $j^-$. The lowest order contribution to the anomaly in $D = 4k+2$ comes from the $(k+2)$-point function of $j^+$ and the $(k+2)$-point function of $j^-$. They are equal and proportional to

$$\int d^D x \hat{\text{tr}}(\hat{C} \star d\hat{A} \star \ldots \star d\hat{A})$$

with $k+1$ $d\hat{A}$ entries. This is exactly the first term of the unique cocycle appropriate for $D = 4k+2$ and confirms what we said above. There are however other possible contributions which may come from mixed correlators with both $j^+$ and $j^-$ entries. The latter are proportional to traces of the type $\hat{\text{tr}}(G^{A_1} \ldots (G^{B_1})^T \ldots)$ in which some of the entries are transposed matrices. Thanks to eq. (6) we can regroup all the transposed matrices on the right. Then applying repeatedly formulas (4) as well as (7) it is easy to see that these traces are reducible. For instance

$$\hat{\text{tr}}(G^A G^B (G^C)^T (G^D)^T) = \frac{1}{N^2} \hat{\text{tr}}(G^A G^B) \hat{\text{tr}}(G^C G^D).$$

But we know that this cannot correspond to any BRST cocycle. Therefore mixed correlators with both $j^+$ and $j^-$ entries cannot contribute to the anomaly. Therefore our
Conclusion is that in $D = 4k + 2$ dimensions the chiral adjoint anomaly is $2N$ times the chiral anomaly in the fundamental representation.

The uniqueness of the anomaly cocycle is a distinctive element of noncommutative chiral gauge theories as compared to the ordinary ones. To see this let us take as an example $D = 6$ and let us analyse the distinct nontrivial ordinary cocycles contained in the only noncommutative one. The latter is determined by $h^{ABCD} = \text{tr}(t^A t^B t^C t^D)$. It should be noticed that when dealing with the ordinary $U(N)$ gauge theory only the completely symmetric part $h^{(ABCD)}$ of $h^{ABCD}$ is relevant, as the other parts of the cocycle vanish. It is then easy to list the independent ordinary cocycles. They are determined by:

- $h^{0000}$, which gives rise to the pure $U(1)$ anomaly;
- $h^{(0abc)} \sim d^{abc}$, which gives rise to a mixed $U(1) \times SU(N)$ anomaly;
- $h^{(00ab)} \sim \delta^{ab}$ which gives rise to another mixed $U(1) \times SU(N)$ anomaly;
- $h^{(abcd)}$ which is a combination of $\delta^{(ab}\delta^{cd)}$ and of $d^{(abc}d^{cd)}$; the first gives rise to a reducible $SU(N)$ anomaly, the second to the irreducible $SU(N)$ anomaly.

All these independent ordinary cocycles coalesce to form a unique noncommutative cocycle. Finally, it is interesting to remark that the ratio between the irreducible ordinary $SU(N)$ anomaly in the adjoint and in the fundamental representation is again $2N$.

5 Conclusion

In this paper, using the principle of nc–locality, we have calculated the anomalies in a chiral noncommutative $U(N)$ gauge theories for all admissible representations of the group of gauge transformations in any even dimension. We have shown in particular that, differently from ordinary theories,

- there do not exist reducible anomalies;
- there do not exist mixed anomalies;
- there exist only one possible anomaly for each even dimension.

Moreover we have explicitly calculated the chiral anomaly in the adjoint representation and found that

- it vanishes in $D = 4k$, as in ordinary theories;
- in $D = 4k + 2$ it equals $2N$ times the anomaly in the fundamental representation; in ordinary theories this property holds generally only for the irreducible part.
We add a short comment concerning covariant anomalies. Covariant anomalies are particular cases of consistent ones. They can be calculated from a nonchiral (i.e. with Dirac fermions) gauge theory by coupling it to a vector plus a pseudovector potential $V_\mu + \gamma A_\mu$, and computing the anomaly of the chiral current, \cite{15}. This anomaly satisfies WZ consistency conditions (more complicated than the above ones, see for instance \cite{16}). The covariant anomaly is obtained by eventually setting $A_\mu = 0$. There is therefore a one-to-one correspondence between covariant and consistent anomalies, they are determined by the same (unsymmetrized) traces of generators in the appropriate representations and the former (in a vector theory) vanish when and only when the latter (in the corresponding chiral theory) do.

Finally the implications of the results recently obtained for anomaly calculations on the lattice in \cite{17} are not clear to us.

Acknowledgments A.S. would like to thank SISSA for the hospitality extended to him during this research. L.B. would like to thank M.Henneaux and A.Schwimmer for an exchange of e-mail messages. This research was partially supported by the Italian MIUR under the program “Fisica Teorica delle Interazioni Fondamentali” and by the RFBR Grant no. 99-02-18417.

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