MULTIPLICATIVE FUNCTION MEAN VALUES:
ASYMPTOTIC ESTIMATES

P. D. T. A. ELLIOTT

1. Statement of results

For many studies in analytic number theory a natural object against which to measure
the mean-value of a complex-valued multiplicative arithmetic function \( n \to g(n) \) is the
mean-value of its attendant function \( n \to |g(n)| \).

This reflects the decomposition \( n \to |g(n)| \exp(i \arg g(n)) \) of a non-vanishing completely
multiplicative function into essentially a unitary character on the multiplicative group of
the positive rationals, and a homomorphism \( n \to \log |g(n)| \) of the positive rationals into the
additive reals.

Some fifty years ago, papers of Delange [3] 1961, Wirsing [11] 1961, [12] 1967, Halász [8]
1968, catalysed the general study of multiplicative functions and moved the field seriously
forward.

In the present paper I re-examine the theorems of Wirsing in the light of more recent
developments and apply related ideas to the consideration of two open-ended questions.

The following four cumulative theorems will be established, all new. Several auxiliary
propositions are also of independent interest.

Theorem 1. Let \( g \) be a non-negative multiplicative function, uniformly bounded on the
primes, for which the series \( \sum q^{-1}g(q) \), taken over the prime-powers \( q = p^k \) with \( k \geq 2 \),
converges, and for which the sums \( y^{-1} \sum_{q \leq y} g(q) \log q \), \( y \geq 2 \), are uniformly bounded.

Let \( h(n) \) be a complex-valued multiplicative function that satisfies \( |h(n)| \leq g(n) \).

Set \( G(x) = \sum_{n \leq x} n^{-1}g(n) \), \( H(x) = \sum_{n \leq x} n^{-1}h(n) \), \( x \geq 1 \).

Then

\[
H(x) = \left( \prod_{p \leq x} \left( 1 + \frac{h(p)}{p} + \frac{h(p^2)}{p^2} + \cdots \right) \left( 1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \cdots \right)^{-1} + o(1) \right) G(x)
\]

as \( x \to \infty \).

Remark. If the series \( \sum p^{-1}(g(p) - \Re h(p)) \) diverges or a sum \( \sum_{k=1}^{\infty} p^{-k}h(p^k) \) has the value
\(-1\), then the product over the primes may be omitted. Otherwise, the product has the
form \( AL(\log x) \), where \( A \) is a non-zero constant and \( L(y) \) a non-vanishing slowly oscillating
function of \( y \).
Theorem 2. Let \( g \) be a non-negative multiplicative function that is uniformly bounded on the primes. Assume that for a positive \( c \), and each \( b, 0 < b < 1 \),
\[
\liminf_{x \to \infty} ((1 - b) \log x)^{-1} \sum_{x^b < p \leq x} p^{-1} g(p) \log p \geq c.
\]

Then for some positive \( c_0 \) and all \( x \geq 2 \),
\[
\sum_{n \leq x} g(n) \geq \frac{c_0 x}{\log x} \prod_{p \leq x} \left( 1 + \frac{g(p) - \log p}{p} \right).
\]

Remark. Under the further assumptions on \( g \) in Theorem I, there is a similar upper bound.

For each positive real \( \tau \), \( \Delta(\tau) \) will denote a compact star-shaped region of the complex plane that contains the origin, has a representation
\[
\{ \rho e^{i\theta}, 0 \leq \theta < 2\pi, 0 \leq \rho \leq w(\theta) \},
\]
with average radius
\[
(2\pi)^{-1} \int_0^{2\pi} w(\theta) \, d\theta, \quad w(2\pi) = w(0),
\]
strictly less than \( \tau \).

Theorem 3. Let the multiplicative function \( g \) satisfy the hypotheses of Theorems I and II and let \( h \) be a complex-valued multiplicative function with \( |h(n)| \leq g(n) \) and values in \( \Delta(\tau) \).

Set
\[
A(x) = \sum_{n \leq x} g(n), \quad B(x) = \sum_{n \leq x} h(n).
\]

Then
\[
B(x) = \left( \prod_{p \leq x} \left( 1 + \frac{h(p)}{p} + \frac{h(p^2)}{p^2} + \cdots \right) \right) \left( \prod_{p \leq x} \left( 1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \cdots \right)^{-1} + o(1) \right) A(x)
\]
as \( x \to \infty \).

Theorem 4. Let the multiplicative function \( g \) satisfy the hypotheses of Theorems I and II and let \( h \) be a complex-valued multiplicative function with \( |h(n)| \leq g(n) \).

Then there are two possibilities.

(i) For some real \( t \) the series \( \sum p^{-1} (g(p) - \text{Re} \, h(p)p^it) \), taken over the primes, converges;
\[
B(x) = (1-it)^{-1} x^{-it} \prod_{p \leq x} \left( 1 + h(p)p^it + \cdots \right) \left( 1 + g(p)p^{-1} + \cdots \right)^{-1} A(x) + o(A(x)), \quad x \to \infty.
\]

(ii) There is no such \( t \), and
\[
B(x) = o(A(x)), \quad x \to \infty.
\]

Of particular interest in Theorems I and II is that beyond dominance by \( g \), there is no non-structural constraint upon the complex values of the function \( h \).
2. **Background**

Two central theorems of Wirsing’s 1967 paper run as follows.

**Satz 1.1.** Let \( \lambda(n) \) be a non-negative multiplicative function, uniformly bounded on the primes, that for a positive \( \tau \) satisfies

\[
\sum_{p \leq x} p^{-1} \log p \lambda(p) \sim \tau \log x, \quad x \to \infty.
\]

Assume further that the series \( \sum q^{-1} \lambda(q) \), taken over the prime-powers \( q = p^k \) with \( k \geq 2 \), converges, and that if \( \tau \leq 1 \) then \( \sum_{q \leq x} \lambda(q) \ll x(\log x)^{-1} \) holds for \( x \geq 2 \).

Then

\[
\sum_{n \leq x} \lambda(n) \sim \frac{e^{-\gamma \tau}}{\Gamma(\tau)} x \prod_{p \leq x} \left( 1 + \frac{\lambda(p)}{p} + \frac{\lambda(p^2)}{p^2} + \cdots \right), \quad x \to \infty,
\]

where \( \gamma \) is Euler’s constant.

**Satz 1.2.** Let \( \lambda(n) \) be a multiplicative function that satisfies the conditions of Satz 1.1. Let \( \lambda^*(n) \) be multiplicative, with values in \( \Delta(\tau) \) and satisfy \( |\lambda^*(n)| \leq \lambda(n) \).

Then

\[
\sum_{n \leq x} \lambda^*(n) = \frac{e^{-\gamma \tau}}{\Gamma(\tau)} x \prod_{p \leq x} \left( 1 + \frac{\lambda^*(p)}{p} + \frac{\lambda^*(p^2)}{p^2} + \cdots \right) + o \left( \sum_{n \leq x} \lambda(n) \right)
\]

as \( x \to \infty \).

In what follows, a product of the form

\[
\prod_{p \leq x} \left( 1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \cdots \right),
\]

when meaningful, may be denoted by \( \prod_x (f) \).

The two theorems of Wirsing may be compared to the following result of Elliott and Kish [6], subsuming ideas from Wirsing and Halász, loc. cit.

**Theorem 5.** Let \( 3/2 \leq Y \leq x \). Let \( g \) be a complex-valued multiplicative function that for positive constants \( \beta, c, c_1 \) satisfies \( |g(p)| \leq \beta; \)

\[
\sum_{w < p \leq x} p^{-1}(|g(p)| - c) \geq -c_1, \quad Y \leq w \leq x,
\]

on the primes. Suppose, further, that the series

\[
\sum_{q} q^{-1}|g(q)|(\log q)^\kappa, \quad \kappa = 1 + c\beta(c + \beta)^{-1},
\]

taken over the prime-powers \( q = p^k \) with \( k \geq 2 \), converges.

Then with

\[
\lambda = \min_{|t| \leq T} \sum_{Y < p \leq x} p^{-1}(|g(p)| - \Re g(p)p^it),
\]
\[
\sum_{n \leq x} g(n) \ll x (\log x)^{-1} \prod_{p \leq x} (1 + p^{-1}|g(p)|) \left( \exp(-\lambda c (c + \beta)^{-1}) + T^{-1/2} \right)
\]

uniformly for \(Y, x, T > 0\), the implied constant depending at most upon \(\beta, c, c_1\) and a bound for the sum of the series over higher prime-powers.

An extension of Theorem 5, a proof of which will be given following that for Theorem 4, obviates the awkward condition involving the factor \((\log q)^\kappa\).

**Theorem 6.** If the estimate in Theorem 5 is weakened to
\[
\sum_{n \leq x} g(n) \ll x (\log x)^{-1} \prod_{x} (|g|) \left( \exp(-\lambda c (c + \beta)^{-1}) + T^{-1/2} \right)^{c/(3c+1)},
\]

then the condition on the prime-power values \(g(p^k), k \geq 2\), may be relaxed to the convergence of the series \(\sum_{p, k \geq 2} p^{-k} |g(p^k)|\) and a uniform bound for the sums \(y^{-1} \sum_{p^k \leq y} |g(p^k)| \log p^k,\) \(y \geq 2\).

For the multiplicative function \(\lambda_0(n)\) defined to be \(\alpha, \beta\) with \(0 < \alpha < \beta\), on the primes in alternate intervals \((\exp(2^k), \exp(2^{k+1})], k = -1, 0, 1, 2, \ldots\), and to be zero on all other prime-powers,
\[
\lim_{x \to \infty} (\log x)^{-1} \sum_{p \leq x} p^{-1} \lambda_0(p) \log p
\]
does not exist, eliminating direct application of Sätze 1.1 and 1.2.

The lower bound of Theorem 2 is obtained in Elliott and Kish [6], Lemma 21, subject to the existence of a positive constant \(c_2\) so that for all large \(x\), \(\sum_{p \leq x} g(p) \log p \geq c_2 x\). By modifying \(\lambda_0\) to be zero on intervals \((y(\log y)^{-2}, y], y = \exp(2^k)\), we obtain a multiplicative function \(\lambda_1\) that will not satisfy such a criterion for any positive \(c_2\).

Never-the-less, Theorems 1, 2 and 4 may be applied to \(\lambda_0, \lambda_1\) with any dominated complex-valued multiplicative function, \(h\).

### 3. Proof of Theorem 4

It is convenient to introduce several preliminary results.

**Lemma 1.** The estimate
\[
\sum_{2 \leq n \leq x} g(n) \leq \left( \frac{x}{\log x} + \frac{10x}{(\log x)^2} \right) \bar{\Delta} \sum_{n \leq x} \frac{g(n)}{n}
\]

with
\[
\bar{\Delta} = \sup_{1 \leq y \leq x} y^{-1} \sum_{q \leq y} g(q) \log q,
\]

where \(q\) denotes a prime-power, holds uniformly for all non-negative real multiplicative functions \(g\), and all \(x \geq 2\).
A proof of Lemma 1 may be found in Elliott [4], Chapter 2, Lemma 2.2. It is immediate that
\[
\sum_{n \leq x} n^{-1} g(n) \leq \prod_{p \leq x} \left( 1 + \sum_{k \leq \log x / \log p} p^{-k} g(p^k) \right) \\
\leq \exp \left( \sum_{q \leq x} q^{-1} g(q) \right).
\]

A proof of the following qualitative corresponding lower bound, a result first obtained by Barban [1] using a different method, may be found in Lemma 20 of Elliott and Kish, [6].

**Lemma 2.** To each positive \( \beta \) there is a further positive \( c(\beta) \) so that a non-trivial non-negative multiplicative function, \( g \), that satisfies \( g(p) \leq \beta \) on the primes, also satisfies
\[
\sum_{n \leq x} g(n)n^{-1} \geq c(\beta) \prod_{p \leq x} (1 + p^{-1} g(p))
\]
uniformly for \( x \geq 1 \).

**Lemma 3.** Let \( g \) be a non-trivial non-negative multiplicative function uniformly bounded on the primes, for which the series \( \sum q^{-1} g(q) \), taken over the prime-powers \( q = p^k \) with \( k \geq 2 \), converges, and for which the sums \( y^{-1} \sum_{q \leq y} g(q) \log q \), \( y \geq 2 \), are uniformly bounded.

Then
\[
\sum_{u \leq n \leq v} \frac{g(n)}{n} \ll \left( \log \left( \frac{\log v}{\log u} \right) + \frac{1}{\log x} \right) \sum_{n \leq x} \frac{g(n)}{n}
\]
uniformly for \( x^{1/2} \leq u \leq v \leq x^{3/2} \), \( x \geq 2 \).

**Proof of Lemma 3** In view of the hypothesis on \( g \), Lemma 1 delivers the uniform estimate
\[
\sum_{n \leq y} g(n) \ll \frac{y}{\log y} \prod_{p \leq x^{3/2}} \left( 1 + \frac{g(p)}{p} \right), \quad 2 \leq y \leq x^{3/2},
\]
which Lemma 2 shows to be \( \ll y(\log y)^{-1} G(x) \). The asserted result then follows from an integration by parts.

For better appreciation the following theorem is given in both its abelian and tauberian aspects. A proof may be found, together with a history of the result from Feller [7] to Stadtmtüller and Trautner [10], in Bingham, Goldie and Teugels [2], Chapter 2, Theorem 2.10.1, pp. 116–118, and Korevaar [9], Chapter IV, Theorem 10.1, pp. 197–199.

Let \( C(y) \), \( D(y) \) be non-negative real-valued functions on the non-negative reals, non-decreasing and right continuous. To each corresponds a Laplace transform, typically
\[
s \to \tilde{C}(s) = \int_{0}^{\infty} e^{-sy} dC(y),
\]
here assumed to be defined for \( s > 0 \).
Lemma 4. Assume that for each $y > 1$

$$D^*(y) = \limsup_{n \to \infty} D(u)^{-1}D(uy)$$

is finite, $D$ implicitly assumed not to be identically zero.

If, for some constant $A$ and slowly-oscillating function $L(y)$,

$$C(y) = (AL(y) + o(1))D(y), \quad y \to \infty,$$

then

$$\tilde{C}(s) = (AL(s^{-1}) + o(1))\tilde{D}(s), \quad s \to 0^+.$$

Further, if $D^*(y) \to 1$ as $y \to 1+$, then the converse is valid.

Remark. The non-decreasing nature of $D$ ensures that $\lim D^*(y), y \to 1$, exists.

Completion of the proof of Theorem 1 We apply Lemma 4 to the pair $2G(e^x) + \Re(H(e^x))$, $G(e^x)$; to the pair with $\Im(H(e^x))$ in place of $\Re(H(e^x))$; and to the pair $G(e^x), G(e^{e^x})$.

Computation with Euler products shows $\tilde{C}(s), \tilde{D}(s)$, the Laplace transforms of the first pair, to exist for all positive $s$ and satisfy $\tilde{C}(s) = f(s)\tilde{D}(s)$, where

$$f(s) - 2 = \Re\left(\prod_p \left(1 + \sum_{k=1}^{\infty} p^{-k(1+s)}h(p^k)\right) \left(1 + \sum_{m=1}^{\infty} p^{-m(1+s)}g(p^m)\right)^{-1}\right).$$

In particular,

$$|f(s) - 2| \ll \exp\left(-\sum_p p^{-1-s}(g(p) - \Re h(p))\right),$$

so that if the series in the exponent diverges for $s = 0$, then $f(s) \to 2$ as $s \to 0+$, and we may apply Lemma 4 with $A = 2$, $L$ identically 1.

We may therefore assume the series $\sum p^{-1}(g(p) - \Re h(p))$ to converge.

From the Chebyshev bound $\pi(y) \ll y(\log y)^{-1}$, integration by parts shows the series $\sum_{p > x^\varepsilon} p^{-1} \exp(-\log p/\log x)$ to be bounded in terms of $\varepsilon$ alone. Since

$$|g(p) - h(p)|^2 \leq 2g(p)(g(p) - \Re h(p)),$$

an application of the Cauchy-Schwarz inequality, confined to the primes on which $g$ does not vanish, shows that

$$\sum_{p > x^\varepsilon} p^{-1-1/\log x}|g(p) - h(p)| \ll \left(\sum_{p > x^\varepsilon} g(p)p^{-1-1/\log x}\right)^{1/2} \left(\sum_{p > x^\varepsilon} p^{-1}(g(p) - \Re h(p))\right)^{1/2}$$

and $o(1)$ as $x \to \infty$.

Moreover,

$$\sum_{p \leq x^\varepsilon} (p^{-1} - p^{-1-1/\log x}) \ll \sum_{p \leq x^\varepsilon} p^{-1} \log p/\log x \ll \varepsilon,$$

the implied constant absolute for all values of $x$ sufficiently large in terms of $\varepsilon$. 
Letting $x \to \infty$, $\varepsilon \to 0+$, we see that as $x \to \infty$

$$f \left( (\log x)^{-1} \right) - 2 = \text{Re} \left( B \exp \left( \sum_{p \leq x} p^{-1} \text{Im} \left( h(p) \right) \right) \right) + o(1),$$

with $B$ the product of

$$\prod_p \left( 1 + \sum_{k=1}^{\infty} p^{-k} h(p^k) \right) \exp \left( -p^{-1} h(p) \right) \prod_p \left( 1 + \sum_{m=1}^{\infty} p^{-m} g(p^m) \right)^{-1} \exp \left( p^{-1} g(p) \right)$$

and $\exp(- \sum_p p^{-1} (g(p) - \text{Re} h(p)))$. Its genesis in terms of Euler products ensures that $|B| \leq 1$; moreover, $B$ will vanish only if for some prime $p$ the sum $1 + \sum_{k=1}^{\infty} p^{-k} h(p^k)$ vanishes.

Note that for any $\beta \geq 1$, the above argument shows that

$$\sum_{x < p \leq x^\beta} p^{-1} \text{Im} \left( h(p) \right) = - \sum_{x < p \leq x^\beta} p^{-1} \text{Im} \left( g(p) - h(p) \right)$$

$$\ll \left( \sum_{x < p \leq x^\beta} p^{-1} \right) \left( \sum_{p > x} p^{-1} |g(p) - h(p)|^2 \right)^{1/2} = o(1)$$

as $x \to \infty$, so that $\exp(\sum_{p \leq x^\beta} p^{-1} \text{Im} \left( h(p) \right))$ is a slowly oscillating function of $s$.

In view of Lemma 3

$$\lim_{y \to 1^+} \limsup_{u \to \infty} G(e^x)^{-1} G(e^{yx}) = 1.$$ Three applications of Lemma 4 in its Tauberian aspect, typically with $A = 1$,

$$L(s) = 2 + \text{Re} \left( B \exp \left( \sum_{p \leq e^x} p^{-1} \text{Im} \left( h(p) \right) \right) \right),$$

delivers the asymptotic estimate

$$H(e^x) = \left( f(x^{-1}) + o(1) \right) G(e^x), \quad x \to \infty,$$

from which Theorem 1 follows rapidly.

4. PROOF OF THEOREM 2

Again a preliminary result is advantageous.

Let $0 \leq g(p) \leq \beta$ for each prime, $p$.

If, for some $\tau > 0$,

$$\sum_{p \leq y} p^{-1} g(p) \log p \sim \tau \log y, \quad y \to \infty,$$

then for each $\varepsilon$, $0 < \varepsilon < 1$,

$$\liminf_{x \to \infty} (\varepsilon \log x)^{-1} \sum_{x^{1-\varepsilon} < p \leq x} p^{-1} \log p \geq \tau.$$ The converse need not be true, as may be seen from the example $\lambda_0$ in section 2.
However, the following converse is valid.

**Lemma 5.** Assume that for \( c > 0 \) and each \( \varepsilon, 0 < \varepsilon < 1 \), the function \( g(p) \), uniformly bounded on the primes, satisfies

\[
\liminf_{x \to \infty} (\varepsilon \log x)^{-1} \sum_{x^{1-\varepsilon} < p \leq x} p^{-1} g(p) \log p \geq c.
\]

Then for each \( \alpha, 0 < \alpha < c \), there is a subsequence of primes, \( r \), such that

\[
\lim_{x \to \infty} (\log x)^{-1} \sum_{r \leq x} r^{-1} g(r) \log r = \alpha.
\]

**Proof of Lemma 5.** We begin with an outline of the argument. Fix a prime \( t \) for which

\[
\sum_{p \leq t} p^{-1} g(p) \log p \geq \alpha \log t.
\]

We define a function \( \tilde{g}(p) \) by choosing, for each prime \( p \), to retain \( g(p) \) or to replace it by zero. For ease of notation \( \sum_{p \leq y} p^{-1} \tilde{g}(p) \log p \) will be denoted by \( S(y) \).

We choose \( \tilde{g}(p) = g(p) \) for \( p \leq t \).

The primes \( y_1 < y_2 < \cdots \) are defined successively as follows. We replace \( g(p) \) by zero on the primes following \( t \) until, for the first time, \( S(y)/\log y \) falls strictly below \( \alpha \). The corresponding value of \( y \) is \( y_1 \).

We choose \( \tilde{g}(p) = g(p) \) on the primes \( p > y_1 \) until, for the first time with \( y > y_1 \), the ratio \( S(y)/\log y \) climbs above \( \alpha \). The corresponding value of \( y \) is \( y_2 \); and so on.

Our initial aim is to show the turning values \( y_j \) not to be logarithmically far apart.

A few preliminary remarks are helpful.

Let \( 0 < \theta < 1, x \geq 2, 3/2 \leq y \leq x^\theta \). With \( 0 < \varepsilon < 1 - \theta \) determine the integer \( k \) by \( x^{(1-\varepsilon)k} < y \leq x^{(1-\varepsilon)(k-1)} = \psi \), so that \( k \geq 2 \). Assume that for all sufficiently large values of \( w \)

\[
\sum_{w^{1-\varepsilon} < p \leq w} p^{-1} g(p) \log p \geq \varepsilon c \log w.
\]

By partitioning the interval \( (x^{(1-\varepsilon)k}, x] \) into adjoining subintervals \( (x^{(1-\varepsilon)m}, x^{(1-\varepsilon)(m-1)}], m = 1, 2, \ldots, k \), we see that provided \( x^{(1-\varepsilon)k} \) is sufficiently large in terms of \( \varepsilon \),

\[
\sum_{y < p \leq x} p^{-1} g(p) \log p \geq c \log(x/\psi) \geq c(\log(x/y) - \log(\psi/y))
\]

\[
geq c \left(1 - \varepsilon(1-\theta)^{-1}\right) \log(x/y),
\]

since \( \log(\psi/y) \leq \log(\psi/\psi^{1-\varepsilon}) = \varepsilon \log \psi \leq \varepsilon \log x \leq \varepsilon(1-\theta)^{-1} \log(x/y) \).

For the purposes of proving Lemma 5, we may therefore replace its lower-bound hypothesis by:

*For each \( \varepsilon, 0 < \varepsilon < 1 \),

\[
\sum_{y < p \leq x} p^{-1} g(p) \log p \geq c \log(x/y)
\]

uniformly for \( 1 \leq y \leq x^{1-\varepsilon} \) and all \( x \) sufficiently large in terms of \( \varepsilon \).
It is clear that the initial prime \( t \) exists.

As a second preliminary remark, if \( 2 \leq y \leq w \), then
\[
(\log w)^{-1}S(w) - (\log y)^{-1}S(y) = ( (\log w)^{-1} - (\log y)^{-1} ) S(y) + (\log w)^{-1} \sum_{y < p \leq w} p^{-1}\overline{g}(p) \log p.
\]
Hence
\[
\left| (\log w)^{-1}S(w) - (\log y)^{-1}S(y) \right| \leq (\log w \log y)^{-1}S(y) \log(w/y) + c_0 (\log w)^{-1} \sum_{y < p \leq w} p^{-1}\log p \leq c_1 (\log(w/y) + 1)(\log w)^{-1}
\]
with a positive constant \( c_1 \) dependent at most upon the upper bound for the \( g(p) \). Here we have employed the elementary estimate \( \sum_{p \leq y} p^{-1}\log p = \log y + O(1) \), \( y \geq 2 \).

In particular, if \( y \) is a prime adjacent to a turning value \( y_k \), then
\[
S(y)/\log y - S(y_k)/\log y_k \ll (|\log(y_k/y)| + 1)/\log y_k \ll 1/\log y_k,
\]
since the ratio of successive increasing primes approaches 1.

We now show the \( y_j \) not to increase too rapidly.

Suppose that \( S(y_k)/\log y_k < \alpha \), so that for the next prime \( p > y_k \), \( g(p) \) is kept. In particular \( S(y_k) \geq \alpha \log y_k + O(1) \). If \( y_k < (\frac{1}{2}y_k+1)^{1-\varepsilon} < \frac{1}{2}y_k+1 \) and \( y_k \) is sufficiently large, then \( \frac{1}{2}y_k+1 y_k^{-1} > y_k \),
\[
S(\frac{1}{2}y_k+1) = S(\frac{1}{2}y_k+1) - S(y_k) + S(y_k) \\
\geq c \log(\frac{1}{2}y_k+1 y_k^{-1}) + \alpha \log y_k + O(1) \\
= \alpha \log(\frac{1}{2}y_k+1) + (c - \alpha) \log(\frac{1}{2}y_k+1 y_k^{-1}) + O(1).
\]
With \( w \) a nearest prime to \( \frac{1}{2}y_k+1 \), \( S(w)/\log w > \alpha \) before the next change point, \( y_{k+1} \).

Thus \( y_k \geq (\frac{1}{2}y_k+1)^{1-\varepsilon} \).

If \( S(y_k) \geq \alpha \log y_k \), then again \( S(y_k) = \alpha \log y_k + O(1) \), and \( \overline{g}(p) = 0 \) on the primes in the interval \( (y_k, \frac{1}{2}y_k+1) \). Hence
\[
S(\frac{1}{2}y_k+1)(\log(\frac{1}{2}y_k+1))^{-1} = S(y_k)(\log(\frac{1}{2}y_k+1))^{-1} \\
= \alpha \log y_k(\log y_k+1)^{-1} + O((\log y_k)^{-1}).
\]
If, now, \( y_k < y_k^{1-\varepsilon} \) and \( y_k \) is sufficiently large then
\[
S(\frac{1}{2}y_k+1)(\log(\frac{1}{2}y_k+1))^{-1} \leq \alpha(1 - \varepsilon) + O((\log y_k)^{-1}),
\]
again leading to a premature change point.

In this case \( y_k \geq y_k^{1-\varepsilon} \).

For all large values of \( y_k, \frac{1}{2}y_k+1^{1-\varepsilon} \leq y_k \leq y_{k+1} \). As a consequence
\[
S(y_{k+1})/\log y_{k+1} - S(y_k)/\log y_k \ll \log(y_{k+1}/y_k)/\log y_{k+1} \ll \varepsilon,
\]
the implied constant independent of \( \varepsilon \). Since \( S(y_k)/\log y_k = \alpha + O(1/\log y_k) \), \( S(y)/\log y - \alpha \ll \varepsilon \) for all sufficiently large values of \( y \), first for prime values then for otherwise arbitrary real values.
The construction of the function \( g \) does not depend upon the value of \( \varepsilon \) and we may apply the argument with \( \varepsilon = 2^{-m}, m = 1, 2, 3, \ldots \), in turn.

Lemma 5 is established.

Completion of the proof of Theorem 2

Let \( 0 < \alpha < c \) and let \( r \) run through a sequence of primes for which \( \sum_{r \leq y} r^{-1}g(r) \log r \sim \alpha \log y, \ y \to \infty \).

Define multiplicative functions \( g_j, j = 1, 2 \), by
\[
g_1(p) = \begin{cases} g(p) & \text{if } p \neq r, \\ 0 & \text{if } p = r, \end{cases} \quad g_2(p) = \begin{cases} 0 & \text{if } p \neq r, \\ g(p) & \text{if } p = r, \end{cases}
\]
and \( g_j(p^k) = 0 \) on all other prime powers.

On squarefree integers \( g \) coincides with \( g_1 * g_2 \), the Dirichlet convolution of \( g_1 \) and \( g_2 \); hence
\[
\sum_{n \leq x} g(n) \geq \sum_{u \leq \sqrt{x}} g_1(u) \sum_{v \leq x/u} g_2(v).
\]

Satz 1.1 of Wirsing (c.f. §2) gives for a typical innersum the asymptotic estimate
\[
(1 + o(1)) \frac{e^{-\gamma \alpha}}{\Gamma(\alpha)} \frac{x}{u \log(x/u)} \prod_{p \leq x/u} \left(1 + \frac{g_2(p)}{p}\right), \ x/u \to \infty.
\]
The doublesum thus exceeds a constant multiple of
\[
\frac{x}{\log x} \prod_{p \leq x} \left(1 + \frac{g_2(p)}{p}\right) \sum_{u \leq \sqrt{x}} \frac{g_1(u)}{u}.
\]
An appeal to Lemma 2 completes the proof.

5. PROOF OF THEOREM 3

Choose a real \( \alpha \) to lie strictly between the average radius of \( \Delta(c) \), and \( c \).

Choose a subsequence of primes \( r \) for which
\[
\sum_{r \leq y} r^{-1}g(r) \log r \sim \alpha \log y, \ y \to \infty.
\]

We define multiplicative functions \( g_j, j = 1, 2 \), by
\[
g_1(p^k) = \begin{cases} g(p^k) & \text{if } p \neq r, \\ 0 & \text{otherwise}, \end{cases} \quad g_2(p^k) = \begin{cases} g(p^k) & \text{if } p = r, \\ 0 & \text{otherwise}. \end{cases}
\]
The function \( g \) has a Dirichlet convolution representation \( g_1 * g_2 \).

We likewise define multiplicative functions \( h_j, j = 1, 2 \), so that \( h = h_1 * h_2, |h_j| \leq g_j, j = 1, 2 \). There is a representation
\[
M = \sum_{n \leq x} h(n) = \sum_{u \leq x} h_1(u) \sum_{v \leq x/u} h_2(v).
\]
Let \(0 < \varepsilon < 1/2\). We remove the contribution from the terms with \(u \leq x^\varepsilon\) and \(x^{1-\varepsilon} < u \leq x\). Typically, by Lemma 1

\[
\sum_{u \leq x^\varepsilon} \frac{g_1(u)}{u} \sum_{v \leq x/u} g_2(v) \ll \sum_{u \leq x^\varepsilon} \frac{g_1(u)}{u} \frac{x}{u \log(x/u)} \prod_{p \leq x/u} \left(1 + \frac{g_2(p)}{p}\right) + \ldots
\]

\[
\ll \frac{x}{\log x} \prod_{x} (g_2) \sum_{u \leq x^\varepsilon} \frac{g_1(u)}{u}.
\]

Moreover,

\[
\sum_{u \leq x^\varepsilon} \frac{g_1(u)}{u} \ll \prod_{p \leq x^\varepsilon} \left(1 + \frac{g_1(p)}{p}\right) \ll \prod_{x^\varepsilon < p \leq x} \left(1 + \frac{g_1(p)}{p}\right)^{-1}.
\]

From the lower bound hypothesis on \(g\) and the construction of the sequence \(r\), an integration by parts shows that

\[
\sum_{x^\varepsilon < p \leq x} \frac{1}{p} g_1(p) \geq \frac{1}{2} (c - \alpha) \log \frac{1}{\varepsilon} + O(1).
\]

The contribution to \(M\) from the terms with \(u \leq x^\varepsilon\) is

\[
\ll \varepsilon^{(c-\alpha)/2} x (\log x)^{-1} \prod_{x} (g), \quad x \to \infty.
\]

For the range \(x^{1-\varepsilon} < u \leq x, v \leq x^\varepsilon\) and we may invert summations, replacing \((c - \alpha)/2\), as the exponent of \(\varepsilon\), by \(\alpha/2\).

We are reduced to the estimation of

\[
M_\varepsilon = \sum_{x^\varepsilon < u \leq x^{1-\varepsilon}} h_1(u) \sum_{v \leq x/u} h_2(v).
\]

Since \(h_2\) inherits its properties relative to \(g_2\) from \(h\), applied to the innersum in \(M_\varepsilon\), Satz 1.2 delivers the asymptotic estimate

\[
\frac{e^{-\gamma\alpha} x/u}{\Gamma(\alpha) \log(x/u)} \left( \prod_{x/u} (h_2) + o \left( \prod_{x/u} (g_2) \right) \right), \quad x \to \infty,
\]

uniformly for \(x^\varepsilon \leq u \leq x^{1-\varepsilon}\).

Introducing factors \(\exp(-p^{-1} h_2(p))\), \(\exp(-p^{-1} g_2(p))\), respectively, the ratio \(\prod_y (h_2) \prod_y (g_2)^{-1}\) has an estimate

\[
(B + o(1)) \exp \left( - \sum_{p \leq y} p^{-1} (g_2(p) - h_2(p)) \right), \quad y \to \infty,
\]

with

\[
B = \prod_p \left( \sum_{k=0}^{\infty} p^{-k} h_2(p^k) \exp(-p^{-1} h_2(p)) \right) \prod_p \left( \sum_{m=0}^{\infty} p^{-m} g_2(p^m) \right)^{-1} \exp(p^{-1} g_2(p)).
\]
If the series $\sum p^{-1}(g_2(p) - \text{Re}(h_2(p)))$ diverges, then uniformly for $x^\varepsilon \leq u \leq x^{1-\varepsilon}$,
\[
\prod_{x/u} (h_2) \prod_{x/u} (g_2)^{-1} = \prod_{x} (h_2) \prod_{x} (g_2)^{-1} + o(1), \quad x \to \infty,
\]
since both product ratios asymptotically vanish.

If the series $\sum p^{-1}(g(p) - \text{Re}(h_2(p)))$ converges, then we may argue as in the proof of Theorem 1. For each positive real $\tau, \, 0 < \tau \leq 1$,
\[
\sum_{x^\varepsilon < p \leq x} p^{-1}(g_2(p) - h_2(p)) \to 0, \quad x \to \infty,
\]
and we formally obtain the same asymptotic equality of ratios.

Likewise, there is a representation
\[
(x/y)^{-\alpha} \prod_{y} (g_2) = (C + o(1)) \exp \left( \sum_{p \leq y} \frac{1}{p} g_2(p) - \alpha \log \log y \right), \quad y \to \infty,
\]
with
\[
C = \prod_{p} \left( \sum_{m=1}^{\infty} p^{-m} g_2(p^m) \right) \exp(-p^{-1}g_2(p)).
\]
An integration by parts shows that for each $\tau, \, 0 < \tau < 1$,
\[
\sum_{x^\varepsilon < p \leq x} p^{-1}g_2(p) + \alpha \log \tau \to 0, \quad x \to \infty,
\]
so that
\[
(x/y)^{-\alpha} \prod_{x/u} (g_2) = (x/y)^{-\alpha} \prod_{x} (g_2) + o(1), \quad x \to \infty.
\]
Altogether, the innersum of $M_\varepsilon$ has the estimate
\[
\frac{e^{-\gamma\alpha}}{\Gamma(\alpha)} \frac{x}{u(\log x)^\alpha} \cdot \frac{1}{(\log(x/u))^{1-\alpha}} \left( \prod_{x} (h_2) + o \left( \prod_{x} (g_2) \right) \right), \quad x \to \infty,
\]
uniformly for $x^\varepsilon \leq u \leq x^{1-\varepsilon}$.

The error terms contribute towards $M_\varepsilon$
\[
o \left( \frac{x}{\log x} \prod_{x} (g) \sum_{x^\varepsilon < u \leq x^{1-\varepsilon}} \frac{g_1(u)}{u} \right) = o \left( \frac{x}{\log x} \prod_{x} (g) \right), \quad x \to \infty,
\]
within which $M_\varepsilon$ has the estimate
\[
\frac{e^{-\gamma}}{\Gamma(\alpha)} \frac{x}{(\log x)^\alpha} \prod_{x} (h_2) \sum_{x^\varepsilon < u \leq x^{1-\varepsilon}} \frac{h_1(u)}{u(\log(x/u))^{1-\alpha}}.
\]

Setting
\[
H_1(y) = \sum_{n \leq y} h_1(n)n^{-1}, \quad G_1(y) = \sum_{n \leq y} g_1(n)n^{-1},
\]
an integration by parts gives a representation
\[
\sum_{x^\varepsilon < u \leq x^{1-\varepsilon}} \frac{h_1(u)}{u(\log(x/u))^{1-\alpha}} = \frac{H_1(x^{1-\varepsilon})}{(\varepsilon \log x)^{1-\alpha}} \frac{H_1(x^\varepsilon)}{((1-\varepsilon) \log x)^{1-\alpha}} - \left(1-\alpha\right) \int_{x^\varepsilon}^{x^{1-\varepsilon}} \frac{H_1(y)}{y(\log(y/x))^{2-\alpha}} dy,
\]
provided \( x^\varepsilon, x^{1-\varepsilon} \) are not positive integers, a situation that we may avoid by choosing a slightly larger value of \( x \).

According to Theorem 1,

\[
H_1(y) = \left( \prod_y (h_1) \prod_y (g_1)^{-1} + o(1) \right) G_1(y), \quad y \to \infty,
\]

where, as above, we may replace the products \( \prod_y \) by \( \prod_x \), uniformly for \( x^{\varepsilon} \leq y \leq x^{1-\varepsilon} \), \( x \to \infty \).

As a consequence,

\[
\sum_{x^{\varepsilon} < u \leq x^{1-\varepsilon}} \frac{h_1(u)}{u \log(x/u)}^{1-\alpha} = \left( \prod_x (h_1) \prod_x (g_1)^{-1} + o(1) \right) \sum_{x^{\varepsilon} < u \leq x^{1-\varepsilon}} \frac{g_1(u)}{u \log(x/u)}^{1-\alpha}, \quad x \to \infty.
\]

Once again, the argument is expedited by considering \( 2G_1(x) + \text{Re} (H_1(x)) \), \( 2G_1(x) + \text{Im} (H_1(x)) \).

Rewinding,

\[
M_\varepsilon = \prod_x (h_1) \prod_x (g_1)^{-1} \frac{e^{-\gamma \alpha}}{\Gamma(\alpha)} \frac{x}{(\log x)^\alpha} \prod_x (h_2) \sum_{x^{\varepsilon} < u \leq x^{1-\varepsilon}} \frac{g_1(u)}{u \log(x/u)}^{1-\alpha} + o \left( \frac{x}{\log x} \prod_x (g) \right)
\]

\[
= \prod_x (h) \prod_x (g)^{-1} \sum_{x^{\varepsilon} < u \leq x^{1-\varepsilon}} \frac{x}{\log(x/u)} \prod_{x/u} (g_2) \frac{g_1(u)}{u} + o \left( \frac{x}{\log x} \prod_x (g) \right)
\]

\[
= \prod_x (h) \prod_x (g)^{-1} \sum_{x^{\varepsilon} < u \leq x^{1-\varepsilon}} \frac{g_1(u) \sum_{v \leq x/u} g_2(v)}{\log(x/u)} + o \left( \frac{x}{\log x} \prod_x (g) \right)
\]

\[
= \prod_x (h) \prod_x (g)^{-1} \sum_{n \leq x} g(n) + O \left( \varepsilon^{\nu} \sum_{n \leq x} g(n) \right)
\]

with \( \nu = \min((\varepsilon - \alpha)/2, \alpha/2) \) and, for all sufficiently large values of \( x \), an implied constant independent of \( \varepsilon \).

A similar estimate holds for \( M \).

Letting \( x \to \infty, \varepsilon \to 0+ \) completes the proof.

6. Proof of Theorem [4]

Case (i). From the assumption that the series \( \sum p^{-1}(g(p) - \text{Re} (h(p)p^it)) \) converges, for each positive \( \delta \) the series taken over the primes \( p \) for which \( g(p) - \text{Re} (h(p)p^it) > \delta \) also converges.

On the remaining primes

\[
|g(p) - h(p)p^it|^2 \leq 2g(p)(g(p) - \text{Re} (h(p)p^it)) \leq 2\beta \delta.
\]

The values of \( h(p)p^it \) lie in a box about the real axis, with corners at \((-2\beta \delta)^{1/2}, \pm(2\beta \delta)^{1/2})\), \((\beta + (2\beta \delta)^{1/2}, \pm(2\beta \delta)^{1/2})\), and area \( 2(2\beta \delta)^{1/2}(\beta + 2(2\beta \delta)^{1/2}) \).
Assuming that $\delta$ is sufficiently small and, in particular, that $2(2\beta\delta)^{1/2} \leq \beta$, this is a region of the type $\Delta(\tau)$ with an average radius
\[
\frac{1}{2\pi} \int_0^{2\pi} w(\theta) \, d\theta \leq \left( \frac{1}{2\pi} \int_0^{2\pi} w(\theta)^2 \, d\theta \right)^{1/2} \leq \left( 4\pi^{-1}(2\beta^3\delta)^{1/2} \right)^{1/2}
\]
that can be fixed at a value as small as desired.

We may follow the proof of Theorem 3 first selecting a subsequence of primes $r$ for which $(\log x)^{-1} \sum_{r \leq x} r^{-1} g(r) \log r \to \alpha$, $x \to \infty$, then removing from that subsequence those primes for which $h(p) p^{it}$ does not belong to a region $\Delta(\alpha)$ defined by a value of $\delta$ that satisfies $4\pi^{-1}(2\beta^3\delta)^{1/2} < \alpha^2$.

The removal of these exceptional primes does not affect the existence or the value of the asymptotic limit for $(\log x)^{-1} \sum_{r \leq x} r^{-1} g(r) \log r$.

The upshot is an asymptotic estimate
\[
\sum_{n \leq x} h(n) n^{it} = \prod_{p \leq x} (1 + h(p) p^{it-1} + \cdots) \left( \prod_x (g) \right)^{-1} \sum_{n \leq x} g(n) + o \left( \sum_{n \leq x} g(n) \right), \quad x \to \infty.
\]

We would like to integrate by parts and remove the weight $n^{it}$ from $h(n) n^{it}$, but have insufficient control over the values of the function $h$. Since, in some sense, we are considering the ratio $h(n) n^{it} (g(n))^{-1}$, at an appropriate moment we switch the weight $n^{it}$ from $h$ to $g$ and consider the ratio $h(n) (g(n)n^{-it})^{-1}$.

Following the argument for Theorem 3 the study of the sum $\sum_{n \leq x} h(n)$ is reduced to that of
\[
\widetilde{M}_\varepsilon = \sum_{x^{\varepsilon} < u \leq x^{1-\varepsilon}} h_1(u) \sum_{v \leq x/u} h_2(v),
\]
where Theorem 3 is applicable to the pair $h_2(n)n^{it}$, $g_2(n)$. There is a corresponding estimate
\[
\sum_{n \leq y} h_2(n) n^{it} = L(\log y) \sum_{n \leq y} g_2(n) + o \left( \sum_{n \leq y} g_2(n) \right), \quad y \to \infty,
\]
with
\[
L(\log y) = \prod_{p \leq y} \left( 1 + h_2(p) p^{it-1} + \cdots \right) \left( \prod_y (g_2) \right)^{-1}, \quad y \geq 2.
\]

Set
\[
H_2(y) = \sum_{n \leq y} h_2(n)n^{it}, \quad G_2(y) = \sum_{n \leq y} g_2(n), \quad y \geq 1/2.
\]
An integration by parts gives a representation
\[
\sum_{n \leq y} h_2(n) = y^{-it} H_2(y) + it \int_{1/2}^y w^{-it-1} H_2(w) \, dw,
\]
provided $y$ is not an integer. Since $G_2(w) \ll w(\log w)^{-1} \prod_w(g_2)$, $w \geq 2$,
\[
\int_2^x w^{-1}G_2(w) \, dw \ll \prod_x(g_2) \int_2^x (\log w)^{-1} \, dw \\
\leq x(\log x)^{-1} \prod_x(g_2) \ll G_2(x), \quad x \geq 2.
\]
Hence
\[
\sum_{n \leq y} h_2(n) = y^{-it}L(\log y)G_2(y) + it \int_2^y w^{-it-1}L(\log w)G_2(w) \, dw + o(G_2(y)), \quad y \to \infty.
\]
As in the proof of Theorem 3 within an acceptable error $L(\log w)$, for $y^\varepsilon \leq w \leq y$, may be replaced by $L(\log y)$ and factored out of the representation:
\[
\sum_{n \leq y} h_2(n) = L(\log y) \left( y^{-it}G_2(y) + it \int_2^y w^{-it-1}G_2(w) \, dw \right) + o(G_2(y)), \quad y \to \infty.
\]
We appeal to the asymptotic estimate
\[
G_2(x) = (1 + o(1)) \frac{e^{-\gamma \alpha}}{\Gamma(\alpha)} \frac{x}{\log x} \prod_x(g_2), \quad x \to \infty,
\]
vouchsafed by Satz 1.1. Once again, as for Theorem 3, we employ the slow oscillation of the function $\prod_x(g_2)(\log x)^{-\alpha}$ to obtain a representation
\[
\sum_{n \leq y} h_2(n) = \frac{e^{-\gamma \alpha}}{\Gamma(\alpha)} L(\log x) \prod_x(g_2) \frac{y^1-it}{(\log y)^1-\alpha} + it \int_2^y \frac{w^{-it}}{(\log w)^{1-\alpha}} \, dw + o(G_2(y))
\]
\[
\quad = \frac{e^{-\gamma \alpha}}{\Gamma(\alpha)} L(\log x) \prod_x(g_2) \frac{y^1-it}{(1-it)(\log y)^{1-\alpha}} + o(G_2(y)),
\]
uniformly for $x^\varepsilon \leq y \leq x$, as $x \to \infty$; stepping from $w$ to $y$ to $x$.

Accordingly,
\[
\widetilde{M}_x = \frac{e^{-\gamma \alpha}}{\Gamma(\alpha)} \frac{x^1-it}{1-it} L(\log x) \prod_x(g_2) (\log x)^{\alpha} \sum_{x^\varepsilon \leq u \leq x^{1-\varepsilon}} \frac{h_1(u)u^{it}}{u(\log x/u)^{1-\alpha}} + o(G(x)), \quad x \to \infty.
\]
We may now formally follow the argument for Theorem 3 the rôle of $h_1(n)$ there here played by $h_1(n) n^{it}$, although on a slightly different set of primes. Eventually only the extra factor $x^{-it}(1-it)^{-1}$ remains.

Case (ii). The series $\sum p^{-1}(g(p) - \Re (h(p)p^{it}))$ diverges for every real $t$. The partial sums of this series are non-decreasing in $x$ and continuous in $t$. Divergence of the series is uniform on every compact interval $|t| \leq T$ and Theorem 3 follows from an application of Theorem 6.

Remark. Under the hypothesis of Case (i) the series $\sum p^{-1}|g(p) - h(p)p^{it}|^2$ converges. The series $\sum p^{-1}g(p) - |h(p)||^2$ and $\sum p^{-1}g(p)|1-e^{it}p^{it}|^2$, where $h(p) = |h(p)|e^{it}$, then also converge.
Suppose further that, for some positive integer \( k \), \( h(p)^k \) is real. The inequality \( |1 - z^k| \leq k|1 - z| \), valid for every \( z \) in the complex unit disc, guarantees the series \( \sum p^{-1} g(p)|1 - p^{2ikt}|^2 \) to converge.

In the present circumstances \( \sum_{p \leq x} p^{-1} g(p) \geq (c + o(1)) \log \log x \) as \( x \to \infty \) and an application of Lemma 15 from Elliott and Kish [5] shows that \( t = 0 \).

A simple example is given by \( h(n) = g(n) \chi(n) \), where \( \chi \) is a Dirichlet character.

The argument of this remark may be given a topological aspect by defining a metric \( \sigma(f,g) = \left( \sum p^{-1} |f(p) - g(p)|^2 \right)^{1/2} \) on equivalence classes of multiplicative functions that coincide of the primes, and restricting study to those functions \( g \) whose distance \( \sigma(g,g_0) \) to a fixed multiplicative function \( g_0 \) is defined, i.e. finite. The topological space of complex-valued multiplicative functions is in this manner locally metrised and correspondingly disconnected.

7. Proof of Theorem 6

We assume the new, weaker restraints upon \( g \). If \( g \) is exponentially multiplicative, i.e. \( g(p^k) = g(p)^k/k! \), and \( |g(p)| \leq \beta \), then for any \( \gamma \) the series \( \sum_{p,k \geq 2} p^{-k} |g(p^k)| (\log p^k)^\gamma \) converges, so that Theorem 4 is applicable. Indeed, for such functions the original exposition of Elliott and Kish, [6] Theorem 2, already contains a proof.

In general, we define an exponentially multiplicative function \( g_1 \) by \( g_1(p) = g(p) \), and a complementary multiplicative function \( g_2 \) by Dirichlet convolution: \( g = g_1 * g_2 \).

Calculation with Euler products shows that \( g_2(p) = 0 \) and for \( k \geq 2 \),

\[
g_2(p^k) = \sum_{r=0}^{k} \frac{(r)!}{(r! - 1)} (-g(p))^r g(p^{k-r}).
\]

In particular,

\[
|g_2(p^k)| \leq \sum_{r=0}^{k} (r! - 1)^{1/\beta^r} |g(p^{k-r})|, \quad k \geq 2.
\]

As a consequence

\[
\sum_{p,k \geq 2} p^{-k} |g_2(p^k)| \leq \sum_{r=0}^{\infty} (r! - 1)^{1/\beta^r} \sum_{p,k \geq 2} p^{-k} |g(p^{k-r})|
\]

\[
\leq \left( \frac{3}{2} \beta^2 + \frac{1}{4} \beta^3 \right) \sum p^{-2} + \left( 1 + \frac{1}{2} \beta^2 \right) \sum_{p,k \geq 2} p^{-k} |g(p^k)|,
\]

and converges.

Moreover,

\[
\sum_{p^k \leq y} |g_2(p^k)| \leq \sum_{r=0}^{\infty} (r! - 1)^{1/\beta^r} \sum_{p^k \leq y, k \geq 2} |g(p^{k-r})|
\]

\[
\ll \sum_{r=0}^{\infty} (r! - 1)^{1/\beta^r} y (\log y)^{-1} \ll y (\log y)^{-1}
\]

uniformly for \( y \geq 2 \).
We may apply Lemma 1 and obtain for \(|g_2|\) the uniform estimate
\[
\sum_{n \leq y} |g_2(n)| \ll y(\log y)^{-1}, \quad y \geq 2.
\]

With \(\delta\) a real number to be chosen presently in the range \(0 < \delta < 1\),
\[
\rho = \exp \left( -\frac{c}{c + \beta} \lambda \right) + T^{-1/2},
\]
as in the statement of Theorem 5, we define \(w = \exp(\rho^\delta \log x)\), so that \(w\) is effectively a function of \(x\) for \(x \geq 2\).

It is convenient to note that we may assume \(\rho^\delta \leq 1/2\), otherwise Theorem 6 follows directly from Lemma 1.

Moreover, provided \(2\delta \beta c < c + \beta\) and \(Y\) does not exceed a certain fixed power of \(x\), which we may likewise assume, \(Y \leq w\).

For otherwise \(\log x/\log Y \leq \rho^\delta \leq \exp(\delta c \beta + \beta \lambda) \ll \exp(\delta c \beta + \beta \lambda)\).

In particular, uniformly for \(w < y \leq x\),
\[
\min_{|t| \leq T} \sum_{Y < p \leq y} p^{-1}(|g(p)| - \text{Re}(g(p)p^it)) \geq \lambda - 2 \sum_{w < p \leq x} p^{-1}|g(p)| \geq \lambda + 2\delta \beta \log \rho + O(1).
\]

Applied to \(g_1\) over the same range of \(y\)-values, Theorem 5 delivers an estimate
\[
\sum_{n \leq y} g_1(n) \ll \frac{y}{\log y} \prod_{y} (|g_1|) \left( \exp \left( -\frac{c\lambda}{c + \beta} \right) \rho^{-2\delta \beta c/(c + \beta)} + T^{-1/2} \right)
\]
\[
\ll \frac{y}{\log y} \prod_{y} (|g_1|) \rho^{1-2\delta \beta c/(c + \beta)},
\]
this last step somewhat wasteful.

We decompose the mean-value of \(g\) into two sums:
\[
\sum_{n \leq x} g(n) = \sum_{b \leq x/w} g_2(b) \sum_{a \leq x/b} g_1(a) + \sum_{a < w} g_1(a) \sum_{x/w < b \leq x/a} g_2(b).
\]

The first doublesum is
\[
\ll \sum_{b \leq x/w} |g_2(b)| x b^{-1}(\log(x/b))^{-1} \prod_{x/b} (|g_1|) \rho^{1-2\delta \beta c/(c + \beta)}
\]
\[
\ll x(\log x)^{-1} \prod_{x} (|g|) \rho^{1-2\delta \beta c/(c + \beta) - \delta}.
\]

The second doublesum is
\[
\ll \sum_{a < w} |g_1(a)| x a^{-1}(\log(x/a))^{-1}
\]
and \( w \leq x^{1/2} \), so that the bound does not exceed a constant multiple of
\[
x(\log x)^{-1} \prod_{p \leq w} (1 + p^{-1}|g(p)|) \ll x(\log x)^{-1} \prod_{x} (|g|) \exp \left( - \sum_{w < p \leq x} p^{-1}|g(p)| \right).
\]
According to the lower bound hypothesis on \( |g(p)| \) in Theorem 5, still in force in Theorem 6, noting that \( w \geq Y \),
\[
\sum_{w < p \leq x} p^{-1}|g(p)| \geq c \sum_{w < p \leq x} p^{-1} + O(1) \geq -\delta c \log \rho + O(1).
\]
Altogther,
\[
\sum_{n \leq x} g(n) \ll \frac{x}{\log x} \prod_{x} (|g|)(\rho^{1-\delta c_0} + \rho^{\delta c})
\]
with \( c_0 = 2\beta c + \beta)^{-1} + 1 \).

We choose \( \delta \) to satisfy \( 1 - \delta c_0 = \delta c \). The earlier condition \( 2\delta \beta c < \beta + c \) is amply satisfied, \( c_0 \) increases with \( \beta \) and \( \delta c \) descends to a limiting value \( c(3c + 1)^{-1} \).

8. Concluding Remarks

The hypothesis on \( |g| \) in Theorem 6 remains essentially weaker than that on \( g \) in Theorem 4. What might a best-possible condition on \( g \) be in order to guarantee the validity of Theorem 4?

Likewise, what might the weakest hypothesis on \( g \) be in order to guarantee the validity of the lower bound in Theorem 2?

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**Department of Mathematics, University of Colorado Boulder, Boulder, Colorado 80309-0395 USA**

*E-mail address: pdtae@euclid.colorado.edu*