Green’s Function of the Relativistic Coulomb System via
Duru-Kleinert Method

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Abstract

In this paper the fixed-energy amplitude (Green’s function) of the relativistic
Coulomb system is solved by Duru-Kleinert (DK) method. In the course of
the calculations we observe an equivalence between the relativistic Coulomb
system and a radial oscillator.

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I. INTRODUCTION

Undoubtedly, the hydrogen atom is one of the most interesting system in quantum mechanics. In the beginnings of this century, it once symbolized the success of Bohr’s quantum theory and Schrödinger’s new wave mechanics, and later pushing the progress of relativistic quantum mechanics with the fine-structure constant and quantum electrodynamics with the observation of the Lamb-shift.

In the past 70 years, for solving the hydrogen atom many different ways such as the local dynamics approaches [1-5], the global dynamics approaches [5-7], and the symmetry viewpoints [8-10] have been developed. They provide us the diverse viewpoints of this system and stimulate the development of some new domains (e.g. [12]). Indeed, as saying by J. J. Thomson “because a mechanical model is richer in implications than the considerations for which it was advanced, it can suggest new directions of research that may lead to important discoveries” the different approach frequently offer more abundant content than we expected.

In this paper, we apply the DK-method to the relativistic path integral and solve the fixed-energy amplitude (Green’s function) of the relativistic Coulomb system. Different from the former papers [13,14] where the relativistic Coulomb system is solved by the path integral with the KS transformation and a beautiful perturbation technique, respectively. The way presented here is more compact and suitable for arbitrary one dimensional and spherical symmetry systems.

II. DURU-KLEINERT METHOD FOR THE RELATIVISTIC POTENTIAL PROBLEMS

Adding a vector potential $A(x)$ to Kleinert’s path integral for a relativistic particle in a potential $V(x)$ [13], we find that the expression of the fixed-energy amplitude of a relativistic particle in external static electromagnetic fields is given by [15]

$$G(x_b, x_a; E) = \frac{i\hbar}{2mc} \int_0^\infty dL \int \mathcal{D}\rho(\lambda)\Phi[\rho(\lambda)] \int \mathcal{D}^D x(\lambda)e^{-A_E[x,x]/\hbar}$$

(2.1)
with the action

$$A_E [\mathbf{x}, \mathbf{x}'] = \int_{\lambda_a}^{\lambda_b} d\lambda \left[ \frac{m}{2\rho(\lambda)} \mathbf{x}'^2 (\lambda) - i(e/c) \mathbf{A}(\mathbf{x}) \cdot \mathbf{x}'(\lambda) - \rho(\lambda) \frac{(E - V(\mathbf{x}))^2}{2mc^2} + \rho(\lambda) \frac{mc^2}{2} \right].$$

(2.2)

where \( L \) is defined as

$$L = \int_{\lambda_a}^{\lambda_b} d\lambda \rho(\lambda),$$

(2.3)

in which \( \rho(\lambda) \) is an arbitrary dimensionless fluctuating scale variable, and \( \Phi[\rho(\lambda)] \) is some convenient gauge-fixing functional. The only condition on \( \Phi[\rho(\lambda)] \) is that [13-16]

$$\int \mathcal{D}\rho(\lambda) \Phi[\rho(\lambda)] = 1.$$  

(2.4)

\( \hbar/mc \) is the well-known Compton wave length of a particle of mass \( m \), \( \mathbf{A}(\mathbf{x}) \) is the vector potential, \( V(\mathbf{x}) \) is the scalar potential, \( E \) is the system energy, and \( \mathbf{x} \) is the spatial part of the \((D+1)\) vector \( \mathbf{x} = (\mathbf{x}, \tau) \). This path integral forms the basis for studying relativistic potential problems.

The relativistic path integral of Eq. (2.1) in the absence of the vector potential \( \mathbf{A}(\mathbf{x}) \) has a more elegant representation providing the new path integral solutions via well-known ones if the systems are in two-dimensional Minkowski space or rotationally invariant systems in any dimensions. By decomposing the Eq. (2.1) into angular parts and take DK-transformation, we get [16,18,19]

$$G(\mathbf{x}_b, \mathbf{x}_a; E) = \frac{1}{(r_b r_a)^{(D-1)/2}} \sum_{l=0}^{\infty} G_l^{DK}(r_b, r_a; E) \sum_{\hat{m}} Y_{\hat{m}}(\mathbf{x}_b) Y_{\hat{m}}^*(\mathbf{x}_a),$$

(2.5)

where superscript DK indicates that the system has been performed by the DK-transformation, the functions \( Y_{\hat{m}}(\mathbf{x}) \) are the \( D \)-dimensional hyperspherical harmonics and \( G_l^{DK}(r_b, r_a; E) \) is the purely radical transformed fixed-energy amplitude

$$G_l^{DK}(r_b, r_a; E) = \frac{\hbar i}{2mc} f_b^{1/4} f_a^{1/4} G(q_b, q_a; \mathcal{E}).$$

(2.6)

The amplitude \( G(q_b, q_a; \mathcal{E}) \) of the fixed-pseudoenergy \( \mathcal{E} \) is given by [16,18,19]
\[ G(q_b, q_a; \mathcal{E}) \equiv \int_0^\infty dS \int \mathcal{D}\rho(s) \Phi[\rho(s)] \int \mathcal{D}q(s) e^{-A_{DK}^s[q,\dot{q}]}/\hbar \]  

(2.7)

with

\[ A_{DK}^s[q,\dot{q}] = \int_0^s ds \left[ \frac{m}{2\rho(s)} \dot{q}^2(s) + \rho(s)f(q(s)) \right. \]

\[ \left. \times \left( \frac{\hbar^2}{2m} \frac{(l + D/2 - 1)^2 - 1/4}{r^2(q(s))} - \frac{[E - V(r(q))]^2}{2mc^2} + \frac{mc^2}{2} \right) + V_{\text{eff}}(q(s)) \right], \]  

(2.8)

where the effective potential \( V_{\text{eff}} \) has the form

\[ V_{\text{eff}}(q(s)) = -\frac{\rho(s)\hbar^2}{m} \left[ \frac{1}{4} h''(q) - \frac{3}{8} \left( \frac{h''(q)}{h'(q)} \right)^2 \right] \]  

(2.9)

with \( h'(q) \) standing for the derivative \( dh(q)/dq \) and the transformation function \( h(q) \) defined as \( r = h(q) \) which is related to the local space-time transformation function \( f(r) \) \[16,18\] by the following equality

\[ h'^2(q) = f(r). \]  

(2.10)

For the pure relativistic Coulomb system under consideration, the potential \( V(r_C) = -e^2/r_C \) and the relativistic radial path integral reads \[18,20\]

\[ G_{\text{RC}}(r_{Cb}, r_{Ca}; E_{C}) = \frac{hi}{2m_{CC}} \int_0^\infty dL \int \mathcal{D}\rho(\lambda) \Phi[\rho(\lambda)] \int \mathcal{D}r_C(\lambda) \exp \left\{ -\frac{1}{\hbar} A_t[r_C, r'_C] \right\} \]  

(2.11)

with the action

\[ A_t[r_C, r'_C] \]

\[ = \int_{\lambda_0}^{\lambda_6} d\lambda \left[ \frac{m_C}{2\rho(\lambda)} r_C^2(\lambda) + \rho(\lambda) \frac{\hbar^2}{2m_{CC}} (l_C + D_C/2 - 1)^2 - 1/4 - \rho(\lambda) \frac{(E_C + e^2/r_C)^2}{2m_{CC}^2} + \rho(\lambda) \frac{m_{CC}^2}{2} \right], \]  

(2.12)

where the Roman subscript \( C \) specifies the Coulomb system. Let us apply the DK-transformation to this relativistic system by taking the following transformation variables
\[
\begin{align*}
    r_C &= h(x) = e^x, \\
    h'^2(x) &= e^{2x} = f(r_C) = r_C^2,
\end{align*}
\]
which maps the interval \( r \in (0, \infty) \) into \( x \in (-\infty, \infty) \) and leads to the effective potential
\[
V_{\text{eff}}(x(s)) = \frac{\rho(s)h^2}{8m_C}.
\]
(2.14)

It is without losing the generalization to take \( \Phi [\rho(\lambda)] = \delta [\rho - 1] \) and than the transformed fixed-energy amplitude of Eq. (2.6) turns into Morse potential system
\[
G_{lC}^{DK}(r_{Cb}, r_{Ca}; E_C) = \frac{\hbar i}{2m_{CC}f_a^{1/4}f_b^{1/4}} G(x_b, x_a; E_M)
\]
with the action
\[
A_{DK}^{}[x, \dot{x}] = \int_0^\infty ds \left[ \frac{m_C}{2} \dot{x}^2(s) + \frac{\nu^2h^2}{2m_C} (e^{2x} - 2\alpha e^x) - E_M \right].
\]
(2.16)

The parameters associated to the relativistic Coulomb system are given as
\[
\begin{align*}
    v &= \frac{1}{\hbar c} \sqrt{m_C^2c^4 - E_C^2} \\
    \alpha &= \frac{E_C e^2}{m_C c^2 - E_C} \\
    E_M &= -\frac{\hbar^2}{2m_C} [(l_C + D_C/2 - 1)^2 - \alpha^2]
\end{align*}
\]
where the notation \( \alpha = e^2/\hbar c \) is the fine structure constant. To go further, let’s make a trivial transformation by taking
\[
\begin{align*}
    x &= 2x_0, \\
    m_C &= m_0/4.
\end{align*}
\]
(2.18)

This maps the Eq. (2.15) into
\[
G_{lC}^{DK}(r_{Cb}, r_{Ca}; E_C) = \frac{\hbar i}{2m_{CC}} e^{x_0e^{x_0a}} \frac{1}{2} \int_0^\infty dS \int D_x x(s) e^{-A_{DK}^{}[x_0, \dot{x}_0]/\hbar}
\]
with the transformed new action.
\[ A^{DK} [x_O, \dot{x}_O] = \int_0^S ds \left[ \frac{m_O}{2} \dot{x}_O^2 (s) + \frac{2v^2 \hbar^2}{m_O} \left( e^{4x_O} - 2\alpha e^{2x_O} \right) - E_M \right]. \]  

(2.20)

The factor 1/2 in Eq. (2.19) accounts for the fact that the normalized states are related by \(|x⟩ = |x_O⟩/2\). At this place, we can apply the DK-transformation again by taking the following transformation functions

\[
\left\{ \begin{align*}
  x_O &= \ln z = h(z), \\
  h^2(z) &= 1/z^2 = f(x_O) = e^{-2x_O},
\end{align*} \right. \]  

(2.21)

which maps the interval \(x_O \in (-\infty, \infty)\) into \(z \in (0, \infty)\) and leads the effective potential \(V_{\text{eff}}(z)\) to

\[ V_{\text{eff}}(z) = -\frac{\hbar^2}{8m_O z^2}. \]  

(2.22)

The fixed-energy amplitude in Eq. (2.19) becomes

\[ G_{ik}^{DK} (r_{Cb}, r_{Ca}; Ec) = \frac{\hbar}{2mc_c} \frac{1}{2} [z_b z_a] \left\{ \frac{1}{\sqrt{z_b z_a}} \int_0^\infty dS' \int_0^\infty Dz(\tau) e^{-A^{DK}[z, \dot{z}]/\hbar} \right\} \]  

(2.23)

with the action of the radial simple harmonic oscillator

\[ A^{DK} [z, \dot{z}] = \int_0^S d\tau \left[ \frac{m_O}{2} z^2 (\tau) + \frac{\hbar^2}{2m_O} \left( l_0 + D_0/2 - 1 \right)^2 - 1/4 + \frac{m_O \omega^2 z^2}{2} - E_O \right]. \]  

(2.24)

The parameter relations between the Eqs. (2.19) and (2.24) are given as

\[
\left\{ \begin{align*}
  \hbar^2 (l_0 + D_0/2 - 1)^2 &= -2m_O E_M, \\
  m_O \omega^2/2 &= 2v^2 \hbar^2 / m_O, \\
  E_O &= 4\alpha v^2 \hbar^2 / m_O.
\end{align*} \right. \]  

(2.25)

By inserting the relations in Eq. (2.17) into these equality, we obtain the parameter relations between the relativistic Coulomb and radial harmonic oscillator

\[
\left\{ \begin{align*}
  \mu_O &= 2\sqrt{\mu_c^2 - \alpha^2}, \\
  \omega &= \sqrt{m_c^4 e^4 - E_C^2 / 2mc_c}, \\
  E_O &= E_C e^2 / mc_c^2.
\end{align*} \right. \]  

(2.26)
where for simplicity the quantity \((l_O + D_O/2 - 1)\) and \((l_C + D_C/2 - 1)\) have been defined as \(\mu_O\) and \(\mu_C\), respectively. With the well-known fixed-energy amplitude of a radial harmonic oscillator

\[
G_{l_O}(z_b, z_a; \mathcal{E}_O) = -i \frac{1}{\omega} \frac{\Gamma((1 + \mu_O)/2 - \mathcal{E}_O/2\hbar\omega)}{\Gamma(1 + \mu_O)\sqrt{z_bz_a}}
\]

\[
\times W_{\mathcal{E}_O/2\hbar\omega, \mu_O/2}((m_O\omega/\hbar)z_b^2)M_{\mathcal{E}_O/2\hbar\omega, \mu_O/2}((m_O\omega/\hbar)z_a^2),
\]

we finally obtain the exact radial fixed-energy amplitude of the relativistic Coulomb system

\[
G_{l_C}(r_Cb, r_Ca; E_C) = \frac{m_{Cc}}{\sqrt{m_C^2c^4 - E_C^2}}
\]

\[
\times \frac{\Gamma \left( \frac{1}{2} + \sqrt{\left(\frac{l_C + D_C}{2} - 1\right)^2 - \alpha^2} - E_C\alpha/\sqrt{m_C^2c^4 - E_C^2} \right)}{\Gamma \left( 1 + 2\sqrt{\left(\frac{l_C + D_C}{2} - 1\right)^2 - \alpha^2} \right)}
\]

\[
\times W_{E_C\alpha/\sqrt{m_C^2c^4 - E_C^2}, \sqrt{\left(\frac{l_C + D_C}{2} - 1\right)^2 - \alpha^2}} \left( \frac{2}{\hbar c} \sqrt{m_C^2c^4 - E_C^2} r_Cb \right)
\]

\[
\times M_{E_C\alpha/\sqrt{m_C^2c^4 - E_C^2}, \sqrt{\left(\frac{l_C + D_C}{2} - 1\right)^2 - \alpha^2}} \left( \frac{2}{\hbar c} \sqrt{m_C^2c^4 - E_C^2} r_Ca \right).
\]

This complete the discussion of Duru-Kleinert equivalence between the relativistic Coulomb system and a radial harmonic oscillator.

### III. CONCLUDING REMARKS

In this paper, the DK-method is applied to the relativistic path integral. As an interesting application, the fixed-energy amplitude of the relativistic Coulomb system is solved by the DK-equivalence of the relativistic Coulomb and a radial harmonic oscillator. Since the equivalence is between the relativistic and non-relativistic physical problems, it may provides us a more unified viewpoint of the different physical problems. Different from the path
integral approach \cite{13} and the perturbation approach \cite{14}, the method presented in the paper just need to find the appropriate transformation functions. Furthermore, all one dimensional and any higher dimensional system with rotationally invariant systems are suitable. We hope that the method presented here offers us a new way for solving the relativistic potential problems.
REFERENCES

[1] E. Schrödinger, Abhandlungen zur Wellenmechanik (Leipzig, 1927).

[2] P. A. M. Dirac, Proc. Roy. Soc. (London) A 109 642 (1925); Proc. roy. Soc. (London) A 117 610 (1928); part II: A 118 351 (1928).

[3] G. Temple, Proc. Roy. Soc. A 128, 487 (1930).

[4] L. C. Hostler, J. Math. Phys. 5 591 (1964); J. Math. Phys. 5 1235 (1964); J. Math. Phys. 11 2966 (1970); J. Math. Phys. 16 1585 (1975).

[5] J. Meixner, Math. Z. 36 677 (1933).

[6] H. Duru and H. Kleinert, Phys. Lett. B 84 185 (1979); Fortschr. Phys. 30 401 (1982).

[7] R. Ho and A. Inomata, Phys. Rev. Lett. 48, 231 (1982); A. Inomata, Phys. Lett. A 101 253 (1984).

[8] C. Grosche, Fortschr. Phys. 40 675 (1992).

[9] W. Pauli, Z. Phys. 36 336 (1926).

[10] H. Kleinert, Fortschr. Phys. 6 1 (1968).

[11] M. Bander and C. Itzykson, Rev. Mod. Phys. 38 330 (1966); Rev. Mod. Phys. 38 346 (1966).

[12] H. Friedrich and D. Wintgen, Phys. Rep. 183 37 (1989).

[13] H. Kleinert, Phys. Lett. A 212 15 (1996).

[14] D. H. Lin, J. Phys. A 31 7577 (1998).

[15] D. H. Lin, J. Phys. A 31 4785 (1998); hep-th/9708144; hep-th/9709152.

[16] H. Kleinert, Path Integrals in Quantum Mechanics, Statistics and Polymer Physics, World Scientific, Singapore, 1995.
[17] K. Fujikawa, Prog. Theor. Phys. 96, 863 (1996); hep-th/9609029, hep-th/9608052.

[18] D. H. Lin, J. Phys. A 30, 3201 (1997).

[19] D. H. Lin, Path Integral on Relativistic Spinless Potential Problems, talk given at the sixth International Conference on Path-Integrals from peV to TeV 50 Years from Feynman’s Paper, Florence, Italy, 25-29 August 1998, to appear in the Proceedings.

[20] D. H. Lin, J. Phys. A 30 4365 (1997).