Multi-scale exploration of convex functions and bandit convex optimization

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Abstract
We construct a new map from a convex function to a distribution on its domain, with the property that this distribution is a multi-scale exploration of the function. We use this map to solve a decade-old open problem in adversarial bandit convex optimization by showing that the minimax regret for this problem is \( \tilde{O}(\text{poly}(n)\sqrt{T}) \), where \( n \) is the dimension and \( T \) the number of rounds. This bound is obtained by studying the dual Bayesian maximin regret via the information ratio analysis of Russo and Van Roy, and then using the multi-scale exploration to solve the Bayesian problem.

1 Introduction
Let \( \mathcal{K} \subset \mathbb{R}^n \) be a convex body of diameter at most 1, and \( f : \mathcal{K} \to [0, +\infty) \) a non-negative convex function. Suppose we want to test whether some unknown convex function \( g : \mathcal{K} \to \mathbb{R} \) is equal to \( f \), with the alternative being that \( g \) takes a negative value somewhere on \( \mathcal{K} \). In statistical terminology the null hypothesis is

\[ H_0 : g = f, \]

and the alternative is

\[ H_1 : \exists \alpha \in \mathcal{K} \text{ such that } g(\alpha) < -\varepsilon, \]

where \( \varepsilon \) is some fixed positive number. In order to decide between the null hypothesis and the alternative one is allowed to make a single noisy measurement of \( g \). That is one can choose a point \( x \in \mathcal{K} \) (possibly at random) and obtain \( g(x) + \xi \) where \( \xi \) is a zero-mean random variable independent of \( x \) (say \( \xi \sim \mathcal{N}(0,1) \)). Is there a way to choose \( x \) such that the total variation distance between the observed measurement under the null and the alternative is at least (up to logarithmic terms) \( \varepsilon/\text{poly}(n) \)? Observe that without the convexity assumption on \( g \) this distance is always \( O(\varepsilon^{n+1}) \), and thus a positive answer to this question would crucially rely on convexity. We show that \( \varepsilon/\text{poly}(n) \) is indeed attainable by constructing a distribution on \( \mathcal{K} \) which guarantees an exploration of the convex function \( f \) at every scale simultaneously. Precisely we prove the following new result on convex functions. We denote by \( c \) a universal constant whose value can change at each occurrence.
Theorem 1  Let $\mathcal{K} \subset \mathbb{R}^n$ be a convex body of diameter at most 1. Let $f: \mathcal{K} \to [0, +\infty)$ be convex and 1-Lipschitz, and let $\varepsilon > 0$. There exists a probability measure $\mu$ on $\mathcal{K}$ such that the following holds true. For every $\alpha \in \mathcal{K}$ and for every convex and 1-Lipschitz function $g: \mathcal{K} \to \mathbb{R}$ satisfying $g(\alpha) < -\varepsilon$, one has

$$
\mu\left(\left\{ x \in \mathcal{K} : |f(x) - g(x)| > \frac{c}{n^{7.5} \log(1 + n/\varepsilon)} \max(\varepsilon, f(x)) \right\}\right) > \frac{c}{n^3 \log(1 + n/\varepsilon)}.
$$

Our main application of the above result is to resolve a long-standing gap in bandit convex optimization. We refer the reader to Bubeck and Cesa-Bianchi [2012] for an introduction to bandit problems (and some of their applications). The bandit convex optimization problem can be described as the following sequential game: at each time step $t = 1, \ldots, T$, a player selects an action $x_t \in \mathcal{K}$, and simultaneously an adversary selects a convex (and 1-Lipschitz) loss function $\ell_t: \mathcal{K} \mapsto [0, 1]$. The player’s feedback is its suffered loss, $\ell_t(x_t)$. We assume that the adversary is oblivious, that is the sequence of loss functions $\ell_1, \ldots, \ell_T$ is chosen before the game starts. The player has access to external randomness, and can select her action $x_t$ based on the history $H_t = (x_s, \ell_s(x_s))_{s < t}$. The player’s performance at the end of the game is measured through the regret:

$$
R_T = \sum_{t=1}^{T} \ell_t(x_t) - \min_{x \in \mathcal{K}} \sum_{t=1}^{T} \ell_t(x),
$$

which compares her cumulative loss to the best cumulative loss she could have obtained in hindsight with a fixed action, if she had known the sequence of losses played by the adversary. A major open problem since Kleinberg [2004], Flaxman et al. [2005] is to reduce the gap between the $\sqrt{T}$-lower bound and the $T^{3/4}$-upper bound for the minimax regret of bandit convex optimization. In dimension one (i.e., $\mathcal{K} = [0, 1]$) this gap was closed recently in Bubeck et al. [2015] and our main contribution is to extend this result to higher dimensions:

Theorem 2  There exists a player’s strategy such that for any sequence of convex (and 1-Lipschitz) losses one has

$$
\mathbb{E}R_T \leq c n^{11} \log^4(T) \sqrt{T},
$$

where the expectation is with respect to the player’s internal randomization.

We observe that this result also improves the state of the art regret bound for the easier situation where the losses $\ell_1, \ldots, \ell_T$ form an i.i.d. sequence. In this situation the best previous bound was obtained by Agarwal et al. [2011] and is $\tilde{O}(n^{16} \sqrt{T})$.

Using Theorem 1 we prove Theorem 2 in Section 2. Theorem 1 itself is proven in Section 3.

2  Proof of Theorem 2

Following Bubeck et al. [2015] we reduce the proof of Theorem 2 to upper bounding the Bayesian maximin regret (this reduction is simply an application of Sion’s minimax theorem). In other words the sequence $(\ell_1, \ldots, \ell_T)$ is now a random variable with a distribution known to the player. Expectations are now understood with respect to both the latter distribution, and possibly the randomness in the player’s strategy. We denote $\mathbb{E}_t$ for the expectation conditionally on the random variable $H_t$. As in Bubeck et al. [2015] we analyze the Bayesian maximin regret with the information theoretic approach of Russo and Van Roy [2014a], which we recall in the next subsection.
2.1 The information ratio

Let \( \mathcal{K} = \{\bar{x}_1, \ldots, \bar{x}_K\} \) be a \( 1/\sqrt{T} \)-net of \( \mathcal{K} \). Note that \( K \leq (4T)^n \). We define a random variable \( \bar{x}^* \in \mathcal{K} \) such that \( \sum_{t=1}^{T} \ell_t(\bar{x}^*) = \min_{x \in \mathcal{K}} \sum_{t=1}^{T} \ell_t(x) \). Using that the losses are Lipschitz one has

\[
R_T \leq \sqrt{T} + \sum_{t=1}^{T} (\ell_t(x_t) - \ell_t(\bar{x}^*)). \tag{1}
\]

We introduce the following key quantities, for \( x \in \mathcal{K} \),

\[
r_t(x) = \mathbb{E}_t(\ell_t(x) - \ell_t(\bar{x}^*)), \quad \text{and} \quad v_t(x) = \text{Var}_t(\mathbb{E}_t(\ell_t(x)|\bar{x}^*)). \tag{2}
\]

In words, conditionally on the history, \( r_t(x) \) is the (approximate) expected regret of playing \( x \) at time \( t \), and \( v_t(x) \) is a proxy for the information about \( \bar{x}^* \) revealed by playing \( x \) at time \( t \). It will be convenient to rewrite these functions slightly more explicitly. Let \( i^* \in [K] \) be the random variable such that \( \bar{x}^* = \bar{x}_{i^*} \). We denote by \( \alpha^* \) its distribution, which we view as a point in the \( K-1 \) dimensional simplex. Let \( \alpha_t = \mathbb{E}_t \alpha^* \). In words \( \alpha_t = (\alpha_{1,t}, \ldots, \alpha_{K,t}) \) is the posterior distribution of \( x^* \) at time \( t \). Let \( f_{i,t}, f_t : \mathcal{K} \to [0, 1], i \in [K], t \in [T] \), be defined by, for \( x \in \mathcal{K} \),

\[
f_t(x) = \mathbb{E}_t \ell_t(x), \quad f_{i,t}(x) = \mathbb{E}_t(\ell_t(x)|\bar{x}^* = \bar{x}_i).
\]

Then one can easily see that

\[
r_t(x) = f_t(x) - \sum_{i=1}^{K} \alpha_{i,t} f_{i,t}(\bar{x}_i), \quad \text{and} \quad v_t(x) = \sum_{i=1}^{K} \alpha_{i,t} (f_{i,t}(x) - f_{i,t}(x))^2. \tag{3}
\]

The main observation in Russo and Van Roy [2014a] is the following lemma, which gives a bound on the accumulation of information (see also [Appendix B, Bubeck et al. [2015]] for a short proof).

**Lemma 1** One always has \( \mathbb{E} \sum_{t=1}^{T} v_t(x_t) \leq \frac{1}{2} \log(K) \).

An important consequence of Lemma 1 is the following result which follows from an application of Cauchy-Schwarz (and (1)):

\[
\mathbb{E} \sum_{t=1}^{T} r_t(x_t) \leq \sqrt{T} + C \sum_{t=1}^{T} \sqrt{\mathbb{E} v_t(x_t)} \Rightarrow \mathbb{E} R_T \leq 2 \sqrt{T} + C \sqrt{\frac{T}{2} \log(K)}. \tag{4}
\]

In particular a strategy which obtains at each time step an information proportional to its instantaneous regret has a controlled cumulative regret:

\[
\mathbb{E} r_t(x_t) \leq \frac{1}{\sqrt{T}} + C \sqrt{\mathbb{E} v_t(x_t)}, \quad \forall t \in [T] \Rightarrow \mathbb{E} R_T \leq 2 \sqrt{T} + C \sqrt{\frac{T}{2} \log(K)}. \tag{5}
\]

Russo and Van Roy [2014a] refers to the quantity \( \mathbb{E} r_t(x_t)/\sqrt{\mathbb{E} v_t(x_t)} \) as the *information ratio*. They show that Thompson Sampling (which plays \( x_i \) at random, drawn from the distribution \( \alpha_t \)) satisfies \( \mathbb{E} r_t(x_t)/\sqrt{\mathbb{E} v_t(x_t)} \leq K \) (without any assumptions on the loss functions \( \ell_t : \mathcal{K} \to [0, 1] \)).

In Bubeck et al. [2015] it is shown that in dimension one (i.e., \( n = 1 \)), the latter bound can be improved using the convexity of the losses by replacing \( K \) with a polylogarithmic term in \( K \) (Thompson Sampling is also slightly modified). In the present paper we propose a completely different strategy, which is loosely related to the Information Directed Sampling of Russo and Van Roy [2014b]. We describe and analyze our new strategy in the next subsection.
### 2.2 A two-point strategy

We describe here a new strategy to select \( x_t \), conditionally on \( H_t \), and show that it satisfies a bound of the form given in (5). To lighten notation we drop all time subscripts, e.g. one has \( r(x) = f(x) - \sum_{i=1}^{K} \alpha_i f_i(\bar{x}_i) \), and \( v(x) = \sum_{i=1}^{K} \alpha_i (f_i(x) - f(x))^2 \). Our objective is to describe a random variable \( X \in \mathcal{K} \) which satisfies

\[
\mathbb{E} r(X) \leq \frac{1}{\sqrt{T}} + C \sqrt{\mathbb{E} v(X)},
\]

where \( C \) is polylogarithmic in \( K \) (recall that \( K \leq (4T)^n \)).

Let \( x^* \in \arg \min_{x \in \mathcal{K}} f(x) \). We translate the functions so that \( f(x^*) = 0 \) and denote \( L = \sum_{i=1}^{K} \alpha_i f_i(\bar{x}_i) \). If \( L \geq -1/\sqrt{T} \) then \( X := x^* \) satisfies (6), and thus in the following we assume that \( L \leq -1/\sqrt{T} \).

**Step 1:** We claim that there exists \( \varepsilon \in [\|L\|/2, 1] \) such that

\[
\alpha \left( \{ i \in [K] : f_i(\bar{x}_i) \leq -\varepsilon \} \right) \geq \frac{|L|}{2 \log(2/|L|) \varepsilon}.
\]

Indeed assume that (7) is false for all \( \varepsilon \in [\|L\|/2, 1] \), and let \( Y \) be a random variable such that \( \mathbb{P}(Y = -f_i(\bar{x}_i)) = \alpha_i \), then

\[
|L| = \mathbb{E} Y \leq |L|/2 + \int_{|L|/2}^{1} \mathbb{P}(Y \geq x) dx < |L|/2 + \int_{|L|/2}^{1} \frac{|L|}{2 \log(2/|L|) x} dx = |L|,
\]

thus leading to a contradiction. We denote \( I = \{ i \in [K] : f_i(\bar{x}_i) \leq -\varepsilon \} \) with \( \varepsilon \) satisfying (7).

**Step 2:** We show here the existence of a point \( \bar{x} \in \mathcal{K} \) and a set \( J \subset I \) such that \( \alpha(J) > n^{7.5} \log(1+n/\varepsilon) \alpha(I) \) and for any \( i \in J \),

\[
|f(\bar{x}) - f_i(\bar{x})| \geq \frac{c}{n^{7.5} \log(1+n/\varepsilon) \max(\varepsilon, f(\bar{x}))}.
\]

We say that a point is *good* for \( f_i \) if it satisfies (8), and thus we want to prove the existence of a point \( \bar{x} \) which is good for a large fraction (with respect to the posterior) of the \( f_i \)'s. Denote

\[
A_i = \left\{ x \in \mathcal{K} : |f(x) - f_i(x)| \geq \frac{c}{n^{7.5} \log(1+n/\varepsilon) \max(\varepsilon, f(\bar{x}))} \right\},
\]

and let \( \mu \) be the distribution given by Theorem 1. Then one obtains:

\[
\sup_{x \in \mathcal{K}} \sum_{i \in I} \alpha_i \mathbb{1}_{\{ x \in A_i \}} \geq \int_{x \in \mathcal{K}} \sum_{i \in I} \alpha_i \mathbb{1}_{\{ x \in A_i \}} d\mu(x) = \sum_{i \in I} \alpha_i \mu(A_i) \geq \frac{c}{n^{7.5} \log(1+n/\varepsilon)} \alpha(I),
\]

which clearly implies the existence of \( J \) and \( \bar{x} \).

**Step 3:** Let \( X \) be such that \( \mathbb{P}(X = \bar{x}) = \alpha(J) \) and \( \mathbb{P}(X = x^*) = 1 - \alpha(J) \). Then

\[
\mathbb{E} r(X) = |L| + \alpha(J) f(\bar{x}),
\]
and using the definition of \( \bar{x} \) one easily see that:

\[
\sqrt{\mathbb{E}v(X)} \geq \sqrt{\alpha(J)w(\bar{x})} \geq \sqrt{\alpha(J)\sum_{i \in J} \alpha_i(f_i(\bar{x}) - f(\bar{x}))^2} \geq \frac{c}{n^{7.5} \log(1 + n/\varepsilon)} \alpha(J) \max(\varepsilon, f(\bar{x})).
\]

Finally, since \( \alpha(J) \geq \frac{c|J|}{\varepsilon n^{3.5} \log^{2}(1+n/\varepsilon)} \), the two above displays clearly implies (6).

3 An exploratory distribution for convex functions

In this section we construct an exploratory distribution \( \mu \) of a convex function \( f \) which satisfies the conditions of Theorem 1, thus concluding the proof of Theorem 2.

3.1 The one-dimensional case

Since our proof of Theorem 1 will proceed by induction, our first goal is to establish the result in dimension 1. This task will be much simpler than the proof for a general dimension, but already contains some of the central ideas used in the general case. In particular, a (much simpler) multi-scale argument is used.

The main ingredient is the following lemma which is easy to verify by picture (we provide a formal proof for sake of completeness).

Lemma 2 Let \( f, g : \mathbb{R} \to \mathbb{R} \) be two convex functions. Suppose that \( f(x) \geq 0 \). Let \( x_0, \alpha \in \mathbb{R} \) be two points satisfying \( \alpha - 1 < x_0 < \alpha \), and suppose that \( g(\alpha) < -\varepsilon \) for some \( \varepsilon > 0 \) and that

\[
f'(x) \geq 0, \forall x > x_0. \tag{9}
\]

Let \( \mu \) be a probability measure supported on \( [x_0, \alpha] \) whose density with respect to the Lebesgue measure is bounded from above by some \( \beta > 1 \). Then we have

\[
\mu \left( \{ x : |f(x) - g(x)| > \frac{1}{4} \beta^{-1} \max(\varepsilon, f(x)) \} \right) \geq \frac{1}{2}.
\]

Proof We first argue that, without loss of generality, one may assume that \( f \) attains its minimum at \( x_0 \). Indeed, we may clearly change \( f \) as we please on the interval \( (-\infty, x_0) \) without affecting the assumptions or the result of the Lemma. Using the condition (9) we may therefore make this assumption legitimate.

Assume, for now, that there exists \( x_1 \in [x_0, \alpha] \) for which \( f(x_1) = g(x_1) \). By convexity, and since \( f(x_0) \geq 0 \) and \( g(\alpha) < 0 \), if such point exists then it is unique. Let \( h(x) \) be the linear function passing through \((\alpha, g(\alpha))\) and \((x_1, f(x_1))\). By convexity of \( g \), we have that \(|g(x) - f(x)| \geq |h(x) - f(x)| \) for all \( x \in [x_0, \alpha] \). Now, since \( h(\alpha) < -\varepsilon \) and since \( \alpha < x_1 + 1 \), we have \( h'(x_0) < -\varepsilon - f(x_0) \). Moreover, since we know that \( f(x) \) is non-decreasing in \([x_0, \alpha]\), we conclude that

\[
|g(x) - f(x)| \geq |h(x) - f(x)|
= |h(x) - f(x_1)| + |f(x) - f(x_1)|
= (\varepsilon + f(x_1))|x - x_1| + |f(x) - f(x_1)|
\geq \max(\varepsilon, f(x))|x - x_1|, \forall x \in [x_0, \alpha].
\]
It follows that

\[ \{ x : |f(x) - g(x)| < \frac{1}{2} \beta^{-1} \max(\epsilon, f(x)) \} \subset I := [x_1 - \frac{1}{4} \beta^{-1}, x_1 + \frac{1}{4} \beta^{-1}] \]

but since the density of \( \mu \) is bounded by \( \beta \), we have \( \mu(I) \leq \frac{1}{2} \) and we’re done.

It remains to consider the case that \( g(x) < f(x) \) for all \( x \in [x_0, \alpha] \). In this case, we may define

\[ \tilde{g}(x) = g(x) + \frac{f(x_0) - g(x_0)}{\alpha - x_0}(\alpha - x). \]

Note that \( \tilde{g}(x) \geq g(x) \) for all \( x \in [x_0, \alpha] \), which implies that \( |g(x) - f(x)| \geq |\tilde{g}(x) - f(x)| \) for all \( x \in [x_0, \alpha] \). Since \( \tilde{g}(x_0) = f(x_0) \), we may continue the proof as above, replacing the function \( g \) by \( \tilde{g} \).

\[ \blacksquare \]

We are now ready to prove the one dimensional case. The proof essentially invokes the above lemma on every scale between \( \epsilon \) and 1.

**Proof** [Proof of Theorem 1, the case \( n = 1 \)] Let \( x_0 \in \mathcal{K} \) be the point where the function \( f \) attains its minimum and set \( d = \text{diam}(\mathcal{K}) \). Define \( N = \lceil \log_2 \frac{1}{\epsilon} \rceil + 4 \). For all \( 0 \leq k \leq N \), consider the interval

\[ I_k = [x_0 - d2^{-k}, x_0 + d2^{-k}] \cap \mathcal{K} \]

and define the measure \( \mu_k \) to be the uniform measure over the interval \( I_k \). Finally, we set

\[ \mu = \frac{1}{N + 2} \sum_{k=0}^{N} \mu_k + \frac{1}{N + 2} \delta_{x_0}. \]

Now, let \( \alpha \in \mathcal{K} \) and let \( g(x) \) be a convex function satisfying \( g(\alpha) \leq -\epsilon \). We would like to argue that \( \mu(A) \geq \frac{1}{2} \log(1 + 1/\epsilon) \) for \( A = \{ x \in \mathcal{K} : |f(x) - g(x)| \geq \frac{\epsilon}{8} \} \).

Set \( k = \lceil \log_2 (|\alpha - x_0|/d) \rceil \). Define \( Q(x) = x_0 + d2^{-k}(x - x_0) \) and set \( \tilde{f}(x) = f(Q(x)) \), \( \tilde{g}(x) = g(Q(x)) \), \( \tilde{\alpha} = Q^{-1}(\alpha) \) and consider the interval

\[ I = Q^{-1}(I_k) \cap \{ x : (x - x_0)(\alpha - x_0) \geq 0 \} \]

It is easy to check that, by definition \( I \) is an interval of length 1, contained in the interval \( [x_0, \tilde{\alpha}] \). Defining \( \tilde{\mu} = \mu_k \), we have that the density of \( \tilde{\mu} \) with respect to the Lebesgue measure is equal to 1. An application of Lemma 2 for the functions \( \tilde{f}, \tilde{g} \), the points \( x_0, \tilde{\alpha} \) and the measure \( \tilde{\mu} \) teaches us that

\[ \mu_k(A) = \mu_{Q^{-1}(I_k)} \left( \left\{ x : |\tilde{f}(x) - \tilde{g}(x)| \geq \frac{\epsilon}{8} \right\} \right) \geq \frac{1}{2} \mu \left( \left\{ x : |\tilde{f}(x) - \tilde{g}(x)| \geq \frac{\epsilon}{8} \right\} \right) \geq \frac{1}{4}. \]

By definition of the measure \( \mu \), we have that whenever \( k \leq N \), one has

\[ \mu(A) \geq \frac{1}{N + 2} \geq \frac{1}{8 \log(1 + 1/\epsilon)}. \]

Finally, if \( k > N \), it means that \( |\alpha - x_0| < 2^{-N} < \frac{\epsilon}{4} \). Since the function \( g \) is 1-Lipschitz, this implies that \( g(x_0) \leq -\epsilon/2 \) which in turn gives \( |f(x_0) - g(x_0)| \geq \frac{\epsilon}{8} \). Consequently, \( x_0 \in A \) and thus \( \mu(A) \geq \mu(\{ x_0 \}) = \frac{1}{N + 2} \geq \frac{1}{8 \log(1 + 1/\epsilon)} \). The proof is complete.

\[ \blacksquare \]
3.2 The high-dimensional case

We now consider the case where \( n \geq 2 \). For a set \( \Omega \subset \mathbb{R}^n \) and a direction \( \theta \in \mathbb{R}^n \) we denote
\[
S_{\Omega,\theta} = \{ x \in \Omega : \langle x, \theta \rangle \leq 1/4 \}, \text{ and } \mu_\Omega \text{ for the uniform measure on } \Omega. \text{ For a distribution } \mu \text{ we write } \text{Cov}(\mu) = \mathbb{E}_{X \sim \mu}XX^\top.
\]

As we explain in Section 3.3 our construction iteratively applies the following lemma:

**Lemma 3** Let \( \varepsilon > 0 \), \( L \in [1, 2n] \). Let \( \Omega \subset \mathbb{R}^n \) be a convex set with \( 0 \in \Omega \) and \( \text{Cov}(\mu_\Omega) = \text{Id} \). Let \( f : \Omega \to [0, \infty] \) be a convex and \( L \)-Lipschitz function with \( f(0) = 0 \). Then there exists a measure \( \mu \) on \( \Omega \) and a direction \( \theta \in S^{n-1} \) such that for all \( \alpha \in \Omega \setminus S_{\Omega,\theta} \) and for every convex function \( g : \Omega \to \mathbb{R} \) satisfying \( g(\alpha) < -\varepsilon \), one has
\[
\mu \left( \left\{ x \in \Omega : |f(x) - g(x)| > \frac{1}{2^{50} n^{1.5} \log(1 + n/\varepsilon)} \text{max}(\varepsilon, f(x)) \right\} \right) > \frac{1}{16n}. \tag{10}
\]

The above lemma is proven in Section 3.4. A central ingredient in its proof is, in turn, the following Lemma, which itself is proven in Section 3.5.

**Lemma 4** Let \( \varepsilon > 0 \), \( \Omega \subset \mathbb{R}^n \) a convex set with \( \text{diam}(\Omega) \leq M \), and \( f : \Omega \to \mathbb{R}_+ \) a convex function. Assume that there exist \( \delta \in (0, \frac{1}{32n^2}) \), \( z \in \Omega \cap B(0, \frac{1}{10}) \), \( \theta \in S^{n-1} \) and \( t > 0 \) such that
\[
\mu_{B(z,\delta)} \left( (\nabla f)^{-1} \left( B \left( t\theta, \frac{t}{16n^2} \right) \right) \right) \geq 1/2. \tag{11}
\]

Then for all \( \alpha \in \Omega \) satisfying \( \langle \alpha, \theta \rangle \geq \frac{1}{8} \) and \( |\alpha| \leq 2n \) and for all convex function \( g : \Omega \to \mathbb{R} \) satisfying \( g(\alpha) < -\varepsilon \), one has
\[
\mu_{B(z,\delta)} \left( \left\{ x \in \Omega : |f(x) - g(x)| > \frac{\delta}{2^{13} M \sqrt{n}} \text{max}(\varepsilon, f(x)) \right\} \right) > \frac{1}{8}.
\]

3.3 From Lemma 3 to Theorem 1: a multi-scale exploration

An intermediate lemma in this argument will be the following:

**Lemma 5** There exists a universal constant \( c > 0 \) such that the following holds true. Let \( \varepsilon > 0 \), \( \Omega \subset \mathbb{R}^n \) a convex set with \( 0 \in \Omega \) and \( \text{Cov}(\mu_\Omega) = \text{Id} \). Let \( f : \Omega \to [0, \infty] \) be a convex and \( 1 \)-Lipschitz function. Then there exists a measure \( \mu \) on \( \Omega \), a point \( y \in \Omega \) and a direction \( \theta \in S^{n-1} \) such that for all \( \alpha \in \Omega \) satisfying
\[
|\langle \alpha - y, \theta \rangle| \geq \frac{c \varepsilon}{16n^{10}}
\]
and for every convex function \( g : \Omega \to \mathbb{R} \) satisfying \( g(\alpha) < -\varepsilon \), one has
\[
\mu \left( \left\{ x \in \Omega : |f(x) - g(x)| > \frac{c}{n^{1.5} \log(1 + n/\varepsilon)} \text{max}(\varepsilon, f(x)) \right\} \right) > \frac{c}{n^2 \log(1 + n/\varepsilon)}. \tag{12}
\]
3.3.1 From Lemma 5 to Theorem 1

Given Lemma 5, the proof of Theorem 1 is carried out by induction on the dimension. The case $n = 1$ has already been resolved above. Now, suppose that the theorem is true up to dimension $n - 1$, where the constant $c > 0$ is the constant from Lemma 5. Let $K \in \mathbb{R}^n$ and $f$ satisfy the assumptions of the theorem. Denote $Q = \text{Cov}(\mu_K)^{-1/2}$ and define

$$\Omega = Q(K), \quad \tilde{f}(x) = f(Q^{-1}(x))$$

so that $\tilde{f} : \Omega \to \mathbb{R}$. Since $\text{diam}(K) \leq 1$, we know that for all $u \in S^{n-1}$, $\text{Var} [\text{Proj}_u \mu_K] \leq 1$ which implies that $\|Q^{-1}\| \leq 1$. Consequently, the function $\tilde{f}$ is $1$-Lipschitz. We now invoke Lemma 5 on $\Omega$ and $\tilde{f}$ which outputs a measure $\mu_1$, a point $y \in \Omega$ and a direction $\theta$. By translating $f$ and $K$, we can assume without loss of generality that $y = 0$. Fix some linear isometry $T : \mathbb{R}^{n-1} \to \theta^\perp$.

Define

$$\Omega' = T^{-1} \text{Proj}_{\theta^\perp}(\Omega \cap \{x : |\langle x, \theta \rangle| \leq \delta\})$$

where $\delta = \frac{c}{16\log n}$ and $c$ is the universal constant from Lemma 5. Since $\tilde{f}$ is convex, there exists $I \subset \mathbb{R} \times \mathbb{R}^n$ so that

$$\tilde{f}(x) = \sup_{(a, y) \in I} (a + \langle x, y \rangle), \quad \forall x \in \Omega. \quad (13)$$

We may extrapolate $\tilde{f}(x)$ to the domain $\mathbb{R}^n$ by using the above display as a definition. We now define a function $h : \Omega' \to \mathbb{R}$ by

$$h(x) := \sup_{w \in [-\delta, \delta]} \tilde{f}(T(x) + w\theta). \quad (14)$$

It is clear that $\text{diam}(\Omega') \leq 1$. Moreover, $h$ is $1$-Lipschitz since it can be written as the supremum of $1$-Lipschitz functions. We can therefore use the induction hypothesis with $\Omega', h(x)$ to obtain a measure $\mu_2$ on $\Omega'$. Next, for $y \in \mathbb{R}^{n-1}$, define

$$N(y) := \{x \in \Omega : T^{-1}(\text{Proj}_{\theta^\perp}x) = y\}$$

and set

$$\mu(W) = \frac{1}{n} \mu_1(Q(W)) + \frac{n - 1}{n} \int_{\Omega'} \frac{\text{Vol}_1(Q(W) \cap N(u))}{\text{Vol}_1(N(u))} d\mu_2(u)$$

for all measurable $W \subset \mathbb{R}^n$.

Fix $\alpha \in K$, let $g : K \to \mathbb{R}$ be a convex and $1$-Lipschitz function satisfying $g(\alpha) \leq -\varepsilon$. Recall that $c$ denotes the universal constant from Lemma 5. Define

$$A = \left\{ x \in K : |f(x) - g(x)| > \frac{c}{n^{7.5} \log(1 + n/\varepsilon)} \max(\varepsilon, f(x)) \right\}.$$

The proof will be concluded by showing that $\mu(A) \geq \frac{c}{n^{3} \log(1 + n/\varepsilon)}$. 

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Define \( \tilde{g}(x) = g(Q^{-1}(x)) \) and remark that \( \tilde{g} \) is 1-Lipschitz. First consider the case that \(|\langle Q\alpha, \theta \rangle| \geq \delta \), then by construction, we have

\[
\mu(A) \geq \frac{1}{n} \mu_1(Q(A)) \\
= \frac{1}{n} \mu_1 \left( \left\{ x \in \Omega; |\tilde{f}(x) - \tilde{g}(x)| > \frac{c}{n^{7.5} \log(1 + n/\varepsilon)} \max(\varepsilon, f(x)) \right\} \right) \\
\geq \frac{c}{n^{3} \log(1 + n/\varepsilon)},
\]

and we’re done.

Otherwise, we need to deal with the case that \(|\langle Q\alpha, \theta \rangle| < \delta \). Define \( q(x) \) to be the function obtained by replacing \( \tilde{f}(x) \) with \( \tilde{g}(x) \) in equation (14) and consider the set

\[
A' = \left\{ x \in \Omega'; |h(x) - q(x)| > \frac{c}{(n-1)^{7.5} \log(1 + n/\varepsilon)} \max(\varepsilon, h(x)) \right\}.
\]

By construction of the measure \( \mu_2 \) we have \( \mu_2(A') \geq \frac{c}{(n-1)^{7.5} \log(1 + n/\varepsilon)} \). We claim that \( N(A') \subset Q(A) \), which implies that

\[
\mu(A) \geq \frac{n-1}{n} \mu_2(A') \geq \frac{c}{n^3 \log(1 + n/\varepsilon)}
\]

which will complete the proof. Indeed, let \( y \in N(A') \). Define \( z = T^{-1}(\text{Proj}_\theta y) \), so that \( z \in A' \). Let \( w_1, w_2 \in N(z) \) be points such that

\[
h(z) = \tilde{f}(w_1), \quad q(z) = \tilde{g}(w_2).
\]

Such points exist since, by continuity, the maximum in equation (14) is attained. Now, since \( z \in A' \), we have by definition that

\[
|\tilde{f}(w_1) - \tilde{g}(w_2)| > \frac{c}{(n-1)^{7.5} \log(1 + n/\varepsilon)} \max(\varepsilon, \tilde{f}(w_1)).
\]

Finally, since the functions \( \tilde{f}, \tilde{g} \) are 1-Lipschitz, we have that

\[
|\tilde{f}(y) - \tilde{g}(y)| \geq |\tilde{f}(w_1) - \tilde{g}(w_2)| - |\tilde{f}(y) - \tilde{f}(w_1)| - |\tilde{g}(y) - \tilde{g}(w_2)| \\
\geq \frac{c}{(n-1)^{7.5} \log(1 + n/\varepsilon)} \max(\varepsilon, \tilde{f}(w_1)) - |y - w_1| - |y - w_2| \\
\geq \frac{c}{(n-1)^{7.5} \log(1 + n/\varepsilon)} \max(\varepsilon, \tilde{f}(y)) - 4\delta \\
= \frac{c}{(n-1)^{7.5} \log(1 + n/\varepsilon)} \max(\varepsilon, \tilde{f}(y)) - \frac{c\varepsilon}{4n^{10}} \\
\geq \frac{c}{n^{7.5} \log(1 + n/\varepsilon)} \max(\varepsilon, \tilde{f}(y))
\]

which implies, by definition, that \( y \in Q(A) \). The proof is complete.
3.3.2 From Lemma 3 to Lemma 5

We construct below a decreasing sequence of domains $\Omega_0 \supseteq \Omega_1 \supseteq \ldots \supseteq \Omega_N$. Let $x_0 \in \Omega$ be a point where $f(x)$ attains its minimum on $\Omega$. Set $\Omega_0 = \Omega - x_0$. Given $i \geq 0$, we define the domain $\Omega_{i+1}$, given the domain $\Omega_i$, by induction as follows. Define $Q_i = \text{Cov}(\mu_{\Omega_i})^{-1/2}$ and $f_i(x) = f(Q_i^{-1}(x + x_0)) - f(x_0)$. We have

$$|\nabla f_i(x)| = |Q_i^{-1}\nabla f(Q_i^{-1}(x))| \leq \|Q_i^{-1}\|.$$  

Now, by Lemma 8 we know that

$$\text{diam}(\Omega_i) \leq \text{diam}(\Omega) \leq n + 1$$

which implies that $\|Q_i^{-1}\| \leq n + 1$. We conclude that $f_i$ is $(n + 1)$-Lipschitz. We may therefore invoke Lemma 3 for the function $f_i$ defined by on the set $Q_i\Omega_i$, with $L = n + 1$. This lemma outputs a direction $\theta_i$ and a measure $\mu_i$ which we denote by $\theta_i$ and $\mu_i$, respectively. We define

$$\Omega_{i+1} = Q_i^{-1}S_{Q_i,\Omega_i,\theta_i}.$$  

Equation (10) yields that for a universal constant $c > 0$,

$$\mu_i \left( \left\{ x - x_0 : |f(x) - g(x)| > \frac{c}{n^{7.5}\log(1 + n/\varepsilon)} \max(\varepsilon, f(x)) \right\} \right) > \frac{c}{n}$$  

for all functions $g(x)$ such that $g(\alpha) < -\varepsilon$, whenever $\alpha \in \Omega_i \setminus \Omega_{i+1}$.

Fix a constant $\ell > 0$ whose value will be assigned later on. Define $\delta = \frac{\ell \varepsilon}{16\log(n)}$ and let

$$N = \min\{i : \exists \theta \in S^{n-1} \text{ such that } |\langle x, \theta \rangle| < \delta, \forall x \in \Omega_i \}.$$  

In other words, $N$ is the smallest value of $i$ such that $\Omega_i$ is contained in a slab of width $2\delta$. Our next goal is to give an upper bound for the value of $N$. To this end, we claim that

$$\text{Vol}(\Omega_{i+1}) \leq \frac{1}{2}\text{Vol}(\Omega_i),$$  

which equivalently says

$$\text{Vol}(S_{Q_i,\Omega_i,\theta_i}) \leq \frac{1}{2}\text{Vol}(Q_i\Omega_i).$$

Let $X \sim \mu_{Q_i\Omega_i}$ and observe that $\mathbb{P}(|\langle X, \theta_i \rangle| \leq 1/4) = \text{Vol}(S_{\Omega_i,\theta_i})/\text{Vol}(Q_i\Omega_i)$. Clearly $\langle X, \theta_i \rangle$ is a log-concave random variable, and using that $\text{Cov}(\text{Proj}_L, \mu_{Q_i\Omega_i}) = \text{Proj}_L$, together with the fact that $\theta_i \in L_i \cap S^{n-1}$ one also has that $\langle X, \theta_i \rangle$ has variance 1. Using that the density of a log-concave distribution of unit variance is bounded by 1 one gets $\mathbb{P}(|\langle X, \theta_i \rangle| \leq 1/4) \leq 1/2$, which proves (16). It is now a simple application of Lemma 9 to see that for all $i$ there exists a direction $v_i \in S^{n-1}$ such that

$$\langle v_i, \text{Cov}(\Omega_i) v_i \rangle \leq c_1 n^{2^{-2i/n}}.$$  

where $c_1 > 0$ is a universal constant. Together with Lemma 8, this yields

$$\text{diam}(\text{Proj}_{v_i}\Omega_i) \leq 2\sqrt{c_1 n^{5/4}2^{-i/n}}.$$
By definition of $N$, this gives

$$N \leq n \log_{1/2} n^{5/4} + n \log_{1/2} \delta \leq n(12 + 2c_1 + 40 \log(1 + n/\varepsilon) - \log c').$$

Take $c' = \min(\{c, 1\})^{2^{1/3}}$. A straightforward calculation gives

$$\frac{c}{N} > \frac{c'}{n \log(1 + n/\varepsilon)}.$$  \hspace{1cm} (17)

Finally, we define

$$\mu(W) = \frac{1}{N} \sum_{i=1}^{N} \mu_i(W - x_0)$$

for all measurable $W \subset \mathbb{R}^n$.

For $\alpha \in \Omega \setminus \{x : |\langle x - x_0, v_N \rangle| \leq \delta\}$ consider a convex function $g(x)$ satisfying $g(\alpha) < -\varepsilon$. Define $\tilde{\alpha} = \alpha - x_0$ and $\tilde{g}(x) = g(x + x_0) - f(x_0)$ and remark that $\tilde{g}(\tilde{\alpha}) < -\varepsilon$. By definition of $N$, there exists $1 \leq i \leq N$ such that $\tilde{\alpha} \in \Omega_i \setminus \Omega_{i+1}$. Thus, equation (15) gives

$$\mu \left( \left\{ x \in \Omega : |f(x) - g(x)| > \frac{c'}{2n^{7.5} \log(1 + n/\varepsilon)} \max(\varepsilon, f(x)) \right\} \right) > \frac{c}{nN} > \frac{c'}{n^2 \log(1 + n/\varepsilon)}.$$  \hspace{1cm} (17)

The proof is complete.

### 3.4 From Lemma 4 to Lemma 3: covering the space via regions with stable gradients

We say that a $(z, \theta, t)$ is a jolly-good triplet if $|z| \leq \frac{1}{16}$ and (11) is satisfied for some appropriate $\delta$, namely $\delta = \frac{1}{\max_{\Omega} \log(1 + L_0/\varepsilon)}$ with $C > 0$ a universal constant whose value will be decided upon later on. Intuitively given Lemma 4 it is enough to find a polynomial (in $n$) number of jolly-good triplets for which the corresponding set of $\theta$-directions partially covers the sphere $S^{n-1}$. The notion of covering we use is the following: For a subset $H \subset S^{n-1}$ and for $\gamma > 0$, we say that $H$ is a $\gamma$-cover if for all $x \in S^{n-1}$, there exists $\theta \in H$ such that $\langle \theta, x \rangle \geq -\gamma$.

Next we explain how to find jolly-good triplets in Section 3.4.1, and then how to find a $\gamma$-cover with such triplets in Section 3.4.2.

#### 3.4.1 A contraction lemma

The following result shows that jolly-good triplets always exist, or in other words that a convex function always has a relatively big set on which the gradient map is approximately constant. Quite naturally the proof is based on a smoothing argument together with a Poincaré inequality.

**Lemma 6** Let $r, \eta, L > 0$ and $0 < \xi < 1$ such that $L > 2\eta r$. Let $\Omega \subset \mathbb{R}^n$ be a convex set, and $f : \Omega \to \mathbb{R}$ be $L$-Lipschitz and $\eta$-strongly convex, that is

$$\nabla^2 f(x) \succeq \eta \text{Id}, \ \forall x \in \Omega.$$
Let \( x_0 \in \Omega \) such that \( B(x_0, r) \subset \Omega \). Then there exist a triplet \((z, \theta, t) \in B(x_0, r) \times S^{n-1} \times [\eta r/2, +\infty)\) such that
\[
\mu_{B(z, \delta)} \left( (\nabla f)^{-1} (B(t \theta, \xi t)) \right) \geq 1/2
\]
for \( \delta = \frac{\eta r}{16n^2 \log \frac{L}{\eta r}}. \)

**Proof** We consider the convolution \( g = f \ast h \), where \( h \) is defined by
\[
h(x) = \frac{1_{\{x \in B(0, \delta)\}}}{\Vol(B(0, \delta))}.
\]
We clearly have that \( g \) is also \( \eta \)-strongly convex. Let \( x_{\min} \) be the point where \( g \) attains its minimum in \( \Omega \). We claim that
\[
|\nabla g(x)| \geq \eta r/2, \quad \forall x \in \Omega \setminus B(x_{\min}, r/2).
\]
Indeed by strong-convexity of \( g \) we have for all \( y \in \Omega \),
\[
|\nabla g(y)| \geq \frac{1}{|y - x_{\min}|} \langle \nabla g(y), y - x_{\min} \rangle \geq |y - x_{\min}| \eta.
\]
which proves (19).

Next, define \( B_0 = B(x_0, r) \) and \( D = B_0 \setminus B(x_{\min}, r/2) \). It is clear that \( \frac{\Vol(D)}{\Vol(B_0)} \geq \frac{1}{2} \). Let \( \nu \) be the push forward of \( \mu_D \) under \( x \mapsto |\nabla g(x)| \). According to (19) and by the assumption that \( f \) is \( L \)-Lipschitz, we know that \( \nu \) is supported on \([\eta r/2, L]\). Thus, there exists some \( t \in [\eta r/2, L] \) such that \( \nu([t, 2t]) \geq \left( 2 \log \frac{L}{\eta r} \right)^{-1} \). Define
\[
A = \{ x \in B_0 : |\nabla g(x)| \in [t, 2t] \},
\]
so we know that
\[
\frac{\Vol(A)}{\Vol(B_0)} \geq \frac{\Vol(A)}{\Vol(D)} \frac{\Vol(D)}{\Vol(B_0)} \geq \frac{1}{4 \log \frac{L}{\eta r}}.
\]
Recall that \( \frac{\Vol_{n-1}(\partial B(0, r))}{\Vol_n(B(0, r))} = \frac{n+1}{r} \). Using Lemma 10, we now have that
\[
\frac{1}{\Vol(A)} \int_A \Delta g(x) dx \leq t \frac{\Vol_{n-1}(\partial B_0)}{\Vol(A)} = t \frac{\Vol_{n-1}(\partial B_0)}{\Vol(B_0)} \frac{\Vol(B_0)}{\Vol(A)} \leq 8ntr^{-1} \log \frac{L}{\eta r}.
\]
Consequently, there exists a point \( z \in A \) for which \(|\nabla g(z)| \geq t \) and \( \Delta g(z) \leq 8ntr^{-1} \log \frac{L}{\eta r} \). In other words, by the definition of \( g \), we have that
\[
\frac{1}{\Vol(B(z, \delta))} \int_{B(z, \delta)} \Delta f(x) dx \leq 8ntr^{-1} \log \frac{L}{\eta r}.
\]
Fix \( 1 \leq i \leq n \), and define \( w(x) = (\nabla f(x) - \nabla g(z), e_i) \), where \( e_i \) is the \( i \)-th vector of the standard basis. Note that
\[
|\nabla w(x)| = |\nabla^2 f(x)e_i| \leq \Delta f(x).
\]
Recall that the Poincaré inequality for a ball (see e.g., Acosta and Durán [2003]) implies that
\[ \int_{B(z, \delta)} |w(x)| dx \leq \delta \int_{B(z, \delta)} |\nabla w(x)| dx. \]

Thus combining the last three displays, and using that \( \delta = \frac{\xi t}{16n^2 \log \frac{L}{\eta r}} \), one obtains
\[ \frac{1}{\text{Vol}(B(z, \delta))} \int_{B(z, \delta)} |w(x)| dx \leq 8 \delta r^{-1} \log \frac{L}{\eta r} \leq \frac{\xi t}{2n}. \]

By using the fact that \( |\nabla f(x) - \nabla g(z)| \leq \sum_{i=1}^{n} |\langle \nabla f(x) - \nabla g(z), e_i \rangle| \), this yields
\[ \frac{1}{\text{Vol}(B(z, \delta))} \int_{B(z, \delta)} |\nabla f(x) - \nabla g(z)| dx \leq \xi t/4 \leq \xi |\nabla g(z)|/2. \]

Finally applying Markov’s inequality one obtains (18) for the triplet \((z, \frac{\nabla g(z)}{|\nabla g(z)|}, |\nabla g(z)|)\).

\[ \square \]

### 3.4.2 Concluding the proof with the contraction lemma

We first fix some \( \eta > 0 \) and, at this point, suppose that \( \nabla^2 f(x) \succeq \eta \) for all \( x \in \Omega \). Later on we will argue that this assumption can be removed. Define \( h_{\Omega}(x) = \sup_{y \in \Omega} \langle x, y \rangle \), the support function of \( \Omega \). Consider the set
\[ \Theta = \{ \theta \in S^{n-1} : h_{\Omega}(\theta) \leq \frac{1}{8} \} \]
and let \( H \) be set of directions obtained from jolly-good triplets, more precisely,
\[ H = \left\{ \theta \in S^{n-1} : \exists z \in \mathbb{R}^n, t \in (0, 1) \text{ such that (11) is true with } \delta = \frac{1}{2^{28}n^6 \log(1 + LN/\eta)} \right\}. \]

Define \( \gamma = \frac{1}{16n} \). Next, we show that \( H \cup \Theta \) is a \( \gamma \)-cover. Let \( \varphi \in S^{n-1} \). Our objective is to find \( \theta \in H \cup \Theta \) such that \( \langle \theta, \varphi \rangle \geq -\gamma \).

First suppose that \( \varphi \notin 8\Omega \). In that case, by Hahn-Banach and since \( 0 \in \Omega \), there exists \( w \in \mathbb{R}^n \) such that \( \langle \varphi, w \rangle = 1 \) and \( \langle w, y \rangle \leq \frac{1}{8} \) for all \( y \in \Omega \). In other words, we have for \( \theta = \frac{w}{|w|} \) that
\[ h_{\Omega}(\theta) \leq \frac{1}{8|w|} \leq \frac{1}{8}, \]
which implies that \( \theta \in \Theta \). Since \( \langle \varphi, \frac{w}{|w|} \rangle \geq 0 \), we are done.

We may therefore assume that \( \varphi/8 \in \Omega \). Since \( \text{Cov}(\mu_{\Omega}) = \text{Id} \), then by Lemma 8 there exists a point \( w \in \mathbb{R}^n \) such that \( |w| \leq n + 1 \) and \( B(w, 1) \subset \Omega \). Define \( r = \frac{1}{2^{15}n^2} \) and take
\[ B_0 = B(\varphi/32 + rw, r). \]

Note that by convexity and by the fact that \( 0 \in \Omega \), we have that \( B_0 \subset \Omega \). We now use Lemma 6 for the ball \( B_0 \) with \( \xi = \frac{1}{2^{15}n^2} \), and \( \delta = \frac{1}{2^{28}n^6 \log(1 + LN/\eta)} \) to obtain a jolly-good triplet \((z(\theta), \theta, t)\).
Denote \( z = z(\theta) \). We want to show that \( \langle \theta, \varphi \rangle \geq -\gamma \). Observe that by convexity of \( f \) and since \( f \) attains its minimum at \( x = 0 \), one has \( \langle \nabla f(x), x \rangle \geq 0 \) for any \( x \). Thus, by definition of a jolly-good triplet one can easily see that \( \langle \theta, z \rangle \geq -\langle \xi + \delta \rangle \). Also by definition \( z \) is in \( B_0 \) and thus \( |32z - \varphi - 32rw| \leq 32r \). This implies:

\[
\langle \theta, \varphi \rangle = \langle \theta, \varphi - 32z + 32rw \rangle + 32\langle \theta, z \rangle - 32r \langle \theta, w \rangle \\
\geq -|\varphi - 32z + 32rw| - 32|w| - 32\xi - 32\delta \geq -\frac{1}{16n}.
\]

This concludes the proof that \( H \cup \Theta \) is a \( \gamma \)-cover.

Next we use Lemma 11 to extract a subset \( H' \subset H \) such that \( |H'| \leq n + 1 \) and \( H' \cup \Theta \) is also a \( \gamma \)-cover for \( S_{n-1} \). An application of Lemma 12 with \( M = 2n \) now gives that there exists \( v \in S_{n-1} \) such that

\[
\Omega \cap \left( \bigcap_{\theta \in H' \cup \Theta} \{ x : \langle x, \theta \rangle \leq \frac{1}{8} \} \right) = \Omega \cap \left( \bigcap_{\theta \in H'} \{ x : \langle x, \theta \rangle \leq \frac{1}{8} \} \right) \subset S_{\Omega,v}.
\]

Finally, an application of Lemma 4 gives us that for all \( \alpha \in \Omega \setminus S_{\Omega,v} \) and every function \( g \) such that \( g(\alpha) < -\varepsilon \) one has for some \( \theta \in H' \),

\[
\mu_{B(z(\theta), \delta)} \left( \left\{ x \in \Omega : |f(x) - g(x)| > \frac{\delta}{2^{13}M\sqrt{n}} \max(\varepsilon, f(x)) \right\} \right) > \frac{1}{8}.
\]

Defining \( \mu = \frac{1}{|H'|} \sum_{\theta \in H'} \mu_{B(z(\theta), \delta)} \), we get

\[
\mu \left( \left\{ x \in \Omega : |f(x) - g(x)| > \frac{1}{2^{42}n^{7.5} \log(1 + Ln/\eta) \max(\varepsilon, f(x))} \right\} \right) > \frac{1}{16n}. \tag{20}
\]

It remains to remove the uniform convexity assumption. This is done by considering the function

\[
x \mapsto f(x) + \eta|x|^2
\]

in place of \( f \) in the above argument. Since \( |x| \leq M \leq 2n \) for all \( x \in \Omega \), the equation (20) becomes

\[
\mu \left( \left\{ x \in \Omega : |f(x) - g(x)| > \frac{c}{2^{42}n^{7.5} \log(1 + Ln/\eta) \max(\varepsilon, f(x))} \right\} \right) > \frac{1}{16n}.
\]

Finally choosing \( \eta = \left( \frac{c}{2^{42}n^{10}} \right)^2 \) one easily obtains

\[
\mu \left( \left\{ x \in \Omega : |f(x) - g(x)| > \frac{1}{2^{50}n^{7.5} \log(1 + n/\varepsilon) \max(\varepsilon, f(x))} \right\} \right) > \frac{1}{16n},
\]

which concludes the proof.
3.5 Proof of Lemma 4

The main ingredient of the proof is the following technical result.

**Lemma 7** Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying $\text{Diam}(\Omega) \leq M$. Let $f : \Omega \to [0, \infty)$ be a non-negative convex function let $g : \Omega \to \mathbb{R}$ be a convex function satisfying $g(\alpha) < -\varepsilon$, for some $\alpha \in \Omega$. Let $z \in \mathbb{R}^n$ and consider the ball $B = B(z, \delta)$. Let $D \subset B$ be a set satisfying

$$\langle \nabla f(x), \alpha - x \rangle \geq 0, \ \forall x \in D.$$ (21)

Assume also that $\mu_B(D) \geq \frac{1}{2}$ and that $|z - \alpha| \geq n\delta$. Define

$$A = \{ x : |f(x) - g(x)| > \frac{\delta}{2^{13}M^2n} \max(\varepsilon, f(x)) \}.$$ (22)

Then one has $\mu_D(A) \geq 1/4$.

**Proof** For $x \in \Omega$, define $\Theta_\alpha(x) = \frac{x - \alpha}{|x - \alpha|}$ and for $\theta \in S^{n-1}$ write $N(\theta) = \Theta_\alpha^{-1}(\theta)$. Denote by $\lambda_\theta$ the one-dimensional Lebesgue measure on the needle $N(\theta)$. Let $\sigma_B, \sigma_D$ be the push-forward of $\mu_B, \mu_D$ under $\Theta_\alpha$. Moreover, for every $\theta \in S^{n-1}$, the disintegration theorem ensures the existence of a probability measure $\mu_{D,\theta}$ on $N(\theta)$, defined so that for every measurable test function $h$ one has

$$\int h(x) d\mu_D(x) = \int_{S^{n-1}} \int_{N(\theta)} h(x) d\mu_{D,\theta}(x) d\sigma_D(\theta)$$ (22)

(in other words, $\mu_{D,\theta}$ is the normalized restriction of $\mu_D$ to $N(\theta)$). Define the measures $\left(\mu_{B,\theta}\right)_\theta$ in the same manner.

It is easy to verify that $\sigma_D$ is absolutely continuous with respect the the uniform measure on $S^{n-1}$, which we denote by $\sigma$. Denote $q(\theta) := \frac{d\sigma}{d\sigma}\mu_B(\theta)$ and $w(\theta) := \frac{d\sigma}{d\sigma}\mu_D(\theta)$.

Using Lemma 7 we obtain that

$$\frac{d\mu_{D,\theta}}{d\lambda_\theta}(x) = \frac{\zeta_n}{\text{Vol}(D)q(\theta)}|x - \alpha|^{n-1}1_{\{x \in D\}},$$ (23)

and

$$\frac{d\mu_{B,\theta}}{d\lambda_\theta}(x) = \frac{\zeta_n}{\text{Vol}(B)w(\theta)}|x - \alpha|^{n-1}1_{\{x \in B\}},$$ (24)

where $\zeta_n$ is a constant depending only on $n$.

For every $\theta \in S^{n-1}$, define $L(\theta)$ to be the length of the interval $N(\theta) \cap B$. Consider the set

$$\mathcal{L} = \left\{ \theta : L(\theta) > \frac{\delta}{32\sqrt{n}} \right\}.$$ (22)

According to Lemma 13 we have that

$$\int_{S^{n-1}\setminus\mathcal{L}} w(\theta) d\sigma(\theta) \leq \frac{1}{8}.$$ (22)
Now, since $D \subset B$ and $\mu_B(D) \geq \frac{1}{2}$, we have that $q(\theta) \leq 2w(\theta)$ for all $\theta \in S^{n-1}$, which gives

$$\sigma_D(\mathcal{L}) = \int_{\mathcal{L}} q(\theta) d\sigma(\theta) \geq \frac{3}{4}.$$ 

Next, consider the set

$$S = \left\{ \theta \in S^{n-1}; \quad q(\theta) \geq \frac{w(\theta)}{4} \right\}.$$

Since $\int_{S^{n-1}} \frac{q(\theta)}{w(\theta)} d\sigma_B(\theta) = 1$ we have

$$\sigma_D(S) = \int_S \frac{q(\theta)}{w(\theta)} d\sigma_B(\theta) = 1 - \int_{S^{n-1}\setminus S} \frac{q(\theta)}{w(\theta)} d\sigma_B(\theta) \geq \frac{3}{4}.$$ 

Using a union bound, we have that $\sigma_D(\mathcal{L} \cap S) \geq \frac{1}{2}$.

Fix $\theta \in \mathcal{L} \cap S$, we would like to give a lower bound on $\mu_{D,\theta}(A)$. In view of Lemma 2, we thus need an upper bound on the density of $\mu_{D,\theta}$. Recall that $\theta \in S$, implies $q(\theta)w(\theta) \geq \frac{1}{4}$ and that by (23) and (24), we have for all $x \in N(\theta) \cap B$,

$$\frac{d\mu_{D,\theta}}{d\mu_{B,\theta}}(x) = \frac{\text{Vol}(B)w(\theta)}{\text{Vol}(D)q(\theta)} 1_{x \in D} \leq 8. \quad (25)$$

Denote $[a, b] = B \cap N(\theta)$ for $a, b \in \mathbb{R}^n$. Assume that $a$ is the interior of the interval $[\alpha, b]$ (if this is not the case, we simply interchange between $a$ and $b$). By the assumption $\theta \in \mathcal{L}$, we know that $|b - a| \geq \frac{\delta}{32\sqrt{n}}$. Writing $Z = \frac{\text{Vol}(B)w(\theta)}{\text{Vol}(D)q(\theta)}$ so that, according to (24),

$$\frac{d\mu_{B,\theta}}{d\lambda_\theta}(x) = Z|x - \alpha|^{n-1}1_{x \in B},$$

and since $\mu_{B,\theta}$ is a probability measure,

$$Z^{-1} = \int_a^b |x - \alpha|^{n-1} dx$$

where, by slight abuse of notation we assume that $a, b, \alpha \in \mathbb{R}$. Thus,

$$Z \leq \frac{32\sqrt{n}}{\delta |a - \alpha|^{n-1}}.$$

Combined with (25), this finally gives

$$\frac{d\mu_{D,\theta}}{d\lambda_\theta}(x) \leq 2^8 \sqrt{n} |x - \alpha|^{n-1} \leq 2^8 \sqrt{n} \left( \frac{|b - \alpha|}{|a - \alpha|} \right)^{n-1} \leq 2^8 \sqrt{n} \left( \frac{1 + \frac{|b - a|}{|a - \alpha|}}{\delta n\delta - \delta} \right)^{n-1} \leq 2^8 e^2 \frac{\sqrt{n}}{\delta},$$

where in the second to last inequality we used the assumption that $|z - \alpha| \geq n\delta$. 

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Define the map $U : \mathbb{R} \to N(\theta)$ by

$$U(x) = \alpha + M(|\alpha| - x)\theta$$

and consider the functions $\tilde{f}(x) = f(U(x))$ and $\tilde{g}(x) = g(U(x))$. Denote $x_0 = \min U^{-1}(D \cap N(\theta))$ and remark that $x_0 \in [|\alpha| - 1, |\alpha|]$. Note that, thanks to equation (21), the assumption (9) holds for the functions $\tilde{f}, \tilde{g}$ and the points $x_0, |\alpha|$. We can now invoke Lemma 2 for these functions with $\mu$ being the pullback of $\mu_{D, \theta}$ by $U(x)$. According to the above inequality one may take $\beta = 2^8 e^2 M \sqrt{n}$ and obtain

$$\mu_{D, \theta}(A) \geq \frac{1}{2}. $$

Integrating over $\theta \in \mathcal{L} \cap \mathcal{S}$ concludes the proof:

$$\mu_D(A) \geq \int_{\mathcal{S} \cap \mathcal{L}} \mu_{D, \theta}(A) d\sigma_D(\theta) \geq \frac{1}{2} \sigma_D(\mathcal{L} \cap \mathcal{S}) \geq \frac{1}{4}. $$

**Proof** [Proof of Lemma 4] Suppose that $(z, \theta, t)$ satisfy equation (18). Fix $\alpha \in \Omega$ satisfying $\langle \alpha, \theta \rangle \geq \frac{1}{8}$ and a function $g(x)$ satisfying $g(\alpha) < -\varepsilon$. Define $B = B(z, \delta)$ and $D = \{ x \in B : |\nabla f(x) - \theta t| < \frac{1}{16} n^{-2} \}$. Let $\mu_B$ be the uniform measure on $B$. According to (18), we know that $\mu_B(D) \geq \frac{1}{2}$. Now, for all $x \in D$ we have that $\nabla f(x) = t(\theta + y)$ with $|y| < \frac{1}{16} n^{-2}$ so we get

$$\left\langle \nabla f(x), \frac{\alpha - x}{|\alpha - x|} \right\rangle = \frac{t}{|\alpha - x|} (\langle \alpha, \theta \rangle + \langle \alpha - x, y \rangle - \langle x, \theta \rangle)$$

$$\geq \frac{t}{|\alpha - x|} \left( \frac{1}{8} - \frac{1}{16}(|\alpha| + |x|) n^{-2} - |x| \right) \geq 0, \ \forall x \in D$$

where we used the fact that $D \subset B$ and so $|x| < |z| < \delta \leq \frac{1}{16}$ and the fact that $|\alpha| \leq 2n$. Note that the above implies the assumption (21). Moreover remark that

$$|z - \alpha| \geq \frac{1}{4} - \frac{1}{8} \geq \frac{1}{8} \geq n\delta.$$

We can thus now invoke Lemma 7 to get $\mu_B(A) \geq 1/8$ where

$$A = \{ x \in \Omega : |f(x) - g(x)| > \frac{\delta}{2^3 M \sqrt{n}} \max(\varepsilon, f(x)) \}.$$

This completes proof.

**3.6 Technical lemmas**

We gather here various technical lemmas.
Lemma 8  Let $C$ be a convex body in $\mathbb{R}^n$. Then
\[
\text{diam}(C) \leq (n + 1)\|\text{Cov}(\mu_C)\|^{1/2}.
\] (26)

On the other hand, if $\text{Cov}(\mu_C) \succeq \text{Id}$ then $C$ contains a ball of radius 1. Furthermore, for all $v \in \mathbb{S}^{n-1}$ one has
\[
\sup_{x \in C} \langle v, x \rangle - \inf_{x \in C} \langle v, x \rangle \leq (n + 1)\langle v, \text{Cov}(\mu_C), v \rangle^{1/2}.
\]

Proof  The first and second parts of the Lemma are found in [Brazitikos et al., 2014, Section 3.2.1]. For the second part, we write $C' = \text{Cov}(C)^{-1/2}C$ and $u = \frac{\text{Cov}(C)^{1/2}u}{|\text{Cov}(C)^{1/2}u|}$. We have
\[
\sup_{x \in C} \langle v, x \rangle - \inf_{x \in C} \langle v, x \rangle = \sup_{x \in C'} \langle v, \text{Cov}(C)^{1/2}x \rangle - \inf_{x \in C'} \langle v, \text{Cov}(C)^{1/2}x \rangle
\]
\[
= \sup_{x \in C'} \langle \text{Cov}(C)^{1/2}v, x \rangle - \inf_{x \in C'} \langle \text{Cov}(C)^{1/2}v, x \rangle
\]
\[
= |\text{Cov}(C)^{1/2}v| \left( \sup_{x \in C'} \langle u, x \rangle - \inf_{x \in C'} \langle u, x \rangle \right) \overset{(26)}{\leq} (n + 1)|\text{Cov}(C)^{1/2}v|.
\]

Lemma 9  Let $C \subset D \subset \mathbb{R}^n$ be two convex bodies with $0 \in C$. Suppose that $\frac{\text{Vol}(C)}{\text{Vol}(D)} \leq \delta$, then there exists $u \in \mathbb{S}^{n-1}$ such that
\[
\langle u, \text{Cov}(\mu_C)u \rangle \leq c\sqrt{n}\delta^{2/n}\langle u, \text{Cov}(\mu_D)u \rangle.
\] (27)

where $c > 0$ is a universal constant.

Proof  Define $\mu = \mu_D$ and $\nu = \mu_C$. By applying a linear transformation to both $\mu$ and $\nu$, we can clearly assume that $\text{Cov}(\mu) = \text{Id}$. Let $f(x)$ be a log-concave probability density in $\mathbb{R}^n$. According to [Klartag, 2006, Corollary 1.2 and Lemma 2.7], we have that
\[
c_1 \leq \left( \sup_{x \in \mathbb{R}^n} f(x) \right)^{1/n} \left( \det \text{Cov}(f) \right)^{1/2n} \leq c_2 n^{1/4}
\] (28)

where $c_1, c_2 > 0$ are universal constants. Denote by $f(x)$ and $g(x)$ the densities of $\mu$ and $\nu$, respectively. Since $\mu, \nu$ are indicators, we have that
\[
\sup_{x \in \mathbb{R}^n} f(x) = f(0) = \delta g(0) = \delta \sup_{x \in \mathbb{R}^n} g(x).
\]

We finally get
\[
\left( \det \text{Cov}(\nu) \right)^{1/n} \overset{(28)}{\leq} c_2^2 \sqrt{n}g(0)^{-2/n}
\]
\[
= c_2^2 \delta^{2/n} \sqrt{n}f(0)^{-2/n}
\]
\[
\overset{(28)}{\leq} (c_2/c_1)^2 \sqrt{n} \left( \det \text{Cov}(\mu) \right)^{1/n} \delta^{2/n} = (c_2/c_1)^2 \sqrt{n}\delta^{2/n}.
\]

The lemma follows by taking $u$ to be the eigenvector corresponding to the smallest eigenvalue of $\text{Cov}(\nu)$.
Lemma 10 Let \( g \) be a convex function defined on a Euclidean ball \( B \subset \mathbb{R}^n \). Let \( A \subset B \) be a closed set such that \( \forall x \in A, |\nabla g(x)| \leq t \). Then

\[
\int_A \Delta g(x)dx \leq t \text{Vol}_{n-1}(\partial B).
\]

Proof Since \( g \) is convex, we can write

\[
g(x) = \sup_{y \in B} w_y(x)
\]

where \( w_y(x) = \langle x - y, \nabla g(y) \rangle + g(y) \). Define

\[
\tilde{g}(x) = \sup_{y \in A} w_y(x).
\]

Clearly \( \tilde{g} \) is convex and \( \tilde{g}(x) = g(x) \) for all \( x \in A \). Moreover \( |\nabla \tilde{g}(x)| \leq t \) for all \( x \in \mathbb{R}^n \). Using Gauss’s theorem, we have

\[
\int_A \Delta g(x)dx \leq \int_B \Delta \tilde{g}(x)dx = \int_{\partial B} \langle \nabla \tilde{g}(x), n(x) \rangle d\mathcal{H}_{n-1}(x) \leq t \text{Vol}_{n-1}(\partial B),
\]

which concludes the proof.

Let \( \gamma > 0 \). Recall that we say that \( H \subset \mathbb{S}^{n-1} \) is a \( \gamma \)-cover if for all \( x \in \mathbb{S}^{n-1} \), there exists \( \theta \in H \) satisfying

\[
\langle \theta, x \rangle \geq -\gamma.
\]

(29)

Lemma 11 Let \( H \subset \mathbb{S}^{n-1} \) be a \( \gamma \)-cover. Then there exists a subset \( I \subset H \) with \(|I| \leq n + 1 \) such that \( I \) is a \( \gamma \)-cover.

Proof We first claim that there is a point \( y \in \text{Conv}(H) \) with \(|y| \leq \gamma \). Indeed, if we assume otherwise then by Hahn-Banach there exists \( \tilde{\theta} \in \mathbb{S}^{n-1} \) such that \( \langle \theta, \tilde{\theta} \rangle > \gamma \) for all \( \theta \in H \), which means the vector \( -\tilde{\theta} \) violates the assumption (29). By Caratheodory’s theorem, there exists \( I \subset H \) with \(|I| \leq n + 1 \) such that \( y \in \text{Conv}(I) \). Write \( I = (\theta_1, ..., \theta_{n+1}) \). Now let \( x \in \mathbb{R}^n \) with \(|x| \leq 1 \). Then since \( \langle x, y \rangle \geq -\gamma \), we have

\[
\sum_{i=1}^{n+1} \alpha_i \langle x, \theta_i \rangle \geq -\gamma
\]

for some non-negative coefficients \( \{\alpha_i\}_{i=1}^{n+1} \) satisfying \( \sum_{i=1}^{n+1} \alpha_i = 1 \). Thus there exists \( \theta \in I \) for which (29) holds.

Lemma 12 Let \( \Omega \subset \mathbb{R}^n \) be a convex set with \( \text{diam}(\Omega) \leq M \) and such that \( 0 \in \Omega \). Let \( H \) be a \( \gamma \)-cover. Then there exists \( \tilde{\theta} \in \mathbb{S}^{n-1} \) such that

\[
\{\alpha \in \Omega : \forall \theta \in H, \langle \alpha, \theta \rangle < M\gamma\} \subset \{\alpha \in \Omega : |\langle \alpha, \tilde{\theta} \rangle| \leq 2M\gamma\}.
\]
**Proof** Since \( \{ \alpha \in \Omega : \forall \theta \in H, \langle \alpha, \theta \rangle < M\gamma \} \) is a convex set which contains 0, showing that it does not contain a ball of radius \( 2M\gamma \) is enough to show that it is included in some slab \( \{ \alpha \in \Omega : |\langle \alpha, \theta \rangle| \leq 2M\gamma \} \). Now suppose that our set of interest \( \{ \alpha \in \Omega : \forall \theta \in H, \langle \alpha, \theta \rangle < M\gamma \} \) actually contains a ball \( B(x, 2M\gamma) \) with \( |x| \in (0, M) \). Let \( \theta \in H \) be such that \( \langle \theta, \frac{x}{|x|} \rangle \geq -\gamma \), and thus in particular \( \langle x, \theta \rangle \geq -M\gamma \). Then one has by the inclusion assumption that \( \langle \theta, x + 2M\gamma \theta \rangle < M\gamma \), but on the other hand one also has \( \langle \theta, x + 2\gamma M\theta \rangle \geq \gamma M \) which yields a contradiction, thus concluding the proof. 

---

**Lemma 13** Let \( \delta > 0 \), \( x_0 \in \mathbb{R}^n \), \( B = B(x_0, \delta) \) and \( \alpha \in \mathbb{R}^n \backslash B \). For \( x \in \mathbb{R}^n \), define \( \Theta_\alpha(x) = \frac{x - \alpha}{|x - \alpha|} \), and let \( \sigma_B \) be the push-forward of \( \mu_B \) under \( \Theta_\alpha \). For every \( \theta \in \mathbb{S}^{n-1} \), define \( L(\theta) \) to be the length of the interval \( \Theta_\alpha^{-1}(\theta) \cap B \). Then one has

\[
\sigma_B \left( \theta : L(\theta) > \frac{\delta}{32\sqrt{n}} \right) \geq \frac{7}{8}.
\]

**Proof** Note that, by definition,

\[
x \in B \text{ and } x + \frac{\delta}{32\sqrt{n}} \frac{\alpha - x}{|\alpha - x|} \in B \Rightarrow L(\Theta_\alpha(x)) > \frac{\delta}{32\sqrt{n}}.
\]

Furthermore it is easy to show that for all \( y \in B \),

\[
y + \frac{\delta}{32\sqrt{n}} \frac{\alpha - x_0}{|\alpha - x_0|} \in B \Rightarrow y + \frac{\delta}{32\sqrt{n}} \frac{\alpha - y}{|\alpha - y|} \in B.
\]

Thus letting \( X \sim \mu_B \) we see that the lemma will be concluded by showing that

\[
P \left( X + \frac{\delta}{32\sqrt{n}} \frac{\alpha - x_0}{|\alpha - x_0|} \in B \right) \geq \frac{7}{8}.
\]

Defining \( \tilde{B} = B \left( x_0 - \frac{\delta}{32\sqrt{n}} \frac{\alpha - x_0}{|\alpha - x_0|}; \delta \right) \), the statement boils down to proving that \( P(X \in \tilde{B}) \geq 7/8 \). By applying an affine linear transformation to both \( B \) and \( \tilde{B} \), this is equivalent to

\[
\frac{\text{Vol} \left( B \left( -\frac{1}{32\sqrt{n}} e_1, 1 \right) \cap B \left( \frac{1}{32\sqrt{n}} e_1, 1 \right) \right)}{\text{Vol}(B(0, 1))} \geq \frac{7}{8}
\]

where \( e_1 \) is the first vector of the standard basis. Next, by symmetry around the hyperplane \( e_1^\perp \), we have

\[
\frac{\text{Vol} \left( B \left( -\frac{1}{64\sqrt{n}} e_1, 1 \right) \cap B \left( \frac{1}{64\sqrt{n}} e_1, 1 \right) \right)}{\text{Vol}(B(0, 1))} = 2\text{Vol} \left( B \left( -\frac{1}{64\sqrt{n}} e_1, 1 \right) \cap \{ x; \langle x, e_1 \rangle \geq 0 \} \right) \text{Vol}(B(0, 1)).
\]

Thus, it is enough to show that \( P \left( |Z| > \frac{1}{64\sqrt{n}} \right) \geq \frac{7}{8} \) where \( Z = \langle X', e_1 \rangle \) and \( X' \sim \mu_{B(0,1)} \). Observe that \( \text{Var}[Z] \geq \frac{1}{8n} \) and that \( Z \) is log-concave (in particular the density of \( Z/\text{Var}[Z] \)) is bounded by 1). This implies that for any \( t > 0 \)

\[
P \left( |Z| < t\sqrt{\text{Var}[Z]} \right) < 2t,
\]

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and thus the lemma follows by taking $t = \frac{1}{16}$.

**Lemma 14** Let $A \subset \mathbb{R}^n$. For $x \in \mathbb{R}^n$, define $\Theta_\alpha(x) = \frac{x - \alpha}{|x - \alpha|}$, and let $\sigma_A$ be the push-forward of $\mu_A$ under $\Theta_\alpha$. Assume that $\sigma_A$ is absolutely continuous with respect the the uniform measure $\sigma$ on $S^{n-1}$ and denote $q(\theta) := \frac{d\sigma_A}{d\sigma}(\theta)$. Finally let $\mu_{A,\theta}$ be the normalized restriction of $\mu_A$ on $N(\theta) = \Theta^{-1}_\alpha(\theta)$, defined so that for every measurable test function $h$ one has

$$\int h(x) d\mu_{D}(x) = \int_{S^{n-1}} \int_{N(\theta)} h(x) d\mu_{A,\theta}(x) d\sigma_A(\theta).$$  \hfill (30)

Denoting $\zeta_n$ for the $(n-1)$-dimensional Hausdorff measure of $S^{n-1}$ one then obtains

$$\frac{d\mu_{A,\theta}}{d\lambda_\theta}(x) = \frac{\zeta_n}{\text{Vol}(A) q(\theta)} |x - \alpha|^{n-1} 1_{\{x \in B\}}.$$  \hfill (31)

**Proof** First observe that the existence of $\mu_{A,\theta}$ is ensured by the disintegration theorem. Now remark that using the integration by polar coordinates formula we have for every measurable test function $\varphi$,

$$\int_{\mathbb{R}^n} \varphi(x) dx = \zeta_n \int_{S^{n-1}} \int_0^\infty r^{n-1} \varphi(\alpha + r\theta) dr d\sigma(\theta).$$

Now, by definition of $q(\cdot)$, we have for every test function $\varphi$,

$$\int_{S^{n-1}} \int_0^\infty r^{n-1} \varphi(\alpha + r\theta) dr d\sigma(\theta) = \int_{S^{n-1}} \int_0^\infty r^{n-1} q(\theta)^{-1} \varphi(\alpha + r\theta) dr d\sigma_A(\theta).$$

Taking $\varphi(x) = h(x) 1_{x \in A}$, we finally get

$$\int h(x) d\mu_A(x) = \frac{1}{\text{Vol}(A)} \int_A h(x) dx$$

$$= \frac{\zeta_n}{\text{Vol}(A)} \int_{S^{n-1}} \int_0^\infty r^{n-1} q(\theta)^{-1} h(\alpha + r\theta) 1_{\{\alpha + r\theta \in A\}} dr d\sigma_A(\theta).$$

Since the above is true for every measurable function $h$, together with equation (30) we get that for every function $h$ and every $\theta \in S^{n-1}$, one must have

$$\int_{N(\theta)} h(x) d\mu_\theta(x) = \frac{\zeta_n}{\text{Vol}(D) q(\theta)} \int_0^\infty r^{n-1} h(\alpha + r\theta) 1_{\{\alpha + r\theta \in A\}} dr$$

$$= \frac{\zeta_n}{\text{Vol}(A) q(\theta)} \int_{N(\theta)} |x - \alpha|^{n-1} h(x) 1_{\{x \in A\}} d\lambda_\theta(x)$$

and the claimed identity (31) follows.
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