On the Error Probability of Stochastic Decision and Stochastic Decoding

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Abstract—This paper investigates the error probability of a stochastic decision and the way in which it differs from the error probability of an optimal decision, i.e., the maximum a posteriori decision. It is shown that the error probability of a stochastic decision with the a posteriori distribution is at most twice the error probability of the maximum a posteriori decision. Furthermore, it is shown that, by generating an independent identically distributed random sequence subject to the a posteriori distribution and making a decision that maximizes the a posteriori probability over the sequence, the error probability approaches exponentially the error probability of the maximum a posteriori decision as the sequence length increases. Using these ideas as a basis, we can construct stochastic decoders for source/channel codes.

Index Terms—channel coding, decision theory, error probability, maximum a posteriori decision, source coding, source coding with decoder side information, stochastic decision, stochastic decoding

I. INTRODUCTION

This paper considers a decision problem that involves gussing an invisible state \( X \) after observing \( Y \), which is correlated with \( X \). In decision theory [2, Section 1.5.2], an optimal decision rule, which minimizes the decision error probability, is called the Bayes decision rule.

Let \( X \) and \( Y \) be random variables that take values in sets \( \mathcal{X} \) and \( \mathcal{Y} \), respectively, where we call \( X \) a state of nature or a parameter and \( Y \) an observation. Let \( p_{X|Y} \) be the joint distribution of \((X, Y)\). Let \( p_X \) and \( p_Y \) be the marginal distributions of \( X \) and \( Y \), respectively. Let \( p_{X|Y} \) be the conditional distribution of \( X \) for a given \( Y \). It is well known that an optimal strategy for guessing the state \( X \) consists of finding \( \hat{x} \), which maximizes the conditional probability \( p_{X|Y}(\hat{x}|y) \) depending on a given observation \( y \). Formally, by taking an \( \hat{x} \) that maximizes \( p_{X|Y}(\hat{x}|y) \) for each \( y \in \mathcal{Y} \), we can define the function \( f_{\text{MAP}}: \mathcal{Y} \rightarrow \mathcal{X} \) as

\[
f_{\text{MAP}}(y) \equiv \arg \max_{\hat{x}} p_{X|Y}(\hat{x}|y)
\]

which is a Bayes decision rule. It should be noted that the discussion throughout this paper does not depend on choosing states with the same maximum probability.

When the cardinality \( |\mathcal{X}| \) of \( \mathcal{X} \) is small, operations \( \text{MAP} \) and \( \text{ML} \) are tractable by using a brute force search. However, with coding problems, these operations appears to be intractable because \(|\mathcal{X}| \) grows exponentially as the dimension of \( \mathcal{X} \) grows. In this paper, we assume a situation where operations \( \text{MAP} \) and \( \text{ML} \) are intractable.

In source coding, \( X \) corresponds to a source output and \( Y \) corresponds to a codeword and side information. In channel coding, \( X \) corresponds to a codeword and \( Y \) corresponds to a channel output, where the decoding with \( \text{MAP} \) is called maximum a posteriori decoding. On the other hand, the decoding method that maximizes the conditional probability \( p_{X|Y}(y|x) \) of a channel is called maximum likelihood decoding, which is equivalent to maximum a posteriori decoding when \( X \) is generated subject to the uniform distribution. In this paper, we call the decision rule with \( f_{\text{MAP}} \) the maximum a posteriori decision rule.

In this paper, we consider a stochastic decision, where the decision is made randomly subject to a probability distribution. We investigate the relationship between the error probabilities of the stochastic and maximum a posteriori decisions. Then, we introduce the construction of stochastic decoders for source/channel codes.

II. DEFINITIONS OF STOCHASTIC DECISION AND DECISION ERROR PROBABILITY

For a stochastic decision, we use a random number generator to obtain \( \tilde{X} \in \mathcal{X} \) after observing \( Y \) and let \( \hat{X} \) be a decision (guess) about the state \( X \). Formally, we generate \( \tilde{X} \) subject to the conditional distribution \( q_{\tilde{X}|Y}(\cdot|y) \) on \( \mathcal{X} \) depending on an observation \( Y \) and let an output be a decision of \( X \), where \( X \leftrightarrow Y \leftrightarrow \tilde{X} \) forms a Markov chain. The joint distribution \( p_{XY\tilde{X}} \) of \((X, Y, \tilde{X})\) is given as

\[
p_{XY\tilde{X}}(x, y, \tilde{x}) = q_{\tilde{X}|Y}(\tilde{x}|y)p_{X|Y}(x|y)p_Y(y).
\]

Let us call \( q_{\tilde{X}|Y} \) a stochastic decision rule. As a special case, when \( q_{\tilde{X}|Y} \) is given by using a function \( f: \mathcal{Y} \rightarrow \mathcal{X} \) and is defined as

\[
q_{\tilde{X}|Y}(\tilde{x}|y) = \begin{cases} 1 & \text{if } \tilde{x} = f(y) \\ 0 & \text{if } \tilde{x} \neq f(y), \end{cases}
\]

we call \( q_{\tilde{X}|Y} \) or \( f \) a deterministic decision rule. It should be noted that the maximum a posteriori decision rule is deterministic.

We can interpret the right hand side of \( \text{MAP} \) as maximizing the likelihood \( p_{X|Y}(\hat{x}|y) \). For this reason, we might call \( f_{\text{MAP}} \) a maximum-likelihood decision rule. In a series of papers on the coding problem, we have called \( f_{\text{MAP}} \) a maximum-likelihood decoding based on this idea.
Throughout this paper, we assume that $\mathcal{X}$ and $\mathcal{Y}$ are countable sets. It should be noted that the results do not change when $\mathcal{Y}$ is an uncountable set, where the summation should be replaced with the integral. The following lemma guarantees that the right hand side of (1) always exists for every $y \in \mathcal{Y}$. This fact, which is implicitly used in this paper, implies that it is enough to assume that $\mathcal{X}$ is a countable set.

**Lemma 1**: Let $q$ be a probability distribution on a countable set $\mathcal{X}$. Then the maximum of $q$ on $\mathcal{X}$ always exists, that is, there is $\hat{x} \in \mathcal{X}$ such that $q(\hat{x}) \geq q(x)$ for any $x \in \mathcal{X}$.

**Proof**: The lemma is trivial when $\mathcal{X}$ is a finite set. In the following, we assume that $\mathcal{X}$ is a countable infinite set.

Since $q(x) \leq 1$ for all $x \in \mathcal{X}$, then $\sup_x q(x)$ always exists, that is, $q(x') \leq \sup_x q(x)$ for all $x' \in \mathcal{X}$, and for any $q' < \sup_x q(x)$ there is a $x' \in \mathcal{X}$ such that $q(x) < q(x') \leq \sup_x q(x)$.

We prove the lemma by contradiction. Assume that there is no $\hat{x} \in \mathcal{X}$ such that $q(\hat{x}) = \sup_x q(x)$. Since $\sum_x q(x) = 1$ and $q(x) \geq 0$ for all $x \in \mathcal{X}$, there is $x_0 \in \mathcal{X}$ such that $q(x_0) > 0$. From the definition of $\sup_x q(x)$, there is a $x_1 \in \mathcal{X}$ such that $q(x_0) > \sup_x q(x)$, where the second inequality comes from the assumption. By repeating this argument, we have a sequence $\{x_i\}_{i=0}^{\infty}$ such that

$$0 < q(x_0) \leq q(x_1) \leq q(x_2) \leq \cdots \leq \sup_x q(x).$$

This implies $\sum_x q(x) \geq \sum_{i=0}^{\infty} q(x_i) = \infty$, which contradicts $\sum_x q(x) = 1$.

**Remark 1**: When $\mathcal{X}$ is a finite dimensional Euclidian space, we can make the same discussion by quantizing uniformly from $\mathcal{X}$ to a countable set, where the decision is interpreted as guessing $X$ with a finite precision. Then we can apply the results to parameter estimation problems.

Let $\chi$ be a support function defined as

$$\chi(S) = \begin{cases} 1 & \text{if a statement } S \text{ is true} \\ 0 & \text{if a statement } S \text{ is false}. \end{cases}$$

(4)

Then the error probability Error$(q_{\hat{X}|Y})$ of a (stochastic) decision rule $q_{\hat{X}|Y}$ is given as

$$\text{Error}(q_{\hat{X}|Y}) = \sum_y p_Y(y) \sum_x p_{X|Y}(x|y) \sum_{\hat{x}} q_{\hat{X}|Y}(\hat{x}|y)\chi(\hat{x} \neq x)$$

$$= \sum_y p_Y(y) \sum_x p_{X|Y}(x|y)[1 - q_{\hat{X}|Y}(x|y)].$$

(5)

In the last equality, $1 - q_{\hat{X}|Y}(x|y)$ corresponds to the error probability of the decision rule $q_{\hat{X}|Y}$ after the observation $y \in \mathcal{Y}$, and Error$(q_{\hat{X}|Y})$ corresponds to the average of this error probability. When $q_{\hat{X}|Y}$ is defined by using $f : \mathcal{Y} \to \mathcal{X}$ and [3], the decision error probability Error$(f)$ of a deterministic decision rule $f$ is given as

$$\text{Error}(f) \equiv \sum_y p_Y(y) \sum_x p_{X|Y}(x|y)\chi(f(y) \neq x)$$

$$= \sum_y p_Y(y)[1 - p_{X|Y}(f(y)|y)].$$

(6)

In fact, it is sufficient to repeat this argument $[1 + 1/q(x_0)]$ times so that $[1 + 1/q(x_0)] q(x_0) > 1$.

It should be noted that the right hand side of the first equality can be derived directly from (5) and the fact that $q_{\hat{X}|Y}(x|y) = \chi(f(y) = x)$

$$= 1 - \chi(f(y) \neq x).$$

(7)

That is, we have Error$(f) = \text{Error}(q_{\hat{X}|Y})$ when $f$ and $q_{\hat{X}|Y}$ satisfy [3].

**III. OPTIMALITY OF MAXIMUM A POSTERIORI DECISION**

In this section, we discuss the optimality of the maximum a posteriori decision. We introduce the following well-known lemma.

**Lemma 2**: Let $(X, Y)$ be a pair consisting of a state $X$ and an observation $Y$ and $p_{XY}$ be the joint distribution of $(X, Y)$. When we make a stochastic decision with a distribution $q_{\hat{X}|Y}$, an optimal decision rule minimizing the decision error probability satisfies $q_{\hat{X}|Y}(x|y) = 0$ for all $(x, y)$ such that $p_Y(y) > 0$ and $p_{X|Y}(x|y) < \max_x p_{X|Y}(x|y)$. In particular, the maximum a posteriori decision rule $f_{\text{MAP}}$ defined by [1] minimizes the decision error probability, where $q_{\hat{X}|Y}$ is defined by $f \equiv f_{\text{MAP}}$ and [3].

**Proof**: From the definition of Error$(q_{\hat{X}|Y})$, we have [3], which is on the top of the next page, where $p_{X|Y}(f_{\text{MAP}}(y)|y) - p_{X|Y}(x|y) \geq 0$ implies that Error$(q_{\hat{X}|Y})$ is minimized only when $q_{\hat{X}|Y}(x|y) = 0$ for all $(x, y)$ such that $p_Y(y) > 0$ and $p_{X|Y}(f_{\text{MAP}}(y)|y) > p_{X|Y}(x|y)$ (i.e., $x \not= f_{\text{MAP}}(y)$).

**IV. ERROR PROBABILITY OF STOCHASTIC DECISION WITH A POSTERIORI DISTRIBUTION**

In this section, we consider the case where $q_{\hat{X}|Y}(\hat{x}|y) = p_{X|Y}(\hat{x}|y)$ for all $(\hat{x}, y)$, that is, we make a stochastic decision with the conditional distribution $p_{X|Y}$ of a state $X$ for a given observation $Y$. It should be noted that $\hat{X}$ is independent of $X$ for a given $Y$, where the joint distribution $p_{XY\hat{X}}$ of $(X, Y, \hat{X})$ is given as

$$p_{XY\hat{X}}(x, y, \hat{x}) = p_{X|Y}(\hat{x}|y)p_{X|Y}(x|y)p_Y(y).$$

In the following, we call this type of decision a stochastic decision with the a posteriori distribution. It should be noted that it may be unnecessary to know (or compute) the distribution $p_{X|Y}$ to make this type of decision. To make this type of decision, it is sufficient that we have a random number generator subject to the distribution $p_{X|Y}(\cdot|y)$ with arbitrary input $y \in \mathcal{Y}$, where the generated random number is independent of $X$ for a given $y$.

We have the following theorem. In Section VII we apply this theorem to an analysis of stochastic decoders of coding problems.

**Theorem 1**: Let $(X, Y)$ be a pair consisting of a state $X$ and an observation $Y$ and $p_{XY}$ be the joint distribution of $(X, Y)$. When we make a stochastic decision with $p_{X|Y}$, the decision error probability of this rule is at most twice the decision error probability of the maximum a posteriori decision rule $f_{\text{MAP}}$. That is, we have

$$\text{Error}(p_{X|Y}) \leq 2\text{Error}(f_{\text{MAP}}).$$
\[ \text{Error}(q_{\hat{X}|Y}) = \sum_y p_Y(y) \sum_x p_{X|Y}(x|y)[1 - q_{\hat{X}|Y}(x|y)] \]
\[ = 1 - \sum_y p_Y(y) \sum_x p_{X|Y}(x|y)q_{\hat{X}|Y}(x|y) \]
\[ = 1 - \sum_y p_Y(y) \left[ \sum_{x \neq f_{\text{MAP}}(y)} p_{X|Y}(x|y)q_{\hat{X}|Y}(x|y) + p_{X|Y}(f_{\text{MAP}}(y)|y)q_{\hat{X}|Y}(f_{\text{MAP}}(y)|y) \right] \]
\[ = 1 - \sum_y p_Y(y) \left[ \sum_{x \neq f_{\text{MAP}}(y)} p_{X|Y}(x|y)q_{\hat{X}|Y}(x|y) + p_{X|Y}(f_{\text{MAP}}(y)|y) \left[ 1 - \sum_{x \neq f_{\text{MAP}}(y)} q_{\hat{X}|Y}(x|y) \right] \right] \]
\[ = 1 - \sum_y p_Y(y)p_{X|Y}(f_{\text{MAP}}(y)|y) + \sum_y p_Y(y) \sum_{x \neq f_{\text{MAP}}(y)} q_{\hat{X}|Y}(x|y) \left[ p_{X|Y}(f_{\text{MAP}}(y)|y) - p_{X|Y}(x|y) \right]. \tag{8} \]

**Proof:** We have
\[ \text{Error}(p_{X|Y}) \]
\[ = \sum_y p_Y(y) \sum_x p_{X|Y}(x|y)[1 - p_{X|Y}(x|y)] \]
\[ = \sum_y p_Y(y) \left[ 1 - \sum_x p_{X|Y}(x|y) \right] \]
\[ \leq \sum_y p_Y(y) \left[ 1 - p_{X|Y}(f_{\text{MAP}}(y)|y) \right] \]
\[ = \sum_y p_Y(y)[1 - p_{X|Y}(f_{\text{MAP}}(y)|y)][1 + p_{X|Y}(f_{\text{MAP}}(y)|y)] \]
\[ \leq 2 \sum_y p_Y(y)[1 - p_{X|Y}(f_{\text{MAP}}(y)|y)] \]
\[ = 2 \text{Error}(f_{\text{MAP}}), \tag{9} \]
where the second inequality comes from the fact that \( p_{X|Y}(f(x)|y) \leq 1 \) and the fourth equality comes from \((6)\).

Here, we introduce the following inequalities, which come from Lemma 2 and Theorem 1. In these inequalities, if either \( \text{Error}(f_{\text{MAP}}) \) or \( \text{Error}(p_{X|Y}) \) vanishes as the dimension (block length) of \( X \) goes to infinity, then the other one also vanishes.

**Corollary 2:**
\[ \text{Error}(f_{\text{MAP}}) \leq \text{Error}(p_{X|Y}) \leq 2 \text{Error}(f_{\text{MAP}}) \]
\[ \frac{1}{2} \text{Error}(p_{X|Y}) \leq \text{Error}(f_{\text{MAP}}) \leq \text{Error}(p_{X|Y}). \]

Here, we introduce another lemma that comes from Lemma 2 and Theorem 1.

**Corollary 3:** Let \((X, Y)\) be a pair consisting of a state \( X \) and an observation \( Y \) and \( p_{XY} \) be the joint distribution of \((X, Y)\). When we make a stochastic decision with \( p_{X|Y} \), the decision error probability of this rule is at most twice the decision error probability of any decision rule \( q_{\hat{X}|Y} \). That is, we have
\[ \text{Error}(p_{X|Y}) \leq 2 \text{Error}(q_{\hat{X}|Y}) \tag{10} \]
for any \( q_{\hat{X}|Y} \).

**Remark 2:** Let us consider a situation where \( q_{\hat{X}|Y} \) is unknown but \( \text{Error}(q_{\hat{X}|Y}) \) can be estimated empirically. Then the above corollary implies that the error probability of stochastic decision with the a posteriori distribution is upper bounded by \( 2 \text{Error}(q_{\hat{X}|Y}) \). For example, when we know empirically that a human being can guess \( X \) with small error probability, then the error probability of a stochastic decision with the a posteriori distribution is also small because it is at most twice the error probability of her/his decision rule.

**Remark 3:** Inequality \((10)\) is tight in the sense that there is a pair consisting of \( p_{X|Y} \) and \( q_{\hat{X}|Y} \) such that \((10)\) is satisfied with equality. In fact, by assuming that
\[ X = \{0, 1\} \]
\[ p_{X|Y}(0|y) = \frac{1}{2} \]
\[ p_{X|Y}(1|y) = 1 - p_{X|Y}(0|y) \]
\[ q_{\hat{X}|Y}(0|y) = \frac{p_{X|Y}(0|y)}{2p_{X|Y}(0|y) - 1} \]
\[ q_{\hat{X}|Y}(1|y) = 1 - q_{\hat{X}|Y}(0|y) \]
for all \( y \in Y \), we have
\[ \text{Error}(p_{X|Y}) = 2 \sum_y p_Y(y)p_{X|Y}(0|y)[1 - p_{X|Y}(0|y)] \]
and
\[ 2 \text{Error}(q_{\hat{X}|Y}) \]
\[ = 2 \sum_y p_Y(y)p_{X|Y}(0|y)[1 - q_{\hat{X}|Y}(0|y)] \]
\[ + 2 \sum_y p_Y(y)p_{X|Y}(1|y)[1 - q_{\hat{X}|Y}(1|y)] \]
\[ = 2 \sum_y p_Y(y)p_{X|Y}(0|y)[1 - q_{\hat{X}|Y}(0|y)] \]
\[ + 2 \sum_y p_Y(y)[1 - p_{X|Y}(0|y)]q_{\hat{X}|Y}(0|y) \]
\[ = 2 \sum_y p_Y(y)p_{X|Y}(0|y) \]
\[ + 2 \sum_y p_Y(y)[1 - 2p_{X|Y}(0|y)]q_{\hat{X}|Y}(0|y) \]
\[ \geq \frac{1}{2} \text{Error}(p_{X|Y}) \]
[3] For example, she/he can recognize handwritten digits with small error probability but we do not know her/his decision rule \( q_{\hat{X}|Y} \) explicitly.
\[= 2 \sum_y p(y)p_{X|Y}(0|y) - 2 \sum_y p(y)p_{X|Y}(0|y)^2\]
\[= 2 \sum_y p(y)p_{X|Y}(0|y)[1 - p_{X|Y}(0|y)]\]

from (3).

V. STOCHASTIC DECISION BY USING APPROXIMATED DISTRIBUTION

Let \(d(q, q')\) be the variational distance of two probability distributions \(q\) and \(q'\) on the same set as

\[d(q, q') = \frac{1}{2} \sum_x |q(x) - q'(x)|\]

(11)
\[= \max_{X \subset X} \{q(\hat{X}) - q'(\hat{X})\}\]

(12)

(see [3], Eq. (11.137))). We have the following lemma.

Lemma 3: Let \((X, Y)\) be a pair consisting of a state \(X\) and an observation \(Y\) and \(p_{XY}\) be the joint distribution of \((X, Y)\). When we make decisions with two stochastic decision rules \(q(\cdot|y)\) and \(q'(\cdot|y)\) for each \(y \in \mathcal{Y}\), we have

\[|\text{Error}(q) - \text{Error}(q')| \leq d(q \times p_{Y'}, q' \times p_{Y'})\]

where \(q \times p_{Y'}(x, y) \equiv q(x|y)p_{Y'}(y)\) and \(q' \times p_{Y'}(x, y) \equiv q'(x|y)p_{Y'}(y)\).

Proof: The lemma is obtained immediately from (13) by considering the decision error event measured by using the joint probability distributions \(q \times p_{Y'}\) and \(q' \times p_{Y'}\). Formally, we have (13), which appears on the top of the next page, where the second inequality comes from (12) and the last equality comes from (11) as

\[\sum_y p_Y(y)d(q(\cdot|y), q'(\cdot|y))\]
\[= \frac{1}{2} \sum_y p_Y(y)\sum_x |q(x|y) - q'(x|y)|\]
\[= \frac{1}{2} \sum_y \sum_x |q'(x|y)p_{Y'}(y) - q(x|y)p_{Y'}(y)|\]
\[= d(q \times p_{Y'}, q' \times p_{Y'})\].

(14)

Applying Lemma 3 by letting \(q \equiv q_{\mathcal{X}|Y}\) and \(q' \equiv p_{X|Y}\), we have the following theorem from Theorem 1.

Theorem 4: Let \((X, Y)\) be a pair consisting of a state \(X\) and an observation \(Y\) and \(p_{XY}\) be the joint distribution of \((X, Y)\). When we make a stochastic decision with \(q_{\mathcal{X}|Y}\), the decision error probability \(\text{Error}(q_{\mathcal{X}|Y})\) is bounded as

\[\text{Error}(q_{\mathcal{X}|Y}) \leq 2\text{Error}(f_{\text{MAP}}) + d(q_{\mathcal{X}|Y} \times p_Y, p_{X|Y} \times p_Y)\].

VI. STOCHASTIC DECISION WITH RANDOM SEQUENCE

In this section, we assume that a conditional probability \(p_{X|Y}\) is computable. We make a stochastic decision \(F(y)\) from a random sequence \(\mathcal{X}^T \equiv (\hat{X}_1, \ldots, \hat{X}_T)\) as

\[T(y) \equiv \arg \max_{t \in \{1, \ldots, T\}} p_{X|Y}(\hat{X}_t|y)\]

(15)
\[F(y) \equiv \hat{X}_{T(y)}\].

A typical example of a random sequence \(\mathcal{X}^T \equiv (\hat{X}_1, \ldots, \hat{X}_T)\) is that generated by a Markov Chain Monte Carlo method. Here, we assume that \(X\) and \(\hat{X}^T\) are conditionally independent for a given \(Y\), that is, the joint distribution \(p_{XYX^T}\) of \((X, Y, \hat{X}^T)\) is given as

\[p_{XYX^T}(x, y, \mathcal{X}^T) = q_{\mathcal{X}^T|Y}(\mathcal{X}^T|y)p_{X|Y}(x|y)p_Y(y)\],

(17)

where \(q_{\mathcal{X}^T|Y}(\cdot|y)\) is a joint probability distribution of \(\mathcal{X}^T\) for a given \(y \in \mathcal{Y}\). Then, we have a decision error probability \(\text{Error}(F)\) as follows;

\[\text{Error}(F) \equiv E_{Y \times \hat{X}^T}[\chi(F(Y) \neq X)]\]
\[= E_{Y \times \hat{X}^T}\left[1 - p_{X|Y}(\hat{X}_T(Y)|Y)\right]\]
\[= E_{Y \times \hat{X}^T}\left[1 - \max_{t \in \{1, \ldots, T\}} p_{X|Y}(\hat{X}_t|Y)\right].\]

(18)

We have the following theorem.

Theorem 5: Let \((X, Y)\) be a pair consisting of a state \(X\) and an observation \(Y\) and \(p_{XY}\) be the joint distribution of \((X, Y)\). When we make a stochastic decision \(F\) with a random sequence \(\mathcal{X}^T\) defined by (15)–(17), the decision error probability \(\text{Error}(F)\) defined by (18) satisfies

\[\text{Error}(F) \leq \min_{t \in \{1, \ldots, T\}} \text{Error}(q_{\mathcal{X}_t|Y})\],

(19)

where \(q_{\mathcal{X}_t|Y}\) is a conditional marginal distribution given as

\[q_{\mathcal{X}_t|Y}(\mathcal{X}_t|y) \equiv \sum_{(\mathcal{X}_t)_{t \in \{1, \ldots, T\} \setminus \{t\}}} q_{\mathcal{X}^T|Y}(\mathcal{X}^T|y).

Proof: From (18), we have

\[\text{Error}(F) = E_{Y \times \hat{X}^T}\left[1 - \max_{t \in \{1, \ldots, T\}} p_{X|Y}(\hat{X}_t|Y)\right]\]
\[\leq E_{Y \times \hat{X}^T}\left[1 - p_{X|Y}(\hat{X}_1|Y)\right]\]
\[= \sum_y p_Y(y)\sum_{t \in \{1, \ldots, T\}} q_{\mathcal{X}_t|Y}(\mathcal{X}_t|y)[1 - p_{X|Y}(\hat{X}_t|y)]\]
\[= \sum_y p_Y(y)\sum_{\mathcal{X}_t} q_{\mathcal{X}_t|Y}(\mathcal{X}_t|y)[1 - p_{X|Y}(\hat{X}_t|y)]\]
\[= \sum_y p_Y(y)\sum_{\mathcal{X}_t} p_{X|Y}(\hat{X}_t|y)[1 - q_{\mathcal{X}_t|Y}(\mathcal{X}_t|y)]\]
\[= \text{Error}(q_{\mathcal{X}_t|Y})\]

(20)

for any \(t \in \{1, \ldots, T\}\), where the fourth equality comes from the fact that \(p_{X|Y}\) and \(q_{\mathcal{X}_t|Y}\) are probability distributions. From this inequality, we have (19).

Applying Lemma 3 by letting \(q \equiv q_{\mathcal{X}_t|Y}\) and \(q' \equiv p_{X|Y}\), we have the following theorem from Theorem 1. This theorem implies that if \(q_{\mathcal{X}_t|Y} \times p_Y\) tends towards \(p_{X|Y} \times p_Y\) as \(t \to \infty\) (e.g. (19) in Appendix) then the upper bound of error probability \(\text{Error}(F)\) is close to at most twice the error probability \(\text{Error}(f_{\text{MAP}})\) of the maximum a posteriori decision.

Theorem 6: Let \((X, Y)\) be a pair consisting of a state \(X\) and an observation \(Y\) and \(p_{XY}\) be the joint distribution of \((X, Y)\). When we make a stochastic decision \(F\) with a random
\[
|\text{Error}(q) - \text{Error}(q')| = \left| \sum_y p_Y(y) \sum_x p_{X|Y}(x|y)|1 - q(x|y)| - \sum_y p_Y(y) \sum_x p_{X|Y}(x|y)|1 - q'(x|y)| \right|
\]
\[
\leq \left| \sum_y p_Y(y) \sum_x p_{X|Y}(x|y)q(x \setminus \{x\}|y) - \sum_y p_Y(y) \sum_x p_{X|Y}(x|y)q'(x \setminus \{x\}|y) \right|
\]
\[
\leq \sum_y p_Y(y) \sum_x p_{X|Y}(x|y)|q(x \setminus \{x\}|y) - q'(x \setminus \{x\}|y)|
\]
\[
\leq \sum_y p_Y(y) \sum_x p_{X|Y}(x|y)d(q(x|y), q'(x|y))
\]
\[
= \sum_y p_Y(y)d(q(x|y), q'(x|y))
\]
\[
d(q \times p_Y, q' \times p_Y)
\]  
\[
(13)
\]

sequence \(\hat{X}_1, \ldots, \hat{X}_T\) defined by \((15)\)–\((17)\), the decision error probability is bounded as
\[
\text{Error}(F) \leq 2\text{Error}(f_{\text{MAP}}) + \min_{t \in \{1, \ldots, T\}} d(\hat{q}_{\hat{X}_t|Y} \times p_Y \times p_V)\text{.}
\]

In the following, we assume that a random sequence \(\hat{X}_T\) is independent and identically distributed (i.i.d.) with a distribution \(q_{\hat{X}_T|Y}\) for a given \(Y\), and that, the conditional probability distribution \(q_{\hat{X}_T|Y}\) is given as
\[
q_{\hat{X}_T|Y}(\hat{x}_t|y) = \prod_{t=1}^{T} q_{\hat{X}_t|Y}(\hat{x}_t|y)\text{.}
\]  
(21)

Then we have the following theorem. From this theorem with a trivial inequality \(\text{Error}(f_{\text{MAP}}) \leq \text{Error}(F)\), we have the fact that \(\text{Error}(F)\) tends towards the error probability \(\text{Error}(f_{\text{MAP}})\) of the maximum a posteriori decision, where the difference \(\text{Error}(F) - \text{Error}(f_{\text{MAP}})\) is exponentially small as the length \(T\) of a sequence increases.

**Theorem 7:** Let \((X, Y)\) be a pair consisting of a state \(X\) and an observation \(Y\) and \(\rho_{XY}\) be the joint distribution of \((X, Y)\). When we make a stochastic decision \(F\) with an i.i.d. random sequence \(\hat{X}_T\) defined by \((15)\)–\((17)\) and \((21)\), the decision error probability \(\text{Error}(F)\) defined by \((18)\) satisfies
\[
\text{Error}(F) \leq \text{Error}(f_{\text{MAP}}) + \sum_y p_Y(y)|1 - q_{\hat{X}_T|Y}(f_{\text{MAP}}(y)|y)\|_T\text{,}
\]  
(22)

where \(f_{\text{MAP}}\) is given by \((1)\). In particular, when \(q_{\hat{X}_T|Y} = \rho_{XY}\), we have
\[
\text{Error}(F) \leq \text{Error}(f_{\text{MAP}}) + \sum_y p_Y(y)|1 - \rho_{XY}(f_{\text{MAP}}(y)|y)\|_T\text{.}
\]  
(23)

Proof: For a given \(y \in \mathcal{Y}\), let \(\hat{X}_T(y) \subset \hat{X}_T\) be defined as
\[
\hat{X}_T(y) = \left\{ \hat{x}_T ; \rho_{XY}(\hat{x}_t|y) = \max_{x} \rho_{XY}(x|y) \right\}
\]  
Then \((22)\) is shown as \((25)\), which appears on the top of the next page, where the inequality comes from the fact that \(\hat{x}_t \notin \hat{X}_T\) implies \(\rho_{XY}(\hat{x}_t|y) < \max_{x} \rho_{XY}(x|y)\) and \(\hat{x}_t \notin f_{\text{MAP}}(y)\) for all \(t \in \{1, \ldots, T\}\). Inequality \((25)\) is obtained from \((28)\) by letting \(q_{\hat{X}_T|Y} = \rho_{XY}\). Inequalities \((24)\) and \((27)\) are shown by the fact that
\[
\sum_y p_Y(y) \left[ 1 - \rho_{XY}(f_{\text{MAP}}(y)|y) \right]^T
\]
\[
\leq \sum_y p_Y(y) \sup_{y, \rho_{XY}(y)>0} \left[ 1 - \rho_{XY}(f_{\text{MAP}}(y)|y) \right]^T
\]
\[
= \sup_{y, \rho_{XY}(y)>0} \left[ 1 - \inf_{x} \max_{x} \rho_{XY}(x|y) \right]^T\text{.}
\]  
(26)

**Remark 4:** When \(|\mathcal{X}|\) is finite, we have
\[
\text{Error}(F) \leq \text{Error}(f_{\text{MAP}}) + \left[ 1 - \inf_{y, \rho_{XY}(y)>0} \max_{x} \rho_{XY}(x|y) \right]^T\text{.}
\]  
(27)

from \((24)\), where the last inequality comes from the fact that
\[
|\mathcal{X}| \max_{x} \rho_{XY}(x|y) \geq \sum_{x} \rho_{XY}(x|y) = 1
\]
for all \(y \in \mathcal{Y}\) implies
\[
\inf_{y, \rho_{XY}(y)>0} \max_{x} \rho_{XY}(x|y) \geq \frac{1}{|\mathcal{X}|}.
\]

We have the same bound \((27)\) from \((22)\), when \(q_{\hat{X}_T|Y} = \rho_{XY}\) is the uniform distribution on \(\mathcal{X}\) for every \(y \in \mathcal{Y}\). This implies that the stochastic decision with an i.i.d. sequence subject to the a posteriori distribution is at least as good as that subject to the uniform distribution. It should be noted that, when \(|\mathcal{X}|\) increases exponentially as the dimension of \(\mathcal{X}\) increases, \(T\) should also increase exponentially to ensure that the second term \([1 - 1/|\mathcal{X}|]^T\) tends towards zero.
Error($F$)

\[
= E_{Y | X} \left[ 1 - \max_{t \in \{1, \ldots, T\}} p_{X|Y}(\bar{x}_t | Y) \right]
\]

\[
= \sum_{y} p_Y(y) \sum_{x \in \mathcal{X}^T} \left[ 1 - \max_{t \in \{1, \ldots, T\}} p_{X|Y}(\bar{x}_t | y) \right] \prod_{t=1}^{T} q_{\bar{x}_t | Y}(\bar{x}_t | y) + \sum_{y} p_Y(y) \sum_{x \in \mathcal{X}^T} \left[ 1 - \max_{t \in \{1, \ldots, T\}} p_{X|Y}(\bar{x}_t | y) \right] \prod_{t=1}^{T} q_{\bar{x}_t | Y}(\bar{x}_t | y)
\]

\[
\leq \sum_{y} p_Y(y) \sum_{x \in \mathcal{X}^T} \left[ 1 - \max_{\bar{x}} p_{X|Y}(\bar{x} | y) \right] \prod_{t=1}^{T} q_{\bar{x}_t | Y}(\bar{x}_t | y) + \sum_{y} p_Y(y) \sum_{x \in \mathcal{X}^T} \sum_{\bar{x} \neq f_{MAP}(y)} \prod_{t=1}^{T} q_{\bar{x}_t | Y}(\bar{x}_t | y)
\]

\[
= \sum_{y} p_Y(y) \left[ 1 - p_{X|Y}(f_{MAP}(y) | y) \right] + \sum_{y} p_Y(y) \prod_{t=1}^{T} \sum_{\bar{x} \neq f_{MAP}(y)} q_{\bar{x}_t | Y}(\bar{x}_t | y)
\]

\[
= \text{Error}(f_{MAP}) + \sum_{y} p_Y(y) \left[ 1 - q_{X | Y}(f_{MAP}(y) | y) \right]^{T}
\]

(25)

By letting $T = 1$ in (22), we have the following corollary. It should be noted that this corollary includes Theorem 1 as the case $T = 1$ and $q_{X | Y} = p_{X | Y}$.

**Corollary 8:** Let $(X, Y)$ be a pair consisting of a state $X$ and an observation $Y$ and $p_{XY}$ be the joint distribution of $(X, Y)$. When we make a stochastic decision with a distribution $q_{X | Y}$, the decision error probability $\text{Error}(q_{X | Y})$ defined by (3) satisfies

\[
\text{Error}(q_{X | Y}) \leq \text{Error}(f_{MAP}) + \sum_{y} p_Y(y) [1 - q_{X | Y}(f_{MAP}(y) | y)],
\]

where $f_{MAP}$ is given by (1).

**VII. CONSTRUCTION OF STOCHASTIC DECODERS**

This section introduces applications of the stochastic decision with the a posteriori distribution to some coding problems.

For simplicity, we assume that we can obtain an ideal random number subject to a given distribution. For a given function $A$ on $X^n$, let $\text{Im} A \equiv \{ A \bar{x} : \bar{x} \in X^n \}$. For a given $c \in \text{Im} A$, let $C_A(c) \equiv \{ x : A \bar{x} = c \}$.

**A. Fixed-length Lossless Compression**

This section introduces a stochastic decoder for a fixed-length lossless compression with an arbitrary small decoding error probability.

Let $\mu_X$ be the probability distribution of $X^n$ and $A : X^n \to \text{Im} A$ be an encoding map, where $c \equiv A \bar{x}$ is the codeword of $x \in X^n$. Then the joint distribution $p_{XC}$ of a source $X \in X^n$ and a codeword $C \in \text{Im} A$ are given as

\[
p_{XC}(x, c) = \mu_X(x) \chi(A \bar{x} = c).
\]

The decoder receives a codeword $c$. By using a stochastic decoder with the distribution

\[
p_{X|C}(x | c) = \frac{\mu_X(x) \chi(A \bar{x} = c)}{\sum_{x} \mu_X(x) \chi(A \bar{x} = c)} = \frac{\mu_X(x) \chi(A \bar{x} = c)}{\mu_X(C_A(c))},
\]

we have the bound of error probability from Theorem 1. This implies that, when we use the encoding map $A$ such that the decoding error probability by using the maximum a posteriori decoder vanishes as $n \to \infty$, the decoding error probability by using the stochastic decoder with $p_{XC}$ also vanishes as $n \to \infty$. In addition, for a special case of source coding with no side information at the decoder, the fundamental limit $\Pi(X)$ is achievable with this code, where $\Pi(X)$ is the spectrum entropy rate of $X$ (see [4]). It should be noted that the right hand side of the above equality is the output distribution of the constrained-random-number generator introduced in [7].

For a given linear code for an additive noise channel, we can use the constrained-random-number generator as the stochastic decoder by letting $X$ be a channel noise, $A$ be a parity check matrix, and $C$ be the syndrome of $X$. A channel encoder encodes a message to a channel input $z \in \{ z : A \bar{z} = 0 \}$. A channel decoder receives a channel output $y$, reproduces a channel noise $\bar{x} = y - z$ from the syndrome $c \equiv A \bar{y} = A(x + z)$ by using the above scheme, and reproduces a message corresponding to $z = y - \bar{x}$, where the decoding is successful when $\bar{x} = y - z$ is reproduced correctly from $c \equiv A \bar{x}$. From the above discussion, the decoding error probability is at most twice the decoding error probability of the maximal a posteriori decoder.

**B. Fixed-length Lossless Compression with Side Information at Decoder**

This section introduces a stochastic decoder for the fixed-length lossless compression of a source $X$ with an arbitrary small decoding error probability, where the decoder has access to the side information $Y$ correlated with $X$.

Let $\mu_{XY}$ be the joint distribution of $(X, Y)$ and $A : X^n \to \text{Im} A$ be an encoding map, where $c \equiv A \bar{x}$ is the codeword of $x \in X^n$. Then the joint distribution $p_{XYC}$ of a source $X \in X^n$, side information source $Y \in Y^n$, and codeword $C \in \text{Im} A$ is given as

\[
p_{XYC}(x, y, c) = \mu_{XY}(x, y) \chi(A \bar{x} = c).
\]

(30)
The decoder receives a codeword \( c \) and side information \( y \). By using a stochastic decoder with the distribution

\[
p_{X|YC}(x|y,c) = \frac{\mu_{XY}(x,y)\chi(Ax=c)}{\sum_x \mu_{XY}(x,y)\chi(Ax=c)} = \frac{\mu_{X|Y}(x|y)\chi(Ax=c)}{\sum_x \mu_{X|Y}(x|y)\chi(Ax=c)} = \frac{\mu_{X|Y}(x|y)\chi(Ax=c)}{\mu_{X|YC}(A|c|y)}, \tag{31}
\]

we obtain the bound of error probability from Theorem 1. This implies that, when we use the encoding map \( A \) such that the error probability of the maximum a posteriori decoder vanishes as \( n \to \infty \), the error probability of the stochastic decoder with \( p_{X|YC} \) also vanishes as \( n \to \infty \). In addition, the fundamental limit \( \overline{H}(X|Y) \) is achievable with this code [9], where \( \overline{H}(X|Y) \) is the spectrum sup-entropy rate of \( X \) (see Theorems 4 and 5). It should be noted that the right hand side of (31) is the output distribution of the constrained-random-number generator introduced in [7].

C. Channel Coding

This section introduces a stochastic decoder for the channel code introduced in [9].

Let \( X \in \mathcal{X}^n \) and \( Y \in \mathcal{Y}^n \) be random variables corresponding to a channel input and a channel output, respectively. Let \( \mu_{XY} \) be the conditional probability of the channel and \( \mu_X \) be the distribution of the channel input. We consider correlated sources \((X,Y)\) with the joint distribution \( \mu_{XY} \times \mu_X \). Let \( A : \mathcal{X}^n \to \text{Im} A \) be the source code with the decoder side information introduced in the previous section. Let \( p_{XYC} \) be the joint distribution defined by (30). The decoder of this source code obtains the reproduction by using the stochastic decoding with the distribution defined by (31). Let \( \text{Error}(A) \) be the error probability of this source code. We can assume that for all \( r > \overline{H}(X|Y) \), \( \delta > 0 \) and all sufficiently large \( n \) there is a function \( A \) such that

\[
\text{Error}(A) \leq \delta, \tag{32}
\]

where a maximum a posteriori decoder is not assumed for this code.

When constructing a channel code, we prepare a map \( B : \mathcal{X}^n \to \mathcal{M}_n \) and a vector \( c \in \text{Im} A \). We use the stochastic encoder with the distribution

\[
p_{XICM}(x|c,m) = \begin{cases} 
\frac{\mu_X(x)\chi(Ax=c)\chi(Bx=m)}{\mu_X(C_A(c)\cap C_B(m))} & \text{if } \mu_X(C_A(c)\cap C_B(m)) > 0, \\
\text{encoding error} & \text{if } \mu_X(C_A(c)\cap C_B(m)) = 0
\end{cases} \tag{33}
\]

for a message \( m \in \mathcal{M}_n \) generated subject to the uniform distribution on \( \mathcal{M}_n \). It should be noted that the right hand side of the above equality is the output distribution of the constrained-random-number generator introduced in [7]. The decoder reproduces \( x \in \mathcal{X}^n \) satisfying \( Ax = c \) by using the stochastic decoder with the distribution given by (31) and reproduces a message \( By \) by operating \( B \) on \( x \).

In the above channel coding, let us assume that \( r + R < \overline{H}(X) \),

\[
r = \frac{1}{n} \log |\text{Im} A|,
\]

\[
R = \frac{1}{n} \log |\text{Im} B|,
\]

and the balanced-coloring property [8] of an ensemble \((B, p_B)\), where \( \overline{H}(X) \) is the spectrum inf-entropy rate of \( X \) and \( M_n \equiv \text{Im} B \equiv \cup_{B \in \mathcal{B}} \text{Im} B \). Then, from [9] Theorem 1 and (32), we have the fact that for all \( \delta > 0 \) and all sufficiently large \( n \) there are \( B \in \mathcal{B} \) and \( c \in \text{Im} A \) such that the error probability \( \text{Error}(B,c) \) of this channel code is bounded as

\[
\text{Error}(B,c) \leq \text{Error}(A) + \delta \leq 2\delta. \tag{34}
\]

It should be noted that the channel capacity

\[
\sup_{X} [\overline{H}(X) - \overline{H}(X|Y)],
\]

which is derived in [7] Lemma 1, is achievable by letting \( n \to \infty \), \( \delta \to 0 \), \( r \to \overline{H}(X|Y) \), \( R \to \overline{H}(X) - \overline{H}(X|Y) \), and \( X \) be the general source that attains the supremum.

D. Comments on Stochastic Decoding with Random Sequence

Here, we comment on the decoding with a random sequence introduced in Section VI. For decoding, we can use random sequences generated by Markov chains (random walks) that converge to the respective stationary distributions [29] and [31]. In this case, we can apply Theorem 6 to guarantee that the decoding error probability is bounded by twice the error probability of the maximum a posteriori decoding as the sequence becomes longer. We can also use random sequences by repeating stochastic decisions independently with the respective distributions [29] and [31], where we can use independent Markov chains with independent initial states to generate an i.i.d. random sequence. In this case, we can use Theorem 7 to guarantee that the decoding error probability tends to the error probability of the maximum a posteriori decoding as the sequence becomes longer. When implementing [15] for [29] and [31], it is sufficient to calculate the value of the numerator on the right hand side of these qualities because the denominator does not depend on \( x \). The numerator value is easy to calculate when the base probability distribution is memoryless.

VIII. CONCLUDING REMARKS

This paper investigated stochastic decision and stochastic decoding problems. It is shown that the error probability of a stochastic decision with the a posteriori distribution is at most twice the error probability of a maximum a posteriori decision. A stochastic decision with the a posteriori distribution may be sub-optimal but acceptable when the error probability of another decision rule (e.g. the maximum a posteriori decision rule) is small. Furthermore, by generating an i.i.d. random sequence subject to the a posteriori distribution and making a decision that maximizes the a posteriori probability over the sequence, the error probability approaches exponentially the
error probability of the maximum a posteriori decision as the sequence becomes longer.

When it is difficult to make the maximum a posteriori decision but the error probability of the decision is small, we may use the stochastic decision rule with the a posteriori distribution as an alternative. In particular, when the error probability of the maximum a posteriori decoding of source/channel coding tends towards zero as the block length goes to infinity, the error probability of the stochastic decoding with the a posteriori distribution also tends towards zero. The stochastic decoder with the a posteriori distribution can be considered to be the constrained-random-number generator \cite{7}. \cite{8} implemented by using the Sum-Product algorithm or the Markov Chain Monte Carlo method (see Appendix). However, the trade-off between the computational complexity and the precision of these algorithms is unclear. It remains a challenge for the future.

**APPENDIX**

In this section, we introduce the algorithms for the constrained-random-number generator \cite{7}, which generates random numbers subject to a distribution

\[
p_{X_{(1)}|Y_{(1)}}(x_{(1)}, c) = \frac{p_{X_{(1)}|Y_{(1)}=c}}{\sum_{x_{(1)}} p_{X_{(1)}|Y_{(1)}}(x_{(1)}|c)} = (35)
\]

for a given matrix $A$ and vectors $c \in \text{Im} A \equiv \{A x : x \in X^n\}$, $y \in Y^n$, where $p_{X_{(1)}|Y_{(1)}}$ is assumed to be memoryless, that is, there is \( \{p_{X_{(i)}|Y_{(i)}}\}_{i=1}^{n} \) such that

\[
p_{X_{(1)}|Y_{(1)}}(x_{(1)}|c) = \prod_{j=1}^{n} p_{X_{(1)}|Y_{(1)}}(x_{(1)}_{(j)}|y_{(j)})
\]

for all $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. We review the sum-product algorithm and the Markov Chain Monte Carlo method, respectively.

In the following algorithms, the symbol \( \leftarrow \) denotes the substitution.

**A. Constrained-Random-Number Generator Using Sum-Product Algorithm**

This section reviews an algorithm that uses the sum-product algorithm.

Let \( \{J_i\}_{i=1}^{n} \) be a family of subsets of \( \{1, \ldots, n\} \). For a set of local functions \( \{f_i : X^{\mid J_i \mid} \to \{0, 1\}\}_{i=1}^{n} \), the sum-product algorithm \cite{11} \cite{6} calculates a real-valued global function \( g \) on \( X \) defined as

\[
g(x_j) \equiv \frac{\sum_{x \setminus \{x_j\}} \Pi f_i(x_{J_i})}{\sum_{x} \Pi f_i(x_{J_i})}
\]

approximately, where the summation \( \sum_{x \setminus \{x_j\}} \) is taken over all $x \in X^n$ except for the variable $x_j$, and the function $f_i$ depends only on the set of variables $x_{J_i} \equiv \{x_j\}_{j \in J_i}$. It should be noted that the algorithm calculates the global function exactly when the corresponding factor graph has no loop. Let \( \sigma_{x_j \rightarrow f_i(x_j)} \) and \( \sigma_{f_i \rightarrow x_j(x_j)} \) be messages calculated as

\[
\begin{align*}
\sigma_{x_j \rightarrow f_i(x_j)} & \leftarrow \prod_{i' \in \{1, \ldots, n\} \setminus \{i\}; j \in J_i'} \sigma_{f_{i'} \rightarrow x_j(x_j)} \\
\sigma_{f_i \rightarrow x_j(x_j)} & \leftarrow \frac{\sum_{x \setminus \{x_j\}} f_i(x_{J_i}) \Pi_{j' \in J_i \setminus \{i\}} \sigma_{x_j \rightarrow f_i(x_j)}}{\sum_{x} \Pi_{j' \in J_i \setminus \{i\}} \sigma_{x_j \rightarrow f_i(x_j)}}.
\end{align*}
\]

The summation \( \sum_{J_i} \) is taken over all \( \{x_j\}_{j \in J_i} \) satisfying \( j \in J_i \) and \( \sigma_{f_i \rightarrow x_j(x_j)} \equiv f_i(x_j)/\sum_{x} f_i(x_j) \) when \( J_i = \{j\} \). The sum-product algorithm is performed by iterating the above operations for every message \( \sigma_{f_i \rightarrow x_j(x_j)} \) and \( \sigma_{x_j \rightarrow f_i(x_j)} \) in \( J_i \) and finally calculating the approximation of the global function as

\[
g(x_j) \approx \prod_{i \in \{1, \ldots, n\}; j \in J_i} \sigma_{f_i \rightarrow x_j(x_j)},
\]

where we assign initial values to \( \sigma_{x_j \rightarrow f_i(x_j)} \) and \( \sigma_{f_i \rightarrow x_j(x_j)} \) when they appear on the right hand side of the above operations and are undefined.

In the following, we describe an algorithm for a constrained-random-number generator. Let \( A \equiv (a_{i,j}) \) be an \( l \times n \) (sparse) matrix with a maximum row weight \( w \). Then the set \( J_i \equiv \{ j \in \{1, \ldots, n\} : a_{i,j} \neq 0 \} \) satisfies \( \mid J_i \mid \leq w \) for all \( i \in \{1, \ldots, l\} \). Then there is a set \( \{a_i : X^{\mid J_i \mid} \rightarrow \{0, 1\}\}_{i=1}^{l} \) of vectors such that

\[
Ax = (a_1 \cdot x_{J_1}, a_2 \cdot x_{J_2}, \ldots, a_l \cdot x_{J_l}),
\]

where \( a_i \) is a \( \mid J_i \mid \)-dimensional vector and \( a_i \cdot x_{J_i} \) denotes the inner product of vectors \( a_i \) and \( x_{J_i} \). Let \( x_i^l \equiv (x_1, \ldots, x_j) \), with \( x_i^l \) is a null string if \( i > j \). Let \( c \equiv (c_1, \ldots, c_l) \in \mathcal{X}^l. \)

In the following, we assume that \( A \) is a full-rank matrix. It should be noted that, when a matrix \( A \) is not full-rank, the elimination of redundant rows can be used to obtain an equivalent condition represented by a full-rank matrix.

**Constrained-Random-Number Generation Algorithm Using Sum-Product Algorithm:***

**Step 1** Let \( k \leftarrow 1 \).

**Step 2** Calculate the conditional probability distribution \( p_{X_k | X_{k-1} Y_{k-1} C} \) defined as

\[
p_{X_k | X_{k-1} Y_{k-1} C} = \sum_{x_k} \prod_{j=k}^{n} p_{X_{(1)}|Y_{(1)}}(x_{(1)}|y_{(j)}) \prod_{i=1}^{l} \chi(a_i \cdot x_{J_i} = c_i)
\]

\[
\equiv \sum_{x_k} \prod_{j=k}^{n} p_{X_{(1)}|Y_{(1)}}(x_{(1)}|y_{(j)}) \prod_{i=1}^{l} \chi(a_i \cdot x_{J_i} = c_i)
\]

\[
(36)
\]

where \( \chi(\cdot) \) is a support function defined by \cite{4}. It should be noted that the sum-product algorithm can be employed to obtain \( \chi(\cdot) \), where \( \{p_{X_{(1)}|Y_{(1)}}\}_{j=k}^{n} \) and \( \{\chi(a_i \cdot x_{J_i} = c_i)\}_{i=1}^{l} \) are local functions, and we substitute the generated sequence \( x_k^{k-1} \) for \( x_k \). If \( \chi(a_i \cdot x_{J_i} = c_i) \) is a constant after the substitution of \( x_k^{k-1} \), we can record the constant in preparation for the future.
Step 3 Generate and record a random number \( x_k \) corresponding to the distribution \( p_{X_k|Y_k} x_k^{k-1} Y_k c_k \).

Step 4 If \( k = n \), output \( x \equiv x_1^n \) and terminate.

Step 5 If \( k = l \), obtain the unique vector \( x_1^{n+1} \) such that \( x \equiv x_1^n \in \{ x : Ax = c \} \), output \( x \), and terminate.

Step 6 Let \( k \leftarrow k + 1 \) and go to Step 2.

We have the following lemma.

**Lemma 4 (12 Theorem 5):** Assume that (36) is computed exactly. Then the proposed algorithm generates \( x \equiv x_1^n \) subject to the probability distribution given by (35).

**B. Constrained Random Number Generator Using Markov Chain Monte Carlo Method**

This section reviews an algorithm that employs the Gibbs sampling [3], which is a kind of Markov Chain Monte Carlo method. In the following algorithm, it is assumed that \( A \) is a systematic matrix illustrated as

\[
A = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
1 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix},
\]

where the left part \( A \equiv (a_{i,j}) \) is an \( l \times [n - l] \) matrix and the right part of \( A \) is the \( l \times l \) identity matrix. It should be noted that, when a matrix \( A \) is not systematic, the elementary transformation and the elimination of redundant rows can be used to obtain an equivalent condition represented by a systematic matrix \( A \). Let \( a_{i,j} \) be the \( j \)-th column of \( A \) and let \( I_j \equiv \{ i : a_{i-n+l,j} \neq 0 \} \subset \{ n - l + 1, \ldots, n \} \). Let \( k \) be the number of iterations.

In the following, we describe an algorithm for a constrained-random-number generator. It should be noted that Steps 2, 5, and 7 realize the stochastic decision defined by (15) and (16), which may be skipped by outputting \( x \) instead of \( x_{\text{max}} \) in Step 9.

**Constrained-Random-Number Generation Algorithm Using Markov Chain Monte Carlo Method:**

**Step 1** Let \( x \) be an arbitrary initial sequence satisfying \( Ax = c \). For example, we can let

\[
x_j \leftarrow \arg \max_{\hat{x}_j : p_{X_j|Y_j}(\hat{x}_j|x_j)} p_{X_j|Y_j}(\hat{x}_j|y_j)
\]

for \( j \in \{ 1, \ldots, n - l \} \) and

\[
x_i \leftarrow a_{i-n+l} - \sum_{j=1}^{n-l} a_{i-n+l,j} x_j
\]

for \( i \in \{ n - l + 1, \ldots, n \} \).

**Step 2** Let

\[
\Lambda \leftarrow \sum_{j=1}^{n} \log p_{X_j|Y_j}(x_j|y_j)
\]

\[
\Lambda_{\text{max}} \leftarrow \Lambda
\]

It should be noted that the converted systematic matrix may not be sparse even when the original matrix is sparse.

**Step 3** Let \( k \leftarrow 1 \).

**Step 4** Choose \( j \in \{ 1, \ldots, n - l \} \) uniformly at random.

**Step 5** Let

\[
\Lambda \leftarrow \Lambda - \sum_{i \in (j) \cup I_j} \log p_{X_i|Y_i}(x_i|y_i).
\]

**Step 6** Calculate the probability distribution \( \nu \) defined as

\[
\nu(x'_j) = \frac{p_{X_j|Y_j}(x'_j|y_j) \prod_{i \in I_j} p_{X_i|Y_i}(v_{i,j}(x'_j)|y_i)}{\sum_{x_j} p_{X_j|Y_j}(x_j|y_j) \prod_{i \in I_j} p_{X_i|Y_i}(v_{i,j}(x_j)|y_i)},
\]

where

\[
v_{i,j}(x_j) = x_i + a_{i-n+l,j}[x_j - x]
\]

for \( i \in I_j \).

**Step 7** Generate \( x'_j \) subject to the probability distribution \( \nu \) and let

\[
x_i \leftarrow \begin{cases} 
x'_j & \text{if } i = j \\
v_{i,j}(x'_j) & \text{if } i \in I_j \\
x_i & \text{otherwise.}
\end{cases}
\]

It should be noted that the renewed sequence \( x \) satisfies \( Ax = c \).

**Step 8** Let

\[
\Lambda \leftarrow \Lambda + \sum_{i \in (j) \cup I_j} \log p_{X_i|Y_i}(x_i|y_i).
\]

If \( \Lambda < \Lambda_{\text{max}} \), let \( \Lambda_{\text{max}} \leftarrow \Lambda \) and \( x_{\text{max}} \leftarrow x \).

**Step 9** If \( k = \kappa \), output \( x_{\text{max}} \) and terminate. Otherwise, let \( k \leftarrow k + 1 \) and go to Step 3.

We have the following lemma.

**Lemma 5 (8 Theorem 5):** For given \( (y, c) \), let \( p_k(x|y, c) \) be the probability of \( x \) at Step 7 in the above algorithm, where we can take an arbitrary initial sequence satisfying \( Ax = c \) at Step 1. Then

\[
\lim_{k \to \infty} d(P_k([y, c], p_XY_{\mathcal{C}}([y, c])) = 0.
\]

for all \( (y, c) \).

From the above lemma and (4), we have

\[
\lim_{k \to \infty} d(P_k \times p_Y \times p_XY_{\mathcal{C}} \times p_Y \times p_Y \times p_Y \times p_Y)
\]

\[
= \lim_{k \to \infty} \sum_{y, c} p_Y \times p_XY_{\mathcal{C}}([y, c]) d(P_k([y, c], p_XY_{\mathcal{C}}([y, c]))
\]

\[
= \sum_{y, c} p_Y \times p_XY_{\mathcal{C}}([y, c]) \lim_{k \to \infty} d(P_k([y, c], p_XY_{\mathcal{C}}([y, c]))
\]

\[
= 0.
\]

for any \( p_Y \times p_Y \times p_Y \times p_Y \).
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