3-Kenmotsu Manifolds
Hassan Attarchi*

(Submitted by M. A. Malakhaltsev)
School of Mathematics, Georgia Institute of Technology, Atlanta, USA
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Abstract—In this paper, a 3-Kenmotsu structure is defined on a 4n + 1 dimensional manifold where such structure seems to be never studied before.
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1. INTRODUCTION

A (2n + 1)-dimensional smooth manifold M is said to have an almost contact structure if the structural group of its tangent bundle reduces to $U(n) \times 1$, where $U(n)$ is the unitary group of degree n [14]. Equivalently, an almost contact structure is given by a triple $(\phi, \xi, \eta)$ satisfying certain conditions [12–14]. It is well-known that the almost contact metric structure on an odd-dimensional manifold is analogous to an almost Hermitian structure on an even-dimensional manifold. For example, in [16], it is shown that a hypersurface in an almost Hermitian manifold has an almost contact metric structure. Moreover, a Kenmotsu manifold is a locally warped product of an interval of the real line and a Kahler manifold with a special warped function [7]. Also, in [3], it is proven that the warped product of a Sasakian manifold with the real line under a certain conformal change will result in a Kahlerian structure.

However, the situation is completely different when someone wants to construct an odd-dimensional structure analogous to the quaternion spaces. There are two choices, $(4n + 1)$ and $(4n - 1)$-dimensional spaces. In literature, what is known as an almost contact 3-structure is based on a $(4n - 1)$-dimensional space or equivalently a $(4n + 3)$-dimensional space [8]. In this case, the structural group of almost contact 3-structure reduces to $Sp(n) \times I_3$ [5, 8].

On the other hand, what makes this difference more deepening is about constructing an analogous structure of Kenmotsu manifolds on $(4n + 3)$-dimensional spaces. The $(4n + 3)$-dimensional manifolds as underlying manifolds of almost contact 3-structures lead us to have only almost contact 3-structure or 3-Sasakian manifold [1, 6]. This terminology is completely in agreement with the result of [3] which was mentioned earlier. This means one can expect that the warped product of a 3-Sasakian manifold with the real line under a certain conformal change will result in a quaternion Kahler manifold. While it cannot support the strategy behind constructing the Kenmotsu manifolds [7]. To have a structure which is locally the warped product of an interval and a quaternion Kahler manifold, one should study a new structure on $(4n + 1)$-dimensional manifolds where its structural group reduces to $Sp(n) \times I$. To the best of my knowledge, nobody has considered and studied $(4n + 1)$-dimensional manifolds from this perspective. To name one of the most recent works on $(4n + 1)$-dimensional manifolds, one can see [2].

The structure of the paper is the following. In Section 2, we introduce necessary notations and study some properties of Kenmotsu manifolds. Section 3 deals with the definition of a 3-Kenmotsu manifold as an odd-dimensional analogous structure of the quaternion Kahler manifolds. In Section 4, it is proved that 3-Kenmotsu manifolds are Einstein spaces. Moreover, it is shown that the $\varphi_\alpha$-holomorphic sectional curvature $H_\alpha(X)$ of this structure satisfies the equation $\sum_{\alpha=1}^{3} H_\alpha(X) = -3$ for all $X \in \Gamma H$ where $H$ is the contact distribution of Kenmotsu structures. In Section 5, an example of 3-Kenmotsu manifolds is studied.

*E-mail: hattarchi@gatech.edu
2. PRELIMINARIES AND NOTATIONS

Let \((\varphi, \eta, \xi, g)\) be an almost contact metric structure on \((2n + 1)\)-dimensional manifold \(M\), where \(\varphi \in \text{End}(TM)\), \(\xi\) is the Reeb vector field, and \(\eta\) is its dual 1-form with respect to the Riemannian metric \(g\). Also, they satisfy following properties:

\[ \varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \eta(\varphi) = 0, \quad \varphi = 0, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \] \(1\)

where \(X, Y \in \Gamma TM\) [1]. An almost contact metric manifold is called a Kenmotsu manifold if

\[ (\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \] \(2\)

where \(\nabla\) is the Levi-Civita connection of the Riemannian metric \(g\) and \(X, Y \in \Gamma TM\). This implies that

\[ \nabla_X \xi = X - \eta(X)\xi, \quad (\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \] \(3\)

where \(X, Y \in \Gamma TM\) [7]. Moreover, the 1-form \(\eta\) of almost contact structure of a Kenmotsu manifold is closed (i.e. \(d\eta = 0\)). The Kahler form \(\Omega\) is defined on an almost contact metric manifold as follows:

\[ \Omega(X, Y) := g(X, \varphi Y), \] \(4\)

where \(X, Y \in \Gamma TM\). Let \((M, \varphi, \eta, \xi, g)\) be a Kenmotsu manifold of dimension \(2n + 1\). The Kahler form \(\Omega\) of \(M\) satisfies the equation \(d\eta = \eta \wedge \Omega\) [11]. Consider the foliation \(F\) of the Reeb vector field \(\xi\) on \(M\). Then, there are local frames \(\{U; x^0, x^i\}\) adapted to this foliation where \(\xi = \partial/\partial x^0\) on \(U\) [10]. The local vector fields of this coordinate system define local frames \(\{\partial/\partial x^0, \partial/\partial x^i\}\) on \(U\), where

\[ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \eta_i \frac{\partial}{\partial x^0}, \] \(5\)

\(\eta_i := \eta(\partial/\partial x^i)\) for \(i = 1, ..., 2n\). The Riemannian metric \(g\) in the local frames (5) has the following format:

\[ g := \begin{pmatrix} 1 & 0 \\ 0 & \text{diag}(g_{ij}) \end{pmatrix}, \] \(6\)

where \(g_{ij} = g(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})\). It is easy to check that the Lie bracket of these local frames satisfies the following properties:

\[ \left[ \frac{\partial}{\partial x^0}, \frac{\delta}{\delta x^i} \right] = 0, \quad \left[ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right] = 0. \] \(7\)

**Lemma 1.** Let \((M, \varphi, \eta, \xi, g)\) be a Kenmotsu manifold, then components of \((g_{ij})\) defined in (6) satisfy the equation \(\xi g_{ij} = 2g_{ij}\).

**Proof.** Using (3), (7) and the Levi-Civita connection \(\nabla\) on \(M\), they imply that

\[ \xi g_{ij} = g\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = g\left(\nabla_{\xi} \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) + g\left(\frac{\delta}{\delta x^i}, \nabla_{\xi} \frac{\delta}{\delta x^j}\right) = g\left(\nabla_{\frac{\delta}{\delta x^i}} \xi, \frac{\delta}{\delta x^j}\right) + g\left(\frac{\delta}{\delta x^i}, \nabla_{\xi} \frac{\delta}{\delta x^j}\right) \]

\[ = g\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = 2g\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = 2g_{ij}. \]

\[ \square \]

**Theorem 1.** The Levi-Civita connection \(\nabla\) on a Kenmotsu manifold \(M\) has the local components:

\[ \nabla_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = \Gamma_{ij}^k \frac{\delta}{\delta x^k} - g_{ij} \frac{\partial}{\partial x^0}, \quad \nabla_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial x^0} = \frac{\partial}{\partial x^i}, \quad \nabla_{\frac{\partial}{\partial x^0}} \frac{\partial}{\partial x^0} = 0, \]

where \(\Gamma_{ij}^k = \frac{1}{2} g^{kl} (g_{lj}, \frac{\delta g_{ij}}{\delta x^l} - \frac{\delta g_{li}}{\delta x^j}). \)

**Proof.** The proof follows from (5)-(7) and Lemma 1. \[ \square \]
A 3-Kenmotsu manifold is defined as follows:

**Definition 1.** Let $M$ be an $m$-dimensional smooth manifold. Then, $M$ is a 3-Kenmotsu manifold if it is equipped with three Kenmotsu structures $(\varphi_\alpha, \eta, \xi, g)$, where $\alpha = 1, 2, 3$ and

$$\varphi_k = \varphi_i \circ \varphi_j,$$

for all $(i, j, k)$ where they are even permutations of $(1, 2, 3)$.

It is easy to see that any 3-Kenmotsu manifold is a $(4n + 1)$-dimensional manifold. The transversal distribution to the foliation $F$ given by Reeb vector field $\xi$ is integrable because $\eta$ is a closed form. Let denote the transversal distribution by $H$, then

$$H = \{X \in TM \mid \eta(X) = 0\}.$$  

One can define three almost complex structures $J_\alpha = \varphi_\alpha|_H$ for $\alpha = 1, 2, 3$ on the maximal integral submanifolds of $H$. It will be easy to check that $J_\alpha$ for $\alpha = 1, 2, 3$ form an almost quaternion structure on these maximal integral submanifolds of $H$. Thus, the distribution $H$ is a $4n$-dimensional distribution and consequently any 3-Kenmotsu manifold is $(4n + 1)$ dimensional manifold.

Moreover, there is a natural volume form on a 3-Kenmotsu manifold given by $Vol_{3} = \Omega^n \wedge \eta$, where $\Omega$ is a 4-form of maximum rank with the following structure

$$\Omega = \Omega_1 \wedge \Omega_2 \wedge \Omega_3 \wedge \Omega_4,$$

$\Omega_\alpha(\ldots) = g(\ldots, \varphi_\alpha)$ for $\alpha = 1, 2, 3$.

**Theorem 2.** Let $M$ be a 3-Kenmotsu manifold. Then, for any $p \in M$, some neighborhood $U$ of that point is identified with a warped product space $(-\epsilon, +\epsilon) \times_f V$ such that $(-\epsilon, +\epsilon)$ is an open interval, $f(t) = ce^t$, and $V$ is a quaternion Kahler manifold.

**Proof.** All three Kenmotsu structures of a 3-Kenmotsu manifold share the same Reeb vector field, therefore they have the same distribution $H$ defined in (9). Theorem 4 in [7] implies that for each Kenmotsu structure $(\varphi_\alpha, \eta, \xi, g)$ on $M$, the manifold $M$ is locally warped product of an open interval and a Kahler manifold with the warped function $f(t) = ce^t$. This means a 3-Kenmotsu manifold is locally warped product of an open interval and an almost quaternion manifold. Considering the Levi-Civita connection $\nabla$ on a maximal integral submanifold of the foliation $H$ which is locally the same as the almost quaternion manifold in the warped product. By using Eqs. (2) and (4), it will be easy to check that $\nabla J_\alpha = 0$ and $\nabla \Omega_\alpha = 0$ for $\alpha = 1, 2, 3$. Thus, theorem 1.1 in [5] implies that the structure on the maximal integral submanifolds of the foliation $H$ are quaternion Kahler manifolds.

**Corollary 1.** The structural group of the tangent bundle of a 3-Kenmotsu manifold will be reducible to $Sp(n) \times I$.

Similar to the quaternion structures, one can show that there is no fourth Kenmotsu structure $(\varphi_4, \eta, \xi, g)$ on $M$ which satisfies the anti-commutativity conditions with the other three structures [1, 15]. To see this, let $J_\alpha = \varphi_\alpha|_H$ for $\alpha = 1, 2, 3, 4$ be almost complex structures induced on the maximal integral submanifolds of $H$. Then, $J_i \circ J_4 = -J_4 \circ J_i$ for $i = 1, 2, 3$ and

$$J_3 \circ J_4 = J_1 \circ J_2 \circ J_4 = -J_4 \circ J_4 \circ J_2 = J_4 \circ J_1 \circ J_2 = J_4 \circ J_3,$$

which is a contradiction with the anti commutativity condition.

Also, it is well-known if there are two almost complex structures (or two Sasakian structures) on $M$, then under some simple conditions, one can construct a quaternion structure (or 3-Sasakian structure) [1, 8] based on those two structures.

**Theorem 3.** Assume $M$ is a $(4n + 1)$-dimensional differential manifold. If there are two Kenmotsu structures $(\varphi_1, \eta, \xi, g)$ and $(\varphi_2, \eta, \xi, g)$ on $M$ satisfying $\varphi_1 \circ \varphi_2 = -\varphi_2 \circ \varphi_1$, then $M$ will have a 3-Kenmotsu structure.

**Proof.** Let $\varphi_3 = \varphi_1 \circ \varphi_2$. Then to prove $(\varphi_3, \eta, \xi, g)$ is a Kenmotsu structure on $M$, one can show that $\varphi_3$ satisfies the equation (2). Thus,

$$(\nabla_X \varphi_3)Y = (\nabla_X \varphi_1 \circ \varphi_2)Y = \nabla_X (\varphi_1 \circ \varphi_2 Y) - \varphi_1 \circ \varphi_2 \nabla_X Y$$

$$= (\nabla_X \varphi_1)\varphi_2 Y + \varphi_1 \nabla_X \varphi_2 Y - \varphi_1 \circ \varphi_2 \nabla_X Y.$$
\[ = g(\varphi_1 X, \varphi_2 Y)\xi - \eta(\varphi_2 Y)\varphi_1 X + \varphi_1((\nabla_X \varphi_2)Y + \varphi_2 \nabla_X Y) - \varphi_1 \circ \varphi_2 \nabla_X Y \]
\[ = g(\varphi_1 X, \varphi_2 Y)\xi + \varphi_1(g(\varphi_2 X, Y)\xi - \eta(Y)\varphi_2 X) \]
\[ = g(\varphi_1 \circ \varphi_2 X, Y)\xi - \eta(Y)\varphi_1 \circ \varphi_2 (X) = g(\varphi_3 X, Y)\xi - \eta(Y)\varphi_3 (X). \]
Moreover, it is easy to check that \( \varphi_k = \varphi_i \circ \varphi_j \) satisfies for all even permutations \( (i, j, k) \) of \( (1, 2, 3) \). \( \square \)

4. SOME PROPERTIES OF A 3-KENMOTSU MANIFOLD

Let \( \tilde{M} \) be a maximal integral submanifold of the foliation \( H \) in the 3-Kenmotsu manifold \( M \). Let denote the Levi-Civita connections of \( M \) and \( \tilde{M} \) by \( \nabla \) and \( \bar{\nabla} \), respectively. The relation between these two connections is given by the Gauss formula \([9]\) as follows:
\[ \nabla_X Y = \bar{\nabla}_X Y + h(X, Y), \quad (10) \]
where \( X, Y \in \Gamma TM \). It follows from Proposition 1 that
\[ h(X, Y) = g(X, Y)\xi, \quad (11) \]
where \( X, Y \in \Gamma TM \). Let \( R \) and \( \bar{R} \) be the curvature tensors of \( M \) and \( \tilde{M} \), respectively. Then (10) and (11) imply that \( R \) and \( \bar{R} \) satisfy the following equation:
\[ R(X, Y, Z, W) = \bar{R}(X, Y, Z, W) - g(h(X, W), h(Y, Z)) + g(h(X, Z), h(Y, W)) \]
\[ = \bar{R}(X, Y, Z, W) - g(X, W)g(Y, Z) + g(X, Z)g(Y, W), \quad (12) \]
where \( X, Y, Z, W \in \Gamma TM \). Moreover, Ricci tensors \( Ric \) and \( \bar{R}ic \) of curvature tensors \( R \) and \( \bar{R} \), respectively, satisfy the following equation for all \( X, Y \in \Gamma TM \):
\[ Ric(X, Y) = \sum_{k=1}^{4n} R(E_k, X, Y, E_k) + R(\xi, X, Y, \xi) = \sum_{k=1}^{4n} R(E_k, X, Y, E_k) - g(X, Y), \quad (13) \]
where \( \{E_1, E_2, ..., E_{4n}\} \) is an orthonormal local basis of \( \Gamma H \). Then, it follows from (12) and (13) that
\[ \bar{R}ic(X, Y) = \sum_{k=1}^{4n} \bar{R}(E_k, X, Y, E_k) = \sum_{k=1}^{4n} R(E_k, X, Y, E_k) \]
\[ + \sum_{k=1}^{4n} g(X, Y) - \sum_{k=1}^{4n} g(E_k, Y)g(X, E_k) = Ric(X, Y) + 4ng(X, Y). \quad (14) \]

Theorem 4. Let \( M \) be a 3-Kenmotsu manifold of dimension \( \geq 9 \). Then, its Ricci tensor is parallel.

Proof. Lemma 3.1 in \([5]\) implies that the Ricci tensor \( \bar{R}ic \) of the quaternion Kahler manifold \( \bar{M} \) is parallel. Then, it follows from (14) that the Ricci tensor \( Ric \) of 3-Kenmotsu manifold \( M \) is parallel. \( \square \)

Theorem 5. Let \( M \) be a 3-Kenmotsu manifold of dimension \( \geq 9 \). Then, it is an Einstein space.

Proof. Theorem 3.3 in \([5]\) implies that the quaternion Kahler manifold \( \bar{M} \) is an Einstein space. Then, it follows from (14) that the Ricci tensor \( Ric \) of 3-Kenmotsu manifold \( M \) satisfies the Einstein equation. Thus, \( M \) is an Einstein space. \( \square \)

Now, we define the \( \varphi_\alpha \)-holomorphic sectional curvature of \( X \in \Gamma H \) by
\[ H_\alpha(X) = K(X, \varphi_\alpha X) = -\frac{R(X, \varphi_\alpha X, X, \varphi_\alpha X)}{g(X, X)g(\varphi_\alpha X, \varphi_\alpha X) - g(X, \varphi_\alpha X)^2} = -\frac{R(X, \varphi_\alpha X, X, \varphi_\alpha X)}{g(X, X)^2}, \]
where \( \alpha = 1, 2, 3 \).

Theorem 6. The \( \varphi_\alpha \)-holomorphic sectional curvatures \( H_\alpha \) satisfy the following equation on a 3-Kenmotsu manifold: \( H_1(X) + H_2(X) + H_3(X) = -3 \), where \( X \in \Gamma H \).

Proof. Let \( (\varphi, \eta, \xi, g) \) be a Kenmotsu structure. Then, it follows from (2), (3) and Proposition 1 that
\[ R(X, Y)\varphi Z = g(\varphi Y, Z)X - g(\varphi X, Z)Y + g(Y, Z)\varphi X - g(X, Z)\varphi Y + \varphi(R(X, Y)Z). \]
Therefore,
\[ R(X, Y, \varphi Z, \varphi W) = g(R(X, Y)\varphi Z, \varphi W) = g(\varphi Y, Z)g(X, \varphi W) - g(\varphi X, Z)g(Y, \varphi W) \\
+ g(Y, Z)g(\varphi X, \varphi W) - g(X, Z)g(\varphi Y, \varphi W) + g(\varphi (R(X, Y)Z), \varphi W). \]

Considering this equation on a 3-Kenmotsu structure for each \( \varphi \alpha \). Let \( \varphi = \varphi_1, Z = X \) and \( Y = W = \varphi_3 X \), then
\[ -R(X, \varphi_3 X, \varphi_1 X, \varphi_2 X) = -g(X, X)g(\varphi_2 X, \varphi_2 X) + R(X, \varphi_3 X, X, \varphi_3 X) \]
\[ = -g(X, X)^2 + R(X, \varphi_3 X, X, \varphi_3 X). \]

Dividing both sides by \( g(X, X)^2 \), it implies
\[ \frac{R(X, \varphi_3 X, \varphi_2 X, \varphi_1 X)}{(g(X, X))^2} = -1 - H_3(X). \]

Consider even permutations of \( (1, 2, 3) \), one can get two other similar equations for \( H_1(X) \) and \( H_2(X) \). Then, the proof is completed by adding these three equations and using the Bianchi identity. \( \square \)

5. AN EXAMPLE OF 3-KENMOTSU MANIFOLDS

Consider the manifold \( M \) given by \( \{(x_0, x_1, x_2, x_3, x_4) \in \mathbb{R}^5 \mid x_0 \neq 0\} \) and vector fields
\[ \xi = X_0 = -x_0 \frac{\partial}{\partial x_0}, \ X_1 = x_0 \frac{\partial}{\partial x_1}, \ X_2 = x_0 \frac{\partial}{\partial x_2}, \ X_3 = x_0 \frac{\partial}{\partial x_3}, \ X_4 = x_0 \frac{\partial}{\partial x_4}, \]
on \( M \). It is easy to check that these vector fields satisfy following properties
\[ [X_i, X_j] = 0, \quad [\xi, X_i] = -X_i, \]
where \( i, j = 1, 2, 3, 4 \). Let \( g \) be a Riemannian metric on \( M \) such that,
\[ g(X_i, X_j) = \delta_{ij} = \begin{cases} 1, & i = j; \\ 0, & i \neq j, \end{cases} \quad i, j = 0, 1, 2, 3, 4. \]

Let \( \varphi_\alpha \) for \( \alpha = 1, 2, 3 \) be \((1,1)\)-tensor field on \( M \) defined by
\[ \varphi_1(\xi) = 0, \quad \varphi_1(X_1) = X_2, \quad \varphi_1(X_2) = -X_1, \quad \varphi_1(X_3) = X_4, \quad \varphi_1(X_4) = -X_3, \]
\[ \varphi_2(\xi) = 0, \quad \varphi_2(X_1) = X_3, \quad \varphi_2(X_2) = -X_4, \quad \varphi_2(X_3) = -X_1, \quad \varphi_2(X_4) = X_2, \]
\[ \varphi_3(\xi) = 0, \quad \varphi_3(X_1) = X_4, \quad \varphi_3(X_2) = X_3, \quad \varphi_3(X_3) = -X_2, \quad \varphi_3(X_4) = -X_1. \]
It is easy to check for all \((i, j, k)\) as an even permutation of \((1, 2, 3)\) we have \( \varphi_k = \varphi_i \circ \varphi_j \). If we define the \(1\)-form \( \eta \) by \( \eta(\cdot) = g(\xi, \cdot) \), it will be a straightforward calculation to check other properties in (1). In this case, the corresponding Levi-Civita connection of \( g \) on \( M \) will have the following components:
\[ \nabla_\xi \xi = \nabla_\xi X_i = \nabla X_i \xi - X_i = \nabla X_i X_j + \delta_{ij} \xi = 0, \quad (15) \]
where \( i, j = 1, 2, 3, 4 \). To show \((\varphi_\alpha, \eta, \xi, g)\), for \( \alpha = 1, 2, 3 \), is a 3-Kenmotsu structure one should check equation (2) which is easy by the help of (15) and definition of \( \varphi_\alpha \), where \( \alpha = 1, 2, 3 \). Moreover, (15) implies that components of the curvature tensor \( R \) satisfy the following equations:
\[ R(X_i, X_j)\xi = 0, \quad i, j = 1, 2, 3, 4, \]
\[ R(\xi, X_i)X_j = -R(X_i, \xi)X_j = \nabla X_i X_j, \quad i, j = 0, 1, 2, 3, 4, \]
\[ R(X_i, X_j)X_k = g(X_i, X_k)X_j - g(X_j, X_k)X_i, \quad i, j = 0, 1, 2, 3, 4. \]
Since \( \{X_1, X_2, X_3, X_4\} \) is an orthonormal basis of sections on \( H \), we can write all \( X \in \Gamma H \) as \( X = \sum_{i=1}^{4} a_i X_i \), where \( a_i \) is a scalar function on \( M \) for \( i = 1, 2, 3, 4 \). Then, the \( \varphi_\alpha \)-holomorphic sectional curvature \( H_\alpha(X) \) of \( X \in \Gamma H \) for \( \alpha = 1, 2, 3 \) can be calculated as follows:
\[ H_\alpha(X) = -\frac{R(X, \varphi_\alpha X, X, \varphi_\alpha X)}{g(X, X)} \]
\[ = -\frac{g(R(X, \varphi_\alpha X)X, \varphi_\alpha X)}{g(X, X)^2 - g(X, \varphi_\alpha X)^2} \]
\[ = -\frac{g(R(X, X)\varphi_\alpha X - g(\varphi_\alpha X, X)X, \varphi_\alpha X)}{g(X, X)^2 - g(X, \varphi_\alpha X)^2} = -1, \]
which confirms the result of Theorem 6.
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