Stochastic Biasing and Galaxy-Mass Density Relation in the Weakly Non-linear Regime

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ABSTRACT

It is believed that the biasing of the galaxies plays an important role for understanding the large-scale structure of the universe. In general, the biasing of galaxy formation could be stochastic. Furthermore, the future galaxy survey might allow us to explore the time evolution of the galaxy distribution. In this paper, the analytic study of the galaxy-mass density relation and its time evolution is presented within the framework of the stochastic biasing. In the weakly non-linear regime, we derive a general formula for the galaxy-mass density relation as a conditional mean using the Edgeworth expansion. The resulting expression contains the joint moments of the total mass and galaxy distributions. Using the perturbation theory, we investigate the time evolution of the joint moments and examine the influence of the initial stochasticity on the galaxy-mass density relation. The analysis shows that the galaxy-mass density relation could be well-approximated by the linear relation. Compared with the skewness of the galaxy distribution, we find that the estimation of the higher order moments using the conditional mean could be affected by the stochasticity. Therefore, the galaxy-mass density relation as a conditional mean should be used with a caution as a tool for estimating the skewness and the kurtosis.

Subject headings: cosmology:theory—large scale structure of universe, biasing

1. Introduction

The enormous data of the large scale galaxy distribution will be obtained in the future. It is expected that the physics in the early universe can be explored by using these data. According to the standard picture based on the gravitational instability, the primordial fluctuations produced during the inflationary stage have evolved into the large scale structure observed in the galaxy sky survey. To compare the theoretical prediction for the density fluctuation with the observation, we need the relation between the total mass and the galaxies. Due to the lack of our knowledge of the galaxy formation process, however, the statistical uncertainty between the galaxies and the total mass density arises. This uncertainty affects the determination of the cosmological parameters from the observation [Hamilton 1997].
Denoting the fluctuation of total mass as $\delta_m$ and that of galaxy distribution as $\delta_g$, the linear biasing relation

$$\delta_g = b\delta_m,$$

(1)
is frequently used in the literature. Here, $b$ is the biasing parameter. A naive extension of (1) to the quasi non-linear regime is obtained by taking the expansion in powers of $\delta_m$:

$$\delta_g = f(\delta_m) = \sum_{n=1} b_n (\delta_m)^n.$$  

(2)
The non-linear biasing parameters $b_n$ is observationally determined by several authors (Fry & Gaztañaga 1993, Fry 1994, Gaztañaga & Frieman 1994). However, we must note that there is an assumption behind equations (1) and (2) that the statistics of the galaxy distribution is completely characterized by that of the total mass $\delta_m$. In reality, the galaxy formation process could be stochastic (Cen & Ostriker 1992). Stochastic biasing is a notion to treat this situation. Note that the deterministic biasing corresponds to a special case of the stochastic biasing. Stochastic property of the galaxy biasing is recently studied by many authors. Pen (1997) has discussed an observational method to determine the stochasticity. Scherrer & Weinberg (1998) has studied the influences of the non-lineararity and the stochasticity on the statistics for the two point correlation function.

Due to the gravitational dynamics and galaxy formation process, the relation of the galaxies and the total mass is generally non-linear and time dependent. The recent observation at high redshift shows that the galaxies at $z \simeq 3$ are strongly biased. The evidence of the strong biasing at high redshift suggests that the time evolution of the biasing should be considered (Peacock et al. 1998). Furthermore, Magliocchetti et al. (1998) showed that the measurement of the angular correlation function of the radio galaxies obtained from the Faint Images of the Radio Sky at Twenty centimeters is consistently explained by taking into account the evolution of the biasing. The time evolution of the galaxy biasing has been firstly studied by Fry (1996). Tegmark & Peebles (1998) extended it to the stochastic biasing and examined the linear evolution. In the previous paper (Taruya, Koyama & Soda 1999), we have investigated the quasi non-linear evolution of the stochastic biasing.

A general formalism for the stochastic biasing is recently proposed by Dekel & Lahav (1998). The non-linearity and the stochasticity of the galaxy biasing are measured by the conditional mean and its scatter. They performed the numerical simulation and evaluated the conditional mean of $\delta_g$ at a given total mass fluctuation $\delta_m$. Blanton et al. (1998) studied the galaxy-mass density relation using the hydrodynamical simulation and explore the physical origin of the scale dependent biasing on small scales.

The purpose of this paper is to investigate the galaxy-mass density relation and its time evolution analytically. Using the Edgeworth expansion, we will derive a general formula for the conditional mean. Our main result is equation (25). The final expression can be written in terms of the joint moment of the galaxy and the total mass. Since the gravity induces the time evolution
of the joint moments, the galaxy-mass density relation cannot be static. Neglecting the merging and galaxy formation process, we calculate the joint moments using the perturbation theory of the Newtonian cosmology. Then we study the time evolution of the galaxy-mass density relation. We find that the estimation of higher order moments such as skewness using the conditional mean could be affected by the stochasticity.

We organize this paper as follows. In section 2, the basic description of the stochastic biasing is explained. Within the formalism, a general formula for the galaxy-mass density relation is derived. Then we examine the time evolution of the galaxy-mass density relation in section 3. The final section is devoted to the summary and the discussions.

2. Stochastic biasing and galaxy-mass density relation

2.1. Basic description

Let us recall the definitions of $\delta_m$ and $\delta_g$:

$$\delta_m(x) \equiv \frac{\rho(x) - \bar{\rho}}{\bar{\rho}}, \quad \delta_g(x) \equiv \frac{n_g(x) - \bar{n}_g}{\bar{n}_g},$$  

where variables with overbars indicate the homogeneous averaged density. In this paper, we treat the smoothed density fields $\delta_m(R)$ and $\delta_g(R)$ using the spherical top-hat window function $W_R(x)$ with the smoothing radius $R$:

$$\delta_{m,g}(R) \equiv \int d^3x W_R(x) \delta_{m,g}(x).$$

Importantly, the one-point distribution function itself depends on the smoothing scale. As the smoothing scale becomes larger, the distribution approaches to the Gaussian distribution.

The statistical information of the galaxy biasing is obtained from the joint probability distribution function (PDF) of the galaxy and the total mass distribution. In our formulation, $\delta_m$ and $\delta_g$ are regarded as the independent stochastic variables. To represent their stochasticity, we introduce the auxiliary random fields $\Delta_m(x)$ and $\Delta_g(x)$. On large scales, the smoothed fields $\delta_{m,g}(R)$ could be regarded as nearly Gaussian fields. The density fields can be expressed as functions of the Gaussian variables and expanded in powers of these variables whose variances are small enough. Denoting such Gaussian variables as $\Delta_{m,g}$, the distributions $\delta_m$ and $\delta_g$ are given by

$$\delta_m = f(\Delta_m, \Delta_g), \quad \delta_g = g(\Delta_m, \Delta_g).$$

Here, we wrote equation (5) in somewhat schematical way. The ”functions” $f$ and $g$ are not assumed as the local functions. The generating function for the one-point functions $\delta_m(R)$ and $\delta_g(R)$ becomes (Takuya, Koyama & Soda 1999)

$$Z(J_m, J_g) \equiv \left\langle e^{i(J_m \delta_m(R) + J_g \delta_g(R))} \right\rangle.$$
\[\int \int D\Delta_m(x) D\Delta_g(x) \mathcal{N}^{-1} \exp \left[ -\left( \Delta_m, \Delta_g \right) G^{-1} \left( \begin{array}{c} \Delta_m \\ \Delta_g \end{array} \right) \right] \times \exp \left[ i \int d^3x W_R(x) \{ J_m \delta_m(\Delta_m,g) + J_g \delta_g(\Delta_m,g) \} \right]. \]

where \( \mathcal{N} \) is a normalization constant and \( G \) is a \( 2 \times 2 \) matrix which gives the stochastic property of \( \Delta_m \) and \( \Delta_g \). The joint moments of \( \delta_m \) and \( \delta_g \) are deduced from the generating function as

\[\langle [\delta_m(R)]^j [\delta_g(R)]^k \rangle = i^{-j+k} \frac{\partial^{j+k} Z}{\partial J_m \partial J_g} \bigg|_{J_m=J_g=0}. \]

The ensemble average \( \langle \cdots \rangle \) is taken with respect to \( \Delta_m \) and \( \Delta_g \). Once we obtain the joint moments \( (7) \), the joint PDF \( P(\delta_m, \delta_g) \) for the smoothed density fields \( \delta_m(R) \) and \( \delta_g(R) \) is constructed:

\[P(\delta_m, \delta_g) = \frac{1}{(2\pi)^3} \int dJ_m dJ_g \ Z(J_m, J_g) e^{-i(J_m \delta_m + J_g \delta_g)}, \]

where we simply denotes \( \delta_{m,g}(R) \) as \( \delta_{m,g} \).

We should keep in mind that the gravitational instability usually affects the galaxy and the total mass distribution. Accordingly, the functions \( f \) and \( g \) in the assumption \( (5) \) are determined by the gravitational dynamics and they would be generally non-local and time dependent (see section \( \text{[3]} \)). Since the resulting joint PDF also depends on time, we would have the time evolving biasing relation.

### 2.2. Stochastic biasing in the weakly non-linear regime: conditional mean and biasing scatter

To proceed further, we shall employ the perturbative approach, hereafter. Taking the variables \( \Delta_{m,g} \) as seeds of perturbation, \( \delta_g \) and \( \delta_m \) can be expressed as

\[\delta_{m,g}(\Delta_m, \Delta_g) = \delta_m^{(1)} + \delta_m^{(2)} + \cdots, \]

where the \( n \)-th order perturbed quantities \( \delta_{m,g}^{(n)} \) are of the same order as \( [\Delta_{m,g}]^n \). Substituting the expansion \( (3) \) into the joint moment \( (7) \), the non-vanishing lowest order quantities become the second moments. They are characterized by the three parameters:

\[\sigma_m^2 \equiv \langle [\delta_m^{(1)}]^2 \rangle, \quad b^2 \equiv \frac{\langle [\delta_m^{(1)}]^2 \rangle}{\langle [\delta_m^{(1)}]^2 \rangle}, \quad r \equiv \frac{\langle \delta_m^{(1)} \delta_g^{(1)} \rangle}{\sqrt{\langle [\delta_m^{(1)}]^2 \rangle \langle [\delta_g^{(1)}]^2 \rangle}}. \]

\( \sigma_m \) is regarded as the variance of the total mass, \( b \) is the linear biasing parameter and \( r \) is the cross correlation. Note that the parameter \( r \) reflects the stochastic property of \( \delta_m \) and \( \delta_g \). At the lowest order level, the deterministic biasing relation \( (1) \) holds if \( r = 1 \).
The higher order perturbations leads to the non-linear galaxy biasing which describes the relation between the non-Gaussian distributions of galaxy and the total mass. Note that the higher order correction also affects the stochastic property of $\delta_m$ and $\delta_g$. To investigate the galaxy biasing in the weakly non-linear regime, we should analyze the non-linearity and the stochasticity separately.

Several authors proposed to use the conditional mean of the galaxy distribution defined by\textcolor{blue}{(Dekel & Lahav 1998, Blanton et al. 1998)}

$$\langle \delta_g \rangle_{| \delta_m} = \int d\delta_g \delta_g P(\delta_g | \delta_m) ; \quad P(\delta_g | \delta_m) = \frac{P(\delta_g, \delta_m)}{P(\delta_m)},$$

(11)

where $P(\delta_g | \delta_m)$ denotes the conditional PDF at a given $\delta_m$. From the conditional mean $\langle \delta_g \rangle_{| \delta_m}$, we can know the non-linear relation of the biasing as the function of $\delta_m$. We should keep in mind that the conditional PDF has a width around the conditional mean $\langle \delta_g \rangle_{| \delta_m}$. The biasing scatter can be introduced to measure the stochasticity in the biasing relation \textcolor{blue}{(Dekel & Lahav 1998)}:

$$\epsilon = \delta_g - \langle \delta_g \rangle_{| \delta_m}.$$  

(12)

The stochastic property of the non-linear biasing relation is understood from the variance and the higher moments of $\epsilon$.

Consider the density fields on large scales. Owing to the assumption (5) and the perturbative treatment (9), we can obtain the analytic expression for the conditional mean $\langle \delta_g \rangle_{| \delta_m}$. The non-linear galaxy-mass density relation using the Edgeworth expansion leads to the expression\textcolor{blue}{(13)}

$$\langle \delta_g \rangle_{| \delta_m} = c_1 \delta_m + c_2 (\delta_m^2 - \langle \delta_m^2 \rangle) + \cdots.$$  

We will show in the next subsection that the coefficients $c_1$ and $c_2$ are expressed in terms of the joint moments of $\delta_m$ and $\delta_g$ (see eq.[26]).

Before deriving the galaxy-mass density relation, we mention the differences between the relation (13) and (2). From the relation (2), Fry & Gaztaña (1993) derived the useful formulae which relate the higher order moments of the galaxies with that of the total mass. For the third moments, we have\textcolor{blue}{(14)}

$$S_{3,g} = \frac{1}{b}(S_{3,m} + 3b_2),$$

where the linear biasing parameter $b_1$ is simply denoted as $b$. The quantities $S_{3,m}$ and $S_{3,g}$ are the skewness of the total mass and the galaxy distribution, respectively. In the tree level perturbation theory, these can be written as\textcolor{blue}{(15)}

$$S_{3,m} = 3 \frac{\langle [\delta_{(1)}^m \delta_{(2)}^m]^2 \rangle}{\langle [\delta_{(1)}^m]^2 \rangle}, \quad S_{3,g} = 3 \frac{\langle [\delta_{(1)}^g \delta_{(2)}^g]^2 \rangle}{\langle [\delta_{(1)}^g]^2 \rangle}.$$  

For the galaxy-mass density relation (13), we can also obtain the equation similar to (14) (see eqs. 26 and 28). We will see in section 3.2 that the formula (14) is recovered when the cross correlation $r$ becomes unity. Hence, the galaxy-mass density relation (13) can be regarded as a generalization of the relation (2) examined by Fry & Gaztaña (1993).
2.3. Derivation of galaxy-mass density relation: a general formula

We are in a position to derive the galaxy-mass density relation (13) as a conditional mean. In section 2.1, we defined the generating function \(Z(J_m, J_g)\) to construct the joint PDF \(\mathcal{P}(\delta_m, \delta_g)\). Using \(Z\), we obtain the generating function for the connected diagrams:

\[
W(J_m, J_g) \equiv \ln Z = \sum_{j, k=0}^{\lambda_{jk}} \frac{\lambda_{jk}}{j! k!} \left( iJ_m \sqrt{\langle \delta_m^2 \rangle} \right)^j \left( iJ_g \sqrt{\langle \delta_g^2 \rangle} \right)^k,
\]

where we define

\[
\lambda_{jk} = \frac{\langle (\delta_m)^j (\delta_g)^k \rangle_c}{\langle (\delta_m)^{j/2} \rangle \langle (\delta_g)^{k/2} \rangle}.
\]

\(\langle \cdots \rangle_c\) denotes the connected part of the moments. By definition, we have \(\lambda_{02} = \lambda_{20} = 1\). The perturbative treatment (9) implies that \(\lambda_{ij}\) satisfies the following scaling relation. Using the variable \(\sigma_m\) defined by equation (10) and assuming that the lowest order variance of the galaxies \(\langle [\delta_g^{(1)}]^2 \rangle\) is of the same order of magnitude as that of the total mass, we have

\[
\lambda_{jk} = S_{jk} \sigma_m^{j+k-2} + \mathcal{O}(\sigma^{j+k}). \quad (16)
\]

The variable \(S_{ij}\) is of the order of unity and we obtain \(S_{20} = S_{02} = 1\).

As long as the variance \(\sigma_m\) is small, treating \(\sigma_m\) as an expansion parameter, we can expand the generating function \(W\). Thus we proceed to approximate the joint PDF \(\mathcal{P}(\delta_g, \delta_m)\). The expansion of \(W\) is referred to as the Edgeworth expansion (Juszkiewicz et al. 1995, Bernardeau & Kofman 1995, Chodorowski & Lokas 1997). Let us write the conditional mean \(\langle \delta_g | \delta_m \rangle\) as

\[
\langle \delta_g | \delta_m \rangle = \frac{A}{\mathcal{P}(\delta_m)}; \quad A = \int d\delta_g \delta_g \mathcal{P}(\delta_m, \delta_g). \quad (17)
\]

Now, define the normalized variables:

\[
\mu = J_m \sqrt{\langle \delta_m^2 \rangle}, \quad \nu = J_g \sqrt{\langle \delta_g^2 \rangle}, \quad x = \frac{\delta_m}{\sqrt{\langle \delta_m^2 \rangle}}, \quad y = \frac{\delta_g}{\sqrt{\langle \delta_g^2 \rangle}}. \quad (18)
\]

Using the generating function \(W\), the numerator and the denominator of equation (17) are expressed in terms of the above variables:

\[
A = \frac{1}{(2\pi)^2} \left( \frac{\langle \delta_g^2 \rangle}{\langle \delta_m^2 \rangle} \right)^{1/2} \int dy \int d\mu d\nu \ y e^{-i(x+\nu y)} W(\mu, \nu), \quad (19)
\]

\[
\mathcal{P}(\delta_m) = \frac{1}{2\pi} \left( \frac{\langle \delta_m^2 \rangle}{\langle \delta_m^2 \rangle} \right)^{1/2} \int d\mu e^{-i\mu x} W(\mu, 0). \quad (20)
\]

Applying the Edgeworth expansion to the above equations, we can obtain the galaxy-mass density relation. The details of the calculation are described in Appendix A. Assuming that the normalized
variables $x$ and $y$ are of the order of unity and using the scaling relation (16), we obtain the final results of the expansion up to $O(\sigma_m^2)$ (Chodorowski & Lokas 1997, Bernardeau & Kofman 1995):

$$A = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \left( \frac{\langle \delta_y^2 \rangle}{\langle \delta_m^2 \rangle} \right)^{1/2} \left[ \lambda_{11} H_1(x) + \left\{ \frac{\lambda_{21}}{2} H_2(x) + \frac{\lambda_{30}}{6} H_4(x) \right\} + \left\{ \frac{\lambda_{31}}{12} H_3(x) + \left( \frac{\lambda_{11}\lambda_{40}}{24} + \frac{\lambda_{30}\lambda_{21}}{12} \right) H_5(x) + \frac{\lambda_{11}\lambda_{30}^2}{72} H_7(x) \right\} \right], \quad (21)$$

and

$$P(\delta_m) = \frac{e^{-x^2/2}}{\sqrt{2\pi} \sqrt{\langle \delta_m^2 \rangle}} \left[ 1 + \frac{\lambda_{30}}{6} H_3(x) + \left\{ \frac{\lambda_{40}}{24} H_4(x) + \frac{\lambda_{30}^2}{72} H_6(x) \right\} \right], \quad (22)$$

where $H_n(x)$ is the Hermite polynomial defined by

$$H_n(x) = \frac{e^{x^2/2} \left( -\frac{d}{dx} \right)^n e^{-x^2/2}}{\sqrt{2^n \pi^n}}, \quad (23)$$

Substituting the results (21) and (22) into equation (17), the conditional mean $\langle \delta_y \rangle|\delta_m$ in the weakly non-linear regime becomes

$$\langle \delta_y \rangle|\delta_m = \sqrt{\langle \delta_y^2 \rangle} \left[ \lambda_{11} x + F_1(x) + F_2(x) \right]. \quad (24)$$

where $F_n(x)$ denotes the terms of the order $O(\sigma_m^n)$ given by

$$F_1(x) = \frac{\lambda_{21}}{2} H_2(x) + \frac{\lambda_{11}\lambda_{30}}{6} [H_4(x) - x H_3(x)],$$

$$F_2(x) = \frac{\lambda_{31}}{2} H_3(x) + \frac{\lambda_{30}\lambda_{21}}{12} [H_5(x) - H_2(x) H_3(x)] + \frac{\lambda_{11}\lambda_{40}}{24} [H_5(x) - x H_4(x)] + \frac{\lambda_{11}\lambda_{30}^2}{72} \left[ H_7(x) - x H_6(x) - 2 H_3(x) H_4(x) + 2 x H_3(x)^2 \right].$$

We should keep in mind that $\lambda_{ij}$ still contains the terms higher than $O(\sigma_m^{i+j})$ and we do not approximate the variance of galaxy distribution $\langle \delta_y^2 \rangle$ in deriving the expression (24).

We first consider the conditional mean to the order $O(\sigma_m)$. Since we can drop the higher order correction to $\lambda_{ij}$ with $i + j \geq 3$, we simply replace $\lambda_{ij}$ with $S_{ij}\sigma_m^{i+j-2}$. As for the variance and the covariance of $\delta_m$ and $\delta_y$, the definition (10) gives

$$\langle \delta_m^2 \rangle = \sigma_m^2, \quad \langle \delta_y^2 \rangle = b^2 \sigma_m^2, \quad \lambda_{11} = r$$

This is correct within the tree-level analysis. Then the weakly non-linear relation (24) up to $O(\sigma_m)$ is written in terms of the variable $\delta_m$ as

$$\langle \delta_y \rangle|\delta_m = c_1 \delta_m + \frac{c_2}{2} \left( \delta_m^2 - \langle \delta_m^2 \rangle \right) + O(\sigma_m^2). \quad (25)$$

The coefficients $c_1$ and $c_2$ becomes

$$c_1 = b \ r, \quad c_2 = b (S_{21} - r S_{30}). \quad (26)$$
The expressions (25) and (26) are our main results.

To understand the stochastic property in the biasing relation, the mass density-galaxy relation should be noted:

\[
\langle \delta_m \rangle | \delta_g = \int d\delta_m \, \delta_m \mathcal{P}(\delta_m | \delta_g),
\]

\[
= \frac{\mathcal{B}}{\mathcal{P}(\delta_g)} ; \quad \mathcal{B} = \int d\delta_m \, \delta_m \mathcal{P}(\delta_m, \delta_g).
\]

Derivation of non-linear mass density-galaxy relation is similar to that of \( \langle \delta_g \rangle | \delta_m \). For the expression (24), we only have to replace the variables \( \langle \delta_g^2 \rangle, \lambda_{ij} \) and \( x \) with \( \langle \delta_m^2 \rangle, \lambda_{ji} \) and \( y \), respectively. We here write down the result in terms of \( \delta_g \) up to \( \mathcal{O}(\sigma_m) \):

\[
\langle \delta_m \rangle | \delta_g = d_1 \, \delta_g + \frac{d_2}{2} \left( \delta_g^2 - \langle \delta_g^2 \rangle \right) + \mathcal{O}(\sigma_m^2).
\]

Then the coefficients \( d_1 \) and \( d_2 \) become

\[
d_1 = \frac{r}{b} , \quad d_2 = \frac{1}{b^2} (S_{12} - r S_{03}).
\]

We observe that the conditional mean (27) does not coincide with the result (25) due to the presence of the cross correlation \( r \) and the fact that \( S_{ij} \) is not symmetric. It will be shown in the next section that both relations give the same relation when the cross correlation \( r \) becomes unity.

We next clarify the effect of the higher order corrections on the galaxy-mass relation \( \langle \delta_g \rangle | \delta_m \). When we take into account the \( F_2(x) \) term, the \( \mathcal{O}(\sigma_m^2) \) contribution requires the third order perturbations. In this case, we cannot ignore the \( \mathcal{O}(\sigma_m^3) \) correction for \( \lambda_{ij} \) with \( i + j = 2 \). Then the evaluation of the first term in the right hand side of (24) is modified. Using the definition (10), we can write

\[
\sqrt{\langle \delta_g^2 \rangle} \, \lambda_{11} x = \frac{\langle \delta_g \delta_m \rangle}{\langle \delta_g^2 \rangle} \delta_m = \left[ b_0 + \mathcal{O}(\sigma_m^2) \right] \delta_m,
\]

where the order \( \mathcal{O}(\sigma_m^2) \) terms come from the loop corrections:

\[
\sigma_m^{-2} \left[ \langle \delta_g^2 \delta_m^2 \rangle + \langle \delta_g^3 \delta_m \rangle + \langle \delta_g \delta_m^3 \rangle - b r \left( \langle \delta_g^2 \rangle + 2 \langle \delta_g \delta_m \rangle \right) \right].
\]

Hence we redefine equation (24) as \( b_{ren} r_{ren} \delta_m \) to the \( \mathcal{O}(\sigma_m^2) \) terms. The parameters \( b_{ren} \) and \( r_{ren} \) are the renormalized quantities for the biasing parameter \( b \) and the cross correlation \( r \) given by the tree-level perturbation. For \( \lambda_{ij} \) with \( i + j \geq 3 \), the correction terms are higher than \( \mathcal{O}(\sigma_m^3) \), which can be verified from the scaling law (16). Rewriting the expression (24) in terms of \( \delta_m \) by using the definition (23), we get the non-linear galaxy-mass density relation in accuracy of \( \mathcal{O}(\sigma_m^2) \):

\[
\langle \delta_g \rangle | \delta_m = \tilde{c}_1 \, \delta_m + \tilde{c}_2 \left( \delta_m^2 - \langle \delta_m^2 \rangle \right) + \tilde{c}_3 \, \delta_m^3 + \mathcal{O}(\sigma_m^3),
\]

where the coefficients \( \tilde{c}_i \) are

\[
\tilde{c}_1 = b_{ren} r_{ren} + \sigma_m^2 \left[ S_{30} S_{21} - r S_{30}^2 - \frac{1}{2} (S_{31} - r S_{40}) \right].
\]
\[
\tilde{c}_2 = \frac{b}{2}(S_{21} - rS_{30}), \\
\tilde{c}_3 = \frac{b}{6}(S_{31} - rS_{40}) - \frac{b}{2}(S_{30}S_{21} - rS_{30}^2).
\]

We immediately see the difference between the above result (30) and the tree-level result (25). In addition to the contribution of \( \delta_m^3 \), the coefficient \( \tilde{c}_1 \) has the extra terms except for the renormalized parameters, which are absent from the coefficient \( c_1 \). The result says that the linear biasing relation in the weakly non-linear regime is different from the linear theory prediction even for \( \delta_m \ll 1 \). Our results are in good agreement with the analyses of the two point correlation function for the galaxy distribution [Scherrer & Weinberg 1998].

### 3. Time evolution of galaxy-mass density relation

The previous section has been devoted to the discussion of the galaxy-mass density relation. In this section, we investigate the time evolution of it. The evolution equations and the initial conditions are given in section 3.1. Within this model, we calculate the time-dependence of the joint moments \( S_{ij} \) explicitly and the relationship between the conditional means \( \langle \delta_g \rangle|_{\delta_m} \) and \( \langle \delta_m \rangle|_{\delta_g} \) is clarified in Sec 3.2. In section 3.3, we will study the time evolution of the conditional mean \( \langle \delta_g \rangle|_{\delta_m} \) in order to reveal the influence of initial stochasticity.

#### 3.1. Evolution equations and initial conditions

The time evolution of the galaxy biasing induced by the gravity has been studied by several authors (Fry 1996, Tegmark & Peebles 1998, Taruya, Koyama & Soda 1999). In these articles, the density fluctuation \( \delta_m \) is assumed to be evolved following the equation of continuity and the Euler equation. On large scales, the total mass distribution moves along the irrotational velocity flow. It is convenient to define the velocity divergence \( \theta \equiv \nabla \cdot v/(aH) \), where \( a \) is the scale factor of the universe and \( H \) is the Hubble parameter. For the galaxy distribution, we consider the epoch after galaxy formation and assume that the merging process is negligible. The distribution \( \delta_g \) should satisfy the equation of the continuity whose velocity field is determined by the gravitational potential. Then the evolution equations for the total mass and the galaxies become

\[
\frac{\partial \delta_m}{\partial t} + H\theta + \frac{1}{a} \nabla \cdot (\delta_m v) = 0, \tag{31}
\]

\[
\frac{\partial \theta}{\partial t} + \left(1 - \frac{\Omega}{2} + \frac{\Lambda}{3H^2}\right)H\theta + \frac{3}{2}H\Omega\delta_m + \frac{1}{a^2H} \nabla \cdot (v \cdot \nabla)v = 0, \tag{32}
\]

and

\[
\frac{\partial \delta_g}{\partial t} + H\theta + \frac{1}{a} \nabla \cdot (\delta_g v) = 0. \tag{33}
\]
The variable Λ is the cosmological constant and Ω is the density parameter defined by

$$\Omega \equiv \frac{8\pi G}{3} \frac{\bar{\rho}}{H^2}. \quad (34)$$

Equations (31), (32) and (33) are our basic equations for the time evolution of $\delta_m$ and $\delta_g$. The same situation has been studied by Fry (1996) in the case of the deterministic biasing (2).

Next, we consider the initial condition given at an initial time $t_i$. The total mass fluctuation $\delta_m$ is produced during the very early stage of the universe, whose initial distribution may have the random Gaussian statistics. We regard such fluctuation as $\Delta_m(x)$. The gravitational instability induces the deviation from the Gaussian statistics and the galaxy formation does not affect the evolution of $\delta_m$ on large scales. We give the initial condition $\delta_m = f(\Delta_m)$ from the perturbative solutions by dropping the decaying mode, which leads to the non-local form of the function $f(\Delta_m)$ (see Appendix B).

On the other hand, the fluctuation of the galaxy number density is induced by the galaxy formation. To give the initial non-Gaussian distribution $\delta_g = g(\Delta_g)$, we need to know the galaxy formation processes. Currently, it is not formidable. Here, we treat $g(\Delta_g)$ as a parameterized function, whose unknown parameters are determined by the observation of galaxies. Assuming $g(\Delta_g)$ as a local function of $\Delta_g$, we take

$$g(\Delta_g) = \Delta_g + \frac{h}{6}(\Delta_g^2 - \langle \Delta_g^2 \rangle) + \cdots. \quad (35)$$

Due to the assumptions stated above, in the perturbative regime, the relation between the galaxy and the total mass distribution can be characterized completely if the stochastic property of the Gaussian variables $\Delta_m$ and $\Delta_g$ is given. It is expressed in terms of the three parameters:

$$\sigma_0^2 = \langle \Delta_m^2 \rangle, \quad b_i^2 = \frac{\langle \Delta_g^2 \rangle}{\langle \Delta_m^2 \rangle}, \quad r_0 = \frac{\langle \Delta_g \Delta_m \rangle}{\left(\langle \Delta_m^2 \rangle \langle \Delta_g^2 \rangle\right)^{1/2}}, \quad (36)$$

which is equivalent to giving the matrix $G$ in the generating function (3). The variables $b_0$, $r_0$ and $\sigma_0$ correspond to the initial biasing parameter, the initial cross correlation and the initial variance of the total mass, respectively. The variance $\sigma_0$ is related to the initial power spectrum $P(k)$ as

$$\sigma_0^2(R) = \int \frac{d^3k}{(2\pi)^3} \tilde{W}_R^2(kR) \tilde{P}(k) \quad \text{;} \quad \tilde{W}(kR) = \frac{2}{(kR)^3} \left[ \sin(kR) - kR \cos(kR) \right], \quad (37)$$

where $\tilde{W}(kR)$ is the top-hat window function in the Fourier space. Hereafter, we simply assume that the parameters $b_0$ and $r_0$ are constant, which is consistent with the fact that there is no evidence of the scale-dependent biasing on large scales (Mann, Peacock & Heavens 1998).
3.2. Variance, covariance and joint moments

Within the above prescription, we will investigate the time evolution of the galaxy-mass density relation in the weakly non-linear regime. In the lowest order of the perturbation, the conditional means (25) and (27) are written in terms of the variance of the total mass $\sigma_m$, the biasing parameter $b$, the cross correlation $r$ and the joint moments $S_{ij}$ with $i + j = 3$.

The parameters $\sigma_m$, $b$ and $r$ are obtained from the linear order solutions of evolution equations (31), (32) and (33). The solutions satisfying the initial conditions in the previous subsection become

$$\delta_m^{(1)}(x,t) = \Delta_m(x) D(t),$$  \hspace{1cm} (38)

$$\delta_g^{(1)}(x,t) = \Delta_m(x) (D(t) - 1) + \Delta_g(x),$$  \hspace{1cm} (39)

where the function $D(t)$ denotes the solution of growing mode by setting $D(t_i) = 1$, which satisfies (Peebles 1980)

$$\ddot{D} + 2H \dot{D} - \frac{3}{2} H^2 \Omega D = 0.$$  \hspace{1cm} (40)

We have $D(t) = a(t)/a(t_i)$ in Einstein-de Sitter universe ($\Omega = 1$). Substituting (38) and (39) into (40), the time dependent parameters are evaluated from the knowledge (1) and (33) as follows (Fry 1996, Taruya, Koyama & Soda 1999, Tegmark & Peebles 1998):

$$\sigma_m(t) = \sigma_0 D,$$

$$b(t) = \sqrt{\frac{(D-1)^2 + 2b_0r_0(D-1) + b_0^2}{D}},$$  \hspace{1cm} (41)

$$r(t) = b^{-1}(t) \left( \frac{D-1 + b_0r_0}{D} \right).$$

We see that the initial cross correlation $r_0 = 1$ leads to $r = 1$. Thus, in our prescription of the time evolution, the stochasticity in the galaxy biasing comes from the initial conditions.

For the joint moments $S_{ij}$, we must calculate the second order perturbations by solving the evolution equations. In Appendix B, the solutions of second order perturbation are summarized and the computation of $S_{ij}$ is explained. The time evolution of the joint moments are parameterized by the initial conditions, $b_0$, $r_0$, $h$, and the power spectrum $P(k)$. The results in Appendix B are

$$S_{30} = \frac{34}{7} - \gamma,$$  \hspace{1cm} (42)

$$S_{21} = \frac{1}{3b(t)} \left[ \left( \frac{34}{7} - \gamma \right) \left\{ \frac{D^2 - 1}{D^2} + \frac{2(D - 1 + b_0r_0)}{D} \right\} \right.$$  

$$\left. + (6 - \gamma) \frac{(D - 1)(b_0r_0 - 1)}{D^2} + h I_W \frac{(b_0r_0)^2}{D^2} \right].$$  \hspace{1cm} (43)
\[ S_{12} = \frac{1}{3b^2(t)} \left[ \left( \frac{34}{7} - \gamma \right) \left( \frac{(D - 1 + b_0r_0)^2}{D^2} + \frac{2(D^2 - 1)(D - 1 + b_0r_0)}{D^3} \right) \right. \\
+ (6 - \gamma) \left\{ \frac{2(D - 1)^2(b_0r_0 - 1)}{D^3} + \frac{(D - 1)((b_0r_0 - 1)^2 + b_0^2)}{D^3} \right\} \\
+ h I_W \frac{2b_0r_0(D - 1 + b_0^2)}{D^3} \right], \quad (44) \]

\[ S_{03} = \frac{1}{b^3(t)} \left[ \left( \frac{34}{7} - \gamma \right) \left( \frac{(D^2 - 1)(D - 1 + b_0r_0)^2}{D^4} \right) \\
+ (6 - \gamma) \frac{(D - 1)(D - 1 + b_0r_0)}{D^4} \left\{ (b_0r_0 - 1)D + b_0^2 + 1 - 2b_0r_0 \right\} \\
+ h I_W \frac{b_0r_0(D - 1 + b_0^2)^2}{D^4} \right], \quad (45) \]

where the numerical value \( I_W \), which comes from the non-Gaussian initial distribution of the galaxies, is defined by

\[
I_W = \sigma_0^{-4} \int \frac{d^3k_1 d^3k_2}{(2\pi)^6} \tilde{W}_R(k_1 + k_2|R) \tilde{W}_R(k_1R) \tilde{W}(k_2R) P(k_1) P(k_2). 
\]

It has been checked by the Monte Carlo integration that \( I_W \) is almost equal to unity. Hence, we can regard \( h \) as the initial skewness of the galaxy distribution. The variable \( \gamma \) is given by

\[
\gamma = -\frac{d}{d(\log R)} \cdot [\log \sigma_0^2(R)].
\]

For the power spectrum with the single power-law \( P(k) \propto k^n \), we have \( \gamma = n + 3 \).

By definitions (10) and (16), the moment \( S_{30} \) is identical with the skewness of the total mass \( S_{3,m} \) (see (3)). The skewness of the galaxy distribution \( S_{3,g} \) is related to the moment \( S_{03} \) as \( S_{3,g} = S_{03}/b(t) \). In the case of \( r_0 = 1 \), using these facts, and equations (42)-(45), we can get the relation

\[
c_2 = b(S_{21} - rS_{30}) = \frac{b}{3}(bS_{3,g} - S_{3,m}).
\]

Thus, in the deterministic case, the coefficients \( c_1 \) and \( c_2 \) can coincide with \( b \) and \( b_2 \) given by equations (2) and (13). It should be recognized that the galaxy-mass density relation defined by equation (2) does not coincide with the conditional mean \( \langle \delta_g \rangle|_{\delta_m} \) in general.

Using the results (42)-(45), we have another important conclusion for the galaxy-mass density relation. We can obtain

\[
S_{21} - S_{30} = -(S_{12} - S_{03}),
\]

for \( r_0 = 1 \). Taking \( \delta_g = b\delta_m \), this means that the approximation of \( \delta_m \) by the perturbative inversion of \( \langle \delta_g \rangle|_{\delta_m} \) coincides with the conditional mean \( \langle \delta_m \rangle|_{\delta_g} \) (eq. [27]). However, if we take into
account the higher order corrections, we can expect that the conditional means \( \langle \delta_g \rangle |_{\delta_m} \) and \( \langle \delta_m \rangle |_{\delta_g} \) does not become equivalent even in \( r_0 = 1 \) case, because of the higher order contribution to the linear order coefficient of the conditional means (see eq. 30 below).

### 3.3. Evolution of \( \langle \delta_g \rangle |_{\delta_m} \)

Now consider the time evolution of the conditional mean \( \langle \delta_g \rangle |_{\delta_m} \) to see the influence of the initial stochasticity on the galaxy-mass density relation. For brevity, we only describe the analysis in the Einstein-de Sitter universe (\( \Omega_0 = 1 \)).

Using the results (31), (32) and (33), the time evolution of the coefficients \( c_1 \) and \( c_2 \) can be examined. In Fig.1, we plot the coefficients \( c_1 \) and \( c_2 \) given by equation (26) as a function of the expansion factor \( a(t) \). As the initial parameters, we chose \( b_0 = 2.0, \ r_0 = 0.8 \) and \( h = 3.0 \) at \( a(t_i) = 1 \) in Fig.1a. The solid line shows the coefficient \( c_1 \). The long-dashed, the short-dashed and the dotted lines denote the time evolution of the coefficient \( c_2 \) with the spectral index \( n = -2, -1.5 \) and \(-1 \) for the single power-law \( P(k) \propto k^n \), respectively. The parameters in Fig.1b is the same as Fig.1a, except for the initial cross correlation for which we take \( r_0 = 0.1 \). These figures show that the coefficients \( c_1 \) and \( c_2 \) approach to unity and zero respectively, independent of the initial parameters and the spectral index. This asymptotic behavior can be ascribed to the attractivity of the gravitational force (Fry 1996, Tegmark & Peebles 1998, Taruya, Koyama & Soda 1999). Fig.1c shows the illustrative example with the initial parameters \( b_0 = 4.63, \ r_0 = 0.2 \) and \( h = 6.96 \) at \( a(t_i) = 1 \). We set the spectral index \( n = -1.41 \). If we identify the initial time \( a = 1 \) with the redshift parameter \( z = 3 \) assuming the Einstein-de Sitter universe, the skewness and the bi-spectrum at present time (which corresponds to \( a = 4 \) in our case) provide the consistent results with the observation of the Lick catalog (Taruya, Koyama & Soda 1999). Although the initial value of the coefficient \( c_2 \) is comparable to that of \( c_1 \) because of the large initial skewness \( h \), it rapidly decreases and becomes negligible due to the small initial cross correlation \( r_0 \). As a demonstration, we evaluate the galaxy-mass density relation \( \langle \delta_g \rangle |_{\delta_m} \) in the \( (\delta_g, \delta_m) \)-plane with the same initial parameters as Fig.1c. The galaxy-mass relation is plotted in Fig.2 by choosing the initial variance of the total mass \( \sigma_0 = 0.1 \). Each line in Fig.2 represents the snapshot at \( a = 1 \) (solid line), \( a = 2 \) (long-dashed line), \( a = 4 \) (short-dashed line) and \( a = 8 \) (dotted line), respectively.

Fig.1 and Fig.2 say that the coefficient \( c_2 \) is usually smaller than the linear coefficient \( c_1 \). We can also confirm this fact for various initial parameters. This result means that the non-linearity in the conditional mean is negligible on large scales. Therefore the result indicates that the relation between the galaxy and the total mass distribution is well-approximated by the linear relation \( \delta_g = b r \delta_m \). If we apply this fact to the estimation of the higher order moments for galaxy distribution, we expect that the moments of the galaxy are simply related to those of the total mass multiplied by the factor inferred from the linear relation. That is, regarding \( \langle \delta_g \rangle |_{\delta_m} \) as \( \delta_g \), we obtain

\[
\langle (\delta_g)^n \rangle \simeq (b r)^n \langle (\delta_m)^n \rangle.
\] (46)
However, we should keep in mind that there exists the scatter in the galaxy-mass relation. In section 2.2, we have defined the biasing scatter $\epsilon$ (see eq.[12]). In the tree-level analysis, the variance of the biasing scatter becomes

$$\frac{\langle \epsilon^2 \rangle}{\sigma_m^2} = (1 - r^2) + \mathcal{O}(\sigma_m^2).$$

(47)

The variance vanishes only when $r_0 = 1$. We can also calculate the third order moment. The result up to the second order perturbation is

$$\frac{\langle \epsilon^3 \rangle}{\sigma_m^4} = b^3 \left[ S_{03} - r S_{30} - 3 r (S_{12} - r S_{21}) \right] + \mathcal{O}(\sigma_m^2),$$

(48)

where we used $c_1 = b r$. Substituting the expressions (42)-(45) and (41) into equation (48), it is easily shown that the third order moment of the biasing scatter also vanishes in the deterministic case, $r_0 = 1$. From equations (47) and (48), the non-Gaussian scatter is expected to affect the simple relation of the higher order moments (46).

To see the influence of the biasing scatter on the evaluation of higher order moments, we examine the skewness of the galaxy distribution. In Fig.3, we plot the time evolution of $S_{3,g}$ as a function of the expansion factor. For each figure, the solid line is the correct skewness obtained from the joint moment $S_{03}$ divided by $b(t)$ and the dashed line represents the skewness $\tilde{S}_{3,g}$ deduced from the galaxy-mass density relation $\langle \delta_g \rangle|_{\delta_m}$. $\tilde{S}_{3,g}$ can be evaluated by equating the conditional mean $\langle \delta_g \rangle|_{\delta_m}$ with $\delta_g$. The non-linear relation between $\delta_m$ and $\delta_g$ (25) implies

$$\tilde{S}_{3,g} = \frac{1}{c_1} (S_{3,m} + 3 c_2),$$

(49)

which is valid within the tree-level approximation (Fry & Gaztañaga 1993). We have already obtained the skewness of the total mass $S_{3,m} = \frac{34}{7} - \gamma$ and found that $S_{3,m}$ is usually larger than the coefficient $3(c_2/c_1)$. Fig.3a and Fig.3b have the same initial parameters as Fig.1a and Fig.1b, respectively, except for the power spectrum specified as the index $n = -2$. As we know from the previous subsection, $\tilde{S}_{3,g}$ exactly coincides with the skewness $S_{3,g}$ only in the deterministic case, i.e, $r_0 = 1$. The figures show that $\tilde{S}_{3,g}$ differs from $S_{3,g}$ in the presence of stochasticity clearly. For the smaller initial cross correlation $r_0$, the deviation of $S_{3,g}$ from $\tilde{S}_{3,g}$ becomes more significant (Fig.3b). Because of the small contribution of the non-linear coefficient $c_2$, $\tilde{S}_{3,g}$ approaches to the skewness of the total mass $S_{3,m} = 3.86$ more rapidly than $S_{3,g}$ (see eq. [49]). For the same parameterization as used in Fig.1c, Fig.3c shows that $\tilde{S}_{3,g}$ at present ($a = 4$) underestimates the correct value of the skewness for the galaxy distribution due to the rapid relaxation to $S_{3,m}$. At $a = 4$, the skewness $S_{3,g}$ from Fig.3c becomes 4.55, while we have $\tilde{S}_{3,g} = 3.50$. The difference cannot be neglected. Thus there exist the cases that the galaxy-mass density relation as a conditional mean leads to incorrect result. This feature is common to the stochastic biasing.
4. Conclusion

In this paper, we have analytically studied the galaxy-mass density relation in the framework of the stochastic biasing. Under the assumptions (3) and (4), in the weakly non-linear regime, we derived a general formula for the galaxy-mass density relation as a conditional mean. The main result in our analysis is equations (25) and (26). We have seen that the higher order contribution to the weakly non-linear galaxy-mass density relation can shift the coefficient of the linear galaxy-mass density relation, in addition to the non-linear term proportional to $\delta^3_m$ (see eq.[30]). This agrees with the analyses of the two point correlation function (Scherrer & Weinberg 1998).

Using the formula for the conditional mean, we have further investigated the time evolution of the galaxy-mass density relation. To develop the analysis, we have made the assumptions: (i) the galaxy formation and the merging process can be ignored; (ii) the initial distribution of galaxies is given by the local function (eq.[35]), while the total mass has the non-local initial condition. Our conclusions can be summarized as follows:

- The conditional mean could be different from the deterministic biasing relation (compare the relation [14] and [28]). In the presence of the scatter, the perturbative inversion of $\langle \delta_g \rangle|_{\delta_m}$ does not recover the conditional mean $\langle \delta_m \rangle|_{\delta_g}$.

- The time evolution of the conditional mean $\langle \delta_g \rangle|_{\delta_m}$ shows that the non-linear term of the conditional means usually becomes negligible. This suggests that the galaxy-mass density relation could be approximated by the linear relation.

- We have found that the time evolution of the skewness deduced from the conditional mean $\langle \delta_g \rangle|_{\delta_m}$ differs from that of the correct skewness $S_{3,g}$ in the case of the small cross correlation. This indicates that the stochasticity could have an important role in the estimation of the higher order moment of galaxies using the conditional mean.

The assumptions (i) and (ii) might not be valid in more realistic situation. The merging process might become important in a real universe and the galaxy formation would introduce the non-local initial condition. Nevertheless, we believe that our qualitative conclusions will not be altered even if these processes are taken into account. It would be interesting to incorporate these effects into the time evolution of the stochastic biasing. We will attack this issue in the future.

The conditional mean $\langle \delta_g \rangle|_{\delta_m}$ is an important quantity to construct the analytic biasing model which is determined by the clustering properties of the halos (Mo & White 1996, Mo, Jing & White 1997, Catelan, Matarrese & Porciani 1998, Sheth & Lemson 1998). In the presence of the stochasticity, we must treat the conditional mean carefully when comparing it with the observation of the galaxy statistics, and vice versa. That is, the linear and non-linear biasing parameter ($b$, $b_2$) estimated from the observation of the skewness or kurtosis does not necessarily give the mean biasing relation. They might provide a signal for the stochastic biasing. In the previous paper (Taruya, Koyama & Soda 1999), we found that the parameter $b_2$ deduced from
the skewness can become negative in the presence of the scatter. The recent observation from the Southern Sky Redshift Survey shows that the non-linear biasing parameter $b_2$ estimated by the skewness may become negative for the biasing parameter $b > 1$ \cite{Benoist98}. This might give an observational evidence for the stochastic biasing as long as our prescription is correct.

The stochasticity is problematic when we get the relation between the galaxies and the total mass from the observation. As discussed by Dekel & Lahav \cite{Dekel98}, the situation may become more complicated in the redshift space. This makes it difficult to determine the cosmological parameter from the observation of the velocity field such as POTENT \cite{Bertschinger89}. For the analysis in the redshift space, Scoccimarro, Couchman & Frieman \cite{Scoccimarro98} studied the influence of the non-linear biasing on the bi-spectrum of the galaxies in the deterministic case. We must investigate how the stochasticity and the non-linearity affect the galaxy biasing in the redshift space. In particular, the higher order statistics such as the skewness and the bi-spectrum should be explored. Extension of our formalism to the redshift space is straightforward and the analysis is now going on \cite{Taruya99}.

Alternative approach to understand the stochastic property of the galaxy biasing is to seek the physical origin of the stochastic biasing itself. Blanton \textit{et al.} \cite{Blanton98} explored the hidden variable to reduce the stochasticity in the relation between galaxies and total mass using the hydrodynamical simulation. They found that the scatter around the conditional mean of the galaxies becomes small if the temperature dependence is taken into account. To combine this approach with ours is also interesting.

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Appendix A: Calculation of Edgeworth series

In this appendix, we explain the calculation for the derivation of the galaxy-mass density relation as a conditional mean in section 2.3.

We start to write down the generating function $W$ expanded in powers of $\sigma_m$. From the scaling relation (16), the generating function up to the order $O(\sigma_m^3)$ is obtained in terms of the normalized variables (18):

$$W(\mu, \nu) = -\frac{1}{2} (\mu^2 + 2\lambda_{11}\mu\nu + \nu^2)$$

$$+ \frac{1}{6} \left\{ \lambda_{30}(i\mu)^3 + 3\lambda_{21}(i\mu)^2(\nu) + 3\lambda_{12}(i\mu)(i\nu)^2 + \lambda_{03}(i\nu)^3 \right\}$$

$$+ \frac{1}{24} \left\{ \lambda_{40}(i\mu)^4 + 4\lambda_{31}(i\mu)^3(i\nu) + 6\lambda_{22}(i\mu)^2(\nu)^2 + 4\lambda_{13}(i\mu)(i\nu)^3 + \lambda_{04}(i\nu)^4 \right\}$$

$$+ O(\sigma_m^3).$$

Note that the first line in the right hand side of (11) is of the order of unity. The second and the third lines are the terms of $O(\sigma_m)$ and $O(\sigma_m^2)$, respectively. Using the expression (11), the conditional mean $\langle \delta_g | \delta_m \rangle$ given by equation (17) is calculated in the following way. For the numerator $A$, we rewrite the expression (19) to

$$A = \frac{i}{2\pi} \left( \frac{\langle \delta_g^2 \rangle}{\langle \delta_m^2 \rangle} \right)^{1/2} \int \int d\mu d\nu \frac{\partial}{\partial \nu} \delta_D(\nu) \cdot e^{-i\mu x + W(\mu, \nu)}. \quad (2)$$

Integrating by part and substituting the expansion (11) into equation (2), we get

$$A = -\frac{i}{2\pi} \left( \frac{\langle \delta_g^2 \rangle}{\langle \delta_m^2 \rangle} \right)^{1/2} \int d\mu \left. e^{-i\mu x + W(\mu, 0)} \frac{\partial}{\partial \nu} W(\mu, \nu) \right|_{\nu = 0},$$

$$= -\frac{i}{2\pi} \left( \frac{\langle \delta_g^2 \rangle}{\langle \delta_m^2 \rangle} \right)^{1/2} \int d\mu e^{-i\mu x - \mu^2/2} \left\{ 1 + \frac{\lambda_{30}}{6} (i\mu)^3 + \left( \frac{\lambda_{40}}{24} (i\mu)^4 + \frac{\lambda_{31}^2}{72} (i\mu)^6 \right) + O(\sigma_m^3) \right\}$$

$$\times \left\{ -\lambda_{11}\mu + \frac{\lambda_{21}}{2} (i\mu)^2 + \frac{\lambda_{31}}{6} (i\mu)^3 + O(\sigma_m^3) \right\}.$$

Keeping the terms up to $O(\sigma_m^2)$, this becomes

$$A = \frac{1}{2\pi} \left( \frac{\langle \delta_g^2 \rangle}{\langle \delta_m^2 \rangle} \right)^{1/2} \left[ \lambda_{11} \left( -\frac{d}{dx} \right) + \frac{\lambda_{21}}{2} \left( -\frac{d}{dx} \right)^2 + \frac{\lambda_{11}\lambda_{30}}{6} \left( -\frac{d}{dx} \right)^4 \right]$$

$$+ \left( \frac{\lambda_{11}\lambda_{40}}{24} + \frac{\lambda_{30}\lambda_{21}}{12} \right) \left( -\frac{d}{dx} \right)^5 + \lambda_{11} \lambda_{30} \left( -\frac{d}{dx} \right)^7 + O(\sigma_m^3) \right] \int d\mu e^{-i\mu x - \mu^2/2},$$

where we replaced $(i\mu)^n$ with $\left( -\frac{d}{dx} \right)^n$. Performing the integration of $\mu$ and the differentiation with respect to $x$, we obtain the expression (21).
The calculation of the denominator in equation (17) is similar to that of the numerator. Substitution of the expansion (1) into equation (20) yields
\[
\mathcal{P}(\delta_m) = \frac{1}{2\pi \sqrt{\langle \delta_m^2 \rangle}} \int \! d\mu \ e^{-i\mu x - \mu^2/2} \left[ 1 + \frac{\lambda_{30}}{6} (i\mu)^3 + \left\{ \frac{\lambda_{40}}{24} (i\mu)^4 + \frac{\lambda_{30}^2}{72} (i\mu)^6 \right\} + \mathcal{O}(\sigma_m^3) \right].
\]
Repeating the same manipulation as in the above, we have
\[
\mathcal{P}(\delta_m) = \frac{1}{2\pi \sqrt{\langle \delta_m^2 \rangle}} \left[ 1 + \frac{\lambda_{30}}{6} \left( -\frac{d}{dx} \right)^3 \right] + \left\{ \frac{\lambda_{40}}{24} \left( -\frac{d}{dx} \right)^4 + \frac{\lambda_{30}^2}{72} \left( -\frac{d}{dx} \right)^6 \right\} + \mathcal{O}(\sigma_m^3) \] 
\[ \times e^{-x^2/2}. \]
Using the definition of the Hermite polynomials (23), we obtain the expression (22).

Appendix B: Second order perturbation and joint moments

When we investigate the time evolution of the conditional mean \( \langle \delta_g \rangle | \delta_m \) and \( \langle \delta_m \rangle | \delta_g \) in the tree-level analysis, in addition to the biasing parameter \( b \) and the cross correlation \( r \), we need the joint moments \( S_{ij} \) with \( i + j = 3 \):
\[
S_{30} = 3 \frac{\langle \delta_m^{(1)} \delta_g^{(2)} \rangle}{\langle \delta_m^{(1)} \rangle^2}, \quad S_{21} = \frac{\langle \delta_m^{(1)} \delta_g^{(2)} \rangle + 2 \langle \delta_m^{(1)} \delta_g^{(1)} \rangle}{b(t) \langle \delta_m^{(1)} \rangle^2}.
\]
\[
S_{12} = \frac{\langle \delta_m^{(2)} \delta_g^{(2)} \rangle + 2 \langle \delta_m^{(1)} \delta_g^{(1)} \rangle}{b^2(t) \langle \delta_m^{(1)} \rangle^2}, \quad S_{03} = 3 \frac{\langle \delta_g^{(1)} \delta_g^{(2)} \rangle}{b^3(t) \langle \delta_m^{(1)} \rangle^2}.
\]
(1)
If we evaluate these quantities for the smoothed density fields, the integration including the window function \( W_R(x) \) must be computed. Although the integration in real space is difficult for the higher order moments, it is known that the calculation is tractable in the Fourier space because of the useful formulae (Bernardeau 1994). Using these formulae, we shall write down the solutions of second order perturbation in the Fourier space. The variables \( \delta_m, \delta_g \) and \( \theta \) are expanded as
\[
\delta_{m,g}(x,t) = \int \frac{d^3k}{(2\pi)^3} \hat{\delta}_{m,g}(k,t) e^{-ikx}, \quad \theta(x,t) = \int \frac{d^3k}{(2\pi)^3} \hat{\theta}(k,t) e^{-ikx}.
\]
Then the second order solutions satisfying the initial conditions are obtained (Fry 1984, Taruya, Koyama & Soda 1999):
\[
\hat{\delta}_m^{(2)}(k,t) = \int \frac{d^3k'}{(2\pi)^3} \left[ D^2(t) \left( \frac{6}{7} \mathcal{R}(k',k-k') + \frac{1}{7} \mathcal{R}(k-k',k') - \frac{3}{2} \mathcal{L}(k',k-k') \right) \right. \]
\[ + \left. \frac{3}{4} E(t) \mathcal{L}(k',k-k') \right] \hat{\Delta}_m(k-k') \hat{\Delta}_m(k'),
\]
(2)
\[
\hat{\delta}_g^{(2)}(k,t) = \hat{\delta}_m^{(2)}(k,t) - \hat{\delta}_m^{(2)}(k,t_i) \]
\[ + \left( D(t) - 1 \right) \int \frac{d^3k'}{(2\pi)^3} \mathcal{R}(k',k-k') \left( \hat{\Delta}_m(k') \hat{\Delta}_g(k-k') - \hat{\Delta}_m(k') \hat{\Delta}_m(k-k') \right) \]
\[ + \frac{h}{6} \int \frac{d^3k'}{(2\pi)^3} \left( \hat{\Delta}_g(k') \hat{\Delta}_g(k-k') - \hat{\Delta}_g^{(2)}(k') \right),
\]
(3)
where
\[ R(k_1, k_2) = 1 + \frac{(k_1 \cdot k_2)}{|k_1|^2}, \quad \mathcal{L}(k_1, k_2) = 1 - \frac{(k_1 \cdot k_2)^2}{|k_1|^2|k_2|^2}. \]

The solutions (2) and (3) contain the function \( E(t) \) which satisfies \( E(t_i) = 1 \). This is the inhomogeneous solution of the following equation:
\[ \ddot{E} + 2H \dot{E} - \frac{3}{2} H^2 \Omega E = 3H^2 \Omega D^2 + \frac{8}{3} \dot{D}^2. \]

In Einstein-de Sitter universe, we have
\[ E(t) = \frac{34}{21} D^2(t). \]

It is known that the \( \Omega \) and \( \Lambda \) dependences of \( E/D^2 \) is extremely weak (Bernardeau 1994). Therefore, we proceed to evaluate \( S_{ij} \) by replacing \( E(t) \) with \( (34/21)D^2(t) \). Substituting the solutions (2) and (3) into the expression (1) and using the formulae below, we get the results (42)-(45):
\[
\int \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} R(k_1, k_2) P(k_1) P(k_2) \tilde{W}_R(|k_1 + k_2| R) \tilde{W}(k_1 R) \tilde{W}(k_2 R) = \sigma_0^4(R) \left( 1 - \frac{\gamma}{6} \right),
\]
\[
\int \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \mathcal{L}(k_1, k_2) P(k_1) P(k_2) \tilde{W}_R(|k_1 + k_2| R) \tilde{W}(k_1 R) \tilde{W}(k_2 R) = \frac{2}{3} \sigma_0^4(R),
\]

where
\[ \gamma = -\frac{d}{d(\log R)} [\log \sigma_0^2(R)]. \]

\( \tilde{W}_R(kR) \) and \( \sigma_0 \) are the window function in the Fourier space and the initial variance of the total mass defined by equation (37), respectively.
REFERENCES

Benoist, C., Cappi, A., da Costa, L.N., Maurogordato, S., Bouchet, F.R., & Schaeffer, R., 1998, astro-ph/9809080.

Bernardeau, F., 1994, ApJ, 433, 1.

Bernardeau, F., & Kofman, L., 1995, ApJ, 443, 479.

Bertschinger, E., & Dekel, A., 1989, ApJ, 336, L5.

Blanton, M., Cen, R., Ostriker, J.P., & Strauss, M.A., astro-ph/9807028.

Catelan, P., Matarrese, S., & Porciani, C., 1998, ApJ, 502, L1.

Cen, R., & Ostriker, J.P., 1992, ApJ, 399, L113.

Chodorowski, M.J., & Lokas, E.L., 1997, MNRAS, 287, 591; astro-ph/9606088.

Chodorowski, M.J., Lokas, E.L., Pollo, A., & Nusser, A., 1998, astro-ph/9802050.

Dekel, A., & Lahav, O., 1998, astro-ph/9806193.

Fry, J.N., 1984, ApJ, 279, 499.

Fry, J.N., & Gaztañaga, E., 1993, ApJ, 413, 447.

Fry, J.N., 1994, Phys.Rev.Lett, 73, 215.

Fry, J.N., 1996, ApJ, 461, L65.

Gaztañaga, E., & Frieman, J.A., 1994, ApJ, 437, L13.

Hamilton, A.J.S., 1997, in Proceedings of Ringberg Workshop on Large-Scale Structure, Hamilton, D.(ed.), Kluwer Academic, Dordrecht.

Juszkiewicz, R., Weinberg, D.H., Amsterdamski, P., Chodorowski, M., & Bouchet, F.R., 1995, ApJ, 442, 39.

Magliocchetti, M., Maddox, S.J., Lahav, O., & Wall, J.V., 1998, astro-ph/9806342.

Mann, B., Peacock, J., & Heavens, A., 1998, MNRAS, 293, 209.

Mo, H.J., & White, S.D.M., 1996, MNRAS, 282, 347.

Mo, H.J., Jing, Y.P., & White, S.D.M., 1997, MNRAS, 284, 189.

Peacock, J.A., Jimenez, R., Dunlop, J.S., Waddington, I., Spinrad, H., Stern, D., Dey, A, & Windhorst, R.A., 1998, MNRAS, 296, 1089.
Peebles, P.J.E., 1980, *The Large-Scale Structure of the Universe* (Princeton U.P., Princeton)

Pen, U., 1998, ApJ, 504, 601.

Scoccimarro, R., Couchman, H.M.P., & Frieman, J.A., 1998, astro-ph/ 9808305 (ApJ, in press).

Scherrer R.J., & Weinberg, D.H., 1998, ApJ, 504, 607.

Sheth, R.K., & Lemson, G., 1998, astro-ph/ 9808138

Taruya, A., Koyama., K & Soda, J., 1999, ApJ, 510, 541.

Taruya, A., & Soda, J., 1999, work in progress.

Tegmark, M., & Peebles, P.J.E., 1998, ApJ, 500, L79.
Figure Caption

**Fig.1** The time evolution of the coefficients $c_1$ and $c_2$ is evaluated as a function of the expansion factor $a(t)$ in the case of the Einstein-de Sitter universe. For each figure, the solid line shows the coefficient $c_1$. The long-dashed, the short-dashed and the dotted lines in Fig.1a and 1b denote the time evolution of $c_2$ with the spectral index $n = -2, -1.5$ and $-1$, respectively. The dashed line in Fig.1c also shows the coefficient $c_2$, but we set the spectral index $n = -1.41$. The initial parameters for each figure are as follows: (a) $b_0 = 2.0$, $r_0 = 0.8$ and $h = 3.0$; (b) $b_0 = 2.0$, $r_0 = 0.1$ and $h = 3.0$; (c) $b_0 = 4.63$, $r_0 = 0.2$ and $h = 6.96$. Note that the parameters in Fig.1c gives the same skewness and the bi-spectrum as obtained from the Lick catalog data at $a = 4$ if the universe is assumed to be the Einstein-de Sitter universe (see section 3.3).

**Fig.2** The time evolution of the galaxy-mass density relation $\langle \delta_g \rangle |\delta_m|$ is plotted in $(\delta_g, \delta_m)$-space. The initial parameters are the same as Fig.1c. Each line represents the snapshot evaluated at $a = 1$ (solid line), $a = 2$ (long-dashed line), $a = 4$ (short-dashed line) and $a = 8$ (dotted line).

**Fig.3** The time evolution of the skewness for the galaxy distribution is shown as a function of the expansion factor $a$ in the Einstein-de Sitter universe. The solid line is the joint moment $S_{03}$ multiplied by the factor $1/b(t)$, which is identical to the correct value of the skewness $S_{3,g}$. The dashed line denotes the skewness $\tilde{S}_{3,g}$ deduced from the galaxy-mass density relation (25), which is evaluated from equation (49). Fig.3a and 3b respectively have the same initial parameters as Fig.1a and Fig.1b, except for the spectral index specified as $n = -2$. As for Fig.3c, the initial parameters are the same parameters as Fig.1c, which gives the same results as obtained from the Lick catalog data if we identify the initial time $a(t) = 1$ with the redshift parameter $z = 3$ in the Einstein-de Sitter universe.
Fig. 1a
Fig. 1b
Fig. 2
Fig. 3a
Fig. 3b
Fig. 3c