SURVEY ON HOMOLOGICAL FLIPS AND HOMOLOGICAL FLOPS

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Abstract. We give a survey for the results in [13, 14, 15], which attempt s to relate the derived categories under general classes of flips and flops. We indicate how the approach fails because of what appears to be a formal problem. We give some ideas, and record some failed attempts, to fix this problem. We also present some new examples.

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1. Log flips

In this article, we give a survey for an approach developed in [13, 14, 15] to the problem of relating derived categories under flips and flops. The notions of flips and flops are sometimes a source of confusion because in the literature (especially those on derived categories) the terminology is sometimes used to refer to the closely related notions of $K$-dominant and $K$-equivalent birational maps. For the sake of clarity, we will recall the notions of flips and flops. In this section, we will start with the notion of a log flip (which generalizes both flips and flops), and we describe an algebraic setup associated to a log flip. In the next section, we will define flips and flops, and elaborate more on this algebraic setup.

We do not assume knowledge of birational geometry from the reader. Most notions from birational geometry that we use will be reminded. We will also add several “checkpoints” in this article that summarize some essential information of the situation, so that readers less familiar with some parts of birational geometry or homological algebra will be able to read the rest of the article even if (s)he does not follow the arguments that lead to that description.

Definition 1.1. A log flip consists of birational projective morphisms between normal varieties

\[ X^- \xrightarrow{\pi^-} Y \xleftarrow{\pi^+} X^+ \]

such that both $\pi^-$ and $\pi^+$ are small, meaning that $\text{Ex}(\pi^\pm) \subset X^\pm$ have codimension $\geq 2$, together with Weil divisors $D^+, D^-, D^0$ on $X^+, X^-, Y$ respectively, strict transforms of each other, and satisfy

1. $D^+$ is $\mathbb{Q}$-Cartier and $\pi^+$-ample.
2. $-D^-$ is $\mathbb{Q}$-Cartier and $\pi^-$-ample.

Recall that a Weil divisor $D \in \text{WDiv}(X)$ is said to be $\mathbb{Q}$-Cartier if $mD$ is Cartier for some $m \in \mathbb{Z} \setminus \{0\}$. This allows one to discuss (relative) ampleness of $D$. By replacing them with a suitable multiple, one
may take $D^\pm$ to be Cartier. However, notice that $D^0$ is never Cartier (unless when $\pi^{\pm}$ are the identity maps).

Since $\pi^{\pm}$ are small, we have

$$(1.2) \quad \pi^{\pm}_*\mathcal{O}_{X^{\pm}}(iD^\pm) = \mathcal{O}_Y(iD^0) \quad \text{for any } i \in \mathbb{Z}$$

Let $\mathcal{A}$ be the sheaf of $\mathbb{Z}$-graded rings over $Y$ defined by $\mathcal{A} = \bigoplus_{i \in \mathbb{Z}} \mathcal{O}_Y(iD^0)$, then by (1.2) and by relative ampleness of $D^+$ and $-D^-$, we have

$$(1.3) \quad X^+ = \text{Proj}_Y^+(\mathcal{A}) := \text{Proj}_{\mathcal{O}_Y}(\mathcal{A}_{\geq 0}) \quad \text{and} \quad X^- = \text{Proj}_Y^-(\mathcal{A}) := \text{Proj}_{\mathcal{O}_Y}(\mathcal{A}_{\leq 0})$$

Moreover, the reflexive sheaves $\mathcal{O}_{X^{\pm}}(iD^\pm)$ also admit a description in terms of $\mathcal{A}$. Namely, denote by $\text{Gr}(\mathcal{A})$ the category of quasi-coherent sheaves of graded $\mathcal{A}$-modules. Then any $M \in \text{Gr}(\mathcal{A}_{\geq 0})$ is also an object in $\text{Gr}(\mathcal{A}_{\geq 0})$, and hence determines an associated sheaf $\tilde{M} \in \text{Qcoh}(\text{Proj}_Y^+(\mathcal{A}))$. The same is true for the negative direction. Then we have

$$\mathcal{O}_{X^{\pm}}(iD^\pm) \cong \tilde{A}(i) \quad \text{under the identifications (1.3)}.$$

Everything we discuss here is local over $Y$, so we may assume that $Y$ is affine, in which case we summarize the essential information in the following

**Checkpoint 1.4.** Every log flip is locally described by a Noetherian $\mathbb{Z}$-graded ring $A$. Namely, we have $R = A_0$, $Y = \text{Spec } R$, $X^+ = \text{Proj}^+(A)$. Let $\mathcal{O}_{X^\pm}(i) := \tilde{A}(i)$, then we have $\pi^{\pm}_*(\mathcal{O}_{X^{\pm}}(i)) = A_i$.

Here we mentioned that $A$ is Noetherian. It is relevant to our situation because of the following result (see, e.g., [2, Theorem 1.5.5]):

**Proposition 1.5.** Let $A$ be a $\mathbb{Z}$-graded ring. The following are equivalent:

1. $A$ is a Noetherian ring;
2. every graded ideal of $A$ is finitely generated;
3. $A_0$ is Noetherian, and both $A_{\geq 0}$ and $A_{\leq 0}$ are finitely generated over $A_0$;
4. $A_0$ is Noetherian, and $A$ is finitely generated over $A_0$.

We wish to study the derived category of $X^\pm$ in terms of the commutative algebra of $A$. The relation is obtained by Serre’s equivalence. We have already seen above that there is a functor $(-)^- : \text{Gr}(A) \to \text{Qcoh}(X^+)$.

Let $\text{Tor}^+ \subset \text{Gr}(A)$ be the subcategory consisting of graded modules $M$ such that for every $x \in M$ there exists $s \in \mathbb{N}$ such that $x \cdot (A_{>0})^s = 0$. Then the functor $(-)^- : \text{Gr}(A)/\text{Tor}^+ \to \text{Qcoh}(X^+)$

$$(1.6) \quad (-)^- : \text{Gr}(A)/\text{Tor}^+ \to \text{Qcoh}(X^+)$$

The usual statement of Serre equivalence asserts that the functor (1.6) is an equivalence if $A$ is non-negatively graded and generated in degree 0 and 1. If one sharpen each step of the usual proof, one have the following stronger version:

**Theorem 1.7** ([15, Theorem 3.15]). Let $\mathcal{O}_{X^+}(i) := \tilde{A}(i)$ as above. Assume that the natural map

$$(1.8) \quad \mathcal{O}_{X^+}(i) \otimes_{\mathcal{O}_{X^+}} \mathcal{O}_{X^+}(j) \to \mathcal{O}_{X^+}(i+j)$$

is an isomorphism for any $i, j \in \mathbb{Z}$. Then the functor (1.6) is an equivalence.

In the case of log flips, the assumption of this Theorem is satisfied if and only if $D^+$ is Cartier. Motivated by this example, we will say that a Noetherian $\mathbb{Z}$-graded ring $A$ is positively Cartier if (1.8) is an isomorphism for any $i, j \in \mathbb{Z}$.

At the level of derived categories, Serre’s equivalence can be described in terms of a semi-orthogonal decomposition. More precisely, there is a local cohomology SOD (so called because of (1.10)):

$$(1.9) \quad D(\text{Gr}(A)) = \langle D_{\text{triv}}(\text{Gr}(A)), D_{\text{Tor}^+}(\text{Gr}(A)) \rangle$$

1Sometimes one would like to think of $D^\pm$ as part of the structure of a log flip, in which case one will have to tackle the case when $D^\pm$ are not Cartier.
Here, $D(\text{Gr}(A))$ is the unbounded derived category of (not necessarily finitely generated) graded modules. $D_{\text{Tor}^+}(\text{Gr}(A)) \subset D(\text{Gr}(A))$ is the full triangulated subcategory consisting of objects $M \in D(\text{Gr}(A))$ such that each cohomology module $H^i(M)$ is in the Serre subcategory $\text{Tor}^+$. The inclusion $i : D_{\text{Tor}^+}(\text{Gr}(A)) \hookrightarrow D(\text{Gr}(A))$ has a right adjoint denoted as $R \Gamma_{I^+}$, whose kernel is by definition $D_{I^+ - \text{triv}}(\text{Gr}(A))$.

As the notation suggests, $R \Gamma_{I^+}(M)$ is the local cohomology complex with respect to the graded ideal $I^+ := A_{>0} \cdot A$. If we denote the decomposition sequence of the SOD $\text{(1.9)}$ by

\[ \ldots \to R \Gamma_{I^+}(M) \xrightarrow{\xrightarrow{D}} \check{C}_{I^+}(M) \to \ldots \]

then $\check{C}_{I^+}(M)$ can be described as a certain Čech complex, while $R \Gamma_{I^+}(M)$ can be described as a certain extended Čech complex. Explicitly, if $I^+$ is generated by homogeneous elements $f_1, \ldots, f_r \in A_{>0}$, then they are given by

\[ \check{C}_{I^+}(M) := \left[ \prod_{1 \leq i_0 \leq r} M_{f_{i_0}} \to \prod_{1 \leq i_0 < i_1 \leq r} M_{f_{i_0}f_{i_1}} \to \ldots \to M_{f_1 \ldots f_r} \right] \]

\[ R \Gamma_{I^+}(M) := \left[ M \to \prod_{1 \leq i_0 \leq r} M_{f_{i_0}} \to \prod_{1 \leq i_0 < i_1 \leq r} M_{f_{i_0}f_{i_1}} \to \ldots \to M_{f_1 \ldots f_r} \right] \]

where the differentials are the alternating sums of restriction maps.

Here and below, if $X$ is a scheme or stack, then $D(X)$ will denote the unbounded derived category of quasi-coherent sheaves $D(X) := D(\text{QCoh}(X))$.

If $A$ is positively Cartier, then it is a formal consequence of Serre’s equivalence that the functor $(\cdot)^\sim : D(\text{Gr}(A)) \to D(X^+)$ restricts to an equivalence $D_{I^+ - \text{triv}}(\text{Gr}(A)) \to D(X^+)$. This is the basic setting we wish to exploit to pass from homological algebra on $\text{Gr}(A)$ to the derived category of $X^+$. We summarize the situation in the following

**Checkpoint 1.12.** For any Noetherian $\mathbb{Z}$-graded ring $A$, there is a local cohomology SOD described by $\text{(1.9)}, \text{(1.10)}, \text{(1.11)}$. If $A$ is positively Cartier then the semi-orthogonal component $D_{I^+ - \text{triv}}(\text{Gr}(A))$ is equivalent to $D(\text{Gr}(A)^+) := D(\text{QCoh}(X^+))$. The same is true for the negative side.

There is also a closely related stacky picture. While it will not be used in any technical way in this work, it will serve as a useful viewpoint.

Let $\mathfrak{X}$ be the base stack $\mathfrak{X} = [\text{Spec}(A)/\mathbb{G}_m]$, so that $\text{Gr}(A) \simeq \text{QCoh}(\mathfrak{X})$. Let $\mathfrak{X}^\pm \subset \mathfrak{X}$ be the open substacks given by the complements of the closed subsets $V(I^\pm) \subset \text{Spec} A$. Then $\mathfrak{X}^\pm$ are Deligne-Mumford, whose coarse moduli space is $X^\pm$. Serre’s equivalence always holds with $X^\pm$ replaced by $\mathfrak{X}^\pm$. Moreover, $A$ is positively Cartier if and only if the map $X^+ \to X^+$ is an isomorphism. Under this stacky viewpoint, the varieties $X^\pm$ (or rather their stacky versions $\mathfrak{X}^\pm$) are related by a variation of GIT quotients.

**Checkpoint 1.13.** The terms in the local cohomology sequence $\text{(1.10)}$ (at least for $M = A$) admit descriptions from two viewpoints

\[ \begin{array}{c|c|c}
A & \pi^+ : X^+ \to Y & j : X^+ \subset \mathfrak{X} \\
\hline
\check{C}_{I^+}(A) & \check{C}_{I^+}(A)_i = R\pi^+_i(\mathcal{O}_{X^+}(i)) & \check{C}_{I^+}(A) = R\pi^+_j(\mathcal{O}_{X^+}) \\
\hline
R \Gamma_{I^+}(A) & R \Gamma_{I^+}(A)_i[1] = R^3\pi^+_i(\mathcal{O}_{X^+}(i)) & \text{local cohomology along the unstable locus } V(I^+) \\
\end{array} \]

Here and below, a subscript $(-)_i$ will always mean its weight grading $i$ part. Thus, given a complex $M$ of graded $A$-modules, then $M_i$ is a complex of $R$-module, where $R = A_0$. This gives a functor $(-)_i : D(\text{Gr}(A)) \to D(R)$. This explains the meaning of the left column of this table. For example, the middle term means that $\check{C}_{I^+}(A)_i \in D(R)$ and $R \pi^+_i(\mathcal{O}_{X^+}(i)) \in D(Y)$ are canonically identified under the equivalence $D(Y) \simeq D(R)$. Indeed, this follows from the explicit Čech complex description $\text{(1.11)}$, whose weight $i$ part is exactly the same Čech complex that computes $R \pi^+_i(\mathcal{O}_{X^+}(i))$. Notice that we

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2 Since $X^+$ is defined as $\text{Proj}(A_{\geq 0})$, the Čech complex that computes $R \pi^+_i(\mathcal{O}_{X^+}(i))$ is $\check{C}_{I^+}(A_{\geq 0})_i$. However, for any homogeneous element $f$ with $\text{deg}(f) > 0$, we have $(A_{\geq 0})_f = A_f$, so that these two Čech complexes are the same. This is
have already seen in [12], the first item of the left column. The third term is then a formal consequence of the first two (here, the notation $R^{>0} \pi^+$ is a shorthand for the good truncation $\tau^{>0} R\pi^+$).

The equalities of the column on the right means that they are canonically identified under the equivalence $\mathcal{D}(\text{Gr}(A)) \simeq \mathcal{D}(X)$. Indeed, under this equivalence, the description (1.11) is standard.

Generally speaking, when we discuss flips and flops in the next section, we will use the column on the left to extract homological information on $A$ from the situation of flips/flops, and then use the column on the right (as a viewpoint) to relate the derived categories of $X^-$ and $X^+$.

2. Flips and flops

We consider flips and flops in this section. Before getting to that, we recall some results in algebraic geometry. The first is the following result, which may be found in [7, Proposition 5.75]:

**Theorem 2.1.** Let $X$ be a projective normal variety. Then there is an isomorphism $\mathcal{O}(K_X) \cong \omega_X$, where $K_X$ is the canonical divisor, and $\omega_X$ is the dualizing sheaf.

**Remark 2.2.** This statement is still true if $X$ is not projective. Namely, for any variety $X$, take the canonical dualizing complex $\omega_X := \mathcal{H}^{-n}(\omega_X^\bullet)$, where $p : X \to \text{Spec }k$ is the projection to a point, and define $\omega_X := \mathcal{H}^{-n}(\omega_X^\bullet)$ where $n = \dim(X)$. If $X$ is proper (over $k$), then $\omega_X$ can be characterized as the dualizing sheaf. But even if $X$ is not proper, $\omega_X \in \text{Coh}(X)$ is still canonically defined. Moreover, if $X$ is normal then we have $\mathcal{O}(K_X) \cong \omega_X$, where $K_X$ is the canonical divisor. For example, if $X$ is quasi-projective then this follows from Theorem [2.1] by taking a compactification. This implies that $\omega_X$ is reflexive whenever $X$ is a normal variety (since we can argue on each affine open subscheme, which is quasi-projective). Since $\omega_X$ and $\mathcal{O}(K_X)$ are both reflexive and coincide on the smooth locus, they must be isomorphic.

We will also need a version of the Kawamata-Viehweg vanishing theorem:

**Theorem 2.3.** Assume $\text{char}(k) = 0$. Let $\pi : X \to Y$ be a projective birational morphism between varieties, where $X$ has at most rational singularities. Let $\mathcal{L} \in \text{Pic}(X)$ be $\pi$-nef, then we have $R^i\pi_* (\omega_X \otimes \mathcal{L}) = 0$ for all $i > 0$.

Now we recall the notion of flips and flops, which are two mutually exclusive subclasses of log flips. We will say that a log flip is trivial if both $\pi^\pm$ are the identity maps. Otherwise, we say that the log flip is non-trivial.

**Definition 2.4.** A flip is a non-trivial log flip $(X^+ , Y, \pi^+, D^\pm )$ in which $D^\pm = K_{X^\pm}$.

A flop is a non-trivial log flop in which $K_{X^\pm}$ is numerically $\pi^\pm$-trivial.

A Gorenstein flop is a non-trivial log flip in which $K_Y$ is Cartier. By smallness, we then have $(\pi^\pm)^* K_Y = K_{X^\pm}$, and therefore it is a flop.

We will consider flips and Gorenstein flops. In fact, we will impose one more condition:

(2.5) Assume that $X^\pm$ have at most rational Gorenstein singularities.

In the case of flips or Gorenstein flops, by applying Theorem 2.3 to $\pi^- : X^- \to Y$, the assumption (2.5) implies that $Y$ has at most rational singularities, and hence is Cohen-Macaulay. In this case, since $X^\pm$ are instances of the fact that, although $X^+$ is defined as $\text{Proj}(A_{\geq 0})$, it is really about $A$. Namely, it is covered by the affine open subschemes $\text{Spec}(A_{(f)}) = \text{Spec}(A_{\geq 0}(f))$ for $\text{deg}(f) > 0$.

Recall that for a morphism $f : X \to Y$ between separated schemes of finite type over $k$, the functor $f^!$ is defined to be $f^! = j^* \circ g^!$, where $X \xrightarrow{f} \mathbb{P}^n \xrightarrow{g} Y$ is a factorization of $f$ such that $g$ is proper and $j$ is a Zariski open inclusion, and the functor $g^!$ is the right adjoint to $Rg_*$. All of these functors are taken between $\mathcal{D}_{\text{qcoh}}(\mathcal{Z}) \simeq \mathcal{D}(\text{Qcoh}(\mathcal{Z}))$.

See [9, Theorem 1-2-3] for the case when $X$ is smooth. This implies our present case by taking a resolution of singularities $f : X \to X$. Namely, since $X$ has rational singularities, we have $Rf_*(\omega_X) \cong \omega_X$, so that we have $Rf_*(\omega_X \otimes f^* \mathcal{L}) \cong \omega_X \otimes \mathcal{L}$ by the projection formula. Then apply $R(\pi f)_*$ to $\omega_X \otimes \mathcal{L}$. 

(2.5) Assume that $X^\pm$ have at most rational Gorenstein singularities.
and $Y$ are Cohen-Macaulay, their canonical dualizing complexes (see Remark 2.2) is concentrated in degree $-n$, so that $\omega_X^* = \omega_X[1]$ for $X$ being $X^+$ or $Y$.

Thus, in the case of flips satisfying (2.5), we have

$$\Phi(Y) = \omega_X^*$$

(2.9)

Indeed, if $f : X \to Y$ is a proper birational morphism between Gorenstein normal varieties such that $f^*K_Y = K_X$, then we have $f^*(\omega_Y) \cong \omega_X$ by (2.1). Also, we again have $f^!(\omega_Y) \cong \omega_X$ by virtue of $X$ and $Y$ being Cohen-Macaulay. Since $f^!(\mathcal{F}) \cong f^*(\mathcal{F}) \otimes f^!(\omega_Y)$ for perfect complexes $\mathcal{F} \in D_{\text{perf}}(Y)$, we conclude that $f^!(\omega_Y) \cong \omega_X$.

Thus, in both cases, we have the following

**Checkpoint 2.6.** For flips and Gorenstein flops satisfying (2.5), there is a dualizing complex $\omega_Y^*$ on $Y$ such that

$$\Phi(Y) = \omega_X^*$$

(2.10)

where $a = 0$ for Gorenstein flops and $a = 1$ for flips.

We introduce the following degreewise dualizing functor (recall that $R = A_0$ and $Y = \text{Spec } R$).

$$D_Y : \text{D}(\text{Gr}(A))^\text{op} \to \text{D}(\text{Gr}(A)), \quad D_Y(M)_i = R\text{Hom}_R(M_{-i}, \omega_Y^*)$$

We postulate the following as an imitation of (2.6):

**Definition 2.8.** A homological flip/flop consists of a pair of isomorphisms in $\text{D}(\text{Gr}(A))$:

$$\Phi^+ : \hat{\mathcal{C}}_I^+(A)(a) \xrightarrow{\cong} D_Y(\hat{\mathcal{C}}_I^+(A))$$

(2.9)

$$\Phi^- : \hat{\mathcal{C}}_I^-(A)(a) \xrightarrow{\cong} D_Y(\hat{\mathcal{C}}_I^-(A))$$

such that they are compatible, in the sense that the following diagram commutes:

$$\begin{array}{ccc}
A(a) & \xrightarrow{\eta^+} & \hat{\mathcal{C}}_I^+(A)(a) \\
\eta^- & \xrightarrow{\phi^+} & \hat{\mathcal{C}}_I^-(A)(a) \\
\end{array}$$

$$\begin{array}{ccc}
D_Y(\eta^-) & \xrightarrow{\phi^-} & D_Y(\hat{\mathcal{C}}_I^+(A)) \\
D_Y(\eta^+) & \xrightarrow{\cong} & D_Y(\hat{\mathcal{C}}_I^-(A)) \\
\end{array}$$

(2.10)

In the case $a = 0$, we call it a homological flop. In the case $a = 1$, we call it a homological flip.

The existence of the isomorphisms $\Phi^\pm$ should be quite believable if we look at both sides of (2.9) weight by weight. For example, by Grothendieck duality, we have

$$R\text{Hom}_Y(R\pi^+_X(\mathcal{O}_{X^+}(-i)), \omega_Y^*) \cong R\pi^+_X(R\text{Hom}_Y(\mathcal{O}_{X^+}(-i), \pi^!(\omega_Y^*))) \xrightarrow{\cong} R\pi^+_X(\mathcal{O}_{X^+}(i + a))$$

(2.11)

In view of (1.13), the left and right hand side are $D_Y(\hat{\mathcal{C}}_I^+(A))_i$ and $\hat{\mathcal{C}}_I^+(A)_{i+a}$ respectively.

The meaning of the commutative diagram (2.10) is harder to describe. Let us just say that it boils down to the fact that the isomorphism $\Phi_X : \mathcal{O}(K_X) \xrightarrow{\cong} \omega_X$ in Theorem 2.1 can be appropriately chosen.

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6Of course, the weight-by-weight description (2.7) of $D_Y(M)$ is not enough to determine $D_Y(M)$ as an object in $D(\text{Gr}(A))$. To actually define $D_Y$, one could start with $\text{Hom}_{\text{Gr}}(-, \mathcal{F}) : \text{Gr}(A)^{\text{op}} \times \text{Mod}(R) \to \text{Gr}(A)$ and take its derived functor.

7One may call the general notion a homological flap, so that it becomes a flip when $a = 0$, and a flop when $a = 1$. 
so that they satisfy a certain compatibility condition across the maps $\pi^\pm$. More precisely, we may require the following diagram to commute:

\[
\begin{array}{ccc}
\pi^+_* \mathcal{O}(K_X^-) & \xrightarrow{\pi^+_*(\Phi_X^-)} & \mathcal{O}(K_Y) & \xleftarrow{\Phi_Y} & \pi^-_* \mathcal{O}(K_X^+) \\
\pi^+_* \omega_X^- & \xrightarrow{\text{Tr}_{X^-/Y}} & \omega_Y & \xleftarrow{\text{Tr}_{X^+/Y}} & \pi^-_* \omega_X^+
\end{array}
\]

where $\Phi_X^\pm$ and $\Phi_Y$ are the isomorphisms in Theorem 2.1 and $\text{Tr}_{X^\pm/Y}$ are induced from certain adjunction morphisms on dualizing complexes. Here, we have written equalities in the first row because they are equal as subsheaves of the sheaf of rational functions on $Y$.

In fact, to pass from the weight-by-weight description (2.11) to an actual isomorphism (2.10) in $\mathcal{D}(\text{Gr}(A))$, we had to develop from scratch the effect of Grothendieck duality in $\mathcal{D}(\text{Gr}(A))$ (this is performed in [15]). Then, we had to show that, under this formalism, (2.12) indeed translates into (2.10).

For this, we had to check many commutative diagrams. All these checkings are lengthy and tedious (see [13] for details), but doesn’t seem to offer any additional insight, so we will skip it in this article. We summarize this discussion by the following

**Theorem 2.13.** For flips and Gorenstein flops satisfying (2.5), the corresponding graded ring $A$ admits the structure of a homological flip/flop.

We will formulate one more property of $A$ that imitates the Kawamata-Viehweg vanishing theorem.

**Definition 2.14.** A $\mathbb{Z}$-graded ring $A$ is said to satisfy **canonical vanishing** at $a \in \mathbb{Z}$ if we have $R^i \Gamma_{Y^+}(A) \in \mathcal{D}_{<a}(\text{Gr}(A))$ and $R^i \Gamma_{Y^-}(A) \in \mathcal{D}_{>a}(\text{Gr}(A))$.

In view of the table of (1.13), we see that Theorem 2.3 immediately implies the following

**Proposition 2.15.** For flips and Gorenstein flops satisfying (2.5), the corresponding graded ring $A$ satisfies canonical vanishing at $a$ (where $a = 0$ for flops and $a = 1$ for flips).

Next, we explore some consequences of Theorem 2.13 and Proposition 2.15. The following Theorem, as well as its proof, may be regarded as the main result of this work:

**Theorem 2.16.** Suppose $(A, \Phi^+, \Phi^-)$ is a homological flip/flop that satisfies canonical vanishing at $a$ (where $a = 0$ for flops and $a = 1$ for flips), then there is an isomorphism in $\mathcal{D}(\text{Gr}(A))$:

$$
\Psi : R^i \Gamma_{Y^+}(A)(a)[1] \xrightarrow{\cong} \mathcal{D}_Y(R^i \Gamma_{Y^-}(A))
$$

Moreover, $A$ is Gorenstein.

**Proof.** Consider the commutative diagram (2.10). Since the maps $\Phi^\pm$ are isomorphisms, we may regard it as a commutative square. As such, the $3 \times 3$-lemma (see, e.g., [1] Proposition 1.1.11] or [10] Lemma 2.6) asserts that it can be extended to a $3 \times 3$ square. More precisely, the object $Z$ as well as the maps in the dotted lines of the following diagram exists, making each row and column part of a distinguished

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8For flips, there is no ambiguity in choosing $D^\pm$, so that $A$ is canonically defined. For flops, we will assume that $D^\pm$ are chosen to be Cartier, possibly by replacing them by their multiple. Since we work under the assumption (2.5), we see that $D^\pm$ are Cartier in both cases. In fact, this was already implicitly used in the second isomorphism in (2.11), which holds because $\mathcal{O}_{X^+}(-i)$ is locally free. Notice that it also holds if $\mathcal{O}_{X^+}(i) = \mathcal{O}_{X^+}(-iD^+)$ is maximal Cohen-Macaulay, which is satisfied whenever $X^\pm$ have at most log terminal singularities (see [11] Corollary 5.25). Accordingly, the assumption (2.5) can be weakened.
triangle:

\[
\begin{array}{cccc}
A(a) & \rightarrow & \check{C}_{I^+}(A)(a) & \rightarrow \\
\downarrow \gamma^- & & \downarrow \check{\gamma}^- & \rightarrow \\
\check{C}_{I^-}(A)(a) & \rightarrow & \check{\gamma}^- & \rightarrow
\end{array}
\]

(2.17)

By the assumption of canonical vanishing, we have \( R\Gamma_{I^+}(A)(a)[1] \in \mathcal{D}_{<0}(\text{Gr}(A)) \) and \( \mathcal{D}_Y(R\Gamma_{I^-}(A)) \in \mathcal{D}_{<0}(\text{Gr}(A)) \), so that, as the cone of \( \Psi \), we have \( Z \in \mathcal{D}_{<0}(\text{Gr}(A)) \). Likewise, we have \( R\Gamma_{I^-}(A)(a)[1] \in \mathcal{D}_{>0}(\text{Gr}(A)) \) and \( \mathcal{D}_Y(R\Gamma_{I^+}(A)) \in \mathcal{D}_{>0}(\text{Gr}(A)) \), so that as the cone of \( \Psi' \), we have \( Z \in \mathcal{D}_{>0}(\text{Gr}(A)) \).

Since \( a = 0 \) or \( a = 1 \), we must have \( Z = 0 \). i.e., \( \Psi \) is an isomorphism.

To prove that \( A \) is Gorenstein, we will show that it has finite injective dimension. Take the local cohomology sequence

\[ \ldots \rightarrow R\Gamma_{I^+}(A) \rightarrow A \rightarrow \check{C}_{I^+}(A) \rightarrow \ldots \]

Then the terms can be rewritten as \( R\Gamma_{I+}(A) \cong \mathcal{D}_Y(R\Gamma_{I^-}(A))(-a)[-1] \) and \( \check{C}_{I^+}(A) \cong \mathcal{D}_Y(\check{C}_{I^-}(A))(-a) \).

Notice that both \( R\Gamma_{I^-}(A) \) and \( \check{C}_{I^+}(A) \) have finite Tor dimension because of the explicit presentation (1.11). Therefore their \( \mathcal{D}_Y \)-dual have finite injective dimension, hence so does \( A \).

For general classes of flips and flops, it seems quite difficult to describe this duality between local cohomology groups explicitly. There is however one simple example where such a description is possible:

**Example 2.18.** Let \( A = k[x_1, \ldots, x_p, y_1, \ldots, y_q] \), where \( \text{deg}(x_i) = 1 \) and \( \text{deg}(y_i) = -1 \). This corresponds to the standard flip/flop, and \( A \) has \( a = q - p \). By the explicit presentation (1.11), it is easy to see that

\[ R\Gamma_{I^+}(A) = k[y_1, \ldots, y_q] \otimes R\Gamma_{I^+}(k[x_1, \ldots, x_p]) \]

Also, in view of the table in (1.13) again, we see that \( R\Gamma_{I^+}(k[x_1, \ldots, x_p]) \) is simply the higher cohomology of \( \mathbb{P}^{q-1} \) (shifted by 1) of \( \mathcal{O}(i) \). Such a computation is standard (see, e.g., [5 Section III.5]), and we have

\[ R\Gamma_{I^+}(A) = k[y_1, \ldots, y_q] \otimes k[x_1, \ldots, x_p]^*(p)[-p] \]

where we denote \( M^* \) to be the \( k \)-linear dual of \( M \). Similarly, we have

\[ R\Gamma_{I^-}(A) = k[x_1, \ldots, x_p] \otimes k[y_1, \ldots, y_q]^*(-q)[-q] \]

We want to claim that they are \( \mathcal{D}_Y \)-dual to each other. While the description of \( \mathcal{D}_Y \) on \( R = A_0 \) seems to be quite complicated in general, we will only need a special case. Namely, since \( R\Gamma_{I^+}(A) \) corresponds to higher pushforwards of \( \mathcal{O}_X(i) \), it must be supported on the image of the exceptional locus, which is a point. In this case, \( \mathcal{D}_Y \) is easy to describe:

**Lemma 2.19.** Suppose that \( M \in \mathcal{D}^b_{\text{coh}}(R) \) is (set-theoretically) supported on \( \text{Spec} k = \text{Spec} R/I \), and let \( \omega^*_\pi \in \mathcal{D}^b_{\text{coh}}(R) \) be a dualizing complex, normalized so that its cohomology sheaf is concentrated in degree 0. Then we have \( \mathcal{D}_Y(M) \cong M^*[-n] \), where \( n = \dim R = p + q - 1 \).

From this, we see that

\[ \mathcal{D}_Y(R\Gamma_{I^-}(A)) = k[x_1, \ldots, x_p]^* \otimes k[y_1, \ldots, y_q]^*[-p-q+1] \]

Given a graded module \( M \in \text{Gr}(A) \), then it has finite injective dimension in the abelian category \( \text{Gr}(A) \) if and only if it has finite injective dimension in the abelian category \( \text{Mod}(A) \). In fact, we have \( \text{inj dim}_{\text{Gr}(A)}(M) \leq \text{inj dim}_{\text{Mod}(A)}(M) \leq \text{inj dim}_{\text{Gr}(A)}(M) + 1 \) (see, e.g. [2 Proposition 3.6.6]). Hence there is no ambiguity in our discussion.

\[ R^1 \] is Cohen-Macaulay in this example, so that we may assume that \( \mathcal{H}^\bullet(\omega^*_\pi) \) is concentrated in degree 0. An analogous statement holds for more general \( R \) (at least for domains finitely generated over \( k \)), for which we may assume that \( \mathcal{H}^\bullet(\omega^*_\pi) \) is concentrated in degree 0 on the smooth part.
Remark 2.20. This example can be directly generalized to the case $A = k[x_1, \ldots, x_p, y_1, \ldots, y_q]$, where $\deg(x_i) > 0$ and $\deg(y_i) < 0$. In this case, let $\eta^+ = \sum_{i=1}^{p} \deg(x_i)$ and $\eta^- = -\sum_{j=1}^{q} \deg(y_j)$, and take $a = \eta^- - \eta^+$. Notice that $R\Gamma_{I^+}(A)$ is still the same extended Čech complex as in Example 2.18 but with generators in different degrees. As such, the computation in [5] Section III.5] carries through verbatim, and we have

$$R\Gamma_{I^+}(A) = k[y_1, \ldots, y_q] \otimes k[x_1, \ldots, x_p]^*(\eta^+)[p]$$

$$R\Gamma_{I^-}(A) = k[x_1, \ldots, x_p] \otimes k[y_1, \ldots, y_q]^*(-\eta^-)[-q]$$

and hence we still have $D_Y(R\Gamma_{I^-}(A)) \cong R\Gamma_{I^+}(A)(a)$.

We wish to apply the results in Theorem 2.13, Proposition 2.15, and Theorem 2.16 to relate the derived categories of $X^-$ and $X^+$. We may summarize the situation as follows

$$\text{kill } R\Gamma_{I^-} \quad \xrightarrow{\mathcal{D}(\text{Gr}(A))} \quad \text{kill } R\Gamma_{I^+}$$

where the derived categories $\mathcal{D}(X^\pm)$ are obtained by “killing” the endofunctors $R\Gamma_{I^\pm}(M) = M \otimes^\mathbb{L}_{A} R\Gamma_{I^\pm}(A)$. The duality of local cohomology groups in Theorem 2.16 then suggests a way to relate these two derived categories. However, an implementation of this idea seems to be not so straightforward. We present one attempt in the next section. This approach is successful in some useful cases (see Section 4), but fall short in general because of what seems to be a formal problem.

3. Weight truncation

Before describing the actual construction, we would like to suggest a parallelism with a paper [11] of Orlov. Namely, we may summarize our desired picture as follows:

Flip or Gorenstein flop (with good singularities)

$$\implies A \text{ is Gorenstein, and } R\Gamma_{I^+}(A)(a)[1] \cong D_Y(R\Gamma_{I^-}(A))$$

(3.1)

On the other hand, for a Gorenstein projective variety $X$ of dimension $n$, Orlov studied the relation between $D^b_{\text{coh}}(X)$ and the triangulated category of singularities of the projective cone of $X$. More precisely, assume that $\mathcal{O}(1)$ is very ample, and let $A = \bigoplus_{i \geq 0} H^0(X, \mathcal{O}(i))$. Assume that $H^j(X, \mathcal{O}(i)) = 0$ for all $j \neq 0, n$ for all $i \in \mathbb{Z}$. Then Orlov established in [11] the following implications

$$\omega_X \cong \mathcal{O}(-a)$$

$$\implies A \text{ is Gorenstein, and } R\text{Hom}_A(k, A) \cong k(a)[-n]$$

(3.2)

Our goal in this section is to imitate the construction of the step $(\ast)$ in (3.2) and try to establish the step $(?)$ in (3.1). Our construction is also influenced and motivated by the papers [6, 7].

Fix an integer $w \in \mathbb{Z}$ once and for all. Let $D_{\geq w}(\text{Gr}(A))$ be the smallest cocomplete (i.e., closed under arbitrary direct sums) triangulated subcategory of $D(\text{Gr}(A))$ containing the objects $\{A(-i)_{i \geq w}\}$, and let $D_{< w}(\text{Gr}(A))$ be defined as above. i.e., it consists of $M \in D(\text{Gr}(A))$ such that $M_i = 0$ for all $i \geq w$. By Neeman-Brown representability, we have an SOD

$$D(\text{Gr}(A)) = \langle D_{< w}(\text{Gr}(A)), D_{\geq w}(\text{Gr}(A)) \rangle$$

(3.3)
The terminology $D_{\geq w}(\text{Gr}(A))$ refers to those objects that are generated in weight $\geq w$, while $D_{< w}(\text{Gr}(A))$ refers to those that are concentrated in weight $< w$. In the setting of $\text{Gr}^+$, $A$ is non-negatively graded, so that $D_{\geq w}(\text{Gr}(A)) = D_{\geq w}(\text{Gr}(A))$. But in our setting of $\mathbb{Z}$-graded rings, these two are very different. In particular, none contain the other.

The decomposition sequence associated to (3.3) will be denoted as

$$
\cdots \to L_{\geq w}M \to M \to L_{< w}M \to \cdots
$$

We will put together the SODs (1.9) and (3.3). Notice that $D_{< w}(\text{Gr}(A))$ consists of those $M \in D(\text{Gr}(A))$ such that each $H^i(M)$ has weight concentrated in degree $< w$, and hence we have $D_{< w}(\text{Gr}(A)) \subset D_{\text{Tor}^+}(\text{Gr}(A))$. Thus, if we compare (1.9) and (3.3), we would expect that $D_{\text{Tor}^+}(\text{Gr}(A))$ would be “bigger” than $D_{\text{Tor}^+}(\text{Gr}(A))$. In fact, one can show that $D_{\text{Tor}^+}(\text{Gr}(A))$ embeds (via a non-identity functor) as a semi-orthogonal summand of $D_{\geq w}(\text{Gr}(A))$. Combined with (3.3), it then gives us a 3-term semi-orthogonal decomposition of $D(\text{Gr}(A))$. This is summarized in the following result, whose proof is simple once it is formulated precisely as below:

**Theorem 3.5.** The restriction of $L_{\geq w}$ to $D_{\text{Tor}^+}(\text{Gr}(A))$ gives a fully faithful functor $L_{\geq w} : D_{\text{Tor}^+}(\text{Gr}(A)) \to D_{\geq w}(\text{Gr}(A))$, with a left adjoint given by $\mathcal{C}_{\text{Tor}^+}$. Hence there is a semi-orthogonal decomposition

$$
D_{\geq w}(\text{Gr}(A)) = \langle L_{\geq w}D_{\text{Tor}^+}(\text{Gr}(A)), D_{\geq w}, \text{Tor}^+(\text{Gr}(A)) \rangle
$$

Combined with (3.3), there is therefore a 3-term semi-orthogonal decomposition

$$
D(\text{Gr}(A)) = \langle D_{< w}(\text{Gr}(A)), L_{\geq w}D_{\text{Tor}^+}(\text{Gr}(A)), D_{\geq w}, \text{Tor}^+(\text{Gr}(A)) \rangle
$$

where the subcategory $D_{\leq w}(\text{Gr}(A)) \subset D(\text{Gr}(A))$ can be characterized more precisely as

$$
D_{\leq w}(\text{Gr}(A)) = \{ M \in D(\text{Gr}(A)) \mid \text{Tor}^+D_{\leq w}(\text{Gr}(A)) \}
$$

The middle component, called the window subcategory can therefore be characterized as

$$
L_{\geq w}D_{\text{Tor}^+}(\text{Gr}(A)) = D_{\leq w}(\leq w)(\text{Gr}(A)) = D_{\geq w}(\text{Gr}(A)) \cap D_{\leq w}(\text{Gr}(A))
$$

which is equivalent to $D_{\text{Tor}^+}(\text{Gr}(A))$ (and hence to $D(\mathbb{X}^+)$) via the functors $\mathcal{C}_{\text{Tor}^+}$ and $L_{\geq w}$.

Notice that, in this Theorem, we have embedded $D(\mathbb{X}^+)$ into $D(\text{Gr}(A))$ in a non-standard way, i.e., as the window subcategory $D_{\leq w}(\leq w)(\text{Gr}(A))$, instead of the more straightforward $D_{\text{Tor}^+}(\text{Gr}(A))$. Notice that the straightforward embedding to $D_{\text{Tor}^+}(\text{Gr}(A))$ has the disadvantage that $\mathcal{C}_{\text{Tor}^+}(M)$ almost never have coherent cohomology even if $M$ does (it has finite cohomological dimension though!). In contrast, this non-standard “window embedding” have the following advantage:

**Theorem 3.7.** The 3-term SOD (4.3) restrict to a 3-term SOD on $D_{\text{coh}}^-(\text{Gr}(A))$

$$
D_{\text{coh}}^-(\text{Gr}(A)) = \langle D_{\text{coh}, < w}^-(\text{Gr}(A)), D_{\text{coh}, \leq w}, D_{\text{coh}, \geq w}(\text{Gr}(A)) \rangle
$$

where the middle component is equivalent to $D_{\text{coh}}^-(\mathbb{X}^+)$.

A proof may be found in [14] (a revised version is under preparation where the arguments will be simplified and put in a broader scope), and is based on the tensor form (3.8) below. We will skip the proof in this article.

Now we are ready to describe the functor that relate the derived categories under flips/flops. First, we introduce the functor

$$
\mathbb{D} : D(\text{Gr}(A))^\text{op} \to D(\text{Gr}(A)), \quad \mathbb{D}(M) := \mathbb{R}\text{Hom}_A(M, A)
$$

We postulate the following idea:

$$
\text{(3.8) The functor } \mathbb{D} \text{ “should” send the window } D_{\geq w}(\leq w)(\text{Gr}(A)) \text{ in the negative direction.}
$$
The idea does not work as it is stated. But let’s press on with the idea, which may be broken down into two parts:

(3.9) (1) The functor \( D_A \) “should” send \( D_{[>-w]}(\text{Gr}(A)) \) to \( D_{[\leq -w]}(\text{Gr}(A)) \)

(2) The functor \( D_A \) “should” send \( D_{[<w]}(\text{Gr}(A)) \) to \( D_{[>-w]}(\text{Gr}(A)) \)

The first item seems quite reasonable, because \( D_A \) sends the generators \( \{A(-i)\}_{i \geq w} \) of \( D_{[\geq w]}(\text{Gr}(A)) \) to the generators \( \{A(-i)\}_{i \leq -w} \) of \( D_{[\leq -w]}(\text{Gr}(A)) \). However, notice that \( D_A \) sends infinite direct products to infinite direct sums, and \( D_{[\leq -w]}(\text{Gr}(A)) \) seems to be not closed under infinite direct products in general, so that (1) seems to be not true in general. In fact, it seems to be not true even if one restrict to \( D_{\text{coh.}[\geq w]}(\text{Gr}(A)) \). More precisely, one may encounter the following problem:

Suppose \( M \) is a bounded below complex \( M = [0 \to M^n \to M^{n+1} \to \ldots] \) such that each \( M^j \) is a finite direct sum of the free graded modules \( A(-i) \) for \( i \leq -w \), then it is not clear whether \( M \) is in \( D_{[\leq -w]}(\text{Gr}(A)) \).

In contrast, one can actually prove the item (2), when restricted to \( D_{\text{coh}}(\text{Gr}(A)) \):

**Proposition 3.11.** Suppose there is a dualizing complex \( \omega^*_A \) on \( R = A_0 \) such that there is an isomorphism \( R\Gamma_{I^+(A)}(a)[1] \cong D_Y(R\Gamma_{I^-}(A)) \) in \( D(\text{Gr}(A)) \). Then \( D_A \) sends \( D_{\text{coh},[<w]}(\text{Gr}(A)) \) to \( D_{\text{coh},[>-w+a]}(\text{Gr}(A)) \).

**Proof.** Suppose \( M \in D_{\text{coh}}(\text{Gr}(A)) \) satisfies \( R\Gamma_{I^-}(D_A(M)) = R\text{Hom}_A(M, A) \otimes_A R\Gamma_{I^-}(A) \)

\[
\cong \left( R\text{Hom}_A(M, R\Gamma_{I^-}(A)) \right) \cong D_Y(M \otimes_A R\Gamma_{I^+}(A)(-a)[1])
\]

where \((*)\) holds because \( R\Gamma_{I^-}(A) \) has finite Tor dimension and \( M \in D_{\text{coh}}(\text{Gr}(A)) \). \( \square \)

By the results of the last section, we may assume that \( A \) is Gorenstein, so that \( D_A \) is in some sense well-behaved. Thus, if \((3.9)(1)\) holds in some good context, we would have obtained a relation between the derived categories of \( \mathfrak{X}^+ \) and \( \mathfrak{X}^- \), similar to the expected relation \((3.1)\). While this picture of using \( D_A \) to relate the two windows was the first argument along this line that the author discovered (and is still very attractive to the author), it seems to be not the most efficient one. We now seek to reformulate it without using the duality \( D_A \).

In our above argument, the functor that we want to use to relate the derived categories of \( \mathfrak{X}^- \) and \( \mathfrak{X}^+ \) can be written as follows:

(3.12) \( D_{\text{coh}}(\mathfrak{X}^+) \xleftarrow{\sim} \left( \left( j^+ \right)^* \right)_{\geq w} \xrightarrow{\sim} \left( \left( j^+ \right)^* \right)_{\leq w} \xrightarrow{\sim} D_{\text{coh}}(\mathfrak{X}^+) \)

Here, we have marked the middle arrow with a question mark to indicate that it is not clear whether the functor \( D_A \) really sends \( D_{\text{coh},[\geq w],(<w]}(\text{Gr}(A)) \) to \( D_{\text{coh},[\leq w],(>-w]}(\text{Gr}(A)) \). Thus, the question mark in \((3.12)\) is a version of \((3.3)\) where we restrict to \( D_{\text{coh}} \). If it holds, then we see that \((3.12)\) is a composition of three fully faithful functors, and is therefore fully faithful, giving a result along the lines of \((3.1)\).

Whether or not the question mark in \((3.12)\) holds, the functor \((3.12)\) makes sense if we consider

\[
D_{\text{coh}}(\mathfrak{X}^+) \xleftarrow{\sim} \left( \left( j^+ \right)^* \right)_{\geq w} \xrightarrow{\sim} D_{\text{coh},[\geq w],(<w]}(\text{Gr}(A)) \xrightarrow{(j^-)^* \circ D_A} D_{\text{coh}}(\mathfrak{X}^-)^{\text{op}}
\]

Since duality is local on \( D_{\text{coh}}(\mathfrak{X}) \), the last functor may be rewritten as \( (j^-)^* \circ D_A = D_{\mathfrak{X}^-} \circ (j^-)^* \). But \( \mathfrak{X}^- \) is Gorenstein, so that \( D_{\mathfrak{X}^-} \) is a contravariant equivalence on \( D_{\text{coh}}(\mathfrak{X}^-) \), and we may skip that functor. In other words, the desired functor is in fact

(3.13) \( D_{\text{coh}}(\mathfrak{X}^+) \xleftarrow{\sim} \left( \left( j^+ \right)^* \right)_{\geq w} \xrightarrow{\sim} D_{\text{coh},[\geq w],(<w]}(\text{Gr}(A)) \xrightarrow{(j^-)^*} D_{\text{coh}}(\mathfrak{X}^-) \)
Since the duality $D_A$ is not involved, we expect that our argument will not need to use the Gorenstein property. In fact, we will show that the duality $R\Gamma I^+(A)(a)[1] \cong D_Y(R\Gamma I^-(A))$ “almost implies” that this functor is fully faithful, except that a similar formal problem arise. To describe the problem, we start with the following straightforward

**Lemma 3.14.** Given a full triangulated subcategory $\mathcal{E} \subset D(Gr(A))$, then the functor $\check{C}_I^- : \mathcal{E} \to D_{-\text{triv}}(Gr(A))$ is fully faithful if and only if

$$R\text{Hom}_A(M, R\Gamma I^-(N)) := R\text{Hom}_A(M, R\Gamma I^-(N))_0 \cong 0 \quad \text{for all } M, N \in \mathcal{E}$$

We wish to show that the second functor in (3.13) is fully faithful by verifying this condition. Thus, let $M \in D_{\text{coh}(<w)}(Gr(A))$ and $N \in D_{\text{coh}([w]}(Gr(A))$, we compute

$$R\text{Hom}_A(M, R\Gamma I^-(N)) \cong R\text{Hom}_A(M, R\Gamma I^-(A)) \otimes_{A}[N] \cong D_Y(R\Gamma I^+(M)[-a][-1]) \otimes_{A}[N]$$

Notice that $D_Y(R\Gamma I^+(M)) \otimes_{A}[N] \in D_{>-w} \otimes_{A}[N] D_{[w]} \subset D_{>0}$. Therefore, its shift by $a \geq 0$ is zero at weight zero.

We see once again that the argument falls short because of what seems like a formal problem (more precisely, the equality marked with (?) may not hold). However, notice that our arguments in either the formulations (3.8) or (3.13) work if the 3-term SOD restricts to $D_{\text{perf}}(Gr(A))$. In the next section, we will see two examples in which it holds.

**Remark 3.15.** One can “solve” the problem (3.9)(1) by formulating the duality $D_A$ in terms of a perfect pairing of DG categories

$$D_{[w]}(Gr(A)) \times D_{[-w]}(Gr(A)) \to D(k), \quad (M, N) \mapsto (M \otimes_{A}[N])_0$$

which then restricts to a pairing on the window subcategories $D_{[w]}(Gr(A))$ and $D_{[-w]}(Gr(A))$. One can then try to show that it is a perfect pairing. Unravelling the definition (at least in the way the author formulated it), it corresponds to the functor (3.13), except that we do not restrict to $D_{\text{coh}}$.

**Remark 3.16.** The functor (3.13) is $O_Y$-linear.

**Remark 3.17.** By a result [8] of Kawamata, any two birational projective Calabi-Yau varieties with at most $Q$-factorial terminal singularities are connected by a finite number of flops, each of which is obviously a Gorenstein flop, and hence falls into our present setting.

**Remark 3.18.** One might try to repeat the arguments in this section for a different version of $D(Gr(A))$. Some potential candidates are: (1) $\text{Ind} D_{\text{coh}}(Gr(A))$, (2) co-derived or contra-derived categories, (3) homotopy category of (unbounded) complexes of finitely generated projectives. For example, if we work with (3), then problems such as (3.10) seem to not arise (although other problems might arise then). Accordingly, the theory of relative homological algebra (e.g. [4]) might be relevant here, although the author is not very familiar with this part of the literature. If one changes the setting, one should also guarantee that results of Section [2] can be carried over. It is not clear to us whether endeavors in these formal directions will be useful (the author has spent a lot of time trying out various formal modifications, but such effort has not been fruitful so far).

4. Some examples

We will present two examples in which the 3-term SOD (3.6) restricts to $D_{\text{perf}}(Gr(A))$, so that our argument does work. For this to hold, it is certainly necessary that the functor $L_{[w]}$ in (3.14) preserves $D_{\text{perf}}(Gr(A))$. The following result says that this condition is also sufficient. A proof may be found in [13] (a revised version is under preparation, for which the arguments will be simplified).

**Proposition 4.1.** Suppose that the functor $L_{[w]}$ in (3.14) preserves $D_{\text{perf}}(Gr(A))$, then the 3-term SOD (3.6) restricts to $D_{\text{perf}}(Gr(A))$. 

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In this case, the functor \((3.13)\) restricts to

\[
D_{\text{perf}}(\mathcal{X}^+) \xleftarrow{\simeq} \mathcal{L}_{[\geq w]}(\mathbf{R})^+ \xrightarrow{(j^-)^*} D_{\text{perf}}(\mathcal{X}^-)
\]

Moreover, the arguments in the previous section then guarantees that, if the duality \(D_Y(\mathbf{R}I(A)) \cong \mathbf{R}I(A)(a)[1]\) holds for some \(a \geq 0\), then \((4.2)\) is fully faithful.

In general, it seems quite difficult to describe \(\mathcal{L}_{[\geq w]}\) explicitly, as it is defined in terms of the Neeman-Brown representability theorem. The closest to an explicit formula that we managed to get is the following tensor form

\[
\mathcal{L}_{[\geq w]}(M) = M_{[\geq w]} \otimes^L_{\mathcal{F}_{[\geq w]} \mathcal{F}}
\]

Here, \(\mathcal{F}\) is the small pre-additive (i.e., Ab-enriched) category given by \(\text{Ob}(\mathcal{F}) = \mathbb{Z}\) and \(\mathcal{F}(i, j) = A_{i-j}\). Then a graded \(A\)-module is the same as a right module over \(\mathcal{F}\) (recall that a right module over a small pre-additive category \(\mathcal{C}\) means an additive functor \(\mathcal{C}^{op} \to \text{Ab}\)). Let \(\mathcal{F}_{[\geq w]} \subset \mathcal{F}\) be the full subcategory on the object set \(\mathbb{Z}_{[\geq w]} \subset \mathbb{Z}\). Then the inclusion functor \(\mathcal{F}_{[\geq w]} \to \mathcal{F}\) gives rise to restriction and induction functors between the module categories. In particular, for any \(M \in \text{Mod}(\mathcal{F})\), we may restrict it to \(\mathcal{F}_{[\geq w]}\) to obtain \(M_{[\geq w]} \in \text{Mod}(\mathcal{F}_{[\geq w]})\), and then tensor it back to \(\mathcal{F}\) to obtain \(M_{[\geq w]} \otimes^L_{\mathcal{F}_{[\geq w]} \mathcal{F}} F \in \text{Mod}(\mathcal{F})\). The right hand side of \((4.3)\) is its derived functor.

The description \((4.3)\) seems to be of little use in the two examples that we present below. In both cases, the functor \(\mathcal{L}_{[\geq w]}\) is described by some ad-hoc arguments specific to the form of the graded ring \(A\) in question.

The first example is the polynomial ring

\[
A = k[x_1, \ldots, x_p, y_1, \ldots, y_q], \quad \text{deg}(x_i) > 0, \text{deg}(y_j) < 0
\]

Write \(A = A^+ \otimes A^-, \) where \(A^+ = k[x_1, \ldots, x_p]\) and \(A^- = k[y_1, \ldots, y_q]\). Then we have

\[
D_{[\geq w]}(\text{Gr}(A^+)) \otimes A^- \subset D_{[\geq w]}(\text{Gr}(A)) \quad \text{and} \quad D_{[< w]}(\text{Gr}(A^+)) \otimes A^- \subset D_{[< w]}(\text{Gr}(A))
\]

This allows us to compute the weight truncation of objects of the form \(M = M^+ \otimes A^-\) for \(M^+ \in D(\text{Gr}(A^+))\). Namely, we have

\[
\mathcal{L}_{[\geq w]}^A(M^+ \otimes A^-) = \mathcal{L}_{[\geq w]}^A(M^+) \otimes A^-
\]

To show that \(\mathcal{L}_{[\geq w]}^A\) preserves \(D_{\text{perf}}(\text{Gr}(A))\), it suffices to verify it on the split-generating objects \(\{A(-i)\}_{i \in \mathbb{Z}}, \) each of which is of the form \(M^+ \otimes A^-\) for \(M^+ = A^+(-i)\). Hence it suffices to compute \(\mathcal{L}_{[\geq w]}^A(A^+(-i))\). Since \(A^+\) is non-negatively graded, we in fact have \(D_{[\geq w]}(\text{Gr}(A^+)) = D_{[\geq w]}(\text{Gr}(A^+))\), so that the weight truncation is easy to describe:

**Lemma 4.5.** For any \(M \in \text{Gr}(A^+),\) we have

\[
\mathcal{L}_{[\geq w]}^A(M^+) = M_{[\geq w]}, \quad \text{and} \quad \mathcal{L}_{[< w]}^A(M) = M/M_{[\geq w]}
\]

As a result, we have

\[
\mathcal{L}_{[\geq w]}^A(A(-i)) = ((A^+)_{[\geq w-i]} \otimes A^-)(-i)
\]

Since \(A^+\) is smooth, we have \((A^+)_{[\geq w-i]} \in D_{\text{perf}}(\text{Gr}(A^+))\). This shows the following

**Proposition 4.6.** For the polynomial ring \((4.4)\), weight truncation \(\mathcal{L}_{[\geq w]}\) preserves \(D_{\text{perf}}(\text{Gr}(A))\).

**Remark 4.7.** Our argument here works whenever a graded ring can be written as \(A = A^+ \otimes A^-\) where \(A^+\) is non-negatively graded and \(A^-\) is non-positively graded. In this case, if \(A^+\) is smooth, then \(\mathcal{L}_{[\geq w]}\) preserves \(D_{\text{perf}}(\text{Gr}(A))\). Interestingly, if \(A^-\) is smooth, then by the arguments of \((6)\), one can show that \(\mathcal{L}_{[\geq w]}\) preserves \(D^h_{\text{coh}}(\text{Gr}(A))\).
Proposition 4.6 shows that our approach in the previous section works perfectly for the polynomial ring \( R \) (notice that we have established the duality \( \mathcal{D}_+(\mathcal{R} X) \cong \mathcal{R} \mathcal{D}_+(\mathcal{A})(a) \) directly in Remark 2.20) without applying the main Theorems of Section 2. We will now compute the corresponding functor \( \mathcal{F} \). For simplicity, we will now assume \( w = 0 \).

Thus, in particular, for \( 0 \leq i < \eta^+ \), the objects \( A(-i) \) are in the window subcategory \( \mathcal{D}_{[0]}(\mathcal{R} \mathcal{D}_+(\mathcal{A})) \). This implies that

\[
(4.8) \quad \text{For } 0 \leq i < \eta^+, \text{ the functor } 4.2 \text{ for } w = 0 \text{ sends } \mathcal{O}_{X^+}(-i) \text{ to } \mathcal{O}_{X^-}(-i).
\]

In fact, this uniquely characterizes the functor \( 4.2 \). Namely, the arguments of \( 0 \) can be employed to give the following characterization of the window subcategory by the “window length” \( \eta^+ \), which then implies that \( \{ \mathcal{O}_{X^+}(-i) \}_{0 \leq i < \eta^+} \) split generates \( \mathcal{D}_{\text{perf}}(X^+) \).

**Proposition 4.9.** For the polynomial ring \( A \), we have \( \mathcal{D}_{\text{perf}}(\mathcal{A}) = \mathcal{D}_{\text{perf}}(\mathcal{A}) \) as a graded module (\( \mathcal{A}^+ \)).

As a result, the subcategory \( \mathcal{D}_{\text{perf}}(\mathcal{A}) \) is split generated by \( \{ A, A(-1), \ldots, A(-\eta^+ + 1) \} \).

Similarly, we have \( \mathcal{D}_{\text{perf}}(\mathcal{A}) \) as a graded module (\( \mathcal{A}^+ \)).

**Remark 4.11.** The case \( \deg(x_i) = 1 \) and \( \deg(y_{ij}) = -1 \) corresponds to a standard flip/flop (we then have \( X^\pm = X^\pm \) and \( \eta^+ = p, \eta^- = q \)). In this case, the functor \( 4.2 \) is isomorphic to the standard span \( X^- \to X^+ \). Indeed, one can simply verify this on the split generators \( \{ \mathcal{O}_{X^+}(-i) \}_{0 \leq i < p} \) of \( \mathcal{D}_{\text{perf}}(X^+) \). The author thanks Alexander Kuznetsov for help with this remark.

We now move to our second class of examples, which are the graded rings associated to a class of 3-fold flips of type A, worked out by Brown and Reid (see [12 Section 11.2]).

Fix positive integers \( d, e, \alpha, \beta, \lambda, \mu \) with \( \gcd(\lambda, \mu) = 1 \). Take \( \lambda, \mu \) with degrees \( \deg(x_i) = 1 \) and \( \deg(y_{ij}) = -1 \). In this case, the functor \( 4.2 \) is isomorphic to the standard span \( X^- \to X^+ \). Indeed, one can simply verify this on the split generators \( \{ \mathcal{O}_{X^+}(-i) \}_{0 \leq i < p} \) of \( \mathcal{D}_{\text{perf}}(X^+) \). The author thanks Alexander Kuznetsov for help with this remark.

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generated in non-positive degrees. Thus, it gives a non-positive presentation of $A$, in the sense of the following

**Definition 4.15.** A Noetherian graded algebra $A$ is said to have a non-positive presentation if there exists a map $C 	o A$ from a polynomial graded algebra $C$ such that, as an object in $\mathcal{D}(\text{Gr}(C))$, we have $A \in \mathcal{D}_{\leq 0}(\text{Gr}(C))$.

**Proposition 4.16.** If $A$ has a non-positive presentation, then $L_{\geq w}^C$ preserves $\mathcal{D}_{\text{perf}}(\text{Gr}(A))$.

**Proof.** In general, we have $D_{\geq w}(\text{Gr}(C)) \otimes^L_C A \subset D_{\geq w}(\text{Gr}(A))$. The condition on non-positive presentation also guarantees that $D_{<w}(\text{Gr}(C)) \otimes^L_C A \subset D_{<w}(\text{Gr}(A))$. Hence we have $L^A_{\geq w}(M \otimes^L_C A) \cong L^C_{\geq w}(M) \otimes^L_C A$ for all $M \in D(\text{Gr}(C))$. Apply this to $M = C(-i)$ for $i \in \mathbb{Z}$ to conclude the proof. $\square$

**Remark 4.17.** With the help of computer programs, given any finite map $C \to A$ from a polynomial graded algebra, one should be able to compute a free resolution of $A$ as a graded $C$-module, from which one should be able to see if the given map $C \to A$ gives a non-positive presentation. In fact, it suffices to compute the weights of $\text{Tor}^A_*(A, k[z_1, \ldots, z_r])$ to verify that $C \to A$ gives a non-positive presentation.

Combining with the previous results, we then have the following

**Theorem 4.18.** For the class of 3-fold flips described by (4.12), there is a fully faithful functor $D_{\text{perf}}(X^+) \to D_{\text{perf}}(X^-)$ given by (4.2).

**Remark 4.19.** Let $X$ be a Q-Gorenstein variety such that $\mathcal{O}_X(iK_X)$ is maximal Cohen-Macaulay for each $i \in \mathbb{Z}$ (this is satisfied, e.g., if $X$ has at most log terminal singularities, see [7, Corollary 5.25]), one may define its Gorenstein root to be the Deligne-Mumford stack $\mathcal{T} := [\text{Spec}(\bigoplus_{i \in \mathbb{Z}} \mathcal{O}_X(iK_X))/\mathbb{G}_m]$. Theorem 4.18 then relates the derived categories of the Gorenstein roots of the two sides of the flip.

One can also compute the functor (4.2) in the present setting. We will work with (4.13) satisfying (4.14). Let $C = k[x_1, \ldots, x_p, y_1, \ldots, y_q, z_1, \ldots, z_r]$, and let $C^+ = k[x_1, \ldots, x_p]$ and $C^- = k[y_1, \ldots, y_q, z_1, \ldots, z_r]$, so that $C = C^+ \otimes C^-$. Let $\eta^+ = \sum_{i=1}^p \deg(x_i)$. Then we have

(4.20) For $0 \leq i < \eta^+$, the functor (4.2) for $w = 0$ sends $\mathcal{O}_{X^+}(-i)$ to $\mathcal{O}_{X^-}(-i)$.

(4.21) For $i > 0$, the functor (4.2) for $w = 0$ sends $\mathcal{O}_{X^+}(i)$ to the associated sheaf on $X^-$ of the complex of graded modules $((C^+)^{\geq i} \otimes C^-) \otimes^L_C A(i)$.

We may also want an analogue of Proposition 4.9. For that, we will need to replace the assumption (4.14) with the following stronger one (which is satisfied by (4.12)):

(4.22) $\dim(A) = p + q + r - s$, $\dim(A/I^+) = q + r - s$, and $\deg(f_i) \leq 0$ for each $1 \leq i \leq s$.

Then we have the following

**Proposition 4.23.** For $A$ in (4.13) satisfying (4.22), we have $D_{\text{perf},[\leq w]}(\text{Gr}(A)) = D_{\text{perf},[\leq w + \eta^+]}(\text{Gr}(A))$. As a result, the subcategory $D_{\text{perf},[\geq 0,<(\eta^+ + 1)]}(\text{Gr}(A))$ is split generated by $\{A, A(-1), \ldots, A(-\eta^+ + 1)\}$.

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