RATIONAL 6-CYCLES UNDER ITERATION OF QUADRATIC POLYNOMIALS

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Abstract. We present a proof, which is conditional on the Birch and Swinnerton-Dyer Conjecture for a specific abelian variety, that there do not exist rational numbers $x$ and $c$ such that $x$ has exact period $N = 6$ under the iteration $x \mapsto x^2 + c$. This extends earlier results by Morton for $N = 4$ and by Flynn, Poonen and Schaefer for $N = 5$.

1. Introduction

In this note, we present a conditional proof that there do not exist rational numbers $x$ and $c$ such that the sequence defined by $x_0 = x$, $x_{n+1} = x_n^2 + c$ (for $n \geq 0$) has exact period 6. The assumptions we have to make are that the $L$-series of a certain genus 4 curve $X_0^{\text{dyn}}(6)$ extends to an entire function and satisfies the usual kind of functional equation, and that the Jacobian of $X_0^{\text{dyn}}(6)$ satisfies the weak form of the Birch and Swinnerton-Dyer conjecture.

This extends a series of investigations on rational cycles under quadratic iteration. It is easy to see that fixed points and 2-cycles are each parametrized by a rational curve; the same is true for 3-cycles. Morton [Mor] has shown that 4-cycles are parametrized by the modular curve $X_1(16)$; he used this to show that there do not exist rational 4-cycles. Flynn, Poonen, and Schaefer [FPS] proved that there are no rational 5-cycles. The present paper gives a conditional proof that there are no rational 6-cycles. It is conjectured (see [Sil]) that there is a universal bound on the number of rational preperiodic points under quadratic iteration. In view of the results obtained so far, it seems reasonable to expect that there are no rational $N$-cycles when $N > 3$. Poonen [Poo] shows that this would imply that there can be at most 9 rational preperiodic points.

Here is an overview of the proof. Pairs $(x, c)$ such that $x$ is periodic of exact order 6 under the map $x \mapsto x^2 + c$ give rise to points on the affine curve $Y_1^{\text{dyn}}(6)$.
with equation
\[ \Phi_6^*(x, c) := \frac{(f^6(x, c) - x)(x^2 - x + c)}{(f^3(x, c) - x)(f^2(x, c) - x)} = 0, \]
where \( f^{(0)}(x, c) = x, f^{(n+1)}(x, c) = f^{(n)}(x^2 + c, c) \) denote the iterates of \( x \mapsto x^2 + c \).

(For some of the points, the orbit of \( x \) actually has exact order a proper divisor of 6; see [Sil] for details.) We denote by \( X_{1}^{\text{dyn}}(6) \) the smooth projective model of \( Y_{1}^{\text{dyn}}(6) \). This curve has an automorphism \( \sigma \) of order 6 that is induced by the map \((x, c) \mapsto (x^2 + c, c)\) on \( Y_{1}^{\text{dyn}}(6) \). We denote the quotient \( X_{1}^{\text{dyn}}(6)/\langle \sigma \rangle \) by \( X_{0}^{\text{dyn}}(6) \). This is a curve of genus 4. We determine the set of rational points on this curve, from which we can find the set of rational points on \( X_{1}^{\text{dyn}}(6) \). It turns out that all of these points are “cusps”, i.e., they are in the complement of \( Y_{1}^{\text{dyn}}(6) \) and hence do not correspond to pairs \((x, c)\) as above.

We first find a nice model of \( X_{0}^{\text{dyn}}(6) \) (see Section 2 below). There are ten rational points on this curve that are easy to find. We show that they generate a torsion-free subgroup \( G \) of rank 3 in the Mordell-Weil group \( J(\mathbb{Q}) \), where \( J \) is the Jacobian of \( X_{0}^{\text{dyn}}(6) \). We further show that there are no other rational points that map into the saturation of this subgroup in \( J(\mathbb{Q}) \). This is done in Section 3 below. It remains to show that \( G \) is a finite-index subgroup of \( J(\mathbb{Q}) \). It is in this part of the proof that we have to make assumptions on the \( L \)-series, since we want to use the Birch and Swinnerton-Dyer conjecture. We compute enough coefficients of the \( L \)-series to show that its third derivative at \( s = 1 \) does not vanish, which, according to the BSD conjecture, implies that the rank of \( J(\mathbb{Q}) \) is at most 3. See Section 4 below. (Note that a 2-descent on \( J \), which is the usual way to obtain an upper bound on the Mordell-Weil rank for low-genus curves, requires knowledge of the class and unit groups of a number field of degree 119. The necessary computations are utterly infeasible with current technology, even when assuming GRH.)

We have used the MAGMA [M] computer algebra system in order to perform the necessary computations. A script that can be loaded into MAGMA and that performs the relevant computations is available at [Scr].

This curve \( X_{0}^{\text{dyn}}(6) \) appears to be the first curve of genus \( \geq 4 \) that is not very special in some way, e.g., hyperelliptic, or covering a curve of smaller genus, or a modular or Shimura curve, for which the set of rational points could be explicitly determined (assuming reasonable standard conjectures). The methods used here should be applicable in other cases as well, provided

- we can find a finite-index subgroup of the Mordell-Weil group,
- its rank is less than the genus, and
- the conductor is reasonably small.
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2. The Model

In order to obtain a smooth projective model of $X_{0}^{\text{dyn}}(6)$, we first find an equation for $Y_{0}^{\text{dyn}}(6)$ (the image of $Y_{1}^{\text{dyn}}(6)$ in $X_{0}^{\text{dyn}}(6)$) as an affine plane curve. For a point $(x, c) \in Y_{1}^{\text{dyn}}(6)$, we denote the “trace” of its orbit by

$$t = x + (x^2 + c) + f^{(2)}(x, c) + \cdots + f^{(5)}(x, c).$$

The resultant with respect to $x$ of $\Phi^{6}(x, c)$ and $t - (f^{(0)}(x, c) + \cdots + f^{(5)}(x, c))$ is a sixth power; one of its sixth roots is

$$\Psi_6(t, c) = 256(t^3 + t^2 - t - 1)c^3 + 16(9t^5 + 10t^3 + 30t^2 - 19t - 37)c^2$$

$$+ 8(3t^7 + t^6 + 2t^5 - 17t^3 + 69t^2 + 52t - 48)c$$

$$+ t^9 - t^8 + 14t^6 + 49t^5 + 175t^4 + 140t^3 + 196t^2 + 448t.$$

We first resolve the singularities at infinity. Successively getting rid of multiple factors at the edges of the Newton polygon, we arrive at the equation

$$F(u, v) = (u^4 - u^3)v^3 + (-u^5 + 9u^4 + 6u^3 - 17u^2 + 3u)v^2$$

$$+ (4u^4 + 74u^3 - 52u^2 - 54u + 24)v$$

$$+ 4u^4 + 24u^3 + 117u^2 - 261u + 72 = 0.$$  

Here

$$u = \frac{2}{t + 1} \quad \text{and} \quad v = 4(c - 1) + (t - 1)^2 + \frac{2}{t + 1},$$

or

$$t = \frac{2}{u} - 1 \quad \text{and} \quad c = \frac{v}{4} - \frac{1}{u^2} + \frac{2}{u} - \frac{u}{4}.$$

The curve defined by this equation has three singularities at points $(\alpha, \beta)$, where

$$3\beta^3 + 32\beta^2 + 69\beta + 72 = 0 \quad \text{and} \quad 18\alpha = 6\beta^2 + 55\beta + 69.$$

From the Newton polygon of $F$, we see that the regular differentials on the smooth projective model of this curve are contained in the space spanned by
\{\omega_0, u\omega_0, uv\omega_0, u^2\omega_0, u^2v\omega_0, u^3\omega_0, u^3v\omega_0\}, where

\[ \omega_0 = \frac{du}{\partial v} = -\frac{ dv}{\partial u} F(u, v). \]

(See [Kho], in particular the example on page 42.) These differentials are regular everywhere except perhaps at the singularities described above. In order to get something regular there, the polynomial that \(\omega_0\) is multiplied by has to vanish at the singularities. We obtain the following basis of \(\Omega^1_{X_0^{\text{dyn}}(6)}\):

\begin{align*}
\omega_1 &= (u^3v + 2u^3 - 3u^2v - u^2 + 3uv + 6)\omega_0 \\
\omega_2 &= (u^3v + 2u^3 - u^2v + u^2 + 3u - 6)\omega_0 \\
\omega_3 &= (u^3v + 2u^3 - 4u^2v - 3u^2 + 3uv - 3u)\omega_0 \\
\omega_4 &= (3u^3v + 4u^3 - 3u^2v + 6u^2 - 6u)\omega_0
\end{align*}

The canonical model of a curve of genus 4 is the intersection of a quadric and a cubic in \(\mathbb{P}^3\). We see that \(u\) is a rational function of degree 3, which implies that the quadric splits, i.e., it is isomorphic to \(\mathbb{P}^1 \times \mathbb{P}^1\) over \(\mathbb{Q}\). So there is a model of \(X_0^{\text{dyn}}(6)\) that is a smooth curve of bidegree \((3, 3)\) in \(\mathbb{P}^1 \times \mathbb{P}^1\). To find a suitable second coordinate (besides \(u\)), we take the quotient of two differentials vanishing on \(u = 0\). This means that the differentials may not contain \(\omega_0\) or \(uv\omega_0\) with a nonzero coefficient. A possible choice is

\[ w = -\omega_1 - \omega_2 + \omega_3 + 2\omega_4 = -\frac{u^2v + 3uv + 18}{u^2v + 2u^2 + 3u + 6}. \]

In terms of \(u\) and \(w\), we now have \(X_0^{\text{dyn}}(6)\) as a smooth curve in \(\mathbb{P}^1 \times \mathbb{P}^1\), with (affine) equation

\[ G(u, w) = w^2(w + 1)u^3 - (5w^2 + w + 1)u^2 - w(u^2 - 2w - 7)u + (w + 1)(w - 3) = 0. \]

We will denote this curve by \(C\). Note that

\[ c = \frac{(-u^3 - 2u^2 + 5u - 10)uw - u^4 + 3u^3 + 8u^2 - 10u + 12}{4u^2(uw + u - 3)} \]

on this model.

Our model has good reduction except at 2 and at \(p = 8029187\). Mod \(p\), we have a node at \((u, w) = (2937959, 7887180)\) with tangent directions defined over \(\mathbb{F}_p\). This point is regular on the arithmetic surface given by our equation.

Mod 2, there is a node at \((1, 0)\) with tangent directions defined over \(\mathbb{F}_4\) and a tacnode at \((0, 1)\) with local branches again defined over \(\mathbb{F}_4\). Both singularities are non-regular points of the arithmetic surface. Resolving these points gives us the minimal proper regular model over \(\mathbb{Z}_2\). The node resolves into a chain of three \(\mathbb{P}^1\)'s whose ends intersect the original component. Blowing up the tacnode gives a double line, all of whose points are non-regular. Blowing up this line, we obtain a
smooth curve of genus 1, meeting the original components in two (regular) points. Therefore, the special fiber of the minimal proper regular model consists of five components $A, B, C, C', D$, each of multiplicity one. $A$ and $B$ are both elliptic curves with trace of Frobenius $-1$, the other components are $\mathbb{P}^1$'s. $A$, $B$, and $D$ are defined over $\mathbb{F}_2$, $C$ and $C'$ are defined over $\mathbb{F}_4$ and conjugate. The intersection matrix is as follows.

$$
\begin{array}{|c|c|c|c|c|}
\hline
  & A & B & C & C' \\
\hline
  A & -4 & 2 & 1 & 1 & 0 \\
  B & 2 & -2 & 0 & 0 & 0 \\
  C & 1 & 0 & -2 & 0 & 1 \\
  C' & 1 & 0 & 0 & -2 & 1 \\
  D & 0 & 0 & 1 & 1 & -2 \\
\hline
\end{array}
$$

(For some worked examples of how to compute minimal regular models, see for example [F+] or [PSS].)

The two (separate) intersection points of $A$ and $B$ are swapped by the action of Frobenius. We see that the reduction of the Jacobian has a 2-dimensional abelian and a 2-dimensional toric component (since the dual graph of the special fiber has two independent loops, compare [BLR, § 9.2]); Frobenius reverses the orientation of both loops. We can summarize our findings in the following lemma.

**Lemma 1.** The Jacobian of $C = X^\text{dyn}_0(6)$ has conductor $2^2 p$, where $p = 8029187$ is the big prime from above. Its Euler factor at 2 is $(1 + T + 2T^2)(1 + T)^2$.

We end this section by showing that $X^\text{dyn}_0(6)$ does not have any special geometrical properties that might help us.

**Lemma 2.** We have $\text{End}_{\mathbb{Q}} J = \mathbb{Z}$. In particular:

1. The automorphism group of $X^\text{dyn}_0(6)$ is trivial (even over $\overline{\mathbb{Q}}$).
2. The Jacobian of $X^\text{dyn}_0(6)$ is absolutely simple.
3. The curve $X^\text{dyn}_0(6)$ does not cover any other curve of positive genus, except itself (not even over $\overline{\mathbb{Q}}$).

**Proof.** We take inspiration from the proof of Prop. 9 in [FPS]. To make matters more concrete, we formulate a computational lemma.

**Lemma 3.** Let $C$ be a curve of genus $g$ over a number field $K$, with Jacobian $J$, let $v$ be a finite place of $K$ of good reduction for $C$, and let $f(T)$ be the Euler factor (i.e., the numerator of the zeta function) of $C$ at $v$. If $f \in \mathbb{Q}[T]$ is irreducible, and no monic irreducible factor of

$$
h(T) = \frac{\text{Res}_x(f(x), f(Tx))}{(1 - T)^{2g}}
$$

is reducible in $\mathbb{Q}$.
has integral coefficients and constant term 1, then \( \text{End}_K J \) embeds into the number field generated by a root of \( f \).

**Proof.** For the proof, note that the roots of \( h \) are all the quotients \( \alpha/\beta \), where \( \alpha \) and \( \beta \) are distinct roots of \( f \). If one of these quotients is a root of unity, then \( h \) has a monic irreducible factor that has integral coefficients and constant term 1 (namely, some cyclotomic polynomial). Conversely, if there is such an irreducible factor, then its roots are units in the splitting field of \( f \), and they have absolute value 1 in all complex embeddings (since \( |\sigma(\alpha)| = q^{-1/2} \) for all complex embeddings \( \sigma \) and all roots \( \alpha \) of \( f \), where \( q \) is the size of the residue class field \( k_v \)). Hence some \( \alpha/\beta \) is a root of unity.

Now this is the case if and only if, for some \( n \geq 2 \), there are distinct roots \( \alpha \) and \( \beta \) of \( f \) such that \( \alpha^n = \beta^n \). This in turn is equivalent to the Galois orbit of \( \alpha^n \) having size less than \( \deg f = 2g \), which means that the characteristic polynomial of the \( n \)th power of the \( v \)-Frobenius is not irreducible.

Our assumptions therefore imply that all these characteristic polynomials are irreducible. (An argument like this was used in [St1] to show that certain genus 2 Jacobians are absolutely simple.) From [WM Thm. 8], we then see that the endomorphism algebra of \( J \) over \( k_v \) is the number field generated by a root of \( f \), and since the endomorphism ring of \( J \) (over \( \bar{K} \)) embeds into this algebra, the claim is proved. (Note that \( J \) is simple over \( k_v \) since \( f \) is irreducible.) \( \square \)

To prove Lemma 2, we compute the Euler factors at \( p = 5 \) and \( p = 7 \). They are

\[
1 + 3T + 6T^2 + 6T^3 - 8T^4 + 150T^5 + 375T^7 + 625T^8.
\]

and

\[
1 + 7T + 28T^2 + 94T^3 + 276T^4 + 658T^5 + 1372T^6 + 2401T^7 + 2401T^8.
\]

We observe that both polynomials satisfy the assumptions in Lemma 3 and that the number fields they generate are linearly disjoint over \( \mathbb{Q} \). This proves the first claim.

Statement (1) then follows, since any nontrivial automorphism of the curve would induce a nontrivial automorphism of the Jacobian \( J \). But the only nontrivial automorphism of \( J \) is multiplication by \( -1 \), and if it would come form an automorphism of the curve, this would imply that the curve is hyperelliptic, which is not the case. Alternatively, we can use the fact that any automorphism of our curve must extend to an automorphism of \( \mathbb{P}^1 \times \mathbb{P}^1 \). Such automorphisms either perform a Möbius transformation on each of the factors separately, or else this type of automorphism is followed by swapping the two factors. A Gröbner basis computation shows that the only automorphism of \( \mathbb{P}^1 \times \mathbb{P}^1 \) that fixes the curve is the identity.
If statement (2) were false, then the algebra $\mathbb{Q} \cong \mathbb{Q} \otimes \mathbb{Z} \text{End}_{\mathbb{Q}} J$ would have zero divisors, which is not the case.

Finally, if the curve covers another curve of positive genus and the map is not an isomorphism, then the other curve has genus strictly between 0 and 4. But then its Jacobian will be a factor of the Jacobian of $X_0^{\text{dyn}}(6)$, so the latter would have to split, contradicting the fact that $J$ is absolutely simple. $\square$

3. Rational Points

A quick search finds the following ten rational points on $C$.

| $u$ | $w$ | $t$ | $c$ |
|-----|-----|-----|-----|
| $P_0$ | 0 | $\infty$ | $\infty$ |
| $P_1$ | 0 | $-1$ | $\infty$ |
| $P_2$ | 0 | 3 | $\infty$ |
| $P_3$ | $\infty$ | 0 | $-1$ |
| $P_4$ | 1 | 2 | 1 |

| $u$ | $w$ | $t$ | $c$ |
|-----|-----|-----|-----|
| $P_5$ | 2 | 1 | 0 |
| $P_6$ | 1 | $\infty$ | 1 |
| $P_7$ | $\infty$ | $-1$ | 1 |
| $P_8$ | $-1$ | $\infty$ | $-3$ |
| $P_9$ | $-\frac{4}{5}$ | $-1$ | $-\frac{7}{2}$ |

The first five of these are the “cusps”; these are the points that have to be added to $Y_0^{\text{dyn}}(6)$ in order to obtain a smooth projective curve. It is known that all cusps on $X_1^{\text{dyn}}(N)$ and hence also on $X_0^{\text{dyn}}(N)$ are rational points, for all $N$.

The remaining five points correspond to cycles of length 6 for the given value of $c$ that are stable as a set (or as a cycle) under the action of the absolute Galois group of $\mathbb{Q}$. For the special values $c = 0$ and $c = -2$, these cycles are “predictable”; they come from roots of unity. For $N = 6$, we find cycles containing $\zeta_9$ when $c = 0$ and cycles containing $\zeta_{13} + \zeta_{13}^{-1}$ (this is the one whose trace $t$ is $-1$) or $\zeta_{21} + \zeta_{21}^{-1}$ (with $t = 1$) when $c = -2$. (We use $\zeta_n$ to denote a primitive $n$th root of unity; these cycles then contain all possible values of the above expressions.)

For $c = -4$, the points in the cycle live in a sextic abelian number field with discriminant $5^3 \cdot 7^4$ and conductor 35; it is the field $\mathbb{Q}(\sqrt{5}, \cos \frac{2\pi}{12})$. Finally, for $c = -\frac{71}{48}$, we find points defined over the quadratic field $\mathbb{Q}(\sqrt{33})$; one point in the cycle is $x = -1 + \frac{1}{12} \sqrt{33}$. In particular, this means that $X_1^{\text{dyn}}(6)(\mathbb{Q}(\sqrt{33}))$ contains an orbit of six non-cuspidal points.

See also the end of [FPS].

We will now prove the following result.

**Lemma 4.** Let $J$ denote the Jacobian of $C = X_0^{\text{dyn}}(6)$.

1. $J(\mathbb{Q})$ has trivial torsion subgroup.
The subgroup $G$ of $J(\mathbb{Q})$ generated by the classes of divisors supported in the 10 rational points listed above is isomorphic to $\mathbb{Z}^3$.

The subgroup is already generated by divisors supported at the cusps.

**Proof.** We know that the prime-to-$p$ torsion in $J(\mathbb{Q})$ injects into $J(\mathbb{F}_p)$ for primes of good reduction, so the observation that (as computed by MAGMA)

$$#J(\mathbb{F}_7) = 2 \cdot 7 \cdot 11 \cdot 47 \quad \text{and} \quad #J(\mathbb{F}_{13}) = 3 \cdot 17 \cdot 23 \cdot 43$$

shows that $J(\mathbb{Q})$ has trivial torsion subgroup.

The main tool for proving the other assertions is the homomorphism

$$\Phi_S : \bigoplus_{i=0}^9 \mathbb{Z}P_i \longrightarrow \text{Pic}_C \longrightarrow \prod_{p \in S} \text{Pic}_C/F_p,$$

where $S$ is a set of primes of good reduction. We take $S = \{3, 5, 7, 11, 13\}$ and compute the kernel of $\Phi_S$. This kernel is a subgroup of rank 9 in $\mathbb{Z}^{10} = \bigoplus \mathbb{Z}P_i$. We apply LLL to it and find that there are six independent elements with very small coefficients (and three large additional basis vectors). We suspect that the small elements come from actual relations between our points; this can then be verified by exhibiting a suitable rational function. (MAGMA provides the necessary functionality for these computations.) Denoting linear equivalence by ‘∼’, we find the following six independent relations.

\[
\begin{align*}
P_0 + P_6 + P_8 & \sim P_1 + P_7 + P_9 \\
P_0 + P_1 + P_2 & \sim 2P_3 + P_7 \\
& \sim 2P_4 + P_6 \\
P_0 + P_2 + P_7 + P_6 & \sim P_1 + P_3 + P_5 + P_6 \\
2P_0 + P_1 + P_6 & \sim 2P_2 + P_3 + 2P_5 \\
3P_0 + P_3 & \sim P_1 + P_2 + P_6 + P_8
\end{align*}
\]

On the other hand, looking at the image of $\Phi_S$, we see that the degree 0 subgroup of $\mathbb{Z}^{10}$ surjects onto $(\mathbb{Z}/3\mathbb{Z})^3$. Since we know that there is no torsion in $J(\mathbb{Q})$, this implies that the rank of the image of the degree 0 subgroup in $J(\mathbb{Q})$ must be at least 3. The existence of the relations above implies that the rank is at most 3, so the rank is exactly 3, and since there is no torsion, the group must be isomorphic to $\mathbb{Z}^3$.

Finally, from the relations we have given it is easy to verify that $P_3$ and $P_5, \ldots, P_9$ can be expressed in terms of $P_0, P_1, P_2,$ and $P_4$. This means that our subgroup is already generated by divisors supported at the latter four points (all of which are cusps). The only relation between the cusps is

$$5P_0 - 10P_1 - 2P_2 + P_3 + 6P_4 \sim 0;$$
it is perhaps worth noting that this relation is \textit{not} induced by the standard “dynamical units” as provided by [Sil, Thm. 2.33] or [Nar], applied to the coordinate ring of $Y_{1}^{\text{dyn}}(6)$. □

Our next result is as follows.

**Lemma 5.** The ten points $P_0, \ldots, P_9$ are the only rational points whose images in $	ext{Pic}_C$ are in the saturation of the subgroup described in the previous lemma.

**Proof.** We use Chabauty’s method (see for example [Cha, Col, MP, Si2]) for the proof. Recall that there is a pairing

$$\Omega^1_1(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p, \quad (\omega, Q) \longmapsto \int_0^Q \omega$$

that induces a perfect $\mathbb{Q}_p$-bilinear pairing

$$\Omega^1_1(\mathbb{Q}_p) \times J(\mathbb{Q}_p)^1 \otimes \mathbb{Z}_p \mathbb{Q}_p \longrightarrow \mathbb{Q}_p,$$

where $J(\mathbb{Q}_p)^1$ denotes the kernel of reduction. If $G \subset J(\mathbb{Q}_p)$ is a subgroup of rank less than $\dim J = 4$, then there must be a nonzero differential $\omega$ that kills $G$ under this pairing. Note that $\omega$ then also kills the saturation

$$\bar{G} = \{ P \in J(\mathbb{Q}_p) : nP \in G \text{ for some } n \geq 1 \}$$

of $G$. We will apply this with $p = 5$ and $G$ the subgroup generated by the known rational points as above.

For points in the kernel of reduction, the integral can be evaluated by formally integrating the power series representing $\omega$ in terms of a system of local parameters at the origin and then plugging in the values at $Q$ of these parameters. For practical computations, it is more convenient to use the canonical identification $\Omega^1_1(\mathbb{Q}_p) \cong \Omega^1_C(\mathbb{Q}_p)$.

Let $P' \in C(\mathbb{Q})$ be a fixed base-point. Let $Q \in J(\mathbb{Q}_p)^1$. Then $Q$ is represented by a divisor of the form $(Q_1 + Q_2 + Q_3 + Q_4) - 4P'$, where the points $Q_j \in C(\mathbb{Q}_p)$ all reduce to $P'$ modulo the prime above $p$ in their field of definition. Let $\tau$ be a uniformizer at $P'$ that reduces mod $p$ to a uniformizer at the reduction of $P'$. The differential $\omega$ can be written as $\phi(\tau) d\tau$ with a power series $\phi \in \mathbb{Q}_p[T]$. Let

$$\lambda = \lambda_1 T + \lambda_2 T^2 + \ldots$$

be its formal integral. Then

$$\int_0^Q \omega = \sum_{j=1}^4 \lambda(Q_j) = \sum_{n=1}^\infty \lambda_n \sum_{j=1}^4 \tau(Q_j)^n;$$
the series converges in $\mathbb{Q}_p$. Note that the power sums can be computed from the coefficients of the characteristic polynomial

$$(X - \tau(Q_1))(X - \tau(Q_2))(X - \tau(Q_3))(X - \tau(Q_4)),$$

which lie in the field of definition of $\mathcal{O}$. 

In our concrete case, we take $P = P_1$. Applying LLL to the kernel of the reduction map $\bigoplus_{j \neq 1} \mathbb{Z}(P_j - P_1) \to J(\mathbb{F}_5)$, we find a basis of $G \cap J(\mathbb{Q}_5)^1$, given by

$$D_1 = P_7 - P_9$$
$$D_2 = P_0 - 6P_1 + 2P_5 + P_7 + P_8 + P_9$$
$$D_3 = P_0 - 3P_1 + 2P_2 + P_4 + P_6 - P_7 - P_8$$

For each of these, we find $D_j'$ such that $D_j \sim D_j' - 4P'$ and $D_j'$ is effective of degree 4, with points reducing to $P'$. The point $P' = P_1$ has coordinates $(u, w) = (0, -1)$; we can choose $u$ as a uniformizer at $P'$ and its reduction. The space of regular differentials is spanned by

$$\omega_0 = \frac{du}{\partial u} G(u, w), \quad \omega_1 = u \omega_0, \quad \omega_2 = w \omega_0, \quad \text{and} \quad \omega_3 = uw \omega_0.$$

We expand each $\omega_i$ as a power series in $u$ times $du$ and let $\lambda_i \in u\mathbb{Q}[u]$ be its formal integral. Then we evaluate each $\lambda_i$ at each $D_j'$ as described above. We determine the kernel of the resulting matrix, which gives us the differential $\omega$ that kills our subgroup $G$. We find that reduced mod 5, this differential is $\bar{\omega} = \bar{\omega}_2$. It vanishes at the points where $w = 0$ or $u = \infty$. There are two such points in $C(\mathbb{F}_5)$, namely $(\infty, -1)$ and $(\infty, 0)$. At the former, $\bar{\omega}$ vanishes to first order, which implies that there are at most two rational points in that residue class (see for example [St2]).

Since we have the points $P_7 = (\infty, -1)$ and $P_9 = (-4/5, -1)$, these must be all the rational points in this residue class. At $(\infty, 0)$, we compute explicitly that the logarithm $\lambda$ that vanishes on $C(\mathbb{Q})$ on this residue class is

$$\lambda = \gamma \tau \left( 1 - (2 + O(5))5\tau + O(5^2) \right)$$

with some constant $\gamma \neq 0$, where $5\tau$ is the uniformizer $w$ at $(\infty, 0)$. So $\lambda$ has a single zero on this residue class, which is taken care of by $P_3 = (\infty, 0)$. On all other points in $C(\mathbb{F}_5)$, $\bar{\omega}$ does not vanish, hence there can be at most one rational point in each of these residue classes. Since it is easily checked that $\{P_0, \ldots, P_9\} \to C(\mathbb{F}_5)$ is surjective, this shows that there are no other points $P$ in $C(\mathbb{Q})$ such that $P - P'$ is in $G$. In fact, there is no such point that maps into the saturation of $G$ in $J(\mathbb{Q}_5)$ (since $\omega$ kills $G$). So there is no rational point on $C$ mapping into the saturation of $G$ other than those already known.

\[\square\]

**Theorem 6.** If $\text{rank } J(\mathbb{Q}) = 3$, then $X_0^{\text{dyn}}(6)$ has only the ten rational points listed above. In particular, it then follows that the only rational points on $X_1^{\text{dyn}}(6)$ are
the cusps, so that there is no cycle of exact length 6 consisting of rational numbers under an iteration $x \mapsto x^2 + c$.

**Proof.** If $J(\mathbb{Q})$ has rank 3, then $J(\mathbb{Q})$ is the saturation of $G$, the subgroup generated by degree 0 divisors supported on the known rational points, since the latter then has finite index in $J(\mathbb{Q})$. The previous lemma then shows that there are no other rational points on $X^\text{dyn}_0(6)$ than those already known. None of the non-cuspidal points among these lift to a rational point on $X^\text{dyn}_1(6)$, so the latter curve can have no non-cuspidal rational points. A rational 6-cycle would give rise to a non-cuspidal rational point on this curve, so such a rational 6-cycle cannot exist. \qed

4. **Bounding the Rank**

It remains to show that rank $J(\mathbb{Q}) = 3$. We know that the rank is at least 3, so it suffices to show that it is at most 3.

The standard procedure for obtaining an upper bound for the rank is a descent on the Jacobian. However, the complexity of this quickly becomes prohibitive when the genus is not very small and the curve does not have any helpful special features. For example, 2-descent on Jacobians of general non-hyperelliptic genus 3 curves is still in its infancy and so far has been successful in only one example (assuming GRH for the computation). Here, we have a curve of genus 4, and it appears that there are no helpful special properties, see Lemma 2 above. Usually, our best bet is a 2-descent, and for this, the best approach seems to be to look at the odd theta characteristics (whose differences generate the 2-torsion subgroup). On a curve of genus 4, there are 120 of them; they correspond to $(1,1)$-forms on $\mathbb{P}^1 \times \mathbb{P}^1$ that meet the curve tangentially in three points (more precisely, the intersection divisor is twice an effective divisor of degree 3). We can set up the scheme describing these; after a Gröbner basis computation, we find that it has one rational point, and the other 119 points form a single Galois orbit. This means that in order to do anything in the direction of a 2-descent, we would have to compute the ideal class group and fundamental units of a number field of degree 119. Before we are able to perform such computations (even if we allow ourselves to assume GRH), we need very substantial progress in the development of suitable algorithms.

The result on the Galois orbits on the odd theta characteristics can be obtained faster by computing the Gröbner bases of the scheme over $\mathbb{F}_5$ and over $\mathbb{F}_{13}$. We first note that $1 + u + w$ gives rise to the point defined over $\mathbb{Q}$ (the $(1,1)$-forms are of the form $a + bu + cw + duw$). Over $\mathbb{F}_5$, the remaining 119 points split into nine orbits of length 7 and four orbits of length 14, whereas over $\mathbb{F}_{13}$, they split into seven orbits of length 17. Since these partitions must refine the orbit partition over $\mathbb{Q}$, there must be a single orbit of length 119.
We can extract some more information. First note that the theta characteristics can be identified with the 2-torsion subgroup \( J[2] \) (by sending the unique odd theta characteristic that is defined over \( \mathbb{Q} \) to the origin). The Galois action on \( J[2] \) must then have orbits of lengths 1 and 119. We will determine the image of Galois in \( \text{Sp}_8(\mathbb{F}_2) \) and deduce that the remaining 136 elements also form a single orbit.

The Frobenius automorphisms at \( p = 5 \) and 13, acting on \( J[2] \), have orders 14 and 17, as we saw above. There is only one maximal subgroup \( \Gamma \) of \( \text{Sp}_8(\mathbb{F}_2) \) (up to conjugacy) whose order is a multiple of \( 14 \cdot 17 \). There is a maximal subgroup of \( \Gamma \) with this property, but it does not contain elements of order 14. Since the action of the full group \( \text{Sp}_8(\mathbb{F}_2) \) is transitive on \( J[2] \setminus \{0\} \), the image of the Galois group in \( \text{Sp}_8(\mathbb{F}_2) \) must be \( \Gamma \) (which is a subgroup of index 120). Since the action of \( \Gamma \) on \( J[2] \) has orbits of lengths 1, 119, and 136, our claim follows. It can be checked that the smallest faithful permutation representation of \( \Gamma \) has degree 119, so that this is really the smallest possible degree of a number field that we can hope for in a 2-descent computation.

Note also that we showed in Lemma 2 that \( J \) has no endomorphisms other than the multiplication-by-\( n \) maps, so that multiplication-by-2 is the isogeny \( J \to J \) of lowest possible degree that can be used for a descent argument. There are no nontrivial Galois-stable subgroups of \( J[2] \), so there are no 2-isogenies to other abelian varieties either.

A possible alternative approach to obtaining a bound for the rank assumes the (weak) Birch and Swinnerton-Dyer conjecture (plus standard conjectures on analytic continuation and functional equations of \( L \)-series, see for example [Hul, Conjs. 2.8.1. and 3.1.1]). The conjecture predicts that the rank of \( J(\mathbb{Q}) \) is the same as the order of vanishing of the \( L \)-series \( L(J, s) = L(C, s) \) at \( s = 1 \). In order to be able to evaluate the \( L \)-series and its derivatives there, we need to compute its coefficients \( a_n \) for values of \( n \) up to a suitable multiple of the square root of its conductor. Luckily, in our case the conductor \( 2^2 \cdot 8029187 \) is not too large, so that we can actually perform the computation in reasonable time.

We use Tim Dokchitser's \( L \)-series package [Dok] in its MAGMA implementation. We will not need to find the Euler factor at the large bad prime (it is beyond the necessary range of coefficients). For the other bad prime 2, we found the Euler factor in Section 2. For the good primes, we need the Euler factor up to \( T^d \), where \( d = \lfloor \log_p N \rfloor \) and \( N \) is the number of coefficients required. This information can be obtained by counting the number of points in \( \mathcal{C}(\mathbb{F}_p^e) \) for \( e = 1, \ldots, \min\{d, 4\} \). For a precision of \( 10^{-20} \), we need 183997 coefficients (which we can compute in a day or so). We verify numerically that our \( L \)-series satisfies the functional equation it is supposed to satisfy (with sign \(-1\)). Then we find that the \( L \)-series and its first two derivatives vanish at \( s = 1 \) to the given precision, whereas \( L''(C, 1) = 0.83601 \ldots \) is clearly nonzero. Assuming the weak Birch and Swinnerton-Dyer conjecture
for $J$, this implies that $\text{rank } J(\mathbb{Q}) \leq 3$. We therefore obtain our main result below.

**Theorem 7.** Let $J$ be the Jacobian of $X^\text{dyn}_0(6)$. If the $L$-series $L(J, s)$ extends to an entire function and satisfies the standard functional equation, and if the weak Birch and Swinnerton-Dyer conjecture is valid for $J$, then there are no rational cycles of exact length $6$ under $x \mapsto x^2 + c$.

5. What Next?

Without fundamentally new ideas, it seems unlikely that we can make our result unconditional in the foreseeable future. In another direction, it looks rather hopeless to try to get a similar result for $X^\text{dyn}_0(7)$. This curve has genus $16$ and bad reduction at the $35$-digit prime $p = 84562621221359775358188841672549561$ and possibly at $2$. In any case, the conductor will be very large (at least $p$) and so there will be no chance whatsoever to use the $L$-series numerically to obtain information on the rank. It might still be possible to get some information on the subgroup of the Jacobian generated by the cusps (e.g., by making use of dynamical units). It will be very hard, however, to use this information for a Chabauty argument, for example.

Another question is how the large bad primes can be explained or even predicted. We have $3701$ for $N = 5$ (the only bad prime for $X^\text{dyn}_0(5)$, see [FPS]; their model is also bad at $2$, but this can easily be repaired), $8029187$ for $N = 6$ and the $35$-digit prime above for $N = 7$. Note that unless we can shed light on this question, it is likely to be very hard to try and prove algebraically that $Y^\text{dyn}_1(N)$ is smooth, since such a proof must break down when the characteristic is one of these primes.

The following could be a possible line of attack for a proof that $Y^\text{dyn}_1(N)(\mathbb{Q})$ is empty for large $N$. There is a good description of the formal neighborhoods of the cusps on $X^\text{dyn}_1(N)$, using symbolic dynamics. If we could use this to prove, for any odd prime $p$, that the cusp is the only rational point in its residue class mod $p$, and also to prove a similar statement modulo a suitable power of $2$, then this would imply that a parameter $c \in \mathbb{Q}$ that allows for a cycle of exact length $N$ of rational numbers must be essentially integral (more precisely, its denominator must divide a fixed power of $2$). It is then fairly easy to show that $N$ is bounded.

For example, assume that $c \in \mathbb{Z}$ and $x$ is in a cycle. Then $x$ must also be an integer. If $c > 0$, we have $x^2 + c > |x|$, so there cannot be a cycle. For $c = 0$, the only possibilities are $x = 0$ and $x = 1$. For $c = -1$, the only possibilities are $x = 0$ and $x = -1$. If $c < -1$, we have $x^2 + c > |x|$ whenever $|x| \geq \sqrt{|c|} + 1$. So we must have $|x| < \sqrt{|c|} + 1$. But then we also need that $|x^2 + c| < \sqrt{|c|} + 1$, which implies that $\sqrt{|c|} - \sqrt{|c|} - 1 < |x| < \sqrt{|c|} + 1$. This interval has length less
than 2, so there are at most two possible values for $|x|$. This implies that there must be either a fixed point or a cycle of length 2. Indeed, for any $x \in \mathbb{Z}$, $x$ is a fixed point for $c = x - x^2$, and $(x, -x - 1)$ is a 2-cycle for $c = -x^2 - x - 1$.

In the more general case when $m^2c \in \mathbb{Z}$ for some fixed integer $m \geq 1$, we can use similar arguments to show that the cycle must be contained in a union of a bounded number of intervals whose lengths are bounded. Since the possible values of $x$ are in the set $\frac{1}{m}\mathbb{Z}$, there must be a bound on their number.

In the spirit of the methods used in this paper, the necessary result for odd primes $p$ would follow from the following two statements.

1. The cuspidal group (i.e., the group generated by degree zero divisors supported at the cusps) is of finite index in the Mordell-Weil group of the Jacobian of $X_1^{\text{dyn}}(N)$.

2. For every cusp $P$, there is a regular differential on $X_1^{\text{dyn}}(N)$, defined over $\mathbb{Q}_p$, that kills the cuspidal group and whose reduction mod $p$ does not vanish at $P$.

However, we need some additional ideas to approach a proof of either one of these.

References

- [BLR] S. Bosch, W. Lütkebohmert, and M. Raynaud, Néron models, Erg. Math. Grenzgeb. 3. Folge, Bd. 21, Springer, Berlin Heidelberg, 1990.
- [Cha] C. Chabauty, Sur les points rationnels des courbes algébriques de genre supérieur à l’unité (French), C. R. Acad. Sci. Paris 212, 882–885 (1941).
- [Col] R.F. Coleman, Effective Chabauty, Duke Math. J. 52, 765–770 (1985).
- [Dok] T. Dokchitser, Computing special values of motivic $L$-functions, Experiment. Math. 13, 137–149 (2004).
- [F+] E.V. Flynn, F. Leprévost, E.F. Schaefer, W.A. Stein, and J.L. Wetherell, Empirical evidence for the Birch and Swinnerton-Dyer conjectures for modular Jacobians of genus 2 curves, Math. Comp. 70, 1675–1697 (2001).
- [FPS] E.V. Flynn, B. Poonen, and E.F. Schaefer, Cycles of quadratic polynomials and rational points on a genus-2 curve, Duke Math. J. 90, 435–463 (1997).
- [Hul] W.W.J. Hulsbergen, Conjectures in arithmetic algebraic geometry, 2nd revised edition, Friedr. Vieweg & Sohn, Braunschweig/Wiesbaden, 1994.
- [Kho] A.G. Khovanskii, Newton polyhedra and the genus of complete intersections, Funct. Anal. and Appl. 12, 38–46 (1978). (Translated from Russian).
- [M] MAGMA is described in W. Bosma, J. Cannon, and C. Playoust, The Magma algebra system I: The user language, J. Symb. Comp. 24, 235–265 (1997). (Also see the Magma home page at http://www.maths.usyd.edu.au:8000/u/magma/.)
- [MP] W. McCallum and B. Poonen, The method of Chabauty and Coleman, Preprint (2007). Available at http://math.berkeley.edu/~poonen/papers/chabauty.pdf
- [Mor] P. Morton, Arithmetic properties of periodic points of quadratic maps, II, Acta Arith. 87, 89–102 (1998).
- [Nar] W. Narkiewicz, Polynomial cycles in algebraic number fields, Colloq. Math. 58, 151–155 (1989).
[Poo] B. Poonen, *The classification of rational preperiodic points of quadratic polynomials over \( \mathbb{Q} \): a refined conjecture*, Math. Z. 228, 11–29 (1998).

[PSS] B. Poonen, E.F. Schaefer, and M. Stoll, *Twists of \( X(7) \) and primitive solutions to \( x^2+y^3 = z^7 \)*, Duke Math. J. 137, 103–158 (2007).

[Sil] J.H. Silverman, *The arithmetic of dynamical systems*, Springer Graduate Texts in Mathematics 241, 2007.

[St1] M. Stoll, *Two simple 2-dimensional abelian varieties defined over \( \mathbb{Q} \) with Mordell-Weil group of rank at least 19*, C. R. Acad. Sci. Paris 321, Série I, 1341–1345 (1995).

[St2] M. Stoll, *Independence of rational points on twists of a given curve*, Compositio Math. 142, 1201–1214 (2006).

[Scr] M. Stoll, MAGMA script accompanying this paper. Available at [http://www.faculty.jacobs-university.de/mstoll/magma/scripts/Xdyn06.magma](http://www.faculty.jacobs-university.de/mstoll/magma/scripts/Xdyn06.magma)

[WM] W.C. Waterhouse and J.S. Milne, *Abelian varieties over finite fields*, pages 53–64 in *1969 Number Theory Institute (Stony Brook, N.Y., 1969)*, Proc. Sympos. Pure Math. 20, Amer. Math. Soc., Providence, 1971.

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