Toda equation and special polynomials associated with the Garnier system

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Abstract

We prove that a certain sequence of tau functions of the Garnier system satisfies Toda equation. We construct a class of algebraic solutions of the system by the use of Toda equation; then show that the associated tau functions are expressed in terms of the universal character, which is a generalization of Schur polynomial attached to a pair of partitions.

This article is based on the results in the author’s Ph.D thesis [19].
Introduction

The Garnier system is the following completely integrable Hamiltonian system of partial differential equations (see [1, 2, 4]):

\[ \frac{\partial q_i}{\partial s_j} = \frac{\partial H_j}{\partial p_i}, \quad \frac{\partial p_i}{\partial s_j} = -\frac{\partial H_j}{\partial q_i}, \quad (i, j = 1, \ldots, N), \quad (0.1a) \]

with Hamiltonians

\[ s_i(s_i - 1)H_i = q_i \left( \alpha + \sum_j q_j p_j \right) \left( \alpha + \kappa_\infty + \sum_j q_j p_j \right) + s_i p_i (q_i p_i - \theta_i) \]

\[ - \sum_{j \neq i} R_{ij}(q_j p_j - \theta_j) q_j p_j - \sum_{j \neq i} S_{ij}(q_i p_i - \theta_i) q_j p_j \]

\[ - \sum_{j \neq i} R_{ij} q_j p_j (q_i p_i - \theta_i) - \sum_{j \neq i} R_{ij} q_i p_i (q_j p_j - \theta_j) \]

\[ - (s_i + 1)(q_i p_i - \theta_i) q_i p_i + (\kappa_1 s_i + \kappa_0 - 1) q_i p_i, \quad (0.1b) \]

where \( R_{ij} = \frac{s_i(s_j - 1)}{(s_j - s_i)} \), \( S_{ij} = \frac{s_i(s_i - 1)}{(s_i - s_j)} \) and

\[ \alpha = -\frac{1}{2} \left( \kappa_0 + \kappa_1 + \kappa_\infty + \sum_i \theta_i - 1 \right). \quad (0.2) \]

Here the symbols \( \sum_i \) and \( \sum_{i \neq j} \) stand for the summation over \( i = 1, \ldots, N \) and over \( i = 1, \ldots, j - 1, j + 1, \ldots, N \), respectively. System (0.1) contains \( N + 3 \) constant parameters

\[ \vec{\kappa} = (\kappa_0, \kappa_1, \kappa_\infty, \theta_1, \ldots, \theta_N) \in \mathbb{C}^{N+3}, \quad (0.3) \]

so that we often denote it by \( \mathcal{H}_N = \mathcal{H}_N(\vec{\kappa}) = \mathcal{H}_N(q, p, s, H; \vec{\kappa}) \), and so on. The Garnier system governs the monodromy preserving deformation of a Fuchsian differential equation with \( N + 3 \) singularities and is an extension of the sixth Painlevé equation \( P_{VI} \); for \( N = 1 \), (0.1) is equivalent to the Hamiltonian system of \( P_{VI} \) (see [13]), in fact.

In this paper, we prove that a certain sequence of \( \tau \)-functions of the Garnier system satisfies Toda equation. We construct a class of algebraic solutions of the system by using Toda equation; then show that the corresponding \( \tau \)-functions are expressed in terms of the universal character, which is a generalization of Schur polynomial attached to a pair of partitions.

First we introduce a group of birational canonical transformations of the Garnier system \( \mathcal{H}_N \). The group forms an infinite group which contains a translation \( \mathbb{Z} \); see Sect. 1. We define a function \( \tau = \tau(s; \vec{\kappa}) \), called the \( \tau \)-function (see [2, 4]), by

\[ d \log \tau = \sum_i H_i ds_i. \quad (0.4) \]

By the use of birational symmetries of \( \mathcal{H}_N \), we have the
Theorem 0.1. A certain sequence of \( \tau \)-functions \( \{ \tau_n | n \in \mathbb{Z} \} \) satisfies the Toda equation:

\[
XY \log \tau_n = c_n \frac{\tau_{n-1} \tau_{n+1}}{\tau_n^2},
\]

where \( X, Y \) being vector fields such that \([X, Y] = 0\) and \( c_n \) a nonzero constant.

Consider the fixed point of a certain birational symmetry, we obtain an algebraic solution of the Garnier system. For example, if \( \kappa_0 = \kappa_1 = 1/2 \), then \( \mathcal{H}_N \) admits an algebraic solution

\[
(q_i, p_i) = \left( \frac{\theta_i \sqrt{s_i}}{\kappa_\infty}, \frac{\kappa_\infty}{2\sqrt{s_i}} \right), \quad i = 1, \ldots, N.
\]

Applying the action of the group of birational symmetries, we thus have the

Theorem 0.2. If two components of the parameter \( \vec{\kappa} = (\kappa_0, \kappa_1, \kappa_\infty, \theta_1, \ldots, \theta_N) \) are half integers then the Garnier system \( \mathcal{H}_N \) admits an algebraic solution.

Secondly we investigate the \( \tau \)-functions associated with algebraic solutions of the Garnier system. Starting from the \( \tau \)-function corresponding to an algebraic solution, we determine a sequence of \( \tau \)-functions by means of Toda equation. Such a sequence of \( \tau \)-functions is converted to polynomials \( T_{m,n} = T_{m,n}(t) \) \( (m, n \in \mathbb{Z}) \) through a certain normalization, where \( t = (t_1, \ldots, t_N) \) and \( t_i = \sqrt{s_i} \). We call \( T_{m,n} \) special polynomials associated with algebraic solutions of \( \mathcal{H}_N \) (see Sect. 3). Algebraic solutions are explicitly written in terms of the special polynomials.

Theorem 0.3. If \( \kappa_0 = 1/2 + m + n, \kappa_1 = 1/2 + m - n \) \( (m, n \in \mathbb{Z}) \), then \( \mathcal{H}_N \) admits an algebraic solution given by

\[
q_i = t_i \frac{\partial}{\partial t_i} \log \frac{T_{m+1,n}}{T_{m,n+1}} \sum_j t_j \frac{\partial}{\partial t_j} \log \frac{T_{m+1,n}}{T_{m,n+1}} - 2m + 2n - 1,
\]

\[
2q_ip_i = \theta_i + m + n + t_i \frac{\partial}{\partial t_i} \log \frac{T_{m,n}}{T_{m,n+1}}.
\]

(See Theorem 3.3) Note that we immediately obtain also the expressions of the other algebraic solutions in Theorem 0.2 via the birational symmetries of \( \mathcal{H}_N \). Finally we give an explicit formula for \( T_{m,n} \) in terms of the universal character (see [7, 16]), which is a generalization of Schur polynomial.

Theorem 0.4. The special polynomials \( T_{m,n}(t) \) \( (m, n \in \mathbb{Z}) \) is expressed as follows:

\[
T_{m,n}(t) = N_{m,n}S_{\lambda,\mu}(x, y).
\]
Here $S_{[\lambda,\mu]}(x, y) = S_{[\lambda,\mu]}(x_1, x_2, \ldots, y_1, y_2, \ldots)$ denotes the universal character attached to a pair of partitions

$$\lambda = (u, u - 1, \ldots, 2, 1), \quad \mu = (v, v - 1, \ldots, 2, 1),$$

(0.9)

with $u = |n - m - 1/2| - 1/2$, $v = |n + m - 1/2| - 1/2$; $N_{m,n}$ is a certain normalization factor, and

$$x_n = -\kappa_\infty + \sum_i \theta_i t_i^n, \quad y_n = -\kappa_\infty + \sum_i \theta_i t_i^{-n}. \quad (0.10)$$

(See Theorem 3.5 and also Corollary 3.6) Recall that the universal character is the irreducible character of a rational representation of $GL(n)$, while Schur polynomial that of a polynomial representation; see [7]. Hence Theorem 0.4 shows us a relationship between the representation theory of $GL(n)$ and the Garnier system, or the theory of monodromy preserving deformation.

We propose in [16] an infinite dimensional integrable system characterized by the universal characters, called the UC hierarchy; and regard it as an extension of the KP hierarchy. Since all the universal characters are solutions of the UC hierarchy, it would be an interesting problem to construct a certain reduction procedure from the hierarchy to the Garnier system; cf. [18].

In Sect. 11 we present a group of birational canonical transformations of the Garnier system $\mathcal{H}_N$. In Sect. 2 we prove that a certain sequence of $\tau$-functions satisfies Toda equation. In Sect. 3 we construct a class of algebraic solutions of $\mathcal{H}_N$ by using Toda equation; then show that the associated $\tau$-functions are explicitly written in terms of the universal characters. Sect. 4 is devoted to the verification of Theorem 3.5.

1 Birational symmetry

First we introduce a group of birational canonical transformations of the Garnier system $\mathcal{H}_N(\vec{\kappa})$; then see that it forms an infinite group which contains a translation $\mathbb{Z}$.

It is known that $\mathcal{H}_N$ has a symmetry which is isomorphic to the symmetric group.

**Theorem 1.1** (see [2, 5]). The Garnier system $\mathcal{H}_N(\vec{\kappa})$ has birational canonical transformations

$$\sigma_m : (q, p, s, \vec{\kappa}) \mapsto (Q, P, S, \sigma_m(\vec{\kappa})), \quad 1 \leq m \leq N + 2,$$
given in the following table:

| $\sigma_m$ | action on $\vec{\kappa}$ | $Q_i$ | $P_i$ | $S_i$ |
|------------|--------------------------|-------|-------|-------|
| $\sigma_m$ | $\theta_m \leftrightarrow \kappa_0$ | $Q_i = \frac{q_i}{R_{im}} \quad (i \neq m)$, $Q_m = \frac{s_m(1 - g_s)}{s_m - 1}$ | $P_i = R_{im}\left(p_i - \frac{s_m}{s_i}p_m\right)$, $P_m = -(s_m - 1)p_m$ | $S_i = \frac{s_m - s_i}{s_m - 1}$, $S_m = \frac{1}{s_i}$ |
| $\sigma_{N+1}$ | $\kappa_1 \leftrightarrow \kappa_0$ | $Q_i = \frac{q_i}{s_i}$ | $P_i = s_ip_i$ | $S_i = \frac{1}{s_i}$ |
| $\sigma_{N+2}$ | $\kappa_1 \leftrightarrow \kappa_\infty$ | $Q_i = \frac{q_i}{g_1 - 1}$ | $P_i = (g_1 - 1) \times \left(p_i - \alpha - \sum_j q_jp_j\right)$ | $S_i = \frac{s_i}{s_1 - 1}$ |

where $g_1 = \sum_j q_j$, $g_s = \sum_j q_j/s_j$, and $\langle \sigma_1, \ldots, \sigma_{N+2} \rangle \simeq \mathfrak{S}_{N+3}$.

Theorem 1.1 is verified by considering a permutation among $N + 3$ singularities of the associated linear differential equation; see [2, 5]. Combine the above $\mathfrak{S}_{N+3}$-symmetry with the fact that Hamiltonians $H_i$ (see (0.1b)) are invariant under the action

$$\kappa_\infty \mapsto -\kappa_\infty,$$

we obtain also the following birational transformations.

**Theorem 1.2.** The Garnier system $\mathcal{H}_N(\vec{\kappa})$ has the birational canonical transformations

$$R_\Delta : \mathcal{H}_N(\vec{\kappa}) \rightarrow \mathcal{H}_N(R_\Delta(\vec{\kappa})).$$

Here the birational transformations $R_\Delta : (q,p) \mapsto (Q,P)$ are described as follows:

| $R_\Delta$ | action on $\vec{\kappa}$ | $Q_i$ | $P_i$ |
|------------|--------------------------|-------|-------|
| $R_{\kappa_\infty}$ | $\kappa_\infty \mapsto -\kappa_\infty$ | $Q_i = q_i$ | $P_i = p_i$ |
| $R_{\kappa_1}$ | $\kappa_1 \mapsto -\kappa_1$ | $Q_i = q_i$ | $P_i = p_i - \frac{\kappa_1}{g_1 - 1}$ |
| $R_{\kappa_0}$ | $\kappa_0 \mapsto -\kappa_0$ | $Q_i = q_i$ | $P_i = p_i - \frac{s_i(g_s - 1)}{s_i}$ |
| $R_{\theta_j}$ | $\theta_j \mapsto -\theta_j$ | $Q_i = q_i$ | $P_j = p_j - \frac{\theta_j}{q_j}$, $P_i = p_i \quad (i \neq j)$ |

We now introduce another birational transformation of $\mathcal{H}_N(\vec{\kappa})$ which seems to be more nontrivial than the previous ones.

**Theorem 1.3.** The Garnier system $\mathcal{H}_N(\vec{\kappa})$ has the birational canonical transformation

$$R_\tau : \mathcal{H}_N(q,p,s,H;\vec{\kappa}) \rightarrow \mathcal{H}_N(Q,P,s,\tilde{H};R_\tau(\vec{\kappa})).$$
where \(R_\tau(\vec{\kappa}) = (-\kappa_0 + 1, -\kappa_1 + 1, -\kappa_\infty, -\theta_1, \ldots, -\theta_N)\) and

\[
Q_i = \frac{s_ip_i(q_ip_i - \theta_i)}{\left(\alpha + \sum_j q_jp_j\right)\left(\alpha + \kappa_\infty + \sum_j q_jp_j\right)}, \tag{1.1a}
\]

\[
Q_iP_i = -q_ip_i, \tag{1.1b}
\]

\[
\tilde{H}_i = H_i - \frac{q_ip_i}{s_i}. \tag{1.1c}
\]

Let \(G\) be a group of birational canonical transformations of \(\mathcal{H}_N(\vec{\kappa})\) defined by

\[
G = \langle \sigma_1, \ldots, \sigma_{N+2}, R_{\kappa_0}, R_{\kappa_1}, R_{\kappa_\infty}, R_{\theta_1}, \ldots, R_{\theta_N}, R_\tau \rangle. \tag{1.2}
\]

We see that \(G\) forms an infinite group which contains \(\mathbb{Z}\). For instance, let

\[
l = R_{\kappa_1} \circ R_\tau \circ R_{\theta_1} \circ \cdots \circ R_{\theta_N} \circ R_{\kappa_\infty} \circ R_{\kappa_0} \in G,
\]

then \(l\) acts on the parameter as its translation:

\[
l(\vec{\kappa}) = \vec{\kappa} + (1, -1, 0, 0, \ldots, 0),
\]

thus \(\{l^n\} \simeq \mathbb{Z} \subset G\).

Remark 1.4. Group \(G\) might not fill all the birational symmetries of \(\mathcal{H}_N\). If \(\theta_i = 0\) \((i \neq 1)\), then \(\mathcal{H}_N\) admits a particular solution written in terms of solutions of the sixth Painlevé equation \(PVI\); see [14, Theorem 6.1]. However group \(G\) with the restriction to \(\theta_i = 0\) \((i \neq 1)\) does not form the affine Weyl group of type \(D^{(1)}_4\), which is the group of birational symmetries for \(PVI\); see [13]. So the author suspects that there would exist another hidden symmetry of \(\mathcal{H}_N\). Anyway, it is an important problem to determine the group of all birational symmetries of the Garnier system \(\mathcal{H}_N\).

Proof of Theorem 1.3 First we shall verify that the transformation \(R_\tau\) is a canonical transformation of Hamiltonian system \(\mathcal{H}_N\); that is,

\[
\sum_i (dp_i \wedge dq_i - dH_i \wedge ds_i) = \sum_i \left(dP_i \wedge dQ_i - d\tilde{H}_i \wedge ds_i\right). \tag{1.3}
\]

From (1.1b), we have

\[
P_i dQ_i + Q_i dP_i = -p_i dq_i - q_i dp_i. \tag{1.4}
\]

Consider the logarithmic derivative of (1.1a), we have

\[
\frac{dQ_i}{Q_i} = \frac{ds_i}{s_i} + \frac{dp_i}{p_i} + \frac{p_i dq_i + q_i dp_i}{q_ip_i - \theta_i} - \left(\frac{1}{\alpha + \sum_j q_jp_j} + \frac{1}{\alpha + \kappa_\infty + \sum_j q_jp_j}\right) \sum_j d(q_jp_j). \tag{1.5}
\]
By taking the wedge product of (1.4) and (1.5), we obtain
\[ dP_i \wedge dQ_i = dp_i \wedge dq_i - \left( \frac{1}{\alpha + \sum_j q_j p_j} + \frac{1}{\alpha + \kappa_{\infty} + \sum_j q_j p_j} \right) d(q_i p_i) \wedge \sum_{j(\neq i)} d(q_j p_j); \]
hence
\[ \sum_i dP_i \wedge dQ_i = \sum_i dp_i \wedge dq_i - \sum_i d\left( \frac{q_i p_i}{s_i} \right) \wedge ds_i. \] (1.6)

On the other hand, it follows from (1.1c) that
\[ d\tilde{H}_i \wedge ds_i = dH_i \wedge ds_i - d\left( \frac{q_i p_i}{s_i} \right) \wedge ds_i. \] (1.7)

Combining (1.6) and (1.7), we get (1.3).

Secondly we shall prove that
\[ \tilde{H}_i = H_i(Q, P, s, R_{\tau}(\vec{\kappa})). \] (1.8)

Notice that \( s_j S_{ij} = s_i R_{ji} \). By using (1.9a) and (1.1b) we have the formulae:

\[ Q_i \left( -\alpha + \sum_j Q_j P_j \right) = s_i p_i(q_i p_i - \theta_i), \] (1.9a)
\[ s_i P_i(Q_i P_i + \theta_i) = q_i \left( \alpha + \sum_j q_j p_j \right) \left( \alpha + \kappa_{\infty} + \sum_j q_j p_j \right), \] (1.9b)
\[ \sum_{j(\neq i)} R_{ji}(Q_j P_j + \theta_j)Q_i P_j = \sum_{j(\neq i)} S_{ij}(q_i p_i - \theta_i)q_j p_i, \] (1.9c)
\[ \sum_{j(\neq i)} S_{ij}(Q_i P_i + \theta_i)Q_j P_i = \sum_{j(\neq i)} R_{ji}(q_j p_j - \theta_j)q_i p_j. \] (1.9d)

Recall the definition of Hamiltonian \( H_i \); see (0.1b). Then we verify (1.8) by (1.9) immediately.

The proof is now complete.

\[ \blacksquare \]

2 Toda equation

In this section we show that a certain sequence of \( \tau \)-functions satisfies the Toda equation.

Since the 1-form \( \omega = \sum_i H_i ds_i \) is closed, we can define, up to multiplicative constants, a function \( \tau = \tau(s; \vec{\kappa}) \) called the \( \tau \)-function by (see [21])
\[ d \log \tau = \sum_i H_i ds_i. \] (2.1)
Let \( l \) be a birational canonical transformation of \( \mathcal{H}_N \) defined by
\[
 l = R_{\kappa_1} \circ R_\tau \circ R_{\theta_1} \circ \cdots \circ R_{\theta_N} \circ R_{\kappa_\infty} \circ R_{\kappa_0},
\] (2.2)
then \( l \) acts on the parameter \( \vec{\kappa} = (\kappa_0, \kappa_1, \kappa_\infty, \theta_1, \ldots, \theta_N) \) as its translation:
\[
 l(\vec{\kappa}) = \vec{\kappa} + (1, -1, 0, 0, \ldots, 0).
\]

Let \((q_i(s), p_i(s), H_i(s))\) be a solution of the Garnier system \( \mathcal{H}_N(\vec{\kappa}) \) and set
\[
 (q_i^+, p_i^+, H_i^+) = (l(q_i), l(p_i), l(H_i)),
\]
\[
 (q_i^-, p_i^-, H_i^-) = (l^{-1}(q_i), l^{-1}(p_i), l^{-1}(H_i)),
\]
then we have the

**Proposition 2.1.** The triple of Hamiltonians \((H_i^+, H_i(s), H_i^-)\) satisfies the differential equation:
\[
 H_i^+ - 2H_i + H_i^- = \frac{\partial}{\partial s_i} \log F(s),
\] (2.4)
where
\[
 F(s) = \left( \sum_j (s_j - 1) \frac{\partial}{\partial s_j} - 1 \right) \sum_k s_k (s_k - 1) H_k - \kappa_1 (\kappa_0 - 1) + \alpha (\alpha + \kappa_\infty).
\] (2.5)

One can prove the proposition by straightforward computations, via the birational transformations given in Sect. II; see [19], for details.

Let \( \tau^\pm = l^\pm (\tau) \), then we rewrite (2.4) into
\[
 \left( \sum_i (s_i - 1) \frac{\partial}{\partial s_i} - 1 \right) \left( \sum_j s_j (s_j - 1) \frac{\partial}{\partial s_j} \right) \log \tau - \kappa_1 (\kappa_0 - 1) + \alpha (\alpha + \kappa_\infty) = c \frac{\tau^+ \tau^-}{\tau^2},
\] (2.6)
where \( c \) is a nonzero constant. Consider the change of variables \( s_i = \xi_i / (\xi_i - 1) \) and the differential operators:
\[
 A = \sum_i \xi_i \frac{\partial}{\partial \xi_i}, \quad B = \sum_i \frac{\partial}{\partial \xi_i},
\] (2.7)
then we have
\[
 \left( \sum_i (s_i - 1) \frac{\partial}{\partial s_i} - 1 \right) \left( \sum_j s_j (s_j - 1) \frac{\partial}{\partial s_j} \right) = (A - B + 1)A.
\] (2.8)

Note that
\[
 [A, B] = AB - BA = -B.
\] (2.9)

Let
\[
 \psi = \Delta^{-(N-1)},
\] (2.10)
where $\Delta$ denotes the difference product of $(\xi_1, \xi_2, \ldots, \xi_N)$, i.e.,

$$\Delta = \prod_{i>j}(\xi_i - \xi_j) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \xi_1 & \xi_2 & \cdots & \xi_N \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{N-1} & \xi_{N-1} & \cdots & \xi_{N-1} \end{vmatrix}.$$ 

Since

$$A\Delta = \frac{N(N-1)}{2} \Delta, \quad B\Delta = 0,$$

we have

$$A\psi = \psi, \quad B\psi = 0. \quad (2.11)$$

Introduce the vector fields

$$X = \psi(A - B), \quad Y = \psi A. \quad (2.12)$$

One can easily verify that $[X, Y] = 0$,

$$XY = \psi^2 (A - B + 1)A, \quad (2.13)$$

and

$$XY \log \psi = \psi^2. \quad (2.14)$$

by using (2.9) and (2.11).

Let us consider the sequence of $\tau$-functions $\{\tau_n | n \in \mathbb{Z}\}$ defined by

$$\tau_n = \psi^{a_n} l^n(\tau), \quad (2.15)$$

with

$$a_n = -(\kappa_1 - n)(\kappa_0 + n - 1) + \alpha(\alpha + \kappa_\infty). \quad (2.16)$$

Substitute (2.15) into (2.6), by virtue of (2.13) and (2.14), we now arrive at the

**Theorem 2.2.** The sequence $\{\tau_n | n \in \mathbb{Z}\}$ satisfies the Toda equation:

$$XY \log \tau_n = c_n \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2}, \quad (2.17)$$

where $X, Y$ being vector fields such that $[X, Y] = 0$ and $c_n$ a nonzero constant.

**Remark 2.3.** A sequence of $\tau$-functions corresponding to other translations also satisfies the Toda equation. For instance, let us consider the birational transformation $\tilde{t}$ defined by

$$\tilde{t} = R_{\kappa_1} \circ l \circ R_{\kappa_1}, \quad (2.18)$$

which acts on the parameter $\vec{\kappa}$ as its translation:

$$\tilde{t}(\vec{\kappa}) = \vec{\kappa} + (1, 1, 0, 0, \ldots, 0).$$
It is easy to see that
\[ R_{\kappa_1}(\tau) = \tau \prod_i (s_i - 1)^{-\kappa_1 \theta_i}. \] (2.19)

Combine this with (2.6), we obtain
\[ \left( \sum_i \frac{\partial}{\partial s_i} - 1 \right) \left( \sum_j s_j (s_j - 1) \frac{\partial}{\partial s_j} \right) \log \tau + \alpha (\alpha + \kappa_\infty) = c \frac{l^{-1}(\tau)}{\tau^2}. \] (2.20)

Also (2.20) is equivalent to the Toda equation via a similar change of variables as above.

3 Algebraic solutions in terms of universal characters

In this section we construct a class of algebraic solutions of the Garnier system \( \mathcal{H}_N \) and then express it in terms of the universal characters.

3.1 Algebraic solutions

Consider the birational canonical transformation
\[ w_0 = R_\tau \circ R_{\theta_1} \circ \cdots \circ R_{\theta_N} \circ R_{\kappa_\infty}, \] (3.1)
given as follows:
\[ w_0 : \mathcal{H}_N(q, p; \vec{\kappa}) \to \mathcal{H}_N(Q, P; w_0(\vec{\kappa})), \]
where \( w_0(\vec{\kappa}) = (-\kappa_0 + 1, -\kappa_1 + 1, \kappa_\infty, \theta_1, \ldots, \theta_N) \) and
\[ Q_i = \frac{s_i p_i (q_i p_i - \theta_i)}{\left( \alpha + \sum_j q_j p_j \right) \left( \alpha + \kappa_\infty + \sum_j q_j p_j \right)}, \] (3.2a)
\[ Q_i P_i = -q_i p_i + \theta_i. \] (3.2b)

If \( \kappa_0 = \kappa_1 = 1/2 \), the fixed point with respect to the action of \( w_0 \) is
\[ (q_i, p_i) = \left( \frac{\theta_i \sqrt{s_i}}{\kappa_\infty}, \frac{\kappa_\infty}{2 \sqrt{s_i}} \right), \quad i = 1, \ldots, N. \] (3.3)

This is an algebraic solution of \( \mathcal{H}_N \). Applying the birational symmetries \( G \) (see Sect. 1) to (3.3), we obtain a class of algebraic solutions.

**Theorem 3.1.** If two components of the parameter \( \vec{\kappa} = (\kappa_0, \kappa_1, \kappa_\infty, \theta_1, \ldots, \theta_N) \) are half integers then \( \mathcal{H}_N \) admits an algebraic solution.
3.2 Special polynomials

Substituting the algebraic solution, (3.3), into Hamiltonians (see (0.1b)), we have

\[
s_i(s_i - 1)H_i = -\frac{1}{2}\kappa_\infty\theta_i\sqrt{s_i} + \frac{1}{4}\theta_i(s_i - 1) + \frac{1}{2}\sum_j \theta_i\theta_j \frac{\sqrt{s_is_j} + 1}{\sqrt{s_j/s_i + 1}}; \quad (3.4)
\]

and then the corresponding \( \tau \)-function is given as follows:

\[
\tau_{0,0} = \prod_i s_i^{-\theta_i} (\sqrt{s_i + 1})^{\theta_i} \frac{1}{2} (\sqrt{s_i + 1})^{\theta_i} \frac{1}{2} (\sqrt{s_i + 1})^{\theta_i} \frac{1}{2} \prod_i (s_i + 1)^{-\theta_i} \theta_j^{\theta_j/2}. \quad (3.5)
\]

Let us consider the birational transformations \( l \) and \( \tilde{l} \), defined respectively by (2.2) and (2.18), which act on the parameter \( \vec{r} \) as its translations:

\[
l(\vec{r}) = \vec{r} + (1, -1, 0, 0, \ldots, 0),
\]

\[
\tilde{l}(\vec{r}) = \vec{r} + (1, 1, 0, 0, \ldots, 0). \quad (3.6)
\]

Introduce a family of \( \tau \)-functions \( \tau_{m,n} \) \((m, n \in \mathbb{Z})\) defined by

\[
\tilde{l}^m l^n(\tau_{0,0}) = \tau_{m,n}. \quad (3.7)
\]

Let

\[
s_i = t_i^2, \quad (3.8)
\]

then (3.5) is rewritten as

\[
\tau_{0,0} = \prod_i t_i^{-\theta_i} (t_i + 1)^{\theta_i} (\sqrt{s_i} + 1)^{\theta_i} \frac{1}{2} (\sqrt{s_i} + 1)^{\theta_i} \frac{1}{2} \prod_i (t_i + 1)^{-\theta_i} \theta_j^{\theta_j/2}. \quad (3.9a)
\]

Applying the action of \( \tilde{l} \) and \( l \), we see that

\[
\tau_{0,1} = \prod_i t_i^{-\theta_i} \tau_{0,0}, \quad (3.9b)
\]

\[
\tau_{1,0} = \left( \prod_i t_i^{-\theta_i} (t_i + 1)^{\theta_i} (\sqrt{s_i} + 1)^{\theta_i} \right) \left( \sum_j \theta_j t_j - \kappa_\infty \right) \tau_{0,0}, \quad (3.9c)
\]

\[
\tau_{1,1} = \left( \prod_i t_i^{-2\theta_i} (t_i + 1)^{\theta_i} (t_i - 1)^{\theta_i} \right) \left( \kappa_\infty - \sum_j \theta_j t_j^{-1} \right) \tau_{0,0}. \quad (3.9d)
\]

The \( \tau \)-functions, \( \tau_{m,n} \) \((m, n \in \mathbb{Z})\), are determined successively by the use of the Toda equations (2.6) and (2.18), from the above initial values (3.9).

Now let us define the functions, \( T_{m,n} = T_{m,n}(t) \) \((m, n \in \mathbb{Z})\), by

\[
T_{m,n}(t) = \tau_{m,n} \prod_i \left( t_i^{\theta_i + m + n} (t_i + 1)^{\theta_i + m + n - 1} / 2 (t_i + 1)^{-\theta_i} (\sqrt{s_i} + 1)^{\theta_i} (\sqrt{s_i} + 1)^{\theta_i} \right) \prod_i (t_i + 1)^{\theta_i t_i^{-1}}. \quad (3.10)
\]
Substituting (3.10) into (2.6) and (2.20) with \( c = 1/4 \), we thus obtain the recurrence relations for \( T_{m,n} \).

**Proposition 3.2.** The function \( T_{m,n} = T_{m,n}(t) \) \((m, n \in \mathbb{Z})\) satisfies the following recurrence relations:

\[
T_{m+1,n} = \prod_i t_i \left\{ \left( \sum_i \frac{t_i^2 - 1}{t_i} \frac{\partial}{\partial t_i} - 2 \right) \sum_i t_i (t_i^2 - 1) \frac{\partial}{\partial t_i} \log T_{m,n} + \kappa_\infty \sum_i \theta_i \frac{t_i^2 + 1}{t_i} - \frac{1}{2} \sum_{i,j} \theta_i \theta_j \frac{t_i^2 + t_j^2}{t_i t_j} - \kappa_\infty^2 + (2m)^2 \right\} \frac{T_{m,n}^2}{T_{m-1,n}}, \tag{3.11a}
\]
\[
T_{m,n+1} = \prod_i t_i \left\{ \left( \sum_i \frac{t_i^2 - 1}{t_i} \frac{\partial}{\partial t_i} - 2 \right) \sum_i t_i (t_i^2 - 1) \frac{\partial}{\partial t_i} \log T_{m,n} + \kappa_\infty \sum_i \theta_i \frac{t_i^2 + 1}{t_i} - \frac{1}{2} \sum_{i,j} \theta_i \theta_j \frac{t_i^2 + t_j^2}{t_i t_j} - \kappa_\infty^2 + (2n - 1)^2 \right\} \frac{T_{m,n}^2}{T_{m,n-1}}, \tag{3.11b}
\]

Here the initial values are given as follows:

\[
T_{0,0} = T_{0,1} = 1, \quad T_{1,0} = \sum_i \theta_i t_i - \kappa_\infty, \quad T_{1,1} = \prod_i t_i \left( \kappa_\infty - \sum_j \theta_j t_j^{-1} \right). \tag{3.12}
\]

We call \( T_{m,n}(t) \) special polynomials associated with algebraic solutions of \( \mathcal{H}_N \). By the above recurrence relations (3.11), we can only state that \( T_{m,n}(t) \) are rational functions in \( t = (t_1, \ldots, t_N) \). We will show that \( T_{m,n}(t) \) are indeed polynomials; see Theorem 3.3 and Corollary 3.6 below. Note that

\[
T_{-m,n}(t) = T_{m,-n}(t) = (-1)^{m(2n-1)} \prod t_i^{m^2+n(n-1)} T_{m,n}(t^{-1}), \tag{3.13}
\]

which is verified easily by the recurrence relations and initial values. Algebraic solutions of \( \mathcal{H}_N \) are explicitly written in terms of the special polynomials \( T_{m,n}(t) \).

**Theorem 3.3.** If \( \kappa_0 = 1/2 + m + n, \ \kappa_1 = 1/2 + m - n \) \((m, n \in \mathbb{Z})\), then \( \mathcal{H}_N \) admits an algebraic solution given as follows:

\[
q_i = \frac{t_i \frac{\partial}{\partial t_i} \log \frac{T_{m+1,n}}{T_{m,n+1}}}{\sum_j t_j \frac{\partial}{\partial t_j} \log \frac{T_{m+1,n}}{T_{m,n+1}} - 2m + 2n - 1}, \tag{3.14a}
\]
\[
2q_i p_i = \theta_i + m + n + t_i \frac{\partial}{\partial t_i} \log \frac{T_{m,n}}{T_{m,n+1}}. \tag{3.14b}
\]
Proof. By using the birational canonical transformations \( l \) and \( \tilde{l} \), we have

\[
\begin{align*}
  l(H_i) &= H_i - \frac{q_ip_i}{s_i}, \\
  \tilde{l}(H_i) &= H_i - \frac{1}{s_i} \left( q_ip_i - \frac{\kappa_1 q_i}{g_1 - 1} \right) + \frac{\theta_i}{s_i - 1},
\end{align*}
\]

(3.15)

(3.16)

where \( g_1 = \sum_j q_j \). We then obtain the relation between \( \tau \)-functions and canonical variables:

\[
q_i = s_i \frac{\partial}{\partial s_i} \log \frac{\tilde{l}(\tau)}{l(\tau)} - \frac{\theta_i s_i}{s_i - 1},
\]

(3.17a)

\[
q_ip_i = s_i \frac{\partial}{\partial s_i} \log \frac{\tau}{l(\tau)}.
\]

(3.17b)

Here recall the definition of \( \tau \)-function, \( \partial/\partial s_i \log \tau = H_i \). Substitute (3.10) into (3.17) with \( s_i = t_i^2 \), we get (3.14).

3.3 Universal characters

To investigate the special polynomial \( T_{m,n} \) in detail, we have to recall the definition of the universal characters; see [7, 16]. For each pair of partitions \( [\lambda, \mu] = ([\lambda_1, \lambda_2, \ldots, \lambda_l], [\mu_1, \mu_2, \ldots, \mu_{l'}]) \), the universal character \( S_{[\lambda, \mu]}(x, y) \) is a polynomial in \((x, y) = (x_1, x_2, \ldots, y_1, y_2, \ldots)\) defined as follows:

\[
S_{[\lambda, \mu]}(x, y) = \det \left( \begin{array}{c} q_{\mu_{l' - i + j}}(y), & 1 \leq i \leq l' \\
 p_{\lambda_{-i + l' + j}}(x), & l' + 1 \leq i \leq l + l' \end{array} \right)_{1 \leq i, j \leq l + l'}. \quad (3.18)
\]

Here \( p_n(x) \) is determined by the generating function:

\[
\sum_{n=0}^{\infty} p_n(x) z^n = e^{\xi(x, z)}, \quad \xi(x, z) = \sum_{n=1}^{\infty} x_n z^n, \quad (3.19)
\]

and set \( p_n(x) = 0 \) for \( n > 0 \); \( q_n(y) \) is the same as \( p_n(x) \) except replacing \( x \) with \( y \). Note that \( p_n(x) \) is explicitly written as follows:

\[
p_n(x) = \sum_{k_1 + 2k_2 + \cdots + nk_n = n} \frac{x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}}{k_1! k_2! \cdots k_n!}. \quad (3.20)
\]

If we count the degree of each variable \( x_n \) and \( y_n \) \((n = 1, 2, \ldots)\) as

\[
\deg x_n = n \quad \text{and} \quad \deg y_n = -n,
\]

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then the universal character $S_{|\lambda,\mu|}(x,y)$ is a weighted homogeneous polynomial of degree $|\lambda| - |\mu|$, where we let $|\lambda| = \lambda_1 + \cdots + \lambda_l$. Note that the Schur polynomial $S_{\lambda}(x)$ (see e.g. [8]) is regarded as a special case of the universal character:

$$S_{\lambda}(x) = \det (p_{\lambda_i - i+j}(x)) = S_{[\lambda,\emptyset]}(x,y).$$

**Example 3.4.** When $\lambda = (2,1)$, $\mu = (1)$, the universal character is given as follows:

$$S_{[(2,1),(1)]}(x,y) = \begin{vmatrix} q_1 & q_0 & q_{-1} \\ p_1 & p_2 & p_3 \\ p_{-1} & p_0 & p_1 \end{vmatrix} = y_1 \left(\frac{x_1^3}{3} - x_3\right) - x_1^2,$$

which is a weighted homogeneous polynomial of degree $|\lambda| - |\mu| = 2$.

The special polynomial $T_{m,n}(t)$ can be written in terms of the universal character.

**Theorem 3.5.** The special polynomial $T_{m,n}(t)$ $(m, n \in \mathbb{Z})$ is expressed as follows:

$$T_{m,n}(t) = N_{m,n} S_{[\lambda,\mu]}(x,y). \quad (3.21)$$

Here $\lambda = (u,u-1,\ldots,2,1)$, $\mu = (v,v-1,\ldots,2,1)$ with $u = |n-m-1/2| - 1/2$, $v = |n+m-1/2| - 1/2$; and

$$x_n = -\kappa + \sum_n \theta_i t_i^n, \quad y_n = -\kappa + \sum_n \theta_i^{-n}. \quad (3.22)$$

The normalization factor $N_{m,n}$ is given by

$$N_{m,n} = (-1)^{v(v+1)/2} \prod_{i=1}^N t_i^{v(v+1)/2} \prod_{j=1}^u (2j-1)!! \prod_{k=1}^v (2k-1)!!. \quad (3.23)$$

Consequently we have the

**Corollary 3.6.** The special polynomial $T_{m,n}(t)$ is indeed a polynomial of degree $m^2 + n(n-1)$; furthermore $T_{m,n}(t) \in \mathbb{Z}[\kappa, \theta_1, \ldots, \theta_N][t]$.

The proof of Theorem 3.5 is given in Sect. 4.

We show in Figure 1 below how the special polynomials $T_{m,n}(t)$ are arranged on $(m,n)$-lattice. We also give some examples of $T_{m,n}(t)$ of small degrees in the case $N = 1$. 

---

**Remark 3.7.**
Figure 1 Special polynomials $T_{m,n}(t)$.

The special polynomials $T_{m,n}(t)$ for $N = 1$ are as follows:

\[
\begin{align*}
T_{0,0} &= T_{0,1} = 1, \quad T_{1,0} = T_{-1,1} = -\kappa_\infty + \theta t, \quad T_{1,1} = T_{-1,0} = -\theta + \kappa_\infty t, \\
T_{0,2} &= T_{0,-1} = \kappa_\infty \theta + t - \kappa_\infty^2 t - \theta^2 t + \kappa_\infty \theta t^2, \\
T_{1,-1} &= T_{-1,2} = \kappa_\infty - \kappa_\infty^3 + 3\kappa_\infty \theta t - 3\kappa_\infty \theta^2 t^2 - \theta^3 t^2, \\
T_{1,2} &= T_{-1,-1} = \theta - \theta^3 + 3\kappa_\infty \theta^2 t - 3\kappa_\infty^2 \theta^2 t^2 - \kappa_\infty^3 t^3 + \kappa_\infty^3 t^3, \\
T_{2,0} &= T_{-2,1} = -\kappa_\infty \theta + \kappa_\infty^3 \theta + 4\kappa_\infty^2 t - \kappa_\infty^4 t - 3\kappa_\infty \theta^2 t - 6\kappa_\infty \theta t^2 + 3\kappa_\infty \theta^2 t^2 + 3\kappa_\infty \theta^3 t^2 + 4\theta^2 t^3 - 3\kappa_\infty^2 \theta^2 t^3 - \theta^4 t^3 - \kappa_\infty \theta t^4 + \kappa_\infty \theta^3 t^4.
\end{align*}
\]

Remark 3.7. Under the specialization \((3.22)\), we let $p_n(x) = P_n(t)$. Then the generating function \((3.19)\) is rewritten as follows:

\[
\sum_{n=0}^{\infty} P_n(t)z^n = (1 - z)^{\kappa_\infty} \prod_{i}(1 - t_i z)^{-\theta_i}.
\]
Hence \( P_n(t) \) has the following expression:

\[
P_n(t) = \frac{(-\kappa_\infty)_n}{(1)_n} F_D(-n, \theta_1, \ldots, \theta_N, \kappa_\infty - n + 1; t),
\]

where \( F_D \) denotes the Lauricella hypergeometric series and \((a)_n = a(a+1)(a+2) \cdots (a+n-1)\); see e.g. \([2, 12, 15]\).

**Remark 3.8.** If \( N = 1, T_{m,n}(t) \) is equivalent to the Umemura polynomial of \( P_{\nu_1} \), for which Masuda considered its explicit formula in terms of universal characters; see \([10, 11]\). We refer also to the results \([9] \) and \([17]\), where a class of rational solutions of \( P_{\nu} \) and that of the (higher order) Painlevé equation of type \( A_{2g+1} \) \((g \geq 1)\) are obtained in terms of universal characters.

**Remark 3.9.** Several other classes of solutions of the Garnier system have been studied. In \([15]\), a family of rational solutions was obtained by the use of Schur polynomials. In \([6]\), solutions in terms of hyperelliptic theta functions were considered from the viewpoint of algebraic geometry.

### 4 Proof of Theorem 3.5

#### 4.1 A generalization of Jacobi’s identity

First we prepare an identity for determinants, which is regarded as a generalization of Jacobi’s identity. Let \( A = (a_{ij})_{i,j} \) be an \( n \times n \) matrix and \( \xi_f^I = \xi_f^J(A) \) its minor determinant with respect to rows \( I = \{i_1, \ldots, i_r\} \) and columns \( J = \{j_1, \ldots, j_r\} \). For two disjoint sets \( I, J \subseteq \{1, \ldots, n\} \), we define \( \epsilon(I; J) \) by

\[
\epsilon(I; J) = (-1)^{|I\setminus J|}, \quad l(I; J) = \# \{(i, j) \in I \times J \mid i > j\}.
\]

**Theorem 4.1.** Let \( I = \{1, 2, \ldots, n\} \) and \( A = (a_{ij})_{i,j \in I} \). The following quadratic relation among minor determinants of \( A \) holds:

\[
\xi_I^I \xi_-^{I-J_1-J_2} = \sum_{K_1, K_2 \subseteq I \setminus \{I-J_1-J_2\}=\emptyset} \epsilon(K_1; K_2) \xi_{I-K_1}^{I-K_1} \xi_{I-K_2}^{I-K_2},
\]

where \(|J_1| = |K_1| = r_1\) and \(|J_2| = |K_2| = r_2\).

Let \( r_1 = r_2 = 1, J_1 = \{1\} \) and \( J_2 = \{n\} \), then \((4.2)\) recovers Jacobi’s identity (see \([3]\)):

\[
\xi_1^{1\cdots n} \xi_2^{2\cdots n-1} = \xi_2^{2\cdots n} \xi_1^{1\cdots n-1} - \xi_2^{1\cdots n-1} \xi_1^{2\cdots n}.
\]
in fact.

**Proof of Theorem 4.1.** Without loss of generality, we can set $J_1 = \{1, 2, \ldots, r_1\}$ and $J_2 = \{n-r_2+1, \ldots, n-1, n\}$. Let $I = \{1, 2, \ldots, 2n-r_1-r_2\}$. Consider a $(2n-r_1-r_2) \times (2n-r_1-r_2)$ matrix $B = (b_{ij})_{i,j \in \tilde{I}}$ given as follows:

\[
\begin{aligned}
(i) \quad b_{ij} &= a_{ij} & \text{for } & i, j \in I; \\
(ii) \quad b_{ij} &= a_{i+j-n+r_1} & \text{for } & i \in I, j \in \tilde{I} \setminus I; \\
(iii) \quad b_{ij} &= a_{i-n+r_1,j} & \text{for } & i \in \tilde{I} \setminus I, j \in J_1; \\
(iv) \quad b_{ij} &= 0 & \text{for } & i \in \tilde{I} \setminus I, j \in I \setminus J_1; \\
(v) \quad b_{ij} &= a_{i-n+r_1,j-n+r_1} & \text{for } & i \in \tilde{I} \setminus I, j \in \tilde{I} \setminus I,
\end{aligned}
\]

i.e., write $A$ as

\[
A = \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{bmatrix},
\]

then $B$ is written as

\[
B = \begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{12} \\
A_{21} & A_{22} & A_{23} & A_{22} \\
A_{31} & A_{32} & A_{33} & A_{33} \\
A_{21} & 0 & 0 & A_{22}
\end{bmatrix}.
\]

Apply the Laplace expansion with respect to rows $I$ and rows $\tilde{I} \setminus I$, we obtain

\[
\det B = \xi_{I}^{I} \xi_{I - J_1 - J_2}^{I - J_1 - J_2}. \tag{4.5}
\]

On the other hand, by the Laplace expansion with respect to columns $I \setminus J_1$ and columns $(\tilde{I} \setminus I) \cup J_1$, we have

\[
\det B = \sum_{K_1, K_2 \subseteq I; \, \, K_1 \cap (I - J_1 - J_2) = \emptyset; \, \, K_2 \cap (I - J_1 - J_2) = \emptyset} \epsilon(K_1; K_2) \xi_{I - J_1}^{I - K_1} \xi_{I - J_2}^{I - K_2}. \tag{4.6}
\]

Thus we verify (4.2).  

\[\blacksquare\]

### 4.2 Vertex operators

Introduce the vertex operators $V_m(k; x, y)$ $(m \in \mathbb{Z})$ defined by (see [16])

\[
V_m(k; x, y) = e^{m \xi(x-\bar{\partial}_x, k)} e^{-m \xi(\bar{\partial}_x, k^{-1})}, \tag{4.7}
\]

where $\bar{\partial}_x$ stands for $\left(\frac{\partial}{\partial x_1}, \frac{1}{2} \frac{\partial}{\partial x_2}, \frac{1}{3} \frac{\partial}{\partial x_3}, \ldots\right)$ and $\xi(x, k) = \sum_{n=1}^{\infty} x_n k^n$. Define the differential operators $X_n$ and $Y_n$ $(n \in \mathbb{Z})$ by

\[
X(k) = \sum_{n \in \mathbb{Z}} X_n k^n = V_1(k; x, y), \quad Y(k) = \sum_{n \in \mathbb{Z}} Y_n k^{-n} = V_1(k^{-1}; y, x). \tag{4.8}
\]
We have the following lemmas; see [16].

**Lemma 4.2.** The operators $X_n$ and $Y_n$ ($n \in \mathbb{Z}$) are raising operators for the universal characters in the sense that

$$S_{[\lambda,\mu]}(x,y) = X_{\lambda_1} \cdots X_{\lambda_l} Y_{\mu_1} \cdots Y_{\mu_l} \cdot 1. \quad (4.9)$$

**Lemma 4.3.** The following relations hold:

\begin{align*}
X_nX_{n-1}X_{m+1} &= 0, \\
Y_nY_{n-1}Y_{m+1} &= 0, \\
[X_m, Y_n] &= 0, \\
\end{align*}

for $m, n \in \mathbb{Z}$. In particular $X_nX_{n+1} = Y_nY_{n+1} = 0$.

**4.3 Proof of Theorem 3.5**

Introduce the Euler operator

$$E = \sum_{n=1}^{\infty} \left( nx_n \frac{\partial}{\partial x_n} - ny_n \frac{\partial}{\partial y_n} \right), \quad (4.11)$$

and operators $L^+$, $L^-$ given as follows:

\begin{align*}
L^+ &= \frac{x^2}{2} + \sum_{n=1}^{\infty} \left( (n+2)n_{n+2} \frac{\partial}{\partial x_n} - ny_n \frac{\partial}{\partial y_n} \right) - x_1 \frac{\partial}{\partial y_1} - \left( -\kappa_{\infty} + \sum_i \theta_i \right) \frac{\partial}{\partial y_2}, \quad (4.12) \\
L^- &= \frac{y^2}{2} + \sum_{n=1}^{\infty} \left( (n+2)n_{n+2} \frac{\partial}{\partial y_n} - nx_n \frac{\partial}{\partial x_n} \right) - y_1 \frac{\partial}{\partial x_1} - \left( -\kappa_{\infty} + \sum_i \theta_i \right) \frac{\partial}{\partial x_2}. \quad (4.13)
\end{align*}

Note that $E$, $L^+$, and $L^-$ are homogeneous operators of degrees 0, 2, and $-2$, respectively. Consider the change of the variables

$$x_n = \frac{\kappa_{\infty} + \sum_i \theta_i t_i^m}{n}, \quad y_n = \frac{\kappa_{\infty} + \sum_i \theta_i t_i^{-n}}{n}, \quad (4.14)$$

and

$$\tilde{T}_{m,n}(x, y) = (-1)^{-v(v+1)/2} \prod_i t_i^{-v(v+1)/2} T_{m,n}(t), \quad (4.15)$$

where $u = |n - m - 1/2| - 1/2$, $v = |n + m - 1/2| - 1/2$. Substitute this into (3.11), we have the recurrence relations for $\tilde{T}_{m,n}(x, y)$:

\begin{align*}
-\tilde{T}_{m+1,n} \tilde{T}_{m-1,n} &= \left\{ \left( L^+ - E - \frac{y^2}{2} - 2 \right) \left( L^+ - E - \frac{x^2}{2} \right) \log \tilde{T}_{m,n} - x_1y_1 + (2m)^2 \right\} \tilde{T}_{m,n}^2, \quad (4.16a) \\
-\tilde{T}_{m,n+1} \tilde{T}_{m,n-1} &= \left\{ \left( L^- - E - \frac{y^2}{2} - 2 \right) \left( L^- - E - \frac{x^2}{2} \right) \log \tilde{T}_{m,n} - x_1y_1 + (2n - 1)^2 \right\} \tilde{T}_{m,n}^2, \quad (4.16b)
\end{align*}
where the initial values are given by
\[ \tilde{T}_{0,0} = \tilde{T}_{0,1} = 1, \quad \tilde{T}_{1,0} = x_1, \quad \tilde{T}_{1,1} = y_1. \] (4.17)

Note that we have
\[ \tilde{T}_{-m,n}(x, y) = \tilde{T}_{m,-n}(x, y) = \tilde{T}_{m,n}(y, x), \] (4.18)
from (3.13).

Theorem 3.3 follows immediately from the
Proposition 4.4.

Let
\[ \tilde{T}_{m,n}(x, y) = \prod_{j=1}^{u} (2j - 1)!! \prod_{k=1}^{v} (2k - 1)!! S_{\lambda,\mu}(x, y), \] (4.19)
where \( \lambda = (u, u-1, \ldots, 2, 1) \) and \( \mu = (v, v-1, \ldots, 2, 1) \), then \( \tilde{T}_{m,n}(x, y) \) satisfies (4.16) and (4.17).

We prepare some lemmas to verify Proposition 4.4.

Lemma 4.5. The following commutation relations hold for \( n \in \mathbb{Z} \):
\[ [X_n, L^+] = -\left( n + \frac{3}{2} \right) X_{n+2} + 2 \left( x_2 - \frac{\partial}{\partial y_2} \right) X_n, \] (4.20)
\[ [Y_n, L^+] = \left( n - \frac{3}{2} - \kappa_\infty + \sum_i \theta_i \right) Y_{n-2} - Y_n \frac{\partial}{\partial y_2}, \] (4.21)
\[ [X_n, x_2] = -\frac{1}{2} X_{n+2}, \] (4.22)
\[ [Y_n, x_2] = -\frac{1}{2} Y_{n-2}. \] (4.23)

Proof. Notice that for any operators \( A \) and \( B \),
\[ e^A Be^{-A} = e^{\text{ad}(A) B} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \cdots, \]
where \( \text{ad}(A)(B) = [A, B] \). We have
\[ [\xi(x - \tilde{\partial}_y, k), L^+] = -\sum_{m=1}^{\infty} \left\{ (m+2)x_{m+2} - \frac{\partial}{\partial y_{m+2}} \right\} k^m, \]
so that
\[ [e^{\xi}(x - \tilde{\partial}_y, k), L^+] = -\sum_{m=1}^{\infty} \left\{ (m+2)x_{m+2} - \frac{\partial}{\partial y_{m+2}} \right\} k^m e^{\xi(x - \tilde{\partial}_y, k)}. \] (4.24)
On the other hand, we have

\[-\xi(\partial_x, k^{-1}), L^+] = -\left(x_1 - \frac{\partial}{\partial y_1}\right)k^{-1} - \sum_{m=1}^{\infty} k^{-m-2} \frac{\partial}{\partial x_m},\]

\[-\xi(\partial_y, k^{-1}), L^+] = k^{-2},\]

then

\[[e^{-\xi(\partial_y, k^{-1})}, L^+] = \left\{-\left(x_1 - \frac{\partial}{\partial y_1}\right)k^{-1} + \frac{k^{-2}}{2} \sum_{m=1}^{\infty} \frac{\partial}{\partial x_m}\right\}e^{-\xi(\partial_y, k^{-1})}. \quad (4.25)\]

Noticing

\[k^{-1} \frac{\partial}{\partial k} X(k) = \sum_{m=1}^{\infty} \left(mx_m - \frac{\partial}{\partial y_m}\right)k^{m-2} X(k) + e^{\xi(x-\tilde{\partial}_y, k)} \sum_{m=1}^{\infty} k^{-m-2} \frac{\partial}{\partial x_m}e^{-\xi(\partial_y, k^{-1})},\]

from (4.24) and (4.25), we obtain

\[\left\{X(k), L^+\right\} = e^{\xi(x-\tilde{\partial}_y, k)}\left[e^{-\xi(\partial_y, k^{-1})}, L^+\right] + \left[e^{\xi(x-\tilde{\partial}_y, k)}, L^+\right]e^{-\xi(\partial_y, k^{-1})}\]

\[= \left\{-k^{-1} \frac{\partial}{\partial k} + \frac{k^{-2}}{2} + 2 \left(x_2 - \frac{1}{2} \frac{\partial}{\partial y_2}\right)\right\}X(k). \quad (4.26)\]

Take the coefficient of \(k^n\), we verify (4.20).

We have

\[\left\{\xi(y-\tilde{\partial}_x, k^{-1}), L^+\right\} = k^{-1} \frac{\partial}{\partial y_1} + \left(-\kappa_{\infty} + \sum_i \theta_i\right)k^{-2} + \sum_{m=1}^{\infty} \left(my_m - \frac{\partial}{\partial x_m}\right)k^{-m-2},\]

\[\left\{\xi(y-\tilde{\partial}_x, k^{-1}), \xi(y-\tilde{\partial}_x, k^{-1}), L^+\right\} = -k^{-2},\]

\[-\xi(\tilde{\partial}_y, k), L^+] = \sum_{m=1}^{\infty} k^{m} \frac{\partial}{\partial y_{m+2}},\]

so that

\[\left[e^{\xi(y-\tilde{\partial}_x, k^{-1})}, L^+\right] = \left\{k^{-1} \frac{\partial}{\partial y_1} + \left(-\kappa_{\infty} + \sum_i \theta_i - \frac{1}{2}\right)k^{-2} + \sum_{m=1}^{\infty} \left(my_m - \frac{\partial}{\partial x_m}\right)k^{-m-2}\right\},\]

\[\left[e^{-\xi(\tilde{\partial}_y, k)}, L^+\right] = \sum_{m=1}^{\infty} k^{m} \frac{\partial}{\partial y_{m+2}}e^{-\xi(\tilde{\partial}_y, k)}.\]

Thus we obtain

\[\left[Y(k), L^+\right] = e^{\xi(y-\tilde{\partial}_x, k^{-1})}\left[e^{-\xi(\tilde{\partial}_y, k)}, L^+\right] + \left[e^{\xi(y-\tilde{\partial}_x, k^{-1})}, L^+\right]e^{-\xi(\tilde{\partial}_y, k)}\]

\[= \left\{-k^{-1} \frac{\partial}{\partial k} + \left(-\kappa_{\infty} + \sum_i \theta_i + \frac{1}{2}\right)k^{-2}\right\}Y(k) - Y(k) \frac{\partial}{\partial y_2}, \quad (4.27)\]
whose coefficient of $k^{-n}$ yields (4.21). By $[-\xi(\tilde{\partial}_x, k^{-1}), x_2] = -k^{-2}/2$, we have

$$[e^{-\xi(\tilde{\partial}_x, k^{-1})}, x_2] = -\frac{k^{-2}}{2} e^{-\xi(\tilde{\partial}_x, k^{-1})},$$

therefore

$$[X(k), x_2] = -\frac{k^{-2}}{2} X(k), \quad [Y(k), x_2] = -\frac{k^{-2}}{2} Y(k). \quad (4.28)$$

Take the coefficients of $k^n$ and $k^{-n}$, we obtain (4.22) and (4.23) respectively. ■

**Lemma 4.6.** For integers $u, v \geq 0$, the following formulae hold:

$$L^+ S_{[u!, v!]}(x, y) = (2u + 1) S_{[(u+2, u-1, \ldots, 1), v!]}(x, y) - (2u + 1) x_2 S_{[u!, v!]}(x, y), \quad (4.29)$$

$$L^- S_{[u!, v!]}(x, y) = (2v + 1) S_{([u!, (v+2, v-1, \ldots, 1)]}(x, y) - (2v + 1) y_2 S_{[u!, v!]}(x, y), \quad (4.30)$$

$$L^+ S_{[u, (v+2, v-1, \ldots, 1)]}(x, y) = (2u + 1) S_{[(u+2, u-1, \ldots, 1), (v+2, v-1, \ldots, 1)]}(x, y)$$

$$= -(2u + 1) x_2 S_{[u!, (v+2, v-1, \ldots, 1)]}(x, y) - \left( v - u - \kappa_\infty + \sum \theta_i \right) S_{[u!, v!]}(x, y), \quad (4.31)$$

$$L^- S_{[(u+2, u-1, \ldots, 1), v!]}(x, y) = (2v + 1) S_{[(u+2, u-1, \ldots, 1), (v+2, v-1, \ldots, 1)]}(x, y)$$

$$= -(2v + 1) y_2 S_{[(u+2, u-1, \ldots, 1), v!]}(x, y) - \left( u - v - \kappa_\infty + \sum \theta_i \right) S_{[u!, v!]}(x, y). \quad (4.32)$$

Here $u! = (u, u - 1, \ldots, 2, 1)$.

**Proof.** First we shall show that

$$L^+ S_{[u!, 0]}(x, y) = (2u + 1) S_{[(u+2, u-1, \ldots, 1), 0]}(x, y) - (2u + 1) x_2 S_{[u!, 0]}(x, y), \quad (4.33)$$

by induction. Using $S_{[0, 0]}(x, y) = 1$ and $S_{[(2), 0]}(x, y) = x_1^2/2 + x_2$, it is easy to verify for $u = 0$. Assume that (4.33) is true for $u - 1$. Applying $X_u$, we have

$$X_u L^+ S_{[(u-1), 0]}(x, y) = L^+ S_{[u!, 0]}(x, y) + [X_u, L^+] S_{[(u-1), 0]}(x, y)$$

$$= (L^+ + 2x_2) S_{[u!, 0]}(x, y) - \left( u + \frac{3}{2} \right) S_{[(u+2, u-1, \ldots, 1), 0]}(x, y),$$

and

$$X_u \left( (2u - 1) S_{[(u+1, u-2, \ldots, 1), 0]}(x, y) - (2u - 1) x_2 S_{[(u-1), 0]}(x, y) \right)$$

$$= -(2u - 1) x_2 S_{[u!, 0]}(x, y) + \frac{1}{2} (2u - 1) S_{[u!, 0]}(x, y),$$

(4.34)
by using the commutation relations (4.20) and the property $X_k X_{k+1} = 0$. Then, by the assumption, we have the desired equation (4.33) immediately. Applying $Y_v Y_{v-1} \cdots Y_1$ to (4.33), we obtain (4.29). Here we recall the commutation relations (4.21), (4.23), and $Y_k Y_{k+1} = 0$.

Since $L^{-}$ is the same as $L^{+}$ except exchanging $x$ with $y$, we verify (4.30) immediately.

Notice that $S_{[u,v]}(x, y)$ does not depend on $y_{2n} (n = 1, 2, \ldots)$. Applying $Y_{v+3}$ to (4.29), we have

$$Y_{v+3} L^{+} S_{[u,v]}(x, y) = L^{+} S_{[u,(v+3,v,\ldots,1)]}(x, y) + \left( v + \frac{3}{2} - \kappa_{\infty} + \sum_{i} \theta_{i} \right) S_{[u,(v+1)]}(x, y),$$

and

$$Y_{v+3} \left( (2u+1) S_{[(u+2,u-1,\ldots,1),v]}(x, y) - (2u+1)x_2 S_{[u,v]}(x, y) \right)$$

$$= (2u+1) S_{[(u+2,u-1,\ldots,1),(v+3,v,\ldots,1)]}(x, y) - (2u+1)x_2 S_{[u,(v+3,v,\ldots,1)]}(x, y)$$

$$+ \left( u + \frac{1}{2} \right) S_{[u,(v+1)]}(x, y).$$

Thus we verify (4.31). Similarly (4.32) also holds.

**Proof of Proposition 4.4.** For the sake of simplicity, we use the following notations:

\begin{align*}
S &= S_{[u,v]}(x, y), \\
S^{+} &= S_{[(u+2,u-1,\ldots,1),v]}(x, y), \\
S^{-} &= S_{[u,(v+3,v,\ldots,1)]}(x, y), \\
S^{+-} &= S_{[(u+2,u-1,\ldots,1),(v+3,v,\ldots,1)]}(x, y). 
\end{align*}

We have

$$\left( \left( L^{-} + E - \frac{y_1^2}{2} - 2 \right) \left( L^{+} - E - \frac{x_1^2}{2} \right) \log S \right) S^2$$

$$= \left( L^{-} + E - \frac{y_1^2}{2} \right) \left( L^{+} - E - \frac{x_1^2}{2} \right) S \cdot S$$

$$- \left( L^{-} + E - \frac{y_1^2}{2} \right) S \cdot \left( L^{+} - E - \frac{x_1^2}{2} \right) S$$

$$- 2 \left( L^{+} - E - \frac{x_1^2}{2} \right) S \cdot S. \quad (4.35)$$

Since $S_{[\lambda,\mu]}(x, y)$ is a weighted homogeneous polynomial of degree $|\lambda| - |\mu|$, the Euler operator $E$ acts on it as

$$ES_{[\lambda,\mu]}(x, y) = (|\lambda| - |\mu|) S_{[\lambda,\mu]}(x, y). \quad (4.36)$$

Then by Lemma 4.6 we have

$$\left( \left( L^{-} + E - \frac{y_1^2}{2} - 2 \right) \left( L^{+} - E - \frac{x_1^2}{2} \right) \log S - x_1 y_1 \right) S^2$$

$$= (2u+1)(2v+1)S^{+-}S - (2u+1)(2v+1)S^{-}S^{+} - (u-v)^2 S^2. \quad (4.37)$$
Now let us substitute (4.19) into the recurrence relations (4.16). By virtue of (4.18), it is enough to consider the cases (I) \( n - m - 1/2 > 0 \), \( n + m - 1/2 > 0 \); and (II) \( n - m - 1/2 < 0 \), \( n + m - 1/2 > 0 \).

First we deal with the case (I), that is, \( m = (v - u)/2 \), \( n = (u + v + 2)/2 \). Substitute (4.19) into the both sides of (4.16), we have

\[
\begin{align*}
\text{LHS of (4.16a)} & = -(2u + 1)(2v + 1)C_{u,v}S_{[(u+1)!,(v-1)!]} \cdot S_{[(u-1)!,(v+1)!]}, \\
\text{RHS of (4.16a)} & = (2u + 1)(2v + 1)C_{u,v}(S^+ - S - S^+ S^-),
\end{align*}
\]

and

\[
\begin{align*}
\text{LHS of (4.16b)} & = -(2u + 1)(2v + 1)C_{u,v}S_{[(u+1)!,(v+1)!]} \cdot S_{[(u-1)!,(v-1)!]}, \\
\text{RHS of (4.16b)} & = (2u + 1)(2v + 1)C_{u,v}(S^+ - S - S^+ S^- + S^2),
\end{align*}
\]

respectively. Here we put \( C_{u,v} = \left( \prod_{j=1}^{u} (2j - 1)! \prod_{k=1}^{v} (2k - 1)! \right)^2 \). Thus it is sufficient to prove

\[
\begin{align*}
-S_{[(u+1)!,(v-1)!]} \cdot S_{[(u-1)!,(v+1)!]} & = S^+ S - S^+ S^-, \quad (4.38) \\
-S_{[(u+1)!,(v+1)!]} \cdot S_{[(u-1)!,(v-1)!]} & = S^+ S - S^+ S^- + S^2. \quad (4.39)
\end{align*}
\]

By using Lemma 4.7 below, we immediately verify (4.38) and (4.39).

The verification for the case (II) is the same.

**Lemma 4.7.** The following formulae hold:

\[
\begin{align*}
S_{[(u+1)!,(v-1)!]} \cdot S_{[(u-1)!,(v+1)!]} - S_{[(u+1)!,(v-1)!]} \cdot S_{[(u-1)!,(v+1)!]} + S_{[u!,v!]}^2 & = 0, \quad (4.40) \\
S_{[(u+1)!,(v-1)!]} \cdot S_{[(u-1)!,(v+1)!]} - S_{[u!,v+2,v-1,...,1]} \cdot S_{[(u+2,u-1,...,1),v!]} & + S_{[(u+2,u-1,...,1),(v+2,v-1,...,1),v!]} \cdot S_{[u!,v!]} = 0. \quad (4.41)
\end{align*}
\]

**Proof.** Consider a \((u + v + 2) \times (u + v + 2)\) matrix

\[
M = \begin{bmatrix}
q_1 & q_0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\
q_2 & q_1 & q_2 & \cdots & \cdots & q_v & q_{v-1} \\
q_3 & q_2 & \cdots & \cdots & \cdots & q_{v+1} & q_v & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & p_u & p_{u+1} & \cdots \\
q_v & q_{v+1} & q_v & \cdots & \cdots & q_{u+1} & q_u & \cdots \\
\cdots & \cdots & \cdots & p_{u+1} & p_u & \cdots & \cdots & \cdots & \cdots \\
p_{u-1} & p_u & \cdots & \cdots & \cdots & p_2 & p_3 & \cdots & \cdots \\
p_{u} & p_{u-1} & \cdots & \cdots & \cdots & p_2 & p_1 & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \cdots & 0 & 0 & p_0 & p_1 \\
\end{bmatrix}, \quad (4.42)
\]
so that $D = \det M = S_{[(u+1)!, (v+1)!]}(x, y)$. Denote by $D[i_1, i_2, \ldots; j_1, j_2, \ldots]$ its minor determinant removing rows $\{i_k\}$ and columns $\{j_k\}$. It is easy to see that

$$
\begin{align*}
D[1, v + 1, v + 2, u + v + 2; 1, 2, u + v + 1, u + v + 2] &= S_{[(u+1)!, (v+1)!]}(x, y), \\
D[1, v + 1; 1, 2] &= S_{[(u+1)!, (v+1)!]}(x, y), \\
D[v + 2, u + v + 2; u + v + 1, u + v + 2] &= S_{[(u+1)!, (v+1)!]}(x, y), \\
D[1, v + 2; 1, 2] &= D[v + 1, u + v + 2; u + v + 1, u + v + 2] = S_{[u!, v!]}(x, y).
\end{align*}
$$

(4.43)

Applying Theorem 4.1 we have

$$
DD[1, v + 1, v + 2, u + v + 2; 1, 2, u + v + 1, u + v + 2] = D[1, v + 1; 1, 2]D[v + 2, u + v + 2; u + v + 1, u + v + 2] - D[1, v + 2; 1, 2]D[v + 1, u + v + 2; u + v + 1, u + v + 2],
$$

(4.44)

which coincides with (4.40).

Take a $(u + v + 2) \times (u + v)$ matrix

$$
\begin{array}{cccccccc}
q_1 & q_0 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & & & & \\
q_{v-1} & q_{v-2} & & & & & \\
\cdots & q_v & q_{v-1} & \cdots & & & \\
\cdots & q_{v+2} & q_{v+1} & \cdots & & & \\
\cdots & p_{u+1} & p_{u+2} & \cdots & & & \\
\cdots & p_{u-1} & p_u & \cdots & & & \\
0 & \cdots & \cdots & 0 & p_0 & p_1
\end{array}
$$

(4.45)

then

$$
\begin{align*}
D[v, v + 1; \emptyset] &= S_{[(u+1)!, (v+1)!]}(x, y), \\
D[v + 2, v + 3; \emptyset] &= S_{[(u+1)!, (v+1)!]}(x, y), \\
D[v, v + 2; \emptyset] &= S_{[u!, (v+2-v-1, \ldots, 1)]}(x, y), \\
D[v + 1, v + 3; \emptyset] &= S_{[(u+2, u-1, \ldots, 1), v!]}(x, y), \\
D[v, v + 3; \emptyset] &= S_{[(u+2, u-1, \ldots, 1), (v+2-v-1, \ldots, 1)]}(x, y), \\
D[v + 1, v + 2; \emptyset] &= S_{[u!, v!]}(x, y).
\end{align*}
$$

(4.46)

By the Plücker relation, we have

$$
\begin{align*}
D[v, v + 1; \emptyset]D[v + 2, v + 3; \emptyset] - D[v, v + 2; \emptyset]D[v + 1, v + 3; \emptyset]
+ D[v, v + 3; \emptyset]D[v + 1, v + 2; \emptyset] &= 0,
\end{align*}
$$

(4.47)

which coincides with (4.41).

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References

[1] Garnier, R.: Sur des équations différentielles du troisième ordre dont l’intégrale générale est uniforme et sur un classe d’équations nouvelles d’ordre supérieur dont l’intégrale générale a ses points critiques fixes. Ann. Sci. École Norm. Sup. (3) 29, 1–126 (1912)

[2] Iwasaki, K., Kimura, H., Shimomura, S., Yoshida, M.: From Gauss to Painlevé: a modern theory of special functions. Aspects of Mathematics, vol. E16, Braunschweig: Vieweg Verlag, 1991

[3] Jacobi, C. G. J.: De formatione et proprietatibus determinantium. J. reine angew. Math. 22, 285–318 (1841)

[4] Kimura, H., Okamoto, K.: On the polynomial Hamiltonian structure of the Garnier system. J. Math. Pures Appl. (9) 63, 129–146 (1984)

[5] Kimura, H.: Symmetries of the Garnier system and of the associated polynomial Hamiltonian system. Proc. Japan Acad. Ser. A Math. Sci. 66, 176–178 (1990)

[6] Kitaev, A. V., Korotkin, D. A.: On solutions of the Schlesinger equations in terms of Θ-functions. Internat. Math. Res. Notices 1998 no. 17, 877–905 (1998)

[7] Koike, K.: On the decomposition of tensor products of the representations of the classical groups: By means of the universal characters. Adv. Math. 74, 57–86 (1989)

[8] Macdonald, I. G.: Symmetric Functions and Hall Polynomials. 2nd ed., Oxford Mathematical Monographs, New York: Oxford University Press Inc., 1995

[9] Masuda, T., Ohta, Y., Kajiwara, K.: A determinant formula for a class of rational solutions of Painlevé V equation. Nagoya Math. J. 168, 1–25 (2002)

[10] Masuda, T.: On a class of algebraic solutions to the Painlevé VI equation, its determinant formula and coalescence cascade. Funkcial. Ekvac. 46, 121–171 (2003)

[11] Noumi, M., Okada, S., Okamoto, K., Umemura, H.: Special polynomials associated with the Painlevé equations II. In: Integrable Systems and Algebraic Geometry, eds. Saito, M.-H., Shimizu, Y., Ueno, K., Singapore: World Scientific, 1998, pp. 349–372

[12] Okamoto K., Kimura, H.: On particular solutions of the Garnier systems and the hypergeometric functions of several variables. Quart. J. Math. Oxford Ser. (2) 37, 61–80 (1986)

[13] Okamoto, K.: Studies on the Painlevé equations, I. Ann. Mat. Pura Appl. (4) 146, 337–381 (1987)

[14] Tsuda, T.: Birational symmetries, Hirota bilinear forms and special solutions of the Garnier systems in 2-variables. J. Math. Sci. Univ. Tokyo 10, 355–371 (2003)
[15] Tsuda, T.: Rational solutions of the Garnier system in terms of Schur polynomials. Internat. Math. Res. Notices 2003 no. 43, 2341–2358 (2003)

[16] Tsuda, T.: Universal characters and an extension of the KP hierarchy. Commun. Math. Phys. 248, 501–526 (2004)

[17] Tsuda, T.: Universal characters, integrable chains and the Painlevé equations. Submitted to Adv. Math., Preprint: UTMS 2004–14

[18] Tsuda, T.: Universal characters and $q$-Painlevé systems. Submitted to Commun. Math. Phys.

[19] Tsuda, T.: Universal characters and integrable systems. Ph.D. thesis, The University of Tokyo, 2003