STRICHARTZ ESTIMATES FOR THE DIRAC EQUATION ON ASYMMETRICALLY FLAT MANIFOLDS

FEDERICO CACCIAFESTA, ANNE-SOPHIE DE SUZZONI, AND LONG MENG

Abstract. In this paper we prove Strichartz estimates for the Dirac equation on asymptotically flat manifolds. The proof combines the weak dispersive estimates proved in [4] with the Strichartz and smoothing estimates for the wave and Klein-Gordon flows exploiting the results in [11] and [7]-[14] in the same geometrical setting.

1. Introduction

In [4], the authors started the study of dispersive dynamics of the Dirac equation in a non-flat setting, proving local smoothing estimates in the cases of asymptotically flat manifolds and warped products. Later on, in [5] and [1], the authors proved respectively local and global in time weighted Strichartz estimates for spherically symmetric manifolds: in this case, it is indeed possible to take advantage of the so called partial wave decomposition, which is analog of the spherical harmonics decomposition for the Dirac operator, in order to recast the equation into a sum of “radial Dirac equations” with potentials, for which several results are available. The problem of proving Strichartz estimates without the assumption of spherical symmetry on the manifold could not be solved directly, as indeed it was not possible to apply the standard Duhamel trick to deduce them from the ones on $\mathbb{R}^3$, since the equation in this framework is a first order perturbation of the flat one. The purpose of this short paper is to fill this gap, proving in fact Strichartz estimates for the Dirac equation on asymptotically flat manifolds of dimension 3.

We thus consider the following Cauchy problem

\begin{align}
\begin{cases}
    i\partial_t u - D_m u = 0 \\
    u(0, x) = u_0(x), \quad x \in \mathbb{R}^3.
\end{cases}
\end{align}

where $u : \mathcal{M} \to \mathbb{C}^4$, $(\mathcal{M}, g)$ is a manifold with a Lorentzian metrics $g$ that decouples space and time. In other words, we assume that it writes

\begin{align}
g_{jk} = \begin{cases}
    1 & \text{if } j = k = 0 \\
    0 & \text{if } jk = 0 \text{ and } j \neq k \\
    -h_{jk}(x) & \text{otherwise}.
\end{cases}
\end{align}

dowed with a spin structure. We refer to Section 2 in [4] (see also [9], section 5.6) for all the details on the construction and properties of the Dirac operator on a manifold $\mathcal{M}$: here, we very briefly limit ourselves to recall that the Dirac operator with mass $m \geq 0$ can be written as

\begin{align}
D_m = -i\gamma^0 \gamma^a e^a_i D_i - \gamma^0 m.
\end{align}

2010 Mathematics Subject Classification. 35Q41, 42B37.

Keywords and phrases. Dirac equation, Strichartz estimates, asymptotically flat manifolds.
where $\gamma^j$ denote the standard Dirac matrices, that is $\gamma^0 = \beta$ and $\gamma^j = \gamma^0 \alpha_j$ for $j = 1, 2, 3$ with
\[
\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}
\]
and $\sigma_j$ are the Pauli matrices
\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
each is a matrix bundle satisfying
\[
e^a_i = e^a_i \delta^{ab} e^b_j
\]
where $\delta$ is the Kronecker symbol. We call it a dreibein since it is restriction to space of a vierbein. A vierbein is a matrix bundle $e_a^i$ satisfying
\[
e^a_i e^b_j = g^{ij}_\nu; e^\nu_a,
\]
setting $e^a_i = e^a_i$ for $j, a = 1, 2, 3$, we get indeed (6). Note that the existence of such a dreibein is induced by the asymptotic flatness of the manifold. The covariant derivative $D_i$ is given by
\[
D_0 = \partial, \quad D_j = \partial_j + B_j, \quad j = 1, 2, 3
\]
where $B_j$ writes
\[
B_j = \frac{1}{8} [\gamma_a, \gamma_b] \Omega_{ij}^a.
\]
It has a purely geometric part $\omega^a_i$, called the spin connection
\[
\omega^a_i = e^a_i \gamma_a e^{ib} + e^a_i \Gamma^i_{jk} e^{jk}
\]
with the Christoffel symbol (or affine connection) $\Gamma^i_{jk}$ given by
\[
\Gamma^i_{jk} := \frac{1}{2} h^{il}(\partial_j h_{lk} + \partial_k h_{lj} - \partial_l h_{jk}),
\]
and a purely algebraic part $\frac{1}{2} [\gamma_a, \gamma_b]$, which is due to the nature of the particle we consider (here a pair electron-positron) and more specifically to its spin (here $\frac{1}{2} \oplus \frac{1}{2}$).

Finally we recall that the scalar curvature $R_{\alpha \beta \rho \sigma}$ writes
\[
R_{\alpha \beta \rho \sigma} := h^{jk}(\partial_i \Gamma^i_{jk} - \partial_k \Gamma^j_{ki} + \Gamma^j_{ik} \Gamma^k_{ij} - \Gamma^j_{ij} \Gamma^i_{kl}).
\]

For what concerns the manifold $\mathcal{M}$, we assume the following

**Assumptions (A).** Let $(\mathcal{M}, g)$ be a 4-dimensional Lorentzian manifold with a metrics $g$ having the structure given by (2), and $h \in C^\infty(\mathbb{R}^3)$. We assume that there exists a constant $C_h$ and $\sigma \in (0, 1)$ such that for all $\alpha \in \mathbb{N}^3$ such that $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \leq 3$ and all $x$,
\[
|\hat{\phi}^\alpha(h_{ij}(x) - \delta_{ij})| \leq C_h (\langle x \rangle)^{-|\alpha|-1-\sigma}
\]
where $\hat{\phi}^\alpha = \hat{\phi}^{\alpha_1} \hat{\phi}^{\alpha_2} \hat{\phi}^{\alpha_3}$.

**Remark 1.1.** The assumptions above are fairly standard, and manifolds satisfying these are usually referred to as asymptotically flat manifolds. We stress the fact that requiring the constant $C_h$ to be small enough is a sufficient condition to ensure that the manifold is non trapping (see e.g. [12], [3]): thus, we will not have to assume this condition, that is crucial in order to have dispersion. Besides, it is easy to see that condition (11) holds for the inverse matrix of $h$ as well, provided the constant $C_h$ is sufficiently small.
Finally, let us mention the fact that the decay condition (11) might not be optimal; in particular, the power $-|\alpha| - 1 - \sigma$ could be weakened to $-|\alpha| - \sigma$ in the massless case, but we here prefer to provide a unified and much simpler presentation of the results.

**Remark 1.2.** According to (6), one can bound, in the sense of matrices, the square of $e$ with $h$. As $h$ is “close” to the identity, estimate (11) holds true for the matrices $e$. Thus, under assumptions (A), it is possible to prove that there exist constants $C_B, C'_B, C, C_R > 0$ such that

$$|B| \leq C_B C_h \langle x \rangle^{-2-\sigma}, \quad |\partial B| \leq C'_B C_h \langle x \rangle^{-2-\sigma}$$

$$|R_h| \leq C_R C_h \langle x \rangle^{-3-\sigma}, \quad |\Gamma| \leq C \Gamma C_h \langle x \rangle^{-2-\sigma}.$$  

These bounds will be proved in forthcoming Proposition 2.1.

Before stating our Theorem, let us recall the definition of admissible Strichartz triple:

**Definition 1.1.** In dimension 3, the triple $(s, q, r)$ is called wave admissible if

$$\frac{1}{q} = \frac{1}{2} - \frac{1}{r}, \quad 2 \leq q, r \leq \infty, \quad r \neq \infty, \quad s = \frac{1}{2} - \frac{1}{r} + \frac{1}{q}.$$  

The triple $(s, q, r)$ is called Klein-Gordon (or Schrödinger) admissible if

$$\frac{2}{q} = \frac{3}{2} - \frac{3}{r}, \quad 2 \leq q \leq \infty, \quad 2 \leq r \leq 6, \quad s = \frac{1}{2} - \frac{1}{r} + \frac{1}{q}.$$  

We will use the standard notation $L^p$ for Lebesgue spaces; more precisely, the norms in time will be denoted by $L_t^p$ and will be intended to be on $\mathbb{R}^+$, and we define

$$\|f\|_{L_t^p(M_h)} = \int_{\mathbb{R}^3} |f(x)|^p \sqrt{\det h(x)} dx.$$  

Then we define the $H^s_p$ and $\dot{H}^s_p$ norms for $s \in \mathbb{N}$ as

$$\|f\|_{H^s_p(M_h)} := \|(-\Delta_h)^{s/2} f\|_{L_t^p(M_h)}, \quad \|f\|_{\dot{H}^s_p(M_h)} := \|(1 - \Delta_h)^{s/2} f\|_{L_t^p(M_h)}$$  

where $\Delta_h$ is the standard Laplace-Beltrami operator. For negative $s$, we define these spaces by duality and for fractional $s$, we define these spaces by interpolation.

Our main result is then the following:

**Theorem 1.2.** Let $(M, g)$ be as given by assumptions (A). Then the massless Dirac flow satisfies the Strichartz estimate:

$$\|e^{itD} u_0\|_{L_t^q \dot{H}^{1-s}_r(M_h)} \leq \|u_0\|_{H^1(M_h)}$$  

for all wave admissible triple $(s, q, r)$, while in the massive case we have

$$\|e^{itDm} u_0\|_{L_t^q H^{1/2-s}_r(M_h)} \leq \|u_0\|_{H^1(M_h)} \quad (m \neq 0)$$  

for all Klein-Gordon admissible triple $(s, q, r)$ such that $q > 2$.

Let us briefly comment on the strategy of our proof, which is short, but relies on several different recent results. The idea consists in squaring equation (1) in order to obtain a system of wave or Klein-Gordon equations on the manifold $(M, g)$ (depending on whether $m = 0$ or $m > 0$), and then combining the estimates for such flows on asymptotically flat manifolds with the standard argument based on Duhamel formula and local smoothing estimates to control the “perturbative terms”. This trick is in fact widely used for the study of several properties of the Dirac equation, as the Dirac operator is indeed built as a suitable “square root” of the Laplacian. Anyway, we should
stress the fact that this strategy comes with two main difficulties: one is in that the Laplace-Beltrami operator obtained after the squaring procedure is not the standard (or “scalar”) one, but it is a “spinorial” Laplace operator (the covariant derivative is not the same as the covariant derivatives for scalar or vector fields). Therefore, it will not be possible to apply directly the existing results, and we will have somehow to estimate the difference of the solutions to the “scalar” and the “spinorial” wave/Klein-Gordon equations. This difference contains a first order term, and this represents the second difficulty, as indeed we will have to rely on a local smoothing at the “first order” level.

Acknowledgments. F.C. and L.M. acknowledge support from the University of Padova STARS project “Linear and Nonlinear Problems for the Dirac Equation” (LAN-PDE), and AS. dS. is supported by the ANR project ESSED ANR-18-CE40-0028.

2. Proof of Theorem 1.2

We begin with proving the statements in Remark 1.2.

Proposition 2.1. Assume that \( C_h \ll 1 \). The dreibein \( e \) exists and can be chosen such that there exist constants \( C_B \) and \( C'_B \) such that for all \( x \), we have
\[
|B(x)| \leq C_B C_h \langle x \rangle^{-\sigma - 2}, \quad |\partial B(x)| \leq C'_B C_h \langle x \rangle^{-\sigma - 3}.
\]
What is more, there exist constants \( C_R \) and \( C_\Gamma \) such that for all \( x \), we have
\[
|R_h(x)| \leq C_R C_h \langle x \rangle^{-\sigma}, \quad |\Gamma| \leq C_\Gamma C_h \langle x \rangle^{-\sigma - 2}.
\]

Proof. The scalar curvature does not depend on \( e \) and writes
\[
R = h^{jk} (\tilde{\partial} \Gamma_{jk}^i - \tilde{\partial}_k \Gamma_{ji}^j + \Gamma_{jk}^l \Gamma_{li}^i - \Gamma_{ij}^l \Gamma_{kl}^i)
\]
with the affine connection given by (9). Therefore for all \( x \), we have
\[
|R_h(x)| \leq C'_{\Gamma} (|\partial \Gamma(x)| + |\Gamma(x)|^2).
\]
Indeed, we can choose \( C_h \) small enough such that for all \( x \), \( |h^{-1}(x)| \leq 2 \). Therefore,
\[
|\Gamma(x)| \leq C'_{\Gamma} |h'(x)| \leq C_{\Gamma} C_h \langle x \rangle^{-2 - \sigma}
\]
and since \((h^{-1})'(x) = -h^{-1}(x)h'(x)h^{-1}(x)\), choosing \( C_h \ll 1 \), we get
\[
|\partial \Gamma(x)| \leq C'_{\Gamma} (|h'(x)|^2 + |h''(x)|) \leq C'_{\Gamma} C_h \langle x \rangle^{-3 - \sigma}.
\]
We deduce
\[
|R_h(x)| \leq C_R C_h \langle x \rangle^{-3 - \sigma}.
\]
We look for \( e \) a matrix bundle such that
\[
h^{ij}(x) = e^i_a(x) \delta^{ab} e^j_b(x)
\]
for all \( x \). This can be rewritten as
\[
h^{ij}(x) = e^{ia}(x) \delta_{ab} e^{jb}(x).
\]
If we restrict \((e^{ia})_{1 \leq i, a \leq 3} \) to be symmetric, this rewrites as
\[
h = e^2.
\]
As it is well-known, \( e \mapsto e^2 \) seen as a map from the the symmetric matrices to the symmetric matrices is \( C^\infty \) and its differential at the identity is twice the identity of the symmetric matrices. Therefore, it can be reversed into a \( C^\infty \) map \( F \) (a square root) around the identity since \( Id^2 = Id \). We choose \( C_h \) small enough such that for all \( x \),
\(h(x)\) lies in \(K\) a compact subset of the definition set of \(F\) and we choose \(e(x) = F(h(x))\) for all \(x\). By the increment theorem, we have
\[
|e(x) - Id| \leq \sup_K |DF| |h(x) - Id|.
\]

We also have that for all \(x\),
\[
|e'(x)| \leq \sup_K |DF| |h'(x)|, \quad |e''(x)| \leq \sup_K |D^2F| |h'(x)|^2 + \sup_K |DF| |h''(x)|,
\]
\[
|e'''(x)| \leq \sup_K |D^3F| |h'(x)|^3 + 3 \sup_K |D^2F| |h'(x)| |h''(x)| + \sup_K |DF| |h'''(x)|.
\]

Therefore, we deduce that for all \(x\),
\[
|e(x) - Id| \leq C_{eC_h} \langle x \rangle^{-\sigma - 1}, \quad |e'(x)| \leq C_{eC_h} \langle x \rangle^{-2 - \sigma},
\]
\[
|e''(x)| \leq C_{eC_h} \langle x \rangle^{-3 - \sigma}, \quad |e'''(x)| \leq C_{eC_h} \langle x \rangle^{-4 - \sigma}.
\]

We deduce
\[
|\omega(x)| \leq C_{\omega} C_h \langle x \rangle^{-2 - \sigma}, \quad |B(x)| \leq C_{B} C_h \langle x \rangle^{-2 - \sigma}, \quad \text{and} \quad |\partial B(x)| \leq C_{B} C_h \langle x \rangle^{-3 - \sigma}.
\]

We now recall the connection between the Dirac and the wave/Klein-Gordon equations on manifolds, and recall the local smoothing estimate for it. This is the first main ingredient in our proof.

**Theorem 2.2.** If \(u\) is a solution to (1), then it also solves the following
\[
\begin{cases}
\partial_t^2 u + m^2 u - \Delta_h u + \frac{1}{4} R_h u = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
\partial_t u(0, x) = i D_m u_0(x), & u(0, x) = u_0(x), \quad x \in \mathbb{R}^3
\end{cases}
\]
where \(\Delta_h = D^i D_i\) with \(D_i\) given by (7). Besides,
\[
\Delta_h v = \tilde{\Delta}_h v + B^i \partial_i v + \tilde{D}^i B_i v + B^i B_i v
\]
where \(\tilde{\Delta}_h\) is the Laplace-Beltrami operator for scalars tensor the \(4\)-dimensional identity matrix, \(\tilde{D}^i \Psi_k = \partial^i \Psi_k - \Gamma^i_k \Psi_l\) and \(B^i = h^{ij} B_j\).

Let \(u\) be a solution of (1), then
\[
\langle x \rangle^{-3/2} \| u \|_{L^2_t L^2_x(M_h)} + \| \langle x \rangle^{-1/2} \tilde{\nabla} u \|_{L^2_t L^2_x(M_h)} \leq \| D u_0 \|_{L^2(M_h)}
\]
where \(\tilde{\nabla}\) denotes the gradient for scalar fields, that is
\[
\| \langle x \rangle^{-1/2} \tilde{\nabla} u \|_{L^2_t L^2_x(M_h)}^2 := \int_M \langle x \rangle^{-1} h^{ij} \langle \partial_i u, \partial_j u \rangle_{C^4} dx.
\]

**Proof.** The relation between the Dirac equation and system (14) is shown in Corollary 1 and Formula (34) in [4]. Furthermore, by Proposition 4 in [4], we have
\[
D_m^2 = m^2 - \Delta_h + \frac{1}{4} R_h.
\]

Let us now prove the local smoothing estimate (16). It is shown in [4, Theorem 1.1] that
\[
\| \langle x \rangle^{-3/2} u \|_{L^2_t L^2_x(M_h)} + \| \langle x \rangle^{-1/2} \nabla u \|_{L^2_t L^2_x(M_h)} \leq \| D_m u_0 \|_{L^2(M_h)}.
\]
Recall that $\nabla = (D_1, D_2, D_3) = \nabla + B$ where $B = (B_1, B_2, B_3)$. According to Proposition 2.1,
$$\| \langle x \rangle^{-1/2} Bu \|_{L^2_t L^2_x(M_h)} \leq C_B C_h \| \langle x \rangle^{-3/2} u \|_{L^2_t L^2_x(M_h)} \leq \| Dm u_0 \|_{L^2(M_h)}.$$ 
Thus,
$$\| \langle x \rangle^{-3/2} u \|_{L^2_t L^2_x(M_h)} + \| \langle x \rangle^{-1/2} \nabla u \|_{L^2_t L^2_x(M_h)} \leq \| \langle x \rangle^{-3/2} u \|_{L^2_t L^2_x(M_h)} + \| \langle x \rangle^{-1/2} \nabla u \|_{L^2_t L^2_x(M_h)} + \| \langle x \rangle^{-1/2} Bu \|_{L^2_t L^2_x(M_h)}.$$ 
Hence
$$\| \langle x \rangle^{-3/2} u \|_{L^2_t L^2_x(M_h)} + \| \langle x \rangle^{-1/2} \nabla u \|_{L^2_t L^2_x(M_h)} \leq \| Dm u_0 \|_{L^2(M_h)}$$
and this concludes the proof. 

As a second ingredient, we need to provide suitable Strichartz and local smoothing estimates for the solutions to the “auxiliary” systems of wave and Klein-Gordon equations. We state them in the next two Theorems.

**Theorem 2.3** (Strichartz estimates for wave/Klein-Gordon). Let $(M, g)$ be a 4-dimensional Lorentzian manifold satisfying assumptions (A) and let $u$ be a solution to the following system:

$$\begin{cases}
\partial_t^2 u + m^2 u - \widetilde{\Delta} h u = 0, & (t, x) \in (t, x) \in M, \\
\partial_t u(0, x) = u_1(x), & u(0, x) = u_0(x), & x \in \mathbb{R}^3.
\end{cases}$$

If $m = 0$, then $u$ satisfies

$$\| u \|_{L^2_t H^{1/2 - \epsilon}(M_h)} \lesssim \| u_0 \|_{H^1(M_h)} + \| u_1 \|_{L^2(M_h)}$$

for any wave admissible triple $(s, q, r)$. 
If $m > 0$, then $u$ satisfies

$$\| u \|_{L^2_t H^{1/2 - \epsilon}(M_h)} \lesssim \| u_0 \|_{H^{1/2}(M_h)} + \| u_1 \|_{H^{-1/2}(M_h)}$$

for any Klein-Gordon admissible triple $(s, q, r)$.

**Proof.** When $m = 0$, this is just Theorem 1.4 in [11]. When $m > 0$, the Strichartz estimate follows from the global-in-time Strichartz estimate on non-trapping conic manifold (i.e. scattering manifold) in [14, Theorem 1.1]. It is shown in [8, Remark 1.2] that any asymptotically flat space $(\mathbb{R}^3, h)$ with decay estimates $|\partial^\alpha (h - \delta)| \leq C_{\alpha} \langle x \rangle^{-|\alpha| - 1}$ is also asymptotically conic (see also [10, Remark 1.5]). Hence we can deduce the result in the case $m > 0$. 

**Theorem 2.4** (Local smoothing estimates for wave/Klein-Gordon). Let $(M, g)$ be a 4-dimensional Lorentzian manifold satisfying assumptions (A). Then the following estimates hold

$$\| \langle x \rangle^{-1/2} e^{it\sqrt{-\Delta_h}} f \|_{L^2_t L^2_x(M_h)} \lesssim \| f \|_{L^2(M_h)},$$

for any $f \in L^2(M_h)$, and

$$\| \langle x \rangle^{-1/2} e^{it\sqrt{m^2 - \Delta_h}} f \|_{L^2_t L^2_x(M_h)} \lesssim \| (1 - \widetilde{\Delta}_h)^{1/4} f \|_{L^2(M_h)}$$

for any $f$ such that $(1 - \widetilde{\Delta}_h)^{1/4} f \in L^2(M_h)$.
Proof. Let us consider the following unitary transform
\[ \mathcal{V} : L^2(\mathbb{R}^3, \sqrt{\det h(x)} dx) \to L^2(\mathbb{R}^3, dx), \quad v \mapsto (\det h(x))^{1/4} v. \]

The transformation \( \mathcal{V} \) sends \(-\tilde{\Delta}_h\) to
\[ P = -\mathcal{V} \tilde{\Delta}_h \mathcal{V}^{-1} = -(\det h(x))^{1/4} \tilde{\Delta}_h (\det h(x))^{-1/4}. \]

Let us start with the massless case. Let \( u := e^{it\sqrt{-\tilde{\Delta}_h}} f \in L^2(\mathcal{M}, dg) \) and let \( v = \mathcal{V} u \); then \( v \in L^2(\mathbb{R}_+ \times \mathbb{R}^3, dx) \), and
\[ \partial_t^2 v - \tilde{\Delta}_h v = 0 \iff \partial_t^2 v - P v = 0. \]

According to [2, Theorem 1.3] or [11, Page 24, Section 6], we get
\[ \| \langle x \rangle^{-1/2} u \|_{L^2_t L^2(\mathcal{M}_h)} = \| \langle x \rangle^{-1/2} v \|_{L^2(\mathbb{R}^n \times \mathbb{R}^3)} \lesssim \| \langle x \rangle^{-1/2} v(0) \|_{L^2(\mathcal{M}_h)} = \| f \|_{L^2(\mathcal{M}_h)}. \]

For the second estimate (massive case), according to [13, Formula (3.5) and Proposition 3.1] (taking \( V = 0 \)), we know that \( \langle x \rangle^{-1} \) is \(-\tilde{\Delta}_h\)-smooth, i.e.,
\[ \| \langle x \rangle^{-1} e^{-it\tilde{\Delta}_h} f \|_{L^2_t L^2(\mathcal{M}_h)} \lesssim \| f \|_{L^2(\mathcal{M}_h)}. \]

It follows from [7, Theorem 2.2 and Theorem 2.4] that
\[ \| \langle x \rangle^{-1/2} e^{it(-\sqrt{m^2 - \tilde{\Delta}_h})} f \|_{L^2_t L^2(\mathcal{M}_h)} \lesssim \| (m^2 - \tilde{\Delta}_h)^{1/4} f \|_{L^2} \]
and this concludes the proof.

We are now in a position to prove our main result.

Proof of Theorem 1.2. According to (14) and (15), by Duhamel Formula, we can write for any \( m \geq 0 \)
\[ u(t, x) := e^{itD_m} u_0 = \dot{W}_m(t) u_0 + i W_m(t) D_m u_0 + \int_0^t W_m(t - s) (\Omega_1(u)(s) + \Omega_2 u(s)) ds \]
where
\[ W_m(t) = \frac{\sin(t\sqrt{m^2 - \tilde{\Delta}_h})}{\sqrt{m^2 - \tilde{\Delta}_h}}, \quad \dot{W}_m = \partial_t W_m \]
and
\[ (25) \quad \Omega_1(u) := 2B^i \partial_i u, \quad \Omega_2 := \partial^i B_i + B^i B_i - i \Gamma^i_4 B_j - \frac{1}{4} \mathcal{R}_h. \]

We deal with the massless and massive cases separately, starting with the former. According to Christ-Kiselev Lemma [6] and Theorem 2.3,
\[ \left\| \int_0^T W_m(t - s) (\Omega_1(u)(s) + \Omega_2 u(s)) ds \right\|_{L^q_t H^{1-s}(\mathcal{M}_h)} \lesssim \left\| \int_0^T e^{-is\sqrt{-\tilde{\Delta}_h}} (\Omega_1(u)(s) + \Omega_2 u(s)) ds \right\|_{L^q_t H^{1-s}(\mathcal{M}_h)} \lesssim \left\| \int_0^T e^{-is\sqrt{-\tilde{\Delta}_h}} (\Omega_1(u)(s) u(s) + \Omega_2 u(s)) ds \right\|_{L^2(\mathcal{M}_h)}. \]
By the dual form of (22),
\[
\left\| \int_0^T e^{-is\sqrt{-\Delta_h}}(\Omega_1(u)(s)u(s) + \Omega_2u(s))ds \right\|_{L^2(M_h)} \lesssim \left\| \langle x \rangle^{1/2+} \Omega_1(u)(s)u + \Omega_2u \right\|_{L^2(M_h)}.
\]
Then by (16) and Remark 1.2, we have
\[
\left\| \langle x \rangle^{1/2+} \Omega_2u \right\|_{L^2(M_h)} \lesssim \left\| \langle x \rangle^{2+} \Omega \right\|_{L^\infty} \left\| \langle x \rangle^{-3/2-}u \right\|_{L^2(M_h)} \lesssim \| Du_0 \|_{L^2(M_h)}.
\]
Then by (16) and Remark 1.2, we have
\[
\left\| \langle x \rangle^{1/2+} \Omega_1(u) \right\|_{L^2(M_h)} \lesssim \left\| \langle x \rangle^{1+} B \right\|_{L^\infty} \left\| \langle x \rangle^{-1/2-} \nabla u \right\|_{L^2(M_h)} \lesssim \| Du_0 \|_{L^2(M_h)}.
\]
Thus thanks to Lemma A.1,
\[
\left\| u \right\|_{L^2_t H^1_t(M_h)} \lesssim \left\| u \right\|_{H^1(M_h)} + \| Du_0 \|_{L^2(M_h)} \lesssim \left\| u \right\|_{H^1(M_h)}.
\]
Now we consider the massive case. According to Christ-Kiselev lemma [6], since we restrict to the case \( q > 2 \), and Theorem 2.3,
\[
\left\| \int_0^T W_m(t-s)(\Omega_1(u)(s) + \Omega_2u(s))ds \right\|_{L^2_t H^{1/2-}(M_h)} \lesssim \left\| e^{it\sqrt{m^2 - \Delta_h}} \int_0^T e^{-is\sqrt{m^2 - \Delta_h}}(\Omega_1(u)(s) + \Omega_2u(s))ds \right\|_{L^2_t H^{1/2-}(M_h)} \lesssim \left\| \int_0^T e^{-is\sqrt{m^2 - \Delta_h}}(\Omega_1(u)(s) + \Omega_2u(s))ds \right\|_{H^{-1/2}(M_h)}.
\]
By the dual form of (23),
\[
\left\| \int_0^T e^{-is\sqrt{m^2 - \Delta_h}}(\Omega_1(u)(s) + \Omega_2u(s))ds \right\|_{H^{-1/2}(M_h)} \lesssim \left\| \langle x \rangle^{1/2} \left( B^i \partial_i u + \Omega_2u \right) \right\|_{L^2(M_h)}.
\]
Then by (16) and Lemma A.1, we have
\[
\left\| \langle x \rangle^{1/2} \Omega_2u \right\|_{L^2_t L^2(M_h)} \lesssim \left\| \langle x \rangle^{2+} \Omega \right\|_{L^\infty} \left\| \langle x \rangle^{-3/2-}u \right\|_{L^2(M_h)} \lesssim \| Du_0 \|_{L^2(M_h)} \lesssim \| u \|_{H^1(M_h)}.
\]
Then by (16), Remark 1.2 and Lemma A.1, we have
\[
\left\| \langle x \rangle^{1+} \Omega \right\|_{L^2_t L^2(M_h)} \lesssim \left\| \langle x \rangle^{1+} B \right\|_{L^\infty} \left\| \langle x \rangle^{-1/2-} \nabla u \right\|_{L^2(M_h)} \lesssim \| Du_0 \|_{L^2} \lesssim \| u \|_{H^1(M_h)},
\]
so that
\[
\left\| u \right\|_{L^2_t H^{1/2-}(M_h)} \lesssim \left\| u \right\|_{H^1(M_h)} + \left\| Du_0 \right\|_{H^{-1/2}(M_h)} \lesssim \left\| u \right\|_{H^1(M_h)}.
\]
and this concludes the proof.

\section*{Appendix A. Norm estimate}

Here we give some results concerning the relationship between the standard Sobolev norm and the one induced by the Dirac operator, which is used in the proof of Theorem 1.2.

\textbf{Lemma A.1.} Under assumptions (A), for \( m \geq 0 \),
\[
\|(m^2 - \Delta_h)^{1/2}u \|_{L^2(M_h)} \lesssim \| D_m u \|_{L^2(M_h)} \lesssim \|(m^2 - \Delta_h)^{1/2}u \|_{L^2(M_h)}.
\]
Proof. Using that $\tilde{\Delta}_h$ is self-adjoint on $L^2(\mathcal{M}_h)$, we have

$$
\| (m^2 - \tilde{\Delta}_h)^{1/2} u \|_{L^2(\mathcal{M}_h)}^2 = \langle (m^2 - \tilde{\Delta}_h) u, u \rangle_{L^2(\mathcal{M}_h)}.
$$

We also have

$$
- \langle \tilde{\Delta}_h u, u \rangle_{L^2(\mathcal{M}_h)} = h^{ij} \langle \partial_i u, \partial_j u \rangle_{L^2(\mathcal{M}_h)},
$$

so that

$$
- \langle \tilde{\Delta}_h u, u \rangle_{L^2(\mathcal{M}_h)} = \delta^{ij} \langle \partial_i u, \partial_j u \rangle_{L^2(\mathcal{M}_h)} + (h^{ij} - \delta^{ij}) \langle \partial_i u, \partial_j u \rangle_{L^2(\mathcal{M}_h)}.
$$

Using Cauchy-Schwarz inequality and the fact that $|h^{ij} - \delta^{ij}| \ll 1$ yields

$$
\delta^{ij} \langle \partial_i u, \partial_j u \rangle_{L^2(\mathcal{M}_h)} \leq - \langle \tilde{\Delta}_h u, u \rangle_{L^2(\mathcal{M}_h)} \leq \delta^{ij} \langle \partial_i u, \partial_j u \rangle_{L^2(\mathcal{M}_h)}.
$$

As the Dirac operator is self-adjoint on $L^2(\mathcal{M}_h)$, we have

$$
\| \mathcal{D}_m u \|_{L^2(\mathcal{M}_h)}^2 = \langle \mathcal{D}_m^2 u, u \rangle_{L^2(\mathcal{M}_h)},
$$

and thanks to the identity $\mathcal{D}_m^2 = m^2 - \tilde{\Delta}_h + \Omega_1 + \Omega_2$ with $\Omega_1$, $\Omega_2$ given by (25) we get

$$
\| \mathcal{D}_m u \|_{L^2(\mathcal{M}_h)}^2 = \langle (m^2 - \tilde{\Delta}_h) u, u \rangle_{L^2(\mathcal{M}_h)} + \langle (\Omega_1(u) + \Omega_2 u), u \rangle_{L^2(\mathcal{M}_h)}.
$$

Using the skew-symmetry of $B_i$, we get

$$
\langle B^i \partial_i u, u \rangle_{L^2(\mathcal{M}_h)} = - \langle \partial_i u, B^i u \rangle_{L^2(\mathcal{M}_h)}, \quad \langle B_i \partial^i u, u \rangle_{L^2(\mathcal{M}_h)} = - \langle \partial^i u, B_i u \rangle_{L^2(\mathcal{M}_h)}
$$

By Cauchy-Schwarz inequality again we get

$$
| \langle \Omega_1(u), u \rangle_{L^2(\mathcal{M}_h)} | \leq \sum_i \| \partial_i u \|_{L^2(\mathcal{M}_h)} \| B \langle x \rangle \|_{L^\infty} \| \langle x \rangle^{-1} u \|_{L^2(\mathcal{M}_h)}.
$$

Now, as the $L^2(\mathcal{M}_h)$ and the $L^2(\mathbb{R}^3)$ norms are equivalent (due to the assumptions on $h$), we use Hardy inequality to get

$$
| \langle \Omega_1(u), u \rangle_{L^2(\mathcal{M}_h)} | \leq \| B \langle x \rangle \|_{L^\infty} \| (m^2 - \tilde{\Delta}_h)^{1/2} u \|_{L^2(\mathcal{M}_h)}^2.
$$

Similarly,

$$
| \langle \Omega_2(u), u \rangle_{L^2(\mathcal{M}_h)} | \leq \| \langle x \rangle^2 \Omega_2 \|_{L^\infty} \| (m^2 - \tilde{\Delta}_h)^{1/2} u \|_{L^2(\mathcal{M}_h)}^2.
$$

Now, as $\| \langle x \rangle B \|_{L^\infty} \ll 1$ (see Proposition 2.1) and

$$
\| \langle x \rangle^2 \Omega_2 \|_{L^\infty} \leq \| \langle x \rangle^2 (\partial^i B_i + B^i B_i - \Gamma^i{}_{jk} B_j + \frac{1}{4} \mathcal{R}_h) \|_{L^\infty} \ll 1,
$$

we finally get that

$$
\| (m^2 - \tilde{\Delta}_h)^{1/2} u \|_{L^2(\mathcal{M}_h)}^2 \leq \| \mathcal{D}_m u \|_{L^2(\mathcal{M}_h)}^2 \leq \| (m^2 - \tilde{\Delta}_h)^{1/2} u \|_{L^2(\mathcal{M}_h)}^2
$$

and this concludes the proof.
REFERENCES

[1] Jonathan Ben-Artzi, Federico Cacciafesta, Anne-Sophie de Suzzoni, and Junyong Zhang. Global strichartz estimates for the dirac equation on symmetric spaces, https://arxiv.org/abs/2101.09218.
[2] Jean-François Bony and Dietrich Hänner. The semilinear wave equation on asymptotically euclidean manifolds. Communications in Partial Differential Equations, 35(1):23–67, 2009.
[3] Federico Cacciafesta, Piero D’Ancona, and Renato Lucà. Helmholtz and dispersive equations with variable coefficients on exterior domains. SIAM J. Math. Anal., 48(3):1798–1832, 2016.
[4] Federico Cacciafesta and Anne-Sophie de Suzzoni. Weak dispersion for the Dirac equation on asymptotic flat and warped product spaces. Discrete Contin. Dyn. Syst., 39(8):4359–4398, 2019.
[5] Federico Cacciafesta and Anne-Sophie de Suzzoni. Local in Time Strichartz Estimates for the Dirac Equation on Spherically Symmetric Spaces. Int. Math. Res. Not. IMRN, (4):2729–2771, 2022.
[6] Michael Christ and Alexander Kiselev. Maximal functions associated to filtrations. J. Funct. Anal., 179(2):409–425, 2001.
[7] Piero D’Ancona. Kato smoothing and Strichartz estimates for wave equations with magnetic potentials. Comm. Math. Phys., 335(1):1–16, 2015.
[8] Andrew Hassell, Terence Tao, and Jared Wunsch. A Strichartz inequality for the Schrödinger equation on nontrapping asymptotically conic manifolds. Comm. Partial Differential Equations, 30(1-3):157–205, 2005.
[9] Leonard E. Parker and David J. Toms. Quantum field theory in curved spacetime. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 2009. Quantized fields and gravity.
[10] Igor Rodnianski and Terence Tao. Effective limiting absorption principles, and applications. Comm. Math. Phys., 333(1):1–95, 2015.
[11] Christopher D. Sogge and Chengbo Wang. Concerning the wave equation on asymptotically Euclidean manifolds. J. Anal. Math., 112:1–32, 2010.
[12] Daniel Tataru. Parametrices and dispersive estimates for Schrödinger operators with variable coefficients. Amer. J. Math., 130(3):571–634, 2008.
[13] Junyong Zhang and Jiqiang Zheng. Global-in-time Strichartz estimates for Schrödinger on scattering manifolds. Comm. Partial Differential Equations, 42(12):1962–1981, 2017.
[14] Junyong Zhang and Jiqiang Zheng. Strichartz estimate and nonlinear Klein-Gordon equation on nontrapping scattering space. J. Geom. Anal., 29(3):2957–2984, 2019.

Federico Cacciafesta: Dipartimento di Matematica, Università degli studi di Padova, Via Trieste, 63, 35131 Padova PD, Italy
Email address: cacciafe@math.unipd.it

Anne-Sophie de Suzzoni: CMLS, École Polytechnique, CNRS, Université Paris-Saclay, 91128 PALAISEAU Cedex, France
Email address: anne-sophie.de-suzzoni@polytechnique.edu

Long Meng: CERMICS, École des ponts ParisTech, 6 and 8 av. Pascal, 77455 Marne-la-Vallée, France
Email address: long.meng@enpc.fr