Integrable Structures in String Field Theory

L. Bonora\textsuperscript{a} and A.S. Sorin\textsuperscript{b}

\textsuperscript{a}International School for Advanced Studies (SISSA/ISAS), Via Beirut 2, 34014 Trieste, Italy and INFN, Sezione di Trieste bonora@sissa.it

\textsuperscript{b}Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research (JINR), 141980, Dubna, Moscow Region, Russia sorin@thsun1.jinr.ru

Abstract. We give a simple proof that the Neumann coefficients of surface states in Witten’s SFT satisfy the Hirota equations for dispersionless KP hierarchy. In a similar way we show that the Neumann coefficients for the three string vertex in the same theory obey the Hirota equations of the dispersionless Toda Lattice hierarchy. We conjecture that the full (dispersive) Toda Lattice hierarchy and, even more attractively a two–matrix model, may underlie open SFT.

1 Introduction

A remarkable integrable structure underlies Witten’s String Field Theory (SFT), \cite{Witten}. It manifests itself in the Neumann coefficients, which can be shown to satisfy the Hirota bilinear equations. This is true for the Neumann coefficients of the surface states such as the sliver and the butterfly states, \cite{Witten, Bonora, Sorin, Bonora2}. In this case the relevant integrable hierarchy is the dispersionless KP (dKP) hierarchy. But, what is more important, it is also true for the Neumann coefficients that define the three string vertex, a structure which is at the core of Witten’s SFT since it describes the basic interaction and the star product. At the tree level in this case the relevant integrable hierarchy is the dispersionless Toda lattice (dTL) hierarchy.

Following an inspiring suggestion of \cite{Witten}, in this paper we show in a straightforward way that the Neumann coefficients for surface states satisfy the Hirota equations for
the dKP hierarchy and that the three string Neumann coefficients satisfy the Hirota bilinear equations for the dTL hierarchy. Since the dKP hierarchy is a subcase of the dTL hierarchy, it is evident that SFT is based (at the tree level) on the latter.

These results are simple verifications that some identities hold for Neumann coefficients in SFT, but they may have important implications. For it is naturally tempting to conjecture that the full SFT is based on the (dispersive) TL hierarchy. And since the latter is the hierarchy underlying two–matrix models, one is lead to conjecture that at the basis of SFT there may lie a two–matrix model. So the discovery of the integrable structure of Witten’s string field theory opens the way to a new (old) exciting field of research.

The paper is organized as follows. In section 2 we introduce the Hirota equations for dKP and dTL hierarchies. In section 3 we introduce new compact formulas for Neumann coefficients for surface states and verify that they satisfy the Hirota equations for the dKP hierarchy. In section 4 we introduce generating functions for the Neumann coefficients of the three string vertex and verify that they satisfy the bilinear Hirota equations for the dTL hierarchy. Section 5 is devoted to comments and future prospects.

2 dKP and dTL Hirota equations

The essence of integrable hierarchies is contained in the Hirota bilinear equations [7]. We present them here for the dKP and dTL hierarchies [8, 9]. To this end let us introduce the flow parameters $t_0, t_k, \bar{t}_k$ with $k = 1, 2, ..., \infty$ and the differential operators

$$D(z) = \sum_{k=1}^{\infty} \frac{1}{kz^k} \frac{\partial}{\partial t_k}, \quad \bar{D}(\bar{z}) = \sum_{k=1}^{\infty} \frac{1}{k\bar{z}^k} \frac{\partial}{\partial \bar{t}_k}.$$ 

(1)

Next we need the $\tau$–function (free energy) $F = F(\{t_0, t_k, \bar{t}_k\})$. For the dKP $F$ depends only on $\{t_k\}$, $k \geq 1$. Then the Hirota equations for the dKP hierarchy can be written in the following compact elegant form (for more details, see [10, 11] and references therein):

$$(z_1 - z_2)e^{D(z_1)D(z_2)F} + (z_2 - z_3)e^{D(z_2)D(z_3)F} + (z_3 - z_1)e^{D(z_3)D(z_1)F} = 0.$$ 

(2)

The dispersionless Toda Lattice hierarchy depends instead on all the parameters $\{t_0, t_k, \bar{t}_k\}$. The Hirota equations can still be written in compact form

$$(z_1 - z_2)e^{D(z_1)D(z_2)F} = z_1e^{-\theta_0 D(z_1)F} - z_2e^{-\theta_0 D(z_2)F},$$ 

(3)

$$z_1\bar{z}_2 \left(1 - e^{-D(z_1)\bar{D}(\bar{z}_2)F}\right) = e^{\theta_0 (\partial_{\theta_0} + D(z_1) + \bar{D}(\bar{z}_2))F}.$$ 

(4)

and their bar-counterparts.

Our task in this paper is to identify the exponents in the above equations (derivatives of $F$) with suitable generating functions of Neumann coefficients and verify that the relevant Hirota equations are identically satisfied.
3 Neumann coefficient for surface states

Neumann coefficients for surface states were defined in [12]. Given a conformal map $f(z)$ from the upper semidisk, they are defined by

$$N^f_{nm} = \frac{1}{nm} \oint_0 \frac{dz}{2\pi i} \frac{1}{z^n} \oint_0 \frac{dw}{2\pi i} \frac{1}{w^m} \partial_w \partial_{\bar{w}} \ln(f(z) - f(w)).$$  \hspace{1cm} (5)

In the following we actually only need that $f(z)$ admits a Taylor expansion around $z = 0$, with $f'(0) \neq 0$. In (5) the integration contours are counterclockwise about the origin.

Now we define the generating function $N^f(z_1, z_2)$ for large $z_1$ and $z_2$,

$$N^f(z_1, z_2) \equiv \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{z_1^n z_2^m} N^f_{nm}. \hspace{1cm} (6)$$

Using (5), we make this definition more explicit as follows

$$N^f(z_1, z_2) = \oint_0 \frac{dw}{2\pi i} \oint_0 \frac{d\zeta}{2\pi i} \sum_{n,m=1}^{\infty} \frac{1}{n(z_1 w)^n} \frac{1}{m(\zeta)} \partial_w \partial_{\bar{w}} \ln(f(w) - f(\zeta))$$

$$= \oint_0 \frac{dw}{2\pi i} \oint_0 \frac{d\zeta}{2\pi i} \ln(1 - \frac{1}{z_1 w}) \ln(1 - \frac{1}{\zeta}) \partial_w \partial_{\bar{w}} \ln(f(w) - f(\zeta)) \hspace{1cm} (7)$$

$$= \oint_0 \frac{dw}{2\pi i} \oint_0 \frac{d\zeta}{2\pi i} \partial_w \ln(1 - \frac{1}{z_1 w}) \partial_{\bar{w}} \ln(1 - \frac{1}{\zeta}) \ln(f(w) - f(\zeta)).$$

The series converge for $|z_1 w| > 1$ and $|z_2 \zeta| > 1$. The integration contours are chosen in such a way as to surround the logarithmic cuts. Next we insert the decomposition

$$\partial_w \ln(1 - \frac{1}{z_1 w}) \partial_{\bar{w}} \ln(1 - \frac{1}{\zeta}) = \frac{1}{w \zeta} - \frac{1}{w \zeta - z_1^2} - \frac{1}{\zeta w - z_1^2} + \frac{1}{w - z_1^{-1}} + \frac{1}{\zeta - z_1^{-1}}. \hspace{1cm} (8)$$

We see immediately that the term $\frac{1}{w \zeta}$ gives rise to a divergent term in (7). For this reason we regularize $N^f(z_1, z_2)$ by subtracting

$$\oint_0 \frac{dw}{2\pi i} \oint_0 \frac{d\zeta}{2\pi i} \partial_w \ln(1 - \frac{1}{z_1 w}) \partial_{\bar{w}} \ln(1 - \frac{1}{\zeta}) \ln(w - \zeta)$$

from it. Repeating the derivation, inserting (5) and performing the contour integrals, we get

$$N^f(z_1, z_2) = \oint_0 \frac{dw}{2\pi i} \oint_0 \frac{d\zeta}{2\pi i} \sum_{n,m=1}^{\infty} \frac{1}{n(z_1 w)^n} \frac{1}{m(\zeta)} \partial_w \partial_{\bar{w}} \ln \left( \frac{f(w) - f(\zeta)}{w - \zeta} \right)$$

$$= \ln \left( \frac{f'(0)}{z_1 - z_2} \frac{f(z_1^{-1}) - f(z_2^{-1})}{(f(0) - f(z_1^{-1}))(f(z_1^{-1}) - f(0))} \right). \hspace{1cm} (9)$$

We stress that the subtracted term with $\ln(w - \zeta)$ does not change the value of the initial integral because it corresponds to the identity mapping with obviously trivial Neumann coefficients.
Now we identify $N^f(z_1, z_2)$ with $D(z_1)D(z_2)F$ in eq. (2),
\begin{equation}
D(z_1)D(z_2)F = N^f(z_1, z_2),
\end{equation}
and analogously for the other exponents. It is now elementary to prove that (2) is identically satisfied. We simply remark that $N^f(z_1, z_2)$ can be rewritten as
\begin{equation}
N^f(z_1, z_2) = \ln \left( \frac{f'(0)}{z_1 - z_2} \left( \frac{1}{(f(0) - f(z_2^{-1}))} - \frac{1}{(f(0) - f(z_1^{-1}))} \right) \right),
\end{equation}
then plug this equation in (2) and use cyclicity.

The fact that $N^f(z_1, z_2)$ satisfies the Hirota equations means that the corresponding Neumann coefficients $N^f_{nm}$ satisfy all the identities implied by (2), which can be obtained by expanding the latter in powers of $z_1, z_2$, and equating the corresponding coefficients. The first few of these identities have been written down in [11, 6] and will not be repeated here. Instead we would like to point out that formula (11) is very efficient in explicitly computing the relevant Neumann coefficients. A very simple example is given by the so–called Nothing state, [5]. In this case
\begin{equation}
f(z) = f_N(z) \equiv \frac{z}{1 + z^2},
\end{equation}
Plugging in this formula one can obtain the following expression for the generating function $N^{f_N}(z_1, z_2)$ of the Neumann coefficients $N^{f_N}_{nm}$ of the Nothing state
\begin{equation}
N^{f_N}(z_1, z_2) = \ln \left( 1 - \frac{1}{z_1z_2} \right).
\end{equation}
Then, the Neumann coefficients can easily be extracted by expanding $N^{f_N}(z_1, z_2)$ using
\begin{equation}
N^{f_N}(z_1, z_2) \equiv - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{z_1^n z_2^m} \frac{1}{\sqrt{nm}} \delta_{n,m}, \quad N^{f_N}_{nm} = - \frac{1}{\sqrt{nm}} \delta_{n,m}
\end{equation}
and, up to normalization, reproduce correctly the corresponding coefficients given in [5].

As a second example, we mention the series of states characterizing by [5]
\begin{equation}
f_{B_n}(z) = \exp \left( - \frac{z^{n+1}}{n} \frac{\partial}{\partial z} \right) z = \frac{z}{(1 + z^n)^{1/n}} \quad n = 1, 2, \ldots .
\end{equation}
The state at $n = 2$ corresponds to the so–called Butterfly state, while the states with other $n$ are generalizations of the latter. Plugging (13) into eq. (14) we obtain the corresponding generating functions $N^{f_{B_n}}(z_1, z_2)$ of Neumann coefficients for these states
\begin{equation}
N^{f_{B_n}}(z_1, z_2) = \ln \left( \frac{(1 + z_1^n)^{1/n} - (1 + z_2^n)^{1/n}}{z_1 - z_2} \right).
\end{equation}
The Neumann coefficients $N^{f_{B_2}}_{nm}$ for the Butterfly state were calculated explicitly in [5], and it is a simple exercise to verify that they are reproduced from eq. (16) at $n = 2$. Let us remark that, actually, the method, developed in [5] to evaluate $N^{f_{B_2}}_{nm}$, can also be used to evaluate the Neumann coefficients $N^{f_{B_n}}_{nm}$ for any $n$, and we have explicitly verified that they are indeed reproduced by our simple formula (16).
4 Neumann coefficients for three string vertex

The three string vertex $\langle 1, 13, 14 \rangle$ of the Open String Field Theory is given by

$$
|V_3\rangle = \int d^{26}p^{(1)}d^{26}p^{(2)}d^{26}p^{(3)} \delta^{26}(p^{(1)} + p^{(2)} + p^{(3)}) \exp(-E) \langle 0, p \rangle_{123} \tag{17}
$$

where

$$
E = \sum_{r,s=1}^{3} \left( \frac{1}{2} \sum_{m,n \geq 1} \eta_{\mu\nu} a_{m}^{(r)\mu} V_{mn}^{rs} a_{n}^{(s)\nu} + \sum_{n \geq 1} \eta_{\mu\nu} p_{(r)}^{\mu} V_{0n}^{rs} a_{n}^{(s)\nu} + \frac{1}{2} \eta_{\mu\nu} p_{(r)}^{\mu} V_{00}^{rs} p_{(s)}^{\nu} \right). \tag{18}
$$

Summation over the Lorentz indices $\mu, \nu = 0, \ldots, 25$ is understood and $\eta_{\mu\nu}$ denotes the flat Lorentz metric. The operators $a_{m}^{(r)\mu}, a_{m}^{(r)\mu\dagger}$ denote the non-zero modes matter oscillators of the $r$-th string ($r = 1, 2, 3$). $p_{(r)}^{\mu}$ are the center of mass momenta of the $r$-th string. $|p_{(r)}\rangle$ is annihilated by the annihilation operators $a_{m}^{(r)\mu}$ and is eigenstate of the momentum operator $p_{(r)}^{\mu}$ with eigenvalue $p_{(r)}^{\mu}$.

The Neumann coefficients $V_{mn}^{rs}$, $V_{0n}^{rs}$ and $V_{00}^{rs}$ were explicitly computed in [13]. For definiteness, we refer to the coefficients contained in Appendix A of [13] as the standard ones.

Our purpose in this section is to show that these coefficients too obey the Hirota equations. The difference with the previous section is that these Hirota equations are the ones characteristic of the dispersionless Toda Lattice hierarchy. We recall that in the definition of $V_{0n}^{rs}$ there is a gauge freedom, since, due to momentum conservation we are allowed to make the redefinition

$$
V_{0n}^{rs} \rightarrow V_{0n}^{rs} + A_{n}^{s}. \tag{19}
$$

We will see below that the Hirota equations involve only gauge invariant combinations of them.

The strategy is the same as in the previous section. We first define a more efficient method to calculate the Neumann coefficient, by introducing suitable generating functions. A basic difference with the previous section is that in this problem, instead of one, we have three functions involved

$$
\begin{align*}
    f_1(z_1) &= e^{\frac{2\pi i}{3}} \left( \frac{1 + iz_1}{1 - iz_1} \right)^{\frac{2}{3}}, \\
    f_2(z_2) &= \left( \frac{1 + iz_2}{1 - iz_2} \right)^{\frac{2}{3}}, \\
    f_3(z_3) &= e^{\frac{-2\pi i}{3}} \left( \frac{1 + iz_3}{1 - iz_3} \right)^{\frac{2}{3}}.
\end{align*} \tag{20}
$$

We define

$$
\begin{align*}
    N_{rs}^{rs}(z_1, z_2) &= \sum_{n,m=1}^{\infty} \frac{1}{z_1^n z_2^m} N_{nm}^{rs}, \tag{21} \\
    N_{0}^{rs}(z) &= \sum_{n=1}^{\infty} \frac{1}{z^n} N_{0n}^{rs}. \tag{22}
\end{align*}
$$
Now, we insert in these definitions the formulas for $N_{nm}^{rs}$ and $N_{0n}^{rs}$ given by \[10\], and, as in the previous section, we extract the corresponding generating functions. The task is easy for $N_{rr}^{rs}$, because we can limit ourselves to taking the formula of the previous section. In other words we have

$$N_{rr}^{rs}(z_1, z_2) = \ln \left( \frac{f_r'(0)}{z_1 - z_2} \frac{f_r(z_1^{-1}) - f_r(z_2^{-1})}{(f_r(0) - f_r(z_2^{-1}))(f_r(z_1^{-1}) - f_r(0))} \right). \tag{23}$$

For $r \neq s$ we start again from \[10\]

$$N_{nm}^{rs} = -\frac{1}{nm} \oint_0 \frac{dz}{2\pi i} \frac{d w}{2\pi i} \frac{1}{w^m} \partial_z \partial_w \ln (f_r(z) - f_s(w)). \tag{24}$$

Next we insert this formula in \[21\]. We notice that, since $r \neq s$ implies $f_r(0) \neq f_s(0)$, we do not need to do a subtraction as in the previous section, so the method works straight away. The result is

$$N^{rs}(z_1, z_2) = -\ln \left( \frac{f_r(z_1^{-1}) - f_s(z_2^{-1})(f_r(0) - f_s(0)))}{(f_r(0) - f_s(z_2^{-1}))(f_r(z_1^{-1}) - f_s(0))} \right). \tag{25}$$

As for $N_0^{rs}(z)$ we start again from the formula that can be found in \[10\]

$$N_{0n}^{rs} = \frac{1}{m} \oint_0 \frac{dz}{2\pi i} \frac{1}{z^n} \partial_z \ln (f_r(0) - f_s(z)), \tag{26}$$

and plug it into eq. \[22\]. Once again we have to distinguish two cases $r = s$ and $r \neq s$. The first case needs a subtraction, the second does not. The result is

$$N_0^{rr}(z) = \ln \left( z \frac{f_r(0) - f_r(z^{-1})}{-f_r'(0)} \right), \tag{27}$$

while, for the case $r \neq s$, we obtain

$$N_0^{rs}(z) = \ln \left( \frac{f_r(0) - f_s(z^{-1})}{f_r(0) - f_s(0)} \right). \tag{28}$$

From the above definitions it is immediate to see that

$$N^{rs}_{nm} = N^{sr}_{mn}.$$ Moreover, from the explicit form of the functions $f_r$ \[20\], one can easily verify that

$$N^{r+1,s+1} = N^{rs}, \quad N_0^{r+1,s+1} = N_0^{rs}$$

with $r$ and $s$ defined mod 3.

\[1\]Notice the different sign with respect to \[10\].
Now, having all the generating functions at hand, we want to show that they satisfy the Hirota equations (3–4). To this end we proceed to identify the exponents of these equations with the previous generating functions. When zero modes are not involved things are straightforward. We have, for instance,

\[ D(z_1)D(z_2)F = N^{11}(z_1, z_2), \]  
\[ D(z_1)\bar{D}(\bar{z}_2)F = N^{12}(z_1, \bar{z}_2). \]  

The identification for the generating functions involving zero modes is more complicated and can only be by trial and error, because of the gauge freedom (19) mentioned above. After some attempts we came to the following identification:

\[ \partial_0 D(z)F = N^{11}_0(z) - N^{21}_0(z), \]  
\[ \bar{\partial}_0 D(\bar{z})F = N^{22}_0(\bar{z}) - N^{12}_0(\bar{z}). \]  

We see that the combinations in the RHS are gauge invariant!

Now we can proceed to verify that our generating functions satisfy the Hirota equations. Substituting (29) in the LHS of (3) and (31) on the RHS, it is trivial to verify that the equations identically satisfied. Similarly, for eq. (4) we have to insert (29) in its LHS and (30–32) in its RHS. However, in this case, we have an indeterminate: \( F_{00} := \partial_0^2 F. \) After some elementary algebra one finds that eq. (4) is identically satisfied provided

\[ e^{F_{00}} = \frac{f_1(0)f'_2(0)}{(f_1(0) - f_2(0))^2} = \frac{16}{27}. \]  

We see that \( F_{00} \) is identical to the standard coefficient \(-V_{rr}^{rr}\).

The Neumann coefficients that one can extract by expanding the generating function (23) and (25) coincide, up to normalization, with the standard Neumann coefficients \( V_{nm}^{rs} \) (18) [15]. For instance, for \( r \neq s \), let us set \( N^{12}(1/x, 1/y) \equiv H(x, y) \). Then

\[ V_{nm}^{12} = -\frac{\sqrt{nm}}{nlm!} \partial_x^n \partial_y^m H(0, 0). \]  

Here are some examples

\[ V_{11}^{12} = -\frac{16}{27}, \quad V_{12}^{12} = -V_{21}^{12} = -\frac{32}{81} \sqrt{\frac{2}{3}}, \quad V_{34}^{12} = \frac{16}{\sqrt{3} 3^5}, \quad V_{25}^{12} = \sqrt{2 \frac{160}{3^6}}. \]

More generally, on the basis of the above equations, we have the following identifications among second derivatives of \( F \), standard Neumann coefficients and the Neumann coefficients \( N_{nm}^{rs} \):

\[ F_{nm}^{11} = \partial_0 \partial_n F = -\sqrt{nm} V_{nm}^{11} = nm N_{nm}^{11}, \]  
\[ F_{nm}^{12} = \partial_0 \partial_n F = -\sqrt{nm} V_{nm}^{12} = nm N_{nm}^{12}, \]  
\[ F_{0n}^{12} = \partial_0 \partial_n F = \sqrt{\frac{n}{2}} (V_{0n}^{12} - V_{0n}^{22}) = n \left( N_{0n}^{22} - N_{0n}^{12} \right), \]  
\[ F_{0n}^{11} = \partial_0 \partial_n F = \sqrt{\frac{n}{2}} (V_{0n}^{21} - V_{0n}^{11}) = n \left( N_{0n}^{11} - N_{0n}^{21} \right). \]
The Neumann coefficients involving zero modes and computed by expanding (27) and (28), do not coincide in general with the standard ones. This is no surprise since the latter were calculated with a definite gauge choice. As far as the $N_{rn}$ coefficients are concerned we see once again that the Hirota equations only constrain gauge invariant combinations of them.

The Hirota equations generate infinite many relations among the coefficients $N_{rns}$, $N_{rn0}$ and $F_{00}$, relations which one can derive by expanding (3–4) in powers of $z_1, z_2, \bar{z}_2$. For instance, from eq. (3) we get

$$F_{t_1 t_2} = \frac{1}{2} F_{t_0 t_2} - \frac{1}{2} (F_{t_0 t_1})^2,$$  \hspace{1cm} (39)

$$\frac{1}{2} F_{t_1 t_2} = \frac{1}{3} F_{t_0 t_3} + \frac{1}{2} F_{t_0 t_1} F_{t_0 t_2} - \frac{1}{6} (F_{t_0 t_1})^3,$$ \hspace{1cm} (40)

and, from (4),

$$e^{F_{00}} = F_{t_1 t_1},$$ \hspace{1cm} (42)

$$e^{F_{00}} F_{t_0 t_1} = \frac{1}{2} F_{t_1 t_2},$$ \hspace{1cm} (43)

and so on. All these equations are identically satisfied.

5 Discussion

From the above results there can be no remaining doubt that an integrable structure underlies Open String Field Theory. At the tree level this is the dTL hierarchy, which implies in particular that the Neumann coefficients are powerfully constrained. This result in itself may seem at first simply formal, since it was already known how to calculate all these coefficients (but above we have presented new effective formulas to compute them). However this fact opens the way to new possibilities, which seemed to be out of reach until now. For, on the one hand, the dispersionless TL hierarchy is a limiting case of the (dispersive) Toda Lattice Hierarchy (TLH) [17]. This hierarchy describes all genus corrections to the dTL results. On the other hand, we know that two–matrix models, [18], are precisely realizations of TLH, in the sense that any solution of two–matrix models satisfy the Hirota equations of TLH. We recall that the Hirota equations are the same for all two–matrix models, but any two–matrix model is characterized in addition by specific coupling conditions (or string equations). We do not actually know whether any solution of TLH can be incorporated in a definite two–matrix model. However this seems to be very likely, and, anyhow, we have indirect evidence that two–matrix models have to do with string theory. In the past specific two–matrix models were shown to describe topological field theories characterizing string vacua with central charge $c < 1$, [19], or $c = 1$, [20, 21]. It is therefore hard to refrain from conjecturing that a specific two–matrix model underlies open SFT. If true, this would be a powerful tool to extract exact higher genus results,
and perhaps also exact all genus results. In fact we recall that, while the Hirota equations generate relations among the Neumann coefficients (but do not completely determine them), a two–matrix model would allow us, at least in principle, to fully determine them (see, for instance, [21, 22] for some examples in an admittedly very simple case).

Needless to say all we said above rings a bell: two–matrix models are coming back via the Dijkgraaf–Vafa description of Yang–Mills theory, see [23]. Perhaps the parallelism between the two appearances of matrix models is not accidental.

Another line of research the results of this paper open up involves their supersymmetric extension, which is of course relevant for supersymmetric SFT. In this direction the relevant object is likely to be the dispersionless version of the supersymmetric KP hierarchy proposed in [23]. Quite recently, the $N = (1|1)$ supersymmetric dTL hierarchy was proposed in [24] which can also be relevant for this problem. However we do not have yet a full fledged version of the latter hierarchy, e.g. its bilinear form (if any), so the research in this field faces an additional obstacle and requires further investigations.

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