HIGHER REPRESENTATION THEORY AND QUANTUM AFFINE SCHUR-WEYL DUALITY

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Abstract. In this article, we explain the main philosophy of 2-representation theory and quantum affine Schur-Weyl duality. The Khovanov-Lauda-Rouquier algebras play a central role in both themes.

Introduction

The Khovanov-Lauda-Rouquier algebras, introduced by Khovanov-Lauda [28, 29] and Rouquier [33, 34], are a family of Z-graded algebras that provide a fundamental framework for 2-representation theory and quantum affine Schur-Weyl duality.

Let \( H_k(\zeta) \) be the finite Hecke algebra with \( \zeta \) a primitive \( n \)-th root of unity and let \( U_q(A_{n-1}^{(1)}) \) be the quantum affine algebra of type \( A_{n-1}^{(1)} \). In [30], Lascoux-Leclerc-Thibon discovered a recursive algorithm of computing Kashiwara’s lower global basis (=Lusztig’s canonical basis) ([26, 31]) and conjectured that the coefficient polynomials, when evaluated at \( q = 1 \), give the composition multiplicities of simple \( H_k(\zeta) \)-modules inside Specht modules.

In [2], Ariki came up with a proof of the Lascoux-Leclerc-Thibon conjecture using the idea of categorification. More precisely, let \( \Lambda \) be a dominant integral weight associated with the affine Cartan datum of type \( A_{n-1}^{(1)} \) and let \( H_k^\Lambda(\zeta) \) be the corresponding cyclotomic Hecke algebra. Let \( \text{proj}(H_k^\Lambda(\zeta)) \) denote the category of finitely generated projective \( H_k^\Lambda(\zeta) \)-modules and let \( K(\text{proj}(H_k^\Lambda(\zeta))) \) be the Grothendieck group of \( \text{proj}(H_k^\Lambda(\zeta)) \). Then Ariki proved

\[
\bigoplus_{k=0}^\infty K(\text{proj}(H_k^\Lambda(\zeta)))_C \cong V(\Lambda),
\]

where \( V(\Lambda) \) is the integrable highest weight module over \( A_{n-1}^{(1)} \). Moreover, he showed that the isomorphism classes of projective indecomposable modules correspond to the lower global basis of \( V(\Lambda) \) at \( q = 1 \), from which the Lascoux-Leclerc-Thibon conjecture follows.

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The idea of categorification, which was originated from \cite{7}, can be explained as follows. In the classical representation theory, we study the properties of an algebra \( A \) that are reflected on various vector spaces \( V \). That is, we investigate various algebra homomorphisms \( \phi : A \to \text{End}(V) \). We identify \( A \) with a category having a single object and its elements as morphisms. Similarly, we consider \( \text{End}(V) \) as a category with \( V \) as its object and linear operators on \( V \) as morphisms. Then the classical representation theory can be understood as the study of functors from a category to another, whence the 1-representation theory.

We now categorify the classical representation theory. Let \( A = \bigoplus_{\alpha \in Q} A_{\alpha} \) be a graded algebra and let \( V = \bigoplus_{\lambda \in P} V_{\lambda} \) be a graded \( A \)-module, where \( Q \) and \( P \) are appropriate abelian groups. We construct 2-categories \( \mathfrak{A} \) and \( \mathfrak{B} \) whose objects are certain categories \( \mathcal{A}_\alpha \) \((\alpha \in Q)\) and \( \mathcal{B}_\lambda \) \((\lambda \in P)\) such that

\[
\bigoplus_{\alpha \in Q} K(\mathcal{A}_\alpha) \cong A, \quad \bigoplus_{\lambda \in P} K(\mathcal{B}_\lambda) \cong V.
\]

We now investigate the properties of 2-functors \( R : \mathfrak{A} \to \mathfrak{B} \). That is, by categorifying the classical representation theory, we obtain the 2-representation theory, the study of 2-functors from a 2-category to another.

So far, one of the most interesting developments in 2-representation theory is the one via Khovanov-Lauda-Rouquier algebras. The Khovanov-Lauda-Rouquier algebras categorify the negative half of quantum groups associated with all symmetrizable Cartan datum \cite{28, 29, 33, 34}. Moreover, the cyclotomic Khovanov-Lauda-Rouquier algebras give a categorification of all integrable highest weight modules \cite{17}. Hence Khovanov-Lauda-Rouquier’s and Kang-Kashiwara’s categorification theorems provide a vast generalization of Ariki’s categorification theorem. (See also \cite{39}.)

When the Cartan datum is symmetric, as was conjectured by Khovanov-Lauda \cite{29}, Varagnolo-Vasserot proved that the isomorphism classes of simple modules (respectively, projective indecomposable modules) correspond to upper global basis (=dual canonical basis) (respectively, lower global basis) \cite{37}. However, when the Cartan datum is not symmetric, the above statements do not hold in general. It is a very interesting problem to characterize the perfect basis and dual perfect basis that correspond to simple modules and projective indecomposable modules, respectively.

On the other hand, the Khovanov-Lauda-Rouquier algebras can be viewed as a huge generalization of affine Hecke algebras in the context of Schur-Weyl duality. The Schur-Weyl duality, established by Schur and others (see, for example, \cite{35, 36}), reveals a deep connection between the representation theories of symmetric groups and general linear Lie algebras. Let \( V = \mathbb{C}^n \) be the vector representation of the general linear Lie algebra \( gl_n \) and consider the \( k \)-fold tensor product of \( V \). Then \( gl_n \) acts on \( V^\otimes k \) by comultiplication and the symmetric group \( \Sigma_k \) acts on


\(V^{\otimes k}\) (from the right) by place permutation. Clearly, these actions commute with each other. The Schur-Weyl duality states that there exists a surjective algebra homomorphism

\[ \phi_k : C\Sigma_k \longrightarrow \text{End}_{gl_n}(V^{\otimes k}), \]

where \(\text{End}_{gl_n}(V^{\otimes k})\) denotes the centralizer algebra of \(V^{\otimes k}\) under the \(gl_n\)-action. Moreover, \(\phi_k\) is an isomorphism whenever \(k \leq n\).

The Schur-Weyl duality can be rephrased as follows. There is a functor \(F\) from the category of finite dimensional \(\Sigma_k\)-modules to the category of finite dimensional polynomial representations of \(gl_n\) given by

\[ M \mapsto V^{\otimes k} \otimes_{C\Sigma_k} M, \]

where \(M\) is a finite dimensional \(\Sigma_k\)-module. The functor \(F\) is called the Schur-Weyl duality functor and it defines an equivalence of categories whenever \(k \leq n\).

In [15], Jimbo extended the Schur-Weyl duality to the quantum setting: \(\Sigma_k\) is replaced by the finite Hecke algebra \(H_k\) and \(gl_n\) is replaced by the quantum group \(U_q(gl_n)\). Then he obtained the quantum Schur-Weyl duality functor from the category of finite dimensional \(H_k\)-modules to the category of finite dimensional polynomial representations of \(U_q(gl_n)\), which also defines an equivalence of categories whenever \(k \leq n\).

In [4, 5, 10], Chari-Pressley, Cherednik and Ginzburg-Reshetikhin-Vasserot constructed a quantum affine Schur-Weyl duality functor which relates the category of finite dimensional representations of affine Hecke algebra \(H_k^{\text{aff}}\) and the category of finite dimensional integrable \(U'_q(A^{(1)}_{n-1})\)-modules. The main ingredients of their constructions are (i) the fundamental representation \(V(\varpi_1)\), (ii) the \(R\)-matrices on the tensor products of \(V(\varpi_1)\) satisfying the Yang-Baxter equation, (iii) the intertwiners in \(H_k^{\text{aff}}\) satisfying the braid relations.

Using Khovanov-Lauda-Rouquier algebras, one can construct quantum affine Schur-Weyl duality functors in much more generality. In [18], Kang, Kashiwara and Kim constructed such a functor which relates the category of finite dimensional modules over symmetric Khovanov-Lauda-Rouquier algebras and the category of finite dimensional integrable modules over all quantum affine algebras. Roughly speaking, the basic idea can be explained as follows. Using a family of good modules and \(R\)-matrices, we determine a quiver \(\Gamma\) and construct a symmetric Khovanov-Lauda-Rouquier algebra \(R^\Gamma(\beta)\) \((\beta \in Q_+)\). We then construct a \((U'_q(\mathfrak{g}), R^\Gamma(\beta))\)-bimodule \(\tilde{V}^{\otimes \beta}\), a completed tensor power arising from good modules, and define the quantum affine Schur-Weyl duality functor \(F\) by

\[ M \mapsto \tilde{V}^{\otimes \beta} \otimes_{R^\Gamma(\beta)} M, \]

where \(M\) is an \(R^\Gamma(\beta)\)-module.
Various choices of quantum affine algebras and good modules would give rise to various quantum affine Schur-Weyl duality functors. We believe that our general approach will generate a great deal of exciting developments in the forthcoming years.

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1. Quantum Groups

We begin with a brief recollection of representation theory of quantum groups.

Let $I$ be a finite index set. An integral matrix $A = (a_{ij})_{i,j \in I}$ is called a symmetrizable Cartan matrix if (i) $a_{ii} = 2$ for all $i \in I$, (ii) $a_{ij} \leq 0$ for $i \neq j$, (iii) $a_{ij} = 0$ if and only if $a_{ji} = 0$, (iv) there exists a diagonal matrix $D = \text{diag}(d_i \in \mathbb{Z}_{>0} \mid i \in I)$ such that $DA$ is symmetric.

A Cartan datum consists of:

1. A symmetrizable Cartan matrix $A = (a_{ij})_{i,j \in I}$,
2. A free abelian group $P$ of finite rank, the weight lattice,
3. $\Pi = \{\alpha_i \in P \mid i \in I\}$, the set of simple roots,
4. $P^\vee := \text{Hom}(P, \mathbb{Z})$, the dual weight lattice,
5. $\Pi^\vee = \{h_i \in P^\vee \mid i \in I\}$, the set of simple coroots

satisfying the following properties

(i) $\langle h_i, \alpha_j \rangle = a_{ij}$ for all $i, j \in I$,
(ii) $\Pi$ is linearly independent,
(iii) for each $i \in I$, there exists an element $\Lambda_j \in P$ such that

$$\langle h_i, \Lambda_j \rangle = \delta_{ij} \quad \text{for all } i, j \in I.$$ 

The $\Lambda_i$’s ($i \in I$) are called the fundamental weights.

We denote by

$$P^+ := \{\Lambda \in P \mid \langle h_i, \Lambda \rangle \geq 0 \text{ for all } i \in I\}$$

the set of dominant integral weights. The free abelian group $Q := \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ is called the root lattice. Set $Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$. For $\beta = \sum k_i \alpha_i \in Q_+$, we define its height to be $|\beta| := \sum k_i$.

Since $A$ is symmetrizable, there exists a symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}^* := Q \otimes_{\mathbb{Z}} P^\vee$ satisfying

$$(\alpha_i, \alpha_j) = a_{ij}, \quad \langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \quad \text{for all } \lambda \in \mathfrak{h}^*, \ i, j \in I.$$
Let $q$ be an indeterminate and set $q_i = q^{a_i} \ (i \in I)$. For $m, n \in \mathbb{Z}_{\geq 0}$, we define 

$$[n]_i := \frac{q^n_i - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! := \prod_{k=1}^n [k]_i.$$ 

We write $e_i^{(k)} := e_i^k/[k]_i!$, $f_i^{(k)} := f_i^k/[k]_i! \ (k \in \mathbb{Z}_{\geq 0}, \ i \in I)$ for the divided powers.

**Definition 1.1.** The quantum group $U_q(g)$ corresponding to a Cartan datum $(\Lambda, P, \Pi, P^\vee, \Pi^\vee)$ is the associative algebra over $\mathbb{Q}(q)$ generated by the elements $e_i, f_i \ (i \in I), \ q^h \ (h \in P^\vee)$ with defining relations

$$q^0 = 1, \quad q^h q^{h'} = q^{h+h'} \ (h, h' \in P^\vee),$$

$$q^h e_i q^{-h} = q^{(h,a_i)} e_i, \quad q^h f_i q^{-h} = q^{-(h,a_i)} f_i \ (h \in P^\vee, i \in I),$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \ (K_i = q^{d_i h_i}, i \in I),$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(1-a_{ij}-k)} e_j e_i^{(k)} = 0 \ (i \neq j),$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(1-a_{ij}-k)} f_j f_i^{(k)} = 0 \ (i \neq j).$$

Let $U^0_q(g)$ be the subalgebra of $U_q(g)$ generated by $q^h \ (h \in P^\vee)$ and let $U^+_{q}(g)$ (respectively, $U^-_{q}(g)$) be the subalgebra of $U_q(g)$ generated by $e_i$ (respectively, $f_i$) for all $i \in I$. Then the algebra $U_q(g)$ has the triangular decomposition

$$U_q(g) \cong U^-_{q}(g) \otimes U^0_{q}(g) \otimes U^+_{q}(g).$$

Let $A = \mathbb{Z[q, q^{-1}]}$. We define the integral form $U_A(g)$ of $U_q(g)$ to be the $A$-subalgebra of $U_q(g)$ generated by $e_i^{(k)}, f_i^{(k)}, q^h \ (i \in I, h \in P^\vee, k \in \mathbb{Z}_{\geq 0})$. Let $U^0_A(g)$ be the $A$-subalgebra of $U_q(g)$ generated by $q^h \ (h \in P^\vee)$ and let $U^+_A(g)$ (respectively, $U^-_A(g)$) be the $A$-subalgebra of $U_q(g)$ generated by $e_i^{(k)}$ (respectively, $f_i^{(k)}$) $(i \in I, k \in \mathbb{Z}_{\geq 0})$. Then we have

$$U_A(g) \cong U^-_A(g) \otimes U^0_A(g) \otimes U^+_A(g).$$

A $U_q(g)$-module $V$ is called a highest weight module with highest weight $\Lambda \in P$ if there exists a nonzero vector $v_\Lambda$ in $V$, called the highest weight vector, such that

(i) $e_i v_\Lambda = 0$ for all $i \in I$,  
(ii) $q^h v_\Lambda = q^{(h, \Lambda)} v_\Lambda$ for all $h \in P^\vee$,  
(iii) $V = U_q(g) v_\Lambda$.  

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For each $\Lambda \in P$, there exists a unique irreducible highest weight module $V(\Lambda)$ with highest weight $\Lambda$. The integral form of $V(\Lambda)$ is defined to be

$$V^*_A(\Lambda) := U^*_A(g) v_\Lambda,$$

where $v_\Lambda$ is the highest weight vector.

Consider the anti-involution $\phi : U_q(g) \to U_q(g)$ defined by

$$q^h \mapsto q^{-h}, \quad e_i \mapsto f_i, \quad f_i \mapsto e_i \quad (h \in P^\vee, \ i \in I).$$

Then there exists a unique non-degenerate symmetric bilinear form $(\ ,\ )$ on $V(\Lambda)$ satisfying

$$(v_\Lambda, v_\Lambda) = 1, \quad (x u, v) = (u, \phi(x) v) \quad \text{for all } x \in U_q(g), \ u, v \in V(\Lambda).$$

The dual of $V^*_A(\Lambda)$ is defined to be

$$V^*_A(\Lambda)^\vee := \{ v \in V(\Lambda) \mid (u, v) \in A \quad \text{for all } u \in V^*_A(\Lambda) \}.$$

Note that $V^*_A(\Lambda)^\vee = \text{Hom}_A(V^*_A(\Lambda_\lambda, A)$ for all $\lambda \in P$.

The category $\mathcal{O}_\text{int}$ consists of $U_q(g)$-modules $M$ such that

(i) $M = \bigoplus_{\mu \in P} M_\mu$, where $M_\mu := \{ m \in M \mid q^h m = q^{(h,\mu)} m \text{ for all } h \in P^\vee \}$,

(ii) $e_i, f_i (i \in I)$ are locally nilpotent on $M$,

(iii) there exist finitely many elements $\lambda_1, \ldots, \lambda_s \in P$ such that

$$\text{wt}(M) := \{ \mu \in P \mid M_\mu \neq 0 \} \subset \bigcup_{j=1}^s (\lambda_j - Q_+).$$

The following properties of the category $\mathcal{O}_\text{int}$ are well-known. (See, for example, [11, 16, 32].)

**Proposition 1.2.**

(a) The category $\mathcal{O}_\text{int}$ is semisimple.

(b) The $U_q(g)$-module $V(\Lambda)$ with $\Lambda \in P^+$ belongs to $\mathcal{O}_\text{int}$.

(c) Every simple object in $\mathcal{O}_\text{int}$ has the form $V(\Lambda)$ for some $\Lambda \in P^+$.

2. **Khovanov-Lauda-Rouquier algebras**

Let $k$ be a field and let $(A, P, \Pi, P^\vee, \Pi^\vee)$ be a Cartan datum. For each $i \neq j$, set

$$S_{ij} := \{ (p, q) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \mid (\alpha_i, \alpha_i)p + (\alpha_j, \alpha_j)q = -2(\alpha_i, \alpha_j) \}.$$

Define a family of polynomials $Q = (Q_{ij})_{i,j \in I}$ in $k[u, v]$ by

$$(2.1) \quad Q_{ij}(u, v) := \begin{cases} 
0 & \text{if } i = j, \\
\sum_{(p,q) \in S_{ij}} t_{i,j;p,q} u^p v^q & \text{if } i \neq j
\end{cases}$$
for some $t_{i,j;p,q} \in k$ such that $t_{i,j;p,q} = t_{j,i;q,p}$ and $t_{i,j; a_{i,j},0} \in k^\times$. In particular,

$$Q_{ii}(u,v) = 0, \quad Q_{ij}(u,v) = Q_{ji}(v,u) \quad (i \neq j).$$

The symmetric group $\mathfrak{S}_n = \langle s_1, s_2, \ldots, s_{n-1} \rangle$ acts on $I^n$ by place permutation, where $s_i$ denotes the transposition $(i, i+1)$.

**Definition 2.1.** The Khovanov-Lauda-Rouquier algebra $R(n)$ of degree $n \geq 0$ associated with $(A,Q)$ is the associative algebra over $k$ generated by the elements $e(\nu)$ ($\nu \in I^n$), $x_k$ ($1 \leq k \leq n$), $\tau_l$ ($1 \leq l \leq n - 1$) with defining relations

$$e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \quad \sum_{\nu \in I^n} e(\nu) = 1,$$

$$x_k x_l = x_l x_k, \quad x_k e(\nu) = e(\nu)x_k,$$

$$\tau_l e(\nu) = e(s_l(\nu))\tau_l, \quad \tau_l \tau_l = \tau_l \tau_l \text{ if } |k - l| > 1,$$

$$\tau_k^2 e(\nu) = Q_{\nu_k, \nu_{k+1}}(x_k, x_{k+1})e(\nu),$$

$$(\tau_k x_l - x_{s(l)}\tau_k)e(\nu) = \begin{cases} -e(\nu) & \text{if } l = k, \nu_k = \nu_{k+1}, \\ e(\nu) & \text{if } l = k + 1, \nu_k = \nu_{k+1}, \\ 0 & \text{otherwise}, \end{cases}$$

$$(\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k)e(\nu) = \begin{cases} \frac{Q_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) - Q_{\nu_k, \nu_{k+1}}(x_{k+2}, x_{k+1})}{x_k - x_{k+2}}e(\nu) & \text{if } \nu_k = \nu_{k+2}, \\ 0 & \text{otherwise}. \end{cases}$$

The algebra $R(n)$ has a $\mathbb{Z}$-grading by assigning the degrees as follows:

$$\deg e(\nu) = 0, \quad \deg x_k e(\nu) = (\alpha_{\nu_k}, \alpha_{\nu_k}), \quad \deg \tau_l e(\nu) = -(\alpha_{\nu_l}, \alpha_{\nu_{l+1}}).$$

We denote by $q$ the degree-shift functor defined by

$$(qM)_k = M_{k-1},$$

where $M = \bigoplus_{k \in \mathbb{Z}} M_k$ is a graded $R(n)$-module. Also there is an algebra involution $\psi : R(n) \rightarrow R(n)$ given by

$$e(\nu) \mapsto e(\nu'), \quad x_k \mapsto x_{n-k+1},$$

$$\tau_{l} e(\nu) \mapsto \begin{cases} -\tau_{n-l} e(\nu') & \text{if } \nu_l = \nu_{l+1}, \\ \tau_{n-l} e(\nu') & \text{if } \nu_l \neq \nu_{l+1}, \end{cases}$$

where $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ and $\nu' = (\nu_n, \ldots, \nu_2, \nu_1)$.

By the embedding $R(m) \otimes R(n) \hookrightarrow R(m + n)$, we may consider $R(m) \otimes R(n)$ as a subalgebra of $R(m + n)$. For an $R(m)$-module $M$ and an $R(n)$-module $N$, we
define their \textit{convolution product} $M \circ N$ by
\begin{equation}
M \circ N := R(m + n) \otimes_{R(m) \otimes R(n)} (M \otimes N).
\end{equation}

Since $R(m + n)$ is free over $R(m) \otimes R(n)$ (\cite[Proposition 2.16]{28}), the bifunctor $(M, N) \mapsto M \circ N$ is exact in $M$ and $N$.

For $n \geq 0$ and $\beta \in \mathbb{Q}_+$ with $|\beta| = n$, set
\begin{align*}
I^\beta := \{ \nu = (\nu_1, \ldots, \nu_n) \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \beta \}, \\
e(\beta) := \sum_{\nu \in I^\beta} e(\nu).
\end{align*}

Then $e(\beta)$ is a central idempotent in $R(n)$. We also define
\begin{align*}
e(\beta, \alpha_i) &:= \sum_{\nu \in I^{\beta+\alpha_i}} e(\nu), \\
e(\alpha_i, \beta) &:= \sum_{\nu \in I^{\beta+\alpha_i}} e(\nu).
\end{align*}

The algebra
\begin{equation*}
R(\beta) := R(n)e(\beta)
\end{equation*}
is called the \textit{Khovanov-Lauda-Rouquier algebra at} $\beta$.

For a $k$-algebra $R$, we denote by $\text{mod}(R)$ (respectively, $\text{proj}(R)$ and $\text{rep}(R)$) the category of $R$-modules (respectively, the category of finitely generated projective $R$-modules and the category of finite dimensional $R$-modules).

If $R$ is a graded $k$-algebra, we will use $\text{Mod}(R)$ (respectively, $\text{Proj}(R)$ and $\text{Rep}(R)$) for the category of graded $R$-modules (respectively, the category of finitely generated projective graded $R$-modules and the category of finite dimensional graded $R$-modules).

For each $i \in I$, define the functors
\begin{align*}
E_i &: \text{Mod}(R(\beta + \alpha_i)) \rightarrow \text{Mod}(R(\beta)), \\
F_i &: \text{Mod}(R(\beta)) \rightarrow \text{Mod}(R(\beta + \alpha_i))
\end{align*}
by
\begin{align*}
E_i(N) &= e(\beta, \alpha_i) R(\beta + \alpha_i) \otimes_{R(\beta + \alpha_i)} N, \\
F_i(M) &= R(\beta + \alpha_i) e(\beta, \alpha_i) \otimes_{R(\beta)} M
\end{align*}
for $M \in \text{Mod}(R(\beta))$, $N \in \text{Mod}(R(\beta + \alpha_i))$.

By \cite[Proposition 2.16]{28}, the functors $E_i$ and $F_i$ are exact and send finitely generated projective modules to finitely generated projective modules. Hence (2.5) restricts to the functors
\begin{align*}
E_i &: \text{Proj}(R(\beta + \alpha_i)) \rightarrow \text{Proj}(R(\beta)), \\
F_i &: \text{Proj}(R(\beta)) \rightarrow \text{Proj}(R(\beta + \alpha_i)).
\end{align*}
For $1 \leq k < n$, set $b_k := \tau_k x_{k+1}$ and $b'_k := x_{k+1} \tau_k$. Let $w_0 = s_{i_1} \cdots s_{i_r}$ be the longest element in $S_n$ and set

$$b(n) := b_{i_1} \cdots b_{i_r}, \quad b'(n) := b'_{i_1} \cdots b'_{i_1}.$$ 

For each $n \geq 0$, we define the divided powers by

$$E^{(n)}_i := b'(n)E^n_i, \quad F^{(n)}_i := F^n_i b(n).$$

In [28] and [33], Khovanov-Lauda and Rouquier proved the following categorification theorem.

**Theorem 2.2.** [28, 33]

There exists an $A$-algebra isomorphism

$$U^{-}_A(g) \xrightarrow{\sim} K(\text{Proj}(R)) \quad \text{given by} \quad f_i^{(n)} \longmapsto [F_i^{(n)}] \quad (i \in I, \ n \geq 0),$$

where $K(\text{Proj}(R)) := \bigoplus_{\beta \in Q_+} K(\text{Proj}(R(\beta))).$

Thus we have constructed a 2-category $\mathfrak{R}$ such that the objects are the categories $\text{Proj}(R(\beta))$ ($\beta \in Q_+$) and the categories $\mathcal{H}om(\text{Proj} R(\alpha), \text{Proj} R(\beta))$ consist of the monomials $F_{i_1} \cdots F_{i_r}$ ($i_k \in I, \ r \geq 0$) of functors satisfying

$$\alpha_{i_1} + \cdots + \alpha_{i_r} = \begin{cases} \alpha - \beta & \text{if} \ \alpha \geq \beta, \\ \beta - \alpha & \text{if} \ \beta \geq \alpha. \end{cases}$$

The morphisms in $\mathcal{H}om(\text{Proj} R(\alpha), \text{Proj} R(\beta))$ are the natural transformations generated by $x_i : F_i \to F_i, \ \tau_{ij} : F_i F_j \to F_j F_i$ ($i, j \in I$) satisfying the relations

$$\tau_{ij} \circ \tau_{ji} = Q_{ij}(F_j x_i, x_j F_i),$$

$$\tau_{jk} F_i \circ F_j \tau_{ik} \circ \tau_{ij} F_k - F_k \tau_{ij} \circ \tau_{ik} F_j \circ F_i \tau_{jk}$$

$$= \begin{cases} Q_{ij}(x_i F_j x_i, x_j F_i) x_i F_j F_i - F_i F_j x_i & \text{if} \ i = k, \\ 0 & \text{otherwise}, \end{cases}$$

$$\tau_{ij} \circ x_i F_j - F_j x_i \circ \tau_{ij} = \delta_{ij},$$

$$\tau_{ij} \circ F_i x_j - x_j F_i \circ \tau_{ij} = -\delta_{ij}.$$ 

It is straightforward to verify that $\mathfrak{R}$ satisfies all the axioms for 2-categories [33, 34].

For the later use, we define a functor $\overline{T}_i : \text{Mod} (R(\beta)) \to \text{Mod} (R(\beta + \alpha_i))$ by

$$\overline{T}_i(M) := R(\beta + \alpha_i) e(\alpha_i, \beta) \otimes_{R(\beta)} M \quad \text{for} \ i \in I, \ M \in \text{Mod} (R(\beta)).$$

The properties of the functors $E_i$, $F_i$ and $\overline{T}_i$ ($i \in I$) are given in the following proposition.
Proposition 2.3. [17]
(a) We have an exact sequence in $\text{Mod } (R(\beta))$
\[ 0 \rightarrow \overline{F_i} E_i M \rightarrow E_i \overline{F_i} M \rightarrow q^{-(\alpha_i, \alpha_i)} M \otimes k[t_i] \rightarrow 0 \]
which is functorial in $M \in \text{Mod } (R(\beta))$.
(b) There exist natural isomorphisms
\[ E_i F_j \xrightarrow{\sim} F_j E_i, \quad E_i \overline{F_j} \xrightarrow{\sim} \overline{F_j} E_i \quad \text{if } i \neq j, \]
\[ E_i F_i \xrightarrow{\sim} q^{-(\alpha_i, \alpha_i)} F_i E_i \oplus 1 \otimes k[t_i] \quad \text{if } i = j, \]
where $t_i$ is an indeterminate of degree $(\alpha_i, \alpha_i)$ and
\[ 1 \otimes k[t_i] : \text{Mod } (R(\beta)) \rightarrow \text{Mod } (R(\beta)) \]
is the degree-shift functor sending $M$ to $M \otimes k[t_i]$ for $M \in \text{Mod } (R(\beta))$ ($\beta \in Q_+)$.

3. Cyclotomic categorification theorem

Let $\Lambda \in P^+$ and let
\[ a^\Lambda(x_1) := \sum_{\nu \in \mathfrak{t}^\beta} x_1^{(h_v+\Lambda)} e(\nu) \in R(\beta). \]
Then the cyclotomic Khovanov-Lauda-Rouquier algebra $R^\Lambda(\beta)$ ($\beta \in Q_+$) is defined to be the quotient algebra
\[ R^\Lambda(\beta) := R(\beta) / R(\beta) a^\Lambda(x_1) R(\beta). \]
We would like to show that the cyclotomic Khovanov-Lauda-Rouquier algebras provide a categorification of irreducible highest weight $U_q(\mathfrak{g})$-modules in the category $\mathcal{O}_{\text{int}}$.

For each $i \in I$, define the functors
\[ E_i^\Lambda : \text{Mod}(R^\Lambda(\beta + \alpha_i)) \rightarrow \text{Mod}(R^\Lambda(\beta)), \]
\[ F_i^\Lambda : \text{Mod}(R^\Lambda(\beta)) \rightarrow \text{Mod}(R^\Lambda(\beta + \alpha_i)) \]
by
\[ E_i^\Lambda(N) = e(\beta, \alpha_i) R^\Lambda(\beta + \alpha_i) \otimes_{R^\Lambda(\beta + \alpha_i)} N, \]
\[ F_i^\Lambda(M) = R^\Lambda(\beta + \alpha_i) e(\beta, \alpha_i) \otimes_{R^\Lambda(\beta)} M \]
for $M \in \text{Mod}(R^\Lambda(\beta))$, $N \in \text{Mod}(R^\Lambda(\beta + \alpha_i))$. However, since $R^\Lambda(\beta + \alpha_i)$ is not free over $R^\Lambda(\beta)$, there is no guarantee that $E_i^\Lambda$ and $F_i^\Lambda$ send finitely generated projective modules to finitely generated projective modules. To prove this, we need to show that $R^\Lambda(\beta + \alpha_i) e(\beta, \alpha_i)$ is a projective right $R^\Lambda(\beta)$-module.
Let 

$$F^\Lambda := R^\Lambda(\beta + \alpha_i) e(\beta, \alpha_i) = \frac{R(\beta + \alpha_i) e(\beta, \alpha_i)}{R(\beta + \alpha_i) a^\Lambda(x_1) R(\beta + \alpha_i) e(\beta, \alpha_i)},$$

$$K_0 := R(\beta + \alpha_i) e(\beta, \alpha_i) \otimes_{R^\beta} R^\Lambda(\beta) = \frac{R(\beta + \alpha_i) e(\beta, \alpha_i)}{R(\beta + \alpha_i) a^\Lambda(x_1) R(\beta) e(\beta, \alpha_i)},$$

$$K_1 := R(\beta + \alpha_i) e(\alpha_i, \beta) \otimes_{R^\beta} R^\Lambda(\beta) = \frac{R(\beta + \alpha_i) e(\alpha_i, \beta)}{R(\beta + \alpha_i) a^\Lambda(x_2) R^\Lambda(\beta) e(\alpha_i, \beta)},$$

where $R^1(\beta)$ is the subalgebra of $R(\beta + \alpha_i)$ generated by $e(\alpha_i, \nu) \ (\nu \in I^\beta)$, $x_k \ (2 \leq k \leq n + 1)$, $\tau_i \ (2 \leq i \leq n)$. Then $F^\Lambda$, $K_0$ and $K_1$ can be regarded as $(R(\beta + \alpha_i), R^\Lambda(\beta))$-bimodules.

Let $t_i$ be an indeterminate of degree $(\alpha_i, \alpha_i)$. Then $k[t_i]$ acts on $R(\beta + \alpha_i) e(\alpha_i, \beta)$ and $K_1$ from the right by $t_i = x_i e(\alpha_i, \beta)$. On the other hand, $k[t_i]$ acts on $K_0$ and $F^\Lambda$ from the right by $t_i = x_{n+1} e(\beta, \alpha_i)$. Hence all of them have a structure of $(R(\beta + \alpha_i), R(\beta) \otimes k[t_i])$-bimodules. Moreover, $F^\Lambda$, $K_0$ and $K_1$ are in fact $(R(\beta + \alpha_i), R^\Lambda(\beta) \otimes k[t_i])$-bimodules.

In [28], it was shown that $K_0$ and $K_1$ are finitely generated projective right $(R^\Lambda(\beta) \otimes k[t_i])$-modules. Let $\pi : K_0 \to F^\Lambda$ be the canonical projection and let $P : K_1 \to K_0$ be the right multiplication by $a^\Lambda(x_1) \tau_1 \cdots \tau_n$.

The following theorem is one of the main results in [17].

**Theorem 3.1.** [17]

The sequence

$$0 \to K_1 \xrightarrow{P} K_0 \xrightarrow{\pi} F^\Lambda \to 0$$

is exact as $(R(\beta + \alpha_i), R^\Lambda(\beta) \otimes k[t_i])$-bimodules.

Hence we get a projective resolution of $F^\Lambda$ of length 1 as a right $R^\Lambda(\beta)[t_i]$-module. By the following lemma, we conclude that $F^\Lambda$ is a finitely generated projective right $R^\Lambda(\beta)$-module.

**Lemma 3.2.** [17] Let $R$ be a ring and let $f(t)$ be a monic polynomial in $R[t]$ with coefficients in the center of $R$.

If an $R[t]$-module $M$ is annihilated by $f(t)$ and has projective dimension $\leq 1$, then $M$ is projective as an $R$-module.

Thus we obtain the following important theorem.
Theorem 3.3. [17]
(a) $R^{A}(\beta + \alpha_i)e(\beta, \alpha_i)$ is a projective right $R^{A}(\beta)$-module.
(b) $e(\beta, \alpha_i)R^{A}(\beta + \alpha_i)$ is a projective left $R^{A}$-module.
(c) The functors $E_{i}^{A}$ and $F_{i}^{A}$ are exact.
(d) The functors $E_{i}^{A}$ and $F_{i}^{A}$ send finitely generated projective modules to finitely
generated projective modules.

Corollary 3.4. [17]
For all $i \in I$ and $\beta \in Q_{+}$, we have an exact sequence of $R(\beta + \alpha_i)$-modules
\[ 0 \rightarrow q^{(\alpha_i, 2\Lambda - \beta)}F_{i}^\Lambda M \rightarrow F_{i} M \rightarrow F_{i}^\Lambda M \rightarrow 0 \]
which is functorial in $M \in \text{Mod } R^{A}(\beta)$.

To complete the construction of cyclotomic categorification, it remains to show
that the adjoint pair $(F_{i}^{A}, E_{i}^{A})$ gives an $sl_2$-categorification introduced by Chuang-
Rouquier [6].

Theorem 3.5. [17]
(a) For $i \neq j$, there exists a natural isomorphism
\[ q^{-(\alpha_i, \alpha_j)}F_{j}^{A}E_{i}^{A} \sim E_{i}^{A}F_{j}^{A}. \]
(b) Let $\lambda = \Lambda - \beta$ $(\beta \in Q_{+})$.
   (i) If $\langle h_i, \Lambda \rangle \geq 0$, there exists a natural isomorphism
   \[ q_{i}^{-2}F_{i}^{A}E_{i}^{A} \oplus \bigoplus_{k=0}^{\langle h_i, \Lambda \rangle - 1} q_{i}^{2k}1 \sim E_{i}^{A}F_{i}^{A}. \]
   (ii) If $\langle h_i, \Lambda \rangle \leq 0$, there exists a natural isomorphism
   \[ q_{i}^{-2}F_{i}^{A}E_{i}^{A} \sim E_{i}^{A}F_{i}^{A} \oplus \bigoplus_{k=0}^{-\langle h_i, \Lambda \rangle - 1} q_{i}^{2k-2}1. \]

Proof. We will give a very rough sketch of the proof. The assertion (a) can be
proved in a straightforward manner.
To prove (b), note that Theorem 3.1 and Corollary 3.4 yield the following commutative diagram.

\[
\begin{array}{ccccccc}
0 & \rightarrow & q^{(\alpha_i,\Lambda-\beta)} F_i E_i M & \rightarrow & q_i^{-2} F_i E_i M & \rightarrow & q_i^{-2} F_i E_i^\Lambda M & \rightarrow & 0 \\
0 & \rightarrow & q^{(\alpha_i,\Lambda-\beta)} E_i F_i M & \rightarrow & E_i F_i M & \rightarrow & E_i^\Lambda F_i^\Lambda M & \rightarrow & 0 \\
q^{(\alpha_i,\Lambda-2\beta)} k[t_i] \otimes M & \rightarrow & k[t_i] \otimes M & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]

Let \( A : q^{2(\alpha_i,\Lambda-\beta)} k[t_i] \otimes R^\Lambda(\beta) \rightarrow k[t_i] \otimes R^\Lambda(\beta) \) be the \( R^\Lambda(\beta) \)-bilinear map given by chasing the diagram. By a detailed analysis of the above commutative diagram at the kernel level, the Snake Lemma gives the following exact sequence of \( R^\Lambda(\beta) \)-bimodules

\[
0 \rightarrow \text{Ker} A \rightarrow q_i^{-2} F_i E_i^\Lambda R^\Lambda(\beta) \rightarrow E_i^\Lambda F_i^\Lambda R^\Lambda(\beta) \rightarrow \text{Coker} A \rightarrow 0.
\]

If \( \langle h_i, \lambda \rangle \geq 0 \), we have \( \text{Ker} A = 0 \), \( \quad \bigoplus_{k=0}^{a-1} k t_i^k \otimes R^\Lambda(\beta) \rightarrow \text{Coker} A \), and if \( \langle h_i, \lambda \rangle \leq 0 \), then \( \text{Coker} A = 0 \), \( \text{Ker}(A) = q^{2(\alpha_i,\Lambda-\beta)} \bigoplus_{k=0}^{a-1} k t_i^k \otimes R^\Lambda(\beta) \), from which our assertion (b) follows.

Set

\[
K(\text{Proj}(R^\Lambda)) := \bigoplus_{\beta \in Q_+} K(\text{Proj} R^\Lambda(\beta)),
\]

\[
K(\text{Rep}(R^\Lambda)) := \bigoplus_{\beta \in Q_+} K(\text{Rep} R^\Lambda(\beta)).
\]

We define the endomorphisms \( E_i \) and \( F_i \) on \( K(\text{Proj}(R^\Lambda)) \) by

\[
E_i = [q_i^{1-(h_i,\Lambda-\beta)} E_i^\Lambda] : K(\text{Proj} R^\Lambda(\beta + \alpha_i)) \rightarrow K(\text{Proj} R^\Lambda(\beta)),
\]

\[
F_i = [F_i^\Lambda] : K(\text{Proj} R^\Lambda(\beta)) \rightarrow K(\text{Proj} R^\Lambda(\beta + \alpha_i)).
\]

On the other hand, we define \( E_i \) and \( F_i \) on \( K(\text{Rep}(R^\Lambda)) \) by

\[
E_i = [E_i^\Lambda] : K(\text{Rep} R^\Lambda(\beta + \alpha_i)) \rightarrow K(\text{Rep} R^\Lambda(\beta)),
\]

\[
F_i = [q_i^{1-(h_i,\Lambda-\beta)} F_i^\Lambda] : K(\text{Rep} R^\Lambda(\beta)) \rightarrow K(\text{Rep} R^\Lambda(\beta + \alpha_i)).
\]
Let $K_i$ be the endomorphism on $K(\text{Proj} \ R^\lambda(\beta))$ and $K(\text{Rep} \ R^\lambda(\beta))$ given by the multiplication by $q_i^{(h_i,\Lambda^{-\beta})}$ for each $\beta \in Q_+$. Then we have

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \text{ for } i, j \in I.$$ 

Therefore, we obtain the cyclo
tomic categrification theorem for irreducible highest weight $U_q(g)$-modules in the category $O_{\text{int}}$.

**Theorem 3.6. [17]**

For each $\Lambda \in P^+$, there exist $U_A(g)$-module isomorphisms

$$K(\text{Proj} \ R^\lambda) \xrightarrow{\sim} V_A(\Lambda) \text{ and } K(\text{Rep} \ R^\lambda) \xrightarrow{\sim} V_A(\Lambda)^\vee.$$ 

Therefore, for each $\Lambda \in P^+$, we have constructed a 2-category $\mathcal{R}^\lambda$ consisting of $\text{Proj} (R^\lambda(\beta))$ ($\beta \in Q_+$), which gives an integrable 2-representation $\mathcal{R}^\lambda$ of $\mathcal{R}$ in the sense of [33, 34]. (See also [39].)

**Remark 3.7.** There are several generalizations of Khovanov-Lauda-Rouquier algebras and categorification theorems. In [20, 23, 25], the Khovanov-Lauda-Rouquier algebras associated with Borcherds-Cartan data have been defined and their properties have been investigated including geometric realization, categorification and the connection with crystal bases. In [8, 9, 14, 21, 22, 24, 38], various versions of Khovanov-Lauda-Rouquier super-algebras have been introduced and the corresponding super-categorifications have been constructed.

## 4. Quantum affine algebras and $R$-matrices

In this section, we briefly review the finite dimensional representation theory of quantum affine algebras and the properties of $R$-matrices (see, for example, [1, 3, 4, 27]).

Let $(A, P, \Pi, P^\vee, \Pi^\vee)$ be a Cartan datum of affine type with $I = \{0, 1, \ldots, n\}$ the index set of simple roots. Let $0 \in I$ be the leftmost vertex in the affine Dynkin diagrams given in [16, Chapter 4]. Set $I_0 = I \setminus \{0\}$. Take relatively prime positive integers $c_j$'s and $d_j$'s ($j \in I$) such that

$$\sum_{j \in I} c_j a_{ji} = 0, \quad \sum_{j \in I} a_{ij} d_j = 0 \quad \text{for all } i \in I.$$ 

Then the weight lattice can be written as

$$P = \bigoplus_{i \in I} \mathbb{Z} \Lambda_i \oplus \mathbb{Z} \delta,$$

where $\delta := \sum_{i \in I} d_i \alpha_i \in P$. We also define $c := \sum_{i \in I} c_i h_i \in P^\vee$. 

We denote by \( \mathfrak{g} \) the affine Kac-Moody algebra associated with \((A, P, P^\vee, \Pi, \Pi^\vee)\) and let \( \mathfrak{g}_0 \) be the finite dimensional simple Lie algebra inside \( \mathfrak{g} \) generated by \( e_i, f_i, h_i \) \((i \in I_0)\). We will write \( W \) and \( W_0 \) for the Weyl group of \( \mathfrak{g} \) and \( \mathfrak{g}_0 \), respectively.

Let \( U_q(\mathfrak{g}) \) be the corresponding quantum group and let \( U'_q(\mathfrak{g}) \) be the subalgebra of \( U_q(\mathfrak{g}) \) generated by \( e_i, f_i, K_i^{\pm 1} \) \((i \in I)\). The algebra \( U'_q(\mathfrak{g}) \) will be called the quantum affine algebra.

Set \( P_{cl} := P/\mathbb{Z}\delta \) and let \( \text{cl} : P \to P_{cl} \) be the canonical projection. Then we have
\[
P_{cl} = \bigoplus_{i \in I} \mathbb{Z}\text{cl}(\Lambda_i) \quad \text{and} \quad P_{cl}^\vee := \text{Hom}_{\mathbb{Z}}(P_{cl}, \mathbb{Z}) = \bigoplus_{i \in I} \mathbb{Z}h_i.
\]

A \( U'_q(\mathfrak{g}) \)-module \( V \) is integrable if
\[
(i) \ V = \bigoplus_{\lambda \in P_{cl}} V_\lambda, \quad \text{where} \ V_\lambda = \{ v \in V \mid K_i v = q_i^{(h_i, \lambda)} v \ \text{for all} \ i \in I \},
\]
\[
(ii) \ e_i, f_i \ (i \in I) \text{ are locally nilpotent on } V.
\]

We denote by \( \mathcal{C}_{\text{int}} \) the category of finite dimensional integrable \( U'_q(\mathfrak{g}) \)-modules.

Let \( M \) be an integrable \( U'_q(\mathfrak{g}) \)-module. A weight vector \( v \in M_\lambda \ (\lambda \in P_{cl}) \) is called an extremal weight vector if there exists a family of nonzero vectors \( \{v_{w, \lambda} \mid w \in W\} \) such that
\[
v_{i, \lambda} = \begin{cases} f_i^{(h_i, \lambda)} v_\lambda & \text{if } \langle h_i, \lambda \rangle \geq 0, \\ e_i^{-(h_i, \lambda)} v_\lambda & \text{if } \langle h_i, \lambda \rangle \leq 0. \end{cases}
\]

Let \( P_{cl}^0 := \{ \lambda \in P_{cl} \mid \langle c, \lambda \rangle = 0 \} \) and set
\[
\varpi_i := \Lambda_i - c_i \Lambda_0 \quad \text{for} \ i \in I_0.
\]

Then there exists a unique finite dimensional integrable \( U'_q(\mathfrak{g}) \)-module \( V(\varpi_i) \) satisfying the following properties:
\[
(i) \ \text{all the weights of } V(\varpi_i) \text{ are contained in the convex hull of } W_0 \text{cl}(\varpi_i).
\]
\[
(ii) \dim V(\varpi_i)_{\text{cl}(\varpi_i)} = 1,
\]
\[
(iii) \text{for each } \mu \in W_0 \text{cl}(\varpi_i), \text{there exists an extremal weight vector of weight } \mu,
\]
\[
(iv) \ V(\varpi_i) \text{ is generated by } V(\varpi_i)_{\text{cl}(\varpi_i)} \text{ as a } U'_q(\mathfrak{g})\text{-module}.
\]

The \( U'_q(\mathfrak{g}) \)-module \( V(\varpi_i) \) is called the fundamental representation of weight \( \varpi_i \) \((i \in I_0)\).

Let \( M \) be a \( U'_q(\mathfrak{g}) \)-module. An involution on \( M \) is called a bar involution if \( \overline{a} v = \overline{a} \overline{v} \) for all \( a \in U'_q(\mathfrak{g}), v \in M \), where \( \overline{e_i} = e_i, \overline{f_i} = f_i, \overline{K_i} = K_i^{-1} \) \((i \in I)\). A finite \( U'_q(\mathfrak{g}) \)-crystal \( B \) is simple if \( (i) \ wt(B) \subset P_{cl}^0, \ (ii) \text{there exists } \lambda \in wt(B) \text{ such that } \#(B_\lambda) = 1, \ (iii) \text{the weight of every extremal vector of } B \text{ is contained in } W_0 \lambda. \)

A finite dimensional integrable \( U'_q(\mathfrak{g}) \)-module \( M \) is good if
\[
(i) \ M \text{ has a bar involution},
\]
\[
(ii) \ M \text{ has a crystal basis with simple crystal},
\]
(iii) $M$ has a lower global basis.

For example, all the fundamental representations $V(\varpi_i)$ ($i \in I_0$) are good. Every good module is irreducible. For any good module $M$, there exists an extremal weight vector $v$ of weight $\lambda$ such that $\text{wt}(U_q'(\mathfrak{g})v) \subset \lambda - \sum_{i \in I_0} \mathbb{Z}_{\geq 0} \mathfrak{g}(\alpha_i)$. Such $\lambda$ is called a dominant extremal weight and $v$ is called a dominant extremal weight vector.

Take $k = C(q) \subset \bigcup_{M > 0} C((q^{1/m}))$. Let $M_{\text{aff}} = k[z, z^{-1}] \otimes_k M$ be the affinization of $M$. For $v \in M$ and $k \in \mathbb{Z}$, the action of $U_q'(\mathfrak{g})$ on $M_{\text{aff}}$ is given by

$$e_i(z^k \otimes v) = \begin{cases} z^{k+1} \otimes e_0 v & \text{if } i = 0, \\ z^k \otimes e_i v & \text{if } i \neq 0, \end{cases}$$

$$f_i(z^k \otimes v) = \begin{cases} z^{k-1} \otimes f_0 v & \text{if } i = 0, \\ z^k \otimes f_i v & \text{if } i \neq 0, \end{cases}$$

$$K_i^{\pm 1}(z^k \otimes v) = q_i^{\pm (h_i, v_0)} (z^k \otimes v) \quad (i \in I).$$

We define a $U_q'(\mathfrak{g})$-module automorphism $z_M : M_{\text{aff}} \to M_{\text{aff}}$ of weight $\delta$ by

$$z^k \otimes v \mapsto z^{k+1} \otimes v \quad (v \in M, k \in \mathbb{Z}).$$

Let $M_1, M_2$ be good $U_q'(\mathfrak{g})$-modules and let $u_1, u_2$ be dominant extremal weight vectors of $M_1$ and $M_2$, respectively. Set $z_1 = z_{M_1}$ and $z_2 = z_{M_2}$. Then there exists a unique $U_q'(\mathfrak{g})$-module homomorphism

$$R_{M_1, M_2}^{\text{norm}}(z_1, z_2) : (M_1)_{\text{aff}} \otimes (M_2)_{\text{aff}} \longrightarrow k[z_1, z_2] \otimes_k k[z_1^{\pm 1}, z_2^{\pm 1}] (M_2)_{\text{aff}} \otimes (M_1)_{\text{aff}}$$

satisfying

$$R_{M_1, M_2}^{\text{norm}}(u_1 \otimes u_2) = u_2 \otimes u_1,$$

$$R_{M_1, M_2}^{\text{norm}} \circ z_1 = z_1 \circ R_{M_1, M_2}^{\text{norm}},$$

$$R_{M_1, M_2}^{\text{norm}} \circ z_2 = z_2 \circ R_{M_1, M_2}^{\text{norm}}.$$

The homomorphism $R_{M_1, M_2}^{\text{norm}}$ is called the normalized $R$-matrix of $M_1$ and $M_2$.

Note that $\text{Im} R_{M_1, M_2}^{\text{norm}} \subset k(z_2/z_1) \otimes_k k[z_2/z_1] (M_2)_{\text{aff}} \otimes (M_1)_{\text{aff}}$. We denote by $d_{M_1, M_2}(u) \in k[u]$ the monic polynomial of the smallest degree such that

$$\text{Im} (d_{M_1, M_2}(z_2/z_1) R_{M_1, M_2}^{\text{norm}}) \subset (M_2)_{\text{aff}} \otimes (M_1)_{\text{aff}}.$$ 

The polynomial $d_{M_1, M_2}(u)$ is called the denominator of $R_{M_1, M_2}^{\text{norm}}$.

The normalized $R$-matrix satisfies the Yang-Baxter equation. That is, for $U_q'(\mathfrak{g})$-modules $M_1, M_2, M_3$, we have

$$(R_{M_2, M_3}^{\text{norm}} \otimes 1) \circ (1 \otimes R_{M_1, M_2}^{\text{norm}}) \circ (R_{M_1, M_2}^{\text{norm}} \otimes 1) = (1 \otimes R_{M_1, M_2}^{\text{norm}}) \circ (R_{M_1, M_3}^{\text{norm}} \otimes 1) \circ (1 \otimes R_{M_2, M_3}^{\text{norm}}).$$
5. Quantum affine Schur-Weyl duality functor

Let \( \{ V_s \mid s \in \mathcal{S}\} \) be a family of good modules and let \( v_s \) be a dominant extremal weight vector in \( V_s \) with weight \( \lambda_s \) (\( s \in \mathcal{S} \)). Take an index set \( J \) endowed with the maps \( X : J \to \mathbb{k}^x \) and \( s : J \to \mathcal{S} \). For each \( i, j \in J \), let

\[
R_{V_s(i),V_s(j)}^\text{norm}(z_i,z_j) : (V_s(i))_{\text{aff}} \otimes (V_s(j))_{\text{aff}} \to \mathbb{k}(z_i,z_j) \otimes \mathbb{k}[z_i^{\pm 1},z_j^{\pm 1}] (V_s(j))_{\text{aff}} \otimes (V_s(i))_{\text{aff}}
\]

be the normalized \( R \)-matrix sending \( v_s(i) \otimes v_s(j) \) to \( v_s(j) \otimes v_s(i) \).

Let \( d_{V_s(i), V_s(j)}(z_j/z_i) \) be the denominator of \( R_{V_s(i), V_s(j)}^\text{norm}(z_i,z_j) \). We define a quiver \( \Gamma^J \) as follows.

(i) We take \( J \) to be the set of vertices.

(ii) We put \( d_{ij} \) many arrows from \( i \) to \( j \), where \( d_{ij} \) the order of zero of \( d_{V_s(i),V_s(j)}(z_j/z_i) \) at \( z_j/z_i = X(j)/X(i) \).

Define the Cartan matrix \( A^J = (a^J_{ij})_{i,j \in J} \) by

\[
a^J_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -d_{ij} - d_{ji} & \text{if } i \neq j. \end{cases}
\]

Thus we obtain a symmetric Cartan datum \((A^J, P, P^\vee, \Pi, \Pi^\vee)\) associated with \( \Gamma^J \).

Set

\[
Q^J_{ij}(u,v) := \begin{cases} 0 & \text{if } i = j, \\ (u - v)^{d_{ij}}(v - u)^{d_{ji}} & \text{if } i \neq j. \end{cases}
\]

We will denote by \( R^J(\beta) \) (\( \beta \in Q_+ \)) the Khovanov-Lauda-Rouquier algebra associated with \((A^J, Q^J)\).

For each \( \nu = (\nu_1, \ldots, \nu_n) \in J^\beta \), let \( \hat{\mathcal{O}}_{\mathfrak{t}^n,X(\nu)} = \mathbb{k}[[X_1 - X(\nu_1), \ldots, X_n - X(\nu_n)]] \) be the completion of \( \mathcal{O}_{\mathfrak{t}^n,X(\nu)} \) at \( X(\nu) := (X(\nu_1), \ldots, X(\nu_n)) \) and set

\[
V_{\nu} := (V_{s(\nu_1)})_{\text{aff}} \otimes \cdots \otimes (V_{s(\nu_n)})_{\text{aff}},
\]

where \( X_k = z_{V_{s(\nu_k)}} \) (\( k = 1, \ldots, n \)).

We define

\[
\hat{V}_{\nu} := \hat{\mathcal{O}}_{\mathfrak{t}^n,X(\nu)} \otimes \mathbb{k}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] V_{\nu} \quad \text{and} \quad \hat{V}^{\otimes \beta} := \bigoplus_{\nu \in J^\beta} \hat{V}_{\nu} e(\nu).
\]

The following proposition is one of the main results of [18].

**Proposition 5.1.** [18] The space \( \hat{V}^{\otimes \beta} \) is a \((U'_q(\mathfrak{g}), R^J(\beta))\)-bimodule.

Hence we obtain a functor

\[
\mathcal{F}_\beta : \text{mod } (R^J(\beta)) \to \text{mod } U'_q(\mathfrak{g})
\]
defined by
\[ M \mapsto \hat{V}^\otimes \beta \otimes R^J(\beta) M \quad \text{for} \quad M \in \text{mod}(R^J(\beta)). \]

Write \( \text{mod}(R^J) := \bigoplus_{\beta \in Q_+} \text{mod}(R^J(\beta)) \) and set
\[ F = \bigoplus_{\beta \in Q_+} F_\beta : \text{mod}(R^J) \rightarrow \text{mod}U_q'(\mathfrak{g}). \]

The functor \( F \) is called the \textit{quantum affine Schur-Weyl duality functor}. The basic properties of \( F \) are summarized in the following theorem.

**Theorem 5.2.** [18]

(a) The functor \( F \) restricts to \( F : \text{rep}(R^J) \rightarrow C_{\text{int}}, \)

where \( \text{rep}(R^J) := \bigoplus_{\beta \in Q_+} \text{rep}(R^J(\beta)) \) and \( C_{\text{int}} \) denotes the category of finite dimensional integrable \( U_q'(\mathfrak{g}) \)-modules.

(b) For each \( i \in J \), let \( S(\alpha_i) := k u(i) \) be the 1-dimensional graded simple \( R^J(1) \)-module defined by
\[ e(j) u(i) = \delta_{ij} u(i), \quad x_1 u(i) = 0. \]

Then we have
\[ F(S(\alpha_i)) \cong (V_{s(i)})_{X(i)}, \]

where \((V_{s(i)})_{X(i)}\) is the evaluation module of \( V_{s(i)} \) at \( z_i = X(i) \).

(c) \( F \) is a tensor functor; i.e., there exists a canonical \( U_q'(\mathfrak{g}) \)-module isomorphims
\[ F(R^J(0)) \cong k, \quad F(M \circ N) \cong F(M) \otimes F(N) \]

for \( M \in \text{rep}(R^J(m)), N \in \text{rep}(R^J(n)) \).

(d) If the quiver \( \Gamma^J \) is of type \( A_n \) \((n \geq 1), D_n \) \((n \geq 4), E_6, E_7, E_8\), then \( F \) is exact.

6. **The Categories \( \mathcal{T}_N \) and \( \mathcal{C}_N \)**

Take \( k = C(q) \). Let \( \mathfrak{g} = A^{(1)}_{N-1} \) be the affine Kac-Moody algebra of type \( A^{(1)}_{N-1} \) and let \( V = V(\varpi_1) \) be the fundamental representation of \( U_q'(A^{(1)}_{N-1}) \) of weight \( \varpi_1 \).

Set \( S = \{V\}, J = Z \) and let \( X : Z \rightarrow k^\times \) be the map given by \( j \mapsto q^{2j} \) \((j \in Z)\). Then the normalized \( R \)-matrix \( R_{V,V}^{\text{norm}} : V_{z_1} \otimes V_{z_2} \rightarrow V_{z_2} \otimes V_{z_1} \) has the denominator \( d_{V,V}(z_2/z_1) = z_2/z_1 - q^2 \). Hence we have
\[ d_{ij} = \begin{cases} 1 & \text{if } j = i + 1, \\ 0 & \text{otherwise}, \end{cases} \]
which yields the quiver $\Gamma_J$ of type $A_\infty$. Take $P_J = \bigoplus_{k \in \mathbb{Z}} \mathbb{Z} \varepsilon_k$ to be the weight lattice and let $Q_J = \bigoplus_{k \in \mathbb{Z}} \mathbb{Z} (\varepsilon_k - \varepsilon_{k+1})$ be the root lattice. There is a bilinear form on $P_J$ given by $(\varepsilon_a, \varepsilon_b) = \delta_{ab}$.

For $a \leq b$, let $l = b - a + 1$ and let $L(a,b) := k u(a,b)$ be the 1-dimensional graded simple $R_J^l(\varepsilon_a - \varepsilon_{b+1})$-module defined by

\[
x_s u(a,b) = 0, \quad \tau_t u(a,b) = 0 \quad (1 \leq s \leq l, \ 1 \leq t \leq l - 1),
\]

\[
e(\nu) u(a,b) = \begin{cases} u(a,b) & \text{if } \nu = (a,a+1,\ldots,b), \\ 0 & \text{otherwise}. \end{cases}
\]

Then we have

\[
F(L(a,b)) \cong \begin{cases} V(\varepsilon_l(-q)^{a+b}) & \text{if } 0 \leq l \leq N, \\ 0 & \text{if } l > N, \end{cases}
\]

where $F : \text{mod } (R^l(l)) \to \text{mod } U'_q(g)$ is the quantum affine Schur-Weyl duality functor.

Recall that $\text{Rep } (R^l(l))$ is the category of finite dimensional graded $R^l(l)$-modules. Set $\mathcal{R} := \bigoplus_{l \geq 0} \text{Rep } (R^l(l))$ and let $\mathcal{S}$ be the Smallest Serre subcategory of $\mathcal{R}$ such that

(i) $S$ contains $L(a,a+N)$ for all $a \in \mathbb{Z}$,

(ii) $X \circ Y, Y \circ X \in \mathcal{S}$ for all $X \in \mathcal{R}, Y \in \mathcal{S}$.

Take the quotient category $\mathcal{R}/\mathcal{S}$ and let $Q : \mathcal{R} \to \mathcal{R}/\mathcal{S}$ be the canonical projection functor. Then we have:

**Proposition 6.1.** [18]

(a) The functor $F$ factors through $\mathcal{R}/\mathcal{S}$. That is, there is a canonical functor $F_S : \mathcal{R}/\mathcal{S} \to \text{mod } U'_q(g)$ such that the following diagram is commutative.

\[
\begin{array}{ccc}
\mathcal{R} & \xrightarrow{F} & \text{mod } U'_q(g) \\
\downarrow Q & & \\
\mathcal{R}/\mathcal{S} & \xrightarrow{F_S} & \\
\end{array}
\]

(b) The functor $F_S$ sends a simple object in $\mathcal{R}/\mathcal{S}$ to a simple object in $\text{mod } U'_q(g)$.

Let $L_a := L(a,a+N-1)$ and $u_a := u(a,a+N-1) \in L_a (a \in \mathbb{Z})$. Then $F(L_a)$ is isomorphic to the trivial representation of $U'_q(g)$. Let $S : P_J \to P_J$ ($\varepsilon_a \mapsto \varepsilon_{a+N-1}$)
be an automorphism on $P_J$ and let $B$ be the bilinear form on $P_J$ given by
\[ B(x, y) := -\sum_{k>0} (S^k x, y) \text{ for all } x, y \in P_J. \]

We define a new tensor product $\star$ on $\mathcal{R}/S$ by
\[ X \star Y := q^{B(\alpha, \beta)} X \circ Y \text{ for } X \in (\mathcal{R}/S)_\alpha, \ Y \in (\mathcal{R}/S)_\beta. \]

Then there exists an isomorphism $R(a)(X) : L_a \star X \sim X \star L_a$ which is functorial in $X \in \mathcal{R}/S$. Moreover, the isomorphisms
\[ R_a(L_b) : L_a \star L_b \sim L_b \star L_a \text{ and } R_b(L_a) : L_b \star L_a \sim L_a \star L_b \]
are inverses to each other. One can verify that $\{L_a, R_a(L_b) \mid a, b \in \mathbb{Z}\}$ forms a commuting family of central objects in $(\mathcal{R}/S, \star)$ (see [18, Appendix A.6]).

Let $T'_N := (\mathcal{R}/S)[L_a^{-1} \mid a \in \mathbb{Z}]$ be the localization of $\mathcal{R}/S$ by this commuting family and define
\[ T_N := (\mathcal{R}/S)[L_a \cong 1 \mid a \in \mathbb{Z}]. \]

We denote by $\mathcal{P} : \mathcal{R}/S \to T_N$ the canonical functor.

**Theorem 6.2.** [18]

(a) The category $T_N$ is a rigid tensor category; i.e., every object in $T_N$ has a right dual and a left dual.

(b) The functor $\mathcal{F}_S$ factors through $T_N$. That is, there exists a canonical functor $\mathcal{F}_N : T_N \to \text{mod} \ U'_q(\mathfrak{g})$ such that the following diagram is commutative.

(c) The functor $\mathcal{F}_N$ is exact and sends a simple object in $T_N$ to a simple object in $\text{mod} \ U'_q(\mathfrak{g})$.

Let $C_N$ be the smallest full subcategory of $\mathcal{C}_{\text{int}}$ consisting of $U'_q(\mathfrak{g})$-modules $M$ such that every composition factor of $M$ appears as a composition factor of a tensor product of modules of the form $V(\varpi_i)_{q^j}$ ($j \in \mathbb{Z}$). Thus $C_N$ is an abelian category containing all $U'_q(\mathfrak{g})$-modules $V(\varpi_i)_{(-q)^{i+2a-1}}$ for $1 \leq i \leq N-1$ and $a \in \mathbb{Z}$. 
Moreover, $\mathcal{C}_N$ is stable under taking submodules, quotients, extensions and tensor products. Hence $\mathcal{F}_N$ restricts to an exact functor

$$\mathcal{F}_N : \mathcal{T}_N \rightarrow \mathcal{C}_N.$$ 

Let $\text{Irr} (\mathcal{T}_N)$ (respectively, $\text{Irr} (\mathcal{C}_N)$) denote the set of isomorphism classes of simple objects in $\mathcal{T}_N$ (respectively, in $\mathcal{C}_N$). Define an equivalence relation on $\text{Irr} (\mathcal{T}_N)$ by setting $X \sim Y$ if and only if $X \sim q^m Y$ for some $m \in \mathbb{Z}$. Set

$$\text{Irr} (\mathcal{T}_N)|_{q=1} := \text{Irr} (\mathcal{T}_N)/\sim.$$ 

**Theorem 6.3.** [18]

(a) The functor $\mathcal{F}_N$ induces a bijection between $\text{Irr} (\mathcal{T}_N)|_{q=1}$ and $\text{Irr} (\mathcal{C}_N)$.

(b) The exact functor $\mathcal{F}_N$ induces a ring isomorphism

$$\phi_N : K(\mathcal{T}_N)|_{q=1} \sim \rightarrow K(\mathcal{C}_N).$$

Therefore, the category $\mathcal{T}_N$ provides a graded lifting of $\mathcal{C}_N$ as a rigid tensor category.

**7. The category $\mathcal{C}_Q$**

In this section, we deal with affine Kac-Moody algebras $\mathfrak{g}$ of type $A_n^{(1)}$ ($n \geq 1$), $D_n^{(1)}$ ($n \geq 4$), $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$. Let $I = \{0, 1, \ldots, n\}$ be the index set for the simple roots of $\mathfrak{g}$ and set $I_0 = I \setminus \{0\}$. We denote by $\mathfrak{g}_0$ the finite dimensional simple Lie subalgebra of $\mathfrak{g}$ generated by $e_i, f_i, h_i$ ($i \in I_0$). Thus $\mathfrak{g}_0$ is of type $A_n$ ($n \geq 1$), $D_n$ ($n \geq 4$), $E_6$, $E_7$, $E_8$, respectively.

Let $Q$ be the Dynkin quiver associated with $\mathfrak{g}_0$. A function $\xi : I_0 \rightarrow \mathbb{Z}$ is called a height function if $\xi_j = \xi_i - 1$ whenever we have an arrow $i \rightarrow j$.

Set

$$\widehat{I}_0 := \{(i, p) \in I_0 \times \mathbb{Z} \mid p - \xi_i \in 2\mathbb{Z}\}.$$

The repetition quiver $\widehat{Q}$ is defined as follows.

(i) We take $\widehat{I}_0$ to be the set of vertices.

(ii) The arrows are given by

$$(i, p) \rightarrow (j, p + 1), \quad (j, q) \rightarrow (i, q + 1)$$

for all arrows $i \rightarrow j$ and $p, q \in \mathbb{Z}$ such that $p - \xi_i \in \mathbb{Z}$, $q - \xi_j \in \mathbb{Z}$.

For all $i \in I_0$, let $s_i(Q)$ be the quiver obtained from $Q$ by reversing the arrows that touch $i$. A reduced expression $w = s_{i_l} \cdots s_{i_1} \in W_0$ is said to be adapted to $Q$ if $i_k$ is the source of $s_{i_k-1} \cdots s_{i_1}(Q)$ for all $1 \leq k \leq l$. It is known that there is a unique Coxeter element $\tau \in W_0$ which is adapted to $Q$. 

Set $\hat{\Delta} := \Delta_+ \times \mathbb{Z}$, where $\Delta_+$ is the set of positive roots of $\mathfrak{g}_0$. For each $i \in I_0$, let $B(i) := \{j \in I_0 \mid \text{there is a path from } j \text{ to } i\}$ and define $\gamma_i := \sum_{j \in B(i)} \alpha_j$. We define a bijection $\phi : \hat{I}_0 \to \hat{\Delta}$ inductively as follows.

1. We begin with $\phi(i, \xi_i) := (\gamma_i, 0)$.
2. If $\phi(i, p) = (\beta, j)$ is given, then we define
   - $\phi(i, p - 2) := (\tau(\beta), j)$ if $\tau(\beta) \in \Delta_+$,
   - $\phi(i, p - 2) := (-\tau(\beta), j - 1)$ if $\tau(\beta) \in \Delta_-$,
   - $\phi(i, p + 2) := (\tau^{-1}(\beta), j)$ if $\tau^{-1}(\beta) \in \Delta_+$,
   - $\phi(i, p + 2) := (-\tau^{-1}(\beta), j + 1)$ if $\tau^{-1}(\beta) \in \Delta_-$.

Let $w_0$ be the longest element of $W_0$ and fix a reduced expression $w_0 = s_{i_1} \cdots s_{i_l}$ which is adapted to $Q$. Set

$$J := \{(i, p) \in \hat{I}_0 \mid \phi(i, p) \in \Pi_0 \times \{0\}\},$$

where $\Pi_0$ denotes the set of simple roots of $\mathfrak{g}_0$. Take the maps $X : J \to \mathbb{K}^\times$ and $s : J \to \{V(\varpi_i) \mid i \in I_0\}$ defined by

$$X(i, p) = (-q)^{p+h}, \quad s(i, p) = V(\varpi_i) \quad \text{for } (i, p) \in J,$$

where $h$ is the Coxeter number of $\mathfrak{g}_0$.

**Theorem 7.1.** [19]

For any $(i, p), (j, r) \in J$, assume that the normalized $R$-matrix $R_{\text{norm}}^{V(\varpi_i), V(\varpi_j)}(z)$ has a pole at $z = (-q)^{r-p}$ of order at most 1. Then the following statements hold.

(a) The Cartan matrix $A^J$ associated with $(J, X, s)$ is of type $\mathfrak{g}_0$.

(b) There exists a quiver isomorphism

$$Q^{\text{rev}} \xrightarrow{\sim} \Gamma^J, \quad k \mapsto \phi^{-1}(\alpha_k, 0) \quad (k \in I_0),$$

where $Q^{\text{rev}}$ is the reverse quiver of $Q$.

(c) The functor $\mathcal{F} : \text{rep}(R^J) \to \mathcal{C}_{\text{int}}$ is exact and

$$\mathcal{F}(S(\alpha_k)) \cong V(\varpi_i)(-q)^{p+h},$$

where $\phi(i, p) = (\alpha_k, 0)$.

**Remark 7.2.** When $\mathfrak{g}$ is of type $A_1^{(1)}$ ($n \geq 1$) or $D_1^{(1)}$ ($n \geq 4$), then the condition in Theorem 7.1 is satisfied. We conjecture that the same is true of $\mathfrak{g} = E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$.

We now bring out the main subject of our interest in this section, the category $\mathcal{C}_Q$ (cf. [13]). Let $\mathcal{C}_Q$ be the smallest abelian full subcategory of $\mathcal{C}_{\text{int}}$ such that

(i) $\mathcal{C}_Q$ is stable under taking submodules, subquotients, direct sums and tensor products,

(ii) $\mathcal{C}_Q$ contains all $U'_q(\mathfrak{g})$-modules of the form $V(\beta)z/(z-1)^lV(\beta)z$ ($\beta \in \Delta_+, l \geq 1$). Here, $V(\beta) = V(\varpi_i)(-q)^{p+h}$ such that $\phi(i, p) = (\beta, 0)$. 
Let \( \text{Nilrep} (R^J(\beta)) \) be the category of finite dimensional ungraded \( R^J(\beta) \)-modules such that all \( x_k \)'s act nilpotently and set

\[
\text{Nilrep} (R^J) := \bigoplus_{\beta \in \mathbb{Q}_+} \text{Nilrep} (R^J(\beta)).
\]

Note that every module in \( \text{Nilrep} (R^J) \) can be obtained by taking submodules, subquotients, direct sums and convolution products of \( P(\alpha_k)/ (x_1^l) \) \((k \in I_0, l \geq 0)\), where \( P(\alpha_k) \) is the projective cover of \( S(\alpha_k) \). Thus we obtain a well-defined functor

\[
\mathcal{F} : \text{Nilrep} (R^J) \longrightarrow \mathcal{C}_Q,
\]

which satisfies the following properties.

**Theorem 7.3.** [18, 19]

(a) \( \mathcal{F} \) is an exact tensor functor.
(b) \( \mathcal{F} \) sends a simple object in \( \text{Nilrep} (R^J) \) to a simple object in \( \mathcal{C}_Q \).

It is straightforward to verify that \( \mathcal{F} \) is a faithful functor. Since \( \mathcal{C}_Q \) is the smallest abelian full subcategory of \( \mathcal{C}_{\text{int}} \) satisfying the conditions (i) and (ii) given above, we conjecture that \( \mathcal{F} \) is full and defines an equivalence of categories.

**Remark 7.4.** Note that our general approach to quantum affine Schur-Weyl duality applies to all quantum affine algebras and any choice of good modules. Thus we expect there are a lot more exciting developments to come. It is an interesting question whether our general construction can shed a new light on the hidden connection between quantum affine algebras and cluster algebras (cf. [12]).

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