New results on comparison of distributions of Gaussian quadratic forms are presented.

Let $\xi_1, \ldots, \xi_n$ – independent $\mathcal{N}(0, 1)$-Gaussian random variables. For $a = (a_1, \ldots, a_n) \in \mathbb{R}_+^n$ and $x \geq 0$ consider the following probability

$$\beta(x, a) = \Pr\left( \sum_{i=1}^{n} a_i \xi_i^2 < x \right).$$

We are interested in what $a, b \in \mathbb{R}_+^n$ and $x$ the following inequality holds

$$\beta(x, a) \leq \beta(x, b). \quad (1)$$

Below the vector $a \in \mathbb{R}_+^n$ is called monotone, if $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$. As usually, $a \geq b$ means $a_i \geq b_i$, $i = 1, \ldots, n$.

1. Known comparison theorem. In [1, Theorem 1] the following result was proved. Let $a, b \in \mathbb{R}_+^n$ – monotone vectors and the following condition for them is fulfilled

$$\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i, \quad k = 1, \ldots, n. \quad (2)$$

Then for any $x \geq 2 \sum_{i=1}^{n} b_i$ the inequality (1) holds.

**Remark 1.** In order (2) to be valid, we need, in particular, $\max_i a_i \geq \max_i b_i$, what is rather restrictive.

Inequality (1) is useful in problems of detection of stochastic signals in Gaussian noise. But application of Theorem 1 from [1] in such problems is rather difficult because of the restrictive assumption (2) and the requirement $x \geq 2 \sum_{i=1}^{n} b_i$ (in stochastic signals detection problems the inequality (1) is usually required for $x < \sum_{i=1}^{n} b_i$).

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In the paper the inequality (1) is proved under different from (2) assumption. First, a simple similar Proposition 1 is proved, and then it is strengthened using additional arguments (Theorems 1 and 2).

Concerning applications of the inequality (1) it should be mentioned that such results are helpful in problems of detection of Gaussian stochastic signals in the background of independent additive Gaussian noise [2–5]. Consider, for example, the problem of detection of Gaussian stochastic signal vector \( s \), in the background of independent additive Gaussian noise \( \xi \). If the vector \( \sigma \) of the vector \( s \) intensities is known, then the logarithm of the corresponding likelihood ratio in that problem is a Gaussian quadratic form, similar to one considered above. Assume that we know only that the vector \( \sigma \) belongs to the given set \( E \). Then natural question arises: is it possible to replace the set \( E \) by a smaller set \( E_0 \) without loss of detection quality (in particular, to replace \( E \) by a single point \( \sigma_0 \))? Such problem will be considered by author in the paper [6].

Some results showing validity of the inequality (1) can also be found in [7, 1].

2. The first result. For vectors \( a, b \in \mathbb{R}^n_+ \) introduce functions

\[
f(i, a, b) = \frac{1}{b_i} - \frac{1}{a_i}, \quad i = 1, \ldots, n, \quad D(a, b) = \left( \prod_{i=1}^{n} \frac{a_i}{b_i} \right)^{1/2}. \tag{3}
\]

**Proposition 1.** Let \( a, b \in \mathbb{R}^n_+ \) and \( D(a, b) > 1 \). If \( x \) satisfies the condition

\[
x \leq \sup_{\substack{c \leq a \\ d \geq b \atop d(c, d) > 1}} \frac{2 \ln D(c, d)}{d(c, d)}, \quad d(c, d) = \max_i f(i, c, d) > 0, \tag{4}
\]

then \( \beta(x, a) \leq \beta(x, b) \).

**Proof.** Consider first the case \( c = a, d = b \). We have

\[
\beta(x, a) = \mathbb{P} \{ \xi \in A(x, a) \}, \quad A(x, a) = \left\{ y : \sum_{i=1}^{n} a_i y_i^2 \leq x \right\}, \tag{5}
\]

where the ellipsoid \( A(x, a) \in \mathbb{R}^n \) has axes \( \{ \sqrt{x/a_i}, \quad i = 1, \ldots, n \} \) and volume \( V(A(x, a)) \), proportional to \( \left( \prod_{i=1}^{n} a_i \right)^{-1/2} \). In order to compare probabilities \( \beta(x, a) \) and \( \beta(x, b) \), consider the difference (see [5])

\[
(2\pi)^{n/2} \left[ \beta(x, a) - \beta(x, b) \right] = \Delta(x, a, b) = \int_{A(x, a)} \cdots \int_{A(x, b)} e^{-\frac{1}{2} \sum_{i=1}^{n} y_i^2} \, dy - \int_{A(x, a)} \cdots \int_{A(x, b)} e^{-\frac{1}{2} \sum_{i=1}^{n} z_i^2} \, dz.
\]
Then based on various additional arguments.

\[ \beta_{\text{max}}(4) \]

fulfilled where the choice of vectors \( c \) in comparison with \( \beta \) statement 1 of Lemma below in section 3). Moreover, \( c_{\beta} \) and \( d \) satisfy the condition (4).

Changing variables \( z_i = \sqrt{a_i/b_i}, i = 1, \ldots, n \), in the second integral we have \( (D = D(a, b)) \)

\[
\Delta(x, a, b) = \int \cdots \int_{A(x,a)} e^{-\frac{1}{2} \sum_{i=1}^{n} y_i^2} - D \exp \left\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{a_j y_j^2}{b_j} \right\} dy =
\]

\[
\leq (1 - De^{-xd/2}) \int \cdots \int_{A(x,a)} e^{-\frac{1}{2} \sum_{i=1}^{n} y_i^2} dy.
\]

Therefore \( \Delta(x, a, b) \leq 0 \), if \( 1 - De^{-xd/2} \leq 0 \), i.e. if

\[
x \max_i f(i, a, b) \leq 2 \ln D = \ln \frac{W(a)}{W(b)} = \sum_{i=1}^{n} \ln \frac{a_i}{b_i},
\]

where \( W(a) = \prod_{i=1}^{n} a_i, W(b) = \prod_{i=1}^{n} b_i \), from which Proposition 1 for \( c = a, d = b \) follows.

If \( c \leq a \) and \( d \geq b \), then \( \beta(x, a) \leq \beta(x, c) \) and \( \beta(x, d) \leq \beta(x, b) \) for any \( x \) (see the statement 1 of Lemma below in section 3). Moreover, \( \beta(x, c) \leq \beta(x, d) \), if \( x \) satisfies the condition (4). ▲

Remark 2. Auxiliary vectors \( c \) and \( d \) in (4) allow sometimes to relax the constraint (4) in comparison with \( c = a \) and \( b = d \). One of further extensions (Theorem 2) concerns the choice of vectors \( c \) and \( d \) for given \( a \) and \( b \).

Below Proposition 1 is strengthened increasing the right-hand side of the condition (4) based on various additional arguments.

3. Strengthening 1. We use the following auxiliary result.

Lemma 1. Let \( a, b \in \mathbb{R}_+^n \) and for components with indices \( i, j \) we have \( \max\{a_i, a_j\} \geq \max\{b_i, b_j\} \) and \( a_i a_j \geq b_i b_j \). Let also \( a_k \geq b_k \) for all \( k \neq i, k \neq j \). Then \( \beta(x, a) \leq \beta(x, b) \) for any \( x \).

2) If \( b \leq a \), then \( \beta(x, a) \leq \beta(x, b) \) for any \( x \).

3) Assume that it is possible to partition the set of indices \( I = \{1, 2, \ldots, n\} \) on \( k \geq 1 \) parts \( I_1, \ldots, I_k \), such that \( I = \bigcup_{j=1}^{k} I_j \), \( I_i \cap I_j = \emptyset, i \neq j \), and the following conditions are fulfilled

\[
b_i \leq a_{0,j}, \quad i \in I_j, \quad j = 1, \ldots, k,
\]

where

\[
a_{0,j} = \left( \prod_{i \in I_j} a_i \right)^{1/|I_j|}.
\]
Then $\beta(x, a) \leq \beta(x, b)$ for any $x$.

Proof. 1) Assume first that $n = 2$. Then the integration region $\mathcal{A}(x, b) \in \mathbb{R}^2$ for $\beta(x, b)$ has volume $V(\mathcal{A}(x, b))$, proportional to $(b_1 b_2)^{-1/2}$, and $V(\mathcal{A}(x, b)) \geq V(\mathcal{A}(x, a))$. The random vector $\xi = (\xi_1, \xi_2)$ has the distribution density $p(y)$, proportional to $e^{-r^2(y)/2}$, $r^2(y) = y_1^2 + y_2^2$, monotonically decreasing in $r \geq 0$. It can be checked that

$$
\inf_{y \in \mathcal{A}(x,a) \setminus \mathcal{A}(x,b)} r^2(y) \geq \sup_{y \in \mathcal{A}(x,b) \setminus \mathcal{A}(x,a)} r^2(y).
$$

Therefore for given volume $V$ (i.e. for given product $b_1 b_2 = T$) the value $\beta(x, b)$ attains its maximum when $b_1 = b_2 = \sqrt{T}$, and monotonically decreases when $b_1$ deviates from $\sqrt{T}$, from which the inequality (1) follows for any $x$.

If $n > 2$, then the inequality (1) holds for any fixed $\{\xi_k\}$, $k \neq i, k \neq j$ and any $x$, from which necessary assertion follows.

2) If $b \leq a$, then $\mathcal{A}(x, a) \subseteq \mathcal{A}(x, b)$, and therefore $\beta(x, a) \leq \beta(x, b)$ for any $x$.

3) That assertion follows from part 1). It is sufficient to consider the case $k = 2$, $I_1 = \{1, \ldots, m\}$ and $I_2 = \{m + 1, \ldots, n\}$ for some $1 < m < n$. Introduce $n$-vector $a_0 = (a_{0,1}, \ldots, a_{0,1}, a_{0,2}, \ldots, a_{0,2})$, consisting of $m$ components $a_{0,1}$ and $n-m$ components $a_{0,2}$. Then repeatedly applying lemma's part 1) it is possible to show that $\beta(x, a) \leq \beta(x, a_0)$ for any $x$. Since $b \leq a_0$, then $\beta(x, a_0) \leq \beta(x, b)$ for any $x$. ▲

We strengthen the Proposition 1. Setting for convenience $c = a$ and $d = b$, assume that there exists $i \leq n-1$, such that $f(i + 1, a, b) < f(i, a, b)$. Then $a_{i+1} < a_i$ (since $b_{i+1} \leq b_i$) and $b_{i+1} < b_i$. We build the vector $a^{(i)}$ such that $\beta(x, a) \leq \beta(x, a^{(i)})$ for all $x$ and $\beta(x, a^{(i)}) \leq \beta(x, b)$ for $x$, satisfying, generally, a weaker than (1) constraint. For that purpose we use the following procedure.

Decrease $a_i$ down to the value $a^{(i)}_i$ and increase respectively $a_{i+1}$ up to the value $a^{(i)}_{i+1}$, such that the following three conditions are satisfied:

1) $a^{(i)}_i \geq a^{(i)}_{i+1}$;

2) $a_{i+1} = a^{(i)}_i a^{(i)}_{i+1}$;

3) $f(i + 1, a^{(i)}, b) = f(i, a^{(i)}, b)$, where $a^{(i)}$ - obtained that way from $a$ the new monotone vector (it differs from $a$ only in components with indices $i$ and $i + 1$).

Then we have $a^{(i+1)}_i < a^{(i)}_i$, $f(i, a^{(i)}, b) < f(i, a, b)$ and $f(i + 1, a, b) < f(i + 1, a^{(i)}, b)$. We also have $\beta(x, a) \leq \beta(x, a^{(i)})$ for any $x$ (due to Lemma 1). Then after standard calculations we have

$$
f(i + 1, a^{(i)}, b) = f(i, a^{(i)}, b) = \frac{1}{2} \left[ a^{(i-1)}_i + a^{(i)}_{i+1} - \sqrt{(a^{(i-1)}_i + a^{(i)}_{i+1})^2 + z_i} \right] < f(i, a, b),
$$

where

$$
z_i = 4 \left( a^{(i-1)}_i a^{(i)}_{i+1} - b^{(i-1)}_i b^{(i+1)}_i \right), \quad i = 1, \ldots, n. \tag{6}
$$

Note that values $f(i, a, b), f(i, a^{(i)}, b)$ may be negative.

Similarly, instead of the vector $a$ we may change the vector $b$ (but in opposite direction), replacing it by the vector $b^{(i)}$ such that $\beta(x, b) \leq \beta(x, b^{(i)})$ for all $x$ and
\[ \beta(x, a) \leq \beta(x, b^{(1)}) \] for \( x \), satisfying, generally, a weaker than (11) constraint. For that purpose, increase \( b_i \) up to the value \( b_i^{(1)} \) and decrease respectively \( b_{i+1} \) down to the value \( b_{i+1}^{(1)} \), such that the following three conditions are satisfied:

1) \( b_i^{(1)} \geq b_{i+1}^{(1)}; \)
2) \( b_{b_{i+1}} = b_i^{(1)} b_{i+1}^{(1)}; \)
3) \( f(i + 1, a, b^{(1)}) = f(i, a, b^{(1)}), \) where \( b^{(1)} \) - obtained that way from \( b \) the new monotone vector (it differs from \( b \) only in components with indices \( i \) and \( i + 1 \)).

Then we have \( b_{i+1}^{(1)} < b_i^{(1)}, f(i, a, b^{(1)}) < f(i, a, b) \) and \( f(i + 1, a, b) < f(i + 1, a, b^{(1)}) \). We also have \( \beta(x, b^{(1)}) \leq \beta(x, b) \) for any \( x \) (due to Lemma). Then after standard calculations we have

\[
f(i + 1, a, b^{(1)}) = f(i, a, b^{(1)}) = \frac{1}{2} \left( \sqrt{(a_i^{−1} + a_{i+1}^{−1})^2 - z_i - a_i^{−1} - a_{i+1}^{−1}} \right) < f(i, a, b),
\]

where \( z_i \) is defined in (11).

Compare values \( f(i, a^{(1)}, b) \) and \( f(i, a, b^{(1)}) \). It is possible to check that \( f(i, a^{(1)}, b) < f(i, a, b^{(1)}) \) (i.e. changing the vector \( a \) gives better result), if \( b_{b_{i+1}} > a_i a_{i+1} \). If \( b_{b_{i+1}} < a_i a_{i+1} \), then \( f(i, a^{(1)}, b) > f(i, a, b^{(1)}) \), i.e. changing the vector \( b \) gives better result.

After that we apply the procedure described to obtained vectors \( a^{(1)} \) or \( b^{(1)} \) and so on. Unfortunately, the author was not able to investigate the optimal sequence of changes the vectors \( a \) or \( b \). For that reason we limit ourselves to the case when only the vector \( a \) (or only the vector \( b \)) is changing.

Note that if \( z_i \geq 0 \) (i.e. \( b_i b_{i+1} \geq a_i a_{i+1} \)), then \( a_i \leq b_i \) and \( f(i, a, b) \leq 0 \), or \( a_{i+1} \leq b_{i+1} \) and \( f(i + 1, a, b) \leq 0 \).

Denote by \( i_0 = i_0(a, b) \) the minimal of indices \( i \), such that \( f(i_0, a, b) = \max \limits_{i} f(i, a, b) \), and by \( i_1 = i_1(a, b) \geq i_0 \) the minimal of indices \( i \), such that \( f(i_1, a, b) = \max \limits_{i} f(i, a, b) \). If \( i_1(a, b) = n \), then the method used does not improve the condition (11) and then in Proposition 1 we have \( \max \limits_{i} f(i, a, b) = 1/b_n - 1/a_n \). Therefore we assume that \( i_1(a, b) \leq n - 1 \).

Change vector \( a \). First, for fixed monotone \( b \) we change the vector \( a \). Choose arbitrary \( i \geq i_0(a, b) \), such that \( f(i + 1, a, b) < f(i, a, b) \), and apply to \( a \) the procedure described. Then for obtained in that way from \( a \) the new monotone vector \( a^{(1)} \) (it differs from \( a \) only in components with indices \( i \) and \( i + 1 \)) we have \( a_{i+1}^{(1)} < a_i^{(1)}, f(i, a^{(1)}, b) < f(i, a, b) \) and \( f(i + 1, a, b) < f(i + 1, a^{(1)}, b) \). We also have \( \beta(x, a) \leq \beta(x, a^{(1)}) \) for any \( x \) (due to Lemma).

As an initial index \( i \) we may set, for example, \( i = i_1(a, b) \). Then apply the procedure described to the received vector \( a^{(1)} \) (i.e. find a new index \( i_1(a^{(1)}, b) \), corresponding component \( a_i^{(1)} \) and transform components \( a_{i+1}^{(1)} \) and \( a_{i+1}^{(1)} \), such that three conditions above are satisfied). It will give a new monotone vector \( a^{(2)} \). Then apply that procedure to the vector \( a^{(2)} \) and so on.

As a result, we may get a sequence of monotone vectors \( a^m, m = 1, 2, \ldots, \) converging to the monotone vector \( a^0 \). Let \( k_1, 1 \leq k_1 \leq i_0(a, b) \) - minimal of indices \( i \), which were
used on all stages of getting the sequence \( \{a^m\} \). Then for the limiting monotone vector \( a^0 \) the following conditions are satisfied:

\[
\begin{aligned}
&f(i, a^0, b) = f(k_1, a^0, b), \quad k_1 \leq i \leq n, \\
&f(i, a^0, b) < f(k_1, a^0, b), \quad 1 \leq i \leq k_1 - 1,
\end{aligned}
\]

\( (7) \)

\[
\prod_{i=k_1}^{n} a_i^0 = \prod_{i=k_1}^{n} a_i,
\]

i.e. the function \( f(i, a^0, b) \) is constant for \( k_1 \leq i \leq n \). Components \( \{a_i^0, i = 1, \ldots, k_1 - 1\} \) of the vector \( a^0 \) coincide with corresponding components \( \{a_i, i = 1, \ldots, k_1 - 1\} \) of the initial vector \( a \) (they will not participate in getting the vector \( a^0 \)).

We find the value \( f(k_1, a^0, b) \). Since

\[
a_i^0 = \frac{b_i}{1 - f(k_1, a^0, b)b_i}, \quad k_1 \leq i \leq n,
\]

then due to the last of conditions (7) values \( \{f(i, a^0, b)\} \) satisfy also equations

\[
\prod_{i=k_1}^{n} a_i^0 = \prod_{i=k_1}^{n} \frac{b_i}{1 - f(k_1, a^0, b)b_i} = \prod_{i=k_1}^{n} a_i,
\]

or, equivalently

\[
\sum_{i=k_1}^{n} \ln \frac{a_i}{b_i} + \sum_{i=k_1}^{n} \ln[1 - f(k_1, a^0, b)b_i] = 0. \tag{8}
\]

Define the value \( T(k, a, b) \) as the unique root of the equation

\[
\sum_{i=k}^{n} \ln \frac{a_i}{b_i} + \sum_{i=k}^{n} \ln[1 - T(k, a, b)b_i] = 0. \tag{9}
\]

Then (see (8) and (7)) \( T(k_1, a, b) = T(k_1, a^0, b) = f(k_1, a^0, b) \). Note that if \( \sum_{i=k_1}^{n} \ln(a_i/b_i) > 0 \), then \( T(k_1, a, b) > 0 \), and if \( \sum_{i=k_1}^{n} \ln(a_i/b_i) < 0 \), then \( T(k_1, a, b) < 0 \).

The index \( k_1(a, b), 1 \leq k_1 \leq i_0(a, b) \) is defined as follows

\[
k_1(a, b) = \min \{k : f(k, a, b) \geq T(k, a, b)\} = \max \{k : f(k - 1, a, b) < T(k, a, b)\}. \tag{10}
\]

Consider now first \( k_1 - 1 \) coordinates \( a_i^0, i = 1, \ldots, k_1 - 1 \) of the vector \( a^0 \). Again we find for them corresponding values \( i_0', i_1', k_1' \) and replace coordinates \( a_i^0, i = k_1', \ldots, k_1 - 1 \) by corresponding coordinates \( a_i^{0'}, i = k_1', \ldots, k_1 - 1 \), such that the function \( f(i, a^0, b) \) is constant for \( k_1' \leq i \leq k_1 - 1 \). Continuing that process, we get the monotone vector \( a^{(1)} \),
such that $\beta(x, a) \leq \beta(x, a^{(0)})$ for any $x$. Moreover, the function $f(i, a^{(0)}, b)$ is piecewise constant and does not decrease in $i$.

Change vector $b$. Similarly for fixed $a$ we may sequentially change the vector $b$ (but in opposite direction), again using the index $i_1(a, b)$ and getting the sequence of monotone vectors $\{b^{(i)}\}$, $i = 1, 2, \ldots$, such that $\beta(x, b) \geq \beta(x, b^{(1)}) \geq \beta(x, b^{(2)}) \geq \ldots$ for any $x$ (because of Lemma). As in the case of vector $a$, assume that $i_1(a, b) \leq n - 1$. Then $f(i_1 + 1, a, b) < f(i_1, a, b)$ and $b_{i_1 + 1} < b_{i_1}$. Apply to $b$ the procedure described and get new monotone vector $b^{(1)}$ (it differs from $b$ only in components with indices $i$ and $i + 1$).

We have for it $b^{(1)}_{i_1 + 1} < b^{(1)}_i$, $f(i_1, a, b^{(1)}) < f(i_1, a, b)$ and $f(i_1 + 1, a, b) < f(i_1 + 1, a, b^{(1)})$. We also have $\beta(x, b^{(1)}) \leq \beta(x, b)$ for any $x$ (because of Lemma).

Then apply the procedure described to the obtained vector $b^{(1)}$ (i.e. find the new index $i_1(a, b^{(1)})$ and corresponding component $b^{(1)}_i$ and transform components $b^{(1)}_i$ and $b^{(1)}_{i_1 + 1}$, such that three conditions above are satisfied). It will give the vector $b^{(2)}$. Then apply that procedure to the vector $b^{(2)}$ and so on.

As a result, we may get a sequence of monotone vectors $b^m$, $m = 1, 2, \ldots$, converging to the monotone vector $b^0$. Let $k_2$, $1 \leq k_2 \leq i_1(a, b)$ - minimal of indices $i_1$, used on all stages of getting the sequence $\{b^n\}$. Then for the limiting monotone vector $b^0$ the following conditions, similar to (7), are satisfied:

\[
\begin{align*}
 f(i, a, b^{(0)}) &= f(k_2, a, b^{(0)}), \quad k_2 \leq i \leq n, \\
 f(i, a, b^{(0)}) &< f(k_2, a, b^{(0)}), \quad 1 \leq i \leq k_2 - 1, \\
 \prod_{i=1}^{n} b_i^{(0)} &= \prod_{i=1}^{n} b_i,
\end{align*}
\]

i.e. the function $f(i, a, b^{(0)})$ is constant for $k_2 \leq i \leq n$. Components $\{b^{(0)}_i, i = 1, \ldots, k_2 - 1\}$ of the vector $b^{(0)}$ coincide with corresponding components $\{b_i, i = 1, \ldots, k_2 - 1\}$ of the initial vector $b$ (they will not participate in getting the vector $b^0$). Now if $\beta(x, a) \leq \beta(x, b^{(0)})$, then $\beta(x, a) \leq \beta(x, b)$.

Since

\[
b_i^{(0)} = \frac{a_i}{1 + f(i, a, b^{(0)})a_i}, \quad k_2 \leq i \leq n,
\]

then due to the last of conditions (11) values $\{f(i, a, b^{(0)})\}$ satisfy also equations

\[
\prod_{i=k_2}^{n} b_i^{(0)} = \prod_{i=k_2}^{n} \frac{a_i}{1 + f(k_2, a, b^{(0)})a_i} = \prod_{i=k_2}^{n} b_i,
\]

or, equivalently

\[
\sum_{i=k_2}^{n} \ln \frac{a_i}{b_i} - \sum_{i=k_2}^{n} \ln [1 + f(k_2, a, b^{(0)})a_i] = 0.
\]
Define the value $D(k, \mathbf{a}, \mathbf{b})$ as the unique root of the equation

$$
\sum_{i=k}^{n} \ln \frac{a_i}{b_i} - \sum_{i=k}^{n} \ln[1 + D(k, \mathbf{a}, \mathbf{b})a_i] = 0. \tag{12}
$$

Then (see [8] and [7]) $D(k_2, \mathbf{a}, \mathbf{b}) = D(k_2, \mathbf{a}, \mathbf{b}^{(0)}) = f(k_2, \mathbf{a}, \mathbf{b}^{(0)})$. Note that if

$$
\sum_{i=k_2}^{n} \ln(a_i/b_i) > 0, \quad \text{then} \quad D(k_2, \mathbf{a}, \mathbf{b}) > 0, \quad \text{and if} \quad \sum_{i=k_2}^{n} \ln(a_i/b_i) < 0, \quad \text{then} \quad D(k_2, \mathbf{a}, \mathbf{b}) < 0.
$$

The index $k_2(\mathbf{a}, \mathbf{b})$, $1 \leq k_2 \leq i_0(\mathbf{a}, \mathbf{b})$ can be defined as follows

$$
k_2(\mathbf{a}, \mathbf{b}) = \min \{k : f(k, \mathbf{a}, \mathbf{b}) \geq D(k, \mathbf{a}, \mathbf{b})\} = \max \{k : f(k - 1, \mathbf{a}, \mathbf{b}) < D(k, \mathbf{a}, \mathbf{b})\}. \tag{13}
$$

In order to formulate the result obtained note that similarly to Proposition 1 we may additionally introduce arbitrary vectors $\mathbf{c} \leq \mathbf{a}$ and $\mathbf{d} \geq \mathbf{b}$, such that $D(\mathbf{c}, \mathbf{d}) > 1$. Then we have

**Theorem 1.** Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n_+$ - monotone decreasing vectors. If $x$ satisfies condition

$$
x \leq \sup_{c \leq \mathbf{a}, d \geq \mathbf{b}, D(c,d) > 1} \left\{ \frac{2 \ln D(c, d)}{\min \{T(k_1, \mathbf{c}, \mathbf{d}), D(k_2, \mathbf{c}, \mathbf{d})\}} \right\}, \tag{14}
$$

(values $T(k, \mathbf{c}, \mathbf{d})$ and $D(k, \mathbf{c}, \mathbf{d})$ are defined in [10] and [12], and values $k_1(k, \mathbf{d})$ and $k_2(k, \mathbf{d})$ in [10] and [13]) then $\beta(x, \mathbf{a}) \leq \beta(x, \mathbf{b})$.

**Remark 3.** If

$$
\frac{1}{b_i} - \frac{1}{a_i} = \max \left( \frac{1}{b_i}, \frac{1}{a_i} \right),
$$

then $k_1(\mathbf{a}, \mathbf{b}) = k_2(\mathbf{a}, \mathbf{b}) = 1$ and

$$
T(k_1, \mathbf{a}, \mathbf{b}) = T(1, \mathbf{a}, \mathbf{b}), \quad D(k_2, \mathbf{a}, \mathbf{b}) = D(1, \mathbf{a}, \mathbf{b}).
$$

The value $T(k) = T(k, \mathbf{a}, \mathbf{b})$ can be upper bounded, for example, as follows. Using Jensen’s inequality $\mathbf{E} \ln \xi \leq \ln \mathbf{E} \xi$, from [9] we have

$$
\sum_{i=k}^{n} \ln \frac{a_i}{b_i} = - \sum_{i=k}^{n} \ln[1 - T(k)b_i] \geq -(n - k + 1) \ln \left[ 1 - \frac{T(k) \sum_{i=k}^{n} b_i}{(n - k + 1)} \right].
$$

Applying the inequality $\ln(1 + x) \leq x$, we get

$$
T(k, \mathbf{a}, \mathbf{b}) \leq \frac{(n - k + 1)}{\sum_{i=k}^{n} b_i} \left[ 1 - \left( \prod_{i=k}^{n} \frac{b_i}{a_i} \right)^{1/(n-k+1)} \right] \leq \frac{\sum_{i=k}^{n} \ln a_i}{\sum_{i=k}^{n} b_i}. \tag{15}
$$
For the value $D(k, a, b)$ those estimates work in opposite direction:

$$D(k, a, b) \geq \frac{\sum_{i=k}^{n} \ln a_i}{\sum_{i=k}^{n} a_i}.$$  

**Corollary 1.** If $a_1 \geq b_1$, the inequality holds

$$\sum_{i=1}^{n} \ln \frac{a_i}{b_i} \geq 0$$  \hspace{1cm} (16)

and any of the following conditions is valid

$$\sum_{i=1}^{n} \ln \frac{a_i}{b_i} + \sum_{i=1}^{n} \ln \left[ 1 - \left( \frac{1}{b_1} - \frac{1}{a_1} \right) b_i \right] \leq 0,$$  \hspace{1cm} (17)

$$\sum_{i=1}^{n} \ln \frac{a_i}{b_i} - \sum_{i=1}^{n} \ln \left[ 1 + \left( \frac{1}{b_1} - \frac{1}{a_1} \right) a_i \right] \leq 0,$$  \hspace{1cm} (18)

then $\beta(x, a) \leq \beta(x, b)$ for any $x$.

**Proof.** We set $c = a$ and $b = d$. Show that if $a_1 \geq b_1$, and conditions (16) and (17) are satisfied, then $k_1(a, b) = 1$. Indeed, due to (10) for that purpose it is sufficient to have

$$f(1, a, b) = \frac{1}{b_1} - \frac{1}{a_1} \geq T(1, a, b),$$

where $T = T(1, a, b) \geq 0$ (see (19)) – the unique root of the equation

$$F(T) = \sum_{i=1}^{n} \ln \frac{a_i}{b_i} + \sum_{i=1}^{n} \ln(1 - Tb_i) = 0.$$  \hspace{1cm} (19)

Since $F(0) \geq 0$ and $F'(T) < 0$, then the equation (19) has the unique root $T = T(1, a, b) \geq 0$. Moreover, if $F[f(1, a, b)] \leq 0$, then $f(1, a, b) \geq T(1, a, b)$ (and then $k_1(a, b) = 1$), which coincides with the condition (17). We also have

$$f(i, a^{(0)}, b) = \frac{1}{b_i} - \frac{1}{a_i^{(0)}} = \frac{1}{b_1} - \frac{1}{a_1^{(0)}} = T(1, a, b) \geq 0, \hspace{1cm} 1 \leq i \leq n.$$  \hspace{1cm} (20)

Hence $b \leq a^{(0)}$, and therefore $\beta(x, a) \leq \beta(x, a^{(0)}) \leq \beta(x, b)$ for any $x$.

Similarly we can show that if $a_1 \geq b_1$, and also conditions (16) and (18) are satisfied, then $k_2(a, b) = 1$, and therefore $\beta(x, a) \leq \beta(x, b^{(0)}) \leq \beta(x, b)$ for any $x$.  \hspace{1cm} ▲

The following result is also valid.

**Proposition 2.** Let $a, b \in \mathbb{R}^n_+$. If $\beta(x, a) \leq \beta(x, b)$ for any $x > 0$, then $\max_i a_i \geq \max_i b_i$.  \hspace{1cm} 9
Proof. It is sufficient to consider the case \( \max_{i} a_i = a_1, \max_{i} b_i = b_1 \), where \( b_1 > 0 \). Assume that \( a_1 < b_1 \). Show that then the inequality \( \beta(x, a) \leq \beta(x, b) \) can not hold for large \( x \). Denoting \( a_1 = (a_1, \ldots, a_1) \) and \( b_1 = (b_1, 0, \ldots, 0) \), note that \( \beta(x, a_1) \leq \beta(x, a) \) and \( \beta(x, b) \leq \beta(x, b_1) \). Therefore it is sufficient to show that for large \( x \) the inequality \( \beta(x, a_1) \leq \beta(x, b_1) \) does not hold. We may limit ourselves to the case \( b_1 = 1, a_1 < 1 \). Then the inequality, equivalent to (1), takes the form

\[
1 - \beta(x, a_1) = \mathbb{P} \left( a_1 \sum_{i=1}^{n} \xi_i^2 > x \right) \geq \mathbb{P} \left( \xi_1^2 > x \right) = 1 - \beta(x, b_1).
\] (21)

The left side of that formula can be bounded using exponential Chebychev inequality

\[
\mathbb{P} \left( a_1 \sum_{i=1}^{n} \xi_i^2 < x \right) \leq \exp \left\{ -\frac{1}{2} \left( n \ln \frac{na_1}{ex} + \frac{x}{a_1} \right) \right\}, \quad x \geq a_1 n,
\]

and therefore

\[
\ln \mathbb{P} \left( a_1 \sum_{i=1}^{n} \xi_i^2 < x \right) \leq -\frac{x + o(x)}{2a_1}, \quad x \to \infty.
\]

On the other hand, using the standard estimate

\[
\mathbb{P} \left( \xi_1^2 > x \right) = 2\mathbb{P} \left( \xi_1 > \sqrt{x} \right) \geq \frac{2\sqrt{x}}{\sqrt{2\pi(1 + x)}} e^{-x/2}, \quad x \geq 0,
\]

we have

\[
\ln \mathbb{P} \left( \xi_1^2 > x \right) \geq -\frac{x + o(x)}{2} > -\frac{x + o(x)}{2a_1} \geq \ln \mathbb{P} \left( a_1 \sum_{i=1}^{n} \xi_i^2 < x \right), \quad x \to \infty,
\]

from which it follows that the inequality (21) can not hold for large \( x \). ▲

From remark 3 and the estimate (15) we also get

**Corollary 2.** Let \( a, b \in \mathbb{R}^n_+ \) - monotone vectors, such that

\[
\frac{1}{b_i} - \frac{1}{a_i} = \max_{i} \left( \frac{1}{b_i} - \frac{1}{a_i} \right), \quad \prod_{i=1}^{n} a_i \geq \prod_{i=1}^{n} b_i.
\]

Then

\[
\beta(x, a) \leq \beta(x, b), \quad x \leq \sum_{i=1}^{n} b_i.
\]
4. Strengthening 2. For vectors $a, b \in \mathbb{R}^n_+$ we use the function $f(i, a, b)$ from (3). In each of vectors $a, b$ we change in some way numeration of their components $\{i\}$, such that the function $f(i, a, b)$ become monotone increasing in $i$ and, in particular,

$$\max_i f(i, a, b) = b_n^{-1} - a_n^{-1}.$$  

Of course, then the values $\beta(x, a), \beta(x, b)$ will not be changed. It is desirable to have the value $b_n^{-1} - a_n^{-1}$ as small as possible.

Then for $c = a, d = b$ the condition (4) takes the form

$$x \leq G(n, a, b) = \frac{1}{d} \sum_{i=1}^{n} \ln \frac{a_i}{b_i}, \quad d = b_n^{-1} - a_n^{-1}. \quad (22)$$

Note that if in (22) $\sum_{i=1}^{n} \ln(a_i/b_i) > 0$, then $d > 0$. Denote

$$t_1 = t_1(a, b) = \min\{i : b_i^{-1} - a_i^{-1} = b_n^{-1} - a_n^{-1}\}. \quad (23)$$

Then $f(i, a, b) = f(t_1, a, b) = b_{t_1}^{-1} - a_{t_1}^{-1}, \; t_1 \leq i \leq n$.

We try to increase the right-hand side of (22), increasing $b_i, \; t_1 \leq i \leq n$, but not changing $\{a_i\}$. Then the value $d$ will decrease. For some $\varepsilon \geq 0$ we set

$$(b'_i)^{-1} = b_i^{-1} - \varepsilon, \; i = t_1, \ldots, n; \quad b'_i = b_i, \; i < t_1, \quad b' = (b'_1, \ldots, b'_n).$$

Then

$$G(n, a, b') = \frac{1}{(d - \varepsilon)} \left[ \sum_{i=t_1}^{n-1} \ln \frac{a_i}{b_i} + \sum_{i=t_1}^{n} \ln \left[a_i(b^{-1}_i - \varepsilon)\right] \right]$$

and (since $\ln z \geq 1 - 1/z$)

$$G'_\varepsilon(n, a, b') = \frac{1}{(d - \varepsilon)^2} \left[ \frac{1}{(d - \varepsilon)} \sum_{i=t_1}^{n-1} \ln \frac{a_i}{b_i} + \sum_{i=t_1}^{n} \ln \left[a_i(b^{-1}_i - \varepsilon)\right] \right] - \frac{1}{(d - \varepsilon)} \sum_{i=t_1}^{n} \frac{1}{b_i^{-1} - \varepsilon}$$

$$= \frac{1}{(d - \varepsilon)^2} \left[ \sum_{i=1}^{n-1} \ln \frac{a_i}{b_i} + \sum_{i=t_1}^{n} \left( \ln \left[a_i(b^{-1}_i - \varepsilon)\right] - 1 + \frac{1}{a_i(b^{-1}_i - \varepsilon)} \right) \right] \geq$$

$$\geq \frac{1}{(d - \varepsilon)^2} \sum_{i=1}^{t_1-1} \ln \frac{a_i}{b_i}.$$  

Therefore if $\sum_{i=1}^{t_1-1} \ln \frac{a_i}{b_i} > 0$, then decrease all $b_i^{-1}, \; t_1 \leq i \leq n$, on the value $\varepsilon$, such that for the new vector $b' \geq b$ (and then $\beta(x, b') \leq \beta(x, b)$ for any $x$) we will have

$$f(i, a, b') = (b'_i)^{-1} - a_i^{-1} = f(t_1 - 1, a, b) = b_{t_1-1}^{-1} - a_{t_1-1}^{-1}, \quad t_1 - 1 \leq i \leq n,$$
i.e. the function $f(i, a, b')$ becomes constant for $t_1 - 1 \leq i \leq n$. For that purpose we set

$$
\varepsilon = b_n^{-1} - a_n^{-1} - (b_{t_1-1}^{-1} - a_{t_1-1}^{-1}) > 0
$$

and then get

$$
G(n, a, b') = \frac{1}{d'} \sum_{i=1}^{n} \ln \frac{a_i}{b_i'} > G(n, a, b), \quad d' = b_{t_1-1}^{-1} - a_{t_1-1}^{-1} > 0,
$$

$$(b_i')^{-1} = a_i^{-1} + b_{t_1-1}^{-1} - a_{t_1-1}^{-1}, \quad i = t_1, \ldots, n; \quad b_i' = b_i, \quad i = 1, \ldots, t_1 - 2.
$$

Similarly we repeat that procedure for the obtained vector $b'$. Denote

$$
t_2 = t_2(a, b) = t_1(a, b') = \min \{ i : b_i^{-1} - a_i^{-1} = b_{t_1-1}^{-1} - a_{t_1-1}^{-1} \}.
$$

Next, again if $\sum_{i=1}^{t_2-1} \ln \frac{a_i}{b_i} > 0$, then decrease all $(b_i')^{-1}$, $t_2 \leq i \leq n$ on the value $\varepsilon_1$, such that for the new vector $b'' \geq b'$ we have

$$
f(i, a, b'') = (b_i'')^{-1} - a_i^{-1} = f(t_2 - 1, a, b) = b_{t_2-1}^{-1} - a_{t_2-1}^{-1}, \quad t_2 - 1 \leq i \leq n,
$$

i.e. the function $f(i, a, b'')$ become constant for $t_2 - 1 \leq i \leq n$. For that purpose we set

$$
\varepsilon_1 = b_n^{-1} - a_n^{-1} - (b_{t_2-1}^{-1} - a_{t_2-1}^{-1}) > 0.
$$

Then we get

$$
G(n, a, b'') = \frac{1}{d''} \sum_{i=1}^{n} \ln \frac{a_i}{b_i''} > G(n, a, b'), \quad d'' = b_{t_2-1}^{-1} - a_{t_2-1}^{-1} > 0,
$$

$$(b_i'')^{-1} = a_i^{-1} + b_{t_2-1}^{-1} - a_{t_2-1}^{-1}, \quad i = t_2, \ldots, n; \quad b_i'' = b_i, \quad i = 1, \ldots, t_2 - 2.
$$

Repeating that process, we get the following result. Denote

$$
n_1 = n_1(a, b) = \min \left\{ m : \sum_{i=1}^{j} \ln \frac{a_i}{b_i} \geq 0, \quad j = m, \ldots, n \right\}. \quad (24)
$$

Then the following result holds (see [22]).

**Theorem 2.** Let $a, b \in \mathbb{R}_+^n$ and let the function $f(i, a, b)$ monotonically increases in $i$.

1) If $n_1 = n_1(a, b) \geq 2$ and $x$ satisfies the condition

$$
x \leq G(n, a, b), \quad (25)
$$

where

$$
G(n, a, b) = \frac{1}{d} \left( \sum_{i=1}^{n_1-1} \ln \frac{a_i}{b_i} + \sum_{i=n_1}^{n} \ln(1 + da_i) \right), \quad d = b_{n_1}^{-1} - a_{n_1}^{-1}, \quad (26)
$$

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then $\beta(x, a) \leq \beta(x, b)$.

2) If $n_1(a, b) = 1$, then $\beta(x, a) \leq \beta(x, b)$ for all $x$.

Explain only the statement 2 of that Theorem. If $n_1(a, b) = 1$, then $a_1 \geq b_1$. Since the function $f(i, a, b)$ monotonically increases in $i$, then $a_i \geq b_i$ for all $i$, and therefore $a \geq b$. Then from the statement 2 of Lemma we get $\beta(x, a) \leq \beta(x, b)$ for all $x$.

5. Examples. 1. Let $b = 1$. If $\prod_{i=1}^n a_i \geq 1$, then for any $x$ we have (see the statement 3 of Lemma) $\beta(x, a) \leq \beta(x, 1)$, i.e.

$$P \left( \sum_{i=1}^n a_i \xi_i^2 < x \right) \leq P \left( \sum_{i=1}^n \xi_i^2 < x \right).$$  \tag{27}

Compare results which give in that example [1, Theorem 1], Proposition 1 and Theorems 1, 2 of the paper.

If additionally the condition (2) is also satisfied then according to [1, Theorem 1] the inequality (27) holds for $x \geq 2n$.

According to Proposition 1 (i.e. the condition (4)) for $c = a$, $d = 1$, the inequality (27) holds for $x$, satisfying the condition

$$x \leq \frac{a_1}{(a_1 - 1)} \sum_{i=1}^n \ln a_i = \frac{2a_1 \ln D}{a_1 - 1}, \quad D^2 = \prod_{i=1}^n a_i > 1. \tag{28}$$

In order to apply Theorem 1 (i.e. the condition (14)) note that if $a$ - monotone decreasing vector, then the function $f(i, a, 1) = 1 - \frac{1}{a_i}$ also monotonically decreases in $i$, and therefore $k_1(a, 1) = 1$. Using formulas (9) and (10) we get

$$T(1, a, 1) = 1 - D^{-2/n}.$$  

Therefore if $D > 1$, then the condition (14) has the form

$$x \leq \frac{2 \ln D}{T(1, a, 1)} = \frac{2 \ln D}{1 - D^{-2/n}} = n f \left( D^{2/n} \right), \tag{29}$$

where

$$f(z) = \frac{z \ln z}{z - 1}, \quad f'(z) > 0, \quad z > 1.$$  

The function $f(z)$ monotonically increases from $f(1) = 1$ up to $f(\infty) = \infty$. In particular, if $D \geq 1$, then the inequality (29) holds for $x \leq n$. The condition (29) may be much better than the condition (28).

Theorem 2 (i.e. the condition (25)) may give result better than (29), and worse than it as well (depending on the vector $a$).

Next example gives the less obvious inequality, opposite to (27).

2. Let $a = 1$, $b_1 > 1$, $b_1 > b_2 \geq \ldots \geq b_n$ and $b_1 b_2 < 1$. Theorem 1 from [1] is not applicable here, since the condition (2) is not fulfilled. The function $f(i, 1, b) = 1/b - 1$
monotonically increases in $i$ and $\max_i f(i, 1, b) = 1/b_n - 1$. Therefore Theorem 1 does not improve the estimate \( (4) \). If $\sum_i \ln b_i < 0$, then from \( (4) \) we get

$$
P \left( \sum_{i=1}^{n} \xi_i^2 < x \right) \leq P \left( \sum_{i=1}^{n} b_i \xi_i^2 < x \right), \quad x \leq \frac{b_n}{(1-b_n)} \sum_{i=1}^{n} \ln \frac{1}{b_i}. \quad (30)$$

In order to apply Theorem 2 (i.e. the condition \( (25) \)) notice that $n_1 = n_1(a, b) = 2$, $d = 1/b_2 - 1$. Therefore we get

$$
P \left( \sum_{i=1}^{n} \xi_i^2 < x \right) \leq P \left( \sum_{i=1}^{n} b_i \xi_i^2 < x \right), \quad x \leq \frac{b_2}{(1-b_2)} \left[ \ln \frac{1}{b_1} + (n-1) \ln \frac{1}{b_2} \right]. \quad (31)$$

The condition \( (31) \) may be much broader than \( (30) \).

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