Quantization of the Derivative Nonlinear Schrödinger Equation

Diptiman Sen

Department of Physics and Astronomy, McMaster University, Hamilton, Ontario L8S 4M1, Canada
and Centre for Theoretical Studies, Indian Institute of Science, Bangalore 560012, India

Abstract

We study the quantum mechanics of the derivative nonlinear Schrödinger equation which has appeared in many areas of physics and is known to be classically integrable. We find that the \( N \)-body quantum problem is exactly solvable with both bound states (with an upper bound on the particle number) and scattering states. Quantization provides an alternative way to understand various features of the classical model, such as chiral solitons and two-soliton scattering.

PACS numbers: 11.10.Lm, 03.65.Ge

---

\(^1\)Electronic address: diptiman@physun.physics.mcmaster.ca
The derivative nonlinear Schrödinger equation (DNLS) in one dimension has historically found applications in many areas of physics, one example being circularly polarized nonlinear Alfvén waves in a plasma \[1\]. Related models have recently received fresh attention in the context of chiral Luttinger liquids; some of these models can be obtained by a dimensional reduction of a Chern-Simons model defined in two dimensions \[2-5\]. The DNLS has some peculiarities, for instance, it is not Galilean invariant, and it has classical solitons which have an upper bound on the particle number and are chiral (with a particular sign of the momentum). In these respects, the DNLS differs from the usual nonlinear Schrödinger equation (NLS), although both of them are classically integrable (see Refs. \[6, 7\] and references therein). Unlike the DNLS, the usual NLS has been studied semiclassically and quantum mechanically in great detail \[8, 9\]; being equivalent to a Bose gas with a two-body attractive \(\delta\)-function interaction \[10, 11\]. These studies have led to an alternative understanding of various features of the classical NLS, such as the solitons and their scattering.

It therefore seems interesting to similarly analyze a quantum mechanical version of the DNLS. We do so in this Letter generalizing the analysis of the two-body problem given in Refs. \[2-5\]. Besides providing us with a new exactly solvable \(N\)-body quantum mechanical problem, our study leads to a different and perhaps simpler way of deriving various properties of the DNLS, such as the peculiar features of the solitons and the time delay in the scattering of two solitons.

We begin by considering a general Lagrangian density of the form

\[
\mathcal{L} = i\hbar \psi^* \partial_t \psi - \frac{\hbar^2}{2m} \left[ \partial_x \psi^* \partial_x \psi + i\lambda \rho (\psi^* \partial_x \psi - \partial_x \psi^* \psi) + 2\mu \rho^2 + \nu \rho^3 \right],
\]

where \(\rho = \psi^* \psi\) is the density with dimensions of inverse length. We have introduced Planck’s constant \(\hbar\) in Eq. (1) for later convenience, although we will first discuss classical mechanics. The usual NLS is obtained by putting \(\lambda = \nu = 0\) in (1), while \(\mu = \nu = 0\) produces the DNLS. Note that the \(\lambda\) term is not invariant under Galilean transformations \[3, 4\], and it flips sign under parity \((x \to -x)\). It is therefore sufficient to consider the case \(\lambda \geq 0\). The parameters \(\lambda\) and \(\nu\) are dimensionless while \(\mu\) has the dimensions of inverse length; the system is scale invariant if we set \(\mu = 0\).

We will only examine the case \(\nu = 0\) for the following reasons. Firstly, it is known that the Euler-Lagrange equations of motion which follow from
Eq. (1) are classically integrable if $\nu = 0$, regardless of the values of $\lambda$ and $\mu$ [6, 7]. Secondly, it is not clear to us how to handle the $\nu \rho^3$ term in (1) quantum mechanically. This piece would lead to a term like $\rho^2 \Psi$ in the Schrödinger equation, where $\Psi$ is the wave function. If the positions of the particles are denoted by $x_1, x_2, \ldots, x_N$, the density operator is given by $\rho(x_i) = \sum_j \delta(x_i - x_j)$. Hence $\rho^2$ contains highly singular terms like $\delta^2(x_i - x_j)$ as well as three-body terms like $\delta(x_i - x_j) \delta(x_i - x_k)$. (A possible way of interpreting $\delta^2(x_i - x_j)$ is as the limit of a two-body $\delta$-function interaction with infinite strength; this forces the wave function to vanish whenever the coordinates of two particles coincide [3]. However the problem then becomes that of hard core bosons or free fermions in one dimension which is easily solvable). We can avoid such singular interactions by setting $\nu = 0$. This gives us the DNLS with a slight generalization if $\mu$ is nonzero.

For simplicity, we will set $\mu = 0$ in Eq. (1) to start with. Later we will reintroduce $\mu$ and will comment on an interesting phenomenon which occurs in that case. We now briefly summarize the classical one-soliton solution of (1). The equation of motion is

$$i \hbar \partial_t \psi = - \frac{\hbar^2}{2m} \partial_x^2 \psi + \frac{i 2 \hbar^2 \lambda}{m} \rho \partial_x \psi .$$

(2)

The density satisfies the continuity equation $\partial_t \rho + \partial_x j = 0$, where

$$j = - \frac{i \hbar}{2m} (\psi^* \partial_x \psi - \partial_x \psi^* \psi) - \frac{\hbar \lambda}{m} \rho^2 .$$

(3)

Since the system is integrable, there are an infinite number of conserved quantities [3]. Three of these are the particle number, momentum and energy,

$$N = \int dx \, \rho ,$$

$$P = - \frac{i \hbar}{2} \int dx \, (\psi^* \partial_x \psi - \partial_x \psi^* \psi) ,$$

$$E = \frac{\hbar^2}{2m} \int dx \, [ \partial_x \psi^* \partial_x \psi + i \lambda \rho (\psi^* \partial_x \psi - \partial_x \psi^* \psi) ] .$$

(4)

With $\lambda > 0$, the one-soliton solution takes the form [3, 8]

$$\psi(x,t) = \sqrt{\rho(x,t)} \exp \left[ i \left( kx - \omega t + \lambda \int_{-\infty}^{x} dy \, \rho(y,t) \right) \right] ,$$

3
\[
\rho(x,t) = \frac{\alpha^2}{2\lambda(\sqrt{\alpha^2 + 4k^2} \cosh \alpha(x - vt) + 2k)},
\]
\[
k = \frac{mv}{\hbar} \quad \text{and} \quad \omega = \frac{mv^2}{2\hbar} - \frac{\hbar \alpha^2}{8m}.
\]
These expressions contain two independent parameters, the velocity \(v\) and the inverse width of the soliton \(\alpha\). On computing the three conserved quantities, we find that \(0 < N\lambda < \pi\), and
\[
\cos N\lambda = \frac{2k}{\sqrt{\alpha^2 + 4k^2}},
\]
\[
M = \frac{m}{\lambda} \tan N\lambda,
\]
with \(P = Mv\) and \(E = P^2/2M\). We observe that \(P\) is always positive, although \(M\), \(v\) and \(E\) are positive if \(0 < N\lambda < \pi/2\) and negative if \(\pi/2 < \cos N\lambda < \pi\). \((P\) would have been negative if we had chosen \(\lambda < 0\)).

We now study this system quantum mechanically, and show that the classical picture is recovered in the limit \(N \to \infty\). The Schrödinger equation describing \(N\) identical bosons is easily derived \([3, 5]\) after defining the wave function
\[
\Psi(x_1, x_2, \ldots, x_N, t) = \langle 0 | \hat{\Psi}(x_1, t) \hat{\Psi}(x_2, t) \ldots \hat{\Psi}(x_N, t) | N \rangle,
\]
where \(\hat{\Psi}(x, t)\) denotes the second-quantized bosonic field operator. We then obtain the equation
\[
i\hbar \frac{\partial}{\partial t} \Psi = -\frac{\hbar^2}{2m} \sum_i \partial^2_{x_i} \Psi + \frac{i\hbar^2 \lambda}{m} \sum_{i<j} \delta(x_i - x_j)(\partial_{x_i} + \partial_{x_j})\Psi,
\]
where we have normal ordered the Hamiltonian to eliminate divergent self-interaction terms like \(\delta(x_i - x_i) = \delta(0)\). We thus find a two-body \(\delta\)-function interaction whose strength depends on the total momentum of the two particles. Note that we do not need to worry about the relative ordering of \(\partial_{x_i} + \partial_{x_j}\) and \(\delta(x_i - x_j)\) since they commute; thus the \(\delta\)-function interaction does not affect the total momentum of the two particles.

Before proceeding further, it is instructive to examine the two-particle problem. For the configuration \(x_1 < x_2\), let us assume a stationary solution of the form
\[
\Psi = \exp \left[ \frac{i}{\hbar}(c_1x_1 + c_2x_2) \right] + a \exp \left[ \frac{i}{\hbar}(c_2x_1 + c_1x_2) \right],
\]
where $c_1$ and $c_2$ may be complex, and $a$ will be called the reflection amplitude. The wave function for $x_2 < x_1$ is then obtained by Bose symmetry. For convenience, we will refer to the exponentials in Eq. (9) as waves and the $c_n$’s as the particle momenta even if the $c_n$’s are not real. The total momentum of (9) is $P = c_1 + c_2$. We can now go to the center of mass and relative coordinates $X = (x_1 + x_2)/2$ and $x = x_1 - x_2$, and factor out the center of mass wave function. The relative motion is described by the equation

$$i\hbar \partial_t \Psi = \left[ -\frac{\hbar^2}{m} \frac{\partial^2}{\partial x^2} - \frac{\hbar^2 \lambda P}{m} \delta(x) + \frac{P^2}{4m} \right] \Psi . \tag{10}$$

We then find that the amplitude $a$ in Eq. (9) is given by

$$a = \frac{c_2 - c_1 - i\lambda(c_2 + c_1)}{c_2 - c_1 + i\lambda(c_2 + c_1)} . \tag{11}$$

Clearly, there is a bound state if $a = 0$ and $P > 0$.

We now seek a bound state solution of the $N$-particle Schrödinger equation (8),

$$\Psi = \exp \left[ \frac{i}{\hbar} \sum_n c_n x_n \right] \tag{12}$$

for the configuration $x_1 < x_2 < \ldots < x_N$; the wave function for all other configurations can be deduced by Bose symmetry. We find that (12) satisfies the boundary conditions of the $\delta$-function interactions if the reflection amplitude given in Eq. (11) is zero for each neighboring pair of particles, i.e.,

$$c_{n+1} - c_n = i\lambda (c_{n+1} + c_n) \tag{13}$$

for $n = 1, 2, \ldots, N - 1$. (It is because of the absence of the reflected waves that our wave function (12) has only one wave, instead of $N!$ as in the general Bethe ansatz). We now demand that the total momentum should be equal to $P = \sum_n c_n$, and discover that the $c_n$’s are given by

$$c_n = P \frac{\sin \theta}{\sin N\theta} \exp \left[ i(2n - N - 1)\theta \right] , \tag{14}$$

where

$$\exp (i2\theta) = \frac{1 + i\lambda}{1 - i\lambda} . \tag{15}$$
The parameter θ lies in the range \([0, \pi/2]\) for \(\lambda > 0\). The energy is given by

\[ E = \sum_n c_n^2 / 2m = P^2 / 2M, \]

where

\[ M = m \frac{\tan N\theta}{\tan \theta}. \tag{16} \]

We now ask, when do Eqs. (12) and (14) describe a bound state, i.e., a state which is normalizable if we use translation invariance to fix the centre of mass at some particular place? It is easy to show that the necessary and sufficient condition for this is that

\[ P \frac{\sin \theta}{\sin N\theta} \sum_{n=1}^{l} \sin(N + 1 - 2n)\theta = P \frac{\sin(N - l)\theta \sin l\theta}{\sin N\theta} > 0, \tag{17} \]

for all values of \(l\) from 1 to \(N/2\) if \(N\) is even and to \((N - 1)/2\) if \(N\) is odd. The conditions in (17) arise on demanding that the probability \(\Psi^*\Psi\) should go to zero if we take the \(l\) particles on the right \((x_{N+1-l}, x_{N+2-l}, \ldots, x_N)\) to ∞ and the \(l\) particles on the left \((x_1, x_2, \ldots, x_l)\) to \(-∞\) (thereby keeping the centre of mass fixed). For small values of \(N\), we can check whether or not Eqs. (17) are satisfied; we find an intricate pattern of allowed values of \(\theta\) and \(P\) (which sometimes has to be negative) for which a bound state exists. The situation simplifies in the limit \(N \to ∞\) and \(\theta \to 0\) keeping \(N\theta\) fixed. We then find a bound state if and only if \(N\theta < \pi\) and \(P > 0\). Further, in this limit, \(\theta = \lambda\) so that the classical and quantum formulae (6) and (16) for the masses agree; we therefore identify the quantum bound state with the classical soliton. (It would be interesting to derive the soliton profile in Eq. (5) from the wave function in (12) using a technique given in Ref. [8]). We thus see that only a finite number of particles can be bound for a given value of \(\theta\). It is quite remarkable that if \(\pi/2 < N\theta < \pi\), the energy of the bound state can be lowered arbitrarily by giving it sufficiently large momentum.

Next we study the scattering of bound states. As a warmup exercise, consider the scattering of a bound object of \(N - 1\) particles with momentum \(P - p\) with one particle of momentum \(p\), where \(p\) is real. For \(x_1 < \ldots < x_N\), we find that the wave function can be written as a superposition of \(N\) waves,

\[ \Psi = \exp \left[ \frac{i}{\hbar} (c_1 x_1 + \ldots + c_{N-1} x_{N-1} + px_N) \right] + A \exp \left[ \frac{i}{\hbar} (px_1 + c_1 x_2 + \ldots + c_{N-1} x_N) \right] \]
\[ + \sum_{n=1}^{N-2} a_n \exp \left[ \frac{i}{\hbar} (c_1 x_1 + \ldots + c_n x_n + px_{n+1} + \ldots + c_{N-1} x_N) \right] \]  (18)

where the \( c_n \)'s are given by Eq. (14) with \( N \) and \( P \) replaced by \( N - 1 \) and \( P - p \) respectively in that equation. The first two waves in (18) correspond respectively to configurations in which the bound state is entirely to the left and entirely to the right of the free particle; \( A \) is therefore the transmission amplitude for the particle to go through the bound object. We now use Eq. (11) repeatedly to relate all the \( a_n \)'s and \( A \) to each other. \( A \) is found to be a pure phase of the form

\[ A = \frac{C}{C^*}, \]

where

\[ C = \prod_{n=1}^{N-1} \left( \frac{p}{\sin \theta} e^{-i \theta} - \frac{P - p}{\sin(N - 1) \theta} e^{i(2n-N+1) \theta} \right). \]  (19)

We now examine the general scattering of two bound objects (solitons), and again discover that there is only transmission and no reflection. We consider one object with \( N_1 \) particles and momentum \( P_1 \) and another object with \( N_2 \) particles and momentum \( P_2 \); let \( N_1 + N_2 = N \). We introduce \( N_1 \) numbers \( c_n^{(1)} \) satisfying Eq. (14) (with \( N \), \( P \) replaced by \( N_1 \), \( P_1 \)), and \( N_2 \) numbers \( c_n^{(2)} \) satisfying (14) with \( N_2 \), \( P_2 \). For a given configuration \( x_1 < x_2 < \ldots < x_N \), we then find that the wave function is a superposition of several waves, each wave having the particle momenta as some permutation of the numbers \( c_n^{(1)} \) and \( c_n^{(2)} \). We do not get all the \( N! \) possible permutations due to the absence of reflection within the momenta \( c_n^{(1)} \)'s, or within the \( c_n^{(2)} \)'s, as discussed around Eq. (13). The permutations allowed are those in which particles \( i \) and \( j \) can have the momenta \( c_n^{(1)} \) and \( c_n^{(1)} \) (or momenta \( c_n^{(2)} \) and \( c_n^{(2)} \) only if \( i < j \). The number of such permutations is \( N!/N_1!N_2! \). Two of these describe configurations in which the object with \( N_1 \) particles is entirely to the left (or to the right) of the object with \( N_2 \) particles; all the other permutations describe configurations in which the \( N_1 \) particles are interspersed amongst the \( N_2 \) particles. Thus the wave function has the form

\[ \Psi = \exp \left[ \frac{i}{\hbar} \left( \sum_{n=1}^{N_1} c_n^{(1)} x_n + \sum_{n=1}^{N_2} c_n^{(2)} x_{N_1+n} \right) \right] \]

\[ + A \exp \left[ \frac{i}{\hbar} \left( \sum_{n=1}^{N_2} c_n^{(2)} x_n + \sum_{n=1}^{N_1} c_n^{(1)} x_{N_2+n} \right) \right] \]

\[ + \text{all the other waves}, \]  (20)
where $A$ denotes the transmission amplitude for one bound state to pass through the other. We compute $A$ by successively passing each of the $N_2$ particles on the right through each of the $N_1$ particles on the left, and using the expressions in Eq. (11) at each such crossing. We finally discover that $A = \exp(i2\delta) = C/C^*$, where

$$C = \prod_{n=2}^{N_2} \left( \frac{P_2}{\sin N_2\theta} e^{i(2n-2-N_2)\theta} - \frac{P_1}{\sin N_1\theta} e^{iN_1\theta} \right) \cdot \prod_{n=1}^{N_1} \left( \frac{P_2}{\sin N_2\theta} e^{-iN_2\theta} - \frac{P_1}{\sin N_1\theta} e^{i(2n-N_1)\theta} \right).$$ (21)

Let us consider the case in which the velocities $v_i = (2E_i/M_i)^{1/2}$ satisfy $v_1 > v_2 > 0$. For weak coupling ($\theta \to 0$ with $N_1, N_2$ held fixed, so that $M_i = mN_i$), we find the phase shift

$$2\delta = 2\theta N_1N_2 \sqrt{E_1/M_1 + E_2/M_2} / \sqrt{E_1/M_1 - E_2/M_2} + O(\theta^2).$$ (22)

From this, we can compute the time delay suffered by the bound state with energy $E_1$; thus $\Delta t_1 = 2\hbar \partial / \partial E_1$ [12]. The expression in (22) and the resultant time delay $\Delta t_1$ agree with the results in Ref. [5] if we set $N_1 = N_2$.

Let us now include the term proportional to $\mu$ in Eq. (1). This adds $(2\hbar^2 \mu/m) \sum_{i<j} \delta(x_i - x_j)$ to the right hand side of the Schrödinger equation (8). For a bound state of $N$ particles with momentum $P$, we find that the numbers $c_n$ in Eq. (12) are given by

$$c_n = \left( P - \frac{N\hbar \mu}{\tan \theta} \right) \frac{\sin \theta}{\sin N\theta} \exp \left[ i(2n - N - 1)\theta \right] + \frac{\hbar \mu}{\tan \theta}. \quad (23)$$

As before, we find that in the limit $\theta \to 0$ with $N\theta$ held fixed, the bound state exists only if $N\theta < \pi$; in addition, we need $P > N\hbar \mu / \tan \theta$. More interestingly, we observe that the energy $E = \sum_n c_n^2 / 2m$ is minimum at a nonzero value of the momentum given by

$$P_0 = \frac{N\hbar \mu}{\tan \theta} \left( 1 - \frac{\tan N\theta}{N \tan \theta} \right) \quad (24)$$

if $\mu \tan N\theta < 0$. For weak coupling ($N\theta < \pi/2$), we can understand this result as follows. We know that there is a zero momentum soliton if $\mu < 0$
(attractive interaction) even if $\theta = 0$ \cite{8,11}; the energy of this is given by

$$E = -\frac{\hbar^2 \mu^2}{6m} (N^3 - N), \quad (25)$$

as we can see by setting $P = 0$ and taking the limit $\theta \to 0$ in Eq. (23). Now if $\theta$ is small and positive, the strength of the attractive $\delta$-function interaction is increased if all the particles move with positive momentum; this lowers the energy by effectively increasing the value of $\mu^2$ in (25). Thus zero momentum is not the state of lowest energy if $\theta$ is nonzero and $\mu < 0$.

For completeness, we would like to mention the states in which all the particles have real momenta $p_n$. (However these purely scattering states are not the lowest energy states of our system). It is convenient to put the system on a circle with circumference $L$, and consider a particular ordering of the positions $0 < x_1 < \ldots < x_N < L$. The wave function is then given by the general Bethe ansatz with a superposition of $N!$ waves. Following Ref. \cite{10}, we impose periodic boundary conditions

$$\Psi(0, x_2, \ldots, x_N) = \Psi(x_2, \ldots, x_N, L). \quad (26)$$

We then find that the $p_n$'s are related to each other by the $N$ equations

$$\frac{p_n L}{\hbar} = 2\pi I_n + \pi (N - 1) + 2 \sum_{l=1}^{N} \tan^{-1} \left( \frac{p_n - p_l}{\lambda(p_n + p_l) - 2\hbar \mu} \right), \quad (27)$$

where the $I_n$'s are integers. These equations may be solved numerically.

Finally, we ask whether the system has a well-defined ground state in the thermodynamic limit $N, L \to \infty$ keeping $N/L$ fixed. Our earlier analysis indicates that the answer is no, even if $\mu$ is zero or even positive (repulsive). We have seen that a bound state with $N$ particles can have an arbitrarily low energy if its momentum is large and $\pi/2 < N\theta < \pi$. If $N$ is very large, the system can lower its energy arbitrarily by forming a number of large momentum bound objects with particle numbers $N_1, N_2, \ldots, N_k$ (adding up to $N$), such that $\pi/2 < N_i \theta < \pi$ for $i = 1, 2, \ldots, k$.

To conclude, the quantization of the DNLS has produced a rich structure. It would be interesting to consider other classically integrable systems and see if they can be quantized in order to shed new light on them.

I thank Rajat Bhaduri for discussions and the Department of Physics and Astronomy, McMaster University for its hospitality during the course of this
work. This research was supported by the Natural Sciences and Engineering Research Council of Canada.

Note Added:
After writing this paper, I learnt that similar work has been published earlier [13, 14]; I thank A. G. Shnirman for pointing this out.

References

[1] M. Wadati, H. Sanuki, K. Konno and Y.-H. Ichikawa, Rocky Mountain J. Math. 8, 323 (1978); Y.-H. Ichikawa and S. Watanabe, J. de Physique, 38, C6-15 (1977).

[2] S. J. Benetton Rabello, Phys. Rev. Lett. 76, 4007 (1996); (E) 77, 4851 (1996).

[3] U. Aglietti, L. Griguolo, R. Jackiw, S.-Y. Pi and D. Seminara, Phys. Rev. Lett. 77, 4406 (1996).

[4] R. Jackiw, preprint no. hep-th/9611185.

[5] H. Min and Q.-H. Park, preprint no. hep-th/9607242, to appear in Phys. Lett. B.

[6] D. J. Kaup and A. C. Newell, J. Math. Phys. 19, 798 (1978); H. H. Chen, Y. C. Lee and C. S. Liu, Phys. Scr. 20, 490 (1979).

[7] P. A. Clarkson and C. M. Cosgrove, J. Phys. A 20, 2003 (1987).

[8] C. Nohl, Ann. Phys. (N.Y.) 96, 234 (1976).

[9] P. P. Kulish, S. V. Manakov and L. D. Faddeev, Theor. Math. Phys. 28, 615 (1976); L. Dolan, Phys. Rev. D 13, 528 (1976).

[10] E. H. Lieb and W. Liniger, Phys. Rev. 130, 1605 (1963).

[11] J. B. McGuire, J. Math. Phys. 5, 622 (1964).
[12] R. Jackiw and G. Woo, Phys. Rev. D 12, 1643 (1975).

[13] A. G. Shnirman, B. A. Malomed and E. Ben-Jacob, Phys. Rev. A 50, 3453 (1994).

[14] Y. Lai and H. A. Haus, Phys. Rev. A 40, 844 and 854 (1989).