The evolution of complete non-compact graphs by powers of Gauss curvature

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Abstract. We prove the all-time existence of non-compact, complete, strictly convex solutions to the $\alpha$-Gauss curvature flow for any positive power $\alpha$.

1. Introduction

We consider a family of complete non-compact strictly convex hypersurfaces $\Sigma_t$ embedded in $\mathbb{R}^{n+1}$ which evolve by any positive power of their Gauss curvature $K$.

Given a complete and strict convex hypersurface $\Sigma_0$ embedded in $\mathbb{R}^{n+1}$, we let

$$F_0 : M^n \to \mathbb{R}^{n+1}$$

be an immersion with $F_0(M^n) = \Sigma_0$. For a given number $\alpha > 0$, we say that a one-parameter family of immersions $F : M^n \times (0, T) \to \mathbb{R}^{n+1}$ is a solution of the $\alpha$-Gauss curvature flow, if for each $t \in (0, T)$, $F(M^n, t) = \Sigma_t$ is a complete and strictly convex hypersurface embedded in $\mathbb{R}^{n+1}$, and $F(\cdot, t)$ satisfies

$$\frac{\partial}{\partial t} F(p, t) = K^\alpha(p, t) \bar{n}(p, t),$$

$$\lim_{t \to 0} F(p, t) = F_0(p),$$

where $K(p, t)$ is the Gauss curvature of $\Sigma_t$ at $F(p, t)$ and $\bar{n}(p, t)$ is the interior unit normal vector of $\Sigma_t$ at the point $F(p, t)$.
The evolution of compact hypersurfaces by the $\alpha$-Gauss curvature flow is widely studied. The case $\alpha = 1$ corresponds to the classical Gauss curvature flow which was first introduced by Firey in [14] and later studied by Tso in [21]. Other works include [1, 9–11, 15]. The power case was studied by Chow in [7] and also in the works [2–4, 16, 17].

In [11] the optimal regularity of viscosity solutions of (1.1), which are not necessarily strictly convex, was studied. These results are local and apply to both compact or non-compact settings.

In this work we will assume that the initial surface $\Sigma_0$ is a complete convex graph over a domain $\Omega \subset \mathbb{R}^n$, that is, $\Sigma_0 = \{(x, u_0(x)) : x \in \Omega\}$ for some function $u_0 : \Omega \to \mathbb{R}$. The domain $\Omega$ may be bounded or unbounded. In particular, the case where $\Sigma_0$ is an entire graph over $\mathbb{R}^n$ will be included. Since the regularity results of [11] apply in this setting as well, we will be mostly concerned here with the questions of existence rather than those of regularity and we have chosen to take a more classical approach to the problem and assume that the initial hypersurface $\Sigma_0$ is \emph{locally uniformly convex}. Our local a priori estimates will guarantee that the solution is locally uniformly convex for all $t > 0$ and therefore it becomes instantly smooth independently from the regularity of the initial data. In the case of a compact initial data this result was shown in [1, 3].

Ecker and Huisken in [12, 13] studied the evolution of entire graphs by mean curvature. In particular, it was shown in [13] that in some sense the mean curvature flow on \emph{entire graphs} behaves better than the heat equation on $\mathbb{R}^n$, namely an entire graph solution exists for all time independently from the growth of the initial surface at infinity. The initial entire graph is assumed to be only locally Lipschitz. This result is based on a clever local gradient estimate which is then combined with the evolution of $|A|^2$ (the square sum of the principal curvatures) to give a local bound on $|A|^2$, independent from the behavior of the solution at spatial infinity. The latter is achieved by adopting the well-known technique of Caffarelli, Nirenberg and Spruck [6] in this geometric setting. More recently, Sáez and Schnürer [19] showed the existence of complete solutions of the mean curvature flow for an initial hypersurface which is a graph $\Sigma_0 = \{(x, u_0(x)) : x \in \Omega_0\}$ over a bounded domain $\Omega_0$, and $u_0(x) \to +\infty$ as $x \to \partial \Omega_0$.

An open question between the experts in the field is whether the techniques of Ecker and Huisken in [12, 13] can be extended to the fully-nonlinear setting and in particular on entire convex graphs evolving by the $\alpha$-Gauss curvature flow. In this work we will show that this is the case as our main result shows.

**Theorem 1.1.** Let $\Sigma_0$ be a complete, non-compact, and locally uniformly convex (defined below) hypersurface embedded in $\mathbb{R}^{n+1}$. Then, given an immersion $F_0 : M^n \to \mathbb{R}^{n+1}$ such that $\Sigma_0 = F_0(M^n)$, and for any $\alpha \in (0, +\infty)$, there exists a complete, non-compact, smooth and strictly convex solution $\Sigma_t := F(M^n, t)$ of the $\alpha$-Gauss curvature flow (1.1) which is defined for all time $0 < t < +\infty$.

Since we have not assumed any regularity on our initial surface $\Sigma_0$, we give next the definition of a locally uniformly convex hypersurface which is not necessarily of class $C^2$.

**Definition 1.2** (Uniform convexity for a hypersurface). We denote by $C^2(\mathbb{R}^{n+1})$ the class of second-order differentiable complete (either closed or non-compact) hypersurfaces embedded in $\mathbb{R}^{n+1}$. For a convex hypersurface $\Sigma \in C^2(\mathbb{R}^{n+1})$, we denote by $\lambda_{\text{min}}(\Sigma)(X)$
the smallest principal curvature of \( \Sigma \) at \( X \in \Sigma \). Given any complete convex hypersurface \( \Sigma \) and a point \( X \in \Sigma \), we define \( \lambda_{\min}(\Sigma)(X) \) by

\[
\lambda_{\min}(\Sigma)(X) = \sup\{\lambda_{\min}(\Phi)(X) : X \in \Phi, \Phi \in C^2_c(\mathbb{R}^{n+1}), \Sigma \subset \text{the convex hull of } \Phi\}
\]

and we say that

(a) \( \Sigma \) is strictly convex, if \( \lambda_{\min}(\Sigma)(X) > 0 \) holds for all \( X \in \Sigma \);

(b) \( \Sigma \) is uniformly convex, if there is a constant \( m > 0 \) satisfying \( \lambda_{\min}(\Sigma)(X) \geq m \) for all \( X \in \Sigma \);

(c) \( \Sigma \) is locally uniformly convex, if for any compact subset \( A \subset \mathbb{R}^{n+1} \), there is a constant \( m_A > 0 \) satisfying \( \lambda_{\min}(\Sigma)(X) \geq m_A \) for all \( X \in \Sigma \cap A \).

To build the above existence theorem (Theorem 1.1), we will consider the hypersurface \( \Sigma_0 \) as the graph of a function by using the following theorem given in [23]; see Figure 1.

**Theorem 1.3** (Hung-Hsi Wu). Given a complete and locally uniformly convex hypersurface \( \Sigma_0 \) embedded in \( \mathbb{R}^{n+1} \), there exists a function \( u_0 : \Omega \to \mathbb{R} \) defined on a convex open domain \( \Omega \subset \mathbb{R}^n \) such that

(i) \( u_0 \) attains its minimum in \( \Omega \) and \( \inf_\Omega u_0 \geq 0 \);

(ii) if \( \Omega \neq \mathbb{R}^n \), then \( \lim_{x \to x_0} u_0(x) = +\infty \) for all \( x_0 \in \partial \Omega \);

(iii) if \( \Omega \) is unbounded, then \( \lim_{R \to +\infty} (\inf_{\Omega \setminus B_R(0)} u_0) = +\infty \).

In Section 5, we will show the all-time existence of a solution \( u : \Omega \times (0, +\infty) \to \mathbb{R} \) to the following parabolic Monge–Ampère equation:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{(\det D^2 u)^{\alpha}}{(1 + |Du|^2)^{(n+2)\alpha-1}}, \\
\lim_{t \to 0} u(x, t) &= u_0(x),
\end{align*}
\]

where each \( u(\cdot, t) \) satisfies the conditions in Theorem 1.3. This proves Theorem 1.1 because the one-parameter family of the graphs \( \Sigma_t \) of a solution \( u(\cdot, t) \) to (1.2) is a desired complete, non-compact and all-time existing solution to the \( \alpha \)-Gauss curvature flow (1.1).
Remark 1.4 (Translating solitons and asymptotic behavior). Urbas in [22] studies complete non-compact convex solutions of (1.1) which are self-similar. In particular, he shows that for any $\alpha > \frac{1}{2}$ and any convex bounded domain $\Omega \subset \mathbb{R}^n$, there exists a translating soliton solution of (1.1) which is a graph over $\Omega$. Conversely, these are the only complete non-compact convex solutions of (1.1) for $\alpha > \frac{1}{2}$ which move by translation. On the other hand, Urbas shows that for $\alpha \in (0, \frac{1}{2}]$ and any $\lambda > 0$ there exists an entire graph convex translating soliton solution of (1.1) which moves with speed $\lambda$. It would be interesting to see whether these translating solitons appear as asymptotic limits, as $t \to +\infty$, of the solutions $\Sigma_t$ to (1.1) given by Theorem 1.1 in the corresponding range of exponents.

Discussion on the proof. The proof of Theorem 1.1 mainly relies on two local a priori estimates: a local bound from below on the smallest principal curvature $\lambda_{\min}$ of the solution $\Sigma_t$, given in Theorem 3.3, and a local bound from above on the speed $K^\alpha$ of $\Sigma_t$, given in Theorem 4.1. Notice that the first estimate is used to control the speed $K^\alpha$. The first bound follows via a Pogolovelov type computation involving an appropriate cut-off function on $\Sigma_t$. The second bound uses the well-known technique by Caffarelli, Nirenberg and Spruck [6] which was also used by Ecker and Huisken in the context of the mean curvature flow in [13].

One of the challenges comes from the fact that the evolution equations of the position vector $F(p, t)$ and its curvature have the differential linearized operators since $K^\alpha$ is not homogeneous of degree one for $\alpha \neq \frac{1}{2}$. To be specific, while the linearized operator $L := \frac{\partial K^\alpha}{\partial n_i j} \nabla_i \nabla_j$ appears in the evolution of curvature (equations (2.7) and (2.6) below), the position vector $F(p, t)$, determining the evolution of our cut-off functions, satisfies the different equation $\frac{\partial}{\partial t} F = (na)^{-1} L F$. After the linearized operators are matched, the evolution of our cut-off function $\psi$ (equation (2.4)) has a reaction term which depends on the Gauss curvature. To establish local curvature estimates we carefully control the reaction terms which appear from the gap between the differential operators.

Another difficulty arises from the non-concavity of equation (1.1). In the most challenging case where $na \geq 1$, equation (1.1) is neither concave nor convex. In previous works which concern with the compact case (see [2, 4, 7, 16]), global estimates on the smallest principal curvature $\lambda_{\min}$ were shown by using the support function on $S^n$.

A brief outline of the paper is as follows: In Section 2 we derive some basic equations under the $\alpha$-Gauss curvature flow and also obtain our local gradient estimate. Sections 3 and 4 are devoted to our two crucial local a priori bounds, the lower bound on the principal curvatures and the upper bound on the speed $K^\alpha$. In Section 5 we establish the all-time existence of the flow which is done in two steps: first we show the existence of a complete solution $\Sigma_t$ on $t \in (0, T)$, where $T = T_\Omega$ depends on the domain $\Omega$. Then, we construct an appropriate barrier to guarantee that each $\Sigma_t$ remains a graph over the same domain $\Omega$, for all $t \in (0, T)$, implying that $T = +\infty$ independently from the domain $\Omega$.

1.1. Notation. For the convenience of the reader, we summarize the following notation which will be frequently used in what follows.

(i) We recall that $g_{ij} = \langle F_i, F_j \rangle$, where $F_i := \nabla_i F$. Also, we denote as usual by $g^{ij}$ the inverse matrix of $g_{ij}$ and set $F^i = g^{ij} F_j$.

(ii) Assume that $\Sigma_t$ is a strictly convex graph solution of (1.1) in $\mathbb{R}^{n+1}$. Then, we let $\tilde{u}(\cdot, t) : M^n \to \mathbb{R}$ denote the height function $\tilde{u}(p, t) = \langle F(p, t), \tilde{e}_{n+1} \rangle$. 

(iii) Given constants $M$ and $\beta \geq 0$, we define the cut-off function $\psi_{\beta}$ by

$$\psi_{\beta}(p, t) = (M - \beta t - \bar{u}(p, t))_+ = \max\{M - \beta t - \bar{u}(p, t), 0\}.$$ 

In particular, we denote $\psi_0 := (M - \bar{u}(p, t))_+$ by $\psi$ for convenience. Also, given a constant $R > 0$ and a point $Y \in \mathbb{R}^{n+1}$, we define the cut-off function $\overline{\psi}$ by

$$\overline{\psi}(p, t) = (R^2 - |F(p, t) - Y|^2)_+ = \max\{R^2 - |F(p, t) - Y|^2, 0\}.$$ 

(iv) For a strictly convex smooth hypersurface $\Sigma_t$, we denote by $b^{ij}$ the inverse matrix $(h^{-1})^{ij}$ of its second fundamental form $h_{ij}$, namely $b^{ij}h_{jk} = \delta^i_k$. Notice that the eigenvalues of $b^{ij}$ on an orthonormal frame are the principal radii of curvature.

(v) We denote by $\mathcal{L}$ the linearized operator

$$\mathcal{L} = \alpha K^\alpha b^{ij} \nabla_i \nabla_j.$$ 

Furthermore, $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ denotes the associated inner product $\langle \nabla f, \nabla g \rangle_{\mathcal{L}} = \alpha K^\alpha b^{ij} \nabla_i f \nabla_j g$, where $f, g$ are differentiable functions on $M^n$, and $\| \cdot \|_{\mathcal{L}}$ denotes the $\mathcal{L}$-norm given by the inner product $\langle \cdot, \cdot \rangle_{\mathcal{L}}$

(vi) We denote by $v = (\bar{n}, \bar{\nu}_{n+1})^{-1}$ the gradient function (as in [13]).

(vii) $H$ denotes the mean curvature.

2. Preliminaries

In this section we will derive some basic equations under the $\alpha$-Gauss curvature flow and also obtain a local gradient estimate. We begin by showing some basic evolution equations.

**Proposition 2.1.** Assume that $\Omega$ and $\Sigma_0$ satisfy the assumptions in Theorem 1.1 and let $\Sigma_t$ be a complete strictly convex graph solution of (1.1). Then, the following hold:

(2.1) \quad \partial_t g_{ij} = -2K^\alpha h_{ij}.

(2.2) \quad \partial_t g^{ij} = 2K^\alpha h^{ij},

(2.3) \quad \partial_t \bar{n} = -\nabla K^\alpha := -(\nabla_j K^\alpha)F^j.

(2.4) \quad \partial_t \psi = \mathcal{L} \psi + (n\alpha - 1) u^{-1} K^\alpha - \beta,

(2.5) \quad \partial_t \overline{\psi} \leq \mathcal{L} \overline{\psi} + 2(n\alpha + 1)(\lambda_{\min} - R)K^\alpha,

(2.6) \quad \partial_t h_{ij} = \mathcal{L} h_{ij} + \alpha K^\alpha (ab^{kl} b^{mn} - b^{km} b^{nl}) \nabla_i h_{mn} \nabla_j h_{kl}

$$+ \alpha K^\alpha H h_{ij} - (1 + n\alpha) K^\alpha h_{ik} h_{kj},$$

(2.7) \quad \partial_t K^\alpha = \mathcal{L} K^\alpha + \alpha K^{2\alpha} H.

(2.8) \quad \partial_t b^{pq} = \mathcal{L} b^{pq} - \alpha K^\alpha b^{ij} b^{pq} (ab^{kl} b^{mn} + b^{km} b^{nl}) \nabla_i h_{kl} \nabla_j h_{mn}

$$- \alpha K^\alpha H b^{pq} + (1 + n\alpha) K^\alpha g^{pq},$$

(2.9) \quad \partial_t v = \mathcal{L} v - 2v^{-1} \| \nabla v \|^2_{\mathcal{L}} - \alpha K^\alpha Hv.
Proof of (2.1). Observe
\[ \partial_t g_{ij} = \langle \partial_t \nabla_i F, \nabla_j F \rangle + \langle \nabla_i F, \partial_t \nabla_j F \rangle = \langle \nabla_i \partial_t F, \nabla_j F \rangle + \langle \nabla_i F, \nabla_j \partial_t F \rangle. \]
Hence, \( \langle \nabla_i F, \partial_t F \rangle = 0 \) gives
\[ \partial_t g_{ij} = -2 \langle \partial_t F, \nabla_i \nabla_j F \rangle = -2 \langle K^\alpha \bar{n}, h_{ij} \bar{n} \rangle = -2 K^\alpha h_{ij}. \]
\[ \square \]

Proof of (2.2). From \( g^{ij} g_{jk} = \delta^i_k \), we can derive
\[ \partial_t g^{ij} = -g^{ik} g^{jl} \partial_t g_{kl} = 2 K^\alpha g^{ik} g^{jl} h_{kl} = 2 K^\alpha h^{ij}. \]
\[ \square \]

Proof of (2.3). \( \langle \bar{n}, \bar{n} \rangle = 1 \) implies \( \langle \partial_t \bar{n}, \bar{n} \rangle = 0 \). Also, \( \langle \bar{n}, \nabla_i F \rangle = 0 \) leads to
\[ \langle \partial_t \bar{n}, \nabla_i F \rangle = -\langle \bar{n}, \partial_t \nabla_i F \rangle = -\langle \bar{n}, \nabla_i \partial_t F \rangle = -\langle \bar{n}, \nabla_i (K^\alpha \bar{n}) \rangle = -\nabla_i K^\alpha \]
from which (2.3) readily follows.
\[ \square \]

Proof of (2.4). The definition of the linearized operator \( \mathcal{L} := \alpha K^\alpha b^{ij} \nabla_i \nabla_j \) gives
\[ (2.10) \quad \mathcal{L} \bar{u} := \alpha K^\alpha b^{ij} \nabla_i \nabla_j \bar{u} = \alpha K^\alpha b^{ij} h_{ij} \bar{n} = n\alpha K^\alpha \bar{n} = (n\alpha) \partial_t F \]
which yields that \( \mathcal{L} \bar{u} = (n\alpha) \partial_t \bar{u} \). Therefore,
\[ \partial_t \psi_\beta = -\beta - \partial_t \bar{u} = -\mathcal{L} \bar{u} + (n\alpha - 1) \partial_t \bar{u} - \beta \]
holds on the support of \( \psi_\beta := (M - \beta t - \bar{u}(p,t))_+ \). Substituting for
\[ \partial_t \bar{u} = \langle \partial_t F, \bar{e}_{n+1} \rangle = (K^\alpha \bar{n}, \bar{e}_{n+1}) = K^\alpha v^{-1}, \]
where \( v := \langle \bar{n}, \bar{e}_{n+1} \rangle^{-1} \), yields (2.4).
\[ \square \]

Proof of (2.5). By (2.10), on the support of \( \overline{\psi} := (R^2 - |F - Y|^2)_+ \) we have
\[ \partial_t \overline{\psi} = -2 \langle (\mathcal{L} F - (n\alpha) \partial_t F) + \partial_t F, F - Y \rangle \]
\[ = -2 \langle \mathcal{L} F, F - Y \rangle + 2\langle n\alpha - 1 \rangle \langle K^\alpha \bar{n}, F - Y \rangle \]
\[ \leq \mathcal{L} \overline{\psi} + 2 \| \nabla F \|^2_{L^2} + 2 K^\alpha |n\alpha - 1| \| F - Y \| \]
\[ \leq \mathcal{L} \overline{\psi} + 2 n\alpha \lambda_{\min}^{-1} K^\alpha + 2(\alpha + 1) R K^\alpha \]
\[ \leq \mathcal{L} \overline{\psi} + 2(n\alpha + 1)(\lambda_{\min}^{-1} + R) K^\alpha. \]
\[ \square \]

Proof of (2.6). We have
\[ \partial_t h_{ij} = \partial_t \langle \nabla_i \nabla_j F, \bar{n} \rangle \]
\[ = \langle \nabla_i \nabla_j \partial_t F, \bar{n} \rangle + \langle \nabla_i \nabla_j F, \partial_t \bar{n} \rangle \]
\[ = \langle \nabla_i \nabla_j (K^\alpha \bar{n}), \bar{n} \rangle + \langle h_{ij} \bar{n}, \partial_t \bar{n} \rangle \]
\[ = \nabla_i \nabla_j K^\alpha + \langle \nabla_j K^\alpha \nabla_i \bar{n}, \bar{n} \rangle + \langle \nabla_i K^\alpha \nabla_j \bar{n}, \bar{n} \rangle + K^\alpha \langle \nabla_i \nabla_j \bar{n}, \bar{n} \rangle + 0. \]
By \( \langle \nabla_i \tilde{n}, \tilde{n} \rangle = 0 \) and \( \langle \nabla_i \nabla_j \tilde{n}, \tilde{n} \rangle = -\langle \nabla_j \tilde{n}, \nabla_i \tilde{n} \rangle = -h_{im} h_{jm}^m \), we deduce

\[(2.11) \quad \partial_t h_{ij} = \nabla_i \nabla_j K^\alpha - K^\alpha h_{im} h_{jm}^m.\]

Applying

\[K = \det(h_{ij} g^{jk}) = \det(h_{ij}) \det(g^{kl}),\]

\[\nabla \log \det(h_{ij}) = b^{kl} \nabla h_{kl}, \quad \frac{\partial b^{kl}}{\partial h_{mn}} = -b^{km} b^{ln}\]
on the first term on the right-hand side of (2.11) yields

\[
\partial_t h_{ij} = \nabla_i \nabla_j K^\alpha - K^\alpha h_{im} h_{jm}^m \\
= \nabla_i (\alpha K^\alpha b^{kl} \nabla_j h_{kl}) - K^\alpha h_{im} h_{jm}^m \\
= \alpha^2 K^\alpha b^{kl} b^{mn} \nabla_j h_{mn} \nabla_j h_{kl} + \alpha K^\alpha a \frac{\partial b^{kl}}{\partial h_{mn}} \nabla_j h_{mn} \nabla_j h_{kl} \\
+ a K^\alpha h_{kl} \nabla_j h_{kl} - K^\alpha h_{im} h_{jm}^m \\
= a K^\alpha b^{kl} \nabla_j h_{kl} + a K^\alpha (a b^{kl} b^{mn} - b^{km} b^{ln}) \nabla_j h_{mn} \nabla_j h_{kl} - K^\alpha h_{im} h_{jm}^m.
\]

On the other hand,

\[
a K^\alpha b^{kl} \nabla_j h_{kl} = a K^\alpha b^{kl} \nabla_j h_{kl} \\
= a K^\alpha b^{kl} \nabla_j h_{kl} + a K^\alpha b^{kl} R_{kjm} h_{im}^m + a K^\alpha b^{kl} R_{klm} h_{jm}^m \\
= a K^\alpha b^{kl} \nabla_j h_{kl} + a K^\alpha (h_{ij} h_{km} - h_{im} h_{kj}) h_{ij} h_{kl} \\
+ a K^\alpha (h_{ij} h_{km} h_{kl} - h_{im} h_{kl} b^{kl}) h_{jm}^m \\
= \mathcal{L} h_{ij} + a K^\alpha (h_{ij} h_{km} - h_{im} h_{kj}) g^{mk} + a K^\alpha (h_{im} - h_{im}) h_{jm}^m \\
= \mathcal{L} h_{ij} + a K^\alpha h_{im} h_{jm}^m - a K^\alpha h_{im} h_{jm}^m
\]

and (2.6) easily follows.

\[\Box\]

**Proof of (2.7).** From (2.11) we have

\[\mathcal{L} K^\alpha = a K^\alpha b^{ij} (\partial_t h_{ij} + K^\alpha h_{im} h_{jm}^m) = a K^\alpha b^{ij} \partial_t h_{ij} + a K^\alpha H.\]

Also, from \( K = \det(h_{ij}) \det(g^{kl}) \), we derive that

\[\partial_t K^\alpha = a K^\alpha \partial_t (\log(\det h_{ij}) + \log(\det g^{kl})) \]

\[= a K^\alpha b^{ij} \partial_t h_{ij} + a K^\alpha g^{kl} \partial_t g^{kl} \]

\[= \mathcal{L} K^\alpha - a K^\alpha H + a K^\alpha g^{kl} \partial_t g^{kl}. \]

Applying (2.2) on the last term yields (2.7).

\[\Box\]

**Proof of (2.8).** The identity \( b^{ij} h_{jk} = \delta_k^i \) leads to

\[(2.12) \quad \partial_t b^{pq} = -b^{ip} b^{jq} \partial_t h_{ij},\]

\[(2.13) \quad \nabla_m b^{pq} = -b^{ip} b^{jq} \nabla_m h_{ij}.\]
Therefore,  

$\nabla_n \nabla_m b^{pq} = -b^{ij} \nabla_n b^{ip} \nabla_m h_{ij} - b^{ip} \nabla_n b^{jq} \nabla_m h_{ij} - b^{ij} b^{ip} \nabla_n \nabla_m h_{ij}$  

$\quad = 2b^{ij} b^{ik} b^{pl} \nabla_n h_{kl} \nabla_m h_{ij} - b^{ip} b^{jq} \nabla_n \nabla_m h_{ij}.$

Hence, $\mathcal{L} b^{pq} := \alpha K^a b^{nm} \nabla_n \nabla_{nm} b^{pq}$ satisfies  

$\mathcal{L} b^{pq} = 2\alpha K^a b^{nm} b^{ij} b^{kpl} \nabla_n h_{kl} \nabla_m h_{ij} - b^{ip} b^{jq} \mathcal{L} h_{ij}$  

$\quad = 2\alpha K^a b^{ip} b^{jq} b^{ki} b^{nm} \nabla_n h_{kl} \nabla_{kn} h_{im} - b^{ip} b^{jq} \mathcal{L} h_{ij}.$

Combing the last identity with (2.6) and (2.12) yields  

$\partial_t b^{pq} = -b^{ip} b^{jq} (\mathcal{L} h_{ij} + \alpha K^a (\alpha b^{kli} b^{mn} - b^{km} b^{nl}) \nabla_i h_{kl} \nabla_j h_{mn}$  

$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + \alpha K^a H h_{ij} - (1 + n\alpha) K^a h_{ik} h_{ij})$  

$\quad = \mathcal{L} h^{pq} - \alpha K^a b^{ip} b^{jq} (\alpha b^{kli} b^{mn} + b^{km} b^{nl}) \nabla_i h_{kl} \nabla_j h_{mn}$  

$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad - \alpha K^a H b^{pq} + (1 + n\alpha) K^a g^{pq}$

which gives (2.8). \hfill \Box

**Proof of (2.9).** By (2.3) we have $\partial_t v = -v^2 \langle \bar{e}_{n+1}, \partial_t \bar{n} \rangle = v^2 \langle \bar{e}_{n+1}, \nabla K^a \rangle$. Furthermore,  

$\mathcal{L} v = -\alpha K^a b^{ij} \nabla_i (v^2 (\bar{e}_{n+1}, \nabla \bar{n}))$  

$\quad = -2\alpha K^a b^{ij} v \nabla_i v \langle \bar{e}_{n+1}, \nabla \bar{n} \rangle + \alpha K^a b^{ij} v^2 \langle \bar{e}_{n+1}, \nabla_i (h_{jk} F^k) \rangle$  

$\quad = 2v^{-1} \| \nabla v \|_{L^2}^2 + \alpha K^a b^{ij} h_{jk} h_{ij} v^2 \langle \bar{e}_{n+1}, \bar{n} \rangle + v^2 \langle \bar{e}_{n+1}, \alpha K^a b^{ij} (\nabla_i h_{jk}) F^k \rangle$  

$\quad = 2v^{-1} \| \nabla v \|_{L^2}^2 + \alpha K^a H v + v^2 \langle \bar{e}_{n+1}, \nabla K^a \rangle.$

Combining the above yields (2.9). \hfill \Box

We recall the notation  

$v := \langle \bar{n}, \bar{e}_{n+1} \rangle^{-1}$ (gradient function)

and  

$\psi_\beta (p, t) := (M - \beta t - \bar{u}(p, t))_+$ (cut-off function),

where $\bar{u}(p, t) = \langle F(p, t), \bar{e}_{n+1} \rangle$ denotes the height function. We will next show the following local gradient estimate.

**Theorem 2.2** (Gradient estimate). Assume that a smooth hypersurface $\Sigma_0$ satisfies the assumptions in Theorem 1.1 and let $\Sigma_t$ be a complete strictly convex smooth graph solution of (1.1) defined on $M^n \times [0, T]$, for some $T > 0$. Given constants $\beta > 0$ and $M \geq \beta$,  

$v(p, t) \psi_\beta (p, t) \leq M \max \{ \sup_{Q_M} v(p, 0), \beta^{-1} n^{\frac{-1}{\alpha+1}} (n\alpha - 1)_+ \},$

where $Q_M = \{ p \in M^n : \bar{u}(p, 0) < M \}.$
Proof. First use (2.4) and (2.9), that is,
\[ \partial_t \psi_\beta = \mathcal{L} \psi_\beta + (n\alpha - 1)\nu^{-1} K^\alpha - \beta, \]
\[ \partial_t \nu = \mathcal{L} \nu - 2\nu^{-1} \| \nabla \nu \|^2_F - a K^\alpha H \nu, \]
to compute
\[ \partial_t (\psi_\beta \nu) = \psi_\beta \mathcal{L} \nu - 2\psi_\beta \nu^{-1} \| \nabla \nu \|^2_F - \alpha \psi_\beta K^\alpha H \nu + \nu \mathcal{E} \psi_\beta + (n\alpha - 1)K^\alpha - \beta \nu \]
\[ = \mathcal{L} (\psi_\beta \nu) - 2(\nabla \psi_\beta, \nabla \nu)_F - 2\psi_\beta \nu^{-1} \| \nabla \nu \|^2_F - \alpha \psi_\beta K^\alpha H \nu \]
\[ + (n\alpha - 1)K^\alpha - \beta \nu \]
\[ = \mathcal{L} (\psi_\beta \nu) - 2\nu^{-1} (\nabla (\psi_\beta \nu), \nabla \nu)_F - \alpha \psi_\beta K^\alpha H \nu + (n\alpha - 1)K^\alpha - \beta \nu. \]

Since (ii), (iii) in Theorem 1.1 imply that \( \psi_\beta \) is compactly supported, for a fixed \( T \in (0, +\infty) \), the function \( \psi_\beta \nu \) attains its maximum on \( M^n \times [0,T] \) at some \( (p_0,t_0) \). If \( t_0 = 0 \), then we obtain the desired result. Assume that \( t_0 > 0 \). Then, at \( (p_0,t_0) \) we have
\[ \partial_t (\psi_\beta \nu) - \mathcal{L} (\psi_\beta \nu) \geq 0 \]
which is equivalent to
\[ (n\alpha - 1)K^\alpha \geq \alpha \psi_\beta K^\alpha H \nu + \beta \nu = (\alpha \psi_\beta K^\alpha H + \beta) \nu. \]

Hence, an interior maximum can be achieved only if \( n\alpha \geq 1 \). From now on, we assume that \( n\alpha \geq 1 \). If we multiply the last inequality by \( MK^{-\alpha} \) and use that \( M \geq \beta \), we obtain
\[ (n\alpha - 1)M \geq (\alpha M \psi_\beta H + \beta MK^{-\alpha}) \nu \geq (\alpha \psi_\beta H + MK^{-\alpha}) \nu. \]

On the other hand, \( M \geq \psi_\beta \) implies
\[ \alpha \psi_\beta H + MK^{-\alpha} \geq \alpha \psi_\beta H + \psi_\beta K^{-\alpha} \geq (\alpha H + K^{-\alpha}) \psi_\beta. \]

Hence,
\[ (n\alpha - 1)M \geq \beta (\alpha H + K^{-\alpha}) \psi_\beta \nu \geq \beta \frac{\nu}{n} \left( \frac{n\alpha}{n\alpha + 1} H + \frac{1}{n\alpha + 1} K^{-\alpha} \right) \psi_\beta \nu. \]

Next, we apply Young’s inequality
\[ \frac{n\alpha}{n\alpha + 1} H + \frac{1}{n\alpha + 1} K^{-\alpha} \geq H \frac{n\alpha}{n\alpha + 1} K^{-\alpha} = (HK^{-1}) \frac{n\alpha}{n\alpha + 1}. \]

Since \( H \geq n K^{1/n} \), we conclude that in the case \( n\alpha \geq 1 \), \( t_0 > 0 \), the following holds at \( (p_0,t_0) \):
\[ (n\alpha - 1)M \geq \beta n^{-1} \frac{n\alpha}{n\alpha + 1} \psi_\beta \nu = \beta n^{-1} \frac{1}{n\alpha + 1} \psi_\beta \nu, \]
from which the desired inequality readily follows. \( \Box \)
3. Lower bounds on the principal curvatures

In this section we will obtain lower bounds on the principal curvatures, as stated in Theorem 3.3. We will achieve this by using Pogorelov type estimates with respect to $b_{ii}$. Recall that $b_{ij}$ denotes the inverse matrix of the second fundamental form $h_{ij}$. Since $b_{ii}$ depends on charts, we will find the relation between $b_{ii}$ and the principal curvatures as in [8]. We begin with a simple observation holding on every smooth strictly convex hypersurface which we prove here for the reader’s convenience.

**Proposition 3.1 (Euler’s formula).** Let $\Sigma$ be a smooth strictly convex hypersurface and let $F : M^n \to \mathbb{R}^{n+1}$ be a smooth immersion with $F(M^n) = \Sigma$. Then, for all $p \in M^n$ and $i \in \{1, \ldots, n\}$, the following holds:

$$\frac{b_{ii}(p)}{g_{ii}(p)} \leq \frac{1}{\lambda_{\min}(p)}.$$

**Proof.** Let $\{E_1, \ldots, E_n\}$ be an orthonormal basis of $T\Sigma F(p)$ satisfying $L(E_j) = \lambda_j E_j$, where $L$ is the Weingarten map and $\lambda_1, \ldots, \lambda_n$ are the principal curvatures of $\Sigma$ at $p$. Denote by $\{a_{ij}\}$ the matrix satisfying $F_i(p) := \nabla_i F = a_{ij} E_j$ and by $\{c_{ij}\}$ the diagonal matrix $\text{diag}(\lambda_1, \ldots, \lambda_n)$. Then, $LF_i(p) = h_{ij}(p) F^j(p)$ implies

$$b_{ij}(p) LF_j(p) = b_{ij}(p) h_{jk}(p) F^k(p) = F^i(p) = g_{ij}(p) a_{jm} E_m.$$

Observing that for the sum $a_{jk} E_k = \sum_{k=1}^n a_{jk} E_k$ we have

$$L(a_{jk} E_k) = a_{jk} L(E_k) = a_{jk} \lambda_k E_k = a_{jk} c_{km} E_m,$$

it follows that

$$g_{ij}(p) a_{jm} E_m = b_{ij}(p) LF_j(p) = b_{ij}(p) L(a_{jk} E_k) = b_{ij}(p) a_{jk} c_{km} E_m.$$  

Hence,

$$g_{ij}(p) a_{jm} = b_{ij}(p) a_{jk} c_{km}.$$

Denoting by $a^{ij}$ and $c^{ij}$ the inverse matrices of $a_{ij}$ and $c_{ij}$, respectively, we have

$$g^{ij}(p) a_{jm} c^{ml} a^{li} = b^{ij}(p) a_{jk} c_{km} c^{ml} a^{li} = b^{jn}(p).$$

Thus, for each $i \in \{1, \ldots, n\}$, the following holds:

$$b^{ii}(p) = \sum_{j,l,m} g^{ij}(p) a_{jm} c^{ml} a^{li} = \sum_{j,l,m} \langle F^i(p), F^j(p) a_{jm} c^{ml} a^{li} \rangle.$$

By setting $F^i(p) = \tilde{a}_{ij} E_j$, we observe

$$a^{ji} = a^{jk} \langle F_k(p), F^i(p) \rangle = a^{jk} \langle a_{kl} E_l, \tilde{a}_{im} E_m \rangle = a^{jk} a_{kl} \tilde{a}_{im} \delta_{ml} = \tilde{a}_{ij}.$$

Thus, we have

$$\langle F^i(p), F^j(p) \rangle = \langle a^{ki} E_k, a^{rj} E_r \rangle = a^{ki} a^{kj}.$$
which yields
\[
b_{ij}^\alpha(p) = \sum_{j,k,l,m} a_{ij}^k a_{klm} a_{ml} = \sum_{j,k,l,m} a_{ij}^k g_{km} a_{ml}
\]
\[
= \sum_{k,l} a_{ij}^k c_{kl} a_{ml} = \sum_{k} a_{ij}^k \lambda_{k}^{-1} = \sum_{k} (a_{ij}^k)^2 \lambda_{k}^{-1}
\]
\[
\leq \sum_{k} (a_{ij}^k)^2 \lambda_{\min}^{-1} = \lambda_{\min}^{-1} \sum_{k,j} (a_{ij}^k E_k, a_{ij} E_j)
\]
\[
= \lambda_{\min}^{-1} \langle F_i^\alpha(p), F_i^\alpha(p) \rangle = \lambda_{\min}^{-1} g^i(p).
\]

**Remark 3.2.** In [7] Chow obtained lower bounds on the principal curvatures of a closed solution \(\Sigma_t\) of (1.1) for all \(\alpha > 0\) by applying the maximum principle for \(\hat{\nabla}_\xi \hat{\nabla}_\xi S + \hat{g} \hat{\xi} S\) where \(S(\cdot,t)\) is the support function of \(\Sigma_t\), and \(\hat{\nabla}\) is a covariant derivative compatible with a metric \(\hat{g}\) of the unit sphere \(S^n\). See also [2, 4, 16]. Notice that the eigenvalues of \(\hat{\nabla}_i \hat{\nabla}_j S + \hat{g}_{ij} S\) with respect to \(\hat{g}_{ij}\) are the principal radii of curvature as eigenvalues of \(b_{ij}\), and the maximum principle also applies for \(\hat{b}_{ij}^\xi / \hat{g}_{ij}^\xi\) if \(\Sigma_t\) is a closed solution. However, \(\hat{g}_{ij}\) depends on time \(t\) while \(\hat{g}_{ij}\) does not.

We will next show a local lower bound on the principal curvatures of \(\Sigma_t\) in terms of the initial data, which constitutes one of the crucial estimates in this work. We recall the definition of the cut-off function \(\psi_\beta(p,t) := (M - \beta t - \bar{u}(p,t))^+\), where \(\bar{u}(p,t) := \langle F(p,t), \hat{e}_{n+1} \rangle\) denotes the height function.

**Theorem 3.3** (Local lower bound for the principal curvatures). Assume that a smooth hypersurface \(\Sigma_0\) satisfies the assumptions in Theorem 1.1 and let \(\Sigma_t\) be a complete strictly convex smooth graph solution of (1.1) defined on \(M^n \times [0,T]\), for some \(T > 0\). Then, given constants \(\beta > 0\) and \(M \geq \beta\), the following holds:

\[
(\psi_\beta^{-n(1+\frac{1}{\alpha})})_{\lambda_{\min}}(p,t) \geq M^{-n(1+\frac{1}{\alpha})} \min \left\{ \inf_{Q_M} \lambda_{\min}(p,0), \frac{\beta^{(n-1)+\frac{1}{\alpha}}}{(n+1)(n\alpha - 1)_+^{(n-1)+\frac{1}{\alpha}}} \right\}
\]

where \(Q_M = \{ p \in M^n : \bar{u}(p,0) < M \}\), and \((n\alpha - 1)_+^{-1} = +\infty\), if \(n\alpha \leq 1\).

**Proof.** Consider the cut-off function
\[
\psi_\beta := (M - \beta t - \bar{u}(p,t))^+,
\]
where \(\bar{u}(p,t) = \langle F(p,t), \hat{e}_{n+1} \rangle\) denotes the height function. Observe that the conditions (ii), (iii) in Theorem 1.1 imply that \(\psi_\beta\) is compactly supported. Therefore, for a fixed \(T \in (0, +\infty)\), the function \(\psi_\beta^{(1+1/\alpha)\lambda_{\min}^{-1}}\) attains its maximum on \(M^n \times [0,T]\) at a point \((p_0, t_0)\). If \(t_0 = 0\), then we obtain the desired result by the bound \(\psi_\beta \leq M\) and the conditions on our initial data. So, we may assume in what follows that \(t_0 > 0\).

We begin by choosing a chart \((U, \varphi)\) with \(p_0 \in \varphi(U) \subset M^n\), on which the covariant derivatives
\[
\{ \nabla_i F(p_0, t_0) := \partial_i (F \circ \varphi)(\varphi^{-1}(p_0), t_0) \}_{i=1,\ldots,n}
\]
form an orthonormal basis of \((T \Sigma_{t_0}) F(p_{0,t_0})\) satisfying
\[
g_{ij}(p_{0,t_0}) = \delta_{ij}, \quad h_{ij}(p_{0,t_0}) = \delta_{ij} \lambda_i(p_{0,t_0}), \quad \lambda_1(p_{0,t_0}) = \lambda_{\text{min}}(p_{0,t_0}).
\]
In particular, at the point \((p_{0,t_0})\) we have \(b^{11}(p_{0,t_0}) = \lambda_{-1}^{11}(p_{0,t_0})\) and \(g^{11}(p_{0,t_0}) = 1\).

Next, we define the function \(w : \varphi(U) \times [0,T] \rightarrow \mathbb{R}\) by
\[
w := \psi^{(1+\frac{1}{\alpha})} \frac{b^{11}}{g^{11}}.
\]

Notice that on the chart \((U, \varphi)\), if \(t \neq t_0\), then the covariant derivatives \(\{\nabla_i F(p_{0,t})\}_{i=1,\ldots,n}\) may not form an orthonormal basis of \((T \Sigma_t) F(p_{0,t})\). However, since Proposition 3.1 holds for every chart and immersion, we have
\[
w \leq \psi^{n(1+\frac{1}{\alpha})} \lambda_{\text{min}}^{-1}.
\]
Hence, for \((p,t) \in \varphi(U) \times [0,T]\), the following holds:
\[
w(p,t) \leq \psi^{n(1+\frac{1}{\alpha})} \lambda_{\text{min}}^{-1} \leq \psi^{n(1+\frac{1}{\alpha})} \lambda_{\text{min}}^{-1}(p_{0,t_0}) = w(p_{0,t_0})
\]
which shows that \(w\) attains its maximum at \((p_{0,t_0})\).

Observe next that since \(\omega K^\alpha b^{ij}\) and sum the equations over all \(i, j\) to obtain
\[
\frac{\mathcal{L} w}{w} - \frac{\|\nabla w\|_F^2}{w^2} = n\left(1 + \frac{1}{\alpha}\right) \frac{\mathcal{L} \psi}{\psi} - n\left(1 + \frac{1}{\alpha}\right) \frac{\|\nabla \psi\|_F^2}{\psi^2} + \frac{\mathcal{L} b^{11}}{b^{11}} - \frac{\|b^{11}\|_F^2}{(b^{11})^2}.
\]

On the other hand, on the support of \(\psi\), we also have
\[
\frac{\partial_t w}{w} = n\left(1 + \frac{1}{\alpha}\right) \frac{\partial_t \psi}{\psi} + \frac{\partial_t b^{11}}{b^{11}} - \frac{\partial_t g^{11}}{g^{11}}.
\]

Subtract the equations above. Then, \(w^{-2} \frac{\|\nabla w\|_F^2}{w} \geq 0\) implies the following inequality:
\[
\mathcal{L} w - \frac{\partial_t w}{w} \geq n\left(1 + \frac{1}{\alpha}\right) \left(\frac{\mathcal{L} \psi}{\psi} - \frac{\partial_t \psi}{\psi} \right) - n\left(1 + \frac{1}{\alpha}\right) \frac{\|\nabla \psi\|_F^2}{\psi^2} + \frac{\mathcal{L} b^{11}}{b^{11}} - \frac{\partial_t b^{11}}{b^{11}} - \frac{\|b^{11}\|_F^2}{(b^{11})^2} + \frac{\partial_t g^{11}}{g^{11}}.
\]
By (2.2) and (2.4) we have \( \partial_t g^{11} = 2K^\alpha h^{11} \) and \( \mathcal{L} \psi_\beta - \partial_t \psi_\beta = \beta - (n\alpha - 1) \nu^{-1} K^\alpha \), while by (2.8),

\[
\mathcal{L} b^{11} - \partial_t b^{11} = a K^\alpha b^{ij} b^{11} (ab^{kl}b^{mn} + b^{km}b^{nl}) \nabla_i h_{kl} \nabla_j h_{mn} + a K^\alpha h^{11} - (1 + n\alpha) K^\alpha g^{11}.
\]

Combining the above yields

\[
(3.2) \quad \frac{\mathcal{L} w}{w} - \frac{\partial_t w}{w} \geq -n \left(1 + \frac{1}{\alpha}\right) \frac{\| \nabla \psi_\beta \|^2_x}{\psi_\beta^2} - \frac{\| \nabla b^{11} \|^2_x}{(b^{11})^2} + \frac{\alpha K^\alpha b^{ij} b^{11} (ab^{kl}b^{mn} + b^{km}b^{nl})}{b^{11}} \nabla_i h_{kl} \nabla_j h_{mn}
- n \left(1 + \frac{1}{\alpha}\right) (n\alpha - 1) \frac{K^\alpha \nu^{-1}}{\psi_\beta} + n \left(1 + \frac{1}{\alpha}\right) \frac{\beta}{\psi_\beta} + \alpha K^\alpha H
- (1 + n\alpha) K^\alpha \frac{g^{11}}{b^{11}} + 2K^\alpha \frac{h^{11}}{g^{11}}.
\]

Now, at \((p_0, t_0)\) the following holds:

\[
(3.3) \quad a K^\alpha H - (1 + n\alpha) K^\alpha \frac{g^{11}}{b^{11}} + 2K^\alpha \frac{h^{11}}{g^{11}}
\geq n\alpha K^\alpha \lambda_{\min} - (1 + n\alpha) K^\alpha \lambda_{\min} + 2K^\alpha \lambda_{\min} = K^\alpha \lambda_{\min}.
\]

In addition, if \(n\alpha \geq 1\), then \(H \geq n\lambda_{\min}\) gives

\[
a K^\alpha H = \left(\alpha - \frac{1}{n}\right) K^\alpha H + \frac{1}{n} K^\alpha H \geq (n\alpha - 1) K^\alpha \lambda_{\min} + \frac{1}{n} K^\alpha H.
\]

Therefore, in the case that \(n\alpha \geq 1\), we can improve (3.3) to obtain

\[
(3.4) \quad a K^\alpha H - (1 + n\alpha) K^\alpha \frac{g^{11}}{b^{11}} + 2K^\alpha \frac{h^{11}}{g^{11}} \geq \frac{1}{n} K^\alpha H.
\]

Also, at the maximum point \((p_0, t_0)\) of \(w\), \(\nabla w(p_0, t_0) = 0\) holds. So, (3.1) leads to

\[
\frac{n(1 + \alpha)}{\alpha} \frac{\| \nabla \psi_\beta \|^2_x}{\psi_\beta^2} + \frac{\| \nabla b^{11} \|^2_x}{(b^{11})^2} = \left(1 + \frac{\alpha}{n(1 + \alpha)}\right) \frac{\| \nabla b^{11} \|^2_x}{(b^{11})^2}
= \left(1 + \frac{\alpha}{n(1 + \alpha)}\right) \alpha \sum_{i=1}^{n} \frac{b^{ij} K^\alpha |\nabla_i h^{11}|^2}{(b^{11})^2}.
\]

From (2.13), we get \(\nabla_i h^{11} = -b^{1k} h_{kl} \nabla_i h_{kl} = -(b^{11})^2 \nabla_i h^{11}\) at \((p_0, t_0)\). Hence,

\[
(3.5) \quad \frac{n(1 + \alpha)}{\alpha} \frac{\| \nabla \psi_\beta \|^2_x}{\psi_\beta^2} + \frac{\| \nabla b^{11} \|^2_x}{(b^{11})^2} = a \left(1 + \frac{\alpha}{n(1 + \alpha)}\right) \sum_{i=1}^{n} b^{ij} (b^{11})^2 K^\alpha |\nabla_i h^{11}|^2.
\]

At the point \((p_0, t_0)\), we define

\[
I_i = b^{ij} (b^{11})^2 K^\alpha |\nabla_i h^{11}|^2, \quad J_i = b^{ij} \nabla_i h_{ij}.
\]
We may rewrite equation (3.5) as

\[ \frac{n(1 + \alpha)}{\alpha} \frac{\| \nabla \psi \|_2^2}{\psi_\beta^2} + \frac{\| \nabla b^{11} \|_2^2}{(b^{11})^2} = \alpha \left( 1 + \frac{\alpha}{n(1 + \alpha)} \right) \sum_{i=1}^{n} I_i \]

and also write

\[ b^{kl} b^{mn} \nabla_i h_{kl} \nabla_i h_{mn} = |b^{mn} \nabla_i h_{mn}|^2 = \left| \sum_{i=1}^{n} b^{ji} \nabla_i h_{ii} \right|^2 = \left| \sum_{i=1}^{n} J_i \right|^2 \]

which gives

\[ \frac{\alpha^2 K^\alpha b^{ij} b^{kl} b^{mn}}{b^{11}} \nabla_i h_{kl} \nabla_j h_{mn} = \alpha^2 K^\alpha b^{11} b^{kl} b^{mn} \nabla_i h_{kl} \nabla_i h_{mn} \]

\[ = \alpha^2 K^\alpha b^{11} \left| \sum_{i=1}^{n} J_i \right|^2. \]

Since \( \alpha^2 \geq \frac{\alpha^2}{1+\alpha} \), we conclude that

\[ \frac{\alpha^2 K^\alpha b^{ij} b^{kl} b^{mn}}{b^{11}} \nabla_i h_{kl} \nabla_j h_{mn} \geq \frac{\alpha^2}{1+\alpha} K^\alpha b^{11} \sum_{i=1}^{n} J_i^2. \]

Finally, at \((p_0, t_0)\) we also have

\[ \frac{\alpha K^\alpha b^{ij} b^{km} b^{nl}}{b^{11}} \nabla_i h_{kl} \nabla_j h_{mn} = \alpha K^\alpha b^{11} b^{km} b^{nl} \nabla_i h_{kl} \nabla_i h_{mn} \]

\[ = \alpha K^\alpha b^{11} \left( \sum_{i=1}^{n} |b^{ii} \nabla_i h_{ii}|^2 + \sum_{i \neq j} b^{ij} b^{jj} |\nabla_i h_{jj}|^2 \right) \]

\[ \geq \alpha K^\alpha b^{11} \left( \sum_{i=1}^{n} |b^{ii} \nabla_i h_{ii}|^2 + 2 \sum_{i \neq 1} b^{ij} b^{11} |\nabla_i h_{11}|^2 \right) \]

\[ = \alpha K^\alpha b^{11} \sum_{i=1}^{n} |J_i|^2 + 2\alpha \sum_{i \neq 1} I_i. \]

Using \( \alpha \geq \frac{\alpha^2}{1+\alpha} \), we may rewrite the inequality above as

\[ \frac{\alpha K^\alpha b^{ij} b^{kl} b^{mn}}{b^{11}} \nabla_i h_{kl} \nabla_j h_{mn} \]

\[ \geq \frac{\alpha^2}{1+\alpha} K^\alpha b^{11} \sum_{i \neq 1} |J_i|^2 + \alpha K^\alpha b^{11} |J_1|^2 + 2\alpha \sum_{i \neq 1} I_i. \]

Adding (3.7) and (3.8) gives that at \((p_0, t_0)\) we have

\[ \frac{\alpha K^\alpha b^{ij} b^{kl} (ab^{kl} b^{mn} + b^{km} b^{nl})}{b^{11}} \nabla_i h_{kl} \nabla_j h_{mn} \]

\[ \geq \frac{\alpha^2}{1+\alpha} K^\alpha b^{11} \left( \sum_{i=1}^{n} |J_i|^2 + \sum_{i \neq 1} |J_i|^2 \right) + \alpha K^\alpha b^{11} |J_1|^2 + 2\alpha \sum_{i \neq 1} I_i. \]
Using the Cauchy–Schwarz inequality

\[
n\left(\sum_{i=1}^{n} |J_i|^2 + \sum_{i \neq 1} |J_i|^2\right) = (1^2 + (-1)^2 + \cdots + (-1)^2)\left(\sum_{i=1}^{n} |J_i|^2 + \sum_{i \neq 1} |J_i|^2\right)
\]

\[
\geq \left| \sum_{i=1}^{n} J_i + \sum_{i \neq 1} -J_i \right|^2 = |J_1|^2
\]

and \(2\alpha \geq \alpha(1 + \frac{\alpha}{n(1+\alpha)})\), we obtain

\[
\frac{n(1+\alpha) \|\nabla \psi_{\beta}\|^2_{L^2}(\alpha h^{k1}b^{mn} + b^{km}h^{nl})}{\psi_{\beta}^2} \frac{\nabla_{ij}h_{kl}}{\nabla_{ij}h_{mn}} 
\geq \alpha \left(1 + \frac{\alpha}{n(1+\alpha)}\right) \left(K^{\alpha}b^{11} |J_1|^2 + \sum_{i \neq 1} I_i\right).
\]

Combining the last inequality with (3.6) while using that \(K^{\alpha}b^{11} |J_1|^2 = I_1\), yields

\[
(3.9) \quad \frac{n(1+\alpha) \|\nabla \psi_{\beta}\|^2_{L^2} - \|\nabla b^{11}\|^2_{L^2}}{\psi_{\beta}^2 (b^{11})^2} + \frac{n(1+\alpha) \|\nabla \psi_{\beta}\|^2_{L^2}(\alpha h^{k1}b^{mn} + b^{km}h^{nl})}{\psi_{\beta}^2} \frac{\nabla_{ij}h_{kl}}{\nabla_{ij}h_{mn}} \geq 0.
\]

We conclude by (3.2), (3.3) and (3.9) that at \((p_0, t_0)\) the following holds:

\[
0 \geq \frac{\mathcal{L} w}{w} - \frac{\partial_t w}{w} \geq -n\left(1 + \frac{1}{\alpha}\right)(n\alpha - 1) \frac{K^{\alpha - 1} \psi_{\beta}}{\psi_{\beta}^2} + n\left(1 + \frac{1}{\alpha}\right) \frac{\beta}{\psi_{\beta}} + K^{\alpha} \lambda_{\min}.
\]

If \(n\alpha \leq 1\), then the last inequality gives \(n(1 + \frac{1}{\alpha}) \leq 0\), a contradiction. Hence \(t_0 = 0\), and therefore the desired result holds. If \(n\alpha \geq 1\), then we use the improved inequality (3.4) instead of (3.3) and perform the same estimates for all the other terms, so that at \((p_0, t_0)\) we obtain

\[
0 \geq \frac{\mathcal{L} w}{w} - \frac{\partial_t w}{w} \geq -n\left(1 + \frac{1}{\alpha}\right)(n\alpha - 1) \frac{K^{\alpha - 1} \psi_{\beta}}{\psi_{\beta}^2} + n\left(1 + \frac{1}{\alpha}\right) \frac{\beta}{\psi_{\beta}} + \frac{1}{n} K^{\alpha} H.
\]

Hence

\[
n\left(1 + \frac{1}{\alpha}\right)(n\alpha - 1) \frac{\psi_{\beta}}{\psi_{\beta}} \geq n\left(1 + \frac{1}{\alpha}\right) \frac{\beta}{\psi_{\beta}} K^{-\alpha} + \frac{1}{n} H.
\]

Since \(n\alpha \geq 1\), we have \(1 + \frac{1}{\alpha} \leq 1 + n\). Using also that \(\nu \geq 1\), \(\psi_{\beta} \leq M\), and \(M \geq \beta\), we conclude from the previous inequality that

\[
n^2(1 + n)(n\alpha - 1) \psi_{\beta}^{-1} \geq n^2\left(1 + \frac{1}{\alpha}\right) K^{-\alpha} \frac{\beta}{\psi_{\beta}} + H \geq \left(n^2\left(1 + \frac{1}{\alpha}\right) K^{-\alpha} + H\right) \frac{\beta}{M}.
\]

Next, we employ Young’s inequality

\[
\frac{K^{-\alpha}}{(n-1)\alpha + 1} + \frac{(n-1)\alpha H}{(n-1)\alpha + 1} \geq K^{-\alpha} \frac{\beta}{\psi_{\beta}} \frac{H}{n^{\alpha-1} \psi_{\beta}^{-\alpha}} = \left(K^{-1} H^{-1}\right) \frac{\alpha}{n^{\alpha-1}}
\]
and observe the following:

\[ K^{-1} H^{n-1} = \frac{H^{n-1}}{\lambda_1 \lambda_2 \cdots \lambda_n} = \frac{1}{\lambda_1} \frac{H^{n-1}}{\lambda_2 \cdots \lambda_n} \geq \frac{1}{\lambda_1} = \lambda_{\min}^{-1}. \]

Combining the last three inequalities yields

\[ n^2(1 + n)(n \alpha - 1)M^{\beta - 1} \psi^{-1}_\beta \geq n^2(1 + \frac{1}{\alpha})K^{-\alpha} + H \]

\[ \geq \frac{K^{-\alpha}}{(n - 1)\alpha + 1} + \frac{(n - 1)\alpha H}{(n - 1)\alpha + 1} \geq (\lambda_{\min}^{-1})^{\frac{\alpha}{(n - 1)\alpha + 1}}. \]

We conclude that if \( n \alpha \geq 1 \), the following holds at \( (p_0, t_0) \):

\[ \lambda_{\min}^{-1} \psi^{(1 + \frac{1}{\alpha})} \leq (n^2(1 + n)(n \alpha - 1)M^{\beta - 1})^{(n - 1) + \frac{1}{\alpha}} \psi^{1 + \frac{n - 1}{\alpha}}. \]

Thus, \( \psi_\beta \leq M \) gives the desired result.

\[ \square \]

4. Speed estimates

This section will be devoted to the proof of a local speed bound. We recall the definition of the cut-off function, \( \psi(p, t) := (M - \tilde{u}(p, t))_+ \), where \( \tilde{u}(p, t) := (F(p, t), \tilde{e}_{n+1}) \) denotes the height function.

**Theorem 4.1** (Local speed bound). Assume that a smooth hypersurface \( \Sigma_0 \) satisfies the assumptions in Theorem 1.1 and let \( \Sigma_t \) be a complete strictly convex smooth graph solution of (1.1) defined on \( M^n \times [0, T) \). Then, given a constant \( M \geq 1 \),

\[ \left( \frac{t}{1 + t} \right) (\psi^2 K^{\pi})(p, t) \leq (4n\alpha + 1)^2 (2\theta)^{1 + \frac{1}{n\alpha}} (\theta \Lambda + M^2) \]

where \( \theta \) and \( \Lambda \) are constants given by

- \( \theta = \sup \{ v^2(p, s) : \tilde{u}(p, s) < M, s \in [0, t] \} \),
- \( \Lambda = \sup \{ \lambda_{\min}^{-1}(p, s) : \tilde{u}(p, s) < M, s \in [0, t] \} \).

**Proof.** Choosing a fixed time \( T_0 \in (0, T) \), we redefine \( \theta \) and \( \Lambda \) by

- \( \theta = \sup \{ v^2(p, t) : \tilde{u}(p, t) < M, t \in [0, T_0] \} \),
- \( \Lambda = \sup \{ \lambda_{\min}^{-1}(p, t) : \tilde{u}(p, t) < M, t \in [0, T_0] \} \).

Also, we define \( \eta : [0, T) \to \mathbb{R} \) by

\[ \eta(t) = \frac{t}{1 + t} \]

which will be used later in this proof.
Following the well-known idea by Caffarelli, Nirenberg and Spruck in [6] (see also [13] and [8]), we define the function \( \varphi = \varphi(v^2)\), depending on \( v^2 \), by

\[
\varphi(v^2) = \frac{v^2}{2\theta - v^2}.
\]

The evolution equation of \( v \) in (2.9) gives

\[
\partial_t(v^2) = \mathcal{L}(v^2) - 2\alpha K^\alpha H v^2 - 6\|\nabla v\|_X^2.
\]

Then, the evolution equation of \( \varphi(v^2) \) is

\[
\partial_t \varphi = \varphi' (\mathcal{L}v^2 - 2\alpha K^\alpha H v^2 - 6\|\nabla v\|_X^2) + \mathcal{L}\varphi - \varphi''\|\nabla v^2\|_X^2 - \varphi' (2\alpha K^\alpha H v^2 + 6\|\nabla v\|_X^2).
\]

Also, the evolution equation of \( K^\alpha \) in (2.7) leads to

\[
\partial_t K^{\alpha} = \mathcal{L} K^{\alpha} - \frac{1}{2} K^{-2\alpha} \|\nabla K^{\alpha}\|_X^2 + 2\alpha K^{3\alpha} H
\]

implying the following evolution equation for \( K^{2\alpha} \varphi(v^2) \):

\[
\partial_t (K^{2\alpha} \varphi) = \mathcal{L}(K^{2\alpha} \varphi) - 2(\nabla K^{2\alpha}, \nabla \varphi)_X + 2\alpha K^{3\alpha} H(\varphi - \varphi' v^2) - \frac{1}{2} \varphi K^{-2\alpha} \|\nabla K^{2\alpha}\|_X^2 - (4\varphi'' v^2 + 6\varphi') K^{2\alpha} \|\nabla v\|_X^2.
\]

Observe that

\[
-2(\nabla K^{2\alpha}, \nabla \varphi)_X = -(\nabla K^{2\alpha}, \nabla \varphi)_X + \varphi^{-1} K^{2\alpha} \|\nabla \varphi\|_X^2 - \varphi^{-1} (\nabla \varphi, \nabla (K^{2\alpha} \varphi))_X \\
\leq \frac{1}{2} \varphi K^{-2\alpha} \|\nabla K^{2\alpha}\|_X^2 + \frac{3}{2} \varphi^{-1} K^{2\alpha} \|\nabla \varphi\|_X^2 - \varphi^{-1} (\nabla \varphi, \nabla (K^{2\alpha} \varphi))_X.
\]

Hence, the following inequality holds:

\[
\partial_t (K^{2\alpha} \varphi) \leq \mathcal{L}(K^{2\alpha} \varphi) - \varphi^{-1} (\nabla \varphi, \nabla (K^{2\alpha} \varphi))_X + 2\alpha K^{3\alpha} H(\varphi - \varphi' v^2) - (4\varphi'' v^2 + 6\varphi' - 6\varphi^{-1} \varphi^2 v^2) K^{2\alpha} \|\nabla v\|_X^2.
\]

Now, we have

\[
\varphi(v^2) + 1 = \frac{2\theta}{2\theta - v^2}, \quad \varphi'(v^2) = \frac{2\theta}{(2\theta - v^2)^2}, \quad \varphi''(v^2) = \frac{4\theta}{(2\theta - v^2)^3}.
\]

Therefore, by direct calculation we obtain

\[
\varphi - \varphi' v^2 = \frac{v^2}{2\theta - v^2} - \frac{2\theta v^2}{(2\theta - v^2)^2} = -\frac{v^4}{(2\theta - v^2)^2} = -\varphi^2
\]

and

\[
\varphi^{-1} \nabla \varphi = \frac{2\theta - v^2}{v^2} \frac{4\theta v \nabla v}{(2\theta - v^2)^2} = 4\theta \varphi^{-3} \nabla v.
\]
and also
\[ 4\varphi''v^2 + 6\varphi' - 6\varphi^{-1}\varphi^2v^2 = \frac{16\theta v^2}{(2\theta - v^2)^2} + \frac{12\theta}{(2\theta - v^2)^2} - \frac{24\theta^2}{(2\theta - v^2)^3} \]
\[ = \frac{4\theta}{(2\theta - v^2)^2} \varphi. \]

Setting \( f := K^{2\alpha} \varphi \) in (4.1) and using the identities above yields
\[ \partial_t f \leq \mathcal{L} f - 4\alpha \varphi v^{-3}(\nabla v, \nabla f)_\mathcal{L} - 2\alpha f K^{\alpha} H \varphi - \frac{4\theta}{(2\theta - v^2)^2} f \|\nabla v\|_\mathcal{L}^2. \]

On the other hand, (2.4) gives
\[ \partial_t \psi = \mathcal{L} \psi + (n\alpha - 1)v^{-1}K^{\alpha} \leq \mathcal{L} \psi + n\alpha K^{\alpha}. \]

Hence, on the support of \( \psi \), we have
\[ \partial_t \psi^{4\alpha} \leq \mathcal{L} \psi^{4\alpha} - 4n\alpha(4n\alpha - 1)\psi^{4\alpha - 2}\|\nabla \psi\|_\mathcal{L}^2 + 4n^2\alpha^2 K^{\alpha} \psi^{4n\alpha - 1}. \]

Thus, on the support of \( \psi \), the following holds:
\[ \partial_t (f \psi^{4\alpha}) \leq \mathcal{L} (f \psi^{4\alpha}) - 2(\nabla \psi^{4\alpha}, \nabla f)_\mathcal{L} - 4\alpha \varphi v^{-3} \psi^{4\alpha} (\nabla v, \nabla f)_\mathcal{L} - 2\alpha f K^{\alpha} H \varphi \psi^{4\alpha} - \frac{4\theta}{(2\theta - v^2)^2} f \psi^{4\alpha} \|\nabla v\|_\mathcal{L}^2 - 4n\alpha(4n\alpha - 1)f \psi^{4\alpha - 2}\|\nabla \psi\|_\mathcal{L}^2 + 4n^2\alpha^2 K^{\alpha} \psi^{4n\alpha - 1}. \]

Next, we compute
\[ -4\alpha \varphi v^{-3} \psi^{4\alpha} (\nabla v, \nabla f)_\mathcal{L} = -4\alpha \varphi v^{-3}(\nabla v, \nabla (f \psi^{4\alpha} ))_\mathcal{L} + 16n\alpha\theta \varphi v^{-3} f \psi^{4\alpha - 1}(\nabla v, \nabla \psi)_\mathcal{L} \]
\[ \leq -4\alpha \varphi v^{-3}(\nabla v, \nabla (f \psi^{4\alpha} ))_\mathcal{L} + \frac{4\theta}{(2\theta - v^2)^2} f \psi^{4\alpha} \|\nabla v\|_\mathcal{L}^2 + 16n^2\alpha^2 \theta (2\theta - v^2)^2 \varphi^2 v^{-6} f \psi^{4\alpha - 2}\|\nabla \psi\|_\mathcal{L}^2 \]
\[ = -4\alpha \varphi v^{-3}(\nabla v, \nabla (f \psi^{4\alpha} ))_\mathcal{L} + \frac{4\theta}{(2\theta - v^2)^2} f \psi^{4\alpha} \|\nabla v\|_\mathcal{L}^2 + 16n^2\alpha^2 \theta \varphi v^{-2} f \psi^{4\alpha - 2}\|\nabla \psi\|_\mathcal{L}^2. \]

Moreover, we have
\[ -2(\nabla \psi^{4\alpha}, \nabla f)_\mathcal{L} = -2\psi^{-4\alpha}(\nabla \psi^{4\alpha}, \nabla (f \psi^{4\alpha} ))_\mathcal{L} + 32n^2\alpha^2 f \psi^{4\alpha - 2}\|\nabla \psi\|_\mathcal{L}^2. \]

Combining the above gives
\[ \partial_t (f \psi^{4\alpha}) \leq \mathcal{L} (f \psi^{4\alpha}) - (2\psi^{-4\alpha} \nabla \psi^{4\alpha} + 4\alpha \varphi v^{-3}\nabla v, \nabla (f \psi^{4\alpha} ))_\mathcal{L} - 2\alpha f K^{\alpha} H \varphi \psi^{4\alpha} + (32n^2\alpha^2 + 16n^2\alpha^2 \theta \varphi v^{-2} - 4n\alpha(4n\alpha - 1)) f \psi^{4\alpha - 2}\|\nabla \psi\|_\mathcal{L}^2 + 4n^2\alpha^2 K^{\alpha} \psi^{4n\alpha - 1}. \]
In addition, on the support of $\psi$, we have $\nabla \psi = -\nabla \hat{u} = -\nabla (F, \tilde{e}_{n+1})$ which leads to

$$\|\nabla \psi\|_L^2 = \|\nabla (F, \tilde{e}_{n+1})\|_L^2$$

$$\leq \sum_{m=1}^{n+1} \|\nabla (F, \tilde{e}_m)\|_L^2$$

$$= \sum_{m=1}^{n+1} \alpha K^\alpha b^{ij} (F_i, \tilde{e}_m) (F_j, \tilde{e}_m)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha K^\alpha b^{ij} \left( \sum_{m=1}^{n+1} (F_i, \tilde{e}_m) (F_j, \tilde{e}_m) \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha K^\alpha b^{ij} g_{ij} \leq n\alpha K^\alpha \lambda^{-1}_{\text{min}} \leq n\alpha \Lambda K^\alpha.$$ 

Hence, $\nu \geq 1$ implies

$$\left(32n^2\alpha^2 + 16n^2\alpha^2 \theta \nu^{-2} - 4n\alpha(4\alpha - 1)\right) f \psi^{4n\alpha - 2} \|\nabla \psi\|_L^2$$

$$\leq n\alpha(16n^2\alpha^2(\theta + 1) + 4n\alpha) f \psi^{4n\alpha - 2} \Lambda K^\alpha.$$ 

Thus, by the inequalities $H \geq nK^{\frac{1}{\alpha}}$ and $\varphi \geq \frac{1}{2\theta}$, the evolution equation of $f \psi^{4n\alpha}$ can be reduced to

$$\partial_t (f \psi^{4n\alpha}) \leq \mathcal{L}(f \psi^{4n\alpha}) - (2\psi^{-4n\alpha} \nabla \psi^{4n\alpha} + 4\theta \psi \psi^{-3} \nabla \psi, \nabla f \psi^{4n\alpha}) \mathcal{L}$$

$$- n\alpha \theta^{-1} K^{\alpha + \frac{1}{\alpha}} \psi^{4n\alpha}$$

$$+ 4n^2\alpha^2(4n\alpha(\theta + 1) + 1) \Lambda K^\alpha f \psi^{4n\alpha - 2} + 4n^2\alpha^2 K^\alpha \psi^{4n\alpha - 1}.$$ 

Involving $\eta := t(1 + t)^{-1}$ and $\partial \eta_t = (1 + t)^{-2} \leq 1$ yields

$$\partial_t (f \psi^{4n\alpha}) \leq \mathcal{L}(f \psi^{4n\alpha}) - (2\psi^{-4n\alpha} \nabla \psi^{4n\alpha} + 4\theta \psi \psi^{-3} \nabla \psi, \nabla \eta^{2n\alpha} f \psi^{4n\alpha}) \mathcal{L}$$

$$- n\alpha \theta^{-1} K^{\alpha + \frac{1}{\alpha}} \eta^{2n\alpha} f \psi^{4n\alpha}$$

$$+ 4n^2\alpha^2(4n\alpha(\theta + 1) + 1) \Lambda K^\alpha \eta^{2n\alpha} f \psi^{4n\alpha - 2}$$

$$+ 4n^2\alpha^2 K^\alpha \eta^{2n\alpha} f \psi^{4n\alpha - 1} + 2n\alpha \eta^{2n\alpha - 1} f \psi^{4n\alpha}.$$ 

Now, by conditions (ii) and (iii) in Theorem 1.1, $\psi$ is compactly supported. Hence $\eta^{2n\alpha} f \psi^{4n\alpha}$ attains its maximum in $M^n \times [0, T]$ at some $(p_0, t_0)$ with $t_0 > 0$. Then, the last inequality implies that at $(p_0, t_0)$,

$$n\alpha \theta^{-1} K^{\alpha + \frac{1}{\alpha}} \eta^{2n\alpha} f \psi^{4n\alpha} \leq 4n^2\alpha^2(4n\alpha(\theta + 1) + 1) \Lambda K^\alpha \eta^{2n\alpha} f \psi^{4n\alpha - 2}$$

$$+ 4n^2\alpha^2 K^\alpha \eta^{2n\alpha} f \psi^{4n\alpha - 1} + 2n\alpha \eta^{2n\alpha - 1} f \psi^{4n\alpha}.$$ 

Multiplying by $(n\alpha)^{-1} \theta K^{-\alpha} \eta^{-2n\alpha + 1} f^{-1} \psi^{4n\alpha - 2}$ yields the bound

$$\eta K^{\frac{1}{2}} \psi^2 \leq 4n\alpha \theta(4n\alpha(\theta + 1) + 1) \Lambda \eta + 4n\alpha \theta \eta \psi + 2\theta K^{-\alpha} \psi^2.$$
and by $\theta \geq 1, \psi \leq M, 1 \leq M_1$, and $\eta \leq 1$,
\[
\eta K^\frac{1}{2} \psi^2 \leq 4n\alpha \eta((4n\alpha(\theta + \psi) + \psi) + 2\theta \psi^{2+2n\alpha}(\eta K^\frac{1}{2} \psi^2)^{-n\alpha}
\leq 4n\alpha \theta(8n\alpha + 1)(\theta \Lambda + M) + 2\theta M^{2+2n\alpha}(\eta K^\frac{1}{2} \psi^2)^{-n\alpha}
\leq 2\theta(16n^2\alpha^2 + 2n\alpha + M^{2n\alpha}(\eta K^\frac{1}{2} \psi^2)^{-n\alpha})(\theta \Lambda + M^2).
\]
Hence, in the case of $\eta K^\frac{1}{2} \psi^2 \geq M^2$, the last inequality yields
\[
\eta K^\frac{1}{2} \psi^2 \leq 2\theta(16n^2\alpha^2 + 2n\alpha + 1)(\theta \Lambda + M^2) \leq 2\theta(4n\alpha + 1)^2(\theta \Lambda + M^2).
\]
In the other case, we can simply obtain
\[
\eta K^\frac{1}{2} \psi^2 \leq M^2 \leq 2\theta(4n\alpha + 1)^2(\theta \Lambda + M^2).
\]
Thus, at $(p_0, t_0)$,
\[
\eta K^\frac{1}{2} \psi^2 \leq 2\theta(4n\alpha + 1)^2(\theta \Lambda + M^2).
\]
Let $\Psi$ denote the maximum value $\eta^{2n\alpha} \psi \psi^{4n\alpha}(p_0, t_0) = \eta^{2n\alpha} \varphi K^{2\alpha} \psi\psi^{4n\alpha}(p_0, t_0)$. Then, $\varphi \leq 1$ gives
\[
\Psi \leq (\eta K^\frac{1}{2} \psi^2)^{2\alpha n}(p_0, t_0) \leq (2\theta)^{2\alpha n}(4n\alpha + 1)^{4\alpha n}(\theta \Lambda + M^2)^{2\alpha n}.
\]
Using also that $(2\theta)^{-1} \leq \varphi$, we finally conclude that for all $p \in M^n$ and $t \in [0, T_0]$ the following holds:
\[
\frac{\eta^{2n\alpha} K^{2\alpha} \psi\psi^{4n\alpha}(p, t)}{2\theta} \leq \psi \leq \Psi \leq (2\theta)^{2\alpha n}(4n\alpha + 1)^{4\alpha n}(\theta \Lambda + M^2)^{2\alpha n}.
\]
Hence, setting $t = T_0$ yields
\[
(\eta K^\frac{1}{2} \psi^2)(p, T_0) \leq (2\theta)^{1+\frac{1}{2\alpha n}}(4n\alpha + 1)^2(\theta \Lambda + M^2)
\]
and the desired result simply follows by substituting $T_0$ by $t$.

5. Long time existence

In this final section, we will establish the all-time existence of the complete non-compact $\alpha$-Gauss curvature flow (1.1), as stated in our main Theorem 1.1. Our proof will be based on the a priori estimates in Sections 2–4, and the proof will be done in two steps. We will first show, in the next theorem, the existence of a complete solution $\Sigma_t$ on $t \in (0, T)$, where $T = T_\Omega$ depends on the domain $\Omega$. We will then construct an appropriate barrier to guarantee that each $\Sigma_t$ remains as a graph over the same domain $\Omega$, for all $t \in (0, T)$, implying that $T = +\infty$ independently from the domain $\Omega$. 

\[\square\]
Theorem 5.1. Let $\Omega, u_0$ and $\Sigma_0$ satisfy the conditions in Theorem 1.1. Assume that $B_R(x_0) \subset \Omega$ for some $R > 0$. Then, given an immersion $F_0 : M^n \to \mathbb{R}^{n+1}$ with $F_0(M^n) = \Sigma_0$, there is a solution $F : M^n \times (0, T) \to \mathbb{R}^{n+1}$ of (1.1) for some $T \geq (na + 1)R^{na+1}$ such that for each $t \in (0, T)$, the image $\Sigma_t := F(M^n, t)$ is a strictly convex smooth complete graph of a function $u(\cdot, t) : \Omega_t \to \mathbb{R}$ defined on a convex open $\Omega_t \subset \Omega$, and also $u(\cdot, t)$ and $\Omega_t$ satisfy the conditions of $u_0$ and $\Omega$ in Theorem 1.1.

We begin with some extra notation.

Notation 5.2. (i) Given a set $A \subset \mathbb{R}^{n+1}$, we denote by $\text{Conv}(V)$ its convex hull:

$$\text{Conv}(V) := \{tx + (1-t)y : x, y \in A, t \in [0, 1]\}.$$ 

(ii) Let $\Sigma$ be a convex complete (either non-compact or closed) hypersurface. If a set $V$ is a subset of $\text{Conv}(\Sigma)$, we say $V$ is enclosed by $\Sigma$ and use the notation $V \leq \Sigma$.

In particular, if $V \cap \Sigma = \emptyset$ and $V \leq \Sigma$, we use $V \subset \Sigma$.

(iii) For a convex smooth hypersurface $\Sigma$ with a point $X \in \Sigma$, we denote by $K(\Sigma)(X)$ the Gauss curvature of $\Sigma$ at $X$.

(iv) For a convex complete graph $\Sigma$ with a point $X \in \Sigma$, we define $\bar{u}(\Sigma)(X)$ and $\nu_{\min}(\Sigma)(X)$ by

$$\bar{u}(\Sigma)(X) = \langle X, \bar{e}_{n+1} \rangle,$$

$$\nu(\Sigma)(X) = \sup \{\bar{u}_L, \bar{e}_{n+1} \}^{-1} : L \text{ is a hyperplane tangent to } \Sigma \text{ at } X\},$$

where $\bar{u}_L$ is the upward unit normal vector of a hyperplane $L$.

(v) We denote the $(n + 1)$-ball of radius $R$ centered at $Y \in \mathbb{R}^{n+1}$ by

$$B_R^{n+1}(Y) := \{X \in \mathbb{R}^{n+1} : |X - Y| < R\}.$$ 

(vi) For a convex closed hypersurface $\Sigma$, we define the support function $S : S^n \to \mathbb{R}$ by

$$S(v) = \max_{Y \in \Sigma} \langle v, Y \rangle.$$ 

(vii) For a convex hypsersurface $\Sigma$ and $\eta > 0$, we denote the $\eta$-envelope of $\Sigma$ by

$$\Sigma^\eta := \{Y \in \mathbb{R}^{n+1} : d(Y, \Sigma) = \eta, Y \notin \text{Conv}(\Sigma)\},$$

where $d$ is the distance function.

(viii) For $r > 0$ and $(x_0, t_0) \in \mathbb{R}^n \times (0, +\infty)$, we denote the parabolic cube centered at $(x_0, t_0)$ by

$$Q_r((x_0, t_0)) := B_r(x_0) \times (t_0 - r^2, t_0],$$

where $B_r(x_0)$ is the $n$-ball of radius $r$ centered at $x_0 \in \mathbb{R}^n$. Also, for $\beta \in (0, 1)$, $C^{\beta, \beta/2}_{\Sigma, t}(Q_r)$ denotes the standard Hölder space with respect to the parabolic distance.
In the proof of Theorem 5.1 we will use a standard Schauder estimate for equation (1.2). However, since (1.2) is not a concave equation, we cannot directly use the known regularity theory. To detour this difficulty, we slightly modify the standard $C^{2,\beta}$ estimate of Tian and Wang in [20].

**Proposition 5.3 ($C^{2,\beta}$ estimate).** Let $u : Q_r \to \mathbb{R}$ be a strictly convex and smooth solution of (1.2) with $Q_r := Q_r((x_0, t_0))$. Then, there exist some $\beta \in (0, 1)$ such that for any $\sigma \in (0, 1)$, we have

$$
\| D^2 u \|_{C^{0,\beta/2}(Q_{\sigma r})} \leq C(r, \sigma, \alpha, \beta, n, \sup_{Q_r} |u|, \sup_{Q_r} \nu, \sup_{Q_r} K, \inf_{Q_r} \lambda_{\text{min}}),
$$

where $\nu$, $K$, and $\lambda_{\text{min}}$ are, respectively, the gradient function, the Gauss curvature, and the smallest principal curvature of $\Sigma_t = \{(y, u(y, t)) : y \in B_r(x_0)\}$ at $(x, u(x, t))$.

**Proof.** One can easily show that $\sup_{Q_r} \nu$, $\sup_{Q_r} K$, $\inf_{Q_r} \lambda_{\text{min}}$ control the ellipticity constant from above and below of the fully-nonlinear operator in (1.2). Thus, the space-time Hölder estimate for $D^2 u$ can be obtained by Krylov–Safonov’s estimate in [18], as in Step 1 of the proof of [20, Theorem 2.1]. Now, we transform (1.2) into the following equation:

$$
\frac{1}{n} \frac{\partial |D^2 u|^n}{\partial t} = \frac{|D^2 u|^n}{\alpha} \left(1 + |D^2 u|^2\right)^{-\frac{n+2\alpha-1}{2\alpha}}.
$$

Since $(\det D^2 u)^{\frac{1}{n}}$ is a concave operator, the result of Caffarelli in [5] gives us a space Hölder estimate for $D^2 u(\cdot, t)$ for each $t$. Thus, the Hölder estimate for $D^2 u$ in $t$ can be achieved in the same manner as in Step 2 of the proof of [20, Theorem 2.1].

**Proof of Theorem 5.1.** We will obtain a solution $\Sigma_t := \{(x, u(\cdot, t)) : x \in \Omega_t \subset \mathbb{R}^n\}$ as a limit

$$
\Sigma_t := \lim_{j \to +\infty} \Gamma_t^j,
$$

where $\Gamma_t^j$ is a strictly convex closed hypersurface which is symmetric with respect to the hyperplane $x_{n+1} = j$ and also evolves by the $\alpha$-Gauss curvature flow (1.1). Let

$$
\Sigma_t^j := \Gamma_t^j \cap (\mathbb{R}^n \times [0, j])
$$

denote the lower half of $\Gamma_t^j$. Then, the symmetry guarantees that each $\Sigma_t^j$ is a graph over the same hyperplane $\mathbb{R}^n$. Thus, by applying the local a priori estimates shown in Sections 2–4 on compact subsets of $\mathbb{R}^{n+1}$, we obtain uniform $C^\infty$ bounds on the lower half of $\Sigma_t^j$ necessary to pass to the limit.

**Step 1: The construction of the approximating sequence $\Sigma_t^j$.** Let $u_0$, $\Sigma_0$ and $\Omega$ be as in Theorem 1.1 and assume that $\inf_\Omega u_0 = 0$.

For each $j \in \mathbb{N}$, we reflect $\Sigma_0 \cap (\mathbb{R}^n \times [0, j])$ over the $j$-level hyperplane

$$
\mathbb{R}^n \times \{j\} := \{(x, j) : x \in \mathbb{R}^n\}
$$

to obtain a uniformly convex closed hypersurface $\tilde{\Gamma}_0^j$ defined by

$$
\tilde{\Gamma}_0^j = \{(x, h) \in \mathbb{R}^{n+1} : h \in \{u_0(x), 2j - u_0(x)\}, x \in \Omega, u_0(x) \leq j\}. $$
Then, we let $\Gamma^j_0$ denote the $(1/j)$-envelope of $\bar{\Gamma}^j_0$, which is a uniformly convex closed hypersurface of class $C^{1,1}$ and we denote it by $(\Gamma^j_0)^{1/j}$. Now, by [2, Theorem 15], there is a unique convex closed viscosity solution $\Gamma^j_t$ of (1.1) with initial data $\bar{\Gamma}^j_0$ which is defined for $t \in (0, T_j)$, where $T_j$ is its maximal existing time. In addition, the uniqueness guarantees the symmetry of $\Gamma^j_t$ with respect to the hyperplane $\mathbb{R}^n \times \{j\}$. Hence, its lower half 

\[ \Sigma^j_t := \Gamma^j_t \cap (\mathbb{R}^n \times [0, j]) \]

is given by the graph of a function $u^j(\cdot, t)$ defined on a convex set $\Omega^j_t \subset \mathbb{R}^n$, namely

\[ \Sigma^j_t := \partial \left( \bigcup_{j \in \mathbb{N}} \text{Conv}(\Gamma^j_j) \right), \quad \Omega^j_t = \bigcup_{j \in \mathbb{N}} \Omega^j_t, \quad t \in [0, T), \quad \text{where } T = \sup T_j. \]

**Step 2: Properties of the approximating sequence $\Sigma^j_t$.** The following hold:

(P1) $\Gamma^j_t \subset \Gamma^j_{t+1}$, $T_j \leq T_{j+1}$, and $u_0(y) \leq u^j(y, t) \leq u^j(y, t)$ hold for all $j \in \mathbb{N}$, $t < T_j$, and $y \in \Omega^j_t$.

(P2) If $B_R(x_0) \subset \Omega$, then $T \geq (n\alpha + 1)^{-1}R^{n\alpha + 1}$.

**Proof of properties (P1) and (P2).** Since we have $\Gamma^j_0 \leq \Gamma^j_{j+1}$, the comparison principle gives that $\Gamma^j_t \leq \Gamma^j_{j+1}$, which yields $u_0(y) \leq u^{j+1}(y, t) \leq u^j(y, t)$. Moreover, since $\Gamma^j_t$ exists until it converges to a point by [2, Theorem 15], we also have $T_j \leq T_{j+1}$.

Let us next show (P2). $B_R(x_0) \subset \Omega$ means that there is a constant $h_0$ satisfying $B_{R}^{n+1}((x_0, h_0)) \leq \Sigma_0$. Set $X_0 = (x_0, h_0)$ and choose $j \geq R + h_0$ so that $B_R^{n+1}(X_0) \leq \bar{\Gamma}^j_0$ holds. On the other hand, $\partial B_{\rho_j(t)}(X_0)$ is a solution of (1.1), where

\[ \rho_j(t) = (R^{n\alpha + 1} - (n\alpha + 1)t)^{\frac{1}{n\alpha + 1}}. \]

Hence, the comparison principle leads to $\partial B_{\rho_j(t)}(X_0) \leq \Gamma^j_t$, and by [2, Theorem 15], $\Gamma^j_t$ exists while $\rho_j(t) > 0$. Thus, (P2) holds.

**Step 3: A priori estimates for $\Gamma^j_t$.** We verify that the a priori estimates in Sections 2–4 apply $\Gamma^j_t$ for the cut-off $\psi_B$ with $M < j$, even though $\Gamma^j_0$ fails to be smooth.

By considering $(0, j)$ as a origin, we have a positive and symmetric support function $S^j_0$ of $\bar{\Gamma}^j_0$. We recall the compactly supported mollifiers $\varphi_\epsilon$ on $S^n$ in [8, Proposition 6.2], and let $S^j_0*\varphi_\epsilon$ denote the convolution $S^j_0 * \varphi_\epsilon$ on $S^n$. Then, by [8, Proposition 6.3], for a small $\epsilon$, $S^j_0*\varphi_\epsilon$ is the support function of a strictly convex smooth closed hypersurface $\Gamma^j_0$. Let $\Gamma^j_{0, \epsilon}$ be the unique strictly convex smooth closed solution of (1.1) defined for $t \in [0, \bar{T}^j_{0, \epsilon}]$ (cf. [7]). Moreover, the uniqueness and the symmetry of $\Gamma^j_{0, \epsilon}$ guarantee that $\Gamma^j_{0, \epsilon}$ is also symmetric with respect to $\mathbb{R}^n \times \{j\}$. Therefore, the lower half of $\Gamma^j_{0, \epsilon}$ remains as a graph and thus a priori estimates as in Sections 2–4 apply for $\Gamma^j_{0, \epsilon}$ with $M \leq j$.

On the other hand, by [8, Proposition 6.3], $\Gamma^j_{0, \epsilon}$ converge to $\Gamma^j_0$ in $C^1$ sense and the following holds:

\[ \liminf_{\epsilon \to 0} \lambda_{\min}(\Gamma^j_{0, \epsilon})(X_\epsilon) \geq \lambda_{\min}(\Gamma^j_0)(X), \]

where $\{X_\epsilon\}$ is a set of points $X_\epsilon \in \Gamma^j_{0, \epsilon}$ converging to $X \in \Sigma^j$ as $\epsilon \to 0$. 


Thus, for $M < j$ and $Q_M := \mathbb{R}^n \times [-1, M]$, the following hold:

$$\liminf_{\epsilon \to 0} \left( \inf_{x \in Q_M} \lambda_{\min}(\Gamma_{t}^{j}(x)) \right) \geq \inf_{x \in Q_M} \lambda_{\min}^{loc}(\Gamma_{0}^{j})(x),$$

$$\limsup_{\epsilon \to 0} \left( \sup_{x \in Q_M} \nu(\Gamma_{t}^{j}(x)) \right) \leq \sup_{x \in Q_M} \nu(\Gamma_{0}^{j})(x).$$

Hence, Theorem 2.2 and Theorem 3.3 give the uniform gradient estimate and the uniform lower bounds of principal curvatures for $\Gamma_{t}^{j,\epsilon}$, and therefore Theorem 4.1 guarantees the uniform upper bound for principal curvatures for $\Gamma_{t}^{j,\epsilon}$. Now, we let the lower half of $\Gamma_{t}^{j,\epsilon}$ be the graph of a function $u^{j,\epsilon}(0, t)$. Then, we have $|u^{j,\epsilon}| \leq j$ and thus Proposition 5.3 implies the local uniform $C^{2,\beta}$ estimates for $u^{j,\epsilon}$. Hence, the limit $\lim_{\epsilon \to 0} u^{j,\epsilon} = u^j$ satisfies the same estimates, and the lower half of $\Gamma_{t}^{j}$ is a smooth graph for $t > 0$.

**Step 4: Passing $\Gamma_{t}^{j}$ to the limit $\Sigma_{t}$.** First we observe that $\bigcup_{j \in \mathbb{N}} \text{Conv}(\Gamma_{t}^{j})$ is a convex body by property (P1):

$$\Gamma_{t}^{j} \subseteq \Gamma_{t}^{j+1} \subseteq \Sigma_{0}.$$  

Thus, $\Sigma_{t}$ is a complete and convex hypersurface embedded in $\mathbb{R}^{n+1}$.

For any constant $M > 0$, $t_0 > 0$, $\sigma > 0$, and $j > M$, we apply the gradient estimate given in Theorem 2.2 with $\beta = \min\{M, t_0^{-1}\sigma\}$, which yields a uniform gradient bound of $\Gamma_{t}^{j}$ for $j > M$ and $t \in [0, t_0]$ in $\mathbb{R}^n \times [0, M - 2\sigma]$. So, the gradient function $\nu(\Sigma_{t})$ of $\Sigma_{t}$ is bounded in each $\mathbb{R}^n \times [0, M - 2\sigma]$. Thus, $\Sigma_{t}$ is a complete convex graph of a function $u(\cdot, t)$ defined on a convex open set $\Omega_{t}$.

In addition, by (P1), we have $u_0(x) \leq u(x, t)$. Moreover, $u(\cdot, t)$ and $\Omega_{t}$ satisfy the conditions (i), (ii), (iii) of $u_0$ and $\Omega$ in Theorem 1.1.

Moreover, by applying the curvature lower bound in Theorem 3.3 and the speed bound in Theorem 4.1, we have a uniform curvature bound from below and above of $\Gamma_{t}^{j}$ in $[0, M - \sigma]$ for large $j > M$, small $\sigma > 0$, and $t \in [t_0, T]$ with $0 < t_0 < T$.

**Step 5: Passing $u^j$ to the limit $u$.** Recall the sequence $u_j$ of solutions to (1.2) defined in Step 1. For $t \in (0, T)$, the following holds by (P1):

$$u(y, t) = \lim_{j \to \infty} u_j(y, t).$$

We begin by choosing $t_1, t_0 \in (0, T)$ ($t_1 < t_0$) and $y_0 \in \Omega_{t_0}$. Since $\Omega_{t_0}$ is open, there is a small $r > 0$ satisfying $B_r(y_0) \subset \Omega_{t_0}$ and $r^2 < t_0 - t_1$. On the other hand, (P1) gives the monotone convergence of $\Omega_{t_0}^{j}$ to $\Omega_{t_0}$. Hence, for some large $J_0 \in \mathbb{N}$, we have $B_r(y_0) \subset \Omega_{J_0, t_0}$, implying that $u^{J_0}(y, t_0) \leq J_0$, for all $y \in B_r(y_0)$. Thus, for all $j \geq J_0$, $t \in [t_1, t_0]$, and $y \in B_r(y_0)$, the convexity of $u^j$ and (P1) lead to

$$0 \leq u_0(y) \leq u^j(y, t_0) \leq u^j(y, t_0) \leq u^{J_0}(y, t_0) \leq J_0. \quad (5.1)$$

Notice that in Step 4, we have shown uniform estimates for $\lambda_{\min}^{-1}$, $\nu$, and $K$ of $\{\Sigma_{t}^{j}\}_{t \in [t_1, t_0]}$ in $\mathbb{R}^n \times [0, J_0]$ for $j > J_0$. Therefore, Proposition 5.3 and (5.1) imply that $u$ is of class $C^{2,\beta}_{loc}(B_r(y_0) \times (t_1, t_0))$ for some $\beta \in (0, 1)$, and $u(y, t)$ is a strictly convex $C^{2,\beta}$ solution of (1.2). Hence, standard regularity results show that actually $u(y, t)$ is $C^\infty$ smooth. Thus, by (P2), we assure that $\Sigma_{t}$ is the desired solution. This finishes the proof of Theorem 5.1. \(\square\)
To finish with the proof of Theorem 1.1, it remains to show that

\[ \Omega_t = \Omega, \]

which follows from the next theorem.

**Theorem 5.4.** Let \( \Omega, u_0 \) and \( \Sigma_0 \) satisfy the conditions in Theorem 1.1. Assume that \( \Sigma_t = \{(x, u(x, t)) : \Omega_t, t \in [0, T]\} \) is a strictly convex smooth complete graph solution of (1.1) such that \( \Omega_t, u(\cdot, t) \) and \( \Sigma_t \) satisfy the conditions of \( \Omega, u_0 \) and \( \Sigma_0 \) in Theorem 1.1. Then, for any closed ball \( \overline{B_{R_0}(y_0)} \subset \Omega \) and any \( t_0 \in (0, T) \), the following holds:

\[ B_{R_0}(y_0) \subset \Omega_t. \]

**Proof.** Let \( y_0 \in \Omega, R_0 \) and \( t_0 \in (0, T) \) be as in the statement of the theorem. Without loss of generality, we may assume that \( y_0 = 0 \) and \( R_0 < 1 \). The given condition \( \overline{B_R(0)} \subset \Omega \) implies that there exists a constant \( m_0 \geq 0 \) such that \( \overline{B_{R_0}(0)} < L_{m_0}(\Sigma_0) \). For each \( m \geq m_0 + 1 \) and each \( \delta > 0 \) sufficiently small satisfying

\[ \delta + 2^{1+n\alpha} R_0^{-n\alpha} \delta^\alpha t_0 < R_0/2 \]

we define the function \( f^{\delta, m} : [m - 1, m] \times [0, t_0] \rightarrow \mathbb{R} \) by

\[ f^{\delta, m}(h, t) := R_0 - \delta(h - m)^2 - 2^{1+n\alpha} R_0^{-n\alpha} \delta^\alpha t. \]

Let \( f^{-1}(\cdot, t) \) denote the inverse function of \( f(\cdot, t) \) and let \( \Phi^{\delta, m}_t \) denote the graph of the rotationally symmetric function \( \psi(y) = f^{-1}(|y|, t) \), namely

\[ \Phi^{\delta, m}_t := \{(y, h) : |y| = f^{\delta, m}(h, t)\} \]

(see Figure 2). By definition, we have \( \Phi^{0, m}_t < \Sigma_0 \). We will show that \( \Phi^{\delta, m}_t \) defines a supersolution of (1.1) and consequently that

\[ \Phi^{\delta, m}_{t_0} < \Sigma_{t_0} \quad \text{for} \quad m > M_\delta + 1 \]

for some constant \( M_\delta \) depending on \( \delta \). Thus,

\[ B_{f^{\delta, m}(m, t_0)}(0) = L_m(\Phi^{\delta, m}_{t_0}) < L_m(\Sigma_{t_0}) \leq \Omega_{t_0}, \]

and by passing \( \delta \) to zero, we will obtain the desired result.

Let us first show that \( \Phi^{\delta, m}_t \) defines a supersolution of (1.1). For convenience, we denote \( \partial_r(f^{-1})(r, t) \) and \( \partial_{rr}(f^{-1})(r, t) \) by \( f^{-1}_r \) and \( f^{-1}_{rr} \), respectively. Then, the Gauss curvature \( K \) of \( \Phi^{\delta, m}_t \) satisfies

\[ K = \frac{f^{-1}_{rr} |f^{-1}_r|^n}{r^{n-1} (1 + |f^{-1}_r|^2)^{\frac{n+2}{2}}} \]

\[ \leq \frac{f^{-1}_{rr}}{R_0/2 |r^{n-1} (1 + |f^{-1}_r|^2)^{\frac{2}{2}}} \]

\[ = -\frac{2^n f_{hh}}{R_0^{n-1} (1 + f^2_h)^2} \leq \frac{2^n \delta}{R_0^{n-1}} < 2^n R_0^{-n} \delta. \]
Also, the gradient function $\nu$ of $f^{-1}$ on $L_{(m-1,m)}(\Phi^\delta_{t,m})$ satisfies
\[
\nu = (1 + |f^{-1}_{r}|^2)^{\frac{1}{2}} = \left(1 + \frac{1}{4\delta^2(h-m)^2}\right)^{\frac{1}{2}} \leq \left(1 + \frac{4\delta^2(h-m)^2}{2\delta(m-h)}\right)^{\frac{1}{2}} \leq \frac{1}{\delta(m-h)}
\]
since $\delta < R_0/2 < \frac{1}{2}$. Finally, on $L_{(m-1,m)}(\Phi^\delta_{t,m})$, we can derive from $f^{-1}(f(h,t),t) = h$ the following:
\[
\partial_t(f^{-1}) = -f^{-1}_{r}\partial_t f = \frac{2^{\frac{1}{2}+\alpha}R_0^{-\alpha}\delta^\alpha}{2\delta(m-h)} = \frac{2^{\frac{1}{2}+\alpha}R_0^{-\alpha}\delta^\alpha}{\delta(m-h)} > K^\alpha \nu.
\]
Thus, $\Phi^\delta_{t,m}$ is a supersolution of (1.1). In other words, $\Sigma_t$ cannot contact with $\Phi^\delta_{t,m}$ in the interior of $\Phi^\delta_{t,m}$.

If there exists the contact time $t^\delta,m \in (0,t_0]$ such that $\Sigma_t \cap \text{Conv}(\Phi^\delta_{t,m}) = \emptyset$ for $t \in (0,t^\delta,m)$ and $\Sigma_{t^\delta,m} \cap \text{Conv}(\Phi^\delta_{t^\delta,m}) \neq \emptyset$, then we denote the contact set by $Z^\delta,m$:
\[
Z^\delta,m := \Sigma_{t^\delta,m} \cap \text{Conv}(\Phi^\delta_{t^\delta,m}) = \Sigma_{t^\delta,m} \cap \Phi^\delta_{t^\delta,m}.
\]
We have just seen that
\[
Z^\delta,m \subset \partial(\Phi^\delta_{t^\delta,m}) = L_{(m-1)}(\Phi^\delta_{t^\delta,m}) \cup L_m(\Phi^\delta_{t^\delta,m}),
\]
and since $\Sigma_t$ is a graph, it cannot contact $\Phi^\delta_{t,m}$ on $L_m(\Phi^\delta_{t^\delta,m})$. Hence,
\[
Z^\delta,m \subset L_{(m-1)}(\Phi^\delta_{t^\delta,m}).
\]
If there exist a contact point \((z_0, m - 1) \in Z_{\delta,m}\), then \(|z_0| = f_{\delta,m}(m - 1, t_{\delta,m}) < R_0\) and the slope of the graph of \(u\) at this point is at most equal to the slope of \(\Phi_{\delta,m}\) (see Figure 2(c)), hence \(|Du|(z_0, t_{\delta,m}) \leq (2\delta)^{-1}\).

On the other hand, since \(A := \{(y, t) : t \in [0, t_0], y \in \Omega_t, |y| \leq R_0, |Du|(y, t) \leq (2\delta)^{-1}\}\) is a compact set, the function \(u\) attains its maximum in \(A\). Set

\[M := \max \{u(y, t) : (y, t) \in A\}.\]

Since \((z_0, t_{\delta,m}) \in A\), we must have \(m - 1 = u(z_0, t_{\delta,m}) \leq M_{\delta}\). Therefore, if \(m > M_{\delta} + 1\), then

\[Z_{\delta,m} \cap L_{(m-1)}(\Phi_{\delta,m}) = \emptyset\]

concluding that \(\Phi_{\delta,m} \leq \Sigma_{t_0}\) and \(B_{f_{\delta,m}(m,t_0)}(0) \leq \Omega_{t_0}\). By passing \(\delta\) to zero, we obtain the desired result. 

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