Monotone Data Visualization Using Rational Trigonometric Spline Interpolation

Farheen Ibraheem, Maria Hussain, and Malik Zawwar Hussain

1 National University of Computer and Emerging Sciences, Lahore, Pakistan
2 Department of Mathematics, Lahore College for Women University, Lahore 54600, Pakistan
3 Department of Mathematics, University of the Punjab, Lahore 54590, Pakistan

Correspondence should be addressed to Maria Hussain; mariahussain1@yahoo.com

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Monotonicity is an indispensable characteristic of data stemming from many physical and scientific experiments. The relationship between the partial pressure of oxygen and percentage dissociation of hemoglobin, consumption function in economics, concentration of atrazine and nitrate in shallow ground waters, and approximation of couples and quasi couples are few phenomena which exhibit monotone trend.

Efforts have been put in by many researchers and a variety of approaches has been proposed to solve this eminent issue [1–17]. Cripps and Hussain [3] visualized the 2D monotone data by Bernstein-Bézier rational cubic function. The authors in [3] converted the Bernstein-Bézier rational cubic function to $C^1$ cubic Hermite by applying the $C^1$ continuity conditions at the end points of interval. The lower bounds of weights functions were determined to visualize monotone curve as monotone curve. Hussain and Sarfraz [8] have conserved monotonicity of curve data by rational cubic function with four shape parameters, two of which were set free and two were shape parameters. Data dependent constraints on shape parameters were developed which assure the monotonicity but one shape parameter is dependent on the other which makes it economically very expensive. Rational cubic function with two shape parameters suggested by Sarfraz [13] sustained monotonicity of curves but lacked the liberty to amend the curve which makes it inappropriate for interactive design. Piecewise rational cubic function was used by M.
Z. Hussain and M. Hussain [7] to visualize 2D monotone data by developing constraints on the free parameters in the specification of rational cubic function. The authors also extended rational cubic function to rational bicubic partially blended function. Simple constraints were derived on the free parameters in the description of rational bicubic partially blended patches to visualize the 3D monotone data. Three kinds of monotonicity preservation of systems of bivariate functions on triangle were defined and studied by Floater and Peña [5]. Sarfraz et al. [12] developed constraints in the specification of a bicubic function to visualize the shape of 3D monotone data.

This paper is a noteworthy addition in the field of shape preservation when the data under consideration admits monotone trend. The suggested algorithm offers numerous advantages over the prevailing ones. Orthogonality of sine and cosine function compels much smoother visual results as compared to algebraic spline. Derivative of the trigonometric spline is much lower than that of algebraic spline. Moreover, trigonometric splines play an instrumental role in robotic manipulator path planning.

The remainder of the paper is structured as follows. Section 2 is devoted to reviewing the rational trigonometric cubic function developed in [11]. In Section 3, rational trigonometric cubic function is extended to rational trigonometric bicubic function. Section 4 aims to develop monotonicity preserving constraints for 2D data. Section 5 submits a solution to shape preservation of 3D monotone data. In Section 6, numerical examples have been demonstrated. Section 7 draws the conclusion and significance of this research.

2. Rational Trigonometric Cubic Function

In this section, rational trigonometric cubic function [11] is reviewed.

Let \( \{ (x_i, f_i) \}, \quad i = 0, 1, 2, \ldots, n \) be the given set of data points defined over the interval \([a, b]\), where \( a = x_0 < x_1 < x_2 < \cdots < x_n = b \). Piecewise rational trigonometric cubic function is defined over each subinterval \( I_i = [x_i, x_{i+1}] \) as

\[
S_i(x) = \frac{p_i(\theta)}{q_i(\theta)},
\]  

\[
p_i(\theta) = \alpha_i f_i (1 - \sin \theta)^3 + \beta_i f_i (1 - \sin \theta)^2 + \gamma_i f_i (1 - \cos \theta)^2 + \delta_i (1 - \cos \theta)^3,
\]

where \( \theta = (\pi/2)(x - x_i)/h_i \), \( h_i = x_{i+1} - x_i \).

The rational trigonometric cubic function (1) is \( C^1 \); that is, it satisfies the following properties:

\[
S(x_i) = f_i, \quad S(x_{i+1}) = f_{i+1},
\]

\[
S'(x_i) = d_i, \quad S'(x_{i+1}) = d_{i+1}.
\]

Here \( d_i \) and \( d_{i+1} \) are derivatives at the end points of the interval \( I_i = [x_i, x_{i+1}] \). The parameters \( \alpha_i \) and \( \delta_i \) are real numbers used to modify the shape of the curve.

3. Rational Trigonometric Bicubic Partially Blended Function

Let \( \{(x_i, y_j, F_{i,j})\}, \quad i = 0, 1, 2, \ldots, n-1; \quad j = 0, 1, 2, \ldots, m-1 \) be the 3D regular data set defined over the rectangular mesh \( I = [a, b] \times [c, d] \), let \( p : a = x_0 < x_1 < \cdots < x_m = b \) be a partition of \([a, b]\), and let \( q : a = y_0 < y_1 < \cdots < y_n = b \) be a partition of \([c, d]\). Rational trigonometric bicubic function which is an extension of rational trigonometric cubic function (1) is defined over each rectangular patch \([x_i, x_{i+1}] \times [y_j, y_{j+1}] \), where \( i = 0, 1, 2, \ldots, n-1; j = 0, 1, 2, \ldots, m-1 \), as

\[
S(x, y) = -AFB^2,
\]

where

\[
F = \begin{pmatrix} 0 & S(x, y_j) & S(x, y_{j+1}) \\ S(x_i, y) & S(x_i, y_j) & S(x_i, y_{j+1}) \\ S(x_{i+1}, y) & S(x_{i+1}, y_j) & S(x_{i+1}, y_{j+1}) \end{pmatrix},
\]

\[
A = [-1, a_0(\theta), a_1(\theta)], \quad B = [-1, b_0(\theta), b_1(\theta)],
\]

\[
a_0 = \cos^2 \theta, \quad a_1 = \sin^2 \theta,
\]

\[
b_0 = \cos^2 \phi, \quad b_1 = \sin^2 \phi.
\]

\[
S(x, y_j), S(x, y_{j+1}), S(x_i, y), \text{ and } S(x_{i+1}, y) \text{ are rational trigonometric bicubic functions defined on the boundary of rectangular patch } [x_i, x_{i+1}] \times [y_j, y_{j+1}] \text{ as}
\]

\[
S(x, y_j) = \left( A_0(1 - \sin \theta)^3 + A_1 \sin \theta (1 - \sin \theta)^2 \right.
\]

\[
+ A_2 \cos \theta (1 - \cos \theta)^2 + A_3 (1 - \cos \theta)^3 \big) \times (q_1(\theta))^{-1},
\]

where

\[
A_0 = \alpha_{i,j} h_{i,j} F_{i,j}, \quad A_1 = \beta_{i,j} h_{i,j} F_{i,j}^2 + \frac{2\alpha_{i,j} h_{i,j} F_{i,j}^2}{\pi},
\]

\[
A_2 = \gamma_{i,j} h_{i,j} F_{i,j}^3 + \frac{2\delta_{i,j} h_{i,j} F_{i,j}^3}{\pi}, \quad A_3 = \delta_{i,j} h_{i,j} F_{i,j+1},
\]

\[
q_1(\theta) = \alpha_{i,j} (1 - \sin \theta)^3 + \beta_{i,j} \sin \theta (1 - \sin \theta)^2
\]

\[
+ \gamma_{i,j} \cos \theta (1 - \cos \theta)^2 + \delta_{i,j} (1 - \cos \theta)^3,
\]
The curve will be monotone if the rational trigonometric cubic function (1) has a monotone shape of data. Therefore, it is customary that the resulting interpolating curve must retain the monotone shape of data. In this section, constraints on shape parameters in the description of rational trigonometric cubic function (1) have been developed to preserve 2D monotone data. Let \((x_i, y_i), i = 0, 1, 2, \ldots, n\) be the monotone data defined over the interval \([a, b]\); that is,

\[
f_i < f_{i+1}, \quad \Delta_i = \frac{f_{i+1} - f_i}{h_i} > 0, \quad i = 0, 1, 2, \ldots, n-1,
\]

\[
d_i > 0, \quad i = 0, 1, 2, \ldots, n.
\]

The curve will be monotone if the rational trigonometric cubic function (1) satisfies the condition

\[
S'(x), \quad \forall x \in [x_i, x_{i+1}], \quad i = 0, 1, 2, \ldots, n-1.
\]

Now, we have

\[
\delta'(x) = \int_{a}^{b} S'(x) \, dx = \frac{\pi}{2h_i (q_i(\theta))^2} \times \left\{(1 - \sin \theta)^4 \sin \theta \cos \theta B_0 + \cos^2 \theta (1 - \cos \theta)^2 (1 - \sin \theta)^2 B_1 + \cos \theta (1 - \cos \theta)^3 (1 - \sin \theta)^2 B_2 + \sin \theta \cos \theta (1 - \sin \theta) (1 - \cos \theta)^2 B_3 + \sin \theta (1 - \cos \theta)^2 (1 - \sin \theta)^3 B_4 + \sin^2 \theta (1 - \sin \theta)^2 (1 - \cos \theta)^2 B_5 + \sin \theta \cos \theta (1 - \sin \theta)(1 - \cos \theta)^3 B_6 + \sin^2 \theta \cos \theta (1 - \cos \theta) (1 - \sin \theta)^2 B_7 \right\}.
\]
Algorithm 2.

Step 1. Take a monotone data set \( \{(x_i, f_i) : i = 0, 1, 2, \ldots, n\} \).

Step 2. Use the Arithmetic Mean Method \([11]\) to estimate the derivatives \(d_i\)'s at knots \(x_i\)'s (note: Step 2 is only applicable if data is not provided with derivatives).

Step 3. Compute the values of parameters \( \beta_i \)'s and \( \gamma_i \)'s using Theorem 1.

Step 4. Substitute the values of variables from Steps 1–3 in rational trigonometric cubic function (1) to visualize monotone curve through monotone data.

5. Monotone Surface Interpolation

Let \( \{(x_i, y_j, F_{i,j}) : i = 0, 1, 2, \ldots, m - 1; j = 0, 1, 2, \ldots, n - 1\} \) be the monotone data set defined over the rectangular mesh \( I = [x_i, x_{i+1}] \times [y_j, y_{j+1}] \) such that

\[
F_{i,j} < F_{i+1,j}, \quad F_{i,j} < F_{i,j+1},
\]

\[
F_{i,j}^x > 0, \quad F_{i,j}^y > 0,
\]

\[
\Delta_{i,j} > 0, \quad \Delta_{i,j} > 0.
\]

Now, surface patch (4) is monotone if the boundary curves defined in (6)–(12) are monotone.

Now, \( S(x, y) \) is monotone if \( S'(x, y) > 0 \), where

\[
S'(x, y) = \frac{\pi}{2h_i(d_i(\theta))} \times \left(1 - \sin^4 \theta \cos \theta \cos \theta R_0 + \cos^2 \theta (1 - \cos \theta)^2 (1 - \sin \theta)^2 R_1 + \cos \theta (1 - \cos \theta)^3 (1 - \sin \theta)^2 R_2 + \cos \theta (1 - \sin \theta)^5 R_3 + \sin \theta \cos \theta (1 - \cos \theta)^2 (1 - \sin \theta) R_4 + \sin \theta \cos \theta (1 - \sin \theta) (1 - \cos \theta)^3 R_5 \right. \]

\[
+ \sin \theta (1 - \cos \theta)^2 (1 - \sin \theta)^3 R_6 + \sin \theta (1 - \sin \theta)^2 (1 - \cos \theta)^2 R_7 + \sin \theta \cos \theta (1 - \cos \theta)^4 R_8 + \sin \theta (1 - \cos \theta)^5 R_9 + \sin \theta \cos \theta (1 - \cos \theta) (1 - \sin \theta)^3 R_{10} + \sin \theta \cos \theta (1 - \cos \theta) (1 - \sin \theta)^5 R_{11} \}
\]
Now the positivity of $S'(x, y_{j+1})$ entirely depends on $R_i$, $i = 0, 1, 2, \ldots, 11$. The denominator in (22) is always positive. Since the parameter $\theta$ lies in first quadrant therefore the trigonometric basis functions will be positive also. This yields the following constraints on the free parameters:

$$\beta_{i,j} > \frac{2\alpha_{i,j} F_{i,j}^x}{\pi \Delta_{i,j}}, \quad \gamma_{i,j} > \frac{2\delta_{i,j} F_{i+1,j}^x}{\pi \Delta_{i,j}}. \quad (24)$$

$S(x, y_{j+1})$ is monotone if

$$S'(x, y_{j+1}) > 0, \quad (25)$$

where

$$S'(x, y_{j+1}) = \frac{\pi}{2h(q_2(\theta))^2} \times \left(1 - \sin \theta\right)^4 \sin \theta \cos \theta T_0 + \cos^2 \theta (1 - \cos \theta)^2 (1 - \sin \theta)^2 T_1 + \cos \theta (1 - \cos \theta)^3 (1 - \sin \theta)^2 T_2$$

$$+ \cos \theta (1 - \sin \theta)^5 T_3 + \sin \theta \cos^2 \theta (1 - \cos \theta)^2 (1 - \sin \theta) T_4 + \sin \theta \cos (1 - \sin \theta)(1 - \cos \theta) T_5$$

$$+ \sin \theta (1 - \cos \theta)^3 T_6 + \sin^2 \theta (1 - \sin \theta)^2 (1 - \cos \theta)^2 T_7$$

$$+ \sin \theta \cos (1 - \cos \theta)^2 (1 - \sin \theta)^2 T_8 + \sin \theta (1 - \cos \theta)^3 \sin \theta (1 - \sin \theta)^2 T_9$$

$$+ \sin^2 \theta (1 - \cos \theta)(1 - \sin \theta)^2 T_{10} + \sin^2 \theta (1 - \cos \theta)(1 - \sin \theta)^2 T_{11} \right),$$

with

$$T_0 = \frac{2h F_{i+1,j}^x}{\pi} \alpha_{i,j+1}^2,$$

$$T_1 = \left(3\alpha_{i,j+1} + \beta_{i,j+1} \delta_{i,j+1}\right) \Delta_{i,j+1} - \frac{2\beta_{i,j+1} F_{i,j+1}^x}{\pi} \alpha_{i,j+1}^2 + \frac{2\delta_{i,j+1} F_{i,j+1}^x}{\pi} \gamma_{i,j+1}^2$$

$$T_2 = \left(3\gamma_{i,j+1} - \beta_{i,j+1} \delta_{i,j+1}\right) \Delta_{i,j+1} - \frac{2\beta_{i,j+1} F_{i,j+1}^x}{\pi} \gamma_{i,j+1}^2 + \frac{2\delta_{i,j+1} F_{i,j+1}^x}{\pi} \beta_{i,j+1}^2$$

$$T_3 = \frac{2h \alpha_{i,j+1} F_{i+1,j}^x}{\pi},$$

$$T_4 = \frac{2h \alpha_{i,j+1} F_{i+1,j}^x}{\pi} \alpha_{i,j+1}^2 + \frac{2h \alpha_{i,j+1} F_{i+1,j}^x}{\pi} \gamma_{i,j+1}^2$$

$$T_5 = \frac{2h \alpha_{i,j+1} F_{i+1,j}^x}{\pi} \beta_{i,j+1}^2 + \frac{2h \alpha_{i,j+1} F_{i+1,j}^x}{\pi} \gamma_{i,j+1}^2$$

$$T_6 = \frac{2h \alpha_{i,j+1} F_{i+1,j}^x}{\pi} \beta_{i,j+1}^2 + \frac{2h \alpha_{i,j+1} F_{i+1,j}^x}{\pi} \gamma_{i,j+1}^2.$$
\[ T_7 = \beta_{i,j+1} \left( y_{i,j+1} - 3\delta_{i,j+1} \right) \Delta_{i,j+1} + \frac{2\alpha_{i,j+1} y_{i,j+1} F^x_{i,j+1}}{\pi} \]
\[ + \left( \beta_{i,j+1} - 3\alpha_{i,j+1} \right) \frac{2\delta_{i,j+1} F^x_{i+1,j+1}}{\pi}, \]
\[ T_8 = \frac{2h_i \delta_{i+1,j+1} F^x_{i,j+1}}{\pi}, \quad T_9 = \frac{2h_i \delta_{i,j+1} F^x_{i+1,j+1}}{\pi}, \]
\[ T_{10} = y_{i,j+1} \Delta_{i,j+1} - \frac{2\delta_{i,j+1} F^x_{i+1,j+1}}{\pi}, \]
\[ T_{11} = \beta_{i,j+1} y_{i,j+1} \Delta_{i,j+1} - \frac{2\beta_{i,j+1} \delta_{i,j+1} F^x_{i+1,j+1}}{\pi} \]
\[ - \frac{2\alpha_{i,j+1} y_{i,j+1} F^x_{i,j+1}}{\pi}. \]  

(27)

The denominator in (26) is always positive. Moreover, the trigonometric basis functions are also positive for \( 0 \leq \theta \leq \pi/2 \). It follows that the positivity of \( S'_{i}(x, y) \) entirely depends upon \( T_{i,j} \), \( i = 0, 1, 2, \ldots, 11 \). This yields the following constraints on the free parameters:

\[ \beta_{i,j+1} > \frac{2\alpha_{i,j+1} F^x_{i,j+1}}{\pi \Delta_{i+1,j}}, \quad y_{i,j+1} > \frac{2\delta_{i,j+1} F^x_{i+1,j+1}}{\pi \Delta_{i+1,j+1}}. \]  

(28)

\( S(x, y) \) is monotone if \( S'_i(x, y) > 0 \). We have

\[ S'_i(x, y) \]
\[ = \frac{\pi}{2h_i(q_i(\phi))^2} x \left\{ (1 - \sin \phi)^4 \sin \phi \cos \phi U_0 \right. \]
\[ + \cos^2 \phi (1 - \cos \phi)^2 (1 - \sin \phi)^2 U_1 \]
\[ + \cos \phi (1 - \cos \phi)^3 (1 - \sin \phi)^2 U_2 \]
\[ + \cos \phi (1 - \sin \phi)^5 U_3 \]
\[ + \sin \phi \cos^2 \phi (1 - \cos \phi)^2 (1 - \sin \phi) U_4 \]
\[ + \sin \phi \cos \phi (1 - \sin \phi) (1 - \cos \phi)^3 U_5 \]
\[ + \sin \phi (1 - \cos \phi)^2 (1 - \sin \phi) U_6 \]
\[ + \sin^2 \phi (1 - \sin \phi)^3 (1 - \cos \phi)^2 U_7 \]
\[ + \sin \phi \cos \phi (1 - \cos \phi)^4 U_8 + \sin \phi (1 - \cos \phi) U_9 \]
\[ + \sin \phi \cos \phi (1 - \cos \phi) (1 - \sin \phi)^3 U_{10} \]
\[ + \sin^2 \phi \cos \phi (1 - \cos \phi) (1 - \sin \phi)^2 U_{11} \}. \]  

(29)

Where

\[ U_0 = \frac{2h_i F^y_{i,j} \delta_{i,j}^2}{\pi}, \]
\[ U_1 = (3\delta_{i,j} \bar{y}_{i,j} - \beta_{i,j+1} \bar{\delta}_{i,j}) \Delta_{i,j} + \frac{2F^y_{i,j} \delta_{i,j} \bar{y}_{i,j}}{\pi} \]
\[ - (3\delta_{i,j} - \bar{\beta}_{i,j}) \frac{2F^y_{i,j+1} \delta_{i,j+1}}{\pi}, \]
\[ U_2 = (\bar{\beta}_{i,j} - 3\alpha_{i,j}) \Delta_{i,j} - \frac{2F^y_{i,j} \delta_{i,j}}{\pi}, \]
\[ U_3 = \frac{2h_i \delta_{i,j} F^y_{i,j}}{\pi}, \]
\[ U_4 = \bar{\beta}_{i,j} \bar{y}_{i,j} \Delta_{i,j} - \frac{4\delta_{i,j} \bar{y}_{i,j} F^y_{i,j+1}}{\pi} - \frac{4\delta_{i,j} \bar{y}_{i,j} F^y_{i,j}}{\pi}, \]
\[ U_5 = \bar{\beta}_{i,j} \bar{\delta}_{i,j} \Delta_{i,j} - \frac{2\delta_{i,j} F^y_{i,j+1}}{\pi}, \]
\[ U_6 = (3\delta_{i,j} - \bar{y}_{i,j}) \Delta_{i,j} + \frac{2\delta_{i,j} F^y_{i,j+1}}{\pi}, \]
\[ U_7 = \bar{\beta}_{i,j} (\bar{y}_{i,j} - 3\delta_{i,j}) \Delta_{i,j} + \frac{2\delta_{i,j} \bar{y}_{i,j} F^y_{i,j}}{\pi} \]
\[ + (\bar{\beta}_{i,j} - 3\alpha_{i,j}) \frac{2\delta_{i,j} F^y_{i,j+1}}{\pi}, \]
\[ U_8 = \frac{2h_i \delta_{i,j} F^y_{i,j+1}}{\pi}, \]
\[ U_9 = \frac{2h_i \delta_{i,j} F^y_{i,j+1}}{\pi}, \quad U_{10} = \bar{y}_{i,j} \bar{\Delta}_{i,j} - \frac{2\delta_{i,j} F^y_{i,j+1}}{\pi}, \]
\[ U_{11} = \bar{\beta}_{i,j} \bar{y}_{i,j} \Delta_{i,j} - \frac{2\delta_{i,j} \bar{y}_{i,j} F^y_{i,j+1}}{\pi} - \frac{2\delta_{i,j} \bar{y}_{i,j} F^y_{i,j}}{\pi}. \]  

(30)

Since the denominator of (29) is always positive and trigonometric basis functions are positive for so the positivity of \( 0 \leq \phi \leq \pi/2 \). It follows that the positivity of \( S'_{i}(x_{i+1}, y) \) entirely depends upon \( U_{i,j} \), \( i = 0, 1, 2, \ldots, 11 \). This yields the following constraints on the free parameters:

\[ \tilde{\beta}_{i,j} > \frac{2\delta_{i,j} F^y_{i,j+1}}{\pi \bar{\Delta}_{i,j}}, \quad \bar{y}_{i,j} > \frac{2\delta_{i,j} F^y_{i,j+1}}{\pi \bar{\Delta}_{i,j}}. \]  

(31)
Table 1: The varying ability of hemoglobin to carry oxygen.

| Partial pressure of oxygen (kPa) | 0 | 2 | 8 | 10 | 18 |
|----------------------------------|---|---|---|----|----|
| Saturation of hemoglobin (%)     | 0 | 70| 91| 91 | 110|

Table 2: The varying ability of myoglobin.

| Partial pressure of oxygen (kPa) | 0 | 4 | 6 | 8 | 10 |
|----------------------------------|---|---|---|---|----|
| Saturation of myoglobin (%)      | 0 | 100|100|100|115|

Table 3: Numerical results corresponding to Figure 2.

| i | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| d_j | 50.9065 | 19.6819 | 0 | 0 | 5.7983 |
| \(\beta_j\) | 35.01 | 7.17 | 0 | 0.01 | — |
| \(\gamma_j\) | 0.1890 | 0.0100 | 0 | 0.7871 | — |

Table 4: Numerical results corresponding to Figure 4.

| i | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| d_j | 56.25 | 0 | 0 | 0 | 15 |
| \(\beta_j\) | 2.8748 | 9.3179 | 0 | 0.01 | — |
| \(\gamma_j\) | 0.01 | 0.01 | 0 | 0.6466 | — |

\(S(x_{i+1}, y)\) is monotone if \(S'_i(x_{i+1}, y) > 0\). We have

\[
S'_i(x_{i+1}, y) = \frac{\pi}{2h_j(q_4(\varphi))^2}
\times \left\{(1 - \sin \varphi)^4 \sin \varphi \cos \varphi V_0 + \cos^2 \varphi (1 - \cos \varphi)^2 (1 - \sin \varphi)^2 V_1 + \cos \varphi (1 - \cos \varphi)^3 (1 - \sin \varphi)^2 V_2 + \cos \varphi (1 - \sin \varphi)^5 V_3 + \sin \varphi \cos^2 \varphi (1 - \cos \varphi)^2 (1 - \sin \varphi) V_4 + \sin \varphi \cos \varphi (1 - \sin \varphi) (1 - \cos \varphi)^3 V_5 + \sin \varphi (1 - \cos \varphi)^2 (1 - \sin \varphi)^3 V_6 + \sin^2 \varphi (1 - \sin \varphi)^2 (1 - \cos \varphi)^2 V_7 + \sin \varphi \cos \varphi (1 - \cos \varphi)^4 V_8 + \sin \varphi (1 \cos \varphi) V_9 + \sin \varphi \cos \varphi (1 - \cos \varphi) (1 - \sin \varphi)^3 V_{10} + \sin^2 \varphi \cos \varphi (1 - \cos \varphi) (1 - \sin \varphi)^2 V_{11}\right\},
\]

where

\[
V_0 = \frac{2h_j F'_{i+1,j} \delta_i^{2+1,j}}{\pi},
\]

\[
V_1 = \left(3\tilde{\alpha}_{i+1,j} \tilde{\gamma}_{i+1,j} - \tilde{\beta}_{i+1,j} \tilde{\delta}_{i+1,j}\right) \tilde{\Delta}_{i+1,j} + \frac{2F'_{i+1,j} \tilde{\alpha}_{i+1,j} \tilde{\gamma}_{i+1,j}}{\pi},
\]

\[
\tilde{\gamma}_{i+1,j} = \frac{2\tilde{\delta}_{i+1,j} F'_{i+1,j}}{\pi},
\]

\[
V_6 = 2\tilde{\alpha}_{i+1,j} \tilde{\gamma}_{i+1,j} \tilde{\Delta}_{i+1,j} - \frac{4\tilde{\delta}_{i+1,j} F'_{i+1,j}}{\pi},
\]

Finally, \(S'_i(x_{i+1}, y)\) is positive if \(V_i, i = 0, 1, 2, \ldots, 11\) are positive. This yields the following constraints on the free parameters:

\[
\tilde{\beta}_{i+1,j} > \frac{2\tilde{\delta}_{i+1,j} F'_{i+1,j}}{\pi \tilde{\Delta}_{i+1,j}}, \quad \tilde{\gamma}_{i+1,j} > \frac{2\tilde{\delta}_{i+1,j} F'_{i+1,j}}{\pi \tilde{\Delta}_{i+1,j}}.
\]

Theorem 3. The bicubic partially blended rational trigonometric function defined in (4) visualizes monotone data in view of the monotone surface if in each rectangular grid \(I = [x_i, x_{i+1}] \times [y_j, y_{j+1}]\), free parameters \(\beta_{i,j}, \gamma_{i,j}, \delta_{i,j}, \alpha_{i,j}\).
Table 5: A 3D monotone data set.

| y/x | 1     | 2     | 3     | 4     | 5     | 6     |
|-----|-------|-------|-------|-------|-------|-------|
| 1   | 0.3202| 0.5385| 0.7762| 1.0198| 1.2659| 1.5133|
| 2   | 0.4717| 0.6403| 0.8500| 1.0770| 1.3124| 1.5524|
| 3   | 0.6500| 0.7810| 0.9605| 1.1662| 1.3865| 1.6155|
| 4   | 0.8382| 0.9434| 1.0966| 1.2806| 1.4841| 1.7000|
| 5   | 1.0308| 1.1180| 1.2500| 1.4142| 1.6008| 1.8028|
| 6   | 1.2258| 1.3000| 1.4151| 1.5620| 1.7328| 1.9209|

Table 6: A 3D monotone data set.

| y/x | 1     | 2     | 3     | 4     | 5     | 6     |
|-----|-------|-------|-------|-------|-------|-------|
| 1   | 0.6931| 1.6094| 2.3026| 2.8332| 3.2581| 3.6109|
| 2   | 1.6094| 2.0794| 2.5649| 2.9957| 3.3673| 3.6889|
| 3   | 2.3026| 2.5649| 2.8904| 3.2189| 3.5264| 3.8067|
| 4   | 2.8332| 2.9957| 3.2189| 3.4657| 3.7136| 3.9512|
| 5   | 3.2581| 3.3673| 3.5264| 3.7136| 3.9120| 4.1109|
| 6   | 3.6109| 3.6889| 3.8067| 3.9512| 4.1109| 4.2767|

\( \beta_{i,j+1}, \gamma_{i,j+1}, \tilde{\beta}_{i,j}, \tilde{\gamma}_{i,j}, \tilde{\beta}_{i+1,j}, \tilde{\gamma}_{i+1,j} \) satisfy the following constraints:

\[
\beta_{i,j} > \frac{2 \alpha_{i,j} F_{x,i,j}}{\pi \Delta_{i,j}}, \quad \gamma_{i,j} > \frac{2 \delta_{i,j} F_{x,i,j}}{\pi \Delta_{i,j}}, \quad \tilde{\beta}_{i,j} > \frac{2 \tilde{\alpha}_{i,j} F_{x,i,j}}{\pi \Delta_{i,j}}, \quad \tilde{\gamma}_{i,j} > \frac{2 \tilde{\delta}_{i,j} F_{x,i,j}}{\pi \Delta_{i,j}}.
\]

\( \tilde{\gamma}_{i+1,j} = u_{i,j} + \max \left\{ \frac{2 \tilde{\delta}_{i+1,j} F_{y,i+1,j}}{\pi \Delta_{i+1,j}} \right\}, \quad u_{i,j} > 0. \)

Algorithm 4.

Step 1. Take a 3D monotone data set \( \{(x_i, y_j, F_{i,j}), i = 0, 1, 2, \ldots, n; j = 0, 1, 2, \ldots, m\} \).

Step 2. Use the Arithmetic Mean Method to estimate the derivatives \( F_{x,i,j}, F_{y,i,j}, F_{x,y,i,j} \) at knots (note: Step 2 is only applicable if data is not provided with derivatives).

Step 3. Compute the values of parameters \( \beta_{i,j}, \gamma_{i,j}, \beta_{i,j+1}, \gamma_{i,j+1}, \tilde{\beta}_{i,j}, \tilde{\gamma}_{i,j}, \tilde{\beta}_{i+1,j}, \tilde{\gamma}_{i+1,j} \) using Theorem 3.

Step 4. Substitute the values of variables from Steps 1–3 in rational trigonometric cubic function (4) to visualize monotone surface through monotone data.

6. Numerical Example

This section illustrates the monotonicity preserving schemes developed in Sections 4 and 5 with the help of examples. The data in Table 1 is observed by exposing identical samples of hemoglobin to different partial pressures of oxygen which results in varying degree of saturation of hemoglobin with...
Table 7: Numerical values corresponding to Figure 6.

| (x_i, y_j) | 1     | 2     | 3     | 4     | 5     | 6     |
|-----------|-------|-------|-------|-------|-------|-------|
|           |       |       |       |       |       |       |
| Numerical values of F_{x,j} |       |       |       |       |       |       |
| 1         | 0.1382| 0.0823| 0.0555| 0.0413| 0.0328| 0.0271|
| 2         | 0.1649| 0.1213| 0.0921| 0.0732| 0.0603| 0.0511|
| 3         | 0.1832| 0.1515| 0.1233| 0.1018| 0.0858| 0.0738|
| 4         | 0.1904| 0.1685| 0.1448| 0.1240| 0.1071| 0.0936|
| 5         | 0.1938| 0.1783| 0.1593| 0.1407| 0.1243| 0.1105|
| 6         | 0.1962| 0.1856| 0.1709| 0.1550| 0.1396| 0.1259|
| Numerical values of F_{y,j} |       |       |       |       |       |       |
| 1         | 0.2087| 0.2280| 0.2406| 0.2448| 0.2467| 0.2480|
| 2         | 0.1481| 0.1892| 0.2184| 0.2312| 0.2377| 0.2423|
| 3         | 0.1068| 0.1552| 0.1926| 0.2130| 0.2247| 0.2333|
| 4         | 0.0813| 0.1292| 0.1686| 0.1937| 0.2097| 0.2221|
| 5         | 0.0649| 0.1096| 0.1481| 0.1754| 0.1943| 0.2097|
| 6         | 0.0538| 0.0947| 0.1310| 0.1588| 0.1794| 0.1969|
| Numerical values of \(\beta_{x,j}\) |       |       |       |       |       |       |
| 1         | 10.9406| 9.7062| 9.0177| 8.6526| 8.4470| —     |
| 2         | 11.0996| 10.3406| 10.0079| 9.8513| 9.7684| —     |
| 3         | 11.6858| 11.1996| 10.8694| 10.6747| 10.5583| —     |
| 4         | 11.8607| 11.5787| 11.3235| 11.1397| 11.0149| —     |
| 5         | 11.9272| 11.7583| 11.5754| 11.4218| 11.3048| —     |
| 6         | —     | —     | —     | —     | —     | —     |
| Numerical values of \(\gamma_{x,j}\) |       |       |       |       |       |       |
| 1         | 13.0594| 14.2938| 14.9823| 15.3474| 15.5530| —     |
| 2         | 12.3315| 12.9236| 13.3931| 13.7011| 13.8977| —     |
| 3         | 12.1426| 12.4531| 12.7625| 13.0043| 13.1785| —     |
| 4         | 12.0737| 12.2518| 12.4569| 12.6399| 12.7863| —     |
| 5         | 12.0728| 12.2417| 12.4246| 12.5782| 12.6952| —     |
| 6         | —     | —     | —     | —     | —     | —     |
| Numerical values of \(\hat{\beta}_{x,j}\) |       |       |       |       |       |       |
| 1         | 11.4688| 11.5120| 11.8546| 11.9391| 11.9689| —     |
| 2         | 10.5384| 10.8247| 11.5416| 11.7866| 11.8858| —     |
| 3         | 9.7828 | 10.3810| 11.2336| 11.6016| 11.7732| —     |
| 4         | 9.2668 | 10.1222| 10.9942| 11.4274| 11.6537| —     |
| 5         | 8.9258 | 9.9673 | 10.8217| 11.2811| 11.5418| —     |
| 6         | —     | —     | —     | —     | —     | —     |
| Numerical values of \(\gamma_{x,j}\) |       |       |       |       |       |       |
| 1         | 12.5312| 12.1490| 12.0616| 12.0312| 12.0311| —     |
| 2         | 13.4616| 12.4963| 12.2213| 12.1165| 12.1142| —     |
| 3         | 14.2172| 12.8787| 12.4267| 12.2357| 12.2268| —     |
| 4         | 14.7332| 13.2084| 12.6331| 12.3675| 12.3463| —     |
| 5         | 15.0742| 13.4662| 12.8168| 12.4961| 12.4582| —     |
| 6         | —     | —     | —     | —     | —     | —     |
| Numerical values of \(\hat{\beta}_{x,j+1}\) |       |       |       |       |       |       |
| 1         | 11.3239| 10.5207| 10.0947| 9.8548| 9.7100| —     |
| 2         | 12.0640| 11.6759| 11.4932| 11.3965| 11.3401| —     |
| 3         | 13.0662| 12.6810| 12.4538| 12.3180| 12.2329| —     |
| 4         | 13.5085| 13.2108| 12.9963| 12.8508| 12.7519| —     |
| 5         | 13.7180| 13.5047| 13.3254| 13.1890| 13.0885| —     |
| 6         | —     | —     | —     | —     | —     | —     |
Table 7: Continued.

| (x_i, y_j) | 1   | 2   | 3   | 4   | 5   | 6   |
|-----------|-----|-----|-----|-----|-----|-----|
| Numerical values of γ_{i,j+1} |     |     |     |     |     |     |
| 1         | 15.4850 | 16.2308 | 16.6264 | 16.8491 | 16.9836 | —   |
| 2         | 14.0006 | 14.5092 | 14.8428 | 15.0559 | 15.1949 | —   |
| 3         | 13.4909 | 13.8260 | 14.0880 | 14.2768 | 14.4104 | —   |
| 4         | 13.2728 | 13.4950 | 13.6932 | 13.8518 | 13.9731 | —   |
| 5         | 13.2618 | 13.4600 | 13.6264 | 13.7531 | 13.8464 | —   |
| 6         | —     | —     | —     | —     | —     | —   |

| Numerical values of β_{i+1,j} |     |     |     |     |     |     |
| 1         | 9.4938 | 11.1409 | 12.5497 | 13.1538 | 13.4520 | —   |
| 2         | 8.8689 | 10.3644 | 11.8756 | 12.6696 | 13.1076 | —   |
| 3         | 8.6842 | 10.0813 | 11.4747 | 12.3097 | 12.8189 | —   |
| 4         | 8.6336 | 10.0176 | 11.2653 | 12.0687 | 12.5971 | —   |
| 5         | 8.6325 | 10.0429 | 11.1705 | 11.9192 | 12.4370 | —   |
| 6         | —     | —     | —     | —     | —     | —   |

| Numerical values of γ_{i+1,j} |     |     |     |     |     |     |
| 1         | 14.5834 | 13.5377 | 13.2398 | 13.1262 | 13.1238 | —   |
| 2         | 15.4020 | 13.9519 | 13.4623 | 13.2553 | 13.2457 | —   |
| 3         | 15.9609 | 14.3091 | 13.6858 | 13.3981 | 13.3751 | —   |
| 4         | 16.3304 | 14.5884 | 13.8848 | 13.5375 | 13.4964 | —   |
| 5         | 16.5779 | 14.7990 | 14.0514 | 13.6641 | 13.6025 | —   |
| 6         | —     | —     | —     | —     | —     | —   |

oxygen. The sample obtaining the highest amount is said to be saturated. The amount of oxygen combined with the remaining samples is taken as percentage of this maximum value. At a low partial pressure of oxygen, the percentage saturation of hemoglobin is very low; that is, hemoglobin is combined with only a very little oxygen. At high partial pressure of oxygen, the percentage saturation of hemoglobin is very high; that is, hemoglobin is combined with large amounts of oxygen, that is, a monotone relation, so the resulting curve must exhibit the same behavior. Figure 1 represents the curve created by assigning random values to free parameters in description of $C^1$ rational trigonometric cubic function (1) which does not retain the monotone nature of the data. This impediment is removed by applying monotonicity preserving schemes developed in Section 4 and is shown in Figure 2. It is evident from the figure that this curve preserves the monotone shape of hemoglobin dissociation curve. Similar investigation in Table 2 displays a series of results for percentage saturation of myoglobin and partial pressure of oxygen. Figure 3 is produced by assigning random values to free parameters in description of $C^1$ rational trigonometric cubic function (1) which fails to conserve the monotone trend of data. Algorithm 2 developed in Section 4 is applied to remove this drawback and Figure 4 displays the required result. Numerical results corresponding to Figures 2 and 4 are shown in Tables 3 and 4.

\[ F(x, y) = \sqrt{\frac{x^2}{25} + \frac{y^2}{16}}. \]  

\[ F(x, y) = \log \left( x^2 + y^2 \right). \]  

The 3D monotone data set in Tables 5 and 6 are generated from the following functions:

Figure 1: $C^1$ rational trigonometric cubic function with $\alpha_i = 1.0$, $\beta_i = 0.5$, $\gamma_i = 1.0$, $\delta_i = 2.0$.

Figures 5 and 7 are produced by interpolating the monotone data sets in Tables 5 and 6, respectively, by $C^1$ rational
Table 8: Numerical values corresponding to Figure 8.

| (x_i, y_j) | 1     | 2     | 3     | 4     | 5     | 6     |
|-----------|-------|-------|-------|-------|-------|-------|
| 1         | 1.0279| 0.4623| 0.2308| 0.1322| 0.0843| 0.0581|
| 2         | 0.8047| 0.4778| 0.2939| 0.1928| 0.1341| 0.0979|
| 3         | 0.6119| 0.4581| 0.3270| 0.2350| 0.1731| 0.1312|
| 4         | 0.4778| 0.4012| 0.3180| 0.2473| 0.1928| 0.1521|
| 5         | 0.3889| 0.3466| 0.2939| 0.2428| 0.1987| 0.1627|
| 6         | 0.3168| 0.2966| 0.2667| 0.2326| 0.1991| 0.1689|

Numerical values of $F_{x_{ij}}$

| 1         | 1.0279| 0.8047| 0.6119| 0.4778| 0.3889| 0.3168|
| 2         | 0.4623| 0.4778| 0.4581| 0.4012| 0.3466| 0.2966|
| 3         | 0.2308| 0.2939| 0.3270| 0.3180| 0.2939| 0.2667|
| 4         | 0.1322| 0.1928| 0.2350| 0.2473| 0.2428| 0.2326|
| 5         | 0.0843| 0.1341| 0.1731| 0.1928| 0.1987| 0.1991|
| 6         | 0.0581| 0.0979| 0.1312| 0.1521| 0.1627| 0.1689|

Numerical values of $F_{y_{ij}}$

| 1         | 13.4612| 11.8021| 10.5579| 9.7618| 9.2601| 9.2601|
| 2         | 13.9316| 11.8084| 10.3699| 10.1191|      |      |
| 3         | 13.8377| 12.7622| 11.4236| 11.0979|      |      |
| 4         | 13.4933| 12.9563| 11.9764| 11.6602|      |      |
| 5         | 13.2255| 12.9325| 12.5819| 11.9879|      |      |
| 6         |      |      |      |      |      |      |

Numerical values of $\beta_{x_{ij}}$

| 1         | 10.5388| 12.1979| 13.4421| 14.2382| 14.7399|      |
| 2         | 10.5932| 11.3237| 12.0568| 12.6377| 13.0617|      |
| 3         | 10.8043| 11.752 | 11.616 | 12.0237| 12.3602|      |
| 4         | 10.9824| 11.1929| 11.4696| 11.7539| 12.0121|      |
| 5         | 10.7745| 11.0675| 11.481 | 11.7434| 12.0121|      |
| 6         |      |      |      |      |      |      |

Numerical values of $\gamma_{x_{ij}}$

| 1         | 13.4612| 13.9316| 13.8377| 13.4933| 13.2255|      |
| 2         | 11.8021| 11.8084| 12.7622| 12.9563| 12.9325|      |
| 3         | 10.5579| 10.3699| 11.4236| 12.102 | 12.5819|      |
| 4         | 9.7618| 10.3699| 11.4236| 11.9764| 12.2566|      |
| 5         | 9.2601| 10.1191| 11.0979| 11.6602| 11.9879|      |
| 6         |      |      |      |      |      |      |

Numerical values of $\beta_{y_{ij}}$

| 1         | 10.5388| 10.5932| 10.8043| 10.9824| 10.7745|      |
| 2         | 12.1979| 11.3237| 11.752 | 11.929 | 11.0675|      |
| 3         | 13.4421| 12.0568| 11.616 | 11.4696| 11.4181|      |
| 4         | 14.2382| 12.6377| 12.0237| 11.7539| 11.7434|      |
| 5         | 14.7399| 13.0617| 12.3602| 12.0121| 12.0121|      |
| 6         |      |      |      |      |      |      |

Numerical values of $\gamma_{y_{ij}}$

| 1         | 13.7691| 12.3176| 11.3888| 10.8035| 10.4245|      |
| 2         | 13.7765| 12.6436| 12.0982| 11.8056| 11.6334|      |
| 3         | 14.8893| 13.9343| 13.3275| 12.9476| 12.7025|      |
| 4         | 15.1157| 14.4785| 13.9724| 13.6036| 13.3401|      |
| 5         | 15.0879| 14.6788| 14.2994| 13.9859| 13.7398|      |
| 6         |      |      |      |      |      |      |
Table 8: Continued.

| $(x_i, y_j)$ | 1          | 2          | 3          | 4          | 5          | 6          |
|-------------|------------|------------|------------|------------|------------|------------|
|             | **Numerical values of $\gamma_{i,j+1}$** |            |            |            |            |            |
| 1           | 7.0627     | 9.6496     | 12.0876    | 13.2188    | 13.7521    |            |
| 2           | 6.8759     | 8.4746     | 10.6260    | 11.9816    | 12.7945    |            |
| 3           | 7.0547     | 8.2958     | 10.0152    | 11.2619    | 11.7051    |            |
| 4           | 7.2590     | 8.4154     | 9.8191     | 10.8928    | 11.7051    |            |
| 5           | 7.4425     | 8.6142     | 9.8100     | 10.7305    | 11.4556    |            |
| 6           | —          | —          | —          | —          | —          | —          |
|             | **Numerical values of $\hat{\gamma}_{i+1,j}$** |            |            |            |            |            |
| 1           | 13.2144    | 12.2673    | 12.1065    | 12.1257    | 11.9898    |            |
| 2           | 14.5622    | 13.0616    | 13.6908    | 14.1502    | 14.4789    |            |
| 3           | 12.1065    | 12.5841    | 13.0257    | 13.3902    | 13.6766    |            |
| 4           | 12.1257    | 12.3697    | 12.7220    | 13.0131    | 13.2416    |            |
| 5           | 11.9898    | 12.3697    | 12.7220    | 13.0131    | 13.2416    |            |
| 6           | —          | —          | —          | —          | —          | —          |
|             | **Numerical values of $\gamma_{i+1,j}$** |            |            |            |            |            |
| 1           | 13.2144    | 14.5622    | 15.4247    | 15.9682    | 16.3201    | —          |
| 2           | 12.2673    | 13.0616    | 13.6908    | 14.1502    | 14.4789    | —          |
| 3           | 12.1065    | 12.5841    | 13.0257    | 13.3902    | 13.6766    | —          |
| 4           | 12.1257    | 12.3697    | 12.7220    | 13.0131    | 13.2309    | —          |
| 5           | 11.9898    | 12.3697    | 12.7220    | 13.0131    | 13.3246    | —          |
| 6           | —          | —          | —          | —          | —          | —          |

Figure 2: $C^1$ monotone rational trigonometric cubic function with $\alpha_i = 2.6$, $\delta_i = 0.4$.

Figure 3: $C^1$ rational trigonometric cubic function with $\alpha_i = 2.5$, $\beta_i = 0.5$, $\gamma_i = 0.5$, $\delta_i = 2.0$.

Figure 4: $C^1$ monotone rational trigonometric cubic function with $\alpha_i = 2.0$, $\delta_i = 0.5$.

Figure 5: $C^1$ rational trigonometric bicubic function with $\alpha_{i,j} = 14$, $\beta_{i,j} = 15$, $\gamma_{i,j} = 6$, $\delta_{i,j} = 7$, $\alpha_{i,j+1} = 8$, $\beta_{i,j+1} = 8$, $\gamma_{i,j+1} = 4$, $\delta_{i,j+1} = 9$, $\alpha_{i+1,j} = 6$, $\beta_{i+1,j} = 5$, $\gamma_{i+1,j} = 4$, $\delta_{i+1,j} = 8$, $\alpha_{i+1,j+1} = 15$, $\beta_{i+1,j+1} = 2$, $\gamma_{i+1,j+1} = 12$, $\delta_{i+1,j+1} = 8$. 
trigonometric bicubic function for arbitrary values of free parameter. Monotone surfaces in Figures 6 and 8 are produced by interpolating the same data by the monotonicity preserving scheme developed in Section 5. Tables 7 and 8 enclose numerical results against Figures 6 and 8.

7. Conclusion

In this paper, monotonicity of data is retained by developing constraints on free parameters in the specification of rational trigonometric function and bicubic blended function. Authors in [7, 8] used algebraic function while the proposed algorithm applies trigonometric function which gives much smoother result due to orthogonality of sine and cosine function. Shape preserving techniques of Butt and Brodlie [1] required insertion of additional knots. In [12], developed scheme failed to maintain smoothness. The proposed technique is local, affirms smoothness, works well for data with derivatives, and does not require insertion of extra knots. Derivative of trigonometric spline is much lower than that of polynomial spline.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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