ON THE APPROXIMATION OF CONVOLUTIONS BY ACCOMPANYING LAWS IN THE SCHEME OF SERIES

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Abstract. The problem of the approximation of convolutions by accompanying laws in the scheme of series satisfying the infinitesimality condition is considered. It is shown that the quality of approximation depends essentially on the choice of centering constants.

The paper considers the problem of approximation of convolutions by accompanying laws for a triangular array of sums of independent random vectors satisfying an infinitesimality condition. We introduce the necessary notation. We will denote by $E_a$, the distribution concentrated at a point $a \in \mathbb{R}^d$, $E = E_0$. The compound Poisson distribution is given by

$$e(\lambda F) = e^{-\lambda} \sum_{s=0}^{\infty} \frac{\lambda^s F^s}{s!},$$

where $\lambda \geq 0$, and $F$ is a probability distribution. Here and below the products and powers of measures are understood in the convolution sense. The Lévy distance between one-dimensional distributions $F$ and $G$ is defined by

$$L(F, G) = \inf \left\{ \varepsilon : F(x - \varepsilon) - \varepsilon \leq G(x) \leq F(x + \varepsilon) + \varepsilon \quad x \in \mathbb{R} \right\},$$

where $F(\cdot)$ is the distribution function corresponding to the distribution $F$.

The Lévy–Prokhorov distance between the distributions $F$ and $G$ in a complete separable metric space is defined as follows:

$$\pi(F, G) = \inf \left\{ \varepsilon : F\{X\} \leq G\{X^\varepsilon\} + \varepsilon \quad G\{X\} \leq F\{X^\varepsilon\} + \varepsilon \right\}$$

for all Borel sets $X$, where $X^\varepsilon$ is the $\varepsilon$-neighborhood of $X$. It is well known that both Lévy and Lévy–Prokhorov distances metrize the weak convergence of probability distributions.

By the same letter $c$ we denote positive absolute constants which may be different even within a single formula. Writing $A \ll B$ means that $A \leq cB$. If the corresponding constant depends on the dimension $d$, we will use the notation $A \ll_d B$. In the future, $\log^* b = \max\{1, \log b\}$, for $b > 0$.

Consider the classical scheme of the series of sums of independent random variables satisfying the infinitesimality condition (cf. [1, 3, 7]). Let $\{X_{j,k}, j = 1, 2, \ldots; k = 1, \ldots, n_j\}$, be

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independent random variables with distributions \( F_{j,k} = \mathcal{L}(X_{j,k}) \). Denote by

\[
F_j = \prod_{k=1}^{n_j} F_{j,k}, \quad j = 1, 2, \ldots,
\]

(4)

the distributions of sums \( S_j = \sum_{k=1}^{n_j} X_{j,k} \). The infinitesimality condition is usually formulated as follows:

\[
\max_{1 \leq k \leq n_j} \mathbb{P}\{|X_{j,k}| \geq \tau\} \to 0 \quad \text{as} \quad j \to \infty
\]
as \( j \to \infty \) for any \( \tau > 0 \). For each fixed \( j \), the distributions of summands \( X_{j,k} \) are connected with the distributions of summands \( X_{m,s} \) \( m \neq j \) only through the infinitesimality condition. If under this condition the sequence of distributions \( F_j \) converges weakly to a probability distribution \( D (F_j \Rightarrow D \text{ as } j \to \infty) \), then, by Khinchin’s theorem, the distribution \( D \) is infinitely divisible. Conditions of convergence to a given infinitely divisible distribution \( D \) (see, e.g., [3, 7]) are usually formulated as conditions which are equivalent to the convergence to the distribution \( D \) of the so-called accompanying infinitely divisible laws

\[
D_j = \prod_{k=1}^{n_j} (E_{a_{j,k}} \mathcal{L}(F_{j,k}, E_{a_{j,k}})).
\]

(5)

The distributions \( D_j \) depend on the centering constants \( a_{j,k} \) which are defined by the equality

\[
a_{j,k} = \int_{|x| \leq \tau} x F_{j,k} \{dx\}.
\]

(6)

The constant \( \tau \), involved in the definition (6), does not depend on \( j \) and \( k \) and it is arbitrary. For different \( \tau \), the numbers \( a_{j,k} \), are (generally speaking) different, and thus different are the distribution \( D_j \) too. However, if, \( F_j \Rightarrow D \), then also \( D_j \Rightarrow D \) as \( j \to \infty \), for any \( \tau \). It seems that this would imply that the distribution of \( D_j \) may be considered as a good infinitely divisible approximation for the distributions \( F_j \), if the latter are defined through a scheme of series satisfying the infinitesimality condition. However, this is not always the case, at least if the accuracy of the approximation is estimated in the Lévy–Prokhorov metric. In particular, it may be not so if the sequence of distributions of \( F_j \) is not relatively compact (in the topology of weak convergence). A discussion of this circumstance is the subject of this paper.

The approximation of sequences of distributions that are not relatively compact in the Lévy–Prokhorov metric has a special interest in connection with a recent result by Yu. A. Davydov and V. I. Rotar’ [2] on the characterization of sequences distributions which are close each to other in the Lévy–Prokhorov metric in terms of the closeness of integrals of uniformly continuous bounded functions.

First of all, note that the infinitesimality condition may be reformulated as follows:

\[
\max_{1 \leq k \leq n_j} \mathcal{L}(F_{j,k}, E) = \varepsilon_j \to 0 \quad \text{as} \quad j \to \infty.
\]

(7)
This condition of closeness of a distribution to the degenerate distribution was proposed by Kolmogorov [5, 6] when considering the problem of the infinitely divisible approximation of convolutions. Condition (7) is closely connected with the condition of the representability of distributions $F_{j,k}$ in the form
\[ F_{j,k} = (1 - p_{j,k})U_{j,k} + p_{j,k}V_{j,k}, \]
where $0 \leq p_{j,k} \leq 1$, the distributions $U_{j,k}$, $k = 1, \ldots, n_j$, are concentrated on the segments $[-\tau_j, \tau_j]$, $\tau_j \geq 0$, $j = 1, 2, \ldots$, and $V_{j,k}$, $k = 1, \ldots, n_j$, are probability distributions, wherein
\[ \tau_j \to 0 \quad \text{and} \quad p_j = \max_{1 \leq k \leq n_j} p_{j,k} \to 0 \quad \text{as} \quad j \to \infty. \]

This condition of closeness of a distribution to the degenerate distribution was used by I. A. Ibragimov and E. L. Presman [4] when considering the above mentioned problem of Kolmogorov.

It was shown that the natural infinitely divisible approximation for distributions of $F_j = \prod_{k=1}^{n_j} F_{j,k}$ under conditions (8) and (9) is given by the distributions
\[ G_j = \prod_{k=1}^{n_j} \left( E_{b_{j,k}} e \left( F_{j,k} E_{-b_{j,k}} \right) \right). \]
with
\[ b_{j,k} = \int_{\mathbb{R}} x U_{j,k} \{ dx \}. \]

It is easy to see that the distributions (10) have the form (5), but with replacing the $a_{j,k}$ by $b_{j,k}$. In [9], it was shown that under conditions (8), (9) and (11) we have
\[ L(F_j, G_j) \ll p_j + \tau_j \log^* \tau_j^{-1}. \]
and
\[ \pi(F_j, G_j) \ll \sum_{k=1}^{n_j} p_{j,k}^2 + p_j + \tau_j \log^* \tau_j^{-1}. \]

If we assume additionally that the distributions $V_{j,k}$ are the same for all $k = 1, \ldots, n_j$, then
\[ \pi(F_j, G_j) \ll p_j + \tau_j \log^* \tau_j^{-1}. \]

The proofs of inequalities (12)–(14), their discussion and the history of the problem may be also found in the monograph [1]. Inequality (12) an optimal (with respect to order) solution to the problem of Kolmogorov [5, 6] of infinitely divisible approximation of convolutions satisfying condition (7). Inequalities (12)–(14) are optimal with respect to order for the dependence of the right-hand sides on $p_j$ and $\tau_j$, and in general, the summand $\sum p_{j,k}^2$ can not be removed from the the right-hand side of inequality (13). Moreover, in general, the right-hand sides of (12)–(14) can not be significantly reduced if the distribution of $G_j$ is replaced by any other infinitely divisible distributions. Thus, inequalities (12)–(14) can be considered as a quantitative refinement of classical Khinchin’s theorem that the limit distribution for the distribution $F_j$, defined in the scheme of series of independent random
variables satisfying the infinitesimality condition, must be infinitely divisible, of course, if it exists. In fact, if \( F_j \Rightarrow D \), then, by (9) and (12), we have the weak convergence \( G_j \Rightarrow D \) as \( j \to \infty \). The presence of the limit distribution implies the relative compactness of the sequence of distributions \( F_j \). The distribution \( D \) is infinitely divisible, as the limit of infinitely divisible distributions. In the monograph [1], the conditions of convergence of distributions \( D \) are equivalent to the convergence to the distribution \( D \) of accompanying infinitely divisible laws \( G_j \) with \( b_{j,k} \), defined by equality (11).

Thus, if we study the \( F_j = \mathcal{L}(S_j) \) and we are interested in a reasonable infinitely divisible approximation for the distributions \( F_j \), then, under conditions (8) and (9), it is given by the distributions \( G_j \) of the form (10) with \( b_{j,k} \) from (11). At the same time, if \( a_{j,k} \) are defined by (6), the distributions \( F_j \) and \( D_j \) can be not close in the Lévy–Prokhorov metric.

**Example 1.** Consider as a simplest example the distribution

\[
F_{j,k} = (1 - j^{-1})E + j^{-1}E_1, \quad k = 1, \ldots, n_j. \tag{15}
\]

Then conditions (8) and (9) are satisfied with

\[
U_{j,k} = E, \quad V_{j,k} = E_1, \quad p_{j,k} = p_j = j^{-1}, \quad \tau_j = 0.
\]

Furthermore, \( b_{j,k} = 0 \) and and the distribution \( F_j \) is binomial with parameters \( n = n_j \) and \( p = p_j = j^{-1} \). The accompanying infinitely divisible distribution \( G_j = e(n_j p_j E_1) \) is the Poisson distribution with parameter \( n_j p_j \). According to a result of Yu. V. Prokhorov [8], the distance in variation between distributions \( F_j \) and \( G_j \) is bounded from above by \( c p_j = c j^{-1} \) and tends to zero as \( j \to \infty \).

As to the approximating accompanying infinitely divisible distribution \( D_j \), then, as noted above, it depends on the choice of the centering constants \( a_{j,k} \). If \( a_{j,k} \) are chosen by the formula (4) with \( \tau \geq 1 \), then \( a_{j,k} = j^{-1}, \quad k = 1, \ldots, n_j \), and

\[
D_j = \prod_{k=1}^{n_j} \left( E_{a_{j,k}} e(F_{j,k} E_{-a_{j,k}}) \right)
= E_{n_j j^{-1}} e(n_j(1 - j^{-1}) E_{-j^{-1}}) e(n_j j^{-1} E_{1-j^{-1}}).
\tag{16}
\]

Denote

\[
V_j = E_{n_j j^{-1}} e(n_j(1 - j^{-1}) E_{1-j^{-1}}) \quad \text{and} \quad W_j = e(n_j(1 - j^{-1}) E_{-j^{-1}}).
\]

\( D_j = V_j W_j \). Obviously,

\[
D_j \{Z^{1/8}\} = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} 1\{x + y \in Z^{1/8}\} W_j \{dx\} \right) V_j \{dy\}
\leq \sup_{x \in \mathbb{R}} W_j \{Z^{1/8} + x\}.
\tag{17}
\]

Here \( 1\{\cdot\} \) is the indicator function, and \( Z^{1/8} \) means the 1/8-neighborhood of the set of all integers \( Z \). The set \( Z^{1/8} \) consists of real numbers \( y \in \mathbb{R} \) representable as \( y = z + t \), where \( z \in Z \) and \( |t| < 1/8 \).
Let $U_\lambda$ be the Poisson distribution with parameter $\lambda > 0$, and $\xi_\lambda$ be a random variable with distribution $U_\lambda$. Then
\[ P\{\xi_\lambda = s\} = \frac{e^{-\lambda} \lambda^s}{s!}, \quad s = 0, 1, 2, \ldots, \tag{18} \]
and $E\xi_\lambda = D\xi_\lambda = \lambda$.

In order to estimate the right-hand side of (17), we need the following elementary property of the Poisson distribution.

**Lemma 1.** There exist absolute positive constants $c_1$ and $c_2$ such that
\[ \sup_{x \in \mathbb{R}} P\{\delta^{-1}\xi_\lambda \in \mathbb{Z}^{1/8} + x\} \leq \frac{5}{8}, \quad \text{if } \delta \geq c_1 \text{ and } \lambda \geq c_2 \delta^2. \tag{19} \]

Choosing constants $1/8$ and $5/8$ in the statement of the lemma is rather arbitrary, it is important that they are some absolute constants satisfying $0 < 1/8 < 1/4$ and $5/8 < 1$.

**Proof of Lemma.** Let $x \in \mathbb{R}$. The set
\[ A_x = \delta (\mathbb{Z}^{1/8} + x) \]
is a union of open intervals $I_j$ of length $\delta/4$ with centers at the points $\delta(j + x)$, $j \in \mathbb{Z}$. Therefore,
\[ P\{\xi_\lambda \in A_x\} = \sum_{j \in \mathbb{Z}} P\{\xi_\lambda \in I_j\}. \tag{20} \]
Accordingly, the set $\mathbb{R} \setminus A_x$ consists of closed intervals $J_j$ of length $3\delta/4$, located between intervals of the set $A_x$, and
\[ P\{\xi_\lambda \in \mathbb{R} \setminus A_x\} = \sum_{j \in \mathbb{Z}} P\{\xi_\lambda \in J_j\}. \tag{21} \]
We assume that the segments $J_j$ are numbered in such a way that for each $j$ the interval $J_j$ is located directly behind the interval $I_j$, if we move on the real axis in the direction $+\infty$.

Obviously,
\[ P\{\xi_\lambda = s\} \geq P\{\xi_\lambda = s + 1\}, \quad s + 1 \geq \lambda, \tag{22} \]
\[ P\{\xi_\lambda = s\} \geq P\{\xi_\lambda = s - 1\}, \quad \text{if } \lambda \geq s. \tag{23} \]
Choosing the constant $c_1$ being large enough, we can ensure that the number of lattice points in each interval $I_j$ will be less than the number of lattice points in each of the segments $J_m$, $m \in \mathbb{Z}$. Let $I_k$ and $I_{k+1}$ be two intervals $I_j$, located closest to the point $\lambda$. If the second and third by proximity to the point $\lambda$ intervals $I_j$ are from it at the same distance, for definiteness we take one of them, which is located between the $\lambda$ and $+\infty$. Inequalities (22)–(23) imply that
\[ P\{\xi_\lambda \in I_j\} \leq P\{\xi_\lambda \in J_{j-1}\}, \quad \text{if } j \geq k + 2, \tag{24} \]
\[ P\{\xi_\lambda \in I_j\} \leq P\{\xi_\lambda \in J_j\}, \quad \text{if } j \leq k - 1. \tag{25} \]
It is well known that

$$\max_{k \in \mathbb{Z}} \mathbb{P}\{\xi_\lambda = k\} \gg \lambda^{-1/2}, \quad \text{if } \lambda \geq 1. \quad (26)$$

In order to prove (26) it is sufficient to note that

$$U_\lambda = U_\lambda^n \quad \text{for all positive integers } n$$

and to use the Berry–Esséen inequality (see [1, 7]). It is even easier to verify the validity of (26), applying the Stirling formula to the right-hand side of (18) and taking into account (22)–(23).

Inequality (26) implies that

$$\max_{j \in \mathbb{Z}} \mathbb{P}\{\xi_\lambda \in I_j\} \leq c \delta \lambda^{-1/2} \leq 1/8, \quad (27)$$

if the constant \(c_2\) is large enough.

Substituting inequalities (24)–(25) in the equality (20) and using relations (21) and (27), we get

$$\mathbb{P}\{\xi_\lambda \in A_x\} \leq 1/4 + \mathbb{P}\{\xi_\lambda \in \mathbb{R} \setminus A_x\} = 5/4 - \mathbb{P}\{\xi_\lambda \in A_x\}. \quad (28)$$

This implies inequality (19).

Let us return to the evaluation of the right-hand side of (17). It is easy to see that

$$W_j = \mathcal{L}(\delta_j^{-1} \xi_{\lambda_j}) \quad \text{with} \quad \delta_j = j, \quad \lambda_j = n_j(1 - j^{-1}). \quad (29)$$

By choosing \(n_j \geq 2 c_2 j^2\), and by applying Lemma [1] with \(\delta = \delta_j, \lambda = \lambda_j\), we obtain that, for \(j \geq \max\{2, c_1\},\)

$$\sup_{x \in \mathbb{R}} W_j\{\mathbb{Z}^{1/8} + x\} = \sup_{x \in \mathbb{R}} \mathbb{P}\{-\delta_j^{-1} \xi_{\lambda_j} \in \mathbb{Z}^{1/8} + x\} = \sup_{x \in \mathbb{R}} \mathbb{P}\{\delta_j^{-1} \xi_{\lambda_j} \in \mathbb{Z}^{1/8} + x\} \leq 5/8. \quad (30)$$

Inequalities (17) and (30) imply that

$$D_j\{\mathbb{Z}^{1/8}\} \leq 5/8. \quad (31)$$

At the same time,

$$F_j\{\mathbb{Z}\} = 1. \quad (32)$$

According to the definition (3), relations (31) and (32) imply that

$$\pi(F_j, D_j) \geq 1/8. \quad (33)$$

Thus, the Lévy–Prokhorov distance \(\pi(F_j, D_j)\) does not tend to zero as \(j \to \infty\), in contrast to the \(\pi(F_j, G_j)\). Of course, this is due to the fact that the sequence of distributions \(F_j\) is not relatively compact.

Inequalities (12)–(14) and Example 1 allow us to conclude that, in the case of identically distributed summands, the distributions \(G_j\) may be always regarded as a good approximation for the distributions \(F_j\), while the distributions \(D_j\) may be far from the distributions \(F_j\) in
the Lévy–Prokhorov metric. Note that in some cases we have \(a_{j,k} = b_{j,k}\) and \(D_j = G_j\). For example, if all considered distributions are symmetric. Or if, in just considered Example 1, \(\tau < 1\) and \(a_{j,k} = b_{j,k} = 0\).

It is important here is that the distributions \(G_j\) are defined for \(\tau_j \to 0\), so that inequalities \((12)\) and \((14)\) ensure the closeness of distributions \(G_j\) and \(F_j\). At the same time, in the definition \((6)\) of \(a_{j,k}\) the value of \(\tau\) is fixed and does not depend on \(j\).

Another important difference between \(b_{j,k}\) and \(a_{j,k}\) is that even if the distributions \(U_{j,k}\) and \(V_{j,k}\) are concentrated, respectively, on the intervals \([-\tau_j, \tau_j]\) and on the sets \(\mathbb{R}\setminus[-\tau_j, \tau_j]\), then, for \(\tau = \tau_j\), we have the equality \(a_{j,k} = (1 - p_{j,k}) b_{j,k}\). The presence of the factor \((1 - p_{j,k})\) in this equality leads to the fact that distributions \(D_j\) and \(F_j\) may be far from each other in the Lévy–Prokhorov metric, even if in the definition \((6)\) of \(a_{j,k}\) the value of \(\tau = \tau_j\) depends on \(j\) and tends to zero as \(j \to \infty\). This is illustrated by the following Example 2 which is a modification of Example 1.

**Example 2.** Now let

\[
F_{j,k} = (1 - j^{-1})E_{-j^{-1}} + j^{-1}E_{1-j^{-1}}, \quad k = 1, \ldots, n_j.
\]  

Then conditions \((5)\) and \((9)\) are satisfied with

\[
U_{j,k} = E_{-j^{-1}}, \quad V_{j,k} = E_{1-j^{-1}}, \quad p_{j,k} = p_j = j^{-1}, \quad \tau_j = j^{-1}.
\]

Furthermore, \(b_{j,k} = -j^{-1}\) and the distribution \(F_j E_{n_j j^{-1}}\) is binomial with parameters \(n_j\) and \(j^{-1}\). Hence, the distribution \(F_j\) itself is a binomial distribution shifted by \(n_j j^{-1}\) to the left. The accompanying infinitely divisible distribution

\[
G_j = E_{-n_j j^{-1}}e(n_j p_j E_1)
\]

is a similarly shifted Poisson distribution with parameter \(n_j p_j\). In analogy to Example 1, the distance in variation between the distributions \(F_j\) and \(G_j\) is bounded from above by \(c_p j = c_j^{-1}\) and tends to zero as \(j \to \infty\).

As to the approximating accompanying infinitely divisible distribution \(D_j\), then if the values of \(a_{j,k}\) are chosen by the formula \((6)\) with \(\tau = j^{-1}\), \(j \geq 3\), then \(a_{j,k} = -j^{-1}(1 - j^{-1})\), \(k = 1, \ldots, n_j\), and

\[
D_j = \prod_{k=1}^{n_j} \left(E_{a_{j,k}}(F_{j,k} E_{-a_{j,k}})\right)
= E_{-n_j j^{-1}}(1-j^{-1}) e(n_j (1 - j^{-1}) E_{1-j^{-1}}) e(n_j j^{-1} E_{1-j^{-1}}).
\]  

By choosing \(n_j \geq 2 c_2 j^4\) and proceeding by analogy with Example 1, it is easy to show that, for \(j \geq \max\{3, c_1\}\) the Lévy–Prokhorov distance \(\pi(F_j, D_j)\) is separated from the zero and, therefore, does not tend to zero as \(j \to \infty\), in contrast to \(\pi(F_j, G_j)\).

In the multidimensional case the situation does not differ from one-dimensional one. Let now \(\{X_{j,k}, j = 1, 2, \ldots; k = 1, \ldots, n_j\}\) be independent \(d\)-dimensional random vectors with
distributions $F_{j,k} = \mathcal{L}(X_{j,k})$, representable as
\begin{equation}
F_{j,k} = (1 - p_{j,k})U_{j,k} + p_{j,k}V_{j,k},
\end{equation}
where $0 \leq p_{j,k} \leq 1$, the distributions $U_{j,k}$, $k = 1, \ldots, n_j$, are concentrated on the Euclidean balls $B_{\tau_j}$, centered at the origin and of radii $\tau_j \geq 0$, $j = 1, 2, \ldots$, $V_{j,k}$, $k = 1, \ldots, n_j$, are arbitrary distributions, and
\begin{equation}
\tau_j \to 0 \quad \text{and} \quad p_j = \max_{1 \leq k \leq n_j} p_{j,k} \to 0 \quad \text{as} \quad j \to \infty.
\end{equation}
Let the distributions $F_j$ and $G_j$ be defined by (4) and (10), where
\begin{equation}
b_{j,k} = \int_{\mathbb{R}^d} x U_{j,k} \{dx\}. \tag{38}
\end{equation}
In author’s paper [10], it was shown that, under the conditions (36), (37) and (38), we have
\begin{equation}
L(F_j, G_j) \ll_d p_j + \tau_j \log^* \tau_j^{-1}. \tag{39}
\end{equation}
and
\begin{equation}
\pi(F_j, G_j) \ll_d \sum_{k=1}^{n_j} p_{j,k}^2 + p_j + \tau_j \log^* \tau_j^{-1}. \tag{40}
\end{equation}
If we assume additionally that the distributions $V_{j,k}$ are the same for all $k = 1, \ldots, n_j$, then
\begin{equation}
\pi(F_j, G_j) \ll_d p_j + \tau_j \log^* \tau_j^{-1}. \tag{41}
\end{equation}
The multidimensional analog of the Lévy distance is defined just as the Lévy–Prokhorov distance, only the Borel sets should be replaced by the parallelepipeds.

If, in addition to the infinitesimality condition (37), we assume that
\begin{equation}
\sum_{k=1}^{n_j} p_{j,k}^2 \to 0 \quad \text{as} \quad j \to \infty,
\end{equation}
then $\pi(F_j, G_j)$ tends to zero. If the distributions $V_{j,k}$ are the same for all $k = 1, \ldots, n_j$, then for $\pi(F_j, G_j) \to 0$ no additional assumptions are required.

In [11], one can find the estimates for the accuracy of strong approximation of the corresponding random vectors which follow from the estimates of the Lévy–Prokhorov distance (40) and (41).

Yu. A. Davydov and V. I. Rotar’ [2] established, in particular, that the sequences $d$-dimensional distributions approach each other in the Lévy–Prokhorov metric if and only if the integrals over these distributions of uniformly bounded continuous functions do the same. It is clear that the above distributions $F_j$ and $G_j$ of a triangular array satisfying the infinitesimality condition, provide a large number of meaningful examples of approaching each other sequences of distributions, including those which are not relatively compact.

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