Shear-free perfect fluids with a solenoidal electric curvature

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Abstract
We prove that the vorticity or the expansion vanishes for any shear-free perfect fluid solution of the Einstein field equations where the pressure satisfies a barotropic equation of state and the spatial divergence of the electric part of the Weyl tensor is zero.

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1. Introduction

There has been much effort devoted to establishing the \textit{shear-free fluid conjecture} that general relativistic, shear-free perfect fluids which obey a barotropic equation of state \( p = p(\mu) \) such that \( p + \mu \neq 0 \) are either non-expanding (\( \theta = 0 \)) or non-rotating (\( \omega = 0 \)). This conjecture appears as such for the first time in Treciokas and Ellis [19] and has been established in many special cases, but a general proof or counter-example is still lacking. In support, the conjecture is known to hold in the following cases: \( p = \text{const} \) (dust with a cosmological constant) [9, 10, 16]; spatial homogeneity [1, 11]; \( dp/d\mu = 1/3 \) (incoherent radiation) [19]; \( dp/d\mu = -1/3 \) [12, 21] or \( 1/9 \) [21]; \( \omega \) and \( \mu \) are parallel [22]; vanishing of the magnetic part \( H \) [7] or of the electric part \( E \) [8, 12] of the Weyl tensor; \( \theta = \theta(\mu) \) [13] or \( \theta = \theta(\omega) \) [18]; Petrov types N [2] and III [3, 4]; there exists a conformal Killing vector parallel to the fluid flow \( \mu \) [6]. It is noteworthy that there exist Newtonian perfect fluids with a barotropic equation of state, which are rotating, expanding but non-shearing. Hence, if true, such behaviour of fluids would be a purely relativistic effect. Recently, in an attempt to generalize the result established by Collins that \( H = 0 \Rightarrow \omega \theta = 0 \), we managed to prove the conjecture for the case when \( \text{div} H = 0 \) (that is, when the magnetic part of the Weyl tensor is solenoidal), with a \( \gamma \)-law equation of state [20]. This result was further generalized in a sense with our being able to establish the conjecture for the case when \( \text{div} H = 0 \) and there is a sufficiently \textit{generic} equation of state [5].
In this paper, we focus our attention to generalizing the result [8, 12] that the conjecture holds when the electric part of the Weyl tensor vanishes, by considering spacetimes for which the electric part is solenoidal. Specifically, we prove the following.

**Theorem.** Consider any shear-free perfect fluid solution of the Einstein field equations where the fluid pressure satisfies a barotropic equation of state and the spatial divergence of the electric part of the Weyl tensor is zero. Then either the fluid is non-rotating or non-expanding.

2. Notations and conventions

We shall be examining shear-free, perfect fluid solutions of the Einstein field equations

\[ R_{ab} - \frac{1}{2} R g_{ab} = \mu u_a u_b + p h_{ab} , \tag{1} \]

where \( u \) is the future-pointing (time-like) unit tangent vector to the flow, \( \mu \) and \( p \) are the energy density and pressure of the fluid, respectively, and \( h_{ab} = g_{ab} + u_a u_b \) is the projection tensor into the rest space of the observers with 4-velocity \( u \). The vanishing of the shear can be expressed by

\[ u_a ; b = \frac{1}{3} \theta h_{ab} + \omega_{ab} - \dot{u}_a u_b . \tag{2} \]

We shall assume familiarity with the notation and conventions of the orthonormal tetrad formalism as given by MacCallum [15]. We begin the analysis by choosing, what we believe to be, a well-suited tetrad alignment for the physical problem at hand. This is the same as in our previous article [20]. First, \( e_0 \) and \( e_3 \) are aligned with \( u \) and \( \omega \), respectively, such that \( \omega = \omega e_3 \neq 0 \). The relevant variables then become \( \mu, p, \theta, \omega, \dot{u}_\alpha, \Omega_\alpha \) together with the quantities \( n_{\alpha\beta} \) and \( a_\alpha \). Latin indices will be tetrad indices taking the values 1, 2, 3, 4. Greek indices take the values 1, 2, 3, while uppercase Latin indices take the values 1, 2 and any expressions involving these have to be read modulo 3 or 2, respectively. The sum of matter density and pressure will be written as

\[ E = \mu + p . \tag{3} \]

Secondly, it is always possible, using the remaining rotational freedom and making use of the Jacobi identities and field equations, to further specialize the tetrad so as to achieve

\[ \Omega_1 = \Omega_2 = \Omega_3 + \omega = 0 , \quad n_{11} = n_{22} = n . \tag{4} \]

Herewith the tetrad is fixed up to the rotations \( e_1 + \epsilon e_2 \rightarrow e^\omega (e_1 + \epsilon e_2) \) satisfying \( \delta_\alpha \epsilon = 0 \).

Because of computational advantages, we will be replacing \( n_{\alpha\beta} (\alpha \neq \beta) \) and \( a_\alpha \) with the new variables \( q_\alpha \) and \( r_\alpha \) defined by \( n_{a-1 a+1} = (r_a + q_a)/2 \) and \( a_\alpha = (r_\alpha - q_\alpha)/2 \). We will also introduce extension variables, \( z_\alpha \) and \( j \), which are related to the components of the spatial gradient of the expansion, by

\[ \partial_\alpha \theta = -\theta^2 / 3 + 2 \omega^2 - (\mu + 3 p)/2 + j , \]

\[ \partial_\alpha \theta = z_\alpha , \tag{5} \]

with \( j \) being the (3+1) covariant divergence of the acceleration,

\[ j \equiv \ddot{u}_\alpha - \partial_\alpha \dot{u}^\alpha + \dot{u}^\alpha u_\alpha - \dot{u}^\alpha (r_\alpha - q_\alpha) . \tag{6} \]

In order to relate the components of \( E_{ab} \) to the spatial gradient of the acceleration, we will make use of the Ricci identity [14]

\[ E_{(ab)} = D_{(a} u_{b)} - \omega_{(a} \omega_{b)} + \dot{u}_{(a} u_{b)} , \tag{7} \]

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The evolution equation for the spatial derivatives of the acceleration can be obtained from (7), using (6):

\[3\partial_0 r_a = -z_a - \theta (\dot{u}_a + r_a),\]  
\[3\partial_0 q_a = z_a + \theta (\dot{u}_a + q_a),\]

while (A.2) and the (0α) field equations (A.14)–(A.16) give us the spatial derivatives of \(\omega\),

\[\partial_1 \omega = \frac{r}{3}z_2 - \omega (q_1 + 2\dot{u}_1),\]  
\[\partial_2 \omega = -\frac{r}{3}z_1 + \omega (r_2 - 2\dot{u}_2),\]  
\[\partial_3 \omega = \omega (\dot{u}_3 + r_3 - q_3),\]

together with the algebraic restriction

\[n_{33} = \frac{2}{3\omega}z_3.\]

The evolution equation for \(n\) follows from (A.10):

\[\partial_0 n = -\frac{\theta}{3} n.\]

Acting with the commutators \([\partial_0, \partial_q]\) and \([\partial_1, \partial_2]\) on the pressure and using (A.8) together with the conservation laws

\[\partial_0 \mu = -\mathcal{E}_\theta, \quad \partial_0 p = -\mathcal{E}_\theta,\]

leads to a first set of evolution equations for the acceleration and vorticity:

\[\partial_0 \dot{u}_a = p' z_a - G \theta \dot{u}_a,\]  
\[\partial_0 \dot{\omega} = \frac{1}{2} \omega \theta (-2 + 3p'),\]

where we have defined

\[G \equiv \frac{p''}{p'} - p' + \frac{1}{3}.\]

The spatial derivatives of the acceleration can be obtained from (7), using (6):

\[\partial_A \dot{u}_A = \frac{1}{2} (j - \omega^2) - \dot{u}_{A+1} q_{A+1} + \dot{u}_{A-1} r_{A-1} - \dot{u}_A^2 + E_{AA} (A = 1 \text{ or } 2),\]
\[\partial_3 \dot{u}_3 = \frac{2}{3} \omega^2 + \frac{1}{2} j - \dot{u}_1 q_1 + \dot{u}_2 r_2 - \dot{u}_3^2 + E_{33},\]
\[\partial_1 \dot{u}_2 = -p' \omega \theta + q_2 \dot{u}_1 + \frac{1}{2} n_{33} \dot{u}_3 - \dot{u}_1 \dot{u}_2 + E_{12},\]
\[\partial_2 \dot{u}_1 = p' \omega \theta - r_1 \dot{u}_2 - \frac{1}{2} n_{33} \dot{u}_3 - \dot{u}_1 \dot{u}_2 + E_{12},\]
\[\partial_1 \dot{u}_3 = -\frac{1}{2} \dot{u}_2 n_{33} - r_3 \dot{u}_1 - \dot{u}_1 \dot{u}_3 + E_{13},\]
\[\partial_2 \dot{u}_3 = \frac{1}{2} \dot{u}_1 n_{33} + q_3 \dot{u}_2 - \dot{u}_2 \dot{u}_3 + E_{23},\]
\[\partial_1 \dot{u}_1 = -\frac{1}{2} \dot{u}_3 n_{33} + \dot{u}_2 + q_1 \dot{u}_3 - \dot{u}_1 \dot{u}_3 + E_{13},\]
\[\partial_3 \dot{u}_2 = \frac{1}{2} \dot{u}_1 n_{33} - \dot{u}_1 - r_2 \dot{u}_3 - \dot{u}_2 \dot{u}_3 + E_{23}.\]
Next we act with the \([\partial_0, \partial_j]\) commutators on \(\omega\) and \(\theta\) and use the propagation of (13) along \(\mathbf{u}\) in order to obtain expressions for the evolution of \(z_\alpha\) along \(e_0\) and for the spatial gradient of \(j\):

\[
\partial_0 z_1 = \theta(-1 + p')z_1 - \frac{1}{2} \omega(-1 + 9p')z_2 + \frac{1}{2} \theta \omega (9G - 2) \hat{u}_2
\]

(26)

\[
\partial_0 z_2 = \theta(-1 + p')z_2 + \frac{1}{2} \omega(-1 + 9p')z_1 - \frac{1}{2} \theta \omega (9G - 2) \hat{u}_1
\]

(27)

\[
\partial_0 z_3 = \theta(-1 + p')z_3.
\]

(28)

\[
\partial_1 j = p' \theta z_1 - \frac{1}{6} \omega (27p' + 13)z_2 + \frac{1}{3} \hat{u}_1 (18\omega^2 + \theta^2 - 3j - 3\mu) - \frac{\epsilon}{2p'} \hat{u}_1
\]

\[+ \frac{1}{2} \theta \omega (9G - 2) \hat{u}_2 + 4\omega^2 q_1
\]

(29)

\[
\partial_2 j = p' \theta z_2 + \frac{1}{6} \omega (27p' + 13)z_1 + \frac{1}{3} \hat{u}_2 (18\omega^2 + \theta^2 - 3j - 3\mu) - \frac{\epsilon}{2p'} \hat{u}_2
\]

\[+ \frac{1}{2} \theta \omega (9G - 2) \hat{u}_1 - 4\omega^2 r_2
\]

(30)

\[
\partial_3 j = p' \theta z_3 + \frac{1}{3} \hat{u}_3 (\theta^2 - 18\omega^2 - 3j - 3\mu) - \frac{\epsilon}{2p'} \hat{u}_3 - 4(r_1 - q_3) \omega^2.
\]

(31)

Now we may evaluate \(\sum_\alpha [\partial_0, \partial_\alpha] \hat{u}_\alpha\), using (6) and (18), (19), which leads to an expression for the evolution of \(j\) in terms of \(\partial_\alpha z_\alpha\). With the aid of the field equations (A.20)–(A.22), their propagation along \(\mathbf{u}\) and the use of the \([\partial_0, \partial_\alpha]\) commutators on \(r_\alpha, q_\alpha, u_\alpha\) (excluding \(r_1, q_2, u_3\)) together with the \([\partial_1, \partial_2]\) commutators on \(\omega\) and Jacobi equation (A.5), one obtains algebraic expressions for the directional derivative \(\partial_3 z_3\), as well as evolution equations for \(j\) as

\[
\partial_0 j = \frac{\theta (G'E + p' - 2Gp')}{p'} (\hat{u}_1^2 + (1 - 2G)(z_1 \hat{u}_1 + z_2 \hat{u}_2 + z_3 \hat{u}_3))
\]

\[+ \frac{1}{3} \theta (1 + 3G) j - p' \theta (1 - 9p') \omega^2
\]

(32)

\[
\partial_3 z_3 = \frac{\theta (G'E - 2Gp')}{p' (1 + 3p')} (\hat{u}_1^2 - 3\hat{u}_3) - \frac{2G}{1 + 3p'} (z_1 \hat{u}_1 + z_2 \hat{u}_2 - 2z_3 \hat{u}_3) - 2z_3 \hat{u}_3
\]

\[+ \frac{2G + 9p^2 - 6p'}{1 + 3p'} \omega^2 \theta + \frac{\theta (-2 + 3G)}{1 + 3p'} E_{33} + r_2 z_2 - q_1 z_1
\]

(33)

\((G' = \frac{dG}{dp'}\) and \(\hat{u}_1^2 \equiv \hat{u}_1^2 + \hat{u}_2^2 + \hat{u}_3^2\). The solutions of equations (A.5) and (A.17) are then given by

\[
\partial_2 q_1 = \frac{\theta (2Gp' - G'E)}{3\omega p' (1 + 3p')} (\hat{u}_1^2 - 3\hat{u}_3^2) + \frac{2G}{3\omega (1 + 3p')} (z_1 \hat{u}_1 + z_2 \hat{u}_2 - 2z_3 \hat{u}_3)
\]

\[+ \frac{r_1 - 3q_3 + 3z_3}{3\omega} z_3 + \frac{\theta (-2 + 3G)}{3\omega (1 + 3p')} E_{33} - \frac{9p^2 + 2G - 9p'}{3(1 + 3p')} \theta \omega
\]

\[- \frac{1}{3\omega} (r_2 z_2 - q_1 z_1) + (r_1 + q_1) r_2 + n(q_3 + r_3) - E_{12}
\]

(34)

and

\[
\partial_1 r_2 = \frac{\theta (2Gp' - G'E)}{3\omega p' (1 + 3p')} (\hat{u}_1^2 - 3\hat{u}_3^2) + \frac{2G}{3\omega (1 + 3p')} (z_1 \hat{u}_1 + z_2 \hat{u}_2 - 2z_3 \hat{u}_3)
\]

\[+ \frac{3r_3 - q_3 + 3z_3}{3\omega} z_3 + \frac{\theta (-2 + 3G)}{3\omega (1 + 3p')} E_{33} - \frac{9p^2 + 2G - 9p'}{3(1 + 3p')} \theta \omega
\]

\[- \frac{1}{3\omega} (r_2 z_2 - q_1 z_1) - (r_2 + q_2) q_1 - n(q_3 + r_3) + E_{12}.
\]

(35)
From the Ricci identity [14], $H_{(ab)} = 2u_{(a}a_{b)} + D_{(a}a_{b)}$, the components of the (trace-free) magnetic part of the Weyl curvature are given by
\begin{align*}
H_{11} &= -\omega (u_1 + r_3), & H_{22} &= -\omega (u_3 - q_3), \\
H_{13} &= z_2/3 - \omega q_1, & H_{23} &= -z_1/3 + \omega r_2, & H_{12} &= 0.
\end{align*}
(36)

At this stage, we introduce the condition that $E$ is solenoidal, $\text{div} E = 0$, which by the Bianchi identity [14], $(\text{div} E)_a = \frac{1}{2} D_a \mu - 3 \omega^2 H_{ab}$, reduces to
\begin{align*}
\mathcal{E} \dot{u}_1 + 3 \rho^2 \omega (z_2 - 3 \omega q_1) &= 0, \\
\mathcal{E} \dot{u}_2 - 3 \rho^2 \omega (z_1 - 3 \omega r_2) &= 0,
\end{align*}
(37) (38)

\begin{equation*}
(\mathcal{E} + 18 \rho^2 \rho^2) \dot{u}_3 - 9 \omega^2 \rho^2 (-r_3 + q_3) = 0.
\end{equation*}
(39)

Apart from $\rho \omega \rho \mathcal{E} \neq 0$ we shall also assume that $\rho' \neq 0, \pm 1/3, 1/9$, $u_1^2 + u_2^2 \neq 0$, $E_{ab} \neq 0, H_{ab} \neq 0, \theta$ not just a function of either $\mu$ or $\omega$, so as to avoid duplication of known results.

Propagating equations (37) and (38) gives then
\begin{equation*}
3 \rho' \left[ 2 \mathcal{E} - 9(1 - 3 \rho^2) \omega^2 \right] \dot{u}_3 - 6 \mathcal{E} \left( 2 \mathcal{E} - 6 \rho^2 - 2 \right) + 81 \mathcal{E} \rho^2 \omega^2 \dot{u}_3 = 0.
\end{equation*}
(40)

These two equations show that $z_A$ is parallel to $u_A$, unless
\begin{equation*}
2 \mathcal{E} - 9 (1 - 3 \rho^2) \omega^2 = 2 \mathcal{E} (6 \rho^2 - 2) + 81 \mathcal{E} \rho^2 \omega^2 = 0.
\end{equation*}
(41)

3. $u_A$ parallel to $z_A$

Taking the evolution of $\dot{u}_A z_1 - \dot{u}_1 z_2 = 0$ along $u$, this results in
\begin{equation*}
27 \rho' (3 \mathcal{E} + 3 \rho^2 - 1) \omega^2 - \mathcal{E} (-1 + 27 \rho^2 + 9 \rho^2 - 54 \rho^2) = 0.
\end{equation*}

If the coefficient of $\omega^2$ were zero, the latter equation would give
\begin{equation*}
\mathcal{E} (-1 + 9 \rho^2) = 0.
\end{equation*}

Hence, the vorticity is a function of the matter density and one has $\partial_1 \mu \partial_2 \omega = - \partial_2 \omega \partial_1 \mu = 0$, which reduces to $(\dot{u}_1 z_1 + \dot{u}_2 z_2) \mathcal{E} = 0$. It follows that $\dot{u}_1 = \dot{u}_2 = 0$, which is impossible.

4. $u_A$ not parallel to $z_A$

Eliminating $\omega$ from (41) fixes the equation of state to be
\begin{equation*}
9 \mathcal{E} \rho' = 18 \rho^2 + 9 \rho^2 - 1 = 0.
\end{equation*}
(42)

Note that a $\gamma$-law equation of state is then only possible when $\rho' = -1/3$ or $1/9$, so henceforth we will exclude all sub-cases in which $\rho'$ is constant. By (41), the vorticity is again a function of matter density. It follows that
\begin{equation*}
\partial_3 \mu \partial_2 \omega - \partial_2 \omega \partial_3 \mu = 0,
\end{equation*}
which after eliminating $q_1$, $r_2$ and $q_3$ and using equations (37)–(39) yields
\begin{equation*}
\mathcal{E} (-3 \omega \dot{u}_A + z_A + 1) \dot{u}_3 = 0.
\end{equation*}

It follows that $\dot{u}_3$ is zero, as otherwise, propagation of equation (41) along $e_A$ would imply $\mathcal{E} (-1 + 3 \rho^2) \dot{u}_3 = 0$ and again $\dot{u}_A = 0$.

When $\dot{u}_3$ is zero, expressions (16), (22), (23), (19) and (13) show that
\begin{align*}
z_3 &= 0, & n_{33} &= 0, & E_{33} &= \dot{u}_1 r_3, & E_{23} &= -\dot{u}_2 q_3, \\
E_{33} &= -\frac{1}{3} (2 \omega^2 + j) + \dot{u}_1 q_1 - \dot{u}_2 r_2.
\end{align*}
Moreover, the conditions $\text{div}\mathbf{E} = 0$, (37)–(39) can now be rewritten as follows:

$$\mathcal{E}\dot{u}_1 + 3\omega p'(z_2 - 3\omega q_1) = 0,$$

$$\mathcal{E}\dot{u}_2 - 3\omega p'(z_3 - 3\omega r_2) = 0,$$

$$p'(q_3 - r_3)\omega^2 = 0.$$  

(43)

Propagating equation (41) along $e_1$ and $e_2$, using equations (43), yields

$$\mathcal{E}(-\dot{u}_1 + 3p'q_1) = 0,$$

$$\mathcal{E}(\dot{u}_2 + 3p'r_2) = 0.$$  

(44)

(45)

Again propagating the latter equations along $e_1$ and $e_2$ respectively and eliminating $E_{12}$ show then

$$90p'^2(3p' - 1)(18p'^2 - 15p' + 1)j + 9p'^2(3p' - 1)(36p'^2 - 9p' + 5)(\dot{u}_1z_1 + \dot{u}_2z_2)$$

$$- 90(3p' - 1)(162p'^4 + 108p'^3 - 162p'^2 + 21p' - 1)(\dot{u}_1^2 + \dot{u}_2^2)$$

$$+ 6p'^3(27p'^2 - 18p' + 7)\mathcal{E} = 0.$$  

(46)

Now we use $z_3 = 0$ and equations (41)–(44), (45) and (43) to simplify equation (33) to

$$\mathcal{E}(3p' - 1)(162p'^4 + 108p'^3 - 162p'^2 + 21p' - 1)(\dot{u}_1^2 + \dot{u}_2^2) - 6(27p'^2 - 18p' + 7)\mathcal{E}p'^3$$

$$- 9p'^2(3p' - 1)(18p'^2 - 15p' + 1)j = 0.$$  

(47)

This can now be combined with (46) to give

$$p'^2(3p' - 1)(36p'^2 - 9p' + 5)(z_1\dot{u}_1 + z_2\dot{u}_2) = 0.$$  

It follows that $\dot{u}_A$ is orthogonal to $z_A$, after which the propagation of equation (46) along $e_0$ gives

$$2(1296p'^4 - 2106p'^3 + 522p'^2 + 9p' - 5)(3p' - 1)^2(\dot{u}_1^2 + \dot{u}_2^2)$$

$$- 24p'^2\mathcal{E}(p' + 1)(81p'^3 - 207p'^2 + 93p' - 7) = 0.$$  

(48)

Taking the time derivative of the latter equation, eliminating $j, z_1$ using equation (46) and $z_2\dot{u}_2 + z_1\dot{u}_1 = 0$ results in a new equation between $p, p'$ and $\dot{u}_1^2 + \dot{u}_2^2$. Eliminating the acceleration between the latter pair eventually results in

$$\frac{\mathcal{E}p'^3(3p' - 1)(18p'^2 - 15p' + 1)}{1296p'^3 - 2106p'^3 + 522p'^2 + 9p' - 5}$$

$$+ 156249p'^3 - 43065p'^2 + 3267p' - 26) = 0,$$

showing that $p'$ has to be a constant, in contradiction with (42) and our assumptions $p \neq -1/3, 1/9$.

5. Discussion

Newtonian homogeneous cosmologies are known [17] to provide counter-examples to the Newtonian shear-free fluid conjecture. In these models, the spatial divergence of the tidal field (the Newtonian analogue of $E$) automatically vanishes, such that, when searching for possible counter-examples to the relativistic shear-free fluid conjecture, it is tempting to look for candidates within the class of perfect fluids having a solenoidal electric part of the Weyl tensor. The result obtained in this paper shows that this attempt remarkably (or luckily—depending on one’s attitude towards the conjecture) fails: shear-free perfect fluids obeying a barotropic equation of state and having $\text{div}\mathbf{E} = 0$ are all non-rotating or non-expanding, in contrast to their Newtonian brethren. Whether the conjecture in its full generality is true or false still remains an intriguing issue.
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