FOLIATIONS WITH COMPLEX LEAVES AND INSTABILITY FOR HARMONIC FOLIATIONS

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Abstract. In this paper, we study stability for harmonic foliations on locally conformal Kähler manifolds with complex leaves. We also discuss instability for harmonic foliations on compact submanifolds immersed in Euclidean spaces and compact homogeneous spaces.

1. Introduction

In this paper we study stability and instability for harmonic foliations. Let $(M, J, g_M)$ be a Hermitian manifold and $\Omega$ the fundamental 2-form associated with $g_M$. Then $(M, J, g_M)$ is a locally conformal Kähler manifold if there exists a closed 1-form $\omega$, called the Lee form, satisfying $d\Omega = \omega \wedge \Omega$. Besides Kähler manifolds, there are numerous examples of locally conformal Kähler manifolds. For instance, a Vaisman manifold is known to be a locally conformal Kähler manifold with non-exact and parallel Lee form.

The main purpose of this paper is to prove the following stability theorem for harmonic foliations on compact locally conformal Kähler manifolds:

Main Theorem. Let $(M, J, g_M)$ be an $n$-dimensional compact locally conformal Kähler manifold. If $\mathcal{F}$ is a harmonic foliation on $M$ with bundle-like metric $g_M$ foliated by complex submanifolds, then $\mathcal{F}$ is stable.

This is an analogue of the theorem “a holomorphic map between two Kähler manifolds is stable as a harmonic map” (see also Corollary 3.1), where harmonicity for a foliation $\mathcal{F}$ on a Riemannian manifold $(N, g_N)$ is defined by Kamber and Tondeur in [6] as the harmonicity of the canonical projection $\pi$ from $TN$ onto the normal bundle $Q$ for the foliation $\mathcal{F}$. The key of the proof of Main Theorem is the compatibility of the complex structure with the connection on the normal bundle of the foliation (see Lemma 3.2).

We also discuss instability for harmonic foliations on compact homogeneous Riemannian manifolds or compact submanifolds in a Euclidean space. We actually obtain a sufficient condition for a harmonic foliation on compact submanifolds immersed in Euclidean space to be unstable, where its application to the standard sphere allows us to obtain the result

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of Kamber and Tondeur [8]. We also prove that, for a compact homogeneous Riemannian manifold \((N, g_N)\) satisfying \(\lambda_1 < 2s \cdot \dim N\), any harmonic foliation on \(N\) with bundle-like \(g_N\) is unstable, where \(\lambda_1\) and \(s\) denote the first eigenvalue of the Laplacian and the scalar curvature of \(g_N\), respectively. Then instability for harmonic foliations on \(N\) is equivalent to the non-existence of stable harmonic map between \(N\) and any compact Riemannian manifold (see Theorem 4.8). In particular, we determine all simply connected compact irreducible symmetric spaces whose harmonic foliation is unstable (see Theorem 4.9).

This paper is organized as follows. In Section 2, we review the theory of harmonic foliations by Kamber and Tondeur. Then Section 3 is devoted to the proof of Main Theorem above for harmonic foliations. Finally in Section 4, we shall show Theorem 4.8 and 4.9 on instability of harmonic foliations.

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2. The Jacobi operator and a stability of harmonic foliations

Let \((N, g_N)\) be an \(n\)-dimensional compact Riemannian manifold and let \(F\) be a foliation given by an integrable subbundle \(L \subset TN\). We define a torsion free connection \(\nabla\) on normal bundle \(Q = TN/L\) by

\[
\begin{align*}
\nabla_X S &= \pi[X, Y_S], \quad \text{for } X \in \Gamma(L), \ S \in \Gamma(Q) \text{ and } Y_S = \sigma(S) \in \Gamma(\sigma(Q)), \\
\nabla_X S &= \pi(\nabla_X Y_S), \quad \text{for } X \in \Gamma(\sigma(Q)), \ S \in \Gamma(Q) \text{ and } Y_S = \sigma(S) \in \Gamma(\sigma(Q)),
\end{align*}
\]

(2.1)

where \(\sigma : Q \to TN\) is a splitting such that \(\sigma(Q)\) coincides with the orthogonal complement \(L^\perp\) of \(L\) in \(TN\) with respect to \(g_N\). If the normal bundle \(Q\) is equipped with a holonomy invariant fiber metric \(g_Q\), i.e. \(X g_Q(S, T) = g_Q(\nabla_X S, T) + g_Q(S, \nabla_X T)\) for all \(X \in \Gamma(TN)\), the foliation \(F\) is called a Riemannian foliation or an R-foliation. There is a unique metric \(g_Q\) for an R-foliation with a torsion free connection \(\nabla\) on the normal bundle \(Q\). A Riemannian metric \(g_N\) on \(N\) is called a bundle-like metric with respect to the foliation \(F\) if the foliation becomes an R-foliation in terms of the fiber metric \(g_Q\) induced on \(Q\).

For a foliation \(F\) on a Riemannian manifold \((N, g_N)\), the curvature \(R^F\) of the connection \(\nabla\) is an \(\text{End}(Q)\)-valued 2-form on \(N\). Since \(i(X)R^F = 0\) for \(X \in \Gamma(L)\), it follows that the Ricci operator \(R^F(S, T) : Q \to Q\) for \(S, T \in \Gamma(Q)\), is well-defined. Define \(P^F(U, V) : Q \to Q\) by \(P^F(U, V)S = -R^F(U, S)V\) for all \(S \in \Gamma(Q)\). The Ricci curvature \(S^F\) for \(F\) is then \(S^F(U, V) = \text{trace} P^F(U, V)\) which is a symmetric bilinear form. We define the Ricci operator \(\rho^F : Q \to Q\) as the corresponding self-adjoint operator given by \(g_Q(\rho^F U, V) = S^F(U, V)\), where \(g_Q\) denotes the holonomy invariant metric on \(Q\). In terms of an orthonormal basis \(e_{p+1}, \ldots, e_n\) of \(Q_x\) at some \(x \in N\), we have \((\rho^F U)_x = \sum_{\alpha=p+1}^n R^F(U, e_{\alpha}) e_{\alpha}\).
Denoting by $\pi \in \Omega^1(N, Q)$ the canonical projection from $TN$ onto $Q$, we have $d_\pi \pi \in \Omega^2(N, Q)$, $d_\pi^* \pi \in C^\infty(N, Q)$, the Laplacian $\Delta$ on $\Omega^1(N, Q)$ and so forth. Then we have the following fact (Kamber and Tondeur [6, 3.3]).

**FACT.** Let $\mathcal{F}$ be an $R$-foliation on compact oriented Riemannian manifold $N$ with a bundle-like metric. Then the following are equivalent:

(i) $\pi$ is harmonic,
(ii) all leaves for the foliation are minimal submanifolds of $N$,
(iii) $\Delta \pi = 0$.

A foliation is said to be harmonic if it satisfies (i) or (ii) in above fact.

We next study first and second variations of $R$-foliation $\mathcal{F}$ on a compact Riemannian manifold $(N, g_N)$ with bundle-like metric $g_N$. We define the energy of the foliation $\mathcal{F}$ by

$$E(\mathcal{F}) = \frac{1}{2}\|\pi\|,$$

where $\pi$ is the canonical projection from $TN$ onto $Q$ and is considered as a $Q$-valued 1-form on $N$. Let $\{U_\alpha, f^\alpha, \gamma^{\alpha \beta}\}$ be the Haefliger cocycle representing $\mathcal{F}$. Namely, $\{U_\alpha\}$ is an open cover of $N$ with $f^\alpha : U_\alpha \rightarrow \mathbb{R}^q$ such that $\gamma^{\alpha \beta}$ are local isometries on $U_\alpha \cap U_\beta (\neq \emptyset)$ satisfying $f^\alpha = \gamma^{\alpha \beta} f^\beta$. Here $q$ denotes the codimension of $\mathcal{F}$. For $\nu \in \Gamma(Q)$, we put

$$\Phi_t^\alpha(x) = \exp_{f^\alpha(x)}(t\nu^\alpha(x)), \quad x \in U_\alpha, \ t \in (-\varepsilon, \varepsilon),$$

where $\nu^\alpha = \nu|_{U_\alpha}$. We then have a variation $\Phi_t^\alpha$ of $f^\alpha = \Phi_0^\alpha$, where $\varepsilon$ is sufficiently small.

Since $\Phi_t^\alpha(x) = \gamma^{\alpha \beta} \Phi_t^\beta(x)$ on $U_\alpha \cap U_\beta$, the local variations $\{\Phi_t^\alpha\}$ define a variation $\mathcal{F}_t$ of the foliation $\mathcal{F}$. Moreover we have

$$\nabla_{\frac{\partial}{\partial t}|_{t=0}}(\Phi_t^\alpha)_s = \nabla \nu^\alpha \in \Omega^1(U_\alpha, Q). \tag{2.2}$$

To obtain the second variation, we need a 2-parameter variation $\mathcal{F}_{s,t}$ of $\mathcal{F}_{0,0} = \mathcal{F}$ defined locally as $\Phi_{s,t}^\alpha$, where

$$\Phi_{s,t}^\alpha(x) = \exp_{f^\alpha(x)}(s\mu^\alpha(x) + t\nu^\alpha(x)), \quad x \in U_\alpha, \ s, t \in (-\varepsilon, \varepsilon)$$

for $\nu, \mu \in \Gamma(Q)$. Then by (2.2)

$$\begin{align*}
\nabla_{\frac{\partial}{\partial s}|_{s=0,t=0}}(\Phi_{s,t}^\alpha)_s &= \nabla \mu^\alpha, \\
\nabla_{\frac{\partial}{\partial t}|_{s=0,t=0}}(\Phi_{s,t}^\alpha)_s &= \nabla \nu^\alpha.
\end{align*}$$

The second variation formula is now given by

$$\begin{align*}
\frac{\partial^2}{\partial s \partial t}\bigg|_{s=0,t=0} E(\mathcal{F}_{s,t}) &= \frac{\partial^2}{\partial s \partial t}\bigg|_{s=0,t=0} \frac{1}{2} \langle \pi_{s,t}, \pi_{s,t} \rangle = \frac{\partial}{\partial s}\bigg|_{s=0,t=0} \langle \nabla \nu, \pi_{s,t} \rangle \\
&= \langle \nabla_{\frac{\partial}{\partial s}} \nabla \nu, \pi \rangle + \langle \nabla \nu, \nabla \mu \rangle = \langle R^\nu(\mu, \pi) \nu, \pi \rangle + \langle \nabla \nabla_{\frac{\partial}{\partial s}} \nabla \nu, \pi \rangle + \langle d_\nu \nu, d_\nu \mu \rangle \\
&= -\langle R^\nu(\mu, \pi) \nu, \pi \rangle + \langle \nabla_{\frac{\partial}{\partial s}} \nu, d_\nu \pi \rangle + \langle d_\nu d_\nu \mu, \nu \rangle = \langle (\Delta - \rho_\nu) \nu, \mu \rangle + \langle \nabla_{\frac{\partial}{\partial s}} \nu, d_\nu \pi \rangle,
\end{align*}$$

where $\Delta$ is the Laplacian acting on $\Omega^1(U_\alpha, Q)$.
where $R$ and $\rho$ are the curvature and the Ricci operator for $Q$, respectively. For a harmonic foliation $\mathcal{F}$, we have

$$
\left. \frac{\partial^2}{\partial s \partial t} \right|_{s=0,t=0} E(\mathcal{F}_{s,t}) = \langle (\Delta - \rho) \mu, \nu \rangle = \langle J_{\mathcal{F}} \mu, \nu \rangle,
$$

where $J_{\mathcal{F}} = \Delta - \rho$ is the Jacobi operator of $\mathcal{F}$. Note that the Jacobi operator $J_{\mathcal{F}}$ is a self-adjoint and strongly elliptic with real eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots \to \infty$ for $i \to \infty$. Here the dimension of each eigenspace $V_{\lambda}(\mathcal{F}) = \{ \nu \in \Gamma(Q); J_{\mathcal{F}} \nu = \lambda \nu \}$ is finite, i.e. $\dim V_{\lambda}(\mathcal{F}) < \infty$.

**Definition.** The *index* of a harmonic foliation $\mathcal{F}$ is defined by

$$
\text{index}(\mathcal{F}) = \sum_{\lambda_i < 0} \dim V_{\lambda_i}(\mathcal{F})
$$

and a harmonic foliation $\mathcal{F}$ is said to be *stable* if $\text{index}(\mathcal{F}) = 0$, i.e. $\langle J_{\mathcal{F}} \nu, \nu \rangle \geq 0$ for all $\nu \in \Gamma(Q)$.

Note that this definition makes sense for the case of harmonic foliation $\mathcal{F}$ with bundle-like metric $g_N$, because if $g_N$ is not bundle-like, then the equality (2.3) does not hold in general.

### 3. Harmonic foliations on locally conformal Kähler manifolds

The purpose of this section is to prove Main Theorem in Introduction.

For a locally conformal Kähler manifold $(M, J, g_M)$ with $\Omega$ and $\omega$, let $B = \omega^\sharp$ be the Lee vector field, where $\sharp$ denotes the raising of indices with respect to $g_M$.

The case when $\omega$ is identically zero, $(M, J, g_M)$ is a Kähler manifold. Any complex submanifold of a Kähler manifold is also Kähler, and especially, is minimal. Hence, in this case, we have the following:

**Corollary 3.1.** The foliations on compact Kähler manifolds with a bundle-like metric foliated by complex submanifolds are stable.

The following lemma is crucial in the proof of Main Theorem:

**Lemma 3.2.** The connection $\nabla$ on $Q$ defined in (2.1) satisfies $\nabla_X J_Q S = J_Q \nabla_X S$ for all $X \in \Gamma(TM)$ and $S \in \Gamma(Q)$, where $J_Q$ denotes the almost complex structure on $Q$ induced by $J$ on $M$.

**Proof.** We first note that any complex submanifold $N$ of a locally conformal Kähler manifold $M$ is minimal if and only if the Lee vector field $B$ for $M$ is tangent to $N$ (for instance, see Dragomir and Ornea [2, Theorem 12.1]). Let $\nabla^M$ be the Levi-Civita
connection of \((M, g_M)\). Then for all \(X, Y \in \Gamma(TM)\),
\[
\nabla^M_X JY = J\nabla^M_X Y + \frac{1}{2}\{\theta(Y)X - \omega(Y)JX - g_M(X, Y)A - \Omega(X, Y)B\},
\]
where \(\theta = \omega \circ J\) and \(A = -JB\). Then if \(X \in \Gamma(\sigma(Q))\) and \(Y \in \Gamma(Q)\), we have
\[
\nabla_X JQ S - J Q \nabla_X S = \pi(\nabla_X JY S - J \nabla_X Y S)
\]
\[
= \pi\left(\frac{1}{2}\{\theta(Y_S)X - \omega(Y_S)JX - g_M(X, Y_S)A - \Omega(X, Y_S)B\}\right) = 0,
\]
by \(\theta(Y_S) = \omega(Y_S) = 0\). On the other hand, if \(X \in \Gamma(L)\) and \(S \in \Gamma(Q)\), by Proposition 2.2 of Dragomir and Ornea \[2\] (cf. Vaisman \[15\]), we have \([X, JY_S] - J[X, Y_S] \in L\). Then
\[
\nabla_X JQ S - J Q \nabla_X S = \pi([X, JY_S] - J[X, Y_S]) = 0,
\]
and this completes the proof of the lemma. 

We define a linear differential operator \(D : \Gamma(Q) \rightarrow \Gamma(Q \otimes T^*M)\) of first order by
\[
DV(X) = \nabla_{JX} V - J Q \nabla_X V, \quad V \in \Gamma(Q) \text{ and } X \in \Gamma(TM).
\]

**Proof of Main Theorem.** It suffices to show
\[
\langle J_V V, V \rangle = \frac{1}{2} \langle DV, DV \rangle
\]
for all \(V \in \Gamma(Q)\). Let \(\{e_1, \ldots, e_n, f_1, \ldots, f_n\}\) be a local orthonormal frame such that \(J e_i = f_i, J f_i = -e_i, 1 \leq i \leq n\), and that the frame \(\{e_1, \ldots, e_p, f_1, \ldots, f_p\}\) spans \(F\). Then
\[
\langle J_V V, V \rangle = \langle d_{\nabla_V} d_{V} V, V \rangle - \langle R^V(V), V \rangle = \langle d_{\nabla_V} d_{V} V \rangle - \langle R^V(V), V \rangle
\]
\[
= \sum_{i=1}^{n} \left\{ \int_M g_Q(\nabla_{e_i} V, \nabla_{e_i} V) v_M + \int_M g_Q(\nabla_{f_i} V, \nabla_{f_i} V) v_M \right\}
\]
\[
- \sum_{i=p+1}^{n} \left\{ \int_M g_Q(R^V(V, e_i) e_i, V) v_M + \int_M g_Q(R^V(V, f_i) f_i, V) v_M \right\}.
\]
On the other hand, $\langle DV, DV \rangle$ is written as

\[
(3.5) \quad \langle DV, DV \rangle = \sum_{i=1}^{n} \left\{ \int_{M} g_Q(DV(e_i), DV(e_i))v_M + \int_{M} g_Q(DV(f_i), DV(f_i))v_M \right\}
\]

\[
= \sum_{i=1}^{n} \left\{ \int_{M} g_Q(\nabla J_{e_i}V - J\nabla_{e_i}V, \nabla J_{e_i}V - J\nabla_{e_i}V) + g_Q(\nabla J_{f_i}V - J\nabla_{f_i}V, \nabla J_{f_i}V - J\nabla_{f_i}V)v_M \right\}
\]

\[
= \sum_{i=1}^{n} \int_{M} \left\{ g_Q(\nabla J_{e_i}V, \nabla J_{e_i}V) - 2g_Q(\nabla J_{e_i}V, J\nabla_{e_i}V) + g_Q(\nabla e_iV, \nabla e_iV) + 2g_Q(\nabla e_iV, J\nabla J_{e_i}V) + g_Q(\nabla J_{e_i}V, J\nabla J_{e_i}V) \right\}v_M
\]

\[
= \sum_{i=1}^{n} \int_{M} \left\{ g_Q(\nabla J_{e_i}V, \nabla J_{e_i}V) + g_Q(\nabla J_{e_i}V, \nabla J_{e_i}V) - g_Q(\nabla J_{e_i}V, J\nabla J_{e_i}V) \right\}v_M
\]

We also observe that

\[
(3.6) \quad \sum_{i=1}^{n} \int_{M} \left\{ e_i g_Q(V, J\nabla J_{e_i}V) - J e_i g_Q(V, J\nabla J_{e_i}V) - g_Q(V, J\nabla_{[e_i, e_i]}V) \right\}v_M = 0,
\]

because if $X \in \Gamma(TM)$ is defined by $g_M(X, Y) = g_Q(\nabla J_{e_i}V, J\nabla J_{e_i}V)$, then the following computation of $\text{div}(X)$ together with $\int_{M} \text{div}(X)v_M = 0$ allows us to obtain (3.6):

\[
\text{div}(X) = \sum_{i=1}^{n} \left\{ g_M(e_i, \nabla_{e_i}^{M}X) + g_M(Je_i, \nabla_{J e_i}^{M}X) \right\}
\]

\[
= \sum_{i=1}^{n} \left\{ e_i g_M(e_i, X) - g_M(\nabla_{e_i}^{M}e_i, X) + J e_i g_M(J e_i, X) - g_M(\nabla_{J e_i}^{M}J e_i, X) \right\}
\]

\[
= \sum_{i=1}^{n} \left\{ e_i g_Q(\nabla J_{e_i}V, J\nabla J_{e_i}V) - J e_i g_Q(\nabla J_{[e_i, e_i]}V, J\nabla J_{e_i}V) \right\}
\]
\[ \sum_{i=1}^{n} \left\{ e_{i}g_{Q}(\nabla_{e_{i}}V, JV) - Je_{i}g_{Q}(\nabla_{e_{i}}V, JV) \right\} \]

Now by (3.5) and (3.6), we have
\[ \langle DV, DV \rangle = 2 \sum_{i=1}^{n} \int_{M} \left\{ g_{Q}(\nabla_{e_{i}}V, \nabla_{e_{i}}V) + g_{Q}(\nabla_{e_{i}}V, \nabla_{e_{i}}V) - g_{Q}(V, JR^{V}(e_{i}, Je_{i})V) \right\} v_{M}. \]

Then for \( 1 \leq i \leq p, \)
\[ R^{V}(e_{i}, Je_{i})V = \nabla_{e_{i}}\nabla_{e_{i}}V - \nabla_{e_{i}}\nabla_{e_{i}}V - \nabla_{e_{i}, Je_{i}}V = \pi[e_{i}, \pi[Je_{i}, V]] - \pi[Je_{i}, \pi[e_{i}, V]] - \pi[e_{i}, \pi[e_{i}, V]] = \pi[e_{i}, \pi[Je_{i}, V]] + \pi[Je_{i}, \pi[V, e_{i}]] + \pi[V, [e_{i}, Je_{i}]] = 0, \]

because the foliation is involutive satisfying
\[ \pi[e_{i}, \pi^{\perp}[Je_{i}, V]] = 0 = \pi[Je_{i}, \pi^{\perp}[e_{i}, V]], \]

where \( \pi^{\perp} = \text{id} - \pi. \) Furthermore, for \( p + 1 \leq i \leq n, \) the Bianchi identity shows that
\[ JR^{V}(e_{i}, Je_{i})V = -JR^{V}(Je_{i}, V)e_{i} - JR^{V}(V, e_{i})Je_{i} = R^{V}(V, Je_{i})Je_{i} + R^{V}(V, e_{i})e_{i} \]

Thus, by (3.4), (3.7), (3.8) and (3.9), we obtain the required identity (3.3) as follows:
\[ \frac{1}{2} \langle DV, DV \rangle = \sum_{i=1}^{n} \int_{M} \left\{ g_{Q}(\nabla_{e_{i}}V, \nabla_{e_{i}}V) + g_{Q}(\nabla_{e_{i}}V, \nabla_{e_{i}}V) \right\} v_{M} \]
\[ - \sum_{i=p+1}^{n} \int_{M} \left\{ g_{Q}(R^{V}(V, e_{i})e_{i}, V) + g_{Q}(R^{V}(V, Je_{i})Je_{i}, V) \right\} v_{M} \]
\[ = \langle J^{V}V, V \rangle. \]

In the remainder of Section 3, we give an example of a stable harmonic foliation on a locally conformal Kähler manifold. Let \( \lambda \) be a complex number satisfying \( |\lambda| \neq 1. \) Denote by \( \langle \lambda \rangle \) the cyclic group generated by the transformation : \( (z_{1}, \ldots, z_{n}) \mapsto (\lambda z_{1}, \ldots, \lambda z_{n}) \) of \( \mathbb{C}^{n} - \{0\}. \) Since this group acts freely and holomorphically on \( \mathbb{C}^{n} - \{0\}, \) the quotient space \( \mathbb{C}H^{n} := (\mathbb{C}^{n} - \{0\})/\langle \lambda \rangle \) is a complex manifold called a Hopf manifold. Consider the Hermitian metric \( g_{0} = (\sum_{k=1}^{n}dz^{k} \otimes d\bar{z}^{k})/||z||^{2} \) on \( \mathbb{C}^{n} - \{0\}. \) Then \( g_{0} \) gives not only a locally conformal Kähler structure but also a Vaisman manifold structure on \( \mathbb{C}H^{n} \) with Lee form \( \omega_{0} = -\{\sum_{k=1}^{n}(z^{k}dz^{k} + \bar{z}^{k}d\bar{z}^{k})/||z||^{2}. \) It is well-known that \( \mathbb{C}H^{n} \) has a principal \( T^{1}_{C} \) -bundle
structure over the projective space \( \mathbb{C}P^{n-1} \). Then the foliation on \( \mathbb{C}H^n \) defined by the canonical projection \( \pi : \mathbb{C}H^n \to \mathbb{C}P^{n-1} \) is harmonic and is stable by Main Theorem, where the metric on \( \mathbb{C}P^{n-1} \) is the Fubini-Study metric.

**Remark.** (i) More generally, Main Theorem is valid even if \( M \) is (not necessarily Kähler and is) just a compact Hermitian manifold, provided that the connection \( \nabla \) defined by (2.1) satisfies Lemma 3.2.

(ii) As to stable harmonic foliations, there exists an example foliated by fibers of a Riemannian submersion whose base space is not a complex manifold. A typical example is the twistor space of a quaternionic Kähler manifold.

### 4. Instability of harmonic foliations

In this section, we discuss instability for harmonic foliations on Riemannian manifolds. Let \( (N, g_N) \) be a Riemannian manifold. By the Weitzenböck formula we have \( (\Delta \pi) = \nabla^* \nabla \pi + S(\pi) \), where by using a local frame \( \{e_1, \ldots, e_n\} \) for \( TN \), we put

\[
\nabla^* \nabla \pi = -\sum_{i=1}^{n} (\nabla^2_{e_i,e_i} \pi) \quad \text{and} \quad S(\pi)(X) = \sum_{i=1}^{n} \left\{ R^\nabla(e_i, X) \pi(e_i) - \pi(R^N(e_i, X)e_i) \right\}
\]

for all \( X \in \Gamma(TN) \). Here \( R^\nabla \) and \( R^N \) denote the curvature tensors associated to \( \nabla \) and \( \nabla^N \), respectively. We then have

\[
(4.1) \quad \Delta \pi = \nabla^* \nabla \pi - \rho_\Delta \cdot \pi + \pi \cdot \rho_N,
\]

in view of the equality

\[
S(\pi)(X) = -\sum_{i=p+1}^{n} R^\nabla(\pi(X), e_i) e_i + \pi \sum_{i=1}^{n} R^N(X, e_i) e_i = -(\rho_\Delta \pi(X)) + \pi(\rho_N(X)).
\]

Let \( \mathcal{F} \) be a Riemannian and harmonic foliation on \( N \) with bundle-like \( g_N \), i.e., the canonical projection \( \pi : TN \to Q \) satisfies \( \Delta \pi = 0 \). Then (4.1) is expressible as

\[
(4.2) \quad \rho_\nabla \cdot \pi = \nabla^* \nabla \pi + \pi \cdot \rho_N.
\]

On the other hand, by operating the Laplacian on \( \pi(X), X \in \Gamma(TN) \), we obtain

\[
(4.3) \quad \Delta(\pi(X)) = d_\nabla^* d_\nabla(\pi(X)) = -\sum_{i=1}^{n} \nabla^2_{e_i,e_i}(\pi(X))
\]

\[
= (\nabla^* \nabla)(X) + \pi(\nabla^N \nabla^N X) - 2 \sum_{i=1}^{n} (\nabla_e \pi)(\nabla^N_{e_i} X).
\]

By (4.2) and (4.3), \( J_\nabla(\pi(X)) = (\Delta - \rho_\nabla)(\pi(X)) \) is written as

\[
J_\nabla(\pi(X)) = - (\pi \cdot \rho_N)(X) + \pi(\nabla^N \nabla^N X) - 2 \sum_{i=1}^{n} (\nabla_{e_i} \pi)(\nabla^N_{e_i} X).
\]
Assume that $N$ is a compact submanifold immersed in the Euclidean space $\mathbb{E}^n$ with the standard inner product $\langle \cdot, \cdot \rangle$. For each vector $v$ in $\mathbb{E}^n$, we define a smooth function $f_v$ on $N$ by $f_v(x) := \langle v, x \rangle$ for $x \in N$. We denote by $\Psi_t, t \in \mathbb{R}$, the flow generated by $V = \text{grad} f_v$. Simple computations give us

$$\langle \nabla^N_X Y, v \rangle = \langle B(X, Y), v \rangle,$$

(4.4)

$$\langle (\nabla^N)^2_X Z, Y \rangle = -\langle B(X, Y), B(Z, V) \rangle + \langle (\nabla B)(X, Y, Z), v \rangle,$$

(4.5)

where $B$ denotes the second fundamental form for the submanifold $N$ in $\mathbb{E}^n$.

The energy functional for $F$ is defined by $E(F) = (1/2) \int_N \|\pi\|^2$. Consider the associated quadratic form $Q_F$ by setting $Q_F(v) = \frac{d^2}{dt^2} E(\Psi_t)|_{t=0} = \int_N g_N(\mathcal{J}_\alpha(V), \pi(V))$. We shall now compute the trace $\text{Tr}(Q_F)$ of $Q_F$ on $\mathbb{E}^n$. By (4.4) and (4.5),

$$g_Q(\mathcal{J}_\alpha(V), \pi(V)) = -g_Q(\pi_N(V)), \pi(V))
+ \sum_{k,l=1}^n \langle B(e_k, e_l), B(e_k, V) \rangle g_Q(\pi(e_l), \pi(V))
- \sum_{k,l=1}^n \langle (\nabla B)(e_k, e_l), v \rangle g_Q(\pi(e_l), \pi(V))
- 2 \sum_{k,l=1}^n \langle B(e_k, e_l), v \rangle g_Q((\nabla e_k \pi)(e_l), \pi(V)).$$

Hence we have

$$\text{Tr}(Q_F) = \int_N \{ -\sum_{k=1}^n g_Q(\pi_N(e_k)), \pi(e_k))
+ \sum_{k,l,m=1}^n \langle B(e_k, e_l), B(e_l, e_m) \rangle g_Q(\pi(e_k), \pi(e_m)) \}$$

$$= \int_N \sum_{a=p+1}^n \sum_{j=1}^p \langle B(e_a, e_j), B(e_j, e_a) \rangle - \langle \rho_N(e_a), e_a \rangle \}.$$

Let $\eta$ denote the mean curvature vector of the submanifold $N$ in $\mathbb{E}^n$. Then by the equation of Gauss, we obtain

$$\text{Tr}Q_F = \int_N \sum_{a=p+1}^n (n \langle B(e_a, e_a), \eta \rangle - 2 \langle \rho_N(e_a), e_a \rangle).$$

This immediately implies
Lemma 4.6. Let \((N, g_N)\) be an \(n\)-dimensional compact submanifold immersed in the Euclidean space \(E^N\). If \(N\) satisfies
\[
n \ll B(u, u), \eta \gg -2 \ll \rho_N(u), u \gg < 0
\]
for all unit vector \(u\) in \(TN\), then every Riemannian and harmonic foliation on \(N\) with bundle-like \(g_N\) is unstable.

In the case where \(N\) is the standard sphere, the above result was proved by Kamber and Tondeur \[8\]. This lemma is a generalization of a result of Ohnita \[10\] known for harmonic maps. The following is now straightforward from Lemma 4.6.

Theorem 4.7. Let \(N\) be an \(n\)-dimensional compact minimal submanifold of a unit sphere \(S^{N-1}(1)\). If the Ricci curvature \(S_N\) of \(N\) satisfies \(S_N > 2/n\), then every Riemannian and harmonic foliation on \(N\) with bundle-like \(g_N\) is unstable.

It might be of some interest to compare results on instability for harmonic foliations with that of harmonic maps. Hence, by combining Theorem 4.7 above with Theorem 4 of \[10\], we obtain:

Theorem 4.8. Let \((N, g_N)\) be an \(n\)-dimensional compact homogeneous Riemannian manifold with irreducible isotropy representation. For \((N, g_N)\), let \(s\) and \(\lambda_1\) denote the scalar curvature and the first eigenvalue of the Laplacian acting on functions, respectively. Then the following conditions are all equivalent:

1. \(\lambda_1 < 2s/n\).
2. Every Riemannian and harmonic foliation on \(N\) with bundle-like \(g_N\) is unstable.
3. There exist no nonconstant stable harmonic maps from \(N\) to Riemannian manifolds.
4. There exist no nonconstant stable harmonic maps from compact Riemannian manifolds to \(N\).
5. The identity map \(id_N\) of \(N\) onto itself is unstable as a harmonic map.

Proof. The implications (3) \(\implies\) (5) and (4) \(\implies\) (5) are trivial. Since the stability of the point foliation on \(N\) is equivalent to the stability of \(id_N\) as a harmonic map, (2) implies (5). Since \(N\) is an Einstein manifold from a result of Smith \[11\], we have the equivalence (1) \(\iff\) (5). Hence, it suffices to show (1) implies (2), (3) and (4). By virtue of the theorem of Takahashi \[12\], there exists a standard minimal immersion \(\varphi\) of \(N\) into a unit hypersphere \(S^n(1)\) by using an orthonormal basis for the first eigenspace of the Laplacian in such a way that \(\varphi\) is an isometric immersion of \((N, (\lambda_1/n)g_N)\) into \(S^n(1)\). Then the Ricci curvature of \((N, (\lambda_1/n)g_N)\) is greater than \(n/2\). By Theorem 4.7 and Theorem 1 of Ohnita \[10\], we obtain (2), (3) and (4).
\[\Box\]
Remark. Theorem 4.8 is valid even if we replace homogeneous $N$ above by a strongly harmonic manifold. However, for strongly harmonic manifolds, no inhomogeneous examples are known (c.f. Besse [1]).

Compact irreducible symmetric spaces which satisfy $\lambda_1 < 2s/n$ were determined by Smith [11], Nagano [9] and Ohnita [10]. Thus we obtain

**Theorem 4.9.** Let $(N, g_N)$ be a compact irreducible symmetric space. Then the following conditions are equivalent:

1. Any Riemannian and harmonic foliation on $N$ with bundle-like $g_N$ is unstable.
2. $N$ is simply connected and belongs to one of the following:
   - (a) $SU(n)$ ($n \geq 2$)
   - (b) $Sp(n)$ ($n \geq 2$)
   - (c) $SU(2n)/Sp(n)$ ($n \geq 3$)
   - (d) $S^n$ ($n \geq 3$)
   - (e) $G_{p,q}^{(n)} = Sp(p + q)/Sp(p) \times Sp(q)$ ($p \geq q \geq 1$)
   - (f) $E_6/F_4$
   - (g) $P_2(\Omega) = F_4/Spin(9)$.

Applying Lemma 4.6 to product isometric immersion, we have the following:

**Corollary.** If $(N, g_N)$ is a product of simply connected compact irreducible symmetric spaces belonging to the list in (2) of Theorem 4.9, then every Riemannian and harmonic foliation on $N$ with bundle-like $g_N$ is unstable.

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