On Multiple Zeta Values of Even Arguments

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Abstract

For $k \leq n$, let $E(2n,k)$ be the sum of all multiple zeta values with even arguments whose weight is $2n$ and whose depth is $k$. Of course $E(2n,1)$ is the value $\zeta(2n)$ of the Riemann zeta function at $2n$, and it is well known that $E(2n,2) = \frac{3}{4}\zeta(2n)$. Recently Z. Shen and T. Cai gave formulas for $E(2n,3)$ and $E(2n,4)$ in terms $\zeta(2n)$ and $\zeta(2)\zeta(2n-2)$. We give two formulas for $E(2n,k)$, both valid for arbitrary $k \leq n$, one of which generalizes the Shen-Cai results; by comparing the two we obtain a Bernoulli-number identity. We also give an explicit generating function for the numbers $E(2n,k)$.

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1 Introduction and Statement of Results

For positive integers $i_1, \ldots, i_k$ with $i_1 > 1$, we define the multiple zeta value $\zeta(i_1, \ldots, i_k)$ by

$$\zeta(i_1, \ldots, i_k) = \sum_{n_1 > \cdots > n_k \geq 1} \frac{1}{n_1^{i_1} \cdots n_k^{i_k}}.$$  \hfill (1)

The multiple zeta value (1) is said to have weight $i_1 + \cdots + i_k$ and depth $k$. Many remarkable identities have been proved about these numbers, but in this note we will concentrate on the case where the $i_j$ are even integers. Let $E(2n, k)$ be the sum of all the multiple zeta values of even-integer arguments having weight $2n$ and depth $k$, i.e.,

$$E(2n, k) = \sum_{\substack{i_1, \ldots, i_k \text{ even} \\ i_1 + \cdots + i_k = 2n}} \zeta(i_1, \ldots, i_k).$$

Of course

$$E(2n, 1) = \zeta(2n) = \frac{(-1)^{n-1} B_{2n} (2\pi)^{2n}}{2(2n)!},$$  \hfill (2)

where $B_{2n}$ is the $2n$th Bernoulli number, by the classical formula of Euler. Euler also studied double zeta values (i.e., multiple zeta values of depth 2) and in his paper [2] gave two identities which read

$$\sum_{i=2}^{2n-1} (-1)^i \zeta(i, 2n-i) = \frac{1}{2} \zeta(2n)$$

$$\sum_{i=2}^{2n-1} \zeta(i, 2n-i) = \zeta(2n)$$

in modern notation. From these it follows that

$$E(2n, 2) = \frac{3}{4} \zeta(2n),$$

though Gangl, Kaneko and Zagier [3] seem to be the first to have pointed it out in print. Recently Shen and Cai [10] proved the formulas

$$E(2n, 3) = \frac{5}{8} \zeta(2n) - \frac{1}{4} \zeta(2) \zeta(2n-2), \ n \geq 3$$ \hfill (3)

$$E(2n, 4) = \frac{35}{64} \zeta(2n) - \frac{5}{16} \zeta(2) \zeta(2n-2), \ n \geq 4.$$ \hfill (4)
Identity (3) was also proved by Machide [9] using a different method.

This begs the question whether there is a general formula of this type for $E(2n, k)$. The pattern

\[
\frac{3}{4}, \frac{3}{4} \cdot \frac{5}{6} = \frac{5}{8}, \quad \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} = \frac{35}{64}
\]

of the leading coefficients makes one curious. In fact, the general result is as follows.

**Theorem 1.** For $k \leq n$,

\[
E(2n, k) = \frac{1}{2^{2(k-1)}} \binom{2k - 1}{k} \zeta(2n)
\]

\[
- \sum_{j=1}^{\lfloor \frac{k-1}{2} \rfloor} \frac{1}{2^{2k-3}(2j+1)B_{2j}} \binom{2k - 2j - 1}{k} \zeta(2j) \zeta(2n - 2j).
\]

The next two cases after (4) are

\[
E(2n, 5) = \frac{63}{128} \zeta(2n) - \frac{21}{64} \zeta(2) \zeta(2n - 2) + \frac{3}{64} \zeta(4) \zeta(2n - 4)
\]

\[
E(2n, 6) = \frac{231}{512} \zeta(2n) - \frac{21}{64} \zeta(2) \zeta(2n - 2) + \frac{21}{256} \zeta(4) \zeta(2n - 4).
\]

We prove Theorem 1 in §3 below, using the generating function

\[
F(t, s) = 1 + \sum_{n \geq k \geq 1} E(2n, k)t^n s^k.
\]

In §2 we establish the following explicit formula.

**Theorem 2.**

\[
F(t, s) = \frac{\sin(\pi \sqrt{1 - s \sqrt{t}})}{\sqrt{1 - s \sin(\pi \sqrt{t})}}.
\]

Our proof uses symmetric functions. We define a homomorphism $\mathfrak{Z} : \text{Sym} \to \mathbb{R}$, where Sym is the algebra of symmetric functions, and a family $N_{n,k} \in \text{Sym}$ such that $\mathfrak{Z}$ sends $N_{n,k}$ to $E(2n, k)$. We then obtain a formula for the generating functions

\[
\mathcal{F}(t, s) = 1 + \sum_{n \geq k \geq 1} N_{n,k} t^n s^k \in \text{Sym}[[t, s]]
\]
and apply \( \mathbf{3} \) to get Theorem \( \mathbf{2} \).

From the form of \( \mathbf{F}(t, s) \) we show that it satisfies a partial differential equation (Proposition \( \mathbf{1} \) below), which is equivalent to a recurrence for the \( N_{n,k} \). From the latter we obtain a formula for the \( N_{n,k} \) in terms of complete and elementary symmetric functions, to which \( \mathbf{3} \) can be applied to give the following alternative formula for \( E(2n, k) \).

**Theorem 3.** For \( k \leq n \),

\[
E(2n, k) = \frac{(-1)^{n-k-1} \pi^{2n}}{(2n+1)!} \sum_{i=0}^{n-k} \binom{n-i}{k} \binom{2n+1}{2i} 2(2^{2i-1} - 1)B_{2i}.
\]

Note that the sum given by Theorem \( \mathbf{3} \) has \( n-k+1 \) terms, while that given by Theorem \( \mathbf{1} \) has \( \left\lfloor \frac{k-1}{2} \right\rfloor + 1 \) terms. Yet another explicit formula for \( E(2n, k) \) can be obtained by setting \( d = 1 \) in Theorem 7.1 of Komori, Matsumoto and Tsumura \( \mathbf{7} \). That formula expresses \( E(2n, k) \) as a sum over partitions of \( k \), and it is not immediately clear how it relates to our two formulas.

Comparison of Theorems \( \mathbf{1} \) and \( \mathbf{3} \) establishes the following Bernoulli-number identity.

**Theorem 4.** For \( k \leq n \),

\[
\sum_{i=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \binom{2k-2i-1}{k} \binom{2n+1}{2i+1} B_{2n-2i} = (-1)^k 2^{2k-2n} \sum_{i=0}^{n-k} \binom{n-i}{k} \binom{2n+1}{2i} (2^{2i-1} - 1)B_{2i}.
\]

It is interesting to contrast this result with the Gessel-Viennot identity (see \( \mathbf{1} \) Theorem 4.2) valid on the complementary range:

\[
\sum_{i=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \binom{2k-2i-1}{k} \binom{2n+1}{2i+1} B_{2n-2i} = \frac{2n+1}{2} \binom{2k-2n}{k}, \quad k > n. \quad (5)
\]

Note that the right-hand side of equation (5) is zero unless \( k \geq 2n \).
2 Symmetric Functions

We think of \(\text{Sym}\) as the subring of \(Q[[x_1, x_2, \ldots]]\) consisting of those formal power series of bounded degree that are invariant under permutations of the \(x_i\). A useful reference is the first chapter of Macdonald \cite{5}. We denote the elementary, complete, and power-sum symmetric functions of degree \(i\) by \(e_i\), \(h_i\), and \(p_i\) respectively. They have associated generating functions

\[
E(t) = \sum_{j=0}^{\infty} e_j t^j = \prod_{i=1}^{\infty} (1 + tx_i)
\]

\[
H(t) = \sum_{j=0}^{\infty} h_j t^j = \prod_{i=1}^{\infty} \frac{1}{1 - tx_i} = E(-t)^{-1}
\]

\[
P(t) = \sum_{j=1}^{\infty} p_j t^{j-1} = \sum_{i=1}^{\infty} \frac{x_i}{1 - tx_i} = \frac{H'(t)}{H(t)}.
\]

As explained in \cite{5} and in greater detail in \cite{6}, there is a homomorphism \(\zeta : \text{Sym}^0 \rightarrow R\), where \(\text{Sym}^0\) is the subalgebra of \(\text{Sym}\) generated by \(p_2, p_3, p_4, \ldots\), such that \(\zeta(p_i)\) is the value \(\zeta(i)\) of the Riemann zeta function at \(i\), for \(i \geq 2\) (in \cite{5,6} this homomorphism is extended to all of \(\text{Sym}\), but we do not need the extension here). Let \(D : \text{Sym} \rightarrow \text{Sym}\) be the degree-doubling map that sends \(x_i\) to \(x_i^2\). Then \(D(\text{Sym}) \subset \text{Sym}^0\), so the composition \(\zeta D = \zeta D\) is defined on all of \(\text{Sym}\). (Alternatively, we can simply think of \(\zeta\) as sending \(x_i\) to \(1/i^2\): see \cite{8} Ch. 1, §2, ex. 21.) Note that \(\zeta(p_i) = \zeta(2i) \in R\). Further, \(\zeta\) sends the monomial symmetric function \(m_{i_1, \ldots, i_k}\) to the symmetrized sum of multiple zeta values

\[
\frac{1}{|\text{Iso}(i_1, \ldots, i_k)|} \sum_{\sigma \in S_k} \zeta(2i_{\sigma(1)}, 2i_{\sigma(2)}, \ldots, 2i_{\sigma(k)}),
\]

where \(S_k\) is the symmetric group on \(k\) letters and \(\text{Iso}(i_1, \ldots, i_k)\) is the subgroup of \(S_k\) that fixes \((i_1, \ldots, i_k)\) under the obvious action.

Now let \(N_{n,k}\) be the sum of all the monomial symmetric functions corresponding to partitions of \(n\) having length \(k\). Of course \(N_{n,k} = 0\) unless \(k \leq n\), and \(N_{k,k} = e_k\). Then \(\zeta\) sends \(N_{n,k}\) to \(E(2n, k)\). Also, if we define (as in the introduction)

\[
\mathcal{F}(t, s) = 1 + \sum_{n \geq k \geq 1} N_{n,k} t^n s^k,
\]
then \( Z \) sends \( \mathcal{F}(t, s) \) to the generating function \( F(t, s) \). We have the following simple description of \( \mathcal{F}(t, s) \).

**Lemma 1.** \( \mathcal{F}(t, s) = E((s - 1)t)H(t) \).

**Proof.** Evidently \( \mathcal{F}(t, s) \) has the formal factorization

\[
\prod_{i=1}^{\infty} (1 + stx_i + st^2x_i^2 + \cdots) = \prod_{i=1}^{\infty} \frac{1 + (s - 1)t x_i}{1 - tx_i} = E((s - 1)t)H(t).
\]

\( \square \)

**Proof of Theorem 2.** Using the well-known formula for \( \zeta(2, 2, \ldots, 2) \) \[4\, \text{Cor. 2.3}],

\[
\mathfrak{z}(e_n) = \zeta(2, 2, \ldots, 2) = \frac{\pi^{2n}}{(2n + 1)!}.
\]

Hence

\[
\mathfrak{z}(E(t)) = \frac{\sinh(\pi \sqrt{t})}{\pi \sqrt{t}},
\]

and the image of \( H(t) = E(-t)^{-1} \) is

\[
\mathfrak{z}(H(t)) = \frac{\pi \sqrt{-t}}{\sinh(\pi \sqrt{-t})} = \frac{\pi \sqrt{t}}{\sin(\pi \sqrt{t})}.
\]

Thus from Lemma 1, \( F(t, s) = \mathfrak{z}(\mathcal{F}(t, s)) \) is

\[
\mathfrak{z}(E((s - 1)t)H(t)) = \frac{\sinh(\pi \sqrt{(s - 1)t})}{\pi \sqrt{(s - 1)t}} \frac{\pi \sqrt{t}}{\sin(\pi \sqrt{t})} = \frac{\sin(\pi \sqrt{(1 - s)t})}{\sqrt{1 - s} \sin(\pi \sqrt{t})}.
\]

\( \square \)

Taking limits as \( s \to 1 \) in Theorem 2, we obtain

\[
F(t, 1) = \frac{\pi \sqrt{t}}{\sin \pi \sqrt{t}}
\]

and so, taking the coefficient of \( t^n \), the following result.
Corollary 1. For all \( n \geq 1 \),
\[
\sum_{k=1}^{n} E(2n, k) = \frac{2(2^{2n-1} - 1)(-1)^{n-1}B_{2n}π^{2n}}{(2n)!}.
\]

Another consequence of Lemma 1 is the following partial differential equation.

Proposition 1.
\[
t \frac{\partial \mathcal{F}}{\partial t}(t, s) + (1 - s) \frac{\partial \mathcal{F}}{\partial s}(t, s) = tP(t)\mathcal{F}(t, s).
\]

Proof. From Lemma 1 we have
\[
\frac{\partial \mathcal{F}}{\partial t}(t, s) = (s - 1)E'((s - 1)t)H(t) + E((s - 1)t)H'(t)
\]
\[
\frac{\partial \mathcal{F}}{\partial s}(t, s) = tE'((s - 1)t)H(t)
\]
from which the conclusion follows.

Now examine the coefficient of \( t^n s^k \) in Proposition 1 to get the following.

Proposition 2. For \( n \geq k + 1 \),
\[
p_1N_{n-1,k} + p_2N_{n-2,k} + \cdots + p_{n-k}N_{k,k} = (n - k)N_{n,k} + (k + 1)N_{n,k+1}.
\]

It is also possible to prove this result directly via a counting argument like that used to prove the lemma of [6, p. 16].

The preceding result allows us to write \( N_{n,k} \) explicitly in terms of complete and elementary symmetric functions as follows.

Lemma 2. For \( r \geq 0 \),
\[
N_{k+r,k} = \sum_{i=0}^{r} (-1)^i \binom{k+i}{i} h_{r-i}e_{k+i}.
\]

Proof. We use induction on \( r \), the result being evident for \( r = 0 \). Proposition 2 gives
\[
\sum_{i=1}^{r+1} p_i N_{k+r+1-i,k} = (r + 1)N_{k+r+1,k} + (k + 1)N_{k+r+1,k+1},
\]
which after application of the induction hypothesis becomes

\[
\sum_{i=1}^{r+1} \sum_{j=0}^{r+1-j} (-1)^j p_i \binom{k+j}{j} h_{r+1-i-j} N_{k+j+k+j} = \\
(r + 1) N_{k+r+1,k} + (k + 1) \sum_{j=0}^{r} \binom{k+1+j}{j} h_{r-j} N_{k+1+j+k+1+j}.
\]

The latter equation can be rewritten

\[
\sum_{j=0}^{r} (-1)^j \binom{k+j}{j} N_{k+j+k+j} \sum_{i=1}^{r+1-j} p_i h_{r+1-i-j} = \\
(r + 1) N_{k+r+1,k} - (k + 1) \sum_{j=1}^{r+1} (-1)^j \binom{k+j}{j-1} h_{r+1-j} N_{k+j+k+j}.
\]

Now the inner sum on the left-hand side is \((r + 1 - j) h_{r+1-j}\) by the recurrence relating the complete and power-sum symmetric functions, so we have

\[
(r + 1) N_{k+r+1,k} - (r + 1) N_{k,k} h_{r+1} = \\
\sum_{j=1}^{r+1} (-1)^j h_{r+1-j} N_{k+j+k+j} \left( (r + 1 - j) \binom{k+j}{j} + (k + 1) \binom{k+j}{j-1} \right),
\]

and the conclusion follows after the observation that \((k + 1) \binom{k+j}{j-1} = j \binom{k+j}{j}\).

**Proof of Theorem 3.** Rewrite Lemma 2 in the form

\[
N_{n,k} = \sum_{i=0}^{n-k} \binom{n-i}{k} (-1)^{n-k-i} h_i e_{n-i}
\]

and apply the homomorphism \(Z\), using equation (6) and

\[
Z(h_i) = 2(2^{2i-1} - 1)(-1)^{i-1}B_{2i} \pi^{2i}.
\]

\[
3(h_i) = \frac{2(2^{2i-1} - 1)(-1)^{i-1}B_{2i} \pi^{2i}}{(2i)!}.
\]
3 Proof of Theorems 1 and 4

From the introduction we recall the statement of Theorem 1:

$$E(2n, k) = \frac{1}{2^{2(k-1)}} \binom{2k-1}{k} \zeta(2n)$$

$$- \sum_{j=1}^{\lfloor \frac{k-1}{2} \rfloor} \frac{1}{2^{2k-3}(2j+1)B_{2j}} \binom{2k-2j-1}{k} \zeta(2j) \zeta(2n-2j).$$

We note that Euler’s formula (2) can be used to write the result in the alternative form

$$E(2n, k) = \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-1)^j \pi^{2j} \zeta(2n-2j)}{2^{2k-2j-2}(2j+1)!} \binom{2k-2j-1}{k} \zeta(2j) \zeta(2n-2j).$$

which avoids mention of Bernoulli numbers.

We now expand out the generating function $F(t, s)$. We have

$$F(t, s) = \frac{1}{\sqrt{1 - s \sin \pi \sqrt{t}}}$$

$$= \frac{\pi \sqrt{t}}{\sin \pi \sqrt{t}} \sum_{j=0}^{\infty} \frac{(-1)^j \pi^{2j} t^j (1 - s)^j (2j + 1)!}{(2j + 1)!} = \sum_{k=0}^{\infty} s^k G_k(t),$$

where

$$G_k(t) = (-1)^k \frac{\pi \sqrt{t}}{\sin \pi \sqrt{t}} \sum_{j \geq k} \frac{(-1)^j \pi^{2j} t^j}{(2j + 1)! \binom{j}{k}}.$$  (8)

Then Theorem 1 is equivalent to the statement that

$$G_k(t) = \sum_{n \geq k} t^n \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-1)^j \pi^{2j} \zeta(2n-2j)}{2^{2k-2j-2}(2j+1)!} \binom{2k-2j-1}{k} \binom{2n}{2j}.$$
for all $k$. We can write the latter sum as

$$\sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-j}(2j+1)!} \left(\begin{array}{c} 2k - 2j - 1 \\ k \end{array}\right) \sum_{n=j+1}^{k-1} \zeta(2n - 2j)t^{n-j} -$$

$$\sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-j}(2j+1)!} \left(\begin{array}{c} 2k - 2j - 1 \\ k \end{array}\right) \sum_{n=j+1}^{k-1} \zeta(2n - 2j)t^{n-j} =$$

$$\frac{1}{2}(1 - \pi \sqrt{t} \cot \pi \sqrt{t}) \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-j}(2j+1)!} \left(\begin{array}{c} 2k - 2j - 1 \\ k \end{array}\right) \sum_{n=j+1}^{k-1} \zeta(2n - 2j)t^{n-j}, \quad (9)$$

where we have used the generating function

$$\frac{1}{2}(1 - \pi \sqrt{t} \cot \pi \sqrt{t}) = \sum_{i=1}^{\infty} \zeta(2i)t^i.$$

Note that the last sum in (9) is a polynomial that cancels exactly those terms in

$$\frac{1}{2}(1 - \pi \sqrt{t} \cot \pi \sqrt{t}) \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-j}(2j+1)!} \left(\begin{array}{c} 2k - 2j - 1 \\ k \end{array}\right) \sum_{n=j+1}^{k-1} \zeta(2n - 2j)t^{n-j}, \quad (10)$$

of degree less than $k$. Thus, to prove Theorem 1 it suffices to show that

$$G_k(t) = \text{terms of degree } \geq k \text{ in expression } (10).$$

From equation (8) it is evident that

$$G_k(t) = \frac{\pi \sqrt{t}}{\sin \pi \sqrt{t}} \cdot \frac{(-t)^k}{k!} \cdot \frac{d^k}{dt^k} \left(\frac{\sin \pi \sqrt{t}}{\pi \sqrt{t}}\right). \quad (11)$$

We use this to obtain an explicit formula for $G_k(t)$.

**Lemma 3.** For $k \geq 0$, $G_k(t) = P_k(\pi^2 t)\pi \sqrt{t} \cot \pi \sqrt{t} + Q_k(\pi^2 t),$
where $P_k, Q_k$ are polynomials defined by

$$P_k(x) = -\sum_{j=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \frac{(-4x)^j}{2^{2k-1}(2j+1)!} \binom{2k-2j-1}{k}$$

$$Q_k(x) = \sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{(-4x)^j}{2^{2k}(2j)!} \binom{2k-2j-1}{k}.$$

**Proof.** In view of equation (11), the conclusion is equivalent to

$$f^{(k)}(t) = (-1)^k k! t^{-k} P_k(\pi^2 t) \cos \pi \sqrt{t} + (-1)^k k! t^{-k} Q_k(\pi^2 t) f(t),$$

where $f(t) = \sin \pi \sqrt{t}/\pi \sqrt{t}$. Differentiating, one sees that the polynomials $P_k$ and $Q_k$ are determined by the recurrence

$$(k+1)P_{k+1}(x) = kP_k(x) - xP'_k(x) - \frac{1}{2}Q_k(x)$$

$$(k+1)Q_{k+1}(x) = \frac{2k+1}{2} Q_k(x) - xQ'_k(x) + \frac{x}{2} P_k(x)$$

together with the initial conditions $P_0(x) = 0$, $Q_0(x) = 1$. The recurrence and initial conditions are satisfied by the explicit formulas above. □

**Proof of Theorem 1.** Using Lemma 3, we have

$$G_k(t) = -\sum_{j=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-1}(2j+1)!} \binom{2k-2j-1}{k} \pi \sqrt{t} \cot \pi \sqrt{t}$$

$$+ \sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{(-4\pi^2 t)^j}{2^{2k}(2j)!} \binom{2k-2j}{k} =$$

$$\frac{1}{2} (1 - \pi \sqrt{t} \cot \pi \sqrt{t}) \sum_{j=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-2}(2j+1)!} \binom{2k-2j-1}{k}$$

+ terms of degree $< k$,

and this completes the proof. □
Proof of Theorem 4. Using Theorem 1 in the form of equation (1), eliminate \( \zeta(2n - 2j) \) using Euler’s formula (2) and then compare with Theorem 3 to get

\[
\frac{1}{\binom{k-1}{j}} \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(-1)^{n-1} \pi^{2n} B_{2n-2j}}{2^{2k-2n-1}(2n-2j)!(2j+1)!} \binom{2k - 2j - 1}{k} = \frac{(-1)^{n-k-1} \pi^{2n}}{(2n+1)!} \sum_{i=0}^{n-k} \binom{n - i}{k} \binom{2n + 1}{2i} 2^{2i-1} B_{2i}.
\]

Now multiply both sides by \((-1)^{n-1}2^{2k-2n-1}\pi^{-2n}(2n+1)!\) and rewrite the factorials on the left-hand side as a binomial coefficient.

\[\square\]

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