On $\mathbb{Q}$-factorial terminalizations of nilpotent orbits

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November 24, 2008

1 Introduction

Let $\mathfrak{g}$ be a complex simple Lie algebra and $G$ its adjoint group. For a parabolic subgroup $Q \subset G$, we denote by $\mathfrak{q}$ its Lie algebra and $\mathfrak{q} = n(\mathfrak{q}) + l(\mathfrak{q})$ its Levi decomposition. For a nilpotent orbit $O_t$ in $l(\mathfrak{q})$, Lusztig and Spaltenstein \cite{L-S} showed that $G \cdot (n(\mathfrak{q}) + \overline{O}_t)$ is a nilpotent orbit closure, say $\overline{O}$, which depends only on the $G$-orbit of the pair $(l(\mathfrak{q}), O_t)$. The variety $n(\mathfrak{q}) + \overline{O}_t$ is $Q$-invariant and the surjective map

$$\pi : G \times^Q (n(\mathfrak{q}) + \overline{O}_t) \to \overline{O}$$

is generically finite and projective, which will be called a generalized Springer map. When $O_t = 0$ and $\pi$ is birational, we call $\pi$ a Springer resolution. An induced orbit is a nilpotent orbit whose closure is the image of a generalized Springer map. An orbit is called rigid if it is not induced.

Recall that for a variety $X$ with rational Gorenstein singularities, a $\mathbb{Q}$-factorial terminalization of $X$ is a birational projective morphism $p : Y \to X$ such that $Y$ has only $\mathbb{Q}$-factorial terminal singularities and $p^*K_X = K_Y$. When $Y$ is furthermore smooth, we call $p$ a crepant resolution. In \cite{F1}, the author proved that for nilpotent orbit closures in a semi-simple Lie algebra, crepant resolutions are Springer resolutions. In a recent preprint \cite{N3}, Y. Namikawa proposed the following conjecture on $\mathbb{Q}$-factorial terminalizations of nilpotent orbit closures.

**Conjecture 1.** Let $O$ be a nilpotent orbit in a complex simple Lie algebra $\mathfrak{g}$ and $\mathcal{O}$ the normalization of its closure $\mathcal{O}$. Then one of the following holds:
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(1) \(\tilde{O}\) is \(\mathbb{Q}\)-factorial terminal;
(2) every \(\mathbb{Q}\)-factorial terminalization of \(\tilde{O}\) is given by a generalized Springer map. Furthermore, two such terminalizations are connected by Mukai flops (cf. [N1], p. 91).

In [N3], Y. Namikawa proved his conjecture in the case when \(\mathfrak{g}\) is classical. In this paper, we shall prove that Conjecture [1] holds for \(\mathfrak{g}\) exceptional (Theorem 5.1 and Theorem 6.1). Two interesting results are also obtained: one is the classification of nilpotent orbits with \(\mathbb{Q}\)-factorial normalization \(\tilde{O}\) (Proposition 4.4) and the other is the classification of nilpotent orbits with terminal \(\tilde{O}\) (Proposition 6.8).

Here is the organization of this paper. After recalling results from [B-M], we first give a classification of induced orbits which are images of birational generalized Springer maps (Proposition 3.1). Using this result, we completely settle the problem of \(\mathbb{Q}\)-factoriality of the normalization of a nilpotent orbit closure in exceptional Lie algebras (Proposition 4.4), which shows the surprising result that only in \(E_6\), \(\tilde{O}\) could be non-\(\mathbb{Q}\)-factorial. We then prove that for rigid orbits the normalization of its closure is \(\mathbb{Q}\)-factorial and terminal (see Theorem 5.1). For induced orbits whose closure does not admit a Springer resolution, we shall first prove that except four orbits (which have \(\mathbb{Q}\)-factorial terminal normalizations), there exists a generalized Springer map which gives a \(\mathbb{Q}\)-factorial terminalization of \(\tilde{O}\). For the birational geometry, unlike the classical case proven by Y. Namikawa, two new types of flops appear here, which we call Mukai flops of type \(E_{6,I}^l\) and \(E_{6,I}^u\) (for the definition see section 6.1). We shall prove in a similar way as in [P2] that any two \(\mathbb{Q}\)-factorial terminalizations given by generalized Springer maps of \(\tilde{O}\) are connected by Mukai flops of type \(E_{6,I}^l\) or \(E_{6,I}^u\) (Corollary 6.5). Then using a similar argument as in [N3], we prove that every \(\mathbb{Q}\)-factorial terminalization of \(\tilde{O}\) is given by a generalized Springer map. An interesting corollary is a classification of nilpotent orbits in a simple exceptional Lie algebra such that \(\tilde{O}\) has terminal singularities (Proposition 6.8).

Acknowledgements: The author would like to thank Y. Namikawa for his corrections and helpful correspondences to this paper. I thank W. de Graaf for his help on computing in GAP4. I am grateful to M. Brion, S. Goodwin, H. Kraft, G. Röhrle, E. Sommers for their helpful correspondences.
2 Preliminaries

In this section, we shall recall some results from [B-M]. Let $W$ be the Weyl group of $G$. The Springer correspondence ([S2]) assigns to any irreducible $W$-module a unique pair $(\mathcal{O}, \phi)$ consisting of a nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ and an irreducible representation $\phi$ of the component group $A(\mathcal{O}) := G_x/(G_x)^0$ of $\mathcal{O}$, where $x$ is any point in $\mathcal{O}$ and $(G_x)^0$ is the identity component of $G_x$. The corresponding irreducible $W$-module will be denoted by $\rho(x, \phi)$. This correspondence is not surjective onto the set of all pairs $(\mathcal{O}, \phi)$. A pair will be called relevant if it corresponds to an irreducible $W$-module, then the Springer correspondence establishes a bijection between irreducible $W$-modules and relevant pairs in $\mathfrak{g}$. For $G$ exceptional, the Springer correspondence has been completely worked out in [S1] for $G_2$, in [S] for $F_4$ and in [A-L] for $E_n$ ($n = 6, 7, 8$). We will use the tables in [C] (Section 13.3).

Consider a parabolic sub-group $Q$ in $G$. Let $L$ be a Levi sub-group of $Q$ and $T$ a maximal torus in $L$. The Weyl group of $L$ is $W(L) := N_L(T)/T$, where $N_L(T)$ is the normalizer of $T$ in $L$. It is a sub-group of the Weyl group $W$ of $G$. For a representation $\rho$ of $W(L)$, we denote by $\text{Ind}_{W}^{W(L)}(\rho)$ the induced representation of $\rho$ to $W$.

**Proposition 2.1** ([B-M], proof of Corollary 3.9). Let $\pi : G \times Q (n(q) + \bar{O}_t) \to \bar{O}_x$ be the generalized Springer map associated to the parabolic sub-group $Q$ and the nilpotent orbit $\mathcal{O}_t$. Then

$$\deg(\pi) = \sum_{\phi} \text{mtp}(\rho(x, \phi), \text{Ind}_{W(L)}^{W}((\rho(t,1))) \deg \phi,$$

where the sum is over all irreducible representations $\phi$ of $A(\mathcal{O}_x)$ such that $(\mathcal{O}_x, \phi)$ is a relevant pair, $\text{mtp}(\rho(x, \phi), \text{Ind}_{W(L)}^{W}((\rho(t,1)))$ is the multiplicity of $\rho(x, \phi)$ in $\text{Ind}_{W(L)}^{W}((\rho(t,1))$ and $\deg \phi$ is the dimension of the irreducible representation $\phi$.

The multiplicity $\text{mtp}(\rho(x, \phi), \text{Ind}_{W_0}^{W}((\rho))$ has been worked out in [A], for any irreducible representation $\rho$ of any maximal parabolic sub-group $W_0$ of $W$. Note that $\text{Ind}_{W(L)}^{W}((\rho) = \text{Ind}_{W_0}^{W}((\text{Ind}_{W(L)}^{W_0}((\rho)))$ for any sub-group $W_0$ of $W$ containing $W(L)$ and $\text{Ind}_{W_0}^{W}((\rho)$ can be determined by the Littlewood-Richardson rules when $W_0$ is classical and by [A] when $W_0$ is exceptional.

By the remark in section 3.8 [B-M], $\text{mtp}(\rho(x,1), \text{Ind}_{W(L)}^{W}((\rho(t,1))) = 1$, which gives the following useful corollary.
Corollary 2.2. If $\mathcal{O}$ is an induced orbit with $A(\mathcal{O}) = \{1\}$, then every generalized Springer map is birational.

Recall that a complex variety $Z$ of dimension $n$ is called rationally smooth at a point $z \in Z$ if

$$H_i(Z, Z \setminus \{z\}; \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } i = 2n, \\ 0 & \text{otherwise}. \end{cases}$$

For a generalized Springer map $\pi : Z := G \times^Q (n(q) + \bar{O}_t) \rightarrow \bar{O}_x$, an orbit $\mathcal{O}_{x'} \subset \bar{O}_x$ is called $\pi$-relevant if $2 \dim \pi^{-1}(x') = \dim \mathcal{O}_x - \dim \mathcal{O}_{x'}$.

Proposition 2.3 ([B-M], Proposition 3.6). Assume that $Z$ is rationally smooth at points in $\pi^{-1}(x')$. Then $\mathcal{O}_{x'}$ is $\pi$-relevant if and only if

$$\text{mtp}(\rho(x',1), \text{Ind}^W_{\text{W}(L)} \rho(t,1)) \neq 0.$$ When $t = 0$, $Z \simeq T^*(G/P)$ is smooth, $\pi$ is the moment map and $\mathcal{O}_x$ is the Richardson orbit associated to $P$. In this case, $\rho(t,1) = \varepsilon_W(L)$ is the sign representation and we have a geometric interpretation of the multiplicity.

Proposition 2.4 ([B-M], Corollary 3.5). For the map $\pi : T^*(G/P) \rightarrow \bar{O}_x$, the multiplicity $\text{mtp}(\rho(x',1), \text{Ind}^W_{\text{W}(L)} \varepsilon_W(L))$ is the number of irreducible components of $\pi^{-1}(\mathcal{O}_{x'})$ of dimension $\dim \mathcal{O}_x + (\dim G/P - 1/2 \dim \mathcal{O}_x)$.

3 Birational generalized Springer maps

Throughout the paper, we will use notations in [M] (section 5.7) for nilpotent orbits. In this section, we classify nilpotent orbits in a simple exceptional Lie algebra which is the image of a birational generalized Springer map. More precisely, we prove the following proposition.

Proposition 3.1. Let $\mathcal{O}$ be an induced nilpotent orbit in a simple complex exceptional Lie algebra. The closure $\bar{\mathcal{O}}$ is the image of a birational generalized Springer map if and only if $\mathcal{O}$ is not one of the following orbits: $A_2 + A_1, A_4 + A_1$ in $E_7$, $A_4 + A_1, A_4 + 2A_1$ in $E_8$.

By Corollary 2.2, to prove Proposition 3.1, we just need to consider induced orbits with non-trivial $A(\mathcal{O})$ but having no Springer resolutions. The
classification of induced/rigid orbits in exceptional Lie algebras can be found for example in [M] (section 5.7). We will use the tables therein to do a case-by-case check. Note that the $G$ therein is simply-connected, thus $A(x)$ in these tables is $\pi_1(O_x)$. On can get $A(O)$ by just omitting the copies of $\mathbb{Z}/d\mathbb{Z}$, $d = 2, 3$ when it presents. When $A(O)$ is $S_2$ (resp. $S_3$), we will denote by $\epsilon$ (resp. $\epsilon_1, \epsilon_2$) its non-trivial irreducible representations.

### 3.1 $F_4$

There are two orbits to be considered: $B_2$ and $C_3(a_1)$. The orbit $B_2$ is induced from $(C_3, 21^4)$. We have $\rho(t,1) = [1^3 : -]$ and $\rho(x,\epsilon) = \phi_{4,8} = \chi_{4,1}$. By [A] (p. 143), we get $\text{mtp}(\rho(x,\epsilon), \text{Ind}_W^{W(C_3A_1)}(\rho(t,1))) = 0$, thus the degree of the associated generalized Springer map is one. The orbit $C_3(a_1)$ is induced from $(B_3, 2^21^3)$. We have $\rho(t,1) = [ - : 21 ]$ and $\rho(x,\epsilon) = \phi_{4,7} = \chi_{4,4}$. By [A] (p. 147), the degree of $\pi$ is one.

### 3.2 $E_6$

When $g = E_6$, every induced orbit either has $A(O) = \{1\}$ or admits a Springer resolution.

### 3.3 $E_7$

We have four orbits to be considered: $A_3 + A_2, D_5(a_1), A_2 + A_1$ and $A_4 + A_1$.

The orbit $A_3 + A_2$ is induced from $(D_6, 32^21^5)$. A calculus shows that the associated generalized Springer map has degree 2. By a dimension counting, it is also induced from $(D_5 + A_1, [21^6] \times [1^2])$. For this induction, one has $\rho(t,1) = [1 : 4^1 \times 1^2]$ and $\rho(x,\epsilon) = \phi_{84,15} = 84^*$. By [A] (p. 49), one gets $\text{mtp}(84^*, \text{Ind}_W^{W(D_5A_1)}[1 : 4^1 \times 1^2]) = \text{mtp}(84^*, \text{Ind}_W^{W(D_5A_1)}[4 : 1] \times [2]) = 0$, thus the induced generalized Springer map is birational. The orbit $D_5(a_1)$ is a Richardson orbit but its closure has no Springer resolutions ([F2]). By Thm. 5.3 [M], it is induced from $(D_6, 3^22^21^2)$. One finds $\rho(t,1) = [1^2 : 21^2]$ and $\rho(x,\epsilon) = \phi_{336,11} = 336^*$. Now by [M] (p. 43), the degree is one.

The orbit $A_2 + A_1$ has a unique induction (by dimension counting) given by $(E_6, A_1)$. We have $\rho(t,1) = 6^*_{\mu}$ and $\rho(x,\epsilon) = \phi_{105,26} = 105_{\mu}$. By [A] (p. 51), the degree is 2. The orbit $A_4 + A_1$ is a Richardson orbit with no symplectic resolutions ([F2]), i.e. the degree given by the induction $(A_2 + 2A_1, 0)$ is of degree 2. It has three other inductions, given by $(E_6, A_2 + 2A_1), (A_6, 2^21^3)$ and
$(A_5 + A_1, 24^12 + 0)$. One shows that every such induction gives a generalized Springer map of degree 2.

### 3.4 $E_8$

We need to consider the following orbits: $A_3 + A_2, D_5(a_1), D_6(a_2), E_6(a_3) + A_1, E_7(a_5), E_7(a_4), E_6(a_1) + A_1, E_7(a_3), A_4 + A_1$ and $A_4 + 2A_1$. The orbit $A_3 + A_2$ is induced from $(D_7, 2^3 10^2)$. We have $\rho_{(t, 1)} = [1 : 16]$ and $\rho(x, e) = \phi_{972.32} = 972^*$. By $\mathbb{A}$ (p. 105), we get $\deg = 1$. The orbit $D_5(a_1)$ is induced from $(E_7, A_2 + A_1)$ by Thm. 5.3 $\mathbb{A}$. We have $\rho_{(t, 1)} = 120^*$ and $\rho(x, e) = \phi_{2100.28} = 2100^*$. By $\mathbb{A}$ (p. 140), we get $\deg = 1$. The induction from $(E_6, A_1)$ gives a map of degree 2. The orbit $D_6(a_2)$ is induced from $(D_7, 3^2 1^3)$. We have $\rho_{(t, 1)} = [- : 2^3 1]$ and $\rho(x, e) = \phi_{2688, 20} = 2688^*_y$. By $\mathbb{A}$ (p. 106), we get $\deg = 1$. The orbit $E_6(a_3) + A_1$ is induced from $(E_7, 2A_2 + A_1)$.

We have $\rho_{(t, 1)} = \phi_{70, 18} = 70^a$ and $\rho(x, e) = \phi_{1134, 20} = 1134^*_y$. By $\mathbb{A}$ (p. 139), we get $\deg = 1$. The orbit $E_7(a_5)$ has $A(\mathcal{O}) = S_3$ and is induced from $(E_6 + A_1, 3A_1 + 0)$. We have $\rho_{(t, 1)} = \phi_{15, 16} \times [1^2] = 15^*_2 \times [1^2], \rho(x, e_1) = \phi_{5600, 19} = 5600_w, \rho(x, e_2) = \phi_{448, 25} = 448_w$. By $\mathbb{A}$ (p. 136), we get $\deg = 1$. The orbit $E_7(a_4)$ is induced from $(E_7, A_3 + A_2)$. We have $\rho_{(t, 1)} = \phi_{378, 14} = 378^a$ and $\rho(x, e) = \phi_{700, 16} = 700_{xx}$. By $\mathbb{A}$ (p. 139), we get $\deg = 1$. The orbit $E_6(a_1) + A_1$ is induced from $(E_7, A_2 + A_1)$. We have $\rho_{(t, 1)} = \phi_{512, 11} = 512^a$ and $\rho(x, e) = \phi_{4096, 12} = 4096_x$. By $\mathbb{A}$ (p. 141), we get $\deg = 1$. The orbit $E_7(a_3)$ is induced from $(D_6, 3^3 2^2 1^2)$. We have $\rho_{(t, 1)} = [2^2 : 21^2]$ and $\rho(x, e) = \phi_{1296, 13} = 1296^*_x$. By $\mathbb{A}$ (p. 43), we get $\Ind_{W(D_6)}^{W(E_7)}([2^2 : 21^2] = 189_b + 189_e + 315_a + 280_a + 336_a + 216_a + 512_a + 378_a + 420_a$. Now by $\mathbb{A}$ (p. 138, p. 140), we get $\deg = 1$.

The orbit $A_4 + A_1$ has a unique induction given by $(E_6 + A_1, A_1 + 0)$, which gives a generalized Springer map of degree 2. The orbit $A_4 + 2A_1$ has a unique induction, given by $(D_7, 24^16)$. This gives a map of degree 2.

This concludes the proof of Proposition 3.1.

### 4 $\mathbb{Q}$-factoriality

In this section, we study the problem of $\mathbb{Q}$-factoriality of the normalization of a nilpotent orbit closure.

**Lemma 4.1.** Let $\mathcal{O}_x$ be a nilpotent orbit in a complex simple Lie algebra and $(G_x)^c$ the identity component of the stabilizer $G_x$ in $G$. Assume that
the character group \( \chi((G_x)^\circ) \) is finite, then \( \text{Pic}(\mathcal{O}_x) \) is finite and \( \bar{\mathcal{O}}_x \) is \( \mathbb{Q} \)-factorial.

**Proof.** The exact sequence \( 1 \to (G_x)^\circ \xrightarrow{i} G_x \to A(\mathcal{O}_x) := G_x/(G_x)^\circ \to 1 \) induces an exact sequence: \( 1 \to \chi(A(\mathcal{O}_x)) \to \chi(G_x) \to \text{Im}(i^*) \to 1. \) By assumption, \( \chi((G_x)^\circ) \) is finite, so is \( \text{Im}(i^*) \). On the other hand, \( A(\mathcal{O}_x) \) is a finite group, thus \( \chi(A(\mathcal{O}_x)) \) is also finite. This gives the finiteness of \( \chi(G_x) \).

The exact sequence \( 1 \to G_x \to G \xrightarrow{q} \mathcal{O}_x \to 1 \) induces an exact sequence \( 1 \to \chi(G_x) \to \text{Pic}(\mathcal{O}_x) \to \text{Im}(q^*) \to 1 \). As \( \text{Pic}(G) \) is finite, so is \( \text{Im}(q^*) \). This proves that \( \text{Pic}(\mathcal{O}_x) \) is finite. The last claim follows from \( \text{codim}(\bar{\mathcal{O}}_x \setminus \mathcal{O}_x) \geq 2. \)

**Remark 4.2.** It is a subtle problem to work out explicitly the group \( \text{Pic}(\mathcal{O}_x) \), since in general \( q^* \), \( i^* \) are not surjective.

**Lemma 4.3.** Let \( \pi : T^*(G/P) \to \bar{\mathcal{O}} \) be a resolution. Then \( \bar{\mathcal{O}} \) is \( \mathbb{Q} \)-factorial if and only if the number of irreducible exceptional divisors of \( \pi \) equals to \( b_2(G/P) \).

**Proof.** As \( \bar{\mathcal{O}} \) admits a positive weighted \( \mathbb{C}^* \)-action with a unique fixed point, \( \text{Pic}(\bar{\mathcal{O}}) \) is trivial. As a consequence, \( \bar{\mathcal{O}} \) is \( \mathbb{Q} \)-factorial if and only if \( \text{Pic}(\mathcal{O}) \) is finite. Let \( E_i, i = 1, \ldots, k \) be the irreducible exceptional divisors of \( \pi \). We have the following exact sequence:

\[
\bigoplus_{i=1}^k \mathbb{Q}[E_i] \to \text{Pic}(T^*(G/P)) \otimes \mathbb{Q} \to \text{Pic}(\mathcal{O}) \otimes \mathbb{Q} \to 0.
\]

By [N3] (Lemma 1.1.1), the first map is injective. Now it is clear that \( \text{Pic}(\mathcal{O}) \) is finite if and only if \( k = b_2(G/P) \).

**Proposition 4.4.** Let \( \mathcal{O} \) be a nilpotent orbit in a simple exceptional Lie algebra and \( \bar{\mathcal{O}} \) the normalization of its closure \( \mathcal{O} \). Then \( \bar{\mathcal{O}} \) is \( \mathbb{Q} \)-factorial if and only if \( \mathcal{O} \) is not one of the following orbits in \( E_6 \): \( 2A_1, A_2 + A_1, A_2 + 2A_1, A_3, A_3 + A_1, A_4, A_4 + A_1, D_5(a_1), D_5 \).

**Proof.** By Lemma [L1], we just need to check orbits whose type of \( C \) contains a factor of \( T_i \) in the tables of [C] (Chap. 13, p.401-407). This gives that nilpotent orbit closures in \( G_2 \) and \( F_4 \) are \( \mathbb{Q} \)-factorial.

In \( E_6 \), there are in total ten orbits to be considered. The orbit closures of \( 2A_1, A_2 + 2A_1 \) have small resolutions by [N1], thus \( \bar{\mathcal{O}} \) is not \( \mathbb{Q} \)-factorial. As
we will see in section 6.1, the orbit closures of $A_2 + A_1, A_3 + A_1$ have small $\mathbb{Q}$-factorial terminalizations, thus $\tilde{O}$ is not $\mathbb{Q}$-factorial. The six left orbit closures have symplectic resolutions. We will now use Proposition 2.4 to calculate the numbers of irreducible exceptional divisors and then apply Lemma 4.3.

When $\mathcal{O} = A_3$, a symplectic resolution is given by the induction $(A_4, 0)$. The boundary $\tilde{\mathcal{O}} \setminus \mathcal{O} = \tilde{\mathcal{O}}_{A_2 + 2A_1}$ has codimension 2 and $\rho_{(A_2 + 2A_1, 1)} = \phi_{60, 11} = 60^*_p$.

By $[A]$ (p. 31), we get $mtp = 1$ while $b_2(G/P) = 2$, thus $\tilde{O}$ is not $\mathbb{Q}$-factorial. When $\mathcal{O} = D_4(a_1)$, it is an even orbit and a symplectic resolution is given by the induction $(2A_2 + A_1, 0)$. The boundary $\tilde{\mathcal{O}} \setminus \mathcal{O} = \tilde{\mathcal{O}}_{A_4 + A_1}$ has codimension 2.

By $[A]$ (p. 33), we get $mtp = 1 = b_2(G/P)$. This implies that $\tilde{O}$ is $\mathbb{Q}$-factorial. For $\mathcal{O} = A_4$, a symplectic resolution is given by the induction $(A_3, 0)$ and $\tilde{\mathcal{O}} \setminus \mathcal{O} = \tilde{\mathcal{O}}_{D_4(a_1)}$ has codimension 2. We find that $mtp = 2$ while $b_2(G/P) = 3$, thus $\tilde{O}$ is not $\mathbb{Q}$-factorial. For $\mathcal{O} = A_4 + A_1$, a symplectic resolution is given by the induction $(A_2 + 2A_1, 0)$ and $\tilde{\mathcal{O}} \setminus \mathcal{O} = \tilde{\mathcal{O}}_{A_4 + A_1}$ has codimension 2. By $[A]$, we find $mtp = 1$ while $b_2(G/P) = 2$, thus $\tilde{O}$ is not $\mathbb{Q}$-factorial. For $\mathcal{O} = D_5(a_1)$, a symplectic resolution is given by the induction $(A_2 + A_1, 0)$ and $\tilde{\mathcal{O}} \setminus \mathcal{O} = \tilde{\mathcal{O}}_{A_4 + A_1} \cup \tilde{\mathcal{O}}_{D_4}$. Only $\tilde{\mathcal{O}}_{A_4 + A_1}$ has codimension 2. By $[A]$, we find $mtp = 2$ while $b_2(G/P) = 3$, thus $\tilde{O}$ is not $\mathbb{Q}$-factorial. For $\mathcal{O} = D_5$, a symplectic resolution is given by the induction $(2A_1, 0)$ and $\tilde{\mathcal{O}} \setminus \mathcal{O} = \tilde{\mathcal{O}}_{E_6(a_3)}$ has codimension 2. By $[A]$, we find $mtp = 1$ while $b_2(G/P) = 4$, thus $\tilde{O}$ is not $\mathbb{Q}$-factorial.

In $E_7$, there are six orbits to be considered. For $\mathcal{O} = A_4$, a symplectic resolution is given by the induction $(A_1 + D_4, 0)$ and $\tilde{\mathcal{O}} \setminus \mathcal{O} = \tilde{\mathcal{O}}_{A_3 + A_2}$ is of codimension 2. Using $[A]$, we find $mtp = 2 = b_2(G/P)$, thus $\tilde{O}$ is $\mathbb{Q}$-factorial. For $\mathcal{O} = E_6(a_1)$, a symplectic resolution is given by the induction $(4A_1, 0)$ and $\tilde{\mathcal{O}} \setminus \mathcal{O} = \tilde{\mathcal{O}}_{E_7(a_4)}$ is of codimension 2. Using $[A]$, we find $mtp = 3 = b_2(G/P)$, thus $\tilde{O}$ is $\mathbb{Q}$-factorial.

In $E_8$, there are seven orbits to be considered. For $\mathcal{O} = D_5 + A_2$, a symplectic resolution is given by the induction $(A_2 + A_4, 0)$ and $\tilde{\mathcal{O}} \setminus \mathcal{O} = \tilde{\mathcal{O}}_{E_7(a_4)} \cup \tilde{\mathcal{O}}_{A_6 + A_1}$ is of codimension 2. As both orbits are special, they are relevant, so we get $mtp = 2 = b_2(G/P)$, thus $\tilde{O}$ is $\mathbb{Q}$-factorial. For $\mathcal{O} = D_7(a_2)$, a symplectic resolution is given by the induction $(2A_3, 0)$ and $\tilde{\mathcal{O}} \setminus \mathcal{O} = \tilde{\mathcal{O}}_{D_5 + A_3}$ is of codimension 2. Using $[A]$, we find $mtp = 2 = b_2(G/P)$, thus $\tilde{O}$ is $\mathbb{Q}$-factorial. For $\mathcal{O} = D_7(a_1)$, a symplectic resolution is given by the induction $(A_2 + A_3, 0)$ and $\tilde{\mathcal{O}} \setminus \mathcal{O} = \tilde{\mathcal{O}}_{E_7(a_3)} \cup \tilde{\mathcal{O}}_{E_6(a_6)}$ is of pure codimension 2. Using $[A]$, we find $mtp = 3 = b_2(G/P)$, thus $\tilde{O}$ is $\mathbb{Q}$-factorial.

Now we consider the following orbits: $A_3 + A_2, D_5(a_1)$ in $E_7$ and $A_3 + A_2$ in
$E_8$. By the proof of Proposition 3.1, $\tilde{O}$ admits a $\mathbb{Q}$-factorial terminalization given by a generalized Springer map $\pi : Z := G \times^P (n(p) + \tilde{O}_t) \to \tilde{O}$ with $b_2(G/P) = 1$ and $\text{Pic}(\tilde{O}_t) \otimes \mathbb{Q} = 0$. One checks easily that for such $\mathcal{O}$, $\tilde{O} \setminus \mathcal{O}$ contains a unique codimension 2 orbit $\mathcal{O}_{x'}$. We then use [A] to check that $\text{mtp}(\rho_{(x',0)}, \text{Ind}_{W(L)}(\rho_{(x',0)})) \neq 0$. As the variety $Z$ is smooth along $G \times^P (n(p) + \tilde{O}_t)$, one checks that $Z$ is smooth in codimension 3. We can now apply Prop. 2.3 to deduce that the pre-image of $\mathcal{O}_{x'}$ under the generalized Springer map is of codimension 1, thus the map is divisorial. As $b_2(Z) = 1$, this implies that $\tilde{O}$ is $\mathbb{Q}$-factorial.

Now we consider the orbit: $A_2 + A_1$ in $E_7$. By the proof of Proposition 3.1, the induction $(E_7, A_2 + A_1)$ of $\mathcal{O} := \mathcal{O}_{D_8(a_1)}$ in $E_8$ gives a birational map $Z := G \times^P (n(p) + \tilde{O}_{A_2 + A_1}) \to \tilde{O}$. We have $\tilde{O} \setminus \mathcal{O} = \tilde{O}_{D_8(a_1)} \cup \tilde{O}_{A_2 + A_1}$. Only the component $\tilde{O}_{A_2 + A_1}$ is of codimension 2 and one shows that $\pi$ is smooth over points in $\mathcal{O}_{A_2 + A_1}$. By applying the proof of Proposition 2.4, we can show that the number of irreducible exceptional divisors of $\pi$ is equal to the multiplicity $\text{mtp}(\rho_{(A_2 + A_1, 0)}, \text{Ind}_{W(E_7)}(\rho_{(A_2 + A_1, 0)}))$, which is 1 by [A]. On the other hand, $\text{Pic}(\mathcal{O})$ is finite by Lemma 4.1. Applying the arguments in the proof of Lemma 4.3, we get that $\text{Pic}(Z) \otimes \mathbb{Q} = \mathbb{Q}$, which implies that $\text{Pic}(\mathcal{O}_{A_2 + A_1}) \otimes \mathbb{Q} = 0$, thus $\tilde{O}_{A_2 + A_1}$ is $\mathbb{Q}$-factorial.

The claim for the remaining four orbits $(A_4 + A_1$ in $E_7$, $A_4 + A_1, A_4 + 2A_1, E_0(a_1) + A_1$ in $E_8$) is proved by the following Lemma.

For a nilpotent element $x \in \mathfrak{g}$, the Jacobson-Morozov theorem gives an $\mathfrak{sl}_2$-triplet $(x, y, h)$, i.e. $[h, x] = 2x, [h, y] = -2y, [x, y] = h$. This triplet makes $\mathfrak{g}$ an $\mathfrak{sl}_2$-module, so we have a decomposition $\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}_i$, where $\mathfrak{g}_i = \{z \in \mathfrak{g} | [h, z] = iz\}$. The Jacobson-Morozov parabolic sub-algebra of this triplet is $\mathfrak{p} := \oplus_{i \geq 0} \mathfrak{g}_i$. Let $P$ be the parabolic subgroup of $G$ determined by $\mathfrak{p}$, whose marked Dynkin diagram is given by marking the non-zero nodes in the weighted Dynkin diagram of $x$. The Jacobson-Morozov resolution of $\tilde{O}_x$ is given by $\mu : Z := G \times^P \mathfrak{n}_2 \to \tilde{O}_x$, where $\mathfrak{n}_2 := \oplus_{i \geq 2} \mathfrak{g}_i$ is a nilpotent ideal of $\mathfrak{p}$.

**Lemma 4.5.** Let $\tilde{O}$ be one of the following orbits: $A_4 + A_1$ in $E_7$, $A_4 + A_1, A_4 + 2A_1, E_0(a_1) + A_1$ in $E_8$. Then $\tilde{O}$ is $\mathbb{Q}$-factorial.

**Proof.** We will consider the Jacobson-Morozov resolution $\mu : G \times^P \mathfrak{n}_2 \to \tilde{O}$. By [N3] (Lemma 1.1.1), $\tilde{O}$ is $\mathbb{Q}$-factorial if the number of $\mu$-exceptional divisors is equal to $b_2(G/P)$. To find $\mu$-exceptional divisors, we will use the computer algebra system GAP4 to compute the dimension of the orbit $P \cdot z$. 


for $z \in n_2$ (which is the same as $\dim[p, z]$). We denote by $\beta_j$ the root vector corresponding to the $j$-th positive root of $g$ as present in $\text{GAP4}$ (see [dG] Appendix B).

Consider first the orbit $O := O_{A_4 + A_1}$ in $E_7$. Its Jacobson-Morozov parabolic subgroup $P$ is given by marking the nodes $\alpha_1, \alpha_4, \alpha_6$ (in Bourbaki’s ordering). Let $Q_1$ (resp. $Q_2$) be the parabolic subgroup given by marking the nodes $\alpha_1, \alpha_6$ (resp. $\alpha_6$). We have $P \subset Q_1 \subset Q_2$. Let $Z_i := G \times Q_i(Q_1, n_2)$ and $\tilde{Z}_i$ its normalization. The Jacobson-Morozov resolution $\mu$ factorizes through three contractions:

$$Z \overset{\mu_1}{\rightarrow} \tilde{Z}_1 \overset{\mu_2}{\rightarrow} \tilde{Z}_2 \overset{\mu_3}{\rightarrow} \tilde{O}.$$ 

We consider the following three elements in $n_2$: $x_1 := \beta_{20} + \beta_{21} + \beta_{25} + \beta_{29}$, $x_2 := \beta_{21} + \beta_{25} + \beta_{26} + \beta_{27} + \beta_{28} + \beta_{29} + \beta_{47}$, $x_3 := \beta_{20} + \beta_{21} + \beta_{28} + \beta_{29} + \beta_{30} + \beta_{31}$. Let $E_i := G \times P \cdot x_i$. Using $\text{GAP4}$, we find $\dim(E_i) = \dim(G/P) + \dim(P \cdot x_i) = 103$, thus $E_i$ are irreducible divisors in $Z$. By calculating the dimensions of $Q_i \cdot x_j$ using $\text{GAP4}$, we get that $\mu_1$ contracts $E_1$ while $\mu_1(E_2)$ and $\mu_1(E_3)$ are again divisors. The divisor $\mu_1(E_2)$ is contracted by $\mu_2$ while $\mu_2(\mu_1(E_3))$ is again a divisor, which is contracted by $\mu_3$. This shows that the three $\mu$-exceptional divisors $E_i, i = 1, 2, 3$ are distinct, thus $\tilde{O}$ is $Q$-factorial. Using the program in [dG], we find $\mu(E_1) = \tilde{O}_{A_4}$ and $\mu(E_2) = \mu(E_3) = \tilde{O}_{A_3 + A_2 + A_1}$.

For the orbit $A_4 + A_1$ in $E_8$, its Jacobson-Morozov parabolic subgroup $P$ is given by marking the nodes $\alpha_1, \alpha_6, \alpha_8$. Let $Q_1$ (resp. $Q_2$) be the parabolic subgroup given by marking the nodes $\alpha_1, \alpha_8$ (resp. $\alpha_8$). As before, we define $\tilde{Z}_i$ and $\mu_i$. We consider the following three elements in $n_2$: $x_1 := \beta_{42} + \beta_{57} + \beta_{33} + \beta_{43}$, $x_2 := \beta_{29} + \beta_{45} + \beta_{56} + \beta_{57} + \beta_{58} + \beta_{59}$, $x_3 := \beta_{57} + \beta_{56} + \beta_{59} + \beta_{61} + \beta_{45} + \beta_{58}$. We define $E_i$ as before and by using $\text{GAP4}$ we find that $E_i, i = 1, 2, 3$ are divisors in $Z$. The map $\mu_1$ contracts $E_1$, the map $\mu_2$ contracts the divisor $\mu_1(E_2)$ and the map $\mu_3$ contracts the divisor $\mu_2(\mu_1(E_3))$. This shows that $E_i, i = 1, 2, 3$ are distinct, thus $\tilde{O}$ is $Q$-factorial. We have furthermore $\mu(E_1) = \mu(E_2) = \tilde{O}_{A_4}$ and $\mu(E_3) = \tilde{O}_{D_4(a_1) + A_2}$.

For the orbit $A_4 + 2A_1$ in $E_8$, its Jacobson-Morozov parabolic subgroup $P$ is given by marking the nodes $\alpha_4, \alpha_8$. Let $Q_1$ be the parabolic subgroup given by marking the nodes $\alpha_8$. We define similarly $\mu_i, \tilde{Z}_1$. We consider the following elements in $n_2$: $x_1 := \beta_{42} + \beta_{57} + \beta_{33} + \beta_{43} + \beta_{61}$, $x_2 := \beta_{32} + \beta_{42} + \beta_{47} + \beta_{53} + \beta_{57} + \beta_{51}$. As before, we define $E_i, i = 1, 2$, which are divisors by calculating in $\text{GAP4}$. The map $\mu_1$ contracts $E_1$ and the map $\mu_2$ contracts the divisor $\mu_1(E_2)$, thus $E_1 \neq E_2$ and $\tilde{O}$ is $Q$-factorial. We have furthermore $\mu(E_1) = \tilde{O}_{A_4 + A_1}$ and $\mu(E_2) = \tilde{O}_{2A_3}$.
The orbit $\mathcal{O} := E_6(a_1) + A_1$ is induced from $(E_7, A_4 + A_1)$. The generalized Springer map $Z := G \times P (n(p)) + \mathcal{O}_{A_4 + A_1}$ is birational. We have $\mathcal{O} \setminus \mathcal{O} = \mathcal{O}_{E_6(a_1)} \cup \mathcal{O}_{D_7(a_2)}$. Only the component $\mathcal{O}_{D_7(a_2)}$ is of codimension 2 and one shows that $\pi$ is smooth over points in $\mathcal{O}_{D_7(a_2)}$. By applying the proof of Proposition 2.4, we can show that the number of irreducible exceptional divisors of $\pi$ is equal to the multiplicity $\text{mtp}(\rho(A_4 + A_1, 1), \text{Ind}_W(E_8) W(E_7) \rho(A_2 + A_1, 1))$, which is 1 by [A]. On the other hand, we have just proved the $\mathbb{Q}$-factoriality of $\tilde{\mathcal{O}}_{A_4 + A_1}$, thus $\text{Pic}(\mathcal{O}_{A_4 + A_1})$ is finite. This gives that $b_2(G \times P (n(p) + \mathcal{O}_{A_4 + A_1})) = 1$ and $\pi$ contains an exceptional divisor, thus $\tilde{\mathcal{O}}$ is $\mathbb{Q}$-factorial.

5 Rigid orbits

The aim of this section is to prove Conjecture 1 for rigid orbits. The classification of rigid orbits can be found for example in [M] (Section 5.7).

**Theorem 5.1.** Let $\mathcal{O}$ be a rigid nilpotent orbit in a complex simple Lie algebra $\mathfrak{g}$. Then $\tilde{\mathcal{O}}$ is $\mathbb{Q}$-factorial terminal.

**Proof.** When $\mathfrak{g}$ is classical, this is proven in [N3]. From now on, we assume that $\mathfrak{g}$ is exceptional. The $\mathbb{Q}$-factoriality of $\tilde{\mathcal{O}}$ is a direct consequence of Proposition 4.4. As $\tilde{\mathcal{O}}$ is symplectic, it has terminal singularities if $\text{codim}_\mathcal{O}(\mathcal{O} \setminus \mathcal{O}) \geq 4$. Using the tables in [M] (section 5.7, 6.4), we calculate the codimension of $\mathcal{O} \setminus \mathcal{O}$ and it follows that every rigid orbit satisfies $\text{codim}_\mathcal{O}(\mathcal{O} \setminus \mathcal{O}) \geq 4$ except the following orbits: $\tilde{\mathcal{A}}_1$ in $G_2$, $\tilde{\mathcal{A}}_2 + A_1$ in $F_4$, $(A_3 + A_1)'$ in $E_7$, $A_3 + A_1, A_5 + A_1, D_5(a_1) + A_2$ in $E_8$.

Consider first the orbit $\mathcal{O} := \tilde{A}_1$ in $G_2$. Its Jacobson-Morozov parabolic subgroup is given by marking the node $\alpha_1$ (in Bourbaki’s ordering). Consider the Jacobson-Morozov resolution $Z := G \times P n_2 \xrightarrow{\mu} \tilde{\mathcal{O}}$. By [F1], $\tilde{\mathcal{O}}$ has no crepant resolution, thus $\mu$ is not small. As $b_2(G/P) = 1$, there exists one unique $\mu$-exceptional irreducible divisor $E$. The canonical divisor $K_Z$ is then given by $K_Z = aE$ with $a > 0$. This implies that $\tilde{\mathcal{O}}$ has terminal singularities. This fact is already known in [K] by a different method.

We now consider the three orbits in $E_8$. Let $Z := G \times P n_2 \xrightarrow{\mu} \tilde{\mathcal{O}}$ be the Jacobson-Morozov resolution and $p : Z \to G/P$ the natural projection. Let $\omega_1, \cdots, \omega_8$ be the fundamental weights of $E_8$. The Picard group $\text{Pic}(G/P)$ is generated by $\omega_i$ s.t. $\alpha_i$ is a marked node. The canonical bundle of $Z$ is given by $K_Z = p^*(K_{G/P} \otimes \text{det}(G \times P n_2^*)$. Let $\cup_j E_j$ be the exceptional
locus of $\mu$, which is of pure codimension 1 since $\tilde{\mathcal{O}}$ is $\mathbb{Q}$-factorial. We have $K_Z = \sum_j a_j E_j$ with $a_j \geq 0$. Note that if we can show $K_{G/P}^{-1} \otimes \text{det}(G \times P \mathfrak{n}_2)$ is ample on $G/P$, then $a_j > 0$ for all $j$ (since $p$ does not contract any $\mu$-exceptional curve), which will prove that $\tilde{\mathcal{O}}$ has terminal singularities. As $T^*(G/P) \simeq G \times P (\oplus_{k \leq -1} \mathfrak{g}_k)$, the line bundle $K_{G/P}^{-1} \otimes \text{det}(G \times P \mathfrak{n}_2)$ corresponds to the character $\wedge^{\text{top}} (\oplus_{k \leq -1} \mathfrak{g}_k) \otimes \wedge^{\text{top}} (\oplus_{k \geq 2} \mathfrak{g}_k) \simeq \wedge^{\text{top}} \mathfrak{g}_{-1}$ of $P$. An explicit basis of $\mathfrak{g}_{-1}$ and the action of a Cartan subalgebra $\mathfrak{h}$ on it can be computed using GAP4. For the orbit $A_3 + A_1$, we get that $K_Z = p^*(-13\omega_6 - 3\omega_8)$. For the orbit $A_2 + A_1$, we get $K_Z = p^*(-3\omega_1 - 7\omega_4 - 5\omega_8)$. For the orbit $D_5(a_1) + A_2$, we get $K_Z = p^*(-7\omega_3 - 6\omega_6 - 3\omega_8)$. This proves the claim for these three orbits.

In a similar way, for the orbit $\tilde{A}_2 + A_1$ in $F_4$, we find that $K_Z = p^*(3\omega_4 - 2\omega_2)$ and for the orbit $(A_3 + A_1)'$ in $E_7$, we obtain $K_Z = p^*(5\omega_1 - 3\omega_4)$, thus the precedent argument does not apply here. Instead we will use another approach. Recall (\cite{P}) that there exists a 2-form $\Omega$ on $G/P$ which is defined at a point $(g, x) \in G \times \mathfrak{n}_2$ by: $\Omega((g, (x, m), (u, u')) = \kappa([u, u'], x) + \kappa(m', u) - \kappa(m, u'))$, where $\kappa(\cdot, \cdot)$ is the Killing form. The tangent space of $Z$ at the point $(g, x)$ is identified with the quotient

$$\mathfrak{g} \times \mathfrak{n}_2/\{(u, [x, u])|u \in \oplus_{i \geq 0} \mathfrak{g}_i\}.$$  

By Lemma 4.3 in \cite{B}, The kernel of $\Omega_{(g, x)}$ consists of images of elements $(u, [x, u])$ with $u \in \oplus_{i \geq -1} \mathfrak{g}_i$, such that $[x, u] \in \mathfrak{n}_2$. This shows that $\Omega_{(g, x)}$ is non-degenerate if and only if the set $\mathfrak{K}_{x} := \{u \in \mathfrak{g}_{-1}|[x, u] \in \mathfrak{n}_2\}$ is reduced to $\{0\}$. Let $s := \wedge^{\text{top}} \Omega$, then $K_Z = \text{div}(s)$ and $s((g, x)) \neq 0$ if and only if $\mathfrak{K}_{x} = \{0\}$. To prove our claim, we just need to show that for a generic point $x$ in every $\mu$-exceptional divisor, the section $s$ vanishes at $x$, i.e. to show that $\mathfrak{K}_{x} \neq \{0\}$.

For the orbit $\tilde{A}_2 + A_1$ in $F_4$, we consider the two elements in $\mathfrak{n}_2$: $x_1 := \beta_{11} + \beta_{12}$ and $x_2 := \beta_{14} + \beta_{15} + \beta_{16}$. Define $E_i := G \times P \mathfrak{x}_i$, $i = 1, 2$. Using GAP4, we find that $E_1$ and $E_2$ are of codimension 1 in $Z$. We have $\mu(E_1) = \mathcal{O}_{\tilde{A}_2}$ and $\mu(E_2) = \mathcal{O}_{A_2 + A_1}$, which shows that the two divisors are distinct. As $b_2(G/P) = 2$, we get $\text{Exc}(\mu) = E_1 \cup E_2$. Consider the two elements in $\mathfrak{g}_{-1}$: $u_1 := \beta_{28}$ and $u_2 := \beta_{25} - 2\beta_{28}$. Then we have $[x_1, u_1] = 0$ and $[x_2, u_2] = \beta_{12} \in \mathfrak{n}_2$, which proves that $u_1 \in \mathfrak{K}_{x_1}$ and $u_2 \in \mathfrak{K}_{x_2}$. From this we get that $K_Z = a_1 E_1 + a_2 E_2$ with $a_i > 0, i = 1, 2$.

For the orbit $(A_3 + A_1)'$ in $E_7$, we consider the two elements in $\mathfrak{n}_2$: $x_1 := \beta_{20} + \beta_{21} + \beta_{29}$ and $x_2 := \beta_{20} + \beta_{34} + \beta_{35} + \beta_{37} + \beta_{43} + \beta_{45}$. We define in
a similar way $E_1, E_2$ which are divisors by a calculus in GAP4. We have 
$\mu(E_1) = \bar{O}_{A_3}$ and $\mu(E_2) = \bar{O}_{2A_2 + A_1}$, thus $\text{Exc}(\mu) = E_1 \cup E_2$. Consider the 
two elements in $g_{-1}$: $u_1 := \beta_{67}$ and $u_2 := \beta_{64} - \beta_{79} - \beta_{81}$. Then we have 
$[x_1, u_1] = 0$ and $[x_2, u_2] = -\beta_{26} - \beta_{27} + \beta_{40} \in n_2$, which proves that $u_1 \in Kn_{x_1}$
and $u_2 \in Kn_{x_2}$. We deduce that $K_Z = a_1E_1 + a_2E_2$ with $a_i > 0, i = 1, 2$, which concludes the proof.

**Remark 5.2.** The three orbits in $E_8$ can also be dealt with in the same way. Thus in this paper, the essential point where we used GAP4 is to compute 
the dimension of $[p, x]$ (surely we have used it in a crucial way to find the elements $x_i$ in $n_2$ and $u_i$ in $g_{-1}$).

**Corollary 5.3.** The normalization $\bar{O}$ is smooth in codimension 3 for the 
following orbits: $A_1$ in $G_2$, $A_2 + A_1$ in $F_4$, $(A_3 + A_1)'$ in $E_7$, $A_3 + A_1, A_5 + A_1, D_5(a_1) + A_2$ in $E_8$. In particular, the closure $\bar{O}$ of these orbits is non-
normal.

Although the complete classification of $O$ with normal closure is unknown 
in $E_7$ and $E_8$, E. Sommers communicated to the author that the orbits in 
the corollary are known to have non-normal closures.

## 6 Induced orbits

Recall ([F1], [F2]) that a nilpotent orbit closure in a simple Lie algebra 
adopts a crepant resolution if and only if it is a Richardson orbit but not in 
the following list: $A_4 + A_1, D_5(a_1)$ in $E_7$, $E_6(a_1) + A_1, E_7(a_3)$ in $E_8$. On 
the other hand, by [N2], if $\bar{O}$ admits a crepant resolution, then any $Q$-
factorial terminalizations of $\bar{O}$ is in fact a crepant resolution. Furthermore 
the birational geometry between their crepant resolutions are well-understood 
([N1], [F2]). Thus to prove Conjecture [ ], we will only consider induced orbits 
whose closure does not admit any crepant resolution.

**Theorem 6.1.** Let $O$ be an induced nilpotent orbit in a complex simple 
exceptional Lie algebra $g$. Assume that $\bar{O}$ admits no crepant resolution. Then 
(i) The variety $\bar{O}$ has $Q$-factorial terminal singularities for the following 
induced orbits: $A_2 + A_1, A_4 + A_1$ in $E_7$ and $A_4 + A_1, A_4 + 2A_1$ in $E_8$.

(ii) If $O$ is not in the list of (i), then any $Q$-factorial terminalization of 
$\bar{O}$ is given by a generalized Springer map. Two $Q$-factorial terminalizations 
of $\bar{O}$ are connected by Mukai flops of type $E_{6,1}^I$ or $E_{6,1}^{II}$ (defined in section 
6.1).
Remark 6.2. Unlike the classical case proved in [N3], for an orbit $O$ such that $\bar{O}$ has no Springer resolution, the Mukai flops of type $A-D-E_6$ defined in [N1] (p. 91) do not appear here. See Corollary 6.3 and Corollary 6.7.

6.1 Mukai flops

Let $P$ be one of the maximal parabolics in $G := E_6$ corresponding to the following marked Dynkin diagrams:

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• — — — — — — —
   |   |   |
   o   o   o
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The Levi part of $P$ is isomorphic to $D_5$. We denote by $O_I$ (resp. $O_{II}$) the nilpotent orbit in $l(p)$ corresponding to the partition $2^21^6$ (resp. $3^22^11^3$). Then we have two generalized Springer maps $\pi_I, \pi_{II}$ with image being the closures of orbits $A_2 + A_1, A_3 + A_1$ respectively. As the component group $A(O_{A_2+A_1}) = A(O_{A_3+A_1}) = \{1\}$, both maps are birational. By [N3], $\bar{O}_I, \bar{O}_{II}$ are $\mathbb{Q}$-factorial terminal, thus $\pi_I, \pi_{II}$ give $\mathbb{Q}$-factorial terminalizations.

Lemma 6.3. The two maps $\pi_I, \pi_{II}$ are small, i.e. the exceptional locus has codimension at least 2.

Proof. For $\pi_I$, we have $\text{codim}(O_{A_2+A_1} \setminus O_{A_2+A_1}) = 4$. As $\pi_I$ is semi-small, this implies the claim. For $\pi_{II}$, the orbit closure $\bar{O}_{A_3+A_1} = O_{A_3+A_1} \cup \bar{O}_{A_3} \cup \bar{O}_{2A_2+A_1}$. The codimension of $\bar{O}_{A_3}$ in $\bar{O}$ is 4, so its pre-image has codimension at least 2. The codimension of $\bar{O}_{2A_2+A_1}$ in $\bar{O}$ is 2. As one sees easily, $G \times P (n(p) + O_I)$ is smooth over points in $O_{2A_2+A_1}$. By Proposition 2.3, we need to check $mtp(\rho(2A_2+A_1,1), \text{Ind}_{W(D_5)}^{W(E_6)}(\rho(O_{II},1))) = 0$. By [C], we have $\rho(2A_2+A_1,1) = 10$, and $\rho(O_{II},1) = [- : 2^21]$. By [A] (p. 31), we get the claim. □

When $P$ changes from one parabolic to the other, we get two $\mathbb{Q}$-factorial terminations of the same orbit. The birational map between them is then a flop, which we will call Mukai flop of type $E_{6,I}$ and $E_{6,II}$ respectively.

6.2 Proof of the theorem

For an orbit $O$ in list (i) of the theorem, the variety $\bar{O}$ is $\mathbb{Q}$-factorial by Proposition 4.3. One checks using tables in Section 5.7 and 6.4 [M] that
codim(\(\mathcal{O} \setminus \mathcal{O}\)) \(\geq 4\), thus \(\tilde{\mathcal{O}}\) has only terminal singularities. This proves claim (i) in the theorem.

Let now \(\mathcal{O}\) be an induced orbit not in list (i). By Proposition 3.1 we have a birational generalized Springer map

\[\pi : G \times \mathbb{Q} (n(q) + \tilde{\mathcal{O}}_t) \to \tilde{\mathcal{O}}.\]

For orbits listed in the proof of Proposition 3.1 we check from the above and from Theorem 5.1 that for our choice of \(\mathcal{O}_t\), the variety \(\tilde{\mathcal{O}}_t\) is either \(\mathbb{Q}\)-factorial terminal or it admits a \(\mathbb{Q}\)-factorial terminalization given by a generalized Springer map. For orbits with \(A(\mathcal{O}) = \{1\}\), i.e. those not listed above, we can check this using the induction tables in [M] (Section 5.7). This shows that \(\tilde{\mathcal{O}}\) admits a generalized Springer map which gives a \(\mathbb{Q}\)-factorial terminalization.

**Lemma 6.4.** For any orbit \(\mathcal{O}\) not listed in (i), there exists a unique pair \((l(q), \mathcal{O}_t)\) which induces \(\mathcal{O}\) such that the associated generalized Springer gives a \(\mathbb{Q}\)-factorial terminalization of \(\tilde{\mathcal{O}}\).

**Proof.** Note that if the normalization of \(G \times \mathbb{Q} (n(q) + \tilde{\mathcal{O}}_t)\) gives a \(\mathbb{Q}\)-factorial terminalization of \(\tilde{\mathcal{O}}\), then \(\tilde{\mathcal{O}}_t\) is \(\mathbb{Q}\)-factorial terminal. As \(\tilde{\mathcal{O}}\) has no Springer resolution, we have \(\mathcal{O}_t \neq 0\). When \(l(q)\) is of classical type, by the proof of Proposition (2.1.1) [N3], the partition \(d := [d_1, \ldots, d_k]\) of \(\mathcal{O}_t\) has full members, i.e. every integer between 1 and \(d_1\) appears in \(d\). When \(l(q)\) is exceptional, we need to consider \(\mathcal{O}_t\) such that \(\mathcal{O}_t\) is not the image of a birational generalized Springer map. By Proposition 3.1 we may assume \(\mathcal{O}_t\) is rigid or \(\mathcal{O}_t\) is the orbit \(A_2 + A_1\) in \(E_7\).

In [Sp] (Appendix in Chap. II), Spaltenstein reproduced the tables of Elashvili which gives all inductions with \(\mathcal{O}_t\) rigid. For our purpose, when \(l(q)\) is of classical type, there are only two additional cases (both in \(E_8\)) not contained therein: the induction \((D_7, 3^22^31^4)\) for \(E_8(a_7)\) and \((D_6, 3^22^21^2)\) for \(E_7(a_3)\). When \(l(q)\) is of exceptional type, we need to consider the induction \((E_7, A_2 + A_1)\) of \(D_5(a_1)\) in \(E_8\). A case-by-case check gives that we have only a few orbits (only in \(E_7, E_8\)) which admit two inductions from either a rigid orbit or from an orbit listed above.

In \(E_7\), the orbit \(A_3 + A_2\) admits two such inductions from \((D_6, 3^22^31^5)\) and \((D_5 + A_1, 2^16 + 0)\). By section 3.3 only the second gives a birational generalized Springer map.

In \(E_8\), the orbit \(A_3 + A_2\) is induced from \((E_7, (3A_1)')\) and from \((D_7, 2^21^10)\). For the degree of the first, we have \(\rho_{(1,1)} = \phi_{35,31} = 35^*_b\) and \(\rho_{x,e} = 972^*_x\). By
2. \[ \text{(A)} \] only the second induction gives a birational generalized Springer map. The orbit \( E_7(a_5) \) is induced from \( (E_7, A_3 + A_1') \) and from \( (E_6 + A_1, 3A_1 + 0) \). For the degree of the first, we have \( \rho(l, 1) = \phi_2^{501.17} = 280w \) and \( \rho_{x, e_2} = 5600w, \rho_{x, e_1} = 448w \). By \[ \text{(A)} \] (p. 142), we get the degree is 2, thus it is not birational. The orbit \( E_7(a_4) \) is induced from \( (D_6, 32^{213}) \) and from \( (D_5 + A_1, 32^{213} + 0) \). One shows that only the first one gives a birational map. 

3.1. only the second induction gives a birational generalized Springer map. The orbit \( E_7(a_5) \) is induced from \( (E_7, (A_3 + A_1') \) and from \( (E_6 + A_1, 3A_1 + 0) \). For the degree of the first, we have \( \rho(l, 1) = \phi_2^{501.17} = 280w \) and \( \rho_{x, e_2} = 5600w, \rho_{x, e_1} = 448w \). By \[ \text{(A)} \] (p. 142), we get the degree is 2, thus it is not birational. The orbit \( E_7(a_4) \) is induced from \( (D_6, 32^{213}) \) and from \( (D_5 + A_1, 32^{213} + 0) \). One shows that only the first one gives a birational map. 

**Corollary 6.5.** For an orbit \( \mathcal{O} \) in the theorem but not in the list (i), any two \( \mathbb{Q} \)-factorial terminalizations of \( \tilde{\mathcal{O}} \) given by generalized Springer maps are connected by Mukai flops of type \( E_{6, I}^I \) or \( E_{6, I}^{II} \).

**Proof.** Consider a \( \mathbb{Q} \)-factorial terminalization given by the generalized Springer map associated to \( (P, \mathcal{O}) \) with \( \mathcal{O} \neq 0 \). Note that if \( l(p) \) is of type \( A \), then \( \tilde{\mathcal{O}} \) admits a Springer resolution, which contradicts our assumption. This allows us to consider only the following situations (for the other cases, there exists a unique conjugacy class of parabolic subgroups with Levi part being \( l(p) \)):

i) \( l(p) \) is \( D_5 \) in \( E_n, n = 6, 7, 8 \).

Consider case i). In \( E_6 \), this is given by the definition of Mukai flops. In \( E_7 \), the induction \( (D_5, 32^{213}) \) gives two \( \mathbb{Q} \)-factorial terminalization of the orbit closure \( \tilde{\mathcal{O}}_{D_6(a_2)} \), which are connected by a Mukai flop of type \( E_{6, I}^{II} \). The induction \( (D_5, 2^{216}) \) gives the even orbit \( A_4 \). In \( E_8 \), the induction \( (D_5, 32^{213}) \) gives two \( \mathbb{Q} \)-factorial terminalization of the orbit closure \( \tilde{\mathcal{O}}_{E_7(a_2)} \), which are connected by a Mukai flop of type \( E_{6, I}^{II} \), while the induction \( (D_5, 2^{216}) \) gives the even orbit \( E_6(a_1) \).

Consider case ii). The induction \( (D_4 + A_1, 32^{21} + 1^2) \) (resp. \( (D_4 + A_1, 2^{21} + 1^2) \)) gives the even orbit \( E_6(b_4) \) (resp. \( E_8(a_6) \)). The induction \( (D_5 + A_1, 32^{213} + 1^2) \) gives the even orbit \( D_7(a_1) \), while the induction \( (D_5 + A_1, 2^{216} + 1^2) \) of \( E_7(a_4) \) gives a generalized Springer map of degree 2. 

Now we prove that every \( \mathbb{Q} \)-factorial terminalization of \( \mathcal{O} \) not in (i) is given by a generalized Springer map. The following proposition is analogous to Proposition (2.2.1) in [N3].

**Proposition 6.6.** Let \( \mathcal{O} \) be a nilpotent orbit in a simple exceptional Lie algebra such that \( \tilde{\mathcal{O}} \) does not admit a Springer resolution. Suppose that a \( \mathbb{Q} \)-factorial terminalization of \( \tilde{\mathcal{O}} \) is given by the normalization of \( G \times \mathbb{Q} \) (n(q) +
for some parabolic $Q$ and some nilpotent orbit $O_t$ in $l(q)$. Assume that $b_2(G/Q) = 1$ and the $Q$-factorial terminalization is small. Then this generalized Springer map is one of those in Section 6.1.

Proof. Assume that $O$ is neither the orbit $A_2 + A_1$ nor $A_3 + A_1$ in $E_6$. As we only consider $O$ such that $\bar{O}$ has no Springer resolutions, by Proposition 4.3, $\bar{O}$ is $Q$-factorial. This implies that every $Q$-factorial terminalization of $\bar{O}$ is divisorial, which concludes the proof.

Now one can argue as in the proof of Theorem (2.2.2) in [N3] to show that every $Q$-factorial terminalization of $O$ not in (i) is actually given by a generalized Springer map. This concludes the proof of our theorem.

The following corollary is immediate from Theorem 6.1 and the proof of Corollary 6.5.

**Corollary 6.7.** Let $O$ be an induced nilpotent orbit. Assume that $\bar{O}$ has no Springer resolution. Then $\bar{O}$ admits a unique $Q$-factorial terminalization unless $O$ is one of the following orbits: $A_2 + A_1, A_3 + A_1$ in $E_6$, $D_6(a_2)$ in $E_7$, $E_7(a_2)$ in $E_8$, in which case $\bar{O}$ admits exactly two different $Q$-factorial terminalizations.

To conclude this paper, we give the following classification of nilpotent orbits such that $\bar{O}$ has terminal singularities.

**Proposition 6.8.** Let $O$ be a nilpotent orbit in a simple complex exceptional Lie algebra. Then $\bar{O}$ has terminal singularities if and only if $O$ is one of the following orbits:

1. rigid orbits;
2. $2A_1, A_2 + A_1, A_2 + 2A_1$ in $E_6$, $A_2 + A_1, A_4 + A_1$ in $E_7$, $A_4 + A_1, A_4 + 2A_1$ in $E_8$.

Proof. By using tables in Section 5.7 and 6.4 of [M], we get that for the three orbits in $E_6$ of (2), we have $\text{codim}(\bar{O} \setminus O) \geq 4$, thus $\bar{O}$ has terminal singularities. By Theorem 5.1 and Theorem 6.1, this implies that the variety $\bar{O}$ has terminal singularities for orbits in (1) and (2).

Assume now $O$ is not in the list, then by Theorem 6.1, $\bar{O}$ admits a $Q$-factorial terminalization given by a generalized Springer map. This implies that if $\bar{O}$ is $Q$-factorial, then $\bar{O}$ is not terminal. By Proposition 4.3, we may assume that $O$ is one of the following orbits in $E_6$: $A_3, A_3 + A_1, A_4, A_4 +$
$A_1, D_5(a_1)$ and $D_5$. As for these orbits except $A_3 + A_1$, the closure $\mathcal{O}$ admits a symplectic resolution, thus $\tilde{\mathcal{O}}$ is not terminal.

We consider the orbit $\mathcal{O} := A_3 + A_1$ in $E_6$. We will use the method in the proof of Theorem 5.1 to show that $\tilde{\mathcal{O}}$ is not terminal. Consider the Jacobson-Morozov resolution $Z := G \times^P \mathfrak{n}_2 \overset{\mu}{\rightarrow} \bar{\mathcal{O}}$, where $P$ is given by marking the nodes $\alpha_2, \alpha_3, \alpha_5$. We have $\bar{\mathcal{O}} \setminus \mathcal{O} = \bar{\mathcal{O}}_{A_3} \cup \bar{\mathcal{O}}_{2A_2+A_1}$. We consider the following two elements in $\mathfrak{n}_2$: $x_1 := \beta_{17} + \beta_{15} + \beta_{20}$ and $x_2 := \beta_{17} + \beta_{18} + \beta_{20} + \beta_{21} + \beta_{24}$. We define $E_i := G \times^P \mathfrak{P} \cdot x_i, i = 1, 2$, which are irreducible divisors by a calculus in GAP4. We have furthermore $\mu(E_1) = \bar{\mathcal{O}}_{A_3}$ and $\mu(E_2) = \bar{\mathcal{O}}_{2A_2+A_1}$, thus the two divisors are distinct. As $b_2(G/P) = 3$ and $\bar{\mathcal{O}}$ is non-$\mathbb{Q}$-factorial, $E_1, E_2$ are the only two $\mu$-exceptional divisors. Using a calculus in GAP4, we can show that $K_{x_1} := \{ u \in \mathfrak{g}_{-1} | [x_1, u] \in \mathfrak{n}_2 \}$ is reduced to $\{0\}$ and $K_{x_2} \neq \{0\}$. This implies that $K_Z = aE_2$ for some $a > 0$, which proves that $\tilde{\mathcal{O}}$ is not terminal.

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