A NEW PROOF OF GRADIENT ESTIMATES FOR MEAN CURVATURE EQUATIONS WITH OBLIQUE BOUNDARY CONDITIONS

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Abstract. In this paper, we will use the maximum principle to give a new proof of the gradient estimates for mean curvature equations with some oblique derivative problems. In particular, we shall give a new proof for the capillary problem with zero gravity.

1. Introduction. In this note, we mainly consider the following oblique boundary value problem for prescribed mean curvature equation

\[ \text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = f(x,u) \quad \text{in} \quad \Omega, \tag{1} \]

\[ v^{q-1} \frac{\partial u}{\partial \gamma} + \psi(x,u) = 0 \quad \text{on} \quad \partial \Omega, \tag{2} \]

where \( \Omega \subset \mathbb{R}^n \) \((n \geq 2)\) is a bounded domain, \( \gamma \) is the inward unit normal vector to \( \partial \Omega \), \( q \geq 0 \) and \( v = \sqrt{1 + |Du|^2} \).

In (2), for \( q = 0 \), it is corresponding to capillary boundary condition and for \( q = 1 \), it is corresponding to Neumann boundary value condition.

The interior gradient estimates and the Dirichlet problem for the prescribed mean curvature equation have been extensively studied. We refer the reader to see the book [4] written by Gilbarg and Trudinger. More other oblique boundary value problems for the second order elliptic equations can be seen in the literature [10] written by Lieberman and the references therein.

For the mean curvature equation with capillary problem, Ural’tseva [15] first got the boundary gradient estimates and the corresponding existence theorem on the positive gravity case \( (f_u \geq \bar{C}_0 > 0, \bar{C}_0 \text{ is a constant}) \). In the same year, Simon and Spruck [12] and Gerhardt [2] obtained existence theorem on the positive gravity case respectively. For more general quasilinear divergence structure equation with conormal derivative boundary value problem, Lieberman [6] gave the gradient estimate. They all obtained these estimates via test function technique.

Spruck [13] firstly used the maximum principle to obtain boundary gradient estimate in two dimension for positive gravity case. Later, Korevaar [5] generalized his normal variation technique and got the boundary gradient estimates in the
positive gravity case in high dimensional cases. In [7, 8], Lieberman developed his maximum principle approach on the boundary gradient estimates to the quasilinear elliptic equations with oblique derivative boundary value problem and in [9] he got the maximum principle proof for the gradient estimates on the general quasilinear elliptic equations with capillary boundary value problems in zero gravity case ($f_u \geq 0$).

Lieberman ([10], in page 360) proved the gradient estimates of a more general class of quasilinear elliptic equations with the boundary condition (2) for $q = 0$ or $q > 1$. Recently, for the specific problem (1)-(2), Ma and Xu [11] got the boundary gradient estimates of mean curvature equations with Neumann problem for $q = 1$ via the maximum principle and obtained an existence result in positive gravity case.

In this paper, we find a new auxiliary function and use the maximum principle to give a unified proof of the gradient estimates for the problem (1)-(2) with $q > 1$. Let's restate the following two results. For convenience, firstly we consider the following two results. For convenience, firstly we consider the boundary value condition with \( \psi = \psi(x) \).

\[
q-1 \frac{\partial u}{\partial \gamma} + \psi(x) = 0 \quad \text{on} \quad \partial \Omega. \tag{3}
\]

**Theorem 1.1** ([10]). Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain, \( n \geq 2 \), \( \partial \Omega \in C^3 \), and \( \gamma \) be the inward unit normal vector to \( \partial \Omega \). Suppose that \( u \in C^2(\overline{\Omega}) \bigcap C^3(\Omega) \) is a solution of (1)-(3) with \( q > 1 \). \( f(x,z), \psi(x) \) are given functions defined in \( \overline{\Omega} \times [-M_0, M_0] \) and \( \Omega \) respectively. Furthermore we assume there exist positive constants \( M_0, L_1, L_2 \) such that

\[
|u| \leq M_0 \quad \text{in} \quad \overline{\Omega}, \tag{4}
\]

\[
f_z(x,z) \geq 0 \quad \text{in} \quad \overline{\Omega} \times [-M_0, M_0], \tag{5}
\]

\[
|f(x,z)| + |f_z(x,z)| \leq L_1 \quad \text{in} \quad \overline{\Omega} \times [-M_0, M_0], \tag{6}
\]

\[
|\psi(x)|_{C^1(\overline{\Omega})} \leq L_2. \tag{7}
\]

Then there exists a small positive constant \( \mu_0 \) such that

\[
\sup_{\partial \Omega \setminus \partial \Omega_0} |Du| \leq \max\{M_1, M_2\},
\]

where \( M_1 \) is a positive constant depending only on \( n, \mu_0, M_0, L_1 \), which is from the interior gradient estimates; \( M_2 \) is a positive constant depending only on \( n, \Omega, \mu_0, M_0, L_1, L_2 \), and \( d(x) = \text{dist}(x, \partial \Omega), \Omega_{\mu_0} = \{ x \in \Omega : d(x) < \mu_0 \} \).

**Theorem 1.2** ([2, 9]). Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain, \( n \geq 2 \), \( \partial \Omega \in C^3 \), and \( \gamma \) be the inward unit normal vector to \( \partial \Omega \). Suppose \( u \in C^2(\overline{\Omega}) \bigcap C^4(\Omega) \) is a solution of (1)-(3) with \( q = 0 \) and satisfies (4). \( f(x,z), \psi(x) \) are given functions defined in \( \overline{\Omega} \times [-M_0, M_0] \) and \( \Omega \) respectively. Assume \( f(x,z) \) satisfies (5)-(6) and \( \psi(x) \) satisfies (7). Furthermore we assume there exists a positive constant \( \mu_0 \) such that

\[
|\psi(x)|_{C^0(\overline{\Omega})} \leq \mu_0 < 1. \tag{8}
\]

Then there exists a small positive constant \( \mu_0 \) such that

\[
\sup_{\partial \Omega \setminus \partial \Omega_0} |Du| \leq \max\{M_1, M_2\},
\]
where $M_1$ is a positive constant depending only on $n, \mu_0, M_0, L_1$, which is from the interior gradient estimates; $M_2$ is a positive constant depending only on $n, \Omega, \mu_0, M_0, L_1, L_2, b_0$.

As we stated before, there is a standard interior gradient estimates for the mean curvature equation, which can also be obtained by the technique in [16] or in [14].

Remark 1 ([4]). If $u \in C^3(\Omega)$ is a bounded solution for the equation (1) with (4), and if $f \in C^1(\overline{\Omega} \times [-M_0, M_0])$ satisfies the conditions (5)-(6), then for any subdomain $\Omega' \subset \subset \Omega$, the following holds

$$\sup_{\Omega'} |Du| \leq M_1,$$

where $M_1$ is a positive constant depending only on $n, M_0, \text{dist}(\Omega', \partial \Omega), L_1$.

The rest of the paper is organized as follows. In section 2, we first give some notations and prove Theorem 1.1 under the help of Lemma 3.5. This lemma will be proved in section 3. Finally we shall show the proof of Theorem 1.2 in section 4.

In this paper, we follow the summation convention: All repeated indices from 1 to $n$ denote summation.

2. Proof of Theorem 1.1. We denote by $\Omega$ a bounded domain in $\mathbb{R}^n$, $n \geq 2$, $\partial \Omega \in C^3$, set $d(x) = \text{dist}(x, \partial \Omega)$, and

$$\Omega_\mu = \{x \in \Omega : d(x) < \mu\}.$$

Then it is well known that there exists a positive constant $\mu_1 > 0$ such that $d(x) \in C^3(\overline{\Omega}_{\mu_1})$. As in [12] or [10] in page 331, we can take $\gamma = Dd$ in $\Omega_{\mu_1}$ and note that $\gamma$ is a $C^2(\overline{\Omega}_{\mu_1})$ vector field. As mentioned in [9] and in [10], we also have the following formulas

$$|D\gamma| + |D^2\gamma| \leq C(n, \Omega) \quad \text{in} \quad \Omega_{\mu_1},$$
$$\gamma^i D_j \gamma^j = 0, \quad \gamma^i D_i \gamma^j = 0, \quad |\gamma| = 1 \quad \text{in} \quad \Omega_{\mu_1}. \quad (9)$$

As in [10], we define

$$c^{ij} = \delta_{ij} - \gamma^i \gamma^j \quad \text{in} \quad \Omega_{\mu_1}, \quad (10)$$

and for a vector $\zeta \in \mathbb{R}^n$, we write $\zeta'$ for the vector with $i$-th component $c^{ij} \zeta_j$. So

$$|D' u|^2 = c^{ij} u_i u_j. \quad (11)$$

Let

$$a^{ij}(Du) = v^2 \delta_{ij} - u_i u_j, \quad v = (1 + |Du|^2)^{\frac{1}{2}}. \quad (12)$$

Then the equations (1)-(2) are equivalent to the followings

$$a^{ij} u_{ij} = f(x, u) v^3 \quad \text{in} \quad \Omega, \quad (13)$$
$$u_\gamma = -v^{-\frac{1}{2}} q(x, u) \quad \text{on} \quad \partial \Omega. \quad (14)$$

Now we begin to prove Theorem 1.1. Using the techniques developed by Spruck [13], Lieberman [9] and Wang [16], we shall choose an auxiliary function which contains $|D' u|^2$ and other lower order terms. Then we use the maximum principle for this auxiliary function in $\overline{\Omega}_{\mu_0}, 0 < \mu_0 \leq \mu_1$, $\mu_0$ is to be determined later. At last, we get our estimates.
Proof of Theorem 1.1. Let 
\[ P(x) = \log |D' u|^2 e^\sqrt{\alpha_0(M_0+1+u)} e^{a_0 d}, \]
where \( \alpha_0 = 2C_0L_2 + 2C_0 + 2 \) is a constant, and \( C_0 = 10n + 10n^2 \max_{x \in \partial \Omega} |\tilde{A}|, |\tilde{A}| \)
is the modulus of the second fundamental form of \( \partial \Omega \). Setting 
\[ \varphi(x) = \log P(x) = \log \log |D' u|^2 + h(u) + g(d), \]
where in \( q > 1 \) boundary value case, we choose 
\[ h(u) = \sqrt{\alpha_0(M_0+1+u)}, \quad g(d) = \alpha_0 d. \quad (15) \]
We now choose \( \mu_0 = \min\{\frac{1}{2}, \frac{\mu_0}{2}\} \). Assume that \( \varphi(x) \) attains its maximum at \( x_0 \in \partial \Omega_{\mu_0} \).

In the following, we divide three cases to complete the proof of Theorem 1.1.

Case I. If \( \varphi(x) \) attains its maximum at \( x_0 \in \partial \Omega \), then we shall get the bound of \( |Du|(x_0) \).

Case II. If \( \varphi(x) \) attains its maximum at \( x_0 \in \partial \Omega \cap \Omega, \) then we shall get the estimates via the standard interior gradient bound [4].

Case III. If \( \varphi(x) \) attains its maximum at \( x_0 \in \Omega \mu_0, \) then we can use the maximum principle to get the bound of \( |Du|(x_0) \).

Now all computations work at the point \( x_0 \).

Case I. Suppose first that \( x_0 \in \partial \Omega \).

We differentiate \( \varphi \) along the normal direction.

\[ \frac{\partial \varphi}{\partial \gamma} = \frac{(|D' u|^2)_\iota \gamma^i}{|D' u|^2 \log |D' u|^2} + h'u_\gamma + g'. \quad (16) \]

Applying (9) and (11), it follows that 
\[ (|D' u|^2)_\iota \gamma^i = (c^{kl}u_{ki}u^\iota)\gamma^i = 2c^{kl}u_{ki}u^\iota \gamma^i. \quad (17) \]

Differentiating (14) with respect to tangential direction, we have 
\[ c^{kl}(u_\iota)_{\iota} = -c^{kl}(v^{1-q}u)^{\iota}_{\iota}. \quad (18) \]

It follows that 
\[ c^{kl}u_{ki}u^\iota = -c^{kl}u_\iota(\gamma^i)_{\iota} - v^{1-q}c^{kl}D_k\psi - (1-q)\psi v^{-q}c^{kl}v_k. \quad (19) \]

For \( \psi = \psi(x, u) \), denote by 
\[ D_k\psi = \psi_{x_k} + \psi_u u_k. \quad (20) \]

Since 
\[ v^2 = 1 + |D' u|^2 + u^2, \quad (21) \]
differentiating (21) with respect to \( x_k \), we obtain 
\[ v_k = \frac{(|D' u|^2)_k}{2v} + \frac{u_k}{v}(u_{ik}u^\iota + u_\iota(\gamma^i)_{\iota}). \quad (22) \]

Inserting (22) into (16), we have 
\[ c^{kl}u_{ki}u^\iota = -c^{kl}u_\iota(\gamma^i)_{\iota} - \frac{v^{1-q}c^{kl}D_k\psi}{1 - (1-q)v^{2q}u^2} \]
\[ - \frac{1 - q}{2} \cdot \frac{\psi v^{1-q-1}}{1 - (1-q)v^{2q}u^2} c^{kl}(|D' u|^2)_k. \quad (23) \]
Since \(c^{kl}\varphi_k = 0\), and \(c^{kl}\gamma^k = 0\), we obtain
\[
e^{(24)}(|D'u|^2)_k = -|D'u|^2 \log |D'u|^2 c^{kl}(h'u_k + g'\gamma^k) - h'|D'u|^2 \log |D'u|^2 c^{kl}u_k. \quad (24)
\]
Inserting (24) into (23), we have
\[
e^{(25)}u_k\gamma^i = -c^{kl}u_i(\gamma^i)_k - \frac{v^{1-q} c^{kl}D_k\psi}{1 - (1 - q)v^{-2q}\psi^2} + \frac{1 - q}{2} \frac{h'\psi v^{-q-1}}{1 - (1 - q)v^{-2q}\psi^2} |D'u|^2 \log |D'u|^2 c^{kl}u_k. \quad (25)
\]
Putting (25) into (17), combining (16), we have
\[
|D'u|^2 \log |D'u|^2 \frac{\partial \varphi}{\partial \gamma} = g'|D'u|^2 \log |D'u|^2 - 2c^{kl}u_i(\gamma^i)_k - \frac{2v^{1-q} c^{kl}D_k\psi u_i}{1 - (1 - q)v^{-2q}\psi^2} - h'\psi v^{1-q} + (1 - q)v^{-1-q}\frac{1}{1 - (1 - q)v^{-2q}\psi^2} |D'u|^2 \log |D'u|^2. \quad (26)
\]
In the following, for the proof of our theorems in this paper, we only consider special case \(\psi = \psi(x)\).

At first for \(q > 1\), since at \(x_0\),
\[
\psi^2 = v^{2q-2}u_\gamma^2 = (1 + |Du|^2)^{q-1}u_\gamma^2, \quad |Du|^2 = |D'u|^2 + u_\gamma^2,
\]
if we assume
\[
|D'u|^2 < u_\gamma^2, \quad (27)
\]
then
\[
|Du|^2 < (2|\psi|^2_{C^0(\partial\Omega)})^\frac{1}{q}. \quad (28)
\]
Thus we complete this proof. So in the following we assume
\[
|D'u|^2 \geq u_\gamma^2, \quad (29)
\]
then from \(|Du|^2 = |D'u|^2 + u_\gamma^2\), we have
\[
|Du|^2 \leq 2|D'u|^2. \quad (30)
\]
Now assume at \(x_0\),
\[
|Du| \geq \max\{10\sqrt{2}, (4q\sqrt{n}|\psi|_{C^0(\partial\Omega)})^{\frac{1}{q-1}}\}. \quad (31)
\]
Inserting (31) into (26), and by the choice of \(h(u), g(d)\) in (15), we obtain
\[
\frac{\partial \varphi}{\partial \gamma} \geq g' - \frac{2|u_i(\gamma^i)_k c^{kl}u_i|}{|D'u|^2 \log |D'u|^2} - \frac{2v^{1-q}|\psi_k|}{|D'u|^2 \log |D'u|^2} - 2|h'\psi|v^{1-q} + \frac{2|\gamma^i)_k u_i u_k|}{|D'u|^2 \log |D'u|^2} - \frac{2v^{1-q}|\psi|_{C^1(\overline{\Omega})}|D'u|}{|D'u|^2 \log |D'u|^2} - 2|h'\psi|v^{1-q} \quad (32)
\]
\[
\geq \frac{\alpha_0}{2} - 10 \sum_{i,k=1}^n |(\gamma^i)_k| - 10n|\psi|_{C^1(\overline{\Omega})} > 0.
\]
On the other hand, we have
\[
\frac{\partial \varphi}{\partial \gamma}(x_0) \leq 0,
\]
it is a contradiction to (32). So we have
\[
|Du|(x_0) \leq \max\{10\sqrt{2}, (4q\sqrt{n}|\psi|_{C^0(\partial\Omega)})^{\frac{1}{q-1}}\}. 
\]
Since

\[ n, \text{L} \]

constant depending only on \( \mu \) where \( \Omega \)

mentioned before, all the calculations will be done at the fixed point

From the above choices, we shall prove Theorem 1.1 with three steps, as we men-

where \( C \)

Then we have at

Assume \( \tilde{u} \)

From Gilbarg and Trudinger [4] [page 368, formula (15.38)], we have

Assume \( |D\tilde{u}|(x_0) = 0 \) for

In this case, \( x_0 \) is a critical point of \( \varphi \). Now we choose the normal coordinate at

Case III. \( x_0 \in \Omega_{\mu_0} \).

This is due to interior gradient estimates. From Remark 1, we have

\[ \sup_{\partial \Omega_{\mu_0} \cap \Omega} |Du| \leq \tilde{M}_1. \quad (33) \]

where \( \tilde{M}_1 \) is a positive constant depending only on \( n, M_0, \mu_0, L_1 \).

Case III. \( x_0 \in \Omega_{\mu_0} \).

From Gilbarg and Trudinger [4] [page 368, formula (15.38)], we have

\[ \sup_{\Omega} |Du|^2 \leq C_1 (1 + \sup_{\partial \Omega} |Du|^2), \quad (34) \]

where \( C_1 \) is a positive constant depending on \( n, L_1, M_0 \).

From Case I, suppose that the formula (29) holds, otherwise we finish the proof

of Theorem 1.1. Hence,

\[ \sup_{\Omega} |Du|^2 \leq C_1 [1 + 2 \sup_{\partial \Omega \cap \{|D'\tilde{u}| \geq 1\}} |D'\tilde{u}|^2] \leq C_2 \sup_{\partial \Omega \cap \{|D'\tilde{u}| \geq 1\}} |D'\tilde{u}|^2. \quad (35) \]

So we have

\[ \sup_{\Omega_{\mu_0}(M)} |D'\tilde{u}|^2 \leq C_3 \sup_{\Omega_{\mu_0}(M)} |D'\tilde{u}|^2, \quad (36) \]

where \( \Omega_{\mu_0}(M) = \Omega_{\mu_0} \cap \{|D'\tilde{u}| \geq M\}, M > 10 \) is a constant; \( C_3 \) is a positive

constant depending only on \( n, L_1, M_0 \).

Assume \( x_1 \in \Omega_{\mu_0}(M) \) such that

\[ \sup_{\Omega_{\mu_0}(M)} |D'\tilde{u}|^2 = |D'\tilde{u}|^2(x_1). \quad (37) \]

Since \( P(x_0) \geq P(x_1) \), then we have

\[ \log |D'\tilde{u}|^2(x_0) h(u(x_0)) g(d(x_0)) \geq \log |D'\tilde{u}|^2(x_1) h(u(x_1)) g(d(x_1)). \quad (38) \]

It follows that

\[ |D'\tilde{u}|^2(x_1) \leq C_4 |D'\tilde{u}|^2(x_0). \quad (39) \]

where \( C_4 \) is a positive constant depending on \( n, \mu_0, L_1, L_2, M_0 \). However

\[ \sup_{\Omega_{\mu_0}} |Du|^2 \leq C_3 \sup_{\Omega_{\mu_0}(M)} |D'\tilde{u}|^2 = C_3 |D'\tilde{u}|^2(x_1) \leq C_4 |D'\tilde{u}|^2(x_0). \quad (40) \]

Assume \( |D'\tilde{u}|(x_0) \geq M \), otherwise we get the estimate. Hence at \( x_0 \),

\[ u_1^2(x_0) = |Du|^2(x_0) \leq C_4 e^{11} u_1^2(x_0). \quad (41) \]

Then we have at \( x_0 \),

\[ e^{11} \geq \frac{1}{C_4} > 0. \quad (42) \]

From the above choices, we shall prove Theorem 1.1 with three steps, as we men-

Step 1: We first get the formula (68).
Taking the first derivatives of \( \varphi \),
\[
\varphi_i = \frac{(|D'u|^2)_i}{|D'u|^2 \log |D'u|^2} + h'u_i + g'^i.
\] (43)

From \( \varphi_i(x_0) = 0 \), we have
\[
(\varphi_i) = -|D'u|^2 \log |D'u|^2 (h'u_i + g'^i).
\] (44)

Taking the derivatives again for \( \varphi_i \), we have
\[
\varphi_{ij} = \frac{(|D'u|^2)_{ij}}{|D'u|^2 \log |D'u|^2} - (1 + \log |D'u|^2) \frac{(|D'u|^2)_i(|D'u|^2)_j}{(|D'u|^2 \log |D'u|^2)^2}
\]
\[
+ h'_{ij} + h''u_{ij} + g'' \gamma^i \gamma^j + g' (\gamma^i) j.
\] (45)

Using the formula (44), it follows that
\[
\varphi_{ij} = \frac{(|D'u|^2)_{ij}}{|D'u|^2 \log |D'u|^2} + h'_{ij} + [h'' - (1 + \log |D'u|^2) h'^2] u_{ij}
\]
\[
+ [g'' - (1 + \log |D'u|^2) g'^2] \gamma^i \gamma^j - (1 + \log |D'u|^2) h' g' (\gamma^i u_j + \gamma^j u_i)
+ g'(\gamma^i) j.
\] (46)

Then we get
\[
0 \geq a^{ij} \varphi_{ij} =: I_1 + I_2,
\] (47)

where
\[
I_1 = \frac{1}{|D'u|^2 \log |D'u|^2} a^{ij}(|D'u|^2)_{ij},
\] (48)

and
\[
I_2 = a^{ij} \left\{ h'_{ij} + [h'' - (1 + \log |D'u|^2) h'^2] u_{ij}
\right.
\]
\[
+ [g'' - (1 + \log |D'u|^2) g'^2] \gamma^i \gamma^j - 2(1 + \log |D'u|^2) h' g' (\gamma^i u_j + \gamma^j u_i)
+ g'(\gamma^i) j \}. \] (49)

From the choice of the coordinate, we have
\[
a^{11} = 1, a^{ii} = v^2 = 1 + u_2^2 \quad (2 \leq i \leq n), a^{ij} = 0 \quad (i \neq j, 1 \leq i, j \leq n),
\] (50)

and
\[
|D'u|^2 = c^{11} u_1^2, \quad |D'u|^2 \log |D'u|^2 = 2c^{11} u_1^2 \log u_1 + c^{11} (\log c^{11}) u_1^2. \]
(51)

Now we first treat the term \( I_2 \).

From the equations (13), (50), (15) and (51), we have
\[
I_2 = h' f v^3 - 2(h'^2 + c^{11} g'^2) u_1 \log u_1 - [(1 + \log c^{11}) h'^2 + c^{11} (1 + \log c^{11}) g'^2
\]
\[
- g' \sum_{2 \leq i \leq n} (\gamma^i) u_1^2 - 4h' g' \gamma^1 u_1 \log u_1 - 2(1 + \log c^{11}) h' g' \gamma^1 u_1
\]
\[
- 2g'^2 \log u_1 - (1 + \log c^{11}) g'^2 + g' \sum_{1 \leq i \leq n} (\gamma^i) u_1,
\] (52)

So we have from (15)
\[
I_2 \geq \sqrt{n} \alpha_0 f v^3 - 2(n + c^{11}) \alpha_0^2 u_1^2 \log u_1 - C_5 u_1^2,
\] (53)

here we use the expression for \( h(u), g(d) \) in (15), and \( C_5 \) is a positive constant depending only on \( n, \Omega, M_0, \mu_0, L_2 \).

Next, we calculate \( I_1 \) and get the formula (67).
From the equation (11), taking the first derivatives of $|D'u|^2$, we have
\[
(\frac{1}{D'u}^2)_{ij} = (e^{kl})_{ij} u_{kl} + 2 c^{kl} u_{ki} u_{lj}. \tag{54}
\]

Taking the derivatives of $|D'u|^2$ once more, we have
\[
(\frac{1}{D'u}^2)_{ij} = (e^{kl})_{ij} u_{kl} + 2 (e^{kl})_{ij} u_{ki} u_{lj} + 2 (e^{kl})_{ij} u_{ki} u_{lj} + 2 e^{kl} u_{ki} u_{lj}. \tag{55}
\]

By the equations (48) and (55), we can rewrite $I_1$ as
\[
I_1 = \frac{1}{|D'u|^2} \log |D'u|^2 \left[ I_{11} + I_{12} + I_{13} + I_{14} \right], \tag{56}
\]
where
\[
I_{11} = u_1^2 a_{ii} (c^{11})_{ii}, \quad I_{12} = 4 u_1 a_{ii} (e^{kl})_{i} u_{ki},
\]
\[
I_{13} = 2 u_1 e^{kl} a_{ij} u_{ij}, \quad I_{14} = 2 c^{kl} a_{ii} u_{kl}.
\]

In the following, we shall deal with $I_{11}, I_{12}, I_{13}$ and $I_{14}$ respectively.

For the terms $I_{11}$ and $I_{12}$: from (50), we have
\[
I_{11} = \sum_{2 \leq i \leq n} (c^{11})_{ii} u_{i1}^4 + \sum_{1 \leq i \leq n} (c^{11})_{ii} u_{i1}^2, \tag{57}
\]
\[
I_{12} = 4 (c^{11})_{i} u_{11} u_{i1} + 4 u_1 \sum_{2 \leq i \leq n} [(c^{11})_{ii} + v^2 (c^{11})_{ii}] u_{i1} + 4 u_1 v^2 \sum_{2 \leq i \leq n} (c^{11})_{ii} u_{i1}. \tag{58}
\]

For the term $I_{13}$: by the equation (13), we have
\[
u_{11} = f v^3 - v^2 \sum_{2 \leq i \leq n} u_{ii}, \tag{59}
\]
and
\[
\Delta u = f v + \frac{u_1^2}{v^2} u_{11}. \tag{60}
\]

Differentiating the equation (13), we have
\[
a^{ij}_{ijkl} = - a^{ij}_{k1} u_{ik} u_{lj} + v^3 D_k f + 3 f v^2 v_k. \tag{61}
\]

From the equation (12), we have
\[
a^{ij}_{pi1} = 2 u_i \delta_{ij} - \delta_{ii} u_{ij} - \delta_{jl} u_{li}. \tag{62}
\]

By the definition of $v$, we have
\[
v u_k = u_{11} u_{1k}. \tag{63}
\]

Hence, from the equations (60)-(63), we have
\[
a^{ij}_{ijkl} = \frac{2 u_1}{v^2} u_{11} u_{1k} + 2 u_1 \sum_{2 \leq i \leq n} u_{i1} u_{ik} + v^3 D_k f + f v u_1 u_{1k}. \tag{64}
\]

By the equation (64), we get
\[
I_{13} = \frac{4 u_1^2}{v^2} u_{11} e^{kl} u_{1k} + 4 u_1^2 \sum_{2 \leq i \leq n} u_{i1} e^{kl} u_{ki} + 2 f u_1^2 v c^{kl} u_{1k} + 2 u_1 v^3 c^{kl} D_k f. \tag{65}
\]
For the term $I_{14}$:

$$I_{14} = 2u_{11}c^{k_1}u_{k_1} + 2v^2 \sum_{2 \leq i \leq n} u_{1i}c^{k_1}u_{ki} + 2v^2 \sum_{2 \leq i \leq n} c^{1i}u_{1i}u_{ii}$$

$$+ 2u_{11} \sum_{2 \leq i \leq n} c^{1i}u_{1i} + 2 \sum_{2 \leq i,j \leq n} c^{ij}u_{1i}u_{1j} + 2v^2 \sum_{2 \leq i \leq n} c^{ii}u_{ii}^2.$$  \hfill (66)

Combining the equations (57), (58), (65) and (66), it follows that

$$I_1 = \frac{1}{|Du|^2 \log |Du|^2} \left[ \frac{4u_1^2}{v^2} + 2 \right] u_{11}c^{k_1}u_{k_1} + (4u_1^2 + 2v^2) \sum_{2 \leq i \leq n} u_{1i}c^{k_1}u_{ki}$$

$$+ 2v^2 \sum_{2 \leq i \leq n} c^{1i}u_{1i}u_{ii} + 2u_{11} \sum_{2 \leq i \leq n} c^{1i}u_{1i} + 2 \sum_{2 \leq i,j \leq n} c^{ij}u_{1i}u_{1j}$$

$$+ 2fvu_1c^{k_1}u_{1k} + 4(c^{11})_1u_{11}u_{11} + 4u_1 \sum_{2 \leq i \leq n} [(c^{1i})_1 + v^2(c^{1i})_1]u_{1i}$$

$$+ 2v^2 \sum_{2 \leq i \leq n} c^{ii}u_{ii}^2 + 4u_1v^2 \sum_{2 \leq i \leq n} (c^{1i})_1u_{1i} + 2u_1v^3c^{k_1}D_kf$$

$$+ \sum_{2 \leq i \leq n} (c^{11})_{ii}u_{1i}^4 + \sum_{1 \leq i \leq n} (c^{11})_{ii}u_{ii}^2 \right].$$  \hfill (67)

Inserting the equations (52) and (67) into (48), we can obtain the following formula

$$0 \geq a^{ij}\varphi_{ij} =: Q_1 + Q_2 + Q_3,$$  \hfill (68)

where $Q_1$ contains all the quadratic terms of $u_{ij}$; $Q_2$ is the term which contains all linear terms of $u_{ij}$; and the remaining terms are denoted by $Q_3$. Then we have

$$Q_1 = \frac{1}{|Du|^2 \log |Du|^2} \left[ \frac{4u_1^2}{v^2} + 2 \right] u_{11}c^{k_1}u_{k_1} + (4u_1^2 + 2v^2) \sum_{2 \leq i \leq n} u_{1i}c^{k_1}u_{ki}$$

$$+ 2v^2 \sum_{2 \leq i \leq n} c^{1i}u_{1i}u_{ii} + 2u_{11} \sum_{2 \leq i \leq n} c^{1i}u_{1i} + 2 \sum_{2 \leq i,j \leq n} c^{ij}u_{1i}u_{1j}$$

$$+ 2v^2 \sum_{2 \leq i \leq n} c^{ii}u_{ii}^2 \right].$$  \hfill (69)

The linear terms of $u_{ij}$ are

$$Q_2 = \frac{1}{|Du|^2 \log |Du|^2} \left[ 2fvu_1c^{k_1}u_{1k} + 4(c^{11})_1u_{11}u_{11} + 4u_1 \sum_{2 \leq i \leq n} (c^{1i})_1u_{1i}$$

$$+ 4u_1v^2 \sum_{2 \leq i \leq n} (c^{11})_{ii}u_{1i} + 4u_1v^2 \sum_{2 \leq i \leq n} (c^{11})_{ii}u_{ii} \right]$$

and the remaining terms are

$$Q_3 = I_2 + \frac{1}{|Du|^2 \log |Du|^2} \left[ \sum_{2 \leq i \leq n} (c^{11})_{ii}u_{1i}^4 + \sum_{1 \leq i \leq n} (c^{11})_{ii}u_{ii}^2 + 2u_1v^3c^{k_1}D_kf \right].$$  \hfill (70)

From the estimate on $I_2$ in (53), we have

$$Q_3 \geq \sqrt{n} \alpha_0 f v^3 - 2(n + c^{11})\alpha_0^2 u_1^2 \log u_1 - C_0 u_1^2,$$  \hfill (72)
in the computation of $Q_3$, we use the relation $D_k f = f_{u_k} + f_{x_k}$ and $f_u \geq 0$, where $C_6$ is a positive constant which depends only on $n, \Omega, M_0, \mu_0, L_1, L_2$.

**Step 2:** In this step we shall treat the terms $Q_1, Q_2$ using the first order derivative condition $\varphi_1(x_0) = 0$, and let

$$A = |D'u|^2 \log |D'u|^2.$$  

(73)

By the equations (44) and (54), we have

$$c^{k_1} u_{k_1} = -\frac{h'}{2} u_1 A - \frac{g' \gamma^1}{2} A - \frac{(c^{11})_1}{2} u_1, \quad i = 1, 2, \ldots, n.$$  

(74)

Using the equation (74), we get

$$c^{k_1} u_{k_1} = -\frac{g' \gamma^1}{2} A - \frac{(c^{11})_1}{2} u_1,$$  

(75)

and

$$c^{k_1} u_{k_1} = -\frac{g' \gamma^i}{2} A - \frac{(c^{11})_1}{2} u_1, \quad i = 2, 3, \ldots, n.$$  

(76)

Through the equation (76) and the choice of the coordinate at $x_0$, we have

$$u_{1i} = -\frac{c^{11}}{c^{11}} u_{ii} - \frac{g' \gamma^1}{2c^{11}} A - \frac{(c^{11})_1}{2c^{11}} u_1, \quad i = 2, 3, \ldots, n.$$  

(77)

Using the equations (75) and (77), it follows that

$$u_{11} = \sum_{2\leq i\leq n} \frac{(c^{11})^2}{c^{11}} u_{ii} - \frac{h'}{2c^{11}} A - \frac{g' \gamma^1}{c^{11}} u_1 + b u_1,$$  

(78)

where we have let

$$b = \frac{1}{2(c^{11})^2} \sum_{2\leq i\leq n} c^i (c^{11})_i - \frac{(c^{11})_1}{2c^{11}}.$$  

By the equations (59) and (78), we have

$$\sum_{2\leq i\leq n} \left[ (c^{11})^2 v^2 + (c^{11})^2 \right] u_{ii} = (c^{11})^2 f v^3 + \frac{c^{11} h'}{2} A + c^{11} g' \gamma^1 A - (c^{11})^2 b u_1.$$  

(79)

Now we use the formulas (75)-(78) to treat each term in $Q_1, Q_2$. At first, we treat the first five terms of $Q_1$ in (69), and get (80)-(84).

By the equations (75) and (78), we have

$$\left( \frac{4u^2}{v^2} + 2 \right) u_{11} c^{k_1} u_{k_1} = -\left( \frac{2u^2}{v^2} + 1 \right) \left[ h'A + g' \gamma^1 A \frac{u_1}{u_1} + (c^{11})_1 u_1 \right] \sum_{2\leq i\leq n} \frac{(c^{11})^2}{c^{11}} u_{ii}$$  

$$+ \left( \frac{u^2}{v^2} + 1 \right) \frac{h'^2}{c^{11}} A^2 + \left( \frac{u^2}{v^2} + 1 \right) \frac{h' g' \gamma^1}{c^{11}} A^2$$  

$$+ \frac{3v^2 - 2}{2v^2} \left[ (c^{11})_1 - 2b \right] h'u_1 A + \frac{3v^2 - 2}{v^2} \frac{g' \gamma^1}{c^{11}} A^2$$  

$$+ \frac{2u^2}{v^2} + 1) g' \gamma^1 \frac{(c^{11})_1}{c^{11}} - b |A| - \left( \frac{2u^2}{v^2} + 1 \right) (c^{11})_1 b u_1^2.$$  

(80)
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From the equations (76) and (77), we get

\[
(4u_1^2 + 2v^2) \sum_{2 \leq i \leq n} u_{1i} c^{1i} u_{ii} = (2u_1^2 + v^2) \frac{A}{u_1} g' \sum_{2 \leq i \leq n} c^{1i} \gamma^i u_{ii} \\
+ (2u_1^2 + v^2) u_1 \left( \frac{g'}{c^{1i}} (u_1^2) \right) u_{ii} + \frac{3g'^2}{2} A^2 \\
+ \frac{g'}{c^{1i}} \sum_{2 \leq i \leq n} (c^{1i}) \gamma^i (2u_1^2 + v^2) A \\
+ \frac{(2u_1^2 + v^2)}{2c^{1i}} \left( (c^{1i})^2 + \frac{g'^2 A^2}{2u_1^2} \right).
\]  

From the equation (77), we have

\[
2v^2 \sum_{2 \leq i \leq n} c^{1i} u_{1i} u_{ii} = -\frac{2v^2}{c^{1i}} \sum_{2 \leq i \leq n} (c^{1i} u_{ii})^2 - \frac{g'}{c^{1i}} v^2 A \sum_{2 \leq i \leq n} c^{1i} \gamma^i u_{ii} \\
- \frac{\gamma^i u_{1i}^2}{c^{1i}} \sum_{2 \leq i \leq n} c^{1i} (c^{1i})_i u_{ii}.
\]  

By the equations (77) and (78), it follows that

\[
2u_1 \sum_{2 \leq i \leq n} c^{1i} u_{1i} = -\frac{2}{(c^{1i})^2} \left( \sum_{2 \leq i \leq n} (c^{1i})^2 u_{ii} \right)^2 \\
+ \left[ h' A + 3g' \gamma^1 \frac{A}{u_1} - 4c^{11} bu_1 - (c^{11})_1 u_1 \right] \sum_{2 \leq i \leq n} \frac{(c^{1i})^2}{(c^{11})^2} u_{ii} \\
- \frac{h' g' \gamma^1 A^2}{2u_1} + \frac{h'}{2(c^{11})^2} \sum_{2 \leq i \leq n} c^{1i} (c^{1i})_i A u_1 - \frac{g'^2 (\gamma^1)^2 A^2}{c^{11}} u_1^2 \\
+ \frac{g' \gamma^1 \left[ 3b + \frac{(c^{11})_1}{c^{11}} \right] A - b[2bc^{11} + (c^{11})_1] u_1^2}. \tag{83}
\]  

Again by the equations (77) and (10), we get

\[
2 \sum_{2 \leq i \leq n} c^{ij} u_{1i} u_{1j} = \frac{2}{(c^{11})^2} \sum_{2 \leq i, j \leq n} c^{ij} c^{1i} c^{1j} u_{ii} u_{jj} \\
- \left[ \frac{2g' (\gamma^1)^3 A}{(c^{11})^2} \frac{2\gamma^1 u_1}{(c^{11})^2} \sum_{2 \leq j \leq n} \gamma^j (c^{11})_j \right] \sum_{2 \leq i \leq n} \gamma^i u_{ii} \\
+ \frac{2u_1}{(c^{11})^2} \sum_{2 \leq i \leq n} c^{1i} (c^{1i})_i u_{ii} + \frac{1 - c^{11} g' A}{(c^{11})^2} \sum_{2 \leq i \leq n} \gamma^i (c^{11})_i \\
+ \frac{(1 - c^{11}) g'^2 A^2}{2u_1^2} + \frac{1}{2(c^{11})^2} \sum_{2 \leq i, j \leq n} c^{ij} (c^{1i})_i (c^{1j})_j u_{1i}^2. \tag{84}
\]  

Now we treat the first four terms of \(Q_2\) in (70), and get (85)-(88).

From the equation (75), we have

\[
2f v u_1^2 c^{k1} u_{1k} = -h' f A v u_1^2 - f g' \gamma^1 A v u_1 - (c^{11})_1 f v u_1^3. \tag{85}
\]
By the equation (78), we obtain
\[
4(c^{(11)})_1 u_1 u_{11} = 4(c^{(11)})_1 u_1 \sum_{2 \leq i \leq n} \frac{(c^{(i)})^2}{(c^{(11)})^2} u_{ii} - \frac{2(c^{(11)})_1}{c^{(11)}} h'A u_1 \\
- \frac{4g' \gamma^1}{c^{(11)}} (c^{(11)})_1 A + 4(c^{(11)})_1 b u_1^2.
\] (86)

From the equation (77), we have
\[
4u_1 \sum_{2 \leq i \leq n} (c^{(i)})_1 u_{1i} = -4u_1 \sum_{2 \leq i \leq n} (c^{(i)})_1 c^{(i)} u_{ii} - \frac{2g'}{c^{(11)}} \sum_{2 \leq i \leq n} (c^{(i)})_1 \gamma^i A u_i \\
- \frac{2}{c^{(11)}} \sum_{2 \leq i \leq n} (c^{(i)})_1 (c^{(1)})_i u_i^2,
\] (87)

and
\[
4u_1 v^2 \sum_{2 \leq i \leq n} (c^{(i)})_i u_{i1} = -4u_1 v^2 \sum_{2 \leq i \leq n} (c^{(i)})_i c^{(i)} u_{ii} - \frac{2g'}{c^{(11)}} \sum_{2 \leq i \leq n} (c^{(i)})_i \gamma^i A u_i \\
- \frac{2}{c^{(11)}} \sum_{2 \leq i \leq n} ((c^{(i)})_i)^2 u_i^2 u_i^2.
\] (88)

We treat the term $Q_1$ using the relations (80)-(84), and use the formulas (85)-(88) to treat the term $Q_2$. By the formula on $Q_3$ in (71), we can rewrite the formula (68) as the following.
\[
0 \geq a^{ij} \varphi_{ij} =: J_1 + J_2,
\] (89)

where $J_1$ only contains the terms with $u_{ii}$, the other terms belong to $J_2$. Denote
\[
J_1 =: \frac{1}{A} [J_{11} + J_{12}],
\] (90)

here $J_{11}$ contains the quadratic terms of $u_{ii}$ ($i \geq 2$), and $J_{12}$ is the term including linear terms of $u_{ii}$ ($i \geq 2$). It follows that
\[
J_{11} = 2v^2 \sum_{2 \leq i \leq n} c^{(i)} u_i^2 - \frac{2v^2}{c^{(11)}} \sum_{2 \leq i \leq n} (c^{(i)} u_{ii})^2 - \frac{2}{(c^{(11)})^3} \sum_{2 \leq i \leq n} (c^{(i)} u_{ii})^2 \\
+ \frac{2}{(c^{(11)})^2} \sum_{2 \leq i, j \leq n} c^{(i)} c^{(j)} u_{ij} u_{jj}\\n= \frac{2}{(c^{(11)})^3} \sum_{2 \leq i \leq n} d_i c_i u_i^2 + 2 \sum_{2 \leq i < j \leq n} \epsilon^{ij} c^{(i)} c^{(j)} u_{ij} u_{jj},
\] (91)

where
\[
d_i = (c^{(i)})^2 v^2 + (c^{(i)})^2 = (c^{(i)})^2 u_i^2 + (c^{(i)})^2 + (c^{(i)})^2, \quad i = 2, 3, \ldots, n,
\] (92)
\[
\epsilon_i = c^{(i)} c^{(i)} - (c^{(i)})^2 = 1 - (\gamma^1)^2 - (\gamma^1)^2, \quad i = 2, 3, \ldots, n.
\] (93)
In addition,

\[ J_{12} = \left[ \frac{2\gamma^3}{(c_{11})^2} \sum_{2 \leq i \leq n} c_{ij}(c_{11})_j u_i - 2(\gamma^1)^2 h' A u_i^2 \frac{v}{u_i} - 2g'(\gamma^1)^3 A u_i \frac{v}{(c_{11})^2} u_i - 2g'\gamma^1 A u_i \right. \\
\left. - 4b(\gamma^1)^2 \frac{u_i}{(c_{11})^2} u_i + \frac{2g'(\gamma^1)^3}{(c_{11})^2} \frac{A}{v^2 u_i} + 2(\gamma^1)^2 (c_{11})_1 u_i \frac{v}{(c_{11})^2} u_i \right] \sum_{2 \leq i \leq n} (\gamma^i)^2 u_{ii} \\
+ 4u_1^2 \sum_{2 \leq i \leq n} (c_{ij})_i u_{ii} - 4u_1 \sum_{2 \leq i \leq n} c_{ij}(c_{11})_j u_{ii} \\
- \left[ \frac{2u_1^3}{c_{11}} + \frac{4(\gamma^1)^2 - 2}{(c_{11})^2 - u_1} \sum_{2 \leq i \leq n} c_{ij}(c_{11})_j u_{ii} \right]. \tag{94} \]

We write other terms as \( J_2 \), then

\[
J_2 = Q_3 - h' f v u_1^2 + (\frac{u_1^2}{v^2} + \frac{1}{2}) \frac{h'^2}{c_{11}} A + \frac{3g'^2}{2} h - h' \gamma^1 v u_1 - (c_{11})_1 f v u_1^3 \frac{v}{A} \\
+ \frac{g'}{c_{11}} \sum_{2 \leq i \leq n} (c_{11})_i (u_i^2 - 1) - \frac{1}{c_{11}} \sum_{2 \leq i \leq n} (c_{ij})(u_i^2 + 2) u_i^2 \frac{A}{A} \\
+ 3u_1^2 + 1) \frac{h' h}{c_{11}} \frac{A}{u_1} - 2(c_{11})_1 \frac{h' u_1 + (u_1^2 + 2) (c_{11})_1}{c_{11}} A - 2b \frac{h' u_1}{c_{11}} \\
+ \frac{h'}{2(c_{11})^2} \sum_{2 \leq i \leq n} c_{ij}(c_{11})_i u_i - \frac{g'^2(\gamma^1)^2 A}{u_1^2} + (u_1^2 + 1) \frac{g'^2(\gamma^1)^2 A}{c_{11}} u_1^2 \\
+ \frac{g'^2 A}{2} u_1^2 + \frac{1}{2c_{11}^2} \frac{(c_{11})_1}{u_1^2} + g'^1 \frac{3b + (c_{11})_1}{c_{11}} - \frac{2g'}{c_{11}} \sum_{2 \leq i \leq n} (c_{11})_i \gamma^i \tag{95} \\
+ \frac{1}{(c_{11})^2} \sum_{2 \leq i \leq n} \gamma^1 (c_{11})_i + (\frac{u_1^2}{v^2} + 1) g'^1 \frac{(c_{11})_1}{c_{11}} - b \\
- \frac{2u_1^2}{v^2} + 1) (c_{11})_i (u_i^2 - 1) \frac{b}{A} - b \frac{2b c_{11} + (c_{11})_1 u_i^2}{A} - \frac{4g'^1}{c_{11}} (c_{11})_i \\
+ \frac{1}{2(c_{11})^2} \sum_{2 \leq i \leq n} c_{ij}(c_{11})_i (u_i^2 - 1) \frac{u_i^2}{A} + 4(c_{11})_1 b \frac{u_i^2}{A} \\
- \frac{2}{c_{11}} \sum_{2 \leq i \leq n} (c_{11})_i (u_i^2 - 1) \frac{u_i^2}{A}. \]

Using the formula on \( Q_3 \) in (72) and \( I_2 \) in (53), we get the following estimate on \( J_2 \),

\[
J_2 \geq -2(h'^2 + 2c_{11} g'^2) u_1^2 \log u_1 + h' f v + \frac{3h'^2}{2} \frac{A}{c_{11}} A + \frac{3g'^2}{2} A - C_7 u_1^2 \tag{96} \geq [h'^2 + c_{11} g'^2] u_1^2 \log u_1 - C_8 u_1^2. \]

So if we use \( h(u), g(d) \) in (15), then we have

\[ J_2 \geq (n + c_{11}) a_0 u_1^2 \log u_1 - C_8 u_1^2, \tag{97} \]

where \( C_7, C_8 \) and the following \( C_9, \ldots, C_{15} \) are positive constants which only depend on \( n, \Omega, \mu_0, M_0, L_1, L_2 \).
Step 3: In this step, we concentrate on $J_1$. We first treat the terms $J_{11}$ and $J_{12}$ and obtain the formula (104), then we complete the proof of Theorem 1.1 through Lemma 3.5.

By the equation (79), we have

$$u_{22} = : \frac{-1}{d_2} \sum_{3 \leq i \leq n} d_i u_{ii} + \frac{D}{d_2},$$  \hspace{1cm} (98)

where we have let

$$D = (c^{11})^2 f v^3 + \frac{c^{11} h'}{2} A + c^{11} g' \gamma^1 A u_1 - (c^{11})^2 b u_1.$$  \hspace{1cm} (99)

We first treat the term $J_{11}$: using the equation (99) to simplify (91), we get

$$J_{11} = \frac{2}{(c^{11})^3 d^2} \left[ \sum_{3 \leq i \leq n} b_{ii} u_{ii}^2 + 2 \sum_{3 \leq i < j \leq n} b_{ij} u_{ii} u_{jj} - 2c_2 D \sum_{3 \leq i \leq n} d_i u_{ii} - 2(c^2)^2 D \sum_{3 \leq i \leq n} (c^{11})^2 u_{ii} + c_2 D^2 \right],$$  \hspace{1cm} (100)

where

$$b_{ii} = c_2 d_i^2 + e_i d_i d_j - 2c^{12} c^{11} c^{11} d_i = (c^{11})^4 (e_i + e_j) v^4 + c_1 v^2 + A_{2i}, \quad i \geq 3$$

$$b_{ij} = c_2 d_i d_j + d_2 c^i c^j - 2c^{12} c^{11} c^{11} d_j = -c_1 c^j c^{11} c^{11} d_i$$

and

$$A_{1i} = (c^{11})^2 [(c^{11})^2 (e_i + e_j) + c^{11} ((c^{11})^2 + (c^{12})^2)],$$

$$A_{2i} = c^{11} (c^{11})^2 [(c^{11})^2 + (c^{12})^2],$$

$$G_{ij} = c^{11} ((c^{11})^2 + (c^{11})^2) + c^j c^{11} c^{11},$$

$$G_{ij} = c^{11} (c^{11})^2 (c^{11})^2.$$  \hspace{1cm} (102)

Now we simplify the terms in $J_{12}$: by the equation (98), we can rewrite (94) as

$$J_{12} = \left[ \frac{2 \gamma^1}{(c^{11})^2} \sum_{2 \leq j \leq n} c^{11} (c^{11})_j u_{11} - 2 \frac{g' \gamma^1}{c^{11}} A u_{11} - 2 \frac{(\gamma^1)^2 h'}{c^{11} v^2} A u_{11}^2 - 4b \frac{(\gamma^1)^2}{c^{11}} u_{11} \right]$$

$$- \left[ \frac{2 \frac{\gamma^1}{(c^{11})^2} u_{11}}{c^{11}} - \frac{2 \gamma^1}{(c^{11})^2} A u_{11} - \frac{(\gamma^1)^2 h'}{c^{11} v^2} A u_{11}^2 - 4b \frac{(\gamma^1)^2}{c^{11}} u_{11} \right]$$

$$+ \left[ \frac{2 \gamma^1}{(c^{11})^2} \sum_{2 \leq j \leq n} c^{11} (c^{11})_j u_{11} - 2 \frac{g' \gamma^1}{c^{11}} A u_{11} - 2 \frac{(\gamma^1)^2 h'}{c^{11} v^2} A u_{11}^2 - 4b \frac{(\gamma^1)^2}{c^{11}} u_{11} \right]$$

$$- \left[ \frac{2 \gamma^1}{(c^{11})^2} \sum_{2 \leq j \leq n} c^{11} (c^{11})_j u_{11} - 2 \frac{g' \gamma^1}{c^{11}} A u_{11} - 2 \frac{(\gamma^1)^2 h'}{c^{11} v^2} A u_{11}^2 - 4b \frac{(\gamma^1)^2}{c^{11}} u_{11} \right]$$

$$- 4c^{12} (c^{11})^2 u_{11} D \left[ \frac{2 u_{11}^3}{c^{11}} + c^{12} (c^{11})^2 \frac{4c^{11} - 2}{c^{11} u_{11}} A u_{11}^2 \right] \frac{D}{d_2}.$$
Using the equations (100) and (103) to treat (90), we have

\[ J_1 = \frac{2}{A d_2 (c_{11})^3} \left[ \sum_{3 \leq i \leq n} b_{ii} u_{ii}^2 + 2 \sum_{3 \leq i < j \leq n} b_{ij} u_{ii} u_{jj} - u_1^3 \log u_1 \sum_{3 \leq i \leq n} b_{ii} u_{ii} + \sum_{3 \leq i \leq n} K_i u_{ii} \right] + R, \]

where

\[ b_i = 2(c_{11})^5 g' \gamma^1 (e_2 - e_i), \quad i \geq 3, \]

and

\[ K_i = -2 c_2 D d_i - 2(\gamma^2)^2 D (c_{11})^2 - g' \gamma^1 (c_{11})^3 \log c_{11} u_1^5 + c_{11} A u_1 \]

\[ \cdot \left[ \gamma_{ij} (c_{11})^2 u_{ij} - \frac{(\gamma^1)^2 h' c_{11}}{v^2} u_1^2 - \frac{g' \gamma^1 (c_{11})^3}{v^2} \frac{d_i}{d_2} - \frac{2 b (c_{11})^2}{v^2} u_1 \right] \]

\[ + \frac{(\gamma^1)^2 c_{11}^3}{A} \left[ \gamma_{ii} (c_{11})^2 u_{ii} - \frac{2 b (c_{11})^2}{v^2} u_1 \right] \]

\[ - 2 u_1 v^2 (c_{11})^3 d_i (c_{12})^2 - \frac{4 u_1 (c_{11})^3}{c_{11}^2} d_2 c_{11}^2 (c_{11})^3 - \frac{d_i}{d_2} c_{12} (c_{11})^2 \]

\[ - \frac{(c_{11})^3}{2} d_2 \left[ \frac{2 u_1^2}{c_{11}^2} + \frac{4 c_{11} - 2}{(c_{11})^2} u_1 \right] \left[ c_{11} (c_{11})^2 \right] - \frac{d_i}{d_2} c_{12} (c_{11})^2 \]

we also have let

\[ R = \frac{2 c_2 D^2}{(c_{11})^3 d_{2A}} + \frac{2 c_{11}^2}{(c_{11})^2} \sum_{2 \leq j \leq n} c_{ij} (c_{11})^2 \frac{u_{ij}}{A} - \frac{2 b (c_{11})^2}{v^2} \]

\[ - \frac{2 b (c_{11})^2}{v^2} u_1 \frac{u_1}{c_{11}^2} + \frac{2 (\gamma^1)^2 h' (c_{11})^3}{v^2} \frac{d_i}{d_2} \]

\[ + \left[ \frac{2 (c_{11})^3}{c_{11}^2} \frac{2 u_1^2}{c_{11}^2} - \frac{4 c_{11}^2 (c_{11})^3}{c_{11}^2} \frac{u_1}{c_{11}^2} \right] \frac{(\gamma^1)^2 D}{d_2} \]

For \( K_i \) and \( R \), using the formulas on \( D \) in (99); the formula of \( A \) in (73); \( e_i ; d_i \) in (92)-(93), and \( h(u), g(d) \) in (15), we have the following estimates

\[ K_i \leq C_9 u_1^2, \]

\[ R \leq C_{10} u_1^2. \]

Now we use Lemma 3.5, if there is a sufficiently large positive constant \( C_{11} \) such that

\[ \| Du \|_{L^1(x_0)} \geq C_{11}, \]

then we have

\[ J_1 \geq \left( \frac{2}{A d_2 (c_{11})^3} \right) \left[ - (n - 2) (c_{11})^7 g'^2 (\gamma')^2 u_1^6 \log^2 u_1 - C_{12} u_1^6 \log u_1 + C_{10} u_1^2 \right] \]

\[ \geq - (n - 2) c_{11} (1 - c_{11})^2 g'^2 u_1^2 \log u_1 - C_{13} u_1^2, \]

where we use the formulas \((\gamma')^2 = 1 - c_{11}, d_2 \) in (92) and \( A \) in (73).

Using the estimates on \( J_1 \) in (110) and \( J_2 \) in (96), from (89) we obtain

\[ 0 \geq a^{ij} \varphi_{ij} \geq \left\{ k^2 + \left[ (c_{11})^2 (n - 2) - c_{11} (n - 3) \right] g'^2 \right\} u_1^2 \log u_1 - C_{14} u_1^2. \]
By the choice of \( h(u), g(d) \) in (15), it follows that
\[
0 \geq a^{ij} \varphi_{ij} \\
\geq \{ n + [(c^{11})^2(n - 2) - c^{11}(n - 3)]a_0^2 \} u_1^2 \log u_1 - C_{14} u_1^2 \\
\geq 3u_1^2 \log u_1 - C_{14} u_1^2.
\] (112)

By (42), (109) and (112), there exists a positive constant \( C_{15} \) such that
\[
|D' u(x_0) | \leq C_{15}.
\] (113)

So from Case I, Case II, and (113), we have
\[
|D' u(x_0) | \leq C_{16}, \quad x_0 \in \Omega_{\mu_0} \cup \partial \Omega.
\]

Since \( \varphi(x) \leq \varphi(x_0) \), for \( \forall x \in \Omega_{\mu_0} \), there exists \( M_2 \) such that
\[
|Du(x) | \leq M_2, \quad \text{in} \quad \Omega_{\mu_0} \cup \partial \Omega,
\] (114)

where \( M_2 \) depends only on \( n, \Omega, \mu_0, M_0, L_1, L_2 \).

So at last we get the following estimate
\[
\sup_{\Omega_{\mu_0}} |Du| \leq \max{\{M_1, M_2\}},
\]

where the positive constant \( M_1 \) depends only on \( n, \mu_0, M_0, L_1 \); and \( M_2 \) depends only on \( n, \Omega, \mu_0, M_0, L_1, L_2 \). So we complete the proof of Theorem 1.1. \( \square \)

3. Some lemmas. In this section, we prove the main Lemma 3.5 which has been used to get the important estimate (110).

Firstly, we give the definition and some properties of elementary symmetric functions. More details can be seen in Caffarelli-Nirenberg-Spruck [1] or Guan-Ma [3].

**Definition 3.1.** For \( 1 \leq k \leq n \) and \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \), define the \( k \)-order elementary symmetric function as follows:
\[
\sigma_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}.
\]

Moreover, define \( \sigma_0 = 1 \); when \( k > n \) or \( k < 0 \), \( \sigma_k = 0 \).

Denote by \( \sigma_k(\lambda_{\mid i}) = \sigma_k(\lambda)|_{\lambda_i = 0} \) and \( \sigma_k(\lambda_{\mid i}) = \sigma_k(\lambda)|_{\lambda_i = \lambda_j = 0} \).

**Definition 3.2.** Garding cone is defined by
\[
\Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_i > 0, \ \forall 1 \leq i \leq k \} \quad (115)
\]

**Proposition 1.** Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n, \ k = 0, 1, \ldots, n \). Then
\[
\sigma_k(\lambda) = \sigma_k(\lambda_{\mid i}) + \lambda_i \sigma_{k-1}(\lambda_{\mid i}), \quad \forall 1 \leq i \leq n,
\] (116)
\[
\sum_{i=1}^{n} \lambda_i \sigma_{k-1}(\lambda_{\mid i}) = k \sigma_k(\lambda),
\] (117)
\[
\sum_{i=1}^{n} \sigma_k(\lambda_{\mid i}) = (n - k) \sigma_k(\lambda).
\] (118)

**Proposition 2** (MacLaurin inequality). For \( \forall \lambda \in \Gamma_k, \ k \geq l \geq 1 \), then
\[
\left[ \frac{\sigma_k(\lambda)}{C_n^k} \right]^\frac{1}{k} \leq \left[ \frac{\sigma_l(\lambda)}{C_n^l} \right]^\frac{1}{l}.
\] (119)
Proposition 3 (Newton-MacLaurin inequality). For $\forall \lambda \in \Gamma_k$, $k > l \geq 1$, then

$$\frac{\sigma_k(\lambda)/C^k_n}{\sigma_{k-1}(\lambda)/C^{k-1}_n} \leq \frac{\sigma_l(\lambda)/C^l_n}{\sigma_{l-1}(\lambda)/C^{l-1}_n}. \quad (120)$$

In the following, we first prove a simple lemma on elementary symmetric functions for $e = (e_2, e_3, \ldots, e_n) \in \mathbb{R}^{n-1}$, where $e_i$ $(i \geq 2)$ is defined in (93). This lemma will be used in the proof of Lemma 3.5.

**Lemma 3.3.** Assume $e = (e_2, e_3, \ldots, e_n)$, then for $i \geq 3$, we have

$$\sigma_{n-3}(e|i)(e_2 - e_i) - \sum_{k \neq i, k \geq 3} \sigma_{n-3}(e|ik)(e_2 - e_k) = (n - 1)\sigma_{n-2}(e|i) - \sigma_{n-2}(e). \quad (121)$$

**Proof of Lemma 3.3.** When $i \geq 3$, using the formulas (116), (117) and (118) in Proposition 1, we obtain

$$\sum_{k \neq i, k \geq 3} \sigma_{n-3}(e|ik) = e_2 \sum_{k \neq i, k \geq 3} \sigma_{n-4}(e|2ik) = e_2\sigma_{n-4}(e|i2) = \sigma_{n-3}(e|i) - \sigma_{n-3}(e|2i). \quad (122)$$

By the formulas (116) and (117), we have

$$\sum_{k \neq i, k \geq 3} e_k \sigma_{n-3}(e|ik) = e_2 \sum_{k \neq i, k \geq 3} e_k \sigma_{n-4}(e|2ik) = (n - 3)e_2\sigma_{n-3}(e|2i) = (n - 3)\sigma_{n-2}(e|i). \quad (123)$$

Hence from the formulas (122) and (123), we have

$$\sigma_{n-3}(e|i)(e_2 - e_i) - \sum_{k \neq i, k \geq 3} \sigma_{n-3}(e|ik)(e_2 - e_k)$$

$$= e_2\sigma_{n-3}(e|i) - e_i\sigma_{n-3}(e|i) - e_2 \sum_{k \neq i, k \geq 3} \sigma_{n-3}(e|ik) + \sum_{k \neq i, k \geq 3} e_k \sigma_{n-3}(e|ik)$$

$$= e_2\sigma_{n-3}(e|i) - e_i\sigma_{n-3}(e|i) - e_2[\sigma_{n-3}(e|i) - \sigma_{n-3}(e|2i)] + (n - 3)\sigma_{n-2}(e|i)$$

$$= (n - 2)\sigma_{n-2}(e|i) - e_i\sigma_{n-3}(e|i)$$

$$= (n - 2)\sigma_{n-2}(e|i) + \sigma_{n-2}(e|i) - \sigma_{n-2}(e)$$

$$= (n - 1)\sigma_{n-2}(e|i) - \sigma_{n-2}(e).$$

In the third equalities, we used $e_i\sigma_{n-3}(e|i) = \sigma_{n-2}(e) - \sigma_{n-2}(e|i)$, which is from (116), we get the fourth equality.

**Lemma 3.4.** Let $a_i = (\gamma^i)^2$. $\gamma = (\gamma^1, \gamma^2, \ldots, \gamma^n)$ is a unit vector in $\mathbb{R}^n$. Set $a = (a_2, a_3, \ldots, a_n)$, $e = (e_2, e_3, \ldots, e_n)$ and $e_i = \sigma_1(a|i)$, $i \geq 2$. Then the matrix $E = (E_{ij})_{3 \leq i, j \leq n}$ is positive definite, where $E_{ij} = e_2 + e_i\delta_{ij}$.
Proof of Lemma 3.4. We only need to prove that the following determinate of $E$ is positive.

$$\det E = \sigma_{n-2}(e) = \sigma_{n-2}(\sigma_1(a|2), \sigma_1(a|3), \ldots, \sigma_1(a|n))$$

$$\begin{align*}
&= \sum_{2 \leq i_1 < i_2 < \cdots < i_{n-2} \leq n-2} (\sigma_1(a) - a_{i_1})(\sigma_1(a) - a_{i_2}) \cdots (\sigma_1(a) - a_{i_{n-2}}) \\
&= \sum_{0 \leq k \leq n-2} (-1)^k(n-k-1)[\sigma_1(a)]^{n-2-k}\sigma_k(a) \\
&= [\sigma_1(a)]^{n-2} + \sum_{2 \leq k \leq n-2} (-1)^k(n-k-1)[\sigma_1(a)]^{n-2-k}\sigma_k(a).
\end{align*}$$

(124)

In the following we will divide two cases, using the Newton-MacLaurin inequality, then we get our conclusion.

Case 1: if $n = \text{odd}$

$$\begin{align*}
&\sum_{2 \leq k \leq n-2} (-1)^k(n-k-1)[\sigma_1(a)]^{n-2-k}\sigma_k(a) \\
&= \sum_{2 \leq k \leq n-2, 3 \leq k \text{ is even}} [k(\sigma_1(a))^{k-1}\sigma_{n-1-k}(a) - (k-1)(\sigma_1(a))^{k-2}\sigma_{n-k}(a)] \\
&= \sum_{2 \leq k \leq n-2, 3 \leq k \text{ is even}} [\sigma_1(a)]^{k-2}[k\sigma_1(a)\sigma_{n-1-k}(a) - (k-1)\sigma_{n-k}(a)] \\
&\geq \sum_{2 \leq k \leq n-2, 3 \leq k \text{ is even}} [\sigma_1(a)]^{k-2}[(n-1)(n-k) - (k-1)]\sigma_{n-k}(a) \\
&\geq 0.
\end{align*}$$

(125)

Case 2: if $n = \text{even}$

$$\begin{align*}
&\sum_{2 \leq k \leq n-2} (-1)^k(n-k-1)[\sigma_1(a)]^{n-2-k}\sigma_k(a) \\
&= \sum_{3 \leq k \leq n-2, 3 \leq k \text{ is odd}} [k(\sigma_1(a))^{k-1}\sigma_{n-1-k}(a) - (k-1)(\sigma_1(a))^{k-2}\sigma_{n-k}(a)] + \sigma_{n-2}(a) \\
&\geq \sum_{3 \leq k \leq n-2, 3 \leq k \text{ is odd}} [\sigma_1(a)]^{k-2}[k\sigma_1(a)\sigma_{n-1-k}(a) - (k-1)\sigma_{n-k}(a)] + \sigma_{n-2}(a) \\
&\geq \sum_{3 \leq k \leq n-2, 3 \leq k \text{ is odd}} [\sigma_1(a)]^{k-2}[(n-1)(n-k) - (k-1)]\sigma_{n-k}(a) + \sigma_{n-2}(a) \\
&\geq 0.
\end{align*}$$

(126)

Since $\sigma_1(a) = \sum_{2 \leq i \leq n} a_i = c^{11} > 0$, it follows that

$$\det E = \sigma_{n-2}(e) \geq [\sigma_1(a)]^{n-2} > 0.$$  \hspace{1cm} (127)

then the matrix $E$ is positive definite.

Now we prove the main lemma.

Lemma 3.5. We define $(b_{ij})$ as in (101), where $d_i, e_i$ are defined as in (92)-(93) and $A_{1i}, A_{2i}, G_{ij}, \hat{G}_{ij}$ are defined as in (102). And we define $b_i$ as in (105),
\[ v^2 = 1 + u_1^2 \] and \( c^1 \geq \frac{1}{c_4} \). We study the following quadratic form

\[
Q(x_3, x_4, \ldots, x_n) = \sum_{3 \leq i \leq n} b_{ii}x_i^2 + 2 \sum_{3 \leq i < j \leq n} b_{ij}x_ix_j - u_1^5 \log u_1 \sum_{3 \leq i \leq n} b_{ii}x_i + \sum_{3 \leq i \leq n} K_i x_i,
\]

where \( K_i \) defined in (106) and we have the estimate (107) for \( K_i \). Then there exists a sufficiently large positive constant \( C_{16} \) which depends only on \( n, \Omega, \mu_0, \sigma, L_1, L_2 \) such that if

\[
|Du(x_0) - u_1(x_0)| \geq C_{16},
\]

then the followings hold.

(I): The matrix \((b_{ij})\) is positive definite if and only if the matrix \((b_{ij}^1) = (E_{ij}) = (e_2 + e_i \delta_{ij})\) is positive definite.

(II): We have

\[
Q(x_3, x_4, \ldots, x_n) \geq -(n - 2)(c^{11}g^2(\gamma^1)^2 u_1^6 \log^2 u_1 - C_{17}u_1^6 \log u_1),
\]

where positive constant \( C_{17} \) also depends only on \( n, \Omega, \mu_0, \sigma, L_1, L_2 \).

Proof of Lemma 3.5. Let

\[
B = (b_{ij}) = B_1 + B_2, B_1 = (c^{11}u_1^4 b_{ij}), B_2 = (O(u_1^2)\delta_{ij}).
\]

We first prove (I):

\[
\sigma_k(B) = \sigma_k(B_1 + B_2) = \sigma_k(B_1) + \sigma_k(B_1, B_1, \ldots, B_1, B_2) + \cdots + \sigma_k(B_1, B_2, \ldots, B_2, B_2) + \sigma_k(B_2) = (c^{11})^k u_1^4k \sigma_k(b_{ij}^1) + O(u_1^{4k-2}),
\]

so if \( u_1 \) is sufficiently large, then \( \sigma_k(B) > 0 \iff \sigma_k(b_{ij}^1) > 0 \).

Now we prove (II): If \( B_1 = ((c^{11})^4u_1^4 b_{ij})_{3 \leq i, j \leq n} \) is positive definite, from the argument in (I), we get

\[
B^{-1} = (B_1 + B_2)^{-1} = B_1^{-1}(I + B_1^{-1}B_2)^{-1} = \frac{1}{(c^{11})^4u_1^4(b_{ij}^1)^{-1}(1 + o(1))}.
\]

Then we have

\[
(b_{ij}^1)^{-1} = \begin{pmatrix} e_2 + e_3 & e_2 & \cdots & e_2 \\ e_2 & e_2 + e_4 & \cdots & e_2 \\ \vdots & \vdots & \vdots & \vdots \\ e_2 & e_2 & \cdots & e_2 + e_n \end{pmatrix}^{-1} = \frac{1}{\sigma_{n-2}(e)} \begin{pmatrix} \sigma_{n-3}(e|3) & -\sigma_{n-3}(e|34) & \cdots & -\sigma_{n-3}(e|3n) \\ -\sigma_{n-3}(e|43) & \sigma_{n-3}(e|4) & \cdots & -\sigma_{n-3}(e|4n) \\ \vdots & \vdots & \vdots & \vdots \\ -\sigma_{n-3}(e|n3) & -\sigma_{n-3}(e|n4) & \cdots & \sigma_{n-3}(e|n) \end{pmatrix} \]

where \( e = (e_2, e_3, \ldots, e_n) \).
Now we solve the following linear algebra equation

\[
\frac{\partial Q}{\partial x_k} = 0, \quad k = 3, 4, \ldots, n. \tag{134}
\]

We assume that \((\bar{x}_3, \bar{x}_4, \ldots, \bar{x}_n)\) is the extreme point of the quadratic form \(Q(x_3, x_4, \ldots, x_n)\). From the definitions of \(b_{ij}, b_i, K_i\) in (101), (105), (106) respectively and the estimate for \(K_i\) in (107), using the formulas (132) and (133), it follows that

\[
\begin{pmatrix}
\bar{x}_3 \\
\bar{x}_4 \\
\vdots \\
\bar{x}_n
\end{pmatrix}
= \frac{1}{2} u_1^5 \log u_1 B^{-1} \begin{pmatrix}
b_3 \\
b_4 \\
\vdots \\
b_n
\end{pmatrix}
+ O(u_1^5)B^{-1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}
\]

\[
= \frac{1}{2} u_1^5 \log u_1 B^{-1} \begin{pmatrix}
b_3 \\
b_4 \\
\vdots \\
b_n
\end{pmatrix}
+ O(u_1) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \tag{135}
\]

\[
= \frac{c^{11} g' \gamma^1 u_1 \log u_1}{\sigma_{n-2}(e)} \frac{1}{B} \begin{pmatrix} e_2 - e_3 \\ e_2 - e_4 \\ \vdots \\ e_2 - e_n \end{pmatrix}
+ O(u_1) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.
\]

From Lemma 3.3, we have for \(i = 3, 4, \ldots, n\),

\[
\bar{x}_i = \frac{c^{11} g' \gamma^1 u_1 \log u_1}{\sigma_{n-2}(e)} \left[ \sigma_{n-3}(e|i)(e_2 - e_i) - \sum_{k \neq i, k \geq 3} \sigma_{n-3}(e|ik)(e_2 - e_k) \right]
+ O(u_1) \tag{136}
\]

\[
= \frac{c^{11} g' \gamma^1 u_1 \log u_1}{\sigma_{n-2}(e)} [(n - 1)\sigma_{n-2}(e|i) - \sigma_{n-2}(e)] + O(u_1).
\]

It follows that we have the following minimum of the quadratic \(Q\),

\[
Q(\bar{x}_3, \bar{x}_4, \ldots, \bar{x}_n)
= \frac{(c^{11})^6 g^2 (\gamma^1)^2 u_1^6 \log^2 u_1}{\sigma_{n-2}^6(e)} \left\{ \sum_{3 \leq i \leq n} (e_2 + e_i) [(n - 1)\sigma_{n-2}(e|i) - \sigma_{n-2}(e)]^2 
+ 2e_2 \sum_{3 \leq i \leq j \leq n} [(n - 1)\sigma_{n-2}(e|i) - \sigma_{n-2}(e)] [(n - 1)\sigma_{n-2}(e|j) - \sigma_{n-2}(e)]
- 2\sigma_{n-2}(e) \sum_{3 \leq i \leq n} (e_2 - e_i) [(n - 1)\sigma_{n-2}(e|i) - \sigma_{n-2}(e)] \right\} + O(u_1^6 \log u_1). \tag{137}
\]
By Proposition 1, we have

\[
\begin{align*}
&\sum_{3\leq i\leq n} (e_2 + e_i) [(n-1)\sigma_{n-2}(e|i) - \sigma_{n-2}(e)]^2 \\
&+ 2e_2 \sum_{3\leq i<j\leq n} [(n-1)\sigma_{n-2}(e|i) - \sigma_{n-2}(e)] [(n-1)\sigma_{n-2}(e|j) - \sigma_{n-2}(e)] \\
&- 2\sigma_{n-2}(e) \sum_{3\leq i\leq n} (e_2 - e_i) [(n-1)\sigma_{n-2}(e|i) - \sigma_{n-2}(e)] \\
= &\sum_{3\leq i\leq n} (n-1)\sigma_{n-2}(e|i) - \sigma_{n-2}(e)]^2 \\
&- 2e_2\sigma_{n-2}(e) \sum_{3\leq i\leq n} [(n-1)\sigma_{n-2}(e|i) - \sigma_{n-2}(e)] \\
&+ \sum_{3\leq i\leq n} e_i [(n-1)\sigma_{n-2}(e|i) - \sigma_{n-2}(e)]^2 \\
&+ 2\sigma_{n-2}(e) \sum_{3\leq i\leq n} e_i [(n-1)\sigma_{n-2}(e|i) - \sigma_{n-2}(e)] \\
= &\sum_{3\leq i\leq n} [(n-1)^2 \sigma_{n-1}(e) - \sigma_1(e)\sigma_{n-2}(e)]\sigma_{n-2}(e) \\
&\geq - \sigma_1(e)\sigma_{n-2}^2(e) \\
&= - (n-2)\sigma_{n-2}^2(e).
\end{align*}
\]

Using (137) and (138), we at last get the following estimate

\[
Q(x_3, x_4, \ldots, x_n) \geq Q(\bar{x}_3, \bar{x}_4, \ldots, \bar{x}_n) \\
\geq - (n-2)(c_1^{11} g \gamma)^2 u^1 \log^2 u_1 + O(u_1^6 \log u_1).
\]

In this computation, the bounds in the coefficient on \(O(u_1^6 \log u_1), O(u_1^7), O(u_1)\) depend only on \(n, \Omega, M_0, \mu_0, L_1, L_2\). Thus we complete this proof. \(\square\)

4. **Proof of Theorem 1.2.**

**Proof of Theorem 1.2.** As in the proof of Theorem 1.1, let

\[
\tilde{P}(x) = \log |D'u|^2 e^{\sqrt{n}\tilde{a}_0(M_0 + 1 + u)} e^{\tilde{a}_0 d},
\]

where we have let

\[
\tilde{a}_0 = 4nL_2 \sqrt{\frac{1 + a_0}{1 - b_0}} + 2C_0 + 2,
\]

which is a constant, and

\[
a_0 = \max_{x \in \partial \Omega} \frac{2\psi^2}{1 - \psi^2}, \quad C_0 = 10n^2 \max_{x \in \partial \Omega} |\tilde{A}|,
\]

|\tilde{A}| is the modulus of the second fundamental form of \(\partial \Omega\). Similarly, let

\[
\varphi(x) = \log \tilde{P}(x) = \log \log |D'u|^2 + h(u) + g(d),
\]

where in the capillary boundary value case, we choose

\[
h(u) = \sqrt{n}\tilde{a}_0(M_0 + 1 + u), \quad g(d) = \tilde{a}_0 d.
\]

(140)
In the following, we choose \( \mu_0 = \min\{\frac{1}{2}, \frac{\mu_1}{2}\} \). Assume that \( \varphi(x) \) attains its maximum at \( x_0 \in \partial U \).

**Case 1.** \( x_0 \in \partial U \). Similar calculations to case I in the proof of Theorem 1.1, letting \( q = 0 \) in (26), we get

\[
|D' u|^2 \log |D' u|^2 \frac{\partial \varphi}{\partial \gamma}(x_0)
=g'|D' u|^2 \log |D' u|^2 - 2 \sum_{1 \leq i, k, l \leq n} c^{kl} u_i u_l \langle \gamma^i \rangle_k
\]

\[
- \frac{2v}{1 - \psi^2} \sum_{1 \leq k, l \leq n} c^{kl} \psi_k u_l - \frac{h' \psi}{v(1 - \psi^2)} |D' u|^2 \log |D' u|^2.
\]

(141)

Since at \( x_0 \),

\[
u_\gamma^2 = \psi^2 (1 + |Du|^2) = \psi^2 (1 + |D' u|^2 + u_\gamma^2),
\]

then

\[
u_\gamma^2 = \frac{\psi^2}{1 - \psi^2} (1 + |D' u|^2).
\]

(142)

If

\[a_0 |D' u|^2 < u_\gamma^2, \quad a_0 = \max_{x \in \partial U} \frac{2\psi^2}{1 - \psi^2},\]

then we get the estimates

\[(a_0 - \frac{\psi^2}{1 - \psi^2} - 1) |D' u|^2 < 1, \quad |D' u|^2 < \frac{1}{a_0 \frac{1 - \psi^2}{1 - \psi^2} - 1},\]

(144)

and we complete this proof.

So we can assume

\[a_0 |D' u|^2 \geq u_\gamma^2,\]

(145)

then from \(|Du|^2 = |D' u|^2 + u_\gamma^2\), we have

\[|Du|^2 \leq (1 + a_0) |D' u|^2.\]

(146)

Now we assume at \( x_0 \), we have

\[|Du| \geq \max\{10 \sqrt{1 + a_0}, \frac{2\sqrt{n}}{1 - \psi^2} \max_{x \in \partial U} \frac{|\psi|}{1 - \psi^2}\},\]

(147)

then we can get the following estimates at \( x_0 \),

\[|D' u| \geq \max\{10, \frac{2\sqrt{n}}{1 + a_0} \max_{x \in \partial U} \frac{|\psi|}{1 - \psi^2}\}.\]

(148)

Inserting (148) into (141), by the choice of \( h(u), g(d) \) in (140), it follows that

\[\frac{\partial \varphi}{\partial \gamma}(x_0) \geq g' - \frac{2 \sum_{1 \leq i, k, l \leq n} c^{kl} u_i u_l \langle \gamma^i \rangle_k}{|D' u|^2 \log |D' u|^2} - \frac{2v}{1 - \psi^2} \frac{|\sum_{1 \leq k, l \leq n} c^{kl} \psi_k u_l|}{|D' u|^2 \log |D' u|^2}
\]

\[- \frac{|h' \psi|}{v(1 - \psi^2)} \geq a_0 - 10 \sum_{1 \leq i, k \leq n} |\langle \gamma^i \rangle_k| - 2n \sqrt{1 + \frac{|\psi|}{1 - \psi^2}} - \frac{1}{v \sqrt{1 - \psi^2}} \sqrt{n a_0} \frac{|\psi|}{1 - \psi^2} > 0.\]
On the other hand, we have \( \frac{\partial \zeta}{\partial x}(x_0) \leq 0 \), it is a contradiction to (149). Then we have
\[
|Du| \leq \max\{10\sqrt{(1 + a_0)}, 2\sqrt{n} \max_{x \in \partial \Omega} |\psi| \}.
\]

**Case 2.** \( x_0 \in \partial \Omega_{\mu_0} \cap \Omega \). This is due to interior gradient estimates. From Remark 1, we have
\[
\sup_{\partial \Omega_{\mu_0} \cap \Omega} |Du| \leq \tilde{M}_1.
\]
where \( \tilde{M}_1 \) is a positive constant depending only on \( n, M_0, \mu_0, L_1 \).

**Case 3.** \( x_0 \in \Omega_{\mu_0} \). As in the proof of the Case III in Theorem 1.1, we have
\[
\sup_{\Omega} |Du|^2 \leq C_1 (1 + \sup_{\partial \Omega} |Du|^2),
\]
where \( C_1 \) is a positive constant depending on \( n, L_1, M_0 \). From Case 1, we assume (145), otherwise we finish the proof of Theorem 1.2. Hence,
\[
\sup_{\Omega} |Du|^2 \leq C_1 (1 + a_0) \sup_{\partial \Omega \cap \{|D'u| \geq 1\}} |D'u|^2 \leq C_2 \sup_{\partial \Omega \cap \{|D'u| \geq 1\}} |D'u|^2.
\]
So we have
\[
\sup_{\Omega_{\mu_0}} |Du|^2 \leq C_3 \sup_{\Omega_{\mu_0}(M)} |D'u|^2,
\]
where \( \Omega_{\mu_0}(M) = \Omega_{\mu_0} \cap \{|D'u| \geq M\} \), \( M > 10 \) is a positive constant; \( C_3 \) is a positive constant depending on \( n, L_1, b_0, M_0 \).

Assume \( x_1 \in \Omega_{\mu_0}(M) \) such that
\[
\sup_{\Omega_{\mu_0}(M)} |D'u|^2 = |D'u|^2(x_1).
\]
Since \( \tilde{P}(x_0) \geq \tilde{P}(x_1) \), it is easy to obtain
\[
|D'u|^2(x_1) \leq C_3 |D'u|^2(x_0).
\]
where \( C_3 \) is a positive constant depending on \( n, L_1, b_0, L_2, M_0 \). So
\[
\sup_{\Omega_{\mu_0}} |Du|^2 \leq C_2 \sup_{\Omega_{\mu_0}(M)} |D'u|^2 = C_2 |D'u|^2(x_1) \leq C_3 |D'u|^2(x_0).
\]
Assume \( |D'u|(x_0) \geq M > 10 \), otherwise we get the estimate. Hence from (155) at \( x_0 \), we obtain
\[
u_i^2(x_0) = |Du|^2(x_0) \leq C_3 e^{11} u_i^2(x_0),
\]
then we have \( e^{11} \geq \frac{1}{C_3} > 0 \).

Similar calculations to Case III in the proof of Theorem 1.1, and by the choice of \( h(u) \), \( g(d) \) in (140), we can obtain at last
\[
0 \geq a^{ij} \psi_{ij} \geq \{h^2 + [(c^{11})^2(n - 2) - c^{11}(n - 3)] g^2 \} u_i^2 \log u_i - C_4 u_i^2 \geq 3u_i^2 \log u_i - C_5 u_i^2.
\]
There exists a positive constant \( C_6 \) such that
\[
|Du|(x_0) \leq C_6.
\]
So from Case 1, Case 2, Case 3, we complete the new proof of Theorem 1.2.

**Remark 2.** For \( \psi = \psi(x, u) \), we can obtain the gradient estimate if \( q = 0 \) and \( \psi_u \leq 0 \). If \( q > 1 \) and \( \psi_u \) is bounded, the gradient estimate is also easily obtained.
Remark 3. For $q = 1$, $\psi = \psi(x,u)$, we can also use the above similar auxiliary function $\varphi_1 = \log |D'u|^2 e^{1+M_0+ue^{\alpha_0 d}}$ to get the gradient estimates for the Neumann problem. But this auxiliary function is only applicable to deal with $n = 2, 3$ dimensional cases, and it is also different from the auxiliary function $\varphi_2 = \log |Dw|^2 e^{1+M_0+ue^{\alpha_0 d}}$, $w = u - \psi d$ in [11].

Remark 4. From above we can see that the auxiliary function $\log |D'u|^2 h(u)g(d)$ in this note can be unified to deal with the boundary problems for $q > 1$ or $q = 0$ or $q = 1(n = 2, 3)$ cases in (2). But the auxiliary function $\log |Dw|^2 h(u)g(d)$, $w = u - \psi d$ in [11] is only suitable for the Neumann problem or semi-linear oblique value problem.

Remark 5. The method in this paper can be generalized to prove the gradient estimate of Hessian equation with prescribed contact angle boundary condition.

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