The SQG Equation as a Geodesic Equation

PEARCE WASHABAUGH

Communicated by V. Šverák

Abstract

We demonstrate that the surface quasi-geostrophic (SQG) equation given by

$$\theta_t + \langle u, \nabla \theta \rangle = 0, \quad \theta = \nabla \times (-\Delta)^{-1/2} u,$$

is the geodesic equation on the group of volume-preserving diffeomorphisms of a Riemannian manifold $M$ in the right-invariant $\dot{H}^{-1/2}$ metric. We show by example, that the Riemannian exponential map is smooth and non-Fredholm, and that the sectional curvature at the identity is unbounded of both signs.

1. Introduction

As discussed by Choi et al. [4], there are a large number of model equations of the full three dimensional Euler equations that have been investigated analytically. Some of these equations arise naturally as geodesic equations of right-invariant metrics of diffeomorphism and volume-preserving diffeomorphism (volumorphism) groups. For example, a special case of the generalized Constantin-Lax-Majda model first discussed by Okamoto et al. [19] is the Wunsch equation [25],

$$\omega_t + u \omega_x + 2u \omega_x = 0, \quad \omega = Hu_x,$$

which is the geodesic equation on the diffeomorphism group of the circle in the $\dot{H}^{1/2}$ right-invariant metric. For equations arising in such a fashion, it is then natural to investigate their associated geometric properties in the manner initiated by Arnold [1]. In this paper, we demonstrate that the well known surface quasi-geostrophic (SQG) equation is the geodesic equation on the volumorphism group of a two dimensional manifold in the $\dot{H}^{-1/2}$ inner product. The SQG equation on a Riemannian manifold $M$ with metric $\langle \cdot, \cdot \rangle$ is given by

$$\theta_t + \langle u, \nabla \theta \rangle = 0, \quad u = \mathcal{R}^{-1} \theta,$$
where $\mathcal{R}^\perp$ is the perpendicular Riesz transform. Many of the basic mathematical properties of this equation were first investigated by Constantine et al. [5]. Importantly, while this equation is known to have solutions for short time, the global in time existence problem is still open. It is believed by some (see example Constantine et al. [5]) that the blow-up mechanism (should it exist) of this equation may have very similar properties to that of the full three dimensional Euler equations. As Bauer et al. [2] did for the Wunsch equation (1), in this paper we investigate some of the basic geometric properties of the SQG equation (2). We perform the necessary computations in a variety of domains in order to keep the paper as simple as possible. Looking forward, it will be necessary to firmly establish the theory of this equation in a single domain (as in Escher and Koley [9] for positive fractional order Sobolev metrics on the diffeomorphism group of the circle).

The following is a list of the geometric properties associated to the SQG equation we explore:

- **Smoothness of the Riemannian exponential map**
  The Riemannian exponential map on the volumorphism group in a Riemannian metric takes a velocity field (tangent vector) to the solution of the geodesic equation of the metric at time one. In our case, the geodesic equation is equivalent to SQG (2) and the geodesic evaluated at time one is a particle trajectory map. We may then ask whether or not this map is smooth. This question is partially answered by Constantine et al. [6], where the authors demonstrate the analyticity of the particle trajectories. Here, we are also concerned with smooth dependence on the initial data. In this paper we demonstrate that the Lagrangian formulation of SQG has smooth dependence on the initial data in the case that the underlying manifold is $\mathbb{R}^2$. This suggests in general that the Riemannian exponential map will be smooth for the $H^{-1/2}$ right invariant metric on the volumorphism group of any manifold.

- **Non-Fredholmness of the Riemannian exponential map**
  Next, we show that the Riemannian exponential map on $Diff_\mu(S^2)$ in the $H^{-1/2}$ inner product is not a Fredholm map in the sense of Smale [24]. Ebin et al. [8] showed that, for $M$ a compact two dimensional Riemannian manifold without boundary, in the $L^2$ metric on $Diff_\mu(M)$, the exponential map is a nonlinear Fredholm map of index zero. It was also demonstrated that the exponential map is not Fredholm in the three dimensional situation. This points to a significant difference between two dimensional and three dimensional hydrodynamics. Fredholmness has been used to obtain results about the $L^2$ geometry of the two dimensional volumorphism group, such as an infinite dimensional version of the Morse Index Theorem (see Misiolek and Preston [17]), and a version of the Morse–Littauer Theorem (see Misiolek [18]). In this paper, we solve the Jacobi equation along a simple rotational flow to demonstrate the existence of an epiconejugate point that is not monoconjugate (see Grossman [10]); thus the exponential map is non-Fredholm. Preston [21] showed that there is a concrete connection between blow up and the existence of conjugate points, thus our argument here provides evidence that the blow up behavior of two dimensional SQG is similar geometrically to that of three dimensional Euler.