Comparative study of the \textit{LOCV} and the \textit{FHNC} approaches for the \textit{nucleonic} matter problem

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Abstract. The \textit{nucleonic} matter problem is investigated by comparing the lowest order constrained variational (\textit{LOCV}) method with the Fermi \textit{hypernetted chain (FHNC)} theory, emphasizing the role of the \textit{LOCV} correlation functions. In this way, the central correlation functions are used in the \textit{LOCV} formalism, for the \textit{Bethe} homework problem. It is shown that the \textit{LOCV} computations reasonably agree with those of \textit{FHNC}. Moreover, the \textit{FHNC} calculations are performed with the \textit{LOCV} correlation functions. It is found that, assuming the \textit{LOCV} or the parametrized correlation functions, the \textit{FHNC} computations do not change significantly. So, one may conclude that the mentioned consistencies refer to the choice of the \textit{LOCV} correlation functions. Because, the contribution of the many-body cluster terms can be ignored, if the \textit{LOCV} correlation functions satisfy the normalization constraint. Then, using the \textit{AV}18 interaction, the operator-dependent (\textit{OD}) correlation functions are employed in the \textit{LOCV} calculations. Note that the \textit{LOCV OD} correlation functions are obtained by averaging over the states. It turns out that the overall behaviour of the \textit{LOCV OD} correlation functions are similar to those of \textit{FHNC}. Although, due to the many-body effects which are considered in the \textit{FHNC} calculations, the \textit{LOCV} results fairly differ from those of \textit{FHNC}. Finally, it is worth mentioning that, unlike the recent \textit{FHNC} calculations, the spin-orbit-dependent correlation functions are included in the \textit{LOCV} approach.

1. Introduction

The lowest order constrained variational (\textit{LOCV}) approach has been hitherto applied to the \textit{nucleonic} matter problem, at zero and finite temperatures [1–19], with different two-body potentials, e.g. the Reid [20] and the \textit{Argonne} interactions [21–25]. In these calculations [1–19], the energy of the \textit{nucleonic} matter is obtained by truncating the cluster expansion of the energy [26, 27], at the two-body cluster term. Since, the \textit{LOCV} formalism is fully state-dependent, in each channel, the two-body correlation functions are computed by functionally minimizing the energy, subject to the normalization constraint [28]. As it was shown in our previous papers [8, 14, 17], by imposing the normalization constraint, the contribution of the higher cluster terms than the two-body one becomes ignorable. So, it was illustrated that the \textit{LOCV} results [1–19] fairly agree with the corresponding predictions of the sophisticated methods, i.e. the Fermi \textit{hypernetted chain (FHNC)} and the Monte Carlo (\textit{MC}) techniques.

In recent years, we compared the \textit{LOCV} method with the \textit{FHNC} theory, for different nuclear many-body problems [17–19]. These comparisons were based on the role of the correlation functions. At the first step, these two methods were compared against the \textit{Fermionic} prescription of the \textit{Bethe} homework problem [17]. The \textit{LOCV} and the extended \textit{LOCV
(ELOCV) central correlation functions were included in the FHNC and the Fermi chain (FC) formalisms instead of the parametrized Krotscheck and Takahashi [29] (KT) ones. It should be noted that in the ELOCV computations, the cluster expansion of the energy is truncated at the three-body level. So, we showed that including the (E)LOCV or the KT correlation functions, the FHNC(FC) predictions [30,31] do not change considerably. Moreover, it was found that the approximated (E)LOCV results reasonably agree with those FHNC [30,31]. It was concluded that these consistencies are originated from the (E)LOCV correlation functions, which optimally satisfy the normalization constraint. In the next step, we reformulated the LOCV method for a fixed number of particles in a periodic box (PBLOCV) [18]. It was found that the PBLOCV results fairly agree with those of FHNC calculations in the periodic boxes (PBFHNC) [32,33]. This agreement refers to the choice of correlation function in the LOCV method, i.e. the satisfaction of the normalization constraint in the periodic boxes. Furthermore, the finite size effects, which can be used to estimate the finite size effects in the auxiliary field diffusion Monte Carlo (AFDMC) simulations, were investigated in the PBFLOCV approach [18]. As it was expected, by increasing the dimensions of the periodic boxes, the PBLOCV results became independent of the number of particles [18]. Finally, we rewritten the channel-dependent LOCV nucleonic matter correlation functions in terms of the first eight operators of the AV18 potential [19]. In this way, the LOCV operator-dependent correlation functions [19] were compared with those of FHNC at single-operator chain (FHNC/SOC) approximation [34–36]. It was demonstrated that there are some discrepancies between the LOCV and the FHNC/SOC correlation functions. Because, unlike the FHNC/SOC correlation functions, the LOCV results are defined to fulfil the two-body cluster approximation. Similarly, due to the many-body effects, the FHNC/SOC nucleonic matter energies became different from those of LOCV. However, one may hope that similar to the Bethe homework problem [17], inserting the LOCV operator-dependent correlation functions in the FHNC/SOC formalism: (i) The FHNC/SOC energy does not change significantly. (ii) The FHNC/SOC results converge faster.

According to the above works [17–19], we illustrated that the LOCV correlation functions, which satisfy the normalization constraint, have several advantages with respect to those of FHNC(SOC):

(i) The LOCV two-body correlation functions are state-dependent, i.e. they are different in each nucleon-nucleon state. On the other hand, the FHNC/SOC correlation functions are state-independent. Because, using the channel-dependent correlation functions, the many-body calculations become complicated. So, according to the reference [37], it is expected that the LOCV channel-dependent correlation functions, which satisfy the normalization constraint, could be more optimal than those of FHNC/SOC.

(ii) The FHNC/SOC correlation functions can not directly be obtained by the minimization of the many-body energy [34]. The FHNC/SOC correlation functions are calculated at the lowest order approximation [34]. Then, using some variational parameters, the contribution of the many-body terms is included in the FHNC/SOC correlation functions [34]. While, the LOCV correlation functions are obtained by functionally minimizing the energy, at the two-body cluster approximation, subject to the normalization constraint. Note that, unlike the FHNC/SOC technique, in the LOCV formalism, the cluster expansion of the energy is truncated at the lowest order. So, the LOCV energy and correlation function could be reported at the same approximation, i.e. at the two-body cluster approximation.

(iii) The LOCV channel-dependent correlation function includes the spin-orbit-dependent correlation functions. While, in recent FHNC/SOC calculations [38,39], the contribution of the spin-orbit-dependent correlation functions has been ignored. Because, the treatment of the spin-orbit correlation functions in the FHNC/SOC formalism is associated with uncertainties and difficulties [40].
So, in this paper, we present a comparative study of the LOCV [17, 19] and the FHNC approaches [30,31,35,36,39,41], by concentrating on the infinite systems, i.e. the pure neutron (symmetric nuclear) matter. It should be noted that the principal aim of this paper is to illustrate that the approximated LOCV approach is capable of reproducing the FHNC/(SOC) and the Monte Carlo calculations in a more simpler way. However, it is shown that, in the case of the spin-orbit-dependent terms, the LOCV method is more reliable than the sophisticated FHNC/SOC and Monte Carlo approaches. In this way, the plan of this paper is as follows: (i) In the Section 2, first, we give the LOCV formalism for the Bethe homework problem [17]. Then, the LOCV formalism for the nucleonic matter system, using the AV18 interaction, is represented [19]. Also, the LOCV operator-dependent correlation functions are extracted from the LOCV channel-dependent formalism [19]. (ii) In the Section 3, the LOCV results [17, 19] are shown in comparison with those of FHNC/(SOC) [30,31,35,36,39,41] and Monte Carlo [25,39,42]. (iii) In addition, the corresponding discussions and conclusions are given in the Section 4. Note that in this section: (a) The updated versions of our previous calculations for the Bethe homework problem [14], are represented. (b) The LOCV nucleonic matter computations of the reference [16] are reviewed.

2. The LOCV formalisms

2.1. The LOCV formalism for the Bethe homework problem

The Hamiltonian of the Bethe homework problem for \( N \) neutrons can be written as follows [17]:

\[
H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \frac{1}{2} \sum_{i \neq j}^{N} v(ij),
\]

(1)

where \( v(ij) \) is given as [17]:

\[
v(ij) \equiv v_B(r_{ij}) = 9263.1 \frac{e^{-4.9r_{ij}}}{r_{ij}} \text{MeV},
\]

(2)

and \( r \) is measured in fm. Note that the Bethe potential is the repulsive part of the Reid soft-core \( ^1S_0 \) interaction [20]. An ideal Fermi gas type wave function is used in the LOCV (ELOCV) method, for the single particle states. Furthermore, using the variational techniques, the wave function of the interacting system is defined as [4–19]:

\[
\psi = F\phi,
\]

(3)

where

\[
F = S \prod_{i>j} f(ij).
\]

(4)

Note that \( \phi \) is the Slater determinant of plane waves and \( S \) is a symmetrizing operator. Similar to the Bethe potential (equation (2)), the correlation functions \( f(ij) \) are assumed to depend on the relative distance of two interacting particles.

Now, considering the cluster expansion of the energy [26,27], one can expand the expectation value of the equation (1) as follows:

\[
E[f] = \frac{1}{N} \frac{\langle \psi|H|\psi \rangle}{\langle \psi|\psi \rangle} = T_F + E_{MB},
\]

(5)

where, \( T_F \) is the Fermi gas kinetic energy, which is defined in terms of the Fermi momentum \( k_f = (3 \pi^2 \rho)^{\frac{1}{3}} \), and \( E_{MB} \) is the contribution of the many-body clusters, i.e. \( E_{MB} = E_2 + E_3 + ... + E_N \).
In the \(E)LOCV\) method, the cluster expansion of the energy is truncated at the two(three)-body cluster term \(E_{2(3)}\), i.e. \([27]\)

\[
E_2 = \frac{1}{2N} \sum_{ij} \langle ij|V_{eff}|ij\rangle_a, \tag{6}
\]

\[
E_3 = [E_{3(2)}] + E_{3(3)} + E_{4(2)} = [E_{3h} + E_{3hh}] + E_{3t} + E_{4h}, \tag{7}
\]

where \([27]\),

\[
V_{eff}(12) = -\frac{\hbar^2}{2m} \left[ f(12), \left[ \nabla^2_{12}, f(12) \right] \right] + f(12) v(12) f(12), \tag{8}
\]

and \(E_{3h}, E_{3hh}, E_{3t}\) and \(E_{4h}\) are given in the equations (11) to (14) of the reference \([17]\). Note that \(|ij >_a\) means that the neutron-neutron state is asymmetric. Applying the plane waves as the non-interacting wave functions, in the equation (6) and the equations (11) to (14) of the reference \([17]\), one arrives at the equations (3.12) to (3.14) and (3.16) of the reference \([27]\) for the two- and the three-body cluster energies.

As it was mentioned in the introduction, in the LOCV approach, the two-body cluster energy is minimized, subject to the normalization constraint \([28]\), i.e.

\[
\xi \equiv \langle \psi|\psi \rangle = -1 = \rho \int (1 - G_2(r_{12})) \, dr_{12} - 1, \tag{9}
\]

where the radial distribution function \(G_2(r_{12})\) is written as follows \([27]\):

\[
G_2(r_{12}) = f^2(r_{12}) \sum_{n=2}^N [\Delta G_2(r_{12})]_n, \tag{10}
\]

and \([\Delta G_2(r_{12})]_{n=2,3}\) are given in the equations (20) to (21) of the reference \([17]\). Note that in the \(E)LOCV\) calculations, \([\Delta G_2(r_{12})]_{n=2(3)}\) is (are) included in the equation (10). Moreover, assuming the normalization constraint (equation (9)), the Lagrange multiplier \(\lambda\) is introduced in the LOCV formalism. In this way, the functional \(L[f] = \{E_2 + \lambda \langle \psi|\psi \rangle \} [f]\) is minimized with respect to \(f(r)\), by solving the Euler-Lagrange equation, i.e. \([17]\)

\[
\frac{\partial L}{\partial f(r)} - \frac{\partial}{\partial r} \frac{\partial L}{\partial f'(r)} = 0, \tag{11}
\]

which leads to the following differential equation \([17]\):

\[
g''(r) - \left[ \frac{A''(r)}{A(r)} + \frac{m}{\hbar^2} (v(r) - \lambda) \right] g(r) = 0, \tag{12}
\]

where

\[
g(r) = A(r) f(r), \tag{13}
\]

and

\[
A^2(r) = r^2 G_F(r). \tag{14}
\]

In the above equation, \(G_F(r) = (1 - \frac{1}{2} l^2(kfr))\) is the Fermi gas radial distribution function which is defined in terms of the spherical Bessel functions, i.e. \(l(x) = \frac{3 J_1(x)}{x^2}\). It should be stated that regarding the \(ELOCV\) normalization constraint, the Euler-Lagrange equation (12) becomes an integro-differential equation in terms of \(f(r)\).
The Euler-Lagrange differential equation (12) is solved while \((ln(f(r)))' = (ln(f_p(r)))'\). Note that \(f_p(r) = G_p^{-\frac{3}{2}}(r)\) is the Pauli function. Then, for \(r > d\) (d is the healing distance), \(f(r) = f_p(r)\). In a similar way, the ELOCV integro-differential equation can iteratively be solved until the equation (10) is satisfied for \(n = 3\). Then, the contribution of \(E_3\) is added to that of \(E_2\). It should be noticed that \(E_3\) is an estimate to the three-body cluster energy, i.e. \(E_2 + E_3\) can not be considered as the upper bound to the true energy. However, if the contribution of \(E_3\) becomes very small in comparison with that of \(E_2\), then it can be concluded that \(T_F + E_2\) is a good approximation to the upper bound ground state energy of Bethe homework problem.

2.2. The LOCV formalism for the nucleonic matter with the AV18 potential

The LOCV formalism, for the nucleonic matter system, using the AV18 potential, is similar to the formalism of the previous subsection. Although, one should consider the following points: (i) In the equation (2), the AV18 interaction [22] is considered instead of the Bethe potential. (ii) In the equation (4), \(f(ij)\) is assumed to be state-dependent. So, the LOCV nucleonic matter two-body correlation function is defined as follows [19]:

\[
F(12) = \sum_{JSTT_z} |JSTT_z > f_{JSTT_z}(12) < JSTT_z|,
\]

where \(J, S\) and \(T(T_z)\) are the angular momentum, the spin and the \((z\) axis projection of) isospin of the pair of nucleons. Note that \(T, T_z = 1, -1\), in the pure neutron matter computations. Moreover, in the decoupled (DC) channels, i.e. for \(S = 0\) and \(S = 1\), \(J = 0\) \((L)\) states, \(f_{JSTT_z}(12)\) is central, i.e. \(f_{JSTT_z}^{DC}(12) = f_{JSTT_z}^{(1)}(r_{12})\). On the other hand, in the coupled (CP) channels \(3S_1 - 3D_1\) and \(3P_2 - 3F_2\), \(f_{JSTT_z}(12)\) are respectively given as follows [19]:

\[
f_{1100}^{(21)}(12) = f_{1100}^{(31)}(r_{12})Q + f_{1100}^{(31)}(r_{12})\sqrt{Q},
\]

\[
f_{211T_z}^{CP}(12) = f_{211T_z}^{(22)}(r_{12})P^{L=1} + f_{211T_z}^{(32)}(r_{12})P^{L=3},
\]

where \(Q(\equiv 1 - \sqrt{Q}) = \frac{3}{2} + \frac{1}{2}S_{12}\) and \(P^L = |L > L|\). Also, \(f_{JSTT_z}^{(ij)}(r_{12})\) can be written in terms of the radial and the tensor(spin-orbit)-dependent correlation functions, i.e. \([11,19]\),

\[
f_{1100}^{(i)}(r_{12}) = f_{1100}^{r}(r_{12}) + a_1f_{1100}^{l}(r_{12}),
\]

\[
f_{211T_z}^{(i)}(r_{12}) = f_{211T_z}^{r}(r_{12}) + a_2f_{211T_z}^{b}(r_{12}),
\]

where, for \(i = 2\) \((3)\), \(a_1 = 2\) \((-4)\) and \(a_2 = 1\) \((-4)\). Note that similar to the equation (16), one can employ the tensor-dependent correlation function in the \(3P_2 - 3F_2\) channel.

With regard to the equation (15) and the two-body potential AV18, the two-body cluster energy, i.e. the equation (6), can be written as follows [19]:

\[
E_2 = \frac{\rho}{2} \int d^3r_{12}\sum_{JLSTT_zT_1T_2} \varpi v_{eff}^{JLSTT_z}(r_{12})I_{L,T_z}(k_f r_{12}),
\]

where \(\varpi = \frac{k^2}{\pi}\rho | < \frac{1}{2}T_1 \frac{1}{2}T_2 | TT_z > |^2(2J + 1)\frac{1}{2}(1 - (-1)^{L+S+T})\) and \(I_{L,T_z}(k_f r_{12})\) is defined in the reference [16]. In addition, \(v_{eff}^{JLSTT_z}(r_{12})\) for the coupled and decoupled channels were given in the references [11,16,19].
Now, the two-body cluster energy is minimized, subject to the normalization constraint, i.e. equation (9), which for the state-dependent correlation functions, is obtained by the following relation [11,19]:
\[
\xi = \rho \int d^3r_{12} \sum_{JSTT_1T_1T_2} \varpi(f_{p,T_2}^2(r_{12}) - f_{JSTT_2}^2(r_{12}))I_{L,T_2}(kFr_{12}) - 1,
\]
(21)
where \(f_{p,T_2}(r_{12})\) is the Pauli function which defines the long-range behaviour of the LOCV correlation functions. Note that \(f_{p,\pm 1}(r_{12}) = [G_F(r_{12})]^{-\frac{1}{2}}\) and \(f_{p,0}(r_{12}) = 1\). So, minimizing the equation (20), subject to the equation (21), one arrives at the following Euler-Lagrange equations for the decoupled, the \(^3P_2 - ^3F_2\) and the \(^3S_1 - ^3D_1\) channels, respectively [11,16,19]:
\[
G_{JSTT_2}^{(1)}(r_{12}) - G_{JSTT_2}^{(1)}(r_{12})\left\{\frac{a_{JSTT_2}^{(1)}(r_{12})}{a_{JSTT_2}^{(1)}(r_{12})} + \frac{m}{\hbar^2}(v_{JSTT_2}^{(1)}(r_{12}) + \lambda)\right\} = 0,
\]
(22)
\[
G_{211T_2}^{(2)}(r_{12}) - G_{211T_2}^{(2)}(r_{12})\left\{\frac{a_{211T_2}^{(2)}(r_{12})}{a_{211T_2}^{(2)}(r_{12})} + \frac{m}{\hbar^2}(v_{211T_2}^{(2)}(r_{12}) + \lambda)\right\} = 0,
\]
(23)
\[
G_{1100}^{(1)}(r_{12}) - G_{1100}^{(1)}(r_{12})\left\{\frac{a_{1100}^{(1)}(r_{12})}{a_{1100}^{(1)}(r_{12})} + \frac{m}{\hbar^2}(v_{1100}^{(1)}(r_{12}) + \lambda) + \Gamma_{1100}^{(1)}(r_{12})\right\} + \Delta_{1100}^{(1)}G_{1100}^{(1)}(r_{12}) = 0,
\]
(24)
where for \(i = 2, 3, j = 3, 2\) and
\[
G_{JSTT_2}^{(1,i,2)}(r_{12}) = a_{JSTT_2}^{(1,i,2)}(r_{12})f_{JSTT_2}^{(1,i,2)}(r_{12}),
\]
(25)
\[
a_{JSTT_2}^{(1)}(r_{12}) = (kFr_{12})^2I_{L,T_2}(kFr_{12}),
\]
(26)
\[
a_{1100}^{(21,3)}(r_{12}) = (kFr_{12})^2\{\beta(\gamma)I_{J-1,T_2}(kFr_{12}) + \gamma(\beta)I_{J+1,T_2}(kFr_{12})\}.
\]
(27)
Note that \(v_{JSTT_2}^{(1)}(r_{12})\), \(\beta\) and \(\gamma\) were given in the references [11,19]. Also, \(v_{JSTT_2}^{(1,i)}(r_{12})\), \(\Gamma_{1100}^{(i)}(r_{12})\) and \(\Delta_{1100}^{(i)}\) are written in terms of \(\tilde{V}, a_{\alpha}, b_{\alpha}\) and \(d_{\alpha}\) which were defined in the reference [11,16] (see the equations (23) to (25) of the reference [16]).

Similar to the previous section, for \(r \leq d\), the above Euler-Lagrange equations are iteratively solved by changing the parameter \(\lambda\). Then, for \(r > d\), the LOCV correlation functions are defined by the Pauli correlation functions. In this way, the two-body correlation function \(F(12)\) in different channels, i.e. \(f_{JSTT_2}(r_{12})\), are computed in the LOCV framework.

Now, one can rewrite \(F(12)\) (equation (15)) in terms of the first eight operators of the AV18 interaction as follows:
\[
F(12) = \sum_p f_p(r_{12})O_p(12),
\]
(28)
\[
= f_c(r_{12}) + f_s(r_{12}) \sigma_1 \cdot \sigma_2 + f_t(r_{12}) \tau_1 \cdot \tau_2 + f_{sr}(r_{12}) \sigma_1 \cdot \sigma_2 \tau_1 \cdot \tau_2 + f_{rt}(r_{12}) S_{12} + f_{tr}(r_{12}) S_{12} \tau_1 \cdot \tau_2 + f_b(r_{12}) L.S + f_u(r_{12}) L.S \tau_1 \cdot \tau_2,
\]
(29)
where, regarding the equations (B1) to (B6) of the reference [19], \(f_p(r_{12})\) could be obtained by evaluating \(F(12)\) in \(ST = 00, 01, 10, 11\). Since the LOCV correlation functions are \(JSTT_2\)-dependent, we find \(F_{ST}(12)\) by the use of the following averaging expression [19]:
\[
F_{ST}(12) = \frac{\sum_{JLT_2T_2} f_{JSTT_2}(r_{12}) I_{L,T_2}(kFr_{12})}{\sum_{JLT_2T_2} I_{L,T_2}(kFr_{12})}.
\]
(30)
So, considering the equations (B1) to (B10) of the reference [19] and the above relation, one can extract the operator-dependent correlation functions \(f_p(r_{12})\) (equation (28)) from the LOCV channel-dependent formalism.
3. Results, discussions and conclusions

3.1. The Bethe homework problem

In this subsection, we present the $LOCV$ results for the Bethe homework problem in comparison with those of $FHNC$ [30, 31]. To this respect, first, we present the $(E)LOCV$ correlation functions in comparison with the corresponding parametrized correlation function $KT$, i.e. 

$$f_{KT}(r) = \exp(-\frac{b}{2r}\exp(-\frac{r}{b})),$$

which is used in the $FHNC$ calculations of the references [30,31]. Note that parameter $b$ at different densities were given in the Table 1 of the reference [17]. Then, we compare the $(E)LOCV$ equation of state for the Bethe homework problem with the corresponding results of $FHNC$ [30,31].

In the figure 1, the $(E)LOCV$ correlation function at $\rho = 0.6 \, fm^{-3}$ are compared with that of $KT$. To show the effects of the normalization constraint (equation (9)), we impose two different constraints: 1. $\xi(\equiv<\psi|\psi>-1) = 0$. This means that in the cluster expansion of $<\psi|\psi> = (1 + \chi_2 + \chi_3 + ...)$ [8], the two-body (two- and three-body) cluster term(s), i.e. $\chi_2 (\chi_2 + \chi_3)$, is (are) considered to become zero in the $(E)LOCV$ calculations. The corresponding calculations are labelled by $(E)LOCV$. 2. $\xi(\equiv<\psi|\psi>-1) = 1$, i.e. in the cluster expansion of $<\psi|\psi> \equiv \chi_2 (\chi_2 + \chi_3)$, is (are) considered to become 1 in the $(E)LOCV$ calculations. However, assuming $\chi_2 + \chi_3 = 1$, the $ELOCV$ formalism does not work. The $LOCV$ computations, applying $\chi_2 = 1$, are labelled by $LOCV(\chi_2 = 1)$. Note that this figure is an updated version of the Figure 2 of the reference [17].

As it is illustrated in the figure 1, the $(E)LOCV$ correlation functions for $\chi_2 (\chi_2 + \chi_3) = 0$ are short-range in comparison with that of $KT$, which is employed in the $FHNC$ computations. On the other hand, similar to the $KT$ correlation function, considering $\chi_2 = 1$, the corresponding $LOCV$ result, i.e. $LOCV(\chi_2 = 1)$, becomes long-range. So, it can be concluded that assuming the $KT$ correlation function, the contribution of the many-body cluster terms in the cluster expansion of $<\psi|\psi>$ becomes large, i.e. the $KT$ correlation function can not satisfy the normalization constraint (equation (9)). It is obvious that the $KT$ correlation function should be long-range to be able to be used in the many-body calculations. Also, we know that the mentioned function is defined in terms of the parameter $b$, which is fixed by the minimization of the many-body energy. But, unlike the $LOCV$ case, the mentioned minimization has not been done, subjected to any condition, e.g. the normalization constraint. Moreover, as it was mentioned in the previous section, the long-range part of the $(E)LOCV$ correlation function is defined by the Pauli function. So, this boundary condition makes the $(E)LOCV$ correlation functions overshoot to one. This overshoot: (i) maximizes the effect of the nucleon-nucleon interaction in the short inter-particle distances. (ii) makes the two-body distribution function $g(r)$ become exactly 1, for $r > d$ (d is the healing distance).

In the figure 2, the $(E)LOCV$ equation of state for the Bethe homework problem, imposing $\chi_2 (\chi_2 + \chi_3) = 0$, are represented in comparison with the $FHNC$ computations of Zabolitzky [31] and the $LOCV$ predictions of Owen, Bishop and Irvine [7] (OBI). It should be mentioned that in the $FHNC$ calculations of Zabolitzky, the $KT$ correlation function is employed. To clarify the significance of the normalization constraint role, the $LOCV$ calculations, assuming $\chi_2 = 1$ ($LOCV(\chi_2 = 1)$), are also shown in this figure. As it was mentioned above, the $ELOCV$ formalism does not work when $\chi_2 + \chi_3 = 1$. Moreover, the $FHNC$ computations, using the $(E)LOCV$ and the $LOCV(\chi_2 = 1)$ correlation functions, which are respectively labelled by $(E)FHNC$ and $FHNC(\chi_2 = 1)$, are plotted for comparison. Note that: (i) The equations (28) to (31) of the reference [17], are used for the $FHNC$ calculations with the $(E)LOCV$ correlation functions. (ii) This figure should be considered as the updated version of the Figure 3 of the reference [17]. (iii) For comparison with the available $FHNC$ and the Monte Carlo results, our computations are given at high densities, i.e. at densities larger than 6 times nuclear saturation density. (iv) The Monte Carlo calculations of Ceperley, Chester and Kalos (CCK) [42], which
Figure 1. The \((E)\)LOCV Bethe homework problem correlation function, for \(\chi_2(\chi_2 + \chi_3) = 0\), versus those of \(KT\) and \(LOCV(\chi_2 = 1)\). This figure should be considered as the updated version of the Figure 2 of the reference [17].

should be considered as the exact results, are shown by dotted line in this figure.

Referring to the figure 2, it is apparent that the \(LOCV\) results are expectedly similar to those of \(OBI\). Note that \(OBI\) also used the \(LOCV\) formalism. Furthermore, considering \(\chi_2(\chi_2 + \chi_3) = 0\), the \((E)\)LOCV results fairly agree with those of \(Zabolitzky\). Whereas, unlike the \(FHNC\) computations of \(Zabolitzky\), in the \((E)\)LOCV calculations, the cluster expansion of energy is truncated, at the two(three)-body cluster term. This consistency refers to the normalization constraint, which is optimally satisfied at the two(three)-body cluster approximation. So, in the \((E)\)LOCV approach, the contribution of higher cluster terms becomes negligible. In addition, the \(FHNC\) calculations with the \((E)\)LOCV correlation function, for \(\chi_2(\chi_2 + \chi_3) = 0\), fairly agree with the corresponding computations of \(Zabolitzky\). While, the former calculations are obtained with one decimal place accuracy, by the use of a simple mixing iteration method. On the other hand, the \(LOCV\) results, assuming \(\chi_2 = 1\), lie far below of the other results. Similarly, the \(FHNC\) computations with the \(LOCV(\chi_2 = 1)\) correlation function significantly differ from the corresponding results, for which the \(LOCV(\chi_2 = 0)\) correlation
Figure 2. The \textit{LOCV}, \textit{ELOCV}, \textit{FHNC} and \textit{EFHNC} results in comparison with those of \textit{LOCV}(\chi_2 = 1) and \textit{FHNC}(\chi_2 = 1). For a better comparison the \textit{OBI} [7], \textit{Zabolitzky} [31] and \textit{CCK} [42] calculations are also shown by dots, squares and dotted line, respectively. This figure is an updated version of the Figure 3 of the reference [17].

function is included. So, one may realize that the consistency among different approaches disappears when \chi_2 = 1, even if one uses the higher approximation method, i.e. the \textit{FHNC} approach. In the other word, the satisfaction of the normalization constraint, at the two(three)-body cluster term, leads to the reasonable agreements between the \textit{(E)LOCV} and the \textit{FHNC} results. Finally, it is worth mentioning that our approximated \textit{(E)LOCV} results fairly agree with those of \textit{CCK} [42], especially at (low) high densities. So, one may realize that the approximated \textit{LOCV} method is able to achieve reasonable results for the Bethe homework problem in a more simpler way than the \textit{FHNC} and the Monte Carlo approaches.

3.2. The nucleonic matter problem for the AV18 interaction

At the beginning, we compare the \textit{LOCV nucleonic} matter equation of state [19] with those of \textit{FHNC/SOC} [39,41] and \textit{AFDMC} [25,39]. Then, the \textit{LOCV} operator-dependent correlation functions [19] are compared with those of \textit{FHNC/SOC} [35,36].
Figure 3. (a) The LOCV nuclear matter equation of state [19], using the tensor(spin-orbit)-dependent correlation function in the $3P_2 - 3F_2$ channel, is labelled by $MT1(2)$. The corresponding $FHNC/SOC$ ($APR(LBFIS)$) [41] ( [39]) and the $AFDMC$ [25] results are presented for comparison. Note that the $MT1, 2$ and the $APR$ data are obtained, employing the $AV18$ interaction [22]. While, the $LBFIS$ and the $AFDMC$ predictions are achieved, using the $AV'$7, 8 interactions [23, 25]. (b) The LOCV neutron matter equations of state [19] ($MT1, 2$), for the $AV18$ [22] interaction, in comparison with those of $FHNC/SOC$ ($APR(LBFIS)$) [41] ( [39]) and $AFDMC$ [39], for which the $AV'$8 [23] potential is considered.

The LOCV symmetric nuclear (pure neutron) matter equation of states [19] for the $AV18$ interaction, which are labelled by $MT1, 2$, are given in the panel (a(b)) of the figure 3. The $MT1(2)$ results are obtained by including the tensor(spin-orbit)-dependent correlation function in the $3P_2 - 3F_2$ channel. Also, the $FHNC/SOC$ calculations of the references [41] ($APR$) and [39] ($LBFIS$) are plotted for comparison. Note that in the $FHNC/SOC$ computations of $APR(LBFIS)$, the $AV18$ ($AV'$8) [22, 23] is considered. Moreover, the symmetric nuclear (pure neutron) matter $AFDMC$ predictions of the reference [25] ( [39]), using the $AV'$7(8) potential [25] ( [23]), are shown by squares in the mentioned figure.
Figure 4. The LOCV nuclear matter correlation functions $f_p(r)$ [19], using the AV18 [22] interaction, at $\rho = 0.16 (0.24)fm^{-3}$, are labelled by $MT1(2)$. The corresponding FHNC/SOC results [36] ($M1, 2$), employing the AV18 and the UIX potentials [43, 44], are also shown for comparison. Note that according to the first 8 operators of the AV18 interaction [22], $f_p(r)$, for $p = c, s, t, st, T, Tt, b, bt$, are respectively plotted in 8 panels.

As it is demonstrated in the figure 3, the $MT1$ results fairly agree with those of FHNC/SOC [39, 41] and AFDMC [25, 39], especially at low densities. While, the $MT2$ calculations considerably lie below of the other results, particularly at high densities. So, it can be inferred that, including the spin-orbit-dependent correlation function in the $^3P_2 - ^3F_2$ channel, the nucleonic matter equation of state becomes more bound. Furthermore, the LBFIS [39] and the AFDMC [25, 39] predictions disagree with those of $MT1, 2$. Note that: (i) The AV18 interaction is employed in the LOCV method. Whereas, in the LBFIS and the AFDMC calculations [22,35], the AV7, 8 potentials are included. (ii) The LOCV results are plotted
Figure 5. The LOCV neutron matter correlation functions $f_p(r)$ (MT1,2) [19], at $\rho = 0.15 \text{fm}^{-3}$ in comparison with those FHNC/SOC [35] (WFF). Moreover, the $MT1(2)$ results are calculated, including the tensor-(spin-orbit)-dependent correlation function in the $^3P_2 - ^3F_2$ channel. Also, the $WFF(LOCV)$ results are given, for the $Av_{14(18)}$ interaction [21] ( [22]). Note that $f_p(r)$, for $p = c, s, T, b$, are respectively plotted in 4 panels.

at high densities, i.e. at densities larger than 6 times nuclear saturation density, in order to compare the mentioned calculations with the available FHNC/SOC data. (iii) The presented results for the symmetric nuclear matter, do not show the saturation near the empirical point. Because, the three-body potential is not included in the given calculations.

Now, we display the LOCV nucleonic matter operator-dependent correlation functions [19] versus those of FHNC/SOC [35, 36]. In this way, in the figure 4, the LOCV symmetric nuclear matter correlation functions $f_p(r)$, at $\rho = 0.16(0.24) \text{fm}^{-3}$ ($MT1(2)$) are plotted in comparison of the FHNC/SOC calculations of the reference [36] ($M1(2)$). Note that: (i)
$f_p(r)$, for $p = c,s,t,T,Tt,b,bT$, are respectively given in eight panels. (ii) In the LOCV computations, the AV18 interaction is applied. While, in the FHNC/SOC approach, the AV18 and the U1X [43, 44] potentials are included. (iii) In the reference [36], the central and the tensor-isospin-dependent correlation functions were reported.

Considering the figure 4, one may recognize that the overall behaviour of the $MT_{1,2}$ results agrees with those of $M_{1,2}$. Although, the FHNC/SOC results are different from those of LOCV. This discrepancies refer to the inclusion of the three-body potential and the many-body terms in the FHNC/SOC calculations. By increasing the density, the LOCV and the FHNC/SOC correlation functions expectedly become short-range. However, the range of the spin-orbit-dependent correlation functions do not change noticeably. Furthermore, the (non-)central correlation functions correctly tend to (zero) one.

In the figure 5, the LOCV pure neutron matter correlation functions $f_p(r)$ [19], using the AV18 potential, are presented at $\rho = 0.15$ fm$^{-3}$. In the LOCV formalism, once the tensor-dependent correlation function and once again the spin-orbit-dependent correlation function is employed in the $^{3}P_2 - ^{3}F_2$ channel, which are respectively labelled by $MT1$ and $MT2$. The FHNC/SOC correlation functions (WFF) [35], considering the $Au_{14}$ potential [21], are also plotted for comparison. Note that in the reference [35], the correlation functions are obtained by considering the three-body potential $UrbanaVII$ [45]. So, we divide them by the variational parameters $\beta_p$ [35], in order to obtain the appropriate correlation functions for the two-body interaction $Au_{14}$. Furthermore, in the mentioned figure, $f_{c,s,T,b}(r_{12})$ are reported. Because, in the neutron matter calculations, the isospin-dependent correlation functions cannot be distinguished from those of isospin-independent.

As shown in the figure 5, the LOCV results fairly behave like those of WFF. But, the FHNC/SOC spin- and tensor-dependent correlation functions are nearly inconsistent with those of LOCV. Because, unlike the LOCV results, the WFF correlation functions are designed for the many-body calculations. Note that the spin-orbit-dependent correlation function, which was not reported in the FHNC/SOC computations, has the shortest range in comparison with the other ones. In addition, at large inter-particle distances, the (non-)central correlation functions expectedly tend to (zero) one.

In the end it is worth mentioning that:

(i) There are some disagreements between the LOCV and the FHNC/SOC predictions. Because, unlike the approximated LOCV approach, in the FHNC/SOC method, the contribution of the many-body terms is included.

(ii) As it was shown in the previous section, the LOCV correlation functions are channel-dependent, i.e. they depend on $JSTT_2$. Furthermore, as it was illustrated in the figure 2, the ELOCV results lie near those of LOCV, especially at low densities. This means that, applying the normalization constraint, the contribution of the higher cluster terms than the two-body ones becomes negligible, i.e. the two-body cluster approximation of the LOCV calculations is reliable. As a result, referring to the reference [37], one may expect that the state-independent FHNC/SOC correlation functions could not be as optimal as those of LOCV.

(iii) Since, the minimization of the many-body energy with respect to the correlation functions are sophisticated, the FHNC/SOC correlation functions cannot be extracted from the many-body formalism, directly. While, in the LOCV approach, the two-body cluster energy is functionally minimized, subject to the normalization constraint of the radial distribution function, at the two-body cluster approximation. As a result, unlike the FHNC/SOC formalism, the LOCV correlation functions could be extracted from the two-body cluster approximation formalism, directly.
4. Conclusions

In conclusion, we can express that:

1. The $(E)LOCV$ Bethe homework calculations fairly agree with those of $FHNC$ (see figure 2). While, the $(E)LOCV$ results were reported at the two(three)-body cluster approximation. So, one may realize that in the $(E)LOCV$ approach, the contribution of the many-body cluster terms, which is included in the $FHNC$ formalism, can be neglected. Because, in the $(E)LOCV$ computations, we impose the normalization constraint.

2. The $(E)LOCV$ correlation functions have distinguished features (see figure 1). So, the $FHNC$ calculations with these correlation functions lie close to the $FHNC$ calculations, using the parametrized $KT$ correlation functions (see figure 2).

3. The $LOCV$ nucleonic matter computations, using the sophisticated $AV18$ potential, are fairly consistent with those of $FHNC/SOC$ and $AFDMC$ (see figure 3). Whereas, the $LOCV$ channel-dependent calculations were obtained at the lowest order approximation. It should be note that in the $LOCV$ formalism, the tensor- and spin-orbit-dependent correlation functions were included in the coupled channels. While, as it was noted in the Introduction, the spin-orbit correlations have been recently ignored in the $FHNC/SOC$ calculations.

4. The behaviour of the $LOCV$ operator-dependent correlation functions is fairly similar to that of $FHNC/SOC$ (see figures 4 to 5).

5. The many-body effects make the $FHNC/SOC$ correlation functions become different from those of $LOCV$ (see figures 4 to 5). But, as it was noted in the Introduction, one may expect that: (i) The $LOCV$ operator-dependent correlation functions, can improve the convergence of the $FHNC/SOC$ calculations. (ii) Including the $LOCV$ or the $FHNC/SOC$ operator-dependent correlation functions, the $FHNC/SOC$ energy does not vary remarkably.

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