QFT’s With Action of Degree 3 and Higher and Degeneracy of Tensors

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Abstract

In this paper we develop a technique of computation of correlation functions in theories with action being cubic or higher degree form in terms of discriminants of corresponding tensors. These are analogues of formula $\int \exp(iT(x))dx \sim \det T^{-1/2}$ for symmetric tensors of rank two.

The reader who does not need a motivation may proceed directly with the technique starting from Section 2.1. In the paper we use terms ”polynomial”, ”form” and ”tensor” as synonyms, depending on the context.

0 Introduction

Let $Q$ be a quadratic form on elements $v$ of a vector space $V$ (configuration space). Then we may take integrals of functions with respect to the measure of integration $\exp(iQ(v))dv$, i.e. take their correlation functions:

$$< f > := \int_V f(v) \exp(iQ(v))dv$$

In particular, when $V$ is n-dimensional and $Q$ is nondegenerate, the correlation function of constant 1 (partition function) is given by the identity:
\[ Z(Q) := <1> = \int_V \exp(iQ(x)) d^n x = \frac{\pi^{n/2}}{|\det Q|^{1/2}} \exp \left( i \pi \frac{\text{sgn} Q}{4} \right) \]

This formula becomes basic if we introduce a parameter \( k \) into our measure, making it as \( \exp(ikQ(v)) dv \) and then let \( k \to \infty \). Make a change of variables \( v' := v \sqrt{k} \) and write:

\[ <f> = \int_V \frac{f(v')}{k^{n/2}} \exp(iQ(v')) dv' \]

Now as \( k \to \infty \) we expand \( f \) around 0, make a renormalisation (i.e. multiply by \( k^{n/2} \) in order to get a finite number as a limit) and get

\[ <f> = <f(0) + O \left( \frac{1}{\sqrt{k}} \right) > = <1 > f(0) + O \left( \frac{1}{\sqrt{k}} \right) \]

So the main impact into correlation function is made by the neighbourhood of radius \( 1/\sqrt{k} \) of critical points, where \( Q(v) \) with its first derivatives equals to 0. For nondegenerate \( Q \) the only such point is the origin 0. When \( Q \) is degenerate, in the space \( V \) besides 0 there is a distinguished set \( C \) of vectors, which are solutions of variational problem (or stationarity condition):

\[ \delta Q(v) \equiv dQ(v) = 0 \quad \text{(0.2)} \]

This variational problem may be written in the form of homogeneous system of linear equations.

**Example** Let \( \dim V = 2 \). Then

\[ \delta Q(v) = 0 \iff \begin{cases} q_{11}v_1 + q_{12}v_2 = 0 \\ q_{21}v_1 + q_{22}v_2 = 0 \end{cases} \]

These are points where \( Q(v) \) is ”stationary” (i.e. with its first derivatives, Eq.(0.2)) equal to 0, or points of extremum of \( Q \). The leading impact to the integral is made by regions in a neighbourhood of the set \( C \) of these critical points. The larger is the value of parameter \( k \), the smaller is the radius of this leading neighbourhood, as we have seen in nondegenerate case. The integral
becomes divergent, but we may make it to have sense passing to a proper limit and using a proper renormalization procedure (division by the "volume of the space" of solutions $C$). Having this done, we get for such renormalized integration of a function $f(v)$:

$$\int_V f(v) \exp(ikQ(v)) dv \sim \frac{1}{Vol \ C} \int_C f(v) dv + O\left(\frac{1}{\sqrt{k}}\right)$$

The higher order (quantum) terms may be computed again by applying at each critical point $c$ the basic formular (0.1), but now for the restriction of $Q$ onto the space transversal to $T_c(C)$. I want to stress that these considerations are applicable to both finite and infinite dimensional cases. When the space $V$ is infinite dimensional (for example, the space of fields $\psi$ on a manifold), the form $Q$ is called action $S(\psi)$ and our integral with a procedure (if there is any) of assigning a sense to it is called path integral. The condition, that the form is degenerate is formulated in terms of existence of $\psi$, which are solutions for variational problem:

$$\delta S(\psi) = 0$$

This problem is written in the form of field equations for $\psi$ (the analogue of homogeneous linear system in finite dimensional case). In infinite dimensional case the solution of these equations is of particular interest, since using series expansion around these classical states is essential for being able to compute correlation functions. So what we need in order to compute correlation function with the action being quadratic form - is to compute the determinant of this form and in case when this form is degenerate - to compute the determinant of its restriction onto the space, transversal to the set of critical points - solutions of $dQ(v) = 0$. In infinite dimensional case the procedure of computation of determinant needs a special definition (zeta-function definition), which at least must be compatible with finite dimensional case.

1 Actions of higher order and critical points

The process of computing the path integral in infinite dimensional case may be defined as a sequence of finite dimensional integrals and a procedure of finding a limit of this sequence as the number of variables tends to $\infty$. In
particular, when the domain of fields $\psi$ is a compact manifold $M$, then each triangulation of $M$ gives us a finite set of vertices as the set of indices of action form $S(\psi)$. Then we need to compute the integral with action being finite dimensional form and give the procedure of passing to the limit as the grid of triangulation is getting smaller.

So let us take the action functional as a form on $m$-dimensional vector space $V$, i.e. a polynomial (in general nonhomogeneous) of $m$ variables $P(x_1,\ldots,x_m)$. Take an $m$-cube $R$ in $V$ with center at 0 and side of length $l$. We may take the integral over $R$ and try to pass to the limit $l \to \infty$:

$$Z(P) := \lim_{l \to \infty} \int_R \exp(iP(x_1,\ldots,x_m))dx_1\ldots dx_m$$

**Proposition 1.1** The limit $Z(P)$ exists iff the set $C$ of critical points of function $P(x_1,\ldots,x_m)$ on $V$ is compact.

This set of critical points is given as the solution of variational problem:

$$dP(x_1,\ldots,x_m) = 0$$

We may expect $Z(P)$ to be analytic function of coefficients of $P$, which, according to Proposition 1.1, has poles at those points which correspond to forms with noncompact set of critical points.

## 2 Homogeneous forms

Let $T(x)$ be a homogeneous polynomial of $m$ variables $x := (x_1,\ldots,x_m)$. Consider the corresponding variational problem:

$$dT(x) = 0$$

In general the number of equations in this system is more then the number of nonhomogeneous variables, so this system is solvable iff the coefficients of $T$ satisfy certain condition.

**Definition** If the variational problem has solutions except $x \equiv (x_1,\ldots,x_m) = (0,\ldots,0)$ (and, since $T$ is homogeneous, the set of critical point $C$ is noncompact) the form $T$ is called *degenerate*. The set of degenerate forms is called *discriminant*. 

4
The discriminantal set is described by the equation on coefficients of the form:

\[ \text{Dis}(T) = 0 \]

where the expression \( \text{Dis}(T) \) is a polynomial of coefficients of \( T \) called the discriminant. It is an invariant of the form.

The solvability of variational problem may be reformulated in terms of geometric properties of the algebraic manifold, corresponding to the polynomial \( T(x) \) as:

**Proposition 2.1** If \( \text{Dis}(T) = 0 \) then the algebraic hypersurface \( T(x) = 0 \) is singular.

**Example** Let \( T(x) = ax^2 + bx_1x_2 + cx_2^2 \) be a binary form of degree 2. The corresponding variational problem is:

\[
\frac{\partial T(x)}{\partial x_1} = 2ax_1 + bx_2 = 0, \quad \frac{\partial T(x)}{\partial x_2} = bx_1 + 2cx_2 = 0
\]

This system is solvable iff \( \text{Dis}(T) = \begin{vmatrix} 2a & b \\ b & 2c \end{vmatrix} = b^2 - 4ac = 0 \), i.e. the roots of \( T(x) = 0 \) (which are two 1-dimensional subspaces in the 2-dimensional space of homogeneous variables \((x_1, x_2)\)) merge.

### 2.1 \( n \)-linear forms

#### 2.1.1 Basic example

**Proposition 2.2**

\[
\int \int_{\mathbb{R}^1 \times \mathbb{R}^3} \exp(iaxy)dxdy = \frac{2\pi}{a}
\]

For a quadratic form \( Q \) on \( n \)-dimensional vector space \( V \) we may take its polarization - a symmetric bilinear form \( B(u, v) \) on the pair \( U \times V \) of \( n \)-dimensional spaces, such that \( Q(v) = B(v, v) \). Let us write the analogue of formula (0.1) in terms of \( B \). Let \( R_1, R_2 \) be two \( n \)-cubes in \( V \) and \( U \) correspondingly with centers at 0 and sides of length \( l \).
Proposition 2.3

\[
Z(B) := \int_U \int_V \exp(iB(v, u)) dvdu := \lim_{l \to \infty} \int_{R_1} \int_{R_2} \exp(iB(v, u)) dvdu = \frac{(2\pi)^n}{\det B}
\]

So we may understand \( \int_U \int_V \exp(iB(v, u)) dvdu \) as the integral taken in this principal value sense.

**Proof** Make a change of variables \( v' = B^{-1}v, \ u' = u \). Using these variables and Proposition 2.2 we write:

\[
\int_U \int_V \exp(iB(v, u)) dvdu = \int_U \int_V \exp(i \sum_{k=1}^n v'_k u'_k) \frac{dv'}{\det B} du' =
\]

\[
\frac{1}{\det B} \left( \int \int \exp(i xy) dx dy \right)^n = \frac{(2\pi)^n}{\det B}
\]

In bilinear case the condition of degeneracy of \( B \), i.e. \( d_v B(u, v) = 0, \ d_u B(u, v) = 0 \) is a pair of equivalent linear systems, while each of those is in turn equivalent to the system \( dQ(v) = 0 \). This equivalence is a specific of bilinear case.

2.1.2 General construction

Let us take as an action functional a \( d \)-linear form \( T \) on the space \( V_1 \oplus \ldots \oplus V_d \), and let \( \dim V_i = n_i \) for each \( i \). Having this form we get the measure of integration \( \exp(iT(v_1, \ldots, v_d))dv_1 \ldots dv_d \). Take an \( n \)-cubes \( R_i \) with centers at 0 in the corresponding \( V_i \) and sides of length \( l \). We may take the integral over \( R_1 \times \ldots \times R_d \) and try to pass to the limit \( l \to \infty \) as in Section 1:

\[
Z(T) := \lim_{l \to \infty} \int_{R_1} \ldots \int_{R_d} \exp(iT(v_1, \ldots, v_d))dv_1 \ldots dv_d
\]

To decide whether \( C \) is compact we need to write for the form \( T(v_1, \ldots, v_d) \) (as for all the forms considered so far) the stationarity conditions:

\[
\delta T(v) = 0 \iff d_{v_i} T(v_1, \ldots, v_d) = 0, \ldots, d_{v_d} T(v_1, \ldots, v_d) = 0
\]

This is a set of \( d \) (one for each \( v_i \)) nonequivalent (unlike in bilinear case) systems of \( (d-1) \)-linear equations.
2.1.3 Compact set of critical points

Let the form \( T \) be nondegenerate. Then the set of critical points consists of only one point - the origin. The term “compact”, applied to this set will become less redundant in the case of nonhomogeneous forms, considered below. Let \( \deg T \) denote the degree of \( \text{Dis}(T) \).

**Proposition 2.4**

\[
\int_{V_d} \cdots \int_{V_1} \exp(iT(v_1, ..., v_d))dv_1...dv_d := \lim_{l \to \infty} \int_{R_d} \cdots \int_{R_1} \exp(iT(v_1, ..., v_d))dv_1...dv_d = \frac{(2\pi)^{n(d-1)}}{|\text{Dis}(T)|^{n/\deg T}}
\]

Notice, that when \( T \) is bilinear \( \deg T = n \), \( \text{Dis}(T) = \det T \) and we get the formula (0.1).

So we may understand \( Z(T) \) as the integral taken according to this limit procedure.

**Proof** This is done in the following steps:

- for \( d = 2 \) this is the content of Proposition 2.3
- \((d + 1)\)-linear form \( T(v_1, ..., v_d, v_{d+1}) \) is considered as a linear combination of \( d \)-linear forms \( T'(v_1, ..., v_d) \), i.e. it may be viewed as \( d \)-linear form on \( V_1 \oplus \cdots \oplus V_d \) with coefficients \( a'_{i_1...i_d} \) depending on \( v_{d+1} \in V_{d+1} \):
  \[
a'_{i_1...i_d} = \sum_{i_{d+1}} a_{i_1...i_d i_{d+1}} (v_{d+1})_{i_{d+1}}
\]
  where \((v_k)_i\) denotes the \( i \)-th component of vector \( v_k \in V_k \).
- providing, that the statement is true for \( d \)-linear forms integrate over the space \( V_1 \oplus \cdots \oplus V_d \) to get
  \[
  Z(T) = (2\pi)^{n(d-2)} \int_{V_{d+1}} \frac{dv_{d+1}}{|\text{Dis}T'(v_{d+1})|^{n/\deg T'}}
  \]
• consider \( V_{d+1} \) as a real \( n_{d+1} \)-cycle of integration in the space \( \mathbb{C}^{n_{d+1}} \) punctured at infinity and compute the last integral as multidimensional residue of the meromorphic function \( f(v_{d+1}) := 1/\text{DisT}'(v_{d+1}) \).

**Example** Let me illustrate this method in the case of \( 2 \times 2 \times 2 \) form

\[
T(x, y, z) = \sum_{i,j,k=1,2} a_{ijk} x_i y_j z_k.
\]

We will use the following fact from complex analysis of many variables:

**Lemma 1** For a function \( f(z_1, z_2) \) holomorphic in domaine \( R \subset \mathbb{C}^2 \) and \( (w_1, w_2) \in R \)

\[
f(z_1, z_2) = \frac{1}{(2\pi i)^2} \int_{|w_1 - z_1| = r} \int_{|w_2 - z_2|} \frac{f(w_1, w_2)}{(w_1 - z_1)(w_2 - z_2)} dw_1 dw_2
\]

Now for \( (x_1, x_2) \in V, (y_1, y_2) \in U, (z_1, z_2) \in W \) using Proposition 2.3 we have:

\[
\int_V \int_U \int_W \exp(i \sum_{i,j,k} a_{ijk} x_i y_j z_k) d^2 x d^2 y d^2 z
\]

\[
= \int_W \left( \int_V \int_U \exp(i \sum_{i,j} b_{ij} x_i y_j) d^2 x d^2 y \right) d^2 z = (2\pi)^2 \int_W \frac{dz_1 dz_2}{\det B}
\]

where \( b_{ij} := \sum_k a_{ijk} z_k \).

Then

\[
\det B = a z_1^2 + b z_1 z_2 + cz_2^2
\]

where \( a, b, c \) are expressions of degree 2 of \( a_{ijk} \).

Make the change of variables

\[
z_1' := (2b + \sqrt{D}) z_1 + (-2b + \sqrt{D}) z_2 \]
\[
z_2' := (z_1 - z_2) / \sqrt{D}
\]

where \( D := b^2 - 4ac \).

Using these variables we may write

\[
\int_W \frac{dz_1 dz_2}{\det B} = \frac{1}{\sqrt{D}} \int_W \frac{dz_1' dz_2'}{z_1' z_2'}
\]
Considering $W$ as real 2-dimensional cycle in $\mathbb{C}^2$, the value of the last integral is computed according to Lemma as 2-dimensional residue of function $\frac{1}{z_1z_2}$ at 0:

$$\int \int |z'_1| |z'_2| = (2\pi i)^2$$

Substituting these data we get:

$$\int \int \int VUW \exp(i \sum a_{ijk} x_i y_j z_k) d^2 x d^2 y d^2 z = -\frac{(2\pi)^4}{\sqrt{\text{Dis}(a_{ijk})}} =$$

$$= -(2\pi)^4 (a_{111}^2 + a_{112}^2 + a_{121}^2 + a_{211}^2) - 2a_{111}a_{121}a_{221} - 2a_{111}a_{211}a_{122} - 2a_{111}a_{221}a_{212} - 2a_{211}a_{122}a_{121} - 2a_{212}a_{211}a_{122} + 4a_{111}a_{221}a_{121}a_{212} + 4a_{121}a_{221}a_{112}a_{222})^{1/2} = -\frac{(2\pi)^4}{\text{Dis}(a_{ijk})^{1/2}}$$

where $\text{Dis}(a_{ijk})$ is the discriminant of 3-linear form, which is:

- the condition of solvability of the system $d(\sum a_{ijk} x_i y_j z_k) = 0$ of 6 bilinear equations (variational problem)

- the condition for the set of critical points of function $\sum a_{ijk} x_i y_j z_k$ on $V \times U \times W$ being noncompact (not discrete)

- the condition for cubic $\sum a_{ijk} x_i y_j z_k$ being singular.

### 2.1.4 Noncompact set of critical points

Let us take as an action functional a $d$-linear form $T$ on the space $V_1 \oplus \ldots \oplus V_d$. Having this form we get the measure of integration $\exp(i T(v_1, \ldots, v_d)) dv_1 \ldots dv_d$. Let the dimension each $V_i$ be equal to $n_i$. Take an $n_i$-cubes $R_i$ with centers at 0 in the corresponding $V_i$ and sides of length $l$. We may take the integral over $R_1 \times \ldots \times R_d$ and try to pass to the limit $l \to \infty$ as in Section 1:

$$Z(T) := \lim_{l \to \infty} \lim_{R_d \to R_1} \ldots \int \exp(i T(v_1, \ldots, v_d)) dv_1 \ldots dv_d$$

To decide whether $C$ is compact we need to write for the form $T(v_1, \ldots, v_d)$ (as for all the forms considered so far) the stationarity conditions:
\[
\delta T(v) = 0 \iff d_v T(v_1, \ldots, v_d) = 0, \ldots, d_{v_d} T(v_1, \ldots, v_d) = 0
\]

This is a set of \(d\) systems (one for each \(v_i\)) of \((d - 1)\)-linear equations the solutions of which will be critical points for \(T\). As in bilinear case this homogeneous system is not solvable in general. If we want this system to have nontrivial (nonzero) solutions, which (since \(T(v_1, \ldots, v_d)\) is \(d\)-linear) will form a linear subspace in \(V_1 \oplus \ldots \oplus V_d\), then the number of equations must be not more then the number of nonhomogeneous variables, i.e. we have the condition on dimensionality of spaces \(V_i\): \(n_1 + \ldots + n_k + \ldots + n_d - d + 2 < n_k\) for some \(k\). In this case the integral diverges. Since in this case the set \(C\) of critical points is a linear subspace in \(V_1 \oplus \ldots \oplus V_d\) we may consider the form \(T'\) induced on the factor space \(V_1 \oplus \ldots \oplus V_d/C\), on which the form is nondegenerate, and so we get into conditions of Section 2.1.3. In terms of computations, in order to get the finite number as the value of \(Z(T)\) we take the limit:

\[
Z(T) := \lim_{l \to \infty} \frac{1}{Vol(R_1 \times \ldots \times R_d \cap C)} \int_{R_d} \ldots \int_{R_1} \exp(iT(v_1, \ldots, v_d))dv_1 \ldots dv_d
\]

\[
= \frac{\Lambda(n_1, \ldots, n_d)}{|Dis(T')|^{(n_1 + \ldots + n_d)/d \deg_{T'}}}
\]

where \(Vol(R_1 \times \ldots \times R_d \cap C)\) is computed with respect to the volume form on \(C\) induced form the total space \(V_1 \oplus \ldots \oplus V_d\), and \(\Lambda(n_1, \ldots, n_d)\) is a constant.

**Example** Let \(T = a_1 x_1 y + a_2 x_2 y + \ldots + a_n x_n y\) be a bilinear form on \(V \times U\), where \(\dim V = n, \dim U = 1\). This form is degenerate, the set of its critical points is \(C = \{(x_1, \ldots, x_n) \in V \mid a_1 x_1 + \ldots + a_n x_n = 0\} \times U\), and \(\dim C = n - 1\).

**Proposition 2.5**

\[
Z(T) = \frac{1}{Vol(C)} \int \exp(iT(x_1, \ldots, x_n, y))dx_1 \ldots dx_n dy = \frac{2\pi}{(a_1^4 + \ldots + a_n^4)^{1/4}}
\]

where the division procedure is understood as passing to the limit above.

Notice, that if \(n = 1\), this formula gives the result of Proposition 2.2.
2.2 $n$-ary Forms

Action functionals being considered in physics so far are restrictions of $d$-linear forms (on finite or infinite dimensional spaces) onto diagonal of sets $V_1 \oplus \ldots \oplus V_d$, i.e. they are taken as $S(v) = T(v, \ldots, v)$. Taking the variational problem we have to differentiate only on one group of variables $v$ and the resulting equations are not $(d-1)$-linear, but equations of degree $(d-1)$ on $v$. These equations may be obtained from any of $d$ systems of (2.2) by restricting it onto diagonal, i.e. letting $v_1 = \ldots = v_d = v$. This restriction causes a symmetry braking in the value of $Z(S)$ - it gets a nonzero imaginary part.

**Proposition 2.6** Let $\dim V = n$. Then

$$Z(S) := \int_V \exp(iS(v))dv = \frac{\Lambda(n, d)}{|\text{Dis}(S)|^{n/\deg S}} \exp(i\pi \text{sgn}(S))$$

where $\Lambda(n, d)$ is a constant (which still has to be computed for general $n$ and $d$), $\text{Dis}(S)$ is the discriminant of $S$, i.e. the condition of singularity of the algebraic hypersurface $S(v) = 0$ in $V$, and the phase $\text{sgn}(S)$ is a function on the set of connected components of $S^dV^* \setminus \text{Dis}(S)$.

The discriminantal manifold $\text{Dis}(S)$ makes a partition of the space $S^dV^*$ of $n$-ary forms and the phase function is constant on each of these component and undergoes a jump when the set of coefficients of $S$ crosses the discriminantal surface passing to another component. So the ”phase function” $\text{sgn}(S)$ distinguish the components of the complement to discriminantal set. The first example of this phenomenon is Formula 0.1. There $\Lambda = \Lambda(n, 2) = \pi n/2$.

2.2.1 Binary forms

Let $\dim V = 2$ (case of binary forms, when our physical space consists only of 2 points), so $S(v) = a_dx^d + ax^{d-1}y + \ldots + a_0y^d$. In this case $\deg S = 2(d-1)$. Then:

$$Z(S) := \int_V \exp(i(a_dx^d + ax^{d-1}y + \ldots + a_0y^d))dxdy = \frac{\Lambda(2, d)}{|\text{Dis}(S)|^{1/d(d-1)}} \exp(i\pi \text{sgn}(S))$$
here \( \text{Dis}(S) \) is the usual discriminant of polynomial of degree \( d \). For binary forms the number of component of the complement to discriminantal set has a geometric interpretation as the number of real roots of polynomial \( S \) the values of \( sgn(S) \) on these components are rational numbers, which still have to be computed for general case.

**Example** \( S = ax^2 + bxy + cy^2 \). Then \( \Lambda(2, 2) = \pi \) and the discriminantal hypersurface partitions 3-dimensional space \( S^2 V^* \ni (a, b, c) \) of coefficients into the following three connected components:

\[
D_1 = \left\{ \left( \frac{b}{2} \right)^2 - ac < 0, \ a > 0, \ c > 0 \right\} \\
D_2 = \left\{ \left( \frac{b}{2} \right)^2 - ac < 0, \ a < 0, \ c < 0 \right\} \\
D_3 = \left\{ \left( \frac{b}{2} \right)^2 - ac > 0 \right\}
\]

Then:

\[
Z(S) := \int \exp(i(ax^2 + bxy + cy^2))dx\,dy = \frac{\pi}{\left| \left( \frac{b}{2} \right)^2 - ac \right|^{1/2}} \exp(i\pi sgn(S))
\]

where the "phase function" is:

\[
sgn(S) = \begin{cases} 
1/2, & \text{for } (a, b, c) \in D_1 \\
-1/2, & \text{for } (a, b, c) \in D_2 \\
0, & \text{for } (a, b, c) \in D_3
\end{cases}
\]

### 3 Nonhomogeneous Forms

Let the action form \( P(v) \) be nonhomogeneous, i.e. the one with terms of different degrees, for example quadratic and cubic. First notice, that any nonhomogeneous action may be made homogeneous by introducing one additional variable and vice versa. The set \( C \) of solutions of variational problem \( \delta P = 0 \) in general consists of several points, critical points of function \( P \). Notice, that if \( v_0 \) is a critical point of \( P(v) \) then, making a change of variables \( v' := v - v_0 \) we have \( P(v) = c_0 + P'(v') \), where \( c_0 = P(v_0) \) is constant and \( P'(v') \) has terms only of order \( \geq 2 \). So without loss of generality we may consider forms without linear terms. According to Proposition 2.1 we expect \( Z(P) \) to be analytic function of coefficients of \( P \) with poles at those points, where the set of solutions of \( \delta P = 0 \) is noncompact. Let us call such forms
degenerate, as in homogeneous case. Notice, that the degeneracy condition on coefficients of $P$ does not coincide with the degeneracy condition for corresponding homogeneous form and vise versa. To find $Z(P)$ as an analytic function of coefficient of $P$ is an open question in general (even in the “initial” case of $P = a_3x^3 + a_2x^2$), but let us see an example, where the analytic properties of $Z(P)$ have a visual interpretation in terms of geometric properties of algebraic hypersurface $P(v) = 0$.

**Example** If we are given a quadratic form $T = \sum_{i,j=0}^{n} q_{ij} y_i y_j$ on $(n + 1)$-dimensional space $V \ni (y_0, y_1, ..., y_n)$, then we have a nonhomogeneous form of $x_i := y_0$ as $P = \sum_{i,j=1}^{n} a_{ij} x_i x_j + \sum_{i=1}^{n} b_i x_i + c$, where $a_{ij} := q_{ij}, b_i := a_{i0} + a_{0i} = 2a_{i0}, c := a_{00}$. Here in order to give a more symmetric form to formulas we use summation over all values of $i, j$, but, of course, $q_{ij} = q_{ji}$.

The variational problem for $P$ is:

$$\sum a_{ij} x_j = b_i / 2, \ i = 1, ..., n$$

Let $v_0$ be the solution of this nonhomogeneous . Then in terms of $v' = v - v_0$ we may write $P = \sum_{i,j=1}^{n} a_{ij} x_i' x_j' + P(v_0)$. Consider the corresponding partition functions $Z(T)$ and $Z(P)$. The condition of degeneracy of $T$ is $\det(q) = 0$, and that for $P$ is $\det(a) = 0$. So while $Z(T)$ is divergent we still may get a finite answer for $Z(P)$ integrating over nonhomogeneous variables. And $Z(P)$ gets in turn divergent just iff the $n \times n$ principal minor of $(q_{ij})$ is 0.

## 4 How to compute the discriminant of a form

In finite dimensional case the algorithm of computing discriminants of $d$-linear forms (or determinants of $d$-dimensional matrices) is described in [1], on computing discriminants of $n$-ary forms see [2]. Of course for infinite dimensional forms we need a procedure which is the analogue of the definition of zeta-function for quadratic actions. This will be the subject of subsequent paper.

**References**

[1] V.Dolotin, *On Discriminants of Multilinear Forms*, E-print [alg-geom/9511010](http://www.arxiv.org/abs/alg-geom/9511010)
[2] V.Dolotin, *On Invariant Theory*, E-print alg-geom/9512011