ABSTRACT. Let $p$ be a prime number. Mazur proved that a geometrically maximal unramified abelian covering of $X_0(p)$ over $\mathbb{Q}$ is given by the Shimura covering $X_2(p) \to X_0(p)$, that is, a unique subcovering of $X_1(p) \to X_0(p)$ of degree $N_p := (p - 1)/\gcd(p - 1, 12)$. In this short paper, we show that a geometrically maximal abelian covering $X'_2(p) \to X_0(p)$ over $\mathbb{Q}$ unramified outside cusps is cyclic of degree $2N_p$.

1. INTRODUCTION

1.1. Geometrically maximal unramified abelian covering. Let $Y$ be a smooth geometrically integral curve over a number field $k$ that admits a degree one divisor. (We do not assume $Y$ to be proper.) Let $k^{ab}$ be the maximal abelian extension of $k$, and set $G_{k}^{ab} := \text{Gal}(k^{ab}/k)$. Let $\pi_1(Y)^{ab}$ be the maximal abelian quotient of the étale fundamental group of $Y$ (with respect to some geometric point). Hence we have $\pi_1(Y)^{ab} = \text{Gal}(k(Y)^{ur,ab}/k(Y))$, where $k(Y)^{ur,ab}$ is the maximal unramified abelian extension of the function field $k(Y)$ of $Y$. Denote by $\pi_1(Y)^{ab,\text{geo}}$ the kernel of the canonical map $\pi_1(Y)^{ab} \twoheadrightarrow G_{k}^{ab}$, so that we obtain an exact sequence

$$0 \to \pi_1(Y)^{ab,\text{geo}} \to \pi_1(Y)^{ab} \to G_{k}^{ab} \to 0.$$ (1.1.1)

It is shown in [1, Theorem 1] that $\pi_1(Y)^{ab,\text{geo}}$ is finite.

Let us say a finite unramified abelian covering $Y' \to Y$ is geometrically maximal (over $k$) if $Y'$ is geometrically integral over $k$ and if the composition $\pi_1(Y)^{ab,\text{geo}} \hookrightarrow \pi_1(Y)^{ab} \twoheadrightarrow \text{Gal}(Y'/Y)$ is an isomorphism (equivalently, any finite unramified abelian covering $Y'' \to Y$ is a subcovering of $Y' \times_{\text{Spec} k} \text{Spec} k^{ab} \to Y$):

![Diagram](https://via.placeholder.com/150)

Since we have assumed that $Y$ admits a degree one divisor, there exists a geometrically maximal unramified abelian covering (see Proposition 2.4.6 below). When $Y$ has a $k$-rational point $x$, a maximal abelian unramified covering of $Y$ in which $x$ splits completely yields a geometrically maximal covering. There is, however, no such a description if no $k$-rational point is available. Given $Y$, we are interested in finding a geometrically maximal unramified abelian covering.
1.2. Modular curves. Let $p$ be a prime. We consider the modular curves $X_0(p)$ and $X_1(p)$ as geometrically integral smooth proper curves over $\mathbb{Q}$. We choose a model of $X_1(p)$ over $\mathbb{Q}$ such that the cusp at infinity splits completely in a finite cyclic covering $f_p : X_1(p) \to X_0(p)$ of degree $(p - 1)/2$. Note that $f_p$ is possibly ramified. We denote by $f'_p : X'_2(p) \to X_0(p)$ its (unique) subcovering of degree $N_p := (p - 1)/\gcd(p - 1, 12)$, which is the maximal unramified subcovering of $f_p$, called the Shimura covering. We recall an important result due to Mazur.

**Theorem 1.2.1** ([3] Theorem 2). The Shimura covering $f'_p : X'_2(p) \to X_0(p)$ is geometrically maximal over $\mathbb{Q}$.

Let $Y_0(p) \subset X_0(p)$ be an open subscheme such that $X_0(p) \setminus Y_0(p)$ consists of all (two) cusps. In this short note, we shall construct a geometrically maximal unramified covering of $Y_0(p)$ and prove the following result.

**Theorem 1.2.2.** There exists a cyclic covering $f'_p : X'_2(p) \to X_0(p)$ of degree $2N_p$ such that

1. $f'_p|_{Y'_2(p)} : Y'_2(p) \to Y_0(p)$ is a geometrically maximal unramified covering of $Y_0(p)$ over $\mathbb{Q}$, where $Y'_2(p) := f'^{-1}_p(Y_0(p))$; and
2. $f'_p$ factors as $f'_p = f'_p \circ \pi_p$, where $\pi_p : X'_2(p) \to X_2(p)$ is a degree two covering that is unramified outside cusps.

We shall prepare some general facts on abelian coverings of smooth curves in §2. We then prove Theorem 1.2.2 in [3]. The function field $\mathbb{Q}(X'_2(p))$ of $X'_2(p)$ will be obtained as a quadratic extension of $\mathbb{Q}(X_2(p))$ generated by a square root of an explicitly constructed rational function in $\mathbb{Q}(X_2(p))$. A key ingredient for this construction is the generalized Dedekind eta functions, which we recall in [3].

1.3. Notations and conventions. Let $G$ be a profinite group. For a $G$-module $M$, $M^G$ and $M_G$ denote its $G$-invariant and coinvariant part, respectively. We write $G^{ab}$ for the quotient of $G$ by the closure of its commutator subgroup. For a connected scheme $S$, we write $\pi_1(S)^{ab} := \pi_1(S, x)^{ab}$, where $\pi_1(S, x)$ is the étale fundamental group of $S$ with respect to some geometric point $x \to S$ (on which $\pi_1(S, x)^{ab}$ does not depend up to unique isomorphism). We write $H^i(S, F)$ for the étale cohomology for an étale sheaf $F$ on $S$.

Let $k$ be a field. We take an algebraic closure $\bar{k}$ and put $G_k := \text{Gal}(\bar{k}/k)$. For a $k$-scheme $V$, we write $\overline{V} = V \times_{\text{Spec } k} \text{Spec } \bar{k}$. A $G_k$-module $A$ is identified with an étale sheaf on $\text{Spec } k$ and we write $H^i(\text{Spec } k, A) = H^i(k, A)$. For $n \in \mathbb{Z}_{>0}$, the $G_k$-module of $n$-th roots of unity in $\bar{k}$ is denoted by $\mu_n$. We write $\mathbb{Z}/n\mathbb{Z}(r) = \mu_n^\otimes$ if $r \geq 0$, and $\mathbb{Z}/n\mathbb{Z}(r) = \text{Hom}(\mathbb{Z}/n\mathbb{Z}(-r), \mathbb{Z}/n\mathbb{Z})$ if $r < 0$. Let $A$ be a torsion $G_k$-module. We write $A[n] = \{a \in A \mid na = 0\}$ and $A(r) = \lim_{\to} A[n] \otimes \mathbb{Z}/n\mathbb{Z}(r)$. We define its maximal $\mu$-type subgroup by

\[(1.3.1) \quad A^\mu := \{\alpha \in A \mid \sigma(\alpha) = \chi_{\text{cyc}}(\sigma)a \text{ for all } \sigma \in G_k\},\]

where $\chi_{\text{cyc}} : G_k \to \hat{\mathbb{Z}}^\times$ is the cyclotomic character. (In other words, $A^\mu(-1) = A(-1)^{G_k}$.)
For a commutative algebraic group $G$ over $k$, we write $G[n]$ for the group of $n$-torsion points on $G(\overline{k})$, $G_{\text{Tor}} = \lim \rightarrow G[n]$ for the group of all torsion points on $G(\overline{k})$, and $TG = \lim \leftarrow G[n]$ for the full Tate module of $G$. We also write $TG(r) := \lim \rightarrow G[n] \otimes \mathbb{Z}/n\mathbb{Z}(r)$ and $\hat{\mathbb{Z}}(r) := \lim \mathbb{Z}/n\mathbb{Z}(r)$.

2. Abelian coverings of smooth curves

In this section, we collect basic facts about abelian fundamental group of a smooth curve.

2.1. Reminders on the generalized Jacobian. Let $X$ be a smooth proper geometrically integral curve over field $k$ of characteristic zero. Suppose that $X$ admits a degree one divisor. Let $D$ be an effective reduced divisor on $X$ and set $Y := X \setminus D$. Let $\widetilde{J} := \text{Jac}(X, D)$ be the generalized Jacobian of $X$ with modulus $D$ in the sense of Rosenlicht-Serre [5], which is a semi-abelian variety over $k$. It fits in an exact sequence

\[ 0 \to \mathbb{G}_D \to \widetilde{J} \to J \to 0, \]

where $J = \text{Jac}(X)$ is the Jacobian variety of $X$, and

\[ \mathbb{G}_D := \text{Coker}[\mathbb{G}_{m,k} \to \bigoplus_{x \in D} \text{Res}_{k(x)/k} \mathbb{G}_{m,k(x)}]. \]

We also recall that there are isomorphisms

\[ \widetilde{J}(k) \cong \text{Div}^0(Y)/\{\text{div}(f) \mid f \in k(Y)^\times, \text{ord}_x(f - 1) \geq 1 \text{ for any } x \in D\} \]

\[ \cong \ker(H^1(X, \mathbb{G}_{X,D}) \to H^1(X, \mathbb{G}_m)^{\text{deg}} \mathbb{Z}), \quad \mathbb{G}_{X,D} := \ker(\mathbb{G}_{m,X} \to \mathbb{G}_{m,D}). \]

2.2. A Galois module $M_{\text{Tor}}$. We define a divisible torsion $G_k$-module $M_{\text{Tor}}$ by (see §1.3)

\[ M_{\text{Tor}} := \text{Hom}(T\widetilde{J}, \mathbb{Q}/\mathbb{Z}(1)). \]

In the terminology of [6, Chapter V, §3], $M_{\text{Tor}}$ can be interpreted as the group of torsion points of the dual 1-motive $M$ of $\widetilde{J}$. (We will not use this fact.) Let

\[ L_D := \text{Hom}(\mathbb{G}_D, \mathbb{G}_{m,k}) \]

be the character group of $\mathbb{G}_D$ from (2.1.2). This is thus a free abelian group of rank $r_D := \sum_{x \in D}[k(x) : k] - 1$ equipped with a continuous $G_k$-action. Let $g_X$ be the genus of $X$.

**Lemma 2.2.1.** There is an exact sequence of $G_k$-modules

\[ 0 \to J_{\text{Tor}} \to M_{\text{Tor}} \to L_D \otimes \mathbb{Q}/\mathbb{Z} \to 0. \]

In particular, $M_{\text{Tor}} \cong (\mathbb{Q}/\mathbb{Z})^{\oplus(g + r_D)}$ as abelian groups.

**Proof.** The first statement follows from (2.1.1) and an isomorphism

\[ J_{\text{Tor}} \cong \text{Hom}(TJ, \mathbb{Q}/\mathbb{Z}(1)) \]

deduced by the Weil pairing $TJ \times J_{\text{Tor}} \to \mathbb{Q}/\mathbb{Z}(1)$. The last statement follows from (2.2.3). \( \square \)

By taking the Tate twist $(-1)$ and the long exact sequence attached to (2.2.3) (see also (1.3.1)), we get an exact sequence that will be used later

\[ 0 \to J_{\text{Tor}}^\mu(-1) \to M_{\text{Tor}}^\mu(-1) \to (L_D \otimes \mathbb{Q}/\mathbb{Z})^\mu(-1) \to H^1(k, J_{\text{Tor}}(-1)). \]
Example 2.2.2. Suppose that $D = P + Q$ consists of two $k$-rational points. Then we have an exact sequence
\[ 0 \to J_{\text{Tor}} \to M_{\text{Tor}} \to \mathbb{Q}/\mathbb{Z} \to 0. \]
Explicitely, the $G_k$-module $M_{\text{Tor}} = \lim_{\to n} M_{\text{Tor}}[n]$ can be described as follows. We have
\[ M_{\text{Tor}}[n] = J[n] \oplus (\mathbb{Z}/n\mathbb{Z}) \]
as an abelian group, and $G_k$-action is given by
\[ \sigma(\alpha, b) = (\sigma(\alpha) + b(\sigma(\beta) - \beta), b) \]
where $\sigma \in G_k$, $\alpha \in J[n]$, $b \in \mathbb{Z}/n\mathbb{Z}$ and $\beta \in J(\overline{k})$ is a fixed element such that $n\beta$ is the class of $P - Q$ in $J(\overline{k})$.

2.3. Abelian fundamental group. The following result is essentially shown in [1], but it is embedded in a (long) proof, and hence we decide to include a proof to cut out the desired part.

Proposition 2.3.1. The exact sequence $(1.1.1)$ splits (non-canonically), and we have a canonical isomorphism
\[ \pi_1(Y)^{\text{ab,geo}} \cong \text{Hom}(M_{\text{Tor}}^\mu(-1), \mathbb{Q}/\mathbb{Z}). \]

Remark 2.3.2. It is shown in [1, Theorem 1] that $\pi_1(Y)^{\text{ab,geo}}$ is finite if $k$ is finitely generated over its prime subfield.

Proof. Recall that we assumed that there exists a degree one divisor on $X$. By the approximation lemma, it implies the existence of a degree one divisor $E$ supported on $Y$. The spectral sequence $E_{2}^{ij} = H^i(k, H^j(Y, \mathbb{Q}/\mathbb{Z})) \Rightarrow H^{i+j}(Y, \mathbb{Q}/\mathbb{Z})$ yields an exact sequence
\[ 0 \to H^1(k, \mathbb{Q}/\mathbb{Z}) \to H^1(Y, \mathbb{Q}/\mathbb{Z}) \to H^0(k, H^1(Y, \mathbb{Q}/\mathbb{Z})) \]
\[ \to H^2(k, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\phi} H^2(Y, \mathbb{Q}/\mathbb{Z}). \]
The divisor $E$ gives rise to a section of $\phi$, showing the injectivity of $\phi$. By taking the dual sequence, we obtain a short exact sequence
\[ 0 \to (\pi_1(Y)^{\text{ab}})_{G_k} \to \pi_1(Y)^{\text{ab}} \to G_k^{\text{ab}} \to 1, \]
which splits again by the presence of $E$. We have shown $(\pi_1(Y)^{\text{ab}})_{G_k} \cong \pi_1(Y)^{\text{ab,geo}}$. We also have $(T\tilde{J})_{G_k} \cong \text{Hom}(M_{\text{Tor}}^\mu(-1), \mathbb{Q}/\mathbb{Z})$ by the definition (2.2.1).

Hence it remains to show an isomorphism of $G_k$-modules
\[ (2.3.1) \quad \pi_1(Y)^{\text{ab}} \cong T\tilde{J} \]
from which the proposition follows by taking $G_k$-coinvariant parts. By the definitions of $G_{X,D}$ from (2.1.4), there exists an exact sequence of étale sheaves on $\overline{Y}$
\[ 0 \to j_!(\mu_n) \to G_{X,D} \xrightarrow{n} G_{X,D} \to 0 \]
for any $n$, where $j : Y \to X$ is the open immersion. It follows that
\[ (2.3.2) \quad \tilde{J}_{\text{Tor}}[n] \cong H^1_c(\overline{Y}, \mu_n), \quad \tilde{J}_{\text{Tor}} \cong H^1_c(\overline{Y}, \mathbb{Q}/\mathbb{Z}(1)), \quad T\tilde{J} \cong H^1_c(\overline{Y}, \hat{\mathbb{Z}}(1)). \]
The last group is dual to $H^1(\overline{Y}, \mathbb{Q}/\mathbb{Z})$ by the Poincaré duality. On the other hand, $\pi_1(Y)^{\text{ab}}$ is also dual to $H^1(\overline{Y}, \mathbb{Q}/\mathbb{Z})$. This proves the first isomorphism in (2.3.1). \qed
2.4. Covering of \( \mathcal{Y} \). Let \( X' \) be another proper smooth geometrically integral curve over \( k \) admitting a degree one divisor, and let \( f : X' \to X \) be a finite \( k \)-morphism. Put \( Y' := f^{-1}(Y) \) and \( D' := X' \setminus Y' \). (We regard \( D' \) as a reduced effective divisor on \( X' \).) Set \( J' := \text{Jac}(X') \), \( \tilde{J}' := \text{Jac}(X', D') \) and \( M'_{\text{Tor}} := \text{Hom}(T, \tilde{J}', \mathbb{Q}/\mathbb{Z}(1)) \). We have the pull-back maps \( f^* : J \to J' \), \( \tilde{J} \to \tilde{J}' \), \( M_{\text{Tor}} \to M'_{\text{Tor}} \) and the push-forward maps \( f_* : J' \to J \), \( \tilde{J}' \to \tilde{J} \), \( M'_{\text{Tor}} \to M_{\text{Tor}} \).

**Definition 2.4.1.** We define

\[
\Sigma(f) := \ker(f^*: J(\overline{k}) \to J'(\overline{k})), \quad \Sigma_D(f) := \ker(f^*: M_{\text{Tor}} \to M'_{\text{Tor}}).
\]

Note that \( \Sigma(f) \subseteq J[d] \) with \( d := \text{deg}(f) \), because \( f_* \circ f^* = d \). It follows that \( \Sigma(f) \) is finite and hence \( \Sigma(f) \subseteq J_{\text{Tor}} \). Similarly, \( \Sigma_D(f) \subseteq M'_{\text{Tor}}[d] \) is finite too. The canonical maps \( J_{\text{Tor}} \to M_{\text{Tor}} \) and \( J'_{\text{Tor}} \to M'_{\text{Tor}} \) from (2.2.3) induce \( \Sigma(f) \to \Sigma_D(f) \).

**Lemma 2.4.2.** (1) The map \( \Sigma(f) \to \Sigma_D(f) \) is injective.

(2) Suppose that \( \gcd\{e(x'/x) \mid x' \in f^{-1}(x) \} = 1 \) for all \( x \in D \), where \( e(x'/x) \) denotes the ramification index. Then \( \Sigma(f) \to \Sigma_D(f) \) is an isomorphism.

**Proof.** Let \( L_D \) and \( L_{D'} \) be the character groups from (2.2.2). By Lemma 2.2.1 we have a commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \to & J_{\text{Tor}} & \to & M_{\text{Tor}} & \to & L_D \otimes \mathbb{Q}/\mathbb{Z} & \to & 0 \\
\downarrow f^* & & \downarrow f^* & & \downarrow & & \downarrow & \\
0 & \to & J'_{\text{Tor}} & \to & M'_{\text{Tor}} & \to & L_{D'} \otimes \mathbb{Q}/\mathbb{Z} & \to & 0.
\end{array}
\]

This shows (1). For (2), it suffices to show the injectivity of the right vertical map under the stated assumption. We have a commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \to & L_D \otimes \mathbb{Q}/\mathbb{Z} & \to & \bigoplus_{x \in D} \mathbb{Q}/\mathbb{Z} & \text{sum} & \mathbb{Q}/\mathbb{Z} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow \text{deg}(f) & & \downarrow & \\
0 & \to & L_{D'} \otimes \mathbb{Q}/\mathbb{Z} & \to & \bigoplus_{x' \in D'} \mathbb{Q}/\mathbb{Z} & \text{sum} & \mathbb{Q}/\mathbb{Z} & \to & 0.
\end{array}
\]

Here \((x, x')\)-component of the middle vertical map is given by 0 if \( f(x') \neq x \) and by \( e(x'/x) \) if \( f(x') = x \). The lemma follows.

**Example 2.4.3.** Let \( N \in \mathbb{Z}_{>0} \) and consider the canonical map \( f_N : X_1(N) \to X_0(N) \), with respect to divisors consisting of all cusps. If \( N \) is square free, then \( f_N \) is unramified at all cusps, and hence \( \Sigma(f_N) \cong \Sigma_D(f_N) \).

**Lemma 2.4.4.** Let \( f_{X, \text{ab}}^X : X_{\text{ab}} \to X \) (resp. \( f_{Y, \text{ab}}^Y : Y_{\text{ab}} \to Y \)) be the maximal unramified abelian subcovering of \( f : X' \to X \) (resp. \( f : Y' \to Y \)). Then we have

\[
\text{Gal}(f_{X, \text{ab}}^X) \cong \text{Hom}(\Sigma(f), \mathbb{Q}/\mathbb{Z}(1)),
\]

\[
\text{Gal}(f_{Y, \text{ab}}^Y) \cong \text{Hom}(\Sigma_D(f), \mathbb{Q}/\mathbb{Z}(1)).
\]

In particular, \( \Sigma(f) \subseteq J_{\text{Tor}}^\mu \) and \( \Sigma_D(f) \subseteq M_{\text{Tor}}^\mu \) (see (1.3.1)).

**Proof.** Since the first statement is a special case of second (for \( D = \emptyset \)), we only prove the latter. Let \( f_{\text{ab}} : Y_{\text{ab}} \to Y \) be the maximal unramified abelian subcovering of the base change.
Let \( f : Y' \rightarrow Y \) be a Galois closure of \( f \). Let \( \alpha : Y' \rightarrow Y \) be a Galois closure of \( f \). We have a commutative diagram

\[
\begin{array}{c}
Y' \xleftarrow{\alpha} Y \\
\downarrow \circlearrowleft \downarrow \\
Y'^{\text{uab}} \xleftarrow{f} Y^{\text{uab}} \\
\downarrow \circlearrowleft \downarrow \\
Y_f^{\text{uab}} \xleftarrow{f} Y_f^{\text{uab}} \\
\downarrow \circlearrowleft \downarrow \\
Y \xrightarrow{\beta} Y \\
\end{array}
\]

The left two squares are Cartesian since \( Y, Y' \) (and hence \( X_f^{\text{uab}} \)) are geometrically integral over \( k \). It follows that \( \text{Gal}(Y_f^{\text{uab}}) \cong \text{Gal}(f^{\text{uab}}_Y) \). With the notation \((-)^{\vee} = \text{Hom}(-, \mathbb{Q}/\mathbb{Z}(1))\), we have (see (2.3.2) for the second isomorphism)

\[
\Sigma(f)^{\vee} = \ker(\tilde{J}_{\text{Tor}} \rightarrow \tilde{J}_{\text{Tor}})^{\vee} \\
\cong \ker(H^1_c(Y', \mathbb{Q}/\mathbb{Z}(1)) \rightarrow H^1_c(Y', \mathbb{Q}/\mathbb{Z}(1)))^{\vee} \\
\cong \text{coker}(H^1(Y', \mathbb{Z}) \rightarrow H^1(Y', \mathbb{Z})) \\
\cong \text{coker}(\pi_1(Y)^{\text{ab}} \rightarrow \pi_1(Y)^{\text{ab}}) \\
\cong \text{Gal}(\gamma)/\text{Gal}(\beta) \cong \text{Gal}(J_f^{\text{uab}}) \cong \text{Gal}(f^{\text{uab}}).
\]

The last statement follows since \( G_k \) acts on \( \text{Gal}(f^{\text{uab}}) \) trivially. \( \square \)

Note that this lemma implies that \( \Sigma(f) = \Sigma(f_{X_{\text{ab}}}) \) and \( \Sigma_D(f) = \Sigma_D(f_{Y_{\text{ab}}}) \). The following definition is equivalent to the one given in the introduction if \( k \) is a number field:

**Definition 2.4.5.** Suppose that \( f : X' \rightarrow X \) (resp. \( f : Y' \rightarrow Y \)) is a finite abelian unramified covering. We say \( f \) is geometrically maximal if \( \Sigma(f) = J_f^{\mu}(\text{Tor}) \) (resp. \( \Sigma_D(f) = M_f^{\mu}(\text{Tor}) \)).

**Proposition 2.4.6.** Suppose that \( k \) is finitely generated over its prime subfield. Given \( Y \), there exists a geometrically maximal abelian unramified covering \( f : Y' \rightarrow Y \).

**Proof.** This follows from Proposition 2.3.1 and Remark 2.3.2. \( \square \)

### 3. Modular Curves

**3.1. First Reduction.** Let \( p \) be an odd prime and set \( k = \mathbb{Q}, \ X = X_0(p), \ Y = Y_0(p) \). Recall that \( D = X \setminus Y \) consists of two \( \mathbb{Q} \)-rational points (i.e. 0 and \( \infty \) cusps). It follows that \( L_D = \mathbb{Z} \) with trivial \( G_Q \)-action, and hence \( (L_D \otimes \mathbb{Q}/\mathbb{Z})^{\mu}(-1) \cong \mathbb{Z}/2\mathbb{Z} \) (see Example 2.2.2). By (2.2.4), we get an exact sequence

\[
0 \rightarrow J_f^{\mu}(\text{Tor})(-1) \rightarrow M_f^{\mu}(\text{Tor})(-1) \rightarrow \mathbb{Z}/2\mathbb{Z}.
\]

Since \( J_f^{\mu}(\text{Tor})(-1) \) is a cyclic group of order \( N_p \) by Theorem 1.2.1, the order of \( M_f^{\mu}(\text{Tor})(-1) \) is either \( N_p \) or \( 2N_p \). In view of Proposition 2.3.1 and Lemma 2.4.4, this implies that if \( J_f' : X_2'(p) \rightarrow X_2(p) \) is a cyclic covering of degree \( 2N_p \), such that \( X_2'(p) \) is geometrically integral and such that the condition (2) in Theorem 1.2.2 holds, then it automatically satisfies (1) as well. We are going to construct such \( J_f' \). If \( p = 2 \) or 3, we may take \( J_f' \) to be the map \( \mathbb{G}_m \rightarrow \mathbb{G}_m, \ x \mapsto x^2 \) under the identification \( Y_0(p) \cong \mathbb{G}_m = \mathbb{P}^1 \setminus \{0, \infty\} \). Below we assume \( p \geq 5 \). We need a preparation.
3.2. **Generalized Dedekind eta functions.** Let \( N \) be a positive integer. For an integer \( g \) not congruent to 0 modulo \( N \), define the generalized Dedekind eta function \( E_g \) by

\[
E_g(\tau) = q^{NB(g/N)/2} \prod_{m=1}^{\infty} (1 - q^{N(m-1)+g})(1 - q^{Nm-g}),
\]

where \( B(x) = x^2 - x + 1/6 \) is the second Bernoulli polynomial (see [4]). Up to scalars, these functions are also called Siegel functions (see [2]). We have the following properties of \( E_g \).

**Proposition 3.2.1** ([7] Corollaries 2 and 3 and Lemma 2]).

1. We have

\[
E_{g+N} = E_{-g} = -E_g.
\]

2. Let \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \). We have, for \( c = 0 \),

\[
E_{g}(\tau + b) = e^{\pi ibNB(g/N)} E_g(\tau),
\]

and, for \( c \neq 0 \),

\[
E_{g}(\gamma \tau) = \varepsilon(a, b, N, c, d) e^{\pi i(g^2ab/N-gb)} E_{ag}(\tau),
\]

where

\[
\varepsilon(a, b, c, d) = \begin{cases} 
eq g & \text{if } c \text{ is odd}, \\ -ie^{2\pi i(ac^2+bd)} & \text{if } d \text{ is odd}. \end{cases}
\]

3. Suppose that \( \prod g E_{eg} \) is a product of generalized Dedekind eta functions satisfying

\[
\sum g e_g \equiv 0 \mod 12, \quad \sum g e_g \equiv 0 \mod 2, \quad \sum g^2 e_g \equiv 0 \mod 2N.
\]

Then \( \prod g E_{eg} \) is a modular function on \( \Gamma_1(N) \). Moreover, if \( N \) is odd, then the conditions can be reduced to

\[
\sum g e_g \equiv 0 \mod 12, \quad \sum g^2 e_g \equiv 0 \mod N.
\]

4. Given a matrix

\[
\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}),
\]

the Fourier expansion of \( E_g(\gamma \tau) \) starts from \( \zeta q^\delta + (\text{higher powers}) \), where \( \zeta \) is a root of unity and

\[
\delta = \frac{(c, N)^2}{2N} P_2\left( \frac{ag}{(c, N)} \right),
\]

where \( P_2(x) = \{x\}^2 - \{x\} + 1/6 \) is the second Bernoulli function.

3.3. **The modular curve** \( X_2'(p) \). Let \( p \) be a prime with \( p \geq 5 \) and \( g \) be an odd generator of \( (\mathbb{Z}/p\mathbb{Z})^\times \). To ease the notation, set

\[
k = N_p = \frac{p-1}{(p-1,12)}, \quad \ell = \frac{(p-1,12)}{2},
\]

where \( (a, b) := \gcd(a, b) \). Let \( \Gamma_2(p) \) be the group generated by \( \Gamma_1(p) \) and any matrix of the form \( \left( \begin{smallmatrix} g^k & * \\ p & \ast \end{smallmatrix} \right) \) and \( X_2(p) \) be the corresponding modular curve so that \( X_2(p) \to X_0(p) \) is the maximal unramified subcover of \( X_1(p) \to X_0(p) \). We have

\[
[\Gamma_0(p) : \Gamma_2(p)] = k, \quad [\Gamma_2(p) : \Gamma_1(p)] = \ell.
\]
For an integer $h$ not congruent to 0 modulo $p$, using (3.2.1), we set

\[(3.3.1) \quad F_h(\tau) = \left( \prod_{j=0}^{\ell-1} E_{g^{jk_h}}(\tau) \right)^{6/\ell} . \]

**Lemma 3.3.1.** The functions $F_h$ have the following properties.

1. $F_h = F_{-h} = F_{p+h}$.
2. $F_{g^{k_h}} = \begin{cases} -F_h, & \text{if } p \equiv 1 \mod 4, \\ F_h, & \text{if } p \equiv 3 \mod 4. \end{cases}$

**Proof.** The fact that $F_h = F_{-h}$ follows immediately from (3.2.2) in Proposition 3.2.1 since $F_h$ is a product of 6 generalized Dedekind eta functions. By the same property of the generalized Dedekind eta functions, we have

\[ F_{p+h} = \left( \prod_{j=0}^{\ell-1} E_{g^{jk_h}(p+h)} \right)^{6/\ell} = (-1)^{6(1+g^k+\cdots+g^{(\ell-1)k})/\ell} F_h \]

Since $g$ is assumed to be odd, we find that $F_{p+h} = F_h$. We next prove Part (2).

By the definition of $F_h$, we have

\[ F_{g^{k_h}} = \left( \frac{E_{g^{k_h}}}{E_h} \right)^{6/\ell} F_h. \]

Now $g^{k\ell} = g^{(p-1)/2} \equiv -1 \mod p$. Hence from (3.2.2) in Proposition 3.2.1 again, we find that

\[ E_{g^{k_h}} = (-1)^{(g^{k+1}_h)/p} E_{-h} = -E_h \]

It follows that

\[ F_{g^{k_h}} = (-1)^{6/\ell} F_h = \begin{cases} -F_h, & \text{if } p \equiv 1 \mod 4, \\ F_h, & \text{if } p \equiv 3 \mod 4. \end{cases} \]

This completes the proof. \qed

Let $\psi$ be the character of order 2 of $\text{SL}(2, \mathbb{Z})$ defined by

\[ \psi(\gamma) = \begin{cases} (-1)^{a+c-d-1}, & \text{if } c \text{ is odd}, \\ (-1)^{b}, & \text{if } c \text{ is even}, \end{cases} \]

for $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}(2, \mathbb{Z})$. In addition, if $\ell$ is even, i.e., if $p \equiv 1 \mod 4$, there is a unique character $\chi$ of order 2 of $\Gamma_2(p)$ with $\Gamma_1(p) \subset \ker \chi$. Explicitly, for $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_2(p)$, let $n$ be an integer such that $a \equiv g^{n_k} \mod p$. Then $\chi(\gamma)$ is equal to $(-1)^n$. Note that if $p \equiv 5 \mod 8$, then $k$ is odd and $\chi$ is simply the character $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \mapsto \left( \begin{array}{c} d \\ p \end{array} \right)$ of nebentype.

Let $\Gamma_2'(p)$ be the kernel of the character

\[ \begin{cases} \psi \chi, & \text{if } p \equiv 1 \mod 4, \\ \psi, & \text{if } p \equiv 3 \mod 4 \end{cases} \]

on $\Gamma_2(p)$.

**Lemma 3.3.2.** The group $\Gamma_2'(p)$ is a normal subgroup of $\Gamma_0(p)$ and $\Gamma_0(p)/\Gamma_2'(p)$ is cyclic of order $2k$.

**Proof.** Let $\rho$ be a character on $\Gamma_0(p)$ of order $(p-1)/2$ with kernel $\Gamma_1(p)$. By definition, $\Gamma_2'(p)$ is the kernel of the character $\rho^{\ell/2} \psi$ (resp. $\rho^{\ell/2} \chi$) if $\ell$ is odd (resp. even) on $\Gamma_0(p)$ of order $2k$. \qed
Combined with this lemma, Theorem 1.2.2 follows from the following result:

**Proposition 3.3.3.** The modular curve $X'_2(p)$ associated to $\Gamma_2(p)$ admits a geometrically integral model over $\mathbb{Q}$ and the covering $\ell^t_p : X'_2(p) \to X_2(p)$ ramifies precisely at each cusp.

We first prove the proposition assuming $p \not\equiv 11 \mod 12$.

**Lemma 3.3.4.** Assume that $\ell \neq 1$, i.e., that $p \equiv 11 \mod 12$. Let $h$ be an integer not congruent to 0 modulo $p$.

1. For $\gamma = (a \ b \ c \ d) \in \Gamma_0(p)$, we have $F_h(\gamma \tau) = \psi(\gamma) F_{ah}(\tau)$.
2. Let $\gamma = (a \ b \ c \ d) \in \Gamma_2(p)$. Then
   $$F_h(\gamma \tau) = \begin{cases} 
   \psi(\gamma) \chi(\gamma) F_h(\tau), & \text{if } p \equiv 1 \mod 4, \\
   \psi(\gamma) F_h(\tau), & \text{if } p \equiv 3 \mod 4.
   \end{cases}$$

3. For any $h$, $F^2_h$ is a modular function on $X_2(p)$ defined over $\mathbb{Q}$.
4. The order of $F^2_h$ at any cusp of $X_2(p)$ is odd, and is zero elsewhere.

**Proof.** Assume that $\ell \neq 1$, i.e., that $p \equiv 11 \mod 12$. Let $\gamma = (a \ b \ c \ d)$ be a matrix in $\Gamma_0(p)$. By Part (2) of Proposition 3.2.1, for any $j = 0, \ldots, \ell - 1$, $E_{g^j h}(\gamma \tau) = \varepsilon(a, b p, c, d) e^{\pi i (g^j h)^2 a b / p - g^j h} E_{g^j h}(\tau)$. Notice that $\varepsilon(a, b p, c, d)$ is a 12th root of unity independent of $h$ and $j k$. Thus,
   $$F_h(\gamma \tau) = \varepsilon(a, b p, c, d)^6 e^{2 \pi i S / 2 p} F_{ah}(\tau),$$
   where
   $$S = \frac{6 a b h^2}{\ell} \sum_{j=0}^{\ell-1} g^{2 j k}$$

We check directly from the definition of $\varepsilon$ that
   $$\varepsilon(a, b p, c, d)^6 = \varepsilon(a, b, c p, d)^6 = \psi(\gamma) \left( \begin{array}{cc} a & b \\ c p & d \end{array} \right) = \psi(\gamma).$$

Since $\ell$ is assumed to be greater than 1, we have
   $$\sum_{j=0}^{\ell-1} g^{2 j k} \equiv 0 \mod p.$$

Also, $S$ is even since either $6 / \ell$ is even or $\sum_{j=0}^{\ell-1} g^{2 j k}$ is even. We conclude that $(2 p) | S$ and $F_h(\gamma \tau) = \psi(\gamma) F_{ah}(\tau)$.

We next prove Part (2). Assume that $\gamma = (a \ b \ c \ d) \in \Gamma_2(p)$. By Part (1), $F_h(\gamma \tau) = \psi(\gamma) F_{ah}(\tau)$. Let $n$ be a nonnegative integer such that $a \equiv g^{n k} \mod p$. By Lemma 3.3.1, we have
   $$F_{ah} = F_{g^{n k} h} = \begin{cases} 
   (-1)^n F_h = \chi(\gamma) F_h, & \text{if } p \equiv 1 \mod 4, \\
   F_h, & \text{if } p \equiv 3 \mod 4.
   \end{cases}$$

This yields the formula in Part (2). The first statement of Part (3) is an immediate consequence of Part (2), and the second statement follows immediately from the definition (3.3.1).

We now prove Part (4). The statement for non-cusp points are obvious from the definition (3.3.1). Let $a/c$, $(a, c) = 1$, be a cusp of $X_2(p)$. Consider first the case $p \nmid c$. Let $\gamma = (a \ b \ c \ d)$ be a matrix in $\text{SL}(2, \mathbb{Z})$. By Part (4) of Proposition 3.2.1, the Fourier expansion of $E_{g^j h}$ starts from the term $q^{1/12 p}$ for any $j$. Since such a cusp has width $p$ and $F^2_h$ is the product of exactly 12 $E_{g^j h}$, the order of $F_h$ at $a/c$ is $1$. 
Now consider the case \( p | c \). Such a cusp has width 1. By Part (4) of Proposition 3.2.1, the order of \( F_h \) at \( a/c \) is
\[
\frac{12}{\ell} \sum_{j=0}^{\ell-1} \frac{p}{2} \left( \left\{ \frac{ag^{jk}h}{p} \right\}^2 - \left\{ \frac{ag^{jk}k}{p} \right\} + \frac{1}{6} \right).
\]

Observe that for any integer \( x \), if we let \( n = \lfloor x/p \rfloor \), then
\[
\left\{ \frac{x}{p} \right\}^2 - \left\{ \frac{x}{p} \right\} = \left( \frac{x}{p} - n \right)^2 - \frac{x}{p} + n = \frac{x^2}{p^2} - \frac{x}{p} - 2n\frac{x}{p} + n^2 + n,
\]
and hence
\[
\frac{p}{2} \left( \left\{ \frac{x}{p} \right\}^2 - \left\{ \frac{x}{p} \right\} \right) \equiv \frac{p}{2} \left( \frac{x^2}{p^2} - \frac{x}{p} \right) \mod 1.
\]
It follows that
\[
\frac{12}{\ell} \sum_{j=0}^{\ell-1} \frac{p}{2} \left( \left\{ \frac{ag^{jk}h}{p} \right\}^2 - \left\{ \frac{ag^{jk}k}{p} \right\} + \frac{1}{6} \right) \equiv \frac{12}{\ell} \sum_{j=0}^{\ell-1} \frac{p}{12} = p \equiv 1 \mod 2
\]
That is, the order of \( F_h^2 \) at the cusp \( a/c \) is odd. This completes the proof. \( \square \)

It follows from the lemma that \( \mathbb{Q}(X'_2(p)) \) is a quadratic extension of \( \mathbb{Q}(X_2(p)) \) generated by a square root of \( F_h^2 \in \mathbb{Q}(X_2(p)) \), that \( \mathbb{Q} \) is algebraically closed in \( \mathbb{Q}(X'_2(p)) \), and that the covering \( X'_2(p) \to X_2(p) \) ramifies exactly at each cusp, showing Proposition 3.3.3 in this case.

To show the proposition for the case \( p \equiv 11 \mod 12 \), we need a slightly different construction of modular functions. In this case, \( \Gamma_2(p) = \Gamma_1(p) \). Let \( h = (h_1, h_2, h_3) \) be any triplet of integers not congruent to 0 modulo \( p \) such that \( h_1^2 + h_2^2 + h_3^2 \equiv 0 \mod p \) and define \( G_h = (E_{h_1}E_{h_2}E_{h_3})^2 \). Then \( G_h^2 \) is a modular function on \( X_1(p) \) by Proposition 3.2.1. Also, by the same computation as above, the order of \( G_h^2 \) at each cusp is odd. In addition, we can verify as above that for \( \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_1(p) \),
\[
G_h(\gamma \tau) = \psi(\gamma)G_h.
\]
Therefore, the proposition holds for the case \( p \equiv 11 \mod 12 \). \( \square \)

**Remark 3.3.5.** If \( p \not\equiv 1 \mod 8 \), then
\[
z(\tau) = \left( \frac{\eta(\tau)}{\eta(p\tau)} \right)^{12/(p-1,12)}
\]
is also a modular function on \( \Gamma'_2(p) \), but not on \( \Gamma_2(p) \). Thus, the function field of \( X'_2(p) \) can be obtained by adjoining either \( F_h(\tau) \) or \( z(\tau) \). The relation between \( F_h \) and \( z \) is
\[
z(\tau) = \pm \prod_{j=0}^{k-1} F_{g_j}.
\]
(In particular, if \( k \) is odd, i.e., if \( p \not\equiv 1 \mod 8 \), then \( z(\gamma \tau) = \psi(\gamma)z(\tau) \) for \( \gamma \in \Gamma_2(p) \).)

**Acknowledgement.** The first author would like to thank Masataka Chida and Fu-Tsun Wei for fruitful discussion. He is partially supported by JSPS KAKENHI Grant (18K03232). The second author was partially supported by Grant 106-2115-M-002-009-MY3 of the Ministry of Science and Technology, Republic of China (Taiwan).
REFERENCES

[1] Nicholas M. Katz and Serge Lang. Finiteness theorems in geometric classfield theory. *Enseign. Math. (2)*, 27(3-4):285–319 (1982), 1981. With an appendix by Kenneth A. Ribet.

[2] Daniel S. Kubert and Serge Lang. *Modular units*, volume 244 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science]*. Springer-Verlag, New York, 1981.

[3] Barry Mazur. Modular curves and the Eisenstein ideal. *Inst. Hautes Études Sci. Publ. Math.*, (47):33–186 (1978), 1977.

[4] Bruno Schoeneberg. *Elliptic modular functions: an introduction*. Springer-Verlag, New York, 1974. Translated from the German by J. R. Smart and E. A. Schwandt, Die Grundlehren der mathematischen Wissenschaften, Band 203.

[5] Jean-Pierre Serre. *Groupes algébriques et corps de classes*. Publications de l’institut de mathématique de l’université de Nancago, VII. Hermann, Paris, 1959.

[6] John Tate. *Les conjectures de Stark sur les fonctions L d’Artin en s = 0*, volume 47 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1984. Lecture notes edited by Dominique Bernardi and Norbert Schappacher.

[7] Yifan Yang. Transformation formulas for generalized Dedekind eta functions. *Bull. London Math. Soc.*, 36(5):671–682, 2004.

MATHEMATICAL INSTITUTE, TOHOKU UNIVERSITY, AOBÀ, SENDAI 980-8578, JAPAN

E-mail address: ytakao@math.tohoku.ac.jp

DEPARTMENT OF MATHEMATICS, NATIONAL TAIWAN UNIVERSITY AND NATIONAL CENTER FOR THEORETICAL SCIENCES, TAIPEI, TAIWAN 10617

E-mail address: yangyifan@ntu.edu.tw