‘Classical’ coherent state generated by curved surface

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Abstract

Analogous coherent states are deduced from classical optical fields on curved surface in this paper. The Gaussian laser beam, as a fundamental mode, cannot be adequately simulated by coherent states due to their inherent diffraction in flat space. But it differs when propagating on a surface with the uniform curvature, the constant Gaussian curvature surface (CGCS). By generalizing the method of Feynman path integral, an equivalent coherent states solution is demonstrated to describe the beam propagation. The temporal evolution of the Schrödinger equation is analogously translated into a spatial transmission in this derivation, and we obtain the expression of quantized momentum transmitted on curved surface, which is proportional to the square root of the Gaussian curvature $K$. In addition to that, a physical picture of beam propagation that is identical to the squeezed state is also built. We hope this research can offer a new view on the quantum field in curved space.

1. Introduction

Recent experimental observations indicate that our spatial universe may be a universe with slightly negative spatial curvature [1, 2]. Friedmann–Robertson–Walker’s (FRW) space-time is significant as a uniform and isotropic space-time model that satisfies cosmological principles [3]. On the other hand, the electromagnetic fields in curved spacetime exhibit a diverse set of features [4–8], demonstrating that studying light beam propagation in a flat vacuum is insufficient in the field of optics.

Beam propagation in curved space-time is an extensive and comprehensive research field, which is proved that when the radius of curvature being comparable to the wavelength, only the intrinsic curvature of space-time affects the local propagation properties of light [9, 10]. Despite general relativity (GR)’s triumph, gravitational effect is too weak in a laboratory environment. One of the analog models of GR is two dimensional (2D) curved surface embedded in 3D space as a result of the 3+1 membrane paradigm theory [11]. When taking a constant time and extracting the equatorial slice of FRW space-time, the remnant metrics can be regard as a 2D curved surface named constant Gaussian curvature surfaces (CGCS). Ever since Batz and Peschel brought up such concept [12], the light propagation and electromagnetic dynamics on 2D curved surface have flourished both theoretically [10, 13–17] and experimentally [5, 9, 18–20]. Mathematically, the different Laplacian operators introduced by curved spaces in Helmholtz equations can be further simplified by constructing spatial effective potential, which is a convenient analogy to the Schrödinger equation [12–14]. Intriguingly, in this paper, we realize that the method of analogy may not only be a mathematical simplification, but also profoundly physical. We could even construct the classical harmonic oscillator potential to produce an oscillatory non-diffracted light that resembles coherent states.

The coherent states were invented by Erwin Schrödinger in the physical context of the quantum harmonic oscillator. Lately, they were used in quantum optics (QO) by Glauber, Sudarshan and Klauder [21, 22]. The anticipated time-invariant Gaussian wave packet (in quadrature representation) might be a ‘decent’ simulation of the coherent state, even though it is not fully consistent. However, the propagation of Gaussian beam in flat space will usually be diffracted. Theoretically, we can restrict the divergence of the
light beam by setting the initial transverse mode distribution of the optical field or constructing an effective potential energy. In the experiment, the means of controlling the energy transmission of optical field are becoming increasingly abundant [23–25]. The generation ways of coherent states are rich, for example by classical oscillating currents or four-wave mixing [15]. In this paper, we consider theoretically how the bending of space itself modulates the propagation of light beam and how to originally obtain the so-called ‘classical’ coherent states.

2. Path integral method and quantized momentum

This paper principally illustrates the light paraxial transmission on a CGCS and the line element of the light on CGCS can be expressed as follows [26]:

$$dl^2 = dh^2 + \cos^2\left(\sqrt{k}h\right)dr^2.$$  (1)

Here, $\rho$ means the equator line coordinate and $h$ stands for the horizontal distance from the equator.

The Helmholtz equation when neglecting polarization effects can thus be written as $(\Delta H + k^2)\Psi + (H^2 - K)\Psi = 0$. Here, $k = 2\pi/\lambda$. The covariant Laplacian $\Delta H = (1/\sqrt{g})\partial_i\sqrt{g}g^{ij}\partial_j$, with $i, j = h, \rho$ and $g = \det(g_{ij}) = Kr^2$, where $r = \cos(\sqrt{k}h)/\sqrt{k}$, describing the generating line of the surface of rotation, i.e. the horizontal distance from the surface to the axis of rotation. The influence of the extrinsic curvature $H$ is also overlooked in this paper as previous literature did [9, 10, 12–14, 27, 28]. If we construct the wave function as $\Psi = Ar^{-1/2}u(h, \rho) \exp(ik\rho)$, $A$ is the intensity coefficient of light. Using Wenzel–Kramers–Brillouin approximation, the Helmholtz equation can be blackeduced to:

$$2ik\partial_\rho u = -Kr^2\partial_h^2u + V_{\text{eff}}u.$$  (2)

The form of equation (2) is similar to the nonlinear Schrödinger equation, and the effective potential can be expressed under the metric equation (1) as $V_{\text{eff}}(h) \approx (K/2) - K(3K/4 - k^2)h^2$. Evidently, this is a harmonic oscillator potential, the simplest potential field in quantum mechanics. This paper also intends to present a new mathematical method to obtain the analytical solution of this wave equation.

It has been demonstrated that the form of the Schrödinger equation is consistent with that of the Helmholtz equation under the paraxial approximation in the process as above, which implies that it is also desirable to introduce the method of path integration (PI) into the field of optics in curved space.

The classical diffraction integral is consideblack as the relationship between the initial input field $u_a(h_a, \rho_a)$ and the final output field as $u_b(h_b, \rho_b) = \int \exp(ikS(Q))u_a(h_a, \rho_a)dh_a$. $S(Q)$ stands for a classical action, and here it is eikonal in the diffraction integral of light.

In classical mechanics, the path of light is indeed unique while photons can take any path, including geodesics, which has the highest probability, due to quantum randomness. Therefore, the diffraction integral formula for light is going to be rewritten for all possible paths $h(\rho)$ from $a$ to $b$:

$$u(h_b, \rho_b) = \int_Q Q(h_a, \rho_a; h_b, \rho_b)u(h_a, \rho_a)dh_a,$$

$$Q(h_a, \rho_a; h_b, \rho_b) = \int_a^b \exp[ikS(Q)]dh(\rho),$$

$$= \int_a^b \exp\left[ik\sqrt{Q}\int_\rho^\rho_0 dl\right]Dh(\rho).$$  (3)

The convolution factor $Q(a, b)$ is called kernel in PI. In the diffraction integral, it is the exponential form of the eikonal. Here, $Dh(\rho)$ means a functional integral from $a$ to $b$. From a mechanical point of view, the optical path mentioned above actually corresponds to the action of Lagrangian integral. For the line element of CGCS, the path integral kernel $Q(a, b)$ in equation (3) has an analytic solution under paraxial approximation:

$$Q(a, b) = \left(\frac{k\sqrt{K}}{2\pi i \sin(\sqrt{K}\rho)}\right)^{1/2} \exp\left[ik\sqrt{K}\rho_0 \sin\left(\sqrt{K}\rho\right)\left(h_a^2 + h_b^2\right) \cos\left(\sqrt{K}\rho\right) - 2h_a h_b\right],$$  (4)

where $\rho = \rho_b - \rho_a$. It is worth noting that this analytic solution is consistent with that obtained by the optical matrix and Collins formula in reference [26]. Equation (2) is a linear differential equation whose
solutions satisfy the linear superposition property, thus by assuming eigen-solutions $u_n(h, \rho)$, a special set of solutions, we can separated it into the following form:

$$u(h, \rho) = f(\rho) \phi(h)$$

or

$$\frac{df(\rho)/d\rho}{f(\rho)} = -i \frac{K^2 \left[ d^2 \phi(h)/d\rho^2 \right] + \phi(h)}{2K}.$$

Then we can construct $f(\rho) = f_0 \exp\left[-(ip/h)\rho\right]$, where $f_0$ is an arbitrary constant factor. Moreover, the transverse mode field $\phi(h)$ can be expressed as a linear combination of $\phi_n(h)$, which is not only normalized but also orthogonal. That is

$$\phi(h) = \sum_{n=1}^{\infty} a_n \phi_n(h).$$

(5)

And the coefficients $a_n = \int_{-\infty}^{\infty} \phi_n^*(h) \phi(h) dh$. So, the light evolution from $a$ to $b$ satisfies:

$$u(h_\alpha, \rho) = \sum_{n=1}^{\infty} a_n \phi_n(h_\alpha) \exp\left(-i p_n \rho/h\right)$$

$$= \sum_{n=1}^{\infty} \left( \int_{-\infty}^{\infty} \phi_n^*(h_\alpha) \phi(h_\alpha) dh_\alpha \right) \phi_n(h_\beta) \exp\left(-i p_n \rho/h\right)$$

$$= \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \phi_n(h_\alpha) \phi_n^*(h_\alpha) \exp\left(-i p_n \rho/h\right) \phi(h_\alpha) dh_\alpha.$$  

(6)

Compablack equation (6) with equation (3), we can obtain: $Q(h_\alpha, 0; h_\beta, \rho) = \sum \exp\left(-i K^2 \rho/2\right)^{1/2} e^{-i K^2 \rho} \phi_n(h_\beta) \phi_n^*(h_\alpha)$. Thus, combining with equation (4), the eigenvalue $p_n$ is given by tracing the kernel $Q$ as equation (7).

$$\text{tr}(Q(a, b)) = \int Q(h_\alpha, 0; h_\beta, \rho) dh_\alpha$$

$$= \left( \frac{\exp\left(-i \sqrt{K^2} \rho/2\right)}{1 - \exp\left(-i \sqrt{K^2} \rho\right)} \right) \sum_{n=0}^{\infty} \exp\left(-i \left( n + \frac{1}{2} \right) \sqrt{K^2} \rho \right).$$

(7)

Particularly, for the eigenstate $u_n(h, \rho) = \phi_n(h) \exp\left(-i p_n \rho/h\right)$, the eigenvalue $p_n = \left( n + \frac{1}{2} \right) h \sqrt{K}$ is the ‘quantized’ momentum of the beam in CGCSs, which is proportional to the square root of the curvature. In flat space, this quantization disappears and the momentum of the beam can be measublack continuously. Intriguingly, no calculation is based on quantum theory, so far, and only the classical calculations in curved flat space, this quantization disappears and the momentum of the beam can be measublack continuously. Intriguingly, no calculation is based on quantum theory, so far, and only the classical calculations in curved flat space is carried out, so this ‘quantized’ momentum is different from the quantum coherent states whose energy levels are integer spaced. Furthermore, by physical interpretation, this momentum quantization can be derived by the standing wave condition of the interference cancellation. So, the circumference of the light path will be $2\pi r = 2\pi/\sqrt{K} = (2n + 1)\lambda/2$, which is consistent to the ‘quantized’ momentum condition. In fact, this peculiar quantized momentum generated by geometric effects is called geometric momentum and has been elucidated in mathematical physics [29].

Furthermore, due to the properties of Hermitian polynomials, the kernel can be expanded into a summarize form:

$$Q(a, b) = \left( \frac{k \sqrt{K}}{\pi} \right)^{1/2} e^{-i \frac{\sqrt{K}}{2} (a^2 + b^2)} \sum_{n=0}^{\infty} \frac{1}{2^n n!} H_n\left( \sqrt{\sqrt{K} a} \right) H_n\left( \sqrt{\sqrt{K} b} \right) e^{-i\left(n + \frac{1}{2}\right) a \sqrt{K} \rho}.$$  

For the convenience of description, we will use the language of quantum mechanical method to treat the optical field as a photon state. By using the expansion of $Q$, the eigenstates can be obtained based on the Dirac notation:

$$|n\rangle_0 \equiv \phi_n(h) = 2^{n/2} (n!)^{-1/2} \left( \frac{k \sqrt{K}}{\pi} \right)^{1/4} H_n\left( \sqrt{\sqrt{K} h} \right) e^{-i\sum \phi_n(h)}.  \tag{8}$$

Equation (8) actually represents the eigen mode of the optical field. The superposition of eigenstates can be decomposed for any state $|a\rangle$, and its evolution can be written as $|a\rangle = \sum_n a_n |n\rangle = \sum_n a_n |n\rangle_0 e^{-i\phi_n(h)}$. The momentum of any of these states will still satisfy the superposition property.
Figures 1(a)–(d) show the propagation properties of different eigenstates, which can also be called number states because they represent the number of dark lines (the zero intensity of beam). On the other hand, we find that the number state $|n\rangle$ displays $n + 1$ straight bright stripes on the curved surface. Although the stripes are not of equal width, the propagation of the beam shows a good non-diffraction property. As shown in figure 1(e), the evolution of this Gaussian light field oscillates around the transmission axis, which foreshadows the construction of coherent states.

3. ‘Classical’ coherent state

The evolution of light on the surface satisfies the paraxial wave equation equation (2), just as the evolution of the quantum state fulfills the Schrödinger equation. We choose a special CGCS, which introduces a special effective potential—the harmonic oscillator potential. A photon may be thought of as a harmonic oscillator in certain ways, and the eigenstate of its annihilation operator (the falling operator) is a coherent state in the field of QO. In the previous section, we constructed the eigenstate of light transmission on a curved surface. Due to the superposition property of the optical field, similarly, we can also construct coherent states in a classical optical field. Generally, the coherent states $|\alpha\rangle$ are defined by using some superposition of the number states $|n\rangle$:

$$|\alpha\rangle = A e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle,$$

where parameter $|\alpha|^2 = \bar{n}$. By substituting equation (9), we can obtain a representation of the light in coordinates:

$$u_\alpha(h, \rho) = \langle h|\alpha\rangle = \frac{A \exp\left\{-\frac{\bar{n}}{2} \left[ h^2 - 2h_0 e^{-i\sqrt{\bar{n}} \rho} + h_0^2 \left( 1 + e^{-2i\sqrt{\bar{n}} \rho} \right) \right]\right\}}{\sqrt{\cos(\sqrt{\bar{n}} \rho) - i \sin(\sqrt{\bar{n}} \rho)}}. \quad (10)$$

Obviously, this is an off-axis Gaussian beam of an initial condition: $u_\alpha(h, 0) = A \exp\left[ -\sqrt{\bar{n}} k (h - h_0)^2 \right]$. The initial center of the beam is $h_0 = \sqrt{\frac{\bar{n}}{k \sqrt{\bar{n}}}}$ and the spot size is a particular value $1/\sqrt{\sqrt{\bar{n}} k}$. This expression of Gaussian beam with special spot size is a so-called function of coherent state.
Next, without loss of generality, the beams of arbitrary spot size need to be studied. In general, the initial Gaussian field is set as \( u(h,0) = A \exp[-(h - h_0)^2/(2\sigma_0^2)] \), \( \sigma_0 \) is the initial spot, \( z_r = k\sigma_0^2/2 \) is the Rayleigh distance of the beam. According to the diffraction integral formula, the following expression of the transmission field is deduced.

\[
u_c(h,\rho) = \frac{A(k\sigma_0^2\sqrt{K})^{1/2}}{\sqrt{2k\sigma_0^2\sqrt{K} \cos(\sqrt{K}\rho)} + i \sin(\sqrt{K}\rho)} \times \exp\left\{ \frac{k\sqrt{K} - (h_0^2 + h^2) \cos(\sqrt{K}\rho) + 2i[h_0 - ihk\sigma_0^2\sqrt{K} \sin(\sqrt{K}\rho)]]}{2k\sigma_0^2\sqrt{K} \cos(\sqrt{K}\rho) + i \sin(\sqrt{K}\rho)} \right\}.
\]

Figure 2 shows the transmission of Gaussian beam with the same initial spot on the different CGCS. We can explore that the diffraction properties of Gaussian beam contend with the focusing effect of positive \( \phi_h \). Generally, for Gaussian beams of arbitrary spot size, we can obtain the evolution of position and momentum of the beam:

\[
\langle h \rangle = \frac{\int u_c h u_c^* \, dh}{\int u_c u_c^* \, dh} = h_0 \cos(\sqrt{K}\rho),
\]

\[
\langle p_h \rangle = \frac{\int u_c \eta_p u_c^* \, dh}{\int u_c \eta_c^* \, dh} = \hbar k\sqrt{K} h_0 \sin(\sqrt{K}\rho),
\]

where \( u_c \) is the Fourier transform of \( u_c \). According to equation (12), it is found that the transmission of the light beam along axis can be regarded as the rotation in the position-momentum \((h - p_h)\) space, which coincides with our previous discussion that CGCS can be regarded as a fractional Fourier transform system [26, 27]. Furthermore, the coordinate uncertainty (spot size) and momentum uncertainty can be obtained by the second moment:

\[
(\Delta h)^2 = \frac{Kk^2\sigma_0^2 + (1/\sigma_0^2) + (Kk^2\sigma_0^2 - 1/\sigma_0^2) \cos(2\sqrt{K}\rho)}{4Kk^2},
\]

\[
(\Delta p_h)^2 = \frac{Kk^2\sigma_0^2 + (1/\sigma_0^2) - (Kk^2\sigma_0^2 - 1/\sigma_0^2) \cos(2\sqrt{K}\rho)}{4}.\]

Here, the notation \((\Delta \hat{O})^2 = (\langle \hat{O} \rangle - \langle \hat{O} \rangle)^2\). From equation (13), the parameters \( X = h(k\sigma_0\sqrt{K}) \) and \( Y = p_h(k\sigma_0\sqrt{K})^{-1}/\hbar \) satisfy an elliptic equation and the uncertainty principle can be given as follows:

\[
(\Delta h)^2(\Delta p_h)^2 = \frac{1}{4}\hbar^2 + \left( \frac{(Kk^2\sigma_0^2 - 1/\sigma_0^2)(\sin 2\sqrt{K}\rho)}{4\sqrt{K}k} \right) \hbar^2 \geq \frac{1}{4}\hbar^2. \tag{14}
\]
Figure 3. Demonstration of the momentum and position uncertainty. (a) Evolution of the uncertainty product under three different conditions. Orthogonal phase space diagram of (b) ‘classical’ coherent state and (c) ‘classical’ squeezed state.

The equal sign in ‘⩾’ symbol is always true if and only if $\sqrt{K} = 1/(2z_r)$, which is also called non-diffraction condition. Besides, the uncertainty product reaches the minimum value $\Delta h \Delta p_x = \hbar/2$. To better illustrate this point, we can introduce two dimensionless orthogonal parameters $X = h(\kappa_0 \sqrt{K})$ and $Y = p_h (\kappa_0 \sqrt{K})^{-1} / h$ as shown in figures 3(b) and (c). In the ‘classical’ coherent state case, $(\Delta X)^2 = (\Delta Y)^2 = 1/2$.

In addition, in the process of beam evolution, it is interesting that the uncertainty product will periodically reach its minimum at $\sin(2\sqrt{K} \rho) = 0$, as shown in figure 3(a). At these special points, we find that although the product of two uncertainties still satisfies the Heisenberg relation, the uncertainty of one of the momentum or position is compressed as equation (15). This is similar to the ‘squeezed state’ in QO as shown in figure 3(c).

$$\Delta X_s = \Delta X, \quad \gamma \cdot \alpha = \Delta Y_s$$

$$(\Delta X)^2 = (\Delta Y)^2 = 1/2.$$

where $\gamma = 2\sqrt{K} z_r$ is the squeeze factor. Evidently, $\gamma = 1$ when the non-diffraction condition is satisfied, which goes back to the ‘classical’ coherent state.

Coherent states and squeezed states are QO concepts, and in fact, they also directly undergo the similar reciprocal transformation process. To some extent, this classical light field exhibits the quantum properties of the first quantization, but it is clear that there exists difficulty in dealing with the second quantization. Here, we offer a new physical understanding of light transmission on a curved surface, and its evolution is similar to that of coherent states in QO, which provides ideas for us to conduct quantum optical simulation experiments from another perspective.

4. Conclusion

We deal with how classical optical fields, in particular Gaussian beams, behave in a curved surface and how a ‘classical’ coherent states approach may aid in physical understanding of the light transmission on curved surfaces. The idea is to use some quantum techniques, such as the path integral, in this classical setting, to obtain a new set of solutions that do not occur in flat space-time. Under a positive curvature, the CGCS metric can restrict the divergence of Gaussian beams, and both momentum and position uncertainty fluctuate periodically. Surprisingly, it is found that the transmission of a basic classical optical field on curved surface generates one kind of simulated coherent state and naturally yields quantized momentum along the transmission direction in this research. The paper also demonstrated the feasibility of creating such quantum states from spatial curvature theoretically.

According to Einstein, gravity can bend spacetime, as this cross section of FRW space-time can be proved to correspond to a curved surface of constant Gaussian curvature, and its shape can be approximated invariant in the short-term transmission of light. Therefore, the research on this surface can theoretically demonstrate the light transmission in gravitational field, and can be used to calculate the
transmission mode of astronomical information. In addition, it is considered the study is not only out of theoretical interest, but can also construct certain special nanostructures with novel features [18, 30, 31] due to the localization of light on the surface, which can be used to the classical simulation of quantum coherent states.

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