Comprehensive Evaluation and its Application in Mathematical Modeling

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Abstract. This paper first introduces the comprehensive evaluation and its purpose, five elements of comprehensive evaluation, and gives the establishment methods of the comprehensive evaluation model, including linear weighting method, nonlinear weighting method and approximate ideal point method. Secondly, the general steps of comprehensive evaluation are summarized, and the treatment methods of evaluation indexes are given: using minimizing – maximizing, centering – maximizing and interval - maximizing to achieve the consistency of index types; using vector normalization, extreme difference transformation, standard sample transformation and linear proportional transformation to standardize index types; using expert consultation and relative comparison to determine the weight of index. Finally, a comprehensive evaluation model of aircraft type is established.

Keywords: Comprehensive Evaluation; Evaluation Index; Evaluation Model; Weight; Mathematical Modeling.

1. Introduction

Comprehensive evaluation is to establish an index system for evaluation, analyze the collected data by using certain methods or models, and make a quantitative overall judgment on the things to be evaluated [1]. The application field and scope of comprehensive evaluation are very wide. From the field of discipline, it is widely used in the evaluation of the characteristics and properties of various things in natural science, such as comprehensive evaluation of environmental monitoring, drug clinical trials, geological hazards, climate characteristics, product quality, etc.; it is widely used in the comprehensive evaluation of general and individual characteristics in social science, for example, comprehensive evaluation of social security, quality of life, social development, teaching level, human settlements and so on. It is more common in the field of economics, for example, comprehensive economic benefit evaluation, well-off construction process evaluation, economic early warning evaluation analysis, production mode comprehensive evaluation, real estate market prosperity comprehensive evaluation and so on. In the mathematical modeling competition and data analysis, the appearance rate of the comprehensive evaluation model is still relatively high, and the practical application is indeed extensive.
2. Comprehensive Evaluation

2.1. Purpose of the comprehensive evaluation
There are only two types of comprehensive evaluation: evaluation of multiple systems and evaluation of one system. There are essentially two purposes for which multiple systems are evaluated: Whose is this thing — classification; which is good and which is bad —— compare and sort. The purpose of evaluating a system is basically to see it passes or fails — implementation degree. Accurate evaluation of a system often plays a decisive role in its further prediction. A good evaluation system is also critical if we need to predict a system.

2.2. Elements of the comprehensive evaluation
There are five main elements of comprehensive evaluation [2]:

(1) Evaluator: the evaluator can be someone or a group. The given purpose of evaluation, the establishment of evaluation index, the selection of evaluation model and the determination of weight coefficient are all related to the evaluator. Therefore, the role of evaluators in the evaluation process cannot be underestimated.

(2) Evaluated objects: With the development of comprehensive evaluation technology, the field of evaluation has also expanded from the initial comprehensive evaluation of economic statistics of various industries to the later technical level, quality of life, well-off level, social development, environmental quality, competitiveness, comprehensive national strength, performance evaluation and so on. These can constitute the object of evaluation. We usually write down the subject as \( x_1, x_2, \ldots, x_n \).

(3) Evaluation index: the evaluation index system reflects the quantity scale and quantity level of specific evaluation object from many angles and levels. It is a dialectical logical thinking process of "concrete - abstract - concrete ", and the process of people's understanding of the overall quantitative characteristics of phenomena is gradually deepened, refined, perfected and systematized. We usually write down the evaluation index as \( f_1, f_2, \ldots, f_m \).

(4) Index weight: the relative importance of evaluation indicators is different relative to a certain evaluation purpose. Whether the weight coefficient is reasonable or not is related to the credibility of the comprehensive evaluation results. The weight is generally written as \( w_i, w_2, \ldots, w_m \) that conform to \( 0 \leq w_i \leq 1, \ i = 1, 2, \ldots, m \), and \( \sum_{i=1}^{m} w_i = 1 \).

(5) Comprehensive evaluation model: refers to the combination of multiple evaluation index values into a holistic comprehensive evaluation value through a certain mathematical model. The comprehensive evaluation model is generally recorded as \( y = y(w,x) \) [3][4]. If the evaluation

\[
\begin{pmatrix}
  f_1 \\
  f_2 \\
  \vdots \\
  f_m
\end{pmatrix}
\]

matrix is involved in the model, we record it as

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{pmatrix}
= \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1m} \\
  a_{21} & a_{22} & \cdots & a_{2m} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nm}
\end{pmatrix}
\]

Wherein, \( a_{ij} \) represents the value of the scheme \( x_i \) in the index \( f_j \) (\( i = 1, 2, \ldots, n; \ j = 1, 2, \ldots, m \)).
2.3. Method for establishing comprehensive evaluation model

There are many modeling methods for comprehensive evaluation [5]. According to the commonly used mathematical models, we give the following three methods

(1) Linear weighting method

The most basic and simple modeling method is to multiply and sum the weights directly with the corresponding normalized measurements. This modeling method is called a linear weighting function:

\[ y = w_1a_1 + w_2a_2 + \ldots + w_ma_m = \sum_{j=1}^{m} w_ja_j \]

Wherein, \(0 \leq w_j \leq 1, (j = 1, 2, \Lambda, m)\) is weight coefficient and \(\sum_{j=1}^{m} w_j = 1\)

It has the following five characteristics: linear compensability; highlighting the function of "larger" index value or weight value; simple calculation; more suitable for each index to be independent of each other; strong "complementarity".

(2) Nonlinear weighting method

The nonlinear weighting method means that the relationship between the evaluation results and the index values is nonlinear, and the nonlinear function can be used as the comprehensive evaluation model:

\[ y = a_1^{w_1} \times a_2^{w_2} \times \ldots \times a_m^{w_m} = \prod_{j=1}^{m} a_j^{w_j} \]

It has two characteristics: emphasizing the consistency of index value, that is, "barrel principle" (one vote veto); it is sensitive to the evaluation index with smaller value, but slow to the evaluation index with larger value.

Approaching ideal point method

Ideal point method is an evaluation function method and an evaluation function method for solving multi-objective programming problems that makes each objective value as close as possible to its ideal (optimal) value. Supposing the ideal point is \(x^* = (x_1^*, x_2^*, \ldots, x_m^*)\), we can calculate the weight distance between calculation scheme \(x = (x_1, x_2, \ldots, x_m)\) and ideal point \(x^* = (x_1^*, x_2^*, \ldots, x_m^*)\):

\[ y = \sum_{j=1}^{m} w_j (x_j - x_j^*)^2 \]

Obviously, the smaller the weighted distance, the better the scheme.

3. General Steps of Comprehensive Evaluation:

(1). Determine the purpose of comprehensive evaluation (classification, ranking, degree of realization).
(2). Select evaluation indicators.
(3). Measure the evaluation index to establish a measurement matrix.
(4). Preprocess the measurement matrix (consistent, standardize).
(5). Determine the weight.
(6). Determine the evaluation model.

4. Treatment of Evaluation Index

4.1. Harmonization of index types

Generally speaking, there are four types of evaluation index data that we measure directly: maximizing index, minimizing index, centering index and interval index[6]. Maximizing index means that the larger the value, the better the corresponding index of the system; minimizing index means that the smaller the value, the better the corresponding index of the system; The centering index means it performs better as
the value is closer to the intermediate value; interval index means it performs better when closing to a certain interval and the performance is the best within the interval. Before establishing the evaluation model, we should turn all the evaluation indexes into the same type, and generally choose the maximum size. The following is a method for evaluating maximizing index.

1. Minimizing to maximizing: supposing $a_{ij}$ is minimizing index, $b_{ij} = M - a_{ij}$, $(M > a_{ij})$ or $b_{ij} = \frac{1}{a_{ij}}$, $(a_{ij} > 0)$, $(i = 1, 2, L, n)$, we can get the maximizing index of $b_{ij}$.

2. Centering to maximizing: for a column in the measurement matrix, the center value is the best. We can turn it to maximizing type in the following method.

Let

$$b_{ij} = \begin{cases} \frac{2(a_{ij} - m)}{M - m} & m \leq a_{ij} \leq \frac{m + M}{2} \\ \frac{2(M - a_{ij})}{M - m} & \frac{m + M}{2} \leq a_{ij} \leq M \end{cases}$$

$(i = 1, 2L, n)$.

Here $m$ and $M$ are the lower and upper bounds allowed by the index, respectively.

3. Interval to maximizing: The following formula can be used to convert interval data types:

Let

$$b_{ij} = \begin{cases} 1 - \frac{q_1 - a_{ij}}{\max\{q_1 - m, M - q_2\}} & a_{ij} < q_1 \\ 1 & q_1 \leq a_{ij} \leq q_2 \\ 1 - \frac{a_{ij} - q_2}{\max\{q_1 - m, M - q_2\}} & a_{ij} > q_2 \end{cases}$$

$[q_1, q_2]$ is a satisfying interval.

4.2. Standardization of index types

1. Vector normalization method:

For $j$ index $f_j$, $x_{ij} = \frac{a_{ij}}{\sqrt{\sum_{j=1}^{n} a_{ij}^2}}$, $(i = 1, 2, L, n; j = 1, 2L, m)$

$-1 \leq x_{ij} \leq 1$, when $a_{ij} \geq 0$, $0 \leq x_{ij} \leq 1$.

2. Differential transformation method:

For positive index $f_j$, $a_j^* = \max\{a_{ij}\}$, $a_j^0 = \min\{a_{ij}\}$, $x_{ij} = \frac{a_{ij} - a_j^0}{a_j^* - a_j^0}$, $(i = 1, 2, L, n)$

For reverse index $f_j$, $a_j^* = \max\{a_{ij}\}$, $a_j^0 = \min\{a_{ij}\}$, $x_{ij} = \frac{a_{ij} - a_j^0}{a_j^* - a_j^0}$, $(i = 1, 2, L, n)$

For positive and reverse indexes, $0 \leq x_{ij} \leq 1$ after the treatment

3. Standard sample transformation method:
For index $f_j$, $x_{ij} = \frac{a_{ij} - \bar{a}_j}{s_j} \quad (i = 1, 2, \ldots, n), \quad \bar{a}_j = \frac{1}{n} \sum_{i=1}^{n} a_{ij}, \quad s_j = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (a_{ij} - \bar{a}_j)^2}.$

(4). Linear proportional transformation method

For positive index $f_j$, $x_{ij} = \frac{a_{ij}}{a_j} \quad (i = 1, 2, \ldots, n), \quad a_j^* = \max_{1 \leq i \leq n} \{a_{ij}\} \neq 0$; for reverse index $f_k$,

$x_k = \frac{a_k^*}{a_{ik}} \quad (i = 1, 2, \ldots, n), \quad a_k^* = \min_{1 \leq i \leq n} \{a_{ik}\}.$ For positive and reverse indexes, when $a_{ij} \geq 0$,

$0 \leq x_{ij} \leq 1.$

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4.3. Determination of index weights

In mathematical modeling, we usually use expert consultation method and relative comparison method.

(1). Expert consultation method

When there is only one expert, weight $w = (w_1, w_2, \ldots, w_m)$; when there are $k$ experts, weight $w_i = (w_{i1}, w_{i2}, \ldots, w_{im})$, $i = 1, 2, \ldots, k$. The final weight $w_j^* = \frac{1}{k} \sum_{i=1}^{k} w_{ij} \quad (j = 1, 2, \ldots, m)$.

(2). Comparison method

Comparison method is to compare $m$ indexes $f_1, f_2, \ldots, f_m$ according to level 3 scale. We can define

\[ h_{ij} = \begin{cases} 1, & \text{When } f_i \text{ is more important than } f_j \\ 0.5, & \text{When } f_i \text{ is as important as } f_j \\ 0, & \text{When } f_i \text{ is less important than } f_j \end{cases} \]

Matrix $H = (h_{ij})_{m \times m}$ is obtained to calculate the weight of index $f_i$ $w_i = \frac{\sum_{j=1}^{m} h_{ij}}{\sum_{j=1}^{m} (\sum_{i=1}^{m} h_{ij})}$ $(i = 1, 2, \ldots, m)$.

5. Specific applications

Example: (airplane purchase) If an airline buys aircraft in the international market, please make a comprehensive evaluation of different types of aircraft according to the six decision indicators in Table 1, and give the purchase plan.

| Model Index | Maximum velocity (Mach) | Maximum range (km) | Maximum load (kg) | Cost (USD 1 million) | Reliability | Sensitivity |
|-------------|-------------------------|--------------------|------------------|----------------------|-------------|------------|
| $a_1$       | 2.0                     | 1500               | 20000            | 5.5                  | Fair        | Very high  |
| $a_2$       | 2.5                     | 2700               | 18000            | 6.5                  | Low         | Fair       |
| $a_3$       | 1.8                     | 2000               | 21000            | 4.5                  | High        | High       |
| $a_4$       | 2.2                     | 1800               | 20000            | 5.0                  | Fair        | Fair       |
**Analysis:** This is a comprehensive evaluation problem, we give the following steps.

**Step 1:** Quantification of indicator values: We quantify qualitative indicators as shown in Table 2.

**Table 2. Quantification of qualitative indicators**

| Grade         | Very low | Low | Fair | High | Very high |
|---------------|----------|-----|------|------|-----------|
| Score         | 1        | 3   | 5    | 7    | 9         |

The quantized matrix can be obtained:

$$X = (x_{ij})_{6 \times 4} = \begin{pmatrix} 2 & 1500 & 20000 & 5.5 & 5 & 9 \\ 2.5 & 2700 & 18000 & 6.5 & 3 & 5 \\ 1.8 & 2000 & 21000 & 4.5 & 7 & 7 \\ 2.2 & 1800 & 20000 & 5 & 5 & 5 \\ \end{pmatrix}$$

**Step 2:** Harmonization of indicator types: we transform matrix $X$ to a maximizing matrix, then

$$X = (x_{ij})_{6 \times 4} = \begin{pmatrix} 2 & 1500 & 20000 & \frac{1}{5.5} & 5 & 9 \\ 2.5 & 2700 & 18000 & \frac{1}{6.5} & 3 & 5 \\ 1.8 & 2000 & 21000 & \frac{1}{4.5} & 7 & 7 \\ 2.2 & 1800 & 20000 & \frac{1}{5} & 5 & 5 \\ \end{pmatrix}$$

**Step 3:** using linear proportional transformation method, the decision matrix $X = (x_{ij})_{4 \times 6}$ is standardized, then

$$Y = (y_{ij})_{4 \times 6} = \begin{pmatrix} 0.80 & 0.56 & 0.95 & 0.82 & 0.71 & 1.00 \\ 1.00 & 1.00 & 0.86 & 0.69 & 0.43 & 0.56 \\ 0.72 & 0.74 & 1.00 & 1.00 & 1.00 & 0.78 \\ 0.88 & 0.67 & 0.95 & 0.90 & 0.71 & 0.56 \\ \end{pmatrix}$$

(Note: The linear proportional transformation method has the function of consistency at the same time)

**Step 4:** The relative comparison transformation method is used to to determine the weight of the six decision indicators, see Table 3.

**Table 3. Quantification of qualitative indicators**

| Maximum velocity (Mach) | Maximum range (km) | Maximum load (kg) | Cost (USD 1 million) | reliability | sensitivity | Total score | Weight |
|-------------------------|--------------------|-------------------|----------------------|--------------|-------------|-------------|--------|
| 0.5                     | 1                  | 1                 | 1                    | 0.5          | 0           | 4           | 0.22   |
| 0                       | 0.5                | 0.5               | 0.5                  | 0            | 0           | 1.5         | 0.08   |
| 0                       | 0.5                | 0.5               | 0.5                  | 0            | 0           | 1.5         | 0.08   |
| 0                       | 0.5                | 0.5               | 0.5                  | 0            | 0           | 1.5         | 0.08   |
| 0                       | 0.5                | 0.5               | 0.5                  | 0            | 0           | 1.5         | 0.08   |
| 0.5                     | 1                  | 1                 | 1                    | 0.5          | 0           | 4           | 0.22   |
| 1                       | 1                  | 1                 | 1                    | 0.5          | 0           | 5.5         | 0.31   |
The weight vector is \( W = (0.2, 0.1, 0.1, 0.1, 0.2, 0.3)^T \)

**Step 5:** The linear weighted index values of each scheme are obtained

\[
u_i = w_1 a_{i1} + w_2 a_{i2} + \sum_{j=1}^{m} w_m a_{ij}.
\]

After calculation, \( u_1 = 0.835 \), \( u_2 = 0.709 \), \( u_3 = 0.853 \) and \( u_4 = 0.738 \) are obtained.

Therefore, the most satisfying scheme is \( a^* = a_3 \) The ordering results of each scheme of machine purchase problem are \( a_3 > a_1 > a_4 > a_2 \)

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