Laplacian Perturbed by Non-Local Operators

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Abstract
Suppose that \( d \geq 1 \) and \( 0 < \beta < 2 \). We establish the existence and uniqueness of the fundamental solution \( q^b(t, x, y) \) to the operator \( L^b = \Delta + S^b \), where

\[
S^b f(x) := \int_{\mathbb{R}^d} \left( f(x + z) - f(x) - \nabla f(x) \cdot z \mathbb{1}_{|z| \leq 1} \right) b(x, z) \frac{dz}{|z|^{d+\beta}}
\]

and \( b(x, z) \) is a bounded measurable function on \( \mathbb{R}^d \times \mathbb{R}^d \) with \( b(x, z) = b(x, -z) \) for \( x, z \in \mathbb{R}^d \). We show that if for each \( x \in \mathbb{R}^d \), \( b(x, z) \geq 0 \) for a.e. \( z \in \mathbb{R}^d \), then \( q^b(t, x, y) \) is a strictly positive continuous function and it uniquely determines a conservative Feller process \( X^b \), which has strong Feller property. Furthermore, sharp two-sided estimates on \( q^b(t, x, y) \) are derived.

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1 Introduction

It is well known that the fundamental solution of the heat equation for Laplacian operator \( \Delta \) is Gaussian kernel. The study of heat kernel estimates for perturbation of Laplace operator \( \Delta \) by gradient operator has a long history and this subject has been studied in many literatures. In recent years, the study for nonlocal operator and the associated discontinuous Markov process has attracted a lot of interests and much progress has been made in this field. In particular, for the operator \( \Delta \) with a pure nonlocal part in \( \mathbb{R}^d \), Song and Vondracek [11] obtained the two sided estimates of transition density of the independent sum of Brownian motion and symmetric stable process \( X^b \), which has strong Feller property. Furthermore, sharp two-sided estimates on \( q^b(t, x, y) \) are derived.

Throughout this paper, let \( d \geq 1 \) be an integer and \( 0 < \beta < 2 \). Recall that a stochastic process \( Z = (Z_t, \mathbb{P}_x, x \in \mathbb{R}^d) \) is called a (rotationally) symmetric \( \beta \)-stable process on \( \mathbb{R}^d \) if it is a Lévy process having

\[
\mathbb{E}_x \left[ e^{i\xi \cdot (Z_t - Z_0)} \right] = e^{-t|\xi|^\beta} \quad \text{for every } x, \xi \in \mathbb{R}^d.
\]
Let $\hat{f}(\xi) := \int_{\mathbb{R}^d} e^{i\xi \cdot x} f(x) dx$ denote the Fourier transform of a function $f$ on $\mathbb{R}^d$. The fractional Laplacian $\Delta^{\beta/2}$ on $\mathbb{R}^d$ is defined as

$$\Delta^{\beta/2} f(x) = \int_{\mathbb{R}^d} \left( f(x + z) - f(x) - \nabla f(x) \cdot z 1_{\{|z| \leq 1\}} \right) \frac{A(d, -\beta)}{|z|^{d+\beta}} dz$$

(1.1)

for $f \in C^2_0(\mathbb{R}^d)$. Here $A(d, -\beta)$ is the normalizing constant so that $\Delta^{\beta/2} \hat{f}(\xi) = -|\xi|^{\beta} \hat{f}(\xi)$. Hence $\Delta^{\beta/2}$ is the infinitesimal generator for the symmetric $\beta$-stable process on $\mathbb{R}^d$.

Let $\mathbb{Z}_t$ be a finite range symmetric $\beta$-stable process in $\mathbb{R}^d$ with jumps of size larger than 1 removed. It is known that the infinitesimal generator of the process $\mathbb{Z}_t$ is the truncated operator

$$\Delta^{\beta/2} f(x) := \int_{|z| \leq 1} \left( f(x + z) - f(x) - \nabla f(x) \cdot z \right) \frac{A(d, -\beta)}{|z|^{d+\beta}} dz.$$ 

(1.2)

Let $b(x, z)$ be a real-valued bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying

$$b(x, z) = b(x, -z) \quad \text{for every } x, z \in \mathbb{R}^d.$$ 

(1.3)

This paper is concerned with the existence, uniqueness and sharp estimates on the “fundamental solution” of the following operator on $\mathbb{R}^d$,

$$\mathcal{L}^b f(x) = \Delta f(x) + \mathcal{S}^b f(x),$$

where

$$\mathcal{S}^b f(x) := \int_{\mathbb{R}^d} \left( f(x + z) - f(x) - \nabla f(x) \cdot z 1_{\{|z| \leq 1\}} \right) \frac{b(x, z)}{|z|^{d+\beta}} dz, \quad f \in C^2_0(\mathbb{R}^d).$$

(1.4)

We point out that if $b(x, z)$ satisfies condition (1.3), the truncation $|z| \leq 1$ in (1.4) can be replaced by $|z| \leq \lambda$ for any $\lambda > 0$; that is, for every $\lambda > 0$,

$$\mathcal{S}^b f(x) = \int_{\mathbb{R}^d} \left( f(x + z) - f(x) - (\nabla f(x), z) 1_{\{|z| \leq \lambda\}} \right) \frac{b(x, z)}{|z|^{d+\beta}} dz.$$ 

(1.5)

In fact, under condition (1.3),

$$\mathcal{S}^b f(x) = \text{p.v.} \int_{\mathbb{R}^d} \left( f(x + z) - f(x) \right) \frac{b(x, z)}{|z|^{d+\beta}} dz$$

$$= \lim_{\varepsilon \to 0} \int_{\{z \in \mathbb{R}^d : |z| > \varepsilon\}} \left( f(x + z) - f(x) \right) \frac{b(x, z)}{|z|^{d+\beta}} dz.$$ 

(1.6)

The operator $\mathcal{L}^b$ is in general non-symmetric. Clearly, $\mathcal{L}^b = \Delta$ when $b \equiv 0$. $\mathcal{L}^b = \Delta + \Delta^{\beta/2}$ when $b \equiv A(d, -\beta)$ and $\mathcal{L}^b = \Delta + \Delta^{\beta/2}$ when $b(x, z) = A(d, -\beta) 1_{\{|z| \leq 1\}}(z)$. It is known that the above two symmetric operators are the infinitesimal generators of the independent sum of the Brownian process and the symmetric $\beta$-stable process (rep. symmetric finite range $\beta$-stable process). The Lévy measures of $\Delta^{\beta/2}$ and $\Delta^{\beta/2}$ are symmetric in the variable $z$ and do not depend on $x$, the perturbed operator $\mathcal{S}^b$ under the condition (1.3) can be viewed as a nonsymmetric extension of $\Delta^{\beta/2}$ and $\Delta^{\beta/2}$.

Our motivation for the operator $\mathcal{L}^b$ comes from a very recent work [7]. The authors consider the existence and uniqueness of the fundamental solution to the fractional Laplacian operator
\( \Delta^{\alpha/2} \) perturbed by lower order nonlocal operator \( S^b \) defined in (1.5) (i.e. \( \Delta^{\alpha/2} + S^b \) and \( 0 < \beta < \alpha < 2 \)) and further derive the two sided heat kernel estimates. The main method to get the upper bound estimate in [7] is the iterative Duhamel’s formula. However, this method can’t work as well for the operator \( L^b \) as in [7]. The main reason is that the Gaussian kernel \( p_0(t, x, y) \) is an exponential function of \( |x - y|^2 \), after finite number of iterations of recursive Duhamel’s formula, the item \( \exp \left( -c^\alpha |x - y|^2 / t \right) \), where \( 0 < c < 1 \) is a constant and \( n \) is the number of iterations, will appear in the upper bound estimate and thus one will lose the exponential term in the last. So to derive a sharp upper bound estimate, we need a more delicate estimate.

On the other hand, one of main goal of this paper is to study sharp two sided heat kernel estimates for \( \Delta \) under finite range non-local perturbation. To the author’s knowledge, even for the symmetric case \( \Delta + a\Delta^{\beta/2} \), \( a > 0 \), the relevant result is new until now. As we shall see in Theorem 1.5 and Corollary 1.6 below, the two sided heat kernel estimates for such operator under some positivity condition will depend heavily on the transition density function for the truncated \( \beta/2 \)-fractional operator \( \Delta^{\beta/2} \), in addition to the Gaussian kernel. This is different from the fractional Laplacian operator \( \Delta^{\alpha/2} \) under finite range nonlocal perturbation, because Theorem 1.4 of [7] shows that the heat kernel for \( \Delta^{\alpha/2} + S^b \) in this case is comparable with the heat kernel for fractional Laplacian operator \( \Delta^{\alpha/2} \). Furthermore, the upper bound estimate for \( \Delta \) under finite range non-local perturbation in Theorem 1.5 in this paper can’t be obtained by the methods in [7] and related literatures, we use a new probability argument to get it.

For \( a \geq 0 \), denote by \( p_a(t, x, y) \) the fundamental function of \( \Delta + a\Delta^{\beta/2} \). Clearly, \( p_a(t, x, y) \) is a function of \( t \) and \( x - y \), so sometimes we also write it as \( p_a(t, x - y) \). It is known (see (2.3) of Section 2 for details) that on \((0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \),

\[
P_0(t, x, y) = (4\pi t)^{-d/2} e^{-|x - y|^2 / 4t},
\]

(1.7)

\[
c_1 \left( t^{-d/2} \land (at)^{-d/\beta} \right) \land \left( p_0(t, c_2 x, c_2 y) + \frac{at}{|x - y|^{d+\beta}} \right)
\leq p_0(t, x, y) \leq c_3 \left( t^{-d/2} \land (at)^{-d/\beta} \right) \land \left( p_0(t, c_4 x, c_4 y) + \frac{at}{|x - y|^{d+\beta}} \right).
\]

(1.8)

Here we use \( a \lor c \) and \( a \land c \) to denote \( \max\{a, c\} \) and \( \min\{a, c\} \), respectively. Note that \((at)^{-d/\beta} \geq t^{-d/2}\) whenever \( 0 < t \leq \alpha^{-2/(2 - \beta)} \).

To establish the fundamental solution of \( L^b \), we use the method of Duhamel’s formula. Since \( L^b = \Delta + S^b \) is a lower order perturbation of \( \Delta \) by \( S^b \), the fundamental solution (or kernel) \( q^b(t, x, y) \) of \( L^b \) should satisfy the following Duhamel’s formula:

\[
q^b(t, x, y) = p_0(t, x, y) + \int_0^t \int_{\mathbb{R}^d} q^b(t - s, x, z) S^b p_0(s, z, y) dz ds
\]

(1.9)

for \( t > 0 \) and \( x, y \in \mathbb{R}^d \). Here the notation \( S^b p_0(s, z, y) \) means the non-local operator \( S^b \) is applied to the function \( z \mapsto p_0(s, z, y) \). Similar notation will also be used for other operators, for example, \( \Delta_x \). Applying (1.9) recursively, it is reasonable to conjecture that \( \sum_{n=0}^\infty q^b_n(t, x, y) \), if convergent, is a solution to (1.9), where \( q^b_0(t, x, y) := p_0(t, x, y) \) and

\[
q^b_n(t, x, y) := \int_0^t \int_{\mathbb{R}^d} q^b_{n-1}(t - s, x, z) S^b p_0(s, z, y) dz ds \quad \text{for } n \geq 1.
\]

(1.10)

The followings are the main results of this paper. We use the notation \( \|b\|_{\infty} := \|b\|_{L^\infty} \).
It is established in [3] that

\[ \text{Theorem 1.1.} \]

There is a constant \( A_0 > 0 \) so that for every bounded function \( b \) on \( \mathbb{R}^d \times \mathbb{R}^d \) satisfying condition (1.13) with \( \|b\|_\infty \leq A_0 \), there is a unique continuous function \( q^b(t, x, y) \) on \( (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \) that satisfies (1.9) on \( (0, \varepsilon) \times \mathbb{R}^d \times \mathbb{R}^d \) with \( |q^b(t, x, y)| \leq c_1 p_1(t, c_2 x, c_2 y) \) on \( (0, \varepsilon) \times \mathbb{R}^d \times \mathbb{R}^d \) for some \( \varepsilon, c_1, c_2 > 0 \), and that

\[ \int_{\mathbb{R}^d} q^b(t, x, y) q^b(s, y, z) dy = q^b(t + s, x, z) \quad \text{for every } t, s > 0 \text{ and } x, z \in \mathbb{R}^d. \] (1.11)

Moreover, the following holds.

(i) \( q^b(t, x, y) = \sum_{n=0}^{\infty} q^b_n(t, x, y) \) on \( (0, 1 \wedge A_0/\|b\|_\infty)^{2/(2-\beta)} \times \mathbb{R}^d \times \mathbb{R}^d \), where \( q^b_n(t, x, y) \) is defined by (1.10).

(ii) \( q^b(t, x, y) \) satisfies the Duhamel’s formula (1.9) for all \( t > 0 \) and \( x, y \in \mathbb{R}^d \). Moreover, \( \mathcal{S}^b_2 q^b(t, x, y) \) exists pointwise in the sense of (1.6) and

\[ q^b(t, x, y) = p_0(t, x, y) + \int_0^t \int_{\mathbb{R}^d} p_0(t - s, x, z) \mathcal{S}^b_2 q^b(s, z, y) dz ds \] (1.12)

for \( t > 0 \) and \( x, y \in \mathbb{R}^d \).

(iii) For each \( t > 0 \) and \( x \in \mathbb{R}^d \), \( \int_{\mathbb{R}^d} q^b(t, x, y) dy = 1 \).

(iv) For every \( f \in C^2_b(\mathbb{R}^d) \),

\[ T^b_t f(x) - f(x) = \int_0^t T^b_s \mathcal{L}^b f(x) ds, \]

where \( T^b_t f(x) = \int_{\mathbb{R}^d} q^b(t, x, y) f(y) dy \).

(v) Let \( A > 0 \). There are positive constants \( C = C(d, \beta, A) \geq 1 \) and \( 0 < C = C(d, \beta, A) \leq 1 \) so that for any \( b \) with \( \|b\|_\infty \leq A \),

\[ |q^b(t, x, y)| \leq C e^{Ct} p_{\|b\|_\infty}(t, Cx, Cy) \quad \text{on } (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d. \] (1.13)

In general the kernel \( q^b(t, x, y) \) in Theorem 1.1 can be negative. The next theorem gives the necessary and sufficient condition for \( q^b(t, x, y) \geq 0 \).

\[ \text{Theorem 1.2.} \]

Let \( b \) be a bounded function on \( \mathbb{R}^d \times \mathbb{R}^d \) that satisfies (1.3) and that

\[ x \mapsto b(x, z) \text{ is continuous for a.e. } z \in \mathbb{R}^d. \] (1.14)

Then \( q^b(t, x, y) \geq 0 \) on \( (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \) if and only if for each \( x \in \mathbb{R}^d \),

\[ b(x, z) \geq 0 \quad \text{for a.e. } z \in \mathbb{R}^d. \] (1.15)

Let \( \mathcal{P}_\beta(t, x, y) \) be the fundamental solution of the truncated operator \( \Delta^{\beta/2} \) defined in (1.2). It is established in [3] that \( \mathcal{P}_\beta(t, x, y) \) is jointly continuous and enjoys the following two sided estimates:

\[ \mathcal{P}_\beta(t, x, y) \sim t^{-d/\beta} \wedge \frac{t}{|x - y|^{d+\beta}} \] (1.16)
for $t \in (0, 1]$ and $|x - y| \leq 1$, and there are constants $c_k > 0$, $k = 1, 2, 3, 4$ so that

$$c_1 \left( \frac{t}{|x - y|} \right)^{c_2|x-y|} \leq \mathbb{P}_b(t, x, y) \leq c_3 \left( \frac{t}{|x - y|} \right)^{c_4|x-y|} \quad (1.17)$$

for $t \in (0, 1]$ and $|x - y| > 1$.

Next theorem drops the assumption (1.14), gives lower bound estimate on $q^b(t, x, y)$ for $b(x, z)$ satisfying condition (1.15) and establish the Feller process for $\mathcal{L}^b$.

Define $m_b := \inf_x \text{essinf}_z b(x, z)$, $M_b := \text{esssup}_{x, z} b(x, z)$.

**Theorem 1.3.** For every $A > 0$, there are positive constants $C_k = C_k(d, \beta, A), k = 1, \cdots, 4$ such that for any bounded $b$ with (1.3), (1.15) and $\|b\|_\infty \leq A$,

$$C_1p_{m_b}(t, C_2x, C_2y) \leq q^b(t, x, y) \leq C_3p_{M_b}(t, C_4x, C_4y) \quad \text{for } t \in (0, 1] \text{ and } x, y \in \mathbb{R}^d. \quad (1.18)$$

Furthermore, for each $\lambda > 0$ and $\varepsilon > 0$, there are positive constants $C_k = C_k(d, \beta, \varepsilon, \lambda, A), k = 5, 6$ such that for any bounded $b$ on $\mathbb{R}^d \times \mathbb{R}^d$ with $\|b\|_\infty \leq A$ satisfying (1.3), (1.15) and

$$\inf_{x \in \mathbb{R}^d, |x| \leq \lambda} b(x, z) > \varepsilon, \quad (1.19)$$

we have

$$C_5 \left( t^{-d/2} \wedge (p_{m_b}(t, C_6x, C_6y) + \mathbb{P}_b(t, C_6x, C_6y)) \right) \leq q^b(t, x, y) \leq C_3p_{M_b}(t, C_4x, C_4y) \quad (1.20)$$

for $t \in (0, 1]$ and $x, y \in \mathbb{R}^d$. For each bounded function $b$ with (1.3) and (1.15), the kernel $q^b(t, x, y)$ uniquely determines a Feller process $X^b = (X^b_t, t \geq 0, \mathbb{P}_x, x \in \mathbb{R}^d)$ on the canonical Skorokhod space $\mathbb{D}([0, \infty), \mathbb{R}^d)$ such that

$$\mathbb{E}_x \left[ f(X^b_t) \right] = \int_{\mathbb{R}^d} q^b(t, x, y) f(y) dy$$

for every bounded continuous function $f$ on $\mathbb{R}^d$. The Feller process $X^b$ is conservative and has a Lévy system $(J^b(x, y) dy, t)$, where

$$J^b(x, y) = \frac{b(x, y - x)}{|x - y|^{d+\beta}}. \quad (1.21)$$

**Remark 1.4.** The estimates in (1.18) are sharp in the sense that $q^b(t, x, y) = p_0(t, x, y)$ when $b \equiv 0$, and $q^b(t, x, y) = p_1(t, x, y)$ when $b \equiv A(d, -\beta)$. In particular, it follows from (1.18) that for every $A \geq 1$, there are positive constants $\tilde{C}_1, \tilde{C}_2 \geq 1$ so that for any $b$ on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.3) with $1/A \leq b(x, z) \leq A$ a.e.

$$(1/\tilde{C}_1) p_1(t, \tilde{C}_2x, \tilde{C}_2y) \leq q^b(t, x, y) \leq \tilde{C}_1 p_1(t, x/\tilde{C}_2, y/\tilde{C}_2) \quad \text{for } t \in (0, 1] \text{ and } x, y \in \mathbb{R}^d. \quad (1.22)$$

If $b$ is a bounded function satisfying (1.3) and (1.15) so that $b(x, z) = 0$ for every $x \in \mathbb{R}^d$ and $|z| \geq R$ for some $R > 0$; or, equivalently if $\mathcal{L}^b = \Delta + S^b$ is a lower order perturbation of $\Delta$ by finite range non-local operator $S^b$, then we have the following refined upper bound.
Theorem 1.5. For every \( \lambda > 0 \) and \( M \geq 1 \), there are positive constants \( C_k = C_k(d, \beta, M, \lambda), k = 7, 8 \) such that for any bounded \( b \) with (1.3), (1.15) and
\[
\sup_x b(x, z) \leq M \sup_{|z| \leq \lambda} (z),
\]
we have
\[
q^b(t, x, y) \leq C_7 \left[ t^{-d/2} \wedge \left( p_0(t, C_8 x, C_8 y) + \overline{p}_\beta(t, C_8 x, C_8 y) \right) \right] \quad \text{for } t \in (0, 1], x, y \in \mathbb{R}^d. (1.24)
\]

The following follows immediately from Theorem 1.5 and (1.20).

Corollary 1.6. For every \( \lambda > 0 \) and \( M \geq 1 \), there are positive constants \( c_k = c_k(d, \beta, M, \lambda), k = 1, \ldots, 4 \) such that for any bounded \( b \) with (1.3) and
\[
M^{-1} \sup_{|z| \leq \lambda} (z) \leq \inf_x b(x, z) \leq \sup_x b(x, z) \leq M \sup_{|z| \leq \lambda} (z),
\]
we have
\[
c_1 \left[ t^{-d/2} \wedge \left( p_0(t, c_2 x, c_2 y) + \overline{p}_\beta(t, c_2 x, c_2 y) \right) \right] \leq q^b(t, x, y) \leq c_3 \left[ t^{-d/2} \wedge \left( p_0(t, c_4 x, c_4 y) + \overline{p}_\beta(t, c_4 x, c_4 y) \right) \right]
\]
for \( t \in (0, 1] \) and \( x, y \in \mathbb{R}^d \).

Remark 1.7. (i) Corollary 1.6 reveals the two sided heat kernel estimates for Laplacian \( \Delta \) under the finite range non-local perturbation with the condition (1.25). This result seems new even in the symmetric case \( \Delta + a \Delta^{\beta/2}, a > 0 \).

(ii) It looks difficult when we try to use Duhamel’s formula to get the refined upper bound (1.24) in Theorem 1.5 under the assumption that \( b(x, z) \) satisfies (1.23). The main reason is that the heat kernel \( \overline{p}_\beta(t, x, y) \) of the truncated operator \( \Delta^{\beta/2} \) exhibits the Poisson type form when \( |x - y| > 1 \) and \( t \in (0, 1] \) (see (1.17)), which makes big trouble in the iteration of the recursive Duhamel’s formula (1.10). To circumvent this obstacle, we adopt a new probability argument to go through it.

The rest of the paper is organized as follows. In Section 2 we derive some estimates on \( \Delta^{\beta/2} p_0(t, x, y) \) that will be used in later. The existence and uniqueness of the fundamental solution \( q^b(t, x, y) \) of \( \mathcal{L}^b \) is given in Section 3. This is done through a series of lemmas and theorems, which provide more detailed information on \( q^b(t, x, y) \) and \( q^b_n(t, x, y) \). Theorem 1.1 then follows from these results. We show in Section 4 that \( \{ T^b_t ; t > 0 \} \) is a strongly continuous semigroup in \( C_\infty(\mathbb{R}^d) \). We then apply Hille-Yosida-Ray theorem and Courrèges’ first theorem to establish Theorem 1.2. When \( b \) satisfies (1.3), (1.14) and (1.15), \( q^b(t, x, y) \) determines a conservative Feller process \( X^b \). In Section 5, we extend the result to general bounded \( b \) that satisfies (1.3) and (1.15) by approximating it by a sequence of \( \{ b^{(n)} \}, n \geq 1 \) that satisfy (1.3), (1.14) and (1.15). Finally, the lower bound estimate in Theorem 1.3 and Theorem 1.5 are established by the Lévy system of \( X^b \) and some probability arguments.

Throughout this paper, we use the capital letters \( C_1, C_2, \cdots \) to denote constants in the statement of the results, and their labeling will be fixed. The lowercase constants \( c_1, c_2, \cdots \) will denote generic constants used in the proofs, whose exact values are not important and can change from one appearance to another. We will use “:=" to denote a definition. For a differentiable function \( f \) on \( \mathbb{R}^d \), we use \( \partial_i f \) and \( \partial_{ij} f \) to denote the partial derivatives \( \frac{\partial f}{\partial x_i} \) and \( \frac{\partial^2 f}{\partial x_i \partial x_j} \).
2 Preliminaries

Suppose that $Y$ is a Brownian motion, and $Z$ is a symmetric $\beta$-stable process on $\mathbb{R}^d$ that is independent of $Z$. For any $a \geq 0$, we define $Y^a$ by $Y^a_t := Y_t + a^{1/\beta}Z_t$. We will call the process $Y^a$ the independent sum of the Brownian process $Y$ and the symmetric $\beta$-stable process $Z$ with weight $a^{1/\beta}$. The infinitesimal generator of $Y^a$ is $\Delta + a\Delta^{\beta/2}$. Let $p_a(t, x, y)$ denote the transition density of $Y^a$ (or equivalently the heat kernel of $\Delta + a\Delta^{\beta/2}$) with respect to the Lebesgue measure on $\mathbb{R}^d$. Recently it is proven in [4] and [11] that

$$c_1 \left(t^{-d/2} \wedge t^{-d/\beta}\right) \wedge \left(p_0(t, c_2x, c_2y) + \frac{t}{|x-y|^{d+\beta}}\right) \leq p_1(t, x, y) \leq c_3 \left(t^{-d/2} \wedge t^{-d/\beta}\right) \wedge \left(p_0(t, c_4x, c_4y) + \frac{t}{|x-y|^{d+\beta}}\right)$$

(2.1)

for $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

Unlike the case of the Brownian motion $Y := Y^0$, $Y^a$ does not have the stable scaling for $a > 0$. Instead, the following approximate scaling property holds: for every $\lambda > 0$, $\{\lambda^{-1}Y^a_{\lambda^2t}, t \geq 0\}$ has the same distribution as $\{Y^a_{\lambda(2-\beta)t}, t \geq 0\}$. Consequently, for any $\lambda > 0$, we have

$$p_{a\lambda(2-\beta)}(t, x, y) = \lambda^d p_a(\lambda^2 t, \lambda x, \lambda y) \quad \text{for } t > 0 \text{ and } x, y \in \mathbb{R}^d.$$ (2.2)

In particular, letting $a = 1, \lambda = a^{1/(2-\beta)}$, we get

$$p_a(t, x, y) = a^{d/(2-\beta)} p_1\left(a^{2/(2-\beta)} t, a^{1/(2-\beta)} x, a^{1/(2-\beta)} y\right) \quad \text{for } t > 0 \text{ and } x, y \in \mathbb{R}^d.$$ 

So we deduce from (2.1) that there exist constants $c_k, k = 1, \cdots, 4$ depending only on $d$ and $\beta$ such that for every $a > 0$ and $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$

$$c_1 \left(t^{-d/2} \wedge (at)^{-d/\beta}\right) \wedge \left(p_0(t, c_2x, c_2y) + \frac{at}{|x-y|^{d+\beta}}\right) \leq p_a(t, x, y) \leq c_3 \left(t^{-d/2} \wedge (at)^{-d/\beta}\right) \wedge \left(p_0(t, c_4x, c_4y) + \frac{at}{|x-y|^{d+\beta}}\right).$$

(2.3)

In fact, (2.3) also holds when $a = 0$.

Recall that $p_0(t, x - y)$ is the transition density function of Brownian motion $Y$.

Lemma 2.1. There exists a constant $C_9 = C_9(d) > 0$ such that for every $t > 0, x \in \mathbb{R}^d$ and $i, j = 1, \ldots, d$,

$$p_0(t, x) \leq C_9 t^{-d/2} \left(1 \wedge \frac{t^{1/2}}{|x|}\right)^{d+2}, \quad |\partial^2_{ij} p_0(t, x)| \leq C_9 t^{-(d+2)/2} \left(1 \wedge \frac{t^{1/2}}{|x|}\right)^{d+4}.$$

Proof. It is known that

$$p_0(t, x) = (4\pi t)^{-d/2} e^{-|x|^2/4t}.$$

Thus, $p_0(t, x) \leq (4\pi t)^{-d/2}$. On the other hand, by the proof of Lemma 2.1 in [11],

$$p_0(t, x) \leq c \frac{t}{|x|^{d+2}}, \quad t > 0, x \in \mathbb{R}^d.$$ (2.4)
For the reader’s convenience, we spell out the details here. For each \( x \neq 0 \), define \( f : (0, \infty) \to (0, \infty) \) as follows:

\[
f(t) = t^{-1-d/2} \exp \left( -\frac{|x|^2}{4t} \right).
\]

Then \( f(0+) = f(\infty) = 0 \). Further

\[
f'(t) = f(t)t^{-2}(-\frac{d}{2} + 1)t + |x|^2/4).
\]

This derivative is zero for \( t_0 = \frac{|x|^2}{4(d/2+1)} \), positive for \( t < t_0 \) and negative for \( t > t_0 \). Thus, \( \max f(t) \leq f(t_0) = c|x|^{-(d+2)} \) and (2.4) follows from it. Therefore, there exists \( c_1 > 0 \) such that

\[
p_0(t, x) \leq c_1 \left( t^{-d/2} \wedge \frac{t}{|x|^{d+2}} \right).
\]

Next, we prove the inequality about the second derivatives of \( p_0(t, x) \). By simple computation and (2.5), we have

\[
|\partial^2_{ij} p_0(t, x)| \leq \left[ \frac{|x|^2}{t^2} + \frac{2}{t} \right] p_0(t, x)
= (4\pi)^2 |x|^2 p_0^{(d+4)}(t, \tilde{x}_1) + 8\pi p_0^{(d+2)}(t, \tilde{x}_2)
\leq c_2 |x|^2 \left( t^{-(d+4)/2} \wedge \frac{t}{|x|^{d+6}} \right) + c_2 \left( t^{-(d+2)/2} \wedge \frac{t}{|x|^{d+4}} \right)
\leq c_3 \left( t^{-(d+2)/2} \wedge \frac{t}{|x|^{d+4}} \right)
\]

where \( \tilde{x}_1 \in \mathbb{R}^{d+4} \) and \( \tilde{x}_2 \in \mathbb{R}^{d+2} \) such that \( |\tilde{x}_1| = |\tilde{x}_2| = |x| \), \( p_0^{(d+4)} \) and \( p_0^{(d+2)} \) are the transition densities of Laplacian operator in dimension \( d + 4 \) and \( d + 2 \).

Define for \( t > 0 \) and \( x \in \mathbb{R}^d \), the function

\[
|\Delta^\beta/2 p_0(t, x)| \begin{cases}
= \int_{|z| \leq t^{1/2}} |p_0(t, x + z) - p_0(t, x) - \frac{\partial}{\partial z} p_0(t, x) \cdot z| \frac{1}{|z|^{d+\beta}} dz \\
+ \int_{|z| > t^{1/2}} |p_0(t, x + z) - p_0(t, x)| \frac{dz}{|z|^{d+\beta}} \quad \text{for } |x|^2 \leq t,
\end{cases}

\[
= \int_{|z| \leq |x|/2} |p_0(t, x + z) - p_0(t, x) - \frac{\partial}{\partial z} p_0(t, x) \cdot z| \frac{1}{|z|^{d+\beta}} dz \\
+ \int_{|z| > |x|/2} |p_0(t, x + z) - p_0(t, x)| \frac{dz}{|z|^{d+\beta}} \quad \text{for } |x|^2 > t.
\]

Let

\[
f_0(t, x, y) := \left( t^{1/2} \vee |x - y| \right)^{-(d+\beta)} = t^{-(d+\beta)/2} \left( 1 \vee \frac{t^{1/2}}{|x - y|} \right)^{d+\beta}.
\]

Lemma 2.2. There exists a constant \( C_{10} = C_{10}(d, \beta) > 0 \) such that

\[
|\Delta^\beta/2 p_0(t, x, y) \leq C_{10} f_0(t, x, y) \quad \text{on } (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d.
\]
Proof. (i) We first consider the case $|x|^2 \leq t$. In this case,

$$
|\Delta_x^{\beta/2} p_0(t, x) = \int_{|z| \leq t^{1/2}} |p_0(t, x + z) - p_0(t, x) - \frac{\partial}{\partial x} p_0(t, x) \cdot z| \frac{dz}{|z|^{d+\beta}} + \int_{|z| \geq t^{1/2}} |p_0(t, x + z) - p_0(t, x)| \frac{dz}{|z|^{d+\beta}} = I + II.
$$

Note that by Lemma 2.1, 

$$
\sup_{u \in \mathbb{R}^d} \left| \frac{\partial^2}{\partial u_i \partial u_j} p_0(t, u) \right| \leq C_9 t^{-(d+2)/2},
$$

so by Taylor's formula,

$$
I \leq \sup_{u \in \mathbb{R}^d} \left| \frac{\partial^2}{\partial u_i \partial u_j} p_0(t, u) \right| \int_{|z| \leq t^{1/2}} \frac{|z|^2}{|z|^{d+\beta}} dz \leq C_1 t^{-(d+2)/2} t^{(2-\beta)/2} \leq c_1 t^{-(d+\beta)/2}.
$$

For the second item $II$, we have

$$
II \leq \int_{|z| \geq t^{1/2}} (p_0(t, x + z) + p_0(t, x)) \frac{dz}{|z|^{d+\beta}} \leq c_2 t^{-d/2} \int_{|z| \geq t^{1/2}} \frac{1}{|z|^{d+\beta}} dz \leq c_3 t^{-(d+\beta)/2}.
$$

(ii) Next, we consider the case $|x|^2 \geq t$. In this case,

$$
|\Delta_x^{\beta/2} p_0(t, x) = \int_{|z| \leq |x|/2} |p_0(t, x + z) - p_0(t, x) - \frac{\partial}{\partial x} p_0(t, x) \cdot z| \frac{dz}{|z|^{d+\beta}} + \int_{|z| \geq |x|/2} |p_0(t, x + z) - p_0(t, x)| \frac{dz}{|z|^{d+\beta}} = I + II.
$$

Note that $|x + z| \geq |x|/2$ for $|z| \leq |x|/2$. So by Lemma 2.1

$$
\sup_{|z| \leq |x|/2} \left| \frac{\partial^2}{\partial x_i \partial x_j} p_0(t, x + z) \right| \leq c_3 \sup_{|z| \leq |x|/2} t|x + z|^{-(d+4)} \leq 2^{(d+4)} c_8 t|x|^{-(d+4)}.
$$

Hence, by Taylor's formula

$$
I \leq \sup_{|z| \leq |x|/2} \left| \frac{\partial^2}{\partial x_i \partial x_j} p_0(t, x + z) \right| \int_{|z| \leq |x|/2} \frac{|z|^2}{|z|^{d+\beta}} dz \leq c_4 t|x|^{-(d+4)} |x|^{2-\beta} = c_4 t|x|^{-(d+2+\beta)}. \tag{2.8}
$$

As $|x|^2 \geq t$, thus $I \leq c_4 |x|^{-(d+\beta)}$. On the other hand, noting that $p_0(t, y) \leq p_0(t, x)$ if $|y| \geq |x|$. Hence, by the condition that $|x|^2 \geq t$, we obtain

$$
II \leq \int_{|z| \geq |x|/2, |x + z| \geq |x|} 2p_0(t, x) \frac{dz}{|z|^{d+\beta}} + \int_{|z| \geq |x|/2, |x + z| \leq |x|} 2p_0(t, x + z) \frac{dz}{|z|^{d+\beta}} \leq 2p_0(t, x) \int_{|z| \geq |x|/2} \frac{dz}{|z|^{d+\beta}} + 2^{d+1+\beta} |x|^{-(d+\beta)} \int_{z \in \mathbb{R}^d} p_0(t, x + z) dz \leq c_5 t|x|^{-(d+2)} |x|^{-\beta} + 2^{d+1+\beta} |x|^{-(d+\beta)} \leq c_6 |x|^{-(d+\beta)}, \tag{2.9}
$$

This establishes the lemma.
Lemma 2.3. There is a constant $C_{11} = C_{11}(d, \beta) > 0$ such that

$$
\int_0^t \int_{\mathbb{R}^d} f_0(s, z, y)dzds \leq C_{11} t^{1-\beta/2}, \quad t \in (0, \infty), \ y \in \mathbb{R}^d. \quad (2.10)
$$

Proof. By the definition of $f_0$,

$$
\int_0^t \int_{\mathbb{R}^d} f_0(s, z, y)dzds \\
\leq \int_0^t \int_{|y-z| \leq s^{1/2}} s^{-(d+\beta)/2} dzds + \int_0^t \int_{|y-z| > s^{1/2}} \frac{1}{|y-z|^{d+\beta}} dzds \\
\leq c_1 \int_0^t s^{-\beta/2} dzds \leq c_2 t^{1-\beta/2}. \quad \square
$$

Define

$$
h(t, x, y) = t^{-d/2} \wedge \left( p_0(t, x, y) + \frac{t}{|x-y|^{d+\beta}} \right). \quad (2.11)
$$

Then $\int_{\mathbb{R}^d} h(t, x, y) dy \asymp 1$. Moreover, for $t \in (0, 1]$, $h(t, x, y) \asymp t^{-d/2}$ when $|x-y| \leq t^{1/2}$ and $h(t, x, y) \asymp p_0(t, x, y) + \frac{t}{|x-y|^{d+\beta}}$ when $|x-y| > t^{1/2}$. Here for two non-negative functions $f$ and $g$, the notation $f \asymp g$ means that there is a constant $c \geq 1$ so that $c^{-1}f \leq g \leq cf$ on their common domain of definitions.

Lemma 2.4. There exist $C_{12} = C_{12}(d, \beta) > 1$ and $0 < C_{13} = C_{13}(d, \beta) < 1$ such that for any $t \in (0, 1]$,\n
$$
\int_0^t \int_{\mathbb{R}^d} h(t-s, x, z) f_0(s, z, y) dz ds \leq \begin{cases} 
    C_{12} h(t, x, y), & |x-y| \leq t^{1/2}, \text{ or } |x-y| > 1, \\
    C_{12} h(t, C_{13} x, C_{13} y), & t^{1/2} < |x-y| \leq 1 
\end{cases}.
$$

Proof. Denote by $I = \int_0^t \int_{\mathbb{R}^d} h(t-s, x, z) f_0(s, z, y) dz ds$.

(i) Suppose that $|x-y| \leq t^{1/2}$. We write $I$ as

$$
I = \int_0^{t/2} \int_{\mathbb{R}^d} h(t-s, x, z) f_0(s, z, y) dz ds \\
+ \int_{t/2}^t \int_{\mathbb{R}^d} h(t-s, x, z) f_0(s, z, y) dz ds \\
= I_1 + I_2.
$$

If $s \in (0, t/2)$, then $t-s \in [t/2, t)$. Thus $h(t-s, x, z) \leq c_1 t^{-d/2}$. Hence, by Lemma 2.3

$$
I_1 \leq c_1 t^{-d/2} \int_0^{t/2} \int_{\mathbb{R}^d} f_0(s, z, y) dz ds \leq c_2 t^{-d/2}.
$$

When $s \in [t/2, t]$, noting that $f_0(s, z, y) \leq c_3 t^{-(d+\beta)/2}$, hence,

$$
I_2 \leq c_3 t^{-(d+\beta)/2} \int_0^t \int_{\mathbb{R}^d} h(t-s, x, z) dz ds \leq c_4 t^{-d/2}.
$$
We thus conclude from the above that there is a \( c_5 > 0 \) such that \( I \leq c_5 h(t, x, y) \) for every \( t \in (0, 1] \) whenever \( |x - y| \leq t^{1/2} \).

(ii) Next assume that \( |x - y| \geq t^{1/2} \). Then

\[
I = \int_0^t \int_{|x-z| < |x-y|/2} h(t - s, x, z) f_0(s, z, y) \, dz \, ds \\
+ \int_0^t \int_{|x-z| > |x-y|/2} h(t - s, x, z) f_0(s, z, y) \, dz \, ds \\
=: I_1 + I_2.
\]

If \( |x - z| \leq |x - y|/2 \), then \( |y - z| \geq |x - y|/2 > t^{1/2}/2 \). Thus \( f_0(s, z, y) \leq c_8 |x - y|^{-(d + \beta)} \) for \( s \in (0, t) \). Therefore,

\[
I_1 = \int_0^t \int_{|x-z| < |x-y|/2} h(t - s, x, z) f_0(s, z, y) \, dz \, ds \\
\leq c_6 |x - y|^{-(d + \beta)} \int_0^t \int_{\mathbb{R}^d} h(t - s, x, z) \, dz \, ds \\
\leq c_6 \frac{t}{|x - y|^{d + \beta}} \\
\leq c_6 h(t, x, y)
\]

Now we consider \( I_2 \) where \( |x - z| > |x - y|/2 \). If \( |x - y| > 1 \), then by Lemma 2.1, \( p_0(t - s, x, z) \leq c_7 \frac{t}{|x-z|^{d+2}} \leq c_7 2^{d+2} \frac{t}{|x-y|^{d+2}} \). So by (2.11), for \( |x - z| > |x - y|/2 > 1/2 \),

\[
h(t - s, x, z) \leq \left[ p_0(t - s, x, z) + |x - z|^{-(d + \beta)} t \right] \leq c_8 t |x - y|^{-(d + \beta)} \leq c_8 h(t, x, y).
\]

Thus,

\[
I_2 \leq c_8 h(t, x, y) \int_0^t \int_{\mathbb{R}^d} f_0(s, z, y) \, dz \, ds \leq c_9 h(t, x, y), \quad |x - y| > 1.
\]

Therefore, combining the above inequality with (2.12), there exists \( c_{10} > 0 \) so that

\[
I \leq c_{10} h(t, x, y), \quad |x - y| > 1.
\]

On the other hand, if \( t^{1/2} < |x - y| \leq 1 \), then we divide \( I_2 \) into two parts:

\[
I_2 = \int_0^{t/2} \int_{|x-z| > |x-y|/2} h(t - s, x, z) f_0(s, z, y) \, dz \, ds \\
+ \int_{t/2}^t \int_{|x-z| > |x-y|/2} h(t - s, x, z) f_0(s, z, y) \, dz \, ds \\
= I_{21} + I_{22}.
\]

We first consider \( I_{21} \). Noting that \( h(t - s, x, z) \leq c_{11} h(t, x/2, y/2) \) for \( |x - z| > |x - y|/2 \) and \( s \in (0, t/2) \), we have

\[
I_{21} \leq c_{11} h(t, x/2, y/2) \int_0^{t/2} \int_{|x-z| > |x-y|/2} f_0(s, z, y) \, dz \, ds \\
\leq c_{12} h(t, x/2, y/2), \quad t^{1/2} < |x - y| \leq 1.
\]
Next, note that by \((2.1)\), there are constants \(c_{13} > 1\) and \(0 < c_{14} < 1\) so that \(h(s, x, z) \leq c_{13} p_1(s, c_{14} x, c_{14} z)\) for \(s \in (0, 1)\) and \(x, z \in \mathbb{R}^d\). Moreover,
\[
f_0(s, z, y) \leq \frac{1}{s} \left[ s^{-d/2} \wedge \frac{s}{|y - z|^{d + \beta}} \right] \leq \frac{1}{s} h(s, z, y) \leq \frac{2c_{13}}{t} p_1(s, c_{14} z, c_{14} y), \quad s \in (t/2, t),
\]
then we have,
\[
I_{22} \leq 2c_{13} \int_{t/2}^{t} \int_{\mathbb{R}^d} p_1(t - s, c_{14} x, c_{14} z) p_1(s, c_{14} z, c_{14} y) \, dz \, ds \tag{2.16}
\]
where the constant \(c_{17} \) is less than 1 and the last inequality holds due to \((2.1)\). By \((2.12)\) and \((2.14)-(2.16)\), there are \(c_{18} > 1\) and \(0 < c_{19} < 1\) so that
\[
I \leq c_{18} h(t, c_{19} x, c_{19} y), \quad t^{1/2} < |x - y| \leq 1.
\]
Therefore, the proof is complete. \(\square\)

### 3 Fundamental solution

Throughout the rest of this paper, \(b(x, z)\) is a bounded function on \(\mathbb{R}^d \times \mathbb{R}^d\) satisfying condition \((1.3)\). Recall the definition of the non-local operator \(S^b\) from \((1.1)\). Let \(|q^b|_0(t, x, y) = p_0(t, x, y)\), and define for each \(n \geq 1\),
\[
|q^b|_n(t, x, y) = \int_0^t \int_{\mathbb{R}^d} |q^b|_{n-1}(t - s, x, z)|S^b z p_0(s, z, y)| \, dz \, ds.
\]
Note that by Lemma \((2.2)\) and \((1.3)\),
\[
|S^b p_0(s, z, y)| \leq \|b\|_{\infty} |\Delta^{\beta/2} p_0(s, z, y) | \leq C_{10} \|b\|_{\infty} f_0(s, z, y), \quad s > 0, \; z, y \in \mathbb{R}^d. \tag{3.1}
\]
In view of \((2.11)\), there is a constant \(C_{14} > 1\) so that
\[
p_0(t, x, y) \leq C_{14} h(t, x, y), \quad t > 0, \; x, y \in \mathbb{R}^d. \tag{3.2}
\]

**Lemma 3.1.** For each \(n \geq 0\) and every bounded function \(b\) on \(\mathbb{R}^d \times \mathbb{R}^d\) satisfying condition \((1.3)\), there exists a finite constant \(C(n)\) depending on \(n\) so that
\[
|q^b|_n(t, x, y) \leq C(n) h(t, C_{13} x, C_{13} y) < \infty, \quad t \in (0, 1], \; x, y \in \mathbb{R}^d. \tag{3.3}
\]

**Proof.** We prove this lemma by induction. \((3.3)\) clearly holds for \(n = 0\) by \((3.2)\). Suppose that \((3.3)\) holds for \(n = j \geq 0\). By Lemma \((2.4)\) and the fact that \(0 < C_{13} < 1\), we have
\[
\int_0^t \int_{\mathbb{R}^d} h(t - s, x, z) f_0(s, z, y) \, dz \, ds \leq C_{12} h(t, C_{13} x, C_{13} y), \quad t \in (0, 1], \; x, y \in \mathbb{R}^d.
\]
Then by the above inequality and (3.1), for \( t \in (0, 1] \), \( x, y \in \mathbb{R}^d \),

\[
|q_{n+1}^b(t, x, y)| \leq C(j) \int_0^t \int_{\mathbb{R}^d} h(t - s, C_{13}^j x, C_{13}^j z) |S_z^b p_0(s, z, y)| \, dz \, ds
\]

\[
\leq C(j) ||b||_\infty \int_0^t \int_{\mathbb{R}^d} h(t - s, C_{13}^j x, C_{13}^j z) f_0(s, C_{13}^j x, C_{13}^j y) \, dz \, ds
\]

\[
\leq C(j) ||b||_\infty C_{10} C_{12} C_{13}^{-j} h(t, C_{13}^{j+1} x, C_{13}^{j+1} y),
\]

where the second inequality holds due to (3.1) and \( 0 < C_{13} < 1 \). Let \( C(j+1) = C(j) ||b||_\infty C_{10} C_{12} C_{13}^{-j} \), then the proof is complete.

Now we define \( q_n^b : (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) as follows. For \( t > 0 \) and \( x, y \in \mathbb{R}^d \), let \( q_0^b(t, x, y) = p_0(t, x, y) \), and for each \( n \geq 1 \), define

\[
q_n^b(t, x, y) = \int_0^t \int_{\mathbb{R}^d} q_{n-1}^b(t - s, x, z) S_z^b p_0(s, z, y) \, dz \, ds.
\]

Clearly by Lemma 3.1 each \( q_n^b(t, x, y) \) is well defined on \((0, 1] \times \mathbb{R}^d \times \mathbb{R}^d\).

For \( \lambda > 0 \), define

\[
b^{(\lambda)}(x, z) = \lambda^{\beta/2-1} b(\lambda^{-1/2} x, \lambda^{-1/2} z).
\]

For a function \( f \) on \( \mathbb{R}^d \), set

\[
f^{(\lambda)}(x) := f(\lambda^{-1/2} x).
\]

By a change of variable, one has from (1.4) that

\[
\Delta f^{(\lambda)}(x) = \lambda^{-1} (\Delta f)(\lambda^{-1/2} x)
\]

and

\[
S^{b^{(\lambda)}} f^{(\lambda)}(x) = \lambda^{-1} (S^b f)(\lambda^{-1/2} x).
\]

Note that the transition density function \( p_0(t, x, y) \) of the Brownian motion has the following scaling property:

\[
p_0(t, x, y) = \lambda^{-d/2} p_0(\lambda^{-1} t, \lambda^{-1/2} x, \lambda^{-1/2} y)
\]

Recall \( q_n^b(t, x, y) \) is the function defined inductively by (3.5) with \( q_0^b(t, x, y) := p_0(t, x, y) \).

**Lemma 3.2.** Suppose that \( b \) is a bounded function on \( \mathbb{R}^d \times \mathbb{R}^d \) satisfying (1.3). For every integer \( n \geq 0 \),

\[
q_n^{b^{(\lambda)}}(t, x, y) = \lambda^{-d/2} q_n^b(\lambda^{-1} t, \lambda^{-1/2} x, \lambda^{-1/2} y), \quad t \leq 1 \wedge \lambda, x, y \in \mathbb{R}^d;
\]

or, equivalently,

\[
q_n^b(t, x, y) = \lambda^{d/2} q_n^{b^{(\lambda)}}(\lambda t, \lambda^{1/2} x, \lambda^{1/2} y), \quad t \leq 1 \wedge \lambda^{-1}, x, y \in \mathbb{R}^d.
\]
Proof. We prove it by induction. Clearly in view of (3.3), (3.9) holds when \( n = 0 \). Suppose that (3.3) holds for \( n = j \geq 0 \). Then by the definition (3.3), (3.7) and (3.8),

\[
q_{j+1}^{b}(t, x, y) = \int_0^t \int_{\mathbb{R}^d} q_j^{b, \lambda}(t-s, x, z)S_z^b p_0(s, z, y) \, dz \, ds
\]

\[
= \int_0^t \int_{\mathbb{R}^d} \lambda^{-d/2} q_j^{b, \lambda}(\lambda^{-1}(t-s), \lambda^{-1/2}x, \lambda^{-1/2}z) \lambda^{-d/2-1} \left( S_z^b p_0(\lambda^{-1}s, \cdot, \lambda^{-1/2}y) \right) (\lambda^{-1/2}z) \, dz \, ds
\]

\[
= \lambda^{-d/2} \int_0^{\lambda^{-1}t} \int_{\mathbb{R}^d} q_j^{b, \lambda}(\lambda^{-1}t-r, \lambda^{-1/2}x, w) \left( S_w^b p_0(r, \cdot, \lambda^{-1/2}y) \right) (w) \, dw \, dr
\]

\[
= \lambda^{-d/2} q_{j+1}^{b}(\lambda^{-1}t, \lambda^{-1/2}x, \lambda^{-1/2}y).
\]

This proves that (3.9) holds for \( n = j + 1 \) and so, by induction, it holds for every \( n \geq 0 \). □

In the following, we use Lemma 3.2 together with Lemma 2.4 to get the refined upper bound of \( |q_n^{b}(t, x, y)| \).

Lemma 3.3. For each \( A > 0 \) and every bounded function \( b \) on \( \mathbb{R}^d \times \mathbb{R}^d \) satisfying condition (1.3) with \( \|b\|_\infty \leq A \),

\[
|q_n^{b}(t, x, y)| \leq C_{14} (AC_{10}C_{12})^n h(t, x, y), \quad t \in (0, 1], x, y \in \mathbb{R}^d, n \geq 0.
\]

Proof. We prove this lemma by induction. By (3.2), (3.11) clearly holds for \( n = 0 \). Suppose that (3.11) holds for \( n = j \geq 0 \). Then by Lemma 2.4 and (3.1), for \( t \in (0, 1], |x-y| \leq t^{1/2} \) or \( t \in (0, 1], |x-y| > 1 \),

\[
|q_{j+1}^{b}(t, x, y)|
\]

\[
\leq C_{14} (AC_{10}C_{12})^j \int_0^t \int_{\mathbb{R}^d} h(t-s, x, z)|S_z^b p_0(s, z, y)| \, dz \, ds
\]

\[
\leq C_{14} (AC_{10}C_{12})^j C_{10}A \int_0^t \int_{\mathbb{R}^d} h(t-s, x, z)f_0(s, z, y) \, dz \, ds
\]

\[
\leq C_{14} (AC_{10}C_{12})^j h(t, x, y).
\]

On the other hand, by (3.12), (3.10) with \( \lambda = t^{-1} \) and \( \|b(t^{-1})\|_\infty = t^{1-\beta/2}\|b\|_\infty \leq t^{1-\beta/2}A \), noting that \( |x/t^{1/2} - y/t^{1/2}| > 1 \) for \( |x-y| > t^{1/2} \), we have for \( t^{1/2} < |x-y| \leq 1 \),

\[
|q_{j+1}^{b}(t, x, y)| = t^{-d/2} |q_{j+1}^{b, (t^{-1})(1, x/t^{1/2}, y/t^{1/2})}|
\]

\[
\leq t^{-d/2}C_{14}(t^{1-\beta/2}AC_{10}C_{12})^{j+1} h(1, x/t^{1/2}, y/t^{1/2})
\]

\[
\leq C_{14}(AC_{10}C_{12})^{j+1} t^{-d/2}t^{1-\beta/2} h(1, x/t^{1/2}, y/t^{1/2})
\]

\[
\leq C_{14}(AC_{10}C_{12})^{j+1} h(t, x, y).
\]

Therefore, the above two displays prove that (3.11) holds for \( n = j + 1 \) and thus for every \( n \geq 1 \). □

Lemma 3.4. For every \( n \geq 0 \), \( q_n^{b}(t, x, y) \) is jointly continuous on \( (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \).

Proof. We prove it by induction. Clearly \( q_0^{b}(t, x, y) \) is continuous on \( (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \). Suppose that \( q_n^{b}(t, x, y) \) is continuous on \( (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \). For every \( M \geq 2 \), it follows from (3.1), Lemma 3.3 and the dominated convergence theorem that for \( \varepsilon < 1/(2M) \),

\[
(t, x, y) \mapsto \int_0^{t-\varepsilon} \int_{\mathbb{R}^d} q_n^{b}(t-s, x, z)S_z^b p_0(s, z, y) \, dz \, ds.
\]
is jointly continuous on \([1/M, 1] \times \mathbb{R}^d \times \mathbb{R}^d\). On the other hand, it follows from (3.1) that

\[
\sup_{t \in [1/M, 1]} \sup_{x,y} \int_0^t \int_{\mathbb{R}^d} h(t-s, x, z)|S_x^b p_0(s, z, y)| \, dz \, ds \\
\leq c_1 A \sup_{t \in [1/M, 1]} (t - \varepsilon)^{-(d+\beta)/2} \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} h(t-s, x, z) \, dz \, ds \\
\leq c_2 A (2M)^{(d+\beta)/2} \varepsilon,
\]

which goes to zero as \(\varepsilon \to 0\); while by (3.1) and (2.10),

\[
\sup_{t \in [1/M, 1]} \sup_{x,y} \int_0^t \int_{\mathbb{R}^d} h(t-s, x, z)|S_x^b p_0(s, z, y)| \, dz \, ds \\
\leq c_3 \sup_{t \in [1/M, 1]} (t - \varepsilon)^{-d/2} \sup_{y \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} |S_x^b p_0(s, z, y)| \, dz \, ds \\
\leq c_4 (2M)^{d/2} \|b\|_\infty \varepsilon^{1-\beta/2} \to 0
\]
as \(\varepsilon \to 0\). We conclude from Lemma 3.3 and the above argument that

\[
q_{n+1}^b(t, x, y) = \int_0^t \int_{\mathbb{R}^d} q_n^b(t-s, x, z) S_x^b p_0(s, z, y) \, dz \, ds
\]
is jointly continuous in \((t, x, y) \in [1/M, 1] \times \mathbb{R}^d \times \mathbb{R}^d\) and so in \((t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d\) by the arbitrariness of \(M\). This completes the proof of the lemma. \(\square\)

**Lemma 3.5.** There is a constant \(C_{15} = C_{15}(d, \beta) > 0\) so that for every \(A > 0\) and every bounded function \(b\) on \(\mathbb{R}^d \times \mathbb{R}^d\) with \(\|b\|_\infty \leq A\) and for every integer \(n \geq 0\) and \(\varepsilon > 0\),

\[
\left| \int_{\{z \in \mathbb{R}^d : |z| > \varepsilon\}} \left( q_n^b(t, x + z, y) - q_n^b(t, x, y) \right) \frac{b(x, z)}{|z|^{d+\beta}} \, dz \right| \leq (C_{15} A)^{n+1} f_0(t, x, y) \tag{3.13}
\]
for \((t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d\), and \(S_{x}^{b} q_n(t, x, y)\) exists pointwise for \((t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d\) in the sense of (1.6) with

\[
S_{x}^{b} q_{n+1}(t, x, y) = \int_0^t \int_{\mathbb{R}^d} S_x^b q_n^b(t-s, x, z) S_x^b p_0(s, z, y) \, dz \, ds \tag{3.14}
\]
and

\[
|S_{x}^{b} q_n(t, x, y)| \leq (C_{15} A)^{n+1} f_0(t, x, y) \quad \text{on } (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d. \tag{3.15}
\]
Moreover,

\[
q_{n+1}^b(t, x, y) = \int_0^t \int_{\mathbb{R}^d} p_0(t-s, x, z) S_x^b q_n^b(s, z, y) \, dz \, ds \quad \text{for } (t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d. \tag{3.16}
\]

**Proof.** Let \(q(t, x, y)\) denote the transition density function of the symmetric \(\beta\)-stable process on \(\mathbb{R}^d\). Then by (3.1), we have

\[
q(t, x, y) \asymp t^{-d/\beta} \left( 1 \wedge \frac{t^{1/\beta}}{|x-y|^{d+\beta}} \right) \quad \text{on } (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d. \tag{3.17}
\]
Observe that (2.6) and (3.17) yield
\[ f_0(t, x, y) \approx t^{-\beta/2} q(t^{\beta/2}, x, y) \quad \text{on } (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d. \] (3.18)

Hence on \((0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d\),
\[
\begin{align*}
&\int_0^t \int_{\mathbb{R}^d} f_0(t-s, x, z) f_0(s, z, y) ds dz \\
&\times \int_0^t (t-s)^{-\beta/2} s^{-\beta/2} \left( \int_{\mathbb{R}^d} q((t-s)^{\beta/2}, x, z) q(s^{\beta/2}, z, y) dz \right) ds \\
&= \int_0^t (t-s)^{-\beta/2} s^{-\beta/2} q((t-s)^{\beta/2} + s^{\beta/2}, x, y) ds \\
&\times q(t^{\beta/2}, x, y) \int_0^t (t-s)^{-\beta/2} s^{-\beta/2} ds \\
&= q(t^{\beta/2}, x, y) t^{1-(2\beta/2)} \int_0^1 (1-u)^{-\beta/2} u^{-\beta/2} du \\
&\times t^{1-\beta/2} f_0(t, x, y).
\end{align*}
\]

In the second \(\times\) above, we used the fact that
\[ (t/2)^{\beta/2} \leq (t-s)^{\beta/2} + s^{\beta/2} \leq 2t^{\beta/2} \quad \text{for every } s \in (0, t) \]
and the estimate (3.17), while in the last equality, we used a change of variable \(s = tu\). So there is a constant \(c_1 = c_1(d, \beta) > 0\) so that
\[
\left| \int_0^t \int_{\mathbb{R}^d} f_0(t-s, x, z) f_0(s, z, y) ds dz \right| \leq c_1 f_0(t, x, y) \quad \text{for every } t \in (0, 1] \text{ and } x, y \in \mathbb{R}^d. \quad (3.19)
\]

By increasing the value of \(c_1\) if necessary, we may do and assume that \(c_1\) is larger than 1.

We now proceed by induction. Let \(C_{15} := c_1 C_{10}\). Note that
\[
|S_{z}^b p_0(t, x, y)| \leq A |\Delta_x^{\beta/2} p_0(t, x, y)| \leq C_{10} A f_0(t, x, y). \quad (3.20)
\]

When \(n = 0\), (3.16) holds by definition. By Lemma 2.2, 3.13 and 3.15 hold for \(n = 0\). Suppose that (3.13) and (3.15) hold for \(n = j\). Then for every \(\varepsilon > 0\), by the definition of \(q_{j+1}^b\), Lemma 3.8, (3.14) and Fubini’s theorem,
\[
\begin{align*}
&\int_{\{w \in \mathbb{R}^d : |w| > \varepsilon\}} \left( q_{j+1}^b(t, x + w, y) - q_{j+1}^b(t, x, y) \right) \frac{b(x, w)}{|w|^{d+\beta}} dw \\
&= \int_0^t \int_{\mathbb{R}^d} \left( \int_{\{w \in \mathbb{R}^d : |w| > \varepsilon\}} \left( q_{j+1}^b(t-s, x + w, z) - q_{j+1}^b(t-s, x, z) \right) \frac{b(x, w)}{|w|^{d+\beta}} dw \right) \\
&\times S_{z}^b p_0(s, z, y) dz ds \\
&\text{and so}
\end{align*}
\]
\[
\left| \int_{\{w \in \mathbb{R}^d : |w| > \varepsilon\}} \left( q_{j+1}^b(t, x + w, y) - q_{j+1}^b(t, x, y) \right) \frac{b(x, w)}{|w|^{d+\beta}} dw \right| \\
\leq \int_0^t \int_{\mathbb{R}^d} (C_{15} A)^{j+1} f_0(t-s, x, z) |S_{z}^b p_0(s, z, y)| dz ds \\
\leq \int_0^t \int_{\mathbb{R}^d} (C_{15} A)^{j+1} f_0(t-s, x, z) C_{10} A f_0(s, z, y) dz ds \\
\leq (C_{15} A)^{j+2} f_0(t, x, y).
\]

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By (3.21) and Lebesgue dominated convergence theorem, we conclude that
\[
S^n_{x}q^n_{j+1}(t, x, y) := \lim_{\varepsilon \to 0} \int_{\{w \in \mathbb{R}^d : |w| > \varepsilon \}} \left( q^n_{j+1}(t, x + w, y) - q^n_{j+1}(t, x, y) \right) \frac{b(x, w)}{|w|^{d+\beta}} dw
\]
exists and (3.14) as well as (3.15) holds for \( n = 0 \).

On the other hand, in view of (3.15) and (3.16) for \( n = 0 \) for every \( t \), we have by the Fubini theorem,
\[
q^n_{j+1}(t, x, y) = \int_{0}^{t} \int_{\mathbb{R}^d} q^n_j(s, x, z) S^n_{x} p_0(t - s, z, y) dz ds
\]

for \( n = j + 1 \). (The same proof verifies (3.14) when \( n = 0 \).) On the other hand, in view of (3.15) and (3.16) for \( n = j \), we have by the Fubini theorem,
\[
q^n_{j+1}(t, x, y) = \int_{0}^{t} \int_{\mathbb{R}^d} q^n_j(s, x, z) S^n_{x} p_0(t - s, z, y) dz ds
\]

This verifies that (3.16) also holds for \( n = j + 1 \). The lemma is now established by induction. \( \square \)

Lemma 3.6. There is a positive constant \( A_0 = A_0(d, \beta) \) so that if \( \|b\|_{\infty} \leq A_0 \), then for every integer \( n \geq 0 \),
\[
|q^n_b(t, x, y)| \leq C_{14} 2^{-n} h(t, x, y) \quad \text{for } t \in (0, 1] \text{ and } x, y \in \mathbb{R}^d, \quad \text{(3.22)}
\]

(3.13) holds and so \( S^n_{x}q^n_b(t, x, y) \) exists pointwise in the sense of (1.3) with
\[
|S^n_{x}q^n_b(t, x, y)| \leq 2^{-n} f_0(t, x, y) \quad \text{for } t \in (0, 1] \text{ and } x, y \in \mathbb{R}^d, \quad \text{(3.23)}
\]

and
\[
\sum_{n=0}^{\infty} q^n_b(t, x, y) \geq \frac{1}{2} p_0(t, x, y) \quad \text{for } t \in (0, 1] \text{ and } |x - y| \leq 3t^{1/2}. \quad \text{(3.24)}
\]

Proof. We take a positive constant \( A_0 \) so that \( A_0 \leq 1 \wedge \left[ 2C_{10}C_{12} + 2C_{15} \right]^{-1} \). We have by Lemma 3.3 and Lemma 5.5 that for every \( b \) with \( \|b\|_{\infty} \leq A_0 \),
\[
|q^n_b(t, x, y)| \leq C_{14} 2^{-n} h(t, x, y) \quad \text{and} \quad |S^n_{x}q^n_b(t, x, y)| \leq 2^{-n} f_0(t, x, y)
\]

for every \( t \in (0, 1] \) and \( x, y \in \mathbb{R}^d \). This establishes (3.22) and (3.23).
On the other hand, by (2.11), there exists \( c \geq 1 \) so that \( h(t, x, y) \leq cp_0(t, x, y) \propto t^{-d/2} \) for \( |x - y| \leq 3t^{1/2} \) and \( t \in (0, 1) \). Take \( A_0 \) small enough so that \( \sum_{n=1}^{\infty} (A_0C_{10}C_{12})^n \leq \frac{1}{2cC_{14}} \). Then for every \( b \) with \( \|b\|_\infty \leq A_0 \), we have by Lemma 3.6 for \( |x - y| \leq 3t^{1/2} \) and \( t \in (0, 1) \),
\[
\sum_{n=1}^{\infty} |q_n^b(t, x, y)| \leq cC_{14} \sum_{n=1}^{\infty} (A_0C_{10}C_{12})^n p_0(t, x, y) \leq \frac{1}{2}p_0(t, x, y).
\]
Consequently, for \( |x - y| \leq 3t^{1/2} \) and \( t \in (0, 1) \),
\[
\sum_{n=0}^{\infty} q_n^b(t, x, y) \geq p_0(t, x, y) - \sum_{n=1}^{\infty} |q_n^b(t, x, y)| \geq \frac{1}{2}p_0(t, x, y).
\]
\[\square\]

We now extend the results in Lemma 3.6 to any bounded \( b \) that satisfies condition (1.3). Recall that \( A_0 \) is the positive constant in Lemma 3.6.

**Theorem 3.7.** For every \( A > 0 \), there are positive constants \( C_{16} = C_{10}(d, \beta, A) > 1 \) and \( 0 < C_{17} = C_{17}(d, \beta, A) < 1 \) so that for every bounded function \( b \) with \( \|b\|_\infty \leq A \), that satisfies condition (1.3) and \( n \geq 0 \),
\[
|q_n^b(t, x, y)| \leq C_{16} 2^{-n} p_1(t, C_{17}x, C_{17}y) \tag{3.25}
\]
for every \( 0 < t \leq 1 \wedge (A_0/\|b\|_\infty)^{2/(2-\beta)} \) and \( x, y \in \mathbb{R}^d \), and
\[
\sum_{n=0}^{\infty} q_n^b(t, x, y) \geq \frac{1}{2}p_0(t, x, y) \quad \text{for } 0 < t \leq 1 \wedge (A_0/\|b\|_\infty)^{2/(2-\beta)} \text{ and } |x - y| \leq 3t^{1/2}. \tag{3.26}
\]
Moreover, for every \( n \geq 0 \), (3.13) holds and so \( S_x^b q_n^b(t, x, y) \) exists pointwise in the sense of (1.6) with
\[
|S_x^b q_n^b(t, x, y)| \leq 2^{-n} f_0(t, x, y) \tag{3.27}
\]
for every \( 0 < t \leq 1 \wedge (A_0/\|b\|_\infty)^{2/(2-\beta)} \) and \( x, y \in \mathbb{R}^d \). Moreover, (3.14) and (3.16) hold.

**Proof.** Note that there exist \( c_k > 0, k = 1, 2 \) such that \( h(t, x, y) \leq c_1 p_1(t, c_2 x, c_2 y) \) for \( t \in (0, 1] \) and \( x, y \in \mathbb{R}^d \). Thus in view of Lemma 3.6, it suffices to prove the theorem for \( A_0 < \|b\|_\infty \leq A \). Set \( r = (\|b\|_\infty/A_0)^{2/(2-\beta)} \). The function \( b^{(r)} \) defined by (3.6) has the property \( \|b^{(r)}\|_\infty = A_0 \). Thus by Lemma 3.6 for every integer \( n \geq 0 \),
\[
|q_n^{b^{(r)}}(t, x, y)| \leq C_{14} 2^{-n} h(t, x, y) \quad \text{for } t \in (0, 1] \text{ and } x, y \in \mathbb{R}^d, \tag{3.28}
\]
(3.13) holds and so \( S_x^b q_n^{b^{(r)}}(t, x, y) \) exists pointwise in the sense of (1.6) with
\[
|S_x^b q_n^{b^{(r)}}(t, x, y)| \leq 2^{-n} f_0(t, x, y) \quad \text{for } t \in (0, 1] \text{ and } x, y \in \mathbb{R}^d, \tag{3.29}
\]
and
\[
\sum_{n=0}^{\infty} q_n^{b^{(r)}}(t, x, y) \geq \frac{1}{2}p_0(t, x, y) \quad \text{for } t \in (0, 1] \text{ and } |x - y| \leq 3t^{1/2}. \tag{3.30}
\]
We have by (3.10), (3.28) and (2.1) that for every \(0 < t \leq 1/r = (A_0/\|b\|_\infty)^{2/(2-\beta)}\) and \(x, y \in \mathbb{R}^d\),
\[
|q_b^n(t, x, y)| = r^{d/2} |q_b^{(r)}(rt, r^{1/2}x, r^{1/2}y)| \\
\leq C_{14} 2^{-n} r^{d/2} h(rt, r^{1/2}x, r^{1/2}y) \\
\leq C_{14} 2^{-n} \left( t^{-d/2} \wedge \left( p_0(t, x, y) + \frac{r^{1-\beta/2}t}{|x-y|^{d+\beta}} \right) \right) \\
\leq C_{16} 2^{-n} p_1(t, C_{17} x, C_{17} y),
\]
which establishes (3.25). Similarly, (3.26) follows from (3.8), and (3.30), while the conclusion of (3.27) is a direct consequence of (3.7), (3.10) and (3.29). That (3.14) and (3.16) hold follows directly from Lemma 3.5 and Lemma 3.2. 

Recall that \(q^b(t, x, y) := \sum_{n=0}^{\infty} q_b^n(t, x, y)\), whenever it is convergent. The following theorem follows immediately from Lemmas 3.4, 3.6 and Theorem 3.7.

**Theorem 3.8.** For every \(A > 0\), let \(C_{16} = C_{16}(d, \beta, A)\) and \(C_{17} = C_{17}(d, \beta, A)\) be the constants in Theorem 3.7. Then for every bounded function \(b\) with \(\|b\|_\infty \leq A\) that satisfies condition (1.3), \(q^b(t, x, y)\) is well defined and is jointly continuous in \((0,1 \wedge (A_0/\|b\|_\infty)^{2/(2-\beta)}) \times \mathbb{R}^d \times \mathbb{R}^d\). Moreover,
\[
|q^b(t, x, y)| \leq 2C_{16} p_1(t, C_{17} x, C_{17} y)
\]
and \(S_x^b q^b(t, x, y)\) exists pointwise in the sense of (1.6) with
\[
|S_x^b q^b(t, x, y)| \leq 2f_0(t, x, y)
\]
for every \(0 < t \leq 1 \wedge (A_0/\|b\|_\infty)^{2/(2-\beta)}\) and \(x, y \in \mathbb{R}^d\), and
\[
q^b(t, x, y) \geq \frac{1}{2} p_0(t, x, y) \quad \text{for } 0 < t \leq 1 \wedge (A_0/\|b\|_\infty)^{2/(2-\beta)} \text{ and } |x-y| \leq 3t^{1/2}.
\]

Moreover, for every \(0 < t \leq 1 \wedge (A_0/\|b\|_\infty)^{2/(2-\beta)}\) and \(x, y \in \mathbb{R}^d\),
\[
q^b(t, x, y) = p_0(t, x, y) + \int_0^t \int_{\mathbb{R}^d} q^b(t-s, x, z) S_x^b p_0(s, z, y) dz ds \quad \text{(3.32)}
\]
\[
q^b(t, x, y) = p_0(t, x, y) + \int_0^t \int_{\mathbb{R}^d} p_0(t-s, x, z) S_x^b q^b(s, z, y) dz ds. \quad \text{(3.33)}
\]

**Theorem 3.9.** Suppose that \(b\) is a bounded function on \(\mathbb{R}^d \times \mathbb{R}^d\) satisfying (1.3). Let \(A_0\) be the constant in Lemma 3.6. Then for every \(t, s > 0\) with \(t+s \leq 1 \wedge (A_0/\|b\|_\infty)^{2/(2-\beta)}\) and \(x, y \in \mathbb{R}^d\),
\[
\int_{\mathbb{R}^d} q^b(t, x, z) q^b(s, z, y) dz = q^b(t+s, x, y). \quad \text{(3.34)}
\]

**Proof.** In view of Theorem 3.7, we have
\[
\int_{\mathbb{R}^d} q^b(t, x, z) q^b(s, z, y) dz = \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \int_{\mathbb{R}^d} q^b_n(t, x, z) q^b_{j-n}(s, z, y) dz.
\]
So it suffices to show that for every $j \geq 0$,
\[
\sum_{n=0}^{j} \int_{\mathbb{R}^d} q^b_n(t, x, z) q^b_{j-n}(s, z, y) dz = q^b_j(t+s, x, y)
\]  
(3.35)

Clearly, (3.35) holds for $j = 0$. Suppose that (3.35) holds for $j = l \geq 1$. Then we have by Fubini’s theorem and the estimates in (3.1) and Theorem 3.7,

\[
\sum_{n=0}^{l+1} \int_{\mathbb{R}^d} q^b_n(t, x, z) q^b_{l+1-n}(s, z, y) dz \\
= \int_{\mathbb{R}^d} q^b_{l+1}(t, x, z) p_0(s, z, y) dz + \sum_{n=0}^{l} \int_{\mathbb{R}^d} q^b_n(t, x, z) q^b_{l+1-n}(s, z, y) dz \\
= \int_{\mathbb{R}^d} \left( \int_{0}^{t} \int_{\mathbb{R}^d} q^b_{l}(t-r, x, w) S^b_w p_0(r, w, z) dw dr \right) p_0(s, z, y) dz \\
+ \sum_{n=0}^{l} \int_{\mathbb{R}^d} q^b_n(t, x, z) \left( \int_{0}^{s} \int_{\mathbb{R}^d} q^b_{l-n}(s-r, z, w) S^b_w p_0(r, w, y) dw dr \right) dz \\
= \int_{0}^{t} \int_{\mathbb{R}^d} q^b_{l}(t-r, x, w) S^b_w p_0(r+s, w, y) dw dr \\
+ \int_{0}^{s} \int_{\mathbb{R}^d} q^b_{l}(t+s-r, x, w) S^b_w p_0(r, w, y) dw dr \\
= q^b_{l+1}(t+s, x, y).
\]

This proves that (3.35) holds for $j = l + 1$. So by induction, we conclude that (3.35) holds for every $j \geq 0$.

For notational simplicity, denote $1 \land (A_0/\|b\|_\infty)^{2/(2-\beta)}$ by $\delta_0$. In view of Theorem 3.9, we can uniquely extend the definition of $q^b(t, x, y)$ to $t > \delta_0$ by using the Chapman-Kolmogorov equation recursively as follows.

Suppose that $q^b(t, x, y)$ has been defined and satisfies the Chapman-Kolmogorov equation (3.34) on $(0, k\delta_0] \times \mathbb{R}^d \times \mathbb{R}^d$. Then for $t \in (k\delta_0, (k+1)\delta_0]$, define
\[
q^b(t, x, y) = \int_{\mathbb{R}^d} q^b(s, x, z) q^b(r, z, y) dz, \quad x, y \in \mathbb{R}^d
\]  
(3.36)

for any $s, r \in (0, k\delta_0]$ so that $s + r = t$. Such $q^b(t, x, y)$ is well defined on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and satisfies (3.34) for every $s, t > 0$. Moreover, since Chapman-Kolmogorov equation holds for $q^b(t, x, y)$ for all $t, s > 0$, we have by Theorem 3.8 that for every $A > 0$, there are constants $c_i = c_i(d, \beta, A)$, $i = 1, 2, 3$, so that for every $b(x, z)$ satisfying (1.3) with $\|b\|_\infty \leq A$,
\[
|q^b(t, x, y)| \leq c_1 e^{c_2 t} p_1(t, c_3 x, c_3 y) \quad \text{for every } t > 0 \text{ and } x, y \in \mathbb{R}^d.
\]  
(3.37)

By the induction method and the Chapman-Kolmogorov equation for $q^b(t, x, y)$, (3.32) and (3.33) can be extended to every $t > 0$ and $x, y \in \mathbb{R}^d$.

**Theorem 3.10.** $q^b(t, x, y)$ satisfies (3.32) and (3.33) for every $t > 0$ and $x, y \in \mathbb{R}^d$. 
Proof. Let $\delta_0 := 1 \land (A_0/\|b\|_\infty)^{2/(2-\beta)}$. It suffices to prove that for every $n \geq 1$, (3.32) and (3.33) hold for all $t \in (0, n\delta_0]$ and $x, y \in \mathbb{R}^d$.

Clearly, (3.32) holds for $t \in (0, n\delta_0]$ with $n = 1$. Suppose that (3.32) holds for $t \in (0, (k-1)\delta_0]$ with $n = k$. For $t \in (k\delta_0, (k+1)\delta_0]$, take $l, s \in (0, k\delta_0]$ so that $l + s = t$. Then we have by Fubini's theorem, Chapman-Kolmogorov equation of $q^b$, Lemma 2.4, (1.3) and (3.37),

$$q^b(l+s,x,y) = \int_{\mathbb{R}^d} q^b(l,x,z)q^b(s,z,y) \, dz$$

$$= \int_{\mathbb{R}^d} q^b(l,x,z) \left( p_0(s,z,y) + \int_0^s \int_{\mathbb{R}^d} q^b(s-r,r,\omega)S^b_\omega p_0(r,\omega,y) \, d\omega \, dr \right) \, dz$$

$$= \int_{\mathbb{R}^d} p_0(l,x,z)p_0(s,z,y) \, dz$$

$$+ \int_{\mathbb{R}^d} \left( \int_0^l \int_{\mathbb{R}^d} q^b(l-u,u,\eta)S^b_\omega p_0(u,\eta,z) \, d\eta \, du \right) p_0(s,z,y) \, dz$$

$$+ \int_0^s \int_{\mathbb{R}^d} q^b(l+s-r,r,\omega)S^b_\omega p_0(r,\omega,y) \, d\omega \, dr$$

$$= p_0(l+s,x,y) + \int_0^l \int_{\mathbb{R}^d} q^b(l-u,u,\eta)S^b_\omega p_0(u+s,\eta,y) \, d\eta \, du$$

$$+ \int_0^s \int_{\mathbb{R}^d} q^b(l+s-r,r,\omega)S^b_\omega p_0(r,\omega,y) \, d\omega \, dr$$

$$= p_0(l+s,x,y) + \int_0^{l+s} \int_{\mathbb{R}^d} q^b(l+s-r,r,z)S^b_\omega p_0(r,z,y) \, dz \, dr.$$ 

By the similar procedure as above, we can also prove that (3.33) holds for every $t > 0$ and $x, y \in \mathbb{R}^d$. 

\[\Box\]

Theorem 3.11. Suppose that $b$ is a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.3). $q^b(t,x,y)$ is the unique continuous kernel that satisfies the Chapman-Kolmogorov equation (3.34) on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and that for some $\varepsilon > 0, c_1 > 1$ and $0 < c_2 < 1$,

$$|q^b(t,x,y)| \leq c_1 p_1(t,c_2x,c_2y)$$

and (3.32) hold for $(t,x,y) \in (0,\varepsilon] \times \mathbb{R}^d \times \mathbb{R}^d$. Moreover, (3.37) holds for $q^b(t,x,y)$.

Proof. Suppose that $\varphi$ is any continuous kernel that satisfies, for some $\varepsilon > 0$, (3.32) and (3.38) hold for $(t,x,y) \in (0,\varepsilon] \times \mathbb{R}^d \times \mathbb{R}^d$. Without loss of generality, we may and do assume that $\varepsilon < 1 \land (A_0/\|b\|_\infty)^{2/(2-\beta)}$. Using (3.32) recursively, one gets

$$\varphi(t,x,y) = \sum_{j=0}^n \varphi_j(t,x,y) + \int_0^t \int_{\mathbb{R}^d} \varphi(t-s,x,z)(S^b_\omega p_0)^{*n+1}(s,z,y) \, ds \, dz.$$ 

Here $(S^b_\omega p_0)^{*n}(s,z,y)$ denotes the $n$th convolution operation of the function $S^b_\omega p_0(s,z,y)$; that is, $(S^b_\omega p_0)^{*n}(s,z,y) = S^b_\omega p_0(s,z,y)$ and

$$(S^b_\omega p_0)^{*n}(s,z,y) = \int_0^s \int_{\mathbb{R}^d} S^b_\omega p_0(r,z,w) (S^b_\omega p_0)^{*n-1}(s-r,w,y) \, dw \, dr \quad \text{for } n \geq 2.$$ 

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It follows from (3.21) and (3.19) that for every \( A > 0 \) so that \( \|b\|_\infty \leq A \),
\[
|(S^b p_0)_{z}^{s,n}(s, z, y)| \leq (C_{15} A)^n f_0(t, x, y),
\]
where \( C_{15} \) is the constant in Lemma 3.5. Noting that the constant \( A_0 \) defined in Lemma 3.6 satisfies \( A_0 \leq 1/(2C_{15}) \). So for every bounded function \( b \) with \( \|b\|_\infty \leq A_0 \), we have
\[
|(S^b p_0)_{z}^{s,n}(s, z, y)| \leq 2^{-n} f_0(s, z, y), \quad s \in (0, 1).
\]
(3.41)

By the scale change formulas (3.7) and (3.8), when \( b > 0 \),
\[
|(S^b p_0)_{z}^{s,n}(s, z, y)| \leq 2^{-n} f_0(s, z, y), \quad s \in (0, 1 \wedge (A_0/\|b\|_\infty)^{2/(2-\beta)}).
\]
Thus, by the condition (3.38) and Lemma 2.4
\[
\left| \int_0^t \int_{\mathbb{R}^d} \overline{q}(t - s, x, z)(S^b p_0)_{z}^{s,n}(s, z, y) dsdz \right| \leq c_3 2^{-n} p_1(t, c_4 x, c_4 y).
\]
It follows from (3.39) that
\[
\overline{q}(t, x, y) = \sum_{n=0}^\infty q^b_n(t, x, y) = q^b(t, x, y)
\]
for every \( t \in (0, \varepsilon] \) and \( x, y \in \mathbb{R}^d \). Since both \( \overline{q} \) and \( q^b \) satisfy the Chapman-Kolmogorov equation (3.34), \( \overline{q} = q^b \) on \((0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d\). \( \square \)

**Remark 3.12.** It follows from the definition of \( q^b_n(t, x, y) \) and Lemma 3.5 that \( (S^b p_0)^{s,n+1} = S^b q^b_n(s, z, y) \).

In view of Lemma 3.2 and Chapman-Kolmogorov equation, we have

**Theorem 3.13.** Suppose that \( b \) is a bounded function on \( \mathbb{R}^d \times \mathbb{R}^d \) satisfying (1.3). \( q^b(t, x, y) = \lambda^{d/2} q^{b(\lambda)}(\lambda t, \lambda^{1/2} x, \lambda^{1/2} y) \) on \((0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d\), where \( b^{(\lambda)}(x, z) := \lambda^{\beta/2-1} b(\lambda^{-1/2} x, \lambda^{-1/2} z) \).

For a bounded function \( f \) on \( \mathbb{R}^d \), \( t > 0 \) and \( x \in \mathbb{R}^d \), we define
\[
T^b_t f(x) = \int_{\mathbb{R}^d} q^b(t, x, y) f(y) dy \quad \text{and} \quad P^b_t f(x) = \int_{\mathbb{R}^d} p_0(t, x, y) f(y) dy.
\]
The following lemma follows immediately from (3.34) and (3.36).

**Lemma 3.14.** Suppose that \( b \) is a bounded function on \( \mathbb{R}^d \times \mathbb{R}^d \) satisfying (1.3). For all \( s, t > 0 \), we have \( T^b_{t+s} = T^b_t T^b_s \).

**Theorem 3.15.** Let \( b \) be a bounded function on \( \mathbb{R}^d \times \mathbb{R}^d \) satisfying (1.3). Then for every \( f \in C^2_b(\mathbb{R}^d) \),
\[
T^b_t f(x) - f(x) = \int_0^t T^b_s \mathcal{L}^b f(x) ds \quad \text{for every } t > 0, x \in \mathbb{R}^d.
\]
Proof. Note that by Theorem 3.10 for $f \in C^2_b(\mathbb{R}^d)$,

$$T^b_t f(x) = P_t f(x) + \int_0^t T^b_{t-s} S^b P_s f(x) ds = P_t f(x) + \int_0^t T^b_s S^b P_{t-s} f(x) ds. \quad (3.42)$$

Hence

$$T^b_t f(x) - f(x) = P_t f(x) - f(x) + \int_0^t T^b_s S^b f(x) ds + \int_0^t T^b_s S^b (P_{t-s} f - f)(x) ds$$

$$= \int_0^t P_s \Delta f(x) ds + \int_0^t T^b_s S^b f(x) ds + \int_0^t T^b_s S^b (P_{t-s} f - f)(x) ds$$

$$= \int_0^t T^b_s \left( \Delta + S^b \right) f(x) ds - \int_0^t \left( \int_r^t T^b_s S^b (\Delta f)(x) ds \right) dr$$

$$+ \int_0^t T^b_s S^b (P_{t-s} f - f)(x) ds$$

$$= \int_0^t T^b_s \mathcal{L} f(x) ds - \int_0^t T^b_s S^b (P_{t-r} f - f)(x) dr + \int_0^t T^b_s S^b (P_{t-s} f - f)(x) ds$$

$$= \int_0^t T^b_s \mathcal{L} f(x) ds.$$

Here in the third inequality, we used (3.42); while in the fifth inequality we used Lemma 2.2 and (3.37), which allow the interchange of the integral sign $\int_r^t$ with $T^b_s S^b$, and the fact that

$$\int_r^t P_{s-r} (\Delta f)(x) ds = \int_r^t \left( \frac{d}{ds} P_{s-r} f(x) \right) ds = P_{t-r} f(x) - f(x).$$

\[ \square \]

**Theorem 3.16.** Let $b$ be a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.3). Then $q^b(t, x, y)$ is jointly continuous in $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and $\int_{\mathbb{R}^d} q^b(t, x, y) dy = 1$ for every $x \in \mathbb{R}^d$ and $t > 0$.

**Proof.** By Lemma 3.14 we have

$$q^b(t + s, x, y) = \int_{\mathbb{R}^d} q^b(t, z, y) q^b(s, z, y) dz, \quad x, y \in \mathbb{R}^d, s, t > 0. \quad (3.43)$$

Continuity of $q^b(t, x, y)$ in $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ follows from Theorem 3.8, (3.43) and the dominated convergence theorem. For $n \geq 1$ and $t \in (0, T]$, it follows from (3.11), Lemma 2.4, Theorem 3.7 and Fubini’s Theorem that for every $t \in (0, 1 \wedge (A_0/\|b\|_\infty)^{2/(2-\beta)})$,

$$\int_{\mathbb{R}^d} d_n^b(t, x, y) dy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^t q^b_{n-1}(t - s, x, z) S^b_{z,x} p_0(s, z, y) ds dz dy$$

$$= \int_{\mathbb{R}^d} \int_0^t q^b_{n-1}(t - s, x, z) S^b_{z,x} \left( \int_{\mathbb{R}^d} p_0(s, z, y) dy \right) ds dz = 0.$$
Hence we have by Lemma 3.6
\[
\int_{\mathbb{R}^d} q^b(t, x, y) \, dy = \int_{\mathbb{R}^d} p_0(t, x, y) \, dy = 1
\]
for \( t \in (0, 1 \wedge (A_0/\|b\|_\infty)^{2/(2-\beta)}) \). This conservativeness property extends to all \( t > 0 \) by (4.13). \( \square \)

Theorem 1.1 now follows from (2.1), (2.3), Theorems 3.8, 3.10, 3.11, 3.15 and 3.16.

4 \( C^\infty \)-Semigroups and Positivity

Recall that \( A_0 \) is the positive constant in Lemma 3.6.

**Lemma 4.1.** Suppose that \( b \) is a bounded function on \( \mathbb{R}^d \times \mathbb{R}^d \) satisfying condition (1.3). Then \( \{T^b_t, t > 0\} \) is a strongly continuous semigroup in \( C^\infty(\mathbb{R}^d) \).

**Proof.** Note that \( q^b(t, x, y) \) is jointly continuous in \((0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \) and there are constants \( c_k, k = 1, 2, 3 \) so that
\[
|q^b(t, x, y)| \leq c_1 e^{c_2 t} p_1(t, c_3 x, c_3 y) \quad \text{for every } t > 0 \text{ and } x, y \in \mathbb{R}^d.
\]
The proof is a minor modification of that for [11, Proposition 2.3]. \( \square \)

**Lemma 4.2.** Let \( b \) be a bounded function on \( \mathbb{R}^d \times \mathbb{R}^d \) satisfying (1.14). For each \( f \in C^2(\mathbb{R}^d) \), \( \mathcal{L}^b f(x) \) exists pointwise and is in \( C^\infty(\mathbb{R}^d) \).

**Proof.** Suppose that \( f \in C^2(\mathbb{R}^d) \). Denote \( \sum_{j=1}^d |\partial^2_{xj} f(x)| \) by \( |D^2 f(x)| \). Let \( R > 1 \) to be chosen later. Then for each \( x \in \mathbb{R}^d \), we have by Taylor expansion,
\[
\Phi_f(x) := \int_{\mathbb{R}^d} \left| f(x + z) - f(x) - \nabla f(x) \cdot z 1_{\{|z| \leq 1\}} \right| \, \frac{1}{|z|^{d+\beta}} \, dz
\]
\[
\leq \int_{|z| \leq 1} \left| f(x + z) - f(x) - \nabla f(x) \cdot z 1_{\{|z| \leq 1\}} \right| \, \frac{1}{|z|^{d+\beta}} \, dz
\]
\[
+ \int_{1 < |z| \leq R} |f(x + z) - f(x)| \, \frac{1}{|z|^{d+\beta}} \, dz + \int_{|z| > R} |f(x + z) - f(x)| \, \frac{1}{|z|^{d+\beta}} \, dz
\]
\[
\leq \sup_{|y| \leq 1} |D^2 f(x + y)| \int_{|z| \leq 1} |z|^{2-d-\beta} \, dz + \int_{1 < |z| \leq R} |f(x + z) - f(x)| \, \frac{1}{|z|^{d+\beta}} \, dz
\]
\[
+ 2\|f\|_\infty \int_{|z| > R} |z|^{-d-\beta} \, dz
\]
\[
= c \sup_{|y| \leq 1} |D^2 f(x + y)| + \int_{1 < |z| \leq R} |f(x + z) - f(x)| \, \frac{1}{|z|^{d+\beta}} \, dz + cR^{-\beta}\|f\|_\infty.
\]
For any given \( \varepsilon > 0 \), we can take \( R \) large so that \( cR^{-\beta}\|f\|_\infty < \varepsilon/2 \) to conclude that
\[
\lim_{|x| \to \infty} \int_{\mathbb{R}^d} \left| f(x + z) - f(x) - \nabla f(x) \cdot z 1_{\{|z| \leq 1\}} \right| \, \frac{1}{|z|^{d+\beta}} \, dz = 0. \quad (4.1)
\]
Since \( e^{\hat{p}_{158}} \) tells us that satisfies the positive maximum principle. On the other hand, Courrége’s first theorem (see [1, (5)]) that \( \mathcal{L}^b f(x) \) exists for every \( x \in \mathbb{R}^d \) and \( \mathcal{L}^b f \in C_{\infty}(\mathbb{R}^d) \).

\[ \Phi_{f(+z) - f(x_0)} < \varepsilon \quad \text{for every } |z| < \delta. \quad (4.2) \]

It follows from the last two displays, the definition of \( \mathcal{L}^b \) and (1.14) that \( \mathcal{L}^b f(x) \) exists for every \( x \in \mathbb{R}^d \) and \( \mathcal{L}^b f \in C_{\infty}(\mathbb{R}^d) \).

**Proof of Theorem 1.2**. Since \( b \) satisfies condition (1.14), it is easy to verify that \( \mathcal{L}^b f \in C_{\infty}(\mathbb{R}^d) \) for every \( f \in C_{\infty}^2(\mathbb{R}^d) \). Let \( \hat{\mathcal{L}}^b \) denote the infinitesimal generator of the strongly continuous semigroup \( \{T^b_t; t \geq 0\} \) in \( C_{\infty}(\mathbb{R}^d) \), which is a closed linear operator. It follows from Theorem 3.15, Lemmas 3.1 and 4.1 that for every \( f \in C_{\infty}^2(\mathbb{R}^d) \), \( (T^b_t f(x) - f(x))/t \) converges uniformly to \( \mathcal{L}^b f(x) \) as \( t \to 0 \). So

\[ C_{\infty}^2(\mathbb{R}^d) \subset D(\hat{\mathcal{L}}^b) \quad \text{and} \quad \hat{\mathcal{L}}^b f = \mathcal{L}^b f \quad \text{for } f \in C_{\infty}^2(\mathbb{R}^d). \quad (4.3) \]

In view of Theorem 3.8, there are constants \( c_k > 0, k = 1, 2, 3 \) so that (3.37) holds. This implies that

\[ \sup_{x \in \mathbb{R}^d} \int_0^\infty e^{-\lambda t} |T^b_t f|(x) dt \leq c_\lambda \|f\|_{\infty}, \quad f \in C_{\infty}(\mathbb{R}^d), \]

for every \( \lambda > c_2 \). Observe that \( e^{-c_2 t} T^b_t \) is a strongly continuous semigroup in \( C_{\infty}(\mathbb{R}^d) \) whose infinitesimal generator is \( \hat{\mathcal{L}}^b - c_2 \). The above display implies that \( (0, \infty) \) is contained in the residual set \( \rho(\hat{\mathcal{L}}^b - c_2) \) of \( \hat{\mathcal{L}}^b - c_2 \). Therefore by Theorem 3.16 and the Hille-Yosida-Ray theorem [9, p165], \( \{e^{-c_2 t} T^b_t; t \geq 0\} \) is a positive preserving semigroup on \( C_{\infty}(\mathbb{R}^d) \) if and only if \( \hat{\mathcal{L}}^b - c_2 \) satisfies the positive maximum principle. On the other hand, Courrége’s first theorem (see [1, p158]) tells us that \( \hat{\mathcal{L}}^b - c_2 \) satisfies the positive maximum principle if and only if for each \( x \in \mathbb{R}^d \),

\[ b(x, z) \geq 0 \quad \text{for a.e. } z \in \mathbb{R}^d. \]

Since \( e^{-c_2 t} T^b_t \) has a continuous integral kernel \( e^{-c_2 t} q^b(t, x, y) \), it follows that \( q^b(t, x, y) \geq 0 \) on \((0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d\) if and only if for each \( x \in \mathbb{R}^d \), (1.15) holds.

5 Feller process and heat kernel estimates

Suppose that \( b \) is a bounded function satisfying conditions (1.3), (1.14) and (1.15). Then it follows from Theorem 1.2 and Lemma 4.1 that \( T^b \) is a Feller semigroup. So it uniquely determines a conservative Feller process \( X^b = \{X^b_t, t \geq 0, \mathbb{P}_x, x \in \mathbb{R}^d\} \) having \( q^b(t, x, y) \) as its transition density function. Since, by Theorem 3.11 \( q^b(t, x, y) \) is continuous and \( q^b(t, x, y) \leq c_1 e^{c_2 t} p_1(t, c_3 x, c_3 y) \) for some positive constants \( c_k, k = 1, 2, 3 \), \( X^b \) enjoys the strong Feller property.

**Proposition 5.1.** Suppose that \( b \) is a bounded function satisfying conditions (1.3), (1.14) and (1.15). For each \( x \in \mathbb{R}^d \) and \( f \in C_{\infty}^2(\mathbb{R}^d) \),

\[ M^f_t := f(X^b_t) - f(X^b_0) - \int_0^t \mathcal{L}^b f(X^b_s) ds \]

is a martingale under \( \mathbb{P}_x \). So in particular, the Feller process \( (X^b, \mathbb{P}_x, x \in \mathbb{R}^d) \) solves the martingale problem for \((\mathcal{L}^b, C_{\infty}^2(\mathbb{R}^d)) \).
Proof. This follows immediately from Theorem 3.15 and the Markov property of $X^b$. □

We next determine the Lévy system of $X^b$. Recall that

$$J^b(x, y) = \frac{b(x, y - x)}{|x - y|^{d+\beta}}. \quad (5.1)$$

By Proposition 5.1 and the similar argument in [4, Theorem 2.6], we have the following result.

**Proposition 5.2.** Suppose that $b$ is a bounded function satisfying conditions (1.3), (1.14) and (1.15). Assume that $A$ and $B$ are disjoint compact sets in $\mathbb{R}^d$. Then

$$\sum_{s \leq t} 1\{X^{b}_{s-} \in A, X^{b}_{s} \in B\} - \int_0^t 1_A(X^{b}_{s}) \int_B J^b(X^{b}_{s}, y) dy \, ds$$

is a $\mathbb{P}_x$-martingale for each $x \in \mathbb{R}^d$.

Proposition 5.2 implies that

$$\mathbb{E}_x \left[ \sum_{s \leq t} 1_A(X^{b}_{s-}) 1_B(X^{b}_{s}) \right] = \mathbb{E}_x \left[ \int_0^t \int_{\mathbb{R}^d} 1_A(X^{b}_{s}) 1_B(y) J^b(X^{b}_{s}, y) dy \, ds \right].$$

Using this and a routine measure theoretic argument, we get

$$\mathbb{E}_x \left[ \sum_{s \leq t} f(s, X^{b}_{s-}, X^{b}_{s}) \right] = \mathbb{E}_x \left[ \int_0^t \int_{\mathbb{R}^d} f(s, X^{b}_{s}, y) J^b(X^{b}_{s}, y) dy \, ds \right]$$

for any non-negative measurable function $f$ on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ vanishing on $\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : x = y\}$. Finally, following the same arguments as in [5, Lemma 4.7] and [6, Appendix A], we get

**Proposition 5.3.** Suppose that $b$ is a bounded function satisfying conditions (1.3), (1.14) and (1.15). Let $f$ be a nonnegative function on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ vanishing on the diagonal. Then for stopping time $T$ with respect to the minimal admissible filtration generated by $X^b$,

$$\mathbb{E}_x \left[ \int_0^T \int_{\mathbb{R}^d} f(s, X^{b}_{s}, u) J^b(X^{b}_{s}, u) du \, ds \right].$$

To remove the assumption (1.14) on $b$, we approximate a general measurable function $b(x, z)$ by continuous $k_n(x, z)$. To show that $q^{b_n}(t, x, y)$ converges to $q^b(t, x, y)$, we establish equi-continuity of $q^b(t, x, y)$ and apply the uniqueness result, Theorem 3.11.

**Proposition 5.4.** For each $0 < t_0 < T < \infty$ and $A > 0$, the function $q^b(t, x, y)$ is uniform continuous in $(t, x) \in (t_0, T) \times \mathbb{R}^d$ for every $b$ with $\|b\|_\infty \leq A$ that satisfies (1.3) and for all $y \in \mathbb{R}^d$. 

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Proof. In view of Theorem 3.13, it suffices to prove the theorem for $A = A_0$, where $A_0$ is the constant in Lemma 3.4 (or in Theorem 1.1). Using the Chapman-Kolmogorov equation for $q^b(t, x, y)$, it suffices to prove the Proposition for $T = 1$.

Noting that $q^b_n, n \geq 1$ can also be rewritten in the following form:

$$q^b_n(t, x, y) = \int_0^t \int_{\mathbb{R}^d} p_0(t - r, x, z)(S^b p_0)^{s,n}_z(r, z, y) \, dz \, dr.$$  

Here $(S^b p_0)^{s,n}_z(r, z, y)$ is defined in (3.40). Hence, for $T > t > s > t_0, x_1, x_2 \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$, we have

$$|q^b_n(s, x_1, y) - q^b_n(t, x_2, y)| \leq \int_0^s \int_{\mathbb{R}^d} |p_0(s - r, x_1, z) - p_0(t - r, x_2, z)|| (S^b p_0)^{s,n}_z(r, z, y) | \, dz \, dr$$

$$+ \int_s^t \int_{\mathbb{R}^d} p_0(t - r, x_2, z)|| (S^b p_0)^{s,n}_z(r, z, y) | \, dz \, dr$$

$$=: I + II.$$  

It is known that there are positive constants $c_1$ and $\theta$ so that for any $t, s \in [t_0, T]$ and $x_i \in \mathbb{R}^d$ with $i = 1, 2$,

$$|p_0(s, x_1, y) - p_0(t, x_2, y)| \leq c_1 t_0^{-(d + \theta)/2} \left( |t - s|^{1/2} + |x_1 - x_2| \right)^\theta, \quad y \in \mathbb{R}^d,$$

we have by (2.6), (3.41) and Lemma 2.3 for $\rho \in (0, s/2)$,

$$I = \int_0^{s - \rho} \int_{\mathbb{R}^d} |p_0(s - r, x_1, z) - p_0(t - r, x_2, z)|| (S^b p_0)^{s,n}_z(r, z, y) | \, dz \, dr$$

$$+ \int_{s - \rho}^s \int_{\mathbb{R}^d} |p_0(s - r, x_1, z) - p_0(t - r, x_2, z)|| (S^b p_0)^{s,n}_z(r, z, y) | \, dz \, dr$$

$$\leq c_2 2^{-(n-1)} \rho^{-(d + \theta)/2} \left( |t - s|^{1/2} + |x_1 - x_2| \right)^\theta \int_0^{s - \rho} \int_{\mathbb{R}^d} f_0(r, z, y) \, dz \, dr$$

$$+ c_2 2^{-(n-1)} (s - \rho)^{-(d + \theta)/2} \int_{s - \rho}^s \int_{\mathbb{R}^d} (p_0(s - r, x_1, z) + p_0(t - r, x_2, z)) \, dz \, dr$$

$$\leq c_3 2^{-(n-1)} \rho^{-(d + \theta)/2} \left( |t - s|^{1/2} + |x_1 - x_2| \right)^\theta s^{1-\theta/2} + c_2 2^{-(n-1)} (s - \rho)^{-(d + \beta)/2} \rho.$$  

Moreover, since $f_0(r, z, y) \leq s^{-(d + \beta)/2}$ for $r \in (s, t)$, we have

$$II \leq 2^{-(n-1)} \int_s^t \int_{\mathbb{R}^d} p_0(t - r, x_2, z)f_0(r, z, y) \, dz \, dr \leq 2^{-(n-1)} s^{-(d + \beta)/2}|t - s|.$$  

(5.3)

Note that

$$|q^b(s, x_1, y) - q^b(t, x_2, y)| \leq |p_0(s, x_1, y) - p_0(t, x_2, y)| + \sum_{n=1}^\infty |q^b_n(s, x_1, y) - q^b_n(t, x_2, y)|.$$  

Then by taking $|t - s|$ and $|x_1 - x_2|$ small, and then making $\rho$ small in (5.2) and (5.3) yields the conclusion of this Proposition.
Proposition 5.5. For each $0 < t_0 < T < \infty$ and $A > 0$, the function $q^b(t,x,y)$ is uniformly continuous in $y$ for every $b$ with $\|b\|_{\infty} \leq A$ that satisfies (1.3) and for all $(t,x) \in (t_0,T) \times \mathbb{R}^d$.

Proof. By Theorem 3.13 and the Chapman-Kolmogrov equation for $q^b(t,x,y)$, it suffices to prove the theorem for $A = A_0$ and $T = 1$, where $A_0$ is the constant in Lemma 3.6.

Define $P(s,x,y) = p_0(s,x) - p_0(s,y)$. For $s > 0$, we have

$$
|S^b_{\rho}p_0(s,y_1) - S^b_{\rho}p_0(s,y_2)|
\leq c_1 \int_{\mathbb{R}^d} |P(s, y_1 + h, y_2 + h) - P(s, y_1, y_2) - (\nabla_{(y_1, y_2)}P(s, y_1, y_2), h 1_{|h| \leq 1})| \frac{dh}{|h|^{d+\beta}}
\leq c_1 \int_{|h| \leq 1} |h|^2 \sup_{\theta \in (0,1)} |\frac{\partial^2}{\partial y_1} p_0(s, y_1 + \theta h) - \frac{\partial^2}{\partial y_2} p_0(s, y_2 + \theta h)| \frac{dh}{|h|^{d+\beta}}
+ c_1 \int_{|h| > 1} |p_0(s, y_1 + h) - p_0(s, y_2 + h) - p_0(s, y_1) + p_0(s, y_2)| \frac{dh}{|h|^{d+\beta}}
\leq c_2 \sup_y |\frac{\partial^3}{\partial y^3} p_0(s, y)||y_1 - y_2| \int_{|h| \leq 1} |h|^2 \frac{dh}{|h|^{d+\beta}} + c_2 \sup_y |\frac{\partial}{\partial y} p_0(s, y)||y_1 - y_2| \int_{|h| > 1} \frac{dh}{|h|^{d+\beta}}
\leq c_3 |y_1 - y_2|^2 s^{-(d+1)/2} + s^{-(d+1)/2},
$$

where in the fourth inequality, we used $|\frac{\partial^3}{\partial y^3} p_0(s, y)| \leq c_3 s^{-(d+3)/2}$ which can be proved similarly by the argument in Lemma 2.1.

Then for each $n \geq 1$, we have by Lemma 2.3, Lemma 3.6 and (5.4) that for $(t,x,y) \in (t_0,1) \times \mathbb{R}^d \times \mathbb{R}^d$ and $\rho \in (0,t_0/2)$,

$$
|q^b_n(t,x,y_1) - q^b_n(t,x,y_2)|
\leq \int_0^t \int_{\mathbb{R}^d} q^b_{n-1}(t-s, x, z) |S^b_{\rho}p_0(s, z, y_1) - S^b_{\rho}p_0(s, z, y_2)| dz ds
+ \int_0^t \int_{\mathbb{R}^d} q^b_{n-1}(t-s, x, z) |S^b_{\rho}p_0(s, z, y_1) - S^b_{\rho}p_0(s, z, y_2)| dz ds
\leq c_4 2^{-(n-1)} \int_0^t \int_{\mathbb{R}^d} p_1(t-s, x, z) \left| S^b_{\rho}p_0(s, z, y_1) - S^b_{\rho}p_0(s, z, y_2) \right| dz ds
+ c_4 2^{-(n-1)} \int_0^t \int_{\mathbb{R}^d} p_1(t-s, x, z) \left| S^b_{\rho}p_0(s, z, y_1) - S^b_{\rho}p_0(s, z, y_2) \right| dz ds
\leq c_5 2^{-(n-1)} t_0^{d/2} \int_0^t \int_{\mathbb{R}^d} \left( |S^b_{\rho}p_0(s, z, y_1)| + |S^b_{\rho}p_0(s, z, y_2)| \right) dz ds
+ c_5 2^{-(n-1)} \rho^{-(d+3)/2} \int_{t_0}^t \int_{\mathbb{R}^d} p_1(t-s, x, z) dz ds
\leq c_6 2^{-(n-1)} t_0^{-d/2} \rho^{1-\beta/2} + c_6 2^{-(n-1)} \rho^{-(d+3)/2} |y_1 - y_2|.
$$

Therefore we have

$$
|q^b(t,x,y_1) - q^b(t,x,y_2)|
\leq |p_0(t,x,y_1) - p_0(t,x,y_2)| + \sum_{n=1}^{\infty} c_6 2^{-(n-1)} t_0^{-d/2} \rho^{1-\beta/2} + \sum_{n=1}^{\infty} c_6 2^{-(n-1)} \rho^{-(d+3)/2} |y_1 - y_2|.
$$

By first taking $|y_1 - y_2|$ small and then making $\rho$ small yields the desired uniform continuity of $q^b(t,x,y)$.

\[ \square \]
Theorem 5.6. Suppose \( b \) is a bounded function on \( \mathbb{R}^d \times \mathbb{R}^d \) satisfying (1.3) and (1.15). The kernel \( q^b(t, x, y) \) uniquely determines a Feller process \( X^b = (X^b_t, t \geq 0, \mathbb{P}, x \in \mathbb{R}^d) \) on the canonical Skorokhod space \( \mathbb{D}((0, \infty), \mathbb{R}^d) \) such that
\[
\mathbb{E}_x \left[ f(X^b_t) \right] = \int_{\mathbb{R}^d} q^b(t, x, y)f(y)dy
\]
for every bounded continuous function \( f \) on \( \mathbb{R}^d \). The Feller process \( X^b \) is conservative and has a Lévy system \((J^b(x,y)dy, t), \) where
\[
J^b(x,y) = \frac{b(x, y - x)}{|x - y|^{d+\beta}}.
\]

Proof. When \( b \) is a bounded function satisfying (1.3), (1.14) and (1.15), the theorem has already been established via Theorem 1.2 and Propositions 5.1-5.3. We now remove the assumption (1.14). Suppose that \( b(x, z) \) is a bounded function that satisfies (1.3) and (1.15). Let \( \varphi \) be a non-negative smooth function with compact support in \( \mathbb{R}^d \) so that \( \int_{\mathbb{R}^d} \varphi(x)dx = 1 \). For each \( n \geq 1 \), define \( \varphi_n(x) = n^d \varphi(nx) \) and
\[
k_n(x, z) := \int_{\mathbb{R}^d} \varphi_n(x - y)b(y, z)dy.
\]
Then \( k_n \) is a function that satisfies (1.3), (1.14) and (1.15) with \( \|k_n\|_{\infty} \leq \|b\|_{\infty} \). By Theorems 1.1 1.2 Propositions 5.4 and 5.5, \( q^{k_n}(t, x, y) \) is nonnegative, uniformly bounded and equicontinuous on \([1/M, M] \times \mathbb{R}^d \times \mathbb{R}^d \) for each \( M \geq 1 \), then there is a subsequence \( \{n_j\} \) of \( \{n\} \) so that \( q^{k_{n_j}}(t, x, y) \) converges boundedly and uniformly on compacts of \((0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \), to some nonnegative continuous function \( \overline{\varphi}(t, x, y) \), which again satisfies (1.13). Obviously, \( \overline{\varphi}(t, x, y) \) also satisfies the Chapman-Kolmogorov equation and \( \int_{\mathbb{R}^d} \overline{\varphi}(t, x, y)dy = 1 \). By (3.32) and Theorem 3.8
\[
q^{k_{n_j}}(t, x, y) = p_0(t, x, y) + \int_0^t \int_{\mathbb{R}^d} q^{k_{n_j}}(t - s, x, z)S_z^{k_{n_j}}p_0(s, z, y)dzds
\]
and
\[
q^{k_{n_j}}(t, x, y) \leq c_1 p_1(t, c_2 x, c_2 y)
\]
for every \( 0 < t \leq 1 \wedge (A_0/\|b\|_{\infty})^{2/(2-\beta)} \) and \( x, y \in \mathbb{R}^d \), where \( c_k, k = 1, 2 \) are positive constants that depend only on \( d, \beta \) and \( \|b\|_{\infty} \). Letting \( j \to \infty \), we have by (3.1) Lemma 2.4 and the dominated convergence theorem that
\[
\overline{\varphi}(t, x, y) = p_0(t, x, y) + \int_0^t \int_{\mathbb{R}^d} \overline{\varphi}(t - s, x, z)S_z^b p_0(s, z, y)dyds
\]
and \( \overline{\varphi}(t, x, y) \leq c_1 p_1(t, c_2 x, c_2 y) \) for every \( 0 < t \leq 1 \wedge (A_0/\|b\|_{\infty})^{2/(2-\beta)} \) and \( x, y \in \mathbb{R}^d \). Hence we conclude from Theorem 3.11 that \( \overline{\varphi}(t, x, y) = q^b(t, x, y) \). This in particular implies that \( q^b(t, x, y) \geq 0 \). So there is a Feller process \( X^b \) having \( q^b(t, x, y) \) as its transition density function. The proof of Propositions 5.1-5.3 only uses the condition (1.14) through its implication that \( q^b(t, x, y) \geq 0 \). So in view of what we just established, Propositions 5.1-5.3 continue to hold for \( X^b \) under the current setting without the additional assumption (1.14). The proof of the theorem is now complete.

For a Borel set \( B \subset \mathbb{R}^d \), we define \( \tau^b_B = \inf\{t > 0 : X^b_t \notin B\} \) and \( \sigma^b_B := \inf\{t \geq 0 : X^b_t \in B\} \).
Proposition 5.7. For each $A > 0$ and $R_0 > 0$, there exists a positive constant $\kappa = \kappa(d, \beta, A, R_0) < 32/9$ so that for every $b$ satisfying (1.3) and (1.13) with $\|b\|_\infty \leq A$, $x \in \mathbb{R}^d$ and $r \in (0, R_0)$,

$$\mathbb{P}_x \left( \tau_{B(x,r)}^b \leq \kappa r^2 \right) \leq \frac{1}{2}.$$ 

Proof. Let $f$ be a $C^2$ function taking values in $[0,1]$ such that $f(0) = 0$ and $f(u) = 1$ if $|u| \geq 1$. Set $f_{x,r}(y) = f(\frac{|x-r|}{r})$. Note that $f_{x,r}$ is a $C^2$ function taking values in $[0,1]$ such that $f_{x,r}(x) = 0$ and $f_{x,r}(y) = 1$ if $y \notin B(x,r)$. Moreover,

$$\sup_{y \in \mathbb{R}^d} | \frac{\partial^2 f_{x,r}(y)}{\partial y_i \partial y_j} | \leq r^{-2} \sup_{y \in \mathbb{R}^d} | \frac{\partial^2 f(y)}{\partial y_i \partial y_j} |.$$

Denote $\sum_{i,j=1}^d | \frac{\partial^2 f(x)}{\partial y_i \partial y_j} |$ by $|D^2 f(x)|$. By Taylor’s formula, it follows that

$$|\mathcal{L}^b f_{x,r}(u)| \leq |\Delta f_{x,r}(u)| + c_1 \int_{|h| \leq r} |f_{x,r}(u+h) - f_{x,r}(u) - \langle \nabla f_{x,r}(u), h \rangle| \frac{dh}{|h|^{d+\beta}}$$

$$+ \int_{|h| > r} |f_{x,r}(u+h) - f_{x,r}(u)| \frac{dh}{|h|^{d+\beta}} \leq r^{-2} |\Delta f(u)| + c_2 |D^2 f(x)| r^{-2} \int_{|h| \leq 1} |h|^2 \frac{dh}{|h|^{d+\beta}} + c_2 \|f\|_\infty \int_{|h| > r} \frac{dh}{|h|^{d+\beta}} \leq c_3 (r^{-2} + r^{-\beta}) \leq c_4 r^{-2}, \quad r \in (0, R_0),$$

where $c_4 = c_4(d, \beta, A, R_0)$ is a positive constant dependent on $R_0$. Therefore, for each $t > 0$,

$$\mathbb{P}_x(\tau_{B(x,r)}^b \leq t) \leq \mathbb{E}_x \left[ f_{x,r}(X_t^b) - f_{x,r}(x) \right] = \mathbb{E}_x \left[ \int_0^{\tau_{B(x,r)}^b \wedge t} \mathcal{L}^b f_{x,r}(X_s^b) ds \right] \leq c_4 \frac{t}{r^2}.$$

Set $\kappa = 32/9 \wedge (2c_4)^{-1}$, then

$$\mathbb{P}_x(\tau_{B(x,r)}^b \leq \kappa r^2) \leq \frac{1}{2}.$$ 

Recall that $m_b = \inf_x \text{essinf}_z b(x,z)$.

Proposition 5.8. For every $A > 0$ and $R_0 > 0$, there exists a constant $C_{18} = C_{18}(d, \beta, A, R_0) > 0$ so that for every $b$ satisfying (1.3) and (1.13) with $\|b\|_\infty \leq A$, $r \in (0, R_0)$ and $x, y \in \mathbb{R}^d$ with $|x - y| \geq 3r$,

$$\mathbb{P}_x(\sigma_{B(y,r)}^b \leq \kappa r^2) \geq C_{18} r^{d+2} \frac{m_b}{|x-y|^{d+\beta}}.$$ 

Proof. By Proposition 5.7,

$$\mathbb{E}_x \left[ \kappa r^2 \wedge \tau_{B(x,r)}^b \right] \geq \kappa r^2 \mathbb{P}_x(\tau_{B(x,r)}^b \geq \kappa r^2) \geq \frac{1}{2} \kappa r^2.$$
Thus by Proposition 5.3, we have for $|x - y| \geq 3r$,

$$\mathbb{P}_x(\sigma_{B(y,r)}^b < \kappa r^2) \geq \mathbb{P}_x(X^b_{\kappa r^2 \wedge \tau_{B(x,r)}^b} \in B(y,r))$$

$$= \mathbb{E}_x \int_0^{\kappa r^2 \wedge \tau_{B(x,r)}^b} \int_{B(y,r)} J^b(X^b_s, u) \, du \, ds$$

$$\geq c_1 \mathbb{E}_x \left[ \kappa r^2 \wedge \tau_{B(x,r)}^b \right] \int_{B(y,r)} \frac{m_b}{|x - y|^{d+\beta}} \, du$$

$$\geq c_2 \kappa r^{d+2} \frac{m_b}{|x - y|^{d+\beta}}.$$ 

$$\square$$

**Proposition 5.9.** For every $A > 0$, $\lambda > 0$ and $\varepsilon > 0$, there exists a constant $C_{19} = C_{19}(d, \beta, A, \varepsilon, \lambda) > 0$ so that for every bounded $b$ that satisfies (1.3), (1.15) and (1.19) with $\|b\|_\infty \leq A$, and $3r \leq |x - y| \leq \lambda/3$,

$$\mathbb{P}_x(\sigma_{B(y,r)}^b < \kappa r^2) \geq C_{19} \frac{r^{d+2}}{|x - y|^{d+\beta}}.$$ 

**Proof.** By Propositions 5.3 and 5.7, we have for $3r \leq |x - y| \leq \lambda/3$,

$$\mathbb{P}_x(\sigma_{B(y,r)}^b < \kappa r^2) \geq \mathbb{P}_x \left( X^b_{\kappa r^2 \wedge \tau_{B(x,r)}^b} \in B(y,r) \right)$$

$$= \mathbb{E}_x \int_0^{\kappa r^2 \wedge \tau_{B(x,r)}^b} \int_{B(y,r)} J^b(X^b_s, u) \, du \, ds$$

$$\geq c_1 \mathbb{E}_x \left[ \kappa r^2 \wedge \tau_{B(x,r)}^b \right] \int_{B(y,r)} \frac{\varepsilon}{|x - y|^{d+\beta}} \, du$$

$$\geq c_2 \kappa r^{d+2} \frac{\varepsilon}{|x - y|^{d+\beta}},$$

where the second inequality holds due to (1.19) and $|X_s^b - u| \leq 3|x - y| \leq \lambda$ for $u \in B(y,r)$ and $X_s^b \in B(x,r)$.

$$\square$$

**Theorem 5.10.** For every $A > 0$ and any $b$ satisfying (1.3) and (1.15) with $\|b\|_\infty \leq A$, (1.18) holds.

**Proof.** By (1.13), we only need to prove the lower bound. Let $\delta_0 := 1 \wedge (A_0/A)^{2/(2-\beta)}$, (3.31) together with (1.4) yields that for any $\|b\|_\infty \leq A$,

$$q^b(t, x, y) \geq c_0 t^{-d/2} \quad \text{for } t \in (0, \delta_0] \text{ and } |x - y| \leq 3t^{1/2},$$

where $c_0 = c_0(d, \beta)$ is a positive constant. By (5.6) and the usual chain argument, there are positive constants $c_1$ and $c_2$ so that

$$q^b(t, x, y) \geq c_1 p_0(t, c_2 x, c_2 y), \quad x, y \in \mathbb{R}^d, \, t \in (0, \delta_0].$$

(5.7)

For every $t \in (0, \delta_0]$, by Proposition 5.7 and Proposition 5.8 with $R_0 = 1$, $r = t^{1/2}/2$ and the
strong Markov property of the process \( X^b \), we get for \(|x - y| > 3t^{1/2}\),
\[
\mathbb{P}_x(X^b_{t + 2\kappa t} \in B(y, t^{1/2})) \geq \mathbb{P}_x(X^b \text{ hits } B(y, t^{1/2}) \text{ before } \frac{1}{4}\kappa t \text{ and stays there for at least } \frac{1}{4}\kappa t \text{ units of time})
\]
\[
\geq \mathbb{P}_x\left(\sigma_{B(y, t^{1/2})} < \frac{1}{4}\kappa t\right) \inf_{z \in B(y, t^{1/2})} \mathbb{P}_z\left(\tau_{B(y, t^{1/2})} \geq \frac{1}{4}\kappa t\right)
\]
\[
\geq \mathbb{P}_x\left(\sigma_{B(y, t^{1/2})} < \frac{1}{4}\kappa t\right) \inf_{z \in B(y, t^{1/2})} \mathbb{P}_z\left(\tau_{B(z, t^{1/2})} \geq \frac{1}{4}\kappa t\right)
\]
\[
\geq c_3 t^{(d+2)/2} \frac{mb}{|x - y|^{d+\beta}}.
\]  
(5.8)

Here \( c_3 = c_3(d, \beta, A) \) is a positive constant. Hence, by (5.6) and (5.8), for \(|x - y| > 3t^{1/2}\) and \( t \in (0, \delta_0) \),
\[
q^b(t, x, y) \geq \int_{B(y, t^{1/2})} q^b(\frac{1}{4}\kappa t, x, z)q^b((1 - \frac{1}{4}\kappa)t, z, y) \, dz
\]
\[
\geq \inf_{z \in B(y, t^{1/2})} q^b((1 - \frac{1}{4}\kappa)t, z, y)\mathbb{P}_x(X^b_{\frac{1}{4}\kappa t} \in B(y, t^{1/2}))
\]
\[
\geq c_4 t^{-d/2} t^{(d+2)/2} \frac{mb}{|x - y|^{d+\beta}}
\]
\[
\geq c_4 \frac{mt}{|x - y|^{d+\beta}},
\]  
(5.9)

where \( c_4 = c_4(d, \beta, A) > 0 \), the third inequality holds due to (5.6), (5.8) and (5.9), (5.10) and the similar argument in (5.8). Finally, noting that \( a \vee b \approx a + b \), (5.6), (5.7), (5.9), (5.10) and the Chapman-Kolmogorov equation yields the desired lower bound estimate. \( \square \)

**Theorem 5.11.** For every \( \lambda > 0, \varepsilon > 0, A > 0 \) and any bounded \( b \) satisfying (1.3), (1.15) and (1.19) with \( ||b||_\infty \leq A \), (1.20) holds.

**Proof.** Let \( \delta_0 := 1 \wedge (A_0/A)^{2/(2-\beta)} \wedge (\lambda/9)^2 \). By Theorem 5.10 and Chapman-Kolmogorov equation, it suffices to prove there exist \( c_k = c_k(d, \beta, A, \varepsilon, \lambda) > 0, k = 1, 2 \) so that \( q^b(t, x, y) \geq c_k \mathbb{P}_\beta(t, c_2 x, c_2 y) \) for \( t \in (0, \delta_0) \) and \(|x - y| > 3t^{1/2}\).

(i) First, we consider the case \(|x - y| > 3t^{1/2}\). For every \( t \in (0, \delta_0) \), by Proposition 5.9 with \( r = t^{1/2}/2 \) and the similar procedure in (5.8),
\[
\mathbb{P}_x\left(X^b_{t^{1/2}} \in B(y, t^{1/2})\right) \geq c_3 t^{(d+2)/2} \frac{1}{|x - y|^{d+\beta}}, \quad \lambda/3 \geq |x - y| > 3t^{1/2}.
\]  
(5.10)

Here \( c_3 = c_3(d, \beta, A, \varepsilon, \lambda) \) is a positive constant. Hence, by (5.10), (5.11) and the similar argument in (5.9), we have
\[
q^b(t, x, y) \geq c_4 \frac{t}{|x - y|^{d+\beta}}, \quad \lambda/3 \geq |x - y| > 3t^{1/2}
\]  
(5.11)

where \( c_4 = c_4(d, \beta, A, \varepsilon, \lambda) > 0 \).

(ii) Next, we consider the case \(|x - y| > \lambda/3\). The following proof is similar to [3] Theorem 3.6. For the reader’s convenience, we spell out the details here.

Take \( C_* = (\lambda/3)^{-1} \). Let \( R := |x - y| \) and \( c_* = C_* \vee \delta^{-1}_0 \). Let \( l \geq 2 \) be a positive integer such that \( c_+ R \leq l \leq c_+ R + 1 \) and let \( x = x_0, x_1, \ldots, x_l = y \) be such that \(|x_i - x_{i-1}| \asymp R/l \asymp 1/c_+\)
for $i = 1, \ldots, l - 1$. Since $t/l \leq C_s R/l \leq C_s/c_+ \leq \delta_0$ and $R/l \leq 1/c_+ \leq \lambda/3$, we have by (5.6), (5.11) and (5.12),

$$q^b(t/l, x, x_{i+1}) \geq c_5(t/l)^{-d/2} \left( p_0(t/l, c_6 x, c_6 x_{i+1}) + \frac{t/l}{(R/l)^{d+\beta}} \right) \geq c_7 \left( (t/l)^{-d/2} \wedge (t/l) \right) \geq c_7 t/l. \tag{5.12}$$

Let $B_i = B(x_i, \lambda/6)$, by (5.12),

$$q^b(t, x, y) \geq \int_{B_{i-1}} \cdots \int_{B_1} q^b(t/l, x, z_1) \cdots q^b(t/l, z_{l-1}, y) \, dz_1 \cdots dz_{l-1} \geq (c_7 t/l)^l \geq (c_8 t/R)^{c_+ R + 1} \geq c_9 (t/R)^{c_{10} R} \tag{5.13}$$

$$\geq c_9 \left( \frac{t}{|x-y|} \right)^{c_{10} |x-y|}. \tag{5.13}$$

By (5.11), (5.13) and the estimates of $\overline{p}_\beta$ in (1.16)-(1.17), we get the desired conclusion.

\textbf{Proof of Theorem 1.3.} Theorem 1.3 now follows from Theorems 5.6, 5.10 and 5.11.

In the remainder of this section, we prove the result of Theorem 1.5, where $\mathcal{L}^b$ is a perturbation of $\Delta$ by finite range nonlocal operator $\mathcal{S}^b$.

\textbf{Proposition 5.12.} For each $M > 1$, there exists a positive constant $C_{20} = C_{20}(d, \beta, M, \lambda)$ so that for every $b$ satisfying (1.3), (1.15) and (1.23) and $r > 0$,

$$\mathbb{P}_x(\tau_{B(x, r)}^b \leq t) \leq C_{20} \frac{t}{r^2}. \tag{5.14}$$

\textbf{Proof.} Let $f$ be a $C^2$ function taking values in $[0, 1]$ such that $f(0) = 0$ and $f(u) = 1$ if $|u| \geq 1$. Set $f_{x,r}(y) = f\left( \frac{|x-y|}{r} \right)$. Note that $f_{x,r}$ is a $C^2$ function taking values in $[0, 1]$ such that $f_{x,r}(x) = 0$ and $f_{x,r}(y) = 1$ if $y \notin B(x, r)$. Moreover,

$$\sup_{y \in \mathbb{R}^d} \left| \frac{\partial^2 f_{x,r}(y)}{\partial y_i \partial y_j} \right| \leq r^{-2} \sup_{y \in \mathbb{R}^d} \left| \frac{\partial^2 f(y)}{\partial y_i \partial y_j} \right|. \tag{5.15}$$

Denote $\sum_{i,j=1}^{d} |\partial^2 f(x)|$ by $|D^2 f(x)|$. By the conditions (1.15) and (1.23) sup$_x b(x, z) \leq M_1 |z| \leq \lambda(z)$, it follows that

$$|\mathcal{L}^b f_{x,r}(u)| \leq |\Delta f_{x,r}(u)| + M \int_{|h| \leq \lambda} |f_{x,r}(u+h) - f_{x,r}(u) - \langle \nabla f_{x,r}(u), h \rangle| \, \frac{dh}{|h|^{d+\beta}}$$

$$\leq r^{-2} \|D^2 f\|_{\infty} + M \|D^2 f\|_{\infty} r^{-2} \int_{|h| \leq \lambda} |h|^2 \frac{dh}{|h|^{d+\beta}} \tag{5.16}$$

$$\leq cr^{-2}, \tag{5.16}$$

where $c = c(d, \beta, M, \lambda)$ is a positive constant independent of $r$. Therefore, for each $t > 0$,

$$\mathbb{P}_x(\tau_{B(x,r)}^b \leq t) \leq \mathbb{E}_x \left[ f_{x,r}(X_{\tau_{B(x,r)}^b}^b) \right] - f_{x,r}(x)$$

$$= \mathbb{E}_x \left[ \int_0^{\tau_{B(x,r)}^b} \mathcal{L}^b f_{x,r}(X_s^b) \, ds \right] \leq c \frac{t}{r^2}, \quad r > 0. \tag{5.16}$$

\textbf{Theorem 5.13.} For every \( M > 1 \) and \( \lambda > 0 \), there are positive constants \( C_k = C_k(d, \beta, M, \lambda), k = 21, 22 \) such that for any \( b \) with \( (1.3), (1.5), \) and \( (1.23) \),

\[
q^b(t, x, y) \leq C_1 t^{-d/2} \land [p_0(t, C_{22} x, C_{22} y) + \overline{\beta}(t, C_{22} x, C_{22} y)], \quad t \in (0, 1], \ x, y \in \mathbb{R}^d.
\]

\textbf{Proof.} By \( (3.37), (1.8) \), and \( (1.16) \), there are constants \( c_k, k = 1, \cdots, 4 \) so that

\[
q^b(t, x, y) \leq c_1 p_1(t, c_2 x, c_2 y) \leq c_3 t^{-d/2} \land [p_0(t, c_4 x, c_4 y) + \overline{\beta}(t, c_4 x, c_4 y)], \quad |x - y| \leq 1, \ t \in (0, 1].
\]

In the following, we will prove that there exist \( c_5 \) and \( c_6 \) so that

\[
q^b(t, x, y) \leq c_5 \overline{\beta}(t, c_6 x, c_6 y), \quad |x - y| > 1, \ t \in (0, 1].
\]

In fact, by \( (1.17) \),

\[
q^b(t, x, y) \leq \overline{\beta}(t, x, cy) \leq c' t^{-d/2}, \quad |x - y| > 1, \ t \in (0, 1], \text{ so } \overline{\beta}(t, x, cy) \approx t^{-d/2} \land \overline{\beta}(t, cx, cy) \text{ in this case, thus } (5.17) \text{ implies that } (5.16) \text{ holds for } |x - y| > 1 \text{ and } t \in (0, 1].
\]

For each \( \lambda > 0 \), let \( r = \lambda \lor 1 \) be a constant to be chosen later. First we will use induction method to prove that there is a constant \( C_0 \) so that

\[
q^b(t, x, y) \leq C_0 \left( \frac{t}{n} \right)^n, \quad |x - y| \geq nr, \ t \in (0, 1], \ n \geq 1.
\]

By \( (3.37) \) and \( (2.1) \), we can find a constant \( C_0 \) so that

\[
q^b(t, x, y) \leq c_7 p_1(t, c_8 x, c_8 y) \leq C_0 \frac{t}{|x - y|^{d - \beta}} \leq C_0 t, \quad |x - y| > 1, \ t \in (0, 1],
\]

where the second inequality holds since by \( (2.1) \),

\[
p_0(t, x, y) \leq c \frac{t}{|x - y|^{d + \beta}} \leq c' \frac{t}{|x - y|^{d + \beta}} \quad \text{for } |x - y| > 1.
\]

Hence \( (5.18) \) naturally holds for \( n = 1 \). Now fix \( C_0 \) and assume \( (5.18) \) holds for \( n = m \). Then \( q^b(t, x, y) \leq C_0 (t/m)^m \) for \( |x - y| \geq mr \) and \( t \in (0, 1] \). Let \( \tau \) be the first time that \( X^b \) exits \( B(x, r - \lambda) \) starting from \( x \). By the strong Markov property, for \( |x - y| \geq (m + 1)r \),

\[
q^b(t, x, y) = E_x[q^b(t - \tau, X^b_\tau; \tau < t)]
\]

\[
= E_x \left[ q^b(t - \tau, X^b_\tau; \tau < \frac{t}{m + 1} \right]
\]

\[
+ E_x \left[ q^b(t - \tau, X^b_\tau; \tau \geq \frac{t}{m + 1} \leq \tau < t \right]
\]

\[
= I + II.
\]

Noting that the jump of \( X^b \) is not larger than \( \lambda \) by Lévy system formula Proposition 5.3 and the condition \( (1.23) \), so starting from \( x \), \( X^b \) will lie in \( B(x, r) \), which implies the distance between \( X^b \) and \( y \) is bigger than \( mr \) for \( |x - y| > (m + 1)r \). Hence, by Proposition 5.12 and our assumption for \( n = m \),

\[
I \leq P_x \left( \tau < \frac{t}{m + 1} \right) \sup_{z \in B(x, r), s \leq t} q^b(s, z, y)
\]

\[
\leq C_20 \left( \frac{m + 1)(r - \lambda)^2}{m} \right)^{m} \left( \frac{t}{m + 1} \right)^{m+1}
\]

\[
= C_0 C_20 \left( \frac{m + 1}{m} \right)^{m} \left( \frac{t}{m + 1} \right)^{m+1}
\]

\[
= C_0 C_20 K_{\lambda} \left( \frac{t}{m + 1} \right)^{m+1}, \ m \geq 1, \ |x - y| > (m + 1)r
\]
where $K_1 > 0$ is a positive constant independent of $m$ and $C_{20}$ is the constant in Proposition 5.12. On the other hand, by Proposition 5.12 and the assumption for $n = m$,

$$II = \sum_{k=1}^{m} \mathbb{E}_x \left[ q^b(t - \tau, x, y); \frac{kt}{m+1} \leq \tau < \frac{(k+1)t}{m+1} \right]$$

$$\leq \sum_{k=1}^{m} \sup_{z \in B(x, \tau), s \leq (m+1-k)t/(m+1)} q^b(s, z, y) \cdot \mathbb{P}_x \left( \frac{kt}{m+1} \leq \tau < \frac{(k+1)t}{m+1} \right)$$

$$\leq C_0 \sum_{k=1}^{m} \left( \frac{1}{m} \frac{(m+1-k)t}{m+1} \right)^m \mathbb{P}_x \left( \tau < \frac{(k+1)t}{m+1} \right)$$

$$\leq C_0 C_{20} \sum_{k=1}^{m} \left( \frac{t}{m+1} \right)^m \left( \frac{m+1-k}{m} \right)^m \frac{(k+1)t}{m+1} (r-\lambda)^2$$

$$= \frac{C_0 C_{20}}{(r-\lambda)^2} \left( \frac{t}{m+1} \right)^{m+1} \sum_{k=1}^{m} (k+1) \left( \frac{m+1-k}{m} \right)^m$$

$$= \frac{C_0 C_{20}}{(r-\lambda)^2} \left( \frac{t}{m+1} \right)^{m+1} \sum_{k=1}^{m} (m+2-k) \left( \frac{k}{m} \right)^m, \quad |x-y| > (m+1)r.$$ 

Define

$$S_m := \sum_{k=1}^{m} (m+2-k) \left( \frac{k}{m} \right)^m, \quad m \geq 1.$$ 

For each $m \geq 1$, define $f_m(u) = (m+2-u)(u/m)^m, u \in \mathbb{R}$. Noting that $f'_m(u) = \frac{u^{m-1}}{m^m} [m(m+2) - (m+1)u] > 0$ for $u \in (0, m]$. Hence,

$$S_m = 2 + \sum_{k=1}^{m-1} (m+2-k) \left( \frac{k}{m} \right)^m$$

$$\leq 2 + \sum_{k=1}^{m-1} \int_{k}^{k+1} (m+2-u) \left( \frac{u}{m} \right)^m du$$

$$\leq 2 + \frac{1}{m^m} \int_{0}^{m} (m+2-u)u^m du$$

$$= 2 + \frac{1}{m^m} \left[ \frac{(m+2)m^{m+1}}{m+1} - \frac{m^{m+2}}{m+2} \right]$$

$$= 2 + \frac{m(3m+4)}{(m+1)(m+2)} \leq K_2, \quad m \geq 1,$$

where $K_2$ is a positive constant. Then by (5.22) and the above inequality, we have

$$II \leq \frac{C_0 C_{20}}{(r-\lambda)^2} \left( \frac{t}{m+1} \right)^{m+1} S_m \leq \frac{C_0 C_{20} K_2}{(r-\lambda)^2} \left( \frac{t}{m+1} \right)^{m+1}.$$ 

Combining this inequality with (5.20) and (5.21), we have for $|x-y| \geq (m+1)r$,

$$q^b(t, x, y) = I + II \leq \frac{C_0 C_{20}(K_1 + K_2)}{(r-\lambda)^2} \left( \frac{t}{m+1} \right)^{m+1}.$$ 

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Noting that the constant $C_{20}$ in Proposition 5.12 is independent of $r$, so we can take $r$ large enough so that \( \frac{C_{20}(K_1+K_2)}{(r-A)^2} \leq 1 \). Thus, (5.18) holds for $n = m + 1$ and thus for all $n \geq 1$.

When $|x - y| > 2r$, there exists $n$ so that $nr \leq |x - y| < (n + 1)r$, then by (5.18),
\[
q_b(t, x, y) \leq C_0 \left( \frac{t}{n} \right)^n \leq C_0 \left( \frac{n + 1}{n} \frac{rt}{|x - y|} \right)^{\frac{n}{n+1}} \leq C_0 \left( \frac{2rt}{|x - y|} \right)^{\frac{|x - y|}{2r}}, \quad t \in (0, 1].
\]

On the other hand, if $1 < |x - y| \leq 2r$, (5.19) shows that
\[
q_b(t, x, y) \leq C_0 \frac{t}{|x - y|^{d+\beta}} \leq C_0 \frac{t}{|x - y|} \leq C_0 \left( \frac{2rt}{|x - y|} \right)^{\frac{|x - y|}{2r}}, \quad t \in (0, 1].
\]

Hence,
\[
q_b(t, x, y) \leq C_0 \left( \frac{2rt}{|x - y|} \right)^{\frac{|x - y|}{2r}}, \quad |x - y| > 1, \quad t \in (0, 1]. \tag{5.23}
\]

Comparing (5.23) with (1.17), we get (5.17) and the proof is complete. \( \square \)

Theorem 1.5 now follows from Theorem 5.13.

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