Diffeomorphism Invariant Actions for Partial Systems

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(November, 1994)

Abstract

Local action principles on a manifold $M$ are invariant (if at all) only under diffeomorphisms that preserve the boundary of $M$. Suppose, however, that we wish to study only part of a system described by such a principle; namely, the part that lies in a bounded region $R$ of spacetime where $R$ is specified in some diffeomorphism invariant manner. In this case, a description of the physics within $R$ should be invariant under all diffeomorphisms regardless of whether they preserve the boundary of this region. The following letter shows that physics in such a region can be described by an action principle that $i$) is invariant under both diffeomorphisms which preserve the boundary of $R$ and those that do not, $ii$) leaves the dynamics of the part of the system outside the region $R$ completely undetermined, and $iii$) can be constructed without first solving the original equations of motion.
I. INTRODUCTION

Action principles for systems with boundaries are a fundamental tool in studying quantum effects in diffeomorphism invariant systems. They have been used to derive black hole entropy in the semiclassical approximation \[1\] and to discuss the pair creation and annihilation of magnetically charged black holes \[2\]. Recently, Carlip \[3\] has even suggested that the entropy of the 2+1 dimensional Banâdoes, Teitelboim, and Zanelli black hole \[4\] can be derived by counting the states produced by the degrees of freedom that arise through a “restriction of the diffeomorphism invariance” by the presence of a boundary; in that case, the horizon of a black hole. In addition, asymptotically flat and other noncompact spacetimes may be described as systems with a boundary \[5\], and spacelike singularities form a natural past or future boundary for many classical solutions of general relativity and low energy string theory (see, e.g., \[6\]). Finally, to render the action of any system finite, it is generally necessary to consider the system only between two times or between two spacelike hypersurfaces. Thus, there is ample motivation to understand any subtleties that arise in the use of variational principles for bounded generally covariant systems.

The actions for such systems are typically of the local form

\[ S^M_0 = \int_M \mathcal{L} \, d^n x \] (1.1)

where \( M \) is an \( n \)-manifold and \( \mathcal{L} \) is a scalar density on \( M \). Such an action is invariant under any diffeomorphism \( \psi : M \to M \). Note that such a map induces a diffeomorphism of the boundary \( \partial M \) of \( M \) as well. If the map did not preserve \( \partial M \), it would correspond to enlarging (or shrinking) the system considered and would in general change the action. While the addition of the proper boundary term to \[1.1\] can enlarge the gauge invariance of \( S^M_0 \) \[4,5\], this will in general require the full solution of the equations of motion. As a result, if the action \( S^M_0 \) defines the notion of gauge equivalence, an infinitesimal map \( \delta \psi \) is a gauge transformation only if it preserves \( \partial M \).

Suppose now that \( M \) may be embedded in some larger manifold \( M' \). We would like to understand how the notions of gauge invariance defined by \( S^M_0 \) and \( S^{M'}_0 \) relate. Because \( S^{M'}_0 \)
is invariant under diffeomorphisms that move the boundary of $\mathcal{M}$, $S_0^{M'}$ provides a larger set of infinitesimal gauge transformations than does $S_0^M$, even within the image of $\mathcal{M}$. Thus, the class of gauge invariants defined by $S_0^M$ is larger than that defined by $S_0^{M'}$. In particular, if the theory contains a metric $g_{\mu\nu}$ and $\mathcal{M}$ is compact, the quantity

$$\int_{\mathcal{M}} \sqrt{-g} d^n x$$

(1.2)

is invariant with respect to gauge transformations defined by $S_0^M$, but not those defined by $S_0^{M'}$.

When the boundaries of $\mathcal{M}$ are not specified by a physical condition, the action $S_0^M$ and $S_0^{M'}$ seem to describe quite different physics. If, however, the above embedding is chosen in a field dependent but diffeomorphism invariant manner (such as by mapping timelike boundaries to sheets of steel and spacelike boundaries to hypersurfaces defined by the reading of some clock), this picture is physically reasonable as $\int_{\mathcal{M}} \sqrt{-g} d^n x$ may be interpreted as the spacetime volume of the region bounded by the steel sheets for the appropriate clock readings. Such a quantity is gauge invariant as defined by $S_0^{M'}$ as well. We would like to make this connection explicit by describing the part of the system within such boundaries in a way that is invariant under all diffeomorphisms of $\mathcal{M}'$, even those that move $\partial \mathcal{M}$.

The purpose of this letter is to use this physical picture to provide an action principle which achieves these goals without first solving the equations of motion. In particular, given any action principle of the form $\int_{\mathcal{M}} \theta(f) L d^n x$ and any (not necessarily local!) scalar field $f$, we show that variation of the action

$$S^M = \int_{\mathcal{M}} \theta(f) \mathcal{L} d^n x$$

(1.3)

where $\theta$ is the Heaviside step function, yields the same Euler-Lagrange equations as $S_0^M$ in the region where $f > 0$ but provides no other restrictions on the dynamics when varied within a diffeomorphism invariant class of field histories. This property is nontrivial only on the surface $f = 0$, but, as should be expected, follows on this surface only if the variations preserve appropriate ‘boundary conditions’ on the field histories. Note that, provided $f < 0$
on the boundary of $\mathcal{M}$, the action $S^M$ is invariant under a larger class of gauge transformations than $S_0^M$. Effectively, it is invariant under transformations that move the boundary of $\mathcal{M}$. These results will be derived in the next section.

II. THE VARIATIONAL PRINCIPLE FOR $S^M$

We now consider a general coordinate invariant action principle of the form \([.1]\) where the Lagrangian density $\mathcal{L}$ is a function of some collection of fields and their first derivatives. An action of this form yields a well-defined variational principle whenever all fields whose derivatives appear in $\mathcal{L}$ are fixed on the boundary. We refer to such fields as type I; fields whose derivatives do not appear in $\mathcal{L}$ will be referred to as type II. However, for some first order systems such as spinor fields or Chern-Simons fields \([9,10]\) it is only appropriate to specify certain parts of these fields and to do so in a generally covariant manner typically involves complicated nonlocal constructions\(^1\). We do not treat such cases, but we expect that they may be addressed by an analysis similar to what follows. We point out that our analysis is appropriate to standard first order formulations of systems, like gravity in any number of dimensions, which also have a second order formulation. In addition, although not manifestly so, the usual action

$$\frac{1}{16\pi} \int_{\mathcal{M}} \sqrt{-g} R + \frac{1}{8\pi} \int_{\partial \mathcal{M}} K$$  \hspace{1cm} (2.1)

for general relativity is of this form, as the boundary term exactly cancels a total divergence which contains the higher derivatives of $g_{\mu \nu}$. More general variational principles, where various momenta are fixed on the boundary, can be obtained from actions of the above form by adding a total divergence to $\mathcal{L}$. If the Lagrangian is in first order form (see, for example, \([11]\)), the addition of such a divergence leaves $\mathcal{L}$ a function only of the fields and their first derivatives. Thus, by passing through the first order formulation, we see that actions of this type are quite general.

\(^1\) Thanks to Steve Carlip for bringing this to the author’s attention.
Suppose that the Lagrangian depends on a set of fields (labeled by \(a\)) of tensor type \(i = (i_1, i_2)\) and density weight \(j\) which we write as \(\phi^{ija(\nu)}\). Here, \(\{\nu\}\) and \(\{\mu\}\) denote the appropriate collection of abstract tensor indices as specified by the value of \(i\). \(L\) will generally depend on fields of more than one tensor type so that both \(i\) and \(j\) will take multiple values. We employ the summation convention on the indices \(i,j,a\) as well as \(\{\nu\}\) and \(\{\mu\}\) so that

\[
j \frac{\partial L}{\partial \phi^{ija(\nu)}\phi^{ija(\nu)}_{\{\nu\}}}
\]

represents a sum over fully contracted fields of all tensor types and all density weights with the contribution of each field multiplied by its density weight.

It will be convenient to introduce an arbitrary background spacetime connection \(\Gamma^\sigma_{\alpha\beta}\) together with the associated covariant derivative operator \(D_\alpha\) and to write \(L\) as a function of the \(\phi^{ijk(\nu)}\) and the \(D_\alpha\phi^{ijk(\nu)}\). Since this connection was introduced by hand, the fields and derivatives must enter the Lagrangian in such a way that \(L\) is independent of \(\Gamma^\sigma_{\alpha\beta}\). We may thus vary \(\Gamma^\sigma_{\alpha\beta}\) as well in our action principle and, while the resulting equation of motion will be identically satisfied, this provides a convenient way to keep track of the relationships that conspire to make \(L\) independent of \(\Gamma^\sigma_{\alpha\beta}\) and thus make \(S_0^M\) diffeomorphism invariant.

We will need the explicit form of the change of \(L\) under an infinitesimal coordinate transformation \(x^\mu \to x^\mu + \delta x^\mu\):

\[
\delta L = -\frac{\partial L}{\partial \phi^{ija(\nu)}_{\{\nu\}}} \left[ \partial_\alpha \phi^{ija(\nu)}_{\{\mu\}} \delta x^\alpha + j \phi^{ija(\nu)}_{\{\mu\}} \partial_\alpha \delta x^\alpha + \sum_k \phi^{ija(\nu)}_{\{\mu\}_1\alpha(\mu)_2} \partial_{\mu_k} \delta x^\alpha - \sum_k \phi^{ija(\nu)}_{\{\mu\}_1\beta(\nu)_2} \partial_{\beta} \delta x^\nu_k \right] \\
- \frac{\partial L}{\partial (D_\rho \phi^{ija(\nu)}_{\{\mu\}})} \left[ \partial_\alpha (D_\rho \phi^{ija(\nu)}_{\{\mu\}}) \delta x^\alpha + j D_\rho \phi^{ija(\nu)}_{\{\mu\}} \partial_\alpha \delta x^\alpha + D_\alpha \phi^{ija(\nu)}_{\{\mu\}} \partial_\rho \delta x^\alpha \\
+ \sum_k D_\rho \phi^{ija(\nu)}_{\{\mu\}_1\alpha(\mu)_2} \partial_{\mu_k} \delta x^\alpha - \sum_k D_\rho \phi^{ija(\nu)}_{\{\mu\}_1\beta(\nu)_2} \partial_{\beta} \delta x^\nu_k \right] 
\]

(2.3)

where the terms in square brackets are the explicit forms of \(\delta \phi^{ija(\nu)}_{\{\mu\}}\) and \(\delta D_\rho \phi^{ija(\nu)}_{\{\mu\}}\) under a change of coordinates. The notation \(\sum_k \phi^{ija(\nu)}_{\{\mu\}_1\alpha(\mu)_2} \partial_{\mu_k} \delta x^\alpha\) represents a sum whose \(k\)th term has the \(k\)th covariant index in the set \(\{\mu\}\) replaced by \(\alpha\) and is contracted on that index with \(\partial_{\mu_k} \delta x^\alpha\), where \(\mu_k\) is just this missing index. The corresponding notation is
used for the contravariant case as well. Equation 2.3 may be written in the form \( \delta \mathcal{L} = - (\partial_\alpha \mathcal{L} \, \delta x^\alpha + Q^\beta_\alpha \, \partial_\beta \delta x^\alpha) \), from which will follow the identity that captures the coordinate invariance of \( S_0^M \).

Since \( S_0^M \) is unchanged by an arbitrary infinitesimal coordinate transformation, we have that

\[
0 = \delta S_0^M = - \int_{\mathcal{M}} \left[ Q^\beta_\alpha \, \partial_\beta \delta x^\alpha + \partial_\alpha \mathcal{L} \, \delta x^\alpha \right] + \int_{\partial \mathcal{M}} \mathcal{L} n_\beta \delta x^\alpha
\]

\[
= - \int_{\mathcal{M}} \partial_\beta [\mathcal{L} \delta^\beta_\alpha - Q^\beta_\alpha] \, \delta x^\alpha + \int_{\partial \mathcal{M}} [\mathcal{L} \delta^\beta_\alpha - Q^\beta_\alpha] n_\beta \delta x^\alpha
\]

(2.4)

where \( n_\beta \) is the outward pointing normal vector field to \( \partial \mathcal{M} \). Thus, we may conclude that

\[ \partial_\beta [\mathcal{L} \delta^\beta_\alpha - Q^\beta_\alpha] \] vanishes in the interior and that \( n_\beta [\mathcal{L} \delta^\beta_\alpha - Q^\beta_\alpha] \) vanishes on the boundary.

Since the fields themselves are unrestricted on \( \partial \mathcal{M} \) (only the variations of the fields are constrained), we must in fact have

\[ \mathcal{L} \delta^\beta_\alpha - Q^\beta_\alpha = 0 \]

(2.5)

\textit{identically} everywhere. This is just the generalization of the familiar statement that, on a \( 0 + 1 \) dimensional spacetime, the Hamiltonian constructed from a diffeomorphism invariant action for a system of scalar fields vanishes identically.

The result 2.3 may be used to show that the action 1.3 provides an acceptable variational principle when \( S^M \) is varied within a class of histories for which the fields are fixed on the \( f = 0 \) surface. Specifically, fix some \( n - 1 \) manifold \( \Sigma \) and an embedding \( \eta : \Sigma \to \mathcal{M} \) such that \( \mathcal{M} - \Sigma \) has exactly two connected components (which we arbitrarily call the inside and the outside). Note that \( \Sigma \) may have a boundary \( \partial \Sigma \). Consider the class of histories for which the surface defined by \( f = 0 \) gives the above embedding of \( \Sigma \) into \( \mathcal{M} \) up to a diffeomorphism and for which \( f > 0 \) inside and \( f < 0 \) outside. The inside may still contain part of \( \partial \mathcal{M} \), although the most interesting case is when \( \partial \mathcal{M} \) lies completely outside. Furthermore, we will vary the histories only within the class for which all type I fields are fixed (up to a diffeomorphism of \( \mathcal{M} \)) on the part of \( \partial \mathcal{M} \) inside of \( f = 0 \) and on the \( f = 0 \) surface. This may be done, for example, by choosing histories for which some set of scalar fields may be
used as a coordinate system near \( \partial M \) and \( f = 0 \) and for which the components of the tensor fields have some fixed relationship with these scalars and their gradients.

A direct variation of \( S_0 \), together with the usual integrations by parts yields:

\[
\delta S^M = \int_M \left( \theta(f) \left[ \left( \frac{\partial L}{\partial \phi_{i[a]}^{\{\nu\}}} - D_\rho \frac{\partial L}{\partial (D_\rho \phi_{i[a]}^{\{\nu\}})} \right) \delta \phi_{\{\mu\}}^{ija} \right. \right.
\]

\[
+ \left( f \phi_{\{\mu\}}^{ija} \right) \delta \phi_{\{\nu\}}^{ija} + \sum_k \phi_{\mu}^{ija(1)} \delta \phi_{\nu}^{ija(2)} - \sum_k \phi_{\mu}^{ija(1)} \phi_{\nu}^{ija(2)} \right)
\]

\[
\left. \left. + \delta(f) \left[ \frac{\partial L}{D_\rho \phi_{\{\mu\}}^{ija(\nu)}} \partial_\rho f \delta \phi_{\{\mu\}}^{ija(\nu)} \right] \right) \right)
\]

\[
+ \int_{\partial M} \theta(f)n_\rho \left( \frac{\partial L}{D_\rho \phi_{\{\mu\}}^{ija(\nu)}} \delta \phi_{\{\mu\}}^{ija(\nu)} \right) \right)
\]

(2.6)

where \( \delta(f) \) is the Dirac delta-function and is not to be confused with the variation \( \delta f \). The first line in (2.6) contains just the usual Euler-Lagrange equations inside the surface \( f = 0 \) and the second line explicitly displays the conspiracies that make \( L \) independent of the background connection. Line four contains the usual boundary terms on the inside part of \( \partial M \), which vanish for our class of variations. The third line contains the terms of interest; in order that our new variational principle not restrict the dynamics excessively, we must show that these terms vanish when the variations satisfy the boundary conditions given above and when the Euler-Lagrange equations are satisfied on the \( f = 0 \) surface\(^2\).

To do so, note that the variation \( \delta f \) will move the \( f = 0 \) boundary surface. Since all fields, including \( f \) itself, are specified on the \( f = 0 \) surface up to diffeomorphism, the variations \( \delta \phi_{\{\mu\}}^{ija(\nu)} \) must be just those induced by some diffeomorphism \( x^\alpha \to x^\alpha + \delta x^\alpha \), which were explicitly displayed in equation (2.3). Similarly, \( \delta f = -\partial_\alpha \delta x^\alpha \). Thus these variations, when evaluated on the \( f = 0 \) surface, are not independent. The relationships between the \( \delta \phi_{\{\mu\}}^{ija(\nu)} \) will conspire, together with the general covariance of \( S_0^M \), to make the term proportional

\(^2\) As may be seen from a brief study of the free relativistic particle, an attempt to leave these variations arbitrary (as in a naive application of (1.3)) and use their coefficients as additional equations of motion leads to nonsense.
to δ(f) vanish when the usual Euler-Lagrange equations of \( S_0^M \) are imposed on the \( f = 0 \) surface.

Now, a comparison of 2.3 and 2.6 shows that use of the Euler-Lagrange equations for \( \Gamma^\sigma_{\alpha\beta} \) greatly simplifies the term proportional to \( \delta(f) \) since, when contracted with \( \frac{\partial L}{\partial (D_\rho \phi^{ij}_\nu)} \), \( \delta\phi^{ij}_\nu \) may be replaced by \( \partial_\alpha \phi_{(\mu)}^{ij} \delta x^\alpha \). Thus, we find

\[
\delta S^M - \theta(f)\delta S_0^M = \int_\mathcal{M} \delta(f) \left[ L \delta^\beta_\alpha - \frac{\partial L}{\partial (D_\beta \phi^{ij}_\nu)} \partial_\alpha \phi_{(\mu)}^{ij} \right] \partial_\beta f \delta x^\alpha \tag{2.7}
\]

which looks suspiciously like 2.5.

Returning to the definition of \( Q^\beta_\alpha \) in 2.3, the Euler-Lagrange equations for \( \phi^{ij}_\nu \) show that we have

\[
Q^\beta_\alpha = D_\beta \left[ \frac{\partial L}{\partial (D_\rho \phi^{ij}_\nu)} \frac{\partial (\delta \phi^{ij}_\nu)}{\partial (D_\beta \delta x^\alpha)} \right] + \frac{\partial L}{\partial (D_\rho \phi^{ij}_\nu)} D_\alpha \phi^{ij}_\nu \delta^\beta_\rho. \tag{2.8}
\]

where we have written the transformation \( \delta \phi^{ij}_\nu \) displayed in 2.3 as a function of \( \delta x^\sigma \) and \( \partial_\beta \delta x^\alpha \). As before, the term in brackets vanishes by the Euler-Lagrange equations for \( \Gamma^\sigma_{\alpha\beta} \). In the second term, the contraction of these same Euler-Lagrange equations with \( \Gamma^\sigma_{\alpha\gamma} \) on the indices \( \sigma \) and \( \gamma \) shows that \( D_\rho \phi^{ij}_\nu \) may be replaced with \( \partial_\rho \phi^{ij}_\nu \) in 2.8. Equation 2.5 then reads

\[
L \delta^\beta_\alpha - \frac{\partial L}{\partial (D_\rho \phi^{ij}_\nu)} \partial_\alpha \phi^{ij}_\nu \delta^\beta_\rho = 0 \tag{2.9}
\]

so that \( \delta S_0^M \) vanishes when the Euler-Lagrange equations hold inside and on the surface \( f = 0 \).

Thus, the action \( 1.3 \) provides a perfectly valid variational principle for our partial system. This leads to the interesting question of how path integrals based on \( 1.3 \) differ from those based on \( 1.1 \). Note that in a canonical framework, and as opposed to the traditional approach of \( 2.1 \), use of an action of the form \( 1.3 \) allows the lapse \( N \) to be completely fixed along with the gauge freedom. It seems likely that path integrals of both types are appropriate to the study of diffeomorphism invariant systems, though with different interpretations which are yet to be fully understood.
ACKNOWLEDGMENTS

The author would like to express his thanks to Steve Carlip, Fay Dowker, Jim Hartle, Gary Horowitz, and Jorma Louko for sharpening his thinking on this subject and for suggesting important references. This work was supported by NSF grant PHY-908502.
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