Lesche Stability of $\kappa$-Entropy

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Abstract

The Lesche stability condition for the Shannon entropy [B. Lesche, J. Stat. Phys. 27, 419 (1982)], represents a fundamental test, for its experimental robustness, for systems obeying the Maxwell-Boltzmann statistical mechanics. Of course, this stability condition must be satisfied by any entropic functional candidate to generate non-conventional statistical mechanics. In the present effort we show that the $\kappa$-entropy, recently introduced in literature [G. Kaniadakis, Phys. Rev. E 66, 056125 (2002)], satisfies the Lesche stability condition.

Key words: Generalized entropies, Lesche stability.

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It is widely accepted that the Boltzmann-Gibbs statistical mechanics, while succeeding to describe systems in a stationary state characterized by an ergodicity consistent with the thermal equilibrium, fails to depict of the statistical properties of anomalous systems as found in surface growth, anomalous diffusion in porous media, gravitation, Lévy flights, fractals, turbulence physics, and economics [1,2].

Generalized statistical mechanics have been developed with the purpose of dealing with anomalous systems by means of the same mathematical tools used in the conventional statistical mechanics. This can be accomplished, in the case of trace-form entropies, with the introduction of an appropriate deformed logarithm. Examples of trace-form entropies are the Tsallis entropy [3] and the Abe entropy [4]. In [5,6] some general properties of the deformed logarithms, leading to a generalization of the Boltzmann-Gibbs distributions, have been studied.

Clearly, there are also entropies which are not of trace-form, e.g. the Rényi entropy [7], the Landsberg-Vedral entropy [8] and the escort entropy [9]. In
these cases the definition of these generalized entropies does not necessitate
the preliminary introduction of a deformed logarithm.

Recently, a non-conventional statistical mechanics has been introduced in
[10,11] which preserves the epistemological and thermodynamics structure of
the standard Boltzmann-Shannon-Gibbs theory. The proposed theory is based
on the following trace-form entropy ($k_b = 1$)

$$S_{\kappa}(p) = -\sum_{i=1}^{N} p_i \ln_{(\kappa)}(p_i) ,$$  \hfill (1)

being $p \equiv \{p_i, i = 1, \ldots, N\}$ a discrete probability distribution normalized as

$$\sum_{i=1}^{N} p_i = 1 ,$$  \hfill (2)

and the $\kappa$-logarithm is defined by

$$\ln_{(\kappa)}(x) = \frac{x^{\kappa} - x^{-\kappa}}{2\kappa} .$$  \hfill (3)

The $\kappa$-logarithm is a monotonically increasing and concave function for $\kappa \in (-1, 1)$, being $d\ln_{(\kappa)}(x)/dx > 0$ and $d^2\ln_{(\kappa)}(x)/dx^2 < 0$, and satisfies the relation

$$\ln_{(\kappa)}(x) = -\ln_{(\kappa)}\left(\frac{1}{x}\right) .$$  \hfill (4)

In the limit of $\kappa \to 0$ Eq. (3) reduces to the standard logarithm and conse-
quently Eq. (1) converges to the well-known Boltzmann-Shannon-Gibbs en-
tropy.

The inverse function of the $\kappa$-logarithm, namely the $\kappa$-exponential, is defined as

$$\exp_{(\kappa)}(x) = \left(\sqrt{1 + \kappa^2 x^2 + \kappa x}\right)^{1/\kappa} .$$  \hfill (5)

It is a positive, monotonically increasing and convex function which reduces
to the standard exponential for $\kappa \to 0$ and satisfies the relation

$$\exp_{(\kappa)}(x) \exp_{(\kappa)}(-x) = 1 .$$  \hfill (6)
After maximizing the $\kappa$-entropy given by Eq. (1) with the constraints (2) and
\begin{equation}
\sum_{i=1}^{N} E_i p_i = \langle E \rangle ,
\end{equation}
being $E_i$ the energy of the $i$-th level with probability of occupation $p_i$, we obtain the probability distribution
\begin{equation}
p_i = \alpha \exp_{\{\kappa\}} \left( -\frac{\beta}{\lambda} (E_i - \mu) \right) ,
\end{equation}
where $\beta = 1/T$ and $-\beta \mu$ are the Lagrange multipliers associated to the constraints given by Eqs. (2) and (7), while the constants $\alpha$ and $\lambda$ depend on the parameter $\kappa$ and are given respectively by $\alpha = [(1 - \kappa)/(1 + \kappa)]^{1/2}$ and $\lambda = \sqrt{T - \kappa^2}$.

We recall now briefly the physical motivations which justify the introduction of $\kappa$-statistical mechanics. For a more detailed discussion we refer the reader to [10].

First we remark that the $\kappa$-exponential and the $\kappa$-logarithm have the two following mathematical properties
\begin{eqnarray}
\exp_{\{\kappa\}} (x) \exp_{\{\kappa\}} (y) & = & \exp_{\{\kappa\}} (x \oplus \kappa y) , \\
\ln_{\{\kappa\}} (xy) & = & \ln_{\{\kappa\}} (x) \oplus \kappa \ln_{\{\kappa\}} (y) ,
\end{eqnarray}
where the $\kappa$-sum $x \oplus \kappa y$ is defined as
\begin{equation}
x \oplus \kappa y = x \sqrt{1 + \kappa^2 y^2} + y \sqrt{1 + \kappa^2 x^2} .
\end{equation}

The above properties in the limit $\kappa \to 0$ reduce to the well-known relations $\exp(x) \exp(y) = \exp(x + y)$ and $\ln(xy) = \ln(x) + \ln(y)$ being $x \oplus 0 y = x + y$. Of course the $\kappa$-sum plays a central role in the construction of the theory. In Ref. [10,11] it has been shown that the algebra constructed starting from the $\kappa$-sum induces a $\kappa$-analysis. It results that the $\kappa$-derivative admits as eigenstate the $\kappa$-exponential just as the ordinary exponential emerges as eigenstate of the ordinary derivative. At this point is evident that the physical motivation of the theory is strictly related with the physical mechanism originating the $\kappa$-deformation. In Ref. [10] it has been shown that the $\kappa$-sum emerges naturally within the Einstein special relativity. More precisely the $\kappa$-sum is related to the relativistic sum of the velocities
\begin{equation}
v_1 \oplus^c v_2 = \frac{v_1 + v_2}{1 + v_1 v_2/c^2} ,
\end{equation}
where ^c indicates the relativistic addition. In the limit $\kappa \to 0$ the $\kappa$-sum reduces to the Lorentz addition of the velocities.

\[ \square \]
through the relation

\[ p(v_1)^\kappa p(v_2) = p(v_1 \oplus c v_2), \]  

(13)

with \( \kappa = 1/m c \), where

\[ p(v_i) = \frac{m v_i}{\sqrt{1 - v_i^2/c^2}}, \]  

(14)

is the relativistic momentum of a particle of rest mass \( m \). Only in the classic limit \( c \to \infty \) the parameter \( \kappa \) approaches zero and both the \( \kappa \)-sum and the relativist sum of the velocities reduce to the ordinary sum. Thus the \( \kappa \)-deformation is originated from the finite value of light speed and results to be a purely relativistic effect.

Clearly when we consider a statistical system of relativistic particles the \( \kappa \)-sum continues to play an important role. Due to the finite values of the light speed \( c \neq \infty \), any signal and then any information propagates with a finite velocity in the system and therefore it results \( \kappa \neq 0 \).

The cosmic rays represents the most important example of relativistic statistical system which manifestly violates the Boltzmann statistics. In Ref. [10] it has been shown that the \( \kappa \)-statistics can predict very well the experimental cosmic ray spectrum. This is an important test for the theory because the cosmic rays spectrum has a very large extension (13 decades in energy and 33 decades in flux).

Let us consider now statistical systems (physical, natural, economical, etc.) in which is involved a limiting quantity, like the light speed in the relativistic particle system. For these systems where the information propagates with finite velocity, it is reasonable suppose that the \( \kappa \)-deformation can appear, so that the \( \kappa \)-statistics results to be the most appropriate theory to describe these systems.

An important question is if the \( \kappa \)-entropy can be associated to a physical system in an experimentally detectable state. The problem can be stated in the following way. Let \( \mathcal{O}(\{x\}) \) be a physical observable, depending on a set of parameters \( \{x\} \equiv \{x_i, i = 1, \cdots, N\} \) with \( \sup[\mathcal{O}(\{x\})] \) its maximal value achieved while varying \( \{x\} \). \( \mathcal{O}(\{x\}) \) is stable if, under a small change of the parameters \( \{x\} \to \{x'\} \), with \( |\{x'\} - \{x\}| < \delta \), the relative discrepancy

\[ \Delta = \frac{\mathcal{O}(\{x'\}) - \mathcal{O}(\{x\})}{\sup[\mathcal{O}(\{x\})]}, \]  

(15)

can be reduced to zero.

This question has been considered, for the first time, by Lesche [12] in 1982 both for the Boltzmann-Shannon-Gibbs entropy and for the Rényi entropy,
showing that the former is stable while the latter is unstable. Successively, Abe [13] showed that the Tsallis entropy is also consistent with the stability condition, while the Landsberg-Vedral entropy does not satisfy the stability condition. In [14] it has been shown that also the escort entropy does not satisfy the experimental robustness criterium. Finally, in [6] it has been considered the problem of Lesche stability condition from an arbitrary distribution function.

We remark that the Lesche inequality represents a condition for the stability of a system under particular sorts of perturbations. Clearly, the Rényi entropy, the Landsberg-Vedral entropy as well as the escort entropy, which fail to satisfy the Lesche stability condition, are worthy of investigation because present many other interesting properties.

The purpose of the present contribution is to show that the $\kappa$-entropy defined by Eq. (1) satisfies the Lesche stability condition and thus it can represent a well defined physical observable.

Let us begin by introducing the following auxiliary function

$$A_\kappa(p, s) = \sum_{i=1}^{N} \left[ p_i - \alpha \exp_{(\kappa)}(-s) \right]_+ ,$$

being $s$ a real positive parameter. The function $[x]_+$ is defined in terms of the Heaviside unit step function $\theta(x)$ through $[x]_+ = x \theta(x)$ so that it results $[x]_+ = x$ for $x > 0$ and $[x]_+ = 0$ for $x \leq 0$.

We remark that the function $A_\kappa(p, s)$ has been considered already in the literature. Originally, in Ref. [12] $A_\kappa(p, s)$ is defined using the $\exp(-s)$ at the place of $\alpha \exp_{(\kappa)}(-s)$. In Ref. [13] the function $A_\kappa(p, s)$ is defined starting from the Tsallis exponential $\exp_q(s)$. Very recently, in Ref. [15], it was considered a very large family of functions having the structure of (16) where at the place of $\alpha \exp_{(\kappa)}(-s)$ appears an arbitrary positive real function $f(s)$.

First we observe that from the relation $|[x]_+ - [y]_+| \leq |x - y|$ it follows that $\forall s$ holds the inequality

$$\left| A_\kappa(p, s) - A_\kappa(q, s) \right| \leq \|p - q\|_1 ,$$

being

$$\|p - q\|_1 = \sum_{i=1}^{N} |p_i - q_i| .$$

Second, following the procedure described in [12,13] we have that from the
definition of \( A_\kappa (p, s) \) follows
\[
\sum_{i=1}^{N} \left[ p_i - \alpha \exp_{\{\kappa\}} (-s) \right]_+ < \sum_{i=1}^{N} p_i ,
\]
(19)

which implies \( A_\kappa (p, s) < 1 \), being \( \{p_i\} \) a probability distribution normalized to one.

On the other hand, by making use of the relation \( \sum_i [x_i]_+ \geq [\sum_i x_i]_+ \) we have
\[
\sum_{i=1}^{N} \left[ p_i - \alpha \exp_{\{\kappa\}} (-s) \right]_+ \geq \left[ \sum_{i=1}^{N} \left( p_i - \alpha \exp_{\{\kappa\}} (-s) \right) \right]_+ = \left[ 1 - \alpha N \exp_{\{\kappa\}} (-s) \right]_+ ,
\]
(20)

and posing \( s \geq \ln_{\{\kappa\}} (N) \), being \( \alpha < 1 \), we can drop the notation \([\cdot]_+ \) in the r.h.s of Eq. (20) obtaining
\[
A_\kappa (p, s) \geq 1 - \alpha N \exp_{\{\kappa\}} (-s) .
\]
(21)

Combining Eqs (19) and (21) we obtain
\[
1 - \alpha N \exp_{\{\kappa\}} (-s) \leq A_\kappa (p, s) < 1 .
\]
(22)

Of course, this condition remains true for any other discrete probability distribution \( \{q_i\} \). Then, after changing the sign of Eq. (22), we obtain
\[
-1 < -A_\kappa (q, s) \leq -1 + \alpha N \exp_{\{\kappa\}} (-s) .
\]
(23)

Summing Eqs (22) and (23) follows
\[
- \alpha N \exp_{\{\kappa\}} (-s) < A_\kappa (p, s) - A_\kappa (q, s) < \alpha N \exp_{\{\kappa\}} (-s) ,
\]
(24)

and then
\[
\left| A_\kappa (p, s) - A_\kappa (q, s) \right| < \alpha N \exp_{\{\kappa\}} (-s) ,
\]
(25)

holding for \( s \geq \ln_{\{\kappa\}} (N) \). Eqs (17) and (25) express two very important properties of the function \( A_\kappa (p, s) \) which will be used in the following.

Now we show that the entropy \( S_\kappa (p) \) can be expressed in terms of the function \( A_\kappa (p, s) \). In fact, from the definition (1) and taking into account the identity \( \alpha^\kappa (1 + \kappa) = \lambda \), we have
\[ S_\kappa(p) = -\sum_{i=1}^{N} p_i \left( \frac{p_i^\kappa - p_i^{-\kappa}}{2\kappa} \right) \]
\[ = -\frac{\lambda}{2\kappa} \sum_{i=1}^{N} p_i \left[ \frac{1}{1+\kappa} \left( \frac{p_i}{\alpha} \right)^\kappa - \frac{1}{1-\kappa} \left( \frac{p_i}{\alpha} \right)^{-\kappa} \right] \]
\[ = -\frac{\lambda}{2\kappa} \sum_{i=1}^{N} p_i \left[ \left( \frac{p_i}{\alpha} \right)^\kappa - \left( \frac{p_i}{\alpha} \right)^{-\kappa} \right] \]
\[ + \frac{\lambda}{2} \sum_{i=1}^{N} p_i \left[ \frac{1}{1+\kappa} \left( \frac{p_i}{\alpha} \right)^\kappa + \frac{1}{1-\kappa} \left( \frac{p_i}{\alpha} \right)^{-\kappa} \right] \]
\[ = \lambda \sum_{i=1}^{N} \int_{0}^{L_i} p_i \, ds + \alpha \lambda \sum_{i=1}^{N} \int_{0}^{\infty} x \frac{d}{dx} \ln_{\kappa}(x) \, dx , \] (26)

where \( L_i = -\ln_{\kappa}(p_i/\alpha) \). After introducing the substitution \( s = -\ln_{\kappa}(x) \) we obtain

\[ S_\kappa(p) = \lambda \sum_{i=1}^{N} \int_{0}^{L_i} p_i \, ds + \lambda \sum_{i=1}^{N} \int_{0}^{\infty} \alpha \exp_{\kappa}(-s) \, ds \]
\[ = \lambda \int_{0}^{\infty} ds - \lambda \sum_{i=1}^{N} \int_{L_i}^{\infty} \left[ p_i - \alpha \exp_{\kappa}(-s) \right] \, ds , \] (27)

and finally we can write the entropy \( S_\kappa(p) \) in terms of the function \( A_\kappa(p, s) \) as follows

\[ S_\kappa(p) = \lambda \int_{-1/\lambda}^{\infty} \left[ 1 - A_\kappa(p, s) \right] \, ds - 1 . \] (28)

Using Eq. (28) the difference of \( \kappa \)-entropy for two probability distributions, namely \( p = \{p_i\} \) and \( q = \{q_i\} \) can be written

\[ \left| S_\kappa(p) - S_\kappa(q) \right| = \lambda \int_{-1/\lambda}^{\infty} \left[ A_\kappa(p, s) - A_\kappa(q, s) \right] \, ds \]
\[ \leq \lambda \int_{-1/\lambda}^{\infty} \left| A_\kappa(p, s) - A_\kappa(q, s) \right| \, ds . \] (29)

After splitting the integration interval in two parts, namely \([-1/\lambda, +\infty) = [-1/\lambda, \ell) \cup [\ell, +\infty)\), Eq. (29) becomes
\[
S_\kappa(p) - S_\kappa(q) \leq \lambda \int_{-1/\lambda}^{\ell} |A_\kappa(p, s) - A_\kappa(q, s)| \, ds + \lambda \int_{\ell}^{\infty} |A_\kappa(p, s) - A_\kappa(q, s)| \, ds ,
\]

where the splitting point \( \ell \) will be defined below. Now, using inequality (17) in the first integral of Eq. (30) and inequality (25) in the second integral of Eq. (30), we obtain:

\[
|S_\kappa(p) - S_\kappa(q)| \leq \lambda G_\kappa(\ell) ,
\]

where the function \( G_\kappa(\ell) \) is defined by

\[
G_\kappa(\ell) = \int_{-1/\lambda}^{\ell} \|p - q\|_1 \, ds + N \alpha \int_{\ell}^{\infty} \exp(\kappa s) \, ds .
\]

After performing the integrations, Eq. (32) can be written as

\[
G_\kappa(\ell) = \alpha N \left\{ \frac{\exp(\kappa \ell)^{1+\kappa}}{1 + \kappa} + \frac{\exp(\kappa \ell)^{1-\kappa}}{1 - \kappa} \right\} + \|p - q\|_1 \left( \ell + \frac{1}{\lambda} \right) .
\]

Let us choose for the parameter \( \ell \) the value \( \ell_0 \) which minimizes the function \( G_\kappa(\ell) \), therefore after posing

\[
\left. \frac{dG_\kappa(\ell)}{d\ell} \right|_{\ell=\ell_0} = 0 ,
\]

we obtain

\[
\ell_0 = \ln(\kappa) \left( \frac{\alpha N}{\|p - q\|_1} \right) .
\]

The condition \( s \geq \ln(\kappa)(N) \), holding for Eq. (25), now requires \( \ell_0 \geq \ln(\kappa)(N) \) and then from Eq. (35) we obtain

\[
\|p - q\|_1 \leq \alpha .
\]

By posing \( \ell = \ell_0 \) in the expression of \( G_\kappa(\ell) \), the inequality (31) becomes
\[ |S_\kappa(p) - S_\kappa(q)| \leq \|p - q\|_1 \left[ \frac{1}{2} \left( \frac{\|p - q\|_1}{N} \right) ^\kappa + \frac{1}{2} \left( \frac{\|p - q\|_1}{N} \right) ^{-\kappa} - \lambda \ln_\{\kappa\} \left( \frac{\|p - q\|_1}{\alpha N} \right) + 1 \right], \]  

(37)

and can be written in the compact form

\[ |S_\kappa(p) - S_\kappa(q)| \leq \|p - q\|_1 \left[ \ln_\{\kappa\} \left( \frac{N}{\|p - q\|_1} \right) + 1 \right]. \]  

(38)

At this point we observe that \( f(x) = -x \ln_\{\kappa\} (x) \) is a positive and increasing function in the interval \([0, \alpha]\). Furthermore we pose \( \|p - q\|_1 < \delta \leq \alpha \), consistently with Eq. (36).

By introducing the maximum value of \( S_\kappa(p) \) corresponding to the uniform distribution \( p \equiv \{p_i = 1/N, \ i = 1, \ldots, N\} \)

\[ S^{\text{max}}_\kappa = \ln_\{\kappa\} (N), \]  

(39)

from Eq. (38) we can evaluate the entropy relative discrepancy as

\[ \left| \frac{S_\kappa(p) - S_\kappa(q)}{S^{\text{max}}_\kappa} \right| \leq \frac{\delta^{1-\kappa}}{1 - N^{-2\kappa}} - \frac{\delta^{1+\kappa}}{N^{2\kappa - 1}} + \frac{2 \kappa \delta}{N^{2\kappa - 1 - \kappa}}. \]  

(40)

Now, as customary, we evaluate the inequality given by Eq. (40) in the thermodynamic limit \( N \to \infty \) and obtain

\[ \left| \frac{S_\kappa(p) - S_\kappa(q)}{S^{\text{max}}_\kappa} \right| \leq \delta^{1-|\kappa|}. \]  

(41)

Finally, by introducing in Eq. (41) the positive quantity \( \epsilon = \delta^{1-|\kappa|} \), we obtain the Lesche inequality

\[ \left| \frac{S_\kappa(p) - S_\kappa(q)}{S^{\text{max}}_\kappa} \right| \leq \epsilon. \]  

(42)

Clearly in Eq. (42), \( \epsilon \) is a continuous function of \( \delta \), approaching 0 for \( \delta \to 0 \) when \(-1 < \kappa < 1\) and therefore the \( \kappa \)-entropy \( S_\kappa \) satisfies the stability condition.
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