ASYMPTOTIC BEHAVIOR OF RANDOM FITZHUGH-NAGUMO SYSTEMS DRIVEN BY COLORED NOISE

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Abstract. In this paper, we prove the existence and uniqueness of random attractors for the FitzHugh-Nagumo system driven by colored noise with a nonlinear diffusion term. We demonstrate that the colored noise is much easier to deal with than the white noise for studying the pathwise dynamics of stochastic systems. In addition, we show the attractors of the random FitzHugh-Nagumo system driven by a linear multiplicative colored noise converge to that of the corresponding stochastic system driven by a linear multiplicative white noise.

1. Introduction. This paper is concerned with the asymptotic behavior of the random non-autonomous FitzHugh-Nagumo system driven by a colored noise in a bounded domain $\Omega$:

$$\begin{cases}
\frac{\partial u}{\partial t} - \Delta u + \alpha v = f(t, x, u) + g(t, x) + R(t, x, u)\zeta_\delta(\theta_t \omega), \\
\frac{\partial v}{\partial t} + \sigma v - \beta u = h(t, x) + \lambda \zeta_\delta(\theta_t \omega)v,
\end{cases}$$

where $\alpha, \beta, \sigma$ and $\lambda$ are positive constants, $g$ and $h$ are in $L^2_{\text{loc}}(\mathbb{R}, L^2(\Omega))$, $f$ and $R$ are nonlinear functions which satisfy certain dissipative conditions.

Given $\delta > 0$, the process $\zeta_\delta$ in (1) is an Ornstein-Uhlenbeck (O-U) process (also known as a colored noise). To describe this process, we introduce a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \{ \omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0 \}$ with the open compact topology, $\mathcal{F}$ is its Borel $\sigma$-algebra, and $\mathbb{P}$ is the Wiener measure. The classical transformation $\{ \theta_t \}_{t \in \mathbb{R}}$ on $\Omega$ is given by $\theta_t \omega(\cdot) = \omega(t + \cdot) - \omega(t)$ for $\omega \in \Omega$. Let $W$ be a two-sided real-valued Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$. For each $\delta > 0$, define a random variable $\zeta_\delta : \Omega \rightarrow \mathbb{R}$ by $\zeta_\delta(\omega) = \frac{1}{\delta} \int_{-\infty}^{0} e^{\frac{t}{\delta}} dW$. Then the process $O_\delta(t, \omega) = \zeta_\delta(\theta_t \omega)$ is called an O-U process, which is a stationary Gaussian process with $\mathbb{E}(\zeta_\delta) = 0$ and is the unique stationary solution of the stochastic equation

$$d\zeta_\delta + \frac{1}{\delta} \zeta_\delta dt = \frac{1}{\delta} dW.$$
The O-U process $O_\delta$ is also called a colored noise because its power spectrum is not flat compared to the white noise, see for instance [4, 25, 42, 43, 55]. Furthermore, the O-U process is the only existing Markovian Gaussian colored noise, see, e.g., [17] and [43].

In general, the Wiener process $W$ can be chosen as a stochastic process to represent the position of the Brownian particle, but the velocity of the particle cannot be obtained from the Wiener process because of the nowhere differentiability of the sample paths of $W$. In such a case, the O-U process was originally constructed in [42, 55] to approximately describe the stochastic behavior of the velocity and hence it can be further used to determine the position of the particle. On the other hand, as demonstrated in [43], in many complex systems, stochastic fluctuations are actually correlated and hence should be modeled by colored noise rather than white noise.

Indeed, one of the most crucial issues in studying stochastic dynamics arises from the modeling of the random forcing. To represent such random forcing, we need to consider two time scales: the time scale $\tau_d$ of the deterministic system and the time scale $\tau_r$ of the random forcing. The stochastic forcing is modeled in different ways based on the ratio of $\tau_r/\tau_d$. If $\tau_r/\tau_d \gg 1$, the dynamical system is very slow with respect to the temporal variability of its random drivers, and hence the random forcing could be modeled as white noise. If $\tau_r/\tau_d \simeq 1$, then the dynamics of the system is sensitive to the autocorrelation of the random forcing, and hence the random forcing should be modeled by colored noise. Based on these considerations, the colored noise has been used in many publications to study the dynamics of physical and biological systems, see, e.g., [4, 25, 34, 35, 38, 42, 43, 55] and the references therein.

In this paper, we will study the dynamics of system (1) driven by colored noise. For a wide class of nonlinear functions $R$, we will prove the random system (1) is pathwise well-posed in $L^2(O) = L^2(O) \times L^2(O)$ and hence generates a continuous non-autonomous cocycle. Moreover, this cocycle possesses a unique tempered random attractor in $L^2(O)$ (see Theorem 2.8). This is in sharp contrast with the corresponding stochastic system driven by a white noise:

\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u + \alpha v &= f(t, x, u) + g(t, x) + R(t, x, u) \circ \frac{dW}{dt}, \\
\frac{\partial v}{\partial t} + \sigma v - \beta u &= h(t, x) + \lambda v \circ \frac{dW}{dt},
\end{align*}
\]

where the symbol $\circ$ indicates the system is understood in the sense of Stratonovich’s integration. As far as the authors are aware, currently, we can only define a random dynamical system for (2) when the diffusion term $R(\cdot, \cdot, u)$ is a linear function in $u \in \mathbb{R}$. In other words, for a general nonlinear function $R$, we are unable to define a random dynamical system for (2) and hence cannot investigate the dynamics of the stochastic equations by the random dynamical systems approach. This is the reason why there is no result available in the literature on the existence of random attractors for (2) with a nonlinear function $R$. Our results indicate that the colored noise is much easier to handle than the white noise for studying pathwise dynamics of stochastic equations.

In the present paper, when $R(t, x, u) \equiv u$, we will also investigate the limiting behavior of system (1) as $\delta \to 0$. In this special case, we will show the solutions of (1) converges to that of (2) in $L^2(O)$ as $\delta \to 0$ (see Corollary 3), and the random attractors of (1) converge to that of (2) in terms of the Hausdorff semi-distance in $L^2(O)$ as $\delta \to 0$ (see Theorem 4.10). This demonstrates that, when $R(t, x, u) \equiv u,$
the random system (1) is is closely related to the stochastic system (2) for sufficiently small $\delta > 0$.

The FitzHugh-Nagumo system was proposed as a mathematical model to describe the signal transmission across axons in neurobiology in [20, 41]. The goal of this paper is to study random attractors of this system driven by colored noise. The concept of random attractors was introduced in [18, 21, 44] and further studied in [6, 7, 8, 9, 11, 12, 13, 15, 16, 18, 22, 23, 24, 26, 32, 37, 40, 50, 51] for the autonomous stochastic equations; and in [2, 14, 19, 27, 28, 52, 53, 54] for the non-autonomous stochastic equations.

We mention that colored noise has already been extensively used in the literature to study the solutions of random equations, see e.g., [1, 25, 34, 35, 38, 43] and the references therein. However, there is no result available regarding the existence of random attractors for equations like (1) driven by colored noise. We also remark that the Wong-Zakai approximations can be used to study the solutions of random equations, see e.g., [1, 25, 34, 35, 38, 43] for more details.

This paper is organized as follows. In the next section, we prove the existence and uniqueness of tempered random attractors for system (1) with a nonlinear diffusion term $R$. We then prove the existence of such attractors for the stochastic system (2) when $R(t, x, u) \equiv u$. In the last section, we prove the convergence of solutions and random attractors of system (1) with $R(t, x, u) \equiv u$, as $\delta \to 0$.

Hereafter, we will denote the inner product and norm of $L^2(\mathcal{O})$ by $(\cdot, \cdot)$ and $\| \cdot \|$, respectively. We also write $L^2(\mathcal{O}) = L^2(\mathcal{O}) \times L^2(\mathcal{O})$ and $\|w\|_{L^2(\mathcal{O})}^2 = \|u\|^2 + \|v\|^2$ for $w = (u, v) \in L^2(\mathcal{O})$. Similarly, we write $F(t, x) = (g(t, x), h(t, x))$ for $t \in \mathbb{R}$ and $x \in \mathcal{O}$, and $\|F(t)\|^2 = \|g(t)\|^2 + \|h(t)\|^2$. The letters $c$ and $c_i$ are used for positive constants whose values may change from line to line.

2. Attractors of random FitzHugh-Nagumo systems. In this section, we study the dynamics of the random FitzHugh-Nagumo system driven by a colored noise. We first define a continuous non-autonomous cocycle for the system, and then prove the existence of pullback random attractors in $L^2(\mathcal{O})$ for a wide class of nonlinear functions $f$ and $R$.

2.1. Definition of continuous cocycles. Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^n$ and $\tau, \delta \in \mathbb{R}$ with $0 < \delta \leq 1$. Consider the following non-autonomous random FitzHugh-Nagumo system defined in $\mathcal{O}$ with homogeneous Dirichlet boundary condition and initial condition:

$$\begin{align*}
\frac{\partial u}{\partial t} - \Delta u + \alpha v &= f(t, x, u) + g(t, x) + R(t, x, u)\zeta_\delta(\theta_\omega), \quad x \in \mathcal{O}, t > \tau, \\
\frac{\partial v}{\partial t} + \sigma v - \beta u &= h(t, x) + \lambda\zeta_\delta(\theta_\omega) v, \quad x \in \mathcal{O}, t > \tau, \\
u(t, x) &= v(t, x) = 0, \quad x \in \partial\mathcal{O}, t > \tau, \\
u(\tau, x) &= u_\tau(x), \quad v(\tau, x) = v_\tau(x), \quad x \in \mathcal{O},
\end{align*}$$

where $g, h \in L^2_{loc}(\mathbb{R}, L^2(\mathcal{O}))$, $\zeta_\delta(\theta_\omega)$ is a colored noise and $f, R : \mathbb{R} \times \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ are continuous functions such that for all $t, s \in \mathbb{R}$ and $x \in \mathcal{O}$,

(G) $f(t, x, s)$ is differentiable in $x, s$ and there exist positive constants $p > 2$, $\alpha_1, \alpha_2$ and $\alpha_3$ such that

$$\begin{align*}
\text{(G1)} & \quad |f(t, x, s)| \leq \alpha_1 |s|^p + \psi_1(t, x), \quad \text{with } \psi_1 \in L^\infty(\mathbb{R}, L^1(\mathcal{O})), \\
\text{(G2)} & \quad |f(t, x, s)| \leq \alpha_2 |s|^{p-1} + \psi_2(t, x), \quad \text{with } \psi_2 \in L_{loc}^p(\mathbb{R}, L^p(\mathcal{O})) \text{ and } \frac{1}{p_1} + \frac{1}{p} = 1,
\end{align*}$$
(G3) \( \frac{\partial f}{\partial x}(t, x, s) \leq -\alpha_3|s|^{p-2} + \psi_3(t, x), \) with \( \psi_3 \in L^\infty_0(\mathbb{R}, L^\infty(\mathcal{O})) \),

(G4) \( \frac{\partial f}{\partial x}(t, x, s) \leq \psi_4(t, x), \) with \( \psi_4 \in L^2_0(\mathbb{R}, L^2(\mathcal{O})) \).

(H) \( R(t, x, s) \) is differentiable in \( x, s \) and there exist positive constants \( 2 \leq q < p \), \( \alpha_4 \) and \( \alpha_5 \) such that

(H1) \( |R(t, x, s)| \leq \alpha_4 |s|^{q-1} + \psi_5(t, x), \) with \( \psi_5 \in L^\infty(\mathbb{R}, L^p(\mathcal{O})) \),

(H2) \( \frac{\partial R}{\partial x}(t, x, s) \leq \alpha_5 |s|^{q-2} + \psi_6(t, x), \) with \( \psi_6 \in L^\infty_0(\mathbb{R}, L^\infty(\mathcal{O})) \),

(H3) \( \frac{\partial R}{\partial x}(t, x, s) \leq \psi_7(t, x), \) with \( \psi_7 \in L^2_0(\mathbb{R}, L^2(\mathcal{O})) \).

Recall that there exists a \( \{\theta_t\}_{t \in \mathbb{R}} \)-invariant subset of full measure (see, e.g., [5]), which is still denoted by \( \Omega \), such that for \( \omega \in \Omega \),

\[
\lim_{t \to \pm \infty} \frac{\omega(t)}{t} = 0. \tag{4}
\]

Throughout this paper, for every \( \omega \in \Omega \) and \( \delta \in (0, 1] \), we write

\[
\zeta_\delta(\omega) = \frac{1}{\delta} \int_{-\infty}^0 e^{\delta s}dW = -\frac{1}{\delta^2} \int_{-\infty}^0 e^{\delta s}\omega(s)ds. \tag{5}
\]

Then \( \zeta_\delta(\theta_t \omega) \) is the so-called O-U process (also known as colored noise) with \( \mathbb{E}(\zeta_\delta) = 0 \). In addition, this process has the following properties.

**Lemma 2.1.** (i). For every \( \omega \in \Omega \), the mapping \( t \to \zeta_\delta(\theta_t \omega) \) is continuous, and for every \( 0 < \delta \leq 1 \),

\[
\lim_{t \to \pm \infty} \frac{|\zeta_\delta(\theta_t \omega)|}{t} = 0. \tag{6}
\]

(ii). For every \( \omega \in \Omega \),

\[
\lim_{t \to \pm \infty} \frac{1}{t} \int_0^t \zeta_\delta(\theta_s \omega)ds = 0 \quad \text{uniformly for } 0 < \delta \leq 1. \tag{7}
\]

**Proof.** (i). By (5) we get

\[
\zeta_\delta(\theta_t \omega) = -\frac{1}{\delta^2} \int_{-\infty}^0 e^{\delta(s + t) - \omega(t)}ds = -\frac{1}{\delta^2} \int_{-\infty}^0 e^{\delta s}\omega(s + t)ds + \frac{1}{\delta} \omega(t), \tag{8}
\]

and hence

\[
\frac{\zeta_\delta(\theta_t \omega)}{t} = -\frac{1}{\delta t} \int_{-\infty}^0 e^{\delta s}\omega(s + t)ds + \frac{\omega(t)}{\delta t}. \tag{9}
\]

By (4), we find that for every \( \varepsilon > 0 \), there exists \( T = T(\omega, \varepsilon) > 0 \) such that for all \( |s| \geq T \),

\[
|\omega(s)| \leq \varepsilon |s|. \tag{10}
\]

We first prove \( \lim_{t \to -\infty} \frac{1}{t} \int_{-\infty}^0 e^{\delta s}\omega(s + t)ds = 0. \) By (10), we get for all \( t \leq -T \) and \( s \leq 0 \),

\[
|\omega(s + t)| \leq \varepsilon |s + t|,
\]

which implies that for all \( t \leq -T \),

\[
\left| \frac{1}{t} \int_{-\infty}^0 e^{\delta s}\omega(s + t)ds \right| \leq \frac{\varepsilon}{|t|} \int_{-\infty}^0 e^{\delta |s + t|}ds \leq \frac{\varepsilon}{|t|} \int_{-\infty}^0 e^{\delta |s|}ds + \delta \varepsilon,
\]

and hence

\[
\lim_{t \to -\infty} \frac{1}{t} \int_{-\infty}^0 e^{\delta s}\omega(s + t)ds = 0. \tag{11}
\]
We now prove \( \lim_{t \to +\infty} \frac{1}{t} \int_{-\infty}^{0} e^{\frac{t}{\delta}} \omega(s + t)ds = 0 \). Note that
\[
\frac{1}{t} \int_{-\infty}^{0} e^{\frac{t}{\delta}} \omega(s + t)ds = \frac{1}{t} \int_{T-t}^{0} e^{\frac{t}{\delta}} \omega(s + t)ds + \frac{1}{t} \int_{-\infty}^{T-t} e^{\frac{t}{\delta}} \omega(s + t)ds.
\]
Since for \( t \geq T \) and \( s \in (T-t, 0) \), we have \( s + t \geq T \), which together with (10) implies, for \( t \geq T \),
\[
\left| \frac{1}{t} \int_{T-t}^{0} e^{\frac{t}{\delta}} \omega(s + t)ds \right| \leq \frac{\varepsilon}{t} \int_{T-t}^{0} e^{\frac{t}{\delta}} |s + t|ds \leq \frac{\varepsilon}{t} \int_{-\infty}^{0} e^{\frac{t}{\delta}} |s|ds + \varepsilon. \tag{12}
\]
By (4) again, we find that there exists \( T_0 = T_0(\omega) > 0 \) such that
\[
|\omega(t)| \leq |t|, \quad \forall |t| \geq T_0. \tag{13}
\]
On the other hand, by the continuity of \( \omega \), there exists \( c_1(\omega) > 0 \) such that
\[
|\omega(t)| \leq c_1(\omega), \quad \forall |t| \leq T_0. \tag{14}
\]
Combining (13) with (14), we get
\[
|\omega(t)| \leq c_1(\omega) + |t|, \quad \forall t \in \mathbb{R}. \tag{15}
\]
Now, due to (15), we get, for \( t \geq T \),
\[
\left| \frac{1}{t} \int_{-\infty}^{T-t} e^{\frac{t}{\delta}} \omega(s + t)ds \right| \leq \frac{1}{t} \int_{-\infty}^{T-t} e^{\frac{t}{\delta}} (c_1(\omega) + |s| + t)ds
\]
\[
\leq \frac{\delta c_1(\omega)}{t} e^{\frac{t}{\delta}} + \delta e^{\frac{t}{\delta}} + \frac{1}{t} \int_{-\infty}^{0} e^{\frac{t}{\delta}} |s|ds. \tag{16}
\]
Then, by (12) and (16), we obtain
\[
\lim_{t \to +\infty} \frac{1}{t} \int_{-\infty}^{0} e^{\frac{t}{\delta}} \omega(s + t)ds = 0, \tag{17}
\]
which along with (11) implies (6).

(ii). By (8) we have
\[
\zeta_{\delta}(\theta \omega) = -\frac{1}{\delta^2} e^{-\frac{t}{\delta}} \int_{-\infty}^{t} e^{\frac{t}{\delta}} \omega(r)dr + \frac{1}{\delta} \omega(t). \tag{18}
\]
This implies that
\[
\frac{1}{t} \int_{0}^{t} \zeta_{\delta}(\theta \omega)ds = \frac{1}{t} \int_{0}^{t} \left( -\frac{1}{\delta^2} e^{-\frac{t}{\delta}} \int_{-\infty}^{s} e^{\frac{t}{\delta}} \omega(r)dr + \frac{1}{t\delta} \int_{0}^{t} \omega(s)ds \right)ds
\]
\[
= \frac{1}{t^2} \int_{-\infty}^{t} e^{\frac{\tau}{\delta}} \omega(\tau)dr - \frac{1}{t\delta} \int_{-\infty}^{0} e^{\frac{\tau}{\delta}} \omega(\tau)dr. \tag{18}
\]
We first show \( \frac{1}{t} \int_{0}^{t} \zeta_{\delta}(\theta \omega)ds \to 0 \) as \( t \to -\infty \), uniformly for \( 0 < \delta \leq 1 \). By (18) we find
\[
\frac{1}{t} \int_{0}^{t} \zeta_{\delta}(\theta \omega)ds = \frac{1}{t} \int_{0}^{t} e^{\delta \omega(t + \delta s)}ds - \frac{1}{t} \int_{-\infty}^{0} e^{\delta \omega(\delta s)}ds. \tag{19}
\]
Then for \( t \leq -T, s \leq 0 \) and \( 0 < \delta \leq 1 \), we obtain that \( t + \delta s \leq t \leq -T \) and hence by (10),
\[
|\omega(t + \delta s)| < \varepsilon |t + \delta s| \leq \varepsilon |t| + \varepsilon \delta |s| \leq \varepsilon |t| + \varepsilon |s|. \]
So, we get, for all $t \leq -T$ and $0 < \delta \leq 1$,
\[
\left| \frac{1}{T} \int_{-\infty}^{0} e^{*} \omega(t + \delta s)ds \right| \leq \frac{\varepsilon}{|t|} \int_{-\infty}^{0} e^{*}(|t| + |s|)ds \leq \varepsilon + \frac{\varepsilon}{|t|} \int_{-\infty}^{0} e^{*}|s|ds. \tag{20}
\]

Now, choose $T_1$ large enough such that
\[
\frac{1}{T_1} \int_{-\infty}^{0} e^{*}|s|ds < 1.
\]

Let $T_2(\varepsilon, \omega) = \max\{T, T_1\}$. Then for all $t \leq -T_2$ and $0 < \delta \leq 1$, we get from (20) that
\[
\left| \frac{1}{T} \int_{-\infty}^{0} e^{*} \omega(t + \delta s)ds \right| < 2\varepsilon. \tag{21}
\]

Furthermore, by (15), we see that for $0 < \delta \leq 1$,
\[
\left| \frac{1}{T} \int_{-\infty}^{0} e^{*} \omega(\delta s)ds \right| \leq \frac{1}{|T|} \int_{-\infty}^{0} e^{*}(c_1(\omega) + |s\delta|)ds \leq \frac{c_1(\omega)}{|T|} + \frac{1}{|T|} \int_{-\infty}^{0} e^{*}|s|ds,
\]
which implies that there exists $T_3 = T_3(\varepsilon, \omega) > 0$ such that for all $|t| \geq T_3$,
\[
\left| \frac{1}{T} \int_{-\infty}^{0} e^{*} \omega(\delta s)ds \right| < \varepsilon. \tag{22}
\]

Now, by (21) and (22), we get from (19) that for all $t \leq \min\{-T_2, -T_3\}$ and $0 < \delta \leq 1$,
\[
\left| \frac{1}{T} \int_{0}^{t} \zeta_\delta(\theta_s \omega)ds \right| < 3\varepsilon,
\]
which implies that $\frac{1}{t} \int_{0}^{t} \zeta_\delta(\theta_s \omega)ds \to 0$ as $t \to -\infty$, uniformly for $0 < \delta \leq 1$. We remain to show that $\frac{1}{t} \int_{0}^{t} \zeta_\delta(\theta_s \omega)ds \to 0$ as $t \to +\infty$ uniformly for $0 < \delta \leq 1$. By (18), we know
\[
\frac{1}{t} \int_{0}^{t} \zeta_\delta(\theta_s \omega)ds = \frac{1}{t \delta} \int_{-\infty}^{t} e^{\frac{s-t}{\delta}} \omega(r)dr - \frac{1}{t} \int_{-\infty}^{0} e^{*} \omega(\delta s)ds
\]
\[
= \frac{1}{t \delta} \int_{-\infty}^{T} e^{\frac{s-t}{\delta}} \omega(r)dr + \frac{1}{t \delta} \int_{T}^{t} e^{\frac{s-t}{\delta}} \omega(r)dr - \frac{1}{t} \int_{-\infty}^{0} e^{*} \omega(\delta s)ds. \tag{23}
\]

Due to (10), we get for $t \geq T$ and $0 < \delta \leq 1$,
\[
\left| \frac{1}{t \delta} \int_{T}^{t} e^{\frac{s-t}{\delta}} \omega(r)dr \right| \leq \frac{\varepsilon}{t \delta} \int_{T}^{t} e^{\frac{s-t}{\delta}} r dr \leq \frac{\varepsilon}{t} \int_{T}^{t} e^{*}(t + \delta s)ds
\]
\[
\leq \frac{\varepsilon}{t} \int_{T}^{t} e^{*}(t + |s|)ds \leq \varepsilon + \frac{\varepsilon}{t} \int_{-\infty}^{0} e^{*}|s|ds. \tag{24}
\]

Then for $t \geq \max\{T, T_1\}$, we get for all $0 < \delta \leq 1$,
\[
\left| \frac{1}{t \delta} \int_{T}^{t} e^{\frac{s-t}{\delta}} \omega(r)dr \right| < 2\varepsilon. \tag{25}
\]

Note that
\[
\frac{1}{t \delta} \int_{-\infty}^{T} e^{\frac{s-t}{\delta}} \omega(r)dr = \frac{1}{t} \int_{-\infty}^{T} e^{*} \omega(t + \delta s)ds,
\]
which along with (15) implies that for all $t \geq T$ and $0 < \delta \leq 1$,
\[
\left| \frac{1}{t \delta} \int_{-\infty}^{T} e^{\frac{r - t}{\delta}} \omega(r) dr \right| \leq \frac{1}{t} \int_{-\infty}^{T} e^{c_1(\omega) + |t + \delta s|} ds
\]
\[
\leq \frac{1}{t} \int_{-\infty}^{T} e^{c_1(\omega) + |t + s|} ds \leq \frac{c_1(\omega)}{t} e^{T - t} + e^{T - t} + \frac{1}{t} \int_{-\infty}^{0} e^{s} |s| ds.
\]  

(26)

Since $0 < \delta \leq 1$, we get $1 \leq \frac{1}{\delta} < \infty$. Then for all $t \geq T$ and $0 < \delta \leq 1$, we have $\frac{T - t}{\delta} \leq T - t$, and hence, by (26),
\[
\left| \frac{1}{t \delta} \int_{-\infty}^{T} e^{\frac{r - t}{\delta}} \omega(r) dr \right| \leq \frac{c_1(\omega)}{T} e^{T - t} + e^{T - t} + \frac{1}{t} \int_{-\infty}^{0} e^{s} |s| ds,
\]
which shows that there exists $T_4 = T_4(\epsilon, \omega) > 0$ such that for all $t \geq T_4$,
\[
\left| \frac{1}{t \delta} \int_{-\infty}^{T} e^{\frac{r - t}{\delta}} \omega(r) dr \right| < 2 \epsilon.
\]  

(27)

Now, by (22), (25) and (27), we obtain from (23) that there exists $T_5 = T_5(\epsilon, \omega) > 0$ such that for all $t \geq T_5$ and $0 < \delta \leq 1$,
\[
\left| \frac{1}{t} \int_{0}^{t} \zeta_\delta(\theta_s \omega) ds \right| < 5 \epsilon,
\]
which implies that $\frac{1}{t} \int_{0}^{t} \zeta_\delta(\theta_s \omega) ds \to 0$ as $t \to +\infty$ uniformly for $0 < \delta \leq 1$. This completes the proof.

For later sections, we now prove the following uniform convergence of colored noise on a finite interval.

**Lemma 2.2.** Let $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $T > 0$. Then for every $\epsilon > 0$, there exists $\delta_0 = \delta_0(\tau, \omega, T, \epsilon) > 0$ such that for all $0 < \delta < \delta_0$ and $t \in [\tau, \tau + T]$,
\[
\left| \int_{0}^{t} \zeta_\delta(\theta_s \omega) ds - \omega(t) \right| < \epsilon.
\]  

(28)

**Proof.** First, by (18), we have
\[
\int_{0}^{t} \zeta_\delta(\theta_s \omega) ds = \frac{1}{\delta} \int_{-\infty}^{T} e^{\frac{r - t}{\delta}} \omega(r) dr - \frac{1}{\delta} \int_{-\infty}^{0} e^{\frac{r}{\delta}} \omega(r) dr
\]
\[
= \int_{-\infty}^{0} e^{\omega(t + \delta s) ds} - \int_{-\infty}^{0} e^{\omega(s \delta) ds}.
\]  

(29)

We first prove
\[
\lim_{\delta \to 0} \int_{-\infty}^{0} e^{\omega(s \delta) ds} = 0.
\]  

(30)

Note that for every $s \leq 0$,
\[
\lim_{\delta \to 0} e^{\omega(s \delta)} = 0.
\]  

(31)

By (15), we get
\[
|e^{\omega(s \delta)}| \leq e^{c_1(\omega) + |s \delta|} \leq e^{c_1(\omega) + |s|}.
\]  

(32)
Since the following integral is convergent, i.e.,
\[
\int_{-\infty}^{0} e^{s}(c_{1}(\omega) + |s|)ds < \infty. \tag{33}
\]
Then (30) follows from (31)-(33) and the Lebesgue Dominated Convergence Theorem. We now prove
\[
\lim_{\delta \to 0} \int_{-\infty}^{0} e^{s}(\omega(t + \delta s)ds = \omega(t), \text{ uniformly for } t \in [\tau, \tau + T]. \tag{34}
\]
Note that
\[
\int_{-\infty}^{0} e^{s}(\omega(t + \delta s)ds - \omega(t)) = \int_{-\infty}^{0} e^{s}(\omega(t + \delta s) - \omega(t))ds
\]
\[
= \int_{-\infty}^{-T_{0}} e^{s}(\omega(t + \delta s) - \omega(t))ds + \int_{T_{0}}^{0} e^{s}(\omega(t + \delta s) - \omega(t))ds,
\tag{35}
\]
where \(T_{0}\) is a positive number to be specified later. First, by (15), we get
\[
\left| \int_{-\infty}^{-T_{0}} e^{s}(\omega(t + \delta s) - \omega(t))ds \right| \leq \int_{-\infty}^{-T_{0}} e^{s}(|\omega(t + \delta s)| + |\omega(t)|)ds
\]
\[
\leq \int_{-\infty}^{-T_{0}} e^{s}(c_{1}(\omega) + |t + \delta s| + |\omega(t)|)ds
\]
\[
\leq \int_{-\infty}^{-T_{0}} e^{s}(c_{1}(\omega) + |t| + |s| + |\omega(t)|)ds. \tag{36}
\]
Since \(t \in [\tau, \tau + T]\) and \(\omega\) is continuous in \(t\), we obtain from the above that there exists \(c_{2} = c_{2}(\tau, \omega, T) > 0\) such that
\[
\left| \int_{-\infty}^{-T_{0}} e^{s}(\omega(t + \delta s) - \omega(t))ds \right| \leq \int_{-\infty}^{-T_{0}} e^{s}(c_{2} + |s|)ds = c_{2}e^{-T_{0}} + \int_{-\infty}^{-T_{0}} e^{s}|s|ds,
\]
which implies that for every \(\epsilon > 0\), there exists \(T_{0} = T_{0}(\tau, \omega, \epsilon, T) \geq 1\) such that
\[
\left| \int_{-\infty}^{-T_{0}} e^{s}(\omega(t + \delta s) - \omega(t))ds \right| < \epsilon. \tag{37}
\]
Also, we know \(\omega(t)\) is continuous in \(t\) and hence is uniformly continuous on \([\tau - 1, \tau + T]\). This fact shows that there exists \(\delta_{1} = \delta_{1}(\tau, \omega, \epsilon, T) \in (0, 1)\) such that for all \(s_{1}, s_{2} \in [\tau - 1, \tau + T]\) with \(|s_{1} - s_{2}| \leq \delta_{1}\),
\[
|\omega(s_{1}) - \omega(s_{2})| < \epsilon. \tag{38}
\]
Let \(\delta_{2} = \frac{\delta_{1}}{T_{0}}\). Then for all \(0 < \delta < \delta_{2}, t \in [\tau, \tau + T]\) and \(s \in [-T_{0}, 0]\), we have \(|s\delta| \leq T_{0}\delta_{2} \leq \delta_{1}\), and hence by (38),
\[
|\omega(t + s\delta) - \omega(t)| < \epsilon. \tag{39}
\]
Now, by (39) we get
\[
\left| \int_{-T_{0}}^{0} e^{s}(\omega(t + \delta s) - \omega(t))ds \right| \leq \epsilon \int_{-T_{0}}^{0} e^{s}ds \leq \epsilon. \tag{40}
\]
Therefore, by (37) and (40) we obtain from (35) that, for all \(0 < \delta < \delta_{2}\) and \(t \in [\tau, \tau + T]\),
\[
\left| \int_{-\infty}^{0} e^{s}\omega(t + \delta s)ds - \omega(t) \right| \leq 2\epsilon, \tag{41}
\]
which implies (34). Then the desired result follows from (29), (30) and (34) directly.

As an immediate consequence of Lemma 2.2 and the continuity of \( \omega \), we have the following estimates.

**Corollary 1.** Let \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( T > 0 \). Then there exist \( \delta_0 = \delta_0(\tau, \omega, T) > 0 \) and \( M = M(\tau, \omega, T) > 0 \) such that for all \( 0 < \delta < \delta_0 \) and \( t \in [\tau, \tau + T] \),

\[
\left| \int_0^t \zeta_\delta(\theta, \omega) d\theta \right| \leq M. \tag{42}
\]

Note that (3) is a deterministic equation parametrized by \( \omega \in \Omega \), by the Galerkin method as in [49], we can prove that for every \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( w = (u_\tau, v_\tau) \in L^2(O) \), system (3) possesses a unique solution

\[
w(\cdot, \tau, \omega, w) = (u(\cdot, \tau, u_\tau), v(\cdot, \tau, v_\tau)) \in C([\tau, \infty), L^2(O)).
\]

Furthermore, the solution is continuous with respect to initial conditions in \( L^2(O) \) and is \( (\mathcal{F}, \mathcal{B}(L^2(O))) \)-measurable in \( \omega \in \Omega \). Then, we can define a continuous cocycle \( \Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(O) \to L^2(O) \) such that for all \( t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega \) and \( w_\tau \in L^2(O) \),

\[
\Phi(t, \tau, \omega, w_\tau) = w(t + \tau, \tau, \theta_{-\tau} \omega, w_\tau). \tag{43}
\]

Let \( D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) be a family of bounded nonempty subsets of \( L^2(O) \) with the property: for every \( \gamma > 0, \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[
\lim_{t \to -\infty} e^{\gamma t} \| D(\tau + t, \theta_\tau \omega) \|_{L^2(O)} = 0. \tag{44}
\]

Such a family is called tempered in \( L^2(O) \). Let \( D \) be the collection of all tempered families of bounded nonempty subsets of \( L^2(O) \), i.e.

\[
D = \{ D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} : D \text{ satisfies } (44) \}. \tag{45}
\]

Recall Poincare’s inequality on \( O \): there exists a positive constant \( \lambda_1 \) such that

\[
\| \nabla \phi \|^2 \geq \lambda_1 \| \phi \|^2 \quad \text{for all } \phi \in H^1_0(O). \tag{46}
\]

Let \( \kappa = \min \{ \lambda_1, \sigma \} \) and \( \kappa_0 \) be a fixed number in \( (0, \kappa) \). Hereafter, we assume

\[
\int_{-\infty}^0 e^{\kappa_0 s} \| \mathbf{F}(s + \tau, \cdot) \|^2 ds < \infty, \quad \text{for every } \tau \in \mathbb{R}, \tag{47}
\]

and for every positive number \( c \),

\[
\lim_{t \to -\infty} e^{ct} \int_{-\infty}^0 e^{\kappa_0 s} \| \mathbf{F}(s + \tau, \cdot) \|^2 ds = 0. \tag{48}
\]

### 2.2. Existence of pullback random attractors.

In this subsection, we prove the existence of tempered pullback random attractors for system (3). We first prove the existence of tempered absorbing sets and then show the pullback asymptotic compactness of solutions.

**Lemma 2.3.** Under the conditions of (G)-(H) and (47), for every \( \theta, \tau \in \mathbb{R}, \omega \in \Omega \) and \( D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D \), there exists \( T = T(\tau, \omega, D, \theta, \delta) > 0 \) such that
that for all \( t \geq T \), the solution of system (3) satisfies
\[
\|w(t,\tau-t,\theta-t\omega,w_{\tau-t})\|_{L^2(O)}^2 + \int_{\tau-t}^t e^{\int_{s-t}^{s}(\kappa-2\lambda\xi(t,\omega))} ds \|
abla u(s)\|^2 ds \\
+ \int_{\tau-t}^t e^{\int_{s-t}^{s}(\kappa-2\lambda\xi(t,\omega))} ds (u(s))^p + \|w(s)\|_{L^2(O)}^2 ds \\
\leq M_1 + M_1 \int_{-\infty}^0 e^{\int_{0}^{t}(\kappa-2\lambda\xi(t,\omega))} ds (1 + \|F(s + \tau, \cdot)\|^2 + \eta_\delta(\theta,\omega)) ds, \quad (49)
\]
where \( w_{\tau-t} \in D(\tau-t,\theta-t\omega), \eta_\delta(\theta,\omega) = |\xi(t,\omega)|^{2\beta} + |\xi(t,\omega)|^{2\alpha} + |\xi(t,\omega)|^{p_1}, \)
and \( M_1 \) is a positive constant independent of \( \tau, \omega, D \) and \( \vartheta \).

**Proof.** Taking the inner product of (3) with \( (2\beta u, 2\alpha v) \) in \( L^2(O) \) we obtain
\[
\frac{d}{dt}(\beta\|u\|^2 + \alpha\|v\|^2) + 2\beta\|\nabla u\|^2 + 2\alpha\sigma\|v\|^2 \\
= 2\beta \int_O f(t,x,u)udx + 2\beta(g,u) + 2\alpha(h,v) \\
+ 2\beta\xi(t,\omega) \int_O R(t,x,u)udx + 2\alpha\lambda\xi(t,\omega)||v||^2. \quad (50)
\]
We infer from (G1), (H1) and Young’s inequality that
\[
2\beta \int_O f(t,x,u)udx \leq -2\alpha_1\beta\|u\|^p_p + 2\beta\|\psi_1\|_1, \quad (51)
\]
\[
2\beta\xi(t,\omega) \int_O R(t,x,u)udx \leq 2\beta\alpha_4|\xi(t,\omega)| \int_O |u|^q dx + 2\beta \int_O |\xi(t,\omega)| \psi_4 |u| dx \\
\leq \alpha_1\beta\|u\|_p^p + c|\xi(t,\omega)|^{2\beta \alpha_4} + c|\xi(t,\omega)|^{p_1}, \quad (52)
\]
\[
2\beta|(g,u)| + 2\alpha|(h,v)| \leq \frac{\lambda_1\beta}{4}\|u\|^2 + \frac{3\alpha_\sigma}{4}\|v\|^2 + \frac{4\beta}{\lambda_1}\|g(t,\cdot)\|^2 + \frac{4\alpha}{3\sigma}\|h(t,\cdot)\|^2, \quad (53)
\]
and
\[
2\beta\lambda \int_O |\xi(t,\omega)| u|^2 dx \leq \frac{\alpha_1\beta}{2}\|u\|_p^p + c|\xi(t,\omega)|^{2\beta \alpha_4}. \quad (54)
\]
By (50)-(54) and \( \kappa = \min\{\lambda_1, \sigma\} \) we get
\[
\frac{d}{dt}(\beta\|u\|^2 + \alpha\|v\|^2) + \frac{1}{4}\kappa(\beta\|u\|^2 + \alpha\|v\|^2) \\
+ (\kappa - 2\beta\xi(t,\omega)) (\beta\|u\|^2 + \alpha\|v\|^2) + \frac{\beta}{2}\|\nabla u\|^2 + \frac{\alpha_1\beta}{2}\|u\|_p^p \\
\leq \eta_\delta(\theta,\omega) + \frac{4\beta}{\lambda_1}\|g(t,\cdot)\|^2 + \frac{4\alpha}{3\sigma}\|h(t,\cdot)\|^2 + 2\beta\|\psi_1(t,\cdot)\|_1, \quad (55)
\]
where \( \eta_\delta(\theta,\omega) = |\xi(t,\omega)|^{2\beta \alpha_4} + |\xi(t,\omega)|^{2\alpha} + |\xi(t,\omega)|^{p_1}. \) Applying Gronwall’s Lemma to (2.2) over \( (r, \vartheta) \) with \( \vartheta \geq r \) and for every \( \omega \in \Omega \), we get
\[
\beta\|u(\vartheta,r,\omega,\omega_t)\|^2 + \alpha\|v(\vartheta,r,\omega,\omega_t)\|^2 + \frac{\beta}{2} \int_r^\theta e^{\int_s^r(\kappa-2\lambda\xi(s,\omega))} ds \|\nabla u(s)\|^2 ds \\
+ \frac{\alpha_1\beta}{2} \int_r^\theta e^{\int_s^r(\kappa-2\lambda\xi(s,\omega))} ds \|u(s)\|_p^p ds
\]
which implies that there exists $t_r$.

Now, replacing $r$ by $\tau - t$ and $\omega$ by $\theta - \omega$ in (56), we get

$$\|w(\tau - t, \tau - \theta; w_{\tau - t})\|_{L^2(\Omega)}^2 + \int_{\tau - t}^\tau e^{\int_{\sigma}^{\tau - \tau} (\kappa - 2\zeta_s(\theta, \omega))d\sigma} \|\nabla u(s)\|^2 ds$$

$$+ \int_{\tau - t}^\tau e^{\int_{\sigma}^{\tau - \tau} (\kappa - 2\zeta_s(\theta, \omega))d\sigma} (\|w(s)\|^2 + \|\theta(s)\|^2) ds$$

$$\leq C e^{\int_{\tau - t}^{\tau - \tau} (\kappa - 2\zeta_s(\theta, \omega))d\sigma} \|w_{\tau - t}\|_{L^2(\Omega)}^2 + c \int_{\tau - t}^{\tau - \tau} e^{\int_{\sigma}^{\tau - \tau} (\kappa - 2\zeta_s(\theta, \omega))d\sigma} \eta_\theta(\theta, \omega) ds$$

$$+ c \int_{\tau - t}^{\tau - \tau} e^{\int_{\sigma}^{\tau - \tau} (\kappa - 2\zeta_s(\theta, \omega))d\sigma} (\|F(s + \tau, \cdot)\|^2 + \|\psi(s + \tau, \cdot)\|^2) ds$$

Next, we estimate every term on the right-hand side of (57). First, by Lemma 2.1, we find

$$\lim_{s \to -\infty} \frac{1}{s} \int_0^s (\kappa - 2\zeta_s(\theta, \omega)) ds = \kappa > 0,$$

which implies that there exists $s_0 < 0$ such that for all $s \leq s_0$,

$$\int_0^s (\kappa - 2\zeta_s(\theta, \omega)) ds < \kappa s_0.$$

Also we know from (47) that

$$\int_{-\infty}^{s_0} e^{\int_{\sigma}^{\tau - \tau} (\kappa - 2\zeta_s(\theta, \omega))d\sigma} (1 + \|F(s + \tau, \cdot)\|^2) ds$$

$$= e^{\int_{-\infty}^{s_0} (\kappa - 2\zeta_s(\theta, \omega))d\sigma} \int_{-\infty}^{s_0} e^{\int_{\sigma}^{\tau - \tau} (\kappa - 2\zeta_s(\theta, \omega))d\sigma} (1 + \|F(s + \tau, \cdot)\|^2) ds$$

$$\leq e^{\int_{-\infty}^{s_0} (\kappa - 2\zeta_s(\theta, \omega))d\sigma} \int_{-\infty}^{s_0} e^{\kappa s_0} (1 + \|F(s + \tau, \cdot)\|^2) ds$$

$$\leq e^{\int_{-\infty}^{s_0} (\kappa - 2\zeta_s(\theta, \omega))d\sigma} \int_{-\infty}^{0} e^{\kappa s_0} (1 + \|F(s + \tau, \cdot)\|^2) ds < \infty.$$

The second term on the right-hand side of (57) is well-defined due to (6), and for the first term, since $w_{\tau - t} \in D(\tau - t, \theta, \omega)$ and $D \in D$, we have

$$e^{\int_{\tau - t}^{\tau - \tau} (\kappa - 2\zeta_s(\theta, \omega))d\sigma} \|w_{\tau - t}\|_{L^2(\Omega)}^2 \leq e^{\int_{\tau - t}^{\tau - \tau} (\kappa - 2\zeta_s(\theta, \omega))d\sigma} D(\tau - t, \theta, \omega) \|w_{\tau - t}\|_{L^2(\Omega)}^2 \to 0,$$

as $t \to \infty$. Thus, there exists $T = T(\tau, \omega, D, \theta, \delta) > 0$ such that for all $t \geq T$,

$$e^{\int_{\tau - t}^{\tau - \tau} (\kappa - 2\zeta_s(\theta, \omega))d\sigma} \|w_{\tau - t}\|_{L^2(\Omega)}^2 \leq 1,$$
which along with (60) completes the proof. □

As a consequence of Lemma 2.3, we get the existence of $\mathcal{D}$-pullback absorbing sets for system (3).

**Corollary 2.** Suppose (G), (H) and (47)-(48) hold. Then the cocycle $\Phi$ associated with system (3) possesses a $\mathcal{D}$-pullback absorbing set $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ in $L^2(\Omega)$ which is given by, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$K(\tau, \omega) = \{w \in L^2(\Omega) : \|w\|_{L^2(\Omega)}^2 \leq R(\tau, \omega)\},$$

where

$$R(\tau, \omega) = M_1 + M_1 \int_{-\infty}^{0} e^{\int_{s}^{\tau} (\kappa - 2\lambda \xi_s(\theta, \omega))ds} (1 + \|F(s + \tau, \cdot)\|^2 + \eta_0(\theta, \omega))ds$$

with the same constant $M_1$ as in (49).

**Proof.** We infer from Lemma 2.3 with $\theta = \tau$ that for $\omega \in \Omega$ and $D \in \mathcal{D}$, there exists $T = T(\tau, \omega, D, \delta) > 0$ such that for all $t \geq T$,

$$\Phi(t, \tau - t, \theta - t \omega, D(\tau - t, \theta - t \omega)) = w(\tau, \tau - t, \theta - t \omega, D(\tau - t, \theta - t \omega)) \subseteq K(\tau, \omega).$$

(62)

Next, we show $K \in \mathcal{D}$. Indeed, let $\gamma$ be an arbitrary positive constant and consider

$$\lim_{t \to -\infty} e^{\gamma t} \|K(\tau + t, \theta t \omega)\|^2 = \lim_{t \to -\infty} e^{\gamma t} R(\tau + t, \theta t \omega)$$

$$= \lim_{t \to -\infty} M_1 e^{\gamma t} + \lim_{t \to -\infty} M_1 e^{\gamma t} \int_{-\infty}^{0} e^{\int_{s}^{\tau} (\kappa - 2\lambda \xi_s(\theta, \omega))ds} ds$$

$$+ \lim_{t \to -\infty} M_1 e^{\gamma t} \int_{-\infty}^{0} e^{\int_{s}^{\tau} (\kappa - 2\lambda \xi_s(\theta, \omega))ds} \|F(s + \tau + t, \cdot)\|^2 ds$$

$$+ \lim_{t \to -\infty} M_1 e^{\gamma t} \int_{-\infty}^{0} e^{\int_{s}^{\tau} (\kappa - 2\lambda \xi_s(\theta, \omega))ds} \eta_0(\theta_t, \omega) ds.$$

Due to (18), we find

$$-2\lambda \int_{0}^{s} \xi_s(\theta + t \omega) ds = -2\lambda e^{\frac{s}{2}} \int_{-\infty}^{0} e^{\tau} \omega(s + t + \delta r) dr + 2\lambda e^{\frac{s}{2}} \int_{-\infty}^{0} e^{\tau} \omega(t + \delta r) dr.$$  

(64)

Let $c_0 = \min\left\{\frac{\kappa - \frac{\kappa_0}{2}}{\lambda}, \frac{\kappa_0}{\lambda}\right\}$. By (4), we know that there exists $T_1 = T_1(\omega) > 0$ such that for all $|t| \geq T_1$,

$$|\omega(t)| < c_0|t|.$$  

(65)

Then for $t \leq -T_1$, $r, s \leq 0$ and $0 < \delta \leq 1$, we have $s + t + \delta r \leq t \leq -T_1$, and hence

$$|\omega(s + t + \delta r)| < c_0|s + t + \delta r| \leq c_0|t| + c_0|s| + c_0|r|.$$  

(66)

Now, for $t \leq -T_1$, we get

$$\left|e^{\frac{s}{2}} \int_{-\infty}^{0} e^{\tau} \omega(s + t + \delta r) ds\right| \leq c_0 e^{-\frac{T_1}{2}} \int_{-\infty}^{0} e^{\tau} |s + |t| + |r|| dr$$

$$\leq c_0 |t| + c_0 |s| + c_1 = -c_0 t - c_0 s + c_1.$$  

(67)
Similarly, for \( t \leq -T_1 \) and \( r \leq 0 \), we have \( t + \delta r \leq t + \delta r \leq -T_1 \), and hence \( |\omega(t + \delta r)| < c_0|t| + c_0|r| \). This implies that for all \( t \leq -T_1 \),
\[
|e^{\frac{t}{2}} \int_{-\infty}^{0} e^{r} \omega(t + \delta r) dr| \leq -c_0 t + c_1.
\]
(68)
It follows from (64), (67) and (68) that for all \( t \leq -T_1 \), \( s \leq 0 \) and \( \delta \in (0, 1] \),
\[
-2 \lambda \int_{0}^{s} \zeta_t(\theta_t + \delta \omega) dr \leq 2 \lambda(-2c_0 t - c_0 s + 2c_1).
\]
(69)
Let \( \gamma_0 = \min\{\kappa_0, \frac{\gamma}{2}\} \). We deduce from (63) and (69) that, for all \( t \leq -T_1 \) and \( 0 < \delta \leq 1 \),
\[
e^{\gamma t} \|K(\tau + t, \theta_t \omega)\|^2
\leq M_1 e^{\gamma t} + c e^{4\gamma_1 \lambda} e^{2t} + M_1 e^{4\gamma_1 \lambda} e^{2t} \int_{-\infty}^{0} e^{\kappa s} \|F(s + \tau + t, \cdot)\|^2 ds
\]
\[
+ M_1 e^{4\gamma_1 \lambda} \int_{-\infty}^{0} e^{\gamma(s+t)} \eta_0(\theta_{t+s} \omega) ds
\]
\[
\leq M_1 e^{\gamma t} + M_1 e^{4\gamma_1 \lambda} e^{2t} + M_1 e^{4\gamma_1 \lambda} e^{2t} \int_{-\infty}^{0} e^{\kappa s} \|F(s + t, \cdot)\|^2 ds
\]
\[
+ M_1 e^{4\gamma_1 \lambda} \int_{-\infty}^{t} e^{\gamma s} \eta_0(\theta_s \omega) ds,
\]
which along with (48) and Lemma 2.1 implies
\[
\lim_{t \to -\infty} e^{\gamma t} \|K(\tau + t, \theta_0 \omega)\|^2 = 0.
\]
This completes the proof.

Also, we can obtain the following estimates from Lemma 2.3 for later purpose.

**Lemma 2.4.** Suppose (G)-(H) and (47) hold, for every \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D} \), there exists \( T = T(\tau, \omega, D, \delta) \geq 1 \) such that for all \( t \geq T \), the solution of system (3) satisfies
\[
\int_{\tau - 1}^{\tau} \left( \|u(s, \tau - t, \theta_{-s} \omega, u_{-s} - t)\|^2_{H^1_0(\mathcal{O})} + \|v(s, \tau - t, \theta_{-s} \omega, v_{-s} - t)\|^2_{H^1_0(\mathcal{O})} \right) ds
\]
\[
+ \int_{\tau - 1}^{\tau} \|u(s, \tau - t, \theta_{-s} \omega, u_{-s} - t)\|^2_{H^1_0(\mathcal{O})} ds
\]
\[
\leq M_2 + M_2 \int_{-\infty}^{\tau} e^{\int_{\tau}^{\tau} (\kappa - 2\lambda \zeta_t(\theta, \omega)) dr} (\|F(s + \tau, \cdot)\|^2 + \eta_0(\theta_s \omega)) ds,
\]
where \( w_{\tau - t} \in D(\tau - t, \theta_{-s} \omega) \) and \( M_2 \) is a positive constant independent of \( \tau \) and \( D \).

Next, we derive uniform estimates of \( u \) in \( H^1_0(\mathcal{O}) \).

**Lemma 2.5.** Suppose (G)-(H) and (47) hold. Then for every \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D} \), there exists \( T = T(\tau, \omega, D, \delta) \geq 1 \) such that for all \( t \geq T \), the solution of system (3) satisfies
\[
\|\nabla u(\tau - t, \theta_{-s} \omega, u_{-s} - t)\|^2
\leq M_3 + M_3 \int_{-\infty}^{\tau} e^{\int_{\tau}^{\tau} (\kappa - 2\lambda \zeta_t(\theta, \omega)) dr} (\|F(s + \tau, \cdot)\|^2 + \eta_0(\theta_s \omega)) ds,
\]
where \( w_{\tau - t} \in D(\tau - t, \theta_{-t} \omega) \) and \( M_3 \) is a positive constant depending on \( \tau \) and \( \omega \).

Proof. By (3) we find
\[
\frac{d}{dt} \| \nabla u \|^2 + 2 \| \Delta u \|^2 = -2 \int_{\mathcal{O}} f(t, x, u) \Delta u dx - 2 (g, \Delta u) + 2 \alpha (\Delta u, v) - 2 \zeta_{\delta}(\theta_{t} \omega) \int_{\mathcal{O}} R(t, x, u) \Delta u dx.
\] (70)

For the first term of the right-hand side of (70), by (G2) and (G4) we get
\[
-2 \int_{\mathcal{O}} f(t, x, u) \Delta u dx = 2 \int_{\mathcal{O}} \frac{\partial f}{\partial x} (t, x, u) \nabla u dx + 2 \int_{\mathcal{O}} \frac{\partial f}{\partial u} (t, x, u) |\nabla u|^2 dx \\
\leq (1 + 2 \| \psi_3(t, \cdot) \|_{L^\infty(\mathcal{O})}) \| \nabla u \|^2 + \| \psi_4(t, \cdot) \|^2 \leq 2 \alpha_3 \int_{\mathcal{O}} |u|^{p-2} |\nabla u|^2 dx.
\] (71)

By Young’s inequality, we have
\[
-2 (g, \Delta u) + 2 \alpha (\Delta u, v) \leq \| \Delta u \|^2 + 2 \| g(t, \cdot) \|^2 + 2 \alpha \| v \|^2. 
\] (72)

By Young’s inequality again, we get
\[
|u|^{p-2} |\zeta_{\delta}(\theta_{t} \omega)| \leq \frac{\alpha_3}{2 \alpha_3} |u|^{p-2} + c \| \zeta_{\delta}(\theta_{t} \omega) \|_{L^\infty(\mathcal{O})} \| \nabla u \|^2,
\]

which along with (H2) and (H3) shows that
\[
-2 \zeta_{\delta}(\theta_{t} \omega) \int_{\mathcal{O}} R(t, x, u) \Delta u dx \\
= 2 \zeta_{\delta}(\theta_{t} \omega) \int_{\mathcal{O}} \frac{\partial R}{\partial x} (t, x, u) \nabla u dx + 2 \zeta_{\delta}(\theta_{t} \omega) \int_{\mathcal{O}} \frac{\partial R}{\partial u} (t, x, u) |\nabla u|^2 dx \\
\leq c (1 + |\zeta_{\delta}(\theta_{t} \omega)\|_{L^\infty(\mathcal{O})} + |\zeta_{\delta}(\theta_{t} \omega)\|_{L^\infty(\mathcal{O})} \| \nabla u \|^2) \\
+ |\zeta_{\delta}(\theta_{t} \omega)|^2 \| \psi_7(t, \cdot) \|^2 + \alpha_3 \int_{\mathcal{O}} |u|^{p-2} |\nabla u|^2 dx.
\] (73)

By (70)-(73), we obtain
\[
\frac{d}{dt} \| \nabla u \|^2 \leq c \left( 1 + |\zeta_{\delta}(\theta_{t} \omega)\|_{L^\infty(\mathcal{O})} + |\zeta_{\delta}(\theta_{t} \omega)\|_{L^\infty(\mathcal{O})} \| \nabla u \|^2 \right) + |\psi_3(t, \cdot)\|_{L^\infty(\mathcal{O})} \\
+ 2 \alpha \| v \|^2 + \| \psi_4(t, \cdot) \|^2 + 2 \| g(t, \cdot) \|^2 + |\zeta_{\delta}(\theta_{t} \omega)|^2 \| \psi_7(t, \cdot) \|^2.
\] (74)

Given \( t \in \mathbb{R}^+, \tau \in \mathbb{R} \) and \( s \in (\tau - 1, \tau) \), by integrating (74) on \((s, \tau)\) we know that
\[
\| \nabla u(s, \tau - t, \omega, \mathbf{u}_{\tau - t}) \|^2 \leq \| \nabla u(s, \tau - t, \omega, \mathbf{u}_{\tau - t}) \|^2 \\
+ c_1 \int_s^\tau \| \nabla u(r, \tau - t, \omega, \mathbf{u}_{\tau - t}) \|^2 dr \\
+ 2 \alpha \int_s^\tau \| v(r, \tau - t, \omega, \mathbf{v}_{\tau - t}) \|^2 dr + 2 \int_s^\tau \| g(r, \cdot) \|^2 dr + c_1.
\] (75)
where \( c_1 = c_1(\tau, \omega) > 0 \). We now integrate (75) with respect to \( s \) on \((\tau, \tau)\) and replace \( \omega \) by \( \theta_{-\tau} \omega \) to obtain
\[
\| \nabla u(\tau, \tau, \theta_{-\tau} \omega, u_{\tau-t}) \|^2 \leq c_2 \int_{\tau-t}^{\tau} \| \nabla v(\tau, \tau, \theta_{-\tau} \omega, v_{\tau-t}) \|^2 \, ds + 2c_3 \int_{\tau-t}^{\tau} \| v(\tau, \tau, \theta_{-\tau} \omega, v_{\tau-t}) \|^2 \, ds + c_3,
\]
where \( c_2 = c_2(\tau, \omega) > 0 \) and \( c_3 = c_3(\tau, \omega, \theta) > 0 \), which along with Lemma 2.4 implies the desired result.

Next, we will show the asymptotic compactness of solutions of system (3). To that end, we write \( v = v_1 + v_2 \) where \( v_1 \) and \( v_2 \) solve the following systems
\[
\frac{dv_1}{dt} + \sigma v_1 = \lambda \zeta_\delta(\theta_t \omega)v_1, \quad v_1(\tau) = v_\tau,
\]
and
\[
\frac{dv_2}{dt} + \sigma v_2 = \beta u + h(t, x) + \lambda \zeta_\delta(\theta_t \omega)v_2, \quad v_2(\tau) = 0.
\]
By (7) and (77) we find that for \( t \in \mathbb{R}^+ \), \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and \( v_{\tau-t} \in D(\tau, t, \theta_{-\tau} \omega) \),
\[
\| v_1(\tau, \tau - t, \theta_{-\tau} \omega, v_{\tau-t}) \|^2 \leq e^{-2\sigma t - 2\lambda \int_0^t \zeta_\delta(\theta_s \omega) \, ds} \| D(\tau - t, \theta_{-i} \omega) \|^2 \rightarrow 0 \text{ as } t \rightarrow \infty.
\]

Next, we derive uniform estimates of \( v_2 \) in \( H^1_0(\Omega) \) for which we further assume
\[
\int_{-\infty}^{0} e^{\kappa s} \| h(s + \tau, \cdot) \|^2 \| H^1_0(\Omega) \, ds < \infty, \text{ for every } \tau \in \mathbb{R}.
\]

**Lemma 2.6.** Let (G) and (80) hold. Then for every \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and \( D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D \), there exists \( T = T(\tau, \omega, D, \delta) \geq 1 \) such that for all \( t \geq T \), the solution of equation (78) satisfies
\[
\| \nabla v_2(\tau, \tau - t, \theta_{-\tau} \omega, 0) \|^2 + \frac{\sigma}{2} \int_{\tau-t}^{\tau} e^{\int_0^s (\kappa - 2\lambda \zeta_\delta(\theta_s \omega)) \, ds} \| \nabla v_2(s) \|^2 \, ds
\leq M_4 + M_4 \int_{-\infty}^{0} e^{\int_0^s (\kappa - 2\lambda \zeta_\delta(\theta_s \omega)) \, ds} \left( 1 + \| h(s + \tau, \cdot) \|^2 \right) + \eta_\delta(\theta_s \omega) + \| h(s + \tau, \cdot) \|^2 \| H^1_0(\Omega) \) \, ds,
\]
where \( M_4 \) is a positive constant independent of \( \tau, \omega \) and \( D \).

**Proof.** Multiplying (78) by \(-\Delta v_2 \), after some calculations we get
\[
\frac{d}{dt} \| \nabla v_2 \|^2 + (\kappa - 2\lambda \zeta_\delta(\theta_t \omega)) \| \nabla v_2 \|^2 + \frac{\sigma}{2} \| \nabla v_2 \|^2 \leq c \| \nabla u \|^2 + c \| h(t, \cdot) \|^2 \| H^1_0(\Omega).\]

Applying Gronwall’s inequality to (82) over \((\tau - t, t)\) with \( t \in \mathbb{R}^+ \) and replacing \( \omega \) by \( \theta_{-\tau} \omega \), we obtain
\[
\| \nabla v_2(\tau, \tau - t, \theta_{-\tau} \omega, 0) \|^2 + \frac{\sigma}{2} \int_{\tau-t}^{\tau} e^{\int_0^s (\kappa - 2\lambda \zeta_\delta(\theta_s \omega)) \, ds} \| \nabla v_2(s) \|^2 \, ds
\leq c \int_{\tau-t}^{\tau} e^{\int_0^s (\kappa - 2\lambda \zeta_\delta(\theta_s \omega)) \, ds} \| \nabla u(s) \|^2 \, ds
+ c \int_{-\infty}^{0} e^{\int_0^s (\kappa - 2\lambda \zeta_\delta(\theta_s \omega)) \, ds} \| h(s + \tau, \cdot) \|^2 \| H^1_0(\Omega) \, ds,
which along with Lemma 2.3 implies the desired result. \hfill \Box

We now establish the $\mathcal{D}$-pullback asymptotic compactness of $\Phi$ in $\mathbb{L}^2(\mathcal{O})$. For this purpose, we need to split $\Phi$ as follows. For every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $w_\tau = (u_\tau, v_\tau) \in \mathbb{L}^2(\mathcal{O})$, we define

$$
\Phi_1(t, \tau, \omega, w_\tau) = (0, v_1(t + \tau, \tau, \theta_{-\tau}\omega, v_\tau))
$$

and

$$
\Phi_2(t, \tau, \omega, w_\tau) = (u(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau), v_2(t + \tau, \tau, \theta_{-\tau}\omega, 0)),
$$

where $v_1$ and $v_2$ are the solutions of equations (77) and (78), respectively. By (43), we find for every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $w_\tau = (u_\tau, v_\tau) \in \mathbb{L}^2(\mathcal{O})$,

$$
\Phi(t, \tau, \omega, w_\tau) = \Phi_1(t, \tau, \omega, w_\tau) + \Phi_2(t, \tau, \omega, w_\tau).
$$

Now, we can obtain the $\mathcal{D}$-pullback asymptotic compactness of $\Phi$ in $\mathbb{L}^2(\mathcal{O})$.

**Lemma 2.7.** Suppose (G), (H), (47) and (80) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, the sequence $\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, w_{0,n})$ has a convergent subsequence in $\mathbb{L}^2(\mathcal{O})$ provided $t_n \to \infty$ and $w_{0,n} \in D(\tau - t_n, \theta_{-t_n}\omega)$.

**Proof.** First, for $w_0 \in D(\tau - t, \theta_{-t}\omega)$, by Lemmas 2.3, 2.5 and 2.6, there exist $T_1 = T_1(\tau, \omega, D, \delta) \geq 1$ and $c_1(\tau, \omega, \delta) > 0$ such that for all $t \geq T_1$,

$$
\|\Phi_2(t, \tau - t, \theta_{-t}\omega, w_0)\|_{H^1_0(\mathcal{O})}^2 \leq c_1.
$$

Let $N_1 = N_1(\tau, \omega, D, \delta) \geq 1$ be large enough such that $t_n \geq T_1$ for $n \geq N_1$. Then by (86) we find that, for all $n \geq N_1$,

$$
\|\Phi_2(t_n, \tau - t_n, \theta_{-t_n}\omega, w_{0,n})\|_{H^1_0(\mathcal{O})}^2 \leq c_1.
$$

By the compactness of embedding $H^1_0(\mathcal{O}) \times H^1_0(\mathcal{O}) \hookrightarrow \mathbb{L}^2(\mathcal{O})$, it follows from (87) that there is $\bar{w} \in \mathbb{L}^2(\mathcal{O})$ such that, up to a subsequence

$$
\Phi_2(t_n, \tau - t_n, \theta_{-t_n}\omega, w_{0,n}) \to \bar{w} \quad \text{strongly in} \quad \mathbb{L}^2(\mathcal{O}).
$$

On the other hand, by (79) we have

$$
\Phi_1(t_n, \tau - t_n, \theta_{-t_n}\omega, w_{0,n}) \to 0 \quad \text{in} \quad \mathbb{L}^2(\mathcal{O}),
$$

which together with (88) and (85) implies

$$
\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, w_{0,n}) \to \bar{w} \quad \text{strongly in} \quad \mathbb{L}^2(\mathcal{O}),
$$

as desired. \hfill \Box

We now prove the existence of $\mathcal{D}$-pullback attractors of $\Phi$.

**Theorem 2.8.** Let (G), (H), (47)-(48) and (80) hold. Then the cocycle $\Phi$ associated with problem (3) has a unique $\mathcal{D}$-pullback attractor $A = \{A(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ in $\mathbb{L}^2(\mathcal{O})$.

**Proof.** Since $\Phi$ has a closed measurable $\mathcal{D}$-pullback absorbing set $K \in \mathcal{D}$ by Corollary 2 and is $\mathcal{D}$-pullback asymptotically compact in $\mathbb{L}^2(\mathcal{O})$ by Lemma 2.7, then the existence and uniqueness of $\mathcal{D}$-pullback attractor $A$ of $\Phi$ follows immediately. \hfill \Box
3. Attractors of stochastic FitzHugh-Nagumo system. In this section, we investigate the dynamics of the stochastic FitzHugh-Nagumo system driven by a linear multiplicative noise. More precisely, we will prove the existence and uniqueness of tempered random attractors for the system. The results of this section will be used for studying the limiting behavior of solutions of the random system (3) when $\delta \to 0$. Indeed, in the next section, we will prove, under certain conditions, the limiting dynamics of system (3) is governed by that of a stochastic system as $\delta \to 0$.

Given $\tau \in \mathbb{R}$, consider the stochastic FitzHugh-Nagumo system
\begin{align}
\begin{cases}
\frac{\partial u}{\partial t} - \Delta u + \alpha v &= f(t,x,u) + g(t,x) + u \circ dW_t, \quad x \in \mathcal{O}, t > \tau, \\
\frac{\partial v}{\partial t} + \sigma v - \beta u &= h(t,x) + v \circ dW_t, \quad x \in \mathcal{O}, t > \tau,
\end{cases}
\end{align}
where $f, g$ and $h$ are the same as in the previous section.

As usual, to study the pathwise dynamics of (89), we need to transform the stochastic equations into random ones parametrized by $\omega \in \Omega$. Let
\[ \bar{w}(t, \tau, \omega) = (\bar{u}(t, \tau, \omega), \bar{v}(t, \tau, \omega)) = e^{-\omega(t)}(u(t, \tau, \omega), v(t, \tau, \omega)). \]
By (89) and (90), we get
\begin{align}
\begin{cases}
\frac{\partial \bar{u}}{\partial t} - \Delta \bar{u} + \alpha \bar{v} &= e^{-\omega(t)}f(t,x,e^{\omega(t)}\bar{u}) + e^{-\omega(t)}g(t,x), \quad x \in \mathcal{O}, t > \tau, \\
\frac{\partial \bar{v}}{\partial t} + \sigma \bar{v} - \beta \bar{u} &= e^{-\omega(t)}h(t,x), \quad x \in \mathcal{O}, t > \tau,
\end{cases}
\end{align}
As in [49], one can show that under condition (G), for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, system (91) has a unique solution $\bar{w}(\cdot, \tau, \omega, \bar{w}_\tau) \in C(\tau, \infty, \mathbb{L}^2(\mathcal{O}))$, which is continuous in $\bar{w}_\tau$ and measurable in $\omega \in \Omega$. Based on the solution of (91), one can define a continuous cocycle $\Phi_0$ for the stochastic system (89) in $\mathbb{L}^2(\mathcal{O})$:
\[ \Phi_0(t, \tau, \omega, \bar{w}_\tau) = \bar{w}(t + \tau, \tau, \theta - \tau, \omega, \bar{w}_\tau) = e^{\omega(t) - \omega(-\tau)} \bar{w}(t + \tau, \tau, \theta - \tau, \omega, \bar{w}_\tau). \]
We will prove $\Phi_0$ has a unique $\mathcal{D}$-pullback attractor $\mathcal{A}_0$ in $\mathbb{L}^2(\mathcal{O})$. First, we derive uniform estimates of solutions in $\mathbb{L}^2(\mathcal{O})$.

**Lemma 3.1.** Suppose (G) and (47) hold. Then
(i). For every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $T > 0$, there exists $L = L(\tau, \omega, T) > 0$ such that for all $t \in [\tau, \tau + T]$,
\[ \|\bar{w}(t, \tau, \omega, \bar{w}_\tau)\|_{\mathbb{L}^2(\mathcal{O})} + \int_{\tau}^{t} \|\bar{u}(s, \tau, \omega, \bar{u}_\tau)\|_p^p ds \leq L(1 + \|\bar{w}_\tau\|_{\mathbb{L}^2(\mathcal{O})}^2 + \int_{\tau}^{t} \|\mathbf{F}(s, \cdot)\|_p^2). \]
(ii). For every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D \in \mathcal{D}$, there exists $T_1 = T_1(\tau, \omega, D) > 0$ such that for all $t \geq T_1$,
\[ \|\bar{w}(t, \tau - t, \theta - \tau \omega, \bar{w}_{\tau - t})\|_{\mathbb{L}^2(\mathcal{O})} + \int_{\tau - t}^{\tau} e^{\kappa(s-\tau)}(\|\bar{u}(s)\|^2 + \|\nabla \bar{u}(s)\|^2 + \|\bar{v}(s)\|^2) ds \leq R_0(\tau, \omega), \]
where \( e^{\omega(-t)-\omega(-\tau)} \bar{w}_{\tau-t} \in D(\tau-t, \theta_{-t}\omega) \) and

\[
\bar{R}_0(\tau, \omega) = L_1 \int_{-\infty}^{0} e^{\kappa s - 2\omega(s) + 2\omega(-\tau)} (1 + \|F(s + \tau, \cdot)\|^2) \, ds
\]

with \( L_1 \) being a positive constant independent of \( \tau \) and \( \omega \).

**Proof.** (i). Taking the inner product of (91) with \((2\beta \bar{u}, 2\alpha \bar{v})\), by (G1) we get

\[
\frac{d}{dt}(\beta \|\bar{u}\|^2 + \alpha \|\bar{v}\|^2) = 2\beta e^{-\omega(t)} \langle f(t, x, \omega(t) \bar{u}), \bar{u} \rangle + 2\beta e^{-\omega(t)} \langle g(t, x), \bar{u} \rangle + 2\alpha e^{-\omega(t)} \langle h(t, x), \bar{v} \rangle
\]

\[
\leq -2\beta \alpha_1 e^{(p-2)\omega(t)} \|\bar{u}\|^p + 2\beta e^{-\omega(t)} \|\psi_1(t, \cdot)\|_1 + \frac{\beta \lambda_1}{4} \|\bar{u}\|^2 + \frac{3\alpha \sigma}{4} \|\bar{v}\|^2
\]

\[
+ \frac{4}{\kappa} e^{-2\omega(t)} (\beta \|g(t, \cdot)\|^2 + \alpha \|h(t, \cdot)\|^2),
\]

which implies that

\[
\frac{d}{dt}(\beta \|\bar{u}\|^2 + \alpha \|\bar{v}\|^2) + \kappa(\|\bar{u}\|^2 + \alpha \|\bar{v}\|^2) + \frac{\beta}{2} \|
abla \bar{u}\|^2
\]

\[
+ \frac{\kappa}{4} (\|\bar{u}\|^2 + \alpha \|\bar{v}\|^2) + 2\beta \alpha_1 e^{(p-2)\omega(t)} \|\bar{u}\|^p
\]

\[
\leq 2\beta e^{-2\omega(t)} \|\psi_1(t, \cdot)\|_1 + \frac{4}{\kappa} e^{-2\omega(t)} (\beta \|g(t, \cdot)\|^2 + \alpha \|h(t, \cdot)\|^2).
\]

For every \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( T > 0 \), integrating (95) over \((\tau, t)\) for \( t \in [\tau, \tau + T] \), we obtain (93) immediately.

(ii). Applying Gronwall’s inequality to (95) over \((\tau, \tau - t)\) with \( \omega \) replaced by \( \theta_{-t}\omega \), after simple computations, we find

\[
\beta \|\bar{u}(\tau, \tau - t, \theta_{-t}\omega, \bar{u}_{\tau-t})\|^2 + \alpha \|\bar{v}(\tau, \tau - t, \theta_{-t}\omega, \bar{v}_{\tau-t})\|^2
\]

\[
+ \frac{\beta}{2} \int_{\tau-t}^{T} e^{\kappa(s-\tau)} \|\nabla \bar{u}(s)\|^2 ds + \frac{\kappa}{4} \int_{\tau-t}^{T} e^{\kappa(s-\tau)} (\|\bar{u}(s)\|^2 + \alpha \|\bar{v}(s)\|^2) ds
\]

\[
\leq e^{-\kappa t} (\|\bar{u}_{\tau-t}\|^2 + \alpha \|\bar{v}_{\tau-t}\|^2)
\]

\[
+ c_1 \int_{-\infty}^{0} e^{\kappa s - 2\omega(s) + 2\omega(-\tau)} (\|F(s + \tau, \cdot)\|^2 + \|\psi_1(s + \tau, \cdot)\|_1) ds
\]

\[
\leq (\alpha + \beta) e^{-\kappa t} e^{2\omega(-\tau) - 2\omega(t)} \|D(\tau - t, \theta_{-t}\omega)\|_{L^2(O)}^2 + c_2 \int_{-\infty}^{0} e^{\kappa s - 2\omega(s) + 2\omega(-\tau)} (1 + \|\bar{F}(s + \tau, \cdot)\|^2) ds,
\]

where \( c_2 \) is independent of \( \tau, \omega \) and \( \delta \). Since \( D \in \mathcal{D} \), by (4) we find that there exists \( T_1 = T_1(\tau, \omega, D) > 0 \) such that for all \( t \geq T_1 \),

\[
(\alpha + \beta) e^{-\kappa t} e^{2\omega(-\tau) - 2\omega(t)} \|D(\tau - t, \theta_{-t}\omega)\|_{L^2(O)}^2
\]

\[
\leq \int_{-\infty}^{0} e^{\kappa s - 2\omega(s) + 2\omega(-\tau)} (1 + \|\bar{F}(s + \tau, \cdot)\|^2) ds.
\]

Then (98) follows from (3) and (3). This completes the proof. 

By Lemma 3.1 we obtain the following estimates.
Lemma 3.2. Suppose (G) and (47) hold. Then for every \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and \( D \in \mathcal{D} \), there exists \( T = T(\tau, \omega, D) > 0 \) such that for all \( t \geq T \),
\[
\| \mathbf{w}(\tau, \tau - t, \theta_{-\tau}, \mathbf{w}_{\tau-t}) \|_{L^2(\mathcal{O})}^2 \leq R_0(\tau, \omega),
\]
where \( \mathbf{w}_{\tau-t} \in D(\tau - t, \theta_{-\omega}) \) and
\[
R_0(\tau, \omega) = L_1 \int_{-\infty}^{0} e^{\kappa s - 2\omega(s)}(1 + \| \mathbf{F}(s + \tau, \cdot) \|^2) ds
\]
with \( L_1 \) being the same constant as in Lemma 3.1 which is independent of \( \tau \) and \( \omega \).

Proof. By (90), we get
\[
\mathbf{w}(\tau, \tau - t, \theta_{-\tau}, \mathbf{w}_{\tau-t}) = e^{-\omega(-\tau)}\bar{w}(\tau, \tau - t, \theta_{-\tau}, \mathbf{w}_{\tau-t})
\]
with \( \bar{w}_{\tau-t} = e^{\omega(-\tau) - \omega(-t)}\bar{w}_{\tau-t} \). Then (98) follows from Lemma 3.1 immediately. \( \square \)

Next, we derive uniform estimates of \( \bar{u} \) in \( H^1_0(\mathcal{O}) \).

Lemma 3.3. Suppose (G) and (47) hold. Then for every \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and \( D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D} \), there exists \( T = T(\tau, \omega, D) > 1 \) such that for all \( t \geq T \), the solution of system (91) satisfies
\[
\| \nabla \bar{u}(\tau, \tau - t, \theta_{-\tau}, \bar{u}_{\tau-t}) \|^2 \leq L_2 + L_2 \int_{-\infty}^{0} e^{\kappa s - 2\omega(s) + 2\omega(-\tau)}(1 + \| \mathbf{F}(s + \tau, \cdot) \|^2) ds,
\]
where \( e^{\omega(-t) - \omega(-\tau)}\bar{w}_{\tau-t} \in D(\tau - t, \theta_{-\omega}) \) and \( L_2 \) is a positive constant depending on \( \tau \) and \( \omega \).

Proof. By (91) we find
\[
\frac{d}{dt} \| \nabla \bar{u} \|^2 + 2 \| \Delta \bar{u} \|^2
= -2e^{-\omega(t)} \int_{\mathcal{O}} f(t, x, u) \Delta \bar{u} dx - 2e^{-\omega(t)}(g, \Delta \bar{u}) + 2\alpha(\Delta \bar{u}, \bar{v}).
\]  

For the first term of the right-hand side of (99), by (G2) and (G4) we get
\[
-2e^{-\omega(t)} \int_{\mathcal{O}} f(t, x, u) \Delta \bar{u} dx
= 2e^{-\omega(t)} \int_{\mathcal{O}} \frac{\partial f}{\partial x}(t, x, u) \nabla \bar{u} dx + 2e^{-\omega(t)} \int_{\mathcal{O}} \frac{\partial f}{\partial u}(t, x, u) |\nabla \bar{u}|^2 dx
\leq (1 + 2e^{-\omega(t)} \| \psi_3(t, \cdot) \|_{L^\infty(\mathcal{O})}) \| \nabla \bar{u} \|^2 + e^{-2\omega(t)} \| \psi_4(t, \cdot) \|^2
- 2e^{-\omega(t)}\alpha \int_{\mathcal{O}} |u|^{p-2} |\nabla \bar{u}|^2 dx.
\]  

By Young’s inequality, we have
\[
-2e^{-\omega(t)}(g, \Delta \bar{u}) + 2\alpha(\Delta \bar{u}, \bar{v}) \leq \| \Delta \bar{u} \|^2 + 2e^{-2\omega(t)} \| g(t, \cdot) \|^2 + 2\alpha^2 \| \bar{v} \|^2.
\]  

By (99)-(101), we see that
\[
\frac{d}{dt} \| \nabla \bar{u} \|^2 + \| \Delta \bar{u} \|^2
\leq (1 + 2e^{-\omega(t)} \| \psi_3(t, \cdot) \|_{L^\infty(\mathcal{O})}) \| \nabla \bar{u} \|^2 + 2\alpha^2 \| \bar{v} \|^2 + e^{-2\omega(t)} \| \psi_4(t, \cdot) \|^2 + 2\| g(t, \cdot) \|^2.
\]  

(102)
Given $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $s \in (\tau - 1, \tau)$, by integrating (3) on $(s, \tau)$ we know that
\[
\|
\nabla \bar{u}(\tau, \tau - t, \omega, \bar{u}_{\tau-t})\|^2 \leq \|
\nabla \bar{u}(s, \tau - t, \omega, \bar{u}_{\tau-t})\|^2 
\]
\[
+ \int_s^\tau (1 + 2e^{-\omega(r)}\|\psi_3(r, \cdot)\|_{L^\infty(\Omega)})\|
\nabla \bar{u}(r, \tau - t, \omega, \bar{u}_{\tau-t})\|^2 dr 
\]
\[
+ 2\alpha^2 \int_s^\tau \|\bar{v}(r, \tau - t, \omega, \bar{v}_{\tau-t})\|^2 dr 
\]
\[
+ 2 \int_s^\tau e^{-2\omega(r)}(\|g(r, \cdot)\|^2 \psi_4(r, \cdot))dr. 
\]
We now integrate (103) with respect to $s$ on $(\tau - 1, \tau)$ and replace $\omega$ by $\theta_{-\tau}\omega$ to obtain
\[
\|
\nabla \bar{u}(\tau, \tau - t, \theta_{-\tau}\omega, \bar{u}_{\tau-t})\|^2 
\]
\[
\leq c_1\int_{\tau-1}^\tau \|
\nabla \bar{u}(s, \tau - t, \theta_{-\tau}\omega, \bar{u}_{\tau-t})\|^2 + \|
\bar{v}(s, \tau - t, \theta_{-\tau}\omega, \bar{v}_{\tau-t})\|^2 ds + c_1, 
\]
where $c_1 = c_1(\tau, \omega) > 0$. By (104) and Lemma 3.1, we obtain the desired estimates.

Similarly to the decomposition method used in Section 2, by (91), we write $\bar{v} = \bar{v}_1 + \bar{v}_2$ where $\bar{v}_1$ and $\bar{v}_2$ solve the following systems
\[
\frac{d\bar{v}_1}{dt} + \sigma\bar{v}_1 = 0, \quad \bar{v}_1(\tau) = \bar{v}, 
\]
and
\[
\frac{d\bar{v}_2}{dt} + \sigma\bar{v}_2 = \beta\bar{u} + e^{-\omega(t)}h(t, x), \quad \bar{v}_2(\tau) = 0.
\]
By (105) we find that for $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $e^{\omega(-t)-\omega(-\tau)}\bar{v}_{\tau-t} \in D(\tau - t, \theta_{-\tau}\omega)$,
\[
\|
\bar{v}_1(\tau, \tau - t, \theta_{-\tau}\omega, \bar{v}_{\tau-t})\|^2 \leq e^{-2\sigma t+2\omega(-\tau)-2\omega(-t)}\|\bar{D}(\tau - t, \theta_{-\tau}\omega)\|^2 \to 0 \text{ as } t \to \infty.
\]

Next, we derive uniform estimates of $\bar{v}_2$ in $H_0^1(\Omega)$.

**Lemma 3.4.** Let (G) and (80) hold. Then for every $\tau \in \mathbb{R}$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D}$, there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \geq T$, the solution of equation (106) satisfies
\[
\|
\nabla \bar{v}_2(\tau, \tau - t, \theta_{-\tau}\omega, 0)\|^2 
\]
\[
\leq L_3 \int_{-\infty}^0 e^{\kappa s-2\omega(s)+2\omega(-\tau)}(1 + \|g(s + \tau, \cdot)\|^2 + \|h(s + \tau, \cdot)\|^2_{H_0^1(\Omega)})ds, 
\]
where $L_3$ is a positive constant depending on $\tau$ and $\omega$.

**Proof.** The proof is similar to that of Lemma 2.6 and hence omitted here.

We now establish the $\mathcal{D}$-pullback asymptotic compactness of $\Phi_0$ in $L^2(\Omega)$. First, we need to split $\Phi_0$ as before. For every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $w_\tau = (u_\tau, v_\tau) \in L^2(\Omega)$, we define
\[
\Phi_{0,1}(t, \tau, \omega, w_\tau) = (0, e^{\omega(t)-\omega(-\tau)}\bar{v}_1(t + \tau, \tau, \theta_{-\tau}\omega, e^{\omega(-\tau)}v_\tau)) 
\]
and
\[
\Phi_{0,2}(t, \tau, \omega, w_\tau) = (u(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau), e^{\omega(t)-\omega(-\tau)}\bar{v}_2(t + \tau, \tau, \theta_{-\tau}\omega, 0)), 
\]
where $\tilde{v}_1$ and $\tilde{v}_2$ are the solutions of equations (105) and (106), respectively. By (92), we find for every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $w_\tau = (u_\tau, v_\tau) \in L^2(\mathcal{O})$,

$$
\Phi_0(t, \tau, w_\tau) = \Phi_{0,1}(t, \tau, w_\tau) + \Phi_{0,2}(t, \tau, w_\tau).
$$

We now prove the $D$-pullback asymptotic compactness of solutions of (89).

**Lemma 3.5.** Suppose (G) and (47), (80) hold. Then the cocycle $\Phi_0$ associated with the stochastic system (89) is $D$-pullback asymptotically compact in $L^2(\mathcal{O})$, that is, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$, the sequence $\Phi_0(t_n, \tau - t_n, \theta_{t_n} \omega; w_{0,n})$ has a convergent subsequence in $L^2(\mathcal{O})$ provided $t_n \to \infty$ and $w_{0,n} \in D(\tau - t_n, \theta_{t_n} \omega)$.

**Proof.** The proof is similar to that of Lemma 2.7, and hence omitted here. $\square$

We now in the position to show the existence of $D$-pullback attractors of $\Phi_0$.

**Theorem 3.6.** Let (G), (47)-(48) and (80) hold. Then the cocycle $\Phi_0$ associated with problem (89) has a unique $D$-pullback attractor $A_0 = \{A_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$ in $L^2(\mathcal{O})$.

**Proof.** Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, define a subset $K_0(\tau, \omega)$ by

$$
K_0(\tau, \omega) = \{ w \in L^2(\mathcal{O}) : \|w\|^2_{L^2(\mathcal{O})} \leq R_0(\tau, \omega) \},
$$

where $R_0(\tau, \omega)$ is defined in (98). Then by Lemma 3.1 we know that for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D \in \mathcal{D}$, there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \geq T$,

$$
\Phi_0(t, \tau - t, \theta_{-t} \omega; D(\tau - t, \theta_{-t} \omega)) = w(\tau, \tau - t, \theta_{-t} \omega, D(\tau - t, \theta_{-t} \omega)) \subseteq K_0(\tau, \omega).
$$

Moreover, by (48), one can verify that $K_0 \in D$. Therefore, $K_0$ is a closed measurable $D$-pullback absorbing set, which together with Lemma 3.5 implies the existence and uniqueness of $D$-pullback attractor $A_0$ of $\Phi_0$. $\square$

4. **Convergence of random attractors.** In this section, we study the limiting behavior of solutions of the random system (3) when $\delta \to 0$. Under certain conditions, we will show the solutions and attractors of system (3) converge to that of the corresponding stochastic system when $\delta \to 0$.

Consider the following random system

$$
\begin{align*}
\frac{\partial u_\delta}{\partial t} - \Delta u_\delta + a_v u_\delta &= f(t, x, u_\delta) + g(t, x) + \zeta_\delta(\theta_t \omega) u_\delta, \\
\delta_0 u_\delta^\prime + \sigma v_\delta - \beta u_\delta &= h(t, x) + \zeta_\delta(\theta_t \omega) v_\delta, \\
u_0(t, x) = v_0(t, x) = 0, \\
u_\delta(t, x) = u_\delta(t, x), v_\delta(t, x) = v_\delta(t, x),
\end{align*}
$$

(114)

From now on, we write the solution of (114) as $w_\delta = (u_\delta, v_\delta)$ to indicate its dependence on $\delta$. Note that system (114) is a special case of (3) and can be obtained by formally replacing $W(t)$ by $\int_0^t \zeta_\delta(\theta_t \omega)dr$ in (89). We will establish the relations between the solutions of systems (89) and (114) and show that the limiting behavior of system (114) is governed by the stochastic system (89) as $\delta \to 0$.

As indicated in Section 2, for every $\delta \in (0,1]$, system (114) generates a continuous cocycle $\Phi_\delta$ in $L^2(\mathcal{O})$ which possesses a unique $D$-pullback attractor $A_\delta$. To compare the solutions of (114) and (89), we introduce a new variable $\tilde{w}_\delta$ given by

$$
\tilde{w}_\delta(t, \tau, \omega) = (\tilde{u}_\delta(t, \tau, \omega), \tilde{v}_\delta(t, \tau, \omega)) = e^{-\int_0^t \zeta_\delta(\theta_t \omega)dr} w_\delta(t, \tau, \omega).
$$

(115)
By (114) and (115), we obtain

\[
\begin{aligned}
\partial_t \bar{u}_\delta + \Delta \bar{u}_\delta + \alpha \bar{u}_\delta &= e^{-\int_0^t \zeta(t,\omega)dr} f(t, x, e^{\int_0^t \zeta(t,\omega)dr} \bar{u}_\delta) \\
+ e^{-\int_0^t \zeta(t,\omega)dr} g(t, x), \quad x \in \Omega, t \geq \tau, \\
\partial_t \bar{u}_\delta + \sigma \bar{u}_\delta - \beta \bar{u}_\delta &= e^{-\int_0^t \zeta(t,\omega)dr} h(t, x), \quad x \in \Omega, t > \tau, \\
\bar{u}_\delta(t, x) &= \bar{u}_\delta(t, x) = 0, \quad x \in \partial \Omega, t \geq \tau, \\
\bar{u}_\delta(t, x) &= \bar{u}_\delta(x) = e^{-\int_0^t \zeta(t,\omega)dr} u_{\delta,T}(x), \\
\bar{u}_\delta(t, x) &= \bar{u}_\delta(x) = e^{-\int_0^t \zeta(t,\omega)dr} v_{\delta,T}(x), \quad x \in \Omega.
\end{aligned}
\tag{116}
\]

First, we derive the uniform estimates on the solutions of system (116) on finite time intervals.

**Lemma 4.1.** Let (G) hold true. For every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $T > 0$, there exist $\delta_0 = \delta_0(\tau,\omega,T) > 0$ and $L_4 = L_4(\tau,\omega,T) > 0$ such that for all $0 < \delta < \delta_0$ and $t \in [\tau, \tau + T]$, the solution of system (116) satisfies

\[
\|\bar{w}_\delta(t, \tau, \omega, \bar{w}_{\delta,T})\|_{L^2(\Omega)}^2 + \int_\tau^t \|\bar{u}_\delta(s, \tau, \omega, \bar{u}_{\delta,T})\|_p^p ds \\
\leq L_4(1 + \|\bar{w}_{\delta,T}\|_{L^2(\Omega)}^2 + \int_\tau^t \|\bar{F}(s, \cdot, \cdot)\|_1^2 ds). \tag{117}
\]

**Proof.** By (116) and (G1) we have

\[
\begin{aligned}
\frac{d}{dt} \beta \|\bar{u}_\delta\|^2 + \alpha \|\bar{v}_\delta\|^2 &+ 2\beta \|\nabla \bar{u}_\delta\|^2 + 2\alpha \|\bar{v}_\delta\|^2 \\
= 2\beta e^{-\int_0^t \zeta(t,\omega)dr} (f(t, x, e^{\int_0^t \zeta(t,\omega)dr} \bar{u}_\delta), \bar{u}_\delta) \\
+ 2\beta e^{-\int_0^t \zeta(t,\omega)dr} (g(t, x), \bar{u}_\delta) + 2\alpha e^{-\int_0^t \zeta(t,\omega)dr} (h(t, x), \bar{v}_\delta) \\
&\leq -2\beta \alpha_1 \|\bar{u}_\delta\|^2 + 2\beta \|\bar{v}_\delta\|^2 + 2\alpha \|\bar{v}_\delta\|^2 + \frac{\beta \lambda_1}{4} \|\bar{u}_\delta\|^2 \\
&+ \frac{3\alpha \beta}{4} \|\bar{v}_\delta\|^2 + \frac{4}{\kappa} e^{-2\int_0^t \zeta(t,\omega)dr} \|\bar{F}(t, \cdot, \cdot)\|_1^2 + \alpha \|h(t, \cdot)\|^2,
\end{aligned}
\tag{118}
\]

which implies

\[
\begin{aligned}
\frac{d}{dt} \beta \|\bar{u}_\delta\|^2 + \alpha \|\bar{v}_\delta\|^2 &+ \kappa (\beta \|\bar{u}_\delta\|^2 + \alpha \|\bar{v}_\delta\|^2) + \frac{\beta}{2} \|\nabla \bar{u}_\delta\|^2 \\
&+ \frac{\kappa}{4} (\beta \|\bar{u}_\delta\|^2 + \alpha \|\bar{v}_\delta\|^2) + 2\beta \alpha_1 e^{-2\int_0^t \zeta(t,\omega)dr} \|\bar{u}_\delta\|_p^p \\
&\leq 2\beta e^{-\int_0^t \zeta(t,\omega)dr} \|\bar{F}(t, \cdot, \cdot)\|_1^2 + \frac{4}{\kappa} e^{-2\int_0^t \zeta(t,\omega)dr} \|\bar{F}(t, \cdot, \cdot)\|_1^2 + \alpha \|h(t, \cdot)\|^2.
\end{aligned}
\tag{119}
\]

For all $\omega \in \Omega$ and $t \geq \tau$ with $\tau \in \mathbb{R}$, integrating (119) from $\tau$ to $t$, we find

\[
\begin{aligned}
\|\bar{w}_\delta(t, \tau, \omega, \bar{w}_{\delta,T})\|_{L^2(\Omega)}^2 + \int_\tau^t e^{\kappa(s-\tau)} \|\nabla \bar{u}_\delta(s)\|^2 ds \\
+ \int_\tau^t e^{\kappa(s-\tau)} e^{-2\int_0^t \zeta(t,\omega)dr} \|\bar{u}_\delta(s)\|_p^p ds \\
\leq C e^{\kappa(t-\tau)} \|\bar{w}_{\delta,T}\|_{L^2(\Omega)}^2 \\
+ c \int_\tau^t e^{\kappa(s-\tau)-2\int_0^t \zeta(t,\omega)dr} (\|\bar{F}(s, \cdot, \cdot)\|^2 + \|\bar{F}(s, \cdot, \cdot)\|_1^2) ds,
\end{aligned}
\tag{120}
\]

which along with (42) implies (117). \qed
Next, we derive uniform estimates of the solutions when $t \to \infty$.

**Lemma 4.2.** Let (G) and (47) hold. Then for every $0 < \delta \leq 1$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau, \omega, D, \delta) > 0$ such that for all $t \geq T$, the solution of system (116) satisfies

$$
\left\| \bar{w}_\delta(\tau, t, \theta_{-\tau}, \bar{w}_{\delta, \tau-\cdot}) \right\|^2_{L^2(\Omega)} + \int_{\tau-\cdot}^{\tau} e^{\kappa(s-\cdot)}(\|\bar{u}_\delta\|^2 + \|\nabla \bar{u}_\delta\|^2 + \|\bar{v}_\delta\|^2)ds 
\leq R_\delta(\tau, \omega),
$$

where $e^{\int_{\tau-\cdot}^{\tau} \zeta(s, \theta, \omega)dr} \bar{w}_{\delta, \tau-\cdot} \in D(\tau - t, \theta_{-\cdot})$ and

$$
\bar{R}_\delta(\tau, \omega) = L_5 \int_{-\infty}^{0} e^{\kappa(s-\cdot)} \zeta(s, \theta, \omega)dr (1 + \|F(s + \tau, \cdot)\|^2)ds,
$$

with $L_5$ being a positive constant independent of $\tau$, $\omega$ and $\delta$.

**Proof.** For $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, integrating (119) from $\tau$ to $\tau$, we get

$$
\int_{\tau}^{\tau} e^{\kappa(s-\cdot)} \|\nabla \bar{u}_\delta(s)\|^2ds + \int_{\tau}^{\tau} e^{\kappa(s-\cdot)}(\|\bar{u}_\delta(s)\|^2 + \|\bar{v}_\delta(s)\|^2)ds 
\leq e^{-\kappa t}(\|\bar{u}_\delta, \tau-\cdot\|^2 + \|\bar{v}_\delta, \tau-\cdot\|^2) + c_1 \int_{-\infty}^{0} e^{\kappa(s-\cdot)} \zeta(s, \theta, \omega)dr (\|F(s + \tau, \cdot)\|^2 + \|\psi_1(s + \tau, \cdot)\|_1)ds 
\leq (\alpha + \beta)e^{-\kappa t+2\int_{-\cdot}^{\tau} \zeta(s, \theta, \omega)dr} \|D(\tau - t, \theta_{-\cdot})\|^2_{L^2(\Omega)} + c_2 \int_{-\infty}^{0} e^{\kappa(s-\cdot)} \zeta(s, \theta, \omega)dr (1 + \|F(s + \tau, \cdot)\|^2)ds,
$$

where $c_2$ is a positive constant independent of $\tau$, $\omega$ and $\delta$. Due to (7), we find

$$
\int_{-\infty}^{0} e^{\kappa(s-\cdot)} \zeta(s, \theta, \omega)dr (1 + \|F(s + \tau, \cdot)\|^2)ds < \infty.
$$

On the other hand, since $D \in \mathcal{D}$, again by (7), we see that there exists $T = T(\tau, \omega, D, \delta) > 0$ such that for all $t \geq T$,

$$
(\alpha + \beta)e^{-\kappa t+2\int_{-\cdot}^{\tau} \zeta(s, \theta, \omega)dr} \|D(\tau - t, \theta_{-\cdot})\|^2_{L^2(\Omega)} 
\leq \int_{-\infty}^{0} e^{\kappa(s-\cdot)} \zeta(s, \theta, \omega)dr (1 + \|F(s + \tau, \cdot)\|^2)ds,
$$

which together with (4) and (122) completes the proof. \qed

As an immediate consequence of Lemma 4.2, we obtain the following estimates.

**Lemma 4.3.** Let (G) and (47) hold. Then for every $0 < \delta \leq 1$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau, \omega, D, \delta) > 0$ such that for all $t \geq T$, the solution of system (114) satisfies

$$
\left\| w_\delta(\tau, t, \theta_{-\tau}, \omega) \right\|^2_{L^2(\Omega)} \leq R_\delta(\tau, \omega),
$$

where $w_{\delta, \tau-\cdot} \in D(t - \tau, \theta_{-\cdot})$ and

$$
R_\delta(\tau, \omega) = L_5 \int_{-\infty}^{0} e^{\kappa(s-\cdot)} \zeta(s, \theta, \omega)dr (1 + \|\bar{F}(s + \tau, \cdot)\|^2)ds,
$$

(124)
with $L_5$ being the same positive constant as in Lemma 4.2 which is independent of $\tau$, $\omega$ and $\delta$.

**Proof.** Due to (115), we have
\[
\|w_\delta(\tau, t, \theta_{-t} \omega, w_\delta, \tau-t)\|_{L^2(\Omega)}^2
\]
which along with Lemma 4.2 implies the desired estimates.
\[
\square
\]

Based on Lemma 4.3, we find a $\mathcal{D}$-pullback absorbing set for equation (114).

**Lemma 4.4.** Suppose (G) and (47)-(48) hold. Then for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the continuous cocycle $\Phi_\delta$ associated with equation (114) possesses a $\mathcal{D}$-pullback absorbing set $K_\delta \in \mathcal{D}$ given by
\[
K_\delta(\tau, \omega) = \{w_\delta \in L^2(\Omega) : \|w_\delta\|_{L^2(\Omega)} \leq R_\delta(\tau, \omega)\},
\] where $R_\delta(\tau, \omega)$ is given by (124), and
\[
\lim_{\delta \to 0} R_\delta(\tau, \omega) = L_5 \int_{-\infty}^0 e^{\kappa s - 2\omega(s)} (1 + \|F(s + \tau, \cdot\|)^2)ds.
\]

**Proof.** As in Lemma 2.5, one can verify $K_\delta \in \mathcal{D}$, which along with Lemma 4.3 implies that $K_\delta$ given by (125) is a $\mathcal{D}$-pullback absorbing set of $\Phi_\delta$. We now prove (126). Given $s, \tau \in \mathbb{R}$, $\delta > 0$ and $\omega \in \Omega$, let
\[
\tilde{R}_\delta(\tau, \omega, s) = e^{\kappa s - 2\omega(s)} \int_0^s \zeta_\delta(\theta_r \omega)dr (1 + \|F(s + \tau, \cdot\|)^2).
\]

By Lemma 2.2 we see
\[
\lim_{\delta \to 0} \tilde{R}_\delta(\tau, \omega, s) = e^{\kappa s - 2\omega(s)} (1 + \|F(s + \tau, \cdot\|)^2).
\]

By (7) we infer that there exists $T_1 = T_1(\omega) > 0$ such that for all $s \leq -T_1$ and $0 < \delta \leq 1$,
\[
2 \left| \int_0^s \zeta_\delta(\theta_r \omega)dr \right| < -(\kappa - \kappa_0)s.
\]

By (127) and (129) we obtain, for all $s \leq -T_1$ and $0 < \delta \leq 1$,
\[
\tilde{R}_\delta(\tau, \omega, s) \leq e^{\kappa s} (1 + \|F(s + \tau, \cdot\|)^2).
\]

By (47), (128), (130) and the Lebesgue Dominated Convergence Theorem we get
\[
\lim_{\delta \to 0} \int_{-\infty}^{-T_1} \tilde{R}_\delta(\tau, \omega, s)ds = \int_{-\infty}^{-T_1} e^{\kappa s - 2\omega(s)} (1 + \|F(s + \tau, \cdot\|)^2)ds.
\]

On the other hand, by Lemma 2.2, $\int_0^s \zeta_\delta(\theta_r \omega)dr \to \omega(s)$ uniformly on $[-T_1, 0]$ as $\delta \to 0$, and hence we obtain
\[
\lim_{\delta \to 0} \int_{-T_1}^0 \tilde{R}_\delta(\tau, \omega, s)ds = \int_{-T_1}^0 e^{\kappa s - 2\omega(s)} (1 + \|F(s + \tau, \cdot\|)^2)ds.
\]

By (131) and (132) we obtain
\[
\lim_{\delta \to 0} \int_{-\infty}^0 \tilde{R}_\delta(\tau, \omega, s)ds = \int_{-\infty}^0 e^{\kappa s - 2\omega(s)} (1 + \|F(s + \tau, \cdot\|)^2)ds,
\]
which yields (126). \qed
Now, we establish the convergence of solutions of (114) as \( \delta \to 0 \). For that purpose, we further assume that there exists \( \psi_8 \in L^\infty_{loc}(\mathbb{R}, L^\infty(\mathcal{O})) \) such that for all \( t, s \in \mathbb{R} \) and \( x \in \mathcal{O} \),
\[
\frac{\partial f}{\partial s}(t, x, s) \leq \psi_8(t, x)(1 + |s|^{p-2}). \tag{134}
\]

**Lemma 4.5.** Suppose (G) and (134) hold and let \( B \) be a bounded set in \( L^2(\mathcal{O}) \) with \( w_{\delta, \tau}, w_{\tau} \in B \). If \( w_{\delta} \) and \( w \) are the solutions of (114) and (89) with initial data \( w_{\delta, \tau} \) and \( w_{\tau} \), respectively, then for every \( t \in [\tau, T + \tau] \),
\[
\|w_\delta\|_{L^2(\mathcal{O})} \leq c\|w_\delta - w\|_{L^2(\mathcal{O})} + c\varepsilon. \tag{135}
\]

**Proof.** Let \( \xi = (\xi_1, \xi_2) = (\bar{u}_\delta - \bar{u}, \bar{u}_\delta - \bar{v}) \). We infer from (116) and (91) that
\[
\frac{\partial f}{\partial t}(\beta|\xi_1|^2 + \alpha|\xi_2|^2) + 2\beta|\nabla \xi_1|^2 + 2\alpha|\xi_2|^2
\]
\[
= 2\beta \int_{\mathcal{O}} \left( e^{-\int_0^t \zeta_s(\theta, \omega)dr} \theta f(t, x, e^{\int_0^t \zeta_s(\theta, \omega)dr} \bar{u}_\delta) - e^{-\omega(t)} f(t, x, e^{\omega(t)} \bar{u}) \right) \xi_1 dx + 2\beta(\varepsilon \int_0^t \zeta_s(\theta, \omega)dr - e^{-\omega(t)} (g, \xi_1) + 2\alpha(e^{-\int_{\omega(t)}^t \zeta_s(\theta, \omega)dr - e^{-\omega(t)} (h, \xi_2).
\]

By Lemma 2.2 we know that for \( \varepsilon > 0 \) and \( T > 0 \), there exists \( \delta_1 = \delta_1(\varepsilon, \tau, \omega, T) > 0 \) such that for all \( 0 < \delta < \delta_1 \) and \( t \in [\tau, T + \tau] \),
\[
|e^{-\int_0^t \zeta_s(\theta, \omega)dr - e^{-\omega(t)}| < \varepsilon \quad \text{and} \quad |e^{\int_0^t \zeta_s(\theta, \omega)dr - e^{-\omega(t)} - 1| < \varepsilon. \tag{137}
\]

First using (G3), then using (G2) and (134) to estimate the nonlinear function in (136), we find that there exists \( c_1 = c_1(\tau, \omega, T) > 0 \) such that for all \( 0 < \delta < \delta_1 \) and \( t \in [\tau, T + \tau] \),
\[
\int_{\mathcal{O}} \theta f(t, x, e^{\int_0^t \zeta_s(\theta, \omega)dr} \bar{u}_\delta) - e^{-\omega(t)} f(t, x, e^{\omega(t)} \bar{u}) \xi_1 dx
\]
\[
= \int_{\mathcal{O}} e^{-\int_0^t \zeta_s(\theta, \omega)dr} (f(t, x, e^{\int_0^t \zeta_s(\theta, \omega)dr} \bar{u}_\delta) - f(t, x, e^{\int_0^t \zeta_s(\theta, \omega)dr} \bar{u})) \xi_1 dx + \int_{\mathcal{O}} e^{-\omega(t)} (f(t, x, e^{\int_0^t \zeta_s(\theta, \omega)dr} \bar{u})) \xi_1 dx \tag{138}
\]
\[
\leq c_1|\xi_1|^2 + c_2\varepsilon(1 + ||\bar{u}||_{p_1}^2 + ||\bar{u}_\delta||_{p_1}^2 + ||\psi_2(t, \cdot)||_{p_1}^2).
\]

In addition, we have
\[
|e^{-\int_0^t \zeta_s(\theta, \omega)dr - e^{-\omega(t)}(g, \xi_1) \leq \frac{1}{2}\varepsilon||\xi_1||^2 + \frac{1}{2}\varepsilon||g(t, \cdot)||^2 \tag{139}
\]
and
\[
|e^{-\int_0^t \zeta_s(\theta, \omega)dr - e^{-\omega(t)}(h, \xi_2) \leq \frac{1}{2}\varepsilon||\xi_2||^2 + \frac{1}{2}\varepsilon||h(t, \cdot)||^2. \tag{140}
\]

By (138), (139) and (136), we find that for all \( \varepsilon \in (0, 1) \), \( 0 < \delta < \delta_1 \) and \( t \in [\tau, T + \tau] \),
\[
\frac{d}{dt}||\xi||^2 \leq c_2||\xi||^2 + c_2\varepsilon(1 + ||\bar{u}||_{p_1}^2 + ||\bar{u}_\delta||_{p_1}^2 + ||F(t, \cdot)||^2 + ||\psi_2(t, \cdot)||_{p_1}^2). \tag{141}
\]
Solving (141) from $\tau$ to $t$ with $t \in [\tau, \tau + T]$, we get
\[
\|\xi(t)\|^2 \leq e^{c_2(t-\tau)}\|\xi(\tau)\|^2 + c_2\epsilon e^{c_2(t-\tau)} \int_{\tau}^{t} (1 + \|u(s)\|_{p} + \|\bar{u}_{\delta}(s)\|_{p} + \|F(s, \cdot)\|^2 + \|w_2(s, \cdot)\|_{p}^{1})ds.
\] (142)

By (142), (93) and Lemma 4.1 we find that there exist $\delta_2 \in (0, \delta_1)$ and $c_3 = c_3(\tau, \omega, T) > 0$ such that for all $0 < \delta < \delta_2$ and $t \in [\tau, \tau + T]$,
\[
\|\tilde{w}_{\delta}(t, \tau, \omega, \bar{w}_{\delta, \tau}) - \tilde{w}(t, \tau, \omega, \bar{w}_{\tau})\|_{L^2(\Omega)}^2 \leq c_3\|\tilde{w}_{\delta, \tau} - \bar{w}_{\tau}\|_{L^2(\Omega)}^2 + c_3\epsilon.
\] (143)

Note that
\[
\begin{aligned}
\tilde{w}_{\delta}(t, \tau, \omega, \bar{w}_{\delta, \tau}) - \tilde{w}(t, \tau, \omega, \bar{w}_{\tau}) &= e^{\int_{\tau}^{t} \zeta(\theta, \omega)d\theta} \tilde{w}_{\delta}(t, \tau, \omega, \bar{w}_{\delta, \tau}) - e^{\omega(t)} \tilde{w}(t, \tau, \omega, \bar{w}_{\tau}) \\
&= e^{\int_{\tau}^{t} \zeta(\theta, \omega)d\theta} (\tilde{w}_{\delta}(t, \tau, \omega, \bar{w}_{\delta, \tau}) - \tilde{w}(t, \tau, \omega, \bar{w}_{\tau})) \\
&\quad + (e^{\int_{\tau}^{t} \zeta(\theta, \omega)d\theta} - e^{\omega(t)}) \tilde{w}(t, \tau, \omega, \bar{w}_{\tau}),
\end{aligned}
\] (144)

where $\tilde{w}_{\delta, \tau} = e^{\int_{\tau}^{t} \zeta(\theta, \omega)d\theta}$ and $w_{\tau} = e^{\omega(t)} \tilde{w}_{\tau}$. It follows from (42) and (144) that there exist $\delta_3 \in (0, \delta_2)$ and $c_4 = c_4(\tau, \omega, T) > 0$ such that for all $0 < \delta < \delta_3$ and $t \in [\tau, \tau + T]$,
\[
\|\tilde{w}_{\delta}(t, \tau, \omega, \bar{w}_{\delta, \tau}) - \tilde{w}(t, \tau, \omega, \bar{w}_{\tau})\|_{L^2(\Omega)}^2 \leq c_4\|\tilde{w}_{\delta, \tau} - \bar{w}_{\tau}\|_{L^2(\Omega)}^2 + c_4\epsilon,
\]
which completes the proof. $\square$

As an immediate consequence of Lemma 4.5, we obtain the convergence of solutions of (114) as $\delta \to 0$.

**Corollary 3.** Assume (G), (134) and $\delta_n \to 0$. Let $w_{\delta_n}$ and $w$ be the solutions of (114) and (89) with initial data $w_{\delta_n, \tau}$ and $w_{\tau}$, respectively. If $w_{\delta_n, \tau} \to w_{\tau} \in L^2(\Omega)$ as $n \to \infty$, then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t > \tau$,
\[
w_{\delta_n}(t, \tau, \omega, w_{\delta_n, \tau}) \to w(t, \tau, \omega, w_{\tau}) \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad n \to \infty.
\] (145)

To establish the uniform compactness of random attractors, we need the following estimates.

**Lemma 4.6.** Suppose (G) and (47) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \subset D$, there exists $\delta_0 = \delta_0(\tau, \omega) \in (0, 1)$ such that for all $0 < \delta < \delta_0$, there exists $T = T(\tau, \omega, D, \delta) \geq 1$ such that for all $t \geq T$, the solution of system (116) satisfies
\[
\|\nabla \tilde{u}_{\delta}(t, \tau-t, \theta_{-t}\omega, \bar{w}_{\delta, \tau-t})\|^2 \leq L_6 + L_6 \int_{-\infty}^{0} e^{\alpha s} f_{+}^{*} \zeta(s, \theta, \omega) ds \left(1 + \|F(s + \tau, \cdot)\|^2\right)ds,
\]
where $e^{\int_{\tau}^{t} \zeta(\theta, \omega)d\theta} \tilde{w}_{\delta, \tau-t} \in D(\tau-t, \theta_{-t}(\omega))$ and $L_6$ is a positive constant depending on $\tau$ and $\omega$, but independent of $\delta$.

**Proof.** By (116) we find
\[
\frac{d}{dt} \|\nabla \tilde{u}_{\delta}\|^2 + 2\|\Delta \tilde{u}_{\delta}\|^2 = -2e^{-\int_{0}^{t} \zeta(\theta, \omega)d\theta} \int_{\Omega} f(t, x, u_{\delta})\Delta \tilde{u}_{\delta} dx \\
- 2e^{-\int_{0}^{t} \zeta(\theta, \omega)d\theta} (g, \Delta \tilde{u}_{\delta}) + 2\alpha(\Delta \tilde{u}_{\delta}, v_{\delta}).
\] (146)
Following the proof of (3), we can obtain from (146) that
\[
\frac{d}{dt} \|\nabla \tilde{u}_{\delta}\|^2 \leq (1 + 2e^{-\int_0^t \zeta_i(\theta, \omega)dr} \|\psi_4(t, \cdot)\|_{L^\infty(O)}) \|\nabla \tilde{u}_{\delta}\|^2 \\
+ 2\alpha_2 \|\tilde{u}_{\delta}\|^2 + e^{-\int_0^t \zeta_i(\theta, \omega)dr} (\|\psi_4(t, \cdot)\|^2 + 2\|g(t, \cdot)\|^2).
\] (147)

Given \( t \in \mathbb{R}^+ \), \( \tau \in \mathbb{R} \) and \( s \in (\tau - 1, \tau) \), first integrating (4) with respect to \( t \) on \( (s, \tau) \), then integrating with respect to \( s \) on \( (\tau - 1, \tau) \), after replacing \( \omega \) by \( \theta_{-\tau_\omega} \), we get
\[
\|\nabla \tilde{u}_{\delta}(\tau, \tau - t, \theta_{-\tau_\omega}, \tilde{u}_{\tau - t})\|^2 \\
\leq \int_{\tau - 1}^0 \|\nabla \tilde{u}_{\delta}(s + \tau, \tau - t, \theta_{-\tau_\omega}, \tilde{u}_{\delta, \tau - t})\|^2 ds \\
+ \int_{\tau - 1}^0 (1 + 2e^{-\int_0^s \zeta_i(\theta_\omega)dr} \|\psi_4(s + \tau, \cdot)\|_{L^\infty(O)}) \|\nabla \tilde{u}_{\delta}(s + \tau)\|^2 ds \\
+ 2\alpha_2 \int_{\tau - 1}^0 \|\tilde{u}_{\delta}(s + \tau, \tau - t, \theta_{-\tau_\omega}, \tilde{u}_{\delta, \tau - t})\|^2 ds \\
+ \int_{\tau - 1}^0 e^{-\int_0^\tau \zeta_i(\theta_\omega)dr} (\|\psi_4(s + \tau, \cdot)\|^2 + 2\|g(s + \tau, \cdot)\|^2)ds.
\] (148)

By (42) and (148) we find that there exist \( \delta_0 = \delta_0(\tau, \omega) > 0 \) and \( c_1 = c_1(\tau, \omega) > 0 \) such that for all \( 0 < \delta < \delta_0 \),
\[
\|\nabla \tilde{u}_{\delta}(\tau, \tau - t, \theta_{-\tau_\omega}, \tilde{u}_{\tau - t})\|^2 \\
\leq c_1 + c_1 \int_{\tau - 1}^0 (\|\nabla \tilde{u}_{\delta}(s + \tau, \tau - t, \theta_{-\tau_\omega}, \tilde{u}_{\delta, \tau - t})\|^2 + \|\tilde{u}_{\delta}(s + \tau)\|^2)ds
\]
which along with Lemma 4.2 implies that there exists \( T_1 = T_1(\tau, \omega, D, \delta) \geq 1 \) such that for \( 0 < \delta < \delta_0 \) and \( t \geq T_1 \),
\[
\|\nabla \tilde{u}_{\delta}(\tau, \tau - t, \theta_{-\tau_\omega}, \tilde{u}_{\tau - t})\|^2 \\
\leq c_1 + c_1 L_5 \int_{-\infty}^0 e^{\alpha s - 2\int_0^s \zeta_i(\theta_\omega)dr} (1 + \|F(s + \tau, \cdot)\|^2)ds,
\]
which completes the proof.

Again, we need to decompose \( \tilde{u}_{\delta} \) as \( \tilde{u}_{\delta} = \tilde{u}_{\delta, 1} + \tilde{u}_{\delta, 2} \), where \( \tilde{u}_{\delta, 1} \) and \( \tilde{u}_{\delta, 2} \) solve the following systems
\[
\frac{d\tilde{u}_{\delta, 1}}{dt} + \sigma \tilde{u}_{\delta, 1} = 0, \quad \tilde{u}_{\delta, 1}(\tau) = \tilde{u}_{\delta, \tau}, \quad (149)
\]
and
\[
\frac{d\tilde{u}_{\delta, 2}}{dt} + \sigma \tilde{u}_{\delta, 2} = \beta \tilde{u}_{\delta} + e^{-\int_0^t \zeta_i(\theta_\omega)dr} h(t, x), \quad \tilde{u}_{\delta, 2}(\tau) = 0. \quad (150)
\]

By (149) we find that for \( t \in \mathbb{R}^+ \), \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and \( e^{\int_0^\tau \zeta_i(\theta_\omega)dr} \tilde{u}_{\delta, \tau - t} \in D(\tau - t, \theta_{-\tau_\omega}) \),
\[
\|\tilde{u}_{\delta, 1}(\tau, \tau - t, \theta_{-\tau_\omega}, \tilde{u}_{\delta, \tau - t})\|^2 \\
\leq e^{-2\sigma t - 2\int_0^t \zeta_i(\theta_\omega)dr} \|D(\tau - t, \theta_{-\tau_\omega})\|^2 \to 0 \text{ as } t \to \infty. \quad (151)
\]

For \( \tilde{u}_{\delta, 2} \), we have the following uniform estimates in \( H_0^1(O) \).
Lemma 4.7. Let $(G)$, (47) and (80) hold. Then for every $0 < \delta \leq 1$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$, there exists $T = T(\tau, \omega, D, \delta) > 0$ such that for all $t \geq T$, the solution of equation (150) satisfies
\[
\|\nabla \Phi_{\delta,2}(\tau - t, \theta_{-t}\omega, 0)\|^2 \leq L_7 \int_{-\infty}^{0} e^{\kappa s - 2} f^*_C(\zeta_s(\theta, \omega) dr \left(1 + \|g(s + \tau, \cdot)\|^2 + \|h(s + \tau, \cdot)\|_{H^1_0(\Omega)}^2\right) ds,
\]
where $L_7$ is a positive constant independent of $\tau, \omega, D$ and $\delta$.

Proof. The proof is similar to Lemma 2.6. Of course, Lemma 4.2 instead of Lemma 2.3 should be used this time, and the details are omitted. \hfill \Box

As before, we split $\Phi_{\delta}$ as follows. For every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $w_{\delta,\tau} = (u_{\delta,\tau}, v_{\delta,\tau}) \in L^2(\mathcal{O})$, we define
\[
\Phi_{\delta,1}(t, \tau, \omega, w_{\delta,\tau}) = (0, e^{\int_0^t \zeta_s(\theta, \omega) dr} v_{\delta,1}(t + \tau, \tau - \omega, e^{\int_0^t \zeta_s(\theta, \omega) dr} v_{\delta,\tau}))
\]
and
\[
\Phi_{\delta,2}(t, \tau, \omega, w_{\delta,\tau}) = (u_{\delta}(t + \tau, \tau - \omega, u_{\delta,\tau}), e^{\int_0^t \zeta_s(\theta, \omega) dr} v_{\delta,2}(t + \tau, \tau - \omega, 0)),
\]
where $v_{\delta,1}$ and $v_{\delta,2}$ are the solutions of equations (149) and (150), respectively. Then for every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $w_{\delta,\tau} = (u_{\delta,\tau}, v_{\delta,\tau}) \in L^2(\mathcal{O})$,
\[
\Phi_{\delta}(t, \tau, \omega, w_{\delta,\tau}) = \Phi_{\delta,1}(t, \tau, \omega, w_{\delta,\tau}) + \Phi_{\delta,2}(t, \tau, \omega, w_{\delta,\tau}).
\]
Recall that for each $\delta > 0$, $\mathcal{A}_\delta$ is the unique $\mathcal{D}$-pullback attractor of $\Phi_{\delta}$ in $L^2(\mathcal{O})$. To obtain the uniform compactness of these attractors with respect to $\delta$, we need further estimates on $\Phi_{\delta,\tau}$ as given below.

Lemma 4.8. Suppose $(G)$, (47) and (80) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$, there exists $\delta_0 = \delta_0(\tau, \omega) \in (0, 1)$ such that for every $0 < \delta < \delta_0$, there exists $T = T(\tau, \omega, D, \delta) \geq 1$ such that for all $t \geq T$,
\[
\|\Phi_{\delta,2}(t, \tau - t, \theta_{-t}\omega, \omega_{\delta,\tau})\|_{H^1_0(\mathcal{O}) \times H^1_0(\mathcal{O})}^2 \leq L_8 + L_8 \int_{-\infty}^{0} e^{\kappa s - 2}\omega(s) \left(1 + \|g(s + \tau, \cdot)\|^2 + \|h(s + \tau, \cdot)\|_{H^1_0(\mathcal{O})}^2\right) ds,
\]
where $\omega_{\delta,\tau} \in D(\tau - t, \theta_{-t}\omega)$ and $L_8$ is a constant depending on $\tau$ and $\omega$, but independent of $\delta$.

Proof. By (154) we get
\[
\Phi_{\delta,2}(t, \tau - t, \theta_{-t}\omega, \omega_{\delta,\tau}) = e^{\int_0^t \zeta_s(\theta, \omega) dr} (u_{\delta}(\tau - t, \theta_{-t}\omega, u_{\delta,\tau}), v_{\delta,2}(\tau - t, \theta_{-t}\omega, v_{\delta,\tau})),
\]
which along with Lemmas 4.2, 4.6 and 4.7 implies that there exists $\delta_1 = \delta_1(\tau, \omega) > 0$ such that for every $0 < \delta < \delta_1$, there exists $T_1 = T_1(\tau, \omega, D, \delta) \geq 1$ such that for all $t \geq T_1$,
\[
\|\Phi_{\delta,2}(t, \tau - t, \theta_{-t}\omega, \omega_{\delta,\tau})\|_{H^1_0(\mathcal{O}) \times H^1_0(\mathcal{O})}^2 \leq c_1 e^{\int_{-\infty}^{0} e^{\kappa s - 2} \zeta_s(\theta, \omega) dr} \left(1 + \|g(s + \tau, \cdot)\|^2 + \|h(s + \tau, \cdot)\|_{H^1_0(\mathcal{O})}^2\right) ds,
\]
(156)
where $c_1 = c_1(\tau, \omega) > 0$. By (42) we find that there exist $\delta_2 = \delta_2(\tau, \omega) \in (0, \delta_1)$ and $c_2 = c_2(\tau, \omega) > 0$ such that for all $0 < \delta < \delta_2$,
\[\left| \int_{-\tau}^{0} \zeta_\delta(\theta, \omega) \, d\tau \right| \leq c_2. \tag{157}\]
On the other hand, by the arguments of (133), we can obtain
\[
\lim_{\delta \to 0} \int_{-\infty}^{0} e^{\kappa s - 2} f_0' \zeta_\delta(\theta, \omega) \, d\tau \left( 1 + \|g(s + \tau, \cdot)\|^2 + \|h(s + \tau, \cdot)\|^2 \right)_{H^{1}_0(\mathcal{O})} \, ds
\]
\[= \int_{-\infty}^{0} e^{\kappa s - \omega(s)} \left( 1 + \|g(s + \tau, \cdot)\|^2 + \|h(s + \tau, \cdot)\|^2 \right)_{\mathcal{H}^1_0(\mathcal{O})} ds,
\]
and hence there exists $\delta_3 = \delta_3(\tau, \omega) \in (0, \delta_2)$ such that for all $0 < \delta < \delta_3$,
\[
\int_{-\infty}^{0} e^{\kappa s - 2\omega(s)} \left( 1 + \|g(s + \tau, \cdot)\|^2 + \|h(s + \tau, \cdot)\|^2 \right)_{\mathcal{H}^1_0(\mathcal{O})} ds
\]
\[\leq 1 + \int_{-\infty}^{0} e^{\kappa s - 2\omega(s)} \left( 1 + \|g(s + \tau, \cdot)\|^2 + \|h(s + \tau, \cdot)\|^2 \right)_{\mathcal{H}^1_0(\mathcal{O})} ds. \tag{158}\]
It follows from (156)-(4) that for all $0 < \delta < \delta_3$ and $t \geq T_1$, there exists $c_3 = c_3(\tau, \omega) > 0$ such that
\[
\|\Phi_{\delta,2}(t, \tau - t, \theta_{-t} \omega, w_{\delta, \tau - t})\|^2_{H^1_0(\mathcal{O})} \leq c_3 + c_3 \int_{-\infty}^{0} e^{\kappa s - 2\omega(s)} \left( 1 + \|g(s + \tau, \cdot)\|^2 + \|h(s + \tau, \cdot)\|^2 \right)_{\mathcal{H}^1_0(\mathcal{O})} ds.
\]
This completes the proof. \qed

Next, we establish the uniform compactness of random attractors with respect to $\delta$.

**Lemma 4.9.** Suppose (G), (47) and (80) hold. Then for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there exists $\delta_0 = \delta_0(\tau, \omega) \in (0, 1)$ such that the set $\bigcup_{0 < \delta < \delta_0} \mathcal{A}_\delta(\tau, \omega)$ is precompact in $L^2(\mathcal{O})$.

**Proof.** Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, by Lemma 4.8, we know that there exists $\delta_1 = \delta_1(\tau, \omega) > 0$ such that for every $0 < \delta < \delta_1$ and $D \in \mathcal{D}$, there exists $T_1 = T_1(\tau, \omega, D, \delta) \geq 1$ such that for all $t \geq T_1$,
\[
\|\Phi_{\delta,2}(t, \tau - t, \theta_{-t} \omega, w_{\delta, \tau - t})\|_{H^1_0(\mathcal{O})}^2 \leq c_1,
\]
where $w_{\delta, \tau - t} \in D(\tau - t, \theta_{-t} \omega)$ and $c_1 = c_1(\tau, \omega) > 0$. Let $t_n \to \infty$ and $w_0 \in \mathcal{A}_\delta(\tau, \omega)$ for some $\delta \in (0, \delta_1)$. By the invariance of $\mathcal{A}_\delta$, for each $n$, there exists $w_n \in \mathcal{A}_\delta(\tau - t_n, \theta_{-t_n} \omega)$ such that
\[
w_0 = \Phi_{\delta,1}(t_n, \tau - t_n, \theta_{-t_n} \omega, w_n).
\]
Since $\mathcal{A}_\delta \in \mathcal{D}$ and $t_n \to \infty$, by (159) we see that there exists $N_1 = N_1(\tau, \omega, \delta) \geq 1$ such that for all $n \geq N_1$,
\[
\|\Phi_{\delta,2}(t_n, \tau - t_n, \theta_{-t_n} \omega, w_n)\|_{H^1_0(\mathcal{O})}^2 \leq c_1.
\]
By (161), we find that there exists $\bar{w}_0 \in H^1_0(\mathcal{O}) \times H^1_0(\mathcal{O})$ such that, up to a subsequence
\[
\Phi_{\delta,2}(t_n, \tau - t_n, \theta_{-t_n} \omega, w_n) \to \bar{w}_0 \text{ weakly in } H^1_0(\mathcal{O}) \times H^1_0(\mathcal{O}),
\]
\[\tag{162}\]
as \( n \to \infty \). Moreover
\[
\| \tilde{w}_0 \|^2_{H^1_0(\Omega) \times H^1_0(\Omega)} \leq c_1.
\]
(163)

On the other hand, by (151) and (153) we get
\[
\Phi_{\delta,1}(t_n, \tau - t_n, \theta-t_n, \omega, w_n) \to 0 \quad \text{in} \quad L^2(\Omega).
\]
(164)

By (4), (162) and (164) we obtain \( w_0 = \tilde{w}_0 \), which along with (163) shows that
\[
\| w_0 \|^2_{H^1_0(\Omega) \times H^1_0(\Omega)} \leq c_1 \quad \text{for all} \quad w_0 \in A_\delta(\tau, \omega),
\]
(165)

By (165), the set \( \bigcup_{0 < \delta < \bar{\delta}_1} A_\delta(\tau, \omega) \) is bounded in \( H^1_0(\Omega) \times H^1_0(\Omega) \), and hence precompact in \( L^2(\Omega) \). This completes the proof.

We finally establish the upper semicontinuity of random attractors as \( \delta \to 0 \).

**Theorem 4.10.** Let (G), (47)-(48), (80) and (134) hold. Then for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),
\[
\lim_{\delta \to 0} \text{dist}_{L^2(\Omega)}(A_\delta(\tau, \omega), A_0(\tau, \omega)) = 0.
\]
(166)

**Proof.** Let \( K_0 = \{ K_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) and \( K_\delta = \{ K_\delta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) be the \( D \)-pullback absorbing sets of \( \Phi_0 \) and \( \Phi_\delta \) given by (112) and (125), respectively. By (126) we have
\[
\lim_{\delta \to 0} \| K_\delta(\tau, \omega) \| = \| K_0(\tau, \omega) \| \quad \text{for all} \quad \tau \in \mathbb{R} \quad \text{and} \quad \omega \in \Omega.
\]
(167)

Then by (167), Corollary 3 and Lemma 4.9, we obtain (166) from Theorem 3.1 in [53] immediately.

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