Resolution of curvature singularities from quantum mechanical and loop perspective

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Abstract We analyze the persistence of curvature singularities when analyzed using quantum theory. First, quantum test particles obeying the Klein-Gordon and Chandrasekhar-Dirac equation are used to probe the classical timelike naked singularity. We show that the classical singularity is felt even by our quantum probes. Next, we use loop quantization to resolve singularity hidden beneath the horizon. The singularity is resolved in this case.

Keywords Singularity resolution · global monopole · loop quantization

1 Introduction

One of the important predictions of the Einstein’s theory of general relativity is the formation of spacetime singularities. In classical general relativity, singularities are defined as points in which the evolution of timelike or null geodesics is not defined after a finite proper time. According to the classification of the classical singularities devised by Ellis and Schmidt [1], scalar curvature singularities are the most strongest one in the sense that the spacetime posses incomplete geodesics ending in them and all the physical quantities such as the gravitational field (scalars formed from curvature tensor), energy density and tidal forces diverge at the singular point.

But such divergence of physical quantities signify the breakdown of predictive power of classical general relativity. If these singularities are covered by horizon (as supposed by Cosmic Censorship Conjecture) then at least the physically most relevant region of spacetime is under control. Naked singularities (those not covered by horizon), on the other hand, provide an observer with causal access to the region of diverging quantities and should be avoided. However, even singularities covered by the horizon can be accessed by an infalling observer and, more importantly, we would like to have a theory that lacks divergences, at least effectively.

The natural direction for resolving the problem of singularities in classical theory is investigating their persistence in quantum picture. Although we do not have a final quantum theory of gravity we still have several tools for analyzing quantum singularities. The first approach relies on examining properties of quantum particle wave functions on the background represented by the studied geometry. This is a frequently used technique based on well understood properties of operators on a Hilbert space. To move further, one might proceed to using quantum fields and possibly even the backreaction of background geometry using semiclassical Einstein equations with suitably regularized stress energy tensor. Finally, one can apply quantization of the geometry itself. The last approach is in principle the most precise but relies on the selected quantization method and we have no generally accepted one in case of gravity.

Quantum singularities were studied for different specific situations (and using also generalizations), mainly using the first approach. We will apply two of the above mentioned approaches for analysis of singularity in case of the general metric of global monopole [16], which is determined by two parameters - one characterizing the "Schwarzschild-type mass" and the other one the deficit of solid angle. The
singularity is generally covered by single horizon but the
class of metrics also contains, as a special case, a naked
singularity which is analyzed from quantum mechanical
point of view using the technique of Horowitz and
Marolf \cite{17} (who continued the pioneering work of Wald
\cite{18}). This method for analyzing timelike singularities
is based on investigation of self-adjoint extensions of the
evolution operator associated with the given wave
equation. If it is unique the spacetime is deemed quan-
tum mechanically non-singular. The analysis is carried
out for relativistic quantum particle wave equations on
a fixed background. Specifically, we review the previous
results for Klein-Gordon equation and show the calcu-
lation using Newman-Penrose formalism for the Dirac
equation, both in the case of pure global monopole with
naked singularity for which the method was developed.
But as already mentioned, the most reliable method
when trying to investigate the possible removal of the
singularities from geometry is quantum gravity. Here
we have selected loop quantization method inspired by
\cite{19,20,21}, where the spacetime beneath the horizon (in
the non-naked subclass) is isometric to the Kantowski-
Sachs cosmology. Then one can apply methods from
Loop Quantum Cosmology (LQC), that are based on
loop quantization on the restricted configuration space.
In this way, the results for resolution of initial cosmo-
logical singularity are translated to statements about
the singularity at the origin \( r = 0 \).

2 The General Metric for Global monopole

It is well known that different types of non-standard
topological objects may have been formed during ini-
tial Universe evolution, such as domain walls, cosmic
strings and monopoles \cite{16,22}. The basic idea is that
these topological defects have formed as a result of a
breakdown of local or global gauge symmetries. The simplest model that gives rise to global monopole is
described by the Lagrangian

\[
L = \frac{1}{2} \partial_{\mu} \phi^a \partial^{\mu} \phi^a, \tag{1}
\]

where \( \phi^a \) is a triplet of scalar fields, \( a = 1, 2, 3 \). The model has a global \( O(3) \) symmetry, which sponta-
nceously broken to \( U(1) \). The field configuration describ-
ing the monopole is

\[
\phi^a = \eta \frac{x^a}{r},
\]

where \( x^a x^a = r^2 \). We assume that underlying geometry
is general static spherically symmetric described by the
line element

\[
ds^2 = -B(r) dt^2 + \frac{dr^2}{A(r)} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \tag{2}
\]

with the usual relation between the spherical coordi-
nates, \( r, \theta, \phi \) and the Cartesian coordinates \( x^a \). The La-
grangian for the above given field configuration simpli-
fies in the following way

\[
L = \frac{1}{2} (\partial_{\mu} \phi^a \partial^\mu \phi^a + \partial_{\nu} \phi^a \partial^\nu \phi^a) = \frac{\eta^2}{r^2}, \tag{3}
\]

and the diagonal energy momentum tensor is given by
these components

\[
T^t_t = T^r_r = -\frac{\eta^2}{r^2}, \quad T^\theta_\theta = T^\phi_\phi = 0. \tag{4}
\]

The general solution of the Einstein equations with this
\( T^\mu_\mu \) is

\[
B = A^{-1} = 1 - 8\pi G \eta^2 - \frac{2GM}{r}, \tag{5}
\]

where \( M \) is a constant of integration. The metric
describes a black hole of mass \( M \), carrying a global monopole
charge characterized by \( \eta \). Such a black hole can be
formed if a global monopole is swallowed by an ordi-
ary black hole \cite{16}.

The Kretschmann scalar which indicates the formation
of curvature singularity is given by

\[
K = \frac{48M^2G^2}{r^6} + \frac{128\pi \eta^2G^2}{r^5} + \frac{256\pi^2G^2\eta^4}{r^4}. \tag{6}
\]

It is obvious that \( r = 0 \) is a typical central curvature
singularity (scalar curvature singularity according to
above mentioned classification) and the dominant con-
tribution comes from term corresponding to black hole
mass \( M \). If \( M > 0 \) the singularity is evidently spacelike
and covered by a single horizon.

3 Global monopole and its singularity

If we assume that the mass term is negligible on the
astrophysical scale or vanishing, we will have

\[
ds^2 = - (1 - 8\pi G \eta^2) dt^2 + \frac{dr^2}{(1 - 8\pi G \eta^2)^2} + r^2 d\Omega^2, \tag{7}
\]

For simplicity we choose \( \alpha^2 = 1 - 8\pi G \eta^2 \) and by rescal-
ing \( r \) and \( t \) variables, we can rewrite the monopole met-
ic as

\[
ds^2 = -dt^2 + dr^2 + \alpha^2 r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \tag{8}
\]
If we calculate the Kretschmann scalar,
\[ K = 4 \left( \frac{\alpha^2 - 1}{r^4} \right)^2, \]
still there is a weaker singularity at \( r = 0 \). From the metric (4), one can immediately see that the singularity is timelike. This time, because our simplified metric does not have the horizon the singularity is naked.

4 Naked Singularity

As mentioned in the Introduction naked singularity poses a serious problems and its resolution would be desirable. In this section, the occurrence of naked singularities in global monopole will be analyzed from quantum mechanical point of view. In probing the singularity, quantum test particles obeying the Klein-Gordon and Dirac equations are used. The reason for using two different types of fields is to clarify whether the classical singularity is sensitive to spin of the fields.

4.1 Klein - Gordon Fields

The Klein-Gordon equation for a free particle satisfies
\[ \Box \psi = \frac{\partial^2 \psi}{\partial t^2} - \nabla \cdot \nabla \psi = M^2 \psi, \]
with the solution
\[ \psi(t) = \exp \left[ -it\sqrt{A} \right] \psi(0). \]
If \( A \) is not essentially self-adjoint, the future time evolution of the wave function (12) is ambiguous. Then, HM criterion defines the spacetime as quantum mechanically singular. However, if there is only one self-adjoint extension, the operator \( A \) is said to be essentially self-adjoint and the quantum evolution described by equation (12) is uniquely determined by the initial conditions. According to the HM criterion, this spacetime is said to be quantum mechanically non-singular. In order to determine the number of self-adjoint extensions, the concept of deficiency indices is used. The deficiency subspaces \( N_\pm \) are defined by (see Ref. [26] for a detailed mathematical background),
\[ N_+ = \{ \psi \in D(A^*), A^* \psi = Z_+ \psi, \quad Im Z_+ > 0 \} \quad \text{with dimension } n_+ \]
\[ N_- = \{ \psi \in D(A^*), A^* \psi = Z_- \psi, \quad Im Z_- < 0 \} \quad \text{with dimension } n_- \]
The dimensions \( (n_+, n_-) \) are the deficiency indices of the operator \( A \). The indices \( n_+(n_-) \) are completely independent of the choice of \( Z_+(Z_-) \) depending only on whether \( Z \) lies in the upper (lower) half complex plane. Generally one takes \( Z_+ = i\lambda \) and \( Z_- = -i\lambda \), where \( \lambda \) is an arbitrary positive constant necessary for dimensional reasons. The determination of deficiency indices then reduces to counting the number of solutions of \( A^* \psi = Z \psi \); (for \( \lambda = 1 \),
\[ A^* \psi \pm i \psi = 0 \]
that belong to the Hilbert space \( \mathcal{H} \). If there are no square integrable solutions (i.e. \( n_+ = n_- = 0 \)), the operator \( A \) possesses a unique self-adjoint extension and it is essentially self-adjoint. Consequently, a sufficient condition for the operator \( A \) to be essentially self-adjoint is to find only solutions satisfying Eq. (14) that do not belong to the Hilbert space.
For the metric (16), the Klein-Gordon equation becomes,

\[
\frac{\partial^2 \psi}{\partial t^2} = -\left \{ \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2 \alpha^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{1}{r^2 \alpha^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \varphi^2} \right \} + \left \{ \frac{\cos \theta}{r^2 \alpha^2 \sin \theta} \frac{\partial \psi}{\partial \theta} + \frac{2}{r} \frac{\partial \psi}{\partial r} \right \}.
\]

In analogy with the equation (10), the spatial operator \( A \) is

\[
A = \left \{ \frac{\partial^2}{\partial r^2} + \frac{1}{r^2 \alpha^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2 \alpha^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right \} + \left \{ \frac{\cos \theta}{r^2 \alpha^2 \sin \theta} \frac{\partial}{\partial \theta} + \frac{2}{r} \frac{\partial}{\partial r} \right \}.
\]

and the equation to be solved is \((A^* \pm i) \psi = 0\). Using separation of variables, \( \psi = R(r) Y_l^m(\theta, \varphi) \), we get the radial portion of equation (14) as,

\[
\frac{d^2 R(r)}{dr^2} + \frac{2}{r} \frac{dR(r)}{dr} + \left( -\frac{l(l+1)}{r^2 \alpha^2} \pm i \right) R(r) = 0.
\] 

The square integrability of the above solution is checked by calculating the squared norm of the above solution in which the function space on each \( t = \) constant hypersurface \( \Sigma \) is defined as \( \mathcal{H} = L^2(\Sigma, \mu) \) where \( \mu \) is the measure given by the spatial metric volume element.

We easily recover the results showed in [8]: The spacetime of global monopole remains singular in the view of relativistic quantum mechanics: the future of a given initial wave packet obeying the Klein-Gordon equation is not generally well determined, similarly to the future of a classical particle which reaches the classical singularity at \( r = 0 \).

4.2 Dirac Fields

The Newman-Penrose formalism will be used here to analyze massless Dirac particle propagating in the space of global monopole. The signature of the metric (16) is changed to \(-2\) in order to use the Dirac equation in Newman-Penrose formalism. Thus, the metric is given by,

\[
ds^2 = dt^2 - dr^2 - r^2 \alpha^2 \left (d\theta^2 + \sin^2 \theta d\varphi^2 \right ).
\] 

The Chandrasekhar-Dirac (CD) [10] equations in Newman-Penrose formalism are given by

\[
(D + \epsilon - \rho) F_1 + (\delta + \pi - \alpha) F_2 = 0,
\]

\[
(\nabla + \mu - \gamma) F_2 + (\delta + \beta - \tau) F_1 = 0,
\]

\[
(D + \bar{\epsilon} - \bar{\rho}) G_2 - (\delta + \bar{\pi} - \bar{\alpha}) G_1 = 0,
\]

\[
(\nabla + \bar{\mu} - \bar{\gamma}) G_1 - (\bar{\delta} + \bar{\beta} - \bar{\tau}) G_2 = 0,
\]

where \( F_1, F_2, G_1 \) and \( G_2 \) are the components of the wave function, \( \epsilon, \rho, \pi, \alpha, \mu, \gamma, \beta, \tau \) are the spin coefficients to be found and the "bar" denotes complex conjugation. The null tetrads vectors for the metric (16) are defined by

\[
l^a = (1, 1, 0, 0),
\]

\[
n^a = \left( \frac{1}{2}, \frac{1}{2}, 0, 0 \right),
\]

\[
m^a = \frac{1}{\sqrt{2}} \left( 0, 1, \frac{1}{\alpha r}, \frac{i}{\alpha \sin \theta} \right).
\]

The directional derivatives in the Dirac equation are defined by \( D = l^a \partial_a, \nabla = n^a \partial_a \) and \( \delta = m^a \partial_a \). We define operators in the following way

\[
D_0 = D
\]

\[
D_0 = -2\nabla
\]

\[
L_0 = \sqrt{2r} \alpha \delta \text { and } L_1^+ = L_0 + \frac{m}{2} \frac{\partial}{\partial r}
\]

\[
L_0 = \sqrt{2r} \alpha \delta \text { and } L_1 = L_0 + \frac{m}{2} \frac{\partial}{\partial r}
\]

The nonzero spin coefficients are,

\[
\mu = -\frac{1}{2r} \rho = -\frac{1}{r} \beta = -\alpha = \frac{1}{2 \sqrt{2}} \cot \theta.
\]

Substituting nonzero spin coefficients and the definitions of the operators given above into the CD equations leads to

\[
\left( D_0 + \frac{1}{r} \right) F_1 + \frac{1}{r \sqrt{2}} L_1 F_2 = 0,
\]

\[
-\frac{1}{2} \left( D_0 + \frac{1}{r} \right) F_2 + \frac{1}{r \sqrt{2}} L_1^+ F_1 = 0,
\]

\[
\left( D_0 + \frac{1}{r} \right) G_2 - \frac{1}{r \sqrt{2}} L_1^+ G_1 = 0,
\]

\[
\frac{1}{2} \left( D_0 + \frac{1}{r} \right) G_1 + \frac{1}{r \sqrt{2}} L_1 G_2 = 0.
\] 

For the solution of the CD equations, we assume separable solution in the form of

\[
F_1 = f_1(r) Y_l(\theta) e^{i(k t + m \varphi)},
\]

\[
F_2 = f_2(r) Y_l(\theta) e^{i(k t + m \varphi)},
\]

\[
G_1 = g_1(r) Y_l(\theta) e^{i(k t + m \varphi)},
\]

\[
G_2 = g_2(r) Y_l(\theta) e^{i(k t + m \varphi)}.
\]
Here \( \{f_1, f_2, g_1, g_2\} \) and \( \{Y_1, Y_2, Y_3, Y_4\} \) are functions of \( r \) and \( \theta \) respectively, \( m \) is the azimuthal quantum number and \( \kappa \) is the frequency of the Dirac spinor, which is assumed to be positive and real. By substituting (25) in (24) we will see that with these assumptions \( f_1(r) = g_2(r) \) and \( f_2(r) = g_1(r) \), \( Y_1(\theta) = Y_3(\theta) \) and \( Y_2(\theta) = Y_4(\theta) \)

Dirac equation reduces to two equations. The radial part of the Dirac equations become

\[
\begin{align*}
(D_0 + \frac{1}{r}) f_1(r) &= \frac{\lambda}{r \alpha \sqrt{2}} f_2(r), \\
\frac{1}{2} \left( D_0^0 + \frac{1}{r} \right) f_2(r) &= \frac{\lambda}{r \alpha \sqrt{2}} f_1(r),
\end{align*}
\]

where \( \lambda \) comes from separation of variables. We further assume that

\[
\begin{align*}
f_1(r) &= \frac{\Psi_1(r)}{r}, \\
f_2(r) &= \frac{\sqrt{2} \Psi_2(r)}{r},
\end{align*}
\]

then equation (28) transforms into,

\[
\begin{align*}
D_0 \Psi_1 &= \frac{\lambda'}{r} \Psi_2, \\
D_0^0 \Psi_2 &= \frac{\lambda'}{r} \Psi_1,
\end{align*}
\]

where \( \lambda' = \frac{\lambda}{\alpha} \), so we will have

\[
\begin{align*}
\left( \frac{d}{dr} + ik \right) \Psi_1 &= \frac{\lambda'}{r} \Psi_2, \\
\left( \frac{d}{dr} - ik \right) \Psi_2 &= \frac{\lambda'}{r} \Psi_1,
\end{align*}
\]

In order to write the above equation in a more compact form we combine the solutions in the following way,

\[
Z_+ = \Psi_1 + \Psi_2, \\
Z_- = \Psi_2 - \Psi_1.
\]

After doing some calculations we end up with a pair of one-dimensional Schrödinger-like wave equations with effective potentials,

\[
\begin{align*}
\left( \frac{d^2}{dr^2} + \lambda^2 \right) Z_\pm &= V_\pm Z_\pm, \\
V_\pm &= \frac{\lambda'^2}{r^2} \mp \frac{\lambda'}{r^2}.
\end{align*}
\]

In analogy with the equation (10), the spatial operator \( A \) for the massless case is

\[
A = - \frac{d^2}{dr^2} + V_\pm,
\]

so we have

\[
\left( \frac{d^2}{dr^2} - \frac{\lambda'^2}{r^2} \mp i \right) Z_\pm = 0.
\]

The solutions of the above equations are expressible using Bessel functions of the first and second kind in the following way

\[
Z_+ = C_1 \sqrt{r} J_{\lambda'} \left( \lambda' - \frac{1}{2} \frac{r}{\sqrt{2}} (1 - i) \right) + C_2 \sqrt{r} Y_{\lambda'} \left( \lambda' - \frac{1}{2} \frac{r}{\sqrt{2}} (1 - i) \right), \\
Z_- = C_1' \sqrt{r} J_{\lambda'} \left( \lambda' + \frac{1}{2} \frac{r}{\sqrt{2}} (1 + i) \right) + C_2' \sqrt{r} Y_{\lambda'} \left( \lambda' + \frac{1}{2} \frac{r}{\sqrt{2}} (1 + i) \right).
\]

Using the asymptotic formulas for Bessel functions when \( r \to \infty \) \((Y(\kappa, z) \approx z^{-1/2} \sin(z - \kappa \pi /2 - \pi /4) \) and \( J(\kappa, z) \approx z^{-1/2} \cos(z - \kappa \pi /2 - \pi /4) \)) and noting the complex argument in both solutions one can find a combination of constants \( C_1 \), \( C_2 \) or \( C'_1 \), \( C'_2 \) which is square integrable near infinity. (But, it is also possible to choose the constants differently so that both solutions are not square integrable!).

When \( r \to 0 \) the approximate expressions for Bessel functions \((Y(\kappa, z) \approx z^{-\kappa} \) for \( \kappa \neq 0 \), \( Y(0, z) \approx \ln(z/2) \)) and \( J(\kappa, z) \approx z^\kappa \) imply that for \( C_2 = 0 \) and \( C'_2 = 0 \) we have square integrable solution near zero. (Here again if we suppose \( C_1 = 0 \) and \( C'_1 = 0 \), for \( \kappa \geq 3/2 \), the solutions are not square integrable!. One could restrict an analysis to only certain wave modes and purposely choose the modes to be quantum regular).

But since we have a solution of equations valid on the whole domain (not just asymptotic forms of equations) we can match the behaviour at zero and infinity. Based on the results we can have solution square integrable over the whole domain and therefore our deficiency indices are nonzero. The operator is not essentially self-adjoint and the spacetime is quantum mechanically singular.

5 Quantum Gravity

Now we are going to investigate the singularity of general global monopole using techniques from loop quantization in the manner of \([20]\). Consider equation (2), for \( r < \frac{2GM}{1-\kappa^2G\eta^2} \). This metric describes spacetime inside
the horizon of a black hole. The coordinate \( t \) is timelike and the coordinate \( r \) is spatial there; for convenience we rename them as \( r \equiv T \) and \( t \equiv t \) with \( T \in [0, \frac{2GM}{1-\alpha^2b}] \) and \( r \in [-\infty, +\infty] \) and the metric becomes

\[
d s^2 = -\left( \alpha^2 - \frac{2GM}{T} \right) dr^2 + \frac{dT^2}{(\alpha^2 - \frac{2GM}{T})} + T^2 dt^2,
\]

we eliminate the coefficient of \( dT^2 \) by defining a new temporal variable \( \tau \) via

\[
d \tau = \frac{dT}{\sqrt{2GM - \alpha^2}}.
\]

Accordingly, the metric becomes

\[
d s^2 = -d\tau^2 + \left( \frac{2GM}{T} - \alpha^2 \right) dr^2 + T^2 (d\theta^2 + \sin^2 \theta d\varphi^2).
\]

We introduce two functions \( a^2(\tau) \equiv \frac{2GM}{T} - \alpha^2 \) and \( b^2(\tau) \equiv T^2(\tau) \) and redefine \( \tau \equiv t \). The metric becomes

\[
d s^2 = -d\tau^2 + a^2(t) \, dr^2 + b^2(t) \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right),
\]

this metric describes a homogeneous, anisotropic Kantowski-Sachs cosmological model with spatial section having topology \( \mathbb{R} \times S^2 \). From this observation comes the motivation to use LQC approach. In our case \( a(t) \) is a function of \( b(t) \).

5.1 Classical observables

The corresponding action for gravity minimally coupled with scalar field can be written in the form

\[
S = \frac{1}{16\pi G} \int dt \, dx \, N \sqrt{h^{1/2}} \left[ K_{ij} K^{ij} - K^2 + \frac{\alpha^2 G}{b^2} \right],
\]

by considering the metric (38), the action becomes

\[
S = -\frac{1}{8\pi G} \int dt \int_0^R dr \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta ab^2 \times \left[ \frac{b^2}{b^2} + \frac{2\dot{b}b}{ab} - \frac{\alpha^2}{b^2} \right],
\]

by using the relation between \( a \) and \( b \), we will be able to write the action in terms of a single function

\[
S = \frac{R\alpha^2}{2G} \int dt \, \sqrt{\frac{b}{2GM}} \left( 1 - \frac{\alpha^2 b}{2GM} \right)^{-1/2} \times \left[ \frac{b^2}{b^2} + \frac{2GM}{b} \left( 1 - \frac{\alpha^2 b}{2GM} \right) \right].
\]

Now, we will compute the Hamiltonian (Hamiltonian constraint). The momentum associated to the chosen configuration variable is

\[
p_b = \frac{R\alpha^2 b}{2G} \sqrt{\frac{b}{2GM}} \left( 1 - \frac{\alpha^2 b}{2GM} \right)^{-1/2},
\]

and therefore we obtain

\[
H = p_b \dot{b} - L \equiv \frac{R\alpha^2 b}{2G} \sqrt{\frac{b}{2GM}} \left( 1 - \frac{\alpha^2 b}{2GM} \right)^{-1/2} \times \left[ \frac{b^2}{b^2} + \frac{2GM}{b} \left( 1 - \frac{\alpha^2 b}{2GM} \right) \right] = 0,
\]

and immediately get the following solution

\[
\dot{b}^2 = \frac{2GM}{b} - \alpha^2,
\]

which is exactly the equation (36). When the horizon radius, \( r_h = \frac{2GM}{\alpha^2} \), is much larger than the scale on which we are probing the singularity, we can write

\[1 - \frac{\alpha^2 b}{2GM} \sim 1\]

so the Hamiltonian would be

\[
H = \frac{R\alpha^2 b}{2G} \sqrt{\frac{b}{2GM}} \left[ \frac{GP}{2R\alpha^2} - \frac{R\alpha^2}{2G} \right].
\]

The volume

\[
V = \int \, dr \, d\theta \, d\varphi \sqrt{h} = 4\pi Rab^2
\]

simplifies when using the above approximation and we obtain

\[
V = l_0b^{3/2}
\]

\[
l_0 = 4\pi R\sqrt{2GM}
\]

The canonical pair is given by \( b \equiv x \) and \( p_b \), with Poisson bracket \( \{x, p_b\} = 1 \).

For isotropic models, only holonomies evaluated in isotropic connections \( A_{\mu}^i = \bar{\psi}^i_{\mu} \) appear. Along straight lines in the direction of translation symmetries \( X_\mu^i = \psi (\partial / \partial x^\mu)^i \), holonomies exp (\( \int X_\mu^i A_{\mu}^i \)) in the fundamental representation of \( SU(2) \) have matrix elements of the form exp (\( i \mu c \)), where \( \mu \) depends on the length.
A straightforward calculation gives

$$U_\gamma(p) \equiv \exp \left( 8\pi G \frac{i}{\hbar L} p \right)$$

(49)

where $\gamma$ is a real parameter and $L$ fixes the length scale. The parameter $\gamma$ determines the separation of momentum points in the phase space.

The pair $(x, U_\gamma(p))$ has the following Poisson bracket algebra

$$\{x, U_\gamma(p)\} = 8\pi G \frac{i}{\hbar L} U_\gamma(p)$$

(50)

A straightforward calculation gives

$$U^{-1}_\gamma \{V^n, U_\gamma\} = U^{-1}_0 \left\{ x^{3n/2}, U_\gamma \right\}$$

$$= \frac{8\pi G l_0^{3n/2} \gamma}{L} \operatorname{sgn}(x) |x|^{3n/2-1}$$

(51)

We are concerned with the quantity $\frac{1}{|x|}$ which can serve as an indicator for singularity presence because classically it diverges for $|x| \to 0$ thus producing singularity. From this moment we choose $n = 1/3$

$$\frac{\operatorname{sgn}(x)}{\sqrt{|x|}} = -\frac{2Li}{8\pi G l_0^{1/3} \gamma} U^{-1}_\gamma \left\{ V^{1/3}, U_\gamma \right\}.$$  

(52)

5.2 Quantization

We will use the basis of Hilbert space introduced in [19] [20], which is formed by eigenstates of $\hat{x}$. This implies the existence of a self-adjoint operator $\hat{x}$, acting on the basis states according to

$$\hat{x} |\mu\rangle = L\mu |\mu\rangle$$

(53)

Next, we want to promote the classical momentum function $U_\gamma = e^{(8\pi G \frac{i}{\hbar L} p)}$ to operator. We can do so by defining the action of $\hat{U}_\gamma$ on the basis states with the help of the definition equation (53) and using commutation relation based on the Poisson bracket between $x$ and $U_\gamma$ we obtain

$$\hat{U}_\gamma |\mu\rangle = |\mu - \gamma\rangle, \quad \left[ \hat{x}, \hat{U}_\gamma \right] = -\gamma L\hat{U}_\gamma.$$  

(54)

Using canonical quantization of Poisson bracket $[,] \to ib\{,\}$, and using equation (51) we get a relation for the length scale

$$L = \sqrt{8\pi l_p}$$

(55)

5.3 Volume operator and disappearance of the singularity

In the vicinity of the singularity we assume the approximate equation (48). Then the volume operator acts in the following way on the basis states

$$\hat{V} |\mu\rangle = l_0 |x|^{3/2} |\mu\rangle = l_0 |\mu|^{3/2} |\mu\rangle$$

(56)

Using the equation (22) and promoting the Poisson brackets to commutators, while setting $\gamma = 1$, we find

$$\frac{\hat{U}}{|x|} = \frac{1}{2\pi l_p^2} \left( \hat{U}^{-1}_\gamma \left[ \hat{V}^{1/3}, \hat{U}_\gamma \right] \right)^2.$$  

(57)

On the basis states this operator acts in the following way

$$\hat{U}^{-1}_\gamma \left[ \hat{V}^{1/3}, \hat{U}_\gamma \right] |\mu\rangle = \left( \hat{U}^{-1}_\gamma \hat{V}^{1/3} \hat{U}_\gamma - \hat{U}^{-1}_\gamma \hat{U}_\gamma \hat{V}^{1/3} \right) |\mu\rangle = l_0^{1/3} l_p^{2/3} \sqrt{\mu - 1 - \sqrt{\mu}} |\mu\rangle$$

(58)

so finally we get

$$\frac{1}{|x|} |\mu\rangle = \sqrt{\frac{2}{\pi l_p^2}} \left( \sqrt{\mu - 1} - \sqrt{\mu} \right)^2 |\mu\rangle.$$  

(59)

We can see that the spectrum is bounded from above and so the singularity is resolved in the quantum theory (the theory gives finite predictions for observables related to singularity). In fact, the eigenvalue of operator $\frac{1}{|x|}$ corresponding to the state $|0\rangle$ which probes the classical singularity is equal to $\sqrt{\frac{2}{\pi l_p^2}}$, which is the highest eigenvalue of the spectrum. Specifically, the operator corresponding to the curvature invariant

$$\mathcal{R}_{\mu\nu\rho\sigma} \mathcal{R}^{\mu\nu\rho\sigma} \equiv \frac{48M^2 \pi^6}{r^6} + \frac{128M \pi G^2 \eta^2}{r^5} + \frac{256G^2 \pi^2 \eta^4}{r^4}$$

(60)

is then automatically finite in quantum mechanics. Promoting it to operator and evaluating on $|0\rangle$ we get

$$\mathcal{R}_{\mu\nu\rho\sigma} \mathcal{R}^{\mu\nu\rho\sigma} |0\rangle = \left( \frac{48M^2 G^2}{|x|^6} + \frac{128M \pi G^2 \eta^2}{|x|^5} + \frac{256G^2 \pi^2 \eta^4}{|x|^4} \right) |0\rangle = \left( \frac{384M^2 G^2}{\pi^4 l_p^2} + \frac{512M \pi G^2 \eta^2}{\pi^3 l_p^2} + \frac{1024\pi G^2 \pi^2 \eta^4}{\pi^2 l_p^2} \right) |0\rangle$$

(61)

On the other hand, when $|\mu| \to \infty$ the eigenvalue of $\frac{1}{|x|}$ goes to zero which is natural behaviour for large $|x|$. Also, it is possible to show that the quantum Hamiltonian constraint gives a discrete difference equation for the coefficients of the physical states.
6 Conclusion

We have seen that we have not been successful in removing the naked singularity by using relativistic quantum mechanics (for both Klein-Gordon and Dirac equations). On the other hand we have shown that the curvature singularity of general global monopole is resolved when the geometry is quantized using loop techniques. Unfortunately, one cannot directly compare the results because the loop quantization relied on radial coordinate being timelike beneath the horizon which is not the case for naked singularity of pure monopole. But still, this might be an indication that the first method is not reliable for determining the fate of singularities in quantum theory and one should rather focus on quantization of the geometry itself. But even the approach using loop quantization that relied on restricted class of geometries should not be trusted completely. One should allow, e.g., for deviations from spherical symmetry to be completely sure about the fate of singularities.

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Appendix A: Geometric quantities

The spatial metric is

\[ h_{ij} = (a^2(t), b^2(t), b^2(t) \sin^2 \theta), \]

The extrinsic curvature is 

\[ K_{ij} = -\frac{1}{2} \frac{\partial h_{ij}}{\partial t}, \]

so that 

\[ K = K_{ij} h^{ij} = - \left( \frac{\dot{a}}{a} + \frac{2 \dot{b}}{b} \right) \]

\[ K_{ij} K^{ij} = \frac{\dot{a}^2}{a^2} + 2 \frac{\dot{b}^2}{b^2} \]

\[ K_{ij} K^{ij} - K^2 = -2 \left( \frac{\dot{b}^2}{b^2} + 2 \frac{\dot{a} \dot{b}}{ab} \right) \]

The Ricci curvature for the space section is

\[ (3) R = \frac{2 \dot{b}^2}{b^2}. \]

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