NONLINEAR DYNAMICS AND STABILITY OF THE SKATEBOARD

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Abstract. In this paper the further investigation and development for the simplified mathematical model of a skateboard with a rider are obtained. This model was first proposed by Mont Hubbard [12, 13]. It is supposed that there is no riders control of the skateboard motion. To derive equations of motion of the skateboard the Gibbs-Appell method is used. The problem of integrability of the obtained equations is studied and their stability analysis is fulfilled. The effect of varying vehicle parameters on dynamics and stability of its motion is examined.

1. Introduction. A significant body of research in nonholonomic mechanics has been developed in the area of vehicle system dynamics including dynamics of various devices for extreme sports. In particular there are many papers describing the motion of a unicycle with rider [31, 32], the snakeboard motion [6, 19, 26], the bicycle dynamics [10, 24] etc. The main results from these papers have been summarized in the book by A.M. Bloch [3].

While the snakeboard has received quite a bit attention in recent literature, the skateboard (which is of course more popular) is poorly represented in the literature. At the late 70th - early 80th of the last century Mont Hubbard [12, 13] proposed two mathematical models describing the motion of a skateboard with the rider. To derive equations of motion of the models he used the principal theorems of dynamics. In our paper we give the further development of the models proposed by Hubbard.

In 1996 Yu. G. Ispolov and B.A. Smolnikov [14] discussed various problems of a skateboard dynamics. In particular, they studied the possibility to accelerate a skateboard using reactions of nonholonomic constraints (so called nonholonomic acceleration). However the model of a skateboard proposed in [14] is two-dimensional while the papers [12, 13] deal with more complicated (and more interesting) three-dimensional models. The results obtained in [14] were essentially used by J. Ackermann and M. Strobel in their diploma thesis [1].

The paper by A. Endrueweit and P. Ermanni [8] deals with the design and the experimental and numerical analysis of a flex slalom skateboard from glass fibre epoxy material for rider body masses from 60 to 80 kg. Elastic properties of the skateboard wheels and various friction models, describing the contact of the skateboard wheels with the ground are discussed in the presentation [11].

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In 2004 A.E. Österling, student of School of Mathematical Sciences (University of Exeter, UK) presented his graduate study [25] where he investigated, using the principal theorems of dynamics, one of the models proposed in [12, 13]. Together with the papers [12, 13] this graduate study gives the most interesting analysis of a skateboard dynamics. In our paper we will refer many times to the papers [12, 13] and [25] and we will compare the results obtained in these papers with our results.

The recent paper by M. Wisse and A. L. Schwab [30] gives the brief review of the main results obtained in [12, 13]. Thus the papers [12, 13] can be considered as keywords on a skateboard dynamics.

2. Problem formulation. Basic coordinate systems. The skateboard typically consists of the board, a set of two trucks and four wheels (Fig. 1). The modern board is generally from 78-83 cm long, 17-21 cm wide and 1-2 cm thick [12].

![Figure 1. Side view of Skateboard.](image)

The most essential elements of a skateboard are the trucks, connecting the axles to the board. Angular motion of both the front and rear axles is constrained to be about their respective nonhorizontal pivot axes (Fig. 1), thus causing a steering angle of the wheels whenever the axles are not parallel to the plane of the board (Fig. 2). The vehicle is steered by making use of this static relationship between steering angles and tilt of the board.

![Figure 2. Rear and Top view of Skateboard.](image)

Following the papers [12, 13] suppose also that there is a torsional spring that exerts a restoring torque between the wheelset and the board proportional to the tilt of the board relative to the wheelset (Fig. 2); the corresponding stiffness coefficient is denoted by $k_1$. 
Our simplest skateboard model assumes that the rider, modeled as a rigid body, remains fixed and perpendicular with respect to the board during all times of the motion (Fig. 2). Thus if the board tilts through $\gamma$, the rider tilts through the same angle relative to the vertical. Let us introduce an inertial coordinate system $OXYZ$ in the ground plane. The origin $O$ of this system is at any point of the ground plane, and the $OZ$-axis is directed perpendicularly to the ground plane. We denote the unit vectors of the coordinate system $OXYZ$ by $e_x$, $e_y$, $e_z$. Let $AB = a$ be the distance between the two axle centers $A$ and $B$ of a skateboard. The position of line $AB$ with respect to $OXYZ$ system is defined by $X$ and $Y$ coordinates of its centre $G$ and by the angle $\theta$ between this line and the $OX$-axis (Fig 3).

![Coordinate system](image)

The tilt of the board through $\gamma$ is accompanied by rotation of the front wheels clockwise through $\delta_f$ and rotation of the rear wheels anticlockwise through $\delta_r$. The wheels of a skateboard are assumed to roll without lateral sliding. This condition is modeled by constraints which may be shown to be nonholonomic

$$
\begin{align*}
-\dot{X} \sin (\theta - \delta_f) + \dot{Y} \cos (\theta - \delta_f) + \frac{a}{2} \ddot{\theta} \cos \delta_f &= 0, \\
-\dot{X} \sin (\theta + \delta_r) + \dot{Y} \cos (\theta + \delta_r) - \frac{a}{2} \ddot{\theta} \cos \delta_r &= 0.
\end{align*}
$$

(1)

We can solve equations (1) with respect to $\dot{X}$ and $\dot{Y}$:

$$
\begin{align*}
\dot{X} &= -\frac{a \ddot{\theta}}{2 \sin (\delta_f + \delta_r)} \left[ \cos \delta_f \cos (\theta + \delta_r) + \cos \delta_r \cos (\theta - \delta_f) \right], \\
\dot{Y} &= -\frac{a \ddot{\theta}}{2 \sin (\delta_f + \delta_r)} \left[ \cos \delta_f \sin (\theta + \delta_r) + \cos \delta_r \sin (\theta - \delta_f) \right].
\end{align*}
$$

(2)

Thus, the velocities of the points $A$ and $B$ will be directed horizontally and perpendicularly to the axles of the wheels.

**Lemma 2.1.** In this case there is a point $P$ on the line $AB$ which has zero lateral velocity.

**Proof.** Let us introduce three unit vectors: $e_1$-vector is directed along the line $AB$ in the direction of motion, $e_3$-vector is directed perpendicularly to the ground plane.
and $e_2$-vector forms right-hand system with $e_1$ and $e_3$. The components of the velocity of the point $A$ using the vectors $e_1$, $e_2$, $e_3$ will be

$$v_A = \left( \dot{X} \cos \theta + \dot{Y} \sin \theta \right) e_1 + \left( \dot{Y} \cos \theta - \dot{X} \sin \theta + \frac{a}{2} \dot{\theta} \right) e_2.$$ 

Suppose that the desired point $P$ is located a distance $z$ from the front truck (point $A$) and its forward velocity equals $u$. Then according to the well-known Euler’s formula

$$v_A = v_P + [\omega \times \overrightarrow{PA}].$$

Taking into account that the angular velocity of the basis $e_1$, $e_2$, $e_3$ relative to $e_x$, $e_y$, $e_z$ equals $\dot{\theta} e_z = \dot{\theta} e_3$ we get that

$$v_A = u e_1 + \left[ \dot{\theta} e_3 \times z e_1 \right]$$

and therefore

$$u = \dot{X} \cos \theta + \dot{Y} \sin \theta, \quad \dot{\theta} z = \dot{Y} \cos \theta - \dot{X} \sin \theta + \frac{a}{2} \dot{\theta}.$$ 

Substituting in these formulae for $\dot{X}$ and $\dot{Y}$ their expressions from (2) we finally find:

$$u = -a \dot{\theta} \cos \delta_f \cos \delta_r, \quad \dot{\theta} = -u \sin (\delta_f + \delta_r), \quad z = \frac{a \sin \delta_f \cos \delta_r}{\sin (\delta_f + \delta_r)}.$$ 

(3)

Thus we found the explicit formula for the distance from point $A$ to the desired point $P$. 

Let us place the origins of three vectors $e_1$, $e_2$, $e_3$ at this point $P$. We denote the obtained coordinate system by $P x_1 x_2 x_3$. Further we will investigate the motion of the skateboard with respect to two coordinate systems: the system $OXYZ$ and the system $P x_1 x_2 x_3$.

3. Kinematic constraints. Now we find the relationship between the steering angles $\delta_f$ and $\delta_r$ and the tilt of the board $\gamma$. Let us denote by $\lambda_f$ and $\lambda_r$ the angles which the front and the rear pivot axis makes with the horizontal (Fig. 1).

**Theorem 3.1.** The steering angles $\delta_f$ and $\delta_r$ are related with the tilt of the board $\gamma$ by the following formulae

$$\tan \delta_f = \tan \lambda_f \sin \gamma, \quad \tan \delta_r = \tan \lambda_r \sin \gamma.$$ 

(4)

*Proof.* To prove these formulae we will use a theory of finite rotations (see, e.g. [18, 21]). Consider the front skateboard truck. The motion of this truck will be considered with respect to $P x_1 x_2 x_3$ system. Suppose that the axle length equals $2b$, then distances $AW_1$ and $AW_2$ from the axle center $A$ to the centers of wheels $W_1$ and $W_2$ equal $AW_1 = AW_2 = b$. If the board is not tilted, then the vector $AW_1$ has the following components with respect to the system $P x_1 x_2 x_3$

$$(0, b, 0)^T.$$

Assume a finite rotation $\eta$ of the forward axle about its pivot axle through $\eta_f$. The unit vector $e$ of the pivot axle has the following components onto the system $P x_1 x_2 x_3$:

$$e = -\cos \lambda_f e_1 - \sin \lambda_f e_3,$$
We know from the theory of finite rotations that if we rotate some vector $\mathbf{\rho}$ through $\chi$ about the axis with directing vector $\mathbf{i}$, then the vector $\mathbf{\rho}$ will be transformed to the vector
\[
\mathbf{\rho}' = (\mathbf{i} \cdot \mathbf{\rho}) \mathbf{i} + (\mathbf{\rho} - (\mathbf{i} \cdot \mathbf{\rho}) \mathbf{i}) \cos \chi + [\mathbf{i} \times \mathbf{\rho}] \sin \chi,
\]
where $(\mathbf{i} \cdot \mathbf{\rho})$ is a scalar product and $[\mathbf{i} \times \mathbf{\rho}]$ is a vector product of the vectors $\mathbf{i}$ and $\mathbf{\rho}$. Consequently if we rotate the vector $\mathbf{AW}_1$ through $\eta_f$ about the pivot axle with the directing vector $\mathbf{e}$ then it will be transformed to the vector
\[
(\mathbf{AW}_1')_{\eta} = \mathbf{AW}_1'' = (b \sin \eta_f \sin \lambda_f, b \cos \eta_f, -b \sin \eta_f \cos \lambda_f)^T.
\]

The angles $\gamma$ and $\delta_f$ are defined as follows: if we make two sequential rotations of the vector $\mathbf{AW}_1$ – the first one through $-\gamma$ about the axis with directing vector $\mathbf{e}_1$ and the second one through $-\delta_f$ about the axis with directing vector $\mathbf{e}_3'$ (which is a transformation of the vector $\mathbf{e}_3$ after the first rotation), then as a result the vector $\mathbf{AW}_1'$ should be transformed to the same vector $\mathbf{AW}_1''$ as after the rotation about the pivot axis (Fig. 4).

After rotation of the vector $\mathbf{AW}_1''$ through $-\gamma$ about $\mathbf{e}_1$ it will be transformed to the vector
\[
(\mathbf{AW}_1'')_{-\gamma} = \mathbf{AW}_1''' = (0, b \cos \gamma, -b \sin \gamma)^T,
\]
and the vector $\mathbf{e}_3$ will be transformed to the vector
\[
\mathbf{e}_3' = (0, \sin \gamma, \cos \gamma)^T.
\]

Then after rotation of the vector $\mathbf{AW}_1'''$ through $-\delta_f$ about $\mathbf{e}_3'$ the vector $\mathbf{AW}_1''$ will be transformed to the vector
\[
(\mathbf{AW}_1'')_{-\delta_f} = (b \sin \delta_f, b \cos \gamma \cos \delta_f, -b \sin \gamma \cos \delta_f)^T.
\]

Taking into account that two vectors $(\mathbf{AW}_1')_{\eta}$ and $(\mathbf{AW}_1'')_{-\delta_f}$ should have equal components we get the following system of equations
\[
\sin \eta_f \sin \lambda_f = \sin \delta_f, \quad \cos \eta_f = \cos \gamma \cos \delta_f, \quad \sin \eta_f \cos \lambda_f = \sin \gamma \cos \delta_f. \tag{5}
\]

Expressing $\sin \eta_f$ from the first equation of the system (5) and substituting it in the third equation we finally obtain the first formula in (4). Similarly for the rear skateboard truck we have the second formula (4).
For the first time formulae (4) have been obtained by a slightly different way in the graduate study by Österling [25]. Before this work Hubbard in his papers [12, 13] derived the relations between $\gamma$, $\delta_f$ and $\delta_r$ assuming that all these angles are small. In this case it is possible to use the theory of infinitesimal rotations [21].

4. Absolute velocity of the board and the rider center of mass. In the previous paragraph we proved that the board tilt $\gamma$ is related to the steering angles $\delta_f$ and $\delta_r$ by the formulae (4). Taking into account these formulae we can conclude that the distance $z \equiv AP$ from the front truck $A$ to the point $P$ may be rewritten as follows:

$$z = \frac{a \sin \delta_f \cos \delta_r}{\sin (\delta_f + \delta_r)} = \frac{a \tan \delta_f}{\tan \delta_f + \tan \delta_r} = \frac{a \tan \lambda_f}{\tan \lambda_f + \tan \lambda_r} = \text{const.}$$

In other words, under the constraints (4) the point $P$ will be located at a constant distance from point $A$. This fact is one of the basic features of a skateboard construction. For the first time this fact has been proved in [25] and the author of [25] asserted that it can be verified experimentally.

Suppose that the board of a skateboard is located a distance $h$ above the line $AB$. Since the tilt of the board through $\gamma$ causes rotation of the entire vehicle (the board and the rider) about the line $AB$, then this tilt produces a translation of the central line of the board relative to the line $AB$. The radius vector of point $P$ on the central line which has been located above point $P$ before the tilt has a form:

$$\overrightarrow{PD} = h \cos \gamma \overrightarrow{e_3} - h \sin \gamma \overrightarrow{e_2}.$$ 

Suppose that the length of the board is also equal to $a$ and the board’s center of mass $C$ is located at its center. Therefore, when the board is not tilted its center of mass $C$ is located a distance $h$ above the center $G$ of the line $AB$. Vector $\overrightarrow{DC}$ from the point $D$ to the board’s center of mass $C$ has the form

$$\overrightarrow{DC} = \frac{a \sin (\delta_f - \delta_r)}{2 \sin (\delta_f + \delta_r)} \overrightarrow{e_1} = \frac{a (\tan \lambda_f - \tan \lambda_r)}{2 (\tan \lambda_f + \tan \lambda_r)} \overrightarrow{e_1}$$

and finally the radius vector of a point $C$ with respect to the system $Px_1x_2x_3$ has the form:

$$\overrightarrow{PC} = \overrightarrow{PD} + \overrightarrow{DC} = \frac{a (\tan \lambda_f - \tan \lambda_r)}{2 (\tan \lambda_f + \tan \lambda_r)} \overrightarrow{e_1} - h \sin \gamma \overrightarrow{e_2} + h \cos \gamma \overrightarrow{e_3}.$$ 

Concerning the rider we assume for more generality that the rider’s center of mass $R$ is not located above the board’s center of mass, but it is located over the central line of the board a distance $d$ from the front truck. We denote by $E$ the point on the central line where the rider stands. Let $l$ be the height of the rider’s center of mass above point $P$. Vector $\overrightarrow{DE}$ has the form

$$\overrightarrow{DE} = \frac{(a - d) \sin \delta_f \cos \delta_r - d \sin \delta_r \cos \delta_f}{\sin (\delta_f + \delta_r)} \overrightarrow{e_1} = \frac{a - d \tan \lambda_f - d \tan \lambda_r}{\tan \lambda_f + \tan \lambda_r} \overrightarrow{e_1}.$$ 

Therefore the radius vector of point $R$ has the form:

$$\overrightarrow{PR} = \overrightarrow{PD} + \overrightarrow{DE} + \overrightarrow{ER} = \frac{(a - d) \tan \lambda_f - d \tan \lambda_r}{\tan \lambda_f + \tan \lambda_r} \overrightarrow{e_1} - l \sin \gamma \overrightarrow{e_2} + l \cos \gamma \overrightarrow{e_3}.$$ 

and the translational velocity of point $R$ equals

$$\overrightarrow{v}_R = \overrightarrow{v}_P + \left[ \omega \times \overrightarrow{PR} \right].$$
Let us find now the formula for velocity of the board’s center of mass

**Lemma 4.1.** The absolute velocity of the board’s center of mass equals:

\[
v_C = u \left( 1 - \frac{(\tan \lambda_f + \tan \lambda_r) h}{a} \sin^2 \gamma \right) e_1 - \frac{u}{2} (\tan \lambda_f - \tan \lambda_r) \sin \gamma e_2 - h \dot{\gamma} \cos \gamma e_2 - h \dot{\gamma} \sin \gamma e_3.\]

**(6)**

**Proof.** To prove this formula we will use the law of composition of velocities. Translational velocity of point C has the form

\[
v^t_C = v_P + [\omega \times \overrightarrow{PC}].
\]

Taking into account that the velocity of point P equals \(v_P = ue_1\) and the angular velocity of the system \(P \times x_1x_2x_3\) equals

\[
\omega = \dot{\theta} e_3 = -\frac{u \sin (\delta_f + \delta_r)}{a \cos \delta_f \cos \delta_r} e_3 = -\frac{u (\tan \lambda_f + \tan \lambda_r) \sin \gamma}{a} e_3,
\]

we finally get:

\[
v^t_C = u \left( 1 - \frac{(\tan \lambda_f + \tan \lambda_r) h}{a} \sin^2 \gamma \right) e_1 - \frac{u}{2} (\tan \lambda_f - \tan \lambda_r) \sin \gamma e_2.
\]

The relative velocity of point C has the form

\[
v^r_C = -h \dot{\gamma} \cos \gamma e_2 - h \dot{\gamma} \sin \gamma e_3.
\]

Taking the sum of translational and relative velocities of point C we finally obtain formula the absolute velocity of the board’s center of mass in the form (6).

Using the similar ideas it is easy to prove the following lemma

**Lemma 4.2.** The absolute velocity of the rider’s center of mass equals:

\[
v_R = u \left( 1 - \frac{(\tan \lambda_f + \tan \lambda_r) l}{a} \sin^2 \gamma \right) e_1 - \frac{u}{2} (\tan \lambda_f - \tan \lambda_r) \sin \gamma e_2 - \frac{u}{a} ((a - d) \tan \lambda_f - d \tan \lambda_r) \sin \gamma e_2 - l \dot{\gamma} \sin \gamma e_3.
\]

**(7)**

Angular velocities of the board and the rider are equal to each other. We can write the angular velocity in the form

\[
\Omega = \dot{\theta} e_3 + \dot{\theta} e_3 = \dot{\theta} e_1 - \frac{u}{a} (\tan \lambda_f + \tan \lambda_r) \sin \gamma e_3.
\]

Thus the formulae for the absolute velocity of the board’s center of mass C and the rider’s center of mass R are given by (6)-(7).

5. Equations of motion. We derive now equations of motion of the given model of a skateboard in the form of the Gibbs – Appell equations. We choose variables \(u\) and \(\dot{\gamma}\) as a pseudovelocities for this problem. In order to derive differential equations of skateboard motion in the Gibbs-Appell form, let us obtain the Gibbs function [2]. This function is known also as the energy of acceleration [21]. It is well known that the Gibbs function of a rigid body can be calculated using the formula (see [21])

\[
S = \frac{m}{2} \mathbf{W}_C^2 + \frac{1}{2} \left( \dot{\Omega} \cdot \Theta_C \dot{\Omega} \right) + \left( \mathbf{\Omega} \cdot [\mathbf{\Omega} \times \Theta_C \mathbf{\Omega}] \right).
\]

**(8)**
where \( m \) is the mass of the body, \( \mathbf{W}_C \) is acceleration of the body's center of mass, \( \Theta_C \) is the moment of inertia tensor of the body and \( \mathbf{\Omega} \) and \( \mathbf{\Omega} \) are the angular velocity and angular acceleration of the body.

We will assume that the directions of the principal axes of inertia both the board and the rider are defined by the unit vectors \( \mathbf{e}_\xi, \mathbf{e}_\eta, \mathbf{e}_\zeta \) of the coordinate system \( P \xi \eta \zeta \), which is constructed by rotating of the system \( P x_1 x_2 x_3 \) through \(-\gamma\) about \( P x_1 \). Thus the \( P \xi \)-axis is always normal to the plane of the board. The unit vectors \( \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \) are expressed through \( \mathbf{e}_\xi, \mathbf{e}_\eta, \mathbf{e}_\zeta \) by the formulae

\[
\mathbf{e}_1 = \mathbf{e}_\xi, \quad \mathbf{e}_2 = \mathbf{e}_\eta, \quad \mathbf{e}_3 = \mathbf{e}_\zeta.
\]

In the principal central axes of inertia, the moment of inertia tensors of the board and the rider, respectively, are given by:

\[
\Theta_b = \begin{pmatrix}
I_{bx} & 0 & 0 \\
0 & I_{by} & 0 \\
0 & 0 & I_{bz}
\end{pmatrix}, \quad \Theta_r = \begin{pmatrix}
I_{rxx} & 0 & 0 \\
0 & I_{ryy} & 0 \\
0 & 0 & I_{rz}
\end{pmatrix}.
\]

Using expressions (6), (7) for the absolute velocity of the board’s center of mass and the rider’s center of mass we can calculate the absolute acceleration of these points. For the absolute acceleration of the board’s center of mass we have the following expression

\[
\mathbf{w}_C = w_{C1}\mathbf{e}_1 + w_{C2}\mathbf{e}_2 + w_{C3}\mathbf{e}_3,
\]

\[
w_{C1} = \dot{u} \left( 1 - \frac{(\tan \lambda_f + \tan \lambda_r) h}{a} \sin^2 \gamma \right) - \frac{u^2 (\tan^2 \lambda_f - \tan^2 \lambda_r)}{2a} \sin^2 \gamma - \frac{3u \dot{\gamma} (\tan \lambda_f + \tan \lambda_r) h}{a} \sin \gamma \cos \gamma,
\]

\[
w_{C2} = h \left( \dot{\gamma}^2 \sin \gamma - \ddot{\gamma} \cos \gamma \right) - \frac{(\tan \lambda_f - \tan \lambda_r)}{2} (\dot{u} \sin \gamma + u \dot{\gamma} \cos \gamma)
\]

\[
w_{C3} = -h \left( \dot{\gamma} \sin \gamma + \ddot{\gamma} \cos \gamma \right).
\]

Similarly for the acceleration of the rider’s center of mass we have the following formula

\[
\mathbf{w}_R = w_{R1}\mathbf{e}_1 + w_{R2}\mathbf{e}_2 + w_{R3}\mathbf{e}_3,
\]

\[
w_{R1} = \dot{u} \left( 1 - \frac{(\tan \lambda_f + \tan \lambda_r) l}{a} \sin^2 \gamma \right) - \frac{3u \dot{\gamma} (\tan \lambda_f + \tan \lambda_r) l}{a} \sin \gamma \cos \gamma
\]

\[
- \frac{u^2}{a^2} ((a - d) \tan \lambda_f - d \tan \lambda_r) (\tan \lambda_f + \tan \lambda_r) \sin^2 \gamma,
\]

\[
w_{R2} = l \left( \dot{\gamma}^2 \sin \gamma - \ddot{\gamma} \cos \gamma \right) - \frac{(a - d) (\tan \lambda_f - d \tan \lambda_r)}{a} (\dot{u} \sin \gamma + u \dot{\gamma} \cos \gamma)
\]

\[
w_{R3} = -l \left( \dot{\gamma} \sin \gamma + \ddot{\gamma} \cos \gamma \right).
\]

The angular acceleration of the system has the form

\[
\mathbf{\ddot{\Omega}} = \dot{\gamma}\mathbf{e}_1 - \frac{u (\tan \lambda_f + \tan \lambda_r)}{a} \dot{\gamma} \sin \gamma \mathbf{e}_2 - \frac{(\tan \lambda_f + \tan \lambda_r)}{a} (\dot{u} \sin \gamma + u \dot{\gamma} \cos \gamma) \mathbf{e}_3.
\]
Using expressions of $e_1, e_2, e_3$ in terms of $e_\xi, e_\eta, e_\zeta$, we get for $\dot{\Omega}$ the following expression:

$$\dot{\Omega} = \dot{\gamma}e_\xi - \frac{(\tan \lambda_f + \tan \lambda_r)}{a} (\dot{\gamma} \sin \gamma + 2u^2 \cos \gamma \sin \gamma) e_\eta$$

$$- \frac{(\tan \lambda_f + \tan \lambda_r)}{a} (\dot{\gamma} \sin \gamma \cos \gamma + (\cos^2 \gamma - \sin^2 \gamma) u^2) e_\zeta.$$

Let $m_b$ be the mass of the board and $m_r$ be the mass of the rider. Denote also

$$I_x = I_{bx} + I_{rx}, \quad I_y = I_{by} + I_{ry}, \quad I_z = I_{bz} + I_{rz}.$$

Substituting expressions for the acceleration of the board’s center of mass, the rider’s center of mass and the angular acceleration of the system into formula (8) we get the Gibbs function of the system:

$$S = \frac{1}{2} \left( \left( A_1 + (C_1 - 2D_1) \sin^2 \gamma + F_1 \sin^4 \gamma \right) \dot{\gamma}^2 + E_1 \dot{\gamma}^2 \right)$$

$$+ (C_1 - 3D_1 + 3F_1 \sin^2 \gamma) \dot{\gamma} \dot{\gamma} \sin \gamma \cos \gamma + \left( D_1 - F_1 \sin^2 \gamma \right) u^2 \dot{\gamma} \sin \gamma \cos \gamma$$

$$+ B_1 \left( \dot{\gamma} \gamma \cos \gamma + u^2 \dot{\gamma} \gamma \cos^2 \gamma - \dot{\gamma} \dot{\gamma} \sin^2 \gamma \right).$$

Here we introduce the following notations:

$$A_1 = m_b + m_r, \quad D_1 = \frac{(\tan \lambda_f + \tan \lambda_r)}{a} (m_b h + m_r l),$$

$$B_1 = \frac{m_b h}{2} (\tan \lambda_f - \tan \lambda_r) + \frac{m_r l}{a} ((a - d) \tan \lambda_f - d \tan \lambda_r),$$

$$C_1 = \frac{m_b}{4} (\tan \lambda_f - \tan \lambda_r)^2 + \frac{I_x}{a^2} (\tan \lambda_f + \tan \lambda_r)^2$$

$$+ \frac{m_r}{a^2} ((a - d) \tan \lambda_f - d \tan \lambda_r)^2, \quad E_1 = I_x + m_b h^2 + m_r l^2,$$

$$F_1 = \frac{(\tan \lambda_f + \tan \lambda_r)^2}{a^2} \left( I_y + m_b h^2 + m_r l^2 - I_z \right).$$

The potential energy of the system consists of the potential due to gravity and the potential due to the torsional spring, connecting the truck with the axles of the wheels:

$$V = \frac{k_1 \dot{\gamma}^2}{2} + m_b g h \cos \gamma + m_r g l \cos \gamma.$$

The Gibbs – Appell equations, describing the dynamics of the given model of a skateboard have the form

$$\frac{\partial S}{\partial \dot{u}} = 0, \quad \frac{\partial S}{\partial \dot{\gamma}} = -\frac{\partial V}{\partial \gamma}.$$
or, in explicit form
\[ (A_1 + (C_1 - 2D_1) \sin^2 \gamma + F_1 \sin^4 \gamma) \dot{u} + B_1 (\dot{\gamma} \cos \gamma - \dot{\gamma}^2 \sin \gamma) \sin \gamma \]
\[ + (C_1 - 3D_1 + 3F_1 \sin^2 \gamma) \ddot{\gamma} \sin \gamma \cos \gamma = 0, \]
\[ E_1 \ddot{\gamma} + (D_1 - F_1 \sin^2 \gamma) \ddot{u} \sin \gamma \cos \gamma + B_1 (\ddot{\gamma} \ddot{u} + \dot{\gamma} \dot{u} \cos \gamma) \cos \gamma \]
\[ + k_1 \gamma - (m_b h + m_r l) g \sin \gamma = 0. \]

The system (9) will be the main object of our further investigations. This system admits the energy integral
\[ H = A_1 + (C_1 - 2D_1) \sin^2 \gamma + F_1 \sin^4 \gamma \frac{u^2}{2} + \frac{E_1}{2} \dot{\gamma}^2 + B_1 \ddot{u} \gamma \sin \gamma \cos \gamma \]
\[ + \frac{k_1}{2} \gamma^2 + (m_b h + m_r l) g \cos \gamma = c_0. \]

Thus for integrability of the system (9) we need another additional first integral. The problem of integrability of the system (9) will be investigated below.

6. Stability of uniform straight-line motion of a skateboard. Equations (9) have the particular solution
\[ u = u_0 = \text{const}, \quad \gamma = 0, \]
which corresponds to uniform straight-line motion of a skateboard. Consider the problem of stability of this particular solution.

**Theorem 6.1.** The uniform straight-line motion (11) of a skateboard is stable under the conditions
\[ \left[ \frac{m_b h}{2} \tan \lambda_f - \tan \lambda_r + \frac{m_r l}{a} ((a - d) \tan \lambda_f - d \tan \lambda_r) \right] u_0 > 0, \]
\[ k_1 + \left( \tan \lambda_f + \tan \lambda_r \frac{u_0^2}{a} - g \right) (m_b h + m_r l) > 0. \]

**Proof.** Setting \( u = u_0 + \xi \) and keeping for \( \gamma \) its notation we write the equations of the perturbed motion
\[ E_1 \ddot{\gamma} + B_1 u_0 \dot{\gamma} + (D_1 u_0^2 + k_1 - (m_b h + m_r l) g) \gamma = \Gamma, \quad \dot{\xi} = \Xi. \]

Here \( \Gamma \) and \( \Xi \) are functions of \( \gamma, \dot{\gamma} \) and \( \xi \), whose development as a series in powers of said variables starts with terms of at least the second order. Moreover, these functions identically vanish with respect to \( \xi \) when \( \gamma = 0 \) and \( \dot{\gamma} = 0 \) (this fact can be verified manually):
\[ \Gamma(0, 0, \xi) = 0, \quad \Xi(0, 0, \xi) = 0. \]

The characteristic equation corresponding to the linearized system (14) has the form:
\[ \lambda \left( E_1 \lambda^2 + B_1 u_0 \lambda + D_1 u_0^2 + k_1 - (m_b h + m_r l) g \right) = 0. \]

When conditions
\[ E_1 > 0, \quad B_1 u_0 > 0, \quad D_1 u_0^2 + k_1 - (m_b h + m_r l) g > 0 \]
are fulfilled, equation (15) has one zero-root and two roots in the left half plane. Since the functions $\Gamma$ and $\Xi$ identically vanish for $\gamma = 0$, $\dot{\gamma} = 0$, then under conditions (16) we have the critical case of one-zero root \([7, 22]\) and solution (11) is stable with respect to $\gamma$, $\dot{\gamma}$ and $u$ (asymptotically stable with respect to $\gamma$ and $\dot{\gamma}$).

Since a condition $E_1 > 0$ is valid for all values of parameters then conditions of stability (16) may be finally written in the form (12)-(13).

**Remark 1.** If at least one of the conditions (12)-(13) is not fulfilled then equation (15) has the root in the right-half plane and solution (11) will be unstable.

**Remark 2.** The expression in the left-hand side of inequality (12) contains $u_0$ as a multiplier. This means that the stability of motion depends on its direction. If one direction of motion is stable the opposite direction is necessary unstable.

The effect that stability of motion depends on its direction is known for many nonholonomic systems. First of all we can mention here the problem of motion of a rattleback (aka wobblestone or celtic stone, see e.g. \([4, 9, 15, 20, 23, 29]\)). In this problem the stability of permanent rotations of a rattleback also depend on the direction of rotation.

**Corollary 1.** Suppose that $u_0 > 0$, $\lambda_f = \lambda_r = \lambda$ and condition (13) is valid. Then the stability of motion (12) depends on the location of the rider. If the rider stands closer to the front truck ($d < a/2$), the motion will be stable and when the rider stands closer to the rear truck ($d > a/2$) it will be unstable.

Let us find now conditions of stability of the equilibrium position of a skateboard, i.e. the particular solution

$$u_0 = 0, \quad \gamma = 0.$$  

When $u_0 = 0$ the characteristic equation (15) has one zero-root and two pure imaginary roots under the condition

$$k_1 - (m_bl + m_rl)g > 0. \quad (17)$$

**Theorem 6.2.** Inequality (17) gives the necessary and sufficient condition for stability of equilibrium position of a skateboard.

*Proof.* Indeed, $\gamma = 0$ is a critical point of the potential energy of the system. Then using the theorems about stability of equilibria of nonholonomic systems, proved by V.V. Rumyantsev \([27, 28]\) (see also \([17]\)) we can conclude that the equilibrium position of a skateboard will be stable with respect to $\gamma$ and $\dot{\gamma}$ if the second variation of the potential energy will be positive for $\gamma = 0$. This condition has the same form that inequality (17). Thus we can conclude that the equilibrium position of a skateboard will be stable if the torsional spring constant is sufficient to overcome the destabilizing gravity torque. \[\square\]

7. **Existence of an invariant measure and the integrable case.** Let us find here the necessary conditions for existence of an invariant measure with analytical density for equations (9) near the uniform straight-line motion of a skateboard.

**Theorem 7.1.** Equations (9) have an invariant measure with analytical density only in the case when

$$B_1 = 0.$$
Proof. Following the results of the paper by V.V. Kozlov [16] we can conclude that the necessary condition of existence of an invariant measure with analytical density for system (9) has the form

\[ \text{tr } \Lambda = 0, \]

where \( \Lambda \) is the matrix of linear part of system (9). Using the characteristic equation (15) we can compute this expression in the explicit form:

\[ \text{tr } \Lambda = -\frac{B_1u_0}{E_1}. \]

Therefore system (9) has an invariant measure only in the case \( B_1 = 0 \). The condition \( B_1 = 0 \) is valid, for example, when the skateboard is symmetric \((\lambda_f = \lambda_r)\) and the rider stands in the center of the board \((d = a/2)\). \( \square \)

**Remark 3.** When \( B_1 = 0 \) the first equation of (9) can be integrated separately.

**Theorem 7.2.** In the case \( B_1 = 0 \) we can completely solve equations (9) in terms of quadratures.

**Proof.** From the condition \( B_1 = 0 \) we can find the explicit expression for \( \tan \lambda_r \) through other parameters of the system

\[ \tan \lambda_r = \tan \lambda_f + \frac{2(a - 2d) m_r l}{m_b h a + 2m_r l d} \tan \lambda_f. \]

Substituting this expression for \( \tan \lambda_r \) into \( A_1, B_1, \ldots, F_1 \), we get

\[ A_1 = m_b + m_r, \quad D_1 = \frac{2(m_h h + m_r l)^2}{m_b h a + 2m_r l d} \tan \lambda_f, \quad E_1 = I_z + m_b h^2 + m_r l^2, \]

\[ C_1 = \frac{m_b m_r (m_h h^2 + m_r l^2) (a - 2d)^2}{(m_b h a + 2m_r l d)^2} \tan^2 \lambda_f + \frac{4I_z (m_b h + m_r l)^2}{(m_b h a + 2m_r l d)^2} \tan^2 \lambda_f, \]

\[ B_1 = 0, \quad F_1 = \frac{4(m_b h + m_r l)^2 (I_y - I_z + m_b h^2 + m_r l^2)}{(m_b h a + 2m_r l d)^2} \tan^2 \lambda_f. \]

Equations (9) will have the form

\[ (A_1 + (C_1 - 2D_1) \sin^2 \gamma + F_1 \sin^4 \gamma) \dot{u} \]

\[ + (C_1 - 3D_1 + 3F_1 \sin^2 \gamma) u \dot{\gamma} \sin \gamma \cos \gamma = 0, \quad (18) \]

\[ E_1 \ddot{\gamma} + (D_1 - F_1 \sin^2 \gamma) u^2 \sin \gamma \cos \gamma + k_1 \gamma - (m_b h + m_r l) g \sin \gamma = 0, \]

with the energy integral obtained from (10)

\[ H = \frac{A_1 + (C_1 - 2D_1) \sin^2 \gamma + F_1 \sin^4 \gamma}{2} u^2 + \frac{E_1}{2} \gamma^2 \]

\[ + \frac{k_1}{2} \dot{\gamma}^2 + (m_b h + m_r l) g \cos \gamma = c_0. \quad (19) \]

Passing from the first of equations (18) to a new independent variable – the angle \( \gamma \) we can rewrite this equation in the form:

\[ \frac{du}{d\gamma} = \frac{(3D_1 - C_1 - 3F_1 \sin^2 \gamma) u \sin \gamma \cos \gamma}{(A_1 + (C_1 - 2D_1) \sin^2 \gamma + F_1 \sin^4 \gamma)}. \quad (20) \]
Equation (20) is a differential equation with separate variables. By solving this equation, we find the velocity $u$ as a function of $\gamma$: $u = u(\gamma)$. Substituting this function into the energy integral (19) we get differential equation for $\gamma$:

$$
\dot{\gamma}^2 = \frac{2}{E_1} \left( c_0 - (m_b h + m_r l) g \cos \gamma \right) - \frac{k_1}{E_1} \gamma^2 - \frac{(A_1 + (C_1 - 2D_1) \sin^2 \gamma + F_1 \sin^4 \gamma)}{E_1} u^2(\gamma).
$$

(21)

Equation (21) is also differential equation with separate variables. By solving this equation we find $\gamma$ as a function of time: $\gamma = \gamma(t)$. Substituting this function in the expression for $u = u(\gamma)$ we find $u = u(\gamma(t)) = u(t)$. Further we can find all remaining variables as functions of time. Indeed for the angle $\theta$ we have the following differential equation

$$
\dot{\theta} = -\left( \frac{\tan \lambda_f + \tan \lambda_r}{a} \right) u \sin \gamma = -\frac{2(m_b h + m_r l) \tan \lambda_f}{m_b ha + 2m_r ld} u \sin \gamma,
$$

from which, having expressions $u = u(t)$ and $\gamma = \gamma(t)$ we find by integrating

$$
\theta(t) = \theta_0 - \frac{2(m_b h + m_r l) \tan \lambda_f}{m_b ha + 2m_r ld} \int_0^t u(\tau) \sin \gamma(\tau) \, d\tau.
$$

(22)

For the coordinates $X$ and $Y$ we have the following differential equations:

$$
\dot{X} = u \cos \theta - \frac{(a - 2d) m_r l \tan \lambda_f}{m_b ha + 2m_r ld} u \sin \gamma \sin \theta,
$$

$$
\dot{Y} = u \sin \theta + \frac{(a - 2d) m_r l \tan \lambda_f}{m_b ha + 2m_r ld} u \sin \gamma \cos \theta.
$$

Since we already have expressions for $u = u(t), \gamma = \gamma(t)$ and $\theta = \theta(t)$ we find from these equations

$$
X = X_0 + \int_0^t u(\tau) \cos \theta(\tau) \, d\tau - \frac{(a - 2d) m_r l \tan \lambda_f}{m_b ha + 2m_r ld} \int_0^t u(\tau) \sin \gamma(\tau) \sin \theta(\tau) \, d\tau,
$$

(23)

$$
Y = Y_0 + \int_0^t u(\tau) \sin \theta(\tau) \, d\tau + \frac{(a - 2d) m_r l \tan \lambda_f}{m_b ha + 2m_r ld} \int_0^t u(\tau) \sin \gamma(\tau) \cos \theta(\tau) \, d\tau.
$$

(24)

Thus for $B_1 = 0$ all unknown functions in the problem can be expressed by quadratures (21)-(24).

In the next paragraph we study the behavior of the system (9) near the equilibrium point

$$
u_0 = 0, \quad \gamma = 0$$

in general case when $B_1 \neq 0$. 

\[ \Box \]
8. Nonlinear dynamics of the skateboard near the equilibrium position.

8.1. Reduction to the normal form. In this paragraph we come back to the case $B_1 \neq 0$. Let for the skateboard we have

$$u_0 = 0, \quad \gamma = 0,$$

i.e. the skateboard is in the equilibrium position on the plane. According to the previous results, inequality (17) provides the necessary and sufficient condition for stability of this equilibrium. Suppose that this condition is fulfilled.

Solving equations (9) with respect to $\dot{u}$ and $\ddot{\gamma}$ and assuming that $u$, $\gamma$ and $\dot{\gamma}$ are small, we can write equations of perturbed motion taking into account the terms which are quadratic in $u$, $\gamma$ and $\dot{\gamma}$:

$$\dot{u} = \frac{B_1 \Omega^2}{A_1} \gamma^2, \quad \ddot{\gamma} + \Omega^2 \gamma = -\frac{B_1 u \dot{\gamma}}{E_1},$$

where we introduce the following notation

$$\Omega^2 = k_1 - (m_b h + m_r l) g / E_1.$$

Note, that the linear terms in the second equation of the system (25) have a form which corresponds to a normal oscillations. For investigation of nonlinear system (25) we reduce it to a normal form [5, 23]. To obtain the normal form of the system (25) first of all we make a change of variables and introduce two complex-conjugate variables $z_1$ and $z_2$:

$$\gamma = \frac{z_1 - z_2}{2i}, \quad \dot{\gamma} = \frac{z_1 + z_2}{2} \Omega, \quad u = z_3.$$

In variables $z_k$, $k = 1, 2, 3$ the linear part of the system (25) has a diagonal form and the derivation of its normal form reduces to separating of resonant terms from the nonlinearities in the right-hand sides of the transformed system (25). Finally, the normal form of the system (25) may be written as follows:

$$\dot{z}_1 = i \Omega z_1 - \frac{B_1}{2 E_1} z_1 z_3, \quad \dot{z}_2 = -i \Omega z_2 - \frac{B_1}{2 E_1} z_2 z_3, \quad \dot{z}_3 = \frac{B_1 \Omega^2}{2 A_1} z_1 z_2.$$  

(26)

Introducing real polar coordinates according to the formulae

$$z_1 = \rho_1 (\cos \sigma + i \sin \sigma), \quad z_2 = \rho_1 (\cos \sigma - i \sin \sigma), \quad z_3 = \rho_2$$

we obtain from the system (26) the normalized system of equations of perturbed motion which is then split into two independent subsystems:

$$\dot{\rho}_1 = -\frac{B_1}{2 E_1} \rho_1 \rho_2, \quad \dot{\rho}_2 = \frac{B_1 \Omega^2}{2 A_1} \rho_2^2,$$

$$\dot{\sigma} = \Omega.$$  

(27)

(28)

Terms of order higher than the second in (27) and those higher than the first in $\rho_k$, $k = 1, 2$ in (28) have been omitted here.

In the $\varepsilon$-neighborhood of the equilibrium position the right-hand sides of equations (27) and (28) differ from the respective right-hand sides of the exact equations of perturbed motion by quantities of order $\varepsilon^3$ and $\varepsilon^2$ respectively. The solutions of the exact equations are approximated by the solutions of system (27)-(28) with an error of $\varepsilon^2$ for $\rho_1$, $\rho_2$ and of order $\varepsilon$ for $\sigma$ in a time interval of order $1/\varepsilon$. Restricting the calculations to this accuracy, we will consider the approximate system (27)-(28) instead of the complete equations of perturbed motion.
8.2. Integration of the normal form. Equation (28) is immediately integrable. We obtain

\[ \sigma = \Omega t + \sigma_0. \]

System (27) describes the evolution of the amplitude \( \rho_1 \) of the board oscillations and also the evolution of the velocity \( \rho_2 \) of a straight-line motion of the skateboard. One can see that this system has the first integral

\[ E_1 \rho_1^2 + \frac{A_1}{\Omega^2 \rho_2^2} = A_1 n_1^2, \quad (29) \]

where \( n_1 \) is a constant, specified by initial conditions. We will use this integral for solving of the system (27) and for finding the variables \( \rho_1 \) and \( \rho_2 \) as functions of time: \( \rho_1 = \rho_1(t), \rho_2 = \rho_2(t) \). Expressing \( \rho_2^2 \) from the integral (29) and substitute it to the second equation of the system (27) we get

\[ \dot{\rho}_2 = \frac{B_1}{2E_1}(\Omega^2 n_1^2 - \rho_2^2). \quad (30) \]

The general solution of equation (30) has the following form:

\[ \rho_2(t) = \frac{\Omega n_1}{1 - n_2 \exp\left(-\frac{B_1 \Omega n_1 t}{E_1}\right)} \left(1 - n_2 \exp\left(\frac{B_1 \Omega n_1 t}{E_1}\right)\right), \quad (31) \]

where \( n_2 \) is a nonnegative arbitrary constant. Now, using the integral (29), we can find the explicit form of the function \( \rho_1(t) \):

\[ \rho_1(t) = 2\sqrt{\frac{A_1 n_1^2 n_2}{E_1} \frac{\exp\left(\frac{B_1 \Omega n_1 t}{2E_1}\right)}{1 + n_2 \exp\left(-\frac{B_1 \Omega n_1 t}{E_1}\right)}}. \quad (32) \]

8.3. Qualitative analysis of a skateboard motion. Let us consider the properties of the solutions (31), (32) of system (27) and their relations with the properties of motion of the skateboard. System (27) has an equilibrium position

\[ \rho_1 = 0, \quad \rho_2 = \Omega n_1 \quad (33) \]

(these particular solutions can be obtained from general functions (31)-(32) if we suppose in these functions \( n_2 = 0 \)). An arbitrary constant \( n_1 \) can be both positive and negative. The positive values of this constant correspond to a straight-line motions of the skateboard with small velocity in stable direction and the negative ones – in unstable direction. Indeed, if we linearize equations (27) near the equilibrium position (33) we get

\[ \dot{\rho}_1 = -\frac{B_1}{2E_1} \Omega n_1 \rho_1, \quad \dot{\rho}_2 = 0. \]

Thus, for \( n_1 > 0 \) the equilibrium position (33) is stable and for \( n_1 < 0 \) it is unstable.

Evolution of the functions \( \rho_1 \) and \( \rho_2 \) gives the complete description of behavior of a skateboard with small velocities. Let us suppose, that at initial instant the system is near the stable equilibrium position \( (n_1 > 0) \) and \( \rho_2(0) \geq 0 \), i.e. \( n_2 \leq 1 \) (the case when \( n_1 > 0, n_2 > 1 \) is similar to the case \( n_1 < 0, n_2 < 1 \), which will be investigated below). These initial conditions correspond the situation when at initial instant the skateboard takes the small velocity

\[ \rho_2(0) = \Omega n_1 \frac{1 - n_2}{1 + n_2} \]
in the stable direction. Then in the course of time the amplitude of oscillations of the board $\rho_1$ decreases monotonically from its initial value

$$\rho_1 (0) = \frac{2n_1}{1 + n_2} \sqrt{\frac{A_1 n_2}{E_1}}$$

to zero, while the velocity of a skateboard $\rho_2$ increases in absolute value. In the limit the skateboard moves in stable direction with a constant velocity $\Omega n_1$ (see Fig. 5-6).

Suppose now that at initial instant the system is near the unstable equilibrium position $n_1 < 0$. Suppose again, that at initial instant $n_2 < 1$, i.e. $\rho_2 (0) < 0$ (the case $n_1 < 0$, $n_2 > 1$ is similar to the case $n_1 > 0$, $n_2 < 1$ which was considered above). These initial conditions correspond the situation when at initial instant the skateboard takes the small velocity

$$\rho_2 (0) = \Omega n_1 \frac{1 - n_2}{1 + n_2}$$
in unstable direction. In this case the limit of the system motions is the same as when $\rho_2(0) \geq 0$ but the evolution of the motion is entirely different. When

$$0 < t < t_* = \frac{E_1 \ln(n_2)}{B_1 n_2 \Omega n_1}$$

the absolute value of the oscillation amplitude $\rho_1$ increases monotonically and the skateboard moves in unstable direction with decreasing velocity. At the instant $t = t_*$ the velocity vanishes and the oscillation amplitude $\rho_1$ reaches its maximum value

$$\rho_1(t_*) = \sqrt{\frac{A_1 n_2^2}{E_1}}$$

When $t > t_*$ the skateboard already moves in stable direction with an increasing absolute value of its velocity and the oscillation amplitude decreases monotonically. Thus when $\rho_2(0) < 0$ during the time of evolution of the motion a change in the direction of motion of the skateboard occurs (Fig. 7-8). The similar nonlinear effects (in particular the change of the direction of motion) were observed earlier.
in other problems of nonholonomic mechanics (for example in a classical problem of dynamics of a rattleback [4, 9, 15, 20, 23, 29]). Thus, we describe here the basic features of dynamics of the simplest skateboard model, proposed in [12, 13] and developed by us.

9. Conclusions. In this paper the problem of motion of a skateboard with a rider was examined. This problem has many common features with other problems of nonholonomic dynamics. In particular it was shown that the stability of motion of the skateboard depends on the direction of motion. The similar effects have been found earlier in the classical problem of a rattleback dynamics. It was found also the integrable case in the problem. Note that the integrable problems are very rare in nonholonomic mechanics and therefore these results seem to be interesting and helpful.

The considered model of a skateboard can be developed by various ways: for example, we can construct a more complicated model of a skateboard taking into account the wheels and the various types of their contact with the ground. We can construct also a more complicated (and more realistic) model of the truck. We also can construct a more complicated model of the rider and taking into account the possibility of the rider to control the skateboard (it will be already control system). All these ideas are very interesting and attractive and we will certainly consider them in the future.

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