On the simplest static and stationary vacuum quadrupolar metrics

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Abstract

In the present paper we argue that a special case of the Bach-Weyl metric describing a static configuration of two Schwarzschild black holes gives rise, after extending its parameter space to complex values, to a very simple 2-parameter model for the gravitational field of a static deformed mass. We compare this model, which has no restrictions on the quadrupole parameter, with the well-known Zipoy-Voorhees $\delta$-metric and show in particular that the mass quadrupole moment in the latter solution cannot take arbitrary negative values. We subsequently add an arbitrary angular momentum to our static model and study some properties of the resulting 3-parameter stationary solitonic spacetime, which permits us to introduce the notion of the Fodor-Hoenselaers-Perjés relativistic multipole moments.

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I. INTRODUCTION

The exterior gravitational field of a static deformed mass can be described in general relativity by various 2-parameter solutions of Einstein’s equations, and recently a comparative analysis of some of these has been carried out in the paper [1]. The well-known Zipoy-Voorhees (ZV) solution [2, 3] (sometimes called the $\delta$ or $\gamma$ metric) is promoted in [1] as the simplest quadrupole metric, advantageous over other models in the context of physical applications, and it is worth noting that during the last decade the ZV metric has been analyzed by various authors. Thus, for instance, in the papers [4, 5] this metric was used for the analysis of the geodesic motion of test particles in the presence of naked singularities, while the recent work [6] studied it in the context of quasinormal modes of deformed compact objects; mention also that the optical properties of the ZV metric have been analyzed in [7], and the recent paper [8] considers the harmonic oscillations of test particles in this spacetime. However, a seemingly simple form of the ZV metric is in reality rather deceptive as the mass-quadrupole moment in it is a function of the varying number of (many) Schwarzschild constituents, which in particular is reflected in the form of the known stationary generalizations of this static solution [9–11]. Moreover, the real situation seems to be even worse because, as will be demonstrated in the present paper, the dimensionless quadrupole moment of the ZV metric cannot take arbitrary negative values determining the
oblateness of the source, and it is not quite clear to which extent this restriction persists in
the stationary versions of the δ metric too.

Motivated by the non-generic nature of the ZV solution as a quadrupolar metric, in the
present paper we will discuss another 2-parameter model of a static deformed mass, the one
not mentioned in [1], that arises as equatorially symmetric specialization of the Bach-Weyl
solution [12] in which the extension of the parameters must be additionally carried out. The
parameters of the mass and mass-quadrupole moment can be introduced explicitly into this
model instead of the original parameter set, and the new physical parameters will have no
any restrictions on their values. We shall also consider a simple stationary generalization
of our static solution and compare it with two known 3-parameter stationary spacetimes.
This will allow us to touch the question of the most suitable definition of the relativistic
multipole moments.

The rest of the paper is organized as follows. We start Sec. II with comments on the
restrictions that exist in the ZV metric with regard to the quadrupole deformations, and then
construct and analyze a simple 2-solitonic model of a static deformed mass. In Sec. III we
consider a stationary 3-parameter generalization of our static model and obtain its concise
form in the equatorial plane and in the extreme limit. Here we also briefly comment on
the mass-quadrupole moment in the stationary generalizations of the ZV spacetime, and on
the most appropriate definition of multipole moments in the context of solution generating
techniques. Concluding remarks are given in Sec. IV.

II. THE EXTENDED 2-PARAMETER STATIC VACUUM SOLUTION

As is well known, the static axisymmetric vacuum gravitational fields in Einstein’s general
theory of relativity are described by the Weyl line element

$$ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f dt^2,$$

where the functions $f$ and $\gamma$ depend on the coordinates $(\rho, z)$ only and satisfy the differential
equations

$$f(f_{,\rho\rho} + \rho^{-1}f_{,\rho} + f_{,\rho\rho}) = f_{,\rho}^2 + f_{,z}^2,$$

$$4\gamma_{,\rho} = \rho f^{-2}(f_{,\rho}^2 - f_{,z}^2),$$

$$2\gamma_{,z} = \rho f^{-2}f_{,\rho}f_{,z}.$$  

(2)
The ZV solution of the system (2) has the form

\[
\begin{align*}
  f &= (R_+ + R_- - 2m) \delta, \\
  c^{2\gamma} &= \left[\frac{(R_+ + R_- - 4m^2)}{4R_+ R_-}\right]^{\delta^2}, \\
  R_\pm &= \sqrt{\rho^2 + (z \pm m)^2},
\end{align*}
\]

(3)

where \(m\) and \(\delta\) are two real parameters. Since the Schwarzschild solution of mass \(m\) is contained in (3) as the particular \(\delta = 1\) case, the constant \(\delta\) may be considered as a deformation parameter describing the deviation of the ZV geometry from spherical symmetry.

After the redefinition \(\delta = 1 + p\) proposed in [4] by Quevedo, one obtains the following formulas for the total mass \(M\) and mass-quadruple moment \(Q\) of the ZV source [4]:

\[
M = m(1 + p), \quad Q = -\frac{1}{3} m^3 p(1 + p)(2 + p),
\]

(4)

and we emphasize the importance of the ‘minus’ sign in the expression for \(Q\). Although at first glance one may think that \(Q\) in (4) can take arbitrary negative values (determining the oblateness of the source) independently of the positive values of \(M\), this is not really the case. Indeed, by inverting formulas (4) and redefining \(Q = qM^3\), we get

\[
m = M \sqrt{1 + 3q}, \quad p = -1 + \frac{1}{\sqrt{1 + 3q}},
\]

(5)

whence it follows, taking into account the reality of the constant \(p\), that the negative values of the dimensionless quadrupole moment \(q\) are restricted by the inequality

\[-\frac{1}{3} < q < 0,
\]

(6)

the corresponding values of \(p\) varying from 0 to \(+\infty\). This obviously invalidates the ZV metric as a generic 2-parameter model for the exterior field of a static deformed mass.

It appears that the well-known Bach-Weyl (BW) solution [12] for two nonequal Schwarzschild masses permits us to elaborate a more attractive model for a static deformed mass in which the mass-quadrupole moment would not have already any restrictions on its negative values.

A starting point in the construction of our model is a special case of the BW spacetime in which the separation parameter is set equal to zero, so that the resulting configuration

1 Note that in the paper [6] the quadrupole moment is given with incorrect sign, which could affect some of the results obtained in that paper.
of two overlapping sources becomes symmetric with respect to the equatorial plane, the corresponding metric functions \( f \) and \( \gamma \) having the form

\[
f = \frac{(R_+ + R_- - 2m_1)(r_+ + r_- - 2m_2)}{(R_+ + R_- + 2m_1)(r_+ + r_- + 2m_2)},
\]

\[
e^{2\gamma} = \frac{(m_1 + m_2)^2[(R_+ + R_-)^2 - 4m_2^2][(r_+ + r_-)^2 - 4m_2^2]}{16(m_1 - m_2)^2R_+R_-r_+r_-}
\times \left[ \frac{m_2(R_+ + R_-) - m_1(r_+ + r_-)}{m_2(R_+ + R_-) + m_1(r_+ + r_-)} \right]^2,
\]

\[
R_\pm = \sqrt{\rho^2 + (z \pm m_1)^2}, \quad r_\pm = \sqrt{\rho^2 + (z \pm m_2)^2}.
\]

(7)

The two arbitrary real parameters of this solution are \( m_1 \) and \( m_2 \), and by analogy with the general case they can be interpreted as individual masses of the Schwarzschild constituents. On the upper part of the symmetry axis (\( \rho = 0, z > \max\{m_1, m_2\} \)) the function \( \gamma \) vanishes, while the function \( f \) takes the form

\[
f(\rho = 0, z) = \frac{(z - m_1)(z - m_2)}{(z + m_1)(z + m_2)},
\]

whence we readily get, via the procedure [13] of calculating the Geroch-Hansen (GH) multipole moments [14, 15], the expressions for the total mass and mass-quadrupole moment:

\[
M = m_1 + m_2, \quad Q = -m_1m_2(m_1 + m_2).
\]

(9)

Now, inverting the above formulas and introducing the dimensionless quadrupole moment \( q = Q/M^3 \), we get

\[
m_1 = \frac{M}{2}(1 + d), \quad m_2 = \frac{M}{2}(1 - d), \quad d = \sqrt{1 + 4q},
\]

(10)

and hence, accounting for the reality of \( m_1 \) and \( m_2 \), one might think that the allowed negative values of \( q \) lie in the interval \((-\frac{1}{4}, 0)\), thus being even more restrictive than in the case of the ZV solution. However, it is easy to see that for all \( q < -\frac{1}{4} \) the parameters \( m_1 \) and \( m_2 \) in (10) become complex conjugate quantities, \( m_2 = \bar{m}_1 \), and these preserve the reality of the axis expression (8), so that the reality of the metric functions \( f \) and \( \gamma \) in (7) is also preserved. Therefore, after changing in (7) the parameters \( (m_1, m_2) \) to \( (M, q) \) by means of (10), we finally arrive at the extended version of our 2-parameter model satisfying the system (2), in which the quadrupole moment \( q \) can take arbitrary real values:

\[
f = \frac{[R_+ + R_- - M(1 + d)][r_+ + r_- - M(1 - d)]}{[R_+ + R_- + M(1 + d)][r_+ + r_- + M(1 - d)]}.
\]
\[ e^{2\gamma} = \frac{[(R_+ + R_-)^2 - M^2(1 + d)^2][(r_+ + r_-)^2 - M^2(1 - d)^2]}{16(1 + 4q)R_+ R_- r_+ r_-} \times \left[ \frac{(1 - d)(R_+ + R_-) - (1 + d)(r_+ + r_-)}{(1 - d)(R_+ + R_-) + (1 + d)(r_+ + r_-)} \right]^2, \]

\[ R_\pm = \sqrt{\rho^2 + [z \pm M(1 + d)/2]^2}, \quad r_\pm = \sqrt{\rho^2 + [z \pm M(1 - d)/2]^2}. \tag{11} \]

Note that in terms of the new parameters \( M \) and \( q \) the axis expression \( (8) \) takes the extended form

\[ f(\rho = 0, z) = \frac{z^2 - Mz - M^2q}{z^2 + Mz - M^2q}, \tag{12} \]

thus being arguably the most concise and elegant axis expression of the function \( f \) containing explicitly the mass monopole and quadrupole relativistic moments as arbitrary parameters.

To the best of our knowledge, the extension of the parameters \( m_1 \) and \( m_2 \) to the complex values in the BW solution has never been attempted earlier in the literature, most probably because such an extension involving ‘imaginary’ masses might be erroneously taken for unphysical by the researchers.

It is clear that the extended solution \( \text{(11)} \) contains the entire non-extended BW solution \( \text{(7)} \) as a particular subextreme case defined by \( q > -\frac{1}{4} \), and in the limit \( q = -\frac{1}{4} \) it reduces to the ZV \( \delta = 2 \) spacetime. The hyperextreme part of our solution corresponds to \( q < -\frac{1}{4} \), in which case the functions \( R_\pm \) and \( r_\pm \) take complex values, and it is worth noting in this respect that the condition for fixing uniquely the branch of the square roots \( r_i = \sqrt{\rho^2 + (z - \alpha_i)^2} \) is \( \text{Re}(r_i) > 0 \), independently of whether \( \alpha_i \) are real or imaginary quantities.

III. THE EXTENDED 3-PARAMETER STATIONARY VACUUM SOLUTION

We now turn to discussing stationary generalizations of the 2-parameter solutions considered in the previous section. Concerning the ZV metric, the main question to examine is whether the introduction of the angular momentum is able to affect somehow the admissible negative values of its mass-quadrupole moment; on the other hand, the static model \( \text{(11)} \) is likely to be given the simplest possible generalization to the stationary case.

Undoubtedly, the most renowned stationary generalization of the ZV metric is the Tomimatsu-Sato (TS) family \([9, 10]\) of solutions for spinning masses which was originally constructed for integral \( \delta \) only, and later extended to arbitrary real \( \delta \) too \([16, 17]\). The complexity of the TS solutions grows rapidly with growing \( \delta \) \([10]\), which makes them hardly
recommended for the use as a simple model describing the field of a spinning mass with arbitrary quadrupole deformation. At the same time, it is not difficult to demonstrate that the mass-quadrupole moment in this important family of stationary spacetimes defines larger oblateness of the source than in the static ZV solution. Indeed, the total mass $M$, the quadrupole moment $Q$ and the total angular momentum $J$ of the TS solutions are given by the formulas \[18, 19\]

\[
M = \frac{\delta \sigma}{p_0}, \quad Q = -M^3 \left( \frac{\delta^2 - 1}{3 \delta^2} p_0^2 + q_0^2 \right), \quad J = M^2 q_0, \tag{13}
\]

where $\sigma$ is an arbitrary positive constant, while the real parameters $p_0$ and $q_0$ are subject to the constraint $p_0^2 + q_0^2 = 1$. Solving the system (13) for $\delta, \sigma$ and $q_0$ and taking into account that $q_0$ represents the dimensionless angular momentum, we obtain, after introducing $q = Q/M^3$ and redefining $\delta = 1 + p$, a stationary generalization of formulas (5):

\[
\sigma = M \sqrt{1 + 3q + 2q_0^2}, \quad p = -1 + \frac{\sqrt{1 - q_0^2}}{\sqrt{1 + 3q + 2q_0^2}}, \tag{14}
\]

whence it follows at once that the negative values of the dimensionless quadrupole moment $q$ are determined by the inequalities

\[
-\frac{1}{3}(1 + 2q_0^2) < q < 0, \quad 0 \leq q_0^2 < 1, \tag{15}
\]

and hence the angular momentum in the TS solutions indeed enlarges the oblateness of the massive sources.

Another well-known stationary generalization of the ZV metric is the Hoenselaers-Kinnersley-Xanthopoulos (HKX) solution \[11, 20\], some variations of which have been analyzed by different authors in application to various problems involving a spinning deformed mass \[21–24\]. The simplest asymptotically flat 3-parameter HKX solution possessing equatorial symmetry is defined by the Ernst complex potential $\mathcal{E}$ \[25\] of the form ($p = \delta - 1$)

\[
\mathcal{E} = \left( \frac{x - 1}{x + 1} \right)^p \frac{A_-}{A_+}, \quad A_\pm = (x \mp 1)(x^2 - 1)^{2p} - \alpha^2 (x \pm 1)(x^2 - y^2)^{2p} - i\alpha (x^2 - 1)^p[(y \pm 1)(x + y)^{2p} + (y \mp 1)(x - y)^{2p}], \tag{16}
\]

where the real constant $\alpha$ is the rotation parameter, and the spheroidal coordinates $x$ and $y$ are related to the coordinates $\rho$ and $z$ by the formulas

\[
x = \frac{1}{2\sigma}(r_+ + r_-), \quad y = \frac{1}{2\sigma}(r_+ - r_-), \quad r_\pm = \sqrt{\rho^2 + (z \pm \sigma)^2}, \tag{17}
\]
σ being a positive real constant. The ZV solution is contained in (16) as the particular \( \alpha = 0 \) case.

Although the HKX version of a stationary \( \delta \)-metric is more compact and simple than the respective TS version with nonintegral \( \delta \), the expressions of the physical quantities \( M, J \) and \( Q \) defined by (16) turn out to be more complicated than formulas (13) of the generalized TS family. This can be seen with the aid of the form of the potential (16) on the upper part of the symmetry axis (\( \rho = 0, z > \sigma \)), namely,

\[
E(\rho = 0, z) = \frac{(z - \sigma)^p[z - \sigma - \alpha^2(z + \sigma)] - 2i\sigma\alpha(z + \sigma)^p}{(z + \sigma)^p[z + \sigma - \alpha^2(z - \sigma)] - 2i\sigma\alpha(z - \sigma)^p},
\]

whence the desired multipole moments of the HKX spacetime can be obtained by means of the procedure [13], finally yielding

\[
M = \sigma \left( p + \frac{1 + \alpha^2}{1 - \alpha^2} \right),
\]

\[
J = \frac{2\sigma^2\alpha[1 + \alpha^2 + 2p(1 - \alpha^2)]}{(1 - \alpha^2)^2},
\]

\[
Q = -\frac{\sigma^3}{3(1 - \alpha^2)^3}\{12\alpha^2(1 + \alpha^2) + p(1 - \alpha^2)[(1 + p)(2 + p)
+ 2\alpha^2(16 - p^2) + \alpha^4(1 - p)(2 - p)]\},
\]

and it seems impossible to resolve the algebraic system (19) analytically for \( \sigma, p \) and \( \alpha \). Nonetheless, it is still possible to demonstrate numerically that, thanks to rotation, the oblateness of the HKX stationary source can be larger than that of the ZV static source. For this purpose, one has first to pass in (19) to the dimensionless angular momentum and quadrupole moment, \( j = J/M^2 \) and \( q = Q/M^3 \) respectively, and then assign particular values to \( M, j \) and \( q \) in order to finally get the corresponding meaningful values of \( \sigma, p \) and \( \alpha \) through the resolution of the system (19) numerically. Thus, for the particular choice \( M = 1, j = 0.65, q = -0.5 \) we obtain \( \sigma \approx 0.562, p \approx 0.406, \alpha \approx 0.397 \) (numerical values are given up to three decimal places); by further leaving \( M \) and \( q \) unchanged and varying only \( j \), we find for \( j = 0.68 \) the numerical solution \( \sigma \approx 0.645, p \approx 0.172, \alpha \approx 0.399 \), while the value \( j = 0.7 \) gives \( \sigma \approx 0.692, p \approx 0.043, \alpha \approx 0.409 \). Since \( q = -0.5 \) in the above numerical solutions is greater in absolute value than the absolute value of the minimal dimensionless quadrupole moment \( q \) of the ZV metric, we have shown, on the one hand, that oblateness of the HKX spinning source can be larger compared with that of the ZV static source. On the other hand, the dependence \( p(j) \) in the above examples, when a smaller \( p \) corresponds
to a larger $j$, simply means that a static ZV source with a smaller intrinsic oblateness needs a larger angular momentum to be deformed beyond the limiting value $-1/3$ than the static source with a larger intrinsic oblateness, which looks quite natural.

We would like to emphasize that both the TS and the HKX families of solutions are physically meaningful spacetimes which over the years have been widely discussed in the literature as legitimate examples representing the exterior field of a spinning mass. At the same time, it is also clear that these solutions can hardly be advocated as the simplest generic models for the exterior geometry around compact spinning objects, with advantages over other known exact solutions. Actually, in what follows we are going to point out a special member of the extended 2-soliton stationary solution which generalizes in a very simple way the static solution (11) from the previous section and contains explicitly the multipoles $M, J$ and $Q$ as three arbitrary real parameters.

We note that in the paper [26] a physical representation of the general 4-parameter metric for the exterior field of a neutron star was obtained in terms of multipole moments. So, taking into account that the static solution (11) is a 2-parameter specialization of that metric, it would be logic to search for its simplest stationary generalization within the same generic metric too. A thorough analysis of the axis expression of the Ernst potential defining the general 4-parameter solution has eventually led us to the particular 3-parameter spacetime with the following remarkably simple axis data:

$$E(\rho = 0, z) \equiv e(z) = \frac{z^2 - Mz - M^2q - iM^2j}{z^2 + Mz - M^2q + iM^2j},$$  \hspace{1cm} (20)

where the parameters $q$ and $j$, as before, are the dimensionless mass-quadrupole moment and dimensionless angular momentum, respectively. The potential $\mathcal{E}$ of the new 3-parameter extended solution in the entire space has the form

$$\mathcal{E} = (A - B)/(A + B),$$

$$A = (\sigma_+ + \sigma_-)^2(R_+ - R_-)(r_+ - r_-) - 4\sigma_+\sigma_-(R_+ + r_-)(R_- + r_+),$$

$$B = M d[\sigma_-(R_+ - R_-) + \sigma_+(R_+ - r_-)],$$

$$R_\pm = \frac{\pm \sigma_+ + i j}{d + 1} \sqrt{\rho^2 + (z \pm M\sigma_+)^2}, \hspace{1cm} r_\pm = \frac{\pm \sigma_- + i j}{d - 1} \sqrt{\rho^2 + (z \pm M\sigma_-)^2},$$

$$\sigma_\pm = \sqrt{q + (1 \pm d)/2}, \hspace{1cm} d = \sqrt{1 + 4(q + j^2)},$$  \hspace{1cm} (21)

and it satisfies the Ernst equation [25]

$$(\mathcal{E} + \bar{\mathcal{E}})(\mathcal{E}_{,\rho,\rho} + \rho^{-1}\mathcal{E}_{,\rho} + \mathcal{E}_{,z,z}) = 2(\mathcal{E}_{,\rho}^2 + \mathcal{E}_{,z}^2).$$  \hspace{1cm} (22)
The corresponding full metric is given by the line element

\[ ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega d\varphi)^2, \]  

(23)

with the metric functions \( f, \gamma \) and \( \omega \) defined by the expressions

\[ f = \frac{A\bar{A} - B\bar{B}}{(A + B)(A + B)}, \quad e^{2\gamma} = \frac{A\bar{A} - B\bar{B}}{K_0 K_0 R_+ R_- r_+ r_-}, \quad \omega = -\frac{2\text{Im}[G(A + B)]}{AA - BB}, \]

\[ G = Md[\sigma_-(R_+ - R_-)(r_+ + r_- - z) - \sigma_+(r_+ - r_-)(R_+ + R_- + z) + M\sigma_+\sigma_-(R_+ + R_- + r_+ + r_-)], \]

\[ K_0 = 4\sigma_+\sigma_-/(q + j^2), \] 

(24)

which have been worked out with the aid of the general formulas of the paper [26]. Note that the Kerr metric [27] is contained in the above formulas as the \( q = -j^2 \) particular case. Note also that the singularities of the solution are defined by the equation \( A + B = 0 \), and they lie on the stationary limit surface \( f = 0 \), which means in particular that the singularities are not expected to show up when the solution is used as the exterior part in a physically meaningful global model of a compact object.

The solution (21) has a remarkable multipole structure. Its first four mass and angular momentum relativistic moments, obtainable from (20), have the form

\[ M_0 = M, \quad M_1 = 0, \quad M_2 = M^3 q, \quad M_3 = 0, \]

\[ J_0 = 0, \quad J_1 = M^2 j, \quad J_2 = 0, \quad J_3 = M^4 q j, \] 

(25)

and it can be shown that the complex coefficients \( m_n \) in the expansion of the function

\[ \xi(z) = \frac{1 - e(z)}{1 + e(z)} = \sum_{n=0}^{\infty} m_n z^{-n-1}, \]

(26)

when \( z \to \infty \), which play a key role in the procedure [13], are defined in the case of the solution (21) by the following very concise generic formulas:

\[ m_{2k} = M^{2k+1} q^k, \quad m_{2k+1} = iM^{2k+2} q^k j, \quad k = 0, 1, 2, \ldots \] 

(27)

It is well known [13] that only the first four quantities \( m_n \) coincide with the GH complex multipoles \( P_n = M_n + iJ_n \), the Kerr solution being an exclusive stationary vacuum spacetime for which \( P_n = m_n \) for all \( n \), so that we easily get from (27), after setting \( q = -j^2 \), the simple formula defining the multipole moments of the Kerr spacetime [15]:

\[ P_n = M(iMj)^n, \quad n = 0, 1, 2, \ldots \] 

(28)
When $q \neq -j^2$, the calculation of the GH multipole moments of the solution (21) in the general case requires finding corrections to the coefficients $m_n$, $n \geq 4$, which would involve combinations of the lower multipoles. Since these additional terms must inevitably spoil the exceptional multipole structure (27) determined by the progressive powers of $q$, we find it necessary to briefly reexamine the issue of multipole moments in the context of the modern solution generating techniques.

We first note that the GH multipole moments were defined as values of certain tensorial quantities “at infinity”, and their practical calculation actually represents a complicated task. That is why Fodor et al. in their paper [13] developed for the stationary axisymmetric asymptotically flat spacetimes an algorithm, based on expansion of the axis expression of the Ernst complex potential $\xi$, which considerably simplifies the computation of the GH multipoles $P_n$. The authors of [13] magnanimously interpreted the discrepancies between the quantities $m_n$ and $P_n$ in favor of the latter, and found the explicit form of the first eleven $P_n$ in terms of $m_n$; however, they failed to discern in $m_n$ an independent definition of relativistic multipole moments which has various advantages over the GH one. From now on we will refer to the complex quantities $m_n$ as the Fodor-Hoenselaers-Perjés (FHP) multipole moments, the real part of which determines the mass multipoles and the imaginary part – the rotational multipoles. It should be noted that the axis value of the Ernst complex potential is the key ingredient in the modern solution generating techniques, and its FHP multipoles $m_n$ determine it uniquely; the subsequent construction of the solution in the whole space can be carried out for instance with the aid of Sibgatullin’s integral method [28]. At the same time, the construction of an exact solution with a specified number of GH multipole moments would require identification of the corresponding infinite set of the FHP moments $m_n$ as a preliminary step for finding the respective axis data, which generically does not look realizable in principle. Our solution (21) illustrates well the above said: we had no problems in finding the general formula for the FHP multipoles (27), while we see no way of getting the corresponding general expression for $P_n$, even though the axis data (20) contains only three parameters.

The adoption of the FHP multipole moments instead of the GH ones makes it possible to rectify the physical interpretation of some known exact solutions. For example, the limit $q = 0$ in the solution (21) leads to the 2-parameter $M-j$ spacetime possessing only two nonzero FHP moments $m_0$ and $m_1$, and therefore describing the exterior gravitational field.
of a rigidly rotating sphere. Interestingly, this particular case of the extended 2-soliton solution \[29\] was briefly analyzed in the paper \[30\] and found physically deficient due to a specific behavior of its higher GH multipole moments. Since the Ernst potential of the $M$-$j$ solution was given in \[30\] with errors, it will be worth noting that the case $q = 0$ in formulas \((21)\) is determined by $d$ and $\sigma_\pm$ of the form

\[
d = \sqrt{1 + 4j^2}, \quad \sigma_\pm = \sqrt{(1 \pm d)/2} = \frac{1}{2}(\sqrt{1 + 2ij} \pm \sqrt{1 - 2ij}). \tag{29}\]

It is clear that within the framework of the GH definition of multipole moments the correct physical interpretation of this special case would be impossible.

The 3-parameter solution \((21)\) has a simple form suggesting the suitability of the solution for astrophysical applications and gravitational experiment. Since a considerable part of such applications is restricted to the analysis of different physical processes and effects in the equatorial plane, it is desirable to have the representation of the metric functions \((24)\) of our solution in the limit $z = 0$. It is not difficult to show, using the results of \[26\], that the $M$-$q$-$j$ metric in the equatorial plane takes the following concise form:

\[
f = \frac{A - B}{A + B}, \quad e^{2\gamma} = \frac{A^2 - B^2}{(r_+ + r_-)^4r_\pm^2}, \quad \frac{\omega}{M} = -\frac{2jB}{A - B},
\]

\[
A = (r_+ + r_-)^2r_+r_- + q + j^2, \quad B = (r_+ + r_-)(r_+r_- + r + q), \quad r_\pm = \sqrt{r + q + \frac{1}{2}} \left[1 \pm \sqrt{1 + 4(q + j^2)}\right], \quad r \equiv \rho^2/M^2, \tag{30}\]

and, remarkably, all three metric functions in \((30)\) are determined exclusively by the $z = 0$ value of the Ernst potential.

We also note that the extreme limit in the solution \((21)\) occurs when $d = 0 \Leftrightarrow q = -\frac{1}{4} - j^2$, and the form of the solution then can be worked out with the aid of the general formulas of Ref. \[31\]. In this case it is convenient to introduce the spheroidal coordinates $x$ and $y$ by the formulas

\[
x = \frac{1}{2M\sigma}(r_+ + r_-), \quad y = \frac{1}{2M\sigma}(r_+ - r_-), \quad r_\pm = \sqrt{\rho^2 + (z \pm M\sigma)^2}, \quad \sigma = \sqrt{\frac{1}{4} - j^2}, \tag{31}\]

in terms of which the potentials $\mathcal{E}$ can be written as

\[
\mathcal{E} = (A - B)/(A + B),
\]
\[ A = \lambda^2 + \frac{1}{2} \lambda^2 + i j \sigma x y (1 - y^2), \]
\[ B = (\sigma x + i j y) \lambda + \frac{1}{2} i j y (1 - y^2), \] (32)

while for the metric coefficients \( f, \gamma \) and \( \omega \) we readily get the expressions
\[ f = \frac{N}{D}, \quad e^{2\gamma} = \frac{N}{\sigma^8 (x^2 - y^2)^4}, \quad \omega = \frac{M (y^2 - 1) W}{N}, \]
\[ N = \lambda^4 - \sigma^2 (x^2 - 1) (1 - y^2) \nu^2, \]
\[ D = N + \lambda^2 \kappa + (1 - y^2) \nu \chi, \]
\[ W = \sigma^2 (x^2 - 1) \nu \kappa + \lambda^2 \chi, \]
\[ \lambda = \sigma^2 (x^2 - 1) - j^2 (1 - y^2), \]
\[ \nu = j y^2, \]
\[ \kappa = 2 \sigma^2 (\sigma x + 1) (x^2 - y^2) + \frac{1}{2} [\sigma x (y^2 + 1) + y^2], \]
\[ \chi = 2 j \sigma^2 (\sigma x + 1) (x^2 - y^2) + \frac{1}{2} j \sigma x (1 - y^2). \] (33)

In view of a not quite accurate statement made in the paper [31] concerning the relation of the extreme vacuum potential (24) of [31] to the well-known Kinnnersley-Chitre (KC) 5-parameter solution [32], we find it instructive to reexamine this issue in more detail. As we have been able to find out recently, the desired relation can be only established for nonzero values of the KC parameter \( \beta \), which may look strange recalling that this parameter is responsible for counterrotation (see, e.g., [33]), while the solution (24) of [31], as well as the solution (32) of this paper, is equatorially symmetric. However, as we have discovered to our big surprise, under certain choices of other parameters in the KC solution, \( \beta \) can describe the corotating case too. Thus, by setting \( \alpha = Q = 0, P = 1, \beta = -p_0 q_0 \) in the KC solution for which we use notations of Ref. [34], we get the following axis value of the corresponding Ernst potential:
\[ \mathcal{E}(\rho = 0, z) = \frac{e_-}{e_+}, \quad e_+ = z^2 + \frac{2 \sigma}{p_0} z + (1 + q_0^2) \frac{\sigma^2}{p_0^2} \pm 2 i q_0 \frac{\sigma^2}{p_0^2}, \] (34)

where \( p_0 \) and \( q_0 \) satisfy the relation \( p_0^2 + q_0^2 = 1 \), and \( \sigma \) is a real constant. Then, taking into account that the total mass \( M \) and angular momentum \( J \) of the KC solution in the particular case (34) are defined by the formulas \( M = 2 \sigma / p_0 \) and \( J = -M^2 q_0 / 2 \), we can introduce \( j = -q_0 / 2 \) and thus arrive at the axis expression of the solution (32) in which the values of the dimensionless angular momentum \( j \) are restricted by the inequality \( |j| < 1/2 \).
since $|q_0| \leq 1$. Therefore, the extreme solution (32) of this section, and the solution (24) of [31] represent the analytically extended versions of the KC subcase [34]. Mention, that the values $|j| > 1/2$ can be also covered by the KC solution, but only after a complex continuation of the parameters $p \to ip, \sigma \to i\sigma, q_0^2 - p_0^2 = 1$, so that $|q_0| \geq 1$.

We emphasize that the extreme metric (32)-(33) describes an oblate object.

IV. CONCLUSIONS

In the present paper we have shown that the ZV metric, regarded by various authors as the simplest model for a static deformed mass, in reality has restrictions on the negative values of the dimensionless quadrupole moment and as such can hardly be considered superior to other known 2-parameter solutions for a non-spherical mass [35-38]. At the same time, in the context of the “simplest model” we have proposed a static 2-parameter metric, obtainable as analytic extension of the BW spacetime, in which the two arbitrary parameters are explicitly the mass monopole and quadrupole moments. This metric was subsequently generalized to the stationary case, to include the angular momentum parameter. The analysis of the multipole structure of our 3-parameter spacetime has led us to the conclusion that the GH multipole moments distort generically the physical interpretation of exact solutions and therefore have to be substituted for what we have called the FHP multipoles, the latter being better adjusted to the modern solution generating methods and to the intrinsic structure of stationary axisymmetric spacetimes. In this respect we would like to observe that Hansen himself [15] pointed out the ambiguity in the definition of multipole moments, so that there should be no surprise when a more precise definition eventually replaces the old one.

Apparently, the use of the FHP multipole moments instead of the GH ones will be able to considerably simplify the multipole analysis of the stationary axially symmetric solutions, facilitating in particular comparison of the analytical and numerical models of astrophysical interest. It is worth noting in this regard that the FHP mass-hexadecapole moment of the 4-parameter solution for the exterior geometry of a neutron star [26] will be strictly quartic in angular momentum, thus lending full support to the Yagi et al. no-hair conjecture for neutron stars [39].
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