GROTHENDIECK TOPOLOGIES
AND DEFORMATION THEORY II

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0. Introduction

0.1. In the present paper we continue the study of deformation theory of algebras using the approach of [Ga]. We will extend the main results of [Ga] to the global case. Namely, we pose and solve the following problem: what cohomological machinery controls deformations of a sheaf of algebras over a scheme? This question has already been studied by many authors [Ill],[Ge],[H-Sch],[Schl].

0.2. Let first $A$ be an associative algebra over a ring. Consider the category of all algebras over $A$, let us call it $C(A)$. One can observe that every question concerning the deformation theory of $A$ can be formulated in terms of this category.

Our first step will be to apply a linearization procedure to $C(A)$, in other words we will endow it with a Grothendieck topology and then we will consider sheaves of abelian groups on it. It will turn out that deformations of $A$ are controlled by cohomologies of certain sheaves on this site. Cohomologies arise naturally as classes attached to torsors and gerbes. All this was done in [Ga].

When $A$ is no longer an algebra over a ring but rather a quasi-coherent sheaf of algebras over a scheme $X$, the definition of $C(A)$ must be modified in order to take into account possible localization with respect to $X$, since the appropriate cohomology theory would incorporate algebra cohomology of $A$ and scheme cohomology of $X$. In this case instead of working with the whole category of sheaves on our site, we single out a subcategory which we call the category of quasi-coherent sheaves. This category will have properties similar to those of the category of quasi-coherent sheaves of $A$--bimodules among all sheaves of $A$--bimodules and it will be more manageable.

The second step will be to find a connection between the category of sheaves on $C(A)$ and the category of quasi-coherent sheaves of $A$--bimodules on our scheme $X$. This connection will be described by two mutually adjoint functors, which would enable us to rewrite the cohomology groups that control deformations of $A$ in terms of cohomologies of some canonical object $T^\bullet(A)$ of the derived category of quasi-coherent sheaves of $A$--bimodules. The object $T^\bullet(A)$ will be called the cotangent complex of $A$. Another approach to the construction of the cotangent complex in a slightly different situation was used by Illusie [Ill].

0.3. Let us now describe the contents of the paper.

In Section 1 we present a brief exposition of some well known facts and results from the theory of sites. For a more detailed discussion the reader is referred to [Ar,Gr]. In the remaining sections we will freely operate with the machinery of...
sheaves, cohomologies, direct and inverse images; therefore the reader is advised to 
look through this section in order to become familiar with the notation.

In Section 2 we define the site $C_X(A)$ along with its variants for affine schemes. 
We introduce also the appropriate categories of sheaves and functors between them. 
The central results are

1. Theorem 2.3.3 with its corollaries, that insure that the category $Sh^{qc}(A)$ is 
well defined
2. Theorem 2.5 that says that cohomologies of quasi-coherent sheaves com-
puted inside the quasi-coherent category and inside the category of all
sheaves give the same answers.

In Section 3 we introduce functors $\Im$ and $\mathcal{L}$ that establish connection between 
the category $Sh^{qc}(X)$ and the category $A_{qc} - mod$. Let us remark that it would 
be possible to work with the category of all sheaves on $C_X(A)$ without introducing 
quasi-coherent sheaves explicitly. We, nevertheless, decided to that, since to our 
mind, introducing this category and basic functors that are connected to it reflects 
the nature of the things and clarifies the exposition.

Finally, Section 4 is devoted to deformation theory. Theorem 4.1.2 describes 
how to pass from deformations to cohomology of sheaves on $C_X(A)$ via torsors and 
gerbes, and in 4.2 we translate the assertions of this theorem to the language of 
cohomology of quasi-coherent sheaves of $A$-bimodules.

0.4. The results of the present paper can be easily generalized to the case of alge-
bras over an arbitrary operad (cf. [Ga,H-Sch]). We opted for treating the case of 
associative algebras only in order to simplify the exposition. One can also develop 
a similar theory for operad co-algebras.

0.5. In recent years there have been a lot of interest in deformati-
on. We have 
to mention the works [H-Sch,Ge-Sch,Ma,St-Schl,Fo]. Our approaches connected to 
that of [Ill]. Let us also point out that one of the central ideas of the present paper: 
to resolve an algebra $A$ by free algebras (at least locally) goes back probably to 
Quillen and to Grothendieck [Qu].

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1. preliminaries on Grothendieck topologies

1.0. In this section we will review certain notions from the theory of sites. Proofs 
will be given mostly in cases when our exposition differs from the standard one.

1.1. Let $C$ be a category possessing fiber products. A Grothendieck topology (cf. 
[Gr]) on it (or a structure of a site) is a collection of morphisms that are called 
covering maps if it satisfies the following three conditions:

1. Any isomorphism is a covering.
2. If $\phi : U \to V$ and $\psi : V \to W$ are coverings, then their composition 
$\psi \circ \phi : U \to W$ is a covering too.
3. If $\phi : U \to V$ is a covering and if $\alpha : V_1 \to V$ is an arbitrary morphism, 
then the base change map $\phi_1 : U \times_V V_1 \to V_1$ is a covering.

1.1.1 Examples.
1. For any category $C$ there exists the minimal Grothendieck topology: the only coverings are isomorphisms. This site will be denoted by $(C, \text{min})$.

2. Let $\text{Set}$ be the category of sets. We introduce the structure of a site on it by declaring surjections to be the covering maps.

3. Let $\text{Set}^o$ be the category opposite to $\text{Set}$. We introduce a Grothendieck topology by declaring $\phi : X \rightarrow Y$ to be a covering if the corresponding map of sets $Y \rightarrow X$ is an injection.

4. Constructions similar to the above ones can be carried out when the category $\text{Set}$ is replaced by an abelian category, in particular, by the category $\text{Ab}$, the category of abelian groups.

5. Let $X$ be a topological space. Let $C(X)$ be the category whose objects are finite disjoint unions of open subspaces of $X$.

$$\text{Hom}(U, V) \overset{\text{def}}{=} \text{maps from } U \text{ to } V \text{ compatible with an embedding to } X.$$ 

A map $\phi \in \text{Hom}(U, V)$ is a covering if it is surjective.

6. Let a site $C$ have a finite object $X_0$ and let $X$ be any other object of $C$. We can define a new site $C_X$ whose underlying category is the category of "objects of $C$ over $S_0$", with morphisms being compatible to the projection to $X$. A morphism $\phi$ in $C_X$ is declared to be a covering if it is a covering in $C$.

1.1.2. Let $C_1$ and $C_2$ be two sites. A functor $F$ between the underlying categories is said to be a functor between sites if the following holds:

1. $F$ maps coverings to coverings.

2. If $A, B, D$ are three objects in $C_1$ with $A, B$ mapping to $D$, then the canonical map $F(A \times B) \rightarrow F(A) \times F(D) \rightarrow F(B)$ is a covering in $C_2$.

We say that a functor $F$ between two sites is strict if it preserves fiber products, i.e. if the map in (2) is an isomorphism.

1.2. **Definition.** A sheaf of sets (resp. of abelian groups) on a site $C$ is a functor $S$ between the sites $C$ and $\text{Set}^o$ (resp. $\text{Ab}^o$), the latter considered with the topology specified in the Example 3 above.

Morphisms between sheaves are by definition natural transformations between such functors.

**Definition.** A presheaf of sets on $C$ (resp. of abelian groups) is sheaf on $C$ when the latter is considered with the minimal topology.

It is an easy exercise to verify that the above definition of a sheaf coincides with the traditional one. From now on by a sheaf we will mean a sheaf of abelian groups. It will be left to the reader to make appropriate modifications for sheaves of sets.

The category of sheaves will be denoted by $\text{Sh}(C)$. This category possesses a natural additive structure and is in fact an abelian category. If $S$ is a sheaf, and if $X \in C$, $S(X)$ will be denoted by $\Gamma(X, S)$ and will be called the set of sections of $S$ over $X$. The map $\Gamma(X, S) \rightarrow \Gamma(Y, S)$ for a map $Y \rightarrow X$ will be called the restriction map.
1.3. Let $F : C_1 \to C_2$ be a functor between sites. We have then the natural functor (called direct image) $F_* : Sh(C_2) \to Sh(C_1)$. This functor is always left exact. It is also right exact if the following condition is satisfied:

For any covering $Z \to F(X)$ there exists a covering $\phi : Y \to X$, endowed with a map $\alpha : F(Y) \to Z$ such that the composition $F(Y) \to Z \to F(X)$ coincides with $F(\phi)$.

The functor $F_*$ has a left adjoint (called the inverse image): $F^* : Sh(C_1) \to Sh(C_2)$. The functor $F^*$ is always right exact and it is also left exact if the functor $F$ is strict in the sense of 1.1.1.

1.3.1 Examples.

1. Let $\text{Forget} : (C, \text{min}) \to C$ be the canonical functor of sites. The above constructions yield the embedding functor from sheaves to presheaves and its left adjoint, which is called the functor of associating a sheaf to a presheaf. It is a good exercise to describe the associated sheaf explicitly.

2. Let $pt$ be the category of one object and one morphism. If $C$ is a site, for any $X \in C$ we have a functor $pt_X : pt \to C$, that sends the unique object of $pt$ to $X$. We have the canonical constant sheaf $Const$ on $pt$. Let by definition $Const_X = pt^*_X(Const)$. This sheaf will be called the constant sheaf corresponding to $X$. By definition we have: $\text{Hom}(Const_X, S) = \Gamma(X, S)$ functorially with respect to $S \in Sh(C)$.

3. Let $F : C_1 \to C_2$ be a functor between sites and let $X \in C_1$. Then

$$F^*(Const_X) \simeq Const_{F(X)}.$$ 

4. Recall the situation of 1.1.1 Example 6. We have the natural embedding functor $i : C_X \to C$ and its right adjoint $\text{Cart} : Y \to Y \times X$. We claim then, that the functors $i_*$ and $\text{Cart}^*$ are canonically isomorphic. We denote this functor by $S \to S|C_X$ and call it the functor of restriction of a sheaf to $C_X$. By definition, for $Y \in C_X$ we have $\Gamma(Y, S|C_X) \simeq \Gamma(Y, S)$.

If now $X \to X_0$ is a covering, the functor $S \to S|C_X$ is exact and faithful.

1.4 Cohomology of sheaves. Along with the abelian category $Sh(C)$ one considers also the corresponding derived categories $D(Sh(C))$, $D^+(Sh(C))$, $D^-(Sh(C))$ and $D^b(Sh(C))$. It can be shown [Ar,Gr] that the category $Sh(C)$ has enough injective objects. In particular, any left exact functor admits a right derived functor. If $X \in C$, $R^i\Gamma(X, S)$ will be denoted by $H^i(X, S)$.

1.4.1 Čech complexes. Let $C$ and $C_X$ be as in 1.3.1 Example 4 above. Put $U_0 = X$ and let $U_i$ denote the $i$-th fiber product of $X$ with itself over $X_0$. Let also $j_i$ denote the canonical map $j_i : U_i \to X_0$.

For any sheaf $S \in Sh(C)$ we can form a canonical complex:

$$0 \to S \to j_0* j_0^*(S) \to j_1* j_1^*(S) \to j_2* j_2^*(S) \to \ldots$$

Claim. This complex is exact.

To prove this statement, we restrict this complex to $X = U_0$ and this enables us to write down an explicit homotopy operator.

The complex

$$0 \to j_0* j_0^*(S) \to j_1* j_1^*(S) \to j_2* j_2^*(S) \to \ldots$$

will be called the Čech complex of $S$. 


1.5 Torsors and Gerbes.

1.5.0. Let now our category possess a final object $X_0$ and let $S$ be a sheaf of abelian groups. $H^i(S)$ will denote $R^i\Gamma(X_0, S)$.

1.5.1. Before defining torsors and gerbes in the sheaf-theoretic context we need to recall several definitions.

Let $\Gamma$ be an abelian group and let $\Gamma$ act on a set $\tau$. We say that $\tau$ is a torsor over $\Gamma$ if this action is simply transitive. Torsors over a given group form a rigid monoidal category (cf. [DM]) under $\tau_1 \otimes \tau_2 \to \tau_1 \times \tau_2/\Gamma$ with the anti-diagonal action of $\Gamma$.

Let now $O$ be a monoidal category and let $M$ be an arbitrary category. We say that $O$ acts on $M$ if we are given

1. A functor $\text{Action} : O \times M \to M$.

2. A natural transformation between the two functors $O \times O \times M \to M$:

$$
\begin{array}{ccc}
O \times O \times M & \xrightarrow{\text{Action}} & O \times M \\
\downarrow & & \downarrow \text{Action} \\
O \times M & \xrightarrow{\text{Action}} & M
\end{array}
$$

such that the obvious "pentagon" identity is satisfied.

We say that $M$ is a gerbe bound by $O$, if for any $X \in M$ the functor $O \to M$ given by $A \to \text{Action}(A \times X)$ is an equivalence of categories.

If $O$ is a groupoid and if $M$ is a gerbe bound by $O$, then $M$ is also a groupoid and $\pi_0(M)$ is a torsor over $\pi_0(O)$.

1.5.2. A sheaf of sets $\Upsilon$ is called a torsor over $S$ if

1. $S$ viewed as a group-like object in the category of sheaves of sets acts on the object $\Upsilon$, i.e if for every $X \in C$, $\Gamma(X, S)$ acts on $\Gamma(X, \Upsilon)$ in a way compatible with restrictions.

2. For every $X \in C$, $\Gamma(X, \Upsilon)$ is a torsor over $\Gamma(X, S)$, whenever the former is nonempty.

3. For some covering $X$ of $X_0$, the set $\Gamma(X, \Upsilon)$ is nonempty.

Let $T_S$ denote the category of torsors over $S$. From 1.5.1 it is easy to deduce that the groupoid $T_S$ possesses a structure of a rigid monoidal category.

Claim. The group $\pi_0(T_S)$ is canonically isomorphic to $H^1(S)$.

Proof. In fact, we claim more: Consider the category $\text{Ext}(\text{Const}_{X_0}, S)$, whose objects are short exact sequences $0 \to S \to E \to \text{Const}_{X_0} \to 0$. Then this category is canonically equivalent to $T_S$.

Indeed, for any such extension $0 \to S \to E \to \text{Const}_{X_0} \to 0$ we associate a torsor $\Upsilon$ by setting for every $X$ over $X_0$

$$
\Gamma(X, \Upsilon) = \text{splittings: } \Gamma(X, \text{Const}_{X_0}) \to \Gamma(X, E)
$$

This functor is easily seen to be an equivalence of (monoidal) categories and $\pi_0(\text{Ext}(\text{Const}_{X_0}, S)) \simeq H^1(S)$.

QED
1.5.3. We are heading towards the definition of gerbes, but first we need to recollect the notion of a stack.

Let $C$ be a site with a final object $X_0$. Suppose that for each $X \in C$ we are given a category $G(X)$, for each map $\alpha : Y \to X$ we are given a functor $G_\alpha : G(X) \to G(Y)$ and for each composition of maps $\alpha : Y \to X$ and $\beta : Z \to Y$, we are given a natural transformation $F_\alpha \circ F_\beta \to F_{\alpha \circ \beta}$, such that all the data are compatible with respect to two-fold compositions. This collection is called a stack if the following two axioms are satisfied:

1. Let $X \in C$, and let us consider the category $C_X$ as in 1.1.1 Example 6. Let also $s_1, s_2$ be two objects of $G(X)$. We can consider the presheaf of sets on $C_X$: $Y \in C_X \mapsto \text{Hom}(s_1|Y, s_2|Y)$. We require that this presheaf is a sheaf for each $X \in C$.

2. Let $\phi : Y \to X$ be a covering. Consider the category of descent data on $Y$ with respect to $X$, whose objects are pairs $s \in G(Y)$ and an isomorphism $p_1^*(s) \to p_2^*(s)$, where $p_1, p_2$ are the two projections from $Y \times Y$ to $Y$, such that the above isomorphism satisfies the obvious cocycle condition on the three-fold fiber product of $Y$ with itself over $X$. Morphisms in this category are defined to be maps $s_1 \to s_2$ compatible with isomorphisms between their pull-backs on $Y \times Y$. We have the obvious functor from $G(X)$ to this category of descent data. We require that this functor is an equivalence of categories.

Examples.

1. All sites that we will be working with in this paper will have the following property: $G(X) := C_X$ is a stack.

2. If $S$ is a sheaf of groups, we can define $G(X) = T_{S|X}$ (torsors over $S$ restricted to the category $C_X$). This is a stack too.

1.5.4 Gerbes. Let once again $C$ be a site with a final object $X_0$ and let $S$ be a sheaf of abelian groups. Let $G$ be a stack on $C$ endowed with the following additional structure:

1. Each $G(X)$ is acted on by the monoidal category $T_{S|X}$.

2. For each $\alpha : Y \to X$ we are given a natural transformation between two functors $T_{S|X} \times G(X) \to G(Y)$:

$$G_\alpha \circ \text{Action}_X \to \text{Action}_Y \circ (T_{S|X} \alpha \times G_\alpha),$$

which is compatible with the natural transformations of 1.5.1(2) and with composition of restrictions.

Suppose that for each $X \in C$, $G(X)$ is a gerbe over $T_{S|X}$ and that there exists a covering $X$ of $X_0$ such that $G(X)$ is nonempty. We say then that $G$ is a gerbe bound by $S$.

Functors between gerbes bound by a sheaf of abelian groups $S$ and natural transformations between such functors are defined in a natural fashion.

Remark. Let $S$ be a sheaf of abelian groups and let $G$ be a stack such that if $s_1, s_2 \in G(X)$, there exists a covering $Y$ of $X$ such that the pull-backs of $s_1$ and $s_2$ on $G(Y)$ become isomorphic. Then $G$ is a gerbe bound by $S$ if for every $X_0 \in G(X)$, $\text{Aut}(s)$ is isomorphic to $\Gamma(X, S)$ functorially in $X$ and in $s$. 
Examples.

1. Let $S$ be in $Sh(C)$. A basic example of a gerbe bound by $S$ is provided by setting $G(X) = T_{S|X}$. This gerbe of $S$-torsors we denote by abuse of notation by $T_S$.

It is an easy observation that a gerbe $G$ is equivalent to $T_S$ if and only if $G(X_0)$ is nonempty.

2. Let $S_1 \to S_2$ be a map of sheaves of abelian groups. If $G$ is a gerbe bound by $S_1$, we can construct an induced gerbe $G'$ bound by $S_2$.

3. (cf. [D-III,BB]) Let $0 \to S \to K_1 \to K_2 \to Const_{X_0} \to 0$ be an exact sequence of sheaves on $C$. Let $K^\bullet$ denote the 2-complex $K_1 \to K_2$. To this 2-complex we can associate a gerbe $G(K_1 \to K_2)$ bound by $S$ in a canonical way by setting: $G(K_1 \to K_2)(X) =$ the category of extensions $0 \to K_1|C_X \to E \to Const_X \to 0$ of sheaves over $C_X$ endowed with a map $E \to K_2|C_X$ commuting with the projection to $Const_{X_0}|C_X \simeq Const_X$. It is easy to verify that $G(K_1 \to K_2)$ defined in this way is indeed a gerbe.

If now $\alpha : K^\bullet \to K'^\bullet$ is a quasi-isomorphism of 2-complexes, we get a canonical functor between the corresponding gerbes $G(K^\bullet)$ and $G(K'^\bullet)$. This means that the operation of assigning a gerbe to a 2-complex is well defined on the derived category $D(Sh(C))$.

1.5.5. The following proposition is not difficult to prove:

Proposition. The assignment $K^\bullet \to G(K^\bullet)$ establishes a one-to-one correspondence between the set isomorphism classes of objects $K^\bullet$ in $D(Sh(C))$ with non-trivial cohomologies only in degrees 0 and 1, such that $H^0(K^\bullet) \simeq S$, $H^1(K^\bullet) \simeq Const_{X_0}$ and the set of equivalence classes of gerbes $G$ bound by $S$.

In particular, since the set of isomorphism classes of 2-complexes of the above type in the derived category is $Ext^2(Const_{X_0},S) \simeq H^2(S)$, to any gerbe $G$ bound by $S$ we can associate a well defined class in $H^2(S)$ that vanishes if and only if $G(X_0)$ of this gerbe is nonempty.

2. THE SITE $C_X(A)$

2.0. As it has been explained in the introduction, our bridge between deformations and cohomology is based on considering sheaves on the site $C_X(A)$ which we are about to define. Throughout this paper, by a scheme we will mean a separated scheme. It is not difficult, however, to generalize all our results to the case of an arbitrary scheme.

2.1. Let $X$ be a scheme and let $Zar_X$ denote the Zariski site of $X$, whose objects are disjoint finite unions of open subsets of $X$ and whose morphisms are maps of schemes over $X$. A morphism in $Zar_X$ is a covering if it is surjective. Let $A$ be a quasi-coherent sheaf of associative algebras on $X$.

2.1.1 Definition of $C_X(A)$.

Objects: triples $(U, B_U, \phi)$, with $U \in Zar_X$, $B_U$ is a quasi-coherent sheaf of associative algebras on $U$ and $\phi : B_U \to A|U$ is a map of sheaves of associative algebras. Here $A|U$ is the restriction of $A$ on $U$. When no confusion can be made, we will omit $\phi$. 
Morphisms: $Hom((V,C_V),(U,B_U))$ is a set of pairs $(j,\alpha)$, where $j \in Hom_{Zar}(V,U)$ and $\alpha : C_V \to B_U|V$ (restriction by means of $j$).

The category $C_X(A)$ is easily seen to have fiber products.

Topology: $(j,\alpha) : (V,C_V) \to (B,B_U)$ is said to be a covering map if $j$ is a covering in $Zar_X$, and if $\alpha$ is an epimorphism.

Sometimes when no confusion can be made we will write $A$ instead of $\Gamma(X,A)$ for $X$ being an affine scheme.

2.1.2 Variant. When $X$ is an affine scheme $X = Spec(R)$, the site $C_X(A)$ will often be denoted by $C_X^{new}(A)$ to emphasize the difference between $C_X^{new}(A)$ and $C_X^{old}(A)$:

**Definition of $C_X^{old}(A)$**.

Objects: $R$-algebras $B$ with a map to $\Gamma(X,A)$.

Morphisms: Algebra homomorphisms commuting with structure maps to $\Gamma(X,A)$.

Topology: Covering maps are defined to be just epimorphisms of algebras.

2.1.3. $Sh_X(A)$ will denote the category of sheaves of abelian groups over $C_X(A)$. For $X$ affine, $X = Spec(R)$, this category will also be denoted by $Sh_X^{new}(A)$, whereas $Sh_X^{old}(A)$ will denote the category of sheaves of abelian groups over $C_X^{old}(A)$.

**Example 1.** Let $(U,B_U) \in C_X^{new}(A)$. According to 1.3.1(2) we can consider the sheaf $Const_{(U,B_U)} \in Sh_X^{new}(A)$. It follows from the Example 1.5.3(1) that $Const_{(U,B_U)}$ is given by $(V,C_V) \mapsto \bigoplus_{Hom((V,C_V),(U,B_U))}(\mathbb{Z})$.

**Example 2.** A similar construction can be carried out in the case of an affine scheme $X = Spec(R)$ for $C_X^{old}(A)$. For a projective $R$-module $V$ let $FreeR(V)$ denote the free associative algebra built on $V$. If now $Free(V) \in C_X^{old}(A)$, the sheaf $const_{Free(V)}$ is a projective object of $Sh_X^{old}(A)$. This is because every covering of $const_{Free(V)}$ admits a section.

2.2. Let $f : Y \to X$ be a morphism of schemes. Let us be given quasi-coherent sheaves of algebras $A$ on $X$ and $A'$ on $Y$. Assume also be given a map of sheaves of algebras $\phi : A' \to f^*(A)$. (Here $f^*$ denotes the pullback.) We say then that $(f,\phi)$ is a map from the pair $(Y,A')$ to the pair $(X,A)$.

2.2.1. We have a functor denoted $(f,\phi)$ or just $f$:

$$C_X(A) \to C_Y(A') : (U,B_U) \mapsto (U \times_X Y, f^*B \times_{f^*(A|U)} (A'|U \times_X Y))$$

This functor is strict if $f$ is flat.

In the case when both $X$ and $Y$ are affine schemes, we have also the functor $C_X^{old}(A) \to C_Y^{old}(A')$. Having said this, we possess the following collection of functors between categories of sheaves on $X$ and on $Y$.

1. $f_* : Sh_Y(A') \to Sh_X(A)$.
2. The left adjoint of $f_* : f^* : Sh_X(A) \to Sh_Y(A')$. This functor is exact if $f$ is flat.
3. (for $X$ and $Y$ affine) $f_* : Sh_Y^{old}(A') \to Sh_X^{old}(A)$.
4. (also for $X$ and $Y$ affine) The left adjoint of the previous functor, denoted by $f^*$. This functor is also exact if $f$ is flat.
2.3. Our next goal will be to define a certain subcategory $S_{C}^{qc}(A)$ in $S_{X}(A)$ which we will call the category of quasi-coherent sheaves. The category $S_{C}^{qc}(A)$ will have properties analogous to those of the category of quasi-coherent sheaves of $O(X)$-modules inside the category of all sheaves of $O(X)$-modules over a scheme $X$. It will turn out that for an affine scheme, the category of quasi-coherent sheaves is equivalent to $S_{C}^{qc}(A)$.

2.3.1. Let $X$ be an affine scheme. We have the natural inclusion functor $C_{X}^{old}(A) \rightarrow C_{X}^{new}(A) : B \rightarrow (X, B)$. In this case there is the direct image functor denoted $(new \rightarrow old) : S_{X}^{new}(A) \rightarrow S_{X}^{old}(A) : \Gamma(B, (new \rightarrow old)(S)) = \Gamma((X, B), S)$, and the left adjoint of $(new \rightarrow old)$, denoted by $(old \rightarrow new)$. The functor $(old \rightarrow new)$ is exact and the functor $(new \rightarrow old)$ is left exact. $R\bullet(new \rightarrow old)$ will denote the right derived functor of $(new \rightarrow old)$.

2.3.2. Let now $f$ be a map $(Y, A') \rightarrow (X, A)$ with $X$ and $Y$ affine.

Lemma.

(1) The functors $f_{*} \circ (new \rightarrow old)$ and $(new \rightarrow old) \circ f_{*} : S_{X}^{new}(A') \rightarrow S_{X}^{old}(A)$ are canonically isomorphic.

(2) The same for the functors $(old \rightarrow new) \circ f^{*}$ and $f^{*} \circ (old \rightarrow new)$ from $S_{X}^{old}(A)$ to $S_{X}^{new}(A')$.

The proof is obvious.

2.3.3. We will now describe the functor $(old \rightarrow new)$ more explicitly. The next result can be considered as an analog of Serre’s lemma.

Theorem.

Let $X$ be an affine scheme. Then the adjunction morphism of functors $Id_{S_{X}^{old}(A)} \rightarrow (new \rightarrow old) \circ (old \rightarrow new)$ is an isomorphism.

Proof of the Theorem.

Let $X$ be an affine scheme and consider a full subcategory $C_{X}^{new}(A)_{aff}$ of $C_{X}^{new}(A)$ formed by pairs $(U, B_{U})$ with $U$ affine. This subcategory carries a natural Grothendieck topology. Let $emb$ be the embedding functor $emb : C_{X}^{new}(A)_{aff} \rightarrow C_{X}^{new}(A)$. It is clearly a functor between sites in the sense of 1.1.2.

The following lemma is easy to prove.

Lemma. The functor $emb_{*} : Sh(C_{X}^{new}(A)) \rightarrow Sh(C_{X}^{new}(A)_{aff})$ is an equivalence of categories. In particular,

$$emb_{*} \circ emb^{*} \simeq Id$$

Let us start now with a sheaf $S \in S_{X}^{old}(A)$ and consider the following presheaf $S'$ on $C_{X}^{new}(A)_{aff}$: for $j : (U, B_{U}) \rightarrow (X, A)$ we set

$$\Gamma((U, B_{U}), S') \simeq \Gamma((U, B_{U}), j^{*}(S)).$$

We claim that this presheaf is in fact a sheaf canonically isomorphic to $emb_{*}(old \rightarrow new)(S)$. Indeed, without restricting generality it suffices to check that if $j : (U, B_{U}) \rightarrow (X, A)$ is a covering, the complex

$$0 \rightarrow \Gamma((X, A), S') \rightarrow \Gamma((U, B_{U}), S') \rightarrow \Gamma((U, B_{U}), S')$$

is exact. This follows by the definition of the presheaf $S'$.
is exact at first two places, where \( j_1 : (U_1, B_{U_1}) \to (X, A) \) denotes the fiber product of \((U, B_U)\) with itself over \((X, A)\).

However, we know by 1.4.1, that the complex of sheaves in \( Sh_{\text{old}}^X(A) \)
\[
0 \to S \to j_* j^*(S) \to j_1* j_1^*(S)
\]
is exact at first two places. The complex of groups, whose exactness we are proving, is just the complex of global sections of this complex of sheaves and it is exact at first two places because the functor of taking sections is left exact.

In order to establish the isomorphism \( S' \to \text{emb}_\bullet (\text{old} \to \text{new})(S) \), we must exhibit an isomorphism:
\[
\text{Hom}_{Sh(C_{\text{new}}^X(A)_{\text{aff}})}(S', \text{emb}_\bullet (M)) \to \text{Hom}_{Sh_{\text{old}}^X(A)}(S, (\text{old} \to \text{new})(M))
\]
for any \( M \in Sh_{\text{new}}^X(A) \). The latter is, however, clear from the construction of \( S' \).

In order to finish the proof of the theorem it remains to show that \((\text{new} \to \text{old}) \circ \text{emb}_\bullet (S)' \simeq S\) but this is obvious.

2.3.4.

Let us now present several corollaries of the above theorem.

Corollary 1. Let \( X \) be an affine scheme and let \( S \in Sh_{\text{old}}^X(A) \). Let also \( j : (U, B_U) \to (X, A) \) be an object of \( C_{\text{new}}^X(A) \) with \( U \) affine. Then
\[
\Gamma((U, B_U), (\text{old} \to \text{new})(S)) \simeq \Gamma((U, B_U), j^*(S))
\]

Proof of Corollary 1. We have
\[
\Gamma((U, B_U), (\text{old} \to \text{new})(S)) \simeq \Gamma((U, B_U), (\text{old} \to \text{new})(S|U, B_U)) \simeq \\
\Gamma((U, B_U), j^*(\text{old} \to \text{new})(S)) \simeq \Gamma((U, B_U), (\text{old} \to \text{new})(j^*(S))) \simeq \\
\Gamma((U, B_U), (\text{new} \to \text{old})(\text{old} \to \text{new})(j^*(S))) \simeq \Gamma(B_U, j^*(S))
\]
Here the first two isomorphisms follow from 1.3.1 Example 4, the third one is a consequence of Lemma 2.3.2, the fourth isomorphism is the definition of the functor \((\text{new} \to \text{old})\) and the last one follows from the Theorem.

QED

Corollary 2. The functor \((\text{old} \to \text{new})\) realizes \( Sh_{\text{old}}^X(A) \) as a full abelian subcategory of \( Sh_{\text{new}}^X(A) \) stable under extensions.

This is a formal consequence of the Theorem.

Corollary 3. For any map \( f : (Y, A') \to (X, A) \) with \( Y \) and \( X \) affine and for any \( S' \in Sh_{\text{new}}^X(A) \) the canonical morphism
\[
f^* \circ \text{(new} \to \text{old})(S') \to \text{(new} \to \text{old}) \circ f^*(S')
\]
is an isomorphism provided that \( S' \) is isomorphic to \((\text{old} \to \text{new})(S)\) for some \( S \in Sh_{\text{new}}^X(A) \).

Proof of Corollary 3.
The canonical natural transformation:

\[ f^* \circ (new \to old) \to (new \to old) \circ f^* \]

follows from the standard mutual adjunction properties of the functors \( f^*, f_*, f^*; f_*, new \to old, old \to new \).

If we put now \( S' = (old \to new)(S) \), we will get on the left hand side \( f^* \circ (new \to old)(old \to new)(S) \simeq f^*(S) \) whereas on the right hand side we get

\( (new \to old) \circ f^* \circ (old \to new)(S) \simeq (old \to new) \circ (old \to new) \circ f^*(S) \)

and it is easy to verify, that under these identifications the above natural transformation yields the identity morphism on \( f^*(S) \).

QED

**Corollary 4.**

The converse of the previous corollary is true: if \( X \) is an affine scheme and if \( S \in Sh_X^{new}(A) \), then \( S \in Sh_X^{old}(A) \) if and only if for every pair \((Y, A')\) mapping to \((X, A)\) by means of \( f \) with \( Y \) being also affine, the canonical map \( f^* \circ (new \to old)(S) \to (new \to old) \circ f^*(S) \) is an isomorphism.

**Proof of Corollary 4.** Let \((old \to new)(new \to old)(S) \to S\) be the canonical adjunction morphism. It follows then from Corollary 1 and from the assumption, that for any \((U, B_U) \in C^{new}_X(A)\) with \( U \) affine the above map induces an isomorphism \( \Gamma((U, B_U), (old \to new)(new \to old)(S)) \simeq \Gamma((U, B_U), S) \).

Hence \( (old \to new)(new \to old)(S) \simeq S \) and the assertion follows.

QED

**Corollary 5.** Let \( X = \text{Spec}(R) \) be an affine scheme.

1. \( R^i(new \to old)(old \to new)(S) = 0 \) for any \( S \in Sh_X^{old}(A) \) and for any \( i \geq 1 \).
2. The functor \((old \to new) = \mathbb{1} : D(Sh_X^{old}(A)) \to D(Sh_X^{new}(A))\) is fully faithful.

**Proof of Corollary 5.**

To prove the first point, we must show that if

\[ 0 \to S \to K_1 \to K_2 \to 0 \]

is an exact sequence of sheaves in \( Sh_X^{new}(A) \) with \( S \in Sh_X^{old}(A) \), the sequence

\[ 0 \to S \to (new \to old(K_1)) \to (new \to old(K_2)) \to 0 \]

is exact in \( Sh_X^{old}(A) \).

For this it suffices to check that if \( B \in C_X^{old}(A) \) is a free algebra built on a projective \( R \)-module, the complex of sections

\[ 0 \to \Gamma(B, S) \to \Gamma(B, K_1) \to \Gamma(B, K_2) \to 0 \]

is exact.

Now, since \( Sh_X^{old}(A) \in Sh_X^{new}(A) \) is stable under extensions, \( Ext^1_{Sh_X^{old}(A)}(\text{Const}(X, B), S) = 0 \) implies \( H^1_{Sh_X^{new}(A)}((X, B), S) = 0 \) and the assertion follows.

The second point readily follows from the first one.

QED
2.3.5. It is worthwhile to notice the following fact:

Lemma. Let $X$ be as in the Proposition.

1. A sheaf $S \in Sh_X^{\text{new}}(A)$ comes from $Sh_X^{\text{old}}(A)$ if and only if the canonical map $(\text{old} \to \text{new}) \circ (\text{new} \to \text{old})(S) \to S$ is an isomorphism.
2. A sheaf $S \in Sh_X^{\text{new}}(A)$ comes from $Sh_X^{\text{old}}(A)$ if and only if for any/some pair $(Y, A')C_X^{\text{old}}(A)$ covering $(X, A)$, $f^\bullet(S)$ comes from $Sh_Y^{\text{old}}(A')$.

Proof. The first point is clear and it implies the second one because of 2.3.2 and corollary 3.

QED

2.4. We arrive now to the definition of a quasi-coherent sheaf in $Sh_X(A)$.

Definition. Let $X$ be first an affine scheme. A sheaf $S \in Sh_X^{\text{new}}(A)$ is said to be quasi-coherent if it belongs to $Sh_X^{\text{old}}(A)$.

Let $X$ now be an arbitrary scheme. A sheaf $S \in Sh_X(A)$ is said to be quasi-coherent if for some (in fact, for any, by Lemma 2.3.5) pair $(U, B)$ covering $(X, A)$ with $U$ affine, the restriction of $S$ onto $(U, B_U)$ is quasi-coherent in the sense of the previous definition.

Quasi-coherent sheaves form a full abelian subcategory in $Sh_X(A)$, which will be denoted by $Sh_X^{\text{qc}}(A)$. For $X$ affine, this category coincides with $Sh_X^{\text{old}}(A)$.

2.4.1 Example. Let $(X, B) \in C_X(A)$, then $\text{Const}_{(X, B)} \in Sh_X^{\text{qc}}(A)$.

2.4.2. The category $Sh_X^{\text{qc}}(A)$ has enough injective objects because every $Sh_X^{\text{qc}}(B_U)$ with $U$ affine does (cf. [Ar,Gri]).

2.5. We have the natural functor $\mathfrak{q}: D(Sh_X^{\text{qc}}(A)) \to D(Sh_X(A))$ that sends $D(Sh_X^{\text{qc}}(A))$ to the subcategory $D_{q}Sh_X(A)$ that consists of objects of $D(Sh_X(A))$ with quasi-coherent cohomologies.

Theorem. The above functor induces an equivalence of categories:

$\mathfrak{q}: D^b(Sh_X^{\text{qc}}(A)) \to D^b_{q}Sh_X(A)$

Corollary 5 of Theorem 2.3.3 implies the assertion for $X$ affine, as well as the following lemma:

Lemma. Let $f: (Y, A') \to (X, A)$ be a map such that

1. $f: Y \to X$ is an affine morphism of schemes.
2. $A' \to f^*(A)$ is an isomorphism.

Then $R^if_\bullet(S) = 0$ for any $i \geq 1$ and for any $S \in Sh_Y^{\text{qc}}(A')$.

Proof of the Lemma.

$R^if_\bullet(S)$ is a sheaf associated to the presheaf

$$(U, B_U) \to H^i((f^{-1}(U), f^*(A)|(f^{-1}(U)), S)).$$

Now, for every $U$ each $B_U$ can be covered by a free one and Corollary 5 of 2.3.3 finishes the proof.

QED

Proof of the Theorem.
Since any complex is glued from its cohomologies, it suffices to prove that for $S_1, S_2 \in \text{Sh}^{qc}_X(A)$ the map $\mathfrak{q} : \text{Ext}^i(S_1, S_2) \to \text{Ext}^i(\mathfrak{q}(S_1), \mathfrak{q}(S_2))$ is an isomorphism.

Choose $j : U \to X$ to be a covering in $\text{Zar}_X$ with $U$ affine. Then $j : U \to X$ is an affine morphism since $X$ is a separated scheme. Choose also an embedding $j^*(S_2) \to I$ where $I$ is an injective object of $\text{Sh}^{qc}_Y(j^*(A))$.

$S \to j_*(I)$ is an injection and let $K$ denote the cokernel. By the above Lemma,

$$R^*j_*(\mathfrak{q}(I)) \simeq \mathfrak{q}(R^*j_*(I)) \simeq \mathfrak{q}(j_*(I))$$

we have a commutative diagram

$$
\begin{array}{ccc}
\text{Ext}^{i-1}(S_1, K) & \xrightarrow{\sim} & \text{Ext}^i(S_1, S_2) \\
\mathfrak{q} & & \mathfrak{q} \\
\text{Ext}^{i-1}(\mathfrak{q}(S_1), \mathfrak{q}(K)) & \xrightarrow{\sim} & \text{Ext}^i(\mathfrak{q}(S_1), \mathfrak{q}(S_2))
\end{array}
$$

and the assertion follows by induction on $i$.

QED

3. $A$-bimodules and sheaves on $C_X(A)$

3.0. In this section we will study the connection of the category of quasi-coherent sheaves of $A$-bimodules to that of quasi-coherent sheaves on $C_X(A)$. The material here is parallel to the one of Section 3 in [Ga]. The category of quasi-coherent sheaves of $A$-bimodules will be denoted by $A_{qc-mod}$.

3.1. Let us recall several definitions from [Ga]. If $B$ is a quasi-coherent sheaf of algebras on a scheme $X$, we denote by $I_B$ the sheaf of $B$-bimodules given by $I_B = \ker(B \otimes B \to B)$ (the map here is the multiplication).

If $M$ is a quasi-coherent sheaf of $B$-bimodules, we will denote by $\Omega(B, M)$ the group $\text{Hom}(I_B, M)$.

3.2. Let now $X$ be a scheme and let $A$ be a quasi-coherent sheaf of associative algebras on $X$. We will construct a localization functor $\mathfrak{z} : A_{qc-mod} \to Sh^{qc}_X(A)$:

Let $M \in A_{qc-mod}$. Consider the presheaf $\mathfrak{z}(M)$ on $C_X(A)$ given by

$$\Gamma((U, B_U), \mathfrak{z}_M) = \Omega(B_U, M|U).$$

The following Lemma is proven by a straightforward verification.

**Lemma.** This presheaf is in fact a sheaf.

3.2.1. If $X$ is an affine scheme, similar constructions can be carried out in the category $C^{old}_X(A)$. In this case we denote the localization functor by $\mathfrak{z}^{old}$.

**Lemma.** The functors

$$(\text{new} \to \text{old}) \circ \mathfrak{z} \text{ and } \mathfrak{z}^{old} : A_{qc-mod} \to Sh^{qc}_X(A)$$

are canonically isomorphic.

The proof is obvious.
3.3. Let us describe $\mathcal{S}_M$ in a slightly different way. Consider the sheaf of algebras $A \oplus M$ over $X$, $(X, A \oplus M) \in C_X(A)$. Then it is a group-like object in this category:

$$\text{Hom}((U, B_U), (X, A \oplus M)) = \Omega(B_U, M(U),$$

and $\mathcal{S}(M)$ is a sheaf given by $\Gamma((U, B_U), \mathcal{S}(M))) = \text{Hom}((U, B_U), (X, A \oplus M))$. In other words, $\mathcal{S}(M)$ is a group like object in the category of sheaves of sets with $\mathcal{S}(M) = \text{Const}_{A \oplus M}$. (Warning: note the difference between $\text{Const}_X$ and $\text{Const}_A$: the former is the constant sheaf corresponding to $X$ in the category of sheaves of sets, whereas the latter is the constant sheaf in the category of the sheaves of groups, cf. 1.2 and 1.3.1(2).)

3.3.1. Proposition.

(1) Let $f : (Y, A') \to (X, A)$ be a map with $Y$ and $X$ affine. Then the functors

$$\mathcal{S}^\text{old} \circ f^* \text{ and } f^* \circ \mathcal{S}^\text{old} : A_{qc} - \text{mod}(X) \to \text{Sh}_{A'}^\text{old}(Y)$$

are canonically isomorphic.

(2) For $X$ and $Y$ arbitrary and for any $f : (Y, A') \to (X, A)$ the functors

$$\mathcal{S} \circ f^* \text{ and } f^* \circ \mathcal{S} : A_{qc} - \text{mod}(X) \to \text{Sh}_{A'}(Y)$$

are canonically isomorphic.

(3) Let once again $X$ be affine. Then the functors

$$(\text{old} \to \text{new}) \circ \mathcal{S}^\text{old} \text{ and } \mathcal{S} : A_{qc} - \text{mod} \to \text{Sh}_A^\text{new}(X) : A_{qc} - \text{mod}(X) \to \text{Sh}_A(X)$$

are canonically isomorphic.

(4) For $X$ arbitrary the functor $\mathcal{S} : A_{qc} - \text{mod} \to \text{Sh}_A(X)$ takes values in $\text{Sh}_A^\text{qc}(A)$.

(5) The functor $\mathcal{S} : A_{qc} - \text{mod} \to \text{Sh}_X^\text{qc}(A)$ is exact and faithful.

Proof of the Proposition. First two points are immediately deduced from the following general Lemma:

Lemma 1. Let $F : C_1 \to C_2$ be a functor between two sites. Let $A \in C_1$ be an (abelian) group-like object and let also $\text{Const}_A$ be the sheaf of abelian groups associated with it. Suppose that $F(A)$ is an (abelian) group-like object in $C_2$ as well and that $F : \text{End}(A) \to \text{End}(F(A))$ is a homomorphism of groups. Then $F^*(\text{Const}_A) \simeq \text{Const}_{F(A)}$.

Proof of the Lemma 1. The proof follows from the following observation: for any $S \in \text{Sh}(C)$,

$$\text{Hom}(\text{Const}_A', S) = \{ \gamma \in \Gamma(A, S) | n \cdot \gamma = n^* \gamma \} \text{ for any } n \in \mathbb{Z},$$

where $n$ on the right hand side denotes the endomorphism $n \cdot \text{Id}_A \in \text{End}(A)$.

QED

The third point of the Proposition follows from the first two and from the Corollary 1 of 2.3.3.

(4) is an immediate consequence of (2) and of (3). In order to prove (5) we may assume $X$ to be affine, where the assertion follows from the following lemma, whose proof is a straightforward verification.
Lemma 2. Let $X = \text{Spec}(R)$ and let $V$ be an $R$-module. Let also $\text{Free}_R(V) \in C^\text{qcd}_X(A)$ be the free associative algebra built on $V$. Then we have a canonical isomorphism of functors: $A_{qc} - \text{mod} \to \text{Ab}$

$$M \to \text{Hom}_R(V, M) \text{ and } M \to \Gamma(\text{Free}_R(V), \exists(M)).$$

QED

3.3.2. The following assertion is easy:

Lemma. Let $f$ be a map from a pair $(Y, A' \simeq f^*(A))$ to the pair $(X, A)$. We have then the direct image functor $f_* : A'_{qc} - \text{mod} \to A_{qc} - \text{mod}$. The functors $\exists_X \circ f_*$ and $f_* \circ \exists_Y$ are canonically isomorphic.

3.4 Lemma-Definition. The functor $\exists : A_{qc} - \text{mod} \to Sh^c_X(A)$ admits a left adjoint denoted by $L : Sh^c_X(A) \to A_{qc} - \text{mod}$

Proof. For any $S \in Sh^c_X(A)$ we must construct a quasi-coherent $A$-module $L(S)$, satisfying $\text{Hom}(L(S), M) \simeq \text{Hom}(S, \exists_M)$ functorially in $M$. Because of 3.3.1, it will suffice to construct $L(S)$ locally on $X$. This reduces us to the situation when $X$ is affine. In this case, any object in $Sh^c_X(A)$ is a quotient of a one of the type $\text{Const}_{\text{Free}(V)}$. However, for $S = \text{Const}_{\text{Free}(V)}$, Lemma 2 of 3.3.1 implies $L(S) = F(V)$, where $F(V)$ denotes the free $A$-module built on $V$. We finish the proof by quoting the following general (and trivial)

Sub-lemma. Let $C_1$ and $C_2$ be two abelian categories and let $F : C_1 \to C_2$ be an additive left exact functor between them. Suppose that $F$ admits a partially defined left adjoint functor which is however defined on a large collection of objects in $C_2$ (i.e. any object in $C_2$ is a quotient of a one from this collection). Then this left adjoint is defined on the whole of $C_2$.

QED

3.4.1. From 3.3.2 and the above Lemma-definition we deduce by adjunction the following result:

Lemma. Let $f$ be a map from a pair $(Y, A' \simeq f^*(A))$ to the pair $(X, A)$. The functors

$$L_Y \circ f^* \text{ and } f^* \circ L_X : Sh^c_A(X) \to A'_{qc} - \text{mod}(Y)$$

are canonically isomorphic.

3.4.2. Since the functor $\exists$ is exact, it can be prolonged to a functor between the corresponding derived categories: $D_{qc}(A) \overset{\text{def}}{=} D(A_{qc} - \text{mod}) \to D(Sh^c_X(A))$ which will be also denoted by $\exists$. Our next aim is to show that the functor $L$ (which is obviously right exact) can also be derived into the functor $L^* : D^-(Sh^c_X(A)) \to D_{qc}(A)$, which will be the left adjoint functor to $\exists : D_{qc}(A) \to D^-(Sh^c_X(A))$. When $X$ is affine, the argument of the above lemma proves everything, since the sheaves $\text{Const}_{\text{Free}(V)}$ with $V$ being a projective $O(X)$-module form a set of projective generators of $Sh^c_X(A)$. However, in order to treat the general case an additional argument is needed, since objects of the derived category cannot be reconstructed just from the local information.
3.4.3 Theorem. Let $X$ be an arbitrary scheme and $A$ be a quasi-coherent sheaf of algebras on $A$. Then the functor $\mathcal{L}$ can be derived into the functor $L^\bullet \mathcal{L}: D^-(Sh^qc_X(A)) \to D^+_q(A)$ which moreover satisfies

$$\text{Hom}(L^\bullet \mathcal{L}(S^\bullet), M^\bullet) \simeq \text{Hom}(S^\bullet, \mathfrak{Z}(M^\bullet))$$

functorially in $S^\bullet \in D^-(Sh^qc_X(A))$ and in $M^\bullet \in D^+_q(A)$. 

Remark. The category $Sh^qc_X(A)$ is lacking objects that would be acyclic for the functor $\mathcal{L}$. The situation here is similar to that in [Bo,Be], when one wants to define the direct image functor for $D$-modules. As in [Bo], there are at least two ways to overcome this difficulty: a more straightforward one is to go beyond the category $Sh^qc_X(A)$ and work with arbitrary sheaves. In this case there are enough acyclic objects for the functor $\mathcal{L}$, but the drawback of this approach is that we will have to rely on the equivalence of the categories $D_qc(A)$ and $D(A - \text{mod})$ with quasi-coherent cohomologies as well as on Theorem 2.6. Another way is the one described below:

Proof of the Theorem. Consider first a pair $(Y, A')$ with $Y$ is affine and let $S$ be a quasi-coherent sheaf on $C_Y(A')$. We will construct a canonical sheaf $\text{Can}'(S)$ mapping surjectively onto $S$ with $\text{Can}'(S)$ being acyclic for the functor $\mathcal{L}$. Namely, $\text{Can}(S) = \oplus \Gamma(B, S) \otimes \text{Const}_B$, the sum being taken over isomorphism classes of objects in $C_{Xq}(A)$ with $B$ a free algebra on a projective $O$-module. This construction has the following two properties:

1) For any map of sheaves $S \to S'$ there is a canonical map $\text{Can}'(S) \to \text{Can}'(S')$.

2) If $f: (Z, f^*(A)) \to (Y, A)$ is a morphism of pairs with $Y$ and $Z$ affine, there exists a canonical map $f^*(\text{Can}'(S)) \to \text{Can}'(f^*(S))$.

Thus any complex $S^\bullet$ of sheaves bounded from above in $Sh^qc_X(A)$ admits a acyclic resolution $\text{Can}(S^\bullet)$ by a complex consisting of sheaves acyclic with respect to the functor $\mathfrak{Z}$.

Let now $S^\bullet$ be a complex bounded from above on $X$ giving rise to an object of $D^-(Sh^qc_X(A))$, and choose $j: U \to X$ to be a covering in $\text{Zar}_X$ with $U$ affine. Put $A' \simeq j^*(A)$ and let also $U_i$ be as in 1.4.1. All these schemes are affine since $X$ is assumed to be separated.

For each $U_i$, fix the canonical resolution $\text{Can}(S^\bullet | U_i)$ of $S^\bullet | U_i$ as above.

Then for each $i$ we can form a complex $\mathcal{L}(\text{Can}(j_i^*(S^\bullet)))$ of quasi-coherent sheaves of $A[U_i]$-bimodules on $U_i$.

For each $i, j$ we have a boundary map $p_j^i: U_i \to U_{i-1}$

and therefore we have a map of complexes:

$$p_j^i \mathcal{L}(\text{Can}(j_{i-1}^*(S^\bullet))) \to \mathcal{L}(\text{Can}(j_i^*(S^\bullet)))$$

which is easily seen to be a quasi-isomorphism by 3.4.1, since the functor

$$p_j^i: Sh^qc_{A_{i-1}}(U_{i-1}) \to Sh^qc_{A_i}(U_i)$$

is exact.

We have a complex of complexes:

$$0 \to \mathcal{L}(\text{Can}(j_0^*(S^\bullet))) \to \mathcal{L}(\text{Can}(j_1^*(S^\bullet))) \to \mathcal{L}(\text{Can}(j_2^*(S^\bullet))) \to \cdots$$

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or, in other words, a double complex $L'^{\cdots}(S^\bullet)$ in $A_{qc} - mod$, whose associated complex we denote by $Ass(L'^{\cdots}(S^\bullet))$.

It is now a standard exercise to check, that for each $i$ the canonical map of complexes

$$
L(Can(j^\bullet(S^\bullet))) \to Ass(L'^{\cdots}(S^\bullet))|U
$$

is a quasi-isomorphism. This in turn implies that the functor

$$
S^\bullet \to Ass(L'^{\cdots}(S^\bullet))
$$

is a well-defined functor $D^{-}(Sh^{qc}_X(A)) \to D^{-}_{qc}(A)$.

To prove that this functor is in fact $L^\bullet L$ we are looking for, it remains to verify the adjunction property:

$$
Hom(L^\bullet L(S^\bullet), M^\bullet) \simeq Hom(S^\bullet, \Im(M^\bullet)) \quad (\ast)
$$

For this we must construct the adjunction morphisms

$$
S^\bullet \to \Im \circ Ass(L'^{\cdots}(S^\bullet))
$$

and

$$
Ass(L'^{\cdots}(\Im(M^\bullet))) \to M^\bullet
$$

This is done in the following way:

$$
\begin{array}{ccc}
Ass[j_i j_i^\bullet(Can(S^\bullet))] & \leftarrow & Can(S^\bullet) & \rightarrow & S^\bullet \\
\downarrow & & & \downarrow & \\
Ass[j_i Can(j_i^\bullet S^\bullet)] & \rightarrow & \Im \circ Ass[j_i L(Can(j_i^\bullet S^\bullet))] & \rightarrow & M^\bullet
\end{array}
$$

for the first adjunction map, and

$$
\begin{array}{ccc}
Ass[j_i L(j_i^\bullet(\Im(M^\bullet)))] & \leftarrow & Ass[j_i L \circ Can(j_i^\bullet(\Im(M^\bullet)))] \\
\downarrow & & \downarrow \\
Ass[j_i j_i^\bullet(\Im(M^\bullet))] & \rightarrow & Ass[j_i j_i^\bullet(M^\bullet)] & \rightarrow & M^\bullet
\end{array}
$$

for the second one. It is now easy to verify, that the adjunction maps constructed above give rise to $(\ast)$.

QED

3.4.4. The following statement readily follows from Lemmas 3.3.2 and 2.2.5 by adjunction and is implicit in the Theorem:

**Lemma.** Let $(Y, A') \to (X, A)$ be a map such that $f : Y \to X$ is affine and flat and such that $A' \to f^*(A)$ is an isomorphism. Then

$$
L^\bullet L_Y \circ f^* \simeq f^* \circ L^\bullet L_X
$$

as functors $D^{-}_{qc}(A) \to D^{-}_{qc}(A')$. 

3.5 Definition. Let \((X, A)\) be as before: a scheme with a quasi-coherent sheaf of algebras on it. We define \(T^\bullet(A)\) to be the object of \(D_{qc}(A)\) given by \(L^\bullet\mathcal{L}(\text{Const}_A)\).

From the fact that \(\mathcal{L}\) is left exact we infer that \(H^i(T^\bullet(A))\) vanishes for \(i > 0\) and that \(H^0(T^\bullet(A)) = I_A\).

\(T^\bullet(A)\) will be called the cotangent complex of \(A\). If \(M\) is a quasi-coherent sheaf of \(A\)-bimodules, we denote by \(H^i_A(M)\) the groups \(\text{Ext}^i(T^\bullet(A), M)\).

3.5.1 Example. Suppose that \(A\) is flat over \(O(X)\). It follows from the results of Quillen [Qu], that \(T^\bullet(A) \cong I_A\). Indeed, this is true for \(X\) affine, and then we apply 3.4.4.

4. Deformation Theory

4.0. This section is almost a word by word repetition of [Ga], after we adopt certain modifications connected with the fact that we are working over a scheme.

4.1.0. For a scheme \(X\), \(O_1(X)\) will denote the sheaf \(O[t]/t^{i+1} \cdot O(X)\).

4.1.1. Let \(A\) be a quasi-coherent sheaf of associative algebras on \(X\).

Definition.

The category \(\text{Deform}^i(A)\) is defined to have as objects quasi-coherent sheaves of associative \(O_i(X)\)-algebras \(A_i\), endowed with an isomorphism \(A_i/A_i \cdot t \cong A\) such that \(\text{Tor}^1_{O_i(X)}(A_i, O(X)) = 0\). (In other words, we need that

\[\ker(t : A_i \to A_i) = \text{im}(t^i : A_i \to A_i)\] identifies under a natural map with \(A\).

Morphisms in this category are just \(O_i(X)-\)algebras homomorphisms respecting the identifications with \(A\) modulo \(t\). This category is obviously a groupoid. It is called the category of \(i\)-th level deformations of \(A\).

We have natural functors \(\text{Deform}^{i+1}(A) \to \text{Deform}^i(A)\) given by reduction modulo \(t^{i+1}\). If \(A_i\) is an object in \(\text{Deform}^i(A)\), we denote by \(\text{Deform}^{i+1}_{A_i}(A)\) the category-fiber of the above functor. This category, which is obviously a groupoid too, is called the category of prolongations of \(A_i\) onto the \(i + 1\)-st level.

4.1.2. We are now ready to state the main result of the present paper:

Theorem.

1. The category \(\text{Deform}^1(A)\) is equivalent to the category \(T_\mathfrak{Z}(A)\) of \(\mathfrak{Z}(A)\)-torsors on \(C_X(A)\).
2. If \(A_i \in \text{Deform}^i(A)\), and if \(A_{i+1} \in \text{Deform}^{i+1}_{A_i}(A)\), \(\text{Aut}(A_{i+1})\) is canonically isomorphic to \(\Omega(A, A) = \Gamma(\text{Const}_{(X, A)}, \mathfrak{Z}(A))\).
3. To any \(A_i \in \text{Deform}^i(A)\) one can associate a gerbe \(G_{A_i}\) bound by \(\mathfrak{Z}(A)\) on \(C_X(A)\) in such a way that \(G_{A_i}((X, A))\) is canonically equivalent to \(\text{Deform}^{i+1}_{A_i}(A)\).

Remarks. It is easy to notice that the first two points in the statement of the Theorem are special cases of the third one.

Proof.

1. The functor \(T : \text{Deform}^1(A) \to T(\mathfrak{Z}(A))\) is given by:

\[\Gamma((U, R_1), T(A_i)) = O(U)\ - \text{algebras homomorphisms } R_1 \to A \cdot |U|\]
and it is easy to show that it is an equivalence of categories.

(2) This is a direct verification.

(3) We define the gerbe $G_{A_i}$ as follows: $G_{A_i}((U, B_U))$ is the groupoid of $O_i(U)$-algebras $B_{U,i+1}$ with an isomorphism $B_{U,i+1}/t^i·B_{U,i+1} ≃ B_U × (A_i)|U$ such that

$\ker(t^i : B_{U,i+1} → B_{U,i+1}) = \text{im}(t : B_{U,i+1} → B_{U,i+1})$ and identifies under a natural morphism with $(A_i)|U$.

Functors $G_{A_i}((U, B_U)) → G_{A_i}((V, C_V))$ for maps $(V, C_V) → (U, B_U)$ are given by taking fiber products.

The fact that $G_{A_i}$ is indeed a gerbe bound by $ℐ(A_i)$ over $C_X(A)$ and that its fiber over $(X, A)$ is equivalent to $\text{Deform}_{A_i}^{i+1}(A)$ can be easily verified.

QED

4.2. We will now translate the assertions of the above theorem into cohomological terms.

Remark. It has been proven (Theorem 2.5) that cohomologies of a quasi-coherent sheaf over $C_X(A)$ computed either in the category of all sheaves or in the category of quasi-coherent sheaves coincide. Hence classes of gerbes and torsors over quasi-coherent sheaves can be computed inside the quasi-coherent category.

1-st Level Deformations. The groupoid $\text{Deform}^1(A)$, which is a priori endowed with a monoidal structure is a Picard category with $\pi_0(\text{Deform}^1(A))$ (the set of isom. classes of objects) being a group isomorphic to $\text{Ext}^1(\text{Const}_{(X,A)}, ℐ(A)) ≃ H^1_A(A)$. This follows from 1.5.2, from 3.4 and from 4.1.2(1).

Prolongation of Deformations 1. If $A_i$ is an $i$-th level deformation, there exists a canonical class in $H^2_A(A)$ which is zero if and only if there exists a prolongation $A_{i+1}$ of $A_i$. This follows from 1.5.5, from 3.4, and from 4.1.2(3).

Prolongation of Deformations 2. Suppose that for a given $i$-th level deformation $A_i$, the category $\text{Deform}_{A_i}^{i+1}(A)$ has an object. Then $\pi_0$ of this category is a torsor over the abelian group $H^1_A(A)$. This follows from 1.5.1, from 3.4. and from 4.1.2(3).

4.3 Example. Suppose now that the sheaf $A$ is flat as a sheaf of $O(X)$–modules. From Example 3.5.1, it follows that deformations of $A$ are controlled by $\text{Ext}^i(I_A, A)$ (Exts being taken in the category of quasi-coherent sheaves of $A$-bimodules) for $i = 1, 2$.

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