VISIBLE ACTIONS ON SPHERICAL NILPOTENT ORBITS IN COMPLEX SIMPLE LIE ALGEBRAS

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Abstract. This paper studies nilpotent orbits in complex simple Lie algebras from the viewpoint of strongly visible actions in the sense of T. Kobayashi. We prove that the action of a maximal compact group consisting of inner automorphisms on a nilpotent orbit is strongly visible if and only if it is spherical, namely, admitting an open orbit of a Borel subgroup. Further, we find a concrete description of a slice in the strongly visible action. As a corollary, we clarify a relationship among different notions of complex nilpotent orbits: actions of Borel subgroups (sphericity); multiplicity-free representations in regular functions; momentum maps; and actions of compact subgroups (strongly visible actions).

1. Introduction

This paper studies nilpotent orbits, and bridging the two notions, “spherical varieties” studied by D. Panyushev [19, 20] and “visible actions” introduced by T. Kobayashi [10]. We shall prove that

“spherical nilpotent orbits = visible nilpotent orbits”,

and give some structural results (“slice” coming from the right-hand side).

T. Kobayashi established a new theory on multiplicity-freeness for unitary representations of Lie groups by introducing the notion of visible actions. We recall briefly from [10, 11, 14] the propagation theory of multiplicity-freeness property. Let \( G \) be a Lie group, and \( V \) a \( G \)-equivariant Hermitian holomorphic vector bundle over a complex manifold \( D \). Then, we have a natural action of \( G \) on the space \( \mathcal{O}(D, V) \) of holomorphic sections, which is not necessarily irreducible when \( G \) does not act transitively on \( D \). Suppose we are given a unitary representation \( \mathcal{H} \) of \( G \) from which there exists a continuous injective homomorphism to \( \mathcal{O}(D, V) \). In general, \( \mathcal{H} \) may not be multiplicity-free even if...
each fiber $V_x$ is multiplicity-free as a representation of the isotropy sub-
group $G_x$ ($x \in D$). However, $\mathcal{H}$ becomes multiplicity-free whenever $G$
acts on the base space $D$ strongly visibly. This is the propagation theory
in \cite{14}, which yields a unified explanation of multiplicity-freeness
for various kinds of multiplicity-free representations which have been
studied by different approaches (see \cite{10, 11, 14}).

A holomorphic action of a Lie group $G$ on a connected complex
manifold $D$ is called strongly visible if there exist a real submanifold
$S$ in $D$ and an anti-holomorphic diffeomorphism $\sigma$ of $D$ such that the
following conditions are satisfied:

\begin{eqnarray}
D &=& G \cdot S, \\
\sigma |_S &=& \text{id}_S, \\
\sigma &\text{preserves each } G\text{-orbit in } D.
\end{eqnarray}

We say that the submanifold $S$ is a slice. It is automatically totally
real, namely, $J_x(T_xS) \cap T_xS = \{0\}$ for any $x \in S$ (see \cite{11} Remark
3.3.3). Here, $J$ stands for the complex structure of $D$. We are particu-
larly interested in a slice of minimal dimension, namely, which coincides
with the codimension of generic $G$-orbits in $D$.

We remark that the original definition \cite{11} Definition 3.3.1] of strongly
visible actions is wider slightly, namely, it allows a complex manifold
$D$ containing a non-empty $G$-invariant open set satisfying (V.1)–(S.2).
For the propagation theory of multiplicity-freeness property, this wider
definition is sufficient. However, since we shall see that these two de-
finitions are equivalent for $G_u$-actions on complex nilpotent orbits, we
adopt the above definition for simplicity for the rest of this paper.

Strongly visible actions arise from many different geometric settings
(cf. \cite{10, 11, 23}). Recently, a classification theory of strongly visi-
ble actions has been developed for Hermitian symmetric spaces \cite{13},
generalized flag varieties \cite{12, 27, 28, 29, 30}, and linear spaces \cite{22, 24}.

This paper deals with a new case where a complex manifold is a
nilpotent orbit in a complex simple Lie algebra. In order to state our
main results, we fix notation. Let $g$ be a finite-dimensional complex
simple Lie algebra and $G_C := \text{Int } g$ the inner automorphism group of
$g$. We denote by $O_X$ a $G_C$-orbit through $X \in g$. Then we have a
$G_C$-isomorphism of complex manifolds $O_X \simeq G_C/(G_C)_X$ where $(G_C)_X$
stands for the isotropy subgroup at $X$. We say that $O_X$ is a nilpotent
orbit if $X \in g$ is a nilpotent element, and is spherical if a Borel subgroup
of $G_C$ has an open orbit in $O_X$. Let $G_u$ be a compact real form of $G_C$.
We prove:

**Theorem A.** If $O_X$ is nilpotent and spherical, then the $G_u$-action on
$O_X$ is strongly visible.

The idea of our proof of Theorem A is based on the induction theo-
rem of strongly visible actions which is first formulated by Kobayashi
[11] Theorem 20] for Type A group. We generalize this idea for arbitrary complex simple Lie groups. For this, we choose an \( \mathfrak{sl}_2 \)-triple \( \{H, X, Y\} \) containing \( X \) as a nilpositive element. The semisimple element \( H \) defines the \( \mathbb{Z} \)-grading of \( \mathfrak{g} \), denoted by \( \mathfrak{g} = \bigoplus_{m \in \mathbb{Z}} \mathfrak{g}(m) \). Then the complex subalgebra \( \mathfrak{l} := \mathfrak{g}(0) \) is reductive. Let \( L_C \) be an analytic subgroup of \( G_C \) with Lie algebra \( \mathfrak{l} \). Taking a conjugation if necessary, we may and do assume that \( L_u := L_C \cap G_u \) is a compact real form of \( L_C \). We set a nilpotent subalgebra by \( \mathfrak{n} := \bigoplus_{m \geq 2} \mathfrak{g}(m) \). Then, the nilpotent orbit \( O_X \) can be realized via the following map:

\[
G_u \times_{L_u} \mathfrak{n} \to \overline{O_X}, \quad (g, Z) \mapsto g \cdot Z,
\]

in particular, the closure \( \overline{O_X} \) is equal to \( G_u \cdot \mathfrak{n} = \{g \cdot Z : g \in G_u, Z \in \mathfrak{n}\} \).

In this setting, we generalize the induction theorem of strongly visible actions:

**Theorem B** (see Theorem [11]). If the \( L_u \)-action on \( \mathfrak{n} \) is strongly visible, then the \( G_u \)-action on \( \overline{O_X} \) is strongly visible.

Theorem [B] means that the strong visibility for non-linear action on \( O_X \) is induced from the strong visibility for linear action on \( \mathfrak{n} \) via (1.1). We then can apply the previous results [22, 24] for the classification of linear visible actions to the \( L_u \)-action on \( \mathfrak{n} \), and thus prove:

**Theorem C.** If \( O_X \) is spherical, then the \( L_u \)-action on \( \mathfrak{n} \) is strongly visible.

Therefore, Theorem [A] follows from Theorems [B] and [C].

Our proof of Theorem [C] applies a case-by-case analysis by using Panyushev [19]. Moreover, we give an explicit description of a slice and an anti-holomorphic diffeomorphism for the \( L_u \)-action on \( \mathfrak{n} \) when \( O_X \) is spherical.

Together with the earlier results [19, 31, 32], we summarize:

**Corollary D.** The following five conditions on a nilpotent orbit \( O_X \) in a complex simple Lie algebra \( \mathfrak{g} \) are equivalent:

(i) \( O_X \) is spherical.

(ii) The height of \( O_X \) equals two or three.

(iii) The space of regular functions on \( O_X \) is multiplicity-free as a representation of \( G_C = \text{Int} \mathfrak{g} \).

(iv) The \( L_u \)-action on the nilpotent subalgebra \( \mathfrak{n} \) is strongly visible.

(v) The \( G_u \)-action on \( O_X \) is strongly visible.

By spherical nilpotent orbits, we mean that it is a nilpotent orbit on which a maximal compact subgroup of \( G_C \) acts strongly visibly.

Here, the height of \( O_X \) is defined by the maximum of \( m \in \mathbb{Z} \) satisfying \( \mathfrak{g}(m) \neq \{0\} \) (see Definition 5.2).
The equivalence between (ii) and (iii) is proved by Panyushev [19]. The equivalence (i) ⇔ (iii) is due to Vinberg–Kimelfeld [31] and Vinberg [32]. The implication (i) ⇒ (iii) is a special case of the propagation theory of multiplicity-freeness property by Kobayashi [10, 11, 14]. The implication (i) ⇒ (iv) and (iv) ⇒ (v) hold by Theorems C and B, respectively.

Corollary [13] has some connection with “small infinite-dimensional representations” of complex reductive Lie groups $G_C$. If $\mathcal{O}_\pi$ is the associated variety of an admissible representation $\pi$ of $G_C$ (see [33]), then the $G_u$-type in $\pi$ is asymptotically the same with the $G_u$-type in the space of regular functions in $\mathcal{O}_\pi$ by [15, Proposition 3.3].

This paper is organized as follows. In Section 2, we review basic notion of nilpotent orbits in complex semisimple Lie algebras. In Section 3, we explain a key theorem for the proof of Theorem C (Theorem 3.6), namely, properties of our choice of a slice and an anti-holomorphic diffeomorphism in the strongly visible $L_u$-action on $\mathfrak{n}$ if $\mathcal{O}_X$ is spherical. In Section 4, we show the induction theorem of strongly visible actions, namely, Theorem B. In Section 5, we give a proof of Theorem 3.6.

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2. Preliminaries

In this section, we review structural theories on nilpotent orbits in complex semisimple Lie algebras which is based on [4].

2.1. $\mathfrak{sl}_2$-triple of nilpotent orbit. Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra and $G_C := \text{Int} \mathfrak{g}$ the inner automorphism group of $\mathfrak{g}$. An element $X \in \mathfrak{g}$ is called a nilpotent element if $\text{ad}(X) \in \mathfrak{g}$.
End(\mathfrak{g}) satisfies \text{ad}(X)^N = 0 for some \(N \in \mathbb{N}\), and called a semisimple element if \text{ad}(X) is diagonalizable. We write \(\mathcal{N}\) for the cone of nilpotent elements of \(\text{ad}(X)\) and \(\mathcal{S}\) for the set of semisimple elements of \(\mathfrak{g}\).

We denote by \(\mathcal{O}_X = G_C \cdot X\) the \(G_C\)-orbit through \(X \in \mathfrak{g}\). An orbit \(\mathcal{O}_X\) is called a nilpotent (resp. semisimple) orbit if \(X \in \mathcal{N}\) (resp. \(X \in \mathcal{S}\)), from which \(\mathcal{O}_X \subset \mathcal{N}\) (resp. \(\mathcal{O}_X \subset \mathcal{S}\)). We write \(\mathcal{N}^*/G_C\) (\(\mathcal{N}^* := \mathcal{N} \setminus \{0\}\)) and \(\mathcal{S}/G_C\) for the set of non-zero nilpotent orbits and of semisimple orbits, respectively.

Suppose we are given \(X \in \mathcal{N}^*\). By Jacobson–Morozov, there exist \(H, Y \in \mathfrak{g}\) such that \(\{H, X, Y\}\) forms an \(\mathfrak{sl}_2\)-triple, namely,

\[
[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.
\]

Then, \(H\) is called a neutral element, \(X\) a nilpositive element, and \(Y\) a nilnegative element. We remark that \(H \in \mathcal{S}\) and \(X, Y \in \mathcal{N}\).

Two \(\mathfrak{sl}_2\)-triples \(\{H, X, Y\}, \{H', X', Y'\}\) in \(\mathfrak{g}\) are said to be conjugate if there exists \(g \in G_C\) such that \(H' = g \cdot H, \ X' = g \cdot X, \) and \(Y' = g^{-1} \cdot Y\). Clearly, two elements \(X, X' \in \mathcal{N}^*\) are conjugate if two \(\mathfrak{sl}_2\)-triples \(\{H, X, Y\}, \{H', X', Y'\}\) in \(\mathfrak{g}\) are conjugate. The opposite is also true by Kostant [16]. Then, this gives rise to the following bijection

\[
(2.1) \quad \mathcal{N}^*/G_C \sim \{\mathfrak{sl}_2\text{-triples in } \mathfrak{g}\}/G_C, \quad \mathcal{O}_X \mapsto \{H, X, Y\}.
\]

Moreover, it follows from Mal’cev [18] that \(\{H, X, Y\}\) and \(\{H', X', Y'\}\) are conjugate if the corresponding neutral elements \(H, H'\) are conjugate. This implies that the map

\[
(2.2) \quad \{\mathfrak{sl}_2\text{-triples in } \mathfrak{g}\}/G_C \to \mathcal{S}/G_C, \quad \{H, X, Y\} \mapsto \mathcal{O}_H
\]

is injective. Hence, composing (2.1) and (2.2) yields the injective map from the set of nilpotent orbits to that of semisimple ones:

\[
(2.3) \quad \Phi : \mathcal{N}^*/G_C \to \mathcal{S}/G_C, \quad \mathcal{O}_X \mapsto \mathcal{O}_H.
\]

2.2. \(\mathbb{Z}\)-grading of \(\mathfrak{g}\). In this subsection, we consider the semisimple transformation \(\text{ad}(H)\) on \(\mathfrak{g}\) for \(H \in \Phi(\mathcal{O}_X)\).

By the general theory on finite-dimensional representations of \(\mathfrak{sl}(2, \mathbb{C})\), all \(\text{ad}(H)\)-eigenvalues are integers. We write \(\mathfrak{g}(m)\) for the \(\text{ad}(H)\)-eigenspace with eigenvalue \(m\), namely,

\[
(2.4) \quad \mathfrak{g}(m) := \{Z \in \mathfrak{g} : \text{ad}(H)Z = mZ\} \quad (m \in \mathbb{Z}).
\]

Then, \(\mathfrak{g}\) is decomposed into the finite sum of \(\text{ad}(H)\)-eigenspaces as

\[
(2.5) \quad \mathfrak{g} = \bigoplus_{m \in \mathbb{Z}} \mathfrak{g}(m).
\]

Since the inclusion \([\mathfrak{g}(m), \mathfrak{g}(n)] \subset \mathfrak{g}(m + n)\) holds for \(m, n \in \mathbb{Z}\), the decomposition (2.5) defines a \(\mathbb{Z}\)-grading of \(\mathfrak{g}\).

Let us take another element \(H' \in \Phi(\mathcal{O}_X)\) and consider the \(\text{ad}(H')\)-eigenspace \(\mathfrak{g}'(m)\). We note that \(H' = g_0 \cdot H\) for some \(g_0 \in G_C\). Then, we have:
Lemma 2.1. $g'(m) = g_0 \cdot g(m) = \{g_0 \cdot Z : Z \in g(m)\}$ for any $m \in \mathbb{Z}$.

Proof. For $Z \in g(m)$, we observe $g_0 \cdot (\text{ad}(H)Z)$ as follows. First, by the definition of $g(m)$, we have

$$g_0 \cdot (\text{ad}(H)Z) = g_0 \cdot (mZ) = m(g_0 \cdot Z).$$

(2.6)

Second, in view of $g_0 \cdot [H, Z] = [g_0 \cdot H, g_0 \cdot Z]$, we express as

$$g_0 \cdot (\text{ad}(H)Z) = \text{ad}(g_0 \cdot H)(g_0 \cdot Z) = \text{ad}(H')(g_0 \cdot Z).$$

(2.7)

Comparing (2.6) and (2.7), we obtain $\text{ad}(H') (g_0 \cdot Z) = m(g_0 \cdot Z)$, from which we have shown $g_0 \cdot g(m) \subset g'(m)$. Therefore, we have proved $g'(m) = g_0 \cdot g(m)$. □

Lemma 2.1 shows that the $\mathbb{Z}$-grading (2.5) is determined by $\mathcal{O}_X$, particularly, independent on the choice of semisimple elements of $\Phi(\mathcal{O}_X)$.

2.3. Nilpotent subalgebra. We define a parabolic subalgebra $\mathfrak{q}$ arising from the $\mathbb{Z}$-grading (2.5) by

$$\mathfrak{q} := \bigoplus_{m \geq 0} g(m)$$

with Levi decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ where

$$\mathfrak{l} := \mathfrak{z}_g(H) = g(0)$$

is a Levi subalgebra and $\mathfrak{u} = \bigoplus_{m > 0} g(m)$ is a nilradical. Then, $\mathfrak{l}$ is a complex reductive Lie algebra.

Let $\mathfrak{n}$ be a nilpotent subalgebra by

$$\mathfrak{n} := \bigoplus_{m \geq 2} g(m).$$

(2.10)

Clearly, $[\mathfrak{q}, \mathfrak{n}] \subset \mathfrak{n}$. As $X \in g(2)$, we have $[\mathfrak{q}, X] \subset \mathfrak{n}$. Further, the opposite inclusion also holds by the representation theory of $\mathfrak{sl}(2, \mathbb{C})$, from which we obtain:

Lemma 2.2. $\mathfrak{n} = [\mathfrak{q}, X]$.

2.4. Realization of $\mathcal{O}_X$ via momentum map. Let $Q_\mathbb{C}$ be a parabolic subgroup of $G_\mathbb{C}$ with Lie algebra $\mathfrak{q}$, which acts on $\mathfrak{n}$. We set

$$\mathfrak{n}^\circ := Q_\mathbb{C} \cdot X.$$  

(2.11)

By Lemma 2.2, $\mathfrak{n}^\circ$ is an open set in $\mathfrak{n}$, in particular, its closure $\overline{\mathfrak{n}^\circ}$ is equal to $\mathfrak{n}$.

We define a $G_\mathbb{C}$-equivariant smooth map $\varphi$ from the holomorphic vector bundle $G_\mathbb{C} \times_{Q_\mathbb{C}} \mathfrak{n}$ on the flag manifold $G_\mathbb{C}/Q_\mathbb{C}$ to $\mathfrak{g}$ by

$$\varphi : G_\mathbb{C} \times_{Q_\mathbb{C}} \mathfrak{n} \to \mathfrak{g}, \ (x, Z) \mapsto x \cdot Z.$$  

(2.12)

Lemma 2.3. $\mathcal{O}_X = \varphi(G_\mathbb{C} \times_{Q_\mathbb{C}} \mathfrak{n}^\circ)$.
Proof. It follows from (2.11) that
\[
\varphi(G_C \times_{Q_C} n^0) = \varphi(G_C \times_{Q_C} (Q_C \cdot X)) = G_C \cdot X = O_X.
\]
Hence, Lemma 2.3 has been proved. \(\square\)

Next, let \(L_C\) be a Levi subgroup of \(Q_C\) with Lie algebra \(l\). Taking a conjugation if necessary, we may and do assume that \(L_u := L_C \cap G_u\) is a compact real form of \(L_C\). Then, the inclusion map \(G_u \hookrightarrow G_C\) induces a biholomorphic diffeomorphism \(G_u/L_u \simeq G_C/Q_C\). This gives rise to an isomorphism as a \(G_u\)-equivariant holomorphic vector bundle as follows:
\[
G_u \times_{L_u} n \simeq G_C \times_{Q_C} n.
\]
(2.14)
We use the same letter \(\varphi\) to denote the \(G_u\)-equivariant map \(G_u \times_{L_u} n \to g\) via the isomorphism (2.14). Then, we have:

**Proposition 2.4.** \(O_X = G_u \cdot n^0\).

**Proof.** By Lemma 2.3 and (2.14), we have
\[
O_X = \varphi(G_u \times_{L_u} n^0) = G_u \cdot n^0,
\]
from which Proposition 2.4 has been proved. \(\square\)

**Remark 2.5 (cf. [11, Theorem 20]).** We can regard \(\varphi\) as the restriction of the momentum map of the Hamiltonian \(G_u\)-action on the flag manifold \(G_u/L_u\).

Let \(g_u\) and \(l_u\) be the Lie algebras of \(G_u\) and \(L_u\), respectively. Identifying \(g_u/l_u\) with \(u\), the nilpotent subalgebra \(n\) seems to be an \(L_u\)-submodule of \(g_u/l_u\). Then, the principal vector bundle \(G_u \times_{L_u} (g_u/l_u)\) is isomorphic to the cotangent bundle \(T^*(G_u/L_u)\). Via the identification of \(g\) with the dual space \(g^*\) by the Killing form on \(g\), the map
\[
\psi : G_u \times_{L_u} u \to g, \quad (g, Z) \mapsto g \cdot Z
\]
is essentially a momentum map of the Hamiltonian \(G_u\)-action on the flag manifold \(G_u/L_u\). It turns out that \(\varphi = \psi|_{G_u \times_{L_u} n}\).

3. Visible actions on nilpotent subalgebras

Let us retain the setting of Section 2. A nilpotent orbit \(O_X\) is written as \(O_X = G_u \cdot n^0\). In view of this realization, it is crucial for the study on the \(G_u\)-action on \(O_X\) to understand the \(L_u\)-action on \(n^0\). Here, the \(Q_C\)-orbit \(n^0\) through \(X\) is equal to the closure of the nilpotent subalgebra \(n\). In this section, we focus on the \(L_u\)-action on \(n\).

3.1. Normal real form of complex simple Lie algebra. First, we define an anti-holomorphic involution of a complex simple Lie algebra.

For a reductive Lie algebra \(g'\), we denote by \(\text{rank}_R g'\) the real rank of \(g'\). A real form \(g'_R\) of a complex reductive Lie algebra \(g\) is called normal if \(\text{rank}_R g'_R = \text{rank}_R g\). Normal real forms of a complex simple Lie algebra exist and are unique up to isomorphism.
Let \( g \) be a complex simple Lie algebra. We take a normal real form \( g_{\mathbb{R}} \) of \( g \). We denote by \( \sigma \) the complex conjugation of \( g \) with respect to \( g_{\mathbb{R}} \), namely,
\[
\sigma(X + \sqrt{-1}Y) = X - \sqrt{-1}Y \quad (X, Y \in g_{\mathbb{R}}).
\]

Let \( \mathfrak{k}_{\mathbb{R}} \) be a maximal compact subalgebra of \( g_{\mathbb{R}} \) and \( g_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} + \mathfrak{p}_{\mathbb{R}} \), the corresponding Cartan decomposition. Then, the Lie algebra \( \mathfrak{k}_{\mathbb{R}} + \sqrt{-1}\mathfrak{p}_{\mathbb{R}} \) is a \( \sigma \)-stable compact real form of \( g \). Since compact real forms are unique up to isomorphism, we may and do assume that \( g_{\mathbb{R}} := \mathfrak{k}_{\mathbb{R}} + \sqrt{-1}\mathfrak{p}_{\mathbb{R}} \) is the Lie algebra of \( G_{\mathbb{R}} \) by taking a conjugation of \( g_{\mathbb{R}} \) if necessary.

3.2. \( \sigma \)-stability of subalgebra. Let \( a_{\mathbb{R}} \) be a maximal abelian subspace in \( \mathfrak{p}_{\mathbb{R}} \) and
\[
a := a_{\mathbb{R}} + \sqrt{-1}a_{\mathbb{R}}.
\]
Then, \( a \) is \( \sigma \)-stable, and a Cartan subalgebra of \( g \) since \( \text{rank}_{\mathbb{R}} g_{\mathbb{R}} = \text{rank} \ g \). Then, \( a_{\mathbb{R}} \) is a Cartan subalgebra of \( g_{\mathbb{R}} \).

Let \( \Delta \equiv \Delta(g, a) \) be a root system of \( g \) with respect to \( a \). We denote by \( g_{\alpha} \) the root space corresponding to \( \alpha \in \Delta \). Then, we have:

**Lemma 3.1.** The root space \( g_{\alpha} \) is \( \sigma \)-stable for any \( \alpha \in \Delta \).

**Proof.** According to (3.2), we write \( A \in a \) as \( A = A_1 + \sqrt{-1}A_2 \ (A_1, A_2 \in a_{\mathbb{R}}) \). As \( a_{\mathbb{R}} \subset g_{\mathbb{R}} \), we have \( \sigma(A) = A_1 - \sqrt{-1}A_2 \). Hence,
\[
\alpha(\sigma(A)) = \alpha(A_1 - \sqrt{-1}A_2) = \alpha(A_1) - \sqrt{-1}\alpha(A_2).
\]
We remark that all of the roots are real on \( a_{\mathbb{R}} \) (cf. [8, Section 4]). This means that \( \alpha(A_1), \alpha(A_2) \in \mathbb{R} \), and then, \( \alpha(\sigma(A)) = \alpha(A_1) - \sqrt{-1}\alpha(A_2) = \alpha(A) \). Since \( \sigma \) is anti-linear, we have
\[
\sigma(\alpha(A)Z) = \alpha(\sigma(A))\sigma(Z) = \alpha(\sigma(A))\sigma(Z) \quad (Z \in g).
\]

Let \( Z_{\alpha} \) be a root vector of \( g_{\alpha} \). It follows from (3.3) that
\[
\sigma([A, Z_{\alpha}]) = \sigma(\alpha(A)Z_{\alpha}) = \alpha(\sigma(A))\sigma(Z_{\alpha}).
\]
On the other hand, we have \( \sigma([A, Z_{\alpha}]) = [\sigma(A), \sigma(Z)] \) since \( \sigma \) is an involution on \( g \). Replacing \( A \) with \( \sigma(A) \) in the equality (3.4) shows
\[
[A, \sigma(Z_{\alpha})] = \alpha(A)\sigma(Z_{\alpha}).
\]
This means that \( \sigma(Z_{\alpha}) \) lies in \( g_{\alpha} \). Hence, Lemma 3.1 has been proved. \( \square \)

We recall the well-known facts that all elements contained in a Cartan subalgebra are semisimple and that two Cartan subalgebras of \( g \) are conjugate by \( G_{\mathbb{C}} \). This means that \( \Phi(\mathcal{O}_X) \cap a \neq \emptyset \). Hence, we take a semisimple element \( H \) in \( \Phi(\mathcal{O}_X) \cap a \). In this setting, we have:

**Lemma 3.2.** The \( \text{ad}(H) \)-eigenspace \( g(m) \) is \( \sigma \)-stable for any \( m \in \mathbb{Z} \).
Proof. Since $[H, Z_\alpha] = \alpha(H)Z_\alpha$ holds for any $Z_\alpha \in g_\alpha$, we have $g_\alpha \subset g(\alpha(H))$. This implies that the eigenspace $g(m)$ is written as $g(m) = \bigoplus_{\alpha \in \Delta_m} g_\alpha$ for some subset $\Delta_m \subset \Delta \cup \{0\}$. Hence, the $\sigma$-stability of $g(m)$ follows from Lemma 3.1. \hfill \Box

The parabolic subalgebra $q$, the Levi subalgebra $l$, and the nilpotent subalgebra $n$, respectively, consist of some $\text{ad}(H)$-eigenspaces. Then, the following lemma is an immediate consequence of Lemma 3.2.

**Lemma 3.3.** The subalgebras $q$, $l$, and $n$ of $g$ are $\sigma$-stable.

By Lemma 3.3, the restriction of $\sigma \in \text{Aut} g$ to $l$ becomes an anti-holomorphic involution of $l$. Here, we set a real form of $l$ by $l_R = l^\sigma = l \cap g_R$. Then, we have:

**Lemma 3.4.** The real form $l_R$ of $l$ is normal.

Proof. By definition, let us show the equality $\text{rank}_R l_R = \text{rank} l$. Here, the inequality $\text{rank}_R l_R \leq \text{rank} l$ holds in general. Then, it is sufficient to see $\text{rank}_R l_R \geq \text{rank} l$.

For this, we consider the maximal abelian $a_R \subset p_R$. The semisimple element $H \in a$ satisfies $[H, a] = \{0\}$, from which $a_R \subset a \subset l$. Then, we obtain $a_R \subset l \cap p_R = l_R \cap p_R$. As $l_R = (l_R \cap \mathfrak{z}_R) + (l_R \cap p_R)$ is a Cartan decomposition of $l_R$, this inclusion implies that $\text{rank}_R l_R \geq \text{dim} a_R$. Since $\text{rank} g \geq \text{rank} l$ holds in general and $g_R$ is a normal real form of $g$, we conclude

$$\text{rank}_R l_R \geq \text{rank} g \geq \text{rank} l.$$  

Therefore, Lemma 3.3 has been proved. \hfill \Box

### 3.3. Compatible automorphism

We lift the anti-linear involution $\sigma$ on $g$ to an anti-holomorphic involution $\tilde{\sigma}$ on the complex simple Lie group $G_C = \text{Int} g$. More precisely, we write $\tilde{\sigma}(g) := \sigma g \sigma \in G_C \ (g \in G_C)$. Then, we have:

$$\sigma(g \cdot Z) = \tilde{\sigma}(g) \cdot \sigma(Z) \quad (g \in G_C, \ Z \in g). \tag{3.6}$$

**Definition 3.5** (see [6][14]). We say that an anti-holomorphic involution $\tilde{\sigma}$ on $G_C$ is a compatible automorphism with the anti-holomorphic diffeomorphism $\sigma$ for the $G_C$-action on $g$ if the equality (3.6) holds.

In view of Lemma 3.3, the Levi subgroup $L_C$ is $\tilde{\sigma}$-stable, and the restriction of $\sigma \in \text{Aut} g$ to $n$ induces an anti-holomorphic diffeomorphism on $n$. Hence, $\tilde{\sigma}$ also becomes a compatible automorphism with $\sigma$ for the $L_C$-action on $n$, namely,

$$\sigma(k \cdot Z) = \tilde{\sigma}(k) \cdot \sigma(Z) \quad (k \in L_C, \ Z \in n). \tag{3.7}$$

The compact real form $G_u$ of $G_C$ is $\tilde{\sigma}$-stable because its Lie algebra $g_u$ is $\sigma$-stable. It follows from Lemma 3.3 that $l_u := l \cap g_u$ is $\sigma$-stable, from which $L_u = L_C \cap G_u$ is $\tilde{\sigma}$-stable.
3.4. Visible actions on nilpotent subalgebra. We are ready to state our theorem on the $L_u$-action on $n$ as follows.

**Theorem 3.6.** If $O_X$ is nilpotent and spherical, then one can find $S_0 \subset n$ such that the following properties are satisfied:

(a) $S_0$ is a real vector space.
(b) $n = L_u \cdot S_0$.
(c) $\sigma|_{S_0} = \text{id}_{S_0}$.

Our proof of Theorem 3.6 uses a case-by-case analysis for each spherical $O_X$, which will be given in Section 5 separated from this section.

3.5. Proof of Theorem C. Let us see that Theorem C follows from Theorem 3.6.

**Proof of Theorem C.** Suppose that $O_X$ is nilpotent and spherical. By Theorem 3.6, there exists a real vector subspace $S_0$ such that $n = L_u \cdot S_0$ and $\sigma|_{S_0} = \text{id}_{S_0}$. We will verify that the data $(S_0, \sigma)$ satisfies the definition of strongly visible actions (see Section 1).

The properties (1) and (2) coincide with (V.1) and (S.1), respectively. To see (S.2), we take $Z \in n$, and write $Z = k \cdot Z_0$ for some $k \in L_u$ and $Z_0 \in S_0$ according to (V.1). By the relation (3.7) and the property (1), we have

$$\sigma(Z) = \sigma(k \cdot Z_0) = \tilde{\sigma}(k) \cdot \sigma(Z_0) = \tilde{\sigma}(k) \cdot Z_0 = \tilde{\sigma}(k \cdot k^{-1} \cdot Z).$$

The element $\tilde{\sigma}(g)k^{-1}$ lies in $L_u$, from which $\sigma(Z) \in L_u \cdot Z$. Hence, we have verified (S.2).

Consequently, Theorem C has been proved. □

4. Induction theorem of strongly visible actions

In this section, we show Theorem B. We again reformulate Theorem B as follows:

**Theorem 4.1.** Let $O_X$ be a nilpotent orbit, and $I = g(0)$ and $n = \bigoplus_{m \geq 2} g(m)$ are defined by the $\mathfrak{sl}_2$-triple as in (2.9) and (2.10), respectively. If the $L_u$-action on $n$ is strongly visible, then the $G_u$-action on $O_X$ is strongly visible.

**Remark 4.2.** This theorem generalize induction theorem of strongly visible actions [11, Theorem 20] for Type A case. It produces new strongly visible actions out of known strongly visible actions (linear visible actions [22, 24]).

Suppose that the $L_u$-action on $n$ is strongly visible. Then, one can take a real submanifold $S_0$ in $n$ such that $n = L_u \cdot S_0$, $\sigma|_{S_0} = \text{id}_{S_0}$. We set

$$S := S_0 \cap n^0. \quad (4.1)$$

Then, $S$ is a real submanifold of $n^0$ since $n^0$ is open in $n$. 


Lemma 4.3. \( n^o = L_u \cdot S \).

Proof. In view of the equality \( n^o = n \cap n^o \), we have
\[
\begin{align*}
n^o &= (L_u \cdot S_0) \cap n^o = (L_u \cdot S_0) \cap (L_u \cdot n^o) = L_u \cdot (S_0 \cap n^o) = L_u \cdot S.
\end{align*}
\]
Hence, Lemma 4.3 has been proved.

Combining Proposition 2.4 with Lemma 4.3, we have
\[
\begin{align*}
O_X &= G_u \cdot n^o = G_u \cdot (L_u \cdot S) = (G_u L_u) \cdot S = G_u \cdot S.
\end{align*}
\]
Therefore, we have proved:

Proposition 4.4. The submanifold \( S \) satisfies the condition (V.1).

Next, we define an anti-holomorphic diffeomorphism of \( O_X \), which arises from \( \sigma \) defined by (3.1) as follows.

Let \( Z \) be an element of \( O_X \), and write \( Z = g \cdot Z_0 \) for some \( g \in G_u \) and \( Z_0 \in S \) due to Proposition 4.4. It is obvious from \( S \subset S_0 \) and \( \sigma |_{S_0} = \text{id}_{S_0} \) that
\[
\sigma |_{S} = \text{id}_S.
\]
Then, the relation (3.7) shows
\[
\sigma(Z) = \sigma(g \cdot Z_0) = \tilde{\sigma}(g) \cdot \sigma(Z_0) = \tilde{\sigma}(g) \cdot Z_0.
\]
Since \( G_u \) is \( \tilde{\sigma} \)-stable (see Section 3.3), the element \( \tilde{\sigma}(g) \) lies in \( G_u \). Then, the equality (4.3) means that \( \sigma(Z) \in G_u \cdot S = O_X \). Hence, \( O_X \) is \( \sigma \)-stable. This implies that the restriction of \( \sigma \) to \( O_X \) becomes an anti-holomorphic diffeomorphism on \( O_X \), which we use the same letter to denote.

Now, we give a proof of Theorem 4.1.

Proof of Theorem 4.1. It is clear that (V.1) and (S.1) hold by Proposition 4.4 and (4.2), respectively. Let \( Z \in O_X \) and write \( Z = g \cdot Z_0 \in G_u \cdot S \). Then, we have
\[
\sigma(Z) = \tilde{\sigma}(g)g^{-1} \cdot (g \cdot Z_0) = \tilde{\sigma}(g)g^{-1} \cdot Z \in G_u \cdot Z.
\]
Hence, we have verified (S.2).

Therefore, Theorem 4.1 has been proved.

5. Proof of Theorem 3.6

This section is devoted to the proof of Theorem 3.6.

First of all, we give a short summary of our proof. The Dynkin–Kostant theory explains that a nilpotent orbit \( O_X \) in a complex simple Lie algebra \( \mathfrak{g} \) is characterized by the weighted Dynkin diagram, denoted by \( \Omega(O_X) \), which is the Dynkin diagram of \( \mathfrak{g} \) with numerical labels. A classification of nilpotent orbits is given in terms of the weighted Dynkin diagrams. Moreover, \( O_X \) defines the height, denoted by ht\((O_X)\). D. Panyushev provides a criterion for \( O_X \) to be spherical
by its height. Since $ht(O_X)$ can be calculated from $\Omega(O_X)$, the table of spherical nilpotent orbits is obtained.

Under these theories, we apply case-by-case analysis on the $L_u$-action on $n$ for each spherical $O_X$. Indeed, we clarify a semisimple element $H \in \Phi(O_X) \cap a$ from $\Omega(O_X)$ and express the $L_C$-action on $n$. By using the classification of strongly visible linear actions, we verify the strong visibility for the $L_u$-action on $n$, and give an explicit description of $S_0$ satisfying (a)–(c).

5.1. **Weighted Dynkin diagram of nilpotent orbit.** In this subsection, we review the weighted Dynkin diagram corresponding to a nilpotent orbit $O_X$ in a complex semisimple Lie algebra $g$. See [4] for survey on weighted Dynkin diagrams in complex semisimple Lie algebras.

Let us retain the setting of Sections 2 and 3. Let $b$ be a Borel subalgebra of $g$ such that $b$ contains the Cartan subalgebra $a$ and is contained in the parabolic subalgebra $q$. We fix a positive system $\Delta^+ \equiv \Delta^+(g,a)$ satisfying $b = a + \bigoplus_{\alpha \in \Delta^+} g_{\alpha}$ and set a closed Weyl chamber by

$$ a_+ = \{ A \in a : \alpha(A) \geq 0 \ (\forall \alpha \in \Delta^+) \}. $$

As mentioned in Section 3.2, the intersection of $a$ and the semisimple orbit $\Phi(O_X)$ is non-empty. In particular, $\Phi(O_X) \cap a_+$ is a singleton set. Then, we define an injective map by

$$ \Psi : N^*/G_C \to a_+, \quad O_X \mapsto H \in \Phi(O_X) \cap a_+. $$

**Definition 5.1** (cf. [19]). We say that $\Psi(O_X) \in \Phi(O_X) \cap a_+$ is the characteristic element of $O_X$.

For the rest of this paper, we fix $H$ to be $H = \Psi(O_X) \in a_+$.

We write $\alpha_1, \ldots, \alpha_r \in \Delta^+$ for the simple roots of $\Delta$ ($r := \text{rank} g = \text{dim}_C a$). As $b \subset q$, the number $\alpha_j(H)$ is a non-negative integer. Moreover, it follows from the representation theory that $\alpha_j(H) \in \{0, 1, 2\}$ for $j = 1, 2, \ldots, r$. Then, we define the injective map as follows:

$$ \Omega : N^*/G_C \to \{0, 1, 2\}^r, \quad \Omega(O_X) := (\alpha_1(H), \ldots, \alpha_r(H)). $$

We label the node of the Dynkin diagram of $g$ corresponding to each simple root $\alpha_j$ with $\alpha_j(H)$. The Dynkin diagram with such labels is called the weighted Dynkin diagram of $O_X$.

We recall from [2.1] that there is a one-to-one correspondence between nilpotent orbits and conjugacy classes of $\mathfrak{sl}_2$-triples. Thanks to Dynkin’s work on $\mathfrak{sl}_2$-triples [5], the injective map (5.2) provides a characterization of nilpotent orbit in $g$ by corresponding weighted Dynkin diagrams.
Next, the $\mathbb{Z}$-grading (2.5) defined by $O$ introduces a function on $N^*/G_{\text{C}}$ as follows:
\[(5.3) \quad h_t : N^*/G_{\text{C}} \to \mathbb{Z}, \quad O \mapsto \max\{m \in \mathbb{Z} : g(m) \neq \{0\}\}.
\]
Since $X \in \mathfrak{g}(2)$, we obtain $h_t(O_X) \geq 2$ for any $O_X \in N^*/G_{\text{C}}$.

**Definition 5.2** ([19, Section 2]). We say that the positive integer $h_t(O_X)$ is the **height** of a nilpotent orbit $O_X$.

D. Panyushev gives a necessary and sufficient condition for a nilpotent orbit to be spherical.

**Fact 5.3** ([19, Theorem 3.1]). For a nilpotent orbit $O_X$, the following two conditions are equivalent:

(i) $O_X$ is spherical.

(ii) $h_t(O_X) \leq 3$.

In view of Fact 5.3, we consider how to calculate $h_t(O_X)$ by the weighted Dynkin diagram $\Omega(O_X)$. Let $\beta \in \Delta^+$ be the highest root.

Then, we have:

**Lemma 5.4.** $h_t(O_X) = \beta(H) = k_1\alpha_1(H) + \cdots + k_r\alpha_r(H)$.

**Proof.** By the proof of Lemma 3.2, we formulate
\[(5.4) \quad h_t(O_X) = \max\{m \in \mathbb{Z} : g(m) \neq \{0\}\} = \max\{\alpha(H) : \alpha \in \Delta\}.
\]
Let $\alpha \in \Delta$ and write $\alpha = l_1\alpha_1 + \cdots + l_r\alpha_r$ ($l_1, \ldots, l_r \in \mathbb{Z}$). Since $\beta$ is the highest root, the inequality $l_j \leq k_j$ holds for any $j = 1, 2, \ldots, r$. As $\alpha_j(H) \geq 0$ (see (5.2)), we estimate
\[\begin{align*}
\alpha(H) &= l_1\alpha_1(H) + \cdots + l_r\alpha_r(H) \\
&\leq k_1\alpha_1(H) + \cdots + k_r\alpha_r(H) \\
&= \beta(H).
\end{align*}
\]
This means that
\[(5.5) \quad \max\{\alpha(H) : \alpha \in \Delta\} = \beta(H).
\]
Combining (5.4) and (5.5), we get $h_t(O_X) = \beta(H)$.

Using Lemma 5.4 we list all weighted Dynkin diagrams $\Omega(O_X)$ with $h_t(O_X) = 2, 3$ in the first and second columns of Tables 5.3 and 5.4.

### 5.2. Visible linear actions.

In this subsection, we recall the recent works on strongly visible linear actions, see [22, 24] for details.

Let $K_{\text{C}}$ be a connected complex reductive Lie group and $V$ a vector space over $\mathbb{C}$. Suppose we are given a holomorphic representation of $K_{\text{C}}$ on $V$. Then, we have naturally the representation of $K_{\text{C}}$ on the polynomial ring $\mathbb{C}[V]$ defined by $f(v) \mapsto f(g^{-1} \cdot v)$. We say that the $K_{\text{C}}$-action on $V$ is a **multiplicity-free action**, or, $V$ is a **multiplicity-free $K_{\text{C}}$-space** if $\mathbb{C}[V]$ is multiplicity-free as a representation of $K_{\text{C}}$. 

Multiplicity-free actions are classified by Kac, Benson–Ratcliff, and Leahy [3, 7, 17] up to geometrically equivalences. Here, two holomorphic representations \((\pi, V)\) and \((\pi', V')\) of connected complex reductive Lie groups \(K_C\) and \(K'_C\), respectively, are geometrically equivalent if the image of \(\pi\) coincides with that of \(\pi'\) under some linear isomorphism from \(V\) to \(V'\).

Let \(K_u\) be a compact real form of \(K_C\). Then, we have:

**Fact 5.5** ([22, 24]). For a holomorphic representation of \(K_C\) on \(V\), the followings are equivalent:

(a) The \(K_C\)-action on \(V\) is a multiplicity-free action.
(b) The \(K_u\)-action on \(V\) is strongly visible.

Fact 5.5 gives a classification of strongly visible linear actions. During them, as we will see in the proof of Theorem 3.6, we need only eight series of multiplicity-free actions, which are listed in Table 5.1.

| \(K_C\) | \(V\) |
|--------|--------|
| (1) \(\mathbb{C}^\times\) | \(\mathbb{C}\) |
| (2) \(SL(p, \mathbb{C})\) | \(\mathbb{C}^p\) |
| (3) \(SL(p, \mathbb{C}) \times \mathbb{C}^\times\) | \(\text{Sym}(p, \mathbb{C})\) |
| (4) \(SL(2p, \mathbb{C}) \times \mathbb{C}^\times\) | \(\text{Alt}(2p, \mathbb{C})\) |
| (5) \(SL(p, \mathbb{C}) \times SL(p, \mathbb{C}) \times \mathbb{C}^\times\) | \(M(p, \mathbb{C})\) |
| (6) \(SO(p, \mathbb{C}) \times \mathbb{C}^\times\) | \(\mathbb{C}^p\) |
| (7) \(E_6(\mathbb{C}) \times \mathbb{C}^\times\) | \(\mathcal{J}_C\) |
| (8) \(SL(2p, \mathbb{C}) \times \mathbb{C}^\times\) | \(\text{Alt}(2p, \mathbb{C}) \oplus \mathbb{C}^{2p}\) |

**Table 5.1.** Multiplicity-free \(K_C\)-action on \(V\)

We pin down some restrictions on integer \(p\) in Table 5.1 as follows: In (2), (3), (5), and (6), \(p \geq 2\); In (4) and (8), \(p \geq 1\).

In (1), the multiplicative group \(\mathbb{C}^\times = \text{GL}(1, \mathbb{C})\) acts on \(\mathbb{C}\) as the standard complex multiplication. The special linear group \(SL(p, \mathbb{C}) = \{g \in M(p, \mathbb{C}) : \det g = 1\}\) and the special complex orthogonal group \(SO(p, \mathbb{C}) = \{g \in SL(p, \mathbb{C}) : g^t g = I_p\}\) act linearly on \(\mathbb{C}^p\), respectively, where \(g^t\) denotes the transposed matrix of \(g\). In (3), \(SL(p, \mathbb{C})\) acts on the space \(\text{Sym}(p, \mathbb{C})\) of complex symmetric matrices by \(g \cdot A = gA^t g\). In (4), \(SL(2p, \mathbb{C})\) acts on the space \(\text{Alt}(2p, \mathbb{C})\) of complex alternating matrices by \(g \cdot A = gA^t g\). In (5), \(SL(p, \mathbb{C}) \times SL(p, \mathbb{C})\) acts on \(M(p, \mathbb{C})\) by \((g, h) \cdot A = gAh^{-1}\). In (7), \(\mathcal{J}_C = \mathcal{J} \otimes \mathbb{R} \mathbb{C} = \text{Herm}(3, \mathbb{C}) \otimes \mathbb{R} \mathbb{C} = \text{Herm}(3, \mathbb{C}_C)\) is the complexified exceptional Jordan algebra, namely, it consists of Hermitian matrices of degree three whose entries are the complexified Cayley algebra \(\mathbb{C}_C = \mathcal{C} \otimes \mathbb{R} \mathbb{C}\). Then, \(\mathcal{J}_C\) is a vector space over \(\mathbb{C}\) with dimension 27. We denote by \(E_6(\mathbb{C})\) the connected and simply connected complex simple Lie group of exceptional type. Then, \(E_6(\mathbb{C})\) acts on \(\mathcal{J}_C\) as automorphisms.
In (3)–(7), the center $C^\times$ of $K_u$ acts on $V$ as the scalar multiplication. In (8), the semisimple part $SL(2p, \mathbb{C})$ of $K_u$ acts on $\mathbb{C}^{2p} \oplus \text{Alt}(2p, \mathbb{C})$ by $g \cdot (v, A) = (gv, gA^t g)$, and the center $C^\times$ acts by $s \cdot (v, A) = (s^3 v, s^2 A)$.

For $(K_C, V)$ in Table 5.1, we present:

**Lemma 5.6 (22, 24).** Let a multiplicity-free $K_C$-action on $V$ be one of cases in Table 5.1. Then, one can take a real vector subspace $T$ and an anti-holomorphic diffeomorphism $\sigma$ for the strongly visible $K_u$-action on $V$ satisfying the following conditions:

(a) $V = K_u \cdot T$.
(b) $\sigma|_T = \text{id}_T$.
(c) The dimension of the vector space $T$ over $\mathbb{R}$ is equal to the support of the semigroup of highest weights occurring in $\mathbb{C}[V]$.

Indeed, we choose $T$ and $\sigma$ as in Table 5.2. Then, we can verify that $(T, \sigma)$ satisfies Lemma 5.6.

| $(K_C, V)$ | $T$     | $\sigma$ | $\tilde{\sigma}$ |
|-----------|---------|----------|-----------------|
| (1)       | $\mathbb{R}$ | $\sigma_1$ | $\tilde{\sigma}_0$ |
| (2)       | $T_1$    | $\sigma_1$ | $\tilde{\sigma}_1$ |
| (3)       | $D_p$    | $\sigma_1$ | $\tilde{\sigma}_1 \otimes \tilde{\sigma}_0$ |
| (4)       | $A_p$    | $\sigma_1$ | $\tilde{\sigma}_1 \otimes \tilde{\sigma}_0$ |
| (5)       | $D_p$    | $\sigma_1$ | $\tilde{\sigma}_1 \otimes \tilde{\sigma}_1 \otimes \tilde{\sigma}_0$ |
| (6)       | $D_{1,1}$ | $\sigma_2$ | $\tilde{\sigma}_2 \otimes \tilde{\sigma}_0$ |
| (7)       | $D_3$    | $\sigma_1$ | $\tilde{\sigma}_1 \otimes \tilde{\sigma}_0$ |
| (8)       | $A_p \oplus T_p$ | $\sigma_1 + \sigma_1$ | $\tilde{\sigma}_1 \otimes \tilde{\sigma}_0$ |

**Table 5.2.** Our choice of $T$, $\sigma$, $\tilde{\sigma}$ for the $K_u$-action on $V$

Here, let us explain the notation used in Table 5.2 as follows.

First, let $\{e_1, \ldots, e_N\}$ be the standard basis of $\mathbb{C}^N$. We define two real subspaces $T_p, D_{1,1}$ in $\mathbb{C}^N$ by

\[
T_p := \mathbb{R}e_1 \oplus \mathbb{R}e_3 \oplus \mathbb{R}e_5 \oplus \cdots \oplus \mathbb{R}e_{2N'} - 1,
\]

\[
D_{1,1} := \mathbb{R}e_1 \oplus \sqrt{-1} \mathbb{R}e_2,
\]

where $N' := \lfloor \frac{N+1}{2} \rfloor$ denotes the maximam of integers which are not greater than $\frac{N+1}{2}$. Second, we denote by $D_N$ the real subspace of $M(N, \mathbb{C})$ consisting of diagonal matrices whose entries are all real, namely,

\[
D_N := \{\text{diag}(r_1, \ldots, r_N) \in M(N, \mathbb{C}) : r_1, \ldots, r_N \in \mathbb{R}\}.
\]

Third, we set a real subspace $A_p$ in $\text{Alt}(2p, \mathbb{C})$ by

\[
A_p := \{J(r_1, \ldots, r_p) \in \text{Alt}(2p, \mathbb{C}) : r_1, \ldots, r_p \in \mathbb{R}\}.
\]
where \( J(r_1, \ldots, r_p) \in \text{Alt}(2p, \mathbb{C}) \) stands for the following block diagonal matrix
\[
J(r_1, \ldots, r_p) := \text{diag}(r_1J_1, \ldots, r_pJ_1), \quad J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

On the other hand, we denote by \( \sigma_1 \) and \( \sigma_2 \) the complex conjugations of \( \mathbb{C}^N \) with respect to real forms \( V_1 \) and \( V_2 \), respectively, where \( N' = \lfloor \frac{N+1}{2} \rfloor \), \( N'' = \lceil \frac{N}{2} \rceil \) and
\[
V_1 := \bigoplus_{i=1}^{N} \mathbb{R}e_i, \quad V_2 := \bigoplus_{i=1}^{N'} \mathbb{R}e_{2i-1} \bigoplus \bigoplus_{i=1}^{N''} \sqrt{-1} \mathbb{R}e_{2i}.
\]

By using the coordinate with respect to the standard basis \( \{e_1, \ldots, e_N\} \), we write \( \sigma_1 \) and \( \sigma_2 \), respectively, as
\[
\sigma_1(v) = \overline{v}, \quad \sigma_2(v) = I_a \overline{v} \quad (v \in \mathbb{C}^N).
\]
where \( \varepsilon_i := (-1)^{i+1} (i = 1, 2, \ldots, N) \) and
\[
(5.6) \quad I_a := \text{diag}(\varepsilon_1, \ldots, \varepsilon_N) \in M(N, \mathbb{C}).
\]

For the standard basis \( \{E_{ij} : 1 \leq i, j \leq N\} \) of \( M(N, \mathbb{C}) \), we also define the standard real form \( \bigoplus_{1 \leq i, j \leq N} \mathbb{R}E_{ij} = M(N, \mathbb{R}) \). With respect to \( M(N, \mathbb{R}) \), we define the complex conjugation of \( M(N, \mathbb{C}) \), which we use the same notation \( \sigma_1 \) to denote.

Suppose we take \( T \) and \( \sigma \) as in Table 5.2 for a multiplicity-free \( K_\mathbb{C} \)-action on \( V \) listed in Table 5.1. Then, we have:

**Lemma 5.7.** There exists a compatible automorphism \( \tilde{\sigma} \in \text{Aut} K_\mathbb{C} \) with respect to \( \sigma \) for the \( K_\mathbb{C} \)-action on \( V \) (see Definition 3.3) such that \( \tilde{\sigma} \) stabilizes \( K_u \) and \( \text{rank}_\mathbb{R} \text{Lie}(K_\mathbb{C}^2) = \text{rank} \text{Lie}(K_\mathbb{C}) \).

**Proof.** For convenience, we let put \( K_1 := SL(p, \mathbb{C}) \) and \( K_2 := SO(p, \mathbb{C}) \).

By using our matrix realization of \( SL(p, \mathbb{C}) \) and \( SO(p, \mathbb{C}) \), we define anti-holomorphic involutions \( \tilde{\sigma}_i \) on \( K_i \) \( (i = 1, 2) \) by
\[
\tilde{\sigma}_1(k) := \overline{k} \quad (k \in K_1), \quad \tilde{\sigma}_2(k) := I_a^{-1} \overline{k} I_a \quad (k \in K_2).
\]

Then, the fixed point set \( K_1^{\tilde{\sigma}_1} \) coincides with \( SL(p, \mathbb{R}) \), and \( K_2^{\tilde{\sigma}_2} \) is isomorphic to the indefinite special orthogonal group \( SO(\lfloor \frac{p}{2} \rfloor, \lceil \frac{p+1}{2} \rceil) \).

Clearly, \( \text{rank}_\mathbb{R} \text{Lie}(K_1^{\tilde{\sigma}_1}) = p = \text{rank} \text{Lie}(K_1) \) and \( \text{rank}_\mathbb{R} \text{Lie}(K_2^{\tilde{\sigma}_2}) = \lfloor \frac{p}{2} \rfloor = \text{rank} \text{Lie}(K_2) \). Similarly, we define an anti-holomorphic involution \( \tilde{\sigma}_0 \) on \( \mathbb{C}^\times \) by \( \tilde{\sigma}_0(s) = \overline{s} \) \( (s \in \mathbb{C}^\times) \). Then, \( (\mathbb{C}^\times)^{\tilde{\sigma}_0} = \mathbb{R}^\times \).

The right column of Table 5.2 gives our choice of anti-holomorphic involution \( \tilde{\sigma} \) on \( K_\mathbb{C} \) for each strongly visible \( K_u \)-action on \( V \). The direct computation shows that \( \tilde{\sigma} \) satisfies Lemma 5.7. \( \square \)

Concerning to our choice of \( \tilde{\sigma} \) as in Table 5.2, we denote by \( \mu \boxtimes \mu' \) for involutions \( \mu \) and \( \mu' \) on complex Lie groups \( H_\mathbb{C} \) and \( H'_\mathbb{C} \), respectively, the involution on \( H_\mathbb{C} \times H'_\mathbb{C} \) defined by \( (\mu \boxtimes \mu')(h, h') = (\mu(h), \mu'(h')) \).
Remark 5.8. Lemma 5.7 holds for all multiplicity-free actions, on which we will discuss in forthcoming paper [25].

5.3. Procedure. In this subsection, we explain the procedure of our proof of Theorem 5.6.

The standard basis \( \{e_1, \ldots, e_N\} \) of \( \mathbb{C}^N \) defines the standard real form \( V_\mathbb{R} = V_1 = \mathbb{R}^N \) and the standard Hermitian inner product \( \langle \cdot, \cdot \rangle \) satisfying \( \langle e_i, e_j \rangle = \delta_{ij} \) \((1 \leq i, j \leq N)\). The dual \( a_\mathbb{R}^* \) of the Cartan subalgebra \( a_\mathbb{R} \subset g_\mathbb{R} \) is realized as a subspace of \( V_\mathbb{R} \), denoted by \( V \). Let \( \{E_1, \ldots, E_N\} \) be the dual basis of \( \{e_1, \ldots, e_N\} \) with \( \langle e_i, E_j \rangle = \delta_{ij} \) \((1 \leq i, j \leq N)\). Then, \( a_\mathbb{R} \) is isomorphic to \( V^* \subset V_\mathbb{R}^* = \mathbb{R}E_1 \oplus \cdots \oplus \mathbb{R}E_N \). Hence, \( a = a_\mathbb{R} + \sqrt{-1}a_\mathbb{R} \cong V^* + \sqrt{-1}V^* \).

We have seen in Lemma 5.4 that the complex conjugation \( \sigma \) with respect to \( g_\mathbb{R} \) stabilizes the root space \( g_\alpha \) \((\alpha \in \Delta)\). Then, we decompose \( g_\alpha \) into the \( \sigma \)-eigenspaces as \( g_\alpha = (g_\alpha \cap g_\mathbb{R}) + (g_\alpha \cap \sqrt{-1}g_\mathbb{R}) \), equivalently, \( g_\alpha \) is the complexification of the Lie algebra \( g_\alpha \cap g_\mathbb{R} \). As \( \dim g_\alpha = 1 \), we have \( \dim (g_\alpha \cap g_\mathbb{R}) = 1 \). Thus, we express \( g_\alpha \cap g_\mathbb{R} = \mathbb{R}E_{\alpha} \) for some root vector \( E_{\alpha} \).

Under this setting, we carry out for each spherical nilpotent orbit \( \mathcal{O}_X \) in \( g \) as follows:

1. Specify the characteristic element \( H \in a_+ \) from the corresponding weighted Dynkin diagram \( \Omega(\mathcal{O}_X) \) (see 5.2).
2. Write \( l = g(0) \), \( g(2) \), and \( g(3) \), respectively, as a direct sum of root spaces (see Lemma 5.2).
3. Verify that the \( L_\mathbb{C} \)-action on \( n \) is a multiplicity-free action comparing with Table 5.1. Then, the \( L_\mathbb{R} \)-action on \( n \) is strongly visible (Fact 5.3).
4. Give a slice \( S_0 \) for the \( L_\mathbb{R} \)-action on \( n \) explicitly by using Table 5.2 In particular, describe \( S_0 \) as \( S_0 = \bigoplus_{\alpha \in \Delta^+(\mathcal{O}_X)} \mathbb{R}E_\alpha \) for some subset \( \Delta^+(\mathcal{O}_X) \) in \( \Delta^+ \).

Owing to Lemmas 5.6 and 5.7, Theorem 3.6 holds for the subspace \( S_0 \) which is constructed according to the above procedure.

5.4. Type \( A_{n-1} \). We begin with the case \( g = \mathfrak{sl}(n, \mathbb{C}) \) for integer \( n \geq 2 \). In this case, \( g_\mathbb{R} = \mathfrak{sl}(n, \mathbb{R}) \). Then, \( a_\mathbb{R}^* = \{a_1 e_1 + \cdots + a_n e_n : a_1, \ldots, a_n \in \mathbb{R}, a_1 + \cdots + a_n = 0\} \). A root system \( \Delta \equiv \Delta(g, a) \) is \( \Delta = \{\pm(e_i - e_j) : 1 \leq i < j \leq n\} \). We fix a positive system as \( \Delta^+ = \{e_i - e_j : 1 \leq i < j \leq n\} \). The simple roots \( \alpha_1, \ldots, \alpha_{n-1} \) are given by \( \alpha_i = e_i - e_{i+1} \) \((1 \leq i \leq n - 1)\). The highest root \( \beta \) is written as \( \beta = e_1 - e_n = \alpha_1 + \cdots + \alpha_{n-1} \).

Let \( \mathcal{O}_X \) be a nilpotent orbit with characteristic element \( H = h_1 E_1 + \cdots + h_n E_n \in a_+ \) where \( h_1 + \cdots + h_n = 0 \) and \( h_1 \geq h_2 \geq \cdots \geq h_n \). Then, \( \alpha_i(H) = h_i - h_{i+1} \) \((1 \leq i \leq n-1)\). Hence, the weighted Dynkin diagram \( \Omega(\mathcal{O}_X) = (m_1, m_2, \ldots, m_{n-1}) \) is given by \((h_1 - h_2, h_2 - h_3, \ldots, h_{n-1} - h_n) \).

A nilpotent orbit \( \mathcal{O}_X \) in \( \mathfrak{sl}(n, \mathbb{C}) \) is spherical if and only if \( \Omega(\mathcal{O}_X) \) coincides with either (A) or (A').
\( \alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_{n-2} \quad \alpha_{n-1} \)

Figure 5.1. Weighted Dynkin diagram for \( \mathfrak{sl}(n, \mathbb{C}) \)

(A) \( \Omega(\mathcal{O}_X) = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0) \) for \( 1 \leq p < \frac{n}{2} \), namely,
\[
m_p = m_{n-p} = 1 \quad \text{and} \quad m_i = 0 \quad \text{for} \quad i \neq p, n-p.
\]

(A') \( n = 2p \) and \( \Omega(\mathcal{O}_X) = (0, \ldots, 0, 2, 0, \ldots, 0) \), namely, \( m_p = 2 \) and \( m_i = 0 \) for \( i \neq p \).

For each case, it follows from Lemma 5.4 that its height \( h_t(\mathcal{O}_X) \) equals two. Then, the nilpotent subalgebra \( \mathfrak{n} \) coincides with \( \mathfrak{g}(2) \).

5.4.1. Case (A). Let \( \Omega(\mathcal{O}_X) \) satisfy Case (A) for \( 1 \leq p < \frac{n}{2} \). Since \( h_p - h_{p+1} = h_{n-p} - h_{n-p+1} = 1 \) and \( h_i - h_{i+1} = 0 \) \( (i \neq p, n-p) \), \( H \) forms
\[
( E_1 + \cdots + E_p ) - ( E_{n-p+1} + \cdots + E_n ).
\]

The nilpotent orbit \( \mathcal{O}_X \) with the above \( \Omega(\mathcal{O}_X) \) consists of complex matrices of degree \( n \) with Jordan type \((2^p, 1^{n-2p})\).

The Levi subalgebra \( \mathfrak{l} = \mathfrak{g}(0) \) is given as follows:
\[
\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{1 \leq i < j \leq p} \mathfrak{g}_{\pm(e_i - e_j)} \oplus \bigoplus_{p+1 \leq i < j \leq n-p} \mathfrak{g}_{\pm(e_i - e_j)} \oplus \bigoplus_{n-p+1 \leq i < j \leq n} \mathfrak{g}_{\pm(e_i - e_j)}.
\]

This means that
\[
\mathfrak{l} \simeq \mathfrak{sl}(p, \mathbb{C}) \oplus \mathfrak{sl}(n-2p, \mathbb{C}) \oplus \mathfrak{sl}(p, \mathbb{C}) \oplus \mathbb{C}^2.
\]

The \( \text{ad}(H) \)-eigenspace \( \mathfrak{g}(2) \) is written as
\[
\mathfrak{g}(2) = \bigoplus_{1 \leq i, j \leq p} \mathfrak{g}_{e_i - e_{n-p+j}}.
\]

Then, \( \mathfrak{g}(2) \) is isomorphic to \( M(p, \mathbb{C}) \), from which
\[
\mathfrak{n} \simeq M(p, \mathbb{C}).
\]

The semisimple part \( SL(p, \mathbb{C}) \times SL(n-2p, \mathbb{C}) \times SL(p, \mathbb{C}) \) of the Levi subgroup \( L \) acts on \( M(p, \mathbb{C}) \) by
\[
(g_1, h, g_2) \cdot A = g_1 A g_2^{-1},
\]
and the center \( (\mathbb{C}^*)^2 \) of \( L \) as the scalar multiplication as follows:
\[
(s, t) \cdot A = s t A.
\]

Then, the \( L \)-action on \( \mathfrak{n} \) is geometrically equivalent to the irreducible action of \( SL(p, \mathbb{C}) \times SL(p, \mathbb{C}) \times \mathbb{C}^* \) on \( M(p, \mathbb{C}) \). It follows from (5) of Table 5.1 that this action is a multiplicity-free action.
We take the subset $S_0$ in $\mathfrak{n}$ as
\begin{equation}
S_0 := \bigoplus_{1 \leq i \leq p} \mathbb{R}e_{i-e_{n-p+i}}.
\end{equation}
Then, $S_0$ is isomorphic to the slice $D_p$ of Table 5.2 for the strongly visible $(SU(p) \times SU(p) \times T)$-action on $M(p, \mathbb{C})$. By Lemma 5.6, the vector space $S_0$ satisfies $\mathfrak{n} = L_u \cdot S_0$. Therefore, we have verified Theorem 3.6 for Case (A).

5.4.2. Case (A'). In case of $n = 2p$ and $\Omega(\mathcal{O}_X)$ satisfies Case (A'), the characteristic element $H \in \mathfrak{a}$ is of the form
\begin{equation}
H = (E_1 + \cdots + E_p) - (E_{p+1} + \cdots + E_{2p}).
\end{equation}
Then, $l \simeq \mathfrak{sl}(p, \mathbb{C}) \oplus \mathfrak{sl}(p, \mathbb{C}) \oplus \mathbb{C}^2$ and $\mathfrak{n} \simeq M(p, \mathbb{C})$. Hence, the $L_{\mathbb{C}}$-action on $\mathfrak{n}$ is a multiplicity-free action. Therefore, the subset $S_0$ defined by (5.8) satisfies Theorem 3.6 from which we have verified for Case (A').

5.5. Type $B_n$. In this subsection, we give a proof of Theorem 3.6 for $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$ for positive integer $n \geq 2$. In this case, $\mathfrak{g}_\mathbb{R}$ is isomorphic to $\mathfrak{so}(n+1, n)$. Then, we have $\mathfrak{a}_\mathbb{R}^+ = V_{\mathbb{R}}$. A root system $\Delta \equiv \Delta(\mathfrak{g}, \mathfrak{a})$ is $\Delta = \{\pm e_i \pm e_j : 1 \leq i < j \leq n\} \sqcup \{\pm e_i : 1 \leq i \leq n\}$. We fix a positive system as $\Delta^+ = \{e_i \pm e_j : 1 \leq i < j \leq n\} \sqcup \{e_i : 1 \leq i \leq n\}$. The simple roots $\alpha_1, \ldots, \alpha_n$ are given by $\alpha_i = e_i - e_{i+1}$ $(1 \leq i \leq n-1)$, and $\alpha_n = e_n$. The highest root $\beta$ is written as $\beta = e_1 + e_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n$.

Let $\mathcal{O}_X$ be a nilpotent orbit with characteristic element $H = h_1 E_1 + \cdots + h_n E_n \in \mathfrak{a}_+$ with $h_1 \geq \cdots \geq h_n \geq 0$. Then, we have $\alpha_1(H) = h_1 - h_{i+1}$ $(1 \leq i \leq n-1)$, and $\alpha_n(H) = h_n$. Thus, the weighted Dynkin diagram $\Omega(\mathcal{O}_X) = (m_1, \ldots, m_{n-1}, m_n)$ is given by $(h_1 - h_2, \ldots, h_{n-1} - h_n, h_n)$.

![Figure 5.2. Weighted Dynkin diagram of $\mathcal{O}_X$ in $\mathfrak{so}(2n+1, \mathbb{C})$](image)

A nilpotent orbit $\mathcal{O}_X$ in $\mathfrak{so}(2n+1, \mathbb{C})$ is spherical if and only if $\Omega(\mathcal{O}_X)$ forms one of the following cases:

(B1) $\Omega(\mathcal{O}_X) = (2, 0, \ldots, 0)$, namely, $m_1 = 2$ and $m_i = 0$ $(i \neq 1)$.
(B2) $\Omega(\mathcal{O}_X) = (0, \ldots, 0, 1, 0, \ldots, 0)$ for $1 \leq p \leq \frac{n}{2}$, namely, $m_{2p} = 1$ and $m_i = 0$ $(i \neq 2p)$.
(B3) \( \Omega(O_X) = (1, 0, \ldots, 0, 1, 0, \ldots, 0) \) for \( 1 \leq p \leq \frac{n-1}{2} \), namely,
\[
m_1 = m_{2p+1} = 1 \text{ and } m_i = 0 \quad (i \neq 1, 2p + 1).
\]
By Lemma 5.4, its height \( \text{ht}(O_X) \) equals two for Cases (B1), (B2), and three for Case (B3).

5.5.1. \textit{Case (B1)}. Let us consider the case \( \Omega(O_X) = (2, 0, \ldots, 0) \). Then, \( H \) is given by
\[
H = 2E_1.
\]
This \( O_X \) consists of complex matrices with Jordan type \((3, 1^{2n-2})\).
The Levi subalgebra \( \mathfrak{l} = \mathfrak{g}(0) \) is given by
\[
\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{2 \leq i < j \leq n} \mathfrak{g}_{\pm e_i \pm e_j} \oplus \bigoplus_{2 \leq i \leq n} \mathfrak{g}_{\pm e_i}.
\]
Then,
\[
\mathfrak{l} \simeq \mathfrak{so}(2n - 1, \mathbb{C}) \oplus \mathbb{C}.
\]
The \( \text{ad}(H) \)-eigenspace \( \mathfrak{g}(2) \) is written as
\[
\mathfrak{g}(2) = \bigoplus_{2 \leq j \leq n} \mathfrak{g}_{e_1 \pm e_j} \oplus \mathfrak{g}_{e_1}.
\]
Then, \( \mathfrak{g}(2) \) is isomorphic to \( \mathbb{C}^{2n-1} \). As \( \text{ht}(O_X) = 2 \), the nilpotent subalgebra \( \mathfrak{n} \) coincides with \( \mathfrak{g}(2) \), namely,
\[
\mathfrak{n} \simeq \mathbb{C}^{2n-1}.
\]
The semisimple part \( SO(2n - 1, \mathbb{C}) \) of the Levi subgroup \( L_{\mathbb{C}} \) acts on \( \mathbb{C}^{2n-1} \) as the standard action, namely,
\[
g \cdot v = gv,
\]
and its center \( \mathbb{C}^x \) acts as the scalar multiplication. This implies that the \( L_{\mathbb{C}} \)-action on \( \mathfrak{n} \) is geometrically equivalent to the \( (SO(2n - 1, \mathbb{C}) \times \mathbb{C}^x) \)-action on \( \mathbb{C}^{2n-1} \). It follows from (6) of Table 5.1 that this action is a multiplicity-free action.
We take the subset \( S_0 \) in \( \mathfrak{n} \) as
\[
S_0 := \mathbb{R}E_{e_1 + e_2} \oplus \mathbb{R}E_{e_1 - e_2}.
\]
Then, \( S_0 \) is isomorphic to the slice \( D_{1,1} \) of Table 5.2 for the strongly visible \( (SO(2n - 1) \times \mathbb{T}) \)-action on \( \mathbb{C}^{2p-1} \). By Lemma 5.6, \( S_0 \) satisfies \( \mathfrak{n} = L_u \cdot S_0. \) Therefore, we have verified Theorem 3.6 for Case (H1).
5.5.2. Case \((B_2)\). Let \(\Omega(O_X)\) satisfy \(m_{2p} = 1\) and \(m_i = 0\) \((i \neq 2p)\) for \(1 \leq p \leq \frac{n}{2}\). Then,

\[
H = E_1 + E_2 + \cdots + E_{2p}.
\]

This \(O_X\) consists of complex matrices with Jordan type \((2^{2p}, 1^{2n-4p+1})\). In particular, \(O_X\) with \(\Omega(O_X) = (0, 1, 0, \ldots, 0)\) \((p = 1)\) is the minimal nilpotent orbit.

The Levi subalgebra \(l = \mathfrak{g}(0)\) is given by

\[
l = \mathfrak{a} \oplus \bigoplus_{1 \leq i < j \leq 2p} \mathfrak{g}_{\pm(e_i - e_j)} \oplus \bigoplus_{2p+1 \leq i < j \leq n} \mathfrak{g}_{\pm e_i \pm e_j} \oplus \bigoplus_{2p+1 \leq i \leq n} \mathfrak{g}_{\pm e_i}.
\]

This means that

\[
l \simeq \mathfrak{sl}(2p, \mathbb{C}) \oplus \mathfrak{so}(2n - 4p + 1, \mathbb{C}) \oplus \mathbb{C}.
\]

The \(\text{ad}(H)\)-eigenspace \(\mathfrak{g}(2)\) is written as

\[
\mathfrak{g}(2) = \bigoplus_{1 \leq i < j \leq 2p} \mathfrak{g}_{e_i + e_j}.
\]

Then, \(\mathfrak{g}(2)\) is isomorphic to \(\text{Alt}(2p, \mathbb{C})\) As \(\text{ht}(O_X) = 2\),

\[
\mathfrak{n} = \mathfrak{g}(2) \simeq \text{Alt}(2p, \mathbb{C}).
\]

The semisimple part of the Levi subgroup \(L_\mathbb{C}\) is isomorphic to \(SL(2p, \mathbb{C}) \times SO(2n - 4p + 1, \mathbb{C})\). Then, \(SL(2p, \mathbb{C})\) acts on \(\text{Alt}(2p, \mathbb{C})\) by

\[
g \cdot A = g A^t g,
\]

and \(SO(2n - 4p + 1, \mathbb{C})\) acts trivially. Further, its center \(\mathbb{C}^\times\) acts as the scalar multiplication. This implies that the \(L_\mathbb{C}\)-action on \(\mathfrak{n}\) is geometrically equivalent to the action of \(SL(2p, \mathbb{C}) \times \mathbb{C}^\times\) on \(\text{Alt}(2p, \mathbb{C})\). It follows from \(4\) of Table \(5.1\) that this action is a multiplicity-free action.

We take \(S_0\) as

\[
S_0 := \bigoplus_{1 \leq i \leq p} \mathbb{R} E_{e_{2i-1} + e_{2i}}.
\]

Then, \(S_0\) is isomorphic to the slice \(A_p\) of Table \(5.2\) for the \((SU(2p) \times \mathbb{T})\)-action on \(\text{Alt}(2p, \mathbb{C})\). By Lemma \(5.6\), Theorem \(3.6\) holds for Case \((B_2)\).

5.5.3. Case \((B_3)\). Let \(\Omega(O_X)\) satisfy \(m_1 = m_{2p+1} = 1\) and \(m_i = 0\) for \(1 \leq p \leq \frac{n-1}{2}\). Then, \(\text{ht}(O_X) = 3\) and

\[
H = 2E_1 + E_2 + E_3 + \cdots + E_{2p+1}.
\]

This \(O_X\) consists of complex matrices with Jordan type \((3, 2^{2p}, 1^{2n-4p-2})\). We divide Case \((B_3)\) into two cases: \(n \neq 2p - 1\); and \(n = 2p - 1\).
First, let us consider the general case \( n \neq 2p - 1 \). Then, the Levi subalgebra \( \mathfrak{l} = \mathfrak{g}(0) \) is given by

\[
\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{2 \leq i < j \leq 2p + 1} \mathfrak{g}(e_i - e_j) \oplus \bigoplus_{2p + 2 \leq i < j \leq n} \mathfrak{g}(e_i + e_j) \oplus \bigoplus_{2p + 2 \leq i \leq n} \mathfrak{g}(e_i).
\]

This means that

\[
\mathfrak{l} \simeq \mathfrak{sl}(2p, \mathbb{C}) \oplus \mathfrak{so}(2n - 4p - 1, \mathbb{C}) \oplus \mathbb{C}^2.
\]

The \( \text{ad}(H) \)-eigenspace \( \mathfrak{g}(2) \) is written as

\[
\mathfrak{g}(2) = \mathfrak{g}(e_i) \oplus \bigoplus_{2p + 2 \leq j \leq n} \mathfrak{g}(e_j) \oplus \bigoplus_{2 \leq i < j \leq 2p + 1} \mathfrak{g}(e_i + e_j).
\]

Then, \( \mathfrak{g}(2) \) is isomorphic to \( \mathbb{C}^{2n - 4p - 1} \oplus \text{Alt}(2p, \mathbb{C}) \). Further, \( \mathfrak{g}(3) \) is of the form

\[
\mathfrak{g}(3) = \bigoplus_{2 \leq j \leq 2p + 1} \mathfrak{g}(e_1 + e_j) \simeq \mathbb{C}^{2p}.
\]

Hence, \( \mathfrak{n} \) is isomorphic to

\[
(5.9) \quad \mathfrak{n} = \mathfrak{g}(2) \oplus \mathfrak{g}(3) \simeq \mathbb{C}^{2n - 4p - 1} \oplus \text{Alt}(2p, \mathbb{C}) \oplus \mathbb{C}^{2p}.
\]

The semisimple part \( \text{SL}(2p, \mathbb{C}) \times \text{SO}(2n - 4p - 1, \mathbb{C}) \) of \( L_C \) acts on \( \mathbb{C}^{2n - 4p - 1} \oplus \text{Alt}(2p, \mathbb{C}) \oplus \mathbb{C}^{2p} \) by

\[
(g, h) \cdot (v, A, w) = (hv, gA^t g, gw),
\]

and its center \( (\mathbb{C}^*)^2 \) acts by

\[
(s, t) \cdot (v, A, w) = (sv, t^2 A, stw).
\]

Then, the \( L_C \)-action on \( \mathfrak{n} \) is geometrically equivalent to the decomposable action consisting of the indecomposable \( (\text{SL}(2p, \mathbb{C}) \times \mathbb{C}^*) \)-action on \( \text{Alt}(2p, \mathbb{C}) \oplus \mathbb{C}^{2p} \) ((8) of Table 5.1) and the irreducible \( (\text{SO}(2n - 4p - 1, \mathbb{C}) \times \mathbb{C}^*) \)-action on \( \mathbb{C}^{2n - 4p - 1} \) ((6) of Table 5.1). Hence, this action is a multiplicity-free action.

Our slice for this action is the direct sum the slices \( S'_0 \) and \( S''_0 \) are isomorphic to the slice \( A_p \oplus T_p \) of Table 5.2 for the action of \( \text{SU}(2p) \times \mathbb{T} \) on \( \text{Alt}(2p, \mathbb{C}) \oplus \mathbb{C}^{2p} \) and \( D_{1,1} \) for the action of \( \text{SO}(2n - 4p - 1) \times \mathbb{T} \) on \( \mathbb{C}^{2n - 4p - 1} \), respectively. In fact, we define

\[
(5.10) \quad S'_0 := \bigoplus_{1 \leq i \leq p} \mathbb{R}E_{e_{2i} + e_{2i+1}} \oplus \bigoplus_{1 \leq j \leq p} \mathbb{R}E_{e_1 + e_j},
\]

and

\[
S''_0 := \mathbb{R}E_{e_1 + e_{2p+2}} \oplus \mathbb{R}E_{e_1 - e_{2p+2}}.
\]

By Lemma 5.6 the subspace

\[
(5.11) \quad S_0 := S'_0 \oplus S''_0 \simeq (D_{1,1} \oplus A_p) \oplus T_p,
\]

satisfies \( \mathfrak{n} = L_u \cdot S_0 \). Therefore, Theorem 3.6 has been proved for Case (1B) with \( n \neq 2p - 1 \).
In the special case where \( g = \mathfrak{so}(4p+3, \mathbb{C}) \) and \( \Omega(\mathcal{O}_X) = (1, 0, \ldots, 0, 1) \), The Levi subalgebra \( \mathfrak{l} \) is given by
\[
\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{2 \leq i < j \leq 2p+1} \mathfrak{g}_{\pm(e_i-e_j)} \cong \mathfrak{sl}(2p, \mathbb{C}) \oplus \mathbb{C}^2.
\]
The ad(\( H \))-eigenspaces \( \mathfrak{g}(2) \), \( \mathfrak{g}(3) \) are written as
\[
\mathfrak{g}(2) = \mathfrak{g}_{e_1} \oplus \bigoplus_{2 \leq i < j \leq 2p+1} \mathfrak{g}_{e_i+e_j} \cong \mathbb{C} \oplus \text{Alt}(2p, \mathbb{C}),
\]
\[
\mathfrak{g}(3) = \bigoplus_{2 \leq j \leq 2p+1} \mathfrak{g}_{e_1+e_j} \cong \mathbb{C}^{2p}.
\]
We take \( S'_0 \cong T_3^p \oplus T_p \) as in (5.10) and \( S''_0 \) as
\[
S''_0 := \mathbb{R}E_{e_1} \cong \mathbb{R}.
\]
By Lemma 5.6, the vector space
\[
S_0 := S'_0 \oplus S''_0 \cong (\mathbb{R} \oplus A_p) \oplus T_p
\]
satisfies \( \mathfrak{n} = L_u \cdot S_0 \). Hence, Theorem 3.6 has been proved for Case (B3).

5.6. Type C\(_n\). In this subsection, we give a proof of Theorem 3.6 for \( g = \mathfrak{sp}(n, \mathbb{C}) \). In this case, \( \mathfrak{g}_R = \mathfrak{sp}(n, \mathbb{R}) \). Then, \( \mathfrak{a}_R^* \cong V_R \). A root system \( \Delta \equiv \Delta(\mathfrak{g}, \mathfrak{a}) \) is \( \Delta = \{ \pm e_i \pm e_j : 1 \leq i < j \leq n \} \cup \{ \pm 2e_i : 1 \leq i \leq n \} \). We fix a positive system \( \Delta^+ \) as \( \Delta^+ = \{ e_i \pm e_j : 1 \leq i < j \leq n \} \cup \{ 2e_i : 1 \leq i \leq n \} \). The simple roots \( \alpha_1, \ldots, \alpha_n \) is given by \( \alpha_i = e_i - e_{i+1} (1 \leq i \leq n-1) \) and \( \alpha_n = 2e_n \). The highest root \( \beta \) is written as \( \beta = 2e_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n \).

Let \( \mathcal{O}_X \) be a nilpotent orbit with characteristic element \( H = h_1E_1 + \cdots + h_nE_n \in \mathfrak{a}_+ \) with \( h_1 \geq \cdots \geq h_n \geq 0 \). Then, \( m_i = h_i - h_{i+1} (i = 1, 2, \ldots, n-1) \), and \( m_n = 2h_n \). Hence, the weighted Dynkin diagram \( \Omega(\mathcal{O}_X) = (m_1, \ldots, m_n) \) is given by \( \Omega(\mathcal{O}_X) = (h_1 - h_2, \ldots, h_{n-1} - h_n, 2h_n) \).

![Figure 5.3. Weighted Dynkin diagram of \( \mathcal{O}_X \) in \( \mathfrak{sp}(n, \mathbb{C}) \)](image)

A nilpotent orbit \( \mathcal{O}_X \) in \( \mathfrak{sp}(n, \mathbb{C}) \) is spherical if and only if \( \Omega(\mathcal{O}_X) \) satisfies either Case (C) or Case (C’):
\[
(C) \quad \Omega(\mathcal{O}_X) = (0, \ldots, 0, 1, 0, \ldots, 0) \text{ for } 1 \leq p < n, \text{ namely, } m_p = 1 \text{ and } m_i = 0 (i \neq p).
\]
\[
(C') \quad \Omega(\mathcal{O}_X) = (0, \ldots, 0, 2), \text{ namely, } m_n = 2 \text{ and } m_i = 0 (i \neq n).
\]
Then, the characteristic element $H \in \mathfrak{a}$ is written as
\begin{equation}
H = E_1 + \cdots + E_p
\end{equation}
for each $\Omega(\mathcal{O}_X)$. It follows from Lemma 5.4 that the height of $\mathcal{O}_X$ equals two. Further, Such $\mathcal{O}_X$ consists of complex matrices with Jordan type $(2^p, 1^{2n-2p})$ ($1 \leq p \leq n$). In particular, $\mathcal{O}_X$ with weighted Dynkin diagram $\Omega(\mathcal{O}_X) = (1, 0, \ldots, 0)$ is the minimal nilpotent orbit.

First, let us consider the general $p \neq n$. The Levi subalgebra $\mathfrak{l} = \mathfrak{g}(0)$ is given by
\begin{align*}
\mathfrak{l} &= \mathfrak{a} \oplus \bigoplus_{1 \leq i < j \leq p} \mathfrak{g}_{e_i - e_j} \oplus \bigoplus_{p+1 \leq i < j \leq n} \mathfrak{g}_{e_i \pm e_j} \oplus \bigoplus_{p+1 \leq i \leq n} \mathfrak{g}_{\pm 2e_i}.
\end{align*}
This means that
\begin{align*}
\mathfrak{l} &\simeq \mathfrak{sl}(p, \mathbb{C}) \oplus \mathfrak{sp}(n-p, \mathbb{C}) \oplus \mathbb{C}.
\end{align*}

The $\text{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as
\begin{align*}
\mathfrak{g}(2) &= \bigoplus_{1 \leq i < j \leq p} \mathfrak{g}_{e_i + e_j} \oplus \bigoplus_{1 \leq i \leq p} \mathfrak{g}_{2e_i}.
\end{align*}
Then, $\mathfrak{g}(2)$ is isomorphic to $\text{Sym}(p, \mathbb{C})$, from which $\mathfrak{n} \simeq \text{Sym}(p, \mathbb{C})$.

The action of the semisimple part $\text{SL}(p, \mathbb{C}) \times \text{Sp}(n-p, \mathbb{C})$ on $\text{Sym}(p, \mathbb{C})$ is written as follows: $\text{SL}(p, \mathbb{C})$ acts by
\begin{align*}
g \cdot A &= gA^t g,
\end{align*}
and $\text{Sp}(n-p, \mathbb{C})$ acts trivially. Its center $\mathbb{C}^\times$ acts as the scalar multiplication. Then, the $L_\mathbb{C}$-action on $\mathfrak{n}$ is geometrically equivalent to the $(\text{SL}(p, \mathbb{C}) \times \mathbb{C}^\times)$-action on $\text{Sym}(p, \mathbb{C})$. It follows from (3) of Table 5.1 that this action is a multiplicity-free action.

We take the subset $S_0$ as
\begin{align*}
S_0 &= \bigoplus_{1 \leq i \leq p} \mathbb{R}E_{2e_i}.
\end{align*}
Then, $S_0$ is isomorphic to the slice $D_p$ of Table 5.2 for the strongly visible $(\text{SU}(p) \times T)$-action $\text{Sym}(p, \mathbb{C})$. By Lemma 5.6, this $S_0$ satisfies $\mathfrak{n} = L_\mathbb{C} \cdot S_0$.

In case of $p = n$, the Levi subalgebra $\mathfrak{l}$ is
\begin{align*}
\mathfrak{l} &= \mathfrak{a} \oplus \bigoplus_{1 \leq i < j \leq n} \mathfrak{g}_{e_i - e_j} \simeq \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{C},
\end{align*}
and $\mathfrak{g}(2)$ is
\begin{align*}
\mathfrak{g}(2) &= \bigoplus_{1 \leq i < j \leq n} \mathfrak{g}_{e_i + e_j} \oplus \bigoplus_{1 \leq i \leq n} \mathfrak{g}_{2e_i} \simeq \text{Sym}(n, \mathbb{C}).
\end{align*}
Then, the $L_{\mathbb{C}}$-action on $\mathfrak{n}$ is geometrically equivalent to the $(SL(n, \mathbb{C}) \times \mathbb{C}^\times)$-action on $\text{Sym}(n, \mathbb{C})$. Hence, the equation $\mathfrak{n} = L_{\mathbb{C}} : S_0$ holds for the subset $S_0$ defined by (5.13).

Therefore, Theorem 3.6 has been verified for $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$.

5.7. Type $D_n$. In this subsection, we give a proof of Theorem 3.6 for $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ for integer $n \geq 4$. In this case, $\mathfrak{g}_{\mathbb{R}}$ is isomorphic to $\mathfrak{so}(n, n)$. Then, $\mathfrak{a}_{\mathbb{R}} = V_{\mathbb{R}}$. A root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ is $\Delta = \{ \pm e_i \pm e_j : 1 \leq i < j \leq n \}$. We fix a positive system $\Delta^+$ as $\Delta^+ = \{ e_i \pm e_j : 1 \leq i < j \leq n \}$. The simple roots $\alpha_1, \ldots, \alpha_n$ is given by $\alpha_i = e_i - e_{i+1}$ $(1 \leq i \leq n - 1)$ and $\alpha_n = e_{n-1} + e_n$. The highest root $\beta$ is written as $\beta = e_1 + e_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$.

Let $\mathcal{O}_X$ be a nilpotent orbit in $\mathfrak{so}(2n, \mathbb{C})$ with characteristic element $H = h_1E_1 + \cdots + h_nE_n \in \mathfrak{a}_{\mathbb{R}}$, with $h_1 \geq \cdots \geq h_{n-1} \geq |h_n|$. Then, the weighted Dynkin diagram $\Omega(\mathcal{O}_X) = (m_1, \ldots, m_n)$ is given by $(h_1 - h_2, \ldots, h_{n-1} - h_n, h_{n-1} + h_n)$.

![Figure 5.4. Weighted Dynkin diagram of $\mathcal{O}_X$ in $\mathfrak{so}(2n, \mathbb{C})$](image)

A nilpotent orbit $\mathcal{O}_X$ in $\mathfrak{so}(2n, \mathbb{C})$ is spherical if and only if $\Omega(\mathcal{O}_X)$ satisfies one of the following cases:

(D1) $\Omega(\mathcal{O}_X) = (2, 0, \ldots, 0)$, namely, $m_1 = 1$ and $m_i = 0$ ($i \neq 1$).

(D2) $\Omega(\mathcal{O}_X) = (0, \ldots, 0, 1, 0, \ldots, 0)$ for $1 \leq p \leq \frac{n}{2} - 1$, namely, $m_{2p} = 1$ and $m_i = 0$ ($i \neq 2p$).

(D2') $n = 2p + 1$ and $\Omega(\mathcal{O}_X) = (0, \ldots, 0, 1, 1)$, namely, $m_{2p} = m_{2p+1} = 1$ and $m_i = 0$ ($i \neq 2p, 2p + 1$).

(D2'') $n = 2p$ and $\Omega(\mathcal{O}_X) = (0, \ldots, 0, 2)$, namely, $m_{2p} = 2$ and $m_i = 0$ ($i \neq 2p$).

(D3) $\Omega(\mathcal{O}_X) = (1, 0, \ldots, 0, 1, 0, \ldots, 0)$ for $1 \leq p < \frac{n}{2} - 1$, namely, $m_1 = m_{2p+1} = 1$ and $m_i = 0$ ($i \neq 1, 2p + 1$).

(D3') $n = 2p + 2$ and $\Omega(\mathcal{O}_X) = (1, 0, 0, \ldots, 0, 1, 1)$, namely, $m_1 = m_{2p+1} = m_{2p+2} = 1$ and $m_i = 0$ ($i \neq 1, 2p + 1, 2p + 2$).

By Lemma 5.4, the height of $\mathcal{O}_X$ equals two if for Cases (D1)–(D2''), and three for Cases (D3), (D3').
5.7.1. Case (D1). Let us consider the case \( \Omega(\mathcal{O}_X) = (2, 0, \ldots, 0) \). Then, the characteristic element \( H \) is of the form
\[
H = 2E_1.
\]
This \( \mathcal{O}_X \) consists of all complex matrices with Jordan type \( (3, 1^{2n-3}) \).

The Levi subalgebra \( \mathfrak{l} \) is given by
\[
\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{2 \leq i < j \leq n} \mathfrak{g}_{\pm e_i \pm e_j}.
\]
This means that
\[
\mathfrak{l} \simeq \mathfrak{so}(2n - 2, \mathbb{C}) \oplus \mathbb{C}.
\]
The ad\((H)\)-eigenspace \( \mathfrak{g}(2) \) is written as
\[
\mathfrak{g}(2) = \bigoplus_{2 \leq j \leq n} \mathfrak{g}_{e_1 \pm e_j}.
\]
Then, \( \mathfrak{g}(2) \) is isomorphic to \( \mathbb{C}^{2n-2} \), from which
\[
\mathfrak{n} \simeq \mathbb{C}^{2n-2}.
\]
Similarly to Case (B1), it turns out that the \( L_C \)-action on \( \mathfrak{n} \) is geometrically equivalent to the \( (SO(2n - 2, \mathbb{C}) \times \mathbb{T})\)-action on \( \mathbb{C}^{2n-2} \). It follows from (6) of Table 5.1 that this action is a multiplicity-free action.

We take the subset \( S_0 \) in \( \mathfrak{n} \) as
\[
S_0 = \mathbb{R}E_{e_1 + e_2} \oplus \mathbb{R}E_{e_1 - e_2}.
\]
Then, \( S_0 \) is isomorphic to the slice \( D_{1,1} \) of Table 5.2 for the \( (SO(2n - 2, \mathbb{C}) \times \mathbb{T})\)-action on \( \mathbb{C}^{2n-2} \). By Lemma 5.6, \( S_0 \) satisfies \( \mathfrak{n} = L_u \cdot S_0 \).

Therefore, Theorem 3.6 has been verified for Case (D1).

5.7.2. Case (D2). Let \( \Omega(\mathcal{O}_X) \) satisfy \( m_2 = 1 \) and \( m_i = 0 \) \((i \neq 2p)\) for \( 1 \leq p \leq \frac{n}{2} - 1 \). Then,
\[
(5.14) \quad H = E_1 + E_2 + \cdots + E_{2p}.
\]
This \( \mathcal{O}_X \) consists of all complex matrices with Jordan type \( (2^{2p}, 1^{2n-4p}) \).

In particular, \( \mathcal{O}_X \) with Jordan type \( (2^2, 1^{2n-4}) \) \((p = 1)\) is the minimal nilpotent orbit in \( \mathfrak{so}(2n, \mathbb{C}) \).

The Levi subalgebra \( \mathfrak{l} \) is given by
\[
\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{1 \leq i < j \leq 2p} \mathfrak{g}_{\pm (e_i - e_j)} \oplus \bigoplus_{2p+1 \leq i < j \leq n} \mathfrak{g}_{\pm e_i \pm e_j}.
\]
This means that
\[
\mathfrak{l} \simeq \mathfrak{sl}(2p, \mathbb{C}) \oplus \mathfrak{so}(2n - 4p - 2, \mathbb{C}) \oplus \mathbb{C}.
\]
The ad\((H)\)-eigenspace \( \mathfrak{g}(2) \) is written as
\[
\mathfrak{g}(2) = \bigoplus_{1 \leq i < j \leq 2p} \mathfrak{g}_{e_i + e_j}.
\]
Then, $\mathfrak{g}(2)$ is isomorphic to $\text{Alt}(2p, \mathbb{C})$, from which

$$\mathfrak{n} \simeq \text{Alt}(2p, \mathbb{C}).$$

The semisimple part $SL(2p, \mathbb{C}) \times SO(2n - 4p - 2, \mathbb{C})$ of the Levi subgroup $L_C$ acts on $\text{Alt}(2p, \mathbb{C})$ as follows: $SL(2p, \mathbb{C})$ by

$$g \cdot A = gAg^t,$$

and $SO(2n - 4p - 2, \mathbb{C})$ trivially. Its center $\mathbb{C}^\times$ acts as scalar multiplications. Then, the $L_C$-action on $\mathfrak{n}$ is geometrically equivalent to the $(SL(2p, \mathbb{C}) \times \mathbb{C}^\times)$-action on $\text{Alt}(2p, \mathbb{C})$. It follows from (4) of Table 5.1 that this action is a multiplicity-free action.

We take the subset $S_0$ in $\mathfrak{n}$ as

$$S_0 := \bigoplus_{j=1}^p \mathbb{R}E_{e_{2j-1}+e_{2j}}. \tag{5.15}$$

Then, $S_0$ is isomorphic to our slice $A_p$ of Table 5.2 for the $(SU(2p) \times \mathbb{T})$-action on $\text{Alt}(2p, \mathbb{C})$. By Lemma 5.6, we have $\mathfrak{n} = L_u \cdot S_0$. Therefore, Theorem 3.6 has been verified for Case (D2).

5.7.3. Case (D2'). Let us consider the case where $\mathfrak{g} = \mathfrak{so}(4p + 2, \mathbb{C})$ ($n = 2p + 1$) and $\Omega(\mathcal{O}_X) = (0, \ldots, 0, 1)$. This nilpotent orbit is the set of all complex matrices with Jordan type $(2^{2p}, 1^2)$. Then, the proof for Case (D2') can be given similarly to Case (D2). In fact, the characteristic element $H$ forms

$$H = E_1 + \cdots + E_{2p}$$

which is the same as in (5.14).

The Levi subalgebra $\mathfrak{l}$ is given by

$$\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{1 \leq i < j \leq 2p} \mathfrak{g}_{\pm(e_i,-e_j)} \simeq \mathfrak{sl}(2p, \mathbb{C}) \oplus \mathbb{C}^2.$$

The $\text{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as

$$\mathfrak{g}(2) = \bigoplus_{1 \leq i < j \leq 2p} \mathfrak{g}_{e_i+e_j} \simeq \text{Alt}(2p, \mathbb{C}).$$

Then, the $L_C$-action on $\mathfrak{n}$ is geometrically equivalent to the $(SL(2p, \mathbb{C}) \times \mathbb{C}^\times)$-action on $\text{Alt}(2p, \mathbb{C})$. Similarly to the previous case, $\mathfrak{n} = L_u \cdot S_0$ holds for the subset $S_0$ defined by (5.15). Hence, Theorem 3.6 for Case (D2') has been verified.

5.7.4. Case (D2''). Let us treat the case where $\mathfrak{g} = \mathfrak{so}(4p, \mathbb{C})$ and $\Omega(\mathcal{O}_X) = (0, \ldots, 0, 2)$. This nilpotent orbit $\mathcal{O}_X$ is very even, namely,
\( \mathcal{O}_X \) is the set of all complex matrices with Jordan type \((2^{2p})\). The characteristic element \( H \) forms the same as in (5.14). Then,

\[
I = a \oplus \bigoplus_{1 \leq i < j \leq 2p} g_{(e_i - e_j)} \simeq \mathfrak{sl}(2p, \mathbb{C}) \oplus \mathbb{C},
\]

\[
n = g(2) = \bigoplus_{1 \leq i < j \leq 2p} g_{e_i + e_j} \simeq \text{Alt}(2p, \mathbb{C}).
\]

Similarly to Cases (D2) and (D2'), the equality \( n = L_u \cdot S_0 \) holds for \( S_0 \) defined by (5.15). Hence, Theorem 3.6 for Case (D2′′) has been checked.

5.7.5. Case (D2''''). There are two weighted Dynkin diagrams corresponding to nilpotent orbits \( O_X \) in \( \mathfrak{so}(4p, \mathbb{C}) \) with Jordan type \((2^{2p})\). One is Case (D2'''), the other is \( \Omega(O_X) = (0, \ldots, 0, 2, 0) \), namely, Case (D2'''). For the latter case, \( H \) is of the form

\[
H = E_1 + \cdots + E_{2p-1} - E_{2p}
\]

which is slightly different from the form in (5.14).

The Levi subalgebra \( l \) is given by

\[
l = a \oplus \bigoplus_{1 \leq i < j \leq 2p-1} g_{(e_i - e_j)} \oplus \bigoplus_{1 \leq i \leq 2p-1} g_{(e_i + e_{2p})}.
\]

This means that

\[
l \simeq \mathfrak{sl}(2p, \mathbb{C}) \oplus \mathbb{C}.
\]

The \( \text{ad}(H) \)-eigenspace \( g(2) \) is written as

\[
g(2) = \bigoplus_{1 \leq i < j \leq 2p-1} g_{e_i + e_j} \oplus \bigoplus_{1 \leq i \leq 2p-1} g_{e_{i-2p}}.
\]

Then,

\[
n \simeq \text{Alt}(2p, \mathbb{C}).
\]

Hence, the \( L_C \)-action on \( n \) is geometrically equivalent to the \((SL(2p, \mathbb{C}) \times \mathbb{C}^\times)\)-action on \( \text{Alt}(2p, \mathbb{C}) \), which is a multiplicity-free action.

We take \( S_0 \) as

\[
S_0 = \bigoplus_{1 \leq i \leq p-1} \mathbb{R}E_{e_{2i-1} + e_{2j}} \oplus \mathbb{R}E_{e_{2p-1} - e_{2p}}.
\]

Then, \( S_0 \) is isomorphic to our slice \( A_\mathfrak{p} \) of Table 5.2 for the \((SU(2p) \times \mathbb{T})\)-action via \( n \simeq \text{Alt}(2p, \mathbb{C}) \). By Lemma 5.6 \( n = L_u \cdot S_0 \). Hence, Theorem 3.6 has been verified.
5.7.6. Case (L\text{I}). Let \( \Omega(\mathcal{O}_X) \) satisfy \( m_1 = m_{2p+1} = 1 \) and \( m_i = 0 \) \((i \neq 1, 2p + 1)\) for \( 1 \leq p \leq \frac{n}{2} - 1 \). Then, \( H \) is of the form

\[
H = 2E_1 + E_2 + \cdots + E_{2p+1}.
\]

This \( \mathcal{O}_X \) is the set of all complex matrices with Jordan type \((3, 2^{2p}, 1^{2n-4p-3})\).

The Levi subalgebra \( \mathfrak{l} \) is given by

\[
\mathfrak{l} = a \oplus \bigoplus_{2 \leq i < j \leq 2p+1} \mathfrak{g}_{\epsilon_i - \epsilon_j} \oplus \bigoplus_{2p+2 \leq i < j \leq n} \mathfrak{g}_{\epsilon_i + \epsilon_j},
\]

from which \( \mathfrak{l} \) is isomorphic to

\[
\mathfrak{l} \simeq \mathfrak{sl}(2p, \mathbb{C}) \oplus \mathfrak{so}(2n - 4p - 2, \mathbb{C}) \oplus \mathbb{C}^2.
\]

Next, the \( \text{ad}(H) \)-eigenspace \( \mathfrak{g}(2) \) is written as

\[
\mathfrak{g}(2) = \bigoplus_{2 \leq i < j \leq 2p+1} \mathfrak{g}_{\epsilon_i + \epsilon_j} \oplus \bigoplus_{2p+2 \leq i < j \leq n} \mathfrak{g}_{\epsilon_i - \epsilon_j}.
\]

Then, \( \mathfrak{g}(2) \) is isomorphic to the direct sum \( \text{Alt}(2p, \mathbb{C}) \oplus \mathbb{C}^{2n-4p-2} \). The eigenspace \( \mathfrak{g}(3) \) forms

\[
\mathfrak{g}(3) = \bigoplus_{2 \leq j \leq 2p+1} \mathfrak{g}_{\epsilon_{1} + \epsilon_{j}} \simeq \mathbb{C}^{2p}.
\]

This implies that

\[
\mathfrak{n} \simeq (\text{Alt}(2p, \mathbb{C}) \oplus \mathbb{C}^{2n-4p-2}) \oplus \mathbb{C}^{2p}.
\]

The semisimple part \( \text{SL}(2p, \mathbb{C}) \times \text{SO}(2n - 4p - 2, \mathbb{C}) \) of the Levi subgroup \( L_{\mathbb{C}} \) acts on \( \text{Alt}(2p, \mathbb{C}) \oplus \mathbb{C}^{2n-4p-2} \oplus \mathbb{C}^{2p} \) by

\[
(g, h) \cdot (A, v, w) = (gA^t g, hv, gw),
\]

and its center \((\mathbb{C}^\times)^2\) acts by

\[
(s, t) \cdot (A, v, w) = (t^2 A, st^2 v, st^3 w).
\]

Then, the \( L_{\mathbb{C}} \)-action on \( \mathfrak{n} \) is geometrically equivalent to the decomposable action consisting of the indecomposable \((\text{SL}(2p, \mathbb{C}) \times \mathbb{C}^\times\text{-action on Alt}(2p, \mathbb{C}) \oplus \mathbb{C}^{2p} ((8) of Table 5.1\)) and the irreducible \((\text{SO}(2n - 4p - 2, \mathbb{C}) \times \mathbb{C}^\times\text{-action on \mathbb{C}^{2n-4p-2} ((6) of Table 5.1)\). Hence, this action is a multiplicity-free action.

Here, we define

\[
S'_0 := \bigoplus_{1 \leq i \leq p} \mathbb{R} E_{e_{2i+1} + e_{2i+2}} \oplus \bigoplus_{1 \leq j \leq p} \mathbb{R} E_{e_1 + e_{2j}},
\]

\[
S''_0 := \mathbb{R} E_{e_1 + e_{2p+2}} \oplus \mathbb{R} E_{e_1 - e_{2p+2}}
\]

Then, \( S'_0 \) is isomorphic to the slice \( A_p \oplus T_p \) of Table 5.2 for the \((\text{SU}(2p) \times \mathbb{T})\text{-action on Alt}(2p, \mathbb{C}) \oplus \mathbb{C}^{2p}, \) and \( S''_0 \) to the slice \( D_{1,1} \) for the \((\text{SO}(2n - 4p - 2) \times \mathbb{T})\text{-action on \mathbb{C}^{2n-4p-2} \). We set

\[
S_0 := S'_0 \oplus S''_0 \simeq (A_p \oplus D_{1,1}) \oplus T_p.
\]
Then, it follows from Lemma 5.6 that \( n = L_u \cdot S_0 \). Therefore, Theorem 3.6 has been verified for Case (1B).

5.7.7. Case (1B'). Let us consider the case where \( \mathfrak{g} = \mathfrak{so}(4p+4, \mathbb{C}) \) and \( \Omega(\mathcal{O}_X) = (1, 0, \ldots, 0, 1, 1) \). Then, the characteristic element \( H \) forms the same as in (5.16). This \( \mathcal{O}_X \) is the set of all nilpotent elements with Jordan type \((3, 2^{2p}, 1)\).

The Levi subalgebra \( \mathfrak{l} \) is given by

\[
\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{2 \leq i < j \leq 2p+1} \mathfrak{g}_{\pm(e_i-e_j)} \cong \mathfrak{sl}(2p, \mathbb{C}) \oplus \mathbb{C}^3.
\]

The \( \text{ad}(H) \)-eigenspace \( \mathfrak{g}(2) \) is written as

\[
\mathfrak{g}(2) = \bigoplus_{2 \leq i < j \leq 2p+1} \mathfrak{g}_{e_i+e_j} \oplus \mathfrak{g}_{e_i \pm e_{2p+2}} \cong \text{Alt}(2p, \mathbb{C}) \oplus \mathbb{C}^2.
\]

Further, \( \mathfrak{g}(3) \) is of the form

\[
\mathfrak{g}(3) = \bigoplus_{2 \leq j \leq 2p+1} \mathfrak{g}_{e_1 + e_j} \cong \mathbb{C}^{2p}.
\]

This implies that the nilpotent subalgebra \( \mathfrak{n} \) is isomorphic to

\[
\mathfrak{n} \cong (\text{Alt}(2p, \mathbb{C}) \oplus \mathbb{C}^2) \oplus \mathbb{C}^{2p}.
\]

The subset \( S_0 \) in \( \mathfrak{n} \) defined by (5.17) satisfies \( \mathfrak{n} = L_u \cdot S_0 \). Therefore, Theorem 3.6 has been verified.

5.8. Type \( E_6 \). In Sections 5.8, 5.12, we deal with \( \mathfrak{g} \) of exceptional type. In this subsection, we give a proof of Theorem 5.5 for \( \mathfrak{g} = \mathfrak{e}_6(\mathbb{C}) \). In this case, \( \mathfrak{g}_S \cong \mathfrak{e}_6(\mathbb{C}) \). Then, \( \mathfrak{a}_S^\mathbb{C} = \{ v \in \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_8 : \langle v, e_7 - e_8 \rangle = \langle v, e_7 - e_6 \rangle = 0 \} = \mathbb{C}(e_8 - e_7 - e_6) \oplus \mathbb{C}e_5 \oplus \cdots \oplus \mathbb{C}e_1 \). A root system \( \Delta = \Delta(\mathfrak{g}, \mathfrak{a}) \) is given by \( \Delta = \{ \pm(e_j - e_j) : 1 \leq j < i \leq 5 \} \cup \{ \pm(xe_8 - e_7 - e_6 + \sum_{j=1}^{5}(-1)^{n(j)} e_j) : \sum_{j=1}^{5} n(j) = 0, 2, 4 \} \), where \( n(1), \ldots, n(5) \in \{ 0, 1 \} \). We fix a positive system \( \Delta^+ \) of \( \mathfrak{g} \) as follows:

\[
\Delta^+ = \{ e_i - e_j : 1 \leq j < i \leq 5 \} \cup \{ \frac{i}{2}(e_8 - e_7 - e_6 + \sum_{j=1}^{5}(-1)^{n(j)} e_j) : \sum_{j=1}^{5} n(j) = 0, 2, 4 \}.
\]

The simple roots \( \alpha_1, \ldots, \alpha_6 \) are given by \( \alpha_1 = \frac{1}{2}(e_8 - e_7 - e_5 - e_6 - e_4 - e_3 - e_2 - e_1) \), \( \alpha_2 = e_2 + e_1 \), and \( \alpha_i = e_{i-1} - e_{i-2} \) (3 \leq i \leq 6). The highest root \( \beta \) is written as \( \beta = \frac{1}{2}(e_8 - e_7 - e_5 + e_4 + e_3 + e_2 + e_1) = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 5\alpha_5 + \alpha_6 \).

Let \( \mathcal{O}_X \) be a nilpotent orbit in \( \mathfrak{e}_6(\mathbb{C}) \) with characteristic element \( H = h_0(E_8 - E_7 - E_6) + h_5E_5 + \cdots + h_1E_1 \in \mathfrak{a}_+ \). Then, \( \alpha_1(H) = \frac{1}{2}(3h_6 - h_5 - h_4 - h_3 - h_2 + h_1) \), \( \alpha_2(H) = h_2 + h_1 \), and \( \alpha_i(H) = h_{i-2} = h_{i-3} \) (3, 4, 5, 6). Hence, the weighted Dynkin diagram \( \Omega(\mathcal{O}_X) = (m_1, m_2, m_3, m_4, m_5, m_6) \) is given by \( \frac{1}{2}(3h_6 - h_5 - h_4 - h_3 - h_2 + h_1, h_2 + h_1, h_2 - h_1, h_3 - h_2 - h_1, h_3 - h_4) \).

A nilpotent orbit \( \mathcal{O}_X \) is spherical if and only if \( \Omega(\mathcal{O}_X) \) satisfies one of the following cases:

\[
(E_6 1) \quad \Omega(\mathcal{O}_X) = (0, 1, 0, 0, 0, 0).
\]
(E_62) \( \Omega(O_X) = (1, 0, 0, 0, 0, 1) \).
(E_63) \( \Omega(O_X) = (0, 0, 0, 1, 0, 0) \).

By Lemma 5.4, the height of \( O_X \) equals two for Cases (E_61), (E_62), and three for Case (E_63).

5.8.1. Case (E_61). We consider the case \( \Omega(O_X) = (0, 1, 0, 0, 0, 0) \). Then, \( H \in \mathfrak{a} \) is written by

\[
H = \frac{1}{2}(E_8 - E_7 - E_6 + E_5 + E_4 + E_3 + E_2 + E_1).
\]

This \( O_X \) is the minimal nilpotent orbit with dimension 22.

The Levi subalgebra \( \mathfrak{l} \) is given by

\[
\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{1 \leq j < i \leq 5} \mathfrak{g}_{\pm(e_i - e_j)} \oplus \bigoplus_{\sum_{j=1}^{5} n(j) = 4} \mathfrak{g}_{\pm \frac{1}{2}(e_8 - e_7 - e_6 + \sum_{j=1}^{5} (-1)^{n(j)} e_j)}.
\]

This means that

\[
\mathfrak{l} \simeq \mathfrak{sl}(6, \mathbb{C}) \oplus \mathbb{C}.
\]

The \( \text{ad}(H) \)-eigenspace \( \mathfrak{g}(2) \) is written as

\[
\mathfrak{g}(2) = \mathfrak{g}_{\frac{1}{2}(e_8 - e_7 - e_6 + e_5 + e_4 + e_3 + e_2 + e_1)} \simeq \mathbb{C}.
\]

Hence, the semisimple part \( SL(6, \mathbb{C}) \) of the Levi subgroup \( L_C \) acts trivially on \( \mathbb{C} \), and the center \( \mathbb{C}^\times \) by

\[
s \cdot z = s^2 z.
\]

Then, the \( L_C \)-action on \( \mathfrak{n} \) is geometrically equivalent to the \( \mathbb{C}^\times \)-action on \( \mathbb{C} \). It follows from (1) of Table 5.1 that this action is a multiplicity-free action.

We take the subspace \( S_0 \) as

\[
S_0 := \mathbb{R}E_{\frac{1}{2}(e_8 - e_7 - e_6 + e_5 + e_4 + e_3 + e_2 + e_1)}.
\]

Then, \( S_0 \) is isomorphic to the slice \( \mathbb{R} \) of Table 5.1 for the T-action on \( \mathbb{C} \). By Lemma 5.5 \( \mathfrak{n} = L_u \cdot S_0 \). Hence, Theorem 3.6 has been verified for Case (E_61).
5.8.2. Case \((E_8)\). Let \(\Omega(\mathcal{O}_X)\) be \((1, 0, 0, 0, 0, 1)\). Then, \(H \in \mathfrak{a}\) is
\[
H = E_8 - E_7 - E_6 + E_5.
\]

The Levi subalgebra \(\mathfrak{l}\) is given by
\[
\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{1 \leq j < i \leq 4} \mathfrak{g}_{\pm e_i \pm e_j}.
\]
This means that
\[
\mathfrak{l} \simeq \mathfrak{so}(8, \mathbb{C}) \oplus \mathbb{C}^2.
\]

The \(\text{ad}(H)\)-eigenspace \(\mathfrak{g}(2)\) is written as
\[
\mathfrak{g}(2) = \bigoplus_{\sum_{j=1}^{4} n(j) = 0, 2, 4} \mathfrak{g}_{\frac{1}{2}(e_8 - e_7 - e_6 + e_5 + e_4 + e_3 + e_2 + e_1)}^\pm \mathfrak{g}_{\frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 - e_2 - e_1)}^\pm.
\]
Then, the semisimple part \(\text{Spin}(8, \mathbb{C})\) of \(L_{\mathbb{C}}\) acts on
\[
\mathfrak{n} \simeq \mathbb{C}^8
\]
as the half-spin representation. We know that this action is geometrically equivalent to the \(SO(8, \mathbb{C})\)-action on \(\mathbb{C}^8\). On the other hand, the center \((\mathbb{C}^\times)^2\) of \(L_{\mathbb{C}}\) acts on \(\mathfrak{n}\) by
\[
(s, t) \cdot v = a^2 t^3 v.
\]
Hence, the \(L_{\mathbb{C}}\)-action on \(\mathfrak{n}\) is geometrically equivalent to the \((SO(8, \mathbb{C}) \times \mathbb{C}^\times)^2\)-action on \(\mathbb{C}^8\). It follows from (6) of Table 5.1 that this action is a multiplicity-free action.

We take the subspace \(S_0\) in \(\mathfrak{n}\) as
\[
S_0 = \mathbb{R}E_{\frac{1}{2}(e_8 - e_7 - e_6 + e_5 + e_4 + e_3 + e_2 + e_1)} \oplus \mathbb{R}E_{\frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 - e_2 - e_1)}.
\]
Then, \(S_0\) is isomorphic to the slice \(D_{1,1}\) of Table 5.2 for the \((SO(8) \times \mathbb{T})\)-action on \(\mathbb{C}^8\). By Lemma 5.6, \(\mathfrak{n} = L_u \cdot S_0\). Hence, Theorem 3.6 for Case \((E_8)\) has been verified.

5.8.3. Case \((E_7)\). Let \(\Omega(\mathcal{O}_X)\) be \((0, 0, 0, 1, 0, 0)\). Then, \(H\) is written by
\[
H = E_8 - E_7 - E_6 + E_5 + E_4 + E_3.
\]

The Levi subalgebra \(\mathfrak{l}\) is given by
\[
\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{3 \leq j < i \leq 5} \mathfrak{g}_{\frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 - e_3)}^\pm \mathfrak{g}_{\frac{1}{2}(e_2 - e_1)}^\pm \bigoplus_{3 \leq j < i \leq 5} \mathfrak{g}_{\pm (e_i - e_j)}.
\]
This means that
\[
\mathfrak{l} \simeq \mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C}) \oplus \mathbb{C}.
\]

The \(\text{ad}(H)\)-eigenspace \(\mathfrak{g}(2)\) is written as
\[
\mathfrak{g}(2) = \bigoplus_{3 \leq j < i \leq 5} \mathfrak{g}_{e_i + e_j} \bigoplus \bigoplus_{\sum_{j=3}^{5} n(j) = 1} \mathfrak{g}_{\frac{1}{2}(e_8 - e_7 - e_6 + \sum_{j=3}^{5} (-1)^{n(j)} e_j) \pm \frac{1}{2}(e_2 - e_1)}.
\]
Then, \( g(2) \) is isomorphic to the vector space \( M(3, \mathbb{C}) \). Further, the eigenspace \( g(3) \) forms
\[
g(3) = g_{\frac{1}{2}}(e_8 - e_7 - e_6 + e_5 + e_4 + e_3) \pm \frac{1}{2}(e_2 + e_1),
\]
which is isomorphic to \( \mathbb{C}^2 \). Hence,
\[
n = g(2) \oplus g(3) \simeq M(3, \mathbb{C}) \oplus \mathbb{C}^2.
\]

The semisimple part \( SL(3, \mathbb{C}) \times SL(2, \mathbb{C}) \times SL(3, \mathbb{C}) \) of the Levi subgroup \( L_{\mathbb{C}} \) acts on \( M(3, \mathbb{C}) \oplus \mathbb{C}^2 \) as follows: \( SL(3, \mathbb{C}) \times SL(3, \mathbb{C}) \) acts by
\[
(g_1, g_2) \cdot (A, v) = (g_1 A g_2^{-1}, v),
\]
and \( SL(2, \mathbb{C}) \) acts by
\[
h \cdot (A, v) = (A, hv).
\]
The center \( \mathbb{C}^x \) acts on \( M(3, \mathbb{C}) \oplus \mathbb{C}^2 \) by
\[
s \cdot (A, v) = (s^2 A, s^3 v).
\]
Then, the \( L_{\mathbb{C}} \)-action on \( n \) is geometrically equivalent to the decomposable action consisting of two irreducible actions; the \( (SL(3, \mathbb{C}) \times SL(3, \mathbb{C}) \times \mathbb{C}^x) \)-action on \( M(3, \mathbb{C}) \) \((5) \) of Table 5.1; the \( SL(2, \mathbb{C}) \)-action on \( \mathbb{C}^2 \) \((2) \) of Table 5.1. Thus, this action is a multiplicity-free action.

We take the subspace \( S'_0 \) in \( g(2) \) as
\[
S'_0 := \mathbb{R}E_{\frac{1}{2}}(e_8 - e_7 - e_6 + e_5 + e_4 + e_3 - e_2 - e_1) \oplus \mathbb{R}E_{\frac{1}{2}}(e_8 - e_7 - e_6 + e_5 - e_3 - e_2 + e_1) \oplus \mathbb{R}E_{e_4 + e_3}.
\]
Then, \( S'_0 \) is isomorphic to the slice \( D_3 \) of Table 5.2 for the \( (SU(3) \times SU(3) \times \mathbb{T}) \)-action on \( M(3, \mathbb{C}) \). It follows from Lemma 5.6 that \( g(2) = (SU(3) \times SU(3) \times \mathbb{T}) \cdot S'_0 \). Similarly, the subspace \( S''_0 \) in \( g(3) \) given by
\[
S''_0 := \mathbb{R}E_{-\frac{1}{2}}(e_8 - e_7 - e_6 + e_5 + e_4 + e_3 - e_2 - e_1)
\]
is isomorphic to \( T_1 \) of Table 5.1 and then \( g(3) = SU(2) \cdot S''_0 \). Thus,
\[
S_0 := S'_0 \oplus S''_0 \simeq D_3 \oplus T_1.
\]
satisfies \( n = L_u \cdot S_0 \). Therefore, Theorem 3.6 for Case \( (E_6, 3) \) has been verified.

5.9. Type \( E_7 \). In this subsection, we give a proof of Theorem 3.6 for \( g = e_7(\mathbb{C}) \). In this case, \( g_2 \simeq e_7(\mathbb{T}) \). Then, \( a_8 = \{ v \in \mathbb{R}e_1 + \cdots + \mathbb{R}e_8 : \langle v, e_8 + e_7 \rangle = 0 \} = \mathbb{R}(e_8 - e_7) \oplus \mathbb{R}e_6 \oplus \cdots \oplus \mathbb{R}e_1 \). A root system \( \Delta \equiv \Delta(g, a) \) is \( \Delta = \{ \pm e_i \pm e_j : 1 \leq j < i \leq 6 \} \cup \{ \pm (e_8 - e_7) \} \cup \{ \pm \frac{1}{2}(e_8 - e_7 + \sum_{j=1}^{6}(-1)^{n(j)}e_j) : \sum_{j=1}^{6} n(j) = 1, 3, 5 \} \) where \( n(1), \ldots, n(6) \in \{0, 1\} \).

We fix a positive system \( \Delta^+ \), as \( \Delta^+ = \{ e_i \pm e_j : 1 \leq j < i \leq 6 \} \cup \{ e_8 - e_7 \} \cup \{ \frac{1}{2}(e_8 - e_7 + \sum_{j=1}^{6}(-1)^{n(j)}e_j) : \sum_{j=1}^{6} n(j) = 1, 3, 5 \} \). The simple roots \( \alpha_1, \ldots, \alpha_7 \) are given by \( \alpha_1 = \frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 - e_2 + e_1) \),
\( \alpha_2 = e_2 + e_1 \), and \( \alpha_{j+2} = e_{j+1} - e_j \) \((j = 1, \ldots, 5)\). The highest root is written as \( \beta = e_8 - e_7 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 \).

Let \( \mathcal{O}_X \) be a nilpotent orbit in \( \mathfrak{e}_7(\mathbb{C}) \) with characteristic element \( H = h_7(E_8 - E_7) + h_6E_6 + \cdots + h_1E_1 \in \mathfrak{a}_+ \). Then, \( \alpha_1(H) = 1/2(2h_7 - h_6 - h_5 - h_4 - h_3 - h_2 + h_1) \), \( \alpha_2(H) = h_2 + h_1 \), and \( \alpha_i(H) = h_{i-1} - h_{i-2} \) \((i = 3, 4, 5, 6, 7)\). Hence, the weighted Dynkin diagram \( \Omega(\mathcal{O}_X) = (m_1, \ldots, m_7) \) is given by \( \frac{1}{2}(2h_7 - h_6 - h_5 - h_4 - h_3 - h_2 + h_1), h_2 + h_1, h_2 - h_1, h_3 - h_2, h_4 - h_3, h_5 - h_4, h_6 - h_5 \).

![Figure 5.6. Weighted Dynkin diagram of \( \mathcal{O}_X \) in \( \mathfrak{e}_7(\mathbb{C}) \)](image)

A nilpotent orbit \( \mathcal{O}_X \) is spherical if and only if \( \Omega(\mathcal{O}_X) \) satisfies one of the following cases:

\( (E_71) \quad \Omega(\mathcal{O}_X) = (1, 0, 0, 0, 0, 0, 0) \).
\( (E_72) \quad \Omega(\mathcal{O}_X) = (0, 0, 0, 0, 0, 1, 0) \).
\( (E_73) \quad \Omega(\mathcal{O}_X) = (0, 0, 0, 0, 0, 2) \).
\( (E_74) \quad \Omega(\mathcal{O}_X) = (0, 0, 1, 0, 0, 0) \).
\( (E_75) \quad \Omega(\mathcal{O}_X) = (0, 1, 0, 0, 0, 1) \).

By Lemma 5.4, the height of \( \mathcal{O}_X \) equals two for Cases \((E_71)-(E_73)\) and three for Cases \((E_74), (E_75)\).

5.9.1. *Case* \((E_77)\). Let \( \Omega(\mathcal{O}_X) = (1, 0, 0, 0, 0, 0, 0) \). Then, \( H \) is of the form

\[
H = E_8 - E_7.
\]

This \( \mathcal{O}_X \) is the minimal nilpotent orbit with dimension 34.

The Levi subalgebra \( \mathfrak{l} \) is given by

\[
\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{1 \leq j < i \leq 6} \mathfrak{g}_{\pm e_i \pm e_j}.
\]

This means that

\[
\mathfrak{l} \cong \mathfrak{so}(12, \mathbb{C}) \oplus \mathbb{C}.
\]

The \( \text{ad}(H) \)-eigenspace \( \mathfrak{g}(2) \) is written as

\[
\mathfrak{g}(2) = \mathfrak{g}_{e_8 - e_7} \cong \mathbb{C}.
\]

Hence, the \( L_{\mathbb{C}} \)-action on \( \mathfrak{n} \) is geometrically equivalent to the \( \mathbb{C}^\times \)-action on \( \mathbb{C} \). It follows from (1) of Table 5.1 that this action is a multiplicity-free action.
We take
\[ S_0 = \mathbb{R}E_{e_8-e_7} \simeq \mathbb{R} . \]
By Lemma 5.6, \( S_0 \) satisfies \( n = L_u \cdot S_0 \). Hence, Theorem 3.6 for Case \((E_7,1)\) has been verified.

5.9.2. Case \((E_7,2)\). We consider the case \( \Omega(\mathcal{O}_X) = (0,0,0,0,0,1,0) \).

Then,
\[ H = E_8 - E_7 + E_6 + E_5 . \]

The Levi subalgebra \( l \) is given by
\[ l = a \oplus \bigoplus_{1 \leq j < i \leq 4} \mathfrak{g}_{\pm e_i \pm e_j} \oplus \bigoplus_{\sum_{j=1}^{4} n(j) = 1,3} \mathfrak{g}_{\pm \frac{1}{2}(e_8-e_7-e_6-e_5+\sum_{j=1}^{4}(-1)^{n(j)}e_j)} . \]

This means that
\[ l \simeq \mathfrak{so}(10, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C} . \]

The ad\((H)\)-eigenspace \( \mathfrak{g}(2) \) is written as
\[ \mathfrak{g}(2) = \mathfrak{g}_{e_8-e_7} \oplus \mathfrak{g}_{e_6+e_5} \oplus \bigoplus_{\sum_{j=1}^{4} n(j) = 0,2,4} \mathfrak{g}_{\pm \frac{1}{2}(e_8-e_7+e_6+e_5+\sum_{j=1}^{4}(-1)^{n(j)}e_j)} . \]

Then, \( \mathfrak{g}(2) \) is isomorphic to \( \mathbb{C}^{10} \), from which
\[ n \simeq \mathbb{C}^{10} . \]

The action of the semisimple part \( SO(10, \mathbb{C}) \times SL(2, \mathbb{C}) \) of the Levi subgroup \( L_{\mathbb{C}} \) on \( \mathbb{C}^{10} \) is given as follows: \( SO(10, \mathbb{C}) \) acts by the standard representation; \( SL(2, \mathbb{C}) \) acts trivially, and the action of the center \( \mathbb{C}^\times \) is the scalar multiplication. Then, the \( L_{\mathbb{C}} \)-action on \( n \) is geometrically equivalent to the \((SO(10, \mathbb{C}) \times \mathbb{C}^\times)\)-action on \( \mathbb{C}^{10} \). It follows from (6) of Table 5.1 that this action is a multiplicity-free action.

We take a subspace \( S_0 \) as
\[ S_0 := \mathbb{R}E_{e_8-e_7} \oplus \mathbb{R}E_{e_6+e_5} . \]

Then, \( S_0 \) is isomorphic to the slice \( D_{1,1} \) of Table 5.2 for the \((SO(10) \times \mathbb{T})\)-action via \( n \simeq \mathbb{C}^{10} \). By Lemma 5.6, \( n = L_u \cdot S_0 \). Hence, Theorem 3.6 for Case \((E_7,2)\) has been verified.

5.9.3. Case \((E_7,3)\). Let \( \Omega(\mathcal{O}_X) = (0,0,0,0,0,0,2) \). Then,
\[ H = E_8 - E_7 + 2E_6 . \]

The Levi subalgebra \( l \) is given by
\[ l = a \oplus \bigoplus_{1 \leq j < i \leq 5} \mathfrak{g}_{\pm e_i \pm e_j} \oplus \bigoplus_{\sum_{j=1}^{5} n(j) = 0,2,4} \mathfrak{g}_{\pm \frac{1}{2}(e_8-e_7-e_6+\sum_{j=1}^{5}(-1)^{n(j)}e_j)} . \]
This means that
\[ l \simeq \mathfrak{c}_6(\mathbb{C}) \oplus \mathbb{C}. \]

The ad\((H)\)-eigenspace \( \mathfrak{g}(2) \) is written as
\[ \mathfrak{g}(2) = \bigoplus_{1 \leq j \leq 5} \mathfrak{g}_{e_6 \pm e_j} \oplus \mathfrak{g}_{e_8 - e_7} \oplus \bigoplus_{\sum_{j=1}^5 n(j) = 0, 2, 4} \mathfrak{g}_{\frac{1}{2}(e_8 - e_7 + e_6 + \sum_{j=1}^5 (-1)^n(e_j))}. \]

Then, \( \mathfrak{g}(2) \) is isomorphic to the complexified Jordan algebra \( \mathfrak{J}_C \), from which
\[ n \simeq \mathfrak{J}_C. \]

This implies that the \( L_C \)-action on \( n \) is geometrically equivalent to the \((E_6(\mathbb{C}) \times \mathbb{C}^\times)\)-action on \( \mathfrak{J}_C \). It follows from (7) of Table 5.1 that this action is a multiplicity-free action.

We take the subspace \( S_0 \) in \( n \) as
\[ S_0 = \mathbb{R}E_{e_8 - e_7} \oplus \mathbb{R}E_{e_6 + e_5} \oplus \mathbb{R}E_{e_6 - e_5}. \]

Then, this is isomorphic to the slice \( D_3 \) of Table 5.2 for the \((E_6 \times T)\)-action on \( \mathfrak{J}_C \). By Lemma 5.6, \( S_0 \) satisfies \( n = L_u \cdot S_0 \). Hence, Theorem 3.6 for Case \( (E_7^3) \) has been verified in this case.

5.9.4. Case \((E_7^4)\). We consider the case \( \Omega(\mathcal{O}_X) = (0, 0, 1, 0, 0, 0, 0) \). Then, the characteristic element \( H \) of \( \mathcal{O}_X \) is expressed by
\[ H = \frac{1}{2}(3E_8 - 3E_7 + E_6 + E_5 + E_4 + E_3 + E_2 - E_1). \]

The Levi subalgebra \( l \) is given by
\[ l = \mathfrak{a} \oplus \bigoplus_{2 \leq i \leq 6} \mathfrak{g}_{\pm(e_i + e_1)} \oplus \bigoplus_{2 \leq j < i \leq 6} \mathfrak{g}_{\pm(e_i - e_j)} \oplus \mathfrak{g}_{\pm\frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 - e_2 + e_1)}. \]

This means that
\[ l \simeq \mathfrak{sl}(6, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}. \]

The ad\((H)\)-eigenspace \( \mathfrak{g}(2) \) is written as
\[ \mathfrak{g}(2) = \bigoplus_{\sum_{j=2}^6 n(j) = 1} \mathfrak{g}_{\frac{1}{2}(e_8 - e_7 + e_6 + \sum_{j=2}^6 (-1)^n(e_j))} \oplus \bigoplus_{\sum_{j=2}^6 n(j) = 2} \mathfrak{g}_{\frac{1}{2}(e_8 - e_7 + e_6 + \sum_{j=2}^6 (-1)^n(e_j))}. \]

Then, \( \mathfrak{g}(2) \) is isomorphic to \( \text{Alt}(6, \mathbb{C}) \). Further, \( \mathfrak{g}(3) \) forms
\[ \mathfrak{g}(3) = \mathfrak{g}_{e_8 - e_7} \oplus \mathfrak{g}_{\frac{1}{2}(e_8 - e_7 + e_6 + e_5 + e_4 + e_3 + e_2 - e_1)} \simeq \mathbb{C}^2. \]

Thus,
\[ n \simeq \text{Alt}(6, \mathbb{C}) \oplus \mathbb{C}^2. \]
The semisimple part $SL(6, \mathbb{C}) \times SL(2, \mathbb{C})$ of the Levi subgroup $L_{\mathbb{C}}$ acts on $\text{Alt}(6, \mathbb{C}) \oplus \mathbb{C}^2$ by
\[(g, h) \cdot (A, v) = (gA^t g, hv),\]
and its center $\mathbb{C}^*$ acts by
\[s \cdot (A, v) = (s^2 A, s^3 v).\]
Then, the $L_{\mathbb{C}}$-action on $\mathfrak{n}$ is geometrically equivalent to the decomposable action consisting of the $(SL(6, \mathbb{C}) \times \mathbb{C}^*)$-action on $\text{Alt}(6, \mathbb{C})$ ((4) of Table 5.1) and the $SL(2, \mathbb{C})$-action on $\mathbb{C}^2$ ((2) of Table 5.1). It follows from Lemma 5.6 that this action is a multiplicity-free action.

We take the subspace $S'_0$ in $\mathfrak{g}(2)$ as
\[S'_0 := \mathbb{R}E_{e_8 - e_7} \oplus \mathbb{R}E_{e_6 + e_5 - e_4 - e_3 + e_2 - e_1} \oplus \mathbb{R}E_{\frac{1}{2}(e_8 - e_7 + e_5 - e_4 + e_3 + e_2 - e_1)}.
\]
Then, $S'_0$ is isomorphic to the slice $A_3$ of Table 5.2 for the $(SU(6) \times T)$-action on $\text{Alt}(6, \mathbb{C})$. By Lemma 5.6, $\mathfrak{g}(2) = (SU(6) \times T) \cdot S'_0$. Similarly, the subspace
\[S''_0 := \mathbb{R}E_{e_8 - e_7},
\]
in $\mathfrak{g}(3)$ is isomorphic to $T_1$ and satisfies $\mathfrak{g}(2) = SU(2) \cdot S''_0$. Hence,
\[S_0 := S'_0 \oplus S''_0 \simeq A_3 \oplus T_1
\]
satisfies $\mathfrak{n} = L_{\mathbb{C}} \cdot S_0$. Therefore, Theorem 3.6 has been verified for Case (E74).

5.9.5. Case (E75). Let $\Omega(\mathcal{O}_X)$ be $(0, 1, 0, 0, 0, 0, 1)$. Then,
\[H = \frac{1}{2}(3E_8 - 3E_7 + 3E_6 + E_5 + E_4 + E_3 + E_2 + E_1).
\]
The Levi subalgebra $\mathfrak{l}$ is given by
\[\mathfrak{l} = a \oplus \bigoplus_{1 \leq j < i \leq 5} \mathfrak{g}_{e_i - e_j} \oplus \bigoplus_{\sum_{j=1}^{5} n(j) = 4} \mathfrak{g}_{\frac{1}{2}(e_8 - e_7 + \sum_{j=1}^{5} (-1)^{n(j)} e_j)} \oplus \mathfrak{g}_{\frac{1}{2}(e_8 - e_7 + e_5 + e_4 + e_3 + e_2 + e_1)}.
\]
This implies that
\[\mathfrak{l} \simeq \mathfrak{sl}(6, \mathbb{C}) \oplus \mathbb{C}^2.
\]
The $\text{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as
\[\mathfrak{g}(2) = \bigoplus_{1 \leq j \leq 5} \mathfrak{g}_{e_6 + e_j} \oplus \bigoplus_{\sum_{j=1}^{5} n(j) = 3} \mathfrak{g}_{\frac{1}{2}(e_8 - e_7 + e_5 + \sum_{j=1}^{5} (-1)^{n(j)} e_j)} \oplus \mathfrak{g}_{\frac{1}{2}(e_8 - e_7 + e_5 + e_4 + e_3 + e_2 + e_1)}.
\]
Then, \( g(2) \) is isomorphic to the direct sum \( \text{Alt}(6, \mathbb{C}) \oplus \mathbb{C} \). Further, \( g(3) \) is of the form

\[
g(3) = g_{e_8 - e_7} \oplus \bigoplus_{\sum_{j=1}^5 n(j) = 1} \mathfrak{g}_{(e_8 - e_7 + e_6 + \sum_{j=1}^5 (-1)^{n(j)} e_j)} \cong \mathbb{C}^6.
\]

Hence, \( n \) is isomorphic to

\[
n \cong \text{Alt}(6, \mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}^6.
\]

The semisimple part \( SL(6, \mathbb{C}) \) of the Levi subgroup \( L_\mathbb{C} \) acts on \( \text{Alt}(6, \mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}^6 \) by

\[
g \cdot (A, z, v) = (gA^t g, z, gv),
\]

and its center \( (\mathbb{C}^\times)^2 \) acts by

\[
(s, t) \cdot (A, z, v) = (s^2 t A, s^2 t^3 z, s^3 t^2 v).
\]

This implies that the \( L_\mathbb{C} \)-action on \( n \) is geometrically equivalent to the decomposable action which consists two actions: the indecomposable \( (SL(6, \mathbb{C}) \times \mathbb{C}^\times) \)-action on \( \text{Alt}(6, \mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}^6 \) ((8) of Table 5.1); the \( \mathbb{C}^\times \)-action on \( \mathbb{C} \) ((1) of Table 5.1). By Lemma 5.6 this action is a multiplicity-free action.

We take the subspace \( S'_0 \) as

\[
S'_0 := \mathbb{R} E_{1}(e_8 - e_7 + e_6 + e_5 + e_3 - e_2 - e_1)
\]

\[
\oplus \mathbb{R} E_{\frac{1}{2}}(e_8 - e_7 + e_6 + e_5 + e_3 - e_2 - e_1) \oplus \mathbb{R} E_{6},
\]

and \( S'_1 \) as

\[
S'_1 := \mathbb{R} E_{e_8 - e_7} \oplus \mathbb{R} E_{\frac{1}{2}}(e_8 - e_7 + e_6 + e_5 + e_3 - e_2 + e_1)
\]

\[
\oplus \mathbb{R} E_{\frac{1}{2}}(e_8 - e_7 - e_6 - e_5 - e_4 + e_3 + e_2 + e_1).
\]

Then, the direct sum \( S'_0 \oplus S'_1 \) is isomorphic to the slice \( A_3 \oplus T_3 \) of Table 5.2 for the \( (SU(6) \times T) \)-action on \( \text{Alt}(6, \mathbb{C}) \oplus \mathbb{C} \). Further, the subspace

\[
S''_0 := \mathbb{R} E_{\frac{1}{2}}(e_8 - e_7 - e_6 - e_5 + e_4 - e_3 - e_2 + e_1)
\]

in \( g(2) \) is isomorphic to the slice \( \mathbb{R} \) for the \( T \)-action on \( \mathbb{C} \). We set

\[
S_0 := (S'_0 \oplus S''_0) \oplus S'_1 \simeq (A_3 \oplus \mathbb{R}) \oplus T_3.
\]

By Lemma 5.6 \( S_0 \) satisfies \( n = L_a \cdot S_0 \). Therefore, Theorem 3.6 has been verified for Case (E.1).
and $\alpha_i = e_{i-1} - e_{i-2}$ ($3 \leq i \leq 8$). The highest root $\beta$ is written as $\beta = e_8 + e_7 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$.

Let $\mathcal{O}_X$ be a nilpotent orbit in $\mathfrak{e}_8(\mathbb{C})$ with characteristic element $H = h_8E_8 + \cdots + h_1E_1 \in \mathfrak{a}_+$. Then, $\alpha_1(H) = \frac{1}{2}(h_8 - h_7 - h_6 - h_5 - h_4 - h_3 - h_2 + h_1)$, $\alpha_2(H) = h_2 + h_1$, and $\alpha_i(H) = h_{i-1} - h_{i-2}$ ($i = 3, 4, \ldots, 8$).

Hence, the weighted Dynkin diagram $\Omega(\mathcal{O}_X) = (m_1, \ldots, m_8)$ is given by $(\frac{1}{2}(h_8 - h_7 - h_6 - h_5 - h_4 - h_3 - h_2 + h_1), h_2 + h_1, h_2 - h_1, h_3 - h_2, h_4 - h_4, h_5 - h_4, h_6 - h_5, h_7 - h_6)$.

![Figure 5.7. Weighted Dynkin diagram of $\mathcal{O}_X$ in $\mathfrak{e}_8(\mathbb{C})$](image)

A nilpotent orbit $\mathcal{O}_X$ in $\mathfrak{e}_8(\mathbb{C})$ is spherical if and only if $\Omega(\mathcal{O}_X)$ satisfies one of the following cases:

- (E₈1) $\Omega(\mathcal{O}_X) = (0, 0, 0, 0, 0, 0, 0, 1)$.
- (E₈2) $\Omega(\mathcal{O}_X) = (1, 0, 0, 0, 0, 0, 0, 0)$.
- (E₈3) $\Omega(\mathcal{O}_X) = (0, 0, 0, 0, 0, 1, 0, 0)$.
- (E₈4) $\Omega(\mathcal{O}_X) = (0, 1, 0, 0, 0, 0, 0, 0)$.

It follows from Lemma [5.4] that the the height of $\mathcal{O}_X$ equals two for Cases (E₈1), (E₈2), and three for Cases (E₈3), (E₈4).

5.10.1. Case (E₈7). Let $\Omega(\mathcal{O}_X)$ be $(0, 0, 0, 0, 0, 0, 0, 1)$. Then, $H \in \mathfrak{a}$ is given by

$$H = E_8 + E_7.$$  

This $\mathcal{O}_X$ is the minimal nilpotent orbit in $\mathfrak{e}_8(\mathbb{C})$ with dimension 58.

The Levi subalgebra $\mathfrak{l} = \mathfrak{g}(0)$ is given by

$$\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{1 \leq j < i \leq 6} \mathfrak{g}_{\pm(e_i \pm e_j)} \oplus \mathfrak{g}_{\pm(e_8 - e_7)} \oplus \bigoplus_{\sum_{j=1}^{6} n(j) = 1, 3, 5} \mathfrak{g}_{\pm \frac{1}{2}(e_8 - e_7 + \sum_{j=1}^{6} n(j)e_j)}.$$

This implies that

$$\mathfrak{l} \simeq \mathfrak{e}_7(\mathbb{C}) \oplus \mathbb{C}.$$

The $\text{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is

$$\mathfrak{g}(2) = \mathfrak{g}_{e_8 - e_7} \simeq \mathbb{C}.$$
Thus, the action of the Levi subgroup $L_C$ on $n$ is geometrically equivalent to the $C^*$-action on $C$. It follows from (1) of Table 5.1 that this action is a multiplicity-free action.

We take the subspace $S_0$ as

$$S_0 := \mathbb{R}E_{e_8-e_7} \simeq \mathbb{R}.$$ 

Then, $n = L_u \cdot S_0$. Therefore, Theorem 3.6 has been verified in this case.

5.10.2. Case $(E_8\mathbb{R})$. Let $\Omega(O_X)$ be $(1,0,0,0,0,0,0,0)$. Then, $H = 2E_8$.

The Levi subalgebra $l = g(0)$ is given by

$$l = a \oplus \bigoplus_{1 \leq j < i \leq 7} g_{\pm e_i \pm e_j}.$$ 

This means that

$$l \simeq \mathfrak{so}(14, \mathbb{C}) \oplus \mathbb{C}.$$ 

The $\text{ad}(H)$-eigenspace $g(2)$ is written as

$$g(2) = \bigoplus_{1 \leq j \leq 7} g_{e_8 \pm e_j}.$$ 

Then, $g(2)$ is isomorphic to $C^{14}$, from which

$$n \simeq C^{14}.$$ 

This implies that the $L_C$-action on $n$ is geometrically equivalent to the $(SO(14, \mathbb{C}) \times \mathbb{C}^\times)$-action on $C^{14}$. It follows from (6) of Table 5.1 that this action is a multiplicity-free space.

We take the subspace in $n$ as

$$S_0 := \mathbb{R}E_{e_8+e_7} \oplus \mathbb{R}E_{e_8-e_7}.$$ 

Then, $S_0$ is isomorphic to the slice $D_{1,1}$ of Table 5.2 for the $(SO(14) \times T)$-action on $C^{14}$. By Lemma 5.6, $n = L_u \cdot S_0$. Therefore, Theorem 3.6 for Case $(E_8\mathbb{R})$ has been verified.

5.10.3. Case $(E_8\mathbb{R})$. Let $\Omega(O_X)$ be $(0,0,0,0,0,0,0,1,0)$. Then, $H$ forms

$$H = 2E_8 + E_7 + E_6.$$ 

The Levi subalgebra $l$ is given by

$$l = a \oplus g_{(e_7-e_6)} \oplus \bigoplus_{1 \leq j \leq 5} g_{\pm e_i \pm e_j} \oplus \bigoplus_{\sum_{j=1}^{5} n(j)=0,2,4} g_{\pm \frac{1}{2}(e_8-e_7-e_6+\sum_{j=1}^{5}(-1)^{n(j)}e_j)}.$$ 

This implies that

$$l \simeq e_6(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}.$$
The \( \text{ad}(H) \)-eigenspace \( g(2) \) is written as
\[
g(2) = g(e_7 + e_6) \oplus \bigoplus_{1 \leq j \leq 5} g(e_8 \pm e_j) \oplus \bigoplus_{\sum_{j=1}^5 n(j) = 0, 2, 4} g_{\frac{1}{2}(e_8 + e_7 + e_6 + \sum_{j=1}^5 (-1)^j n(j)e_j)},
\]
Then, \( g(2) \) is isomorphic to the complexified Jordan algebra \( \mathcal{J}_C \). Further,
\[
g(3) = g(e_8 + e_7) \oplus g(e_8 + e_6) \simeq \mathbb{C}^2.
\]
Hence,
\[
n \simeq \mathcal{J}_C \oplus \mathbb{C}^2.
\]
The semisimple part of the Levi subgroup \( L_C \) is isomorphic to \( E_6(\mathbb{C}) \times SL(2,\mathbb{C}) \), which acts on \( \mathcal{J}_C \oplus \mathbb{C}^2 \) diagonally, namely,
\[
(g, h) \cdot (v, w) = (gv, hw).
\]
Further, the center \( \mathbb{C}^\ast \) acts by
\[
s \cdot (v, w) = (s^2 v, s^3 w).
\]
This implies that the \( L_C \)-action on \( n \) is geometrically equivalent to the decomposable action consisting of the irreducible \( (E_6(\mathbb{C}) \times \mathbb{C}^\ast) \)-action on \( \mathbb{C}^{27} \) (7) of Table 5.1 and the irreducible \( SL(2,\mathbb{C}) \)-action on \( \mathbb{C}^2 \) (2) of Table 5.1. By Lemma 5.6, this action is a multiplicity-free action.

We take the subspace in \( g(2) \) as
\[
S'_0 := \mathbb{R}E_{e_8 + e_5} \oplus \mathbb{R}E_{\frac{1}{2}(e_8 + e_7 + e_6 - e_5 + e_4 + e_3 + e_2 - e_1)} \oplus \mathbb{R}E_{\frac{1}{2}(e_8 + e_7 + e_6 - e_5 - e_3 - e_2 + e_1)}.
\]
Then, \( S'_0 \) is isomorphic to the slice \( D_3 \) for the \( (E_6 \times \mathbb{T}) \)-action on \( \mathbb{C}^{27} \).
By Lemma 5.6, \( g(2) = (E_6 \times \mathbb{T}) \cdot S'_0 \). Similarly, the subspace
\[
S''_0 = \mathbb{R}E_{e_8 + e_7},
\]
in \( g(3) \) is isomorphic to \( T_1 \), and then \( g(3) = SU(2) \cdot S''_0 \). Hence,
\[
S_0 := S'_0 \oplus S''_0 \simeq D_3 \oplus T_1
\]
satisfies \( n = L_\alpha \cdot S_0 \), from which Theorem 3.6 for Case (E_8[2]) has been verified.

5.10.4. Case (E_8[4]). Let \( \Omega(O_X) \) be \( (0, 1, 0, 0, 0, 0, 0, 0) \). Then,
\[
H = \frac{1}{2}(5E_8 + E_7 + E_6 + \cdots + E_1).
\]
The Levi subalgebra \( l = g(0) \) is given by
\[
l = a \oplus \bigoplus_{1 \leq j < i \leq 7} g_{\pm(e_i - e_j)} \oplus \bigoplus_{\Sigma_{j=1}^7 n(j) = 6} g_{\frac{1}{2}(e_8 + \sum_{j=1}^7 (-1)^j n(j)e_j)}.
\]
This means that
\[
l \simeq \mathfrak{sl}(8, \mathbb{C}) \oplus \mathbb{C}.
\]
The ad($H$)-eigenspace $\mathfrak{g}(2)$ is written as
\[
\mathfrak{g}(2) = \bigoplus_{1 \leq j \leq 7} \mathfrak{g}_{e_8-e_j} \oplus \bigoplus_{\sum_{j=1}^{7} n(j)=2} \mathfrak{g}_{\frac{1}{2}(e_8+\sum_{j=1}^{7}(-1)^{n(j)}e_j)}.
\]
Then, $\mathfrak{g}(2)$ is isomorphic to $\text{Alt}(8, \mathbb{C})$. Moreover, the eigenspace $\mathfrak{g}(3)$ forms
\[
\mathfrak{g}(3) = \bigoplus_{1 \leq j \leq 7} \mathfrak{g}_{e_8+e_j} \oplus \mathfrak{g}_{\frac{1}{2}(e_8+e_7+e_6+e_5+e_4+e_1+e_2+e_1)} \simeq \mathbb{C}^8.
\]
Hence,
\[
\mathfrak{n} \simeq \text{Alt}(8, \mathbb{C}) \oplus \mathbb{C}^8.
\]
The semisimple part $SL(8, \mathbb{C})$ of the Levi subgroup $L_\mathbb{C}$ acts on $\text{Alt}(8, \mathbb{C}) \oplus \mathbb{C}^8$ diagonally, namely,
\[
g \cdot (A, v) = (gA^t g, gv),
\]
and its center $\mathbb{C}^\times$ acts by
\[
s \cdot (A, v) = (s^2 A, s^3 v).
\]
Then, the $L_\mathbb{C}$-action on $\mathfrak{n}$ is geometrically equivalent to the indecomposable $(SL(8, \mathbb{C}) \times \mathbb{C}^\times)$-action on $\text{Alt}(8, \mathbb{C}) \oplus \mathbb{C}^8$. It follows from (8) of Table 5.2 that this action is a multiplicity-free action.

We take the subspace $S'_0 \subset \mathfrak{g}(2)$ as
\[
S'_0 := \mathbb{R}E_{e_8-e_1} \oplus \mathbb{R}E_{\frac{1}{2}(e_8+e_7+e_6+e_5+e_4+e_3-e_2+e_1)} \\
\quad \oplus \mathbb{R}E_{\frac{1}{2}(e_8+e_7+e_6-e_4+e_3+e_1+e_2+e_1)} \oplus \mathbb{R}E_{\frac{3}{2}(e_8-e_7+e_6+e_4+e_3+e_2+e_1)}
\]
and $S''_0 \subset \mathfrak{g}(3)$ as
\[
S''_0 := \mathbb{R}E_{e_8+e_7} \oplus \mathbb{R}E_{e_8+e_5} \oplus \mathbb{R}E_{e_8+e_3} \oplus \mathbb{R}E_{e_8+e_1}.
\]
We set
\[
S_0 := S'_0 + S''_0.
\]
Then, $S_0$ is isomorphic to the slice $A_4 \times T_4$ of Table 5.2 for the $(SU(8) \times T)$-action on $\text{Alt}(8, \mathbb{C}) \oplus \mathbb{C}^8$. It follows from Lemma 5.6 that $\mathfrak{n} = L_\mathbb{C} \cdot S_0$. Therefore, Theorem 3.6 has been verified.

5.11. **Type $F_4$.** In this subsection, we give a proof of Theorem 3.6 for $\mathfrak{g} = \mathfrak{f}_4(\mathbb{C})$. In this case, $\mathfrak{g}_\mathbb{R} \simeq \mathfrak{f}_4(\mathbb{R})$. Then, $\mathfrak{a}_\mathbb{R} = \mathfrak{e}_1 \oplus \cdots \oplus \mathfrak{e}_4$.

A root system $\Delta$ is $\Delta = \{ \pm e_i \pm e_j : 1 \leq i < j \leq 4 \} \cup \{ \pm e_i : 1 \leq i \leq 4 \}$. We fix a positive system $\Delta^+$ as
\[
\Delta^+ = \{ e_i \pm e_j : 1 \leq i < j \leq 4 \} \cup \{ e_i : 1 \leq i \leq 4 \} \cup \{ \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4) \}.
\]
The simple roots $\alpha_1, \ldots, \alpha_4$ are given by $\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$, $\alpha_2 = e_4$, $\alpha_3 = e_3 - e_4$, and $\alpha_4 = e_2 - e_3$. The highest root $\beta = e_1 + e_2$ is written as $\beta = 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4$.

Let $O_X$ be a nilpotent orbit in $\mathfrak{f}_4(\mathbb{C})$ with characteristic element $H = h_1 H_1 + \cdots + h_4 H_4 \in \mathfrak{a}_+$. Then, the weighted Dynkin diagram
\[ \Omega(O_X) = (m_1, \ldots, m_4) \text{ is given by } \left( \frac{1}{2}(h_1 - h_2 - h_3 - h_4), h_4, h_3 - h_4, h_2 - h_3 \right). \]

**Figure 5.8.** Weighted Dynkin diagram of \( O_X \) in \( f_4(\mathbb{C}) \)

A nilpotent orbit \( O_X \) is spherical if and only if \( \Omega(O_X) \) satisfies one of the following cases:

- \((F_4\,1)\) \( \Omega(O_X) = (0, 0, 0, 1) \).
- \((F_4\,2)\) \( \Omega(O_X) = (1, 0, 0, 0) \).
- \((F_4\,3)\) \( \Omega(O_X) = (0, 0, 1, 0) \).

The height of \( O_X \) equals two for Cases \((F_4\,1)\) and \((F_4\,2)\), and three for Case \((F_4\,3)\).

5.11.1. **Case \((F_4\,1)\).** Let \( \Omega(O_X) = (0, 0, 0, 1) \). Then, \( H \in \mathfrak{a} \) is given by

\[ H = E_1 + E_2. \]

This \( O_X \) is the minimal nilpotent orbit with dimension 16.

The Levi subalgebra \( \mathfrak{l} \) is given by

\[ \mathfrak{l} = \mathfrak{a} \oplus g_{\pm(\pm(e_1-e_2)\pm e_3 \pm e_4)} \oplus g_{\pm(\pm e_1-e_2)} \oplus g_{\pm e_3} \oplus g_{\pm e_4}. \]

This means that

\[ \mathfrak{l} \simeq \mathfrak{sp}(3, \mathbb{C}) \oplus \mathbb{C}. \]

The \( \text{ad}(H) \)-eigenspace \( \mathfrak{g}(2) \) is written as

\[ \mathfrak{g}(2) = g_{e_1+e_2} \simeq \mathbb{C}. \]

This implies that the action of the Levi subgroup \( L_C \) on \( \mathfrak{n} \) is geometrically equivalent to the \( \mathbb{C}^* \)-action on \( \mathbb{C} \). It follows from (1) of Table 5.11 that this action is a multiplicity-free actions.

We take

\[ S_0 = \mathbb{R}E_{e_1+e_2}. \]

Then, it follows from Lemma 5.6 that \( \mathfrak{n} = L_a \cdot S_0 \). Therefore, Theorem 3.6 has been verified for Case \((F_4\,1)\).

5.11.2. **Case \((F_4\,2)\).** Let \( \Omega(O_X) = (1, 0, 0, 0) \). Then,

\[ H = 2E_1. \]

The Levi subalgebra \( \mathfrak{l} \) is given by

\[ \mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{2 \leq i < j \leq 4} g_{\pm e_i \pm e_j} \oplus \bigoplus_{2 \leq i \leq 4} g_{\pm e_i} \simeq \mathfrak{so}(7, \mathbb{C}) \oplus \mathbb{C}. \]
The eigenspace $g(2)$ is of the form:

$$g(2) = g_{e_1} \oplus \bigoplus_{2 \leq i < j \leq 4} g_{e_1 \pm e_j}.$$  

Then,

$$n \simeq \mathbb{C}^7.$$  

This implies that the $L_C$-action on $n$ is geometrically equivalent to the irreducible $(SO(7, \mathbb{C}) \times \mathbb{C}^\times)$-action on $\mathbb{C}^7$. It follows from (6) of Table 5.1 that this action is a multiplicity-free action.

We take the subspace $S_0$ as

$$S_0 = \mathbb{R}E_{e_1+e_2} \oplus \mathbb{R}E_{e_1-e_2}.$$  

Then, this is isomorphic to the slice $D_{1,1}$ of Table 5.2 for the $(SO(7) \times T)$-action on $\mathbb{C}^7$. By Lemma 5.6, $n = L_u \cdot S_0$. Hence, Theorem 3.6 has been verified for Case (F $\bar{4}$ 2).

5.11.3. Case (F $\bar{4}$ 3). Let $\Omega(O_X)$ be $(0, 0, 1, 0)$. Then,

$$H = 2E_1 + E_2 + E_3.$$  

The Levi subalgebra $l$ is given by

$$l = a \oplus g_{e_1} \oplus g_{e_2} \oplus g_{e_3} \oplus \frac{1}{2}(e_1+e_2+e_3) \oplus \frac{1}{2}e_4.$$  

This means that

$$l \simeq \mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}.$$  

The $\text{ad}(H)$-eigenspace $g(2)$ is written as

$$g(2) = g_{e_1} \oplus g_{e_2+e_3} \oplus g_{e_1 \pm e_4} \oplus \frac{1}{2}(e_1+e_2+e_3) \pm \frac{1}{2}e_4.$$  

Then, $g(2)$ is isomorphic to $\text{Sym}(3, \mathbb{C})$. Further, $g(3)$ forms

$$g(3) = g_{e_1+e_2} \oplus g_{e_1+e_3} \simeq \mathbb{C}^2.$$  

Hence,

$$n \simeq \text{Sym}(3, \mathbb{C}) \oplus \mathbb{C}^2.$$  

The semisimple part $SL(3, \mathbb{C}) \times SL(2, \mathbb{C})$ of the Levi subgroup $L_C$ acts on $\text{Sym}(3, \mathbb{C}) \oplus \mathbb{C}^2$ by

$$(g, h) \cdot (A, v) = (gA^tg, hv),$$

and its center $\mathbb{C}^\times$ by

$$s \cdot (A, v) = (sA, s^3v).$$

Then, the $L_C$-action on $n$ is geometrically equivalent to the decomposable action consisting of the $(SL(3, \mathbb{C}) \times \mathbb{C}^\times)$-action on $\text{Sym}(3, \mathbb{C})$ ((3) of Table 5.1) and the $SL(2, \mathbb{C})$-action on $\mathbb{C}^2$ ((2) of Table 5.1). Hence, this action is a multiplicity-free action.

We take the subspace in $g(2)$ as

$$S'_0 := \mathbb{R}E_{e_1+e_4} \oplus \mathbb{R}E_{e_1-e_4} \oplus \mathbb{R}E_{e_2+e_3}.$$
Then, $S_0'$ is isomorphic to the slice $D_3$ of Table 5.2. By Lemma 5.6, $g(2) = (SU(3) \times T) \cdot S_0'$. Similarly,

$$S_0'' := \mathbb{R}E_{e_1 + e_2}$$

is isomorphic to $T_1$ of Table 5.1 and then $g(3) = SU(2) \cdot S_0''$. We set

$$S_0 = S_0' \oplus S_0'' \simeq D_3 \oplus T_1.$$ 

Then, the linear space $S_0$ satisfies $\mathfrak{n} = L_u \cdot S_0$. Therefore, Theorem 3.6 has been verified for Case (F).

5.12. Type $G_2$. In this subsection, we give a proof of Theorem 3.6 for $\mathfrak{g} = \mathfrak{g}_2(\mathbb{C})$. In this case, $\mathfrak{g}_\mathbb{R} \simeq \mathfrak{g}_2(\mathbb{R})$. Then, $a_+^\mathbb{R} = \{v \in \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3 : \langle v, e_1 + e_2 + e_3 \rangle = 0\}$. A root system $\Delta = \Delta(\mathfrak{g}, a)$ is $\Delta = \{\pm(e_i - e_j) : 1 \leq i < j \leq 3\} \cup \{\pm(-2e_1 + e_2 + e_3), \pm(e_1 - 2e_2 - e_3), \pm(-e_1 - e_2 + 2e_3)\}$. We fix a positive system $\Delta^+$ as $\Delta^+ = \{e_1 - e_2, -e_2 + e_3, -e_1 + e_3, -2e_1 + e_2 + e_3, e_1 - 2e_2 + e_3, -e_1 - e_2 + 2e_3\}$. The simple roots are $\alpha_1 := e_1 - e_2$ and $\alpha_2 := -2e_1 + e_2 + e_3$. The highest root $\beta = -e_1 - e_2 + 2e_3$ is written as $\beta = 3\alpha_1 + 2\alpha_2$.

Let $\mathcal{O}_X$ be a nilpotent orbit in $\mathfrak{g}_2(\mathbb{C})$ with characteristic element $H = h_1 E_1 + h_2 E_2 + h_3 E_3 \in a_+$ with $h_1 + h_2 + h_3 = 0$. Then, $\alpha_1(H) = h_1 - h_2$ and $\alpha_2(H) = -2h_1 + h_2 + h_3$. Hence, the weighted Dynkin diagram $\Omega(\mathcal{O}_X) = (m_1, m_2)$ is given by $(h_1 - h_2, -2h_1 + h_2 + h_3)$.

![Figure 5.9. Weighted Dynkin diagram of $\mathcal{O}_X$ in $\mathfrak{g}_2$](image)

A nilpotent orbit $\mathcal{O}_X$ is spherical if and only if $\Omega(\mathcal{O}_X)$ satisfies either Case (G21) or Case (G22):

- (G21) $\Omega(\mathcal{O}_X) = (0, 1)$.
- (G22) $\Omega(\mathcal{O}_X) = (1, 0)$.

It follows from Lemma 5.4 that the height of $\mathcal{O}_X$ equals two if $\Omega(\mathcal{O}_X) = (0, 1)$ (Case (G21)), and three if $\Omega(\mathcal{O}_X) = (1, 0)$ (Case (G22)).

5.12.1. Case (G21). Let $\Omega(\mathcal{O}_X)$ be $(0, 1)$. This $\mathcal{O}_X$ is the minimal nilpotent orbit with dimension six. Then, $H \in a$ is of the form

$$H = \frac{1}{3}(-E_1 - E_2 + 2E_3).$$

The Levi subalgebra $\mathfrak{l}$ is given by

$$\mathfrak{l} = a \oplus \mathfrak{g}_{\pm(e_1 - e_2)}.$$ 

This means that

$$\mathfrak{l} \simeq \mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}.$$
The $\text{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is
\[ \mathfrak{g}(2) = \mathfrak{g}_{-e_1-e_2+2e_3} \simeq \mathbb{C}. \]
Then, the action of the Levi subgroup $L_C$ on $\mathfrak{n}$ is geometrically equivalent to the $\mathbb{C}^*$-action on $\mathbb{C}$. It follows from (1) of Table 5.1 that this action is a multiplicity-free action.

We take the subspace $S_0$ to be
\[ S_0 = \mathbb{R}E_{-e_1-e_2+2e_3}. \]
Then, $\mathfrak{n} = L_u \cdot S_0$. Therefore, Theorem 3.6 has been verified for Case $(G_2^1)$.

5.12.2. Case $(G_2^2)$.

Let $\Omega(O_X)$ be $(1, 0)$. Then,
\[ H = -E_2 + E_3. \]

The Levi subalgebra $\mathfrak{l}$ is given by
\[ \mathfrak{l} = \mathfrak{a} \oplus \mathfrak{g}_{-2e_1+e_2+e_3}. \]
This means that
\[ \mathfrak{l} \simeq \mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}. \]

The $\text{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as
\[ \mathfrak{g}(2) = \mathfrak{g}_{-e_2+e_3} \simeq \mathbb{C}. \]
Further, the eigenspace $\mathfrak{g}(3)$ is
\[ \mathfrak{g}(3) = \mathfrak{g}_{e_1-2e_2+e_3} \oplus \mathfrak{g}_{-e_1+e_2+2e_3} \simeq \mathbb{C}^2. \]
Then,
\[ \mathfrak{n} \simeq \mathbb{C} \oplus \mathbb{C}^2. \]

The semisimple part $\text{SL}(2, \mathbb{C})$ of $L_C$ acts on $\mathbb{C}$ trivially and on $\mathbb{C}^2$ as the standard representation. Its center $\mathbb{C}^*$ acts by
\[ s \cdot (z, v) = (s^2 z, s^3 v). \]
Then, the $L_C$-action on $\mathfrak{n}$ is geometrically equivalent to the decomposable action consisting of the $\mathbb{C}^*$-action on $\mathbb{C}$ ((1) of Table 5.1) and the $\text{SL}(2, \mathbb{C})$-action on $\mathbb{C}^2$ ((2) of Table 5.1). Hence, this action is a multiplicity-free action.

We take
\[ S'_0 := \mathbb{R}E_{-e_2+e_3} \simeq \mathbb{R}. \]
Then, $S'_0$ satisfies $\mathfrak{g}(2) = T \cdot S''_0$. Further, we take the subset $S''_0$ of $\mathfrak{g}(3)$ as
\[ S''_0 := \mathbb{R}E_{-e_1-e_2+2e_3}. \]
Then, $S''_0$ is isomorphic to the slice $T_1$ for the $SU(2)$-action on $\mathbb{C}^2$. It follows from Lemma 5.6 that $\mathfrak{g}(3) = SU(2) \cdot S''_0$. Hence, we set
\[ S_0 := S'_0 \oplus S''_0 \simeq \mathbb{R} \oplus T_1. \]
Then, $S_0$ satisfies $\mathfrak{n} = L_u \cdot S_0$. Therefore, Theorem 3.6 has been verified.

5.13. **Proof of Theorem 3.6.**

In Sections 5.4–5.12, we have given $S_0$ for the $L_u$-action on $\mathfrak{n}$ satisfying the properties (M)–(G) explicitly for each $O_X$ in $\mathfrak{g}$. Then, Theorem 3.6 follows from Lemmas 5.6 and 5.7.

5.14. **Corollary of proof for Theorem 3.6.** Finally, we give two corollaries of the proof of Theorem 3.6 on $O_X$ with $\text{ht}(O_X) = 2$.

The first corollary below is concerned to the property on the $L_u$-action on $\mathfrak{n}$.

**Corollary 5.9.** Let $O_X$ be a nilpotent orbit in a complex simple Lie algebra $\mathfrak{g}$. If $\text{ht}(O_X) = 2$, then we have:

1. The $L_u$-action on $\mathfrak{n}$ is geometrically equivalent to the isotropy representation for some non-compact irreducible Hermitian symmetric space $G/K$, namely, $L_u$ is locally isomorphic to $K$ and the $L_u$-action on $\mathfrak{n}$ is isomorphic to the $K$-action on the tangent space $T_{eK}(G/K)$ at the origin $eK \in G/K$.
2. One can take a slice $S$ for the strongly visible $G_u$-action on $O_X$ satisfying $\dim \mathbb{R} S = \text{rank } G/K$.

**Proof.** The first statement follows from the proof of Theorem 3.6 given in Sections 5.4–5.12 and [22]. The second one is an immediate consequence of [13].

A special case of height two nilpotent orbits is the minimal nilpotent orbit. Here, a nilpotent orbit $O_X$ in the complex semisimple Lie algebra $\mathfrak{g}$ is called *minimal* if the closure of $O_X$ is contained in that of any non-zero nilpotent orbit in $\mathfrak{g}$, namely, $\overline{O_X} \subset \overline{O}$ for any $O' \in \mathcal{N}^*/G_C$. It is known that there exists uniquely the minimal nilpotent orbit in a complex simple Lie algebra $\mathfrak{g}$.

The second corollary gives a new characterization for a complex nilpotent orbit to be minimal by the nilpotent subalgebra $\mathfrak{n}$ as follows.

**Corollary 5.10.** For a nilpotent orbit $O_X$ in a complex simple Lie algebra $\mathfrak{g}$, the following two conditions are equivalent:

1. $O_X$ is minimal.
2. $\dim_{\mathbb{C}} \mathfrak{n} = 1$.

Moreover, the $G_u$-action on the minimal $O_X$ is strongly visible with one-dimensional slice.

**Proof.** Table 5.5 gives a list of the minimal nilpotent orbit for each complex simple Lie algebra $\mathfrak{g}$. This shows that $\dim_{\mathbb{C}} \mathfrak{n} = \dim_{\mathbb{C}} \mathfrak{g}(2) = 1$ for minimal $O_X$. By Theorem 3.6, the $L_u$-action on $\mathfrak{n}$ is strongly visible with slice $S \simeq \mathbb{R}$. Hence, $S = S_0 \cap \mathfrak{n}^\circ \simeq \mathbb{R}^\times$ becomes a slice for the $G_u$-action on $O_X$. 


| \( \mathfrak{g} \) | \( \Omega(\mathcal{O}_X) = (m_1, m_2, \ldots) \) | \( l = \mathfrak{g}(0) \) | \( \mathfrak{g}(2) \) | \( \mathfrak{g}(3) \) | \( S_0 \) |
|---|---|---|---|---|---|
| \( a_{n-1} \) | \( m_p = m_{n-p} = 1 \ (p < \frac{n}{2}) \) | \( \mathfrak{sl}(p, \mathbb{C}) \oplus \mathfrak{sl}(n - 2p, \mathbb{C}) \oplus \mathfrak{sl}(p, \mathbb{C}) \oplus \mathbb{C}^2 \) | \( M(p, \mathbb{C}) \) | \( \{0\} \) | \( \mathbb{R}^p \) |
| \( a_{2p-1} \) | \( m_p = 2 \) | \( \mathfrak{sl}(p, \mathbb{C}) \oplus \mathfrak{sl}(p, \mathbb{C}) \oplus \mathbb{C}^2 \) | \( M(p, \mathbb{C}) \) | \( \{0\} \) | \( \mathbb{R}^p \) |
| \( b_n \) | \( m_1 = 1 \) | \( \mathfrak{so}(2n - 1, \mathbb{C}) \oplus \mathbb{C} \) | \( \mathbb{C}^{2n-1} \) | \( \{0\} \) | \( \mathbb{R}^2 \) |
| \( b_n \) | \( m_{2p} = 1 \) | \( \mathfrak{sl}(2p, \mathbb{C}) \oplus \mathfrak{so}(2n - 4p + 1, \mathbb{C}) \oplus \mathbb{C} \) | \( \text{Alt}(2p, \mathbb{C}) \) | \( \{0\} \) | \( \mathbb{R}^p \) |
| \( b_n \) | \( m_1 = m_{2p+1} = 1 \ (p < \frac{n-1}{2}) \) | \( \mathfrak{sl}(2p, \mathbb{C}) \oplus \mathfrak{so}(2n - 4p - 1, \mathbb{C}) \oplus \mathbb{C}^2 \) | \( \mathbb{C}^{2n-4p-1} \oplus \text{Alt}(2p, \mathbb{C}) \) | \( \mathbb{C}^{2p} \) | \( (\mathbb{R}^2 \oplus \mathbb{R}^p) \oplus \mathbb{R}^p \) |
| \( b_{2p+1} \) | \( m_1 = m_{2p+1} = 1 \) | \( \mathfrak{sl}(2p, \mathbb{C}) \oplus \mathbb{C}^2 \) | \( \mathbb{C} \oplus \text{Alt}(2p, \mathbb{C}) \) | \( \mathbb{C}^{2p} \) | \( (\mathbb{R} \oplus \mathbb{R}^p) \oplus \mathbb{R}^p \) |
| \( c_n \) | \( m_p = 1 \ (p < n) \) | \( \mathfrak{sl}(p, \mathbb{C}) \oplus \mathfrak{sp}(n - p, \mathbb{C}) \oplus \mathbb{C} \) | \( \text{Sym}(p, \mathbb{C}) \) | \( \{0\} \) | \( \mathbb{R}^p \) |
| \( c_n \) | \( m_n = 2 \) | \( \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{C} \) | \( \text{Sym}(n, \mathbb{C}) \) | \( \{0\} \) | \( \mathbb{R}^p \) |
| \( d_n \) | \( m_1 = 2 \) | \( \mathfrak{so}(2n - 2, \mathbb{C}) \oplus \mathbb{C} \) | \( \mathbb{C}^{2n-2} \) | \( \{0\} \) | \( \mathbb{R}^p \) |
| \( d_n \) | \( m_{2p} = 1 \ (p < \frac{n-2}{2}) \) | \( \mathfrak{sl}(2p, \mathbb{C}) \oplus \mathfrak{so}(2n - 4p, \mathbb{C}) \oplus \mathbb{C} \) | \( \text{Alt}(2p, \mathbb{C}) \) | \( \{0\} \) | \( \mathbb{R}^p \) |
| \( d_{2p+1} \) | \( m_2p = m_{2p+1} = 1 \) | \( \mathfrak{sl}(2p, \mathbb{C}) \oplus \mathbb{C}^2 \) | \( \text{Alt}(2p, \mathbb{C}) \) | \( \{0\} \) | \( \mathbb{R}^p \) |
| \( d_{2p} \) | \( m_2p = 2 \) | \( \mathfrak{sl}(2p, \mathbb{C}) \oplus \mathbb{C} \) | \( \text{Alt}(2p, \mathbb{C}) \) | \( \{0\} \) | \( \mathbb{R}^p \) |
| \( d_{2p} \) | \( m_2p-1 = 2 \) | \( \mathfrak{sl}(2p, \mathbb{C}) \oplus \mathbb{C} \) | \( \text{Alt}(2p, \mathbb{C}) \) | \( \{0\} \) | \( \mathbb{R}^p \) |
| \( d_n \) | \( m_1 = m_{2p+1} = 1 \ (p < \frac{n-2}{2}) \) | \( \mathfrak{sl}(2p, \mathbb{C}) \oplus \mathfrak{so}(2n - 4p - 2, \mathbb{C}) \oplus \mathbb{C}^2 \) | \( \mathbb{C}^{2n-4p-2} \oplus \text{Alt}(2p, \mathbb{C}) \) | \( \mathbb{C}^{2p} \) | \( (\mathbb{R}^2 \oplus \mathbb{R}^p) \oplus \mathbb{R}^p \) |
| \( d_{2p+2} \) | \( m_1 = m_{2p+1} = m_{2p+2} = 1 \) | \( \mathfrak{sl}(2p, \mathbb{C}) \oplus \mathbb{C}^3 \) | \( \mathbb{C}^2 \oplus \text{Alt}(2p, \mathbb{C}) \) | \( \mathbb{C}^{2p} \) | \( (\mathbb{R}^2 \oplus \mathbb{R}^p) \oplus \mathbb{R}^p \) |

Table 5.3. Spherical nilpotent orbits and slices for the \( L_n \)-action on \( \mathfrak{n} : \mathfrak{g} \) is of classical type
| $\mathfrak{g}$ | $\Omega(\mathcal{O}_X)$ | $l = \mathfrak{g}(0)$ | $\mathfrak{g}(2)$ | $\mathfrak{g}(3)$ | $S_0$ |
|---|---|---|---|---|---|
| $\mathfrak{e}_6$ | (0, 1, 0, 0, 0, 0) | $\mathfrak{sl}(6, \mathbb{C}) \oplus \mathbb{C}$ | $\mathbb{C}$ | \{0\} | $\mathbb{R}$ |
| $\mathfrak{e}_6$ | (1, 0, 0, 0, 0, 1) | $\mathfrak{so}(8, \mathbb{C}) \oplus \mathbb{C}^2$ | $\mathbb{C}^8$ | \{0\} | $\mathbb{R}^2$ |
| $\mathfrak{e}_6$ | (0, 0, 0, 1, 0, 0) | $\mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}$ | $M(3, \mathbb{C})$ | $\mathbb{C}^2$ | $\mathbb{R}^3 \oplus \mathbb{R}$ |
| $\mathfrak{e}_7$ | (1, 0, 0, 0, 0, 0) | $\mathfrak{so}(12, \mathbb{C}) \oplus \mathbb{C}$ | $\mathbb{C}$ | \{0\} | $\mathbb{R}$ |
| $\mathfrak{e}_7$ | (0, 0, 0, 0, 0, 1) | $\mathfrak{so}(10, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}$ | $\mathbb{C}^{10}$ | \{0\} | $\mathbb{R}^2$ |
| $\mathfrak{e}_7$ | (0, 0, 0, 0, 0, 2) | $\mathfrak{e}_6(\mathbb{C}) \oplus \mathbb{C}$ | $\mathbb{C}^{27}$ | \{0\} | $\mathbb{R}^3$ |
| $\mathfrak{e}_7$ | (0, 0, 0, 0, 0, 0) | $\mathfrak{sl}(6, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}$ | $\text{Alt}(6, \mathbb{C})$ | $\mathbb{C}^2$ | $\mathbb{R}^3 \oplus \mathbb{R}$ |
| $\mathfrak{e}_7$ | (0, 1, 0, 0, 0, 1) | $\mathfrak{sl}(6, \mathbb{C}) \oplus \mathbb{C}^2$ | $\mathbb{C} \oplus \text{Alt}(6, \mathbb{C})$ | $\mathbb{C}^6$ | $(\mathbb{R} \oplus \mathbb{R}^3) \oplus \mathbb{R}^3$ |
| $\mathfrak{e}_8$ | (0, 0, 0, 0, 0, 0, 1) | $\mathfrak{e}_7(\mathbb{C}) \oplus \mathbb{C}$ | $\mathbb{C}$ | \{0\} | $\mathbb{R}$ |
| $\mathfrak{e}_8$ | (1, 0, 0, 0, 0, 0, 0) | $\mathfrak{so}(14, \mathbb{C}) \oplus \mathbb{C}$ | $\mathbb{C}^{14}$ | \{0\} | $\mathbb{R}^2$ |
| $\mathfrak{e}_8$ | (0, 0, 0, 0, 0, 0, 1, 0) | $\mathfrak{e}_6(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}$ | $\mathbb{C}^{27}$ | \{0\} | $\mathbb{R}^3 \oplus \mathbb{R}$ |
| $\mathfrak{e}_8$ | (0, 1, 0, 0, 0, 0, 0, 0) | $\mathfrak{sl}(8, \mathbb{C}) \oplus \mathbb{C}$ | $\text{Alt}(8, \mathbb{C})$ | $\mathbb{C}^8$ | $\mathbb{R}^4 \oplus \mathbb{R}^4$ |
| $\mathfrak{f}_4$ | (0, 0, 0, 1) | $\mathfrak{sp}(3, \mathbb{C}) \oplus \mathbb{C}$ | $\mathbb{C}$ | \{0\} | $\mathbb{R}$ |
| $\mathfrak{f}_4$ | (1, 0, 0, 0) | $\mathfrak{so}(7, \mathbb{C}) \oplus \mathbb{C}$ | $\mathbb{C}^7$ | \{0\} | $\mathbb{R}^2$ |
| $\mathfrak{f}_4$ | (0, 0, 1, 0) | $\mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}$ | $\text{Sym}(3, \mathbb{C})$ | $\mathbb{C}^2$ | $\mathbb{R}^3 \oplus \mathbb{R}$ |
| $\mathfrak{g}_2$ | (0, 1) | $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}$ | $\mathbb{C}$ | \{0\} | $\mathbb{R}$ |
| $\mathfrak{g}_2$ | (1, 0) | $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}$ | $\mathbb{C}$ | \{0\} | $\mathbb{R} \oplus \mathbb{R}$ |

Table 5.4. Spherical nilpotent orbits and slices for the $L_u$-action on $\mathfrak{n}$: $\mathfrak{g}$ is of exceptional type.
Conversely, suppose that \( \dim_{\mathbb{C}} \mathfrak{n} = 1 \). Then, the height of \( \mathcal{O}_X \) has to be equal to two. Indeed, if \( \text{ht}(\mathcal{O}_X) = d > 2 \), then \( \mathfrak{n} \) contains the complex vector subspace \( \mathfrak{g}(2) \oplus \mathfrak{g}(d) \). Obviously, \( \dim_{\mathbb{C}}(\mathfrak{g}(2) \oplus \mathfrak{g}(d)) \geq 2 \), from which we have \( \dim_{\mathbb{C}} \mathfrak{n} \geq 2 \). Let us assume that \( \text{ht}(\mathcal{O}_X) = 2 \). From our case-by-case analysis on the \( L_n \)-action on \( \mathfrak{n} \) (see also Tables 5.3 and 5.4), it turns out that \( \dim_{\mathbb{C}} \mathfrak{n} \neq 1 \) if \( \mathcal{O}_X \) is not minimal. Therefore, we have proved that \( \dim_{\mathbb{C}} \mathfrak{n} = 1 \) only if \( \mathcal{O}_X \) is minimal.

Consequently, Corollary 5.10 has been proved. \( \square \)

| \( \mathfrak{g} \) | \( \Omega(\mathcal{O}_X) \) | \( \mathfrak{g}(2) \) | \( S_0 \) | Section |
|---|---|---|---|---|
| \( \mathfrak{a}_{n-1} \) | (1, 0, \ldots, 0, 1) | \( \mathbb{C} \) | \( \mathbb{R} \) | 5.4.1 |
| \( \mathfrak{b}_n \) | (0, 1, 0, \ldots, 0) | \( \mathbb{C} \) | \( \mathbb{R} \) | 5.5.2 |
| \( \mathfrak{c}_n \) | (1, 0, \ldots, 0) | \( \mathbb{C} \) | \( \mathbb{R} \) | 5.6 |
| \( \mathfrak{d}_n \) | (0, 1, 0, \ldots, 0) | \( \mathbb{C} \) | \( \mathbb{R} \) | 5.7.2 |
| \( \mathfrak{e}_6 \) | (0, 1, 0, 0, 0, 0) | \( \mathbb{C} \) | \( \mathbb{R} \) | 5.8.3 |
| \( \mathfrak{e}_7 \) | (1, 0, 0, 0, 0, 0, 0) | \( \mathbb{C} \) | \( \mathbb{R} \) | 5.9.1 |
| \( \mathfrak{e}_8 \) | (0, 0, 0, 0, 0, 0, 0, 1) | \( \mathbb{C} \) | \( \mathbb{R} \) | 5.10.1 |
| \( \mathfrak{f}_4 \) | (0, 0, 0, 1) | \( \mathbb{C} \) | \( \mathbb{R} \) | 5.11.1 |
| \( \mathfrak{g}_2 \) | (0, 1) | \( \mathbb{C} \) | \( \mathbb{R} \) | 5.12.1 |

Table 5.5. Minimal nilpotent orbit in \( \mathfrak{g} \)

REFERENCES

[1] P. Bala, R. W. Carter, Classes of unipotent elements in simple algebraic groups. I. *Math. Proc. Cambridge Philos. Soc.* 79 (1976), 401–425.
[2] P. Bala, R. W. Carter, Classes of unipotent elements in simple algebraic groups. II. *Math. Proc. Cambridge Philos. Soc.* 80 (1976), 1–17.
[3] C. Benson, G. Ratcliff, A classification of multiplicity free actions, *J. Algebra* 181 (1996), 152–186.
[4] D. Collingwood, W. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold Mathematics Series, Van Nostrand Reinhold Co., New York, 1993.
[5] E. B. Dynkin, Semisimple subalgebras of semisimple Lie algebras, *Mat. Sbornik N.S.* 30 (1952), 349–462 (3 plates).
[6] J. Faraut, E. G. F. Thomas, Invariant Hilbert spaces of holomorphic functions, *J. Lie Theory* 9 (1999), 383–402.
[7] V. Kac, Some remarks on nilpotent orbits, *J. Algebra* 64 (1980), 190–213.
[8] A. W. Knapp, *Lie groups beyond an introduction, Second edition*, Progress in Mathematics, 140, Birkhäuser Boston, Inc., Boston, MA, 2002.
[9] T. Kobayashi, Multiplicity free theorem in branching problems of unitary highest weight modules, in *Proceedings of Symposium on Representation Theory 1997* (ed. K. Mimachi) (1997), 9–17.
[10] T. Kobayashi, Geometry of multiplicity-free representations of \( GL(n) \), visible actions on flag varieties, and trinity, *Acta. Appl. Math.* 81 (2004), 129–146.
[11] T. Kobayashi, Multiplicity-free representations and visible actions on complex manifolds, *Publ. Res. Inst. Math. Sci.* 41 (2005), 497–549, special issue commemorating the fortieth anniversary of the founding of RIMS.
[12] T. Kobayashi, A generalized Cartan decomposition for the double coset space $(U(n_1) \times U(n_2) \times U(n_3))/U(n)/(U(p) \times U(q))$, J. Math. Soc. Japan 59 (2007), 669–691.

[13] T. Kobayashi, Visible actions on symmetric spaces, Transform. Group 12 (2007), 671–694.

[14] T. Kobayashi, Propagation of multiplicity-freeness property for holomorphic vector bundles, in Lie Groups: structure, actions, and representations, 113–140, Progress in Mathematics, 306, Birkhäuser Boston, Inc., Boston, MA, 2013.

[15] T. Kobayashi, Y. Oshima, Classification of symmetric pairs with discretely decomposable restrictions of $(g, K)$-modules, J. Reine Angew. Math. 703 (2015), 201–223.

[16] B. Kostant, The principal three-dimensional subgroup and Betti numbers of a complex simple Lie group, Amer. J. Math. 81 (1959), 973–1032.

[17] A. Leahy, A classification of multiplicity free representations, J. Lie Theory 8 (1998), 367–391.

[18] A. I. Mal’cev, On semi-simple subgroups of Lie groups, Amer. Math. Soc. Translation 1950 (1950), 43 pp.

[19] D. Panyushev, Complexity and nilpotent orbits, Manuscripta Math. 83 (1994), 223–237.

[20] D. Panyushev, On spherical nilpotent orbits and beyond, Ann. Inst. Fourier 49 (1999), 1455–1476.

[21] D. Panyushev, Some amazing properties of spherical nilpotent orbits, Math. Z. 245 (2003), 557–580.

[22] A. Sasaki, Visible actions on irreducible multiplicity-free spaces, Int. Math. Res. Not. IMRN (2009), 3445–3466.

[23] A. Sasaki, A characterization of non-tube type Hermitian symmetric spaces by visible actions, Geom. Dedicata 145 (2010), 151–158.

[24] A. Sasaki, Visible actions on reducible multiplicity-free spaces, Int. Math. Res. Not. IMRN (2011), 885–929.

[25] A. Sasaki, Compatible automorphisms and visible actions on linear spaces in preparation.

[26] T. Springer, R. Steinberg, Conjugacy classes, in Seminar on Algebraic Groups and Related Finite Groups, Lecture Notes in Mathematics, 131, Springer, Berlin, 1970, 167–266.

[27] Y. Tanaka, Visible actions on flag varieties of type C and a generalization of the Cartan decomposition, Tohoku Math. J. 65 (2013), 281–295.

[28] Y. Tanaka, Visible actions on flag varieties of type D and a generalization of the Cartan decomposition, J. Math. Soc. Japan 65 (2013), 931–965.

[29] Y. Tanaka, Visible actions on flag varieties of type B and a generalization of the Cartan decomposition, Bull. Aust. Math. Soc. 88 (2013), 81–97.

[30] Y. Tanaka, Visible actions on flag varieties of exceptional groups and a generalization of the Cartan decomposition, J. Algebra 399 (2014), 170–189.

[31] E. B. Vinberg, B. N. Kimelfeld, Homogeneous domains on flag manifolds and spherical subgroups of semisimple Lie groups, Funct. Anal. Appl., 12 (1978), 168–174.

[32] E. B. Vinberg, Complexity of actions of reductive groups, Functional Anal. Appl. 20 (1986), 1–11.

[33] D. Vogan, Associated varieties and unipotent representations, Progress in Mathematics 101, 315–388, Birkhäuser Boston, Inc., Boston, MA, 1991.
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