The motion of relativistic strings in curved space-times

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Abstract

This paper concerns the motion of a relativistic string in a curved space-time. As a general framework, we first analyze relativistic string equations, i.e., the basic equations for the motion of a one-dimensional extended object in a curved enveloping space-time \((\mathcal{M}, \tilde{g})\), which is a general Lorentzian manifold, and then investigate some interesting properties enjoyed by these equations. Based on this, under suitable assumptions we prove the global existence of smooth solutions of the Cauchy problem for relativistic string equations in the curved space-time \((\mathcal{M}, \tilde{g})\). In particular, we consider the motion of a relativistic string in the Ori's space-time, and give a sufficient and necessary condition guaranteeing the global existence of smooth solutions of the Cauchy problem for relativistic string equations in the Ori's space-time.

Key words and phrases: Curved space-time, Ori’s space-time, relativistic string equations, quasilinear hyperbolic system, Cauchy problem, global smooth solution.

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1 Introduction

This paper concerns the nonlinear dynamics of a relativistic string moving in a curved space-time. It is well known that, in particle physics, the string model is frequently used to study the structure of hardrons. In fact, a free string is a one-dimensional physical object whose motion is represented by a time-like surface. On the other hand, in mathematics, extremal surfaces in a physical space-time include the following four types: space-like, time-like, light-like or mixed types. For the case of space-like extremal (minimal or maximal) surfaces in the Minkowski space-time, we refer to the classical papers by Calabi [5] and by Cheng and Yau [7]. The case of time-like extremal surfaces in the Minkowski space-time has been investigated by several authors (e.g., [3] and [26]). Barbashov, Nesterenko and Chervyakov in [3] study nonlinear partial differential equations describing extremal surfaces in the Minkowski space-time and provide examples with exact solutions. Milnor [26] generates examples that display considerable variety in the shape of entire time-like extremal surfaces in the 3-dimensional Minkowski space-time $\mathbb{R}^{1+2}$ and shows that such surfaces need not be planar.

For the case of extremal surfaces of mixed type, Gu investigates the extremal surfaces of mixed type in the $n$-dimensional Minkowski space-time (cf. [12]) and constructs many complete extremal surfaces of mixed type in the 3-dimensional Minkowski space-time (cf. [13]). Recently, Kong et al. re-study the equations for time-like extremal surfaces in the Minkowski space-time $\mathbb{R}^{1+n}$, which corresponds to the motion of an open string in $\mathbb{R}^{1+n}$ (see [18]-[19]). For the multidimensional version, Hoppe et al. derive the equation for a classical relativistic membrane moving in the Minkowski space-time $\mathbb{R}^{1+3}$, which is a nonlinear wave equation corresponding to the extremal hypersurface equation in $\mathbb{R}^{1+3}$, and give some special classical solutions (cf. [1], [16]). The Cauchy problem with small initial data for the time-like extremal surface equation in the Minkowski space-time has been studied successfully by Lindblad [25] and, by Chae and Huh [8] in a more general framework. Using the null forms in Christodoulou and Klainerman’s style (cf. [8] and [17]), they prove the global existence of smooth solutions for sufficiently small initial data with compact support.

In [22], Kong et al investigate the dynamics of relativistic (in particular, closed) strings moving in the multidimensional Minkowski space-time $\mathbb{R}^{1+n}$ ($n \geq 2$). They first derive a system with $n$ nonlinear wave equations of Born-Infeld type which governs the motion of the string. This system can also be used to describe the extremal surfaces in $\mathbb{R}^{1+n}$. Then they show that this system enjoys some interesting geometric properties. Based on this, they give a sufficient and necessary condition guaranteeing the global existence of extremal surfaces without space-like point for given initial data. This result corresponds to the global propagation of nonlinear waves for the system.
describing the motion of the string in $\mathbb{R}^{1+n}$. Moreover, a great deal of numerical analysis are investigated, and the numerical results show that, in phase space, various topological singularities develop in finite time in the motion of the string. More recently, Kong and Zhang furthermore study the motion of relativistic strings in the Minkowski space $\mathbb{R}^{1+n}$ (see [21]). Surprisingly, they obtain a general solution formula for this complicated system of nonlinear wave equations. Based on this solution formula, they successfully prove that the motion of closed strings is always time-periodic. Moreover, they further extend the solution formula to finite relativistic strings.

However, in a curved space-time there are only few results to obtain (see [10] and Sections 24 and 32 in [2]). Gu [10] shows that the motion of a string can be determined by constructing a certain wave map from the Minkowski plane to the enveloping space-time which is a given Lorentzian manifold. Later, Gu investigates the Cauchy problem for wave maps from $\mathbb{R}^{1+1}$ to $\mathbb{S}^{1+1}$ and proves a theorem on the existence of global smooth solutions (see [11]). Recently, in [15] we consider the motion of relativistic strings in the Schwarzschild space-time, and under suitable assumptions we prove a global existence theorem on smooth solutions of the Cauchy problem for the equations for the motion of relativistic strings with small arc length.

In this paper we consider the nonlinear dynamics of a relativistic string moving in a curved space-time. As a general framework, we first analyze relativistic string equations, i.e., the basic equations for the motion of a one-dimensional extended object in a curved enveloping space-time $(\mathcal{N}, \tilde{g})$ which stands for a general Lorentzian manifold, and then investigate some interesting properties enjoyed by these equations. Based on this, under suitable assumptions we prove the global existence of smooth solutions of the Cauchy problem for relativistic string equations in the curved space-time $(\mathcal{N}, \tilde{g})$. Since the Ori’s space-time has recently received much attention mainly due to the fact that it is a time-machine solution with a compact vacuum core of the Einstein’s field equations (see [27]), we particularly consider the motion of a relativistic string in the Ori’s space-time, and give a sufficient and necessary condition guaranteeing the global existence of smooth solutions of the Cauchy problem for relativistic string equations in the Ori’s space-time.

The paper is organized as follows. In Section 2, we investigate the nonlinear dynamics of a relativistic string moving in a general curved space-time. Section 3 is devoted to the study on the motion of a relativistic string in the Ori’s space-time. The conclusion and discussion are given in Section 4.
2 The motion of relativistic strings in general curved space-times

In this section, we investigate the motion of a one-dimensional extended object in the enveloping space-time \((\mathcal{N}, \tilde{g})\), which stands for a given general Lorentzian manifold.

Since the world sheet of the one-dimensional extended object corresponds to a two-dimensional extremal sub-manifold, denoted by \(\mathcal{M}\), we may choose the local coordinates \((\zeta^0, \zeta^1)\) in \(\mathcal{M}\). For simplicity, we also denote \(\zeta^0 = t, \zeta^1 = \theta\). Let the position vector in the space-time \((\mathcal{N}, \tilde{g})\) be

\[
X(t, \theta) = (x^0(t, \theta), x^1(t, \theta), \cdots, x^n(t, \theta)).
\]

Denote

\[
x^A_{\mu} = \frac{\partial x^A}{\partial \zeta^\mu} \quad \text{and} \quad x^A_{\mu\nu} = \frac{\partial^2 x^A}{\partial \zeta^\mu \partial \zeta^\nu} \quad (A = 0, 1, \cdots, n; \mu, \nu = 0, 1).
\]

Then the induced metric of the sub-manifold \(\mathcal{M}\) can be written as

\[
g_{\mu\nu} = \tilde{g}_{AB} x^A_{\mu} x^B_{\nu} \quad (A, B = 0, 1, \cdots, n; \mu, \nu = 0, 1).
\]

As a result, the corresponding Euler-Lagrange equations for the one-dimensional extended object moving in the space-time \((\mathcal{N}, \tilde{g})\) read

\[
g^\mu\nu \left( x^C_{\mu\nu} + \tilde{\Gamma}^C_{AB} x^A_{\mu} x^B_{\nu} - \Gamma^\rho_{\mu\nu} x^C_{\rho} \right) = 0 \quad (C = 0, 1, \cdots, n),
\]

where \(g^{-1} \equiv (g^{\mu\nu})\) is the inverse of the metric \(g\), \(\tilde{\Gamma}^C_{AB}\) and \(\Gamma^\rho_{\mu\nu}\) stand for the connections of the metric \(\tilde{g}\) and the induced metric \(g\), respectively.

Since we are only interested in the physical motion, we may assume that the sub-manifold \(\mathcal{M}\) is \(C^2\) and time-like, i.e.,

\[
\Delta \triangleq \det g < 0.
\]

This implies that the world sheet of the extended object is time-like, and then the motion satisfies the causality, or say, the motion is physical.

Under the assumption (2.5), the global solution to (2.4) is diffeomorphic to the global solution of the following equations provided with the same initial data (see [1] and [14])

\[
E_C \triangleq g^{\mu\nu} \left( x^C_{\mu\nu} + \tilde{\Gamma}^C_{AB} x^A_{\mu} x^B_{\nu} \right) = 0 \quad (C = 0, 1, \cdots, n).
\]

So in this paper it suffices to investigate the global existence of smooth solutions of the system (2.6) instead of (2.4).
Remark 2.1 The mapping \( \phi \) described by the system (2.6) is essentially a wave map from the Minkowski space \( \mathbb{R}^{1+1} \) to the Lorentzian manifold \( (\mathcal{M}, \tilde{g}) \). Gu \[9\] proved that the solution of the Cauchy problem for the harmonic map \( \phi : \mathbb{R}^{1+1} \to \mathcal{M} \) exists globally, where \( \mathcal{M} \) is a Riemannian manifold. According to the authors’ knowledge, there exists only a few results on the wave map from \( \mathbb{R}^{1+1} \) to a general Lorentzian manifold.

Notice that the system (2.6) can be written in the following form

\[
g_{11}x_t^C - 2g_{01}x_t^\theta + g_{00}x_t^\theta + g_{11}\tilde{\Gamma}_A^C x_t^A x_t^B - 2g_{01}\tilde{\Gamma}_A^B x_t^A x_\theta + g_{00}\tilde{\Gamma}_A^B x_\theta x_\theta = 0. \tag{2.7}
\]

Introduce

\[
u = X, \ v = X_t, \ w = X_\theta \tag{2.8}
\]

and denote

\[
U \triangleq (u, v, w)^T. \tag{2.9}
\]

Then the system (2.7) can be equivalently rewritten as

\[
U_t + AU_\theta + B = 0, \tag{2.10}
\]

where

\[
A = \begin{pmatrix}
0 & 0 & 0 \\
0 & -\frac{2g_{01}}{g_{11}} I_{n+1} & \frac{g_{00}}{g_{11}} I_{n+1} \\
0 & -I_{n+1} & 0
\end{pmatrix}, \tag{2.11}
\]

and

\[
B = (-v^T, \tilde{B}^T, 0)^T_{3(n+1) \times 1}, \tag{2.12}
\]

in which \( \tilde{B} = (\tilde{B}^0, \tilde{B}^1, \ldots, \tilde{B}^n)^T \) and

\[
\tilde{B}^C = \tilde{\Gamma}_A^C x_t^A x_t^B - \frac{2g_{01}}{g_{11}} \tilde{\Gamma}_A^B x_t^A x_\theta + \frac{g_{00}}{g_{11}} \tilde{\Gamma}_A^B x_\theta x_\theta.
\]

By a direct calculation, the eigenvalues of the matrix \( A \) read

\[
\begin{cases}
\lambda_1 = \cdots = \lambda_{n+1} \triangleq \lambda_0 = 0, \\
\lambda_{n+2} = \lambda_{n+3} = \cdots = \lambda_{2n+2} \triangleq \lambda_- = -\frac{g_{01} - \sqrt{g_{01}^2 - 9g_{00}g_{11}}}{g_{11}}, \\
\lambda_{2n+3} = \lambda_{2n+4} = \cdots = \lambda_{3n+3} \triangleq \lambda_+ = -\frac{g_{01} + \sqrt{g_{01}^2 - 9g_{00}g_{11}}}{g_{11}}.
\end{cases}
\]

The right eigenvector corresponding to \( \lambda_i \) (\( i = 1, 2, \ldots, 3n+3 \)) can be chosen as

\[
\begin{cases}
r_i = (e_i, 0, 0)^T \quad (i = 1, \ldots, n+1), \\
r_i = (0, -\lambda_- e_i, -e_i)^T \quad (i = n+2, \ldots, 2n+2), \\
r_i = (0, -\lambda_+ e_i, e_i)^T \quad (i = 2n+3, \ldots, 3n+3),
\end{cases}
\]

where \( e_i \) is the standard unit vector in \( \mathbb{R}^{n+1} \).
where
\[ e_i = (0, \cdots, 0, 1, 0, \cdots, 0) \quad (i = 1, \cdots, n + 1). \]

While, the left eigenvector corresponding to \( \lambda_i \) \((i = 1, 2, \cdots, 3n+3)\) can be taken as
\[
\begin{align*}
    l_i &= (e_i, 0, 0) \quad (i = 1, \cdots, n + 1), \\
    l_i &= (0, e_i-(n+1), \lambda_i e_i-(n+1)) \quad (i = n + 2, \cdots, 2n + 2), \\
    l_i &= (0, e_i-(2n+2), \lambda_i e_i-(2n+2)) \quad (i = 2n + 3, \cdots, 3n + 3).
\end{align*}
\] (2.16)

**Proposition 2.1** Under the assumption (2.5), the system (2.10) is a non-strictly hyperbolic system with \(3(n + 1)\) eigenvalues (see (2.14)), and the corresponding right (resp. left) eigenvectors can be chosen as (2.15) (resp. (2.16)).

**Proposition 2.2** Under the assumption (2.5), the system (2.10) is linearly degenerate in the sense of Lax (see [24]).

**Proof.** Obviously, it holds that
\[
\nabla \lambda_0 \cdot r_i = 0 \quad (i = 1, 2, \cdots, n + 1).
\]

We next calculate the invariants \( \nabla \lambda_- r_i \) \((i = n + 2, \cdots, 2n + 2)\) and \( \nabla \lambda_+ r_i \) \((i = 2n + 3, \cdots, 3n + 3)\).

In fact, for every \( C \in \{0, 1, 2, \cdots, n\}\), by a direct calculation we obtain
\[
\begin{align*}
    \frac{\partial \lambda_-}{\partial v_C} &= \tilde{g}_{CB} w^B \frac{\lambda_-}{\sqrt{g_{01} - g_{00}g_{11}}} + \tilde{g}_{CB} v^B \frac{1}{\sqrt{g_{01} - g_{00}g_{11}}} \\
    \frac{\partial \lambda_-}{\partial w_C} &= \tilde{g}_{CB} w^B \frac{\lambda_-^2}{\sqrt{g_{01} - g_{00}g_{11}}} + \tilde{g}_{CB} v^B \frac{\lambda_-}{\sqrt{g_{01} - g_{00}g_{11}}}.
\end{align*}
\] (2.17) and (2.18)

Then we have
\[
\nabla \lambda_- \cdot r_{n+2+C} = -\lambda_+ \frac{\partial \lambda_-}{\partial v_C} + \frac{\partial \lambda_-}{\partial w_C} = 0 \quad (C = 0, 1, 2, \cdots, n). \] (2.19)

Similarly, we can prove
\[
\nabla \lambda_+ \cdot r_{2n+3+C} = -\lambda_- \frac{\partial \lambda_+}{\partial v_C} + \frac{\partial \lambda_+}{\partial w_C} = 0 \quad (C = 0, 1, 2, \cdots, n). \] (2.20)

Thus, the proof is completed. \(\square\)

**Theorem 2.1** Under the assumption (2.5), \( \lambda_- \) (resp. \( \lambda_+ \)) is a Riemann invariant corresponding to \( \lambda_+ \) (resp. \( \lambda_- \)). Moreover, these two Riemann invariants satisfy
\[
\begin{align*}
    \frac{\partial \lambda_-}{\partial t} + \lambda_+ \frac{\partial \lambda_-}{\partial \theta} &= 0, \\
    \frac{\partial \lambda_+}{\partial t} + \lambda_- \frac{\partial \lambda_+}{\partial \theta} &= 0.
\end{align*}
\] (2.21)
Proof. Multiplying (2.10) by the left eigenvectors given by (2.16) leads to

\[
\begin{align*}
\begin{cases}
v_t^C + \lambda_- v_\theta^C + \lambda_+ (w_t^C + \lambda_- w_\theta^C) + \bar{B}_C^C = 0, \\
v_t^C + \lambda_+ v_\theta^C + \lambda_- (w_t^C + \lambda_+ w_\theta^C) + \bar{B}_C^C = 0.
\end{cases}
\end{align*}
\tag{2.22}
\]

Noting (2.19) and using (2.22), we have

\[
\begin{align*}
\frac{\partial \lambda_-}{\partial t} + \lambda_+ \frac{\partial \lambda_-}{\partial \theta} &= \frac{\partial}{\partial t}(v^C + \lambda_- w^C) + \frac{\partial}{\partial \theta}(v^C + \lambda_+ w^C) \\
&= \frac{\partial}{\partial t}(v^C + \lambda_+ w^C) - \frac{\partial}{\partial t}(v^C + \lambda_- w^C).
\end{align*}
\tag{2.23}
\]

By a direct calculation,

\[
\frac{\partial \lambda_-}{\partial u^C} = \frac{1}{2\sqrt{g_{01} - g_{00}g_{11}}} \frac{\partial \tilde{\Gamma}_{AB}}{\partial u^C}(v^A + \lambda_- w^A)(v^B + \lambda_- w^B).
\tag{2.24}
\]

On the other hand, notice that (2.13) can be rewritten as

\[
\bar{B}_C^C = \tilde{\Gamma}_{AB}^C(v^A + \lambda_- w^A)(v^B + \lambda_+ w^B).
\tag{2.25}
\]

Then, substituting (2.17), (2.24) and (2.25) into (2.23) yields

\[
\frac{\partial \lambda_-}{\partial t} + \lambda_+ \frac{\partial \lambda_-}{\partial \theta} = 0.
\]

Similarly, we can prove

\[
\frac{\partial \lambda_+}{\partial t} + \lambda_- \frac{\partial \lambda_+}{\partial \theta} = 0.
\]

Thus, the proof is completed. \(\square\)

Remark 2.2 The system (2.21) is a 2 \(\times\) 2 quasilinear hyperbolic system with linearly degenerate characteristic fields, it plays an important role in our argument.

Introduce

\[
p^C = v^C + \lambda_- w^C, \quad q^C = v^C + \lambda_+ w^C \quad (C = 0, 1, \ldots, n).
\tag{2.26}
\]

Then the system (2.10) can be equivalently rewritten as

\[
\begin{align*}
\begin{cases}
\frac{\partial p^C}{\partial t} + \lambda_+ \frac{\partial p^C}{\partial \theta} &= -\tilde{\Gamma}_{AB}^C(u^A)p^B, \\
\frac{\partial q^C}{\partial t} + \lambda_- \frac{\partial q^C}{\partial \theta} &= -\tilde{\Gamma}_{AB}^C(u^A)q^B.
\end{cases}
\end{align*}
\tag{2.27}
\]

Remark 2.3 Noting (2.21) and (2.27), we observe that, once one can solve \(\lambda_\pm\) from the system (2.21), then (2.27) becomes a semilinear hyperbolic system of first order.
Moreover, by calculations we have
\[
\tilde{g}_{AB} p^A p^B = \tilde{g}_{AB} (v^A + \lambda w^A)(v^B + \lambda w^B) \\
= \tilde{g}_{AB} v^A v^B + \lambda \tilde{g}_{AB} w^A v^B + \lambda \tilde{g}_{AB} v^A w^B + \lambda^2 \tilde{g}_{AB} w^A w^B \\
= g_{00} + 2\lambda g_{01} + \lambda^2 g_{11} \equiv 0
\]
and
\[
\tilde{g}_{AB} q^A q^B \equiv 0.
\]
Thus we have proved

**Proposition 2.3.** \( p \) and \( q \) are two null vectors, i.e., it holds that
\[
\tilde{g}_{AB} p^A p^B \equiv 0 \quad \text{and} \quad \tilde{g}_{AB} q^A q^B \equiv 0. \tag{2.28}
\]

At the end of this section, we consider the global existence of smooth solutions of the relativistic string equations in a general curved space-time.

Consider the Cauchy problem for the system (2.6) (or equivalently, (2.7)) with the initial data
\[
x^C(0, \theta) = \varphi^C(\theta), \quad x^C_t(0, \theta) = \psi^C(\theta) \quad (C = 0, 1, \ldots, n),
\]
where \( \varphi^C(\theta) \) are \( C^2 \)-smooth functions with bounded \( C^2 \)-norm, while \( \psi^C(\theta) \) are \( C^1 \)-smooth functions with bounded \( C^1 \)-norm. In physics, \( \varphi(\theta) = (\varphi^0(\theta), \varphi^1(\theta), \ldots, \varphi^n(\theta)) \) and \( \psi(\theta) = (\psi^0(\theta), \psi^1(\theta), \ldots, \psi^n(\theta)) \) stand for the initial position and the initial velocity of the string under consideration, respectively.

Introduce
\[
\Lambda_{\pm}(\theta) = \frac{-g_{01}[\varphi, \psi](\theta) \pm \sqrt{(g_{01}[\varphi, \psi](\theta))^2 - g_{00}[\varphi, \psi](\theta)g_{11}[\varphi, \psi](\theta)}}{g_{11}[\varphi, \psi](\theta)}
\]
and
\[
\mathcal{L}(\theta) = g_{00}[\varphi, \psi](\theta)g_{11}[\varphi, \psi](\theta) - (g_{01}[\varphi, \psi](\theta))^2,
\]
where
\[
g_{00}[\varphi, \psi](\theta) = \tilde{g}_{AB}(\varphi)\varphi^A \psi^B, \quad g_{01}[\varphi, \psi](\theta) = \tilde{g}_{AB}(\varphi)\varphi^A \psi^B, \quad g_{11}[\varphi, \psi](\theta) = \tilde{g}_{AB}(\varphi)\varphi^A \psi^B.
\]

In fact, in physics \( \Lambda_{\pm}(\theta) \) stand for the characteristic propagation speeds of the point \( \theta \) at the initial time, and \( \mathcal{L}(\theta) \) denotes the Lagrangian energy density.

For given \( \varphi \) and \( \psi \), we consider the Cauchy problem for the system (2.21) with the initial data
\[
t = 0 : \quad \lambda_{\pm} = \Lambda_{\pm}(\theta), \quad \tag{2.33}
\]
where $\Lambda_{\pm}(\theta)$ are defined by (2.30). Since we only consider the physical motion, it is natural to assume that

$$\Lambda_{-}(\theta) < \Lambda_{+}(\theta), \quad \forall \theta \in \mathbb{R}. \quad (2.34)$$

The condition (2.34) is equivalent to the fact that the assumption (2.5) is satisfied at the initial time, i.e., the motion is physical at the time $t = 0$.

Under the assumption (2.34), by Kong and Tsuji [20], the Cauchy problem (2.21), (2.33) has a unique global $C^1$ solution $\lambda_{\pm} = \lambda_{\pm}(t, \theta)$ defined on $\mathbb{R}^+ \times \mathbb{R}$ if and only if, for every fixed $\theta_2 \in \mathbb{R}$, it holds that

$$\Lambda_{-}(\theta_1) < \Lambda_{+}(\theta_2), \quad \forall \theta_1 < \theta_2. \quad (2.35)$$

In fact, the condition (2.35) guarantees that the motion is always physical for all time $t \in \mathbb{R}^+$.

By the same method as in He and Kong [15], we can prove the following theorem.

**Theorem 2.2** Suppose that $\tilde{g}$ is a Lorentzian metric, $\varphi(\theta)$ is a $C^2$-smooth vector-valued function with bounded $C^2$-norm and $\psi(\theta)$ is a $C^1$-smooth vector-valued function with bounded $C^1$-norm. Suppose furthermore that the assumptions (2.34) and (2.35) are satisfied. Then there exists a positive constant $\varepsilon$ such that the Cauchy problem (2.6), (2.29) admits a unique global $C^2$-smooth solution $x^C = x^C(t, \theta)$ for all $t \in \mathbb{R}^+$, provided that

$$\int_{-\infty}^{\infty} \left| \frac{d\varphi^C(\theta)}{d\theta} \right| d\theta \leq \varepsilon \quad \text{and} \quad \int_{-\infty}^{\infty} \left| \psi^C(\theta) \right| d\theta \leq \varepsilon. \quad (2.36)$$

**Remark 2.4** In Theorem 2.2, the constant $\varepsilon$ only depends on the $C^2$-norm of $\varphi$ and the $C^1$-norm of $\psi$. The first inequality in (2.36) implies that the BV-norm of $\varphi^C(\theta)$ is small, that is, the arc length of the initial string is small; while the second inequality in (2.36) implies that the $L^1$-norm of the initial velocity is small. The physical meaning of Theorem 2.2 is as follows: for a string with small arc length, the smooth motion exists globally (or say, no singularity appears in the whole motion process), provided that the $L^1$-norm of the initial velocity is small. In geometry, Theorem 2.2 essentially gives a global existence result on smooth solutions of a wave map from the Minkowski space-time $\mathbb{R}^{1+1}$ to a general curved space-time (cf. [10]).

### 3 The motion of relativistic strings in Ori’s space-time

Since the Ori’s space-time has recently received much attention mainly due to the fact that it is a time-machine solution with a compact vacuum core of the Einstein’s field equations, in this section we mainly investigate the motion of a relativistic string in the Ori’s space-time, and give
a sufficient and necessary condition guaranteeing the global existence of smooth solutions of the Cauchy problem for relativistic string equations in the Ori’s space-time.

As discussed in Section 2, under the assumption (2.34), the Cauchy problem (2.21), (2.33) has a unique global $C^1$ solution $\lambda_\pm(\theta,t)$ on $\mathbb{R}^+ \times \mathbb{R}$ if and only if, for every fixed $\theta_2 \in \mathbb{R}$, (2.35) holds. Moreover, on the existence domain of the solution, it always holds that

$$\lambda_-(t,\theta) < \lambda_+(t,\theta), \quad \forall (t,\theta) \in \mathbb{R}^+ \times \mathbb{R}. \quad (3.1)$$

See Kong and Tsuji [20]. Therefore, in what follows, we assume that (2.34) and (2.35) are always satisfied.

On the other hand, it follows from Serre [28] that the solution $(\lambda_-, \lambda_+)$ of the Cauchy problem (2.21), (2.33) satisfies the following identity

$$\partial_t \left( \frac{2}{\lambda_+ - \lambda_-} \right) + \partial_\theta \left( \frac{\lambda_+ + \lambda_-}{\lambda_+ - \lambda_-} \right) = 0. \quad (3.2)$$

This allows us to introduce the following transformation of the variables

$$(t,\theta) \rightarrow (t,\vartheta), \quad (3.3)$$

where $\vartheta = \vartheta(t,\theta)$ is given by

$$\begin{align*}
\vartheta(0,\theta) &= \Theta_0(\theta) \triangleq \int_{\theta_0}^{\theta} \frac{2}{\lambda_+(\zeta) - \lambda_-(\zeta)} d\zeta, \quad \forall \theta \in \mathbb{R},
\end{align*} \quad (3.4)$$

The following lemma comes from He and Kong [15].

**Lemma 3.1** Under the assumptions (2.34) and (2.35), the mapping defined by (3.3)-(3.4) is globally diffeomorphic; moreover, it holds that

$$\frac{\partial}{\partial t} + \lambda_+ \frac{\partial}{\partial \vartheta} = \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial t} + \lambda_- \frac{\partial}{\partial \vartheta} = \frac{\partial}{\partial t} - \frac{\partial}{\partial \theta}. \quad (3.5)$$

**Remark 3.1** The mapping defined by (3.3)-(3.4) is somewhat similar to the transformation between the Euler version and Lagrange version for one-dimensional gas dynamics.

By Lemma 3.1, under the coordinates $(t,\vartheta)$, the system (2.24) can be equivalently rewritten as

$$\begin{align*}
\frac{\partial \bar{p}^C}{\partial t} + \frac{\partial \bar{p}^C}{\partial \vartheta} &= -\tilde{\Gamma}_{AB}^C(\bar{u}) \bar{p}^A q^B, \\
\frac{\partial \bar{q}^C}{\partial t} + \frac{\partial \bar{q}^C}{\partial \vartheta} &= -\tilde{\Gamma}_{AB}^C(\bar{u}) \bar{p}^A q^B, \quad (C = 0,1,\ldots, n),
\end{align*} \quad (3.6)$$

where $\bar{u}^C(t,\vartheta) = u^C(t,\theta)$, $\bar{p}^C(t,\vartheta) = p^C(t,\theta)$ and $\bar{q}^C(t,\vartheta) = q^C(t,\theta)$, respectively.
Introduce the light-cone coordinates

\[ \xi = t + \vartheta, \quad \eta = t - \vartheta. \]  

(3.7)

Then, under the coordinates \((\xi, \eta)\), the system (3.6) can be rewritten in the following form

\[
\begin{align*}
\frac{\partial \tilde{u}^C}{\partial \xi} + \tilde{\Gamma}_{AB}(\tilde{u}) \tilde{p}^A \tilde{q}^B = 0, \\
\frac{\partial \tilde{q}^C}{\partial \eta} + \tilde{\Gamma}_{AB}(\tilde{u}) \tilde{p}^A \tilde{q}^B = 0
\end{align*}
\]

(3.8)

where \(\tilde{u}^C(\xi, \eta) = \bar{u}^C(t, \vartheta)\), \(\tilde{p}^C(\xi, \eta) = \bar{p}^C(t, \vartheta)\) and \(\tilde{q}^C(\xi, \eta) = \bar{q}^C(t, \vartheta)\), respectively.

On the other hand, noting (2.8), (2.26), (3.5) and using (3.7) gives

\[ \bar{p}(\xi, \eta) = \bar{u}_\eta, \quad \bar{q}(\xi, \eta) = \bar{u}_\xi. \]

(3.9)

Then, the system (3.8) can be equivalently rewritten as

\[
\frac{\partial^2 \bar{u}^C}{\partial \xi \partial \eta} + \bar{\Gamma}_{AB}(\bar{u}) \bar{u}_\eta ^A \bar{u}_\xi ^B = 0 \quad (C = 0, 1, \cdots , n).
\]

(3.10)

The equations in (3.10) are nothing but the equations for the wave map from the Minkowski plane \(\mathbb{R}^{1+1}\) to the Lorentzian manifold \((\mathcal{N}, \bar{g})\). Here we would like to mention that, when the target manifold is Riemannian instead of Lorentzian, the global existence of smooth solutions to the system (3.10) has been proved by Gu [9] successfully.

Recently, Ori [27] presented a class of curved space-time vacuum solutions which develop closed timelike curves at some particular moment, and then used those vacuum solutions to construct a time-machine model. His solution reads

\[ ds^2 = dx^2 + dy^2 - 2dzdt + |f(x, y, z) - t|dz^2, \]

(3.11)

where \(f(x, y, z)\) is an arbitrary function (probably periodic in \(z\)) satisfying

\[ f_{xx} + f_{yy} = 0. \]

(3.12)

In metric (3.11), \((t, x, y, z)\) stands for the local coordinates \((x^0, x^1, x^2, x^3)\) of the enveloping space-time, i.e., the Ori’s space-time in the present situation, and by (3.8), also the coordinates \((u^0, u^1, u^2, u^3)\).

For the case of the Ori’s space-time, the system (3.8) becomes

\[
\begin{align*}
\frac{\partial \bar{p}^0}{\partial \xi} + \frac{1}{2} \left( \bar{p}^0 \bar{q}^3 + \bar{p}^3 \bar{q}^0 \right) - \frac{1}{2} f_x (\bar{p}^1 \bar{q}^3 + \bar{p}^3 \bar{q}^1) \\
- \frac{1}{2} f_y (\bar{p}^2 \bar{q}^3 + \bar{p}^3 \bar{q}^2) + \frac{1}{2} (t - f - f_z) \bar{p}^3 \bar{q}^3 = 0, \\
\frac{\partial \bar{p}^1}{\partial \xi} + \frac{1}{2} f_x \bar{p}^3 \bar{q}^3 = 0, \\
\frac{\partial \bar{p}^2}{\partial \xi} - \frac{1}{2} f_y \bar{p}^3 \bar{q}^3 = 0, \\
\frac{\partial \bar{p}^3}{\partial \xi} + \frac{1}{2} \bar{p}^3 \bar{q}^3 = 0
\end{align*}
\]

(3.13)
and

\[
\begin{align*}
\frac{\partial \bar{u}^0}{\partial \eta} + \frac{1}{2}(\bar{p}^0 \bar{q}^3 + \bar{p}^3 \bar{q}^0) - \frac{1}{2} f_x(\bar{p}^1 \bar{q}^3 + \bar{p}^3 \bar{q}^1) \\
- \frac{1}{2} f_x(\bar{p}^2 \bar{q}^3 + \bar{p}^3 \bar{q}^2) + \frac{1}{2}(t - f - f_z) \bar{p}^3 \bar{q}^3 = 0,
\end{align*}
\]

(3.14)

while the system (3.10) becomes

\[
\begin{align*}
\frac{\partial^2 \bar{u}^0}{\partial \xi \partial \eta} + \frac{1}{2}(\bar{u}^0 \bar{q}_\xi^3 + \bar{u}^3 \bar{q}_\xi^0) - \frac{1}{2} f_x(\bar{u}^1 \bar{q}_\xi^3 + \bar{u}^3 \bar{q}_\xi^1) \\
- \frac{1}{2} f_x(\bar{u}^2 \bar{q}_\xi^3 + \bar{u}^3 \bar{q}_\xi^2) + \frac{1}{2}(t - f - f_z) \bar{u}^3 \bar{q}_\xi^3 = 0,
\end{align*}
\]

(3.15)

As in Ori [27], for concreteness we now specialize to a simple example. Take

\[
f = a(x^2 - y^2)
\]

(3.16)

for some positive constant \(a\). This yields an empty curved space-time, locally isometric to a linearly polarized plane wave. For this concrete situation, the system (3.15) becomes

\[
\begin{align*}
\frac{\partial^2 \bar{u}^0}{\partial \xi \partial \eta} + \frac{1}{2}(\bar{u}^0 \bar{q}_\xi^3 + \bar{u}^3 \bar{q}_\xi^0) - a \bar{u}^1 (\bar{u}^1 \bar{q}_\xi^3 + \bar{u}^3 \bar{q}_\xi^1) \\
+ a \bar{u}^2 (\bar{u}^2 \bar{q}_\xi^3 + \bar{u}^3 \bar{q}_\xi^2) + \frac{1}{2}(\bar{u}^0 - a \left[(\bar{u}^1)^2 - (\bar{u}^2)^2\right]) \bar{u}^3 \bar{q}_\xi^3 = 0,
\end{align*}
\]

(3.17)

The key point to solve the system (3.17) is to solve \(\bar{u}^3\) from the last equation in (3.17). In fact, once one solves \(\bar{u}^3\) from the last equation, then the second and third equations in (3.17) become linear, and then one can easily solve the unknown functions \(\bar{u}^1\) and \(\bar{u}^2\) from the second and third equations in (3.17). After solving \(\bar{u}^1\), \(\bar{u}^2\) and \(\bar{u}^3\), by substituting them into the first equation in (3.17), one can find that the first equation in (3.17) also becomes linear, and then one can easily solve \(\bar{u}^0\) from it. Therefore, in what follows, it suffices to consider the last equation in (3.17).
In order to solve the last equation in (3.17), denoted by (3.17)4, we first gives the corresponding initial data.

In fact, the half plane \{ (t, \vartheta) \mid t \geq 0, \vartheta \in \mathbb{R} \} in the coordinates \((t, \vartheta)\) becomes the half plane \{ (\xi, \eta) \mid \xi + \eta \geq 0 \} in the coordinates \((\xi, \eta)\), while the initial line \(t = 0\) becomes the line \(\xi + \eta = 0\).

See Figure 1.

\[\begin{array}{c}
\text{Figure 1: The } (\xi, \eta)\text{-plane}
\end{array}\]

Therefore, by (3.7), it follows from the initial data (2.29) defined on \(t = 0\) that

\[\bar{u}^3(\xi, -\xi) = \bar{\varphi}^3(2\xi), \quad \bar{u}^3_\eta(\xi, -\xi) = \bar{p}_0^3(2\xi)\]  \hspace{1cm} (3.18)

on the initial line \(\xi + \eta = 0\).

We next solve the Cauchy problem (3.17)4, (3.18) on the half plane \{ (\xi, \eta) \mid \xi + \eta \geq 0 \}.

Notice that (3.17)4 can be rewritten as

\[\frac{\partial \bar{u}^3_\eta}{\partial \xi} = \frac{1}{2} \bar{u}^3 \bar{u}^3_\eta.\]  \hspace{1cm} (3.19)

Fixing \(\eta\) and integrating (3.19) with respect to \(\xi\) from \(-\eta\) gives

\[\begin{align*}
\frac{\partial \bar{u}^3}{\partial \eta} (\xi, \eta) &= \frac{\partial \bar{u}^3}{\partial \eta} (-\eta, \eta) \times \exp \left\{ \frac{1}{2} \int_{-\eta}^{\xi} \frac{\partial \bar{u}^3}{\partial s} (s, \eta) ds \right\} \\
&= \bar{p}_0^3(-2\eta) \times \exp \left\{ \frac{1}{2} \bar{u}^3(\xi, \eta) \right\} \times \exp \left\{-\frac{1}{2} \bar{\varphi}^3(-2\eta) \right\}. \\
&= \bar{p}_0^3(-2\eta) \times \exp \left\{ \frac{1}{2} \bar{u}^3(\xi, \eta) \right\} \times \exp \left\{-\frac{1}{2} \bar{\varphi}^3(-2\eta) \right\}. \\
\end{align*}\]  \hspace{1cm} (3.20)

Obviously, (3.20) can be rewritten as

\[\begin{align*}
\frac{\partial}{\partial \eta} \left( \exp \left\{-\frac{1}{2} \bar{u}^3(\xi, \eta) \right\} \right) &= -\frac{1}{2} \bar{p}_0^3(-2\eta) \times \exp \left\{-\frac{1}{2} \bar{\varphi}^3(-2\eta) \right\}. \\
&= -\frac{1}{2} \bar{p}_0^3(-2\eta) \times \exp \left\{-\frac{1}{2} \bar{\varphi}^3(-2\eta) \right\}.
\end{align*}\]  \hspace{1cm} (3.21)
Fixing $\xi$ and integrating (3.21) with respect to $\eta$ from $-\xi$ leads to
\[
\exp\left\{ -\frac{1}{2} \bar{u}^3(\xi, \eta) \right\} = \exp\left\{ -\frac{1}{2} \bar{\varphi}^3(2\xi) \right\} - \frac{1}{2} \int_{-\xi}^{\eta} \bar{p}_0^3(-2s) \exp\left\{ -\frac{1}{2} \bar{\varphi}^3(-2s) \right\} ds, \tag{3.22}
\]
namely,
\[
\bar{u}^3(\xi, \eta) = -2 \ln \left\{ \exp\left\{ -\frac{1}{2} \bar{\varphi}^3(2\xi) \right\} - \frac{1}{2} \int_{-\xi}^{\eta} \bar{p}_0^3(-2s) \exp\left\{ -\frac{1}{2} \bar{\varphi}^3(-2s) \right\} ds \right\}. \tag{3.23}
\]
In the coordinates $(t, \vartheta)$, (3.23) becomes
\[
\bar{u}^3(t, \vartheta) = -2 \ln \left\{ \exp\left\{ -\frac{1}{2} \bar{\varphi}^3(t + \vartheta) \right\} - \frac{1}{2} \int_{t}^{\vartheta} \bar{p}_0^3(-2s) \exp\left\{ -\frac{1}{2} \bar{\varphi}^3(-2s) \right\} ds \right\}. \tag{3.24}
\]
Noting that
\[
\bar{p}_0^3 = \bar{\psi}^3 - \bar{\varphi}_\vartheta^3, \tag{3.25}
\]
we obtain from (3.24) that
\[
\bar{u}^3(t, \vartheta) = -2 \ln \left\{ \exp\left\{ -\frac{1}{2} \bar{\varphi}^3(t + \vartheta) \right\} - \frac{1}{2} \int_{t}^{\vartheta} \bar{\psi}^3(2s) \exp\left\{ -\frac{1}{2} \bar{\varphi}^3(2s) \right\} ds \right\}. \tag{3.26}
\]
Summarizing the above arguments yields

**Theorem 3.1** Under the assumptions (2.34) and (2.35) (i.e., the physical motion assumptions), the Cauchy problem for the relativistic string equations in the Ori’s space-time with the initial data (2.29) admits a unique global $C^2$ smooth solution on $\mathbb{R}^+ \times \mathbb{R}$ if and only if, in the coordinates $(t, \vartheta)$ the initial data satisfies
\[
\exp\left\{ -\frac{1}{2} \bar{\varphi}^3(\vartheta + t) \right\} + \exp\left\{ -\frac{1}{2} \bar{\varphi}^3(\vartheta - t) \right\} > \int_{\vartheta}^{\vartheta+2} \bar{\bar{\psi}}^3(2s) \exp\left\{ -\frac{1}{2} \bar{\varphi}^3(2s) \right\} ds, \quad \forall t > 0, \forall \vartheta \in \mathbb{R}, \tag{3.27}
\]
where $\varphi^3$ and $\bar{\psi}^3$ are the corresponding initial data in the coordinates $(t, \vartheta)$ of $\varphi^3$ and $\bar{\psi}^3$ in the original coordinates $(t, \theta)$, respectively.

**Corollary 3.1** If $\bar{\psi}^3(\theta) \leq 0$ for all $\theta \in \mathbb{R}$, then the Cauchy problem for the relativistic string equations in the Ori’s space-time with the initial data (2.29) admits a unique global $C^2$ smooth solution on $\mathbb{R}^+ \times \mathbb{R}$.

On the other hand, noting (3.24), we have

**Corollary 3.2** If $\bar{p}_0^3(\theta) \leq 0$ for all $\theta \in \mathbb{R}$, then the Cauchy problem for the relativistic string equations in the Ori’s space-time with the initial data (2.29) admits a unique global $C^2$ smooth solution on $\mathbb{R}^+ \times \mathbb{R}$.

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Remark 3.2 It is obvious that \((3.17)_4\) can also be rewritten as
\[
\frac{\partial \bar{u}^3}{\partial \eta} = \frac{1}{2} \bar{u}^3 \bar{u}_\xi,
\]
(3.28)
in this case we have a similar discussion by means of \(\bar{q}^3_0\) instead of \(\bar{p}^3_0\). That is to say, in this way we can also prove Theorem 3.1.

Thus, using \(\text{(3.24)}\) and a similar one by means of \(\bar{q}^3_0\), we have

Corollary 3.3 If it holds that
\[
\|\bar{p}^3_0\|_{L^1} \ll 1 \quad \text{and} \quad \|\bar{q}^3_0\|_{L^1} \ll 1,
\]
then the Cauchy problem for the relativistic string equations in the Ori’s space-time with the initial data \(\text{(2.29)}\) admits a unique global \(C^2\) smooth solution on \(\mathbb{R}^+ \times \mathbb{R}\).

4 Conclusion and discussion

It is well known, in particle physics, the string model is used to consider the structure of hardrons. A free string is a one-dimensional physical object whose motion is represented by a time-like extremal surface in the physical space-times. The extremal surfaces play an important role in both mathematics and physics, in particular, in the theoretical apparatus of elementary particle physics. The theory on the motion of a relativistic string in the Minkowski spacetime has been studied extensively, many and beautiful results have been obtained. However, the study on the motion of a relativistic string in curved space-times is vastly open, there are a lot of fundamentally important problems needed to solve. The main difficulty is that the PDEs for such a motion are essentially nonlinear.

In fact, the motion of a string in a curved enveloping space-time \((\mathcal{M}, \bar{g})\), which stands for a given general Lorentzian manifold, can be determined by constructing a certain wave map from the Minkowski plane to \((\mathcal{M}, \bar{g})\). When the target manifold is Riemannian, the global existence of smooth solutions to the system \(\text{(3.10)}\) has been proved by Gu \([9]\). On the other hand, when the enveloping space-time \((\mathcal{M}, \bar{g})\) takes some special cases, for example, \(S^{1+1}\) or the Schwartzchild space-time, some results on the global existence of smooth solutions to the corresponding wave map equations have also been obtained (see \([11], [15]\)). In the present paper, we consider another special but important case that the enveloping space-time is Ori’s: (1) as a general framework, we first analyze relativistic string equations in a curved enveloping space-time \((\mathcal{M}, \bar{g})\) which stands for a general Lorentzian manifold, and then investigate some interesting properties enjoyed by these
equations; (2) based on this, under suitable small assumptions we prove the global existence of smooth solutions of the Cauchy problem for relativistic string equations in $(\mathcal{N}, \tilde{g})$; (3) in particular, we investigate the motion of a relativistic string in the Ori’s space-time, and give a sufficient and necessary condition guaranteeing the global existence of smooth solutions of the Cauchy problem for relativistic string equations in the Ori’s space-time.

Our ultimate goal is to study the global existence or breakdown phenomena of smooth solutions of the relativistic string equations in a general curved enveloping space-time $(\mathcal{N}, \tilde{g})$ without any small assumption. This is a hard task. However, in the Gaussian coordinates, the Lorentzian metric of the (curved) space-time described by the Einstein’s field equations can be written, at least locally, as

$$\tilde{g} = \begin{pmatrix} -1 & 0 \\ 0 & h \end{pmatrix},$$

(4.1)

where $=(h_{ij})_{n \times n}$ stands for a Riemannian metric. See Kossowski and Kriele [23] for the details.

In the present situation, the system (3.6) becomes

$$\begin{align*}
\frac{\partial p^0}{\partial t} + \frac{\partial p^0}{\partial \vartheta} &= - \frac{1}{2} \frac{\partial h_{ij}}{\partial x^0} p^i p^j, \\
\frac{\partial p^k}{\partial t} + \frac{\partial p^k}{\partial \vartheta} &= - \frac{1}{2} h_{kl} \frac{\partial h_{li}}{\partial x^0} p^l p^i q^j - \frac{1}{2} h_{kl} \frac{\partial h_{li}}{\partial x^0} p^l p^j q^0 - \tilde{\Gamma}^k_{ij} p^i p^j (k = 1, \cdots, n),
\end{align*}$$

(4.2)

where $\tilde{\Gamma}^k_{ij}$ stand for the connections corresponding to the Riemannian metric $h$. Similarly, the equations satisfied by $q$ can be obtained. It is easy to see that the system (4.2) somewhat possesses a special form with some geometric structures, perhaps this will shed light on solving our ultimate problem. This is worthy to be studied seriously in the future.

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