Cross section, final spin and zoom-whirl behavior in high-energy black hole collisions

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We study the collision of two highly boosted equal-mass, nonrotating black holes with generic impact parameter. We find such systems to exhibit zoom-whirl behavior when fine tuning the impact parameter. Near the threshold of immediate merger the remnant black hole Kerr parameter can be near maximal ($a/M \gtrsim 0.95$) and the radiated energy can be as large as $35 \pm 5\%$ of the center-of-mass energy.

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I. Introduction. As the major foundation of experimental particle physics, high energy collisions enable us to probe the fundamental characteristics of short range interactions: the larger the center-of-mass (CM) energy of the collision, the shorter the distance one can probe. The mass-energy equivalence of relativity, however, imposes a distinct limit on this method. All forms of energy gravitate and collisions at trans-Planckian energies are inputs for the Monte-Carlo simulators used to search for different boost magnitudes. Unless stated otherwise, we use geometrical units $G = c = 1$.

II. Setup. The simulations presented in this work have been obtained with the LEAN code [12], which is based on the CACTUS computational toolkit [13]. The code employs mesh refinement (provided by CARPET [14]) and the apparent horizon (AH) finder AHFINDERDIRECT [15]. Puncture initial data are provided by a spectral solver [16]. The simulations are performed as in the head-on case [10], but here we use equatorial instead of octant symmetry and employ higher resolution near the BHs.

We set up a coordinate system such that the BHs start on the $x$-axis separated by a coordinate distance $d$ and with radial (tangential) momentum $P_r$ ($P_y$). The impact parameter is $b = L/P = P_y d/P$, where $P$ is the linear momentum of either BH, and $L$ is the initial orbital angular momentum. We extract gravitational radiation by computing the Newman-Penrose scalar $\Psi_4$ at different radii $r_{ex}$ from the center of the collision. $\Psi_4$ is decomposed into multipoles $\psi_{lm}$ using spin-weight $-2$ spherical harmonics: $\Psi_4(t, r_{ex}, \theta, \phi) = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} -2 Y_{lm}(\theta, \phi) \psi_{lm}(t, r_{ex})$, where $\theta$ is measured relative to the $z$-axis. The estimated spurious or “junk” radiation in the initial data is quite insensitive to the impact parameter and comparable to that present in the head-on case [10]; we remove it from reported results in the same manner. Errors due to discretization and finite extraction radius are comparable to those reported in [10]. The estimated uncertainties in radiated quantities are 3\% and 15\% for low and high boost, respectively, and the phase error in the GW signal used in the zoom-whirl analysis below is 0.2 rad.

Our analysis of grazing collisions is based on three one-parameter sequences of numerical simulations of equal-mass, nonspinning BH binaries. Each sequence is characterized by fixed initial coordinate separation $d$ and Lorentz boost $\gamma \equiv (1 - v^2)^{-1/2}$ of the holes, while the impact parameter $b$ is varied. These sequences are (1) $\gamma = 1.520$ ($v = 0.753$) and $d/M = 174.1$; (2) $\gamma = 1.520$ ($v = 0.753$) and $d/M = 62.4$; (3) $\gamma = 2.933$ ($v = 0.940$) and $d/M = 23.1$, where $M$ is the total BH mass.

III. Results. The results of our study are most conveniently presented in terms of three distinct regimes we encounter as the impact parameter is increased starting from the head-on limit $b = 0$: (i) immediate mergers, (ii) nonprompt mergers and (iii) the scattering regime where no common AH forms. These regimes are separated by two special values of $b$: the threshold of immediate merger $b^*$ and the scattering threshold $b_{scat}$. The remarkable features of BH binaries in these different regimes will be
described in detail in the remainder of this section.

The scattering threshold $b_{\text{scat}}$ is defined such that the two BHs merge for $b < b_{\text{scat}}$ and scatter to infinity for $b > b_{\text{scat}}$. By analyzing collisions with CM velocity $v \lesssim 0.90$, Shibata et al. [11] estimate $b_{\text{scat}}/M \sim (2.5 \pm 0.05)/v$. The analysis of sequences 1 and 2 shows that merger occurs only for $b_{\text{scat}}/M \leq 3.4$, consistent with [11], but for sequence 3 we find $2.3 \lesssim b_{\text{scat}}/M \lesssim 2.4$, indicating that [11] may overestimate $b_{\text{scat}}$ for large $\gamma$. Previous studies in the literature, which are based on the Penrose construction, look for AHs in the union of two shock waves and find $b_{\text{scat}}/M = 1.685$ as $v \rightarrow 1$ [17]. Our results suggest that estimates of BH production cross sections ($\propto b_{\text{scat}}^2$) obtained through that construction are accurate to within a factor $< 2$. We further emphasize the surprising agreement between our simulations and the point-particle approximation: for example, a cross section from high-energy scattering off a Kerr BH with $j \sim 0.95$ gives $b_{\text{scat}}/M \sim 2.36$ [18].

The above definition of the scattering threshold is purely based on the nature of the end state of the binary but ignores details of the interaction. A closer look at these details reveals the existence of a **threshold of immediate merger** $b^*$ [19]. Roughly speaking, for $b < b^*$ merger occurs within the first encounter, whereas for $b^* < b < b_{\text{scat}}$ it does not, but sufficient energy is radiated to put the binary into a bound state that **eventually** results in a merger. A more precise definition arises in the context of the geodesic limit. The argument is this: if a BH production cross section is consistent with the zoom-whirl picture, and if an expression of the form of Eq. (1) holds. To this end we estimate the number of orbits $n$ in the whirl phase in two ways: (a) using the puncture trajectories ($n_p$), and (b) using the GW flux measured far from the impact ($n_{\text{GW}}$). For method (a), in the scattering cases we define $n_p$ as the total angle divided by $2\pi$ traversed by the puncture from the initial position until it reaches a distance from the origin of twice the minimum distance (i.e., roughly twice the whirl radius). For merger cases, $n_p$ is counted in a like manner until the puncture crosses a distance $1/2$ the whirl radius. Our current bracket gives $b^*/M = 3.35 \pm 0.01$, close to which we already see a “blurring” of the threshold, as binaries with $b > b^*$ do not separate to twice the whirl distance before merging (see the inset of Fig. 1). For method (b), we define $n_{\text{GW}}$ to be the number of GW cycles divided by 2 in the $l = m = 2$ component of the wave, from the initial time until the time when the $l = 2$ mode luminosity reaches $1/2$ its peak: see Fig. 1 [11] for an example. We expect this estimate to be decent because the luminosity seems to be largest and roughly constant during the whirl phase. The “1/2 criterion” is somewhat arbitrary, but as long as it is applied consistently it should have little effect on our estimate of the slope $\Gamma$ in Eq. (1).

\[
n = C - \Gamma \ln |b - b^*| ,
\]

where $C$ is a family-dependent constant and $\Gamma$ is a constant inversely proportional to the Lyapunov instability exponent of the limiting spherical orbit. Although unstable spherical orbits are a formal idealization, not realized in practice because of GW energy loss, BH mergers do indeed approach a whirl-like configuration near the threshold of immediate merger. Such a configuration can in principle be sustained until all the excess kinetic energy, roughly equal to $2m_{\text{err}}(\gamma - 1)$, is lost in GWs, where $m_{\text{err}}$ is the irreducible mass of each black hole. This threshold is blurred to a size $\delta b \sim e^{(C-n)/\Gamma}$ in parameter space.

We now analyze sequence 1 in more detail to determine if it is consistent with the zoom-whirl picture, and if an expression of the form of Eq. (1) holds. The two estimates $n_p$ and $n_{\text{GW}}$ are shown in Fig. 2 for immediate merger and nonprompt merger impact parameters about $b^*$. The plots indicate that a relationship...
The uncertainties in the fitted slopes to Eq. (1) are purely from the linear regression analysis.

of the form (1) is valid, with a slope $\Gamma \approx 0.2$ to within $\sim 50\%$ (ignoring systematic and computational errors we have not been able to account for); this is a factor of 2-3 smaller than the analogous geodesic problem of a high-speed point particle scattering (prograde) off a Kerr BH with $j \approx 0.95$. Given the large range of relevant $\Gamma$’s in the geodesic case (cf. Fig. 9 of [19]) this provides reasonable evidence for zoom-whirl-like behavior in high-speed, comparable-mass collisions.

The threshold of immediate merger $b^*$ appears to play a special role when we consider the amount of GW energy radiated and the final spin of the post-merger BH. To illustrate this point, in Fig. 3 we plot these quantities as functions of the impact parameter for sequence 2.

For $v \sim 0.75$ the radiated energy increases by about one order of magnitude, from $\sim 2.2\%$ for $b = 0$ to $\geq 23\%$ for $b \approx b^*$. Two points are particularly noteworthy in this regard. (i) Even at this comparatively small boost $v \sim 0.75$ we comfortably exceed the maximum of $14\pm3\%$ reported for the ultrarelativistic limit of head-on collisions [10]. Grazing collisions with larger boosts, in turn, radiate enormous amounts of gravitational radiation: for one run of sequence 3 with $v \sim 0.94$ and $b \sim b^*$ the radiated energy is $\sim 45 \pm 5\%$ of the CM energy. (ii) The maximum radiation as well as the maximum final spin (cf. below) is obtained near the threshold of immediate merger $b^*$ as opposed to $b_{\text{scat}}$. The notion that excess kinetic energy drives zoom-whirl behavior seems to be consistent with the data, in that the maximum total energy radiated near $b^*$ is approximately equal to the initial kinetic energy minus the spin energy of the final BH.

These surprisingly large amounts of GW energy correspond to huge luminosities. Ref. [10] showed that the high-energy, head-on collision of two nonspinning BHs could generate luminosities up to $dE/dt \sim 0.01$. For non-head-on collisions and nonprompt mergers we observe even higher luminosities. For instance, for $v = 0.75$ and $b \approx b^*$ the maximum luminosity is $\sim 0.02$, and extrapolation to $v = 1$ indicates that one might reach luminosities $\gtrsim 0.1$, corresponding in physical units to $\sim 3.6 \times 10^{58}\text{erg s}^{-1}$. This is the largest luminosity from a BH merger known to date, approaching in order of magnitude the universal limit $dE/dt \lesssim 1$ suggested by Dyson [22].

We conclude our analysis with a discussion of the final spin resulting from the merger. It has been argued in Washik et al. [23] that “no merger of equal-mass BHs can lead to a final BH with maximal spin parameter $j_{\text{fin}} \approx 1^*$, as long as the BHs are nonspinning. The maximum spin reported for the sequence studied in that work is 0.823. Larger final spins have been obtained from binary simulations with nonzero initial spin. Dain et al. [24] report the largest value measured so far ($j_{\text{fin}} = 0.922$), although Fig. 3 in [24] might imply an even larger final spin. The latter work expresses doubts, however, whether merging equal-mass BHs can “produce a very-close-to-maximal final BH”. Our simulations suggest a different outcome for the merger of equal-mass, nonspinning BH binaries. In Table I we report the largest final spin measured so far in any numerical BH merger simulation. We further conjecture that even equal-mass, nonspinning binaries can result in a final spin arbitrarily close to the Kerr limit $j = 1$. It is important in this context to bear in mind the difficulties in measuring the final spin with high accuracy (cf. [26]). These difficulties were our main reason to generate sequence 2. The high oscillation frequency of the ringdown signal as $j \rightarrow 1$ requires high resolution in the GW extraction zone. By using a smaller initial separation, sequence 2 enables us to meet this requirement at tolerable computational cost.

We have checked our results by calculating the final spin in a variety of ways: (i) we used energy balance arguments to find $j_{\text{fin}} = J_{\text{fin}}/M_{\text{fin}}^2 = (J_{\text{ini}} - J_{\text{rad}})/(M_{\text{ADM}} - E_{\text{rad}})^2$; (ii) we fit the quasinormal mode (QNM) frequency and damping time of the final BH and inverted them to obtain $j_{\text{QNM}}$ (see e.g. [27]); (iii) we used the
equatorial circumference of a Kerr BH \( C_c = 4\pi M \) to find 
\( 2\pi A_{\text{AH}}/C_c^2 = 1 + \sqrt{1 - j_{\text{AH}}^2} \), where \( A_{\text{AH}} \) is the AH area.

These different estimates are compared in Table 1 for three selected impact parameters leading to merger, and shown in Fig. 3 for sequence 2 runs. Within our uncertainty estimates (~3% for \( j_{\text{QNM}}, j_{\text{AH}} \) and ~8% for \( j_{\text{in}} \)) we observe good agreement throughout sequence 2. For \( 2.7 \lesssim b/M \lesssim b/M \) we find \( j_{\text{in}} > 0.9 \), and for \( b \lesssim b^* \) our estimated final spins can be quite close to extremal.

For example, for \( b = 3.04M \) we directly measure \( j = 0.96 \pm 0.03 \), and we expect further fine tuning of \( b \) to yield even larger values of \( j \). Our estimates are substantially larger than the ones quoted by Shibata et al. [11].

Given the difficulties in achieving the necessary numerical accuracy, perhaps the apparent discrepancies are due to our increased resolution.

IV. Conclusions. High-energy BH collisions are fertile ground for testing many ideas and conjectures in general relativity and high energy physics. We find that these collisions can radiate at least 35 ± 5% of the CM energy in GWs, that a merger can lead to the remnant BH spinning very close to extremal, and we display near-threshold phenomena akin to zoom-whirl in geodesics. We find no evidence of cosmic censorship violation. Our results are crucial for BH event generators in TeV-scale gravity: for four-dimensional spacetimes our results are consistent with Penrose’s construction to within a factor < 2. For the first time we show the dependence of the spin of the BH remnant on the impact parameter, a direct input in BH event generators.

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[1] S. B. Giddings, Gen. Rel. Grav. 34, 1775 (2002).
[2] M. W. Choptuik and F. Pretorius, arXiv:0908.1780 [gr-qc].
[3] T. Banks and W. Fischler, hep-th/9906038.
[4] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Rev. D 66, 044011 (2002).
[5] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 3370 (1999); Phys. Rev. Lett. 83, 4690 (1999).
[6] S. B. Giddings and S. D. Thomas, Phys. Rev. D 65, 056010 (2002); S. Dimopoulos and G. Landsberg, Phys. Rev. Lett. 87, 161602 (2001); J. L. Feng and A. D. Shapere, Phys. Rev. Lett. 88, 021303 (2001).
[7] P. Kanti, Lect. Notes Phys. 769, 387 (2009).
[8] J. A. Frost et al., arXiv:0904.0979 [hep-ph]; M. Cavaglia et al., Comput. Phys. Commun. 177, 506 (2007); D. C. Dai et al., Phys. Rev. D 77, 076007 (2008).
[9] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998); E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998); S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B 428, 105 (1998).
[10] H. Nastase, arXiv:hep-th/0501068.
[11] U. Sperhake et al., Phys. Rev. Lett. 101, 161101 (2008).
[12] M. Shibata, H. Okawa and T. Yamamoto, Phys. Rev. D 78, 101501(R) (2008).