UPPER BOUND ON THE NUMBER OF SYSTEMS OF HECKE EIGENVALUES FOR SIEGEL MODULAR FORMS (MOD $p$)

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Abstract. We derive an explicit upper bound for the number $\mathcal{N}(g, N, p)$ of systems of Hecke eigenvalues coming from Siegel modular forms (mod $p$) of dimension $g$ and level $N$ relatively prime to $p$. In the special case of elliptic modular forms ($g = 1$), our result agrees with recent work of G. Herrick.

1. Introduction

We prove the following

**Theorem 1.** Fix an integer $g \geq 1$, a prime $p$ and an integer $N \geq 3$ not divisible by $p$. The number of systems of Hecke eigenvalues coming from Siegel modular forms (mod $p$) of dimension $g$ and level $N$ satisfies

$$\mathcal{N}(g, N, p) \leq c_g \cdot \# \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z}) \cdot p^{(g+2)(g-1)/2} \cdot (p^2 - 1) \cdot \prod_{j=1}^{g} (p^j + (-1)^j),$$

where

$$c_g = (1)^{\frac{g(g+1)}{2}} \cdot 2^{-g} \cdot \prod_{j=1}^{g} \zeta(1 - 2j) = \frac{1}{2^{2g}g!} \cdot \prod_{j=1}^{g} B_{2j}.$$

**Corollary 2 (Asymptotics).**

(a) Fix $g$ and $p$, then

$$\mathcal{N}(g, N, p) = O \left( N^{2g^2+g+1} \right).$$

(b) Fix $g$ and $N$, then

$$\mathcal{N}(g, N, p) = O \left( p^{g^2+g+1} \right).$$

The constants are effectively computable.

**Proof.** Part (a) follows from the fact that the algebraic group GSp_{2g} has dimension $2g^2+g+1$. Part (b) is obvious.

Combined with Theorem 1.1 of [Ghi04], Theorem 1 gives

**Corollary 3 (Algebraic modular forms).** Let $B/\mathbb{Q}$ be the quaternion algebra ramified at $p$ and $\infty$. The number of systems of Hecke eigenvalues coming from algebraic modular forms (mod $p$) of level $N$ on GU$_g(B)$ satisfies the inequality of Theorem 1.
Remarks. (a) In the case of classical modular forms \((g = 1)\) with \(\Gamma(N)\) level structure we get \(\mathcal{N}(1, N, p) = O(p^3)\); the author recently learned of an upper bound with the same asymptotics obtained by Graham Herrick for \(\Gamma_0(N)\) and \(\Gamma_1(N)\) level structures. The previously known bound was \(O(p^4)\) and can be found in [Joc82].

(b) A more general approach in the context of algebraic modular forms is presented in Section 7 of [Gro96].

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2. The setup

We fix the dimension \(g\), the prime \(p\) and the level \(N\) not divisible by \(p\). We define (see Sections 2.2 and 2.3 of [Ghi04] for details)

\[
\Sigma = \{ [A, \lambda] : A \text{ superspecial abelian variety over } \mathbb{F}_p, \lambda \text{ principal polarization} \},
\]

\[
\Sigma(N) = \{ [A, \lambda, \alpha] : \text{as above, } \alpha \text{ symplectic level } N \text{ structure on } (A, \lambda) \},
\]

\[
\tilde{\Sigma}(N) = \{ [A, \lambda, \alpha, \eta] : \text{as above, } \eta \text{ hermitian basis of invariant differentials over } \mathbb{F}_{p^2} \text{ on } (A, \lambda) \}.
\]

The notation \([x]\) stands for the isomorphism class of the object \(x\).

It follows from the proof of Theorem 4.5 of [Ghi04] that the restriction of Siegel modular forms to the superspecial locus \(\Sigma(N)\) induces a bijection on the sets of systems of Hecke eigenvalues. Therefore the number that we want to estimate is the number of systems of Hecke eigenvalues occurring in the spaces of superspecial modular forms

\[
S_\tau(N) = \left\{ f : \tilde{\Sigma}(N) \to W_\tau \mid f([A, \lambda, \alpha, M\eta]) = \tau(M)^{-1} f([A, \lambda, \alpha, \eta]) \text{ for all } M \in \text{GU}_g(\mathbb{F}_{p^2}) \right\},
\]

as the weight \(\tau : \text{GU}_g(\mathbb{F}_{p^2}) \to \text{GL}(W_\tau)\) runs through the set of irreducible representations of \(\text{GU}_g(\mathbb{F}_{p^2})\) over \(\mathbb{F}_p\).

Note that for a fixed class \([A, \lambda, \alpha] \in \Sigma(N)\), a superspecial modular form \(f \in S_\tau(N)\) is completely determined by the value it takes on \([A, \lambda, \alpha, \eta]\) for any choice of \(\eta\), and therefore

\[
\dim S_\tau(N) \leq \#\Sigma(N) \cdot \dim W_\tau.
\]

We get the following upper bound for the number we are interested in:

\[
\mathcal{N}(g, N, p) \leq \#\Sigma(N) \cdot \sum_\tau \dim W_\tau.
\]

3. The cardinality of \(\Sigma(N)\)

We write

\[
\#\Sigma(N) = \sum_{[A, \lambda, \alpha] \in \Sigma(N)} 1 = \sum_{[A, \lambda] \in \Sigma} \#\{[A, \lambda, \alpha]\}.
\]

So we fix \((A, \lambda)\) and we want to count the number of level \(N\) structures \(\alpha\), up to \(\mathbb{F}_p\)-isomorphism. If we ignore the isomorphisms, there are precisely \(#\text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})\) level \(N\) structures. But by definition, two level \(N\) structures \(\alpha\) and \(\alpha'\) on \((A, \lambda)\) are isomorphic if
and only if \( \alpha' = \alpha \circ \varphi \) for some automorphism \( \varphi \) of \((A, \lambda)\). Hence we can continue our computation from (1) as follows:

\[
\#\Sigma(N) = \sum_{[A, \lambda] \in \Sigma} \#\{[A, \lambda, \alpha]\} = \sum_{[A, \lambda] \in \Sigma} \frac{\#\text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})}{\#\text{Aut}(A, \lambda)} = c_g \cdot \#\text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z}) \cdot \prod_{j=1}^{g} (p^j + (-1)^j).
\]

Here we used the following mass formula due to Ekedahl (page 159 of [Eke87], see also Corollary 9.5 in [vdG99]):

\[
\sum_{[A, \lambda] \in \Sigma} \frac{1}{\#\text{Aut}(A, \lambda)} = c_g \cdot \prod_{j=1}^{g} (p^j + (-1)^j).
\]

4. Representations of \( \text{GU}_g(\mathbb{F}_p^2) \)

We need to estimate the sum of the dimensions of the irreducible representations of \( \text{GU}_g(\mathbb{F}_p^2) \) defined over \( \mathbb{F}_p \). For this we use the theory of finite groups with split \((B, N)\)-pairs, as explained in [Cur70].

The cardinality of the group \( \text{GU}_g(\mathbb{F}_p^2) \) is

\[
\# \text{GU}_g(\mathbb{F}_p^2) = p^{g(g-1)/2}(p-1) \prod_{j=1}^{g} (p^j - (-1)^j).
\]

Therefore the size of a \( p \)-Sylow subgroup is \( p^{g(g-1)/2} \), and so Corollary 5.11 of [Cur70] says that

\[
\dim W_\tau \leq p^{g(g-1)/2} \quad \text{for any irreducible } \tau.
\]

The rank of \( \text{GU}_g(\mathbb{F}_p^2) \) is \( g \), and so by Proposition 6.1 of [Gro96] we know that the number of irreducible representations of \( \text{GU}_g(\mathbb{F}_p^2) \) over \( \mathbb{F}_p \) is \( p^{g-1}(p^2 - 1) \). Therefore we get the inequality

\[
\sum_{\tau} \dim W_\tau \leq p^{g-1}(p^2 - 1)p^{g(g-1)/2} = p^{(g+2)(g-1)/2} \cdot (p^2 - 1).
\]

5. The end

It remains to put (2) and (3) together to get

\[
\mathcal{N}(g, N, p) \leq c_g \cdot \#\text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z}) \cdot p^{(g+2)(g-1)/2} \cdot (p^2 - 1) \cdot \prod_{j=1}^{g} (p^j + (-1)^j),
\]

which is precisely the content of Theorem 1.
REFERENCES

[Cur70] Charles W. Curtis. Modular representations of finite groups with split \((B, N)\)-pairs. In Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69), pages 57–95. Springer, Berlin, 1970.

[Eke87] Torsten Ekedahl. On supersingular curves and abelian varieties. Math. Scand., 60(2):151–178, 1987.

[Ghi04] Alexandru Ghitza. Hecke eigenvalues of Siegel modular forms \((\text{mod } p)\) and of algebraic modular forms. J. Number Theory, 106(2):345–384, 2004. Also available from arXiv:math.NT/0309006.

[Gro96] Benedict H. Gross. Modular Galois representations, January 1996. Preprint.

[Joc82] Naomi Jochnowitz. Congruences between systems of eigenvalues of modular forms. Trans. Amer. Math. Soc., 270(1):269–285, 1982.

[vdG99] Gerard van der Geer. Cycles on the moduli space of abelian varieties. In Moduli of curves and abelian varieties, Aspects Math., E33, pages 65–89. Vieweg, Braunschweig, 1999.

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