A test of the instanton vacuum chiral quark model with axial anomaly low-energy theorems

M.M. Musakhanov\textsuperscript{1} and F. C. Khanna \textsuperscript{2}

1 Theoretical Physics Dept, Tashkent State University, 700095, Uzbekistan

2 Department of Physics, University of Alberta, Edmonton, Canada T6G2J1
   and TRIUMF, 4004 Wesbrook Mall, Vancouver, BC, Canada, V6T2A3
   E-mail: yousuf@iaph.silk.glas.apc.org, khanna@phys.ualberta.ca

Abstract

The QCD+QED axial anomaly low-energy theorems are applied to estimate the accuracy of an instanton vacuum - based chiral quark model. The low-energy theorems give an exact relation between the matrix elements of the gluon and photon parts of the QCD+QED axial anomaly operator equation. The matrix elements between vacuum and two photon states and between vacuum and two gluon states are calculated for arbitrary $N_f$. It is shown that this model does satisfy the low-energy theorems with an accuracy of $\sim 17\%$.

We estimate also the contribution of the nonperturbative conversion of gluons into photons to the decay $\eta' \rightarrow 2\gamma$ and compare with experimental data.

PACS number(s): 11.15.Tk , 12.38.Lg.
1 Introduction

In gauge theories the symmetries of the initial classical lagrangian may be destroyed by quantum fluctuations. One of the most important examples is the famous axial anomaly. The axial anomaly leads to many interesting nonperturbative phenomena in physics. Among them are $B$-violation processes in electroweak (EW) physics, $U_A(1)$ problem in QCD etc. The solution of these problems is intimately related to the topologically nontrivial structure of the vacuum in the gauge theories.

In this paper we apply the axial anomaly low-energy theorems to test the chiral quark model which is based on the instanton model of QCD vacuum. We find that this model does satisfy these theorems with an accuracy of $\sim 17\%$. This conclusion provides solid background to calculate different amplitudes of nonperturbative conversion of gluons into hadrons and photons.

We estimate also the contribution of the nonperturbative conversion of gluons into photons to the decay $\eta' \to 2\gamma$. By using experimental date for the width of $\eta' \to 2\gamma$ we conclude that $\eta'$-singlet axial current coupling constant is given as $f_0 = 1.16 f_\pi$.

1.1 Low-energy theorems from axial anomaly

The axial anomaly in the divergence of the singlet axial current in QCD + QED leads to a low-energy theorem for the matrix elements of this operator equation over vacuum and two-photon states:

$$\langle 0| N_f \frac{g^2}{32\pi^2} G\tilde{G}|2\gamma\rangle = N_c \frac{e^2}{8\pi^2} \sum_f Q_f^2 F^{(1)}\tilde{F}^{(2)}$$

at $q^2 = 0$. Here, $N_f$ is the number of the flavors, $g$ is the QCD coupling constant with $G\tilde{G} = \frac{1}{2} \varepsilon_{\mu\nu\lambda\sigma} G^a_{\mu\nu} G^a_{\lambda\sigma}$ and $G^a_{\mu\nu}$ being the operator of the gluon field strengths, $N_c$ is the number of the colors, $e$ and $Q_f$ are QED coupling constant and the electric charges of the quarks respectively, $2 F^{(1)}\tilde{F}^{(2)} = \varepsilon^{\mu\nu\lambda\sigma} F^{(1)}_{\mu\nu} F^{(2)}_{\lambda\sigma}$, $F^{(i)}_{\mu\nu} = \varepsilon^{(i)}_{\mu} q_{\nu} - \varepsilon^{(i)}_{\nu} q_{\mu}$, $\varepsilon^{(1,2)}_{\mu}$, $q_{1,2}$ are polarizations and momenta of photons respectively and $q = q_1 + q_2$. This relation is a consequence of the absence of a massless singlet pseudoscalar boson. Here the contribution of the quark masses is neglected.

As it is evident, gluons can interact with photons only through quark loops. In perturbation theory it leads to at least $\sim g^4$ results for the left hand side of Eq. (1) (see e.g. recent discussion of the higher-loop contributions to the axial anomaly [2]). So, the solution of this theorem is related only to the nonperturbative phenomena connected with the structure of QCD vacuum.

Another nontrivial low-energy theorem concerns the matrix element over vacuum and two-gluon states:

$$\langle 0| g^2 G\tilde{G}|2\text{gluons}\rangle = 0$$

at the limit $q^2 = 0$.

These matrix elements are calculated in the instanton vacuum generated N-JL type quark model [3], [4] for arbitrary $N_f$.

1.2 The instanton vacuum of QCD

The instanton is the solution of gluodynamics in the Euclidian space[5]:
The anti-instanton solution has the same form as in Eq. (3) but with $\eta_{\mu\nu}^b$ instead of $\bar{\eta}_{\mu\nu}$. Here $O_{I(I)}$ is the orientation matrix of the instanton $I$ (anti-instanton $\bar{I}$) in color space, $\bar{\eta}(\eta) = \text{t'Hooft factors}$, $\rho$ is the size and $z$ is the position of the instanton. For large interinstanton distances $R \gg \rho$ the sum of the $N_+$ instantons and the $N_-$ anti-instantons is also an approximate solution. The calculation of the action for gluon fields leads to a sum of actions of free instantons and classical interinstanton potential $V(R, \rho_1, \rho_2, O)$, where $O = O_I^T O_2$ is a matrix of relative orientation. The most important part is the instanton-antiinstanton potential $V_{I\bar{I}}$. It is well-known that at large distances it has a form of the dipole-dipole interaction potential and may be attractive.

The main assumption of the model concerns small distances $R \sim \rho$. At small distances it is assumed that there is a repulsion. This assumption is supported by both phenomenological and theoretical considerations. This leads to a stabilization of the size and density of instantons. The quantities mean number of instantons $\langle N \rangle$ and a mean instanton size $\bar{\rho}$ include effects of the instanton interactions. In the following calculations, for simplicity, all instanton sizes are considered to be $\bar{\rho}$.

Both the phenomenological estimates and variational calculations lead to a mean interinstanton distance of $\bar{R} = \left(\frac{\langle N \rangle}{N} \right)^{1/4} \sim 1 \text{ fm}$ and a mean instanton size of $\bar{\rho} \sim 1/3 \text{ fm}$. The small packing parameter $(\bar{\rho}/\bar{R})^4 = 0.012$ provides a possibility for the independent averaging over positions and orientations of instantons.

### 1.3 The chiral quark model

The main assumption of this model is an interpolation formula for the quark propagator in the single instanton field. It is approximated as the sum of a free propagator and an explicit contribution of the zero mode $\xi_{I(I)}$, $\xi_{I(I)}$.

$$(i\hat{\nabla}(\xi_{I(I)}) + im)^{-1}_{1-\text{inst}} \approx (i\hat{\partial})^{-1} - \frac{\Phi_{\pm}(x; \xi_{I(I)})\Phi_{\pm}^*(y; \xi_{I(I)})}{im}.$$  (4)

Here, $\hat{\nabla} = \hat{\partial} - ig\hat{A}$, $\Phi_{\pm}(x; \xi_{I(I)})$ is the zero mode wave function of the fermion in the background of one $I(I)$. It depends on the collective instanton variables $\xi_{I(I)}$ – the size $\rho$, the position $z$ and the orientation $O$ of the instanton.

This interpolating formula should be accurate both at small momenta ($\rho \ll 1/\bar{\rho}$), where the zero mode is dominant, and at large momenta ($\rho \gg 1/\bar{\rho}$), where the propagator reduces to the free one. In the background of an $N_-$-instanton configuration and keeping in mind the low density of the instanton media this formula leads to the partition function of the model:

$$Z_N = \int D\psi D\psi^\dagger \exp(\int d^4x \psi^\dagger i\hat{\partial}\psi) W_+^{N_+} W_-^{N_-},$$  (5)

where

$$W_\pm = \left(-\frac{4\pi^2 \bar{\rho}^2}{N_c} \right)^{N_f} \int d^4z \frac{1}{V} \det J_\pm(z),$$  (6)

$$J_\pm(z)_{fg} = \int \frac{d^4kd^4l}{(2\pi)^8} \exp(-i(k-l)z) F(k)F(l) \psi_f^\dagger(k) \frac{1}{2}(1 \pm \gamma_5)\psi_g(l),$$  (7)
and the contribution of the current quark masses is neglected. The form-factor $F$ is related to the zero–mode wave function in momentum space $\Phi_{\pm}(k; \xi_{I(\bar{I})})$ and is equal to:

$$F(k) = -t \frac{d}{dt} [I_0(t)K_0(t) - I_1(t)K_1(t)] \rightarrow \begin{cases} 1 & t \to 0 \\ \frac{3}{4}t^{-3} & t \to \infty \end{cases},$$

with $t = \frac{1}{2} k \bar{\rho}$.

The formula:

$$(ab)^N = \int d\lambda \exp(Nln\frac{aN}{\lambda} - N + \lambda b)$$

(9)

provides the final expression for the partition function [4]:

$$Z_N = \int D\psi D\psi^\dagger \exp(\int \psi^\dagger \hat{D}_G \psi + Y_+ + Y_-),$$

(10)

where

$$Y_\pm = (i)^{N_f} \lambda \int d^4 z \det J_\pm(z) = \left(\frac{2V}{N}\right)^{N_f-1} (iM)^{N_f} \int d^4 z \det J_\pm(z).$$

(11)

The self-consistency condition at the saddle point in Eqs. (10), and (11) leads to

$$4N_c V \int \frac{d^4 k}{(2\pi)^4} \frac{M^2 F^4(k)}{M^2 F^4(k) + k^2} = N.$$  

(12)

### 2 Calculations with the low-energy theorems

In the quasiclassical(saddle-point) approximation any gluon operator receives its main contribution from instanton background. As an example, for one instanton (anti-instanton) $I(\bar{I})$,

$$g^2 G^2(x) = \frac{192 \rho^4}{[\rho^2 + (x - z)^2]^4} = f(x - z),$$

and

$$g^2 \tilde{G}(x) = \pm f(x - z).$$

In this particular case, the calculation of the matrix element can be reduced to the calculation of the partition function

$$\hat{Z}_N[\kappa, a] = Z_N^{-1} \int D\psi D\psi^\dagger \exp(-\hat{S}_{eff}),$$

(15)

with an effective action, $\hat{S}_{eff}$, in the presence of an external electromagnetic field, $a_\mu$, and an external field $\kappa(x)$ is given as

$$-\hat{S}_{eff} = \int \psi^\dagger \check{D} \psi + Y_+ + Y_- + \int dx \left( Y_{G\tilde{G}}(x) + Y_{\tilde{G}G}(x) \right) \kappa(x),$$

$$Y_{G\tilde{G}}(x) = \pm \left(\frac{2V}{N}\right)^{N_f-1} (iM)^{N_f} \int d^4 z f(x - z) \det J_\pm(z),$$

(16)

where $\check{D} = \hat{D} - ieQf \hat{\mu}$. Finally, Eq. (16) can be rewritten as:

$$-\hat{S}_{eff} = \int \psi^\dagger \check{D} \psi +$$

$$\left(\frac{2V}{N}\right)^{N_f-1} \int dz \det(iM_+(z)J_+(z)) + \left(\frac{2V}{N}\right)^{N_f-1} \int dz \det(iM_-(z)J_-(z)),$$

(17)
where

\[ M_{\pm}(z) = \left( 1 \pm \int dx \kappa(x) f(x-z) \right)^{(N_f-1)^{-1}} M. \]

Another remarkable formula

\[ \exp(\lambda \det[iA]) = \int d\mathcal{M} \exp \left[ -(N_f-1)\lambda^{-\frac{1}{N_f-1}} (\det \mathcal{M})^{\frac{1}{N_f-1}} + itr(MA) \right] \tag{18} \]

is used in the following discussion. It is possible to check this by the saddle point approximation of the integral.

By using Eq. (18) it is easy to show that the effective bosonized action \( \hat{S}_{\text{eff}}[\mathcal{M}_{\pm}, a, \kappa] \), describing mesons (the matrices \( \mathcal{M}_{\pm} \)) in the presence of the external fields \( a_{\mu} \) and \( \kappa \) is

\[ -\hat{S}_{\text{eff}}[\mathcal{M}_{\pm}, a, \kappa] = \int dz \left( -(N_f-1) \left( \frac{2\kappa}{N_f} \right)^{-1} (\det \mathcal{M}_{\pm})^{\frac{1}{N_f-1}} \right) + \\
Tr \ln \left( i\hat{D} + i\mathcal{M}_{+}MF^2 (1 + (\kappa f))^{N_f^{-1}} \frac{1}{2}(1 + \gamma_5) + i\mathcal{M}_{-}MF^2 (1 - (\kappa f))^{N_f^{-1}} \frac{1}{2}(1 - \gamma_5) \right) \tag{19} \]

For the processes without mesons\( (\mathcal{M}_\pm = 1) \) the partition function is:

\[ \hat{Z}_N[\kappa, a] = \exp Tr \ln \left(i\hat{D} + iMF^2(1 + (\kappa f))^{\frac{1}{2}}(1 + \gamma_5) + iMF^2(1 - (\kappa f))^{\frac{1}{2}}(1 - \gamma_5)\right) \times (i\hat{D} + iMF^2)^{-1}, \tag{20} \]

where \((\kappa f) = \int dx \kappa(x)f(x-z)\).

### 2.1 The low-energy theorems for the matrix element between vacuum and two-photons states

The matrix element is generated by

\[ \frac{\delta \hat{Z}_N[\kappa, a]}{\delta \kappa(x) \delta a_{\mu}(x_1) \delta a_{\nu}(x_2)} |_{\kappa, a = 0}. \]

As it is clear from Eq. (20) the factors in the vertices in the corresponding Feynman loop-diagram are \( eQ_f \gamma_{\mu} \) and \( iMfF^2 \gamma_5 N_f^{-1} \).

We must calculate \( \Delta(q^2) \) (in the limit \( q^2 \to 0 \)), which is defined by:

\[ (2\pi)^4 \delta(q - q_1 - q_2) \Delta(q^2) = \int dx \exp(-iqx) \langle 0 | g^2 G\bar{G}(x) | 2\gamma \rangle = \\
e^2 e^{(1)} e^{(2)} f(0) |T(g^2 G\bar{G}(x) j_{\mu}^{em}(x_1) j_{\nu}^{em}(x_2))|0\rangle \exp(i(-iqx + q_1 x_1 + q_2 x_2)) dx dx_1 dx_2. \tag{21} \]

It is clear that

\[ \Delta(q^2) = e^{(1)} e^{(2)} f(q^2) N_c e^2 \sum_f Q_f^2 \\
\times Tr\left[ \int\frac{d^4p}{(2\pi)^4} \frac{iMF_1 F_2 \gamma_5 (\bar{p} - \bar{q}_1 + iMF_3^2) \gamma_\mu (\bar{p} + \bar{q}_2 + iMF_2^2)}{(p - q_1)(p + q_2)(p^2 + M^2 F_1^2)((p + q_2)^2 + M^2 F_2^2)} \right. \\
+ \left. (\mu \leftrightarrow \nu, q_1 \leftrightarrow q_2) \right]. \tag{22} \]

Here \( F_1 = F((p - q_1), F_2 = F(p + q_2), F_3 = F(p) \) and

\[ f(q^2) = \int dx \exp(-iqx)f(x) \tag{23} \]
Table 1. The accuracy of the model [3] for the different values of the parameter $r = M\rho$ in the Eq. (27) to satisfy axial anomaly low-energy theorems.

| $r$  | .0 | .1 | .2 | .3 | .4 | .5 | .6 | .7 | .8 | .9 | 1.0 |
|------|----|----|----|----|----|----|----|----|----|----|-----|
| $\delta(r)$ | .00 | .05 | .09 | .13 | .15 | .17 | .19 | .20 | .21 | .22 | .23 |

is the form-factor of the one-instanton contribution to $g^2G\tilde{G}$. At $q^2 = 0$

$$f(q^2 = 0) = 32\pi^2.$$ 

It is easy to show that the trace in Eq. (22) can be reduced to

$$8M^2\epsilon_{\mu\nu\lambda\sigma}q_{1\lambda}q_{2\sigma}\Gamma(q^2, r),$$

where

$$\Gamma(q^2, r) = \int \frac{d^4p}{(2\pi)^4} \frac{F_1 F_2 F_3}{((p - q_1)^2 + M^2 F_1^4)(p^2 + M^2 F_2^4)((p + q_2)^2 + M^2 F_3^4)} + ... \quad (24)$$

and $r = M\rho$. It is possible to calculate this integral analytically if we put $r = 0$ in Eq. (24) (In this case $F = 1$). In this approximation and at a small values of $q^2$, we have

$$\Gamma(q^2, r = 0) = \frac{1}{32\pi^2 M^2} \left(1 - \frac{q^2}{12M^2}\right) \quad (25)$$

As a result, in the $r = 0$ approximation the left side of the low-energy theorem, Eq. (1), is

$$(N_f \frac{g^2}{32\pi^2}) (\frac{4e^2 N_c}{g^2 N_f} \sum_f Q_f^2) F^{(1)} F^{(2)} \quad (26)$$

and this coincides with the right side of Eq. (1).

The calculations of the integrals in Eq. (24) can be performed also with an account of the form–factor $F(p)$. The deviation of results of the calculation from the approximate results of the $r = 0$ tell us about the accuracy of this model. We need to calculate

$$\Gamma(q^2 = 0, r) = M^2 \int \frac{d^4p}{(2\pi)^4} \frac{F^2(p)(F^2(p) - p^2 \frac{dF^2(p)}{dp})}{(p^2 + M^2 F_4^4(p))^3}$$

An evolution of $\Gamma(q^2 = 0, r)$ was performed numerically. Table 1 presents the factor $\delta(r) = 1 - \Gamma(0, r)/\Gamma(0, r = 0)$, where $r = M\rho$.

### 2.2 The low-energy theorem for the matrix element between vacuum and two-gluons states

Here we present, without details, calculations related to Eq. (2). This matrix element can be written in the form:

$$\langle 0 | g^2 G\tilde{G} | g(\epsilon^{(1)}, q_1), g(\epsilon^{(2)}, q_2) \rangle = \epsilon_{\mu_1}^{(1)\alpha_1} \epsilon_{\mu_2}^{(2)\alpha_2}$$

$$\times \int \partial_2^S \partial_1^S \langle 0 | Tg^2 G\tilde{G} A_{\mu_1}^{a_1}(x_1) A_{\mu_2}^{a_2}(x_2) | 0 \rangle \exp i(q_1 x_1 + q_2 x_2) dx_1 dx_2. \quad (28)$$
Here $A_\mu^a(x)$ is a total gluon field, $\epsilon^{(i)a}_{\mu}q_i$ are the polarization and the momentum of gluons respectively.

As usual, we expand the total field $A_\mu^a(x)$ around the instanton background. The main term in Eq. (28) is the contribution of the instanton background and is $\sim O(g^{-2})$. The next term is the contribution of the perturbative fluctuations over instanton background and is $\sim O(g^2)$. It is easy to see from previous considerations that $O(g^{-2})$ term is given by the formula

$$Z_N^{-1} \int D\psi D\psi^\dagger \exp(-S_{eff}) \left( \left( Y_{G\tilde{G}AA^+(x)} + Y_{G\tilde{G}AA^-(x)} \right) Q \right),$$

where

$$Y_{G\tilde{G}AA^\pm} = \pm \left( \frac{2V}{N} \right)^{N_f-1} (iM)^{N_f} \int dz f(x - z)$$

$$\times \int d\sigma (-\partial_\sigma^2)A_{\mu_1}^{(i)a_1}(x_1)(-\partial_\sigma^2)A_{\mu_2}^{(i)a_2}(x_2) \det J_\pm(z),$$

Here the instanton(anti-instanton) is located at the point $z$ with its orientation $O$.

Repeating the bosonization trick leads to the result for the $O(g^{-2})$ contribution which is proportional to

$$Tr[(i\tilde{\sigma} + iMF^2)^{-1}iMF^2\gamma_5].$$

It is clear that this $Tr$ and as a consequence $O(g^{-2})$ term are equal to zero.

The next order ($O(g^2)$) term is the contribution of the two diagrams. The first diagram is the direct contribution of the operator $g^2G\tilde{G}$ which is equal to

$$-g^2G^{(1)}\tilde{G}^{(2)},$$

where $2G^{(1)}\tilde{G}^{(2)} = \epsilon^{\mu\nu\lambda\sigma}G^{(1)a}_{\mu\nu}\epsilon^{(2)a}_{\lambda\sigma}$, $G^{(i)a}_{\mu\nu} = \epsilon^{(i)}_{\mu}q_{i\nu} - \epsilon^{(i)}_{\nu}q_{i\mu}$.

The factors in the vertices of the second loop–diagram are $g\lambda_a/2\gamma_5$ and $iMfF^2\gamma_5N_f^{-1}$. An account of the contribution from all flavors gives the coefficient $N_f$.

A comparison with the previous calculations (Eq. (21), (23)) leads to the result that the contribution of the second loop–diagram is equal in magnitude but opposite in sign to the contribution of the first diagram at $q^2 = 0$.

Then, terms of $O(g^2)$ are equal to zero in the limit $q^2 \to 0$. This conclusion is valid in $r = 0$ approximation. Thus we conclude that the instanton vacuum generated chiral quark model satisfies the low-energy theorems with an accuracy given in Table 1.

### 3 Nonperturbative conversion of gluons into photons and $\eta' \to 2\gamma$ decay

One of the first applications of QED axial anomaly was electromagnetic decays of pseudoscalar mesons \[4\]. Here we consider the application of the QCD+QED axial anomaly to the same kind of decays. Let us consider the matrix element of the divergence of the axial singlet current $\partial_\mu j^5_\mu$ between vacuum and two–photons states at nonzero but small $q^2$. We will compare this one with similar matrix element of the divergence of the third component of the axial isovector current $\partial_\mu j^5_\mu$. The definition of currents are $j^5_\mu = \bar{q}\gamma_\mu\gamma_5q$ and $j^{i3}_\mu = \bar{q}\gamma_\mu\gamma_5(\tau_i/2)q$. The matrix elements of these currents $<0|j^5_\mu|\eta'(p)> = 3^{1/2}if_\eta p_\mu$ and $<0|j^{i3}_\mu|\pi_j(p)> = \delta_{ij}if_\pi p_\mu$ defines the couplings $f_\eta$ and $f_\pi$. 


The decay constant of charged pions is $f_\pi = 93.3\, MeV$. In contrast to this one, $f_0$ has not been measured experimentally. Our approach provides a method for estimating $f_0$ from the data on $\eta' \to 2\gamma$ decay.

The matrix element under consideration is

$$\langle 0 | \partial_\mu j_\mu^5 | 2\gamma \rangle = \langle 0 | N_f \frac{g^2}{32\pi^2} G \bar{G} | 2\gamma \rangle - N_c \frac{e^2}{8\pi^2} \sum_f Q_f^2 F^{(1)} \tilde{F}^{(2)}$$  \tag{31}$$

We neglect here the masses of the $u, d, s$ quarks.

The lowest intermediate meson state in this equation is $\eta'$ - meson with the non-zero mass $m_{\eta'}$ due to axial anomaly, as was mentioned above. With definition $\langle \eta' | 2\gamma \rangle = f_{\eta' \to 2\gamma} F^{(1)} \tilde{F}^{(2)}$ it is easy to conclude that $O(q^2)$ terms provide the equation

$$3^{1/2} f_0 f_{\eta' \to 2\gamma} = N_c \frac{e^2}{8\pi^2} \sum_f Q_f^2 \rho \bigg( \frac{m_{\eta'}^2}{12M^2} + \frac{1}{4} m_{\eta'}^2 \rho^2 \bigg).$$  \tag{32}$$

In the $r = 0$ approximation we conclude that right side of this equation is equal to

$$N_c \frac{e^2}{8\pi^2} \sum_f Q_f^2 \bigg( \frac{m_{\eta'}^2}{12M^2} + \frac{1}{4} m_{\eta'}^2 \rho^2 \bigg).$$  \tag{33}$$

On the other hand, as it is well known, the QED axial anomaly in the divergence of the third component of the isovector axial current gives the expression

$$f_\pi f_{\pi_0 \to 2\gamma} = N_c \frac{e^2}{8\pi^2} (Q_u - Q_d^2),$$  \tag{34}$$

where $f_{\pi_0 \to 2\gamma}$ is defined analogously with $f_{\eta' \to 2\gamma}$. Theoretical value of the width $\Gamma_{\pi_0 \to 2\gamma} = 7.25\, KeV$ is rather close to the experimental one 7.95$\, KeV$.

With experimental value for the width $\Gamma_{\eta' \to 2\gamma} = 4.296\, KeV$, $M = 340\, MeV$ and $\rho = 0.3\, fm$ in (33) we estimate that

$$f_0 = 1.16 f_\pi.$$  \tag{35}$$

This quantity was estimated previously by another method using $SU_f(3)$-symmetry type arguments and assumptions about the $J/\psi \to \gamma \eta(\eta')$ decay mechanism \[11, 12, 13\]. Here the axial anomaly equation, eq. (31), is used again but for the matrix elements over the vacuum and $\eta$ and $\eta'$ states. As a result, neglecting the contributions of the current masses of the light $(u, d, s)$ quarks we get

$$m_{\eta'}^2 3^{1/2} f_0 = N_f \frac{g^2}{16\pi^2} < 0|G \bar{G}|\eta' >$$  \tag{36}$$

The matrix element on the right side can be extracted from data on the transitions and decays of heavy quarkonium systems\[11\]. Taking into account theoretical relation $\[11, 14\]$ and experimental widths of the $J/\psi \to \gamma \eta(\eta')$ decays$\[12\]$

$$\frac{| < 0|G \bar{G}|\eta' > |^2 p_{\eta'}^3}{| < 0|G \bar{G}|\eta > |^2 p_\eta^3} = \frac{\Gamma(J/\psi \to \gamma \eta' (p_{\eta'}))}{\Gamma(J/\psi \to \gamma \eta (p_\eta))} = 5$$

it is easy to find that $\[10\]$

$$\frac{| < 0|G \bar{G}|\eta' > |}{| < 0|G \bar{G}|\eta > |} = 2.46.$$
On the other hand

\[ N_f \frac{g^2}{16\pi^2} < 0|G\tilde{G}|\eta> = -2im_s < 0|\bar{s}\gamma_5 s|\eta> = (\frac{2}{3})^{1/2}f_sm^2_{\eta}. \]

\( SU_f(3) \) symmetry arguments give \( f_S = 1.86f_\pi \).

This type of the estimates gives

\[ f_0 = 1.06f_\pi. \] (37)

The estimates, Eqs.(35) and (37) agree with each other even better than the usual accuracy of the \( SU_f(3)-\)symmetry. It is crucial to repeat all of these estimates with account of the \( SU_f(3)-\)symmetry breaking.

4 Conclusion

The solution of the axial anomaly low-energy theorems very nontrivially related with nonperturbative instanton structure of the QCD vacuum. The accuracy of the instanton vacuum based chiral quark model \( \text{[3]} \) to satisfy these theorems is \( \sim 17\% \) and, probably, is related with the accuracy of the interpolating formula \( \text{[1]} \) for the quark propagator.

This approach provides a solid background to calculating the different amplitudes of the nonperturbative conversion of gluons into hadrons and photons. As example, we applied this approach to the \( \eta' \to 2\gamma \) decay process and, in result, successfully estimated \( \eta'-\)singlet axial current coupling constant, the Eq.(35).

We are planning to apply this approach to the hadronic and photonic transitions in a heavy quark systems and the scattering of these systems on hadrons.

We hope to provide further discussions of all of these problems in a separate publication.

5 Acknowledgments

One of the authors(M.M.) is grateful to I. Musatov and E. Shuryak for useful discussions about the instantons and acknowledges the hospitality of the Theory Group of CEBAF, the Theoretical Physics Institute of the University of Alberta and the Department of Physics of SUNY at Stony Brook. The work of M. Musakhanov is supported in part by the grant INTAS-93-0239. The work of F.Khanna is supported in part by the Natural Sciences and Engineering research Council of Canada.

References

[1] M. A. Shifman, Sov. Phys. Usp. 32 (1989) 289.
[2] B. Faizullaev, M. M. Musakhanov, N. K. Pak, Phys. Lett. B361 (1995) 155.
[3] D. Diakonov and V. Petrov, Nucl. Phys. B272 (1986) 457.
[4] D. I. Diakonov, M. V. Polyakov, C. Weiss, Hadronic matrix elements of gluon operators in the instanton vacuum, hep/ph/9510232.
[5] A. Belavin, A. Polyakov, A. Schwartz and Yu. Tiupkin, *Phys. Lett.* **59 B** (1975) 85.

[6] G. ’t Hooft, *Phys. Rev.* **D 14** (1976) 3432; *ibid.* **D 18** (1978) 2199.

[7] E. V. Shuryak, *Nucl. Phys.* **B203** (1982) 93, 116; *Phys. Rev. D** **52** (1995) 5370.

[8] D. Diakonov and V. Petrov, *Nucl. Phys.* **B245** (1984) 259.

[9] S. L. Glashow, R. Jackiw, S. S. Shei, *Phys. Rev.* **187** (1969) 1916.

[10] E. V. Shuryak, *Rev. Mod. Phys.* **65** (1993) 1.

[11] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, V. I. Zakharov, *Nucl. Phys. B165* (1980) 55.

[12] J. J Hernangez et al., *Phys. Lett.* **B239** (1990) 1.

[13] Patricia Ball, J.-M. Frere, M. Tytgat, Phenomenological evidence for the gluon content of $\eta$ and $\eta'$, [hep-ph/9508359](http://arxiv.org/abs/hep-ph/9508359).