Disentanglement and Inseparability correlation: in two-qubit system

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Started from local universal isotropic disentanglement, a threshold inequality on reduction factors is proposed, which is necessary and sufficient for this type of disentanglement processes. Furthermore, we give the conditions realizing ideal disentanglement processes provided that some information on quantum states is known. In addition, based on fully entangled fraction, a concept called inseparability correlation is presented. Some properties on inseparability correlation coefficient are studied.

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I. INTRODUCTION

Entanglement describes a system composed of two or more particles, which exhibits the astonishing property that the results of measurements on one particle can not be specified independently of the parameters of the measurements on the other particles. As one of the most striking features of the quantum mechanics, entanglement is playing a more and more important role in the young field of quantum information [1].

Recently, there has been growing interest in disentanglement of quantum states. Usually an ideal disentanglement process can be described as: transforming a state of two or more subsystems into a disentangled state (generally a mixture of product states) such that the reduced density matrices of the subsystems are unaffected, i.e.

\[ \rho_{12} \rightarrow \rho'_{12} = \sum_i \omega_i \rho_i^{(1)} \otimes \rho_i^{(2)} \quad (1) \]

with

\[ T_{r_1} \rho_{12} = T_{r_1} \rho'_{12}, \]
\[ T_{r_2} \rho_{12} = T_{r_2} \rho'_{12}, \quad (2) \]

where the positive weights \( \omega_i \) satisfy \( \sum_i \omega_i = 1 \), and \( \rho_i^{(1)} \) and \( \rho_i^{(2)} \) refer to the density matrices of the subsystem 1 and 2 respectively.

In an extreme case of disentanglement, both quantum entanglement and classical correlations are eliminated, i.e., the final state becomes

\[ \rho_{12} \rightarrow \rho'_{12} = \rho^{(1)} \otimes \rho^{(2)}. \quad (3) \]

It was recently shown by Terno [2] that an arbitrary state can not be disentangled into a tensor product of its reduced density matrices (i.e. to satisfy Eqs (2) and (3)). Otherwise, in the process to distinguish an quantum state secretly chosen from a certain set of two entangled states, the probability that the observer makes a wrong guess will be lower than Helstrom’s minimal error probability, which can not be allowed by quantum mechanics. Subsequently, Mor proved that an ideal universal disentangling machine to satisfy Eqs (2) and (3) can not exist [4]. In other words, an arbitrary quantum entangled state can not be disentangled. Similar to quantum no-cloning theorem [3], this result also stems from the linearity of quantum mechanics. Since ideal universal disentanglement processes can not be realized, a natural question is then what level of disentanglement can be reached by a universal disentangling machine? Does there exist an optimal universal disentangling machine? Started from Peres-Horodecki’s separability criterion [5], S. Bandyopadhyay et. al have constructed a type of universal disentangling machine by using local

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cloning operations. Afterwards, they discussed optimal universal disentangling machines for two-qubit system and obtained a result identical to the case of local cloning operations. In the present paper, we shall study the essential limits to the universal processes of disentanglement under the condition of local operations. In addition, arising from a class of special local operations, the varies of correlation feature of the given composite system are also considered.

In section 2, by analyzing the changes of Horodecki’s T matrix induced by local operations, a necessary condition for local universal disentangling machines is derived. Furthermore, we prove that the necessary condition are also sufficient for this type of disentanglement processes. We find that Bandyopadhyay’s results are only two special cases to satisfy this threshold boundary. Moreover, for a certain set of two-qubit pure entangled states on which one knows partial information, we put forward a necessary and sufficient condition realizing ideal disentanglement by manipulating only one part of states. In section 3, based on fully entangled fraction (see ref.), a concept called inseparability correlation is proposed and its properties under some local operations are studied. In the last section, we give a brief argument on some aspects of disentanglement and entanglement measure.

II. DISENTANGLEMENT BY LOCAL OPERATIONS

A. Local universal isotropic disentanglement processes

Let $\rho_{\text{ent}}$ be an entangled density matrix of two-qubit system; and $\rho_1$ and $\rho_2$ be the reduced density matrices of the subsystem 1 and 2, respectively. Then an universal isotropic disentanglement process is defined as:

$$\rho_{\text{ent}} \rightarrow \rho_{\text{disent}},$$

(4)

together with

$$\rho_i' = \text{Tr}_j (\rho_{\text{disent}}) = \eta_i \rho_i + \left(1 - \frac{\eta_i}{2}\right) I,$$

(5)

where the parameter $\eta_i$, called reduction factor, describes the shrinking of the ith reduced density matrix. To meet the requirements of universality in the disentanglement process, it is reasonable to require that the reduced density matrices isotropically shrink. Due to the reduced density matrices of the entangled system are mixed states, in general, the fidelity between the reduced density matrices $\rho_i$ and $\rho_i'$:

$$F(\rho_i, \rho_i') = \left(\text{Tr} \sqrt{\rho_i' \rho_i \rho_i' \rho_i}\right)^2$$

cannot remain a constant. Fidelity is no longer a suitable standard to quantify of disentanglement. So, we use the reduction factors to define a quality factor of disentanglement:

$$Q = \frac{\eta_1 + \eta_2}{2}.$$  

(6)

when Q reaches its maximum, the corresponding universal disentanglement process is optimal.

For a two-qubit system, it is convenient to use the Hilbert-Schmidt space representation of the density matrices:

$$\rho_{12} = \frac{1}{4} \left( I \otimes I + \vec{r} \cdot \vec{\sigma} \otimes I + I \otimes \vec{s} \cdot \vec{\sigma} + \sum_{m,n=1}^{3} t_{mn} \sigma_m \otimes \sigma_n \right),$$

(7)

where $\rho_{12}$ acts on the Hilbert space $H = H_1 \otimes H_2 = C^2 \otimes C^2$. $I$ stands for the identity operator, {$\sigma_n$}$^{3}_{n=1}$ are the standard Pauli matrices, and $\vec{r}$ and $\vec{s}$ are vectors in $R^3$ called Bloch Vectors with $\vec{r} \cdot \vec{s} = \sum_{i=1}^{3} r_i \sigma_i$. The coefficients $t_{mn} = \text{Tr} (\rho_{12} \sigma_m \otimes \sigma_n)$ form a real matrix which we shall denote by $T(\rho_{12})$. Each part of reduced density matrices is the following:

$$\rho_1 = \text{Tr}_2 \rho_{12} = \frac{1}{2} (I + \vec{r} \cdot \vec{\sigma}),$$

$$\rho_2 = \text{Tr}_1 \rho_{12} = \frac{1}{2} (I + \vec{s} \cdot \vec{\sigma}).$$

(8)

(9)

For any local general measurements, a particularly useful description is the so-called operator-sum representation.
\[ \rho_1' = \sum_i A_i \rho_1 A_i^+, \quad (10) \]
\[ \rho_2' = \sum_j B_j \rho_2 B_j^+. \quad (11) \]

The two sets of operators satisfy the completeness relations \( \sum_i A_i^+ A_i = I \) and \( \sum_j B_j^+ B_j = I \). For simplicity of expression, we use two completely positive and trace-preserving superoperators \( \tilde{V}_i \) (i = 1, 2) to characterize the above maps. In view of the requirements of Eqs (10) and (11), the map relations can be written as
\[ \rho_1' = \tilde{V}_1 (\rho_1) = \frac{1}{2} (I + \eta_1 \vec{\tau} \cdot \vec{\sigma}) , \quad (12) \]
\[ \rho_2' = \tilde{V}_2 (\rho_2) = \frac{1}{2} (I + \eta_2 \vec{\tau} \cdot \vec{\sigma}) . \quad (13) \]

Due to the arbitrariness of the vectors \( \vec{\tau} \) and \( \vec{\sigma} \) (\(| \vec{\tau} |, | \vec{\sigma} | < 1 \)), we can further deduce the map relations satisfied by the identity operator and the Pauli operators:
\[ \left\{ \begin{array}{l}
\tilde{V}_i (I) = I , \\
\tilde{V}_i (\sigma_j) = \eta_i \sigma_j \quad (i = 1, 2; \; j = 1, 2, 3).
\end{array} \right. \quad (14) \]

Thus, after the joint action described by \( \sum_{i,j} A_i \otimes B_j \), the whole density matrix of the two-qubit system becomes
\[ \rho_{12}' = \frac{1}{4} \left( I \otimes I + \eta_1 \vec{\tau} \cdot \vec{\sigma} \otimes I + \eta_2 I \otimes \vec{\tau} \cdot \vec{\sigma} + \sum_{m,n=1}^{3} \eta_1 \eta_2 t_{mn} \sigma_m \otimes \sigma_n \right) . \quad (15) \]

**B. Horodecki's tetrahedron**

For a long time, it is a puzzling question to check if a quantum system is entangled or not. Peres and Horodecki et. al have already systematically studied this question [5]. Horodecki et. al proved that Peres’s criterion is necessary and sufficient condition for separability of 2×2 and 2×3 systems.

For a two-qubit system, the correlations between subsystems are embodied in the matrix \( T (\rho) \) [10]:
\[ E (\vec{\tau}, \vec{\sigma}) = Tr \left( \rho \vec{\tau} \cdot \vec{\sigma} \otimes \vec{\tau} \cdot \vec{\sigma} \right) = \left( \vec{\tau}, T \vec{\sigma} \right) . \quad (16) \]

In accordance with Eq. (14), we can always select Pauli’s operators in a suitable representation to diagonalize the matrix \( T \):
\[ T' = O_1 T O_2^+ = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} , \quad (17) \]

where \( O_i \) stands for a real orthonormal matrix. In this representation the initial Bloch vectors \( \vec{\tau} \) and \( \vec{\sigma} \) are changed into:
\[ \vec{\tau}' = O_1 \vec{\tau} \quad \text{and} \quad \vec{\sigma}' = O_2 \vec{\sigma} . \quad (18) \]

Then, the matrix \( T \) is connected with the vector \( \vec{T} = (t_1, t_2, t_3) \) (\( \vec{T} \in R^3 \)). Horodecki et. al have already systematically analyzed the space of the characteristic vectors. For an arbitrary quantum state, all the characteristic vectors of its matrix \( T \) are contained in the tetrahedron \( \tilde{T} \) with vertices \( \tilde{t}_0 = (-1, -1, -1) \), \( \tilde{t}_1 = (-1, 1, 1) \), \( \tilde{t}_2 = (1, -1, 1) \), \( \tilde{t}_3 = (1, 1, -1) \). For all separable quantum states, their characteristic vectors belong to the octahedron \( \tilde{L} \) with the vertices \( \tilde{a}_{\pm} = (\pm 1, 0, 0) \), \( \tilde{a}_{\pm} = (0, \pm 1, 0) \), \( \tilde{a}_{\pm} = (0, 0, \pm 1) \) (see Fig. 1). (It is worthy to stress that the vector \( \vec{T} \) relevant to the matrix \( T \) is not unique because different orthonormal matrices can be selected to diagonalize the matrix \( T \). It
where the maximum is over all completely entangled states from [20]: defined "fully entangled fraction" of a density matrix $\rho$ correlation:

$$T$$

tall the quantum states with characteristic vectors being located in $\tilde{\text{ABC}}$ at dot $\rightarrow$ fraction is concerned, these states can not be different from separable quantum states. $\tilde{\text{tetrahedron}}$ $\rho$, $\rightarrow$ is an arbitrary dot in inseparability correlation region. In the hexahedron $\tilde{\text{tetrahedron}}$ $\tilde{T}$ is mapped to the hexahedron $\tilde{H}$ with vertices $\tilde{\sigma} = (0,0,0)$, $\tilde{A} = (1,0,0)$, $\tilde{B} = (0,1,0)$, $\tilde{C} = (0,0,1)$, $\tilde{D} = (1,1,1)$ (see Fig. 2).

All the map relations are summarized in the following:

$$
\begin{align*}
\tilde{T} = (t_1, t_2, t_3) &\rightarrow \tilde{m} = (|t_1|, |t_2|, |t_3|) \\
T \rightarrow H &\\
L \rightarrow \tilde{T}_{oABC}
\end{align*}
$$

(20)

C. The threshold condition for disentanglement

For a $2 \times 2$ composite system, people pay more attention to the correlation between subsystems. C. H. Bennett et. al defined "fully entangled fraction" of a density matrix $\rho$ [14], which can act as an index to characterize nonlocal correlation:

$$f (\rho) = \max \langle e | \rho | e \rangle ,$$

(21)

where the maximum is over all completely entangled states $|e\rangle$. Fully entangled fraction can be indicated as another from [20]:

$$f (\rho) = \frac{1}{4} (1 + N (\rho)) ,$$

(22)

where $N(\rho)=\text{Tr}M(\rho)$ (see ref [10][11]). Although, relying on different representations, a density matrix $\rho$ normally can correspond to different matrix $M(\rho)$, the trace of matrix $M(\rho)$ is unique.

In Fig 2, for all separable quantum states, the characteristic vectors of their matrices $M$ are contained in the tetrahedron $T_{oABC}$, while, all the dots in the tetrahedron $T_{ABCD}$ correspond to inseparable quantum states. We call $T_{oABC}$ and $T_{ABCD}$ separability and inseparability correlation region respectively. Here, it is worthy to note that not all the quantum states with characteristic vectors being located in $T_{oABC}$ are separable, but, as far as fully entangled fraction is concerned, these states can not be different from separable quantum states.

$Q$ is an arbitrary dot in inseparability correlation region $T_{ABCD}$ (see Fig. 2). Line segment $\rightarrow Q$ intersects plane $ABC$ at dot $Q'$. Let us define the inseparability correlation coefficient $I_c$ of the dot $Q$:

$$I_c = \frac{\rho Q - \rho Q'}{\rho Q'} = N (\rho Q) - 1.$$

(23)

We stipulate that inseparability correlation coefficients of all the dots in the region $T_{oABC}$ are zero. Thereby, for any two-qubit density matrix $\rho$, its inseparability correlation coefficient has the following form:

$$I_c (\rho) = \begin{cases} 
0 & (N (\rho) \leq 1) \\
N (\rho) - 1 & (N (\rho) > 1)
\end{cases} .$$

(24)

At a geometric intuitive angle, inseparability correlation coefficient depicts the minimal distance from the characteristic vector of given two qubits system to separability correlation region. In the hexahedron $\tilde{H}$, all the dots in the identical plane perpendicular to the orientation $(1,1,1)$ have the same inseparability correlation coefficient. The maximum of $I_c$ is in the dot $D$, $I_c (\text{max}) = 2$, corresponding to four Bell states.

By observing Eq (13), we find local universal isotropic disentangling operations cause the matrix $T (\rho)$ shrink by the ratio $\eta_1 \eta_2$. In view of inseparability correlation coefficient $I_c$, we can achieve the necessary condition realizing local universal isotropic disentangling operations:

$$\eta_1 \eta_2 \leq \frac{1}{1 + I_c (\text{max})} = \frac{1}{3} .$$

(25)
In the following part, we will prove that the necessary condition is also sufficient for this type of disentangling operations.

For an arbitrary pure entangled state $|\Psi\rangle_i$, without loss of generality, it can be indicated as $|\Psi\rangle_i = \cos \theta |00\rangle + \sin \theta |11\rangle$ in a suitable representation. In this case we have:

$$\rho^i = \begin{pmatrix} \cos^2 \theta & 0 & 0 & \sin \theta \cos \theta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sin \theta \cos \theta & 0 & 0 & \sin^2 \theta \end{pmatrix}.$$ (26)

We can use the coefficients of group operators in Hilbert Schmidt Space to represent the above density matrix:

$$\rho^i = \frac{1}{4} \begin{pmatrix} 1 + t_{33} + s_3 + r_3 & 0 & 0 & t_{11} - t_{22} \\ 0 & 1 - t_{33} + s_3 - r_3 & t_{11} + t_{22} & 0 \\ 0 & t_{11} + t_{22} & 1 - t_{33} - s_3 + r_3 & 0 \\ t_{11} - t_{22} & 0 & 0 & 1 + t_{33} - s_3 - r_3 \end{pmatrix}.$$ (27)

(Here, we leave out the coefficients equal to zero.) These coefficients satisfy the following relations:

$$\begin{cases} s_3 = r_3 = s, \\ t_{11} = -t_{22} = t, \\ t_{33} = 1, \\ \sum_{i=1}^3 t_{ii}^4 + s_i^2 + r_i^2 = 3. \end{cases}$$ (28)

After a local universal isotropic disentangling operation to satisfy map relation (24), the density matrix $\rho^0$ is changed into:

$$\rho^0 = \frac{1}{4} \begin{pmatrix} 1 + \eta_1 \eta_2 + (\eta_1 + \eta_2) s & 0 & 0 & 2\eta_1 \eta_2 t \\ 0 & 1 - \eta_1 \eta_2 + (\eta_1 + \eta_2) s & 0 & 0 \\ 0 & 0 & 1 - \eta_1 \eta_2 - (\eta_1 - \eta_2) s & 0 \\ 2\eta_1 \eta_2 t & 0 & 0 & 1 + \eta_1 \eta_2 - (\eta_1 + \eta_2) s \end{pmatrix}.$$ (29)

We define matrix $\rho^T$ as the partial transposition of $\rho^0$ (see ref [3]). Thus, $\rho^T$ has the following form:

$$\rho^T = \frac{1}{4} \begin{pmatrix} 1 + \eta_1 \eta_2 + (\eta_1 + \eta_2) s & 0 & 0 & 2\eta_1 \eta_2 t \\ 0 & 1 - \eta_1 \eta_2 + (\eta_1 - \eta_2) s & 0 & 2\eta_1 \eta_2 t \\ 0 & 0 & 1 - \eta_1 \eta_2 - (\eta_1 - \eta_2) s & 0 \\ 0 & 2\eta_1 \eta_2 t & 0 & 1 + \eta_1 \eta_2 - (\eta_1 + \eta_2) s \end{pmatrix}.$$ (30)

Based on Peres-Horodecki criterion, the necessary and sufficient condition satisfying $\rho^0$ is separable follows that none of the eigenvalue of $\rho^T$ is negative, i.e.:

$$[1 - \eta_1 \eta_2 + (\eta_1 - \eta_2) s][1 - \eta_1 \eta_2 - (\eta_1 - \eta_2) s] - 4\eta_1^2 \eta_2^2 t^2 \geq 0.$$ (31)

Under the conditions (28), we can derive inequality:

$$(1 - \eta_1^2)(1 - \eta_2^2) + \left[(\eta_1 - \eta_2)^2 - 4\eta_1^2 \eta_2^2\right]t^2 \geq 0.$$ (32)

when $t$ reaches maximum ($t_{max} = 1$), the threshold inequality for local universal isotropic disentanglement processes is obtained:

$$\eta_1 \eta_2 \leq \frac{1}{3}.$$ (33)

In addition, we know all the separable states still keep separable after being performed local operations. Hence, we complete the proof.

S. Bandyopadhyay et. al have studied the disentanglement processes by using local cloning operations [6]. Their main results can be summarized as that it is possible to disentangle any two-qubit entangled state either by applying local cloning on one of its qubits provided the reduction factor of the isotropic cloner is less than or equal to $\frac{1}{3}$, or by local cloning of the individual subsystems provided the reduction factor of the isotropic cloner used is less than or
equal to $\frac{1}{2^2}$. Obviously, the two kinds of cloning operations are only special cases to satisfy the threshold condition \(Q_2\) for local universal isotropic disentangling operations.

At the first part of this section, we have defined the quality factor of disentanglement \(Q\). Under local isotropic disentangling operations, based on the threshold inequality \(Q_2\), the maximal \(Q\) can be obtained: \(Q_{\text{max}} = \frac{2}{3}\), corresponding to the case \(\eta_1 = 1, \eta_2 = \frac{1}{2}\).

Finally, let us consider the following question: if the local universal disentanglement processes satisfying the threshold limit \(Q_2\) eliminate quantum entanglement of the given system, can they fully eliminate classical correlations between subsystems and make the density matrix of the whole system become the product of local density matrices? The answer is negative unless \(\eta_1\eta_2 = 0\). It is shown in Fig 2 that for any pure entangled state \(\rho\), none of the eigenvalues of the matrix \(M(\rho)\) is zero (see ref \([4]\)). Hence, after the disentanglement process each eigenvalue of the matrix \(M(\rho)\) is only shrunk by the ratio \(\eta_1\eta_2\) ( \(\eta_1\eta_2 \neq 0\)), while the determinant of \(M(\rho)\) is still nonzero. However, it is easy to prove that a necessary condition for arbitrary quantum state with the product form is that its matrix \(M\) has zero determinant. As a result, when the factor \(\eta_1\eta_2\) is not equal to zero, local universal isotropic disentangling operations can not completely cut off classical correlations between subsystems.

D. Ideal disentanglement in some special cases

As we know, there are no universal ideal disentanglement processes. There appears another question. Provided that some information about quantum states is obtained, can we perform ideal disentangling operations for these quantum states? Clearly, if we know all the information about an entangled state, it can be disentangled completely. Suppose that Alice has a bipartite system: \(|\Psi\rangle_{AB} = \cos \alpha |0\rangle_A |1\rangle_B + e^{i\vartheta} \sin \alpha |1\rangle_A |0\rangle_B \). She sends particle B to Bob. If Bob has known the wavefunction of the biparticle state and which particle is sent, he can prepare a pair of entangled particles identical to Alice’s: \(|\Psi\rangle'_{A'B'} = \cos \alpha |0\rangle_{A'} |1\rangle_{B'} + e^{i\vartheta} \sin \alpha |1\rangle_{A'} |0\rangle_{B'} \). Bob swaps particle B with particle B’, and traces out the particles B and A’. As a result, the state of the remaining particles is a product of mixed states. This is a trivial result.

In general, we do not know all the information on quantum states. Assume that Alice has a set of known two-qubit pure entangled states \(S = \{|\Psi^{(i)}_{AB}\rangle\}_{i=1}^{n}\). She randomly singles out quantum states from this set and transmits part B of these states. Here, the reduced density matrices of different states satisfy the relation \(\rho_B^i \neq \rho_B^j\) ( \(i \neq j\)). Provided that a particle B is captured by eavesdropper Eve in its transmitting process, in what case Eve can perform an ideal disentangling operation for this entangled state by only manipulating the particle B? To solve this question, here, we introduce the following theorem:

Theorem 1 : The state secretly chosen from the set of two-qubit pure entangled states \(S = \{|\Psi^{(i)}_{AB}\rangle\}_{i=1}^{n}\) can be perfectly disentangled by only manipulating the particle B if the reduced density matrices of the part B of all the states in this set commute with each other, i.e. \(\rho_B^i, \rho_B^j = 0\) ( \(i \neq j\); \(i, j = 1,..., n\)).

Proof: If all the reduced density matrices \(\rho_B^i, \rho_B^j\) commute with each other, their spectral decompositions have the form:

\[
\rho_B^i = \lambda^i |s\rangle \langle s| + (1 - \lambda^i) |\perp\rangle \langle \perp|,
\]

(34)

where \(|s\rangle, |\perp\rangle\) is a set of orthonormal basis for the qubit system. One can use the particle B as the control bit and an additional qubit as the target to perform a CNOT ( Controlled NOT ) operation:

\[
\begin{align*}
|s\rangle |0\rangle \rightarrow |s\rangle |0\rangle, \\
|\perp\rangle |0\rangle \rightarrow |\perp\rangle |1\rangle,
\end{align*}
\]

(35)

where \(|0\rangle\) and \(|1\rangle\) are orthonormal states. After taking the trace over the additional qubit, \(\rho_B^{(i)}\) becomes a mixture of product states, while its reduced density matrices keep invariant, i.e., a perfect disentangling process is fulfilled.

In the special cases that the eavesdropper only knows some information on the state set, we get another theorem:

Theorem 2 : For a set of two-qubit pure entangled states \(S = \{|\Psi^{(i)}_{AB}\rangle\}_{i=1}^{n}\), the reduced density matrices of the part B are assumed to have the form \(\rho_B^i = \lambda^i |s^i\rangle \langle s^i| + (1 - \lambda^i) |\perp^i\rangle \langle \perp^i|\). If one only knows the orthonormal basis of \(\rho_B^i\)’s spectral decompositions \(|s^i\rangle, |\perp^i\rangle\) but not spectral coefficients \(\{\lambda^i\}_{i=1}^{n}\), he can perform an ideal deterministic disentangling operation for this set by only manipulating the particle B, if and only if \(\rho_B^i, \rho_B^j = 0\) ( \(i \neq j\); \(i, j = 1,..., n\)).
Proof: The sufficient condition can be easily obtained from the theorem 1. Our task remains to prove the converse. Suppose that there are two density matrices $\rho_A$ and $\rho_B$ with $[\rho_A, \rho_B] \neq 0$. We introduce an additional system C in the state $|e_0 \rangle$. (if the state of the system is mixed, without loss of generality, we can view it as a pure state in a larger space). Then, a joint unitary evolution between the particle $B$ and the additional system is performed:

$$e^{i\Delta^{(i)}} \sqrt{\lambda^i}|s^{iA}\rangle U_{BC} (|s^{iB}\rangle |e_0 \rangle) + e^{i\Delta^{(i)}} \sqrt{1 - \lambda^i}|s^{iA}\rangle U_{BC} (|s^{iB}\rangle |e_0 \rangle).$$

(36)

An ideal disentanglement process requires that the reduced density matrices of the original state remain unchanged, i.e.,

$$\rho_A = \text{Tr}_{BC}(\rho_{ABC}) = \lambda^i |s^{iA}\rangle \langle s^{iA}| + (1 - \lambda^i) |s^{iA}\rangle \langle s^{iA}|,$$

(37)

$$\rho_B = \text{Tr}_{C}(\rho_{ABC})$$

$$= \text{Tr}_{C} \{\lambda^i U_{BC} (|s^{iB}\rangle |e_0 \rangle \langle e_0|) U_{BC}^+ + (1 - \lambda^i) U_{BC} (|s^{iB}\rangle |e_0 \rangle \langle e_0|) U_{BC}^+ \}$$

$$= \lambda^i |s^{iB}\rangle \langle s^{iB}| + (1 - \lambda^i) |s^{iB}\rangle \langle s^{iB}|.$$

(38)

Let us set $\sigma_3^{(iB)} = |s^{iB}\rangle \langle s^{iB}| - |s^{iB}\rangle \langle s^{iB}|$. Then $\rho_B$ has the following form:

$$\rho_B = \frac{1}{2} \left(I + 2 (\lambda^i - \frac{1}{2}) \sigma_3^{(iB)} \right).$$

(39)

We can use a completely positive and trace-preserving superoperator $\tilde{V}$ to characterize the operation for particle $B_i$:

$$\tilde{V} (|s^{iB}\rangle \langle s^{iB}|) = \frac{1}{2} (I + a_i \tilde{\sigma}_{i1}),$$

(40)

$$\tilde{V} (|s^{iB}\rangle \langle s^{iB}|) = \frac{1}{2} (I + b_i \tilde{\sigma}_{i2}),$$

(41)

where $\tilde{\sigma}_{i1}$ and $\tilde{\sigma}_{i2}$ satisfy $\tilde{\sigma}_{i1} \cdot \tilde{\sigma}_{i1} = I$ and $\tilde{\sigma}_{i2} \cdot \tilde{\sigma}_{i2} = I$, respectively. The $a_i$ and $b_i$ are real numbers satisfying $|a_i| \leq 1$ and $|b_i| \leq 1$. The superoperator $\tilde{V}$ maps the reduced density matrix $\rho_B$ to itself:

$$\tilde{V} (\rho_B) = \lambda^i \tilde{V} (|s^{iB}\rangle \langle s^{iB}|) + (1 - \lambda^i) \tilde{V} (|s^{iB}\rangle \langle s^{iB}|)$$

$$= \lambda^i \left[ \frac{1}{2} (I + a_i \tilde{\sigma}_{i1}) \right] + (1 - \lambda^i) \left[ \frac{1}{2} (I + b_i \tilde{\sigma}_{i2}) \right].$$

(42)

So, we obtain the following relation

$$\lambda_i a_i \tilde{\sigma}_{i1} + (1 - \lambda_i) b_i \tilde{\sigma}_{i2} = 2 \left( \lambda^i - \frac{1}{2} \right) \sigma_3^{(iB)}.$$

(43)

Since $\lambda_i$ is unknown, by comparing two sides of the above equation, we get further relations

$$\tilde{\sigma}_{i1} = \tilde{\sigma}_{i2} = \sigma_3^{(iB)};$$

(44)

$$a_i = 1, \ b_i = -1.$$  

(45)

Thus the joint unitary evolution must be
where $|e_1\rangle$ and $|e_2\rangle$ are two normalized states satisfying $\langle e_1 | e_1 \rangle = 1$ and $\langle e_2 | e_2 \rangle = 1$, respectively. After the joint unitary evolution, the inseparability correlation coefficient of the system AB is $I_c (\rho^{(i)'}_{AB}) = 4 \sqrt{\lambda_i (1 - \lambda_i)} |\langle e_1 | e_2 \rangle|$. To meet the requirement of $I_c = 0$, $|e_1\rangle$ and $|e_2\rangle$ must be orthonormal. As the final result, we find that $|e_1\rangle$ and $|e_2\rangle$ are two orthonormal states. Thereby, the reduced density matrices of the additional system and the particle $B_i$ have the same form. In essence, a universal broadcasting process of the mixed states is fulfilled. This result is conflict with the result that noncommuting mixed states cannot be broadcast [12].

If one performs a projection measurement on the additional system, an uncontrolled change of the reduced density matrix of the particle $A_i$ will lead to the failure of ideal disentanglement. While, any operation in quantum mechanics can be represented by a generalized unitary evolution, together with a measurement. Hence, we complete the proof.

### III. PROPERTIES OF INSEPARABILITY CORRELATION COEFFICIENT

In this section, we will embark on analyzing the properties of inseparability correlation coefficient under some local operations. For $I_c$ coefficient we have:

Prop 1 : $I_c (\rho) = 0$ if $\rho$ is separable. When the given system is in pure state or Bell mixed state $I_c = 0$ can be used as necessary and sufficient criterion of separability.

Prop 2 : Local unitary operations leave $I_c (\rho)$ invariant, i.e. $I_c (\rho) = I_c (U_A \otimes U_B \rho U_A^+ \otimes U_B^+)$. For Prop 1, we can refer to [11] and [13]. Prop 2 is evident because local unitary transformations cannot give rise to the change of three eigenvalues of the matrix $M$. However, we are more concerned with the properties of inseparability correlation coefficient under local general operations. Does it have the properties similar to the entanglement measures?

For a qubit system, its density matrix has the following form:

$$\rho = \frac{1}{2} (I + \vec{\tau} \cdot \vec{\sigma}).$$

(48)

A general operation maps $\rho$ to $\rho'$ [13]:

$$\rho' = \tilde{V} (\rho) = \frac{1}{2} (I + \tilde{\vec{\tau}} \cdot \tilde{\vec{\sigma}}),$$

(49)

$$\tilde{\vec{\sigma}} = \tilde{\vec{\tau}} + \vec{\tau} \cdot B,$$

(50)

where $\vec{\tau}$ is a constant vector in $R^3$, and $B$ is a $3 \times 3$ real matrix. This is an affine map, mapping the Bloch sphere into itself. Under the action of the superoperator $\tilde{V}$, the identity and Pauli operators have the following map relations:

$$\left\{ \begin{array}{l} \tilde{V} (I) = I + \tilde{\vec{\tau}} \cdot \tilde{\vec{\sigma}} \\ \tilde{V} (\sigma_i) = \sum_{j=1}^{3} B_{ij} \sigma_j \end{array} \right..$$

(51)

For a $2 \times 2$ system described by Eq(7), we perform a general operation on the first subsystem. Here, we only consider the case of $\vec{\tau} = 0$, in which adopting the following trick will excuse us from complicated calculations.

We can introduce an auxiliary Bell mixed state $\rho_B$ corresponding to the state $\rho_{12}$ described by Eq(7):

$$\rho_B = \frac{1}{4} \left( I \otimes I + \sum_{m=1}^{3} t_m \sigma \otimes \sigma \right).$$

(52)

With the limit of $\vec{\tau} = 0$, we have:

$$T (\rho'_{12}) = B^T T (\rho_{12}) = B^T T (\rho_B) = T (\rho_B').$$

(53)
Because inseparability correlation coefficient only depends on the matrices $T$ we can study the varies of Bell mixed state $\rho_2$ instead of the state $\rho_{12}$.

With regard to Bell mixed states, their entanglement of formation has been given by C. H. Bennett et. al [14]:

$$E(\rho) = h(f(\rho)),$$

where $f(\rho)$ refers to fully entangled fraction. The function $h(f)$ is defined as:

$$h(f) = \begin{cases} H\left[\frac{1}{2} + \sqrt{f(1-f)}\right], & (f \geq \frac{1}{2}) \\ 0, & (f < \frac{1}{2}) \end{cases}$$

(55)

Here, $H(x) = -x \log_2 x - (1-x) \log_2 (1-x)$. In accordance with Eqs (22, 54, 55), we find that $E(\rho)$ is a monotonically increasing function of $N(\rho)$ for the Bell mixed state $\rho$ in inseparability correlation region. While, an accepted fact is that entanglement cannot increase under local operations and classical communications. Therefore, $N(\rho)$ cannot increase. As far as the Bell mixed states in separability correlation region are concerned, $N(\rho)$ cannot exceed one under local operations. Thus, we conclude that inseparability correlation coefficient $I_c(\rho)$ can not increase under a class of special local operations ($\vec{\delta} = 0$) and classical communications. i.e.

$$I_c(\rho_{12}) \geq I_c(\tilde{V}(\rho_{12})) = I_c(\rho'_{12}).$$

(56)

In the case of $\vec{\delta} \neq 0$, the properties of inseparability correlation coefficient are not clear yet. We guess, under more general operations, $I_c(\rho)$ cannot increase. We will go on with this problem in our future work.

**IV. DISCUSSION AND CONCLUSION**

Under local operations, to achieve general isotropic disentanglement, a threshold limit has been shown in section 2. A further question is that if nonlocal operations are allowed, to make reduced density matrices shrink isotropically, what threshold limit will be attained? S. Ghosh et. al hold that local disentangling machines are better than nonlocal ones [1]. But they do not give a theoretical proof. We think it is still an open question.

Local cloning operations can be used for broadcasting entanglement [1] and for disentangling. Recently, the N→M optimal universal quantum cloning has been studied [16, 17] by Bruss et. al and Werner, respectively. For the 1→M-qubit cloning operations, the optimal isotropic reduction factor is $\eta_{\text{max}} = \frac{M+1}{2M}$. If symmetrically performing local cloning operations on an unknown two-qubit system, based on the threshold inequality $\eta^2 \leq \frac{1}{3}$, one can derive $M \geq 3$, i.e. the 1→3 cloning operations on each subsystem will disentangle the initial two-qubit system. Furthermore, let us consider nonlocal cloning cases in which the two-qubit is cloned as a whole. The maximal reduction factor is $\eta'_{\text{max}} = \frac{M+1}{6M}$. (Note that here $\eta'$ is a reduction factor in 2×2 Hilbert space. The original matrix $T$ is shrunk by $\eta$. ) Using the method similar to Section 2, we can conclude that: under this type of operations, the necessary and sufficient condition realizing universal disentanglement is $\eta' \leq \frac{1}{3}$. Therefore, the result of $M \geq 6$ is derived, i.e. under nonlocal cloning operations, the 1→6-pair entanglement broadcast will disentangle any two-qubit system. By contrast, we can conclude that quantum inseparability can be copied better by a nonlocal copier than by a local copier.

In the quantum information field, the quantitative degree of entanglement is attracting more and more attention. In recent years, several entanglement measures have been proposed [14, 15]. Bennett et. al use Bell singlet state as a quantitative standard and introduce the concepts of entanglement of formation and entanglement of distillation [14], which characterize two major aspects about given ensemble of entanglement. Vedral et. al define entanglement measure as the minimum distance between given density matrix and the subset containing all disentangled states [15]. This scientific description provides a clear distinction between quantum entanglement and classical correlations. Inseparability correlation coefficient, which may act as a sufficient criterion for entanglement, can not be used to quantify of entanglement. Because the entanglement hidden in the mixed states is not only determined by the matrix $T(\rho)$ but also embodied in the relations between $\tilde{T}$, $\tilde{\delta}$ and $T(\rho)$ [20]. Nevertheless, we believe that there exist close connections among the matrix $T(\rho)$, Bloch vectors of the subsystems and all kinds of definitions on entanglement measure. Studying the relationship is still a meaningful work.

In conclusion, we study local disentanglement processes in two-qubit system, and give the conditions realizing ideal disentanglement provided that some information on quantum states is known. Furthermore, under some local operations, we analyze the behaviors of inseparability correlation coefficient. We believe that the results of the present paper can help in deeper understanding of the connection between entanglement and disentanglement.
Note added: Recently, we found that Ghosh et al. replaced their paper [7], in which the threshold inequality $\eta_1 \eta_2 \leq \frac{1}{3}$ is also obtained.

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FIG. 1. Geometrical representation of characteristic vectors of the matrices $T$, the tetrahedron $\tilde{T}$ represents all states in two-qubit system and the bold-line-contoured octahedron contains the subset of separable states. Here, $A=(-1,-1,-1)$, $B=(1,1,-1)$, $C=(1,-1,1)$, $D=(-1,1,1)$.

FIG. 2. The space of characteristic vectors of the matrices $M$, the tetrahedron $\tilde{T}_{oABC}$ contains characteristic vectors of all separable states. Here, $Q$ is a dot in inseparability correlation region $\tilde{T}_{ABCD}$. Line segment $oQ$ intersects plane $ABC$ at dot $Q'$. 
