Change-Point Analysis of Time Series with Evolutionary Spectra∗

ALESSANDRO CASINI†  PIERRE PERRON‡
University of Rome Tor Vergata  Boston University

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Abstract

This paper develops change-point methods for the spectrum of a locally stationary time series. We focus on series with a bounded spectral density that change smoothly under the null hypothesis but exhibits change-points or becomes less smooth under the alternative. We address two local problems. The first is the detection of discontinuities (or breaks) in the spectrum at unknown dates and frequencies. The second involves abrupt yet continuous changes in the spectrum over a short time period at an unknown frequency without signifying a break. Both problems can be cast into changes in the degree of smoothness of the spectral density over time. We consider estimation and minimax-optimal testing. We determine the optimal rate for the minimax distinguishable boundary, i.e., the minimum break magnitude such that we are able to uniformly control type I and type II errors. We propose a novel procedure for the estimation of the change-points based on a wild sequential top-down algorithm and show its consistency under shrinking shifts and possibly growing number of change-points. Our method can be used across many fields and a companion program is made available in popular software packages.

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†Corresponding author at: Department of Economics and Finance, University of Rome Tor Vergata, Via Columbia 2, Rome, 00133, IT. Email: alessandro.casini@uniroma2.it.
‡Department of Economics, Boston University, 270 Bay State Road, Boston, MA 02215, US. Email: perron@bu.edu.
1 Introduction

Classical change-point theory focuses on detecting and estimating structural breaks in the mean or regression coefficients. Early contributions include, among others, Hinkley (1971), Yao (1987), Andrews (1993), Horváth (1993) and Bai and Perron (1998), who assume the presence of a single or multiple change-points in the parameters of an otherwise stationary time series model; see the reviews of Aue and Hórvath (2013) and Casini and Perron (2019) for more details. More recently there has been a growing interest about functional and time-varying parameter models where the latter are characterized by infinite-dimensional parameters which change continuously over time [see, e.g., Dahlhaus (1997), Neumann and von Sachs (1997), Hörmann and Kokoszka (2010), Dette, Preuß, and Vetter (2011), Zhang and Wu (2012), Panaretos and Tavakoli (2013), Aue, Dubart Nourinho, and Hormann (2015) and van Delft and Eichler (2018)]. Several authors have extended the scope of the stationarity tests originally introduced by Priestley and Subba Rao (1969), and further developed by, e.g., Dwivedi and Subba Rao (2010), Jentsch and Subba Rao (2015) and Bandyopadhyay, Carsten, and Subba Rao (2017), to these settings. In the context of locally stationary time series, Paparoditis (2009) proposed a test based on comparing a local estimate of the spectral density to a global estimate and Preuß, Vetter, and Dette (2013) proposed a test for stationarity using empirical process theory. In the context of functional time series, tests for stationarity were considered by Horváth, Kokoszka, and Rice (2014) and Aue, Rice, and Sönmez (2018) using time domain methods, and by Aue and van Delft (2020) and van Delft, Characiejus, and Dette (2018) using frequency domain methods.

We develop inference methods about the changes in the degree of smoothness of the spectrum of a locally stationary time series, and hence, about change-points in the spectrum as a special case. The key parameter is the regularity exponent that governs how smooth the path of the local spectral density is over time. We address two local problems. The first is the detection of discontinuities (or breaks) in the spectrum at unknown date and frequency. The second involves the detection of abrupt yet continuous changes in the spectrum over a short time period at an unknown frequency without signifying a break. For example, the spectrum becomes rougher over a short time period, meaning that the paths are less smooth as quantified by the regularity exponent. This can occur for a stationary process whose parameters start to evolve smoothly according to Lipschitz continuity, or for a locally stationary process with Lipschitz parameters that change to continuous but non-differentiable functional parameters. For example, the volatility of high-frequency stock prices or of other macroeconomic variables is known to become rougher (i.e., less smooth) without signifying a structural break after central banks’ official announcements, especially in periods of high market uncertainty. In seismology, earthquakes are made up of several seismic waves that
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arrive at different times and so changes in the smoothness properties of each wave is important for locating the epicenter and identifying what materials the waves have passed through. We consider minimax-optimal testing and estimation for both problems, following Ingster (1993). We determine the optimal rate for the minimax distinguishable boundary, i.e., the minimum break magnitude such that we are still able to uniformly control type I and type II errors.

The problem of discriminating discontinuities from a continuous evolution in a nonparametric framework has received relatively less attention than the classical change-point problem with a few exceptions [Müller (1992), Spokoiny (1998), Müller and Stadtmüller (1999), Wu and Zhao (2007) and Bibinger, Jirak, and Vetter (2017)]. These focused on nonparametric regression and high-frequency volatility, and considered time domain methods while we consider frequency domain methods. This adds a difficulty in that, e.g., the search for a break or smooth change has to run over two dimensions, the time and frequency indices at which the change-point occurs. Our test statistics are the maximum of local two-sample $t$-tests based on the local smoothed periodogram. We construct statistics that allows the researcher to test for a change-point in the spectrum at a prespecified frequency and others that allow to detect a break in the spectrum without prior knowledge about the frequencies. These test statistics can detect both discontinuous and smooth changes, and therefore are useful for both inference problems discussed above. The asymptotic null distribution follows an extreme value distribution. In order to derive this result, we first establish several asymptotic results, including bounds for higher-order cumulants and spectra of locally stationary processes. These results are complementary to some in Dahlhaus (1997), Panaretos and Tavakoli (2013), Aue and van Delft (2020) and Casini (2021), and extend some classical frequency domain results for stationary processes [see, e.g., Brillinger (1975)] to locally stationary processes.

Change-point problems have also been studied in the frequency domain in several fields, though with less generality. Adak (1998) investigated the detection of change-points in piecewise stationary time series by looking at the difference in the power spectral density for two adjacent regimes. He compared several distance metrics such as the Kolmogorov-Smirnov, Crámer-Von Mises and CUSUM-type distance proposed by Coates and Diggle (1986). Last and Shumway (2008) focused on detecting change-points in piecewise locally stationary series. They exploited some of the results in Kakizawa, Shumway, and Taniguchi (1998) and Huang, Ombao, and Stoffer (2004) to propose a Kullback-Liebler discrimination information but did not derive the null distribution of the test statistic. We provide a general change-point analysis about the time-varying spectrum of a time series and establish the relevant asymptotic theory of the proposed test statistics under both the null and alternative hypotheses.

We also address the problem of estimating the change-points, allowing their number to increase with the sample size and the distance between change-points to shrink to zero. We propose
a procedure based on a wild sequential top-down algorithm that exploits the idea of bisection combining it with a wild resampling technique similar to the one proposed by Fryzlewicz (2014). We establish the consistency of the procedure for the number of change-points and their locations. We compare the rate of convergence with that of standard change-point estimators under the classical setting [e.g., Yao (1987), Bai (1994), Casini and Perron (2021a), Casini and Perron (2020a) and Casini and Perron (2020b)]. We verify the performance of our methods via simulations which show their benefits. The advantage of using frequency domain methods to detect change-points is that they do not require to make assumptions about the data-generating process under the null hypothesis beyond the fact that the spectrum is bounded. Furthermore, the method allows for a broader range of alternative hypotheses than time domain methods which usually have good power only against some specific alternatives. For example, tests for change-points in the volatility do not have power for change-points in the dependence and vice versa. Our methods are readily available for use in many fields such as speech processing, biomedical signal processing, seismology, failure detection, economics and finance. It is also used as a pre-test before constructing the recently introduced double kernel long-run variance estimator that accounts more flexibly for nonstationarity [cf. Casini (2021), Casini and Perron (2021b) and Casini, Deng, and Perron (2021)].

The rest of the paper is organized as follows. Section 2 introduces the statistical setting and the hypothesis testing problems. Section 3 presents the test statistics and state their null limit distributions. Section 4 addresses the consistency of the tests and their minimax optimality. Section 5 discusses the estimation of the change-points while Section 6 provides details for the implementation of the methods. Section 7 develops the asymptotic results for the higher-order cumulants and spectra of locally stationary processes that are needed in the proofs of the main results and are also of independent interest in the analysis of locally stationary processes. The results of some Monte Carlo simulations are presented in Section 8. Section 9 reports brief concluding comments. An online supplement [cf. Casini and Perron (2021c)] contains all mathematical proofs. The code to implement our procedures is provided in Matlab, R and Stata languages through a Github repository.

2 Statistical Environment and the Testing Problems

Section 2.1 introduces the statistical setting and Section 2.2 presents the hypotheses testing problems. We work in the frequency domain under the locally stationary framework introduced by Dahlhaus (1997). Casini (2021) extended his framework to allow for discontinuities in the spectrum which then results in a segmented locally stationary process. This corresponds to the relevant process under the alternative hypothesis of breaks in the spectrum. Since local stationarity is a
special case of segmented local stationarity we begin with the latter. We use an infill asymptotic setting whereby we rescale the original discrete time horizon \([1, T]\) by dividing each \(t\) by \(T\).

### 2.1 Segmented Locally Stationary Processes

Suppose \(\{X_t\}_{t=1}^T\) is defined on an abstract probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where \(\Omega\) is the sample space, \(\mathcal{F}\) is the \(\sigma\)-algebra and \(\mathbb{P}\) is a probability measure. Let \(i \triangleq \sqrt{-1}\). We use the notation \(\overline{A}\) for the complex conjugate of \(A \in \mathbb{C}\).

**Definition 2.1.** A sequence of stochastic processes \(\{X_{t,T}\}_{t=1}^T\) is called segmented locally stationary (SLS) with \(m_0 + 1\) regimes, transfer function \(A_0\) and trend \(\mu\) if there exists a representation

\[
X_{t,T} = \mu_j \left(\frac{t}{T}\right) + \int_{-\pi}^{\pi} \exp(i\omega t) A_0^j(t,T)(\omega) \, d\xi(\omega), \quad \left( t = T_{j-1}^0 + 1, \ldots, T_j^0 \right),
\]

for \(j = 1, \ldots, m_0 + 1\), where by convention \(T_0^0 = 0\) and \(T_{m_0+1}^0 = T\) \((T \triangleq \{T_1^0, \ldots, T_{m_0}^0\})\), and the following holds:

(i) \(\xi(\omega)\) is a stochastic process on \([-\pi, \pi]\) with \(\overline{\xi(\omega)} = \xi(-\omega)\) and

\[
\text{cum}\{d\xi(\omega_1), \ldots, d\xi(\omega_r)\} = \varphi \left( \sum_{j=1}^r \omega_j \right) g_1(\omega_1, \ldots, \omega_{r-1}) \, d\omega_1 \cdots d\omega_r,
\]

where \(\text{cum}\{\cdot\}\) is the cumulant of \(r\)th order, \(g_1(\omega_1, \ldots, \omega_{r-1})\) is \(2\pi\)-periodic and \(\varphi(\omega) = \sum_{j=-\infty}^{\infty} \delta(\omega + 2\pi j)\) is the period \(2\pi\) extension of the Dirac delta function \(\delta(\cdot)\).

(ii) There exists a constant \(K > 0\) and a piecewise continuous function \(A: [0, 1] \times \mathbb{R} \to \mathbb{C}\) that, for each \(j = 1, \ldots, m_0 + 1\), there exists a \(2\pi\)-periodic function \(A_j: (\lambda_{j-1}^0, \lambda_j^0] \times \mathbb{R} \to \mathbb{C}\) with \(A_j(u, -\omega) = A_j(u, \omega)\), \(\lambda_j^0 \triangleq T_j^0 / T\) and, for all \(T\),

\[
A(u, \omega) = A_j(u, \omega) \quad \text{for} \quad \lambda_{j-1}^0 < u \leq \lambda_j^0;
\]

\[
\sup_{1 \leq j \leq m_0 + 1} \sup_{T_{j-1}^0 < t \leq T_j^0} \sup_{\omega \in [-\pi, \pi]} \left| A_0^j(t,T)(\omega) - A_j(t/T, \omega) \right| \leq K T^{-1}.
\]

(iii) \(\mu_j(t/T)\) is piecewise continuous.

The smoothness properties of \(A\) in \(u\) guarantees that \(X_{t,T}\) has a piecewise locally stationary behavior. We refer to Casini (2021) for several theoretical properties of SLS processes. Zhou (2013) considered piecewise locally stationary processes in a time domain setting but his notion is less general.
2.2 The Testing Problem

We focus on time-varying spectra that are bounded, thereby excluding unit root and long memory processes. We consider the following class of time-varying spectra under the null hypothesis,

\[ F(\theta, D) = \left\{ \{ f(u, \omega) \}_{u \in [0,1], \omega \in [-\pi, \pi]} : \sup_{\omega \in [-\pi, \pi], u, v \in [0,1], |v-u|<h} |f(u, \omega) - f(v, \omega)| \leq Dh^\theta \right\}, \]

(2.4)

for \( D < \infty \). The key parameter of the testing problem under the null hypotheses is \( \theta > 0 \). This is the regularity exponent of \( f \) in the time dimension. For \( \theta > 1 \), \( f \) is constant in \( u \) and reduces to the spectral density of a stationary process. For \( \theta = 1 \), \( f \) is Lipschitz continuous in \( u \). For \( \theta < 1 \), \( f \) is \( \theta \)-Hölder continuous. Local stationarity corresponds to \( \theta > 0 \) and \( f \) being differentiable [see Dahlhaus (1996b)]. The latter is the setting that we consider under the null hypothesis. To avoid redundancy, we do not require differentiability directly for the functions in \( F(\theta, D) \) since below we assume that the transfer function \( A(u, \omega) \) is differentiable in \( u \) which in turn implies that \( f \) is differentiable in \( u \). Since most of the applied works concerning locally stationary processes rely on Lipschitz continuity (i.e., \( \theta = 1 \)), our specification of the null hypothesis is more general and encompasses them.

We now discuss features that are relevant under the alternative hypothesis. Our focus is on (i) discontinuities of \( f \) in \( u \) which correspond to \( \theta = 0 \) and (ii) decreases in the smoothness of the trajectory \( u \mapsto f(u, \omega) \) for each \( \omega \) which correspond to a decrease in \( \theta \). Both cases refer to the properties of the spectral density and thus to the second-order properties of \( X_{t,T} \).

Case (i) involves a break in the spectrum, i.e., there exits \( \lambda_0^b \in (0, 1) \) such that \( \Delta f(\lambda_0^b, \omega) \triangleq (f(\lambda_0^b, \omega) - \lim_{u \downarrow \lambda_0^b} f(u, \omega)) \neq 0 \) for some \( \omega \in [-\pi, \pi] \).

Case (ii) involves a fall in the regularity exponent from \( \theta \) to \( \theta' \in (0, \theta) \) after some \( \lambda_0^b \) for some period of time for some \( \omega \in [-\pi, \pi] \); i.e., the spectrum becomes rougher after some \( \lambda_0^b \in (0, 1) \) for some time period before returning to \( \theta \)-smoothness. The case of an increase in \( \theta \) is technically more complex to handle (see Section 4 for details). As an example, consider a locally stationary AR(1),

\[ X_{t,T} = a(t/T)X_{t-1,T} + \sigma(t/T)e_t, \quad t = 1, \ldots, T, \]

where \( a : [0,1] \rightarrow (-1, 1) \) and \( \sigma : [0,1] \rightarrow \mathbb{R}_+ \) are functional parameters satisfying a Lipschitz condition and \( \{e_t\} \) is an i.i.d. sequence with zero mean and unit variance. Additionally, if \( \sup_{u \in [0,1]} |\sigma(u)| < \infty \) and the initial condition satisfies some regulation condition, then \( f(u, \omega) \) is uniformly bounded and \( \theta = 1 \). Problem (ii) refers to either \( a(\cdot) \) or \( \sigma(\cdot) \), or both, becoming or
smooth, i.e., we have a change from $\theta = 1$ to some $\theta'$ such that $\theta' \in (0, 1)$. In this case, $X_{t,T}$ becomes an AR(1) process with functional parameters that are still continuous but less smooth and may not be differentiable.

Case (i) has received most attention so far in the time series literature although under much stronger assumptions [e.g., $f(u, \omega) = f(\omega)$]. Case (ii) is a new testing problem and can be of considerable interest in several fields even though it requires larger sample sizes than problem (i). We show below that our tests are consistent and have minimax optimality properties for both cases. Note that case (ii) is a local problem. In this paper, we do not consider more global problems where for example the spectrum is such that a fall in $\theta$ to $\theta' \in (0, \theta)$ occurs on $(\lambda^0_b, 1]$. This represents a continuous change in the smoothness of the spectrum that persists until the end of the time interval. Different test statistics are needed for this case, as will be discussed later.

As discussed by Last and Shumway (2008), an important question is which magnitude of the discontinuity in the time-varying spectrum can be detected. Or equivalently, how much the time-varying spectrum can change over a short time period without indicating a discontinuity. We introduce the quantity $b_T$, called the detection boundary or simply “rate”, which is defined as the minimum break magnitude $\Delta f (\lambda^0_b, \omega)$ such that we are still able to uniformly control the type I and type II errors as indicated below. To address the minimax-optimal testing [cf. Ingster (1993)], we first restrict our attention to case (i) described by a break and defer a more general treatment to Section 4.

Testing Problem for Case (i)

Given the discussion above, for some fractional break point $\lambda^0_b \in (0, 1)$ and frequency $\omega_0$, and a decreasing sequence $b_T$, we consider the following class of alternative hypotheses:

$$ F_{1,\lambda^0_b,\omega_0} (\theta, b_T, D) = \{ \{ f(u, \omega) \}_{u \in [0,1], \omega \in [-\pi, \pi]} | (f(u, \omega) - \Delta f(u, \omega))_{u \in [0,1]} \in F(\theta, D) ; |\Delta f(\lambda^0_b, \omega_0)| \geq b_T \} . $$

We can then present first the hypothesis testing problem that we wish to address:

$$ H_0 : \{ f(u, \omega) \}_{u \in [0,1], \omega \in [-\pi, \pi]} \in F(\theta, D) $$

$$ H^B_1 : \exists \lambda^0_b \in (0, 1) and \omega_0 \in [-\pi, \pi] with \{ f(u, \omega) \}_{u \in [0,1], \omega \in [-\pi, \pi]} \in F_{1,\lambda^0_b,\omega_0} (\theta, b_T, D) . $$

Observe that $H^B_1$ requires at least one break but allows for multiple breaks even across different $\omega$. For the testing problem (2.6), we establish the minimax-optimal rate of convergence of the tests.
suggested [see Ch. 2 in Ingster and Suslina (2003) for an introduction]. A conventional definition is the following. For a nonrandomized test \( \psi \) that maps a sample \( \{X_t\}_{t \geq 0} \) to zero or one, we consider the maximal type I error

\[
\alpha_\psi (\theta) = \sup_{\{f(u, \omega)\}_{u \in [0,1], \omega \in [-\pi, \pi]} \in F(\theta, D)} \mathbb{P}_f (\psi = 1),
\]

and the maximal type II error

\[
\beta_\psi (\theta, b_T) = \sup_{\lambda^0 \in (0,1), \omega_0 \in [-\pi, \pi]} \sup_{\{f(u, \omega)\}_{u \in [0,1], \omega \in [-\pi, \pi]} \in F_1, \lambda^0_0, \omega_0} \mathbb{P}_f (\psi = 0),
\]

and define the total testing error as \( \gamma_\psi (\theta, b_T) = \alpha_\psi (\theta) + \beta_\psi (\theta, b_T) \). The notion of asymptotic minimax-optimality is as follows. We want to find sequences of tests and rates \( b_T \) such that \( \gamma_\psi (\theta, b_T) \to 0 \) as \( T \to \infty \). The larger is \( b_T \) the easier it is to distinguish between \( \mathcal{H}_0 \) from \( \mathcal{H}_1^B \) but we may incur at the same time a larger type II error \( \beta_\psi (\theta, b_T) \). The optimal value \( b_T^{opt} \), named the minimax distinguishable rate, is the minimum value of \( b_T > 0 \) such that \( \lim_{T \to \infty} \inf_{\psi} \gamma_\psi (\theta, b_T) = 0 \).

A sequence of tests \( \psi_T \) that satisfies the latter relation for all \( b_T \geq b_T^{opt} \) is called minimax-optimal.

Minimax-optimality has been considered in other change-point problems. Loader (1996) and Spokoiny (1998) considered the nonparametric estimation of a regression function with break size fixed. Bibinger, Jirak, and Vetter (2017) considered breaks in the volatility of semimartingales under high-frequency asymptotics while we focus on breaks in the spectral density and thus we work in the frequency domain. Another difference from previous work is that we do not deal with i.i.d. observations; we cannot use the same approach to derive the minimax lower bound as in Bibinger, Jirak, and Vetter (2017) because their information-theoretic reductions exploit independence. We need to rely on approximation theorems [cf. Berkes and Philipp (1979)] to establish that our statistical experiment is asymptotically equivalent in a strong Le Cam sense to a high dimensional signal detection problem. This allows us to derive the minimax bound using classical arguments based on the results in Ingster and Suslina (2003), Ch. 8. The relevant results are stated in Section 4.

3 Tests for Changes in the Spectrum and Their Limiting Distributions

Section 3.1 introduces the test statistics while Section 3.2 presents the results concerning their asymptotic distributions under null hypothesis. These results apply also to the case of smooth
alternatives which we discuss formally in Section 4.

3.1 The Test Statistics

We first define the quantities needed to define the tests. Let $h : \mathbb{R} \to \mathbb{R}$ be a data taper with $h(x) = 0$ for $x \notin [0, 1)$,

$$H_{k,T}(\omega) = \sum_{s=0}^{T-1} h\left(\frac{s}{T}\right)^k \exp(-i\omega s),$$

and (for $n_T$ even),

$$d_{L,h,T}(u, \omega) \triangleq \sum_{s=0}^{n_T-1} h\left(\frac{s}{n_T}\right) X_{\lfloor Tu \rfloor - n_T + s, T} \exp(-i\omega s), \quad I_{L,h,T}(u, \omega) \triangleq \frac{1}{2\pi H_{n_T}(0)} |d_{L,h,T}(u, \omega)|^2,$$

$$d_{R,h,T}(u, \omega) \triangleq \sum_{s=0}^{n_T-1} h\left(\frac{s}{n_T}\right) X_{\lfloor Tu \rfloor + n_T - s, T} \exp(-i\omega s), \quad I_{R,h,T}(u, \omega) \triangleq \frac{1}{2\pi H_{n_T}(0)} |d_{R,h,T}(u, \omega)|^2,$$

where $I_{L,h,T}(u, \omega)$ (resp., $I_{R,h,T}(u, \omega)$) is the local periodogram over a segment of length $n_T \to \infty$ that uses observations to the left (resp. right) of $\lfloor Tu \rfloor$. The smoothed local periodogram is defined as

$$f_{L,h,T}(u, \omega) = \frac{2\pi}{n_T} \sum_{s=1}^{n_T-1} W_T\left(\omega - \frac{2\pi s}{n_T}\right) I_{L,h,T}\left(u, \frac{2\pi s}{n_T}\right),$$

with $f_{R,h,T}(u, \omega)$ defined similarly to $f_{L,h,T}(u, \omega)$ but with $I_{R,h,T}(u, \omega)$ in place of $I_{L,h,T}(u, \omega)$, where $W_T(\omega)$ ($-\infty < \omega < \infty$) is a family of weight functions of period $2\pi$,

$$W_T(\omega) = \sum_{j=-\infty}^{\infty} b_{W,T}^{-1}(\omega + 2\pi j) W\left(\frac{b_{W,T}^{-1}(\omega + 2\pi j)}{2}\right),$$

with $b_{W,T}$ a bandwidth and $W(\beta)$ ($-\infty < \beta < \infty$) a fixed function. We define

$$\tilde{f}_{L,r,T}(\omega) = M_{S_r}^{-1} \sum_{j \in S_r} f_{L,h,T}(j/T, \omega) \quad \text{and} \quad \tilde{f}_{R,r,T}(\omega) = M_{S_r}^{-1} \sum_{j \in S_r} f_{R,h,T}(j/T, \omega),$$

where

$$S_r = \{rm_T - m_T/2 + [n_T/2] + 1, rm_T - m_T/2 + [n_T/2] + 1 + m_{S,T},$$

$$\ldots, rm_T + [n_T/2] + 1 + m_{S,T} M_{S,T}/2\},$$
with $m_{ST} = \left\lfloor \frac{m_T}{2} \right\rfloor$ and $M_{ST} = \left\lfloor \frac{M_T}{m_{ST}} \right\rfloor$. \( \tilde{f}_{a,r,T}(\omega) \) \( (a = L, R) \) denotes the average local spectral density around time \( rm_T \) computed using \( f_{a,h,T}(j/T, \omega) \) where \( r = 1, \ldots, M_T = \left\lfloor \frac{T}{m_T} \right\rfloor - 1 \). We do not use all the \( m_T \) local spectral densities \( f_{a,h,T}(j/T, \omega) \) \( (a = L, R) \) in the block \( r \) but only those separated by \( m_{ST} \) points. Thus, \( S_r \) is a subset of the indices in the block \( r \). We need to consider a sub-sample of the \( f_{a,h,T}(j/T, \omega) \)'s \( (a = L, R) \) because there is strong dependence among the adjacent terms, e.g., \( f_{a,h,T}(j/T, \omega) \) and \( f_{a,h,T}((j + 1)/T, \omega) \) \( (a = L, R) \). A large deviation between \( \tilde{f}_{L,r,T}(\omega) \) and \( \tilde{f}_{R,r+1,T}(\omega) \) suggests the presence of a break in the spectrum close to time \( (r + 1) m_T \) at frequency \( \omega \).

We first present a test statistic for the detection of a change-point in the spectrum \( f(\cdot, \omega) \) for a given frequency \( \omega \). A second test statistic that we consider detects change-points in \( f(\cdot, \omega) \) occurring at any frequency \( \omega \in [-\pi, \pi] \). The latter is arguably more useful in practice because often the practitioner does not know a priori at which frequency the spectrum is discontinuous.

We begin with the following test statistic,

\[
S_{\text{max},T}(\omega) \triangleq \max_{r = 1, \ldots, M_T - 2} \left| \frac{\tilde{f}_{L,r,T}(\omega) - \tilde{f}_{R,r+1,T}(\omega)}{\sigma_{L,r}(\omega)} \right|, \quad \omega \in [-\pi, \pi],
\]

\( (3.1) \)

where \( \sigma_{L,r}^2(\omega) \triangleq \text{Var}(\sqrt{M_{ST}} \tilde{f}_{L,r,T}^*(\omega)) \) and

\[
\tilde{f}_{L,r,T}^*(\omega) = M_{ST}^{-1} \sum_{j \in S_r} f_{L,h,T}(j/T, \omega)
\]

with \( f_{L,h,T}(j/T, \omega) = f_{L,h,T}(j/T, \omega) - E(f_{L,h,T}(j/T, \omega)) \). Test statistics of the form of \( (3.1) \) were also used in the time domain in the context of nonparametric change-point analysis [cf. Wu and Zhao (2007) and Bibinger, Jirak, and Vetter (2017)] and forecasting [cf. Casini (2018)]. The derivation of the null distribution uses a (strong) invariance principle for nonstationary processes [see, e.g., and Wu and Zhou (2011)].

The test statistic \( S_{\text{max},T}(\omega) \) aims at detecting a break in the spectrum at some given frequency \( \omega \). An alternative would be to consider a double-sup statistic which takes the maximum over \( \omega \in [-\pi, \pi] \). Theorem 7.4 below shows that \( I_{h,T}(u, \omega) \) and \( I_{h,T}(u, \omega_k) \) are asymptotically independent if \( 2\omega_j, \omega_k \pm \omega_k \neq 0 \pmod{2\pi} \). However, the smoothing over frequencies introduces short-range dependence over \( \omega \). Thus, we specify a framework based on an infill procedure over the frequency domain \([-\pi, \pi]\) by assuming that there are \( n_\omega \) frequencies \( \omega_1, \ldots, \omega_{n_\omega} \), with \( \omega_1 = -\pi \) and \( \omega_{n_\omega} = \pi - \epsilon, \epsilon > 0 \), and \( |\omega_j - \omega_{j+1}| = O(n_\omega^{-1}) \) for \( j = 1, \ldots, n_\omega - 2 \). Assume that \( n_\omega \to \infty \) as \( T \to \infty \).

Let \( \Pi \triangleq \{\omega_1, \ldots, \omega_{n_\omega}\} \). The maximum is taken over the following set of frequencies

\[
\Pi' \triangleq \{\omega_1, \omega_2 + \lfloor n_T b_{W,T} \rfloor, \ldots, \omega_{n_\omega - \lfloor n_T b_{W,T} \rfloor - 1}, \omega_{n_\omega}\}.
\]

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Let \( n'_\omega = \lfloor n_\omega / ([n_T b_{W,T}] + 1) \rfloor \). Note that \( \Pi' \subset \Pi \). Due to the short-range dependence introduced by the smoothed periodogram, we cannot consider the maximum over all frequencies in \( \Pi \) because the statistics would not be independent. This then leads to the double-sup statistic,

\[
S_{D_{\text{max},T}} \triangleq \max_{\omega_k \in \Pi'} \sqrt{\log (M_T)(M_{S,T}^{1/2}S_{\text{max},T} (\omega_k) - \gamma_{M_T}) - \log (n'_\omega)}.
\] (3.2)

The double-sup form of \( S_{D_{\text{max},T}} \) is a new feature for change-point testing using frequency domain methods.

Next, we consider alternative test statistics that are self-normalized such that one does not need to estimate \( \sigma^2_{L,r} (\omega) \). We consider the following test statistic,

\[
R_{\text{max},T} (\omega) \triangleq \max_{r=1,\ldots,M_T-2} \left| \frac{\tilde{f}_{L,r,T} (\omega)}{f_{R,r+1,T} (\omega)} - 1 \right|,
\] (3.3)

where \( \omega \in [-\pi, \pi] \). We can define a test statistic corresponding to \( S_{D_{\text{max},T}} \) by

\[
R_{D_{\text{max},T}} \triangleq \max_{\omega_k \in \Pi'} \sqrt{\log (M_T)(M_{S,T}^{1/2}R_{\text{max},T} (\omega_k) - \gamma_{M_T}) - \log (n'_\omega)}.
\]

### 3.2 The Limiting Distribution Under the Null Hypothesis

Let \( X_{t,T} = (X_{t,T}^{(a_1)}, \ldots, X_{t,T}^{(a_p)}) \) with finite \( p \geq 1 \). Denote by \( \kappa_{X_{t,T}}^{(a_1,\ldots,a_r)} (k_1,\ldots,k_{r-1}) \) the time-\( t \) cumulant of order \( r \) of \( (X_{t+k_1}^{(a_1)}, \ldots, X_{t+k_{r-1}}^{(a_{r-1})}, X_t^{(a_r)}) \) with \( r \leq p \).

**Assumption 3.1.** (i) \( \{X_{t,T}\} \) is a mean-zero locally stationary process (i.e., \( m_0 = 0 \)); (ii) for all \( j = 1,\ldots,p \), \( A^{(a_j)} (u, \omega) \) is \( 2\pi \)-periodic in \( \omega \) and the periodic extensions are differentiable in \( u \) and \( \omega \) with uniformly bounded derivative \( \partial / \partial u \) \( (\partial / \partial \omega) A (u, \omega) \); (iii) \( g_1 \) is continuous.

**Assumption 3.2.** There is an \( l \geq 0 \) such that

\[
\sum_{k_1,\ldots,k_{r-1}=-\infty}^{\infty} (1 + |k_r|^l) \sup_{1 \leq t \leq T} |\kappa_{X_{t,T}}^{(a_1,\ldots,a_r)} (k_1,\ldots,k_{r-1})| < \infty,
\] (3.4)

for \( j = 1,\ldots,r-1 \), and any \( r \)-tuple \( a_1,\ldots,a_r \) for \( r = 2, 3, \ldots \)

**Assumption 3.3.** (i) The data taper \( h : \mathbb{R} \rightarrow \mathbb{R} \) with \( h (x) = 0 \) for \( x \notin [0, 1) \) is bounded and of bounded variation; (ii) The sequence \( \{n_T\} \) satisfies \( n_T \rightarrow \infty \) as \( T \rightarrow \infty \) with \( n_T / T \rightarrow 0 \); (iii) \( W (\beta) (-\infty < \beta < \infty) \) is real-valued, even, of bounded variation, and satisfies \( \int_{-\infty}^{\infty} W (\beta) d\beta = 1 \) and \( \int_{-\infty}^{\infty} |W (\beta)| d\beta < \infty \).

Assumption 3.1 requires \( \{X_{t,T}\} \) to be locally stationarity. Without loss of generality, we assume that \( \{X_{t,T}\} \) has zero mean. All results go through when the mean is non-zero or when
using demeaned series. The differentiability assumption on \( A(u, \omega) \) implies that \( f(u, \omega) \) is also differentiable. This means that under the null hypothesis we require \( f(u, \omega) \) to be differentiable in \( u \) and to have some regularity exponent \( \theta > 0 \). The differentiability of \( A(u, \omega) \) in \( u \) can be relaxed at the expense of more complex proofs to establish the results in Section 7 on high-order cumulants and spectra. Without differentiability, for any regularity exponent \( \theta > 0 \) the test statistics above follow the same asymptotic distribution as when differentiability holds, though we do not discuss this case formally. Assumption 3.2 is the usual cumulant condition applied to locally stationary processes. Assumption 3.3-(i,ii) are standard in the nonparametric estimation literature while Assumption 3.3-(iii) is also used for spectral density estimation under stationarity [see, e.g., Brillinger (1975)].

We need to impose some conditions on the temporal dependence of the observations in the sub-samples \( S_r (r = 1, \ldots, M_T - 2) \). Let \( \{e_t\}_{t \in \mathbb{Z}} \) be a sequence of i.i.d. random variables and \( I^*_T (j/T, \omega) = I_{h,T}(j/T, \omega) - \mathbb{E}(I_{h,T}(j/T, \omega)) \). Assume \( I^*_{h,T}(j/T, \omega) = H_h(j/T, \mathcal{F}_{j+n_T/2}) \) where \( \mathcal{F}_t \triangleq \{ \ldots, e_{t-1}, e_t \} \) and \( H_h : [0, 1] \times \mathbb{R}^\infty \mapsto \mathbb{R} \) is a measurable function. We use the dependence measure introduced by Wu (2005, 2007) for stationary processes and extended to nonstationary processes by Wu and Zhou (2011). Let \( \{e_t'\}_{t \in \mathbb{Z}} \) be an independent copy of \( \{e_t\}_{t \in \mathbb{Z}} \) and \( \mathcal{L}^q \) denote the space generated by the \( q \)-norm, \( q > 0 \). For all \( j \), assume \( I^*_{h,T}(j/T, \omega) \in \mathcal{L}^q \). For \( w \geq 0 \) define the dependence measure,

\[
\phi_{w,q} = \sup_{j \in \{S_r : r = 1, \ldots, M_T - 2\}} \left\| I^*_{h,T}(j/T, \omega) - I^*_{h,T}\{w\}(j/T, \omega) \right\|_q
\]

\[
\sup_{j \in \{S_r : r = 1, \ldots, M_T - 2\}} \left\| H_h\left(j/T, \mathcal{F}_{j+n_T/2}\right) - H_h\left(j/T, \mathcal{F}_{j+n_T/2,w}\right) \right\|_q,
\]

where \( \mathcal{F}_{j+n_T/2,w} \) is a coupled version of \( \mathcal{F}_{j+n_T/2} \) with \( e_{j+n_T/2-w} \) replaced by an i.i.d. copy \( e'_{j+n_T/2-w} \). Assume \( \Upsilon_{n,q} = \sum_{j=1}^n \phi_{j,q} < \infty \) for some \( n \in \mathbb{Z} \). Let \( \tau_T = T^{\vartheta_1} (\log(T))^{\vartheta_2} \) where \( \vartheta_1 = (1/2 - 1/q + \gamma/q) / (1/2 - 1/q + \gamma) \) and \( \vartheta_2 = (\gamma + \gamma/q) / (1/2 - 1/q + \gamma) \) for some \( \gamma > 0 \).

**Assumption 3.4.** Let \( 2 < q \leq 4 \) and assume either (i) \( \Upsilon_{n,q} = O(n^{-\gamma}) \) or (ii) \( \phi_{w,q} = O(\rho^w) \) for some \( \rho \in (0, 1) \).

For standard technical reasons inherent to the proofs, we need to assume that the spectral density is strictly positive. Theorem 7.6 shows that the variance of \( f_{L,h,T}(u, \omega) \) depends on \( f(u, \omega) \). Thus, the denominator of the test statistic depends on \( f(u, \omega) \). Assumption 3.5 requires the latter to be bounded away from zero. In practice, if one suspects that at some frequencies \( f(u, \omega) \) can be close to zero, then one can add a small number \( \epsilon_f > 0 \) to the denominator of the test statistic to guarantee numerical stability.

**Assumption 3.5.** \( f_- = \min_{\omega \in [0, 1], \omega \in [-\pi, \pi]} f(u, \omega) > 0. \)
The next assumption ensures that the local spectral density estimates are asymptotically independent when evaluated at some given frequencies [see Theorem 7.6 below]. It is used to derive the asymptotic null distribution of the double-sup test statistics $S_{D_{\text{max},T}}$ and $R_{D_{\text{max},T}}$.

**Assumption 3.6.** Assume that $2\omega_j, \omega_j \pm \omega_k \not\equiv 0 \pmod{2\pi}$ for $\omega_j, \omega_k \in \Pi$.

**Condition 1.** (i) The sequence $\{m_T\}$ satisfies $m_T \to \infty$ as $T \to \infty$, and

$$M_1^{1/2}m_T^bT^{-\theta} (\log (M_T))^{1/2} + \tau^2_T \log (M_T) M_T^{-1}$$

$$+ M_T n_T^2 \log (M_T) T^{-4} + M_T (\log (n_T))^2 \log (M_T) n_T^{-2} \to 0;$$

(ii) $b_{W,T} \to 0$ such that $Tb_{W,T} \to \infty$ and $\log (M_T) M_T b_{W,T}^4 \to 0$.

Part (i) imposes lower and upper bounds on the growth condition of the sequence $\{m_T\}$. The upper bound relates to the smoothness of $A(u, \omega)$ under the null hypotheses and to $n_T$.

Let $\gamma_{M_T} = [4 \log (M_T) - 2 \log (M_T))]^{1/2}$ and $\mathcal{V}$ denote a random variable with an extreme value distribution defined by $P(\mathcal{V} \leq v) = \exp(-\pi^{-1/2} \exp(-v))$.

**Theorem 3.1.** Let Assumption 3.1, 3.2 with $l = 0$ and $r = 2, 3.3-3.5$ and Condition 1 hold. Under $H_0$, $\sqrt{\log (M_T)}(M_1^{1/2}S_{\text{max},T}(\omega) - \gamma_{M_T}) \Rightarrow \mathcal{V}$ for any $\omega \in [-\pi, \pi]$.

Theorem 3.1 shows that the asymptotic null distribution follows an extreme value distribution. The derivation of the null distribution uses a (strong) invariance principle for nonstationary processes [see, e.g., and Wu and Zhou (2011)]. To make the test operational, we need a uniformly consistent estimate of $\sigma_L^2(\omega)$; recall (3.1). This is discussed in Section 6. The following theorems shows that the asymptotic null distribution of the remaining tests $S_{D_{\text{max},T}}, R_{\text{max},T}(\omega)$ and $R_{D_{\text{max},T}}$ also follows an extreme value distribution, though the additional Assumption 3.6 and the extra factor $\log (n_T')$ are needed for $S_{D_{\text{max},T}}$ and $R_{D_{\text{max},T}}$.

**Theorem 3.2.** Let Assumption 3.1, 3.2 with $l = 0$ and $r = 2, 3.3-3.6$ and Condition 1 hold. Under $H_0$ we have $S_{D_{\text{max},T}} \Rightarrow \mathcal{V}$.

**Theorem 3.3.** Let Assumption 3.1, 3.2 with $l = 0$ and $r = 2, 3.3-3.5$ and Condition 1 hold. Under $H_0$, $\sqrt{\log (M_T)}(M_1^{1/2}R_{\text{max},T}(\omega_k) - \gamma_{M_T}) \Rightarrow \mathcal{V}$ and, in addition if Assumption 3.6 holds, then $R_{D_{\text{max},T}} \Rightarrow \mathcal{V}$.

### 4 Consistency and Minimax Optimal Rate of Convergence

In this section, we discuss the consistency and minimax-optimal lower bound. We consider alternative hypotheses where $f$ is less smooth than under $H_0$, including the case of breaks as a
special case. Suppose that under $\mathcal{H}_0$ the spectrum $f(u, \omega)$ is differentiable in both arguments and behaves until time $T\lambda_0^0$ as specified in $F(\theta, D)$ for some $\theta > 0$ and $D < \infty$. After $T\lambda_0^0$, the regularity exponent $\theta$ drops to some $\theta'$ with $0 < \theta' < \theta$ for some non-trivial period of time. That is, since $F(\theta, D) \subset F(\theta', D)$, we need that $f$ behaves as $\theta'$-regular for some period of time such that there exists a $\omega$ with $\{f(u, \omega)\}_{u \in [0,1]} \notin F(\theta, D)$. This guarantees that $\mathcal{H}_0$ and $\mathcal{H}_1^\text{S}$ (to be defined below) are well-separated. To this end, define for some function $g_u$ with $u \in [0,1]$, $\Delta_h' g_u = (g_{u+h} - g_u) / |h|^{\theta'}$ for $h \in [-u, 1-u]$. The set of possible alternatives is then defined as

$$\mathbf{F}_{1,\lambda_0^0,\omega_0}'(\theta, \theta', b_T, D) = \left\{ \{f(u, \omega)\}_{u \in [0,1], \omega \in [-\pi, \pi]} \in F(\theta', D) \big| \inf_{|h| \leq 2m_T/T} \Delta_h' f(\lambda_0^0, \omega_0) \geq b_T \quad \text{or} \quad \sup_{|h| \leq 2m_T/T} \Delta_h' f(\lambda_0^0, \omega_0) \leq -b_T \right\}.$$ 

Note that $\mathbf{F}_{1,\lambda_0^0,\omega_0}'$ depends on $m_T$ but since $m_T$ depends on $\theta$ we can omit it from the argument of $\mathbf{F}_{1,\lambda_0^0,\omega_0}'$. This leads to the following testing problem,

$$\begin{align*}
\mathcal{H}_0 : \{f(u, \omega)\}_{u \in [0,1]} & \in F(\theta, D) \\
\mathcal{H}_1^\text{S} : \exists \lambda_0^0 \in (0, 1) \text{ and } \omega_0 \in [-\pi, \pi] \text{ with } \{\{f(u, \omega)\}_{u \in [0,1], \omega \in [-\pi, \pi]}\} & \in \mathbf{F}_{1,\lambda_0^0,\omega_0}'(\theta, \theta', b_T, D).
\end{align*}$$

Note that $\mathcal{H}_1^\text{S}$ allows for multiple changes. $\mathcal{H}_1^\text{P}$ is a special case of $\mathcal{H}_1^\text{S}$ since it can be seen as the limiting case of $\mathcal{H}_1^\text{S}$ as $\theta' \to 0$. In the context of infinite-dimensional parameter problems one faces the issue of distinguishability between the null and the alternative hypotheses. It is evident that one cannot test $f \in F(\theta, D)$ versus $f \in F(\theta', D)$ for $\theta > \theta'$. First, since $F(\theta, D) \subset F(\theta', D)$, one has at least to remove the set of functions in $F(\theta, D)$ from those in $F(\theta', D)$. Still, as discussed by Ingster and Suslina (2003), this would not be enough since the two hypotheses are still too close. That explains why we focus on spectral densities $f$ that belong to $\mathbf{F}_{1,\lambda_0^0,\omega_0}'(\theta, \theta', b_T, D)$ under $\mathcal{H}_1^\text{S}$. These are rough enough so as not to be close to functions in $F(\theta, D)$. This is captured by the requirement that the difference quotient $\Delta_h' f$ exceeds the so-called rate $b_T$. As $T \to \infty$ the requirement becomes less stringent since $b_T \to 0$. See Hoffmann and Nickl (2011) and Bibinger, Jirak, and Vetter (2017) for similar discussions in different contexts.

We assume that $X_{t,T}$ is segmented locally stationary with transfer function $A(u, \omega)$ satisfying the following smoothness properties.

**Assumption 4.1.** (i) $\{X_{t,T}\}$ is a mean-zero segmented locally stationary process; (ii) $A(u, \omega)$ is twice continuously differentiable in $u$ at all $u \neq \lambda_j^0$ ($j = 1, \ldots, m_0 + 1$) with uniformly bounded derivatives $(\partial / \partial u) A(u, \cdot)$ and $(\partial^2 / \partial u^2) A(u, \cdot)$; (iii) $A(u, \omega)$ is twice left-differentiable in $u$ at $u = \lambda_j^0$ ($j = 1, \ldots, m_0 + 1$) with uniformly bounded derivatives $(\partial / \partial - u) A(u, \cdot)$ and $(\partial^2 / \partial - u^2) A(u, \cdot)$. 
Assumption 4.2. (i) $A(u, \omega)$ is twice differentiable in $\omega$ with uniformly bounded derivatives $(\partial/\partial \omega) A(., \omega)$ and $(\partial^2/\partial \omega^2) A(., \omega);$ (ii) $g_4(\omega_1, \omega_2, \omega_3)$ is continuous in its arguments.

We now move to the derivation of the minimax lower bound. As explained before, we restrict attention to a strictly positive spectral density in the frequency dimension at which the null hypotheses is violated. That is, $f_-(\omega_0) = \inf_{u \in [0, 1]} f(u, \omega_0) > 0.$ Such restriction is not imposed on $f(u, \omega)$ for $\omega \neq \omega_0.$

Theorem 4.1. Let Assumption 3.2 with $l = 0$ and $r = 2,$ 3.3-3.4, 4.1-4.2 and $f_-(\omega_0) > 0$ hold. Consider either set of hypotheses $\{H_0, H_1^B\}$ with $\theta' = 0$ or $\{H_0, H_1^S\}$ with $0 < \theta' < \theta.$ Then, for

$$b_T \leq (T/\log(M_T))^{-\frac{\theta - \theta'}{2\theta' + 1}} D^{-\frac{2\theta' + 1}{2\theta' + 1}} f_-(\omega_0),$$

we have $\lim_{T \to \infty} \inf_\psi \gamma_\psi(\theta, b_T) = 1.$

The theorem implies the need for

$$b_T^{\text{opt}} > (T/\log(M_T))^{-\frac{\theta - \theta'}{2\theta' + 1}} D^{-\frac{2\theta' + 1}{2\theta' + 1}} f_-(\omega_0),$$

otherwise there cannot exist a minimax-optimal test yielding $\lim_{T \to \infty} \inf_\psi \gamma_\psi(\theta, b_T) = 0.$ Note that the lower bound does not depend on $\omega.$ In Theorem 4.2 we establish a corresponding upper bound. From the lower and upper bounds we deduce the optimal rate for the minimax distinguishable boundary. We can also derive tests based on $b_T^{\text{opt}}.$ For example, using the test statistic (3.1) for $\{H_0, H_1^B\}$ we obtain the following test $\psi^*: \psi^*(\{X_t\}_{1 \leq t \leq T}) = 1$ if $S_{\text{max},T}(\omega) \geq 2D^\ast \sqrt{\log(M_T^\ast)/m_T^\ast}$ for $\omega \in [-\pi, \pi]$ where $D^\ast > 2,$ $m_T^\ast = \sqrt{\log(M_T^\ast) T^\theta/D)}$ and $M_T^\ast = \lfloor T/m_T^\ast \rfloor.$ Hence, in order to construct such a test we need knowledge of $\theta$ under $H_0.$ We discuss this in Section 6.

Next, we establish the optimal rate for minimax distinguishability. Note that either alternatives $H_1^B$ or $H_1^S$ allows for multiple breaks which may occur close to each other. For technical reasons one has to either assume that the breaks do not cancel each other or assume that they cannot be too close. Here, we assume the latter which is implied by the definition of segmented locally stationary. Note that the following results require further restrictions on the relation between $n_T$ and $m_T.$

Theorem 4.2. Let Assumption 3.2 with $l = 0$ and $r = 2,$ 3.3-3.4, 4.1-4.2 hold. Consider either alternative hypotheses $H_1^B$ with $\theta' = 0$ and $\lambda_j^0 < \lambda_{j+1}^0$ for $j = 1, \ldots, m_0,$ or $H_1^S$ with $0 < \theta' < \theta.$ If

$$\left(\sqrt{\log(M_T^\ast)/m_T^\ast}\right)^{-1} \left((m_T^\ast/T)^\theta + (n_T/T)^2 + \log(n_T)/n_T + b_T^{W,T}\right) \to 0,$$ (4.2)
and
\[ b_T^* > \left( 4D^* \sup_{u \in [0,1]} f(u, \omega_0) + 2 \right)^{- \frac{\theta + \theta'}{2(\theta + 1)}} (T / \log (M_T))^{- \frac{\theta + \theta'}{2(\theta + 1)}} D^{- \frac{2\theta'}{2(\theta + 1)}}. \tag{4.3} \]

then \( \lim_{T \to \infty} \gamma_{b^*} (\theta, b_T^*) = 0 \) and \( b_T^{opt} \propto (T / \log (M_T))^{- \frac{\theta + \theta'}{2(\theta + 1)}}. \)

The theorem shows that a smooth change in the regularity exponent \( \theta \) cannot be distinguished from a break of magnitude smaller than \( b_T^{opt} \) because the change from \( \theta \) to \( \theta' \) has to persist for some time. This is also indicated by the restriction \( \theta' > 0 \). The minimax bound is similar to the one established by Bibinger, Jirak, and Vetter (2017) for the volatility of a Itô semimartingale. The theorem suggests that knowledge of the frequency \( \omega_0 \) at which the spectrum changes regularity is irrelevant for the determination of the bound. However, we conjecture that if the spectrum exhibits a break or smooth change of the form discussed above simultaneously across multiple frequencies then the lower bound may be further decreased as one can pool additional information from inspection of the spectrum for the set of frequencies subject to the change. The key assumption would be that the change occurs at the same time \( \lambda_0^0 \) for a given set of frequencies \( \omega \). This may be of interest for economic and financial time series since they often exhibit a break simultaneously at high and low frequencies. We leave this to future research.

5 Estimation of the Change-Points

We now discuss the estimation of the break locations for the case of discontinuities in the spectrum (i.e., \( \mathcal{H}_1^{B,m_0} \) where \( m_0 \) is the number of breaks, recall Definition 2.1). The same estimator is valid for the locations of the smooth changes as under \( \mathcal{H}_1^S \). For the latter case we later provide intuitive remarks about the consistency result and the conditions needed for the result. We first consider the case of a single break (i.e., \( \mathcal{H}_1^{B,1} \)) and then present the results for the case of multiple breaks (i.e., \( \mathcal{H}_1^{B,m_0} \)).

5.1 Single Break Alternatives \( \mathcal{H}_1^{B,1} \)

Let
\[ D_{r,T} (\omega) \triangleq M_{S,T}^{-1/2} \left| \sum_{j \in S_{L,r}} f_{L,h,T} (j/T, \omega) - \sum_{j \in S_{R,r}} f_{R,h,T} (j/T, \omega) \right|, \quad \omega \in [-\pi, \pi]. \]
where

\[ S_{L,r} = \{ r - m_T + 1, r - m_T + 1 + m_{S,T}, \ldots, r - m_T + 1 + m_{S,T}M_{S,T} \}, \]
\[ S_{R,r} = \{ r + 1, r + 1 + m_{S,T}, \ldots, r + 1 + m_{S,T}M_{S,T} \}, \]

and \( r = 2m_T, 3m_T, \ldots \) with \( r < (M_T - 1)m_T - n_T \). Note that the maximum of the statistics \( D_{r,T}(\omega) \) is a version of \( S\text{max}_{r,T} \) that does not involve the normalization \( \sigma_{f,L,r}(\omega) \). The change-point estimator is defined as

\[ T^{\hat{\lambda}}_{b,T} = \arg\max_{r=2m_T,3m_T,\ldots} \max_{\omega \in [-\pi, \pi]} D_{r,T}(\omega). \]

Recall that we consider the following alternative hypothesis:

\[ H_{B,1} : \{ f(T_0^b/T, \omega_0) - f(T_0^{b,+}/T, \omega_0) = \delta_T \neq 0, \ \omega_0 \in [-\pi, \pi] \}, \]

where \( T_0^{b,+} = \lim_{s \uparrow T_0^b, s > T_0^b} s \). The break magnitude can be either fixed or converge to zero as specified by the following assumption.

**Assumption 5.1.** \( \delta_T \to 0 \) and \( \delta_T M_{S,T}^{1/2} / \sqrt{\log(T)} \to (0, \infty) \).

**Proposition 5.1.** Let Assumption 3.2 with \( l = 0 \) and \( r = 2, 3.3-3.4, 4.1 \) with \( m_0 = 1 \) and Condition 1 hold. Under \( H_{B,1} \), if \( \delta_T \) is fixed or satisfies Assumption 5.1, we have \( \hat{\lambda}_{b,T} - \lambda_0^b = O_{\mathbb{P}}(m_{S,T} \sqrt{M_{S,T} \log(T)}/(T\delta_T)) \).

It is useful to compare the rate of convergence in Proposition 5.1 with that of classical change-point estimators of a break fraction in the mean. For fixed shifts, the latter rate of convergence is \( O_{\mathbb{P}}(T^{-1}) \) while for shrinking shifts it is \( O_{\mathbb{P}}((T\delta_T^2)^{-1}) \) where \( \delta_T \to 0 \) with \( \delta_T T^{1/2 - \vartheta} \) for some \( \vartheta \in (0, 1/2) \) [cf. Yao (1987)]. Unlike the classical change-point problem where the mean is constant except for the break, our problem involves a spectrum that can vary smoothly under the null. Hence, for fixed shifts, the rate of convergence in our problem is slower. The smallest break magnitude allowed by Proposition 5.1 is \( \delta_T = O(\sqrt{\log(T)}/M_{S,T}^{1/2}) \). Under this condition the convergence rate for the classical change-point estimator is \( O_{\mathbb{P}}(M_{S,T}(T \log(T))^{-1}) \) which is faster by a factor \( O(m_{S,T} \sqrt{\log(T)}) \) than the one suggested by Proposition 5.1. In addition, in classical change-point setting \( \delta_T \to 0 \) is allowed at a faster rate. This is obvious since in our setting a small break can be confounded with a smooth local change.

Under the smooth alternative \( H_{1}^{S} \) the estimator is consistent only when \( \theta \)-regularity is violated in a small interval around \( \lambda_0^b \) or when \( \theta \)-regularity is violated only once. If \( \theta \)-regularity is violated only once and the length of this interval exceeds \( O(m_{S,T} \sqrt{M_{S,T} \log(T)}/T\delta_T) \), or is violated at
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Under the smooth alternative $H^S_1$ the estimator is consistent when $\theta$-regularity is violated only once in the sample and also when the violation occurs in a small interval around $\lambda^0$ which does not exceed $O(m_sT^{1/2}T^{-1/2} \log(T)/T\delta_T)$. If that interval is longer then this becomes a global problem which cannot be addressed by the estimation method considered in this section. This also relates to the discussion in Section 4 that one cannot perfectly separate functions with $\theta'$-smoothness from functions with $\theta'$-smoothness such that $\theta' < \theta$.

5.2 Multiple Breaks Alternatives $H^{B,m_0}_1$

Let us assume that there are $m_0 > 1$ break points in $f(u, \omega)$. Let $0 < \lambda^0_1 < \ldots < \lambda^0_{m_0} < 1$. We consider the following class of alternative hypotheses:

$$H^{B,m_0}_1 : \{ f \left( \frac{T^0_l}{T}, \omega_l \right) - f \left( \frac{T^0_{l^-}}{T}, \omega_l \right) = \delta_l,T \neq 0, \quad \omega_l \in [-\pi, \pi] \text{ for } 1 \leq l \leq m_0 \}.$$ 

We provide a consistency result for both $m_0$ and the actual locations of the breaks $\lambda^0_l$ ($1 \leq l \leq m_0$). Let $I \subseteq \{2m_T, 3m_T, \ldots, (M_T - 1)m_T - n_T\}$ denote a generic index set. One can test for a break at some time index in $I$ by using the test $\psi^* \left( \{X_r\}_{r \in I} \right)$ based on $\max_{\omega_k \in \Pi} S_{\max,T}(\omega_k)$ and if the test rejects one can estimate the break location using

$$T_{\hat{\lambda}_T}(I) = \arg\max_{r \in I} \max_{\omega \in [-\pi, \pi]} D_{r,T}(\omega). \quad (5.1)$$

We can then update the set $I$ by excluding a $v_T$-neighborhood of $T_{\hat{\lambda}_T}$ and repeat the above steps. This is a sequential top-down algorithm exploiting the classical idea of bisection. However, this procedure may not be efficient. For example, consider the first step of the algorithm in which we test for the first break; this is associated with the largest break magnitude ($\delta_1,T > \delta_l,T$ for all $l = 2, \ldots, m_0$). If the true break date $T^0_1$ falls in between two indices in $I$, say $r_1$ and $r_2 = r_1 + m_T$, then this does not maximize either power or precision of the location estimate because one would need to compare two adjacent blocks exactly separated at $T^0_1$ but $T^0_1 \notin I$ since $T^0_1 \in (r_1, r_2)$. Hence, we introduce a wild sequential top-down algorithm.

Continuing with the above example, we draw randomly without replacement $K \geq 1$ separation points $r^\diamond$ from the interval $(r_1, r_2)$ and for each separation point compute $D_{r^\diamond,T}(\omega)$ where $r^\diamond \in (r_1, r_2)$. We take the maximum value. Then, we update $I$ by removing $r_1$ and adding $r^\diamond$. We repeat this for all indices in $I$. Because the $K$ separation points are drawn randomly, there is always some probability to pick up the separation point that guarantees the highest power.
natural question is why not take all integers between $r_1$ and $r_2$ and compute $D_{r^*,T}(\omega)$ for each. The reason is that in applications involving high frequency data (e.g., weakly, daily, and so on) that would be highly computationally intensive especially with multiple breaks as one wishes to change $m_T$ when searching for an additional break.

We are now ready to present the algorithm. Guidance as to a suitable choice of $K$ will be given below. Let $\nu_T \to \infty$ with $\nu_T/T \to 0$ and $m_T/\nu_T \to 0$. Consider the test $\psi(\{X_t\}_{1 \leq t \leq T}, I) = 1$ if $S_{\text{max},T}(I) \geq 2D^* \sqrt{\log(M_T^*)/m_T^*}$ where

$$S_{\text{max},T}(I) \triangleq \max_{r \in \mathcal{I}} \max_{\omega_k \in \mathcal{H}} \left| \sum_{j \in s_L} f_{L,h,T}(j/T, \omega_k) - \sum_{j \in s_R} f_{R,h,T}(j/T, \omega_k) \right| \sigma_{L,r}(\omega_k),$$

and $D^*$, $m_T^*$ and $M_T^*$ as defined in Section 4.

**Algorithm 1.** Set $\mathcal{I} = \{2m_T, 3m_T, \ldots, (M_T - 1)m_T - n_T\}$ and $\mathcal{F} = \emptyset$.

1. For $r \in \mathcal{I} \setminus \{2m_T\}$ uniformly draw (without replacement) $K$ points $r_0, \ldots, r_K$ from $I(r) = \{r - m_T + 1, \ldots, r\}$ and compute $\mathcal{P}^* = \arg\max_{k=1, \ldots, K} \max_{\omega \in [-\pi, \pi]} D_{r_k,T}(\omega)$; set $\mathcal{I} = (\mathcal{I} \setminus \{r\}) \cup \{\mathcal{P}^*\}$.

2. If $\psi(\{X_t\}_{1 \leq t \leq T}, I) = 0$ return $\mathcal{F} = \emptyset$. Otherwise proceed with step (3).

3. Estimate the change-point $T\lambda_T(\mathcal{I})$ via (5.1) using $\mathcal{I}$.

4. Set $\mathcal{\hat{I}} = \mathcal{I} \setminus \{T\lambda_T(\mathcal{I}) - \nu_T, \ldots, T\lambda_T(\mathcal{I}) + \nu_T\}$ and $\mathcal{\hat{F}} = \mathcal{F} \cup \{T\lambda_T(\mathcal{I})\}$. Return to step (1).

Finally, arrange the estimated change-points $\lambda_{l,T}$ in $\mathcal{\hat{F}}$ in chronological order and use the symbol $|S|$ for the cardinality of a set $S$. To each $\lambda_{l,T}$ the procedure can return the frequency $\bar{\omega}_l$ at which the break is found.

**Assumption 5.2.** We have $\delta_{l,T} \to 0$ with $\inf_{1 \leq l \leq m_0} \delta_{l,T} \geq 2D^* \sqrt{\log(T)}^{2/3}$. For $\nu_T \to \infty$ with $\nu_T = O(T/\nu_T)$, it holds that $\inf_{1 \leq l \leq m_0 - 1} |\lambda_{l+1}^0 - \lambda_l^0| \geq \nu_T^{-1}$.

Assumption 5.2 allows for shrinking shifts and possibly growing number of change-points as long as $m_0/\nu_T \to 0$. The following proposition presents the consistency result for the number of change-points $m_0$ and for the change-point locations $\lambda_l^0$ ($l = 1, \ldots, m_0$), and the rate of convergence of their estimates.

**Proposition 5.2.** Let Assumption 3.2 with $l = 0$ and $r = 2$, 3.3-3.4, 4.1 with $m_0 = 1$ and Condition 1 hold. Then, under $\mathcal{H}_{B,m_0}$ we have (i) $\mathbb{P}(|\mathcal{\hat{F}}| = m_0) \to 1$ and sup$_{1 \leq l \leq m_0} |\hat{\lambda}_{l,T} - \lambda_l^0| = o_T(1)$, and (ii) sup$_{1 \leq l \leq m_0} |\hat{\lambda}_{l,T} - \lambda_l^0| = O_T(m_{S,T} \sqrt{M_{S,T}} \log(T)/(T \inf_{1 \leq l \leq m_0} \delta_{l,T}))$. Furthermore, if $K = O(a_T m_T)$ with $a_T \in (0, 1]$ such that $a_T \to 1$, then the breaks are detected in decreasing order of magnitude.

The number of draws $K$ may be fixed or increase with the sample size. However, the algorithm can return the change-point dates in decreasing order of the break magnitudes only if $K$
is sufficiently large. Note that at each loop of the algorithm it is not possible to know to which \( \lambda_l^0 \) \((l = 1, \ldots, m_0)\) the estimate \( \hat{\lambda}_T \) is consistent for. Only after all breaks are detected and we rearrange the estimated change-points in \( \hat{\nu} \) in chronological order, we can learn such information. The same procedure can be applied for the case of multiple smooth local changes, though the notation becomes cumbersome and so we omit it.

6 Implementation

In this section we explain how to consistently estimate \( \sigma_{L, \gamma}(\omega) \) and how to choose the tuning parameters \( m_T, n_T, b_{W,T}, v_T, b_{1,T} \) and \( K \). Let \( \hat{f}_{L,h,T}(j/T, \omega) = f_{L,h,T}(j/T, \omega) - \bar{f}_{L,h,T}(\omega) \) for \( j \in S_r \). Let \( S_{r,+}(j) = \{S_r/\{\ldots, rm_T - m_T/2 + 1 + m_{ST}(j-1)\}\} \) for \( j \geq 0 \) and \( S_{r,-}(j) = \{S_r/\{\ldots, rm_T - m_T/2 + 1 + m_{ST}(-j+1)\}\} \) for \( j < 0 \). Define \( \hat{\sigma}^2_{L,r}(\omega) = \sum_{j=-m_{ST}+1}^{M_{S,T}-1} K_1(b_{1,T}j) \hat{\Gamma}_r(j) \) where

\[
\hat{\Gamma}_r(j) = \begin{cases} 
M_{S,T}^{-1} \sum_{t \in S_{r,+}(j)} \hat{f}_{L,h,T}(t/T, \omega) \hat{f}_{L,h,T}((t-jm_{ST})/T, \omega), & j \geq 0 \\
M_{S,T}^{-1} \sum_{t \in S_{r,-}(j)} \hat{f}_{L,h,T}(t/T, \omega) \hat{f}_{L,h,T}((t+jm_{ST})/T, \omega), & j < 0 
\end{cases}
\]

The quantity \( \hat{\sigma}^2_{L,r}(\omega) \) is a local long-run variance estimator where \( K_1 \) is a kernel and \( b_{1,T} \) is the associated bandwidth. The uniform consistency result follows from the results in Casini (2021).

The choice of the sequences \( m_T \) and \( n_T \) can be based on a mean-squared error (MSE) criterion or cross-validation exploiting results derived for locally stationary series. For example, data-dependent methods for bandwidths in the context of locally stationary processes were investigated by, among others, Casini (2021), Dahlhaus (2012), Dahlhaus and Giraitis (1998) and Richter and Dahlhaus (2019). The optimal amount of smoothing depends on the regularity exponent \( \theta \), on the boundness of the moments and on the extent of the dependence of \{\( X_t \)\}. In this work we focus on the optimal order of the bandwidths, neglecting the constants. We relegate to future work a more detailed analysis of data-dependent methods for this problem for which multiple smoothing directions are present.

For spectral densities satisfying Lipschitz continuity, \( \theta = 1 \) so that \( m_T \propto T^{2/3-\epsilon} \) while for \( \theta = 1/2 \) we have \( m_T \propto T^{1/2-\epsilon} \) where in both cases \( \epsilon > 0 \). In applied work, it is common to work under stationarity \( (\theta > 1) \) or locally stationary with Lipschitz smoothness \( (\theta = 1) \). Hence, we use the optimal bandwidths for \( \theta = 1 \) which works well for both stationary and locally stationary cases. Of course, if one has prior knowledge about the smoothness properties of the parameters of the data-generating process, one can choose a suitable \( \theta \). Assuming \( q = 4 \) and \( \gamma \) large enough we have \( \tau_T \propto T^{1/4} \) and so the optimal values that satisfy Condition 1 are \( m_T = T^{0.66}, n_T \propto T^{0.62} \) and \( b_{W,T} = n_T^{-1/6} \).
The regularity exponent \( \theta \) also affects the test \( \psi(\{X_t\}_{1 \leq t \leq T}, I) \) in Algorithm 1. It is possible to get an estimate of \( \theta \) under the null hypothesis as follows. Compute \( S_{D_{\max},T} \) where the maximum is taken among the indices of the blocks such that the null hypothesis is not violated and name it \( s^*_{D_{\max}} \). Then \( s^*_{D_{\max}} \) is the maximum value of \( S_{D_{\max},T} \) under the null. Solve 
\[
s^*_{D_{\max}} = 2\sqrt{\log (M^*_T) / m^*_T}
\]
for \( \theta \), where recall that \( m^*_T \) and \( M^*_T \) depend on \( \theta \). This yields a preliminary estimate of \( \theta \) which can then be used for the test \( \psi(\{X_t\}_{1 \leq t \leq T}, I) \). Similarly, \( b^{opt}_T \) depends on \( \theta \) and \( \theta' \). Using the same approach, for a given \( \theta' \) one can solve \( s^*_{D_{\max}} = b^{opt}_T \) for \( \theta \) as function of \( \theta' \). If one is interested in the alternative \( \mathcal{H}_1^B \), we have \( \theta' = 0 \) and so this immediately yields an estimate for \( \theta \). If one is interested in the alternative \( \mathcal{H}_1^S \), then one can try a few values of \( \theta' \) in the range \( (0, \theta) \). However, note that in order to use Algorithm 1 only \( \theta \) is needed. The knowledge of \( \theta' \) under \( \mathcal{H}_1^S \) is only needed to obtain \( b^{opt}_T \).

We employ the rectangular data taper \( h(t/T) = 1 \) for all \( 0 \leq t \leq T \). Also we set \( v_T = T^{0.666} \) so as to guarantee the condition \( m_T/v_T \to 0 \). We follow the results in Casini (2021) that suggest \( b_{1,T} = M_{S,T}^{-1/3} \). Our default recommendation is \( K = 10 \). Our simulations with different data-generating processes and sample sizes show that this choice strikes a good balance between the precision of the change-point estimates and computing time. For \( T > 1000 \) we recommend setting \( K = \lfloor m_T / 3 \rfloor \).

The test statistics \( S_{max,T}(\omega) \) and \( R_{max,T}(\omega) \) depend on \( \omega \). The choice of \( \omega \) is, of course, important as it involves different frequency components and hence different periodicities. If the user does not have a priori knowledge about the frequency at which the spectrum has a change-point, our recommendation is to run the tests for multiple values of \( \omega \in [0, \pi] \). Even if the change-point occurs at some \( \omega_0 \) and one selects a value of \( \omega \) close but not equal to \( \omega_0 \) the tests should still be able to reject the null hypothesis given the differentiability of \( f(u, \omega) \). Thus, one can select a few values of \( \omega \) evenly spread on \( [0, \pi] \).

### 7 Results About High-Order Cumulants and Spectra of Locally Stationary Series

This section establishes asymptotic results about second and high-order cumulants and spectra for locally stationary series. These are used to derive the limiting distributions of the test statistics for the change-point problems introduced in Section 3. They are also of independent interest in the literature related to locally stationary and nonstationary processes more generally. We consider the tapered finite Fourier transform, the local periodogram and the smoothed local periodogram.
Let
\[
d_{h,T}(u, \omega) \triangleq \sum_{s=0}^{n_T-1} h \left( \frac{s}{n_T} \right) X_{[Tu]-n_T/2+s+1,T} \exp(-i\omega s),
\]
\[
I_{h,T}(u, \omega) \triangleq \frac{1}{2\pi H_{2,n_T}(0)} |d_{h,T}(u, \omega)|^2,
\]
where \(I_{h,T}(u, \omega)\) is the periodogram over a segment of length \(n_T\) with midpoint \([Tu]\). The smoothed local periodogram is defined as
\[
f_{h,T}(u, \omega) = \frac{2\pi}{n_T} \sum_{s=1}^{n_T-1} W_T \left( \omega - \frac{2\pi s}{n_T} \right) I_{h,T}(u, \frac{2\pi s}{n_T}),
\]
where \(W_T(\omega)\) and \(b_{W,T}\) are defined in Section 3. Note that \(d_{L,h,T}(u, \omega), I_{L,h,T}(u, \omega)\) and \(f_{L,h,T}(u, \omega)\) considered in Section 3 are asymptotically equivalent to \(d_{h,T}(u, \omega), I_{h,T}(u, \omega)\) and \(f_{h,T}(u, \omega)\), respectively.

If (3.4) holds for \(l = 0\), then we can define the \(r\)th order cumulant spectrum at the rescale time \(u \in (0, 1)\),
\[
f^{(a_1, \ldots, a_r)}_{X}(u, \omega_1, \ldots, \omega_{r-1}) = (2\pi)^{r-1} \sum_{k_1, \ldots, k_{r-1} = -\infty}^{\infty} \kappa^{(a_1, \ldots, a_r)}_{X,Tu}(k_1, \ldots, k_{r-1}) \exp \left( -i \sum_{j=1}^{k-1} \omega_j k_j \right),
\]
for any \(r\) tuple \(a_1, \ldots, a_r\) with \(r = 2, 3, \ldots\)

7.1 Local Finite Fourier Transform

We first present the asymptotic expression for the joint cumulants of the finite Fourier transform. Next, we use this result to obtain the limit distribution of the transform. This result is subsequently used to derive the second-order properties of the local periodogram and smoothed local periodogram in the next subsections. Corresponding results for a stationary series can be found in Brillinger (1975) and references therein. Let \(d_{h,T}(u, \omega) = [d_{h,T}^{(a_j)}(u, \omega)] (j = 1, \ldots, r),\)
\[
H^{(a_1, \ldots, a_r)}_{n_T}(\omega) = \sum_{s=0}^{n_T-1} \left( \prod_{j=1}^{r} h_{a_j} \left( s/n_T \right) \right) \exp(-i\omega s), \quad \text{and}
\]
\[
H^{(a_1, \ldots, a_r)}(\omega) = \int \left( \prod_{j=1}^{r} h_{a_j}(t) \right) \exp(-i\omega t) dt.
\]
Let $\mathcal{N}_p^C (\mathbf{c}, \Sigma)$ denote the complex normal distribution for some $p$-dimensional vector $\mathbf{c}$ and $p \times p$ Hermitian positive semidefinite matrix $\Sigma$.

**Theorem 7.1.** Let Assumption 3.1, 3.2 with $l = 0$ and Assumption 3.3-(ii) hold. Let $h_{a_j} (x)$ satisfy Assumption 3.3-(i) for all $j = 1, \ldots, p$. We have

$$\begin{align*}
\text{cum} \left( d_{h, T}^{(a_1)} (u, \omega_1), \ldots, d_{h, T}^{(a_r)} (u, \omega_r) \right) \\
= (2\pi)^{r-1} H_{n_T}^{(a_1, \ldots, a_r)} \left( \sum_{j=1}^r \omega_j \right) f_X^{(a_1, \ldots, a_r)} (u, \omega_1, \ldots, \omega_{r-1}) + \varepsilon_T,
\end{align*}$$

where $\varepsilon_T = o(n_T)$ uniformly in $\omega_j (j = 1, \ldots, r)$. If Assumption 3.2 holds with $l = 1$, then $\varepsilon_T = O (n_T/T)$ uniformly in $\omega_j (j = 1, \ldots, r)$. Furthermore,

$$f_X^{(a_1, \ldots, a_r)} (u, \omega_1, \ldots, \omega_{r-1}) = A^{(a_1)} ([Tu], \omega_1) \cdots A^{(a_p)} ([Tu], \omega_p) g_p (\omega_1, \ldots, \omega_{p-1}),$$

i.e., the spectrum that corresponds to the spectral representation (2.1) with $m_0 = 0$.

**Theorem 7.2.** Let Assumption 3.1, 3.2 with $l = 0$ and Assumption 3.3-(ii) hold. Let $h_{a_j} (x)$ satisfy Assumption 3.3-(i) for all $j = 1, \ldots, p$. We have: (i) If $2\omega_j, \omega_j \pm \omega_k \not\equiv 0 \pmod{2\pi}$ for $1 \leq j < k \leq J_\omega$ with $1 \leq J_\omega < \infty$, $d_{h, T} (u, \omega_j) (j = 1, \ldots, J_\omega)$ are asymptotically independent $\mathcal{N}_p^C (0, 2\pi n_T [H^{(a_1, a_r)} (0) f_X^{(a_1, a_r)} (u, \omega_j)]) (l, r = 1, \ldots, p)$ variables; (ii) If $\omega = 0, \pm \pi, \pm 2\pi, \pm 3\pi, \ldots$, $d_{h, T} (u, \omega)$ is asymptotically $\mathcal{N}_p (0, 2\pi n_T [H^{(a_1, a_r)} (0) f_X^{(a_1, a_r)} (u, \omega)]) (l, r = 1, \ldots, p)$ independently from the previous variates.

### 7.2 Local Periodogram

We now study several properties of the tapered local periodogram. We begin with the finite-sample bias and variance. We then present results about its asymptotic distribution which allow us to conclude that the local periodogram evaluated at distinct ordinates results in estimates that are asymptotically independent thereby mirroring the stationary case. This result is exploited when deriving the limit distribution of the test statistics that do not require knowledge of the frequency at which the change-point occurs.

**Theorem 7.3.** Let Assumption 3.1, 3.2 with $l = 0$ and Assumption 3.3-(i,ii) hold. We have for $-\infty < \omega < \infty$,

$$\mathbb{E} (I_{h, T} (u, \omega)) = \left( \int_{-\pi}^\pi |H_{n_T} (\alpha)|^2 d\alpha \right)^{-1} \int_{-\pi}^\pi |H_{n_T} (\alpha)|^2 f_X (u, \omega - \alpha) d\alpha + O \left( \frac{\log (n_T)}{n_T} \right)$$

(7.2)
The first equality shows that the expected value of $I_{h,T}(u, \omega)$ is a weighted average of the local spectral density at rescaled time $u$ with weights concentrated in a neighborhood of $\omega$ and relative weights determined by the taper. The second equality shows that $I_{h,T}(u, \omega)$ is asymptotically unbiased for $f_X(u, \omega)$ and provides a bound on the asymptotic bias.

**Theorem 7.4.** Let Assumption 3.1, 3.2 with $l = 0$ and Assumption 3.3-(i,ii) hold. We have: (i) For $-\infty < \omega_j, \omega_k < \infty$,

\[
\operatorname{Cov}\{I_{h,T}(u, \omega_j), I_{h,T}(u, \omega_k)\} = |H_{2,n_T}(0)|^{-2} \left( |H_{2,n_T}(\omega_j - \omega_k)|^2 + |H_{2,n_T}(\omega_j + \omega_k)|^2 \right) f_X(u, \omega_j)^2 + O\left(\frac{n_T^{-1}}{n_T}\right),
\]

where $O(n_T^{-1})$ is uniform in $\omega_j$ and $\omega_k$; (ii) If $2\omega_j, \omega_j \pm \omega_k \neq 0 (\mod 2\pi)$ with $1 \leq j < k \leq J_\omega$, the variables $I_{h,T}(u, \omega_j)$ ($j = 1, \ldots, J_\omega$) are asymptotically independent $f_X(u, \omega_j)\chi_2^2/2$ variates. Also, if $\omega = \pm\pi, \pm3\pi, \ldots$, $I_{h,T}(u, \omega)$ is asymptotically $f_X(u, \omega)\chi_1^2$, independently of the previous variates.

### 7.3 Smoothed Local Periodogram

We now extend Theorem 7.3-7.4 to the smoothed local periodogram. Since our test statistics are based on it, these results are directly employed to derive their limiting null distributions.

**Theorem 7.5.** Let Assumption 3.1, 3.2 with $l = 0$ and Assumption 3.3 hold. Let $b_{W,T} \to 0$ as $T \to \infty$ with $b_{W,T}n_T \to \infty$. Then,

\[
\mathbb{E}(f_{h,T}(u, \omega)) = \int_{-\infty}^{\infty} W(\beta) f_X(u, \omega - b_{W,T}\beta) \, d\beta + O\left(\left(n_Tb_{W,T}\right)^{-1}\right) + O\left(\log(n_T)n_T^{-1}\right) \quad (7.4)
\]

\[
= f_X(u, \omega) + \frac{1}{2}\left(\frac{n_T}{T}\right)^2 \left(\int_0^1 h^2(x) \, dx\right)^{-1} \int_0^1 x^2h^2(x) \, dx \frac{\partial^2}{\partial u} f_X(u, \omega)
\]

\[
+ \frac{1}{2}b_{W,T}^2 \int_0^1 x^2W(x) \, dx \frac{\partial^2}{\partial \omega} f_X(u, \omega) + O\left(\left(n_T/T\right)^{-2}\right) + O\left(\log(n_T)n_T^{-1}\right) + o\left(\frac{b_{W,T}^2}{n_T}\right).
\]

The error terms are uniform in $\omega$.

**Theorem 7.6.** Let Assumption 3.1, 3.2 with $l = 0$ and Assumption 3.3-(i,ii) hold. Let $b_{W,T} \to 0$ as $T \to \infty$ with $b_{W,T}n_T \to \infty$. Then, $f_{h,T}(u, \omega_1), \ldots, f_{h,T}(u, \omega_{J_\omega})$ are asymptotically jointly normal
satisfying

\[
\lim_{T \to \infty} n_T b_{W,T} \text{Cov} \left( f_{h,T} (u, \omega_j), f_{h,T} (u, \omega_k) \right) = 2\pi \left[ \eta \{\omega_j - \omega_k\} + \eta \{\omega_j + \omega_k\} \right] \int h(t)^4 dt \left[ \int h(t)^2 dt \right]^{-2} \int W(\alpha)^2 d\alpha f_X(u, \omega_j)^2. \tag{7.5}
\]

Consistency of the spectral density estimates of a stationary time series was obtained by Grenander and Rosenblatt (1957) and Parzen (1957). Asymptotic normality was considered by Rosenblatt (1959), Brillinger and Rosenblatt (1967), Hannan (1970) and Anderson (1971). Theorem 7.6 presents corresponding results for the locally stationary which highlight the effect of the smoothing time in addition to the frequency domain. Panaretos and Tavakoli (2013) established similar results for functional stationary processes while Aue and van Delft (2020) established some results for functional locally stationary processes using a different notion of local stationarity.

8 Small-Sample Evaluations

In this section, we conduct a Monte Carlo analysis to evaluate the properties of the proposed methods. We first discuss the detection of the change-points and then their localization. We investigate different types of changes and consider the test statistics \( S_{\text{max},T}(\omega) \), \( S_{\text{Dmax},T}(\omega) \), \( R_{\text{max},T}(\omega) \), \( R_{\text{Dmax},T}(\omega) \) proposed here and the test statistic \( \hat{D} \) proposed by Last and Shumway (2008). The latter is included for comparison since it applies to the same problems. We consider the following data-generating processes where in all models the innovation \( e_t \) is a Gaussian white noise \( e_t \sim \mathcal{N}(0, 1) \). Models M1 involves a stationary AR(1) process \( X_t = \rho X_{t-1} + e_t \) with \( \rho = 0.3 \) and 0.6, while M2 involves a locally stationary AR(1) \( X_t = \rho(t/T) X_{t-1} + e_t \) where \( \rho(t/T) = 0.4 \cos (0.8 - \cos (2t/T)) \). Note that \( \rho(t/T) \) varies smoothly from 0.1389 to 0.3920. Model M1 and M2 are used to verify the finite-sample size of the tests. We verify the power in models M3 and M4 using the specification in model M1 and M2, respectively, for the first regime and consider two additional regimes with different specifications. Hence, two breaks are present. In model M3,

\[
X_t = \begin{cases} 
0.3X_{t-1} + e_t, & 1 \leq t \leq [T\lambda_1^0] \\
0.6X_{t-1} + 0.7e_t, & [T\lambda_1^0] + 1 \leq t \leq [T\lambda_2^0] \\
0.6X_{t-1} + e_t, & [T\lambda_2^0] + 1 \leq t \leq T
\end{cases}
\]
while, for model M4

\[
X_t = \begin{cases} 
\rho(t/T) X_{t-1} + 0.7 e_t, & 1 \leq t \leq \lfloor T\lambda_1^0 \rfloor \\
0.8 X_{t-1} + e_t, & \lfloor T\lambda_1^0 \rfloor + 1 \leq t \leq \lfloor T\lambda_2^0 \rfloor \\
\rho(t/T) X_{t-1} + 0.7 e_t, & \lfloor T\lambda_2^0 \rfloor + 1 \leq t \leq T
\end{cases}
\]

where \(\rho(t/T)\) is as in model M2. In model M3, the second regime involves higher serial dependence while in the third regime the variance doubles relative to the second regime. In model M4, the second regime involves a stationary autoregressive process with strong serial dependence while in the third regime \(X_t\) assumes the same dynamics as in the first regime. Models M3-M4 feature alternative hypotheses in the forms of breaks in the spectrum.

We consider the alternative hypothesis of more rough variation without signifying a break (i.e., \(H_1^S\) defined in Section 4) in model M5 given by \(X_t = \sigma(t/T) e_t\) where \(\sigma^2(t/T) = \max\{1.5, \, \bar{\sigma}^2 + \cos(1 + \cos(10t/T))\}\) with \(\bar{\sigma}^2 = 1\). Note that even though \(\sigma^2(\cdot)\) is locally stationary, the degree of smoothness alternates throughout the sample. It starts from \(\sigma^2(\cdot) = 1.5\) and maintains this value for some time, then within a short period it increases slowly to \(\sigma^2(\cdot) = 2\) and falls slowly back to \(\sigma^2(\cdot) = 1.5\). It keeps this value until the final part of the sample where it increases slowly to \(\sigma^2(\cdot) = 2\) in a short period. Thus, \(\sigma^2(\cdot)\) alternates between periods where it is constant (i.e., \(\theta > 1\)) and periods where it becomes non-constant but less smooth (i.e., \(\theta = 1\)). Importantly, no break occurs; only a change in the smoothness as specified in \(H_1^S\). In unreported simulations we also considered the case where \(\theta\) changes from Lipschitz continuity (i.e., \(\theta = 1\)) to the continuity-path of Wiener processes (i.e., \(\theta \approx 1/2\)) with results that are similar to those reported here. For the test statistic \(\hat{D}\) of Last and Shumway (2008), we obtain the critical value by simulations. As suggested by the authors we compute the finite-sample distribution of \(\hat{D}\) by simulating a white noise under the null hypotheses with a sample size \(T = 1000\) and then obtain the critical value. We consider the three sample sizes \(T = 250, 500\) and 1000. The significance level is \(\alpha = 0.05\). For the test statistics \(S_{\text{max},T}(\omega)\) and \(R_{\text{max},T}(\omega)\), we use as a default value \(\omega = 0\) given that the interest is often in low frequency analysis. We set \(\lambda_1^0 = 0.33\) and \(\lambda_2^0 = 0.66\) throughout. The number of simulations is 5,000 for all cases.

The results are reported in Table 1-2. We first discuss the size of the tests. The tests proposed in this paper have good empirical size for both models and all sample sizes. The test statistics \(S_{D_{\text{max},T}}\) and \(R_{D_{\text{max},T}}\) are slightly undersized for \(T = 250\) but their empirical size improves for \(T = 500\) and 1000. The test statistics \(S_{\text{max},T}\) and \(R_{\text{max},T}\) share accurate empirical sizes in all cases. In contrast, the test statistic \(\hat{D}\) of Last and Shumway (2008) is largely oversized for \(T = 250\) and 500. For \(T = 1000\) it works better but it is still oversized. This means that the finite-sample
distribution of $\hat{D}$ has high variance and changes substantially across different sample sizes. Since the simulated critical value is obtained with a sample size $T = 1000$ it works better for this sample size than for the others for which the size control is poor.

Turning to the power of the tests, we note that it is not fair to compare the proposed tests with the test $\hat{D}$ when $T = 250$ and 500 since the latter is largely oversized in those cases. In model M3, all the proposed tests have good power which increases with the sample size. The tests $S_{D_{\text{max}},T}$ and $R_{\text{max},T}$ have the highest power, followed by $S_{\text{max},T}$ and lastly $R_{D_{\text{max}},T}$. The power differences are not large except those involving $R_{D_{\text{max}},T}$ for $T = 250$ which has substantially lower power. It is important to note that for $T = 1000$ the proposed tests have higher power than the test $\hat{D}$ of Last and Shumway (2008) even though the latter was oversized. For $T = 250$ and 500, where the $\hat{D}$ test was largely oversized, the proposed tests only have slightly lower power. This confirms that the proposed tests have very good power. Similar comments apply to model M4.

Model M5 involves changes in the smoothness without involving a break. This constitutes a more challenging alternative hypothesis, and as expected, the power for each test is lower than in models M3-M4. The test with the highest power is $S_{D_{\text{max}},T}$, for $T = 1000$ the test with the lowest power is $\hat{D}$, for $T = 250$ the $\hat{D}$ test has higher power again due to its oversize problem. Overall, the results show that the proposed tests have accurate empirical size even for small sample sizes and have good power against different forms of breaks or smooth changes.

Next, we consider the estimation of the number of change-points ($m_0$) and their locations. We consider the following two models, both with $m_0 = 2$. The model M6 is given by

$$X_t = \begin{cases} 
0.7e_t, & 1 \leq t \leq \lfloor T\lambda_1^0 \rfloor \\
0.6X_{t-1} + 0.7e_t, & \lfloor T\lambda_1^0 \rfloor + 1 \leq t \leq \lfloor T\lambda_2^0 \rfloor \\
0.6X_{t-1} + e_t, & \lfloor T\lambda_2^0 \rfloor + 1 \leq t \leq T 
\end{cases}$$

while model M7 is the same as model M4. We set $\lambda_1^0 = 0.33$ and $\lambda_2^0 = 0.66$ and $T = 1000$ throughout. Table 3 reports summary statistics for $\hat{m} - m_0$. It displays the percentage of times with $\hat{m} = m_0$, the median, and the 25% and 75% quantile of the distribution of $\hat{m}$. We only consider Algorithm 1. We do not report the results for the corresponding procedure of Last and Shumway (2008) because it is based on $\hat{D}$ which is oversized and so it finds many more breaks than $m_0$. Table 3 shows that $\hat{m} = m_0$ occurs for about 85% of the simulations with model M6 and about 80% with model M7. This suggests that Algorithm 1 is quite precise. As expected it performs better in model M6 since the specification of the alternative is farther from the null. The quantiles of the empirical distribution also suggest that the change-point estimates $\hat{T}_1$ and $\hat{T}_2$ are accurate. For example the median is very close to the their respective true value $T_1^0 = 333$ and $T_2^0 = 666$. 


Similar conclusions arise from different models and sample sizes, in unreported simulations.

9 Conclusions

We develop a theoretical framework for inference about the smoothness of the spectral density over time. We provide frequency domain statistical tests for the detection of discontinuities in the spectrum of a segmented locally stationary time series and for changes in the regularity exponent of the spectral density over time. We provide different test statistics depending on whether prior knowledge about the frequency component at which the change-point occurs is available. The null distribution of the test follows an extreme value distribution. We rely on the theory on minimax-optimal testing developed by Ingster (1993). We determine the optimal rate for the minimax distinguishable boundary, i.e., the minimum break magnitude such that we are still able to uniformly control type I and type II errors. We propose a novel procedure to estimate the change-points based on a wild sequential top-down algorithm and show its consistency under shrinking shifts and possibly growing number of change-points. The advantage of using frequency domain methods to detect change-points is that it does not require to make assumptions about the data-generating process under the null hypothesis beyond the fact that the spectrum is differentiable and bounded. Furthermore, the method allows for a broader range of alternative hypotheses compared to time domain methods which usually have power against a limited set of alternatives. Overall, our results show the usefulness of our method.


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10 Appendix

10.1 Tables

Table 1: Empirical small-sample size for models M1-M2

|                  | Model M1 |                  |                  |                  |
|------------------|----------|------------------|------------------|------------------|
|                  | $\alpha = 0.05$ | $T = 250$ | $T = 500$ | $T = 1000$ |
| $S_{\max,T}(0)$ | 0.039    | 0.043            | 0.053            |
| $S_{D\max,T}$   | 0.029    | 0.049            | 0.047            |
| $R_{\max,T}(0)$ | 0.040    | 0.054            | 0.042            |
| $R_{D\max,T}$   | 0.025    | 0.032            | 0.038            |
| $\hat{D}$ statistic | 0.581   | 0.471            | 0.068            |

|                  | Model M2 |                  |                  |                  |
|------------------|----------|------------------|------------------|------------------|
|                  |          | $T = 250$ | $T = 500$ | $T = 1000$ |
| $S_{\max,T}(0)$ | 0.061    | 0.059            | 0.057            |
| $S_{D\max,T}$   | 0.035    | 0.055            | 0.058            |
| $R_{\max,T}(0)$ | 0.036    | 0.035            | 0.039            |
| $R_{D\max,T}$   | 0.025    | 0.032            | 0.035            |
| $\hat{D}$ statistic | 0.731   | 0.583            | 0.102            |
Table 2: Empirical small-sample power for models M3-M5

| Model M3 | $\alpha = 0.05$ | $T = 250$ | $T = 500$ | $T = 1000$ |
|----------|----------------|------------|------------|------------|
| $S_{\text{max},T}(0)$ | 0.694 | 0.850 | 0.889 |
| $S_{\text{Dmax},T}$ | 0.734 | 0.890 | 0.921 |
| $R_{\text{max},T}(0)$ | 0.768 | 0.940 | 0.973 |
| $R_{\text{Dmax},T}$ | 0.456 | 0.752 | 0.874 |
| $\hat{D}$ statistic | 0.961 | 0.967 | 0.790 |

| Model M4 |
|----------|
| $T = 250$ | $T = 500$ | $T = 1000$ |
| $S_{\text{max},T}(0)$ | 0.868 | 0.964 | 0.973 |
| $S_{\text{Dmax},T}$ | 0.938 | 0.988 | 0.996 |
| $R_{\text{max},T}(0)$ | 0.927 | 0.997 | 0.999 |
| $R_{\text{Dmax},T}$ | 0.775 | 0.983 | 0.998 |
| $\hat{D}$ statistic | 1.000 | 1.000 | 1.000 |

| Model M5 |
|----------|
| $T = 250$ | $T = 500$ | $T = 1000$ |
| $S_{\text{max},T}$ | 0.223 | 0.475 | 0.565 |
| $S_{\text{Dmax},T}$ | 0.325 | 0.801 | 0.918 |
| $R_{\text{max},T}$ | 0.028 | 0.237 | 0.369 |
| $R_{\text{Dmax},T}$ | 0.025 | 0.189 | 0.304 |
| $\hat{D}$ statistic | 0.834 | 0.695 | 0.172 |

Table 3: Summary statistics for the empirical distribution of $\hat{m} - m_0$

| Summary of $\hat{m} - m_0$ | Percent time $\hat{m} = m_0$ | $Q_{0.25}$ | Median | $Q_{0.75}$ |
|-----------------------------|-------------------------------|------------|--------|------------|
| Model M6 | Algorithm 1 | 85.50 | $\hat{T}_1$ | 299 | 333 | 352 |
| | | | $\hat{T}_2$ | 632 | 663 | 688 |
| Model M7 | Algorithm 1 | 80.12 | $\hat{T}_1$ | 317 | 336 | 359 |
| | | | $\hat{T}_2$ | 623 | 655 | 685 |
Abstract

This supplemental material contains the Mathematical Appendix which includes all proofs of the results in the paper. Section S.A.1 presents a few preliminary lemmas. In Section S.A.2 we provide the proofs of the results of Section 7 which are used in the proofs of the main results in the paper. Section S.A.3-S.A.5 present the proofs of the results of Section 3-5, respectively.
S.A  Mathematical Appendix

S.A.1  Preliminary Lemmas

Let \( L_T : \mathbb{R} \to \mathbb{R}, \ T \in \mathbb{R}_+ \) be the \( 2\pi \)-periodic extension of

\[
L_T (\omega) \triangleq \begin{cases} 
T, & |\omega| \leq 1/T, \\
1/|\omega|, & 1/T \leq |\omega| \leq \pi.
\end{cases}
\]

For a complex-valued function \( w \) define \( H_{nT} (w (\cdot), \omega) = \sum_{s=0}^{nT-1} w (s) \exp (-i\omega s) \), and, for the taper \( h (x) \), \( H_{k,nT} (\omega) = H_{nT} \left( h^k \left( \frac{s}{nT} \right), \omega \right) \), and \( H_{nT} (\omega) = H_{1,nT} (\omega) \).

**Lemma S.A.1.** Let \( \Pi \triangleq (-\pi, \pi] \). With a constant \( K \) independent of \( T \) the following properties hold: (i) \( L_T (\omega) \) is monotone increasing in \( T \) and decreasing in \( \omega \in [0, \pi] \); (ii) \( \int_{\Pi} L_T (\alpha) d\alpha \leq K \ln T \) for \( T > 1 \).

**Proof of Lemma S.A.1.** See Lemma A.4 in Dahlhaus (1997). \( \square \)

**Lemma S.A.2.** Suppose \( h (\cdot) \) satisfies Assumption 3.3 and \( \vartheta : [0, 1] \to \mathbb{R} \) is differentiable with bounded derivative. Then we have for \( 0 \leq t \leq nT \),

\[
H_{nT} \left( \vartheta \left( \frac{\cdot}{T} \right) h \left( \frac{\cdot}{nT} \right), \omega \right) = \vartheta \left( \frac{t}{T} \right) H_{nT} (\omega) + O \left( \sup_x |d \vartheta (x) / dx| \frac{nT}{T} L_{nT} (\omega) \right)
\]

\[
= O \left( \sup_{x \leq nT/T} |\vartheta (x)| L_{nT} (\omega) + \sup_x |d \vartheta (x) / dx| L_{nT} (\omega) \right).
\]

The same holds, if \( \vartheta (\cdot/T) \) is replaced on the left side by numbers \( \psi_{s,T} \) with \( \sup_s |\vartheta_{s,T} - \vartheta (s/T)| = O (T^{-1}) \).

**Proof of Lemma S.A.2.** Dahlhaus (1997) proved this result under differentiability of \( h (\cdot) \). By Abel’s transformation [cf. Exercise 1.7.13 in Brillinger (1975)],

\[
H_{nT} \left( \vartheta \left( \frac{\cdot}{T} \right) h \left( \frac{\cdot}{nT} \right), \omega \right) - \vartheta \left( \frac{t}{T} \right) H_{nT} (\omega) = \sum_{s=0}^{nT-1} \left[ \vartheta \left( \frac{s}{T} \right) - \vartheta \left( \frac{t}{T} \right) \right] h \left( \frac{s}{nT} \right) \exp (-i\omega s)
\]

\[
= -\sum_{s=0}^{nT-1} \left[ \vartheta \left( \frac{s}{T} \right) - \vartheta \left( \frac{s-1}{T} \right) \right] H_s \left( \frac{s}{nT}, \omega \right)
\]

\[
+ \left[ \vartheta \left( \frac{nT-1}{T} \right) - \vartheta \left( \frac{t}{T} \right) \right] H_{nT} \left( \frac{t}{nT}, \omega \right). \quad (S.1)
\]

By repeated applications of Abel’s transformation,

\[
H_s \left( h \left( \frac{\cdot}{nT} \right), \omega \right) = \sum_{t=0}^{s-1} h \left( \frac{t}{s} \right) \exp (-i\omega t)
\]

\[
= \sum_{t=0}^{s-1} \left( h \left( \frac{t}{s} \right) - h \left( \frac{t-1}{s} \right) \right) H_t (1, \omega)
\]

\[
+ h \left( \frac{nT-1}{nT} \right) H_{nT} (1, \omega)
\]

\[
= \sum_{t=0}^{s-1} \left( h \left( \frac{t}{s} \right) - h \left( \frac{t-1}{s} \right) \right) H_t (1, \omega) + 0,
\]

S-1
where we have used $h((n_T - 1)/n_T) - h(1) = O\left(\frac{1}{n_T}\right)$ and $h(x) = 0$ for $x \notin [0, 1)$. Since $h(\cdot)$ is of bounded variation, if $|\omega| \leq 1/s$ we have

$$\sum_{t=0}^{s-1} \left| \left( h\left(\frac{t}{s}\right) - h\left(\frac{t-1}{s}\right) \right) \right| |H_t(1, \omega)| \leq \sum_{t=0}^{s-1} \left| \left( h\left(\frac{t}{s}\right) - h\left(\frac{t-1}{s}\right) \right) \right| \leq (s - 1) \sum_{t=0}^{s-1} \left| \left( h\left(\frac{t}{s}\right) - h\left(\frac{t-1}{s}\right) \right) \right|$$

$$\leq C (s - 1),$$

whereas if $1/s \leq |\omega| \leq \pi$ we have,

$$\sum_{t=0}^{s-1} \left| \left( h\left(\frac{t}{s}\right) - h\left(\frac{t-1}{s}\right) \right) \right| |H_t(1, \omega)| \leq \sum_{t=0}^{s-1} \left| \left( h\left(\frac{t}{s}\right) - h\left(\frac{t-1}{s}\right) \right) \right| \leq C \frac{1}{|\omega|} \sum_{t=0}^{s-1} \left| \left( h\left(\frac{t}{s}\right) - h\left(\frac{t-1}{s}\right) \right) \right| \leq C \frac{1}{|\omega|}.$$

Thus, $H_n(h(\cdot/n_T), \omega) \leq L_s(\omega) \leq L_{n_T}(\omega)$ where the last inequality follows by Lemma S.A.1-(i). It follows from (S.1) that,

$$H_{n_T} \left( \frac{\vartheta}{n_T} \right) - \vartheta \left( \frac{t}{T} \right) H_{n_T}(\omega) = O \left( \sup_{x \leq n_T/T} |\vartheta(x)| L_{n_T}(\omega) + \sup_x |d\vartheta(x)/dx| L_{n_T}(\omega) \right).$$

**Lemma S.A.3.** Assume that $h^{(a_j)}(x)$ satisfies Assumption 3.3-(i) for all $j = 1, \ldots, p$, then we have for some $C$ with $0 < C < \infty$,

$$\left| \sum_{s=0}^{n_T-1} h_T^{(a_1)}(s + k_1) \cdots h_T^{(a_p)}(s + k_{p-1}) h_T^{(a_1)}(s) \exp(-i\omega s) - H_T^{(a_1, \ldots, a_p)}(\omega) \right| \leq C \left( |k_1| + \ldots + |k_{p-1}| \right).$$

**Proof of Lemma S.A.3.** See Lemma P4.1 in Brillinger (1975). □

**Lemma S.A.4.** Let $\{Y_T\}$ be a sequence of $p$ vector-valued random variables, with (possibly) complex components, and such that all cumulants of the variate $(Y_T^{(a_1)}, Y_T^{(a_2)}, \ldots, Y_T^{(a_p)})$ exist and tend to the corresponding cumulants of a variate $(Y^{(a_1)}, Y^{(a_2)}, \ldots, Y^{(a_p)})$ that is determined by its moments. Then $Y_T$ tends in distribution to a variate having components $Y^{(a_1)}, \ldots, Y^{(a_p)}$.

**Proof of Lemma S.A.4.** It follows from Lemma P4.5 in Brillinger (1975). □

### S.A.2 Proofs of the Results of Section 7

#### S.A.2.1 Proof of Theorem 7.1

For $|Tu| - n_T/2 + 1 \leq t_1, \ldots, t_p \leq |Tu| + n_T/2 - 1$,

$$\text{cum}(X_{t_1,T}, \ldots, X_{t_p,T})$$

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\[
= \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \exp(it_1\omega_1 + \cdots + it_p\omega_p) \\
\times A_{t_1,T}^0(\omega_1) \cdots A_{t_p,T}^0(\omega_p) \eta \left( \sum_{j=1}^{p} \omega_j \right) g_p(\omega_1, \ldots, \omega_{p-1}) d\omega_1 \cdots d\omega_p.
\]

We can replace \( A_{t_j,T}^0(\omega_j) \) by \( A(t_j/T, \omega_j) \) using (2.3), and then replace \( A(t_j/T, \omega_j) \) by \( A([Tu], \omega_j) \) using the smoothness of \( A(u, \cdot) \). Altogether, this gives an error \( O(n_T/T) \). Let \( t_1 = t_p + k_1, \ldots, t_{p-1} = t_p + k_{p-1} \). We have

\[
cum (X_{t_1,T}, \ldots, X_{t_p,T}) \\
= \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \exp(i((\omega_1 + \cdots + \omega_{p-1})t_p + \omega_1k_1 + \cdots + \omega_{p-1}k_{p-1} + t_1\omega_p)) \\
\times A([Tu], \omega_1) \cdots A([Tu], \omega_p) \eta \left( \sum_{j=1}^{p} \omega_j \right) g_p(\omega_1, \ldots, \omega_{p-1}) d\omega_1 \cdots d\omega_p + O(n_T/T)
\]

\[
= \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \exp(i((\omega_1 + \cdots + \omega_{p-1} + \omega_p)t_p + \omega_1k_1 + \cdots + \omega_{p-1}k_{p-1})) \\
\times A([Tu], \omega_1) \cdots A([Tu], \omega_p) \eta \left( \sum_{j=1}^{p} \omega_j \right) g_p(\omega_1, \ldots, \omega_{p-1}) d\omega_1 \cdots d\omega_p + O(n_T/T)
\]

\( \Delta \kappa_{Tu,t_p}(k_1, \ldots, k_{p-1}) + O(n_T/T). \) (S.2)

This shows that \( cum (X_{t_1,T}, \ldots, X_{t_p,T}) \) depends on \( t_p \) only through \( \exp(i(\omega_1 + \cdots + \omega_{p-1} + \omega_p)t_p) \). The cumulant of interest in Theorem 7.1 has the following form,

\[
cum \left( d_{h,T}^{(a_1)}(u, \omega_1), \ldots, d_{h,T}^{(a_p)}(u, \omega_p) \right) \\
= \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} H_{n_T} \left( A_{[Tu]-n_T/2+1+,T}^{0(a_1)}(\gamma_1) h_{a_1} \left( \frac{\cdot}{n_T} \right), \omega_1 - \gamma_1 \right) \\
\times H_{n_T} \left( A_{[Tu]-n_T/2+1+,T}^{0(a_2)}(\gamma_2) h_{a_2} \left( \frac{\cdot}{n_T} \right), \omega_2 - \gamma_2 \right) \\
\times \cdots \\
\times H_{n_T} \left( A_{[Tu]-n_T/2+1+,T}^{0(a_p)}(\gamma_p) h_{a_p} \left( \frac{\cdot}{n_T} \right), \omega_p - \gamma_p \right) \\
\times \exp \left( i((\gamma_1 + \cdots + \gamma_p)[Tu]) \right) \eta \left( \sum_{j=1}^{p} \gamma_j \right) g_p(\gamma_1, \ldots, \gamma_{p-1}) d\gamma_1 \cdots d\gamma_p + o(1).
\]

By Lemma S.A.2, the latter is equal to

\[
\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} A^{(a_1)}(u, \gamma_1) \cdots A^{(a_p)}(u, \gamma_p) \\
\times H^{(a_1)}_{n_T}(\omega_1 - \gamma_1) \cdots H^{(a_p)}_{n_T}(\omega_p - \gamma_p) \\
\times \exp \left( i((\gamma_1 + \cdots + \gamma_p)[Tu]) \right) \eta \left( \sum_{j=1}^{p} \gamma_j \right) g_p(\gamma_1, \ldots, \gamma_{p-1}) d\gamma_1 \cdots d\gamma_p.
\] (S.3)

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plus a remainder term \( R_u \) with

\[
|R_u| \leq C \frac{n_T}{T} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} L_{n_T} (\omega_1 - \gamma_1) \cdots L_{n_T} (\omega_p - \gamma_p) \exp \left( i ((\gamma_1 + \cdots + \gamma_p) |Tu|) \right) 
\times \eta \left( \sum_{j=1}^{p} \gamma_j \right) g_p (\gamma_1, \ldots, \gamma_{p-1}) d\gamma_1 \cdots d\gamma_p
\leq C \frac{n_T}{T} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} L_{n_T} (\omega_1 - \gamma_1) \cdots L_{n_T} (\omega_p - \gamma_p) d\gamma_1 \cdots d\gamma_p
\leq C \frac{n_T}{T} (\ln n_T)^p,
\]

where we have used \( g_p (\gamma_1, \ldots, \gamma_{p-1}) \leq \text{const}_p \), the fact that \( \int_{-\pi}^{\pi} \exp \{ i (\gamma |Tu|) \} d\gamma = 2 \sin (\pi |Tu|) / |Tu| \), and the third inequality follows from Lemma S.A.1-(ii).

Next, note that the function \( H_{n_T} (\omega) \) will have substantial magnitude only for \( \omega \) near some multiple of \( 2\pi \). Thus, by continuity of \( A (\cdot, \omega), g_p \), and of the exponential function we yield that (S.3) is equal to

\[
\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} A^{(a_1)} (u, \omega_1) \cdots A^{(a_p)} (u, \omega_p)
\times H^{(a_1)}_{n_T} (\omega_1 - \gamma_1) \cdots H^{(a_p)}_{n_T} (\omega_p - \gamma_p)
\times \exp \left( i ((\omega_1 + \cdots + \omega_p) |Tu|) \right) \eta \left( \sum_{j=1}^{p} \omega_j \right) g_p (\omega_1, \ldots, \omega_{p-1}) d\gamma_1 \cdots d\gamma_p.
\]

By Lemma S.A.3,

\[
\left| \sum_{s=0}^{n_T-1} h_{a_1} \left( \frac{s + k_1}{n_T} \right) \cdots h_{a_{p-1}} \left( \frac{s + k_{p-1}}{n_T} \right) h_{a_p} \left( \frac{s}{n_T} \right) \exp \left( i \sum_{j=1}^{p} \omega_j s \right) - H^{(a_1, \ldots, a_p)} (\sum_{j=1}^{p} \omega_j) \right|
\leq C \left| \sum_{s=0}^{n_T-1} h_{a_1} \left( \frac{s + k_1}{n_T} \right) \cdots h_{a_{p-1}} \left( \frac{s + k_{p-1}}{n_T} \right) h_{a_p} \left( \frac{s}{n_T} \right) \exp \left( i \sum_{j=1}^{p} \omega_j s \right) \right|
\leq C (|k_1| + \cdots + |k_{p-1}|).
\]

Thus, (S.5) is equal to

\[
\sum_{k_1=-n_T}^{n_T} \cdots \sum_{k_{p-1}=-n_T}^{n_T} \exp \left( -i \sum_{j=1}^{p-1} \omega_j k_j \right)
\times \left( \kappa^{(a_1, \ldots, a_p)}_{T u, \lambda} (k_1, \ldots, k_{p-1}) H^{(a_1, \ldots, a_p)}_{T u, \lambda} \left( \sum_{j=1}^{p} \omega_j \right) + O (n_T/T) \right) + \varepsilon_T,
\]

where \( \kappa^{(a_1, \ldots, a_p)}_{T u, \lambda} (k_1, \ldots, k_{p-1}) \) is cumulant \( (X_{1_{k_1 T}}^{(a_1)}, \ldots, X_{p_{k_{p-1} T}}^{(a_p)}) \) + \( (n_T/T) \) and

\[
|\varepsilon_T| \leq C \sum_{k_1=-n_T}^{n_T} \cdots \sum_{k_{p-1}=-n_T}^{n_T} \kappa^{(a_1, \ldots, a_p)}_{T u, \lambda} (k_1, \ldots, k_{p-1}) (|k_1| + \cdots + |k_p|) < \infty.
\]

Note that \( |\varepsilon_T| / n_T \to 0 \) since \( (|k_1| + \cdots + |k_p|) / n_T \to 0 \). Thus, \( \varepsilon_T = o (n_T) \) uniformly in \( \omega_j (j = 1, \ldots, p) \).
Altogether we have
\[
\begin{align*}
\text{cum} \left( d_{h,T}^{(a_1)} (u, \omega_1), \ldots, d_{h,T}^{(a_p)} (u, \omega_p) \right) &= (2\pi)^{p-1} H_{a_1,\ldots,a_p}^{(a_1,\ldots,a_p)} \left( \sum_{j=1}^p \omega_j \right) f_{X}^{(a_1,\ldots,a_p)} (u, \omega_1, \ldots, \omega_{p-1}) + \varepsilon_T,
\end{align*}
\]

where \( f_{X}^{(a_1,\ldots,a_p)} (u, \omega_1, \ldots, \omega_{p-1}) \) is given in (2.1). The proof for \( r \)th cumulant of \( d_{h,T}^{(a_1)} (u, \omega_1) (j = 1, \ldots, r) \) with \( r < p \) is the same as for the \( p \)th cumulant.

Note that from (S.6) we have
\[
\begin{align*}
\sum_{k_1=-n_T}^{n_T} \cdots \sum_{k_{p-1}=-n_T}^{n_T} \exp \left( -i \sum_{j=1}^{p-1} \omega_j k_j \right) \kappa_{Tu,t_p} (k_1 \ldots, k_{p-1})
&= \sum_{k_1=-n_T}^{n_T} \cdots \sum_{k_{p-1}=-n_T}^{n_T} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \exp( i(\gamma_1 + \cdots + \gamma_{p-1} + \gamma_p) t_p \\
&+ (\omega_1 - \gamma_1) k_1 + \cdots + (\omega_{p-1} - \gamma_{p-1}) k_{p-1}))
\times A^{(a_1)} (\lfloor Tu \rfloor, \gamma_1) \cdots A^{(a_p)} (\lfloor Tu \rfloor, \gamma_p) g_p (\gamma_1, \ldots, \gamma_{p-1}) d\gamma_1 \cdots d\gamma_p.
\end{align*}
\]

Since \( \sum_{j=1}^p \gamma_j \equiv 0 \mod 2\pi \), \( \gamma_p \) is normalized and so the latter is equivalent to
\[
\begin{align*}
\sum_{k_1=-n_T}^{n_T} \cdots \sum_{k_{p-1}=-n_T}^{n_T} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \exp( i(\gamma_1 k_1 + \cdots + \gamma_{p-1} k_{p-1}))
\times A^{(a_1)} (\lfloor Tu \rfloor, \omega_1) \cdots A^{(a_p)} (\lfloor Tu \rfloor, \omega_p) g_p (\omega_1, \ldots, \omega_{p-1}) d\gamma_1 \cdots d\gamma_{p-1},
\end{align*}
\]

where we have used the continuity of \( A (\cdot, \omega) \) and \( g_p \). Then,
\[
A^{(a_1)} (\lfloor Tu \rfloor, \omega_1) \cdots A^{(a_p)} (\lfloor Tu \rfloor, \omega_p) g_p (\omega_1, \ldots, \omega_{p-1}) = f_{X}^{(a_1,\ldots,a_p)} (u, \omega_1, \ldots, \omega_{p-1}) \quad (S.7)
\]
is the spectrum that corresponds to the spectral representation (2.1) with \( m_0 = 0 \). In view of the following identities [see e.g., Exercise 1.7.5-(c,d) in Brillinger (1975)],
\[
\sum_{k=-n_T}^{n_T} \exp (-i \omega k) = \frac{\sin (n_T + 1/2) \omega}{\sin \omega/2}, \quad \int_{-\pi}^{\pi} \sin (n_T + 1/2) \omega \sin \omega/2 d\omega = 2\pi,
\]
we yield
\[
\begin{align*}
\text{cum} \left( d_{T}^{(a_1)} (u, \omega_1), \ldots, d_{T}^{(a_p)} (u, \omega_p) \right) &= (2\pi)^{p-1} H_{T}^{(a_1,\ldots,a_p)} \left( \sum_{j=1}^p \omega_j \right) f_{X}^{(a_1,\ldots,a_p)} (u, \omega_1, \ldots, \omega_{p-1}) + \varepsilon_T,
\end{align*}
\]
which verifies (S.7). □
S.A.2.2 Proof of Theorem 7.2

Proof of Theorem 7.2. We have,
\[
\mathbb{E} (d_{h,T} (u, \omega)) = \sum_{s=0}^{n_T - 1} \exp (-i \omega s) \mathbb{E} (X_{[Tu] - n_T/2 + s, T}) = 0.
\]
By Theorem 7.1 we deduce
\[
n^{-1} \text{Cov} \left( d_{h,T}^{(a_j)} (u, \pm \omega_j), \ d_{h,T}^{(a_k)} (u, \pm \omega_k) \right) = n^{-1} 2\pi H_{n_T}^{(a_1, a_r)} (\pm \omega_j \mp \omega_k) f_X^{(a_1, a_r)} (u, \pm \omega_j (n_T)) + o(1) + O\left( n^{-1} \right).
\]
Note that [see, e.g., Lemma P4.6 in Brillinger (1975)],
\[
\left| H_{n_T}^{(a_1, \ldots, a_p)} (\omega) \right| \leq \frac{C}{|\sin (\omega/2)|},
\]
where \( C \) is a constant with \( 0 < C < \infty \). If \( \omega_j \pm \omega_k \not\equiv 0 \text{ (mod 2\pi)} \) the first term on the right-hand side of (S.8) tends to zero using (S.9). If \( \pm \omega_j \equiv \omega_k \equiv 0 \text{ (mod 2\pi)} \) the right-hand side of (S.8) tends to
\[
2\pi H_{n_T}^{(a_1, a_r)} (0) f_X^{(a_1, a_r)} (u, \pm \omega_j) = 2\pi \left( \int h^{(a_1)} (t) h^{(a_r)} (t) \, dt \right) f_X^{(a_1, a_r)} (u, \pm \omega_j).
\]
This shows that the second-order cumulants behave as indicated by the theorem. By Theorem 7.1 for \( r > 2 \),
\[
n^{-r/2} \text{cum} \left( d_{h,T}^{(a_1)} (u, \pm \omega_j_1), \ldots, d_{h,T}^{(a_r)} (u, \pm \omega_j_r) \right) = n^{-r/2} (2\pi)^{r-1} H_{n_T}^{(a_1, \ldots, a_r)} (\pm \omega_{j_1} \pm \cdots \pm \omega_{j_r}) f_X^{(a_1, \ldots, a_r)} (u, \pm \omega_j_1, \ldots, \pm \omega_j_{r-1}) + o\left( n^{-r/2} \right).
\]
The latter tends to 0 as \( n_T \to \infty \) if \( r > 2 \) because \( H_{n_T}^{(a_1, \ldots, a_r)} (\omega) = O\left( n_T \right) \). Thus, also the cumulants of order higher than two behave as indicated by the theorem. This implies that the cumulants of the considered variables and the conjugates of those variables tend to the cumulants of Gaussian random variable. Since the distribution of the latter is fully determined by its moments, the theorem follows from Lemma S.A.4. The second part of the theorem follows from the fact that \( \sin (\omega) = 0 \) for \( \omega = 0, \pm \pi, \pm 2\pi, \pm 2\pi, \ldots \). □

S.A.2.3 Proof of Theorem 7.3

The proof of the second equality in (7.2) is similar to Dahlhaus (1996a) who proved the result under stronger assumptions on the data taper. Using the spectral representation (2.1),
\[
\text{cum} \left( d_{h,T} (u, \omega), \ d_{h,T} (u, -\omega) \right) = \sum_{t=0}^{n_T - 1} \sum_{s=0}^{n_T - 1} h \left( \frac{t}{T} \right) h \left( \frac{s}{T} \right) \int_{-\pi}^{\pi} \exp (-i (\omega - \eta) (s - t)) A_0^0 (\sin (T - t) \ (\eta)) A_0^0 (\sin (T - t + s) \ (-\eta)) \, d\eta.
\]
We use Abel’s transformation to replace $A^0_{[Tu]-nT/2+t}(\eta)$ by $A(u, \omega)$,

$$
\frac{1}{nT} \sum_{t=0}^{nT-1} h\left( \frac{t}{nT} \right) \left( A^0_{[Tu]-nT/2+t}(\eta) - A(u, \omega) \right) \exp(-i(\omega - \eta)t)
$$

$$
= \frac{1}{nT} \sum_{t=0}^{nT-1} \left( A^0_{[Tu]-nT/2+t}(\eta) - A^0_{[Tu]-nT/2+t-1}(\eta) \right) H_t\left( h\left( \frac{t}{nT}, \omega - \eta \right) \right)
$$

$$
+ \left( A^0_{[Tu]-nT/2+nT-1}(\eta) - A(u, \omega) \right) H_{nT}\left( h\left( \frac{1}{nT}, \omega - \eta \right) \right)
$$

$$
\leq O\left( \frac{nT}{T} \right) L_{nT}(\omega - \eta) + O\left( \frac{nT}{T} + O(|\omega - \eta|) \right) L_{nT}(\omega - \eta),
$$

where the inequality follows from using Lemma S.A.2,

$$
\left| H_t\left( h\left( \frac{t}{nT}, \omega - \eta \right) \right) \right| \leq L_t(\omega - \eta) \leq L_{nT}(\omega - \eta). \tag{S.10}
$$

Since we are dividing by $\sum_{s=0}^{nT-1} h(s/nT)^2 \sim nT$ we get,

$$
nT^{-1} \sum_{t=0}^{nT-1} h\left( \frac{t}{nT} \right) \left( A^0_{[Tu]-nT/2+t}(\eta) - A\left( u + \frac{t-nT/2}{T}, \omega \right) \right) \exp(-i(\omega - \eta)t)
$$

$$
\leq O\left( \frac{1}{T} \right) L_{nT}(\omega - \eta) + O\left( \frac{1}{T} + nT^{-1}O(|\omega - \eta|) \right) L_{nT}(\omega - \eta)
$$

$$
\leq C < \infty
$$

where we have used the fact that $L_{nT}(\omega - \eta) \leq nT$ and that

$$
|\omega - \eta| L_{nT}(\omega - \eta) = \begin{cases} 
|\omega - \eta| nT, & |\omega - \eta| \leq 1/nT \\
|\omega - \eta| / |\omega - \eta|, & 1/nT \leq |\omega - \eta| \leq \pi.
\end{cases}
$$

Using Lemma S.A.2 and (S.10) we have,

$$
nT^{-1} \sum_{s=0}^{nT-1} \left( s/T \right) \exp(i(\omega - \eta)s) A^0_{[Tu]-nT/2+s}(-\eta) \, dn
$$

$$
= nT^{-1} \left| A\left( \left[ Tu \right] - nT/2 \right), -\eta \right) H_{nT}(-\omega + \eta) + O(T^{-1})
$$

$$
= nT^{-1} O\left( \sup_{u \in [0, 1]} A(u, -\eta) \right) L_{nT}(-\omega + \eta) + O(T^{-1}).
$$

Thus, after integration over $\eta$ we yield that the error in replacing $A^0_{[Tu]-nT/2+t}(\eta)$ by $A(u, \omega)$ is $O((\log nT)/nT)$. Next, we replace $A^0_{[Tu]-nT/2+s}(-\eta)$ by $A(u, \omega)$ and integrate over $\eta$ using the relation

$$
A(u, \omega) A(u, -\omega) = |A(u, \omega)|^2 = f_X(u, \omega).
$$

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In view of
\[
\int_{-\pi}^{\pi} |H_{nT}(\alpha)|^2 \, d\alpha = 2\pi \sum_{t=0}^{n_T-1} \left( \frac{t}{n_T} \right)^2, \tag{S.11}
\]
we then yield
\[
\mathbb{E}(I_{h,T}(u, \omega)) = \frac{1}{2\pi H_{2,n_T}(0)} \sum_{t=0}^{n_T-1} \sum_{s=0}^{n_T-1} h\left( \frac{t}{T} \right) h\left( \frac{s}{T} \right) \int_{-\pi}^{\pi} \exp(-i(\omega-\alpha)(s-t)) f_X(u, \alpha) \, d\alpha + O\left( \frac{\log n_T}{n_T} \right)
\]
\[
= \frac{1}{\int_{-\pi}^{\pi} |H_{nT}(\alpha)|^2 \, d\alpha} \int_{-\pi}^{\pi} H_{nT}(\omega-\alpha)^2 f_X(u, \alpha) \, d\alpha + O\left( \frac{\log n_T}{n_T} \right)
\]
\[
= \frac{1}{\int_{-\pi}^{\pi} |H_{nT}(\alpha)|^2 \, d\alpha} \int_{-\pi}^{\pi} H_{nT}(\alpha)^2 f_X(u, \omega-\alpha) \, d\alpha + O\left( \frac{\log n_T}{n_T} \right).
\]
This shows the first equality of (7.2). For the second equality of (7.2) replace \(A_{0,T_u}^{n_T-2/t+T,\omega}(-\eta)\) by \(A(u + (t - n_T/2)/T, \omega)\) and \(A_{0,T_u}^{n_T-2/t+T,\omega}(-\eta)\) by \(A(u + (t - n_T/2)/T, -\omega)\) so that (S.12) holds with \(f_X(u + (t - n_T/2)/T, \omega)\) in place of \(f_X(u, \alpha)\). Then take a second-order Taylor expansion of \(f_X\) around \(u\) to obtain
\[
\mathbb{E}(I_{h,T}(u, \omega)) = \frac{1}{2\pi H_{2,n_T}(0)} \sum_{t=0}^{n_T-1} h\left( \frac{t}{T} \right)^2 f_X\left(u + \frac{t-n_T/2}{T}, \omega\right) + O\left( \frac{\log n_T}{n_T} \right)
\]
\[
= f_X(u, \omega) + \frac{1}{2} \left( \frac{n_T}{T} \right)^2 \int_0^1 x^2 h^2(x) \, dx \frac{\partial^2}{\partial u^2} f_X(u, \omega)
\]
\[
+ o\left( \left( \frac{n_T}{T} \right)^2 \right) + O\left( \frac{\log n_T}{n_T} \right). \quad \Box
\]

**S.A.2.4 Proof of Theorem 7.4**

By Theorem 2.3.1-(ix) in Brillinger (1975), \(\text{Cov}(Y_j, Y_k) = \text{cum}(Y_j, Y_k)\) for possibly complex variables \(Y_j\) and \(Y_k\). Thus,
\[
\text{Cov}(d_{h,T}(u, \omega_j), d_{h,T}(u, -\omega_j), d_{h,T}(u, \omega_k), d_{h,T}(u, -\omega_k)) = \text{cum}(d_{h,T}(u, \omega_j), d_{h,T}(u, -\omega_j), d_{h,T}(u, \omega_k), d_{h,T}(u, -\omega_k)).
\]

By the product theorem for cumulants [cf. Brillinger (1975), Theorem 2.3.2], we have to sum over all indecomposable partitions \(\{P_1, \ldots, P_m\}\) with \(|P_i| = \text{card}(P_i) \geq 2\) of the two-way table,
\[
\begin{array}{c|cc}
& a_{j,1} & a_{j,2} \\
\hline
a_{k,1} & a_{k,2} \\
\end{array}
\]
where \(a_{j,1}\) and \(a_{j,2}\) stand for the position of \(d_{h,T}(u, \omega_j)\) and \(d_{h,T}(u, -\omega_j)\), respectively. This results in,
\[
\text{cum}(d_{h,T}(u, \omega_j), d_{h,T}(u, -\omega_j), d_{h,T}(u, \omega_k), d_{h,T}(u, -\omega_k)) = \text{cum}(d_{h,T}(-\omega_j), d_{h,T}(-\omega_j), d_{h,T}(\omega_k), d_{h,T}(-\omega_k))
\]
\[
+ \text{cum}(d_{h,T}(\omega_j)) \text{cum}(d_{h,T}(-\omega_j), d_{h,T}(\omega_k), d_{h,T}(-\omega_k)).
\]
+ three similar terms
+ \sum (d_{h,T}(\omega_j)) \sum (d_{h,T}(\omega_k)) \sum (d_{h,T}(-\omega_j), d_{h,T}(-\omega_k))
+ three similar terms
+ \sum (d_{h,T}(\omega_j), d_{h,T}(-\omega_j)) \sum (d_{h,T}(-\omega_j), d_{h,T}(-\omega_k))
+ \sum (d_{h,T}(\omega_j), d_{h,T}(-\omega_k)) \sum (d_{h,T}(-\omega_j), d_{h,T}(\omega_k)).

Then, by Theorem 7.1,

\begin{align}
\sum (d_{h,T}(u, \omega_j) \sum (d_{h,T}(u, \omega_k)) \sum (d_{h,T}(u, -\omega_j), d_{h,T}(u, -\omega_k)) \\
= (2\pi)^3 H_{4,n_T} (0) f_X (u, \omega_j, -\omega_j, \omega_k) + O (1)
\end{align}

+ three similar terms
+ \sum (d_{h,T}(\omega_j) \sum (d_{h,T}(-\omega_j)) \sum (d_{h,T}(-\omega_j), d_{h,T}(-\omega_k))
+ \sum (d_{h,T}(\omega_j), d_{h,T}(-\omega_j)) \sum (d_{h,T}(-\omega_j), d_{h,T}(\omega_k)).

Given

\begin{align}
H_{2,n_T} (0) = \sum_{T=0}^{n_T-1} h^2 (t/T) \sim n_T \int h^2 (\alpha) d\alpha
\end{align}

and

\begin{align}
H_{2,n_T} (\omega_j - \omega_k) H_{2,n_T} (-\omega_j + \omega_k) = |H_{2,n_T} (\omega_j - \omega_k)|^2
\end{align}

the result of the theorem follows because

\begin{align}
n_T^{-2} (2\pi)^3 H_{4,n_T} (0) f_X (u, \omega_j, -\omega_j, \omega_k) = O \left( n_T^{-1} \right),
\end{align}

and because the \( O (1) \) terms on the right-hand side of (S.13) becomes negligible when multiplied by \( H_{2,n_T}^{-2} (0) \).

Next, we prove the second result of the theorem. Recall that \( z \sim \mathcal{N}_p^C (\mu, \Sigma) \) means that the \( 2p \) vector

\[
\begin{bmatrix}
\Re z \\
\Im z
\end{bmatrix}
\]

is distributed as

\[
\mathcal{N}_{2p} \left( \begin{bmatrix} \Re \mu_z \\ \Im \mu_z \end{bmatrix}; \frac{1}{2} \begin{bmatrix} \Re \Sigma_z & -\Im \Sigma_z \\ -\Im \Sigma_z & \Re \Sigma_z \end{bmatrix} \right),
\]

where \( \Sigma_z \) is a \( p \times p \) hermitian positive semidefinite matrix. By Theorem 7.2 we know that \( \Re d_{h,T}(\omega_j) \) and \( \Im d_{h,T}(\omega_j) \) are asymptotically independent \( \mathcal{N}_2 (0, \pi n_T f_X (u, \omega_j)) \) variates. Hence, by the Mann-Wald Theorem,

\[
I_{h,T} (u, \omega_j (n_T)) = (2\pi n_T)^{-1} \left\{ (\Re d_{h,T}(u, \omega_j (n_T)))^2 + (\Im d_{h,T}(\omega_j (n_T)))^2 \right\},
\]

is asymptotically distributed as \( f_X (u, \omega_j) \chi_2^2 / 2 \) if \( 2\omega_j \equiv 0 \pmod{2\pi} \). This proves part (i). For part (ii), if \( \omega = \pm \pi, \pm 3\pi, \ldots \) then \( I_{h,T} (u, \omega) \) is asymptotically distributed as \( f_X (u, \omega) \chi_2^2 \), independently from the
previous variates. □

**S.A.2.5 Proof of Theorem 7.5**

Using Theorem 7.3 we have

\[
\mathbb{E} (f_{h,T} (u, \omega)) = \frac{2\pi}{nT} \sum_{s=0}^{nT-1} W_T \left( \omega - \frac{2\pi s}{nT} \right) \mathbb{E} \left( I_{h,T} \left( u, \frac{2\pi s}{nT} \right) \right) \\
= \frac{2\pi}{nT} nT - 1 \sum_{s=0}^{nT-1} W_T \left( \omega - \frac{2\pi s}{nT} \right) \mathbb{E} \left( f X \left( u, \frac{2\pi s}{T} \right) \right) + O \left( nT^{-1} \right) + O \left( \log (nT) nT^{-1} \right).
\]

The first term on the right-hand side is

\[
\frac{2\pi}{nT} nT - 1 \sum_{s=0}^{nT-1} W_T \left( \omega - \frac{2\pi s}{nT} \right) \mathbb{E} \left( f X \left( u, \frac{2\pi s}{T} \right) \right) \\
= \int_0^{2\pi} W_T \left( \omega - \alpha \right) f X \left( u, \alpha \right) d\alpha + O \left( (nTb_T)^{-1} \right) + O \left( \log (nT) nT^{-1} \right) \\
= \int_0^{2\pi} \sum_{j=-\infty}^{\infty} b_T^{-1} W \left( b_T^{-1} \left( \omega - \alpha + 2\pi j \right) \right) f X \left( u, \alpha \right) d\alpha + O \left( (nTb_T)^{-1} \right) + O \left( \log (nT) nT^{-1} \right) \\
= \int_{-\infty}^{\infty} W \left( \beta \right) f X \left( u, \omega - \beta b_T \right) d\beta + O \left( (nTb_T)^{-1} \right) + O \left( \log (nT) nT^{-1} \right),
\]

where the last equality follows from the change in variable \( \beta = b_T^{-1} (\omega - \alpha) \). This yields the first equality of (7.4). The second equality follows from the first and Theorem 7.3 along with a Taylor expansion. □

**S.A.2.6 Proof of Theorem 7.6**

Let

\[
c_T (u, k) = H_{2,T} (0)^{-1} \sum_{s=0}^{nT-1} h \left( \frac{s + k}{T} \right) h \left( \frac{s}{T} \right) X_{[Tu] - nT/2 + s + k + 1,T} X_{[Tu] - nT/2 + s + 1,T}.
\]

We can rewrite \( I_{h,T} (u, \omega) \) using \( c_T (u, k) \) as follows,

\[
I_{h,T} (u, \omega) = (2\pi)^{-1} \sum_{k=-\infty}^{\infty} \exp (-i\omega k) c_T (u, k).
\]

Note that

\[
f_{h,T} (u, \omega) = \int_0^{2\pi} W_{2,T} \left( \omega - \alpha \right) I_{h,T} (u, \alpha) d\alpha + O \left( (nTb_{2,T})^{-1} \right),
\]

where \( W_{2,T} (\omega) = \sum_{k=-\infty}^{\infty} w (b_{2,T} k) \exp (-i\omega k) \) and \( w (k) = \int_{-\infty}^{\infty} W_{2,T} (\alpha) \exp (i\alpha k) d\alpha \) for \( k \in \mathbb{R} \). From Theorem 7.4,

\[
\text{Cov} (f_{h,T} (u, \omega_j) , f_{h,T} (u, \omega_k))
\]
\[
\int_0^{2\pi} \int_0^{2\pi} W_{2,T}(\omega_j - \alpha)W_{2,T}(\omega_k - \beta) \text{Cov}(I_{h,T}(u, \alpha), I_{h,T}(u, \beta)) \, d\alpha d\beta = H_{2,n_T}^{-1}(0) H_{2,n_T}^{-1}(0)^{-1} \int_0^{2\pi} \int_0^{2\pi} W_{2,T}(\omega_j - \alpha)W_{2,T}(\omega_k - \beta) \\
\times \{|H_{2,n_T}(\alpha - \beta)|^2 + |H_{2,n_T}(\alpha + \beta)|^2\} |f(u, \alpha)|^2 \, d\alpha d\beta + O\left(n_T^{-1}\right),
\]

We now show that
\[
\int_0^{2\pi} W_{2,T}(\omega_k - \beta) |H_{2,n_T}(\alpha - \beta)|^2 \, d\beta = 2\pi W_{2,T}(\omega_k - \alpha) \sum_{s=0}^{n_T-1} h^4(s) + O(b_{W,T}^{-2}),
\]
uniformly in \(\alpha\). We can expand (S.14) as follows,
\[
\sum_{t=0}^{n_T-1} \sum_{s=0}^{n_T-1} h^2(t/n_T) h^2(s/n_T) \int_0^{2\pi} W_{2,T}(\omega_k - \beta) \times \exp\{-i(\alpha - \beta) t + i(\alpha - \beta) s\} \, d\beta \\
= \sum_{t=0}^{n_T-1} \sum_{s=0}^{n_T-1} h^2(t) h^2(s) \int_0^{2\pi} \sum_{k=-\infty}^{\infty} w(b_{W,T}k) \exp\{-i(\omega_k - \beta) k\} \\
\times \exp\{-i(\alpha - \beta) t + i(\alpha - \beta) s\} \, d\beta \\
= \sum_{t=0}^{n_T-1} \sum_{s=0}^{n_T-1} h^2(t/n_T) h^2(s/n_T) w(b_{W,T}(t-s)) \exp(i(\omega_k - \alpha)(t-s)) \\
= \sum_{k=-\infty}^{\infty} w(b_{W,T}k) \exp(i(\omega_k - \alpha) k) \sum_{s=0}^{n_T-1} h^2((s+k)n_T) h^2(s/n_T) \\
= 2\pi W_{2,T}(\omega_k - \alpha) \sum_{s=0}^{n_T-1} h^4(s/n_T) + R_T,
\]
where we have applied Lemma S.A.3 to \(\exp(i(\omega_k - \alpha) k) \sum_{s=0}^{n_T-1} h^2(s+k) h^2(s)\) to yield,
\[
\left| \exp(i(\omega_k - \alpha) k) \sum_{s=0}^{n_T-1} h^2(s+k) h^2(s) - \exp(i(\omega_k - \alpha) k) \sum_{s=0}^{n_T-1} h^4(s/n_T) \right| \leq C |k| ,
\]
and
\[
|R_T| \leq C \sum_{k=-\infty}^{\infty} |w(b_{W,T}k)| |k| \sim C b_{W,T}^{-2} \int |x| |w(x)| \, dx,
\]
for \(0 < C < \infty\). The latter result follows because
\[
C \sum_{k=-\infty}^{\infty} |w(b_{W,T}k)| |k| = C b_{W,T}^{-2} b_{W,T} \sum_{k=-\infty}^{\infty} |w(b_{W,T}k)| |b_{W,T}k|
\]

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Finally, we consider the magnitude of the joint cumulants of order \( r \). We have

\[
\text{cum} \left( f_{h,T}(u, \omega_1), \ldots, f_{h,T}(u, \omega_r) \right) = 2\pi \left\{ H_{2,n_T}(0) \right\}^{-r} \times \sum_{t_1=0}^{n_T-1} \sum_{t_2=0}^{n_T-1} w(b_T(t_1 - t_2)) \cdots w(b_T(t_{2r-1} - t_{2r})) \times \exp (-i\omega_1 (t_1 - t_2) - \ldots - i\omega_r (t_{2r-1} - t_{2r})) h_{n_T}(t_1) \cdots h_{n_T}(t_{2r}) \times \text{cum} \left( X_{[T u]} - n_T / 2 + t_1 + 1, T X_{[T u]} - n_T / 2 + t_2 + 1, T, \ldots, X_{[T u]} - n_T / 2 + t_{2r-1} + 1, T X_{[T u]} - n_T / 2 + t_{2r} + 1, T \right).
\]

Equation (7.5) follows from

\[
n_T b_{W,T} \text{Cov} \left( f_{h,T}(u, \omega_j), f_{h,T}(u, \omega_k) \right) = b_{W,T} 2\pi n_T H_{2,T}(0)^{-1} n_T H_{2,T}(0)^{-1} n_T^{-1} \sum_{l=0}^{n_T-1} h(t/n_T)^4 \times \int_0^{2\pi} \left\{ \sum_{l=-\infty}^{\infty} b_{W,T}^{-1} W \left( b_{W,T}^{-1} (\omega_j - \alpha + 2\pi l) \right) \right. \times \sum_{l=-\infty}^{\infty} b_{W,T}^{-1} W \left( b_{W,T}^{-1} (\omega_k - \alpha + 2\pi l) \right) \left| f(u, \alpha) \right|^2 + \sum_{l=-\infty}^{\infty} b_{W,T}^{-1} W \left( b_{W,T}^{-1} (\omega_j - \alpha + 2\pi l) \right) \times \sum_{l=-\infty}^{\infty} b_{W,T}^{-1} W \left( b_{W,T}^{-1} (\omega_k + \alpha + 2\pi l) \right) \left| f(u, \alpha) \right|^2 \left. \right\} d\alpha + O \left( (n_T b_{W,T})^{-1} \right) + O \left( b_{W,T} \right) = 2\pi \left( \int h^2(t) \, dt \right)^{-2} \int h^4(t) \, dt \int_0^{2\pi} \left[ \eta \{ \omega_j - \omega_k \} \left| f(u, \omega_j) \right|^2 + \eta \{ \omega_j + \omega_k \} \left| f(u, \omega_j) \right|^2 \right] W^2(\alpha) \, d\alpha + O \left( (n_T b_{W,T})^{-1} \right) + O \left( b_{W,T} \right).
\]

for a finite \( 0 < C < \infty \). A similar result holds for the second term involving \( |H_{2,n_T}(\alpha + \beta)|^2 \). Overall, we have

\[
\text{Cov} \left( f_{h,T}(u, \omega_j), f_{h,T}(u, \omega_k) \right) = 2\pi n_T H_{2,T}(0)^{-2} \sum_{s=0}^{n_T-1} h(s/n_T)^4 \int_0^{2\pi} \{ W_{2,T} (\omega_j - \alpha) W_{2,T} (\omega_k - \alpha) \left| f(u, \alpha) \right|^2 + W_{2,T} (\omega_j - \alpha) W_{2,T} (\omega + \alpha) \left| f(u, \alpha) \right|^2 \} d\alpha + O \left( b_{W,T}^{-2} n_T^{-2} \right) + O \left( n_T^{-1} \right).
\]

Finally, we consider the magnitude of the joint cumulants of order \( r \). We have

\[
\text{cum} \left( f_{h,T}(u, \omega_1), \ldots, f_{h,T}(u, \omega_r) \right) = 2\pi \left\{ H_{2,n_T}(0) \right\}^{-r} \times \sum_{t_1=0}^{n_T-1} \sum_{t_2=0}^{n_T-1} w(b_T(t_1 - t_2)) \cdots w(b_T(t_{2r-1} - t_{2r})) \times \exp (-i\omega_1 (t_1 - t_2) - \ldots - i\omega_r (t_{2r-1} - t_{2r})) h_{n_T}(t_1) \cdots h_{n_T}(t_{2r}) \times \text{cum} \left( X_{[T u]} - n_T / 2 + t_1 + 1, T X_{[T u]} - n_T / 2 + t_2 + 1, T, \ldots, X_{[T u]} - n_T / 2 + t_{2r-1} + 1, T X_{[T u]} - n_T / 2 + t_{2r} + 1, T \right).
\]
Note that
\[
\sum_{\nu} c_{X \ldots X}(u; t_j, j \in v_1) \cdots c_{X \ldots X}(u; t_j, j \in v_p),
\]
where \(c_{X \ldots X}(u; t_j, j \in v_1)\) is the time-\(T_u\) cumulant involving the variables \(X_{t_j}\) for \(j \in v_1\) and where the summation is over all indecomposable partitions \(\nu = (v_1, \ldots, v_p)\) of the table

\[
\begin{array}{cc}
1 & 2 \\
3 & 4 \\
\vdots & \vdots \\
2r - 1 & 2r \\
\end{array}
\]

As the partition is indecomposable, in each set \(v_p\) of the partition we may find an element \(t^*_p\) such that none of \(t_j - t^*_p, j \in v_p\) (\(p = 1, \ldots, P\)) is \(t_{2l-1} - t_{2l}, l = 1, 2, \ldots, r\). Define \(2r - P\) new variables \(k_1, \ldots, k_{2r-P}\) as the nonzero \(t_j - t^*_p\). Eq. (S.15) is now bounded by

\[
C^r n_T^{-r} \sum_{\nu} \sum_{t^*_1} \cdots \sum_{t^*_1} \sum_{k_{2r-P}} \cdots \sum_{k_{2r-P}} w(b_{W, T}(k_{\alpha_1} + t^*_{\beta_1} - \alpha_1 - t^*_{\beta_2}) \cdots w(b_{W, T}(k_{\alpha_r} + t^*_{\beta_r} - \alpha_r - t^*_{\beta_r})) \times |h(t^*_r/n_T)|^{2r}|c_{X \ldots X}(u; k_1, \ldots) \cdots c_{X \ldots X}(u; \ldots, k_{2r-P})|,
\]

for some finite \(C\) where \(\alpha_1, \ldots, \alpha_{2r}\) are selected from \(1, \ldots, 2r\) and \(\beta_1, \ldots, \beta_{2r}\) from \(1, \ldots, P\). By Lemma 2.3.1 in Brillinger (1975) there are \(P - 1\) linearly independent differences among the \(t_{\beta_1}^* - t_{\beta_2}^*, \ldots, t_{\beta_{2r-1}}^* - t_{\beta_{2r}}^*\). Suppose these are \(t_{\beta_1}^* - t_{\beta_2}^*, \ldots, t_{\beta_{2r-2}}^* - t_{\beta_{2r-1}}^*\). Making the change of variables

\[
s_1 = k_{\alpha_1} + t^*_{\beta_1} - \alpha_1 - t^*_{\beta_2} \\
\vdots \\
s_{p-1} = k_{\alpha_{2p-3}} + t^*_{\beta_{2p-3}} - \alpha_{2p-3} - t^*_{\beta_{2p-2}},
\]

the cumulant (S.15) is bounded by

\[
C^r n_T^{-r} \sum_{\nu} \sum_{s_1} \sum_{s_{p-1}} \cdots \sum_{s_1} \sum_{s_{p-1}} \sum_{k_{2r-P}} \cdots \sum_{k_{2r-P}} |w(b_{W, T}s_1) \cdots w(b_{W, T}s_{p-1})| |h(t^*_r/n_T)|^{2r}|c_{X \ldots X}(u; k_1, \ldots) \cdots c_{X \ldots X}(u; \ldots, k_{2r-P})| \\
\leq C^r n_T^{-r+1}b_{W, T}^{-P-1} \sum_{\nu} C_{n_2, 1} \cdots C_{n_2, P} \\
= O \left( n_T^{-r+1}b_{W, T}^{-P-1} \right),
\]

where \(P \leq r\) and \(C_{n_2, j} = \sup_{u \in [0, 1]} \sum_{t_1, \ldots, t_{n_2, j}} |c_{X \ldots X}(u; t_1, \ldots, t_{n_2, j})|\) with \(n_{2, j}\) denoting the number of elements in the \(j\)th set of the partition \(\nu\). It follows that for \(r > 2\),

\[
\sum \left( n_T b_{W, T} \right)^{1/2} f_{h, T}(u, \omega_1), \ldots, (n_T b_{W, T})^{1/2} f_{h, T}(u, \omega_r) \to 0.
\]
Thus, the variates \( f_{h,T}(u, \omega_1), \ldots, f_{h,T}(u, \omega_r) \) are asymptotically normal with the moment structure given in the theorem. \( \square \)

**S.A.3 Proof of the Results of Section 3**

**S.A.3.1 Preliminary Lemmas**

**Lemma S.A.5.** Assumption 3.1, 3.2 with \( l = 0 \) and \( r = 2 \), 3.3-3.5 and Condition 1 hold. Under \( \mathcal{H}_0 \) we have \( \sqrt{\log(M_T)} M^{1/2}_{S,T}(S_{\max,T}(\omega) - \tilde{S}_{\max,T}(\omega)) \overset{p}{\rightarrow} 0 \) for any \( \omega \in [-\pi, \pi] \) where

\[
\tilde{S}_{\max,T}(\omega) \triangleq \max_{r=1,\ldots,M_T-2} \left| \frac{\tilde{f}_{r,T}(\omega) - \tilde{f}_{r+1,T}(\omega)}{\sigma_{f,r}(\omega)} \right|.
\]

**Proof of Lemma S.A.5.** Note that for arbitrary sequences of numbers \((a_i)_{i=1,\ldots,N}\) and \((b_i)_{i=1,\ldots,N}\) with \( N \geq 1 \), we have for any \( i \),

\[
|a_i| \leq |a_i - b_i| + |b_i| \leq \max_{i=1,\ldots,N} |a_i| + \max_{i=1,\ldots,N} |b_i|.
\]

(S.16) The inequality still holds if on the left-hand side we replace \( |a_i| \) by \( \max_{i=1,\ldots,N} |a_i| \). We have

\[
S_{\max,T}(\omega) - \tilde{S}_{\max,T}(\omega)
\]

\[
= \max_{r=1,\ldots,M_T-2} \left| \frac{\tilde{f}_{L,r,T}(\omega) - \tilde{f}_{R,r+1,T}(\omega)}{\sigma_{L,r}(\omega)} \right| - \max_{r=1,\ldots,M_T-2} \left| \frac{\tilde{f}_{r,T}(\omega) - \tilde{f}_{r+1,T}(\omega)}{\sigma_{f,r}(\omega)} \right|.
\]

(S.17) Using (S.16) the right-hand side of (S.17) is less than or equal to

\[
\max_{r=1,\ldots,M_T-2} \left| \frac{\tilde{f}_{L,r,T}(\omega) - \tilde{f}_{R,r+1,T}(\omega)}{\sigma_{L,r}(\omega)} \right| + \max_{r=1,\ldots,M_T-2} \left| \frac{\tilde{f}_{r,T}(\omega) - \tilde{f}_{r+1,T}(\omega)}{\sigma_{f,r}(\omega)} \right| - \max_{r=1,\ldots,M_T-2} \left| \frac{\tilde{f}_{r,T}(\omega) - \tilde{f}_{r+1,T}(\omega)}{\sigma_{f,r}(\omega)} \right|.
\]

The second line converges to zero in probability given the uniform asymptotic equivalence of \( \sigma_{L,r}(\omega) \) and \( \sigma_{f,r}(\omega) \) with an error \( O(T^{-1}) \). Thus, it is sufficient to show

\[
\max_{r=1,\ldots,M_T-2} \left| \frac{\tilde{f}_{L,r,T}(\omega) - \tilde{f}_{R,r+1,T}(\omega)}{\sigma_{L,r}(\omega)} \right| \overset{p}{\rightarrow} 0.
\]

We use the following decomposition,

\[
\sqrt{\log(M_T)} M^{1/2}_{S,T} \left| \frac{\tilde{f}_{L,r,T}(\omega) - \tilde{f}_{R,r+1,T}(\omega)}{\sigma_{L,r}(\omega)} \right| \leq \sqrt{\log(M_T)} M^{1/2}_{S,T} \left| \frac{\tilde{f}_{L,r,T}(\omega) - \tilde{f}_{r+1,T}(\omega)}{\sigma_{L,r}(\omega)} \right|.
\]

(S.18)
\[ + \sqrt{\log (M_T) M_{S,T}^{1/2}} \left| \frac{f_{r,T}(\omega) - f_{r+1,T}(\omega)}{\sigma_{L,r}(\omega)} \right| . \]

Let us consider the first term on the right-hand side of (S.18). Note that for all \( \epsilon > 0 \) and all constants \( C > 0 \), we have

\[
P \left( \max_{r=1,\ldots,M_T-2} \left| \sqrt{\log (M_T) M_{S,T}^{1/2}} \left( f_{L,r,T}(\omega) - f_{R,r+1,T}(\omega) \right) \sigma_{L,r}(\omega) \right| > \epsilon \right) \\
\leq P \left( \max_{r=1,\ldots,M_T-2} \left| \sqrt{\log (M_T) M_{S,T}^{1/2}} \left( f_{L,r,T}(\omega) - f_{R,r+1,T}(\omega) \right) \right| \cdot \max_{r=1,\ldots,M_T-2} \frac{1}{\sigma_{L,r}(\omega)} > \epsilon \right) \\
\leq P \left( \max_{r=1,\ldots,M_T-2} \left| \sqrt{\log (M_T) M_{S,T}^{1/2}} (nTb_{W,T})^{1/2} \left| f_{L,r,T}(\omega) - f_{R,r+1,T}(\omega) \right| \right| > \frac{\epsilon}{C} \right) \quad (S.19) \\
+ P \left( \max_{r=1,\ldots,M_T-2} \frac{1}{(nTb_{W,T})^{1/2} \sigma_{L,r}(\omega)} > C \right). \]

Theorem 7.5 implies that

\[ E (f_{h,T}(u, \omega)) = f(u, \omega) + O \left( (nT/T)^{-2} \right) + O \left( \log (nT) n_T^{-1} \right) + o \left( b_{W,T}^2 \right). \]

The same result holds for \( f_{L,h,T}(u, \omega) \) and \( f_{R,h,T}(u, \omega) \). By Assumption 4.1 we have

\[ f((r + 1) m_T + j)/T, \omega) - f((rm_T + j)/T, \omega) = O \left( (mt/T)^\theta \right), \quad \text{uniformly in } r \text{ and } j. \quad (S.20) \]

Thus, it follows that

\[
\sqrt{\log (M_T) M_{S,T}^{1/2}} (nTb_{W,T})^{1/2} \left| f_{L,r,T}(\omega) - f_{R,r+1,T}(\omega) \right| \\
= \sqrt{\log (M_T) M_{S,T}^{1/2}} \left( O \left( (mt/T)^\theta \right) + O \left( (nT/T)^{-2} \right) + O \left( \log (nT) n_T^{-1} \right) + O \left( b_{W,T}^2 \right) \right) \\
= O \left( 1 \right),
\]

where the last equality uses Condition 1. By using Markov’s inequality, this shows that

\[
P \left( \max_{r=1,\ldots,M_T-2} \sqrt{\log (M_T) M_{S,T}^{1/2}} (nTb_{W,T})^{1/2} \left| f_{L,r,T}(\omega) - f_{R,r+1,T}(\omega) \right| > \frac{\epsilon}{C} \right) \to 0.
\]

The second term of (S.19) also converges to zero because by Assumption 3.5 it follows that \((nTb_{W,T})^{1/2} \sigma_{L,r}(\omega)\) is bounded below by \(f_+ > 0\). For example, choose \( C = 3/f_- \). Altogether we yield that the right-hand side of (S.19) converges to zero. The argument for the second term of (S.18) is analogous. \( \square \)

**Lemma S.A.6.** Assumption 3.1, 3.2 with \( l = 0 \) and \( r = 2 \), 3.3-3.5 and Condition 1 hold. Under \( H_0 \) we have \( \sqrt{\log (M_T) M_{S,T}^{1/2}} (R_{max,T}(\omega) - \bar{R}_{max,T}(\omega)) \to 0 \) for any \( \omega \in [-\pi, \pi] \) where

\[
\bar{R}_{max,T}(\omega) \triangleq \max_{r=1,\ldots,M_T-2} \left| \frac{f_{r,T}(\omega)}{f_{r+1,T}(\omega)} - 1 \right|.
\]

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Proof of Lemma S.A.6. Using (S.16) we have

\[
\left| R_{\max,T}(\omega) - \tilde{R}_{\max,T}(\omega) \right| \leq \max_{r=1,\ldots,M_{T}-2} \left| \frac{\tilde{f}_{L,r,T}(\omega)}{f_{R,r+1,T}(\omega)} - 1 - \left( \frac{\tilde{f}_{r,T}(\omega)}{f_{R,r+1,T}(\omega)} - 1 \right) \right|
\]

\[
\leq \max_{r=1,\ldots,M_{T}-2} \left| \frac{\tilde{f}_{L,r,T}(\omega)}{f_{R,r+1,T}(\omega)} \left( \frac{1}{f_{R,r+1,T}(\omega)} - \frac{1}{\tilde{f}_{r,T}(\omega)} \right) \right|
\]

\[
+ \max_{r=1,\ldots,M_{T}-2} \left| \frac{\tilde{f}_{L,r,T}(\omega) - \tilde{f}_{r,T}(\omega)}{\tilde{f}_{r,T}(\omega)} \right|.
\]

Let us consider the second term on the right-hand side of (S.21). Note that for all \( \epsilon > 0 \) and all constants \( C > 0 \), we have

\[
P \left( \max_{r=1,\ldots,M_{T}-2} \sqrt{\log(M_T)M_{S,T}^{1/2}} \left| \tilde{f}_{L,r,T}(\omega) - \tilde{f}_{r,T}(\omega) \right| > \epsilon \right)
\]

\[
\leq P \left( \max_{r=1,\ldots,M_{T}-2} \sqrt{\log(M_T)M_{S,T}^{1/2}} \left| \tilde{f}_{L,r,T}(\omega) - \tilde{f}_{r,T}(\omega) \right|, \max_{r=1,\ldots,M_{T}-2} \left| \frac{1}{\tilde{f}_{r,T}(\omega)} \right| > \epsilon \right)
\]

\[
\leq P \left( \max_{r=1,\ldots,M_{T}-2} \sqrt{\log(M_T)M_{S,T}^{1/2}} \left| \tilde{f}_{L,r,T}(\omega) - \tilde{f}_{r,T}(\omega) \right| > \frac{\epsilon}{C} \right)
\]

\[
\leq P \left( \max_{r=1,\ldots,M_{T}-2} \left| \frac{1}{\tilde{f}_{r,T}(\omega)} \right| > C \right). \tag{S.22}
\]

By using the same argument as in Lemma S.A.5,

\[
\max_{r=1,\ldots,M_{T}-2} \sqrt{\log(M_T)M_{S,T}^{1/2}} \left| \tilde{f}_{L,r,T}(\omega) - \tilde{f}_{r,T}(\omega) \right| = o_P(1).
\]

By using Markov's inequality, this shows that

\[
P \left( \max_{r=1,\ldots,M_{T}-2} \sqrt{\log(M_T)M_{S,T}^{1/2}} \left| \tilde{f}_{L,r,T}(\omega) - \tilde{f}_{R,r+1,T}(\omega) \right| > \frac{\epsilon}{C} \right) \to 0. \tag{S.23}
\]

By Theorem 7.6, \( \tilde{f}_{r+1,T}(\omega) = f((r + 1) m_T, \omega) + o_P(1) \). Thus, the second term of (S.22) also converges to zero for example by choosing \( C = 3/f_{-} \). Altogether we yield that the right-hand side of (S.22) converges to zero. Next, we consider the first term of (S.21). For any \( \epsilon > 0 \) and any \( C > 0 \), we have

\[
P \left( \max_{r=1,\ldots,M_{T}-2} \sqrt{\log(M_T)M_{S,T}^{1/2}} \left| \tilde{f}_{L,r,T}(\omega) \left( \frac{1}{\tilde{f}_{R,r+1,T}(\omega)} - \frac{1}{\tilde{f}_{r+1,T}(\omega)} \right) \right| > \epsilon \right)
\]

\[
\leq P \left( \max_{r=1,\ldots,M_{T}-2} \sqrt{\log(M_T)M_{S,T}^{1/2}} \left| \tilde{f}_{L,r,T}(\omega) \left( \frac{1}{\tilde{f}_{r+1,T}(\omega)} \right) \right| > \frac{\epsilon}{C} \right)
\]

\[
+ P \left( \max_{r=1,\ldots,M_{T}-2} \left| \frac{1}{\tilde{f}_{r+1,T}(\omega)} \right| > C \right). \tag{S.24}
\]

The first term on the right-hand side above is less than or equal to,

\[
P \left( \max_{r=1,\ldots,M_{T}-2} \left| \tilde{f}_{L,r,T}(\omega) \right| > C_2 \right)
\]

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\[ + \mathbb{P} \left( \max_{r=1, \ldots, M_T-2} \sqrt{\log (M_T) M_{S,T}^{1/2}} \left| \tilde{f}_{r+1,T} (\omega) - \tilde{f}_{R,r+1,T} (\omega) \right| > \frac{\epsilon}{C \cdot C_2} \right) \]

for all \( C_2 > 0 \). We can choose \( C_2 \) large enough such that the first term above converges to zero. The second term above converges to zero by the same argument as in (S.23). The second term on the right-hand side of (S.24) can be expanded as follows,

\[
\mathbb{P} \left( \max_{r=1, \ldots, M_T-2} \left| \frac{1}{\tilde{f}_{r+1,T} (\omega)} \tilde{f}_{R,r+1,T} (\omega) \right| > C \right) \\
\leq \mathbb{P} \left( \min_{r=1, \ldots, M_T-2} \left| \tilde{f}_{R,r+1,T} (\omega) \right| < C^{-1/2} \right) + \mathbb{P} \left( \min_{r=1, \ldots, M_T-2} \left| \tilde{f}_{r+1,T} (\omega) \right| < C^{-1/2} \right) \\
\leq \mathbb{P} \left( \min_{r=1, \ldots, M_T-2} \left| \tilde{f}_{R,r+1,T} (\omega) \right| < C^{-1/2} \right) + \mathbb{P} \left( \min_{r=1, \ldots, M_T-2} \left| \tilde{f}_{r+1,T} (\omega) \right| < 2C^{-1/2} \right) \\
+ \mathbb{P} \left( \max_{r=1, \ldots, M_T-2} \left| \tilde{f}_{R,r+1,T} (\omega) - \tilde{f}_{r+1,T} (\omega) \right| > 2C^{-1/2} \right).
\]

The first two terms on the right-hand side have been already discussed above. The third term has also been discussed above even with the factor \( \sqrt{\log (M_T) M_{S,T}^{1/2}} \) in front. \( \square \)

**S.A.3.2 Proof of Theorem 3.1**

From Lemma S.A.5 it is sufficient to show the result for \( \tilde{S}_{\text{max},T} (\omega) \) since the latter is asymptotically equivalent to \( S_{\text{max},T} (\omega) \). Define \( f_{h,T}^* (j/T, \omega) = f_{h,T} (j/T, \omega) - \mathbb{E} (f_{h,T} (j/T, \omega)) \). For \( \omega \in [-\pi, \pi] \) let \( S_{r+1} (\omega) = \sum_{j \in (S_s, s=1, \ldots, r+1)} f_{h,T}^* (j/T, \omega) \) and

\[
R_{r,T} (\omega) = \frac{1}{M_{S,T}} \left( S_{r+1} (\omega) - \sum_{j \in (S_s, s=1, \ldots, r+1)} \mathcal{W}_j (\omega) - \left( S_r (\omega) - \sum_{j \in (S_s, s=1, \ldots, r)} \mathcal{W}_j (\omega) \right) \right),
\]

where \( \mathcal{W}_j (\omega) = \sigma_j (\omega) Z_j \) with \( Z_j \sim \text{i.i.d. } \mathcal{N} (0, 1) \). Write

\[
\tilde{f}_{r,T} (\omega) = M_{S,T}^{1/2} \sum_{j \in S_r} f_{h,T} (j/T, \omega) = M_{S,T}^{1/2} \sum_{j \in S_r} \left( f_{h,T}^* (j/T, \omega) + \mathbb{E} (f_{h,T} (j/T, \omega)) \right) = \frac{1}{M_{S,T}} \sum_{j \in S_r} \mathcal{W}_j (\omega) + R_{r,T} + \frac{1}{M_{S,T}} \sum_{j \in S_r} \mathbb{E} (f_{h,T} (j/T, \omega)).
\]

Under Assumption 3.4-(i), Theorem 1 in Wu and Zhou (2011) yields \( \max_{0 \leq r \leq M_{S,T}-1} |R_{r,T}| = O_p (\tau_T / M_{S,T}) \). The same bound holds under Assumption 3.4-(ii) by Corollary 1 in Wu and Zhou (2011). By Theorem 7.5,

\[
\mathbb{E} (f_{h,T} (j/T, \omega)) = f (j/T, \omega) + O \left( (n_T / T)^2 \right) + O \left( b_{W,T}^2 \right) + O (\log (n_T) / n_T).
\]

Using (S.20) we yield

\[
\sqrt{M_{S,T}} \left( \tilde{f}_{r+1,T} (\omega) - \tilde{f}_{r,T} (\omega) \right)
\]
\[
= \frac{1}{\sqrt{M_{S,T}}} \left( \sum_{j \in S_{r+1}} \mathcal{W}_j (\omega) - \sum_{j \in S_r} \mathcal{W}_j (\omega) \right) \\
+ O \left( M_{S,T}^{1/2} n_T^{3/2} \right) + O_P \left( \tau_T / M_{S,T}^{1/2} \right) + O_P \left( M_{S,T}^{1/2} n_T^2 + M_{S,T}^{1/2} V_{W,T} + M_{S,T}^{1/2} \log (n_T) / n_T \right) \\
= \frac{1}{\sqrt{m_T}} \left( \sum_{j \in S_{r+1}} \mathcal{W}_j (\omega) - \sum_{j \in S_r} \mathcal{W}_j (\omega) \right) \\
+ o_P \left( (\log M_T)^{-1/2} \right).
\]

The result then follows from Lemma 1 in Wu and Zhao (2007). \(\square\)

S.A.3.3 Proof of Theorem 3.2

Lemma S.A.7. Let \(Y(\omega)\) denote a random variable defined by \(\mathbb{P}(Y(\omega) \leq v) = \exp(-\pi^{-1/2} \exp(-v))\) for \(\omega \in \Pi\). Assume that for \(\omega, \omega' \in \Pi\) the variables \(Y(\omega)\) and \(Y(\omega')\) are independent. Let \(Y^* := \max_{\omega \in \Pi} Y(\omega) = \log(n_\omega)\). Then, \(\mathbb{P}(Y^* \leq v) = \exp(-\pi^{-1/2} \exp(-v))\).

Proof. Since \(Y(\omega)\) is independent from any \(Y(\omega')\) with \(\omega \neq \omega'\), we have

\[
\log \mathbb{P}(Y^* \leq v) = \sum_{j=1}^{n_\omega} \log \mathbb{P}(Y_j(\omega) \leq (\log(n_\omega) + v)) \\
= \sum_{j=1}^{n_\omega} (-\pi^{-1/2} \exp \left( \log \left( n_\omega^{-1} \right) \right) \exp(-v)) \\
= -\pi^{-1/2} \exp(-v).
\]

Thus, \(\mathbb{P}(Y^* \leq v) = \exp(-\pi^{-1/2} \exp(-v))\). \(\square\)

Proof of Theorem 3.2. From Theorem 7.6 it follows that \(f_{h,T}(u, \omega_j)\) and \(f_{h,T}(u, \omega_k)\) are asymptotically independent if \(\omega_k = \omega_k \neq 0 \bmod(2\pi), 1 \leq j < k \leq n_\omega\). The result then follows from Lemma S.A.5 and S.A.7, and Theorem 3.1. \(\square\)

S.A.3.4 Proof of Theorem 3.3

Due to the self-normalization nature of the test statistic, we can use Lemma S.A.6 and similar steps to Proposition A1-A.3 in Bibinger, Jirak, and Vetter (2017) to show that it is sufficient to consider the behavior of

\[
\widehat{R}^*(\omega) = \max_{r=1, \ldots, M_T-2} \left| M_{S,T}^{-1} \sum_{j \in S_r} \mathcal{G}_r(j/T, \omega) - M_{S,T}^{-1} \sum_{j \in S_{r+1}} \mathcal{G}_r(j/T, \omega) \right|,
\]

where \(\mathcal{G}_r(j/T, \omega)\) are random variables with mean \(\mathbb{E}(f_{h,T}(j/T, \omega))\), unit variance and satisfy Assumption 3.4. For \(\omega \in [-\pi, \pi]\) let \(S_{r+1}(\omega) = \sum_{j \in S_\omega(s=1, \ldots, r+1)} \left( \mathcal{G}_r(j/T, \omega) - \mathbb{E}(f_{h,T}(j/T, \omega)) \right)\) and

\[
R_{r,T}(\omega) = \frac{1}{M_{S,T}} \left( S_{r+1}(\omega) - \sum_{j \in S_\omega(s=1, \ldots, r+1)} \mathcal{W}_j (\omega) - \left( S_r (\omega) - \sum_{j \in S_\omega(s=1, \ldots, r)} \mathcal{W}_j (\omega) \right) \right),
\]

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where \( W_j(\omega) = Z_j \) with \( Z_j \sim \text{i.i.d.} \mathcal{N}(0, 1) \). Write

\[
M_{S,T}^{-1} \sum_{j \in S_r} g_T(j/T, \omega) = M_{S,T}^{-1} \sum_{j \in S_r} ((g_T(j/T, \omega) - \mathbb{E}(g_T(j/T, \omega))) + \mathbb{E}(g_T(j/T, \omega)))
\]

(S.27)

\[
= \frac{1}{M_{S,T}} \sum_{j \in S_r} W_j(\omega) + R_{r,T} + \frac{1}{M_{S,T}} \sum_{j \in S_r} \mathbb{E}(g_T(j/T, \omega)).
\]

As in the proof of Theorem 3.1 we have \( \max_{0 \leq r \leq M_{S,T} - 1} |R_{r,T}| = O_\mathbb{P}(\tau_T/M_{S,T}) \). By Theorem 7.5, \( \mathbb{E}(g_T(j/T, \omega)) = f(j/T, \omega) + O((n_T/T)^2) + O(b^2_{W,T}) + O(\log(n_T)/n_T) \). Using (S.20) we yield

\[
\sqrt{M_{S,T}} \left( M_{S,T}^{-1} \sum_{j \in S_{r+1}} g_T(j/T, \omega) - M_{S,T}^{-1} \sum_{j \in S_r} g_T(j/T, \omega) \right)
\]

(S.28)

\[
= \frac{1}{\sqrt{M_{S,T}}} \left( \sum_{j \in S_{r+1}} W_j(\omega) - \sum_{j \in S_r} W_j(\omega) \right)
\]

\[
+ O\left( M_{S,T}^{1/2} b_{T}^{T}/T^{\theta} \right) + O_\mathbb{P}\left( \tau_T/M_{S,T}^{1/2} \right) + O_\mathbb{P}\left( M_{S,T}^{1/2} (n_T/T)^2 + M_{S,T}^{1/2} b^2_{W,T} + M_{S,T}^{1/2} \log(n_T)/n_T \right)
\]

\[
= \frac{1}{\sqrt{m_T}} \left( \sum_{j \in S_{r+1}} W_j(\omega) - \sum_{j \in S_r} W_j(\omega) \right)
\]

\[
+ o_\mathbb{P}\left( \left( \log M_T \right)^{-1/2} \right).
\]

The result about \( R_{\max,T}(\omega) \) follows from Lemma 1 in Wu and Zhao (2007). The result concerning \( R_{D_{\max,T}} \) follows by using the same argument as in the proof of Theorem 3.2. \( \square \)

### S.A.4 Proofs of the Results in Section 4

For a sequence of random variables \( \{\xi_j\} \), let \( \mathbb{P}(\xi_j) \) denote the law of the observations \( \{\xi_j\} \). Let \( \|\mathbb{P}(\xi_j) - \mathbb{P}(\xi_j')\|_{TV} \) define the total variation distance between the probability measures \( \mathbb{P}(\xi_j) \) and \( \mathbb{P}(\xi_j') \). For two random variables \( Y \) and \( X \) with distributions \( \mathbb{P}_Y \) and \( \mathbb{P}_X \), respectively, denote the Kullback-Leibler divergence by \( D_{KL}(Y||X) = D_{KL}(\mathbb{P}_Y||\mathbb{P}_X) = \int \log(\mathbb{P}_Y/d\mathbb{P}_X) \ d\mathbb{P}_Y \).

### S.A.4.1 Proof of Theorem 4.1

The proof is based on several steps of information-theoretic reductions that allow us to show the asymptotic equivalence in the strong Le Cam sense of our statistical problem to a special high-dimensional signal detection problem. The minimax lower bound is then obtained by using classical arguments as in Ingster and Suslina (2003). Information-theoretic reductions were also used by Bibinger, Jirak, and Vetter (2017) to establish a minimax lower bound for change-point testing in volatility in the context of high-frequency data. Our derivations differ from theirs in several ways because we deal with serially correlated observations while they had independent observations. Furthermore, our testing problem is more complex because our observations have an unknown distribution while their observations are squared of standard normal variables.
We first consider alternatives as in $H_{i1}^B$. Throughout the proof we set
\[
m_T = C_T \left( \sqrt{\log (M_T)^\theta / D} \right)^{\frac{1}{\theta m_{1/2}}},
\]
with a constant $C_T > 0$. We begin by granting the experimenter additional knowledge thereby focusing on a simpler sub-model. This additional knowledge can only decrease the lower bound on minimax distinguishability and therefore such lower bound carries over to the original model. We restrict attention to a sub-class of $F_{1,\lambda_0^0,\omega_0}(\theta, b_T, D)$ which is characterized by a break at time $\lambda_0^0 \in (0, 1)$ with $|f(\lambda_0^0, \omega_0) - f(\lambda_0^0+, \omega_0)| \geq b_T$, where $f(\lambda_0^0+, \omega) = \lim_{s \uparrow \lambda_0^0} f(s, \omega)$. We further assume that the break point is an integer multiple of $m_T$, i.e., $T\lambda_0^0m_T^{-1} \in \{1, 2, \ldots, \lfloor T/m_T \rfloor - 1\}$.

In order to simplify the proof, we consider a simplified version of the problem following Bibinger, Jirak, and Vetter (2017). We set $f_-(\omega_0) = 1$ and let
\[
f(j/T, \omega_0) = \begin{cases} 1 + (m_T - j \mod m_T)^\theta T^{-\theta}, & T\lambda_0^0 < j \leq T\lambda_0^0 + m_T, \\ 1, & \text{else} \end{cases}
\]
We discuss the general case $f_-(\omega_0) \neq 1$ at the end of this proof. Eq. (S.30) specifies that the spectrum at frequency $\omega_0$ exhibits a break of order $b_T$ at $\lambda_0^0$ and then decays on the interval $(\lambda_0^0, \lambda_0^0 + T^{-1}m_T]$ smoothly with regularity $\theta$ and is constant elsewhere. Name this sub-class $F_{\lambda_0^0,\omega_0}^+$. Note that here the location of $\lambda_0^0$ is still unknown. To establish the lower bound, it suffices to focus on the sub-class of the above form.

Next, we introduce a stepwise approximation to $f(j/T, \omega_0)$. Define, for a given sequence $a_T$ with $a_T \rightarrow \infty$ and $a_Tm_T^{-1} = o(1/\log (M_T))$,
\[
\tilde{f}(j/T, \omega_0) = \begin{cases} 1 + (m_T - la_T)^\theta T^{-\theta}, & T\lambda_0^0 + (l - 1)a_T < j \leq T\lambda_0^0 + la_T, \\ 1, & \text{else} \end{cases}, \quad 1 \leq l \leq m_T/a_T.
\]
We are given the observations $I_{L,h,T}(j/T, \omega)$ for $j = n_T + 1, \ldots, T$ and $\omega \in [-\pi, \pi]$. Assume without loss of generality that $\omega_0 \neq \pm \pi, \pm 3\pi, \ldots$. By Theorem 7.4(ii), $I_{L,h,T}(j/T, \omega_0)$ is approximately $f(j/T, \omega_0) \chi_{2/2}^2$ for $j/T \neq \lambda_0^0$. For $j/T = \lambda_0^0$, $I_{L,h,T}(j/T, \omega_0)$ is approximately $f(j/T, \omega_0) \chi_{2/2}^2$ which also follows from Theorem 7.4(ii) since Assumption 4.1 is continuous from the left at $\lambda_0^0$. However, note that $I_{L,h,T}(j/T, \omega_0)$ is not asymptotically independent of $I_{L,h,T}(l/T, \omega_0)$ for $l = j - n_T + 1, \ldots, j$. Let $S_j = \{n_T + 1, n_T + 1 + m_{S,T}, \ldots\}$. Let $\zeta_j = f(j/T, \omega_0) \chi_{2/2}^2$ and $\zeta_j^* = f(j/T, \omega_0) \chi_{2/2}^2$ where $\zeta_j^*$ are independent across $j$. Define $\zeta_j^* = \tilde{f}(j/T, \omega_0) \chi_{2/2}^2$ where $\zeta_j^*$ are independent across $j$.

We distinguish between two cases: (i) $\theta > 1/2$ and (ii) $\theta \leq 1/2$.

(i) Case $\theta > 1/2$. Let us consider the following distinct experiments:

$\mathcal{E}_1$ : Observe $\{\zeta_j\}_{j=n_T+1}^T$ and information $T\lambda_0^0m_T^{-1} \in \{1, 2, \ldots, \lfloor T/m_T \rfloor - 1\}$ is provided.

$\mathcal{E}_2$ : Observe $\{\zeta_j^*\}_{j=n_T+1}^T$ and information $T\lambda_0^0m_T^{-1} \in \{1, 2, \ldots, \lfloor T/m_T \rfloor - 1\}$ is provided.

$\mathcal{E}_3$ : Observe $\{\zeta_j\}_{j=n_T+1}^T$ and information $T\lambda_0^0m_T^{-1} \in \{1, 2, \ldots, \lfloor T/m_T \rfloor - 1\}$ is provided.

$\mathcal{E}_4$ : Observe $\chi = ((f jm_T/T, \omega_0) \chi_{2m_T+j}^2)_{j \in I_1}$, $(f(\lambda_0^0 + (j - 1)a_T + 1)/T, \omega_0) \chi_{2m_T+j}^2)_{j \in I_2}$, where $I_1 = \{1, \ldots, \lambda_0^0 Tm_T^{-1}, \lambda_0^0 Tm_T^{-1} + 2, \ldots, \lfloor T/m_T \rfloor\}$, $I_2 = \{1, 2, \ldots, m_Ta_T^{-1}\}$, and $\chi_{2m_T+j}^2)_{j \in I_1}$, and $\chi_{2m_T+j}^2)_{j \in I_2}$ are i.i.d. sequences of chi-square random variables with $2m_T$ and $2a_T$ degrees of freedom, respectively. Further, information $T\lambda_0^0m_T^{-1} \in \{1, 2, \ldots, \lfloor T/m_T \rfloor - 1\}$ is provided.

$\mathcal{E}_5$ : Observe $\xi = (\{m_{T}^{1/2} \zeta_j f jm_T/T, \omega_0) + \tilde{f}(jm_T/T, \omega_0)\}_{j \in I_1}$, $(a_T^{1/2} \zeta_j f(\lambda_0^0 + (j - 1)a_T + 1)/T, \omega_0) + \tilde{f}(jm_T/T, \omega_0)\}_{j \in I_1}$.

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\( f(\lambda_j^0 + ((j - 1)a_T + 1)/T, \omega_0))_{j \in \mathcal{I}_2} \), where \( \{\zeta_j\}_{j \in \mathcal{I}_1} \) and \( \{\tilde{\zeta}_j\}_{j \in \mathcal{I}_2} \) are i.i.d. standard normal random variables. Further, information \( T \lambda_j^0 m_T^{-1} \in \{1, 2, \ldots, \lfloor T/m_T \rfloor - 1\} \) is provided.

We assume that \( \{\zeta_j\} \) and \( \{\tilde{\zeta}_j\} \) are realized on the same probability space which is rich enough to allow for both sequences to be realized there. This is richer then the probability space in which \( \{\zeta_j\} \) is realized. Thus, the latter probability space is extended in the usual way using product spaces. The symbol \( \approx \) denotes asymptotic equivalence while \( \sim \) denotes strong Le Cam equivalence. Our proof consists of showing the following strong Le Cam equivalence of statistical experiments:

\[
\mathcal{E}_1 \approx \mathcal{E}_2 \approx \mathcal{E}_3 \sim \mathcal{E}_4 \approx \mathcal{E}_5. \tag{S.31}
\]

Therefore, given the relation (S.31), the lower bound for \( \mathcal{E}_5 \) carries over to the less informative experiment \( \mathcal{E}_1 \). We prove (S.31) in steps.

Step 1: \( \mathcal{E}_1 \approx \mathcal{E}_2 \). Given \( \zeta_j = f(j/T, \omega_0) \chi_j^2/2 \) and the boundness of \( f(\cdot, \cdot) \), Theorem 1 in Berkes and Philipp (1979) implies that there exists a sequence \( \{\tilde{\zeta}_j\}_{j \in \mathcal{J}_j} \) of independent random variables such that \( \tilde{\zeta}_j \) has the same distribution as \( \zeta_j \) and \( \mathbb{P}(|\zeta_j - \tilde{\zeta}_j| \geq \nu_j) \leq \nu_j \) with \( \nu_j > 0 \). In view of Assumption 3.4 we have \( \sum_{j=1}^{\infty} \nu_j < \infty \) which in turn yields,

\[
\sum_{j=1}^{\infty} |\zeta_j - \tilde{\zeta}_j| < \infty \quad \mathbb{P} - \text{almost surely.} \tag{S.32}
\]

Note that

\[
|S_j|^{-1} \sum_{j \in S_j} |\zeta_j - \tilde{\zeta}_j| = |S_j|^{-1} \sum_{j=mT+1}^{J_1} |\zeta_j - \tilde{\zeta}_j| + |S_j|^{-1} \sum_{j \in S_j, j > J_1} |\zeta_j - \tilde{\zeta}_j|.
\]

Choose \( J_1 \) large enough such that \( \sum_{j \in S_j, j > J_1} |\zeta_j - \tilde{\zeta}_j| \to 0 \mathbb{P}\)-almost surely. This implies that \( \|\mathbb{P}_i - \mathbb{P}_j\|_{TV} \to 0 \). The latter shows that \( \mathcal{E}_1 \approx \mathcal{E}_2 \).

Step 2: \( \mathcal{E}_2 \approx \mathcal{E}_3 \). Note that \( c \chi_j^2 \) with \( c > 0 \) is approximately distributed as \( \Gamma(1, 2c) \) where \( \Gamma(a, b) \) is the Gamma distribution with parameters \( (a, b) \). The Kullback-Leibler divergence of \( \Gamma(1, 2c) \) from \( \Gamma(1, 2\tilde{c}) \) is given by

\[
D_{KL}(\mathbb{P}_c \| \mathbb{P}_{\tilde{c}}) = (\log c - \log \tilde{c}) + \frac{\tilde{c} - c}{c}.
\]

For \( c = \tilde{c} + \delta \) with \( \delta \to 0 \), we obtain

\[
D_{KL}(\mathbb{P}_c \| \mathbb{P}_{\tilde{c}}) = \log \left( \frac{\tilde{c} + \delta}{\tilde{c}} \right) + \frac{\tilde{c} - (\tilde{c} + \delta)}{\tilde{c} + \delta} = -\frac{\delta^2}{2\tilde{c}^2} + O\left(\delta^2\right) + O\left(\delta^3\right). \tag{S.33}
\]

By Pinsker’s inequality,

\[
\left\| \mathbb{P}_i(\{\zeta_i\}) - \mathbb{P}_j(\{\tilde{\zeta}_i\}) \right\|_{TV}^2 \leq \frac{1}{2} D_{KL}(\mathbb{P}_i(\{\zeta_i\}) \| \mathbb{P}_j(\{\tilde{\zeta}_i\})).
\]

Thus, using (S.33) and the additivity of Kullback-Leibler divergence for independent distributions, we
have
\[
D_{KL}\left(\mathbb{P}_{\zeta^*} \parallel \mathbb{P}_{\zeta_j}\right) = C \sum_{s=1}^{m_T} \sum_{j=1}^{a_T} \left(jT^{-1}\right)^{2\theta} = CO\left(a_T T^{-1}\right)^{2\theta} m_T.
\]

This tends to zero in view of (S.29) and \(m_T^{-1}a_T \to 0\).

Step 3: \(E_3 \sim E_4\). The vector of averages
\[
\left(\sum_{s=1}^{m_T} \zeta_{jm_T+s-1}^* \right)_{j \in I_1}, \left(\sum_{s=1}^{a_T} \zeta_{T\lambda_b^0+(j-1)a_T+s-1} \right)_{j \in I_2},
\]
forms a sufficient statistic for \(\{\tilde{f}(j/T, \omega_0)\}_{(j/T) \in [0,1]}\). Hence, by Lemma 3.2 of Brown and Low (1996) this yields the strong Le Cam equivalence.

Step 4: \(E_4 \approx E_5\). Let
\[
\chi^* = (m_T^{-1/2}(\tilde{f}(jm_T/T, \omega_0)(\chi^2_{2m_T,j} - 2m_T))_{j \in I_1},
\]
\[
a_T^{-1/2}(\tilde{f}(\lambda_b^0 + (j-1)a_T + 1)/T, \omega_0)(\chi^2_{2a_T,j} - 2a_T))_{j \in I_2},
\]
\[
\xi^* = ((\xi_j \tilde{f}(jm_T/T, \omega_0))_{j \in I_1}, (\tilde{\xi}_j \tilde{f}(\lambda_b^0 + (j-1)a_T + 1)/T, \omega_0))_{j \in I_2}.
\]

Note that \(\|\mathbb{P}_\chi - \mathbb{P}_{\xi^*}\|_{TV}^2 = \|\mathbb{P}_\chi - \mathbb{P}_{\xi^*}\|_{TV}^2\). By Pinsker’s inequality and independence,
\[
\|\mathbb{P}_\chi^* - \mathbb{P}_{\xi^*}\|_{TV}^2 \leq 2^{-1} D_{KL}\left(\mathbb{P}_\chi^* \parallel \mathbb{P}_{\xi^*}\right) \\
\leq 2^{-1} \sum_{j \in I_1} D_{KL}\left(m_T^{-1/2}(\tilde{f}(jm_T/T, \omega_0)(\chi^2_{2m_T,j} - 2m_T)) \parallel \xi_j \tilde{f}(jm_T/T, \omega_0)\right) \\
+ 2^{-1} \sum_{j \in I_2} D_{KL}(a_T^{-1/2}(\tilde{f}(\lambda_b^0 + (j-1)a_T + 1)/T, \omega_0)(\chi^2_{2a_T,j} - 2a_T)) \\
\|\xi_j \tilde{f}(\lambda_b^0 + (j-1)a_T + 1)/T, \omega_0)\).
\]

We now apply Theorem 1.1 in Bobkov, Chistyakov, and Götze (2013) with \(c_1 = 12^{-1}\kappa_3^2\) in (1.3) there, where \(\kappa_3\) is the third-order cumulant of the variable in question. This gives the following bounds,
\[
D_{KL}\left((m_T)^{-1/2}(\tilde{f}(jm_T/T, \omega_0)(\chi^2_{2m_T,j} - 2m_T)) \parallel \xi_j \tilde{f}(jm_T/T, \omega_0)\right) = \frac{1}{12} \left(\frac{8}{2m_T}\right) + o\left(\frac{1}{m_T \log m_T}\right),
\]
and
\[
D_{KL}(a_T^{-1/2}(\tilde{f}(\lambda_b^0 + (j-1)a_T + 1)/T, \omega_0)(\chi^2_{2a_T,j} - 2a_T)) \\
\|\xi_j \tilde{f}(\lambda_b^0 + (j-1)a_T + 1)/T, \omega_0)\) = \frac{1}{12} \left(\frac{8}{2a_T}\right) + o\left(\frac{1}{a_T \log a_T}\right).
\]

Hence, \(\|\mathbb{P}_\chi - \mathbb{P}_{\xi^*}\|_{TV}^2 = O(Tm_T^{-2}) + O(m_T a_T^{-2})\). Since \(\theta > 1/2\) we have \(Tm_T^{-2} \to 0\). Finally, since \(m_T^{-1}a_T \to 0\) we can choose \(a_T\) sufficiently fast such that \(m_T a_T^{-2} \to 0\). Thus, we have \(\|\mathbb{P}_\chi - \mathbb{P}_{\xi^*}\|_{TV} \to 0\).

By step 1-4, it is sufficient to establish the minimax lower bound for experiment \(E_5\). After adding an
additional drift $\xi$, which gives an equivalent problem, we cast the problem as a high dimensional location signal detection problem [cf. Ingster and Suslina (2003)] from which the bound can be derived using classical arguments. Consider the observations

$$
\xi^* = ((m_T^{-1/2} \xi_j \tilde{f}(jm_T/T, \omega_0) + \tilde{f}(jm_T/T, \omega_0) - 1)_{j \in \mathcal{I}_1},
$$

and the hypothesis

$$
\mathcal{H}_0: \sup_j \left( \tilde{f}(j/T, \omega_0) - 1 \right) = 0 \quad \text{versus} \quad \mathcal{H}_1: \sup_j \left( \tilde{f}(j/T, \omega_0) - 1 \right) \geq b_T. \quad (S.34)
$$

The goal is to find the maximal value $b_T \to 0$ such that the hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$ are non-distinguishable in the minimax sense or $\lim_{T \to \infty} \inf_{\psi} \gamma_{\psi}(\theta, b_T) = 1$. Here the detection rate is $b_T \propto (T^{-1} m_T)^{\theta} \propto T^{-\frac{\theta}{2d+1}}$.

Consider the product measures $\mathbb{P}_{\mathcal{H}_0} = \mathbb{P}_{\xi^*} \times \mathbb{P}_{\omega_0}$ and $\mathbb{P}_{\mathcal{H}_1} = \mathbb{P}_{\xi^*} \times \mathbb{P}_{\lambda_0^T}$ where $\mathbb{P}_{\xi^*}$ is the probability law of $\xi^*$ and $\mathbb{P}_{\omega_0}$ is the measure for the no break case. Thus, $\mathbb{P}_{\mathcal{H}_0}$ is the probability measure under $\mathcal{H}_0$ while $\mathbb{P}_{\mathcal{H}_1}$ is the probability measure under $\mathcal{H}_1$ which draws a break at time $\lambda_0^T$ with $T \lambda_0^T m_T^{-1} \in \{1, 2, \ldots, \lfloor T/m_T \rfloor - 1\}$ uniformly from this set. From similar derivations that yield eq. (2.20)-(2.22) in Ingster and Suslina (2003), it follows that

$$
\inf_{\psi} \gamma_{\psi}(\theta, b_T) \geq 1 - \frac{1}{2} \left\| \mathbb{P}_{\mathcal{H}_1} - \mathbb{P}_{\mathcal{H}_0} \right\|_{TV} \geq 1 - \frac{1}{2} \left\| \mathbb{E}_{\mathbb{P}_{\mathcal{H}_0}} \left( \mathcal{L}_{0,1}^2 - 1 \right) \right\|^{1/2},
$$

where $\mathcal{L}_{0,1} = \frac{d\mathbb{P}_{\mathcal{H}_1}}{d\mathbb{P}_{\mathcal{H}_0}}$ is the likelihood ratio between $\mathbb{P}_{\mathcal{H}_1}$ and $\mathbb{P}_{\mathcal{H}_0}$.

It remains to consider the case $\theta \leq 1/2$. In a different setting, Bibinger, Jirak, and Vetter (2017) considered separately the case where their regularity exponent $\alpha$ satisfies $\alpha \leq 1/2$ to obtain the minimax lower bound. The same arguments can be applied in our context which lead to the same result as for the case $\theta > 1/2$.

The general case with $f_-(\omega_0) > 0$ rather than with $f_-(\omega_0) = 1$ as discussed above follows from the same arguments after we rescale the equations in (S.34). The only difference is the form of the detection rate which is now $b_T \leq f_-(\omega_0) D (T^{-1} m_T)^{\theta}$.

The proof for the lower bound for the alternative $\mathcal{H}_1^S$ is similar to the proof discussed above. The minor differences in the proof outlined by Bibinger, Jirak, and Vetter (2017) also apply here. □

**S.A.4.2 Proof of Theorem 4.2**

We present the proof for the statistic $\mathcal{S}_{\text{max,T}}$. The proof for the other test statistics discussed in Section 3 is similar and omitted. Without loss of generality we assume that $\omega_0 \neq \pm \pi$. Let $M_{\mathcal{S},T} = m_T/m_{\mathcal{S},T}^T$ and $m_{\mathcal{S},T}/m_T^T \to [0, \infty)$. If $\lfloor T \lambda_0^T \rfloor \notin \{\{S_r\} \cup \{S_{r+1}\}\}$ or if $\omega \neq \omega_0$ then

$$
\left| \frac{\tilde{f}_{L,r,T}(\omega) - \tilde{f}_{R,r+1,T}(\omega)}{\sigma_{L,r}(\omega)} \right| = \left( M_{\mathcal{S},T} \right)^{-1} \sum_{j \in S_r} \left( f_{L,h,T}(j/T, \omega) + \mathbb{E}(f_{L,h,T}(j/T, \omega)) \right) \frac{1}{\sigma_{L,r}(\omega)}
$$
where the last inequality follows from (4.2). As in the proof of Theorem 3.1, we have \( \sqrt{M_{r,T}} f_{r,T}(\omega) = O_P(1) \) for \( 1 \leq r \leq M_T^* - 2 \). This can be used to obtain the following inequality, if \( [T \lambda_0^r] \in \{\{S_r\} \cup \{S_{r+1}\}\} \) and \( \omega = \omega_0 \),

\[
S_{\text{max},T}(\omega_0) \geq -\hat{f}_{r,T}(\omega_0) + \frac{T}{m_T} \left| \int_{\lambda_0^0}^{\lambda_0^1} f(u, \omega_0) du - \int_{\lambda_0^0}^{(r+1)m_T^* + n_T/2 + m_{S,T}M_{S,T}/2)/T} f(u, \omega_0) du \right| \\
\times \left(1 - o_P(1)\right) \sup_u \hat{f}(u, \omega_0)
\]

\[
\geq -O_P\left((m_T^*)^{-1/2}\right)
\]

\[
+ \frac{T}{m_T} \left| \int_{(rm_T^* - m_T^*/2 + n_T/2 + 1)/T}^{(r+1)m_T^* + n_T/2 + m_{S,T}M_{S,T}/2)/T} f(u, \omega_0) du - \int_{\lambda_0^0} f(u, \omega_0) du \right| \\
\times \left(1 - o_P(1)\right) \sup_u \hat{f}(u, \omega_0).
\]

Note that \( \gamma_{\phi^*}(\theta, b_T^r) \to 0 \) follows from

\[
P\left(S_{\text{max},T}(\omega) < 2D^* \sqrt{\log(M_T^*)/m_T^*}\right) \to 1, \quad \text{for all } \omega \in [-\pi, \pi], \quad \text{under } \mathcal{H}_0 \quad (S.36)
\]

\[
P\left(S_{\text{max},T}(\omega) \geq 2D^* \sqrt{\log(M_T^*)/m_T^*}\right) \to 1, \quad \text{for some } \omega \in [-\pi, \pi], \quad \text{under } \mathcal{H}_1^B \text{ or } \mathcal{H}_1^S. \quad (S.37)
\]

We first show (S.36). Note that

\[
2D^* \sqrt{\log(M_T^*)/m_T^*} \geq 2\sqrt{\log(M_T^*)/m_T^*} + D(m_T^*/T)^\theta.
\]

Under \( \mathcal{H}_0 \), since \( \theta' < \theta \) we have for all \( \omega \in [-\pi, \pi] \),

\[
S_{\text{max},T}(\omega) \leq \max_{1 \leq r \leq M_T^* - 2} \hat{f}_{r,T}(\omega) + D(m_T^*/T)^{\theta'} + O_P\left(n_T^2 + \log(n_T)/n_T + o\left(b_{W,T}^2\right)\right).
\]
Given (4.2) to conclude the proof we have to show

\[ P \left( \max_{1 \leq r \leq M_{S,T}^2} \hat{f}_{r,T}(\omega_0) \leq \sqrt{\log (M_{S,T}^2)/m_{S,T}^2} \right) \to 1. \]

The latter result follows from \( \sqrt{\log (M_{S,T}^2)/m_{S,T}^2} \leq \sqrt{\log (M_{S,T}^2)/m_{S,T}^2} \) which is implied by Theorem 3.1.

We now prove (S.37) under \( H_B^S \). We have to show that the second term on the right hand side of (S.37) is greater than or equal to \( 2D^* \sqrt{\log (M_{S,T}^2)/m_{S,T}^2} \). The term in question is larger than \( b_{S,T}^r - 2D (m_{S,T}^2/T)^\theta \). In view of (4.3) with \( \theta' = 0 \) the result follows.

We now prove (S.37) under \( H_B^S \). For \( h \leq 2m_{S,T}^2/T \) we have \( f(\lambda_0^h + h, \omega_0) \geq f(\lambda_0^h, \omega_0) + b_{S,T}^r h^{\theta'} \) or \( f(\lambda_0^h + h, \omega_0) \leq f(\lambda_0^h, \omega_0) - b_{S,T}^r h^{\theta'} \). Thus,

\[
\frac{T}{m_{S,T}^2} \int_{\lambda_0^h + m_{S,T}^2/T}^{\lambda_0^h + 2m_{S,T}^2/T} \left( f(u, \omega_0) - f(u - m_{S,T}^2/T, \omega_0) \right) du \geq b_{S,T}^r (m_{S,T}^2/T)^{\theta'} \geq 2D^* \sqrt{\log (M_{S,T}^2)/m_{S,T}^2},
\]

where the second equality follows from (4.3). \( \square \)

**S.A.5 Proofs of the Results of Section 5**

**S.A.5.1 Proof of Proposition 5.1**

The following lemma is simple to verify.

**Lemma S.A.8.** Let \( C(u) \) and \( d(u) \) be functions on \([0, \lambda_0^h]\) such that \( d(u) \) is increasing. As long as

\[ d(\lambda_0^h) - d(\lambda_0^h - \kappa) \geq \sup_{0 \leq u \leq \lambda_0^h} |C(u)| \text{ for some } \kappa \in [0, \lambda_0^h], \]

we have that,

\[ \argmax_{0 \leq u \leq \lambda_0^h} (d(u) + C(u)) \geq \lambda_0^h - \kappa. \]   (S.38)

An analogous result holds if \( C(u) \) and \( d(u) \) are functions on \([\lambda_0^h, 1]\) and \( d(u) \) is decreasing.

**Proof of Proposition 5.1.** For \( \lambda_0^h \in (0, 1) \) define \( \tau_b = \left[ T \lambda_0^h + 1 \right] \), i.e., the smallest integer such that \( \tau_b/T \) is larger or equal than \( \lambda_0^h + 1/T \). Denote by \( \{ f(u, \omega_0) \}_{u \in [0, 1]} \) the path of the spectrum \( f(\cdot, \omega_0) \) without the break: \( f(\tau/T, \omega) = \hat{f}(\tau/T, \omega) + \delta_T \mathbf{1}_{\{ \tau \geq \tau_b \}} \). Without loss of generality, we assume \( \delta_T > 0 \). Define \( d(\tau/T, \omega) = 0 \) for \( \omega \neq \omega_0 \) and

\[
d(\tau/T, \omega_0) = \begin{cases} 0 & \text{if } r + m_T < \tau_b, \\ (r + m_T - \tau_b) m_{S,T}^{-1/2} M_{S,T}^{-1/2} & \text{if } r = \tau_b - m_T, \tau_b - m_T + m_{S,T}, \ldots, \tau_b, \\ M_{S,T}^{1/2} \delta_T & \text{if } r > \tau_b, \end{cases}
\]

and \( \{ d(u, \omega_0) \}_{u \in [0, 1]} \) is the associated piecewise constant increasing step function. By Lemma S.A.5 it is sufficient to consider

\[ D'_{r,T}(\omega) = M_{S,T}^{-1/2} \left| \sum_{j \in S_{L,r}} f_{h,T}(j/T, \omega) - \sum_{j \in S_{R,r}} f_{h,T}(j/T, \omega) \right|, \quad \omega \in [-\pi, \pi]. \]   (S.39)
For \( r = m_T, 2m_T, \ldots \) write
\[
\sum_{j \in S_{L,r}} f_{h,T}(j/T, \omega_0) - \sum_{j \in S_{R,r}} f_{h,T}(j/T, \omega_0) \\
= \sum_{j \in S_{L,r}} (f_{h,T}(j/T, \omega_0) - \mathbb{E}(f_{h,T}(j/T, \omega_0))) - \sum_{j \in S_{R,r}} (f_{h,T}(j/T, \omega_0) - \mathbb{E}(f_{h,T}(j/T, \omega_0))) \\
+ \sum_{j \in S_{L,r}} \left( \mathbb{E}(f_{h,T}(j/T, \omega_0)) - \tilde{f}(j/T, \omega_0) \right) - \sum_{j \in S_{R,r}} \left( \mathbb{E}(f_{h,T}(j/T, \omega_0)) - f(j/T, \omega_0) \right) \\
+ \sum_{j \in S_{L,r}} \tilde{f}(j/T, \omega_0) - \sum_{j \in S_{R,r}} \tilde{f}(j/T, \omega_0) - \sum_{j \in S_{R,r}} \left( f(j/T, \omega_0) - \tilde{f}(j/T, \omega_0) \right).
\]

For \( r = 2m_T, \ldots, \tau_b \) let \( C(r/T, \omega) = D'_{r,T}(\omega) \) for \( \omega \neq \omega_0 \) and
\[
C(r/T, \omega_0) = M_{S,T}^{-1/2} \left( \sum_{j \in S_{L,r}} f_{h,T}(j/T, \omega_0) - \sum_{j \in S_{R,r}} f_{h,T}(j/T, \omega_0) \\
+ \sum_{j \in S_{R,r}, j > \tau_b} \left( f(j/T, \omega_0) - \tilde{f}(j/T, \omega_0) \right) \right),
\]
for \( \omega = \omega_0 \). Note that \( C(s/T, \omega) \) does not involve any break for any \( \omega \). Thus, we can proceed similarly as in the proofs of Section 3. That is, we exploit the smoothness of \( f(\cdot, \cdot) \) under \( \mathcal{H}_0 \) to yield \( \sup_{u \in [0, \lambda_0]} \sup_{\omega \in [-\pi, \pi]} |C(u, \omega)| = O_P(\sqrt{\log(T)}) \). This combined with the definition of \( d(r/T, \omega_0) \) implies that for each \( r = \tau_b - [m_T/B], \ldots, \tau_b \) where \( B \) is any finite integer with \( B > 1 \),
\[
|d(r/T, \omega_0)| > \max_{\omega \in [-\pi, \pi]} (|C(r/T, \omega)|) > 0,
\]
with probability approaching one and
\[
D_{r,T}(\omega) = |d(r/T, \omega) + C(r/T, \omega)| = d(r/T, \omega) + \text{sign}(C(r/T, \omega)) |C(r/T, \omega)|.
\]
By definition of \( d(\cdot, \omega_0) \), for \( \kappa_T \in [0, m_T/(BT)] \),
\[
d(\tau_b/T, \omega_0) - d(\tau_b/T - \kappa_T, \omega_0) = [\kappa_T] m_{S,T}^{-1} \delta_T M_{S,T}^{-1/2}.
\]
In order to apply Lemma S.A.8, we need to choose \( \kappa_T \) such that \( [\kappa_T] m_{S,T}^{-1} \delta_T M_{S,T}^{-1/2} \sqrt{\log(T)} \geq 1 \) or \( \sqrt{M_{S,T} \log(T)} m_{S,T} / (\delta_T T) = o(\kappa_T) \). Lemma S.A.8 then yields
\[
\frac{\tau_b}{T} \geq \max_{r=2m_T, 3m_T, \ldots; r < \tau_b, \omega \in [-\pi, \pi]} T^{-1} D_{r,T}(\omega) = \max_{r=m_T, \ldots, \tau_b} T^{-1} D_{r,T}(\omega_0) \geq \frac{\tau_b}{T} - \kappa_T.
\]
The case \( r > \tau_b \) can be treated similarly by symmetry. It results in
\[
\frac{\tau_b}{T} \leq \max_{r=\tau_b, \ldots, T-m_T, \omega \in [-\pi, \pi]} T^{-1} D_{r,T}(\omega) = \max_{r=\tau_b, \ldots, T-m_T} T^{-1} D_{r,T}(\omega_0) \leq \frac{\tau_b}{T} + \kappa_T.
\]
Therefore, we conclude \( |\hat{\lambda}_b - \tau_b/T| = O_P(\kappa_T) \rightarrow 0 \). □
S.A.5.2 Proof of Proposition 5.2

Set \( \hat{I} = \{2m_T, 3m_T, \ldots, (M_T - 1)m_T - n_T\} \setminus \{2m_T\} \) and \( \tilde{I} = \emptyset \). Under \( \mathcal{H}_{1,M} \), the arguments in the proof of Theorem 3.2 yields,

\[
\max_{r \in \hat{I} \setminus \{r_{L,l}, r_{R,l}\}} D_{r,T}(\omega) = O_{\bar{F}}\left(\sqrt{\log(T)}\right).
\]

Let \( r_{L,l}, r_{R,l} \in \hat{I} \) \((l = 1, \ldots, m_0)\) such that \( r_{R,l} - r_{L,l} = m_T \) and \( r_{L,l} \leq T_0 < r_{R,l} \). For any \( \omega \) we have

\[
\max_{r \in \hat{I} \setminus \{r_{L,l}, r_{R,l}, \ldots, r_{L,m_0}, r_{R,m_0}\}} D_{r,T}(\omega) = O_{\bar{F}}\left(\sqrt{\log(T)}\right).
\]

For each \( r \in \hat{I} \), we draw \( K \) points \( \omega_k^{\circ} \) with \( k = 1, \ldots, K \) uniformly (without replacement) from \( \mathcal{I}(r) \). Consider the following events,

\[
\mathcal{D}_1 = \{\forall r \in \hat{I} \text{ and } \forall k = 1, \ldots, K, (\exists! 1 \leq l \leq m_0) \text{ s.t. } T_0 \in \left[r_k^{\circ} - m_T, r_k^{\circ} + m_T\right]\}
\]

\[
\mathcal{D}_2 = \{\forall l = 1, \ldots, m_0 \exists r \in \hat{I} \text{ s.t. } \exists k = 1, \ldots, K, \text{ s.t. } T_0 - r_k^{\circ} = Cm_T \text{ for some } C \in [0, 1]\}.
\]

Let \( A^c \) denote the complement of a set \( A \). Note that \( \mathbb{P}(\mathcal{D}_1^c \cap \mathcal{D}_2^c) = \mathbb{P}(\mathcal{D}_2^c) \) by Assumption 5.2 and that \( \mathbb{P}(\mathcal{D}_2^c) = 0 \) if there are still undetected breaks.

The remaining arguments will be valid on the set \( \mathcal{D}_1 \cap \mathcal{D}_2 \) as long as there are undetected breaks. Let \( r_l, r_{l+1} \in \hat{I} \) be such that \( T_0 \in [r_l, r_{l+1}) \). As in the proof of Proposition 5.1,

\[
D_{r_l,T}(\omega_l) = |O_{\bar{F}}\left(M_{S,T}^{-1/2} \delta_{l,T} \left(M_{S,T} - (r_l - T_0) 1\{r_{l-1} < T_0 \leq r_l\} + (r_{l+1} - T_0) 1\{r_l < T_0 < r_{l+1}\}\right)\right)|.
\]

Note that if \( D_{r_l,T}(\omega_l) / (\delta_{l,T} \sqrt{M_{S,T}}) \xrightarrow{P} 0 \) then we must have \( D_{r_{l+1},T}(\omega_l) = O_{\bar{F}}(\delta_{l,T} \sqrt{M_{S,T}}) \). Using a similar argument as in Lemma S.A.5 one can show that \( S_{D_{max,T}}(\hat{I}) \) is asymptotically equivalent to

\[
\max_{r \in \hat{I}} \max_{k \in K} \max_{\omega \in [-\pi, \pi]} D_{r_k^{\circ},r}(\omega). \]

Thus, in step (2) \( \psi(\{X_t\}_{1 \leq t \leq T}, \hat{I}) = 1 \) because for large enough \( T \),

\[
\max_{r \in \hat{I}} \max_{k \in K} \max_{\omega \in [-\pi, \pi]} D_{r_k^{\circ},r}(\omega_0) \geq \max_{r \in \hat{I}} \max_{\omega \in [-\pi, \pi]} D_{r,T}(\omega)
\]

\[
= |\delta_{l,T} O_{\bar{F}}(\sqrt{M_{S,T}})|
\]

\[
\geq \inf_{1 \leq l \leq m_0} |\delta_{l,T} O_{\bar{F}}(\sqrt{M_{S,T}})|
\]

\[
= 2D^* (\log(T))^{2/3}
\]

\[
> 2D^* \sqrt{\log(M_T)},
\]

where the last equality follows from Assumption 5.2. We now move to step (3). By the arguments in the proof of Proposition 5.1, there exists \( 1 \leq l \leq m_0 \) such that \( |\lambda_0^0 - \lambda_T^0(\hat{I})| \leq m_T / T \). Since \( \inf_{1 \leq l \leq m_0 - 1} |\lambda_{l+1}^0 - \lambda_l^0| \geq v_T^{-1} \) and \( m_T / v_T \to 0 \) there can exist exactly one \( l \) that satisfies \( |\lambda_l^0 - \lambda_T^0(\hat{I})| \leq m_T / T \). For such a \( \lambda_l^0 \) define \( \tau_{l,b} = \lceil T\lambda_l^0 + 1 \rceil \), the smallest integer such that \( \tau_{l,b} / T \) is larger or equal than \( \lambda_l^0 + 1/T \). Denote by \( \{\tilde{f}(u, \omega)\}_{u \in [0, 1]} \) the path of the spectrum \( f(\cdot, \omega) \) without the break \( \delta_{l,T} \):

\[
f(r/T, \omega_l) = \tilde{f}(r/T, \omega_l) + \delta_{l,T} 1\{r \geq \tau_{l,b}\}.
\]

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Without loss of generality, we assume \( \delta_{l,T} > 0 \). Define \( d_l (r/T, \omega) = 0 \) for \( \omega \neq \omega_l \) and

\[
d_l (r/T, \omega_l) = \begin{cases} 
0 & \text{if } r + m_T < \tau_{l,b}, \\
(r + m_T - \tau_{l,b}) m_{S,T}^{-1} M_{S,T}^{-1/2} \delta_{l,T} & \text{if } r = \tau_{l,b} = -m_T, \tau_{l,b} - m_T + m_{S,T}, \ldots, \tau_{l,b}, \\
M_{S,T}^{1/2} \delta_{l,T} & \text{if } r > \tau_{l,b},
\end{cases}
\]

for \( \omega = \omega_l \). Let \( \{d (u)\}_{u \in [0,1]} \) be the associated piecewise constant increasing step function. For any \( r \in \hat{T} \), write

\[
\sum_{j \in S_{L,r}} f_{h,T} (j/T, \omega) - \sum_{j \in S_{R,r}} f_{h,T} (j/T, \omega) = \sum_{j \in S_{L,r}} (f_{h,T} (j/T, \omega) - \mathbb{E} (f_{h,T} (j/T, \omega))) - \sum_{j \in S_{R,r}} (f_{h,T} (j/T, \omega) - \mathbb{E} (f_{h,T} (j/T, \omega))) \\
+ \sum_{j \in S_{L,r}} \left( \mathbb{E} (f_{h,T} (j/T, \omega)) - \tilde{f} (j/T, \omega) \right) - \sum_{j \in S_{R,r}} \left( \mathbb{E} (f_{h,T} (j/T, \omega)) - \tilde{f} (j/T, \omega) \right) \\
+ \sum_{j \in S_{L,r}} \tilde{f} (j/T, \omega) - \sum_{j \in S_{R,r}} \tilde{f} (j/T, \omega) - \sum_{j \in S_{R,r}} \left( f (j/T, \omega) - \tilde{f} (j/T, \omega) \right).
\]

For \( r = 2m_T, 3m_T, \ldots, T \), let \( C_l (r/T, \omega) = D'_{r,T} (\omega) \) for \( \omega \neq \omega_l \) where \( D'_{r,T} (\omega) \) is given in (S.39) and

\[
C_l (r/T, \omega_l) = M_{S,T}^{-1/2} \left( \sum_{j \in S_{L,r}} f_{h,T} (j/T, \omega_l) - \sum_{j \in S_{R,r}} f_{h,T} (j/T, \omega_l) \\
+ \sum_{j \in S_{R,r}, j > \tau_{l,b}} \left( f (j/T, \omega_l) - \tilde{f} (j/T, \omega_l) \right) \right),
\]

for \( \omega = \omega_l \). We proceed as in the proof of Proposition 5.1. We have

\[
d (r/T, \omega_l) \geq \max_{\omega \in \{-\pi, \pi\}/\{\omega_l, \ldots, \omega_{m_0}\}} |d (r/T, \omega)| > 0,
\]

with probability approaching one. Exploiting the smoothness on \((\lambda_{l-1}, \lambda_0)\), we have

\[
\sup_{u \in (\lambda_{l-1}, \lambda_0)} \sup_{\omega \in [-\pi, \pi]} |C_l (u, \omega)| = O_p \left( \sqrt{\log (T)} \right).
\]

This implies

\[
D_{r,T} (\omega) = |d_l (r/T, \omega) + C_l (r/T, \omega)| = (d_l (r/T, \omega) + \text{sign} (C_l (r/T, \omega)) |C_l (r/T, \omega)|),
\]

for each \( r = \tau_{l,b} - \lfloor m_T/B \rfloor, \ldots, \tau_{l,b} \) where \( B \) is any integer with \( 1 < B < \infty \). By definition of \( d_l (\cdot, \omega_l) \), for \( \kappa_T \in [0, m_T/(BT)] \) we have

\[
d_l (\tau_{l,b}/T, \omega_l) - d_l (\tau_{l,b}/T - \kappa_T, \omega_l) = |\kappa_T T| m_{S,T}^{-1} \delta_{l,T} M_{S,T}^{-1/2}.
\]

In order to apply Lemma S.A.8, we need to choose \( \kappa_T \) such that \[|\kappa_T T| \delta_{l,T} M_{S,T}^{-1/2} / m_{S,T} \sqrt{\log (T)} \geq 1\] or
\[ m_{S,T} \sqrt{\frac{\log(T)}{\delta_{1,T}}} = o(\kappa_T). \] Lemma S.A.8 then yields
\[
\frac{T_{lb}}{T} \geq \max_{r \in (\hat{T} \setminus \{r : r \geq \tau_{lb}\})} \tau_r - \kappa_T.
\]
The case \( r > \tau_{lb} \) can be treated similarly by symmetry. It results in
\[
\frac{T_{lb}}{T} \leq \max_{r \in (\hat{T} \setminus \{r : r < \tau_{lb}\})} \tau_r - \kappa_T.
\]
Therefore, we conclude \( |\hat{\lambda}_T - \tau_{r,b}/T| = O_F(\kappa_T) \to 0. \) Now set \( \hat{T} = \hat{T} \setminus \{T \hat{\lambda}_T(\hat{T}) - v_T, \ldots, T \hat{\lambda}_T(\hat{T}) + v_T\} \) and \( \hat{T} = \hat{T} \cup \{T \hat{\lambda}_T(\hat{T})\}. \) Since \( \mathbb{P}(D_2) = 0 \) if there are still undetected breaks, we can repeat the above steps (1)-(4). The final results are \( \mathbb{P}(|\hat{T}| = m_0) \to 1 \) and, after ordering the elements of \( \hat{T} \) in chronological order, \( \sup_{1 \leq t \leq m_0} |\hat{\lambda}_{t,T} - \lambda_t^0| = O_F(m_{S,T} \sqrt{\log(T)/(T \inf_{1 \leq t \leq m_0} \delta_{t,T})}). \)

Assume without loss of generality that \( \delta_{1,T} \geq \delta_{2,T} \geq \cdots \geq \delta_{m_0,T}. \) Let \( \hat{\lambda}_T^{(q)} (q = 1, \ldots, m_0) \) denote the \( q \)th break detected by the procedure. It remains to prove that if \( K \to \infty \) then \( \hat{\lambda}_T^{(q)} \) is consistent for \( \lambda_q^0 (q = 1, \ldots, m_0). \) Consider the first break \( \lambda_1^0. \) In order for the algorithm to return \( \hat{\lambda}_T^{(1)} \) such that \( |\hat{\lambda}_T^{(1)} - \lambda_1^0| \to 0 \) we need the following event to occur with sufficiently high probability, \( W = \{\forall l = 1 \exists r \in \hat{T} \text{ and } k = 1, \ldots, K \text{ s.t. } r_{r,k} \in T_1^0\}. \) Note that
\[
W^c = \{T_1^0 \text{ not sampled in } K \text{ draws from } T_1^0 - m_T + 1, \ldots, T_1^0 \text{ without replacement}\}.
\]
Thus,
\[
1 - \mathbb{P}(W^c) = 1 - \frac{m_T - 1}{m_T} \times \frac{m_T - 2}{m_T - 1} \times \cdots \times \frac{m_T - K}{m_T - K + 1}
\]
\[
= 1 - \frac{m_T - K}{m_T}
\]
\[
\to 1,
\]
only if \( K = O(a_T m_T) \) with \( a_T \in (0, 1] \) such that \( a_T \to 1. \) Note that \( K \leq m_T \) by construction. The same argument can be repeated for \( l = 2, \ldots, m_0. \) \( \Box \)