Geometric phases between biorthogonal states

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We investigate the evolution of a state which is dominated by a finite-dimensional non-Hermitian time-dependent Hamiltonian operator with a nondegenerate spectrum by using a biorthonormal approach. The geometric phase between any two states, biorthogonal or not, are generally derived by employing the generalized interference method. The counterpart of Manini-Pistolesi non-diagonal geometric phase in the non-Hermitian setting is taken as a typical example.

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where $\zeta, \zeta' \in \mathbb{C}$ are complex numbers, and they satisfy the requirement $\zeta = \zeta'$. It should be stressed here that only the state vector $|\psi(t)\rangle \in \text{Span}\{n(t)\}_{n=1}^{\infty}$ rather than its dual $\langle \tilde{\psi}(t) | \rangle \in \text{Span}\{\langle n(t) | \rangle\}_{n=1}^{\infty}$ describes a quantum state of the physical system, although they stand equally from Eqs. (3) and (4).

In conventional Hermitian quantum mechanics, the maximum interference formula between any two non-orthogonal normalized rays $A, B$ is written by

$$I_{\text{max}} = \sup_{\alpha \in \mathbb{R}} |\langle A|e^{i\alpha} + B\rangle|^2$$

$$= \sup_{\alpha \in \mathbb{R}} \langle A|e^{-i\alpha} + B|A^\dagger e^{i\alpha} + B\rangle, \quad (8)$$

which induces the Pancharatnam connection $A^P = \text{Im}\langle A|B \rangle$. In order to investigate the issue of geometric phase in the above non-Hermitian setting, the Pancharatnam connection need modifying by generalizing the maximum interference formula Eq. (5). According to the bi-normalization Eq. (6) with local gauge transformation Eq. (7), the generalized interference formula $\mathcal{G}^2$ between any two non-biorthonormal rays $\psi_1, \psi_2$ is defined as

$$\mathcal{G}^2 = \frac{\langle \tilde{\psi}_1 e^{-i\theta} + \tilde{\psi}_2 | e^{i\theta} \rangle}{\langle \tilde{\psi}_1 | \tilde{\psi}_2 \rangle} = \langle \tilde{\psi}_1 e^{-i\theta} + \tilde{\psi}_2 | e^{i\theta} \rangle, \quad \theta \in \mathbb{C}. \quad (9)$$

**Definition.** When the generalized interference intensity $\mathcal{G}^2$ is stationary with respect to the complex-valued phase $\theta$ such that $\sqrt{\langle \tilde{\psi}_1 | \tilde{\psi}_2 \rangle} = 1$, then $|\psi_1\rangle$ and $|\psi_2\rangle$ are said to be “in phase” or parallel.

According to the definition, the generalized Pancharatnam connection $A^{GP}$ is given by

$$A^{GP} = \sqrt{\langle \tilde{\psi}_1 | \tilde{\psi}_2 \rangle} - 1. \quad (10)$$

As the infinitesimal version of the real-valued Pancharatnam connection $\text{Im}\langle B(s)|B(s+ds) \rangle = 0$ can induce a parallel transport law $\text{Im}\langle B(s)|B(s) \rangle = 0$ by

$$\langle B(s)|B(s+ds) \rangle = 1 + \langle B(s)|\frac{d}{ds}B(s) \rangle ds + \mathcal{O}(ds^2), \quad (11)$$

the counterpart of the generalized Pancharatnam connection $\sqrt{\langle \tilde{\psi}(s)|\psi(s+ds) \rangle\langle \psi(s+ds)|\psi(s) \rangle} - 1 = 0$ can also give a parallel transport law in the non-Hermitian setting

$$\langle \tilde{\psi}(s)|\frac{d}{ds}\psi(s) \rangle = 0, \quad \langle \frac{d}{ds}\tilde{\psi}(s)|\psi(s) \rangle = 0, \quad (12)$$

by

$$\sqrt{\langle \tilde{\psi}(s)|\psi(s+ds) \rangle\langle \psi(s+ds)|\psi(s) \rangle} = 1 + \langle \tilde{\psi}(s)|\frac{d}{ds}\psi(s) \rangle ds + \mathcal{O}(ds^2)$$

$$= 1 - \langle \frac{d}{ds}\tilde{\psi}(s)|\psi(s) \rangle ds + \mathcal{O}(ds^2), \quad (13)$$

here bi-normalization Eq. (6) has been used. It should be noted that the parallel transport law $\sqrt{\langle \tilde{\psi}(s)|\psi(s+ds) \rangle\langle \psi(s+ds)|\psi(s) \rangle} - 1$ must be equal to 0 rather than any other complex numbers, because any complex number (except 0) can be expressed as an exponential of a complex number which will be involved into the complex-valued phase $\theta$. Moreover, the infinitesimal version of the generalized Pancharatnam connection Eq. (10) gives

$$A^{GP}(s) = \langle \tilde{\psi}(s)|\frac{d}{ds}\psi(s) \rangle, \quad (14)$$

which transform under the gauge transformation Eq. (7) as follow,

$$A^{GP}(s) \rightarrow A^{GP}(s) + i \frac{dc}{ds}. \quad (15)$$

The tangent vector $\langle \frac{d}{ds}\psi(s) \rangle$ is not gauge covariant,

$$\langle \frac{d}{ds}\psi(s) \rangle \rightarrow e^{i\zeta} \langle \frac{d}{ds}\psi(s) \rangle + i \frac{dc}{ds} \langle \psi(s) \rangle \rangle. \quad (16)$$

One can check that the covariant derivative $\frac{D}{ds}$ can be defined as

$$\frac{D}{ds} \langle \psi(s) \rangle = \left( \frac{d}{ds} - A^{GP}(s) \right) \langle \psi(s) \rangle. \quad (17)$$

Likewise, the duals of Eqs. (9)-(17) can be obtained by the operation of tilde Eq. (2). Hence, there exists a gauge invariant quantity $\langle \tilde{D}\psi(s)|\tilde{D}\psi(s) \rangle$ which can be used to defined a metric on ray space,

$$\frac{d}{ds} = \langle \tilde{D}\tilde{\psi}(s)|\tilde{D}\tilde{\psi}(s) \rangle \frac{d}{ds} \langle \tilde{D}\tilde{\psi}(s)|\tilde{D}\tilde{\psi}(s) \rangle \frac{d}{ds}. \quad (18)$$

The metric Eq. (18) then determins the geodesic in ray space by variation of the length $I(C)$

$$I(C) = \int_C d\mathcal{L} = \int_C \sqrt{d\mathcal{L}^2} = \int_C \sqrt{\langle \tilde{D}\tilde{\psi}(s)|\tilde{D}\tilde{\psi}(s) \rangle \frac{d}{ds} \langle \tilde{D}\tilde{\psi}(s)|\tilde{D}\tilde{\psi}(s) \rangle \frac{d}{ds}} \, ds, \quad (19)$$

from which one can obtain a pair of geodesic equations,

$$\frac{D^2}{ds^2} \langle \psi(s) \rangle = 0, \quad \frac{D^2}{ds^2} \langle \psi(s) \rangle = 0. \quad (20)$$

Equations in Eq. (20) are gauge covariant under local gauge transformation Eq. (7). It should be stressed that the equations in Eq. (20) must hold simultaneously. When the generalized interference intensity $\mathcal{G}^2$ in Eq. (9) is stationary with respect to the complex-valued phase $\theta$, the generalized Pancharatnam phase can be obtained,

$$\theta_{1,2}^{GP} = -\frac{i}{2} \log \frac{\langle \tilde{\psi}_1 | \tilde{\psi}_2 \rangle}{\langle \tilde{\psi}_2 | \tilde{\psi}_1 \rangle}. \quad (21)$$
Theorem. Let the two non-biorthogonal states $|\psi_1\rangle, |\psi_2\rangle$ be connected by a geodesic $G^{1,2}$ satisfying Eq. (20), then the generalized Pancharatnam phase $\theta_{1,2}^{GP}$ is given by

$$\theta_{1,2}^{GP} = -i \int_{G^{1,2}} A^{GP}$$

where $A^{GP} = \langle \bar{\psi} | d\psi \rangle$ is the connection 1-form.

Proof. Consider a geodesic $|\varphi(s)\rangle$ starting from $|\psi(0)\rangle = |\psi_1\rangle$ and ending in the ray $|\psi_2\rangle$ by $|\varphi(1)\rangle$ satisfying $A^{GP}(s) = 0$, then geodesic equation Eq. (20) reduces to $\frac{d^2}{ds^2} |\varphi(s)\rangle = 0$, whose solution is a straight line described by

$$|\varphi(s)\rangle = (1-s)|\psi_1\rangle + s|\varphi(1)\rangle, \forall s \in [0,1]. \quad (22)$$

Let $q(s) = \sqrt{\langle \varphi(s) | \varphi(s) \rangle} - 1$. It can be verified that $q(0) = 0$ and $q(0) = 0$ due to $A^{GP}(s) = 0$ on the geodesic $|\varphi(s)\rangle$. By inserting Eq. (22) into $q(0) = 0$, one can obtain

$$\langle \bar{\psi}_1 | \psi(1) \rangle = \langle \bar{\varphi}(1) | \psi_1 \rangle, \quad (23)$$

where we use the binormalization condition $\langle \bar{\psi}_1 | \psi_1 \rangle = \langle \bar{\psi}_1 | \psi_1 \rangle = 1$. By inserting Eqs. (22) and (23) into $q(s)$, then we have $q(s) = 0, \forall s \in [0,1]$, which means $|\psi_1\rangle$ and $|\varphi(s)\rangle$ are “in phase”. Due to the gauge covariance of the geodesic equation Eq. (20), let $|\psi(s)\rangle = e^{i\theta(s)}|\varphi(s)\rangle, \forall s \in [0,1]$, with the boundary condition $\theta(0) = 0$ and $\theta(1) = \theta_{1,2}^{GP}$, then $|\psi(s)\rangle$ is still a geodesic linking $|\psi_1\rangle$ to $|\psi_2\rangle$, which is denoted by $G^{1,2}$. And finally, $-i \int_{G^{1,2}} A^{GP} = \int_0^1 \frac{d}{ds} \theta(s) ds = \theta_{1,2}^{GP}$. \qed

According to the theorem, one can link $N$ vertices $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_N\rangle$ one-by-one by $N - 1$ geodesics $G^{1,2}, G^{2,3}, \ldots, G^{N-1,N}$ to obtain the accumulated generalized Pancharatnam phase $\theta_{1,2,\ldots,N}^{GP}$ along the continuous curve $G^{open} = G^{1,2} + G^{2,3} + \cdots + G^{N-1,N}$ by

$$\theta_{1,2,\ldots,N}^{GP} = -i \int_{G^{open}} A^{GP} = -i \sum_{n=1}^{N-1} \int_{G^{n,n+1}} A^{GP} = \sum_{n=1}^{N-1} \theta^{GP}_{n,n+1} + \theta^{GP}_{1,2,\ldots,N}.$$ \quad (24)

It should be noted that $\theta_{1,2,\ldots,N}^{GP}$ is not gauge invariant under local gauge transformation Eq. (7) and hence involves in any possible phase including dynamical phase. However, if $|\psi_N\rangle$ is linked back to $|\psi_1\rangle$ by a geodesic $G^{N,1}$, then the curve $G^{closed} = G^{1,2} + G^{2,3} + \cdots + G^{N-1,N} + G^{N,1}$ is continuous and closed such that $\theta_{1,2,\ldots,N}^{GP}$ is gauge invariant,

$$\theta_{1,2,\ldots,N,1}^{GP} = \sum_{n=1}^{N} \theta^{GP}_{n,n+1} \mod N.$$ \quad (25)

Here, the geodesic $G^{N,1}$ is added to remove any possible phase, including dynamical phase, which can be produced or removed by local gauge transformation Eq. (7). Hence, the accumulated generalized Pancharatnam phase $\theta_{1,2,\ldots,N}^{GP}$ is purely geometrical. As $N \rightarrow \infty$, the curve $G^{open}$ becomes smooth while $G^{N,1}$ is still unchanged. The geometric phase difference between $|\psi(1)\rangle = |\psi_N\rangle$ and $|\psi(0)\rangle = |\psi_1\rangle$, \lim$_{N \rightarrow \infty} \theta_{1,2,\ldots,N,1}^{GP}$, can be calculated by

$$\lim_{N \rightarrow \infty} \theta_{1,2,\ldots,N,1}^{GP} = \sum_{n=1}^{N} \theta^{GP}_{n,n+1} \mod N.$$ \quad (26)

where \lim$_{N \rightarrow \infty} \theta_{N,N-1,\ldots,1}^{GP}$ represents the geometric phase difference between $|\psi(0)\rangle$ and $|\psi(1)\rangle$.

Based on the above preparation, we now discuss the geometric phase between initial and final states no matter whether they are non-biorthogonal or biorthogonal, i.e., $\langle \psi(0) | \psi(t) \rangle \neq 0$ or $\langle \psi(0) | \psi(t) \rangle = 0$.

Non-biorthogonal case — The evolving state $|\psi(t)\rangle$ starting from the initial state $|\psi(0)\rangle$ is governed by the non-Hermitian Schrödinger equation Eq. (3). The geometric phase $\gamma^{geo}(0,t)$ between $|\psi(0)\rangle$ and $|\psi(t)\rangle$ can be calculated by Eq. (26),

$$\gamma^{geo}(0,t) = \frac{1}{2} \log \left( \frac{\langle \bar{\psi}(0) | \psi(t) \rangle}{\langle \bar{\psi}(0) | \psi(0) \rangle} \right) + \int_0^t \left( \frac{d}{dt} |\psi(t)\rangle |\psi(t)\rangle \right) dt'$$

$$= \frac{1}{2} \log \left( \frac{\langle \bar{\psi}(0) | \psi(t) \rangle}{\langle \bar{\psi}(0) | \psi(0) \rangle} \right) + \int_0^t \frac{d}{dt} \langle \bar{\psi}(t') | H(t) | \psi(t') \rangle dt'$$

$$= \theta^{GP}(0,t) - \gamma^{geo}(0,t).$$ \quad (27)

It should be noted in Eq. (27) that the existence of $\gamma^{geo}(0,t)$ as well as $\theta^{GP}(0,t)$ merely depends on whether the initial state is biorthogonal to the final state rather
than any internationally traveled state, while the dynamical phase $\gamma_{\text{dyn}}(0, t)$ continuously exists.

**Bireorthogonal case** — Due to the bireorthogonality between the initial state $|\psi(0)\rangle$ and the final state $|\psi(t)\rangle$, i.e., $\langle \psi(0)|\psi(t)\rangle = 0$, the geometric phase between them can not be evaluated by Eq. (27) directly for log 0 is not defined mathematically. However, if an internationally traveled state $|\psi(t_1)\rangle$ is non-bireorthogonal to both the initial and the final states, then the geometric phase $\gamma_{geo}(0, t)$ between the initial state $|\psi(0)\rangle$ and the final state $|\psi(t)\rangle$ can still be calculated indirectly by Eq. (27),

$$\gamma_{geo}(0, t) = \gamma_{geo}(0, t_1) + \gamma_{geo}(t_1, t)$$

$$= -\frac{i}{2} \log \frac{\langle \tilde{\psi}(0)|\tilde{\psi}(t_1)\rangle}{\langle \tilde{\psi}(t)|\tilde{\psi}(t_1)\rangle}$$

$$+ \int_0^t \langle \tilde{\psi}(t')|H(t)|\tilde{\psi}(t')\rangle \, dt'.$$  

(28)

Here, the internationally traveled state $|\psi(t_1)\rangle$ acts as a torchbearer to guarantee that the geometric phase difference can be preserved and delivered from the initial state to the final bireorthogonal state. Besides, $|\psi(t_1)\rangle$ does not interrupt the process of the state evolution. Seen from another perspective, both the initial and the final states are projected onto the internationally traveled state, and the total geometric phase difference is equal to the difference between $\gamma_{geo}(0, t_1)$ and $\gamma_{geo}(t_1, t)$,

$$\gamma_{geo}(0, t) = \gamma_{geo}(0, t_1) - \gamma_{geo}(t_1, t).$$

(29)

Hence, the internationally traveled state $|\psi(t_1)\rangle$ is unnecessary because it can be replaced with any state $|a\rangle$ which is non-bireorthogonal to both the initial and final states to implement Eq. (27),

$$\gamma_{geo}(0, t) = -\frac{i}{2} \log \frac{\langle \tilde{\psi}(0)|\tilde{\psi}(t)\rangle}{\langle \tilde{\psi}(t)|\tilde{\psi}(0)\rangle}$$

$$+ \int_0^t \langle \tilde{\psi}(t')|H(t)|\tilde{\psi}(t')\rangle \, dt'.$$  

(30)

The first term in Eq. (30) can be obtained by modifying Eq. (2),

$$\gamma^2 = \left< e^{-i\theta} (\tilde{\psi}(0)|a\rangle a + \langle \tilde{\psi}(t)|a\rangle a \mid a\rangle |\tilde{\psi}(0)\rangle e^{i\theta} + a(\tilde{\psi}(t)) \right>.$$  

(31)

The second term in Eq. (30) is to remove the dynamical phase off the final state $|\psi(t)\rangle$.

As a typical example, we consider the off-diagonal geometric phases in the non-Hermitian setting by using Eq. (30): two states $|j(0)\rangle$ and $|k(0)\rangle$ evolve adiabatically to $|j(t)\rangle$ and $|k(t)\rangle$, respectively, such that $\langle j(0)|j(t)\rangle = 0$ and $\langle k(0)|k(t)\rangle = 0$. We can find a state $|a\rangle$ which is not bireorthogonal to $|j(0)\rangle$, $|j(t)\rangle$, $|k(0)\rangle$, or $|k(t)\rangle$. Then the off-diagonal geometric phases $\gamma_{jk}^{\text{geo}}$ is given by

$$\gamma_{jk}^{\text{geo}} = \gamma_{\text{geo}}[|j(0)\rangle, |a\rangle, |k(t)\rangle] + \lambda_{\text{geo}}[|k(0)\rangle, |a\rangle, |j(t)\rangle]$$

$$+ \gamma_{j}^{\text{geo}}(0, t) + \gamma_{k}^{\text{geo}}(0, t),$$

(32)

where

$$\gamma_{jk}^{\text{geo}}[|j(0)\rangle, |a\rangle, |k(t)\rangle] = -\frac{i}{2} \log \frac{\langle j(0)|a\rangle \langle a|k(t)\rangle \langle k(t)|j(0)\rangle}{\langle j(0)|a\rangle \langle a|j(0)\rangle \langle j(0)|k(t)\rangle \langle k(t)|j(0)\rangle},$$

$$\gamma_{jk}^{\text{geo}}[|k(0)\rangle, |a\rangle, |j(t)\rangle] = -\frac{i}{2} \log \frac{\langle k(0)|a\rangle \langle a|j(t)\rangle \langle j(t)|k(0)\rangle}{\langle k(0)|a\rangle \langle a|j(t)\rangle \langle j(t)|k(0)\rangle \langle k(0)|j(t)\rangle \langle j(t)|k(0)\rangle},$$

(33)

and

$$\gamma_{j}^{\text{geo}}(0, t) = -\frac{i}{2} \log \frac{\langle j(0)|a\rangle \langle a|j(t)\rangle}{\langle j(0)|a\rangle \langle a|j(0)\rangle \langle j(0)|k(t)\rangle \langle k(t)|j(0)\rangle} + \int_0^t \langle j(t')|H(t)|j(t')\rangle \, dt',$$

$$\gamma_{k}^{\text{geo}}(0, t) = -\frac{i}{2} \log \frac{\langle k(0)|a\rangle \langle a|k(t)\rangle}{\langle k(0)|a\rangle \langle a|k(0)\rangle \langle k(0)|j(t)\rangle \langle j(t)|k(0)\rangle} + \int_0^t \langle k(t')|H(t)|k(t')\rangle \, dt'.$$  

(34)

For more than two bireorthogonal states, the similar procedure can be performed.

In conclusion, we investigated and also suggested a general formalism for the geometric phases between any two states, bireorthogonal or not, in a finite-dimensional non-Hermitian quantum dynamical system with a non-degenerate spectrum. Based on the generalized interference formula, we re-defined the concept of “in-phase” in the non-Hermitian setting, which contributed to the discussion of geometric aspects. Finally, we gave the counterpart of Manini-Pistolesi non-diagonal geometric phase in the non-Hermitian setting as a typical example.

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