Some Ergodic Theorems for Random Rotations on Wiener Space

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Abstract
In this paper we study ergodicity and mixing property of some measure preserving transformations on the Wiener space \((W, H, \mu)\) which are generated by some random unitary operators defined on the Cameron-Martin space \(H\).

1 Introduction
Although the Wiener measure is one of the most popular and studied probability measures, there are surprisingly few results about the ergodicity of the measure preserving transformations of the Wiener paths. In fact since the early work of Maruyama [3] and that of Wiener and Akutowicz [6] it is difficult to find any work about the subject in the literature. Having studied the general Wiener-measure preserving transformations in [3] (cf. also [4]), we were led to look at the ergodicity of these transformations under the light of the powerful techniques developed by the Stochastic Calculus of Variations of Paul Malliavin. In fact even about the classical Wiener-Ito decomposition we have a better knowledge now and using this latter technique, one can characterize rather easily the ergodicity and the mixing property of the transformations which are the second quantizations of the deterministic unitary transformations on the Cameron-Martin space \(H\). When we take a random unitary transformation \(R(w)\) of the Cameron-Martin space with the property that, for any \(h \in H\), \(\nabla Rh\) is a quasi-nilpotent operator on \(H\), then \(R\) generates also a Wiener-measure preserving transformation, called rotation. However the ergodicity in this case is much more difficult to characterize, since the randomness of \(R\) induces a very strong non-linearity and we can not use anymore the Wiener chaos technique as easily as in the deterministic...
case. In this paper we study the ergodicity and the strong mixing property of this kind of random rotations in some special cases: the first is the case where the randomness enters as an input which is independent of the Wiener “paths”. In the second situation we replace this independence hypothesis by another interesting hypothesis, namely we assume that the (Gaussian) divergence of the resolution of the identity associated to the unitary operator defines a cylindrical martingale (indexed with the associated spectrum) having the chaotic representation property. The next section presents some necessary conditions for ergodicity. Finally we derive a sufficient condition for strong mixing (which is also necessary when \( R \) is deterministic) and give two generic examples of strongly mixing transformations. Let us now explain in detail the plan of the paper.

Let \((W,H,\mu)\) be an abstract Wiener space, i.e. \(W\) is a separable Banach space, \(H\) is a Hilbert space, whose continuous dual is identified with itself and it is densely and continuously injected in \(W\). For any \(e \in W^*\),

\[
\int_W e^{i\langle e,w \rangle} \mu(dw) = E[\exp i \langle e,w \rangle] = \exp -|j(e)|_H^2/2,
\]

where \(j\) denotes the injection \(W^* \hookrightarrow H\). Let \(T : W \rightarrow W\) be a measurable, invertible and measure preserving transformation on \(W\), the problem considered in this paper is the ergodicity of this transformation. As an example of such a transformation consider the classical Wiener space: let \(B_t, t \in [0,1]\) be a standard Wiener process taking values in \(\mathbb{R}^n\), consider \(\gamma(t,w)\) which for every \(t\) in \([0,1]\) takes value in the class of unitary \(n \times n\)-matrices. Then, if \(\gamma\) is non-random or under suitable measurability assumptions on \(\gamma(t,w)\), the process \(Y(t,w)\) defined as \(Y(t,w) = \int_0^t \gamma(\tau,w)dB_{\tau}\) exists as an Ito integral and, due to the celebrated theorem of Paul Lévy, \((t,w) \rightarrow Y(t,w)\) is also a standard Brownian motion in \(\mathbb{R}^n\).

The class of transformations on Wiener space that will be considered is as follows. Let \((e_n, n \in \mathbb{N})\) be a complete, orthonormal basis of \(H\) and \(e_n \in W^*\) for all \(n \in \mathbb{N}\). By the Ito-Nisio theorem (cf. [1, 4] \footnote{In the sequel, as long as there is no confusion, we shall not distinguish the elements of \(W^*\), from their images in \(H\).})

\[
w = \sum_{n=1}^{\infty} (\delta e_n(w))e_n
\]
\( \mu \)-almost surely in the sense that

\[
\left\| w - \sum_{1}^{N} (\delta e_{n}(w))e_{n} \right\|_{W} \to 0
\]

\( \mu \)-almost surely as \( N \to \infty \) where \( \delta e(w) =_{W<} e, w >_{W} \) is the abstract version of the Wiener integral of \( j(e) \).

Consider first the case where \( R \) is a non-random unitary transformation on \( H \), then

\[
w \to T(w) = \sum_{n=1}^{\infty} \delta (Re)_{n}(w) e_{n}
\]

is a measure preserving transformation of \( W \). The ergodicity of this class of transformations was characterized in [3]. In this paper we consider the problem of ergodicity for the case where \( R = R(w) \) is random. In [3] (cf. also [4]) we have already shown that if \( R(w) \) is almost surely a unitary transformation on \( H \), then under additional (non-trivial) assumptions, the mapping defined by

\[
w \to T(w) = \sum_{n=1}^{\infty} \delta (R(w)e)_{n}(w) e_{n} \tag{1.1}
\]

(where \( \delta (Re) \) denotes the ‘divergence’ or ‘Skorohod integral’) is a measure preserving transformation on the Wiener space. Hence the problem of ergodicity of such transformations is natural.

In the next section we summarize some relevant results from the Malliavin calculus. In the third section we give necessary and sufficient conditions for two classes of random rotations in terms of their resolution of identity. The first class consists of the rotations whose randomness are independent of the underlying Wiener paths. The second class maybe described as the set of rotations whose (random) resolutions of identity define cylindrical martingales (indexed with the Cameron-Martin space \( H \)) with chaotic representation property.

The case of general rotations is considered in Section 4 and necessary conditions for ergodicity is derived. A sufficient condition for strong mixing for a general class of rotations is derived in Section 5.
2 Preliminaries

Let \((W, H, \mu)\) be an abstract Wiener space, a mapping \(\varphi\) from \(W\) into some separable Hilbert space \(X\) will be called a cylindrical function if it is of the form \(\varphi(w) = f(<v_1, w>, \ldots, <v_n, w>)\) where \(f \in C_0^\infty(\mathbb{R}^n, X)\), \(v_i \in W^*\) for \(i = 1, \ldots, n\). For such a \(\varphi\), we define \(\nabla \varphi\) as

\[
\nabla \varphi(w) = \sum_{i=1}^{n} \partial_i f(<v_1, w>, \ldots, <v_n, w>) \tilde{v}_i
\]

where \(\tilde{v}_i\) is the image of \(v_i\) under the injection \(W^* \hookrightarrow H\). It follows that \(\nabla\) is a closable operator on \(L^p(\mu, X)\), \(p \geq 1\) and we will denote its closure with the same notation. The powers \(\nabla^k\) of \(\nabla\) are defined by iteration. For \(p > 1\), \(k \geq 1\), we denote by \(D_{p,k}(X)\) the completion of \(X\)-valued cylindrical functions with respect to the norm:

\[
\|\varphi\|_{D_{p,k}(X)} \equiv \|\varphi\|_{p,k} = \sum_{i=0}^{k} \|\nabla^i \varphi\|_{L^p(\mu, X^\otimes H^\otimes i)}
\]

Let us denote by \(\delta\) the formal adjoint of \(\nabla\) with respect to the Wiener measure \(\mu\) and define \(L\) as \(\delta \circ \nabla\). The well-known result of P. A. Meyer assures that the norm defined above is equivalent to

\[
\|\varphi\|_{p,k} = \|\varphi\|_{p,k} \equiv \|(I + L)^{k/2} \varphi\|_{L^p(\mu, X)},
\]

and \(L\) is called the Ornstein-Uhlenbeck operator or the number operator. Note that, due to its self adjointness, its non-integer powers are well-defined. Moreover we can also define \(D_{p,k}(X)\) for negative \(k\)’s using the second norm and we denote by \(D(X) = \cap_{p>1} \cap_{k \in \mathbb{N}} D_{p,k}(X)\) and, \(D'(X) = \cup_{p>1} \cup_{k \in \mathbb{Z}} D_{p,k}(X)\). In case \(X = \mathbb{R}\) we write simply \(D_{p,k}(\mathbb{R})\), \(D(\mathbb{R})\), \(D'(\mathbb{R})\). Let us recall that

\[
\nabla : D_{p,k}(X) \rightarrow D_{p,k-1}(X \otimes H)
\]

and

\[
\delta : D_{p,k}(X \otimes H) \rightarrow D_{p,k-1}(X)
\]

are continuous linear operators for any \(p > 1\), \(k \in \mathbb{Z}\).
2.1 Rotations

Rotation Theorem: Let $R$ be a strongly measurable random variable on $W$ with values in the space of bounded linear operators on $H$. Assume that $R$ is almost surely an isometry on $H$ (i.e., $|R(w)h|_H = |h|_H \mu$-almost surely, for any $h \in H$). Further assume that for some $p > 1$ and for all $h \in H$, $Rh \in D_{p,2}(H)$ and $\nabla Rh \in D_{p,1}(H \otimes H)$ is a quasi-nilpotent operator on $H^2$.

If moreover, either

a) for any $h \in H$,

$$(I + i \nabla Rh)^{-1} \cdot Rh \in L^q(\mu, H), \; q > 1$$

(Here $q$ may depend on $h \in H$) or,

b) $Rh \in D(H)$ for any $h \in H$,

then

$$E \left[ \exp i \delta(Rh) \right] = e^{-\frac{1}{2} |h|^2}.$$

(2.2)

Besides, for any complete, orthonormal basis $(e_i, i \geq 1)$ of $H$, the sum

$$T(w) = \sum_{i=1}^{\infty} \delta(Re_i)(w)e_i$$

converges almost surely in the strong topology of $W$, the result is almost surely independent of any particular choice of $(e_i, i \geq 1)$, consequently $T$ defines a measure preserving transformation of $W$ which is called the rotation associated to $R$.

Remark: In fact it suffices to assume (H) the above hypothesis for any $h \in H_1$, where $H_1$ is any arbitrary dense vector subspace of $H$.

\[\text{From this theorem it follows that } H, \]

$$\delta(Re_i), i \geq 1$$

are independent, identically distributed (i.i.d.) $N(0,1)$-random variables and the equation (H) defines a measure preserving transformation of $W$ thanks

\[\text{This means that } \lim_{n \to \infty} \|(\nabla Rh)^n\|_{L(H,H)}^{1/n} = 0 \text{ almost surely or, equivalently, trace } (\nabla Rh)^n = 0 \text{ almost surely, for all } n \geq 2.\]
to the Itô-Nisio theorem (cf. [1, 4]). The random isometry \( R \) satisfying the conditions for this theorem under (b) will be said to satisfy the rotation conditions. Let us remark that, to an operator \( R \) with the above properties, for any fixed \( k \in H \), it corresponds another one, satisfying the same properties, defined as \( w \rightarrow R(w + tk), t \in [0, 1] \), that we shall denote by \( R_{t,k} \).

With this notion we define a new operator as

\[
X_k^RF(w) = \frac{d}{dt}F(T_{t,k}(w))|_{t=0},
\]

where \( T_{t,k}w \) is defined as

\[
T_{t,k}w = \sum_{i=1}^{\infty} \delta(R_{t,k}e_i)(w)e_i.
\]

\( X^R \) is closable ([4]) and we have

\[
\nabla_k(F \circ T) = (R(\nabla F \circ T), k)_H + X_k^RF
\]

for any cylindrical \( F \). This operator plays an important role in the analysis of random rotations:

**Lemma A** Let \( u : W \rightarrow H \) be any cylindrical map, then one has

\[
\delta u \circ T = \delta(R(u \circ T)) + \text{trace}(RX^Ru).
\]

**Proof:** Let \((e_i, i \in \mathbb{N})\) be a complete, orthonormal basis of \( H \). We have, using the relation (2.3) and denoting \((u, e_i)_H\) by \( u_i \),

\[
\delta u \circ T = \sum_{i=1}^{\infty} \{u_i \circ T \delta(Re_i) - ((\nabla_{e_i}u_i) \circ T)\}
\]

\[
= \sum_{i=1}^{\infty} \{u_i \circ T \delta(Re_i) - (R\nabla_{u_i} \circ T, Re_i)_H\}
\]

\[
= \sum_{i=1}^{\infty} \{u_i \circ T \delta(Re_i) - (\nabla(u_i \circ T) - X^R_{u_i}, Re_i)_H\}
\]

\[
= \delta(R(u \circ T)) + \sum_{i=1}^{\infty} (X^R_{u_i}, Re_i)_H.
\]

**Remark:** Since \( \delta u \circ T \) and \( \delta(R(u \circ T)) \) are independent of the choice of \((e_i, i \in \mathbb{N})\), so does \( \text{trace}(RX^R u) \).
2.2 Traces

Let $H$ be a separable Hilbert space and let $\varphi = (\varphi_i, i \geq 1)$ be a fixed complete orthonormal basis of $H$. We will use $\varphi^{(n)} = (\varphi_1^{(n)})$ to denote the complete orthonormal basis on $H^\otimes n$ induced by $\varphi$, i.e. $\mathbf{i} = (i_1, \ldots, i_n) \in \mathbb{N}^n$, $\varphi_1^{(n)} = \varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_n}$ and the sequence $\varphi^{(n)}_1$’s are arranged in lexicographical order. For a bounded operator $A$ on $H^\otimes n$ we define the $\varphi$-trace as:

\[
\text{trace} \varphi A = \sum_{\mathbf{i}} (A\varphi_1^{(n)}, \varphi_1^{(n)})_{H^\otimes n} \tag{2.4}
\]

where the summation is in lexicographical order and provided the series converges. From now on we will delete the $\varphi$ and denote the series defined by (2.4) as trace $A$.

Let $u \in \mathbb{D}_{2,1}(H)$, the $\varphi$-Ogawa integral of $u$ is defined as

\[
\delta \varphi \circ u = \sum_i (u, \varphi_i) \delta \varphi_i
\]

provided that the series converges in $L^2$. Then \[4\], $\delta \varphi \circ u$ exists iff $\text{trace} \varphi \nabla u$ exists and then $\delta u = \delta \varphi \circ u - \text{trace} \varphi \nabla u$.

The following two lemmas will be needed later.

**Lemma B** Let $R$ satisfy the rotation conditions, and for some fixed complete orthonormal basis $\varphi = (\varphi_i, i \geq 1)$, $\mu$-almost surely

\[
\text{trace} \varphi \nabla R(w)h = 0,
\]

for any $h \in H$. Then

\[
\sum_i (\delta \varphi_i) R^* \varphi_i = \sum_i (R \varphi_i) \varphi_i.
\]

**Remark:** Note that the right hand side is independent of $\varphi$, but equality holds only if $\text{trace} \varphi \nabla Rh = 0$.

**Proof:** Let $h$ be an element of $H$, then

\[
\sum_i \delta \varphi_i(R^* \varphi_i, h) = \sum_i \delta \varphi_i(\varphi_i, Rh)_H = \delta \varphi \circ (Rh) = \delta (Rh) = \sum_i (\delta R \varphi_i) \cdot (\varphi_i, h)_H.
\]
Lemma C Let $R$ and $\varphi$ be as in the lemma above, and let $v \in \mathbb{D}(H)$, trace $\varphi \nabla v = 0$ \textit{and} trace $\varphi \nabla (R(w)v(Tw)) = 0$. Then

$$(\delta v) \circ T = \delta \left( R(v \circ T) \right).$$

If only trace $\varphi \nabla v = 0$ then

$$(\delta v) \circ T = \delta \circ \left( R(v \circ T) \right) \quad (2.5)$$

Proof: Note that

$$\delta h \circ T = \sum \delta(R\varphi_i)(\varphi_i, h)_H = \delta Rh. \quad (2.6)$$

Now, by (2.6) and Lemma A,

$$\delta v \circ T = (\delta^\varphi \circ v \circ \text{trace } \varphi \nabla v) \circ T$$

$$= (\delta^\varphi \circ v) \circ T$$

$$= \sum ((v, \varphi_i)_H \delta \varphi_i) \circ T$$

$$= \sum (v \circ T, \varphi_i)_H \delta R\varphi_i$$

$$= \sum (v \circ T, R^* \varphi_i)_H \delta \varphi_i$$

$$= \delta \circ (R(v \circ T)) \quad \text{(this proves (2.5))}$$

$$= \delta (R(v \circ T))$$

\square

3 Chaos representation and ergodic rotations

Let $(W, H, \mu)$ be an abstract Wiener space. Let $(p_\theta, \theta \in [0, 2\pi])$ be a right continuous resolution of identity on $H$ and let $\mathcal{R}$ denote the class of non random unitary operators on $H$ which are represented by it:

$$\mathcal{R} = \left\{ R : R = \int_0^{2\pi} e^{i\varphi(\theta)} dp_\theta \right\}$$
where $\varphi(\cdot)$ is real valued, right continuous on $[0,2\pi]$. Note that the elements of $R$ commute. Further assume that $\int_A dp_\theta \neq 0$ for all $\theta$ sets $A$ of positive Lebesgue measure.

Let $(M,M,P)$ be a probability space, independent of $W$. Let $(R_i(m), i \in \mathbb{N})$ be an i.i.d. sequence taking values in $R$ and $(\psi_i(\theta,m), i \geq 1)$ are $\mathcal{M}$-measurable i.i.d. continuous functions on $[0,2\pi]$. Consider the product space $(W \times M, \mathcal{B}(W) \otimes \mathcal{M}, \mu \times P)$. Set

$$R_n(m) = \int_0^{2\pi} e^{i\varphi_n(\theta,m)} dp_\theta.$$ 

Let us define $T^i(w,m)$ as

$$T^i(w,m) = \sum_{i=1}^{\infty} \delta(R_1(m)R_2(m) \ldots R_n(m)e_i) e_i.$$ 

By ergodicity we mean that for any square integrable $F$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} F(T^i(w,m)) = E[F]$$

$\mu \times P$-almost surely. Recall that from Lemma A

$$T^1(w,m) = \sum_i \delta e_i \cdot R^*_1(m)e_i$$

and

$$T^n(w,m) = \sum_i (\delta e_i) R^*_n(m) R^*_{n-1}(m) \ldots R^*_1(m)e_i.$$ 

**Theorem 1** Under the above assumptions on $R$ and $T$, the following is necessary and sufficient for the ergodicity of $T = (T^i, i \in \mathbb{N})$:

1. $\theta \rightarrow (p_\theta h, h)_H$ is continuous on $[0,2\pi]$ for all $h \in H$.

2. For all $\eta$ in $[0,2\pi]$, the inequality

$$\left|E\left[\exp i(\psi_1(\theta,m) - \eta)\right]\right| < 1$$

holds for almost all $\theta$ in $[0,2\pi]$ with respect to the Lebesgue measure.
Proof: Starting with necessity, assume that $\theta \to p_\theta$ is discontinuous at $\theta = \theta_0$. If $h$ is in the invariant subspace defined by the projection $p_{\theta_0} - p_{\theta_0 -}$, then

$$R_n(m)h = e^{i\psi_n(\theta_0, m)}h$$

for any $n \geq 1$, hence, for a.a. $m$

$$|\delta R_n h| = |\delta h|$$

and $T$ is not ergodic. Similarly, assume that

$$E[e^{i\psi_1(\theta, m)}] = e^{i\eta}$$

holds for almost all $\theta \in A$, where $A$ is a measurable subset of $[0, 2\pi]$ of positive Lebesgue measure. Then $\psi_1(\theta, m) = \eta$ almost surely on $A \times M$. Set $H_A$ to be the invariant subspace of the projection $\int_A dp_\theta$. For any $h \in H_A$ and $n \geq 1$, we have

$$\delta h = e^{-i\eta} \delta R_n h$$

and, again, this result implies that $T$ is non ergodic.

Turning now to sufficiency, assume that $T$ is not ergodic, then for some square integrable $F$,

$$F(w, m) = F(T^1(w, m)) \quad \text{a.s. } \mu \times P \quad (3.2)$$

fixing $m$ and developing $F$ in a multiple Wiener-Ito series:

$$F = \sum_{n=0}^{\infty} I_n(K_n) ,$$

the relation (3.2) yields

$$I_n(K_n^{(m)}) = I_n(R_1^{\otimes n}(m)K_n^{(m)})$$

$\mu \times P$-almost surely. Hence, $P$-almost surely

$$0 = \int_{[0, 2\pi]^n} \left| 1 - \prod_{j=1}^{n} \exp i\psi_1(\theta_j, m) \right|^2 d\left( (p_\theta_1 \otimes \ldots \otimes p_\theta_n)K_n^{(m)}, K_n^{(m)} \right)_{H^\otimes n}$$

$$= \int_{[0, 2\pi]^n} \left| 1 - \prod_{j=1}^{n} \exp i\psi_1(\theta_j, m) \right|^2 \rho(d\theta_1, \ldots, d\theta_n, m) \quad (3.3)$$
where $\rho$ is a continuous (atomless) positive measure (cf. [5]). Hence

$$0 = \int_{[0,2\pi]^n} \left| 1 - \cos \sum_{j=1}^{n} \psi_1(\theta_j, m) \right| \rho(d\theta_1, \ldots, d\theta_n, m).$$

Consequently

$$\sum_{j=1}^{n} \psi_1(\theta_j, m) = 0 \mod 2\pi \quad (3.4)$$

$P$-almost surely and the support of $\sum_{j=1}^{n} \psi_1(\theta_j, m)$ lies in a sub-manifold of $[0,2\pi]^n$ whose dimension is at most $n - 1$. Since the measure $\rho$ is continuous, it vanishes on the lower dimensional manifolds. Consequently (3.4) is impossible and $T$ is ergodic.

The following result is almost a corollary of Theorem 1:

**Theorem 2** Let $R$ be a weakly measurable random variable with values in the set of unitary operators on $H$ satisfying the rotation condition. Assume that it has a representation as

$$R = \int_{[0,2\pi]} e^{i\theta} dp_\theta(w),$$

where $(p_\theta(w), \theta \in [0,2\pi])$ is a weakly measurable resolution of identity on $H$. Assume furthermore that, for any $h \in H$,

$$\theta \rightarrow \delta p_\theta h = m_\theta(h)$$

is a martingale with respect to the filtration $(D_\theta, \theta \in [0,2\pi])$, whose predictable increasing process, denoted by $(a_\theta(h,h), \theta \in [0,2\pi])$ is deterministic, where $D_\theta$ denotes the right continuous filtration generated by $\{\delta p_\tau h, h \in H, \tau \leq \theta\}$. Then the transformation $T : W \rightarrow W$ is ergodic if and only if the vector measure defined by $\theta \rightarrow d\theta E[(p_\theta h, h)_H]$ has no atom.

**Proof:** Let us note that, since $\delta h = m_{2\pi}(h)$, the cylindrical martingale $(m_\theta, \theta \in [0,2\pi])$ has the chaotic representation property. Besides, we have

$$E[m_\theta(h)^2] = E[E[\delta h | D_\theta]^2] = E[\delta h \delta p_\theta h] = E[(h, p_\theta h)_H],$$
hence $a_\theta(h, h) = E[(p_\theta h, h)_H]$ for any $h \in H$ and $\theta \in [0, 2\pi]$. The chaotic representation property means that any square integrable random variable $F$ can be represented as

$$F = E[F] + \sum_{n=1}^\infty \int_{[0,2\pi]^n} (f_n(t_1, \ldots, t_n), dm_{t_1} \otimes \cdots \otimes dm_{t_n})_{H^\otimes n},$$

where $f_n : [0, 2\pi]^n \rightarrow H^\otimes n$ is measurable, symmetric with respect to $(t_1, \ldots, t_n)$ and

$$E[F - E[F]]^2 = \sum_{n=1}^\infty n! \int_{[0,2\pi]^n} d((a_{t_1} \otimes \cdots \otimes a_{t_n}) f_n, f_n)_{H^\otimes n}.$$

Note that

$$F \circ T = E[F] + \sum_{n=1}^\infty \int_{[0,2\pi]^n} e^{i\sum_{k=1}^n t_k} (f_n(t_1, \ldots, t_n), dm_{t_1} \otimes \cdots \otimes dm_{t_n})_{H^\otimes n},$$

hence the rest of the proof goes exactly as the proof of Theorem 1. 

4 A necessary condition for ergodicity of non-independent rotations

Let $R : W \rightarrow O(H)$ be a random unitary operator satisfying the rotation condition. Suppose that it has a representation given as

$$R(w) = \int_0^{2\pi} e^{i\psi(\theta, w)} d\theta,$$

where the random function $\psi$ takes values in the class of continuous Lebesgue measurable functions from $[0, 2\pi]$ to $[0, 2\pi]$.

**Proposition 1** Assume that $\psi(\theta, w) \in D_{2,1}(L^2([0, 2\pi], d\theta))$ and that $\nabla \psi(\theta, w)$ is orthogonal to the subspace induced by $p_{\theta} - p_{\theta_+}$ for every $\theta$. Then the following conditions

(a) $p_{\theta}$ is continuous on $[0, 2\pi]$

(b) If $A$ is nonrandom Lebesgue measurable subset of $[0, 2\pi]$ such that for some $\eta$ and for a.a. $\theta \in A$, $\psi(\theta, w) = \eta$ almost surely then the Lebesgue measure of $A$ is zero.
are necessary for the ergodicity of $T$ which is generated by $R$.

Remarks:

1. Equation (4.1) implies that $R$ and $R \circ T$ commute.

2. The requirement that $\nabla \psi(\theta, w)$ is orthogonal to $p_\theta - p_{\theta_-}$ is satisfied if $\psi(\theta, w)$ is predictable with respect to the $\sigma$-field generated by $\{\delta(p_\theta h), h \in H\}$.

3. Condition (b) is a necessary condition under (4.1) even if the orthogonality condition for $p_\theta - p_{\theta_-}$ is not satisfied.

Proof: Assume that $p_\theta$ is discontinuous at $\theta = \theta_0$. Then there exists $h \in H$ such that $(p_{\theta_0} - p_{\theta_0_-})h = h$ and then

$$
\delta(Rh) = \delta(e^{i\psi(\theta_0,w)}h) = e^{i\psi(\theta_0,w)} \delta(h)
$$

by the orthogonality assumption for $\nabla \psi$. Hence

$$
|\delta Rh| = |\delta h|
$$

and $|\delta h|$ is a nontrivial eigenfunction with eigenvalue 1, therefore $T$ can not be ergodic. Similarly, assume that for some $\eta$ and a set $A$ of positive Lebesgue measure

$$
\exp i\psi(\theta, w) = e^{i\eta}
$$

for a.a. $\theta \in A$, then for $\pi_A = \int_A d\theta p_\theta$ and $h$ invariant with respect to $\pi_A$

$$
\delta Rh = e^{i\eta} \delta h
$$

and again $|\delta h|$ is invariant hence $T$ is not ergodic.

5 A condition for mixing

In this section we give a sufficient condition for the strong mixing property of some random rotations.
**Theorem 3** Let $R$ satisfy the rotation condition, define inductively the sequence of operators $(Q_n, n \geq 1)$ as $Q_1(w) = R(w)$ and

$$Q_n(w) = R(w) \cdot R(Tw) \cdots R(T^{n-1}w)$$

for $n \geq 2$. Assume that for all $n \in \mathbb{N}$ and all $k, h \in H$

$$\delta h \circ T^n = \delta(Q_n h)$$  \hspace{1cm} (5.1)

almost surely and the random variable

$$w \to \delta(Q_n(\cdot + k)h)(w)$$  \hspace{1cm} (5.2)

has a Gaussian distribution with variance $|h|^2_H$. Then $T$ is strongly mixing if, for any $h, k \in H$,

$$\lim_{n \to \infty} (k, Q_n h)_H = 0$$  \hspace{1cm} (5.3)

in probability.

**Remark 1** Before the proof of the theorem, let us give some typical examples of situations in which the conditions (5.1) and (5.2) hold:

1. By Lemma B (Section 2), if for all $n \in \mathbb{N}$ and $h \in H$

$$\text{trace} \, \varphi \nabla Q_n h = 0$$  \hspace{1cm} (5.4)

then condition (5.1) holds.

2. If, for any $h \in H$, $\nabla Q_n h$ is quasi-nilpotent or if $Q_n h$ is adapted to the standard Wiener filtration, then the condition (5.2) holds.

3. Assume that $W = C([0, 1], \mathbb{R}^d)$ and that $\sigma : [0, 1] \times W \to O(\mathbb{R}^d)$ (orthogonal transformations of $\mathbb{R}^d$) is an optional process. Define $R$ as

$$R(w)h(t) = \int_0^t \sigma(s, w)\dot{h}(s)ds, \quad h \in H.$$  

Then the transformation $T$ defined as

$$T(w) = \sum_{i=1}^{\infty} \delta(Re_i) e_i$$

satisfies the hypothesis (5.1) and (5.2) (cf. also the example at the end of this section).
In fact to see the last claim assume that $\dot{u}$ is a $dt \times d\mu$-square integrable, smooth optional step process and let $u$ be the $H$-valued random variable whose Lebesgue density is $\dot{u}$. Then we have

$$\delta u \circ T = \sum_i (\dot{u}_{s_i}, W_{s_{i+1}} - W_{s_i}) \circ T$$

$$= \sum_i (\dot{u}_{s_i} \circ T, W_{s_{i+1}} \circ T - W_{s_i} \circ T)$$

$$= \sum_i \left( \dot{u}_{s_i} \circ T, \delta(RU_{[s_i,s_{i+1}]}) \right),$$

where $U_{[s_i,s_{i+1}]}$ denotes the image in $H$ of the indicator function of the interval $[s_i, s_{i+1}]$ under the usual injection of $L^2([0,1])$ into $H$, i.e. $f(s) \to \int_0^1 f(s) ds$ and $(W_t, t \in [0,1])$ is the $d$-dimensional Wiener process. We also have from Lemma A

$$\left( RU_{[s_i,s_{i+1}]}, \nabla(\dot{u}_{s_i} \circ T) \right)$$

$$= (RU_{[s_i,s_{i+1}]}, R\nabla\dot{u}_{s_i} \circ T) + (RU_{[s_i,s_{i+1}]}, X^R\dot{u}_{s_i})$$

$$= (U_{[s_i,s_{i+1}]}, \nabla\dot{u}_{s_i} \circ T) + (RU_{[s_i,s_{i+1}]}, X^R\dot{u}_{s_i})$$

$$= 0,$$ (5.5)

where the first term at (5.5) is zero because $\dot{u}_{s_i}$ is $\mathcal{F}_{s_i}$-measurable, hence its derivative has its support in the interval $[0, s_i]$. For the second term, it suffices to take $\dot{u}_{s_i}$ of the form $f(\delta l)$, where $f$ is a smooth function, $l \in H$ such that the support of $l$ is in $[0, s_i]$. Then we have

$$\left( RU_{[s_i,s_{i+1}]}, X^R\dot{u}_{s_i} \right) = f'(\delta Rl)(\delta Rl, RU_{[s_i,s_{i+1}]})$$

$$= f'(\delta Rl)\delta(RU_{[s_i,s_{i+1}]}, Rl)$$

and it is immediate to see that $\nabla RU_{[s_i,s_{i+1}]} Rl = 0$ because of the special form of $R$. Hence, we see that $(\delta u) \circ T = \delta(R(u \circ T))$, then the general case follows by a limiting argument.

**Proof of the theorem:** We will show that $\lim_{n \to \infty} E[F \circ T^n] = 0$ for all square integrable $F$ such that $E[F] = 0$ and this implies mixing. Since the span of the Wick exponentials is dense in $L^2(\mu)$, it suffices to show that

$$E \left[ \rho(\delta k) \rho \left( \delta(Q_n h) \right) \right] \xrightarrow{n \to \infty} 1$$ (5.6)
for all $h, k \in H$, where $\rho(\delta k) = \exp(\delta k - \frac{1}{2} |k|_H^2)$ and

$$\rho(\delta(Q_n h)) = \exp\left\{ \delta(Q_n h) - \frac{1}{2} |h|_H^2 \right\} = \rho(\delta h) \circ T^n.$$ 

Again by a density argument, it suffices to show that

$$E\left[ (\delta k)^l \rho(\delta(Q_n h)) \right] \xrightarrow{n \to \infty} E[(\delta k)^l],$$

for any $l \in \mathbb{N}$. By Theorem 3.5.4 and Corollary 3.6.1 of [4]

$$E[(\delta k)^l] = E \left[ (\delta k)^l (w + Q_n h) \rho(-\delta Q_n h) \right]$$

$$= E \left[ (\delta k + (k, Q_n h)_H)^l \cdot \rho(-\delta Q_n h) \right].$$

Since $(k, Q_n h)_H$ is bounded and converges to zero in probability

$$\lim_{n \to \infty} E \left[ (\delta k)^l \rho(-\delta(Q_n h)) \right] = \lim_{n \to \infty} E \left[ (\delta k + (k, Q_n h)_H)^l \cdot \rho(-\delta Q_n h) \right]$$

$$= E[(\delta k)^l].$$

Remark: Note that the condition (5.3) is also necessary when $R$ is a deterministic operator. More generally, if $T$ is strongly mixing in the frame of a classical Wiener space, the Ito representation theorem implies that

$$\lim_{n \to \infty} (Q_n h, k)_H = 0$$

in the weak $L^p$-topology for any $p \geq 1$.

An example for a rotation satisfying condition (5.1) (via (5.4)) is the following:

Assume that

(a) $R(w) = \int_0^{2\pi} e^{i\psi(\theta, w)} d\theta$ and

(b) $\psi(\theta, w)$ is adapted to $\mathcal{F}_\theta = \sigma\{\delta(p_\theta h), h \in H\}$, then $\psi(\theta, Tw)$ is also $\mathcal{F}_\theta$ adapted.
Now,
\[ Q_n(w) = \int_0^{2\pi} \exp \left\{ i \psi(\theta, T^j w) d\theta \right\}. \]  

(5.7)

Then
\[ \nabla Q_n(w) h = \sum_{k=1}^{n} \int_0^{2\pi} i \exp i \sum_{j=1}^{n} \psi(\theta, T^j w) \nabla \psi(\theta, T^k w) dp\theta. \]

Under suitable smoothness conditions and since \( \psi \) is adapted it holds that
\[ \nabla \psi(\theta, T^k w) \perp (p_{\theta_2} - p_{\theta_1}) h, \]  

if \( \theta_2 > \theta_1 \geq \theta, \)

hence \( \text{trace} \, \nabla Q_n(w) h = 0 \) and (5.11) also holds.

This result can be generalized as the following theorem, the proof of which goes exactly along the same lines as the proof of Theorem \( \text{3} \), hence it will be omitted:

**Theorem 4** Assume that \( (Q_n, n \geq 1) \) is a sequence of random isometries of \( H \) such that \( Q_n \) is in the domain of the divergence operator and \( \delta(Q_n h) \) is an \( N_1(0, |h|^2_H) \)-Gaussian random variable for any \( h \in H \). Assume moreover that the shift defined as \( w \to w + Q_n(w) h \) satisfies the Girsanov identity, in the sense that
\[ E \left[ F(w + Q_n(w) h) \exp \left\{ -\delta(Q_n h) - \frac{1}{2} |h|^2_H \right\} \right] = E[F] \]

for any \( F \in C_b(W) \). Denote by \( T_n \) the measure preserving transformation of \( W \), defined by \( Q_n \), i.e. \( \delta h \circ T_n = \delta(Q_n h), h \in H \). Then a sufficient condition for the strong mixing property of \( (T_n, n \geq 1) \) is that
\[ \lim_{n \to \infty} (Q_n h, k)_H = 0 \]

in probability, for any \( h, k \in H \).

Here is an application of Theorem \( \text{3} \):

**Example 1** Let \( W = C_0([0, 1], \mathbb{R}^d) \), then the Cameron-Martin space is the space of the \( \mathbb{R}^d \)-valued, absolutely continuous functions on \( [0, 1] \), with the square integrable derivatives. Assume that \( R \) is given by
\[ Rh(t) = \int_0^t R(h'(t)) dt, \]
where $R_t$ is an $\mathbb{R}^d \otimes \mathbb{R}^d$-valued, adapted process such that, for any $x \in \mathbb{R}^d$, $|R_t^* R_t x| = |x|$ almost surely. Define $T$ as to be $\delta h \circ T = \delta(Rh)$, $h \in H$. Assume that $R_t \otimes R_s$ is independent of $(R_t \otimes R_s) \circ T \ldots (R_t \otimes R_s) \circ T^{n-1}$ for any $n \geq 2$, $s < t \in [0, 1]$-ds \times dt almost surely. Assume moreover that the two point function $A_{s,t} = E[R_s \otimes R_t]$ satisfies the following:

$$\lim_{n \to \infty} (A^n_{s,t} x, y)_{\mathbb{R}^{2d}} = 0$$

almost surely for any $x, y \in \mathbb{R}^{2d}$, $s < t \in [0, 1]$. Then $T$ is strongly mixing.

Let us give another example:

**Example 2** Assume that $W = C_0([0, 1], \mathbb{R})$, with the corresponding Cameron-Martin space. Assume also that $((b_i^t, t \in [0, 1]), i \geq 1)$ is a sequence of one-dimensional Wiener processes, independent of $W$. Define $(T_n, n \geq 1)$ inductively as

$$T_1 w(t) = w_1(t) = \int_0^t \text{sign}(b^1_s) dw_s,$$

$$T_{n+1} w(t) = w_{n+1} = \int_0^t \text{sign}(b^n_s) dw_n(s)$$

and regard $T_n$ as a function of the Wiener path although it depends also on $b^1, \ldots, b^n$. Then it is a measure preserving transformation of $W$. We have

$$\delta h \circ T_n = \delta Q_n h$$

$$= \int_0^1 \text{sign}(b^n_s) \ldots \text{sign}(b^1_s) h'(s) dw_s.$$

Since $ds \times dt$-almost surely, $|E[\text{sign}(b^n_s) \text{sign}(b^i_s)]| < 1$, we have, for any $h, k \in H$, $\lim_{n \to \infty} (Q_n h, k)_H = 0$ in $L^2$, hence the sequence $(T_n, n \geq 1)$ is strongly mixing.

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