Quantum Corrections on Fuzzy Sphere

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Abstract

We investigate quantum corrections in non-commutative gauge theory on fuzzy sphere. We study translation invariant models which classically favor a single fuzzy sphere with $U(1)$ gauge group. We evaluate the effective actions up to the two loop level. We find non-vanishing quantum corrections at each order even in supersymmetric models. In particular the two loop contribution favors $U(n)$ gauge group over $U(1)$ contrary to the tree action in a deformed IIB matrix model with a Myers term. We further observe close correspondences to 2 dimensional quantum gravity.

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1 Introduction

Matrix models are a promising approach to understand non-perturbative aspects of superstring/M-theory\cite{1}\cite{2}. In this approach, space-time and matter fields may emerge from matrix degrees of freedom. A well-known prototype of such a possibility is the following non-commutative D-brane solutions in the large $N$ limit

\[ A_{1}^{cl} = \hat{p}, \quad A_{2}^{cl} = \hat{q}, \]
\[ [\hat{p}, \hat{q}] = -i. \tag{1.1} \]

Non-commutative (NC) gauge theory is obtained from matrix models around such solutions\cite{3}\cite{4}\cite{5}. In string theory, NC gauge theory on flat space is realized with constant $B_{\mu\nu}$ field\cite{6}. NC gauge theory exhibits UV-IR mixing which is a characteristic feature of string theory\cite{7}. The advantage of matrix model construction of NC gauge theory is that it maintains the manifest gauge invariance under $U(N)$ transformations

\[ A_{\mu} \rightarrow U A_{\mu} U^{\dagger}. \tag{1.2} \]

The gauge invariant observables of NC gauge theory, the Wilson lines were constructed through matrix models\cite{8}\cite{9}. In bosonic string theory, they are the emission vertex operators for the entire closed string modes\cite{10}\cite{11}.

To further elucidate the dynamics of the matrix models for superstring/M-theory, it may be useful to investigate matrix models on homogeneous spaces. Experimentally it is now very likely that we live in a homogeneous space, namely de-Sitter space. A homogeneous space is realized as $G/H$ where $G$ is a Lie group and $H$ is a closed subgroup of $G$. NC gauge theories on homogeneous spaces are realized by matrix models. In string theory, they may appear with non-constant $B_{\mu\nu}$ field\cite{12}\cite{13}. It is an interesting problem on its own to study NC gauge theories on curved manifolds.

We have formulated a general procedure to construct fuzzy homogeneous spaces $G/H$ in \cite{14}. We first consider a representation of $G$ which contains a (highest weight) state which is invariant under $H$ modulo a local gauge group. We also require that fuzzy $R^{n}$ where $n$ is the dimension of $G/H$ is realized in the local patch around such a state. We are thus restricted to symplectic manifolds. It is because the $\star$ product on such a manifold can be reduced to that of a flat manifold (Moyal product) locally by choosing the Darboux coordinates. Since the Kähler form serves as the symplectic form, Kähler manifolds such as $CP^{n}$ satisfy this requirement\cite{15}. 


We embed the Lie generators of $G$ into $N$ dimensional Hermitian matrices where $N$ is the dimension of the representation. The gauge fields on fuzzy $G/H$ are constructed as bi-local fields\cite{16}. The bi-local fields are the tensor products of the relevant representation and the complex conjugate of it. Since they are reducible, we can decompose them into the irreducible representations. They are guaranteed to form the complete basis of $N \times N$ Hermitian matrices by construction.

In this paper we investigate quantum aspects of matrix models on fuzzy $S^2$ up to the two loop level. Although $S^2$ has been chosen for simplicity, it may be possible to draw implications for generic homogeneous spaces from it. We can address such physically interesting questions as the SUSY breaking effects and correspondences to quantum gravity in a concrete setting.

Let us briefly summarize the basic facts of fuzzy $S^2$. $S^2$ is a homogeneous space since $S^2 = SU(2)/U(1)$. In SU(2), we have the Hermitian operators $\hat{j}_x, \hat{j}_y, \hat{j}_z$ which satisfy the commutation relations of the angular momentum:

$$[\hat{j}_x, \hat{j}_y] = i \hat{j}_z. \quad (1.3)$$

Contrary to $R^2$ case, such a commutation relation can be realized with finite size matrices since $S^2$ is compact. The raising and lowering operators $\hat{j}^+, \hat{j}^-$ can be formed from $\hat{j}_x, \hat{j}_y$ which satisfy

$$\hat{j}^\pm = \frac{1}{\sqrt{2}} (\hat{j}_x \pm i \hat{j}_y),$$

$$[\hat{j}^+, \hat{j}^-] = \hat{j}_z, \quad [\hat{j}_z, \hat{j}^\pm] = \pm \hat{j}^\pm. \quad (1.4)$$

Let us adopt an $N = 2l + 1$ dimensional representation of spin $l$. We further consider the semiclassical limit where $l$ is assumed to be large. For the localized states around the north pole, we can approximate $\hat{j}_z \sim l$. By rescaling the operators, we obtain the following commutation relations from (1.4),

$$\tilde{a} = \frac{1}{\sqrt{l}} \hat{j}_+, \quad \tilde{a}^\dagger = \frac{1}{\sqrt{l}} \hat{j}^-, \quad \tilde{1} = \frac{1}{l} \hat{j}_z,$$

$$[\tilde{a}, \tilde{a}^\dagger] = \tilde{1}, \quad [\tilde{1}, \tilde{a}] = \frac{1}{l} \tilde{a}, \quad [\tilde{1}, \tilde{a}^\dagger] = -\frac{1}{l} \tilde{a}^\dagger. \quad (1.5)$$

Since we obtain the identical algebra with fuzzy $R^2$, flat fuzzy plane is locally realized. Note that the radius of $S^2$ is $\sqrt{l}$ after fixing the non-commutativity scale to be 1.
The gauge fields are constructed as the bi-local fields. They are the tensor product of the spin $l$ representations

$$l \otimes l = \sum_{j=0}^{2l} j.$$

From the above decomposition, we can see that a group of representations with spins up to $2l$ form the complete basis of $N \times N$ Hermitian matrices.

The organization of this paper is as follows. We initiate our investigation of quantum effects in NC gauge theory on fuzzy sphere through matrix models in section 2. We summarize our results up to the two loop level and discuss their physical implications in section 3. We conclude in section 4 with discussions. We delegate the detailed evaluation of the two loop effective action to Appendix A.

2 Quantum effects in matrix models

NC gauge theories on compact homogeneous spaces can be constructed through matrix models. A minimal model which realizes NC gauge theories on fuzzy sphere is constructed in[17]. It is a reduced super Yang-Mills theory in 3d with a Myers term:

$$S = Tr \left( -\frac{1}{4}[A_\alpha, A_\beta][A_\alpha, A_\beta] + \frac{i}{3} f^{\alpha\beta\gamma}[A_\alpha, A_\beta]A_\gamma + \frac{1}{2} \bar{\psi} \sigma^\alpha [A_\alpha, \psi] \right),$$

(2.1)

where $A_\alpha$ and $\psi$ are $N \times N$ Hermitian matrices. $\psi$ is a two component Majorana spinor field and $\sigma^\alpha$ are Pauli matrices. The action is invariant under the following supersymmetry:

$$\delta A_\alpha = i \bar{\epsilon} \sigma_\alpha \psi,$$

$$\delta \psi = \frac{i}{2} ([A_\alpha, A_\beta] - i f_{\alpha\beta\gamma} A_\gamma) \sigma^{\alpha\beta} \epsilon.$$

(2.2)

We can also deform IIB matrix model in an analogous way [14]

$$S_{IIB} \rightarrow S_{IIB} + \frac{i}{3} f_{\mu\nu\rho} Tr[A_\mu, A_\nu]A_\rho,$$

(2.3)

where $f_{\mu\nu\rho}$ is the structure constant of a compact Lie group $G$. The perturbation term may represent non-constant $B_{\mu\nu}$ field [19]. Since there are 10 Hermitian matrices $A_\mu$ in IIB matrix model, the number of the Lie generators of $G$ cannot exceed 10 in this construction.

Although this model no longer preserves SUSY, it possesses many similarities to (2.1) when $G = SU(2)$. 5

5Although it is possible to construct a deformed IIB matrix model with SUSY when $G = SU(2)[18]$, fuzzy $S^2$ solutions break SUSY at the classical level in that model.
Since these models possess the translation invariance
\[ A_\mu \to A_\mu + c_\mu \tag{2.4} \]
and also
\[ \psi \to \psi + \epsilon \tag{2.5} \]
we remove these zero-modes by restricting \( A_\mu \) and \( \psi \) to be traceless.

The equation of motion is
\[ [A_\mu, [A_\mu, A_%\nu]] + i f_{\mu\rho\nu} [A_\mu, A_\rho] = 0 \] \tag{2.6}
The nontrivial classical solution is
\[ A^{cl}_{\alpha} = t^\alpha, \quad \text{other } A^{cl}_\mu = 0 \tag{2.7} \]
where \( t^\alpha \)'s satisfy the Lie algebra of \( G \).

We consider the following classical solution for \( G = SU(2) \) case.
\[ A^{cl}_{\alpha} = fj_\alpha \otimes 1_{n \times n}, \quad A^{cl}_i = 0 \tag{2.8} \]
where \( j_\alpha \) are angular momentum operators in the spin \( l \) representation and \( i \) denotes the orthogonal directions to the 3 dimensional space in which \( S^2 \) resides. The fixed parameters of the matrix models are \( N \) (the dimension of the matrices) and \( f \) (the coefficient of the Myers term). Since different \( U(n) \) gauge groups could be realized as long as \( N \) is divisible by \( n \), gauge groups are dynamically determined in the matrix models. The classical action associated with this solution is
\[ -\frac{f^4}{6} nl(l+1)(2l+1). \tag{2.9} \]
For a large but fixed \( N = 2l_{\text{max}} + 1 \), the irreducible representation of spin \( l_{\text{max}} \) is selected by minimizing the classical action[12].

In the large \( N \) limit, the fuzzy \( R^2 \) may be realized locally as we recalled it in the preceding section. The local momenta are introduced as \( j_\alpha = \sqrt{l} \hat{p}_\alpha \). We expand the action around the classical solution as \( A_\alpha = f \sqrt{l}(\hat{p}_\alpha + \hat{a}_\alpha), A_i = f \sqrt{l} \phi_i \). In this paragraph, \( \alpha \) denotes the tangential directions to \( R^2 \) while \( i \) denotes the transverse directions to it. Here we have fixed the non-commutativity scale to be 1. After using the Moyal-Weyl correspondence,
\[ \hat{a} \to a(x), \]
\[ \hat{ab} \to a(x) * b(x), \]
\[ Tr \to \left( \frac{1}{2\pi} \right) \int d^2 x tr, \tag{2.10} \]
we obtain the following NC gauge theory from (2.3)
\[-f^4l^2 \left( \frac{1}{2\pi} \right) \int d^2x \text{tr} \left( \frac{1}{4} [D_\alpha, D_\beta]^2 + \frac{1}{2} [D_\alpha, \phi_i]^2 + \frac{1}{4} [\phi_i, \phi_j]^2 \right. \\
\left. + \frac{1}{2} \bar{\psi} \Gamma^\alpha [D_\alpha, \psi] + \frac{1}{2} \bar{\psi} \Gamma_\gamma [\phi_\gamma, \psi] \right) \]
(2.11)

\( \text{tr} \) denotes the trace operation over \( U(n) \) gauge group. The Chern-Simons terms are suppressed by \( 1/\sqrt{l} \). In this way, we identify the coupling constant of NC gauge theory at the non-commutativity scale as
\[ g_{NC}^2 = 2\pi \left( \frac{1}{lf^2} \right)^2. \]
(2.12)
The classical action (2.9) is \( O(N) \) for a finite gauge coupling
\[ \frac{2\pi nl}{3g_{NC}^2} \sim \frac{2\pi N}{6g_{NC}^2}. \]
(2.13)
In order to obtain a finite gauge coupling \( g_{NC} \), we need to choose \( f^2 \sim 1/l \). Since we find \( N \sim O(l) \) in 2 dimensions, we need to let \( f \) vanish in the large \( N \) limit as \( f \sim O(1/\sqrt{N}) \). From (2.11), we can see that SUSY is locally recovered in this limit.

In order to obtain NC gauge theory, we expand the matrices around the classical solution:
\[ A_\alpha = f(j_\alpha + a_\alpha), \quad A_i = f \phi_i \quad (i = 1 \sim 7), \quad \psi \rightarrow f^{3/2} \psi. \]
(2.14)
Here we have separated bosonic fluctuations into vector (triplet) \( a_\alpha \) and scalar (singlet) \( \phi_i \) fields. After this procedure, the action (2.3) becomes

\[ f^4 Tr \left[ -\frac{1}{4} (L_\alpha a_\beta - L_\beta a_\alpha + [a_\alpha, a_\beta])^2 - \frac{1}{2} (L_\alpha \phi_i + [a_\alpha, \phi_i])^2 - \frac{1}{4} [\phi_i, \phi_j]^2 \right. \\
\left. + \frac{i}{2} \epsilon_{\alpha \beta \gamma} (L_\alpha a_\beta - L_\beta a_\alpha) a_\gamma + \frac{i}{3} \epsilon_{\alpha \beta \gamma} [a_\alpha, a_\beta] a_\gamma \right. \\
\left. + \frac{1}{2} \bar{\psi} \Gamma^\alpha (L_\alpha \psi + [a_\alpha, \psi]) + \frac{1}{2} \bar{\psi} \Gamma_\gamma [\phi_\gamma, \psi] \right]. \]
(2.15)
where \( L_\alpha X \equiv [j_\alpha, X] \).

After the gauge fixing, the bosonic quadratic action is simply given by
\[ \frac{f^4}{2} Tr \left( a_\alpha L^2 a_\alpha + \phi_i L^2 \phi_i \right). \]
(2.16)
\( L^2 \) is the Laplacian on fuzzy \( S^2 \) which acts on matrix spherical harmonics \( Y_{jm} \) as
\[ L^2 Y_{jm} = j(j + 1) Y_{jm}. \]
(2.17)
In our matrix model, $j$ is constrained as $1 \leq j \leq 2l$. We thus expand bosonic quantum fluctuations in terms of $2l + 1$ dimensional matrix spherical harmonics $Y_{jm}$:

$$a = \sum_{jm} Y_{jm} a_{jm}, \quad \phi = \sum_{jm} Y_{jm} \phi_{jm},$$

(2.18)

where the expansion coefficients $a_{jm}, \phi_{jm}$ are $n$ dimensional Hermitian matrices in turn.

The modes with spin $j = 0$ are zero-modes since the inverse propagators vanish for them. There are $n^2 - 1$ of them since we have removed the trace part already using the translation invariance. They are the moduli of $n$ coincident fuzzy spheres. Our strategy is to integrate massive modes first to obtain the Wilsonian effective action. Apart from the constant term which measures the free energy of the local vacuum, it is also the functional of zero-modes.

We argue that it must be a non-polynomial matrix model of $n \times n$ Hermitian matrices which govern the dynamics of the moduli fields. In this paper we compute the constant part of the effective action up to the two loop level since it is an important step to understand the dynamics of the model. We are content to briefly discuss one loop effective action for zero modes.

The quadratic action for $\psi$ is

$$\frac{1}{2} \bar{\psi} \Gamma^\alpha L_\alpha \psi.$$  

(2.19)

In the 3d model of (2.1), we can adopt

$$\Gamma^0 = \sigma^2, \quad \Gamma^1 = \sigma^3, \quad \Gamma^2 = \sigma^1.$$  

(2.20)

For Majorana-Weyl spinors in ten dimension, we can effectively factorize $\Gamma$ matrices as

$$\Gamma^\mu = \tilde{\gamma}^0 \gamma^\mu,$$  

(2.21)

where

$$\tilde{\gamma}^0 = \sigma^2 \otimes 1_8, \quad \tilde{\gamma}^9 = -\sigma^1 \otimes 1_8, \quad \tilde{\gamma}^8 = \sigma^3 \otimes 1_8,$$

$$\tilde{\gamma}^i = -1_2 \otimes \gamma^i_8 \quad (1 \leq i \leq 7),$$

(2.22)

and $\gamma^i_8$ are real and antisymmetric. With the choice of $\epsilon^{\alpha \beta \gamma} = \epsilon^{098} = 1$, the fermionic kinetic term of (2.3) boils down to that of (2.1) with the multiplicity of 8.

Let us expand $\psi$ in terms of $Y_{jm}$

$$\psi = \sum_{jm} \psi_{jm} Y_{jm}.$$  

(2.23)
Since $\psi$ carries spin $\frac{1}{2}$, the total angular momentum of $\psi_{jm}$ is either $j + \frac{1}{2}$ or $j - \frac{1}{2}$. For each state, the eigenvalue of $\sigma^\alpha L_\alpha$ is either $j$ with the multiplicity of $2j + 2$ or $-(j + 1)$ with the multiplicity of $2j$ respectively. It is because

$$\sigma^\alpha L_\alpha = (L_\alpha + \frac{\sigma^\alpha}{2})^2 - L^2 - \frac{3}{4}. \quad (2.24)$$

In our calculation of the partition function, we divide out the following gauge volume of $SU(N)/Z_N$ by gauge fixing

$$2^{N^2 + N - 1} \pi^\frac{N-1}{2} \frac{1}{\sqrt{N}} \frac{1}{\prod_{k=1}^{N-1} k!}, \quad (2.25)$$

which appeared as the universal factor in [21]. We recall that the ‘exact’ free energy of IIB matrix model is as follows in this normalization[22]

$$-\log(\sum_{n|N} \frac{1}{n^2}). \quad (2.26)$$

We have 8 real bosonic degrees of freedom after subtracting the ghost contribution. We have also 8 $SU(2)$ doublet real fermionic degrees of freedom as we have just seen. On the other hand, the degrees of freedom is one each in (2.1). Since each field is an $n$ dimensional Hermitian matrix, the one loop determinant takes the following form.

$$\left( (1 \cdot 2)^3 (2 \cdot 3)^5 \cdots (j - 1 \cdot j)^{2j-1} (j \cdot j + 1)^{2j+1} \cdots \right)^{-4n^2} \left( 1 \cdot 2^2 \cdot 2^6 \cdot 3^4 \cdots (j - 1)^{2j} \cdot j^{2j-2} \cdot j^{2j+2} \cdot (j + 1)^{2j} \cdots \right)^{4n^2}. \quad (2.27)$$

Let us focus on the multiplicity of the eigenvalue of $j$ in the above expression. In the fermionic sector, the multiplicity is $(2j + 2) + (2j - 2) = 4j$. The multiplicity in the bosonic sector is also $(2j + 1) + (2j - 1) = 4j$. Therefore the bosonic and fermionic contributions cancel in general. However we need to recall that we have the upper cut-off for $j$ as it cannot exceed $2l$. Concerning the largest possible factor $2l + 1$ in the one loop determinant, the bosonic and fermionic multiplicity is $4l + 1$ and $4l$ respectively. Since they do not match, we find the following constant term in the effective action

$$4n^2 \log(2l + 1). \quad (2.28)$$

At one loop level, we can obtain the effective action with a generic background $p_\mu$ in a closed form as

$$\frac{1}{2} Tr \log(P^2 \delta_{\mu\nu} - 2i F_{\mu\nu} - 2i e_{\mu\nu\rho} P^\rho) - Tr \log(P^2)$$

$$-\frac{1}{4} Tr \log \left( (P^2 + i 2 F_{\mu\nu} \Gamma^{\mu\nu}) (\frac{1 + \Gamma_{11}}{2}) \right), \quad (2.29)$$
where

\[ [p_\mu, X] = P_\mu X, \]
\[ [f_{\mu\nu}, X] = F_{\mu\nu} X, \quad f_{\mu\nu} = i[p_\mu, p_\nu]. \] (2.30)

With our choice of \( p_\mu = j_\mu \), we find \( F_{\mu\nu} = -\epsilon_{\mu\nu\rho} L^\rho \). In order to evaluate the contributions from large eigenvalues, we may expand this expression into the power series of \( F_{\mu\nu} \). The leading correction is found to be

\[ 2Tr \frac{1}{L^2} = 2n^2 \sum_{j=1}^{2l} \frac{(2j + 1) - 1}{j(j + 1)} \sim 4n^2 \log(2l). \] (2.31)

We find that it is consistent with the constant term (2.28).

The one loop quantum correction in the model of (2.1) is precisely \( 1/8 \) of this result. We thus find non-vanishing one loop quantum corrections even in a supersymmetric matrix model. Since SUSY transformation of \( \psi \) (2.2) vanishes in these backgrounds, one might expect that fuzzy spheres do not receive quantum corrections. We observe that this expectation failed because the bosonic and fermionic degrees of freedom do not match at the upper cut-off. This miss-match can be further traced to the fact that bosonic fields \( a_\mu \) are in the integer and the fermionic fields \( \psi \) are in the half integer representation of \( SU(2) \).

This fact might sound contradictory to the presence of SUSY transformation (2.2) in this model. However we can argue that it must be broken since there cannot be unbroken supersymmetry in de-Sitter space[20]. Since there is no positive conserved energy in de-Sitter space, there is no supercharge \( Q \) whose square equals the Hamiltonian. Since the Euclidean continuation of an \( n \)-dimensional de-Sitter space \( dS_n \) is an \( n \) sphere \( S^n \), our conclusion follows. In other words, it is a thermal SUSY breaking effect due to a de-Sitter temperature \( O(1/l) \). We further note that KMS conditions are satisfied since ‘spin-statistics’ relations are respected as we just mentioned.

Let us reflect on the one loop free energy from quantum gravity point of view[23][24]. The matter contribution to the geometric entropy in 2d supergravity can be obtained through conformal anomaly on \( S^2 \):

\[ \frac{\delta S}{\delta \varphi} = \frac{c}{4}, \] (2.32)

where \( c \) denotes the matter central charge. Semiclassically we can integrate this equation as

\[ -S = -\frac{c}{4} \log(A), \] (2.33)

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where $A$ is the ratio of short and long distance cut-offs.

On the other hand we may rewrite (2.31) as

$$-4n^2 \log(2l) + 8n^2 \log(2l) \quad (2.34)$$

The second term in this expression can be subtracted by renormalizing the cosmological constant term. It is because we can attribute it to the fermion mass term. The first term is the remaining conformal anomaly (geometric entropy). With the identification $A = (2l)^2$ and $c = 8n^2$, we confirm that the one loop correction in NC gauge theory on $S^2$ can be understood from 2d quantum gravity point of view.

In the case of matrix quantum mechanics, the situation is different since we have the time like dimension in addition. Although we have analogous difficulties to construct a model with time independent SUSY, NC gauge theories on fuzzy spheres with time dependent SUSY are constructed in the context of M-theory on a pp-wave in [25]. The fully supersymmetric solutions of the model are fuzzy spheres represented by $N$ dimensional (reducible) representations of $SU(2)$. Since the Hamiltonian can be related to the squares of the supercharges, the vacuum energy vanishes for all such configurations[26].

Although we are mostly content to compute the constant term of the effective action in this paper, it also depends on the zero modes. At the one loop level, we can estimate the effective action for bosonic zero-modes $\bar{a}$ by putting $p_\mu = j_\mu + \bar{a}_\mu$ in (2.29). The leading term is

$$-2Tr\left[ \frac{1}{P_2^\alpha} P_\alpha \frac{1}{P_2^\alpha} P_\alpha \right] - 2iTr\left[ \frac{1}{P_2^\alpha} [P_\alpha, P_\beta] \frac{1}{P_2^\gamma} \varepsilon^{\alpha\beta\gamma} P_\gamma \right]$$

$$= 4n^2 \log(2l + 1) + 4tr\bar{a}^\alpha \bar{a}^{\alpha} - 8tr\bar{\phi}_i \bar{\phi}_i + \cdots. \quad (2.35)$$

We find that the coincident $n$ spheres are unstable since $\bar{\phi}_i$ become tachyonic at this order. When two fuzzy spheres are separated by a large distance $\phi$, there is a logarithmic repulsion between the two

$$8\log\left( \frac{2l}{\phi} \right). \quad (2.36)$$

Such an instability is absent in 3d model of (2.1) since there are no $\phi_i$ fields.

### 3 Two loop effective action

In this section we summarize our two loop effective actions and discuss their physical implications. We delegate the detailed evaluation of them to Appendix A.

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6We recall that the minimally coupled fermionic kinetic term on $S^2$ is $\sigma \cdot L + 1$. 

9
In the case of $U(1)$ gauge group, the two loop effective action is
\[
F(l) = -36 \frac{1}{f^4} (F_3^p(l) - F_3^{np}(l)).
\] (3.1)

In the case of 3d model, we obtain
\[
F_{3d}(l) = - \frac{1}{f^4} (F_3^p(l) - F_3^{np}(l)),
\] (3.2)

where $F_3^p(l)$ and $F_3^{np}(l)$ are defined in (A.24). We have found numerically that
\[
F_3^p(l) \sim F_3^{np}(l) \sim 2 \frac{1}{N},
\]
\[
F_3^p(l) - F_3^{np}(l) \sim 6 \frac{1}{N^2}.
\] (3.3)

Therefore we find
\[
F_{3d}(l) = -6 \frac{2}{f^4 N^2} = -1.5 \frac{g_{NC}^2}{2\pi}.
\] (3.4)

Since it is $O(1)$ for fixed $g_{NC}^2$, it is much smaller than the tree $O(N)$ or one loop $O(\log(N))$ contributions. It is because the amplitude is convergent in our model due to SUSY and the theory becomes free (ordinary $U(1)$ gauge theory with adjoint matter) in the infrared limit.

The situation is different for $U(n)$ gauge group. For $U(n)$ gauge group, we obtain
\[
F(l) = 36 \frac{1}{f^4} \left( n^3 (-F_3^p(l) + \frac{5}{4} F_5(l)) - n (-F_3^{np}(l) + \frac{5}{4} F_5(l)) \right),
\] (3.5)

where $F_5(l)$ is defined in (A.41). We have numerically found that
\[
F(l) \sim -27 \frac{1}{f^4 (2l + 1)} n(n^2 - 1) + O(1/N^2).
\] (3.6)

The total effective action up to the two loop to the leading order of $1/N$ is
\[
-\frac{f^4}{6} n(l + 1)(2l + 1) + 4n^2 \log(2l + 1) - 27 \frac{1}{f^4 (2l + 1)} n(n^2 - 1).
\] (3.7)

If we put $f^4 = 2\pi n/l^2 \lambda^2$, the effective action becomes
\[
n^2 V \left( -\frac{2\pi}{6\lambda^2} + 4\log(V)/V - \frac{27}{8\pi} \lambda^2 (1 - 1/n^2) \right).
\] (3.8)

In this form, we can see that the effective action is proportional to the volume $V = 2l + 1$ and $n^2$. It also possesses the standard $1/n$ expansion where $\lambda^2 = ng_{NC}^2$ is the 't Hooft coupling.

However in a finite matrix model, we may fix $N$ and $f$ not $V = N/n$ and $\lambda^2$. In this case a gauge group $U(n)$ is dynamically determined by minimizing the effective action. So we need to minimize the following expression with respect to $n$
\[
-\frac{f^4}{24n^2} N^3 + 4n^2 \log(N/n) - 27 \frac{1}{f^4 N} n^2 (n^2 - 1).
\] (3.9)
For $U(n)$ gauge group, the tree and the two loop contributions are comparable when $f^4 N^2 \sim g_{NC}^2 \sim 1$. Although the tree action is minimized when $n = 1$, we observe that the two loop contribution could easily overwhelm it if we increase $n$. Therefore we conclude that the classically favorable NC $U(1)$ gauge theory on fuzzy sphere is unstable at the two loop level in a deformed IIB matrix model with a Myers term. Since this model contains 8 transverse degrees of freedom, this instability may correspond to the $c = 1$ barrier in 2d quantum gravity.

In the case of the 3d model, the corresponding expression is

$$F_{3d}(l) = \frac{1}{f^4} \left( n^3 (-F^p_3(l) + 3F_5(l)) - n(-F^{up}_3(l) + 3F_5(l)) \right).$$

We find numerically

$$F_{3d}(l) \sim \frac{1}{f^4(2l + 1)} n(n^2 - 1) + O(1/N^2).$$

The total effective action up to the two loop to the leading order of $1/N$ is

$$-\frac{f^4}{6} nl(l + 1)(2l + 1) + \frac{1}{2} n^2 \log(2l + 1) + \frac{1}{f^4(2l + 1)} n(n^2 - 1).$$

We find again that the two loop correction is $O(N)$ for finite $g_{NC}^2$ which is the same order with the tree action. As it is recalled shortly, the situation is very different from gauge theory in flat space where SUSY cancellations take place irrespectively of the gauge group. Since SUSY transformation of $\psi$ field vanishes classically for any gauge group $U(n)$, the existence of quantum corrections signals SUSY breaking effects on $S^2$. However the two loop contributions for $U(n)$ gauge groups are positive in sign and there is no instability of $U(1)$ gauge theory in this model. The stability of the model is consistent with 2d quantum gravity since it naively corresponds to $c = 1$ supergravity.

**commutative sphere limit**

In this subsection, we show that NC gauge theory on fuzzy sphere reduces to ordinary gauge theory on $S^2$ in the formal semiclassical (or infrared) limit. This correspondence implies

$$Tr \to V \int \frac{d\Omega}{4\pi} tr,$$

where $d\Omega = d\cos(\theta)d\varphi$. $V = N/n$ is the volume of $S^2$ and $tr$ is over $U(n)$ gauge group. The action (2.15) gives the following result in the semiclassical limit:

$$\frac{V f^4}{4\pi} \int d\Omega tr \left( -\frac{1}{4}(L_\alpha a_\beta - L_\beta a_\alpha + [a_\alpha, a_\beta])^2 - \frac{1}{2}(L_\alpha \phi_i + [a_\alpha, \phi_i])^2 \right)$$
\[-\frac{1}{4} [\phi_i, \phi_j]^2 + \frac{i}{2} \epsilon_{\alpha\beta\gamma} (L_\alpha a_\beta - L_\beta a_\alpha) a_\gamma + \frac{i}{3} \epsilon_{\alpha\beta\gamma} [a_\alpha, a_\beta] a_\gamma \]
\[+ \frac{1}{2} \bar{\psi} \gamma^\alpha (L_\alpha \psi + [a_\alpha, \psi]) + \frac{1}{2} \bar{\psi} \gamma^\phi [\phi_i, \psi]. \tag{3.14} \]

They are nothing but the ordinary gauge theory action on $S^2$. The gauge coupling $g_{CM}^2$ is identified with
\[g_{CM}^2 = \frac{2\pi}{f^4} = lg_{NC}^2. \tag{3.15} \]

It is larger than $g_{NC}^2$ in (2.12) by a factor $l$. It is because the gauge coupling is scale dependent in 2 dimension. The factor $l$ can be understood if we recall that the radius of the sphere is $O(\sqrt{l})$ with respect to the non-commutativity scale and $g_{CM}^2$ is the coupling constant at the infrared cut-off scale.

This correspondence can be proven at the Feynman rules level. In the perturbative investigation of ordinary gauge theory on $S^2$, we expand the fields in terms of the spherical harmonics
\[\phi = \frac{1}{\sqrt{V}} \sum_{jm} \phi_{jm} Y_{jm}(\theta, \varphi), \tag{3.16} \]

where
\[\int \frac{d\Omega}{4\pi} Y_{j_1 m_1}(\theta, \varphi) Y_{j_2 m_2}(\theta, \varphi) = \delta_{j_1, j_2} \delta_{m_1, -m_2} (-1)^{m_1}. \tag{3.17} \]

The propagators are
\[< \phi_{j_1 m_1} \phi_{j_2 m_2} > = \frac{1}{f^4 j_1 (j_1 + 1)} \frac{1}{\sqrt{V}} \delta_{j_1, j_2} \delta_{m_1, -m_2} (-1)^{m_1}. \tag{3.18} \]

The three point vertices are given in terms of $3j$ symbols since
\[\frac{f^4}{4\pi \sqrt{V}} \int d\Omega Y_{j_1 m_1}(\theta, \varphi) Y_{j_2 m_2}(\theta, \varphi) Y_{j_3 m_3}(\theta, \varphi) = f^4 \sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)} \]
\[\times \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right) \frac{1}{\sqrt{V}} \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{array} \right). \tag{3.19} \]

The Feynman rules of NC gauge theory on fuzzy sphere can be found in Appendix A. We can see that the propagators are identical in the both cases as it is well known. On the other hand, the interaction vertices of NC gauge theory contain $6j$ symbols. In the large $l$ limit, they can be related to $3j$ symbols:
\[\left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ l & l & l \end{array} \right\} \rightarrow \frac{1}{\sqrt{2l}} \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{array} \right). \tag{3.20} \]

with $j_1, j_2, j_3$ being fixed. Hence the interaction vertices of NC gauge theory (A.4) reduces to those of ordinary gauge theory expression (3.19) as $V \sim 2l$. We can thus observe that the former reduces to the latter in the formal semi-classical limit at the Feynman rules level.
**U(1) Yang-Mills theory on non-commutative \( R^2 \)**

In this section, we have investigated the 2-loop effective action of the matrix models on fuzzy sphere. In the remainder we compare the results with the 2-loop 1PI contribution to vacuum energy in U(1) Yang-Mills theory on fuzzy \( R^2 \).

We quote the action:

\[
S = \frac{1}{\gNC^2} \int d^2x \mathcal{L},
\]

\[
\mathcal{L} = -\frac{1}{2} A^\mu \partial^2 A_\mu - b \partial^2 c - \frac{i}{2} tr \bar{\Psi} \Gamma^\mu \partial_\mu \Psi
\]

\[
+ i \partial_\mu b \ast [c, A^\mu] - \frac{1}{2} tr \bar{\Psi} \ast \Gamma^\mu [A_\mu, \Psi]_\ast
\]

\[
- \frac{i}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu) \ast [A_\mu, A^\nu]_\ast - \frac{1}{4} [A_\mu, A_\nu]_\ast [A_\mu, A^\nu]_\ast.
\]  

(3.21)

One can obtain this action locally from the matrix models as it is explained in the preceding section after the identification of \( \gNC^2 = 2\pi/f^4l^2 \). In this theory vacuum energy contributions from the 2-loop 1PI graphs can be evaluated as follows:

- **2-loop 1PI from 4-gauge boson vertex:**

  \[
  -45 \gNC^2 V \cdot \int \frac{d^2p}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} \frac{1}{p^2q^2} \frac{1}{p^2q^2} (1 - \cos(p \wedge q)),
  \]  

  (3.22)

  where \( V \) denotes the volume of \( R^2 \).

- **2-loop 1PI from 3-gauge boson vertices:**

  \[
  9 \gNC^2 V \cdot \int \frac{d^2p}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} \left( \frac{1}{p^2q^2} - \frac{p \cdot q}{p^2q^2(p + q)^2} \right) (1 - \cos(p \wedge q)).
  \]  

  (3.23)

- **2-loop 1PI from ghost-gauge vertices:**

  \[
  \gNC^2 V \cdot \int \frac{d^2p}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} \frac{p \cdot q}{p^2q^2(p + q)^2} (1 - \cos(p \wedge q)).
  \]  

  (3.24)

- **2-loop 1PI from fermion-gauge vertices:**

  \[
  -64 \gNC^2 V \cdot \int \frac{d^2p}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} \frac{p \cdot q}{p^2q^2(p + q)^2} (1 - \cos(p \wedge q)).
  \]  

  (3.25)

The total amplitude is

\[
-36 \gNC^2 V \cdot \int \frac{d^2p}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} \frac{1}{p^2q^2} (1 - \cos(p \wedge q))
\]

\[
-72 \gNC^2 V \cdot \int \frac{d^2p}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} \frac{p \cdot q}{p^2q^2(p + q)^2} (1 - \cos(p \wedge q)).
\]

(3.26)
The first and the second term cancel each other since
\[ \frac{p \cdot q}{p^2 q^2 (p + q)^2} = \frac{1}{2} \left( \frac{1}{p^2 q^2} - \frac{1}{q^2 (p + q)^2} - \frac{1}{p^2 (p + q)^2} \right). \tag{3.27} \]

By comparing these amplitudes with those in Appendix A, we can find the following correspondence between the amplitudes of NC $U(1)$ gauge theory on $R^2$ and $S^2$:
\[ \int \frac{d^2 p}{(2\pi)} \int \frac{d^2 q}{(2\pi)} \frac{1}{p^2 q^2} (1 - \cos(p \wedge q)) \Leftrightarrow \frac{l}{2} (F^p_1(l) - F^{np}_1(l)), \]
\[ \int \frac{d^2 p}{(2\pi)} \int \frac{d^2 q}{(2\pi)} \frac{p \cdot q}{p^2 q^2 (p + q)^2} (1 - \cos(p \wedge q)) \Leftrightarrow \frac{l}{2} (F^p_2(l) - F^{np}_2(l)), \tag{3.28} \]
where $F^{p(np)}_1(l)$ and $F^{p(np)}_2(l)$ are defined in (A.16)

Non-planar phases: $\cos(p \wedge q)$ in NC gauge theories on $R^2$ correspond to $(-1)^{(j_1+\ldots)}$ in those on $S^2$. We point out here that the $6j$ symbols vanish unless the non-planar phases are trivial in the semiclassical (infrared) limit. Therefore the non-commutativity plays no role in the infrared limit in both theories. We further observe the identical power counting rules in both theories. Namely logarithmically divergent amplitudes and convergent amplitudes correspond to each other.

The amplitudes of NC gauge theory on fuzzy $S^2$, however, are different from those on $R^2$ in the following points:

- In NC gauge theory on fuzzy $R^2$, the cancellation of 2-loop effective action occurs in the planer and non-planer sectors separately. In NC gauge theory on fuzzy $S^2$, the corresponding cancellation does not occur if we consider planar or non-planar contributions separately.

- In NC gauge theory on fuzzy $S^2$, the fermionic contribution gives rise to extra terms represented by $F^p_3$ and $F^{np}_3$. They result in the non-vanishing effective action even for $U(1)$ gauge theory. The origin of this new term may be thought as the spin connection of fermions in a curved space.

### 4 Conclusions and Discussions

In this paper, we have investigated quantum corrections in NC gauge theory on fuzzy sphere up to the two loop level. Such theories are realized by matrix models with a Myers term. The classical solutions of the matrix models are reducible representations of $SU(2)$ which
represent a group of fuzzy spheres. The irreducible representation which represents a single fuzzy sphere minimizes the classical action. These backgrounds are supersymmetric in the sense that the SUSY transformation of $\psi$ field classically vanishes.

We have found that the quantum corrections do not vanish at each order. At the one loop level, it does not vanish due to the presence of the cut-off for the angular momentum. Due to the mismatch between the bosonic and fermionic degrees of freedom at the largest angular momentum, we find $O(\log(N))$ one loop contribution. At the two loop level, we find comparable quantum corrections to the tree action with finite gauge coupling $g_{NC}$ in the case of $U(n)$ gauges group. It is because SUSY cancellations do not take place in planar and non-planar sectors separately contrary to flat theory. In a deformed IIB matrix model with a Myers term, we find this effect destabilize the classically favorable $U(1)$ gauge theory.

Our results are consistent with the fact that there cannot be unbroken supersymmetry in de-Sitter space since the Euclidean continuation of an $n$-dimensional de-Sitter space $dS_n$ is an $n$ sphere $S^n$. We may put forward a generic argument as follows. In any field theory on a homogeneous space $G/H$, we need to identify the Hamiltonian with one of the Killing vectors which form the Lie algebra of $G$. Since there is no positive Killing vectors in this case, there is no positive conserved energy. It in turn implies that there is no supercharge $Q$ whose square equals the Hamiltonian. Consequently there cannot be unbroken supersymmetry in compact homogeneous spaces.

One of the most important questions concerning NC gauge theory is its relation to quantum gravity. Various similarities have been pointed out such as the absence of local gauge invariant operators, UV-IR mixing and Wilson lines which couple to closed strings. In this paper we have found that NC gauge theory on $S^2$ allows 2d quantum gravitational interpretation. We have reproduced the semiclassical free energy of 2d supergravity. We have further seen a possible signal for $c = 1$ barrier. It is therefore tempting to conjecture that NC gauge theory on $S^2$ belongs to the same universality class of 2d quantum gravity.

Since we can locally recover flat theory with SUSY in these models, the SUSY breaking effect must come from those degrees of freedom which is sensitive to the curvature. They are either infra-red (IR) or ultra-violet (UV) degrees of freedom. Since the two loop amplitudes are finite due to SUSY in our case, we are mostly sensitive to IR degrees of freedom. In the case of $U(1)$ gauge theory, we find little quantum corrections since it becomes free theory in IR regime.

In the case of higher dimensional spaces like $CP^2$ or $S^2 \times S^2$, the effective actions are no
longer convergent in the sense that they are dominated by IR contributions. In that case, we expect that UV degrees of freedom also contribute to the SUSY breaking effect. It will be interesting to generalize our investigations to higher dimensional homogeneous spaces. It is also interesting to explore the relevance of these spaces to IIB matrix model in the spirit of [28][29].

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Appendix A

In this appendix, we evaluate the two loop effective action of NC gauge theory on fuzzy sphere. We consider NC gauge theory with $U(1)$ gauge group in the context of a deformed IIB matrix model with a Myers term. We also explain how to modify our results in the case of the 3d model or different gauge groups.

There are 5 diagrams to evaluate which are illustrated in Figure 1. (a),(b) and (c) represent contributions from gauge fields. (a) and (b) are of different topology while (c) involves the Myers type interaction. (d) involves ghost and (e) fermions respectively.

![Figure 1: Feynman diagrams of 2 Loop corrections to the effective action](image)

We expand matrices in terms of matrix spherical harmonics:

$$A_\mu = p_\mu + \sum_{jm} a^\mu_{jm} Y_{jm},$$
\[ \psi = \sum_{jm} \psi_{jm} Y_{jm}, \]  
(A.1)

where \( p_{\alpha} = j_{\alpha} \) and other \( p_{\mu}'s = 0 \). We adopt the following representation of \( Y_{jm} \):

\[ (Y_{jm})_{s's'} = (-1)^{l-s} \left( \begin{array}{ccc} l & j & l \\ -s & m & s' \end{array} \right) \sqrt{2j+1}. \]  
(A.2)

where they are normalized as

\[ \text{Tr} Y_{j_1m_1} Y_{j_2m_2} = (-1)^{m_1} \delta_{j_1,j_1} \delta_{m_1,-m_2}. \]  
(A.3)

The cubic couplings of the matrix spherical harmonics can be evaluated as

\[ Tr[Y_{j_1m_1} Y_{j_2m_2} Y_{j_3m_3}] \\
= (-1)^{2l} \sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)} \times \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right) \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ l & l & l \end{array} \right). \]  
(A.4)

We refer to [27] for (3j) and \{6j\} symbols.

**bosonic propagators**

From the quadratic terms in the gauge fixed action, we can read propagators of gauge boson modes \( a_{jm}^\mu \) and ghost modes \( b_{jm} \), \( c_{jm} \) as follows

\[ \langle a_{j_1m_1}^\mu a_{j_2m_2}^\nu \rangle = \frac{1}{f^4} \frac{(-1)^{m_1}}{j_1(j_1+1)} \delta_{\mu\nu} \delta_{j_1j_2} \delta_{m_1,-m_2}, \]  
(A.5)

\[ \langle c_{j_1m_1} b_{j_2m_2} \rangle = \frac{1}{f^4} \frac{(-1)^{m_1}}{j_1(j_1+1)} \delta_{j_1j_2} \delta_{m_1,-m_2}. \]  
(A.5)

In terms of fields

\[ a_{st}^\mu = \sum_{jm} a_{jm}^\mu (Y_{jm})_{st}, \]

\[ b_{st} = \sum_{jm} b_{jm} (Y_{jm})_{st}, \]

\[ c_{st} = \sum_{jm} c_{jm} (Y_{jm})_{st}, \]  
(A.6)

propagators become

\[ \langle a_{st}^\mu a_{uv}^\nu \rangle = \frac{1}{f^4} \sum_{jm} \frac{(-1)^{m}}{j_1(j_1+1)} \delta_{\mu\nu} (Y_{jm})_{st} (Y_{j-m})_{uv}, \]

\[ \langle c_{st} b_{uv} \rangle = \frac{1}{f^4} \sum_{jm} \frac{(-1)^{m}}{j_1(j_1+1)} (Y_{jm})_{st} (Y_{j-m})_{uv}. \]  
(A.7)
contribution from 3-gauge boson vertices (b)

Firstly we calculate 2-loop 1PI contribution from 3-gauge boson vertices:

\[
V_3 = \frac{f^4}{2} \text{Tr} [p_\mu, a_\nu][a_\mu, a_\nu] - \frac{f^4}{2} \text{Tr} [p_\nu, a_\mu][a_\mu, a_\nu].
\] (A.8)

There are 3! = 6 type contractions following Wick’s theorem in the calculation of \( \frac{1}{2} V_3 V_3 \). The result is that:

\[
\frac{10 - 1}{2} \frac{f^4}{2} \sum_{j_1j_2j_3=1}^{2l} \sum_{m_1m_2m_3} (-1)^{m_1+m_2+m_3} j_1(j_1+1)j_2(j_2+1)j_3(j_3+1)
\times \text{Tr} ([p_\mu, Y_{j_1m_1}][Y_{j_2m_2}, Y_{j_3m_3}])
\times \left( \text{Tr} ([p_\mu, Y_{j_1-1m_1}][Y_{j_2-m_2}, Y_{j_3-m_3}]) + \text{Tr} ([p_\mu, Y_{j_2-m_2}][Y_{j_1-m_1}, Y_{j_3-m_3}]) \right).
\] (A.9)

3j-, 6j- symbol expression of the results

We can further evaluate the matrix representation (A.9) in terms of 3j and 6j symbols. The adjoint \( P_\alpha \equiv [p_\alpha, \ ] \) act on \( Y_{jm} \) as

\[
\begin{align*}
[p_+, Y_{jm}] &= \sqrt{(j-m)(j+m+1)} Y_{jm+1}, & p_+ &= p_1 + ip_2, \\
[p-, Y_{jm}] &= \sqrt{(j+m)(j-m+1)} Y_{jm-1}, & p_- &= p_1 - ip_2, \\
[p_3, Y_{jm}] &= m Y_{jm}.
\end{align*}
\] (A.10)

Using this formula and the expression:

\[
\text{Tr} (Y_{j_1m_1}[Y_{j_2m_2}, Y_{j_3m_3}])
= (1 - (-1)^{j_1+j_2+j_3})(-1)^{2l} \sqrt{(2j_1+1)(2j_2+1)(2j_3+1)} \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right) \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ l & l & l \end{array} \right\}^2,
\] (A.11)

one can obtain

\[
\begin{align*}
9 \cdot \frac{1}{f^4} \sum_{j_1j_2j_3=1}^{2l} \sum_{m_1m_2m_3} (1 - (-1)^{j_1+j_2+j_3}) & \left( \frac{(2j_1+1)(2j_2+1)(2j_3+1)}{j_2(j_2+1)j_3(j_3+1)} \right) \\
\times & \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right)^2 \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ l & l & l \end{array} \right\}^2 \\
+ 9 \cdot \frac{1}{f^4} \sum_{j_1j_2j_3=1}^{2l} \sum_{m_1m_2m_3} (1 - (-1)^{j_1+j_2+j_3}) & \left( \frac{(2j_1+1)(2j_2+1)(2j_3+1)}{j_1(j_1+1)j_2(j_2+1)j_3(j_3+1)} \right) \\
\times & \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right)^2 \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ l & l & l \end{array} \right\}^2 \\
\times & \left[ -\frac{1}{2} \sqrt{(j_1-m_1)(j_1+m_1+1)(j_2+m_2)(j_2-m_2+1)} \right]
\end{align*}
\]
\[ \times \left( \frac{j_1}{m_1+1} \frac{j_2}{m_2} \frac{j_3}{m_3} \right) \left( \frac{j_1}{-m_1-1} \frac{j_2}{-m_1} \frac{j_3}{-m_3} \right) \]
\[-\frac{1}{2} \sqrt{(j_1 + m_1)(j_1 - m_1 + 1)(j_2 - m_2)(j_2 + m_2 + 1)} \]
\[ \times \left( \frac{j_1}{m_1 - 1} \frac{j_2}{m_2} \frac{j_3}{m_3} \right) \left( \frac{j_1}{-m_2} \frac{j_2}{-m_1} \frac{j_3}{-m_3} \right) \]
\[-m_1 m_2 \left( \frac{j_1}{m_1} \frac{j_2}{m_2} \frac{j_3}{m_3} \right) \left( \frac{j_1}{-m_2} \frac{j_2}{-m_1} \frac{j_3}{-m_3} \right) \]
\[ (A.12) \]

In the first line, one can use the property of 3j-symbol:
\[ \sum_{m_3} \sum_{m_1 m_2} \left( \frac{j_1}{m_1} \frac{j_2}{m_2} \frac{j_3}{m_3} \right)^2 = \sum_{m_3} \frac{1}{2j_3 + 1} = 1, \] (A.13)
and the property of 6j-symbol:
\[ \sum_{j_1=1}^{2l} (2j_1 + 1) \left\{ \frac{j_1}{l} \frac{j_2}{l} \frac{j_3}{l} \right\}^2 = \frac{1}{2l + 1} - \frac{1}{(2l + 1)(2j_2 + 1)} \delta_{j_2,j_3}, \]
\[ \sum_{j_1=1}^{2l} (-1)^{j_1} (2j_1 + 1) \left\{ \frac{j_1}{l} \frac{j_2}{l} \frac{j_3}{l} \right\}^2 = \left\{ \frac{l}{l} \frac{l}{l} \frac{j_2}{j_3} \right\} - \frac{1}{(2l + 1)(2j_2 + 1)} \delta_{j_2,j_3}. \] (A.14)

Here we need to recall that SU(2) singlet states are absent in the gluon propagator which results in the extra terms on the right-hand side of (A.14). Although they cancel each other for U(1) case, it is not the case for U(n) gauge groups.

In this way one can rewrite (A.9) as
\[ \frac{9}{f^3} \left( F_1^p(l) - F_1^{np}(l) - F_2^p(l) + F_2^{np}(l) \right), \] (A.15)

where
\[ F_1^p(l) = \frac{1}{2l + 1} \sum_{j_1j_2=1}^{2l} \frac{(2j_1 + 1)(2j_2 + 1)}{j_1(j_1 + 1)j_2(j_2 + 1)}, \]
\[ F_1^{np}(l) = \sum_{j_1j_2=1}^{2l} (-1)^{j_1+j_2} \frac{(2j_1 + 1)(2j_2 + 1)}{j_1(j_1 + 1)j_2(j_2 + 1)} \left\{ \frac{l}{l} \frac{l}{l} \frac{j_2}{j_1} \right\}, \]
\[ F_2^p(l) = \sum_{j_1j_2j_3=1}^{2l} \sum_{m_1 m_2 m_3} \frac{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)}{j_1(j_1 + 1)j_2(j_2 + 1)j_3(j_3 + 1)} \left\{ \frac{j_1}{l} \frac{j_2}{l} \frac{j_3}{l} \right\}^2 \]
\[ \times \left[ \frac{1}{2} \sqrt{(j_1 - m_1)(j_1 + m_1 + 1)(j_2 + m_2)(j_2 - m_2 + 1)} \left( \frac{j_1}{m_1 + 1} \frac{j_2}{m_2} \frac{j_3}{m_3} \right) \left( \frac{j_1}{m_1} \frac{j_2}{m_2 + 1} \frac{j_3}{m_3} \right) \right] \]
\[ + \frac{1}{2} \sqrt{(j_1 + m_1)(j_1 - m_1 + 1)(j_2 - m_2)(j_2 + m_2 + 1)} \left( \frac{j_1}{m_1 - 1} \frac{j_2}{m_2} \frac{j_3}{m_3} \right) \left( \frac{j_1}{m_1} \frac{j_2}{m_2 - 1} \frac{j_3}{m_3} \right) \]
The interaction vertex which involves 4 gauge bosons is

\[ \text{contribution from 4-gauge boson vertex (a)} \]

It has the same form as the 2nd term in (A.9).

We secondly calculate the contribution from the ghost-gauge boson vertex:

\[ \text{ghost contribution (d)} \]

The result is that:

\[ \text{It gives rises to the following contribution} \]

\[ \text{contribution from 4-gauge boson vertex (a)} \]

The interaction vertex which involves 4 gauge bosons is

\[ \text{It gives rises to the following contribution} \]

\[ \text{The overall coefficient 9 = 10 - 1 changes to 2 for the 3d model since 2 = 3 - 1.} \]

\[ \text{ghost contribution (d)} \]

We secondly calculate the contribution from the ghost-gauge boson vertex:

\[ V_{gh} = -f^4 \text{Tr}[p_{\mu}, b][c, a_{\mu}]. \] (A.17)

The result is that:

\[ \frac{1}{2} \sum_{j_1,j_2,j_3=1}^{2i} \sum_{m_1,m_2,m_3} \frac{(-1)^{m_1+m_2+m_3}}{j_1(j_1+1)j_2(j_2+1)j_3(j_3+1)} \]

\[ \times \text{Tr}([p_{\mu}, Y_{j_1,m_1}][Y_{j_2,m_2}, Y_{j_3,m_3}]) \cdot \text{Tr}([p_{\mu}, Y_{j_2,-m_2}][Y_{j_3,-m_3}, Y_{j_3,m_3}]). \] (A.18)

It has the same form as the 2nd term in (A.9).
\[
\begin{align*}
&= -45 \frac{1}{f^4} \sum_{j_1,j_2,j_3=1}^{2l} \sum_{m_1,m_2,m_3} j_1(j_1+1) \frac{(2j_1+1)(2j_2+1)(2j_3+1)}{j_1(j_1+1)j_2(j_2+1)j_3(j_3+1)} \\
&\quad \times (1 - (-1)^{j_1+j_2+j_3}) \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\
& m_1 & m_2 & m_3 \end{array} \right)^2 \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\
& l & l & l \end{array} \right\}^2 \\
&= -45 \frac{1}{f^4} (F_p^l - F_{np}^l). \tag{A.20}
\end{align*}
\]

In the 3d model, the corresponding amplitude is
\[
-3 \frac{1}{f^4} (F_p^l - F_{np}^l). \tag{A.21}
\]

**contribution from cubic vertices (c)**

The cubic vertex which contains the structure constant of \( SU(2) \) is
\[
V_{\text{cubic}} = \frac{i}{3} f^4 \epsilon_{\mu\nu\rho} \text{Tr} \left[ a_\mu, a_\nu \right] a_\rho. \tag{A.22}
\]

Their contribution is
\[
< \frac{1}{2!} V_{\text{cubic}} V_{\text{cubic}} > 1\text{PI-2loop} \]
\[
= 2 \frac{1}{f^4} \sum_{j_1,j_2,j_3=1}^{2l} \sum_{m_1,m_2,m_3} (\text{Tr} \left[ Y_{j_1m_1} [Y_{j_2m_2}, Y_{j_3m_3}] \right])^2 \\
&\quad \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\
& m_1 & m_2 & m_3 \end{array} \right)^2 \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\
& l & l & l \end{array} \right\}^2 \\
&= \frac{4}{f^4} (F_3^p(l) - F_3^{np}(l)), \tag{A.23}
\]

where
\[
F_3^p(l) = \sum_{j_1,j_2,j_3=1}^{2l} \left( \frac{(2j_1+1)(2j_2+1)(2j_3+1)}{j_1(j_1+1)j_2(j_2+1)j_3(j_3+1)} \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\
& l & l & l \end{array} \right\} \right)^2,
\]
\[
F_3^{np}(l) = \sum_{j_1,j_2,j_3=1}^{2l} (-1)^{j_1+j_2+j_3} \left( \frac{(2j_1+1)(2j_2+1)(2j_3+1)}{j_1(j_1+1)j_2(j_2+1)j_3(j_3+1)} \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\
& l & l & l \end{array} \right\} \right)^2. \tag{A.24}
\]
fermion propagators and vertices

The fermionic action is

\[ S^\psi = -\frac{f^4}{2} \text{Tr} \left( \bar{\psi} \Gamma^\mu [A_\mu, \psi] \right), \]  

(A.25)

where we have scaled out \( f \).

Specially we can represent the \( \Gamma^\mu \) matrices as follows

\[
\Gamma^0 = \begin{pmatrix} 0 & i1_{16} \\ -i1_{16} & 0 \end{pmatrix}, \quad \Gamma^9 = \begin{pmatrix} 0 & \gamma^9 \\ \gamma^9_T & 0 \end{pmatrix}, \quad \Gamma^8 = \begin{pmatrix} 0 & \gamma^8 \\ \gamma^8_T & 0 \end{pmatrix},
\]

\[
\Gamma^i = \begin{pmatrix} 0 & \gamma^i \\ \gamma^i_T & 0 \end{pmatrix} \quad (i = 1, 2 \ldots, 7),
\]

(A.26)

where

\[
\gamma^8 = \begin{pmatrix} 0 & 1_8 \\ 1_8 & 0 \end{pmatrix}, \quad \gamma^9 = \begin{pmatrix} 1_8 & 0 \\ 0 & -1_8 \end{pmatrix},
\]

\[
\gamma^i = \begin{pmatrix} 0 & \gamma^i_8 \\ -\gamma^i_8 & 0 \end{pmatrix} \quad (i = 1, 2 \ldots, 7),
\]

(A.27)

and \( \gamma^i_8 \) are real and antisymmetric.

The fermion \( \psi \) is written by using chiral projection

\[
\psi = \frac{1 + \Gamma}{2} \begin{pmatrix} \psi_16 \\ \psi_{16}' \end{pmatrix},
\]

\[
\Gamma = -i\Gamma^0 \Gamma^1 \cdots \Gamma^9,
\]

(A.28)

We can rewrite the fermion action as

\[ S^\psi = \frac{f^4}{2} \text{Tr} \left[ \bar{\psi}_{16} \bar{\gamma}^0 \gamma^\mu [A_\mu, \psi_{16}] \right]. \]

(A.29)

The fermion kinetic term is

\[ S^\psi_0 = \frac{f^4}{2} \text{Tr} \left[ \bar{\psi}_{16} \bar{\gamma}^0 \gamma^\alpha [p_\alpha, \psi_{16}] \right], \]

(A.30)

where

\[
\bar{\gamma}^0 = \sigma^2 \otimes 1_8, \quad \bar{\gamma}^9 = -\sigma^1 \otimes 1_8, \quad \bar{\gamma}^8 = \sigma^3 \otimes 1_8,
\]

\[
\bar{\gamma}^i = -1_2 \otimes \gamma^i_8 \quad (i = 1, \ldots, 7).
\]

(A.31)

The interaction vertices are

\[ S^\psi_1 = \frac{f^4}{2} \text{Tr} \left[ \bar{\psi}_{16} \bar{\gamma}^0 \gamma^\alpha [a_\alpha, \psi_{16}] \right] + \frac{f^4}{2} \text{Tr} \left[ \bar{\psi}_{16} \bar{\gamma}^0 \gamma^i [\phi_i, \psi_{16}] \right]. \]

(A.32)
The kinetic term can be rewritten as
\[ S_0^\psi = \frac{f^4}{2} \text{Tr} \left[ \psi_a^T \sigma^2 \sigma^\alpha [p_\alpha, \psi_a] \right], \] (A.33)
where \( \psi_a \) are 8 SU(2) doublets. From (A.33), the fermion propagator is
\[ \langle (\psi_a)_{\beta, st} (\psi_b \sigma^2)_{\delta, uv} \rangle = \frac{1}{f^4} \sum_{j_1=1}^{2}\sum_{m=-(j_1+\frac{1}{2})}^{j_1+\frac{1}{2}} \left\{ -\frac{1}{j_1+1} C_- (j_1, j_2, \delta, m) + \frac{1}{j_1} C_+ (j_1, j_2, \delta, m) \right\} \]
\[ \times \left( Y_{j_1, m-\beta} \right)_{st} \left( Y_{j_1, m-\delta} \right)_{uv} \delta_{ab}, \] (A.34)
where \( s, t, u, v \) are matrix indices, \( \beta, \delta \) are \( SU(2) \) spin indices and \( a, b \) are 8 ‘flavor’ indices. \( C_- , C_+ \) are defined as follows:
\[ C_+(j_1, j_2, \delta, m) = -C_-(j_1, j_2, \delta, m) = \frac{j_1 \pm m + \frac{1}{2}}{2j_1 + 1}, \]
\[ C_-(j_1, j_2, \delta, m) = -\sqrt{(j_1 - m + \frac{1}{2})(j_1 + m + \frac{1}{2})}, \]
\[ C_+(j_1, j_2, \delta, m) = \sqrt{(j_1 - m + \frac{1}{2})(j_1 + m + \frac{1}{2})}. \] (A.35)

**fermionic contribution (e)**

2-loop (1PI) contribution to the effective action from the fermion-boson vertex is
\[ \frac{1}{2!} \langle (\psi^\dagger \sigma^2 \psi)^2 \rangle_{\text{2loop-1PI}} \]
\[ = \frac{1}{f^4} \sum_{j_1=1}^{2}\sum_{m=-(j_1+\frac{1}{2})}^{j_1+\frac{1}{2}} \left( 4 \cdot (7 - 3) \frac{\text{Tr}\{(Y_{j_1 m_1 jerkm_2 m_3})^2\}}{j_1(j_1 + 1)j_2(j_2 + 1)j_3(j_3 + 1)} \right) \]
\[ + 4 \cdot (7 + 1) \frac{\text{Tr}\{(p_\mu, Y_{j_1 m_1 jerkm_2 m_3})^2\}}{j_1(j_1 + 1)j_2(j_2 + 1)j_3(j_3 + 1)} \times \text{Tr}[p_\mu, Y_{j_2 m_2} Y_{j_3 m_3}] \]
\[ = \frac{32}{f^4} \left[ F_3^p (l) - F_3^{np} (l) - 2F_2^p (l) + 2F_2^{np} (l) \right]. \] (A.36)

We listed the contributions which involve \( \phi_i \) and \( a_\alpha \) exchanges separately.

The fermionic contribution in the 3d model is
\[ \frac{1}{f^4} \left[ -3F_3^p (l) + 3F_3^{np} (l) - 2F_2^p (l) + F_2^{np} (l) \right]. \] (A.37)
2-loop effective action

By combining these contributions, we find the total 2-loop free energy $F(l)$ of $U(1)$ NC gauge theory on fuzzy sphere as follows

$$-F(l) = -36 \frac{1}{f^4}(F_1^p(l) - F_1^{np}(l) + 2F_2^p(l) - 2F_2^{np}(l) - F_3^p(l) + F_3^{np}(l))$$

$$= 36 \frac{1}{f^4}(F_3^p(l) - F_3^{np}(l)).$$  \hspace{1cm} (A.38)

It is because we can prove that $F_1^p(l) + 2F_2^p(l) = F_1^{np}(l) + 2F_2^{np}(l) = F_5(l)$ by using the Jacobi-identity. In the case of 3d model, our result is

$$-F_{3d}(l) = -\frac{1}{f^4}(F_1^p(l) - F_1^{np}(l) + 2F_2^p(l) - 2F_2^{np}(l) - F_3^p(l) + F_3^{np}(l))$$

$$= \frac{1}{f^4}(F_3^p(l) - F_3^{np}(l)).$$  \hspace{1cm} (A.39)

For $U(n)$ gauge group, we obtain

$$-F(l) = -36 \frac{1}{f^4} \left( n^3(F_1^p(l) + 2F_2^p(l) - F_3^p(l) + \frac{1}{4}F_5(l)) - n(F_1^{np}(l) + 2F_2^{np}(l) - F_3^{np}(l) + \frac{1}{4}F_5(l)) \right)$$

$$= -36 \frac{1}{f^4} \left( n^3(-F_3^p(l) + \frac{5}{4}F_5(l)) - n(-F_3^{np}(l) + \frac{5}{4}F_5(l)) \right),$$  \hspace{1cm} (A.40)

where

$$F_5(l) = \frac{1}{2l+1} \sum_{j=1}^{2l} \frac{2j+1}{(j(j+1))^2} \to \frac{1}{2l+1} \quad (l \to \infty).$$  \hspace{1cm} (A.41)

$F_5(l)$ takes care of the absence of the $SU(2)$ singlet state in the gluon propagator. In the 3d model, the two loop effective action for $U(n)$ gauge group is

$$-F_{3d}(l) = -\frac{1}{f^4} \left( n^3(F_1^p(l) + 2F_2^p(l) - F_3^p(l) + 2F_5(l)) - n(F_1^{np}(l) + 2F_2^{np}(l) - F_3^{np}(l) + 2F_5(l)) \right)$$

$$= -\frac{1}{f^4} \left( n^3(-F_3^p(l) + 3F_5(l)) - n(-F_3^{np}(l) + 3F_5(l)) \right).$$  \hspace{1cm} (A.42)
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