Local quantum measurement and no-signaling imply quantum correlations

H. Barnum, S. Beigi, S. Boixo, M. B. Elliott, and S. Wehner

1 Perimeter Institute for Theoretical Physics, 31 Caroline St. N, Waterloo, ON, N2L 2Y5 Canada
2 Institute for Quantum Information, California Institute of Technology, Pasadena, CA 91125, USA

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We show that, assuming that quantum mechanics holds locally, the finite speed of information is the principle that limits all possible correlations between distant parties to be quantum mechanical as well. Local quantum mechanics means that a Hilbert space is assigned to each party, and then all local POVM measurements are (in principle) available; however, the joint system is not necessarily described by a Hilbert space. In particular, we do not assume the tensor product formalism between the joint systems. Our result shows that if any experiment would give non-local correlations beyond quantum mechanics, quantum theory would be invalidated even locally.

Quantum correlations between space-like separated systems are, in the words of Schrödinger, “the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought” [1]. Indeed, the increasing experimental support [2] for correlations violating Bell inequalities [3] is at odds with local realism. Quantum correlations have been investigated with increasing success [4], but what is the principle that limits them [5]?

Consider two experimenters, Alice and Bob, at two distant locations. They share a preparation of a bipartite physical system, on which they locally perform one of several measurements. This shared preparation may thereby cause the distribution over the possible two outcomes to be correlated. In nature, such non-local correlations cannot be arbitrary. For example, it is a consequence of relativity that information cannot propagate faster than light. The existence of a finite upper bound on the speed of information is known as the principle of no-signaling. This principle implies that if the events corresponding to Alice’s and Bob’s measurements are separated by space-like intervals, then Alice cannot send information to Bob by just choosing a particular measurement setting. Equivalently, the probability distribution over possible outcomes on Bob’s side cannot depend on Alice’s choice of measurement setting, and vice versa. Quantum mechanics, like all modern physical theories, obeys the principle of no-signaling.

But is no-signaling the only limitation for correlations observed in nature? Bell [3] initiated the study of these limitations based on inequalities, such as the CHSH expression [7]. It is convenient to describe this inequality in terms of a game played by Alice and Bob. Suppose we choose two bits \(x, y \in \{0, 1\}\) uniformly and independently at random, and hand them to Alice and Bob respectively. We say that the players win if, they are able to return answers \(a, b \in \{0, 1\}\) respectively, such that \(x \cdot y = a + b \mod 2\). Alice and Bob can agree on any strategy beforehand, that is, they can choose to share any preparation possible in a physical theory, and choose any measurements in that theory, but there is no further exchange of information during the game. The probability that the players win is

\[
\frac{1}{4} \sum_{x, y \in \{0, 1\}} \sum_{a, b \in \{0, 1\}} p(a, b|M_A^x, M_B^y) \tag{1}
\]

where \(p(a, b|M_A^x, M_B^y)\) denotes the probability that Alice and Bob obtain measurement outcomes \(a\) and \(b\) when performing the measurements \(M_A^x\) and \(M_B^y\) respectively (any pre- or post-processing can be taken as part of the measurement operation). Classically, i.e. in any local realistic theory, this probability is bounded by [6]

\[
p_{\text{classical}} \leq 3/4 . \tag{2}
\]

Such an upper bound is called a Bell inequality.

Crucially, Alice and Bob can violate this inequality using quantum mechanics [4]. The corresponding bound is [7]

\[
p_{\text{quantum}} \leq 1 - \frac{1}{2} + \frac{1}{2\sqrt{2}} , \tag{3}
\]

and there exists a shared quantum state and measurements that achieve it [6]. Further, there is now compelling experimental evidence that nature violates Bell inequalities and does not admit a local realistic description [2]. Yet, there exist stronger no-signaling correlations (outside quantum mechanics) which achieve success probability \(p_{\text{nosignal}} = 1\) [5]. So why, then, isn’t nature more non-local [3]?

Studying limitations on non-local correlations thus forms an essential element of understanding nature. On one hand, it provides a systematic method to both theoretically and experimentally compare candidate physical theories [9]. On the other hand, it crucially affects our understanding of information in different settings such as cryptography and communication complexity [10–13]. For example, if nature would admit \(p_{\text{nosignal}} = 1\), any two-party communication problem could be solved using only a single bit of communication, independent of its size [11]. Also, for the special case of the CHSH inequality, it is known that the bound (3) is a consequence
of information theoretic constraints such as uncertainty relations [12] or the recently proposed principle of information causality [13]. However, characterizing general correlations remains a difficult challenge [14], and it is interesting to consider what other constraints may impose limits on quantum correlations.

Result. We forge a fundamental link between local quantum theory and non-local quantum correlations. In particular, we show that if Alice and Bob are locally quantum, then relativity theory implies that their non-local correlations admit a quantum description. The assumption of being locally quantum may thus provide another “reason” why the correlations we observe in nature are restricted by more than the principle of no-signaling itself. Figure 1 states our result.

Let us explain more formally what we mean by being locally quantum (see also Figure 1). We say that Alice is locally quantum, if her physical system can be described by means of a Hilbert space $\mathcal{H}_A$ of some fixed finite dimension $d$, on which she can perform any local quantum measurement (POVM) $M_A = \{Q_a\}_a$ given by bounded operators $Q_a \in \mathcal{B}(\mathcal{H}_A)$. The probability $p(a|M_A)$ that she obtains an outcome $a$ for measurement $M_A = \{Q_a\}_a$ is given by a function $\mathcal{B}(\mathcal{H}_A) \rightarrow [0,1]$ applied to the POVM elements. Gleason’s theorem for POVM elements implies that the state of Alice’s system is then described by a state $\rho_A \in \mathcal{B}(\mathcal{H}_A)$, and similarly for Bob, where we use $\mathcal{H}_B$ and $M_B = \{R_b\}_b$ to denote his Hilbert space and measurements respectively. Conceptually, this means that quantum mechanics describes Alice and Bob’s local physical systems.

However, we make no a priori assumption about the nature of the joint system held by Alice and Bob. In particular, we do not assume that it is described by a tensor product of their local Hilbert spaces, or that their joint system is quantum mechanical. This means that Alice and Bob can share any possible preparation which assigns probabilities to local POVM measurements. That is, their preparation is simply a function $\omega$ such that the probabilities of observing outcomes $a$ and $b$ for measurements $M_A = \{Q_a\}_a$ and $M_B = \{R_b\}_b$ are given by $p(a,b|M_A, M_B) = \omega(Q_a, R_b)$. In particular, the state of their joint system may not be described by any density matrix.

Nevertheless, we are able to show that just from the assumptions that Alice and Bob are locally quantum and that the no-signaling principle is obeyed, it follows that there exist a Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, a state $\rho_{AB} \in \mathcal{B}(\mathcal{H}_{AB})$ and measurements $M_A = \{Q_a\}_a$ and $M_B = \{R_b\}_b$ for Alice and Bob, such that

$$p(a,b|M_A, M_B) = \omega(Q_a, R_b) = \text{tr}(\hat{Q}_a \otimes \hat{R}_b \rho_{AB}) \quad (4)$$

That is, all correlations can be reproduced quantum mechanically.

Implications. Our result solves an important piece of the puzzle of understanding non-local correlations, and their relation to the rich local phenomena we encounter in quantum theory such as Bohr’s complementarity principle, Heisenberg uncertainty and Kochen-Specker non-contextuality. In particular, it implies that if we obey local quantum statistics we can never hope to surpass a Tsirelson-type bound on $p_{\text{quantum}}$ like that of [5], ruling out the possibility of such striking differences with respect to information processing as those pointed out in [11]. Indeed, if we were able to surpass such bounds, then the local systems of Alice and Bob could not be quantum.

Other recent works also attempt to explain the limitations of quantum correlations. For example, the principle of information causality [13] starts with the assumption that nature demands that certain communication tasks should be hard to solve. Together with the assumption of the no-signaling principle, this allows one to obtain Tsirelson’s bound for the special case of the CHSH inequality. In our work, we also assume the no-signaling principle, but combine it with a different assumption, namely, that the world is locally quantum, that is, quantum mechanics correctly describes the laws of nature of local physical systems. Making this assumption we recover the quantum limit on all possible non-local correlations (not only the Tsirelson’s bound for the CHSH inequality).

Proof. To prove our result, we now proceed in two steps. First, we explain a known characterization of all no-signaling probability assignments to local quantum measurements [10–18]. Second, we use this characterization to show that the resulting correlations can be obtained in quantum mechanics.

From local quantum measurements to POPT states: Fix two finite dimensional Hilbert spaces on Alice and Bob’s sides. A local quantum measurement (or POVM) consists of a pair of measurements $M_A$ and $M_B$ with outcome labels $\{a\}$ and $\{b\}$ respectively on Alice and Bob’s Hilbert spaces. Such POVMs are described by complex Hermitian matrices $M_A = \{Q_a\}_a$, $M_B = \{R_b\}_b$, $Q_a, R_b \geq 0$, which sum to the identity, i.e., $\sum_a Q_a = \sum_b R_b = 1$ (see Figure 1). A preparation shared between Alice and Bob assigns outcome probabilities $p(a,b|M_A, M_B)$ to any choice of measurements $M_A$ and $M_B$. More precisely, it corresponds to a function $\omega$ on the pair of POVM elements such that $p(a,b|M_A, M_B) = \omega(Q_a, R_b)$.

Kläy, Randall, and Foulis [17] have shown (see Appendix) that assuming no-signaling, the shared preparations (or equivalently the functions $\omega$) are in one-to-one correspondence with matrices $W_{AB}$ such that $\text{tr}(W_{AB}) = 1$ and

$$p(a,b|M_A, M_B) = \text{tr}((Q_a \otimes R_b) W_{AB}) \geq 0. \quad (5)$$

The matrices $W_{AB}$ are called positive on pure tensors (POPT) states. All quantum states are POPT states,
In order to do so, we associate to each POPT state \( W \) a map \( \mathcal{W} \) from matrices to matrices using the Choi-Jamiołkowski isomorphism. Explicitly, \( W_{AB} \) is obtained from \( \mathcal{W} \) by acting on Bob’s side of the (projection on the) maximally entangled state \(|\Phi\rangle\)

\[
W_{AB} = \mathbb{1} \otimes \mathcal{W}(|\Phi\rangle\langle\Phi|) .
\]  

Because \( W_{AB} \) is a POPT, the associated map \( \mathcal{W} \) is positive, i.e., it sends positive matrices to positive ones, but it may not be an admissible quantum operation. Nevertheless, if \( \mathcal{W} \) still maps POVMs to POVMs we can obtain the POPT correlations by moving the action of \( \mathcal{W} \) from the maximally entangled state to the measurement elements. In particular, if \( \mathcal{W} \) is unital (\( \mathcal{W}(\mathbb{1}) = \mathbb{1} \)), the map

\[
f : Q_a \mapsto \tilde{Q}_a = \mathcal{W}(Q_a^T)^T ,
\]

maps POVM measurements to POVM measurements. We then show that (7) holds with \( \sigma_{AB} = |\Phi\rangle\langle\Phi| \). Let \( d \) be the local dimension of Alice and Bob. If \( \mathcal{W} \) is unital we have

\[
\text{tr} \left( (Q_a \otimes R_b)W_{AB} \right) = \text{tr} \left( (Q_a \otimes R_b)\mathbb{1} \otimes \mathcal{W}(|\Phi\rangle\langle\Phi|) \right) = \text{tr} \left( \mathcal{W}(|\Phi\rangle\langle\Phi|) (Q_a \otimes W^* (R_b)) \right) = \frac{1}{d} \text{tr} \left( (Q_a^T W^* (R_b)) \right) = \frac{1}{d} \text{tr} \left( \mathcal{W}(Q_a^T) R_b \right) = \text{tr} \left( (\tilde{Q}_a \otimes R_b) |\Phi\rangle\langle\Phi| \right) ,
\]

where \( W^* \) denotes the adjoint of \( \mathcal{W} \). This establishes (7) in the unital case.

In general, \( \mathcal{W} \) can be decomposed into a unital map and another map. This other map gives a quantum state \( \sigma_{AB} \) by acting on \( |\Phi\rangle \). Then \( f \) is defined in terms of the unital map as before. We finish the proof by showing that \( \sigma_{AB} \) is well-normalized and (7) is satisfied. For a general positive map, let \( M \) be the image of the identity, i.e., \( \mathcal{W}(\mathbb{1}) = M \). The matrix \( M \) is normalized, \( \text{tr}(M)/d = \).
\( \text{tr}(W_{AB}) = 1 \). We assume initially that \( M \) is invertible, and define
\[
\tilde{W}(\cdot) = M^{-1/2}W(\cdot)M^{-1/2}.
\]

The map \( \tilde{W} \) is unital. Further, the quantum state \( \sigma_{AB} = |\psi\rangle\langle\psi| \) given by
\[
|\psi\rangle = (M^{1/2})^T \otimes 1|\Phi\rangle
\]
is well-normalized, that is, \( \text{tr}(\sigma_{AB}) = \text{tr}(M^T)/d = 1 \). Thus by defining \( f \) as in [9] but in terms of \( W \) we conclude
\[
\text{tr}((Q_a \otimes R_b)W_{AB}) = \frac{1}{d} \text{tr}(W(Q_a^T)R_b)
= \frac{1}{d} \text{tr}\left( (M^{1/2}\tilde{W}(Q_a^T)M^{1/2}) R_b \right)
= \text{tr}\left( (\tilde{Q}_a \otimes R_b)\sigma_{AB} \right).
\]

If \( M \) is not invertible, in order to define \( \tilde{W} \), one can start with the map \( (1 - \epsilon)W(\cdot) + \epsilon 1\text{tr}(\cdot) \), and then take the limit \( \epsilon \to 0 \).

**Conclusion.** We have shown that being locally quantum is sufficient to ensure that all non-local correlations between distant parties can be reproduced quantum mechanically, if the principle of no-signaling is obeyed. This gives us a natural explanation of why quantum correlations are weaker than is required by the no-signaling principle alone, i.e., given that one can describe local physics according to quantum measurements and states, then no-signaling already implies quantum correlations.

It would be interesting to know whether our work can be used to derive more efficient tests for non-local quantum correlations than those proposed in [14]. Finally, it is an intriguing question whether one can find new limits on our ability to perform information processing *locally* based on the limits of non-local correlations, which we now know to demand local quantum behavior.

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**Appendix**

We include a derivation of the POPT states for completeness. We follow the more general version in [13]. The outline is the following: using no-signalling, we apply Gleason’s theorem on both sides, Alice and Bob. This implies that the no-signaling POPT state is bilinear on Alice and Bob measurements, which gives its form.

We denote the local POVMs by \( M_A = \{Q_a\}_a \) and \( M_B = \{R_b\}_b \). The joint probability distribution is given by a function \( \omega \) acting on POVM elements
\[
p(a, b|M_A, M_B) = \omega(Q_a, R_b).
\]

Notice that for any pair of POVMs
\[
\sum_{a, b} \omega(Q_a, R_b) = 1,
\]
but \( \omega \) is not assumed to be bilinear at this point. No-signalling implies that for all \( M_B \)
\[
\sum_b \omega(Q_a, R_b) = \sum_b p(a, b|M_A, M_B) = p(a|M_A) = \omega(Q_a).
\]

That is, the marginal distribution is well defined.

For any POVM element \( Q_a \) on Alice’s side we can define a corresponding function \( \omega_a \) which acts on Bob’s POVM elements. The function \( \omega_a \) is defined by its action on any POVM element \( R_b \) with the equation
\[
\omega_a(R_b) = \omega(Q_a, R_b).
\]

Notice that, for every POVM \( M_B \) on Bob’s side, no-signalling from Bob to Alice implies that
\[
\sum_b \omega_a(R_b) = \sum_b \omega(Q_a, R_b) = \omega(Q_a).
\]

Because \( \omega_a \) adds to the constant value \( \omega(Q_a) \) when it is summed over any POVM, we can use Gleason’s theorem [19][21] to identify \( \omega_a \) with an *unnormalized* quantum state \( \tilde{\sigma}_a \) on Bob’s side. Specifically, for any POVM element \( R_b \), we have
\[
\omega_a(R_b) = \omega(Q_a, R_b) = \text{tr}(\tilde{\sigma}_a R_b).
\]

The previous equation allows us to define, for any given POPT \( \omega \), a map \( \tilde{\omega} \) from POVM elements \( Q_a \) on Alice’s side to unnormalized quantum states on Bob’s side
\[
\tilde{\omega}(Q_a) = \tilde{\sigma}_a.
\]

Now choose an informationally complete POVM \( M_B = \{R_b\} \) on Bob’s side. Then \( \tilde{\omega} \) is given by the functions \( \omega^b \) defined by
\[
\omega^b(Q_a) = \omega(Q_a, R_b) = \text{tr}(\tilde{\sigma}_a R_b).
\]

We use no-signalling from Alice to Bob to apply Gleason’s theorem to each function \( \omega^b \) from the informationally complete POVM, as we did before with no-signalling in the other direction. The action of \( \omega^b \) is then given by an unnormalized quantum state, which implies that it is linear. This proves that \( \tilde{\omega} \) is linear.
Once we have established the linearity of $\hat{\omega}$ we can identify it with the operator $W$ introduced in the text according to

$$\hat{\omega}(Q_a) = \frac{1}{d} W(Q_a^T) .$$

(22)

Finally, we can write

$$\omega(Q_a, R_b) = \text{tr}(\hat{\omega}(Q_a) R_b) = \frac{1}{d} \text{tr}(W(Q_a^T) R_b)$$

(23)

$$= \text{tr}((Q_a \otimes R_b) W_{AB}).$$