Two-loop heavy top corrections to the $b \to s\gamma$ decay

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Abstract

We compute the two-loop corrections to the coefficient of the $b \to s\gamma$ magnetic penguin present in the limits of heavy top and/or heavy higgs. This kind of corrections affects in a significant way the observables measured at LEP. On the contrary we find that, due to a numerical accident, the correction to $\text{BR}(B \to X_s\gamma)$ is negligible (below the 1% level for any possible value of the higgs mass) when the leading order result is expressed in the usual way in terms of the semileptonic $\text{BR}(B \to X_c e\nu)$.

1 Introduction

The $b \to s\gamma$ decay seems the more promising observable in $B$, $K$ and $D$ physics to search for a supersymmetric effect. Some observables (like the $K \to \pi\nu\bar{\nu}$ decays and various tree level processes) are predictable with remarkable accuracy within the SM [1], but are expected to receive too small supersymmetric corrections. In other cases (like in $B$ mixing, or in charm-less $B$ decays) the supersymmetric corrections can be large enough to affect the experimental result, but remain masked by too large hadronic uncertainties in the SM prediction [2][3][4]. Even if the minimal supersymmetric SM (MSSM) Lagrangian contains the new sources of flavour and CP violation that can be left at low energy by unification-scale physics [2], it does not seem possible to disentangle the new supersymmetric effects in $B$-physics from the SM background [3]. More optimistic conclusions can only be obtained assigning ad hoc values to the $\sim 100$ parameters of the MSSM [2], or if $R$-parity is broken in appropriate way. Various supersymmetric effects become larger if a stop state is light (this possibility allows baryogenesis at the electroweak scale [7], but requires an unnaturally large fine tuning [8]) or if $\tan\beta$ is large (this possibility can be realized naturally with some appropriate hierarchy between the parameters of the higgs potential). In both cases also the supersymmetric correction to $b \to s\gamma$ is enhanced.

In all cases, it is interesting to study the $b \to s\gamma$ magnetic penguin and it is reasonable to expect that its Wilson coefficient, usually named $C_7$, receives a relatively large and hopefully detectable $O(10\%)$ supersymmetric correction. Consequently, it is useful to perform accurate multi-loop computations of this observable.

In order to obtain a sufficiently accurate SM prediction, the next-to-leading (NLO) QCD corrections, of relative order $g_2^3/(4\pi)$, have been computed for all relevant values of the $\overline{MS}$ scale $\mu$: for $\mu$ near the electroweak scale $\mu_W$, for $\mu$ near the $B$ scale $\mu_B$, and for $\mu$ between $\mu_W$ and $\mu_B$. All these three stages give numerically relevant QCD corrections of order 20% to the decay rate. Also the supersymmetric contributions to $C_7$ are affected by relevant QCD corrections. However, only the QCD corrections to the charged higgs mediated contribution are known [4]. It is difficult to present in a compact and accurate form the SUSY-QCD correction to the remaining relevant supersymmetric contribution, mediated by charginos and squarks.

Electroweak corrections, of relative order $g_2^2/(4\pi)^2$ and $\ln(\mu_W^2/\mu_B^2) \cdot e^2/(4\pi)^2$, have partly been computed in [4] and found to be relevant.

In this paper we compute, for arbitrary value of the Higgs mass, the heavy top corrections of relative order $g_t^2/4\pi$ where $g_t$ is the top quark Yukawa coupling in the SM. We also compute the corrections of relative order $\lambda^2/4\pi$ where $\lambda$ is the quartic higgs coupling in the SM. More precisely the couplings $g_t$ and $\lambda$ induce corrections that depend on the top, higgs and $W$ boson squared masses $(m_t^2, m_h^2, m_W^2)$, and increase when $m_t$ or $m_h$ become large, as summarised in table [4]. We compute these potentially relevant corrections in the limit $m_W \to 0$. In the same approximation, the ‘heavy top corrections’ and/or the ‘heavy higgs corrections’ to the precision electroweak observables measured at LEP have been computed in [4] and/or [5]. The heavy
top corrections to LEP observables are so relevant that also the sub-leading terms, suppressed by a power of $m_H^2/m_W^2$, have recently been computed \[10\]. From these results it is possible to derive the heavy top corrections to various $B$ and $K$ decays \[7\] generated by ‘electric’ effective operators. On the contrary, a new computation is necessary to obtain the corrections to the $b \to s\gamma$ magnetic penguin.

This computation is outlined in section 2 (various details are confined to the appendices) and is more cumbersome than in the other mentioned cases. It is not possible to relate the magnetic penguin to a simpler vertex, as in \[14\]. As a consequence more than 50 two loop diagrams, shown in fig. 2 need to be evaluated. Renormalization is a complicated task: we cannot set the $b$-quark mass $m_b$ to zero, we have to deal with CKM mixing, and we have to connect $b \to s\gamma$ with the measurable $B \to X_s\gamma$. A particularly appropriate way of doing the renormalization is described in sections 2.3 and 2.4. The final result is discussed in section 3. We separately give few % corrections. The presence of accidental cancellations suggests that the approximation $m_W \to 0$ could be not a good one. In any case it appears unlikely that the neglected corrections, suppressed by powers of $m_H^2/m_W^2$, be sufficiently large to make the heavy top and/or heavy higgs effects relevant.

2 Computation

In this section we outline the computation. For simplicity we will refer to the heavy top and/or heavy higgs corrections to LEP observables are so relevant that also the sub-leading terms, suppressed by a power of $m_H^2/m_W^2$, have recently been computed \[10\]. From these results it is possible to derive the heavy top corrections to various $B$ and $K$ decays \[7\] generated by ‘electric’ effective operators. On the contrary, a new computation is necessary to obtain the corrections to the $b \to s\gamma$ magnetic penguin.

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2.1 One loop result

We begin with recalling the structure of the one-loop result. The contribution of the first generation is negligible, so that the unitarity constraint on the CKM matrix becomes $V_{cb}V_{cs}^\ast + V_{tb}V_{ts}^\ast = 0$. It is also unnecessary (even it would be immediate) to compute the $\mathcal{O}(g_s, \lambda)^2$ corrections to terms suppressed by the ratio $m_s/m_b$ between the masses of the strange and the bottom quarks. The effective Hamiltonian for the $b \to s\gamma$ decay is

$$
\mathcal{H}_{\text{eff}} = - \frac{G_F}{\sqrt{2}} V_{tb} V_{ts}^\ast \sum_{i=1}^{8} C_i \mathcal{O}_i
$$

where $G_F$ is the Fermi constant, $C_i$ are the Wilson coefficients and, for our purposes, the relevant operators are

$$
\mathcal{O}_2 \equiv 4 \langle \bar{s}\gamma_\mu P_L c\rangle \langle \bar{c}\gamma_\mu P_L b \rangle, \quad \mathcal{O}_7 \equiv 4 \frac{e}{(4\pi)^2} m_b \langle \bar{s}\gamma_\mu P_R b \rangle F_{\mu\nu}, \quad \mathcal{O}_8 \equiv 4 \frac{g_3}{(4\pi)^2} m_b \langle \bar{s}\gamma_\mu P_R b \rangle G_{\mu\nu}^a
$$

where $i, j$ are colour indexes. At leading order, and before including QCD corrections, $C_2 = 1$. The leading contribution to $C_7$ is given, in the the renormalizable Feynman-Fujikawa gauge \[18\], by the one-loop graphs (named “W”, “I” and “E”) shown in fig. 1:

$$
C_7 = \left[ P_W(x_t) - P_W(x_c) \right] + \left[ - \frac{x_t}{2} P_I(x_t) + \frac{x_t}{2} P_E(x_t) \right] = \frac{3}{2} x_t P_E(x_t) = -0.2
$$

where $x_t = m_t^2/m_W^2$, $x_c = m_c^2/m_W^2 \approx 0$ and the three penguin loop functions $P(x)$ are given in appendix C. In the limit $m_t \gg m_W$ (in which we will compute the $\mathcal{O}(g_s, \lambda)^2$ corrections) $P_W(x_t \to \infty) = 0$ and the one-loop result \[3\] simplifies to

$$
C_7(m_t \gg m_W) = \left[ 0 - \frac{23}{36} + \frac{5}{12} \frac{1}{9} \right] = -\frac{1}{3}.
$$

2.2 Two loop $\mathcal{O}(g_s, \lambda)^2$ graphs

We have seen that the one-loop $b \to s\gamma$ decay, in the limit of heavy top, does not receive any contribution from the graphs “W” of fig. \[4\] with a $W$-boson and a top quark in the loop. It is easy to understand this fact with a simple dimensional analysis and a look at the vertices, explicitly written in fig. \[4\]. For the same reason \[4\] also at two loops the heavy top and/or heavy higgs corrections can be computed in the gauge-less limit. In the same way the contribution mediated by the $W$ boson and the charm quark does not receive heavy top and/or heavy higgs corrections. At the light of these considerations, all the two-loops graphs that give $\mathcal{O}(g_s, \lambda)^2$ corrections are shown in fig. \[5\] (the 12 graphs named ‘B’, ‘h’ and ‘χL’ vanish in the limit of zero higgs mass). We have not plotted other 10 graphs of tadpole type (we will discuss them in the following).
2.3 Renormalization procedure

Unrenormalized tree level quantities will be denoted with a superscript $0$. The renormalization factor of a field $\phi$ ($\phi = \{t_L, t_R, b_L, b_R, \phi_R, \chi, \ldots\}$) is named $Z_\phi$, and $\phi^0 = \phi/\sqrt{Z_\phi}$. The renormalization factor of a parameter $\varphi$ ($\varphi = \{m_t, m_b, v, V_{ij}, \ldots\}$) is named $Z_\varphi$, and $Z_{\varphi} \psi^0 = \psi^0$ plus the quantum corrections to it. All the renormalization $Z$ factors are defined in a more precise way in appendix B.

Renormalization would be a very cumbersome task if done adding counterterm diagrams to cancel the one-loop divergences. It is more convenient to proceed in a different way. Renormalize amounts to express the quantum-corrected $b \to s\gamma$ amplitude in terms of quantum-corrected physical quantities: the bare parameters must be substituted with measurable quantities, and the $b, s, \gamma$ fields have to be re-normalized. In the limit $m_t \gg m_W$ the one loop $b \to s\gamma$ amplitude depends on the following bare parameters: $g_t^0$ (or $m_t^0$), $g_b^0$ (or $m_b^0$), $v^0$ (or $G_F^0$, that appears as the overall dimensional factor in $B$), the electric charge $e$ (that do not get any $O(g_t, \lambda)^2$ correction, because it must be the same for any quark or lepton) and the CKM mixing angle between the two heavy generations. We define the renormalized parameters in terms of precisely measured quantities:

- the pole top mass, $m_t$, given by

$$m_t^0 = m_t \frac{Z_{m_t}}{Z_{t_L} Z_{t_R}}$$

In the limit $m_t \gg m_W$ the one loop contribution to $C_7$ depends on $m_t$ only through terms that vanish as $\varepsilon \to 0$, so that only the divergent parts of the $Z$ give a contribution. It is thus irrelevant to distinguish between the top pole mass and top running mass parameter.

- the pole bottom mass, $m_b$, given by

$$m_b^0 = m_b \frac{Z_{m_b}}{Z_{b_L} Z_{b_R}}$$

Since the one-loop $O(g_t^2)$ correction to the $b$ propagator is mediated by a heavy top, these is no difference between pole (on-shell) and zero-momentum bottom mass.

- the Fermi constant measured in $\mu$ decay, $G_F$. Including the $O(g_t, \lambda)^2$ corrections, $G_F$ (or the $W$ mass term at zero momentum, $m_W$) is linked to the higgs vacuum expectation value (vev) $v$ by

$$4G_F \sqrt{2} = \frac{g_2^2}{2 m_W^2} = \frac{1}{v^2 Z_{\chi}}$$

All the graphs contain a (bottom quark $b$)-(top quark $t$)-(unphysical charged higgs $\chi$) vertex. The SM Lagrangian (A.3) contains two different vertices of this kind:

$$g_b \bar{b}_L t_L \chi^- \quad \text{and} \quad -g_t \bar{b}_R t_R \chi^-.$$

With the exception of the `$h_L$', `$\eta_L$' and `$\chi_L$' graphs (marked with a $\otimes$ in fig. 2), both vertices contribute to the $b_R \to s_L \gamma$ magnetic penguin, and give two different loop structures. The difference is clear from the one-loop graphs of fig. 1, where we have separately plotted the two contributions, named "I" and "E". The "E"-type graphs are usually more cumbersome to compute, give smaller contributions and generate a different $\gamma$ factor. All the graphs contain a (bottom quark $b$)-(top quark $\bar{t}$)-($\chi$) vertex. We can employ naive dimensional regularization (i.e. anticommuting $\gamma_5$) with $\overline{\text{MS}}$ renormalization scale $\bar{\mu}$.

Figure 1: The one-loop $b \to s\gamma$ graphs. The symbol $\otimes$ denotes a $m_b$ factor. All possible attachments of the photon have been shown.
|                        | asymptotic dependence of the corrections to | SM electroweak observables | large $m_t$ limit | large $m_h$ limit |
|------------------------|--------------------------------------------|-----------------------------|-------------------|-------------------|
|                        | one loop | two loops | one loop | two loops |
| $b \to s\gamma$ magnetic penguin | $m_t^2$ | $m_t^2$ | $m_t^2$ | $m_t^2$ |
| $m_0^2 t$               | $m_0^2 t$ | $m_0^2 t$ | $m_0^2 t$ | $m_0^2 t$ |

Table 1: Asymptotic dependence of the heavy top and of the heavy higgs corrections to a typical observable measured ad LEP, and to the coefficient of the $b \to s\gamma$ magnetic penguin.

Figure 2: The two loop-diagrams that give $O(g_t, \lambda)^2$ corrections. The thick (thin) continuous lines represent the top ($b$ and $s$) quark. The dotted (dashed) thin lines represent the charged (neutral) unphysical higgses $\chi$ ($\eta$). The thick dashed line represent the SM higgs boson $h$. The thin wavy lines represent all possible attachments of the photon. In the graphs marked with a * the correction on the $\chi$ propagator has to be subtracted at zero $\chi$ momentum. Tadpole diagrams are not shown.
where $Z_{\chi}$ is precisely defined in appendix B.3. The relation (3), valid in the gauge-less limit, can be obtained considering the boson vectors as external currents and noticing that the Ward identities of the SU(2)$_L$ global symmetry imply that the effective Lagrangian contains the following term

$$\mathcal{L}_{\text{eff}} = Z_{\chi} \left| \partial_{\mu} \chi - i \frac{g_2}{v} \sqrt{2} W_{\mu} \right|^2 + \cdots = m_W^2 |W|^2 + \cdots$$

- The higgs vev $v$ must be determined from the minimisation condition of the effective potential. In the gauge-less limit this is equivalent to require that the unphysical higgses $\chi$ and $\eta$ be massless Goldstone bosons:

$$m_\chi^2 = \mu^2 - 2\lambda v^2 + \text{quantum corrections of order} \frac{g_2^2 m_\chi^2}{(4\pi)^2}$$

At this point we encounter a technical problem: there are corrections to $v^2$ of order $g_2^4 v^2$. When $v_0$ is expressed in terms of $v$, the term in the one loop coefficient of the $b \to s\gamma$ penguin suppressed by one power of $\lambda_t^{-1} = m_W^2/m_t^2$, $C_7 = -1/3 + 3/(\ln x_t)/4x_t + \cdots$, would give a heavy top correction, infrared (IR) divergent in the limit $m_W \to 0$, that cancels a similar IR divergence present in the two loop graphs. To avoid these infrared problems, it is more convenient to cancel the quantum corrections to the Goldstone masses in eq. (3) adding appropriate counterterms in order to obtain $v = v_0$. This procedure requires to include the new $b \to s\gamma$ Feynman graphs that contain these counterterms. This is equivalent to compute the two-loop graphs named “F” and “B” and marked with an * in fig. 2 with the prescription that the correction to the $\chi$ propagator has to be subtracted at zero $\chi$ momentum. When this is done, all infrared divergences disappear and the graphs can be safely computed at $m_W = 0$.

- Finally we have to renormalize the CKM mixing angles. Since only the mixing between 2$^\text{nd}$ and 3$^\text{rd}$ generation is relevant, the mixing is described by only one parameter. Furthermore we may set $V_{ts} = 1$, $V_{tb} = s_{23} \ll 1$: we can neglect the $O(g_1, \lambda^2)$ corrections to terms suppressed by powers of the Wolfenstein parameter $\lambda_t^2 \approx 0.05$. At this point, we remember that normalising the $b \to s\gamma$ decay rate to the semi-leptonic $b \to c\ell \nu$ one allows to eliminate a large hadronic uncertainty coming from a $m_t^2$ phase-space factor. For this reason, we will extract the renormalized value of $s_{23}$ from $\text{BR}(B \to X_c \ell \nu)$.

We now show how is it possible to accomplish the last point. It is very convenient to renormalize a non standard form of the bare Lagrangian that is respected by the quantum corrections that we are considering. Keeping only the bare interactions and the bare fields necessary to describe flavour mixing, we write the SU(2)$_L$-symmetric bare Lagrangian as

$$\mathcal{L}_0 = \sum_q q_i^d i\gamma^\mu \gamma^\nu \bar{q}^{0} \gamma^\mu \gamma^\nu q^{0} + g_1 t_{R}^0 Q_3^0 H^{*0} + g_1 b_{R}^0 (Q_3^0 - s_{23}^0 Q_2^0) H^{*0}$$

where $q = \{ Q_1^0, Q_2^0, t_{R}^0, b_{R}^0 \}$. The down components of the quark doublets, $Q_3^0$, are not mass eigenstates. Had we started from the usual flavour basis of mass eigenstates, complicated flavour redefinitions would be necessary to obtain a unitary renormalized CKM matrix. The tree level Lagrangian and the divergent part of the quantum corrections have a SU(2)$_L$ global symmetry. This symmetry is useful to understand some cancellations (for example why $Z_{m_t}$ is not divergent) and to fix some dangerous signs. It is easy to verify that the 1PI effective Lagrangian, corrected only by the divergent part of the quantum corrections is

$$\mathcal{L}_{\text{effective}} = \sum_q Z_q q_i^d i\gamma^\mu \gamma^\nu \bar{q}^{0} \gamma^\mu \gamma^\nu q^{0} + Z_{t_{R}} g_1 t_{R}^0 Q_3^0 H^{*0} + g_1 b_{R}^0 (Z_{Q_3} Q_3^0 - s_{23}^0 Q_2^0) H^{*0}$$

and has the same form as the bare Lagrangian (5) (its last term does not get $O(g_1^2)$ corrections). The $Z$ factors are

$$Z_{Q_3} = 1 + \frac{1}{2} \frac{g_1^2}{(4\pi)^2} \frac{1}{\varepsilon}, \quad Z_{t_{R}} = Z_{b_{R}} = 1 + \frac{g_1^2}{(4\pi)^2} \frac{1}{\varepsilon}, \quad Z_{Q_2} = Z_{b_{R}} = Z_{b_{R}} = 1.$$

We are now ready to renormalize also the finite part of the quantum corrections. Performing, in the effective Lagrangian, the following redefinitions of the fields

$$t_{R}^0 = \frac{t_{R}}{\sqrt{Z_{t_{R}}}}, \quad b_{R}^0 = \frac{b_{R}}{\sqrt{Z_{b_{R}}}}, \quad Q_3^0 = \left( t_{L} / \sqrt{Z_{t_{R}}} \right), \quad Q_2^0 = \left( c_{L} \right).$$

\text{and couplings}

$\text{For this reason we have not plotted the graphs with a tadpole correction to the } \chi \text{ propagator: they are cancelled by this procedure. The tadpole graph in which the external photon is attached to the tadpole loop also gives a zero contribution to the } b \to s\gamma \text{ magnetic moment.}$

$\text{We have checked that the more cumbersome procedure of keeping } m_W \text{ as an infrared regulator gives the same final result.}$

When doing the computation at $m_W = 0$, it would be useful to know if the 1/2 $\varepsilon$ poles are due to infrared (IR) or to ultraviolet (UV) divergences. However some of the graphs in fig. 2 contain terms like $\varepsilon / \varepsilon_{\text{UV}}\varepsilon_{\text{IR}}$, so that we have not been able to distinguish IR from UV divergences using dimensional regularization only.
the renormalized parameters \( m_t, m_b \) and \( s_{23} \) coincide with the physical quantities previously chosen (the meaning of the \( Z \) factors should be clear from the notation; they are precisely defined and computed in appendix B). Notice that, only at this stage, we have performed a unitary flavour rotation in the down components of the quark doublets \( Q \), so that the renormalized quark fields are mass eigenstates. The \( V_{cb} \) CKM mixing angle that appears at the gauge \( c_L \, W b_L \) vertex is \( s_{23} \) (the gauge interaction of \( Q_3^c \) does not get \( O(g_t, \lambda)^2 \) contributions, so that the link between \( B \to X_c \gamma \) and the semi-leptonic decay \( B \to X_c \ell \nu \) decay can now be easily extracted.

### 2.4 Renormalization of \( C_7 \)

We are now ready to express the one-loop \( b \to s \gamma \) magnetic penguin in terms of renormalized parameters and fields. The magnetic penguin obtained from the (unusual) tree level Lagrangian (8) is

\[
\mathcal{H}^{1 \text{ loop}}_{\text{top penguins}} = - \left\{ \frac{5}{12} g_t^0 g_b^0 m_t^{1/2} \mathcal{O}[\bar{Q} R, Q_{3L}] - \frac{1}{9} g_t^0 m_t^{1/2} \mathcal{O}[Q_{3L}, i \partial Q_{3L}] \right\} \left( \frac{\bar{\mu}}{m_t^2} \right)^{2 \epsilon}
\]

where \( \mathcal{O}[a, b] = \left( \frac{e}{(4 \pi)^2} (\bar{a} \gamma_{\mu} \nu F)_{\mu \nu} \right) \). Notice that no explicit mixing angle appears at this stage. The two contributions arise respectively from the one-loop graphs named “I” and “E” in fig. 1 and give rise to two different operators, that are equivalent only on-shell. Since the graph “E” is not one-particle irreducible, we must take this fact into account in the renormalization. Operating the redefinitions (6) and (9), and using the renormalized Dirac equation \( i \partial b = m_b b \) to set the \( b \)-quark on-shell, we obtain

\[
\mathcal{H}^{1 \text{ loop}}_{\text{top penguins}} = - \frac{g_F}{\sqrt{2}} s_{23} \left\{ \frac{5}{12} Z_{m_t} Z_{b_L} - \frac{1}{9} Z_{b_L} \right\} \left[ 1 + \epsilon \ln \left( \frac{m_t^2}{Z_{t_L} Z_{t_R}} \right) + O(\epsilon^2) \right] \mathcal{O}_7.
\]

The final formula for the coefficient \( C_7 \) of the \( b \to s \gamma \) magnetic penguin, that includes all renormalized \( O(g_t, \lambda)^2 \) corrections, is thus

\[
C_7 = \frac{23}{36} + \frac{5}{12} Z_{m_t} Z_{b_L} - \frac{1}{9} Z_{b_L} - \frac{g_t^2}{(4 \pi)^2} \frac{11}{24} \left( \text{two-loop diagrams} \right) + \left( \text{plotted in figure 2} \right).
\]

We have added the charm contribution, \(-23/36\), that does not get \( O(g_t, \lambda)^2 \) corrections. In conclusion the heavy top and/or heavy higgs correction to \( C_7 \), computed in the limit \( m_W \to 0 \), is

\[
C_7(m_t^2 > m_W^2) = -\frac{1}{3} + \frac{g_t^2}{(4 \pi)^2} C_{77}^{g_2}
\]

with

\[
C_{77}^{g_2} = \frac{-16 + 39 r - 11 r^2 - 26 r^3}{144 r} - \frac{-16 + 2 r - 36 r^2 + 74 r^4 - 45 r^5 + 2 r^6}{144 r^2} + \frac{-8 - 2 r - 6 r^2 + 52 r^3 + 85 r^4 - 33 r^5 + 2 r^6}{72 r^3} \text{Li}_2(1 - r) - \frac{80 + 68 r - 262 r^2 + 134 r^3 - 25 r^4 + 2 r^5}{288 r} \Phi(r \frac{r}{4}) + \frac{+ 8 - 17 r - 2 r^2 - 14 r^3}{72 r^2} \ln r - \frac{74 - 45 r + 2 r^2}{288 r^2} \ln^2 r \approx 0.14 - 0.0024 r - 0.046 \ln r
\]

where \( r = m_h^2 / m_t^2 \), the functions \( \text{Li}_2 \) and \( \Phi \) are defined in appendix [5], and the approximation holds for any reasonable value of \( m_h \in [60, 1000] \) GeV. Since, instead of using \( \overline{\text{MS}} \) subtractions, we have expressed the result in terms of physical quantities, no dependence on the \( \overline{\text{MS}} \) scale \( \bar{\mu} \) is left. In the limits \( m_h \ll m_t \) and \( m_h \gg m_t \) \( C_7 \) reduces to

\[
C_{77}^{g_2} (r \to 0) = \frac{\pi^2}{36} - \frac{1}{9} - \left( \frac{\pi^2}{36} - \frac{1}{16} \right), \quad C_{77}^{g_2} (r \to \infty) = \left( \frac{19}{144} \ln r + \frac{17}{432} \right) \frac{\pi^2}{24}
\]

where \( N_c = 3 \) is the number of colours. In table [6] the asymptotic dependences of these corrections in the limits of heavy top and of heavy higgs are compared to the corresponding limits of the corrections to the observables measured at LEP.
corrections the usual way (see [11] and references therein), that we now briefly recall. Neglecting, for simplicity, QCD Higgs mass in the limit of heavy top ad/or higgs.

\[
\text{Figure 3: The percentage correction to BR}(B \to X_s \gamma)\text{ for } m_t(\text{pole}) = (175 \pm 5)\text{ GeV and as function of the Higgs mass in the limit of heavy top ad/or higgs.}
\]

2.5 Branching ratio

With our renormalization scheme the \( B \to X_s \gamma \) branching ratio can be linked to the \( b \to s \gamma \) decay width in the usual way (see [11] and references therein), that we now briefly recall. Neglecting, for simplicity, QCD corrections

\[
\text{BR}(B \to X_s \gamma) = \frac{\Gamma(b \to s \gamma)}{\Gamma(b \to c \bar{c} \nu)} \text{BR}(B \to X_c \bar{c} \nu)
\]

where \( \text{BR}(B \to X_c \bar{c} \nu) = 0.105 \pm 0.05 \) has been measured [21].

\[
\Gamma(b \to s \gamma) = \frac{C_F^2}{32 \pi^3} m_b^5 |s_{23} c_{23}|^2 \frac{e^2}{4 \pi^2} |C_7(\bar{m}_b)|^2,
\]

\[
\Gamma(b \to c \bar{c} \nu) = \frac{C_F^2}{32 \pi^3} m_b^5 |s_{23}|^2 \frac{g(z)}{6}, \quad z \equiv \frac{m_c^2}{m_b^2}
\]

and \( g(z) \) is a phase space factor [11]. The \( \mathcal{O}(g_t, \lambda)^2 \) corrections to the semileptonic width are entirely contained in our definitions of the renormalized parameters. As previously said the complete CKM mixing can be reinserted neglecting the \( \mathcal{O}(g_t, \lambda)^2 \) corrections to the terms suppressed by powers of \( \lambda_W^2 : (s_{23} c_{23})^2 / (s_{23})^2 \to |V_{tb} V_{ts} / V_{cb}|^2 = 1 + \lambda_W^2 (2 \rho_W - 1) + \mathcal{O}(\lambda_W^4), \) (\( \lambda_W \approx 0.22 \) and \( \rho_W \) are Wolfenstein parameters [20]).

These expressions receive important QCD corrections (perturbative and non perturbative), that can be added in the usual way. Infact, the \( \mathcal{O}(g_t, \lambda)^2 \) corrections that we are considering, only affect the values of the Wilson coefficients at the electroweak scale, \( C_i(\mu_W), \) \( i = 1, \ldots, 8. \) We have computed the correction to \( C_7. \) Other relevant \( \mathcal{O}(g_t, \lambda)^2 \) corrections can be present in \( C_2 \) (the coefficient of the \((\bar{s} \gamma_\mu P_L c) (\bar{c} \gamma_\mu P_L b)\) operator) and in \( C_8 \) (the coefficient of the \( b \to s g \) chromo-magnetic penguin). In a formal expansion in powers of \( \alpha_3, \) the corrections to \( C_2 \) and \( C_8 \) enter at the same order as the correction to \( C_7. \)

- However, the correction to \( C_8 \) cannot have a significant effect on the \( B \to X_s \gamma \) decay, in view of the small mixing coefficient between \( C_7 \) and \( C_8: \)

\[
C_7(\bar{m}_b) = -0.155 + 0.65 C_7(\bar{\mu}_W) + 0.085 C_8(\bar{\mu}_W),
\]

where \( \bar{m}_b \sim m_b \) and \( \bar{\mu}_W \sim m_W. \) In any case, it is immediate to obtain the renormalized correction to \( C_8 \) from a subset of the two-loop graphs of fig[4]. In the heavy top limit \( C_8 \) has a weak dependence on the higgs mass \( m_h \) and can be approximated with its value at \( m_h = 0, \)

\[
C_8(\bar{\mu}_W) = -1/8 + g_t^2 / (4\pi^2)(\pi^2 / 12 - 13/32).
\]

- The \( \mathcal{O}(g_t, \lambda)^2 \) correction to \( C_2 \) vanishes in our renormalization scheme.

We remember that the coefficient \( C_7 \) can also be extracted from the spectrum of the \( B \to X_s \ell^+ \ell^- \) decays [4].
3 Numerical result

Due to a numerical accident, the potentially relevant heavy top correction to the $b \to s \gamma$ magnetic penguin turns out to be negligible, if the leading order result is expressed in terms of the renormalized parameters $G_F$ and $\text{BR}(B \to X_s \ell \nu)$. The percentage correction to BR($B \to X_s \gamma$) is plotted in fig. 3 as function of the higgs mass, $m_h$, and for $m_t = (175 \pm 5)$ GeV, where $m_t$ is the pole top mass. In the plot we have included the QCD corrections at NLO order and all the heavy top and/or heavy higgs corrections (also the small ones, not fully given in the text, due to the mixing with the chromo-magnetic penguin).

We see that the correction is never larger than 1% for any possibly interesting value of the higgs mass. We remember that small values of the Higgs mass are preferred. The electroweak precision measurements give $m_h < 250$ GeV at 90% C.L. [22]. Moreover, the minimal supersymmetric extension of the SM (MSSM) predicts a light higgs: at tree level it requires that $m_h < m_Z$. Adding the one-loop corrections [23] and barring an unnatural fine-tuning larger than 20, the MSSM bound on the higgs mass becomes $m_h < 120$ GeV [23].

As said, the smallness of the correction is due to a numerical accident. Before having done this computation a relevant heavy top correction, at the 5% level or even larger could not be excluded. However, due to accidental cancellations between different contributions these corrections are smaller than 1%.

4 Conclusions

We have computed the heavy top and the heavy higgs corrections to the Wilson coefficient of the $b \to s \gamma$ magnetic penguin in the limit $m_W \to 0$. These potentially relevant corrections turn out to be very small. Since there are accidental cancellations between different contributions, it is possible that the limit $m_W \to 0$ be not a good approximation. However it appears unlikely that the neglected corrections, suppressed by powers of $m_W^2/m_t^2$, be sufficiently large to make the heavy top effects relevant. For these reasons it seems safe to conclude that these kind of corrections can be neglected.

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A The relevant Lagrangian

As explained in the text we can work in the gauge-less limit. In a flavour basis where the Yukawa matrices $g_u$ and $g_d$ are diagonal, the relevant Lagrangian is

$$\mathcal{L} = \bar{q}i\gamma\partial q + |\partial H|^2 + \mu^2 |H|^2 - \lambda |H|^4 - (\bar{u}_R g_u Q H^* + \bar{d}_R g_d V^\dagger Q H + \text{h.c.}) \quad (A.1)$$

where $V$ is the CKM matrix, $d = \{d, s, b\}$, $u = \{u, c, t\}$, $q = \{u, d\}$. The contractions of the $SU(2)_L$ doublets $Q$ and $H$, expressed in terms of their up ($\uparrow$) and down ($\downarrow$) components, are

$$\bar{u}_R Q H^\uparrow = \bar{u}_R (Q^\dagger H^\uparrow + Q^\dagger H^\downarrow) \quad \text{and} \quad \bar{d}_R Q H = \bar{d}_R (Q_4 H^\uparrow - Q^\dagger H^\downarrow)$$

so that the mass terms of the quarks have conventional sign. Decomposing the fields into mass eigenstates (we denote the unphysical higgses with greek characters)

$$H = \begin{pmatrix} v + (h + i\eta)/\sqrt{2} \\ \chi^- \end{pmatrix}, \quad Q = \begin{pmatrix} u_L \\ V \bar{d}_L \end{pmatrix} \quad (A.2)$$

one obtains

$$\mathcal{L} = \mathcal{L}_\text{free} + \mathcal{L}_\text{Yuk} - V_{\text{cubic}} - V_{\text{quartic}} + \sqrt{2}v(\mu^2 - 2\lambda v^2)h \quad (A.3)$$

where

$$\mathcal{L}_\text{free} = \bar{q}i(\partial - m_q)q + |\partial H|^2 + \frac{(\partial h)^2}{2} + \frac{(\partial \eta)^2}{2} - \frac{(6\lambda v^2 - \mu^2)}{2} \frac{h^2}{2} + \frac{\mu^2 - 2\lambda v^2}{2} \frac{\eta^2}{2} + |\chi|^2$$

$$\mathcal{L}_\text{Yuk} = -g_b \left[ \frac{i h - i\gamma_5 \eta}{\sqrt{2}} \bar{b} \left( - (\chi^- - V_{1d} d_L t_R + \text{h.c.}) \right) + \bar{b} \left( - (\chi^- - V_{1d} d_L t_R + \text{h.c.}) \right) \right] - g_\mu \left[ \frac{i h - i\gamma_5 \eta}{\sqrt{2}} \bar{b} \left( \chi^- - V_{1d} d_L t_R + \text{h.c.} \right) \right]$$

Even if $g_2(\mu_W) \approx 1.2$ is not significantly larger than $g_2 \approx 1.0$, the well known NLO QCD effects of relative order $g_2^2/(4\pi)^2$ give significantly larger corrections (at the 20% level) because they have ‘more colour’ and ‘more spin’ circulating in the single Feynman graphs.

4The ones enhanced by a $N_c$ factor have recently been computed [13]. Including QCD corrections they give another $-1\%$ correction to $\text{BR}(B \to X_s \gamma)/\text{BR}(B \to X_c \ell \nu)$. 

\text{Page 8}
We write the inverse 1-loop propagator of a b quark with momentum $p_b$ as $p_b [Z_{bL} P_L + Z_{bR} P_R] - Z_{mb} m_b$ where $P_{L,R} = (1 \mp \gamma_5)/2$. Since there are no infrared divergences even in the limit $m_W = 0$, the constants $Z$ depend on $p_b$ only through irrelevant terms suppressed by powers of $p_b^2/m_b^2$. Their values are: $Z_{bL} = 1$, $Z_{bR} = 1$, $Z_{mb} = 1 + \frac{g_t^2}{(4\pi)^2} \left[ \frac{1}{\varepsilon} + \ln \frac{\tilde{m}_t^2}{m_b^2} + 1 \right]$, where $\varepsilon$ denotes the finite parts that we do not need.

We write the inverse 1-loop propagator of a t quark with momentum $p_t$ as $p_t [Z_{tL} P_L + Z_{tR} P_R] - Z_{mt} m_t$. We need only the divergent parts of the $Z$ constants. These divergent parts do not depend on $p_t$ and can be more easily computed in the limit of unbroken SU(2)$_L$ global symmetry. We find:

$$Z_{mt} = 1 + \frac{g_t^2}{(4\pi)^2} \left[ \frac{0}{\varepsilon} + \cdots \right], \quad Z_{tL} = 1 + \frac{g_t^2}{(4\pi)^2} \left[ \frac{1}{\varepsilon} + \cdots \right], \quad Z_{tR} = 1 + \frac{g_t^2}{(4\pi)^2} \left[ \frac{1}{2\varepsilon} + \cdots \right]$$

where $\cdots$ denotes the finite parts that we do not need.

We write the one-particle irreducible (1PI) propagator of a unphysical charged higgs $\chi$ with momentum $p_\chi$ as $i/(q^2 Z_{\chi} (p_\chi^2) - m_\chi^2 Z_{m_\chi} (p_\chi^2))$. As explained in the text we compute the vanishing of Goldstone masses, $m_\chi^2 = 0$, inserting appropriate counterterms (equivalent to the non 1PI tadpole graphs that shift the vev $v$ to the quantum corrected minimum). We only need the constant $Z_\chi$ at zero momentum. At one loop we find:

$$Z_\chi = 1 + \frac{1}{(4\pi)^2} \left[ \lambda + g_t^2 N_c \left( \frac{1}{\varepsilon} + \ln \frac{\tilde{m}_t^2}{m_\chi^2} + \frac{1}{2} \right) \right]$$

with $\lambda = +g_t^2 v/4$.

Our final result contains the bi-logarithmic function $Li_2(z)$ defined as $Li_2(x) \equiv -\int_0^1 \ln(1 - xt) \, dt/t$ and the function

$$\Phi(z) \equiv \begin{cases} \sqrt{\frac{\pi}{1-z}} \, 4 \, \text{Im} \, Li_2 \exp(2i \arcsin \sqrt{z}) & \text{for } 0 < z < 1 \\ \sqrt{\frac{\pi}{z-1}} \, \left( \frac{\pi}{3} - \ln^2(4z) + 2 \ln^2 \frac{1-\sqrt{1-1/z}}{2} - 4Li_2 \frac{1-\sqrt{1-1/z}}{2} \right) & \text{for } z > 1 \end{cases}$$

The one loop penguin functions used in the text are

$$P_W(x) = \frac{23 + 67 x - 50 x^2}{36(x-1)^3} + \frac{2 - 7 x + 6 x^2}{6(x-1)^2} \, x \ln x$$

$$P_I(x) = \frac{3 - 5 x}{6(x-1)^2} + \frac{3 x - 2}{3(x-1)^3} \ln x$$

$$P_E(x) = \frac{7 - 5 x - 8 x^2}{36(x-1)^3} + \frac{3 x - 2}{6(x-1)^2} \, x \ln x$$

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5The corresponding expression given by Barbieri et al in [4] contains a misprint (not present in their final result).
References

[1] See for example G. Buchalla, A. Buras and M. Lautenbacher, Rev. Mod. Phys. 68 (1996) 1125 (hep-ph/9512380) and references therein.

[2] S. Bertolini, F. Borzumati, A. Masiero and G. Ridolfi, Nucl. Phys. B353 (1991) 591; A. Ali, G. Giudice and T. Mannel, Z. Phys. C67 (1995) 417 (hep-ph/9408211).

[3] A. Ali, G. Giudice and T. Mannel, Z. Phys. C67 (1995) 417 (hep-ph/9408211).

[4] T. Goto, T. Nihei and Y. Okada, Phys. Rev. D53 (1996) 523 (hep-ph/9510306).

[5] T. Goto, T. Okada, Y. Shimizu and M. Tanaka, Phys. Rev. D55 (1997) 4273 (hep-ph/9609513).

[6] J. Hewett and J. Wells, Phys. Rev. D55 (1997) 5549 (hep-ph/9610328); see also ref. [7].

[7] Y. Grossman and M. Worah, Phys. Lett. B395 (1997) 241 (hep-ph/9612260); M. Worah, hep-ph/9711265.

[8] L.J. Hall, V.A. Kostelecky and S. Raby, Nucl. Phys. B267 (1986) 415;

[9] R. Barbieri and L.J. Hall, Phys. Lett. B338 (1994) 212;

[10] R. Barbieri, L. Hall and A. Strumia, Nucl. Phys. B449 (1995) 437 (hep-ph/9501326);

[11] J. Hisano, T. Moroi, K. Tobe and M. Yamaguchi, Phys. Rev. D53 (1996) 4312 (hep-ph/9510306);

[12] M.E. Gómez and H. Goldberg, Phys. Rev. D53 (1996) 5244 (hep-ph/9510307).

[13] G. Buchalla and A.J. Buras, Phys. Rev. D49 (1994) 4945 (hep-ph/9308320).

[14] C. Greub and T. Hurt, Phys. Rev. D56 (1997) 2934 (hep-ph/9704326). M. Ciuchini et al. in [7]; A.J. Buras, A. Kwentkowski and N. Pott, hep-ph/9711334.

[15] J. Fleischer, O.V. Tarasov and F. Jegerlehner, Phys. Rev. D51 (1995) 2934 (hep-ph/9510335).

[16] R. Barbieri et al., Phys. Lett. B361 (1995) 146 (hep-ph/9506379); N. Pott, Phys. Rev. D54 (1996) 938 (hep-ph/9512254).

[17] C. Greub, T. Hurth and D. Wyler, Phys. Lett. B380 (1996) 385 (hep-ph/9602282) and Phys. Rev. D54 3364 (1996) (hep-ph/9603304).

[18] A. Ali and C. Greub, Z. Phys. C49 (1991) 431; A. Ali and C. Greub, Phys. Lett. B361 (1995) 146 (hep-ph/9506379); N. Pott, Phys. Rev. D54 (1996) 938 (hep-ph/9512254).

[19] M. Ciuchini et al. in [7]; A.J. Buras, A. Kwentkowski and N. Pott, hep-ph/9711334.

[20] M. Ciuchini et al. in [7]; A.J. Buras, A. Kwentkowski and N. Pott, hep-ph/9711334.

[21] A. Ali, G. Giudice and T. Mannel, Z. Phys. C67 (1995) 417 (hep-ph/9408211).

[22] M. Ciuchini, E. Franco, G. Martinelli, A. Masiero and L. Silvestrini, Phys. Rev. Lett. 79 (1997) 978 (hep-ph/9704278); S.A. Abel, W.N. Cottingham and I.B. Whittingham, hep-ph/9803401.

[23] M. Carena, M. Quiroés and C.E.M. Wagner, Phys. Lett. B380 (1996) 81 (hep-ph/9603422); A. Riotto, hep-ph/9803357.

[24] A. Strumia, Phys. Lett. B397 (1997) 204 (hep-ph/9609286).

[25] K. Adel and Y-P. Yao, Phys. Rev. D49 (1994) 4945 (hep-ph/9308320).

[26] G. Degrassi, P. Gambino and G. Giudice, hep-ph/9710312. The first reference only contains all terms relevant in the MSSM; the other references also include the negligible corrections requested by a formal NLO expansion.

[27] M. Ciuchini, G. Degrassi, P. Gambino and G. Giudice, hep-ph/9710333.

[28] F.M. Borzumati and C. Greub, hep-ph/9802391. The one-loop QCD corrections to the matrix elements of $$\mathcal{O}_2$$ and $$\mathcal{O}_5$$ and the two-loop QCD corrections to the matrix element of $$\mathcal{O}_2$$ have never been recalculated. The NLO corrections to the matrix elements of the remaining operators are expected to be irrelevant and have never been computed.

[29] A. Czarnecki and W.J. Marciano, hep-ph/9804252.

[30] R. Barbieri et al., Nucl. Phys. B409 (1993) 105; J. Fleischer, O.V. Tarasov and F. Jegerlehner, Phys. Rev. D51 (1995) 3820.

[31] J. Van der Bij and M. Veltman, Nucl. Phys. B231 (1984) 205;

[32] R. Barbieri et al., Phys. Lett. B317 (1993) 381.

[33] G. Degrassi, P. Gambino and A. Sirìln, Phys. Lett. B394 (1997) 188 (hep-ph/9611364).

[34] G. Buchalla and A.J. Buras, Phys. Rev. D57 (1998) 216 (hep-ph/9707241).

[35] K. Fujikawa, B.W. Lee and A. Sanda, Phys. Rev. D6 (1972) 1923.

[36] S. Wolfram, The Mathematica book, 3rd ed. (Wolfram Media/Cambridge University Press, 1996).

[37] L. Wolfenstein, Phys. Rev. Lett. 51 (1983) 1945.

[38] CLEO collaboration, Phys. Rev. Lett. 76 (1996) 1570. The LEP experiments give slightly higher (and more questionable) values for the semileptonic branching ratio.

[39] For a review, see G. Altarelli, R. Barbieri and F. Caravaglios, Int. J. Mod. Phys A13 (1998) 1031 (hep-ph/9712365). The most recent analysis can be found at the LEP Electroweak Working Group Home Page, www.cern.ch/LEPEWWG.

[40] Y. Okada, M. Yamaguchi and T. Yanagida, Progr. Theor. Phys. 85 (1990) 1;

[41] M. Carena, J.R. Espínosa, M. Quiroés, C.E.M. Wagner, Phys. Lett. B355 (1995) 209 (hep-ph/9504315).

[42] G.W. Anderson, D.J. Castaño and A. Riotto, Phys. Rev. D55 (1997) 2950 (hep-ph/9609467);

[43] R. Barbieri and A. Strumia, hep-ph/9801353.