Enhancing Parameter Estimation Precision in a Dissipative Environment with Two-Photon Driving

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1. Introduction

Improving the precision of parameter measurement plays an unparalleled role in the development of basic science and technology.[1–6] In classical physics, the best metrology precision, known as the shot noise limit (SNL), scales as $1/\sqrt{N}$ with $N$ being the number of resources employed in the measurements. Quantum effects can help us overcome the SNL, such as squeezing[7–11] and entanglement.[12–15] Notably, the quantum effect of light can enhance the imaging resolution[16] in many important fields, such as radar[17,18] and gravitational wave detection.[19,20]

Although quantum effects can improve measurement accuracy, they are quite unstable because of environmental interference. A realistic physical system interacts with the surrounding environment, leading to loss of information from the system to the environment. When the decoherence induced by the environment is severe, the measurement accuracy should not exceed or be far below the SNL. Many works have shown that for noisy quantum metrology,[21–24] initial quantum entangled, or squeezed states cannot help in obtaining better precision than the SNL. Some methods have been proposed to prevent environmental interference. Dynamic decoupling is an active and effective method,[25,26] and it has been applied to suppress the decoherence in optical fibers[27,28] and in superconducting Qubits.[29] Generally, the strategy of dynamic decoupling has been primarily studied in the δ-pulse regime.[30,31] In dynamic decoupling, unitary control pulses are instantaneously applied to the system at specific times. However, while dynamic decoupling prevents environmental interference, it also prevents the encoding of parameter information. Therefore, dynamic decoupling cannot perform well in quantum metrology. It was found that the non-Markovian effect[32] can make the precision surpass the SNL in dephasing noise conditions with an Ohmic spectral density. However, as the encoding time increases, the precision worsens. In some special cases, the quantum advantages can be recovered with a long-time limit. Correlated environments[33] can make the precision surpass the SNL; however, symmetrical correlated environments are not common. Recently, Wang et al.[34] and Bai et al.[35] revealed that the Heisenberg limit and the ideal Zeno limit could be recovered in a long-encoding-time condition, which is attributed to the non-Markovian effect and the formation of a bound state between the quantum probe and its dissipative environment. However, a unique coupling spectral density is required to form the bound state. Hence, it is necessary to further investigate the ultrasensitive measurement in a typical dissipative Markovian environment under the long-encoding-time condition. It is worth noting that when regaining the SNL, the measurement accuracy is high enough under the long-encoding-time condition.

In this article, we use two-photon driving[36–38] to improve the frequency precision of the optical field in a typical unavoidable dissipative environment. Focusing on the long-encoding-time condition, in the phenomenological description,[39,40] the photons are lost; therefore, frequency information cannot be obtained. We investigate the functions of different magnitudes of the parametric drive in frequency estimation. Our analysis reveals that the two-photon drive can regain the SNL, and the uncertainty of frequency can be minimized with an appropriate magnitude of the parametric drive in the long-encoding-time condition. Our result shows that two-photon driving can realize ultrasensitive measurement in a dissipative environment by forming effective non-Hermitian parity-time (PT) symmetry dynamics.[41–46]

The article is organized as follows. In Section 2, we introduce the physical model and the mathematical description of two measurement methods. In Section 2.1, we consider the case of a small magnitude of the parametric drive in the long-encoding-time condition. In Section 2.2, the case of a large magnitude is discussed with the initial coherent state. The case of a specific magnitude is discussed in Section 2.3. In Section 3, a simple explanation is given. We make a brief conclusion and outlook in Section 4. In Appendix A, we give a detailed analytical derivation.
ment is unavoidable. This leads to greater uncertainty over frequency.

Figure 1. Diagram of estimating the frequency $\omega$. $|\psi_{in}\rangle$ represents the initial state. a) represents the direct photon detection, b) represents the homodyne detection. Dissipative noise has an impact during the process of encoding parameter $\omega$.

2. Frequency Measurement in Dissipative Noises

Typically, the process of estimating the parameter of a system can be classified into three steps: first, prepare the initial probe state; second, encode the parameter information; third, make the final measurement to obtain the parameter.

Here, we want to measure the frequency $\omega$ of an optical cavity field, as shown in Figure 1. We prepare the photons with the initial pure state $|\psi_{in}\rangle$. Thereafter, the information of the frequency $\omega$ is encoded. In ideal metrology, the measurement uncertainty scales are $1/t$, where $t$ denotes the interrogation time. Therefore, for a long-encoding-time, one can obtain $\omega$ with high accuracy. However, during the encoding, the dissipative environment is unavoidable. This leads to greater uncertainty over time. The precision of frequency $\omega$ worsens. We will utilize two-photon driving to improve the result. In the last step, the final detection obtains the information of $\omega$. In Figure 1, we consider that there are two feasible detection schemes: (a) represents direct photon detection; (b) represents homodyne detection. Both detection schemes are not optimal, which cannot saturate the Cramér-Rao bound governed by quantum Fisher information. However, it sufficiently demonstrates the superiority of a two-photon drive in an experimental setup.

In the reference frame rotating at the coherent pump frequency $\omega_p$, the total Hamiltonian is described by ($\hbar = 1$ throughout this article)

\[
\hat{H} = \Delta \hat{a}^\dagger \hat{a} + \sum_k [\Delta \omega_k \hat{b}_k^\dagger \hat{b}_k + g_k (\hat{a} \hat{b}_k^\dagger + \hat{a}^\dagger \hat{b}_k)] + \frac{\lambda^2}{2} (\hat{a}^\dagger - \hat{a}^2)
\]

where $\Delta = (\omega - \omega_p)$ is the cavity-pump detuning, $\Delta \omega_k = (\omega_k - \omega_p)$ is the environment-pump detuning, $\hat{a}$ is the annihilation operator of the optical cavity field with frequency $\omega$, $\hat{b}_k$ is the annihilation operator of the $k$th environmental mode with frequency $\omega_k$, and $g_k$ denotes its coupling strength to the optical field. $\lambda$ is the magnitude of the two-photon drive, which can be realized in down-conversion processes in nonlinear optics. The environment structure can be characterized by the spectral density

\[
J(\omega') = \sum_k g_k^2 \delta(\omega' - \Delta \omega_k).
\]

In the Heisenberg picture, we can obtain $\hat{a}(t) = G(t)\hat{a}(0) + \int_0^t d\tau G(t)\hat{a}\hat{b}(\tau) + \sum_k [\mu_k(t)\hat{b}_k(\tau) + v_k(\tau)\hat{b}_k^\dagger(\tau)]$, with $|G(t)|^2 + |L(t)|^2 + \sum_k (|\mu_k(t)|^2 + |v_k(\tau)|^2) = 1$. At the same time, $G(t)$, $L(t)$, $\mu_k(t)$, and $v_k(t)$ must satisfy following equations

\[
G(t) = \lambda L(t) - i\Delta G(t) - \int_0^t ds K(t - s) G(s)
\]

\[
\dot{L}(t) = iG(t) + i\Delta L(t) - \int_0^t ds K^*(t - s) L(s)
\]

\[
\dot{\mu}_k(t) = \lambda v_k - i\Delta \mu_k(t) - \int_0^t ds K(t - s) \mu_k(s) - ig_k e^{i\omega t}
\]

\[
\dot{v}_k(t) = \lambda \mu_k - i\Delta v_k(t) - \int_0^t ds K^*(t - s) v_k(s)
\]

where $K(t-s) = \int_0^\infty d\omega J(\omega')e^{i\omega(t-s)}$ is the noise correlation function and the initial conditions $G(0) = 1$, $L(0) = 0$, $\mu_k(0) = 0$, and $v_k(0) = 0$ (see Appendix A). We suppose that the bandwidth of the interaction spectrum is much larger than the coupling strength. Therefore, the Wigner–Weisskopf approximation can be used, which has been proved to be equivalent to the Markov approximation. By replacing $J(\omega')$ with $J(\Delta)$ and extending the lower limit of the integral to be $-\infty$, we can obtain

\[
K(t-s) = 2\gamma \delta(t-s)
\]

where $\gamma = \pi J(\Delta)$. Then we can obtain the analytical results of $G(t)$, $L(t)$, $\mu_k(t)$, and $v_k(t)$ by the Laplace transform

\[
G(t) = e^{-t\gamma} \left( \cosh[t\sqrt{\lambda^2 - \Delta^2}] - \frac{i\Delta \sinh[t\sqrt{\lambda^2 - \Delta^2}]}{\sqrt{\lambda^2 - \Delta^2}} \right)
\]

\[
L(t) = \lambda e^{-t\gamma} \frac{\omega \sinh[t\sqrt{\lambda^2 - \Delta^2}]}{\sqrt{\lambda^2 - \Delta^2}}
\]

The detail analytical results of $\mu_k(t)$ and $v_k(t)$ are shown in Appendix A.

Then, the best precision of estimating $\Delta$ can be evaluated by the error propagation formula,

\[
\delta\omega = \frac{\delta M}{\delta \omega} = \frac{\delta M}{\delta M/\delta \Delta}
\]
\[ \delta M = \sqrt{\langle \psi(0)|\psi(0) \rangle - \bar{M}^2} \quad \bar{M} = \langle \psi(0)|e^{-iHt}\rangle \] (9)

where \( \langle \psi(0)|=\langle 0| \) and \( \psi(0) \) is the initial state. We assume that the noise is initially in a vacuum state \( \langle |0\rangle = |0\rangle \). For direct photon detection, the measurement operator is the field quadrature \( \hat{M}_d = (\hat{a}^\dagger + \hat{a})/2 \), where the measurement angle \( \theta \) can be controlled by the local oscillator.\[54\]

2.1. Small Magnitude of the Parametric Drive in the Long-Encoding-Time Condition

In ideal metrology, the uncertainty of parameter \( \omega \) is proportional to \( 1/\tau \), where \( \tau \) denotes the interrogation time. The measurement precision of \( \omega \) is almost 0 for the long-encoding-time. However, in the Markovian noise, the information of the parameter disappears for a long time.\[33\] In this section, we consider that the precision is obtained with a small magnitude \( \lambda \) of the parametric drive in the long-encoding-time condition. When \( \lambda^2 < \gamma^2 + \Delta^2 \), we define it as a small magnitude.

For the long-encoding-time, \( t |\gamma - (\lambda^2 - \Delta^2)^{1/2} | \gg 1 \)

we obtain

\[ \hat{\lambda}(t) = \sum_{k=0}^{\infty} \frac{g_k(-i\gamma - \Delta - \Delta_k)e^{-i\Delta_k t}}{\lambda^2 - \Delta^2 - (\gamma - i\Delta_k)^2} \hat{b}_k + \frac{g_k A e^{i\Delta_k t}}{\lambda^2 - \Delta^2 - (\gamma + i\Delta_k)^2} \hat{b}_k^\dagger \] (10)

The information of \( \omega \) can be obtained by direct photon detection, but not by homodyne detection because of \( \bar{M}_d = 0 \). With direct photon detection, we can derive that (see Appendix B)

\[ \bar{M}_d = \langle \hat{a}^\dagger(t)\hat{a}(t) \rangle = \frac{\lambda^2}{-\lambda^2 + \Delta^2 + \gamma^2} \] (11)

\[ \delta M_d = \langle \hat{a}^\dagger(t)\hat{a}(t) \rangle^{1/2} = \frac{\lambda^2}{-\lambda^2 + \Delta^2 + \gamma^2} \] (12)

Substituting the above equations into Equation (10), the uncertainty of \( \omega \) is derived

\[ \delta\omega^2 \approx \frac{(-\lambda^2 + \gamma^2 + \Delta^2)[(-\lambda^2 + \gamma^2 + \Delta^2)]}{4\lambda^2 \Delta^2} \] (13)

We can see that without the two-photon drive \( (\lambda = 0) \) the uncertainty \( \delta\omega = \infty \), which means that the information of parameter disappears. With the two-photon drive \( (\gamma \neq 0) \) the uncertainty is a finite value, which means that we can still obtain the information of the parameter. When \( \lambda^2 \rightarrow \gamma^2 + \Delta^2 \), the uncertainty is \( \delta\omega \approx 0 \). It shows that the parameter \( \omega \) can be measured accurately, and the estimation precision of \( \omega \) is independent of the initial state of the optical cavity field.

2.2. Large Magnitude of the Parametric Drive

In this section, we investigate the estimation precision with a large magnitude, \( \lambda^2 > \gamma^2 + \Delta^2 \). For a long-encoding-time, \( t [(\lambda^2 - \Delta^2)^{1/2} - \gamma] \gg 1 \), we obtain

\[ \hat{\lambda}(t) = e^{(\sqrt{\gamma^2 - \lambda^2} - \gamma)} \left\{ \frac{1}{2} \left( 1 - \frac{i\Delta}{\sqrt{\lambda^2 - \Delta^2}} \right) \hat{a}^\dagger + \frac{\lambda}{2\sqrt{\lambda^2 - \Delta^2}} \hat{a}^\dagger + \sum_{k} \hat{b}_k \right\} \] (14)

where

\[ \hat{b}_k = \frac{-g_k[i\lambda^2 + (-i\Delta + i\Lambda)(\gamma + \Delta) - \Delta_k)]}{2\Lambda(\lambda^2 - (\gamma + i\Delta_k)^2)} \hat{b}_k + \frac{i\lambda g_k[\gamma+\Lambda+i\Delta_k]}{2\Lambda(\lambda^2 - (\gamma + i\Delta_k)^2)} \hat{b}_k^\dagger \]

with \( \Lambda = \sqrt{\lambda^2 - \Delta^2} \). When the initial state is the coherent state \( |\psi(0)\rangle = |\alpha\rangle \), \( \alpha^2 \gg 1 \), we use two methods to measure the parameter \( \Delta \). For direct photon detection, the uncertainty is described by (see Appendix B)

\[ \delta\omega^2 \approx \frac{\lambda(\lambda - \Delta)(\lambda + \Delta)}{4Nt^2\Delta(\lambda + \sqrt{\lambda^2 - \Delta^2})} \] (15)

where \( N = \alpha^2 \) denotes the number of input photons. We can see that \( \delta\omega^2 \ll 1/Nt^2 \), recovering the scale in ideal metrology. From the above equation, we can see that the uncertainty of \( \omega \) will increase with the magnitude \( \lambda \). This shows that the optimal magnitude should be close to \( \sqrt{\gamma^2 + \Delta^2} \).

From Equation (15), direct photon detection obtained low precision for a small value of \( \Delta \), especially for \( \Delta \approx 0 \). We consider homodyne detection to solve this question. Using a similar calculation (see Appendix B), we obtain the uncertainty of \( \omega \) with homodyne detection

\[ \delta\omega^2 \approx \frac{\lambda^2}{16Nt^2\Delta^2} \quad \text{for } \theta = 0 \] (16)

\[ \delta\omega^2 \approx \frac{\lambda^2}{8Nt^2} \quad \text{for } \theta = \pi/2 \] (17)

where we consider \( \lambda \gg \Delta > 0 \) and \( \Delta^2t \gg \lambda^2 \). This result shows that the homodyne detection with the angle \( \theta = 0 \) performs like the case of direct photon detection. Homodyne detection with the angle \( \theta = \pi/2 \) can perform very well with a low value of the parameter \( \Delta \) (as shown in Figure 2). It is worth mentioning that for \( \Delta \approx 0 \), the estimation precision is given by a similar calculation

\[ \delta\omega^2 \approx 0 \quad \text{for } \theta = \pi/2 \] (18)
The parameter can perform better than in the case for the angle $\theta = \pi/2$ of the optical cavity field in the Heisenberg picture is described. We discuss the case of $\Lambda = \pi$, in which, $\Delta = \pi$ and $\Delta = \pi/2$. By using the initial coherent state $|\alpha\rangle$, we can obtain the estimation precision for $\gamma > 0$ (see Appendix B)

$$\delta \omega^2 \approx \frac{\gamma^2(7\gamma^2 + 3\Delta^2)}{4N\Delta^2(\gamma + \sqrt{\gamma^2 + \Delta^2})}, \quad \text{for } \theta = 0$$

$$\delta \omega^2 \approx \frac{\gamma^2(7\gamma^2 + 3\Delta^2)}{4N\Delta^2}, \quad \text{for } \theta = \pi/2$$

For $\Delta = 0$, the estimation precision can be given by $\delta \omega^2 \approx 7\gamma^2/(4N)$ with the measurement angle $\theta = \pi/2$.

Comparing the results in Sections 2.1 and 2.2, we find that only for a small value of $\gamma$ ($\gamma \ll \Delta$), the estimation with the magnitude $\lambda = \sqrt{\gamma^2 + \Delta^2} \approx \Delta$ is optimal for estimating the frequency $\omega$. With the homodyne detection, the measurement precision obtained at the specific magnitude $\lambda = \sqrt{\gamma^2 + \Delta^2}$ is slightly lower than that obtained near the specific magnitude $\lambda = \sqrt{\gamma^2 + \Delta^2}$ for a large value $\gamma$. For any value of $\gamma$, it is worth further exploring whether the optimal measurement precision can be obtained at point $\lambda = \sqrt{\gamma^2 + \Delta^2}$ by optimizing the measurement scheme (we leave it as an open question).

In conclusion, with the two-photon drive ($\lambda > 0$), the uncertainty of the frequency is finite for a long time, which is in contrast to the results without a drive. When $\gamma$ is close to or equal to $\sqrt{\gamma^2 + \Delta^2}$, the uncertainty of the frequency $\omega$ can be close to 0 for $\Delta \neq 0$ and long $t$. This means that direct photon detection and homodyne detection approach the optimal measurement scheme under the long-encoding-time condition and $\Delta \neq 0$.

### 3. Discussion

We have investigated the estimation precision of the frequency $\omega$ for long encoding time. The results show that the two-photon drive can help improve the estimation. When $\gamma \ll \Delta$, the optimal magnitude is at a specific point $\lambda = \sqrt{\gamma^2 + \Delta^2} \approx \Delta$; otherwise, the magnitude $\lambda = \Delta$ is not optimal in estimating the frequency $\omega$. We can give an explanation with the PT symmetry dynamics. Defining a vector of operators

$$|\tilde{a}(t)\rangle = (\tilde{a}(t), \tilde{a}^\dagger(t))^T$$

the Heisenberg equations of motions can be written as

$$i\partial_t |\tilde{a}(t)\rangle = \hat{H}_{\text{eff}}|\tilde{a}(t)\rangle + (\hat{F}, \hat{F}^\dagger)^T$$

where $\hat{F} = \sum_k g_k \hat{b}_k(t)$ and the effective Hamiltonian $\hat{H}_{\text{eff}}$ is

$$\hat{H}_{\text{eff}} = \left(\begin{array}{cc} \Delta & i\lambda \\ i\lambda & -\Delta \end{array}\right)$$

This effective Hamiltonian is non-Hermitian PT symmetrical. The eigenvalues are written as $\pm \sqrt{\Delta^2 - \lambda^2}$. The exceptional point is at $\Delta = \lambda$. Many works have shown that the exceptional point can improve the estimation precision. Therefore, for $\gamma \ll \Delta$, in Section 2.3, we prove that the optimal magnitude is $\lambda = \sqrt{\Delta^2 + \gamma^2} \approx \Delta$. However, when $\sqrt{\Delta^2 + \gamma^2}$ is not close to $\Delta$, the magnitude $\lambda = \Delta$ is not optimal. It is because that quantum-limited signal to noise at EPs is proportional to the perturbation from the environment.

According to the calculations in Sections 2.1–2.3, when the magnitude $\lambda$ is near $\sqrt{\Delta^2 + \gamma^2}$, the uncertainty is close to 0. The reason is that when the magnitude is close to $\sqrt{\Delta^2 + \gamma^2}$, the number of exciting photons $(\partial t |\tilde{a}(t)\rangle)$ is close to infinity. Therefore, considering the cost of energy (or time), the optimal estimation precision of $\Delta$ should not be 0.

Our model is feasible over a wide range of parameters, and readily realizable in current quantum experimental equipment. The two-photon driving can be controlled by using standard nonlinear wave-mixing techniques. Although our research is in quantum optical systems, it can be extended to general Bose
4. Conclusion and Outlook

We have investigated the function of the two-photon drive in improving the estimation precision of the field frequency. By applying the Wigner–Weisskopf approximation, we obtain the analytical results of the dynamics of the field operator in the Heisenberg picture. With direct photon detection and homodyne detection, we reveal that the detrimental results from the dissipation noise can be suppressed. With the two-photon drive, the uncertainty of frequency is finite for a long time, which is in contrast to the results without a drive. Moreover, with a long encoding time, the estimation uncertainty can be close to 0 for the magnitude $\lambda$ close to $\sqrt{\Delta^2 + \gamma^2}$.

The present study is expected to impact the understanding of two-photon driving significantly. It is helpful for practical high-precision sensor design. The upcoming work is on the role of quantum resources, such as quantum entangled or the squeezed state, in improving the accuracy of parameter measurements in noisy environments using two-photon drives. Concurrently, the role of the two-photon drive in non-Markov environments is also worth investigating.

Appendix A

We give the detailed derivation of the dynamical equation under the total Hamiltonian

$$\hat{H} = \Delta \hat{a}^\dagger \hat{a} + \sum_k [\Delta \omega_k \hat{b}_k^\dagger \hat{b}_k + g_k (\hat{a} \hat{b}_k^\dagger + \hat{a}^\dagger \hat{b}_k) + i(\hat{a}^\dagger \hat{a}^\dagger - \hat{a}^\dagger \hat{a})]$$

(A1)

In the Heisenberg picture, the equations of motion of the field operator read

$$\dot{\hat{a}}(t) = \lambda \hat{a}(t) - i \Delta \hat{a}(t) - i \sum_k g_k \hat{b}_k(t)$$

(A2)

$$\dot{\hat{b}}_k(t) = -i \Delta \omega_k \hat{b}_k(t) - ig_k \hat{a}(t)$$

(A3)

Substituting the formal solution of Equation (27)

$$\hat{b}_k(t) = \hat{b}_k(0) e^{-i \Delta \omega_k t} - ig_k \int_0^t e^{-i \Delta \omega_k (t-s)} \hat{a}(s) ds$$

(A4)

into Equation (26), we obtain

$$\dot{\hat{a}}(t) = \lambda \hat{a}(t) - i \Delta \hat{a}(t) - \int_0^t ds K(t-s) \hat{a}(s) - i \sum_k g_k \hat{b}_k(0) e^{-i \Delta \omega_k t}$$

(A5)

where $K(t-s) = \int_0^\infty d\nu f(\nu) e^{-i \nu(t-s)}$ is the noise correlation function. Using the Wigner–Weisskopf approximation $K(t-s) = \pi J(\Delta \hat{a}(t-s))$. The linearity of Equation (24) implies that a general field operator $\hat{a}(t)$ can be expanded as

$$\hat{a}(t) = G(t) \hat{a}(0) + L^\ast(t) \hat{a}(0) + \sum_k \mu_k(t) \hat{b}_k(0) + \nu_k(t) \hat{b}_k^\dagger(0)$$

(A6)

We can obtain $|G(t)|^2 + |L(t)|^2 + \sum_k |\mu_k(t)|^2 + |\nu_k(t)|^2 = 1$ from the commutation relation $[\hat{a}(t), \hat{a}^\dagger(t)] = 1$. Substituting this expansion into Equation (28), Equations (2)–(5) can be solved analytically

$$G(t) = e^{\gamma t} \left[ \cosh(t \Lambda) - \frac{i \Delta \sinh(t \Lambda)}{\Lambda} \right], \quad L(t) = \frac{\lambda e^{\gamma t} \Delta \sinh(t \Lambda)}{\Lambda}$$

(A7)

$$\mu_k(t) = \left[ i g_k (\nu + \Delta - i \Delta \omega_k) \Delta^2 \left[ e^{-\nu(t-\nu - t)} - e^{\nu(t-\nu + t)} \right] \right]$$

$$+ \left[ i e^{2\Delta t} \Delta - 1 + e^{2\nu t} \Delta \right] \frac{2 \lambda \Delta^2}{(\Lambda^2 - (\nu - i \Delta \omega_k)^2)}$$

$$\nu_k(t) = -i g_k e^{-\nu(t+\Delta)} \times \Lambda + e^{2\Delta t} \Delta + e^{2\nu t} \Delta \Lambda + (\nu - i \Delta \omega_k)$$

(A8)

where $\Lambda = \sqrt{\lambda^2 - \Delta^2}$.

Appendix B

For a small magnitude and long duration, the field operator can be simplified as

$$\hat{a}(t) = \sum_k \frac{g_k \lambda e^{-i \Delta \omega_k t}}{\lambda^2 - \Delta^2 - (\nu - i \Delta \omega_k)^2} \hat{b}_k^\dagger$$

(B1)

The variance of direct photon detection $M_d$ can be calculated as

$$\delta M_d = \langle (\hat{a}^\dagger(t) \hat{a}(t))^2 \rangle - \langle (\hat{a}^\dagger(t) \hat{a}(t))^2 \rangle^2$$

(B2)

where $\langle \cdot \rangle = \langle \Psi(0) | \langle \cdot \rangle | \Psi(0) \rangle$. We use the decoupling relation to calculate the above equation, which is written as

$$\langle A B C D \rangle \approx \langle A B \rangle \langle C D \rangle + \langle A \hat{D} \rangle \langle B \hat{C} \rangle + \langle A \hat{C} \rangle \langle B \hat{D} \rangle$$

$$- 2 \langle \hat{A} \hat{B} \rangle \langle \hat{C} \hat{D} \rangle$$

(B3)

Using the Wigner–Weisskopf approximation, we can obtain

$$\sum_k \left| \frac{g_k \lambda e^{-i \Delta \omega_k t}}{\lambda^2 - \Delta^2 - (\nu - i \Delta \omega_k)^2} \right|^2$$

$$= \int_0^\infty d\Delta \omega_j \frac{\left| \left(-i \nu - \Delta - i \Delta \omega_j \right) \right|^2}{\lambda^2 - \Delta^2 - (\nu - i \Delta \omega_j)^2}$$

$$= 1 + \frac{\lambda^2}{2(\nu^2 + \Delta^2 - \lambda^2)}$$

(B4)
\[
\sum_i \left| \frac{g_i \lambda e^{i \omega t}}{\lambda^2 - \Delta^2 - (\gamma + i \Delta \omega_t)^2} \right|^2 = \int_{-\infty}^{\infty} d\Delta \omega_t J(\Delta) \left| \frac{\lambda}{\lambda^2 - \Delta^2 - (\gamma + i \Delta \omega_t)^2} \right|^2 = \frac{\lambda^2}{2(\gamma^2 + \Delta^2 - \lambda^2)} (B5)
\]

\[
\sum_i \frac{g_i (-i \gamma - \Delta - \Delta \omega_t) e^{i \omega t}}{\lambda^2 - \Delta^2 - (\gamma - i \Delta \omega_t)^2} + \frac{g_i \lambda e^{i \omega t}}{\lambda^2 - \Delta^2 - (\gamma + i \Delta \omega_t)^2} = \frac{\lambda(\gamma + i \Delta)}{2(\gamma^2 + \Delta^2 - \lambda^2)} (B6)
\]

Using the above equations, we can obtain the uncertainty of parameter \(\omega\)

\[
\delta \omega^2 \approx \frac{(-\lambda^2 + \gamma^2 + \Delta^2)^2 [-\lambda^2 + 3(\gamma^2 + \Delta^2)]}{4\lambda^2 \Delta^2} (B7)
\]

Utilizing similar methods and \(\hat{a}[\Delta \omega_t] = a[\alpha] (\alpha\ \text{is\ real})\), we can obtain the results for a large magnitude \(\lambda > \sqrt{\Delta^2 + \gamma^2}\). With direct photon detection, the estimation precision is given by

\[
\delta \omega^2 \approx \frac{\lambda(\lambda^2 - \Delta^2)^2 (\lambda + \sqrt{\lambda^2 - \Delta^2})}{N \Delta^2 \{ -2 \lambda^4 t + 2 \lambda^2 t \Delta^2 + \sqrt{\lambda^2 - \Delta^2} \}^2} (B8)
\]

where \(N = a^2\) denotes the number of input photons.

With the homodyne detection, the estimation precision is given by

\[
\delta \omega^2 \approx \frac{\lambda(\lambda^2 - \Delta^2) (\lambda + \Delta)^2 [(\lambda^2 - \Delta^2 - 2 \gamma \sqrt{\lambda^2 - \Delta^2}) \cos \theta + (-2 \gamma + \sqrt{\lambda^2 - \Delta^2} (\lambda + \Delta \sin \theta)]}{2N(-\gamma + \sqrt{\lambda^2 - \Delta^2} \{ \Delta(\lambda - 2 \lambda^2 t + 2 t \Delta^2 - 2 \lambda t \sqrt{\lambda^2 - \Delta^2}) - (\lambda^2 - 2 t \Delta^2 \sqrt{\lambda^2 - \Delta^2} \sin \theta) \}^2} (B9)
\]

At the specific magnitude \(\lambda = \sqrt{\Delta^2 + \gamma^2}\), using a similar calculation, we can derive

\[
\delta \omega^2 \approx \frac{\gamma^2 (7 \gamma^2 + 3 \Delta^2 + 2 \Delta \sqrt{\gamma^2 + \Delta^2} \sin \theta)}{4N \{ \Delta[\gamma t^2 + (-1 + \lambda t) \sqrt{\gamma^2 + \Delta^2}] \cos \theta - [\gamma^2 + (1 - \lambda t) \Delta^2] \sin \theta \}^2} (B10)
\]

Under different specific conditions, the above equations can be further simplified in Sections 2.2 and 2.3.

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**Conflict of Interest**

The authors declare no conflict of interest.

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[1] V. Giovannetti, S. Lloyd, L. Maccone, Science 2004, 306, 1330.
[2] V. Giovannetti, S. Lloyd, L. Maccone, Phys. Rev. Lett. 2006, 96, 010401.
[3] K. Bongs, R. Launay, M. A. Kasevich, Appl. Phys. B 2006, 84, 599.
[4] M. G. A. Paris, Int. J. Quant. Inf. 2009, 7, 125.
[5] M. A. Taylor, J. Janousek, V. Daria, J. Knittel, B. Hage, H.-A. Bachor, W. P. Bowen, Nat. Photonics 2013, 7, 229.
[6] V. Giovanetti, S. Lloyd, L. Maccone, Science 2004, 306, 1330.
[7] A. Asasi et al., Nat. Photon. 2013, 7, 613.
[8] P. Cappellaro, J. Emerson, N. Boulant, C. Ramanathan, S. Lloyd, D. G. Cory, Phys. Rev. Lett. 2005, 94, 020502.
[9] T. Nagata, R. Okamoto, J. L. O’Brien, K. Sasaki, S. Takeuchi, Science 2007, 316, 726.
[10] X. -Y. Luo, Y.-Q. Zou, L.-N. Wu, Q. Liu, M.-F. Han, M. K. Tey, L. You, Science 2017, 355, 620.
[11] Y. Israel, S. Rosen, Y. Silberberg, Phys. Rev. Lett. 2014, 112, 103604.
[21] U. Dorner, R. Demkowicz-Dobrzanski, B. J. Smith, J. S. Luneed, W. Wasilewski, K. Banaszek, I. A. Walmsley, Phys. Rev. Lett. 2009, 102, 040403.

[22] R. Demkowicz-Dobrzanski, U. Dorner, B. J. Smith, J. S. Luneed, W. Wasilewski, K. Banaszek, I. A. Walmsley, Phys. Rev. A 2009, 80, 013825.

[23] F. Hudelist, J. Kong, C. Liu, J. Jing, Z. Y. Ou, W. Zhang, Nat. Commun. 2014, 5, 3049.

[24] Y.-S. Wang, C. Chen, J.-H. An, Phys. Rev. A 2017, 80, 013825.

[25] F. Hudelist, J. Kong, C. Liu, J. Jing, Z. Y. Ou, W. Zhang, Nat. Commun. 2014, 5, 3049.

[26] M. Heinz, R. König, Phys. Rev. A 2019, 80, 013825.

[27] A. Biswas, D. A. Lidar, Phys. Rev. A 2006, 74, 062303.

[28] S. Longhi, Phys. Rev. Lett. 2019, 123, 010408.

[29] A. W. Chin, S. F. Huelga, M. B. Plenio, Phys. Rev. Lett. 2012, 109, 233601.

[30] D. Xie, A. Wang, Phys. Rev. Lett. 2014, 112, 2079.

[31] J. Ren, H. Hodaei, G. Harari, A. U. Hassan, W. Chow, M. Soltani, D. Christodoulides, M. Khajavikhan, Opt. Lett. 2017, 42, 1556.

[32] V. Weisskopf, E. Wigner, Eur. Phys. J. A 1930, 63, 54.

[33] W. Louisell, Quantum Statistical Properties of Radiation, Wiley, New York 1973, pp. 418–428.

[34] H. Wiseman, G. Milburn, Quantum Measurement and Control, Cambridge University Press, New York 2010.

[35] S. F. Huelga, C. Macchiavello, T. Pellizzari, A. K. Ekert, M. B. Plenio, J. I. Cirac, Phys. Rev. Lett. 1997, 79, 3865.

[36] L.-P. Yang, C. Y. Cai, D. Z. Xu, W.-M. Zhang, C. P. Sun, Phys. Rev. A 2013, 87, 021210.

[37] S. Longhi, Phys. Rev. Lett. 2018, 120, 150501.

[38] Y. Shen, The Principles of Nonlinear Optics, Wiley, New York 1984.

[39] C. Macklin, K. O’Brien, D. Hover, M. E. Schwartz, V. Bolkhovsky, X. Zhang, W. D. Oliver, I. Siddiqi, Science 2015, 350, 307.

[40] J. Naskoo, K. Thapliyal, A. Pathak, S. Banerjee, Phys. Rev. A 2018, 97, 063840.