ALEXANDROV-BAKELMAN-PUCCI TYPE ESTIMATE FOR PLURISUBHARMONIC FUNCTIONS

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ABSTRACT. We prove an optimal Alexandrov-Bakelman-Pucci type estimate for plurisubharmonic functions without assuming their continuity. This generalizes a result of Y. Wang. As a corollary we generalize an estimate from [DD19]. We also address a problem posed in [Wan12].

1. Introduction

The Alexandrov weak maximum principle is a basic tool in modern PDE theory. In its classical version for a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ living in a bounded domain $\Omega$ it reads:

\[
\sup_{\Omega} u \leq \sup_{\partial \Omega} u + \frac{\text{diam}(\Omega)}{\omega_n^{1/n}} \left( \int_{\{ -u = \Gamma_{-u} \}} |\det D^2 u| \right)^{\frac{1}{n}},
\]

see Lemma 9.2 in [GT01]. Recall that $\omega_n$ above stands for the volume of the unit ball in $\mathbb{R}^n$ and $\{ -u = \Gamma_{-u} \}$ is the so-called contact set (see below). This inequality is especially fundamental in the viscosity theory of nonlinear elliptic second order equations (see [CIL92] for the notions of viscosity theory). In particular it is instrumental in the proof of the more general Alexandrov-Bakelman-Pucci estimate, which establishes a uniform bound on the viscosity supersolutions $u$ of the equation

\[
F(D^2 u) = f,
\]

with $F$ being a uniformly elliptic second order differential operator and $f \in C(\Omega)$. The Alexandrov-Bakelman-Pucci estimate, or ABP for short, reads:

\[
\sup_{\Omega} u^- \leq C \text{diam}(\Omega) \left( \int_{\Omega \cap \{ u = \Gamma_u \}} (f^+)^n \right)^{\frac{1}{n}},
\]

(see Theorem 3.6 in [CC95]), where $u$, which is continuous and non-negative on the boundary, is a viscosity supersolution of (2), $u^-$ denotes $\max \{ -u, 0 \}$, $f^+ := \max \{ f, 0 \}$ and $C$ is a universal constant.

Nowadays many improvements of this estimate exist under special assumptions on $u$ or on the equation (see [Cab95], [CCKS96], [AIM06] to mention just a few). In any case a Sobolev regularity of order at least $W^{2,p}_{loc}, p > \frac{n}{2}$ is required for $u$ for the theory to work for general second order nonlinear equations. Note that by the Sobolev embedding this forces $u$ to be continuous.

Recently viscosity methods were applied for the complex Monge-Ampère equation, see [Zer13]. In [Wan12] Y. Wang, extending results from [Blo05] and [CP92], proved in this setting that if $u$ is plurisubharmonic, (PSH for short) and continuous one has the following bound:

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\[ \sup_{\Omega} u^{-} \leq C d \left\| f \cdot \chi_{\{u = \Gamma_u\}} \right\|_{L^2(\Omega)}^{\frac{1}{n}}, \]

where \( \Omega \subset \subset B_d \) (\( B_d \) is a ball of radius \( d \)), \( u \in C(\overline{\Omega}) \) satisfies \((dd^c u)^n \leq f\) in the viscosity sense, \( u_{|\partial \Omega} \geq 0 \), and \( f \geq 0 \) is a real valued function from \( C(\overline{\Omega}) \). Note that in this bound no convexity assumptions on \( \Omega \) are made and, more importantly, the \( L^2 \) norm on the right hand side is taken only over a special subset of \( \Omega \).

The standard definitions in viscosity theory require that viscosity supersolutions have to be lower semicontinuous - [CIL92, Zer13]. On the other hand plurisubharmonic functions are axiomatically upper semicontinuous. Hence the continuity assumption in Wang’s result is natural from the viscosity point of view.

On the other hand there are many PSH functions \( u \) which fail to be continuous, yet the Monge-Ampère operator \((dd^c u)^n\) is well-defined in the sense of pluripotential theory. In fact Bedford and Taylor defined \((dd^c u)^n\) as a non-negative Borel measure for a continuous plurisubharmonic function \( u \) in [BT76], and then generalized the construction to \( u \in L^\infty_{loc} \) in [BT82]. We recall that this passage is not just a matter of technicalities. It requires delicate potential theoretic arguments, but the construction allowed the resolution of several long-standing open problems (see [BT82] for more details). Later on Błocki [Blo04, Blo06] found the exact conditions on \( u \) under which Bedford and Taylor’s definition can be applied. In fact many discontinuous PSH functions have measures with smooth densities - any discontinuous PSH function dependent on a fewer than \( n \) variables would do. There are also other types of maximal PSH functions which are discontinuous (see for example [Sic81]). We shall also provide such examples with almost everywhere positive densities (see Example 17 below).

This clearly shows that there is a discrepancy between pluripotential and viscosity supersolutions - a fact that has been observed already in [EGZ11]. On the bright side pluripotential and viscosity subsolutions are equivalent (see [EGZ11, Zer13]). We refer the reader to the recent paper [GLZ], where inequalities for mixed Monge-Ampère measures are studied from viscosity and pluripotential viewpoint. It is worth pointing out that in [GLZ] the lack of continuity is a serious source of troubles (inequalities for mixed Monge-Ampère measures of continuous PSH functions can be studied using much simpler tools, see [Blo96]).

Despite these discrepancies there are also results linking both theories. In fact an easy argument (see [Zer13]) shows that a pluripotential supersolution \( u \) with continuous right-hand side becomes viscosity supersolution once the lower semicontinuous regularization

\[ u_* (w) := \liminf_{z \to w} u(z) \]

is applied (we use the convention \( \liminf_{z \to w} = \liminf_{z \to w} \)). We note that for continuous up to the boundary \( u \) this regularization keeps \( u \) fixed and hence continuous pluripotential supersolutions are also viscosity supersolutions. We also note that for a generic lower semicontinuous function \( u \) it holds that \( u_* \geq u \), but in general \( u_* \neq u \) (even more: \( (u_*)_s \) need not be equal to \( u_* \)), whereas if \( u \) is upper semicontinuous, in particular plurisubharmonic, then \( u_* \leq u \).

Yet another subtle issue is the continuity up to the boundary and the right notion of boundary values. The standard assumption \( u \in C(\overline{\Omega}) \) resolves all these issues in the continuous setting. It is thus worth pointing out that Wang’s estimate fails dramatically if one merely assumes \( u \in C(\Omega) \) and \( u \) is defined on \( \partial \Omega \) by \( u(z) := \limsup_{\Omega \ni w \to z \in \partial \Omega} u(w) \),
which is the standard potential-theoretic extension making $u$ upper semicontinuous on $\Omega$, (see Example 16). Discarding the boundary continuity assumption has further negative consequences. For example one can no longer use the uniqueness of solutions to the Dirichlet problem or various versions of the comparison principle.

The aim of this paper is to investigate whether one can relax the continuity assumptions in Wang’s argument (with suitable modifications) and prove an Alexandrov-Bakelman-Pucci type estimate in the special case of bounded plurisubharmonic $u$ and right hand side function $f \in L^2(\Omega)$. The affirmative answer is summarized in the following main theorem:

**Theorem 1.** Let $u$ be a bounded plurisubharmonic function (not necessarily continuous) in a bounded domain $\Omega \subseteq \mathbb{C}^n$. Suppose that $(dd^c u)^n \leq f$ as measures for some non-negative function $f \in L^2(\Omega)$. Then the following Alexandrov-Bakelman-Pucci type inequality holds:

$$
\sup_{\Omega} u^- \leq \sup_{\partial \Omega} (u_*)^- + C \text{diam}(\Omega) \| f \cdot \chi_{\Gamma_{u^\star = u^\star}} \|^2_{L^2(\Omega)},
$$

where $C$ is a numerical constant dependent only on the dimension $n$.

**Remark 2.** If $u$ is plurisubharmonic and defined in a larger domain $U$ containing $\Omega$, then one can use $\liminf_{\Omega \ni z \to \partial \Omega} u$ rather than $u_*$ on the boundary of $\Omega$ which results in a slightly better bound in the above inequality - see Example 17.

Here and below whenever the measure $(dd^c u)^n$ is absolutely continuous with respect to the Lebesgue measure we will write, abusing the notation slightly, $(dd^c u)^n = f$, where $f$ is the density of the measure.

As one application of this generalization we mention that the following theorem was proved in [DD19] (Theorem 31) with the extra assumption, that $u$ is continuous.

**Theorem 3.** Let $U \subseteq \mathbb{C}^n$ be a domain that contains the ball $B_R(z_0) = \{ z \in \mathbb{C}^n : \| z - z_0 \| \leq R \}$. Assume that a continuous $u \in PSH(U)$ obeys the conditions:

1. For some $\Lambda > 0$ it holds that $\Lambda d\lambda^{2n} \geq (dd^c u)^n \geq 0$ on $B_R(z_0)$ as measures, where $\lambda^{2n}$ is the Lebesgue measure.
2. There exists $\sigma > 0$ such that $u(z) \geq \sigma \| z - z_0 \|^2$ when $z \in B_R(z_0)$ and $u(z_0) = 0$. Then, there exists a constant $c = c(n, \Lambda)$ such that $u(z) \leq \frac{1}{\sigma n^2} \| z - z_0 \|^2$ for all $z \in B_R(z_0)$.

The only place where we needed the continuity was Lemma 29 in [DD19], which now can be substituted by Corollary 18 below, so the continuity assumption can be dropped.

**Theorem 4.** The conclusion of Theorem 3 holds even without the continuity assumption on $u$.

In [Wan12], Wang posed the following problem (see Remark 12 there): Kołodziej’s estimate yields that if $0 \leq f \in L^p(\Omega)$, for some $p > 1$, then the plurisubharmonic solution $u$ of $(dd^c u)^n = f$, $u \geq 0$ on $\partial \Omega$ satisfies

$$
\sup_{\Omega} u^- \leq C(p, n, \text{diam}(\Omega)) \| f \|^\frac{1}{p} \text{diam}(\Omega),
$$

Comparing this with (1) or (5) one wonders whether or not $\| f \chi_{\Gamma_{u^\star = u^\star}} \|_{L^2(\Omega)}$ can control $\| f \|^\frac{1}{p} \text{diam}(\Omega)$ or vice versa. We show that the answer is negative in general, see Example 20 below. Kołodziej’s estimate itself will be treated in a subsequent paper.

We also present some examples, further remarks, and applications of Theorem 1.
2. Proof of the main theorem

We refer to [Kol05, Blo96] for the basics of pluripotential theory, in particular for the construction of the Monge-Ampère measures for locally bounded plurisubharmonic functions. For the viscosity theory good sources are [CIL92, CC95] and for the special case of the viscosity theory of the complex Monge-Ampère operator we refer to the survey [Zei13].

Recall that for a convex function $v$ defined on a domain $Ω$ (treated as a subdomain of $\mathbb{C}^n$ identified with $\mathbb{R}^{2n}$) the gradient image is defined as follows:

$$\partial v(x) := \{ p \in \mathbb{R}^{2n} : v(y) \geq v(x) + \langle p, y - x \rangle, \forall y \in Ω \},$$

with $\langle p, q \rangle$ denoting the usual Euclidean inner product. Note that, by convexity, it does not matter whether the inequality holds in the whole $Ω$ or just locally around $x, \text{that is, the definition of the gradient image is independent of } Ω. \text{More generally for a Borel set } A \text{ the gradient image of } A \text{ is defined by}

$$\partial v(A) := \bigcup_{x \in A} \partial v(x).$$

It is a classical fact (see Lemma 1.1.12 [Gut01] for instance) that for almost every vector $p \in \partial v(A)$ there is a unique $x \in A$ such that $p$ is in the gradient image of the point $x$. This fact leads to the classical construction of Alexandrov’s Monge-Ampère measure of a convex function (see Section 1.1 in [Gut01] for a modern exposition):

**Theorem 5.** Given a convex function $v$ on a domain $Ω$ and any Borel subset $A \subseteq Ω$ the set function

$$Mv(A) := \lambda^{2n}(\partial v(A))$$

is a Borel measure, which is finite on compact sets.

The real Monge-Ampère measure can also be defined, still for a convex function $v$, through analytic methods - see [RT77]. A simple but fundamental observation - Proposition 3.4 in [RT77], states that both constructions are in fact equivalent:

**Theorem 6.** Let $v$ be a convex function defined in a domain $Ω$. Then the Alexandrov and the weak Monge-Ampère measures of $v$ agree on Borel subsets of $Ω$.

Let $U$ be a fixed bounded domain and $u$ be a lower semicontinuous real valued function on $U$, which is bounded below on $U$, and such that

$$\liminf_{U \ni z \to w} u(z) \geq 0, \text{ for any } w \in \partial U.$$  

Fix a ball $B_d$ of radius $d$ such that $U \subseteq B_d$ and let $B_{2d}$ be a concentric ball of radius $2d$. Denote by $Γ_u$ the convex envelope of $u$ defined as follows: We extend $\min\{u, 0\} = -u^-$ by zero from $U$ to $B_{2d}$ and call this extension $\tilde{u}$. Also

$$Γ_u(x) = Γ_{u,B_d}(x) := \sup \{ l(x) : l \text{ is affine}, l \leq \tilde{u} \text{ in } B_{2d} \}, \text{ } x \in B_{2d},$$

and $C_u = C_{u,B_d} := \{ Γ_u = \tilde{u} \}$ is the so-called contact set of $u$. Note that in [Wan12] there is a typo in Definition 4, seeming to imply that $l \leq \tilde{u}$ only in $U$, not in $B_{2d}$. Unless $\tilde{u} = 0$, we have $C_u \subseteq U$ and $C_u = \{ Γ_u = u \}$ and we will assume this from now on.

Usually some extra assumptions such as continuity ([GT01, CC95]) are made on $u$, but just lower semicontinuity is needed to ensure that the contact set is closed. Note that condition (7) guarantees that $\tilde{u}$ is lower semicontinuous, whenever $u$ is. The function $Γ_u$ is convex, hence continuous, and the supremum in (8) is attained at every point, since graphs of convex functions allow supporting hyperplanes at every point. Even if $u$ is convex in $U$ then $u \neq Γ_u$ and $C_u \neq U$, unless $u \equiv 0.$
For lower semicontinuous functions $u$ such that $\liminf_{U_{2z+u}} u(z)$ is negative for some $w \in \partial U$, we first extend $u$ as a lower semicontinuous function on $\bar{U}$, which we also denote by $u$. This is done by setting $u(w) = \liminf_{U_{2z+u}} u(z)$ for $w \in \partial U$. Next we define $\Gamma_u$ and $C_u$ as $\Gamma_u = \sup_{\partial U} u^-$ and $\{ \Gamma_u \sup_{\partial U} u^- = u + \sup_{\partial U} u^- \}$ respectively. Note that the estimate we want to prove is not completely invariant with respect to adding constants to $u$, since the contact set may change, but choosing so will give us the sharpest form of the estimate.

Clearly $u + \sup_{\partial U} u^-$ satisfies the condition (7). Of course $(u + \sup_{\partial U} u^-)^- \neq u^- - \sup_{\partial U} u^-$, but

$$\sup_{\partial U}(u + \sup_{\partial U} u^-) = \sup_{\partial U}(u^- - \sup_{\partial U} u^-) = \sup_{\partial U}(-u^-).$$

If $u \in PSH(U)$ then $u_*$ is lower semicontinuous on $\bar{U}$ and $\sup_{\partial U} u^- = \sup_{\partial U} u_*^-$. This is not true for $\sup_{\partial U} u^-$ and $\sup_{\partial U} u_-$ as shown by Example 14.

The following two lemmas are well-known to the experts - see Lemma 1.4.4 in [Gut01], where a continuous version is proven. We include a sketch for the sake of completeness:

**Lemma 7.** Let $v$ be a lower semicontinuous function on the closure of a bounded domain $\Omega$ contained in a ball $B_d$. Let also $v \geq 0$ on $\partial \Omega$ while $v(x_0) < 0$ for some $x_0 \in \Omega$. Define $V(x_0) := \{ q \in \mathbb{R}^{2n} : v(x_0) + \langle q, \xi - x_0 \rangle < 0, \forall \xi \in \overline{B}_{2d} \}$. Then

$$V(x_0) \subseteq \partial \Gamma_v(\{ \Gamma_v = \hat{v} \}).$$

**Proof.** Assume that the vector $q$ belongs to $V(x_0)$. Note that the supremum $\lambda_0 := \sup\{ \lambda : \lambda + \langle q, \xi - x_0 \rangle \leq \hat{v}(\xi), \forall \xi \in \overline{B}_{2d} \}$ is attained as $\hat{v}(\xi) - \langle q, \xi - x_0 \rangle$ is lower semicontinuous. Then $\lambda_0 \leq v(x_0) < 0$ as the evaluation at $x_0$ shows. Furthermore still by the lower semicontinuity of $\hat{v}$ there exists a point $\hat{\xi} \in \overline{B}_{2d}$, such that $\hat{v}(\hat{\xi}) = \lambda_0 + \langle q, \hat{\xi} - x_0 \rangle$. As $q \in V(x_0)$ we have that $\hat{v}(\hat{\xi}) < 0$ and $\hat{v} = 0$ on $\overline{B}_{2d} \setminus \Omega$ now implies that $\hat{\xi} \in \Omega$. But then $\hat{\xi} \in \{ \Gamma_v = \hat{v} \}$ and finally $q \in \partial \Gamma_v(\{ \Gamma_v = \hat{v} \})$, as claimed.

The lemma implies that $V(x_0) \subseteq \partial \Gamma_v(\{ \Gamma_v = \hat{v} \})$. On the other hand it is easy to see that the ball $B_{-\frac{v(x_0)}{\lambda_0}}(0)$ is contained in $V(x_0)$, hence

$$\omega_{2n} \frac{(-v(x_0))^2n}{(2d)^{2n}} \leq \lambda^{2n}(V(x_0)) \leq \lambda^{2n}(\partial \Gamma_v(\{ \Gamma_v = \hat{v} \})).$$

As a corollary we obtain the following weak Alexandrov maximum principle (compare with Theorem 1.4.5 in [Gut01]):

**Lemma 8.** Let $v$ be a lower semicontinuous function on the closure of a bounded domain $\Omega$. Then

$$\sup_{\Omega}(v^-) \leq \sup_{\partial \Omega}(v^-) + C \ diam(\Omega) \ \lambda^{2n}(\partial \Gamma_v(\{ \Gamma_v = \hat{v} \})^{\frac{1}{2n}}.$$<ref>As a result, the Alexandrov-Bakelman-Pucci estimate boils down to establishing a bound on the volume of the gradient image of the contact set. In [Wan12] this is done by exploiting the fact that for continuous plurisubharmonic $u$ and continuous right hand side $f$, the function $\Gamma_u$ is a viscosity supersolution to $(dd^c u)^n = f^2 \chi_{\{u=\hat{u}\}}.$</ref>
In the viscosity approach one would look for a lower differential tests at points of the contact set. In our setting no viscosity tools are available since the right hand side is merely measurable.

Instead we shall construct a different function in the following crucial lemma:

**Lemma 9.** Let \( u \) be a bounded plurisubharmonic function in a domain \( \Omega \) such that \((dd^c u)^n \leq f\) as measures for some \( f \in L^2(\Omega)\). Let then \( \Gamma_{u_*} \) be the convex envelope of \( \tilde{u}_* \) and \( z_0 \in \{ \Gamma_{u_*} = \tilde{u}_* \} \). Fix small positive \( r < \text{dist}(z_0, \partial \Omega) \). Let finally the convex function \( v \) solve the real Monge-Ampère Dirichlet problem

\[
\begin{aligned}
  v \in C(\overline{B_r(z_0)}); \\
  \det D^2 v = \frac{f^2}{4^n (n!)^2}; \\
  v|_{\partial B_r(z_0)} = \Gamma_{u_*}.
\end{aligned}
\]

Then \( v \leq \Gamma_{u_*} \) in \( B_r(z_0) \).

**Proof.** Note that \( v \) need not agree with \( u_* \) at \( z_0 \). Observe that if \( v \) was additionally smooth then \((dd^c v)^n = 4^n n! \det(v_{ij}) \geq 2^n n! \sqrt{\det D^2 v} \geq f \) from a comparison result of the real and complex Hessians of a convex function - see [Blo05]. But a possibly singular \( v \) is locally a uniform limit of smooth convex approximants \( v_j \) (standard convolutions with smoothing kernel would do), and passing to the limit we obtain \((dd^c v)^n \geq f\) as measures for any such convex solution \( v \).

Next, \( \Gamma_{u_*} \leq u_* \leq u \) together with \((dd^c u)^n \leq f\) gives that \( v \leq u \) in \( B_r(z_0) \), by the comparison principle for plurisubharmonic functions (see [BTS2] or [Kol05]). But now \( v \) is continuous, hence \( v \leq u_* \). Note also that \( v|_{\partial B_r(z_0)} \leq 0 \), hence \( v \) is non-positive in the interior of the ball. Thus \( v \leq \tilde{u}_* \) and finally \( v \leq \Gamma_{u_*}. \)

The main theorem now follows in the following way: As \( v \leq \Gamma_{u_*} \) on \( B_r(z_0) \) with equality on the boundary, \( \partial \Gamma_{u_*}(B_r(z_0)) \subseteq \partial v(B_r(z_0)) \), by Lemma 1.4.1 in [Gut01]. Hence

\[
\lambda^{2n}(\partial \Gamma_{u_*}(B_r(z_0))) \leq \lambda^{2n}(\partial v(B_r(z_0))) = \int_{B_r(z_0)} \frac{f^2}{4^n (n!)^2},
\]

where we used Theorem 1 to justify the last equality.

In particular this means that the Alexandrov measure of \( \Gamma_{u_*} \), restricted to the contact set is majorized by \( \frac{f^2}{4^n (n!)^2} \). As a result

\[
\lambda^{2n}(\partial \Gamma_{u_*}(\{\Gamma_{u_*} = \tilde{u}_*\})) \leq \frac{1}{2 \sqrt{n!}} \left( \int_{\Gamma_{u_*} = \tilde{u}_*} f^2 \right)^{\frac{1}{2n}},
\]

and coupling this with Lemma 8 applied for \( v = u_* \) the main result follows.

3. Applications and remarks

**Remark 10.** The a priori assumption of boundedness on \( u \) can not be dropped, since it is not true that \( f \in L^2(\Omega) \) yields \( u \in L^\infty \), as the example of a pluricomplex Green function \( u \) on \( \Omega \) shows, where \( \Omega = \Omega_1 \setminus \{ w \} \), with \( w \) being the pole of \( u \). For unbounded \( u \), the notion of convex contact set is no longer meaningful, at least if one keeps the standard definition.

**Remark 11.** Following our proof carefully, we get that the constant in [5] can be taken as \( C = \frac{1}{2 \sqrt{\pi n!}} \). One can not get a smaller constant as the following example shows.

Take \( u(z) \) such that \( u \) is plurisubharmonic, \( u \geq 0 \) on \( \partial \Omega \), where \( \Omega = B_d(0) \), \( u(z) > \frac{\sqrt{d^2 - a^2}}{\sqrt{4d^2 - a^2}} (\| z \| - 2d) \) on \( B_d(0) \setminus B_{\frac{d^2}{2a}}(0) \), and \( u(z) = -\sqrt{d^2 - \| z \|^2} \) on a neighborhood of
The contact set is \( B_{2\varepsilon}(0) \), for some \( 0 < a \leq d \). The contact set is \( B_{2\varepsilon}(0) \) and \( f = (dd^c u)^n = 4^n n! \det(u_{ij}) = 4^n n! \frac{1}{2^{n+1}} \frac{2a^2 - \|z\|^2}{(a^2 - \|z\|^2)^{n+2}} \) there. The integral of \( f^2 \) over the contact set is a complicated expression, fortunately the real Hessian of \( u \), which is \( \det D^2 u = \frac{a^2}{(a^2 - \|z\|^2)^{n+1}} \), is both comparable and easily explicitly integrable there. We have

\[
1 \leq \frac{f^2}{4^n n!^2 \det D^2 u} \leq 1 + \frac{a^4}{64d^4 - 16a^2 d^2}
\]
on the contact set, so (5) yields

\[
a \leq 2dC 2n \int_{B_{2\varepsilon}(0)} f^2 \leq 2dC 2n \left(1 + \frac{a^4}{64d^4 - 16a^2 d^2}\right) \int_{B_{2\varepsilon}(0)} 4^n n!^2 \det D^2 u
\]

\[
= 2dC 2n \sqrt{1 + \frac{a^4}{64d^4 - 16a^2 d^2}} \left(\int_{B_{2\varepsilon}(0)} 4^n n!^2 \frac{a^2}{(a^2 - \|z\|^2)^{n+1}}\right)
\]

Now letting \( a \to 0^+ \) proves the claim.

Pursuing the task of obtaining the best constant possible, we can modify our construction by assuming \( \Omega \subseteq B_d \subseteq B_{d+\varepsilon} \) and taking the convex envelope with respect to \( B_{d+\varepsilon} \) instead of \( B_{2\varepsilon} \) (the definition of contact sets changes accordingly). With a few modifications of the proof we get an ABP estimate with the constant \( C = \frac{d+\varepsilon}{4d\sqrt{\pi} \cdot 4^n n!^2} \) and the same example as above shows that it is optimal. In the limit when \( \varepsilon \to 0^+ \) we get a slightly better constant than would directly correspond to (4).

**Remark 12.** The same example demonstrates that it is not possible to obtain the ABP estimate with optimal constant while integrating over a set which is essentially smaller than the contact set.

The next example shows that the exponent 2 in (5) is optimal, that is, one cannot substitute the \( L^2 \) norm of \( f \) on the contact set with a \( L^p \) norm for any \( 1 < p < 2 \).

**Example 13.** Let \( \Omega \) be the unit ball in \( \mathbb{C}^n \) and let \( u(z) = \|z\|^\alpha - 1 \in PSH(\Omega) \), for \( 2 > \alpha > 0 \). It is a matter of routine calculus to check that

\[
(dd^c u)^n = f(z) = 2^{n-1} n! \alpha^{n+1} \|z\|^{\alpha - 2n}.
\]

Switching to polar coordinates, one sees that \( f \in L^p(\Omega) \), for any \( 1 < p < \frac{2}{2 - \alpha} \). On the other hand it cannot be true, that

\[
\sup_{\Omega} u^- \leq \sup_{\partial \Omega} (u_+)^- + C \text{diam}(\Omega) \|f \chi_{\Gamma_u = \hat{u}}\|_{L^p(\Omega)}\frac{1}{n!},
\]

since if \( \alpha \leq 1 \) then \( u(z) \geq \|z\| - 1 \) and hence \( \{\Gamma_u = \hat{u}\} \) consists of the sole origin.

Let us remark that the problem of defining a correct notion of boundary values for \( u \in PSH(\Omega) \) is a subtle one, as already noted in [BT76]. Interestingly, there the authors remark that sufficiently general uniqueness theorem for the Monge-Ampère equation would imply nonexistence of nontrivial inner functions in the unit ball of \( \mathbb{C}^n \), \( n > 1 \). However, the existence of such functions was later proven by Aleksandrov- [Ale82] and
Hakim-Sibony-Løw\cite{HS82,Løw82}. The nontrivial inner functions can in fact be used to show that it is necessary to consider the lower semicontinuous regularization of plurisubharmonic functions on the boundary in our considerations:

**Example 14.** Let $\Omega$ be the unit ball in $\mathbb{C}^n$, $n > 1$, and let $F$ be a nontrivial holomorphic inner function on $\Omega$. As the radial limits of $F$ exist almost everywhere on $\partial \Omega$ and their absolute values are equal to 1 again almost everywhere, it is obvious that

$$\limsup_{\Omega \ni z \to \partial \Omega} |F(z)|^2 - 1 = 0$$

everywhere on $\partial \Omega$. It is well-known (see Theorem 19.1.3 in \cite{Rud08}) that in turn

$$\liminf_{\Omega \ni z \to \partial \Omega} |F(z)|^2 - 1 = -1.$$ 

Consider the maximal plurisubharmonic function $u(z) := |F(z)|^2 - 1$. Then $\sup_{\Omega} u^- = 1$, the last term in (5) vanishes, and it is obvious that Alexandrov-Bakelman-Pucci type estimate is possible only if $\liminf$-boundary values are taken into consideration.

**Remark 15.** The same example shows the lack of uniqueness of the solution of the Dirichlet problem without boundary continuity: both $u_1(z) = |F(z)|^2 - 1$ and $u_2(z) = 0$ are plurisubharmonic and satisfy $\limsup_{\Omega \ni z \to \partial \Omega} u_i(z) = 0$, and $(dd^c u_i)^n = 0$. Also the global comparison principle fails in this generality.

Our next example demonstrates the difference of taking the lim inf-boundary values only from within the considered domain as compared to the lower regularization, where approach from outside is allowed:

**Example 16.** Consider the function

$$u(z) := \sum_{n=3}^{\infty} a_n \log \left| z - \frac{1}{2} - \frac{1}{n} \right|,$$

defined on the unit disc in $\mathbb{C}$, where the constants $a_n > 0$ are chosen so small that $u\left(\frac{1}{2}\right) = \min_{\{|z| \leq \frac{1}{2}\}} u \geq -1$. Let $\{\theta_j\}_{j=1}^{\infty}$ be a dense sequence of angles in $[0, 2\pi)$. Define

$$v(z) := \sum_{j=1}^{\infty} \frac{1}{2^n} u(e^{i\theta_j} z).$$

By construction $v|_{\{|z| \leq \frac{1}{2}\}} \geq -1$, while $v|_{\{|z| \leq \frac{1}{4}\}} \equiv -\infty$. Taking $\hat{v}(z) := e^v(z)$ results in a bounded subharmonic function (the boundedness from above is clear) with constantly zero lim inf-boundary values. Finally note that

$$\hat{V}(z_1, z_2) := \hat{v}(z_1)$$

is a maximal plurisubharmonic function defined in, say, the unit ball in $\mathbb{C}^2$. Let the domain $\Omega$ be the ball centered at zero of radius $\frac{1}{2}$. Taking $\liminf_{\Omega \ni z \to \partial \Omega} \hat{V}(z)$ rather than $\hat{V}_i(z_0)$ results in a sharper Alexandrov-Bakelman-Pucci inequality.

Example 16 also shows that is easy to produce discontinuous maximal plurisubharmonic functions. These are, however, not very useful in our considerations, since the Alexandrov-Bakelman-Pucci type inequality holds trivially for maximal plurisubharmonic functions. In turn non-maximal discontinuous $PSH$ functions with non-negative densities do not seem to be studied thoroughly in the literature. We believe that ABP type estimates in the discontinuous setting can be helpful in their study. But first of all one wants to know if such functions do exist. Hence we provide an example:

Let $K$ be a planar compact set, which is non-polar, contained in the imaginary axis $\{z : Re z = 0\}$ and is irregular in the sense of potential theory (see \cite{Ran95} for these
An explicit construction is possible by choosing a sequence of intervals accumulating at 0, with controlled lengths and suitably situated with respect to each other (see [Sic97] for details). Irregularity can be established by using the Wiener criterion. Let $V^*_K$ be the extremal function associated to the set $K$ (or Green function for the complement of $K$ with pole at infinity). It is known that $V^*_K$ is positive and harmonic outside $K$, subharmonic in $\mathbb{C}$, and $\Delta V^*_K$ is a positive Borel measure supported on $K$. Because $K$ is irregular and non-polar, $V^*_K$ fails to be continuous.

**Example 17.** Let $\Omega \subset \subset \mathbb{C}^2$ be a bounded domain, contained in $\{(z, w) \in \mathbb{C}^2 : |w| < 1\}$, $K \times \{0\} \subseteq \Omega$ and $V^*_K$ is as above. Then

$$u(z, w) := V^*_K(z) + (\text{Re}z)^2(1 + |w|^2)$$

is a bounded discontinuous plurisubharmonic function on $\Omega$ such that $(dd^c u)^2 = f$, where $f \geq 0$ is not everywhere zero and is smooth.

**Proof.** The discontinuity and boundedness are clear. Computing the complex Hessian, at a point outside $\{(z, w) \in \mathbb{C}^2 : \text{Re}z = 0\}$, that is, near which $u$ is $C^2$, gives one

$$\begin{pmatrix} u_{zz} & u_{z\bar{w}} \\ u_{wz} & u_{\bar{w}\bar{z}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \Delta V^*_K + \frac{1}{2}(1 + |w|^2) & w\text{Re}z \\ w\text{Re}z & (\text{Re}z)^2 \end{pmatrix},$$

so the determinant is $\frac{1}{2}(\text{Re}z)^2(1 - |w|^2)$, which extends to a non-negative and smooth function on $\Omega$. On the other hand $(dd^c u)^2$ can put no mass on $\{(z, w) \in \mathbb{C}^2 : \text{Re}z = 0\}$ since $\frac{1}{2} \Delta V^*_K$ is killed by the term $(\text{Re}z)^2$.

Theorem 1 has several immediate corollaries.

**Corollary 18.** Let $u$ be as in Theorem 1. Suppose moreover that $f \in L^\infty(\Omega)$. Then for any relatively compact subdomain $U \subseteq \Omega$ the following estimate holds:

$$\sup_U u^- \leq \sup_{\partial U} (u^-) + Cdiam(U)V^1(U)^{(1/2n)} ||f||_{L^n(U)}^{1/n},$$

where $C$ is a numerical constant dependent only on the dimension $n$.

**Proof.** This follows trivially by estimating the last term in $\Sigma$ by $V^1(U)^{(1/2n)} ||f||_{L^n(U)}^{1/n}$.

**Corollary 19.** Under the assumptions of Theorem 1 and if the supremum of $u^-$ is not attained on the boundary, then the contact set $\{\Gamma_u = \bar{u}_e\}$ of the function $u$ has positive Lebesgue measure. In a sense such plurisubharmonic functions have "pointwise convex" lower semicontinuous regularizations (their graphs allow supporting real hyperplanes) on a big set.

Concerning Wang’s problem we observe the following.

**Example 20.** The $L^2$ norm over the contact set can not control the $L^p$ norm over the whole domain and vice versa, which is demonstrated by the the examples below (in each case we put $(dd^c u)^n = f$ and $(dd^c u_e)^n = f_e$):

1. Let $u$ be a function of the type considered in Remark 17, namely $u(z) = -\sqrt{d^2 - ||z||^2}$. We have $f \not\in L^1(B_d(0))$, so we perturb $u$ near the boundary of the ball by setting $u_e = u$ on $B_{d-2\epsilon}(0)$, $u_e = \sqrt{d^2 - (d-\epsilon)^2}(||z|| - d)$ on $B_d(0) \setminus B_{d-\epsilon}(0)$ and $u_e$ is extended by using a smooth transition function on $B_{d-\epsilon}(0) \setminus B_{d-2\epsilon}(0)$, keeping convexity. The contact set of $u_e$ is $\partial B_d(0)$.

Then $\|f_e \chi_{\{u_e = u_e\}}\|_{L^2(\Omega)}$ stays fixed, whereas $\|f_e\|_{L^p(B_d(\Omega))}$ is arbitrarily big for any $p \geq 1$. 


(2) Let $u$ be the function form Example 13, namely $u(z) = \|z\|^{\alpha} - 1, 0 < \alpha < 1$. We construct $u_{\epsilon}$ by first properly normalizing $u$, that is putting $\epsilon \left( \left( \frac{\|z\|}{d} \right)^{\alpha} - 1 \right)$ and after that truncating the "tip" of its graph in a small ball. This is done by setting $u_{\epsilon} = -\sqrt{\epsilon^2 - \|z\|^2} \in$ in a neighborhood of $B_{\frac{3d}{4}}(0)$ and patching this function smoothly with the normalized $u$ over $B_{\frac{3d}{4}}(0) \setminus B_{\frac{3d}{8}}(0)$, while keeping the plurisubharmonicity. This is again done by using a smooth transition function. The patching is possible because $u_{\epsilon}\left( \frac{\epsilon^2}{d} \right) = \epsilon \left( \left( \frac{\epsilon^2}{d} \right)^{\alpha} - 1 \right) > -\sqrt{\epsilon^2 - \left( \frac{\epsilon^2}{2d} \right)^2} = u_{\epsilon}\left( \frac{\epsilon^2}{2d} \right)$ if $\epsilon$ is small enough. As in Remark 12, the contact set is $B_{\frac{3d}{8}}(0)$ and

$$\|f_{\epsilon}\chi_{\{\Gamma_{u_{\epsilon}} = \bar{u}_{\epsilon}\}}\|_{L^2(\Omega)} \sim C \left( \frac{\epsilon}{\sqrt{4d^2 - \epsilon^2}} \right)^n \to 0,$$

whereas, using Example 13

$$\|f_{\epsilon}\|_{L^p(\Omega)} > \|f_{\epsilon}\|_{L^p\left( B_d(0) \setminus B_{\frac{3d}{4}}(0) \right)} \sim C_1 \epsilon^n + C_2 \left( \epsilon^{\frac{p}{p+n-1}} \right).$$

Hence $\|f_{\epsilon}\|_{L^p(\Omega)}$ is either bounded and separated from zero if $1 \leq p < \frac{4}{3-2\alpha}$ or arbitrarily big if $p \geq \frac{4}{3-2\alpha}$.

(3) Let $u_{\epsilon}$ be defined as follows. Let $u_{\epsilon} = \log \frac{\|z\|}{d}$ on $B_d(0) \setminus B_{\frac{3d}{4}}(0)$, $u_{\epsilon} = \left[ \log \frac{\epsilon}{d} \right] \|z\|^2 - \left[ \log \frac{\epsilon}{d} \right] \frac{3d}{4} \frac{\epsilon}{d}$ on $B_{\frac{3d}{4}}(0)$. Note that $u_{\epsilon}(w) > u_{\epsilon}(v)$ if $\|w\| = \frac{3d}{4}, \|v\| = \frac{\epsilon}{d}$ and $\epsilon$ is small enough. Now we extend $u_{\epsilon}$ on $B_{\frac{3d}{4}}(0) \setminus B_{\frac{3d}{8}}(0)$ in such a way that $u_{\epsilon}$ is increasing with $\|z\|$, smooth, plurisubharmonic and $f_{\epsilon}$ is decreasing. The contact set is the closed ball of radius $2d - \sqrt{4d^2 - \epsilon^2} \gtrsim \frac{\epsilon}{d}$. Now

$$\|f_{\epsilon}\chi_{\{\Gamma_{u_{\epsilon}} = \bar{u}_{\epsilon}\}}\|_{L^2(\Omega)} > \|f_{\epsilon}\|_{L^2\left( B_{\frac{3d}{8}}(0) \right)} \sim C \left| \log \frac{\epsilon}{d} \right|^n \to \infty,$$

whereas

$$\|f_{\epsilon}\|_{L^p(\Omega)} = \|f_{\epsilon}\|_{L^p\left( B_{\frac{3d}{8}}(0) \right)} \leq C \epsilon^{\frac{n(2-p)}{p}} \left| \log \frac{\epsilon}{d} \right|^n \to 0,$$

for any $1 \leq p < 2$.

**Remark 21.** Following the proof of Theorem 1, one sees that the assumption $f \in L^2(\Omega)$ can be changed to just $f \in L^2(\{\Gamma_{u_{\epsilon}} = \bar{u}_{\epsilon}\})$ and the first of the Examples 21 shows that under such assumption the ABP estimate applies to a wider range of plurisubharmonic functions.

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