The Quantization of the Generalized mKdV Equations for \( \hat{sl}_2 \)

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Abstract

We construct quantum deformations of the integrals of motion of the generalized mKdV equations for \( \hat{sl}_2 \). For this, we give the relevant vertex operator algebra and prove quantum Serre relations for vertex operators, it allows to construct a \( q \)-BGG resolution and to deform the classical integrals of motion in a commutativ family.
1 Introduction

The generalized mKdV equations were introduced in [13]. They are associated to an arbitrary Kac-Moody algebra and an integer $\ell$. The Lax operator has the form:

$$L = \partial_x + p - \ell + \sum_{-\ell \leq i \leq 0} \sum_p u_{i,p}(x)e_{i,p} = \partial_x + L,$$

where for all $i$, $(e_{i,p})_p$ is a basis of the principal degree $i$ part of the affine Lie algebra $\hat{g}$. In [1], we studied the simplest case of these equations when $g = \hat{\mathfrak{sl}}_2$ and $\ell = 3$. In that case, $L = p_{-3} + H_{-2}e_{-1} + F_{-1}f_{-1} + H_0h_0$ (we use the convention that $x_n$ has principal degree $-n$). The main theorem was a geometric proof of the commutation of the classical integrals of motion (already proved in [7]). For that aim, one of our results was that the group-theoretic interpretation of [8] – interpretation of the space of jets of $H_0, E_{-1}, F_{-1}, H_{-2}$ with the functions on a quotient $N_+/A_+$ of subgroups of the Lie group associated with $\hat{\mathfrak{sl}}_2$ – could be carried out in this situation, if we set the central element $d_{-1} = (E_{-1} + F_{-1} + H^{2}_{-2})/2$ and all its derivatives equal to zero. The action of the derivation $\partial_x$ on:

$$\pi_0^0 = \mathbb{C}[H_0^{(n)}, E_{-1}^{(n)}, F_{-1}^{(n)}, H_{-2}^{(n)}]/(d_{-1}, d'_{-1}, \ldots, d^{(n)}_{-1}, \ldots),$$

was then identified to the right action of $p_{-3}$, the principal degree $-3$ element of $\mathfrak{sl}_2$, on $N_+/A_+$, where $N_+$ is viewed as a submanifold of the flag space of $\mathfrak{sl}_2$. This allowed us, by studying a BGG complex (defined in [2] [3]), to give a cohomological interpretation of the integrals of motion similar to that of [8]. We have then showed in [1] the commutation of the classical integrals of motion.

**Theorem 1.1.** The $d_{2i+1}$, which are in $\mathbb{C}[\pi_0^0]$, are in involution for the Poisson brackets.

The $d_{2i+1}$ are some polynomials in $E_{-1}, F_{-1}, H_{-2}, H_0$ of the generalised mKdV equations, the $d_{2i+1}$ form the classical integrals of motion, which commute with each other as in [8], and so generate a family of commuting vector fields.

In the present paper, we study the quantization of the system and we construct quantum deformations of the classical integrals of motion. The main theorem is (theorem [5],[4]):
Theorem 1.2. The integrals of motion $\oint d_{2i+1}$ admit quantum deformations, which belong to a completed universal enveloping algebra and which commute with each other and have the same degrees as the integrals of motion.

For this, we construct a Lie algebra naturally associated to the classical Poisson algebra and show, only after taking its quotient by the quantum deformation of $d_{-1}$ (definition 3.1) and derivatives, a VOA structure (theorem 4.1), the three axioms for a VOA (defined in [8]) are proved, field-state correspondence (proposition 4.2), Sugawara field (proposition 4.3), locality (proposition 4.4). The classical screening charges are replaced by vertex operators (introduced in [4] [12]) $\bar{V}_+ = \sum_{i \in \mathbb{Z}} E_{-1}[-2/3 + i] \otimes V_+[-1 - i]$ and $\bar{V}_- = \sum_{i \in \mathbb{Z}} F_{-1}[-2/3 + i] \otimes V_-[-1 - i]$. We show (theorem 5.1) that $\bar{V}_+$ and $\bar{V}_-$ satisfy the quantum Serre relations. This generalizes a result of [5]; this result is obtained by a reasoning using the analogy with a lattice model. Finally, we construct a quantum BGG resolution and use deformation arguments to show that the classical integrals of motion admit quantizations, which commute to each other (theorem 6.1).

2 The Lie algebra $\tilde{g}$, associated fields

2.1 The Lie algebra $\tilde{g}$

The Poisson brackets at level of the variables of the generalized mKdV equations are the following ones in [1] [3]:

$$\{H_0(x), H_0(y)\} = \frac{1}{2} \partial_x \delta_{x,y},$$

$$\{H_0(x), E_{-1}(y)\} = 0, \{H_0(x), F_{-1}(y)\} = 0, \{H_0(x), H_{-2}(y)\} = 0,$$

$$\{E_{-1}(x), E_{-1}(y)\} = 0, \{F_{-1}(x), F_{-1}(y)\} = 0,$$

$$\{E_{-1}(x), F_{-1}(y)\} = -4H_{-2}(x)\delta_{x,y},$$

$$\{E_{-1}(x), H_{-2}(y)\} = 2\delta_{x,y}, \{F_{-1}(x), H_{-2}(y)\} = -2\delta_{x,y},$$

$$\{H_{-2}(x), H_{-2}(y)\} = 0.$$

The Lie algebra $\tilde{g}$ is defined by its generators, $E_{-1}[n], F_{-1}[n]$ and $H_{-2}[n]:$

$$E_{-1}[n], n \in -2/3 + \mathbb{Z},$$
\[ F_{-1}[n], n \in -2/3 + \mathbb{Z}, \]
\[ H_{-2}[n], n \in -1/3 + \mathbb{Z}, \]

and relations, with \( \beta \) a deformation parameter:
\[
[E_{-1}[n], F_{-1}[m]] = -4\beta H_{-2}[n + m],
\]
\[
[E_{-1}[n], H_{-2}[m]] = 2\beta \delta_{n+m,0},
\]
\[
[F_{-1}[n], H_{-2}[m]] = -2\beta \delta_{n+m,0},
\]
\[
[H_{-2}[n], H_{-2}[m]] = 0.
\]

Let \( \mathfrak{h} \) be the algebra generated by:
\[ H_0[n], n \in \mathbb{Z}, \]

and relations:
\[
[H_0[n], H_0[m]] = (1/2)(n)\beta \delta_{n+m,0}.
\]

The algebra \( \tilde{\mathfrak{g}} \) is the direct product of \( \mathfrak{g} \) and \( \mathfrak{h} \).
\[
[H_0[n], E_{-1}[m]] = 0, [H_0[n], F_{-1}[m]] = 0, [H_0[n], H_{-2}[m]] = 0.
\]

The corresponding fields are:
\[
E_{-1}(z) = \sum_{n \in -2/3 + \mathbb{Z}} E_{-1}[n]z^{-n-2/3},
\]
\[
F_{-1}(z) = \sum_{n \in -2/3 + \mathbb{Z}} F_{-1}[n]z^{-n-2/3},
\]
\[
H_{-2}(z) = \sum_{n \in -1/3 + \mathbb{Z}} H_{-2}[n]z^{-n-1/3},
\]
\[
H_0(z) = \sum_{n \in -1 + \mathbb{Z}} H_0[n]z^{-n-1}.
\]

The sums are over the ring \( \mathbb{Z} \) shifted by a certain number such that \( z \) has an integer power.
2.2 The normal ordering of two fields

The normal ordering of two fields $A, B$ of conformal weights $\Delta_A$ and $\Delta_B$, is defined by the following way:

$$A(z) = \sum_{n \in \mathbb{Z}} A[n] z^{-n-\Delta_A},$$

$$B(z) = \sum_{n \in \mathbb{Z}} B[n] z^{-n-\Delta_B},$$

$$A_{reg}(z) = \sum_{n = -\Delta_A \geq 0} A[n] z^{-n-\Delta_A},$$

$$A_{sing}(z) = \sum_{n = -\Delta_A < 0} A[n] z^{-n-\Delta_A},$$

$$: AB : (z) = A_{reg}(z)B(z) + B(z)A_{sing}(z).$$

3 $d_{-1}^3$ quantum central element

3.1 Definition of $d_{-1}^3$

The following element is defined:

Definition 3.1.

$$d_{-1}^3(z) = E_{-1}(z) + F_{-1}(z) + : H_{-2} H_{-2} : (z).$$  \hspace{1cm} (2)

3.2 Centrality

Lemma 3.1. The Fourier coefficients of $d_{-1}^3$ are central in the enveloping algebra $U \mathfrak{g}$.

Indeed, the normal ordering is developed and gives a decomposition of the delta function.

Definition 3.2. Let $U \mathfrak{g}^{(0)}$ be the quotient of $U \mathfrak{g}$ by the ideal generated by the $d_{-1}^3[n]$. 

5
3.3 The degree of $d_{-1}^\beta$

The element $d_{-1}^\beta$ is homogeneous:

$$d_{-1}^\beta = E_{-1} + F_{-1} : H_{-2} :,$$

as, $\deg(E_{-1}) = \deg(F_{-1}) = -2, \deg(H_{-2}) = -1$, it gives:

$$\deg(d_{-1}^\beta) = -2.$$

The conformal weight is $2/3$.

4 The VOA associated with $\mathfrak{g}$

The goal of this section is to show the following theorem:

Theorem 4.1. There exists a VOA structure associated with $\mathfrak{g}$ and $d_{-1}^\beta$.

The vacuum module is $\pi^\beta_0$ and the space of fields $\mathcal{F}$, as defined above [8] [12].

4.1 The quantum module of the states

Definition 4.1. Let $U^+$ be the sub-algebra of $(U\mathfrak{g})^{(0)}$ generated by the $X[n], n \geq 0, X = E_{-1}, F_{-1}, H_{-2}$ and let $U^-$ be the one generated by the $X[n], n < 0, X = E_{-1}, F_{-1}, H_{-2}$.

The quantum module $\pi^\beta_0$ is defined by:

Definition 4.2.

$$\pi^\beta_0 = (U\mathfrak{g})^{(0)} \otimes U^+ \mathbb{C} \cong U^-,$$

with $U\mathfrak{g}^+$ acting trivially over $\mathbb{C}$ as a character, putting all $X[n]$ to zero.

Let $|\Omega\rangle$ be the following element:

$$|\Omega\rangle = 1 \otimes 1,$$

then the application $U^- \rightarrow \pi^\beta_0, T \mapsto T|\Omega\rangle$ is an isomorphism of vector spaces.

A degree is defined over $\pi^\beta_0$ by:

$$\deg(|\Omega\rangle) = 0, \deg(E_{-1}[n]) = n, \deg(F_{-1}[n]) = n, \deg(H_{-2}[n]) = n.$$
4.2 The quantum algebra of the fields

The algebra of the fields $\mathcal{F}$ is defined by:

$$\mathcal{F} \hookrightarrow \widehat{U}_g^{(0)}[[z, z^{-1}]],$$

where $\mathcal{F}$ is freely generated by $E_{-1}(z)$, $F_{-1}(z)$ and $H_{-2}(z)$, and the derivatives of the fields, taking the normal orderings in the completed space of $U_g^{(0)}[[z, z^{-1}]]$. As $d_{-1}^3$ is homogeneous, a degree is defined over $\mathcal{F}$ by:

$$\deg(E_{-1}(z)) = -2/3, \quad \deg(F_{-1}(z)) = -2/3, \quad \deg(H_{-2}(z)) = -1/3,$$

and:

$$\deg(\cdot \ AB \cdot) = \deg(A) + \deg(B),$$

and:

$$\deg(A') = \deg(A) - 1.$$

The degrees introduced in the [1] are $-3$ times the one defined here. The conformal weight is the opposite of the above degree.

**Proposition 4.1**. The normal product from the left to the right of the following expressions:

$$E_{-1}(z)^{e_0} E'_{-1}(z)^{e_1} \ldots E_{-1}^{(n)}(z)^{e_n} \ldots$$

$$F_{-1}(z)^{f_0} F'_{-1}(z)^{f_1} \ldots F_{-1}^{(n)}(z)^{f_n} \ldots$$

$$H_{-2}(z)^{h_0} H'_{-2}(z)^{h_1} \ldots H_{-2}^{(n)}(z)^{h_n} \ldots,$$

define a basis of $\mathcal{F}$.

*Proof:* this proposition is proved using associativity and commutativity of normal ordered products, up to terms of lower degree.

4.3 The fields-states correspondence

It is first given the following lemma:

**Lemma 4.1**. If both limits $\lim_{z \to 0} A(z)|\Omega\rangle$ and $\lim_{z \to 0} B(z)|\Omega\rangle$ exist, then the limits $\lim_{z \to 0} :AB:(z)|\Omega\rangle$ and $\lim_{z \to 0} A'(z)|\Omega\rangle$ also exist.
Proposition 4.2. There exists a correspondence:

\[ \pi_0^\beta \overset{\phi}{\cong} F, \]

\[ X \mapsto \phi_X, \]

so that:

\[ \lim_{z \to 0} \phi_X(z)|\Omega\rangle = X. \]

A derivation is defined over the fields by the usual derivation and too for the states \( v \) by:

\[ \partial|\Omega\rangle = 0, \]

\[ \partial(E_{-1}[n]v) = (-n + 1/3)E_{-1}[n - 1]v + E_{-1}[n]\partial v, \]

\[ \partial(F_{-1}[n]v) = (-n + 1/3)F_{-1}[n - 1]v + F_{-1}[n]\partial v, \]

\[ \partial(H_{-2}[n]v) = (-n + 2/3)H_{-2}[n - 1]v + H_{-2}[n]\partial v. \]

It gives:

\[ \frac{\partial}{\partial z} \circ \phi = \phi \circ \partial. \] (3)

4.4 The derivation of the VOA

Set:

\[ T(z) = -1/4 : E'_{-1}H_{-2} : (z) - : F'_{-1}H_{-2} : (z), \]

\[ T(z) = \sum_n L_n z^{-n-2}. \] (4)

Proposition 4.3.

\[ [L_0, X(z)] = X'(z), X \in F. \]

The computation is standard but uses \( d'_{-1} \beta = 0. \)

Remark 1. The field \( T(z) \) obeys the relations of the Virasoro algebra without central charge.
4.5 The locality of the fields of the VOA

**Proposition 4.4.** All fields of the space $\mathcal{F}$ are local with respect to each other.

The two fields $E_{-1}$ and $F_{-1}$ are considered, and their correlation function over two vectors $|\xi\rangle$ and $|v\rangle$.

\[
\langle \xi | : E_{-1}(z) F_{-1}(w) : |v\rangle = \\
\langle \xi | E_{-1_{\text{reg}}}(z) F_{-1}(w) + F_{-1}(w) E_{-1_{\text{sing}}}(z) |v\rangle, \\
\langle \xi | E_{-1}(z) F_{-1}(w) |v\rangle = \langle \xi | E_{-1}(z) F_{-1}(w) : |v\rangle + \\
\sum_{-n-2/3 < 0, m} \langle \xi | [E_{-1}[n], F_{-1}[m]] |v\rangle z^{-n-2/3} w^{-m-2/3},
\]

with the formula of the commutator,

\[
\sum_{-n-2/3 < 0, m} \langle \xi | (-4\beta H_{-2}[n + m]) |v\rangle z^{-n-2/3} w^{-m-2/3},
\]

the sum is the expansion for $w$ small with respect to $z$ of:

\[
\langle \xi | H_{-2}(w) |v\rangle / (z - w),
\]

then the same holds with:

\[
\langle \xi | F_{-1}(w) E_{-1}(z) |v\rangle,
\]

for $z$ chosen small with respect to $w$. Therefore, both expressions $\langle \xi | E_{-1}(z) F_{-1}(w) |v\rangle$ and $\langle \xi | F_{-1}(w) E_{-1}(z) |v\rangle$ give expansions of the same rational function on $\mathbb{C}^2$ in different spaces of formal series. This proves the axiom of locality.

4.6 The VOA associated with $\tilde{g}$

**Definition 4.3.** Let $\mathfrak{h}$ be the algebra generated by the $H_0[n]$. Let $\pi_0^{b,\beta}$ be the Fock vacuum module.

The tensor product:

\[
\tilde{\pi}_0^{\beta} = \pi_0^{b,\beta} \otimes \pi_0^{\beta},
\]

admits a VOA structure.
It exists a VOA associated with $\tilde{g}$ which is the tensor product of the VOA associated with $g$ by the one attached to $h$, it possesses a vacuum module $\tilde{\pi}_0^\beta$ and a space of fields $\tilde{F}$.

The Virasoro field of the algebra tensor product is:

$$T_0(z) \otimes 1 + 1 \otimes T(z),$$

with $T_0(z)$, the Virasoro field of the free fields.

5 The quantum Serre relation

5.1 The vertex operators

We will define quantum analogues $\tilde{V}_\pm$ of the classical screening charges $Q_+$ and $Q_-$ which appeared in [1].

**Definition 5.1.** Let the algebra $g'$ be the semidirect product of $g$ by $p$ so that:

$$[H_0[0], p] = 1,$$
$$[X[n], p] = 0,$$

for $X[n] \neq H_0[0]$. Let the algebra $g'_+$ be constructed by $H_0[0], p, X[n]$, for positive $n$.

**Definition 5.2.** A character $\chi_n$ is defined over this algebra by:

$$p \mapsto n$$
$$H_0[0] \mapsto 0,$$
$$X[n] \mapsto 0.$$

**Definition 5.3.** The quantum modules $\pi_n^\beta$ are:

$$\pi_n^\beta = U_{\tilde{g}'} \otimes_{U_{\tilde{g}'_+}} \mathbb{C}_{\chi_n}.$$
Definition 5.4. Two operators $\bar{V}_+$ and $\bar{V}_-$ are defined as endomorphisms of each module of highest weight for $\tilde{\mathfrak{g}}$, the semi-direct product of the enveloping algebra of $\tilde{\mathfrak{g}}$ by $e^{\pm p}$, elements which commute with all the generators of $\tilde{\mathfrak{g}}$ excepted $H_0[0]$, and such that:

$$e^{\pm p}H_0[0]e^{\mp p} = H_0[0] \pm \beta.$$ 

The sum of $\pi_\beta$ is such a module of highest weight.

$$h_0(z) = \int z H_0, \quad V_+(z) = e^{h_0^+(z)} e^{h_0^+(z)},$$

with $h_0^\pm$:

$$h_0^+(z) = - \sum_{n \in -\mathbb{N}^*} \frac{H_0[n]}{n} z^{-n},$$

$$h_0^-(z) = - \sum_{n \in \mathbb{N}^*} \frac{H_0[n]}{n} z^{-n},$$

$$\bar{V}_+ = \int E_{-1}(z)V_+(z)dz =$$

$$= \int \sum_n E_{-1}[n] z^{-n-2/3} V_+(z) dz,$$

with:

$$V_+(z) = \sum_n V_+[n] z^{-n},$$

$$\bar{V}_- = \int E_{-1}(z)V_-(z)dz =$$

$$= \int \sum_n E_{-1}[n] z^{-n-2/3} V_-(z) dz,$$

with:

$$V_-(z) = \sum_n V_-[n] z^{-n},$$

$$\bar{V}_+ = \sum_{i \in \mathbb{Z}} E_{-1}[\frac{-2}{3} + i] \otimes V_+[-1 - i],$$

$$\bar{V}_- = \sum_{i \in \mathbb{Z}} F_{-1}[\frac{-2}{3} + i] \otimes V_-[-1 - i].$$
5.2 The quantum Serre relation

The following quantum Serre relation ($q$-Serre) between the operators $\bar{V}_+$ and $\bar{V}_-$ is:

**Theorem 5.1.** The following quantum Serre relation holds:

$$S = \bar{V}_-\bar{V}_+^3 - 3[\bar{V}_-\bar{V}_+^2] + 3[\bar{V}_+^3\bar{V}_-] - \bar{V}_+^3\bar{V}_- = 0,$$

with:

$$[3] = q + q^{-1} + 1,$$

**Proof:** It gives four terms:

$$S = S_1 - [3]S_2 + [3]S_3 - S_4,$$

$$S_1 = \sum_{i,j_1,j_2,j_3 \in \mathbb{Z}} E_{-1}[i - 2/3]F_{-1}[j_1 - 2/3]F_{-1}[j_2 - 2/3]F_{-1}[j_3 - 2/3] \otimes V_-[-i - 1]V_+[-j_1 - 1]V_+[-j_2 - 1]V_+[-j_3 - 1]$$

$$S_2 = \sum_{i,j_1,j_2,j_3 \in \mathbb{Z}} F_{-1}[j_1 - 2/3]E_{-1}[i - 2/3]F_{-1}[j_2 - 2/3]F_{-1}[j_3 - 2/3] \otimes V_+[-j_1 - 1]V_-[-i - 1]V_+[-j_2 - 1]V_+[-j_3 - 1],$$

$$S_3 = \sum_{i,j_1,j_2,j_3 \in \mathbb{Z}} F_{-1}[j_1 - 2/3]F_{-1}[j_2 - 2/3]E_{-1}[i - 2/3]F_{-1}[j_3 - 2/3] \otimes V_+[-j_1 - 1]V_-[-i - 1]V_+[-j_2 - 1]V_+[-j_3 - 1],$$

$$S_4 = \sum_{i,j_1,j_2,j_3 \in \mathbb{Z}} F_{-1}[j_1 - 2/3]F_{-1}[j_2 - 2/3]F_{-1}[j_3 - 2/3]E_{-1}[i - 2/3] \otimes V_+[-j_1 - 1]V_-[-j_2 - 1]V_+[-j_3 - 1]V_-[-i - 1].$$

It is then possible to reorganize the four terms using the relations of the Lie algebra.

A family of elements of $(U\tilde{g})^{(0)}$ is chosen:

$$E_{-1}[i - 2/3]F_{-1}[j_1 - 2/3]F_{-1}[j_2 - 2/3]F_{-1}[j_3 - 2/3],$$

for $i \geq 0$,

$$F_{-1}[j_1 - 2/3]F_{-1}[j_2 - 2/3]F_{-1}[j_3 - 2/3]E_{-1}[i - 2/3],$$
for $i < 0$, 
$$H_{-2}[i - 4/3]F_{-1}[j_1 - 2/3]F_{-1}[j_2 - 2/3],$$
for $i \geq 0$, 
$$F_{-1}[j_1 - 2/3]F_{-1}[j_2 - 2/3]H_{-2}[i - 4/3],$$
for $j$, 
$$F_{-1}[j - 2/3].$$

This family of terms is free. The identities obtained by the elements in the expression of $[5.1]$ are the following ones:

for $E_{-1}[i - 2/3]F_{-1}[j_1 - 2/3]F_{-1}[j_2 - 2/3]F_{-1}[j_3 - 2/3]$, and $F_{-1}[j_1 - 2/3]F_{-1}[j_2 - 2/3]F_{-1}[j_3 - 2/3]E_{-1}[i - 2/3]$,

$$(V_-[-i - 1]V_+[-j_1 - 1]V_+[-j_2 - 1]V_+[-j_3 - 1] -$$

$- [3]V_+[-j_1 - 1]V_-[-i - 1]V_+[-j_2 - 1]V_+[-j_3 - 1] +$

$+ [3]V_+[-j_1 - 1]V_+[-j_2 - 1]V_-[-i - 1]V_+[-j_3 - 1] -$

$- V_+[-j_1 - 1]V_+[-j_2 - 1]V_+[-j_3 - 1]V_-[-i - 1]) = 0,$

because of the $q$-Serre relation between $V_\pm$.

For the other expressions, the terms are then put in the form of contour integrals. $z_2$ from 1 to 1 rotating around 0 in the trigonometric way, $z_1$ too, around $z_2$, $z$ around $z_1$ and $z_2$, and endly $z'$ around $z_1$, $z_2$ and $z$.

$$- \int V_-(z')V_+(z)V_+(z_1)V_+(z_2)[1/(z - z')][z^{i - 1}^1 z_1^{j_1} z_2^{j_2}]dzdz'dz_1dz_2 -$$

$$- \int V_-(z')V_+(z_1)V_+(z)V_+(z_2)[1/(z - z')][z^{i - 1}^1 z_1^{j_1} z_2^{j_2}]dzdz'dz_1dz_2 -$$

$$- \int V_-(z')V_+(z_1)V_+(z_2)V_+(z)[1/(z - z')][z^{i - 1}^1 z_1^{j_1} z_2^{j_2}]dzdz'dz_1dz_2 +$$

$+ [3] \int V_+(z)V_-(z')V_+(z_1)V_+(z_2)[1/(z - z')][z^{i - 1}^1 z_1^{j_1} z_2^{j_2}]dzdz'dz_1dz_2 +$

$+ [3] \int V_+(z_1)V_-(z')V_+(z)V_+(z_2)[1/(z - z')][z^{i - 1}^1 z_1^{j_1} z_2^{j_2}]dzdz'dz_1dz_2 +$

$+ [3] \int V_+(z_1)V_-(z')V_+(z_2)V_+(z)[1/(z - z')][z^{i - 1}^1 z_1^{j_1} z_2^{j_2}]dzdz'dz_1dz_2 -$
\[-3\int V_+(z_1)V_+(z)V_-(z')V_+(z_2)[1/(z-z')][z^{i-1}z_1^{i_1}z_2^{i_2}]dzdz'dz_1dz_2-
\]
\[-3\int V_+(z)V_+(z_1)V_-(z')V_+(z_2)[1/(z-z')][z^{i-1}z_1^{i_1}z_2^{i_2}]dzdz'dz_1dz_2-
\]
\[-3\int V_+(z_1)V_+(z_2)V_-(z')V_+(z)[1/(z-z')][z^{i-1}z_1^{i_1}z_2^{i_2}]dzdz'dz_1dz_2+
\]
\[+\int V_+(z_1)V_+(z_2)V_+(z)V_-(z')[1/(z-z')][z^{i-1}z_1^{i_1}z_2^{i_2}]dzdz'dz_1dz_2+
\]
\[+\int V_+(z_1)V_+(z)V_+(z_2)V_-(z')[1/(z-z')][z^{i-1}z_1^{i_1}z_2^{i_2}]dzdz'dz_1dz_2+
\]
\[+\int V_+(z)V_+(z_1)V_+(z_2)V_-(z')[1/(z-z')][z^{i-1}z_1^{i_1}z_2^{i_2}]dzdz'dz_1dz_2+
\]
\[+(z_1 \leftrightarrow z_2) = 0.\]
\[\int V_-(z')V_+(\xi)V_+(z)V_+(z_1)[1/(z-z')][1/(\xi-z')][z_1^{i_1}dzdz'd\xi d\xi_1+
\]
\[+\int V_-(z')V_+(z_1)V_+(\xi)V_+(z)[1/(z-z')][1/(\xi-z')][z_1^{i_1}dzdz'd\xi d\xi_1+
\]
\[+\int V_-(z')V_+(\xi)V_+(z_1)V_+(z)[1/(z-z')][1/(\xi-z')][z_1^{i_1}dzdz'd\xi d\xi_1-
\]
\[-3(\int V_+(\xi)V_-(z')V_+(z)V_+(z_1)[1/(z-z')][1/(\xi-z')][z_1^{i_1}dzdz'd\xi d\xi_1+
\]
\[+\int V_+(z_1)V_-(z')V_+(\xi)V_+(z)[1/(z-z')][1/(\xi-z')][z_1^{i_1}dzdz'd\xi d\xi_1+
\]
\[+\int V_+(\xi)V_-(z')V_+(z_1)V_+(z)[1/(z-z')][1/(\xi-z')][z_1^{i_1}dzdz'd\xi d\xi_1+
\]
\[+3(\int V_+(\xi)V_+(z)V_-(z')V_+(z_1)[1/(z-z')][1/(\xi-z')][z_1^{i_1}dzdz'd\xi d\xi_1+
\]
\[+\int V_+(z_1)V_+(\xi)V_-(z')V_+(z)[1/(z-z')][1/(\xi-z')][z_1^{i_1}dzdz'd\xi d\xi_1+
\]
\[+\int V_+(\xi)V_+(z_1)V_-(z')V_+(z)[1/(z-z')][1/(\xi-z')][z_1^{i_1}dzdz'd\xi d\xi_1-
\]
\[-(\int V_+(z_1)V_+(\xi)V_+(z)V_-(z')[1/(z-z')][1/(\xi-z')][z_1^{i_1}dzdz'd\xi d\xi_1+
\]
If the first expression vanishes for $V_0$ (zero modes), the same manipulation implies the vanishing for all the $\alpha_+$ and $\alpha_-$ powers of $z$, $z'$. By linear combination, it implies the general equality. We will therefore prove the identity for zero modes.

$$
+ \int V_+(\xi) V_+(z_1) V_+(z) V_-(z') [1/(z - z')][1/(\xi - z)] z_1^j dzdz'd\xi dz_1 + \\
+ \int V_+(\xi) V_+(z) V_+(z_1) V_-(z') [1/(z - z')][1/(\xi - z)] z_1^j dzdz'd\xi dz_1 + [ \\
-(\int V_-(z') V_+(z) V_+(\xi) V_+(z_1) [1/(z - z')][1/(\xi - z)] z_1^j dzdz'd\xi dz_1 + \\
+ \int V_-(z') V_+(z_1) V_+(\xi) V_+(z) [1/(z - z')][1/(\xi - z)] z_1^j dzdz'd\xi dz_1 + \\
+ \int V_-(z') V_+(z_1) V_+(z) V_+(\xi) [1/(z - z')][1/(\xi - z)] z_1^j dzdz'd\xi dz_1 + \\
+ [3] \int V_+(z) V_-(z') V_+(\xi) V_+(z_1) [1/(z - z')][1/(\xi - z)] z_1^j dzdz'd\xi dz_1 + \\
+ [3] \int V_+(z) V_-(z') V_+(z_1) V_+(\xi) [1/(z - z')][1/(\xi - z)] z_1^j dzdz'd\xi dz_1 + \\
+ [3] \int V_+(z_1) V_-(z') V_+(z) V_+(\xi) [1/(z - z')][1/(\xi - z)] z_1^j dzdz'd\xi dz_1 - \\
- [3] \int V_+(z_1) V_+(z) V_-(z') V_+(\xi) [1/(z - z')][1/(\xi - z)] z_1^j dzdz'd\xi dz_1 - \\
- [3] \int V_+(z) V_+(\xi) V_-(z') V_+(z_1) [1/(z - z')][1/(\xi - z)] z_1^j dzdz'd\xi dz_1 + \\
- [3] \int V_+(z) V_+(\xi) V_-(z') V_+(z_1) [1/(z - z')][1/(\xi - z)] z_1^j dzdz'd\xi dz_1 - \\
-(\int V_+(z) V_+(\xi) V_-(z') V_+(z_1) [1/(z - z')][1/(\xi - z)] z_1^j dzdz'd\xi dz_1 - \\
- \int V_+(z_1) V_+(z) V_+(\xi) V_-(z') [1/(z - z')][1/(\xi - z)] z_1^j dzdz'd\xi dz_1 - \\
- \int V_+(z) V_+(z_1) V_+(\xi) V_-(z') [1/(z - z')][1/(\xi - z)] z_1^j dzdz'd\xi dz_1] = 0.
$$

If the first expression vanishes for $V_+(z)$ and $V_-(z')$ in the case of the zero modes, the same manipulation implies the vanishing for all the $\alpha_+ V_+(z)$ and $\alpha_- V_-(z')$, using for $\alpha_+$ and $\alpha_-$ powers of $z$, $z'$. By linear combination, it implies the general equality. We will therefore prove the identity for zero modes.
To show the vanishing, the contours are displaced toward the unity circle and each of the integrals are decomposed with the angular sectors.

\[
\int_S V_-(z')V_+(z)V_+(z_1)V_+(z_2)[1/(z - z')]dzdz'dz_1dz_2,
\]

with \( S \) a certain angular sector.

**Lemma 5.1.** For the contour integrals, the rule of calculus is the following one, for \( 0 < \arg(z) < \arg(z') < 2\pi \) and \( |z| > |z'| \):

\[
V_+(z)V_-(z') = (z - z')^\alpha : V_+(z)V_-(z') :,
\]

it gives:

\[
V_+(z)V_-(z') = e^{i\alpha\pi}V_-(z')V_+(z),
\]

with:

\[
q^{-1} = e^{i\alpha\pi}, \alpha = \beta^2.
\]

The vanishing of the coefficients must be showed with each of the angular sectors, using the standard integrals by the rules of calculus. For this aim, it is made use of a lattice. Because of the independance of the angular sectors for a lattice (lemma 5.2), the vanishing of the coefficients for the angular sectors of the lattice implies that the coefficients vanish for the ones of the contour integrals, and so proves the identity.

### 5.3 The lattice

The lattice is given by \( \mathbb{Z} \) and the variables are \( x^+_i, x^-_i \), with:

\[
x^+_i x^+_i' = q^{e'e_i} x^+_i x^+_i,
\]

for \( i < i', i, i' \in \{1, 2, ..., N\} \) and \( e, e' = +, - \); it allows to give a definition of the discreet analogue of the charge operators,

\[
V^d_\pm = \sum_{i=1}^N x^\pm_i.
\]

For a permutation of \( \{z, w_1, w_2, w_3\} \), \( \sigma \), and for a fixed angular sector, the integration over it is:
\[
\int_S V_{\text{sig}(z)}(\sigma(z))V_{\text{sig}(w_1)}(\sigma(w_1))V_{\text{sig}(w_2)}(\sigma(w_2))V_{\text{sig}(w_3)}(\sigma(w_3))
\]
\[
= c(\sigma) \int_{\sigma^{-1}(S)} V_-(z)V_+(w_1)V_+(w_2)V_+(w_3),
\]
with \(\text{sig}\), the sign corresponding with the permutated term, and with \(c(\sigma)\), a coefficient which depends on the permutation.

Also for the permutation of \(\{i, j_1, j_2, j_3\}\), the sum over the corresponding lattice sector is:
\[
\sum_{i,j,k \in S'} x_{\sigma(i)}^{\text{sig}(i)} x_{\sigma(j_1)}^{\text{sig}(j_1)} x_{\sigma(j_2)}^{\text{sig}(j_2)} x_{\sigma(j_3)}^{\text{sig}(j_3)} = c(\sigma) \sum_{i,j,k \in \sigma^{-1}(S')} x_i^- x_{j_1}^+ x_{j_2}^+ x_{j_3}^+,
\]
the rules of permutation are the same.

The vanishing of the coefficients of the lattice sums is proved using an identity for the lattice (5.3) (lemma 5.3), after having showed the independance of the angular sectors over the lattice (lemma 5.2). The vanishing of the coefficients of the standard integrals over the sectors is showed and the identity (5.2).

The sectors is showed and the identity (5.2).

**Lemma 5.2.** The angular sectors over the lattice are independant.

**Lemma 5.3.** The following identity for the lattice is satisfied:
\[
\sum_{j,j',j_1,j_2} x_j^+ x_{j_1}^+ x_{j_2}^+ e(j,j') - x_j^- x_{j_1}^+ x_{j_2}^+ e(j,j') - x_j^+ x_{j_1}^- x_{j_2}^+ e(j,j') +
\]
\[
\sum_{j,j',j_1,j_2} x_j^+ x_{j_1}^- x_{j_2}^+ e(j,j') + x_j^+ x_{j_1}^+ x_{j_2}^- e(j,j') +
\]
\[
+ [3] \sum_{j,j',j_1,j_2} x_j^+ x_{j_1}^- x_{j_2}^+ e(j,j') + x_j^+ x_{j_1}^+ x_{j_2}^- e(j,j') +
\]
\[
+ [3] \sum_{j,j',j_1,j_2} x_j^+ x_{j_1}^- x_{j_2}^+ e(j,j') - x_j^+ x_{j_1}^+ x_{j_2}^- e(j,j') -
\]

\[\text{Lemma 5.3.} \quad \text{The following identity for the lattice is satisfied:}
\]

\[\text{Lemma 5.2.} \quad \text{The angular sectors over the lattice are independant.}
\]

\[\text{Lemma 5.3.} \quad \text{The following identity for the lattice is satisfied:}
\]
The identity (5.3) is obtained.

Proof:
e is decomposed in elementary antisymmetric functions; so we can have \( e(j, j') = 1 \) and the other terms are zero.

Some derivations of the ring \( C[x_i^\pm] \) are used:

Lemma 5.4. The following formulas define derivations of the ring \( C[x_i^\pm] \):

\[
\begin{align*}
\delta_i^+ &= \delta_i, \\
\delta_i^+ (x_j^\varepsilon) &= \delta_{\varepsilon \delta_{ij}} x_j^\varepsilon, \\
\delta_i^- (x_j^\varepsilon) &= \delta_{\varepsilon \delta_{ij}} x_j^\varepsilon.
\end{align*}
\]

The derivation of the \( q \)-Serre relation \( S^r = 0 \) for the lattice, which is proved by use of the coproduct of \( U_q\hat{s}l_2 \), is then:

\[
\begin{align*}
S^r &= V_-V_+^3 - [3]V_+V_-^2V_+ + [3]V_+^2V_-V_+ - V_+^3V_- = 0, \\
\delta_j^+ \delta_j^- (S) &= \delta_j^+(x_j^- V_+^3) - [3]V_+x_j^- V_+^2 + [3]V_+^2x_j^- V_+ - V_+^3x_j^-
\end{align*}
\]

\[
= \sum_{j_1, j_2} -x_j^- x_j^+ x_j^+ x_j^- x_j^- x_j^+ - x_j^- x_j^- x_j^+ x_j^+ x_j^+ x_j^- + [3] \sum_{j_1, j_2} x_j^- x_j^+ x_j^+ x_j^- x_j^- x_j^+ + x_j^- x_j^- x_j^+ x_j^+ x_j^+ x_j^- + [3] \sum_{j_1, j_2} -x_j^- x_j^+ x_j^- x_j^- x_j^+ x_j^- x_j^- x_j^+ + x_j^- x_j^- x_j^+ x_j^+ x_j^- x_j^- x_j^+ + \sum_{j_1, j_2} x_j^- x_j^+ x_j^- x_j^- x_j^+ x_j^- + x_j^- x_j^- x_j^- x_j^+ x_j^- x_j^- x_j^+ + x_j^- x_j^- x_j^- x_j^- x_j^- x_j^- x_j^+ + x_j^- x_j^- x_j^- x_j^- x_j^- x_j^- x_j^- x_j^+ + (j_1 \leftrightarrow j_2) = 0.
\]

The identity (5.3) is obtained.

\( \Box \)

A same calculation holds in the case \( F_{-1} \), the derivations \( \delta_i^\pm \) are applied three times.
6 Calculations of cohomology

6.1 The quantum resolution BGG

The quantum resolution BGG ($q$-BGG) is constructed with Verma modules $\mathbb{M}$:

$$B_j^q(\mathfrak{g}) = \bigoplus_{l(s)=j} M^q_{\rho-s(\rho)},$$

with $M^q_{\rho-s(\rho)}$ a module of Verma.

**Definition 6.1.**

$$\tilde{\pi}^\beta_0 = \bar{\pi}^\beta_0,$$

if $n \neq 0$,

$$\tilde{\pi}^\beta_n = \bar{\pi}^\beta_n \oplus \bar{\pi}^{\beta}_{n+1}.$$

**Definition 6.2.** A degree $d_{\text{VOA}}$ is defined for $\tilde{\pi}^\beta_n$, $\nu^\beta_n$ being the vector of highest weight:

$$d_{\text{VOA}}\nu^\beta_n = \frac{n^2}{3},$$

$$d_{\text{VOA}}X[i] = i.$$

d respects the degrees.

**Notation 1.** For all $n$, define $\tilde{H}_k(n)^\beta$ as the cohomology $H_k(\tilde{\pi}^\beta(n)).$

6.2 The degrees of the $\tilde{\pi}_n$

**Lemma 6.1.** $\bar{V}_\pm$ are morphisms of $\tilde{\pi}^\beta_n$ towards $\tilde{\pi}^\beta_{n+1}$, if $\tilde{\pi}^\beta_n$ has for degree VOA $d_{\text{VOA}} - n^2/3$ and $\tilde{\pi}^\beta_{n+1}$, $d_{\text{VOA}} - (n+1)^2/3$, then they are of degree $1/3$.

At level of $q$-BGG, the cohomology of the resolution is $\Lambda^*a^*$. The degree of the resolution gives $\tilde{\pi}^\beta_0$ of degree 0, $\tilde{\pi}^\beta_1$ is of degree 1, $\tilde{\pi}^\beta_{-1}$ of degree $-2/3$. The differential is of degree 0.
6.3 The character of the resolution

Let us fix $n \in [1/3] \mathbb{Z}$. The Euler characteristic of the resolution is for the degree $n$ component of the VOA:

$$
\chi^\beta(n)(\tilde{F}_s, d) = \sum_i \dim(\tilde{F}_{2i}(n)) - \sum_i \dim(\tilde{F}_{2i+1}(n)) =
$$

$$
= \sum_i \dim(\tilde{H}_{2i}(n)^\beta) - \sum_i \dim(\tilde{H}_{2i+1}(n)^\beta).
$$

The character is the same in classical as in quantum. The character of the quantum cohomology is the same as that of the quantum complex, as the character of the cohomology is the one of the complex, it is obtained that the character is the same in classical as in quantum.

The classical cohomology is zero in odd degree if $3n$ is even, $n$ being $d_{VOA}$, and zero in even degree if $3n$ is odd.

In a point of specialisation, the cohomology increases; so that:

$$
dim(\tilde{H}_{2i}(n)^\beta) \leq dim(\tilde{H}_{2i}(n)),
$$

$$
dim(\tilde{H}_{2i+1}(n)^\beta) \leq dim(\tilde{H}_{2i+1}(n)).
$$

As:

$$
dim(\tilde{H}_{2i+1}(n)) = 0,
$$

$$
dim(\tilde{H}_{2i+1}(n)^\beta) = 0.
$$

And:

$$
dim(\tilde{H}_{2i}(n)^\beta) \leq dim(\tilde{H}_{2i}(n)).
$$

As the character is the same in quantum as in classical:

$$
\sum_i dim(\tilde{H}_{2i}(n)^\beta) = \sum_i dim(\tilde{H}_{2i}(n)),
$$

it is obtained:

$$
dim(\tilde{H}_{2i}(n)^\beta) = dim(\tilde{H}_{2i}(n)).
$$

Spectral sequence \[14\] must be considered, the vertical cohomology is $\Lambda^*\alpha^*$, with a horizontal differential of degree 1. The vertical
differential of the spectral sequence is of degree \((0, 1)\), the horizontal one is of degree \((1, 0)\).

The cohomology of the total complex is \(E^{p,q}_2\).

For \(p = 0, 1\), \(E^{p,q}_2 = \Lambda^q a^*\), for \(p \neq 0, 1\), \(E^{p,q}_2 = 0\).

The cohomology of the total complex is \(\sum_{p+q=n} E^{p,q}_2\). The \(H^1\) of the total complex is \(E^{1,0}_2 \oplus E^{0,1}_2\), it gives, \(\mathbb{C} \oplus a^*\).

### 6.4 The quantum integrals of motion

**Theorem 6.1**. The integrals of motion \(\oint d_{2i+1}\) admit quantum deformation, which belong to a completed universal envelopping algebra and which commute with each other and have the same degrees as the integrals of motion.

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