Composite particle hydrodynamics from dyonic black branes

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Abstract

We construct an effective hydrodynamics of composite particles in (2+1) dimensions carrying magnetic flux, employing a holographic approach. The hydrodynamics is obtained by perturbation of the dyonic black brane solutions in the derivative expansion. We introduce a consistent way to avoid mixing of different orders in the expansion. Thanks to this method, it is possible to take the strong external magnetic field limit in the dual field theory. To compare our result with the composite particle system, we study several cases which correspond to special solutions of Einstein’s equation and Maxwell’s equations.
1 Introduction

The AdS/CFT correspondence tells us that strongly coupled CFT’s can be described by supergravity or string theory in higher dimensional AdS space [1]. This nontrivial relation provides many possible techniques to study strongly coupled field theories, which are very difficult to investigate using conventional field theory methods. For this reason, the AdS/CFT relation has been applied to diverse topics: for instance supersymmetric quiver gauge theories, quantum chromodynamics, condensed matter theory etc. In such cases, the AdS/CFT may tackle easily the region of strong coupling, high density or finite temperature. Another interesting problem which can be analyzed by AdS/CFT is the physics in a strong magnetic field. This is our main interest in the present work.

When the external magnetic field is applied to an electron system, the spectrum is given by the Landau levels. Including interactions between the electrons in 2+1 dimensions, one expects the quantum Hall effect in the system. Although the situation is not easy to understand, a nice tool has been proposed to figure out such an effect. It is a quasi-particle excitation around the nontrivial vacuum. The appearance of quasi-particles such as phonons, magnons, and Cooper pair is commonplace in condensed matter physics. For the physics under strong magnetic field the appropriate tool is composite fermion which we find more advantageous to describe the dynamics. The composite fermion is a very useful concept to study quantum Hall system in particular, just like Cooper pairs for describing the mechanism of superconductors [2]. A nice review for composite fermion can be found in [3]. See e.g. [4] for a more thorough exposition.

When the filling fraction $\tilde{\nu}$ is smaller than 1, all the electrons occupy the lowest Landau level, and it is the energy level which describes most of the system. Also the mixing with other level can be ignored in a sufficiently strong magnetic field. As we describe the system using the Landau level the kinetic energy is a mere constant. This means that the only relevant part of the hamiltonian is the interaction energy between electrons. In order to understand the system, one should diagonalize the hamiltonian, which in general is not easy. To make the matter worse, since the only scale in the problem is given by the interaction energy, the perturbative analysis is not available. However, there is an interesting proposal which can help us understand the system. It has been known for some time that some wave-functions conjectured with the help of experimental data describe the system very well. They have a common factor which implies the situation where every electron sees vortices at positions which are occupied by other electrons. It tells us that the effective particle of the system is the composite fermion which is a bound
state of an electron and quantum vortices.

Intuitively, this idea provides a nice way to understand the physics under strong magnetic field. Near the half filling case, the vortices contained in each electron cancel out the external field and the resulting magnetic field must be absent or very small. In this case, we may treat the system as a collection of composite fermions with small external magnetic field and not in terms of Landau levels, which behave as the composite-fermion’s fermi sea \[2\]. Following this reasoning, the particles contain not only electric charge but also magnetic flux. This additional ingredient should manifest itself when we calculate physical quantities. In this paper, we show the effect in the current and the energy-momentum tensor of hydrodynamics through holographic approach.

This point of view already has been exploited in the holographic approach \[5\], where the authors identified a composite fermion system with a dyonic black brane in 3 + 1 dimensional anti-de Sitter spacetime. Some basic quantum properties on both sides of the duality are studied. It is assumed that the magnetic charge of black brane is proportional to its electric charge with filling fraction \(\tilde{\nu}\). This is the starting point for our gravity dual.

On the other hand, another important ingredient of this paper is the fluid/gravity correspondence, which was suggested in \[6\]. The authors derived the boundary relativistic hydrodynamics from the bulk Einstein’s equations with negative cosmological constant. There have been many generalizations of this work: in different dimensions, different black holes , different boundary conditions and models including higher curvature terms \[7, 8, 9, 10, 11, 12, 13, 14, 15, 16\]. For the problem of our interest here, quantum hall fluid, we need to consider the dyonic black holes as in \[5\]. One is naturally led to expect that the dyonic black hole could be dual to the magnetohydrodynamics. This duality relation has been considered in \[17, 18, 19, 20, 21, 22, 23, 24, 25\].

For the study of holographic hydrodynamics, we start with Einstein-Maxwell action with the negative cosmological constant in 3 + 1 dimension. Using the AdS/CFT dictionary, dyonic black brane solution describes a finite temperature conformal field theory with finite charge density in an external magnetic field. Macroscopic thermodynamic quantities such as temperature (Hawking temperature), charge density (electric charge) and magnetic field (magnetic charge) appear as the parameters of the black brane. We also consider a boost parameter \(\vec{\beta}\) and the thermodynamic quantities as functions of boundary coordinates. Performing the derivative expansion order by order, one can find new fluctuating black brane solution. We consider only
the leading order result here and construct the boundary energy momentum tensor and current when the external magnetic field is present. The expression contains transport coefficients of the fluid. In addition to these boundary tensors, the bulk Einstein’s equation and Maxwell’s equation provide additional constraints for the tensors. These equations play a role of the relativistic Navier-Stokes equations at the boundary. Combining them, our result describes a magnetohydrodynamics in 2+1 dimensional spacetime.

Adopting the approach of the fluid/gravity correspondence for the dyonic black branes, one should be careful of perturbation for the magnetic field. Since the interpretation of the magnetic field is the external field in the dual field theory, one can’t perturb it without physically plausible reason. In our case, we don’t want to introduce further dynamical degrees of freedom, and so the magnetic part should be kept as the original constant field. In other words, the boosted magnetic field is written in terms of constant velocity field and constant magnetic charge, and these should be replaced by functions of the boundary coordinates in the next step of the fluid/gravity approach. It seems that the external field is taken as a dynamical field. To avoid this situation, we add a field $\delta F_{\mu\nu}$, which is given by difference between the original constant field and the fluctuation field. In conclusion, we don’t touch the external field in the dual field theory. So our computation is safe from the inconsistent situation.

In addition, there is another technical problem. We may choose a gauge for the magnetic field, where the gauge potential is linear in the boundary coordinate, $x^\mu$, for example, $A^\text{magnetic}_\mu \sim \hat{H} \epsilon_{\mu\nu\lambda} x^\nu \bar{u}^\lambda$. Following the method of the fluid/gravity correspondence, one can easily see that the $x^\mu$ makes some trouble for the expansion. It causes mixing of the different derivative orders and makes the magnetic field linear in the boundary coordinate. So it is difficult to deal with a strong magnetic field. This is one of reason why most of work of the fluid/gravity approach are concentrated on the weak magnetic field case. We try to keep off this problem by taking into account not the gauge potential but the field strength, and then some part of field strength has a constraint looks like a conservation equation. This, however, does not mean the external field is dynamical in the boundary field theory, because the total field strength is still constant. So the constraint is nothing but an identity, $dF_{\text{ext}} = 0$. We will discuss this in more detail.

Following this method, we obtained the general solution which is consistent with the derivative expansion and the zeroth order(strong) magnetic field. This bulk solution is dual to the magnetohydrodynamics. As in [5], we substituted $Q(x)/\bar{\nu}$ for the magnetic field $H(x)$ and then we provide the transport coefficients depending on the filling fraction. In this case, the result resembles the magnetohydrodynamics with the first order(weak) external field. We point out
that this effective description is holographic dual to the transformation from the particle system
in a strong magnetic field to the composite particle system in a weak magnetic field. In addition,
we consider an example where the effective external magnetic field vanishes.

This paper is organized as follows. In section 2, we discuss the derivation of hydrodynamics
from the derivative expansion of Einstein equation. In section 3, the first order solution is
investigated for various cases. Then, we discuss the composite particle case in section 4. We
summarize our result and conclude in section 5.

2 General structure of metric and field strength

It is our goal to study strongly coupled 2+1 dimensional CFT system with a charge density in
a strong magnetic field. So we introduce a model which has Maxwell’s field and Einstein-Hilbert
action with negative cosmological constant as follows.

\[ S = \frac{1}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g} \left( R + \frac{6}{l^2} \right) + \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^3x \sqrt{-\gamma} \Theta - \frac{1}{4g_c^2} \int_{\mathcal{M}} d^4x \sqrt{-g} F^2 + I_C, \]

where \( \Theta \) is the trace of the extrinsic curvature, so called Gibbons-Hawking term, and \( I_C \) is a
counter term, which cancels divergence of the action. We can write down the expression for our
configuration explicitly as follows

\[ I_C = \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^3x \frac{2}{l} \sqrt{-\gamma} \left( 1 + \frac{l^2}{R(\gamma_{\mu\nu})} \right), \]  

(1)

where \( \gamma_{\mu\nu} \) is the induced metric which is defined in A.2. These counter terms were first intro-
duced in [26, 27], and the general results with matter fields can be found in [28, 29]. From the
above action, one can derive the equations of motion as following,

\[ R_{IJ} - \frac{1}{2}g_{IJ}R - \frac{3}{l^2}g_{IJ} - \frac{16\pi G}{2g_c^2}(F_{KI}F^{KJ} - \frac{1}{4}g_{IJ}F^2) = 0, \]  

(2)

\[ \nabla_J F^J_I = 0. \]  

(3)

These are Einstein’s equations and Maxwell’s equations. For convenience’s sake, we define
functions related to the equations of motion.

\[ W_{IJ} \equiv R_{IJ} + \frac{3}{l^2}g_{IJ} - \frac{16\pi G}{2g_c^2}(F_{KI}F^{KJ} - \frac{1}{4}g_{IJ}F^2), \]  

(4)

\[ W_I \equiv \nabla_J F^J_I. \]  

(5)

Using these functions, we may express the equations of motion as \( W_I = 0 \) and \( W_{IJ} = 0. \)
Now we would like to introduce an external magnetic field to the 2+1 dimensional system. For this, it is natural to consider the dyonic black brane solution in AdS spacetime as a gravity dual. The dyonic uniform black brane solution solving the equations of motion is as follows.

\[ ds^2 = -r^2 f(r) dv^2 + 2 dv dr + r^2 (dx^2_1 + dx^2_2), \]
\[ f(r) = (1 - \frac{\bar{M}}{r^3} + \frac{\bar{Q}^2 + \bar{H}^2}{4r^4}), \]
\[ \mathcal{F} = \frac{\bar{Q}}{r^2} dr \wedge dv + \bar{H} dx_1 \wedge dx_2, \]
\[ A = \frac{\bar{Q}}{r} dv + \left( \int x^i \delta_{ij} \bar{H} \right) dx^j. \]

where \( i \) indices are running in \((x_1, x_2)\). We have rescaled \( F_{IJ} \) by \( \frac{16\pi G}{g^2} \) and took \( l = 1 \). The metric function \( f(r) \) has two zeros, outer horizon and inner horizon which are denoted by \( r_+ \) and \( r_- \), respectively. Following the approach in [6], we consider following boosted solution,

\[ ds^2 = -r^2 f(r)(\bar{u}_\mu dx^\mu)^2 - 2 \bar{u}_\mu dx^\mu dr + r^2 P_{\mu\nu} dx^\mu dx^\nu, \]
\[ \mathcal{F} = -\bar{Q} \bar{u}_\mu dx^\mu \wedge dr - \frac{1}{2} \bar{H} \bar{u}^\lambda \epsilon_{\lambda\mu\nu} dx^\mu \wedge dx^\nu, \]
\[ \bar{u}^0 = \frac{1}{\sqrt{1 - \beta^2_i}}, \quad \bar{u}^i = \frac{\beta_i}{\sqrt{1 - \beta^2_i}}, \quad P_{\mu\nu} = \eta_{\mu\nu} + \bar{u}_\mu \bar{u}_\nu, \]

where we chose orientation \( \epsilon^{012} = 1 \) and \( \bar{u}^\mu \) is a timelike vector from boost parameter \( \vec{\beta} \). In [5], the authors considered \( \bar{H} = \bar{Q}/\bar{\nu} \) as a holographic dual of a composite fermion system. We will discuss it later on.

As a second step, we perturb this solution with keeping the macroscopic thermodynamic structure in the dual field theory, so we regard the mass \( \bar{M} \), the electric charge \( \bar{Q} \), magnetic charge \( \bar{H} \) and boost parameter \( \vec{\beta} \) as functions of the boundary coordinate \( x^\mu \), denoted by \( M(x), Q(x), H(x) \) and \( \vec{\beta}(x) \). In the dual field theory, they are interpreted as energy, charge density, external magnetic field and fluid velocity, respectively. For the field strength except for magnetic part, one may choose gauge field ansatz, \( -\frac{\bar{Q}}{r^2} u_\mu(x) dx^\mu \), which produces same field strength when all functions are constant. It is, however, not easy to find regular gauge field ansatz for the magnetic part. One way is to consider \( F_{\text{mag}} = -\frac{1}{2} H(x) u^\lambda(x) \epsilon_{\lambda\mu\nu} dx^\mu \wedge dx^\nu \) as

\[ \bar{M} = r_+^3 + r_+ r_+^2 + r_+^2 r_- + r_-^3 \quad \text{and} \quad \bar{Q}^2 + \bar{H}^2 = 4r_- r_+ (r_+^2 + r_+ r_- + r_-^2) \]
the magnetic flux with imposing $d\mathcal{F}_{\text{mag}} = 0$. This setup is equivalent to taking the field strength and the metric as follows.

$$ds^2 = -r^2 f(r)(u_\mu dx^\mu)^2 - 2u_\mu dx^\mu dr + r^2 P_{\mu\nu}(x)dx^\mu dx^\nu,$$  

where $H(x)u^\mu(x)$ satisfies $\partial_\mu (Hu^\mu) = 0$ due to the Bianchi identity. Coming back to constant functions, this becomes the previous solution \[7\]. As we explained in the introduction, we considered the field strength with the constraint instead of the gauge field like $A_\mu = -\tilde{H}\bar{u}^\lambda\epsilon_{\lambda\mu\nu}x^\nu$, which produces singular field strength after replacing $\tilde{H}$ and $\bar{u}^\lambda$ with $H(x)$ and $u^\mu(x)$.

However, this ansatz makes another problem discussed in the introduction. Unlike $u^\mu(x)$, the magnetic flux $H(x)$ is not a dynamical field. Therefore we need to compensate the variation of $H(x)$ by an additional field which does not appear in the zeroth order. We denote such a field as $\delta F_{\mu\nu}$, defined by

$$\delta F_{\mu\nu}(x) \equiv \partial_\mu \delta A_\nu(x) - \partial_\nu \delta A_\mu(x) = \frac{1}{2}\tilde{H}\bar{u}^\lambda\epsilon_{\lambda\mu\nu} + \frac{1}{2}H(x)u^\lambda(x)\epsilon_{\lambda\mu\nu}.$$  

Adding this to \[9\], Our starting field strength becomes

$$\mathcal{F} = -\frac{Q}{r^2}u_\mu dx^\mu \wedge dr - \frac{\partial_\mu (Q u_\nu)}{r} dx^\mu \wedge dx^\nu - \frac{1}{2}Hu^\lambda\epsilon_{\lambda\mu\nu}dx^\mu \wedge dx^\nu + \frac{1}{2}\delta F_{\mu\nu}dx^\mu \wedge dx^\nu,$$  

where sum of the last two terms is nothing but original constant magnetic field, thus we clearly do not perturb the magnetic part and do not introduce any further dynamical degrees of freedom. The advantage of this compensation is that we may take strong magnetic field which is consistent with the derivative expansion.

Now we are ready for the next step of \[6\]. Let us suppose that the parameters have boundary coordinate dependence, then the metric \[8\] and the field strength \[11\] don’t satisfy the equations of motion any more. Accordingly, $W_{IJ}$ and $W_I$ are not equal to zero but proportional to derivatives of these functions. We call them source terms denoted by $-S_{IJ}^{(1)}$ and $-S_{IJ}^{(2)}$. To obtain first order solution, we may add first order terms to the metric and the gauge field with following ansatz \[3\]

$$ds^{(n)2} = \frac{k_{(n)}(r)}{r}dv^2 + 2h_{(n)}(r)dvdr + r^2j_{(n)}(r)dx^i dx^i + r^2(\alpha_{ij}^{(n)} - h_{(n)}(r)\delta_{ij})dx^i dx^j,$$

$$A^{(n)} = a_{\nu}^{(n)}(r)dv + a_{i}^{(n)}(r)dx^i,$$  

\[12\]Where $\delta^{ij}\alpha_{ij} = 0$, and we took a gauge choice, $g_{rr} = 0$, $g_{r\mu} \propto u_\mu$, $Tr[g^{(0)}]^{-1}g^{(n)} = 0$, $\forall n > 0$. In addition, we take a convenient frame which can be always recovered to the covariant form with $u_\mu$, $P_{\mu\nu}$ by the boost.
where we used \((n)\) instead of \((1)\) for general order, since this ansatz is available in the higher orders. Plugging the above terms into \(W_{IJ}\) and \(W_I\), one can get additional terms. We call them “correction terms” denoted by \(C_{IJ}^{(1)}\) and \(C_I^{(1)}\). Using the source terms and the correction terms, the first order Einstein’s equation and the Maxwell’s equation can be expressed by a simple form as \(S_{IJ}^{(1)} = C_{IJ}^{(1)}\) and \(S_I^{(1)} = C_I^{(1)}\).

In general, if we know the \((n-1)\)-th order solution, then one can easily evaluate \(S_{IJ}^{(n)}\) and \(S_I^{(n)}\), \(C_{IJ}^{(n)}\) and \(C_I^{(n)}\). Therefore, the \(n\)-th order equations of motion are given by

\[
W_{IJ}^{(n)} = C_{IJ}^{(n)} - S_{IJ}^{(n)} = 0, \tag{13}
\]

\[
W_I^{(n)} = C_I^{(n)} - S_I^{(n)} = 0, \tag{14}
\]

where the equations are differential equations. The unknown functions in (12) can be obtained by solving the above equations. Not all of the correction terms, however, are independent. For example, \(C_v^{(n)}\) and \(C_r^{(n)}\) are same up to some factor. The relation among the components is given by

\[
C_v^{(n)} + r^2 f(r) C_r^{(n)} = 0, \tag{15}
\]

\[
C_{vv}^{(n)} + r^2 f(r) C_{vr}^{(n)} = 0, \tag{16}
\]

\[
\left\{ r^2 \left( r^2 f(r) C_{ri}^{(n)} + C_{vi}^{(n)} \right) \right\} + \frac{H}{2g} \epsilon_{ij} C_j^{(n)} = 0. \tag{17}
\]

The above identities restrict the source terms as follows.

\[
(S_u^{(n)} + r^2 f(r) S_r^{(n)}) = 0, \tag{18}
\]

\[
(S_{uv}^{(n)} + r^2 f(r) S_{vr}^{(n)}) = 0, \tag{19}
\]

\[
r^4 f(r) S_{ri}^{(n)}(r) + r^2 S_{vi}^{(n)}(r) + \int_{r_+}^{r} \frac{H}{2} \epsilon_{ij} S_j^{(n)}(r') dr' = \frac{H}{2} (Q \epsilon_{ik} + H \delta_{ik}) j_k^{(n)}(r_+). \tag{20}
\]

These constraint equations provide the hydrodynamics equations (the relativistic Navier-Stokes equations).

### 3 First order hydrodynamics from gravity

Let us compute the first order solution explicitly. In fact, the computation is performed in a suitable frame and at the origin, where the \(\vec{\beta}(0)\) vanishes. After all the computation in

\[ C_{IJ}^{(n)} \text{ and } C_I^{(n)} \text{ are listed in Appendix A.1.2} \]

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this frame, one can easily recover the corresponding covariant form. This clever method was suggested in [6] for the first time. To see details of the computation, we recommend visiting Appendix A.1. Using zeroth order solution (8), (11) and derivative expansion, one can find the first order source terms as listed in Appendix A.1.1. Putting these source terms into constraint (18), it gives

\[ \partial_v Q + Q \partial_i \beta_i = 0. \]  

(21)

Changing this equation to a covariant form, we obtain a conservation equation of the current as following

\[ \partial_{\mu}(Q u^\mu) = 0. \]  

(22)

Next constraint (19) gives

\[ (2 \partial_v M + 3M \partial_i \beta_i) - \frac{4}{r} [H(\partial_v H + H \partial_i \beta_i) + Q(\partial_v Q + Q \partial_i \beta_i)] = 0. \]  

(23)

Since \( M, H \) and \( \beta_i \) depend on only the boundary coordinates \( x^\mu \), this is equivalent to

\[ H(\partial_v H + H \partial_i \beta_i) + Q(\partial_v Q + Q \partial_i \beta_i) = 0, \]  

(24)

\[ 2\partial_v M + 3M \partial_i \beta_i = 0. \]  

(25)

By (21), the first constraint (24) is nothing but another equation \( \partial_{\mu}(Hu^\mu) = 0 \), which already appeared as an assumption in the zeroth order expression (11). So this is not just assumption but a result of equation of motion. The last constraint (20) is more complicated form as follows

\[ \partial_i M + 3M \partial_v \beta_i = Q \delta F_{vi} - He^{ij} \delta F_{vj} + He^{ij}(Q \delta_{jk} - H \epsilon_{jk})j_k(r_+) \]  

\[ -H \epsilon^{ij} [\epsilon_{jk} \partial_k H - \delta_{jk} \partial_k Q + (H \epsilon_{jk} - Q \delta_{jk}) \partial_v \beta_k]. \]  

(26)

The covariant form of (25) and (26) gives the equations for a magnetohydrodynamics. We will discuss this soon.

After the calculations in Appendix A.1.3, and up to undetermined integration constants \( C_3 \) and \( D_i \), the first order solutions of Einstein’s equations and Maxwell’s equations can be written. Firstly, the metric is given by

\[ ds^2 = -r^2 f(r)(u_{\mu}dx^\mu)^2 - 2u_{\mu}dx^\mu dr + r^2 P_{\mu\nu}dx^\mu dx^\nu + 2r^2 \alpha(r)\sigma_{\mu\nu}dx^\mu dx^\nu \]  

\[ + (r \partial_v u^\nu + \frac{1}{4r^2} Hu^\lambda \epsilon^{\lambda\mu\nu} \delta F_{\mu\nu} - C_3 \frac{Q}{2r^2}) (u_{\mu}dx^\mu)^2 - 2r^2 j_1(r)u_{\mu}dx^\mu dx^\nu, \]  

(27)
where the current and the external field are given by

$$\sigma_{\mu\nu} = \frac{1}{2} P_{\mu\alpha} P_{\nu\beta} \left[ \partial^\alpha u^\beta + \partial^\beta u^\alpha - P^{\alpha\beta}(\partial \cdot u) \right],$$

$$j^{(1)}_\mu (r) = f(r) \int_\infty^r \frac{dx}{x^4 f(x)^2} \left\{ -x^2 u^\nu \partial_\nu u_\mu + \mathcal{D}_\mu + \frac{1}{x} \left[ \frac{2 M^2 + 8 M r^3}{3 M} - \frac{16 r^6}{3 M} \right] u^\nu \partial_\nu u_\mu + M + 2 r^3 \right\} ,$$

$$+ \frac{M + 2 r^3}{3 r_+ M} (Q P_{\mu\nu} - H u^\lambda \epsilon_{\mu\lambda\nu})(\partial^\nu Q + u^\alpha \epsilon_{\alpha\nu\beta} \partial^\beta H) - \frac{4 r_+(M - r^3)}{3 M} \mathcal{D}_\mu \right\} .$$

Here the $\alpha(r)$ is given in Appendix A.1.3 and $\mathcal{D}_\mu$ is the covariant form of $\mathcal{D}_i$, so it satisfies $\mathcal{D}_\mu u^\mu = 0$. The gauge field is obtained by an integration. The solution is

$$\mathcal{A}_\mu = \frac{\delta A_\mu}{r} - \frac{1}{2} \left( \int^r H u^\lambda \epsilon_{\lambda\mu\nu} dx^\nu \right) - \left( \frac{C_3}{r} - \frac{H}{2 r_+^2} u^\lambda \epsilon_{\lambda\mu\nu} \partial^\nu u^\mu \right) u_\mu$$

$$- (\partial_\mu Q + u^\lambda \epsilon_{\lambda\mu\nu} \partial^\nu H + Q u^\nu \epsilon_{\lambda\mu\nu} \partial_\lambda u^\mu) \int^r \frac{dy}{y^2 f(y)} \left( \frac{1}{r} - \frac{1}{y} \right)$$

$$+ (Q P_{\mu\nu} + H u^\lambda \epsilon_{\lambda\mu\nu}) \int^r \frac{dy}{y^2 f(y)} \left( j^{(1)\nu}_\mu \right) .$$

Putting these result into formulæ (31) and (32), we get the energy-momentum tensor and the current in the boundary theory as follows.

$$T_{\mu\nu} = M (\eta_{\mu\nu} + 3 u_{\mu} u_{\nu}) + \mathcal{D}_\mu u_\nu + \mathcal{D}_\nu u_\mu - 2 r_+^2 \sigma_{\mu\nu} ,$$

$$J^\mu = Q u^\mu + C_3 u^\mu + u^\lambda \delta F_\lambda^\mu + \frac{1}{3 M} (Q P_{\mu\nu} + H u^\lambda \epsilon_{\lambda\mu\nu}) \mathcal{D}_\nu$$

$$- \frac{1}{3 r_+^2} \left( 1 + 2 r_+^2 \right) \left( P_{\mu\nu} \partial_\nu Q + u^\lambda \epsilon_{\lambda\mu\nu} \partial_\nu H \right)$$

$$- \frac{2}{3 r_+^2} \left( 1 + 2 r_+^2 \right) \left( Q P_{\mu\nu} + H u^\lambda \epsilon_{\lambda\mu\nu} \right) u^\nu \partial_\sigma u_\nu .$$

Using these expressions (31) and (32), The constraints (18), (19) and (20) are written as

$$\partial_\mu J^{(0)\mu} = 0 , \partial_\nu (H u^\mu) = 0$$

$$\partial_\mu T^{(0)\mu\nu} = J^{(1)\nu}_\mu (F^{(0)}_{\mu\nu}) + J^{(0)}_\mu (F^{(1)}_{\mu\nu}) ,$$

where the current and the external field are defined by

$$J^{(0)\mu} = Q u^\mu , T^{(0)\mu\nu} = M (\eta_{\mu\nu} + 3 u_{\mu} u_{\nu}) ,$$

$$F^{(0)}_{\mu\nu} = - \frac{1}{2} H u^\lambda \epsilon_{\lambda\mu\nu} dx^\mu \wedge dx^\nu , F^{(1)}_{\mu\nu} = \frac{1}{2} \delta F_{\mu\nu} dx^\mu \wedge dx^\nu .$$

More explicitly, $(F_{\mu\nu})_{ext} = (F^{(0)}_{\mu\nu})_{ext} + (F^{(1)}_{\mu\nu})_{ext} = - H(x) u^\lambda(x) \epsilon_{\lambda\mu\nu} + \delta F_{\mu\nu}(x) = - H u^\lambda \epsilon_{\lambda\mu\nu}$. 

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5 More explicitly, $(F^{(0)}_{\mu\nu})_{ext} + (F^{(1)}_{\mu\nu})_{ext} = - H(x) u^\lambda(x) \epsilon_{\lambda\mu\nu} + \delta F_{\mu\nu}(x) = - H u^\lambda \epsilon_{\lambda\mu\nu}$. 

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In first order, they are nothing but the magnetohydrodynamic equations with a Bianchi identity for the external field,

\[ \partial_\mu J^\mu = 0 , \quad dF_{\text{ext}} = 0 , \]
\[ \partial_\mu T^{\mu\nu} = J_\mu F^{\mu\nu}_{\text{ext}} . \]  

(36)

3.1 Pure electric black brane case

If we consider the case without the magnetic field \( H = 0 \) and take the Landau frame, the \( C_3 \) and the \( D_i \) vanish. The current and the spacial component of the energy-momentum conservation equation turn out to be

\[ J^0 = Q , \quad J^i = (F^{(1)}_{\text{ext}})_{vi} - \frac{1}{3r_+} (1 + \frac{2r_3}{M} ) (\partial_i Q + 2Q\partial_v\beta_i) , \]
\[ \partial_i M + 3M\partial_v\beta_i = Q(F^{(1)}_{\text{ext}})_{vi} . \]  

(37)

(38)

From these two equations, one can easily find different expression of the current which is standard in the hydrodynamics,

\[ J^\mu = Qu^\mu + \sigma_1 \left( u_\nu (F^{(1)}_{\text{ext}})^{\nu\mu} - TP^{\mu\nu} \partial_\nu \frac{\mu}{T} \right) , \]  

(39)

where \( T \) and \( \mu = \frac{Q}{r_+} \) are the temperature and the chemical potential of the dual field theory, respectively. The electric conductivity function \( \sigma_1 \) is given by

\[ \sigma_1 = \left( \frac{12r_+^4 - Q^2}{3(4r_+^4 + Q^2)} \right)^2 . \]  

(40)

The boundary energy momentum tensor and the hydrodynamics equations are given by

\[ T_{\mu\nu} = M(\eta_{\mu\nu} + 3u_\mu u_\nu) - 2r_+^2 \sigma_{\mu\nu} \]
\[ \partial_\mu T^{\mu\nu} = J_\mu (F^{(1)}_{\text{ext}})^{\nu\mu} , \quad \partial_\mu J^\mu = 0 . \]  

(41)

(42)

This result is the (2+1)-dimensional version of [11].

3.2 Small velocity and constant magnetic field limit

In [20], the authors studied a dyonic black brane with small velocity and constant magnetic field. It is possible to cover this limit using our result. In order to get their result, we take \( (F^{(1)}_{\text{ext}})_{\mu\nu} = 0, \ \tilde{\beta} = 0 \) and constant \( H \). From the energy-momentum conservation equation  

\[ \tilde{\beta} \]  

This is given by the Hawking temperature in the bulk solution by AdS/CFT
we can get current \( J_i = \frac{1}{\pi r_+^2} \epsilon_{ij} \partial_i M \). And we already have the current expression (32). In this case, the current becomes

\[
J_i = -\frac{1}{3r_+} \left( 1 + \frac{2r_+^3}{M} \right) (\partial_i Q - \epsilon_{ij} \partial_j H) + \frac{1}{3M} (Q \delta_{ij} - H \epsilon_{ij}) \partial_j .
\] (43)

Subtracting the current, one can find \( D_i \) which is just \( T_{vi} \),

\[
D_i = T_{vi} = \frac{3M}{Q^2 + H^2} \left[ \frac{Q}{H} \epsilon_{ij} \partial_j M - \partial_i M + \frac{1}{3r_+} \left( 1 + \frac{2r_+^3}{M} \right) (Q \partial_i Q + H \epsilon_{ij} \partial_j Q) \right] .
\] (44)

This is consistent with the result in [20].

4 Fluid dynamics for composite particles

In this section, we present the transport coefficients of the fluid dynamics corresponding to the dyonic black brane. In what follows we assign the filling fraction as the ratio of the charge density to the external field as in [5]. Such an identification of the magnetic field shows how the coefficients depend on the filling fraction. In addition, we provide an example dual to a special solution of the bulk equation in which the current is proportional to the Poincare dual of the external field.

4.1 Magnetohydrodynamics corresponding to dyonic black brane

Now we are ready to get the magnetohydrodynamics from (31), (32) and (36). The conventional form of the current and the energy-momentum tensor doesn’t contain \( u^\sigma \partial_\sigma u^\nu \) and so we have to substitute for it. Using conservation equations (36), one can replace it with derivatives of the macroscopic thermodynamics functions. Taking the Landau frame with \( C_3 = 0 \) and \( D_\mu = 0 \), the boundary current and the energy momentum tensor are as follows.

\[
T_{\mu \nu} = M (\eta_{\mu \nu} + 3u_\mu u_\nu) - 2r_+^2 \sigma_{\mu \nu},
\]

\[
J^\mu = Q u^\mu + \left( \Sigma_1 P^{\mu \nu} - \Sigma_2 u_\lambda \epsilon^{\lambda \mu \nu} \right) \partial_\nu \left( \frac{\mu}{T} \right) + \left( \Sigma_3 P^{\mu \nu} - \Sigma_4 u_\lambda \epsilon^{\lambda \mu \nu} \right) \partial_\nu \left( \frac{m}{T} \right),
\]

where the energy density \( M \) is given by \( r_+^4 + \frac{Q^2 + H^2}{4r_+} \), and \( \mu = \frac{Q}{r_+} \) and \( m = \frac{H}{r_+} \) are the chemical potential and the magnetization. The temperature \( T \) and the outer horizon \( r_+ \) are given by

\[
T = \frac{12r_+^2 - \mu^2 - m^2}{16\pi r_+^3}, \quad r_+ = 4\pi T + \sqrt{(4\pi T)^2 + 3(\mu^2 + m^2)}.
\] (47)
The transport coefficients are presented in the Appendix A.3. The magnetohydrodynamics equations are as follows.

\[ \partial_\mu T^{\mu\nu} = J_\mu (F_{\text{ext}})^{\mu\nu}, \quad \partial_\mu J^\mu = 0 , \quad (48) \]

where \( (F_{\text{ext}})^{\mu\nu} = -H \epsilon_{\mu\nu\lambda} u^\lambda + \delta F^{\mu\nu} = -\bar{H} \epsilon_{\mu\nu\lambda} \bar{u}^\lambda \), the constant external field.

### 4.2 Composite particle system

Since we are interested in the fluid which consists of the composite particles, let us employ the identification of magnetic field, \( \bar{H} = \bar{Q} / \tilde{\nu} \), in [5] and look at what the effect is in the formulation. The \( \tilde{\nu} \) is the filling fraction related to a Chern-Simon level in the dual field theory. Keeping \( \tilde{\nu} \) constant, one can follow the fluid/gravity approach. The result is easily obtained by substituting \( Q(x)/\bar{\nu} \) for \( H(x) \) in the previous subsection. Under this substitution, the right hand side of (48) shows interesting procedure as follows.

\[ J_\mu (F_{\text{ext}})^{\mu\nu} = (J^{(0)})_\mu (F_{\text{ext}})^{(1)}_{\mu\nu} + (J^{(1)})_\mu (F_{\text{ext}})^{(0)}_{\mu\nu} \]

\[ = Q u_\mu \delta F^{\mu\nu} - (J^{(1)})_\mu \frac{1}{\bar{\nu}} Q u_\lambda \epsilon^{\lambda\mu\nu} \]

\[ = Q u_\mu \left( \delta F^{\mu\nu} + \frac{1}{\bar{\nu}} \epsilon^{\lambda\mu\nu} (J^{(1)})_\lambda \right) . \quad (49) \]

Thus the hydrodynamics equation becomes effectively following form,

\[ \partial_\mu T^{\mu\nu} = J_\mu (F_{\text{eff}})^{\mu\nu}, \quad \partial_\mu J^\mu = 0 , \quad (50) \]

where \( (F_{\text{eff}})^{\mu\nu} \) is given by \( \delta F^{\mu\nu} + \frac{1}{\bar{\nu}} \epsilon^{\mu\nu\lambda} (J^{(1)})^\lambda \) which plays a role of the first order external field.

The current and the energy-momentum tensor are rearrange as follows.

\[ J^\mu = (Q + C_3) u^\mu - \sigma_2 T^{\mu\nu} \partial_\nu \frac{\mu}{T} + \frac{1}{1 + 1/\bar{\nu}^2} \sigma_2 u_\nu (F_{\text{eff}})^{\nu\mu} - \frac{1}{\bar{\nu}} \frac{1}{1 + 1/\bar{\nu}^2} u_\lambda \epsilon^{\lambda\mu\nu} u^\sigma (F_{\text{eff}})^{\sigma\nu} + \frac{Q}{3M} \partial_\mu , \]

\[ T_{\mu\nu} = M (\eta_{\mu\nu} + 3 u_\mu u_\nu) - 2 r_+^2 \sigma_{\mu\nu} + \partial_\mu u_\nu + \partial_\nu u_\mu , \quad (51) \]

where the conductivity \( \sigma_2 \) is given by

\[ \sigma_2 = \left( \frac{12 r_+^4 - (1 + 1/\bar{\nu}^2) Q^2}{3(4 r_+^4 + (1 + 1/\bar{\nu}^2) Q^2)} \right)^2 . \quad (52) \]

Replacing \( Q \) with \( Q/\sqrt{1 + 1/\bar{\nu}^2} \), this quantity becomes the conductivity for the pure electric case in [40]. One can see that, in this composite particle picture, the magnetohydrodynamics
with the strong (zeroth order) magnetic field can be changed into the fluid dynamics with weak (first order) magnetic field in \[3.1\], except for the Hall current part. In addition, one can write down the current with \( \delta F_{\mu\nu} \) instead of \( (F_{\text{eff}})^{\mu\nu} \) in the Landau frame as follows.

\[
T_{\mu\nu} = M(\eta_{\mu\nu} + 3u_{\mu}u_{\nu}) - 2r_{+}^{2}\sigma_{\mu\nu},
\]

\[
J^{\mu} = Qu^{\mu} + (\Sigma_{a}P^{\mu\nu} - \Sigma_{b}u_{\lambda}e^{\lambda\mu\nu})\partial_{\nu}\frac{\mu}{T} + (\Sigma_{c}P^{\mu\nu} - \Sigma_{d}u_{\lambda}e^{\lambda\mu\nu})u_{\nu}\delta F_{\sigma\nu},
\]

The transport coefficients are given in the Appendix A.3. The above current form is a special case of more general consideration in [25]. One can compare it to this expression.

### 4.3 Example with vanishing \((F_{\text{eff}})^{\mu\nu}\)

Now we consider a special case where the effective external field vanishes. This situation is simply realized when \( \delta F_{\mu\nu} = -\frac{1}{\tilde{\nu}} e^{\nu\lambda}(J^{(1)}{,})_{\lambda} \). Incorporating with \( H = Q/\tilde{\nu} \), one can write down a relation between the external field and the current as follows.

\[
J^{\mu} = \frac{\tilde{\nu}}{2}e^{\nu\lambda}(F_{\text{ext}})^{\mu\lambda},
\]

It is well known that this condition is the equation of motion of a Chern-Simon theory whose action is like \( S_{\text{eff}} \sim \int d^{3}x (\tilde{\nu}A_{\text{ext}} \wedge dA_{\text{ext}} + J^{\mu}(A_{\text{ext}})^{\mu} + \cdots) \). Plugging \[55\] into \[51\], then the current and the momentum become

\[
J^{\mu} = Qu^{\mu} + \frac{\tilde{\nu}}{2}e^{\nu\lambda}(F_{\text{ext}})^{\mu\lambda},
\]

\[
T_{\mu\nu} = M(\eta_{\mu\nu} + 3u_{\mu}u_{\nu}) - 2r_{+}^{2}\sigma_{\mu\nu} + D_{\mu}u_{\nu} + D_{\nu}u_{\mu},
\]

where \( M \) and \( D_{\mu} \) are given by

\[
M = r_{+}^{3} + \left( 1 + \frac{1}{r_{+}^{2}} \right) \frac{Q^{2}}{4r_{+}}
\]

\[
D^{\mu} = \frac{(12r_{+}^{4} - (1 + 1/\tilde{\nu}^{2})Q^{2})(4r_{+}^{3} + (1 + 1/\tilde{\nu}^{2})Q^{2})}{192Q\pi r_{+}^{3}(4r_{+}^{4} + (1 + 1/\tilde{\nu}^{2})Q^{2})}P^{\mu\nu}\partial_{\nu}\mu - \frac{3\tilde{\nu}(4r_{+}^{4} + (1 + 1/\tilde{\nu}^{2})Q^{2})}{4r_{+}Q}u_{\nu}e^{\mu\nu\lambda}u_{\sigma}\delta F_{\sigma\lambda}.
\]

The spatial part of current form \[56\] gives exact the Hall conductivity \( \sigma_{xy} = \tilde{\nu} \) and the temporal component gives us the charge density corrected by the Chern-Simon constraint as follows.

\[
J^{0} = Qu^{0} + \tilde{\nu}\delta F_{i2} = \tilde{\nu}\tilde{H}u^{0}.
\]

The energy momentum tensor provides non-vanishing \( T_{0i} \) component. One can see that the second term of \( D_{\mu} \) plays a role of the Hall momentum. And the magnetohydrodynamics equation
\( \partial_\mu J^\mu = 0 \), \( \partial_\mu T^{\mu\nu} = 0 \). 

As we intended, the energy momentum tensor is divergence free. This reminds us of the well known fact discussed in the introduction. One advantage of the composite fermion paradigm is that one can transform the electron system with external magnetic field into the composite particle system without external magnetic field or small magnetic field. Such a phenomenon can be described by a Chern-Simon theory with an external field. Since we are dealing with same situation by (55), the change in our case is a holographic version for the transformation [4].

5 Discussion

In this paper, we have investigated a strongly coupled field theory system in an external magnetic field. Using the holographic approach, this system is realized by the dyonic black branes. The electric and magnetic charge of the dyonic branes correspond to the charge density and the external field in the dual field theory, respectively. Since we consider 3+1 dimensional AdS spacetime, the dual system is the 2+1 dimensional and this bulk solution could be dual to the quantum Hall systems.

The important property and the interesting phenomena of the quantum Hall system are captured by transport coefficients which can be measured in laboratories. This measurement is nicely encoded in a hydrodynamics, a purely phenomenological effective model. In addition, it is well known that the hydrodynamics for Super Yang-Mill fluid can be derived from the long wave length limit of fluctuating black holes [6]. This is one of successful result in AdS/CFT. We have followed the method and derived the hydrodynamics for the dyonic black brane.

In the fluid/gravity correspondence approach, the magnetohydrodynamics type equation was studied for the first time in [11]. The authors considered Reissner-Nordstrom black brane in 4+1 dimensional AdS spacetime. They started with boundary condition on the bulk gauge field, \((A_{ext})_\mu = \lim_{r \to \infty} A_\mu (r)\), which has no contribution to the zeroth order solution.

Giving boundary coordinates dependence like \((A_{ext})_\mu (x)\), this makes a correction to the hydrodynamics equation and finally one can obtain the magnetohydrodynamics. The external field, however, is give by \((F_{ext})_{\mu\nu} = \partial_\mu (A_{ext})_\nu - \partial_\nu (A_{ext})_\mu\), and this field strength is first order in derivative, i.e, weak external field limit. Thus our work is devoted to introducing the zeroth order or strong external field. In order to realize such a configuration, we chose the field strength
in (11). For the constant $\bar{H}$ and $\bar{u}_\mu$, it is easy to find corresponding gauge potential. While one takes them as functions of the boundary coordinates $x^\mu$, the existence of gauge potential requires $\partial_\mu (\bar{H} u^\mu) = 0$ in the first order. As we pointed out in section 3, this has nothing to do with a dynamical current. It is just a part of the trivial equation, $dF_{ext} = 0$, and consistent with one of the equations of motion (23) and so we don’t need further constraint.

In order to check our result with other works, we discussed some examples. Our first example is for a pure electric black brane case without the magnetic field. It is presented in the subsection 3.1. This first order RN black brane solution is dual to a fluid with a conserved current. We obtained the electric conductivity in the 2+1 dimensional system. This is a four-dimensional version of [11]. And second example is the energy momentum tensor in [20]. In their paper, they considered a configuration where fluid velocity vanishes and the zeroth order external field $H$ is constant. Our result reproduces their first order result.

Our goal is to investigate the relation between the composite fermion or particle system and the dyonic black brane through the fluid/gravity approach. This duality was conjectured in [5], where the authors considered magnetic field, $Q/\tilde{\nu}$. In our consideration, we obtained magnetohydrodynamics equation (50) with the effective external field $(F_{eff})^{\mu\nu} = \delta F_{ext}^{\mu\nu} + \frac{1}{\tilde{\nu}} \epsilon^{\mu\nu\lambda} (J^{(1)})_{\lambda}$, where $(J^{(1)})_{\lambda}$ is the first order part of the current. The current and the energy momentum tensor is given in (51). In the expression, we show that the current is very similar to the pure electric case except for the Hall current in the Landau frame. And the electric conductivity has same structure but, with different electric charge $Q_{eff} = \sqrt{1 + 1/\tilde{\nu}^2} Q$.

As a next consideration, we investigate a very special situation, where $F_{eff}$ vanishes. Then such a condition is equivalent to (55), which could be regarded as an equation of motion for a Chern-Simon theory with the filling fraction, $\tilde{\nu}$. By construction, this describes the Quantum Hall fluid exactly. The hall conductivity is given by the fraction as we see in (56). In addition we observed that there exists the Hall current type momentum flow in the energy momentum tensor (57). In this example, we would like to point out that the hydrodynamics equation is the magnetohydrodynamics type like $\partial_\mu T^{\mu\nu} = J_\mu (F_{ext})^{\mu\nu}$ in general, however the RHS vanishes when the fluid satisfies the equation of motion (55) for the Chern-Simon theory. This is very familiar effect which appears in the quantum Hall system. To understand such a system, one need to introduce the composite fermions which are bound states of a charged fermion and magnetic fluxes. Using this quasi particle as a basic degrees of freedom, the external magnetic field can be canceled by the fluxes effectively at some fixed filling fraction. Thus electron system in strong magnetic field can be described by composite fermion system without magnetic field.
We realized this cancelation holographically by showing that final hydrodynamics equations are conservation equations without the external field contribution, \( \partial_\mu T^{\mu\nu} = 0 \) and \( \partial_\mu J^\mu = 0 \). Thus we may interpret this system as composite particle fluid without external field.

As future directions, it is valuable to study the interesting duality transformation among transport coefficients in [5], but we didn’t include it at the moment to consider minimal setup. In order to consider the duality, one has to include an axion term, \( \int \theta F \wedge F \), in our action [5] and it produces an important change in the fluid/gravity correspondence approach. In addition, another interesting problem is about role of \( \int R \wedge R \) in the magnetohydrodynamics. In the boundary field theory, this is related to the Hall viscosity and the angular momentum density which are another important topics in the hydrodynamics [31] [32].

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A Appendix

A.1 Solving equations of motion

A.1.1 First order source terms

Let us compute first order solution explicitly. We follow a clever way suggested in [6]. Using the field strength and the metric in (5), we can find first order source terms. One efficient way for this calculation is to take one special position in \( x^1-x^2 \) plane. Most convenient choice is the origin and such a choice doesn’t lose generality. As second step, we may take an appropriate frame, where \( \beta^i(0) = 0 \), then the calculation in this frame becomes much easier. Expanding

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and putting them into $W_{IJ}$ and $W_I$, one can calculate the source terms from scalar and tensor parts of Einstein’s equations as follows.

\[ S^{(1)}_{rr} = 0, \]  
\[ S^{(1)}_{rv} = \frac{He^{ij}Q\partial_i\beta_j}{2r^3} - \frac{He^{ij}\delta F_{ij}}{4r^4} + \frac{\partial_i\beta_i}{r}, \]  
\[ S^{(1)}_{vo} = r\partial_v f(r) + \frac{He^{ij}\delta F_{ij}f(r)}{4r^3} - \frac{(r^2 f(r))'}{2} \partial_i\beta_i - \frac{He^{ij}Q\partial_i\beta_j f(r)}{2r^3}, \]  
\[ S^{(1)}_{ij} = \left\{ \frac{He^{kl}\delta F_{kl}}{4r^2} - \frac{QHe^{kl}\partial_k\beta_l}{2r^3} + r\partial_i\beta_i \right\} \delta_{ij} + r(\partial_i\beta_j + \partial_j\beta_i), \]

And the source terms for the vector parts of the Einstein’s equations and the Maxwell’s equations are given by

\[ S^{(1)}_{vi} = \frac{4r^4 + 2rM + Q^2 - H^2}{4r^3}\partial_v\beta_i - \frac{He^{ij}\partial_jQ + He^{ij}Q\partial_v\beta_j - r\partial_iM + H\partial_iH}{2r^3} \]  
\[ S^{(1)}_{ri} = -\frac{\partial_v\beta_i}{r}, \]  
\[ S^{(1)}_{i} = -\frac{\partial_iQ - He^{ij}\partial_jH + Q\partial_v\beta_i - He^{ij}\partial_v\beta_j}{r^2}. \]

In addition, the other scalar part of the Maxwell equations are

\[ S^{(1)}_{r} = -\frac{He^{ij}\partial_i\beta_j}{r^3}, \]
\[ S^{(1)}_{\theta} = \frac{\partial_\theta Q + Q\partial_v\beta_i + He^{ij}\partial_jf(r)}{r^2}. \]
A.1.2 General correction terms

Putting (12) into $W_{IJ}$ and $W_I$, we can evaluate $C_{IJJ}^{(n)}$ and $C_{I}^{(n)}$ given by

$$C_{vv}^{(n)} = f(r) \left\{ -\left(6r^2 + \frac{H^2}{r^2}\right)h^{(n)}(r) - r^2 (r^2 f(r))' h^{(n)}(r) - \frac{rk^{(n)(n)}(r)}{2} + \frac{Q}{2} a_v^{(n)}(r) \right\},$$

(71)

$$C_{vr}^{(n)} = \left(6 + \frac{H^2}{r^2}\right)h^{(n)}(r) + \frac{(r^2 f(r))' h^{(n)}(r)}{r} + \frac{k^{(n)(n)}(r)}{2r} - \frac{Q}{2r^2} a_v^{(n)}(r),$$

(72)

$$C_{rr}^{(n)} = \frac{1}{r^4} \left\{ r^4 h^{(n)}(r) \right\}',$$

(73)

$$C_{vi}^{(n)} = \frac{H}{2r^2} (H \delta_{ij} + Q \epsilon_{ij}) j^{(n)}(r) - f(r) \left\{ \frac{1}{2} \left( r^4 j^{(n)}(r) \right)' - \frac{Q}{2} a_{ij}^{(n)}(r) \right\},$$

(74)

$$C_{ri}^{(n)} = \frac{1}{2r^2} \left( r^4 j^{(n)}(r) \right)' - \frac{Q \delta_{ij} + H \epsilon_{ij}}{2r^2} a_j^{(n)}(r),$$

(75)

$$C_{ij}^{(n)} = \frac{-\left\{ r^4 f(r) a_{ij}^{(n)}(r) \right\}'}{2} + \delta_{ij} \left\{ \left(6r^2 - \frac{H^2}{r^2}\right)h^{(n)}(r) + \frac{r^8 f(r) h^{(n)}(r)'}{2r^4} + k^{(n)}(r) + \frac{Q}{2} a_v^{(n)}(r) \right\},$$

(76)

and

$$C_v^{(n)} = f(r) \left\{ r^2 a_v^{(n)}(r) + 2Q h^{(n)}(r) \right\}',$$

(77)

$$C_r^{(n)} = -\frac{1}{r^2} \left\{ r^2 a_v^{(n)}(r) + 2Q h^{(n)}(r) \right\}',$$

(78)

$$C_i^{(n)} = \left\{ r^2 f(r) a_i^{(n)}(r) - (Q \delta_{ij} - H \epsilon_{ij}) j^{(n)}(r) \right\}',$$

(79)

where we have taken $l = 1$.

A.1.3 First order solutions

Given the source terms in Appendix A.1.1 one can find the correction terms (12) in the metric and the gauge field by simple integrations. The scalar and the tensor parts are related to the
source terms $S_{rr}^{(1)}$, $S_r^{(1)}$, $\delta^{ij}S_{ij}^{(1)}$ and $S_{ij}^{(1)} - \frac{1}{2}\delta_{ij}S_{kk}^{(1)}$. For these combinations, we obtained

\begin{align}
  h^{(1)}(r) &= C_1 + \frac{C_2}{r^3}, \\
  a_{ij}^{(1)}(r) &= \frac{H}{2r^2}\delta^{ij}\beta_j + C_2\frac{Q}{2r^4} + \frac{C_3}{r} + C_4, \\
  k^{(1)}(r) &= -\frac{1}{4r}He^{ij}\delta F_{ij} + r^2\partial_r\beta_i, \\
  \alpha_{ij}^{(1)}(r) &= \alpha(r) \left( \partial_r\beta_j + \partial_j\beta_i - \delta_{ij}(\partial_k\beta^k) \right),
\end{align}

where $\alpha(r)$ and its asymptotic behavior are given by

$$
\alpha(r) = -\int_{\infty}^{r} \left( \frac{y^2 - r^2}{y^2f(y)} \right) dy = \frac{1}{r} - \frac{r^2}{3y^2} + \frac{M}{4r^4} - \frac{Q^2 + H^2}{20r^5} + O \left( \frac{1}{r^6} \right).
$$

In the above result, we have many integration constants. Since first integration constant $C_1$ will give non-normalizable mode, this should vanish. For $C_2$, one can always set this zero by coordinate transformation of $r$. $C_4$ can be taken as zero, because this is pure gauge. Non-vanishing $C_5$ means non-vanishing trace of energy-momentum tensor, this must vanish because of the conformal symmetry. Thus the only remaining constant is $C_3$.

Considering vector part equations $W_{r1}$ and $W_i$, we got a second order differential equation as follows.

$$
\frac{r^2f(r)\left( r^2j_i^{(1)}(r) \right)'' - r^2j_i^{(1)}(r) \left( r^2f(r) \right)''}{r^2} = \frac{1}{r^2}\zeta_i(r),
$$

where $\zeta_i(r)$ is given by

$$
\zeta_i(r) = 2r^4f(r)S_{r1}^{(1)}(r) - (H^2 + Q^2)j_i^{(1)}(r_+) + (Q\delta_{ij} + H\epsilon_{ij}) \int_{r_+}^{r} dxS_{ij}^{(1)}(r)
$$

$$
= -2r^3f(r)\partial_r\beta_i - (H^2 + Q^2)j_i^{(1)}(r_+)
$$

$$
+ \left( \frac{1}{r} - \frac{1}{r_+} \right) (Q\delta_{ij} + H\epsilon_{ij}) \{ \partial_j Q - \epsilon_{jk}\partial_k H + Q\partial_i\beta_j - H\epsilon_{jk}\partial_k\beta_i \}.
$$

The general solution is

$$
j_i^{(1)}(r) = D_{1i}f(r) + f(r)\int_{r_1}^{r} dx \frac{1}{x^4f(x)^2} \left\{ \int_{r_0}^{x} dy \frac{\zeta_i(y)}{y^2} + D_{2i} \right\}.
$$

$r = \infty$ boundary condition gives $D_{1i} = 0$ and $r_1 = \infty$, then the solution becomes

$$
j_i^{(1)}(r) = f(r)\int_{\infty}^{r} dx \frac{1}{x^4f(x)^2} \left\{ \Delta_i(x) + D_i \right\},
$$

where $\Delta_i(x)$ is$^{(1)}$. 
where $\Delta_i(r)$ is a little bit complicate form as

$$
\Delta_i(r) = -r^2 \partial_v \beta_i + \frac{1}{r}(H^2 + Q^2)j_i^{(1)}(r_+) \\
+ \frac{1}{r} \left[ \frac{1}{r_+} (Q\delta_{ij} + H\epsilon_{ij}) (\partial_j Q - \epsilon_{jk}\partial_k H) + 2(M - 2r^3)\partial_v \beta_i \right] \\
- \frac{1}{2r^2} \left[ (Q\delta_{ij} + H\epsilon_{ij}) (\partial_j Q - \epsilon_{jk}\partial_k H) + (H^2 + Q^2)\partial_v \beta_i \right].
$$

(89)

When we compute boundary tensors, i.e., the energy momentum tensor and the current, the tensors are given by asymptotic form of metric and field. So one has to look at the asymptotic form of $j_i^{(1)}$, which is as follows

$$
\frac{1}{r} \int dx \frac{j_i^{(1)}(x) - j_i^{(1)}(r_+)}{x^2 f(x)} + \int dy \frac{dy}{y^2 f(y)} \int dx S_i^{(1)}(x)
$$

(91)

where $S_i^{(1)} = -2rQ - \epsilon_{ij}\partial_j H + Q\partial_v \beta_i - H\epsilon_{ij}\partial_v \beta_j$, and have taken the boundary condition $a_i^{(1)}(r = \infty) = 0$. The asymptotic form of gauge field near boundary is

$$
a_i^{(1)}(r) = \frac{1}{r} \left[ Q\delta_{ij} - H\epsilon_{ij} \right] j_j^{(1)}(r_+) + \frac{1}{r} \left( \partial_j Q - \epsilon_{ij}\partial_j H + Q\partial_v \beta_i - H\epsilon_{ij}\partial_v \beta_j \right) + O \left( \frac{1}{r^2} \right).
$$

(92)

Using the asymptotic expressions, we can read the boundary tensors $J^\mu$ and $T^\mu\nu$ as defined in section A.2.

In order to find $j_i^{(1)}(r_+)$, we should consider regularity conditions for the metric. The metric should be smooth at horizon, so $D_i$ can be determined by

$$
D_i = -3M j_i^{(1)}(r_+) + \frac{n_i}{r_+^2} + 3\partial_v \beta_i r_+^2,
$$

(93)

where $n_i$ is

$$
n_i = -\frac{1}{2} (Q\delta_{ij} + H\epsilon_{ij}) \left\{ \partial_j Q - \epsilon_{jk}\partial_k H \right\} - \frac{H^2 + Q^2}{4}\partial_v \beta_i.
$$

(94)

One can check that the metric smooth at the horizon by this $D_i$. Thus $j_i^{(1)}(r_+)$ can be expressed in terms of $D_i$,

$$
j_i^{(1)}(r_+) = \frac{n_i}{3Mr_+^2} + \frac{r_+^2}{M}\partial_v \beta_i - \frac{D_i}{3M}.
$$

(95)
A.2 Boundary stress energy tensor and current

In this section, we briefly describe the prescription of boundary stress-energy tensor and current. We will follow prescription in [26, 27, 28, 29] to get the boundary stress energy tensor. Our metric can be decomposed under the ADM decomposition

$$ ds^2 = \gamma_{\mu\nu} (dx^\mu + V^\mu dr)(dx^\nu + V^\nu dr) + N^2 dr^2, $$

then one can obtain the stress energy tensor from the action (1) through variation of the boundary metric $\gamma_{\mu\nu}$. The boundary stress energy tensor is given by

$$ T_{\mu\nu} \equiv \lim_{r \to \infty} r^2 \frac{-2}{\sqrt{-\gamma}} \frac{\delta S_{cl}}{\delta \gamma_{\mu\nu}} = \lim_{r \to \infty} r [-2(\Theta_{\mu\nu} - \Theta \gamma_{\mu\nu} + \frac{2}{l} \gamma_{\mu\nu} \pm lG_{\mu\nu})], $$

where the last two terms came from the counter term in (1) and we have considered the on-shell action $S_{cl}$. The extrinsic curvature $\Theta_{\mu\nu}$ is

$$ \Theta_{\mu\nu} = \frac{1}{2N} [\partial_\nu \gamma_{\mu\nu} - D_\mu V_\nu - D_\nu V_\mu]. $$

The boundary current can be obtained through variation of the boundary gauge field [30].

$$ J^\mu = \lim_{r \to \infty} r^3 \frac{-1}{\sqrt{-\gamma}} \frac{\delta S_{cl}}{\delta \tilde{A}_\mu} = \lim_{r \to \infty} r^3 NF^{r\mu} $$

$$ = \lim_{r \to \infty} r^3 \frac{1}{N} [\gamma^{\mu\lambda} (A'_\lambda - \partial_\lambda A_\nu) - V^\lambda \gamma^{\mu\sigma} (\partial_\lambda A_\sigma - \partial_\sigma A_\lambda)], $$

where $\tilde{A}_\mu$ is the gauge field projected on the boundary.

A.3 General transports coefficients

The coefficients in (46) is given by

$$ \Sigma_1 = \Sigma_0^{-1} [(1 + \sigma_G)\sigma_A - \sigma_H\sigma_B], $$

$$ \Sigma_2 = \Sigma_0^{-1} [(1 + \sigma_G)\sigma_B + \sigma_H\sigma_A], $$

$$ \Sigma_3 = \Sigma_0^{-1} [(1 + \sigma_G)\sigma_C - \sigma_H\sigma_D], $$

$$ \Sigma_4 = \Sigma_0^{-1} [(1 + \sigma_G)\sigma_D + \sigma_H\sigma_C], $$

$$ \Sigma_5 = \Sigma_0^{-1} [(1 + \sigma_G)(1 + \sigma_E) - \sigma_H\sigma_F], $$

$$ \Sigma_6 = \Sigma_0^{-1} [(1 + \sigma_G)\sigma_F + \sigma_H(1 + \sigma_E)], $$

(100)
where $\Sigma_0 = (1 + \sigma_G)^2 + \sigma_H^2$, and

\[
\begin{align*}
\sigma_A &= \sigma_0 \left( 1 + \frac{2r_+^3}{M} \right) \left( \frac{\mu}{T} \sigma_0 - \frac{T}{\mu} \right), \\
\sigma_B &= \sigma_0 \left( 1 + \frac{2r_+^3}{M} \right) \frac{\mu}{T} \sigma_0, \\
\sigma_C &= \sigma_0 \left( 1 + \frac{2r_+^3}{M} \right) m \sigma_0, \\
\sigma_D &= \sigma_0 \left( 1 + \frac{2r_+^3}{M} \right) \left( \frac{m}{T} \sigma_0 - \frac{T}{m} \right), \\
\sigma_E &= -\frac{2Q^2}{9Mr_+} \left( 1 + \frac{2r_+^3}{M} \right), \\
\sigma_F &= +\frac{2HQ}{9Mr_+} \left( 1 + \frac{2r_+^3}{M} \right), \\
\sigma_G &= \frac{2H^2}{9Mr_+} \left( 1 + \frac{2r_+^3}{M} \right), \\
\sigma_H &= +\frac{2HQ}{9Mr_+} \left( 1 + \frac{2r_+^3}{M} \right),
\end{align*}
\]

(101)

as well as

\[
\sigma_0 = \frac{2T^2(36r_+^4 + H^2 + Q^2)r_+^2}{2(4r_+^4 + H^2 + Q^2)(12r_+^4 + H^2 + Q^2)}
\]

(102)

And the coefficients in (54) is obtained through

\[
\begin{align*}
\Sigma_a &= \Sigma_1 + \Sigma_3 / \nu, \\
\Sigma_b &= \Sigma_2 + \Sigma_4 / \nu,
\end{align*}
\]

(103)

where we also identify $H \equiv Q / \nu$.

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