A CHERN–SIMONS THEORY VIEW OF NONCOMMUTATIVE SCALAR FIELD THEORY

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Abstract. We show that a version of Abelian gauge theory on $\mathbb{R}^3_\lambda$, when restricted to a single fuzzy sphere, reduces in the large $N$ limit to the Langmann–Szabo–Zarembo (LSZ) matrix model, which originally emerges in the study of scalar field theory on the Moyal plane. We then prove that the LSZ matrix model is actually equivalent to the matrix model of $U(N)$ Chern–Simons theory on $S^3$. The correspondence holds in a generalized sense: depending on the spectra of the two external matrices of the LSZ model, the Chern–Simons matrix model either describes the Chern–Simons partition function, the unknot invariant, given by quantum dimensions, or the Hopf link invariant. Equivalently, the evaluation of the LSZ model can be written in terms of the $S$ and $T$ modular matrices of the WZW model.

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1. Introduction

The idea of introducing non-zero commutators among position or momentum coordinates in different directions goes back to the 1940s [1]. The consequences of applying ideas and results of noncommutative geometry in quantum field theory are far-reaching but also considerably involved with both several non-trivial results and difficulties as well.

In the late 1990s there was a boost of interest in noncommutative (NC) field theories in great part due to the fact that low energy string theory can be related to NC field theory [2, 3]. However, soon it was established that the expectation that washing out the space-time points could weaken UV divergences in quantum field theory, and consequently simplify renormalization, did not work as expected. Rather the opposite turned out to hold: renormalization gets
harder due to noncommutativity, because in planar diagrams of a perturbative expansion the UV divergences simply persist. Second, in the non-planar diagrams, they tend to “mix” with IR divergences [4]. We refer to [3, 6] for classical reviews of the topic.

The study of Abelian gauge theories in noncommutative $\mathbb{R}^3$ has been carried out in [7–12]. The deformation of three-dimensional euclidean space considered is the so-called $\mathbb{R}^3_\lambda$, which takes advantage of the commutation relations of generators of the $\mathfrak{su}(2)$ algebra. The discrete eigenvalues of the quadratic Casimir produce a “foliation” in terms of fuzzy spheres of different radii.

In [10, 11] it has been found the most general action satisfying three (weak enough) conditions: gauge invariance, stability of the vacuum, positivity. The second condition is tightly related to the choice of fundamental field for the theory; in particular, two different approaches, namely expanding a covariant gauge field around a gauge invariant flat connection [10] or around the null configuration [11] lead to dissimilar pictures. In both cases, after convenient gauge fixing, the gauge theories reduce to a sheaf of scalar theories with quartic interaction, one on each fuzzy sphere foliating $\mathbb{R}^3_\lambda$. On one hand, the first model yields a partition function which appears to be related to a string model with a background $B$-field enforcing string noncommutativity [13]. On the other hand, the formal equivalence of the action in the latter case with the Langmann–Szabo–Zarembo (LSZ) model [14, 15] was highlighted [11].

The LSZ model [15] is a matrix model representation of certain scalar field theories on the Moyal plane. It has the explicit form

$$Z_N(E, \bar{E}) = \int \mathcal{D} M \mathcal{D} M^\dagger \exp \left( -N \text{Tr} \left\{ M E M^\dagger + M^\dagger \bar{E} M + \hat{V} \left( M^\dagger M \right) \right\} \right),$$

where $M$ is a $N \times N$ complex matrix and $E, \bar{E}$ are external matrices whose eigenvalues are determined by the model, and $\hat{V} \left( M^\dagger M \right)$ will be a polynomial potential with linear and quadratic terms in $M^\dagger M$. We will first argue here that, following [10–12] but with slightly modified derivations of $\mathbb{R}^3_\lambda$, the reduced model on the fuzzy sphere coincides with the LSZ model exactly, in the large radius limit. This will lead us to a discussion on how the matrix model describes noncommutative scalar field theories on the Moyal plane and on the fuzzy $S^2$.

We stress that the scalar theory on the fuzzy sphere that we consider here is different in its construction from other models previously studied in the literature, as we will explain in detail in Section 3.

On the other hand, a matrix model description of several gauge theories in three dimensions has been developed since early 2000s and especially in the last decade. This is the case for topological and supersymmetric gauge theories and the first results for the case of Chern–Simons theory on Seifert manifolds were given in [16], generalizing previous work [17]. The simplest case is that of $S^3$, where the partition function admits the expression [16]

$$Z_{CS} = \int_{[\infty, \infty]^N} \prod_{j=1}^N dx_j \ e^{-\frac{1}{2\pi g_s} \sum_{j=1}^N x_j^2} \prod_{1 \leq j < k \leq N} \left( 2 \sinh \left( \frac{x_j - x_k}{2} \right) \right)^2,$$

with $g_s$ a coupling constant, which can be related to the level $k \in \mathbb{Z}$ of Chern–Simons theory by $g_s = \frac{2\pi i}{N+k}$. A review of early results is [18]. The matrix model description can also be obtained and further understood by using different types of localization of the path integral [19–21]. In the case of supersymmetric theories in three dimensions the localization results [22] have lead to a wealth of new developments.

Interestingly, the case of $S^3$, given by expressions such as (1.1), or expressions with Schur polynomial insertions, describing then Wilson loop observables, are completely solvable in terms
of standard random matrix theory methods \[23, 25\]. We will connect the exact solvability of the
matrix models for the observables of $U(N)$ Chern–Simons theory on $S^3$ directly to that of the
LSZ matrix model. In turn this gives analytical evaluations of the latter model.

In this way, different insertions of external matrices $E, \tilde{E}$ will correspond to the calculation
of different Chern–Simons observables. We will finally exactly solve the matrix model giving a
characterization in terms of elements of the modular matrices $S, T$ of the Wess–Zumino–Witten
(WZW) model \[26, 18\].

The article is organized as follows. In the next Section, after a review of the basics of the
physics of the noncommutative plane with a magnetic field, we discuss the LSZ model. In Section
3, we study a scalar field theory on the fuzzy sphere, using the construction of an Abelian gauge
theory on noncommutative $R^3$. We show then how the LSZ model emerges in the large radius
limit. In Section 4, we analyze the LSZ matrix model and show its equivalence with the Chern–
Simons matrix model. We emphasize that the equivalence holds in a generalized sense, as the
external matrices are not restricted to come from a kinetic operator, and different sets of spectra
of the external matrices eigenvalues correspond to different observables in Chern–Simons theory.

2. Noncommutative scalar theory with background field

In this section we briefly review the LSZ model: in the first subsection, the geometric con-
struction of the Moyal plane is sketched, in the presence of a background magnetic field, while
the second subsection is dedicated to the construction of a scalar field theory with quartic
interaction.

2.1. Moyal plane with magnetic field. The Moyal plane is defined through the commutation
relations

\[ [q^i, q^j] = i\theta\epsilon^{ij}, \]

where $\theta$ is the essential parameter of the theory, with dimension of length squared. A Moyal
plane can always be seen as an harmonic system, in the sense that passing to dimensionless
complex coordinates one has $[z, \bar{z}] = 1$. The noncommutative plane needs not to arise from
a modification of space-time. In fact, an example is given by a particle moving on a plane
with a magnetic field of intensity $B$ in the transverse direction; momentum space then becomes
noncommutative $R^2$, as momentum operators modify according to:

\[ p_i \mapsto P_i := p_i - \frac{1}{2} B\epsilon_{ij} q^j. \]

In this case the covariant momenta $P_i$ satisfy the commutation relation $[P_i, P_j] = -iB\epsilon_{ij}$.

If the two frameworks are put together, that is, a transverse magnetic field is plug in over a
noncommutative plane, three possible harmonic oscillator pictures arise:

(i) on the two-dimensional position space, with annihilation and creation operators given by
the complex coordinates as above;
(ii) on the two-dimensional momentum space, with annihilation and creation operator defined
analogously;
(iii) a pair of canonical harmonic oscillators, one on each phase space plane.

However, the most suitable choice is none of them, and we will take a mixture of all those
ingredients to form two commuting copies of annihilation and creation operators, in such a way
that the problem decouples into two one-dimensional harmonic systems. To do so, define:

\[ z := \frac{q^1 + i q^2}{\sqrt{2\theta}}, \quad \bar{z} := \frac{q^1 - i q^2}{\sqrt{2\theta}}, \]

\[ v := \frac{p_1 + i p_2}{\sqrt{2\theta^{-1}}}, \quad \bar{v} := \frac{p_1 - i p_2}{\sqrt{2\theta^{-1}}}, \]

and use them to introduce the operators:

\[ a_1 = \frac{z + iv}{\sqrt{2}}, \quad a_1^\dagger = \frac{\bar{z} - i\bar{v}}{\sqrt{2}}, \]

\[ a_2 = \frac{\bar{z} + i\bar{v}}{\sqrt{2}}, \quad a_2^\dagger = \frac{z - iv}{\sqrt{2}}. \]

Straightforward calculations provide:

\[ [a_\alpha, a^\dagger_\beta] = \delta_{\alpha\beta}, \]

\[ [a_\alpha, a_\beta] = 0 = [a_\alpha^\dagger, a_\beta^\dagger], \]

for \( \alpha, \beta = 1, 2 \), hence we got a pair of decoupled harmonic oscillators.

**Remark.** The lifting of the obstruction \( \theta \) shifts the canonical symplectic structure on the cotangent bundle. It turns out that such shifted 2-form is still symplectic. One can then rotate to Darboux coordinates so that the new symplectic structure on the phase space \( T^* \mathbb{R}^2 \) is block-diagonal. Calculations above are precisely the explicit change of coordinates.

Consider now the differential operator \( D_i \) associated to the covariant momenta \( P_i \), and \( \tilde{D}_i \) analogous but carrying a reflected magnetic field \( -B \). If we take the arbitrary combination \( -\sigma D^2 - \tilde{\sigma} \tilde{D}^2 \) and evaluate it at the symmetric point \( \sigma = \tilde{\sigma} = \frac{1}{2} \), we obtain:

\[ \left( -\sigma D^2 - \tilde{\sigma} \tilde{D}^2 \right)_{\sigma = \tilde{\sigma} = \frac{1}{2}} = \tilde{p}^2 + \frac{B^2}{4} q^2 = \theta^{-1} \left( \frac{B^2 \theta^2}{4} \{z, \bar{z}\} + \{v, \bar{v}\} \right), \]

where the bracket in the right-hand side stands for anticommutation. On the other hand, in terms of the harmonic oscillators description, we have:

\[ \sum_{\alpha=1}^{2} a_\alpha^\dagger a_\alpha = \frac{1}{2} \left( \{z, \bar{z}\} + \{v, \bar{v}\} - i[v, \bar{z}] + i[z, \bar{v}] \right), \]

which means

\[ \left( -\sigma D^2 - \tilde{\sigma} \tilde{D}^2 \right)_{\sigma = \tilde{\sigma} = \frac{1}{2}} = \frac{2}{\theta} \sum_{\alpha=1}^{2} \left( a_\alpha^\dagger a_\alpha + \frac{1}{2} \right) \]

at points \( B^2 \theta^2 = 4 \).

**Remark.** The preferred curves \( \frac{B^2 \theta^2}{4} = 1 \) correspond to the self-dual points of the Langmann–Szabo symmetry [27]. The theory is independent of the actual choice of curve in parameter space we restrict to, namely \( B = \pm 2\theta^{-1} \). In fact, the two theories we obtain are equivalent in the Seiberg–Witten sense, i.e., they transform into the same theory. The invariance reflects the fact that the operators \( D_i, \tilde{D}_i \) only differ by a reflection \( B \mapsto -B \), thus the symmetric choice \( \sigma = \tilde{\sigma} \) drops the dependence on the sign of the magnetic field.
2.2. **LSZ model.** Given a scalar field \( \Psi \) on the Moyal plane, we can expand it in terms of the Landau basis, consisting of eigenstates of both harmonic oscillators, as:

\[
(2.4) \quad \Psi = \sum_{\ell_1, \ell_2 = 1}^{\infty} M_{\ell_1 \ell_2} |\ell_1, \ell_2\rangle.
\]

This expression naturally defines an infinite matrix \( M \) associated to the field \( \Psi \). Now recall the kinetic operator in (2.3); using the property

\[
a_\alpha^\dagger a_\alpha |\ell_1, \ell_2\rangle = (\ell_\alpha - 1) |\ell_1, \ell_2\rangle, \quad \ell_\alpha = 1, 2, \ldots, \alpha = 1, 2,
\]

it is possible to write:

\[
\left( \sum_{\alpha=1}^{2} \left( a_\alpha^\dagger a_\alpha + \frac{1}{2} \right) \right) \Psi = \sum_{\ell_1, \ell_2} M_{\ell_1 \ell_2} \left\{ \left( \ell_1 - \frac{1}{2} \right) + \left( \ell_2 - \frac{1}{2} \right) \right\} |\ell_1, \ell_2\rangle
\]

\[
= \sum_{\ell_1, \ell_2} \left( \ell_1 - \frac{1}{2} \right) \{ M_{\ell_1 \ell_2} |\ell_1, \ell_2\rangle + M_{\ell_2 \ell_1} |\ell_2, \ell_1\rangle \}.
\]

Therefore one obtains:

\[
(2.5) \quad \langle \Psi^\dagger, \left( \sum_{\alpha=1}^{2} \left( a_\alpha^\dagger a_\alpha + \frac{1}{2} \right) \right) \Psi \rangle = \sum_{\ell_1, \ell_2} \left( \ell_1 - \frac{1}{2} \right) \{ M_{\ell_2 \ell_1}^\dagger M_{\ell_1 \ell_2} + M_{\ell_2 \ell_1} M_{\ell_1 \ell_2}^\dagger \}
\]

\[
= \text{Tr} \left\{ M^\dagger K M + M K M^\dagger \right\},
\]

where in the last line we introduced the diagonal matrix

\[
(2.6) \quad E_{\ell_1 \ell_2} = \left( \ell_1 - \frac{1}{2} \right) \frac{4\pi}{N} \delta_{\ell_1 \ell_2}.
\]

Consider the action \([14, 15]\)

\[
S_{LSZ} [\Psi, \Psi^\dagger] = \int_{\mathbb{R}^2} \left\{ \frac{1}{2} \Psi^\dagger \left( -\sigma D^2 - \tilde{\sigma} \tilde{D}^2 \right) \Psi + \frac{1}{2} \Psi \left( -\sigma D^2 - \tilde{\sigma} \tilde{D}^2 \right) \Psi^\dagger + m_0^2 \Psi^\dagger \Psi + \frac{\tilde{g}^2}{2} (\Psi^\dagger \Psi)^2 \right\},
\]

Evaluated at the symmetric point \( \sigma = \tilde{\sigma} = \frac{1}{2} \) it becomes:

\[
(2.8) \quad S_{LSZ} [\Psi, \Psi^\dagger] = N \text{Tr} \left\{ M^\dagger E M + M E M^\dagger + \hat{m}^2 M^\dagger M + \frac{\hat{g}^2}{2} \left( M^\dagger M \right)^2 \right\}
\]

where we introduced the dimensionless couplings

\[
\hat{m}^2 = \left( \frac{2\pi \theta}{N} \right) m_0^2, \quad \hat{g}^2 = \left( \frac{2\pi \theta}{N} \right) g_0^2.
\]

We also have used eqs. \([2.3]\) and \([2.5]\), and the matrix \( E \) defined in \([2.6]\).

Notice that, to regularize the integral, we truncate the matrix \( M \) to its top-left \( N \times N \) block, which introduces a finite cutoff at short distance \( \sqrt{\frac{2\pi \theta}{N}} \). The full theory is recovered in the large \( N \) limit. As expected from general features of noncommutative field theory, the original noncommutativity of the phase space is eventually encoded in the noncommutativity of matrix multiplication. Consistently, the space-time integral becomes a trace.
3. Scalar field on the fuzzy sphere

We now focus our attention on the construction of a scalar field theory with quartic interaction on a fuzzy sphere. The aim is to prove that such theory is equivalent to the LSZ model in the large radius and large \( N \) limit.

Scalar theories on noncommutative spaces and in particular on the fuzzy sphere had been extensively studied, mainly looking at them as regularized UV/IR mixing-free version of commutative theories \([28, 30]\). In the standard setting the kinetic sector prevents the angular degrees of freedom to be integrated out and to reduce the path integral to an integral over eigenvalues. A perturbative approach in the kinetic term was proposed in \([31, 32]\), equivalent to a high-temperature expansion, and the presence of phase transitions, together with a triple point, was suggested. This procedure was generalized to \(\mathbb{C}P^N\) \([33]\). See \([34]\) for a review, describing both theoretical predictions and numerical results, paying special attention to the phase transition. An extended scheme, allowing both a high-temperature (large-interaction) and small-interaction analysis was presented in \([35]\), providing further understanding of the phase structure of the model. Other nonperturbative aspects are still under investigation: in \([36, 37]\) the behaviour of correlation functions of the matrix model in the disordered phase is analyzed.

However, all these works lead to a matrix model different from the one studied here, the main distinction being that in present work we start with a gauge theory on three-dimensional noncommutative euclidean space, and project on the fuzzy sphere after gauge-fixing \([10–12]\), instead of straightforwardly use the adjoint action of \(\text{su}(2)\) generators on a fuzzy scalar to obtain the kinetic term. That is, we do not start with a scalar theory on the fuzzy sphere, and instead we construct a field theory in the ambient space \(\mathbb{R}^3_\lambda\), which eventually reduces to the desired model on each fuzzy sphere of the foliation. The difference in the approach also implies a different scaling limit at large \( N \).

In what follows, we first give a very brief review of properties of the deformation of \(\mathbb{R}^3\) known as \(\mathbb{R}^3_\lambda\) and of the fuzzy sphere. Then, the second subsection is dedicated to the development of gauge theory in \(\mathbb{R}^3_\lambda\), following \([11, 12]\). Eventually, we reduce it to a scalar theory on a fuzzy sphere and study the large radius limit \( r \to \infty \).

3.1. From \(\mathbb{R}^3_\lambda\) to fuzzy spheres. We start with a noncommutative version of the three-dimensional euclidean space, imposing the coordinates to satisfy commutation relations related to the ones of \(\text{su}(2)\). Such space is known in the literature as \(\mathbb{R}^3_\lambda\). Then, as we will see, irreducible representations of \(\text{su}(2)\) determine a foliation of \(\mathbb{R}^3_\lambda\) in terms of fuzzy spheres \([38]\). We refer to \([39, 3]\) for detailed insights in \(\mathbb{R}^3_\lambda\).

Consider the hermitian operators \(\{x^\mu\}_{\mu=1}^3\) satisfying:

\[
[x^\mu, x^\nu] = i\lambda \epsilon^{\mu\nu\rho} x^\rho,
\]

with \(\lambda\) a length parameter. Coordinates in \(\mathbb{R}^3_\lambda\) are then the generators of \(\text{su}(2)\) up to a length scale factor \(\lambda\). Irreducible representation of \(\text{su}(2)\) in terms of \((2n+1) \times (2n+1)\) matrices are labelled by half-integers \(n\); whenever \(n\) is fixed the Casimir relation implies:

\[
x^2 := \sum_{\mu=1}^3 (x^\mu)^2 = \lambda^2 n(n+1),
\]

which corresponds to pick a sphere of fixed radius \(r^2 = \lambda^2 n(n+1)\), denoted by \(S^2_n\). One could interpret this construction of the fuzzy sphere as analogous to the usual embedding of \(S^2\) into \(\mathbb{R}^3\), but replacing coordinates with noncommutative hermitian operators.
It is well-known that the large $N = 2n + 1$ limit of the fuzzy sphere at fixed radius gives the commutative sphere. Conversely, the scaling limit $r \to \infty$ keeping the parameter

$$\theta = \frac{r^2}{n}$$

fixed leads to the Moyal plane. We focus on this latter setting and write:

$$\lambda^2 = \frac{\theta}{n + 1}; \quad r^2 = \theta n.$$

As shown in [9–11], there exists a matrix bases on each fuzzy sphere, which allows to identify fields in $\mathbb{R}^3_\lambda$ with a stack of matrices of increasing size. Noncommutativity is then encoded in $N \times N$ matrix multiplication, with $N = 2n + 1$, at each level $n \in \frac{1}{2} N$.

Integration over $\mathbb{R}^3_\lambda$ is defined as

$$\int_{\mathbb{R}^3_\lambda} F = 2\pi \sum_{n \in \frac{1}{2} N} \lambda^3(n + 1) \text{Tr}_N \left\{ f^{(N)} \right\} = 2\pi \theta^{3/2} \sum_{n \in \frac{1}{2} N} \frac{1}{\sqrt{n + 1}} \text{Tr}_N \left\{ f^{(N)} \right\},$$

where $f^{(N)}$ is the matrix representation of the function $F$ at level $n$, and $\text{Tr}_N$ denotes the usual trace operation for size $N = 2n + 1$ square matrices. This integral has the expected behaviour at large $N$, giving the volume of a sphere of radius $\sim \sqrt{\theta n}$. Dropping the radial degree of freedom, the integral reduces to

$$\int_{S^2_n} F = 2\pi \theta \text{Tr}_N \left\{ f^{(N)} \right\}$$
on a specific fuzzy sphere. The index $N$ will be understood from now on.

3.2. Setup of the model. We consider the Hilbert space of states spanned by eigenstates of angular momentum operators. Elements of the canonical basis are labelled by $n, k$, where $n \in \frac{1}{2} N$ is the eigenvalue of the Casimir, parametrizing the radial degree of freedom, and $k = -n, \ldots, n$, as usual. Any choice of irreducible representation of $\mathfrak{su}(2)$ quenches the radial degree of freedom.

At this point, we reproduce the model in [11, 12] with only a slight modification: we change the derivations. We introduce the preferred forms

$$\tau_\mu := \frac{x^\nu}{\lambda} \delta_{\nu \mu},$$

which yield associated derivations defined as

$$D_\mu := \text{Ad}_{\tau_\mu} = i [\tau_\mu, \cdot],$$

satisfying the ring relation

$$[D_\mu, D_\nu] = -\frac{1}{r} \epsilon_{\mu \nu \rho} D_\rho.$$

It is also possible to introduce the preferred 1-form $\Theta$ given by $\Theta(D_\mu) = -i \tau_\mu$. It satisfies the property:

$$d\Theta(D_\mu, D_\nu) + [\Theta(D_\mu), \Theta(D_\mu)] = 0.$$ 

By virtue of this latter expression, $\Theta$ is a flat connection. The most general gauge-invariant action in this framework includes a term coupling the gauge field to $\Theta$. However, in our analysis, such term would only provide a constant correction to the mass term; we can thus reabsorb it in the definition of bare mass.

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1We adapt the prescription of [4], to treat $\lambda, r$ as functions of $n$. 
We now pass to the construction of an Abelian gauge theory on $\mathbb{R}^3$. Consider a $U(1)$-connection $A$, represented by the anti-hermitian gauge fields $A_{\mu}$,

$$\nabla_{D\mu} = D_\mu + A_\mu.$$  

Its curvature $F^A$ has components:

$$F_{\mu\nu}^A := F^A (D_\mu, D_\nu) = \left[ \nabla_{D\mu}, \nabla_{D\nu} \right] - \nabla_{[D\mu, D\nu]} = D_\mu A_\nu - D_\nu A_\mu + [A_\mu, A_\nu] + \frac{1}{p} \epsilon_{\mu\nu\rho} A_\rho.$$  

We see that the curvature has the usual form plus an extra term, reminiscent of the underlying noncommutativity.

One can introduce covariant coordinates $C_\mu$ using the gauge-invariant flat connection $\Theta$:

$$C_\mu := \nabla_{D\mu} - \nabla_{D\mu}^{(inv.)} = A_\mu - \Theta_\mu.$$  

It is possible to check that they actually transform covariantly under gauge transformations, and the corresponding curvature is:

$$F_{\mu\nu}^C = [C_\mu, C_\nu] + \frac{1}{p} \epsilon_{\mu\nu\rho} C_\rho.$$  

At this point, two options disclose to set up an Abelian gauge model: we ought to chose either $A$ or $C$ as the fundamental field of the theory. They differ by a flat connection: we expect the two resulting theories to be related by a redefinition of the vacuum. The two different procedures can be retrieved, respectively, in [10] and [11]. We follow the second picture, taking $C_\mu$ as variables, and we pursue the most general action such that:

(i) it is gauge invariant and at most quartic in the fundamental variable;

(ii) it does not involve tadpoles at classical order, which means not to include linear terms in the fundamental variable;

(iii) it is positive.

The second condition is essential in order to have a stable vacuum. Taking such family of gauge-invariant actions and writing down the classical equations of motion, one finds out that the absolute minimum\(^2\) is the null configuration $C_\mu = 0$.

We do not write explicitly the family of actions here, and just sketch the procedure. We rewrite the action in terms of Hermitian fields as $C_\mu := i \Phi_\mu$ and remove the redundant degree of freedom fixing the gauge $\Phi_3 = \tau_3$. Rearranging the remaining real scalar fields into complex ones $\Phi, \Phi^\dagger$, the gauge-fixed action finally reads\(^4\):

$$S \left[ \Phi, \Phi^\dagger \right] = \int_{\mathbb{R}^3} \left( \Phi^\dagger Q \Phi + \Phi Q \Phi^\dagger + \frac{g_2}{2} \Phi^\dagger \ast \Phi \ast \Phi^\dagger \ast \Phi \right),$$  

with the kinetic operator defined as:

$$Q := m_0^2 Id + \frac{8}{3} \tau_3^2 - i \frac{8}{3} \tau_3 D_3.$$  

The $\ast$-product is the matrix multiplication when the fields are represented in the natural matrix basis at each level $n$. As we already mentioned, the most general action would include a positive constant term $\sim \Phi^\dagger \Phi$, which has been reabsorbed in $m_0^2$.

---

\(^2\)Other local minima are present. However, due to the special form of the derivations in the present work, all the “commuting vacua” are constant configurations.

\(^3\)It is the Coulomb gauge $A_3 = 0$ in terms of usual gauge fields.

\(^4\)We restrict the parameter space where the potential is of the form $\sim (\Phi^\dagger \Phi)$, which corresponds to the $\Omega = \frac{1}{3}$ choice of [12].
3.3. **Large $N$ limit.** At this point, we project the system onto a single fuzzy sphere. This means we fix a half-integer $n$ and restrict to those states spanned by the $n$-th eigenstate of the Casimir radial operator. The remaining degree of freedom is the degeneracy at fixed $n$, labelled by $k = -n, \ldots, n$. We recall that the radius is $\sqrt{\theta n}$, with $\theta$ to be kept fixed at large $N$, where $N = 2n + 1$. Fields are projected into $N \times N$ matrices, according to the foliation of $\mathbb{R}^3 \lambda$. We then calculate the matrix elements appearing in the action:

$$
\langle k | Q \phi | k' \rangle = m_0^2 \langle k | \phi | k' \rangle + \frac{8}{3} \pi \theta \left( \langle k | (x^3)^2 \phi | k' \rangle + \langle k | x^3 [x^3, \phi] | k' \rangle \right)
$$

$$
= \langle k | \phi | k' \rangle \left\{ m_0^2 + \frac{8 n k}{3 \theta} (2k - k') \right\}.
$$

We multiply this expression by a factor $2\pi \theta$ which will arise from the integral, and obtain:

$$
2\pi \theta \langle k | Q \phi | k' \rangle = (2n + 1) \langle k | \phi | k' \rangle \left\{ \tilde{m}^2 + \frac{16\pi k (2k - k')}{3 n(2n + 1)} \right\},
$$

where we have defined the dimensionless parameter

$$
\tilde{m}^2 := \left( \frac{2\pi \theta}{2n + 1} \right) m_0^2
$$

by scaling the bare mass with the cutoff induced by finite $N$. Hence, denoting by $S_n$ the restriction of the action to a single fuzzy sphere, one has:

$$
S_n \left[ \phi, \phi^\dagger \right] = 2\pi \theta \text{Tr} \left\{ \phi^\dagger Q \phi + \phi Q \phi^\dagger + \frac{g_0^2}{2} (\phi^\dagger \phi)^2 \right\},
$$

with

$$
2\pi \theta \text{Tr} \left\{ \phi^\dagger Q \phi + \phi Q \phi^\dagger \right\} = (2n + 1) \sum_{k, k' = -N}^N \left\{ \langle k' | \phi^\dagger | k \rangle \langle k | \phi | k' \rangle \left( \tilde{m}^2 + \frac{16\pi k (2k - k')}{3 n(2n + 1)} \right) \right.
$$

$$
+ \langle k' | \phi | k \rangle \langle k | \phi^\dagger | k' \rangle \left( \tilde{m}^2 + \frac{16\pi k (2k - k')}{3 n(2n + 1)} \right) \right\}.
$$

Switching the labels of the dummy variables $k, k'$ in the last summand, we obtain

$$
(2n + 1) \sum_{k, k' = -n}^n \left\{ \tilde{m}^2 \left( \langle k' | \phi^\dagger | k \rangle \langle k | \phi | k' \rangle + \langle k' | \phi | k \rangle \langle k | \phi^\dagger | k' \rangle \right) \right.
$$

$$
+ \langle k' | \phi^\dagger | k \rangle \langle k | \phi | k' \rangle \left( \frac{32\pi k^2 + k'^2 - kk'}{3 n(2n + 1)} \right) \right\}.
$$

At this point, in order to compare with the LSZ model, we need to pass from the angular momentum to the harmonic oscillator description. This is done using the relation $k = n - \ell$ for $\ell = 0, 1, \ldots 2n = N - 1$. We thus define the rearranged matrix $M$ as

$$
M_{\ell_1 \ell_2} := \langle n - \ell_1 | \phi | n - \ell_2 \rangle.
$$

In terms of the new indices we have

$$
k^2 + k'^2 - kk' = n^2 - n (\ell_1 + \ell_2) + \ell_1^2 + \ell_2^2 - \ell_1 \ell_2.
$$
and the action is rewritten as:

\[ S_n \left[ \phi, \phi^\dagger \right] = (2n + 1) \sum_{\ell_1, \ell_2 = 0}^{2n} \left\{ \tilde{m}^2 \left( M^\dagger_{\ell_2 \ell_1} M_{\ell_1 \ell_2} + M_{\ell_2 \ell_1} M^\dagger_{\ell_1 \ell_2} \right) + M^\dagger_{\ell_2 \ell_1} M_{\ell_1 \ell_2} \frac{32\pi}{3} n^2 - \frac{\ell_1^2 + \ell_2^2 - \ell_1 \ell_2}{n(2n + 1)} \right\} + \left( \frac{2\pi \theta}{2n + 1} \right) \frac{g_0^2}{2} \text{Tr} \left\{ \left( M^\dagger M \right)^2 \right\}. \]

Eventually, rescaling in the usual manner \( g_0^2 \) into a dimensionless parameter \( \tilde{g}^2 \), and splitting the second line into the sum of two terms, the action on the fuzzy sphere reads:

\[ S_n \left[ \phi, \phi^\dagger \right] = N \text{Tr} \left\{ M^\dagger Q M + MQ M^\dagger + \left( \tilde{m}^2 + \frac{8\pi}{3} \right) M^\dagger M + \frac{\tilde{g}^2}{2} \left( M^\dagger M \right)^2 \right\}, \]

where the kinetic matrix \( Q \) is defined as:

\[ Q_{\ell_1 \ell_2} = \left( \tilde{m}^2 + \frac{8\pi}{3} - \frac{32\pi}{3N} \left( \ell_1 + \frac{1}{4} \right) \right) \delta_{\ell_1 \ell_2} + \mathcal{O} \left( \frac{1}{N^2} \right). \]

Therefore, we found out that, in the large \( N \) limit, this model coincides with the LSZ at the self-dual point, with magnetic field scaled by a factor \(-\frac{g_0^2}{2}\).

Thus, two in principle unrelated models, one describing a scalar field on a noncommutative plane with magnetic field, and the other the projection of a gauge model in noncommutative three-dimensional space, coincide at large \( N \). Note however that we have restricted to preferred points in parameter space for the LSZ model, the self-dual points of the Langmann–Szabo symmetry. Then, for the symmetric choice \( \sigma = \hat{\sigma} \), setting the theory in the curves \( B^2 \theta^2 = 4 \) produces a collapse of the parameter space, and only the dependence on \( N \) and two dimensionless couplings remains. The same simplification occurs in the restriction of the gauge model in \( \mathbb{R}^3 \) to a specific fuzzy sphere, where \( \theta \) starts to contribute at order \( 1/N^2 \). These cancellations seem to occur due to the large amount of symmetry of the LSZ model at self-dual points.

4. EXACT SOLUTION OF THE MATRIX MODEL

As we have seen throughout the previous sections, different noncommutative field theories reduce to a matrix model of the form \[ 10 \]

\[ Z_N \left( E, \tilde{E} \right) = \int DMD\bar{D}M^\dagger \exp \left( -N \text{Tr} \left\{ MEM^\dagger + M^\dagger \tilde{E} M + \tilde{V} \left( M^\dagger M \right) \right\} \right) \]

in terms of \( N \times N \) complex matrices, depending on the insertion of two external matrices. We now consider a quadratic polynomial with dimensionless coefficients

\[ \tilde{V} \left( M^\dagger M \right) = \tilde{m}^2 \left( M^\dagger M \right) + \frac{\tilde{g}^2}{2} \left( M^\dagger M \right)^2. \]

We also let the external fields \( E, \tilde{E} \) have arbitrary eigenvalues which we write as \( \frac{1}{N} \eta_\ell, \frac{1}{N} \tilde{\eta}_\ell \) respectively, for \( \ell = 1, \ldots, N \), to be consistent with their form in the noncommutative field theories above. In particular, the standard LSZ as in \[ 2.6 \] corresponds to external matrices with equal eigenvalues \( \eta_\ell = \tilde{\eta}_\ell \) given by consecutive integers plus a constant shift. The unconventional presence of \( \tilde{E} \) explicitly breaks \( U(N) \) symmetry, but the system is still tractable.

We now find exact solutions for this matrix model in terms of the spectra of the external fields. The results can be written in terms of the observables of \( U(N) \) Chern–Simons theory in \( \mathbb{S}^3 \), with \( q \) real.
4.1. General solution. As shown in [15], we can approach the solution using the polar coordinate decomposition of a generic $N \times N$ complex matrix $M$:

$$M = U_1 \text{diag} (\lambda_1, \ldots, \lambda_N) U_2,$$

with $U_{\alpha=1,2}$ unitary matrices and $\lambda_\ell \geq 0$. The Jacobian of this transformation is

$$\mathcal{D}M \mathcal{D}M^\dagger = [dU_1] [dU_2] \prod_{\ell=1}^N dy_\ell \Delta_N [y]'^2,$$

where $[dU_\alpha]$ is the invariant Haar measure over $U(N)$, and $y_\ell := \lambda_\ell^2$ and

$$\Delta_N [y] = \prod_{1 \leq \ell < \ell' \leq N} (y_\ell - y_{\ell'})$$

is the Vandermonde determinant.

As shown in [15], the use of this transformation implies that integrations over $U_{\alpha=1,2} \in U(N)$ decouple over the two types of external field terms. Denoting by $\Lambda^2 = \text{diag} (y_1, \ldots, y_N)$, the angular degrees of freedom $U_\alpha$ can be integrated out using the Harish-Chandra–Itzykson–Zuber (HCIZ) formula [42, 43]:

$$(4.2) \quad \int_{U(N)} [dU_2] \exp \left\{ -N \text{Tr} \left( EU_2^\dagger \Lambda^2 U_2 \right) \right\} = C_N \frac{\det_{1 \leq \ell, \ell' \leq N} (e^{-4\pi \eta_{\ell, \ell'}})}{\Delta_N [\eta] \Delta_N [y]},$$

where we use the explicit form of the eigenvalues of $E, \tilde{E}$ and denoted

$$C_N := (4\pi)^{-\frac{N(N-1)}{2}} \prod_{j=1}^{N-1} j! (4\pi)^{-\frac{N(N-1)}{2}} G(N+1),$$

where $G(\cdot)$ is a Barnes $G$-function.

Analogous expression is obtained for $U_1$ replacing $E$ by $\tilde{E}$. For a generic potential $\tilde{V} (M^\dagger M)$ as in (4.1), one gets:

$$(4.3) \quad Z_N \left( E, \tilde{E} \right) = \int \prod_{j=1}^N dy_j \Delta_N [y]'^2 e^{-\sum_{j=1}^N \left( \tilde{m}_j^2 y_j + \frac{\tilde{\gamma}}{2} y_j^2 \right)}$$

$$\times \int [dU_2] \exp \left\{ -N \text{Tr} \left( EU_2^\dagger \Lambda^2 U_2 \right) \right\} \int [dU_1] \exp \left\{ -N \tilde{E} U_1^\dagger \Lambda^2 U_1 \right\}$$

This was already done in [15] but now, after applying the HCIZ formula, we will not expand the resulting determinants into sums of permutations of $N$ objects. Instead, plugging HCIZ (4.2) into (4.3), we note that the partition function for the matrix model is:

$$Z_N \left( E, \tilde{E} \right) = C_N \int \prod_{\ell=1}^N dy_\ell \Delta_N [y]'^2 e^{-\sum_{\ell=1}^N \left( \tilde{m}_\ell^2 y_{\ell} + \frac{\tilde{\gamma}}{2} y_{\ell}^2 \right)} \frac{\det_{1 \leq \ell, \ell' \leq N} (e^{-4\pi \eta_{\ell, \ell'}})}{\Delta_N [y]'^2 \Delta_N [\eta] \Delta_N [\tilde{\eta}]},$$

where we recall that $\eta_{\ell}, \tilde{\eta}_{\ell}$ stand for the eigenvalues of $E, \tilde{E}$ respectively, up to a factor $\frac{4\pi}{N}$. At this point, we do not expand the determinant and cancel a Vandermonde squared instead. That is, after a suitable rescaling of the integration variables:

$$(4.4) \quad Z_N \left( E, \tilde{E} \right) = \frac{C_N'}{\Delta_N [\eta] \Delta_N [\tilde{\eta}]} \int \prod_{\ell=1}^N dy_\ell e^{-\sum_{\ell=1}^N \left( m_\ell^2 y_{\ell} + \frac{\gamma}{2} y_{\ell}^2 \right)} \frac{\det_{1 \leq \ell, \ell' \leq N} (e^{-\eta_{\ell, \ell'}})}{1 \leq \ell, \ell' \leq N} \frac{\det_{1 \leq \ell, \ell' \leq N} (e^{-\tilde{\eta}_{\ell, \ell'}})}{1 \leq \ell, \ell' \leq N}.$$

\[\text{See [44] for comments on this parametrization with regards to the more usual one in terms of eigenvalues [41].}\]

In any case, this transformation is much used and very useful when studying complex matrix models, such as the LSZ.
with coefficients redefined as:

$$C'_N = (4\pi)^{-N} C_N = \frac{2^{-N(N+1)/2}}{\text{vol} \left( U(N) \right)}, \quad m^2 = \frac{\hat{m}^2}{4\pi}, \quad g^2 = \frac{\hat{g}^2}{(4\pi)^2},$$

where vol is the volume of the gauge group. Notice that this normalization is essentially the partition function of a Gaussian matrix model (GUE ensemble [41]).

In the theory of non-intersecting Brownian motion, the determinants in (4.4) are very familiar. This whole theory of determinantal processes is known to be directly related to $U(N)$ Chern–Simons theory on $S^3$ and with the Wess–Zumino–Witten (WZW) model [44], where such connection was shown to follow from specializations of the determinants

$$\det_{1 \leq \ell, \ell' \leq N} \left( e^{-\eta_\ell y_{\ell'}} \right), \quad \det_{1 \leq \ell, \ell' \leq N} \left( e^{-\bar{\eta}_{\ell'} y_{\ell}} \right)$$

in (4.4). However, it is more direct to show the relation through the corresponding matrix model formulation. Recall for this the definition of a Schur polynomial [42]

$$s_\mu (x_1, \ldots, x_N) = \frac{\det_{1 \leq \ell, \ell' \leq N} \left( x_\ell^{\mu_\ell + N - \ell'} \right)}{\det_{1 \leq \ell, \ell' \leq N} \left( x_\ell^{N - \ell} \right)},$$

and hence, if we rewrite the scaled eigenvalues $\eta_\ell, \bar{\eta}_{\ell'}$ of the external matrices $E, \bar{E}$, assuming they are integers, as:

$$-\eta_\ell = \mu_\ell + N - \ell,$$
$$-\bar{\eta}_{\ell'} = \nu_\ell + N - \ell,$$

for $\ell = 1, \ldots, N$, then we immediately have:

$$Z_N \left( E, \bar{E} \right) = \frac{C'_N}{\Delta_N \left( \eta \right) \Delta_N \left( \bar{\eta} \right)} \int_{[-\infty, \infty]^N} \prod_{\ell=1}^N d\eta_\ell \, \Delta_N \left( e^{y_\ell} \right) e^{-\sum_{\ell=1}^N \left( m^2 y_\ell + \frac{\hat{m}^2}{2} y_{\ell'}^2 \right)} \times s_\mu \left( e^{y_1}, \ldots, e^{y_N} \right) s_\nu \left( e^{y_1}, \ldots, e^{y_N} \right).$$

We henceforth adopt the shorthand notation

$$\hat{Z}_N (\mu, \nu) := \Delta_N \left( \eta \right) \Delta_N \left( \bar{\eta} \right) Z_N \left( E, \bar{E} \right),$$

stressing the dependence on the partitions $\mu, \nu$. Then, using

$$\prod_{1 \leq \ell < \ell' \leq N} \left( e^{y_\ell} - e^{y_{\ell'}} \right)^2 = \prod_{1 \leq \ell < \ell' \leq N} \left( 2 \sinh \left( \frac{y_\ell - y_{\ell'}}{2} \right) \right)^2 \prod_{m=1}^N e^{(N-1)y_m},$$

we obtain

$$\hat{Z}_N (\mu, \nu) = C'_N \int_{[-\infty, \infty]^N} \prod_{\ell=1}^N d\eta_\ell \, e^{-\sum_{\ell=1}^N \left( \beta y_\ell + \frac{\hat{m}^2}{2} y_{\ell'}^2 \right)} \prod_{1 \leq \ell < \ell' \leq N} \left( 2 \sinh \left( \frac{y_\ell - y_{\ell'}}{2} \right) \right)^2 \times s_\mu \left( e^{y_1}, \ldots, e^{y_N} \right) s_\nu \left( e^{y_1}, \ldots, e^{y_N} \right),$$

with $\beta = m^2 - N + 1$. This latter expression is close to the general version of the $U(N)$ Chern–Simons on $S^3$ matrix model, with two different insertions of Schur polynomials, whose evaluation gives the Hopf link invariant [13, 40]. One just needs to have one of the two Schur polynomials conjugated. For this, one set of Schur variables should now be inverted and we
give the corresponding mathematical details in the Appendix. We obtain the matrix model representation

\[
\hat{Z}_N (\mu, \nu) = A (N, |\mu|, |\nu|) \int_{[-\infty, \infty]^N} dx_1 \cdots dx_N \prod_{j=1}^N e^{-\frac{x_j^2}{2} \Delta_j} \prod_{1 \leq j < k \leq N} \left( 2 \sinh \left( \frac{x_j - x_k}{2} \right) \right)^2 \times \delta \left( e^{x_1}, \ldots, e^{x_N} \right) s_{\nu^*} \left( e^{-x_1}, \ldots, e^{-x_N} \right),
\]

where \( \nu^* := (\nu_1 - \nu_N, \nu_1 - \nu_{N-1}, \ldots, 0) \) and

\[
A (N, |\mu|, |\nu|) := C_N \exp \left( \frac{\beta^2 N}{2g^2} + \frac{\beta}{g^2} (|\mu| - |\nu^*|) \right),
\]

with \( \beta = \beta - \nu_1 \) and the number \( |\mu| = \sum_j \mu_j \) is the size of the partition \( \mu \) (and likewise for the other partition). Notice that \( |\nu^*| \neq |\nu| \), in particular \( |\nu^*| = N\nu_1 - |\nu| \).

Notice also that the numerical prefactors in (4.6), which depend exclusively on the eigenvalues of the external matrices, can be written, using Weyl’s denominator formula (see Appendix), as

\[
\Delta_N [\eta] = \prod_{1 \leq \ell < \ell' \leq N} (\mu_\ell - \mu_\ell' - \ell + \ell') = G(N + 1) \dim \mu,
\]

\[
\Delta_N [\eta] = \prod_{1 \leq \ell < \ell' \leq N} (\nu_\ell - \nu_\ell' - \ell + \ell') = G(N + 1) \dim \nu,
\]

where \( G(\cdot) \) is again the Barnes \( G \)-function. Therefore, we finally obtain

\[
Z_N \left( E, \hat{E} \right) = C \cdot \langle W_{\mu^*} \rangle,
\]

with

\[
C := \frac{\exp \left( \frac{\beta^2 N}{2g^2} + \frac{\beta}{g^2} (|\mu| - |\nu^*|) \right)}{(4\pi)^{N(N+1)/2} G(N+1) \dim \mu \dim \nu} S_{00},
\]

with \( \langle W_{\mu^*} \rangle \) the Hopf link and \( S_{00} = Z_{CS} \) being the \( U(N) \) Chern–Simons partition function which is a quantum topological invariant of \( S^3 \), also known as Witten–Reshetikhin–Turaev invariant \[47, 48\], whose explicit expression we give below. Notice that even if we start with two identical external matrices \( E = \hat{E} \) (that is, if the two partitions are identical), this does not lead to a diagonal part of the Hopf link average only, because of the emergence of the starred partition in the Chern–Simons interpretation.

Recall that Chern–Simons observables depend on framing \[49\]: the relation between such dependence and the modular matrices is given in \[50\]. Observables in the matrix model description are not in the canonical framing, and the Hopf link average in the present case is:

\[
\langle W_{\mu^*} \rangle = \langle TST \rangle_{\mu^*}
\]

in terms of the modular \( S, T \) matrices. In the general case, the framed Hopf link comes as \( T^n S T^m \); if \( n = m = 0 \), that is, the canonical framing in \( S^3 \), it is exactly the modular \( S \) matrix, while if \( n + m = 2 \), the framing is the \( U(1) \)-invariant Seifert framing \[51\]. This is the case of the matrix model description.

4.2. Quantum dimensions. An important particular case of the above general setting is when one of the partitions in (4.5) is void. That is, one has an external matrix with the equispaced spectra and the other one generalized with a partition. This case corresponds, as we shall see, to quantum dimensions in the Chern–Simons interpretation \[24\]. Quantum dimensions of a
representation associated to the partition $\mu$ is given by the following hook-content formula \cite{45,24}

\[
\dim_{q \mu} = \prod_{x \in \mu} \frac{|N + c(x)|_q}{[h(x)]_q},
\]

where for each box $x \equiv (j, k)$ of the Young diagram determined by $\mu$, the quantity $h(x) := \mu_j + \mu'_k - j - k + 1$ is the hook length, with the prime meaning conjugate diagram, and $c(x) := j - k$ is known as the content of the box $x$. The operation $[\cdot]_q$ denotes the symmetric $q$-number, that is

\[
[n]_q = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}.
\]

In Chern–Simons theory on $S^3$, the unknot invariant is given by quantum dimensions \cite{18,24}. Since one of the two external matrices has harmonic oscillator spectrum the matrix model above reduces to

\[
\hat{Z}_N (\mu, \nu) = A(N, |\mu|, 0) \int_{[-\infty, \infty]^N} \prod_{j=1}^N dx_j \, e^{-\frac{g^2}{2} \sum_{j=1}^N x_j^2} \prod_{1 \leq j < k \leq N} \left( 2 \sinh \left( \frac{x_j - x_k}{2} \right) \right)^2 \times s_{\mu} (e^{x_1}, \ldots, e^{x_N}),
\]

whose exact evaluation \cite{24} leads to

\[
Z_N (E, \tilde{E}) = \frac{A(N, |\mu|, 0)}{G(N + 1)^2 \dim \mu} \frac{Z_{CS} \cdot q^{-\frac{1}{2} C_2(\mu)} \dim_{q \mu}}{G(N + 1) \dim \mu} \exp \left( \frac{\beta^2 N}{2 g^2} + \frac{\beta |\mu|}{g^2} \right) \frac{Z_{CS} \cdot q^{-\frac{1}{2} C_2(\mu)} \dim_{q \mu}}{(4\pi)^{N(N+1)/2} G(N + 1) \dim \mu}
\]

where $q = e^{-1/g^2}$, and $Z_{CS} = S_{00}$ is the partition function of $U(N)$ Chern–Simons theory on $S^3$, discussed below and which is given by the same matrix model but with no Schur insertion.

Furthermore, in the expression above the term

\[
C_2 (\mu) = (N + 1) |\mu| + \sum_j (\mu_j^2 - 2j \mu_j)
\]

is the $U(N)$ Casimir of the representation $\mu$, labelled by the Young diagram associated to the partition $\mu$, with $\mu_j$ boxes in the $j$-th row, with rows understood to be aligned on the left.

Again, the result of (4.9) can be written in terms of $S$ and $T$ matrices since $\dim_{q \mu} = S_{0\mu}/S_{00}$ and $T_{\mu\lambda} \propto q^{-\frac{1}{2} C_2(\mu)} \delta_{\mu\lambda}$, but there are the same additional normalization factors as above. Indeed, the result for this case, which can be computed directly, also follows from the result above by directly choosing one partition to be void.

The appearance of quantum dimensions is interesting in that they appear as well in the study of noncommutative gauge theories through the analysis of WZW D-branes \cite{13,52}. However, our previous discussion of noncommutative scalar field theory only leads to the simpler setting, described in what follows, where the two external matrices are equal and have harmonic oscillator spectra.

\footnote{In Chern–Simons theory $g_s = \frac{1}{2} = \frac{2\pi i}{4\pi}$. The real string coupling constant $g_s$ is used when describing topological strings. That is the same type of description here, since $g^2$ is real.}
4.3. Chern–Simons and LSZ partition function. We have seen in previous Sections that our study of noncommutative scalar field theory naturally leads to a LSZ matrix model with $E = \tilde{E}$ and spectra $\frac{\pi}{N} (\ell - \frac{1}{2})$ for $\ell = 1, ..., N$ (see (3.6)), or the same spectra but shifted (see (2.17)) for the fuzzy sphere. Thus, we consider now the case in which both partitions are void: this corresponds to the two external matrices having harmonic oscillator spectra. In particular, from (4.5), we have that $\eta_\ell = \ell - N$ for $\ell = 1, ..., N$. The fact that the two spectra have an overall energy shift only has an impact at the level of renormalization of the mass parameter. This follows immediately from a simple property of Schur polynomials, given in the Appendix. More precisely, we conveniently change variables in order to relabel the eigenvalues $\eta_\ell$ according to $\ell \mapsto N + 1 - \ell$, and reabsorb the constant shift into the definition of the mass, as explained in the Appendix. This corresponds to reordering the Landau levels, assigning the highest energy level to the first eigenvalue and the lowest to the $N$-th eigenvalue.

Then one obtains the matrix model without both Schur polynomial insertions, and the corresponding matrix model integral is the one for the Chern-Simons partition function, given in (1.2) (recall that $g_s = 1/g^2$). The matrix model has the exact solution (2.2):

\[
Z_{CS} = \left(\frac{2\pi}{g^2}\right)^{N/2} N! e^{N(N^2+1)/6g^2} \prod_{j=1}^{N-1} (1 - q^j)^{N-j},
\]

with $q = e^{-1/g^2}$ as above. The product can also be written as a $q$-deformed Barnes function, which, in the limit $g \to \infty$ (which is $q \to 1$), reduces to the Barnes $G$-function. Then, the LSZ matrix model partition function is

\[
Z_N \left( E, \tilde{E} \right) = \exp \left( \frac{(m^2 - N + 1)^2 N}{2g^2} \right) \frac{Z_{CS}}{(4\pi)^{N(N+1)/2}} G(N + 1).
\]

Therefore, in the limit $q \to 1$, the non-trivial products in the numerator and denominator of (1.12) will cancel and only prefactor contributions will remain. Likewise, consider also the large $N$ limit with the double scaling $N/g^2 = \text{cte}$ of the free energy $F_{CS} = \ln Z_{CS}$ of Chern–Simons theory, which has a well-known topological string interpretation \cite{15, 16}. Notice again, in addition to the prefactors, the normalization by the Barnes $G$-function, and the fact that, splitting the Chern–Simons free energy into non-perturbative and perturbative contributions, the non-perturbative part is essentially the semiclassical, $q \to 1$, value, $F_{np} = \log \left( G(N + 1) g_s^{N^2} (2\pi)^{(N(N+1)/2)} \right)$, in canonical framing. Thus, that normalization subtracts part of the non-perturbative contribution.

The absence of a preferred temporal direction in the Moyal plane as well as in the fuzzy sphere prevents us from a meaningful Hilbert space picture of the partition functions (1.12). However, we can still provide an interpretation in terms of energy density levels. The system in facts splits into two clearly separated contributions: a Landau Hamiltonian density and an interaction term. Throughout the solution, the former is encoded in the Vandermonde determinants, while the interaction appears as a Gaussian measure.

In the pure LSZ case, with both external matrices with harmonic oscillator spectra, the Landau levels are $\frac{2\ell - 1}{g^2}$. Introducing equal partitions $\mu = \nu$ to generalize the external matrices would correspond to a distortion of the Landau spectrum. As an example, the insertion of an antisymmetric partition $\mu = (1, \ldots, 1)$ shifts the whole spectra by one level in the negative direction; as shown in the Appendix, this corresponds to a renormalization of the mass. Remarkably, the symmetric partition $\mu = (N, 0, \ldots, 0)$ determines the same spectrum, although obtained by taking the highest energy level and sending it to the bottom. Moreover, the triangular partition $\mu = (N-1, N-2, \ldots, 1, 0)$ sends all the Landau levels to the lowest one. Generally,
any triangular partition with the first rows with decreasing number of boxes from \(N - n_0\) to 1, \(1 \leq n_0 \leq N\), and the remaining void corresponds to introduce a cutoff at the \(n_0\)-th Landau level of energy density \(\frac{2n_0-1}{n_0}\) and project all the higher energy states onto it.

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Appendix A. Schur polynomials

Here we include explicit calculations involving Schur polynomials \[45\]. First, to deal with the inversion of variables appearing in a Schur polynomial, we can use the identity

\[
s_\nu(x_1^{-1}, \ldots, x_N^{-1}) = \prod_{j=1}^{N} x_j^{-\nu_j} s_\nu^*(x_1, \ldots, x_N),
\]

where the starred partition is defined as \(\nu^* := (\nu_1 - \nu_N, \nu_1 - \nu_{N-1}, \ldots, 0)\). Then the following holds:

\[
\begin{align*}
\int \prod_{j=1}^{N} dz_j \Delta_N^{(\text{hyp})}(z) e^{-\frac{1}{2g_s} \sum_{j=1}^{N} z_j^2} s_\mu(e^{z_1}, \ldots, e^{z_N}) s_\nu(e^{z_1}, \ldots, e^{z_N}) \\
= \int \prod_{j=1}^{N} dz_j \Delta_N^{(\text{hyp})}(z) e^{-\frac{1}{2g_s} \sum_{j=1}^{N} z_j^2} \sum_{j=1}^{N} e^{\nu_j z_j} s_\mu(e^{z_1}, \ldots, e^{z_N}) s_\nu^*(e^{-z_1}, \ldots, e^{-z_N}) \\
= e^{\frac{N \nu_1 g_s}{2}} \int \prod_{j=1}^{N} dw_j \Delta_N^{(\text{hyp})}(w) e^{-\frac{1}{2g_s} \sum_{j=1}^{N} w_j^2} s_\mu(e^{w_1+\nu_1 g_s}, \ldots, e^{w_N+\nu_1 g_s}) s_\nu^*(e^{-w_1-\nu_1 g_s}, \ldots, e^{-w_N-\nu_1 g_s}) \\
= \tilde{A}(N, |\mu|, |\nu^*|) \int \prod_{j=1}^{N} dw_j \Delta_N^{(\text{hyp})}(w) e^{-\frac{1}{2g_s} \sum_{j=1}^{N} w_j^2} s_\mu(e^{w_1}, \ldots, e^{w_N}) s_\nu^*(e^{-w_1}, \ldots, e^{-w_N}),
\end{align*}
\]

where \(\Delta_N^{(\text{hyp})}(z) \equiv \prod_{1 \leq j < k \leq N} 2 \sinh\left(\frac{z_j - z_k}{2}\right)\) and \(\tilde{A}(N, |\mu|, |\nu^*|) := e^{\frac{N \nu_1 g_s}{2}} e^{\nu_1 g_s (|\mu| - |\nu^*|)}\). Note that, in contrast to the text, we started without a linear term if the potential of the matrix model.

A.1. Spectral shift and rectangular Schur. A simple identity of Schur polynomials quickly shows what occurs if, in the case of a equiphased, harmonic oscillator spectra, we have a global overall shift in the spectrum (that is, a different zero point energy). If we have a rectangular partition of length \(N\), \((l, l, \ldots, l)\) which we denote by \(l^N = (l^N)\) then, assuming that \(\lambda\) is a partition of length equal or lower than \(N\), it holds

\[
(A.1) \quad s_{\lambda+l^N}(e^{x_1}, \ldots, e^{x_N}) = \prod_{i=1}^{N} e^{l x_i} s_\lambda(e^{x_1}, \ldots, e^{x_N}),
\]

Therefore, an overall spectral shift by an integer \(l\) in one external matrix, corresponds to a renormalization of the mass parameter \(\hat{m}^2 \mapsto \hat{m}^2 - l\).

\footnote{We underline that such cutoff is intrinsically different from the naturally induced short-distance one \(\sqrt{\hat{m}^2}\).}
A.2. Dimensions. The value of $s_\lambda(1, \ldots, 1)$ gives the dimension of the irreducible representation of $U(N)$ with highest weight $\lambda$. Using Weyl’s denominator formula

$$s_\lambda(1, \ldots, 1) = \frac{\prod_{i<j}(\mu_i - \mu_j)}{\prod_{i<j}((i-j))},$$

where $\mu_i = \lambda_i + N - i$. Thus, it can also be written as

$$s_\lambda(1, \ldots, 1) = \dim \lambda = \frac{1}{G(N+1)} \prod_{i<j}(\lambda_i - \lambda_j - i + j),$$

where $G(\cdot)$ is the Barnes $G$-function.

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