From Grassmann complements to Hodge-duality

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Introduction

Hodge duality is a central concept of 20th century algebraic and analytic geometry and plays a non-negligible role also in recent mathematical physics. At first sight one might expect that its origins lie in the 1930s when its name-giving protagonist, William V.D. Hodge, started his mathematical research. On the other hand, a close link between Hodge’s theory and the Maxwell equation has sometimes been claimed not only from a systematic point of view but also historically.\(^1\) The question of how dense this connection was in the historical sense leads back to the late 19th and early 20th century development of electrodynamics. Of course we are well advised not to take systematic correspondences too easily as an indicator of a historically effective relationships. Historical mis-readings of F. Klein’s early attempt at reconstructing Riemann’s study of harmonic functions and meromorphic forms in complex dimension \(n = 1\) by linking it to the study of harmonic flows on surfaces (Klein, 1882) may serve as a warning.

If one undertakes a journey to the late 19th and early 20th century with this question in mind, an unexpected author comes into sight: Hermann Grassmann. Grassmann was of some influence for the early understanding of the linear algebraic background for duality concepts in electromagnetism and elsewhere. He and the authors following him in this respect, did not speak about “duality” but used the language of “complements” of alternating products of “extensive” (later vectorial) quantities. This explains the thematic arc spanned in this paper, indicated already in the title.

Our report starts with a glance at Grassmann’s so-called complements of alternating products. Readers who are acquainted with Hodge duality will immediately perceive it as a linear algebraic template (“precursor”) for the later Hodge *-operator (section 1). The first two subsections (1.1 and 1.2) discuss how Grassmann proceeded; subsection 1.3 puts the Grassmannian complements into the context of the vectorial operations developed simultaneously, but independently, from Grassmann’s theory. Readers who are not so well versed with Hodge duality and the Hodge operator can slowly approach the topic by getting acquainted with this elementary twin.

Section 2 jumps into the history of electromagnetism from an extremely selective viewpoint dictated by our topic. It does not try to trace all kinds of duality relations in electromagnetism, which early on in its history played an important heuristic role and were often linked to philosophical, sometimes vague speculations on dualism between electricity and magnetism. Here we concentrate on that phase of electromagnetism in which duality in the sense of Grassmann complements became visible and was expressed in a clear mathematical form. That means we jump into the history of electrodynamics when it became relativistic. Readers who want to get a broader perspective of electromagnetism and even only the Maxwell equation have to consult the respective literature in the history of physics.\(^2\) Like so many fundamental concepts in special relativity the

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\(^1\) In his report on dualities in mathematics and physics M. Atiyah, e.g., remarks: “Maxwell’s equations actually motivated Hodge for his work on harmonic forms in general. As indicated, Maxwell’s equations are about forms of degree 2 in 4 dimensions and Hodge went to forms of any degree \(q\) in any dimension \(n\)” (Atiyah, 2008, p. 77).

\(^2\) See in particular (Darrigol, 2000; Hunt, 1991).
introduction of Grassmann complements into the formulation of the Maxwell equations stems from Hermann Minkowski (sec. 2.1). The linear algebraic duality relation contained in Minkowski’s relativistic electrodynamics was taken over into the more involved “curved” background of general relativity, i.e. was adapted to the infinitesimal structures (the tangent and cotangent spaces/bundles and their sections, in more recent terminology) of Riemannian geometry. Several authors played a role, among them notably W. Pauli, F. Kottler, and T. de Donder (sec. 2.2). Einstein did not belong to this group; he preferred to formulate Maxwellian electrodynamics without an (explicit) reference to this type of duality relation.

Elie Cartan, on the other hand, did so (sec. 2.3). It may come as a surprise that Cartan even used a generalized version of Grassmann complements not only for expressing the Maxwell equation, but also for his reformulation and generalization of Einstein gravity. In his attempt at bridge-building between Einstein’s theory and the generalized theory of elasticity of E. and F. Cosserat Cartan needed a transmutation of translational curvature into one represented as a rotational momentum (torsion) and vice versa. In three-dimensional space, which served Cartan as a guide to his intuition, a Grassmann type duality was just the right tool; Cartan could then use it also for formulating his generalized theory of gravity in dimension \( n = 4 \).

The third section discusses the origins of Hodge duality. In order to do justice to Hodge, a short introduction to the background of Riemann’s theory of Abelian integrals is necessary (sec. 3.1). Hodge’s research started with the goal to do the “same” (or something similar) for \( n \) dimensions, which meant as a first step to define what it means for a differential form on a manifold \( M \) with metric (a Riemannian manifold) to be “harmonic”. A short outline of the early stage of this research program is given in sec. 3.2. Hodge’s research soon let him state an intriguing relationship, although not immediately proven in a valid form, between the numbers of linearly independent harmonic \( p \)-forms and the topology of \( M \), viz. its Betti numbers. This led him to new insights into the theory of algebraic surfaces (sec. 3.3).

In the mid 1930s and 1940s Hodge brought his program into a first stage of maturity, documented in his book on The Theory and Application of Harmonic Integrals (Hodge, 1941). Here all the basic topics we are interested in can be found in sufficient clarity: Hodge \( * \)-operation, the definition of harmonic forms, and a discussion of the relation to the Maxwell equation (sec. 3.4). It may be disillusioning to see, how elementary and reduced to simple special cases his discussion of the Maxwell equation remained. That seems to be characteristic for Hodge’s research profile; he was rather a (pure) mathematician than truly interested in the link to mathematical physics.

Subsection 3.5 gives a condensed picture of how Hodge’s methods and results were assimilated by the wider community of complex algebraic geometers as far as transcendent (i.e. analytic) methods were concerned. This section should not be misunderstood of claiming to be a full history of this assimilation; it can only give short glimpses into the phase in which Hodge’s theory was generalized and turned into what since then became to be known as Hodge theory, with the Hodge-Laplacian and the study of the

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\(^3\)See the separate and more detailed discussion in (Scholz, 2019).
Hodge structure of holomorphic and anti-holomorphic forms on a complex Kählerian manifold.

Developments after mid-century are only touched at in the outlook of section 4. It would be inadequate, however, to stop short in our report without at least mentioning the shift Hodge theory underwent with the introduction of sheaf cohomological methods by P. Dolbeault, J.-P. Serre and others. In the end, Serre’s famous duality theorem was a generalization of Hodge duality, fraught with lasting consequences for mathematics in the second half of the twentieth century (sec. 4.1). The final subsection 4.2 turns back to the question of how Hodge’s duality concept was taken up in physics. The strongest role for it arose within Yang-Mills theory.

In the 1970s and 1980s, the decades of the rising standard model of elementary particles, the manifest success for Young-Mills type gauge fields in physics had strong repercussions also in mathematics. It inspired mathematicians to study the vacuum solutions of Yang-Mills equations. This lead to a kind of generalized theory of harmonic forms and became a fruitful tool in differential topology, in particular for manifolds of dimension $n = 4$. The remarks of our outlook on this topic remain of course incomplete. They cannot be more than first hints to a field of recent mathematics the history of which has still to be written.

1 Grassmann’s complement of “extensive quantities”

1.1 Grassmann’s Ausdehnungslehre (1862)

Hermann Günther Grassmann (1809–1877) is today often recalled as the founder of $n$-dimensional vector spaces and the algebra of their exterior products, clad in a peculiar geometrico-algebraic theory which he called the doctrine of extension (“Ausdehnungslehre”).

He got into acquainted with vector-like ideas through his father’s, Justus Günther Grassmann’s, publications on crystallography and gave a condensed review of them (Grassmann, 1839) before he embarked on the project of his general theory of extension. He published the ideas on his Ausdehnungslehre in two book-length general publications (Grassmann, 1844, 1862), the second one now also available in English (Grassmann, 1862/2000), and several articles on more specialized topics.

For present day readers the exposition in the second book is easier to follow than the first one. Here Grassmann introduced an extensive quantity (“Ausdehnungsgrösse”) $a$ of an extensive domain of level $n$ (“Ausdehnungsgebiet $n$-ter Stufe”) as a linear combination of a finite number of independent generators $e_1, \ldots, e_n$ (Grassmann’s symbols), called by him the original units (“ursprüngliche Einheiten”)

$$a = \sum_{i=1}^{n} \lambda_i e_i ,$$

See among others (Petsche, 2009; Schubring, 1996)

(Scholz, 1996)

For the articles see (Gräßmann, 1894–1911, vols. 2.1, 2.2, 3.1). The first Ausdehnungslehre (1844) is now also available in French (Grassmann, 1994).
where $\lambda_i$ denote real, rational or later even complex coefficients. On this basis he explained the operations (addition, scalar multiplication) and their fundamental laws (Grassmann, 1862, chap. 1, part I), which were later condensed into the axioms of a vector space (Peano, 1888).\footnote{Dorier, 1995} In this way a Grassmannian extensive domain of level $n$ may essentially be understood as an $n$-dimensional vector space $V$ over the rationals, the real or the complex numbers.

He discussed bi- and multilinear products of two or more extensive quantities $a, b, \ldots$ in general (chap. 2) – in later terminology tensor products – and showed particular interest for alternating products of quantities written by him in square brackets (without commas) $[ab]$ or, in the case of several, say $k$, independent quantities $a_1, \ldots, a_k$, $(1 \leq k \leq n)$ as

$$A = [a_1 \ldots a_k].$$

Grassmann called this a combinatorial product or exterior product ("äusseres Produkt") (chap. 3, part I). Such products formed quantities of the $k$-th level ("Grössen $k$-ter Stufe") and were interpreted by him geometrically – we would say as an equivalence class of paralleloptopes of dimension $k$. These quantities were thus of a new type; by linear combination they established a new extensive domain of $k$-th level in a principal domain of level $n$ ("Hauptgebiet $n$-ter Stufe"). They could be generated by "simple" quantities of $k$-th level of form $[e_{i_1} \ldots e_{i_k}]$ with original units $e_{i_1}, \ldots, e_{i_k}$. In modernized symbolism such an extensive domain of $k$-th level corresponded to the $k$-th exterior product $\wedge^k V$ of $V$ with elements $A$ linearly generated by quantities of the type $[a_1 \ldots a_k]$.

Of course Grassmann realized that for $k = n$ the domain of the $n$-th level is generated by the combinatorial product of all original units. He therefore identified $\Lambda^n V$ with the "numerical quantities" $\mathfrak{N}$ by the following rule (Grassmann, 1844, §89)\footnote{Today one would substitute the number field $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$ for $\mathfrak{N}$.}

**Unit convention:** The combinatorial product of all original units is considered to be equal to the "numerical unit":

$$[e_1 e_2 \ldots e_n] = 1 \quad (2)$$

On such a basis he studied the properties of the alternating algebra over $V$ in quite some detail, e.g., the rules of the operation which we would write as $\Lambda^k V \times \Lambda^l V \rightarrow \Lambda^{k+l} V$. He did so even for the case $k + l > n$ by reducing the result "modulo $\Lambda^n V$" and the unit convention. Today such a structure is called a Grassmann algebra.

Only in the next step of developing his theory (chap. 4, part I) Grassmann introduced an interior product ("inneres Produkt") in an extensive domain (see below). He did so in such a way that the original units $e_1, \ldots, e_n$ turned into an orthonormal basis of the extensive domain, the vector space $V$, to use later terminology. The introduction was built upon the properties of the complement ("Ergänzung") of an extensive quantity. This is not all we find in (Grassmann, 1862). In chapter 5 (part I) of the book the author sketched how to study the classical geometrical, i.e. Euclidean, space as an extensive domain of level 3 with exterior/combinatorial and interior products. In part II he studied
what we would call analysis in \( n \)-dimensional real vector spaces with differentiation, infinite series and integration.

We need not look at these interesting parts of his book here; for our purpose it suffices to concentrate on the introduction of Grassmann’s (dual) complement in an extensive domain.

1.2 Dual complement (Ergänzung)

In §4 of his chapter 3 on combinatorial products Grassmann introduced a new operation by forming the complement of a given quantity (“Ergänzung der Grössen”, article 89). For \( E \) a unit of the \( k \)-th level, in slightly modernized notation (using double subscripts) \( E = [e_{i_1} e_{i_2} \ldots e_{i_k}] \), he introduced the

**Definition**: The complement \( E' \) of a unit of the \( k \)-th level \( E = [e_{i_1} e_{i_2} \ldots e_{i_k}] \) is the combinatorial product of all units not appearing in \( E \), say \( E' = \pm [e_{j_1} e_{j_2} \ldots e_{j_{n-k}}] \), where the sign is chosen in such a way that

\[
[EE'] = [e_1 e_2 \ldots e_n]
\]

is the “absolute unit”.

The standard notation for the complement of \( E \), introduced and used by Grassmann, is \( |E|^9 \).

For any quantity \( A \) of \( k \)-th level, say \( A = \alpha_1 E_1 + \ldots \alpha_l E_l \) (with \( E_1, \ldots, E_l \) units of the \( k \)-the level) the complement is defined by linear continuation:

\[
|A| = \alpha_1 |E_1| + \ldots \alpha_l |E_l|
\]

Using the later notation \( \wedge \) for Grassmann’s combinatorial/exterior products, the complement of \( E = e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k} \) (where the \( e_{i_i} \)'s are pairwise distinct) is given by

\[
|E| = \text{sg}(i_1, \ldots, i_k, j_1, \ldots, j_{n-k}) e_{j_1} \wedge \ldots \wedge e_{j_{n-k}}
\]

where \( \{i_1, \ldots, i_k, j_1, \ldots, j_{n-k}\} = \{1, \ldots, n\} \) and \( \text{sg}(\ldots) \) denotes the sign of the permutation (\( \ldots \)).

Grassmann studied the operation

\[
|:\wedge^k V \longrightarrow \wedge^{n-k} V
\]

in quite some detail. Among others he showed:

\[
||A| = (-1)^{k(n-k)} A \quad \text{(Grassmann, 1862, art. 92)}
\]

In particular for \( n \) uneven \( ||A| = A \), while for \( n \) even \( ||A| = (-1)^k A \). Up to sign, the complement of the Grassmann complement is the original quantity. After the turn to the

---

9a... nenne ich ‘Ergänzung von \( E \’) diejenige Grösse \( E' \), welche dem kombinatorischen Produkte \( E' \) aller in \( E \) nicht vorkommenden Einheiten gleich oder entgegengesetzt ist, je nachdem \( [EE'] \) der absoluten Einheit gleich oder entgegengesetzt ist” (Grassmann, 1862, p. 57, art. 89).
20th century this would be considered as a kind of duality and $A$ was accordingly called the dual complement (“duale Ergänzung”), for example in (Pauli, 1921, §12); see section 2.2.

For the Grassmann complement of a product, $A \in \wedge^k V, B \in \wedge^l V$ (and $\dim V = n$) our author showed

$$||AB|| = ||A||B||$$ (5)

if $k + l < n$ (Graßmann, 1862, art. 94). Using his definition of the combinatorial product for $k + l > n$ by factorizing out products $[e_1 \ldots e_n] \equiv 1$ Grassmann extended the products and the complements to the general case. In this constellation he spoke of a “regressive product”, in contrast to a “progressive product” in the case $k + l \leq n$, and found that (5) can be extended to the general case (Graßmann, 1862, art. 97) and to an arbitrary number of factors (art. 98). In particular for extensive quantities $a, b \ldots f$ of the first level (for us, elements of $V$)

$$|[a|b|\ldots|f]| = |ab|\ldots|f|.$$ (6)

Using the dual complement and the external product Grassmann finally defined the interior product of two extensive quantities $A, B$ (of any level) as

$$[A|B]$$ (Graßmann, 1862, art. 138).

The level of the interior product is $n + k - l$ for $l > k$ and $k - l$ otherwise. The interior product of quantities of the same level is thus a numerical quantity (“Zahlgrösse”, art. 141). For $k = l = 1$ and $n = 3$ it coincided with Hamilton’s scalar product of vectors; for general dimension $n$ Grassmann had his version of what later would be called a positive definite bilinear form, or Euclidean scalar product.

1.3 Vector operations in $\mathbb{R}^3$ from Grassmann’s perspective

Grassmann’s theory of extensive quantities was not immediately absorbed by mathematicians and/or physicists. As a symbolical tool for mathematical physics the vectorial calculus in three-dimensional space arising from Hamilton’s was a strong competitor during the last third of the 19th century. But the reception of Grassmann’s theory was not exceptionally slow if one compares it with other general mathematical theories (Rowe, 1996; Tobies, 1996).11

Early on, i.e. already in the years 1844ff., also Hamilton insisted on the usefulness of the non-commutative product of the vector part of quaternions, inherited from the product structure of the quaternions. What he called the vector product stood in close relation to Grassmann’s complement of the exterior product of two extensive quantities in 3-space. His (commutative) scalar product was equivalent to a special case of Grassmann’s interior product. At the turn to the 1880s Josia W. Gibbs was one of the influential authors who separated the 3-dimensional vector calculus from the quaternionic calculus. The

10(Crowe, 1967; Reich, 1996)

11See, in particular (Rowe, 1996, p. 132).
other one was Oliver Heaviside, see (Crowe, 1967, chap. 5). Gibbs knew Grassmann’s approach well and was aware of the relation between the vector product in the sense of Hamilton and the (dual complement of the) exterior product for three-dimensional “extensive quantities”.

Here the special type of Grassmann duality in 3-dimensional space, in short $V \cong \mathbb{R}^3$ endowed with the Euclidean interior product and “units” (orthonormal basis vectors) $e_1, e_2, e_3$ came into play. In terms of Grassmann’s exterior/combinatorial product and his dual complement the vector product $w$ of $u, v \in V$ is nothing but

$$w = u \times v = | [u \, v] |$$

Grassmann knew of course the relation – valid in all dimensions – which we would write as mappings

$$| : \wedge^{n-1} V \xrightarrow{\cong} V \quad | : [e_1 \ldots e_{n-1}] \mapsto e_n \text{ etc.} \quad (8)$$

(with etc. indicating that cyclical permutations of indices are permitted) and could easily use it for introducing an alternating product in 3-dimensions. For an orthonormal basis this implies in particular the identity $e_1 \times e_2 = | [e_1 e_2] | = e_3$ etc.

In the last third of the 19th century a varied scene of vector algebra and analysis developed among mathematicians and physicists. In this heterogeneous milieu Grassmann’s exterior product of vectors (his “extensive quantities of 2-nd level”) in 3-space were often assimilated to vectors, often suppressing the duality operation $| \text{mediating between the two types of quantities. But also in the physics tradition a distinction between the two was realized to be important. The reason was that point inversions, which became important in crystallography, have different effects on elements in $V \cong \mathbb{R}^3$ and in $\wedge^2 V$. In the first case an extensive quantity changes sign, in the second case it does not (corresponding to the eigenvalues of an inversion $-1$ respectively $+1$ in later terminology).

In his Kompendium der theoretischen Physik (1895/96) Woldemar Voigt introduced the terminology of a polar vector for an element in $V \cong \mathbb{R}^3$ and axial vector for one arising as the complement of an external product, i.e. an element of $| \wedge^2 V$ (where, of course, the notation mixing Grassmann’s symbol $|$ with a modern one for the alternating product is mine). From a physical point of view the most important elementary case of an axial vector was a “couple of vectors” $(u, v)$ acting on a rigid body. The combined action of a couple (a rotational momentum in later terminology) could best be symbolized as $u \times v$ or $| [uv] |$, respectively. This terminology entered the survey article on “Basic concepts of geometry” in Enzyklopädie der Mathematischen Wissenschaften, written by Max Abraham (Abraham, 1901, p. 6ff.). It can thus be considered as part of the general knowledge at the turn to the 20th century.

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\[12\] In the draft of a letter to Schlegel in 1888 Gibbs wrote: “My acquaintance with Grassmann’s work was also due to the subject of E. [electricity] & in particular to the note wh (sic!) he published in Crelle’s Jour, in 1877 calling attention to the fact that the law of the mutual action of two elements of currents wh Clausius had just published had been given 1845 by himself. . . . that law is so very simply expressed by means of the external product” cited from (Crowe, 1967, p. 153). Gibbs’ referred her to the original publication (Grassmann, 1845).

\[13\] (Crowe, 1967, chaps. 5–7), (Reich, 1996).

\[14\] See (Abraham, 1901, p. 10, fn. 17) and (Reich, 1996, p. 203).
2 Dual complements in physics of the early 20th century

2.1 Minkowski’s “six-vectors” and their duals in relativistic electrodynamics

The field theoretic formulation of electric and magnetic phenomena due mainly to Michael Faraday and James Clerk Maxwell was a major achievement of physics in the 19th century (Darrigol, 2000). As we have seen already, Gibbs’ interest in Grassmann was partially due to the latter’s first steps in the direction of a vector analytic representation of magnetic and electrical phenomena. In the study of electromagnetism certain aspects of “duality” in the wider sense came up. Electric and magnetic phenomena were a striking, and heuristically successful example for the search of “duality” in sense of natural philosophy from Oerstedt to Faraday and found an expression in the pairing of the electric field strength \(E\) (“electromotive force”) with the magnetic field \(B\) (“magnetic induction” mentally depicted by Maxwell as a system of directed vortices formed by the magnetic fluid) and their excitations \(D\) (“electric displacement”) and \(H\) (“magnetic intensity”). All of them were directed quantities in space, which slowly came to be known as fields.\(^{15}\)

Close to the end of the 19th century the (non-relativistic) Maxwell equations were written in different notations (in components, quaternionic, or vector analysis) and with changing conventions for the proportionality constants. In slightly modified notation (for the field strengths and excitations) and using 3-dimensional vector calculus they can be resumed in a form similar to the one used in (Hertz, 1890), and (Föppl, 1894) (which Einstein knew well).\(^{16}\)

\[
\begin{align*}
\text{curl } E + \frac{1}{c} \frac{\partial}{\partial t} B &= 0, \quad \text{div } B = 0 \quad (9) \\
\text{curl } H - \frac{1}{c} \frac{\partial}{\partial t} D &= J, \quad \text{div } D = \rho \quad (10)
\end{align*}
\]

where \(\rho\) and \(J = (J_1, J_2, J_3)\) denote the charge and current densities respectively and \(c\) the velocity of light. The two equations of (9) are a refined mathematical expression for Faraday’s induction law (Steinle, 2011, p. 449ff.) and the non-existence of magnetic charges, while (10) can be read as a dynamicized version of Oerstedt’s and Ampère’s characterization of the magnetic effects of an electric current (ibid., p. 453ff.) and the existence of charges as the source of electric fields. The quantities \((E, H)\), usually considered as the electric respectively magnetic field strength, and their respective excitations \((D, B)\) are assumed to be proportional,\(^{17}\) with (material dependent) coefficients the dielectric constant \(\epsilon\) and the magnetic permeability \(\mu\), such that

\[
D = \epsilon E, \quad B = \mu H.
\]

Moreover \(\epsilon_0 \mu_0 = c^{-2}\) for the vacuum values \(\epsilon_0, \mu_0\) of the coefficients.

\(^{15}\)(McMullin, 2002; Steinle, 2013, 2016, 2011)

\(^{16}\)Similarly in papers by O. Heaviside in the second half of the 1880s. For details see (Hunt, 1991, p. 125ff.) and (Darrigol, 2000).

\(^{17}\)In the literature a variety of terminology is to be found; often \(D\) is called the electric displacement, \(B\) the magnetic induction. Here both are being considered to be excitations of the corresponding field strengths.
This way of writing the four Maxwell equations suggests a close analogy between the pairs \((E, B)\) and \((D, H)\), expressed in the two sets (9) and (10) of the Maxwell equations. The analogy seemed far away from Grassmann duality, but that changed with the relativistic reformulation and generalization of Maxwellian electrodynamics. The step from the non-relativistic to the relativistic theory of the Maxwell-Lorentz theory is a highly involved story which deserves more interest than can be invested here; in our discussion we focus on an extremely selective aspect, the introduction of duality concepts in the sense of Grassmann.

We enter this story with a short glance at Albert Einstein’s conceptual analysis of the Lorentzian relativity principle in his famous paper on “The electrodynamics of moving bodies” (Einstein, 1905). There he considered, among others, the transformation of an electric field \(E = (E_1, E_2, E_3)\) and a magnetic field given by \(B = (B_1, B_2, B_3)\) from one system of reference \(\mathcal{S}\) to another one \(\mathcal{S}'\) which is in linear uniform motion with respect to \(\mathcal{S}\). An electric charge \(q\) resting in \(\mathcal{S}'\) will be in motion, if considered from the point of view of \(\mathcal{S}\). This leads to magnetic forces induced by the motion of \(q\) with regard to \(\mathcal{S}\) in addition to those of \(B\). Einstein considered the transformation laws for the combined field quantities \((E, B)\) and concluded that

...the magnetic and electric forces have no existence independent of the state of motion of the coordinate system (Einstein, 1905, p. 910)

Einstein thus indicated an underlying unity of the fields \((E, B)\), and similarly \((D, H)\), but he was not able to cast the intended unification into a proper mathematical form beyond the transformation formulas. This was achieved a few years later by Hermann Minkowski.

Minkowski proposed a unified mathematical representation of the (special) relativistic electromagnetic field by two antisymmetric matrices \(F = (F_{ij})\) and \(f = (f_{ij})\). The first one \(F = (F_{ij})\), later on called the Faraday tensor, contained the 6 components of \((E, B)\) in a proper arrangement, and \(f\) combined the components of \((D, H)\), where \(F_{ij} = -F_{ji}\) and \(f_{ij} = -f_{ji}\) (Minkowski, 1908, pp. 356, 168).\(^{18}\) Moreover, Minkowski used units such that \(c = 1\) (at least for the numerical value). Then \(\varepsilon_0 = \mu_0^{-1}\). and the two matrices were proportional,

\[
f = \mu_0^{-1} F. \tag{12}\]

He explained how these matrices operate as antisymmetric bilinear forms on vectors in spacetime (which later became to be known as Minkowski space) \(x = (x_1, \ldots, x_4)\) and \(u = (u_1, \ldots, u_4)\), and how they have to be modified under Lorentz transformations. In other words he used the matrices to represent antisymmetric tensors or alternating 2-forms over spacetime.\(^{19}\) Minkowski neither used the terminology of tensors, nor did he mention Grassmann’s alternating products explicitly. He rather talked about his matrices

\(^{18}\)\(F_{12} = B_3, F_{13} = -B_2, F_{23} = B_1\) and \(F_{j4} = -\sqrt{-1} E_j, \; j = 1, \ldots, 3;\) similarly for \(f\) with \(B_j, E_j\) replaced with \(H_j\) and \(D_j\) respectively. The factor \(-\sqrt{-1}\) is due to Minkowski’s notation of the Lorentz metric with imaginary time components. With the exception of \(E\) Minkowski used different symbols, \(m = \mathscr{M}, \mathfrak{m} = \mathscr{B}\) and \(e = \mathcal{G}\).

\(^{19}\)In short notation \(f(x, u) = \mathcal{T} \cdot f \cdot u\) and \(F(x, u) = \mathcal{T} \cdot F \cdot u\) (Minkowski, 1908, eq. (23) (24), p. 364f.).
as vectors of the second kind. This language may have been inspired by the Grassmann tradition in 3-dimensional space, if not by Grassmann himself. Here one called “axial vectors” like angular velocity, rotational momenta vectors of the second kind. In 4-dimensional space the identification with the original vector space was no longer feasible; accordingly Minkowski’s antisymmetric tensors became to be known as “six-vectors" for the next few years before the tensor terminology took roots with the general theory of relativity.

Minkowski introduced a symbolism of his own for the vector analysis on Minkowski space, extending, e.g., the \( \nabla \) operator of 3-dimensional vector analysis to an operator he called \( \text{lor} = (\partial_1, \ldots, \partial_4) \) which had to be transformed under Lorentz transformations like a vector in Minkowski space. For the second set of the Maxwell equations (10) he got in slightly adapted notation

\[
\text{div } f = s, \quad (13)
\]

in the sense of \( \sum_j \partial_j f_{ij} = s_i \), where \( s \) denotes the 4-current with \( s_k = J_k \) for \( j = 1, 2, 3 \) and, essentially, \( s_4 = \rho \). Moreover we have to keep in mind that he wrote the equation as \( \text{lor } f = \sum_j \partial_j f_{ij} = -s_i \) (Minkowski, 1908, p. 384).

In order to bring the first set of the Maxwell equations (9) into a comparable form, he introduced the dual matrix (“duale Matrix”) \( f^* \) (Minkowski’s terminology and symbolism) of a skew symmetric matrix \( f \) as

\[
f^* = (f^*_{ij}), \quad \text{where } f^*_{ij} = \text{sgn}(ijkl) f_{kl}, \quad (14)
\]

with \( \text{sgn}(ijkl) \) the sign of the permutation. Then the first set of the Maxwell equation became in adapted notation

\[
\text{div } F^* = 0, \quad (15)
\]

which in Minkowski’s own 4-dimensional vector analysis read as \( \text{lor } F^* = 0 \). In the result Minkowski had shown how his new 4-dimensional symbolism could be used to give a unified expression of the Maxwell equations. In order to achieve this he introduced an apparently new type of duality \( A \mapsto A^* \) for skew symmetric matrices \( A \).

But how new was this type of duality? Seen from the perspective of Grassmann’s theory this duality was already inherent in the complement of combinatorial products. One only needed to read Minkowski’s matrices as a pragmatic notation for combinatorial products of 1-forms over Minkowski space considered as an “extensive domain” generated by \( e_1, \ldots, e_4 \) (with respect to an orthonormal reference system \( \mathcal{G} \)). Denoting the 1-forms as \( e^*_i (1 \leq i \leq 4) \) with \( e^*_i(e_j) = \delta^*_i j \), the matrices stand for

\[
F = \sum_{1 \leq i < j \leq 4} F_{ij}[e^*_i e^*_j], \quad f = \sum_{1 \leq i < j \leq 4} f_{ij}[e^*_i e^*_j], \quad (16)
\]

in the more recent Cartan symbolism of differential forms \( F = \sum F_{ij} \, dx^i \wedge dx^j \) and \( f = \sum f_{ij} \, dx^i \wedge dx^j \). Minkowski, however, did not mention Grassmann while he mentioned \(^{20}\) I delete the imaginary factor \( \sqrt{-1} \) used by Minkowski for timelike components in order to formally assimilate the Minkowski metric to a Euclidean signature.
Hamilton’s quaternions but did not find his calculus useful for his purpose (Minkowski, 1908, p. 375, fn.).

Arnold Sommerfeld, an important early promoter of Minkowski’s approach to special relativity (Walter, 2010), explained carefully and with an explicit reference to Grassmann how one could perceive Minkowski’s vectors of the second kind geometrically as a linear combination of “surface elements of content 1” in 4-dimensional spacetime (Sommerfeld, 1910, p. 750). As they have the dimension 6 he introduced the terminology of six-vectors, i.e. elements of $\Lambda^2 M$ in our notation (where $M$ denotes the Minkowski space with coordinates $(x_1, \ldots, x_4)$).$^{21}$ Projecting such a “surface element of content 1” $\varphi$ into the $(x_i, x_j)$-plane gave him the coordinates $\varphi_{ij}$. That allowed him to define the complement (“Ergänzung”) $\varphi^*$ in agreement with Grassmann and with Minkowski by (Sommerfeld, 1910, p. 756)

$$\varphi^*_{ij} = \varphi_{kl} \text{ with } \text{sg}(ijkl) = 1.$$ 

In this way Sommerfeld geometrized Minkowski’s algebraic (matrix) approach to the Maxwell field a step further, brought it into an explicit relation with Grassmann’s concepts, and prepared the way for a generalization to a varying (“curved”) metric on spacetime.

### 2.2 Dual complements in the general theory of relativity

This did not mean that Minkowski/Sommerfeld’s dualization found unanimous approval among physicists. Einstein reproached such a representation of the Maxwell field quantities as “involved and confusing”. In the introduction to his paper on electrodynamics in the general theory of relativity he described Minkowski’s characterization of the electrodynamic field by a “six-vector” and a second dual one “whose components have (...) the same values as the first one, but are distinct in the way the components are associated with the four coordinate axes” (Einstein, 1916). Then the ...

\[ \text{...two systems of Maxwellian equations are obtained by setting the divergence of the first one equal to zero, and the divergence of the other one equal to the four-vector of the electric current. (Einstein, 1916, p. 184)} \]

He went on criticizing:

The introduction of the dual six-vector makes its covariance-theoretical representation relatively involved and confusing. Especially the derivations of the conservation theorems of momentum and energy are complicated, particularly in the case of the general theory of relativity, because it also considers the influence of the gravitational field upon the electromagnetic field (ibid.).

---

$^{21}$According to Sommerfeld, Emil Wiechert had already characterized the magnetic excitation $H$ (here called “magnetische Feldstärke”) as a Grassmannian quantity of the second kind in classical space (Sommerfeld, 1910, p. 750).
Einstein proposed a different perspective. He started from the observation that a derivation of the electromagnetic field tensor $F = (F_{ij})$ from a 4-vector potential $\phi = (\phi_i)$ is always possible, in modernized notation $F = d\phi$. Then Minkowski’s equation (15), the first set of the Maxwell equations, boils down to

$$dF = dd\phi = 0$$  \hspace{1cm} (17)

(with $d$ the exterior differential). Of course Einstein wrote this in the form of 4 component equations (Einstein, 1916, equ. (2a)), but still his equations were easier to grasp and to apply than Minkowski’s.

For the other set of the Maxwell equations Einstein proposed to “stay with the generalization of Minkowki’s scheme”, i.e. to write it as a divergence equation, although now in curved spacetime. In order to do so, he introduced the contravariant tensor density $\mathfrak{F}$ with components

$$\mathfrak{F}^{\mu\nu} = \sqrt{|\det g|} \sum_{\alpha,\beta} g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}.$$  \hspace{1cm} (18)

Then the second set of Maxwell equations became a generalization of Minkowski’s (13),

$$\sum_\nu \partial_\nu \mathfrak{F}^{\mu\nu} = \mathfrak{J}^\mu,$$  \hspace{1cm} (19)

with $\mathfrak{J}$ the electric current density 4-vector (Einstein, 1916, eq. (5)). Using the covariant derivative $\nabla$ and using the Einstein summation convention, this equation can also be written as

$$\nabla_\nu \mathfrak{F}^{\mu\nu} = \mathfrak{J}^\mu.$$  \hspace{1cm} (20)

Einstein’s proposal was widely influential. One might expect that Minkowski’s dualization idea was filtered out with the transition from the special to the general theory of relativity. But this was not the case. In an important part of the literature on GRT dual complements continued to play a role, although a subordinate one in comparison with other, more central, concepts for the theory. Let us just review a sample of three authors, W. Pauli, T. de Donder, and F. Kottler.

**Wolfgang Pauli** included a section of its own on dual complements (“duale Ergänzung”), in the English translation “dual tensors” (Pauli, 1958), in his well-known review of the theories of relativity for *Enzyklopädie der Mathematischen Wissenschaften* (Pauli, 1921, §12). In agreement with his context ($\text{dim}M = 4$) he restricted the discussion to what he called surface and spatial tensors (“Flächen- und Raumtensoren”), i.e. to alternating forms of the second and third level $\xi \in \Lambda^k V = 2, 3$. He formulated them already in a differential geometric setting needed for the general theory of relativity (in later notation $\xi$ in the sections of $\Lambda^2(TM)$). Like Sommerfeld, he introduced the duality concept via normalized orthogonal surface elements.\(^{23}\) For an alternating 2-tensor $\xi$ with

\(^{22}\)Einstein called it a “contravariant $\sqrt{-6}$-vector” (Einstein, 1916, p. 186).

\(^{23}\)“With every surface element (…) in a four-dimensional manifold can be associated another, normal to it, which has the property that all straight lines in the one are perpendicular to all straight lines in the other. Such a surface element is called dual to (the first one) if, in addition, it is of the same magnitude.” (Pauli, 1958, p. 33)
coefficients $\xi^i$ he concluded that the dual $\xi^*$ can be given by the corresponding 2-form with coefficients
\[
\xi^*_{ij} = \sqrt{|\det g|} \xi^{kl} \quad \text{with} \quad (ijkl) \sim (1234),
\]
where $\sim$ signifies transformation by an even number of transpositions (Pauli, 1921, eq. (54b)). The coefficients of the dual 2-tensor can easily be calculated by "lifting of indices" with the metric.

For an alternating 3-tensor $\xi$, on the other hand, he similarly arrived at
\[
\xi^*_{i} = \sqrt{|\det g|} \xi^{jkl} \quad \text{with} \quad (ijkl) \sim (1234),
\]
vice versa the dualization of a vector (field) $\xi$ results in an alternating 3-form
\[
\xi^*_{ijk} = \sqrt{|\det g|} \xi^{l} \quad \text{with} \quad (ijkl) \sim (1234),
\]
This allowed to denote integrations of 2- or 3-forms (in later terminology) over surfaces or spatial domains by expressions using tensor or vector densities, i.e. quantities which transform similar to tensors/vectors but with and additional coefficient $\sqrt{|\det g|}$ (Pauli, 1921, §19).

Dual complements remained for Pauli a subordinate formal tool for specific calculations. Although he mentioned Minkowski’s usage of dual complements in his discussion of the special relativistic Maxwell equations, he clearly sided with Einstein’s preference to do without (Pauli, 1921, §28). In the end, Pauli’s adaptation of Grassmann duality to the differential geometric constellation of general relativity remained without further consequences for his overall presentation of the theory. For other authors of the early 1920s, among them F. Kottler (1922) and T. de Donder (1923), this was different. They explored the possibilities for expressing general relativistic Maxwell theory in the terms of dual complements further.

Théophile de Donder, e.g., adapted Minkowski’s presentation of electrodynamics to the context of Einstein gravity. He considered the Maxwell tensor $M$ with three of its components $M_{\alpha\beta}$ encoding the electric displacement and the other three the magnetic field. He introduced the components $M^*_{\mu \nu}$ (notation by de Donder) of the Faraday tensor as the duals of the functions $M_{\alpha\beta}$ ("les dualistiques des fonctions") similar to Minkowski. Because of the Lorentzian metric $g_{\alpha\beta}$ of Einstein gravity he explained these “duals” by a formula similar to Pauli’s. Rewritten in the light of his commentary his formula boils down to
\[
M^*_{\alpha\beta} = (-1)^{\mu+\nu+1} \sqrt{|\det g|} M^{\mu \nu},
\]
where $\alpha < \beta$ are “the two numbers of the permutation 1,2,3,4, if one suppresses $\mu, \nu \ (\mu < \nu)$” (de Donder, 1923, p. 60). He then introduced an alternating sum of partial derivatives
\[
M^{\beta} = (-1)^{\alpha} \partial_\alpha M^{\beta \alpha},
\]
which would correspond to the dual of a Cartanian (exterior) differential $*dM$ and formulated the Maxwell equations in these terms (ibid. p. 64). In this way de Donder generalized Minkowski’s Grassmann duals to the context of general relativity and applied them in his presentation of electrodynamics theory, but he did not formulate the
Maxwell equations in a form which would boil down to exterior differentials, if expressed in later terminology.

This was different for Friedrich Kottler. He gave dual complements a clearer and important conceptual role in a structural study of Maxwell’s theory. Kottler made a great step towards a foundational study, how far electrodynamics can be formulated without the use of an underlying metrical structure, and at which point metrical aspects come in.\textsuperscript{24} Kottler carved out the dual nature of the two (systems of) Maxwell’s equation by opposing the two 4-dimensional field tensors $F = (F_{ij})$ like in (Minkowski, 1908) and

$$E := H^*,$$  \hspace{1cm} (24)

the dual complement of $H = (H_{ij})$ which was his notation for Minkowski’s tensor $f$.\textsuperscript{25} All three were (and are) alternating covariant tensors.

$F$ and $H$ will here be called the first and the second electromagnetic field tensors, or also Faraday tensors; and $E$ the Maxwell tensor. For $c = 1$ the proportionality (12) in vacuum implies $H = \mu_0^{-1}F$. For $\mu_0 = 1$ the two Faraday tensors become (numerically) equal, $F = H$ and accordingly the Maxwell tensor its dual complement, $E = F^*$.\textsuperscript{26} Because of its role in the source equation (II) of the Maxwell theory, Kottler’s $E$, the Maxwell tensor, is now sometimes called the field excitation and $F$ the field strength.\textsuperscript{27}

By playing the game of dual complementation in the inverse direction with respect to Minkowski, Kottler brought the special relativistic Maxwell equations in a form particularly well adapted to its invariant properties with regard to integration. With $S$ the dual complement, $S = s^*$, of the “covariant” current (i.e. 1-form) $s$ (Kottler, 1922, p. 125), he could rewrite Minkowski’s variant of the vacuum Maxwell equations (15), (13) with $\mu_0 = \epsilon_0 = 1$ in a form which can be stated without change of content as

$$(I) \quad dF = 0, \quad (II) \quad dE = dH^* = S,$$  \hspace{1cm} (25)

where $S$ is a current density 3-form.

Of course Kottler did not use exterior differentials but wrote the equations in terms of sums/differences of partial derivatives (Kottler, 1922, pp. 122, 125), e.g. (I) as

$$\partial_k F_{ml} + \partial_l F_{mk} + \partial_m F_{kl} = 0,$$

similarly for (II). But even so he remarked that the existence of (local) 4-potentials $\alpha$ for the Farad tensor, $F = d\alpha$, and the continuity equation for the 4-current, $dS = 0$, are direct consequences of these equations (Kottler, 1922, p. 127).

As an advantage of this form Kottler (1922, p. 124f.) emphasized that the corresponding integral relations have an invariant and physically informative meaning. If we allow us again to use a modernized notation in terms of integrals of differential forms

\textsuperscript{24}Today this is called a premetric approach to electrodynamics. For a recent study deepening Kottler’s approach see (Hehl, 2016).

\textsuperscript{25}This means $E_{12} = D_3, E_{13} = -D_2, E_{23} = D_1$, $E_{14} = H_1, E_{24} = H_2, E_{34} = H_3$; cf. fn 18.

\textsuperscript{26}Kottler called $F$ “magnetoelectric six-vector” and $E$ as the “electromagnetic six-vector” field tensor (Kottler, 1922, pp. 123, 127).

\textsuperscript{27}E.g. in (Hehl, 2003).
over a 3-dimensional compact submanifold $A$ of Minkowski space, respectively its smooth boundary $\partial A$, Kottler could now write the integral version of the Maxwell equations as:

\[
\begin{align*}
(I') \int_{\partial A} dF &= 0, \\
(II') \int_{\partial A} H^* &= \int_A E = \int_A dE = \int_A S \tag{26}
\end{align*}
\]

The first electromagnetic (Faraday) tensor is source free, while the dual complement of the second tensor, i.e. the Maxwell tensor for proportionality constants set to 1, is sourced by the electric 4-current. Kottler was convinced that (25) can be considered as the archetype (“Urgestalt”) of the Maxwell equations (ibid. p. 129). He took it as a starting point for exploring the foundations of electrodynamics also in general relativity.

He proposed to consider “generalized complements” of alternating covariant 2-tensors $E = (E_{ij})$ and of alternating 3-tensors $S = (S_{ijk})$ defined with regard to any “vector of 4-th level” $e = (e_{1234})$ which “of course may vary from place to place”, i.e. an alternating 4-form $e = e_{1234}dx^1 \wedge \ldots \wedge dx^4$, if we use later notation.\(^{28}\) He defined the generalized complements of $E$ and of $S$ as the alternating contravariant 2-tensors $E^* = (E^*_{ij})$, respectively as contravariant 1-tensor $S^* = (S^i)$ with components given by

\[
\begin{align*}
E_{ij} &= e_{ijkl} E^{* kl}, \\
S_{ijk} &= e_{ijkl} S^l. \tag{27}
\end{align*}
\]

This transformation of an alternating contravariant tensor into an alternating covariant tensor was peculiar for Kottler (neither Grassmann, nor Minkowski or Pauli had considered such a relation). In the following it will be called Kottler complement.

This feature allowed to formulate an abstract version of the Maxwell equations for generalized Faraday tensors $F$ and $E^*$ which live in a pre-metric structure characterized by the data $(M, e)$ on a differentiable manifold $M$ and a volume form $e$ (Kottler, 1922, p. 130). Written in exterior differential notation the equations mimicked (25) closely. (I) remained unchanged, while (II) had to account for corrections by a value given in terms of the volume form; and both acquired a generalized meaning:

\[
\begin{align*}
dF &= 0, \\
dE &= S \tag{28}
\end{align*}
\]

After such an extreme generalization of the Maxwell equations he turned towards explaining why “a metric enters in the usual presentation” of electromagnetism:

The reason lies in the connecting relations (“Verknüpfungsrelationen”) of the two vectors $E$ and $F$ (Kottler, 1922, p. 130).

Such a “connecting relation” could be linear like in (11) or, in general relativity, be established by a metric tensor $g = (g_{ij})$ with volume form $e = \sqrt{|\det g|} dx^1 \wedge \ldots \wedge dx^4$. Then the Kottler complement (27) for alternating 2-tensors became in components, e.g. for $F$ and $F^*$ (Kottler, 1922, p. 132),

\[
F_{ij} = \sqrt{|\det g|} e_{ijkl} F^{* kl}. \tag{29}
\]

\(^{28}\)Kottler used the notation $\epsilon_{1234}$ for $e_{1234}$ which disagrees with the present widespread notation of the “epsilon-symbol”, used also below.
with
\[ \epsilon_{ijkl} = \begin{cases} \text{sgn}(ijkl) & \text{if } (ijkl) \text{ is a permutation of } (1234) \\ 0 & \text{otherwise} \end{cases} \] (30)

In particular for a choice of units such that the numerical value of the permeability of the vacuum becomes \( \mu_0 = 1 \), one can identify \( h = H = F \) (cf. (12)). Kottler’s second Faraday tensor then turned into \( E = F^* \), from which with (29)
\[ F_{pq} = \sqrt{|\det g|} \epsilon_{pqkl} E^{kl}. \] (31)

Lowering indices, Kottler (1922, p. 133) wrote this relation in the form
\[ E_{ij} = \sum_{k,l} \epsilon_{pqkl} (|\det g|)^{-\frac{1}{2}} g_{ik}g_{jl} F_{pq}. \]

In terms of the later Hodge-star operator (31) is nothing but \( F = *E \) and thus \( E = \pm *F \).\(^{29}\) In other words, Kottler’s complement (27) contained what later would be identified as the Hodge duality for alternating 2-forms or 1-forms in a 4-dimensional (pseudo-) Riemannian manifold as a special case, although he introduced it in an, at the outset, premetric approach which presupposed only a (globally given) volume form.

### 2.3 Cartan

Kottler was not the only one to realize that a part of relativistic Maxwell theory could be formulated independent of metrical concepts. Élie Cartan also did so more or less at the same time (1921–1922) when he turned towards studying general relativity from the point of view of differential forms, infinitesimal group operations and connections. This gave him the opportunity to develop his generalized theory of differential geometric spaces, which he called “espaces non-holonomes” (later Cartan spaces), and a peculiar view of Einstein gravity. He read Einstein’s theory in analogy with E. and F. Cosserat’s generalized theory of elasticity which allowed for hypothetical torque (rotational momenta) in elastic media, in addition to the ordinary stress forces. Motivated by this idea Cartan formulated a generalized view of gravity theory which included torsion in addition to the usual curvature known from Riemannian geometry. Later this theory would be called Einstein-Cartan gravity.\(^{30}\)

Cartan knew Grassmann’s work well, he appreciated in particular Grassmann’s combinatorial products. They helped him to formulate his calculus of differential forms, e.g. \( \omega = \sum_i a_i dx_i, \check{\omega} = \sum_i b_i dx_i \) and in particular the alternating product of differential forms,\(^{31}\) which in his notation appeared as, e.g.,
\[ [\omega \check{\omega}] = \sum_{i<j} (a_i b_j - a_j b_i) [dx_i dx_j] \]

\(^{29}\)Upper sign for Riemannian, lower sign for a Lorentzian manifold.

\(^{30}\)For a survey of Einstein-Cartan theory see (Trautman, 2006), for the development of Cartan’s thought (Chorlay, 2010b; Nabonnand, 2016; Scholz, 2019).

\(^{31}\)Cf. (Katz, 1999, 1985).
He also introduced the exterior differential $\omega'$ of a form $\omega$, e.g.,

$$\omega' = \left( \frac{\partial a_i}{\partial x_j} - \frac{\partial a_j}{\partial x_i} \right) [dx_i dx_j].$$

In the following Cartan’s differentials will be written in the more recent form $d\omega$ in place of $\omega'$, while sticking to his notation of the alternating product by square bracket without comma.

Cartan’s described the infinitesimal neighbourhoods (later understood as tangent spaces $T_pM$) of a point $p$ with coordinates $x_1, \ldots, x_n$ by using differential forms $\omega_1, \ldots, \omega_n$ (later understood as the elements of a coframe of $T_pM$) and characterized the geometry in the neighbourhood by infinitesimal group operations which mimicked, in a specific way, the Kleinian approach to geometry with a principal group $G$. This led him to characterize the geometry by a connection in the group, given by a set of differential 1-forms $\omega_{kl}$. The indices $k, l$ indicate the components of the infinitesimal group. As as a whole, i.e. considering the complete matrix $(\omega_{kl})$, this collection can be understood as a 1-form with values in the infinitesimal group (the Lie algebra) of $G$ which served as a generalized rotational group for the geometry. Moreover he also considered the coframe given by the $\omega_j$ as a kind of translational connection, complementary to the rotational connection given by the $\omega_{kl}$. He analysed the deviation of such a structure from the corresponding (global) Kleinian geometry and characterized it by two types of curvatures, the rotational curvature $\Omega_{kl}$ and the translational curvature $\Omega_i$. Both were constructed as differential 2-forms, the second one as a whole, $(\Omega_i)$ vector valued, and the first one, considered as a matrix $(\Omega_{kl})$ with values in the Lie algebra of $G$. This opened up the way to studying a new class of differential geometric spaces, later called Cartan spaces.\(^{32}\) In the papers considered here he considered special orthogonal groups of Euclidean or Lorentzian signature in the dimensions $n = 3, 4$.

Cartan made use of Grassmannian complements at different places in his work, most clearly in his large study *Sur les variétés à connexion affine et la théorie de la relativité généralisée* (Cartan, 1923/1924, 1925).\(^{33}\) He introduced the dualization of alternating products in an exemplary way in a case by case discussion.

In the 3-dimensional case, for example, described by mobile orthogonal but not normalized 3-frames $(e_1, e_2, e_3)$ and the associated metric $g = \text{diag}(g_{11}, g_{22}, g_{33})$, he mentioned in passing that any *bivector* (a Grassmannian quantity of the second level) can be represented in the form

$$\Omega^{13} [e_1 e_3] + \Omega^{31} [e_3 e_1] + \Omega^{12} [e_1 e_2],$$

or just as well by *polar vector* of the same measure ("de la même mesure") (Cartan, 1923/1924, §79, p. 16)

$$\frac{1}{\sqrt{g_{11}g_{22}g_{33}}}(\Omega_{23} e_1 + \Omega_{31} e_2 + \Omega_{12} e_3).$$

\(^{32}\)For a recent mathematical textbook see (Sharpe, 1997), for historical literatur see fn. 30.

\(^{33}\)Reprint of both parts in (Cartan, 1955a), English in (Cartan, 1986).
Clearly this kind of relationship was reciprocal (ibid, §60, p.400). In dimension $n = 4$ there exists an “invariant correspondence” between a vector (field) $\xi^i e_i$ and a “polar trivector”

$$\sum_{i=1}^{4} \epsilon_{ijkl} \xi_i [e_j e_k e_l] \quad \text{(where } j < k < l),$$

where the symbol $\epsilon_{ijkl}$ is used like in (30); similarly also between “a system of bivectors” $\xi^{ij} [e_i e_j]$ and the system of “polar bivectors”

$$\sum_{i<j} \epsilon_{ijkl} \xi_{ij} [e_k e_l] \quad \text{(with } k < l).$$

We have seen above that in other parts of the literature such Grassmann-type reciprocal relationships were called dual complements. Although it was not Cartan’s terminology we will call them so in the sequel. For Cartan’s interpretation of the Einstein tensor, for his generalizations of the theory of gravity, and in the discussion of the relativistic Maxwell equation dual complements turned out to be of central importance.

When Cartan started studying Einstein’s theory he looked for an analogy to the generalized theory of elasticity proposed by the brothers E. and F. Cosserat. Cartan developed a peculiar interpretation of the Einstein equation which in ordinary symbolism of differential geometry is

$$G = \kappa T,$$

(32)

with the Einstein tensor $G = \text{Ric} - \frac{R}{2} g$ formed from curvature expressions of spacetime, the stress-energy-momentum tensor $T$, and the gravitational constant $\kappa$ of Einstein theory. Cartan considered the right hand side of (32) as a quantity describing the dynamical state of matter (“quantité mouvement mass” (Cartan, 1922b, p. 437, first emphasis ES, second emphasis in the original)). The equation itself seemed to allow him an identification of a geometrical curvature quantity with a matter quantity.

As already indicated, in Cartan’s approach the curvature was expressed by a set of differential 2-forms $\Omega_{ij}$ the indices of which indicate components of the “infinitesimal rotations” (the Lie algebra of the special orthogonal group). He rewrote the Einstein tensor as an $(n - 1)$-form in his curvatures (rotational and translational) for $n = 3$ and $n = 4$. Here we need not discuss the derivation of the curvature forms and Cartan’s way

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34 Cartan wrote the sum term by term with explicit signs (Cartan, 1923/1924, §60, p. 401).
35 See (Brocato, 2009; Scholz, 2019).
36 $\text{Ric}$ denotes the Ricci tensor $g$ the metric, $R$ the scalar curvature of spacetime, $\kappa = 8\pi G_N$ with the Newton constant $G_N$ and $c = 1$.
37 On sait que, dans la théorie de la relativité généralisée d’Einstein, le tenseur qui caractérise complètement l’état de la matière au voisinage d’un point d’Univers est identifié à un tenseur faisant intervenir uniquement les propriétés géométriques de l’Univers au voisinage de ce point” (Cartan, 1922b, p. 437, first emphasis ES, second emphasis in the original).
38 For an analogue to the Einstein equation in dimension $n = 3$ he considered the rotation group of Euclidean space, for $n = 4$ the one of Minkowski space, i.e. the Lorentz group.
of writing the Einstein tensor. The stress-energy tensor $T = (T^i_j)$ of Einstein theory is a vector valued 1-form (for both $n = 3, 4$). A transformation from $(n - 1)$-forms to 1-forms was possible, in principle, by means of a Grassmann type duality. Cartan, however, did not care about such a move; he found it more natural to express the dynamical state of matter by a 2-form ($n = 3$) or 3-form ($n = 4$) respectively, which assigns forces and rotational momenta to a surface or spatial element of spacetime. But the infinitesimal rotations and translations (the values of the curvature forms $(\Omega_{kl})$ and $(\Omega_i)$) had to be transmuted into vectors or momenta, which could be interpreted as the stress force or torque acting on the respective surface element.

For dimension $n = 3$ this was easy to achieve; infinitesimal rotations in dimension $n = 3$ were represented in vector analysis by vector products anyhow; they could be associated to a vector (at least implicitly by a Grassmann duality as we have seen in section 1.3). On the other hand, the infinitesimal translations of $(\Omega_i)$ could be transmuted into a “bivector”, a Grassmannian quantity of the second level. Then they could be interpreted as the geometrical expression of rotational momenta. This was the background for Cartan’s, at first sight quite surprising, choice to call the translational curvature of his new type of geometry torsion, which he announced already in his Comptes Rendus notes:

To any closed infinitesimal loop there is generally an associated translation; in this case one can say that the given space differs from Euclidean space in two respects: 1. by a curvature in the sense of Riemann, which is expressed by the rotation; 2. by a torsion which is expressed by the translation (emphasis in the original). (Cartan, 1922c, 594f.)

For $n = 4$ the case was more complicated. In a long discussion Cartan found that there arose problem for the enhancement of the curvature by a translational component (Cartan, 1923/1924, §78–§83). In fact, it would be inconsistent with the Maxwell-Lorentz theory of electrodynamics if one did not assume an additional term for the mass-energy (“quantité mouvement mass”). This led him to adding an expression to the the energy momentum tensor (respectively 3-form), which much later would become known as a spin term.

We need not go here to the heart of the problem Cartan found in the Maxwell-Lorentz theory for his generalized theory of gravity. For our purpose it will suffice to shed a glance at his discussion of the relativistic Maxwell equation. To start with, he rewrote special relativistic Maxwell theory coherently in terms of differential forms and exterior derivations. In particular, the (first) Faraday tensor and the Maxwell tensor were represented

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39 For more details see (Scholz, 2019). Error note for the journal version of this article: All capital $\Omega$s on pp. 50–52 have to be replaced with small $\omega$s, also those on the right hand side of eqs. (4),(5). Moreover in in eq.s (15, 16, 18, 19, 20, 23) – not in (22) – capital $\Omega$s with one lower index have to be replaced with small $\omega$s, e.g., $\Omega_k$ should be $\omega_k$ etc. The arXiv version is correct.

40 “Dans les cas général où il y a une translation associée à tout contour fermé infiniment petit, on peut dire que l’espace donné se différencie de l’espace euclidien de deux manières: 1° par une courbure au sens de Riemann, qui se traduit par la rotation; 2° par une torsion, qui se traduit par la translation” (Cartan, 1922c, 594f.).

41 See the commentary by A. Trautman in (Cartan, 1986) and with regard to history (Scholz, 2019).
as 2-forms $\Omega, \overline{\Omega}$ with

$$\Omega = \sum_{(ijk)} B_i \left(dx_j \wedge dx_k\right) + \sum_i E_i \left(dx_i \wedge dt\right), \quad (33)$$

$$\overline{\Omega} = \sum_{(ijk)} D_i \left(dx_j \wedge dx_k\right) + \sum_i H_i \left(dx_i \wedge dt\right), \quad (34)$$

where $(ijk)$ indicates the summation over all cyclic permutations of $(123)$, and the charge/current density $S$ as a 3-form

$$S = \rho \left(dx_1 \wedge dx_2 \wedge dx_3\right) - \sum_{(ijk)} J_i \left(dx_j \wedge dx_k \wedge dt\right)$$

Taking account of his sign conventions the Maxwell equations came out as

$$d\Omega = 0, \quad d\overline{\Omega} = -4\pi S,$$  \quad (35)

that is like Kottler’s equation (28), up to sign and with units such that numerically $\mu_0 = 4\pi$ (Cartan, 1923/1924, §80).

Because of his method of orthonormal frames it was not too difficult for Cartan to import Maxwell theory into a general relativistic framework. Assuming $n = 4$ differential forms $\omega^0, \ldots, \omega^n$ which at every point $p$ represented an orthonormal Lorentzian co-frame to some basis (frame) $e_0, \ldots, e_3$ of the infinitesimal vector space at $p$. The Faraday and Maxwell tensors of special relativity turned into 2-forms on the generalized space, which in slightly adapted notation were

$$\Omega = \sum_{i<j} H_{ij} \left[\omega^i \wedge \omega^j\right], \quad \overline{\Omega} = \sum_{i<j, k<l} \epsilon_{ijkl} H^{ij} \left[\omega^k \wedge \omega^l\right], \quad (36)$$

where $i, j, k, l \in 0, \ldots, 3$ and $\epsilon_{ijkl}$ like in (30). Apparently Cartan understood the lifted indices for $H_{ij}$ like in the Ricci calculus. In his orthonormal frames the metric is diagonalized of Lorentzian signature, $g = \text{diag} (+1, -1, -1, -1)$. The lifting of indices thus has only consequences for the sign of the coefficient. $\overline{\Omega}$ denoted thus a Grassmann type dual complement written in Cartan symbolism. He concluded that with an adapted definition of the electric current,

$$S = \sum_{ijkl} \epsilon_{ijkl} J^i \left[\omega^j \wedge \omega^k \wedge \omega^l\right],$$

the Maxwell equations (35) are still valid. Moreover

$$\ldots \text{they don’t make use of the affine connection of the universe (Cartan, 1923/1924, §82, p. 20).}$$

So Cartan was well aware of the fact that Maxwellian electrodynamics depends only weakly on the metric of spacetime, and not at all on the affine connection. The equations

\textsuperscript{42}Cartan spoke of “système de référence de Galilée” (Cartan, 1923/1924, §82).
themselves can be formulated in terms of exterior differentials; they do not depend on
the metric at all. The interrelation between the Faraday tensor and the Maxwell tensor
can be expressed in terms of a Grassmann type duality, later called Hodge duality;
it thus depends indirectly on metrical concepts of spacetime. Moreover we have seen,
how Grassmann type complements were used by Cartan also for other purposes in his
investigations of gravity.

3 The birth of Hodge duality

It is now time to turn towards the main topic of our historical report, Hodge theory.
William V.D. Hodge (1903–1975) started the study of algebraic surfaces and varieties
at the end of the 1920, at a time when modern methods of algebra were increasingly
introduced into algebraic geometry and the combinatorial topology (analysis situs) of
manifolds and complexes. He became a central figure for enriching this field by gener-
alizing Riemann’s transcendent methods in the study of compact Riemann surfaces to
higher dimensions. Before we look at his work we have to shed a short glance at the
background knowledge from which he started.

3.1 Background: Riemann’s theory of Abelian integrals

Riemann had taken advantage of the fact that the real and imaginary parts \( udz \) and
\( vdz \) of a holomorphic (or meromorphic) form \( \omega = udz + v dz \) (\( z \) complex coordinate)
are harmonic, i.e., the coefficient functions satisfy the partial differential equation \( \Delta u = 0 \), \( \Delta v = 0 \). In his influential paper on the theory of Abelian functions (Riemann, 1857)
he characterized the (real) harmonic forms on a Riemann surface \( \mathcal{F} \), after cutting the
latter along a number of curves into a simply connected surface, by boundary value
problems and proposed to solve them by applying the Dirichlet principle. At his time
this method was not yet mathematically well developed, and it took half a century and
the work of many mathematicians to fix the loose ends. At the end point of this story
stands Hilbert’s vindication of the Dirichlet principle (Hilbert, 1904) and Weyl’s famous
book on the Idee der Riemannschen Fläche (Weyl, 1913).\(^{43}\) A crucial outcome of his
approach were deep insights into the interrelation of the topology of a compact Riemann
surface \( \mathcal{F} \), its complex structure, and its algebraic description. In particular Riemann
studied closed curves (by later authors called “retrosections”) which neither partially nor
in total bound a part of the surface and found that their maximal number is even, \( 2p \)
(Riemann, 1857, §3, p. 104). He called \( 2p + 1 \) the “order of connectivity” of \( \mathcal{F} \). Later
one would speak of \( b_1 = 2p \) as the first Betti number of the surface.

He studied integrals \( w_j = \int_\gamma \omega_j \) over a curve \( \gamma \) (closed or not) with holomorphic \( \omega \),
so-called Abelian integral of the first kind (of the second and third kind for meromorphic
differentials without, respectively with poles of first order), and found that there are only
finitely many linearly independent ones \( w_1, \ldots, w_q \) on a given surface \( \mathcal{F} \) (Riemann, 1857,

\(^{43}\)The problems of this approach and its solution have been discussed historically at many places, see in
particular (Monna, 1975), (Bottazzini, 2013, chaps. 5, 6, 7.7), (Gray, 2015, chaps. 16–18).
Due to his usage of the Dirichlet principle for the real and imaginary parts of Abelian integrals \( w = u + iv \), the maximal number of independent holomorphic forms turned out to be determined by the structure of dissection of \( \mathcal{F} \) into a simply connected surface. It agreed with the number identified in the analysis situs study of the surface, \( q = p \). Riemann considered such an integral \( w_j \) as a multivalued function on \( \mathcal{F} \), the values of which differ by (integer) linear combinations of \( a_{ij} = \int_{\gamma_i} \omega_j \), \( i = 1, \ldots, 2p \), the "Periodicitätsmoduln" Riemann (1857, p. 105 etc.) later simply called periods.

He also showed that a compact Riemann surface can be represented as a complex algebraic curve with an algebraic equation \( F(z, w) = 0 \) in two complex variables \( z, w \) in various ways. He argued that two representations of this type can be transformed into another algebraically (by what later would be called birational transformations) and showed that for a curve of order \( n \) with \( w \) simple branch points (and no ones of higher order) the number \( p \) of Abelian integrals satisfies the relation

\[
p = \frac{w}{2} - (n - 1)
\]

and can thus be read off from the algebraic representation of the Riemann surface as a curve (Riemann, 1857, §7, p. 114).

This opened the way for an algebraic study of compact Riemann surfaces, viz. complex algebraic curves embedded in the complex projective plane. It was pursued further by A. Clebsch, M. Noether, A. Brill and many other mathematicians. These authors were able to replace Riemann’s analytic ("transcendent") methods by algebraic ones and characterized in particular the Abelian differentials/integrals by algebraic expressions ("adjoint polynomials/curves"). In this context Clebsch introduced the name genus ("Geschlecht") for the algebraically determined number (37). For him and mathematicians in his circle it must have been clear (already in the 1860s and 1870s) that this number was a birational invariant and that it characterized the analysis situs (topological) property of the corresponding Riemann surface. F. Klein explained this connection between different aspects of Riemann’s theory to a broader (mathematical) audience (Klein, 1882).

The situation became much more complicated, when the attempt was made to apply Riemann’s integrated approach to the study of complex algebraic surfaces. Clebsch (1868) introduced the genus \( p \) of an algebraic surface \( \mathcal{S} \) possessing only simple singularities as the number of linearly independent double integrals of a special type, defined algebraically. Cayley (1871) attempted to find a formula analogous to (37) for the genus of a surface, but realized that the number so defined did not always agree with Clebsch’s value \( p \). M. Noether therefore called it the arithmetical genus \( p_a \) in contrast to Clebsch’s, which he now called geometrical genus \( p_g \). Both types of genus turned out to be birational invariants.\(^{47}\)

\(^{44}\)The corresponding differential forms \( \omega_1, \ldots, \omega_q \) are independent if no linear combination of them is a total differential \( dW \) of a univalued function \( W \) on \( \mathcal{F} \).

\(^{45}\)See also the annotation (2) by H. Weber in 1876, the editor of the first edition of Riemann’s Werke.

\(^{46}\)Lê (2020) also emphasizes this role of Klein’s book, but imputes that for Clebsch and the mathematicians of his generation the interrelationship between the different aspects of Riemann’s theory was unclear or doubtful.

\(^{47}\)See (Brigaglia, 2004b, p. 313f.).
About this time Betti (1871) indicated how Riemann’s analysis situs concepts, in particular the order of connectivity, can be generalized to higher dimensions. For a manifold of dimension \( n \) he introduced connectivity numbers \( b_k \) (later called Betti numbers) for any dimension \( 1 \leq k \leq n \). But it remained completely unclear whether or how the different types of genera of a complex algebraic surface \( \mathcal{S} \) (with real dimension \( \dim_{\mathbb{R}} \mathcal{S} = 4 \)) had something to do with these topological invariants (the \( b_k \)). From now on the different aspects of Riemann’s integrated research program combining analysis situs, algebraic geometry, differential forms and their integrals (even specified to the case of holomorphic or meromorphic form), which to a certain degree even tied up with the differential geometry of manifolds,\(^{48}\) started to be developed in different research traditions with weak overlap or interchange. This remained essentially so for about half a century.

### 3.2 Hodge’s first steps towards generalizing Riemann’s theory of Abelian integrals

When Hodge entered research in mathematics, several of the fields treated by Riemann had been developed further and stood at the brink of becoming mathematical subdisciplines of their own. Hodge assimilated a wide area of literature with different approaches to what would become the geometry and topology of the 20th century.\(^{49}\) Riemannian differential geometry and its generalizations got an immense push with the rise of the general theory of relativity.\(^{50}\) On the other hand, the algebraic geometry of complex varieties had accumulated a rich corpus of insights in particular established by the Italian school of geometers; although it was not always based on reliable foundations.\(^{51}\)

The analysis situs study of manifolds and complexes, at beginning of the new century often called combinatorial topology, was being reshaped by modern algebraic concepts and was turning into algebraic topology.\(^{52}\) The integration of differential forms over submanifolds, in particular closed ones (so-called “cycles”) had been introduced in higher dimensions by E. Picard, H. Poincaré and E. Cartan. By generalizing the theorem of Stokes these authors realized that the integral \( \int_c \omega \) of a differential form \( \omega \) of degree \( k \) with vanishing derivative (\( d\omega = 0 \)), taken over a closed submanifold \( c \) with \( \dim c = k \) is the same for any homologically equivalent submanifold \( \tilde{c} \), \( \int_{\tilde{c}} \omega = \int_c \omega \). Thus the periods \( \int_{\tilde{c}} \omega \) of such a differential form with respect to a generating system \( c_1, \ldots, c_m \) of the \( k \)-th homology could be considered as belonging to the analysis situs of the manifold, long before the idea of cohomology theory was shaped. During the 1920s E. Severi and S. Lefschetz were the main protagonists of using this type of analysis situs for the study of algebraic surfaces.

At the turn to the 1930s de Rham (1931) made in his PhD dissertation a decisive step forward towards establishing a (dualizing) analogy between the forming the boundary \( \partial c \) of a (differentiable) complex \( c \) and the exterior derivative \( d\omega \) of a \( k \)-form \( \omega \).\(^{53}\) He

\(^{48}\)Cf. (Scholz, 1980).
\(^{49}\)See, e.g., the literature cited in (Hodge, 1932, 1934a).
\(^{50}\)See among others (Bourguignon, 1992; Bottazzini, 1999; Scholz, 1999b, 2019; Reich, 1994, 1992)
\(^{51}\)(Brigaglia, 2004a, b; Schappacher, 2015)
\(^{52}\)See (James, 1999a, b; Epple, 1999; Herreman, 1997, 2000; Scholz, 1999a; Volkert, 2002) and the respective contributions to this volume.
\(^{53}\)For de Rham and his environment see (Chatterji, 2013).
introduced homological terminology into the treatment of differential forms: \( \omega \) is *closed* ("fermée") if \( d\omega = 0 \) (written by him in Cartan’s notation of the time \( \omega' = 0 \)); it is *homologue zero*, \( \omega \sim 0 \) (notation used by de Rham), if it is the exterior differential of another form \( \tilde{\omega} \) of degree \( k - 1 \), \( \omega = \tilde{\omega}' \) etc. (p. 176). He showed:

- Every closed form is homologue to a linear combination of finitely many "formes élémentaires" (de Rham, 1951, p. 180).
- A closed form with all periods zero is itself homologue zero, \( \omega \sim 0 \) (p. 185).
- Given \( m \) homologically independent \( k \)-cycles \( c_j \) and \( m \) rel values \( a_1, \ldots, a_m \) on can find a closed \( k \)-form \( \omega \) such that \( \int_{c_j} \omega = a_j \) for \( 1 \leq j \leq m \) (p.186).
- And finally, the maximal number of homologically independent closed \( k \)-forms \( q_k \) coincides with the \( k \)-th Betti number \( q_k = p_k \) (p. 187).

In this sense de Rahm established the basic insights into what later would become the (de Rham) cohomology of differentiable manifolds, although he stopped short of introducing the cohomology groups of differential forms themselves.\(^{54}\)

In the late 1920s Hodge started to study the periods of rational forms on algebraic varieties. He joined and expanded the research program of E. Severi and S. Lefschetz, who promoted a (re-) integration of analysis situs methods into the theory not only of complex algebraic surfaces but also of algebraic varieties \( V \) of higher dimensions. In these attempts integrals \( \int_c \omega \) like above were studied, but here with rational \( k \)-forms \( \omega = R(x_1, \ldots, x_k)dx_1 \ldots dx_k \) which lead to finite integrals over analytically defined sub-varieties or even complexes \( c \) ("complexes analytiques") of dimension \( \text{dim}_\mathbb{R} c = k \). Their integrals, called of the *first kind* like in the case of algebraic curves, were expected to give important information on the variety \( V \); but their study led to much more complications than in Riemann’s context.\(^{55}\) Already for \( n = 2, k = 2 \) it was unclear whether there are integrals of the first kind all periods of which are zero. Severi asked for their number and neither he nor Lefschetz expected that it might be zero by general reasons.\(^{56}\)

Using analytic considerations, i.e. by using “transcendent” methods rather than relying on exclusively algebraic ones, Hodge concluded that this number is zero; but with the present analytic tools at hand the proof was difficult (Hodge, 1930). Only after an initially strong opposition by Lefschetz Hodge’s claim was accepted (Atiyah, 1976, p. 175f.). For Hodge the dissertation of de Rham came just at the right time for evolving his approach. From 1931 onward he could build upon the the methods of general differential forms and concentrate on specifying them for the case of *harmonic forms*, in order to generalize Riemann’s analytic theory to higher dimensions. But how could one characterize harmonic forms for complex algebraic varieties of dimension \( n > 1 \)? If one wanted to use the Beltrami-Laplace operator \( \Delta \) for characterizing harmonic functions \( u \),

\[
\Delta u = \frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g} g^{ij}) \partial_j u = 0 ,
\]

\(^{54}\)See (Katz, 1985, 1999; Massey, 1999).

\(^{55}\)On ne sait pas grand’ chose sur ces intégrales, mais il est probables qu’elles ont une importance considérable pour la théorie algébro-arithmetique de \( V_{\mathbb{C}} \)" (Lefschetz, 1929, p. 55). Lefschetz considered the number \( p_g^{(k)} \) of independent \( k \)-fold integrals of the first kind as a generalization of Clebsch/Noether’s geometric genus \( p_g \) of curves.

\(^{56}\)(Lefschetz, 1921, p. 350), (Lefschetz, 1929, p. 28ff.).
a Riemannian metric \( g = (g_{ij}) \) on the manifold had to be presupposed.\(^{57}\)

In his first sketches of his theory and the announcement of it Hodge (1932) avoided the problem and argued with “euclidean n-cells”, i.e. he assumed the cell complexes studies as embedded in a Euclidean space. But already in the paper (Hodge, 1934a) written in 1932, although published only two years later, and in an outline of his theory (Hodge, 1933b) he explained how a complex algebraic variety \( V \) of dimension \( m \) could be endowed with a Riemannian metric. He assumed that \( V \) can be given in a singularity free form and embedded in a projective space of sufficiently high dimension \( r \).\(^{58}\) He then used a method by G. Mannoury for endowing the projective space \( P^r(\mathbb{C}) \), and with it also the subvariety \( V \), with a Riemannian metric.\(^{59}\) Mannoury proposed to embed the projective space \( P^r(\mathbb{C}) \) in an Euclidean space of even higher dimension \( s = (r + 1)^2 \) and to use the induced metric of the embedding \( P^r(\mathbb{C}) \hookrightarrow \mathbb{E}^s \) in the Euclidean space of dimension \( s \).\(^{60}\)

Hodge apparently found it clear that different, but birationally equivalent presentations of \( V \) ought to lead to the same invariants for the algebraic manifold. At least he did not hesitate to speak of the submanifold \( M \) which corresponds to the points of \( V \) as the Riemannian manifold \( M \) of \( V \) and considered \( M \) as a higher dimensional analogue of the Riemann surface \( F \) of an algebraic curve. In this expectation Hodge turned out to be right, but it took a long way to go until this hope could be justified. In his look back Atiyah emphasizes the problem by talking of Hodge’s “apparently strange idea of introducing an auxiliary metric into algebraic geometry”, which could be vindicated only much later.\(^{61}\) The problem is:

For Riemann surfaces the complex structure defines a conformal structure and hence the Riemannian metric is not far away, but in higher dimensions this relation with conformal structures breaks down and makes Hodge’s success all the more surprising. (Atiyah, 1976, p. 187f.)

It thus was a daring move combined perhaps with a visionary perspective and a bit of luck, which allowed Hodge to anticipate crucial insights into the role which harmonic differential forms and their integrals could play as an intermediary between topology and complex algebraic geometry.

\(^{57}\)In modernized notation the formula boils down to \( \Delta u = \nabla_i \partial^i u = 0 \) with \( \nabla \) the covariant derivative associated to \( g \).

\(^{58}\)The general theorem has not yet been proved, but we shall assume its truth, or better, we shall confine our attention to varieties which can be transformed into varieties without multiple points, and we shall suppose that \( V \) is a variety of \( m \) dimensions without singularities, lying in a complex projective space of \( r \) dimensions \((0, \ldots, z_r)\) (Hodge, 1933b, p. 304).

\(^{59}\)(Mannoury, 1900)

\(^{60}\)With \( X_h, X_{hk} = X_{kh}, Y_{hk} = -Y_{kh}, \ (h, k = 0, \ldots, r) \) coordinates of \( \mathbb{E}^s \), one sets \( X_h = \sqrt{2z_h z_{\bar{h}}}, \ X_{hk} = z_h z_{\bar{k}} + z_k z_{\bar{h}}, \ Y_{hk} = i(z_h z_{\bar{k}} - z_k z_{\bar{h}}) \), where the projective coordinates are constrained by \( \sum_{j=0}^{r} |z_j|^2 = 1 \) (Hodge, 1933b, p. 304).

\(^{61}\)According to Atiyah (1976, p. 187f.) a first vindication of Hodge’s application of his theory to algebraic manifolds resulted from his proof that the decomposition of the space of harmonic forms \( H^r = \sum_{p+q=r} H^{p,q} \) (see below) the dimensions \( h^{p,q} = \dim H^{p,q} \) are invariants of the complex structure of \( V \). An intrinsic definition of the Hodge numbers \( h^{p,q} \) became available only in the 1950s after the introduction of sheaf theory (see sec. 4.1).
3.3 The \(*\)-operation, the Hodge theorem and algebraic surfaces

Early in the 1930s our author announced a central theorem of his new theory and showed how it could applied to the study of algebraic surfaces (Hodge, 1933a).

**Theorem 1** On an orientable Riemannian manifold with Betti numbers \(p_m\) there are exactly \(p_m\) harmonic \(p\)-forms \((1 \leq m \leq n)\).

This was the first version of what later would become known as Hodge’s theorem. The proof followed in two papers in the *Proceedings of the London Mathematical Society* (Hodge, 1934a,b). In these papers Hodge mentioned de Rham’s dissertation and profited from its insight with regard to the dual character of (closed) differential forms and the corresponding cycles in the manifold. He introduced a basis \(\{w_i\}\) of the harmonic forms of degree \(m\), related to a basis of the \(m\)-th homology represented by cycles \(\{\Gamma_j\}\) \((1 \leq i, j \leq p_k)\), by the condition \(\int_{\Gamma_j} w_i = \delta_{ij}\). If we use the later notation \(H_k(M, \mathbb{R})\) for the homology and \(\mathcal{H}^m(M)\) for the harmonic forms (of the first kind) of grade \(k\) on the manifold \(M\), Hodge clearly noticed a duality relation between the two, even though he did not yet use the word. In a follow up paper read in February 1934 he took stock of what was achieved and set out to give “an account of the principles on which the method is based” (Hodge, 1935, p. 249).

If we use an ex-post notation of de Rham’s insight into the relation between “homolog differentials” and “cycles” as a duality relation between homology and harmonic forms we can resume Hodge’s theorem in retrospect as

\[
\mathcal{H}^k(M) \cong H_k(M, \mathbb{R})^*. 
\]  

(39)

In the first paper, (Hodge, 1933a), the general theorem was stated without using the terminology of harmonic forms, while it appeared prominently in the next publications. In 1933 Hodge rather used the language of “linear independent skew symmetric tensors” \(B_{i_1...i_m}\) satisfying two equations (1), (2). Using an abbreviated notation \(B\) for Hodge’s alternating tensor (or form), his equation (1) expressed a vanishing exterior derivative, \(dB = 0\). The second one demanded the vanishing of “contravariant derivative”, i.e. a covariant derivative with lifted index \(\nabla^r B = 0\). Only in the review paper (Hodge, 1935) he introduced an equivalent formulation of equation (2) by expressing it as the vanishing of the exterior derivative of the Hodge dual \(d \ast B = 0\) (see below).64 In the 1933 paper he introduced the terminology “harmonic integrals” and “harmonic forms” only a in the

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62 In Hodge’s formulation, an “analytic construct of \(n\) dimensions which has the topological properties of an orientable absolute manifold ... which has attached to it a Riemannian (positive definite metric)” (Hodge, 1933a, p. 312).

63 (Hodge, 1934a, p. 257) (Hodge, 1934b, p. 90f.)

64 The tensor equations in (Hodge, 1933a, p. 312) (with original equ. numbers) were:

\[
\sum_{r=1}^{m+1} (-1)^{r-1} B_{i_1...i_{r-1}i_{r+1}...i_{m+1}i_r} = 0 \quad (1)
\]

\[
g^{rs} B_{i_1...i_m,s} = 0 \quad (2)
\]
context of double integrals on algebraic surfaces \((n = 4, m = 2)\), (Hodge, 1933a, p. 315ff.).

For a detailed proof of the announced theorem he referred to (Hodge, 1934a); in 1933 he gave only an outline of the argument, but emphasized its importance for understanding the geometric genus \(p_g\) of a (singularity free) algebraic surfaces \(\mathcal{X}\). We cannot go into details of this interesting history; but certain aspects of Hodge’s discussion of algebraic surfaces shed light on our story. Here, for dimension \(n = 4\) and \(m = 2\), he showed that for a skew symmetric tensor now called \(\phi = B_{ij}dx^i dx^j\) satisfying the two equations (1), (2) of his theorem another form \(\phi’\) could be defined, which he called “conjugate” and for which \(d\phi’ = 0\) in addition to \(d\phi = 0\). Up to a differential form \(\omega\) this “conjugate” had the form of a Grassmann dual as it was used in the general relativistic Maxwell equation:

\[
\phi’ = \frac{1}{2} \sqrt{\text{det} gg}^{\alpha \beta} B_{\alpha \beta} dx^k dx^l + c \omega, \quad (40)
\]

where \(\omega\) is independent of \(\phi\) and exact (a “total differential”), \(c\) a constant.

Moreover, also \(\omega\) is a harmonic form of the first kind, and \(\phi\) and \(\phi’\) are the real and imaginary parts of complex differentials of the first kind, of which there are \(p_g\) this and the main theorem showed that the second Betti number is

\[
p_2 = 2p_g + 1. \quad (41)
\]

The topological nature of the geometric genus for surfaces was thus already visible.

Hodge did not stop at this point. Using harmonic forms he showed that the intersection matrix \(A\) of a base of the second homology of the surface is non-degenerate symmetric and the number \(q\) of negative elements in the signature \(\text{sig} A = (p, q)\) (its negative index of inertia) is also \(p_2\) and thus can be expressed in terms of the geometrical genus by \(q = 2p_g + 1\). The upshot of the argument was that in even stronger sense than by (41) “…\(p_g\) is expressed as a topological invariant of the manifold”, namely by the signature of its intersection matrix (Hodge, 1933a, p. 318).

The result may appear unspectacular today; at the time it was not. It was a striking evidence that also for complex dimensions \(n > 1\) the theory of harmonic differential forms (and their integrals) promised further insights into the connection between topological and birational invariants of algebraic varieties. In the words of Atiyah:

This was a totally unexpected result and, when published (8) [(Hodge, 1933a), ES], it created quite a stir in the world of algebraic geometers. In particular it convinced even the most sceptical of the importance of Hodge’s theory, and it became justly famous as ‘Hodge’s signature theorem’.

Twenty years later it played a key role in Hirzebruch’s work on the Riemann-Roch theorem and it remains one of the highlights of the theory of harmonic forms. (Atiyah, 1976, p. 178)

Of course the paper (Hodge, 1933a) stood not alone. It was a whole series of papers in the first half of the 1930s, which taken together “created the stir” alluded to by Atiyah.

\(^{65}\)Hodge wrote this sum term by term, but used Einstein summation convention at other places of the same publication.
3.4 Hodge’s definition of harmonic forms, Hodge duality, and the Maxwell equation

Let us come back to the question how Hodge characterized harmonic forms after 1933. In the resumée paper (Hodge, 1935) he explained his differential geometric calculations in more detail. He defined a harmonic \( p \)-form on a differentiable Riemannian manifold \( M \) of dimension \( m \), as an antisymmetric tensor \( P \) with components \( P_{i_1\ldots i_p} \) which satisfies two conditions which in modernized notation are

\[
\begin{align*}
(I) & \quad dP = 0, \\
(II) & \quad \nabla^j P_{i_1\ldots i_{p-1}j} = 0.
\end{align*}
\] (42)

Hodge formulated them as “integrability conditions” and remarked that they are the same as those stated already in (Hodge, 1933a, eq. (1), (2)). He also added another form of the second condition in terms of the tensor operations used in the general relativistic literature for the Grassmann complement and introduced a tensorial dualization denoted by an upper asterix (Hodge, 1935, p. 260). He soon noticed that the prescription given here was too generous and refined it in a follow up paper by the definition

\[
P^*_{j_1\ldots j_{n-p}} = \frac{1}{p!} \sqrt{g} \varepsilon_{i_1\ldots i_p j_1\ldots j_{n-p}} P^{i_1\ldots i_p},
\] (43)

where \( \sqrt{g} \) was the abbreviation for \( \sqrt{\det g} \) used in the physics literature, index lifting was understood like in Ricci calculus, the \( \varepsilon \)-symbol denoted the sign of the permutation, and the Einstein summation convention was assumed (Hodge, 1936, p. 485). Concatenated with his own name the designation Hodge \( * \)-operator for the dualization (43) would in the following years replace the former Grassmann dual complement and become the generally used expression for it.

Hodge remarked that also \( P^* \) is an antisymmetric covariant tensor, although now in \( (2m - p) \) components, and a “total differential”, i.e. closed. Condition (II) of (42) can then be rewritten in terms of \( P^* \). A differential form \( P \) is thus harmonic in Hodge’s sense (Hodge, 1935, p. 260f.) if

\[
\begin{align*}
(I) & \quad dP = 0, \\
(II) & \quad dP^* = 0.
\end{align*}
\] (44)

Moreover here

\[
(P^*)^* = (-1)^p P.
\] (45)

Remember \( \dim M = 2m \); this simplifies the general relation \( (P^*)^* = (-1)^{p(n-p)} P \) for Riemannian manifolds of dimension \( n \).

Because of this relation (the word “duality” was not yet used in the 1930s by Hodge) it was clear that the harmonic forms of grade \( p \) and of grade \( (n-p) \) stand in 1:1 correspondence, i.e. in later notation

\[
\mathcal{H}^p \cong \mathcal{H}^{n-p}, \quad \text{by} \quad P \mapsto P^*.
\] (46)

\(^{66}\)Hodge wrote the following two identities in terms of the corresponding integrals.
In his influential book of 1941 Hodge introduced the language of $P^*$ as the “dual of the form $P$” (Hodge, 1941, p. 110ff.). The relation (46) would later become to be known as Hodge duality; after 1955 usually with an additional dualization of vector spaces,

$$\mathcal{H}^p \cong (\mathcal{H}^{n-p})^*,$$

resulting from the bilinear pairing between $p$-forms $\alpha$ and $(n-p)$-forms $\beta$ given by

$$(\alpha, \beta) \mapsto \int_M \alpha \wedge \beta.$$ 

The equations (I) and (II) of (44) can easily be identified as a generalization of the vacuum Maxwell equation: Equation (II) of (42) generalizes Einstein’s form of the second Maxwell equation (20) with the right hand side equal zero; equ (II) of (44) corresponds to Kottler’s equ. (25) and/or Cartan’s (35). In fact Hodge used the notation of the general relativistic literature. One may be tempted to consider this as an indication that he defined harmonic forms with the analogy to the vacuum solution of the Maxwell equation in mind. The remark by Atiyah on Hodge’s motivation quoted in the introduction (footnote 1) seems to do so.

Hodge neither said so in his publications of the 1930s nor in his book (Hodge, 1941). The closest he came to allude to such a relation was, as far as I can see, an explanation of harmonic forms (“tensors”) in the book as “the analogues of the electrical intensity and magnetic induction” (Hodge, 1941, p. 112). With this remark Hodge referred to an analogy to classical (non-relativistic) electromagnetism, involving the electric field written as a 1-form $E = E_i dx^i$ and the magnetic induction written as a 2-form $B = B_{ij} dx^i dx^j$ where $B_{ij} = \frac{1}{\sqrt{|g|}} \epsilon_{ijk} B^k$ with $g = (g_{ij})$ some positive definite metric in dimension $n = 3$ (Hodge, 1941, p. 111). The 2-form $B$ was thus written as a Grassmann-Hodge dual of the corresponding vector field with components $B^i$. Hodge remarked that in the case of the Euclidean metric the classical equations of vector analysis, $\text{curl} E = 0$ and $\text{div} B = 0$ were usually considered as giving rise to a scalar potential $\phi$ for the electric field and a vector potential $A = (A^i)$ for the magnetic induction. He emphasized that this is true only locally (in simply connected regions), while “in the large” only the vanishing of the exterior differential can be stated, $dE = 0$, $dB = 0$.\footnote{Hodge used a symbolic notation sui generis: $E \to 0$, $B \to 0.$}

Generously omitting the difference of $E$ and $B$ he found it justified to continue:

We now define harmonic tensors to be the analogues of the electrical intensity and magnetic induction in the large, and we are thus led to the following definition: A $p$-form $P$ is a harmonic form if (1) it is regular everywhere on $M$, and (2) it satisfies everywhere the conditions $P \to 0$, $P^* \to 0$ [i.e. $dP = 0$, $dP^* = 0$, ES]. (Hodge, 1941, p. 112. emph. in original)

He again neither mentioned any relation to the relativistic Maxwell vacuum equation nor to the wave equation.

A bit earlier, before he presented this electromagnetic analogy, Hodge explained the route he had taken towards generalizing the harmonicity condition $\Delta \phi = 0$ from functions

67Hodge used a symbolic notation sui generis: $E \to 0$, $B \to 0.$
φ to differential forms, and motivated his idea how to proceed from flat (Euclidean) space to Riemannian geometry, again without mentioning Minkowski space or pseudo-Riemannian manifolds. In the case of a (positive definite) metric \( g = (g_{ij}) \) with Levi-Civita covariant derivative \( \nabla \) and for a function \( u \) he proposed to replace the condition for the flat Laplacian \( \Delta \)

\[
div \text{grad} \phi = \Delta \phi = 0
\]

by the Beltrami-Laplace operator \( \Delta_g \) and to call a function \( \phi \) harmonic if

\[
\Delta_g \phi = \nabla_i \partial^i \phi = \frac{1}{\sqrt{\det g}} \sum_{i=1}^{n} \partial_i (\sqrt{\det g} g^{ij} \phi_j) = 0,
\]

with \( \phi_j = \partial_j \phi \) (Hodge, 1941, p. 108). The last equality of (47) can be read as

...the condition that \((n-1)\)-form

\[
\frac{1}{(n-1)!} \sqrt{\det g} g^{ij} \phi_j \epsilon_{i_1 \ldots n-1} dx^{i_1} \ldots dx^{n-1}
\]

should be closed. This geometrical form of the condition suggests the generalisation of the notion of a harmonic function which we are seeking. (Hodge, 1941, p. 109)

By this observation Hodge motivated the introduction of \( P^* \) like in (43) for an alternating \( p \)-form \( P \) as a step towards his definition of harmonic forms. He now talked explicitly about the dual of the form \( P \) (p. 110) and showed that it has the property

\[
P^{**} = (-1)^{p(n-p)} P.
\]

Like in the first half of the 1930s he defined a harmonic \( p \)-form \( P \) to be one which is “regular everywhere on \( M \)” such that both \( P \) and \( P^* \) are closed; i.e. the two equations of (44) are satisfied (Hodge, 1941, p. 112).

At the time when Hodge developed his theory of harmonic forms it was well known that the vacuum Maxwell equation in flat (Minkowski) space leads to a wave equation \( \Delta u_j = 0 \) for all components of the 4-potential \( u = (u_j) \) of the Faraday tensor \( F = du \).\(^{68}\)

The vacuum Maxwell equation could thus have supplied a striking example for Hodge’s definition of a harmonic differential forms. The example had even already been adapted to “curved” spaces in the literature of general relativity, to which Hodge referred indirectly by using the symbolism of the Ricci calculus with the Einstein sum convention and other notational details.\(^{69}\) If the physics literature was anything more than a quarry for notations Hodge must have noticed that the Maxwell equation was a paradigmatic example for defining the harmonicity of differential forms in general. But to my knowledge he never mentioned the general relativistic vacuum Maxwell equation in his publications.

\(^{68}\)With Grassmann complement \( F^* \) and \( dF^* = 0 \) one finds for \( u = \sum_j u_j dx^j \) that \( d^* du = 0 \iff \Delta u_j = 0 \) for all components \( j \).

\(^{69}\)In his bibliographic references Hodge did not include literature of theoretical physics.
of the 1930s or in (Hodge, 1941). This sheds some doubts on the historical reliability of Atiyah’s remark that the Maxwell equation served as a motivation for Hodge’s harmonic forms. But we can also not exclude that Hodge avoided to discuss such a relation in the written work, because for an open motivational argument two technical difficulties would have come to the fore and had, perhaps, to be discussed. First the relativistic Maxwell equation assumes a metric with Lorentzian signature; the resulting Beltrami-Laplace operator thus turns into a Beltrami-d’Alembertian and accordingly the partial differential equation becomes hyperbolic rather than elliptic. Hodge had good reasons to stay with the elliptic case.

Secondly there is a quite complicated interrelation between the Beltrami-Laplace characterization of the harmonicity condition for components of differential forms and Hodge’s definition. Already for a 1-form \( u = u_j dx^j \) like the 4-potential of the Maxwell field \( F = du \) a closer inspection of the Maxwell equations in a (pseudo-) Riemannian space with Levi-Civita derivative \( \nabla \) and Ricci curvature \( Ric = (R_{ij}) \) shows that the differential geometric equivalent of \( dF^* = d * d u = 0 \) is

\[
\nabla_j \nabla^j u_i - R_{ij} u^j = 0 \quad \iff \quad \Delta_g u_i - \nabla^j (\Gamma^{j}_{ik} u^k) - R_{ij} u^j = 0.
\]

The Beltrami-Laplace harmonicity condition is thus “deformed” by additional terms depending on the Ricci curvature terms, the Levi-Civita connection and its derivatives (Frankel, 1997, p. 370). Hodge’s approach was perfectly designed to avoid the analysis of such complications. Maybe he was aware that there was such a difficulty, without necessarily having gone into their details. Then the analogue to the potential of classical electromagnetism in (Hodge, 1941) might be read as the expression of a didactical (over-) simplification.\textsuperscript{70}

### 3.5 Short remarks on the further development of Hodge’s theory

The proof of Hodge’s main theorem (39) remained a problem for more than a decade. His first approach in (Hodge, 1934\textsuperscript{a}) left gaps which he tried to fill according to a hint of H. Kneser (Hodge, 1936). The second proof found wider readership when it was reproduced in his book (Hodge, 1941, chap. 3). It was discussed in a Princeton seminar and H.F. Bohnenblust noticed that even the improved version was based on a problematic limit argument.\textsuperscript{71} But shortly later, “building on the formal foundations laid by Hodge”, Hermann Weyl showed how the problem can be fixed (Weyl, 1943). Independently Kunihiko Kodaira developed a proof of Hodge’s theorem in his PhD dissertation, using orthogonal decomposition of the space of \( p \)-forms (see the passage on de Rham below). After the war it was published in the Annals of Mathematics (Kodaira, 1949) and brought him an invitation to the United States.

\textsuperscript{70} Against such a generous interpretation speaks another remark of Atiyah on his mathematical teacher: “In fact Hodge knew little of the relevant analysis, no Riemannian geometry, and only a modicum of physics. His insight came entirely from algebraic geometry, where many other factors enter to complicate the picture” (Atiyah, 1976, p. 186). This agrees better with the discussed source texts than Atiyah’s statement on the motivational role of the Maxwell equation for Hodge.

\textsuperscript{71} Bohnenblust gave a counter example to the limit argument, documented in (Weyl, 1943, p. 1).
Weyl reformulated Hodge’s main theorem: For any any differential form $f$ there exists a uniquely determined form $\eta \sim f$, i.e. homologously equivalent in the sense of de Rham, for which $d\eta^* = 0$. If $f$ is closed, $\eta$ is harmonic in the sense of Hodge. Weyl commented:

The new proposition shows at once that for any rank $p$ the space of closed forms modulo null may be identified with the space of harmonic forms. (Weyl, 1943, p. 6)

In the slightly later terminology and notation Hodge’s main theorem can thus be stated as the fact, that every cohomology class $f$ of the de Rham cohomology $H^k_{dR}(M, \mathbb{R})$ (for an at least twice differentiable manifold $M$) can be uniquely represented by a harmonic form, or, if $\mathcal{H}^k(M)$ denotes the the vector space of harmonic $k$-forms on $M$,$^{72}$

$$\mathcal{H}^k(M) \cong H^k_{dR}(M, \mathbb{R}). \quad (49)$$

Only a few years after Hodge’s book appeared, Kunihiko Kodaira (1944) and independently Georges de Rham coauthored by Pierre Bidal (1946) introduced a Laplacian operator $\Delta_H$ adapted to the framework of Hodge’s theory. Both publications introduced a new operation for $p$-forms $\omega$, denoted $\delta$ by Bidal/de Rham and called codifferential,$^{73}$

$$\delta \omega = *d* \omega = (d\omega^*)^*. \quad (50)$$

In a manifold of dimension $n$ they defined the Laplace-Hodge operator as

$$\Delta_H = (-1)^{(p+1)n}d\delta + (-1)^{pn}\delta d. \quad (51)$$

The Swiss authors explained that this definition can be understood as a covariant generalization of the Laplacian of a vector field $A = (A^i)$ which has been used in Euclidean space in classical electromagnetism since the late 19th century,

$$\Delta A = \text{grad} \text{div} A - \text{curl} \text{curl} A. \quad (52)$$

In fact, after replacing the vector field by the corresponding 1-form $\alpha = A_idx^i$ one finds that $\delta \alpha$ corresponds to $\text{div} A$ and similarly $d\delta \alpha = \text{grad} \text{div} A$, $\delta d \alpha = \text{curl} \text{curl} A$ (Bidal, 1946, p. 11).

Both (groups of) authors showed that a form $\omega$ on a closed manifold $M$ (compact without boundary) satisfying $\Delta_H \omega = 0$ is harmonic in the sense of Hodge.$^{76}$ Bidal/de Rham gave an elegant proof by introducing a scalar product $(\ , \ )$ on the vector space of $p$-forms

$$(\alpha, \beta) = \int_M \alpha \beta^*. \quad (53)$$

$^{72}$ According to Massey (1999, p. 581) de Rham’s theory was explicitly formulated as a cohomology theory by H. Cartan as late as 1948 in seminars at Harvard and in his own Paris Séminaire.

$^{73}$ Kodaira used different notations and language and a divergence expression similar to Hodge’s (42) in place of $d*$.  

$^{74}$ (Bidal, 1946, p. 11) and up to sign (Kodaira, 1944, p. 193).  

$^{75}$ The components of $\Delta A$ are just $\Delta A_i = \sum_j \partial_j^2 A_i$.  

$^{76}$ (Kodaira, 1944, Thm. 7), (Bidal, 1946, p. 12).
Then $d$ and $\delta$ turned out to be adjoint operators, $(d\alpha, \beta) = (\alpha, \delta\beta)$ and vice versa. They found and exploited mutual orthogonality relations between forms of the types harmonic $(\Delta_H \alpha = 0)$, homolog zero $(\alpha = d\omega$ for some $(p-1)$-form $\omega)$, “cohomolog” zero $(\alpha = \delta\varphi$ for some $(p+1)$-form $\varphi)$ and showed that every $C^2$ differential form can be decomposed in three summands, $\alpha = \alpha_1 + \alpha_2 + \alpha_3$, which are respectively homolog zero, cohomolog zero, and harmonic. The whole article was written in an impressingly clear language in the style of the Bourbaki group, applied to the geometry on differential manifolds and elliptic operators. It would go beyond the scope of the present paper, however, to discuss it in more detail.

About the same time André Weil opened the study of Hodge theory on complex Hermitian manifolds with a positive definite metric given by

$$ds^2 = \sum_{\nu=1}^{n} \omega_\nu \overline{\omega}_\nu = h_{\alpha\beta} \, dz^\alpha d\overline{z}^\beta \quad (h_{\beta\alpha} = \overline{h}_{\alpha\beta}),$$

where $\nu, \alpha, \beta$ run between 1 and $n$ and the $\omega_\nu$ are $n$ independent linear combinations of the complex coordinate differentials $dz_1, \ldots, dz_n$ (Weil, 1947). By demanding that the 2-form associated to the metric

$$\omega = \sum_{\nu} \omega_\nu \wedge \overline{\omega}_\nu = \sum \omega_{\alpha\beta} \, dz_\alpha \wedge d\overline{z}_\beta \quad \text{with} \quad \omega_{\alpha\beta} = i \, h_{\alpha\beta} \quad (\omega_{\beta\alpha} = -\omega_{\alpha\beta}),$$

is closed, $d\omega = 0$, Weil specialized to a Kählerian metric. This allowed him to introduce a $\ast$-operator on complex differential forms and also the codifferential $\delta$ and Laplacian $\Delta$ like in (Bidal, 1946) without going back to the real structure. His main interest was directed towards studying not only holomorphic differential forms, but also meromorphic ones, i.e. those with singularities. He proposed to generalize what Hodge had started to do with holomorphic forms on complex algebraic manifolds embedded in a projective space and endowed with a Riemannian metric (see above and (Hodge, 1941, p. 188ff.)) to complex analytic manifold with Kählerian metric and meromorphic forms. His proposals were soon taken up by Eckmann and Gugenheimer in a series of notes in the *Comptes Rendus* and continued by Kodaira.

It did not take long that also Hodge took up the thread. In (Hodge, 1951b) and (Hodge, 1951a) he undertook a systematic study of the holomorphic and antiholomorphic forms and of mixed type $(p, q)$:

$$\omega = \omega_{\alpha_1 \ldots \alpha_p, \beta_1 \ldots \beta_q} \, d\overline{z}^{\alpha_1} \wedge \ldots \wedge d\overline{z}^{\alpha_p} \wedge d\overline{z}^{\beta_1} \wedge \ldots \wedge d\overline{z}^{\beta_q},$$

where $d\overline{z}^{\beta}$ denotes the differential with regard to the complex conjugate of the coordinate $z^\beta$. The exterior differential could then be decomposed in a contribution of the derivative with regard to the holomorphic differentials and $\overline{\partial}$ with regard to the antiholomorphic differentials, $d = \partial + \overline{\partial}$ and similarly for the codifferential $\delta$. This allowed

77Because of orthogonality of the summands $\delta d\alpha$ and $d\delta \alpha$, the harmonicity of $\alpha$ implies that both summands are zero (and vice versa). Moreover as $(d\alpha, d\alpha) = \pm (\alpha, \delta d\alpha)$, the vanishing of $\delta d\alpha$ implies $d\alpha = 0$; similarly $\delta \alpha = 0$ (Bidal, 1946, p. 12).

78A crucial paper for the study of Hermitian manifolds, cited by Weil, was (Chern, 1946).
to introduce harmonic forms of type \((p, q)\) and the decomposition of the harmonic forms of rank \(k\) into harmonic forms of mixed type \((p, k - p)\) (Hodge, 1951b, p. 106). Let us denote the vector spaces of the latter by \(\mathcal{H}^{(p,q)}(M, \omega)\) and their (real) dimensions by \(h^{(p,q)}\). Hodge showed that the dimensions depend only on the complex structure, not on \(\omega\). Up to isomorphism the vector spaces thus depend only on the manifold, \(\mathcal{H}^{(p,q)}(M)\) (Hodge, 1951b, p. 109). Moreover

\[
\mathcal{H}^k(M) = \bigoplus_{p+q=k} \mathcal{H}^{(p,q)}(M),
\]

The dimensions \(h^{(p,q)}(M)\) became to be known as Hodge numbers, the vector space including their subdivision (52) as the Hodge structure of \(M\). Because of Hodge’s main theorem \(h^k = p_k\), the corresponding Betti number, while in general \(h^k \neq h^{(k,0)}\).

Finally our author formulated a necessary condition for a compact Kählerian manifold to be analytically isomorphic to an algebraic variety and called those which satisfy it Kählerian manifolds of “restricted type” (Hodge, 1951b, p. 107, 110), later called Hodge manifold. Three years later Kodaira (1954) was able to prove the sufficiency of this condition. Hodge’s criterion turned out to be an “intrinsic characterization” of algebraic varieties from the standpoint of analytic manifolds.

4 A short outlook on Hodge duality in mathematics and physics after 1950

4.1 Hodge theory and Hodge duality become sheaf cohomological

Hodge’s work had an important influence on the differential geometry and topology of manifolds, cohomology theory and, of course, the geometric theory of complex functions in several variables. The latter turned into what has been called the analytic geometry of the 20th century. In his address to the 1954 International Congress of Mathematics Hermann Weyl called Hodge’s theory “one of the great landmarks in the history of our science in the present century” because it made the fruitfulness of Riemann’s “transcendental method” evident (Weyl, 1954, p. 616). It would be overconfident trying to give a resumée of the developments resulting from it. Only a few glimpses into one aspect of the further developments be given here, in particular the reformulation of Hodge theory in sheaf cohomological terms.\(^{79}\)

Kodaira’s extension of Hodge’s approach to meromorphic forms (Abelian differentials of the second and third kind) bore fruit. After having constructed such forms with prescribed periods and singularities Kodaira (1949) proved the analogue of the famous Riemann-Roch theorem for compact 2-dimensional Riemannian manifolds and showed that any compact Kähler surface with two algebraically independent meromorphic functions can be represented as an algebraic surface.\(^{80}\) Five years later he extended the theory

\(^{79}\)For more information one may like to consult (Weyl, 1954), (Dieudonné, 1989, pp. 254ff., 580ff.), (Atiyah, 1976). The “glimpses” are selected from these publications.

\(^{80}\)The classical Riemann-Roch theorem deals with compact Riemann surfaces \(S\) of genus \(p\). It states a relation between the dimension \(l\) of the (complex) vector space of meromorphic functions with divisor
and proved the result mentioned above that every Hodge manifold (Kählerian manifold of restrictive type) is bianalytically equivalent to an algebraic submanifold of a complex projective space (Kodaira, 1954). In collaboration with D. Spencer he was able to show that two birational invariants of an algebraic varieties \( V \) introduced by Severi, called the genera \( p_a \) and \( P_a \) in allusion to the arithmetical genus of curves introduced by Clebsch et al., have an underpinning in the Hodge structure of \( V \).\(^{81}\) He and Spencer introduced the arithmetic genus \( a(M) \) of a compact Kählerian manifold \( M \) as the alternating sum of the numbers of holomorphic forms,

\[
a(M) = \sum_{j=0}^n (-1)^j h^{(j,0)}(M),
\]

and showed that for \( M = V \), a singularity free algebraic variety of dimension \( n \), the three genera are essentially the same, \( P_a = p_a = (-1)^n(a(V) - 1) \). This showed that in particular Severi’s genera are analytic and even birational invariants.\(^{82}\) In their work the authors already used the recently introduced method of sheaf cohomology.

It is impossible to sketch here the rise of sheaf theory. But it ought to be said that the assimilation of Hodge structures to sheaf theory played an non-negligible role in its early history, although it has not yet found the corresponding attention in the historical literature.\(^{83}\) Soon after Jean Leray and Henri Cartan introduced sheaves in the late 1940s, Pierre Dolbeault considered the cohomology with coefficients in the sheaf of germs of holomorphic \( p \)-forms on \( X \), abbreviated by \( \Omega^p \).\(^{84}\) Denoting the resulting sheaf cohomology by \( H^q(X, \Omega^p) \) he showed that in the case of a Kähler manifold it coincides with Hodge’s harmonic forms of mixed type (Dolbeault, 1953):

\[
H^q(X, \Omega^p) \cong H^{(p,q)}(X)
\]  

He formulated his theorem more generally for any analytic variety in which case he had to characterize the right hand side of (54)) by a second sheaf theoretical cohomology denoted

\[^{81}\text{The equality } p_a = P_a \text{ was known for the dimensions } n = 1, 2; \text{ for } n = 3 \text{ Severi (1909) had sketched an incomplete proof. Zariski (1952) proved a conditional equality for even } n, \text{ if it is true for } n - 1.\]

\[^{82}\text{At first Kodaira called the alternating sum of the Hodge numbers the “virtual arithmetic genus”; after the proof of the identities with Severi’s genera the attribute “virtual” was omitted (Kodaira, 1953b).}\]

\[^{83}\text{A partial exception is the study of the origins of sheaf theory in (Chorlay, 2010a), for more technical reviews in the Bourbaki style of history one may consult (Houzel, 1990, 1998). On the relation to Hodge theory see the passages in (Dieudonné, 1989) indicated in fn. 79.}\]

\[^{84}\text{He did not specify which construction of cohomology he referred to. Since 1936 several authors had developed different approaches to cohomology theories for general spaces } X. \text{ The most well known were probably Čech cohomology introduced by (Dowker, 1937) and a cohomology theory derived from } k\text{-cochains defined by functions on ordered } (k + 1) \text{ sets of points of } X \text{ (Spanier, 1948). Not much later Hurewicz, Dugundji and Dowker showed that Spencer’s cohomology and Čech-type cohomology lead to isomorphic homology modules (Massey, 1999, p. 592).}\]
$H^{(p,q)}(X)$, the so-called Dolbeault cohomology. It was derived from co-chains $A^{(p,q)}(X)$ on a complex analytic manifold $X$ with coefficients in the germs of mixed holomorphic-antiholomorphic differential forms with distributional coefficients, so-called currents of type $(p, q)$. Here the differential operator $d$ is decomposed into its holomorphic and its antiholomorphic components, $d = \partial + \overline{\partial}$, like in the Rham cohomological interpretation of the Hodge structure. Like in (54) then

$$H^q(X, \Omega^p) \cong H^{(p,q)}(X)$$

(Dolbeault’s theorem).

We cannot discuss its derivation in more detail here, but see below that it became important for the later work, in particular for Serre’s duality theorem. Moreover it showed that $H^q(X, \Omega^p)$ may be considered as a representation of a generalized Hodge structure also in the general case of a complex analytic manifold without presupposing a Kählerian metric as auxiliary device. Atiyah emphasized this conceptual shift in his report on Hodge:

For Riemann surfaces the complex structure defines a conformal structure and hence the Riemannian metric is not far away, but in higher dimensions this relation with conformal structures breaks down and makes Hodge’s success all the more surprising. Only in the 1950s, with the introduction of sheaf theory, was an alternative and more intrinsic definition given for the Hodge numbers, namely

$$h^{p,q} = \dim H^q(X, \Omega^p)$$

where $\Omega^p$ is the sheaf of holomorphic $p$-forms. (Atiyah, 1976, p. 187f.)

This was an important basis for the generalization of Hodge duality by Jean Pierre Serre for any paracompact complex analytic manifold $X$. Serre (1955) considered the sheaves of germs of differential forms $A^{(p,q)}$ of type $(p, q)$ with coefficients in differentiable functions and those with distributional coefficients, called $K_\ast^{(p,q)}$, and a coboundary operator $d = \partial + \overline{\partial}$ like in de Rham cohomology. All this was defined not only for the manifold $X$ itself but for any complex vector bundle $V$ over $X$. The resulting cohomology $H^q(X, \Omega^p(V))$ corresponded to Dolbeault’s generalized Hodge structure and worked not only for holomorphic differential forms on $X$ but also for meromorphic ones with singularities encoded by a divisor $D$, respectively the line bundle $V = L(D)$ associated to it. Let the dual vector bundle be denoted by $V^*$; it is associated to the divisor $K - D$ with $K$ the canonical divisor class. In this general setting $A^{(p,q)}(V), K^{(p,q)}(V)$

---

85) Paracompactness of $X$ (i.e. every open covering has a locally finite refinement) is important in this context for constructing Čech type cohomology theories.

86) The lower star notation $K_\ast^{(p,q)}$ was used by Serre only from p. 17 onward; they expressed a duality relation to the $A^{(p,q)}$.

87) Already Kodaira and Spencer used a line bundle $L(D)$ associated to a divisor for a compact Kählerian manifold for dealing with meromorphic forms (Kodaira, 1953a).

88) A divisor $D$ consists of a finite collection $D_1, \ldots, D_l$ of analytic subvarieties of codimension 1 in $X$, endowed with integral weights $n_j \neq 0$ and written as $D = \sum n_j D_j$. The degree of the divisor is $|D| = \sum n_j$. The subsets $D_j$ with $n_j < 0$ encode the loci of singularities (with poles of order $\leq |n_j|$), those with $n_j > 0$ zeroes of order $\leq n_j$. For any (non-zero) meromorphic function $f$ on $X$ the zeroes and poles of $f$ define a divisor $(f)$, called canonical. The collection of all canonical divisors is the canonical divisor class, usually denoted by $K = \{(f)\}$. 37
and $K^{(p,q)}_*(V^*)$ were infinite dimensional vector spaces with a Frechet topology. This made the analysis much more demanding than in the case of a compact manifold $X$.

Serre stated the following generalized duality theorem (Serre, 1955, thm. 2, p. 20):

**Theorem 2** Let $X$ be a paracompact complex analytic manifold of dimension $n$ and $V \to X$ an analytic vector bundle with dual $V^* \to X$. If in the construction of chain complexes $A^{(p,q)}(V)$ and $K^{(p,q)}_*(V^*)$, indicated above, the antiholomorphic coboundary operator $\partial$ consists of vector space homomorphisms, the topological dual of the Frechet space $H^q(X, \Omega^p(V))$ constructed from the first complex $A^{(p,q)}(V)$ is isomorphic to the cohomology $H^{n-q}_*(X, \Omega^{n-p}(V^*))$ constructed from the second complex $K^{(p,q)}_*(V^*)$. In short:

$$H^q(X, \Omega^p(V))^* \cong H^{n-q}_*(X, \Omega^{n-p}(V^*))$$

The theorem was a feast in dualities. It gathered up at least three (according to how one counts even five) dualities and intertwined them into one whole: the dual relation between $p$-forms and $(n-p)$-forms typical for the Hodge $*$-operator went together with the duality between the distributional coefficients and the $K^{(p,q)}_*$ to those of the $A^{(p,q)}$. The dualization $V^*$ of the vector bundle $V$ demanded the substitution of the divisor $D$ by $K - D$; finally the Frechet space $H^q(X, \Omega^p(V))$ had to be dualized.

It was a long path to go from the original Hodge duality (46) to this theorem which is rather demanding already in its formulation. Serre did not hesitate to show that it covered special cases relevant for the study of complex analytic manifolds. Aside from a specialization for Stein manifolds (thm. 3)\(^89\) he explicated that for a compact complex analytic manifold $X$ the situation becomes close to what Hodge had done. Because in this case the vector spaces are finite dimensional, the theorem specializes to

$$H^q(X, \Omega^p(V)) \cong (H^{n-q}_*(X, \Omega^{n-p}(V^*)))^*$$

for all $0 \leq q \leq n$ (Serre, 1955, thm. 4).

In the light of Dolbeault’s relation (54) the specialization (55) of Serre’s duality theorem was clearly a generalization of Hodge’s duality (46) to meromorphic forms. It did not need any recourse to a metrical structure on $X$, which could be circumvented by using dual pairings of vector spaces. Serre accomplished for Hodge duality what Dolbeault had done for the core of Hodge theory, in particular Hodge’s theorem. In this sense (55) completed the transfer of Hodge theory to sheaf cohomology.

It became an input for a generalization of the classical theorem of Riemann-Roch to algebraic manifolds of any dimension by F. Hirzbruch.\(^90\) Soon later both theorems, Hirzebruch’s and Serre’s, were even more generalized by A. Grothendieck, but both

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\(^89\) A **Stein manifold** is a complex analytic manifold which is bianalytically embeddable in a $\mathbb{C}^r$. An intrinsic characterization by holomorphic separability and holomorphic convex hulls of compact subsets is possible. Non-compact Riemann surfaces are Stein manifolds.

\(^90\) Hirzebruch (1956) replaced the left hand side of the Riemann-Roch theorem (see fn. 80) by the arithmetic genus (53) of a vector bundle $V \to X$ (in particular $V = L(D)$ for a divisor $D$) and discovered how to express the right hand side by topological invariants of $X$ and $V$ (by a polynomial in Chern classes applied to the orientation class of $H_2(X)$).
(theorems and authors) continued to play an important role in mathematical research of the late 20th century; the theorems continue to do so, Serre duality, e.g., for the deformation theory of complex analytic manifolds.

4.2 From Hodge duality in physics via Yang-Mills theory back to mathematics

Hodge was not really interested in applying his dualization in physics. Even though he mentioned a reference to electrodynamics in his book on *The Theory and Application of Harmonic Integrals* (Hodge, 1941) the references to physics remained elementary and remained at the level of the classical Maxwell equations in 3-dimensional space with a Riemannian metric. No wonder that the main momentum and influence of Hodge’s work was to be felt in mathematics. But it did not remain without repercussions in theoretical physics.

During the course of years the Hodge $\ast$-operation took the place of what had formerly been Grassmann type complements of differential forms in relativistic electrodynamics, although not always with explicit reference to Hodge, e.g. (Misner, 1973, p. 108ff.). Then the Maxwell equations acquired a form close to the one of de Donder, Kottler and Cartan in the 1920s (see sec. 2.2):

$$dF = 0, \quad d \ast F = 4\pi J$$

with $F$ the Faraday tensor, $\ast F$ the corresponding Maxwell tensor and $\mathcal{J}$ the current density 3-form related to the current density co-vector $\mathcal{J}$ by $\mathcal{J} = \ast \mathcal{J}$.\footnote{This view of the Maxwell equation is particularly illuminating for a *premetric* approach to electrodynamics, which tries to avoid the reference to a metrical structure on spacetime as far as possible. At a foundational level the Faraday tensor $F$, the Maxwell tensor $H$, and their fundamental equations $dF = 0$, $dH = 4\pi \mathcal{J}$ are introduced separately, before different “constitutive” relations between the two tensors are postulated and studied like in Kottler’s approach (see sec. 2.1).}

For relativistic Maxwell theory a metrical structure enters through the Hodge $\ast$-operator and the constitutive relation is given by

$$H = \ast F.$$  \hspace{1cm} (57)

In the premetric approach this is only one among different alternatives, and the question may be posed whether the “origin” of the metrical structure on spacetime can be grounded on electromagnetism rather (or in addition to) gravity (Hehl, 2003). Einstein gravity works the other way round; and the Hodge operator expresses a basic physical law, the constitutive relation (57) of relativistic Maxwell electrodynamics.

Hodge $\ast$-duality also entered gravity theory, although only in a subordinate role, e.g. for the classification of the Weyl curvature tensor (Kopczyński, 1992, p.136ff.). That seems to have happened only after the rise of Yang-Mills theories to prominence, even for authors who emphasized a Cartan geometric approach to gravity like in (Hehl, 1989, p. 1096ff.). As far as I can see, it played no conceptual role, comparable to the one in Cartan’s considerations (section 2.3).

\footnote{For a concise discussion see (Frankel, 1997, p. 366ff.).}
A major field for using Hodge type dualization arose in physics from the growing acceptance of Yang-Mills gauge theory in the standard model of elementary particle physics from 1970s onward. This is a story of its own which still has to be told from a historical perspective.\footnote{There are many specialized articles focusing on separate themes; for sources on the origins of gauge theory see (O’Raifeartaigh, 2000), for those on gauge theories of gravitation (Blagojević, 2013). An accessible mathematical introduction is (Nielsen, 2005), a popular account of the overall story (Crease, 1996).} In their paper (1954) Chen Ning Yang and Robert Mills proposed a field theory for strong interactions between Dirac spinor fields $\psi$ extended by (i.e. tensorized by) dynamical degrees of freedom in a representation space of the special unitary group $SU(2)$ (the “isospin space”) describing different but related states of elementary particles. The theory was modelled after the example of electromagnetism and worked with a field potential given by a differential form $A = A_\mu dx^\mu$ on (special relativistic) spacetime with values in $\mathfrak{su}(2)$, up to equivalence under so-called gauge transformations, where we abbreviate the system of coefficients as $A_\mu = (A^\alpha_{\beta \mu})$.

From a mathematical point of view the Yang-Mills potential $A$ can be understood as a connection $\Gamma = \Gamma(A)$ on a $SU(2)$ principal fibre bundle and properly formed associated vector bundles used to characterize matter fields. It took some time, before this interpretation became common knowledge among mathematicians and physicists.\footnote{According to Yang he learned from differential geometers at Stony Brooks about the geometrical interpretation of gauge fields at the end of the 1960s, but started to appreciate it only five years later, in the mid 1970s (Yang, 1983, pp. 73–75).} The field strength, called the Yang-Mills field of the potential, is then given by a differential 2-form $F = F_{\mu\nu} dx^\mu dx^\nu$ with values in $\mathfrak{su}(2)$. It arises as a covariant exterior derivative of the potential, $F = d\Gamma A$ and can be understood geometrically as the curvature of the connection $\Gamma(A)$,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

The dynamics of the Yang-Mills field is assumed to be governed by a Lagrangian analogous to the one of Maxwell theory. This leads to dynamical equations for the interaction field, called Yang-Mills equations, which can be written in a form similar to the Maxwell equation,

$$d\Gamma F = 0, \quad \delta_\Gamma F = *d_\Gamma* F = 4\pi J,$$

with $\delta_\Gamma$ the co-differential analogous to (50). In contrast to (56), the 2-form $F$ and the source 1-form $J$, called the Dirac current of the present equation, have values in $\mathfrak{su}(2)$.

Solutions of the vacuum Yang-Mills equations, i.e. those with $F = 0$, are analogue to harmonic forms. This is the case for (generalized) Yang-Mills theories with any compact group $G$ in place of $SU(2)$. In this sense “the general Yang-Mills theory can be considered a ‘non-abelian Hodge theory’” (Bourguignon, 1982, p. 404), which brings us back from physics to mathematics.

In 4-dimensional manifolds the 2-forms $\Lambda^2$ decompose into eigenspaces $\pm 1$ of the Hodge operator, $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$. A vacuum Yang-Mills field $F$ decomposes correspondingly into $F = F^+ + F^-$, its so-called self-dual and anti-self-dual components. Those $F$ which reduce to the (anti-) self-dual component have been baptised instantons. Shortly after
the “November revolution” (Pickering) of the standard model, which had shown that the paradigm of perturbatively quantized gauge field theories promised to open up a path toward an effective model of the elementary constitution of matter, physicists started to study the (anti-) self-dual solutions of Yang-Mills equations in 4-dimensional Euclidean space $\mathbb{E}^4$. They spoke of “pseudoparticle solutions” and realized that these are related to topological properties of a 3-sphere bundle over $\mathbb{E}^4$ (Belavin, 1976). Mathematicians soon joined, Hodge’s former PhD student Michael Atiyah was among the first. This led to the study of module spaces of instanton solutions on 4-dimensional manifolds, which became an important subfield of differential topology of 4-manifolds in the last two decades of the 20th century. It may be considered as a second chapter of the history of Hodge theory with its interplay between physics and geometry. But it is definitely a story of its own, to be told elsewhere.

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