Edge and Bulk of the Fractional Quantum Hall Liquids

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(October 8, 2018)

Abstract

An effective Chern-Simons theory for the Abelian quantum Hall states with edges is proposed to study the edge and bulk properties in a unified fashion. We impose a condition that the currents do not flow outside the sample. With this boundary condition, the action remains gauge invariant and the edge modes are naturally derived. We find that the integer coupling matrix $K$ should satisfy the condition $\sum_I (K^{-1})_{IJ} = \nu/m$ ($\nu$: filling of Landau levels, $m$: the number of gauge fields) for the quantum Hall liquids. Then the Hall conductance is always quantized irrespective of the detailed dynamics or the randomness at the edge.

74.10.-d, 73.20.Dx, 73.40.Hm
Recently the chiral Tomonaga-Luttinger (TL) liquid [1] realized at the edges of fractional quantum Hall liquid (FQHL) [2] has been proposed to be a promising ideal one-dimensional system among the ones which have ever been studied. In the quantum wire with zero magnetic field, the randomness causes backward scatterings and hence the localization. In contrast, it can be avoided in the QHL due to the following reason [3]. The probability of the backward scatterings is proportional to the overlap of the wavefunctions of the edge modes with the opposite chiralities. When the chiralities of the edge modes in one edge of the sample are all the same and the other edge is spatially well separated, this probability is exponentially small and can be neglected. Hence the Hall conductance of the edge is robust against the randomness and is quantized, which is the explanation of the (F)QHE in terms of the edge picture. When one intentionally introduces the backward scatterings between the two edges by making a point contact, the system is expected to be described as the TL model with potential barriers [4,5]. This idea [5] beautifully explains the recent experiment in the $\nu = 1/3$ FQHL [6] ($\nu$ is the filling of the Landau levels).

In the case $\nu = 2/3$, on the other hand, it has been proposed that there are two edge modes for each edge corresponding to the $\nu = 1$ QHL and $\nu = -1/3$ FQHL [2,7]. In this case the two edge modes are not spatially separated and generally interact with each other. Recently Kane et al. [8] studied this case and found that the Hall conductance due to these edge modes is not quantized and takes a nonuniversal value depending upon the coupling constant, which contradicts the experiments observing the quantized Hall conductance plateau at $\nu = 2/3$. The resolution of this puzzle they proposed is that the randomness at the edge makes the neutral mode massive and only the charged mode remains massless, which gives the quantized Hall conductance.

Haldane [9] have opposed to their conclusion by showing that the Hall conductance is quantized without any randomness by taking the anomaly into account. He also proposed the idea of topological (T-) stability of the chiral edge modes, without which the quasi-particles with opposite chiralities are generated by the randomness at the edge and the FQHL becomes unstable.
If one regards the FQHE as the bulk phenomenon, however, it should be insensitive to the details at the edge and the Hall conductance should be always quantized. In addition, other sets of experiments show the importance of the bulk currents. One example is the measurement of the Hall voltage in the sample, which shows the existence of the voltage drop in the bulk and hence the Hall current \[10\]. Another is the sample size dependence of the critical current \(I_c\) for the breakdown the QHE [11]. It is proportional to the width of the sample, which shows that the current is distributed mainly in the bulk.

In this paper we develop a theoretical framework which treats the edge and bulk on an equal footing in order to resolve the puzzles mentioned above. Our theory is based on the Chern-Simons effective theory of FQHL [12–15]. Following the hierarchy construction by Jain [16] and by Blok and Wen [13], we decompose each of the original electrons into fictitious fermions. Each fermion is represented as a composite particle of a boson and a Chern-Simons gauge flux. The external magnetic field \(B_0\) is cancelled with the Chern-Simons gauge flux on average, and the bosons are condensed to superfluidity \[12\]. This condensation can be regarded as the hidden off-diagonal long range ordering (ODLRO). According to this picture the FQHE is the bulk phenomenon similar to the superconductivity, and the edge current is analogous to the supercurrent which is localized near the surface of the sample [17]. The conserved current density \(J_I^\mu\) of the bose condensate is expressed in terms of the gauge field \(a_I^\mu\) \((I = 1, 2, \cdots, m)\) in two spatial dimensions as \(J_I^\mu = \frac{1}{2\pi} \varepsilon^{\mu\nu\lambda} \partial_\nu a_{I\lambda}\). A quasi-particle is represented as a vortex in the bose condensate, which is defined as a zero point of the amplitude of the boson order parameter. Thus the phase of the boson order parameter there is singular. The quasi-particle current density \(j_I^\mu\) is defined as \(j_I^0(r) = \sum_\ell \delta(r - R_I^\ell(t))\), and \(j_I^\alpha(r) = \sum_\ell \frac{dR_I^\ell(t)}{dt} \delta(r - R_I^\ell(t)) \) \((\alpha = x, y)\), where \(R_I^\ell(t)\) is the center of the \(\ell\)th vortex for the \(I\)th bose condensate. The effective Lagrangian for the Abelian FQHL is written in terms of the gauge fields and the quasi-particle currents as [12–15]

\[
L = \int_S d^2 r \sum_I \left[ \sum_j \frac{1}{4\pi} K_{I,j} \varepsilon^{\mu\nu\lambda} a_{I\mu} \partial_\nu a_{I\lambda} - a_{I\mu} j_I^\mu - \frac{1}{2\pi} A_{I\mu} \varepsilon^{\mu\nu\lambda} \partial_\nu a_{I\lambda} - \frac{1}{2\pi} V(r) \varepsilon^{\alpha\beta} \partial_\alpha a_{I\beta} - \frac{1}{g_I f_I} f_I^{\mu\nu} j_I^\mu \right],
\]

(1)
where $K_{IJ} = K_{JI}$ is the $IJ$ component of the integer-valued symmetric matrix $K$ representing the coupling between the $I$th and $J$th Bose condensates which uniquely specifies the topological structure of the Abelian FQHL. It also gives the filling by $\nu = \sum_{IJ} (K^{-1})_{IJ}$ (see e.g. [12]). $V(r)$ is an arbitrary potential for the electrons. The Maxwell term $(1/g_I) f_{I\mu\nu} f_{I}^{\mu\nu}$ $(f_{I\mu\nu} = \partial_{\mu} a_{I\nu} - \partial_{\nu} a_{I\mu})$ in (1) is explicitly written as $(2/g_I) [c_{I}^{2} f_{Ixy}^{2} - f_{Iox}^{2} - f_{Ioy}^{2}]$, and naturally arises in the duality mapping [12]. The coupling constant $g_I$ is given by $16\pi^{2} \rho_{I}s/m_{I}$ where $\rho_{I}s$ is the superfluidity density and $m_{I}$ is the mass of the bosons. The velocity $c_{I}$ of the Bogoliubov mode for the $I$th Bose condensate is given by $c_{I}^{2} = \rho_{I}s V_{I}/(2m_{I})$ where $V_{I}$ is the short-range repulsive interaction between the bosons. The hard-core condition is realized by the effective $V_{I}$ with $m_{I} V_{I} \sim 1$ in low energies. The vector potential $A_{\mu}$ of the electromagnetic field is coupled to the $\mu$ component of the physical current density $J_{I}^{\mu} = \sum_{J} J_{IJ}^{\mu}$. Note that the vector potential for the constant external magnetic field $B_0$ has been already taken into account in the structure of the $K$ matrix, and is not included in $A_{\mu}$. Similarly $a_{I\mu}$ and the density $J_{I}^{0}$ are measured from their average values in the following discussion.

The integral is over the sample $S$, and on the boundary $\partial S$ we impose

$$\sum_{\alpha=x,y} J_{I}^{\alpha} n_{\alpha}|_{\partial S} = 0,$$  \hspace{1cm} (2)

where $\vec{n} = (n_{x}, n_{y})$ is the unit vector normal to the boundary. This boundary condition simply expresses the physical condition that the current can not flow through the boundary $\partial S$. Since it is a physical requirement, it is obviously invariant with respect to a gauge transformation $a_{I\mu} \rightarrow a_{I\mu} + \partial_{\mu} \phi_{I}$. A remarkable fact is that the Chern-Simons term in the Lagrangian (1) is also gauge invariant with the boundary condition (2).

We now derive the equation of motion by requiring that the variation of the action $A = \int dt L$ vanishes. The result is $\sum_{J} K_{IJ} \varepsilon_{\mu\nu\lambda} \partial_{\nu} a_{IJ\lambda} = 2\pi J_{I}^{\mu} + \varepsilon_{\mu\nu\lambda} \partial_{\nu}(A_{\lambda} + \delta_{\lambda 0} V) + 8\pi \partial_{\nu} f_{I}^{\mu\nu}/g_{I}$ which is expressed in terms of the current density only and gauge invariant. It is written

$$\begin{bmatrix}
K & -8\pi g^{-1} \partial_{t} & -8\pi c^{2} g^{-1} \partial_{y} \\
8\pi g^{-1} \partial_{t} & K & 8\pi c^{2} g^{-1} \partial_{x} \\
-8\pi g^{-1} \partial_{y} & 8\pi g^{-1} \partial_{x} & K
\end{bmatrix}
\begin{bmatrix}
J^{x} \\
J^{y} \\
J^{0}
\end{bmatrix}
= \frac{1}{2\pi}
\begin{bmatrix}
-(E_{y} + \partial_{y} V) \mathbf{q} \\
(E_{x} + \partial_{x} V) \mathbf{q} \\
B \mathbf{q}
\end{bmatrix}
+ \begin{bmatrix}
j^{x} \\
j^{y} \\
j^{0}
\end{bmatrix}$$  \hspace{1cm} (3)
where \( K = \{ K_{ij} \}, \quad g^{-1} = \text{diag}(g_1^{-1}, \cdots, g_m^{-1}) \), and \( c^2 = \text{diag}(c_1^2, \cdots, c_m^2) \) are \( m \times m \) matrices and \( J^x = [J^x_1, \cdots, J^x_m] \), \( j^x = [j^x_1, \cdots, j^x_m] \), \( q = [1, \cdots, 1] \), etc. are vectors with \( m \) components.

Here we have introduced the ”charge vector” \( q \) representing the coupling to the external electromagnetic field [9]. Equations (2) and (3) together with the Maxwell equations for \( A_\mu \) constitute the fundamental equations, from which all the results below are obtained. In the following sections 1 and 2 we shall neglect quasi-particles, i.e., vortices, and set \( j^x_\mu \) to be zero. In section 3 the effects of the quasi-particles will be considered.

1. Quantization of the Hall Conductance

In order to study the quantization it is enough to consider the stationary case \( \partial_t J^\mu_t = \partial_t A_\mu = 0 \). Then (3) becomes \( K J^x - 8\pi c^2 g^{-1} \partial_y J^0 = -(E_y + \partial_y V) q/(2\pi) \) and \( K J^y + 8\pi c^2 g^{-1} \partial_x J^0 = (E_x + \partial_x V) q/(2\pi) \). Define the charging energy \( U(J^0_1, \cdots, J^0_m) \equiv \int d^2 r \sum f^y \partial_y^2 g_I = \int d^2 r \sum 8\pi c^2 J^0_I/g_I \), which is present in the Maxwell term of the Lagrangian (1). Then the physical current \( J^\alpha = \sum_I J^\alpha_I \) is given by

\[
J^\alpha(r) = -\varepsilon^{\alpha\beta} \partial_\beta R(r),
\]

where \( R(r) \) is defined by

\[
R(r) \equiv \frac{1}{2\pi} \left[ \sum_{IJ} (K^{-1})_{IJ}(A_0(r) + V(r)) - \sum_{IJ} (K^{-1})_{IJ} \frac{\delta U}{\delta J^0_I(r)} \right].
\]

Let us now integrate (4) from \( A \) to \( B \) along the contour \( C \) in Fig.1. The result is that the total current \( I \) flowing across the contour \( C \) is \( I = \Delta R \) where \( \Delta R = R(B) - R(A) \).

The Hall conductance \( \sigma_H \) is given by \( I = \sigma_H \Delta \mu \), where \( \Delta \mu = \mu(B) - \mu(A) \) and \( \mu(r) = A_0(r) + V(r) - \delta E(\{J^0_I(r)\})/\delta J^0(r) \) is the chemical potential. If \( \Delta R \) is proportional to \( \Delta \mu \), the Hall conductance is given by their ratio \( \sigma_H = \Delta R/\Delta \mu \). In fact we shall show that this is the case if a condition on \( K \) matrix is satisfied. The physically realized charging energy \( E(J^0) \) is obtained by minimizing the function \( U(J^0_1, \cdots, J^0_m) \) with the constraint \( \sum_I J^0_I = J^0 \). Then it can be shown that \([9]\)

\[
\frac{\delta E(J^0)}{\delta J^0} = \frac{1}{m} \sum_I \frac{\delta U(J^0_1, \cdots, J^0_m)}{\delta J^0_I}
\]
for quite general functional \( U \). The comparison of (5) and (6) leads that \( R(r) \) is proportional to \( \mu(r) \) if and only if the condition
\[
\sum_{I} (K^{-1})_{IJ} = \frac{1}{m} \sum_{IJ} (K^{-1})_{IJ} = \frac{\nu}{m}
\]
is satisfied, and \( \sigma_H = \nu/2\pi \). This is a striking result of the effective theory in the following respects.

(i) \( \sigma_H \) is always quantized for arbitrary shape of the sample and potential \( V(r) \) as long as the quasi-particle current \( j_I^\mu \) is absent. In Fig.1 we can move the point \( B \) (or \( A \)) freely as long as it does not cross the current terminal, and the integral of (4) along \( C \) remains the same. This concludes that the the chemical potential \( \mu(r) \) remains the same along the edge. The chemical potential drop \( \Delta \mu \) occurs only at the current terminals. Similarly the configuration of the 4 terminals \( D,E,F,G \) at one of the edge in Fig.1 is topologically almost equivalent to that in the above discussion. As before we integrate (4) along the path \( C' \) in the bulk connecting \( D \) and \( E \), and we obtain the quantized Hall conductance \( \sigma_{xy} = I/\Delta \mu = \nu/2\pi \).

(ii) The relation \( \sigma_H = \Delta R/\Delta \mu = \nu/2\pi \) is derived without assuming the infinitesimal \( I \) or \( \Delta \mu \). Therefore we believe that our theory goes beyond the linear response theory, and remains valid for current \( I \) less than the critical value \( I_c \) for the breakdown phenomenon.

(iii) In the integral of (4) along the path \( C \) or \( C' \) in Fig.1, the bulk current contributions can not be neglected as shown below. Therefore the quantization of \( \sigma_H \) occurs only when both the bulk and edge currents are treated in a self-consistent way. Our picture is that the quantization is the bulk phenomenon like the superconductivity \([12]\), and hence robust against the perturbations at the edge as shown below.

The condition (7) is equivalent to a condition that the ”charge vector” \( q \) is an eigenvector of the \( K \) matrix in the notation of Ref. \([9]\). The condition (7) is satisfied for the type \( K = I + pP \) where \( I \) is the unit matrix, \( p \) is an even integer, and \( P \) is the matrix with all the matrix elements being 1. This \( K \) matrix describes the FQHL with \( \nu = m/(1 + mp) \) including \( \nu = 2/3, 2/5, \) etc \([13, 15]\). Obviously this condition (7) is not satisfied for general...
\(K\) matrices and gives a criterion for the realization of each Abelian quantum Hall state.

2. Edge Effects and Anomaly

The above discussion does not explicitly distinguish between the bulk and edge effects. Now we concentrate on the edge properties.

**Current and Charge Distribution near the Edge** - For the steady state we set \(\partial_t J_I^\mu = \partial_t A_\mu = 0\) in (3). Consider the sample extending the semi-infinite plane \((x < 0)\) with the straight edge at \(x = 0\) and \(V(r) = 0\). Since there is no spatial dependence in the \(y\)-direction, \(J_x^I = E_y = 0\) in the stationary state. For simplicity let us consider the case where \(\nu = 1/(2n + 1) \equiv \eta^{-1} = K^{-1}\) (\(n: \text{integer}\)). Only one gauge field is enough (\(m = 1\)) in this case, and (3) is in the simplest form in which the \(m \times m\) matrices and vectors are \(c\)-numbers. Then it becomes \(J_0 - \kappa^{-2} d^2 J_0 / dx^2 = (8\pi / g\eta^2) d^2 A_0 / dx^2\), where \(A_0(x)\) is the scalar potential and related to the electric field \(E_x\) as \(E_x = -dA_0 / dx\). \(\kappa^{-1} \equiv 8\pi c / g\eta\) is the characteristic length scale of the spatial change. The magnitude of \(\kappa^{-1}\) is estimated as \(\kappa^{-1} \sim \ell_B \sim \sqrt{\hbar/eB_0}\) (magnetic length). This together with \(A_0(x) = -2 e^2 \int dx' \ln |x - x'| J_0^y(x')\) constitutes self-consistent equations. These equations without the term \(-\kappa^{-2} d^2 J_0 / dx^2\) has been already studied by several authors [19]. They obtained the charge and voltage drop localized near the edge, but the localization length \(W\) is of the order of \(W \sim \sqrt{L_x e^2 / \hbar \omega_c}\) with \(\omega_c\) being the cyclotron frequency and \(L_x\) is the sample width. In our case \(L_x \to \infty\) and the current distribution has power-law tail (\(\propto x^{-2}\)) without any length scale. For a typical sample size \(W\) is much larger than the magnetic length \(\ell_B \sim \kappa^{-1}\), and \(-\kappa^{-2} d^2 J_0 / dx^2\) can be safely neglected and our analysis is consistent with the previous ones [19]. Thus the current distribution is not localized near the edge within the length scale of \(\ell_B\) where the edge mode is localized as shown shortly. Therefore the bulk current can never be neglected when one takes the Coulomb interactions into account.

**Edge Mode as a Self-Induced Eigenmode of the FQHL** [17] - We set \(E_x = E_y = B = 0\) in (3). Let us first consider the case of \(\nu = 1/(2n + 1)\) (\(m = 1\)) and the the semi-infinite sample \((x < 0)\) described above. The edge mode is derived by assuming \(J_x = 0, J_y = J_0^y e^{\gamma_x} e^{-i\omega t + ik_y y}\), and \(J^0 = J_0^0 e^{\gamma_x} e^{-i\omega t + ik_y y}\). Putting these into (3), we obtain two decoupled equations. One
is from the 1st row of (3) and is given as

$$\omega J_0^y - c^2 k_y J_0^0 = 0.$$ (8)

The other is from the 2nd and 3rd rows and is the eigenvalue problem for the inverse of the penetration length \(\gamma\) as

$$\begin{bmatrix} \eta & 8\pi c^2 g^{-1}\gamma \\ 8\pi g^{-1}\gamma & \eta \end{bmatrix} \begin{bmatrix} J_0^y \\ J_0^0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$ (9)

which gives \(\gamma = \pm \kappa = \pm g\eta/8\pi c\) and \(-J_0^y = \pm cJ_0^0\). Here we take the solution with positive \(\gamma\) in order that it is non-diverging and localized near the edge. Putting this solution into (8) we obtain the dispersion relation of this mode \(\omega = -ck_y\), where the velocity \(c\) is that of the Bogoliubov mode for the bose condensate. Hence we have obtained the chiral edge mode with the linear dispersion relation microscopically from (2) and (3), which has been assumed previously [2]. This derivation can be generalized to the hierarchy cases \((m > 1)\) in a straightforward way, but the calculation is rather complicated. The electromagnetic responses, however, can be analyzed without solving the eigenvalue problem as will be discussed in the following section.

**Electromagnetic Responses of the Edge Modes** - We calculate the response \(J^\mu(x, k_y, \omega)\) to the dynamical external field \(A_\mu\) with the frequency \(\omega\) and the wavenumber \(k_y\) along the edge. We first study the simplest case of \(\nu = 1/(2n + 1)\) \((m = 1)\), and later the hierarchy cases \((m > 1)\). Then (3) with the boundary condition (2) can be solved as

$$J_0^0(x, k_y, \omega) = \frac{1}{2\pi \eta} \kappa e^{\kappa x} i E_y(x = 0, k_y, \omega) \frac{k_y + \omega/c}{k_y + \omega/c} + \text{terms containing } E_x, B$$

$$J_0^y(x, k_y, \omega) = -\frac{1}{2\pi \eta} \kappa e^{\kappa x} i E_y(x = 0, k_y, \omega) \frac{k_y + \omega/c}{k_y + \omega/c} + \text{terms containing } E_x, B.$$ (10)

We interpret the term containing \(E_y(x = 0)\) as the charge and current of the edge channel because they are localized near the edge and also proportional to the electric field along the edge [3]. Therefore the edge channel contribution was not included in the discussion above where \(E_y\) was set to be zero. The terms containing \(E_x\) and \(B\) are regarded as the bulk
current and charge. The total edge current $I_{\text{edge}}^y$ and the edge charge $I_{\text{edge}}^0$ are the integrals of the above edge contributions over $x$. We obtain $I_{\text{edge}}^y = -(\nu/2\pi)iE_y(x = 0)/(k_y - \omega/c)$, which implies that the conductance of the edge channel is $\nu/2\pi$ when considering the case $\omega = 0$ and $E_y = -\partial_y A_0 = -ik_y A_0$.

For the hierarchy cases ($m > 1$), it can also be shown that the edge channel contributions can be singled out by setting $E_x = B = 0$ and $E_y(x = 0) \neq 0$ as in the case of $\nu = 1/(2n+1)$ ($m = 1$). Then we integrate the second and third row of (3) along the $x$ direction to obtain $K I_{\text{edge}}^y \equiv K \int^0 dx J^y = -8\pi c^2 g^{-1} J^0(x = 0)$ and $K I_{\text{edge}}^0 \equiv K \int^0 dx J^0 = -8\pi g^{-1} J^y(x = 0)$. Putting these into the first row of (3) at $x = 0$, we obtain the anomaly equation $K(\partial_t I_{\text{edge}}^0 + \partial_y I_{\text{edge}}^y) = E^y(x = 0)q/2\pi$. For the physical edge charge $I_{\text{edge}}^0$ and current $I_{\text{edge}}^y$ the anomaly equation is

$$\partial_t I_{\text{edge}}^0 + \partial_y I_{\text{edge}}^y = (\nu/2\pi)E_y,$$

which implies that the edge conductance $\sigma_{\text{edge}}$ is $\nu/2\pi$ because this equation becomes $\partial_y(I_{\text{edge}}^y + (\nu/2\pi)A_0) = 0$ when one considers the stationary flow. This $\sigma_{\text{edge}}$ is, however, different from the physical Hall conductance $\sigma_H$ due to the following reasons: (i) The edge is the equi-potential line in the steady state, and the voltage drop occurs at the current terminals as shown before, which are not taken into account in (10) and need special treatment. (ii) Even though the current terminals are taken into account, the Coulomb repulsion spreads the current distribution much deeper into the sample than the penetration depth $\kappa^{-1} \sim \ell_B$ of the edge mode. (iii) The voltage drop is not the same as the chemical potential difference since the charging energy is present.

$\sigma_{\text{edge}}$ is not measured by the Hall conductance but rather by some optical experiments. Then it is remarkable that $\sigma_{\text{edge}} = \sigma_H$ when $V(r) = 0$. $\sigma_{\text{edge}}$ is susceptible to the randomness at the edge as stressed by Kane et al. $\sigma_H$, on the other hand, is robust against random potential $V(r)$ if $j_I^\mu = 0$. Thus it is possible that the values of $\sigma_{\text{edge}}$ and $\sigma_H$ are different.

3. Quasi-Particles
We now consider the effects of the quasi-particles which can not be neglected when we take into account a random potential at the edge. In the presence of the nonzero quasi-particle (vortex) current $j^\mu_I$ in the r.h.s. of (3), the quantization of the Hall conductance breaks down. Thanks to the Maxwell term the spatial dependence of the fields are smooth even near the edge, and the only possible singularity is that of the vortex (quasi-particle). Therefore it is easy to distinguish between the current due to the quasi-particle and the bose condensate. According to this criterion the so called ”quasi-particle” in the edge channel is represented as the kink of the condensate field variables and have no singularity, and hence does not contribute to the quasi-particle current density $j^\mu_I$. This can be viewed in the following way. Consider a vortex inside the sample. This has the finite energy in the FQHL at plateau, and have accordingly tends to be excluded to the outside of the sample like a magnetic flux in the superconductor. When the center of the vortex $R_I\ell$ goes out of the sample [20], it will leave a smooth phase change nearby, which can be identified as a kink in the edge mode. With backward scatterings by the randomness near the edge, pair creations of the vortices at different edge modes occurs. The center of the vortices, however, will be repelled from the sample. These processes have been described in terms of the charged operators like $e^{i\phi}$ where $\phi$ is the bose field for the chiral edge mode [8]. In this case the quasi-particle current $j^\mu_I$ is localized near the random potential. This does not mean the effects of the randomness is also localized. It is possible that the edge modes becomes massive. If we can make the path where $j^\mu_I = 0$ connecting the two voltage terminals, however, the discussion above is not modified. The Hall conductance remains to be quantized even if the edge modes becomes massive. In the case of the point contact which has been studied in [6], on the other hand, a path of the integral (4) always crosses the region where the quasi-particle current $j^\mu$ is nonzero. Therefore the quantization of the Hall conductance breaks down and we expect the voltage drop across the point contact where the quasi-particle tunneling between the two edges of the sample occurs.

In summary, we have proposed an effective theory of fractional quantum Hall liquid with edges in the dual representation, which describes the edge and bulk in a unified way. The
Maxwell term cures the pathology of the topological theory which has no Hamiltonian. The edge modes with anomaly are derived, and the distribution of the current, charge, and Hall voltage can be also calculated. With the condition $\sum_I (K^{-1})_{IJ} = \nu/m$ and no quasi-particle in the bulk, we obtain the quantization of the Hall conductance irrespective of the detailed dynamics and the randomness at the edge.

**ACKNOWLEDGMENTS**

We are grateful to T.K.Ng, A.Zee, X.G.Wen, and Y.S. Wu for useful discussions. This work is supported by Grant-in-Aid for Scientific Research No. 04240103 from the Ministry of Education, Science, and Culture of Japan.
FIGURES

Hall bar with voltage terminals at $A, B$ and $D, E$, and the current terminals at $F, G$. $I$ is the total current flowing in and out while $J$ is the current density in the sample.
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[20] Of course the vortex outside of the sample is a fictitious object expressing the phase configuration inside the sample.