Ramification Points of Seiberg-Witten Curves

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1. Introduction
   M-theory description of the Seiberg-Witten theory
   Multiple M5-branes wrapping a punctured Riemann surface

2. Ramification points of Seiberg-Witten curves
   Ramification of an M5-brane over a Riemann sphere
   $SU(2) \times SU(2)$ SCFT and the ramification point
   $SU(3)$ pure gauge theory and Argyres-Douglas fixed points

3. Conclusion
   Discussion and Outlook
   Summary
Outline

1. Introduction
   M-theory description of the Seiberg-Witten theory
   Multiple M5-branes wrapping a punctured Riemann surface

2. Ramification points of Seiberg-Witten curves
   Ramification of an M5-brane over a Riemann sphere
   $SU(2) \times SU(2)$ SCFT and the ramification point
   $SU(3)$ pure gauge theory and Argyres-Douglas fixed points

3. Conclusion
   Discussion and Outlook
   Summary
The following brane configuration of type IIA string theory gives a four-dimensional $\mathcal{N} = 2$ $SU(2)$ superconformal field theory (SCFT). [Witten, 1997]
When $u = 0$, lifting the brane system to M-theory gives

$$x^4 + ix^5 = u$$

After turning on $u \neq 0$, these M5-branes become a single M5-brane wrapping a complex curve $f(t, v) = 0$, where

$$f(t, v) = (t - t_1)(t - t_2)v^2 - ut, \quad t = \exp(-s).$$
Seiberg-Witten curve and differential from M-theory

**Seiberg-Witten curve $C_{SW}$ from an M5-brane**

- The M5-brane from the M-theory lift of the IIA brane system wraps a complex one-dimensional algebraic curve $f(t, v) = 0$.
- This curve is identified with $C_{SW}$ of the four-dimensional theory from the IIA brane system.

**Seiberg-Witten differential $\lambda$ and M2-branes**

- When $\lambda$ is integrated over a 1-cycle $\gamma$ on $C_{SW}$, it gives the mass of the corresponding BPS state.
- The mass can also be calculated from an M2-brane ending on the M5-brane along the boundary $\gamma$, which gives us a unique $\lambda = \frac{v}{t} dt$. [Fayyazuddin-Spalinski, 1997], [Henningson-Yi, 1997], [Mikhailov, 1997]
1. **Introduction**
   - M-theory description of the Seiberg-Witten theory
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2. **Ramification points of Seiberg-Witten curves**
   - Ramification of an M5-brane over a Riemann sphere
   - $SU(2) \times SU(2)$ SCFT and the ramification point
   - $SU(3)$ pure gauge theory and Argyres-Douglas fixed points

3. **Conclusion**
   - Discussion and Outlook
   - Summary
Gaiotto’s description of $\mathcal{N} = 2$ gauge theory

- The four-dimensional $\mathcal{N} = 2$ $SU(2)$ SCFT comes from two M5-branes wrapping a Riemann sphere with four punctures, $C_G$. [Gaiotto, 2009]

- The punctures are at the poles of $\lambda = \frac{v(t)}{t} \, dt$.
- The locations of the punctures depend only on the gauge coupling parameter.
- The Coulomb branch parameter corresponds to the deformations of the M5-branes along the fiber of $T^* C_G$. 

![Diagram: Multiple M5-branes wrapping a punctured Riemann surface]
For the $C_{SW}$ from the following IIA brane configuration, there are two ways of describing it as a covering space of a complex plane.

- When the $v$-plane is the base, $C_{SW}$ is a hyperelliptic curve.
- When the $t$-plane is the base, $C_{SW}$ is a three-sheeted covering of a Riemann sphere.  

[Martinec-Warner, 1995]  
[Hollowood, 1997]
Classification of punctures

- Each puncture can be characterized by a Young tableau.

- For an $SU(N)$ gauge theory, a Young tableau of a puncture has $N$ boxes.
Young tableaux characterize ramification

- Young tableaux can describe the ramification structure of a covering space.

- Question: can we use this to understand the geometric origin of the classification of punctures?
1. Introduction
   M-theory description of the Seiberg-Witten theory
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2. Ramification points of Seiberg-Witten curves
   Ramification of an M5-brane over a Riemann sphere
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   $SU(3)$ pure gauge theory and Argyres-Douglas fixed points

3. Conclusion
   Discussion and Outlook
   Summary
Ramification points and branch points

- Consider a 2-sheeted covering map \( \pi : C \to B, \ s \mapsto s^2 \).

- When there is a nontrivial ramification in a covering map, there is a **ramification point** in the **covering space**.
- A ramification point is mapped by the covering map to a **branch point** in the **base space**.
\( C_{SW} \) wrapping a Riemann sphere

- Consider \( C_{SW} \) of \( SU(2) \) SCFT,
  \[
  f(t, v) = (t - 1)(t - t_1)v^2 - ut. 
  \]
- From the curve we can get \( C_{SW} \), a compact Riemann surface.
- \( \pi \circ \phi^{-1} : C_{SW} \to C_B \) is the covering map we want.
- We only need a local description of \( \pi \) around each \( p_i \).
- Near a point \( p_i \in C_{SW} \),
  \[
  \phi_{p_i}(s) = (t(s), v(s)) \\
  \Rightarrow \pi_{p_i}(s) = t(s). 
  \]
• When analyzing the ramification of an algebraic curve, it is convenient to introduce a ramification divisor $R_\pi$,

\[
R_\pi = \sum_{p \in C_{SW}} (\nu_p(\pi) - 1)[p] = \sum_i (\nu_{p_i}(\pi) - 1)[p_i],
\]

where $\nu_p(\pi)$ is the ramification index of a point $p$,

\[
\pi_p(s) - \pi_p(0) \propto s^{\nu_p(\pi)}, \quad \nu_p(\pi) \in \mathbb{Z}^+.
\]

• The Riemann-Hurwitz formula provides a relation between $\deg(\pi)$, $R_\pi$, and $g(C_{SW})$.

\[
\chi_{C_{SW}} = \deg(\pi) \cdot \chi_{\mathbb{C}P^1} - \deg(R_\pi)
\]

\[
\iff \deg(R_\pi) = 2(g(C_{SW}) + \deg(\pi) - 1).
\]
Example: $SU(2)$ SCFT

$$\pi_{p_1}(s) = s^2, \pi_{p_2}(s) = 1 + c_2 s^2, \pi_{p_3}(s) = t_1 + c_3 s^2, \pi_{p_4}(r^{-1}) = r^2.$$  

$$\Rightarrow R_\pi = 1 \cdot [p_1] + 1 \cdot [p_2] + 1 \cdot [p_3] + 1 \cdot [p_4].$$

• Each branch point has the same Young tableau as the corresponding puncture!
• Consistent with the Riemann-Hurwitz formula,
  $$\deg(R_\pi) = 1 + 1 + 1 + 1 = 4 = 2(g(C_{SW}) + \deg(\pi) - 1).$$
1. Introduction
   M-theory description of the Seiberg-Witten theory
   Multiple M5-branes wrapping a punctured Riemann surface

2. Ramification points of Seiberg-Witten curves
   Ramification of an M5-brane over a Riemann sphere
   $SU(2) \times SU(2)$ SCFT and the ramification point
   $SU(3)$ pure gauge theory and Argyres-Douglas fixed points

3. Conclusion
   Discussion and Outlook
   Summary
Consider the following IIA brane configuration that gives a four-dimensional $\mathcal{N} = 2$ $SU(2) \times SU(2)$ SCFT.

\[ f(t, v) = (t - 1)(t - t_1)(t - t_2)v^2 - u_1 t^2 - u_2 t. \]
\( \mathcal{C}_{SW} \) of \( SU(2) \times SU(2) \) SCFT and its ramification

- The ramification of \( \mathcal{C}_{SW} \) over \( \mathcal{C}_B \) is

\[ \pi(q) = -u_2/u_1 : \text{branch point of a different kind} \]

- Comes from solving \( dt(q) = 0 \) along the curve \( f(t, v) = 0 \).
- Does not correspond to any puncture.
- Depends on the Coulomb branch parameters \( \{u_1, u_2\} \).
- Consistent with the Riemann-Hurwitz formula,
\[
\deg(R_\pi) = 1 + 1 + 1 + 1 + 1 + 1 = 6 = 2(g(\mathcal{C}_{SW}) + \deg(\pi) - 1).
\]
Two different kinds of branch points

- \(\{\pi(p_i)\}\) and \(\pi(q)\) are from the ramification points of \(C_{SW}\).
- When considering \(C_{SW}\), \(\{\pi(p_i)\}\) are from the points we added to compactify it, whereas \(\pi(q)\) is from the ramification point of \(C_{SW}\), which is located at \(dt = 0\) along \(C_{SW}\).
Physical interpretation of the branch points

- A branch point from the ramification point of $C_{SW}$ has a physical interpretation of being a contact point of M5-branes. [Gaiotto-Moore-Neitzke, 2009]

- A puncture is understood as a real codimension two defect on M5-branes from a transversal M5-brane. [Gaiotto, 2009]
Moving the branch point over $C_B$

- Changing the Coulomb branch parameters moves the location of $\pi(q)$ on $C_B$.

- We can use this to illustrate various changes of Coulomb branch parameters.
Collision of two branch points

- When \( \pi(q) \) collides with \( \pi(p_i) \), the original Seiberg-Witten curve factors into two curves.

- The new branch point is a useful tool to visualize various interesting limits of Coulomb branch parameters.
1. Introduction
   M-theory description of the Seiberg-Witten theory
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2. Ramification points of Seiberg-Witten curves
   Ramification of an M5-brane over a Riemann sphere
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   $SU(3)$ pure gauge theory and Argyres-Douglas fixed points

3. Conclusion
   Discussion and Outlook
   Summary
Brane configuration of $SU(3)$ pure gauge theory

- The following IIA brane configuration gives a four-dimensional $\mathcal{N} = 2$ $SU(3)$ pure gauge theory.

- $C_{SW}$ of the theory is the zero locus of

$$f(t, v) = t^2 + (v^3 - u_2 v - u_3)t + 1.$$
By analyzing the ramification of $C_{SW}$ over $C_B$, we get

- $\pi(p_1) = 0$, $\pi(p_2) = \infty$.
- $\pi(q_{ab}) = \left( v_{2a}^3 + \frac{u_2}{2} \right) + b \sqrt{\left( v_{2a}^3 + \frac{u_2}{2} \right)^2 - 1}$, $v_{2a} = a \sqrt{u_2}/3$. 

$C_{SW}$ of $SU(3)$ pure gauge theory and its ramification
Branch points near the limit of AD fixed points

- When the Coulomb branch parameters approach one of the Argyres-Douglas fixed points, \[ \text{[Argyres-Douglas, 1995]} \]

\[ u_2 = 0, \quad u_3 = \pm 2, \]

\( \{\pi(q_{ab})\} \) gather together around \( t = 1 \).

\[ u_2 \to 0, \quad u_3 \to 2 \]

\[ u_2 = 3\epsilon^2\rho, \quad u_3 = 2 + 2\epsilon^3 \]
Rescaling of parameters

- Zoom in on the part of $C_B$ near $t = 1$, that is, redefine the variables as

$$
t = 1 + i\epsilon^{3/2}w, \quad v = \epsilon z, \quad u_2 = 0 + 3\epsilon^2 \rho, \quad u_3 = 2 + 2\epsilon^3,
$$

and take $\epsilon \to 0$.

- Then $f(t, v)$ becomes

$$
f(t, v) = (t - 1)^2 + \epsilon^3(z^3 - 3\rho z - 2)t
\approx (-w^2 + z^3 - 3\rho z - 2)\epsilon^3 + \mathcal{O}(\epsilon^{9/2}),
$$
Appearance of the small torus

- The genus 1 curve given by \( w^2 = z^3 - 3\rho z - 2 \) is the small torus that appears at the Argyres-Douglas fixed points.

\[ \text{Argyres-Douglas, 1995} \]

\( w_\pm = \pm \sqrt{z^3 - 3\rho z - 2}, \)
\( z_i^3 - 3\rho z_i - 2 = 0. \)
1. Introduction
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   Multiple M5-branes wrapping a punctured Riemann surface

2. Ramification points of Seiberg-Witten curves
   Ramification of an M5-brane over a Riemann sphere
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3. Conclusion
   Discussion and Outlook
   Summary
How to construct the global picture of $C_{SW}$ from $C_B$

- So far we analyzed only the local picture of $C_{SW}$ around its ramification points. How can we have the global description of it?
- How can we build the whole $C_{SW}$ from $C_B$?
M2-brane ending at a ramification point

- A D2(M2)-brane ending on the IIA brane system gives a surface operator. [Alday-Gaiotto-Gukov-Tachikawa-Verlinde, 2009]

- Use this M2-brane to study the ramification point.
- Take the M2- and M5-brane system and turn it into IIB geometry plus D-brane. [Dijkgraaf-Hollands-Sułkowski-Vafa, 2007]
- Study the effective 2D theory on the M2-brane. [Hanani-Hori, 1997; Dorey, 1998; Gaiotto-Moore-Neitzke, 2011]
M5-branes wrapping a punctured Riemann surface give us a correspondence between the four-dimensional $\mathcal{N} = 2$ gauge theory and the two-dimensional CFT on the Riemann surface. [Alday-Gaiotto-Tachikawa, 2009]

Various collisions of branch points give us the factorization of the base Riemann surface. This may be observed, through the AGT conjecture, as $Z \rightarrow Z_1 Z_2$, and/or $\mathcal{F} \rightarrow \mathcal{F}_1 \mathcal{F}_2$. 
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2. Ramification points of Seiberg-Witten curves
   Ramification of an M5-brane over a Riemann sphere
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3. Conclusion
   Discussion and Outlook
   Summary
• When the Seiberg-Witten curve of a four-dimensional $\mathcal{N} = 2$ gauge theory wraps a Riemann surface as a multi-sheeted cover, the curve develops ramification points that are mapped to branch points on the Riemann surface.

• The branch points are different from the punctures of Gaiotto, and their locations on the Riemann surface depend in general on every parameter of the theory, including Coulomb branch parameters.

• These branch points can help us to explore interesting physics in various limits of the parameters, including Argyres-Douglas fixed points.