On a certain asymptotic relationship involving 
\( \vartheta(t) - \lfloor t \rfloor \) and \( t^{1/2} \)

Hisanobu Shinya

July 21, 2008

Abstract

Let \( \lfloor t \rfloor \) denote the greatest positive integer less than or equal to a given positive real number \( t \) and \( \vartheta(t) \) the Chebyshev \( \vartheta \)-function. In this paper, we prove a certain asymptotic relationship involving 
\( \vartheta(t) - \lfloor t \rfloor \) and \( t^{1/2} \).

Email address: shinyah18@yahoo.co.jp
Keywords: Chebyshev \( \vartheta \)-function, power series, Riemann zeta function, Riemann Hypothesis.
2000 MSC: 11N37, 30B10.

1 Introduction

Let \( \psi(t) \) and \( \vartheta(t) \) denote the Chebyshev \( \psi \)-function and the Chebyshev \( \vartheta \)-function, respectively; as always, if \( p \) denotes primes and \( w \) positive integers, then using the Mangoldt \( \Lambda \)-function which is

\[
\Lambda(n) := \begin{cases} 
\log p & : n = p^w \\
0 & : \text{otherwise},
\end{cases}
\]
the former is defined by
\[ \psi(t) := \sum_{n \leq t} \Lambda(n), \]
and the latter by
\[ \vartheta(t) := \sum_{p \leq t} \log p. \]
Let \([t]\) denote the greatest positive integer less than or equal to a given positive real number \(t\) and
\[ \eta(t) := \vartheta(t) - [t]. \]

In this discussion, we prove a certain asymptotic relationship involving \(\eta(t)\) and \(t^{1/2}\).

Classical analysis of arithmetical functions has brought forth a number of concise asymptotic formulas such as [1, Theorem 4.9]
\[ \psi(x) = O(x) \quad \text{as } x \to \infty \]  
(1)

or [1, Theorem 4.11]
\[ \sum_{n \leq x} \psi\left(\frac{x}{n}\right) = x \log x + O(x) \quad \text{as } x \to \infty. \]  
(2)

At the time when the prime number theorem was yet a conjecture, formulas such as (1) and (2) may have been considered as evidences for the theorem. History, as in the case of the prime number theorem, suggests that while asymptotic formulas do not directly put an end to unsolved problems, they may offer some evidences for such problems.

Given an analytic function \(f(s)\), we denote the \(n\)th derivative of \(f(s)\) by \(f^{(n)}(s)\). We define
\[ E(s) := \int_{1}^{\infty} \frac{\eta(t)dt}{t^{s+1}}, \]
and so
\[ E^{(n)}(s) = (-1)^n \int_1^\infty \frac{\eta(t)(\log t)^n}{t^{s+1}} dt, \]
which are valid for Re(s) > 1 because \[1, \text{Theorem 4.10}\]
\[ \eta(t) = \vartheta(t) - [t] = O(t) \quad \text{as } t \to \infty. \]

The validity of the integral representation of the derivative of \( E(s) \) can be shown with arguments in Section 11.7 of \[1\], taking some care with the fact that the integrand is piecewise continuous.

We denote the Riemann zeta function with \( \zeta(s) \), which is defined in the traditional manner by
\[ \zeta(s) := \sum_{n=1}^\infty \frac{1}{n^s}, \quad \text{for } \text{Re}(s) > 1. \quad \tag{3} \]

Let \( D(s_0; h) \) \((h > 0)\) be any disk of radius \( h \) centered at \( s_0 \) (Re\( (s_0) > 1 \)) such that
1. \( 1/2 \in D(s_0; h) \);
2. for all \( s \in D(s_0; h) \), we have Re\( (s) > 1/3 \) and \( \zeta(s), \zeta(2s) \neq 0 \).

The existence of such a disk \( D(s_0; h) \) follows from the fact that the magnitude of the imaginary part of any nontrivial root \( \rho \) of the \( \zeta \)-function is greater than 10 \[2, \text{Chapter 6}\]. For instance, consider choosing \( s_0 = 1 + q \) and \( h = 1/2 + q' \), where \( q' > q > 0 \) and \( q' \) is arbitrarily small. Then it is easy to see that for any \( s \in D(1+q; 1/2+q'), \text{Re}(s) > 1/3, |\text{Im}(2s)| < 10 \) and \( 1/2 \in D(1+q; 1/2+q') \).

Throughout the paper, the symbols \( s_0 \) and \( D(s_0; h) \) have the same meanings as defined above.

Being motivated by the optimistic vision on the study of asymptotic number-theoretic relationships described above, we shall address the following theorem.
Theorem 1. Let $s_0$ (Re$(s_0) > 1$) be a complex number such that for some $h > 0$, the disk $D(s_0; h)$ satisfies the conditions 1 and 2. Then we have

$$E^{(n)}(s_0) \sim (-1)^{n+1} n! (s_0 - 1/2)^{-n - 1} \text{ as } n \to \infty.$$ 

Without assuming the Riemann Hypothesis, which is equivalent [2, Chapter 5] to the formula

$$\eta(t) = O(t^{1/2 + \epsilon}) \text{ as } t \to \infty, \text{ for each } \epsilon > 0, \quad (4)$$

it is generally hard to obtain results concerning the function $\eta(t)$, the main reason being that few methods for elaborating formulas such as (5) which do not depend on the distribution of nontrivial roots of the $\zeta$-function have been widely known. Since for Re$(s) > 1/2$,

$$\frac{d^n}{ds^n}[(s - 1/2)^{-1}] = \frac{d^n}{ds^n} \left[ \int_1^\infty \frac{t^{1/2} dt}{t^{s+1}} \right] = (-1)^n \int_1^\infty \frac{t^{1/2} (\log t)^n dt}{t^{s+1}}$$

and

$$\frac{d^n}{ds^n}[(s - 1/2)^{-1}] = (-1)^n n! (s - 1/2)^{-n - 1},$$

if $s_0$ is as defined in Theorem 1 then the theorem is

$$\lim_{n \to \infty} \int_1^\infty \frac{\eta(t)(\log t)^n dt}{(s_0)^{n+1}} = -1.$$

Hence, Theorem 1 may have some implications for the Riemann Hypothesis (i.e., the equation (4)), but we are technically not ready for such an analysis at present. Hence, in this paper, we focus on Theorem 1.

To prove Theorem 1, we use the fact that the function

$$\frac{-1}{s} \left( \frac{\zeta'(s)}{\zeta(s)} + \zeta(s) \right)$$

is analytic (i.e., Lemma 2) on $D(s_0; h)$ and another fact that the function $\Delta(s)$ extends to a meromorphic function on $D(s_0; h)$ with a simple pole at
$s = 1/2$ and residue 1 at $s = 1/2$ (i.e., Lemma 4). Other than these results of analytic number theory, we employ only basic results on analytic functions (i.e., Lemmas 5 and 6).

We finish this section with the following preliminary lemmas.

Let
\[ \delta(t) := \psi(t) - \vartheta(t) \]

and
\[ \Delta(s) := \int_{1}^{\infty} \frac{\delta(t)dt}{t^{s+1}}. \]

Since [1, Theorem 4.1] \[ \delta(t) = O(t^{1/2}(\log t)^2) \quad \text{as } t \to \infty, \]

the integral representation for $\Delta(s)$ is valid for $\text{Re}(s) > 1/2$.

The following lemma gives a relationship between $\eta(t)$, $\delta(t)$, and $\zeta(s)$, and becomes the starting point for a proof of Theorem 1.

**Lemma 1.** For $\text{Re}(s) > 1$, we have
\[
-\frac{1}{s} \left( \frac{\zeta'(s)}{\zeta(s)} + \frac{\zeta(s)}{s} \right) = E(s) + \Delta(s).
\] (5)

**Proof.** The following formula [1, Exercise 1, Chapter 11]
\[
-\frac{\zeta'(s)}{s \zeta(s)} = \int_{1}^{\infty} \frac{\psi(t)dt}{t^{s+1}}, \quad \text{Re}(s) > 1
\] (6)

is well-known. By the definition $\delta(t) = \psi(t) - \theta(t)$, we write (6) as
\[
-\frac{\zeta'(s)}{s \zeta(s)} = \int_{1}^{\infty} \frac{(\vartheta(t) + \delta(t))dt}{t^{s+1}}, \quad \text{Re}(s) > 1.
\] (7)

Rewriting [1, Exercise 1, Chapter 11] (3) as
\[
\frac{\zeta(s)}{s} = \int_{1}^{\infty} \frac{|t|dt}{t^{s+1}}, \quad \text{Re}(s) > 1,
\] (8)
and taking the difference of the left and right members of (7) and (8), we have
\[-\frac{1}{s} \left( \frac{\zeta'(s)}{\zeta(s)} + \zeta(s) \right) = \int_1^\infty \eta(t) \frac{dt}{t^{s+1}} + \int_1^\infty \delta(t) \frac{dt}{t^{s+1}}, \quad \text{Re}(s) > 1.\]

This completes the proof of the lemma.

Lemma 2. The function
\[-\frac{1}{s} \left( \frac{\zeta'(s)}{\zeta(s)} + \zeta(s) \right)\]
is analytic on $D(s_0; h)$.

Proof. Since there exists no nontrivial root of $\zeta(s)$ on $D(s_0; h)$, both of the functions
\[-\frac{1}{s} \left( \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s - 1} \right) \quad \text{and} \quad -\frac{1}{s} \left( \zeta(s) - \frac{1}{s - 1} \right)\]
are analytic on $D(s_0; h)$. (Proofs of the analyticity of both functions at $s = 1$ are given in [1 Theorem 13.8] for that of the former and in [3 Theorem 1.2, Chapter 16] for that of the latter.) It is an elementary fact that the sum of two functions analytic on $D(s_0; h)$ is analytic on $D(s_0; h)$; hence, the sum of two functions described above is analytic on $D(s_0; h)$. This completes the proof of the lemma.

Lemma 3. [3, Lemma 1.2, pp. 374] Let $\{f_n\}$ be a sequence of analytic functions on an open set $S$ and let $f_n(s) = 1 + h_n(s)$. Suppose that
\[\sum_{n=1}^{\infty} h_n(s)\]
converges uniformly and absolutely on $S$. Let $K$ be any compact subset of $S$ not containing any of the zeros of the functions $f_n$ for all $n = 1, 2, \ldots$. Then the product $\prod_{n=1}^{\infty} f_n(s)$ converges to an analytic function $f$ on $S$, and for $s \in K$ we have

$$
\frac{f'(s)}{f(s)} = \sum_{n=1}^{\infty} \frac{f'_n(s)}{f_n(s)}.
$$

(9)

**Lemma 4.** The function $\Delta(s)$ extends to a meromorphic function on the disk $D(s_0; h)$ with only a simple pole at $s = 1/2$. The residue of $\Delta(s)$ at $s = 1/2$ is 1.

*Proof.* Let $p_n$ denote the $n$th prime. For $\text{Re}(s) > 1$, choose

$$
f_n(s) = \frac{1}{1 - p_n^{-s}}
$$

in Lemma 3. Then we have [3, Proof of Theorem 1.3, pp. 443]

$$
- \frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{\log p}{p^s} + \sum_{n=2}^{\infty} \sum_p \frac{\log p}{p^{ns}}.
$$

(10)

It is easy to show (see [1, Theorem 4.2]) that the first series on the right side of (10) is the Dirichlet series representation for the function

$$
\int_1^{\infty} \frac{\vartheta(t)dt}{ts+1}.
$$

With [6] and the definition of $\delta(t)$, it is easy to see that the second series on the right side of (10) is the Dirichlet series representation for the function $s\Delta(s)$, valid for $\text{Re}(s) > 1/2$ as noted above.

Hence, to prove the lemma, it is sufficient to show that the analytic continuation of the function

$$
\frac{1}{s} \sum_{n=2}^{\infty} \sum_p \frac{\log p}{p^{ns}}
$$

7
to $D(s_0; h)$ is meromorphic on $D(s_0; h)$ with only a simple pole at $s = 1/2$ and residue 1 at $s = 1/2$. We separate the series as

$$\Delta(s) = \frac{1}{s} \sum_{n=2}^{\infty} \sum_{p} \frac{\log p}{p^{ns}} = \frac{1}{s} \sum_{p} \frac{\log p}{p^{2s}} + \frac{1}{s} \sum_{n=3}^{\infty} \sum_{p} \frac{\log p}{p^{ns}}. \quad (11)$$

The second series on the extreme right side of (11) converges uniformly and absolutely on $D(s_0; h)$ (by the condition 2 for the definition of $D(s_0; h)$), and so it is analytic there.

To analyze the first series on the extreme right side of (11), we choose

$$f_n(s) = \frac{1}{1 - p_n^{-2s}}$$

in Lemma 3 for $\text{Re}(s) > \frac{1}{2}$. Then we have

$$f_n'(s) = -2p_n^{-2s} \log p_n (1 - p_n^{-2s})^{-2},$$

and

$$\frac{f_n'(s)}{f_n(s)} = -2p_n^{-2s} \log p_n = -\frac{2 \log p_n}{p_n^{2s} - 1}.$$  

Using the expansion

$$\frac{1}{p_n^{2s} - 1} = \frac{1}{p_n^{2s}} \frac{1}{1 - p_n^{-2s}} = \frac{1}{p_n^{2s}} \left( 1 + \frac{1}{p_n^{2s}} + \frac{1}{p_n^{4s}} + \cdots \right) = \frac{1}{p_n^{2s}} + \frac{1}{p_n^{4s}} + \cdots,$$

it is easy to see that

$$-\frac{1}{2s} \frac{d}{ds} [\zeta(2s)] = \sum_{p} \frac{\log p}{p^{2s}} + \sum_{n=2}^{\infty} \sum_{p} \frac{\log p}{p^{2ns}},$$

or multiplying by $\frac{1}{s}$,

$$-\frac{1}{2s} \frac{d}{ds} [\zeta(2s)] = \frac{1}{s} \sum_{p} \frac{\log p}{p^{2s}} + \frac{1}{s} \sum_{n=2}^{\infty} \sum_{p} \frac{\log p}{p^{2ns}}. \quad (12)$$
Since \( \frac{d}{ds} \zeta(2s) \) has a simple pole at \( s = 1/2 \) with residue \(-1\) (see [3, Lemma 1.4, pp. 180]) and no other singularities on \( D(s_0; h) \), \( \zeta(2s) \neq 0 \) on \( D(s_0; h) \), and the remaining series on the right side of (12) converges uniformly and absolutely for \( \text{Re}(s) > 1/4 \), we find out that the analytic continuation of the function \( \frac{1}{s} \sum_p \frac{\log p}{p^s} \) to \( D(s_0; h) \) is meromorphic on \( D(s_0; h) \) with only a simple pole at \( s = 1/2 \) and residue 1 at \( s = 1/2 \). By the first equality in (11), together with the information on the function \( \frac{1}{s} \sum_p \frac{\log p}{p^s} \), the proof of the lemma completes.

\[ \square \]

**Lemma 5.** [3] Chapters 2 and 3] Let \( f \) be analytic on a closed disk \( \bar{D}(z_0; R) \) of radius \( R > 0 \) centered at \( z_0 \). Then \( f \) has the unique power series expansion

\[
f(s) = \sum_{n=0}^{\infty} a_n (s - z_0)^n,
\]

where

\[
a_n = \frac{f^{(n)}(z_0)}{n!}.
\]

The radius of the convergence of the series is \( \geq R \), and the convergence is absolute.

**Lemma 6.** [3] Chapter 5] If \( f \) is analytic on some disk \( D'(z_0; R) \) centered and punctured at \( z_0 \) and has a simple pole at \( s = z_0 \), then \( f \) has the Laurent series expansion

\[
f(s) = \frac{a_{-1}}{s - z_0} + \sum_{n=0}^{\infty} a_n (s - z_0)^n,
\]

which is valid on \( D'(z_0; R) \).

## 2 The proof of Theorem 1

All the symbols have the same meanings as defined in the previous section. In this section, we give a proof of Theorem [1]
By Lemmas 1, 2, and 4, it is plain that both of the functions

\[ E(s) \quad \text{and} \quad \Delta(s) \]

are meromorphic on \( D(s_0; h) \) with only a simple pole at \( s = 1/2 \).

Now define \( H(s) \) by

\[ H(s) := E(s) + (s - 1/2)^{-1}. \tag{13} \]

We show that \( H(s) \) is analytic on \( D(s_0; h) \). By Lemmas 4 and 6, the function

\[ U(s) := (s - 1/2)^{-1} - \Delta(s) \]

is analytic on \( D(s_0; h) \).

With Lemma 1, we have

\[ \frac{-1}{s} \left( \frac{\zeta'(s)}{\zeta(s)} + \zeta(s) \right) = E(s) + \Delta(s) = E(s) + (s - 1/2)^{-1} - U(s), \]

and so rewriting this expression with (13) as

\[ H(s) = U(s) - \frac{1}{s} \left( \frac{\zeta'(s)}{\zeta(s)} + \zeta(s) \right), \]

Lemma 2 guarantees that \( H(s) \) is indeed analytic on \( D(s_0; h) \).

With Lemma 5, we write

\[ H(s) = \sum_{n=0}^{\infty} h_n (s - s_0)^n, \]

\[ E(s) = \sum_{n=0}^{\infty} E^{(n)}(s_0)(s - s_0)^n \frac{1}{n!}, \]

and

\[ (s - 1/2)^{-1} = \sum_{n=0}^{\infty} (-1)^n (s_0 - 1/2)^{-n-1} (s - s_0)^n. \]
With these expressions, we rewrite (13) as

\[
\sum_{n=0}^{\infty} h_n(s - s_0)^n = \sum_{n=0}^{\infty} \left( \frac{E^{(n)}(s_0)}{n!} + (-1)^n(s_0 - 1/2)^{-n-1} \right) (s - s_0)^n. \tag{14}
\]

Lemma 5 implies that for each \(n = 0, 1, 2, \ldots\),

\[
h_n = \frac{E^{(n)}(s_0)}{n!} + (-1)^n(s_0 - 1/2)^{-n-1}. \tag{15}
\]

Now in (14), let \(s = 1/2\). Since \(H(s)\) is analytic on \(D(s_0; h)\), the series

\[
\sum_{n=0}^{\infty} |h_n||1/2 - s_0|^n
\]

converges, which implies that

\[
\lim_{n \to \infty} |h_n||1/2 - s_0|^n = 0. \tag{16}
\]

To prove Theorem 1, for each \(n = 0, 1, 2, \ldots\), we write

\[
\left| \frac{E^{(n)}(s_0)}{n!} + (-1)^n(s_0 - 1/2)^{-n-1} \right| = \lambda_n|s_0 - 1/2|^{-n-1}, \tag{17}
\]

or with (15),

\[
|h_n| = \lambda_n|s_0 - 1/2|^{-n-1}. \tag{18}
\]

Substituting (18) to (16), we obtain

\[
\lim_{n \to \infty} |h_n||1/2 - s_0|^n = \lim_{n \to \infty} \lambda_n|s_0 - 1/2|^{-1} = 0, \tag{19}
\]

which implies

\[
\lim_{n \to \infty} \lambda_n = 0. \tag{20}
\]

Dividing (17) by \(|s_0 - 1/2|^{-n-1}\), we have

\[
\left| \frac{E^{(n)}(s_0)}{n!(s_0 - 1/2)^{-n-1}} + (-1)^n \right| = \left| \frac{E^{(n)}(s_0)}{(-1)^n n!(s_0 - 1/2)^{-n-1}} + 1 \right| = \lambda_n, \tag{21}
\]

11
where in the first equality, we have used the simple observation

\[ |X + (-1)^n| = |(-1)^n((-1)^n X + 1)| = |(-1)^n X + 1|, \]

valid for any complex number \( X \). The proof of Theorem 1 completes by (20) and (21).

**Acknowledgements** I thank Jonathan Sondow for his advice on the exposition and the trick of substituting \( s = \frac{1}{2} \) to (14) in the proof of Theorem 1 which has shortened the argument. This technique also appears in the proof of Lemma 2 of the paper [4], and in fact Theorem 1 is a corollary to the lemma of Sondow-Zlobin.

**References**

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, New York, 1976.

[2] H. M. Edwards, *Riemann’s Zeta Function*, Dover, New York, 2001. (First published 1974.)

[3] S. Lang, *Complex Analysis*, 4th ed., Springer, New York, 1999.

[4] J. Sondow and S. Zlobin, Integrals Over Polytopes, Multiple Zeta Values and Polylogarithms, and Euler’s Constant, arXiv:math.NT/0705.0732, v2.