Invertible extensions of symmetric operators and
the corresponding generalized resolvents.

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1 Introduction.

Let \( A \) be a closed symmetric invertible operator in a Hilbert space \( H \). The
domain of \( A \) is not supposed to be necessarily dense in \( H \). Let \( \tilde{A} \) be a self-
adjoint extension of \( A \), acting in a Hilbert space \( \tilde{H} \supseteq H \). Recall that an
operator-valued function \( R_\lambda \), given by the following relation:

\[
R_\lambda = R_\lambda(A) = R_{s;\lambda}(A) = P_{\tilde{H}} \left( \tilde{A} - \lambda E_{\tilde{H}} \right)^{-1}|_H, \quad \lambda \in \mathbb{R},
\]

is said to be a generalized resolvent of the symmetric operator \( A \) (corresponding to the extension \( \tilde{A} \)). Fix an arbitrary point \( \lambda_0 \in \mathbb{R} \). An arbitrary
generalized resolvent \( R_{s;\lambda} \) of the operator \( A \) is given by Shtraus’s formula:

\[
R_{s;\lambda} = \left\{ \begin{array}{l}
(A_F(\lambda) - \lambda E_H)^{-1}, \quad \lambda \in \Pi_{\lambda_0}, \\
(A_{F^*}(\lambda) - \lambda E_H)^{-1}, \quad \lambda \in \Pi_{\lambda_0}^*,
\end{array} \right.
\]

(1)

where \( F(\lambda) \) is a function from \( S;\lambda_0(\Pi_{\lambda_0} ; N_{\lambda_0}(A), N_{\lambda_0}^*(A)) \). Conversely, an
arbitrary function \( F(\lambda) \in S;\lambda_0(\Pi_{\lambda_0} ; N_{\lambda_0}(A), N_{\lambda_0}^*(A)) \) defines by relation (1) a
generalized resolvent \( R_{s;\lambda} \) of the operator \( A \). Moreover, for different functions from \( S;\lambda_0(\Pi_{\lambda_0} ; N_{\lambda_0}(A), N_{\lambda_0}^*(A)) \) there correspond different
generalized resolvents of the operator \( A \). By \( S;\lambda_0(\Pi_{\lambda_0} ; N_{\lambda_0}(A), N_{\lambda_0}^*(A)) = S(\Pi_{\lambda_0} ; N_{\lambda_0}(A), N_{\lambda_0}^*(A)) \) we denote a set of all operator-valued functions
\( F(\lambda) \in S(\Pi_{\lambda_0} ; N_{\lambda_0}(A), N_{\lambda_0}^*(A)) \), which are \( \lambda_0 \)-admissible with respect to the
operator \( A \). Throughout this paper we shall use the notations from the survey paper [1].

Our main aim here is to characterize those generalized resolvents \( R_{s;\lambda} \), which are generated by an (at least one) invertible self-adjoint extension. Such generalized resolvents we shall call generalized I-resolvents. A
generalized I-resolvent will be described by Shtraus’s formula (1), where the
parameter \( F(\lambda) \) satisfies a boundary condition at 0. As a by-product, using
the generalized Neumann’s formulas we describe all invertible extensions of
a symmetric operator inside \( H \).

Notations. As it was already mentioned, in this paper we use notations from [1].
2 Invertible extensions.

Let $A$ be a closed symmetric invertible operator in a Hilbert space $H$. Let $z$ from $\mathbb{C}_{-} (\mathbb{C}_{+})$ be a fixed number. Recall that the following formulas (see [1, Theorem 3.13])

$$D(B) = D(A) \oplus (T - E_H)D(T),$$

$$B(f + T\psi - \psi) = Af + zT\psi - \overline{\psi}, \quad f \in D(A), \ \psi \in D(T),$$

establish a one-to-one correspondence between all admissible with respect to $A$ isometric operators $T$, $D(T) \subseteq \mathcal{N}(A)$, $R(T) \subseteq \mathcal{N}(A)$, and all symmetric extensions $B$ of the operator $A$. We have

$$D(T) = \mathcal{N}(A) \cap R(B - zE_H),$$

$$T \subseteq (B - \overline{\pi}E_H)(B - zE_H)^{-1}.$$  \hspace{1cm} (5)

Formulas (2), (3) define a one-to-one correspondence between all admissible with respect to $A$ non-expanding operators $T$, $D(T) \subseteq \mathcal{N}(A)$, $R(T) \subseteq \mathcal{N}(A)$, and all dissipative (respectively accumulative) extensions $B$ of the operator $A$. Relations (4), (5) hold in this case, as well.

Consider the Cayley transformation of the operator $A$:

$$U_z = U_z(A) = (A - \overline{\pi}E_H)(A - zE_H)^{-1} = E_H + (z - \overline{\pi})(A - zE_H)^{-1}. \hspace{1cm} (6)$$

The operator $B$ may be also determined by the following relations:

$$W_z = (B - \overline{\pi}E_H)(B - zE_H)^{-1} = E_H + (z - \overline{\pi})(B - zE_H)^{-1},$$

$$B = (zW_z - \overline{\pi}E_H)(W_z - E_H)^{-1} = zE_H + (z - \overline{\pi})(W_z - E_H)^{-1}.$$  \hspace{1cm} (8)

$$W_z = U_z(A) \oplus T.$$  \hspace{1cm} (9)

It is readily checked that

$$\mathcal{M}(A) = \mathcal{M}(A^{-1}), \quad \mathcal{N}(A) = \mathcal{N}(A^{-1}), \quad \lambda \in \mathbb{R};$$

$$U_{\lambda}(A) = \frac{\overline{\lambda}}{\lambda}U_{\lambda}(A^{-1}), \quad \lambda \in \mathbb{R}.$$  \hspace{1cm} (11)

Theorem 1 Let $A$ be a closed symmetric invertible operator in a Hilbert space $H$, and $z \in \mathbb{R}$ be a fixed point. Let $T$ be an admissible with respect to $A$ non-expanding operator with $D(T) \subseteq \mathcal{N}(A)$, $R(T) \subseteq \mathcal{N}(A)$. The following two conditions are equivalent:
(i) The operator $B$, defined by (2) and (3), is invertible;
(ii) The operator $zT$ is $\frac{1}{z}$-admissible with respect to $A^{-1}$.

**Proof.** From relation (8) it follows that $B$ is invertible if and only if the operator $\tilde{z}W_2$ has no non-zero fixed elements. By (9),(11) we see that
\[
\tilde{z}W_2 = \tilde{z}U(A) \oplus \tilde{z}T = U_1(A^{-1}) \oplus \tilde{z}T.
\]
It remains to apply Theorem 3.6 from [1]. ✷

**Remark 1** By the definition of an admissible operator condition (ii) of the last theorem is equivalent to the following condition:
(iii) the operator $T - \frac{\tilde{z}}{z}X_1(A^{-1})$ is invertible,
where $X_\lambda(A^{-1})$ is the forbidden operator for $A^{-1}$.

**Theorem 2** Let $A$ be a closed symmetric invertible operator in a Hilbert space $H$. Suppose that $A$ has finite defect numbers. Then there exists an invertible self-adjoint operator $\tilde{A} \supseteq A$ in a Hilbert space $\tilde{H} \supseteq H$.

**Proof.** Let the operator $A$ from the formulation of the theorem have the deficiency index $(n, m)$, $n, m \in \mathbb{Z}_+$, $n + m \neq 0$. Set $\mathcal{H} = H \oplus H$ and $\mathcal{A} = A \oplus (-A)$. The closed symmetric operator $\mathcal{A}$ has equal defect numbers. Fix an arbitrary number $z \in \mathbb{R}_e$. Let $\{f_k\}_{k=1}^{n+m}$ be an orthonormal basis in $\mathcal{N}_z(A)$. Set
\[
T(\alpha f_1) = \alpha h, \quad \alpha \in \mathbb{C},
\]
where $h \in \mathcal{N}_z(A)$ is an arbitrary element such that $\|h\|_{\mathcal{H}} = 1$, and $h \neq \frac{\tilde{z}}{z}X_1(A^{-1})f_1$ if $f_1 \in D(X_1(A^{-1}))$; $h \neq X_\lambda(A)f_1$ if $f_1 \in D(X_\lambda(A))$. The operator $T$ with the domain $\text{span}\{f_1\}$ is $z$-admissible with respect to $\mathcal{A}$ and condition (iii) holds for $\mathcal{A}$. By Theorem 1 the symmetric extension $B$, corresponding to $T$, is invertible. The defect numbers of $B$ are equal to $n + m - 1$. If $B$ is not self-adjoint, we can take $B$ instead of $\mathcal{A}$ in the above construction to obtain a closed symmetric extension with the deficiency index $(n + m - 2, n + m - 2)$. Repeating this procedure we shall construct a self-adjoint invertible extension of $\mathcal{A}$. ✷

3 Generalized resolvents.

Consider a closed symmetric invertible operator $A$ in a Hilbert space $H$. Choose and fix an arbitrary point $\lambda_0 \in \mathbb{R}_e$. A function $F(\lambda) \in \mathcal{S}_{\mathbb{R};\lambda_0}(\Pi_{\lambda_0} ; \mathcal{N}_{\lambda_0}(A) , \mathcal{N}^{\perp}_{\lambda_0}(A))$
is said to be $\lambda_0$-I-admissible with respect to the operator $A$, if the validity of
\[
\lim_{\lambda \in \Pi_{\lambda_0}, \lambda \to 0} F(\lambda) \psi = \frac{\lambda_0}{\lambda_0} X_{\lambda_0} (A^{-1}) \psi,
\]
\[
\lim_{\lambda \in \Pi_{\lambda_0}, \lambda \to 0} \left[ \frac{1}{|\lambda|} (\|\psi\|_H - \|F(\lambda)\psi\|_H) \right] < +\infty,
\]
for some $\varepsilon$: $0 < \varepsilon < \frac{\pi}{2}$, implies $\psi = 0$.

A set of all operator-valued functions $F(\lambda) \in S_{a;\lambda_0}(\Pi_{\lambda_0}; \mathcal{N}_{\lambda_0}(A), \mathcal{N}_{\lambda_0}(A))$ which are $\lambda_0$-I-admissible with respect to the operator $A$, we shall denote by
\[
S_{I;\lambda_0}(\Pi_{\lambda_0}; \mathcal{N}_{\lambda_0}(A), \mathcal{N}_{\lambda_0}(A)) = S_{I}(\Pi_{\lambda_0}; \mathcal{N}_{\lambda_0}(A), \mathcal{N}_{\lambda_0}(A)).
\]

In the case $R(A) = H$, we have $D(X_{\lambda_0} (A^{-1})) = \{0\}$ and therefore
\[
S_{I;\lambda_0}(\Pi_{\lambda_0}; \mathcal{N}_{\lambda_0}(A), \mathcal{N}_{\lambda_0}(A)) = S_{a;\lambda_0}(\Pi_{\lambda_0}; \mathcal{N}_{\lambda_0}(A), \mathcal{N}_{\lambda_0}(A)).
\]

**Theorem 3** Let $A$ be a closed symmetric invertible operator in a Hilbert space $H$, and $\lambda_0 \in \mathbb{R}_e$ be an arbitrary point. An arbitrary generalized I-resolvent $R_\lambda$ of the operator $A$ has the following form:
\[
R_\lambda = \begin{cases} 
(A_{F(\lambda)} - \lambda E_H)^{-1}, & \lambda \in \Pi_{\lambda_0} \\
(A_{F(\lambda)}^* - \lambda E_H)^{-1}, & \bar{\lambda} \in \Pi_{\lambda_0}. 
\end{cases}
\]

where $F(\lambda)$ is a function from $S_{I;\lambda_0}(\Pi_{\lambda_0}; \mathcal{N}_{\lambda_0}(A), \mathcal{N}_{\lambda_0}(A))$. Conversely, an arbitrary function $F(\lambda) \in S_{I;\lambda_0}(\Pi_{\lambda_0}; \mathcal{N}_{\lambda_0}(A), \mathcal{N}_{\lambda_0}(A))$ defines by relation (14) a generalized I-resolvent $R_\lambda$ of the operator $A$. Moreover, for different functions from $S_{I;\lambda_0}(\Pi_{\lambda_0}; \mathcal{N}_{\lambda_0}(A), \mathcal{N}_{\lambda_0}(A))$ there correspond different generalized I-resolvents of the operator $A$.

**Proof.** Let $A$ be a closed symmetric invertible operator in a Hilbert space $H$, $\lambda_0 \in \mathbb{R}_e$.

Let us prove the first statement of the theorem. Let $R_\lambda$ be an arbitrary generalized I-resolvent of the operator $A$. The generalized I-resolvent $R_\lambda$ is generated by a self-adjoint invertible operator $\tilde{A} \supseteq A$ in a Hilbert space $\tilde{H} \supseteq H$. Repeating the arguments from the proof of Theorem 3.34 in [1] we obtain a representation (14), where $F(\lambda)$ is a function from $S_{a;\lambda_0}(\Pi_{\lambda_0}; \mathcal{N}_{\lambda_0}(A), \mathcal{N}_{\lambda_0}(A))$. It remains to check that $F(\lambda) \in S_{I;\lambda_0}(\Pi_{\lambda_0}; \mathcal{N}_{\lambda_0}(A), \mathcal{N}_{\lambda_0}(A))$.

The self-adjoint extension $\tilde{A}^{-1}$ of the operator $A^{-1}$ in a Hilbert space $\tilde{H} \supseteq H$ generates the following objects, see Subsection 3.6 in [1]: the operator-valued function $\mathfrak{B}_\lambda (A^{-1}, \tilde{A}^{-1})$, $\lambda \in \mathbb{R}_e$; the operator $\mathfrak{B}_\infty (A^{-1}, \tilde{A}^{-1})$;
the operator-valued function \( \Phi(\lambda; \frac{1}{\lambda_0}, A^{-1}, \tilde{A}^{-1}) \), \( \lambda \in \Pi_{\lambda_0} \); the operator \( \Phi(\lambda; \frac{1}{\lambda_0}, A^{-1}, \tilde{A}^{-1}) \).

On the other hand, the operator \( A \) can be identified with the operator

\[
A = A \oplus A_e,
\]

with \( A_e = o_{H} \), in the Hilbert space

\[
\tilde{H} = H \oplus H_e, \tag{16}
\]

where \( H_e := \tilde{H} \ominus H \), see Subsection 3.5 in [1]. Then

\[
A^{-1} = A^{-1} \oplus A_{e}^{-1} = A^{-1} \oplus A_e. \tag{17}
\]

By the generalized Neumann’s formulas, for \( \tilde{A} \supseteq A \) there corresponds an isometric operator \( T, \quad D(T) = N_{\lambda_0}(A), \quad R(T) = N_{\lambda_0}(\tilde{A}) \), which is \( \lambda_0 \)-admissible with respect to \( A \). Moreover, since the operator \( \tilde{A} \) is invertible, then by Theorem [1] we conclude that the operator \( \frac{\lambda_0}{\lambda_0}T \) is \( \frac{1}{\lambda_0} \)-admissible with respect to \( A^{-1} \).

Applying Theorem 3.16 in [1] for the operator \( A^{-1} \), with \( z = \frac{1}{\lambda_0} \), we obtain that the operator \( \Phi(\frac{1}{\lambda_0}; A^{-1}, \tilde{A}^{-1}T) \) is \( \frac{1}{\lambda_0} \)-admissible with respect to \( A^{-1} \).

By the generalized Neumann’s formulas, for \( \tilde{A}^{-1} \supseteq A^{-1} \) there corresponds an isometric operator \( V, \quad D(V) = N_{\lambda_0}(A^{-1}) = N_{\lambda_0}(A), \quad R(V) = N_{\lambda_0}(A^{-1}) = N_{\lambda_0}(\tilde{A}) \), which is \( \frac{1}{\lambda_0} \)-admissible with respect to \( A^{-1} \). Consider the following operator \( \mathfrak{B}(\frac{1}{\lambda_0}; A^{-1}, V) \), see (3.38) in [1]:

\[
\mathfrak{B}(\frac{1}{\lambda_0}; A^{-1}, V)h = P_{\tilde{H}}(A^{-1})Vh = P_{\tilde{H}}\tilde{A}^{-1}h, \quad h \in D((A^{-1})V) \cap H = D(\tilde{A}^{-1}) \cap H.
\]

Comparing this definition with the definition of the operator \( \mathfrak{B}_{\infty}(A^{-1}, \tilde{A}^{-1}) \) we conclude that

\[
\mathfrak{B}(\frac{1}{\lambda_0}; A^{-1}, V) = \mathfrak{B}_{\infty}(A^{-1}, \tilde{A}^{-1}). \tag{18}
\]

Applying Theorem 3.20 in [1] for the operator \( \tilde{A}^{-1} \), with \( z = \frac{1}{\lambda_0} \), we get:

\[
\mathfrak{B}(\frac{1}{\lambda_0}; A^{-1}, V) = (A^{-1})\Phi(\frac{1}{\lambda_0}; A^{-1}, V), \frac{1}{\lambda_0}.
\]
On the other hand, by the definition of $\Phi_\infty(\frac{1}{\lambda_0}; A^{-1}, \tilde{A}^{-1})$ we have:

$$\mathfrak{B}_\infty(A^{-1}, \tilde{A}^{-1}) = (A^{-1})_{\Phi_\infty(\frac{1}{\lambda_0}; A^{-1}, \tilde{A}^{-1}), \frac{1}{\lambda_0}}.$$  

Then

$$\Phi_\infty(\frac{1}{\lambda_0}; A^{-1}, \tilde{A}^{-1}) = \Phi(\frac{1}{\lambda_0}; A^{-1}, V). \quad (19)$$

By the generalized Neumann’s formulas (4), (5) we get:

$$D(T) = N_{\lambda_0}(A) \cap R(\tilde{A} - \lambda_0 E_{\tilde{H}}),$$

$$T \subseteq (\tilde{A} - \lambda_0 E_{\tilde{H}})(\tilde{A} - \lambda_0 E_{\tilde{H}})^{-1};$$

$$D(V) = N_{\lambda_0}(A^{-1}) \cap R(\tilde{A}^{-1} - \frac{1}{\lambda_0} E_{\tilde{H}}) = N_{\lambda_0}(A) \cap R(\tilde{A} - \lambda_0 E_{\tilde{H}}),$$

$$V \subseteq (\tilde{A}^{-1} - \frac{1}{\lambda_0} E_{\tilde{H}})(\tilde{A}^{-1} - \frac{1}{\lambda_0} E_{\tilde{H}})^{-1}$$

$$= \lambda_0(\tilde{A} - \lambda_0 E_{\tilde{H}})(\tilde{A} - \lambda_0 E_{\tilde{H}})^{-1}.$$  

Therefore

$$V = \frac{\lambda_0}{\lambda_0} T.$$  

By (19) we get:

$$\Phi_\infty(\frac{1}{\lambda_0}; A^{-1}, \tilde{A}^{-1}) = \Phi(\frac{1}{\lambda_0}; A^{-1}, \lambda_0 T). \quad (20)$$

Therefore the operator $\Phi_\infty(\frac{1}{\lambda_0}; A^{-1}, \tilde{A}^{-1})$ is $\frac{1}{\lambda_0}$-admissible with respect to $A^{-1}$.

The self-adjoint operator $\tilde{A} \supseteq A$ in a Hilbert space $\tilde{H} \supseteq H$ generates the operator-valued function $\mathfrak{B}_\lambda(A, \tilde{A})$, $\lambda \in \mathbb{R}$, and the operator-valued function $\mathfrak{F}(\lambda; \lambda_0, A, \tilde{A})$, $\lambda \in \Pi_{\lambda_0}$ (see [1, Subsection 3.6]). Denote (see [1, p. 222])

$$\mathfrak{L}_\lambda(A, \tilde{A}) = \left\{ h \in D(\tilde{A}) : (\tilde{A} - \lambda E_{\tilde{H}})h \in H \right\}, \quad \lambda \in \mathbb{C}; \quad (21)$$

$$\mathfrak{L}_\lambda(A, \tilde{A}) = P_{\tilde{H}} \mathfrak{L}_\lambda, \quad \lambda \in \mathbb{C}.$$  

We shall also need sets $\mathfrak{L}_\lambda(A^{-1}, \tilde{A}^{-1})$, $\mathfrak{L}_\lambda(A^{-1}, \tilde{A}^{-1})$, $\lambda \in \mathbb{C}$, constructed by (21) with the operators $A^{-1}, \tilde{A}^{-1}$ instead of $A, \tilde{A}$.  

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Choose an arbitrary element \( h \in \tilde{\Sigma}_\lambda(A, \tilde{A}), \lambda \in \mathbb{R}_e \). Then \( \tilde{A}h \in R(\tilde{A}) \), and
\[
(\tilde{A}^{-1} - \frac{1}{\lambda} E_H)\tilde{A}h = -\frac{1}{\lambda}(\tilde{A} - \lambda E_H)h \in H.
\]
Therefore \( \tilde{A}h \in \tilde{\Sigma}_\lambda(A^{-1}, \tilde{A}^{-1}) \), and
\[
\tilde{A}\tilde{\Sigma}_\lambda(A, \tilde{A}) \subseteq \tilde{\Sigma}_\lambda(A^{-1}, \tilde{A}^{-1}), \quad \lambda \in \mathbb{R}_e.
\]
In order to obtain the equality:
\[
\tilde{A}\tilde{\Sigma}_\lambda(A, \tilde{A}) = \tilde{\Sigma}_\lambda(A^{-1}, \tilde{A}^{-1}), \quad \lambda \in \mathbb{R}_e,
\] (22)
it remains to apply the proved inclusion for the operators \( A^{-1}, \tilde{A}^{-1} \) instead of \( A, \tilde{A} \), and with \( \frac{1}{\lambda} \) instead of \( \lambda \).

The operator \( P_H \mid \tilde{\Sigma}_\lambda(A, \tilde{A}), \lambda \in \mathbb{R}_e \), is invertible, see considerations below (3.48) in [1]. By the definition of \( \mathfrak{B}_\lambda(A, \tilde{A}) \) we may write:
\[
\mathfrak{B}_\lambda(A, \tilde{A}) = P_H \tilde{A} \left( P_H \mid \tilde{\Sigma}_\lambda(A, \tilde{A}) \right)^{-1}, \quad \lambda \in \mathbb{R}_e.
\] (23)
Applying this representation for the operators \( A^{-1}, \tilde{A}^{-1} \) instead of \( A, \tilde{A} \), and with \( \frac{1}{\lambda} \) instead of \( \lambda \) we get:
\[
\mathfrak{B}_\lambda(A^{-1}, \tilde{A}^{-1}) = P_H \tilde{A}^{-1} \left( P_H \mid \tilde{\Sigma}_\lambda(A^{-1}, \tilde{A}^{-1}) \right)^{-1}, \quad \lambda \in \mathbb{R}_e.
\]
Observe that
\[
R(\mathfrak{B}_\lambda(A, \tilde{A})) = P_H \tilde{A} \tilde{\Sigma}_\lambda(A, \tilde{A}) = \tilde{\Sigma}_\lambda(A^{-1}, \tilde{A}^{-1}), \quad \lambda \in \mathbb{R}_e,
\]
and
\[
\mathfrak{B}_\lambda(A^{-1}, \tilde{A}^{-1}) \mathfrak{B}_\lambda(A, \tilde{A}) g = g, \quad g \in \tilde{\Sigma}_\lambda(A, \tilde{A}), \lambda \in \mathbb{R}_e.
\]
Therefore
\[
\mathfrak{B}_\lambda(A, \tilde{A})^{-1} = \mathfrak{B}_\lambda(A^{-1}, \tilde{A}^{-1}), \quad \lambda \in \mathbb{R}_e.
\] (24)
By the definition of the operator-valued function \( \mathfrak{F}(\lambda) \) and by (24) we may write:
\[
\mathfrak{F}(\lambda; \lambda_0, A, \tilde{A}) = (\mathfrak{B}_\lambda(A, \tilde{A}) - \lambda_0 E_H)(\mathfrak{B}_\lambda(A, \tilde{A}) - \lambda_0 E_H)^{-1} \bigg|_{\mathcal{A}_{\lambda_0}(A)}, \quad \lambda \in \Pi_{\lambda_0};
\]
and, also for $\lambda \in \Pi_{\lambda_0}$, we have:

\[
\mathfrak{F}(\frac{1}{\lambda}; \frac{1}{\lambda_0}, A^{-1}, \tilde{A}^{-1})
\]

\[= (\mathfrak{B}_{\frac{1}{\lambda}}(A^{-1}, \tilde{A}^{-1}) - \frac{1}{\lambda_0}E_H)(\mathfrak{B}_{\frac{1}{\lambda}}(A^{-1}, \tilde{A}^{-1}) - \frac{1}{\lambda_0}E_H)^{-1}\bigg|_{N_{\lambda_0}(A)}
\]

\[= \frac{\lambda_0}{\lambda_0}(\mathfrak{B}_\lambda(A, \tilde{A}) - \lambda_0E_H)(\mathfrak{B}_\lambda(A, \tilde{A}) - \lambda_0E_H)^{-1}\bigg|_{N_{\lambda_0}(A)}.
\]

Therefore

\[
\mathfrak{F}(\frac{1}{\lambda}; \frac{1}{\lambda_0}, A^{-1}, \tilde{A}^{-1}) = \frac{\lambda_0}{\lambda_0}\mathfrak{F}(\lambda; \lambda_0, A, \tilde{A}), \quad \lambda \in \Pi_{\lambda_0}.
\] (25)

Suppose that relations (12), (13) with $F(\lambda) = \mathfrak{F}(\lambda; \lambda_0, A, \tilde{A})$ hold for some $\varepsilon: 0 < \varepsilon < \frac{\pi}{2}$. Then using the change of a variable $y = \frac{1}{\lambda}$ we get:

\[
\lim_{y \in \Pi_{\lambda_0}, \ y \to \infty} \mathfrak{F}(\frac{1}{y}; \lambda_0, A, \tilde{A})\psi = \frac{\lambda_0}{\lambda_0}X_{\frac{1}{\lambda_0}}(A^{-1})\psi,
\] (26)

\[
\lim_{y \in \Pi_{\lambda_0}, \ y \to \infty} \left[|y|\|\psi\|_H - \|\mathfrak{F}(\frac{1}{y}; \lambda_0, A, \tilde{A})\psi\|_H\right] < +\infty.
\] (27)

By (25) we obtain that

\[
\mathfrak{F}(\frac{1}{y}; \lambda_0, A, \tilde{A}) = \frac{\lambda_0}{\lambda_0}\mathfrak{F}(\frac{1}{y}; \lambda_0, A, \tilde{A}), \quad \lambda_0 \in \Pi_{\lambda_0}.
\]

Substituting this expression in relations (26), (27) we get:

\[
\lim_{y \in \Pi_{\lambda_0}, \ y \to \infty} \mathfrak{F}(y; \frac{1}{\lambda_0}, A^{-1}, \tilde{A}^{-1})\psi = X_{\frac{1}{\lambda_0}}(A^{-1})\psi,
\] (28)

\[
\lim_{y \in \Pi_{\lambda_0}, \ y \to \infty} \left[|y|\|\psi\|_H - \|\mathfrak{F}(y; \frac{1}{\lambda_0}, A^{-1}, \tilde{A}^{-1})\psi\|_H\right] < +\infty.
\] (29)

By Theorem 3.32 in [1] we conclude that $\psi \in D(\Phi_{\infty}(\frac{1}{\lambda_0}; A^{-1}, \tilde{A}^{-1}))$, and

\[
\Phi_{\infty}(\frac{1}{\lambda_0}; A^{-1}, \tilde{A}^{-1})\psi = X_{\frac{1}{\lambda_0}}(A^{-1})\psi.
\]
Since $\Phi_\infty(\frac{1}{\lambda_0}; A^{-1}, \tilde{A}^{-1})$ is $\frac{1}{\lambda_0}$-admissible with respect to $A^{-1}$, we obtain that $\psi = 0$. Consequently, $F(\lambda) \in \mathcal{S}_{f, \lambda_0}(\Pi_{\lambda_0}; N_{\lambda_0}(A), N_{\lambda_0'}(A))$.

Let us check the second statement of the theorem. Let $F(\lambda)$ be an arbitrary function from $\mathcal{S}_{f, \lambda_0}(\Pi_{\lambda_0}; N_{\lambda_0}(A), N_{\lambda_0'}(A))$. We shall use the following lemma.

**Lemma 1** Let $n, m \in \mathbb{Z}_+ \cup \{\infty\}$: $n + m \neq 0$. There exists a closed symmetric invertible operator $A$ in a Hilbert space $H$, $\overline{D(A)} = H$, $\overline{R(A)} = H$, which has the deficiency index $(n, m)$.

**Proof.** Let $H_0$ be an arbitrary Hilbert space, and $\mathfrak{A} = \{f_k\}_{k=0}^\infty$ be an orthonormal basis in $H_0$. Consider the following operator (unilateral shift):

$$V_0h = \sum_{k=0}^\infty \alpha_k f_{k+1}, \quad h = \sum_{k=0}^\infty \alpha_k f_k \in H_0, \quad \alpha_k \in \mathbb{C},$$

with $D(V_0) = H_0$. The operator $V_0$ is closed, isometric, and its deficiency index is $(0, 1)$. The condition $V_0g = \pm g$, for an element $g \in H_0$, implies $g = 0$. Consequently, the inverse Cayley transformation:

$$A_0 = i(V_0 + E_{H_0})(V_0 - E_{H_0})^{-1},$$

is a closed symmetric invertible operator in $H_0$, with the deficiency index $(0, 1)$. If $h \in H_0$ and $h \perp (V_0 \pm E_{H_0})H_0$, then $V_0^*h = \mp h$. The condition $V_0^*h = V_0^{-1}p_{H_0}^{H_0}h = \mp h$, for an element $h \in H_0$, implies $h = 0$. Therefore,

$$\overline{D(A_0)} = H_0, \quad \overline{R(A_0)} = H_0.$$ 

Set

$$H_l = \bigoplus_{j=0}^l H_0, \quad W_l = \bigoplus_{j=0}^l V_0, \quad l \in \mathbb{Z}_+ \cup \{\infty\}.$$

$W_l$ is a closed isometric operator in $H_l$. The deficiency index of $W_l$ is equal to $(0, l + 1)$. If $W_lh = \pm h$, or $W_l^*h = \pm h$, then $h = 0$. Then

$$A_l = i(W_l + E_{H_l})(W_l - E_{H_l})^{-1},$$

is a closed symmetric invertible operator in $H_l$, with the deficiency index $(0, l + 1)$, $l \in \mathbb{Z}_+$. Moreover, we have $\overline{D(A_l)} = H_l$, $\overline{R(A_l)} = H_l$. Observe that the operator $-A_l$ has the deficiency index $(l + 1, 0)$, $l \in \mathbb{Z}_+$. If $m > 0, n > 0$, we set

$$H = H^{m-1} \oplus H^{n-1}, \quad A = (-A_{m-1}) \oplus A_{n-1}.$$
Let us return to the proof of the theorem. If the operator $A$ is self-adjoint, then the set $S_{\iota; \lambda}$ consists of a unique function $F(\lambda) = o_H$. Of course, this function generates the resolvent of $A$ by (14). Thus, we can assume that $A$ is not self-adjoint.

We shall use the scheme of the proof of the corresponding statement in Theorem 3.34 in [1]. By Lemma 1 there exists a closed symmetric invertible operator $A_1$ in a Hilbert space $H_1$, $D(A_1) = H_1$, $R(A_1) = H_1$, which has the same defect numbers as $A$. Let $U$ and $W$ be arbitrary isometric operators, which map respectively $N_{\lambda}(A)$ and $N_{\lambda}(A)$ on $N_{\lambda}(A)$. Set $F_1(\lambda) = W^{-1}F(\lambda)U$, $\lambda \in \Pi_{\lambda}$. Since $F_1(\lambda)$ belongs to $S(\Pi_{\lambda};N_{\lambda}(A),N_{\lambda}(A))$, by Shtraus’s formula it generates a generalized resolvent $R_\lambda(A_1)$ of $A_1$.

Let us check that $R_\lambda(A_1)$ is generated by a self-adjoint invertible operator $\hat{A}_1 \supseteq A_1$ in a Hilbert space $\hat{H}_1 \supseteq H_1$. Suppose that $R_\lambda(A_1)$ is generated by a self-adjoint operator $\hat{A}_1 \supseteq A_1$ in a Hilbert space $\hat{H}_1 \supseteq H_1$. Suppose that $U_i(\hat{A}_1)h = -h$, for an element $h \in \hat{H}_1$. Then

\[(h, U_i(\hat{A}_1)g)_{\hat{H}_1} = -(U_i(\hat{A}_1)h, U_i(\hat{A}_1)g)_{\hat{H}_1} = -(h, g)_{\hat{H}_1}, \quad g \in \hat{H}_1,
\]

\[0 = (h, (U_i(\hat{A}_1) + E_{\hat{H}_1})g)_{\hat{H}_1}, \quad g \in \hat{H}_1.
\]

In particular, $h$ is orthogonal to $(U_i(A_1) + E_{H_1})D(U_i(A_1)) = R(A_1)$. Then $h \in \hat{H}_1 \ominus H_1$. Set

\[\hat{H}_0 = \{h \in \hat{H}_1 : U_i(\hat{A}_1)h = -h\}.
\]

Observe that $\hat{H}_0$ is a subspace of $\hat{H}_1 \ominus H_1$. Then

\[\hat{H}_1 = H_1 \oplus (\hat{H}_1 \ominus H_1) = H_1 \oplus \left((\hat{H}_1 \ominus H_1) \ominus \hat{H}_0\right) \ominus \hat{H}_0 = \hat{H}_1 \ominus \hat{H}_0,
\]

where $\hat{H}_1 := H_1 \oplus \left((\hat{H}_1 \ominus H_1) \ominus \hat{H}_0\right)$.

Notice that $U_i(A_1)\hat{H}_0 = \hat{H}_0$ and $U_i(A_1)\hat{H}_1 = \hat{H}_1$. Set $W_1 = U_i(\hat{A}_1)|_{\hat{H}_1}$. There are no non-zero elements $g \in \hat{H}_1$ such that $W_1g = -g$. Then the inverse Cayley transformation

\[\tilde{A}_1 := i(W_1 + E_{\hat{H}_1})(W_1 - E_{\hat{H}_1})^{-1},
\]

is invertible. Since $\tilde{A}_1 \subseteq \hat{A}_1$, then

\[(\tilde{A}_1 - \lambda E_{\hat{H}_1})^{-1} \subseteq (\hat{A}_1 - \lambda E_{\hat{H}_1})^{-1}.
\]
Therefore $\tilde{A}_1$ generates $R_\lambda(A_1)$.

Set

$$H_e := \tilde{H}_1 \ominus H_1.$$  

Starting from the same formula, we repeat the rest of the arguments in the proof of Theorem 3.34 in [1]. In what follows, we shall use notations and constructions from this proof without additional references. We shall obtain a self-adjoint operator $\tilde{A} \supseteq A$ in a Hilbert space $\tilde{H} \supseteq H$, which generates a generalized resolvent $R_\lambda$ of $A$. This generalized resolvent is related to $F(\lambda)$ by (14). It remains to check that the operator $\tilde{A}$ is invertible.

Since the operator $\tilde{A}_1$ is invertible, then by Theorem 1 we obtain that the operator $\frac{1}{\lambda_0} T$ is $\frac{1}{\lambda_0}$-admissible with respect to $A_1^{-1}$. By Theorem 3.16 in [1] we conclude that the operators $\Phi(\frac{1}{\lambda_0}; A_1^{-1}, \frac{\lambda_0}{\lambda_0} T)$ and $\frac{\lambda_0}{\lambda_0} T_{22}$ are $\frac{1}{\lambda_0}$-admissible with respect to $A_1^{-1}$ and $A_1^{-1} = A_1$, respectively.

Comparing the domains of $\Phi(\frac{1}{\lambda_0}; A_1^{-1}, \frac{\lambda_0}{\lambda_0} T)$ and $\Phi(\frac{1}{\lambda_0}; A^{-1}, \frac{\lambda_0}{\lambda_0} V)$ we conclude that

$$D(\Phi(\frac{1}{\lambda_0}; A_1^{-1}, \frac{\lambda_0}{\lambda_0} V)) = UD(\Phi(\frac{1}{\lambda_0}; A_1^{-1}, \frac{\lambda_0}{\lambda_0} T)).$$  

Using Remark 3.15 and formula (3.28) in [1, p. 218] for $\Phi(\frac{1}{\lambda_0}; A^{-1}, \frac{\lambda_0}{\lambda_0} V)$ and $\Phi(\frac{1}{\lambda_0}; A_1^{-1}, \frac{\lambda_0}{\lambda_0} T)$, we get:

$$\Phi(\frac{1}{\lambda_0}; A^{-1}, \frac{\lambda_0}{\lambda_0} V) = W \Phi(\frac{1}{\lambda_0}; A_1^{-1}, \frac{\lambda_0}{\lambda_0} T) U^{-1}.$$  

We can apply the arguments in the proof of the already proved first statement of the theorem for the operator $A := A_1$; the point $\lambda_0$; the generalized $I$-resolvent $R_\lambda := R_\lambda(A_1)$ of $A_1$, which is generated by the self-adjoint invertible operator $\tilde{A} := \tilde{A}_1$ in $\tilde{H}_1 \supseteq H_1$.

$$\Phi_\infty(\frac{1}{\lambda_0}; A_1^{-1}, \tilde{A}_1^{-1}) = \Phi(\frac{1}{\lambda_0}; A_1^{-1}, \frac{\lambda_0}{\lambda_0} T).$$  

$$\tilde{\Phi}(\frac{1}{\lambda_0}; A_1^{-1}, \tilde{A}_1^{-1}) = \frac{\lambda_0}{\lambda_0} \tilde{\Phi}(\lambda; \lambda_0, A_1, \tilde{A}_1), \quad \lambda \in \Pi_{\lambda_0}.$$  

Then

$$\tilde{\Phi}(y; \frac{1}{\lambda_0}, A_1^{-1}, \tilde{A}_1^{-1}) = \frac{\lambda_0}{\lambda_0} \tilde{\Phi}(\frac{1}{y}; \lambda_0, A_1, \tilde{A}_1), \quad y \in \Pi_{\lambda_0}.$$  

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By Theorem 3.32 in [1] we have:

\[ D(\Phi^\infty(\frac{1}{\lambda}; A_1^{-1}, \tilde{A}_1^{-1})) = \{ \psi \in \mathcal{N}_{\frac{1}{\lambda}}(A_1^{-1}) : \]

\[ \lim_{\lambda \to \infty} \left[ |\lambda| \left( \|\psi\|_{H_1} - \|\mathfrak{F}(\lambda; \frac{1}{\lambda}, A_1^{-1}, \tilde{A}_1^{-1})\psi\|_{H_1} \right) \right] < +\infty \}, \quad (35) \]

where 0 < ε < π/2.

Using (30), (32), (35), (36), (34) and the change of a variable: \( y = \frac{1}{\lambda} \), we obtain that

\[ D(\Phi(\frac{1}{\lambda}; A^{-1}, \frac{\lambda}{\lambda_0} V)) = \{ \psi \in \mathcal{N}_{\frac{\lambda}{\lambda_0}}(A) : \]

\[ \lim_{y \to 0} \left[ \left| \frac{1}{y} \right| \left( \|\psi\|_{H} - \|F(y)\psi\|_{H} \right) \right] < +\infty \}; \quad (37) \]

\[ \Phi(\frac{1}{\lambda_0}; A^{-1}, \frac{\lambda_0}{\lambda_0} V)\psi = \frac{\lambda_0}{\lambda_0} \lim_{y \to 0} F(y)\psi, \quad \psi \in D(\Phi(\frac{1}{\lambda_0}; A^{-1}, \frac{\lambda_0}{\lambda_0} V)). \]

(38)

Suppose that there exists an element \( \psi \in D(\Phi(\frac{1}{\lambda_0}; A^{-1}, \frac{\lambda_0}{\lambda_0} V) \cap X_{\frac{\lambda_0}{\lambda_0}}(A^{-1}) \)

such that the following equality holds:

\[ \Phi(\frac{1}{\lambda_0}; A^{-1}, \frac{\lambda_0}{\lambda_0} V)\psi = X_{\frac{\lambda_0}{\lambda_0}}(A^{-1})\psi. \]

By (37), (38) this means that \( \psi \in \mathcal{N}_{\frac{\lambda}{\lambda_0}}(A) \) and

\[ \lim_{y \to 0} \left[ \left| \frac{1}{y} \right| \left( \|\psi\|_{H} - \|F(y)\psi\|_{H} \right) \right] < +\infty, \]

\[ \frac{\lambda_0}{\lambda_0} \lim_{y \to 0} F(y)\psi = X_{\frac{\lambda_0}{\lambda_0}}(A^{-1})\psi. \]

Since \( F(\lambda) \) is \( \lambda_0 \)-I-admissible with respect to the operator \( A \), we get \( \psi = 0 \). This means that \( \Phi(\frac{1}{\lambda_0}; A^{-1}, \frac{\lambda_0}{\lambda_0} V) \) is \( \frac{1}{\lambda_0} \)-admissible with respect to \( A^{-1} \).

Since \( \frac{\lambda_0}{\lambda_0} T_{22} \) is \( \frac{1}{\lambda_0} \)-admissible with respect to \( A_c \), then by Theorem 3.16 in [1] we obtain that the operator \( \frac{\lambda_0}{\lambda_0} V \) is \( \frac{1}{\lambda_0} \)-admissible with respect to \( A^{-1} \). By Theorem [1] we conclude that the operator \( A V = \tilde{A} \) is invertible.

The last statement of the theorem follows directly from Shtraus’s formula. □
References

[1] S.M. Zagorodnyuk, Generalized resolvents of symmetric and isometric operators: the Shtraus approach. — *Annals of Functional Analysis*, 4, No. 1 (2013), 175-285.

Invertible extensions of symmetric operators and the corresponding generalized resolvents.

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In this paper we study invertible extensions of a symmetric operator in a Hilbert space $H$. All such extensions are characterized by a parameter in the generalized Neumann’s formulas. Generalized resolvents, which are generated by the invertible extensions, are extracted by a boundary condition among all generalized resolvents in the Shtraus formula.