A note on PL-disks and rationally slice knots

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Abstract. We give infinitely many examples of manifold-knot pairs \((Y, J)\) such that \(Y\) bounds an integer homology ball, \(J\) does not bound a non-locally-flat PL-disk in any integer homology ball, but \(J\) does bound a smoothly embedded disk in a rational homology ball. The proof relies on formal properties of involutive Heegaard Floer homology.

Every knot \(K\) in \(S^3\) bounds a non-locally-flat PL-embedded disk in \(B^4\), obtained by taking the cone over \(K\). (Throughout, we will not require PL-disks to be locally-flat.) The analogous statement does not hold for knots in more general manifolds. Adam Levine [9, Theorem 1.2] found examples of manifold-knot pairs \((Y, J)\) such that \(Y\) bounds a contractible 4-manifold and \(J\) does not bound a PL-disk in any homology ball \(X\) with \(\partial X = Y\); see also [7].

The main result of this note concerns rationally slice knots in homology spheres bounding integer homology balls:

**Theorem 1.** There exist infinitely many manifold-knot pairs \((Y, J)\) where \(Y\) is an integer homology sphere and

1. \(Y\) bounds an integer homology 4-ball,
2. \(J\) does not bound a PL-disk in any integer homology 4-ball,
3. \(J\) does bound a smoothly embedded disk in a rational homology 4-ball.

Throughout, let \(Y\) be an integer homology sphere. Recall that a knot \(J \subset Y\) is rationally slice if \(J\) bounds a smoothly embedded disk in a rational homology 4-ball \(W\) with \(\partial W = Y\). Two manifold-knot pairs \((Y_0, J_0)\) and \((Y_1, J_1)\) are integrally (respectively rationally) homology concordant if \(J_0\) and \(J_1\) are concordant in an integral (respectively rational) homology cobordism between \(Y_0\) and \(Y_1\). A knot \(J \subset Y\) is integrally (respectively rationally) homology concordant to a knot \(K\) in \(S^3\) if and only if \(J \subset Y\) bounds a PL-disk in an integer (respectively rational) homology ball.

Theorem [1] is an immediate consequence of the following theorem, where \(\overline{V}_0\) and \(\overline{V}_0\) are the involutive knot Floer homology invariants of [5] and \(V_0\) the knot Floer homology invariant defined in [10, Section 2.2] (see also [13], [11]):

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THEOREM 2. Let $K$ be a negative amphichiral rationally slice knot in $S^3$ with $V_0 \geq 1$ and $V_0 = \overline{V}_0 = 0$ and let $\mu$ be the core of surgery in $M = S^3_{1/\ell}(K)$, where $\ell$ is an odd positive integer. Consider $J = \mu \# U \subset M \# -M$, where $U$ denotes the unknot in $-M$. Then $(M \# -M, J)$ is rationally slice, hence rationally homology concordant to a knot in $S^3$, but $(M \# -M, J)$ is not integrally homology concordant to any knot in $S^3$.

REMARK 3. The figure-eight satisfies the hypotheses of Theorem 2 by [3] (see also [1], Section 3] and [5], Theorem 1.7]). More generally, the genus one knots $K_n$ with $n$ positive full twists in one band and $n$ negative full twists in the other band, $n$ odd, also satisfy the hypotheses of Theorem 2; see Figure 1. By [2], Theorem 4.16, $K_n$ is rationally slice. (Alternatively, $K_n$ is strongly negative amphichiral, hence rationally slice [8], Section 2].) Furthermore, $\sigma(K_n) = 0$ since $K_n$ is amphichiral. The knot $K_n$ has Seifert form

\[
\begin{pmatrix} n & 1 \\ 0 & -n \end{pmatrix}
\]

which implies that $\text{Arf}(K_n) = 1$ if and only if $n$ is odd. Since $K_n$ is alternating, it now follows from [5], Theorem 1.7 that for $n$ odd, $V_0 = 1$ and $\overline{V}_0 = \overline{V}_0 = 0$.

Remark 4. Note that $M$ does not bound an integer homology ball (since, for instance, $d(M) = 2V_0 \neq 0$), but $M \# -M$ does.

The proof of Theorem 2 is inspired by the proof of [7], Theorem 1.1(1)]. Our proof relies on the following result from [4] relating the involutive correction term $d$ [5], Section 5] with the ordinary Heegaard Floer correction term $d$ [12], Section 4], for even denominator surgery on knots in $S^3$:

PROPOSITION 5 ([4], Proposition 1.7]). Let $K$ be a knot in $S^3$ and let $p, q > 0$ be relatively prime integers, with $p$ odd and $q$ even. Then

\[ d(S^3_{p/q}(K), [p/2q]) = d(S^3_{p/q}(K), [p/2q]) \]

where $[p/2q]$ denotes the unique self-conjugate Spin$^c$ structure on $S^3_{p/q}(K)$.

The key feature from the above proposition is that for even denominator surgery on a knot in $S^3$, we have that $d$ is equal to $d$ for the unique self-conjugate Spin$^c$ structure on the surgery. More generally, we have the following corollary of Proposition 5:
Corollary 6. Let $J$ be a knot in an integer homology sphere $Y$ and let $p, q > 0$ be relatively prime integers, with $p$ odd and $q$ even. If $(Y, J)$ is integrally homology concordant to a knot in $S^3$, then

$$d(Y_{p/q}(J), [p/2q]) = d(Y_{p/q}(J), [p/2q])$$

where $[p/2q]$ denotes the unique self-conjugate Spin$^c$ structure on $Y_{p/q}(J)$.

Proof. If $(Y, J)$ is integrally homology concordant to a knot $(S^3, K)$, then $Y_{p/q}(J)$ and $S^3_{p/q}(K)$ are integrally homology cobordant; the homology cobordism is given by surgering along the concordance annulus from $(Y, J)$ to $(S^3, K)$. Since $d$ and $d$ are invariants of integer homology cobordism, the result follows from Proposition 5.

The proof of Theorem 2 relies on finding manifold-knot pairs $(Y, J)$ where $d$ and $d$ of even denominator surgery along $J$ differ; the result then follows from Corollary 6.

Proof of Theorem 2. We first show that $(M \# -M, J)$ is rationally slice. Since $K$ is rationally slice, the core of surgery in $M = S^3_{1/\ell}(K)$ is rationally homology concordant to the core of surgery in $S^3_{1/\ell}(U)$, which is the unknot in $S^3$; that is, $(M, \mu)$ is rationally slice. Hence $(M \# -M, J)$ is also rationally slice.

We now show that $(M \# -M, J)$ is not integrally homology concordant to any knot in $S^3$. Since $\mu$ is the core of surgery in $S^3_{1/\ell}(K)$, we have that

$$M_{1/n}(\mu) = S^3_{1/(n-\ell)}(K).$$

Choose an even positive integer $n$ such that $n > \ell$. Since $\ell$ is odd, $n$ is even, and $n - \ell > 0$, by Proposition 1.7 we have that

$$d(M_{1/n}(\mu)) = d(S^3_{1/(n-\ell)}(K)) = -2V_0(K)$$

and

$$d(M_{1/n}(\mu)) = d(S^3_{1/(n-\ell)}(K)) = -2V_0(K) = 0.$$ 

Since $J = \mu \# U \subset M \# -M$, we have that

$$(M \# -M)_{1/n}(J) = M_{1/n}(\mu) \# -M.$$ 

Note that $-M = S^3_{1/\ell}(-K) = S^3_{1/\ell}(K)$, where the last equality follows from the fact that $K$ is negative amphichiral. Since $\ell > 0$, Proposition 1.7 implies that

$$d(-M) = -2V_0(K) \quad \text{and} \quad d(-M) = \overline{d}(-M) = 0.$$ 

Recall that Proposition 1.3 states that if $Y_1$ and $Y_2$ are integer homology spheres, then

$$d(Y_1 \# Y_2) = d(Y_1) + \overline{d}(Y_2).$$

Hence $d(M_{1/n}(\mu) \# -M) \leq -2V_0(K)$. Since $d$ is additive under connected sum, we have that $d(M_{1/n}(\mu) \# -M) = 0$.

We have shown that

$$d((M \# -M)_{1/2}(J)) \leq -2V_0(K) \quad \text{and} \quad d(((M \# -M)_{1/2}(J)) = 0.$$ 

Recall that $V_0(K) \geq 1$. Now by Corollary 4 it follows that $(M \# -M, J)$ is not integrally homology concordant to any knot in $S^3$. 

□
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