COHOMOLOGY OF BURNSIDE RINGS

BENEN HARRINGTON

Department of Mathematics, University of York, York YO10 5DD, UK

Abstract.
Let $G$ be a finite group and $A(G)$ its Burnside ring. For $H \subset G$ let $\mathbb{Z}_H$ denote the $A(G)$-module corresponding to the mark homomorphism associated to $H$. When the order of $G$ is square-free we give a complete description of the $A(G)$-modules $\text{Ext}^l_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J)$ and $\text{Tor}^l_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J)$ for any $H, J \subset G$ and $l \geq 0$. We show that if the order of $G$ is not square-free then there exist $H, J \subset G$ such that $\text{Ext}^l_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J)$ and $\text{Tor}^l_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J)$ have unbounded rank as finite groups.

Preliminaries.
Let $G$ be a finite group. The isomorphism classes of finite $G$-sets form a commutative semi-ring where addition is given by disjoint union and multiplication is given by cartesian product. The Burnside ring $A(G)$ is the Grothendieck ring associated to this semi-ring. For a finite $G$-set $X$, we write $[X]$ for the corresponding isomorphism class in $A(G)$. We first recall some facts about the Burnside ring, see [6] (Chapter 1) for proofs and further details.

Given a subgroup $H \subset G$, the set of left cosets $G/H$ has the natural structure of a $G$-set, where for $g, g' \in G$ and $gH \in G/H$, we have $g \cdot g'H = gg'H$. Given $J \subset G$, the $G$-sets $G/H$ and $G/J$ are isomorphic if and only if $H$ is conjugate to $J$ in $G$. Each transitive $G$-set $X$ is isomorphic to $G/H$ for some $H \subset G$, and the Burnside ring is free on the set of isomorphism classes of transitive $G$-sets. Write $\text{ccs}(G)$ for the set of conjugacy classes of subgroups of $G$. For $H \subset G$, write $(H)$ for the conjugacy class of subgroups to which $H$ belongs.

For a subgroup $H \subset G$ and finite $G$-set $X$, let $X^H$ denote the subset of $X$ of points fixed by $H$. The mark homomorphism $\pi_H : A(G) \to \mathbb{Z}$ associated to $H$ is defined by putting $\pi_H([X]) = |X^H|$ for each finite $G$-set $X$ and extending to $A(G)$. Write $\mathbb{Z}_H$ for the left $A(G)$-module structure on $\mathbb{Z}$ defined by $a \cdot n = \pi_H(a)n$ for $a \in A(G)$ and $n \in \mathbb{Z}$.

Lemma 1. Let $H, J$ be subgroups of $G$.

1. $\pi_H([G/H]) = [N_G H : H]$.
2. $\pi_J([G/H]) \neq 0$ if and only if $J$ is conjugate to a subgroup of $H$.
3. $\pi_H = \pi_J$ if and only if $(H) = (J)$.

For each $(H) \in \text{ccs}(G)$ we then have a well-defined homomorphism $\pi(H) = \pi_H$. We have a homomorphism of rings

$$
\pi : A(G) \to \prod_{(H) \in \text{ccs}(G)} \mathbb{Z}
$$

E-mail address: bh885@york.ac.uk
where for $a \in A(G)$,
\[ a \mapsto (\pi_H(a))_{(H) \in \ccs(G)}. \]
This is an embedding of rings. The ring $\prod_{(H) \in \ccs(G)} \mathbb{Z}$ is called the ghost ring of $G$.
Most of our results on the Burnside ring hold more generally for arbitrary subrings of the ghost ring, and in the next section we introduce the notion of a $B$-ring in order to provide the appropriate setting for stating these results.

Let $R$ be a commutative ring. Let $\text{Ab}$ be the category of abelian groups and $R$-$\text{Mod}$ the category of left $R$-modules. For $R$-modules $M$, $N$, the functors $\text{Hom}_R(M, -) : R$-$\text{Mod} \to \text{Ab}$ and $\text{Hom}_R(-, N) : R$-$\text{Mod} \to \text{Ab}$ are left exact. For $l$ a non-negative integer, we write $\text{Ext}^l_R(M, N)$ for the $l$th right derived functor of $\text{Hom}_R(M, -)$ applied to the module $N$, or equivalently for the $l$th right derived functor of $\text{Hom}_R(-, N)$ applied to the module $M$. The functor $- \otimes_R N : R$-$\text{Mod} \to \text{Ab}$ is right-exact, and we write $\text{Tor}^l_R(M, N)$ for the $l$th left derived functor of $- \otimes_R N$ applied to the module $M$. Since $R$ is a commutative ring, each $\text{Ext}^l_R(M, N)$ and $\text{Tor}^l_R(M, N)$ is naturally endowed with the structure of a left $R$-module. In what follows all modules over a commutative ring are left modules.

**$B$-rings.**

For $I$ a finite set, define $\text{Gh}(I) = \prod_{i \in I} \mathbb{Z}$. Let $R$ be a subring of $\text{Gh}(I)$ and for each $i \in I$ let $\pi_i$ be the corresponding projection $R \to \mathbb{Z}$. For $r \in R$, write $r(i)$ for $\pi_i(r)$.

**Definition 2.** Say that $R \subset \text{Gh}(I)$ is a $B$-ring if for each distinct pair $i, j \in I$ there exists an $r \in R$ with $r(i) \neq 0$ and $r(j) = 0$.

If some subring $R \subset \text{Gh}(I)$ is not a $B$-ring, with $i, j \in I$ a pair for which the above condition fails, then it is clear that $r(i) = r(j)$ for all $r \in R$. Then $R$ is isomorphic to the ring $S \subset \text{Gh}(I - \{j\})$ obtained by omitting the factor corresponding to $j$. Repeating this process if necessary we obtain a subset $I' \subset I$ and a $B$-ring $R' \subset \text{Gh}(I')$ with $R$ isomorphic to $R'$.

We give an intrinsic definition of these rings as follows.

**Proposition 3.** A ring $S$ is isomorphic to a $B$-ring $R \subset \text{Gh}(I)$ for some finite set $I$ if and only if $S$ is a commutative ring which is of finite rank and torsion-free as a $\mathbb{Z}$-module, with $\mathbb{Q} \otimes \mathbb{Z} R$ a product of $|I|$ copies of $\mathbb{Q}$.

**Proof.** If $R \subset \text{Gh}(I)$ is a $B$-ring then it is certainly commutative and torsion-free, since $\text{Gh}(I)$ is. As a $\mathbb{Z}$-module $\text{Gh}(I)$ is finitely generated, so $R$ is of finite rank. For each pair $i, j$ of distinct elements of $I$, let $r_{i,j}$ be an element of $R$ satisfying $r(i) \neq 0$ and $r(j) = 0$. Then putting $s_i = \prod_{j \neq i} r_{i,j}$ for each $i \in I$, we have $s_i(j) \neq 0$ if and only if $i = j$. Let $N = \prod_{i \in I} s_i(i)$ and $N_i = N / s_i(i)$. For $i \in I$ write $e_i$ for the corresponding primitive idempotent of $\text{Gh}(I)$. Then
\[ N \cdot e_i = N_i s_i \in R, \]
and so $N \cdot \text{Gh}(I) \subset R \subset \text{Gh}(I)$. Hence
\[ \mathbb{Q} \otimes \mathbb{Z} R \simeq \mathbb{Q} \otimes \mathbb{Z} \text{Gh}(I) \simeq \prod_{i \in I} \mathbb{Q}, \]
i.e. $\mathbb{Q} \otimes R$ is isomorphic to a product of $|I|$ copies of $\mathbb{Q}$. 
Suppose $S$ is a commutative ring which is of finite rank and torsion-free as a $\mathbb{Z}$-module, with $\mathbb{Q} \otimes \mathbb{Z} S$ a product of finitely copies of $\mathbb{Q}$. Then we have an isomorphism

$$\theta : \mathbb{Q} \otimes S \to \prod_{\ell \in I'} \mathbb{Q}$$

for some finite indexing set $I'$.

Since $S$ is torsion-free, $\mathbb{Q} \otimes S$ contains a copy of $S$ as the subring $1 \otimes S \subset \mathbb{Q} \otimes S$. Denote the image $\theta(1 \otimes S)$ by $S' \subset \prod_{\ell \in I'} \mathbb{Q}$, and for each $i \in I'$ let $\hat{\pi}_i$ denote the projection map $\prod_{\ell \in I'} \mathbb{Q} \to \mathbb{Q}$ onto the $i$th factor. Write $\pi_i$ for the restriction of $\hat{\pi}_i$ to $S'$. Since $S'$ is of finite rank as a $\mathbb{Z}$-module we must have that $\pi_i(S) \subset \mathbb{Z} \subset \mathbb{Q}$ for each $i \in I'$. We can then regard $S'$ as sitting inside $\text{Gh}(I') = \prod_{\ell \in I'} \mathbb{Z} \subset \prod_{\ell \in I'} \mathbb{Q}$. We claim that this embedding defines a $B$-ring. For $s \in S$, write $s'$ for the element $\theta(1 \otimes s)$ of $S'$. It remains to show that for each distinct pair $i, j \in I'$ we can find an element $s \in S$ such that $\pi_i(s') \neq 0$ and $\pi_j(s') = 0$.

Let $f_1, \ldots, f_n$ be the primitive idempotents of $\text{Gh}(I')$, and note that $\hat{\pi}_j(f_i) = 1$ if $j = i$ and $\hat{\pi}_j(f_i) = 0$ otherwise. For each $i \in I$, we have $f_i = \theta(q_i \otimes t_i)$ for some $q_i \in \mathbb{Q}$ and $t_i \in S$. Then $t'_i = \theta(1 \otimes t_i) = (1/q_i) \cdot \theta(q_i \otimes t_i) = (1/q_i)f_i$, and so $\pi_i(t'_i) = 1/q_i \neq 0$ and $\pi_j(t'_i) = 0$ for each $j \neq i$. Thus $t'_i$ satisfies the condition of Definition 2 for any $j \neq i$, and the embedding $S' \subset \text{Gh}(I')$ defines a $B$-ring.

Let $R \subset \text{Gh}(I)$ be a $B$-ring. For each $i \in I$ we have an $R$-module $\mathbb{Z}_i$ defined by letting $R$ act on the set $\mathbb{Z}$ by $r \cdot n = r(i)n$ for $r \in R$ and $n \in \mathbb{Z}$.

**Lemma 4.** The $R$-modules $\text{Ext}_R^l(\mathbb{Z}_i, \mathbb{Z}_j)$ and $\text{Tor}_R^l(\mathbb{Z}_i, \mathbb{Z}_j)$ are finite for any $l \geq 1$ and $i, j \in I$.

**Proof.** Note that $R_{\mathbb{Q}} := \mathbb{Q} \otimes Z R$ is semisimple. Then

$$\text{Ext}_{R_{\mathbb{Q}}}^l(\mathbb{Q} \otimes \mathbb{Z}_i, \mathbb{Q} \otimes \mathbb{Z}_j) = 0$$

for any $l \geq 1$ and $i, j \in I$. But

$$\mathbb{Q} \otimes \mathbb{Z} \text{Ext}_R^l(\mathbb{Z}_i, \mathbb{Z}_j) \simeq \text{Ext}_{R_{\mathbb{Q}}}^l(\mathbb{Q} \otimes \mathbb{Z}_i, \mathbb{Q} \otimes \mathbb{Z}_j)$$

(see e.g. [7] Proposition 3.3.10) and so $\text{Ext}_R^l(\mathbb{Z}_i, \mathbb{Z}_j)$ is torsion for any $l \geq 1$ and $i, j \in I$. Since it is also finitely generated, it is finite. The result for $\text{Tor}_R^l(\mathbb{Z}_i, \mathbb{Z}_j)$ follows similarly. 

Let $R \subset \text{Gh}(I)$ be a $B$-ring and $M$ a finitely generated $R$-module. Let $\mathcal{F}_i$ be a free $R$-module resolution of $M$, where $\mathcal{F}_i = R^{\oplus m_i}$ for $l \geq 0$. Applying $\text{Hom}_R(\cdot, \mathbb{Z}_j)$ for some $j \in I$ gives a chain complex where each term is of the form $\text{Hom}_R(R^{\oplus m_i}, \mathbb{Z}_j) \simeq \mathbb{Z}_j^{\oplus m_i}$. Applying $- \otimes_R \mathbb{Z}_j$ gives a chain complex where each term is of the form $R^{\oplus m_i} \otimes \mathbb{Z}_j \simeq \mathbb{Z}_j^{\oplus m_i}$. $\text{Ext}_R^l(M, \mathbb{Z}_j)$ and $\text{Tor}_R^l(M, \mathbb{Z}_j)$ are then isomorphic to subquotients of $\mathbb{Z}_j^{\oplus m_i}$, and it follows that the $R$-module structure of each $\text{Ext}_R^l(M, \mathbb{Z}_j)$ and $\text{Tor}_R^l(M, \mathbb{Z}_j)$ is given by $r \in R$ acting by $r(j)$. It follows that for $i, j \in I$ and $l \geq 0$, any direct sum decomposition of $\text{Ext}_R^l(\mathbb{Z}_i, \mathbb{Z}_j)$ or $\text{Tor}_R^l(\mathbb{Z}_i, \mathbb{Z}_j)$ as an abelian group is automatically a decomposition as an $R$-module.

By instead considering the functor $\mathbb{Z}_i \otimes_R -$ for $i \in I$, we note that for any $R$-module $N$, the $R$-module structure of $\text{Tor}_R^l(\mathbb{Z}_i, N)$ for each $l \geq 0$ is given by $r$ acting by $r(i)$. Similarly, by considering an injective resolution of $N$, we have that the $R$-module structure of each $\text{Ext}_R^l(\mathbb{Z}_i, N)$ is given by $r \in R$ acting by $r(i)$. 


Definition 5. For distinct $i, j \in I$, define $d(i, j)$ to be the greatest positive integer such that $r(i) \equiv r(j) \mod d(i, j)$ for each $r \in R$.

Since the $R$-module structure of each $\text{Ext}^l_R(M_i, Z_j)$ and $\text{Tor}^l_R(M_i, Z_j)$ is given by both $r$ acting by $r(i)$ and $r$ acting by $r(j)$, it follows that each indecomposable $R$-module summand of $\text{Ext}^l_R(M_i, Z_j)$ and $\text{Tor}^l_R(M_i, Z_j)$ must be of the form $Z_i/mZ_i \cong Z_j/mZ_j$ for some $m | d(i, j)$.

Fix some rational prime $p$ and put $k = \mathbb{F}_p$. For a commutative ring $S$ write $\overline{S}$ for the quotient ring $S/pS \cong S \otimes \mathbb{Z} k$. For an $S$-module $M$, write $\overline{M}$ for the $\overline{S}$-module $M/pM$. If $M$ is annihilated by $p$, we will also denote the associated $\overline{S}$-module by $M$.

Lemma 6. Let $S$ be a commutative ring which is free as a $\mathbb{Z}$-module. Let $M$ be a torsion-free $S$-module and $N$ an $S$-module annihilated by $pS$. Then

$$\text{Ext}^l_S(M, N) \cong \text{Ext}^l_{\overline{S}}(\overline{M}, N)$$

and

$$\text{Tor}^l_S(M, N) \cong \text{Tor}^l_{\overline{S}}(\overline{M}, N)$$

for each $l \geq 0$.

Proof. For any $S$-module $X$, any homomorphism of $S$-modules $\phi : X \to N$ must vanish on $pX$, and so we have an induced homomorphism $\overline{\phi} : \overline{X} \to N$. Similarly, any homomorphism of $\overline{S}$-modules $\psi : \overline{X} \to N$ lifts to a homomorphism of $S$-modules $X \to N$. It follows that

$$\text{Hom}_S(X, N) \cong \text{Hom}_{\overline{S}}(\overline{X}, N).$$

Let $(\mathcal{F}_\bullet, \partial_\bullet)$ be a free $S$-module resolution of $M$. In particular $\mathcal{F}_\bullet$ is a free $\mathbb{Z}$-module resolution of the $\mathbb{Z}$-module $M$, so applying $- \otimes \mathbb{Z} k$ gives a chain complex over $\overline{M}$ with homology groups $\text{Tor}^l_S(M, k)$. But $M$ is torsion-free so the homology groups vanish and the chain complex is exact. Since $\mathcal{F}_\bullet$ is a free $S$-module resolution, $\mathcal{F}_\bullet \otimes \mathbb{Z} k$ is a free $\overline{S}$-module resolution.

Applying $\text{Hom}_{\overline{S}}(-, N)$ to $\mathcal{F}_\bullet \otimes \mathbb{Z} k$ and computing cohomology then computes the groups $\text{Ext}^l_S(M, N)$. But $\text{Hom}_{\overline{S}}(\overline{S}, N) \cong \text{Hom}_S(S, M)$, so this is the same as applying $\text{Hom}_S(-, N)$ to $\mathcal{F}_\bullet$ and taking cohomology, i.e. computing the groups $\text{Ext}^l_S(M, N)$. The proof for $\text{Tor}^l_S(M, N)$ is analogous. \hfill \Box

For a $B$-ring $R \subset \text{Gl}(I)$ and $i \in I$, write $k_i$ for the $R$-module where $r \in R$ acts on the field $k$ by $r(i)$. Put $\overline{R} = R \otimes \mathbb{Z} k$.

For each $j \in I$ we have a short exact sequence of $R$-modules

$$(\dag) \quad 0 \to Z_j \xrightarrow{p_i} Z_j \to k_j \to 0,$$

where the map $Z_j \to Z_j$ is given by multiplication by $p$. Applying $\text{Hom}_R(Z_i, -)$ gives a long exact sequence beginning with

$$0 \to \text{Hom}_R(Z_i, Z_j) \xrightarrow{p} \text{Hom}_R(Z_i, Z_j) \to \text{Hom}_R(k_i, k_j)$$

$$\xrightarrow{\text{Ext}^1_R(Z_i, Z_j)} \text{Ext}^1_R(Z_i, Z_j) \to \text{Ext}^1_R(k_i, k_j)$$

$$\xrightarrow{\text{Ext}^2_R(Z_i, Z_j)} \text{Ext}^2_R(Z_i, Z_j) \to \text{Ext}^2_R(k_i, k_j) \to \cdots$$
where we make use of the additivity of Ext$^l_R (\mathbb{Z}_l, -)$ for each $l \geq 1$ to identify each map Ext$^l_R (\mathbb{Z}_l, \mathbb{Z}_j) \to$ Ext$^l_R (\mathbb{Z}_i, \mathbb{Z}_j)$ as multiplication by $p$, and make use of Lemma 6 above to replace each Ext$^l_R (\mathbb{Z}_i, k_j)$ with Ext$^l_{\mathbb{Z}_l} (k_i, k_j)$.

For each $l \geq 1$ and each rational prime $q$, let $M_{l,q}$ be the submodule of Ext$^l_R (\mathbb{Z}_i, \mathbb{Z}_j)$ annihilated by some power of $q$. Since Ext$^l_R (\mathbb{Z}_i, \mathbb{Z}_j)$ is finite, it follows that we have a decomposition of $R$-modules

$$\text{Ext}^l_R (\mathbb{Z}_i, \mathbb{Z}_j) \simeq \bigoplus_q M_{l,q}$$

where the sum is over all rational primes $q$.

For $l \geq 1$, let $a_l$ be the rank of $M_{l,p}$, and let $b_l$ be the $k$-dimension of Ext$^l_{\mathbb{Z}_l} (k_i, k_j)$. Note that $a_l$ is equal to the number of summands appearing in a decomposition of $M_{l,p}$ as a sum of indecomposable $R$-modules. For a non-zero indecomposable summand $\mathbb{Z}/p^a\mathbb{Z}$ of $M_{l,p}$, the map $\mathbb{Z}/p^a\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^a\mathbb{Z}$ has kernel $p^{a-1}\mathbb{Z}/p^a\mathbb{Z} \simeq \mathbb{Z}/p\mathbb{Z}$. So for $l \geq 1$, the kernel of the map $\left( \text{Ext}^l_R (\mathbb{Z}_i, \mathbb{Z}_j) \xrightarrow{p} \text{Ext}^l_R (\mathbb{Z}_i, \mathbb{Z}_j) \right)$ is a $k$-vector space with dimension $a_l$. Similarly, the cokernel of this map is a $k$-vector space of dimension $b_l$, and so the image of the connecting homomorphism Ext$^l_{\mathbb{Z}_l} (k_i, k_j) \to$ Ext$^{l+1}_R (\mathbb{Z}_i, \mathbb{Z}_j)$ has dimension $b_l - a_l$. Hence

$$a_{l+1} = \dim_k \ker \left( \text{Ext}^{l+1}_R (\mathbb{Z}_i, \mathbb{Z}_j) \xrightarrow{p} \text{Ext}^{l+1}_R (\mathbb{Z}_i, \mathbb{Z}_j) \right)$$

$$= \dim_k \text{im} \left( \text{Ext}^l_{\mathbb{Z}_l} (k_i, k_j) \to \text{Ext}^{l+1}_R (\mathbb{Z}_i, \mathbb{Z}_j) \right)$$

$$= b_l - a_l$$

for $l \geq 1$. In order to determine the sequence $a_l$, it is then sufficient to compute the sequence $b_l$.

Applying $\mathbb{Z}_i \otimes_R - \to (\dag)$ gives a long exact sequence ending with

$$\ldots \longrightarrow \text{Tor}^R_2 (\mathbb{Z}_i, \mathbb{Z}_j) \longrightarrow \text{Tor}^{\mathbb{Z}_l}_2 (k_i, k_j) \longrightarrow \text{Tor}^R_1 (\mathbb{Z}_i, \mathbb{Z}_j) \longrightarrow \text{Tor}^{\mathbb{Z}_l}_1 (k_i, k_j) \longrightarrow \mathbb{Z}_i \otimes \mathbb{Z}_j \longrightarrow \mathbb{Z}_i \otimes \mathbb{Z}_j \longrightarrow k_i \otimes \mathbb{Z}_j \longrightarrow 0.$$  

Write $z_l$ for the $p$-rank of $\text{Tor}^R_l (\mathbb{Z}_i, \mathbb{Z}_j)$ and $y_l$ for the dimension $\text{Tor}^{\mathbb{Z}_l}_l (k_i, k_j)$. Then repeating the same argument as above gives

$$z_l = y_{l+1} - z_{l+1}$$

for $l \geq 1$.

**$B$-rings modulo a prime.**

Since a $B$-ring $R \subset \text{Gh}(I)$ is of finite rank as a $\mathbb{Z}_i$-module, $\overline{R}$ is a finite-dimensional $k$-algebra, and we have an $\overline{R}$-module decomposition of $\overline{R}$ as a direct sum of finitely many indecomposable projective $\overline{R}$-modules. Since $\overline{R}$ is commutative, this is a decomposition of commutative local $k$-algebras. By the usual block theory considerations (see e.g. II Chapter II.5), studying the cohomology of $\overline{R}$ reduces to studying the cohomology of these indecomposable summands.
Define a relation $\sim_p'$ on $I$ by putting $i \sim_p' j$ if and only if $p \mid d(i, j)$ for $i \neq j$. Note that by the definition of $d(i, j)$ this relation is symmetric and transitive, and we write $\sim_p$ for the equivalence relation defined by taking its reflexive closure. Let $E$ denote the set of equivalence classes of $I$ with respect to $\sim_p$. For an equivalence class $E \in E$, write $k_E$ for the (well-defined) $R$-module which is $k$ as an abelian group and where $r \cdot m = r(i)m$ for $m \in k$ and where $i$ is any element of $E$.

**Lemma 7.** For each $E \in E$ there exists an $r \in R$ such that $r(i) \equiv 1 \mod p$ for each $i \in E$ and $r(j) \equiv 0 \mod p$ for each $j \notin E$.

**Proof.** For each equivalence class $E'$ distinct from $E$, we have an $r_{E'} \in R$ with $r_{E'}(i) \neq r_{E'}(j) \mod p$ for $i \in E$ and $j \in E'$. Subtracting $r_{E'}(j)$ if required, we can assume $r_{E'}(j) = 0$, and hence $p \mid r_{E'}(i)$. Replacing $r_{E'}$ by $r_{E'}^{-1}$ if required, we can assume $r_{E'}(i) \equiv 1 \mod p$. Then putting $r = \prod_{E', E} r_{E'}$ it is clear that $r$ has the claimed properties. \hfill $\square$

**Proposition 8.** We have a surjective homomorphism of $k$-algebras

$$\theta : \overline{R} \to \prod_{E \in E} k_E$$

with kernel the radical of $\overline{R}$.

**Proof.** We have a homomorphism of rings

$$R \to \prod_{E \in E} k_E$$

given by

$$r \mapsto (r(i) \mod p)_{E \in E}$$

where $i \in E$. By Lemma 7 this homomorphism is surjective. The kernel of this map contains $pR$, and we let $\theta$ be the induced surjective homomorphism of $k$-algebras. Since $\prod_{E \in E} k_E$ is semisimple, the kernel of $\theta$ certainly contains the radical of $\overline{R}$. It remains to show the reverse inclusion.

As in the proof of Proposition 3 choose for each $i \in I$ an element $s_i \in R$ such that $s_i(j) \neq 0$ if and only if $j = i$. Let $s_i(i) = p^t n_i$ where $n_i$ is coprime to $p$; put $t = \max_i t_i$ and put $N = \prod_{i \in I} n_i$. Let $r \in R$ be such that $r(i) \equiv 0 \mod p$ for each $i \in I$. Putting $q = N^t r^{t+1}$, we have that $ps_i(i) \mid q(i)$ for each $i \in I$, and so we can define integers $m_i \in \mathbb{Z}$ by requiring $q(i) = pm_i s_i(i)$. It follows that

$$q = p \cdot \sum_{i \in I} m_i s_i$$

and so $q \in pR$. Then the image of $q$ in $\overline{R}$ is zero, and hence the image of $r^{t+1}$ in $\overline{R}$ is zero, since $N$ is coprime with $p$.

It follows that the kernel of $\theta$ is nilpotent, and hence equal to the radical of $\overline{R}$. \hfill $\square$

**Corollary 9.** i. $\overline{R}$ is the direct sum of $|E|$ indecomposable $k$-algebras;

ii. the set $\{k_E\}_{E \in E}$ is a complete irredundant set of irreducible modules for $\overline{R}$;

iii. the dimension of the indecomposable summand corresponding to $k_i$ is $|E|$, the cardinality of the $\sim_p$-equivalence class of $i \in I$. The maximal ideal of each indecomposable summand has codimension 1.
Proof. The only part that does not follow immediately is iii. For an equivalence class $E$, let $R'$ be the $B$-ring $R' \subset \prod_{i \in E} \mathbb{Z}$ induced by $R$, and $\pi_E : R \to R'$ the corresponding homomorphism. Then $\pi_E$ descends to a map $R \to \overline{R}'$ and this is a surjection of $k$-algebras. Applying part i to $\overline{R}'$, the $k$-algebra $\overline{R}$ is indecomposable of dimension $|E|$, and so the indecomposable summand of $\overline{R}$ corresponding to $E$ has dimension $\geq |E|$. Since we can do this for each equivalence class in $\mathcal{E}$, the summand must have dimension $|E|$. Since the radical of $\overline{R}$ has dimension $\dim R - |\mathcal{E}|$, the radical of the summand corresponding to $E$ must have dimension $|E| - 1$. □

It follows that each indecomposable summand of $\overline{R}$ is a commutative local finite-dimensional $k$-algebra $S$ with maximal ideal $\mathcal{M}$ satisfying $S/\mathcal{M} \simeq k$. There is a unique $S$-module structure on $k$ where $\mathcal{M} \cdot k = 0$, and we denote this $S$-module by $k$.

The following result is standard (see [4] Chapter 2, §3).

Lemma 10. Let $S$ be as above. Then $\operatorname{Tor}^S_l(k,k) \simeq \operatorname{Ext}^l_S(k,k)$ as $S$-modules for each $l \geq 0$.

Corollary 11. For $i,j \in I$, write $a_l$ for the $p$-rank of $\operatorname{Ext}^l_R(\mathbb{Z}_i,\mathbb{Z}_j)$ and $z_l$ for the $p$-rank of $\operatorname{Tor}^l_R(\mathbb{Z}_i,\mathbb{Z}_j)$. Then $z_l = a_{l+1}$ for all $l \geq 1$.

Proof. We have recurrence relations

$$a_{l+1} = b_l - a_l$$

and

$$z_l = y_{l+1} - z_{l+1}$$

for $l \geq 1$, where $b_l$ and $y_l$ are the dimensions of the $k$-vector spaces $\operatorname{Ext}^l_R(k_i,k_j)$ and $\operatorname{Tor}^l_R(k_i,k_j)$ respectively. If $i \neq j$ then $\operatorname{Ext}^l_R(k_i,k_j) = \operatorname{Tor}^l_R(k_i,k_j) = 0$ for each $l$ and the result is trivially true. Otherwise, by Lemma 10 we have $b_l = y_l$.

Suppose $i = j$. It is clear that $\operatorname{Hom}_R(\mathbb{Z}_i,\mathbb{Z}_i) \simeq \mathbb{Z}_i$ and $\operatorname{Hom}_R(\mathbb{Z}_i,\mathbb{k}_i) \simeq \mathbb{k}_i$ so the long exact sequence associated to (†) begins with the short exact sequence

$$0 \to \operatorname{Hom}_R(\mathbb{Z}_i,\mathbb{Z}_i) \to \operatorname{Hom}_R(\mathbb{Z}_i,\mathbb{k}_i) \to \operatorname{Hom}_R(\mathbb{k}_i,\mathbb{k}_i) \to 0.$$ 

It follows that the map $\operatorname{Ext}^1_R(\mathbb{Z}_i,\mathbb{Z}_i) \to \operatorname{Ext}^1_R(\mathbb{Z}_i,\mathbb{k}_i)$ is injective and hence $\operatorname{Ext}^1_R(\mathbb{Z}_i,\mathbb{Z}_i)$ has zero $p$-part and $a_1 = 0$. Similarly, $z_1 = b_1$, and the recurrence gives $z_l = a_{l+1}$ for all $l \geq 1$ as claimed.

Suppose $i \neq j$ with $i \sim_p j$. It is clear that $\operatorname{Hom}_R(\mathbb{Z}_i,\mathbb{Z}_j) = 0$ and $\operatorname{Hom}_R(\mathbb{Z}_i,\mathbb{k}_j) \simeq k_j$, so by inspection of the long exact sequence associated to (†) we have $a_1 = 1$. Similarly we have $\mathbb{Z}_i \otimes \mathbb{Z}_j \simeq \mathbb{Z}_j/d(i,j)\mathbb{Z}_j$ and $\mathbb{Z}_i \otimes k_j \simeq k_j$ from which we obtain $z_1 = b_1 - 1$, and once more we have $z_l = a_{l+1}$ for each $l \geq 1$. □

Corollary 12. Suppose $p \nmid d(i,j)$ for all distinct $i,j \in I$. Then $\overline{R}$ is semisimple.

Proof. Since $p \nmid d(i,j)$ for all distinct $i,j \in I$, we have $|\mathcal{E}| = |I|$ and the homomorphism $\theta$ of Proposition 8 is an isomorphism. □

Corollary 13. Let $i,j \in I$ be distinct with $d(i,j) = 1$. Then

$$\operatorname{Ext}^l_R(\mathbb{Z}_i,\mathbb{Z}_j) = 0$$

for all $l \geq 0$. 
\textbf{Corollary 14.} Suppose that distinct elements \(i, j\) of \(I\) are such that \(\{i, j\}\) is an equivalence class for the relation \(\sim_p\) on \(I\). Write \(a_l\) and \(a'_l\) for the \(p\)-ranks of \(\Ext^l_R(\mathbb{Z}_i, \mathbb{Z}_i)\) and \(\Ext^l_R(\mathbb{Z}_i, \mathbb{Z}_j)\) respectively. Then

\[
\begin{align*}
a_l &= \begin{cases} 
0 & \text{if } l \text{ odd} \\
1 & \text{if } l \text{ even}
\end{cases} \\
a'_l &= \begin{cases} 
1 & \text{if } l \text{ odd} \\
0 & \text{if } l \text{ even}
\end{cases}
\end{align*}
\]

for all \(l \geq 1\).

\textit{Proof.} The algebra \(\overline{R}\) has an indecomposable 2-dimensional \(k\)-algebra summand corresponding to the pair \(\{i, j\}\). Any indecomposable local \(k\)-algebra of dimension 2 with maximal ideal of codimension 1 is isomorphic to \(A = k[x]/(x^2)\). We have a free \(A\)-module resolution of \(k\) given by

\[
\ldots \to A \to \ldots \to A \to A \to k \to 0
\]

where each map \(A \to A\) is given by \(1_A \mapsto x\), and so \(\Ext^l_A(k, k) \simeq k\) for all \(l \geq 0\), i.e. in the notation of our recurrence relations we have \(b_l = 1\) for all \(l \geq 1\). Now \(a_{l+1} = b_l - a_l\) and \(a'_{l+1} = b_l - a'_l\), where \(a_1 = 0\) and \(a'_1 = 1\), from which the corollary follows immediately. \(\square\)

The Burnside ring as \(B\)-ring.

\textbf{Proposition 15.} The embedding \(\pi : A(G) \to \Gh(\text{ccs}(G))\) defines a \(B\)-ring.

\textit{Proof.} We need to show that for non-conjugate subgroups \(H, J \subset G\) we can find an \(a \in A(G)\) with \(\pi_H(a) \neq 0\) and \(\pi_J(a) = 0\). Now if \(J\) is not conjugate to a subgroup of \(H\), then \(a = [G/J] \text{ suffices. Otherwise, if } J \text{ is conjugate to a subgroup of } H \text{ then } H \text{ is not conjugate to a subgroup of } J\), and putting \(a = [N_J : J][G/J] - [G/J]\) we have \(\pi_J(a) = 0\) and \(\pi_H(a) = [N_GJ : J] \neq 0\). \(\square\)

For \(H \subset G\) and \(p\) a prime, let \(O^p(H)\) denote the smallest normal subgroup \(K \triangleleft H\) such that \(H/K\) is a \(p\)-group. We recall the following result due to Dress (Proposition 1).

\textbf{Proposition 16.} Let \(H, J\) be subgroups of \(G\). Then \(\pi_H(a) \equiv \pi_J(a) \mod p\) for each \(a \in A(G)\) if and only if \(O^p(H)\) is conjugate to \(O^p(J)\).

Since \(A(G)\) is a \(B\)-ring, we have integers \(d((H), (J))\) defined for each distinct pair \((H), (J)\) \(\in \text{ccs}(G)\). For \(H, J \subset G\) non-conjugate subgroups, we write \(d(H, J)\) for \(d((H), (J))\). It follows immediately from Proposition 16 that \(p \mid d(H, J)\) if and only if \(O^p(J)\) is conjugate to \(O^p(H)\).
Lemma 18. For all $d$ the relation $\sim$ group of $H_2$. For $l$ even, it remains to show that each $G \cap N$ in $p$ whenever $l$ conjugacy classes of subgroups $G$.

Proof. Suppose we can construct an $s$ power summand only when $l$ is odd since $\pi d$ is clear since $\pi 2$. Since $\pi d$ has cardinality $\pi d$ if and only if $\pi d$ is zero for some $l$. If $(H) \in E_i$ for some $i$ then $M_{H,l}$ is non-zero whenever $l$ is even, and if $\{(H),(J)\} = E_i$ for some $i$ then $M_{H,l}$ is non-zero whenever $l$ is odd. Otherwise $(H) \notin E_i$ for each $i$, in which case by part i $\{(H)\}$ is an equivalence class in $E$ and so the block corresponding to $H$ is 1-dimensional and $M_{H,l} = 0$ for all $l \geq 1$.

For $l$ and $l'$, we know by Corollary 14 that if $(H) \in E_i$ for some $i$ then $M_{H,l}$ is non-zero whenever $l$ is even, and if $\{(H),(J)\} = E_i$ for some $i$ then $N_{H,l,l}$ is non-zero whenever $l$ is odd. Otherwise $(H) \notin E_i$ for each $i$, in which case by part i $\{(H)\}$ is an equivalence class in $E$ and so the block corresponding to $H$ is 1-dimensional and $M_{H,l} = 0$ for all $l \geq 1$. Similarly, if $\{(H),(J)\} \neq E_i$ for each $i$, then $(H)$ and $(J)$ belong to distinct equivalence classes and $N_{H,l,l} = 0$ for all $l \geq 0$.

Suppose $\{(H),(J)\}$ is an equivalence class in $E$. By Corollary 14, $M_{H,l}$ has a $p$-power summand only when $l$ is even, and $N_{H,l,l}$ has a $p$-power summand only when $l$ is odd. It remains to show that each $p$-power summand is in fact $Z/pZ$. For $N_{H,l,l}$ this is clear since $d(H,J)$ annihilates $N_{H,l,l}$ and $p^2 \nmid d(H,J)$ by Lemma 18. For $M_{H,l}$ note that by dimension shifting this is the $p$-part of $\text{Ext}^{l-1}_{A(G)}(K_H,Z_H)$ where $K_H = \ker \pi_H$. Since $p^2 \nmid d(H,J')$ for any $J' \subset G$, and since $p \nmid d(H,J')$ if and only if $(J') = (J)$, we can construct an $s_H \in A(G)$ such that $\pi_{J'}(s_H) \neq 0$ if and only if $(H) = (J')$, and such that $p^2 \nmid \pi_H(s_H)$. Now $s_H$ annihilates $K_H$, so it follows that $s_H$ annihilates $\text{Ext}^l_{A(G)}(Z_H,Z_H)$, and so a fortiori $s_H$ annihilates $M_{H,l}$. So any $p$-power summand of $M_{H,l}$ must be of the form $Z/pZ$. □
Theorem 20. Suppose $|G|$ is square-free. Then for all $H, J \subset G$ and $l \geq 1$, we have an isomorphism of $A(G)$-modules $\text{Ext}^l_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J) \simeq \text{Ext}^{l+2}_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J)$.

In the remainder we establish the converse, by showing that if $|G|$ is not square-free then there exists $H, J \subset G$ such that the rank of $\text{Ext}^l_{p^2}(\mathbb{Z}_H, \mathbb{Z}_J)$ is unbounded as $l \to \infty$. This is an easy consequence of the following two results. Recall that for a field $F$, a commutative $F$-algebra $S$ is said to be symmetric if there exists an $F$-linear map $\lambda : S \to F$ such that $\ker \lambda$ contains no non-zero ideals of $S$.

Theorem 21 (Gustafson [5]). If $p^2 \mid |G|$ and $F$ is a field of characteristic $p$ then the $F$-algebra $A(G) \otimes_\mathbb{Z} F$ is not symmetric.

Theorem 22 (Gulliksen [3]). Let $S$ be a commutative noetherian local ring with maximal ideal $\mathcal{M}$ and with residue field $F = S/\mathcal{M}$. Then the sequence $(\dim \text{Tor}_i^S(F, F))_{i \in \mathbb{N}}$ is bounded if and only if
\[
d(S) \geq \dim \mathcal{M}/\mathcal{M}^2 - 1,
\]
where $d(S)$ denotes the Krull dimension of $S$.

Theorem 23. If $|G|$ is not square-free then there exist subgroups $H, J \subset G$ such that the groups $\text{Ext}^l_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J)$ have unbounded rank.

Proof. Let $p$ be a prime with $p^2 \mid |G|$ and put $k = F_p$. The algebra $\overline{A(G)} = A(G) \otimes_\mathbb{Z} k$ is not symmetric by Theorem 21 so it has an indecomposable $k$-algebra summand $S$ which is not symmetric. Now $S$ is finite-dimensional so the maximal ideal $\mathcal{M}$ of $S$ is nilpotent. Then $\mathcal{M}$ is the only prime of $S$ and $d(S) = 0$. It follows by Theorem 22 that the sequence $(\dim \text{Tor}_i^S(k, k))_{i \in \mathbb{N}}$ is unbounded if and only if $\dim \mathcal{M}/\mathcal{M}^2 > 1$.

If $\dim \mathcal{M}/\mathcal{M}^2 = 0$ then $S = k$ is clearly symmetric. Suppose $\dim \mathcal{M}/\mathcal{M}^2 = 1$. Let $t \in S$ generate $\mathcal{M}$ and note that $\{1_t, t, \ldots, t^q\}$ is a vector space basis for $S$ for some $q \geq 1$. Define $\lambda : S \to k$ by putting
\[
\lambda \left( \sum_{i=0}^q a_i t^i \right) = a_q.
\]
Now if $J \subset S$ is a non-zero ideal then we can choose some non-zero element $s = \sum_{i=0}^q b_i t^i$ in $J$, and choose $m$ to be minimal such that $b_m \neq 0$. Then $t^{q-m} s = b_m t^q \in J$ is not in $\ker \lambda$, so $\ker \lambda$ contains no non-zero ideals and $S$ is symmetric, a contradiction. It follows that $\dim \mathcal{M}/\mathcal{M}^2 > 1$ and the sequence $(\dim \text{Tor}_i^S(k, k))_{i \in \mathbb{N}} = (\dim \text{Ext}_i^S(k, k))_{i \in \mathbb{N}}$ is unbounded.

By Corollary 8 the summand $S$ corresponds to some equivalence class $E \in \mathcal{E}$, and we have
\[
\dim \text{Ext}_i^S(k, k) = \dim \text{Ext}_i^S(k_E, k_E).
\]
Let $H, J$ be subgroups of $G$ with $(H), (J) \in \mathcal{E}$, let $\alpha_l$ be the $p$-rank of $\text{Ext}^l_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J)$, and let $\beta_l$ be the dimension of $\text{Ext}^l_{A(G)}(k_E, k_E)$. We have a recurrence
\[
\alpha_{l+1} = \beta_l - \alpha_l
\]
for $l \geq 1$. Since $\beta_l$ is unbounded, it follows immediately that $\alpha_l$ is unbounded, and hence the groups $\text{Ext}_l^G(\mathbb{Z}_H, \mathbb{Z}_J)$ have unbounded rank. \qed
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