Weight one Jacobi forms and umbral moonshine

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Abstract

We analyze holomorphic Jacobi forms of weight one with level. One such form plays an important role in umbral moonshine, leading to simplifications of the statements of the umbral moonshine conjectures. We prove that nonzero holomorphic Jacobi forms of weight one do not exist for many combinations of index and level, and use this to establish a characterization of the McKay–Thompson series of umbral moonshine in terms of Rademacher sums.

Keywords: umbral moonshine, Weil representations, Jacobi forms

1. Introduction

Umbral moonshine [1, 2] attaches distinguished vector-valued mock modular forms to automorphisms of the Niemeier lattices. To be specific, let $X$ be the root system of a Niemeier lattice (i.e. any self-dual even positive definite lattice of rank 24, other than the Leech lattice). Define $G^X := \text{Aut}(N^X)/W^X$, where $N^X$ is the self-dual lattice associated with $X$ by Niemeier’s classification [3] (see also [4]), and $W^X$ is the subgroup of $\text{Aut}(N^X)$ generated by reflections in the root vectors. Then [2] describes an assignment $g \mapsto H^X_g$ of vector-valued holomorphic functions—the umbral McKay–Thompson series—to elements $g \in G^X$. (A very explicit description of this assignment appears in section B of [5].)

The situation is analogous to monstrous moonshine [6], where holomorphic functions $T_m$—the monstrous McKay–Thompson series—are attached to monster elements $m \in \mathbb{M}$. In this case the $T_m$ are distinguished in that they are the normalized principal moduli (i.e.
normalized hauptmodules) attached to the genus zero groups $\Gamma_m < SL_2(\mathbb{R})$. Thanks to the work of Borcherds [7], we know that they are also the graded trace functions arising from the action of $M$ on the graded infinite-dimensional $M$-module $V^2 = \bigoplus_{\nu \geq -1} V^2_{\nu}$ constructed by Frenkel–Lepowsky–Meurman [8–10].

The conjectures are formulated in section 6 of [2], in order to identify analogues for the $H^X_\ell$ of these two properties of the monstrous McKay–Thompson series. For the analogue of the normalized principal modulus property (also known as the genus zero property of monstrous moonshine), the notion of optimal growth is formulated in [2], following the work [11] of Dabholkar–Murthy–Zagier.

Recall from section 6.3 of [2] that a vector-valued function $H = (H_r)$ is called a mock modular form of optimal growth for $\Gamma_0(n)$ with multiplier $\nu$, weight $\frac{1}{2}$ and shadow $S$ if

\[
(1.1) \quad H(\gamma) = H(\gamma \tau) \nu(\gamma)(c\tau + d)^{-\frac{1}{2}} + e\left(-\frac{1}{8}\right) \int_{-\infty}^{\infty} (z + \tau)^{-\frac{1}{2}} S(z) dz
\]

for $\gamma = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)$. Note that $e(x) := e^{2\pi ix}$. Roughly, the content of (i) of (1.1) is that $H$ is a mock modular form for $\Gamma_0(n)$. The other two conditions strongly restrict the growth of the components of $H$ near the cusps.

Conjecture 6.5 of [2] predicts that $H^X_\ell$ is the unique, up to scale, mock modular form of weight $\frac{1}{2}$ for $\Gamma_0(n)$ with optimal growth, for a suitably chosen $n$, multiplier system and shadow. This conjecture is known to be true in special cases, but we now appreciate that the conjecture is false in general due to the existence of nonzero holomorphic Jacobi forms of weight one. This conjecture is formulated in section 6 of [2], in order to identify analogues for the usual weight 1, index $m$ and level $N$ if it is invariant for the usual weight 1, index $m$ action of $\Gamma_0(N) := \Gamma_0(N) \times \mathbb{Z}^2$,

\[
\xi(\tau, z) := \sum_{r \mod 2m} h_r(\tau) \theta_{m,r}(\tau, z)
\]

remains bounded near every cusp of $\Gamma_0(N)$. Say that $\xi$ is a Jacobi cusp form if the $h_r$ all tend to 0, near all the cusps of $\Gamma_0(N)$. In (1.4) we write $\theta_{m,r}$ for the theta functions.
attached to the even lattice \( \sqrt{2m}\mathbb{Z} \), where \( q = e(\tau) \) and \( y = e(z) \). For later use we define

\[
\theta_{m,r}^{\pm}(\tau, z) := \theta_{m,-r}(\tau, z) \pm \theta_{m,r}(\tau, z),
\]

and note that \( \theta_{m,h}(hr, hz) = \theta_{mh, rh}(\tau, z) \) when \( h \) is a positive integer.

Write \( J_{1,m}(N) \) for the space of holomorphic Jacobi forms of weight 1, index \( m \) and level \( N \). Skoruppa proved \([13]\) that

\[
\text{nonzero examples.}
\]

Thus it was a surprise to us when we discovered some nonzero examples. Indeed, examples may be extracted from the existing literature. Recall the Dedekind eta function, \( \eta(\tau) := q^\frac{1}{2} \prod_{n > 0}(1 - q^n) \). The function \( \eta(\tau)\theta_{2,1}^{-1}(\tau, \frac{1}{2}z) \) appears in section 1.6, example 1.14 of \([15]\). (Note that \( \theta_{2,1}(\tau, \frac{1}{2}z) \) is the work cited.) It is a Jacobi form of weight 1, index \( \frac{1}{2} \) and level 1, with a nontrivial multiplier system. Observe that if \( h \) is a positive integer and \( \xi(\tau, z) \) is a Jacobi form of index \( m \) and level \( N \) with some multiplier system, then \( \xi(\tau, hz) \) is a Jacobi form of index \( mh^2 \) with the same level, and \( \xi(h\tau, hz) \) is a Jacobi form of index \( mh \) and level \( Nh \). In both cases the weight is unchanged and the multiplier system transforms in a controlled way. From the explicit description of the multiplier system of \( \eta(\tau)\theta_{2,1}^{-1}(\tau, \frac{1}{2}z) \) in \([15]\), we deduce that

\[
\xi_{1,12}(\tau, z) := \eta(6\tau)\theta_{2,1}^{-1}(6\tau, 6z) = (y^{-6} - y^6)q + O(q^7)
\]

is a nonzero element of \( J_{1,12}(36) \). This proves proposition 1.1. Indeed, it proves the a priori stronger statement that weight one cusp forms of level 36 exist, since \( \eta(6\tau) \) vanishes at all cusps of \( \Gamma_0(36) \). In a similar way we obtain a nonzero, noncuspidal element

\[
\xi_{1,8}(\tau, z) := \theta_{8,1}(\tau, 0)\theta_{8,1}^{-1}(\tau, z) = (y^4 - y^{-4})q + O(q^5)
\]

of \( J_{1,8}(32) \), by considering the function \( \eta(2\tau)^2\eta(\tau)^{-1}\theta_{2,1}^{-1}(\tau, \frac{1}{2}z) \) which appears in theorem 1.4 of \([16]\). (Note that \( \theta_{8,1}(\tau, 0) = \eta(8\tau)^{2}\eta(4\tau)^{-1} \).

An infinite family of examples may be extracted from corollary 4.9 of \([17]\). For \( a, b \in \mathbb{Z}^+ \) define the theta quark\(^6\)

\[
Q_{a,b}(\tau, z) := \eta(\tau)^{-1}\theta_{2,1}^{-1}(\tau, \frac{1}{2}az)\theta_{2,1}(\tau, \frac{1}{2}bz)\theta_{2,1}(\tau, \frac{1}{2}(a+b)z).
\]

Then \( Q_{a,b} \) (denoted \( Q_{a,b} \) in \([17]\)) is a Jacobi form of weight 1, index \( m = a^2 + ab + b^2 \) and level \( N = 1 \), with a multiplier system satisfying

\[
Q_{a,b}|_{1,m}(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, (\lambda, \mu)) = e\left(\frac{1}{3}\right)Q_{a,b}, \quad Q_{a,b}|_{1,m}(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, (\lambda, \mu)) = Q_{a,b},
\]

for \( \lambda, \mu \in \mathbb{Z} \). From \((1.10)\) we deduce that \( Q_{a,b}(3\tau, 3z) \) is a nonzero element of \( J_{1,3m}(9) \) for \( m = a^2 + ab + b^2 \), for any \( a, b \in \mathbb{Z}^+ \). The first of these theta-quarks, \( Q_{1,1} \), will play an important role in the sequel. To set this up we define

\(^5\)We are grateful to Ken Ono for pointing out to us that nonzero holomorphic weight one Jacobi forms should exist.

\(^6\)Theta quarks belong to a more general theory of theta blocks due to Gritsenko–Skoruppa–Zagier \([18]\).
Theorem 1.2 ([20]). Let \( m \) and \( N \) be positive integers and assume that \( N \) is square-free. Let \( N_m \) be the smallest divisor of \( N \) such that \( m \) and \( N \) are relatively prime. If \( \frac{N}{N_m} \) is odd then \( J_{1,m}(N) = J_{1,m}(N_m) \). If \( \frac{N}{N_m} \) is even then \( J_{1,m}(N) = J_{1,m}(2N_m) \). Also, if \( m \) is odd then \( J_{1,m}(2) = \{0\} \).

The fact that \( J_{1,m}(N) = \{0\} \) when \( N \) is square-free and \( (m,N) = 1 \) is obtained by taking theorem 1.2 together with Skoruppa’s earlier result [13] that \( J_{1,m}(1) = \{0\} \) for all \( m \).

In this article we revisit the conjectures of [2], in light of the existence of holomorphic weight one Jacobi forms, and we establish the vanishing of \( J_{1,m}(N) \) for many \( m \) and \( N \). For example, we prove the following result, which is a specialization of lemma 3.11.

Lemma 1.3. Let \( m \) and \( N \) be positive integers. If all of the prime divisors of \( mN \) are congruent to 1 modulo 4 then \( J_{1,m}(N) = \{0\} \).

As we explain in section 2.1, the particular Jacobi forms (1.8) and (1.11) play special roles in moonshine, the latter leading us to a simplification (conjecture 2.1) of the umbral moonshine module conjecture. It is this simplified form of the umbral moonshine module conjecture which is described in the recent review [21] and proven in [5].

In section 2.2 we formulate a characterization (conjecture 2.2) of the umbral McKay–Thompson series in terms of Rademacher sums. This serves as a natural analogue of the genus zero property of monstrous moonshine, and a replacement for conjecture 6.5 of [2], which as we have discussed is false in general. We point out a geometric interpretation of the theta quark \( Q_{1,1} \) in section 2.3, and also indicate a possible connection to physics.

The remainder of the paper is devoted to a proof of the Rademacher sum characterization, conjecture 2.2. This follows (corollary 3.2) from our main result, theorem 3.1, which states that a certain family of holomorphic Jacobi forms of weight one vanishes identically. The proof of theorem 3.1 is given in section 3.4. We obtain it by applying a representation theoretic approach that has been developed by Skoruppa (see [22]). We review the relevant background in sections 3.1 and 3.3. The methods of section 3.4 work best for levels that are not divisible by high powers of 2 or 3. In section 3.2 we use a more elementary technique to prove (mostly) complementary vanishing results for some spaces with small index.

We attach four appendices to this article. In appendix C we present revised coefficient tables for the umbral McKay–Thompson series \( H^e_8 \) with \( X = A^3_6 \) (tables C1 and C2). Using these together with the character table of \( G^X \) (table B1) in appendix B one can compute the (revised) multiplicities of irreducible \( G^X \)-modules that appear in appendix D. The proof of our main result (theorem 3.1) is obtained by applying the lemmas we prove in section 3.4 to the tables in appendix A (tables A1–A21).
2. Umbral moonshine

2.1. Umbral moonshine modules

Conjecture 6.1 of [2] predicts the existence of umbral analogues $K^X$ of $V$, for each Niemeier root system $X$. At the time this conjecture was formulated we had identified candidate functions $H^X_{g,r}$ for $g \in G^X$, which implied that $K^X$ would have peculiar properties in the case that $X = A^3_2$. Nonetheless, we did not think that there could be other possibilities for the $H^X_{g,r}$ with $X = A^3_2$, in light of Schmidt’s result [14] discussed above.

It turns out that $\xi_{g}^{(9)}(\tau, z)$ (see (1.11)) and $\phi_{g}^{(9)}(\tau, z) := \sum_{r \text{ mod } 18} H_{g,r}^{X} (\tau) \theta_{9r}(\tau, z)$ have the same multiplier system, for $X = A^3_2$ and $o(g) = 0 \text{ mod } 3$, on $\Gamma_0(3)$ for $g \in 3A$, and on $\Gamma_0(6)$ in the case that $g \in 6A$. Thus we may use these holomorphic Jacobi forms to revise our prescription for $H^X_{g}$, for $X = A^3_2$, for the two conjugacy classes $[g] \subset G^X$ with $o(g) = 0 \text{ mod } 3$.

Doing this we are led to a reformulation of conjecture 6.1 of [2] that is uniform with respect to the choice of Niemeier root system $X$. To be precise, let us redefine $H^X_{3A}$ and $H^X_{6A}$ for $X = A^3_2$, by setting

$$H^X_{g}(\tau) := H^X_{g,old}(\tau) + t^{(9)}_{g}(\tau),$$

(2.1)

where $t^{(9)}_{g} = (t^{(9)}_{g,r})$ is the vector-valued theta series of weight $\frac{1}{2}$, defined by

$$t^{(9)}_{3A,r}(\tau) := \begin{cases} 0, & \text{if } r = 0 \text{ mod } 9, \text{ or } r \neq 0 \text{ mod } 3, \\ \pm \theta_{3,3}(\tau, 0), & \text{if } r \pm 3 \text{ mod } 18, \\ \pm \theta_{3,0}(\tau, 0), & \text{if } r \pm 6 \text{ mod } 18, \end{cases}$$

(2.2)

$$t^{(9)}_{6A,r}(\tau) := \begin{cases} 0, & \text{if } r = 0 \text{ mod } 9, \text{ or } r \neq 0 \text{ mod } 3, \\ \pm \theta_{3,3}(\tau, 0), & \text{if } r \pm 3 \text{ mod } 18, \\ \pm \theta_{3,0}(\tau, 0), & \text{if } r \pm 6 \text{ mod } 18. \end{cases}$$

(2.3)

Note that $t^{(9)}_{g}$ is just the vector-valued modular form whose components are the theta coefficients of $\xi_{g}^{(9)}$ (see (1.11)).

We present the corresponding coefficient tables (revisions of tables 75–82 in [2]) in appendix C. This leads us to revised decomposition tables (revisions of tables 170 and 171 in [2]), which we present in appendix D. Motivated by these new decompositions we reformulate conjecture 6.1 of [2] as follows.

**Conjecture 2.1.** Let $X$ be a Niemeier root system and let $m$ be the Coxeter number of $X$. There exists a naturally defined $\mathbb{Z}/2m\mathbb{Z} \times \mathbb{Q}$-graded super-module

$$K^X = \bigoplus_{r \text{ mod } 2m} K^X_r = \bigoplus_{r \text{ mod } 2m} \bigoplus_{D \in \mathbb{Z} \text{ mod } 6m} K_{r,-\frac{D}{2m}}^X$$

(2.4)

for $G^X$, such that the graded super-trace attached to an element $g \in G^X$ is recovered from the vector-valued mock modular form $H^X_{g}$ via

$$H^X_{g,r}(\tau) = \sum_{D \in \mathbb{Z} \text{ mod } 6m} \text{str}_{K^X_{r,-\frac{D}{2m}}}(g) q^{-\frac{D}{2m}}.$$  

(2.5)
Moreover, $K_r^X = \{0\}$ for $r = m \mod 2m$. If $0 < r < m$, then the homogeneous component $K^X_{r,d}$ of $K^X$ is purely even for $d \geq 0$, and purely odd for $d < 0$. If $-m < r < 0$, then the homogeneous component $K^X_{r,d}$ is purely odd for $d \geq 0$, and purely even for $d < 0$.

The existence of $K^X$ for $X = \mathcal{A}_1^2$ was established by Gannon in [23]. More recently, the existence of all the $G^X$-modules $K^X$ satisfying the specifications of conjecture 2.1 was established in [5]. We note here that the proof of the positivity of multiplicities in section 3 of the work cited depends upon our corollary 3.2.

The formulation of conjecture 2.1 that appears in [5] (see also section 9.3 of [21]) is slightly different from the above, avoiding the use of superspaces and supertraces. We now explain the equivalence.

Recall that $H_{r,f}^X = -H_{-r,f}^X$ for all $X, g \in G^X$ and $r \in \mathbb{Z}/2m\mathbb{Z}$, so to prove conjecture 2.1 it suffices to construct the $K^X_r$ for $0 < r < m$. If the highest rank irreducible component of $X$ is not of type A, then there are further symmetries amongst the $H_{r,f}^X$ that further reduce the problem. Namely, if $X$ has a type D component but no type A components then $m = 2 \mod 4$, we have $H_{r,f}^X = 0$ for $r = 0 \mod 2$, and $H_{r,f}^X = H_{-r,f}^X$. So it suffices to consider the $K^X_r$ for $r \in \{1, 3, 5, \ldots, \frac{1}{2}m\}$. If $X$ has no components of type A or D then either $X = E_6^3$ or $X = E_8^3$. For $X = E_6^3$ it suffices to consider $r \in \{1, 4, 5\}$, and for $X = E_8^3$ we need only $r \in \{1, 7\}$.

With this in mind let us define $I^X \subset \mathbb{Z}/2m\mathbb{Z}$ by setting $I^X := \{1, 2, 3, \ldots, m-1\}$ in case $X$ has a type A component. Set $I^X \setminus \{1\} := \{1, 3, 5, \ldots, \frac{1}{2}m\}$ in case $X$ has a type D component but no type A components, and set $I^X := \{1, 4, 5\}$ for $X = E_6^3$, and $I^X := \{1, 7\}$ for $X = E_8^3$.

Define $H^X_{r,\nu}(\tau)$ to be the $I^X$-vector-valued function (i.e. the vector-valued function with components indexed by $I^X$) whose components are the $H_{r,f}^X$ for $r \in I^X$. The $H^X_{r,\nu}$ inherit good modular properties from the $H^X_{r,f}$. For example, if $X$ has a simple component $A_{m-1}$ then the multiplier system of $H^X_{r,\nu}$ is the inverse of that associated with

$$S_{m-1}(\tau) := (S_{m,1}(\tau), \ldots, S_{m,m-1}(\tau)), \quad (2.6)$$

where $S_{m,\nu}(\tau) := \sum_{k \in \mathbb{Z}} (2km + r)q^{\frac{1}{2}(2km + r)^2}$. For $X = E_8^3$ the multiplier system of $H^X_{r,\nu}$ is the inverse of that associated with

$$S_{8}(\tau) := ((S_{30,1} + S_{30,11} + S_{30,19} + S_{30,29})(\tau), (S_{30,7} + S_{30,13} + S_{30,17} + S_{30,23})(\tau)) \quad (2.7)$$

(See section 4.1 of [2].) More generally, if the mock modular form $H^X_{r,\nu}$ transforms under $\Gamma_0(n_p)$ with multiplier system $\rho^X_{r,\nu}$ and shadow $S^X_{r,\nu}$, then there is a directly related multiplier system $\rho^X_{r,\nu}$ such that $H^X_{r,\nu}$ transforms under $\Gamma_0(n_p)$ with multiplier $\rho^X_{r,\nu}$ and shadow $S^X_{r,\nu}$, where $S^X_{r,\nu}$ is the $I^X$-vector-valued cusp form whose components are the $S^X_{r,\nu}$ for $r \in I^X$.

Note that only one component of $H^X_{r,\nu}$ has a pole. Namely,

$$H_{r,\nu}^{X,1}(\tau) = H_{r,\nu}^{X,1}(\tau) = -2q^{-\frac{r}{d}} + O(q^{1-\frac{1}{d}}). \quad (2.8)$$

In terms of the $H^X_{r,\nu}$, the conjecture 2.1 may be rephrased as the statement that there exist naturally defined bi-graded $G^X$-modules

$$K^X = \bigoplus_{r \in I^X} K^X_{r,d} \bigoplus_{d = 2, 4, 8 \mod 12} K^X_{r,-\frac{d}{2}} \quad (2.9)$$

such that the graded trace attached to an element $g \in G^X$ is recovered from the vector-valued mock modular form $H^X_{r,\nu}$ via
\[ H^{K}_{g,r}(\tau) = -2q^{-\frac{1}{2}} \delta_{r,1} + \sum_{\substack{D \in \mathbb{Z} \cap \mathbb{N} \text{ mod } 4 \neq 0 \atop \text{ } \Gamma_{D} \in \Gamma_{K}}} \text{tr}_{g,r}(g) q^{-\frac{\Gamma_{D}}{2}}. \]  

(2.10)

This is the form in which conjecture 2.1 has been expressed in [5, 21].

Conjecture 6.11 from [2] also concerns the umbral moonshine modules. Now we may simplify it by removing the last sentence, concerning \( X = A_{4}^{3} \), as the newly defined \( H^{K}_{g} \) lead (conjecturally) to representations \( K^{X}_{g} \), which are doublets for \( G^{X} \) whenever \( D \neq 0 \). This is what we would expect, given the discussion in section 6.4 of [2], since there are no values of \( n \) attached to \( X = A_{4}^{3} \) in table 10 of [2].

To conclude this section we note that the example (1.8) also plays a special role in moonshine, because it develops [24] that \( \xi_{1,8}\left(\frac{1}{2}\tau, \frac{1}{2}z\right) = \theta_{2,1}(\tau, 0)\theta_{2,1}(\tau, z) \) is the only nonzero Jacobi form appearing in generalized Mathieu moonshine that is not related to one of the Mathieu moonshine forms by the action of \( SL_{2}(\mathbb{Z}) \). A few further holomorphic Jacobi forms appear in generalized umbral moonshine; they are described explicitly in table 2 of [25]. Conjecturally [25], there are infinite-dimensional modules for certain deformations of the Drinfel’d doubles of the umbral groups \( G^{X} \) that underly the functions of generalized umbral moonshine.

### 2.2. Umbral mock modular forms

We need to modify the statement of conjecture 6.5 of [2], in light of the discussion in section 1. Since the notion of optimal growth is too weak to determine the umbral McKay–Thompson series in general, we recast our reformulation in terms of Rademacher sums, thus generalizing conjecture 5.4 of [1]. The \( H^{K}_{g} \) are well adapted to this, as they each have a pole in exactly one component.

Write \( \Gamma_{\infty} \) for the group of upper-triangular matrices in \( SL_{2}(\mathbb{Z}) \). For \( \alpha \in \mathbb{R} \) and \( \gamma \in SL_{2}(\mathbb{Z}) \) define \( r_{\frac{1}{2}}^{[\alpha]}(\gamma, \tau) := 1 \) if \( \gamma \in \Gamma_{\infty} \). For \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) not in \( \Gamma_{\infty} \) set

\[ r_{\frac{1}{2}}^{[\alpha]}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau) := e\left(\frac{\alpha}{c(c\tau+d)}\right) \sum_{k \geq 0} \left(\begin{array}{c} -2\pi i \frac{\alpha}{c(c\tau+d)} \\ \Gamma(n+\frac{1}{2}) \end{array}\right)^{n+\frac{1}{2}}, \]

(2.11)

where \( e(x) := e^{2\pi ix} \) and we use the principal branch to define \( z^{\frac{1}{2}} \) for \( z \in \mathbb{C} \). Suppose that \( \nu \) is a multiplier system for vector-valued modular forms of weight \( \frac{1}{2} \) on \( \Gamma = \Gamma_{0}(n) \) for some \( n \), and suppose that \( \nu = (\nu_{(g)}) \) satisfies \( \nu_{(g)}\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = e\left(\frac{1}{2}m\right) \) for some basis \( \{\nu_{i}\} \), for some positive integer \( m \). To this data we attach the Rademacher sum

\[ R_{\nu}(\tau) := \lim_{K \to \infty} \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma_{K}^{2}} \nu_{(g)}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\frac{1}{2}(c\tau+d)\right) e\left(-\frac{1}{4m} \frac{a\tau+b}{c\tau+d}\right) e\left(\frac{a\tau+b}{4m(c\tau+d)}\right) \tau, \]

(2.12)

where \( \Gamma_{K}^{2} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\} \in \Gamma \mid 0 \leq c < K, \mid d < K^{2} \right\} \). See [26] for an introduction to Rademacher sums, and [27] for a general and detailed discussion of the vector-valued case. It is nontrivial to verify the convergence of (2.12). An argument which covers the cases of
A case of conjecture 2.2 was proven first in [12], via

\[ M \text{ is denoted } \]  

are supported on perfect square exponents, so asymptotically, 100% of them vanish.

the 8-vector-valued theta series whose

\[ \text{for } 0 \text{ to be the 8-vector-valued theta series whose} \]

\[ \text{and } \sigma \text{ (} \in \mathbb{Z} \text{ on functions} \]

are bounded, and almost always\(^7\) zero.

the case that \( X = A_3^1 \) does not satisfy \( o(g) = 0 \mod 3 \), then we have

\[ \tilde{H}^X_g(\tau) = -2R^X_{\Gamma_0(3)_d\cdot t}(\tau). \]  

If \( X = A_3^1 \) and \( g \in G^X \) satisfies \( o(g) = 0 \mod 3 \) then

\[ \tilde{H}^X_{g,r}(\tau) = -2R^X_{\Gamma_0(3)_d\cdot t}(\tau) + i_g(\tau). \]  

According to the discussion of section 5.2 of [1], conjecture 2.2 is a natural analogue of the genus zero property of monstrous moonshine. It also naturally generalizes conjecture 5.4 of [1]. Evidently the case \( X = A_3^1 \) requires special treatment from the point of view of Rademacher sums, but the difference between \( \tilde{H}^X_g \) and \( -2R^X_{\Gamma_0(3)_d\cdot t} \) for \( X = A_3^1 \) is slight, for the coefficients of \( i_g(\tau) \) are bounded, and almost always\(^7\) zero.

Implicit in conjecture 2.2 is the statement that the expressions defining the Rademacher sums \( R^X_{\Gamma_0(3)_d\cdot t} \) converge. As we have alluded to above, this is verified in section 3 of [5]. Given this, we obtain conjecture 2.2 as a consequence of our main result, theorem 3.1, in section 3 (see corollary 3.2). The \( X = A_3^1 \) case of conjecture 2.2 was proven first in [12], via different methods.

2.3. Paramodular forms

The content of sections 2.1 and 2.2 demonstrate the importance of the theta quark \( Q_{1,1} \) (see (1.9)) to umbral moonshine. In this short section we point out a relation between \( Q_{1,1} \) and the geometry of complex surfaces, and a possible connection to physics.

To prepare for this recall the (degree 2) Siegel upper half-space, defined by

\[ H_2 := \left\{ Z = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix} \in M_2(\mathbb{C}) \mid \Im(Z) > 0 \right\}, \]  

(2.15)

which is acted on naturally by the symplectic group \( \text{Sp}_4(\mathbb{R}) \). For \( t \) as a positive integer, define the paramodular group \( \Gamma_t < \text{Sp}_4(\mathbb{Q}) \) by setting

\[ \tau^t := \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & 1 & * \\ * & 1 & * & * \end{pmatrix} \in \text{Sp}_4(\mathbb{Q}) \mid \text{all } * \text{ in } \mathbb{Z} \right\}. \]  

(2.16)

Then \( \mathcal{A}_t := \Gamma_t \backslash \mathbb{H}_2 \) is a coarse moduli space for (1, t)-polarized abelian surfaces (see [28], where \( \Gamma_t \) is denoted \( \Gamma[t] \)).

For \( k \) an integer the weight \( k \) action of \( \Gamma_t \) on functions \( F : \mathbb{H}_2 \rightarrow \mathbb{C} \) is defined by setting

\[ (F|_k\gamma)(Z) := \det(CZ + D)^{-k} F \left( (AZ + B)(CZ + D)^{-1} \right) \]  

(2.17)

\(^7\) The coefficients of \( i_g^9 \) are supported on perfect square exponents, so asymptotically, 100% of them vanish.

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for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_t$. Holomorphic functions that are invariant for this action are called paramodular forms of weight $k$ for $\Gamma_t$, or Siegel modular forms in case $t = 1$. Write $M_k(\Gamma_t)$ for the vector space they comprise. Then the space of the holomorphic sections of the canonical line bundle on $A_t$ may be identified with $M_1(\Gamma_t)$. According to Hilfsatz 3.2.1 of [29] a holomorphic differential on $A_t$ represented by $F \in M_1(\Gamma_t)$ extends to a holomorphic differential on a nonsingular model of a compactification of $A_t$ if and only if $F$ is cuspidal.

Suppose $F \in M_k(\Gamma_t)$ is a Siegel modular form. Then setting $p = e(\sigma)$ and writing $F(Z) = \sum_{m \geq 0} \phi_m(\tau, z) p^m$, it follows from the invariance of $F$ under the action (2.17) that $\phi_m$ is a Jacobi form of weight $k$ and index $m$. In particular, $\phi_1$ has index 1. Maass discovered [31] a lifting $J_{k,1} \rightarrow M_k(\Gamma_t)$ which reverses this process, assigning a Siegel modular form $F \in M_k(\Gamma_t)$ to a Jacobi form $\phi \in J_{k,1}$ in such a way that $\phi$ is the coefficient of $p$ in $F$. Gritsenko introduced a generalization $J_{k,2} \rightarrow M_k(\Gamma_t)$ of the Maass lift in [32] (see also [15, 28]), and a further generalization adapted to Jacobi forms with level and character appears as theorem 2.2 in [16].

Applying theorem 2.2 of [16] to the theta quark $Q_{1,1}$ we obtain a paramodular form

$$X_{1,9}(Z) := \sum_{m \in \mathbb{Z}} \tilde{Q}_{1,1} T_m^{(1)}(m)(Z)$$

(2.18)
of weight 1 for $\Gamma_9$ with a character of order 3. (See the work cited for the operators $\phi \mapsto \tilde{\phi}_m T_m^{(N)}(m)$.) So the cube of $X_{1,9}$ defines a holomorphic differential on $A_9$.

It was shown by O’Grady [33] (see also [34]) that the Satake compactification of $A_9$ is rational, so there are no cusps of weight 3 for $\Gamma_9$. So $X_{1,9}$ is an example of a paramodular form of an odd weight that is not a cusp form. It is in some sense the first example of a non-cuspidal paramodular form with odd weight, because it can be shown by restriction to the 1-dimensional cusps of $\Gamma_9$ that all forms in $M_k(\Gamma_t)$ are cuspidal when $k$ is odd and $t$ is neither divisible by 16, nor divisible by the square of any odd prime. So all odd weight paramodular forms for $\Gamma_t$ are cuspidal if $t < 9$.

We refer to [35] for a detailed analysis of the cusps of $\Gamma_t$. Theorem 8.3 of [36] gives a more general construction of the paramodular forms of weight 3, and we can recover $X_{1,9}$ by taking $a_i = b_i = 1$ there. Robust methods for computing the spaces of paramodular forms precisely are developed and applied in [37].

It is interesting to note that $A_t$ also appears as a moduli space describing massless degrees of freedom in certain compactifications of heterotic string theory, and associated paramodular forms have been shown [38–40] to govern one-loop corrections of their interaction terms. This may be a good setting in which to understand the physical significance of $Q_{1,1}$. A discussion of umbral mock modular forms in this context appears in section 5.5 of [1].

3. Weight one Jacobi forms

In this section we prove our main result. To formulate it set $\phi^X_g := \sum_r H^{X,g}_{r,m} \theta_{m,r}$ for $X$ a Niemeier root system and $g \in G^X$. In general, $\phi^X_g$ is a (weak) mock Jacobi form of weight 1 and index $m$, where $m = m^X$ is the Coxeter number of any simple component of $X$. We refer to [11] or [41] for background on mock Jacobi forms.

---

8 This is one of the main motivations for the notion of the Jacobi form. See [30] for an early analysis together with applications to hyperbolic Kac–Moody Lie algebras.

9 We thank Cris Poor and David Yuen for explaining this.
Theorem 3.1. Let $X$ be a Niemeier root system and let $g \in G^X$. If $\xi$ is a holomorphic Jacobi form of weight 1 and index $m^X$ with the same level and multiplier system as $\phi^X_g$, then $\xi = 0$, except possibly if $X = A_4^X$ and $o(g) = 0 \mod 3$.

For us, a main application of theorem 3.1 is the verification of conjecture 2.2.

Corollary 3.2. Conjecture 2.2 is true.

Proof of corollary 3.2. Let $X$ be a Niemeier root system and let $g \in G^X$. The convergence of the Rademacher sum $R = R_{(\nu_0(nr),S_\nu_0)}^X \phi^X_g$ is proven in section 3 of [5], so writing $R_r$ for $r \in I^X$ for the components of $R$ we may define a $2m$-vector-valued function $\hat{R} = (\hat{R}_r)$ with components indexed by $\mathbb{Z}/2m\mathbb{Z}$, as follows. For $X \notin \{E_6^X, E_7^X\}$ we set $\hat{R}_r := \pm R_r$ for $\pm r \in I^X$, and $\hat{R}_r := 0$ for $\pm r \notin I^X$. In case $X = E_6^X$ set $R_7 := R_{10}$, $R_8 := R_4$, $R_{11} := R_5$ and $\hat{I}^X := \{1, 4, 5, 7, 8, 11\}$, and then define $\hat{R}_r := \pm R_r$ for $\pm r \in \hat{I}^X$, and $\hat{R}_r := 0$ for $\pm r \notin \hat{I}^X$. For $X = E_7^X$. set $R_7 := R_9 := R_{11} := R_{13}$, $R_{23} := R_{17} := R_{13} := R_5$, and $\hat{I}^X := \{1, 7, 11, 13, 17, 19, 23, 29\}$, and then define $\hat{R}_r$ in analogy with the case $X = E_6^X$. Then $\rho := \sum \hat{R}_r \theta_{m, r}$ is a weak mock Jacobi form of weight 1 and index $m$ with the same level and multiplier system as $\phi^X_g$. Also, the polar parts of $-2\hat{R}$ and $H^X_g$ are the same by construction, so $\xi := \phi^X_g + 2\rho$ is a holomorphic mock Jacobi form of weight 1 with some level.

We claim that the shadow of $\xi$ must vanish, so that $\xi$ is actually a holomorphic Jacobi form. To see this let $\xi = \sum h_r \theta_{m, r}$ be the theta-decomposition of $\xi$ and let $\sum \hat{g}_r \theta_{m, r}$ be the shadow of $\xi$. Then $g = (g_r)$ is a $2m$-vector-valued cusp form of weight $\frac{1}{2}$ transforming with the conjugate multiplier to that of $H^X_g = (H^X_{g_r})$. The proof of proposition 3.5 in [43] shows that if $g' = (g'_r)$ is another vector-valued cusp form of weight $\frac{1}{2}$ transforming in the same way as $g$ then the Petersson inner product $(g, g')$ vanishes unless some of the $h_r$ have nonzero polar parts. But all the $h_r$ are bounded at all cusps by construction. So $g = (g_r)$ is orthogonal to all cusp forms and vanishes identically.

So $\xi$ is a holomorphic Jacobi form as claimed. Applying theorem 3.1 to $\xi$ we conclude that $\xi$ vanishes identically unless $X = A_4^X$ and $o(g) = 0 \mod 3$. So $\phi^X_g = -2\rho$, which is the prediction of conjecture 2.2, for all $g \in G^X$ for all $X$, except when $X = A_4^X$ and $g$ belongs to the $3A$ or $6A$ conjugacy classes of $G^X$. To complete the proof of conjecture 2.2 we need to show that for $X = A_4^X$ and $g \in 3A \cup 6A$, the $r = 3$ and $r = 6$ components of the Rademacher sum $R$ vanish identically. This follows from the fact that if $\nu = (\nu_{ij})_{0 \leq i, j < 9}$ denotes the multiplier system of the 8-vector-valued modular form $S^{0h}$ (see (2.6)), and if $\gamma \in \Gamma_0(3)$, then the matrix entries $\nu_{ij}(\gamma)$ vanish whenever $i = 0 \mod 3$ and $j \neq 0 \mod 3$, or $j = 0 \mod 3$ and $i \neq 0 \mod 3$. In other words, the multiplier of $S^{0h}$ becomes block diagonal, with blocks indexed by $\{1, 2, 4, 5, 7, 8\}$ and $\{3, 6\}$, when restricted to $\Gamma_0(3)$. It follows from this, and the description of the multiplier systems for $3A$ and $6A$ in [2], or the explicit descriptions of the $H^X_g$ in [5], that the corresponding Rademacher sums have vanishing $r = 3$ and $r = 6$ components. The prediction of conjecture 2.2 follows for $X = A_4^X$ and $o(g) = 0 \mod 3$. This completes the proof.

We prove theorem 3.1 in section 3.4. To prepare for this we review the metaplectic double cover of the modular group, and some results from [13] in section 3.1, and we review some facts about the characters of the Weil representations following [22] in section 3.3. In section 3.2 we describe an approach that, although not powerful enough to prove our main theorem, can rule out the nonzero holomorphic Jacobi forms of weight one for infinitely many
indexes and levels. Our presentation (and, in particular, our notation) is similar to that which appears in section 2 of [41].

3.1. Weil representations

Background references for this section include [13, 22, 44]. The theory we review here may be situated within the much broader setting of Howe duality and the theta correspondence. We refer to [45, 46] for introductory surveys of these topics.

Write $\widetilde{SL}_2(\mathbb{Z})$ for the metaplectic double cover of $SL_2(\mathbb{Z})$. We may realize $\widetilde{SL}_2(\mathbb{Z})$ as the set of pairs $(\gamma, v)$ where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $v : \mathbb{H} \to \mathbb{C}$ is a holomorphic function satisfying $v(\tau)^2 = c\tau + d$. Then the product in $\widetilde{SL}_2(\mathbb{Z})$ is given by $(\gamma, v(\tau))(\gamma', v'(\tau)) = (\gamma\gamma', v(\gamma')v'(\tau))$. For the generators we may take $T := (T, 1)$ and $S := (S, \tau^2)$ where $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

According to [47], for example, if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ is such that $c = 0 \mod 4$ and $d = 1 \mod 4$ then $j(\gamma, \tau) := \theta_{1,0}(\gamma\tau, 0)\theta_{1,0}(\tau, 0)^{-1}$ satisfies $j(\gamma, \tau)^2 = c\tau + d$. So we may consider the pairs $(\gamma, j(\gamma, \tau))$ where $\gamma$ belongs to the principal congruence subgroup $\Gamma(4m)$, being the kernel of the natural map $SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/4m\mathbb{Z})$. These constitute a normal subgroup of $\tilde{SL}_2(\mathbb{Z})$, which we denote $\Gamma(4m)^*$. The corresponding quotient $\widetilde{SL}_2(\mathbb{Z})/\Gamma(4m)^*$ is a double cover—let us denote it $\tilde{SL}_2(\mathbb{Z}/4m\mathbb{Z})$—of $SL_2(\mathbb{Z}/4m\mathbb{Z})$.

We will be concerned with the representations of $\tilde{SL}_2(\mathbb{Z})$ arising, via the construction of Weil, from (cyclic) finite quadratic spaces. A finite quadratic space is a finite abelian group $A$ equipped with a function $Q : A \to \mathbb{Q}/\mathbb{Z}$ such that $Q(na) = n^2Q(a)$ for $n \in \mathbb{Z}$ and $a \in A$, and such that $B(a, b) := Q(a + b) - Q(a) - Q(b)$ is a $\mathbb{Z}$-bilinear map. Let $\mathcal{C}A = \bigoplus_{a \in A} \mathbb{C}e^a$ be the group algebra of $A$. The Weil representation associated with $(A, Q)$ is the left $\tilde{SL}_2(\mathbb{Z})$-module structure on $\mathcal{C}A$ defined by requiring that

$$
\tilde{T}e^a = e(Q(a))e^a,
$$
$$
\tilde{S}e^a = \sigma_A|A|^{-\frac{1}{2}} \sum_{b \in A} e(-B(a, b))e^b,
$$

(3.1)

for $a \in A$, where $\sigma_A := |A|^{-\frac{1}{2}} \sum_{a \in A} e(-Q(a))$. In general, $\sigma_A$ is an eighth root of unity. The action (3.1) factors through $SL_2(\mathbb{Z})$ if and only if $\sigma_A^4 = 1$.

The $\theta_{m,r}$ (see (1.5)) furnish concrete realizations of such Weil representations. To explain this let $\Theta_m$ denote the vector space spanned by the $\theta_{m,r}$ for $r \in \mathbb{Z}/2m\mathbb{Z}$, and define a right $\tilde{SL}_2(\mathbb{Z})$-module structure on $\Theta_m$ by setting

$$
(\phi|_{\mathbb{Z}/m}(\gamma, v))(\tau, z) := \phi \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) \frac{1}{v(\tau)} e \left( -m\frac{cz^2}{c\tau + d} \right)
$$

(3.2)

for $\phi \in \Theta_m$ and $(\gamma, v) \in \tilde{SL}_2(\mathbb{Z})$, when $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\Theta_m$ is isomorphic to the dual of the Weil representation attached to $D_m := (\mathbb{Z}/2m\mathbb{Z}, x \mapsto x^2)$). Explicitly, we may associate $\theta_{m,r}$ with the linear map $e^r \mapsto \delta_{r,m}$ on $\mathbb{C}D_m$. (Note that $\sigma_A = e(-\frac{1}{8})$ in (3.1) for $A = D_m$, for any $m$.) Also, $\Theta_m$ is trivial for the action of $\Gamma(4m)^*$, so it may be regarded as a representation of $\tilde{SL}_2(\mathbb{Z}/4m\mathbb{Z})$. 
Except when \( m = 1 \), the \( \Theta_m \) are not irreducible. We will make use of the explicit decomposition into irreducible \( \widetilde{SL}_2(\mathbb{Z}) \)-modules, which is given in Satz 1.8 of [13]. To describe this, note first that the orthogonal group \( O_m := \{ a \in \mathbb{Z}/2m\mathbb{Z} \mid a^2 = 1 \mod 4m \} \) of \( D_m \) acts naturally on \( \Theta_m \), via \( \theta_{m,r} : a := \theta_{m,ra} \) for \( r \in \mathbb{Z}/2m\mathbb{Z} \) and \( a \in O_m \), and this commutes with the action of \( SL_2(\mathbb{Z}) \). Next, define a Hermitian inner product on \( \Theta_m \) by requiring that the \( \theta_{m,r} \) furnishes an orthonormal basis. Then the action of \( \widetilde{SL}_2(\mathbb{Z}) \) on \( \Theta_m \) is unitary with respect to this inner product, according to lemma 1.4 of [13]. For \( d \) an integer, the assignment \( U_d : \phi(\tau, z) \mapsto \phi(\tau, dz) \) defines an \( \widetilde{SL}_2(\mathbb{Z}) \)-module map \( \Theta_m \to \Theta_{md} \). Define \( \Theta_m^{\text{new}} \) to be the orthogonal complement in \( \Theta_m \) of the subspace spanned by the images of the maps \( U_d \) for \( d \) positive and \( d^2 \mid m \). The action of \( O_m \) preserves \( \Theta_m^{\text{new}} \), so for \( \alpha \in \hat{O}_m := \text{hom}(O_m, \mathbb{C}^\times) \) we may define a sub \( \widetilde{SL}_2(\mathbb{Z}) \)-module of \( \Theta_m^{\text{new}} \) by setting

\[
\Theta_m^{\text{new},\alpha} := \{ \phi \in \Theta_m^{\text{new}} \mid \phi \cdot \alpha(a) = \phi(a) \}.
\]  

(3.3)

Then \( \Theta_m^{\text{new},\alpha} \) is irreducible for \( \widetilde{SL}_2(\mathbb{Z}) \) and the \( \widetilde{SL}_2(\mathbb{Z}) \)-irreducible submodules of \( \Theta_m \) are exactly the \( \Theta_m^{\text{new},\alpha} \mid U_d \) for \( d^2 \mid m \), according to Satz 1.8 of [13]. Note that the decomposition \( \Theta_m = \Theta_m^{\text{new}} \oplus \Theta_m^{\text{old}} \) is preserved by \( \widetilde{SL}_2(\mathbb{Z}) \), where \( \Theta_m^{\pm} \) denotes the span of the \( \theta_{m,r} \) (see (1.6)). Indeed, the maps \( U_d \) furnish an \( \widetilde{SL}_2(\mathbb{Z}) \)-module isomorphism

\[
\Theta_m^{\pm} \simeq \bigoplus_{d^2 \mid m} \bigoplus_{\alpha \in \hat{O}_m} \Theta_m^{\text{new},\alpha}_{m/d^2}.
\]  

(3.4)

3.2. Exponents

In this short section we present a simple criterion (lemma 3.4), which although not powerful enough to handle all the cases of theorem 3.1, can be used to prove the vanishing of \( J_{1,m}(N) \) for many cases in which \( N \) is not square-free.

To prepare for the proof, note that since \( \Gamma(4m)^* \) acts trivially on the \( \theta_{m,r} \), the theta-coefficients \( h_\gamma \) (see (1.4)) of a holomorphic Jacobi form \( \xi(\tau, z) \) of weight 1, index \( m \) and level \( N \) belong to the space \( M_2(\Gamma_0(N) \cap \Gamma(4m)) \). Here \( M_2(\Gamma) \), for \( \Gamma \) a subgroup of some \( \Gamma(4m) \) is the vector space of the holomorphic functions \( h : \mathbb{H} \to \mathbb{C} \) such that \( h|_2(\gamma, j(\gamma, \tau)) = h \) for \( \gamma \in \Gamma \), and \( h|_2(\gamma, v) \) remains bounded as \( \Im(\tau) \to \infty \) for every \( (\gamma, v) \in \widetilde{SL}_2(\mathbb{Z}) \), where

\[
(h|_2(\gamma, v))(\tau) := h \left( \frac{a\tau + b}{c\tau + d} \right) \frac{1}{v(\tau)}.
\]  

(3.5)

Since \( \Gamma(4m)^* \) is normal in \( \widetilde{SL}_2(\mathbb{Z}) \), the space \( M_2(\Gamma(4m)) \) is naturally an \( \widetilde{SL}_2(\mathbb{Z}) \)-module. Also, \( U_0 : \phi(\tau, z) \mapsto \phi(\tau, 0) \) defines a map \( \Theta_m \to M_2(\Gamma(4m)) \).

**Proposition 3.3.** Suppose that \( \xi \in J_{1,m}(N) \) for some positive integers \( m \) and \( N \). Choose \( M \) so that \( m|M \) and \( N|4M \). Then the theta-coefficients of \( \xi \) belong to \( \sum_{M|M} \Theta_M | U_0 \).

**Proof.** Assume that \( m, N \) and \( M \) are as in the statement of the proposition. Then the theta-coefficients of \( \xi \) belong to \( M_2(\Gamma(4M)) \). According to Satz 5.2 of [13] we have

\[
M_2(\Gamma(4M)) = \bigoplus_{M'|M} \bigoplus_{\alpha' \in \hat{O}_{M'}} \Theta_{M'}^{\text{new},\alpha'} | U_0.
\]  

(3.6)

The claimed result follows. \( \square \)
Proposition 3.3 puts restrictions on the exponents that can appear in the theta-coefficients of a candidate nonzero Jacobi form of weight 1 with a given index and level. On the other hand, the fact that the Fourier coefficients of such a Jacobi form can only involve integer powers of $q$ also imposes restrictions. Taking these together we can rule out all but the zero function in many instances, especially when the index is small.

**Lemma 3.4.** Suppose that $M$ is a positive integer and $m$ is a positive divisor of $M$. Set $m' = \frac{M}{m}$. If the equation

$$r^2m' + s^2t = 0 \mod 4M$$

(3.7)

has no solutions $(r, s, t)$ with $0 < r < m$ and $t$ a divisor of $M$, then $J_{1, m}(4M) = \{0\}$.

**Proof.** According to proposition 3.3, if $\xi = \sum h_r \theta_{mr}$ is the theta decomposition of a Jacobi form $\xi \in J_{1, m}(4M)$ then $h_r$ belongs to $\sum_{M' | M} \Theta_{M'}|U_0$ for all $r$. Let $M' | M$ and set $t = \frac{M}{M'}$. Then $\theta_{M', s}(4M \tau, 0) \in q^{-rM'}\mathbb{Z}[q^{-M'}]$. On the other hand $h_r(4r \tau) \in q^{-rM}\mathbb{Z}[q^{-M}]$. So in order for $h_r$ to be a nonzero linear combination of the $\theta_{M', s}(\tau, 0)$ with $M' | M$, we require that (3.7) holds, for some $s$. That is, if $h_r$ is a nonzero element of $\sum_{M' | M} \Theta_{M'}|U_0$ then we must have (3.7) for some $r$ and $s$, and some $t | M$. We have $h_r = -h_{-r}$ since $\xi$ has weight 1, so we may assume that $0 < r < m$. This proves the claim.

The next result serves as an application of lemma 3.4.

**Proposition 3.5.** We have $J_{1, 2}(2^a) = \{0\}$, $J_{1, 3}(4 \cdot 3^a) = \{0\}$ and $J_{1, 4}(2^e) = \{0\}$ for every non-negative integer $a$.

**Proof.** We obtain the vanishing of $J_{1, 2}(2^a)$ by applying lemma 3.4 with $m = 2$ and $M$ an arbitrary positive power of 2. Indeed, in this case $r = 1$ so taking $M = 2^a$ in (3.7) we obtain $2b^2 + 2a^{-1} = 0 \mod 2^{a+2}$ where $0 \leq b \leq a$. If $s$ is a solution then $2b^2 = 2a^{-1}d$ for some $d = 7 \mod 8$, but 7 is neither even nor a square modulo 8 so there is no such $s$. So $J_{1, 2}(2^{a+2})$ vanishes according to lemma 3.4. The vanishing of $J_{1, 3}(4 \cdot 3^a)$ and $J_{1, 4}(2^e)$ is obtained very similarly, by taking $m = 3$ and $m = 4$, respectively, and $M = m^a$.

3.3. Prime power parts

The module $\Theta_{m^{\infty,\infty}}$ factors through $\overline{SL}_2(\mathbb{Z}/4m\mathbb{Z})$ so it is natural to consider its prime power parts. To explain what this means note that if $m_p$ denotes the largest power of $p$ dividing $m$ then the natural map

$$\overline{SL}_2(\mathbb{Z}/4m\mathbb{Z}) \to \overline{SL}_2(\mathbb{Z}/4m_p\mathbb{Z}) \times \prod_{p \text{ odd prime}} SL_2(\mathbb{Z}/m_p\mathbb{Z})$$

(3.8)

is an isomorphism. Thus any irreducible $\overline{SL}_2(\mathbb{Z}/4m\mathbb{Z})$-module can be written as an external tensor product of irreducible modules for $\overline{SL}_2(\mathbb{Z}/4m_p\mathbb{Z})$ and the $SL_2(\mathbb{Z}/m_p\mathbb{Z})$. At the level of characters, if $\chi$ is an irreducible character for $\overline{SL}_2(\mathbb{Z}/4m\mathbb{Z})$ then there are corresponding characters $\chi_p$ of $SL_2(\mathbb{Z}/4m_p\mathbb{Z})$ and the $SL_2(\mathbb{Z}/m_p\mathbb{Z})$ such that

$$\chi(\gamma) = \prod_{p | 4m} \chi_p(\gamma_p)$$

(3.9)
for $\gamma \in \overline{SL}_2(\mathbb{Z}/4m\mathbb{Z})$, where $\gamma_p$ is the $\overline{SL}_2(\mathbb{Z}/4m_2\mathbb{Z})$ or $\overline{SL}_2(\mathbb{Z}/m_p\mathbb{Z})$ component of the image of $\gamma$ under the map (3.8). Call $\chi_p$ the $p$-part of $\chi$.

Even though $CD_m$—being the left $\overline{SL}_2(\mathbb{Z})$-module dual to the right $\overline{SL}_2(\mathbb{Z})$-module structure on $\Theta_m$—is generally not irreducible, it has well-defined $p$-parts. Specifically, if we set $D_m(a) := \left(\mathbb{Z}/2m\mathbb{Z}, \frac{a \gamma}{m}\right)$ for a coprime to $m$, and $L_m(a) := \left(\mathbb{Z}/m\mathbb{Z}, \frac{a \gamma}{m}\right)$ for a coprime to $m$ and $m$ odd, then we have

$$CD_m := CD_{m_2}(a_2) \otimes \bigotimes_{p \text{ odd prime}} CL_{m_p}(a_p)$$

(3.10)

for certain $a_p \in \mathbb{Z}$, where the factors in (3.10) correspond to the factors in (3.8). A calculation reveals that $a_2$ is an inverse to $\frac{m}{m_2}$ modulo $4m_2$, and $a_p$ is an inverse to $\frac{m}{m_p}$ modulo $m_p$.

Write $CD_m(a)^\pm$ for the submodule of $CD_m(a)$ spanned by the $e^a \pm e^{-a}$ for $a \in \mathbb{Z}/2m\mathbb{Z}$. For $m$ odd write $CL_m(a)^\pm$ for the submodule of $CL_m(a)$ spanned by the $e^a \pm e^{-a}$ for $a \in \mathbb{Z}/m\mathbb{Z}$. If $m = p^k$ for $p$ as an odd prime and $k \geq 2$ then $CL_{p^k}(a)^\pm$ admits a natural embedding by $CL_{p^{k-1}}(a)^\pm$. Let $CL_{p^k}(a)^{\text{new,} \pm}$ denote the orthogonal complement of the submodule $CL_{p^{k-1}}(a)^\pm$ in $CL_{p^k}(a)^\pm$. Set $CL_{p^k}(a)^{\text{new,} \pm} := CL_{p^k}(a)^{\pm}$.

**Lemma 3.6 [22]** For $p$ an odd prime, $k$ a positive integer and a coprime to $p^k$, the $\overline{SL}_2(\mathbb{Z})$-modules $CL_{p^k}(a)^{\text{new,} \pm}$ are irreducible.

Let $\Theta_m^\pm$ be the submodule of $\Theta_m$ spanned by the $\theta_{m, \pm}^\pm$ (see 1.6). Set $L_{p^k}^{\text{new,} \pm} := L_{p^k}(1)^{\text{new,} \pm}$ for $p$ an odd prime. We will employ the following notation for the characters of the $\overline{SL}_2(\mathbb{Z})$-modules we have defined

$$\vartheta_m(\gamma) := \text{tr}(|\Theta_m|)$$

$$\vartheta_{m, \pm}^\pm(\gamma) := \text{tr}(|\Theta_{m, \pm}^\pm|)$$

$$\nu_{m, \pm}^\pm(\gamma) := \text{tr}(|\Theta_{m, \pm}^{\text{new,} \pm}|)$$

$$\lambda_{p^k}(\gamma) := \text{tr}(|L_{p^k}^{\text{new,} \pm}|).$$

To describe the $p$-parts of the $\nu_{m, \pm}^\pm$ it is convenient to abuse notation and evaluate $\alpha$ on primes. Specifically, if $m_p$ denotes the highest power of $p$ dividing $m$, then there is a unique $a \mod 2m$ such that $a = -1 \mod 2m_p$ and $a = 1 \mod \frac{m}{m_p}$, and we write $\alpha(p)$ instead of $\alpha(a)$. With this convention it follows from (3.10) that the 2-part of $\nu_{m, \pm}^\pm$ takes the form $\sigma(\vartheta_{m_p}^\pm)$ where $\alpha(2) = \pm 1$ and $\sigma$ is the Galois automorphism of $\mathbb{Q}(e(\frac{1}{m_p}))$ mapping $e(\frac{1}{m_p})$ to $e(\frac{m}{2m_p})$, for $a_2$ as in (3.10). For $p$ odd the $p$-part of $\nu_{m, \pm}^\pm$ is $\sigma(\lambda_{m_p}^\pm)$ where $\alpha(p) = \pm 1$, and now $\sigma$ maps $e(\frac{1}{m_p})$ to $e(\frac{m}{2m_p})$ where $a_p$ is as in (3.10).

**3.4. Proof of the main result**

We present the proof of theorem 3.1 in this section. Our main technical tool is lemma 3.11, which we establish by applying a specialization (proposition 3.8) of theorem 8 in [22].

Note that if $m$ and $m'$ are both positive divisors of some integer $M$, then the product $\vartheta_m \vartheta_{m'}$ (see (3.11)) factors through $\overline{SL}_2(\mathbb{Z}/4M\mathbb{Z})$. It will be useful to know when an irreducible
constituent of \( \partial_m \partial_{m'} \) can factor through \( SL_2(\mathbb{Z}/4M\mathbb{Z})/\{\pm I\} \). Our first result in this section is an answer to this question.

**Lemma 3.7.** Let \( m \) and \( m' \) be positive integers and suppose that \( \sigma \) and \( \sigma' \) are Galois automorphisms of \( \mathbb{Q}(e(\frac{1}{m})) \) and \( \mathbb{Q}(e(\frac{1}{m'})) \), respectively. If \( \sigma(i) = \sigma'(i) \) then the character \( \sigma(\partial_m^-)\sigma'(\partial_{m'}^-) \) factors through \( SL_2(\mathbb{Z})/\{\pm I\} \), but \(-I\) acts nontrivially on every irreducible constituent of \( \sigma(\partial_m^-)\sigma'(\partial_{m'}^-) \) and \( \sigma(\partial_m^+)\sigma'(\partial_{m'}^+) \). If \( \sigma(i) = -\sigma'(i) \) then the characters \( \sigma(\partial_m^-)\sigma'(\partial_{m'}^+) \) and \( \sigma(\partial_m^+)\sigma'(\partial_{m'}^-) \) factor through \( SL_2(\mathbb{Z})/\{\pm I\} \), but \(-I\) acts nontrivially on every irreducible constituent of \( \sigma(\partial_m^-)\sigma'(\partial_{m'}^+) \).

**Proof.** The dimension of the \( i \)th eigenspace for the action of \( S \) on \( \Theta_m^- \otimes \Theta_{m'}^- \) (for any choice of signs) is

\[
\frac{1}{4} \sum_{k=0}^{3} (-i)^{ak} \partial_m^+ (\tilde{S}^k) \partial_{m'}^+ (\tilde{S}^k). 
\]

(3.12)

Set \( \zeta := e(\frac{1}{2}) \). From the definition (3.1), using identities for the quadratic Gauss sums, we compute that

\[
\partial_m^+ (\tilde{S}^k) = \begin{cases} (m \pm 1)(-i)^k \zeta^{\pm k} & \text{if } k = 0 \text{ mod } 2, \\ (-i)^k \zeta^{\pm k} & \text{if } k = 1 \text{ mod } 2 \text{ and } m = 0 \text{ mod } 2, \\ 0 & \text{if } k = 1 \text{ mod } 2 \text{ and } m' = 1 \text{ mod } 2. \end{cases} 
\]

(3.13)

Applying (3.13) to (3.12) we find that the \( \pm i \) eigenspaces for the action of \( S \) on \( \Theta_m^- \otimes \Theta_{m'}^- \) are trivial, as are the \( \pm 1 \) eigenspaces for the actions of \( S \) on \( \Theta_m^- \otimes \Theta_{m'}^- \) and \( \Theta_m^+ \otimes \Theta_{m'}^+ \). This proves the claimed result for \( \sigma \) and \( \sigma' \) both trivial. The general case is very similar. \( \Box \)

The next result is a slight refinement of a specialization of theorem 8 in [22].

**Proposition 3.8 ([22]).** Let \( m \) and \( N \) be positive integers, and let \( M \) be a positive integer such that \( m \mid M \) and \( N \) divides \( 4M \). Then we have

\[
\dim J_{1,m}(N) = \sum_{\mu \mid M \text{ square--free}} \langle \tilde{\mu} - \hat{\Theta}_m^+, \hat{1}_N \rangle 
\]

(3.14)

where \( \hat{1}_N \) is the character of \( SL_2(\mathbb{Z}) \) obtained via induction from the trivial character of \( \Gamma_0(N) \).

The hypothesis on \( M \) ensures that \( \hat{1}_N \) in (3.14) factors through \( SL_2(\mathbb{Z}/4M\mathbb{Z}) \). So we may regard the inner product \( \langle \cdot, \cdot \rangle \) in (3.14) as the usual scalar product on class functions for \( SL_2(\mathbb{Z}/4M\mathbb{Z}) \).

**Proof of proposition 3.8.** In the notation of [22] we take \( F \) to be the \( 1 \times 1 \) matrix \((m)\), we set \( \Gamma' = \Gamma_0(N) \), and let \( V \) be the trivial representation of \( \Gamma' \). Applying theorem 8 to these choices we obtain that \( J_{1,m}(N) \) is isomorphic to the space of \( SL_2(\mathbb{Z}) \)-invariant vectors in

\[
\bigoplus_{\mu \mid M \text{ square--free}} CD_m(-1)^+ \otimes CD_m(-1) \otimes \hat{V} 
\]

(3.15)

where \( \hat{V} \) is the space of \( SL_2(\mathbb{Z}/4M\mathbb{Z}) \)-invariant vectors in \( V \), and \( CD_m(-1)^+ \) denotes the subspace of \( CD_m(-1) \) spanned by the vectors \( e^a + e^{-a} \) for \( a \in \mathbb{Z}/2m'\mathbb{Z} \), and \( V \) is the
$\text{SL}_2(\mathbb{Z})$-module obtained via induction from $V$. By our hypotheses, all the modules in (3.15) factor through $\text{SL}_2(\mathbb{Z}/4\mathbb{Z})$. For this group, the character of $\text{CD}_m(-1)^+$ is $\vartheta^+_m$, and the character of $\text{CD}_m(-1)$ is $\vartheta_m$. So if we write $I_N$ for the character of $V$, and write $I$ for the trivial character of $\text{SL}_2(\mathbb{Z}/4\mathbb{Z})$, then the space of $\text{SL}_2(\mathbb{Z})$-invariants in the summand $\text{CD}_m(-1)^+ \otimes \text{CD}_m(-1) \otimes V$ is the scalar product $(1, \vartheta^+_m \vartheta_m \bar{I}_N) = (\vartheta^+_m, \bar{I}_N)$ in the ring of class functions on $\text{SL}_2(\mathbb{Z}/4\mathbb{Z})$. Now $\bar{I}_N$ factors through $\text{SL}_2(\mathbb{Z}/4\mathbb{Z})/\{\pm I\}$ by construction, whereas $-I$ is nontrivial on every irreducible constituent of $\vartheta^+_m \vartheta_m$ by lemma 3.7. So $(\vartheta^+_m \vartheta_m, \bar{I}_N) = 0$ and $(\vartheta^+_m \vartheta_m, \bar{I}_N) = (\vartheta^+_m \vartheta_m, \bar{I}_N)$. The claimed formula follows. □

Before presenting our main application of proposition 3.8 we require some results about the characters of $\text{SL}_2(\mathbb{Z}/64\mathbb{Z})$ and $\text{SL}_2(\mathbb{Z}/p^2\mathbb{Z})$ for $p$ an odd prime. The next lemma was obtained by working directly with the character table of $\text{SL}_2(\mathbb{Z}/64\mathbb{Z})$, which we constructed on a computer using GAP [48].

**Lemma 3.9.** Suppose that $k, k' \in \{1, 2, 4, 8, 16\}$, and $\sigma$ and $\sigma'$ are Galois automorphisms of $Q(e(\frac{1}{16}))$ such that $\sigma(i) = \sigma'(i')$. Then $(\sigma(\vartheta^+_k)\sigma'(\vartheta^+_k), \bar{I}_{16}) = 0$. If $(k, k')$ is not $(8, 2)$ or $(16, 16)$ then $(\sigma(\vartheta^+_k)\sigma'(\vartheta^+_k), \bar{I}_{16}) = 0$.

**Lemma 3.10.** If $p$ is an odd prime then $\bar{I}_p$ has four irreducible constituents. One of these is the trivial character, another has degree $p$, and the remaining two have degree $\frac{1}{2}(p^2 - 1)$.

**Proof.** Let $p$ be an odd prime. Set $G = \text{SL}_2(\mathbb{Z}/p^2\mathbb{Z})/\{\pm I\}$ and $G = \text{SL}_2(\mathbb{Z}/p\mathbb{Z})/\{\pm I\}$, let $B$ be the image of $\Gamma_0(p^2)$ in $G$, and let $B$ be the image of $\Gamma_0(p)$ in $G$. Then $\bar{I}_p$ is the character of the permutation module $\mathbb{C}[G/B] := \mathbb{C}G \otimes_{\mathbb{C}B} \mathbb{C}$, where $C$ is the right trivial module for $B$, and $\bar{I}_p$ is the character of $\mathbb{C}[G/B]$. Let $W$ be the kernel of the natural map $\mathbb{C}[G/B] \to \mathbb{C}[G/B]$. Then $I_p = \bar{I}_p$ is the character of $W$. The space $\mathbb{C}[G/B]$ realizes the permutation module for the action of $\text{PSL}_2(\mathbb{Z}/p\mathbb{Z})$ on the points of the projective line over $\mathbb{Z}/p\mathbb{Z}$, and is well known to have two irreducible constituents, one being trivial. So $\bar{I}_p$ is the sum of the trivial character and an irreducible character of degree $p$.

So we need to find the degrees of the irreducible constituents of $W$. We may construct these constituents explicitly. To do this, define cosets $u_x := \left(\begin{array}{cc} x & -1 \\ 1 & 0 \end{array}\right)B$ and $v_y := \left(\begin{array}{cc} 1 & 0 \\ y & 1 \end{array}\right)B$ for $x, y \in \mathbb{Z}/p^2\mathbb{Z}$. Then the set $\{u_x, v_y \mid y = 0 \text{ mod } p\}$ is a basis for $\mathbb{C}[G/B]$. For $x \in \mathbb{Z}/p^2\mathbb{Z}$ and $a \in \mathbb{Z}/p\mathbb{Z}$ define 1-dimensional vector spaces $U_x^a$ and $U_{\infty}^a$ in $\mathbb{C}[G/B]$ by setting

$$U_x^a := \text{Span} \left\{ \sum_{k \mod p} e\left(-\frac{ak}{p}\right) u_{x+kp} \right\}, \quad U_{\infty}^a := \text{Span} \left\{ \sum_{k \mod p} e\left(-\frac{ak}{p}\right) v_{-kp} \right\}. \quad (3.16)$$

Then $U_x^a$ only depends on $x \mod p$, and $W$ is the direct sum of the $U_x^a$ with $a \neq 0$. With $S = \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right)$ and $T = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)$ we have $T(U_x^a) = U_{x+1}^a$ and $S(U_x^a) = U_{-x^2}^a$, where $a+x^2$ means $a$ in the case of $x \in [0, \infty)$, and is the usual product $ax^2$ otherwise. So if $W$ is the sum of the $U_x^a$ where $a$ is restricted to the nonzero quadratic residues, and if $W''$ is the sum of the
\( U_a \) where \( a \) runs over the nonzero nonquadratic residues, then \( W = W' \oplus W'' \) as \( G \)-modules. We may observe directly that \( W' \) and \( W'' \) are irreducible, or apply Frobenius reciprocity to show that \( \#(B \setminus G/B) = 4 \) is an upper bound on the number of constituents of \( \tilde{1}_p \). We have \( \dim W' = \dim W'' = \frac{1}{2}(p^2 - 1) \), so the lemma has been proved.

The next result is our main application of proposition 3.8, and our main technical tool for proving theorem 3.1.

**Lemma 3.11.** Let \( m \) and \( N \) be positive integers. Assume that none of \( m \), \( N \) or \( mN \) are divisible by 32. Then \( J_{1,m}(N) \) is irreducible, or apply Frobenius reciprocity to \( W = \tilde{1}_p \) is similarly. By our conditions on \( l \) is odd and \( m \) and \( N \) are positive integers. Assume that none of \( m \), \( N \) or \( mN \) are divisible by 32. Then \( J_{1,m}(N) \) is divisible by 16. Let \( P \) be the product of the prime divisors of \( (m,N) \) that are congruent to 3 modulo 4. Then taking \( M = \frac{1}{2}mN \) in proposition 3.8 we see that the dimension of \( J_{1,m}(N) \) is a sum of the scalar products

\[
\langle \nu^{\alpha_1}_p \nu^{\alpha_2}_p, \tilde{1}_N \rangle
\]

(see (3.11)), to be computed in the space of class functions on \( SL_2(\mathbb{Z}/4M\mathbb{Z}) \), where \( l \) is a divisor of \( m \), and \( l' \) divides \( \frac{1}{2}mN \).

Suppose that \( p = 3 \mod 4 \) is a prime dividing \( mN \). Then \( p \) may occur as a divisor of \( l \) or \( l' \), orpossibly both, in a summand (3.17). We claim that such a summand (3.17) must vanish unless \( l \) and \( l' \) are both exactly divisible by \( p \), or both exactly divisible by even powers of \( p \). Write \( l_p \), for the highest power of \( p \) dividing \( l \), and interpret \( l'_p \) similarly. By our conditions on \( m \) and \( N \) the remaining possibilities are that \( (l_p,l'_p) = (p^2,p) \), \( (p,p^2) \), \( (p,1) \) or \( (1,p) \). Also, in the former two cases, \( N \) must be coprime to \( p \). If \((l_p,l'_p) = (p^2,p)\) then the \( p \)-part of \( \nu^{\alpha_1}_p \nu^{\alpha_2}_p \) is \( \sigma(\lambda^+_p)\sigma'(\lambda^-_p) \) (see (3.11)) for some Galois automorphisms \( \sigma,\sigma' \) of \( \mathbb{Q}(e(\frac{1}{p})) \), and the \( p \)-part of \( \tilde{1}_N \) is trivial. But \( \sigma(\lambda^+_p) \) and \( \sigma'(\lambda^-_p) \) are irreducible to different degrees, so the trivial character cannot arise as a constituent of their product. The case that \((l_p,l'_p) = (p,p^2)\) is similar. For \((l_p,l'_p)\) equal to \((p,1)\) or \((1,p)\) the \( p \)-part of \( \nu^{\alpha_1}_p \nu^{\alpha_2}_p \) is \( \sigma(\lambda^+_p) \) for some Galois automorphism \( \sigma \) of \( \mathbb{Q}(e(\frac{1}{p})) \). However, \( \tilde{1}_N \) is now the permutation character of \( SL_2(\mathbb{Z}) \) arising from its action on the cosets of \( \Gamma_0(N_p) \) where \( N_p \) is the largest power of \( p \) dividing \( N \) (namely, \( 1, p \) or \( p^2 \)). As such, the irreducible constituents of \( \tilde{1}_N \) have degrees \( 1 \) (the trivial character), \( p \), or \( \frac{1}{2}(p^2 - 1) \), according to lemma 3.10. Certainly \( \frac{1}{2}(p + 1) \) is not 1 or \( p \) or \( \frac{1}{2}(p^2 - 1) \) for any odd prime \( p \), and \( \frac{1}{2}(p^2 - 1) \) is not one of these degrees when \( p > 3 \). Although \( CL_{13}(a) \) is 1-dimensional it is not trivial. So we may conclude that the summand (3.17) vanishes unless \( p = 3 \mod 4 \) either divides both \( l \) and \( l' \) exactly, or divides both \( l \) and \( l' \) with even multiplicity.

So for the remainder of the proof we may restrict to summands (3.17) such that the odd parts of \( l \) and \( l' \) (i.e., \( \frac{1}{2}l \) and \( \frac{1}{2}l' \)) are congruent modulo 4. For now, let us also restrict ourselves to the case where \( m \) is odd and \( N = 32 \mod 64 \). Then the \( l \) in a summand (3.17) is odd, and \( l' \neq 0 \mod 16 \). In the case in which \( l' \) is odd we have \( l = l' \mod 4 \), so the 2-parts of \( \nu^{\alpha_1}_p \) and \( \nu^{\alpha_2}_p \) are both \( \sigma(\nu^{\alpha}_p) = \sigma(\nu^{\alpha}_p) \), where \( \sigma \) is the Galois automorphism of \( \mathbb{Q}(e(\frac{1}{2})) \) that maps \( e(\frac{1}{2}) \)
to \(e(\frac{q}{2})\). By lemma 3.7, no irreducible constituent of \(\sigma(\hat{\theta}_{\nu}^{+})^2\) factors through \(SL_2(\mathbb{Z})/\{\pm I\}\), so (3.17) vanishes in this case. If \(l' = 2 \mod 4\) then the 2-part of \(\nu_{p'}^\sigma\) is \(\sigma'(\hat{\theta}_{\nu}^{\pm})\) (we have \(\nu_{p'}^{\pm} = \hat{\theta}_{\nu}^{\pm}\)) where \(\sigma'\) is the Galois automorphism of \(\mathbb{Q}(e(\frac{q}{2}))\) such that \(\sigma(e(\frac{q}{2})) = e(\frac{q}{2})\) for \(d' = \frac{q}{2} \mod 8\), and the sign in \(\hat{\theta}_{\nu}^{\pm}\) is determined by \(\alpha'\). We claim that the products \(\sigma(\hat{\theta}_{\nu}^{+})\sigma'(\hat{\theta}_{\nu}^{+})\) have no constituents in common with \((\hat{1}_{N})_2 = \hat{1}_{32}\). Since \(l = \frac{q}{2} \mod 4\), this follows from lemma 3.7 for \(\sigma(\hat{\theta}_{\nu}^{+})\sigma'(\hat{\theta}_{\nu}^{+})\), and follows from lemma 3.9 for \(\sigma(\hat{\theta}_{\nu}^{+})\sigma'(\hat{\theta}_{\nu}^{+})\). A directly similar argument holds for \(l' = 4 \mod 8\) and \(l' = 8 \mod 16\), wherein the 2-part of \(\nu_{p'}^\sigma\) is \(\sigma'(\nu_{p'}^{\pm})\) for \(k' = 4\) or \(k' = 8\), and \(\sigma'\) is the Galois automorphism of \(\mathbb{Q}(e(\frac{q}{24}))\) mapping \(e(\frac{q}{24})\) to \(e(\frac{q}{24})\) for \(\sigma'\) as an inverse to the \(\frac{q}{2}\) modulo \(4k'\). Just as above, now using \(l = \frac{q}{2} \mod 4\), lemmas 3.7 and 3.9 show that \(\sigma(\hat{\theta}_{\nu}^{+})\sigma'(\hat{\theta}_{\nu}^{+})\) has no constituents in common with \(\hat{1}_{32}\). We conclude that \(\dim J_{1,m}(N) = 0\) when \(m\) is odd and \(N = 32 \mod 64\).

The argument for \(m = 4 \mod 8\) and \(N = 32 \mod 64\) is the same except that we also have to consider the products \(\sigma(\nu_{p}^{\pm})\sigma'(\hat{\theta}_{\nu}^{+})\) where \(k' \in \{1, 2, 4, 8\}\) and \(\sigma\) is now a Galois automorphism of \(\mathbb{Q}(e(\frac{q}{8}))\) such that \(\sigma(i) = \sigma'(i)\). Lemmas 3.7 and 3.9 show that such products have no constituents in common with \(\hat{1}_{32}\). Indeed, these same lemmas show that the \(\sigma(\nu_{p}^{\pm})\sigma'(\hat{\theta}_{\nu}^{+})\) and \(\sigma(\hat{\theta}_{\nu}^{+})\sigma'(\hat{\theta}_{\nu}^{+})\) have no constituents in common with \(\hat{1}_{16}\) when \(k, k' \in \{1, 2, 4, 8, 16\}\), and \(\sigma\) and \(\sigma'\) are suitable Galois automorphisms satisfying \(\sigma(i) = \sigma'(i)\). This handles the case in which neither \(m\) or \(N\) are divisible by 32, and completes the proof of the claim. 

Note that \(\nu_{p}^{\pm}\) does occur as a constituent of \(\hat{1}_{N}\) when \(p\) is an odd prime and \(p^2|N\). This is the main reason for our restriction on odd primes congruent to \(3\ mod \ 4\) in the statement of lemma 3.11. Lemma 3.9 explains our restrictions on powers of 2.

Lemma 3.11 can handle all but a few of the cases of theorem 3.1. For the remainder we need to show the vanishing of \(J_{1,m}(N)\) in some instances where \(mN\) is divisible by 27. We obtain this via three more specialized applications of proposition 3.8, which we now present. Our proofs will apply properties of the character tables of \(SL_2(\mathbb{Z}/9\mathbb{Z})\) and \(SL_2(\mathbb{Z}/25\mathbb{Z})\) which we verified using GAP [48].

**Lemma 3.12.** We have \(J_{1,3}(144) = \{0\}\).

**Proof.** Applying proposition 3.8 with \(m = 3, M = 36\) and \(N = 144\) we see that the dimension of \(J_{1,3}(144)\) is a sum of terms

\[
\langle \nu_{p}^{\pm} \nu_{p'}^{\pm}, \hat{1}_{144} \rangle
\]

(see (3.11)) where \(l'\) is a divisor of 36 and \(\alpha'(2)\alpha'(3) = 1\). The 3-part of \(\hat{1}_{144}\) is \((\hat{1}_{144})_3 = \hat{1}_9\) and has no nontrivial constituents with a degree less than 2 (see lemma 3.10). But the 3-part of \(\nu_{p}^{\mp}\) is \(\lambda_{3}^{\pm}\), which is a nontrivial character of degree 1. So if 3 does not divide \(l'\) then the 3-part of \(\nu_{p'}^{\pm}\) is trivial and (3.18) has a factor \(\langle \lambda_{3}^{\pm}, \hat{1}_9 \rangle\), which vanishes. So we may assume that 3 divides \(l'\). Suppose that 9 does not divide \(l'\). Then taking \(k'\) to be the highest power of 2 dividing \(l'\) we have that \(k' \in \{1, 2, 4\}\) and the 2-part of \(\nu_{p}^{\pm} \nu_{p'}^{\pm}\) is \(\sigma(\hat{\theta}_{\nu}^{+})\sigma'(\hat{\theta}_{\nu}^{+})\), where \(\sigma\) and \(\sigma'\) satisfy \(\sigma(i) = \sigma'(i) = -i\). No such products have a constituent in common with \((\hat{1}_{144})_2 = \hat{1}_{16}\).
by lemmas 3.7 and 3.9, so we may assume that \( l' \) is divisible by 9. Then the 3-part of \( \nu_5 \hat{\nu}^{a'} \) is \( \lambda_5 \hat{\sigma}'(\lambda_5^g) \) where the sign is such that \( \alpha'(3) = \pm 1 \), and \( \sigma'(e(\frac{1}{3})) = e(\frac{1}{2^2}) \) for \( a' \) an inverse to \( 4k' \) modulo 9. The products \( \lambda_5 \hat{\sigma}'(\lambda_5^g) \) have no constituents in common with \( I_9 \) because \( S \) only has \( \pm i \) for eigenvalues on the corresponding representations. By direct computation we find that \( \langle \lambda_3 \hat{\sigma}'(\lambda_3^g), I_9 \rangle = 0 \) for all choices of \( \sigma' \). We conclude that \( J_{1,3}(144) \) vanishes, as required.


\[ \text{Lemma 3.13.} \quad \text{We have } J_{1,6}(36) = \{0\}. \]

\textbf{Proof.} The proof is similar to that of lemma 3.12, but a little more involved. We apply proposition 3.8 to \( m = 6, N = 36 \) and \( M = 18 \) to see that the dimension of \( J_{1,6}(36) \) is a sum of terms

\[ \langle \nu_6^a \nu_6^{a'}, \hat{I}_{36} \rangle \tag{3.19} \]

where \( \alpha(2)\alpha(3) = -1 \), \( l' \) is a divisor of 18 and \( \alpha'(2)\alpha'(3) = 1 \). The 3-part of \( \hat{I}_{36} \) is \( \hat{I}_9 \) and has no nontrivial constituents with a degree less than 2, but the 3-part of \( \nu_6^a \) takes the form \( \sigma(\lambda_3^g) \) and is thus a nontrivial character of degree 1 or 2. So arguing as for lemma 3.12 we restrict ourselves to the case in which 3 divides \( l' \). If \( l' \) is not divisible by 9 then the 2-part of \( \nu_6^a \nu_6^{a'} \) takes the form \( \sigma(\nu_2^a) \sigma'(\nu_2^{a'}) \) for \( k' \in \{1, 2\} \) where \( \sigma(i) = \sigma'(i) = -i \). No such products have constituents in common with \( I_4 \) according to lemmas 3.7 and 3.9, so we may assume that \( l' \) is either 9 or 18. But if \( l' = 9 \), then the 2-part of \( \nu_6^a \nu_6^{a'} \) is \( \sigma(\nu_2^a)\nu_1^i \) where \( \sigma(i) = -i \).

Lemma 3.7 tells us that only \( \sigma(\nu_2^a)\nu_1^i \) can have a constituent in common with \( \langle \hat{I}_{36} \rangle = \hat{I}_9 \), but it is irreducible of degree \( 6 \), and \( \hat{I}_4 \) is degree \( 6 \) but not irreducible.

So we are left with the case that \( l' = 18 \). Now the 2-part of \( \nu_6^a \nu_6^{a'} \) is \( \sigma(\nu_2^a)\nu_1^i \) where \( \sigma(i) = -i \) as before. Applying lemma 3.7 we restrict ourselves to \( \sigma(\nu_2^a)\nu_1^i \) and \( \sigma(\nu_2^a)\nu_1^i \), the latter of these has degree 1 but is not trivial, so it is not a constituent of \( I_4 \), whereas the former does have a degree 2 constituent in common with \( \hat{I}_4 \). So we have reduced ourselves to considering \( \langle \nu_6^a \nu_6^{a'}, \hat{I}_{36} \rangle \), where \( \alpha(2) = 1, \alpha(3) = -1, \alpha'(2) = 1 \) and \( \alpha'(3) = 1 \). The 3-part of this is \( \langle \sigma(\lambda_3^g)\lambda_3^g, I_9 \rangle \), but this vanishes for both choices of \( \sigma \). We conclude that \( J_{1,6}(36) \) also vanishes.

\[ \text{Lemma 3.14.} \quad \text{We have } J_{1,30}(36) = \{0\}. \]

\textbf{Proof.} Apply proposition 3.8 with \( m = 30, N = 36 \) and \( M = 90 \) to see that the dimension of \( J_{1,30}(36) \) is a sum of terms

\[ \langle \nu_{30}^a \nu_{30}^{a'}, \hat{I}_{36} \rangle \tag{3.20} \]

where \( \alpha(2)\alpha(3) = -1 \), \( l' \) is a divisor of 90 and \( \alpha'(2)\alpha'(3)\alpha'(5) = 1 \). The 3-part of \( I_{36} \) is trivial, but the 5-part of \( \nu_{30}^a \) is not, so we may assume that the 5-part of \( \nu_{30}^a \) is not trivial either. That is, we may assume that 5 divides \( l' \). The 3-part of \( \nu_{30}^a \) is a nontrivial character of degree 1 or 2 so we can restrict ourselves to the case in which 3 divides \( l' \) just as we did for lemma 3.13. Also, as in the proof there, consideration of the 2-parts shows that \( l' \) must be
divisible by 9, so \( l' \) is either 45 or 90, and further consideration of the 2-parts rules out \( l' = 45 \).

So, we may assume that \( l' = 90 \). Now the 5-part of \( \nu_{30} \nu_{90}' \) is \( \sigma(\lambda_5^\pm) \sigma'(\lambda_5^\pm) \) where \( \sigma(e(\frac{1}{3})) = e(\frac{2}{3}) \) and \( \sigma'(e(\frac{1}{5})) = e(\frac{4}{5}) \). Inspecting the characters we see that \( \sigma(\lambda_5^\pm) \) is not dual to \( \sigma'(\lambda_5^\pm) \) for these particular \( \sigma \) and \( \sigma' \). We conclude that \( J_{1,30}(36) \) vanishes, as we were required to show.

We are now prepared to present the proof of theorem 3.1.

**Proof of theorem 3.1.** The proof is summarized in appendix A, where we display each of the conjugacy classes \([g]\) of each nontrivial group \( G_X \) arising in umbral moonshine, together with levels \( N_g \) such that the mock Jacobi form \( \phi_X^g = \sum H^g_{\alpha} \theta_{m,r} \) corresponding to \([g]\) has the same multiplier system as a Jacobi form of weight 1, index \( m = m_X \) and level \( N = N_g \). The validity of the given \( N_g \) can be verified using the explicit descriptions of the \( H^g_X \) in [5]. Inspecting the tables in appendix A we find that lemma 3.11 implies the vanishing of \( J_{1,m}(N) \) for all \([g]\) except for the cases shaded in yellow or orange. The orange cases are the two classes with \( X = \mathbb{A}^3_4 \) for which theta series contributions to \( H^g_X \) exist. The vanishing of the \( J_{1,m}(N) \) corresponding to the yellow classes is obtained by applying lemma 3.12 for \( X = \mathbb{A}^3_4 \) and \([g] \in \{3B, 6B, 12A\}\), lemma 3.13 for \( X = \mathbb{D}^6_4 \) and \([g] \in \{3C, 6C\}\) and lemma 3.14 for \( X = \mathbb{E}^3_8 \) and \([g] = 3A\). For \( g = e \) in any \( G_X \), the corresponding level is 1 and the vanishing of \( J_{1,m}(1) \) was proven much earlier in [13]. The proof of the theorem is complete.

A number of cases of theorem 3.1 can also be established using proposition 3.5, or more refined applications of proposition 3.3.

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\(^{11}\) Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.
## Appendix A. Levels

### Table A1. Conjugacy classes and levels at $\ell = 2$, $X = A_2^2$.

| $[g]$ | 1A | 2A | 2B | 3A | 3B | 4A | 4B | 4C | 5A | 6A | 6B |
|-------|----|----|----|----|----|----|----|----|----|----|----|
| $n_g | h_g$ | 1 | 1 | 2 | 1 | 2 | 2 | 3 | 1 | 3 | 3 | 4 |
| $N_g$ | 1 | 2 | 4 | 3 | 9 | 8 | 4 | 16 | 5 | 6 | 36 |

### Table A2. Conjugacy classes and levels at $\ell = 3$, $X = A_2^3$.

| $[g]$ | 1A | 2A | 2B | 4A | 2B | 3A | 3B | 6A | 6B | 4B | 4C |
|-------|----|----|----|----|----|----|----|----|----|----|----|
| $n_g | h_g$ | 1 | 1 | 2 | 8 | 2 | 1 | 2 | 2 | 3 | 3 | 4 |
| $N_g$ | 1 | 4 | 16 | 2 | 4 | 3 | 12 | 9 | 36 | 8 | 4 |

### Table A3. Conjugacy classes and levels at $\ell = 4$, $X = A_2^4$.

| $[g]$ | 1A | 2A | 2B | 4A | 2B | 3A | 3B | 6A | 6B | 4B | 4C |
|-------|----|----|----|----|----|----|----|----|----|----|----|
| $n_g | h_g$ | 1 | 1 | 2 | 2 | 4 | 4 | 2 | 1 | 3 | 3 | 4 |
| $N_g$ | 1 | 2 | 4 | 8 | 16 | 2 | 3 | 6 | 12 | 32 | 4 | 7 |

### Table A4. Conjugacy classes and levels at $\ell = 5$, $X = A_2^5$.

| $[g]$ | 1A | 2A | 2B | 3A | 6A | 5A | 10A | 4A | 4B | 4C | 12AB |
|-------|----|----|----|----|----|----|-----|----|----|----|-----|
| $n_g | h_g$ | 1 | 1 | 4 | 2 | 2 | 1 | 3 | 3 | 2 | 6 | 12 |
| $N_g$ | 1 | 2 | 4 | 8 | 16 | 2 | 3 | 6 | 12 | 32 | 4 | 7 |

### Table A5. Conjugacy classes and levels at $\ell = 6$, $X = A_2^6D_4$.

| $[g]$ | 1A | 2A | 2B | 4A | 3A | 6A | 8AB |
|-------|----|----|----|----|----|----|-----|
| $n_g | h_g$ | 1 | 1 | 2 | 1 | 2 | 2 | 3 | 1 | 3 | 2 | 4 |
| $N_g$ | 1 | 2 | 2 | 4 | 3 | 6 | 8 |

### Table A6. Conjugacy classes and levels at $\ell = 6 + 3$, $X = D_4^3$.

| $[g]$ | 1A | 3A | 2A | 6A | 3B | 3C | 4A | 12A | 5A | 15AB | 2B | 2C | 4B | 6B | 6C |
|-------|----|----|----|----|----|----|----|-----|----|-------|----|----|----|----|----|
| $n_g | h_g$ | 1 | 1 | 3 | 2 | 1 | 2 | 3 | 1 | 4 | 2 | 4 | 6 | 5 | 3 | 2 | 1 | 2 | 1 | 4 | 1 | 6 | 1 |
| $N_g$ | 1 | 3 | 2 | 6 | 3 | 9 | 8 | 24 | 5 | 15 | 2 | 4 | 6 | 36 |
| $[g]$ | 1A | 2A | 2B | 2C | 3A | 6A |
|------|----|----|----|----|----|----|
| $n_g | h_k$ | 1/1 | 1/4 | 2/8 | 3/1 | 3/4 |
| $N_g$ | 1   | 4   | 16  | 3   | 12  |

**Table A8.** Conjugacy classes and levels at $\ell = 8$, $X = A_2^2D_2^2$.

| $[g]$ | 1A | 2A | 2B | 2C | 4A |
|------|----|----|----|----|----|
| $n_g | h_k$ | 1/1 | 1/2 | 2/1 | 2/1 | 2/4 |
| $N_g$ | 1   | 2   | 2   | 2   | 8   |

**Table A9.** Conjugacy classes and levels at $\ell = 9$, $X = A_3^1$.

| $[g]$ | 1A | 2A | 2B | 2C | 3A | 6A |
|------|----|----|----|----|----|----|
| $n_g | h_k$ | 1/1 | 1/4 | 2/1 | 2/2 | 3/3 | 3/12 |
| $N_g$ | 1   | 4   | 2   | 4   | 9   | 36  |

**Table A10.** Conjugacy classes and levels at $\ell = 10$, $X = A_2^2D_6$.

| $[g]$ | 1A | 2A | 4A |
|------|----|----|----|
| $n_g | h_k$ | 1/1 | 1/2 | 2/2 |
| $N_g$ | 1   | 2   | 4   |

**Table A11.** Conjugacy classes and levels at $\ell = 10 + 5$, $X = D_5^1$.

| $[g]$ | 1A | 2A | 3A | 2B | 4A |
|------|----|----|----|----|----|
| $n_g | h_k$ | 1/1 | 2/2 | 3/1 | 2/1 | 4/4 |
| $N_g$ | 1   | 4   | 3   | 2   | 16  |

**Table A12.** Conjugacy classes and levels at $\ell = 12$, $X = A_{11}D_7E_6$.

| $[g]$ | 1A | 2A |
|------|----|----|
| $n_g | h_k$ | 1/1 | 1/2 |
| $N_g$ | 1   | 2   |

**Table A13.** Conjugacy classes and levels at $\ell = 12 + 4$, $X = E_6^4$.

| $[g]$ | 1A | 2A | 2B | 4A | 3A | 6A | 8A |
|------|----|----|----|----|----|----|----|
| $n_g | h_k$ | 1/1 | 1/2 | 2/1 | 2/4 | 3/1 | 3/2 | 4/8 |
| $n_g | h_k$ | 1   | 2   | 2   | 8   | 3   | 6   | 32  |
Table A14. Conjugacy classes and levels at \( \ell = 13 \), \( X = A_{12}^2 \).

| \( [g] \) | 1A | 2A | 4AB |
|-------|----|----|------|
| \( n_g | h_g \) | 1|1 | 1|4 | 2|8 |
| \( N_g \) | 1 | 4 | 16 |

Table A15. Conjugacy classes and levels at \( \ell = 14 + 7 \), \( X = D_8^3 \).

| \( [g] \) | 1A | 2A | 3A |
|-------|----|----|----|
| \( n_g | h_g \) | 1|1 | 2|1 | 3|3 |
| \( N_g \) | 1 | 2 | 9 |

Table A16. Conjugacy classes and levels at \( \ell = 16 \), \( X = A_{15}D_9 \).

| \( [g] \) | 1A | 2A |
|-------|----|----|
| \( n_g | h_g \) | 1|1 | 1|2 |
| \( N_g \) | 1 | 2 |

Table A17. Conjugacy classes and levels at \( \ell = 18 \), \( X = A_{17}E_7 \).

| \( [g] \) | 1A | 2A |
|-------|----|----|
| \( n_g | h_g \) | 1|1 | 1|2 |
| \( N_g \) | 1 | 2 |

Table A18. Conjugacy classes and levels at \( \ell = 18 + 9 \), \( X = D_{10}E_7^2 \).

| \( [g] \) | 1A | 2A |
|-------|----|----|
| \( n_g | h_g \) | 1|1 | 2|1 |
| \( N_g \) | 1 | 2 |

Table A19. Conjugacy classes and levels at \( \ell = 22 + 11 \), \( X = D_{12}^2 \).

| \( [g] \) | 1A | 2A |
|-------|----|----|
| \( n_g | h_g \) | 1|1 | 2|2 |
| \( N_g \) | 1 | 2 |
Appendix B. Characters at $\ell = 9$

Table B1. Character table of $G^X \simeq \text{Dih}_6$, $X = A_3^3$.

| $[g]$ | FS | 1A | 2A | 2B | 2C | 3A | 6A |
|-------|----|----|----|----|----|----|----|
| $[g^2]$ | 1A | 1A | 1A | 1A | 3A | 3A |
| $[g^3]$ | 1A | 2A | 2B | 2C | 1A | 2A |
| $\chi_1$ | + | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | + | 1 | 1 | -1 | -1 | 1 | 1 |
| $\chi_3$ | + | 2 | 2 | 0 | 0 | -1 | -1 |
| $\chi_4$ | + | 1 | -1 | -1 | 1 | 1 | -1 |
| $\chi_5$ | + | 1 | -1 | 1 | -1 | 1 | -1 |
| $\chi_6$ | + | 2 | -2 | 0 | 0 | -1 | 1 |

Table B2. Twisted Euler characters and frame shapes at $\ell = 9$, $X = A_3^3$.

| $[g]$ | 1A | 2A | 2B | 2C | 3A | 6A |
|-------|----|----|----|----|----|----|
| $n_g | h_g$ | 1 | 1 | 4 | 2 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 12 |
| $\bar{\chi}_X^{\bar{\chi}}$ | 3 | 3 | 1 | 1 | 0 | 0 |
| $\chi_X^{\chi}$ | 3 | -3 | 1 | -1 | 0 | 0 |
| $\bar{\Pi}_X^{\bar{\chi}}$ | $1^3$ | $1^3$ | $1^{21}$ | $1^{21}$ | $3^1$ | $3^1$ |
| $\bar{\Pi}_g^{\bar{\chi}}$ | $1^{24}$ | $2^{12}$ | $1^{28}$ | $2^{12}$ | $3^8$ | $6^5$ |
## Appendix C. Coefficients at $\ell = 9$

Table C1. $H^{(9)}_{\ell,1}, X = A_3^1$.

| $|g|$ | 1 | 2A | 2B | 2C | 3A | 6A |
|-----|---|----|----|----|----|----|
| 1   | 1 | 1  | 4  | 2  | 2  | 3  |
| 2   | 1 | 2  | 2  | 0  | 0  | 0  |
| 3   | 3 | 3  | 3  | 3  | 3  | 3  |

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Table C2. $H_{32}^{(9)}$, $X = A_3$.

| $\Gamma_\varepsilon$ | 1A | 2A | 2B | 2C | 3A | 6A |
|----------------------|----|----|----|----|----|----|
| 32                   | 4  | -4 | 0  | 0  | -2 | 2  |
| 68                   | 6  | -6 | 2  | -2 | 0  | 0  |
| 104                  | 8  | -8 | 0  | 0  | 2  | -2 |
| 140                  | 16 | -16| 0  | 0  | -2 | 2  |
| 176                  | 18 | -18| -2 | 2  | 0  | 0  |
| 212                  | 30 | -30| 2  | -2 | 0  | 0  |
| 248                  | 40 | -40| 0  | 0  | -2 | 2  |
| 284                  | 54 | -54| 2  | -2 | 0  | 0  |
| 320                  | 74 | -74| -2 | 2  | 2  | -2 |
| 356                  | 102| -102| 2 | -2 | 0  | 0  |
| 392                  | 126| -126| -2| 2  | 0  | 0  |
| 428                  | 174| -174| 2 | -2 | 0  | 0  |
| 464                  | 218| -218| -2| 2  | -4 | 4  |
| 500                  | 284| -284| 4 | -4 | 2  | -2 |
| 536                  | 356| -356| -4| 4  | 2  | -2 |
| 572                  | 460| -460| 4 | -4 | -2 | 2  |
| 608                  | 564| -564| -4| 4  | 0  | 0  |
| 644                  | 716| -716| 4 | -4 | 2  | -2 |
| 680                  | 884| -884| -4| 4  | -4 | 4  |
| 716                  | 1098| -1098| 6 | -6 | 0  | 0  |
| 752                  | 1342| -1342| -6| 6  | 4  | -4 |
| 788                  | 1658| -1658| 6 | -6 | -4 | 4  |
| 824                  | 2000| -2000| -8| 8  | 2  | -2 |
| 860                  | 2456| -2456| 8 | -8 | 2  | -2 |
| 896                  | 2960| -2960| -8| 8  | -4 | 4  |
| 932                  | 3582| -3582| 10| -10| 0  | 0  |
| 968                  | 4294| -4294| -10| 10| 4  | -4 |
| 1004                 | 5174| -5174| 10| -10| -4 | 4  |
| 1040                 | 6156| -6156| 12| 12 | 0  | 0  |
| 1076                 | 7378| -7378| 14| -14| 4  | -4 |
| 1112                 | 8748| -8748| 12| 12 | -6 | 6  |
| 1148                 | 10400| -10400| 16| -16| 2  | -2 |
| 1184                 | 12280| -12280| -16| 16| 4  | -4 |
| 1220                 | 14544| -14544| 16| -16| -6 | 6  |
| 1256                 | 17084| -17084| 20| 20 | 2  | -2 |
| 1292                 | 20140| -20140| 20| -20| 4  | -4 |
| 1328                 | 23590| -23590| -22| 22| -8 | 8  |
| 1364                 | 27656| -27656| 24| -24| 2  | -2 |
Table C3. $H_{13}^{(9)}, X = A_3^3$.

| $|g|$ | $\Gamma_x$ | 1A | 2A | 2B | 2C | 3A | 6A |
|-----|-----------|----|----|----|----|----|----|
|     | 1         | 1/4| 2  | 2/2| 3/3| 3|12 |
| 27  | 4         | 4  | 0  | 0  | -2 | -2 |
| 63  | 6         | 6  | -2 | -2 | 0  | 0  |
| 99  | 12        | 12 | 0  | 0  | 0  | 0  |
| 135 | 18        | 18 | 2  | 2  | 0  | 0  |
| 171 | 24        | 24 | 0  | 0  | 0  | 0  |
| 207 | 36        | 36 | 0  | 0  | 0  | 0  |
| 243 | 52        | 52 | 0  | -2| -2 |
| 279 | 72        | 72 | 0  | 0  | 0  | 0  |
| 315 | 96        | 96 | 0  | 0  | 0  | 0  |
| 351 | 126       | 126| -2 | -2| 0  |
| 387 | 168       | 168| 0  | 0  | 0  | 0  |
| 423 | 222       | 222| 2  | 2  | 0  | 0  |
| 459 | 288       | 288| 0  | 0  | 0  | 0  |
| 495 | 366       | 366| -2 | -2| 0  |
| 531 | 468       | 468| 0  | 0  | 0  | 0  |
| 567 | 594       | 594| 2  | 2  | 0  | 0  |
| 603 | 744       | 744| 0  | 0  | 0  | 0  |
| 639 | 930       | 930| -2 | -2| 0  |
| 675 | 1156      | 1156| 0  | -2| -2 |
| 711 | 1434      | 1434| 2  | 2  | 0  |
| 747 | 1764      | 1764| 0  | 0  | 0  |
| 783 | 2160      | 2160| 0  | 0  | 0  |
| 819 | 2640      | 2640| 0  | 0  | 0  |
| 855 | 3210      | 3210| 2  | 2  | 0  |
| 891 | 3888      | 3888| 0  | 0  | 0  |
| 927 | 4692      | 4692| -4 | -4| 0  |
| 963 | 5652      | 5652| 0  | 0  | 0  |
| 999 | 6786      | 6786| 2  | 2  | 0  |
| 1035| 8112      | 8112| 0  | 0  | 0  |
| 1071| 9678      | 9678| -2 | -2| 0  |
| 1107| 11520     | 11520| 0  | 0  | 0  |
| 1143| 13674     | 13674| 2  | 2  | 0  |
| 1179| 16188     | 16188| 0  | 0  | 0  |
| 1215| 19116     | 19116| -4 | -4| 0  |
| 1251| 22536     | 22536| 0  | 0  | 0  |
| 1287| 26508     | 26508| 4  | 4  | 0  |
| 1323| 31108     | 31108| 0  | -2| -2 |
| 1359| 36438     | 36438| -2 | -2| 0  | 0  |
Table C4. \( H^{(9)}_{E_d}, X = A_3^\dagger \).

| \([g]\) | 1A | 2A | 2B | 2C | 3A | 6A |
|---|---|---|---|---|---|---|
| \( g \) | 1 | 1\(|4\) | 2 | 2\(|2\) | 3\(|3\) | 3\(|12\) |
| 20 | 2 | −2 | −2 | 2 | 2 | −2 |
| 56 | 8 | −8 | 0 | 0 | 2 | −2 |
| 92 | 10 | −10 | −2 | 2 | −2 | 2 |
| 128 | 18 | −18 | 2 | −2 | 0 | 0 |
| 164 | 26 | −26 | −2 | 2 | 2 | −2 |
| 200 | 42 | −42 | 2 | −2 | 0 | 0 |
| 236 | 50 | −50 | −2 | 2 | 2 | −2 |
| 272 | 78 | −78 | 2 | −2 | 0 | 0 |
| 308 | 100 | −100 | −4 | 4 | −2 | 2 |
| 344 | 140 | −140 | 4 | −4 | 2 | −2 |
| 380 | 180 | −180 | −4 | 4 | 0 | 0 |
| 416 | 244 | −244 | 4 | −4 | −2 | 2 |
| 452 | 302 | −302 | −6 | 6 | 2 | −2 |
| 488 | 404 | −404 | 4 | −4 | 2 | −2 |
| 524 | 502 | −502 | −6 | 6 | −2 | 2 |
| 560 | 648 | −648 | 8 | −8 | 0 | 0 |
| 596 | 806 | −806 | −6 | 6 | 2 | −2 |
| 632 | 1024 | −1024 | 8 | −8 | −2 | 2 |
| 668 | 1250 | −1250 | −10 | 10 | 2 | −2 |
| 704 | 1574 | −1574 | 10 | −10 | 2 | −2 |
| 740 | 1916 | −1916 | −12 | 12 | −4 | 4 |
| 776 | 2372 | −2372 | 12 | −12 | 2 | −2 |
| 812 | 2876 | −2876 | −12 | 12 | 2 | −2 |
| 848 | 3530 | −3530 | 14 | −14 | −4 | 4 |
| 884 | 4240 | −4240 | −16 | 16 | 4 | −4 |
| 920 | 5168 | −5168 | 16 | −16 | 2 | −2 |
| 956 | 6186 | −6186 | −18 | 18 | −6 | 6 |
| 992 | 7460 | −7460 | 20 | −20 | 2 | −2 |
| 1028 | 8894 | −8894 | −22 | 22 | 2 | −2 |
| 1064 | 10664 | −10664 | 24 | −24 | −4 | 4 |
| 1100 | 12634 | −12634 | 26 | 26 | 4 | −4 |
| 1136 | 15070 | −15070 | 26 | −26 | 4 | −4 |
| 1172 | 17790 | −17790 | −30 | 30 | −6 | 6 |
| 1208 | 21080 | −21080 | 32 | −32 | 2 | −2 |
| 1244 | 24794 | −24794 | −34 | 34 | 2 | −2 |
| 1280 | 29250 | −29250 | 38 | −38 | −6 | 6 |
| 1316 | 34248 | −34248 | −40 | 40 | 6 | −6 |
| 1352 | 40234 | −40234 | 42 | −42 | 4 | −4 |
| $|g|$ | $1A$ | $2A$ | $2B$ | $2C$ | $3A$ | $6A$ |
|-----|-----|-----|-----|-----|-----|-----|
| $\Gamma_g$ | 18 | 18 | 2 | 2 | 3 | 3 | 12 |
| 11 | 4 | 4 | 0 | 0 | −2 | −2 |
| 47 | 6 | 6 | 2 | 2 | 0 | 0 |
| 83 | 12 | 12 | 0 | 0 | 0 | 0 |
| 119 | 16 | 16 | 0 | 0 | −2 | −2 |
| 155 | 24 | 24 | 0 | 0 | 0 | 0 |
| 191 | 36 | 36 | 0 | 0 | 0 | 0 |
| 227 | 52 | 52 | 0 | 0 | −2 | −2 |
| 263 | 68 | 68 | 0 | 0 | 2 | 2 |
| 299 | 96 | 96 | 0 | 0 | 0 | 0 |
| 335 | 130 | 130 | 2 | 2 | −2 | −2 |
| 371 | 168 | 168 | 0 | 0 | 0 | 0 |
| 407 | 224 | 224 | 0 | 0 | 2 | 2 |
| 443 | 292 | 292 | 0 | 0 | −2 | −2 |
| 479 | 372 | 372 | 0 | 0 | 0 | 0 |
| 515 | 480 | 480 | 0 | 0 | 0 | 0 |
| 551 | 608 | 608 | 0 | 0 | −4 | −4 |
| 587 | 764 | 764 | 0 | 0 | 2 | 2 |
| 623 | 962 | 962 | 2 | 2 | 2 | 2 |
| 659 | 1196 | 1196 | 0 | 0 | −4 | −4 |
| 695 | 1478 | 1478 | −2 | −2 | 2 | 2 |
| 731 | 1832 | 1832 | 0 | 0 | 2 | 2 |
| 767 | 2248 | 2248 | 0 | 0 | −2 | −2 |
| 803 | 2744 | 2744 | 0 | 0 | 2 | 2 |
| 839 | 3348 | 3348 | 0 | 0 | 0 | 0 |
| 875 | 4064 | 4064 | 0 | 0 | −4 | −4 |
| 911 | 4910 | 4910 | 2 | 2 | 2 | 2 |
| 947 | 5924 | 5924 | 0 | 0 | 2 | 2 |
| 983 | 7116 | 7116 | 0 | 0 | −6 | −6 |
| 1019 | 8516 | 8516 | 0 | 0 | 2 | 2 |
| 1055 | 10184 | 10184 | 0 | 0 | 2 | 2 |
| 1091 | 12132 | 12132 | 0 | 0 | −6 | −6 |
| 1127 | 14404 | 14404 | 0 | 0 | 4 | 4 |
| 1163 | 17084 | 17084 | 0 | 0 | 2 | 2 |
| 1199 | 20202 | 20202 | 2 | 2 | −6 | −6 |
| 1235 | 23824 | 23824 | 0 | 0 | 4 | 4 |
| 1271 | 28054 | 28054 | −2 | −2 | 4 | 4 |
| 1307 | 32956 | 32956 | 0 | 0 | −8 | −8 |
| 1343 | 38626 | 38626 | 2 | 2 | 4 | 4 |
| \([g]\) | 1A | 2A | 2B | 2C | 3A | 6A |
|---|---|---|---|---|---|---|
| \(\Gamma_3\) | 1 | 1/4 | 2 | 2/2 | 3/3 | 3/12 |
| 0 | 1 | -1 | -1 | 1 | 1 | -1 |
| 36 | 6 | -6 | 2 | -2 | 0 | 0 |
| 72 | 6 | -6 | -2 | 2 | 0 | 0 |
| 108 | 14 | -14 | 2 | -2 | 2 | -2 |
| 144 | 18 | -18 | -2 | 2 | 0 | 0 |
| 180 | 30 | -30 | 2 | -2 | 0 | 0 |
| 216 | 36 | -36 | -4 | 4 | 0 | 0 |
| 252 | 60 | -60 | 4 | -4 | 0 | 0 |
| 288 | 72 | -72 | -4 | 4 | 0 | 0 |
| 324 | 108 | -108 | 4 | -4 | 0 | 0 |
| 360 | 132 | -132 | -4 | 4 | 0 | 0 |
| 396 | 186 | -186 | 6 | -6 | 0 | 0 |
| 432 | 230 | -230 | -6 | 6 | 2 | -2 |
| 468 | 312 | -312 | 8 | -8 | 0 | 0 |
| 504 | 384 | -384 | -8 | 8 | 0 | 0 |
| 540 | 504 | -504 | 8 | -8 | 0 | 0 |
| 576 | 618 | -618 | -10 | 10 | 0 | 0 |
| 612 | 798 | -798 | 10 | -10 | 0 | 0 |
| 648 | 972 | -972 | -12 | 12 | 0 | 0 |
| 684 | 1236 | -1236 | 12 | -12 | 0 | 0 |
| 720 | 1494 | -1494 | -14 | 14 | 0 | 0 |
| 756 | 1872 | -1872 | 16 | -16 | 0 | 0 |
| 792 | 2256 | -2256 | -16 | 16 | 0 | 0 |
| 828 | 2790 | -2790 | 18 | -18 | 0 | 0 |
| 864 | 3348 | -3348 | -20 | 20 | 0 | 0 |
| 900 | 4098 | -4098 | 22 | -22 | 0 | 0 |
| 936 | 4896 | -4896 | -24 | 24 | 0 | 0 |
| 972 | 5942 | -5942 | 26 | -26 | 2 | -2 |
| 1008 | 7068 | -7068 | -28 | 28 | 0 | 0 |
| 1044 | 8514 | -8514 | 30 | -30 | 0 | 0 |
| 1080 | 10080 | -10080 | -32 | 32 | 0 | 0 |
| 1116 | 12060 | -12060 | 36 | -36 | 0 | 0 |
| 1152 | 14226 | -14226 | -38 | 38 | 0 | 0 |
| 1188 | 16920 | -16920 | 40 | -40 | 0 | 0 |
| 1224 | 19884 | -19884 | -44 | 44 | 0 | 0 |
| 1260 | 23520 | -23520 | 48 | -48 | 0 | 0 |
| 1296 | 27540 | -27540 | -52 | 52 | 0 | 0 |
| 1332 | 32424 | -32424 | 56 | -56 | 0 | 0 |
Table C7. \( H_{k,7}^{(9)} \), \( X = A_3^1 \).

| \([g]\) | 1A | 2A | 2B | 2C | 3A | 6A |
|---|---|---|---|---|---|---|
| \( \Gamma_6 \) | 1 | \( \frac{1}{4} \) | 2 | \( \frac{1}{2} \) | \( \frac{3}{2} \) | 3 | \( \frac{3}{12} \) |
| 23 | 2 | 2 | -2 | -2 | 2 | 2 |
| 59 | 4 | 4 | 0 | 0 | -2 | -2 |
| 95 | 8 | 8 | 0 | 0 | 2 | 2 |
| 131 | 12 | 12 | 0 | 0 | 0 | 0 |
| 167 | 18 | 18 | -2 | -2 | 0 | 0 |
| 203 | 24 | 24 | 0 | 0 | 0 | 0 |
| 239 | 38 | 38 | 2 | 2 | 2 | 2 |
| 275 | 52 | 52 | 0 | 0 | -2 | -2 |
| 311 | 66 | 66 | -2 | -2 | 0 | 0 |
| 347 | 92 | 92 | 0 | 0 | 2 | 2 |
| 383 | 124 | 124 | 0 | 0 | -2 | -2 |
| 419 | 156 | 156 | 0 | 0 | 0 | 0 |
| 455 | 206 | 206 | -2 | -2 | 2 | 2 |
| 491 | 268 | 268 | 0 | 0 | -2 | -2 |
| 527 | 338 | 338 | 2 | 2 | 2 | 2 |
| 563 | 428 | 428 | 0 | 0 | 2 | 2 |
| 599 | 538 | 538 | -2 | -2 | -2 | -2 |
| 635 | 672 | 672 | 0 | 0 | 0 | 0 |
| 671 | 842 | 842 | 2 | 2 | 2 | 2 |
| 707 | 1040 | 1040 | 0 | 0 | -4 | -4 |
| 743 | 1274 | 1274 | -2 | -2 | 2 | 2 |
| 779 | 1568 | 1568 | 0 | 0 | 2 | 2 |
| 815 | 1922 | 1922 | 2 | 2 | -4 | -4 |
| 851 | 2328 | 2328 | 0 | 0 | 0 | 0 |
| 887 | 2824 | 2824 | -4 | -4 | 4 | 4 |
| 923 | 3416 | 3416 | 0 | 0 | -4 | -4 |
| 959 | 4106 | 4106 | 2 | 2 | 2 | 2 |
| 995 | 4936 | 4936 | 0 | 0 | 4 | 4 |
| 1031 | 5906 | 5906 | -2 | -2 | -4 | -4 |
| 1067 | 7040 | 7040 | 0 | 0 | 2 | 2 |
| 1103 | 8392 | 8392 | 4 | 4 | 4 | 4 |
| 1139 | 9960 | 9960 | 0 | 0 | -6 | -6 |
| 1175 | 11784 | 11784 | -4 | -4 | 0 | 0 |
| 1211 | 13936 | 13936 | 0 | 0 | 4 | 4 |
| 1247 | 16434 | 16434 | 2 | 2 | -6 | -6 |
| 1283 | 19316 | 19316 | 0 | 0 | 2 | 2 |
| 1319 | 22680 | 22680 | -4 | -4 | 6 | 6 |
| \([g]\) | \(1A\) | \(2A\) | \(2B\) | \(2C\) | \(3A\) | \(6A\) |
|------|------|------|------|------|------|------|
| 8    | 2    | -2   | 2    | -2   | 2    | -2   |
| 44   | 2    | -2   | -2   | 2    | 2    | -2   |
| 80   | 6    | -6   | 2    | -2   | 0    | 0    |
| 116  | 2    | -2   | -2   | 2    | 2    | -2   |
| 152  | 12   | -12  | 4    | -4   | 0    | 0    |
| 188  | 10   | -10  | -2   | 2    | -2   | 2    |
| 224  | 20   | -20  | 4    | -4   | 2    | -2   |
| 260  | 20   | -20  | -4   | 4    | 2    | -2   |
| 296  | 36   | -36  | 4    | -4   | 0    | 0    |
| 332  | 38   | -38  | -6   | 6    | 2    | -2   |
| 368  | 66   | -66  | 6    | -6   | 0    | 0    |
| 404  | 70   | -70  | -6   | 6    | -2   | 2    |
| 440  | 104  | -104 | 8    | -8   | 2    | -2   |
| 476  | 120  | -120 | -8   | 8    | 0    | 0    |
| 512  | 172  | -172 | 8    | -8   | -2   | 2    |
| 548  | 194  | -194 | -10  | 10   | 2    | -2   |
| 584  | 276  | -276 | 12   | -12  | 0    | 0    |
| 620  | 316  | -316 | -12  | 12   | -2   | 2    |
| 656  | 418  | -418 | 14   | -14  | 4    | -4   |
| 692  | 494  | -494 | -14  | 14   | 2    | -2   |
| 728  | 640  | -640 | 16   | -16  | -2   | 2    |
| 764  | 746  | -746 | -18  | 18   | 2    | -2   |
| 800  | 960  | -960 | 20   | -20  | 0    | 0    |
| 836  | 1124 | -1124| -20  | 20   | -4   | 4    |
| 872  | 1408 | -1408| 24   | -24  | 4    | -4   |
| 908  | 1658 | -1658| -26  | 26   | 2    | -2   |
| 944  | 2054 | -2054| 26   | -26  | -4   | 4    |
| 980  | 2398 | -2398| -30  | 30   | 4    | -4   |
| 1016 | 2952 | -2952| 32   | -32  | 0    | 0    |
| 1052 | 3450 | -3450| -34  | 34   | -6   | 6    |
| 1088 | 4186 | -4186| 38   | -38  | 4    | -4   |
| 1124 | 4898 | -4898| -42  | 42   | 2    | -2   |
| 1160 | 5892 | -5892| 44   | -44  | -6   | 6    |
| 1196 | 6864 | -6864| -48  | 48   | 6    | -6   |
| 1232 | 8220 | -8220| 52   | -52  | 0    | 0    |
| 1268 | 9558 | -9558| -54  | 54   | -6   | 6    |
| 1304 | 11348| -11348| 60  | -60  | 8    | -8   |
## Appendix D. Decompositions at $\ell = 9$

### Table D1. $K_4^{(9)}$, $X = A_3^1$

| $\chi_1$ | $\chi_2$ | $\chi_3$ |
|----------|----------|----------|
| -1       | -2       | 0        | 0        |
| 35       | 0        | 0        | 0        |
| 71       | 0        | 0        | 2        |
| 107      | 0        | 0        | 2        |
| 143      | 0        | 2        | 2        |
| 179      | 2        | 2        | 4        |
| 215      | 2        | 2        | 6        |
| 251      | 4        | 4        | 6        |
| 287      | 4        | 6        | 10       |
| 323      | 6        | 6        | 14       |

### Table D2. $K_4^{(9)}$, $X = A_3^1$

| $\chi_1$ | $\chi_2$ | $\chi_3$ |
|----------|----------|----------|
| 11       | 0        | 0        | 2        |
| 47       | 2        | 0        | 2        |
| 83       | 2        | 2        | 4        |
| 119      | 2        | 2        | 6        |
| 155      | 4        | 4        | 8        |
| 191      | 6        | 6        | 12       |
| 227      | 8        | 8        | 18       |
| 263      | 12       | 12       | 22       |
| 299      | 16       | 16       | 32       |
| 335      | 22       | 20       | 44       |

### Table D3. $K_4^{(9)}$, $X = A_3^1$

| $\chi_1$ | $\chi_2$ | $\chi_3$ |
|----------|----------|----------|
| 23       | 0        | 2        | 0        |
| 59       | 0        | 0        | 2        |
| 95       | 2        | 2        | 2        |
| 131      | 2        | 2        | 4        |
| 167      | 2        | 4        | 6        |
| 203      | 4        | 4        | 8        |
| 239      | 8        | 6        | 12       |
| 275      | 8        | 8        | 18       |
| 311      | 10       | 12       | 22       |
| 347      | 16       | 16       | 30       |
Table D4. $K_{2}^{(9)}$, $X = A_{3}^{1}$.

|   | $\chi_5$ | $\chi_6$ | $\chi_7$ |
|---|---------|---------|---------|
| 32 | 0       | 0       | 2       |
| 68 | 0       | 2       | 2       |
| 104 | 2       | 2       | 2       |
| 140 | 2       | 2       | 6       |
| 176 | 4       | 2       | 6       |
| 212 | 4       | 6       | 10      |
| 248 | 6       | 6       | 14      |
| 284 | 8       | 10      | 18      |
| 320 | 14      | 12      | 24      |
| 356 | 16      | 18      | 34      |

Table D5. $K_{4}^{(9)}$, $X = A_{3}^{1}$.

|   | $\chi_5$ | $\chi_6$ | $\chi_7$ |
|---|---------|---------|---------|
| 20 | 2       | 0       | 0       |
| 56 | 2       | 2       | 2       |
| 92 | 2       | 0       | 4       |
| 128 | 2      | 4       | 6       |
| 164 | 6      | 4       | 8       |
| 200 | 6      | 8       | 14      |
| 236 | 10     | 8       | 16      |
| 272 | 12     | 14      | 26      |
| 308 | 18     | 14      | 34      |
| 344 | 22     | 26      | 46      |

Table D6. $K_{6}^{(9)}$, $X = A_{3}^{1}$.

|   | $\chi_5$ | $\chi_6$ | $\chi_7$ |
|---|---------|---------|---------|
| 8  | 0       | 2       | 0       |
| 44 | 2       | 0       | 0       |
| 80 | 0       | 2       | 2       |
| 116 | 2      | 0       | 0       |
| 152 | 0      | 4       | 4       |
| 188 | 2      | 0       | 4       |
| 224 | 2      | 6       | 6       |
| 260 | 6      | 2       | 6       |
| 296 | 4      | 8       | 12      |
| 332 | 10     | 4       | 12      |
Table D7. $K_A^{(0)}(9, X = A_3^1)$

| $\chi^2$ | $\chi^3$ | $\chi^4$ | $\chi^5$ | $\chi^6$ |
|---------|---------|---------|---------|---------|
| 27      | 0       | 0       | 2       |
| 63      | 0       | 2       | 2       |
| 99      | 2       | 2       | 4       |
| 135     | 4       | 2       | 6       |
| 171     | 4       | 4       | 8       |
| 207     | 6       | 6       | 12      |
| 243     | 8       | 8       | 18      |
| 279     | 12      | 12      | 24      |
| 315     | 16      | 16      | 32      |
| 351     | 20      | 22      | 42      |
| 387     | 28      | 28      | 56      |
| 423     | 38      | 36      | 74      |
| 459     | 48      | 48      | 96      |
| 495     | 60      | 62      | 122     |
| 531     | 78      | 78      | 156     |
| 567     | 100     | 98      | 198     |
| 603     | 124     | 124     | 248     |
| 639     | 154     | 156     | 310     |
| 675     | 192     | 192     | 386     |

Table D8. $K_A^{(0)}(9, X = A_3^1)$

| $\chi^2$ | $\chi^3$ | $\chi^4$ | $\chi^5$ | $\chi^6$ |
|---------|---------|---------|---------|---------|
| 27      | 0       | 1       | 0       | 0       |
| 63      | 0       | 0       | 2       | 2       |
| 99      | 2       | 2       | 0       | 2       |
| 135     | 4       | 4       | 2       | 4       |
| 171     | 4       | 0       | 2       | 6       |
| 207     | 6       | 6       | 4       | 10      |
| 243     | 8       | 8       | 4       | 12      |
| 279     | 12      | 12      | 6       | 20      |
| 315     | 16      | 16      | 12      | 24      |
| 351     | 20      | 22      | 20      | 36      |
| 387     | 28      | 28      | 24      | 44      |
| 423     | 38      | 36      | 34      | 62      |
| 459     | 48      | 48      | 36      | 76      |
| 495     | 60      | 62      | 56      | 104     |
| 531     | 78      | 78      | 60      | 128     |
| 567     | 100     | 98      | 88      | 168     |
| 603     | 124     | 124     | 98      | 206     |
| 639     | 154     | 156     | 138     | 266     |
| 675     | 192     | 192     | 156     | 324     |

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References

[1] Cheng M C N, Duncan J F R and Harvey J A 2014 Umbral moonshine Commun. Number Theory Phys. 8 101–242
[2] Cheng M C N, Duncan J F R and Harvey J A 2014 Umbral moonshine and the Niemeier lattices Res. Math. Sci. 1 1–81
[3] Niemeier H-V 1973 Definite quadratische Formen der Dimension 24 und Diskriminante 1 J. Number Theory 5 142–78
[4] Conway J H and Sloane N J A 1999 Sphere Packings, Lattices and Groups (Grundlehren der Mathematischen Wissenschaften vol 290) (Fundamental Principles of Mathematical Sciences) 3rd edn, ed E Bannai et al (New York: Springer)
[5] Duncan J F R, Griffin M J and Ono K 2015 Proof of the umbral moonshine conjecture Res. Math. Sci. 2 11
[6] Conway J H and Norton S P 1979 Monstrous moonshine Bull. Lond. Math. Soc. 11 308–39
[7] Borcherds R E 1992 Monstrous moonshine and monstrous Lie superalgebras Inventiones Math. 109 405–44
[8] Frenkel I B, Lepowsky J and Meurman A 1984 A natural representation of the Fischer–Griess monster with the modular function J as character Proc. Natl Acad. Sci. USA 81 3256–60
[9] Frenkel I B, Lepowsky J and Meurman A 1985 A Moonshine Module for the Monster Vertex Operators in Mathematics and Physics (Berkeley, California, 1983) (Mathematical Sciences Research Institute Publications vol 3) (New York: Springer) pp 231–73
[10] Frenkel I B, Lepowsky J and Meurman A 1988 Vertex Operator Algebras and the Monster (Pure and Applied Mathematics vol 134) (Boston, MA: Academic)
[11] Dabholkar A, Murthy S and Zagier D 2012 Quantum black holes, wall crossing, and mock modular forms (arXiv:1208.4074 [hep-th])
[12] Cheng M C N and Duncan J F R 2012 On Rademacher sums, the largest Mathieu group, and the holographic modularity of moonshine Commun. Number Theory Phys. 6 697–758
[13] Skoruppa N-P 1985 Über den Zusammenhang Zwischen Jacobiformen und Modulformen Halbganzen Gewichts (Bonner Mathematische Schriften) (Bonn: Bonn Mathematical Publications) p 159 (Universität Bonn Mathematisches Institut, Bonn Dissertation, Rheinische Friedrich–Wilhelms–Universität, Bonn, 1984)
[14] Schmidt R 2009 A remark on a paper of Ibukiyama and Skoruppa Abh. Math. Semi. Univ. Hamburg 79 189–91
[15] Gritsenko V A and Nikulin V V 1998 Automorphic forms and Lorentzian Kac–Moody algebras. II Int. J. Math. 9 201–75
[16] Cléry F and Gritsenko V 2011 Siegel modular forms of genus 2 with the simplest divisor Proc. Lond. Math. Soc. 102 1024–52
[17] Cléry F and Gritsenko V 2013 Modular forms of orthogonal type and Jacobi theta-series Abh. Math. Semin. Univ. Hamburg 83 187–217
[18] Gritsenko V, Skoruppa N-P and Zagier D Theta blocks (in preparation)
[19] Ibukiyama T and Skoruppa N-P 2007 A vanishing theorem for Siegel modular forms of weight one Abh. Math. Semin. Univ. Hamburg 77 229–35
[20] Ibukiyama T and Skoruppa N-P Correction to a vanishing theorem for Siegel modular forms of weight one (in preparation)
[21] Duncan J F R, Griffin M J and Ono K 2015 Moonshine Res. Math. Sci. 2 11
[22] Skoruppa N-P 2008 Jacobi Forms of Critical Weight and Weil Representations Modular Forms on Schiermonnikoog (Cambridge: Cambridge University Press) pp 239–66
[23] Gannon T 2016 Much ado about Mathieu Adv. Math. 301 322–58
[24] Gaberdiel M R, Persson D, Ronellenfitsch H and Volpato R 2013 Generalized Mathieu moonshine Commun. Number Theory Phys. 7 145–223
[25] Cheng M C N, de Lange P and Whalen D P Z 2016 Generalised umbral moonshine (arXiv: 1608.07835 [math.RT])
[26] Cheng M C N and Duncan J F R 2014 Rademacher Sums and Rademacher Series Conformal Field Theory, Automorphic Forms and Related Topics (Contributions in Mathematical and Computational Sciences vol 8) ed W Kohnen and R Weissauer (Berlin: Springer) pp 143–82
[27] Whalen D 2014 Vector-valued Rademacher sums and automorphic integrals (arXiv:1406.0571 [math.NT])
[28] Gritsenko V 1994 Irrationality of the moduli spaces of polarized abelian surfaces Int. Math. Res. Not. https://doi.org/10.1155/S1073792894000267
[29] Freitag E 1977 Siegelsche modulfunktionen Jahresbericht Dtsch. Math.-Ver. 79 79–86
[30] Feingold A J and Frenkel I B 1983 A hyperbolic Kac–Moody algebra and the theory of Siegel modular forms of genus 2 J. Math. Ann. 263 87–144
[31] Maß H 1980 Über ein Analogon zur Vermutung von Saito-Kurokawa Inventiones Math. 60 85–104
[32] Gritsenko V 1995 Arithmetical Lifting and its Applications Number Theory (Paris, 1992–3) (London Mathematical Society Lecture Note Series vol 215) (Cambridge: Cambridge University Press) pp 103–26
[33] O’Grady K G 1989 On the Kodaira dimension of moduli spaces of abelian surfaces Compos. Math. 72 121–63
[34] Gross M and Popescu S 2001 The moduli space of (1, 11)-polarized Abelian surfaces is unirational Compos. Math. 126 1–23
[35] Poor C and Yuen D S 2013 The cusp structure of the paramodular groups for degree two J. Kor. Math. Soc. 50 445–64
[36] Gritsenko V, Poor C and Yuen D S 2015 Borcherds products everywhere J. Number Theory 148 164–95
[37] Breeding J I, Poor C and Yuen D S 2016 Computations of spaces of paramodular forms of general level J. Kor. Math. Soc. 53 645–89
[38] Lopes Cardoso G 1997 Perturbative gravitational couplings and Siegel modular forms in D = 4, N = 2 string models Nucl. Phys. Proc. Suppl. 56B 94–101
[39] Neumann C D D 1999 The elliptic genus of Calabi–Yau 3- and 4-folds, product formulae and generalized Kac–Moody algebras J. Geom. Phys. 29 5–12
[40] Nazaroglu C 2013 Jacobi forms of higher index and paramodular groups in N = 2, D = 4 compactifications of string theory J. High Energy Phys. JHEP12(2013)074
[41] Cheng M C N and Duncan J F R 2016 Optimal Mock Jacobi theta functions (arXiv:1605.04480 [math.NT])
[42] Griffin M J and Mertens M H 2016 A proof of the Thompson moonshine conjecture Res. Math. Sci. 3 36
[43] Brumer A and Funke J 2004 On two geometric theta lifts Duke Math. J. 125 45–90
[44] Borcherds R E 1998 Automorphic forms with singularities on Grassmannians Inventiones Math. 132 491–562
[45] Prasad D 1993 Weil Representation, Howe Duality, and the Theta Correspondence Theta Functions: from the Classical to the Modern (CRM Proc. Lecture Notes) (Providence, RI: American Mathematical Society) pp 105–27
[46] Prasad D 1998 A brief survey on the theta correspondence Number Theory (Tiruchirapalli, 1996), Contemporary Mathematics vol 210 (Providence, RI: American Mathematical Society) pp 171–93
[47] Shimura G 1973 On modular forms of half integral weight Ann. Math. (2) 97 440–81
[48] The GAP Group 2005 GAP—Groups, Algorithms, and Programming, Version 4.4 www.gap-system.org