ON A RESULT OF MIYANISHI-MASUDA

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1. Introduction

Let $X$ be a smooth affine surface over $\mathbb{C}$ with an affine ruling (an $\mathbb{A}^1$-fibration) $\rho : X \to \mathbb{A}^1_\mathbb{C}$. Assume that $\rho$ is surjective, has a unique degenerate fiber, and this fiber is irreducible. In [3] such a surface $X$ is called affine pseudo-plane. It is of class $ML_1$ if $\rho$ is essentially unique, that is for any other affine ruling $\rho' : X \to \mathbb{A}^1_\mathbb{C}$, the general fibers of $\rho$ and $\rho'$ are the same. In [3] the following classification result is obtained.

Theorem 1.1. (Miyanishi-Masuda) Suppose that $X$ is an affine pseudo-plane of class $ML_1$. If $X$ admits an effective $\mathbb{C}^*$-action then the following hold.

(i) This $\mathbb{C}^*$-action is necessarily hyperbolic.

(ii) The universal covering $\tilde{f} : \tilde{X} \to X$ is a cyclic covering of degree $d$, where $d$ is the multiplicity of the unique degenerate fiber of $\rho$.

(iii) $\tilde{X}$ is an affine hypersurface in $\mathbb{A}^3_\mathbb{C} = \text{Spec} \mathbb{C}[x, y, z]$ with equation $x^m y = z^d - 1$ for some $m > 1$.

(iv) The Galois group $\mathbb{Z}_d = \langle \zeta \rangle$ of the covering $\tilde{f} : \tilde{X} \to X$, where $\zeta = \zeta_d$ is a primitive $d$-th root of unity, acts on $\tilde{X}$ via $\zeta \cdot (x, y, z) = (\zeta x, \zeta^{-m} y, \zeta^e z)$, where $\gcd(e, d) = 1$.

(v) The $\mathbb{C}^*$-action $\lambda \cdot (x, y, z) := (\lambda x, \lambda^{-m} y, \lambda^e z)$ ($\lambda \in \mathbb{C}^*$) on $\tilde{X}$ descends to the given $\mathbb{C}^*$-action on $X$, up to replacing $\lambda$ by $\lambda^{-1}$.

Let us add some remarks. An affine ruling on $X$ induces an affine ruling $\tilde{\rho} : \tilde{X} \to \mathbb{A}^1_\mathbb{C}$ with a unique degenerate fiber consisting of $d$ disjoint components isomorphic to $\mathbb{A}^1_\mathbb{C}$. In case $m > 1$ there is an essentially unique such affine ruling on $\tilde{X}$, defined by the restriction $x|\tilde{X}$. However, for $m = 1$, $y|\tilde{X}$ gives a second independent affine ruling, which also descends to $X = \tilde{X}/\mathbb{Z}_d$. Thus in this case $X$ cannot be a $ML_1$ surface.

If we want the $\mathbb{Z}_d$-action on $\tilde{X}$ to be free, the exponents $e$ and $d$ above must be coprime. Indeed, otherwise $\zeta^b = 1$ for some $b$ with $0 < b < d$, and we would have $\zeta^b (0, 0, z) = (0, 0, z)$ for every $d$-th root of unity $z$.

On the other hand, for every triple $(d, e, m)$ with $d \geq 1, m \geq 2$ and $\gcd(e, d) = 1$, (iii)-(v) determine a smooth affine pseudo-plane $X$ of class $ML_1$ with an effective $\mathbb{C}^*$-action. Thus Theorem 1.1 provides indeed a complete classification of these surfaces.

Here we give an alternative proof of Theorem 1.1 based on the results in [1, 2].

2. The proof

Under the assumptions of Theorem 1.1 $X \not\cong \mathbb{A}^2_\mathbb{C}$, since otherwise $X$ would admit another affine ruling $\rho' : X \to \mathbb{A}^1_\mathbb{C}$ with general fibers different from those of $\rho$, which contradicts the condition $ML_1$.

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A smooth affine surface $X$ with an elliptic $\mathbb{C}^*$-action is always isomorphic to $\mathbb{A}^2_k$, so this case is impossible. If $X$ is smooth and the $\mathbb{C}^*$-action on $X$ is parabolic then according to Proposition 3.8(b) in [1], $X = \text{Spec } A_0[D]$ for an integral divisor $D$ on a smooth affine curve $C = \text{Spec } A_0$. The existence of an affine ruling $\rho$ on $X$ with the base $\mathbb{A}^2_k$ implies that $C \cong \mathbb{A}^1_{\mathbb{C}}$. Hence $D$ is a principal divisor. By Theorem 3.2(b) in [1], we have again $X \cong \mathbb{A}^2_{\mathbb{C}} = \text{Spec } A_0[0]$ with $A_0 = \mathbb{C}[t]$, which is impossible.

Thus the $\mathbb{C}^*$-action on $X = \text{Spec } A$ is necessarily hyperbolic. Accordingly we can write

$$A = A_0[D_+, D_-]$$

with a pair of $\mathbb{Q}$-divisors $D_{\pm}$ on a smooth affine curve $C = \text{Spec } A_0$ satisfying $D_+ + D_- \leq 0$, see Theorem 4.3 in [1]. The remainder of the proof is based on Lemmas 2.1 and 2.2 below.

**Lemma 2.1.** Under the assumptions of Theorem 1.1, $A \cong A_0[D_+, D_-]$, where $A_0 = \mathbb{C}[t]$ and

$$D_+ = -\frac{e'}{d} [0], \quad D_- = \frac{e'}{d} [0] - \frac{1}{m} [1].$$

**Proof of Lemma 2.1.** By Lemmas 1.6 and 2.1 in [2], $X$ admits an affine ruling over an affine base if and only if it admits a non-trivial $\mathbb{C}^*$-action defined by a non-zero homogeneous locally nilpotent derivation $\partial \in \text{Der}(A)$. Moreover, $A_0 = \mathbb{C}[t]$ in [1]; and, up to an automorphism $\lambda \mapsto \lambda^{-1}$ of $\mathbb{C}^*$ (thus switching $(D_+, D_-) \leftrightarrow (D_-, D_+)$) we may assume that $e = \deg \partial \geq 0$. By Lemma 3.5 and Corollary 3.27 in [2], $e = 0$ implies that $X \cong \mathbb{A}^1_{\mathbb{C}} \times \mathbb{C}^*$, so the induced affine ruling $X \to \mathbb{C}^*$ is essentially unique and has the base $\mathbb{C}^*$, which contradicts our assumption. Thus $e > 0$.

According to Corollary 3.23 in [2], the latter implies that the fractional part $\{D_+\} = D - [D]$ is zero or is supported on one point, and we can choose this point to be $0 \in \mathbb{A}^1_{\mathbb{C}}$. Such a surface $X = \text{Spec } A$ is of class ML$_1$ if and only if the fractional part $\{D_-\}$ is supported on at least 2 points, see [2, Theorem 4.5].

Replacing $(D_+, D_-)$ by the equivalent pair $(\{D_+\}, D_- + [D_+])$ (see Theorem 4.3(b) in [1]) we may suppose that $D_+ = \{D_+\} = -e'/d[0]$, where $\gcd(e', d) = 1$ and $d > 0$.

For any affine pseudo-plane $X$, the Picard group Pic$X$ is a torsion group [3, Ch. 3, 2.4.4]. On the other hand, for a $\mathbb{C}^*$-surface $X$ as above, $\text{rk}_\mathbb{Q} (\text{Pic}X \otimes \mathbb{Q}) \geq l - 1$, where $l$ is the number of points $b_j \in \mathbb{A}^1_{\mathbb{C}}$ such that $(D_+ + D_-)(b_j) < 0$, see Corollary 24.24 in [2]. Hence $l \leq 1$ and so, $\exists p \in \mathbb{A}^1_{\mathbb{C}} : (D_+ + D_-)(q) = 0 \forall q \neq p$.

Since $D_+ (q) = 0 \forall q \neq 0$ we have $D_-(q) = 0 \forall q \neq 0, p$. It follows that $\text{supp}(D_-) = \text{supp}(\{D_-\}) = \{0, p\}$ with $p \neq 0$. After an automorphism of $\mathbb{A}^1_{\mathbb{C}}$ we may assume that $p = 1$. Thus finally

$$D_\pm (0) = \mp e'/d, \quad D_+(1) = 0, \quad D_-(1) = a/m \not\in \mathbb{Z} \quad \text{and} \quad D_\pm (q) = 0 \forall q \neq 0, 1,$$

where $\gcd(a, m) = 1$ and $m > 0$. The smoothness of $X$ forces $a = -1$, see Theorem 4.15 in [1]. This proves Lemma 2.1.

Next we use the following description [2, Corollary 3.30], where for a $\mathbb{Q}$-divisor $D$, $d(D)$ denotes the minimal positive integer $d$ such that $dD$ is integral.

**Lemma 2.2.** We let $A = \mathbb{C}[t][D_+, D_-]$, where $D_+ + D_- \leq 0$, $d(D_+) = d$, $d(D_-) = k$. We assume that $D_+ = -\frac{e'}{d}[0]$ and $D_-(0) = -\frac{1}{k}$, and we let $\partial \in \text{Der}(A)$ be a homogeneous locally nilpotent derivation with $e = \deg \partial > 0$. Then there exists a
unitary polynomial $Q \in \mathbb{C}[t]$ with $Q(0) \neq 0$ and $\text{div}(t^i Q(t)) = -kD_-$ such that, if $A' = A_{k,P}$ is the normalization of $A_{k,P}$, then

$$B_{k,P} = \mathbb{C}[u,v,s]/\left(u^k v - P(s)\right), \quad \text{where} \quad P(s) = Q(s^d)s^{ke'+dl},$$

then the group $\mathbb{Z}_d = \langle \zeta \rangle$ acts on $B_{k,P}$ and also on $A'$ via

$$\zeta(u,v,s) = (\zeta^{e'}u, v, \zeta s),$$

so that $A \cong A'^{\mathbb{Z}_d}$. Furthermore, $ee' \equiv 1 \mod d$ and $\partial \equiv cu \frac{\partial}{\partial s}$ for some constant $c \in \mathbb{C}^*$. With this result we can complete the proof of Theorem 1.1 as follows. We may assume that $A = A_0[D_+, D_-]$ with $A_0 = \mathbb{C}[t]$ and $(D_+, D_-)$ as in Lemma 2.1. With $k := \text{lcm}(d, m)$ let us write $k = mm' = dd'$ and $l = -e'd'$, so that

$$D_+ = -e'd'[0] = \frac{l}{k}[0], \quad D_- = \frac{e'}{d'}[0] - \frac{1}{m}[1] = -\frac{l}{k}[0] - \frac{m'}{k}[1].$$

Thus Lemma 2.2 can be applied in our setting with $Q = (t - 1)^{m'}$. By this lemma, $A = A'^{\mathbb{Z}_d}$, where $A'$ is the normalization of

$$B = \mathbb{C}[u,v,s]/\left(u^k v - (s^d - 1)^{m'}\right),$$

with the action of $\mathbb{Z}_d$ as in [3] and with the $\mathbb{C}^*$-action $\lambda(u,v,s) = (\lambda u, \lambda^{-k} v, s)$.

The element $w = \frac{s^d - 1}{u_m} \in \text{Frac}(B)$ satisfies $w^{m'} = v$ and so is integral over $B$, hence

$$A' \cong \mathbb{C}[u,w,s]/(u^m w - (s^d - 1)).$$

Because of [3] we have $\zeta w = \zeta^{-me'}w$. Thus after applying an automorphism $\zeta \mapsto \zeta^{e'}$ of $\mathbb{Z}_d$, both the $\mathbb{Z}_d$-action and the $\mathbb{C}^*$-action on $\tilde{X} = \text{Spec} A' \subseteq \mathbb{A}^3_\mathbb{C} = \text{Spec} \mathbb{C}[u,w,s] \cong \text{Spec} \mathbb{C}[x,y,z]$ have the claimed form

$$\zeta(u,w,s) = (\zeta u, \zeta^{-m}w, \zeta^{e'} s) \quad \text{respectively}, \quad \lambda(u,w,s) = (\lambda u, \lambda^{-m} w, s).$$

This proves the theorem. □

REFERENCES

[1] H. Flenner, M. Zaidenberg: Normal affine surfaces with $\mathbb{C}^*$-actions. Osaka J. Math. 40, 2003, 981–1009.
[2] H. Flenner, M. Zaidenberg: Locally nilpotent derivations on affine surfaces with a $\mathbb{C}^*$-action. Prépublication de l’Institut Fourier de Mathématiques, 638, Grenoble 2004; math.AG/0403215; to appear in Osaka J. Math.
[3] K. Masuda, M. Miyanishi: Affine Pseudo-planes with torus actions. Preprint, 2005.
[4] M. Miyanishi: Open algebraic surfaces. CRM Monograph Series, 12. Amer. Math. Soc., Providence, RI, 2001.

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