Review of Computational Schemes in Inverse Heat Conduction Problems

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ABSTRACT

This paper aims to present a review of the current computational methods and applications of inverse heat conduction problems (IHCPs) in different fields. Generally speaking, there are two major solving categories of this issue: mesh methods and meshless algorithms of their strengths and weaknesses are also discussed in this study. Finally, the challenges and future trends related to the computational schemes of IHCPs are interpreted.

1. Brief Introduction

Heat conduction problems often divide into two distinct categories: direct and inverse problems. Direct heat conduction problems are primarily concerned with the computation of the temperature distribution inside the solid bodies for known boundary and initial conditions, heat generation rates, and thermophysical properties. On the other hand, the determination of surface temperatures, heat source rates, and thermophysical properties using measured temperatures inside solid bodies are categorized as inverse heat conduction problems (IHCPs). Direct heat conduction problems are contemplated to be well-posed.

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On the other hand, IHCPs are pondered to be mathematically ill-posed, that is, that unavoidable random errors (noise) in the measured information may cause errors increased by several orders of magnitudes in the estimated unknowns [1].

Many researchers are interested in tackling the IHCPs by utilizing mesh methods and meshless approaches. During the last few decades, many mesh schemes were investigated in [1], the space marching difference schemes [2], the slowly divergent space marching schemes [3], the mollification method [4], the mollified space marching finite differences algorithm (MSFDA) for one-dimensional (1D) IHCP [5], the stable space MSFDA [6], the MSFDA for two-dimensional (2D) IHCP [7], the control volume algorithm combined with a digital filter method [8], the difference approximation method [9], the finite difference method (FDM) and Fourier transform techniques [10], the finite element method (FEM) [11], the multi-node unsteady surface element method [12], the extension of Beck’s method [13], the boundary element method (BEM) [14–26], the dual reciprocity boundary element method (DRBEM) [27], the sequential function specification method [28], the new type of basic functions of FEM [29], the differential quadrature method (DQM) and Fourier series approach [30], the wavelet regularization method [31], the non-iterative FEM [32], the FEM and optimization method [33], the maximum entropy method (MEM) [34], the Bayesian inference approach [35], the dual reciprocity BE-based sequential function specification solution method [36], the modified sequential approach [37], the direct transformation matrices method [38], the Bayesian statistical inference method [39], the discrete wavelet transform method [40], the input estimation method including finite element scheme [41], the group preserving scheme [42], the damped heat wave algorithm [43], the FDM with least-squares scheme [44], the hybrid regularization method [45], two regularization strategies (one is based on the modification of the equation and the other is based on the truncation of high-frequency components) [46], the semi-Markovian concept with a Bayesian estimation technique [47], the lattice free FDM [48], the modified Tikhonov regularization (TR) method [49], the adjoint-weighted variational formulation [50], the variable metric methods [51], the TR method and Fourier regularization method [52], the parametric study and optimal algorithm [53], the trained proper orthogonal decomposition–radial basis function (RBF) network inverse method [54], the wavelet denoising algorithm [55], the TR and conjugate gradient methods (CGMs) together with a discrepancy stopping rule [56], the CGM and adjoint state equations [57], the optimization of predictions [58], the steepest descent method [59], the transformation method [60], the optimal modified method [61], the splitting-based CGM [62], the FDMs and backward Euler and Crank–Nicolson schemes [63], the BEM and TR method [64], different finite element approaches [65], the differential-difference regularization method [66], the CGM [67], the regularization term method, the differential equation method, the gradient integration method, and the sequential gradient method [68], the hybrid scheme based on the Laplace transform, change of variables, and the least-squares scheme [69], the adaptive wavelet methods and sparsity reconstruction [70], the method of lines and a quasi-reversibility method [71], the DQM [72], the high-fidelity thermocouple model [73], the sequential algorithm and error sensitivity analysis [74], the quantum-behaved particle swarm optimization (PSO) [75], the modified sequential PSO algorithm [76], the global time method [77], the mollification and wavelet prefiltering methods [78], the lattice Boltzmann method [79], the model function approach [80], the filled function method and the Bryden–Fletcher–Goldfarb–Shanno algorithm [81], the PSO-based algorithms [82], the complex-variable differentiation method [83], the regularization techniques and shielded thermocouple analysis [84], the parameterized gradient integration method [85], the BEM coupled with the CGM [86], the wavelet regularization method and [87], the iterative regularization method [88], the modified version of the classical TR technique together with the corresponding error estimates [89], the Ritz–Galerkin method [90], the method of lines with numerical differentiation of the sequential temperature–time histories [91], the estimation metrics and optimal regularization in a Tikhonov digital filter [92], the low cost surrogate model-based evolutionary optimization solvers [93], the derivative regularization method [94], the least-squares method and the genetic algorithm (sequential and multi-core parallelization approach) [95], the filter technique [96], the sequential specification function method and CGM [97], the Levenberg–Marquardt algorithm (LMA) and Galerkin FEM [98], the CGM [99], the variational iteration method [100], the analysis of the conditioning of methods [101], the CGM with adjoint equation [102], the enhanced particle swarm optimization (EPSO) algorithm [103], the Tikhonov digital filter technique [104], the control volume FEM and control volume method [105], the BEM based on potential theory [106], the digital filter form of TR method [107], the mollified marching scheme [108], the finite element formulation incorporated with modified cubic spline method [109], the coupling method [110], the digital filters methods [111], the modified TR method [112], the finite volume method and a boundary fitted mesh [113], the sequential Beck approach [114], the TR method [115], the CGM and FEM [116], the CGM, the LMA and gradient projection method [117], the wavelet regularization method.
[118], and the modified LMA [119]. We will describe the above-mentioned mesh methods in detail in Section 3.

For the meshless approaches, the method of fundamental solution (MFS) [120,121], the Legendre polynomials method [122], the meshless local Petrov–Galerkin (MLPG) method [123], the RBFs and TR method [124], the Green’s function [125], the global space-time multiquadric method [126], the local Petrov–Galerkin approach [127], the modified MFS [128], a Trefftz method with exponential basis functions (EBFs) [129], the singular boundary method [130], the complex variable reproducing kernel particle method (CVRKPM) [131], an analytical transfer function method [132], a meshless method based on the fundamental solution and RBF [133], the Gaussian RBFs method [134], the discrete Fourier transform method (DFTM) [135], the calibration integral equation method (CIEM) [136], the boundary integral equation method [137], the multiple-scale polynomial Trefftz method (MSPTM) [138]. We will also depict the above-mentioned schemes in detail in Section 3.

This study is arranged as follows. Section 2 demonstrates the IHCPs. Then, in Section 3, we interpret the computational schemes for IHCPs. At last, some conclusions, current challenges and future work are drawn in Section 4.

2. Inverse Problem Statement

The sideways heat equation is a model of an IHCP that when the temperature data from a measurement device located inside a body is available, we endeavor to determine the outside temperature of the body. Here, we are interested in the numerical solutions of

\[ u_t = \nu u_{xx}, \quad \ell \geq x \geq 0, \quad t \geq 0, \tag{1} \]

with a left side boundary condition

\[ u(0, t) = a(t), \quad t \geq 0, \tag{2} \]

and initial condition

\[ u(x, 0) = h(x), \quad \ell \geq x \geq 0, \tag{3} \]

in which \( \nu \) is the thermal conductivity of a heat-conducting rod with length \( \ell \).

When it is impossible to measure the temperature on the surface directly, the sideways heat equation often takes place in engineering applications, in which one wants to deal with the surface temperature from measurements inside a heat-conducting object. The issue setup with its physical model is displayed in Figure 1. Assume that we can insert a thermocouple inside the rod to measure the temperature at a position \( x = a < \ell \), and the data are indicated by:

\[ u(a, t) = \beta(t), \quad t \geq 0. \tag{4} \]

The inverse problem is hence, to choose the temperature field inside the rod by utilizing Equations (1–4).

3. Computational Schemes for IHCPs

Many researchers are concerned about coping with the IHCPs by employing mesh schemes and meshless algorithms. We will interpret these methods in detail as follows:

3.1. Mesh Methods

Many finite difference discretizations for the sideways heat equation of IHCP were discussed in [1], and many are explained and compared by [2,3]. In addition, stability standpoints of those finite difference schemes in connection with mollification are considered in a series of papers; see, e.g. [4–8]. In these successful computations with the space-marching, FDMs are also described. For example, Eldén [9,10] discretized the sideways heat equation of IHCP by a differential-difference equation and interpreted the approximation properties of time-discrete approximations by utilizing the Fourier transform techniques. He found that the time discretization has a ‘regularizing effect,’ that is, the high-frequency noise is prohibited from

![Figure 1. Determination of surface temperature from interior measurements. The thermocouple could be superseded by any thermal sensor.](image-url)
In addition, Lesnic et al. [23] also extended the results of the temperature and heat flux on the remaining boundary. Energy technique to solve the IHCP, namely to determine the leading coefficient in the heat equation with an extra condition at the terminal. They showed the efficiency and the rapid convergence of the methods. Furthermore, Krutz et al. [11], Lithouhi and Beck [12], Reinhardt [13], Cialkowski [29], Ling et al. [32], Chen and Tuan [41], Grysa and Leśniewska [65], Dehghan et al. [90], Min et al. [98], Duda [105], Kanjanakijkasem [109] and Zhu et al. [116] employed the FEM, the multi-node unsteady surface element method, the extension of Beck’s method, new type of basic functions of FEM, the non-iterative FEM, the input estimation method including finite element scheme, the different finite element approaches, the Ritz–Galarkin method, the LMA and Galerkin FEM, the control volume FEM and control volume method, the finite element formulation incorporated with modified cubic spline method, and the CGM and FEM to solve the IHCP, respectively. Disadvantages of utilizing finite elements and finite differences are that they usually produce instabilities in the numerical algorithms and need a large number of cells or elements.

The BEM has been contemplated in the study of IHCP and employed by many researchers. Brebbia [14] first proposed the BEM to evade the additional finite differencing obtained in the conventional FEM for heat transfer issues. Since then, several researchers have started to utilize various BEM formulations to tackle the ill-posed problem existing in this area; see, e.g. [15–21]. After that, Lesnic et al. [22] addressed the BEM along with the minimum energy technique to solve the IHCP, namely to determine the temperature and heat flux on the remaining boundary. In addition, Lesnic et al. [23] also extended the results of [24] for solving the IHCP when no boundary condition is assigned and employed the BEM to reveal the solution numerically. Besides, Al-Najem et al. [25] established the singular value decomposition (SVD) with BEM and the least-squares approach with integral transform approach to tackle 2D steady-state IHCPs. Shen [26] even presented two BEMs, a collocation scheme and a weighted one, to resolve the IHCP. Furthermore, Singh and Tanaka [27] also investigated an application of the in conjunction with iterative regularization for the solution of time-dependent IHCPs. On the basis of the dual reciprocity boundary element along with sequential function specification scheme, Behbahani-nia and Kowsary [36] have lately developed an algorithm to cope with 2D IHCPs involving unknown time and space varying boundary heat flux estimation. Because there is no need on domain discretization in the BEM, the DRBEM, the BEM and TR method, the BEM coupled with the CGM and the BEM based on potential theory, the location of interior points, in which the temperature data are gathered, can be selected in a rather arbitrary way; see, e.g. [14–28,64,86,106].

About the regularization and optimization methods of IHCPs, Fu et al. [31] addressed the wavelet regularization method to resolve the IHCP and obtained acceptable numerical results. Usual regularization approaches are equivalent to filtering out high-frequency components, in a uniform way on the whole stretch of data. This sets a limit to the trade-off that can be acquired between efficient noise reduction and good resolution. To overcome this limit, many researchers developed the other different regularization methods; see, e.g. [33,40,45,46,49,52,56–59,61,62,66–69,71,75,76,81,82,84,85,87–89,92–97,99,102–104,107,111,112,115,117–119].

For the different solving techniques of IHCP, Kim and Lee [34] proposed the MEM to tackle 2D IHCPs and found the solution which maximizes the entropy functional under the given temperature measurements. The current scheme converted the inverse problem to a non-linear constrained optimization issue. The constraint of the optimization problem was the statistical consistency between the measured temperature and the estimated temperature. They showed the considerable enhancement in resolution for stringent examples in comparison with a conventional approach. Later, Wang and Zabaras [35] addressed a Bayesian inference approach to solve the IHCP and captured very well the probability distribution of the unknown heat flux. Lin et al. [37] used a modified sequential approach to eliminate the leading error caused by adding the use of future information in the process of preliminary estimation and effectively reduced the average relative error when adding the large random measurement error. After that, the ill-posed IHCP was analyzed by pondering the stability of the semi-discretization numerical methods. Then, the resulting ordinary
differential equations at the discretized times were numerically integrated towards the spatial direction by the group preserving scheme [42]. It was shown that this algorithm was quite effective and better than other numerical solvers, including the fourth-order Runge–Kutta method.

3.2. Meshless Methods

Lately, many meshless approaches have been proposed to tackle the IHCPs, such as Hon and Wei [120] addressed a new meshless and integration-free numerical scheme to solve IHCPs. To regularize the resultant ill-conditioned linear system of equations, they applied successfully both the Tikhonov regularization technique and the L-curve method to obtain a stable numerical solution. The approach was readily extendable to solve high-dimensional problems under irregular domain. Nevertheless, they did not contemplate the numerical solution with noisy effect. Later, Hon and Wei [121] developed the MFS to solve multidimensional IHCPs. To tackle the ill-conditioning problem of the resultant linear system of equations, they used the Tikhonov regularization method based on the generalized cross-validation (GCV) criterion for choosing the regularization parameter and acquired a stable approximation solution. Shidfar and Pourgholi [122] employed the Legendre polynomials and a linear transformation to tackle the IHCPs. After that, Sladek, Sladek and Hon [123] utilized a MLPG method to solve stationary and transient IHCPs in 2D and three-dimensional (3D) axisymmetric bodies. They applied SVD method to deal with the ill-conditioned linear system of algebraic equations obtained from the local integral equations after moving least-squares approximation. Later, Shidfar et al. [124] showed the RBF and Tikhonov regularization method to cope with the IHCP, and they claimed that the numerical results for these issues demonstrated the efficiency of the developed method. Moreover, determination of regularization parameter was based on L-curve technique. In [125], the applicability of the 3D transient analytical solution based on Green’s function has been shown to tackle the IHCPs. Nevertheless, they did not deliberate the numerical disturbance effect. Apart from this, Li and Mao [126] established a radial basis collocation method based on the global space-time multiquadratic method to tackle the IHCPs. The least-square technique was introduced to find the solution of the overdetermined linear system. This present study investigated two types of the ill-posed heat conduction problems: the IHCP to recover the surface temperature and heat flux history on a source point from the measurement data at interior locations, and the backward heat conduction problem to retrieve the initial temperature distribution from the known temperature distribution at a given time. Sladek et al. [127] interpreted that using the MLPG approach to the inverse transient heat conduction problems in three-dimensional solids with continuously inhomogeneous and anisotropic material properties. After that, Li et al. [128] proposed a modified MFS to tackle the reconstruction problem of IHCP. This scheme showed efficient and stable numerical results. Later, Movahedian et al. [129] addressed a Trefftz method based on using EBFs to resolve the one-dimensional and 2D IHCPs. Their numerical results were accurate and efficient. After that, Gu et al. [130] developed the singular boundary method to cope with inverse anisotropic heat conduction problems. They claimed that the method cured the perplexing fictitious boundary issue associated with the MFs while inheriting the merits of the latter of being truly meshless, integration free and easy to implement. Owing to its boundary-only discretization and semi-analytical nature, the method can be viewed as an ideal candidate for the solution of inverse problems. Weng et al. [131] used the CVRKPM to solve the 2D IHCPs and obtained good computational accuracy. Later, Fernandes et al. [132] employed an analytical transfer function (or impulse response) method to resolve the IHCPs, which was based on Green’s function and the equivalence between thermal and dynamic systems. From the temperature profile (hypothetical or experimental temperature far from the heat source) and knowing the transfer function it was possible to estimate the heat flux by different approaches: deconvolution, spectral densities estimation or inverse fast Fourier procedure. This study was concluded with the application of the technique in an experimental case of temperature estimation at the tool-work-piece interface during a machining process. After that, Arghand and Amirfakhrian [133] utilized a meshless method based on the fundamental solution and RBF to deal with a backward IHCP. Since the coefficients matrix was ill-conditioned, they applied the TR method to solve the resulted system of linear equations and addressed the GCV criterion to choose a regularization parameter. Zhang and Li [134] proposed a recursion numerical technique to tackle the IHCPs with an unknown time-dependent heat source and the Neumann boundary conditions. Because the coefficients matrix was badly ill-conditioned, they also applied the TR method to resolve the resulted system of linear equations and used the GCV criterion to obtain stable numerical results. Later, two methods of solving the IHCP with employment of the discrete Fournier transform are presented in [135]. The first one operated similarly to the SVD algorithm and consisted in reducing the number of components of the discrete Fournier transform which were taken into account to determine the solution to the inverse problem. The second method was related to the regularization of the solution to the inverse problem in the discrete Fournier
transform domain. Results of calculations made in both ways brought very good outcomes and confirmed the usefulness of applying the discrete Fournier transform to solving IHCPs. Chen et al. [136] constructed the CIEM to investigate in the context of 2D IHCP. This work utilized a Volterra integral equation of the first kind that was pertinent to many applications where the total surface heat transfer was required. Numerical results indicate the merit of the approach when examining two dissimilar isotropic materials. Later, Garshasbi and Hassani [137] established the boundary integral equation method to cope with an inverse boundary value problem for 2D heat equation in an annular domain. Since the resulting system of linear algebraic equations was ill-posed, the Tikhonov first-order regularization procedure was employed to obtain a stable solution. Determination of regularization parameter was based on L-curve technique. After that, Liu et al. [138] developed the MSPTM to acquire the effective and accurate results, even for those of severely ill-posed inverse problems under very large noises.

4. Conclusions, Current Challenges and Future Work

In this article, on the basis of our mentioned literature studies, we realize that the development of meshless methods become the mainstream trend. Furthermore, numerous efforts to achieve this trend has been attempted by many researchers. Note that many challenges need to be overcome for different applications of IHCPs, especially peak shape. The stability analysis of approaches is also discussed. In the future, we will propose several schemes to solve the 3D non-linear IHCPs and obtain accurate results. In addition, the current challenges are how to tackle the complex geometry engineering problems.

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