Polarized Image of a Rotating Black Hole in Scalar–Tensor–Vector–Gravity Theory

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Abstract

The polarized images of a synchrotron emitting ring are studied in the spacetime of a rotating black hole in the scalar–tensor–vector–gravity (STVG) theory. The black hole owns an additional dimensionless modified gravity (MOG) parameter described as its deviation from a Kerr black hole. The effects of the MOG parameter on the observed polarization vector and Stokes Q − U loops depend heavily on the spin parameter, the magnetic field configuration, the fluid velocity, and the observation inclination angle. For the fixed MOG parameter, the changes of the polarization vector in the image plane are similar to those in the Kerr black hole case. The comparison of the polarization images between the Kerr–MOG black hole and M87* implies that there remains some possibility for the STVG–MOG theory.

Unified Astronomy Thesaurus concepts: Black holes (162)

1. Introduction

The black hole images of M87* (Akiyama et al. 2019a, 2019b, 2019c, 2019d, 2019e, 2019f) and Sgr A* (Akiyama et al. 2022), photographed by the Event Horizon Telescope, strongly confirm the existence of black holes in our universe, which have greatly stimulated the study of black hole physics, both theoretically and experimentally. These images achieving a diffraction limited angular resolution bring us a lot of information from the strong field region near a black hole. In the first polarized images of the black hole M87* (Akiyama et al. 2021a, 2021b), the twisting polarization pattern with a prominent rotationally symmetric mode reveals that there is a strong magnetic field around the black hole in terms of electron synchrotron radiations. Moreover, the analysis shows that the polarization patterns depend also on the strongly curved spacetime near the black holes. Thus, the study of polarized images of black holes is beneficial to probe the matter distribution and the related physical process around black holes (Connors et al. 1980; Bromley et al. 2001; Li et al. 2009; Shcherbakov et al. 2012; Dexter 2016; Gold et al. 2017; Jiménez-Rosales & Dexter 2018; Marin et al. 2018; Moscbrodzka 2020; Palumbo et al. 2020; Gelles et al. 2021; Moscibrodzka et al. 2021; Narayan et al. 2021; Qin et al. 2021; Zhang et al. 2021, 2022; Hu et al. 2022; Liu et al. 2022; Zhu & Guo 2022), and even check theories of gravity.

Exact description for polarized images of black holes must resort to numerical simulations, which is generally computationally expensive due to the broad parameter surveys and the complicated couplings among astrophysical and relativistic effects. Recently, a simple model of an equatorial ring of magnetized orbiting fluid has been developed to investigate the polarized images of synchrotron emission around the Schwarzschild black hole (Narayan et al. 2021) and Kerr black hole (Gelles et al. 2021). Although only the emission from a single radius $r_s$ is considered, this model can clearly reveal dependence of the polarization signatures on magnetic field configuration, black hole spin, and observer inclination. Moreover, with this model, the image of a finite thin disk can be produced by simply summing contributions from individual radii. Thus, this model has been recently applied to study the polarized image of an equatorial emitting ring in various spacetimes, such as, 4D Gauss–Bonnet black holes (Qin et al. 2021), regular black holes (Liu et al. 2022), Schwarzschild–Melvin black holes (Zhu & Guo 2022), and so on.

The observations of galaxies (Rubin et al. 1965; Rubin & Ford 1970) reveal a discrepancy between the observed dynamics and the amount of luminous matter. Theoretically, this discrepancy can be explained by introducing exotic dark matter. However, to date, there is no exact evidence to confirm the existence of dark matter. Another possible resolution to the discrepancy is a modification of gravity theory. STVG (Moffat 2006b) is a kind of fully covariant modified gravity (MOG) theory. This MOG theory contains three gravitational fields: the Einstein metric related to massless tensor graviton, a massless scalar graviton, and a massive vector graviton. The STVG–MOG theory (Moffat 2006b) has successfully explained the rotation curves of galaxies (Moffat & Rahvar 2013; Moffat & Toth 2015) and the dynamics of galactic clusters (Moffat 2006a; Brownstein & Moffat 2007; Moffat & Rahvar 2014). Moreover, the gravitational waves in this STVG–MOG theory have been studied (Moffat 2016; Green et al. 2018). The Schwarzschild-like and Kerr-like black hole solutions in STVG–MOG theory are obtained in Moffat (2015a). These black hole solutions have an additional MOG parameter that yielded a variable gravitational constant. It is of interest to study the observational effects of black holes in STVG–MOG theory because they could help understand this modified gravity theory and some fundamental issues in physics. Effects of the MOG parameter on the quasi-normal modes (Manfredi et al. 2017), black hole shadow (Moffat 2015b; Wang et al. 2019), black hole thermodynamics (Mureika et al. 2016), and other physical processes (Hussain & Jamil 2015; Lee & Han 2017; Moffat 2017; Pérez et al. 2017; Sharif & Shahzadi 2017; Sheoran et al. 2018; Wei & Liu 2018)
have been studied in the spacetime of Schwarzschild–MOG and Kerr–MOG black holes. This paper aims to study the polarization information in the image of a synchrotron emitting ring around a Kerr–MOG black hole (Moffat 2015a) and to probe the effects of the MOG parameter on the polarization image.

The paper is organized as follows: Section 2 briefly introduces the Kerr–MOG black hole and presents formulas to calculate the observed polarization vector in the image plane of an emitting ring in this spacetime. Section 3 presents the polarization images of a synchrotron emitting ring and probes the effects of the MOG parameter on the polarization image. Finally, this paper ends with a summary.

2. Observed Polarization Field in a Kerr–MOG Black Hole Spacetime

The STVG theory is a covariant modified theory of gravity and its action is composed of scalar, tensor, and vector fields (Moffat 2006b)

\[ S = S_G + S_\phi + S_\nu, \]  

with

\[ S_G = -\frac{1}{16\pi} \int \sqrt{-g} \frac{R}{G} d^4x, \]

\[ S_\phi = -\frac{1}{4\pi} \int \sqrt{-g} \left[ \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + V(\phi) \phi^\nu \right] d^4x, \]

\[ S_\nu = -\int \sqrt{-g} \left[ \frac{1}{2} \phi^\mu \left( \nabla_\nu G_{\mu\nu} G + \nabla_\nu \phi^\mu \right) \right] d^4x + \frac{V(G)}{G^2} + \frac{V(\phi)}{\phi^2} \]

\[ + \frac{V(\nu)}{\nu^2} \]  

(2)

\( S_G \) corresponds to the Einstein gravity action and \( R \) is the Ricci scalar, \( S_\phi \) is the action of the massive vector field \( \phi_\mu \), and \( B_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu \); \( G \) is a scalar field \( G = G_N(1 + \alpha) \) corresponding to a spin 0 massless graviton, \( G_N \) is Newton’s gravitational constant, and \( \alpha \) is a dimensionless parameter. In this modified theory, the mass of the vector field \( \phi_\mu \) is also an effective spin 0 scalar field \( \nu \); \( V(G) \) and \( V(\nu) \) are self-interaction potentials of the \( \mu \) and \( G \) fields, respectively. The spacetime of a rotating black hole in the STVG–MOG theory (Moffat 2006b) can be described by the so-called Kerr–MOG metric with the

Figure 1. Effects of the MOG parameter \( \alpha \) on the polarized vector and EVPA in the Kerr–MOG black hole spacetime (3) for the magnetic field owned only the radial component \( B_r \). Here we set \( M = 1 \), \( r_s = 6 \), \( \theta_o = 20^\circ \), \( \beta_\nu = 0.3 \), and \( \chi = -90^\circ \).
form in Boyer–Lindquist coordinate (Moffat 2015a)

\[ ds^2 = -\frac{\Delta \rho^2}{\Xi} dt^2 + \frac{\rho^2}{\Xi} dr^2 + \Sigma d\theta^2 + \frac{\Xi \sin \theta^2}{\rho^2} (d\phi - \omega dt)^2, \]

where

\[ \Delta = r^2 - 2GMr + a^2 + \alpha G_{\text{NGM}}, \quad \rho^2 = r^2 + a^2 \cos \theta^2, \]
\[ \omega = \frac{a(a^2 + r^2 - \Delta)}{\Xi}, \quad \Xi = (r^2 + a^2)^2 - \Delta a^2 \sin \theta^2. \]

Here \( M \) and \( a \) are the mass parameter and the spin parameter of the Kerr–MOG black hole, respectively. The Kerr–MOG metric can reduce to the Kerr metric in the limit \( \alpha = 0 \). Thus, the parameter \( \alpha \) can be used to measure the deviation of the Kerr–MOG black hole in STVG–MOG theory from the Kerr black hole in general relativity. The Arnowitt–Deser–Misner (ADM) mass and the angular momentum of the Kerr–MOG black hole are given by Sheoran et al. (2018)

\[ M = (1 + \alpha)M, \quad J = Ma. \]

With the ADM mass \( M \), the function \( \Delta \) can be rewritten as

\[ \Delta = r^2 - 2GMr + a^2 + \frac{\alpha}{1 + \alpha} G_N^2 \mathcal{M}^2. \]

The outer and inner horizon radii are the roots of \( \Delta = 0 \), i.e.,

\[ r_{\pm} = M \pm \sqrt{\frac{\mathcal{M}^2}{1 + \alpha} - a^2}. \]

Here we set \( G_N = 1 \) without loss of generality. The extremal limit for the Kerr–MOG black hole is \( \mathcal{M}^2 = (1 + \alpha)a^2 \). Similarly, the Kerr–MOG metric is singular at \( r^2 = 0 \) as in the Kerr case.

Now, the polarization vectors are to be studied for photons emitted from the ring with radius \( r_s \) around a Kerr–MOG black hole. A synchrotron emitting ring is assumed to lie in the equatorial plane \( (S^2, q^2) \). In the local zero-angular-momentum-observer (ZAMO) frame of the point \( P \) in the ring, the four-vector components \( V^{(a)} \) of the emitter are related to the vector \( V_\mu \) in Kerr–MOG spacetime by Narayan et al. (2021) and Gelles et al. (2021)

\[ V^{(a)} = \eta^{(a)(b)} e^{\mu}_{(b)} V_\mu, \]
where $\eta^{(a)(b)}$ is the flat Minkowski metric and $e^\mu_{(b)}$ is the ZAMO tetrad

$$e^\mu_{(b)} = \begin{bmatrix} \frac{1}{r_s} \sqrt{\Delta_s} & 0 & \frac{\omega_s}{r_s} \sqrt{\Delta_s} & 0 \\ 0 & \Delta r & 0 & 0 \\ 0 & 0 & \frac{r_s}{\sqrt{-\xi}} & 0 \\ 0 & 0 & 0 & -\frac{1}{r_s} \end{bmatrix}. \quad (9)$$

For a boosted emitter in the local orthonormal ZAMO frame, its boosting velocity is assumed to be in the $r - \phi$ plane and has a form

$$\mathbf{\beta} = r_s \mathbf{\beta}_s [\cos \chi (\hat{r}) + \sin \chi (\hat{\phi})]. \quad (10)$$

The vectors of the point emitter in the boosted orthonormal frame can be obtained by a Lorentz transformation $\Lambda^{(a)}_{(b)}$ from the ZAMO frame, i.e.,

$$V^\mu_{(a)} = \Lambda^{(a)}_{(b)} V^\mu_{(b)}, \quad (11)$$

where

$$\Lambda^{(a)}_{(b)} = \begin{bmatrix} \gamma & -\beta_s \gamma \cos \chi & -\beta_s \gamma \sin \chi \\ -\beta_s \gamma \cos \chi & \gamma - 1 & \cos \chi \cos \chi \\ -\beta_s \gamma \sin \chi & \gamma - 1 & \sin \chi \cos \chi \\ 0 & 0 & \gamma - 1 & \sin \chi \sin \chi \end{bmatrix}. \quad (12)$$

and $\gamma$ is the Lorentz factor. Thus, the vector of the point source in black hole spacetime can be obtained from the boosted orthonormal frame by an inverse transformation

$$V^\mu = e^{\mu}_{\nu(c)} \Lambda^{(c)}_{(a)} V^\nu, \quad (13)$$

where $\Lambda^{(a)}_{(b)}$ is the inverse matrix of $\Lambda^{(a)}_{(b)}$. Here, the orientation $(\hat{x}, \hat{y}, \hat{z})$ is equivalent to $(\hat{r}, \hat{\phi}, \hat{\theta})$.

To study the polarized image of an equatorial emitting ring around a Kerr–MOG black hole, the null geodesic describing

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**Figure 3.** Effects of the MOG parameter $\alpha$ on the polarized vector and EVPA in the Kerr–MOG black hole spacetime (3) for the equatorial magnetic field. Here $r_s = 6$, $a = -0.3$, $\theta_o = 20^\circ$, $\beta_s = 0.3$, and $\chi = -90^\circ$.
photon propagation must be solved first. Using the Hamilton–Jacobi approach, the null geodesic equation in Kerr–MOG black hole spacetime can be expressed as

\[
\frac{\rho^2}{E} p^\rho = \frac{\Delta}{r^2 + a^2 - a \lambda^2 - a \lambda + a(\lambda - a \sin^2 \theta)},
\]
\[
\frac{\rho^2}{E} p^\theta = \frac{a}{\Delta} (r^2 + a^2 - a \lambda) + \frac{\lambda}{\sin^2 \theta} - a,
\]
\[
\frac{\rho^2}{E} p^r = \pm \rho \sqrt{\mathcal{R}(r)},
\]
\[
\frac{\rho^2}{E} p^\phi = \pm \rho \sqrt{\Theta(\theta)}. \tag{14}
\]

The conserved quantities \( \lambda \) and \( \eta \) correspond to the energy-rescaled angular momentum parallel to the axis of symmetry and Carter constant, respectively. The radial potential \( \mathcal{R}(r) \) and the angular potential \( \Theta(\theta) \) can be expressed as

\[
\mathcal{R}(r) = (r^2 + a^2 - a \lambda^2 - \Delta \eta + (a - \lambda)^2),
\]
\[
\Theta(\theta) = \eta + a^2 \cos \theta^2 - \lambda^2 \cot \theta^2. \tag{15}
\]

The photon trajectory is determined by its initial position, the conserved quantities \( \lambda, \eta \), and the signs \( \pm r, \pm \phi \) of its initial motion. The photon’s four-momentum can be given by

\[
p_\mu dx^\mu = -dt \pm r \frac{\sqrt{\mathcal{R}(r)}}{\Delta(r)} dr \pm \eta \sqrt{\Theta(\theta)} d\theta + \lambda d\phi. \tag{16}
\]

In the Kerr–MOG black hole spacetime (3), the celestial coordinates \((x, y)\) for the photon’s arrival position on the observer’s screen are

\[
x = -\frac{\lambda}{\sin \theta_o}, \quad y = \pm \rho \sqrt{\Theta(\theta)}, \tag{17}
\]

where angle \( \theta_o \) is the observer’s polar inclination from the normal direction of the accretion disk and \( \pm \rho \) is the sign of \( \rho^\phi \).

As the photon emitted from the initial position \((r_s, \theta_s)\) moves along the null geodesic to the observer at position \((r_o, \theta_o)\), its trajectory is given by the null geodesic Equation (14) (Gralla & Lupsasca 2020; Gelles et al. 2021)

\[
I_r \equiv \int_{r_s}^{r_o} \frac{dr}{r \pm \rho \sqrt{\mathcal{R}(r)}} = \int_{\theta_s}^{\theta_o} \frac{d\theta}{\pm \rho \sqrt{\Theta(\theta)}} \equiv G_{r_0}. \tag{18}
\]

Here, the slash denotes that the sign of \( \pm \) is \( \text{sign}(\rho^\phi) \) or \( \pm \rho \text{sign}(\rho^\phi) \) in the path integral changes at a radial or angular turning point. For a photon’s trajectory with \( n \) turning
points in $\theta$ and $\theta_i = \frac{\pi}{2}$, the null geodesic equation can be simplified as (Gralla & Lupsasca 2020; Gelles et al. 2021)

$$\sqrt{-u_{a}a^{2}L} + \text{sign}(\beta)F_{\omega} = 2mK\left(\frac{u_{+}}{u_{-}}\right). \quad (19)$$

with

$$F_{\omega} = F\left(\arcsin\frac{\cos\theta_i}{\sqrt{u_{+}}}, \frac{u_{+}}{u_{-}}\right), \quad u_{\pm} = \Delta_{\beta} \pm \frac{\Delta^{2}_{\beta} + \eta^{2}}{a^{2}},$$

$$\Delta_{\beta} = \frac{1}{2}\left(1 - \frac{\eta^{2} + \lambda^{2}}{a^{2}}\right), \quad \Delta = \frac{1}{2}\left(1 - \frac{\eta^{2} + \lambda^{2}}{a^{2}}\right), \quad (20)$$

where $F$ and $K$ denote the first-kind incomplete and complete elliptic integrals, respectively. Actually, from the geodesic Equation (19), an inversion formula for the emission radius $r_s(L_s)$ can be obtained (Gralla & Lupsasca 2020)

$$r_s(L_s) = \frac{r_3 r_4 - r_3 r_4 \sin^{2}\left(\frac{1}{2}\sqrt{r_3 r_4} L - F_{\omega}k\right)}{r_3 - r_4 \sin^{2}\left(\frac{1}{2}\sqrt{r_3 r_4} L - F_{\omega}k\right)}, \quad (21)$$

where

$$F_{\omega} = F\left(\arcsin\sqrt{\frac{r_{31}}{r_{41}}}, k\right), \quad k = \frac{r_{32} r_{41}}{r_{31} r_{42}},$$

$$r_{ij} = r_{i} - r_{j}. \quad (22)$$

Here $r_1$, $r_2$, $r_3$, $r_4$ are the four roots of radial potential $R(r)$ and $sn(q|k)$ is Jacobi elliptic function. In the Kerr–MOG black hole spacetime (3), the four roots $r_1$, $r_2$, $r_3$, $r_4$ can also expressed as

$$r_{1,2} = -z \pm \sqrt{-\frac{A}{2} - z^2 + \frac{B}{4z}},$$

$$r_{3,4} = z \pm \sqrt{-\frac{A}{2} - z^2 + \frac{B}{4z}}, \quad (23)$$

where

$$z = \sqrt{\frac{\omega_{\pm} + \omega_{-} - \frac{A}{6}}{2} > 0},$$

$$\omega_{\pm} = \frac{\sqrt{Q}}{2} \pm \sqrt{\left(\frac{\beta}{3}\right)^{3} + \left(\frac{Q}{2}\right)},$$

$$P = -\frac{A^{3}}{12} - C, \quad Q = -\frac{A}{3}\left[\left(\frac{A}{6}\right)^{2} - C\right] - \frac{B^{2}}{8}, \quad (24)$$

with

$$A = a^{2} - \eta - \lambda^{2},$$

$$B = 2M \left[\eta + (\lambda - a)^{2} + \alpha[\eta + (\lambda - a)^{2}]\right],$$

$$C = -a^{2}\eta - \frac{\alpha M^{2}}{1 + \alpha}[\eta + (\lambda - a)^{2}]. \quad (25)$$

Combining Equations (19) and (21), one can numerically compute the set of celestial coordinates $(x,y)$ for a photon emitted on an equatorial ring with a radius $r_s$. The photon four-momentum at the source $(r_s, \theta_i = \frac{\pi}{2})$ can be expressed as

$$p_{t} = -1, \quad p_{r} = \pm_{r} \sqrt{R(r_s)},$$

$$p_{\theta} = \pm_{r} \sqrt{\eta}, \quad p_{\phi} = \lambda, \quad (26)$$

Figure 5. Effects of the MOG parameter $\alpha$ on the polarized vector and EVPA in the Kerr–MOG black hole spacetime (3) for the different observer inclination angle $\theta_o$. Here $r_s = 6$, $a = -0.3$, $\beta = 0.3$, $\chi = -90^\circ$, $B_r = 0.87$, $B_\phi = 0.5$, and $B_\theta = 0.$
where the sign of $p^\theta$ at the source $\pm_s = (-1)^{m_s} \pm_r$. The sign of $\pm_r$ can be computed by a semianalytic calculation (Gralla & Lupsasca 2020). With the determined sign of $p^\theta$, the four-momentum $p^\mu$ of photon at the source can be calculated by

$$
p^\mu = \frac{1}{r_s} \left[ \frac{r_s^2 + a^2}{\Delta_s} (r_s^2 + a^2 - a \lambda) + a(\lambda - a) \right].
$$

Converting the four-momentum $p^\mu$ to the local frame of the emitter by Equation (11), the local photon polarization at the source can be obtained. In the local frame, one has $f^\tau = 0$. For the synchrotron radiation, the spatial components of the photon polarization vector at the source $f = (f^\tau, f^\rho, f^\phi)$ are related to the local three-momentum $p = (p^\tau, p^\rho, p^\phi)$ and the local magnetic field $B = (B^\tau, B^\rho, B^\phi)$ by

$$
f = \frac{p \times B}{|p|}.
$$

The photon polarization vector at the source $f^\mu$ in Boyer–Linquist coordinates can be computed through the Equation (13). With the angle $\zeta$ between $p$ the magnetic field $B$

$$
\sin \zeta = \frac{|p \times B|}{|p| |B|},
$$

the normalized polarization vector satisfies

$$
f^\mu f_\mu = \sin \zeta^2 |B|^2.
$$

In the propagation of photons in the Kerr–MOG black hole spacetime (3), the polarization vector $f^\mu$ obeys

$$
f^\mu p_\mu = 0, \quad p^\mu \nabla_\mu f^\nu = 0.
$$

According to the Walker–Penrose theorem (Walker & Penrose 1970; Chandrasekhar 1983), along null geodesic in the Kerr–MOG geometry (3), there is a conserved complex quantity

$$
\kappa = p^\mu f^\nu \left( l_i n_j - j_i n_j - m_i \bar{m}_j + \bar{m}_i m_j \right) \Psi^{\frac{1}{2}}.
$$

Figure 6. Effects of the MOG parameter $\alpha$ on the polarized vector and EVPA in the Kerr–MOG black hole spacetime (3) for different $a$ in the case with the magnetic field owned only the vertical component $B_\theta$. Here $r_s = 6$, $\theta_o = 20^\circ$, $\chi = -90^\circ$, $j_\rho = 0.3$, and $B_\theta = 1$. 

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with \[
\Psi_2 = \frac{\kappa}{(r - ia \cos \theta)^2} \left[ 1 - \frac{\alpha \kappa}{(1 + \alpha)(r + ia \cos \theta)} \right].
\]

Making use of the celestial coordinates \((x, y)\) and the Walker–Penrose constant \(\kappa\) at the source \(r_s, \theta_s = \frac{\pi}{2}\), the polarization vector on the observer’s screen can be obtained by Chandrasekhar (1983) and Himwich et al. (2020)

\[
f^x = \frac{\gamma \kappa_2 - \mu \kappa_1}{\mu^2 + y^2}, \quad f^y = \frac{\gamma \kappa_1 + \mu \kappa_2}{\mu^2 + y^2}, \quad \mu = -(x + a \sin \theta_0).
\]

In general, the intensity of linear polarization for synchrotron radiation that reaches the observer from the source position can be approximated as (Gelles et al. 2021; Narayan et al. 2021)

\[
|I| = g^{3 + \alpha_\nu} l_p |B|^{|1 + \alpha_\nu} \sin \zeta^1 + \alpha_\nu,
\]

where \(g\) is the redshift factor measured by the ratio of the photon energies at the observer \(E_o = 1\) and at the emitter \(E_s = p^\lambda = 1\). The power \(\alpha_\nu\) depends on the properties of the accretion disk. Here, we set \(\alpha_\nu = 1\) as in Gelles et al. (2021) and Narayan et al. (2021). The quantity \(l_p = \frac{\rho^\nu}{\rho^\nu_0} H\) is the geodesic path length through the emitting material. \(H\) is the height of the disk, which can be taken to be a constant for simplicity. Thus, the observed components of photon polarization vector are

\[
f^{x}_{\text{obs}} = \sqrt{\rho^\nu} g^2 |B| \sin \zeta^x, \quad f^{y}_{\text{obs}} = \sqrt{\rho^\nu} g^2 |B| \sin \zeta^y.
\]

Finally, the total polarization intensity and the electric vector position angle (EVPA) can be given by

\[
I = (f^{x}_{\text{obs}})^2 + (f^{y}_{\text{obs}})^2, \quad \text{EVPA} = \frac{1}{2} \arctan \frac{U}{Q},
\]

where \(Q \text{ and } U\) are the Stokes parameters

\[
Q = (f^{x}_{\text{obs}})^2 - (f^{y}_{\text{obs}})^2, \quad U = -2f^{x}_{\text{obs}} f^{y}_{\text{obs}}.
\]
Figure 8. Effects of the MOG parameter $\alpha$ on the polarized vector and EVPA in the Kerr–MOG black hole spacetime (3) for different $\theta_o$ in the case where magnetic field has only the vertical component $B_\theta$. Here $r_s = 6$, $a = -0.3$, $\beta_\nu = 0.3$, $\chi = -90^\circ$, and $B_\theta = 1$.

Figure 9. Effects of the MOG parameter $\alpha$ on the $Q$–$U$ diagram for different equatorial magnetic fields in the Kerr–MOG black hole (3). Here $r_s = 6$, $\theta_o = 20^\circ$, $a = -0.3$, $\beta_\nu = 0.3$, and $\chi = -90^\circ$. The blue dashed line and the red solid line correspond to the cases with the MOG parameter $\alpha = 9$ and $\alpha = 0$, respectively. Black crosshairs indicate the origin of each plot.
For the Kerr–MOG black hole spacetime (3), combining photon geodesic with Equations (32), (36), (37), (38), and (39), the polarization intensity and EVPA in the pixel related to the point source can be obtained. Repeating similar operations along the emitting ring, the total polarization image of the emitting ring around a Kerr–MOG black hole and the corresponding effects of a MOG parameter can be presented.

3. Effects of the MOG Parameter on the Polarized Image of an Equatorial Emitting Ring around a Kerr–MOG Black Hole

Figures 1–8 present the polarization vector distribution in the image of the emitting ring with radius $r_s = 6$ around a Kerr–MOG black hole. Results show that the polarization vector distribution in the image depends on not only the magnetic field configuration, the motion of fluid particle, and the observer’s inclination angle, but also on the spin parameter and the MOG parameter $\alpha$ of black hole.

Figures 1–5 present the change of the polarized vector and EVPA with the MOG parameter $\alpha$ in the case that the magnetic field lies in the equatorial plane for the fixed parameters $r_s = 6$, $\theta_o = 20^\circ$, $\beta_v = 0.3$, and $\chi = -90^\circ$. For the case with only radial magnetic field, Figure 1 shows that the polarization intensity and the EVPA increase with the MOG parameter $\alpha$. The quantity $\Delta EVPA \equiv EVPA - EVPA_K$, described the EVPA difference between the Kerr–MOG and Kerr spacetimes, increases with the parameter $\alpha$. In the cases with different spin parameters, the change of the polarization image feature with $\alpha$ is qualitatively similar in the Kerr–MOG spacetime. For the case with only angular magnetic field, Figure 2 indicates that the polarization intensity increases with the MOG parameter $\alpha$. However, EVPA increases with $\alpha$ as the azimuthal coordinate $\phi$ lies in the region $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$, but decreases with $\alpha$ as the azimuthal coordinate $\phi \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. For the equatorial magnetic field with nonzero radial and angular components, from Figure 3, the polarization intensity still increases with $\alpha$. The change of EVPA becomes more complicated. With the increasing of the ratio $B_r/B_\phi$, the region where EVPA increases with $\alpha$ becomes broad so that EVPA finally becomes an increasing function of $\alpha$.

Figure 4 presents the effects of MOG parameter $\alpha$ on the polarized images for the different fluid direction angle $\chi$. For the different $\chi$, the polarization intensity still increases with $\alpha$. However, as the angle $\chi$ changes from $-120^\circ$ to $\chi = -180^\circ$, the region where EVPA increases with $\alpha$ becomes narrow and then EVPA finally becomes a decreasing function of $\alpha$. The effects of the MOG parameter $\alpha$ on the polarized vector and EVPA for the different observer inclination angle $\theta_o$ are shown.
in Figure 5. It is shown that with the increase of the observer inclination angle $\theta_o$, the region where polarization intensity and EVPA increases with $\alpha$ becomes narrow.

Figures 6–8 also present the dependence of the polarization intensity and the EVPA on the MOG parameter $\alpha$ in the case where the magnetic field is perpendicular to the equatorial plane. Results show that the effects of MOG parameter $\alpha$ on the polarized vector and EVPA also depend on the azimuthal coordinate $\phi$ of the point in the emitting ring, the black hole spin parameter, the fluid direction angle, and the observer inclination angle. The dependence of polarization intensity and EVPA with the MOG parameter $\alpha$ varies periodically with the azimuthal coordinate $\phi$. For the fixed $\theta_o = 20^\circ$ and $\chi = -90^\circ$, as $\phi$ varies in the range $(0, 2\pi)$, the polarization intensity and EVPA first decrease with $\alpha$, then increase, and finally decrease once again. Similarly, for the different spin parameter $a$, the
dependence of polarization intensity and EVPA on $\alpha$ is also qualitatively similar in this case. With the increase of $\chi$, the region of the polarization intensity increasing with $\alpha$ increases, but the region for the EVPA increasing with $\alpha$ decreases. As the observer inclination angle $\theta_o$ increases, the polarization intensity gradually becomes a monotonically increasing function of $\alpha$, but the change tendency of EVPA with $\alpha$ is similar to that in the case of the low inclination angle $\theta_o$. Moreover, the region where EVPA increases with $\alpha$ becomes narrow in the high inclination angle case.

Figures 9–11 show the effects of the MOG parameter $\alpha$ on Stokes $Q - U$ loop patterns in the image of the emitting ring. In general, there are two loops enclosing the origin in the $Q - U$ plane. Effects of the MOG parameter $\alpha$ on the $Q - U$ diagram depend heavily on the magnetic field configuration, the fluid velocity, the observation inclination angle, and the spin parameter of black hole. As the magnetic field lies in the equatorial plane, the observed two Stokes $Q - U$ loops increase with the MOG parameter $\alpha$ for the observer with $\theta_o = 20^\circ$ and different fluid direction angle $\chi$. For the high observation inclination angle, the inner loop vanishes and the change of loop size with the MOG parameter $\alpha$ becomes more complicated. For the fixed $\alpha$, as the spin parameter increases, Figure 12 shows that the inner loop increases for different magnetic fields. However, the outer loop decreases with $a$ as the magnetic field lies in the equatorial plane, and it no longer changes monotonously with $a$ in the case of the vertical magnetic field. Moreover, for the fixed $\alpha$, the changes of the $Q - U$ loop patterns with the magnetic field configuration, the fluid velocity, and the observation inclination angle are similar to those in the Kerr black hole case.

Finally, Figure 13 gives a comparison of the polarization image between a Kerr–MOG black hole and the black hole M87* (Akiyama et al. 2021a, 2021b). Here $\theta = 17^\circ$, $\beta = 0.4$, $\chi = -150^\circ$, $B_r = 0.87$, $B_\theta = 0.5$, and $B_\phi = 0$.

![Figure 13](attachment:image.png)

**Figure 13.** Comparison between the polarimetric image of Kerr–MOG black hole and the black hole M87* (Akiyama et al. 2021a, 2021b). Here $\theta = 17^\circ$, $\beta = 0.4$, $\chi = -150^\circ$, $B_r = 0.87$, $B_\theta = 0.5$, and $B_\phi = 0$. The small difference in the polarization images in Figure 13 implies that there remains some possibility for the STVG–MOG theory.
4. Summary

This study investigated the polarized images of the emission ring around a Kerr–MOG black hole with an additional dimensionless parameter $\alpha$. The results show that for the fixed MOG parameter $\alpha$, the change of the polarization vector in the image plane with the spin parameter, the magnetic field configuration, the fluid velocity, and the observation inclination angle are similar to those in the Kerr black hole case. The effects of the MOG parameter $\alpha$ on the observed polarization vector depend on the black hole parameters, the material distribution around the black hole, and the observation inclination angle. For the cases where the magnetic field lies in the equatorial plane, the polarization intensity increases with the MOG parameter $\alpha$ in the lower observation inclination angle case, and no longer varies monotonously in the higher observation inclination angle case. The change of EVPA with $\alpha$ becomes more complicated and also depends on the ratio between the radial component and the azimuthal component of magnetic field $B_R/B_\phi$. As the magnetic field is perpendicular to the equatorial plane, the dependence of polarization intensity and EVPA with the MOG parameter $\alpha$ varies periodically with the azimuthal coordinate $\phi$ in the lower observation inclination angle case. As the observer inclination angle $\theta_\circ$ increases, the polarization intensity gradually becomes a monotonically increasing function of $\alpha$, but the change tendency of EVPA with $\alpha$ is similar to that in the case of the low inclination angle $\theta_\circ$ and the region where EVPA increases with $\alpha$ becomes more narrow.

Effects of the MOG parameter $\alpha$ on the $Q - U$ diagram also depend heavily on the magnetic field configuration, the fluid velocity, the observation inclination angle, and the spin parameter of the black hole. In the case with the lower observed inclination, the size of the $Q - U$ loop increases with $\alpha$ as the magnetic field lies in the equatorial plane, and the size of the outer loop increases and the inner loop decreases as the magnetic field is vertical to the equatorial plane. In the higher inclination angle case, the sizes of two loops increase with $\alpha$ although the inner loop dramatically shrinks as the magnetic field lies in the equatorial plane; however, as the magnetic field is vertical to the equatorial plane, the inner loop vanishes and the change of loop size with the MOG parameter $\alpha$ becomes more complicated. Moreover, for the fixed $\alpha$, as the spin parameter increases, the inner loop increases for different magnetic fields. However, the outer loop decreases with $\alpha$ as the magnetic field lies in the equatorial plane, and it no longer changes monotonously with $\alpha$ in the case of the vertical magnetic field. Finally, making a comparison of the polarization images between the Kerr–MOG black hole and M87*, some similar spiral structures appeared in the polarimetric images of two black holes. Our result also implies that there remains some possibility for the STVG–MOG theory.

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