Multi-object Tracking in Unknown Detection Probability with the PMBM Filter
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Abstract—This paper focuses on the joint multi-object tracking (MOT) and the estimate of detection probability with the Poisson multi-Bernoulli mixture (PMBM) filter. In a majority of multi-object scenarios, the knowledge of detection probability is usually uncertain, which is often estimated offline from the training data. In such cases, online filtering is not allowed or believable, otherwise, significant parameter mismatch will result in biased estimates (state and cardinality of objects). Consequently, the ability of adaptively estimating the detection probability is essential in practice. In this paper, we detail how the detection probability can be estimated accompanied with the state estimates. Besides, closed-form solutions to the proposed method are derived by approximating the intensity of Poisson random finite set (RFS) to a Beta-Gaussian mixture form and density of Bernoulli RFS to a single Beta-Gaussian form. Simulation results show the effectiveness and superiority of the proposed method.

Index Terms—Detection probability, Poisson multi-Bernoulli mixture, Beta-Gaussian mixture.

I. INTRODUCTION

Multi-object tracking (MOT) has been an increasingly hot topic both in military and civilian areas in the last few years. The aim of the MOT is to jointly estimate the state and cardinality of objects synchronously from the monitored scenario. So far, MOT has been widely adopted in many fields, such as environmental monitoring, battlefield surveillance and distributed sensor network [1]–[5].

A common difficulty for MOT is the association problem between objects and observations. Amongst currently studied algorithm, Joint Probabilistic Data Association (JPDA) [1], Multiple Hypotheses Tracking (MHT) [2], and Random Finite Set (RFS) [6], [7] are the main schemes for MOT. Recently, RFS approach receives a high degree of attention because of the effective solutions of the association problem. Under the RFS framework, some filters based on the RFS theory are developed mainly including unlabeled and labeled filters. The unlabeled RFS-based filters mainly consist of probability hypothesis density (PHD) [7], Cardinality-PHD (CPHD) [8], multi-Bernoulli (MB) [9], [10] and the recently developed Poisson MB mixture (PMBM) [11]–[13] filters, while the labeled RFS-based filters include labeled MB (LMB) [14], generalized LMB (GLMB) [15], [16], labeled MBM (LMBM) [17], and marginalized δ-GLMB (Mδ-GLMB) [18] filters. Moreover, Gaussian mixture (GM) [19], [20] and sequential Monte Carlo (SMC) [21], [22] are the main implementation tools for these filters.

Comparing to the other unlabeled RFS-based filters (PHD, CPHD, and MB), a unique and important characteristic of the PMBM filter [12], [13] is the conjugacy property like the labeled RFS-based GLMB [15] and LMBM [17] filters, which means that the posterior distribution has the same functional form as the prior. The reason why conjugacy property is important is that it allows the posterior to be written in terms of some single-object predictions and updates, which provides a convenient computation method compared with the multi-object predictions and updates. Moreover, the PMBM filter also shows advantages in low-detection scenarios [12], [13], [23], [24]. As a result, the PMBM filter receives a lot of attention and has been increasingly adopted in many applications [25]–[27].

In the actual MOT environment, there is a significant source of certainly, detection model, in addition to the dynamic model whereas the detection model is usually assumed to be known by the offline estimate from the training data in most algorithms. In such cases, online filtering process is not feasible, otherwise, significant mismatch in detection model will cause erroneous estimates of state and cardinality of objects. In order to make the filters more adaptable to the environment, Mahler et al. have proposed a new CPHD filter by online estimating the unknown detection probability [28] in which object state is augmented with a parameter of detection model and the augmented state model is propagated and estimated along with CPHD recursion. The application of the proposed CPHD filter in [28] has been applied to track cell microscopy data with unknown background parameters [29]. Afterwards, a similar strategy has also been applied to the MB filter [30] and GLMB filter [31]. Both of them show the effectiveness of the strategy in [28]. Moreover, some other method by exploiting the Inverse Gamma Gaussian mixture (IGGM) distribution to implement the PHD/CPHD filters is also proposed in [32], and a method for multistatic Doppler radar with unknown detection probability based on the GLMB filter is proposed in [33]. To the best of our knowledge, the research on the PMBM filter with unknown detection probability hasn’t been realized yet.

Considering the attractive characteristics of the PMBM filter such as conjugacy property and low detection tolerance, we explore a PMBM recursion with unknown detection probability which can jointly estimate the state of object and detection probability. The main contributions of the paper are described as follows: For the scenarios with unknown detection probability, we propose an effective method immune to the mismatch of detection model by online estimating the detection probability, and also provide the analytic implementation by introducing a Beta function to model the
detection probability. Two groups of simulation experiments with different detection probabilities are given to verify the effectiveness of the proposed method.

The outline of the rest of the paper is as follows. Section II introduces the background knowledge, and Section III describes the PMBM filter with unknown detection probability and its detailed implementation. Simulation results are provided in Section IV, and the conclusions are drawn in Section V.

II. BACKGROUNDD

In this section, some notations are introduced. Moreover, the review of multi-object Bayes filter, the PMBM RFS and conventional PMBM filter is provided.

A. Notations

In this paper, lower case letters (e.g., \(x\) and \(z\)) denote state and observation of single-object while upper case letters (e.g., \(X\) and \(Z\)) denote states and observations of multi-object, respectively. Suppose there are \(N\) objects and \(M\) observations at time \(k\), then the multi-object state and multi-object observation can be represented as

\[
X_k = \{x_{k,1}, \ldots , x_{k,N}\} \subset \mathcal{X}
\]

\[
Z_k = \{z_{k,1}, \ldots , z_{k,M}\} \subset \mathcal{Z}
\]

where \(\mathcal{X}\) and \(\mathcal{Z}\) denote the state space and observation space respectively. Each single-object state \(x_{k,i} = [x_{k,i}^T, x_{k,v}^T]^{\top}\) comprises the position \(x_{k,p}^T\) and velocity \(x_{k,v}^T\), where \(\cdot^{\top}\) denotes the transpose.

B. Multi-object Bayes filter

Given multi-object transition function \(f_{k|k-1}(\cdot|\cdot)\) and multi-object likelihood function \(g_k(\cdot)\), the prediction and update steps of the multi-object Bayes filter can be given by

\[
f_{k|k-1}(X) = \int f_{k|k-1}(X|\xi) f_{k-1}(\xi) d\xi
\]

\[
f_k(X) = \frac{g_k(Z_k|X)}{\int g_k(Z_k|X) f_{k|k-1}(X) d\xi}
\]

where the involved integral is the set integral which is defined in [3].

C. PMBM RFS

Conditioned on the observation set \(Z_{1:k} = (Z_1, \ldots , Z_k)\), the multi-object state RFS \(X_k\) is modeled as the union of independent RFS \(X_{k1}^u\) (undetected objects) and \(X_{k1}^d\) (potentially detected object), respectively. Hence, the posterior density of the PMBM RFS can be denoted by the finite set statistics (FISST) convolution as

\[
f_k(X) = \sum_{Y \subseteq X} f_k^P(Y) f_{k|k}(X|Y).
\]

\(^1\)Undetected objects: exist at the current time but have never been detected.

\(^2\)Potentially detected object: a new observation may be a new object for the first detection and can also correspond to another previously detected object or clutter. Considering that it may exist or not, we refer to it as potentially detected object.

\(f_k^P(\cdot)\) is a Poisson density which is given by

\[
f_k^P(X) = e^\lambda \prod_{i=1}^n \lambda f_k(x_i)
\]

\[
= e^{-\int \mu_k(x) dx} \left[\mu_k(\cdot)\right]^X
\]

where \(\mu_k(x) = \lambda f_k(x)\) is the intensity function and \(\lambda\) the Poisson rate as well as \(f_k(\cdot)\) a probability density function (pdf) of a single object. Moreover, \(f_{k|k}(\cdot)\) is a PMBM which is given by

\[
f_{k|k}^{mbm}(X) \propto \sum_{j \in X_{1:k-1}^u} \prod_{i=1}^n \omega_{j,k,f_{j,i}}(X_i)
\]

when there is only one global hypothesis with \(|\mathcal{I}| = 1\).

D. PMBM recursion

Here, the review of the recursive process of the PMBM filter is given.

1) Prediction Process: Poisson density \(f_k^P(\cdot)\) and the multi-Bernoulli mixture density \(f_{k|k-1}^{mbm}(\cdot)\) are predicted separately.

Suppose the intensity function of Poisson density at time \(k - 1\) is \(\mu_{k|k-1}(x)\), then the predicted intensity at time \(k\) is

\[
\mu_{k|k-1}(x) = \gamma_k(x) + \int f_{k|k-1}(x|\xi) p_{S,k}(\xi) \mu_{k-1}(\xi) d\xi
\]

where \(\gamma_k(x)\) is the intensity of birth model at time \(k\) and \(f_{k|k-1}(x|\xi)\) and \(p_{S,k}(\cdot)\) denote the state transition function of single object and survival probability, respectively.

Given the \(i\)-th object in \(j\)-th global hypothesis at time \(k - 1\) with \(\omega_{j,k}^{i,1}, \tau_{j,k}^{i,1}, p_{k-1}^{i,1}(x)\), then the prediction process of MB components is given by

\[
\omega_{k|k-1}^{i,1} = \omega_{k-1}^{i,1}
\]

\[
\tau_{k|k-1}^{i,1} = \int p_{k-1}^{i,1}(\xi) p_{S,k}(\xi) d\xi
\]

\[
p_{k|k-1}^{i,1}(x) \propto \int f_{k|k-1}(x|\xi) p_{S,k}(\xi) p_{k-1}^{i,1}(\xi) d\xi
\]

where \(\omega_{k|k-1}^{i,1}, \tau_{k|k-1}^{i,1}, p_{k|k-1}^{i,1}(x)\) denote the predicted hypothesis weight, existence probability, and pdf of the \(i\)-th Bernoulli component in the \(j\)-th global hypothesis, respectively.

2) Update Process: The update process mainly consists of the following parts:

- update for undetected objects;
- update for potential objects detected for the first time;
- misdetection for previously potentially detected objects;
- update for previously potentially detected objects using received observation set.
The detailed descriptions are discussed below.

1) Update for undetected objects:
\[ \mu_k(x) = (1 - p_{D,k}(x)) \mu_{k|k-1}(x) \]  
where \( p_{D,k}(\cdot) \) is the detection probability.

2) Update for potential objects detected for the first time:
\[ \begin{align*}
    r^P_k(z) &= e_k(z) / \rho^P_k(z) \\
    p^P_k(x|z) &= p_{D,k}(x) g_k(z|x) \mu_{k|k-1}(x) / e_k(z)
\end{align*} \]

3) Misdetection for previous potentially detected objects:
\[ \begin{align*}
    \omega^M_{j,i}(\emptyset) &= \omega^M_{j,i-1}(1 - p_{D,k}(x)) r^M_{j,i} \\
    r^M_{j,i} &= r^M_{j,i-1}(1 - p_{D,k}(x)) / (1 - p_{D,k}(x) r^M_{j,i}) \\
    \rho^M_k(z) &= \rho^M_{j,i-1}(x).
\end{align*} \]

4) Update for previous potentially detected objects using received observation set:
\[ \begin{align*}
    \omega^{P}_{j,i}(z) &= \omega^{P}_{j,i-1} r^{P}_{j,i-1} \\
    r^{P}_{j,i}(z,x) &= p_{D,k}(x) g_k(z|x) \rho^{P}_{j,i-1}(x) \\
    \rho^{P}_k(z) &= \rho^{P}_{j,i}(x,z) \propto p_{D,k}(x) g_k(z|x) \rho^{P}_{j,i-1}(x).
\end{align*} \]

It can be seen that the update process is also separate where the undetected objects are just preserved by multiplying the weight by a misdetection probability shown in (14) of part 1 and the potential targets are updated from three parts (parts 2-4)).

Another factor is the generation of global hypothesis. In theory, the global hypothesis should go through all possible data association based on all single-object hypotheses. Because of the bottleneck of computation, Murty’s algorithm \[44\] is considered in which a cost matrix is constructed by the calculated weight in (13, 16) and (22). The detailed implementation steps can be referred to \[13\].

III. PMBM FILTER WITH UNKNOWN DETECTION PROBABILITY

In this section, the joint estimates of state of objects and detection probability are provided. First, the basic construction method is introduced by augmenting the unknown detection probability to state of object. Hereafter, the PMBM recursion for the augmented state model is derived. The analytic implementation based on Beta Gaussian mixture is finally given.

A. Augmented state model
Following the approach in \[28\], a variable \( a \in [0,1] \) is augmented to the state,
\[ \hat{x} = (x,a) \]  
The integral of the augmented state \( \hat{x} \) is adjusted into a double integral,
\[ \int f(\hat{x})d\hat{x} = \int \int f(x,a)dadx \]

Meanwhile, the state transition and observation models are the same as the conventional case, except that we focus on the augmented state model, which are given by
\[ f_{k|k-1}(\hat{x}|\hat{z}) = f_{k|k-1}(x,a|\zeta,\alpha) \]
\[ g_k(z|x) = g_k(z|x,a) = g_k(z|x, a) \]
\[ p_{S,k}(\hat{x}) = p_{S,k}(x,a) \]
\[ p_{D,k}(\hat{x}) = p_{D,k}(x, a) \]

Furthermore, the birth model with augmented model is denoted as an intensity \( \lambda^k_0(x,a) \).

B. Recursion

The derivation of the PMBM filter recursion for the augmented state model featuring the unknown detection probability is straightforward by substituting the augmented state model into the conventional PMBM recursion. Next, the direct consequences of derivations are given by Propositions 1 and 2. The new recursion has the same complexity of the conventional recursion.

**Proposition 1.** If at time \( k - 1 \), the intensity of Poisson RFS \( \mu_{k-1}(\hat{x}) \) and MB RFS with \( \{ w_{k-1}^{j,i}, r_{k-1}^{j,i}, p_{k-1}^{j,i} \} \) are given, which denote the undetected objects and potential objects respectively, then the intensity of Poisson process and density of MBM process can be given by

\( \begin{align*}
    \mu_{k|k-1}(x,a) &= \gamma_k(x,a) \\
    &+ \int \int f_{k|k-1}(x|\zeta) f_{k|k-1}(a|\alpha) p_{S,k}(\zeta) \mu_{k-1}(\zeta, \alpha) d\zeta d\alpha
\end{align*} \]

\( \begin{align*}
    w_{k|k-1}^{j,i} &= w_{k-1}^{j,i} \\
    r_{k|k-1}^{j,i} &= r_{k-1}^{j,i} \int \int p_{S,k}(\zeta) p_{k-1}^{j,i}(\zeta, \alpha) d\zeta d\alpha
\end{align*} \]

**Proposition 2.** If at time \( k \), the predicted PMBM filter with parameters \( \{ \mu_{k|k-1}(\hat{x}), w_{k|k-1}^{j,i}, r_{k|k-1}^{j,i}, p_{k|k-1}^{j,i} \} \) are given, then for a given observation set \( Z_k \), the updated intensity of Poisson process and density of MBM process can be given by four parts. (a) Update for undetected objects:
\[ \mu_{k|k}(x,a) = (1 - a) \mu_{k|k-1}(x,a) \]
(b) Update for potential objects for the first time:
\[ r_k^j(z) = e_k(z)/\rho_k^j(z) \] (35) 
\[ p_k^j(x, a|z) = ag(z|x)\mu_{k|k-1}(x, a)/e_k(z) \] (36) 

where 
\[ \rho_k^j(z) = e_k(z) + c(z) \] (37) 
\[ e_k(z) = \int_0^1 g(x|\alpha)\mu_{k|k-1}(x, \alpha)d\alpha dx \] (38) 

(c) Misdetection for potentially detected objects:
\[ w_k^{j,i}(\emptyset) = w_k^{j,i} \times \] 
\[ (1 - r_k^{j,i}) + r_k^{j,i} \int_0^1 p_k^{j,i}(x, \alpha)(1 - \alpha)d\alpha dx \] (39) 
\[ r_k^{j,i}(\emptyset) \] 
\[ = \frac{r_k^{j,i}}{1 - r_k^{j,i} + r_k^{j,i} \int_0^1 p_k^{j,i}(x, \alpha)(1 - \alpha)d\alpha dx} \] (40) 
\[ p_k^{j,i}(\emptyset, a) = \frac{p_k^{j,i}}{\int_0^1 p_k^{j,i}(x, \alpha)(1 - \alpha)d\alpha dx} \] (41) 

(d) Update for previous potentially detected objects using received observation set:
\[ w_k^{j,i}(\emptyset) = w_k^{j,i}r_k^{j,i} \] 
\[ \times \int_0^1 ag(z|x)p_k^{j,i}(x, \alpha)d\alpha dx \] (42) 
\[ r_k^{j,i}(\emptyset) = \frac{1}{\int_0^1 ag(z|x)p_k^{j,i}(x, \alpha)d\alpha dx} \] (43) 
\[ p_k^{j,i}(x, a|z) = \frac{ag(z|x)p_k^{j,i}(x, \alpha)}{\int_0^1 ag(z|x)p_k^{j,i}(x, \alpha)d\alpha dx} \] (44) 

After all possible new single-object hypotheses are calculated, another important step is to form the global hypotheses for the next recursion. In order to avoid all possible data hypotheses for each previous global hypothesis, we still adopt the construction strategy based upon the Marty’s algorithm [53]. Assume there are \( n_o \) old tracks in the global hypothesis \( j \) and \( m \) observations \( \{z_1, \ldots, z_M\} \), which indicates that there are \( M \) potential detected objects. Then the cost matrix at time \( k \) can be formed as follows.
\[ C_j = -ln \begin{bmatrix} \omega_{j,1}^{\beta,1} & \omega_{j,1}^{\beta,2} & \cdots & \omega_{j,1}^{\beta,1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \omega_{j,M}^{\beta,1} & \omega_{j,M}^{\beta,2} & \cdots & 0 & \cdots & \omega_{j,M}^{\beta,M} \end{bmatrix} \] (45)
where \( \omega_{j,m}^{\beta,m} (m \in \{1, \ldots, M\}) \) denotes the weight of the \( m \)-th potential detected object given by (37) and \( \omega_{j}^{m,n} \) is the weight after the \( m \)-th observation updates the \( n \)-th old track in the \( j \)-th global hypothesis, which is denoted as
\[ \omega_{j}^{m,n} = w_k^{j,i} r_k^{j,i}(z)/\rho_k^{j,i}(\emptyset) \] (46)

with \( \rho_k^{j,i}(z) \) given by
\[ \rho_k^{j,i}(z) = r_k^{j,i} \int p_{D,k}(\xi)g_k(z|\xi)p_k^{j,i-1}(\xi)d\xi \] (47)
and \( \rho_k^{j,i}(\emptyset) \) given by (39).

C. Beta-Gaussian mixture Implementation

In this section, a closed-form implementation for the proposed PMBM recursion with unknown detection probability is derived based on the Beta-Gaussian mixture. The Gaussian distribution is used to model the state of object same as the conventional PMBM filter while the Beta function is used to model the detection probability. Before the implementation is given, some introductions about the Beta distribution are first provided as follows.

Definition 1. For \( 0 \leq a \leq 1 \), the shape parameters \( s > 1, t > 1 \), are a power function of variable \( a \) and of its reflection \((1-a)\) as follows.
\[ \beta(a; s, t) = \frac{a^{s-1}(1-a)^{t-1}}{\int_0^1 a^{s-1}(1-a)^{t-1}d\alpha} = \frac{a^{s-1}(1-a)^{t-1}}{B(s, t)} \] (48)
with mean \( \mu_{\beta} = \frac{s}{s+t} \) and covariance \( \sigma_{\beta}^2 = \frac{s t}{(s+t)^2(s+t+1)} \).

For the Beta distribution \( \beta(a; s, t) \), some of its properties are summarized which will be used in the following derivations.
\[ (1-a)\beta(a; s, t) = \frac{B(s+1, t+1)}{B(s, t)} \beta(a; s, t+1) \] (49)
\[ a\beta(a; s, t) = \frac{B(s+1, t)}{B(s, t)} \beta(a; s+1, t) \] (50)
\[ \frac{s}{s+t} = \frac{B(s+1, t)}{B(s, t)} \] (51)
\[ \frac{\mu_{\beta}}{s+t} = \frac{B(s+t+1)}{B(s, t)} \] (52)

Moreover, the prediction of the Beta distribution satisfies
\[ \beta(a_{+} s_{+} t_{+}) = \int \beta(a; s, t) f_{+} (a_{+} |a) da \] (53)
with
\[ s_{+} = \left( \frac{\mu_{\beta} + (1 - \mu_{\beta} + \sigma_{\beta}^2)}{\sigma_{\beta}^2} \right) \mu_{\beta_{+}} \]
\[ t_{+} = \left( \frac{\mu_{\beta} + (1 - \mu_{\beta} + \sigma_{\beta}^2)}{\sigma_{\beta}^2} \right) (1 - \mu_{\beta_{+}}) \]
\[ \sigma_{\beta_{+}}^2 = k_{\beta} \sigma_{\beta}^2, k_{\beta} \geq 1 \]

For the considered standard linear Gaussian model, some observations are given as follows.

- Each object follows a linear Gaussian dynamical model, i.e.,
\[ f_k(z|x) = \mathcal{N}(z; F_k - 1, Q_{k-1}) \] (54)
\[ g_k(z|x) = \mathcal{N}(z; H_k x, R_k) \] (55)
where $F_{k-1}$ and $Q_{k-1}$ denote the state transition matrix and process noise covariance, and $H_k$ and $R_k$ are the observation matrix and observation noise covariance, respectively.

- The survival probability for each object is state independent, i.e.,

$$p_{S,k}(x) = p_{S,k}$$

(56)

- The intensity of newborn model is a Beta-Gaussian mixture of the form

$$\gamma_k(x, a) = \sum_{i=1}^{J_k^g} \eta_{r,k}^i \beta(a; s_{r,k}^i, t_{r,k}^i) \mathcal{N}(x; m_{r,k}^i, P_{r,k}^i)$$

(57)

where $J_k^g$, $\eta_{r,k}^i$, $s_{r,k}^i$, $t_{r,k}^i$, $m_{r,k}^i$, $P_{r,k}^i$, $i = 1, \ldots, J_k^g$ are given model parameters.

Then, the analytic solution to the PMBM filter with unknown detection probability can be represented in Propositions 3 and 4.

**Proposition 3.** If at time $k-1$, the intensity of Poisson process $\mu_{k-1}(x)$ is a Beta-Gaussian mixture form

$$\mu_{k-1}(x, a) = \sum_{i=1}^{J_k^p} \eta_{b,k}^{i, \mu} \beta(a; s_{b,k}^{i, \mu}, t_{b,k}^{i, \mu}) \mathcal{N}(x; m_{b,k}^{i, \mu}, P_{b,k}^{i, \mu})$$

(58)

and the density of Bernoulli component is a single Beta-Gaussian form

$$p_{k-1}^{j,i}(x, a) = \beta(a; s_{k-1}^{j,i}, t_{k-1}^{j,i}) \mathcal{N}(x; m_{k-1}^{j,i}, P_{k-1}^{j,i})$$

(59)

then, the predicted intensity of Poisson process and density of MBM process are given by

(a) Poisson process:

$$\mu_{k|k-1}(x, a) = \gamma_k(x, a)$$

$$+ \sum_{i=1}^{J_k^p} \eta_{b,k}^{i, \mu} \beta(a; s_{b,k}^{i, \mu}, t_{b,k}^{i, \mu}) \mathcal{N}(x; m_{b,k}^{i, \mu}, P_{b,k}^{i, \mu})$$

(60)

with

$$s_{b,k|k-1}^{i, \mu} = \left( \frac{\mu_{\beta,k|k-1}^{i, \mu} (1 - \mu_{\beta,k|k-1}^{i, \mu})}{[\sigma_{\beta,k|k-1}^{i, \mu}]^2} - 1 \right) \mu_{\beta,k|k-1}^{i, \mu}$$

(61)

$$t_{b,k|k-1}^{i, \mu} = \left( \frac{\mu_{\beta,k|k-1}^{i, \mu} (1 - \mu_{\beta,k|k-1}^{i, \mu})}{[\sigma_{\beta,k|k-1}^{i, \mu}]^2} - 1 \right) \mu_{\beta,k|k-1}^{i, \mu}$$

(62)

$$\times (1 - \mu_{\beta,k|k-1}^{i, \mu})$$

(63)

$$m_{b,k|k-1}^{i, \mu} = F_{k-1} m_{b,k-1}^{i, \mu}$$

(64)

$$P_{b,k|k-1}^{i, \mu} = Q_{k-1} + F_{k-1} P_{b,k-1}^{i, \mu} F_{k-1}^T$$

(65)

$$\mu_{\beta,k|k-1}^{i, \mu} = \mu_{\beta,k-1}^{i, \mu} = \frac{s_{b,k|k-1}^{i, \mu}}{s_{k-1}^{i, \mu} + t_{k-1}^{i, \mu}}$$

(66)

$$[\sigma_{\beta,k|k-1}^{i, \mu}]^2 = k_{\beta} [\sigma_{\beta,k-1}^{i, \mu}]^2$$

$$= \frac{s_{k-1}^{i, \mu} t_{k-1}^{i, \mu}}{(s_{k-1}^{i, \mu} + t_{k-1}^{i, \mu})^2 (s_{k-1}^{i, \mu} + t_{k-1}^{i, \mu} + 1)}$$

(67)

(b) MBM process:

$$w_{k|k-1}^{j,i} = w_{k-1}^{j,i}$$

(68)

$$r_{k|k-1}^{j,i} = p_{S,k} r_{k-1}^{j,i}$$

(69)

$$p_{k|k-1}^{j,i}(x, a) = \beta(a; s_{k|k-1}^{j,i}, t_{k|k-1}^{j,i}) \mathcal{N}(x; m_{k|k-1}^{j,i}, P_{k|k-1}^{j,i})$$

(70)

with

$$s_{k|k-1}^{j,i} = \left( \frac{\mu_{\beta,k|k-1}^{j,i} (1 - \mu_{\beta,k|k-1}^{j,i})}{[\sigma_{\beta,k|k-1}^{j,i}]^2} - 1 \right) \mu_{\beta,k|k-1}^{j,i}$$

(71)

$$t_{k|k-1}^{j,i} = \left( \frac{\mu_{\beta,k|k-1}^{j,i} (1 - \mu_{\beta,k|k-1}^{j,i})}{[\sigma_{\beta,k|k-1}^{j,i}]^2} - 1 \right) \mu_{\beta,k|k-1}^{j,i}$$

(72)

$$\times (1 - \mu_{\beta,k|k-1}^{j,i})$$

(73)

$$m_{k|k-1}^{j,i} = F_{k-1} m_{k-1}^{j,i} + m_{k-1}^{i, \mu}$$

(74)

$$P_{k|k-1}^{j,i} = Q_{k-1} + F_{k-1} P_{k-1}^{j,i} F_{k-1}^T$$

(75)

$$\mu_{\beta,k|k-1}^{j,i} = \mu_{\beta,k-1}^{j,i} = \frac{s_{k-1}^{i, \mu}}{s_{k-1}^{i, \mu} + t_{k-1}^{i, \mu}}$$

(76)

$$[\sigma_{\beta,k|k-1}^{j,i}]^2 = k_{\beta} [\sigma_{\beta,k-1}^{j,i}]^2$$

$$= \frac{s_{k-1}^{i, \mu} t_{k-1}^{i, \mu}}{(s_{k-1}^{i, \mu} + t_{k-1}^{i, \mu})^2 (s_{k-1}^{i, \mu} + t_{k-1}^{i, \mu} + 1)}$$

(77)

$$[\sigma_{\beta,k|k-1}^{j,i}]^2 = k_{\beta} [\sigma_{\beta,k-1}^{j,i}]^2$$

$$= \frac{s_{k-1}^{i, \mu} t_{k-1}^{i, \mu}}{(s_{k-1}^{i, \mu} + t_{k-1}^{i, \mu})^2 (s_{k-1}^{i, \mu} + t_{k-1}^{i, \mu} + 1)}$$

(78)

**Remark 1.** The proof is straightforward by substituting the Beta-Gaussian mixture form into the prediction equations in Proposition 1. The resultant expressions are also the Beta-Gaussian mixture form where the intensity of Poisson density is a Beta-Gaussian mixture form and the density of Bernoulli component is a single Beta-Gaussian form. The prediction of Gaussian distribution is the same as the conventional prediction while that of Beta distribution is based on the properties (see (49)-(53)) of Beta distribution.

**Proposition 4.** If at time $k$, the predicted intensity of Poisson density $\mu_{k|k-1}(x, a)$ is given by the following Beta-Gaussian mixture form

$$\mu_{k|k-1}(x, a)$$

$$= \sum_{i=1}^{J_k^p} \eta_{b,k|k-1}^{i, \mu} \beta(a; s_{b,k|k-1}^{i, \mu}, t_{b,k|k-1}^{i, \mu}) \mathcal{N}(x; m_{b,k|k-1}^{i, \mu}, P_{b,k|k-1}^{i, \mu})$$

(79)

and the predicted density of Bernoulli component is given by a Beta-Gaussian form

$$p_{k|k-1}^{j,i}(x, a) = \beta(a; s_{k|k-1}^{j,i}, t_{k|k-1}^{j,i}) \mathcal{N}(x; m_{k|k-1}^{j,i}, P_{k|k-1}^{j,i})$$

(80)
Then, given a observation set $Z_k$, the update of Poisson process and MBM process are given by four following parts.

(a) Update for undetected objects:

$$
\mu_k(x, a) = \sum_{i=1}^{J_k} \eta_{i,k}^{a,\mu} \beta(a; s_{i,k}^{a,\mu}, t_{i,k}^{a,\mu}) N(x; m_{i,k}^{a,\mu}, P_{i,k}^{a,\mu})
$$

where

$$
\eta_{i,k}^{a,\mu} = \frac{B(s_{i,k}^{a,\mu}, t_{i,k}^{a,\mu}+1)}{B(s_{i,k}^{a,\mu-1}, t_{i,k}^{a,\mu})}
$$

$$
s_{i,k}^{a,\mu} = s_{i,k}^{a,\mu-1} + 1
$$

$$
t_{i,k}^{a,\mu} = t_{i,k}^{a,\mu} + 1
$$

$$
m_{i,k}^{a,\mu} = m_{i,k}^{a,\mu-1} + K(z - H_k m_{i,k}^{a,\mu-1})
$$

$$
P_{i,k}^{a,\mu} = (I - K H_k) P_{i,k}^{a,\mu-1}
$$

(b) Update for potential objects for the first time:

$$
\rho_k^p(z) = e_k(z) / p_k^p(z)
$$

$$
p_k^p(x, a|z) = \frac{1}{e_k(z)} \sum_{i=1}^{J_k} \eta_{i,k}^{a,\mu} \frac{B(s_{i,k}^{a,\mu}, t_{i,k}^{a,\mu}+1)}{B(s_{i,k}^{a,\mu-1}, t_{i,k}^{a,\mu})} \times \beta(a; s_{i,k}^{a,\mu}, t_{i,k}^{a,\mu}) q(z) N(x; m_{i,k}^{a,\mu}, P_{i,k}^{a,\mu})
$$

where

$$
e_k(z) = \sum_{i=1}^{J_k} \eta_{i,k}^{a,\mu} \frac{s_{i,k}^{a,\mu}}{s_{i,k}^{a,\mu-1} + t_{i,k}^{a,\mu}} q(z)
$$

$$
q(z) = N(z; H_k m_{i,k}^{a,\mu-1}, H_k P_{i,k}^{a,\mu-1} H_k^T + R_k)
$$

$$
m_{i,k}^{a,\mu} = m_{i,k}^{a,\mu-1} + K(z - H_k m_{i,k}^{a,\mu-1})
$$

$$
P_{i,k}^{a,\mu} = (I - K H_k) P_{i,k}^{a,\mu-1}
$$

$$
K = P_{i,k}^{a,\mu} H_k^T (H_k P_{i,k}^{a,\mu} H_k^T + R_k)^{-1}
$$

$$
s_{i,k}^{a,\mu} = s_{i,k}^{a,\mu-1} + 1
$$

$$
t_{i,k}^{a,\mu} = t_{i,k}^{a,\mu} + 1
$$

(c) Misdetection for potentially detected objects:

$$
w_{i,k}^{j,i}(\tilde{\theta}) = w_{i,k}^{j,i} (1 - \frac{r_{i,k}^{j,i}}{s_{i,k}^{a,\mu-1} + t_{i,k}^{a,\mu}})
$$

$$
r_{i,k}^{j,i}(\tilde{\theta}) = \frac{s_{i,k}^{a,\mu-1} + t_{i,k}^{a,\mu}}{s_{i,k}^{a,\mu-1} + t_{i,k}^{a,\mu}}
$$

$$
p_k^p(x, a|z) = \beta(a; s_{i,k}^{a,\mu}, t_{i,k}^{a,\mu}) q(z) N(x; m_{i,k}^{a,\mu}, P_{i,k}^{a,\mu})
$$

(d) Update for previous potentially detected objects using received observation set:

$$
w_{i,k}^{j,i}(z) = \frac{s_{i,k}^{a,\mu-1}}{s_{i,k}^{a,\mu-1} + t_{i,k}^{a,\mu}} q(z)
$$

$$
r_{i,k}^{j,i}(z) = 1
$$

$$
p_k^p(x, a|z) = \beta(a; s_{i,k}^{a,\mu}, t_{i,k}^{a,\mu}) q(z) N(x; m_{i,k}^{a,\mu}, P_{i,k}^{a,\mu})
$$

where

$$
q(z) = N(z; H_k m_{i,k}^{a,\mu-1}, H_k P_{i,k}^{a,\mu-1} H_k^T + R_k)
$$

$$
m_{i,k}^{a,\mu} = m_{i,k}^{a,\mu-1} + K(z - H_k m_{i,k}^{a,\mu-1})
$$

$$
P_{i,k}^{a,\mu} = (I - K H_k) P_{i,k}^{a,\mu-1}
$$

$$
K = P_{i,k}^{a,\mu} H_k^T (H_k P_{i,k}^{a,\mu} H_k^T + R_k)^{-1}
$$

$$
s_{i,k}^{a,\mu} = s_{i,k}^{a,\mu-1} + 1
$$

$$
t_{i,k}^{a,\mu} = t_{i,k}^{a,\mu} + 1
$$

In terms of the update for potential objects for the first time, to make the form of the Bernoulli component consistent, we approximate the Beta-Gaussian mixture to a single Beta-Gaussian form by performing moment matching as follows.

$$
p_k^p(x, a|z) = \beta(a; s_{i,k}^{a,\mu}, t_{i,k}^{a,\mu}) q(z) N(x; m_{i,k}^{a,\mu}, P_{i,k}^{a,\mu})
$$

Furthermore, after each update is finished, component merging is performed by using the Hellinger distance for the Poisson process, meanwhile, component pruning is performed by a predetermined threshold for both Poisson process and MBM process. The detailed approximation technology is can be found in \ref{23}.

D. Estimates

In the process of estimates, the global hypothesis of the MBM process with the highest weight is selected

$$
\tilde{\gamma} = \arg \max_j \prod_i w_{i,k}^{j,i}
$$

Then, those Bernoulli components whose weights are above a pre-set threshold $\Gamma$,

$$
\tilde{\gamma} = \{ i : r_{k,i}^{j,i} > \Gamma \}
$$

are selected as estimated state of objects, and the estimate for the number of objects is $\tilde{N}_k = \sum_i r_{k,i}^{j,i}$. Moreover, the estimate of detection probability can be extracted from the mean of Beta distributions of the selected Bernoulli components,

$$
\tilde{a} = \frac{1}{|\tilde{N}_k|} \sum_i \frac{\tilde{r}_{k,i}^{j,i}}{\tilde{r}_{k,i}^{j,i} + \tilde{s}_{k,i}^{j,i}}
$$

IV. Performance Assessment

In this section, we test the proposed method and compare it with the CPHD filter \ref{23} in terms of the Optimal SubPattern Assignment (OSPA) error \ref{53} with $c = 100m$ and $p = 1$.

Consider a two-dimensional scenario space $4500m \times 4500m$ in which twelve objects move at the nearly constant velocity (NCV) model in the surveillance area. Each object state consists of 2-dimension position and velocity, i.e., $x = [p_x, p_y, v_x, v_y]^T$ and each observation polluted by noise is
a vector of planar position \( z = [x, y]^\top \). Moreover, the parameters of the model are given by

\[
F_k = \begin{bmatrix} I_2 & \Delta t I_2 \\ 0_2 & I_2 \end{bmatrix}, \quad Q_k = \sigma_u^2 \begin{bmatrix} \Delta^2 t^2 I_2 & \Delta^2 t I_2 \\ \Delta^2 t I_2 & \Delta^2 I_2 \end{bmatrix},
\]

\[
H_k = \begin{bmatrix} I_2 & 0_2 \end{bmatrix}, \quad R_k = \sigma_z^2 I_2,
\]

where \( I_n \) and \( 0_n \) denote the \( n \times n \) identity and zero matrices respectively. \( \sigma_u^2 = 5 \text{ms}^{-2} \) and \( \sigma_z^2 = 10 \text{m} \) are the standard deviations of process noise and observation noise. The sampling rate is \( \Delta = 1 \text{s} \). The probability of survival for each object is \( p_{S,k} = 0.90 \). In addition, the monitored time is \( T = 80 \text{s} \).

The threshold of Poisson component pruning is \( T_P = 10^{-5} \) and that of Bernoulli component pruning is \( T_B = 10^{-5} \). The parameters setting of the CPHD filter is the same as those in [28]. Moreover, the threshold when extracting object state is set to \( \Gamma = 0.8 \).

The birth model is a Beta-Gaussian mixtures form with twelve Beta-Gaussian components:

\[
\gamma_k(x, a) = \sum_{i=1}^{12} \eta_b \beta(a; s_b, t_b) \mathcal{N}(x; m_{\gamma,k}^i, P_b)
\] (92)

All Beta-Gaussian components share the same probability of existence of \( \eta_b = 0.05 \) and same parameters of Beta distribution of \( s_b = t_b = 1 \), but having the different Gaussian densities. All the Gaussian components have the same covariance matrix of \( P_b = \text{diag}(\{40, 40, 40\})^\top \) but different means, \( m_{\gamma,k}^{(1)} = \{1000, 2300, 0, 0\}^\top, m_{\gamma,k}^{(2)} = \{3000, 1200, 0, 0\}^\top, \ldots \)

\[
\begin{align*}
&\{2000, 2000, 0, 0\}^\top, m_{\gamma,k}^{(4)} = \{3000, 1200, 0, 0\}^\top, m_{\gamma,k}^{(5)} = \{2000, 2000, 0, 0\}^\top, m_{\gamma,k}^{(6)} = \{3000, 1200, 0, 0\}^\top, m_{\gamma,k}^{(7)} = \{2000, 2000, 0, 0\}^\top, m_{\gamma,k}^{(8)} = \{3000, 1200, 0, 0\}^\top, m_{\gamma,k}^{(9)} = \{2000, 2000, 0, 0\}^\top, m_{\gamma,k}^{(10)} = \{3000, 1200, 0, 0\}^\top, m_{\gamma,k}^{(11)} = \{2000, 2000, 0, 0\}^\top, m_{\gamma,k}^{(12)} = \{3000, 1200, 0, 0\}^\top.
\end{align*}
\]

Furthermore, clutter is modeled as a Poisson RFS with clutter rate \( \lambda_c = 20 \), which means there are 20 points per scan. Object-originated observations are generated according to a constant detection probability \( p_{D,k} \), which is unknown in the simulation experiments. The observation region and trajectories are presented in Fig. 1. Next, two cases with different detection probabilities are studied and compared from both OSPA error and cardinality estimate as well as estimate of detection probability. Meanwhile, the comparisons of different observation noise are also provided in two cases: \( p_{D} = 0.95 \) and \( p_{D} = 0.65 \). All of the results are averaged over 100 independent Monte Carlo (MC) runs.

### A. Case 1: \( p_{D} = 0.95 \)

In this scenario, the actual but unknown detection probability is set to \( p_{D} = 0.95 \). The comparisons of OSPA error and cardinality between the CPHD and PMBM filters are shown in Figs. 2 and 3. Overall, the performance of the PMBM filter is much better than that of the CPHD filter even though the error is a little relatively large when the objects appear (at 20s, 40s and 50s) or disappear (at 60s). Meanwhile, the cardinality of the PMBM filter is almost same as the true
number of object and the covariance of cardinality estimate of the PMBM filter is smaller than that of the CPHD filter. Furthermore, the estimate of detection probability is given in Fig. 4. It can be seen that the PMBM filter performs much better than the CPHD filter with a more precise estimate of detection probability, which approaches to the true value.

B. Case 2: \( p_D = 0.65 \)

Different from case 1, the lower detection probability with \( p_D = 0.65 \) is considered. The comparisons of OSPA error and cardinality are shown in Figs. 5 and 6. It shows that both OSPA error and cardinality of the PMBM filter is still much better than that of the CPHD filter, and the gap between two filters is greater compared with case 1. Besides, the estimate of detection probability is given in Fig. 7. Fig. 7 shows the estimate of the detection probability with the CPHD filter is far worse than the PMBM filter whereas the estimate with the PMBM filter approaches to the true value.

C. Comparison of covariance of observation noise

Moreover, changing the covariance of observation noise, we compare the OSPA error for both cases given in Table I. It shows that the OSPA error increases as the covariance of observation noise increases for both filters in two cases. As expected, the performance of the PMBM filter is always better than that of the CPHD filter under the same parameters.

| \( \sigma_{\varepsilon} (p_D = 0.95) \) | 5  | 10 | 15 | 20 | 25 |
|---------------------------------|----|----|----|----|----|
| CPHD                           | 33.22 | 36.18 | 38.59 | 42.63 | 46.36 |
| PMBM                           | 8.29 | 12.22 | 15.84 | 18.15 | 21.82 |

| \( \sigma_{\varepsilon} (p_D = 0.65) \) | 5  | 10 | 15 | 20 | 25 |
|---------------------------------|----|----|----|----|----|
| CPHD                           | 59.36 | 61.38 | 63.25 | 65.27 | 67.30 |
| PMBM                           | 26.00 | 28.04 | 30.43 | 38.68 | 41.28 |

V. Conclusions

In this paper, we mainly research the multi-object tracking (MOT) in unknown detection probability with the PMBM filter. First, a construction strategy by augmenting the state with a parameter of detection probability is presented. Then, the recursive expressions are presented: prediction and update processes. Moreover, the detailed implementation by using...
a Beta function to represent the detection model and representing the intensity of Poisson process with a Beta-Gaussian mixture form as well as the density of Bernoulli component with a single Beta-Gaussian form. The effectiveness and robustness of the proposed approach has been analyzed and also compared with the CPHD filter by means of simulations.

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Cov of measurement noise

OSPA/m

CPHD
PMBM

Cov of measurement noise