Sobolev Mapping Properties of the Scattering Transform for the Schrödinger Equation

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Abstract. We consider the scattering transform for the Schrödinger equation with a singular potential and no bound states. Using the Riccati representation for real-valued potentials on the line, we obtain invertibility and Lipschitz continuity of the scattering transform between weighted and Sobolev spaces. Our approach exploits the connection between scattering theory for the Schrödinger equation and scattering theory for the ZS–AKNS system.

1. Introduction

The purpose of this paper is to study Sobolev space mapping properties of the direct and inverse scattering maps for the one-dimensional Schrödinger equation with a potential of low regularity and no bound states. One of our motivations is to use the scattering maps for the Schrödinger equation to construct and study solutions of the KdV equation on the line with initial data of low regularity using the inverse scattering method. This paper presents first steps toward this goal which we will continue in [9].

In this paper, we will describe a new representation for singular potentials on the line, the Riccati representation, inspired by the work of Kappeler, Perry, Shubin, and Topalov [10] on the Miura map [18]. As we will see, the Sobolev mapping properties of the scattering map are particularly transparent when this representation is used. An analogous representation for Schrödinger operators on the circle appears in the work of Kappeler and Topalov [11,14] on well-posedness of the periodic KdV and mKdV equations.

If $q$ is a real-valued distribution on the real line belonging to the space $H^{-1}(\mathbb{R})$, the Schrödinger operator $-d^2/dx^2 + q$ may be defined as the self-adjoint operator
associated to the closure of the semibounded quadratic form
\begin{equation}
q(\varphi) = \int |\varphi'(x)|^2 \, dx + \left\langle q, |\varphi|^2 \right\rangle
\end{equation}
with domain $C_0^\infty(\mathbb{R})$ (see Appendix B in [10] and references therein). It is natural to begin by considering such singular potentials without negative-energy bound states, i.e., distributions $q$ for which the quadratic form (1.1) is non-negative. As shown in [10], such a distribution can be presented in the form
\[ q = u' + u^2 \]
where $u \in L^2_{\text{loc}}(\mathbb{R})$ is the logarithmic derivative of a positive solution $y \in H^1_{\text{loc}}(\mathbb{R})$ of the zero-energy Schrödinger equation $-y'' + qy = 0$. The function $u$ is called a Riccati representation for the distribution $q$.

There is a one-to-one correspondence between Riccati representatives $u$ and strictly positive solutions $y$ to the zero-energy Schrödinger equation, normalized so that $y(0) = 1$. This latter set consists either of a single point or a one-parameter family of solutions. Explicitly, $y = \theta y_- + (1 - \theta)y_+$, where $y_{\pm}$ are the unique, normalized, positive solutions with the property that
\[ \int_0^\infty \frac{ds}{y_+(s)} = \int_{-\infty}^0 \frac{ds}{y^2(s)} = \infty \]
(see §5 of [10]). If we set $u_{\pm} = \frac{d}{dx} \log y_{\pm}$, these “extremal” Riccati representatives $u_{\pm}$ have the property that $v := u_- - u_+$ is a nonnegative, Hölder continuous function and is either strictly positive, if $u_+ \neq u_-$, or identically zero, if $u_+ = u_-$. We can now describe the class of potentials we will study and define the Riccati representation for such potentials that will play a central role in our work. Denote by $Q$ the set of real-valued distributions $q \in H^{-1}(\mathbb{R})$ with the properties that
(i) the quadratic form (1.1) is non-negative, and
(ii) the Riccati representatives $u_{\pm}$ obey $u_{\pm} \in L^1(\mathbb{R}^\pm) \cap L^2(\mathbb{R})$.
We have $Q = Q_0 \cup Q_>$ where $Q_0$ is the set of all $q \in Q$ with $v(0) = 0$, and $Q_>$ is the set of all such distributions with $v(0) > 0$. This class includes the usual Faddeev–Marchenko class [7] but also positive measures with suitable decay, certain highly oscillating potentials, and sums of delta functions with positive weight (see §1 of [7] and §2 of [8] for further examples). The set $Q_0$ is very unstable under perturbations so that potentials in the sets $Q_0$ and $Q_>$ are referred to respectively as “exceptional” and “generic” potentials.

A distribution $q \in Q$ is uniquely determined by the data
\begin{equation}
\begin{pmatrix} u_-|_{(-\infty,0)} & u_+|_{(0,\infty)} & v(0) \end{pmatrix} \in X^- \times X^+ \times [0,\infty),
\end{equation}
where $X^\pm = L^2(\mathbb{R}^\pm) \cap L^1(\mathbb{R}^\pm)$ (see [8], Lemma 2.3). We will call the triple $\begin{pmatrix} u_-|_{(-\infty,0)} & u_+|_{(0,\infty)} & v(0) \end{pmatrix}$ the Riccati representation of $q$. Note that $q \in Q_0$ has a unique Riccati representative $u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

For $q \in Q$, it was shown in [7] (for $q \in Q_0$) and [8] (for $q \in Q_>$) that the usual formulation of scattering theory for the Schrödinger equation carries over. First,

\footnote{That is, real-valued measurable functions $q$ with $\int (1 + |x|) |q(x)| \, dx < \infty$.}
there exist Jost solutions $f_{\pm}(x,k)$, asymptotic as $x \to \pm \infty$ to $\exp(\pm ikx)$. Second, one can use these solutions to define reflection coefficients $r_{\pm}(k)$ that describe scattering. The scattering maps $S_{\pm}$ are then defined as

$$S_{\pm} : q \mapsto r_{\pm}.$$ 

We will study the scattering maps, parameterizing their domain using the Riccati representation.

The Riccati representation connects the scattering problem for the Schrödinger equation to the scattering problem for the ZS–AKNS system (see Zakharov–Shabat [21] and Ablowitz–Kaup–Newell–Segur [1]):

$$\frac{d}{dx} \Psi = i k \sigma_3 \Psi + Q(x) \Psi,$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$Q(x) = \begin{pmatrix} 0 & u(x) \\ u(x) & 0 \end{pmatrix},$$

where $u$ is a Riccati representative for $q$. If $q \in Q_0$, then the Schrödinger scattering problem is in fact equivalent to the scattering problem for (1.3) with potential (1.4), and the scattering maps can be studied using techniques developed for the ZS–AKNS system (see [6] and [7]). On the other hand, if $q \in Q_>$, one can construct Jost solutions $f_+$ and $f_-$ for the Schrödinger equation from scattering solutions associated to ZS–AKNS systems (1.3), where the potential $Q$ is given by (1.4) respectively with $u = u_+$ and $u = u_-$. The Riccati representation is particularly well-suited to studying Sobolev space mapping properties of the scattering map. We first consider the case of $q \in Q_0$, where $q$ is specified uniquely by a single real-valued Riccati representative $u \in X$, with $X$ denoting the Banach space $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with norm

$$\|u\|_X = \|u\|_{L^1(\mathbb{R})} + \|u\|_{L^2(\mathbb{R})}.$$ 

We will write $X_\mathbb{R}$ for the real Banach space of real-valued functions $u \in X$. Denote by $\widehat{X}$ and $\widehat{X}_\mathbb{R}$ the images of $X$ and $X_\mathbb{R}$ under the Fourier transform, set $\|\hat{u}\|_{\widehat{X}} = \|\hat{u}\|_{X_\mathbb{R}}$, and let

$$\widehat{X}_1 := \{ r \in \widehat{X}_\mathbb{R} : \|r\|_{\infty} < 1 \}.$$ 

Note that $r(-k) = r(k)$ for any $r \in \widehat{X}_\mathbb{R}$. It was shown in [6], [7] that the scattering maps $S_{\pm}$ are invertible, locally bi-Lipschitz maps from $X_\mathbb{R}$ onto $\widehat{X}_1$. Since the maps $S_{\pm}$ in the Riccati variable are scattering maps for the ZS–AKNS system, one can use techniques of Zhou [22] to prove the following refined Sobolev mapping property. For $s \geq 0$, let

$$L^{2,s}(\mathbb{R}) := \{ u \in L^2(\mathbb{R}) : (1 + |x|)^s u \in L^2(\mathbb{R}) \}$$

and denote by $H^s(\mathbb{R})$ the image of $L^{2,s}(\mathbb{R})$ under the Fourier transform. Note that, for $s > 1/2$, $L^{2,s}(\mathbb{R}) \subset X$ and $H^s(\mathbb{R})$ consists of continuous functions. If we set

$$H^s_1(\mathbb{R}) := \{ r \in H^s(\mathbb{R}) \cap \widehat{X}_\mathbb{R} : \|r\|_{\infty} < 1 \},$$

one can prove the following refined mapping property.
Let’s studied Sobolev space mapping properties of the scattering map, defined as follows. In terms of $\mathbb{R}^n$ space locally bi-Lipschitz continuous maps from $L^s$ authors impose weighted $\mathbb{R}$ and denote variables are given by $(1.6)$ $d_8$ in $\mathbb{R}$. The space $S$ from $X$ and Remark 9); in general, as shown in $\mathbb{R}$, one has $r_\pm(k) = -1 + O(k^2)$ as $k \to 0$ (see, for example, $\mathbb{R}$, §2, Theorem 1, Part V and Remark 9); in general, as shown in $\mathbb{R}$, one has the weaker condition that the functions $\frac{1 - |r_\pm(k)|^2}{k^2}$ belong to $\mathbb{R}$ and do not vanish at $k = 0$. The direct scattering maps in the Riccati variables are given by $S_\pm : \left( u_-|_{(-\infty,0)}, u_+|_{(0,\infty)}, v(0) \right) \mapsto r_\pm$. For $r \in \mathbb{R}$, we shall write $r(k) = \frac{1 - |r(k)|^2}{k^2}$ and denote $R_\geq := \left\{ r \in \mathbb{R} : r(0) = -1, |r(k)| < 1 \text{ if } k \neq 0, \tilde{r} \in \mathbb{R}, \tilde{r}(0) \neq 0 \right\}.$ The space $R_\geq$ is a metric space when equipped with the metric $d(r_1, r_2) = \|r_1 - r_2\|_\mathbb{R} + \|\tilde{r}_1 - \tilde{r}_2\|_\mathbb{R}.$ In $\mathbb{R}$, it was shown that the maps $S_\pm$ are locally bi-Lipschitz continuous onto maps from $X^- \times X^+ \times (0,\infty)$ onto $R_\geq$ equipped with the metric $(1.6)$. We will prove a finer mapping property, analogous to Theorem 1.1, for the scattering map on generic potentials. We set $R_s = \{ r \in R_\geq \cap H^s(\mathbb{R}) : \tilde{r} \in H^s(\mathbb{R}) \}$ and equip $R_s$ with the metric $d_s(r_1, r_2) = \|r_1 - r_2\|_{H^s(\mathbb{R})} + \|\tilde{r}_1 - \tilde{r}_2\|_{H^s(\mathbb{R})}.$

**Theorem 1.1.** For any $s > 1/2$, the restrictions $S_\pm : L^{2,s}(\mathbb{R}) \cap X \to H^s_+(\mathbb{R})$ are onto, invertible, locally bi-Lipschitz continuous maps.

We will not give the details of the proof but rather concentrate on the more challenging case where $q \in Q_\geq$. To formulate our main theorem we first recall some results from $\mathbb{R}$.

If $q \in Q_\geq$, the reflection coefficients $r_\pm$ belong to $\mathbb{R}$, but $r_\pm(0) = -1$ and $|r(k)| < 1$ for $k \neq 0$. For smooth, compactly supported generic potentials, one has $r_\pm(k) = -1 + O(k^2)$ as $k \to 0$ (see, for example, $\mathbb{R}$, §2, Theorem 1, Part V and Remark 9); in general, as shown in $\mathbb{R}$, one has the weaker condition that the functions $\frac{1 - |r_\pm(k)|^2}{k^2}$ belong to $\mathbb{R}$ and do not vanish at $k = 0$. The direct scattering maps in the Riccati variables are given by $S_\pm : \left( u_-|_{(-\infty,0)}, u_+|_{(0,\infty)}, v(0) \right) \mapsto r_\pm$. For $r \in \mathbb{R}$, we shall write $r(k) = \frac{1 - |r(k)|^2}{k^2}$ and denote $R_\geq := \left\{ r \in \mathbb{R} : r(0) = -1, |r(k)| < 1 \text{ if } k \neq 0, \tilde{r} \in \mathbb{R}, \tilde{r}(0) \neq 0 \right\}.$ The space $R_\geq$ is a metric space when equipped with the metric $d(r_1, r_2) = \|r_1 - r_2\|_\mathbb{R} + \|\tilde{r}_1 - \tilde{r}_2\|_\mathbb{R}.$ In $\mathbb{R}$, it was shown that the maps $S_\pm$ are locally bi-Lipschitz continuous onto maps from $X^- \times X^+ \times (0,\infty)$ onto $R_\geq$ equipped with the metric $(1.6)$. We will prove a finer mapping property, analogous to Theorem 1.1, for the scattering map on generic potentials. We set $R_s = \{ r \in R_\geq \cap H^s(\mathbb{R}) : \tilde{r} \in H^s(\mathbb{R}) \}$ and equip $R_s$ with the metric $d_s(r_1, r_2) = \|r_1 - r_2\|_{H^s(\mathbb{R})} + \|\tilde{r}_1 - \tilde{r}_2\|_{H^s(\mathbb{R})}.$

**Theorem 1.2.** For any $s > 1/2$, the direct scattering maps $S_\pm$ are invertible, locally bi-Lipschitz continuous maps from $L^{2,s}(\mathbb{R}) \times L^{2,s}(\mathbb{R}) \times (0,\infty)$ onto the space $R_s$. Fourier-type mapping properties of the map $q \mapsto r$ have been studied by many authors, including Cohen, Deift and Trubowitz, and Faddeev. These authors impose weighted $L^1$ assumptions on $q$ and obtain regularity results for $r$ in terms of $\infty$-norms of $r$ and its derivatives. Kappeler and Trubowitz studied Sobolev space mapping properties of the scattering map, defined as follows. Let $s(k) = 2ikr(k)/t(k)$, where $r$ is the reflection coefficient and $t$ is the transmission coefficient, and introduce the weighted Sobolev spaces $H_{n,\alpha} = \left\{ f \in L^2 : x^\beta \partial_x^j f \in L^2, 0 \leq j \leq n, 0 \leq \beta \leq \alpha \right\},$ $H_{n,\alpha}^\# = \left\{ f \in H_{n,\alpha} : x^\beta \partial_x^{\alpha+1} f \in L^2, 1 \leq \beta \leq \alpha \right\}.$
Kappeler and Trubowitz show that the map $q \mapsto s$ takes potentials $q \in H_{N,N}$ without bound states to scattering functions $s$ belonging to $H_{N-1,N}^\#$ for $N \geq 3$. They extend their results to potentials with finitely many bound states in $[16]$. They also prove analyticity and investigate the differential of the scattering map.

Our results are similar to those of Kappeler and Trubowitz in that we study $L^2$-based Sobolev spaces, which leads to a more symmetrical formulation of the mapping properties. In our case, we examine the scattering map in the Riccati variables (1.2) and so treat potentials which are more singular than the class treated by Kappeler and Trubowitz. In a subsequent paper [9], we will extend the methods developed here to consider mapping properties between weighted fractional Sobolev spaces which preserve the KdV flow.

This paper is organized as follows. In section 2, we first review the connection between Jost solutions to the Schrödinger and ZS–AKNS equations. In section 3 we obtain estimates on the direct scattering map using a Fourier representation for the Jost solutions derived in [7]. In section 4, we use the representation formulas of [8], derived from Gelfand–Levitan–Marchenko equation for the ZS–AKNS system, to analyze the inverse scattering map. Finally, in section 5, we give the proof of the main theorem.

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2. Schrödinger Scattering and the ZS–AKNS System

In this section, we recall how the Jost solutions and reflection coefficients for a Schrödinger operator with Miura potential may be computed by solving the associated ZS–AKNS equations with potentials $u_+$ and $u_-$. We assume throughout that $u_\pm \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}^\pm)$ are real-valued.

First, we recall the connection between the Schrödinger equation with a Miura potential and the ZS–AKNS system. If $u \in L^2_{\text{loc}}(\mathbb{R})$ and $q = u^2 + u$ then the Schrödinger equation
\begin{equation}
- y'' + qy = k^2 y
\end{equation}
is equivalent to the system
\begin{equation}
\frac{d}{dx} \begin{pmatrix} y \\ y^{[1]} \end{pmatrix} = \begin{pmatrix} u & 1 \\ -k^2 & -u \end{pmatrix} \begin{pmatrix} y \\ y^{[1]} \end{pmatrix}
\end{equation}
where $y^{[1]} := y' - uy$ is the quasi-derivative of $y$. Note that $y$ and $y^{[1]}$ are absolutely continuous, and the initial value problem for (2.2) has a unique solution. For a given choice of $u$ and solutions $g$ and $h$ of (2.1), the Wronskian
\begin{equation}
[f, g] = g(x)h^{[1]}(x) - g^{[1]}(x)h(x)
\end{equation}
is independent of $x$.

The Jost solutions $f_\pm(x, k)$ satisfy (2.4) with respective asymptotic conditions
\begin{equation}
\lim_{x \to \pm \infty} \begin{pmatrix} f_\pm(x) - e^{\pm ikx} \\ f_\pm^{[1]}(x) + ike^{\pm ikx} \end{pmatrix} = 0
\end{equation}
where
\begin{equation}
f_\pm^{[1]} := f_\pm - u_\pm f_\pm.
\end{equation}
If \([\cdot,\cdot]_\pm\) denotes the Wronskian (2.3) with \(u = u_\pm\), it follows from the asymptotics (2.4) that
\[-[f_+(x,k),f_+(x,-k)]_\pm = [f_-(x,k),f_-(x,-k)]_- = 2ik.\]
Thus, for \(k \neq 0\), there are coefficients \(a(k)\) and \(b(k)\) so that
\[f_+(x,k) = a(k)f_-(x,-k) + b(k)f_-(x,k).\]

By standard arguments,
\[
|a(k)|^2 - |b(k)|^2 = 1,
\]
and the reality conditions
\[a(-k) = \overline{a(k)}, \quad b(-k) = \overline{b(k)}\]
hold. Moreover,
\[a(k) = \frac{[f_+(x,k),f_-(x,k)]_-}{[f_-(x,-k),f_-(x,k)]_-}\]
and
\[b(k) = \frac{[f_+(x,k),f_-(x,-k)]_-}{[f_-(x,k),f_-(x,-k)]_-}.\]

The reflection coefficients \(r_\pm\) are given by
\[r_-(k) = b(k)/a(k), \quad r_+(k) = -b(-k)/a(k),\]
so that \(|r_+(k)| = |r_-(k)|\). The transmission coefficient is given by \(t(k) = 1/a(k)\), and the involution
\[r(k) \mapsto -\frac{t(k)}{t(-k)} r(-k)\]
maps \(r_-\) to \(r_+\) and vice versa.

To compute the Jost solutions \(f_\pm\) we exploit the following connection between the Schrödinger equation with potential \(q = u' + u^2\) and the ZS–AKNS system
\[
\frac{d}{dx} \Psi = ik\sigma_3 \Psi + Q(x) \Psi
\]
with potential
\[Q(x) = \begin{pmatrix} 0 & u(x) \\ u(x) & 0 \end{pmatrix}.\]

If \(\Psi = (\psi_1, \psi_2)^T\) is a vector-valued solution of (2.12) with potential (2.13), then
\[
\begin{pmatrix} \psi_1 + \psi_2 \\ ik(\psi_1 - \psi_2) \end{pmatrix}
\]
solves the system (2.12). In particular, if \(\Psi_+\) and \(\Psi_-\) are the unique matrix-valued solutions of the respective problems
\[
\frac{d}{dx} \Psi_\pm = ik\sigma_3 \Psi_\pm + \begin{pmatrix} 0 & u_\pm(x) \\ u_\pm(x) & 0 \end{pmatrix} \Psi_\pm,
\]
\[
\lim_{x \to \pm \infty} |\Psi_\pm(x) - e^{ixk\sigma_3}| = 0,
\]
and \(r_\pm\) are given by
\[r_-(k) = b(k)/a(k), \quad r_+(k) = -b(-k)/a(k),\]
so that \(|r_+(k)| = |r_-(k)|\). The transmission coefficient is given by \(t(k) = 1/a(k)\), and the involution
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\[
\frac{d}{dx} \Psi_\pm = ik\sigma_3 \Psi_\pm + \begin{pmatrix} 0 & u_\pm(x) \\ u_\pm(x) & 0 \end{pmatrix} \Psi_\pm,
\]
\[
\lim_{x \to \pm \infty} |\Psi_\pm(x) - e^{ixk\sigma_3}| = 0,
\]
then the formulas

\begin{align}
(2.14) \quad f_+(x, k) &= \psi_{11}^+(x, k) + \psi_{21}^+(x, k) \\
(2.15) \quad f_+^{[1]}(x, k) &= ik \left( \psi_{11}^+(x, k) - \psi_{21}^+(x, k) \right) \\
(2.16) \quad f_-(x, k) &= \overline{\psi_{11}^+(x, k) + \psi_{21}^+(x, k)} \\
(2.17) \quad f_-^{[1]}(x, k) &= -ik \left( \psi_{11}^+(x, k) - \psi_{21}^+(x, k) \right)
\end{align}

hold, where the bar denotes complex conjugation. A short computation with (2.18)–(2.19) leads to the formulas

\begin{align}
(2.18) \quad a(k) &= \begin{vmatrix} \psi_{11}^+(x, k) & \overline{\psi_{21}^+(x, k)} \\ \psi_{21}^+(x, k) & \psi_{11}^+(x, k) \end{vmatrix} - \frac{v(x)}{2ik} f_+(x, k) f_-(x, k), \\
(2.19) \quad b(k) &= \begin{vmatrix} \psi_{11}^+(x, k) & \overline{\psi_{11}^+(x, -k)} \\ -\psi_{21}^+(x, k) & -\psi_{21}^+(x, -k) \end{vmatrix} + \frac{v(x)}{2ik} f_+(x, k) f_-(x, k),
\end{align}

where, for a $2 \times 2$ matrix $A$, $|A|$ denotes the determinant.

These two formulas lie at the heart of our analysis for the direct problem. They show explicitly the singularity at $k = 0$ that occurs when $u_+ \neq u_-$. The singularity is always nonzero in this case since $v$ is strictly nonzero and $f_\pm(x, 0)$ are positive solutions of the zero-energy Schrödinger equation.

To study the scattering map via the formulas (2.18)–(2.19), we will use integral representations for the solutions $\Psi^\pm$. These integral representations give $\Psi^\pm$ as Fourier transforms of functions given by explicit multilinear series in $u_\pm$. Let $\Psi^\pm(x, k) = \exp(ikx)N^\pm(x, k)$ and denote by $n^\pm_{ij}$ the entries of $N^\pm$. In order to compute the Jost solutions from (2.14)–(2.17), it suffices to study $n^+_{11}$ and $n^+_{21}$. We will describe only the integral representations for $n^+_{11}$ and $n^+_{21}$ and their properties since those of $n^-_{11}$ and $n^-_{21}$ are very similar.

From [7], section 3.1, equations (3.14) and (3.15) and following, we have

\begin{align*}
n^+_{11}(x, k) - 1 &= \int_0^\infty A(x, \zeta)e^{ik\zeta} \, d\zeta, \\
n^+_{21}(x, k) &= \int_0^\infty B(x, \zeta)e^{ik\zeta} \, d\zeta.
\end{align*}

Here $A$ and $B$ have multilinear expansions of the form

\begin{align*}
A(x, \zeta) &= \sum_{n=1}^\infty A_n(x, \zeta), \quad B(x, \zeta) = \sum_{n=1}^\infty B_n(x, \zeta)
\end{align*}

with

\begin{align*}
A_n(x, \zeta) &= \int_{\Omega_{2n}(\zeta)} u_+(y_1) \ldots u_+(y_{2n}) \, dS_{2n}, \\
B_n(x, \zeta) &= \int_{\Omega_{2n-1}(\zeta)} u_+(y_1) \ldots u_+(y_{2n}) \, dS_{2n-1},
\end{align*}
where, for \( \zeta \in \mathbb{R} \), \( \Omega_n(\zeta) \) is the set of all \( y = (y_1, \ldots, y_n) \) in \( \mathbb{R}^n \) with \( x \leq y_1 \leq \cdots \leq y_n \) and

\[
(2.20) \quad \sum_{j=0}^{n-1} (-1)^j y_{n-j} = \zeta,
\]

while \( dS_n \) is surface measure on the hyperplane \((2.20)\).

For each fixed \( x \) we have

\[
(2.21) \quad \left\| n_{11}^+(x, \cdot) - 1 \right\|_{H^s(\mathbb{R})} \leq \| A(x, \cdot) \|_{L^2, s(\mathbb{R})},
\]

\[
(2.22) \quad \left\| n_{21}^+(x, \cdot) \right\|_{H^s(\mathbb{R})} \leq \| B(x, \cdot) \|_{L^2, s(\mathbb{R})}.
\]

Thus, to estimate the \( H^s \)-norms of \( n_{11}^+ \) and \( n_{21}^+ \) as functions of \( k \), it suffices to obtain summable estimates on \( \| A_n(x, \cdot) \|_{L^2, s(\mathbb{R})} \) and \( \| B_n(x, \cdot) \|_{L^2, s(\mathbb{R})} \).

To do this, we first note the identity

\[
\| \psi \|_{L^2, s(\mathbb{R})} = \sup \left\{ \left\| (1 + |\zeta|)^s \psi(\zeta) \right\| d\zeta : \| \varphi \|_{L^2} = 1 \right\}.
\]

Next, setting \( y := (y_1, \ldots, y_n) \), \( dy := dy_1 \cdots dy_n \), \( U(y) := u_+(y_1) \cdots u_+(y_n) \), and defining

\[
\zeta_n(y) := \sum_{j=0}^{n-1} (-1)^j y_{n-j},
\]

we find that

\[
(2.23) \quad \int_0^\infty \varphi(\zeta) \int_{\Omega_{2n}(\zeta)} U(y)dS_{2n}d\zeta = \int_{x_1, x_1 \leq \cdots \leq y_{2n}} U(y)\varphi(\zeta_n(y))dy,
\]

\[
(2.24) \quad \int_x^\infty \varphi(\zeta) \int_{\Omega_{2n-1}(\zeta)} U(y)dS_{2n-1}d\zeta = \int_{x_1, x_1 \leq \cdots \leq y_{2n-1}} U(y)\varphi(\zeta_n(y)-1)dy.
\]

Observe that for \( y \) obeying \( 0 \leq x \leq y_1 \leq \cdots \leq y_n \) the estimate \( |\zeta_n(y)| \leq y_n \) holds. We then get from the integral representation for \( A_n \) and \((2.23)\) that, for any \( \varphi \in L^2(\mathbb{R}) \),

\[
\left| \int_0^\infty (1 + \zeta)^s \varphi A_n(x, \zeta) \right| d\zeta \leq \left| \int_{x_1, x_1 \leq \cdots \leq y_{2n-1}} \varphi \right| d\zeta \left| \int_x^\infty (1 + y_{2n})^s \varphi(\zeta_n(y))\right| d\zeta \left| \int_{x_1, x_1 \leq \cdots \leq y_{2n-1}} \varphi \right| d\zeta \leq \frac{\| u_+ \|_{L^2(\mathbb{R})}}{2n-1} \| \varphi \|_{L^2(\mathbb{R})}.
\]

Therefore

\[
\| A_n(x, \cdot) \|_{L^2, s(\mathbb{R})} \leq \frac{\| u_+ \|_{L^2(\mathbb{R})}}{(2n-1)!} \| \varphi \|_{L^2(\mathbb{R})},
\]

and similar estimates give

\[
\| B_n(x, \cdot) \|_{L^2, s(\mathbb{R})} \leq \frac{\| u_+ \|_{L^2(\mathbb{R})}}{(2n-2)!} \| \varphi \|_{L^2(\mathbb{R})}.
\]

Since \( A_n(x, \cdot) \) and \( B_n(x, \cdot) \) are multilinear functions of \( u_+ \) and the series for \( A(x, \cdot) \) and \( B(x, \cdot) \) converge absolutely in \( L^{2, s}(\mathbb{R}) \), standard arguments show that, for
every fixed \( x \geq 0 \), \( A(x, \cdot) \) and \( B(x, \cdot) \) depend analytically in \( L^{2,s}(\mathbb{R}) \) on \( u_+ \in X^+ \).

Hence:

**Proposition 2.1.** Assume that \( s > 1/2 \) and that \( u_+ \in L^{2,s}(\mathbb{R}^+) \). Then \( n_{11}^+(x, \cdot) - 1 \) and \( n_{21}^+(x, \cdot) \) belong to \( H_s(\mathbb{R}) \) for each fixed \( x \geq 0 \), depend analytically therein on \( u_+ \in X^+ \), and the estimates

\[
\sup_{x \geq 0} \left( \| n_{11}^+(x, \cdot) - 1 \|_{H_s(\mathbb{R})} + \| n_{21}^+(x, \cdot) \|_{H_s(\mathbb{R})} \right) \leq \| u_+ \|_{L^{2,s}(\mathbb{R}^+)} \exp\left\{ \| u_+ \|_{L^1(\mathbb{R}^+)} \right\}
\]

hold.

A similar analysis, based on the integral representations for \( n_{11}^- \) and \( n_{21}^- \), shows:

**Proposition 2.2.** Assume that \( s > 1/2 \) and that \( u_- \in L^{2,s}(\mathbb{R}^-) \). Then \( n_{11}^-(x, \cdot) - 1 \) and \( n_{21}^-(x, \cdot) \) belong to \( H_s(\mathbb{R}) \) for each fixed \( x \leq 0 \), depend analytically therein on \( u_- \in X^- \), and the estimates

\[
\sup_{x \leq 0} \left( \| n_{11}^-(x, \cdot) - 1 \|_{H_s(\mathbb{R})} + \| n_{21}^-(x, \cdot) \|_{H_s(\mathbb{R})} \right) \leq \| u_- \|_{L^{2,s}(\mathbb{R}^-)} \exp\left\{ \| u_- \|_{L^1(\mathbb{R}^-)} \right\}
\]

hold.

3. The Direct Problem

We now consider the mappings \((u_-, u_+, v(0)) \mapsto r_\pm\). In order to study the mapping properties we introduce the auxiliary functions

\[
\tilde{a}(k) = \frac{k}{k+i} a(k),
\]

\[
\tilde{b}(k) = \frac{k}{k+i} b(k),
\]

\[
\tilde{\tau}(k) = \frac{1 - |r_\pm(k)|^2}{k^2},
\]

and note the relations

\[
r_-(k) = \frac{\tilde{b}(k)}{\tilde{a}(k)}, \quad r_+(k) = \frac{i - k\tilde{b}(-k)}{i + k\tilde{a}(k)},
\]

and

\[
\tilde{\tau}(k) = \frac{1}{k^2 + 1} \left( \frac{1}{|\tilde{a}(k)|^2} \right).
\]

**Proposition 3.1.** Suppose that \( u_\pm \in L^{2,s}(\mathbb{R}^\pm) \) for some \( s > 1/2 \). Then \( r_\pm \in H^s(\mathbb{R}) \) and \( \tilde{\tau} \in H^s(\mathbb{R}) \) with \( \tilde{\tau}(0) \neq 0 \), and the maps

\[
L^{2,s}(\mathbb{R}^+) \times L^{2,s}(\mathbb{R}^-) \times \mathbb{R}^+ \to H^s(\mathbb{R})^3
\]

\((u_+, u_-, v(0)) \mapsto (r_-, r_+, \tilde{\tau})\)

are locally Lipschitz continuous.

**Proof.** From the representation formulae (2.18) and (2.19) evaluated at \( x = 0 \) we have

\[
\tilde{a}(k) = \frac{k}{k+i} \begin{vmatrix} n_{11}^+(0,k) & n_{21}^+(0,k) \\ n_{21}^-(0,k) & n_{11}^-(0,k) \end{vmatrix} - \frac{1}{k+i} \frac{v(0)}{2i} f_+(0,k) f_-(0,k)
\]

\[
\tilde{b}(k) = \frac{k}{k+i} \begin{vmatrix} n_{11}^+(0,k) & n_{21}^+(0,k) \\ n_{21}^-(0,k) & n_{11}^-(0,k) \end{vmatrix} - \frac{1}{k+i} \frac{v(0)}{2i} f_+(0,k) f_-(0,k)
\]

\[
\tilde{\tau}(k) = \frac{1}{k^2 + 1} \left( \frac{1}{|\tilde{a}(k)|^2} \right)
\]
Proposition 3.2. The mapping 
\[ \mathcal{I}_s : r \mapsto -\frac{t(k)}{t(-k)} r(-k) \]

is a continuous involution from \( \mathcal{R}_s \) to itself.

We omit the proof, since it is completely analogous to that of Proposition 3.3 in [8], except that the Banach algebra \( 1 + \tilde{X} \) is replaced with the Banach algebra \( 1 + H^s(\mathbb{R}) \).

4. The Inverse Problem

In this section, we assume given a function \( r \in \mathcal{R}_s \) and set \( r^\# = \mathcal{I}_s r \). It follows from [8] that there exists a unique distribution \( q \in H^{-1}(\mathbb{R}) \) with Riccati representatives \( u = u_+ \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}^+) \) and \( u^\# = u_- \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}^-) \) so that the corresponding Schrödinger operator has \( r \) and \( r^\# \) as its right and left reflection coefficients, respectively. We wish to show that the Riccati representatives \( u \) and \( u^\# \) reconstructed from \( r \) and \( r^\# \) belong respectively to \( L^{2,s}(\mathbb{R}^+) \) and \( L^{2,s}(\mathbb{R}^-) \).
do so, we will recall the reconstruction formulas for \( u \) and \( u^\# \) derived in [8] from the Gelfand–Levitan–Marchenko equations. Let us define

\[
F(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} r(k) e^{2ikx} \, dk, \\
F^\#(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} r^\#(k) e^{-2ikx} \, dk.
\]

Note that \( F \) and \( F^\# \) belong to \( L^2_{\text{loc}}(\mathbb{R}) \). Setting

\[
\Omega(x) = \begin{pmatrix} 0 & F(x) \\ F(x) & 0 \end{pmatrix}, \quad \Omega^\#(x) = \begin{pmatrix} 0 & F^\#(x) \\ F^\#(x) & 0 \end{pmatrix},
\]

the right and left Gelfand–Levitan–Marchenko equations are respectively

\[
\Omega(x + \zeta) + \Gamma(x, \zeta) + \int_0^\infty \Gamma(x, t) \Omega(x + \zeta + t) \, dt = 0, \quad \zeta > 0, \\
\Omega^\#(x + \zeta) + \Gamma^\#(x, \zeta) + \int_0^\infty \Gamma^\#(x, t) \Omega^\#(x + \zeta + t) \, dt = 0, \quad \zeta < 0,
\]

for the \( 2 \times 2 \) matrix-valued kernels \( \Gamma \) and \( \Gamma^\# \). The right and left Riccati representatives are reconstructed via

\[
u(x) = -\Gamma_{12}(x, 0), \\
u^\#(x) = \Gamma^\#_{12}(x, 0).
\]

Let \( \gamma(x, \zeta) = \Gamma_{12}(x, \zeta) \) and \( \gamma^\#(x, \zeta) = \Gamma^\#_{12}(x, \zeta) \). Let \( T_F \) and \( T_{F^\#} \) be the integral operators (depending parametrically on \( x \))

\[
(T_F \psi)(\zeta) = \int_0^\infty F(x + \zeta + t) \psi(t) \, dt, \\
(T_{F^\#} \psi)(\zeta) = \int_{-\infty}^0 F^\#(x + \zeta + t) \psi(t) \, dt.
\]

Then, as vectors in \( L^2(\mathbb{R}^+) \) (resp. in \( L^2(\mathbb{R}^-) \)) for each fixed \( x \),

\[
(I - T_F^2) \gamma(x, \cdot) = -F(x + \cdot), \\
(I - T_{F^\#}^2) \gamma^\#(x, \cdot) = -F^\#(x + \cdot).
\]

As shown in the proof of [8], Proposition 4.2, the operator \( (I - T_F^2)^{-1} \) is bounded from \( L^2(\mathbb{R}^+) \) to itself (resp. \( (I - T_{F^\#}^2)^{-1} \) is bounded from \( L^2(\mathbb{R}^-) \) to itself). From these equations and the reconstruction formulas, it is not difficult to see that

\[
u(x) = F(x) - G(x), \\
u^\#(x) = -F^\#(x) + G^\#(x),
\]

where

\[
G(x) = \int_0^\infty F(x + t) H(x, t) \, dt, \\
G^\#(x) = \int_{-\infty}^0 F^\#(x + t) H^\#(x, t) \, dt.
\]
and

\[ H(x, \, \cdot) = (I - T_{F}^{2})^{-1}((T_{F}F)(x + \, \cdot)), \]

\[ H^\#(x, \, \cdot) = (I - T_{F^\#}^{2})^{-1}((T_{F^\#}F^\#)(x + \, \cdot)). \]

We are interested in estimating the behavior of \( G \) as \( x \to +\infty \) (resp. of \( G^\# \) as \( x \to -\infty \)). It suffices to consider \( x > x_0 \) (resp. \( x < -x_0 \)) for sufficiently large \( x_0 \). Choosing \( x_0 \) so large that

\[ \int_{x_0}^{\infty} |F(s)| \, ds < 1/2, \quad \int_{-\infty}^{x_0} |F^\#(s)| \, ds < 1/2, \]

we have \( \|T_{F}\|_{L^{p}} \to L^{p} < 1 \) for \( p = 1, 2 \), and similarly for \( T_{F^\#} \). Note that we can make such a choice of fixed \( x_0 \) in a small neighborhood of a given \( F \in L^{2,s}(\mathbb{R}) \) since \( L^{2,s}(\mathbb{R}) \subset L^{1}(\mathbb{R}) \) for \( s > 1/2 \). We can then obtain convergent multilinear expansions for \( G \) and \( G^\# \) valid respectively for \( x > x_0 \) and \( x < -x_0 \). These multilinear expansions can be estimated, much as in the previous section, to obtain the required weighted estimates. We will give the analysis for \( G \) since the analysis for \( G^\# \) is very similar.

For \( x > x_0 \) we have the expansion

\[ H(x, \, \cdot) = \sum_{j=0}^{\infty} \left(T_{F}^{2j+1}[F(x + \, \cdot)]\right)(\, \cdot) \]

convergent in \( L^{2}(\mathbb{R}^+) \). From this expansion and the Cauchy–Schwarz inequality it follows that

\[ G(x) = \sum_{n=1}^{\infty} G_n(x) \]

in \( L^{\infty}(x_0, \infty) \), where

\[ G_n(x) = \int_{\mathbb{R}^{2n}_+} F(x + t_1)F(x + t_1 + t_2) \ldots F(x + t_{2n-1} + t_{2n})F(x + t_{2n}) \, dt \]

and \( dt := dt_1 \ldots dt_{2n} \). We will show that, for \( x_0 > 0 \),

\[ \int_{x_0}^{\infty} (1 + x)^{2s} |G_n(x)|^2 \, dx \leq \|F\|_{L^{1}(x_0, \infty)}^{4n} \int_{x_0}^{\infty} (1 + x)^{2s} |F(x)|^2 \, dx, \]

from which it follows that \( \int_{0}^{\infty} (1 + x)^{2s} |G(x)|^2 \, dx < \infty \). Let \( f(x) := |F(x)| \) and \( \bar{f}(x) := (1 + x)^{2s} f^2(x) \). Since \( x \leq x + t_1 \) in the range of integration for \( G_n \), it follows from the Cauchy–Schwarz inequality that

\[ \int_{x_0}^{\infty} (1 + x)^{2s} |G_n(x)|^2 \, dx \leq \int_{x_0}^{\infty} I_n(x)J_n(x) \, dx, \]

where

\[ I_n(x) := \int_{\mathbb{R}^{2n}_+} \bar{f}(x + t_1)f(x + t_1 + t_2) \ldots f(x + t_{2n-1} + t_{2n})f(x + t_{2n}) \, dt \]

and

\[ J_n(x) := \int_{\mathbb{R}^{2n}_+} f(x + t_1 + t_2) \ldots f(x + t_{2n-1} + t_{2n})f(x + t_{2n}) \, dt. \]
Theorem 1.2. Let \( \mathcal{A} \) be a unital algebra with respect to pointwise addition and multiplication. Then \( \mathcal{A} \) has range contained in \( \mathcal{R}_s \) and that \( \mathcal{S}_x \) are locally Lipschitz continuous maps from \( L^2,\star(\mathbb{R}^+) \times L^2,\star(\mathbb{R}^+) \times (0,\infty) \) into the space \( \mathcal{R}_s \). On the other hand, given a reflection coefficient \( r \in \mathcal{R}_s \), Proposition 4.1 shows that the Riccati representatives reconstructed from \( r \) and \( r^\# \) satisfy \( u \in L^2,\star(\mathbb{R}^+) \) and \( u^\# \in L^2,\star(\mathbb{R}^-) \) and are locally Lipschitz continuous as respective functions of \( r \) and \( r^\# \). It follows from the analysis of section 4 in [8] that \( u \) and \( u^\# \) are the unique right- and left-hand Riccati representatives of a real-valued distribution \( q \in H^{-1}(\mathbb{R}) \) having reflection coefficients \( r \) and \( r^\# \). This shows that \( \mathcal{S}_x \) are onto \( \mathcal{R}_s \) and completes the proof of Theorem 1.2.

5. Proof of the Main Theorem

We now give the proof of Theorem 1.2. Proposition 3.1 shows that \( \mathcal{S}_x \) have range contained in \( \mathcal{R}_s \) and that \( \mathcal{S}_x \) are locally Lipschitz continuous maps from \( L^2,\star(\mathbb{R}^-) \times L^2,\star(\mathbb{R}^+) \times (0,\infty) \) into the space \( \mathcal{R}_s \). On the other hand, given a reflection coefficient \( r \in \mathcal{R}_s \), Proposition 4.1 shows that the Riccati representatives reconstructed from \( r \) and \( r^\# \) satisfy \( u \in L^2,\star(\mathbb{R}^+) \) and \( u^\# \in L^2,\star(\mathbb{R}^-) \) and are locally Lipschitz continuous as respective functions of \( r \) and \( r^\# \). It follows from the analysis of section 4 in [8] that \( u \) and \( u^\# \) are the unique right- and left-hand Riccati representatives of a real-valued distribution \( q \in H^{-1}(\mathbb{R}) \) having reflection coefficients \( r \) and \( r^\# \). This shows that \( \mathcal{S}_x \) are onto \( \mathcal{R}_s \) and completes the proof of Theorem 1.2.

Appendix A. \( H^s(\mathbb{R}) \) as a Banach algebra

Throughout this appendix, we shall write \( L^p \) and \( H^s \) for the spaces \( L^p(\mathbb{R}) \) and \( H^s(\mathbb{R}) \), respectively. We refer the reader to the book by Runst and Sickel [20] for the properties of the Sobolev spaces \( H^s \) and to the book by Rudin [19] for the basic notions of the Banach algebras.

For \( s > \frac{1}{2} \), the space \( H^s \) is a closed algebra with respect to pointwise addition and multiplication. Thus, upon introducing an equivalent norm, \( H^s \) becomes a Banach algebra. We denote by \( 1 + H^s \) the extension of \( H^s \) to a unital algebra; \( 1 + H^s \) consists of functions of the form \( g := c \cdot 1 + f \) with \( c \in \mathbb{C} \) and \( f \in H^s \). We recall that the spectrum \( \sigma(g) \) of an element \( g \in 1 + H^s \) is the set of all \( \lambda \in \mathbb{C} \) such that \( g - \lambda \cdot 1 \) is not invertible in \( 1 + H^s \).

Lemma A.1. Assume that \( s > \frac{1}{2} \). Then for every \( g \in 1 + H^s \), the spectrum \( \sigma(g) \) is contained in the closure \( \overline{\text{ran}} g \) of the range \( \text{ran} g \).
PROOF. It suffices to prove the implication

\[ g \in 1 + H^s \quad \text{and} \quad 0 \not\in \text{ran } g \implies \frac{1}{g} \in 1 + H^s. \]

Without loss of generality, we may assume that \( g = 1 + f \) with \( f \in H^s \). Also, we set \( C := \|1/g\|_{L^\infty} \).

Consider first the case \( s \in (\frac{1}{2}, 1) \). Recall that then \( \phi \in L^2 \) belongs to \( H^s \) if and only if

\[ \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{1+2s}} \, dx \, dy < \infty. \]

Setting

\[ \phi(x) = \frac{1}{g(x)} - 1 = -\frac{f(x)}{g(x)} \in L^2 \]

and observing that

\[ |\phi(x) - \phi(y)| \leq C^2 |f(x) - f(y)|, \]

we easily conclude that \( \phi \in H^s \).

Next, for \( s = 1 \) we find that

\[ \left( \frac{1}{g} \right)' = -\frac{f'}{g^2} \in L^2, \quad \frac{1}{g} - 1 = -\frac{f}{g} \in L^2, \]

so that \( 1/g \in 1 + H^s \).

Finally, let \( s = n + \alpha \), where \( n \in \mathbb{N} \) and \( \alpha \in (0, 1) \). Then

\[ \left( \frac{1}{g} \right)^{(n)} = \frac{f^{(n)}}{g^2} + \psi \]

where \( \psi \in H^1 \). Since \( f^{(n)} \in H^\alpha \) and \( 1/g^2 \in H^1 \) by the above, we conclude that \( (1/g)^{(n)} \in H^{\alpha}(\mathbb{R}) \). Hence \( 1/g \in 1 + H^s \), and the proof is complete. \( \square \)

We now have the following analogue of the Wiener–Levy theorem for the algebra \( 1 + H^s \).

**Corollary A.2.** Assume that \( \Omega \) is an open subset in \( \mathbb{C} \) and that \( \phi \) is a complex-valued function that is analytic on \( \Omega \). Denote by \( M_\Omega \) the set of all elements \( g \) of \( 1 + H^s \) such that \( \text{ran } g \subset \Omega \). Then, for every \( g \in M_\Omega \), the composition \( \phi \circ g \) belongs to \( 1 + H^s \) and the mapping

\[ M_\Omega \ni g \mapsto \phi \circ g \in 1 + H^s \]

is locally Lipschitz continuous.

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