JORDAN GROUPS AND ALGEBRAIC SURFACES

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Abstract. We prove that an analogue of Jordan’s theorem on finite subgroups of general linear groups holds for the groups of biregular automorphisms of algebraic surfaces. This gives a positive answer to a question of Vladimir L. Popov.

1. Introduction

Throughout this paper, $k$ is an algebraically closed field of characteristic zero and $\mathbb{P}^1$ is the projective line over $k$. Let $U$ be an algebraic variety over $k$ [14, Vol. 2, Ch. VI, Sect. 1]. Then $U(k)$ and $\text{Aut}(U)$ stand for its set of $k$-points and the group of biregular $k$-automorphisms respectively. Unless otherwise stated, by a point of $U$ we mean a $k$-point. If $U$ is irreducible then we write $k(U)$ and $\text{Bir}(U)$ for its field of rational functions and the group of birational $k$-automorphisms respectively; $\text{Aut}(U)$ is a subgroup of $\text{Bir}(U)$. By an elliptic curve we mean an irreducible smooth projective curve of genus 1 over $k$. If $X$ is an elliptic curve and $T \subset X(k)$ is a nonempty finite set of points on $X$ then the (sub)group

$$\text{Aut}(X, T) = \{ u \in \text{Aut}(X) \mid u(T) = T \} \subset \text{Aut}(X)$$

is finite, since $X \setminus T$ is a hyperbolic curve. If $S$ is a smooth irreducible projective surface over $k$ then an irreducible closed curve $C$ in $S$ is called a $(-1)$-curve if it is smooth rational and its self-intersection index is $-1$.

The following definition was inspired by the classical theorem of Jordan [2 Sect. 36] about finite subgroups of general linear groups over fields of characteristic zero.

Definition 1.1 (Definition 2.1 of [9]). A group $B$ is called a Jordan group if there exists a positive integer $J_B$ such that every finite subgroup $B_1$ of $B$ contains a normal commutative subgroup, whose index in $B_1$ is at most $J_B$.

Remark 1.2. Clearly, a subgroup of a Jordan group is also Jordan. If a Jordan group $G_1$ is a subgroup of finite index in a group $G$ then $G$ is also Jordan.

V. L. Popov [9 Sect. 2], see also [10] posed a question whether $\text{Aut}(S)$ is a Jordan group when $S$ is an algebraic surface over $k$. He obtained a positive answer to his question for almost all surfaces. (The case of rational surfaces was treated earlier by J.-P. Serre [12 Sect. 5.4]). The only remaining case is when $S$ is birationally (but not biregularly) isomorphic to a product $X \times \mathbb{P}^1$ of an elliptic curve $X$ and the projective line. In [10] the second named author proved that $\text{Aut}(S)$ is
a Jordan group if $S$ is a projective surface. The aim of this paper is to extend this result to the case of arbitrary algebraic surfaces. Our main result is the following statement, which gives a positive answer to Popov’s question.

**Theorem 1.3.** If $X$ is an elliptic curve over $k$ and $S$ is an irreducible normal algebraic surface that is birationally isomorphic to $X \times \mathbb{P}^1$ then $\text{Aut}(S)$ is a Jordan group.

**Remark 1.4.** The group $\text{Bir}(X \times \mathbb{P}^1)$ is not Jordan [15].

**Remark 1.5.** Suppose that $S$ is a non-smooth irreducible normal surface. Since it is normal, there are only finitely many singular points on $S$. Then, by [10, Sect. 2, Cor. 8], $\text{Aut}(S)$ is Jordan. This implies that in the course of the proof of Theorem 1.3 we may assume that $S$ is smooth. On the other hand, by a theorem of Zariski [17, Cor. II.2.6 on p. 53], every irreducible smooth surface is quasi-projective. This implies that in the course of the proof of Theorem 1.3 we may assume that $S$ is smooth quasi-projective.

**Corollary 1.6.** Suppose that $V$ is an irreducible normal algebraic variety over $k$. If $\dim(V) \leq 2$ then $\text{Aut}(V)$ is Jordan.

**Proof of Corollary 1.6.** We have $\text{Aut}(V) \subset \text{Bir}(V)$. If $V$ is not birationally isomorphic to a product of the projective line and an elliptic curve then $\text{Bir}(V)$ is Jordan ([9, Th. 2.32]) and therefore its subgroup $\text{Aut}(V)$ is also Jordan. If $V$ is birationally isomorphic to a product of the projective line and an elliptic curve then $\dim(V) = 2$ and Theorem 1.3 implies that $\text{Aut}(V)$ is Jordan.

**Theorem 1.7.** Let $V$ be an irreducible algebraic variety over $k$. If $\dim(V) \leq 2$ then $\text{Aut}(V)$ is Jordan.

**Proof of Theorem 1.7.** Let $\nu : V^\nu \to V$ be the normalization of $V$ ([8, Ch. III, Sect. 8], [4, Ch. 2, Sect. 2.14]). Here $\nu$ is a birational (surjective) regular map that is called the normalization map for $V$ and $V^\nu$ is an irreducible normal variety (of the same dimension as $V$) over $k$ [8, Th. 4 on p. 203]. The universality property of the normalization map implies that every biregular automorphism of $V$ lifts uniquely to a biregular automorphism of $V^\nu$ [4, Ch. 2, Sect. 2.14, Th. 2.25 on p. 141]. This give rise to the embedding of groups

$$\text{Aut}(V) \hookrightarrow \text{Aut}(V^\nu).$$

By Corollary 1.6 the group $\text{Aut}(V^\nu)$ is Jordan. Since $\text{Aut}(V)$ is isomorphic to a subgroup of Jordan $\text{Aut}(V^\nu)$, it is also Jordan.

**Corollary 1.8.** Let $V$ be an algebraic variety over $k$. If $\dim(V) \leq 2$ then $\text{Aut}(V)$ is Jordan.

**Proof.** Let $V_1, \ldots, V_r$ be all the irreducible components of $V$. Clearly, all $V_i$ are irreducible algebraic varieties with $\dim(V_i) \leq \dim(V) \leq 2$. By Theorem 1.7 all $\text{Aut}(V_i)$ are Jordan. Now Lemma 1 in Section 2.2 of [10] implies that $\text{Aut}(V)$ is also Jordan.

**Remark 1.9.** Suppose that $k$ is the field $\mathbb{C}$ of complex numbers and $X$ is a smooth irreducible quasi-projective non-projective surface. Then $M = X(\mathbb{C})$ carries the natural structure of a connected oriented smooth real noncompact fourfold and the
group Aut(\(X\)) embeds naturally in the group Diff(\(M\)) of the (real) diffeomorphisms of the fourfold \(M\). While Aut(\(X\)) is always Jordan, there are examples of connected oriented smooth noncompact real fourfolds, whose group of diffeomorphisms is not Jordan \([11]\).

The paper is organized as follows. In Section 2 we discuss minimal closures of surfaces. In Section 3 we prove Theorem 1.3.

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2. Minimal closures

2.1. Let \(X\) be an elliptic curve over \(k\) and \(S\) be a smooth irreducible quasi-projective surface over \(k\) that is birationally isomorphic to \(X \times \mathbb{P}^1\). There exists an irreducible smooth projective surface \(\bar{S}\) such that its certain Zariski-open subset is biregularly isomorphic to \(S\) (further we identify \(S\) with this open subset). Clearly, the inclusion map \(S \subset \bar{S}\) is a birational morphism. This implies that

\[
\text{Aut}(S) \subset \text{Bir}(S) = \text{Bir}(\bar{S})
\]

and therefore one may view Aut(\(S\)) as a subgroup of Bir(\(S\)). Since \(\bar{S}\) is birationally isomorphic to \(S\), it is also birationally isomorphic to \(X \times \mathbb{P}^1\).

Let us fix a birational isomorphism between \(\bar{S}\) and \(X \times \mathbb{P}^1\). The projection map \(X \times \mathbb{P}^1 \to X\) gives rise to a rational map \(\bar{\pi} : \bar{S} \to X\) with dense image. Since \(\bar{S}\) is smooth and \(X\) becomes abelian variety (after a choice of a base point), it follows from a theorem of Weil [1, Sect. 4.4] that \(\bar{\pi}\) is regular. Since \(\bar{S}\) is projective, \(\bar{\pi} : \bar{S} \to X\) is surjective, because its image is closed.

For each \(x \in X(k)\) we write \(\bar{F}_x\) for the effective divisor \(\bar{\pi}^*(x)\) on \(\bar{S}\) that is the pullback (under \(\bar{\pi}\)) of the divisor \((x)\) on \(S\). Clearly, the support of \(\bar{F}_x\) coincides with the curve \(\bar{\pi}^{-1}(x)\) on \(\bar{S}\). One say that the fiber of \(\bar{\pi}\) over \(x\) is reduced if all irreducible components of the divisor \(\bar{F}_x\) have multiplicity 1. We say that the fiber of \(\bar{\pi}\) over \(x\) is irreducible if the curve \(\bar{\pi}^{-1}(x)\) is irreducible; if this is the case then its multiplicity in \(\bar{F}_x\) is 1 [6, Ch. 1, Sect. 1.4, Lemma 1.4.1(1) on p. 195].

It is known [13, Ch. IV] that for all but finitely many \(x \in X(k)\) the fiber of \(\bar{\pi}\) over \(x\) is irreducible and reduced, and the curve \(\bar{\pi}^{-1}(x)\) is smooth (and irreducible). We call such fibers nonsingular and other fibers singular.

If \(C\) is a rational curve on \(\bar{S}\) then the restriction of \(\bar{\pi}\) to \(C\) must be a constant map, because every map from a rational curve to an elliptic curve is constant. This implies that \(C\) lies in a fiber of \(\bar{\pi}\). (In particular, every \((-1)\)-curve on \(\bar{S}\) lies in a fiber of \(\bar{\pi}\).) This implies that every birational automorphism of \(\bar{S}\) is fiberwise [4, Sect. 13, Th. 2]; see Sect. 2.2 below.

However, if \(x \in X(k)\) and the fiber \(\bar{\pi}^{-1}(x)\) is singular then the corresponding divisor \(\bar{F}_x\) enjoys the following properties [6, Ch. I, Sect. 2.12; Ch. 3, Sect. 1.4, Lemma 1.4.1 on p. 195]] (see also [3]).
(i) Each irreducible component of $\bar{F}_x$ is a smooth rational curve (and the corresponding graph is a tree) \cite[Sect. 3].

(ii) At least, one of the irreducible components of $\bar{F}_x$ is a $(-1)$-curve \cite[Sect. 4.2].

(iii) If one of the irreducible components of $\bar{F}_x$ is an irreducible component of $\bar{F}_x$ (\cite[Sect. 1.4, p. 99]{7}). Clearly, $\pi$ is infinite and therefore is everywhere dense in $\{1\} \rightarrow \text{Aut}_X(\bar{S}) \subset \text{Bir}(\bar{S}) \xrightarrow{\pi} \text{Aut}(X) \rightarrow \{1\}$ where the subgroup $\text{Bir}_X(\bar{S})$ consists of all birational automorphisms $\sigma \in \text{Bir}(\bar{S})$ such that $\bar{\sigma} = \bar{\pi}$ (i.e., $\sigma$ leaves invariant every fiber of $\bar{\pi}$). In addition, $\text{Bir}_X(\bar{S})$ is isomorphic to the projective linear group $\text{PGL}(2, k(X))$ over the field $k(X)$ of rational functions on $X$ \cite[Lecture V, Sect. 1.4, p. 99]{7}.

2.2. If $\sigma \in \text{Bir}(\bar{S})$ then there is a unique biregular automorphism $f(\sigma) : X \rightarrow X$ such that the composition $\bar{\sigma} = \bar{\pi} \circ \pi \circ f(\sigma)$ is regular. Clearly, $\sigma$ sends the fiber $\bar{\pi}^{-1}(x)$ to the fiber $\pi^{-1}(f(\sigma)(x))$ for all $x \in X(k)$. We get a surjective group homomorphism $f : \text{Bir}(\bar{S}) \rightarrow \text{Aut}(X)$, $\sigma \mapsto f(\sigma)$

that fits into a short exact sequence

$$\{1\} \rightarrow \text{Aut}_X(S) \subset \text{Bir}_X(\bar{S}) \xrightarrow{\pi} \text{Aut}(X) \rightarrow \{1\}$$

where $\text{Aut}_X(S)$ is the pullback (under $\pi$) of the divisor $\pi^*(x)$ on $S$ that is the pullback (under $\pi$) of the divisor $\pi^*(x)$ on $S$. Clearly, the support of $\bar{F}_x$ coincides with the curve $\pi^{-1}(x)$ on $S$. It is also clear that the divisor $F_x$ on $S$ is the pullback of the divisor $\bar{F}_x$ on $S$ under the (open) inclusion map $S \subset \bar{S}$. One says that the fiber of $\pi$ over $x$ is reduced if all irreducible components of the divisor $F_x$ have multiplicity $1$. We say that the fiber of $\pi$ over $x$ is irreducible if it is a multiple of a simple divisor, i.e., the curve $\pi^{-1}(x)$ is irreducible. Clearly, if the fiber of $\bar{\pi}$ over $x$ is irreducible (resp. reduced, resp. smooth) then the fiber of $\pi$ over $x$ is irreducible (resp. reduced, resp. smooth). On the other hand, if $\bar{F}_x$ has an irreducible component, say, $\bar{C}$ that appears in $\bar{F}_x$ with multiplicity $m > 1$ and, in addition, $\bar{C}$ meets $S$ then $C := \bar{C} \cap S$ is an irreducible curve in $S$ that is a component of $F_x$ and that appears in $F_x$ with the same multiplicity.
in particular, the fiber of $\pi$ over $x$ is not reduced. Notice also that if $\bar{C}_1$ and $\bar{C}_2$ are distinct irreducible components of $\bar{F}_x$ that meet $F_x$ then $C_1 := \bar{C}_1 \cap S$ and $C_2 := \bar{C}_2 \cap S$ are distinct irreducible components of $F_x$; in particular, the fiber of $\pi$ over $x$ is not irreducible.

It follows from the results about the fibers of $\bar{\pi}$ mentioned in Sect. 2.1 (see also theorems of Bertini [14] vol. 1, Ch. 2, Sect. 6.1 and 6.2) that either all the fibers of $\pi$ are smooth irreducible reduced or the set $T_1$ of points $x \in \pi(S(k)) \subset X(k)$ such that, at least, one of these properties does not hold, is finite. Clearly,

$$f(\text{Aut}(S)) \subset \text{Aut}(X, T_0), \ f(\text{Aut}(S)) \subset \text{Aut}(X, T_1).$$

This implies that if either $T_0$ or $T_1$ is non-empty then $f(\text{Aut}(S))$ is a finite group and $\text{Aut}_X(\bar{S})$ is a subgroup of finite index in $\text{Aut}(S)$.

**2.4.** It follows from the theorem of Jordan that the projective linear group $\text{PGL}(2, k(X))$ is Jordan [9, 10]. Since $\text{Bir}_X(\bar{S})$ is isomorphic to $\text{PGL}(2, k(X))$ (see Sect. 2.2), it is also a Jordan group. This implies in turn that its subgroup $\text{Aut}_X(\bar{S})$ is also Jordan. It follows that if either $T_0$ or $T_1$ is non-empty then $\text{Aut}(S)$ contains the Jordan subgroup $\text{Aut}_X(\bar{S})$ of finite index and therefore is Jordan itself, thanks to Remark 1.2

In order to handle the case of empty $T_0$ and $T_1$, we need additional ideas.

**Definition 2.5.** The projective surface $\bar{S}$ is called a (relative) minimal closure of $S$ if every $(-1)$-curve on $\bar{S}$ meets $S$. See [3, Sect. 4.9]. A minimal closure of $S$ always exists [3, Prop. 4.10]. (Warning: if $\bar{S}$ is a minimal closure then the complement of $S$ in $\bar{S}$ does not have to be a divisor!)

**Lemma 2.6** (Lemma 4.12 of [3]). Assume that $\pi(S) = X$ and all the fibers of $\pi$ are smooth irreducible and reduced.

If $\bar{S}$ is a minimal closure of $S$ then all the fibers of $\bar{\pi} : \bar{S} \to X$ are irreducible.

**Proof.** Suppose that there exists $x \in X(k)$ such that the fiber of $\bar{\pi}$ over $x$ is not irreducible and therefore is singular. Then $\bar{F}_x$ contains an irreducible component $C_1$ with multiplicity $m \geq 1$ (Sect. 2.1). The minimality of $\bar{S}$ implies that $C_1 = \bar{C}_1 \cap S$ is non-empty and therefore is an irreducible component of $F_x$ with the same multiplicity $m$ (Sect. 2.3). Since the fiber of $\pi$ over $x$ is reduced, $m = 1$. This implies that $\bar{F}_x$ contains another irreducible component $\bar{C}_2$ that is also a $(-1)$-curve. Again $C_2 = \bar{C}_2 \cap S$ is an irreducible component of $F_x$ that does not coincide with $C_1$. This implies that the fiber of $\pi$ over $x$ is not irreducible, which is not the case. \qed

**Theorem 2.7.** Assume that $\pi(S) = X$ and all the fibers of $\pi$ are smooth irreducible and reduced. Let $\bar{S}$ be a minimal closure of $S$. Then every biregular automorphism of $\bar{S}$ extends uniquely to a biregular automorphism of $\bar{S}$. In other words,

$$\text{Aut}(S) \subset \text{Aut}(\bar{S}) \subset \text{Bir}(\bar{S}).$$

**Proof.** By Lemma 2.6 every fiber $\bar{F}_x$ is an irreducible curve isomorphic to $\mathbb{P}^1$.

Let $g : S \to S$ be a biregular automorphism of $S$. Let us extend $g$ to a birational map

$$\bar{g} : \bar{S} \to \bar{S}.$$
Assume that $\tilde{g}$ is not a regular map. Let $S'$ be a resolution of the indeterminacies of $\tilde{g}$, i.e. a smooth irreducible surface included into the following commutative diagram.

\[
\begin{array}{cccc}
S' & \xrightarrow{u} & \tilde{S} & \xrightarrow{\tilde{g}} S \\
\downarrow & & \cup & \\
S & \xrightarrow{g} S & \cup & \\
\downarrow & & \downarrow & \\
\pi \downarrow & & \pi & \\
X & \xrightarrow{h} X
\end{array}
\]

where $u$ is a birational morphism that is a composition of finitely many blow ups and induces a biregular isomorphism between $u^{-1}(S)$ and $S$ (such an $u$ exists, because $g$ is defined on $S$), $g'$ and $\tilde{\pi} = \tilde{\pi} \circ u$ are morphisms, and $h = f(g) \in \text{Aut}(X)$ is a biregular automorphism of $X$. (The group homomorphism $f$ is defined in Sect. 2.2.)

Let $D' \subset S'$ be the union of all exceptional curves for $g'$ and let $D = g'(D') \subset \bar{S}$, which is a finite set.

Every point $z$ of $\tilde{S}$ that does not lie on $D$ has only one preimage $g'^{-1}(z) \in S'$ ([14 Ch. 2, Sect. 4, Th. 2]).

Let $B'$ be the union of exceptional curves for $u$. Clearly,

\[ B' \subset S' \setminus u^{-1}(S) \]

This implies that

\[ u(B') \cap S = \emptyset. \]

We want to show that $B' \subset D'$, because then one may contract all components of $B'$ and $\tilde{g}$ would appear to be a morphism.

Let $C'$ be an irreducible component of $B'$. The point $u(C')$ lies in $u(B')$ and therefore does not belong to $S$.

Since $X$ is an elliptic curve, and $C'$ is rational, $\bar{\pi}(g'(C'))$ is a point $x \in X(k)$. Thus, since all the fibers of $\bar{\pi}$ are irreducible (thanks to Lemma 2.6), either

**Case 1.** $g'(C')$ is a point and therefore $C' \subset D'$;

or

**Case 2.** $g'(C') = \bar{F}_x = \bar{\pi}^{-1}(x) \subset \bar{S}$. Let us put $x_1 := h^{-1}(x) \in X(k)$. Then $x = h(x_1) \in X(k)$. Let $s \in F_x \setminus (F_x \cap D) \subset S$ be a point of the fiber $F_x$, which is not in the image of $\bar{D}$. Therefore it has only one preimage $s_1 := g'^{-1}(s)$. Moreover, $s_1 \in u^{-1}(S)$, because $s \in S$. On the other hand, since $g'(C') = \bar{F}_x$, there is a point $c \in C' \subset S' \setminus u^{-1}(S)$ such that $g'(c) = s$. Clearly, $c \neq s_1$ and we get a contradiction that shows that the Case 2 does not occur.

This proves that every $g \in \text{Aut}(S)$ extends to a regular birational map $\tilde{g} : \tilde{S} \to \tilde{S}$. Since the same is true for $\bar{g}^{-1} \in \text{Aut}(S)$, the map $\tilde{g}$ is a biregular automorphism of $\bar{S}$.

\[ \square \]

### 3. Proof of Theorem 1.3

Remark 1.3 tells us that we may assume that $S$ is a smooth quasi-projective surface. In light of results of Section 2.2, we may also assume that every fiber of $\pi$ is smooth irreducible and reduced, and $\pi(S) = X$. Let $\bar{S}$ be a minimal closure of $S$. 

By Theorem 2.7, Aut(S) is a subgroup of Aut(\(\bar{S}\)). Since \(\bar{S}\) is projective, the results of [16] imply that the group Aut(\(\bar{S}\)) is Jordan and therefore its every subgroup is Jordan. It follows that Aut(S) is Jordan.

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