Research Paper

Combining alpha streams with costs

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ABSTRACT

In this paper, we discuss investment allocation to multiple alpha streams traded on the same execution platform with internal trade crossing. We then point out the differences between such a case and investment allocation when alpha streams are traded on separate execution platforms with no crossing. In the latter case, allocation weights are nonnegative; in the former, they can be negative. The effects of linear and nonlinear (impact) costs are different in these two cases due to turnover reduction when the trades are crossed. The turnover reduction depends on the universe of traded alpha streams, so if some alpha streams have zero allocations, turnover reduction needs to be recomputed; hence, an iterative procedure is required. We discuss an algorithm for finding allocation weights with crossing and linear costs. We also discuss a simple approximation when nonlinear costs are added, making the allocation problem tractable while still capturing nonlinear portfolio capacity bound effects. We also define “regression with costs” as a limit of optimization with costs, which is useful in often-occurring cases with a singular alpha covariance matrix.

Keywords: alpha stream; crossing trades; transaction costs; impact; portfolio turnover; investment allocation.
1 MOTIVATION AND SUMMARY

Combining multiple hedge fund alpha streams has the benefit of diversification.\(^1\) We then need to determine how to allocate investment into these different alpha streams \(\alpha_i\) or, mathematically speaking, how to determine the weights \(w_i\) with which the investment should be allocated to individual alphas.\(^2\)

If individual alpha streams are traded on separate execution platforms, then the weights are nonnegative: \(w_i \geq 0\). This applies to the hedge fund of funds vehicles, which take long positions in individual hedge fund alpha streams, as well as long-only mutual funds. This is also irrespective of whether transaction costs are included or not. So, the investment allocation problem is some portfolio optimization problem, whereby we determine the weights \(w_i\) based on an optimization criterion,\(^3\) eg, maximizing the Sharpe ratio, profit and loss (P&L), maximizing the P&L subject to a condition on the Sharpe ratio, etc. Invariably, this portfolio optimization involves the requirement that the weights are nonnegative. Since the weights are nonnegative, if we include linear costs \(L_i\), the P&L is simply given by (\(I\) is the investment level; see Section 2.1 for more detail)

\[
P = I \sum_i (\alpha_i - L_i)w_i.
\]

So, adding the linear cost \(L_i > 0\) simply has the effect of reducing the alpha, \(\alpha_i\).

Combining and trading multiple hedge fund alpha streams on the same execution platform has the further benefit that, by internally crossing the trades between different alpha streams (as opposed to going to the market), we benefit from substantial savings

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\(^1\) For a partial list of hedge fund literature, see, for example, Schneeweis et al (1996), Ackerman et al (1999), Brown et al (1999), Edwards and Liew (1999b), Edwards and Liew (1999a), Fung and Hsieh (1999), Liang (1999), Agarwal and Naik (2000b), Agarwal and Naik (2000a), Fung and Hsieh (2000), Liang (2000), Asness et al (2001), Edwards and Caglayan (2001), Fung and Hsieh (2001), Liang (2001), Lo (2001), Brooks and Kat (2002), Kao (2002), Amin and Kat (2003) and Chan et al (2006), and references therein.

\(^2\) By “alpha” we mean any “expected return”; a priori it need not even be stock based.

\(^3\) For a partial list of portfolio optimization and related literature, see, for example, Markowitz (1952), Charnes and Cooper (1962), Sharpe (1969), Merton (1969), Schaible (1974), Magill and Constantinides (1976), Perold (1984), Davis and Norman (1990), Dumas and Luciano (1991), Adcock and Meade (1994), Shreve and Soner (1994), Bienstock (1996), Cvitanić and Karatzas (1996), Yoshimoto (1996), Atkinson et al (1997), Bertsimas et al (1999), Cadenillas and Pliska (1999), Chang et al (2000), Kellerer et al (2000), Rockafellar and Uryasev (2000), Gondzio and Kouwenberg (2001), Konno and Wijayanayake (2001), Mokkhavesa and Atkinson (2002), Costa and Paiva (2002), Alizadeh and Goldfarb (2003), Best and Hlouskova (2003), Janeček and Shreve (2004), Lobo et al (2007), Zagst and Kalin (2007), Potaptchik et al (2008), Moro et al (2009), Goodman and Ostrov (2010), Bichuch (2012), Mitchell and Braun (2013) and Soner and Touzi (2013), and references therein.
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on transaction costs (for a recent discussion, see Kakushadze and Liew (2014)). In this framework, the weights with which the alphas are combined need no longer be nonnegative. This is because, due to different alphas being correlated with each other, the optimal allocation for the weights can be such that some alphas are traded in reverse, against their originally intended signal. On the one hand, we no longer have the \( w_i > 0 \) bound, which simplifies the optimization problem. On the other hand, when costs are included, this leads to a complication, because the costs are positive whether a given alpha is traded along or against the signal. For example, in the case of linear costs, the P&amp;L now becomes (assuming, for the sake of simplicity, the same linear cost regardless of the direction of trading; see Section 2.1 for more detail):

\[
P = I \sum_i (\alpha_i w_i - L_i |w_i|).
\]

(1.2)

It is the modulus in \( L_i |w_i| \) that complicates the weight-optimization problem, both in the case of linear cost and when nonlinear costs, or the impact of trading on prices, are included.

Yet another issue arises when we account for turnover reduction due to internal crossing. Without internal crossing, turnover \( T \) of the combined portfolio is simply the weighted sum of the individual turnovers \( \tau_i \) (by \( \tau_i \equiv D_i/I_i \), we mean the percentage of the dollar turnover \( D_i \) of the individual alpha stream \( \alpha_i \), with respect to the total dollar investment \( I_i \), into this alpha stream, assuming it is traded separately without any crossing with other alpha streams):

\[
T = \sum_i \tau_i |w_i|.
\]

(1.3)

However, when trades are crossed, turnover reduces, and when the number of alphas is large, the following model is expected to provide a good approximation (Kakushadze 2014):

\[
T \approx \rho_* \sum_i \tau_i |w_i|,
\]

(1.4)

where \( 0 < \rho_* \leq 1 \) is the turnover reduction coefficient. In Kakushadze (2014), we proposed a spectral model for estimating \( \rho_* \), which is based on the correlation

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4 For example, consider two alphas, \( \alpha_1 > 0 \) and \( \alpha_2 > 0 \), with unit variances and correlation \( \rho > 0 \), with no costs. The Sharpe ratio \( S \) is maximized by \( w_1 = \gamma (\alpha_1 - \rho \alpha_2), w_2 = \gamma (\alpha_2 - \rho \alpha_1) \), where \( \gamma \) is fixed from \( |w_1| + |w_2| = 1 \). If \( \alpha_2 < \rho \alpha_1 \), then \( w_2 < 0 \), so \( S \rightarrow \max \) requires “shorting” \( \alpha_2 \).

5 In this regard, optimizing alpha streams in the context of trading them on the same execution platform is different from stock portfolio optimization. With stocks, there is no “internal crossing” or “turnover reduction”. Another difference is that, with stocks, for dollar-neutral portfolios there is a constraint \( \sum_a d_a = 0 \), where \( d_a \) is the dollar holding for the stock labeled by \( a \). For alphas, we have the condition on the weights \( \sum_i |w_i| = 1 \) instead.
matrix of the alphas. It is designed to work when the number of alphas is large and the distribution of individual turnovers, \( \tau_i \), is not skewed. The turnover reduction coefficient \( \rho_* \) depends on the universe of alphas that are being traded – this is the case in the aforementioned spectral model, and it is also expected to be a model-independent property. In this regard, if, upon solving the optimization problem, some weights \( w_i \) turn out to be zero, then \( \rho_* \) needs to be recomputed with the corresponding alphas dropped, and optimization needs to be repeated with the recomputed \( \rho_* \). In fact, this process needs to be repeated iteratively until it converges. This is yet another feature specific to alpha stream optimization with internal crossing.

Thus, determining the optimal allocation of weights when alphas are traded on the same execution platform and trades are internally crossed is a rather different optimization problem than finding the weights when we combine alpha streams traded on separate trading platforms. In this paper, motivated by these differences, we discuss the optimization problem in the framework of combining alphas traded on the same trading platform with internal crossing. We discuss an algorithm (which requires a finite number of iterations) for finding \( w_i \) in the presence of linear costs. The optimization criterion is taken to be maximizing the Sharpe ratio, and the alpha covariance matrix is taken to be of a factor-model form. We also discuss the case when nonlinear (impact) costs are added. We discuss a simple approximation in this case, which makes the optimization problem tractable while still capturing the nonlinear dependence on the investment level that governs the portfolio capacity bounds. We also discuss the case where the alpha covariance matrix is singular, which often occurs in practical applications. We cover how to do “regression with costs”, which is a limit of optimization with costs.

The remainder of this paper is organized as follows. In Section 2, we give our notations and setup. In Section 3, we discuss optimization with linear costs. In Section 4, we discuss optimization with linear costs plus impact. In Section 5, we discuss the regression limit of optimization with costs.

### 2 DEFINITIONS AND SETUP

We have \( N \) alphas \( \alpha_i, i = 1, \ldots, N \). Each alpha is actually a time series \( \alpha_i(t_s), s = 0, 1, \ldots, M \), where \( t_0 \) is the most recent time. Below, \( \alpha_i \) refers to \( \alpha_i(t_0) \).

Let \( C_{ij} \) be the covariance matrix of the \( N \) time series \( \alpha_i(t_s) \). Let \( \Psi_{ij} \) be the corresponding correlation matrix, ie,

\[
C_{ij} = \sigma_i \sigma_j \Psi_{ij}, \tag{2.1}
\]

where \( \Psi_{ii} = 1 \). If \( M < N \), then only \( M \) eigenvalues of \( C_{ij} \) are nonzero, while the remainder have “small” values, which can be positive or negative. These small values
are zeros distorted by computational rounding. In such cases, we can deform the covariance matrix so it is positive-definite (see Section 3.1 of Kakushadze (2014) for a deformation method based on Rebonato and Jäckel (1999)). Still, the off-diagonal elements of the sample covariance matrix (or a deformation thereof) typically are not expected to be too stable out of sample. In this regard, instead of using a computed sample covariance matrix (based on the alpha time series), we can use a much more stable, constructed factor model covariance matrix, which we discuss in Section 3.1.

To begin with, we will ignore trading costs. Alphas $\alpha_i$ are combined with weights $w_i$. Portfolio P&L, volatility and the Sharpe ratio are given by

$$P = I \sum_{i=1}^{N} \alpha_i w_i,$$

$$R = I \sqrt{\sum_{i,j=1}^{N} C_{ij} w_i w_j},$$

$$S = \frac{P}{R},$$

where $I$ is the investment level. Any leverage is included in the definition of $\alpha_i$, ie, if a given alpha labeled by index $\ell \in [1, \ldots, N]$ before leverage is $\tilde{\alpha}_\ell$ (this is a raw, unlevered alpha), and the corresponding leverage is $K_\ell$: 1, then we define $\alpha_\ell \equiv K_\ell \tilde{\alpha}_\ell$. With this definition, the weights satisfy the condition

$$\sum_{i=1}^{N} |w_i| = 1.$$  

Here, we allow the weights to be negative. This is because we are primarily interested in the case where the alphas are traded on the same execution platform and trades between alphas are crossed, so we are actually trading the combined alpha. Since generically there are nonzero correlations between different alphas (that is, at least some off-diagonal elements of the correlation matrix $\Psi_{ij}$ are nonzero), the optimal solution can have some negative weights, ie, it is more optimal to trade some alphas reversed.

In this paper, we will focus on the optimization where we maximize the Sharpe ratio:

$$S \rightarrow \text{max}.$$  

6 Actually, this assumes that there are no N/As in any of the alpha time series. If some or all alpha time series contain N/As in a nonuniform manner, and the correlation matrix is computed by omitting such pairwise N/As, then the resulting correlation matrix may have negative eigenvalues that are not “small” in the sense used above, ie, they are not zeros distorted by computational rounding. The deformation method mentioned below can be applied in this case as well.
We will assume that there are no upper or lower bounds on the weights; our primary goal here is to set the framework for optimization with linear and nonlinear costs.

The solution to (2.6) in the absence of costs is given by

$$w_i = y \sum_{j=1}^{N} C_{ij}^{-1} \alpha_j,$$  \hspace{1cm} (2.7)

where $C^{-1}$ is the inverse of $C$, and the normalization coefficient $y$ is determined from (2.5). Without delving into any details, we simply assume that $C$ is invertible or is made as such (e.g., via a deformation (see, for example, a method discussed in Kakushadze (2014) based on Rebonato and Jäckel (1999))). We will discuss the case where $C$ is singular in Section 5.

If $C_{ij}$ is diagonal and we have all $\alpha_i > 0$, then all $w_i$ are also positive. However, when we have nonzero correlations between alphas, some weights can be negative even if all alphas are positive. A simple example is given in Footnote 4.

2.1 Linear costs

Linear costs can be modeled by subtracting a linear penalty from the P&L:

$$P = I \sum_{i=1}^{N} \alpha_i w_i - LD,$$  \hspace{1cm} (2.8)

where $L$ includes all fixed trading costs (Securities and Exchange Commission fees, exchange fees, broker–dealer fees, etc) and linear slippage.\(^7\) The linear cost assumes no impact, i.e., trading does not affect the stock prices. Each alpha is assumed to trade a large number of stocks. Each individual stock has its own contribution to linear cost, which depends on its liquidity, volatility, etc. When summed over a large number of stocks and a large number of alphas, the linear cost can be modeled (with the caveat mentioned in Footnote 7) as being proportional to the dollar turnover (i.e., the dollar amount traded by the portfolio) $D = IT$, where $T$ is what we refer to simply as the turnover (so the turnover $T$ is defined as a percentage). Details are relegated to Appendix A, which discusses the relation between, on the one hand, $D$, $T$ and $w_i$ (which are the quantities typically used in the optimization discussions), and, on the other hand, the individual stock prices and shares traded (which are the quantities typically used in the transaction cost discussions).

Let $\tau_i$ be the turnovers corresponding to individual alphas $\alpha_i$. If we ignore turnover reduction resulting from combining alphas (or if the internal crossing is switched off),

\(^7\) Here, for simplicity, the linear slippage is assumed to be uniform across all alphas. This is not a critical assumption and can be relaxed, e.g., by modifying the definition of $L_i$ below. In essence, this assumption is made to simplify the discussion of turnover reduction.
then

\[ T = \sum_{i=1}^{N} \tau_i |w_i|. \]  

(2.9)

However, with internal crossing, turnover reduction can be substantial and needs to be taken into account. In Kakushadze (2014), we proposed a model of turnover reduction, according to which when the number of alphas \( N \) is large, the leading approximation (in the \( 1/N \) expansion) is given by

\[ T \approx \rho_* \sum_{i=1}^{N} \tau_i |w_i|. \]  

(2.10)

where \( 0 < \rho_* \leq 1 \) is the turnover reduction coefficient. Let us emphasize that this formula is expected to be a good approximation in the large \( N \) limit (so long as the distribution of individual turnovers \( \tau_i \) is not skewed), regardless of how \( \rho_* \) is modeled.

In Kakushadze (2014), we also proposed a spectral model for estimating \( \rho_* \) based on the correlation matrix \( \Psi_{ij} \):

\[ \rho_* \approx \frac{\psi^{(1)}}{N \sqrt{N}} \left| \sum_{i=1}^{N} \tilde{V}_i^{(1)} \right|, \]  

(2.11)

where \( \psi^{(1)} \) is the largest eigenvalue of \( \Psi_{ij} \), and \( \tilde{V}_i^{(1)} \) is the corresponding eigenvector normalized such that

\[ \sum_{i=1}^{N} (\tilde{V}_i^{(1)})^2 = 1. \]

We then have

\[ P = I \sum_{i=1}^{N} (\alpha_i w_i - L_i |w_i|), \]  

(2.12)

where

\[ L_i \equiv L \rho_* \tau_i > 0. \]  

(2.13)

Note that, under the rescaling \( w_i \to \zeta w_i (\zeta > 0) \), we have \( P \to \zeta P, R \to \zeta R \) and \( S = \text{inv} \). This allows us to recast the Sharpe ratio maximization condition (2.6) into the following minimization problem:

\[ g(w, \lambda) \equiv \frac{\lambda}{2} \sum_{i,j=1}^{N} C_{ij} w_i w_j - \sum_{i=1}^{N} (\alpha_i w_i - L_i |w_i|), \]  

(2.14)

\[ g(w, \lambda) \to \min, \]  

(2.15)

where \( \lambda > 0 \) is a free parameter, which is determined after the minimization with respect to \( w_i \) (with \( \lambda \) fixed) from the requirement (2.5). If it were not for the modulus
in $L_i|w_i|$, this optimization problem would be solvable in closed form. The modulus complicates things a bit. The problem can still be solved; however, it requires a finite iterative procedure, ie, the solution (formally) is exact and obtained after a finite number of iterations.$^8$

3 OPTIMIZATION WITH LINEAR COSTS

Let $J$ and $J'$ be the subsets of the index $i = 1, \ldots, N$, such that

$$w_i \neq 0, \quad i \in J,$$

$$w_i = 0, \quad i \in J'.$$ (3.1)

Let

$$\eta_i \equiv \text{sgn}(w_i), \quad i \in J.$$ (3.2)

Note that, since the modulus has a discontinuous derivative, the minimization equations are not the same as setting the first derivatives of $g(w, \lambda)$ with respect to $w_i$ to zero. More concretely, first derivatives are well-defined for $i \in J$ but not for $i \in J'$. So, we have the following minimization equations for $w_i, i \in J$:

$$\lambda \sum_{j \in J} C_{ij} w_j - \alpha_i \eta_i = 0, \quad i \in J.$$ (3.3)

There are additional conditions for the global minimum$^9$ corresponding to the directions $i \in J'$:

$$\frac{\lambda}{2} \sum_{i,j=1}^{N} C_{ij} (w_i + \epsilon_i)(w_j + \epsilon_j) - \sum_{i=1}^{N} (\alpha_i (w_i + \epsilon_i) - L_i |w_i + \epsilon_i|)$$

$$\geq \frac{\lambda}{2} \sum_{i,j=1}^{N} C_{ij} w_j w_j - \sum_{i=1}^{N} (\alpha_i w_j - L_j |w_j|),$$ (3.4)

where $w_i, i \in J$, are determined using (3.4), while $w_j = 0, i \in J'$. The conditions (3.5) must be satisfied, including for arbitrary infinitesimal $\epsilon_i$. Taking into account (3.4), these conditions can be rewritten as follows$^{10}$:

$$\sum_{j \in J'} \left( \lambda \sum_{i \in J} C_{ij} w_i \epsilon_j - \alpha_j \epsilon_j + L_j |\epsilon_j| \right) \geq 0.$$ (3.5)

$^8$ More precisely, this is the case when the covariance matrix takes a factor-model form (see below).

$^9$ The global optimum conditions are discussed in Appendix B.

$^{10}$ Since here $\epsilon_i$ are taken to be infinitesimal, these are the conditions for a local minimum. In Appendix B, we show that the local minimum we find here is also the global minimum.
Since $\epsilon_j, j \in J'$, are arbitrary (albeit infinitesimal), this gives the following conditions:
\[
\forall j \in J', \quad \left| \lambda \sum_{i \in J} C_{ij} w_i - \alpha_j \right| \leq L_j.
\] (3.7)

These conditions must be satisfied by the solution to (3.4). The solution that minimizes $g(w, \lambda)$ is given by
\[
w_i = \frac{1}{\lambda} \sum_{j \in J} D_{ij} (\alpha_j - L_j \eta_j), \quad i \in J,
\] (3.8)
and $D$ is the inverse matrix of the $N(J) \times N(J)$ matrix $C_{ij}, i, j \in J$, where $N(J) \equiv |J|$ is the number of elements of $J$:
\[
\sum_{k \in J} C_{ik} D_{kj} = \delta_{ij}, \quad i, j \in J.
\] (3.9)

ie, $D$ is not a restriction of the inverse of the $N \times N$ matrix $C_{ij}$ to $i, j \in J$.

Here, the following observation is in order. In the above solution, a priori, we do not know
(i) what the subset $J'$ is,
(ii) what the values of $\eta_i$ are for $i \in J$.

This means, a priori, we have a total of $3^N$ possible combinations (including the redundant, empty $J$ case), so if we go through this finite set, we will solve the problem exactly. However, $3^N$ is a prohibitively large number for any decent number of alphas, which we in fact assume to be large, so we need a cleverer way of solving the problem.

3.1 Factor model

We need to reduce the number of iterations. In this regard, the following observation is useful. Suppose, for a moment, that $C_{ij}$ were diagonal:
\[
C_{ij} = \xi_i^2 \delta_{ij}.
\]

Then (3.6) simplifies, and we have $w_i = 0$ for $i \in J'$, such that $|\alpha_i| \leq L_i$, and for $i \in J$, such that $|\alpha_i| > L_i$. From (3.4), we have $\eta_i = \text{sgn}(\alpha_i)$ and
\[
w_i = \frac{[\alpha_i - L_i \text{sgn}(\alpha_i)]}{\lambda \xi_i^2},
\]

ie, in this case, we do not need any iterations. This suggests that, if we reduce the “off-diagonality” of $C_{ij}$, the number of required iterations should also decrease.
This can be achieved by considering a factor model for alphas. Just as in the case of a stock multifactor risk model, instead of \( N \) alphas, we deal with \( F \) risk factors, and the covariance matrix \( C_{ij} \) is replaced by \( \Gamma_{ij} \), given by

\[
\Gamma \equiv \Xi + \Omega \Phi \Omega^T, \quad (3.10)
\]

\[
\Xi_{ij} \equiv \xi_i^2 \delta_{ij}, \quad (3.11)
\]

where \( \xi_i \) is the specific risk for each \( \alpha_i \), \( \Omega_{iA} \) is an \( N \times F \) factor loadings matrix and \( \Phi_{AB} \) is the factor covariance matrix, \( A, B = 1, \ldots, F \), ie, the random processes \( \gamma_i \) corresponding to \( N \) alphas are modeled via \( N \) random processes \( z_i \) (corresponding to specific risk), together with \( F \) random processes \( f_A \) (corresponding to factor risk):

\[
\gamma_i = z_i + \sum_{A=1}^{F} \Omega_{iA} f_A, \quad (3.12)
\]

\[
\langle z_i, z_j \rangle = \Xi_{ij}, \quad (3.13)
\]

\[
\langle z_i, f_A \rangle = 0, \quad (3.14)
\]

\[
\langle f_A, f_B \rangle = \Phi_{AB}, \quad (3.15)
\]

\[
\langle \gamma_i, \gamma_j \rangle = \Gamma_{ij}. \quad (3.16)
\]

Instead of an \( N \times N \) covariance matrix \( C_{ij} \), we now have an \( F \times F \) covariance matrix \( \Phi_{AB} \). So, below we will set

\[
C = \Gamma \equiv \Xi + \hat{\Omega} \hat{\Omega}^T, \quad (3.17)
\]

\[
\hat{\Omega} \equiv \Omega \hat{\Phi}, \quad (3.18)
\]

\[
\hat{\Phi} \hat{\Phi}^T = \Phi, \quad (3.19)
\]

where \( \hat{\Phi}_{AB} \) is the Cholesky decomposition of \( \Phi_{AB} \), which is assumed to be positive-definite.

There are various approaches to constructing factor models for alpha streams. Here, we simply assume a factor-model form for the covariance matrix without delving into the details of how it is constructed.\(^{11}\) We briefly mention one evident possibility: we can use the first \( F \) principal components of the covariance matrix as the factor loadings matrix. We then need to construct a specific risk and factor covariance matrix (which in itself is nontrivial). This is essentially the arbitrage pricing theory (APT) approach.

\(^{11}\) A more detailed discussion of factor models for alpha streams will appear in a forthcoming paper.
3.2 Optimization with factor model

In the factor-model framework, the optimization problem reduces to solving an $F$-dimensional system as follows. First, let

$$v_A = \sum_{i=1}^{N} w_i \tilde{\Omega}_{iA} = \sum_{i \in J} w_i \tilde{\Omega}_{iA}, \quad A = 1, \ldots, F.$$  \hspace{1cm} (3.20)

Then, from (3.4), we have

$$w_i = \frac{1}{\lambda \xi_i^2} \left( \alpha_i - L_i \eta_i - \lambda \sum_{A=1}^{F} \tilde{\Omega}_{iA} v_A \right), \quad i \in J.$$  \hspace{1cm} (3.21)

Recalling that we have

$$w_i \eta_i > 0, \quad i \in J,$$  \hspace{1cm} (3.22)

we get

$$\eta_i = \text{sgn} \left( \alpha_i - \sum_{A=1}^{F} \tilde{\Omega}_{iA} v_A \right), \quad i \in J,$$  \hspace{1cm} (3.23)

$$\forall i \in J, \quad \left| \alpha_i - \sum_{A=1}^{F} \tilde{\Omega}_{iA} v_A \right| > L_i,$$  \hspace{1cm} (3.24)

$$\forall i \in J', \quad \left| \alpha_i - \sum_{A=1}^{F} \tilde{\Omega}_{iA} v_A \right| \leq L_i,$$  \hspace{1cm} (3.25)

where (3.24) follows from (3.20) and (3.21). The last two inequalities define $J$ and $J'$ in terms of $F$ unknowns $v_A$.

Substituting (3.21) into (3.20), we get the following system of $F$ equations for $F$ unknowns $v_A$:

$$\sum_{B=1}^{F} Q_{AB} v_B = a_A.$$  \hspace{1cm} (3.26)

where

$$Q_{AB} = \delta_{AB} + \sum_{i \in J} \frac{\tilde{\Omega}_{iA} \tilde{\Omega}_{iB}}{\xi_i^2},$$  \hspace{1cm} (3.27)

$$a_A = \frac{1}{\lambda} \sum_{i \in J} \frac{\tilde{\Omega}_{iA}}{\xi_i^2} \left[ \alpha_i - L_i \eta_i \right].$$  \hspace{1cm} (3.28)
so we have

\[ v_A = \sum_{B=1}^{F} Q_{AB}^{-1} a_B, \]  

(3.29)

where \( Q^{-1} \) is the inverse of \( Q \).

Note that (3.29) solves for \( v_A \) given \( \eta_i, J \) and \( J' \). On the other hand, (3.23)–(3.25) determine \( \eta_i, J \) and \( J' \) in terms of \( v_A \). The entire system can then be solved iteratively.

An algorithm for an iterative procedure for solving the system (3.23)–(3.25) and (3.29) is relegated to Appendix C. Let us emphasize that the iterative procedure is finite, ie, it converges in a finite number of iterations.

4 THE EFFECT OF IMPACT ON WEIGHT OPTIMIZATION

Next, let us discuss the effect of impact, ie, nonlinear costs, on weight optimization. Generally, introducing nonlinear impact makes the weight-optimization problem computationally more challenging and requires the introduction of approximation methods.

One way of modeling trading costs is to introduce linear and nonlinear terms:

\[ P = I \sum_{i=1}^{N} \alpha_i w_i - L D - \frac{1}{n} Q D^n, \]  

(4.1)

where \( D = I T \) is the dollar amount traded, \( T \) is the turnover and \( Q \) and \( n > 1 \) are model dependent (and can be measured empirically). If we model turnover using (2.10), then we have

\[ P = I \sum_{i=1}^{N} (\alpha_i w_i - L_i |w_i|) - \frac{\hat{Q}}{n} \left[ \sum_{i=1}^{N} w_i \right]^n, \]  

(4.2)

where the modulus accounts for the possibility of some \( w_i \) being negative, and \( \hat{Q} \) is defined as follows:

\[ \hat{Q} = Q(I \rho)^n. \]  

(4.3)

For general fractional \( n \), which would have to be measured empirically, the weight-optimization problem would have to be solved numerically. Sometimes \( n \) is assumed to be \( \frac{3}{2} \). Here, we keep it arbitrary.

First, note that if individual turnovers \( \tau_i \equiv \tau \) are identical, then the nonlinear cost contribution into \( P \) is independent of \( w_i \) as we have (2.5). In this case, it simply shifts \( P \) by a constant and the problem can be solved exactly as in the previous section.\textsuperscript{12}

\textsuperscript{12} In fact, in this case the contribution of the linear cost also shifts \( P \) by a constant.
If \( \tau_i \) are not all identical, then we need to solve the following problem:

\[
g(w, \lambda) = \frac{1}{2} \sum_{i,j=1}^{N} C_{ij} w_i w_j - \sum_{i=1}^{N} (\alpha_i w_i - L_i |w_i|) + \frac{\tilde{Q}'}{n} \left[ \sum_{i=1}^{N} \tau_i |w_i| \right]^n.
\]  

(4.4)

\[
g(w, \mu, \bar{\mu}) \to \min,
\]

(4.5)

where

\[
\tilde{Q}' \equiv \frac{\tilde{Q}}{I}.
\]  

(4.6)

Here, we can use successive iterations to deal with the nonlinear term, and the various stability issues associated with convergence must be addressed. A simpler approach is to note that the key role of the nonlinear term is to model portfolio capacity\(^{13}\) via its dependence on \( I \), not its detailed structure in terms of individual alphas. In this regard, the following approximation is a reasonable way of simplifying the problem.

Let

\[
\bar{\tau} \equiv \frac{1}{N} \sum_{i=1}^{N} \tau_i,
\]  

(4.7)

\[
\tilde{\tau}_i \equiv \tau_i - \bar{\tau}.
\]  

(4.8)

If the distribution of \( \tilde{\tau}_i \) has a small standard deviation, we can use the following approximation (where we are using (2.5)):

\[
\left[ \sum_{i=1}^{N} \tau_i |w_i| \right]^n \approx \bar{\tau}^n + n \bar{\tau}^{n-1} \sum_{i=1}^{N} \tilde{\tau}_i |w_i|.
\]  

(4.9)

The objective function can be rewritten as (modulo an immaterial constant term)

\[
g(w, \lambda) \approx \frac{1}{2} \sum_{i,j=1}^{N} C_{ij} w_i w_j - \sum_{i=1}^{N} (\alpha_i w_i - \tilde{L}_i |w_i|),
\]  

(4.10)

where

\[
\tilde{L}_i \equiv L_i + \tilde{Q}' \bar{\tau}^{n-1} \tau_i = L_i + Q_p^n I^{n-1} \bar{\tau}^{n-1} \tau_i.
\]  

(4.11)

---

\(^{13}\)By this we mean the value of the investment level \( I = I_* \), for which the P&L \( P_{opt}(I) \) is maximized and where, for any given \( I \), P&L \( P_{opt}(I) \) is computed for the optimized weights \( w_i \). When only linear cost is present, capacity is unbounded. When nonlinear cost is included, \( I_* \) is finite.

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ie, in this approximation, the effect of the nonlinear term reduces to increasing the linear slippage. We can solve this problem as in the previous section. Note, however, that the “effective” linear cost $\tilde{L}_i$ now depends on the investment level $I$ via (4.11), which controls capacity. Thus, for $I$ such that

$$\forall i = 1, \ldots, N, \quad \tilde{L}_i \geq |\alpha_i|,$$

(4.12)

the P&L cannot be positive, so the capacity $I_*$ is finite (see Footnote 13).

5 REGRESSION AS THE LIMIT OF OPTIMIZATION

Let us go back to optimization without costs. The Sharpe ratio is maximized by

$$w_i = \gamma \sum_{j=1}^{N} C_{ij}^{-1} \alpha_j,$$

(5.1)

where $\gamma$ is a normalization constant.

Let $C_{ij}$ have a factor-model form

$$C_{ij} = \nu_i \delta_{ij} + \sum_{A=1}^{K} \Lambda_{iA} \Lambda_{jA},$$

(5.2)

where $\nu_i$ is specific variance and $\Lambda_{iA}, A = 1, \ldots, K$, is the factor loadings matrix in the basis where the factor covariance matrix is the identity matrix.\(^{14}\)

We have

$$w_i = \frac{\gamma}{\nu_i} \left( \alpha_i - \sum_{j=1}^{N} \frac{\alpha_j}{\nu_j} \sum_{A,B=1}^{K} \Lambda_{iA} \Lambda_{jB} Q_{AB}^{-1} \right),$$

(5.3)

where $Q_{AB}^{-1}$ is the inverse of

$$Q_{AB} = \delta_{AB} + \sum_{\ell=1}^{N} \frac{1}{\nu_{\ell}} \Lambda_{\ell A} \Lambda_{\ell B}.$$  

(5.4)

---

\(^{14}\) In other words, the factor covariance matrix is absorbed into the definition of the factor loadings matrix.
Note that for $N = 1$ and $K = 1$ we have

$$w_1 = \frac{\gamma \alpha_1}{v_1 + A^2_{11}},$$

which reproduces (5.1).

### 5.1 Regression limit

Let

$$v_i \equiv \zeta \tilde{v}_i.$$  

Consider the following limit:

$$
\begin{align*}
\zeta & \to 0, \\
\gamma & \to 0, \\
\frac{\gamma}{\zeta} & = \tilde{y} = \text{fixed}, \\
\tilde{v}_i & = \text{fixed}.
\end{align*}
$$

In this limit, we have

$$w_i = \frac{\tilde{y}}{\tilde{v}_i} \left( \alpha_i - \sum_{j=1}^{N} \frac{\alpha_j}{\tilde{v}_j} \sum_{A,B=1}^{K} A_{iA} A_{jB} \tilde{Q}^{-1}_{AB} \right) = \frac{\tilde{y}}{\tilde{v}_i} \epsilon_i,$$

where $\tilde{Q}^{-1}_{AB}$ is the inverse of

$$\tilde{Q}_{AB} \equiv \sum_{\ell=1}^{N} \frac{1}{\tilde{v}_\ell} A_{\ell A} A_{\ell B}.$$  

Note that

$$\sum_{i=1}^{N} w_i A_{iC} = 0, \quad C = 1, \ldots, K.$$  

In fact, $\epsilon_i$ are the residuals of a weighted regression (with weights $1/\tilde{v}_i$) of $\alpha_i$ over $A_{iA}$ (without intercept). If all weights are identical ($\tilde{v}_i \equiv \tilde{v}$), then we have an equally weighted regression:

$$\alpha_i = \sum_{A=1}^{K} A_{iA} \eta_A + \epsilon_i,$$

where $\eta_A$ are the regression coefficients (in matrix notation): $\eta = (A^T A)^{-1} A^T \alpha$. 

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5.2 Regression limit with costs

We can take a similar limit in the solution of Section 3 with costs. In this limit, we have

\[ \xi_i^2 = \zeta \tilde{\xi}_i^2, \quad (5.15) \]
\[ \lambda = \tilde{\lambda}, \quad (5.16) \]
\[ \zeta \to 0, \quad (5.17) \]
\[ \tilde{\xi}_i^2 = \text{fixed}, \quad (5.18) \]
\[ \tilde{\lambda} = \text{fixed}. \quad (5.19) \]

In this limit, (3.21) reduces to

\[ w_i = \frac{\varepsilon_i}{\lambda \tilde{\xi}_i^2}, \quad (5.20) \]

where \( \varepsilon_i \) are the residuals of a weighted regression (with weights \( 1/\tilde{\xi}_i^2 \)) of \( \alpha_i - L_i \eta_i \) over \( \hat{\Omega}_{iA} \) (without intercept). We can use (5.20) (instead of (3.21)) in the iterative procedure discussed at the end of Section 3, which now defines “regression with linear costs” (as opposed to “optimization with linear costs”) and can be useful in cases where the full factor model is not known, but factor loadings \( \hat{\Omega}_{iA} \) can be constructed. An example of this is when the number of observations \( M + 1 \) for alphas is small \( M \ll N \), so the covariance matrix \( C_{ij} \) is singular and has only \( M \) nonvanishing eigenvalues \( e_i \). In this case, we can use, for example, the first \( M \) principal components \( P_{iA} \) (corresponding to the nonzero eigenvalues \( e_i \)) to construct factor loadings via \( \hat{\Omega}_{iA} = \sqrt{e_i} P_{iA} \), and for \( \tilde{\xi}_i^2 \) we can use, for example, \( C_{ii} \) (which are all positive).\(^{15}\)

Finally, note that we can also consider regression with linear and nonlinear costs; the latter is treated using the approximation discussed in Section 4.

APPENDIX A. LINEAR COSTS

In this appendix, we discuss linear costs in more detail, starting with linear costs for underlying individual stocks, which we discuss in terms of the individual stock prices \( P_A \) and the corresponding volumes traded \( Q_{iA} \). Here, the index \( A = 1, \ldots, N_S \) labels stocks, where \( N_S \) is the total number of stocks traded. As before, \( i = 1, \ldots, N \), where \( N \) is the number of alphas. Then, \( Q_{iA} \) is the volume (i.e., the number of shares) for the stock labeled by \( A \) traded by \( \alpha_i \). Here, volumes \( Q_{iA} \) are unsigned quantities,

\(^{15}\) Note that, for the regression, we can actually set \( \hat{\Omega}_{iA} = P_{iA} \), as any transformation of the form \( \hat{\Omega} \to \hat{\Omega} Z \), where \( Z \) is an arbitrary nonsingular \( M \times M \) matrix, does not change the regression residuals (although it affects the regression coefficients).
ie, \( Q_{iA} \geq 0 \) for both buys and sells. Let \( L_{iA} \) be the per-share linear cost of trading the stock \( A \) by \( \alpha_i \). Let us assume that there is no internal crossing. Then, the total linear cost of trading all stocks by all alphas is given by

\[
C_{\text{lin}} = \sum_{i=1}^{N} \sum_{A=1}^{N_S} L_{iA} Q_{iA}.
\]  

(A.1)

This equation, however, is not practical for the purpose of weight optimization. We need to make some simplifying assumptions so we can express \( C_{\text{lin}} \) in terms of the weights \( w_i \). The two simplifying assumptions are as follows. First, we assume that \( L_{iA} \) is independent of the \( i \) index, ie, the cost of trading the stock labeled by \( A \) is independent of which alpha is trading it. This assumption need not hold in the most general case. However, when the number of stocks \( N_S \) is large and the number of alphas \( N \) is large, this is expected to be a reasonable approximation, which can be thought of as setting the \( L_{iA} \) to their mean value (as averaged over all alphas):

\[
L_{iA} \approx L_A = \frac{1}{N} \sum_{i=1}^{N} L_{iA}.
\]  

(A.2)

Second, in optimization we deal with dollar holdings, not share holdings; thus, the total (meaning, long-plus-short) dollar holding for each alpha is \( H_i \equiv I |w_i| \). On the other hand, we can write

\[
H_i = \sum_{A=1}^{N_S} P_A S_{iA},
\]  

(A.3)

where \( S_{iA} \) is the absolute value of shares held by \( \alpha_i \) in stock \( A \). Similarly, to tackle the optimization problem, trading costs should also be given in terms of traded dollar amounts. This is achieved by assuming that \( L_A \) is proportional to the prices \( P_A \), ie, \( L_A \approx LP_A \), where \( L \) is independent of \( A \). We then have

\[
C_{\text{lin}} \approx L \sum_{i=1}^{N} \sum_{A=1}^{N_S} P_A Q_{iA} = LD.
\]  

(A.4)

The equality follows from the definition of the portfolio dollar turnover \( D = \sum_{i=1}^{N} D_i \), where \( D_i = \sum_{A=1}^{N_S} P_A Q_{iA} \) are individual dollar turnovers in the absence of internal crossing. (We discuss turnover reduction in the presence of internal crossing in Section 2.1.) Note that \( D_i = D |w_i| \) and \( T = D/I \). Further, the meaning of (A.4) is that the linear cost is approximately a fixed fraction of the dollar amount traded. This is expected to be a reasonable approximation when linear slippage makes a dominant contribution to the linear cost. Linear slippage for an individual stock is roughly proportional to an average bid–ask spread, which scales linearly with the stock price, so that when the linear cost is summed over a large number of stocks and a large number of alphas, we arrive at the above approximation.
APPENDIX B. CONDITIONS FOR GLOBAL MINIMUM

In Section 3, we gave the conditions for the global minimum:

\[
\frac{\lambda}{2} \sum_{i,j=1}^{N} C_{ij} (w_i + \epsilon_i)(w_j + \epsilon_j) - \sum_{i=1}^{N} (\alpha_i (w_i + \epsilon_i) - L_i |w_i + \epsilon_i|)
\]

\[
\geq \frac{\lambda}{2} \sum_{i,j=1}^{N} C_{ij} w_i w_j - \sum_{i=1}^{N} (\alpha_i w_i - L_i |w_i|),
\]

(B.1)

where \(w_i, i \in J\), are determined using (3.4), while \(w_i = 0, i \in J',\) and \(\epsilon_i\) are arbitrary. In Section 3, we discussed these conditions for arbitrary infinitesimal \(\epsilon_i\), which gave the conditions for a local minimum. Here, we discuss the above conditions for noninfinitesimal \(\epsilon_i\). Taking into account (3.4), we have

\[
\frac{\lambda}{2} \sum_{i,j=1}^{N} C_{ij} \epsilon_i \epsilon_j + \sum_{j \in J'} \left( \frac{\lambda}{2} \sum_{i \in J} C_{ij} \epsilon_i \epsilon_j - \alpha_j \epsilon_j + L_j |\epsilon_j| \right)
\]

\[
+ \sum_{i \in J} L_i (|w_i + \epsilon_i| - |w_i| - \eta_i \epsilon_i) \geq 0.
\]

The first term is manifestly positive-semidefinite as \(C_{ij}\) is positive-definite; the second term is positive-semidefinite due to (3.7), which implies (3.6); the third term is manifestly positive-semidefinite as \(\eta_i = \text{sgn}(w_i)\). So, the local minimum we found in Section 3 is also the global minimum. This is because all \(L_i > 0\).

APPENDIX C. ITERATIVE PROCEDURE

At the initial iteration, we take \(J^{(0)} = \{1, \ldots, N\}\), so that \(J^{(0)}\) is empty and

\[\eta_i^{(0)} = \pm 1, \quad i = 1, \ldots, N.\] (C.1)

While \textit{a priori} the values of \(\eta_i^{(0)}\) can be arbitrary, unless \(F \ll N\), in some cases we might encounter convergence speed issues. However, if we choose

\[\eta_i^{(0)} = \text{sgn}(\alpha_i), \quad i = 1, \ldots, N,\] (C.2)

then the iterative procedure generally is expected to converge rather quickly. Further, note that the solution is actually exact, ie, the convergence criteria are given by (recall from Appendix B that this produces the global optimum)

\[J^{(s+1)} = J^{(s)},\] (C.3)

\[\eta_i^{(s+1)} = \eta_i^{(s)}, \quad \forall i \in J^{(s+1)},\] (C.4)

\[\nu_A^{(s+1)} = \nu_A^{(s)}, \quad \forall A \in \{1, \ldots, F\},\] (C.5)
where $s$ and $s + 1$ label successive iterations. In other words, the iterative procedure is finite: it converges in a finite number of iterations. Finally, note that $w_i$ for $i \in J$ are given by (3.21), while $w_i = 0$ for $i \in J'$.

**Remark C1** Because the alphas $\alpha_i, i \in J'$ are no longer traded, we can drop such alphas, if any, recalculate $\rho_*$ in (2.13) using the corresponding correlation matrix $\psi'_{ij} \equiv \psi_{ij}, i, j \in J$, recalculate $w_i$ using such $\rho_*$ and repeat this procedure until the subset $J$, based on which $\rho_*$ is computed, is the same as the subset for which $w_i \neq 0$, where $w_i$ are computed based on such $\rho_*$.17

**DECLARATION OF INTEREST**

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16 The first two of these criteria are based on discrete quantities and are unaffected by computational (machine) precision effects, while the last criterion is based on continuous quantities and, in practice, is understood as satisfied within computational (machine) precision or preset tolerance.

17 When $N$ is large, this procedure is stable and convergent, as $\rho_*$ does not change much with $N$ (see Kakushadze 2014).
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