NONCOMMUTATIVE RIEMANN HYPOTHESIS

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Abstract. In this note, making use of noncommutative $l$-adic cohomology, we extend the generalized Riemann hypothesis from the realm of algebraic geometry to the broad setting of geometric noncommutative schemes in the sense of Orlov. As a first application, we prove that the generalized Riemann hypothesis is invariant under derived equivalences and homological projective duality. As a second application, we prove the noncommutative generalized Riemann hypothesis in some new cases.

1. Introduction and statement of results

Let $k$ be a global field and $\Sigma_k$ its (infinite) set of non-archimedean places.

Let $X$ be a smooth proper $k$-scheme and $0 \leq w \leq 2 \dim(X)$ an integer. Following Serre's foundational work [19, 20] (consult also Manin [14]), consider the $L$-function $L_w(X; s) := \prod_{\nu \in \Sigma_k} L_w,\nu(X; s)$ of weight $w$. As proved in loc. cit., this infinite product converges absolutely in the half-plane $\text{Re}(s) > \frac{w}{2} + 1$ and is non-zero in this region. Moreover, the following two conditions are expected to hold:

(C1) The $L$-function $L_w(X; s)$ admits a (unique) meromorphic continuation to the entire complex plane.

(C2) When $\text{char}(k) = 0$, the only possible pole of $L_w(X; s)$ is located at $s = \frac{w}{2} + 1$ with $w$ even.

When $\text{char}(k) = 0$, the conditions (C1)-(C2) have been proved in many cases: certain 0-dimensional schemes, certain elliptic curves, certain modular curves, certain abelian varieties, certain varieties of Fermat type, certain Shimura varieties, etc. When $\text{char}(k) > 0$, condition (C1) follows from Grothendieck’s work [5].

The following conjecture, which implicitly assumes condition (C1), goes back to the work [18] of Riemann.

Generalized Riemann hypothesis $R_w(X)$: All the zeros of the $L$-function $L_w(X; s)$ that are contained in the critical strip $\frac{w}{2} < \text{Re}(s) < \frac{w}{2} + 1$ lie in the vertical line $\text{Re}(s) = \frac{w+1}{2}$.

The generalized Riemann hypothesis play a central role in mathematics. For example, when $\text{char}(k) = 0$ and $X = \text{Spec}(k)$, the conjecture $R_0(X)$ reduces to the classical extended Riemann hypothesis ERH$_k$, i.e., all the zeros of the Dedekind zeta function $\zeta_k(s) := \sum_{I \subset O_k} \frac{1}{N(I)}$ that are contained in the critical strip $0 < \text{Re}(s) < 1$ lie in the vertical line $\text{Re}(s) = \frac{1}{2}$; note that in the particular case where $k = \mathbb{Q}$, ERH$_k$ is the famous Riemann hypothesis. The status of the generalized Riemann conjecture depends drastically on the characteristic of $k$. On the one hand, when $\text{char}(k) = 0$, no cases have been proved. On the other hand, when $\text{char}(k) > 0$, the generalized Riemann hypothesis follows from Deligne’s work [3, 4].

Now, let $A$ be a geometric noncommutative $k$-scheme in the sense of Orlov; consult §4 below. A standard example is the canonical dg enhancement $\text{perf}_{dg}(X)$ of the derived category of perfect complexes $\text{perf}(X)$ of a smooth proper $k$-scheme $X$ (consult Keller’s survey [7, §4.6]); consult §3 below for further examples. In §6 below, making use of noncommutative $l$-adic cohomology, we construct the noncommutative counterparts $L_{\text{even}}(A; s) := \prod_{\nu \in \Sigma_k} L_{\text{even},\nu}(A; s)$ and $L_{\text{odd}}(A; s) := \prod_{\nu \in \Sigma_k} L_{\text{odd},\nu}(A; s)$ of the classical $L$-functions. Moreover, we prove the following noncommutative counterpart of Serre’s convergence result:

Theorem 1.1. The infinite product $L_{\text{even}}(A; s)$, resp. $L_{\text{odd}}(A; s)$, converges absolutely in the half-plane $\text{Re}(s) > 1$, resp. $\text{Re}(s) > \frac{1}{2}$, and is non-zero in this region.

Similarly to the above condition (C1), it is expected that the noncommutative $L$-functions $L_{\text{even}}(A; s)$ and $L_{\text{odd}}(A; s)$ admit a (unique) meromorphic continuation to the entire complex plane. Under this assumption, the generalized Riemann hypothesis admits the following noncommutative counterpart:

Noncommutative generalized Riemann hypothesis $R_{\text{even}}(A)$ and $R_{\text{odd}}(A)$: All the zeros of the noncommutative $L$-function $L_{\text{even}}(A; s)$, resp. $L_{\text{odd}}(A; s)$, that are contained in the critical strip $0 < \text{Re}(s) < 1$, resp. $\frac{1}{2} < \text{Re}(s) < \frac{3}{2}$, lie in the vertical line $\text{Re}(s) = \frac{1}{2}$, resp. $\text{Re}(s) = 1$.

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The noncommutative (generalized) Riemann hypothesis was originally envisioned by Kontsevich in his seminal talks [8, 9]. The next result relates this conjecture with the generalized Riemann hypothesis:

**Theorem 1.2.** Given a smooth proper k-scheme $X$, we have the following implications:

\[
\{ \text{R}_w(X) \}_w \text{even} \Rightarrow \text{R}_w(\text{perf}_{dg}(X)) \quad \{ \text{R}_w(X) \}_w \text{odd} \Rightarrow \text{R}_w(\text{perf}_{dg}(X)).
\]

When $\text{char}(k) > 0$, the converse implications of (1.3) hold. Moreover, when $\text{char}(k) = 0$ and the $L$-functions \( \{L_w(X; s)\}_{0 \leq w \leq 2\dim(X)} \) satisfy condition (C2), the converse implications of (1.3) also hold.

Intuitively speaking, Theorem 1.2 shows that the generalized Riemann hypothesis belongs not only to the realm of algebraic geometry but also to the broad setting of geometric noncommutative schemes.

2. Applications to commutative geometry

Let $k$ be a global field; since the generalized Riemann hypothesis holds when $\text{char}(k) > 0$, we restrict ourselves to the case where $\text{char}(k) = 0$. In this section, making use of Theorem 1.2, we prove that the generalized Riemann hypothesis is invariant under derived equivalences and homological projective duality.

**Derived invariance.** Let $X$ and $Y$ be two smooth proper $k$-schemes. In what follows, we assume that the associated $L$-functions \( \{L_w(X; s)\}_{0 \leq w \leq 2\dim(X)} \) and \( \{L_w(Y; s)\}_{0 \leq w \leq 2\dim(Y)} \) satisfy condition (C2).

**Corollary 2.1 (Derived invariance).** If the derived categories of perfect complexes $\text{perf}(X)$ and $\text{perf}(Y)$ are (Fourier-Mukai) equivalent, then we have the following equivalences:

\[
\{ \text{R}_w(X) \}_w \text{even} \Leftrightarrow \{ \text{R}_w(Y) \}_w \text{even} \quad \{ \text{R}_w(X) \}_w \text{odd} \Leftrightarrow \{ \text{R}_w(Y) \}_w \text{odd}.
\]

**Proof.** If the triangulated categories $\text{perf}(X)$ and $\text{perf}(Y)$ are (Fourier-Mukai) equivalent, then the $\text{dg}$ categories $\text{perf}_{dg}(X)$ and $\text{perf}_{dg}(Y)$ are Morita equivalent. Consequently, we obtain the following equivalences:

\[
\text{R}_w(\text{perf}_{dg}(X)) \Leftrightarrow \text{R}_w(\text{perf}_{dg}(Y)) \quad \text{R}_w(\text{perf}_{dg}(X)) \Leftrightarrow \text{R}_w(\text{perf}_{dg}(Y)).
\]

By combining them with Theorem 1.2, we hence obtain the above equivalences (2.2). \(\square\)

In the literature there are numerous examples of smooth proper $k$-schemes $X$ and $Y$ for which the above Corollary 2.1 applies; consult, for example, the book [6] and the references therein.

**Homological Projective Duality.** Homological Projective Duality (=HPD) was introduced by Kuznetsov in [13] as a tool to study the derived categories of perfect complexes of linear sections. Let $X$ be a smooth $k$-scheme equipped with a line bundle $L_X(1)$; we write $X \to \mathbb{P}(V)$ for the associated map, where $V := H^0(X, L_X(1))^\vee$. Assume that we have a Lefschetz decomposition $\text{perf}(X) = \langle A_0, A_1, \ldots, A_{i-1} \rangle$ with respect to $L_X(1)$ in the sense of [13, Def. 4.1]. Following [13, Def. 6.1], let us write $Y$ for the HP-dual of $X$, $L_Y(1)$ for the HP-dual line bundle, and $Y \to \mathbb{P}(V^\vee)$ for the associated map. Given a generic linear subspace $L \subset V^\vee$, consider the smooth linear sections $X_L := X \times_{\mathbb{P}(V)} \mathbb{P}(L^\perp)$ and $Y_L := Y \times_{\mathbb{P}(V^\vee)} \mathbb{P}(L)$. In what follows, we assume that the associated $L$-functions \( \{L_w(X_L; s)\}_{0 \leq w \leq 2\dim(X_L)} \) and \( \{L_w(Y_L; s)\}_{0 \leq w \leq 2\dim(Y_L)} \) satisfy the above condition (C2).

**Theorem 2.3 (HPD-invariance).** Assume that the triangulated category $\mathcal{A}_0$ admits a full exceptional collection. Under this assumption, the following holds:

\[
\text{ERH}_k \Rightarrow \{ \text{R}_w(X_L) \}_{w \text{ even}} \Leftrightarrow \{ \text{R}_w(Y_L) \}_{w \text{ even}} \quad \{ \text{R}_w(X_L) \}_{w \text{ odd}} \Leftrightarrow \{ \text{R}_w(Y_L) \}_{w \text{ odd}}.
\]

**Remark 2.5.** The assumption of Theorem 2.3 is quite mild since it holds in all the examples in the literature.

Roughly speaking, Theorem 2.3 “cuts in half” the difficulty of proving the generalized Riemann hypothesis, i.e., if the generalized Riemann hypothesis holds for a linear section, then it also holds for the HP-dual linear section. In the literature there are numerous examples of homological projective dualities for which the above Theorem 2.3 applies (e.g., Veronese-Clifford duality, Grassmannian-Pfaffian duality, Spinor duality, Determinantal duality, etc); consult, for example, the surveys [12, 24] and the references therein.

3. Applications to noncommutative geometry

Let $k$ be a global field. In this section, making use of Theorem 1.2, we prove the noncommutative generalized Riemann hypothesis in some new cases.
Noncommutative gluings of schemes. Let $X$ and $Y$ be two smooth proper $k$-schemes and $B$ a perfect dg $\text{perf}_{k}(X)$-$\text{perf}_{k}(Y)$ bimodule. Following Orlov [17, Def. 3.5], we can consider the gluing $X \circ_{B} Y$ of the $k$-schemes $X$ and $Y$ via the $k$-bimodule $B$ (Orlov used a different notation). As proved by Orlov in [17, Thm. 4.11], $X \circ Y$ is a geometric noncommutative $k$-scheme.

Theorem 3.1. We have the following implications:

\[
\{R_{w}(X)\}_{w \text{ even}} + \{R_{w}(Y)\}_{w \text{ even}} \Rightarrow \text{Reven}(X \circ_{B} Y) \quad \{R_{w}(X)\}_{w \text{ odd}} + \{R_{w}(Y)\}_{w \text{ odd}} \Rightarrow \text{Rodd}(X \circ_{B} Y).
\]

In particular, the conjectures $\text{Reven}(X \circ_{B} Y)$ and $\text{Rodd}(X \circ_{B} Y)$ hold when $\text{char}(k) > 0$.

Calabi-Yau dg categories associated to hypersurfaces. Let $X \subset \mathbb{P}^{n}$ be a smooth hypersurface of degree $\deg(X) \leq n + 1$. As proved by Kuznetsov in [11, Cor. 4.1], we have a semi-orthogonal decomposition $\text{perf}(X) = \langle T, \mathcal{O}_{X}, \ldots, \mathcal{O}_{X}(n - \deg(X)) \rangle$. Moreover, the full dg subcategory $\mathcal{T}_{\text{dg}}$ of $\text{perf}_{k}(X)$, consisting of the objects of $\mathcal{T}$, is a Calabi-Yau dg category of fractional CY-dimension $\frac{n+1-\deg(X)+2}{\deg(X)}$. Note that $\mathcal{T}_{\text{dg}}$ is a geometric noncommutative $k$-scheme. Note also that $\mathcal{T}_{\text{dg}}$ is not Morita equivalent to a dg category of the form $\text{perf}_{k}(Y)$, with $Y$ a smooth proper $k$-scheme, whenever its CY-dimension is not an integer.

Theorem 3.2. We have the following implications:

\[(3.3) \quad \{R_{w}(X)\}_{w \text{ even}} \Rightarrow \text{Reven}(\mathcal{T}_{\text{dg}}) \quad \{R_{w}(X)\}_{w \text{ odd}} \Rightarrow \text{Rodd}(\mathcal{T}_{\text{dg}}).
\]

In particular, the conjectures $\text{Reven}(\mathcal{T}_{\text{dg}})$ and $\text{Rodd}(\mathcal{T}_{\text{dg}})$ hold when $\text{char}(k) > 0$.

Finite-dimensional algebras of finite global dimension. Let $A$ be a finite-dimensional $k$-algebra of finite global dimension. Examples include path algebras of finite quivers without oriented cycles and their admissible quotients. As proved by Orlov in [17, Cor. 5.4], $A$ is a geometric noncommutative $k$-scheme.

Example 3.4 (Dynkin quivers). Let $\Delta$ be a Dynkin quiver of type $A_{n}$, $D_{n}$, $E_{6}$, $E_{7}$ or $E_{8}$. Recall that its Coxeter number $h$ is equal to $n + 1$, $2(n - 1)$, $12$, $18$ or $30$. It is well-known that the quiver algebra $A := k\Delta$ has fractional CY-dimension $\frac{h}{h+2}$. Consequently, in all these cases the geometric noncommutative $k$-scheme $A$ is not Morita equivalent to a dg category of the form $\text{perf}_{k}(Y)$, with $Y$ a smooth proper $k$-scheme.

Consider the largest semi-simple quotient $A/J$ of $A$, where $J$ stands for the Jacobson radical. Thanks to Artin-Wedderburn’s theorem, $A/J$ is Morita equivalent to the product $D_{1} \times \cdots \times D_{n}$, where $V_{1}, \ldots, V_{n}$ stand for the simple (right) $A/J$-modules and $D_{i} := \text{End}_{A/J}(V_{i}), \ldots, D_{n} := \text{End}_{A/J}(V_{n})$ for the associated division $k$-algebras. Let us denote by $k_{1}, \ldots, k_{n}$ the centers of the division $k$-algebras $D_{1}, \ldots, D_{n}$.

Theorem 3.5. Assume that the quotient $k$-algebra $A/J$ is separable (this holds, for example, when $k$ is perfect). Under this assumption, we have the implication $\sum_{i=1}^{n} \text{Reven}(\text{Spec}(k_{i})) \Rightarrow \text{Reven}(A)$. In particular, the conjecture $\text{Reven}(A)$ holds when $\text{char}(k) > 0$.

Remark 3.6 (Artin $L$-functions). Since $A/J$ is separable, the finite field extension $k_{i}/k$, with $1 \leq i \leq n$, is also separable. Therefore, under the classical Galois-Grothendieck correspondence, the $k$-scheme $\text{Spec}(k_{i})$ corresponds to the finite set $\text{Spec}(k_{i})(\overline{k})$ equipped with the continuous action of the absolute Galois group $\text{Gal}(\overline{k}/k)$. Consequently, the $L$-function $L_{0}(\text{Spec}(k_{i}); s)$ (used in conjecture $\text{Reven}(\text{Spec}(k_{i}))$) reduces to the classical Artin $L$-function $L(\rho_{i}; s)$ associated to the $\mathbb{C}$-linear representation $\rho_{i}: \text{Gal}(\overline{k}/k) \to \text{GL}(\mathbb{C}\text{Spec}(k_{i})(\overline{k}))$.

Finite-dimensional dg algebras. Let $A$ be a smooth finite-dimensional dg $k$-algebra in the sense of Orlov [16]. As proved in [16, Cor. 3.4], $A$ is a geometric noncommutative $k$-scheme. Following [16, Def. 2.3], consider the quotient $A/J_{+}$, where $J_{+}$ stands for the external dg Jacobson radical of $A$.

Theorem 3.7. Assume that the quotient dg $k$-algebra $A/J_{+}$ is separable in the sense of [16, Def. 2.11] (this holds when $k$ is perfect) and that $\text{char}(k) > 0$. Under these assumptions, the conjecture $\text{Reven}(A)$ holds.

Remark 3.8. In the above Theorems 3.5 and 3.7, the conjecture $\text{Rodd}(A)$ also holds; consult §10 below.

4. Preliminaries

Let $k$ be a field. Throughout the note, we will assume some basic familiarity with the language of dg categories (consult Keller’s survey [7]) and will write $\text{dgcat}(k)$ for the category of (small) dg categories.
4.1. Geometric noncommutative schemes. Following Orlov [17, Def. 4.3], a dg category $A$ is called a geometric noncommutative $k$-scheme if there exists a smooth proper $k$-scheme $X$ and an admissible triangulated subcategory $\mathfrak{A}$ of $\text{perf}(X)$ such that $A$ and $\mathfrak{A}_{\text{dg}}$ are Morita equivalent. Every geometric noncommutative $k$-scheme $A$ is, in particular, a smooth proper dg category in the sense of Kontsevich\(^1\) [10].

**Lemma 4.1.** Let $k'/k$ be a field extension. If the dg category $A$ is a geometric noncommutative $k$-scheme, then the dg category $A \otimes_k k'$ is a geometric noncommutative $k'$-scheme.

**Proof.** By definition, there exists a smooth proper $k$-scheme $X$ and an admissible triangulated subcategory $\mathfrak{A}$ of $\text{perf}(X)$ such that $A$ and $\mathfrak{A}_{\text{dg}}$ are Morita equivalent. Consequently, we obtain an induced Morita equivalence between $A \otimes_k k'$ and $\mathfrak{A}_{\text{dg}} \otimes_k k'$. Consider the following Morita equivalence:

$$(4.2) \quad \text{perf}_{\text{dg}}(X) \otimes_k k' \to \text{perf}_{\text{dg}}(X \times_k k').$$

Let us denote by $\mathfrak{A}'$ the smallest full triangulated subcategory of $\text{perf}(X \times_k k')$ containing the objects $\mathcal{F} \times_k k'$, with $\mathcal{F} \in \mathfrak{A}$, and by $\mathfrak{A}'_{\text{dg}}$ the associated full dg subcategory of $\text{perf}_{\text{dg}}(X \times_k k')$. By construction, the above Morita equivalence $(4.2)$ restricts to a Morita equivalence $\mathfrak{A}_{\text{dg}} \otimes_k k' \to \mathfrak{A}'_{\text{dg}}$. Therefore, the proof follows now from the fact that $\mathfrak{A}'$ is an admissible triangulated subcategory of $\text{perf}(X \times_k k')$. \qed

4.2. Additive invariants. Recall from [21, §2.1] that a functor $E: \text{dgcat}(k) \to D$, with values in an additive category, is called an additive invariant if it satisfies the following two conditions:

(i) It sends Morita equivalences to isomorphisms.

(ii) Let $B, C \subseteq A$ be dg categories inducing a semi-orthogonal decompositions $H^0(A) = \langle H^0(B), H^0(C) \rangle$ in the sense of Bondal-Orlov [2, Def. 2.4]. Under these notations, the inclusions $B \subseteq A$ and $C \subseteq A$ induce an isomorphism $E(B) \oplus E(C) \to E(A)$.

**Lemma 4.3.** Let $k'/k$ be a field extension. Given an additive invariant $E: \text{dgcat}(k') \to D$, the composed functor $E(\otimes_k k') : \text{dgcat}(k) \to D$ is also an additive invariant.

**Proof.** Condition (i) follows from the fact that the functor $\otimes_k k'$ preserves Morita equivalences; consult [15, Prop. 7.1]. Concerning condition (ii), let $B, C \subseteq A$ be dg categories inducing a semi-orthogonal decomposition $H^0(A) = \langle H^0(B), H^0(C) \rangle$. The associated dg categories $\text{pre}(B \otimes_k k'), \text{pre}(C \otimes_k k') \subseteq \text{pre}(A \otimes_k k')$, where $\text{pre}(-)$ stands for Bondal-Kapranov’s pretriangulated envelope [1], also induce a semi-orthogonal decomposition $H^0(\text{pre}(A \otimes_k k')) = \langle \text{pre}(B \otimes_k k'), \text{pre}(C \otimes_k k') \rangle$. Therefore, since the canonical dg functors $A \otimes_k k' \to \text{pre}(A \otimes_k k'), B \otimes_k k' \to \text{pre}(B \otimes_k k'),$ and $C \otimes_k k' \to \text{pre}(C \otimes_k k')$, are Morita equivalences, the proof of condition (ii) follows now from the fact that the functor $E$ satisfies condition (ii). \qed

Consult [21, §2.3] for the construction of the universal additive invariant $U: \text{dgcat}(k) \to \text{Hmo}_0(k)$. Given any additive invariant $E$, there exists a unique $\mathbb{Z}$-linear functor $\mathcal{E}$ making the following diagram commute:

$$(4.4) \quad \begin{array}{ccc}
\text{dgcat}(k) & \xrightarrow{E} & D \\
U & \downarrow & \\
\text{Hmo}_0(k) & \xrightarrow{\mathcal{E}} & \\
\end{array}$$

5. Noncommutative $l$-adic cohomology

Let $k$ be a field. Given a prime number $l \neq \text{char}(k)$, recall from [22, §2.5] the construction of the $l$-adic étale $K$-theory functor with values in the (homotopy) category of spectra:

$$(5.1) \quad K^\text{et}(-) : \text{dgcat}(k) \to \text{Spt} \quad A \mapsto \text{holim}_{n \geq 0} K^\text{et}(\mathcal{A}; \mathbb{Z}/l^n).$$

By construction, the homotopy groups $\pi_n(K^\text{et}(\mathcal{A}; l))$ are modules over the ring of $l$-adic integers $\mathbb{Z}_l$.

**Definition 5.2** (Noncommutative $l$-adic cohomology). Given a dg category $A$, its noncommutative $l$-adic cohomology is defined as follows ($\mathcal{K}$ stands for a fixed separable closure of $k$):

$$(5.3) \quad H_{\text{even}, l}(A) := \pi_0(K^\text{et}(\mathcal{A} \otimes_k \mathcal{K})_l) \otimes \mathbb{Z}[1/l] \quad H_{\text{odd}, l}(A) := \pi_1(K^\text{et}(\mathcal{A} \otimes_k \mathcal{K})_l) \otimes \mathbb{Z}[1/l].$$

\(^1\)Orlov asked in [17, Question 4.4] if there exist smooth proper dg categories which are not geometric noncommutative schemes. To the best of the author’s knowledge, this question remains wide open.
Note that, by construction, the noncommutative $l$-adic cohomology groups (5.3) are $\underline{Q}_l$-vector spaces. Moreover, they are equipped with a continuous action of the absolute Galois group $\text{Gal}(\overline{k}/k)$. Consequently, we obtain the following well-defined functors with values in the category of $\underline{Q}_l$-linear $\text{Gal}(\overline{k}/k)$-modules:

$$H_{\text{even}, l}(-), H_{\text{odd}, l}(-): \text{dgcat}(k) \to \text{Gal}(\overline{k}/k)\text{-Mod}. \quad (5.4)$$

**Proposition 5.5.** The functors (5.4) are additive invariants.

**Proof.** The $l$-adic étale $K$-theory functor (5.1) is an additive invariant; consult [22, §2.5]. Therefore, the proof follows from the above general Lemma 4.3. □

**Proposition 5.6.** Given a smooth proper $k$-scheme $X$, we have isomorphisms of $\text{Gal}(\overline{k}/k)$-modules

$$H_{\text{even}, l}(\text{perf}_{dg}(X)) \simeq \bigoplus_{w \text{ even}} H^w_{\text{et}}(X \times_k \overline{k}; \underline{Q}_l(\frac{w}{2})) \quad \text{ and } \quad H_{\text{odd}, l}(\text{perf}_{dg}(X)) \simeq \bigoplus_{w \text{ odd}} H^w_{\text{et}}(X \times_k \overline{k}; \underline{Q}_l(\frac{w-1}{2})).$$

where $H_{\text{et}}(-)$ stands for étale cohomology.

**Proof.** Since the $k$-scheme $X$ is smooth and proper, the associated $\overline{k}$-scheme $X \times_k \overline{k}$ is, in particular, regular and separated. These conditions imply that Thomason’s étale descent spectral sequence [25, Thm. 4.1] is well-defined and degenerates rationally; consult Soulé [26, §3.3.2]. Consequently, we obtain an isomorphism of $\text{Gal}(\overline{k}/k)$-modules between $\pi_0(K^+(\text{perf}_{dg}(X \times_k \overline{k})))$ and the direct sum $\bigoplus_{w \text{ even}} H^w_{\text{et}}(X \times_k \overline{k}; \underline{Q}_l(\frac{w}{2}))$ and between $\pi_1(K^+(\text{perf}_{dg}(X \times_k \overline{k})))$ and the direct sum $\bigoplus_{w \text{ odd}} H^w_{\text{et}}(X \times_k \overline{k}; \underline{Q}_l(\frac{w-1}{2}))$. The proof follows now from the Morita equivalence $\text{perf}_{dg}(X) \otimes_k \overline{k} \to \text{perf}_{dg}(X \times_k \overline{k})$. □

**Lemma 5.7.** Given a geometric noncommutative $k$-scheme $\mathcal{A}$, the associated $\underline{Q}_l$-linear $\text{Gal}(\overline{k}/k)$-modules (5.3) are finite-dimensional.

**Proof.** By definition, there exists a smooth proper $k$-scheme $X$ and an admissible triangulated subcategory $\mathcal{A}$ of $\text{perf}(X)$ such that $\mathcal{A}$ and $\mathcal{A}_{dg}$ are Morita equivalent. Let us denote by $\perp \mathcal{A}$ the left orthogonal complement of $\mathcal{A}$ in $\text{perf}(X)$ and by $\perp \mathcal{A}_{dg}$ the associated full dg subcategory of $\text{perf}_{dg}(X)$. By construction, we have a semi-orthogonal decomposition $H^0(\text{perf}_{dg}(X)) = \langle H^0(\perp \mathcal{A}_{dg}), H^0(\perp \perp \mathcal{A}_{dg}) \rangle$, i.e., $\text{perf}(X) = \langle \mathcal{A}, \perp \mathcal{A} \rangle$. Therefore, making use of Proposition 5.5, we obtain the following computations:

$$H_{\text{even}, l}(\text{perf}_{dg}(X)) \simeq H_{\text{even}, l}(\mathcal{A}) \oplus H_{\text{even}, l}(\perp \perp \mathcal{A}_{dg}) \quad \text{ and } \quad H_{\text{odd}, l}(\text{perf}_{dg}(X)) \simeq H_{\text{odd}, l}(\mathcal{A}) \oplus H_{\text{odd}, l}(\perp \perp \mathcal{A}_{dg}).$$

The proof follows now from Proposition 5.6 and from the fact that the étale cohomology $\underline{Q}_l$-vector spaces $\{H^w_{\text{et}}(X \times_k \overline{k}; \underline{Q}_l(\frac{w}{2}))\}_{w \text{ even}}$ and $\{H^w_{\text{et}}(X \times_k \overline{k}; \underline{Q}_l(\frac{w-1}{2}))\}_{w \text{ odd}}$ are finite-dimensional. □

6. Noncommutative $L$-functions

Let $k$ be a global field. We start by fixing some important notations:

**Notation 6.1.** Given a non-archimedean place $\nu \in \Sigma_k$, let us write $k_{\nu}$ for the completion of $k$ at $\nu$, $\mathcal{O}_\nu$ for the valuation ring of $k_{\nu}$, $\kappa_{\nu}$ for the residue field of $\mathcal{O}_\nu$, $p_\nu$ for the characteristic of $\kappa_{\nu}$, $N_\nu$ for the cardinality of the finite field $\kappa_{\nu}$, $I_\nu$ for the inertia subgroup, i.e., the kernel of the canonical surjective map $\text{Gal}(\overline{k}/k_{\nu}) \to \text{Gal}(\overline{k}/\kappa_{\nu})$, and $\pi_{\nu} \in \text{Gal}(\overline{k}/\kappa_{\nu}) \simeq \hat{\mathbb{Z}}$ for the geometric Frobenius, i.e., the inverse of the arithmetic Frobenius $\lambda \mapsto \lambda^{N_\nu}$.

**Notation 6.2.** Let us choose a prime number $l_{\nu} \neq p_\nu$ and a field embedding $\iota_{\nu}: \underline{Q}_{l_{\nu}} \hookrightarrow \mathbb{C}$ for every $\nu \in \Sigma_k$.

The above Lemmas 4.1 and 5.7 enable the following definition:

**Definition 6.3** (Noncommutative $L$-functions). Given a geometric noncommutative $k$-scheme $\mathcal{A}$, its noncommutative $L$-functions are defined as follows (we are implicitly using the field embedding $\iota_{\nu}$):

\[
L_{\text{even}}(\mathcal{A}; s) := \prod_{\nu \in \Sigma_k} L_{\text{even}, \iota_{\nu}}(\mathcal{A}; s) \quad L_{\text{even}, \iota_{\nu}}(\mathcal{A}; s) := \frac{1}{\det(\text{id} - N_{\nu}^{-s}(\pi_{\nu} \otimes \underline{Q}_{l_{\nu}} \mathbb{C}))} \left| H_{\text{even}, \iota_{\nu}}(\mathcal{A} \otimes_k k_{\nu})^{I_{\nu} \otimes \underline{Q}_{l_{\nu}} \mathbb{C}} \right|
\]

\[
L_{\text{odd}}(\mathcal{A}; s) := \prod_{\nu \in \Sigma_k} L_{\text{odd}, \iota_{\nu}}(\mathcal{A}; s) \quad L_{\text{odd}, \iota_{\nu}}(\mathcal{A}; s) := \frac{1}{\det(\text{id} - N_{\nu}^{-s}(\pi_{\nu} \otimes \underline{Q}_{l_{\nu}} \mathbb{C}))} \left| H_{\text{odd}, \iota_{\nu}}(\mathcal{A} \otimes_k k_{\nu})^{I_{\nu} \otimes \underline{Q}_{l_{\nu}} \mathbb{C}} \right|
\]
Proposition 6.4. Let $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$ be geometric noncommutative $k$-schemes inducing a semi-orthogonal decomposition $H^0(\mathcal{A}) = \langle H^0(\mathcal{B}), H^0(\mathcal{C}) \rangle$. Under these notations, we have the following equalities:
\begin{equation}
L_{\text{even}}(\mathcal{A}; s) = L_{\text{even}}(\mathcal{B}; s) \cdot L_{\text{even}}(\mathcal{C}; s) \quad \text{and} \quad L_{\text{odd}}(\mathcal{A}; s) = L_{\text{odd}}(\mathcal{B}; s) \cdot L_{\text{odd}}(\mathcal{C}; s).
\end{equation}

Proof. Let $\nu \in \Sigma_k$ be a non-archimedean place. Thanks to Proposition 5.5 and Lemma 4.3, we have an isomorphism of $\text{Gal}(\overline{k}/k_\nu)$-modules between $H_{\text{even},\nu}(\mathcal{A} \otimes_k k_\nu)$ and $H_{\text{even},\nu}(\mathcal{B} \otimes_k k_\nu) \otimes H_{\text{even},\nu}(\mathcal{C} \otimes_k k_\nu)$. This implies that $L_{\text{even},\nu}(\mathcal{A}; s) = L_{\text{even},\nu}(\mathcal{B}; s) \cdot L_{\text{even},\nu}(\mathcal{C}; s)$. Consequently, the left-hand side of (6.5) follows from Definition 6.3. The proof of the odd case is similar: simply replace the word “even” by the word “odd”. □

Proposition 6.6. Given a smooth proper $k$-scheme $X$, we have the following equalities:
\begin{equation}
L_{\text{even}}(\text{perf}_{dg}(X); s) = \prod_{w \text{ even}} L_w(X; s + \frac{w}{2}) \quad \text{and} \quad L_{\text{odd}}(\text{perf}_{dg}(X); s) = \prod_{w \text{ odd}} L_w(X; s + \frac{w}{2}).
\end{equation}

Roughly speaking, Proposition 6.6 shows that the noncommutative even/odd $L$-function of $\text{perf}_{dg}(X)$ may be understood as the “weight normalization” of the product of the $L$-functions of $X$ of even/odd weight.

Proof. Recall first that the $L$-function of $X$ of weight $w$ is defined as follows (consult Notations 6.1-6.2):
\begin{equation}
L_w(X; s) := \prod_{\nu \in \Sigma_k} L_{w,\nu}(X; s) = \frac{1}{\det(\text{id} - N_{\nu}^{-s}(\pi_\nu \otimes Q_{\nu}) \mathbb{C})} \cdot H_{et}^w((X \times_k k_\nu) \times_{k_\nu} k_\nu ; Q_{\nu} \mathbb{C})
\end{equation}

Note that we have the following isomorphisms of $\text{Gal}(\overline{k}/k_\nu)$-modules
\begin{equation}
H_{\text{even},\nu}(\text{perf}_{dg}(X) \otimes_k k_\nu)^{I_\nu} \cong \prod_{w \text{ even}} H_{et}^w((X \times_k k_\nu) \times_{k_\nu} k_\nu ; Q_{\nu} \mathbb{C})^{I_\nu}
\end{equation}

where (6.8) follows from the Morita equivalence $\text{perf}_{dg}(X) \otimes_k k_\nu \to \text{perf}_{dg}(X \times_k k_\nu), \mathcal{F} \mapsto \mathcal{F} \times_k k_\nu$, (6.9) from Proposition 5.6, and (6.10) from the fact that the $\text{Gal}(\overline{k}/k_\nu)$-module $Q_{\nu}(\mathbb{F}_q)$ is unramified (i.e., $I_\nu$ acts trivially). Moreover, since the action of the geometric Frobenius on $Q_{\nu}(\mathbb{F}_q)$ is given by multiplication by $N_{\nu}^{-1}$, the $\text{Gal}(\overline{k}/k_\nu)$-module (6.10) may be identified with $\bigoplus_{w \text{ even}} H_{et}^w((X \times_k k_\nu) \times_{k_\nu} k_\nu ; Q_{\nu} \mathbb{C})^{I_\nu}$. This implies the following equalities:
\begin{equation}
L_{\text{even},\nu}(\text{perf}_{dg}(X); s) = \prod_{w \text{ even}} \frac{1}{\det(\text{id} - N_{\nu}^{-s}(\pi_\nu \otimes Q_{\nu}) \mathbb{C})} \cdot H_{et}^w((X \times_k k_\nu) \times_{k_\nu} k_\nu ; Q_{\nu} \mathbb{C})^{I_\nu}
\end{equation}

Consequently, the left-hand side of (6.7) follows now from the following equalities:
\begin{equation}
L_{\text{even}}(\text{perf}_{dg}(X); s) := \prod_{\nu \in \Sigma_k} L_{\text{even},\nu}(\text{perf}_{dg}(X); s) = \prod_{\nu \in \Sigma_k} \prod_{w \text{ even}} L_w(X; s + \frac{w}{2}) = \prod_{w \text{ even}} L_w(X; s + \frac{w}{2}).
\end{equation}

The proof of the odd case is similar: simply replace the word “even” by the word “odd” and $\frac{w}{2}$ by $\frac{w-1}{2}$. □

7. Proof of Theorem 1.1

Recall that $k$ is a finite field extension of $\mathbb{Q}$ (when $\text{char}(k) = 0$) or a finite field extension of $\mathbb{F}_q(t)$ (when $\text{char}(k) > 0$), where $\mathbb{F}_q$ is the finite field with $q$ elements. In what follows, we will write $\mathcal{O}_k \subset k$ for the integral closure of $\mathbb{Z}$ in $k$ (when $\text{char}(k) = 0$) or for the integral closure of $\mathbb{F}_q[t]$ in $k$ (when $\text{char}(k) > 0$).

Recall that since $\mathcal{A}$ is a geometric noncommutative $k$-scheme, there exists a smooth proper $k$-scheme $X$ and an admissible triangulated subcategory $\mathfrak{A}$ of $\text{perf}(X)$ such that $\mathcal{A}$ and $\mathfrak{A}_{dg}$ are Morita equivalent.

Notation 7.1. Given an integer $0 \leq w \leq 2\dim(X)$ and a prime number $l \neq \text{char}(k)$, let us write $\beta_w$ for the dimension of the $Q_\nu$-vector space $H_{et}^w(X \times_k \overline{k}; Q_\nu)$; it is well-known that this dimension is independent of $l$. 


Lemma 7.2. For every non-archimedean place $\nu \in \Sigma_k$, we have the equalities (consult Notation 6.2):

(7.3) \[ \dim_{\mathbb{Q}_l} H_{\text{even},l,\nu}(\text{perf}_{\text{dg}}(X) \otimes_k k_{\nu}) = \sum_{w \text{ even}} \beta_w \] \[ \dim_{\mathbb{Q}_l} H_{\text{odd},l,\nu}(\text{perf}_{\text{dg}}(X) \otimes_k k_{\nu}) = \sum_{w \text{ odd}} \beta_w . \]

Proof. We have the following isomorphisms of $\mathbb{Q}_l$-vector spaces

(7.4) \[ H_{\text{even},l,\nu}(\text{perf}_{\text{dg}}(X) \otimes_k k_{\nu}) \cong H_{\text{even},l,\nu}(\text{perf}_{\text{dg}}(X \times_k k_{\nu})) \]

(7.5) \[ \cong \bigoplus_{w \text{ even}} H^w_{\text{et}}((X \times_k k_{\nu}) \times_{k_{\nu}} k_{\nu}; \mathbb{Q}_l(w/2)) \]

where (7.4) follows from the Morita equivalence $\text{perf}_{\text{dg}}(X) \otimes_k k_{\nu} \to \text{perf}_{\text{dg}}(X \times_k k_{\nu}), F \mapsto F \times_k k_{\nu}$, and (7.5) from Proposition 5.6. Consequently, the left-hand side of (7.3) follows from the (well-known) fact that the canonical homomorphism $H^w_{\text{et}}(X \times_k \mathbb{F}_l; \mathbb{Q}_l) \to H^w_{\text{et}}((X \times_k k_{\nu}) \times_{k_{\nu}} k_{\nu}; \mathbb{Q}_l)$ is invertible. The proof of the odd case is similar: simply replace the word “even” by the word “odd” and $w/2$ by $w-1/2$. \(\square\)

Notation 7.6. Let $S_X$ be the (finite) set of prime ideals of $\mathcal{O}_k$ where $X$ has bad reduction, $\mathcal{O}_k[S_X^{-1}]$ the localized ring, and $\Sigma_X$ the subset of $\Sigma_k$ corresponding to the prime ideals of $\mathcal{O}_k[S_X^{-1}]$. Given a prime ideal $\mathcal{P} \triangleleft \mathcal{O}_k[S_X^{-1}]$, we will denote by $\nu_{\mathcal{P}} \in \Sigma_X$ the corresponding non-archimedean place. Similarly, given a non-archimedean place $
u \in \Sigma_X$, we will denote by $\mathcal{P}_\nu \triangleleft \mathcal{O}_k[S_X^{-1}]$ the corresponding prime ideal.

Lemma 7.7. Let $\nu \in \Sigma_X$ be a non-archimedean place and $\lambda$ an eigenvalue of the automorphism $\pi_{\nu} \otimes_{\mathbb{Q}_{\nu}} \mathbb{C}$ of the $\mathbb{C}$-vector space $H_{\text{even},l,\nu}(\text{perf}_{\text{dg}}(X) \otimes_k k_{\nu})^I_{\nu} \otimes_{\mathbb{Q}_{\nu}} \mathbb{C}$, resp. $H_{\text{odd},l,\nu}(\text{perf}_{\text{dg}}(X) \otimes_k k_{\nu})^I_{\nu} \otimes_{\mathbb{Q}_{\nu}} \mathbb{C}$. Under these notations, we have $|\lambda| = 1$, resp. $|\lambda| = N_{\nu}^{1/2}$.

Proof. As explained in the proof of Proposition 6.6, we have an isomorphism of $\text{Gal}(\overline{\mathbb{Q}}_{\nu}/k_{\nu})$-modules

(7.8) \[ H_{\text{even},l,\nu}(\text{perf}_{\text{dg}}(X) \otimes_k k_{\nu})^I_{\nu} \otimes_{\mathbb{Q}_{\nu}} \mathbb{C} \cong \bigoplus_{w \text{ even}} H^w_{\text{et}}((X \times_k k_{\nu}) \times_{k_{\nu}} k_{\nu})^I_{\nu} \otimes_{\mathbb{Q}_{\nu}} \mathbb{C}, \]

where the automorphism $\pi_{\nu} \otimes_{\mathbb{Q}_{\nu}} \mathbb{C}$ on the left-hand side of (7.8) corresponds to the diagonal automorphism $\bigoplus_{w \text{ even}} (\pi_{\nu} \otimes_{\mathbb{Q}_{\nu}} \mathbb{C}) \cdot N_{\nu}^{1/2}$ on the right-hand side. Let $\lambda$ be an eigenvalue of the automorphism $\pi_{\nu} \otimes_{\mathbb{Q}_{\nu}} \mathbb{C}$ of the $\mathbb{C}$-vector space $H^w_{\text{et}}((X \times_k k_{\nu}) \times_{k_{\nu}} k_{\nu})^I_{\nu} \otimes_{\mathbb{Q}_{\nu}} \mathbb{C}$, with $0 \leq w \leq 2\dim(X)$. Since $X$ has good reduction at the prime ideal $\mathcal{P}_\nu$, it follows from Deligne’s proof of the Weil conjecture (consult [4]) and from the smooth proper base-change property of étale cohomology that $|\lambda| = N_{\nu}^{1/2}$. Consequently, we conclude that $|\lambda| = N_{\nu}^{1/2} N_{\nu}^{-1/2} = 1$. The proof of the odd case is similar: simply replace the word “even” by the word “odd” and the diagonal automorphism $\bigoplus_{w \text{ even}} (\pi_{\nu} \otimes_{\mathbb{Q}_{\nu}} \mathbb{C}) \cdot N_{\nu}^{1/2}$ by $\bigoplus_{w \text{ odd}} (\pi_{\nu} \otimes_{\mathbb{Q}_{\nu}} \mathbb{C}) \cdot N_{\nu}^{-1/2}$. \(\square\)

Now, note that since the set $\Sigma_k \setminus \Sigma_X$ is finite, in order to prove Theorem 1.1, we can (and will) replace the infinite products $\prod_{\nu \in \Sigma_k} L_{\text{even},\nu}(A; s)$ and $\prod_{\nu \in \Sigma_k} L_{\text{odd},\nu}(A; s)$ by the infinite products $\prod_{\nu \in \Sigma_X} L_{\text{even},\nu}(A; s)$ and $\prod_{\nu \in \Sigma_X} L_{\text{odd},\nu}(A; s)$, respectively.

Notation 7.9. Given a non-archimedean place $\nu \in \Sigma_k$ and an integer $n \geq 1$, consider the complex numbers:

\[ \#_{(+,\nu,n)} := \text{trace}(\pi_{\nu} \otimes_{\mathbb{Q}_{\nu}} \mathbb{C}) \cdot n \cdot H_{\text{even},l,\nu}(A \otimes_k k_{\nu})^I_{\nu} \otimes_{\mathbb{Q}_{\nu}} \mathbb{C} \]

\[ \#_{(-,\nu,n)} := \text{trace}(\pi_{\nu} \otimes_{\mathbb{Q}_{\nu}} \mathbb{C}) \cdot n \cdot H_{\text{odd},l,\nu}(A \otimes_k k_{\nu})^I_{\nu} \otimes_{\mathbb{Q}_{\nu}} \mathbb{C}. \]

Proposition 7.10. Given a non-archimedean place $\nu \in \Sigma_X$ and an integer $n \geq 1$, we have the inequalities:

\[ |\#_{(+,\nu,n)}| \leq \sum_{w \text{ even}} \beta_w \quad |\#_{(-,\nu,n)}| \leq (\sum_{w \text{ odd}} \beta_w) \cdot N_{\nu}^{1/2}. \]

Proof. Let us write $\chi_{(+,\nu),\nu}$, resp. $\chi_{(-,\nu),\nu}$, for the dimension of the $\mathbb{C}$-vector space

(7.11) \[ H_{\text{even},l,\nu}(A \otimes_k k_{\nu})^I_{\nu} \otimes_{\mathbb{Q}_{\nu}} \mathbb{C}, \text{ resp. } H_{\text{odd},l,\nu}(A \otimes_k k_{\nu})^I_{\nu} \otimes_{\mathbb{Q}_{\nu}} \mathbb{C}, \]

and $\{\lambda_{(+,\nu,1)}, \ldots, \lambda_{(+,\nu,\chi_{(+,\nu),\nu})}\}$, resp. $\{\lambda_{(-,\nu,1)}, \ldots, \lambda_{(-,\nu,\chi_{(-,\nu),\nu})}\}$, for the set of eigenvalues (with multiplicities) of the automorphism $\pi_{\nu} \otimes_{\mathbb{Q}_{\nu}} \mathbb{C}$ of (7.11). Moreover, let us write $\mathfrak{A}_l$ for the left orthogonal complement of $\mathfrak{A}$ in $\text{perf}(X)$ and $\mathfrak{A}_{\text{dg}}$ for the associated full dg subcategory of $\text{perf}_{\text{dg}}(X)$. By construction, we have
a semi-orthogonal decomposition \( H^0(\text{perf}_d(X)) = \langle H^0(\mathfrak{A}_d), H^0(\mathfrak{A}_{d'}) \rangle \), i.e., \( \text{perf}(X) = \langle \mathfrak{A}, \mathfrak{A} \rangle \). Therefore, by combining Proposition 5.5 with Lemma 4.3, we obtain an isomorphism of Gal\((k_f/k_t)\)-modules between \( H_{\text{even}, d_1}(\text{perf}_d(X) \otimes_k k_t) \) and the direct sum \( H_{\text{even}, d_1}(\mathfrak{A} \otimes_k k_t) \oplus H_{\text{even}, d_1}(\mathfrak{A}_{d'} \otimes_k k_t) \) and between \( H_{\text{odd}, d_1}(\text{perf}_d(X) \otimes_k k_t) \) and the direct sum \( H_{\text{odd}, d_1}(\mathfrak{A} \otimes_k k_t) \oplus H_{\text{odd}, d_1}(\mathfrak{A}_{d'} \otimes_k k_t) \). On the one hand, thanks to Lemma 7.7, this implies that if \( \lambda \) is an eigenvalue of the automorphism \( \pi \otimes \mathbb{Q}_p \), \( \mathbb{C} \) of (7.11), then \(|\lambda| = 1\), resp. \(|\lambda| = \mathbb{N}_p^2\). On the other hand, thanks to Lemma 7.2, this implies that \( \chi_{(+,\nu)} \leq \sum_{w \text{ even}} \beta_{w} \), resp. \( \chi_{(-,\nu)} \leq \sum_{w \text{ odd}} \beta_{w} \). As a consequence, we obtain the following inequalities:

\[
\#_{(+,\nu,n)} = |\lambda_{(+,\nu,1)}^n + \cdots + \lambda_{(+,\nu,\chi_{(+,\nu)})}^n| \leq |\lambda_{(+,\nu,1)}|^n + \cdots + |\lambda_{(+,\nu,\chi_{(+,\nu)})}|^n = \chi_{(+,\nu)} \leq \sum_{w \text{ even}} \beta_{w}
\]

\[
\#_{(-,\nu,n)} = |\lambda_{(-,\nu,1)}^n + \cdots + \lambda_{(-,\nu,\chi_{(-,\nu)})}^n| \leq |\lambda_{(-,\nu,1)}|^n + \cdots + |\lambda_{(-,\nu,\chi_{(-,\nu)})}|^n = \chi_{(-,\nu)} \cdot N_{\nu}^{\frac{\nu}{2}} \leq \left( \sum_{w \text{ odd}} \beta_{w} \right) \cdot N_{\nu}^{\frac{\nu}{2}}.
\]

This concludes the proof of Proposition 7.10.

The following general result, whose proof is a simple linear algebra exercise, is well-known:

**Lemma 7.12.** Given an endomorphism \( f : V \to V \) of a finite-dimensional \( \mathbb{C} \)-linear vector space, we have the equality of formal power series \( \log \left( \frac{1}{1 - \log(f)} \right) = \sum_{n \geq 1} \text{trace}(f^n) \frac{t^n}{n!} \), where \( \log(t) := \sum_{n \geq 1} \frac{1 - t^{-n}}{n} (t - 1)^{n} \).

Given a non-archimedean place \( \nu \in \Sigma_X \), consider the formal power series and their exponentiations:

\[
\phi_{(+,\nu)}(t) := \sum_{n \geq 1} \#_{(+,\nu,n)} \frac{t^n}{n} \quad \varphi_{(+,\nu)}(t) := \exp(\phi_{(+,\nu)}(t)) = \sum_{n \geq 0} a_{(+,\nu,n)} t^n
\]

\[
\phi_{(-,\nu)}(t) := \sum_{n \geq 1} \#_{(-,\nu,n)} \frac{t^n}{n} \quad \varphi_{(-,\nu)}(t) := \exp(\phi_{(-,\nu)}(t)) = \sum_{n \geq 0} a_{(-,\nu,n)} t^n.
\]

Note that, thanks to Lemma 7.12, we have \( \varphi_{(+,\nu)}(N_{\nu}^{-s}) = L_{\text{even},\nu}(A; s) \) and \( \varphi_{(-,\nu)}(N_{\nu}^{-s}) = L_{\text{odd},\nu}(A; s) \) for every non-archimedean place \( \nu \in \Sigma_X \).

**Definition 7.13.** Consider the following multiplicative Dirichlet series

\[
\varphi_{+}(s) := \sum_{I \in \mathcal{O}_k[S_X^{\text{per}}]} \frac{b_{+}(I)}{N(I)^s} \quad \varphi_{-}(s) := \sum_{I \in \mathcal{O}_k[S_X^{\text{per}}]} \frac{b_{-}(I)}{N(I)^s},
\]

where \( I \in \mathcal{O}_k[S_X^{\text{per}}] \) is an ideal, \( b_{+}(I) := a_{(+,\nu,p,r_1)} \cdots a_{(+,\nu,p_m,r_m)} \) and \( b_{-}(I) := a_{(-,\nu,p,r_1)} \cdots a_{(-,\nu,p_m,r_m)} \) are the products associated to the (unique) prime decomposition \( I = \mathcal{P}_1^{r_1} \cdots \mathcal{P}_m^{r_m} \), and \( N(I) \) is the norm of \( I \).

The following general result, concerning the absolute convergence of Dirichlet series, is well-known:

**Lemma 7.14.** We have the following equivalences

\[
\sum_{I} \left| \frac{b_{+}(I)}{N(I)^s} \right| < \infty \Leftrightarrow \prod_{\mathcal{P}} \sum_{n \geq 0} \frac{b_{+}(\mathcal{P}^n)}{N(\mathcal{P}^n)^s} < \infty \Leftrightarrow \sum_{I} \left| \frac{b_{+}(I)}{N(I)^s} \right| < \infty \Leftrightarrow \prod_{\mathcal{P}} \sum_{n \geq 0} \frac{b_{-(\mathcal{P})}}{N(\mathcal{P}^n)^s} < \infty,
\]

where \( \mathcal{P} \in \mathcal{O}_k[S_X^{\text{per}}] \) is a prime ideal. Moreover, if the left-hand side, resp. right-hand side, of (7.15) converges, then we obtain the equality \( \sum_{I} \frac{b_{+}(I)}{N(I)^s} = \prod_{\mathcal{P}} \sum_{n \geq 0} \frac{b_{+}(\mathcal{P}^n)}{N(\mathcal{P}^n)^s} \), resp. \( \sum_{I} \frac{b_{-}(I)}{N(I)^s} = \prod_{\mathcal{P}} \sum_{n \geq 0} \frac{b_{-(\mathcal{P})}}{N(\mathcal{P}^n)^s} \).

Given a prime ideal \( \mathcal{P} \in \mathcal{O}_k[S_X^{\text{per}}] \), recall that its norm \( N(\mathcal{P}) \) is defined as the cardinality of the quotient field \( \mathcal{O}_k[S_X^{\text{per}}]/\mathcal{P} \). Since \( \mathcal{O}_k[S_X^{\text{per}}]/\mathcal{P} \) is isomorphic to the residue field \( \kappa_{\nu_P} \), we hence obtain the following equalities:

\[
\sum_{n \geq 0} \frac{b_{+}(\mathcal{P}^n)}{N(\mathcal{P}^n)^s} = \sum_{n \geq 0} \frac{a_{(+,\nu_P,n)}}{N_{\nu_P}^s} \cdot \varphi_{(+,\nu_P)}(N_{\nu_P}^{-s}) = L_{\text{even},\nu_P}(A; s)
\]

\[
\sum_{n \geq 0} \frac{b_{-}(\mathcal{P}^n)}{N(\mathcal{P}^n)^s} = \sum_{n \geq 0} \frac{a_{(-,\nu_P,n)}}{N_{\nu_P}^s} \cdot \varphi_{(-,\nu_P)}(N_{\nu_P}^{-s}) = L_{\text{odd},\nu_P}(A; s).
\]
Therefore, thanks to Lemma 7.14 and to classical properties of Dirichlet series, in order to prove Theorem 1.1, it suffices to show the following claim: we have $\prod_P \sum_{n \geq 0} \frac{|a_{\nu_P, n}|}{N(P)^{nz}} < \infty$, resp. $\prod_P \sum_{n \geq 0} \frac{|a_{-\nu_P, n}|}{N(P)^{nz}} < \infty$, for every real number $z > 1$, resp. $z > \frac{3}{2}$. Note that, by construction, we have the following inequalities:

\[
\sum_{n \geq 0} \frac{|a_{\nu_P, n}|}{N(P)^{nz}} \leq \exp\left(\sum_{n \geq 1} \frac{|\#(\nu_P, n)|}{n N(P)^{nz}}\right) \leq \exp\left(\sum_{n \geq 1} \frac{|\#(-\nu_P, n)|}{n N(P)^{nz}}\right) \leq \exp\left(\sum_{n \geq 1} \frac{|\#(-\nu_P, n)|}{n N(P)^{nz}}\right).
\]

Moreover, by taking $\exp(-)$ to Lemma 7.17 below, we observe that $\prod_P \exp\left(\sum_{n \geq 1} \frac{|\#(\nu_P, n)|}{n N(P)^{nz}}\right) < \infty$, resp. $\prod_P \exp\left(\sum_{n \geq 1} \frac{|\#(-\nu_P, n)|}{n N(P)^{nz}}\right) < \infty$, for every real number $z > 1$, resp. $z > \frac{3}{2}$. Consequently, by taking the product, over all the prime ideals $P \in \mathcal{O}_k[S^{-1}]$, of the inequalities (7.16), we obtain the aforementioned claim.

**Lemma 7.17.** We have $\sum_P \sum_{n \geq 1} \frac{|\#(\nu_P, n)|}{n N(P)^{nz}} < \infty$, resp. $\sum_P \sum_{n \geq 1} \frac{|\#(-\nu_P, n)|}{n N(P)^{nz}} < \infty$, for every real number $z > 1$, resp. $z > \frac{3}{2}$.

**Proof.** Note that, thanks to Proposition 7.10 and to the equality $N(P) = N_{\nu_P}$, it suffices to show that $\sum_P \sum_{n \geq 1} \frac{1}{N(P)^{nz}} < \infty$, resp. $\sum_P \sum_{n \geq 1} \frac{1}{N(P)^{nz}} < \infty$, for every real number $z > 1$, resp. $z > \frac{3}{2}$. Let us assume first that $\text{char}(k) = 0$. Recall that in this case $k$ is a finite field extension of $\mathbb{Q}$. In what concerns $\sum_P \sum_{n \geq 1} \frac{1}{N(P)^{nz}} < \infty$, with $z > 1$, we have the following (in)equalities (where $p$ stands for a prime number)

\[
\sum_{p} \sum_{n \geq 1} \frac{1}{N(P)^{nz}} \leq \sum_{p} \sum_{n \geq 1} \frac{1}{p^{nz}} \leq [k : \mathbb{Q}] \cdot \sum_{p} \sum_{n \geq 1} \frac{1}{p^{nz}} = [k : \mathbb{Q}] \cdot \sum_{p} \frac{1}{p^z} = 2 \cdot [k : \mathbb{Q}] \cdot \sum_{n \geq 1} \frac{1}{n^z} < \infty,
\]

where (7.18) follows from the fact that the norm $N(P)$ is a power of $p$ whenever $P$ divides $p$, (7.19) from the fact that the number of prime ideals $P$ which divide $p$ is always bounded by the degree $[k : \mathbb{Q}]$, (7.20) from the (convergent) geometric series $\sum_{n \geq 0} \frac{1}{p^{nz}} = \frac{1}{1 - \frac{1}{p^z}}$, (7.21) from a simple inspection, and (7.22) from the fact that the Riemann zeta function $\zeta(s) := \sum_{n \geq 1} \frac{1}{n^z}$ is convergent when $\text{Re}(s) > 1$. Similarly, in what concerns $\sum_P \sum_{n \geq 1} \frac{1}{N(P)^{nz}} < \infty$, with $z > \frac{3}{2}$, we have the (in)equalities:

\[
\sum_{p} \sum_{n \geq 1} \frac{1}{N(P)^{nz}} \leq \sum_{p} \sum_{n \geq 1} \frac{1}{p^{(z-\frac{3}{2})}} = [k : \mathbb{Q}] \cdot \sum_{p} \sum_{n \geq 1} \frac{1}{p^{(z-\frac{3}{2})}} = [k : \mathbb{Q}] \cdot \sum_{p} \frac{1}{p^{z-\frac{3}{2}}} - 1 \leq 2 \cdot [k : \mathbb{Q}] \cdot \sum_{n \geq 1} \frac{1}{n^{z-\frac{3}{2}}} < \infty.
\]

Let us now assume that $\text{char}(k) > 0$. Recall that in this case $k$ is a finite field extension of $\mathbb{F}_q(t)$. In what concerns $\sum_P \sum_{n \geq 1} \frac{1}{N(P)^{nz}} < \infty$, with $z > 1$, we have the following (in)equalities ($(p(t))$, resp. $(q(t))$, stands
for a prime ideal, resp. ideal, of the ring \( \mathbb{F}_q[t] \):

\[
(7.23) \quad \sum_{p} \sum_{n \geq 1} \frac{1}{N(P) n^z} \leq \sum_{(p(t))} \sum_{n \geq 1} \frac{1}{N((p(t))) n^z} \leq [k : \mathbb{F}_q(t)] \cdot \sum_{(p(t))} \sum_{n \geq 1} \frac{1}{N((p(t))) n^z} \\
= [k : \mathbb{F}_q(t)] \cdot \sum_{(p(t))} \frac{1}{N((p(t)))^z - 1} \\
\leq 2 \cdot [k : \mathbb{F}_q(t)] \cdot \sum_{(q(t))} \frac{1}{N((q(t)))^z} < \infty,
\]

where (7.23) follows from the fact that the norm \( N(P) \) is a power of \( N((p(t))) \) whenever \( P \) divides \( (p(t)) \), (7.24) from the fact that the number of prime ideals \( P \) which divide \( (p(t)) \) is always bounded by the degree \( [k : \mathbb{F}_q(t)] \), (7.25) from the (convergent) geometric series \( \sum_{n \geq 0} \frac{1}{N((p(t)))^{nz}} = \frac{1}{1 - \frac{1}{N((p(t)))^z}} \), (7.26) from a simple inspection, and (7.27) from the fact that the classical zeta function \( \sum_{q(t)} \frac{1}{N((q(t)))^z} \) is convergent when Re\((s) > 1\). Similarly, in what concerns \( \sum_{P} \sum_{n \geq 1} \frac{1}{N(P) n^{z - 2}} \leq \sum_{(p(t))} \sum_{n \geq 1} \frac{1}{N((p(t))) n^{z - 2}} \leq [k : \mathbb{F}_q(t)] \cdot \sum_{(p(t))} \sum_{n \geq 1} \frac{1}{N((p(t))) n^{z - 2}} = [k : \mathbb{F}_q(t)] \cdot \sum_{(q(t))} \frac{1}{N((q(t)))^{z - 2}} - 1 \leq 2 \cdot [k : \mathbb{F}_q(t)] \cdot \sum_{(q(t))} \frac{1}{N((q(t)))^{z - 2}} < \infty \). This concludes the proof of Lemma 7.17. 

8. Proof of Theorem 1.2

Note first that if the \( L \)-functions \( \{L_w(X; s)\}_{w \text{ even}} \), resp. \( \{L_w(X; s)\}_{w \text{ odd}} \), satisfy condition (C1), then the shifted \( L \)-functions \( \{L_w(X; s + \frac{w}{2})\}_{w \text{ even}} \), resp. \( \{L_w(X; s + \frac{w-1}{2})\}_{w \text{ odd}} \), also satisfy condition (C1). Thanks to Proposition 6.6, this hence implies that the noncommutative \( L \)-function \( L_{even}(\text{perf}_{dg}(X); s) \), resp. \( L_{odd}(\text{perf}_{dg}(X); s) \), admits a (unique) meromorphic continuation to the entire complex plane. Now, Proposition 6.6 implies moreover the implications (1.3).

Let us assume now that \( \text{char}(k) > 0 \) and that the conjecture \( R_{even}(\text{perf}_{dg}(X)) \), resp. \( R_{odd}(\text{perf}_{dg}(X)) \), holds. It follows from the work of Grothendieck [5] and Deligne [8, 4] that the \( L \)-function \( L_w(X; s) \), with \( 0 \leq w \leq 2\dim(X) \), does not have a pole in the critical strip \( \frac{w}{2} < \text{Re}(s) < \frac{w}{2} + 1 \). Consequently, thanks to Proposition 6.6, we conclude that all the zeros of the \( L \)-function \( L_w(X; s) \) that are contained in the critical strip \( \frac{w}{2} < \text{Re}(s) < \frac{w}{2} + 1 \) lie necessarily in the vertical line \( \text{Re}(s) = \frac{w+1}{2} \) (otherwise, the noncommutative \( L \)-function \( L_{even}(\text{perf}_{dg}(X); s) \), resp. \( L_{odd}(\text{perf}_{dg}(X); s) \), would have a zero outside the vertical line \( \text{Re}(s) = \frac{1}{2} \), resp. \( \text{Re}(s) = 1 \)). In other words, the converse implications of (1.3) hold.

Let us assume now that \( \text{char}(k) = 0 \), that the \( L \)-functions \( \{L_w(X; s)\}_{0 \leq w \leq 2\dim(X)} \) satisfy condition (C2), and that the conjecture \( R_{even}(\text{perf}_{dg}(X)) \), resp. \( R_{odd}(\text{perf}_{dg}(X)) \), holds. Since the \( L \)-function \( L_w(X; s) \), with \( 0 \leq w \leq 2\dim(X) \), does not have a pole in the critical strip \( \frac{w}{2} < \text{Re}(s) < \frac{w}{2} + 1 \), we hence conclude from Proposition 6.6 that all the zeros of \( L_w(X; s) \) that are contained in the critical strip \( \frac{w}{2} < \text{Re}(s) < \frac{w}{2} + 1 \) lie
necessarily in the vertical line \( \text{Re}(s) = \frac{w+1}{2} \) (otherwise, the noncommutative \( L \)-function \( L_{\text{even}}(\text{perf}_{\text{dg}}(X); s) \), resp. \( L_{\text{odd}}(\text{perf}_{\text{dg}}(X); s) \), would have a zero outside the vertical line \( \text{Re}(s) = \frac{1}{2} \), resp. \( \text{Re}(s) = 1 \). In other words, the converse implications of (1.3) also hold.

9. Proof of Theorem 2.3

By definition of the Lefschetz decomposition \( \text{perf}(X) = \langle A_0, A_1(1), \ldots, A_{i-1}(i-1) \rangle \), we have a chain of admissible triangulated subcategories \( A_{i-1} \subseteq \cdots \subseteq A_1 \subseteq A_0 \) with \( A_r(r) := A_r \otimes \mathcal{L}_X(r) \); note that \( A_r(r) \) is equivalent to \( A_r \). Let us write \( a_r \) for the right-orthogonal complement of \( A_{r+1} \) in \( A_r \); these are called the primitive subcategories in \([13, \S4]\). By construction, we have the following semi-orthogonal decompositions:

\[
A_r = \langle a_r, a_{r+1}, \ldots, a_{i-1} \rangle \quad 0 \leq r \leq i-1.
\]

Following \([13, \text{Thm. 6.3}]\), we have a chain of admissible triangulated subcategories \( B_{j-1} \subseteq \cdots \subseteq B_1 \subseteq B_0 \) and an associated HP-dual Lefschetz decomposition \( \text{perf}(Y) = \langle B_{j-1}(-1-j), \ldots, B_1(-1), B_0 \rangle \) with respect to \( \mathcal{L}_Y(1) \). Moreover, we have the following semi-orthogonal decompositions:

\[
B_r = \langle a_0, a_1, \ldots, a_{\dim(Y) - r - 2} \rangle \quad 0 \leq r \leq j-1.
\]

Furthermore, since the linear subspace \( L \subset V^\vee \) is generic, we can assume without loss of generality that the linear sections \( X_L \) and \( Y_L \) are not only smooth but also that they have the expected dimensions, i.e., \( \dim(X_L) = \dim(X) - \dim(L) \) and \( \dim(Y_L) = \dim(Y) - \dim(L^\perp) \). As explained in \([13, \text{Thm. 6.3}]\), this yields the following semi-orthogonal decompositions

\[
\text{perf}(X_L) = \langle C_L, A_{\dim(L)}(1), \ldots, A_{i-1}(i-\dim(L)) \rangle
\]

\[
\text{perf}(Y_L) = \langle B_{j-1}(1) \langle L^\perp \rangle - j), \ldots, B_{\dim(L^\perp)}(-1), C_L \rangle,
\]

where \( C_L \) is a common (triangulated) category. Let us denote by \( A_{r,\text{dg}} \), by \( a_{r,\text{dg}} \), and by \( C_{L,\text{dg}} \), the dg enhancements of \( A_r \), \( a_r \), and \( C_L \), induced from the dg category \( \text{perf}_{\text{dg}}(X_L) \). Similarly, let us denote by \( B_{r,\text{dg}} \) and by \( C'_{L,\text{dg}} \) the dg enhancements of \( B_r \) and \( C_L \), induced from the dg category \( \text{perf}_{\text{dg}}(Y_L) \). Since the functor \( \text{perf}_{\text{dg}}(X_L) \to C_L \to \text{perf}_{\text{dg}}(Y_L) \) is of Fourier-Mukai type, the dg categories \( C_{L,\text{dg}} \) and \( C'_{L,\text{dg}} \) are Morita equivalent.

Note that, by construction, all the aforementioned dg categories are geometric noncommutative \( k \)-schemes. Note that by combining the above semi-orthogonal decompositions (9.1)-(9.4) with Lemma 9.8 below and with an iterated application of Proposition 6.4, we obtain the following equalities

\[
\text{L}_{\text{even}}(\text{perf}_{\text{dg}}(X_L); s) = \text{L}_{\text{even}}(C_{L,\text{dg}}; s) \cdot \zeta_k(s) \cdots \zeta_k(s) \quad \text{L}_{\text{odd}}(\text{perf}_{\text{dg}}(X_L); s) = \text{L}_{\text{odd}}(C_{L,\text{dg}}; s)
\]

\[
\text{L}_{\text{even}}(\text{perf}_{\text{dg}}(Y_L); s) = \zeta_k(s) \cdots \zeta_k(s) \cdot \text{L}_{\text{even}}(C_{L,\text{dg}}; s) \quad \text{L}_{\text{odd}}(\text{perf}_{\text{dg}}(Y_L); s) = \text{L}_{\text{odd}}(C_{L,\text{dg}}; s),
\]

where the number of copies of the Dedekind zeta function \( \zeta_k(s) \) in (9.5), resp. in (9.6), is equal to the sum of the ranks of the (free) Grothendieck groups \( K_{0}(A_{\dim(L)}) \), \ldots, \( K_{0}(A_{i-1}) \), resp. \( K_{0}(B_{j-1}) \), \ldots, \( K_{0}(B_{\dim(L^\perp)}) \). Consequently, since the Dedekind zeta function \( \zeta_k(s) \) satisfies conditions (C1)-(C2), the following holds:

\[
\text{ERH}_k \Rightarrow \left( \text{R}_{\text{even}}(\text{perf}_{\text{dg}}(X_L)) \Rightarrow \text{R}_{\text{odd}}(\text{perf}_{\text{dg}}(Y_L)) \right) \quad \text{R}_{\text{odd}}(\text{perf}_{\text{dg}}(X_L)) \Leftrightarrow \text{R}_{\text{odd}}(\text{perf}_{\text{dg}}(Y_L)).
\]

The proof follows now from the combination of (9.7) with Theorem 1.2.

**Lemma 9.8.** We have the following computations (with \( 0 \leq r \leq i-1 \))

\[
\text{L}_{\text{even}}(a_{r,\text{dg}}; s) = \zeta_k(s) \cdots \zeta_k(s) \quad \text{L}_{\text{odd}}(a_{r,\text{dg}}; s) = 1,
\]

where \( n_r \) stands for the rank of the (free) Grothendieck group \( K_{0}(a_r) \).

**Proof.** We start by recalling from \([21, \S2.3]\) the definition of the additive category \( \text{Hom}_{0}(k) \); consult \S 4.2. Given two dg categories \( A \) and \( B \), let us write \( D(A^{\text{op}} \otimes k) \) for the derived category of \( A-B \)-bimodules and \( \text{rep}(A, B) \) for the full triangulated subcategory of \( D(A^{\text{op}} \otimes k) \) consisting of those \( A-B \)-bimodules \( B \) such that for every object \( x \in A \) the associated right \( B \)-module \( B(x,-) \) belongs to the full triangulated subcategory of compact objects \( D_{c}(B) \). The objects of the category \( \text{Hom}_{0}(k) \) are the (small) dg categories, the abelian groups of morphisms \( \text{Hom}_{0}(k)(U(A), U(B)) \) are given by the Grothendieck groups \( K_{0} \text{rep}(A, B) \), and the composition law is induced by the (derived) tensor product of dg bimodules.
By assumption, we have a full exceptional collection \( \mathcal{A}_0 = \{ \mathcal{E}_1, \ldots, \mathcal{E}_n \} \). As explained in [21, §2.4.2], this implies that \( U(\mathcal{A}_{0,dg}) \simeq U(k)^{\oplus n} \) in the additive category \( \text{Hmono}(k) \). Moreover, since \( a_r \) is an admissible triangulated subcategory of \( \mathcal{A}_0 \), \( U(\mathcal{A}_{r,dg}) \) becomes a direct summand of \( U(\mathcal{A}_{0,dg}) \). Thanks to the computation
\[
\text{Hom}_{\text{Hmono}(k)}(U(\mathcal{A}_0), U(\mathcal{A}_0)) \simeq \text{Hom}_{\text{Hmono}(k)}(U(k)^{\oplus n}, U(k)^{\oplus n}) \simeq \text{Hom}_{\text{Hmono}(k)}(U(k), U(k))^{\oplus n} \times n \simeq K_0 \text{rep}(k) \times (n \times n) \simeq K_0 \text{D}(k)^{\oplus n} \times n
\]
we observe that the endomorphism ring of \( U(\mathcal{A}_{0,dg}) \) is isomorphic to the ring of \((n \times n)\)-matrices with \( Z \)-coefficients. This implies that \( U(\mathcal{A}_{r,dg}) \) is then necessarily isomorphic to \( U(k)^{\oplus n_r} \) for a certain integer \( n_r \leq n \). Moreover, this integer \( n_r \) is equal to the rank of the (free) Grothendieck group \( K_0(\mathcal{A}_r) \) because
\[
\text{Hom}_{\text{Hmono}(k)}(U(k), U(\mathcal{A}_{r,dg})) := K_0 \text{rep}(k, \mathcal{A}_{r,dg}) \simeq K_0(\mathcal{A}_r) \quad \text{and} \quad \text{Hom}_{\text{Hmono}(k)}(U(k), U(k)^{\oplus n_r}) \simeq Z^{\oplus n_r}.
\]
Now, recall from Proposition 5.5 and Lemma 4.3 that, given any non-archimedean place \( \nu \in \Sigma_k \), the associated functors \( H_{\text{even}, l_\nu}(- \otimes_k k_{\nu}) \) and \( H_{\text{odd}, l_\nu}(- \otimes_k k_{\nu}) \) are additive invariants. Consequently, since these additive invariants factor through the additive category \( \text{Hmono}(k) \) (consult the factorization (4.4)), we obtain an isomorphism of \( Q_{\nu} \)-linear \( \text{Gal}(k_{\nu}/k_{\nu}) \)-modules between \( H_{\text{even}, l_\nu}(\mathcal{A}_{r,dg} \otimes_k k_{\nu}) \) and the direct sum \( H_{\text{even}, l_\nu}(k \otimes_k k_{\nu})^{\oplus n_r} \) and between \( H_{\text{odd}, l_\nu}(\mathcal{A}_{r,dg} \otimes_k k_{\nu}) \) and the direct sum \( H_{\text{odd}, l_\nu}(k \otimes_k k_{\nu})^{\oplus n_r} \). Therefore, making use of the canonical Morita equivalence \( k \to \text{perf}_{dg}(\text{Spec}(k)) \), of Proposition 5.6, and of Definition 6.3, we obtain the above computations (9.9). \( \square \)

10. Proof of Theorems 3.1, 3.2, 3.5, and 3.7

Proof of Theorem 3.1. By construction of the noncommutative gluing \( X \otimes Y \), we have the semi-orthogonal decomposition \( H^0(X \otimes B Y) = (H^0(\text{perf}_{dg}(X)), H^0(\text{perf}_{dg}(Y))) \). Consequently, thanks to Proposition 6.4, we conclude that \( L_{\text{even}}(X \otimes B Y; s) = L_{\text{even}}(\text{perf}_{dg}(X); s) \cdot L_{\text{even}}(\text{perf}_{dg}(Y); s) \). This yields the implication \( R_{\text{even}}(\text{perf}_{dg}(X)) \Rightarrow R_{\text{even}}(\text{perf}_{dg}(Y)) \). Therefore, the proof follows now from Theorem 1.2. The proof of the odd case is simply to replace the word “even” by the word “odd”.

Proof of Theorem 3.2. Thanks to the decomposition \( \text{perf}(X) = \{ T_i, \mathcal{O}_X, \ldots, \mathcal{O}_X(n - \text{deg}(X)) \} \), an iterated application of Proposition 6.4 yields the following equalities:
\[
L_{\text{even}}(\text{perf}_{dg}(X); s) = L_{\text{even}}(T_{dg}; s) \cdot \zeta_k(s) \cdot \cdots \cdot \zeta_k(s) \quad \text{and} \quad L_{\text{odd}}(\text{perf}_{dg}(X); s) = L_{\text{odd}}(T_{dg}; s).
\]
On the one hand, since the Dedekind zeta function \( \zeta_k(s) \) satisfies conditions (C1)-(C2), the left-hand side of (10.1) implies that \( R_{\text{even}}(\text{perf}_{dg}(X)) \Rightarrow R_{\text{even}}(T_{dg}) \). On the other hand, the right-hand side of (10.1) implies that \( R_{\text{odd}}(\text{perf}_{dg}(X)) \Leftrightarrow R_{\text{odd}}(T_{dg}) \). Consequently, the proof follows now from Theorem 1.2.

Proof of Theorem 3.5. As proved in [23, Thm. 3.5], since the quotient \( k \)-algebra \( A/J \) is separable, we have an isomorphism \( U(A)_Q \simeq U(k_1)_Q + \cdots + U(k_n)_Q \) in the \( Q \)-linearization \( \text{Hmono}(k)_Q \) of the additive category \( \text{Hmono}(k) \); consult §4.4. Thanks to Proposition 5.5 and Lemma 4.3, given any non-archimedean place \( \nu \in \Sigma_k \), the associated functors \( H_{\text{even}, l_\nu}(- \otimes_k k_{\nu}) \) and \( H_{\text{odd}, l_\nu}(- \otimes_k k_{\nu}) \) are additive invariants. Since these additive invariants take values in the category \( \text{Gal}(k_{\nu}/k_{\nu}) \)-Mod, which is a \( Q \)-linear category, the above isomorphism combined with the factorization (4.4) leads to the following isomorphisms of \( \text{Gal}(k_{\nu}/k_{\nu}) \)-modules:
\[
H_{\text{even}, l_\nu}(A \otimes_k k_{\nu}) \simeq \bigoplus_{i=1}^n H_{\text{even}, l_\nu}(k_i \otimes_k k_{\nu}) \quad \text{and} \quad H_{\text{odd}, l_\nu}(A \otimes_k k_{\nu}) \simeq \bigoplus_{i=1}^n H_{\text{odd}, l_\nu}(k_i \otimes_k k_{\nu}).
\]
Making use of the canonical Morita equivalences \( k_i \to \text{perf}_{dg}(\text{Spec}(k_i)) \), of Proposition 5.6, and of Definition 6.3, we hence conclude that \( L_{\text{even}}(A; s) = \prod_{i=1}^n L_0(\text{Spec}(k_i)) \) and \( L_{\text{odd}}(A; s) = 1. \) This yields the implication \( \sum_{i=1}^n R_0(\text{Spec}(k_i)) \Rightarrow R_{\text{even}}(A) \).

Proof of Theorem 3.7. Recall first that we have a canonical Morita equivalence \( A \to \text{perf}_{dg}(A) \). Following Orlov [17, Cor. 3.4], there exists a smooth proper \( k \)-scheme \( X \) such that \( \text{perf}(A) \) is an admissible triangulated subcategory of \( \text{perf}(X) \). Moreover, since the quotient dg \( k \)-algebra \( A/J_k \) is separable, we have a semi-orthogonal decomposition \( \text{perf}(X) = \langle \text{perf}(D_1), \ldots, \text{perf}(D_n) \rangle \), where \( D_1, \ldots, D_n \) are separable division \( k \)-algebras. Let us denote by \( k_1, \ldots, k_n \) the centers of the \( k \)-algebras \( D_1, \ldots, D_n \) these are separable finite field extensions of \( k \). Making use of [23, Thm. 2.11], we hence conclude that \( U(A)_Q \) becomes a direct
summand of $U(\text{perf}_{dg}(X))_Q \simeq U(\mathbb{A})_Q \oplus \cdots \oplus U(\mathbb{A}_n)_Q$ in the category $\text{Hmo}_0(\mathbb{A})_Q$. Note that, similarly to the proof of Theorem 3.5, this implies that $L_{\text{odd}}(\mathbb{A}; s) = 1$. Now, recall from [21, §4.9] that the classical category of Artin motives (with $\mathbb{Q}$-coefficients) may be identified with the idempotent completion of the full subcategory of $\text{Hmo}_0(\mathbb{A})_Q$ consisting of the objects $\{U(\text{perf}_{dg}(X))_Q | X\}$ is a 0-dimensional $k$-scheme). Under this identification, $U(\mathbb{A})_Q$ corresponds to an Artin motive $\rho: \text{Gal}(\overline{k}/k) \to \text{GL}(V)$, where $V$ is a finite dimensional $\mathbb{Q}$-vector space, and the noncommutative $L$-function $L(\text{even}(\mathbb{A}; s))$ to the classical Artin $L$-function $L(\rho; s)$ of $\rho$. Consequently, the proof follows now from Weil’s work [27, §V].

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