The Modified Hartmann Potential Effects on $\gamma$-rigid Bohr Hamiltonian

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Abstract. In this paper, we present the solution of Bohr Hamiltonian in the case of $\gamma$-rigid for the modified Hartmann potential. The modified Hartmann potential was formed from the original Hartmann potential, consists of $\beta$ function and $\theta$ function. By using the separation method, the three-dimensional Bohr Hamiltonian equation was reduced into three one-dimensional Schrodinger-like equation which was solved analytically. The results for the wavefunction were shown in mathematically, while for the binding energy was solved numerically. The numerical binding energy for the presence of the modified Hartmann potential is lower than the binding energy value in the absence of modified Hartmann potential effect.

1. Introduction
The collective model of atomic nucleus has been introduced by Bohr and Mottelson since more than 60 years ago, was well-known as Bohr-Mottelson hamiltonian or Bohr Hamiltonian. It is important and attractive topics in nuclear physics to explain the properties and behavior of the nucleus. The model of collection motion for atomic nucleus treats the nucleus as a vibrating and rotating liquid drop. In this model, the focus is on some collective parameters like $\gamma$, the angle measuring origin from axial symmetry, $\beta$, which represents deformation coordinate from the spherical shape, $Q_k (k = 1, 2, 3)$, which are the components of angular momentum in the intrinsic frame [1-3]. The solution of Bohr Hamiltonian in various cases have been investigated by some researchers. The solution for Davidson potential was analyzed in 1974 [4] and others paper involving Kratzer potential [5] and Morse potential [6] has been solved. Iachello used in the critical point symmetries E(5) and X(5) [7,8]. In the framework of the Bohr Hamiltonian, applying minimal length effect of considering gravitational influences in the system and it was used to satisfy experimental data. Effect of the presence and the absence of minimal length has been investigated in Coulomb-like potential on $\gamma$-rigid by Alimohammadi et.al in 2016 [3]. And then, in 2017, Alimohammadi et. al analyzed Bohr Hamiltonian in $\gamma$-rigid with position-dependent mass [9]. In this paper, we are interested in comparing these two situations: the collective motion of atomic nucleus in the absence of minimal length effects in the presence and the absence of modified Hartmann potential in $\gamma$-rigid Bohr Hamiltonian. The Hartmann potential is an exactly solvable ring-shaped potential was introduced by H. Hartmann [10-11], is given as:

$$V(r, \theta) = \eta \sigma^2 \epsilon_0 \left( \frac{2q_0}{r} - \frac{1}{r^2} \frac{q \sigma^2}{\sin^2 \theta} \right)$$

(1)
where \( a_0 = \hbar^2 / B_m e^2 \) and \( \epsilon_0 = -B_m e^4 / 2\hbar^2 \) represent the Bohr radius and the ground state energy of the hydrogen atom, respectively, \( B_m \) is rest mass electron, \( e \) is a charge of the electron and \( \eta, \sigma, \varsigma \) are three dimensionless parameters. When \( q = 0, \eta \sigma^2 = Z \), the Hartmann potential was reduced to the Coulomb potential, which \( Z \) is protons numbers of an atom. This ring-shaped potential was introduced to describe ring-shaped molecules like cyclic polyenes and benzene[12]. For double-ring-shaped, this potential is given by

\[
V(r, \theta) = \eta \sigma^2 \epsilon_0 \left( \frac{2a_0}{r} - \frac{1}{\beta^2 \sin^2 \theta} \right) + \frac{\hbar^2}{2B_m} \frac{c}{r^2 \cos^2 \theta}, \quad c \geq 0
\]

where \( c \) is a dimensionless parameter, and \( 0 < r < \infty, 0 \leq \theta \leq \pi \). The Hartmann potential in equation (2) is formed to explain the behavior of electron [10], so we modified it becomes the suitable form which can be applied to explain the collective motion of nucleus using Bohr Hamiltonian by changing the variable \( r \) into deformation coordinate \( \beta \), is given as

\[
V(\beta, \theta) = \eta \sigma^2 \epsilon_1 \left( \frac{2a_0}{\beta^2} - \frac{1}{3\beta^2 \sin^2 \theta} \right) + \frac{\hbar^2}{2B_m} \frac{1}{3\beta^2 \cos^2 \theta}
\]

where \( a_i = r_0 A^i \) and \( \epsilon_i = a_{\text{u}} A + a_{\text{n}} A^{2i} + a_{\text{c}} Z A^i + a_{\text{m}} \left( (N - Z)^2 / A \right) \), which \( B_i \) is nucleon mass, \( A \) is a number of neutrons plus protons, \( N \) is a number of neutrons, \( r_0 \) is a radius of the nucleus \( a_{\text{u}} = -16 \text{ MeV}, a_{\text{n}} = 20 \text{ MeV}, a_{\text{c}} = 0.751 \text{ MeV}, a_{\text{m}} = 21.4 \text{ MeV} \) and \( \eta, \sigma, \varsigma, c \) are three dimensionless parameters. \( a_i \) represent the nucleus radius and \( \epsilon_i \) is binding energy of nucleus from the Bethe-Weizsacker formula [13]. The Hartman potential in equation (2) and the modified Hartmann potential in equation (3) are shapes invariant, so the modification which was applied can be used, mathematically.

2. The Bohr Hamiltonian in the case \( \gamma \)-rigid in the absence of minimal length

The collective Bohr Hamiltonian in the case \( \gamma \)-rigid operator is given as [3]:

\[
\hat{H} = -\frac{\hbar^2}{2B_1} \Delta + \frac{a \hbar^4}{B_1} \Delta^2 + V(\beta, \theta, \phi)
\]

with

\[
\Delta = \left[ \frac{1}{\beta^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} - \frac{\sin^2 \theta}{\sin \theta} \frac{\partial}{\partial \phi} \right]
\]

and \( \Delta_\Omega \) is the angular part of Laplace operator, is given as

\[
\Delta_\Omega = \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right]
\]

For the absence minimal length effect, where \( \alpha = 0 \), the Hamiltonian operator in equation (4) is reduced to

\[
\hat{H} = -\frac{\hbar^2}{2B_1} \Delta + V(\beta, \theta, \phi)
\]

By inserting equations (5-6) into equation (7), and by using \( H \Psi(\beta, \theta, \phi) = E \Psi(\beta, \theta, \phi) \) we obtained

\[
-\frac{\hbar^2}{2B_1} \left[ \frac{1}{\beta^2} \frac{\partial^2}{\partial \beta^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} - \frac{\sin^2 \theta}{\sin \theta} \frac{\partial}{\partial \phi} \right] \Psi(\beta, \theta, \phi) = (E - V(\beta, \theta, \phi)) \Psi(\beta, \theta, \phi)
\]

where

\[
V(\beta, \theta, \phi) = V(\beta) + (V(\theta) / 3\beta^2) + (V(\phi) / 3\beta^2 \sin \theta)
\]

By letting \( \Psi(\beta, \theta, \phi) = \{ \rho(\beta) / \beta \} \phi(\theta) \phi(\phi) \) in equation (8), substituting equation (9) into equation (8) with applying equation (3) and by using separation variable method, we obtained the three one-dimensional differential equations, are given as

\[
\frac{d^2 \rho}{d\beta^2} + \left( \frac{2B_1}{\hbar^2} E - \frac{2a \eta \sigma^2 \varsigma}{\beta} \right) - \frac{\lambda}{\beta^2} R = 0
\]

\[
\frac{d^2 \phi}{d\phi^2} + \left( \cot \theta \frac{d}{d\theta} + \lambda - \frac{\lambda}{\sin \theta} - \frac{2B_1}{\hbar^2} \left( \frac{2a \eta \sigma^2 \varsigma \varsigma}{\sin \theta} + \frac{\lambda}{\cos \theta} \right) \right) \Theta = 0
\]
\[ \frac{d^2 \mathcal{P}}{d\rho^2} + \Lambda^2 \Phi = 0 \]  
(12)
where we have introduced the separation constants \( \lambda = L(L+1) \) and \( \Lambda^2 \). \( L \) denotes the orbital angular momentum quantum number \([9]\).

3. Solution of the Bohr Hamiltonian equation for modified Hartmann potential

3.1. Solution for \( \beta \)-part

In this part, we used equation (10), which is like Coulomb potential. By setting \( \rho = r\beta, \tau = \sqrt{-8B_E/h^2}, s = \left( -2B_E/h^2 \right) \left( 2a_\beta \eta \sigma^2 e_i / \tau \right) \) in equation (10), we get

\[ \frac{d^2 R}{d\rho^2} + \left( \frac{\lambda}{\rho} - \frac{1}{4} - \frac{(L+1)/3}{\rho^3} \right) R = 0 \]  
(13)

By taking the physically acceptable solution \( R(\rho) = \rho^{-(L+1)/3} e^{-\rho/2} f(\rho) \) and substituting into equation (13), yield

\[ \rho \frac{d^2 f}{d\rho^2} + \left( 2\sqrt{\frac{(L+1)/3}{s} + \frac{1}{4} + 1} - \rho \right) \frac{df}{d\rho} - \left( \sqrt{\frac{(L+1)/3}{s} + \frac{1}{4} + 1} - s \right) f = 0 \]  
(14)

Equation (14) is the confluent hypergeometric differential equation, so for the eigenvalue, we obtained from relation \( \left( \sqrt{\frac{(L+1)/3}{s} + \frac{1}{4} + 1} - s \right) - s = -n, n = 0, 1, 2, \ldots \), yield

\[ E_{n,L} = -\frac{1}{2\beta} \left( 2a_\beta \eta \sigma^2 e_i / \sqrt{\frac{(L+1)/3}{s} + \frac{1}{4} + 1 + n} \right)^2 \]  
(15)
which it is in agreement with the result of the energy equation in Ref. 3, but different in potential constants. The numerical energies are shown in Table 1 for isotope Palladium (Pd) with \( n = 0, n = 1 \) by using equation (18), where \( L \) is not dependent in solution for the angular part, so it is not influenced by the angular part of the modified Hartmann potential. The binding energy for isotope Pd increase for the heavier isotopes, while for the increasing value of \( L \) and \( n \), energy decrease. The binding energy gives an explanation about the rigid properties of the nucleus. The heavy nucleus which is indicated by a number of protons plus neutrons \( A \) has the greater binding energy, it shows that the heavy nucleus more rigid than the small nucleus. While, in the greater value of \( n \), which indicates the nucleus orbital, the binding energy decrease, because in the outer orbital, the nucleus force decrease.

### Table 1. Numerical energy spectra (in TeV) for isotope Palladium (\( ^{102}\text{Pd} \)) with \( \eta = 0.2, \sigma = 0.2 \)

| Nucleus | \( E_{0,2} \) | \( E_{0,4} \) | \( E_{0,6} \) | \( E_{0,8} \) | \( E_{1,2} \) | \( E_{1,4} \) | \( E_{1,6} \) | \( E_{1,8} \) |
|---------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| \( ^{102}\text{Pd} \) | 46.312      | 18.909      | 10.137      | 6.296       | 20.583      | 10.861      | 6.657       | 4.488       |
| \( ^{104}\text{Pd} \) | 47.743      | 19.494      | 10.450      | 6.490       | 21.219      | 11.196      | 6.863       | 4.628       |
| \( ^{106}\text{Pd} \) | 49.132      | 20.061      | 10.754      | 6.678       | 21.837      | 11.522      | 7.063       | 4.761       |
| \( ^{108}\text{Pd} \) | 50.423      | 20.588      | 11.037      | 6.855       | 22.410      | 11.825      | 7.249       | 4.887       |
| \( ^{110}\text{Pd} \) | 51.670      | 21.097      | 11.309      | 7.024       | 22.964      | 12.117      | 7.428       | 5.008       |

Then, for the wave function, we use equation (14). Equation (14) is confluent hypergeometric differential equation, so the generalized wave function was obtained in hypergeometric term, is given as

\[ R(\rho) = \rho^{-(L+1)/3} e^{-\rho/2} F_1 \left( \sqrt{\frac{(L+1)/3}{s} + \frac{1}{4} + 1}; a_\beta \eta \sigma^2 e_i \sqrt{\frac{2\beta}{s}}; \frac{2\sqrt{(L+1)/3}}{s} \right) \]  
(16)
The wavefunction in equation (16) can be transformed to $\beta$ function to explain about the behavior nucleus in varying deformation coordinates. Every nucleus has specific deformation coordinate which is considered with the number of protons plus neutrons $A$.

3.2. Solution for $\theta$-part

For the $\theta$-part solution, we used the variable substitution $\cos \theta = x$ into equation (11), yield

$$
(1 - x^2) \frac{d^2 \Theta}{dx^2} - 2x \frac{d \Theta}{dx} + \left( L(L+1) - \frac{d^2}{1-x^2} - \frac{2B_c}{h^2} \right) \Theta = 0
$$

where $d^2 = \Lambda^2 + \left( 2B_c/h^2 \right) q^2 \sigma^2 a_i^2 e_i$. Equation (17) is universal associated-Legendre differential equation when $c = 0, L = k + d, |\Lambda|, k = 0, 1, 2, \ldots$ which has the three regular singular points $x = 0, \pm 1$, so we take the form,

$$
\Theta = \left( 1 - x^2 \right)^{\frac{d}{dx^2}} x^2 y(x)
$$

where $\delta$ is taken as $\delta (d - 1) - \left( 2B_c/h^2 \right) = 0$, yield

$$
\delta = \left\{ \begin{array}{ll}
0 \text{ or } 1, & c = 0; \\
\left[ 1 + \sqrt{1 + \left( \frac{8B_c}{h^2} \right) c} \right]/2, & c > 0
\end{array} \right.
$$

By substituting equation (18) into equation (17), give the differential equation is given as

$$
(1 - x^2) \frac{d^2 y}{dx^2} + \left( \frac{2d}{x} \right) - \left( 2(d + \delta) + 1 \right) x \frac{dy}{dx} + \left( L(L+1) - (d + \delta)(d + \delta + 1) \right) y = 0
$$

To solve equation (20), we used the series method by choosing the form: $y(x) = \sum_{n=0}^{\infty} y_n x^n$

and substituting into equation (20), yield the values of quantum number $L$ are given as,

$$
L = 2k + d + \delta = \left\{ \begin{array}{ll}
2k + \sqrt{\Lambda^2 + \left( \frac{2B_c}{h^2} \right) q^2 \sigma^2 a_i^2 e_i} + \left[ 1 + \sqrt{1 + \left( \frac{8B_c}{h^2} \right) c} \right]/2, & c > 0
\end{array} \right.
$$

and we obtained,

$$
y(x) = \sum_{\kappa=0}^{\infty} \frac{(-1)^{\kappa}}{2^L \kappa! (k-\kappa)!} \frac{\Gamma(k+\delta-\kappa+1)}{\Gamma(2k+2\delta-2\kappa+1)} \times \frac{\Gamma(2L-2\kappa+1)}{\Gamma(L-\kappa+1)} x^{2k-2\kappa}
$$

For the wavefunction, we substituted equation (23) into equation (18), so we obtained

$$
\Theta = \left( 1 - x^2 \right)^{\frac{d}{dx^2}} x^2 \delta \sum_{\kappa=0}^{\infty} \frac{(-1)^{\kappa}}{2^L \kappa! (k-\kappa)!} \frac{\Gamma(k+\delta-\kappa+1)}{\Gamma(2k+2\delta-2\kappa+1)} \times \frac{\Gamma(2L-2\kappa+1)}{\Gamma(L-\kappa+1)} x^{2k-2\kappa}
$$

The solution in $\theta$-part gives two solutions considering of $\delta$ value in equation (19).

3.3. Solution for $\varphi$-part

Equation (12) which is the ordinary differential equation, so we get the solution is given as

$$
\Phi = \left( \frac{1}{\sqrt{2\pi}} \right) e^{iA} \Lambda = 0, \pm 1, \pm 2, \pm 3, \ldots
$$

so, total wavefunction was obtained by using equations (16, 24-25),

$$
\Psi(\beta, \theta, \varphi) = \frac{1}{\beta} \rho^{(2L+1)/2} e^{-\frac{\rho}{2} - \frac{\beta}{2}} F_1 \left( \frac{L(L+1)}{3} + \frac{1}{4}, \frac{L(L+1)}{3} + \frac{1}{4} + 1; a, \eta^2 \sigma^2 \right) \left[ \frac{2}{\sqrt{h^2 E}} \sqrt{\frac{2}{h^2 E}} \right] \times \left( 1 - x^2 \right)^{\frac{d^2}{dx^2}} x^2 \sum_{\kappa=0}^{\infty} \frac{(-1)^{\kappa}}{2^L \kappa! (k-\kappa)!} \frac{\Gamma(k+\delta-\kappa+1)}{\Gamma(2k+2\delta-2\kappa+1)} \times \frac{\Gamma(2L-2\kappa+1)}{\Gamma(L-\kappa+1)} x^{2k-2\kappa}
$$

The wavefunction in equation (26) is depending on $\beta$ function, the polar in $\theta$ function and azimuth in $\varphi$ function which has the related quantum number. In this paper, we only give the mathematical solution in the wavefunction. We solved numerically the binding energy from equation (15) which depend on the polar solution and azimuth solution to compare the result of energy between the
presence and the absence of the modified Hartmann potential. The result is shown in Table 2 for isotope Pd with the value of $k = 0,1; \Lambda = 0, c = 0$ which $L$ is obtained from the $\theta$ -part solution in equation (22).

| Nucleus | $E_{0.\ell_1}$ | $E_{0.\ell_2}$ | $E_{1.\ell_1}$ | $E_{1.\ell_2}$ |
|---------|----------------|----------------|----------------|----------------|
| $^{102}$Pd | 1.262 | 1.258 | 1.248 | 1.244 |
| $^{104}$Pd | 1.253 | 1.250 | 1.240 | 1.236 |
| $^{106}$Pd | 1.245 | 1.241 | 1.232 | 1.228 |
| $^{108}$Pd | 1.237 | 1.233 | 1.224 | 1.220 |
| $^{110}$Pd | 1.229 | 1.225 | 1.216 | 1.213 |

The binding energies in Table 2 show that in the presence of the Hartmann potential in $\theta$ -part give the smaller value of energy if it was compared with the energy result which is shown in Table1 for the absence of the Hartmann potential. It indicated that the effect of Hartmann potential in binding energy gives the decreasing value of energy. For the isotope Palladium, the increase of value $A$ or for the heavier nucleus, the binding energy decrease, it is reverse with the results in the absence of the modified Hartmann potential. The result in Table 2, the value of $L_1$ is smaller than $L_2$ which give the effect in the value of energy, where for the greater value of $L$, the energy decrease. The $\varphi$ -part solution does not give the influence in the value of binding energy, but in wavefunction only, because the modified Hartmann potential which was used in this research does not have the $\varphi$ -part potential. The binding energy of nucleus in the presence of the modified Hartmann potential give the large different with the results in the absence the Hartmann potential effect.

4. Conclusion
This paper has developed to compare the presence and the absence of the modified Hartmann potential effect in Bohr Hamiltonian in the case of gamma rigid and without the minimal length effect. The solution for wavefunction was expressed in mathematically in equation (26). The numerically result for binding energy for some nucleus were solved and there are largely different between the value energy for the presence modified Hartmann potential and without this potential. The modified Hartmann potential give effect in decreasing value of binding energy.

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References
[1] Bohr A 1952 K. Dan. Vidensk. Selks. Mat-Fys. Medd 26 14
[2] Bohr A and Mottelson B 1953 K. Dan. Vidensk. Selks. Mat-Fys. Medd 27 16
[3] Alimohammadi M, Hassanabadi H and Sobhani H 2016 Modern Physics Letter A 31 1650193
[4] Rohozinski S G, Srebrny J and Horbaczewska K 1974 Z. Phys. 268 401
[5] Bonatsos D, Georgoudis P E, Minkov N. Petrellis D and Quesne C 2013 Phys. Rev. C 88 034316
[6] Boztosum I, Bonatsos D and Inci I 2008 Phys. Rev. C 77 044302
[7] Iachello F 2000 Phys. Rev. Lett. 85 3580
[8] Iachello F 2001 Phys. Rev. Lett. 87 052502
[9] Alimohammadi M, Hassanabadi H and Zare S 2017 Nuclear Physics A 960 78
[10] Hartmann H and Schuch D 1980 *Int. J. Quantum Chem* **18** 125
[11] Chang-Yuan C, Fa-Lin L, Dong-Sheng S and Shi-Hai D 2013 *Chin. Phys. B* **22** 100302
[12] Rahbar H nd Sadeghi J 2016 *Theoretical Physics* **17**
[13] Greiner W and Maruhn J A 1996 Nu clear Models Springer-Verlag Berlin Heidelberg New York