Status and Perspectives of Non-perturbative Renormalization in Weak Decays.

M. Talevi\textsuperscript{a} *

\textsuperscript{a}Department of Physics & Astronomy, University of Edinburgh, The King’s Buildings, EH9 3JZ (UK)

We discuss the status and the problems related to the application of the off-shell non-perturbative renormalization method in a fixed gauge to operators relevant to weak decays. In particular, we critically reappraise the method recently proposed for the $\Delta I = 1/2$ rule. We also present a general analysis of the renormalization for the $\Delta I = 3/2$ operators, and apply it to the $\Delta S = 2$ operator.

1. Introduction

Lattice QCD is a unique, systematically improvable, method for computing matrix elements from first principles, and has proven a powerful and appealing approach. In spite of the successes, progress has been slow due to the presence of systematic effects, such as discretization and higher-order renormalization effects. The Symanzik improvement program \cite{Symanzik} is an attrative method which allows to reduce discretization order by order in $a$ in physical quantities. The improvement coefficients were computed at first in perturbation theory (PT) at lowest order, reducing the error from $O(a)$ to $O(g_a^2)$ \cite{PT}, and recently non-perturbatively, achieving a full $O(a^2)$ improvement \cite{NPM}. In parallel, there has been significant progress in the development of non-perturbative (NP) methods \cite{NP}. It is by now generally accepted that NP methods in the renormalization of lattice operators yield reliable and accurate results and should be used whenever possible. In the following, we concentrate on the applications of the non-perturbative renormalization method of ref. \cite{NPM}, hereafter refered to as NPM. The other methods are discussed by Sommer \cite{Sommer} and Kurasinski \cite{Kurasinski} in this Workshop.

Renormalization of lattice operators is a crucial ingredient in the calculation of physical weak matrix elements on the lattice. A physical amplitude $A_{\alpha \rightarrow \beta}$ of a weak transition $\alpha \rightarrow \beta$ is calculated via the Operator Product Expansion (OPE) by

$$A_{\alpha \rightarrow \beta} = C_W(M_W/\mu)\langle \alpha | \hat{O}(\mu) | \beta \rangle$$

where $C_W$ is the Wilson coefficient of the OPE, $M_W$ is the mass of the $W$ boson, $\mu$ is the renormalization scale and $\langle \alpha | \hat{O}(\mu) | \beta \rangle$ is the matrix element of the renormalized operator (at the scale $\mu$) relevant to the physical process. The Wilson coefficient $C_W(M_W/\mu)$ contains the short-distance information and can be calculated in PT in the continuum at the renormalization scale $\mu$. The matrix element contains the long-distance dynamics and thus must be calculated non-perturbatively on the lattice. Renormalization relates the regularized lattice matrix elements to its continuum counterpart.

On the lattice, chiral symmetry is explicitly broken with Wilson-like fermions. The possibility of recovering the chiral symmetry in the continuum limit was shown in \cite{FNR}. The general prescription is to subtract from the bare operator $O(a)$ all the operators of dimension less or equal than $O(a)$, which have the same quantum numbers conserved by the regularization,

$$\langle \alpha | \hat{O}(\mu) | \beta \rangle = \lim_{a \rightarrow 0} \langle \alpha | Z_O(\mu a) O(a) + \sum_i Z_i O_i(a) | \beta \rangle,$$

where the subtracted operators $O_i$ are not constrained to the same chiral representation of $O(a)$. If these operators have lower dimension than $O$, the mixing constants are power-divergent in the cutoff, $Z_i \sim 1/a^d$ with $d = \text{dim} |O| - \text{dim} |O_i| > 0$. However, this procedure is valid only if $O(a)$ is made of operators exactly of dimension $O$. In this case, the mixing terms vanish

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\textsuperscript{*}Invited talk at the International Workshop “Lattice QCD on Parallel Computers”, 10-15 March 1997, Tsukuba (Japan)
These divergent factors can pick up exponentially small contributions in the coupling $\alpha_s$, yielding a finite contribution as $a \to 0$, i.e.

$$\frac{1}{a} e^{-1/\alpha_s(a)} \sim \Lambda_{\text{QCD}}.$$  

These divergences must be subtracted in a completely NP way.

2. Operators and phenomenology

There are a number of four-fermion operators which are relevant to different physical processes of phenomenological interest. They all have in common that their renormalization suffers from power-divergent [11,12].

- LL operators: the $\Delta I = 3/2$ components are necessary for the $B_K$ parameter, which enters in the study of CP violation in $K^0 - \bar{K}^0$ mixing. It is important to obtain a precise and reliable result, as with the measured value of the top quark mass, it enables us to limit the range of values of the CP-violation phase $\delta$.

The $\Delta I = 1/2$ components on the other hand are relevant to the study of the octect enhancement in $K \to \pi\pi$ decay, whose quantitative understanding still defies theorists. The difference with the $\Delta I = 3/2$ case is that the bare operator are allowed to mix with operators of lower dimensionality and hence with coefficients that are power-divergent [11,12].

- LR operators: the $I = 3/2$ part of the LR operators which appear in the effective weak Hamiltonian due to the electromagnetic penguin diagrams are the only operators which give rise to an imaginary part in the $K^+ \to \pi^+\pi^0$ amplitude, thus yielding the dominant contribution to $\epsilon'/\epsilon$. This contribution is usually expressed by the $B_7$ and $B_8$ parameters.

The $I = 1/2$ part of these operators have penguin contractions which make them too hard to handle at present. Note also that in presence of the heavy top quark, there is no GIM suppression.

- $\Delta B = 2$ operators: these are the chiral partners of the $\Delta I = 3/2$, and have recently been proposed in the study of flavour-changing neutral currents in the supersymmetric extensions of the Standard Model [14].

3. The method

In the NPM, the renormalization conditions are applied directly to the Green functions of quarks and gluons, in a fixed gauge, with given off-shell external states of large virtualities [3]. The method mimicks what is usually done in the perturbative calculation, but the Green functions are evaluated in a NP fashion from Monte Carlo simulations.

To give the flavour of the method, let us consider the simplified case of a multiplicatively renormalizable operator, e.g. a two-quark operator $O^{\text{latt}}(a)$, the renormalization condition we impose is [3]

$$Z_{\text{RI}}(\mu a) \langle p | O^{\text{latt}}(a) | p \rangle_{\mu^2 = \mu_0^2}$$

$$= \langle p | O^{\text{latt}}(a) | p \rangle_{\mu^2 = \mu_0^2},$$

where $\langle p | \cdots | p \rangle$ denotes the matrix element of external quarks of momenta $p$ which can be calculated non-perturbatively in the QCD coupling via Monte Carlo simulations [3]. The renormalized operator obtained with the NPM is then

$$\hat{O}_{\text{RI}}(\mu) = Z_{\text{RI}}^{\text{latt}}(\mu a) O^{\text{latt}}(a),$$

which depends on the external states and the gauge, but not on method used to regulate the ultra-violet divergences. To stress this point, we call the NP renormalization scheme Regularization Independent (RI) [5]. The physical operator

$$O^{\text{Phys}}(M_W) = C_{\text{RI}}(M_W/\mu) \hat{O}_{\text{RI}}(\mu)$$

is independent of external momenta and gauge (up to higher orders in continuum PT and lattice systematic effects) if the Wilson coefficient
function $C_{\text{RI}}(M_W/\mu)$ in the RI scheme is calculated with the same external momenta and gauge of $O_{\text{RI}}(\mu)$. The advantage of the RI scheme is that it completely avoids the use of lattice PT, which is expected to have a worse convergence than the continuum expansion. The coefficient function $C_{\text{RI}}(M_W/\mu)$ is instead calculated in continuum PT, which cannot be avoided since the Wilson OPE is defined perturbatively.

The phenomenologically more interesting case of four-fermion operators are in general not multiplicatively renormalizable. The operators which need to be subtracted are dictated by the symmetries of the action: charge conjugation (C), parity (P) and $s \leftrightarrow d$ flavour switching symmetry (S).

The main advantage of the NPM is its generality, being valid for any composite operator, as long as we can find (a posteriori) a window in the range of renormaliztion scales $\mu$ such that $\Lambda_{\text{QCD}} \ll \mu \ll O(1/a)$, in order to keep under control both the higher-order effects in the (continuum) perturbative calculation of $C_{\text{RI}}$ and discretization errors. We stress that this requirement is common to all NP methods on the lattice which work at a single value of the lattice spacing. The alternative would be matching from one value of the coupling to another, as done for example in the method of ref. [3]. The main disadvantage of the NPM is the necessity of gauge-fixing, which leaves a residual less constraining symmetry to dictate the form of the mixing, i.e. BRST-invariance.

4. $\Delta I = 1/2$: Seeking new ideas

A quantitative theoretical understanding of the $\Delta I = 1/2$ rule in $K \rightarrow \pi \pi$ decays has proven to be a formidable task since the calculation of hadronic matrix elements in the low-energy NP regime is needed. Let us review the strategies proposed so far.

In the continuum, with an active charm quark and the GIM mechanism at work, the operator basis given by

$$O_{LL}^\pm = \frac{1}{2}[(\bar{s}d)_L(\bar{u}u)_L \pm (\bar{s}u)_L(\bar{d}d)_L] - (u \rightarrow c).$$

In the framework of lattice QCD with Wilson-like fermions, the renormalization strategy is complicated by chiral symmetry breaking. In fact, the Wilson term induces the mixing of $O_{LL}^\pm$ with lower-dimensional operators, with power-divergent coefficients, which need to be subtracted non-perturbatively. In the following, we shall concentrate only on the octet component of $O_{LL}^\pm$, which we will denote with $O_0^\pm$. The lattice penguin operators, being proportional to $(m_u^2 - m_d^2)a^2 \ll 1$, will be neglected in the following.

We are now faced with the problem of calculating the four-point matrix elements $\langle \pi^+ \pi^- | O_0^\pm | K^0 \rangle$. The standard approach is to rely on lower order chiral PT to relate them to the more tractable three-point and two-point matrix elements:

$$\langle \pi^+ \pi^- | O_0^\pm | K^0 \rangle = i\gamma(\pm) (m_K^2 - m_{\pi}^2) / f_K$$

where $\gamma(\pm)$ are obtained from

$$\langle \pi^+ | O_0^\pm | K^0(q) \rangle = \gamma(\pm)(p \cdot q) - 2\delta(\pm) m_{\pi}^2 / f_K$$
$$\langle 0 | O_0^\pm | K^0 \rangle = i\delta(\pm) (m_K^2 - m_{\pi}^2)$$

In this approach, one relies on the calculation of the $K \rightarrow \pi$ matrix element, which only picks up a contribution from the parity-conserving (PC) part of the operators. Exploiting CPS symmetry, we obtain in the PC sector a renormalization structure of the form:

$$\tilde{O}_{PC}^\pm = Z_{PC}^\pm \left[ O_0^\pm + \sum_i Z_i^\pm O_i^\pm + Z_s^\pm O_s + Z_S^\pm O_S \right]$$

where $O_0^\pm$ are the PC bare operators, $O_i^\pm$, $i = 1, \ldots, 4$ are dimension-six operators of wrong chirality (cf. sec. [1]), $O_s$ is the magnetic operator $\bar{s}\sigma_{\mu\nu}F_{\mu\nu}d$ and $O_S$ is a dimension-three scalar density $\bar{s}d$. By GIM and power-counting, $Z_s^\pm \propto (m_c - m_u)$ and $Z_S^\pm \propto (m_c - m_u)/a^2$. Thus, the coefficient of the magnetic operator can in principle be calculated in PT, though it involves a two-loop calculation and is very complicated. The coefficient of the scalar density is power-divergent and can only be reliably calculated in a NP fashion.

There are in principle several NP approaches for calculating the mixing coefficients:
1. by imposing the Ward Identities on physical hadronic states \[1\];
2. by imposing the Ward Identities on quark states, as done for the \(\Delta S = 2\) operator in \[8\];
3. by the NPM \[12\].

The principal drawback of the method 1. is that it looses predictive power as the number of coefficients to determine gets large, as in the present case. Methods 2. and 3. are equivalent in the re-
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Some examples are

\[s(\bar{q} + m_d)d, \quad s(\bar{d} + m_d)^2d,\]  
\[(10)\]

which, by GIM, generate counterterms proportional to \((m_c - m_u)/a\) and \((m_c - m_u)/a\), respectively. When inserted in correlations, these terms give a finite contribution to the subtraction coefficient \(C_s\) of the scalar density \(\bar{s}d\). In PT, this problem is of course also present, but the spurious contributions can be eliminated by looking at the momentum dependence

\[C_s = C_s^{\text{true}} + f + g(p' + p') + h(p^2 + p'^2 + b\cdot p'),\]

where \(f, g, h\) are calculable functions of \(m_s\) and \(m_d\) and \(p, p'\) are the momenta of the two external legs. In principle, this could also be attempted in a NP approach, but it highly unlikely to be able to achieve the necessary accuracy.

- operators are not gauge invariant, because the off-shell Green functions are calculated in a fixed gauge.

The non-gauge invariant operators that may mix are dictated by lattice BRST symmetry \[19\]. Some of the these operators have been classified in ref. \[20\], e.g.

\[\bar{s}d\bar{A}(\bar{q} + m_d)d, \quad \bar{s}(\bar{q} + m_d)\bar{q}\bar{A}d.\]  
\[(11)\]

Again all of these operators need to be taken into account as they give a finite contribution to the subtraction coefficient.

The application of the NPM without these additional operators has been outlined in ref. \[12\]. Let us recall it to give a flavour of its complexity. According to the NPM, the mixing \(Z_s\) are determined by finding a set of projectors on the tree-
level amputated Green functions (GF), with off-
shell quark and gluon external states, the choice of which depends on the nature of the operators at hand. For the \(\Delta I = 1/2\) operators we choose the following set of external states: \(\bar{q}q, \bar{q}g, q\bar{q}q\), with the momenta given below in eq. \[(12)\]. For each choice of external states, i.e. for each different set of GF, we need different type of projectors. Let us denote with \(\hat{P}_S\) the projector on the \(q\bar{q}\) GF of the operator \(O_S\), with \(\hat{P}_g\) the projector on the \(q\bar{q}g\) GF of the operator \(O_g\), and with \(\hat{P}_j^\pm, j = 1, \ldots, 4\) the set of mutually orthogonal projectors on the operators \(O_i, i = 1, \ldots, 4\) \[23\]. Applying the projectors to the corresponding NP GF of the renormalized operators \(\hat{O}^\pm\), with an appropriate choice of the external states, we require that the renormalized operators be proportional to the bare operators, \(\hat{O}^\pm(\mu) \propto O^\pm_0(a)\) (up to terms of \(O(a)\)), i.e. we impose the following renormalization conditions (trace over colour and spin is understood in the projection operation):

\[\begin{align*}
\hat{P}_S(q(p)|\hat{O}^\pm|q(p)) &= 0 \\
\hat{P}_g(q(p-k)|q(k)|\hat{O}^\pm|q(p)) &= 0 \\
\hat{P}_j^\pm(q(p)|q(p)|\hat{O}^\pm|q(p)) &= 0,
\end{align*}\]  
\[(12)\]

where \(p\) and \(k\) denote the momentum of the external quark and gluon legs. The system of equations \[(12)\] in principle completely determines in a NP way the renormalization constants, as we have six conditions (non-homogeneous due to the matrix elements of \(O^\pm_0\), cf. eq. \[\hat{A}\]) in six unknown mixing constants, \(Z_i^\pm, i = 1, \ldots, 4, Z^\pm, Z^\pm_0\).

Unfortunately, since solving eq. \[(12)\] involves delicate cancellations between large contributions, it results in a very noisy determination, even with large statistics. The main computational difficulty lies in the calculation of the GF with penguin contractions. The need to include the operators which vanish by the equations of
motions and are not gauge invariant renders the application of the NPM, which was already complicated without them, highly impractical. We conclude that the standard positive parity methods are not a viable way of approaching the \( \Delta I = 1/2 \) rule and new ideas are needed \[17,21\].

5. \( \Delta I = 3/2 \): A general analysis of dimension-six mixing

We now turn to discuss the renormalization of \( \Delta I = 3/2 \) operators. They differ from the \( \Delta I = 1/2 \) operators in that there are no lower-dimensional operators with the same flavour content with which they can mix. This implies that we need not take into consideration the operators that vanish by the equations of motion or the non-gauge invariant ones as they can only affect the mixing with the lower dimensional operators.

In order to address this problem, it is convenient to work with 4 distinct fermion flavours \( \psi_f, f = 1, \ldots, 4 \), of degenerate mass. Once the mixing of the dimension-six generic operators with others of the same dimension has been obtained with four distinct flavours, it is straightforward to apply it to the appropriate operators of physical flavours.

We define the generic four fermion operators

\[
O_{T(1)\Gamma(2)} = (\bar{\psi}_1 \Gamma^{(1)} \psi_2)(\bar{\psi}_3 \Gamma^{(2)} \psi_4)
\]

\[
O_{T(2)\Gamma(1)} = (\bar{\psi}_1 \Gamma^{(2)} \psi_2)(\bar{\psi}_3 \Gamma^{(1)} \psi_4)
\]

\[
O_{F(1)\Gamma(2)} = (\bar{\psi}_1 \Gamma^{(1)} \psi_4)(\bar{\psi}_3 \Gamma^{(2)} \psi_2)
\]

\[
O_{F(2)\Gamma(1)} = (\bar{\psi}_1 \Gamma^{(2)} \psi_4)(\bar{\psi}_3 \Gamma^{(1)} \psi_2)
\]

(13)

where \( \Gamma^{(1)} \) and \( \Gamma^{(2)} \) denotes any Dirac matrix, and \( t^a \) the colour matrices. Under renormalization, the operators of eq.(13) can in principle mix with any other dimension-six operator, provided it has the same quantum numbers. The generic QCD Wilson lattice action with 4 degenerate quarks is symmetric under parity \( P \), and charge conjugation \( C \). Moreover, there are three other useful (flavour) symmetries of the action, namely the flavour exchange symmetry \( S \equiv (\psi_2 \leftrightarrow \psi_4) \) and the switching symmetries \( S' \equiv (\psi_1 \leftrightarrow \psi_2, \psi_3 \leftrightarrow \psi_4) \) and \( S'' \equiv (\psi_1 \leftrightarrow \psi_3, \psi_2 \leftrightarrow \psi_3) \). In Table 1, we classify the operators \( O_{T(1)\Gamma(2)} \) or combinations of them, according to the discrete symmetries \( P, C, S' \) and \( S'' \). We adopt the notation

\[
O_{T(1)\Gamma(2)} = \frac{1}{2N} \eta_{\Gamma(1)\Gamma(2)} O_{T(1)\Gamma(2)} \pm O_{T(2)\Gamma(1)}
\]

Note that the results of Table 2 apply also to the operators \( O_{T(1)\Gamma(2)} \), since, upon performing the symmetry transformations, sign differences, resulting from the presence of the colour matrix, disappear because the colour matrices appear quadratically. On the other hand, \( O_{F(1)\Gamma(2)} \) is obtained by applying \( C \) on \( O_{T(1)\Gamma(2)} \). Since \( S \) transforms \( S' \) into \( S'' \), the properties of Table 1 also apply to \( O_{F(1)\Gamma(2)} \), but with all \( S' \) and \( S'' \) columns exchanged. Again, the operator \( O_{F(2)\Gamma(2)} \) has the same properties as \( O_{F(1)\Gamma(2)} \), since the colour matrix \( t^a \) appears quadratically.

Our aim is to find complete bases of operators which mix under renormalization. Thus, besides classifying them according to their symmetries, we must also eliminate the operators which are not independent. This is seen by applying the standard identity of colour matrices

\[
t^a_{AB} t^b_{CD} = -\frac{1}{2N} \delta_{AB} \delta_{CD} + \frac{1}{2} \delta_{AD} \delta_{CB}
\]

(14)

on the \( t^a \)'s of a given operator. For the operator \( O_{T(1)\Gamma(2)} \) the result has the general form

\[
O_{T(1)\Gamma(2)} = -\frac{1}{2N} O_{T(1)\Gamma(2)} + \frac{1}{2} \sum_{n,m} C_{nm} O_{F}^{T(1)\Gamma(2)}
\]

Table 1

| \( O_{T(1)\Gamma(2)} \) | \( \text{P} \) | \( \text{CS} \) | \( \text{CS}' \) | \( \text{CPS} \) | \( \text{CPS}' \) |
|------------------------|----------------|----------------|----------------|----------------|----------------|
| \( O_{VV} \)           | +1             | +1             | +1             | +1             | +1             |
| \( O_{AA} \)           | +1             | +1             | +1             | +1             |
| \( O_{PP} \)           | +1             | +1             | +1             | +1             |
| \( O_{SS} \)           | +1             | +1             | +1             | +1             |
| \( O_{TT} \)           | +1             | +1             | +1             | +1             |
| \( O_{[VA+AV]} \)      | -1             | -1             | -1             | +1             | +1             |
| \( O_{[VA-AV]} \)      | -1             | -1             | +1             | +1             |
| \( O_{[SP+PS]} \)      | -1             | +1             | -1             | -1             |
| \( O_{[SP-PS]} \)      | -1             | +1             | -1             | -1             |
| \( O_{[TF]} \)         | -1             | +1             | -1             | -1             |

| Classification of four-fermion operators according to lattice symmetries. These properties are also valid for the operators \( O_{T(1)\Gamma(2)} \). For the operators \( O_{F(1)\Gamma(2)} \) and \( O_{F(2)\Gamma(1)\Gamma(2)} \) we must exchange the entries of the columns \( \text{CS}' \leftrightarrow \text{CS}'' \) and \( \text{CPS}' \leftrightarrow \text{CPS}'' \).
where the sum runs over all the Dirac matrices obtained by the Fierz transformation of $\Gamma^i\Gamma^j$, and the factors $C_{nm}$ are the appropriate constants of the Fierz transformation. Analogously we can express $O_{i\pm\Gamma^i(=\Gamma^j)}$ in terms of $O_{\Gamma(=\Gamma^j)}$ and $O_{i\pm\Gamma^j}$. Therefore, in the following, it is adequate to limit ourselves to the mixing of $O_{\Gamma(=\Gamma^j)}$'s and $O_{i\pm\Gamma^j}$'s, according to the entries of Table I.

Having eliminated the non-independent operators we proceed in classifying the complete bases of operators which mix under renormalization, according to the following two rules:

1. All operators with identical values of $P$, $CPS'$ and $CPS''$ are allowed to mix with each other. These now form a maximal basis.

2. If possible, the maximal basis must be decomposed into smaller bases, by using the remaining symmetry $S$, in order to form linear combinations of the operators of our basis, which have definite $S$, i.e. $S = +1$ or $S = -1$, that can only mix among themselves.

The first rule is easy to apply in practice, because the $P$, $CPS'$ and $CPS''$ values of the operators can be read off from Table I. As an example of this rule, we note that $O_{[SP-PS]}$ mixes with $O_{[VA-AV]}$, since they both have $P = -1$, $CPS' = -1$ and $CPS'' = +1$. Having thus applied the first rule, we turn to the specific task of reducing the basis, for each case of interest (second rule). This we now do case by case, using $S$ symmetry.

5.1. Parity violating operators

We consider first the parity violating four-fermion operators, cf. Tab. I. None of the four violating operators have identical $CPS'$ and $CPS''$ values. Each of them, however, mixes with some $S$-counterpart, e.g. $O_{[SP+PS]}$ and $O_{[SP+PS]}$ or $O_{[VA-AV]}$, $O_{[SP-PS]}$.

We examine first $O_{[VA-AV]}$ which mixes with $O_{[VA-AV]}$, only, forming a basis of two operators, characterized by $CPS' = CPS'' = +1$. We rotate this basis into

$$O^\pm_{[VA+AV]} \equiv \frac{1}{2} \left[ O_{[VA+AV]} \pm O^F_{[VA+AV]} \right],$$

and note that $O^+_{[VA+AV]}$ has $S = +1$ and $O^-_{[VA+AV]}$ has $S = -1$. Thus, they do not mix with each other. The final result is that the original basis of two operators has been decomposed into two bases of one operator each: the two operators $O^\pm_{[VA+AV]}$ of eq. (15) renormalize multiplicatively.

We now turn to $O_{[VA-AV]}$. It mixes with $O_{[SP-PS]}$, since they both have $CPS' = +1$ and $CPS'' = -1$. Similarly, $O^F_{[VA-AV]}$ and $O_{[SP-PS]}$ have $CPS' = -1$, $CPS'' = +1$. It is convenient to combine the two bases into a product basis of 4 operators:

$$O^\pm_{[VA-AV]} \equiv \frac{1}{2} \left[ O_{[VA-AV]} \pm O^F_{[VA-AV]} \right],$$

None of these operators have definite $CPS'$ or $CPS''$. However, they have definite $S = \pm 1$. Thus, they mix in pairs according to their $S$ value; i.e. we have reduced the original basis of four operators into two bases of two operators each.

Similarly, we rotate the operators $O_{[SP+PS]}$, $O^F_{[SP+PS]}$, $O_{[TT]}$ and $O^F_{[TT]}$ (with $CPS' = CPS'' = -1$), into the new basis

$$O^\pm_{[SP+PS]} \equiv \frac{1}{2} \left[ O_{[SP+PS]} \pm O^F_{[SP+PS]} \right],$$

which, once more, is decomposed into two bases, of two operators each, with definite $S = \pm 1$.

If we introduce, for notational compactness, the notation

$$Q^\pm_1 = O^\pm_{[VA+AV]}, \quad Q^\pm_2 = O^\pm_{[VA-AV]}, \quad Q^\pm_3 = O^\pm_{[SP+PS]}, \quad Q^\pm_4 = O^\pm_{[SP-PS]}, \quad Q^\pm_5 = O^\pm_{[TT]}$$

the renormalization structure becomes

$$\hat{Q}^\pm_i = \tilde{Z}^\pm_i Q^\pm_i \quad (i,j = 1,\ldots,5)$$
where $\hat{Q}^\pm_i$ denote the renormalized operators and $\hat{Z}_{ij}^\pm$ is the renormalization (and mixing) matrix (summation over repeated indices is implied).

Dropping, for simplicity, the ± subscripts, the matrix $\hat{Z}_{ij}$ is a (relatively sparse) block diagonal matrix of the form

$$
\begin{pmatrix}
\hat{Z}_{11} & 0 & 0 & 0 & 0 \\
0 & \hat{Z}_{22} & \hat{Z}_{23} & 0 & 0 \\
0 & \hat{Z}_{32} & \hat{Z}_{33} & 0 & 0 \\
0 & 0 & 0 & \hat{Z}_{44} & \hat{Z}_{45} \\
0 & 0 & 0 & \hat{Z}_{54} & \hat{Z}_{55}
\end{pmatrix}
$$

(20)

It is important to notice that this NP renormalization structure, determined by the symmetries of the action, is the same as in the continuum naive dimensional regularization scheme, or any other regularization that does not break chirality explicitly. In fact, the operators that mix belong to the same chiral representation, and their chiral structures can be obtained from each other by Fierz transformations.

5.2. Parity conserving operators

Let us now pass to the parity-conserving operators, cf. Tab. 1. All of the parity conserving operators $O_{\Pi\Pi}$ are eigenstates of all the discrete symmetries listed, with eigenvalue +1. Thus by rule 1., unlike the parity violating case, they all mix among each other and also with the five $O_{\Pi\Pi}^F$‘s; the complete maximal basis consists of 10 operators. By rule 2., we rotate our basis into a new one:

$$
O_{\Pi\Pi}^\pm = \frac{1}{2} [O_{\Pi\Pi} \pm O_{\Pi\Pi}^F]
$$

(21)

in which the 5 $O_{\Pi\Pi}^\pm$‘s having $S = +1$ and mix only among themselves; the same is true for the $O_{\Pi\Pi}^F$‘s which have $S = -1$. Thus the original basis of 10 operators has been decomposed into two independent bases of 5 operators each.

This result can be used in the renormalization of the operators $O_{[VV+AA]}^\pm$, which are the parity conserving partners of the operators $O_{LL}^\pm$. Clearly, $O_{[VV+AA]}^\pm$ mixes with $O_{VV-AA}^\pm$, $O_{SS}^\pm$, $O_{PP}$, and $O_{TT}^\pm$. Any other linear combination of these mixing operators of “wrong” naive chirality is in principle acceptable; particular choices are a question of convenience. Here we discuss three such options.

The first option is the one which enables a comparison of this mixing to the perturbative calculations of [29] (and also the NP computations of [27, 28, 29]). We call this a **perturbative-inspired (PI)** basis:

$$
\begin{align*}
O_{\Pi\Pi}^1 &= \frac{1}{2} O_{[VV+AA]}^F \\
O_{\Pi\Pi}^2 &= -\frac{1}{8N_e^2} O_{[SS-PP]}^F \\
O_{\Pi\Pi}^3 &= \frac{2(N_c^2 + N_e - 1)}{16N_e} O_{[VV-AA]}^F \\
O_{\Pi\Pi}^4 &= \frac{2}{8N_e} O_{[SS+PP+TT]}^F \\
O_{\Pi\Pi}^5 &= \frac{(N_c - 1)}{8N_e} O_{[SS+PP-\frac{3}{4}TT]}^F \\
\end{align*}
$$

(22)

In this base, $O_{\Pi\Pi}^\pm$ mixes with the operators $O_{\Pi\Pi}^i$, $i = 1, 2, 3$ which already appear at the level of the one-loop perturbative calculation [29], but also with $O_{\Pi\Pi}^4$, which is not present at the one-loop level. The arbitrary numerical overall colour factors of $O_{\Pi\Pi}^i$, $i = 1, 2, 3$ are defined so as to be in agreement with the convention of [29], and the colour factor of $O_{\Pi\Pi}^5$ has been set equal to the one of $O_{\Pi\Pi}^3$. This choice seems natural for the comparison of its relative weight with respect to the other operators present at one-loop.

A second option, which is exploited in [30], consists in taking the basis of the eigenvector of the Fierz matrix. We then call this a **Fierz-inspired (FI)** basis:

$$
\begin{align*}
O_{\Pi\Pi}^1 &= O_{[VV+AA]}^F \\
O_{\Pi\Pi}^2 &= O_{[SS+PP+TT]}^F \\
O_{\Pi\Pi}^3 &= O_{[SS+PP-\frac{3}{4}TT]}^F \\
O_{\Pi\Pi}^4 &= O_{[VV-AA+2(SS-PP)]}^F \\
O_{\Pi\Pi}^5 &= O_{[VV-AA-2(SS-PP)]}^F \\
\end{align*}
$$

(23)

A third option, used in [31], consists in the following basis, which we call **Ward-inspired** basis, as it can obtained, up to signs, from the parity-violating base, eq. (15), with a chiral transformation $\psi_4 \rightarrow \gamma_5\psi_4$:

$$
\begin{align*}
Q_{\Pi\Pi}^1 &= O_{[VV+AA]}^F \\
Q_{\Pi\Pi}^2 &= O_{[VV-AA]}^F \\
Q_{\Pi\Pi}^3 &= O_{[SS-PP]}^F \\
Q_{\Pi\Pi}^4 &= O_{[SS+PP]}^F \\
Q_{\Pi\Pi}^5 &= O_{[TT]}^F \\
\end{align*}
$$

(24)
Whichever basis we choose, upon renormalization the structure will be of the form
\[ \hat{Q}_i^\pm = Z_{ij}^\pm Q_j^\pm \quad (i,j = 1, \ldots, 5), \quad (25) \]
where the matrix \( Z^\pm \) is not sparse as in the PV case.

5.3. Scale dependence

Close to the continuum and chiral limit, the UV divergent elements of the renormalization matrix \( Z \) depend on \( a_\mu \) and \( g_0^2 \) only, \( Z_{ij} = Z_{ij}(a_\mu, g_0^2) \), whereas non-divergent elements are of the form \( Z_{ij}(g_0^2) \). Since the structure of the renormalization matrix \( \bar{Z} \) is the same as in the continuum, all its matrix elements are logarithmically divergent, i.e. \( \bar{Z}_{ij} = \bar{Z}_{ij}(a_\mu, g_0^2) \).

The specification of which elements of the matrix \( Z \) diverge and which are finite, in the limit \( a \to 0 \), can be achieved non-perturbatively with the aid of the axial Ward Identity (WI). We refer to [23] for a detailed presentation while we give here only the prescription for the renormalization. First, we need to construct the subtracted operators
\[ Q^{1\text{ub}}_i = Q_i + \sum_{j=1,2,3} c_{ij} Q_j, \]
\[ Q^{2\text{ub}}_i = Q_i + \sum_{j=1,4,5} c_{ij} Q_j, \quad i = 2, 3 \]
\[ Q^{4\text{ub}}_i = Q_i + \sum_{j=1,2,3} c_{ij} Q_j, \quad i = 4, 5 \]
and, second, renormalize the operators \( Q^{1\text{ub}} \) as in the continuum, i.e. with a matrix of the form given in the PV case, cf. eq. (20).

Before discussing the numerical results, it is worth while to stress in which conditions the direct implementation of the WI, as exploited for the \( \Delta S = 2 \) operator in [8], and the NPM are equivalent. The WI holds for operators with the correct chiral properties, that is multiplicatively renormalizable operators transforming according to a well defined representation of the chiral algebra. In fact, it is by imposing its validity on the renormalized operators one can fix the mixing coefficients of the form factors which stem from the chiral violation due to the Wilson term. In the NPM, this is achieved by imposing that the projections of the renormalized operator \( \hat{O}^{\Delta S=2} \) (cf. eq. (28) for the its explicit form) on the four chiral violating form factors are zero,
\[ P_j(p) \hat{O}^{\Delta S=2}(p) = 0, \quad j = 1, \ldots, 4. \quad (27) \]
But this is true if there are no other causes of chiral symmetry breaking, either due to explicit presence of mass terms or due to spontaneous symmetry breaking in the chiral limit. But both these effect die off in the large momenta region. So the WI method and the NPM are equivalent for sufficiently large values of the renormalization scale \( \mu^2 = p^2 \). In any case, the overall multiplicatively renormalization constant cancels in the WI and thus cannot be determined. Thus the NPM (or some other renormalization method) is needed even if the WI method is used to obtain the mixing coefficients.

6. Numerical results

Our NP Monte Carlo simulations have been performed on an APE machine. We have generated an ensemble of 100 independent gauge-field configurations, using a tree-level improved SW-Clover action on a \( 16^3 \times 32 \) lattice, at \( \beta = 6.0 \). The quark propagator has...
chosen to present the results at the intermediate \( \kappa = 0.1432 \). We have chosen not to extrapolate in \( m_qa \) as we feel that the best one can do with a systematic error is to control it rather than extrapolate in it.

We first consider the renormalization of the operator \( O_0^+ \) of eq. (22), which has the same renormalization properties of parity-conserving part of the \( \Delta S = 2 \) operator \( \bar{s}_\mu \bar{d} \bar{s}_\mu \bar{d} \). The renormalized operator is

\[
\hat{O}^+ = Z_0^+(\mu a)[O_0^+(a) + \sum_{i=1}^4 Z_i^+ O_i^+(a)],
\]

where the operators \( O_1^+, \ldots, O_4^+ \) are given in eq. (22) for the PI basis, and by \( O_2^+, \ldots, O_5^+ \) in eq. (23) for the FI basis.

In fig. 1 we show the results of the mixing \( Z \)'s in the PI basis at different renormalization scales \( \mu^2a^2 \). It is clearly notable that that \( Z_2^+ \) and \( Z_4^+ \) are very well defined and almost scale-independent in a large “window” of \( \mu^2a^2 \), whereas \( Z_1^+ \) and \( Z_3^+ \) present a smaller window, i.e. a more pronounced scale-dependence. Moreover, \( Z_4^+ \) which is absent in 1-loop PT, is not negligible. We stress that the large fluctuations at small \( \mu^2a^2 \) do not spoil the validity of the NPM, since in that region the perturbative matching to a continuum scheme is not reliable, as for any NP lattice method at a fixed lattice spacing [1].

Fig. 2 shows the \( m_qa \) dependence for \( \Delta S = 2 \) mixing \( Z \)'s in PI basis as a function of \( \mu^2a^2 \). The three symbols correspond to the values of \( \kappa = 0.1425, 0.1432, 0.1440 \).

The three values of the hopping parameter \( \kappa = 0.1425, 0.1432, 0.1440 \), corresponding to a pion mass of \( 900 \sim 600 \) MeV. In comparing the NP results with PT, we have used both a standard bare lattice coupling (SPT) \( \alpha_s^{\text{bare}} = g_0^2/4\pi \) and boosted coupling (BPT) of \( \alpha_V = \alpha_s^{\text{bare}}/\langle \text{Tr}\,\Box \rangle \simeq 1.68 \alpha_s^{\text{latt}} [16]. \)

The three values of the hopping parameter allow us to extend the analysis presented in [2] to include the study of the mass dependence of the \( Z \)'s, i.e. the effects of the systematics of \( O(m_qa) \). In general, these effects were expected to be small for the light quark sector [24], though at lower values of \( m_qa \) the statistical errors tend to be larger and at higher values of \( m_qa \) the discretization errors are larger. We have found that indeed the expectations are fulfilled, as can be seen from fig. 3 in which the symbols representing the different \( \kappa \) values can barely be distinguished, at least in the significant region of large \( \mu^2a^2 \). Thus, we have

![Figure 2](image-url)

Figure 2. \( m_qa \) dependence for NP \( \Delta S = 2 \) mixing \( Z \)'s in PI basis as a function of \( \mu^2a^2 \). The three symbols correspond to the values of \( \kappa = 0.1425, 0.1432, 0.1440 \).

### Table 2

| \( \mu^2a^2 \) | \( \alpha \) | \( \beta \) | \( \gamma \) |
|----------------|----------|----------|----------|
| 0.31           | 0.030(18)| 0.27(21)| 0.90(15)|
| 0.62           | -0.027(16)| 0.36(18)| 0.75(13)|
| 0.96           | -0.012(14)| 0.24(17)| 0.69(12)|
| 1.27           | 0.005(13)| 0.14(16)| 0.68(12)|
| 1.39           | -0.009(13)| 0.24(16)| 0.67(12)|
| 1.85           | -0.003(13)| 0.18(16)| 0.66(11)|
| 2.46           | -0.001(12)| 0.24(15)| 0.65(11)|
| 4.01           | -0.002(12)| 0.44(15)| 0.67(11)|
| BPT            | -0.052(12)| 0.16(15)| 0.62(11)|
significant values of $\mu^2 a^2$, for the quark masses we used.

Using the NP $Z^+$'s in PI basis with their mass dependence taken into account (extending the analysis of ref. [12]), we can revisit the study of the chiral behaviour of the $B_K$ parameter [23,25].

We adopt the usual parametrization

$$\langle O^+ \rangle = \alpha + \beta m^2_K + \gamma (p \cdot q) + ...$$  \hspace{1cm} (29)

where $\alpha$ is a lattice artefact that we expect to vanish in the chiral limit. In tab. 3 we present the parameters $\alpha, \beta, \gamma$ obtained for different scales $\mu^2 a^2$. In fig. 3 the result using the NP $Z^+$'s at $\mu^2 = 0.96$ and the bare matrix elements from [23] is shown. Clearly, the use of the mixing $Z'$s in PT (all equal at 1-loop [26]) does not yield the desired behaviour, even if a boosted coupling is used. This is due to the delicate cancellations which occur among the bare matrix elements that can only be resolved beyond 1-loop. On the contrary, using the NP $Z$'s, the intercept is compatible with zero. This behaviour is consistently found at all scale $\mu^2 a^2 \gtrsim 0.96$. Although the use of the complete set of operators ($N = 4$ in fig. 3) yield a better chiral behaviour that the use of only the operators which mix in 1-loop PT ($N = 3$), since the mixing with $O^+_1$ starts at $O(g_0^3)$, we do not expect drastic changes in the chiral behaviour. If the chiral behaviour were sensibly different, we would not trust the matching to the continuum which has an uneludable perturbative uncertainty. Indeed we find that $\alpha_{N=3}$ is also compatible with zero, and compatible with $\alpha_{N=4}$ although with large statistical errors forced by the thinning approximation [25]. We can only state that the value of $B_K$, proportional to $\gamma$, is unaltered and its RGI value at NLO is $\hat{B}_K = 0.85 \pm 0.15$ while the correct chiral behaviour of the continuum, signaled by the vanishing of $\alpha$, is recovered [23].

The chiral behaviour of the $\Delta S = 2$ has also been studied in [8] imposing the chiral WI on quark states using the FI basis. The results for the mixing $Z$'s, obtained with an unimproved Wilson action, seems to show a very stable signal as a function of $\mu^2 a^2$ [10]. In particular, the fluctuations at small scales are much reduced. It
could be concluded that the WI method and the NPM, although conceptually equivalent at large values of $\mu$, are not numerically such. To understand this point in more detail, we have tried taking linear combinations of the $Z$'s in PI basis and expressing them in the FI basis, as shown in fig. 4. It is clear that the $Z$'s, calculated with the NPM, in this new basis show a much greater stability, comparable to the one obtained by the WI method. It must be stressed that although the stability of the $Z$'s depends on the choice of the basis, the physical results do not. In fact, the correct chiral behaviour of the renormalized operator is obtained with either basis. This is due to the fact that the $Z$'s which present greater fluctuations multiply bare matrix elements of the operators that weight less than the ones multiplied by the more stable $Z$'s. So that stability of the chiral behaviour with the PI basis shown in tab. 2 is due to the extremely clean determination of $Z_2$ in fig. 4 and to the fact that $\langle O_2 \rangle \approx -3\langle O_{1,3,4} \rangle$.

As a final flourish, in fig. 5 and fig. 6 we show the overall renormalization constants for $O_{\{VV+AA\}}^\pm$ and $O_{\{VV-AA\}}^\pm$, which are the PC and PV part of the LL operator and, as already stressed, can only be determined with the NPM. The comparison with PT shows that, while $Z^+$ is in good agreement with boosted PT, for $Z^-$ the agreement is better with standard PT.

7. Conclusions

Recently, there has been considerable progress both in the Symanzik improvement program and in the development of non-perturbative renormalization methods. We have presented the application of the off-shell renormalization method using a tree-level improved SW-Clover action to four-fermion operators with light quarks relevant to weak decays. For the $\Delta I = 3/2$ sector we have presented a general analysis of the renormalization structure, and applied it to the $\Delta S = 2$ operator obtaining the correct chiral behaviour for the $B_K$ parameter. On the other hand, for the $\Delta I = 1/2$ sector, the standard positive parity ap-
proach was shown to be unviable, and new ideas are needed.

Acknowledgements

I am grateful to A. Donini, V. Giménez, G. Martinelli, G.C. Rossi, C.T. Sachrajda, S. Sharpe, M. Testa and A. Vladikas, for a most enjoyable and fruitful collaboration on the material presented in this talk. I would like to thank A. Vladikas for reading the manuscript and for his comments.

I would also like to extend a warm thank to the organizers of the Workshop and to the Center for Computational Physics in Tsukuba, for their invitation and for creating a very stimulating and pleasant scientific environment.

I acknowledge support from EPSRC through grant GR/K41663, from PPARC through grant GR/L22744 and partial support from INFN.

REFERENCES

1. K. Symanzik, Nucl. Phys. B226 (1983) 187 and 205.
2. M. Lüscher and P. Weisz, Comm. Math. Phys. 97 (1985) 59, E: Comm. Math. Phys. 98 (1985) 433.
3. B. Sheikholeslami and R. Wohlert, Nucl. Phys. B259 572.
4. G. Heatlie et al., Nucl. Phys. B352 (1991) 266.
5. M. Lüscher et al., Nucl. Phys. B478 (1996) 365.
6. G. Martinelli et al., Nucl. Phys. B445 (1995) 81.
7. K. Jansen et al., Phys. Lett. B372 (1996) 275.
8. S. Aoki et al. (JLQCD Collaboration), Nucl. Phys. B(Proc. Suppl.) 53 (1997) 349.
9. R. Sommer, these proceedings.
10. Y. Kuramashi, these proceedings.
11. L. Maiani et al., Nucl. Phys. B289 (1987) 505.
12. A. Donini et al., Nucl. Phys. B(Proc. Suppl.) 53 (1997) 883.
13. M. Bochicchio et al., Nucl. Phys. B262 (1985) 331.
14. M. Ciuchini et al., BaBar Workshop Proposal, Rome, 11-14 Nov. 1996.
15. M. Ciuchini et al., Z. Phys. C68 (1996) 239.

16. G.P. Lepage and P.B. Mackenzie, Phys. Rev. D48 (1993) 2250.
17. C. Bernard et al., Nucl. Phys. B(Proc. Suppl.) 4 (1988) 483.
18. G. Curci et al., Phys. Lett. B202 (1988) 363.
19. M. Lüscher, Les Houches Summer School, 1988.
20. C.T. Hill and G.G. Ross, Nucl. Phys. B171 (1980) 141.
21. G. Martinelli et al., in preparation.
22. A. Donini et al., Phys. Lett. B360 (1995) 83.
23. A. Donini et al., in preparation.
24. G. Martinelli et al., Nucl. Phys. B397 (1993) 479.
25. M. Crisafulli et al., Phys. Lett. B369 (1996) 325.
26. G. Martinelli, Phys. Lett. B141 (1984) 395; C. Bernard et al., Phys. Rev. D36 (1987) 3224.