MODEL OF THE QUARK MIXING MATRIX

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ABSTRACT

The structure of the Cabibbo-Kobayashi-Maskawa (CKM) matrix is analyzed from the standpoint of a composite model. A model is constructed with three families of quarks, by taking tensor products of sufficient numbers of spin-1/2 representations and imagining the dominant terms in the mass matrix to arise from spin-spin interactions. Generic results then obtained include the familiar relation $|V_{us}| = (m_d/m_s)^{1/2} - (m_u/m_c)^{1/2}$, and a less frequently seen relation $|V_{ub}| = \sqrt{2}(m_s/m_b) - (m_c/m_t)$. The magnitudes of $V_{ub}$ and $V_{td}$ come out naturally to be of the right order. The phase in the CKM matrix can be put in by hand, but its origin remains obscure.

1. Introduction

The pattern of charge-changing weak transitions among quarks undoubtedly is a reflection of deeper physics. In the present article we examine the form this pattern might be expected to take if the underlying physics is that of a composite system. We construct a three-level quantum mechanical model of composite quarks, thereby obtaining an illustration of how the quark mixing matrix, or Cabibbo-Kobayashi-Maskawa (CKM) matrix, might arise from some underlying substructure.

A crucial problem associated with the construction of realistic models is to understand why, if the compositeness mass scale is very high, quarks and leptons are so much lighter than this scale. We assume that the dynamics is such as to solve this problem. We shall imagine that the families of quarks and leptons are the low-mass states (perhaps massless in some limit) of a quantum-mechanical system whose other states lie at the compositeness scale. We shall assume that this scale is independent of the physics giving rise to any of the quark masses.

Because of our familiarity with quark models, we shall use a language which is closely related to the nonrelativistic quark model. We recognize that a proper treatment of deeply bound composite systems may require a relativistic treatment; however, we find nonrelativistic discussions a convenient means of counting states and of dealing with dynamically induced masses of the unwanted (i.e., higher-lying) excitations.

2. Model for three families of quarks

It so happens that one can form precisely three spin-1/2 levels with the
product of three spin-1/2 subunits and a spin-1 subunit. The tensor product in question is

\[
\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes 1 = 3 \left( \frac{1}{2} \right) \oplus 3 \left( \frac{3}{2} \right) \oplus \frac{5}{2}.
\]  

(1)

We shall adopt such a model for quark and lepton families, regarding them as the spin-1/2 members of the set (1). We do not inquire here into the origin of the subunits. The spin-1 subunit may either arise as a result of the composition of two spin-1/2 subunits in the manner mentioned above, or it may be due to some internal orbital angular momentum. In what follows we shall label the spins of the three spin-1/2 subunits by \( S_i \) (\( i = 1, 2, 3 \)) and that of the spin-1 subunit by \( L \). We shall adopt the following basis for description of our states. 1) We imagine \( S_1 \) and \( S_2 \) to be coupled to a total spin \( S_{12} = S_1 + S_2 \). 2) Next, we define \( J_{12} = S_{12} + L \). 3) Finally, we form the total spin \( J = J_{12} + S_3 \). We may then label our three spin-1/2 states by \(| S_{12}, J_{12} \rangle \). The basis states are

\[
| 0, 1 \rangle \equiv | u_0 \rangle; \quad | 1, 1 \rangle \equiv | c_0 \rangle; \quad | 1, 0 \rangle \equiv | t_0 \rangle.
\]  

(2)

We shall take our model Hamiltonian to consist of a constant term and linear combinations of terms proportional to \( S_i \cdot S_j \) and \( L \cdot S_i \). We examine the following simplified form of the mass matrix for quarks,

\[
M(q) = M_0(J) + a S_1 \cdot S_3 + b S_2 \cdot S_3 + c(1 + L \cdot S_3),
\]

(3)

where \( M_0(J) = \text{const.} \times [J(J + 1) - 3/4] \) is chosen so as to make all states with \( J > 1/2 \) arbitrarily heavy, while \( M_0(1/2) = 0 \).

The mass operator (3) leads to the following mass matrix for spin-1/2 quarks:

\[
M_{1/2} = \begin{bmatrix}
0 & \alpha \sqrt{2} & \alpha \\
\alpha \sqrt{2} & \beta & \beta \sqrt{2} \\
\alpha & \beta \sqrt{2} & \gamma
\end{bmatrix},
\]

(4)

where

\[
\alpha \equiv (a - b)/4; \quad \beta \equiv (2c - a - b)/4; \quad \gamma = c
\]  

(5)

With separate mass matrices (4) for up and down quarks, one has six parameters with which to describe six quark masses. This theory has no CP violation.

Without a microscopic theory of CP violation, the best we can do in composite models of the present variety is to put in phases by hand. We let \( \alpha \rightarrow -i\alpha, \gamma \rightarrow -i\gamma \) in the mass matrix for the up quarks. We then assume

\[
M_{U} = \begin{bmatrix}
0 & -i\alpha \sqrt{2} & -i\alpha \\
-i\alpha \sqrt{2} & \beta & \beta \sqrt{2} \\
-i\alpha & \beta \sqrt{2} & -i\gamma
\end{bmatrix}; \quad M_{D} = \begin{bmatrix}
0 & \alpha' \sqrt{2} & \alpha' \\
\alpha' \sqrt{2} & \beta' & \beta' \sqrt{2} \\
\alpha' & \beta' \sqrt{2} & \gamma'
\end{bmatrix},
\]

(6)

for \( Q = 2/3 \) and \( Q = -1/3 \) quarks, respectively. In the limit that \( \alpha \ll \beta \ll \gamma \) we obtain the following approximate CKM matrix elements:

\[
V_{us} \simeq i \left( \frac{-m_d}{m_s} \right)^{1/2} - \left( \frac{-m_u}{m_c} \right)^{1/2}; \quad V_{cb} \simeq \sqrt{2} \left( \frac{m_s}{m_b} - i \frac{m_c}{m_t} \right);
\]

(7a,b)
\[ V_{ub} \approx i \left( \frac{-m_d m_s}{m_b} \right)^{1/2} - \frac{m_s}{m_b} \left( \frac{-2m_u}{m_c} \right)^{1/2} + \frac{(-m_u m_c/2)^{1/2}}{m_t} ; \]  
\[ V_{td} \approx \left( \frac{-m_d m_s}{m_b} \right)^{1/2} + i \frac{m_c}{m_t} \left( \frac{-2m_d}{m_s} \right)^{1/2} + \frac{(-m_u m_c/2)^{1/2}}{m_t} . \]  

We substitute a set of sample quark masses to see the implications of Eqs. (7). We choose  
\[ m_u = 5.2 \text{ MeV} ; \ m_d = 9.2 \text{ MeV} ; \ m_s = 194 \text{ MeV} ; \ m_c = 1.41 \text{ GeV} ; \ m_b = 6.33 \text{ GeV} ; \ m_t = 200 \text{ GeV} . \]  
These are values at a scale of 1 GeV. After rephasing to get to the Wolfenstein parametrization of the CKM matrix, we obtain  
\[ V_{us} = 0.226 ; \ V_{cb} = 0.044 ; \ V_{ub} = 0.0051 - 0.0024 i ; \ V_{td} = 0.0049 - 0.0024 i ; \ \rho = 0.51 ; \ \eta = 0.24 . \]  

The above results are typical consequences of the expressions (7), and have some interesting features.

1) The magnitude of \( V_{cb} \) is governed by the ratios of quark masses, rather than their square roots. As a result, one expects smaller typical values for this quantity than for \( V_{us} \).

2) The orders of magnitude of \( |V_{ub}| \) and \( |V_{td}| \) are correct.

3) The parameters in the mass matrices (6) may be related to those in the mass operator (3) via Eqs. (5). In the solution in which the signs of the masses of the four heaviest quarks are all positive, \( a, b, \) and \( c \) are all of order \( m_t \), with small splittings among these quantities responsible for the masses of \( u \) and \( c \). Similarly, \( a', b', \) and \( c' \) are all of order \( m_b \), with small splittings responsible for \( m_d \) and \( m_s \). The presence of \( S_3 \) in nearly all the terms of the mass operator (3) then would suggest that the spin-1/2 subunit carrying \( S_3 \) changes its identity under a charge-changing weak transition. However, it seems difficult to preserve the connection between \( S_3 \) and the spin of the composite fermion.

4) The origin of the CP violating phase remains a mystery.

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4. References

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