THE CONVOLUTION ALGEBRA STRUCTURE ON 
\( K^G(\mathcal{B} \times \mathcal{B}) \)

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ABSTRACT. We show that the convolution algebra \( K^G(\mathcal{B} \times \mathcal{B}) \) is isomorphic to the Based ring of the lowest two-sided cell of the extended affine Weyl group associated to \( G \), where \( G \) is a connected reductive algebraic group over the field \( \mathbb{C} \) of complex numbers and \( \mathcal{B} \) is the flag variety of \( G \).

INTRODUCTION

We are interested in understanding the equivariant group \( K^G(\mathcal{B} \times \mathcal{B}) \), where \( G \) is a connected reductive algebraic group over \( \mathbb{C} \) and \( \mathcal{B} \) is the flag variety of \( G \).

When \( G \) has simply connected derived subgroup, the Künneth formula \( K^G(\mathcal{B} \times \mathcal{B}) \simeq K^G(\mathcal{B}) \otimes_{R_G} K^G(\mathcal{B}) \) is proved in Proposition 1.6 of [KL] and plays an important role in Kazhdan-Lusztig’s proof of Delinge-Langlands conjecture for affine Hecke algebra associated to \( G \), where \( R_G \) denotes the representation ring of \( G \). Furthermore, by Theorem 1.10 of [Xi], the convolution algebra structure on \( K^G(\mathcal{B} \times \mathcal{B}) \) is isomorphic to the based ring of the lowest two-sided cell of the extended affine Weyl group associated to \( G \).

In general, \( K^G(\mathcal{B} \times \mathcal{B}) \) is not isomorphic to \( K^G(\mathcal{B}) \otimes_{R_G} K^G(\mathcal{B}) \). To set a Deligne-Langlands-Lusztig classification for affine Hecke algebra associated to \( G \), it seems useful to understand the equivariant \( K \)-groups \( K^G(\mathcal{B} \times \mathcal{B}) \). The main result of this paper is Theorem 1.1, which says that the convolution algebra on \( K^G(\mathcal{B} \times \mathcal{B}) \) is isomorphic to the based ring of the lowest two-sided cell of the extended affine Weyl group associated to \( G \). Since the based ring is known explicitly in [Xi], the main result gives an explicit description to the equivariant \( K \)-group \( K^G(\mathcal{B} \times \mathcal{B}) \).

1. Preliminary

1.1. Let \( G \) be a connected reductive algebraic group over \( \mathbb{C} \), \( B \) a Berol subgroup of \( G \) and \( T \) a maximal torus of \( G \), such that \( T \subset B \). The Weyl group \( W_0 = N_G(T)/T \) of \( G \) acts on the character group \( X = \text{Hom}(T, \mathbb{C}^\times) \) of \( T \). Using this action we define the extended affine Weyl group \( W = X \rtimes W_0 \).
By classification theorem for connected reductive algebraic groups, there exists a connected reductive algebraic group $\tilde{G}$ with simply connected derived subgroup such that $G$ is a quotient group of $\tilde{G}$ modulo a finite subgroup of the center of $\tilde{G}$. Denote by $\pi : \tilde{G} \to G$ the quotient homomorphism. Set $\tilde{B} = \pi^{-1}(B)$, $\tilde{T} = \pi^{-1}(T)$, $\tilde{X} = \text{Hom}(\tilde{T}, \mathbb{C}^*)$ and $\tilde{W} = \tilde{X} \times W_0$. Note that $\tilde{X}$ is naturally a subgroup of $\tilde{X}$ of finite index, hence $\tilde{W}$ is a naturally subgroup of $\tilde{W}$ of finite index.

Let $R \subset X$ be the root of $G$ and $\tilde{G}$. Let $R^- \subset R$ to be the set of negative roots determined by $B$. Set $R^+ = R - R^-$. Let $\Delta \subset R^+$ be set of simple positive roots.

Denote by $\lambda_\alpha$ the dominant fundamental weight corresponding to a simple positive root $\alpha \in R^+$. For any $w \in W_0$, define $x_w = w^{-1}(\prod_{\alpha \in \Delta_+ \wedge \alpha < 0} \lambda_\alpha) \in \tilde{X}$. It is known that $\mathbb{Z}[\tilde{X}]$ is a free $\mathbb{Z}[X]^{W_0}$-module with a basis $\{x_w | w \in W_0\}$.

Let $\ell : \tilde{W} \to \mathbb{N}$ be the length function. Note that $\ell(w\lambda) = \ell(w) + \ell(\lambda)$ for any $w \in W_0$ and any dominant weight $\lambda \in \tilde{X}$. Also we have $\ell(\lambda_\alpha s_\alpha) = \ell(\lambda_\alpha) - 1$ for any positive simple root $\alpha \in \Delta$.

1.2. Let $\Sigma = \{wx_w | w \in W_0\}$. Then the lowest two-sided cell $\tilde{c}_0$ of $\tilde{W}$ consists of elements $f^{-1}w_0\chi g$ with $f, g \in \Sigma$ and $\chi \in \tilde{X}^+$. (See [Shi])

Here $w_0$ is the longest element of $W_0$ and $\tilde{X}^+$ is the set of dominant weights of $\tilde{X}$. The lowest two-sided cell of $\tilde{W}$ is $c_0 = \tilde{c}_0 \cap W$. The ring structure of $J_{c_0}$ of $\tilde{c}_0$ is defined in §2 of [L1] and explicitly determined in §4 [Xi]. As a $\mathbb{Z}$-module, it is free with a basis $t_z$, $z \in \tilde{c}_0$. The based ring $J_{c_0}$ of $c_0$ is a subring of $J_{\tilde{Z}_0}$ spanned by all $t_z$, $z \in c_0$.

1.3. For an algebraic group $M$ over $\mathbb{C}$ and an variety $Z$ over $\mathbb{C}$ which admits an algebraic action of $M$, denote by $K^M(Z)$ the Grothendieck group of $M$-equivariant coherent sheaves on $Z$. We refer to Chapter 5 of [CG] for more about the equivariant $K$-group $K^M(Z)$.

There is a natural map $L : \mathbb{Z}[\tilde{X}] \to \tilde{K}^G(B)$ which associates $\chi \in \tilde{X}$ to the unique equivariant line bundle $[L(\chi)]$ on $B$ such that $\tilde{T}$ acts on the fibre $L(\chi)_B$ over $B \in B$ via $\chi$. Here $B = \tilde{G}/\tilde{B} = G/B$ is the flag variety. It is well known that $L$ is an isomorphism. By abuse of notation, we will use $\chi$ and $[L(\chi)]$ interchangeably in the following context.

The convolution on $K^G(B \times B)$ is defined by

$$\tilde{\mathcal{F}} \ast \tilde{\mathcal{G}} = Rp_{13*}(p_{12*}\tilde{\mathcal{F}} \otimes_{\tilde{B} \times \tilde{B} \times \tilde{B}} p_{23*}\tilde{\mathcal{G}}),$$

where $\tilde{\mathcal{F}}, \tilde{\mathcal{G}} \in K^G(B \times B)$ and $p_{12}, p_{13}, p_{23} : B \times B \times B \to B \times B$ are obvious natural projections. Identifying $K^G(B \times B)$ with $K^G(B) \otimes_{\tilde{B}} K^\tilde{G}(\tilde{B}) \simeq \mathbb{Z}[\tilde{X}] \otimes_{\mathbb{Z}[\tilde{X}]} \mathbb{Z}[\tilde{X}]$, the convolution becomes

$$(\chi_1 \otimes \chi_2) \ast (\chi'_1 \otimes \chi'_2) = (\chi_2, \chi'_1)\chi_1 \otimes \chi'_2,$$
where \((,) : Z[\tilde{X}] \otimes_{Z[\tilde{X}]} Z[\tilde{X}] \to Z[\tilde{X}]^{W_0}\) is given by
\[
(\chi_2, \chi'_1) = \delta^{-1} \sum_{w \in W_0} (-1)^{\ell(w)} w(\chi_2 \rho \chi'_1).
\]
Here \(\delta = \prod_{\alpha \in R^+} (\alpha^\frac{1}{2} - \alpha^\frac{-1}{2})\) and \(\rho = \prod_{\alpha \in R^+} \alpha^\frac{1}{2}\).

For \(f = wx_w \in \Sigma\), set \(x_f = x_w\). Since \((,)\) is a perfect pairing (See Proposition 1.6 in [KL]), we can find \(y_f \in Z[\tilde{X}]\) such that \((x_f, y_f) = \delta_i f_{i'}\). The following result is due to N. Xi. (See Theorem 1.10 in [Xi].)

\((*)\) The map \(\sigma : J_{\tilde{c}_0} \to K^G(\mathcal{B} \times \mathcal{B}) \simeq Z[\tilde{X}] \otimes_{Z[\tilde{X}]} Z[\tilde{X}]\) given by \(t_{f^{-1}w_i f'} \mapsto V(\chi)y_f \otimes x_{f'}\) for \(\chi \in \tilde{X}^+, f, f' \in \Sigma\) is an isomorphism of \(R_\tilde{G}\)-algebras. Here \(V(\chi) \in R_\tilde{G}\) stands for the irreducible \(\tilde{G}\)-module with highest weight \(\chi\).

Now we state the main result of this paper.

**Theorem 1.1.** (a) The natural map \(i : K^G(\mathcal{B} \times \mathcal{B}) \to K^\tilde{G}(\mathcal{B} \times \mathcal{B})\) is an injective of homomorphism of algebra.
(b) As a \(Z\)-module, the image of \(i\) is spanned by \(\{V(\chi)y_f \otimes x_{f'}; \chi \in \tilde{X}^+, f, f' \in \Sigma, f^{-1}w_0 \chi f' \in W\}\).
(c) In particular, via the isomorphism \(\sigma\) in \((*)\), \(J_{\tilde{c}_0}\) is isomorphic to the convolution algebra \(K^G(\mathcal{B} \times \mathcal{B})\) as \(R_\tilde{G}\)-algebras.

2. **Proof of Theorem 1.1**

2.1. Set \(\Omega = \tilde{W} / W = \tilde{X} / X = \{\lambda X; \lambda \in \tilde{X}\}\), which is a finite abelian group. For a left coset \(\lambda X\), let \(Z[\lambda X]\) be the \(Z\)-submodule of the group algebra \(Z[\tilde{X}]\) spanned by elements in \(\lambda X\). For any \(A \in Z[\lambda X]\) and \(B \in Z[\mu X]\), we have \(AB \in Z[\lambda \mu X]\). Moreover if there is \(C \in Z[\tilde{X}]\) such that \(A = BC\), then \(C \in Z[\lambda \mu^{-1} X]\).

**Lemma 2.1.** For \(f \in \Sigma\), we have \(y_f \in Z[x_f^{-1} X]\).

**Proof.** For \(f, f' \in \Sigma\), set \(A_{f, f'} = (x_f, x_{f'}) = \delta^{-1} \sum_{w \in W_0} (-1)^{\ell(w)} w(x_f \rho x_{f'})\) which lies in \(Z[x_f x_{f'}] X\). Let \((A_{f, f'})_{(f, f') \in \Sigma \times \Sigma}\) be the inverse matrix of \((A_{f, f'})_{(f, f') \in \Sigma \times \Sigma}\). Then a direct computation shows \(A_{f, f'} \in Z[x_f^{-1} x_{f'}^{-1} X]\). Since \(y_f = \sum_{f' \in \Sigma} A_{f, f'} x_{f'}\), we have \(y_f \in Z[x_f^{-1} X]\). \(\square\)

2.2. For \(w \in W_0\), let \(Y_w \subset \mathcal{B} \times \mathcal{B}\) be the \(G\)-orbit containing \((B, wB)\). Then \(\mathcal{B} \times \mathcal{B} = \bigsqcup_{w \in W_0} Y_w\) and the projection to the first factor \(p_1 : Y_w \to \mathcal{B}\) is an affine bundle of rank \(\ell(w)\). Numbering the elements of \(W_0\) as \(u_1, u_2, \ldots, u_r\) such that \(u_i \neq u_j\) if \(j < i\). Let \(F_i = \bigsqcup_{j \leq i} Y_{u_j}\). Then \(F_i\) is closed in \(\mathcal{B} \times \mathcal{B}\). We have the following commutative diagram
Let \( \tilde{\text{Proposition}} \, 2.2 \).

The commutative diagram of the diagram are exact sequences. Since \( p_1 : Y_{u_i} \rightarrow \mathcal{B} \) is an affine bundle, we have the following commutative diagram

\[
\begin{array}{ccc}
K^G(Y_{u_i}) & \xrightarrow{\sim} & K^G(\mathcal{B}) \\
| & & | \\
z[X] & \xrightarrow{\sim} & z[\tilde{X}]
\end{array}
\]

which shows that the natural morphism \( K^G(Y_{u_i}) \rightarrow K^\tilde{T}(Y_{u_i}) \) is injective. Using induction on \( i \), we see that the natural morphism \( i : K^G(\mathcal{B} \times \mathcal{B}) \rightarrow K^\tilde{T}(\mathcal{B} \times \mathcal{B}) \) is injective. One shows directly that \( i \) is a homomorphism of convolution algebras. Part (a) of Theorem 1.1 is proved.

2.3. For \( w \in W_0 \), let \( X_w = \tilde{B}w\tilde{B}/\tilde{B} \) and \( X^w = wB^+\tilde{B}/\tilde{B} \), where \( B^+ \supset \tilde{T} \) is the opposite of \( \tilde{T} \). Then \( X^w \) is an \( \tilde{T} \)-invariant open neighborhood of \( X_w \) in \( \mathcal{B} \). Set \( X_i = \bigsqcup_{j \leq i} X_{uj} \). Note that \( \mathcal{B} \times \mathcal{B} = \mathcal{B}\setminus(\tilde{G} \times \mathcal{B}) \), where the action of \( \tilde{B} \) on \( \mathcal{G} \times \mathcal{B} \) is given by \( b(g, h\mathcal{B}) = (gb^{-1}, bh\mathcal{B}) \). Hence \( K^\tilde{T}(\mathcal{B} \times \mathcal{B}) \simeq K^\tilde{T}(\mathcal{B}) \simeq K^\tilde{T}(\mathcal{B}) \). Similarly, \( K^\tilde{T}(F_i) \simeq K^\tilde{T}(X_i) \) and \( K^\tilde{T}(Y_{u_i}) \simeq K^\tilde{T}(X_{u_i}) \).

Let \( j_w : Y_w \rightarrow \mathcal{B} \times \mathcal{B} \) be the natural \( G \)-equivariant inclusion. Since \( X_w \) is a \( \tilde{T} \)-equivariant vector bundle over a single point. We have \( K^\tilde{T}(Y_{u_i}) \simeq K^\tilde{T}(X_{u_i}) \simeq z[\tilde{X}] \). Then a direct computation shows that the induced homomorphism \( Lj_w^* \) of equivariant \( K \)-groups is given by

\[
Lj_w^* : K^\tilde{T}(\mathcal{B} \times \mathcal{B}) \rightarrow K^\tilde{T}(Y_{u_i}) \simeq z[\tilde{X}],
\]

\[
x_1 \otimes x_2 \mapsto x_1 w(x_2),
\]

where \( x_1 \otimes x_2 \in z[\tilde{X}] \otimes_{z[\tilde{X}]w_0} z[\tilde{X}] \simeq K^\tilde{T}(\mathcal{B} \times \mathcal{B}) \).

Proposition 2.2. Let \( l \in K^\tilde{T}(\mathcal{B} \times \mathcal{B}) \), then \( l \in K^G(\mathcal{B} \times \mathcal{B}) \) if and only if \( Lj_w^*(l) \in K^G(Y_{u_i}) \) for any \( w \in W_0 \).

Proof. Denote by \( k_w : X_w \hookrightarrow \mathcal{B} \), \( k_i : X_i \hookrightarrow \mathcal{B} \) and \( j_i : F_i \hookrightarrow \mathcal{B} \times \mathcal{B} \) the natural immersions. Then we have

\[
\begin{array}{ccc}
0 & \xrightarrow{\sim} & K^G(F_{i-1}) \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\sim} & K^\tilde{T}(F_{i-1})
\end{array}
\]

where all the morphisms are natural and obvious. By [CG], the rows of the diagram are exact sequences. Since \( p_1 : Y_{u_i} \rightarrow \mathcal{B} \) is an affine bundle, we have the following commutative diagram

\[
\begin{array}{ccc}
K^G(Y_{u_i}) & \xrightarrow{\sim} & K^G(\mathcal{B}) \\
| & & | \\
z[X] & \xrightarrow{\sim} & z[\tilde{X}]
\end{array}
\]

which shows that the natural morphism \( K^G(Y_{u_i}) \rightarrow K^\tilde{T}(Y_{u_i}) \) is injective.
Hence our proposition is equivalent to the following statement

For any \( l \in K^\tilde{T}(\mathcal{B}) \), \( l \in K^T(\mathcal{B}) \) if and only if \( \text{Lk}_{u_i}^*(l) \in K^T(X_{u_i}) \) for any \( w \in W_0 \).

The "only if" part is obviously. We show the "if" part. Note that the support \( \text{supp}(l) \) of \( l \) belongs to some \( X_i \), we argue by induction on \( i \). If \( \text{supp}(l) = \emptyset \), that is, \( l = 0 \), then the statement follows trivially. Suppose the proposition holds for any element in \( K^\tilde{T}(\mathcal{B}) \) whose support belongs to \( X_{i-1} \). We show it also holds for \( z \in K^\tilde{T}(\mathcal{B}) \) with \( \text{supp}(l) \subset X_i \).

Since we have the following \( \tilde{T} \)-equivariant morphism

\[
\begin{array}{ccc}
X_{u_i} & \xrightarrow{l} & X_{u_i} \\
\oplus_{\alpha \in R^- \cup u_i^{-1}(\alpha) > 0} C_\alpha & \xrightarrow{l} & \oplus_{\beta \in R^+ \cup u_i(\beta)}, \\
\end{array}
\]

where \( C_\alpha \) denotes the one dimensional vector space \( \mathbb{C} \) on which \( \tilde{T} \) acts via the character \( \alpha \). By Proposition 5.4.10 in [CG],

\[
\text{Lk}_{u_i}^*(l) = \prod_{\alpha \in R^+, u_i^{-1}(\alpha) > 0} (1 - \alpha^{-1}) l|_{X_{u_i}} \in K^T(X_{u_i}).
\]

Since \( \prod_{\alpha \in R^+, u_i^{-1}(\alpha) > 0} (1 - \alpha^{-1}) \in K^T(X_{u_i}) \), then \( l|_{X_{u_i}} \in K^T(X_{u_i}) \) by 2.1. Since \( X_{u_i} \) is a \( \tilde{T} \)-stable open subset of \( X_i \), there exists \( l' \in K^T(X_i) \) such that \( l|_{X_{u_i}} = l'|_{X_{u_i}} \). Then \( \text{supp}(l - l') \subset X_{i-1} \). Using induction hypothesis, we have \( l - l' \in K^T(\mathcal{B}) \). Hence \( l = (l - l') + l' \in K^T(\mathcal{B}) \) and the proof is finished.

\[\square\]

Corollary 2.3. Let \( z = f^{-1}w_0 \chi f' \) with \( f, f' \in \Sigma \) and \( \chi \in \tilde{X}^+ \). Then \( z \in W \) if and only if \( \sigma(z) = V(\chi)y_f \otimes x_{f'} \in K^G(\mathcal{B} \times \mathcal{B}) \).

\textbf{Proof.} Obviously \( z \in W \) if and only if \( x^{-1}_f \chi f x_f \in X \). On the other hand, \( V(\chi) \in \mathbb{Z}[\chi X] \). By Lemma 2.1, \( y_f \in \mathbb{Z}[x_f^{-1}X] \). Then by Proposition 2.2, \( V(\chi)y_f \otimes x_{f'} \in K^G(\mathcal{B} \times \mathcal{B}) \) if and only if \( V(\chi)y_f x_{f'} \in \mathbb{Z}[X] \), which is equivalent to \( x^{-1}_f \chi x_f \in X \).

\[\square\]

Proof of part (b) and (c) of Theorem 1.1. By Corollary 2.3, \( \sigma(J_0) \subset K^G(\mathcal{B} \times \mathcal{B}) \). It remains to show that \( K^G(\mathcal{B} \times \mathcal{B}) \subset \sigma(J_0) \). Let \( l \in K^G(\mathcal{B} \times \mathcal{B}) \). Due to (*), we can assume that

\[
l = \sum_{f, f' \in \Sigma} a_{f, f'} V(\chi_{f, f'}) y_f \otimes x_{f'},
\]

with \( a_{f, f'} \in \mathbb{Z} \), \( \chi_{f, f'} \in \tilde{X}^+ \). Since \( y_f \otimes x_f = \sigma(t_f^{-1}w_0f) \in K^G(\mathcal{B} \times \mathcal{B}) \) for each \( f \in \Sigma \), we have

\[
(y_f \otimes x_f) * l * (y_{f'} \otimes x_{f'}) = a_{f, f'} V(\chi_{f, f'}) y_f \otimes x_{f'} \in K^G(\mathcal{B} \times \mathcal{B}).
\]
3. SOME RESULTS ON $K^G(\mathcal{P} \times \mathcal{P})$

3.1. Let $I \subset \Delta$ be a subset and $P$ the parabolic subgroup of type $I$ containing $B$. Define $\mathcal{P} = G/P$ be the variety of all parabolic subgroups of type $I$.

Let $\mathcal{D}$ be the set of double cosets of $W_0$ with respect to $W_I \subset W_0$. Here $W_I$ is the parabolic subgroup generated by $I$. For each $w \in W_0$, define

$$Z_w = \{(P, P') \in \mathcal{P} \times \mathcal{P} \mid (P, P') \text{ is conjugate to } (P, wP)\},$$

where $\bar{w}$ denotes the double coset $W_I \bar{w}W_I$. Then we have

$$\mathcal{P} \times \mathcal{P} = \coprod_{d \in \mathcal{D}} Z_d.$$

For any double coset $d \in \mathcal{D}$, there is a unique element $u_d \in d$ such that $u_d$ is the smallest in $d$ under the Bruhat order. Let $d, d' \in \mathcal{D}$, we say $d \geq d'$ if $u_d \geq u_{d'}$ under the Bruhat order.

**Lemma 3.1.** With notations as above, then we have $d \geq d'$ if and only if $Z_d \supseteq Z_{d'}$.

**Proof.** Consider the natural projections $Y_{u_d} \to Z_d$ and $Y_{u_{d'}} \to Z_{d'}$, which are restrictions of the natural projection $p : B \times B \to \mathcal{P} \times \mathcal{P}$ to $Y_{u_d}$ and $Y_{u_{d'}}$ respectively. Since $u_d \geq u_{d'}$, we have $Y_{u_d} \supseteq Y_{u_{d'}}$. Hence $Z_d \supset p(Y_{u_d}) \supset p(Y_{u_{d'}}) = Z_{d'}$. The "if part" follows from the fact that the morphism $p$ above is projective. \hfill \Box

**Proposition 3.2.** Let $d \in \mathcal{D}$. We have the following short exact sequence:

$$0 \to K^G(Z_{<d}) \to K^G(\bar{Z}_d) \to K^G(Z_d) \to 0.$$

Here $Z_{<d} = \bar{Z}_d - Z_d$.

**Proof.** It suffices to prove the injection $K^G(Z_{<d}) \hookrightarrow K^G(\bar{Z}_d)$. Note that we have a natural injective morphism $K^G(Z) \hookrightarrow K^G(T)$ for $Z = Z_{<d}$ or $Z = \bar{Z}_d$, where $T \subset G$ is a maximal torus. Hence it suffices to prove the result for torus $T$. By Lemma 1.6 in [L2], we just have to show that $K^T(Z_d)$ is a free $R_T$-module for all $d \in \mathcal{D}$. Let's consider the projection $q : Z_d \to \mathcal{P}$ given by $(P, P') \to P$. Let $x \in \{w \in W_0 \mid w$ is of minimal length among $wW_I\}$. Define $Z_d^x = q^{-1}(BxP/P)$ which is $T$-stable. Thus it suffices to show $K^T(Z_d^x)$ is a free $R_T$-module. Note that $Z_d^x = B_x\backslash(B \times F_x)$, where $B_x = \{y \in B : yxP = xP\}$ and $F_x = q^{-1}(xP)$. Then we have

$$K^T(Z_d^x) = K^T(B_x \backslash(B \times F_x)) = K^B(B_x \backslash(B \times F_x)) = K^T(F_x).$$
Since $F_\circ = xP/(xP \cap x\alpha P)$, and $xP/(xP \cap x\alpha P)$ contains a Borel subgroup of a Leavy subgroup of $xP$ containing $T$, it admits a partition which are $T$-vector spaces. Hence the proposition follows. □

3.2. Let $w \in d = W_I w W_I$. Choose Borel subgroups $\hat{B}_w \subset P$ and $\hat{B}_w' \subset wP$ such that $\hat{B}_w \cap \hat{B}_w'$ is a Borel subgroup of $P \cap wP$. Assume $(\hat{B}_w, \hat{B}_w') \in Y_u$ for some $u \in d$. Consider the natural projection $p|_{Y_u} : Y_u \to Z_w$. It is easy to see $p^{-1}(P, wP) = P \cap wP/(\hat{B}_w \cap \hat{B}_w') = B_{P \cap wP}$. Here $B_{P \cap wP}$ denotes the flag variety of $P \cap wP$. Hence $p|_{Y_u}$ is a projective morphism. So $u = u_d$ is the minimal length element in the double coset $d$.

Define

$$R_{p*} : K^G(Y_{u_d}) \longrightarrow K^G(Z_w)$$

$$[\mathcal{F}] \mapsto \sum_i (-1)^i [R^i p_* \mathcal{F}].$$

Let $\chi \in X = \text{Hom}(T, \mathbb{C}^*)$. Denote by $\theta_{\chi}$ the $G$-equivariant line bundle over $Y_{u_d}$ such that $T$ acts on the fiber of $\theta_{\chi}$ over $(\hat{B}_w, \hat{B}_w') \in Y_{u_d}$ via $\chi$.

**Proposition 3.3.** With notations in 3.2. The morphism $R(p|_{Y_{u_d}})_* : K^G(Y_{u_d}) \longrightarrow K^G(Z_w)$ defined above is surjective.

**Proof.** Let’s compute $R(p|_{Y_{u_d}})_*([\theta_{\chi}])$. Note that $p$ is smooth and projective, and $Z_w$ is integral (as a scheme). Hence by Corollary 12.9 of [Har], we have $R^i p|_{Y_{u_d}}^*(\theta_{\chi})$ is vector bundle over $Z_w$ and $R^i p_* (\theta_{\chi})|_{(P, wP)} = H^i(p^{-1}(P, wP), \theta_{\chi}|_{P \cap wP})$ for all $i \geq 0$. Hence when $\chi$ is a dominant weight, $R_{p*}(\theta_{\chi})|_{(P, wP)} = V_\chi$, where $V_\chi$ is the irreducible $P \cap wP$-module with highest weight $\chi$. Note that all $[V_\chi]$ with $\chi$ dominant generates $K^{P \cap wP}(pt) = K^G(Z_w)$. Hence $R_{p*}$ is surjective. □

**Corollary 3.4.** The natural morphism $R_{p*} : K^G(B \times B) \to K^G(P \times P)$ is surjective.

**Proof.** Let $l \in K^G(P \times P)$. We show that $l$ lies in the image of $R_{p*}$ by induction on the dimension of the support of $\text{supp}(l)$. If $\text{supp}(l) = \emptyset$, hence $l = 0$, it follows trivially. Now we assume that the statement holds for any $l'$ such that $\text{supp}(l') \subset Z_{<d}$ for some $d \in D$ and any $l' \in K^G(P \times P)$. Assume $\text{supp}(l) \subset Z_{\leq d}$. Let $l_d = i_d^*(l) \in K^G(Z_d)$, where $i_d : Z_d \hookrightarrow Z_{\leq d}$ be the natural open immersion. By Proposition 3.3, there is $f_d \in K^G(Y_{u_d})$ such that $R(p|_{Y_{u_d}})_*(f_d) = l_d$. Now extend $f_d$ to some $\tilde{f}_d \in K^G(Y_{u_d})$. Thanks to 3.2, we have $p^{-1}(Z_d) \cap Y_{u_d} = Y_{u_d}$. Hence $R_{p*} (\tilde{f}_d)|_{Z_d} = l_d$. Then the support $i_d^*(l - R_{p*}(\tilde{f}_d)) = 0$. By Proposition 3.2, we have $\text{supp}(l - R_{p*}(\tilde{f}_d)) \subset Z_{<d}$. By induction hypothesis, the statement follows. □
References

[CG] N. Chriss and V. Ginzburg, *Representation Theory and Complex Geometry*, Birkhauser, Boston, 1997.

[Har] R. Hartshorne, *Algebraic Geometry*, Springer, New York, 1997.

[KL] D. Kazhdan and G. Lusztig, *Proof of The Delign-Langlands Conjecture for Hecke Algebras*, Invent.Math., Vol.87, 1987, 153–215.

[L1] G. Lusztig, *Cells in Affine Weyl Groups, II*, J.Alg., 109(1987), no. 2, 536–548.

[L2] G. Lusztig, *Bases in Equivariant K-theory. II*, Representation Theory, Vol. 3 (1999), 281–353.

[Xi] N. Xi, *The Based Ring of The Lowest Two-Sided Ring of an Affine Weyl Group*, J.Alg., 134(1990), 356–368.

[Shi] J. Shi, *A Two-Sided Cell in an Affine Weyl Group, I, II*, J.London.Math.Soc.,(2), Vol.36(1987), 407–420; Vol.37(1988), 253–264.