ON A LONG RANGE SEGREGATION MODEL

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Abstract. In this work we study the properties of segregation processes modeled by a family of equations

\[ L(u_i)(x) = u_i(x) F_i(u_1, \ldots, u_K)(x) \quad i = 1, \ldots, K \]

where \( F_i(u_1, \ldots, u_K)(x) \) is a non-local factor that takes into consideration the values of the functions \( u_j \)'s in a full neighborhood of \( x \). We consider as a model problem

\[ \Delta u^\varepsilon_i(x) = \frac{1}{\varepsilon^2} u^\varepsilon_i(x) \sum_{i \neq j} H(u^\varepsilon_j)(x) \]

where \( \varepsilon \) is a small parameter and \( H(u^\varepsilon_j)(x) \) is for instance

\[ H(u^\varepsilon_j)(x) = \int_{B_1(x)} u^\varepsilon_j(y) \, dy \]

or

\[ H(u^\varepsilon_j)(x) = \sup_{y \in B_1(x)} u^\varepsilon_j(y). \]

Here the set \( B_1(x) \) is the unit ball centered at \( x \) with respect to a smooth, uniformly convex norm \( \rho \) of \( \mathbb{R}^n \). Heuristically, this will force the populations to stay at \( \rho \)-distance 1, one from each other, as \( \varepsilon \to 0 \).

1. Introduction

Segregation phenomena occur in many areas of mathematics and science: from equipartition problems in geometry, to social and biological processes (cells, bacteria, ants, mammals), to finance (sellers and buyers). There is a large body of literature in connection to our work and we would like to refer to \[ [4] [5] [8–21] [26–29] [31–33] \] and the references therein. We particularly would like to point out the articles \[ [15] [26] [28] [29] [31] \] where spatial separation due to competition for resources is discussed among ant nests, mussels and sessile animals.

They study a family of models arising from different applications whose main two ingredients are: in the absence of competition species follow a “propagation” equation involving diffusion,
transport, birth-death, etc, but when two species overlap, their growth is mutually inhibited by competition, consumption of resources, etc. The simplest form of such models consists, for species $\sigma_i$ with spatial density $u_i$, on a system of equations

$$L(u_i) = u_i F_i(u_1, \ldots, u_K).$$

The operator $L$ quantifies diffusion, transport, etc, while the term $u_i F_i$ does attrition of $u_i$ from competition with the remaining species.

In these models, the interaction is punctual, i.e. $u_i(x)$ interacts with the remaining densities also at position $x$. There are many processes, though where the growth of $\sigma_i$ at $x$ is inhibited by the populations $\sigma_j$ in a full area surrounding $x$.

The purpose of this work is a first attempt to study the properties of such a segregation process. Basically, we consider a family of equations,

$$L(u_i)(x) = u_i(x) F_i(u_1, \ldots, u_K)(x)$$

where $F_i(u_1, \ldots, u_K)(x)$ is now a non-local factor that takes into consideration the values of $u_j$ in a full neighborhood of $x$. Given the previous discussion a possible model problem would be the system

$$\Delta u_i^\varepsilon(x) = \frac{1}{\varepsilon^2} u_i^\varepsilon(x) \sum_{i \neq j} H(u_j^\varepsilon)(x), \quad i = 1, \ldots, K$$

where $\varepsilon$ is a small parameter and $H(u_j^\varepsilon)(x)$ is a non-local operator, for instance

$$H(u_j^\varepsilon)(x) = \int_{B_1(x)} u_j^\varepsilon(y) \, dy$$

or

$$H(u_j^\varepsilon)(x) = \sup_{y \in B_1(x)} u_j^\varepsilon(y).$$

To study the limit configuration when the competition for resources is very high, we consider the limit when $\varepsilon$ tends to 0. Heuristically, the non-local term forces the populations to stay at distance 1, one from each other. As an example, as we will prove, in the case of two populations in dimension two, we will have strips of length precisely one between the regions.
where the populations live. At “edge” points, that we will define as singular points, the angles of the asymptotic cones have to be the same, see Figure 1. Here $S_i = S^1_i \cup S^2_i$, $i = 1, 2$, represents the region where the the population $\sigma_i$ with density $u_i$ exists. Moreover, the ratio between the normal derivatives at regular points across the free boundary, depends on the ratio of the respective curvature $\kappa$. For example, if $Z_1 \in \partial S^1_1$ and $Z_2 \in \partial S^1_2$, $Z_1$ and $Z_2$ are not “edge” points, and $d(Z_1, Z_2) = 1$ then

$$\frac{u^1_1(Z_1)}{u^2_2(Z_2)} = \frac{\kappa(Z_1)}{\kappa(Z_2)} \quad \text{if} \quad \kappa(Z_2) \neq 0,$$

and $u^1_1(Z_1) = u^2_2(Z_2)$ if $\kappa(Z_2) = 0$.

\[\text{Figure 1. Example of a limit configuration for } K = 2, n = 2\]

We will consider instead of the unit ball in the Euclidean norm $B_1(x)$, the translation at $x$ of a general smooth set $B$ that is also uniformly convex, bounded and symmetric with respect to the origin. The set $B$ defines a smooth, uniformly convex norm $\rho$ in $\mathbb{R}^n$.

Let us note that there is some similarity also with the Lasry-Lions model of price formation (see [6,25]) where selling and buying prices are separated by a gap due to transaction cost.
2. Notation and statement of the problem

Let $B$ be an open bounded domain of $\mathbb{R}^n$, convex, symmetric with respect to the origin and with smooth boundary. Then $B$ can be represented as the unit ball of a norm $\rho : \mathbb{R}^n \to \mathbb{R}$, $\rho \in C^\infty(\mathbb{R}^n \setminus \{0\})$, called the defining function of $B$, i.e.,

$$\mathcal{B} = \{ x \in \mathbb{R}^n \mid \rho(x) < 1 \}.$$ 

We assume that $B$ is uniformly convex, i.e., there exists $0 < a \leq A$ such that in $\mathbb{R}^n \setminus \{0\}$

$$aI_n \leq D_2^2 \left( \frac{1}{2} \rho^2 \right) \leq AI_n,$$

where $I_n$ is the $n \times n$ identity matrix. In what follows we denote

$$\mathcal{B}_r := \{ y \in \mathbb{R}^n \mid \rho(y) < r \},$$

$$\mathcal{B}_r(x) := \{ y \in \mathbb{R}^n \mid \rho(x - y) < r \}.$$ 

So through the paper we will always refer to the Euclidean ball as $B$ and to the $\rho$-ball as $\mathcal{B}$.

For a given closed set $K$, let

$$d_{\rho}(\cdot, K) = \inf_{y \in K} \rho(\cdot - y)$$

be the distance function from $K$ associated to $\rho$. Then there exist $c_1, c_2 > 0$ such that

$$c_1d(\cdot, K) \leq d_{\rho}(\cdot, K) \leq c_2d(\cdot, K),$$

where $d(\cdot, K)$ is the distance function associated to the Euclidian norm $\| \cdot \|$ of $\mathbb{R}^n$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We will denote by $(\partial\Omega)_1$ the $\rho$-strip of size 1 around $\partial\Omega$ in the complement of $\Omega$ defined by

$$(\partial\Omega)_1 := \{ x \in \Omega^c : d_{\rho}(x, \partial\Omega) \leq 1 \}.$$ 

For $i = 1, \ldots, K$, let $f_i$ be non-negative functions defined on $(\partial\Omega)_1$ with supports at $\rho$-distance equal or greater than 1, one from each other:

$$d_{\rho}(\text{supp } f_i, \text{supp } f_j) \geq 1, \quad \text{for } i \neq j.$$
We will consider the following system of equations: for \( i = 1, \ldots, K \)
\[
\begin{cases}
\Delta u_\varepsilon^i(x) = \frac{1}{\varepsilon^2} u_\varepsilon^i(x) \sum_{j \neq i} H(u_\varepsilon^j)(x) & \text{in } \Omega, \\
u_\varepsilon^i = f_i & \text{on } (\partial \Omega)_1.
\end{cases}
\] (2.4)

The functional \( H(u_j)(x) \) depends only on the restriction of \( u_j \) to \( B_1(x) \).

We will consider, for simplicity,
\[
H(w)(x) = \int_{B_1(x)} w^p(y)\varphi(\rho(x - y))dy, \quad 1 \leq p < \infty
\] or
\[
H(w)(x) = \sup_{B_1(x)} w
\]
with \( \varphi \) a strictly positive smooth function of \( \rho \), with at most polynomial decay at \( \partial B_1 \):
\[
\varphi(\rho) \geq C(1 - \rho)^q, \quad q \geq 0.
\] (2.7)

In rest of the paper, when we refer to consider \( u_\varepsilon^1, \ldots, u_\varepsilon^K \), viscosity solutions of the problem (2.4), we mean that \( u_\varepsilon^1, \ldots, u_\varepsilon^K \) are continuous functions that satisfy in the viscosity sense the system of equations (2.4). Moreover, we make the following assumptions: for \( i = 1, \ldots, K \),
\[
\begin{dcases}
\varepsilon > 0, \Omega \text{ a bounded Lipschitz domain of } \mathbb{R}^n, \\
f_i : (\partial \Omega) \to \mathbb{R}, f_i \geq 0, f_i \not\equiv 0, f_i \text{ is Hölder continuous}, \\
\exists c > 0 \text{ s. t. } \forall x \in \partial \Omega \cap \text{supp } f_i, |B_r(x) \cap \text{supp } f_i| \geq c|B_r(x)|, \\
(2.3) \text{ holds true}, \\
H \text{ is either of the form } (2.5) \text{ or } (2.6) \text{ and } (2.7) \text{ holds.}
\end{dcases}
\] (2.8)

3. Main results

For the reader’s convenience we present our main results below. Assume that (2.8) holds true, then:

**Existence (Theorem 4.1):**

There exist continuous functions \( u_\varepsilon^1, \ldots, u_\varepsilon^K \), depending on the parameter \( \varepsilon \), viscosity solutions of the problem (2.4).
Limit problem (Corollary 5.6):
There exists a subsequence \((\tilde{u})^{\varepsilon_m}\) converging locally uniformly, as \(\varepsilon \to 0\), to a function \(\tilde{u} = (u_1, \ldots, u_K)\), satisfying the following properties:

i) the \(u_i\)'s are locally Lipschitz continuous in \(\Omega\) and have supports at distance at least 1, one from each other, i.e.

\[ u_i \equiv 0 \quad \text{in the set} \quad \{x \in \Omega \mid d_\rho(x, \text{supp } u_j) \leq 1\} \quad \text{for any } j \neq i. \]

ii) \(\Delta u_i = 0\) when \(u_i > 0\).

Semiconvexity of the free boundary (Corollary 6.2):
If \(x_0 \in \partial\{u_i > 0\}\) there is an exterior tangent \(\rho\)-ball of radius 1 at \(x_0\).

The supports of \(u_i\) are sets of finite perimeter (Corollary 6.5):
The set \(\{u_i > 0\}\) has finite perimeter.

Sharp characterization of the interfaces (Theorem 7.1):
Under the additional assumption that \(p = 1\) in \((2.5)\), the supports of the limit functions are at distance exactly 1, one from each other, i.e., if \(x_0 \in \partial\{u_i > 0\} \cap \Omega\), then there exists \(j \neq i\) such that

\[ \overline{B_1(x_0)} \cap \partial\{u_j > 0\} \neq \emptyset. \]

Classification of singular points in dimension 2 (Lemma 8.9, Theorem 8.10, Corollary 8.11, Corollary 8.12):
For \(n = 2\), under the additional assumption that \(p = 1\) in \((2.5)\), for \(i \neq j\), let \(x_0 \in \partial\{u_i > 0\} \cap \Omega\) and \(y_0 \in \partial\{u_j > 0\} \cap \Omega\) be points such that \(\{u_i > 0\}\) has an angle \(\theta_i\) at \(x_0\), \(\{u_j > 0\}\) has an angle \(\theta_j\) at \(y_0\) and \(\rho(x_0 - y_0) = 1\). Then we have

\[ \theta_i = \theta_j. \]

If \(x_0 \in \partial\{u_i > 0\} \cap \partial\Omega\) and \(y_0 \in \partial\{u_j > 0\} \cap \Omega\), then

\[ \theta_i \leq \theta_j. \]
Moreover, singular points, i.e. points where the free boundaries have corners, are isolated and finite. If the domain is a strip and there are only two populations, under additional monotonicity assumptions on the boundary data, the free boundary sets $\partial\{u_i > 0\}$, $i = 1, 2$, are of class $C^1$.

Lipschitz regularity for free boundary for the obstacle problem associated in dimension 2 (Theorem 8.18):

For $n = 2$, under the additional assumption that $p = 1$ in (2.5), $f_i \equiv 1$ and additional conditions about the regularity of $\partial \Omega$, if $(u_1^\varepsilon, \ldots, u_K^\varepsilon)$ is a particular solution of (2.4) which satisfies the associated obstacle problem (8.49) with $(u_1, \ldots, u_K)$ the limit as $\varepsilon \to 0$, then the free boundaries $\partial\{u_i > 0\}$, $i = 1, \ldots, K$, are Lipschitz curves of the plane.

Free boundary condition (Theorem 9.2):

In any dimension, assume that we have 2 populations, $H$ is defined as in (2.5) with $\varphi \equiv 1$, $p = 1$ and $B_1(x) = B_1(x)$ is the Euclidian ball, $0 \in \partial\{u_1 > 0\}$, $e_n \in \partial\{u_2 > 0\}$, and $\partial\{u_1 > 0\}$ and $\partial\{u_2 > 0\}$ are of class $C^2$ in a neighborhood of 0 and $e_n$ respectively.

Let $\kappa_i(0)$ denote the principal curvatures of $\partial\{u_1 > 0\}$ at 0, where outward is the positive direction and let $\kappa_i(e_n)$ denote the principal curvatures of $\partial\{u_2 > 0\}$ at $e_n$ where now inward is the positive direction. Then, we have the following relation on the exterior normal derivatives of $u_1$ and $u_2$:

$$\frac{u_1^1(0)}{u_2^2(e_n)} = \prod_{i=1}^{n-1} \frac{\kappa_i(0)}{\kappa_i(e_n)} \quad \text{if } \kappa_i(0) \neq 0 \text{ for some } i = 1, \ldots, n-1,$$

and

$$u_1^1(0) = u_2^2(e_n) \quad \text{if } \kappa_i(0) = 0 \text{ for any } i = 1, \ldots, n-1.$$
4. Existence of solutions

This proof follows the same steps as in [30] and it is written below for the reader’s convenience.

**Theorem 4.1.** Assume (2.8). Then there exist continuous positive functions $u_1^\varepsilon, \ldots, u_K^\varepsilon$, depending on the parameter $\varepsilon$, viscosity solutions of the problem (2.4).

**Proof.** The proof uses a fixed point result. Let $B$ be the Banach space of bounded continuous vector-valued functions defined on the domain $\Omega$ with the norm

$$
\|(u_1, u_2, \ldots, u_K)\|_B := \max_i \left( \sup_{x \in \Omega} |u_i(x)| \right).
$$

For $i = 1, \ldots, K$, let $\phi_i$ be the solutions of

$$(4.1) \quad \begin{cases}
\Delta \phi_i = 0 & \text{in } \Omega, \\
\phi_i = f_i & \text{on } \partial \Omega.
\end{cases}$$

Let $\Theta$ be the subset of bounded continuous functions in $\Omega$, that satisfy prescribed boundary data, and are bounded from above and from below as stated below:

$$
\Theta = \left\{ (u_1, u_2, \ldots, u_K) \mid u_i : \Omega \to \mathbb{R} \text{ is continuous, } 0 \leq u_i \leq \phi_i \text{ in } \Omega, u_i = f_i \text{ on } (\partial \Omega)_1 \right\}.
$$

Notice that $\Theta$ is a closed and convex subset of $B$. Let $T^\varepsilon$ be the operator that is defined on $\Theta$ in the following way: $T^\varepsilon ((u_1, u_2, \ldots, u_K)) := (v_1^\varepsilon, v_2^\varepsilon, \ldots, v_K^\varepsilon)$ if for any $i = 1, \ldots, K$, $v_i^\varepsilon$ is solution to the following problem:

$$(4.2) \quad \begin{cases}
\Delta (v_i^\varepsilon)(x) = \frac{1}{\varepsilon^2} v_i^\varepsilon(x) \sum_{j \neq i} H(u_j)(x) & \text{in } \Omega \\
v_i^\varepsilon = f_i & \text{on } (\partial \Omega)_1,
\end{cases}$$

where $u_j, j \neq i$ are given. Observe that if $T^\varepsilon$ has a fixed point

$$
T^\varepsilon ((u_1^\varepsilon, u_2^\varepsilon, \ldots, u_K^\varepsilon)) = (u_1^\varepsilon, u_2^\varepsilon, \ldots, u_K^\varepsilon)
$$

then $(u_1^\varepsilon, u_2^\varepsilon, \ldots, u_K^\varepsilon)$ is a solution of problem (2.4).

In order for $T^\varepsilon$ to have a fixed point, we need to prove that it satisfies the hypothesis of the Schauder fixed point Theorem, see [23]:

1. $T^\varepsilon(\Theta) \subset \Theta$.
Classical existence results guarantee the existence of a viscosity solution \((v_{\varepsilon}^1, v_{\varepsilon}^2, \ldots, v_{\varepsilon}^K)\) of problem (4.2) which is smooth in \(\Omega\). Since \(f_i \geq 0\) and \(f_i \not\equiv 0\), the strong maximum principle implies

\[ v_{\varepsilon}^i > 0 \quad \text{in} \ \Omega. \]

This implies that

\[ \Delta v_{\varepsilon}^i \geq 0 \quad \text{in} \ \Omega, \tag{4.3} \]

and, again from the comparison principle, we have

\[ v_{\varepsilon}^i \leq \phi_i \quad \text{in} \ \Omega. \]

We have proved that \(T_{\varepsilon}((u_1, u_2, \ldots, u_K)) \in \Theta\).

(2) \(T_{\varepsilon}\) is continuous:

Let us assume that \(((u_1)_m, \ldots, (u_K)_m) \to (u_1, \ldots, u_K)\) in \(B\) meaning that when \(m\) tends to \(+\infty\),

\[ \max_{1 \leq i \leq K} \|(u_i)_m - u_i\|_{L^\infty} \to 0. \]

We need to prove that for each fixed \(\varepsilon > 0\)

\[ \|T_{\varepsilon}((u_1)_m, \ldots, (u_K)_m) - T_{\varepsilon}(u_1, \ldots, u_K)\|_B \to 0 \]

when \(m \to +\infty\). Let

\[ T_{\varepsilon}((u_1)_m, \ldots, (u_K)_m) = ((v_{\varepsilon}^1)_m, \ldots, (v_{\varepsilon}^K)_m), \]

then if we prove that there exists a constant \(C_{\varepsilon}\) independent of \(m\), so that we have the estimate, for \(i = 1, \ldots, K\)

\[ \|(v_{\varepsilon}^i)_m - v_{\varepsilon}^i\|_{L^\infty} \leq C_{\varepsilon} \max_j \|(u_j)_m - u_j\|_{L^\infty}, \]

the result follows. For all \(x \in \Omega\) and for fixed \(i\), let \(\omega_m\) be the function

\[ \omega_m(x) = (v_{\varepsilon}^i)_m(x) - v_{\varepsilon}^i(x), \]
and suppose for instance that there exists $y \in \Omega$ such that

\[ \omega_m(y) > r^2 D \max_j \|(u_j)_m - u_j\|_{L^\infty}, \]

for some large $D > 0$, where $r$ is such that $\Omega \subset B_r$, and $B_r$ is the ball centered at 0 of radius $r$ in the Euclidean norm. We want to prove that this is impossible if $D$ is sufficiently large. Let $h_m$ be the concave radially symmetric function

\[ h_m(x) = \gamma_m (r^2 - |x|^2), \]

with $\gamma_m = D \max_j \|(u_j)_m - u_j\|_{L^\infty}$. Observe that:

(a) $h_m(x) = 0$ on $\partial B_r$;

(b) $h_m(x) \leq r^2 D \max_j \|(u_j)_m - u_j\|_{L^\infty}$ for all $x$ in $B_r$;

(c) $0 = \omega_m(x) \leq h_m(x)$ on $\partial \Omega$, since $(v^\varepsilon_i)_m$ and $v^\varepsilon_i$ are solutions with the same boundary data.

Since we are assuming (4.4), there exists a negative minimum of $h_m - \omega_m$ in $\Omega$. Let $x_0 \in \Omega$ be a point where the minimum value of $h_m - \omega_m$ is attained. Then

\[ h_m(x_0) - \omega_m(x_0) < 0 \quad \text{and} \quad \Delta(h_m - \omega_m)(x_0) \geq 0. \]

Then, we have

\[
\begin{align*}
\Delta \omega_m(x_0) &= \Delta((v^\varepsilon_i)_m)(x_0) - \Delta v^\varepsilon_i(x_0) \\
&= \frac{1}{\varepsilon^2} \left( \left( (v^\varepsilon_i)_m(x_0) - v^\varepsilon_i(x_0) \right) \sum_{j \neq i} H((u_j)_m)(x_0) \\
&\quad - v^\varepsilon_i(x_0) \sum_{j \neq i} (H(u_j)(x_0) - H((u_j)_m)(x_0)) \right) \\
&\quad + \frac{1}{\varepsilon^2} \left( \left( (v^\varepsilon_i)_m(x_0) - v^\varepsilon_i(x_0) \right) \sum_{j \neq i} H((u_j)_m)(x_0) \\
&\quad - v^\varepsilon_i(x_0)(K - 1) C \max_j \|(u_j)_m - u_j\|_{L^\infty(\Omega)} \right).
\end{align*}
\]
adding and subtracting $\frac{1}{\varepsilon^2} v_i^\varepsilon(x_0) \sum_{j \neq i} H((u_j)_m)(x_0)$, where $C$ depends on the $f_j$’s and $\varphi$. Then

$$0 \leq \Delta(h_m - \omega_m)(x_0)$$

$$\leq -2\gamma_m n - \frac{1}{\varepsilon^2} \left( (v_i^\varepsilon)_m - v_i^\varepsilon(x_0) \sum_{j \neq i} H((u_j)_m)(x_0) ight)$$

$$- v_i^\varepsilon(x_0) (K - 1) C \max_j \| (u_j)_m - u_j \|_{L^\infty}$$

$$\leq -2 nD \max_j \| (u_j)_m - u_j \|_{L^\infty} + \frac{1}{\varepsilon^2} v_i^\varepsilon(x_0) (K - 1) C \max_j \| (u_j)_m - u_j \|_{L^\infty}$$

$$\leq -2 nD \max_j \| (u_j)_m - u_j \|_{L^\infty} + \frac{\tilde{C}}{\varepsilon^2} \max_j \| (u_j)_m - u_j \|_{L^\infty}$$

because $0 < h_m(x_0) < \omega_m(x_0) = (v_i^\varepsilon)_m - v_i^\varepsilon(x_0)$ and $\sum_{j \neq i} H((u_j)_m)(x_0) \geq 0$ and so

$$- \frac{1}{\varepsilon^2} ((v_i^\varepsilon)_m - v_i^\varepsilon(x_0) \sum_{j \neq i} H((u_j)_m)(x_0) \leq 0.$$

Taking $D = D_\varepsilon > \frac{\tilde{C}}{2n \varepsilon^2}$, we obtain that

$$0 \leq \Delta(h_m - \omega_m)(x_0) < 0$$

which is a contradiction.

(3) $T^\varepsilon(\Theta)$ is precompact:

Let $((u_1)_m, \ldots, (u_K)_m)$ be a bounded sequence in $B$ and let

$$((v_1^\varepsilon)_m, \ldots, (v_K^\varepsilon)_m) = T^\varepsilon((u_1)_m, \ldots, (u_K)_m).$$

Then by standard Hölder estimates for viscosity solutions, $((v_1^\varepsilon)_m, \ldots, (v_K^\varepsilon)_m)$ is bounded in the space of Hölder continuous functions on $\overline{\Omega}$. Since the subset of $\Theta$ of Hölder continuous functions on $\overline{\Omega}$ is precompact in $\Theta$, we can extract from $((v_1^\varepsilon)_m, \ldots, (v_K^\varepsilon)_m)$ a subsequence which is converging in $B$.

We have proven the existence of a solution $(u_1^\varepsilon, \ldots, u_K^\varepsilon)$ of (2.4). The same argument as in (1) shows that $u_i^\varepsilon > 0$ in $\Omega$. This concludes the proof of the theorem. $\Box$
5. Uniform in $\varepsilon$ Lipschitz estimates

In this section we will prove uniform in $\varepsilon$ Lipschitz estimates that will imply the convergence, up to subsequences, of the solution $(u^1_\varepsilon, \ldots, u^K_\varepsilon)$ of (2.4) to a limit function $(u^1, \ldots, u^K)$ as $\varepsilon \to 0$. We will show that the functions $u^i$'s are locally Lipschitz continuous in $\Omega$ and harmonic inside their support. Moreover, $u^i \equiv 0$ in the $\rho$-strip of size 1 of the support of $u^j$ for any $j \neq i$, i.e., the supports of the limit functions are at distance at least 1, one from each other. We start by proving general properties of subsolutions of uniform elliptic equations.

Lemma 5.1. Let:

a) $\omega$ be a subharmonic function in $B_1$, such that
   a1) $\omega \leq 1$ in $B_1$;
   a2) $\omega(0) = m > 0$.

b) $D_0$ be a smooth convex set with bounded curvatures

$$|z_i(\partial D_0)| \leq C_0, \quad i = 1, \ldots, n - 1$$

(like $B_1$ above).

Then, there exists a universal $\tau_0 = \tau_0(C_0, n, \rho)$ such that, if the distance $d_\rho(D_0, 0) \leq \tau_0m$, then

$$\sup_{\partial D_0 \cap B_1} \omega \geq \frac{m}{2}.$$ 

Proof. Assume w.l.o.g. that $0 \notin D_0$ and let $h$ be harmonic in $B_1 \setminus D_0$ and such that

$$\begin{cases} 
  h = 1 & \text{on } (\partial B_1) \setminus D_0 \\
  h = \frac{m}{2} & \text{on } (\partial D_0) \cap B_1.
\end{cases}$$

By assumption (b), the set $B_1 \setminus D_0$ satisfies an exterior uniform ball condition at any point of $\partial D_0 \cap B_1$, therefore, by a standard barrier argument, $h$ grows no more than linearly away from $\partial D_0$ in $B_{\frac{1}{2}}$, i.e., there exist $k_1, k_2 > 0$ depending on $C_0$ and $n$ such that, if $x \in B_{\frac{1}{2}} \setminus D_0$ and $d(x, \partial D_0) \leq k_2$, then $h(x) \leq k_1d(x, \partial D_0) + \frac{m}{2}$. To prove that $h(0) < m$ observe that if $\tau_0 \leq k_2 c_1$, where $c_1$ is given by (2.2), then $d(0, \partial D_0) \leq \tau_0 m/c_1 \leq k_2 m \leq k_2$ and therefore, if in
addition \( \tau_0 \) is so small that \( \frac{k_1}{c_1} \tau_0 \leq \frac{1}{2} \), we have

\[
h'(0) \leq k_1 d(0, \partial D_0) + \frac{m}{2} \leq \frac{k_1}{c_1} \rho_0(0, \partial D_0) + \frac{m}{2} \leq \frac{k_1}{c_1} \rho_0 m + \frac{m}{2} < m.
\]

Hence, we must have \( \sup_{(\partial D_0 \cap B_1)} \omega \geq \frac{m}{2} \), otherwise the comparison principle would imply \( \omega(x) \leq h(x) \) in \( B_1 \setminus D_0 \), which is a contradiction at \( x = 0 \).

\[\square\]

**Lemma 5.2.** Let \( \omega \) be a positive subsolution of a uniformly elliptic equation, \((\lambda^2 I \leq a_{ij} \leq \Lambda^2 I)\)

\[
a_{ij} D_{ij} \omega \geq \theta^2 \omega \quad \text{in } B_r.
\]

Then there exist \( c, C > 0 \) such that

\[
\frac{\omega(0)}{\sup_{B_r} \omega} \leq Ce^{-c\theta r}.
\]

**Proof.** The function

\[
g(x) = \sum_{i=1}^n \cosh \left( \frac{\theta}{\Lambda} x_i \right)
\]

is a supersolution of the equation \( a_{ij} D_{ij}u = \theta^2 u \). Moreover, using the convexity of the exponential function, it is easy to check that it satisfies

\[
g(x) \geq C_1 e^{c\theta r} \quad \text{for any } x \in \partial B_r.
\]

Then, the comparison principle implies

\[
\frac{\omega(x)}{\sup_{B_r} \omega} \leq \frac{g(x)}{C_1 e^{c\theta r}} \quad \text{for any } x \in B_r.
\]

The result follows taking \( x = 0 \).

\[\square\]

The next lemma says that if \( u^\varepsilon_i \) attains a positive value \( \sigma \) at some interior point, then all the other functions \( u^\varepsilon_j, j \neq i \), go to zero exponentially in a \( \rho \) -ball of radius \( 1 + c\sigma \) around that point.

**Lemma 5.3.** Assume \((2.8)\). Let \( (u^\varepsilon_1, \ldots, u^\varepsilon_K) \) be a viscosity solution of the problem \((2.4)\). For \( i = 1, \ldots, K, \sigma > 0 \), and \( 0 < r < 1 \) let

\[
\Gamma^\sigma_i \equiv \{ y \in \Omega : d_\rho(y, \supp f_i) \geq 2r, u^\varepsilon_i = \sigma \}.
\]
and
\[ m := \frac{\sigma}{\sup_{\partial \Omega} f_i}. \]

Then, there exists a universal constant \( 0 < \tau < 1 \) such that, in the sets
\[ \Sigma_{i,j}^{\sigma,r} := \{ x \in \Omega : d_\rho(x, \Gamma_i^{\sigma,r}) \leq 1 + \frac{\tau mr}{2}, d_\rho(x, \text{supp } f_j) \geq \frac{\tau mr}{4} \} \]
we have
\[ u_j^\varepsilon \leq Ce^{-\varepsilon \alpha}, \quad \text{for } j \neq i, \]
for some positive \( \alpha \) and \( \beta \) depending on the structure of \( H \) (\( p \) and \( q \)).

**Proof.** Let \( 0 < \tau < 1 \) to be determined. For \( 0 < r < 1 \), let us consider the set \( \Sigma_{i,j}^{\sigma,r} \) defined above and let \( \bar{x} \in \Sigma_{i,j}^{\sigma,r} \). We want to show that for \( j \neq i \), we have
\[ (5.1) \quad \Delta u_j^\varepsilon \geq \frac{C\sigma r \bar{\rho} \bar{\beta}}{\varepsilon^2} u_j^\varepsilon \quad \text{in } B_{\frac{\tau mr}{4}}(\bar{x}) \]
for some \( \bar{\alpha}, \bar{\beta} > 0 \). Let us prove it for \( \bar{x} \) such that \( d_\rho(\bar{x}, \text{supp } f_j) \geq \frac{\tau mr}{4} \), which is the hardest case. First of all, remark that since \( d_\rho(\bar{x}, \text{supp } f_j) \geq \frac{\tau mr}{4} \), the ball \( B_{\frac{\tau mr}{4}}(\bar{x}) \) does not intersect \( \text{supp } f_j \). Therefore, \( u_j^\varepsilon \) (which is eventually zero in \( B_{\frac{\tau mr}{4}}(\bar{x}) \cap \Omega^c \)) satisfies
\[ (5.2) \quad \Delta u_j^\varepsilon \geq \frac{1}{\varepsilon^2} \sum_{k \neq j} H(u_k^\varepsilon) \quad \text{in } B_{\frac{\tau mr}{4}}(\bar{x}). \]
Next, the ball \( B_{1 - \frac{\tau mr}{2}}(\bar{x}) \) is at distance \( \tau mr \) from a point \( y \in \Gamma_i^{\sigma,r} \). Remark that since \( B_{2r}(y) \cap \text{supp } f_i = \emptyset \), the function \( u_i^\varepsilon \) (which is eventually equal to zero in \( B_{2r}(y) \cap \Omega^c \)) satisfies \( \Delta u_i^\varepsilon \geq 0 \) in \( B_{2r}(y) \). Moreover, since \( u_i^\varepsilon \) is subharmonic in \( \Omega \), it attains its maximum at the boundary of \( \Omega \), so that \( u_i^\varepsilon / \sup_{\partial \Omega} f_i \leq 1 \) in \( \Omega \). In particular \( m = \frac{\sigma}{\sup_{\partial \Omega} f_i} \leq 1 \). Set
\[ (5.3) \quad v(x) := \frac{u_i^\varepsilon(y + rx)}{\sup_{\partial \Omega} f_i}, \]
then \( v \leq 1 \) and \( v(0) = u_i^\varepsilon(y) / \sup_{\partial \Omega} f_i = \sigma / \sup_{\partial \Omega} f_i = m \) and \( \Delta v \geq 0 \) in \( B_1 \). Let
\[ D_0 := B_{1 - \frac{\tau mr}{2}} \left( \frac{\bar{x} - y}{r} \right), \]
then the principal curvatures of \( D_0 \) satisfy
\[ |\kappa_i(\partial D_0)| \leq \frac{C_\rho}{1 - \frac{\tau mr}{2}} = \frac{2rC_\rho}{2 - r\tau m} < 2rC_\rho < 2C_\rho. \]
Moreover $D_0$ is at distance $\tau m$ from 0. Hence, from Lemma [5.1] applied to the function $v$ given by (5.3) with $D_0$ defined as above, if $\tau = \min\{1, \tau_0\}$, where $\tau_0$ is the universal constant given by the lemma, then there is a point $z$ in $\partial B_{1 - \frac{\tau m}{r}}(\overline{x}) \cap B_r(y)$, such that $u_\epsilon^z(z) \geq \sigma / 2$. Next, remark that if $x \in B_{\frac{\tau m}{r}}(\overline{x})$ then

\[\mathcal{B}_1(x) \supset B_{\frac{\tau m}{4}}(z)\]

(since $d_\rho(x, z) \leq d_\rho(x, \overline{x}) + d_\rho(\overline{x}, z) \leq \frac{\tau m}{4} + 1 - \frac{\tau m}{2} = 1 - \frac{\tau m}{4}$).

Let us first consider the case $H$ defined as in (2.6). Then for any $x \in B_{\frac{\tau m}{4}}(\overline{x})$ we have

\[H(u_\epsilon^z)(x) = \sup_{B_1(x)} u_\epsilon^z \geq u_\epsilon^z(z) \geq \frac{\sigma}{2},\]

which, with together (5.2), implies (5.1) with $\alpha = 1$ and $\beta = 0$.

Next, let us turn to the case $H$ defined as in (2.5). Remark that since $z \in B_r(y)$ and $d_\rho(y, \text{supp } f_i) \geq 2r$, we have that $B_r(z) \cap \text{supp } f_i = \emptyset$ and therefore the function $u_\epsilon^z$ (which is eventually equal to zero in $B_r(z) \cap \Omega^c$) satisfies $\Delta u_\epsilon^z \geq 0$ in $B_r(z)$. This implies that $(u_\epsilon^z)^p$, $p \geq 1$, is subharmonic in $B_r(z)$ and by the mean value inequality

\[(5.4) \quad \int_{B_s(z)} (u_\epsilon^z)^p dx \geq \left(\frac{\sigma}{2}\right)^p\]

in any Euclidian ball $B_s(z) \subset B_r(z)$, for any $p \geq 1$. Since $d_\rho$ and the Euclidian distance are equivalent, there is an $s \sim \tau mr$ such that

\[(5.5) \quad B_s(z) \subset B_{\frac{\tau m}{8}}(z) \subset B_{\frac{\tau m}{4}}(z) \subset B_1(x).\]

Moreover, if $y \in B_s(z)$ and $x \in B_{\tau mr}(\overline{x})$, then

\[\rho(y - x) \leq \rho(y - z) + \rho(z - \overline{x}) + \rho(\overline{x} - x) \leq \frac{\tau m r}{8} + \left(1 - \frac{\tau m r}{2}\right) + \frac{\tau m r}{4} = 1 - \frac{\tau m r}{8},\]

that is

\[(5.6) \quad 1 - \rho(y - x) \geq \frac{\tau m r}{8}.\]
Hence, using (5.5), (2.7), (5.6) and (5.4), for all \( x \in B_{\frac{\tau mr}{4}}(\bar{x}) \) we get

\[
H(u_\varepsilon^i)(x) = \int_{B_1(x)} (u_\varepsilon^i)^p(y) \varphi(y - x) dy \\
\geq \int_{B_{\bar{s}}(z)} (u_\varepsilon^i)^p(y) C(1 - \rho(y - x))^q dy \\
\geq \int_{B_{\bar{s}}(z)} (u_\varepsilon^i)^p(y) C \left( \frac{\tau mr}{8} \right)^q dy \\
\geq C \sigma^\alpha r^\beta
\]

where \( \bar{x} \) and \( \bar{\beta} \) depend on \( p, q \) and on the dimension \( n \). This and (5.2) imply (5.1).

Now, by Lemma 5.2 we get

\[
u_\varepsilon^j(\bar{x}) \leq Ce^{-\frac{\alpha \varepsilon^a \beta}{\varepsilon}}
\]

for \( \alpha = \frac{\alpha}{2} + 1 \) and \( \beta = \frac{\beta}{2} + 1 \), and the lemma is proven.

\[\square\]

**Corollary 5.4.** Assume (2.8). Let \((u_\varepsilon^1, \ldots, u_\varepsilon^K)\) be a viscosity solution of the problem (2.4). Let \( y \) be a point in \( \Omega \) such that

\[
u_\varepsilon^i(y) = \sigma, \quad d_\rho(y, \text{supp } f_j) \geq 1 + \tau mr, \quad i \neq j \quad \text{and} \quad d_\rho(y, \partial \Omega) \geq 2r,
\]

where \( m = \frac{\sigma}{\sup_{\partial \Omega} f_i} \), \( 0 < r < 1, \varepsilon \leq \sigma^2 \varepsilon^2 \beta^2 \) and \( \tau, \alpha \) and \( \beta \) are given by Lemma 5.3. Then there exists a constant \( C_0 > 0 \) such that in \( B_{\frac{\tau mr}{4}}(y) \) we have

\[
|\nabla u_\varepsilon^i| \leq \frac{C_0}{r}
\]

and

\[
\Delta u_\varepsilon^i \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{uniformly.}
\]

**Proof.** First of all, remark that \( m \leq 1 \), as \( u_i \) attains its maximum at the boundary of \( \Omega \). Since in addition \( \tau < 1 \), we have that \( B_{\frac{\tau mr}{4}}(y) \subset B_{2r}(y) \subset \Omega \). Therefore, we use (2.4) to estimate \( \Delta u_\varepsilon^i(z) \), for \( z \in B_{\frac{\tau mr}{4}}(y) \). In order to do that, we need to estimate \( H(u_\varepsilon^j)(z) \) for \( j \neq i \). But \( H(u_\varepsilon^j)(z) \) involves points \( x \) at \( \rho \)-distance 1 from \( z \). Let \( x \) be such that \( d_\rho(x, z) \leq 1 \), then
\[ d_\rho(x, y) \leq 1 + \frac{\tau mr}{2}. \] Moreover, since \( d_\rho(y, \text{supp } f_j) \geq 1 + \tau mr \), we have \( d_\rho(x, \text{supp } f_j) \geq \frac{\tau mr}{2}. \)

Hence, by Lemma 5.3 for any \( j \neq i \)

\[ u_\varepsilon^i(x) \leq C e^{-c_\varepsilon \alpha r^3} \quad \text{for } x \in B_1(z). \]

From the previous estimate and (2.4), it follows that for \( z \in B_{\frac{\tau mr}{2}}(y) \) we have

\[ (5.9) \quad 0 \leq \Delta u_\varepsilon^i(z) \leq u_\varepsilon^i(z) \leq u_\varepsilon^i(z) \frac{C e^{-c_\varepsilon \alpha r^3}}{\varepsilon^2} = o(1) \quad \text{as } \varepsilon \to 0, \]

for \( \varepsilon \leq \sigma^{2 \alpha r^2}. \) If we normalize the ball \( B_{\frac{\tau mr}{2}}(y) \) in a Lipschitz fashion:

\[ \overline{u}_\varepsilon(z) := 2 \frac{u_\varepsilon^i \left( \frac{\tau mr}{2} z + y \right)}{\tau mr}, \]

then we have

\[ \overline{u}_\varepsilon(0) = 2 \frac{u_\varepsilon^i(y)}{\tau r} = 2 \sup_{\partial \Omega} f_i, \]

and

\[ 0 \leq \Delta \overline{u}_\varepsilon(z) \leq \frac{\tau mr}{2} \sum_{j \neq i} \frac{1}{\varepsilon^2} H(u_\varepsilon^j) \left( \frac{\tau mr}{2} z + y \right) \quad \text{for } z \in B_1(0), \]

where

\[ \frac{\tau mr}{2} \sum_{j \neq i} \frac{1}{\varepsilon^2} H(u_\varepsilon^j) \left( \frac{\tau mr}{2} z + y \right) \leq \frac{C e^{-c_\varepsilon \alpha r^3}}{\varepsilon^2}. \]

Then, by the Harnack inequality (see e.g. Theorem 4.3 in [3]), we get

\[ \sup_{B_{\frac{1}{2}}(0)} \overline{u}_\varepsilon \leq C_n \left( \inf_{B_{\frac{1}{2}}(0)} \overline{u}_\varepsilon + \frac{C e^{-c_\varepsilon \alpha r^3}}{\varepsilon^2} \right) \leq C_n \left( \frac{2 \sup_{\partial \Omega} f_i}{\tau r} + \frac{C e^{-c_\varepsilon \alpha r^3}}{\varepsilon^2} \right) \leq \frac{C}{r}. \]

Lipschitz estimates then imply that \( |\nabla \overline{u}_\varepsilon| \leq C/r \) in \( B_{\frac{1}{2}}(0) \) and (5.7) follows.

Further, (5.9) implies (5.8). \( \square \)

The next lemma says that in a \( \rho \)-strip of size 1 of support of the \( f_j \)'s, the function \( u_\varepsilon^i, i \neq j \), decays to 0 exponentially.

**Lemma 5.5.** Assume (2.8). Let \( (u_\varepsilon^1, \ldots, u_\varepsilon^K) \) be a viscosity solution of the problem (2.4). For \( j = 1, \ldots, K, \sigma > 0 \), let \( \Gamma_\sigma^j := \{ f_j \geq \sigma \} \subset \Omega^\sigma. \) Then on the sets

\[ \{ x \in \Omega : d_\rho(x, \Gamma_\sigma^j) \leq 1 - r \}, \quad 0 < r < 1 \]
we have
\[ u^\varepsilon_i \leq Ce^{-\frac{\alpha \varepsilon i}{r}}, \quad \text{for } i \neq j, \]
for some positive \( \alpha \) and \( \beta \) depending on the structure of \( H \) (\( p \) and \( q \)) and the modulus of continuity of \( f_j \).

Proof. Let \( \bar{x} \in \Omega \) and \( y \in \Gamma_j \) be such that \( d_\rho(\bar{x}, y) \leq 1 - r \). We want to estimate \( H(u^\varepsilon_j)(x) \), for any \( x \in B_{2r}(\bar{x}) \). Let \( x \in B_{2r}(\bar{x}) \), then
\[ (5.10) \quad d_\rho(x, y) \leq 1 - \frac{r}{2}. \]
Let us first consider the case \( H \) defined as in (2.6). We have
\[ H(u^\varepsilon_j)(x) = \sup_{B_1(x)} u^\varepsilon_j \geq f_j(y) \geq \sigma. \]
Next, let us turn to the case \( H \) defined as in (2.5). Let \( r_0 := \min\{\sigma \gamma, r/4\} \), for some \( \gamma \) depending on the modulus of continuity of \( f_j \), then \( f_j \geq \sigma/2 \) in the set \( B_{r_0}(y) \cap \text{supp } f_j \). Moreover, remark that from (5.10) and \( r_0 \leq r/4 \), we have
\[ B_{r_0}(y) \cap \text{supp } f_j \subset B_{4r}(y) \subset B_{2r}(y) \subset B_1(x), \]
and for any \( z \in B_{r_0}(y) \cap \text{supp } f_j \)
\[ \rho(x - z) \leq \rho(x - y) + \rho(y - z) \leq 1 - \frac{r}{2} + r_0 \leq 1 - \frac{r}{4}. \]
Therefore, using in addition (2.7), and that, by (2.8), \( |B_{r_0}(y) \cap \text{supp } f_j| \geq c|B_{r_0}(y)| \), we get
\[ H(u^\varepsilon_j)(x) = \int_{B_1(x)} (u^\varepsilon_j)^p(z)\varphi(\rho(x - z))dz \]
\[ \geq \int_{B_{r_0}(y) \cap \text{supp } f_j} (u^\varepsilon_j)^p(z)(1 - \rho(x - z))^q dz \]
\[ \geq \int_{B_{r_0}(y) \cap \text{supp } f_j} (f_j)^p(z)C \left( \frac{r}{4} \right)^q dz \]
\[ \geq C_\beta \sigma^p r_0^q, \]
where \( \beta \) depends on \( q \) and on the dimension \( n \).
Then, for $H$ defined as in (2.5) or (2.6), the function $u^i_\varepsilon$, $i \neq j$ (which is eventually zero in $B_{\frac{r}{2}}(\bar{x}) \cap \Omega^c$) is subsolution of

$$\Delta u^i_\varepsilon \geq u^i_\varepsilon \frac{C\sigma^2 r_0^{\beta}}{\varepsilon^2}$$

in $B_{\frac{r}{2}}(\bar{x})$, where $p = 1$ and $\beta = 0$ in the case (2.6). The conclusion follows as in Lemma 5.3 □

The following corollary is a consequence of Lemma 5.3, Corollary 5.4 and Lemma 5.5.

**Corollary 5.6.** Assume (2.8). Let $(u^1_\varepsilon, \ldots, u^K_\varepsilon)$ be a viscosity solution of the problem (2.4).

Then, there exists a subsequence $(u^{\varepsilon}_{i1}, \ldots, u^{\varepsilon}_{iK})$ and continuous functions $(u_1, \ldots, u_K)$ such that,

$$(u^{\varepsilon}_{i1}, \ldots, u^{\varepsilon}_{iK}) \to (u_1, \ldots, u_K) \quad \text{as } l \to +\infty, \quad \text{a.e. in } \Omega$$

and the convergence of $u^{\varepsilon}_{i1}$ to $u_i$ is locally uniform in the set $\{x \in \Omega : d_\rho(x, \supp f_j) > 1, j \neq i\}$.

Moreover, we have:

i) the $u_i$’s are locally Lipschitz continuous in $\Omega$ and have disjoint supports, in particular

$$u_i \equiv 0 \quad \text{in the set } \{x \in \Omega | d_\rho(x, \supp u_j) \leq 1\} \quad \text{for any } j \neq i.$$

ii) $\Delta u_i = 0$ when $u_i > 0$.

**Proof.** Fix an index $i = 1, \ldots, K$. Let us denote

$$\Omega_i := \{x \in \Omega | d_\rho(x, \supp f_j) > 1 \text{ for any } j \neq i\},$$

and

$$B_i := \Omega \setminus \overline{\Omega}_i.$$

**Claim 1:** $u^i_\varepsilon(x) \to 0$ as $\varepsilon \to 0$ for any $x \in B_i$.

Indeed, let $x_0$ belong to $B_i$, then there exists $j \neq i$ such that $d_\rho(x_0, \supp f_j) < 1$. Remark that

$$\{x \in \Omega | d_\rho(x, \supp f_j) < 1\} \subset \cup_{r, \sigma > 0} \{x \in \Omega | d_\rho(x, \Gamma^\sigma_j) \leq 1 - r\},$$

where $\Gamma^\sigma_j = \{f_j \geq \sigma\}$. Therefore, there exist $r, \sigma > 0$ such that $x_0 \in \{x \in \Omega | d_\rho(x, \Gamma^\sigma_j) \leq 1 - r\}$,

and by Lemma 5.5 we have that $u^i_\varepsilon(x_0) \leq C e^{-\frac{\alpha^2 r^{\beta}}{\varepsilon}}$, for some $\alpha, \beta > 0$. Claim 1 follows.
Claim 2: there exists a subsequence \((u_i^\varepsilon)\) locally uniformly convergent in \(\Omega_i\) as \(l \to +\infty\) to a locally Lipschitz continuous function \(u_i\).

Fix, \(0 < r < 1\), and define

\[
\Omega'_i := \{ x \in \Omega_i \mid d_\rho(x, \partial \Omega) > 2r, d_\rho(x, \text{supp } f_j) \geq 1 + \tau r \text{ for any } j \neq i \},
\]

Fix \(\theta < \frac{1}{2\alpha}\) and set \(\sigma_\varepsilon = \varepsilon^\theta > 0\) and consider \(\tau, \alpha\) and \(\beta\) as given by Lemma 5.3. Since \(\varepsilon = \sigma_\varepsilon^{2\alpha} \frac{1}{\beta - 2\alpha} = \sigma_\varepsilon^{2\alpha \varepsilon^\theta (\frac{1}{\beta} - 2\alpha)}\) and \(\frac{1}{\beta} - 2\alpha > 0\), we can fix \(\varepsilon_0 = \varepsilon_0(r)\) so small that for any \(\varepsilon < \varepsilon_0\) we have that \(\varepsilon \leq \sigma_\varepsilon^{2\alpha r^2\beta}\). Then, by Corollary 5.4 the functions

\[
v_i^\varepsilon := (u_i^\varepsilon - \sigma_\varepsilon)_+ = (u_i^\varepsilon - \varepsilon^\theta)_+
\]

are Lipschitz continuous in \(\Omega'_i\). Indeed, when \(u_i^\varepsilon(x) < \varepsilon^\theta\), then \(v_i^\varepsilon(x) = 0\). Next, let \(x\) such that \(u_i^\varepsilon(x) > \varepsilon^\theta\), then \(\nabla v_i^\varepsilon(x) = \nabla u_i^\varepsilon(x)\). Set \(\sigma = u_i^\varepsilon(x)\), then at those points \(x\), we have that \(d_\rho(x, \text{supp } f_j) \geq 1 + \tau r \geq 1 + m \tau r\), where \(m = \sigma / \sup_{\partial \Omega} f_i \leq 1\). Moreover, \(d_\rho(x, \partial \Omega) > 2r\) and \(\varepsilon \leq \sigma_\varepsilon^{2\alpha r^2\beta} \leq 2\alpha \varepsilon r^2\beta\). We can therefore apply Corollary 5.4 and we get that

\[
|\nabla u_i^\varepsilon(x)| \leq \frac{C_0}{r}.
\]

This concludes the proof that the functions \(v_i^\varepsilon\) are Lipschitz continuous in \(\Omega'_i\). Therefore, we can extract a subsequence \((v_i^\varepsilon)\) uniformly convergent to a Lipschitz continuous function \(u_i\) in \(\Omega'_i\) as \(l \to +\infty\). By the definition of the \(v_i\)’s, this implies that there exists a subsequence \((u_i^\varepsilon)\) uniformly convergent to the same function \(u_i\) in \(\Omega'_i\) as \(l \to +\infty\). Taking \(r \to 0\) and using a diagonalization argument, we can find a subsequence of \((u_i^\varepsilon)\) converging locally uniformly to a Lipschitz function \(u_i\) in \(\Omega_i\). This ends the proof of Claim 2.

Claims 1 and 2 yield the convergence, up to a subsequence, of \(u_i^\varepsilon\) to a continuous function \(u_i\) which is locally Lipchitz in both \(\Omega_i\) and \(B_i\). The fact that the supports of the limit functions are at distance greater or equal than 1, is a consequence of Claims 1 and 2 and Lemma 5.3. Moreover, from the proof of Claim 2 and Corollary 5.4, we infer that the limit function \(u_i\) is harmonic inside its support, i.e. (ii). To conclude the proof of (i), we just need to prove that \(u_i\) is Lipschitz in a neighborhood of points belonging to \(\partial B_i = \partial \Omega_i \cap \Omega\). Let \(x_0 \in \partial \Omega_i \cap \Omega\), then
from Claim 1, \( u_i(x_0) = 0 \). If \( x_0 \notin \partial \{ u_i > 0 \} \), then in a neighborhood of \( x_0 \), \( u_i \equiv 0 \) and of course it is Lipschitz there. On the other hand, if \( x_0 \in \partial \{ u_i > 0 \} \), then, since there exists an exterior \( \rho \)-tangent ball of radius 1 at any point of \( \partial \Omega_i \cap \Omega \) and \( u_i \) is harmonic inside its support, a barrier argument implies that there exist \( r_0, C > 0 \) such that \( 0 \leq u_i(x) = u_i(x) - u_i(x_0) \leq C|x - x_0| \) for any \( x \in B_{r_0}(x_0) \). This concludes the proof of (i).

This concludes the proof of the corollary.

\[ \square \]

6. A SEMICONVEXITY PROPERTY OF THE FREE BOUNDARIES

Let \((u_1, \ldots, u_K)\) be the limit of a convergent subsequence of \((u_1^\varepsilon, \ldots, u_K^\varepsilon)\), whose existence is guaranteed by Corollary 5.6. For \( i = 1, \ldots, K \), let us denote
\[
\text{(6.1)} \quad S(u_i) := \{ x \in \Omega : u_i > 0 \}.
\]
(In the next sections, for simplicity this set will be represented by \( S_i \).) Then the sets \( S(u_i) \) have the following semiconvexity property:

**Lemma 6.1.** Given \( S(u_i) \) consider
\[
T(u_i) = \{ x \in \Omega : d_\rho(x, S(u_i)) \geq 1 \}
\]
and
\[
S^*(u_i) = \{ x \in \Omega : d_\rho(x, T(u_i)) > 1 \}
\]
Then \( \partial S(u_i) \subset \partial S^*(u_i) \).

**Proof.** We have that \( S^*(u_i) \supset S(u_i) \). To prove the desired inclusion, for \( \sigma > 0 \) consider the sets
\[
S_\sigma(u_i) := \{ x \in \Omega : u_i > \sigma \},
\]
\[
T_\sigma(u_i) := \{ x \in \Omega : d_\rho(x, S_\sigma(u_i)) \geq 1 \}
\]
and
\[
S^*_\sigma(u_i) := \{ x \in \Omega : d_\rho(x, T_\sigma(u_i)) > 1 \}.
\]

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Notice that, the union of $\rho$-balls centered at points in $S_\sigma(u_i)$ coincides with the union of $\rho$-balls centered at points in $S^*_\sigma(u_i)$, i.e.

a) $(T_\sigma(u_i))^c = \bigcup B_1(x)$ for $x \in S_\sigma(u_i)$ and

b) $(T_\sigma(u_i))^c = \bigcup B_1(x)$ for $x \in S^*_\sigma(u_i)$.

If $x \in S_\sigma(u_i)$, from (i) of Corollary 5.6 we have that $d_\rho(x, \text{supp}\ f_j) > 1$ for $j \neq i$, and the locally uniform convergence of $u_\varepsilon^x$ to $u_i$ and Lemma 5.3 imply that, up to subsequences, $u_\varepsilon^x \leq C e^{-c_\alpha,\beta \varepsilon}$ in $B_1(x)$, where $2r = \min\{d_\rho(x, \text{supp}\ f_i), C(d_\rho(x, \text{supp}\ f_j) - 1)\}$. Now, the set where $u_\varepsilon^x$ decays is the same if we had considered $x \in S^*_\sigma(u_i)$, since from (a) and (b) we have

$$
\bigcup_{x \in S_\sigma(u_i)} B_1(x) = \bigcup_{x \in S^*_\sigma(u_i)} B_1(x).
$$

Therefore, $H(\varepsilon u_\varepsilon^x)$ goes to zero as $\varepsilon$ goes to zero in $S^*_\sigma(u_i)$. It follows that $\Delta u_i \equiv 0$ in $S^*_\sigma(u_i)$, if $S^*_\sigma(u_i)$ is not empty. Now, if $A$ is a connected component of $S_\sigma(u_i)$, then there exists a connected component $A^*$ of $S^*_\sigma(u_i)$ such that $A \subset A^*$. Since $u_i$ is harmonic and non-negative in $A^*$, the strong maximum principle implies that $u_i > 0$ in all $A^*$, that is $A^* \subset A$. We conclude that $A = A^*$. Therefore, any connected component of $S_\sigma(u_i)$ is equal to a connected component of $S^*_\sigma(u_i)$. Passing to the limit as $\sigma \to 0$, we obtain that any connected component of $S(u_i)$ is equal to a connected component of $S^*(u_i)$. In particular, $\partial S(u_i) = \partial S^*(u_i)$. \qed

From Lemma 6.1 we can conclude that the sets $S(u_i)$ have a tangent $\rho$-ball of radius 1 from outside at any point of the boundary, as stated in the following corollary.

**Corollary 6.2.** If $x_0 \in \partial S(u_i) \cap \Omega$ there is an exterior tangent ball, $B_1(y)$ at $x_0$, in the sense that for $x \in B_1(y) \cap B_1(x_0)$, all $u_j(x) \equiv 0$ (including $u_i$).

The following two lemmas about the distance function may be known in the literature and we provide the proof here for the reader’s convenience.
Lemma 6.3. Let $S$ be a closed set. Let $d_\rho(\cdot, S)$ denote the $\rho$-distance function from $S$. Then, in the set $\{x : d_\rho(x, S) > 0\}$, $d_\rho(\cdot, S)$ satisfies in the viscosity sense

$$\Delta d_\rho(\cdot, S) \leq \frac{C}{d_\rho(\cdot, S)},$$

where $C$ is a constant depending on $n$, $\|Dd_\rho(\cdot, S)\|_{L^\infty}$ and the constant $A$ given in (2.1).

Proof. We first prove that there exists a smooth tangent function from above at any point of the graph of $d_\rho(\cdot, S)$ in the set $\{d_\rho(\cdot, S) > 0\}$. For simplicity we will write $d_S(\cdot)$ instead of $d_\rho(\cdot, S)$. Let $y_0$ be a point in the complementary of $S$. Let $x \in \partial S$ be a point where $y_0$ realizes the distance from $S$. Assume, without loss of generality, that $x = 0$. Then $d_\rho(y_0, 0) = \rho(y_0) = d_S(y_0)$. In particular, the ball $B_{\rho(y_0)}(y_0)$ is contained in $S^c$ and tangent to $S$ at $0$. For any $y \in B_{\rho(y_0)}(y_0)$, we have that $d_S(y) \leq d_\rho(y, 0) = \rho(y)$, therefore the cone, graph of the function $y \to \rho(y)$, is tangent from above to the graph of $d_S(\cdot)$ at $(y_0, d_S(y_0))$.

Next, let $\psi$ be a test function touching from below $d_S(\cdot)$ at $y_0$, then $\psi$ touches from below the function $\rho(y)$ at $y_0$. In particular, $\Delta \psi(y_0) \leq \Delta \rho(y_0)$. Let us compute $\Delta \rho$. Using (2.1), we get

$$D^2(\rho) = \frac{1}{\rho} D^2 \left( \frac{1}{2\rho^2} \right) - \frac{D\rho \otimes D\rho}{\rho} \leq \frac{1}{\rho} (AI_n - D\rho \otimes D\rho),$$

which gives

$$\Delta \rho \leq \frac{C}{\rho}.$$

In particular,

$$\Delta \psi(y_0) \leq \frac{C}{\rho(y_0)} = \frac{C}{d_S(y_0)}.$$

This concludes the proof. \qed

Lemma 6.4. Let $S$ be a closed and bounded set. Let us denote by $d_\rho(\cdot, S)$ the $\rho$-distance function from $S$ and by $(S)_1$ the set at $\rho$-distance 1 from $S$. Then $(S)_1$ has finite perimeter.

Proof. For simplicity we will write $d_S(\cdot)$ instead of $d_\rho(\cdot, S)$, as in the previous lemma and first we present an heuristic proof integrating $\Delta d_S^2$ over the set $\{0 < d_S < 1\}$. Since $|Dd_S|$ is bounded,
from Lemma 6.3 we see that

\[ \Delta d_s^2 = 2|Dd_s|^2 + 2d_s \Delta d_s \leq C. \]

Therefore, integrating \( \Delta d_s^2 \), we get

\[
C \geq \int_{\{0 < d_s < 1\}} \Delta d_s^2 dx = \int_{\{d_s = 0\}} 2d_s Dd_s \cdot n d\mathcal{H}^{n-1} + \int_{\{d_s = 1\}} 2d_s Dd_s \cdot n d\mathcal{H}^{n-1}
\]

\[
= \int_{\{d_s = 1\}} 2Dd_s \cdot n d\mathcal{H}^{n-1} \geq c \int_{\{d_s = 1\}} d\mathcal{H}^{n-1} = c\mathcal{H}^{n-1}(\{d_s = 1\}),
\]

where \( n = Dd_s / |Dd_s| \) is the unit exterior normal. This provides an upper bound for \( \mathcal{H}^{n-1}(\{d_s = 1\}) \) and concludes the heuristic proof.

To make the argument precise, we need to correct the regularity problem over the boundary. For that, consider a smooth function \( \eta \) with compact support in \((0, 1)\) such that \( 0 \leq \eta(\xi) \leq \xi \) for any \( \xi \in [0, 1] \), \( \eta(\xi) = \xi \) for \( \xi \in [\delta, 1 - \delta] \), \( |\eta'| \leq c \) on \((0, 1 - \delta)\) and \( \eta'(\xi) \leq -c/\delta \) for \( \xi \in (1 - \delta, 1) \), where \( \delta > 0 \) is a small parameter. Then, in a weak sense we have

\[
(6.2) \quad \text{div}(\eta'(d_s)|Dd_s|) = \eta'(d_s)|Dd_s|^2 + \eta(d_s) \Delta d_s.
\]

Moreover, from Lemma 6.3 in the set \( \{0 < d_s < 1\} \) we have

\[
\eta(d_s) \Delta d_s \leq \eta(d_s) \frac{C}{d_s} \leq C
\]

in the viscosity sense and therefore in the distributional sense (see, e.g., [24] for the equivalence between viscosity solutions and weak solutions). Therefore, since \( \eta(d_s) \) is a function with compact support in \( \{0 < d_s < 1\} \), we get

\[
0 = \int_{\{0 < d_s < 1\}} \text{div}(\eta'(d_s)|Dd_s|) \leq \int_{\{0 < d_s < 1\}} \eta'(d_s)|Dd_s|^2 dx + C
\]

\[
= \int_{\{0 < d_s < 1 - \delta\}} \eta'(d_s)|Dd_s|^2 dx + \int_{\{1 - \delta < d_s < 1\}} \eta'(d_s)|Dd_s|^2 dx + C
\]

\[
\leq \int_{\{1 - \delta < d_s < 1\}} \eta'(d_s)|Dd_s|^2 dx + C
\]

\[
\leq -\frac{c}{\delta} \int_{\{1 - \delta < d_s < 1\}} |Dd_s|^2 dx + C.
\]

Now, using the coarea formula and the inequalities above, we get

\[
\frac{1}{\delta} \int_{1 - \delta}^1 \mathcal{H}^{n-1}(\{d_s = t\}) dt = \frac{1}{\delta} \int_{\{1 - \delta < d_s < 1\}} |Dd_s|^2 dx \leq C.
\]
Finally, taking the limit as $\delta \to 0^+$ and using the lower semicontinuity of the perimeter with respect to the convergence in measure, we infer that

$$\text{Per}(\{d_S = 1\}) \leq \liminf_{\delta \to 0^+} \frac{1}{\delta} \int_{1-\delta}^{1} \mathcal{H}^{n-1}(\{d_S = t\})dt \leq C.$$ 

This concludes the proof of the lemma. \hfill \Box

**Corollary 6.5.** The sets $S(u_i)$, $i = 1, \ldots, K$ have finite perimeter.

**Proof.** The corollary is an immediate consequence of Lemma 6.1 and Lemma 6.4. \hfill \Box

7. A sharp characterization of the interfaces

In Section 5 we proved that the supports of the limit functions $u_i$’s are at distance at least 1, one from each other (see Corollary 5.6). In this section we will prove that they are exactly at distance 1, as stated in the following theorem.

**Theorem 7.1.** Assume (2.8) with $p = 1$ in (2.5). Let $(u_1^\varepsilon, \ldots, u_K^\varepsilon)$ be a viscosity solution of the problem (2.4) and $(u_1, \ldots, u_K)$ the limit as $\varepsilon \to 0$ of a convergent subsequence. Let $x_0 \in \partial\{u_i > 0\} \cap \Omega$, then there exists $j \neq i$ such that

$$(7.1) \quad \overline{B_1(x_0)} \cap \partial\{u_j > 0\} \neq \emptyset.$$ 

**Proof.** It is enough to prove the theorem for a point $x_0$ for which $\partial S(u_i)$ has a tangent $\rho$-ball from inside, since such points are dense on $\partial S(u_i)$. Indeed, let $x$ be any point of $\partial S(u_i)$. Let us consider a sequence of points $(x_k)$ contained in $S(u_i)$ and converging to $x$ as $k \to \infty$. Let $d_k$ be the $\rho$-distance of $x_k$ from $\partial S(u_i)$. Then the $\rho$-balls $B_{d_k}(x_k)$ are contained in $S(u_i)$ and there exist points $y_k \in \partial S(u_i) \cap B_{d_k}(x_k)$ where the $x_k$’s realize the distance from $\partial S(u_i)$. The sequence $(y_k)$ is a sequence of points of $\partial S(u_i)$ that have a tangent $\rho$-ball from the inside and converges to $x$.

Next, remark that from (b) in Corollary 5.6 we have that $d_\rho(x_0, \text{supp } f_j) \geq 1$ for any $j \neq i$. If there is a $j$ such that $d_\rho(x_0, \text{supp } f_j) = 1$, then (7.1) is obviously true. Therefore, we can assume
that \( d_\rho(x_0, \text{supp } f_j) > 1 \) for any \( j \neq i \). Then, for small \( S > 0 \) we have that \( B_{1+S}(x_0) \cap \text{supp } f_j = \emptyset \) and from \((2.4)\), we know that

\[
\Delta u_j^\varepsilon \geq \frac{1}{\varepsilon^2} u_j^\varepsilon \sum_{k \neq j} H(u_k^\varepsilon) \quad \text{in } B_{1+S}(x_0).
\]

We divide the proof in two cases.

a) \( H(u)(x) = \int_{B_1(x)} u(y) \varphi(\rho(x-y)) \, dy \)

and

b) \( H(u)(x) = \sup_{y \in B_1(x)} u(y) \).

**Proof of case a):** Let \( S(u_i) = \{ x \in \Omega : u_i > 0 \} \) as in \((6.1)\). Let \( B_S \) be a small \( \rho \)-ball centered at \( x_0 \in \partial S(u_i) \). Then, as a measure, as \( \varepsilon \to 0 \), up to subsequence

\[
\Delta u_i^\varepsilon \big|_{B_S(x_0)} \longrightarrow \Delta u_i \big|_{B_S(x_0)}
\]

(that has strictly positive mass, since \( u_i \) is not harmonic in \( B_S(x_0) \)).

We bound by below

\[
\int_{B_{1+S}(x_0)} \sum_{j \neq i} \Delta u_j^\varepsilon \, dx \quad \text{by} \quad \int_{B_S(x_0)} \Delta u_i^\varepsilon \, dx.
\]

Indeed

\[
\varepsilon^2 \int_{B_S(x_0)} \Delta u_i^\varepsilon(x) \, dx = \sum_{j \neq i} \int_{B_S(x_0)} \int_{B_1(x)} u_i^\varepsilon(x) \varphi(\rho(x-y)) u_j^\varepsilon(y) \, dy \, dx
\]

\[
= \sum_{j \neq i} \int_{B_S(x_0)} \int_{B_1+s(x_0)} u_i^\varepsilon(x) \chi_{[0,1]}(\rho(x-y)) \varphi(\rho(x-y)) u_j^\varepsilon(y) \, dx \, dy
\]

\[
\leq \sum_{j \neq i} \int_{B_{2+S}(x_0)} \int_{B_1+s(x_0)} u_i^\varepsilon(x) \chi_{[0,1]}(\rho(x-y)) \varphi(\rho(x-y)) u_j^\varepsilon(y) \, dx \, dy
\]

\[
= \sum_{j \neq i} \int_{B_{1+S}(x_0)} \int_{B_1(y)} u_i^\varepsilon(x) \varphi(\rho(x-y)) u_j^\varepsilon(y) \, dx \, dy
\]

\[
\leq \varepsilon^2 \sum_{j \neq i} \int_{B_{1+S}(x_0)} \Delta u_j^\varepsilon(y) \, dy,
\]

where \( \chi_{[0,1]} \) is the indicator function of the set \([0,1]\).
Therefore, for any small positive $S$, taking the limit in $\varepsilon$ we get

$$\int_{B_{1+S}(x_0)} \sum_{j \neq i} \Delta u_j \geq \int_{B_S(x_0)} \Delta u_i > 0$$

which implies that there exists $j \neq i$ such that $u_j$ cannot be identical equal to zero in $B_{1+S}(x_0)$.

Since $S$ small is arbitrary, the result follows.

The case b) is more involved. We may assume $x_0 = 0$. Let $y_0$ be such that $B_\mu(y_0) \subset S(u_i)$ and $0 \in \partial B_\mu(y_0)$. By Corollary 6.2 we know that there exists a $\rho$-ball $B_1(y_1)$ such that $B_1(y_1) \cap S(u_i) = \emptyset$ and $0 \in \partial B_1(y_1)$.

Let us first prove two claims.

Claim 1: There exists $\mu' < \mu$ and $C_1 > 0$ such that in the annulus $\{\mu' < \rho(x-y_0) < \mu\}$ we have

$$u_i(x) \geq C_1 d_\rho(x, \partial B_\mu(y_0))$$

Since any $\rho$-ball $B$ satisfies the uniform interior ball condition, for any point $\bar{x} \in \partial B_\mu(y_0)$ there exists an Euclidian ball $B_{R_0}(z_0)$ of radius $R_0$ independent of $\bar{x}$ contained in $B_\mu(y_0)$ and tangent to $\partial B_\mu(y_0)$ at $\bar{x}$. Let $m > 0$ be the infimum of $u_i$ on the set $\{x \in B_\mu(y_0) \mid d(x, \partial B_\mu(y_0)) \geq R_0/2\}$, where $d$ is the Euclidian distance function, and let $\phi$ be the solution of

$$\begin{cases}
\Delta \phi = 0 & \text{in } \{\frac{R_0}{2} < |x - z_0| < R_0\} \\
\phi = 0 & \text{on } \partial B_{R_0}(z_0) \\
\phi = m & \text{on } \partial B_{\frac{R_0}{2}}(z_0)
\end{cases}$$

i.e., for $n \geq 3$,

$$\phi(x) = C(n)m\left(\frac{R_0^{n-2}}{|x - z_0|^{n-2}} - 1\right).$$

Since $u_i$ is harmonic in $B_\mu(y_0)$ and $u_i \geq \phi$ on $\partial B_{R_0}(z_0) \cup \partial B_{\frac{R_0}{2}}(z_0)$, by comparison principle

$$u_i \geq \phi \text{ in } \{\frac{R_0}{2} < |x - z_0| < R_0\}.$$  

In particular, for any $x \in \{\frac{R_0}{2} < |x - z_0| < R_0\}$ and belonging to the segment between $z_0$ and $\bar{x}$, using that $\phi$ is convex in the radial direction,

$$\frac{\partial \phi}{\partial \nu_i}|_{\partial B_{R_0}(z_0)} = \frac{C(n)(n-2)m}{R_0}$$
where $\nu_i$ is the interior normal at $\partial B_{R_0}(z_0)$, and (2.2), we get

$$u_i(x) \geq \frac{C(n)(n-2)m}{R_0} d(x, \partial B_{R_0}(z_0)) = C(n, R_0)m d(x, \partial B_\mu(y_0)) \geq C_1 d(\rho(x, \partial B_\mu(y_0)).$$

Therefore, letting $\bar{x}$ vary in $\partial B_\mu(y_0)$ we get

$$u_i(x) \geq C_1 d(\rho(x, \partial B_\mu(y_0))) \quad \text{for any } x \in B_\mu(y_0) \text{ with } d(x, \partial B_\mu(y_0)) \leq \frac{R_0}{2}.$$

Using (2.2), Claim 1 follows.

Next, let $e_0 = y_0/\rho(y_0)$ and fix $\sigma < \mu$ so small that $B_\sigma(\sigma e_0) \subset \{\mu' < \rho(x-y_0) < \mu\} \cap B_{1+\delta}(y_1)$.

For $r \in [\sigma - \nu, \sigma + \nu]$ and small $\nu < \sigma$, let us define

$$\bar{u}_i := \inf_{\partial B_r(\sigma e_0)} u_i \quad \text{and} \quad \bar{u}_i := \inf_{\partial B_r(\sigma e_0)} u_i.$$

Since for $r \in [\sigma, \sigma + \nu]$, $\partial B_r(\sigma e_0) \cap (S(u_i))^c \neq \emptyset$ and $u_i \equiv 0$ on $(S(u_i))^c$, we have

$$u_i = 0 \quad \text{for } r \in [\sigma, \sigma + \nu].$$

By Claim 1, we know that in $B_\sigma(\sigma e_0)$ we have

$$u_i(x) \geq C_1 d(\rho(x, \partial B_\mu(y_0))) \geq C_1 d(\rho(x, \partial B_\sigma(\sigma e_0))) = C_1(\sigma - \rho(x - \sigma e_0)).$$

We deduce that for $r \in [\sigma - \nu, \sigma]$

$$\bar{u}_i = \inf_{\partial B_r(\sigma e_0)} u_i \geq \inf_{\partial B_r(\sigma e_0)} C_1(\sigma - \rho(x - \sigma e_0)) = C_1(\sigma - r).$$

From the previous inequality and (7.3), we infer that

$$u_i \geq C_1(\sigma - r)^+, \quad r \in [\sigma - \nu, \sigma + \nu].$$

Next, for $j \neq i$, $r \in [\sigma - \nu, \sigma + \nu]$, let us define

$$\bar{u}_j := \sup_{\partial B_{1+r}(\sigma e_0)} u_j \quad \text{and} \quad \bar{u}_j := \sup_{\partial B_{1+r}(\sigma e_0)} u_j.$$
The functions $u_\varepsilon^i$ and $\bar{u}_\varepsilon^j$ are respectively solutions of

$$
\Delta_r u_\varepsilon^i \leq \frac{1}{\varepsilon^2} u_\varepsilon^i \sum_{i \neq j} \sup_{B_1(\bar{z}_r^i)} u_\varepsilon^j
$$

(7.5)

$$
\Delta_r \bar{u}_\varepsilon^j \geq \frac{1}{\varepsilon^2} \bar{u}_\varepsilon^j \sup_{B_1(\bar{z}_r^j)} u_\varepsilon^i
$$

where

$$\Delta_r u = u_{rr} + \left(\frac{n-1}{r}\right) u_r = \frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left( r^{n-1} \frac{\partial u}{\partial r} \right)$$

and $z_r^i$ and $\bar{z}_r^j$ are respectively the points where the infimum of $u_\varepsilon^i$ on $\partial B_r(\sigma e_0)$ and the supremum of $u_\varepsilon^j$ on $\partial B_{1+r}(\sigma e_0)$ are attained. Note that in spherical coordinates

$$\Delta u = \Delta_r u + \Delta_{\theta} u$$

and that if we are on a point where $u$ attains a minimum value in the $\theta$ for a fixed $r$ then $\Delta_{\theta} u \geq 0$ and the opposite inequality holds if we are on a maximum point. We also remark that

$$\bar{y}_r^j := \sigma e_0 + \frac{r}{r+1} (\bar{z}_r^j - \sigma e_0) \in \partial B_r(\sigma e_0) \cap \partial B_1(\bar{z}_r^j),$$

therefore

$$
\sup_{B_1(\bar{z}_r^j)} u_\varepsilon^i \geq u_\varepsilon^i(\bar{y}_r^j) \geq \bar{u}_\varepsilon^i.
$$

(7.6)

Moreover, since $B_1(\bar{z}_r^j) \subset B_{1+r}(\sigma e_0)$ and $u_\varepsilon^j$ is a subharmonic function, we have

$$
\sup_{B_1(\bar{z}_r^j)} u_\varepsilon^j \leq \sup_{B_{1+r}(\sigma e_0)} u_\varepsilon^j
$$

(7.7)

$$
= \sup_{\partial B_{1+r}(\sigma e_0)} u_\varepsilon^j
$$

$$
= \bar{u}_\varepsilon^j.
$$

From (7.5), (7.6) and (7.7), we conclude that

$$
\Delta_r \bar{u}_\varepsilon^j \leq \Delta_r \left( \sum_{j \neq i} \bar{u}_\varepsilon^j \right).
$$

(7.8)

In other words, for any $\phi \in C_c^\infty(\sigma - \upsilon, \sigma + \upsilon)$, $\phi \geq 0$, we have

$$
\int_{\sigma - \upsilon}^{\sigma + \upsilon} \bar{u}_\varepsilon^i \frac{\partial}{\partial r} \left( r^{n-1} \frac{\partial}{\partial r} \left( \frac{1}{r^{n-1}} \phi \right) \right) dr \leq \int_{\sigma - \upsilon}^{\sigma + \upsilon} \sum_{j \neq i} \bar{u}_\varepsilon^j \frac{\partial}{\partial r} \left( r^{n-1} \frac{\partial}{\partial r} \left( \frac{1}{r^{n-1}} \phi \right) \right) dr.
$$

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Passing to the limit as \( \varepsilon \to 0 \) along a uniformly converging subsequence, we get
\[
\int_{\sigma - \varepsilon}^{\sigma + \varepsilon} u_i \frac{\partial}{\partial r} \left( r^{n-1} \frac{\partial}{\partial r} \left( \frac{1}{r^{n-1} \phi} \right) \right) \, dr \leq \int_{\sigma - \varepsilon}^{\sigma + \varepsilon} \sum_{j \neq i} \bar{u}_j \frac{\partial}{\partial r} \left( r^{n-1} \frac{\partial}{\partial r} \left( \frac{1}{r^{n-1} \phi} \right) \right) \, dr .
\]
The linear growth of \( u_i \) away from the free boundary given by (7.3) and (7.4), implies that \( \Delta_r u_i \) develops a Dirac mass at \( r = \sigma \) and
\[
\int_{\sigma - \varepsilon}^{\sigma + \varepsilon} u_i \frac{\partial}{\partial r} \left( r^{n-1} \frac{\partial}{\partial r} \left( \frac{1}{r^{n-1} \phi} \right) \right) \, dr > 0,
\]
for \( \varepsilon \) small enough. Hence, \( \Delta_r (\sum_{j \neq i} \bar{u}_j) \) is a positive measure in \((\sigma - \varepsilon, \sigma + \varepsilon)\) and therefore there exists \( j \neq i \) such that \( u_j \) cannot be identically equal to zero in the ball \( B_{1+\sigma}(\sigma e_0) \). Since \( \sigma \) small is arbitrary, the result follows.

8. Classification of singular points and Lipschitz regularity in dimension 2

In this section we study singular points in dimension 2. We will always assume (2.8) with \( p = 1 \) in (2.5). From the results of the previous sections we know that the solutions \( u_1^{\varepsilon}, \ldots, u_K^{\varepsilon} \) of system (2.4), through a subsequence, converge as \( \varepsilon \to 0 \) to functions \( u_1, \ldots, u_K \) which are locally Lipschitz continuous in \( \Omega \) and harmonic inside their support. For \( i = 1, \ldots, K \), let us denote the interior of the support of \( u_i \) by \( S_i \) as in (6.1) and the union of the interior of the supports of all the other functions by

\[
C_i := \bigcup_{j \neq i} S_j .
\]

Since the sets \( S_i \) are disjoint we have \( \partial C_i = \bigcup_{j \neq i} \partial S_j \). From Theorem 7.1 we know that \( S_i \) and \( C_i \) are at \( \rho \)-distance 1, therefore for any point \( x \in \partial S_i \) there is a point \( y \in \partial C_i \) such that \( \rho(x - y) = 1 \). We say that \( x \) realizes \( y \) the distance from \( C_i \).

**Definition.** A point \( x \in \partial S_i \) is a **singular** point if it realizes the distance from \( C_i \) to at least two points in \( \partial C_i \). We say that \( x \in \partial S_i \) is a **regular** point if it is not singular.

Geometrically, we can describe regular and singular points as follows. Let \( x \in \partial S_i \) be a singular point and \( y_1, y_2 \in \partial C_i \) points where \( x \) realizes the distance from \( C_i \). Then the balls
$u_1 > 0$ \hspace{1cm} S_1 \hspace{1cm} \partial S_1 \hspace{1cm} x_0$

**Figure 2.** Asymptotic cone at $x_0$

$B_1(y_1)$ and $B_1(y_2)$ are tangent to $\partial S_i$ at $x$. Consider the convex cone determined by the two tangent lines to the two tangent $\rho$-balls $B_1(y_1)$ and $B_1(y_2)$, which does not intersect the two $\rho$-balls. The intersection of all cones generated by all $\rho$-balls of radius 1, tangent at $x$ and with center in $C_i$ defines a convex asymptotic cone centered at $x$, see Figure 2. If $x \in \partial S_i$ is a regular point, then there is only one point $y \in \partial C_i$ where $x$ realizes the distance from $C_i$. In this case, the two tangent balls coincide and therefore, by definition the asymptotic cone at $x \in \partial S_i$ is an half-plane. We will show that at regular points $\partial S_i$ is the graph of a differentiable function.

If $\theta \in [0, \pi]$ is the opening of the cone at $x$, we say that $S_i$ has an angle $\theta$ at $x$. Regular points correspond to $\theta = \pi$. When $\theta = 0$ the tangent cone is actually a semi-line and $S_i$ has a cusp at $x$. We will show, later on in this section, that, assuming additional hypothesis on the boundary data and the domain $\Omega$, the case $\theta = 0$ never occurs and therefore the free boundaries are Lipschitz curves of the plane.

**Lemma 8.1.** Let $C = \{(x_1, x_2) : x_2 \geq \alpha |x_1|\}$, $\alpha \geq 0$, be the asymptotic cone of $S_i$ at $0 \in \partial S_i$. Then there exist $y_1, y_2 \in \partial C_i$ such that the balls $B_1(y_1)$ and $B_1(y_2)$ are tangent respectively to the lines $x_2 = \pm \alpha x_1$ at $0$.

**Proof.** Let $y_1, y_2 \in B_1(0)$ be such that $x_2 = \alpha x_1$ is a tangent line to $B_1(y_1)$ at $0$ and $x_2 = -\alpha x_1$ is a tangent line to $B_1(y_2)$ at $0$. Suppose by contradiction that $y_1, y_2 \notin \partial C_i$. Then, any $y \in C_i$ such
that \( \rho(y-0) = 1 \) must lie in the smaller arc in \( \partial B_1(0) \) between \( y_1 \) and \( y_2 \). Moreover, there exists \( \delta > 0 \) such that all \( \rho \)-balls \( B_1(y) \) have at most as tangent lines at \( 0 \) the lines \( x_2 = \pm(\alpha - \delta)x_1 \). Then the asymptotic cone at \( 0 \) must contain the cone \( \{ (x_1, x_2) : x_2 \geq (\alpha - \delta)|x_1| \} \), which is not possible. \( \square \)

**Lemma 8.2.** Assume that \( S_i \) has an angle \( \theta \in (0, \pi] \) at \( x_0 \in \partial S_i \). Then, there exists a neighborhood \( U \) of \( x_0 \), a system of coordinates \( (x_1, x_2) \) and a locally Lipschitz function \( \psi : (\mathbb{R}, r) \rightarrow \mathbb{R} \), for some \( r > 0 \), such that in the system of coordinates \( (x_1, x_2) \), we have that \( x_0 = (0, 0) \) and

\[
\partial S_i \cap U = \{ (x_1, \psi(x_1) : x_1 \in (-r, r) \}.
\]

If in addition \( \theta = \pi \), then \( \varphi \) is differentiable at \( 0 \).

**Proof.** Let \( C \) be the convex asymptotic cone of \( S_i \) at \( x_0 \). Let us fix a system of coordinates \( (x_1, x_2) \) such that the \( x_2 \) axis coincides with the axis of the cone and is oriented such that the cone is above the \( x_1 \) axis. Then we have that \( x_0 = (0, 0) \) and \( C = \{ (x_1, x_2) : x_2 \geq \alpha|x_1| \} \) with \( \alpha = \tan(\frac{\pi - \theta}{2}) \). To prove that in this system of coordinates, \( \partial S_i \) is the graph of a function in a small neighborhood of \( x_0 \), it suffices to show that there exists a small \( r > 0 \) such that, for any \( |t| < r \), the vertical line \( \{ x_1 = t \} \), intersects \( \partial S_i \cap B_r(0) \) at only one point. Suppose by contradiction that there exists a sequence \( (t_n) \) such that \( t_n \rightarrow 0 \) as \( n \rightarrow +\infty \), and the line \( \{ x_1 = t_n \} \) intersects \( \partial S_i \cap B_r(0) \) at two distinct points \( (t_n, a_n) \) and \( (t_n, b_n) \) with \( b_n > a_n \). Assume, without loss of generality, that \( t_n > 0 \) for any \( n \). By Lemma 8.1 there exist \( y_1, y_2 \in \partial C_i \) that realize the distance from \( 0 \), and such that \( B_1(y_1) \) is tangent to the line \( \{ (x_1, x_2) : x_2 = \alpha x_1 \} \) at \( 0 \) and \( B_1(y_2) \) is tangent to \( \{ (x_1, x_2) : x_2 = -\alpha x_1 \} \) also at \( 0 \). For instance, in the particular case of the Euclidean norm, we would have \( y_1 = \left( \sqrt{\frac{1}{1 + \alpha^2}}, -\alpha \sqrt{\frac{1}{1 + \alpha^2}} \right) \) and \( y_2 = \left( -\sqrt{\frac{1}{1 + \alpha^2}}, -\alpha \sqrt{\frac{1}{1 + \alpha^2}} \right) \). In general, what we can say is that the \( x_2 \) coordinate of \( y_1 \) and \( y_2 \) is a negative value \( -c \). We have that \( B_1(y_1) \cap B_1(y_2) = \emptyset \), since \( \theta > 0 \). Moreover, \( S_i \cap (B_1(y_1) \cup B_1(y_2)) = \emptyset \). Then, both points \( (t_n, a_n) \) and \( (t_n, b_n) \) must be above \( B_1(y_1) \cup B_1(y_2) \).
for $n$ large enough. Next, let $y_n^a, y_n^b \in \partial C_i$ be points where $(t_n, a_n)$ and $(t_n, b_n)$, respectively, realize the distance from $C_i$. Then the $\rho$-balls $B_1(y_n^a)$ and $B_1(y_n^b)$ are exterior tangent balls to $\partial S_i$ at $(t_n, a_n)$ and $(t_n, b_n)$, respectively. Recall that the $\rho$-distance between the points $(t_n, a_n)$ and $(t_n, b_n)$ converges to 0 as $n \to +\infty$, and so, the point $y_n^a$ has to belong to the lower half $\rho$-ball $\partial B_1(t_n, a_n) \cap \{x_2 < a_n\}$ for $n$ large enough. Indeed, if not the tangent $\rho$-ball $B_1(y_n^a)$ would contain $(t_n, b_n)$ for $n$ large enough. Similarly, $y_n^b$ has to belong to the upper half $\rho$-ball $\partial B_1(t_n, b_n) \cap \{x_2 > b_n\}$ for $n$ large enough. This implies that the tangent $\rho$-ball $B_1(y_n^a)$ will converge to a tangent ball to $S_i$ at 0, $B_1(y^b)$, with $y^b \in \{x_2 \geq 0\}$. On the other hand, by the definition of the asymptotic cones, all the centers of the tangent balls at 0 must belong to the set $\partial B_1(0) \cap \{x_2 \leq -c\}$, where $-c < 0$ is the $x_2$ coordinate of the points $y_1, y_2$ defined above. Therefore, we have reached a contradiction. We infer that there exists $r > 0$ such that $\partial S_i$ is the graph of a function $\psi : (-r, r) \to \mathbb{R}$. Since $\partial S_i$ is a closed set, $\psi$ is continuous.

Let us prove that $\psi$ is Lipschitz continuous at 0. If $C = \{x_2 \geq \alpha |x_1|\}$ is the tangent cone of $S_i$ at $x_0$ in the system of coordinates $(x_1, x_2)$, then for $r > 0$ small enough we have

$$\{x_2 \geq 2\alpha |x_1|\} \subset S_i \cap B_r(0) \subset \left\{x_2 \geq \frac{\alpha}{2} |x_1|\right\},$$

that is, for $|x_1| < r$,

$$\frac{\alpha}{2} |x_1| \leq \psi(x_1) = \psi(x_1) - \psi(0) \leq 2\alpha |x_1|.$$

Therefore, $\psi$ is Lipschitz at 0.

Next, assume that $\theta = \pi$. Then, we have that $y_1 = y_2$, and $x_0$ is a regular point. Therefore, $B_1(y_1) \subset \{x_2 < 0\}$ is the unique tangent ball to the graph of $\psi$ at $x_0 = (0, 0)$. Moreover, the tangent cone is the half plane $\{x_2 \geq 0\}$. Let us show that $\psi$ is differentiable at 0. Assume by contradiction that there exists a sequence of positive points $(x_1^n) \in (-r, r)$ such that $x_1^n \to 0$ as $n \to +\infty$ and

$$\lim_{n \to +\infty} \frac{\psi(x_1^n)}{x_1^n} = \beta \neq 0.$$

(8.2)
Since there exists a tangent ball from below to the graph of $\psi$ at 0 contained in $\{x_2 < 0\}$, we must have $\beta > 0$. For any point $(x^n_1, \psi(x^n_1)) \in \partial S_i$ there exists a point $y_n \in \partial C_i$ such that $B_1(y_n)$ is tangent to $S_i$ at $(x^n_1, \psi(x^n_1))$. Let $y_2 \in \partial C_i$ be the limit of a converging subsequence of $(y_n)$. Then the $\rho$-ball $B_1(y_2)$ is an exterior tangent ball at $S_i$ at 0. Equation (8.2) gives $\psi(x^n_1) \geq \frac{\beta}{2} x^n_1$ for $n$ large enough, i.e., the points $(x^n_1, \psi(x^n_1))$ of the free boundary are above the line $\{x_2 = \beta/2|x_1|\}$. This implies that $y_1 \neq y_2$, that is the limit $\rho$-ball $B_1(y_2)$ must be different from $B_1(y_1)$. This is in contradiction with the fact that $x_0$ is a regular point. Therefore we must have

$$\lim_{x_1 \to 0^+} \frac{\psi(x_1)}{x_1} = 0.$$ 

Similarly, one can prove that

$$\lim_{x_1 \to 0^-} \frac{\psi(x_1)}{x_1} = 0.$$ 

We conclude that $\psi$ is differentiable at 0 and $\psi'(0) = 0$.

\[\square\]

**Lemma 8.3.** Assume that there exists an open set $U$ of $\mathbb{R}^2$ such that any point of $U \cap \partial S_i$ is regular. Then $U \cap \partial S_i$ is a $C^1$-curve of the plane.

**Proof.** Let $y_0 \in \partial S_i \cap U$. By Lemma 8.2 there exists a differentiable function $\psi$ and a small $r > 0$, such that, in the system of coordinates $(x_1, x_2)$ centered at $y_0$ and with the $x_2$ axis in the direction of the inner normal of $\partial S_i$ at $y_0$, $\partial S_i \cap B_r(y_0)$ is the graph of $\psi$. Moreover, in this system of coordinates, $\psi(y_0) = \psi'(y_0) = 0$. By Corollary 6.2 there exists a tangent ball from below, with uniform radius, at any point of the graph of $\psi$. This implies that for any $|x_1^0| < r$, there exists a $C^2$ function $\varphi_{x_1^0}$ tangent from below to the graph of $\psi$ at $x_1^0$ and such that $|\varphi_{x_1^0}''| \leq C$, for some $C > 0$ independent of $x_1^0$. Therefore we have, for any $|x_1| < r$,

$$\psi(x_1) \geq \varphi_{x_1^0}(x_1) \geq \varphi_{x_1^0}(x_1^0) + \varphi'_{x_1^0}(x_1^0)(x_1 - x_1^0) - C|x_1 - x_1^0|^2$$

$$= \psi(x_1^0) + \psi'(x_1^0)(x_1 - x_1^0) - C|x_1 - x_1^0|^2.$$
Now, let us show that $\psi$ is of class $C^1$. Fix a point $x_1^0$ and consider a sequence $(x_1^l)$ converging to $x_1^0$ as $l \to +\infty$. Let $p$ be the limit of a convergent subsequence of $(\psi'(x_1^l))$. Passing to the limit in $l$ the inequality,

$$\psi(x_1) \geq \psi(x_1^l) + \psi'(x_1^l)(x_1 - x_1^l) - C|x_1 - x_1^l|^2,$$

we get

$$\psi(x_1) \geq \psi(x_1^0) + p(x_1 - x_1^0) - C|x_1 - x_1^0|^2,$$

for any $|x_1| < r$. Since $\psi$ is differentiable at $x_1^0$, we must have $p = \psi'(x_1^0)$. \hfill \Box

**Lemma 8.4.** Assume that the supports of the boundary data, $f_i$'s, on $(\partial \Omega)_1$ have a finite number of connected components. Then the sets $S_i$'s have a finite number of connected components.

*Proof.* Consider all the connected components of $S_i$, $S_i^j$, $i = 1, \ldots, K$ and $j = 1, 2, 3, \ldots$. Remark that for any $i$ and $j$

$$\partial S_i^j \cap \{x \in (\partial \Omega)_1 : f_i(x) > 0\} \neq \emptyset.$$

Indeed, if not we would have $u_i = 0$ on $\partial S_i^j$ and $\Delta u_i \geq 0$ in $S_i^j$. The maximum principle then would imply $u_i \equiv 0$ in $S_i^j$, which is not possible. Moreover, by continuity, $\partial S_i^j$ must contain one connected component of the set $\{x \in (\partial \Omega)_1 : f_i(x) > 0\}$. For this reasons we say that the components of $S_i$ reach the boundary of $\Omega$. This implies that the connected components of $S_i$ are finite. \hfill \Box

### 8.1. Properties of singular points.

We start by proving three lemmas that will allow to estimate the growth of the solutions near the singular points. The first lemma claims that positive functions which are superharmonic (subharmonic) in a cone and vanish on its boundary, have at least (at most) linear growth away from the boundary of the cone far from the vertex, with a slope that degenerate in a Hölder fashion approaching the vertex. The power just depends on the opening of the cone. The second and third lemmas generalize these estimates to domains which are sets of points at $\rho$-distance greater than 1 from a closed bounded set. Then we prove
that the set of singularities is a set of isolated points and we give a characterization. For the
set $S_i$ which has finite perimeter, we denote by $\partial^* S_i$ the reduced boundary, that is the set of
points whose blow-ups converge to half-planes and the essential boundary, $\partial_0 S_i$, are all points
except points of Lebesgue density zero and one. Moreover, $\mathcal{H}^1(\partial_0 S_i \setminus \partial^* S_i) = 0$. For more
details see \cite{122}.

**Lemma 8.5.** Let $v$ be a nonnegative Lipschitz function defined on $B_1 \subset \mathbb{R}^n$, such that $\Delta v$ is
locally a Radon measure on $B_1$ and such that $v$ smooth on $S = \{ v > 0 \}$. Assume that $S$ is a
set of finite perimeter. Then, for every smooth $\phi$ with compact support contained in $B_1$
\[
\int_{B_1} \Delta v \phi = \int_S \Delta v \phi dx - \int_{\partial^* S} \frac{\partial v}{\partial \nu_S} \phi d\mathcal{H}^{n-1},
\]
where $\nu_S$ is the measure-theoretic outward unit normal and $\partial^* S$ is the reduced boundary.

**Proof.** As a distribution and integrating by parts
\[
\int_{B_1} \Delta v \phi = \int_S v \Delta \phi dx = \int_S \text{div}(v \nabla \phi) - \text{div}(\nabla v \phi) + \Delta v \phi dx.
\]
Applying the generalized Gauss-Green theorem (see \cite{7}, and also \cite{122} for more details) we
obtain the result. \hfill \Box

**Lemma 8.6.** Let $\theta_0 \in (0, \pi]$. Let $C$ be the cone defined in polar coordinates by
\[
C = \{(\rho, \theta) \mid \rho \in [0, +\infty), \ 0 \leq \theta \leq \theta_0\}.
\]
Let $u_1$ and $u_2$ be respectively a superharmonic and subharmonic positive function in the interior
of $C \cap B_{2r_0}$, such that $u_1 \geq u_2 = 0$ on $\partial C \cap B_{2r_0}$. Then for any $r < r_0/3$ there exist $R = R(\theta_0, r)$,
and constants $c, C > 0$ depending on respectively $(\theta_0, u_1, r_0)$ and $(\theta_0, u_2, r_0)$, but independent of
$r$, such that for any $x \in [r, 3r] \times [0, R]$ we have
\begin{enumerate}
\item $u_1(x) \geq cr^\alpha d(x, \partial C)$
\item $u_2(x) \leq Cr^\alpha d(x, \partial C)$
\end{enumerate}
where \( \alpha \) is given by
\[
1 + \alpha = \frac{\pi}{\theta_0}.
\]

Proof. Let us introduce the function
\[
(8.3) \quad v(\varrho, \theta) := \varrho^{1+\alpha} \sin((1 + \alpha)\theta).
\]
Notice that \( v \) is harmonic in the interior of \( C \), since it is the imaginary part of the function \( z^{1+\alpha} \), where \( z = x + iy \), which is holomorphic in the set \( \mathbb{C} \setminus (-\infty, 0] \). Moreover \( v \) is positive inside \( C \) and vanishes on its boundary. By a barrier argument, \( u_1 \) has at least linear growth away from the boundary of \( C \), meaning for \( \rho \in [r_0/2, 3r_0/2] \) (far from the vertex and from \( \partial B_{2r_0} \))
\[
 u_1(x) \geq kd(x, \partial C),
\]
for \( k = c_0 \min_{d(x, \partial C) \geq s_0} u_1 \), and for \( x \in \{ x \in C : r_0/2 < |x| < 3r_0/2, d(x, \partial C) \leq s_0 \} \) where \( c_0 \) and \( s_0 \) depend on \( r_0 \) and \( \theta_0 \). Therefore, we can find a constant \( c > 0 \) depending on \( u_1, r_0 \) and \( \theta_0 \), such that
\[
 u_1 \geq cv \quad \text{on} \quad C \cap \partial B_{r_0}.
\]
Since in addition \( u_1 \geq cv = 0 \) on \( \partial C \cap B_{r_0} \), the comparison principle implies
\[
(8.4) \quad u_1 \geq cv \quad \text{in} \quad C \cap B_{r_0}.
\]
Since \( v \) is increasing in the radial direction and if we are near \( \partial C \) it is also increasing in the \( \theta \) direction, for \( r \leq |x| \leq 3r \), with \( r \) such that \( r \leq \frac{\pi r_0}{3} \) and \( d(x, C) \leq R \) with \( R = r \min \left\{ 1, \tan \left( \frac{\theta_0}{2} \right) \right\} \),
\[
 u_1(x) \geq cv(x) \geq Cr^\alpha d(x, \partial C)
\]
and (a) follows.

To prove (b) similarly, we have
\[
(8.5) \quad u_2 \leq Cv \quad \text{in} \quad C \cap B_{r_0},
\]
where $C$ depends on $(\theta_0, u_2, r_0)$ but it is independent of $r$. In particular, for $r \leq |x| \leq 3r$ and $d(x, C) \leq \frac{R}{2}$

$$u_2(x) \leq Cv(x) \leq \tilde{C}r^\alpha d(x, \partial C).$$

\[\square\]

**Lemma 8.7.** Let $\Omega$ be an open set, $C$ be a closed subset of $\Omega$ and $S = \{x \in \Omega \mid d_\rho(x, C) \geq 1\}$. Let $S_1$ be a connected component of $S$. Assume that $\partial S_1 = \Gamma_1 \cup \Gamma_2$, with $\Gamma_1 \cap \Gamma_2 = \{0\}$ and $S_1$ has an angle $\theta_0 \in (0, \pi]$ at 0 $\in \partial S_1$. Let $u_1$ be a superharmonic positive function in $S_1 \cap B_{2r_0}(0)$, with $u_1 = 0$ on $\partial S_1 \cap B_{2r_0}(0)$. Then, there exists a sequence $(x_h) \subset \Gamma_1$ of regular points convergent to zero, $x_h \to 0$ as $h \to 0$, and there exist balls $B_{R_h}(z_h) \subset S_1$ tangent to $\partial S_1$ at $x_h$, where $R_h \geq c|x_h|$, such that

$$u_1(x) \geq cR_h^{\alpha_\delta} d(x, \partial B_{R_h}(z_h)) \quad \text{for any} \quad x \in B_{R_h}(z_h) \setminus B_{\frac{R_h}{4}}(z_h),$$

where $\alpha_\delta$ is given by

$$1 + \alpha_\delta = \frac{\pi}{\theta_0 - \delta}.$$

**Proof.** Since $\theta_0 \in (0, \pi]$ for any $0 < \delta < \theta_0$, there exist $r_\delta > 0$ and a cone $C^1_\delta$ centered at 0 with opening $\theta_0 - \delta$ such that

$$C^1_\delta \cap B_{r_\delta}(0) \subset S_1 \cap B_{r_\delta}(0).$$

Take a sequence of points $t_h \in \partial C^1_\delta \cap B_{r_\delta}(0)$ converging to 0 as $h \to 0$. Let

$$r_h := d(t_h, 0) \quad \text{and} \quad R_h := r_h \min \left\{1, \tan \left(\frac{\theta_0 - \delta}{2}\right)\right\}.$$

Then, for $h$ small enough, there exist balls $B_{R_h}(s_h) \subset C^1_\delta \cap B_{r_\delta}(0)$ such that $t_h \in \partial B_{R_h}(s_h)$. Consider a system of polar coordinates $(\varrho, \theta)$ centered at 0. Moving the balls $B_{R_h}(s_h)$ along the $\theta$ direction until it touches $\Gamma_1$, we can find a sequence of regular points $x_h$ in that region, such that $d(x_h, 0) \leq cr_h$ and balls $B_{R_h}(z_h) \subset S_1 \cap B_{r_\delta}(0)$ such that $x_h \in \partial B_{R_h}(z_h)$. Observe that the center of the ball, $z_h$, remains inside the cone $C^1_\delta$, that is, for $h$ and $\delta$ small enough, we
have that $z_h \in C^1_\delta$ and $d(z_h, \partial C^1_\delta) \geq \frac{R_h}{2}$. Let us introduce the barrier function
\[ \phi(x) := \frac{m}{\log 4} \log \left( \frac{R_h}{|x - z_h|} \right), \quad \text{where} \quad m = \inf_{\partial B_{\frac{R_h}{4}}(z_h)} u_1. \]

Then $\phi$ satisfies
\[
\begin{cases}
\Delta \phi = 0 & \text{in } B_{R_h}(z_h) \setminus B_{\frac{R_h}{4}}(z_h) \\
\phi = 0 & \text{on } \partial B_{R_h}(z_h) \\
\phi = m & \text{on } \partial B_{\frac{R_h}{4}}(z_h).
\end{cases}
\]

Since $u_1 \geq \phi$ on $\partial B_{R_h}(z_h) \cup \partial B_{\frac{R_h}{4}}(z_h)$ the comparison principle then implies
\[ u_1 \geq \phi \quad \text{in } B_{R_h}(z_h) \setminus B_{\frac{R_h}{4}}(z_h). \]

If $\nu_1$ is the inner normal vector of $B_{R_h}(z_h)$, then for $x \in \partial B_{R_h}(z_h)$,
\[ \frac{\partial \phi}{\partial \nu_1}(x) = \frac{m}{R_h \log 4}. \]

and the convexity of $\phi$ in the radial direction gives, for any $x \in B_{R_h}(z_h) \setminus B_{\frac{R_h}{4}}(z_h)$
\[ u_1(x) \geq \frac{m}{R_h \log 4} d(x, \partial B_{R_h}(z_h)). \]

Let us estimate $m$. Since $d(z_h, \partial C^1_\delta) \geq \frac{R_h}{2}$, we have that $d(x, \partial C^1_\delta) \geq \frac{R_h}{4}$ for any $x \in B_{\frac{R_h}{4}}(z_h)$. As in Lemma 8.6, consider the harmonic function $v(x)$, introduced in (8.3), defined on the cone $C^1_\delta (\alpha = \alpha_\delta)$ and the comparison principle result stated in (8.4). Then
\[ m \geq c \min_{\partial B_{\frac{R_h}{4}}(z_h)} v \geq \min \left\{ v \left( r_h - \frac{R_h}{4}, \frac{\theta_0 - \delta}{8} \right), v \left( \frac{3r_h}{4}, \frac{\pi}{16} \right) \right\} = c_1 \left( \frac{3r_h}{4} \right)^{\alpha_\delta + 1} \]

where $c_1 = c_1(u_1, r_s, \theta_0 - \delta)$. Then, since $\frac{r_h}{R_h} \geq 1$ we conclude that for any $x \in B_{R_h}(z_h) \setminus B_{\frac{R_h}{4}}(z_h)$,
\[ u_1(x) \geq c R_h^{\alpha_\delta} d(x, \partial B_{R_h}(z_h)). \]

This concludes the proof of the lemma. \(\square\)

**Lemma 8.8.** Let $\Omega$ be an open set, $C$ be a closed subset of $\Omega$ and $S = \{ x \in \Omega \mid d_\rho(x, C) \geq 1 \}$. Let $S_1$ be a connected component of $S$. Assume that $S_1$ has an angle $\theta_0 \in [0, \pi]$ at $0 \in \partial S_1$. Let $u_2$ be a subharmonic positive function in $S_1 \cap B_{2r_0}(0)$, with $u_2 = 0$ on $\partial S_1 \cap B_{2r_0}(0)$. 

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Then, for any $0 < \delta < \theta_0$, there exists $r_\delta > 0$ such that for any $r < r_\delta / 5$ there exist

$$R = R(\theta_0, r),$$

and a constant $C > 0$ depending on $(\theta_0 + \delta, u_2, r_\delta)$, but independent of $r$, such that

$$u_2(x) \leq C r^{\beta_\delta} d(x, \partial S_1) \quad \text{for any } x \in (B_{3r}(0) \setminus B_r(0)) \cap \left\{ x \in S_1 : d(x, \partial S_1) \leq \frac{R}{4} \right\}$$

where $\beta_\delta$ is given by

$$1 + \beta_\delta = \frac{\pi}{\theta_0 + \delta}.$$ 

Proof. For any $\delta > 0$, there exist $r_\delta > 0$, a cone $C^2_\delta$ centered at 0 and with opening $\theta_0 + \delta$ such that

$$S_1 \cap B_{r_\delta}(0) \subset C^2_\delta \cap B_{r_\delta}(0).$$

Take any $r < r_\delta$ and let $y \in \partial S \cap (B_{3r}(0) \setminus B_r(0))$ and $r_y := d(y, 0) \in (r, 3r)$. Since $S$ is at $\rho$-distance 1 from $C$, for any point of the boundary of $S_1$ there exists an exterior tangent $\rho$-ball of radius 1. This implies that for $r$ small enough, there exists $w_y$ such that the Euclidian ball $B_{R_y}(w_y)$ is contained in the complement of $S$ and $y \in \partial B_{R_y}(w_y)$, where $R_y$ is defined by

$$R_y = r_y \min \left\{ 1, \tan \left( \frac{\theta_0 + \delta}{2} \right) \right\}.$$

Let us take now as barrier the function

$$\psi(x) := \frac{M}{\log \frac{3}{2}} \log \left( \frac{|w_y - x|}{R_y} \right) \quad \text{with } M = \sup_{\partial B_{\frac{3}{2}R_y}(w_y)} u_2.$$

Then $\psi$ satisfies

$$\begin{cases}
\Delta \psi = 0 & \text{in } B_{\frac{3}{2}R_y}(w_y) \setminus B_{R_y}(w_y) \\
\psi = M & \text{on } \partial B_{\frac{3}{2}R_y}(w_y) \\
\psi = 0 & \text{on } \partial B_{R_y}(w_y).
\end{cases}$$

Using the comparison principle with $u_2$, the concavity of $\psi$ in the radial direction gives that for any $x \in B_{\frac{3}{2}R_y}(w_y) \setminus B_{R_y}(w_y)$

$$u_2 \leq \frac{M}{R_y \log \left( \frac{3}{2} \right)} d(x, \partial B_{R_y}(w_y)).$$
Let us estimate $M$. Consider again a system of polar coordinates $(\rho, \theta)$ centered at 0 and the harmonic function $v(x)$, introduced in (8.3), defined on the cone $C^{2}_\alpha (\alpha = \beta_\delta)$. By definition of $v$, $R_y$, and taking into account (8.5), for $\delta$, $r$ small enough,

$$M \leq C \max_{\partial B_{3R_y}(w_y)} v \leq C v \left( \frac{4r_y}{2}, \frac{\theta_0 + \delta}{2} \right) = C_1 (4r_y)^{\beta_\delta + 1} = C_1 r_y^{\beta_\delta} \min \{1, \tan \left( \frac{\theta_0 + \delta}{2} \right) \}$$

we see that for any $x \in B_{3R_y}(w_y) \setminus B_{R_y}(w_y)$ and belonging to the segment $y + s(y - w_y)$, $s \in (0, \frac{1}{2})$, we have

$$u_2(x) \leq C M d(x, \partial B_{R_y}(w_y)) = C M d(x, \partial S_1) \leq C_1 r_y^{\beta_\delta} d(x, \partial S_1).$$

Letting the tangent ball moving along $\partial S_1 \cap (B_{3r_y}(0) \setminus B_{r_y}(0))$, we get (b).

□

**Lemma 8.9.** Assume (2.8) with $n = 2$ and $p = 1$ in (2.5). Assume in addition that the supports on $\partial \Omega$ of the boundary data $f_i$’s have a finite number of connected components. Let $(u_1^\varepsilon, \ldots, u_K^\varepsilon)$ be a viscosity solution of the problem (2.4) and $(u_1, \ldots, u_K)$ the limit as $\varepsilon \to 0$ of a convergent subsequence. The set of singular points of $\Omega$ is a set of isolated points.

**Proof.** Suppose by contradiction that there exists a sequence of distinct singular points $(y_k)_{k \in \mathbb{N}}$ such that $y_k \in \partial S_j$ and $y_k \to y \in \Omega$ as $k \to +\infty$. Since by Lemma 8.4 the connected components of the sets $S_i$, $i = 1, \ldots, K$ are finite, we may assume without loss of generality that the points $y_k$ belong to the same connected component of $S_j$, which we denote by $S^1_j$. If there exists $\theta_{\text{max}} < \pi$ such that $S^1_j$ has an angle smaller than $\theta_{\text{max}}$ at $y_k$ for any $k$, then, there exists $K$ such that starting from $y_K$, after a finite number of singular points $S^1_j$ would be an isle and not reach the boundary. Therefore we would have $u_j = 0$ on $\partial S^1_j$ and $\Delta u_j = 0$ in $S^1_j$, and the maximum principle would imply $u_j \equiv 0$ in $S^1_j$, which is a contradiction. We infer that there exists a $k \in \mathbb{N}$ such that the angle at $y_k$ is close to $\pi$. In particular, if $x^k_1$ and $x^k_2$ are points in $C_j$ that realize the $\rho$-distance from $S_j$ at $y_k$, then $\rho$-distance between $x^k_1$ and $x^k_2$ is less than one.
Next, suppose that $x_k^i$ and $x_k^j$ belong to the same connected component of $S_i$, for some $i \neq j$. Then, by Theorem 7.1 we know that $\partial S_i \cap B_1(y_k)$ has to contain the arc of the unit $\rho$-ball between $x_k^i$ and $x_k^j$. If not, there would be points in the curve connecting $x_k^i$ and $x_k^j$ which do not realize the distance from $C_i$. Any point inside this arc is a regular point at $\rho$-distance 1 from $y_k$. Consider any of them, for instance the middle point of the arc, denoted by $x_k$. We want to compare the mass of the Laplacian of $u_i$ at $x_k$ with the mass of the Laplacian at $u_j$ at $y_k$, across the free boundaries. Let us first assume $H$ defined as in (2.5). For $\sigma < \frac{1}{8}d_\rho(x_k^i, x_k^j)$ let us define

$$D_\sigma(x_k) := \{x \in B_\sigma(x_k) \mid d(x, \partial C_i) \leq \sigma^2\},$$

where $C_i$ is the asymptotic cone to $S^1_i$ at $x_k$. Note that, since $x_k$ is a regular point, $\partial C_i$ is the tangent line to $\partial S^1_i$ at $x_k$, and so $C_i$ has opening $\pi$. Let $(D_\sigma(x_k))_1$ be the set of points at $\rho$-distance less than 1 from $D_\sigma(x_k)$, then we have that

(8.8)\[ \int_{D_\sigma(x_k)} \Delta u_i \leq \sum_{j \neq i} \int_{(D_\sigma(x_k))_1} \Delta u_j \]

as in (7.2) with $(D_\sigma(x_k))_1$ in place of $B_{1+S}(x_0)$. By the Hopf Lemma, we obtain

(8.9)\[ \int_{D_\sigma(x_k)} \Delta u_i = \int_{\partial S_i \cap D_\sigma(x_k)} \frac{\partial u_i}{\partial \nu_i} d\mathcal{H} \geq c_\mathcal{H}(\partial S_i \cap D_\sigma(x_k)) = \tilde{C}\sigma \]

where $\nu_i$ is the inner normal vector.

Now we estimate $\int_{(D_\sigma(x_k))_1} \Delta u_j$. From Corollary 6.5 we know that $S_j$ is a set of finite perimeter. Therefore by Lemma 8.5 and Lemma 8.8 we obtain the following estimate

(8.10)\[ \int_{(D_\sigma(x_k))_1} \Delta u_j \leq \int_{\partial^* S^1_j \cap (D_\sigma(x_k))_1} \frac{\partial u_j}{\partial \nu_{S^1_j}} d\mathcal{H} \leq C_\sigma^{\beta_\delta} \mathcal{H}(\partial S^1_j \cap (D_\sigma(x_k))_1) \]

where $\nu_{S_j}$ is the measure-theoretic inward unit normal to $S^1_j$ and $\beta_\delta > 0$. Since, for some constant $c$

$$\partial S^1_j \cap (D_\sigma(x_k))_1 \subset \partial S^1_j \cap B_{\sigma\rho}(y_k)$$

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by (2.2), there exists \( \tilde{c}_2 \), that for simplicity we will still name \( c \), such that \( \partial S_j^1 \cap (D_\sigma(x_k))_1 \subset \partial S_j^1 \cap B_{c\sigma}(y_k) \). Then

\[(8.11) \quad \mathcal{H}(\partial^* S_j^1 \cap (D_\sigma(x_k))_1) \leq \text{Per}(\partial S_j^1 \cap B_{c\sigma}(y_k)).\]

To estimate \( \text{Per}(\partial S_j^1 \cap B_{c\sigma}(y_k)) \), consider (6.2) in the distributional sense. Then, take a smooth function \( 0 \leq \phi \leq 1 \) with compact support contained in \( B_{c\sigma}(y_k) \cap \{x : 0 < d(x, S_i) < 1\} \) and such that \( \phi \equiv 1 \) on the set \( B_{c\sigma}(y_k) \cap \{x : 1 - \delta < d(x, S_i) < 1 - \epsilon\} \), for \( 0 < \epsilon < \delta \) and \( \delta \) as introduced in the definition of \( \eta \) in the proof of Lemma 6.4. Then, for \( d_{S_i}(\cdot) = d_\rho(\cdot, S_i) \) we have that

\[
0 = \int_{B_{c\sigma}(y_k) \cap \{x : 0 < d_{S_i} < 1\}} \text{div}(\eta(d_{S_i})Dd_{S_i})\phi = \int_{B_{c\sigma}(y_k) \cap \{x : 0 < d_{S_i} < 1\}} \eta'(d_{S_i})|Dd_{S_i}|^2\phi dx \\
+ \int_{B_{c\sigma}(y_k) \cap \{x : 0 < d_{S_i} < 1\}} \eta(d_{S_i})\Delta d_{S_i}\phi \\
\leq \int_{B_{c\sigma}(y_k) \cap \{x : 0 < d_{S_i} < 1\}} \eta'(d_{S_i})|Dd_{S_i}|^2\phi dx + C\sigma.
\]

Proceeding as in Lemma 6.4 we obtain that

\[(8.12) \quad \text{Per}(\partial S_j^1 \cap B_{c\sigma}(y_k)) \leq C\sigma.\]

Putting together (8.8), (8.9), (8.10), (8.11) and (8.12) we obtain

\[C\sigma^{1+\beta}\delta \geq \tilde{C}\sigma\]

and we get a contradiction for \( \sigma \) small enough. In the case (2.6) the proof follows the same steps using (7.8).

Therefore we must have that \( x_{k_1}^1 \) and \( x_{k_2}^2 \) belong to different components of \( C_j \) for any \( k \geq \tilde{k} \). In particular, since the distance between them is less than one, they must belong to two different components of the same population. Suppose that \( x_{k_1}^1 \in S_i^1 \) and \( x_{k_2}^2 \in S_i^2 \), for \( i \neq j \). Consider the consecutive two points \( x_{k_1}^{1+1} \) and \( x_{k_2}^{2+1} \) which realize the distance at \( y_{k+1} \), and again belong to two different components of \( C_j \). Since \( S_j^1 \) (to which \( y_k \) belongs) and \( S_i^2 \) reach the boundary of \( \Omega \), the point \( x_{k_2}^{2+1} \) must belong to a connected component different from \( S_i^1 \). Iterating the
procedure, we construct a sequence of distinct points belonging to connected components, each
different from the others. This is in contradiction with Lemma 8.4. We conclude that singular
points are isolated.

□

Theorem 8.10. Assume (2.8) with \( n = 2 \) and \( p = 1 \) in (2.5). Let \((u_1^\varepsilon, \ldots, u_K^\varepsilon)\) be a viscosity
solution of the problem (2.4) and \((u_1, \ldots, u_K)\) the limit as \( \varepsilon \to 0 \) of a convergent subsequence.
For \( i \neq j \), let \( x_0 \in \partial S_i \cap \Omega \) and \( y_0 \in \partial S_j \cap \Omega \) be points such that \( S_i \) has an angle \( \theta_i \in [0, \pi] \) at
\( x_0 \), \( S_j \) has an angle \( \theta_j \in [0, \pi] \) at \( y_0 \) and \( \rho(x_0 - y_0) = 1 \). Then we have

\[
\theta_i = \theta_j. \tag{8.13}
\]

If \( x_0 \in \partial S_i \cap \partial \Omega \) and \( y_0 \in \partial S_j \cap \Omega \), then

\[
\theta_i \leq \theta_j. \tag{8.14}
\]

Proof. By Lemma 8.4, the connected components of the sets \( S_i \)'s are finite. Assume \( x_0 \in \overline{\Omega} \)
and \( y_0 \in \Omega \). Without loss of generality we can assume that \( x_0 = 0 \). It suffices to show the
theorem for \( y_0 \) belonging to a region that is side by side with \( S_i \), in the sense that 0 is the limit
as \( h \to 0 \) of interior regular points \( x_h \in \partial S_i \cap \Omega \) with the property that \( x_h \) realizes the distance
from \( S_j \) at \( y_h \in \partial S_j \cap \Omega \) interior points, with \( y_h \to y_0 \) as \( h \to 0 \). Let \( C_i \) be the asymptotic cone
at 0. Let us first suppose for simplicity that \( \partial S_i \) and \( \partial S_j \) are locally a cone around 0 and \( y_0 \)
respectively. In particular, \( \theta_i, \theta_j > 0 \). We will explain later on how to handle the general case.

Proof of Theorem 8.10 when \( \partial S_i \) and \( \partial S_j \) are locally cones. We assume that there exists \( r_0 > 0 \)
such that \( \partial S_i \cap B_{2r_0} = C_i \cap B_{2r_0} \), where \( B_{2r_0} \) is the Euclidian ball centered at 0 of radius \( 2r_0 \).
When \( x_0 \in \partial \Omega \) we are just interested in the side of the cone \( C_i \) contained in \( \Omega \).

If \((\varrho, \theta)\) is a system of polar coordinates in the plane centered at zero, we may assume that
\( C_i \) is the cone given by

\[
C_i = \{(\varrho, \theta) \mid \varrho \in [0, +\infty), 0 \leq \theta \leq \theta_i\}.
\]
Let us first consider the case (2.6). Let us assume that \( x = (2r_h, 0) \), with \( r_h > 0 \). We know that \( r_h \to 0 \) as \( h \to 0 \), then we can fix \( h \) so small that \( r_h < r_0/3 \). By Lemma 8.6 applied to \( u_1 = u_i \), we have

\[
(8.15) \quad u_i(x) \geq cr_h^\alpha d(x, \partial S_i) \quad \text{for any } x \in [r_h, 3r_h] \times [0, R_h],
\]

where

\[
(8.16) \quad 1 + \alpha = \frac{\pi}{\theta_i} \geq 1.
\]

Now, we repeat an argument similar to the one in the proof of Theorem 7.1. We look at \( \inf u_i \) in small circles of radius \( r \) that go across the free boundary of \( u_i \) and we look at \( \sup u_j \) in circles of radius \( r + 1 \) across the free boundary of \( u_j \), then we compare the mass of the correspondent Laplacians. Precisely, there exists a small \( \sigma > 0 \) and \( e \in S_i \) such that \( B_\sigma(e) \subseteq [r_h, 3r_h] \times [0, R_h] \) and \( x_h \in \partial B_\sigma(e) \). In particular, in \( B_\sigma(e) \) the function \( u_i \) satisfies (8.15). For \( \upsilon < \sigma \) and \( r \in [\sigma - \upsilon, \sigma + \upsilon] \), we define

\[
(8.17) \quad u_i := \inf_{\partial B_r(e)} u_i \quad \text{and} \quad \bar{u}_j := \sup_{\partial B_{r+1}(e)} u_j.
\]

In what follows we denote by \( C \) and \( c \) several constants independent of \( h \). For \( r \in [\sigma - \upsilon, \sigma] \), by (8.15) we have

\[
u_i \geq \inf_{\partial B_r(e)} cr_h^\alpha d(x, \partial S_i) \geq \inf_{\partial B_r(e)} Cr_h^\alpha d_\rho(x, \partial S_i) \geq Cr_h^\alpha (\sigma - r).
\]

For \( r \in [\sigma, \sigma + \upsilon] \), the ball \( B_r(e) \) goes across \( \partial S_i \), therefore we have \( u_i = 0 \). Hence

\[
u_i(r) \geq Cr_h^\alpha (\sigma - r) \quad \text{for } r \in [\sigma - \upsilon, \sigma]
\]

\[
u_i(r) = 0 \quad \text{for } r \in [\sigma, \sigma + \upsilon].
\]

Next, let us study the behavior of \( \bar{u}_j \). First of all, let us show that

\[
d_\rho(e, \partial S_j) = \rho(e - y_h) = 1 + \sigma.
\]

Since \( d_\rho(e, \partial S_i) = \sigma \) and \( d_\rho(S_i, S_j) \geq 1 \), it is easy to see that \( d_\rho(x, \partial S_j) \geq 1 + \sigma \). The function \( \rho \) is also called a Minkowski norm and from known results about Minkowski norms, if we denote by \( T \) the Legendre transform \( T : \mathbb{R}^n \to \mathbb{R}^n \) defined by \( T(y) = \rho(y)D\rho(y) \), then \( T \) is a bijection with
inverse $T^{-1}(\xi) = \rho^*(\xi)D\rho^*(\xi)$, where $\rho^*$ is the dual norm defined by $\rho^*(\xi) := \sup\{y \cdot \xi \mid y \in B_1\}$. Now, the ball $B_1(y_h)$ is tangent to $\partial S_i$ at $x_h$ and therefore is also tangent to $B_\sigma(e)$ at $x_h$. This implies that $D\rho(e - x_h) = -D\rho(x_h - e) = D\rho(x_h - y_h)$. Consequently we have

$$e - x_h = T^{-1}(T(e - x_h)) = T^{-1}(\sigma D\rho(e - x_h)) = T^{-1}(\sigma D\rho(x_h - y_h))$$

$$= \sigma T^{-1}(T(x_h - y_h)) = \sigma(x_h - y_h).$$

We infer that

$$e = x_h + \sigma(x_h - y_h) \tag{8.20}$$

and

$$\rho(e - y_h) = (1 + \sigma)\rho(x_h - y_h) = 1 + \sigma,$$

which proves (8.19). As a consequence $\partial B_{1+r}(e) \cap S_j = \emptyset$ for $r \in [\sigma - \nu, \sigma)$, while if $r \in (\sigma, \sigma + \nu]$ then $\partial B_{1+r}(e) \cap S_j \neq \emptyset$ and $\partial B_{1+r}(e)$ enters inside $S_j$ at $\rho$-distance at most $r - \sigma$ from the boundary of $S_j$. In particular we have

$$\bar{u}_j = 0 \quad \text{for } r \in [\sigma - \nu, \sigma]. \tag{8.21}$$

Next, if $\theta_j$ is the angle of $S_j$ at $y_0$, let $\beta$ be defined by

$$1 + \beta = \frac{\pi}{\theta_j} \geq 1. \tag{8.22}$$

Remark that $y_h$ is at $\rho$-distance $2r_h$ from $y_0$. Again by Lemma 8.6 applied to $u_2 = u_j$, (after a rotation and a translation), we have the following estimate

$$u_j(x) \leq Cr_\rho^\beta d(x, \partial S_j) \leq Cr_\rho^\beta d_\rho(x, \partial S_j),$$

in a neighborhood of $y_h$. As a consequence, recalling in addition that the ball $B_{1+r}(e)$ enters in $S_j$ at $\rho$-distance $r - \sigma$ from the boundary, for $r \in [\sigma, \sigma + \nu]$ we get

$$\bar{u}_j = \sup_{\partial B_1+r(e)} u_j \leq Cr_\rho^\beta (r - \sigma).$$

The last estimate and (8.21) imply

$$\bar{u}_j(r) \leq Cr_\rho^\beta (r - \sigma)^+, \quad \text{for } r \in [\sigma - \nu, \sigma + \nu]. \tag{8.23}$$

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Now, we want to compare the mass of the Laplacians of $u_i$ and $\bar{u}_j$. Define as in (8.17)

$$u_\varepsilon^i := \inf_{\partial B_r(e)} u_\varepsilon^i, \quad \bar{u}_\varepsilon^k := \sup_{\partial B_{1+r}(e)} u_\varepsilon^k, \ k \neq i.$$ 

For $\sigma$ and $\upsilon$ small enough, the ball $B_r(e)$ is contained in $\Omega$ for any $r \leq \sigma + \upsilon$, and thus

$$\Delta u_\varepsilon^i = \frac{1}{\varepsilon^2} u_\varepsilon^i \sum_{k \neq i} H(u_\varepsilon^k) \text{ in } B_{r+\sigma}(e).$$ 

On the other hand, since $x_h$ is an interior regular point that realizes its distance from $S_j$ at an interior point, $y_h$, its distance from the support of the boundary data $f_k$ is greater than 1, for any $k \neq i$. We infer that, for $\sigma$ and $\upsilon$ small enough and $r \leq \sigma + \upsilon$,

$$\Delta u_\varepsilon^k \geq \frac{1}{\varepsilon^2} u_\varepsilon^k \sum_{l \neq k} H(u_\varepsilon^l) \text{ in } B_{1+r}(e).$$ 

Hence, arguing as in the proof of Theorem 7.1, we see that

(8.24) $$\Delta_r u_\varepsilon^i \leq \sum_{k \neq i} \Delta_r \bar{u}_\varepsilon^k \text{ in } (\sigma - \upsilon, \sigma + \upsilon),$$

where $\Delta_r u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right)$. Since $x_h$ is a regular point of $\partial S_i$ that realizes the distance from $S_j$ at $y_h \in \partial C_i$, the ball $B_{1+\sigma+\upsilon}(e)$ does not intersect the support of the functions $u_k$ for $k \neq j$ and small $\upsilon$ and $\sigma$. Therefore, multiplying inequality (8.24) by a positive test function $\phi \in C^\infty_c (\sigma - \upsilon, \sigma + \upsilon)$, integrating by parts in $(\sigma - \upsilon, \sigma + \upsilon)$ and passing to the limit as $\varepsilon \to 0$ along a converging subsequence, the only surviving function on the right-hand side is $\bar{u}_j$ and we get

(8.25) $$\int_{\sigma - \upsilon}^{\sigma + \upsilon} \frac{\partial}{\partial r} \left( \bar{u}_j \frac{\partial}{\partial r} \left( \frac{1}{r} \phi \right) \right) dr \leq \int_{\sigma - \upsilon}^{\sigma + \upsilon} \bar{u}_j \frac{\partial}{\partial r} \left( \frac{1}{r} \phi \right) dr.$$

Let us choose a function $\phi$ which is increasing and $(\sigma - \upsilon, \sigma)$ and decreasing in $(\sigma, \sigma + \upsilon)$ and hence with maximum at $r = \sigma$, and let us estimates the left and the right hand-side of the last
inequality. Estimates (8.18) imply that \( \frac{\partial u_i}{\partial r}(\sigma^-) \leq -Cr_h^\alpha \). Therefore, for small \( \upsilon \) we have
\[
\int_{\sigma^-}^{\sigma^+} \upsilon u_i \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \left( \frac{1}{r} \phi \right) \right) \, dr = - \int_{\sigma^-}^{\sigma} \frac{\partial u_i}{\partial r} r \frac{\partial}{\partial r} \left( \frac{1}{r} \phi \right) \, dr
\]
\[
= - \int_{\sigma^-}^{\sigma} \left( \frac{\partial u_i}{\partial r}(\sigma^-) + o_{\sigma^-}(1) \right) r \frac{\partial}{\partial r} \left( \frac{1}{r} \phi \right) \, dr
\]
\[
\geq - \int_{\sigma^-}^{\sigma} \frac{\partial u_i}{\partial r}(\sigma^-) \left( \frac{\partial \phi}{\partial r} - \frac{1}{r} \phi \right) \, dr
\]
\[
- \upsilon(1) \int_{\sigma^-}^{\sigma} \left( \frac{\partial \phi}{\partial r} + \frac{1}{r} \phi \right) \, dr
\]
\[
\geq - \frac{\partial u_i}{\partial r}(\sigma^-) \left[ \phi(\sigma) - \phi(\sigma) \log \left( \frac{\sigma}{\sigma - \upsilon} \right) \right]
\]
\[
- \upsilon(1) \left[ \phi(\sigma) + \phi(\sigma) \log \left( \frac{\sigma}{\sigma - \upsilon} \right) \right]
\]
\[
\geq (Cr_h^\alpha - \upsilon(1))\phi(\sigma).
\]
Similarly, using (8.23) and integrating by parts, we get
\[
\int_{\sigma^-}^{\sigma^+} \tilde{u}_j \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \left( \frac{1}{r} \phi \right) \right) \, dr \leq (Cr_h^\beta + \upsilon(1))\phi(\sigma).
\]
From the previous estimates and (8.25), letting \( \upsilon \) go to 0, we obtain
\[
r_h^\alpha \leq Cr_h^\beta,
\]
and therefore, for \( h \) small enough
\[
\beta \leq \alpha.
\]
Recalling the definitions (8.16) and (8.22) of \( \alpha \) and \( \beta \) respectively, we infer that
\[
\theta_i \leq \theta_j.
\]
This proves (8.14). If \( x_0 = 0 \) is an interior point of \( \Omega \), exchanging the roles of \( u_i \) and \( u_j \), we get the opposite inequality
\[
\theta_j \leq \theta_i,
\]
and this proves (8.13) for \( H \) defined as in (2.6).

Next, let us turn to the case (2.5). Again we compare the mass of Laplacians of \( u_i \) and \( u_j \) across the free boundaries. For \( \sigma < r_h \) let us define
\[
(8.26) \quad D_\sigma(x_h) := \{ x \in B_\sigma(x_h) \mid d(x, \partial S_i) \leq \sigma^2 \}.
\]
Then, if we denote by \((D_\sigma(x_h))_1\) the sets of points at \(\rho\)-distance less than 1, we have that

\[
\int_{D_\sigma(x_h)} \Delta u_i \leq \sum_{k \neq i} \int_{(D_\sigma(x_h))_1} \Delta u_k,
\]

as in (7.2) with \((D_\sigma(x_h))_1\) in place of \(B_{1+S}(x_0)\). By Lemma 8.6 the normal derivative of \(u_i\) with respect to the inner normal \(\nu_i\), at any point on the boundary \(\partial C_i\) with distance to the vertex between \(r_h\) and \(3r_h\) is greater than \(cr_h^{\alpha}\), then

\[
\int_{D_\sigma(x_h)} \Delta u_i = \int_{\partial C_i \cap D_\sigma(x_h)} \frac{\partial u_i}{\partial \nu_i} d\mathcal{H} \geq c \int_{2r_h - c\sigma}^{2r_h + C\sigma} r_h^\alpha dr = Cr_h^{\alpha}\sigma.
\]

Remark that

\[
(D_\sigma(x_h))_1 \cap \partial S_j \subset B_{c\sigma}(y_h) \cap \partial S_j
\]

therefore, for \(\sigma\) small enough, again from Lemma 8.6 we have

\[
\int_{(D_\sigma(x_h))_1} \Delta u_j \leq Cr_h^{\beta}\sigma.
\]

Then for \(r_h\) small enough we obtain that

\[
\beta \leq \alpha
\]

and therefore

\[
\theta_i \leq \theta_j.
\]

If \(x_0 = 0\) is an interior point of \(\Omega\), exchanging the roles of \(u_i\) and \(u_j\) we get the opposite inequality

\[
\theta_j \leq \theta_i.
\]

This concludes the proof of the theorem in the particular case in which \(\partial S_i\) and \(\partial S_j\) are locally a cone around 0 and \(y_0\) respectively.

We are now going to explain how to adapt the proof in the general case.

**Proof of Theorem 8.10 in the general case.** If \(\theta_i = 0\), then \(\theta_i \leq \theta_j\). Assume \(\theta_i \in (0, \pi]\) and \(\theta_j \in [0, \pi]\), then for any \(0 < \delta < \theta_i\), there exist \(r_\delta > 0\), a cone \(C_\delta\) centered at 0 and with opening
\( \theta_i - \delta \), and a cone \( C_j^\delta \) centered at \( y_0 \) and with opening \( \theta_j + \delta \) such that

\[
C_i^\delta \cap B_{r_\delta}(0) \subset S_i \cap B_{r_\delta}(0) \quad \text{and} \quad S_j \cap B_{r_\delta}(y_0) \subset C_j^\delta \cap B_{r_\delta}(y_0).
\]

Let \( (x_h)_h \) be the sequence of regular points on \( \partial S_i \cap \Omega \) given by Lemma 8.7 (consider \( \Gamma_1 \) the closest side to \( S_j \)) and let \( r_h = d(0, x_h) \). Denote by \( y_h \) the point on \( \partial S_j \cap \Omega \) at \( \rho \)-distance 1 from \( x_h \). Then, \( d_\rho(y_h, y_0) \leq cr_h \). Now, the proof of the theorem proceeds like in the previous case and we can compare the mass of the laplacians across the free boundaries of \( u_i \) and \( u_j \).

Let us first consider the case (2.5). For \( \sigma > 0 \) take \( D_\sigma(x_h) \) and \((D_\sigma(x_h))_1\) as defined as in (8.26). For \( \sigma \) small enough, by Lemma 8.9 \( \partial S_i \cap D_\sigma(x_h) \) does not contain singular points and by Lemma 8.3 it is a \( C^1 \) curve of the plane.

By Lemma 8.7

\[
\int_{D_\sigma(x_h)} \Delta u_i = \int_{\partial S_i \cap D_\sigma(x_h)} \frac{\partial u_i}{\partial \nu_i} d\mathcal{H} \geq Cr_h^{\alpha_\delta} \sigma.
\]

Remark that

\[
(D_\sigma(x_h))_1 \cap \partial S_j \subset B_{c\sigma}(y_h) \cap \partial S_j
\]

therefore, for \( \sigma \) small enough, from Lemma 8.8 as in the proof of Lemma 8.9 we have

\[
\int_{(D_\sigma(x_h))_1} \Delta u_j \leq \tilde{C}r_h^{\beta_\delta} \sigma.
\]

Then for \( h \) small enough, we obtain that

\[
\beta_\delta \leq \alpha_\delta
\]

and therefore

\[
\theta_i \leq \theta_j.
\]

If \( x_0 = 0 \) is an interior point of \( \Omega \), exchanging the roles of \( u_i \) and \( u_j \) we get the opposite inequality

\[
\theta_j \leq \theta_i.
\]

Next, let us turn to the case (2.6). Then, we define, for \( r \in [R_h - \nu, R_h + \nu] \),

\[
u_i := \inf_{\partial B_r(z_h)} u_i \quad \text{and} \quad \bar{u}_j := \sup_{\partial B_{1+r}(z_h)} u_j.
\]
Arguing as before, and using the Lemma 8.7 we get
\[ \beta \delta \leq \alpha \delta, \]
and therefore, letting \( \delta \) go to 0, we finally obtain
\[ \theta_i \leq \theta_j. \]

Remark in particular that if \( \theta_i > 0 \) then \( \theta_j > 0 \). If \( x_0 = 0 \) is an interior point of \( \Omega \), exchanging the roles of \( u_i \) and \( u_j \) we get the opposite inequality \( \theta_j \leq \theta_i \).

□

An immediate corollary of Theorem 8.10 is the \( C^1 \)-regularity of the free boundaries when \( K = 2 \) and under the following additional assumptions on \( \Omega, f_1 \) and \( f_2 \):

\( (8.28) \quad \Omega := \{ (x_1, x_2) \in \mathbb{R}^2 \mid g(x_2) \leq x_1 \leq h(x_2), \ x_2 \in [a, b] \}, \quad b - a \geq 4 \)

where

\( (8.29) \quad \begin{cases} g, h : [a, b] \to \mathbb{R} \text{ are Lipschitz functions with} \\ -m_2 \leq g \leq -m_1 \leq M_2 \leq h \leq M_1, \quad M_2 \geq -m_1 + 4; \end{cases} \)

the boundary data are such that

\( (8.30) \quad \begin{cases} f_1 \equiv 1, \ f_2 \equiv 0 \quad \text{on} \quad \{ x_1 \leq g(x_2) \}, \\ f_1 \equiv 0, \ f_2 \equiv 1 \quad \text{on} \quad \{ x_1 \geq h(x_2) \}, \\ f_1 \text{ is monotone decreasing in } x_1 \text{ on } \{ x_2 \leq a \} \cup \{ x_2 \geq b \}, \\ f_2 \text{ is monotone increasing in } x_1 \text{ on } \{ x_2 \leq a \} \cup \{ x_2 \geq b \}. \end{cases} \)

These assumptions imply that \( -u_1 \) and \( u_2 \) are monotone increasing in the \( x_1 \) direction. Then we have the following

**Corollary 8.11.** Assume \( (2.8) \) with \( p = 1 \) in \( (2.5) \). Assume in addition \( K = n = 2 \), \( (8.28) \), \( (8.29) \) and \( (8.30) \). Then the sets \( \partial S_i, i = 1, 2 \), are of class \( C^1 \).

**Proof.** We know that the sets \( \partial S_i \) are curves of the plane at \( \rho \)-distance 1, one from each other. Suppose by contradiction that \( \partial S_1 \) has an angle \( \theta < \pi \) at \( y_0 \). In particular, there exist two \( \rho \)-balls of radius 1, centered at two points \( z, w \in \partial S_2 \) that are tangent to \( \partial S_1 \) at \( y_0 \). Then, by the monotonicity property of the \( u_i \)'s and Theorem 7.1 the arc of the \( \rho \)-ball of radius 1 centered
at \( y_0 \) between the points \( z \) and \( w \) must be all in \( \partial S_2 \). This means that any point inside this arc, which is a regular point of \( \partial S_2 \), is at \( \rho \)-distance 1 from the singular point \( y_0 \in \partial S_1 \). This contradicts Theorem 8.10. We have shown that any point of the free boundaries is regular. Then by Lemma 8.3 the free boundaries are of class \( C^1 \). This concludes the proof. \( \square \)

Another corollary of Theorem 8.10 is that the number of singular points is finite.

**Corollary 8.12.** Assume (2.8) with \( n = K = 2 \) and \( p = 1 \) in (2.5). Assume in addition that the supports on \( \partial \Omega \) of the boundary data \( f_1 \) and \( f_2 \) have a finite number of connected components. Then singular points form a finite set.

**Proof.** From Lemma 8.4, \( S_1 \) and \( S_2 \) have a finite number of connected components. Moreover, we recall that any connected component has to reach the boundary.

Let \( x_0 \) be a singular point belonging to the boundary of the support of one of the limit functions \( u_i \). W.l.o.g. let us assume \( x_0 \in \partial S_1 \). Let \( y_1, y_2 \in \partial S_2 \) two different points where \( x_0 \) realizes the distance from \( S_2 \), \((y_1, y_2 \in \partial B_1(x_0) \cap \partial S_2, \) see Figure 3\). We can choose \( y_1 \) such that \( B_1(x_0) \) is the limit as \( k \to +\infty \) of balls \( B_1(x_k) \) with \( x_k \in \partial S_1 \), tangent to points \( y_k \in \partial S_2 \) with \( y_k \to y_1 \) and \( x_k \to x_0 \) as \( k \to +\infty \). Theorem 8.10 implies that \( S_2 \) has an angle at \( y_1 \) and \( y_2 \) and the intersection of the arc on \( \partial B_1(x_0) \) between \( y_1 \) and \( y_2 \) with \( \partial C_1 \) must have empty interior. This means that near \( y_1 \) there are points on \( \partial S_2 \) outside \( \overline{B_1(x_0)} \). These points are at distance greater than 1 from \( x_0 \) and from any other point of \( \partial S_1 \) close to \( x_0 \) and must realize the distance from \( S_1 \) outside \( B_1(y_1) \), see Figure 3. Therefore if we take a sequence \( z_k \) of such points converging to \( y_1 \) and we consider the corresponding tangent balls centered at points that are in \( \partial S_1 \) where the \( z_k \)'s realize the distance, we obtain a second tangent ball \( B_1(x_1) \) for \( y_1 \) with \( x_1 \neq x_0 \).

Now, let us denote by \( S_1^1 \) the connected component of \( S_1 \) whose boundary contains \( x_0 \). Remember that since \( S_1 \) and \( S_2 \) are at \( \rho \)-distance 1, we have \( u_1 \equiv 0 \) in \( \overline{B_1(y_1)} \cup \overline{B_1(y_2)} \).
Moreover, since the connected components of $S_2$ whose boundaries contain $y_1$ and $y_2$ must reach the boundary of $\Omega$, they separate the components of $S_1$ whose boundaries contain $x_0$ and $x_1$. Therefore $x_1$ must belong to the boundary of different components of $S_1$. The same argument that we have used for $x_1$ and $x_0$ proves also that $y_1$ and $y_2$ must belong to the boundary of different components of $C_1$.

We conclude that a singular point $x_0$ of $S_1$ involves at least four different connected components and there correspond to it another singular point, $x_1$, belonging to a different component of $S_1$ (see Figure 4). Assume w.l.o.g. that $x_1 \in \partial S_1^2$. Since all the connected components must reach the boundary of $\Omega$, $x_1$ is the only singular point of $S_1^2$ corresponding to a singular point of $S_1^1$. Since the connected component of $S_1$ are finite, we infer that there is a finite number of singular points on $\partial S_1^1$. This argument applied to any connected component of $S_1$ shows that singular points of $S_1$ are finite. This concludes the proof of the theorem.

\[ \square \]

8.2. Lipschitz regularity of the free boundaries. In this section, we will show, under some additional assumptions on the domain $\Omega$ and the boundary data $f_i$, that we can construct a
solution of problem \([2.4]\) such that the free boundaries \(S_i\) of the limiting functions have the following properties: if \(S_i\) has an angle \(\theta\) at a singular point, then \(\theta > 0\). This result can be rephrased by saying that the free boundaries are Lipschitz curves of the plane. Let us make the assumptions precise. We assume that the domain \(\Omega\) has the property that for any point of the boundary there are tangent \(\rho\)-balls of radius \(1 + \eta\), with \(\eta > 0\) contained in \(\Omega\) and in its complementary. Precisely:

\[
\begin{align*}
\Omega \text{ is a bounded domain of } \mathbb{R}^2; \\
\exists \eta > 0 \text{ such that } \forall x \in \partial \Omega, \exists B_{1+\eta}(y), B_{1+\eta}(z) \text{ such that } \\
x \in \partial B_{1+\eta}(y) \cap \partial B_{1+\eta}(z), B_{1+\eta}(y) \subset \Omega, \text{ and } B_{1+\eta}(z) \subset \Omega^c.
\end{align*}
\]

On the boundary data \(f_i, i = 1, \ldots, K\), we assume,

\[
\begin{align*}
f_i &\equiv 1 \text{ in } \text{supp } f_i; \\
\exists c > 0 \text{ s. t. } \forall x \in \partial \Omega \cap \text{supp } f_i, |B_r(x) \cap \text{supp } f_i| \geq c|B_r(x)|, \\
d_\rho(\text{supp } f_i, \text{supp } f_j) &\geq 1, i \neq j, \\
d_\rho(\text{supp } f_i \cap \partial \Omega, \text{supp } f_{i+1} \cap \partial \Omega) &\equiv 1, \text{ where } f_{K+1} := f_1; \\
\Gamma_i := \text{supp } f_i \cap \partial \Omega &\text{ is a connected } (C^2) \text{ curve of } \partial \Omega.
\end{align*}
\]

We are going to build a solution of \([2.4]\) such that the support of any limiting function \(u_i\) contains a full neighborhood of \(\Gamma_i\) in \(\Omega\) with Lipschitz boundary. Then we prove that the free
boundaries are Lipschitz. In order to do it, we first prove the existence of a solution \((u_1^\varepsilon, \ldots, u_K^\varepsilon)\) of an obstacle problem associated to system \([2.4]\). Then we show that the functions \(u_i^\varepsilon\)'s never touch the obstacles, implying that \((u_1^\varepsilon, \ldots, u_K^\varepsilon)\) is actually solution of \([2.4]\). We consider obstacle functions \(\psi_i\), for \(i = 1, \ldots, K\) defined as follows. Let \(y_i^1, y_i^2\) be the endpoints of the curve \(\Gamma_i\). For \(0 < \mu < \lambda < 1\), we set:

\[
\Gamma_i^\mu := \{ x \in \Omega^c \mid d(x, \Gamma_i) = \mu \},
\]

\[
\Gamma_i^{\mu, \lambda} := \{ x \in \Gamma_i^\mu \mid d(x, y_i^1), d(x, y_i^2) \geq \lambda \}.
\]

For \(\mu\) and \(\lambda\) small enough, \(\Gamma_i^{\mu, \lambda}\) is a \(C^{1,1}\) curve of \(\Omega^c\) with endpoints \(z_i^1, z_i^2\) such that \(d(z_i^l, y_i^l) = \lambda\), \(l = 1, 2\). We finally set

\[
A_i := \{ x \in \Omega \mid d(x, \Gamma_i^{\mu, \lambda}) < \lambda \} = \Omega \cap \left( \bigcup_{x \in \Gamma_i^{\mu, \lambda}} B_\lambda(x) \right).
\]

Remark that

\[
\partial A_i = \Gamma_i \cup (\partial A_i \cap \Omega),
\]

\[
A_i := \{ x \in \Omega \mid d(x, \Gamma_i^{\mu, \lambda}) < \lambda \} = \Omega \cap \left( \bigcup_{x \in \Gamma_i^{\mu, \lambda}} B_\lambda(x) \right).
\]
where $\partial A_i \cap \Omega$ is given by the union of two arcs contained respectively in the balls $B_\lambda(z_1^i)$ and $B_\lambda(z_2^i)$, and a curve contained in the set of points of $\Omega$ at distance $\lambda - \mu$ from $\Gamma_i$, (see Figure 5). Denote by $\alpha_i^l$ the angle of $A_i$ at $y_i^l$, $l = 1, 2$. Remark that

$$\left\{ \begin{array}{ll}
\alpha_i^l \to \frac{\pi}{2} + o_\lambda(1) & \text{if } \mu \to 0 \\
\alpha_i^l \to 0 & \text{if } \mu \to \lambda,
\end{array} \right. \tag{8.34}$$

where $o_\lambda(1) \to 0$ as $\lambda \to 0$.

We take as obstacles the functions $\psi_i : (\Omega)_1 \to \mathbb{R}$ defined as the solutions of the following problem, for $i = 1, \ldots, K$,

$$\left\{ \begin{array}{ll}
\Delta \psi_i = 0 & \text{in } A_i \\
\psi_i = f_i & \text{on } (\partial \Omega)_1 \\
\psi_i = 0 & \text{in } \Omega \setminus A_i.
\end{array} \right. \tag{8.35}$$

In this section we deal with the solution $(u_1^\varepsilon, \ldots, u_K^\varepsilon)$ of the following obstacle system problem:

for $i = 1, \ldots, K$,

$$\left\{ \begin{array}{ll}
u_i^\varepsilon \geq \psi_i & \text{in } \Omega, \\
\Delta u_i^\varepsilon(x) \leq \frac{1}{\varepsilon^2} u_i^\varepsilon(x) \sum_{j \neq i} H(u_j^\varepsilon)(x) & \text{in } \Omega, \\
\Delta u_i^\varepsilon(x) = \frac{1}{\varepsilon^2} u_i^\varepsilon(x) \sum_{j \neq i} H(u_j^\varepsilon)(x) & \text{in } \{u_i^\varepsilon > \psi_i\} \\
u_i^\varepsilon = f_i & \text{on } (\partial \Omega)_1.
\end{array} \right. \tag{8.36}$$

In the whole section we make the following assumptions:

$$\left\{ \begin{array}{l}
\varepsilon > 0, \tag{8.37}
\end{array} \right.$$

and (8.31) hold true,

$H$ is either of the form (2.5) with $p = 1$, or (2.6) and (2.7) holds true;

For $i = 1, \ldots, K$, $A_i$ and $\psi_i$ are defined by (8.33) and (8.35) respectively.

**Theorem 8.13.** Assume (8.37). Then, there exist continuous positive functions $u_1^\varepsilon, \ldots, u_K^\varepsilon$, depending on the parameter $\varepsilon$, viscosity solutions of the problem (8.36). In particular

$$\Delta u_i^\varepsilon(x) = \frac{1}{\varepsilon^2} u_i^\varepsilon(x) \sum_{j \neq i} H(u_j^\varepsilon)(x) \text{ in } \Omega \setminus A_i. \tag{8.38}$$

Moreover, for $i = 1, \ldots, K$,

$$\Delta u_i^\varepsilon \geq 0 \text{ in } \Omega. \tag{8.39}$$
in the viscosity sense.

Proof. The proof of the existence of a solution \((u_1^\varepsilon, \ldots, u_K^\varepsilon)\) of (8.36) is a slightly modification of the proof of Theorem 4.1. Here

\[
\Theta = \{(u_1, u_2, \ldots, u_K) | u_i : \Omega \to \mathbb{R} \text{ is continuous, } \psi_i \leq u_i \leq \phi_i \text{ in } \Omega, u_i = f_i \text{ on } (\partial\Omega)_1 \}.
\]

In the set \(\Omega \setminus A_i\), we have that \(u_i^\varepsilon > 0 = \psi_i\) which implies (8.38). Inequality (8.39) is a consequence of the following facts: in the set \(\{u_i^\varepsilon > \psi_i\}\) we have \(\Delta u_i^\varepsilon = \frac{1}{\varepsilon^2}u_i^\varepsilon \sum_{j \neq i} H(u_j^\varepsilon) \geq 0\); in the interior of the set \(\{u_i^\varepsilon = \psi_i\}\), \(\Delta u_i^\varepsilon = \Delta \psi_i = 0\); the free boundaries \(\partial \{u_i^\varepsilon > \psi_i\}\) have locally finite \(n-1\)-Hausdorff measure, see [2]. □

**Theorem 8.14.** Assume (8.37). Let \((u_1^\varepsilon, \ldots, u_K^\varepsilon)\) be viscosity solution of the problem (8.36). Then, there exists a subsequence \((u_1^\varepsilon_l, \ldots, u_K^\varepsilon_l)\) and continuous functions \((u_1, \ldots, u_K)\) defined on \(\Omega\), such that

\[
(u_1^\varepsilon_l, \ldots, u_K^\varepsilon_l) \to (u_1, \ldots, u_K) \text{ as } l \to +\infty, \text{ a.e. in } \Omega
\]

and the convergence of \(u_i^\varepsilon_l\) to \(u_i\) is locally uniform in the support of \(u_i\). Moreover, we have:

i) the \(u_i\)'s are locally Lipschitz continuous in \(\Omega\), in particular, there exists \(C_0 > 0\) such that, if \(d_\rho(x, \partial\Omega) \geq r\), then

\[
(8.40) \quad |\nabla u_i(x)| \leq \frac{C_0}{r).
\]

ii) the \(u_i\)'s have disjoint supports, more precisely:

\[
\text{ } u_i \equiv 0 \text{ in the set } \{x \in \Omega | d_\rho(x, \text{supp } u_j) \leq 1\} \text{ for any } j \neq i.
\]

iii) \(\Delta u_i = 0\) when \(u_i > 0\).

iv) \(u_i \geq \psi_i\) in \(\Omega\).

v) \(u_i = f_i\) on \(\partial\Omega\).

Proof. The convergence theorem is again a consequence of Lemma 5.3, Corollary 5.4 and Lemma 5.5 which hold true with \(\text{supp } f_i\) and \(\text{supp } f_j\) replaced respectively by \(\text{supp } \psi_i = A_i\) and
\text{supp } \psi_j = A_j \text{ (in Lemma 5.3 and Corollary 5.4), and } \Gamma_j^\sigma \text{ defined as the set } \{ \psi_j \geq \sigma \} \text{ (in Lemma 5.5). Estimates } (5.7) \text{ of Corollary 5.4 in particular imply } (8.40). \text{ Property (iv) is an immediate consequence of } u_i^\varepsilon \geq \psi_i \text{ in } \Omega. \text{ Finally, (v) is implied by the fact that } \psi_i \leq u_i^\varepsilon \leq \phi_i \text{ in } \Omega, \text{ and } \phi_i = \psi_i = f_i \text{ on } \partial \Omega, \text{ where } \phi_i \text{ is given by } (4.1). \square

As proven in Corollary 6.2, one can show that the free boundaries satisfy the exterior } \rho \text{-ball condition with radius } 1, \text{ that they have finite } 1\text{-Hausdorff dimensional measure and that the distance between the support of two different functions is precisely one. We are now going to prove that, if } \lambda - \mu \text{ is small enough, then any solution of the obstacle problem } (8.36) \text{ never touches the obstacles inside the domain } \Omega. \text{ To this aim, we first need the following lemma:}

\textbf{Lemma 8.15.} Assume \textit{(8.37)}. Then, there exists } c > 0 \text{ such that, for } i = 1, \ldots, K, \text{ we have}
\begin{equation}
\frac{\partial \psi_i}{\partial \nu_i}(x) \leq -\frac{c}{\lambda - \mu} \text{ for any } x \in \partial A_i \cap \Omega,
\end{equation}
\text{where } \nu_i \text{ is the exterior normal vector to the set } A_i.

\textit{Proof.} Fix any point } x_0 \in \partial A_i \cap \Omega. \text{ Then, by definition of } A_i, \text{ there exists a point } z \in \Omega^c \text{ such that } d(z, \partial \Omega) = \mu, B_\lambda(z) \cap \Omega \subset A_i \text{ and } x_0 \in \partial B_\lambda(z). \text{ Consider now the ring } \{ x \mid \mu < |x - z| < \lambda \} \text{ and the barrier function } \phi \text{ solution of}
\begin{equation*}
\begin{cases}
\Delta \phi = 0 & \text{in } \{ x \mid \mu < |x - z| < \lambda \} \\
\phi = 1 & \text{on } \partial B_\mu(z) \\
\phi = 0 & \text{on } \partial B_\lambda(z).
\end{cases}
\end{equation*}
\text{The function } \psi_i \text{ is harmonic in } B_\lambda(z) \cap \Omega, \text{ } \psi_i \geq 0 = \phi \text{ on } \partial B_\lambda(z) \cap \Omega \text{ and } \psi_i = 1 \geq \phi \text{ on } \partial \Omega \cap B_\lambda(z). \text{ Therefore by the comparison principle, we have that } \psi_i(x) \geq \phi(x) \text{ for any } x \in B_\lambda(z) \cap \Omega, \text{ and this implies } (8.41) \text{ at } x = x_0. \square

\textbf{Theorem 8.16.} Assume \textit{(8.37)}. Let } (u_1, \ldots, u_K) \text{ be the limit of a converging subsequence of } (u_1^\varepsilon, \ldots, u_K^\varepsilon), \text{ solution of } (8.36). \text{ Set } a := \lambda - \mu. \text{ Then, there exists } a_0 > 0 \text{ such that for any } a < a_0, \text{ we have, for } i = 1, \ldots, K,
\begin{equation}
u_i > \psi_i \text{ in } A_i \cap \Omega.
\end{equation}
Proof. In order to prove (8.42), it is enough to show that

\[(8.43) \quad u_i(x) > \psi_i(x), \quad \text{for any } x \in \partial A_i \cap \Omega.\]

Indeed, if (8.43) holds true, since by (8.35) and Theorem 8.14 both \(u_i\) and \(\psi_i\) are harmonic in \(A_i\), the strong maximum principle implies \(u_i > \psi_i\) in \(A_i\). This and (8.43) give (8.42). Suppose by contradiction that there exists a point \(x_0 \in \partial A_i \cap \Omega\) such that \(u_i(x_0) = \psi_i(x_0) = 0\). Then, by (8.41), we have that

\[(8.44) \quad \frac{\partial u_i}{\partial \nu_i}(x_0) \leq \frac{\partial \psi_i}{\partial \nu_i}(x_0) \leq -\frac{c}{\lambda - \mu} = -\frac{c}{a}.\]

Assumptions (8.31) imply that if the angles \(\alpha_i^l\) of \(A_i\) at \(y_i^l\), \(l = 1, 2\), are small enough, the sets defined by

\[\Sigma_i = \{y : y = x + \nu_i(x), x \in \partial A_i \cap \Omega\}\]

and

\[\Sigma_i^- = \{y : y = x + t\nu_i(x), x \in \partial A_i \cap \Omega, 0 < t < 1\}\]

are compactly supported in \(\Omega\) and

\[(8.45) \quad d_\rho(x_0, \text{supp } \psi_j) > 1 \quad \text{for any } j \neq i.\]

Therefore, by (8.34), we can choose \(a\) so small that (8.45) holds true. Moreover, from (8.45), there exists a small \(\sigma > 0\) such that \(B_{1+\sigma}(x_0) \cap \text{supp } \psi_j = \emptyset, j \neq i\), and from (8.36), we know that

\[\Delta u_j^\varepsilon \geq \frac{1}{\varepsilon^2} u_j^\varepsilon H(u_i^\varepsilon) \quad \text{in } B_{1+\sigma}(x_0)\]

(consider \(u_j^\varepsilon\) extended by zero if the ball falls out of \(\Omega\)). When \(H\) is defined as in (2.5) with \(p = 1\), arguing as in (8.27) in proof of Theorem 8.10 we obtain that

\[\sum_{j \neq i} \int_{(D_\sigma(x_0))_1} \Delta u_j \geq \int_{D_\sigma(x_0)} \Delta u_i.\]

Now, since \(u_i \geq \psi_i > 0\) in \(A_i\) and \(u_i(x_0) = 0\), the point \(x_0\) belongs to \(\partial \{u_i > 0\} \cap \partial A_i \cap \Omega\). Since \(\partial A_i \cap \Omega\) has an interior tangent ball and \(\partial \{u_i > 0\}\) has a exterior tangent ball, we know
that $x_0$ is a regular point. Since the set of regular points is an open set, see Lemma 8.9 for $\sigma$ small enough we have

\[(8.46) \quad \int_{D_\sigma(x_0)} \Delta u_i \geq - \int_{\partial \{u_i > 0\} \cap D_\sigma(x_0)} \frac{\partial u_i}{\partial \nu_i} d\mathcal{H},\]

where $\nu_i$ is still the exterior normal vector to $A_i$. On another hand, if $y_0$ is the point that realizes the distance one with $x_0$, assume w.l.o.g. that $y_0 \in \partial \text{supp} u_j$, $y_0$ has to be in $\Sigma_i$ and $y_0$ has to be a regular point. Then, for $\rho$ small enough such that $\partial \{u_j > 0\} \cap B_\rho(y_0)$ is $C^1$ we have

\[\int_{B_\rho(y_0)} \Delta u_j = - \int_{\partial \{u_j > 0\} \cap B_\rho(y_0)} \frac{\partial u_j}{\partial \nu_j} d\mathcal{H}.\]

Now, using the fact that for $\sigma$ small enough such that $\rho > c\sigma$, $\text{supp} u_j \cap (D_\sigma(x_0))_1 \subset B_{c\sigma}(y_0)$, we have

\[(8.47) \quad \int_{B_{c\sigma}(y_0)} \Delta u_j \geq \int_{(D_\sigma(x_0))_1} \Delta u_i.\]

Putting all together, dividing (8.46) and (8.47) respectively by $\mathcal{H}(\partial \{u_i > 0\} \cap D_\sigma(x_0))$ and $\mathcal{H}(\partial \{u_j > 0\} \cap B_{c\sigma}(y_0))$, and passing to the limit when $\sigma \to 0$ we obtain

\[(8.48) \quad - \frac{\partial u_i}{\partial \nu_j}(y_0) \geq - \frac{\partial u_i}{\partial \nu_j}(x_0) \geq \frac{\tilde{c}}{a}.\]

We are now going to show that (8.48) yields a contradiction. Indeed, the point $y_0$ realizes its distance from the set $\{u_i > 0\}$ at $x_0$, therefore the ball $B_1(y_0)$ is tangent to $\{u_i > 0\}$ at $x_0$. Moreover, since $A_i \subset \{u_i > 0\}$, the ball $B_1(y_0)$ is tangent to $A_i$ at $x_0$. On the other hand, for $a$ small enough, by assumption (8.31), $B_1(y_0)$ is contained in $\Omega$. In particular, the $\rho$-distance of $y_0$ from $\partial \Omega$ is greater than 1. Therefore, from (8.40), we infer that $|\nabla u_j(y_0)| \leq C_0$, which is in contradiction with (8.48) for $a$ small enough.

When $H$ is defined as in (2.6), we argue as in case (b) in the proof of Theorem 7.1 and similarly, we get a contradiction for $a$ small enough. \hfill \Box
Corollary 8.17. Under the assumptions of Theorem 8.16 if \( a < a_0 \) then \((u_1^\varepsilon, \ldots, u_K^\varepsilon)\) is solution of the following problem

\[
\begin{align*}
& u_i^\varepsilon \geq \psi_i \quad \text{in } \Omega, \\
& \Delta u_i^\varepsilon(x) = \frac{1}{\varepsilon^2} u_i^\varepsilon(x) \sum_{j \neq i} H(u_j^\varepsilon)(x) \quad \text{in } \Omega, \\
& u_i^\varepsilon = f_i \quad \text{on } (\partial \Omega)_1.
\end{align*}
\]

(8.49)

In particular, \((u_1^\varepsilon, \ldots, u_K^\varepsilon)\) is solution of (2.4).

We are now ready to show that free boundaries are Lipschitz.

Theorem 8.18. Let \((u_1^\varepsilon, \ldots, u_K^\varepsilon)\) be the solution of (2.4) given by Corollary 8.17. Let \((u_1, \ldots, u_K)\) be the limit as \( \varepsilon \to 0 \) of a converging subsequence, then the free boundaries \( \partial\{ u_i > 0 \} \), \( i = 1, \ldots, K \), are Lipschitz curves of the plane.

Proof. By contradiction let's assume that the free boundaries are not Lipschitz. This would imply that there exists at least one singular point with asymptotic cone with zero opening.

Let \( x_0 \) be an interior singular point with asymptotic cone with zero angle. W.l.o.g. suppose \( x_0 \in \partial \{ u_1 > 0 \} \). Let \( e_1 \) be the line perpendicular to the cone axis and passing through \( x_0 \), in which we choose an orientation such that the cone is below the axis \( e_1 \). As we proved in Theorem 8.10 and Corollary 8.12 there exist \( y_0 \) and \( y_1 \), with \( y_0, y_1 \in \bigcup_{j \neq 1} \partial \{ u_j > 0 \} \) singular points at distance one from \( x_0 \) with asymptotic cones with zero opening. Also, by Theorem 7.1 for any regular point \( x \in \partial \{ u_1 > 0 \} \cap B_1(x_0) \) there exists a correspondent \( y \in \bigcup_{j \neq 1} \partial \{ u_j > 0 \} \) such that

\[
y = x + \nu(x)
\]

with \( \nu(x) \) the external normal vector to \( \partial \{ u_1 > 0 \} \) at \( x \). Observe that \( y_0, y_1 \) must lie on \( e_1 \). In fact, let \( x_n^l \in \partial \{ u_1 > 0 \} \) be regular points converging to \( x_0, x_n^l \to x_0 \) as \( n \to +\infty \), from the left side of the cone axis and let \( x_n^r \in \partial \{ u_1 > 0 \} \) be the regular points such that \( x_n^r \to x_0 \) as \( n \to +\infty \), from the right side of the cone axis. Then, the limit of the normal vectors \( \nu(x_n^l) \to \nu^l \) and \( \nu(x_n^r) \to \nu^r \), are both on the direction \( e_1 \) since they are orthogonal to the cone axis. Let
Let $y_0$ and $y_1$ be w.l.o.g. the points defined by

$$y_0 = x_0 + \nu^l \quad y_1 = x_0 + \nu^r.$$ 

So we have to have three singular points at distance one, all on the line $e_1$. Repeating the same argument and using $y_1$ as the reference singular point now, we conclude that there must exist another singular point, $y_2$, with $0$ opening cone, at distance one from $y_1$ and also on the axis $e_1$. Iterating, we will be able to proceed until the prescribed boundary of the domain stops us from finding the next point. We will have all singular points with cone with zero opening aligned on the axis $e_1$ and the distance of $y_k$ to the boundary of $\Omega$ along $e_1$ is less or equal than $1$.

Now, there are two cases: either $y_k \in \partial \Omega$ or $y_k \in \Omega$. If $y_k \in \partial \Omega$ assume w.l.o.g. that $y_k \in \partial \{u_1 > 0\}$. Since $u_1 \geq \psi_1$ we have $A_1 \subset \{u_1 > 0\}$ and that $y_k$ must coincide with one of the points $y^l_l, l = 1, 2$, endpoints of the curve $\Gamma_1$. Indeed, by the forth assumption in (8.32), no points of $\partial \{u_1 > 0\}$ are on $\partial \Omega$ between the curves $\Gamma_1$ and $\Gamma_2$, and $\Gamma_1$ and $\Gamma_K$. Assume w.l.o.g. that $y_k = y^1_1$. Let $\theta$ be the angle of $\partial \{u_1 > 0\}$ at $y^1_1$. Then, from (8.14) of Theorem 8.10 applied to $y_k = y^1_1$ and $y_0 = y_{k-1}$, we get $\theta = 0$. On the other hand, since $A_1 \subset \{u_1 > 0\}$ then $\theta \geq \alpha^1_1 > 0$, where $\alpha^1_1$ is the angle of $A_1$ at $y^1_1$. We have obtained a contradiction. Suppose now that $y_k$ is an interior point. Again, assume w.l.o.g. that $y_k \in \partial \{u_1 > 0\}$. Let $z_k \in \partial \Omega$ be the closest point to $y_k$ in the direction $e_1$ and $d(y_k, z_k) = l < 1$. Recall that by (8.31) there is an exterior tangent ball at $z_k$, $B_{1+\eta}$, so once the axis $e_1$ is crossed, $\Omega$ will remain outside of the tangent ball at $z_k$ and so $\partial \Omega$ will not cross again $e_1$ in $B_{1}(y_k)$. We know that $z_k$ cannot belong to $\partial \{u_j > 0\}$ since it does not respect the distance one and also $A_j \subset \{u_j > 0\}$. And by Theorem 7.1 for any point on the free boundary there exists a correspondent point at distance one belonging to the support of another function. Taking in account the previous case, the only
Figure 6. Contradiction in the case $y_k \in \partial \Omega$

Figure 7. Contradiction in the case $y_k \in \Omega$
option is that the point that realizes the distance from $y_k$, $\bar{y}$, belongs to $B_1(y_k)$ and it must be such that the angle between $e_1$ and the line that contains both $y_k$ and $\bar{y}$ is strictly positive, see Figure 7. Therefore, we must conclude that $B_1(\bar{y}) \cap \{u_1 > 0\} \neq \emptyset$.

We have obtained a contradiction. We conclude that the free boundaries cannot have a zero angle at a singular point, therefore they are Lipschitz curves of the plane. □

9. A relation between the normal derivatives at the free boundary

In this section we restrict ourself to the following case:

\[
\begin{align*}
K &= 2 \\
H &= \text{defined like in (2.5), with} \\
p &= 1, \varphi \equiv 1 \text{ and } \rho \text{ the Euclidian norm}.
\end{align*}
\]

Therefore, the system (2.4) becomes

\[
\begin{align*}
\Delta u_1^\varepsilon(x) &= \frac{1}{\varepsilon^2} u_1^\varepsilon(x) \int_{B_1(x)} u_2^\varepsilon(y) \, dy \quad \text{in } \Omega, \\
\Delta u_2^\varepsilon(x) &= \frac{1}{\varepsilon^2} u_2^\varepsilon(x) \int_{B_1(x)} u_1^\varepsilon(y) \, dy \quad \text{in } \Omega,
\end{align*}
\]

where we denote by $B_1(x)$ the Euclidian ball of radius 1 centered at $x$. Let $(u_1, u_2)$ be the limit functions of a converging subsequence that we still denote $(u_1^\varepsilon, u_2^\varepsilon)$ and for $i = 1, 2$ let

\[ S_i := \{u_i > 0\}. \]

From Section 7 we know that the $u_i$’s have disjoint support and that there is a strip of width exactly one that separates $S_1$ and $S_2$. Moreover, Corollary 6.2 guarantees that at any point of the boundary of the two sets, the principal curvatures are less or equal 1. For $i = 1, 2$, let $x_i \in \partial S_i$ be such that $x_1$ is at distance 1 from $x_2$, $\partial S_i$ is of class $C^2$ in a neighborhood of $x_i$, and all the principal curvatures of $\partial S_i$ at $x_i$ are strictly less than 1. Without loss of generality we can assume $x_1 = 0$ and $x_2 = e_n$, where $e_n = (0, \ldots, 1)$. Let us denote by $u_1^\varepsilon(0)$ and $u_2^\varepsilon(e_n)$ the exterior normal derivatives of $u_1$ and $u_2$ respectively at 0 and $e_n$. Note that the two normals have opposite direction. We want to deduce a relation between $u_1^\varepsilon(0)$ and $u_2^\varepsilon(e_n)$. Let us start by recalling some basic properties about the level surfaces of the distance function to a set.
9.1. **Level surfaces of the distance function to a set. Some basic Properties.** Consider a bounded open set $S$ and its boundary $\partial S$, of the class $C^2$. Let $\kappa_i(x)$ be the principal curvatures of $\partial S$ at $x$ (outward is the positive direction). Assume that for any point $x \in \partial S$ there exists a tangent ball $B_R(z)$ to $\partial S$ at $x$ such that $B_R(z) \subset S^c$. In particular the principle curvatures satisfy $\kappa_i(x) \leq 1/R$, $i = 1, \ldots, n - 1$. Then:

a) the distance function to $S$, $d_S(x) = d(x, \overline{S})$, is defined and is $C^2$ as long as $0 < d_S(x) < R$.

In the following lemma, which may be known in the literature, we provide a proof of the $C^{1,1}$-regularity for a more general set, which is not necessary $C^2$, it may have edges as well, but it has the property that for any tangent ball there exists a “clean area”, in the sense explained below. For the $C^2$-regularity in the case of $C^2$-boundaries, see for instance Theorem 14.16 in [23].

Given a bounded closed set $F$, we say that $\Pi$ is a supporting hyperplane at $x \in \partial F$, if $x \in \Pi$ and there exists a ball $B \subset F^c$ such that $B$ is tangent to $\Pi$ at $x$.

**Lemma 9.1.** Let $F$ be a bounded closed set. Assume that there exists $R > 0$ such that, for any $x \in \partial F$ and any supporting hyperplane $\Pi$ at $x$, there is a ball $B_R(z)$ tangent to $\Pi$ at $x$ such that $B_R(z) \subset F^c$. Let us denote by $d_F(x) = d(x, F)$ the distance function from $F$. Then $d_F$ is of class $C^{1,1}$ in the set $\{0 < d_F < R\}$.

**Proof.** Let $y_0 \in \{0 < d_F < R\}$. To prove that $d_F$ is of class $C^{1,1}$ at $y_0$, we show that there are smooth functions whose graphs are tangent from below and above the graph of $d_F$ at $(y_0, d_F(y_0))$. As proven in Lemma (6.3), the distance function from a closed bounded set has always a smooth tangent function from above. Indeed, let $x \in \partial F$ be a point where $y_0$ realizes the distance from $F$. Assume, without loss of generality, that $x = 0$. Then $d(y_0, 0) = |y_0| = d_F(y_0)$. Moreover, the ball $B_{|y_0|}(y_0)$ is contained in $F^c$ and tangent to $F$ at $0$. For any $y \in B_{|y_0|}(y_0)$, we have that $d_F(y) \leq d(y, 0) = |y|$.
Therefore the cone, graph of the function \( y \to |y| \) (which is smooth at \( y_0 \neq 0 \)) is tangent from above to the graph of \( d_F \) at \((y_0, d_F(y_0))\).

Next, we prove the existence of a smooth function tangent from below. Note that the tangent line to \( B_{|y_0|}(y_0) \) at 0 is a supporting hyperplane to \( F \) at 0. Therefore, there exists a ball \( B_R(z) \) tangent to \( F \) at 0. Therefore, there exists a ball \( B_R(y_0) \subset F^c \). We must have \( z = R y_0 / |y_0| \).

Moreover, since \( B_R(R y_0 / |y_0|) \subset F^c \), for any \( y \in B_R(R y_0 / |y_0|) \cap \{0 < d_F < R\} \), we have that

\[
d_F(y) \geq d \left( y, \partial B_R \left( R \frac{y_0}{|y_0|} \right) \right) = R - d \left( y, R \frac{y_0}{|y_0|} \right)
\]

and \( d_F(y_0) = |y_0| = R - d \left( y_0, R \frac{y_0}{|y_0|} \right) \). That is to say, the cone, graph of the function \( y \to R - d \left( y, R \frac{y_0}{|y_0|} \right) \) is tangent by below to the graph of \( d_F \) at \((y_0, d_F(y_0))\). We conclude that \( d_F \) is \( C^{1,1} \) at \( y_0 \).

Let \( S(k) \) denote the surface that is at distance \( k \) from \( S \)

\[
S(k) := \{ x : d_S(x) = k \},
\]

then, for \( k < 1 + \varepsilon \) and \( x \in S(k) \), there is a unique point \( x_0 \in S(0) \), such that \( x = x_0 + k \nu(x_0) \) where \( \nu(x_0) \) is the unit normal vector at \( x_0 \) in the positive direction. More precisely, if we denote \( K := \max \{|\kappa_i(x)| : 1 \leq i \leq n - 1, x \in \partial S\} \) and \( f(x, t) := x + t \nu(x) \), then \( f \) is a diffeomorphism between \( \partial S \times (-k, k) \) and the neighborhood of \( \partial S \), \( N_k(S) = \{ x + t \nu(x) : x \in \partial S, |t| < k \} \) with \( k < \frac{1}{K} \).

b) for all \( x_0 \in \partial S \) if we consider the linear transformation \( x_t = x_0 + t \nu(x_0) \) we obtain \( S(t) \). Hence, since the tangent plane for each \( S(t) \) is always perpendicular to \( \nu(x_0) \), the eigenvectors of the principal curvatures remain constant along the trajectories of \( d_S \), for \( d_S < 1 + \varepsilon \).

c) the curvatures of \( S(k) \) satisfy, see Figure 8

\[
\kappa_i(x_0 + k \nu(x_0)) = \frac{1}{\kappa_i(x)} - k = \frac{\kappa_i(x_0)}{1 - \kappa_i(x_0) k}, \quad i = 1, \ldots, n - 1, \quad k < 1 + \varepsilon
\]
Figure 8. Curvatures relation

for $x_0 \in \partial S$.

d) for $x_0 \in \partial S$, the ball $B_1(x_0)$ touches $S(1)$ at the point $x_0 + \nu(x_0)$, where $\nu$ is the outward normal. Moreover, it separates quadratically from $S(1)$, that is, for any small $r > 0$ and for any $x \in B_r(x_0 + \nu(x_0)) \cap \partial B_1(x_0)$, we have that $d(x, S(1)) \leq Cr^2$, for some $C > 0$.

9.2. Free boundary condition. Following Subsection 9.1, we denote by $\kappa_i(0)$ the principal curvatures of $\partial S_1$ at 0 where outward is the positive direction and by $\kappa_i(e_n) = \frac{\kappa_i(0)}{1 - \kappa_i(0)}$, the principal curvatures of $\partial S_2$ at $e_n$. Remark that since the normal vectors to $S_1$ and $S_2$ respectively at 0 and $e_n$, have opposite directions, for $\kappa_i(e_n)$ the inner direction of $S_2$ is the positive one. The main result of this section is the following:

**Theorem 9.2.** Assume $\Theta$. Let $0 \in \partial S_1$ and $e_n \in \partial S_2$. Assume that $\partial S_1$ is of class $C^2$ in $B_{4h_0}(0)$ and that the principal curvatures satisfy: $\kappa_i(0) < 1$ for any $i = 1, \ldots, n - 1$. Then, we have the following relation:

$$\frac{u_1'(0)}{u_2'(e_n)} = \prod_{i=1, \kappa_i(0) \neq 0}^{n-1} \frac{\kappa_i(0)}{\kappa_i(e_n)} \quad \text{if } \kappa_i(0) \neq 0 \text{ for some } i = 1, \ldots, n - 1,$$

and

$$u_1'(0) = u_2'(e_n) \quad \text{if } \kappa_i(0) = 0 \text{ for any } i = 1, \ldots, n - 1.$$

In order to prove Theorem 9.2, we first prove a lemma that relates the mass of the Laplacians of the limit functions across the interfaces. For a point $x$ belonging to a neighborhood of $\partial S_1$
around 0, let us denote by $\nu(x) = \nu(x_0)$ the exterior normal vector at $x_0 \in \partial S_1$, where $x_0$ is the unique point such that $x = x_0 + t\nu(x_0)$, for some small $t > 0$. From (a) in Subsection 9.1 $\nu(x)$ is well defined.

**Lemma 9.3.** Under the assumptions of Theorem 9.2, for small $h < h_0$, let

$$D_h := B_h(0) \cap \{x : d(x, \partial S_1) \leq h^2\}$$

and

$$E_h := \{y \in \mathbb{R}^n \mid y = x + \nu(x), x \in D_h\}.$$  

Then

$$\int_{D_h} \Delta u_1 = \int_{E_h} \Delta u_2.$$

**Proof.** Remark that the surface $E_h \cap \partial S_2$ is of class $C^2$ for $h$ small enough, being $\kappa_i(0) < 1$ for $i = 1, \ldots, n - 1$, see Subsection 9.1. The Laplacians of the $u_i$’s are positive measures and

$$\int_{D_h} \Delta u_1 = \lim_{\varepsilon \to 0} \int_{D_h} \Delta u_1^\varepsilon(x) \, dx = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{D_h} \int_{B_1(x)} u_1^\varepsilon(x) u_2^\varepsilon(y) \, dy \, dx,$$

and

$$\int_{E_h} \Delta u_2 = \lim_{\varepsilon \to 0} \int_{E_h} \Delta u_2^\varepsilon(y) \, dy = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{E_h} \int_{B_1(y)} u_1^\varepsilon(x) u_2^\varepsilon(y) \, dx \, dy.$$  

Let $s$ be such that $\varepsilon^{\frac{1}{4}} < s < h$, where $\alpha$ is given by Lemma 5.3. We split the set $D_h$ in the following way

$$D_h = D_{h,s}^+ \cup D_{h,s}^- \cup D_{h,s},$$

where

$$D_{h,s}^+ := \{x \in D_h \mid d(x, \partial S_1) > s^2 \text{ and } u_1(x) > 0\},$$

$$D_{h,s}^- := \{x \in D_h \mid d(x, \partial S_1) > s^2 \text{ and } u_1(x) = 0\},$$

$$D_{h,s} := \{x \in D_h \mid d(x, \partial S_1) \leq s^2\}.$$

Similarly

$$E_h = E_{h,s}^+ \cup E_{h,s}^- \cup E_{h,s}.$$
where

\[ E_{h,s}^+ := \{ x \in E_h \mid d(x, \partial S) > s^2 \text{ and } u_2(x) > 0 \}, \]

\[ E_{h,s}^- := \{ x \in E_h \mid d(x, \partial S) > s^2 \text{ and } u_2(x) = 0 \}, \]

\[ E_{h,s} := \{ x \in E_h \mid d(x, \partial S) \leq s^2 \}, \]

see Figure 9. Since \( \partial S_1 \) is a smooth surface around 0, and \( \Delta u_1 = 0 \) in \( S_1 \), we have that \( u_1 \) grows linearly away from the boundary in a neighborhood of 0. This and the uniform convergence of \( u^\varepsilon \) to \( u_1 \), imply that there exists \( c > 0 \) such that \( u^\varepsilon(x) > cs^2 \), for any \( x \in D_{h,s}^+ \) for \( \varepsilon \) small enough. Then, by Lemma 5.3 \( u^\varepsilon(y) \leq ae^{-\frac{b(\varepsilon s^2)^{2\alpha}}{c}}, (a, b \text{ positive constants}) \), for \( y \in B_1(x) \) and any \( x \in D_{h,s}^+ \). In an analogous way, if \( y \in E_{h,s}^+ \), we know that for \( \varepsilon \) small enough \( u^\varepsilon(y) > cs^2 \) and by Lemma 5.3 \( u^\varepsilon(x) \leq ae^{-\frac{b(\varepsilon s^2)^{2\alpha}}{c}} \) for \( x \in B_1(y) \). Since we have chosen \( s \) such that \( s^{2\alpha} > \varepsilon \frac{1}{2} \), we have that \( u^\varepsilon(y) = o(\varepsilon^2) \) uniformly in \( y \), for any \( y \in \bigcup_{x \in D_{h,s}^+} B_1(x) \) and \( u^\varepsilon(x) = o(\varepsilon^2) \) uniformly in \( x \), for any \( x \in \bigcup_{y \in E_{h,s}^+} B_1(y) \). Remark that

\[ D_{h,s}^- \subset \bigcup_{y \in E_{h,s}^+} B_1(y). \]

Therefore we have

\[
\frac{1}{\varepsilon^2} \int_{x \in D_{h,s}} \int_{y \in B_1(x)} u^\varepsilon(x)u^\varepsilon(y) \, dy \, dx = \frac{1}{\varepsilon^2} \int_{x \in D_{h,s}^+} \int_{y \in B_1(x)} u^\varepsilon(x)u^\varepsilon(y) \, dy \, dx \]
\[
\quad + \frac{1}{\varepsilon^2} \int_{x \in D_{h,s}^-} \int_{y \in B_1(x)} u^\varepsilon(x)u^\varepsilon(y) \, dy \, dx 
\]
\[
\quad + \frac{1}{\varepsilon^2} \int_{x \in D_{h,s}^+} \int_{y \in B_1(x)} u^\varepsilon(x)u^\varepsilon(y) \, dy \, dx 
\]
\[
= \frac{1}{\varepsilon^2} \int_{x \in D_{h,s}} \int_{y \in B_1(x)} u^\varepsilon(x)u^\varepsilon(y) \, dy \, dx + o(1). 
\]

Analogously

\[
\frac{1}{\varepsilon^2} \int_{y \in E_{h,s}} \int_{x \in B_1(y)} u^\varepsilon(x)u^\varepsilon(y) \, dx \, dy = \frac{1}{\varepsilon^2} \int_{y \in E_{h,s}} \int_{B_1(y)} u^\varepsilon(x)u^\varepsilon(y) \, dx \, dy + o(1). 
\]

Next, for fixed \( x \in D_{h,s} \), we have

\[ B_1(x) \cap \{ y \mid d(y, \partial S_2) > s^2 \} \subset B_{1+h}(0) \cap \{ y \mid d(y, \partial S_2) > s^2 \} \cap \{ u_2 \equiv 0 \}. \]
Therefore for any $y \in B_1(x) \cap \{y \mid d(y, \partial S_2) > s^2\}$, the ball $B_1(y)$ enters in $S_1 \cap B_{2h}(0)$ at distance at least $s^2$ from $\partial S_1$. Since $\partial S_1 \cap B_{4h}(0)$ is of class $C^2$, $u_1$ has linear growth away from the boundary in $\partial S_1 \cap B_{2h}(0)$ and therefore there exists a point in $B_1(y)$ where $u_1 \geq cs^2$ for some $c > 0$. Like before, Lemma 5.3 implies that $u_2(\varepsilon) = o(\varepsilon^2)$. We infer that

(9.4) \[
\frac{1}{\varepsilon^2} \int_{x \in D_{h,s}} \int_{y \in B_1(x)} u_1^\varepsilon(x) u_2^\varepsilon(y) dy dx = \frac{1}{\varepsilon^2} \int_{x \in D_{h,s}} \int_{y \in B_1(x) \cap \{y \mid d(y, \partial S_2) \leq s^2\}} u_1^\varepsilon(x) u_2^\varepsilon(y) dy dx + o(1).
\]

Finally, remark that (d) of Subsection 9.1 implies that for $x \in D_{h,s}$

(9.5) \[
B_1(x) \cap \{y \mid d(y, \partial S_2) \leq s^2\} \subset E_{h+cs,s}
\]

for some $c > 0$. From (9.2), (9.3), (9.4) and (9.5), we get

\[
\int_{D_h} \Delta u_1^\varepsilon(x) dx = \frac{1}{\varepsilon^2} \int_{x \in D_h} \int_{y \in B_1(x)} u_1^\varepsilon(x) u_2^\varepsilon(y) dy dx
= \frac{1}{\varepsilon^2} \int_{x \in D_{h,s}} \int_{y \in B_1(x) \cap \{y \mid d(y, \partial S_2) \leq s^2\}} u_1^\varepsilon(x) u_2^\varepsilon(y) dy dx + o(1)
\leq \frac{1}{\varepsilon^2} \int_{x \in D_{h,s}} \int_{y \in E_{h+cs,s}} u_1^\varepsilon(x) u_2^\varepsilon(y) dy dx + o(1)
\leq \frac{1}{\varepsilon^2} \int_{y \in E_{h+cs,s}} \int_{x \in B_1(y)} 70 u_1^\varepsilon(x) u_2^\varepsilon(y) dx dy + o(1)
= \int_{E_{h+cs}} \Delta u_2^\varepsilon(y) dy + o(1).
\]
Similar computations give
\[ \int_{E_h} \Delta u_2^\varepsilon(y)dy \leq \int_{D_{h+cs}} \Delta u_1^\varepsilon(x)dx + o(1). \]

Letting first \( \varepsilon \) and then \( s \) go to 0, the conclusion of the lemma follows.

\[ \square \]

**Lemma 9.4.** Under the assumptions of Theorem 9.2, let \( \Gamma_h^1 = \partial S_1 \cap B_h(0) \) and let \( \Gamma_h^2 = \{ x + \nu(x) : x \in \Gamma_h^1 \} \). Then we have the limits

\[ \lim_{h \to 0} \frac{\int_{\Gamma_h^2} dA}{\int_{\Gamma_h^1} dA} = \prod_{i=1}^{n-1} \frac{\zeta_i(0)}{\zeta_i(e_n)} \text{ if } \zeta_i(0) \neq 0 \text{ for some } i = 1, \ldots, n-1, \]

and

\[ \lim_{h \to 0} \frac{\int_{\Gamma_h^2} dA}{\int_{\Gamma_h^1} dA} = 1 \text{ if } \zeta_i(0) = 0 \text{ for any } i = 1, \ldots, n-1. \]

**Proof.** Consider the diffeomorphism \( f_t(x) = f(x,t) = x + t\nu(x) \). Then \( \Gamma_h^2 = f_1(\Gamma_h^1) \) and

\[ \int_{\Gamma_h^2} dA = \int_{\Gamma_h^1} |Jf_1(x)|dA, \]

where \( |Jf_1| \) is the determinant of the Jacobian of \( f_1 \). Taking as basis of the tangent space at 0 the principal directions, \( \tau_i \), then the differential of \( f_1 \) at \( x \) is given by

\[ (df_1)(\tau_i) = \tau_i + (d\nu)(\tau_i) = \tau_i - \zeta_i \tau_i. \]

So,

\[ |Jf_1(x)| = \prod_{i=1}^{n-1} (1 - \zeta_i(x)) \]

and

\[ \frac{\int_{\Gamma_h^2} dA}{\int_{\Gamma_h^1} dA} = \frac{1}{\text{Area}(\Gamma_h^1)} \int_{\Gamma_h^1} \prod_{i=1}^{n-1} (1 - \zeta_i(x))dA. \]

Passing to the limit when \( h \) converges to zero, we obtain

\[ \lim_{h \to 0} \frac{\int_{\Gamma_h^2} dA}{\int_{\Gamma_h^1} dA} = \prod_{i=1}^{n-1} (1 - \zeta_i(0)). \]

Now, if \( \zeta_i(0) \neq 0 \) for some \( i = 1, \ldots, n-1, \) then

\[ \prod_{i=1}^{n-1} (1 - \zeta_i(0)) = \prod_{i=1}^{n-1} \frac{1 - \zeta_i(0)}{\zeta_i(0)} = \prod_{i=1}^{n-1} \frac{\zeta_i(0)}{\zeta_i(e_n)}. \]
and (9.6) follows.

If \( \kappa_i(0) = 0 \) for any \( i = 1, \ldots, n - 1 \), then

\[
\prod_{i=1}^{n-1} (1 - \kappa_i(0)) = 1
\]

and we get (9.7).

\[\Box\]

**Proof of Theorem 9.2**

Let \( \Gamma_1^h = \partial S_1 \cap D_h \) and \( \Gamma_2^h = \partial S_2 \cap E_h \). The Laplacians \( \Delta u_i \), are jump measures along \( \partial S_i \), \( i = 1, 2 \), and satisfy

\[
\int_{D_h} \Delta u_1 = -\int_{\Gamma_1^h} u_1^1 \nu \, dA \quad \text{and} \quad \int_{E_h} \Delta u_2 = -\int_{\Gamma_2^h} u_2^2 \nu \, dA.
\]

Then, using Lemma 9.3 we get

\[
1 = \frac{\int_{D_h} \Delta u_1}{\int_{E_h} \Delta u_2} = \frac{\int_{\Gamma_1^h} u_1^1 \nu \, dA}{\int_{\Gamma_2^h} u_2^2 \nu \, dA},
\]

and so

\[
\frac{\int_{\Gamma_1^h} u_1^1 \nu \, dA}{\int_{\Gamma_2^h} u_2^2 \nu \, dA} = \frac{\int_{\Gamma_2^h} \nu \, dA}{\int_{\Gamma_1^h} \nu \, dA}.
\]

Since, when \( h \to 0 \),

\[
\frac{\int_{\Gamma_1^h} u_1^1 \nu \, dA}{\int_{\Gamma_2^h} u_2^2 \nu \, dA} \to \frac{u_1^1(0)}{u_2^2(e_n)},
\]

by Lemma 9.4 the conclusion of Theorem 9.2 follows.

\[\Box\]

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