Engineering exotic second-order topological semimetals by periodic driving

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Second-order topological semimetals (SOTSMs) are featured with hinge Fermi arcs. How to generate them in different systems has attracted much attention. We propose a scheme to create exotic SOTSMs by periodic driving. Novel Dirac SOTSMs, with a widely tunable number of nodes and hinge Fermi arcs, the adjacent nodes having the same chirality, and the coexisting nodal points and loops, are generated at ease by the periodic driving. When the time-reversal symmetry is broken, our scheme permits us to realize hybrid-order Weyl semimetals with the coexisting hinge and surface Fermi arcs. Our Weyl semimetals possess a rich hybrid of 2D sliced zero- and π/T-mode topological phases, which may be any combination of the normal insulator, Chern insulator, and second-order topological insulator. Enriching the family of topological semimetals, our scheme supplies a convenient way to artificially synthesize exotic topological phases by periodic driving.

I. INTRODUCTION

Topological quantum matters [1–5] including topological insulator, superconductor, and semimetal enrich the paradigm of condensed matter physics. The finding of higher-order topological phases opens up a new frontier of topological physics [6–19]. Featured with hinge and corner states for three (3D)- and two-dimensional (2D) systems, second-order topological insulators (SOTIs) with some fantastic applications [20] have been realized in various systems [21–30]. On the other hand, topological Dirac [31–39] and Weyl [40–54] semimetals also have been widely studied due to their chiral anomaly and close connection with diverse topological phases [54–59]. Second-order topological semimetals (SOTSMs) in both Dirac [60–64] and Weyl types [65, 66] were recently proposed. Different from surface Fermi arc in first-order semimetals, SOTSMs manifest in hinge Fermi arc [64–66]. Although they have been simulated in classical acoustic metamaterials [67, 68], their observation of SOTSMs in electronic materials is hard. One of the difficulties is that the control ways to various interactions in natural materials are limited because their features would not be switched once they are fabricated.

Coherent control via periodic driving dubbed Floquet engineering has become a versatile tool in creating topological phases in systems of ultracold atoms [69, 70], photonics [71, 72], superconductor qubits [73], and graphene [74]. It permits us to artificially synthesize a variety of exotic topological phases [75–79]. Greatly increasing the controllability and reducing the realistic difficulty in generating topological phases by adjusting intrinsic parameters of static systems, Floquet engineering supplies an extra dimension in exploring topological matters. Following the theme of condensed-matter physics in discovering novel quantum matter, one generally desires to know what exotic features can be created in the recently proposed SOTSMs by this novel dimension. Further, one also expects Floquet engineering to enrich the control way to complicated interactions in natural materials such that the difficulty in observing SOTSMs there could be reduced. Although some studies on Floquet engineering to SOTIs have been performed [80–89], those for SOTSMs are still lacking. A challenge is how to establish the bulk-corner correspondence (BCC) under the fact that the periodic-driving induced symmetry breaking and the presence of novel π/T-mode corner states invalidates the BCC of the original static system.

We propose a scheme to artificially create exotic SOTSMs by Floquet engineering. A complete BCC to the SOTSMs induced by periodic driving is estab-
lished. Taking a system of spinless fermions moving on a cubic lattice as an example, we find diverse Dirac SOTSMs with a widely tunable number of nodes and hinge Fermi arcs, the adjacent nodes appearing in pair of same chirality, and the coexisting second-order nodal points and lines, which are absent in the static system. By adding a perturbation to break the time-reversal symmetry, hybrid-order Weyl SOTSMs manifesting in the coexisting hinge and surface Fermi arcs are created by the driving. Our work highlights Floquet engineering as a convenient way to control and explore novel SOTSMs.

**II. STATIC SYSTEM**

We investigate a system of spinless fermions moving on a 3D lattice [see Fig. 1(a)]. Its Hamiltonian [90] reads $H = \sum_k C_k \hat{H}(k) C_k^\dagger$, with $C_k = (c_{1,k}^\dagger, c_{2,k}^\dagger, c_{3,k}, c_{4,k})$ and [6]

$$\hat{H}(k) = [\gamma + \chi(k_z) \cos k_x, \Gamma_5 - \chi(k_z) \sin k_x] \Gamma_3 - [\gamma + \chi(k_z) \cos k_y] \Gamma_2 - \chi(k_z) \sin k_y \Gamma_1,$$

(1)

where $\lambda$, $\gamma$, and $a$ are, respectively, the intercell, intracell, and interlayer hopping rates, $\chi(k_z) = \lambda + a \cos k_z$, $\Gamma_1 = \tau_0 \tau_2 \sigma_y$, and $\Gamma_5 = \tau_2 \tau_0 \sigma_y$, with $\tau_i$ and $\sigma_i$ being Pauli matrices, $\tau_0$ and $\sigma_0$ being identity matrices. It is a 3D generalization of 2D SOTIs in the Benalcazar-Bernevig-Hughes model [91] by considering the interlayer hopping. It is different from the one of Ref. [66] only in the interlayer-hopping way.

The SOTSM is sliced into a family of 2D $k_z$-dependent SOTIs and normal insulators. As a prerequisite for Dirac semimetal, the time-reversal $T = K$, with $K$ being the complex conjugation, and inversion $P = \tau_0 \tau_2 \sigma_y$ symmetries are respected. The system also has the mirror-rotation $M_{xy} = [\tau_0 - \tau_2 \tau_0 \sigma_x - (\tau_0 + \tau_2) \sigma_y]/2$ and chiral $S = \tau_0 \tau_2 \sigma_y$ symmetries. Thus, the 2D SOTIs are well described by $\mathcal{H}(\theta, \theta, k_x)$ along the high-symmetry line $k_y = \theta$, which is diagonalized into $\mathcal{H}^{\text{eff}}(\theta, k_x) = \mathbf{h}^\pm \cdot \mathbf{\sigma}$ and $\mathbf{h}^\pm = \sqrt{2}[\gamma + \chi(k_z) \cos \theta, \pm \chi(k_z) \sin \theta, 0]$. Its topology is characterized by the mirror-graded winding numbers $W(k_z) = (W_+ - W_-)/2$, where $W_{\pm}$ are the winding numbers of $\mathcal{H}^{\text{eff}}(\theta, k_x)$ [24, 90]. The phase diagram in Fig. 1(b) reveals a phase transition at $|\gamma| = |\chi(k_z)|$, where $W(k_z) = -1$ signifies the formation of a SOTI. The energy spectrum in Fig. 1(c) confirms the presence of a $4|W(k_z)|$-fold degenerate zero-mode state distributing at the corners. The corner state contributes to the hinge Fermi arcs [see Fig. 1(d)]. The family of 2D SOTIs forms a 3D SOTSM which hosts the Dirac nodes at the critical points $k_z = \arccos[-(\lambda \pm \gamma)/a]$ of 2D SOTI transition. Each Dirac node carries a chirality $Q$ [55]. The chirality of the Dirac node $k_{z,0}$ equals to the difference of the winding numbers of its separated phases, i.e., $Q = W(k_{z,0} + \delta) - W(k_{z,0} - \delta)$, with $\delta > 0$ being an infinitesimal [90]. Figure 1(b) shows the adjacent Dirac nodes have the opposite $Q$, which explains that only one four-fold degenerate corner state at most is formed.

**III. PERIODIC-DRIVING INDUCED EXOTIC SOTSMs**

A. Dirac SOTSMs via Floquet engineering

We consider that the intracell hopping $\gamma$ periodically changes its strength in a step-like manner within the respective time duration $T_1$ and $T_2$ as

$$\gamma(t) = \begin{cases} \gamma_1 = q_1 f, & t \in [mT, mT + T_1] \\ \gamma_2 = q_2 f, & t \in [mT + T_1, (m + 1)T], \end{cases}$$

(2)

where $m \in \mathbb{Z}$, $T = T_1 + T_2$ is the driving period, and $f$ is an energy scale to make the amplitudes $q_j$ dimensionless. Determined by the overlap of spatial wave-function of the fermion on the relevant sites, $\gamma(t)$ could be realized by applying electric field via gate voltage. A periodic system $\mathcal{H}(t)$ does not have well-defined energies. According to Floquet theorem, the one-period evolution operator $\hat{U}_T = T e^{-i\int_0^T \mathcal{H}(t)dt}$ defines an effective Hamiltonian $\hat{H}_{\text{eff}} \equiv \frac{1}{T} \ln \hat{U}_T$ whose eigenvalues are called the quasienergies [92, 93]. The SOTSMs of our periodic system are defined in such quasienergy spectrum. Applying Floquet theorem on a general four-band $\mathcal{H}_n(k) = \mathbf{n}_n \cdot \Gamma$, we have $\mathcal{H}_{\text{eff}}(k) = \frac{1}{T} \ln [e^{-i\int_0^T \mathcal{H}(t)dt} e^{-i\mathcal{H}_n(k)T}]$ [90]. First, we obtain from $\mathcal{H}_{\text{eff}}(k)$ that the bands close for $k$ and driving parameters satisfying either

$$T_1 E_{\gamma} = z_j \pi, \quad (3)
$$

with $\mathbf{n}_j \equiv \mathbf{n}_j/|\mathbf{n}_j|$, at the quasienergy zero (or $\pi/T$) when $z_j$ are integers with same (or different) parities and $z$ is even (or odd) number. Giving the positions of Dirac nodes, Eqs. (3) and (4) provide a guideline to control the driving parameters for engineering various Dirac nodes at will. Deriving $\mathbf{n}_j$ from Eq. (1) with $\gamma$ driven as Eq. (2), we obtain the conditions for the Dirac nodes as follows. **Case I:** Equation (3) results in that the Dirac nodes present at $k$ satisfying

$$\sqrt{2}[\gamma_j^2 + \chi^2(k_z) + \gamma_j \chi(k_z) (\cos k_x + \cos k_y)]^2 T_j = z_j \pi. \quad (5)$$

Satisfied by three independent parameters $(k_x, k_y, k_z)$, the two constraints in Eqs. (5) result in the band-touching points to form a loop instead of discrete points. Thus, it generally gives a nodal-line semimetal. Defined in the full Brillouin zone (BZ), Eqs. (5) describe the physics beyond the high-symmetry lines. **Case II:** $\mathbf{n}_1, \mathbf{n}_2 = \pm 1$ needs the high-symmetry line $\theta = 0$ or $\pi$. According to Eq. (4), the Dirac nodes present when

$$\sqrt{2}[|\gamma_1 + \chi(k_z) e^{i\theta}| T_1 \pm |\gamma_2 + \chi(k_z) e^{i\theta}| T_2] = z_{\theta, \pm \pi}, \quad (6)$$

Satisfied by three independent parameters $(k_x, k_y, k_z)$, the two constraints in Eqs. (6) result in the band-touching points to form a loop instead of discrete points. Thus, it generally gives a nodal-line semimetal. Defined in the full Brillouin zone (BZ), Eqs. (6) describe the physics beyond the high-symmetry lines.
FIG. 2. Quasienergy spectra as the change of $k_z$ under $x$-(a) and $x,y$-direction (b) open boundary conditions. The red solid (dashed) line is the dispersion relation along the high-symmetry line $\theta = \pi \left( 0 \right)$. (c) Mirror-graded winding numbers for the zero- and $\pi/T$-mode corner states as the change of $k_z$. Hinge Fermi arcs contributed by the zero-($d$) and $\pi/T$-mode ($e$) corner states. Phase diagram characterized by $W_0$ ($f$) and $W_{\pi/T}$ ($g$). Phase boundaries obtained from Eq. (6) with $s_{\pi,-} = 0$, $-2$, $-4$, and $-6$ in ($f$) and $-1$, $-3$, $-5$, and $-7$ in ($g$) by solid lines and with $z_{\pi,-} = 0$ and $-2$ in ($f$) and $-1$ in ($g$) by dashed lines. We use $T_1 = 0.5f^{-1}$, $T_2 = 3.5f^{-1}$, $q_2 = -q_1 = 1.2$, $\lambda = -0.7f$, and $\alpha = 0.55f$.

for \(\text{sgn}[\prod_{j=1}^{2} (\gamma_j + \chi(k_z)e^{i\theta})] = \pm 1\). Satisfied by discrete $\theta$ and $k_z$, it gives a nodal-point semimetal.

It is interesting to see that we not only can control the number and the position of the Dirac nodes but also can create nodal-line semimetal from the static nodal-point one by virtue of the periodic driving.

Second, we establish the BCC from $H_{\text{eff}}(k)$ for our system. Although $H_{\text{eff}}(k)$ inherits the inversion and mirror-rotation symmetries, it does not have the time-reversal and chiral symmetries $[90]$. To recover the symmetries, we make two unitary transformations $G_j(k) = e^{i(-1)^jH_j(k)/T_j/2}$ and obtain $H_{\text{eff,j}}(k) = G_j(k)H_{\text{eff}}(k)G_j^{-1}(k) [76]$. Respecting the time-reversal and chiral symmetries, $H_{\text{eff,j}}(k)$ have well-defined mirror-graded winding numbers $W_j$. Since the transformations do not change the quasienergies, the SOTIs in $H_{\text{eff}}(k)$ at the quasienergies zero and $\pi/T$ are described by $W_j$ as

\[
W_0 = (W_1 + W_2)/2, \quad W_{\pi/T} = (W_1 - W_2)/2. \tag{7}
\]

Reflecting the firm BCC, the numbers of the zero- and $\pi/T$-mode corner states equal to $4|W_0|$ and $4|W_{\pi/T}|$. The $k_z$-dependent SOTIs form a 3D SOTSMs, which hosts second-order Dirac nodes at the critical points of a 2D topological phase transition and hinge Fermi arcs from the distribution of the corner states. Note that it is equivalent to deem that $H_{\text{eff}}(k)$ has hidden time-reversal and chiral symmetries under the redefined operations $G_j^{-1}(-k)T_jG_j(k)$ and $G_j^{-1}(k)S_G(k)$ $[90]$.

We demonstrate the constructive role of the periodic driving in generating novel Dirac nodal-point SOTSMs in Fig. 2. The quasienergy spectrum under the $x$-direction open boundary in Fig. 2(a) shows a topologically trivial phase, while the one under the $x, y$-direction open boundary in Fig. 2(b) shows rich topological phases in both the quasienergies zero and $\pi/T$. It signifies the diverse topological phases trivial in the first order but nontrivial in the second order. The Dirac nodal points in Fig. 2(b) at $k_z = 0.15\pi$, $0.54\pi$, $0.28\pi$, and $0.62\pi$ are well explained by Eq. (6) with $s_{\pi,-} = -2$, $-3$, $s_{\pi,-} = -1$ and 0, respectively. Compared with the static case, the number of the nodal points is dramatically enhanced. It verifies the periodic driving as a useful way to manipulate the nodal points. The 2D SOTIs are completely characterized by the winding number $W_j$ defined in $H_{\text{eff,j}}$. The numbers $4|W_0|$ and $4|W_{\pi/T}|$ correctly count the zero- and $\pi/T$-mode corner states [see Fig. 2(c)]. Another interesting result is that the chiralities of the adjacent Dirac nodal points possess the same sign instead of the opposite sign in the static case. This explains why more corner states than the static case are created by the periodic driving. It also endows the Dirac nodal points in our periodic system with robustness to the perturbation-induced annihilation, which is only sensitive to the nodal points with opposite chiralities $[55]$. Both of the zero- and $\pi/T$-mode corner states contribute the hinge Fermi arcs [see Figs. 2(d) and 2(e)] of the SOTSMs.

To give a global picture of the Dirac nodal-point SOTSMs in our periodic system, we plot in Figs. 2(f) and 2(g) the phase diagram characterized by $W_0$ and $W_{\pi/T}$ in the $k_z-T_2$ plane. Much richer 2D-sliced SOTIs with a widely tunable number of zero- and $\pi/T$-mode corner states than the static case in Fig. 1(b) are created by the periodic driving. The phase boundaries well described by Eq. (6) correspond to the Dirac nodal points of the SOTSMs. Different from the static case, where the Dirac nodal points separate the trivial and SOTIs, the ones in our periodic system also separate the SOTIs with a different number of corner states. A clear tendency with the increase of the period $T_2$ is that the number of the Dirac nodal points increases, which can be analytically understood from Eq. (6).

Next, we create the Dirac nodal-loop SOTSMs from the static nodal-point ones via engineering the periodic driving to satisfy Eqs. (5). The quasienergy spectrum in
Fig. 3(a) reveals that, besides the zero-mode Dirac nodal points at \( k_2 = 0.14\pi \) and \( 1.86\pi \), recoverable by Eq. (6) with \( z_{\pi/0,+} = 2 \), and the \( \pi/T \)-mode ones at \( k_2 = 0.66\pi \) and \( 1.34\pi \), recoverable by Eq. (6) with \( z_{\pi/0,+} = 1 \), there are two extra band-touching points at \( k_2 = 0.29\pi \) and \( 1.71\pi \). Plotting the two surfaces governed by Eqs. (5) in the BZ in Fig. 3(c), we really see two closed intersecting lines at \( k_2 = 0.29\pi \) and \( 1.71\pi \). It confirms the presence of two parallel nodal loops. The associated mirror-graded winding numbers in Fig. 3(b) show that both of the nodal points and loops cause the second-order topological phase transition, which endows them with the second-order feature. All these results verify the formation of a novel SOTSM with coexisting nodal points and loops via periodically driving a static Dirac nodal-point one. Such phase has not been found in static systems. Although a similar semimetal with coexisting nodal points and loops was reported in the static system [94], it is in the first-order Weyl type. However, ours is in the second-order Dirac type and protected by both \( P \) and \( T \) symmetries. The result proves the distinguished role of the periodic driving in creating exotic matters absent in static systems.

B. Weyl SOTSMs via Floquet engineering

Our periodic driving scheme (2) can be used to create novel Weyl SOTSMs by introducing a perturbation \( \Delta \mathcal{H} = i\Gamma_1 \Gamma_3 \) to break the \( T \) symmetry. The quasienergy spectrum in Fig. 4(a) reveals that each Dirac point in Fig. 2(a) splits into two Weyl points with a Chern insulator formed between them. Each Weyl point can be analytically explained by our band-touching condition [90]. The Chern insulator is signified by the gapless chiral boundary states, which can be topologically witnessed by the Wannier center [65]. The Wannier center of the zero- and \( \pi/T \)-mode \( \mathcal{C} \) quasienergy gaps. Coexisting surface and hinge Fermi arcs contributed by the zero-(d) and \( \pi/T \)-mode (e) first-order boundary and second-order corner states. We use \( p = 0.07f \) and the others being same as Fig. 2.

Fig. 4. (a) Quasienergy spectra under the \( x,y \)-direction open boundary condition. Wannier centers for the zero (b) and \( \pi/T \)-mode (c) quasienergy gaps. Coexisting surface and hinge Fermi arcs contributed by the zero-(d) and \( \pi/T \)-mode (e) first-order boundary and second-order corner states. We use \( p = 0.07f \) and the others being same as Fig. 2.
IV. DISCUSSION AND CONCLUSION

Although we only show the generation of the same order of semimetals as the static case, the periodic driving also has the ability to create the SOTSMs from the static first-order semimetal or even normal insulator [90]. The step-like driving protocol is considered just for analytical solvability. Our scheme is generalizable to other driving forms. The SOTSMs have been predicted in Cd$_3$As$_2$, KMgBi, and PtO$_2$ [61, 95] and realized in classical acoustical metamaterials [67, 68]. Periodic driving has exhibited its power in engineering exotic phases in electronic material [74, 96], ultracold atoms [97], superconductor qubits [73], and photonics [71, 72, 98, 99]. The progress indicates that our result is realizable in the recent experimental state of the art.

In summary, we have investigated the exotic SOTSMs induced by periodic driving. It is revealed that the periodic driving provides a sufficient freedom in creating novel SOTSMs absent in static systems. The discovered widely tunable number of nodes and hinge Fermi arcs, the adjacent nodes with same chirality, and the coexisting nodal points and nodal loops in Dirac SOTSMs and the hybrid-order Weyl semimetals with the coexisting hinge and surface Fermi arcs dramatically enrich the family of topological semimetals in natural materials. Our result indicates that the periodic driving supplies a feasible and convenient way to explore the exotic semimetal physics by adding the time periodicity as a novel control dimension. This significantly reduces the difficulties in fabricating specific material structure in static systems.

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CONTENTS

S1. Real-space Hamiltonian and mirror-graded winding number 1

S2. Chirality of Dirac points 1

S3. Floquet Hamiltonian 2

S4. Symmetry analysis 3

S5. Weyl nodes 3

S6. Wilson loop 4

S7. Conversion of different orders of semimetals 4

References 5

S1. REAL-SPACE HAMILTONIAN AND MIRROR-GRADED WINDING NUMBER

We investigate a system of spinless fermions moving on a 3D lattice. Its real-space Hamiltonian reads

\[
\hat{H} = \sum_{\mathbf{r}} \{ \gamma \hat{c}^\dagger_{\mathbf{r},1}(\hat{c}_{\mathbf{r}+\mathbf{a},3} + \hat{c}_{\mathbf{r}+\mathbf{a},4}) + \hat{c}^\dagger_{\mathbf{r},2}(\hat{c}_{\mathbf{r}+\mathbf{a},4} - \hat{c}_{\mathbf{r},3}) \\
+ \lambda [\hat{c}^\dagger_{\mathbf{r},1}(\hat{c}_{\mathbf{r}+\mathbf{a},3} + \hat{c}_{\mathbf{r}+\mathbf{a},4}) + \hat{c}^\dagger_{\mathbf{r},2}(\hat{c}_{\mathbf{r}+\mathbf{a},4} - \hat{c}_{\mathbf{r},3})] \\
+ \frac{\rho}{2} [\hat{c}^\dagger_{\mathbf{r},1}(\hat{c}_{\mathbf{r}+\mathbf{a},3} + \hat{c}_{\mathbf{r}+\mathbf{a},4}) + \hat{c}^\dagger_{\mathbf{r},2}(\hat{c}_{\mathbf{r}+\mathbf{a},4} - \hat{c}_{\mathbf{r},3}) \\
- \hat{c}_{\mathbf{r},3} + \hat{c}_{\mathbf{r},4}] + H.c. \},
\]

(S1)

where \( \hat{c}_{\mathbf{r},i} \) (\( i = 1, 2, 3, 4 \)) is the annihilation operator of the fermion at sublattice \( i \) of unit-cell site \( \mathbf{r} = (x, y, z) \), \( \lambda, \gamma \), and \( \rho \) are the intercell, intracell, and interlayer hopping rates, respectively. Our system is a 3D generalization of the Benalcazar-Bernevig-Hughes (BBH) model [1, 2], which is a 2D second-order topological insulators (SOTIs), by further considering the coupling between different layers. The 3D second-order topological semimetal (SOTSM) can be sliced into the stacking of 2D SOTIs and normal insulators.

The momentum-space Hamiltonian under the periodic boundary condition along all the three directions reads

\[
\hat{H} = \sum_{\mathbf{k}} \hat{C}_{\mathbf{k}} \mathcal{H}(\mathbf{k}) \hat{C}_{\mathbf{k}}^\dagger \text{ with } \hat{C}_{\mathbf{k}} = (\ \hat{c}_{\mathbf{k},1} \ \hat{c}_{\mathbf{k},2} \ \hat{c}_{\mathbf{k},3} \ \hat{c}_{\mathbf{k},4} \ )
\]

and

\[
\mathcal{H}(\mathbf{k}) = [\gamma + \chi(k_z) \cos k_x] \Gamma_5 - \chi(k_z) \sin k_x \Gamma_3 \\
- [\gamma + \chi(k_z) \cos k_y] \Gamma_2 - \chi(k_z) \sin k_y \Gamma_1,
\]

(S2)

where \( \chi(k_z) = \lambda + \alpha \cos k_z, \Gamma_1 = \tau_y \sigma_z, \Gamma_2 = \tau_y \sigma_y, \Gamma_3 = \tau_y \sigma_z, \Gamma_4 = \tau_z \sigma_0, \) and \( \Gamma_5 = \tau_x \sigma_0 \), with \( \tau_i \) and \( \sigma_i \) being Pauli matrices, \( \tau_0 \) and \( \sigma_0 \) being identity matrices.

The second-order topology of the system with mirror-rotation and chiral symmetries can be described by the mirror-graded winding number [3]. The mirror-graded winding number is defined in the Hamiltonian \( \mathcal{H}(\theta, \theta, k_z) \) along the high-symmetry line \( k_x = k_y = \theta \). After diagonalizing \( \mathcal{H}(\theta, \theta, k_z) \) into \( \text{diag}[\mathcal{H}^+(\theta, \theta, k_z), \mathcal{H}^-(\theta, \theta, k_z)] \) with \( \mathcal{H}^\pm(\theta, \theta, k_z) = h^\pm \cdot \sigma \) and \( h^\pm = \sqrt{2}[\gamma + \chi(k_z) \cos \theta, \pm \chi(k_z) \sin \theta, 0] \), we define the mirror-graded winding number as

\[
W(k_z) = (W_+ - W_-)/2.
\]

(S3)

Here \( W_\pm \) are the winding number associated with \( \mathcal{H}^\pm(\theta, k_z) \) as

\[
W_\pm = \frac{i}{2\pi} \int_0^{2\pi} \langle u_\pm(\theta, k_z) | \partial_\theta | u_\pm(\theta, k_z) \rangle d\theta
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \langle h^\mp \times \partial_\theta h^\pm \rangle d\theta,
\]

(S4)

where \( h^\pm = h^\pm /|h^\pm| \) and \( |u_\pm(\theta, k_z)\rangle \) are the eigenstates of \( \mathcal{H}^\pm(\theta, k_z) \).

S2. CHIRALITY OF DIRAC POINTS

Each Dirac node has a well-defined chirality. The chirality for the first-order node has been defined in Ref. [4]. We here give a definition of the chirality for a second-order one. Choosing a closed path \( c \) encircling the Dirac node \((k_0, k_0, k_0)\), we define its chirality as

\[
Q = \frac{-i}{4\pi} \oint_c \left[ \langle u_+(\mathbf{k}) | \nabla_\mathbf{k} | u_+(\mathbf{k}) \rangle - \langle u_-(\mathbf{k}) | \nabla_\mathbf{k} | u_-(\mathbf{k}) \rangle \right] d\mathbf{k},
\]

(S5)

where \( |u_\pm(\mathbf{k})\rangle \) are the eigenstates of \( \mathcal{H}^\pm(\mathbf{k}) \) with the mirror-rotation symmetry.

It can be proven that the chirality of the second-order Dirac node equals exactly to the difference between the
mirror-graded winding numbers of the two topological phases separated by this Dirac node. In order to prove this, we choose for convenience a rectangle path depicted in Fig. S1 as $c_1$: from $(-\pi - \delta, -\pi - \delta, k_z, 0 - \delta)$ to $(\pi - \delta, \pi - \delta, k_z, 0 + \delta)$, $c_2$: from $(\pi - \delta, \pi - \delta, k_z, 0 - \delta)$ to $(\pi - \delta, \pi - \delta, k_z, 0 + \delta)$, $c_3$: from $(\pi - \delta, \pi - \delta, k_z, 0 + \delta)$ to $(-\pi - \delta, -\pi - \delta, k_z, 0 - \delta)$, and $c_4$: from $(-\pi - \delta, -\pi - \delta, k_z, 0 + \delta)$ to $(-\pi - \delta, -\pi - \delta, k_z, 0 - \delta)$, where $\delta$ is an infinitesimal. Then the chirality reads $Q = \sum_{j=1}^{4} Q_j$ with $Q_j$ being the chirality contributed by the path $c_j$. It can be readily seen that $Q_2$ and $Q_4$ have the same integral function but along opposite integral paths $c_2$ and $c_4$. Thus we have $Q_2 + Q_4 = 0$. According to the definition of mirror-graded winding number, we have $Q_1 = W(k_z, 0 + \delta)$ and $Q_3 = -W(k_z, 0 - \delta)$. Therefore, we obtain $Q = W(k_z, 0 + \delta) - W(k_z, 0 - \delta) = 2\pi$.

To numerically confirm this conclusion, we use Eq. (S5) to calculate the chirality of the Dirac point at $k_0 = (0, 0, \frac{\pi}{2})$ when $\gamma = -0.2 \frac{e}{h}$ in Fig. 1(b) of the main text. Making the low-energy expansion to $H^{\pm}(\theta, k_z)$ near the Dirac node $k_0$, we obtain

$$H^{\pm}(\theta, k_z) = \sqrt{2} [\pm (a(k_z - \frac{\pi}{2})\sigma_x \pm \lambda \sigma_y)].$$

(S6)

Their eigenstates $|u_{\pm}(\theta, k_z)\rangle$ corresponding to the respective smaller eigenvalues can be readily calculated. By choosing the closed path $c$ enclosing the Dirac node $k_0$, i.e., $\theta = A \cos \beta$ and $k_z - \frac{\pi}{2} = A \sin \beta$, we can calculate from Eq. (S5)

$$Q_{k_0} = \frac{1}{2\pi} \int_0^{2\pi} \frac{a\lambda}{\alpha^2 \sin^2 \beta + \lambda^2 \cos^2 \beta} d\beta = 1.$$ 

(S7)

It is the same as the value calculating from the difference of the mirror-graded winding numbers of its neighboring topological phases, where $W(k_z, 0 + \delta) = 0$ and $W(k_z, 0 - \delta) = -1$.

### S3. FLOQUET HAMILTONIAN

According to $\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = 2\delta_{ij} I_{4 \times 4}$ and $\Gamma^2 = I_{4 \times 4}$, we have $(\mathbf{n} \cdot \Gamma)^2 = n^2 = E^2$ and $e^{-i\alpha n \Gamma} = \cos(\alpha E) - i \mathbf{n} \cdot \Gamma \sin(\alpha E)$ with $\mathbf{n} = E \mathbf{n}$. Therefore, the one-period evolution operator can be expanded as

$$U_T = e^{-i n \cdot \Gamma T} e^{-i \mathbf{n} \cdot \Gamma T} = \cos(E T_1) \cos(E T_2) \sin(E T_1) \sin(E T_2) \left[ \mathbf{n}_1 \cdot \mathbf{n}_2 + \sum_{j \neq k} n_{1j} n_{2k} \Gamma_j \Gamma_k \right]$$

$$-i \mathbf{n}_1 \cdot \Gamma \sin(E T_1) \cos(E T_2) - i \mathbf{n}_2 \cdot \Gamma \cos(E T_1) \sin(E T_2) \equiv A I_{4 \times 4} - i B,$$

(S8)

where $A$ and $B$ are

$$A = \cos(E T_1) \cos(E T_2) - \sin(E T_1) \sin(E T_2) \mathbf{n}_1 \cdot \mathbf{n}_2,$$

(S9)

$$B = -i \sin(E T_1) \sin(E T_2) \sum_{j \neq k} n_{1j} n_{2k} \Gamma_j \Gamma_k + \mathbf{n}_1 \cdot \Gamma \sin(E T_1) \cos(E T_2) + \mathbf{n}_2 \cdot \Gamma \cos(E T_1) \sin(E T_2).$$

(S10)

The unitariness of $U_T$ requires that $A^2 I_{4 \times 4} + B^2 = I_{4 \times 4}$. It indicates that $B^2 = B^2 I_{4 \times 4}$ and $A^2 + B^2 = 1$. Thus we have

$$U_T = \cos(\arccos A) I_{4 \times 4} - \frac{B}{B} \sin(\arccos A)$$

$$= \exp \left[ -\frac{B}{B} \arccos A \right].$$

(S11)

Then according to $H_{\text{eff}} = \frac{i}{T} \ln U_T$, the effective Hamiltonian reads

$$H_{\text{eff}} = \frac{\arccos A B}{T} B.$$ 

(S12)

Using the fact that the eigenvalues of $B$ are $\pm B$, we readily obtain the eigenvalues of $H_{\text{eff}}$ are

$$\varepsilon = \pm \frac{\arccos A}{T}.$$ 

(S13)
When $A = 1$, the band touching point occurs at $\varepsilon = 0$. When $A = -1$, the band touching point occurs at $\varepsilon = \pi/T$. According to Eq. (S9), the bands touch at $\varepsilon = 0$ for the points of $\mathbf{k}$ and driving parameters which satisfy either

$$T_j E_j = z_j \pi,$$  \hspace{1cm} (S14)

or

$$\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{T_1 E_1 \pm T_2 E_2} = z \pi,$$  \hspace{1cm} (S15)

where $z_1$ and $z_2$ are integers with same parity, and $z$ is even number. The bands touch at $\varepsilon = \pi/T$ for the points of $\mathbf{k}$ and driving parameters which satisfy either

$$T_j E_j = z_j \pi,$$  \hspace{1cm} (S16)

or

$$\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{T_1 E_1 \pm T_2 E_2} = z \pi,$$  \hspace{1cm} (S17)

where $z_1$ and $z_2$ are integers with different parities, and $z$ is odd number.

### S4. SYMMETRY ANALYSIS

It can be verified that the static Hamiltonian (1) in the main text expresses the time-reversal chiral symmetry \( \mathcal{T} \mathcal{H}_j(\mathbf{k}) \mathcal{T}^{-1} = \mathcal{H}_j(-\mathbf{k}) \), the inversion symmetry \( \mathcal{P} \mathcal{H}_j(\mathbf{k}) \mathcal{P}^{-1} = \mathcal{H}_j(-\mathbf{k}) \), and the mirror-rotation symmetry \( \mathcal{M}_{xy} \mathcal{H}_j(\mathbf{k}) \mathcal{M}_{xy}^{-1} = \mathcal{H}_j(k_y, k_z, k_x) \). It can be proven that the latter two symmetries are preserved, while the former two symmetries are broken by our periodic driving. According to \( \mathcal{H}_{eff}(\mathbf{k}) \equiv \frac{i}{\hbar} \ln U_T(\mathbf{k}) \) and \( U_T(\mathbf{k}) = e^{-i\mathcal{H}_2(k)T_2}e^{-i\mathcal{H}_1(-k)T_1} \), we have

$$\mathcal{P}U_T(\mathbf{k})\mathcal{P}^{-1} = e^{i\mathcal{H}_2(-k)T_2}e^{-i\mathcal{H}_1(-k)T_1} = U_T(-\mathbf{k}),$$  \hspace{1cm} (S18)

$$\mathcal{M}_{xy}U_T(\mathbf{k})\mathcal{M}_{xy}^{-1} = e^{i\mathcal{H}_2(k_y,k_z,k_x)T_2}e^{-i\mathcal{H}_1(k_y,k_z,k_x)T_1} = U_T(k_y,k_z,k_x),$$  \hspace{1cm} (S19)

which results in that the inversion and mirror-rotation symmetries are inherited by \( \mathcal{H}_{eff}(\mathbf{k}) \). On the other hand, because

$$\mathcal{T}U_T(\mathbf{k})\mathcal{T}^{-1} = e^{i\mathcal{H}_2(-k)T_2}e^{i\mathcal{H}_1(-k)T_1} \neq U_T(-\mathbf{k}),$$  \hspace{1cm} (S20)

$$\mathcal{S}U_T(\mathbf{k})\mathcal{S}^{-1} = e^{i\mathcal{H}_2(k)T_2}e^{i\mathcal{H}_1(k)T_1} \neq U_T^{-1}(\mathbf{k}),$$  \hspace{1cm} (S21)

the time-reversal and chiral symmetries are broken by \( \mathcal{H}_{eff}(\mathbf{k}) \).

However, the time-reversal and chiral symmetries can be simultaneously recovered by the unitary transformation \( G_1(\mathbf{k}) = e^{-i\mathcal{H}_1(k)/T_1} \) and \( G_2(\mathbf{k}) = e^{i\mathcal{H}_2(k)/T_2} \). Using \( \mathcal{H}_{eff,j}(\mathbf{k}) = \frac{1}{\hbar} \ln \tilde{U}_{T,j}(\mathbf{k}) \) with \( \tilde{U}_{T,j}(\mathbf{k}) = G_j(\mathbf{k})U_T(\mathbf{k})G_j^{-1}(\mathbf{k}) \), we have

$$\mathcal{T}\tilde{U}_{T,1}(\mathbf{k})\mathcal{T}^{-1} = e^{i\mathcal{H}_1(-k)T_1/2}e^{i\mathcal{H}_2(-k)T_2}e^{i\mathcal{H}_1(-k)T_1/2} \neq \tilde{U}_{T,1}^\dagger(-\mathbf{k}),$$  \hspace{1cm} (S22)

$$\mathcal{T}\tilde{U}_{T,2}(\mathbf{k})\mathcal{T}^{-1} = e^{i\mathcal{H}_2(-k)T_2/2}e^{i\mathcal{H}_1(-k)T_1}e^{i\mathcal{H}_2(-k)T_2/2} \neq \tilde{U}_{T,2}^\dagger(-\mathbf{k}),$$  \hspace{1cm} (S23)

which means that the time-reversal symmetry \( \mathcal{T}\mathcal{H}_{eff,j}(\mathbf{k})\mathcal{T}^{-1} = \mathcal{H}_{eff,j}(-\mathbf{k}) \) is recovered in \( \tilde{H}_{eff}(\mathbf{k}) \). In the similar manner, we can prove

$$\mathcal{S}\tilde{U}_{T,1}(\mathbf{k})\mathcal{S}^{-1} = e^{i\mathcal{H}_1(k)T_1/2}e^{i\mathcal{H}_2(k)T_2}e^{i\mathcal{H}_1(k)T_1/2} = \tilde{U}_{T,1}^\dagger(\mathbf{k}),$$  \hspace{1cm} (S24)

$$\mathcal{S}\tilde{U}_{T,2}(\mathbf{k})\mathcal{S}^{-1} = e^{i\mathcal{H}_2(k)T_2/2}e^{i\mathcal{H}_1(k)T_1}e^{i\mathcal{H}_2(k)T_2/2} = \tilde{U}_{T,2}^\dagger(\mathbf{k}),$$  \hspace{1cm} (S25)

which means that the chiral symmetry \( \mathcal{S}\mathcal{H}_{eff,j}(\mathbf{k})\mathcal{S}^{-1} = -\mathcal{H}_{eff,j}(\mathbf{k}) \) is also recovered in \( \tilde{H}_{eff}(\mathbf{k}) \). With these two symmetries recovered, we can use the similar way as the static system to establish the bulk-corner correspondence by defining the proper topological invariants for our periodically driven system.

It is interesting to note that the original effective Hamiltonian \( \mathcal{H}_{eff}(\mathbf{k}) \) equivalently possesses a hidden time-reversal symmetry and a hidden chiral symmetry after performing the inverse unitary transformations. After rewriting Eqs. (S22) and (S23) as

$$\mathcal{T}G_j(k)U_T(k)G_j^{-1}(k)\mathcal{T}^{-1} = G_j(-k)U_T^\dagger(-k)G_j^{-1}(-k),$$  \hspace{1cm} (S26)

we readily obtain

$$G_j^{-1}(-k)\mathcal{T}G_j(k)U_T(k)G_j^{-1}(-k)\mathcal{T}^{-1}G_j(-k) = U_T^{-1}(-k).$$  \hspace{1cm} (S27)

It implies that the original effective \( \mathcal{H}_{eff}(\mathbf{k}) \) possesses the time-reversal symmetry under the newly defined time-reversal operator \( G_j^{-1}(-k)\mathcal{T}G_j(k) \). Similarly, Eqs. (S24) and (S25) result in

$$G_j^{-1}(k)\mathcal{S}G_j(k)U_T(k)G_j^{-1}(k)\mathcal{S}^{-1}G_j(k) = U_T^{-1}(k),$$  \hspace{1cm} (S28)

It implies that \( \mathcal{H}_{eff}(\mathbf{k}) \) possesses the chiral symmetry under the newly defined chiral operator \( G_j^{-1}(k)\mathcal{S}G_j(k) \).

### S5. WEYL NODES

The Hamiltonian along the high-symmetry lines \( k_x = k_y = \theta = 0 \) or \( \pi \) satisfies \( \mathcal{H}_1(\theta,k_z) + i\mu \Gamma_3, \mathcal{H}_2(\theta,k_z) + i\mu \Gamma_1 \Gamma_3 = 0 \). Thus we have

$$\mathcal{H}_{eff}(\theta,k_z) = \sum_{j=1,2} \left[ \gamma_j + \chi(k_z) \cos \theta \right] (\Gamma_5 - \Gamma_2)T_j/T$$

$$+ i\mu \Gamma_1 \Gamma_3 (T_1 + T_2)/T.$$  \hspace{1cm} (S29)

Its eigenvalues are \( \pm \varepsilon_{\pm}(\theta,k_z) \), where

$$\varepsilon_{\pm}(\theta,k_z) = p(T_1 + T_2)/T \pm \sqrt{2} |\gamma_1 T_1 + \gamma_2 T_2 + e^{i\theta} \chi(k_z)(T_1 + T_2)|/T.$$  \hspace{1cm} (S30)

The Weyl points are present if

$$\varepsilon_{\pm}(\theta,k_z) = n\theta, \pm \pi/T.$$  \hspace{1cm} (S31)

We plot in Fig. S2 the four quasienergies \( \pm \varepsilon_{\pm}(\theta,k_z) \) along the high-symmetry line \( \theta = 0 \) and \( \pi \). They explain
FIG. S2. Quasienergies $\pm \varepsilon_{\pm}(\theta,k_z)$ with $\theta = 0$ (a) and $\pi$ (b) of $H_{\text{eff}}(\theta,k_z)$. We use the parameters same as Fig. 4 in the main text.

well the Weyl points formed in Fig. 4(a) of the main text. The Weyl points at $k_z = 0.06\pi$ and $0.51\pi$ are reproduced by Eq. (S31) with $n_{\pi,+} = 2$ and $3$, respectively. The ones at $k_z = 0.21\pi$ and $0.57\pi$ are reproduced by Eq. (S31) with $n_{\pi,-} = -2$ and $-3$, respectively. The ones at $k_z = 0.24\pi$ and $0.59\pi$ are reproduced by Eq. (S31) with $n_{0,-} = -1$ and $0$, respectively. The ones $k_z = 0.31\pi$ and $0.65\pi$ are reproduced by Eq. (S31) with $n_{0,+} = 1$ and $n_{0,-} = 0$, respectively.

**S6. WILSON LOOP**

The Chern insulator is characterized by the Chern number. The Chern number relates to the Wannier center $\frac{2\pi i}{N} \log[W(k_y,k_z)]$ as [5]

$$C(k_z) = -i \int_0^{2\pi} \partial_{k_y} \log[W(k_y,k_z)] dk_y,$$

(S32)

where $W(k_y,k_z)$ is called Wilson loop. The Wilson loop is defined by the multiplication of the discretized Berry connections along $k_z$, i.e.

$$W(k_y,k_z) = \prod_{j=0}^{N-1} (u(k_x+(j+1)\Delta,k_y,k_z)u(k_x+j\Delta,k_y,k_z))$$

(S33)

where $|u(k_x,k_y,k_z)|$ is the eigen state of $H_{\text{eff}}(k)$ and $\Delta = 2\pi/N$. The Wannier center itself can also act as a quantification of the topological phase. If the Wannier center $\frac{2\pi i}{N} \log[W(k_y,k_z)]$ changes from $-0.5$ to $0.5$ when $k_y$ runs over the full Brillouin zone, then the system is a Chern insulator with one pair of chiral boundary state formed.

FIG. S3. (a) Energy spectrum of the static systems $H_1$ and $H_2$ under the open-boundary condition along $x$ and $y$ directions. (b) Quasienergy spectrum of the periodically changed system between $H_1$ and $H_2$ under the open-boundary condition along $x$ and $y$ directions. Hinge Fermi arcs of the zero modes in (c) and the $\pi/T$ mode in (d). The parameters are $T_1 = 0.5f^{-1}$, $T_2 = f^{-1}$, $a = 1.5f$, $\lambda = 0.2f$, and $q_2 = q_1 = -2.5$.

**S7. CONVERSION OF DIFFERENT ORDERS OF SEMIMETALS**

The periodic driving also possesses the ability to realize the inter-conversion of different orders of topological semimetals. More interesting, not only the second-order topological semimetal from the static first-order semimetal, but also the one from the static normal insulator can be realized by the periodic driving.

Figure S3 shows the result that a second-order Weyl semimetal is produced from the static normal insulator by the periodic driving. The energy spectrum of the static Hamiltonian $H_1$ in Fig. S3(a), which is the same as the one of $H_2$, indicates that there is no band touching point and thus both of $H_1$ and $H_2$ are topologically trivial. When $H_1$ and $H_2$ are periodically inter-changed under the protocol (2) in the main text, a series of Dirac nodal points and zero- and $\pi/T$-mode bound states are formed in the quasienergy spectrum [see Fig. S3(b)]. The corresponding probability distributions of the zero- and $\pi/T$-mode bound states in Figs. S3(c) and S3(d) signify the presence of the hinge Fermi arcs. It confirms the second-order nature of the formed Dirac nodal points in Fig. S3(b). Therefore, we succeed in converting the static normal insulator to a second-order Dirac semimetal by our periodic driving protocol.

Figure S3 shows the result that a second-order Weyl semimetal is produced from the static first-order Weyl semimetal by the periodic driving. The energy spectrum of the static system $H_1$ in Fig. S3(a), which is the same as the one of $H_2$, reveals three continuous band touching lines. The 2D Chern insulators characterized by the gap-
FIG. S4. (a) Energy spectrum under the open-boundary condition along $x$ and $y$ directions and (b) Wannier centers for the zero mode of the static systems $H_1$ and $H_2$. (c) Quasienergy spectrum of the periodically changed system between $H_1$ and $H_2$ under the open-boundary condition along $x$ and $y$ directions. Surface Fermi arcs of the zero modes corresponding to the static systems $H_1$ and $H_2$ in (d). Hinge Fermi arcs of the zero modes corresponding to the periodically driven system in (e). The parameters are $T_1 = T_2 = 0.5f^{-1}$, $a = 1.5f$, $\lambda = 0.2f$, $p = 0.5f$, and $q_2 = -q_1 = 1.5$.

less chiral boundary states are formed within these band touching lines. They can be topologically witnessed by the Wannier center in Fig. S4(b), where the jump of the Wannier center from $-0.5$ to $0.5$ when $k_y$ runs from $0$ to $2\pi$ verifies the formation of a first-order Chern band. The probability distribution of the gapless chiral boundary states contributes to the surface Fermi arcs [see Fig. S3(d)]. It demonstrates that both of the static systems $H_1$ and $H_2$ are the first-order Weyl semimetal, which is sliced into a family of 2D Chern insulators and normal insulators separated by the Weyl points. When $H_1$ and $H_2$ are periodically inter-changed under the protocol (2) in the main text, it is interesting to find from the quasienergy spectrum in Fig. S4(c) that a series of 2D zero-mode corner states are formed. The probability distributions of these corner states form the hinge Fermi arcs [see Fig. S3(e)]. Thus, the periodically driven system is a second-order Weyl semimetal. By this example, we succeed in converting the static first-order Weyl semimetal to a second-order one by the periodic driving.

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