Pontryagin maximum principle for the deterministic mean field type optimal control problem via the Lagrangian approach

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Abstract

We study the necessary optimality conditions for the general deterministic mean field type free-endpoint optimal control problem. Our study relies on the Lagrangian approach that treats the mean field type control system as a crowd of infinitely many agents who are labeled by elements of some probability space. Within the framework of the Lagrangian approach, we derive the Pontryagin maximum principle. In particular, it implies that the costate variables are also labeled by the elements of the probability space. Furthermore, we consider the Kantorovich and the Eulerian formalizations those describe the mean field type control system via distribution on the set of trajectories and non-local continuity equation respectively. We prove that each local minimizer in the Kantorovich or Eulerian formulation corresponds to a local minimizer within the Lagrangian approach. Using this, we deduce the general theory, we examine the model system of mean field type linear quadratic regulator. For this system, we show that the optimal strategy is determined by the linear feedback.

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1 Introduction

The main object of the paper is the system consisting of many identical agents who interact via some external media and try to achieve a common goal. We study this system using the mean field approach that comes from the statistical physics and examine the limit system where the number of agents tends to infinity. The latter can be regarded as a dynamical system in the space of probability measures. First, the mean field interacting dynamical systems appeared as models of plasma [49, 50] (see also [42, 47] for the mathematical theory of the mean field interaction systems). Recently, such models found applications in study of crowds and flocks behavior, opinion dynamics, etc [8, 17, 22–24].

The many agent systems with mean field interaction in the presence of controls can be treated in three ways. First, one can assume that each agent chooses his/her control to optimize his/her own utility. This assumption leads to the mean field game theory proposed by Lasry, Lions [38, 39] and (independently) by Huang, Malhame, Caines [30]. The second approach appears if we consider the many agent systems affected by one external control. The mean field type control is mixture of these aforementioned approaches. On one hand, it implies that each agent has his/her own controls. On the other hand, the mean field type control theory assumes that the agents behave collectively to achieve a common goal. Thus, the main object of mean field type control theory is systems of intellectual agents acting cooperatively.

The mean field type control theory inherited such problems as existence of optimal control, dynamic programming and necessary optimality conditions in the Pontryagin maximum principle form from the classical optimal theory. Papers [1, 32] provide the existence of the mean field type optimal control. The dynamical programming principle and the representation of the value function function of the mean field type control problem as the solution of the Bellman equation in the space of probability measures is discussed in [7, 9, 10, 19, 21, 10, 13, 14]. The Pontryagin maximum principle for the stochastic mean field type control problem was derived [4, 16]. Surprisingly, the derivation of the necessary optimality conditions for the deterministic mean field type optimal control is more involved than this problem for
the pure stochastic case. Nowadays, the Pontryagin maximum principle is obtained only under additional regularity assumptions on the profile of controls those imply that either the velocity field is continuous \[14\] or even the controls depend on the agents’ states smoothly \[13, 15\] (see also papers \[12, 45\] where the Pontryagin maximum principle for the systems affected by an external control is discussed). Additionally, the mean field type control theory raises its own questions. Among them is the finite agent approximation problem \[28, 37\] that provides the consistency of the mean field type control theory.

Notice that the dynamic programming principle, Pontryagin maximum principle as well as finite agent approximations of the mean field type control problems require the technique of differential and sub-differential calculus in the space of probability measures. We refer to papers \[18, 29\] for the detailed exposition of various approaches of this field.

The paper is concerned with the Pontryagin maximum principle for the mean field type optimal control problem with the evolution of each agent driven by the ordinary differential equation. As it was mentioned above, in the current literature, the Pontryagin maximum principle is derived for the mean field type control problems with regularity assumption on the profile of agents’ controls. We aims to lift this assumption and derive the Pontryagin maximum principle for the general deterministic mean field type optimal control problem including, in particular, the case of unbounded control space. To this end, we use the Lagrangian approach \[19\] that implies the labeling of agents by elements of some probability space and, formally, reduces the original problem to the certain control problem on the space of functions. Recall \[19\] that the deterministic mean field type control problems can be also formalized within the Eulerian and Kantorovich approaches. The Eulerian approach relies on the description of the evolution of the distribution of agents through the nonlinear continuity equation and regards the mean field type control problem as a control problem in the space of probability measures. Finally, one can consider the mean field type optimal control problem as an optimization problem for distributions on the set of curves under constrain that these distributions are concentrated on the set of admissible curves. This idea leads to the Kantorovich approach. The equivalence between the Kantorovich and Eulerian approaches was proved in \[31, \text{Theorem } 1\] under the convexity assumption. The value functions within all these aforementioned approaches coincide under the same assumption \[19\].

In the paper, we consider the deterministic mean field type optimal control problem with free-endpoint assuming that the dynamics and the payoff functions are continuously differentiable w.r.t. the state of each agent and the measure describing the distribution of all agents. We adopt the concept of intrinsic derivative w.r.t. probability measure proposed in \[18\]. The key result of the paper is the Pontryagin maximum principle for the Lagrangian formulation of the mean field type optimal control problem. In this case, the costate variable is described by a process coupled with the original mean field type control process. Further, we extend results of \[19\] and prove that the local minimizers within the Kantorovich and Eulerian approaches correspond to local minimizers in the Lagrangian framework. Using this, we obtain the Pontryagin maximum principle for the Kantorovich and Eulerian approaches. In the latter case, costate equation is replaced by the continuity equation both on state and costate variables. Additionally, we apply the Pontryagin maximum principle in the Lagrangian framework to analyze the mean field type linear-quadratic regulator. In this model, we assume that the motion of each agent is given by the linear differential equation while the payoff combines the averaged cost of the agents’ controls and the terms describing the collective behavior of all agents. We show that the optimal control in this model problem can be chosen in the feedback form. Moreover, the control of each agent is determined by the mean state of all agents and the deviation of the agent’s state from this mean.

Notice that in the originally the Pontryagin maximum principle was obtained as the necessary condition for the strong extremum \[46\]. Later, it was shown that the Pontryagin maximum principle corresponds to the more subtle notion of extremum called a Pontryagin extremum \[27\]. It lies between the strong and weak extrema. We extend the notion of Pontryagin extremum to the Lagrangian formulation of mean field type optimal control problem. As in the finite-dimensional control theory, the Pontryagin extremum is weaker than the strong extremum.

The paper is organized as follows. In Section \[2\] we introduce the general notation, the state and control spaces. Additionally, in that section we recall the definition of the intrinsic derivative w.r.t. measure variable. Section \[3\] is concerned with the Lagrangian approach to the mean field type control systems. Further, in Section \[4\] we introduce the concepts of strong and Pontryagin local minima for the Lagrangian formalization of the mean field type control problem and give the statement of the Pontryagin...
maximum principle in this case. The next two sections are concerned with the proof of this result. To this end, we study spike variations of the mean field type optimal control processes within the Lagrangian approaches (see Section 8). In Section 6, we derive the costate equation, transversality and maximization conditions those constitute the Pontryagin maximum principle for the Lagrangian formalization. The Kantorovich approach is examined in Section 7. Here we study the relationship between strong local extrema within the Kantorovich and Lagrangian frameworks and derive the Pontryagin maximum principle in the Kantorovich formulation. Using the same scheme, we show that the local Eulerian minimizer corresponds to the Lagrangian one and deduce the Eulerian version of the Pontryagin maximum principle in Section 8. Finally, Section 9 provides the analytical study of the model mean field type linear quadratic regulator.

2 Preliminaries

2.1 General notation

- If $X_1, \ldots, X_n$ are sets, $i_1, \ldots, i_k$ are some indexes from $\{1, \ldots, n\}$, then we denote by $p^{i_1, \ldots, i_k}$ the natural projector from $X_1 \times \ldots \times X_n$ onto $X_{i_1} \times \ldots \times X_{i_k}$, i.e.

$$p^{i_1, \ldots, i_k}(x_1, \ldots, x_n) = (x_{i_1}, \ldots, x_{i_k}).$$

- If $(\Omega', \mathcal{F})$, $(\Omega'', \mathcal{F}'')$ are measurable spaces, $m$ is a probability on $\mathcal{F}'$, $h : \Omega' \rightarrow \Omega''$ is $\mathcal{F}'/\mathcal{F}''$-measurable function, then we denote by $h_\sharp m$ the push-forward measure that is the probability on $\mathcal{F}''$ defined by the rule: for $E \in \mathcal{F}''$,

$$h_\sharp m(E) \triangleq m(h^{-1}(E)).$$

- If $(X, \rho_X)$ is a metric space, then $\mathcal{B}(X)$ stands for the Borel $\sigma$-algebra on $X$. We will assume that the metric space is always endowed with the Borel $\sigma$-algebra.

- If $(\Omega, \mathcal{F})$ is a measurable space, $(X, \rho_X)$ is a metric space, then denote by $B(\Omega, \mathcal{F}; X)$ the set of all $\mathcal{F}/\mathcal{B}(X)$-measurable functions. When $\Omega$ is a metric space and $\mathcal{F}$ is a corresponding Borel $\sigma$-algebra we will omit the dependence on $\mathcal{F}$.

- If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $(X, \rho_X)$ is a metric space, while $g : \Omega \rightarrow X$ is a $\mathcal{F}/\mathcal{B}(X)$-measurable function, then we denote by $E g$ the expectation of $g$ according to the probability $\mathbb{P}$, i.e.,

$$E g \triangleq \int_{\Omega} g(\omega) \mathbb{P}(d\omega).$$

- If $(\Omega, \mathcal{F})$ is a measurable space, $m$ is a measure on $\mathcal{F}$, $(X, \| \cdot \|_X)$ is a normed space, $p \geq 1$, then we denote by $L^p(\Omega, \mathcal{F}, m; X)$ the set of functions $g \in B(\Omega, \mathcal{F}; X)$ such that, for some (equivalently, any) $x_* \in X$,

$$E \|g\|^p = \int_{\Omega} \|g(\omega)\|_X^p m(d\omega) < \infty.$$ 

The norm on $L^p(\Omega, \mathcal{F}, m; X)$ is given by the rule:

$$\|g\|_{L^p} = \left[ E \|g(\omega)\|^p \right]^{1/p} = \left[ \int_{\Omega} \|g\|_X^p m(d\omega) \right]^{1/p}.$$ 

- If $(X, \rho_X)$, $(Y, \rho_Y)$ are Polish spaces, $C(X, Y)$ stands for the set of continuous function from $X$ to $Y$. Further, $C_b(X, Y)$ denotes the set of all continuous and bounded function. We will consider the usual sup-norm on $C_b(X, Y)$. If $Y = \mathbb{R}$, we omit the second argument.
• If \((X, \rho_X)\) is a Polish space, then we denote by \(\mathcal{P}(X)\) the space of all Borel probabilities on it. We endow \(\mathcal{P}(X)\) with the topology of narrow convergence. Recall that a sequence \(\{m_n\}_{n=1}^{\infty} \subset \mathcal{P}(X)\) narrowly converges to \(m\), iff, for every \(\phi \in C_b(X)\),
\[
\int_X \phi(x)m_n(dx) \to \int_X \phi(x)m(dx) \text{ as } n \to \infty.
\]

• If \((X, \rho_X)\) and \((Y, \rho_Y)\) are Polish spaces, \(m_X\) and \(m_Y\) are measures on \(X\) and \(Y\) respectively, then \(m_X \otimes m_Y\) stands for the direct product of measures defined by the rule: for every measurable sets \(E_X \subset X\), \(E_Y \subset Y\),
\[
(m_X \otimes m_Y)(E_X \times E_Y) \triangleq m_X(E_X) \cdot m_Y(E_Y).
\]

• In the case where \((X, \rho_X)\) and \((Y, \rho_Y)\) are Polish spaces, \(m_X\) is a measure on \(X\), while \(h : X \to \mathcal{P}(Y)\) is a weakly measurable function, i.e., for every \(\phi \in C_b(Y)\), the mapping \(t \mapsto \int_Y \psi(y)h(x,dy)\) is measurable, we denote by \(m_X \ast h\) the measure on \(X \times Y\) defined by the rule: for every \(\phi \in C_b(X \times Y)\),
\[
\int_{X \times Y} \phi(x,y)(m_X \ast h)(dx,dy) = \int_X \int_Y \phi(x,y)h(x,dy)m_X(dx).
\]

• If \((X, \rho_X)\) is a Polish space, \(p \geq 1\), then we denote by \(\mathcal{P}^p(X)\) is set of probability measures with the finite \(p\)-th moment, i.e., \(m \in \mathcal{P}(X)\) lies in \(\mathcal{P}^p(X)\) if, for some (equivalently, any) \(x_\ast \in X\),
\[
\mathcal{M}_p^p(m) \triangleq \int (\rho(x,x_\ast))^p m(dx) < \infty.
\]

If \(X\) is Banach, we will choose \(x_\ast = 0\). Below, \(\mathcal{M}_p(m)\) denotes the \(p\)-th root of \(\mathcal{M}_p^p(m)\).

• The space \(\mathcal{P}^p(X)\) is endowed with the \(p\)-th Wasserstein metric defined by the rule: for \(m', m'' \in \mathcal{P}^p(X)\),
\[
W_p(m', m'') \triangleq \left[ \inf \left\{ \int_{X \times X} \rho_p(x', x'') \pi(dx', dx'') : \pi \in \Pi(m', m'') \right\} \right]^{1/p},
\]

where \(\Pi(m'm'')\) stands for the set of all plans between \(m'\) and \(m''\), i.e., \(\pi \in \mathcal{P}(X \times X)\) if \(p^1 \cdot \pi = m'\) and \(p^2 \cdot \pi = m''\). Notice that the sequence \(\{m_n\}_{n=1}^{\infty} \subset \mathcal{P}^p(X)\) converges to \(m \in \mathcal{P}^p(X)\) in the \(p\)-th Wasserstein metric iff \(m_n\) converges to \(m\) narrowly and \(\{m_n\}_{n=1}^{\infty}\) has uniformly integrable \(p\)-th moment [3].

• We assume that \(\mathbb{R}^d\) is the Euclidean space of column-vectors, when \(\mathbb{R}^{d,*}\) stands for the space of row-vectors.

• If \(\phi : \mathbb{R}^d \to \mathbb{R}\) is a \(C^1\)-function, then \(\nabla_x \phi(x)\) denotes the row-vector of its partial derivatives. In the case where \(\phi\) takes values in \(\mathbb{R}^d\), \(\nabla_x \phi\) is assumed to be a matrix.

• \(\lambda\) stands for the Lebesgue measure on the time interval \([0, T]\), \(T > 0\), \(L^p([0, T]; X) \triangleq L^p([0, T], \mathcal{B}([0, T]), \lambda; X)\);

• \(AC^p([0, T], X)\) stands for the set of absolutely continuous functions from \([0, T]\) to \(X\) with the metric derivative lying in \(L^p([0, T], \mathbb{R})\) (see [3], §1.1 for details).

### 2.2 Calculus on the space of probability measures

In the paper, we consider the concept of intrinsic derivative. Let \(\Phi : \mathcal{P}^p(\mathbb{R}^d) \to \mathbb{R}\). The following definition is borrowed from [18].

**Definition 2.1.** The function \(\Phi\) is called of the class \(C^1\) if there exists a continuous function \(\frac{\delta \Phi}{\delta m} : \mathcal{P}^p(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}\) such that, for any \(m' \in \mathcal{P}^p(\mathbb{R}^d)\),
\[
\lim_{s \downarrow 0} \frac{\Phi((1-s)m + sm') - \Phi(m)}{s} = \int_{\mathbb{R}^d} \frac{\delta \Phi}{\delta m}(m, y)(m'(dy) - m(dy)).
\]
The function $\frac{\delta \Phi}{\delta m}$ is called the flat derivative of the function $\Phi$.
For the $C^1$-function $\Phi$, we, in particular, have the following equality:

$$\Phi(m') - \Phi(m) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta \Phi}{\delta m}((1-s)m + sm', y)[m'(dy) - m(dy)]ds.$$ 

Notice that the function $\frac{\delta \Phi}{\delta m}$ is defined up to an additive constant. Following [18], we assume the normalization: for each $m \in \mathcal{P}_p(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \frac{\delta \Phi}{\delta m}(m, y)m(dy) = 0.$$ 

The following definition also is proposed in [18].

**Definition 2.2.** If the function $\mathbb{R}^d \ni y \mapsto \frac{\delta \Phi}{\delta y}(m, y)$ is $C^1$, then the function $\nabla_m \Phi$ defined by the rule

$$\nabla_m \Phi(m, y) \triangleq \nabla_y \frac{\delta \Phi}{\delta m}(m, y)$$

is called an intrinsic derivative of the function $\Phi$.

In the following we assume that $\nabla_m \Phi(m, y) \in \mathbb{R}^{d,*}$. When $\nabla_m \Phi$ exists and is continuous, we say that $\Phi$ is continuously differentiable.

Similarly to the finite dimensional case, the boundness of the derivative w.r.t. probability implies the Lipschitz continuity w.r.t. to the Wasserstein distance. This property is proved in Proposition A.1 (see Appendix A). Additionally, in that Appendix, we compute the intrinsic derivative for two basic cases and derive the formula for the derivative of the superposition of function defined on the space of measures and the push-forward operation.

### 2.3 State and control spaces

It is assumed that the state space for each agent is $\mathbb{R}^d$. We denote the set of all trajectories on $[0, T]$ by $\Gamma$, i.e.,

$$\Gamma \triangleq C([0, T], \mathbb{R}^d).$$

We endow $\Gamma$ with the usual sup-norm denoted by $\| \cdot \|_\infty$. The set of continuous functions defined on $[0, T]$ with values in $\mathbb{R}^{d,*}$ will be denoted by $\Gamma^*$. As above, on $\Gamma^*$ we consider the sup-norm still denoted by $\| \cdot \|_\infty$.

We denote the evaluation operator by $e_t$, i.e, for each $t \in [0, T]$, $e_t : \Gamma \rightarrow \mathbb{R}^d$ acts by the rule:

$$e_t(\gamma) \triangleq \gamma(t).$$

With some abuse of notation, we use the symbol $e_t$ for the evaluation operators defined on $\Gamma^*$ and $\Gamma \times \Gamma^*$.

In those case, $e_t$ takes values either in $\mathbb{R}^{d,*}$ or in $\mathbb{R}^d \times \mathbb{R}^{d,*}$.

If $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, then we say that $x(\cdot) : [0, T] \rightarrow \mathbb{R}^d$ satisfies the differential equation

$$\frac{d}{dt}x(t) = b(t, x(t))$$

if, for every $t \in [0, T]$,

$$x(t) = x(0) + \int_0^t b(\tau, x(\tau))d\tau.$$ 

In the paper, we primary deal with the Lagrangian approach which describe the motion of the mean field type control system as a process $X$ defined on some standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that such processes have continuous paths with the sup-norms lying in $L^p$ for some $p > 1$, i.e, we work with the space

$$\mathcal{X}^p \triangleq L^p(\Omega, \mathcal{F}, \mathbb{P}; \Gamma).$$
The norm on $\mathcal{X}^p$ is equal to
\[
\|X\|_{\mathcal{X}^p} \triangleq (\mathbb{E}[\|X(\cdot)\|_{p^*}^p])^{1/p} = \left( \int_\Omega \left[ \sup_{t \in [0,T]} \|X(t,\omega)\| \right]^p \mathbb{P}(d\omega) \right)^{1/p}.
\]

Further, recall that, for $X \in \mathcal{X}^p$, $t \in [0,T]$, $X(t) \# \mathbb{P}$ denotes the push-forward measure of $\mathbb{P}$ by the mapping $X(t) : \Omega \to \mathbb{R}^d$.

In the paper, we consider the case where the set of instantaneous controls $U$ is a closed subset of some normed space. Generally, the set $U$ can be unbounded (see assumption (H1) below). Thus, it is reasonable to assume that the agents use the controls with the finite distribution of all agents at time $t$. Additionally, $m(t)$ describes the distribution of all agents at time $t$. The initial distribution of agents is assumed to be fixed and equal to $m_0$. The agents tries the averaged individual cost. The latter is equal to
\[
\sigma(x(T), m(T)) + \int_0^T f_0(t, x(t), m(t), u(t)) dt.
\]

We fix $p > 1$. Let $q$ be such that
\[
\frac{1}{p} + \frac{1}{q} = 1.
\]

Below we assume that
(H1) $U$ is a closed subset of a Banach space;
(H2) the functions $f$, $f_0$ and $\sigma$ are continuous;
(H3) there exists a constant $C_\infty$ such that
\[
\|f(t, x, m, u)\| \leq C_\infty (1 + \|x\| + \mathcal{M}_p(m) + \|u\|),
\]
\[
|f_0(t, x, m, u)| \leq C_\infty (1 + \|x\|^p + \mathcal{M}_p^p(m) + \|u\|^p),
\]
\[
|\sigma(x, m)| \leq C_\infty (1 + \|x\|^p + \mathcal{M}_p^p(m));
\]

(H4) there exists a modulus of continuity $\varsigma(\cdot)$ such that
\[
\|f(t_1, x, m, u) - f(t_2, x, m, u)\| \leq \varsigma(t_1 - t_2)(1 + \|x\| + \mathcal{M}_p(m) + \|u\|),
\]
\[
|f_0(t_1, x, m, u) - f_0(t_2, x, m, u)| \leq \varsigma(t_1 - t_2)(1 + \|x\|^p + \mathcal{M}_p^p(m) + \|u\|^p);
(H5) the function $f$ is continuously differentiable w.r.t. $x$ and $m$; its derivatives $\nabla_x f$ and $\nabla_m f$ are bounded by constants $C_x$ and $C_m$ respectively;

(H6) the function $f_0$ is continuously differentiable w.r.t. $x$ and $m$; the derivatives $\nabla_x f_0$ and $\nabla_m f_0$ satisfy the following growth conditions with constants $C_x^0$, $C_m^0$:

\[
\|\nabla_x f_0(t, x, m, u)\|_q \leq C_x^0(1 + \|x\|^p + M_p^x(m) + \|u\|^p),
\]

\[
\|\nabla_m f_0(t, x, m, y, u)\|_q \leq C_m^0(1 + \|x\|^p + \|y\|^p + M_p^m(m) + \|u\|^p);
\]

(H7) the terminal payoff $\sigma$ is continuously differentiable; the functions $\nabla_x \sigma$ and $\nabla_m \sigma$ satisfy the following estimates with some nonnegative constants $C_x^\sigma$, $C_m^\sigma$:

\[
\|\nabla_x \sigma(x, m)\|_q \leq C_x^\sigma(1 + \|x\|^p + M_p^x(m)),
\]

\[
\|\nabla_m \sigma(x, m, y)\|_q \leq C_m^\sigma(1 + \|x\|^p + \|y\|^p + M_p^m(m)).
\]

Hereinafter, $\nabla_m f(t, x, m, y, u)$ denotes the derivative of $f$ w.r.t. measure variable for fixed $t$, $x$ and $u$. Recall that this derivative is a function of extra variable $y \in \mathbb{R}^d$. The same concerns $\nabla_m f_0(t, x, m, y, u)$ and $\nabla_m \sigma(x, y, m)$.

Let us introduce the Lagrangian approach to the mean field type control problems (see [19] for details). It relies on labeling of the agents by elements of the set $\Omega$. In the following, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a standard probability space.

**Definition 3.1.** We say that a pair $(X, u)$, where $X \in \mathcal{X}^p$, $u \in \mathcal{U}^p$, is Lagrangian control process if, for $\mathbb{P}$-a.e. $\omega \in \Omega$, $X(\cdot, \omega)$ solves the differential equation

\[
\frac{d}{dt}X(t, \omega) = f(t, X(t, \omega), X(t)\mathbb{P}, u(t, \omega)).
\]

The payoff function within the Lagrangian approach is computed by the formula:

\[
J_L(X, u) \triangleq \mathbb{E} \left[ \sigma(X(T), X(T)\mathbb{P}) + \int_0^T f_0(t, X(t), X(t)\mathbb{P}, u(t))dt \right].
\]

For the Lagrangian formulation of the optimal control problem we will consider two type of the initial conditions. First, assume that the initial assignment of agents $X_0 \in L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ is given, whilst the second approach fixes only the initial distribution.

**Definition 3.2.** We say that a Lagrangian control process $(X, u)$ meets the initial condition for the given assignment $X_0$ where $X_0\mathbb{P} = m_0$ if

\[
X(0) = X_0, \quad \mathbb{P}\text{-a.s.}
\]

Given $X_0$, we denote the set of control processes satisfying initial assignment condition by $\mathcal{A}_L(X_0)$.

We say that a process $(X, u)$ satisfies the initial distribution conditions if

\[
X(0)\mathbb{P} = m_0.
\]

The set of control processes satisfying initial distribution condition is denoted by $\text{Adm}_L(m_0)$.

Notice that,

\[
\text{Adm}_L(m_0) = \bigcup_{X_0 \in L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) : X_0\mathbb{P} = m_0} \mathcal{A}_L(X_0).
\]

In the paper we examine both strong and Pontryagin minima. In the latter case, we use the concept borrowed from [3].

**Definition 3.3.** Given an initial assignment $X_0 \in L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$, we say that a control process $(X^*, u^*) \in \mathcal{A}_L(X_0)$ is a strong local $L^p$-minimizer at $X_0$ if there exists $\varepsilon > 0$ satisfying the following condition: for any $(X, u) \in \mathcal{A}_L(X_0)$ such that $\|X - X^*\|_{L^p} \leq \varepsilon$,

\[
J_L(X^*, u^*) \leq J_L(X, u).
\]
Definition 3.4. A control process $(X^*, u^*) \in \text{Adm}_L(m_0)$ is called a strong local $W_p$-minimizer at $m_0 \in \mathcal{P}(\mathbb{R})$ if one can find $\varepsilon > 0$ such that (2) holds true for every $(X, u) \in \text{Adm}_L(m_0)$ satisfying $W_p(X(t)\mu, X^*(t)\mu) \leq \varepsilon$ when $t \in [0, T]$.

Definition 3.5. Given an initial assignment $X_0 \in L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$, a control process $(X^*, u^*) \in \mathcal{A}_L(X_0)$ is said to be a Pontryagin local $L^p$-minimizer at $X_0$ if there exists $\varepsilon > 0$ satisfying the following condition: for any $(X, u) \in \mathcal{A}_L(X_0)$ such that $\|X - X^*\|_{L^p} \leq \varepsilon$ and $(\lambda \otimes \mathbb{P})\{(t, \omega) \in [0, T] \times \Omega : u^*(t, \omega) \neq u(t, \omega)\} \leq \varepsilon$, inequality (2) is fulfilled.

Definition 3.6. A control process $(X^*, u^*) \in \text{Adm}_L(m_0)$ is called a Pontryagin local $W_p$-minimizer at $m_0$ if one can find $\varepsilon > 0$ such that (2) holds true for every $(X, u) \in \text{Adm}_L(m_0)$ satisfying $W_p(X(t)\mu, X^*(t)\mu) \leq \varepsilon$ and $(\lambda \otimes \mathbb{P})\{(t, \omega) \in [0, T] \times \Omega : u^*(t, \omega) \neq u(t, \omega)\} \leq \varepsilon$ when $t \in [0, T]$.

Let us notice the relationship between the minima introduced above. Here $X_0 \in L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ and $m_0 \in \mathcal{P}(\mathbb{R})$ are such that $m_0 = X_0\mu$.

- Every strong $W_p$-minimizer is a strong $L^p$-minimizer.
- If $(X^*, u^*)$ is a Pontryagin local $W_p$-minimizer, it is a Pontryagin local $L^p$-minimizer.
- Every Pontryagin local $L^p$-minimizer (respectively, $W_p$-minimizer) is a strong local $L^p$-minimizer (respectively $W_p$-minimizer).

4 Pontryagin maximum principle for the Lagrangian formulation of mean field type optimal control problem

In this section we assume that we are given with a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$, initial assignment $X_0 \in L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$, and a control process $(X^*, u^*)$ defined on this probability space that is a $L^p$-minimizer. Further, $m_0 = X_0\mu$.

To formulate the Pontryagin maximum principle, we use an extension of the notion of the Lebesgue point. Recall that a point $t$ is a Lebesgue point for the function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ if $\phi(t)$ can be approximated by the means of this function over neighborhoods of $t$. The prominent Lebesgue differentiation theorem states that if the function $\phi$ is integrable, then almost every times are Lebesgue points. The maximization condition in the finite-dimensional Pontryagin maximum principle usually is proved at the Lebesgue points. Our concept relies on the property that is close to concept of the Lebesgue point. However, we formulate it in the $L^p$ sense w.r.t. $\omega$.

Definition 4.1. A point $t \in [0, T]$ is called regular for the process $(X, u)$ if

$$\lim_{h \downarrow 0} \mathbb{E} \left\| \frac{1}{h} \int_t^{t+h} f(\tau, X(\tau), X(\tau)\mu, u(\tau))d\tau - f(t, X(t), X(t)\mu, u(t)) \right\|^p = 0,$$

and

$$\lim_{h \downarrow 0} \mathbb{E} \left| \frac{1}{h} \int_t^{t+h} f_0(\tau, X(\tau), X(\tau)\mu, u(\tau))d\tau - f_0(t, X(t), X(t)\mu, u(t)) \right| = 0.$$

Proposition 4.2. Almost every $t \in [0, T]$ are regular.

The proposition directly follows from Corollary [12, 2] proved in the Appendix [13] and the fact that, for a.e. $t \in T$, $\|u(t)\|_{L^p} < \infty$. The latter is a direct consequence of assumption $\|u\|_{L^p} < \infty$.

To formulate the Pontryagin maximum principle, we define the Pontryagin function (Hamiltonian) as follows: for $t \in [0, T]$, $x \in \mathbb{R}^d$, $m \in \mathcal{P}(\mathbb{R})$, $\psi \in \mathbb{R}^{d+*}$, $u \in U$, set

$$H(t, x, m, \psi, u) \triangleq \psi f(t, x, m, u) - f_0(t, x, m, u).$$

(3)
Additionally, let $\mathbb{H} : [0, T] \times L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \times L^q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{d, *}) \times L^p(\Omega, \mathcal{F}, \mathbb{P}; U) \to \mathbb{R}$ be defined by the following rule: for $t \in [0, T]$, $X \in L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$, $\Psi \in L^q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{d, *})$, $u \in L^p(\Omega, \mathcal{F}, \mathbb{P}; U)$,

$$
\mathbb{H}(t, X, \Psi, u) \triangleq \mathbb{E} H(t, X, \Psi, X \sharp \mathbb{P}, u).
$$

**Theorem 4.3.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a standard probability space, $X_0$ be an initial assignment, $(X^*, u^*) \in \mathcal{A}(X_0)$ be a Pontryagin local $L^p$-minimizer. Then there exists a function $\Psi \in L^q(\Omega, \mathcal{F}, \mathbb{P}; \Gamma^*)$ such that the following conditions holds true:

- **costate equation:** for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\Psi(\cdot, \omega)$ solves

$$
\frac{d}{dt} \Psi(t, \omega) = -\Psi(t, \omega) \nabla_x f(t, X^*(t, \omega), X^*(t) \sharp \mathbb{P}, u^*(t, \omega)) + \nabla_x f_0(t, X^*(t, \omega), X^*(t) \sharp \mathbb{P}, u^*(t)) \\
- \int_{\Omega}\nabla_m f(t, X^*(t, \omega'), X^*(t) \sharp \mathbb{P}, X^*(t, \omega), u^*(t, \omega')) \mathbb{P}(d\omega') \\
+ \int_{\Omega}\nabla_m f_0(t, X^*(t, \omega'), X^*(t) \sharp \mathbb{P}, X^*(t, \omega), u^*(t, \omega')) \mathbb{P}(d\omega').
$$

- **transversality condition:** for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$
\Psi(T, \omega) = -\nabla_x \sigma(X^*(T, \omega), X^*(T) \sharp \mathbb{P}) - \int_{\Omega}\nabla_m \sigma(X^*(T, \omega'), X^*(T) \sharp \mathbb{P}, X^*(T, \omega)) \mathbb{P}(d\omega').
$$

- **maximization of the Hamiltonian condition:** at every point $s \in [0, T]$ that is regular for $(X^*, u^*)$,

$$
\mathbb{H}(s, X^*(s), \Psi(s), u^*(s)) = \max_{\nu \in L^p(\Omega, \mathcal{F}, \mathbb{P}; U)} \mathbb{H}(s, X^*(s), \Psi(s), \nu)
$$

or, equivalently,

$$
H(s, X^*(s), \Psi(s), X^*(s) \sharp \mathbb{P}, u^*(s)) = \max_{u \in U} H(s, X^*(s), \Psi(s), X^*(s) \sharp \mathbb{P}, u) \quad \mathbb{P}\text{-a.s.}
$$

**Remark 4.4.** Computing the derivatives according to the formulae given in Propositions A.3, A.4 we arrive at the following the system on state and costate variables in the Hamiltonian form:

$$
\frac{d}{dt} X^*(t) = \nabla_x \mathbb{H}(t, X^*(t), \Psi(t), u^*(t)), \quad X(0) = X_0,
$$

$$
\frac{d}{dt} \Psi(t) = -\nabla_X \mathbb{H}(t, X^*(t), \Psi(t), u^*(t)), \quad \Psi(T) = -\nabla_X \Sigma(X^*(T)).
$$

Here, $\nabla_X \mathbb{H}, \nabla_x \mathbb{H}$ stands for the derivatives w.r.t. $X \in L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ and $\Psi \in L^q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{d, *})$. Additionally,

$$
\Sigma(X) \triangleq \mathbb{E} \sigma(X, X \sharp \mathbb{P}).
$$

Notice that this representation looks like a Pontryagin maximum principle for the processes defined on Banach space $L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ in the case where the controls are defined on $L^p(\Omega, \mathcal{F}, \mathbb{P}; U)$. In the paper, we do not rely on this reducing to the control problem in the Banach space due to fact that this way requires conditions those are stronger than $\{H1\} \{H7\}$ (see $[33, 36]$). In particular, these papers requires the existence of the Frechet derivative of the Hamiltonian $\mathbb{H}$ w.r.t. $X$ and $\Psi$, while we can prove only the Gateaux differentiability of the Hamiltonian (see Proposition A.4). Therefore, we give the direct proof that relies on the definition of derivative w.r.t. probability measure.

Definitions 3.3, 3.4 and Theorem 4.3 imply the following.

**Corollary 4.5.** The conclusion of Theorem 4.3 holds true in the cases when $(X^*, u^*)$ is a Pontryagin local $W_p$-minimizer, a strong local $L^p$-minimizer or strong local $W_p$-minimizer.
5 Spike variations

In this section we introduce and discuss spike variations of the Lagrangian control processes those play the crucial role in the proof of Pontryagin maximum principle in the Lagrangian form.

First, let us mention some properties of the space $X^p$. If $X \in X^p$, then, for each $t \in [0, T]$, $X(t) \in L^p(\Omega, F, \mathbb{P}; \mathbb{R}^d)$

and $M_p^p(X(t)\otimes \mathbb{P}) = \|X(t)\|_{L^p} \leq \|X\|_{X^p}$.

Notice that, if $X \in X^p$, then its restriction on $[s, r]$ lies in $L^p([s, r] \times \Omega, B([s, r]) \otimes F, \lambda \otimes \mathbb{P}; \mathbb{R}^d)$. We denote by $\|X\|_{L^p, s, r}$ the norm of restriction of $X$ on $[s, r]$ regarded as an element of $L^p([s, r] \times \Omega, B([s, r]) \otimes F, \lambda \otimes \mathbb{P}; \mathbb{R}^d)$, i.e.,

$$\|X\|_{L^p, s, r} = \left[ \int_s^r \mathbb{E} \|X(t)\|^p dt \right]^{1/p}.$$

The following relation between $\|X\|_{X^p}$ and $\|X\|_{L^p, s, r}$ is fulfilled:

$$\|X\|_{L^p, s, r} \leq (r - s)^{1/p} \|X\|_{X^p}.$$ Further, assume that a measurable function $X : [s, r] \times \Omega \to \mathbb{R}^d$ is such that:

- for each $t \in [0, T]$, $X(t) \in L^p(\Omega, F, \mathbb{P}; \mathbb{R}^d)$,
- the function $t \mapsto \|X(t)\|_{L^p}$ is bounded.

Then $X \in L^p([s, r] \times \Omega, B([s, r]) \otimes F, \lambda \otimes \mathbb{P}; \mathbb{R}^d)$ for every $s, r \in [0, T], s < r$, and

$$\|X\|_{L^p, s, r} \leq (r - s)^{1/p} \sup_{t \in [s, r]} \|X(t)\|_{L^p}.$$ (8)

Now we define the spike variation of Lagrangian control processes that is the main tool in the proof of Theorem 4.3.

Let $s \in [0, T)$ be a regular point for $(X^*, u^*)$. Choose a function $\nu \in L^p(\Omega, F, \mathbb{P}; U)$. For $h \in [0, T - s]$, set

$$u^h(\nu)(t, \omega) \triangleq \begin{cases} u^*(t, \omega), & t \in [0, s), \\ \nu(\omega), & t \in [s, s + h), \\ u^*(t, \omega), & t \in [s + h, T]. \end{cases}$$

Notice that $u^0 \equiv u^*$.

Denote

$$\Delta^s_\nu f^*(\omega) \triangleq f(s, X^*(s, \omega), X^*(s)\otimes \mathbb{P}, \nu(\omega)) - f(s, X^*(s, \omega), X^*(s)\otimes \mathbb{P}, u^*(s, \omega)),$$ (9)

$$\Delta^s_\nu f^h(\omega) \triangleq f(s, X^*(s, \omega), X^*(s)\otimes \mathbb{P}, \nu(\omega)) - f_0(s, X^*(s, \omega), X^*(s)\otimes \mathbb{P}, u^*(s, \omega)).$$ (10)

Further, let us consider the following system of ODEs:

$$\frac{d}{dt} Z^h_\nu(t, \omega) = f(t, Z^h_\nu(t, \omega), Z^h_\nu(t)\otimes \mathbb{P}, u^h_\nu(t, \omega)), \quad Z^h_\nu(0, \omega) = X_0(\omega).$$ (11)

**Proposition 5.1.** For each $h \in [0, T]$, there exists a unique solution of (11).

The proof this statement directly follows from Proposition C.1. The very construction of $Z^h_\nu$ implies that:

- $Z^h_\nu(0) = X^*(0)$, $t \in [0, T]$;
- for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$Z^h_\nu(t, \omega) = X^*(t, \omega), \quad t \in [0, s].$$
The following statement provides the estimates of the norm of $Z_h(t)$ as well as the distance between $Z_h(t)$ and $X^*(s)$. Its proof will use the Lipschitz continuity of the function $f$ w.r.t. $x$ and $m$. Recall that assumption (H5) and Proposition A.1 yield that, for every $t \in [0, T]$, $x_1, x_2 \in \mathbb{R}^d$, $m_1, m_2 \in \mathcal{P}(\mathbb{R}^d)$, $u \in U$,

$$
\|f(t, x_1, m_1, u) - f(t, x_2, m_2, u)\| \leq C_x \|x_1 - x_2\| + C_m W_p(m_1, m_2).
$$

(12)

Here $C_x$ and $C_m$ are upper bounds for the derivatives of the function $f$ w.r.t. $x$ and $m$ respectively.

**Proposition 5.2.** There exist constants $C_0, C_1, C_2, \bar{h}$ dependent on $(X^*, u^*)$ and $\nu$ such that

1. $\|Z_h(t)\|_{L^p} \leq C_0$ for $h \in [0, T]$, $t \in [s, T]$;
2. $\|Z_h(t) - X^*(s)\|_{L^p} \leq C_1(t - s)$ for $t \in [s, s + \bar{h}]$;
3. $\|Z_h(t) - X^*(t)\|_{L^p} \leq C_2 h$ when $h < \bar{h}$.

**Proof.** First, notice that

$$
\int_0^T \int_\Omega \|u_h^0(t, \omega)\|_{L^p} d\omega dt \leq \int_0^T \int_\Omega \|u^*(t, \omega)\|_{L^p} d\omega dt + \|\nu\|_{L^p} h.
$$

Thus,

$$
\|u_h^0\|_{L^p} \leq C_u \|u^*\|_{L^p} + T^{1/p} \|\nu\|_{L^p}.
$$

(13)

Due to assumption (H3) and equality $Z_h(s) = X^*(s)$, we have the following estimate, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$
\|Z_h(t, \omega) - X^*(s, \omega)\| \leq \int_s^t \|f(\tau, Z_h(\tau, \omega), Z_h(\tau) \in \mathbb{P}, u_h^0(\tau, \omega))\| d\tau
\leq C_\infty (t - s) + C_\infty \int_s^t (\|Z_h(\tau, \omega)\| + \|Z_h(\tau)\|_{L^p} + \|u_h^0(\tau, \omega)\|) d\tau.
$$

Hence, using the triangle inequality, we conclude that, if $t \in [s, T]$,

$$
\|Z_h(t) - X^*(s)\|_{L^p} \leq C_\infty (t - s) + 2C_\infty \int_s^t \|Z_h(\tau)\|_{L^p} d\tau + C_\infty \int_s^t \|u_h^0(\tau)\|_{L^p} d\tau.
$$

(14)

Thanks to (13), we obtain

$$
\|Z_h(t) - X^*(s)\|_{L^p} \leq C_\infty (t - s) + C_\infty C_u + 2C_\infty \int_s^t \|Z_h(\tau)\|_{L^p} d\tau.
$$

(15)

Since $X^*(s) \in L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$, estimate (15) together with the Gronwall’s inequality give the first statement of the proposition.

Estimating the right-hand side of (14) according to the first statement of the proposition, we obtain that, for $t \in [s, s + \bar{h}]$

$$
\|Z_h(t) - X^*(s)\|_{L^p} \leq C_\infty (1 + 2C_0 + \|\nu\|_{L^p}) (t - s).
$$

This proves the second statement of the proposition.

To prove the third statement of the proposition, we use the assumption that $s$ is the regular point for $(X^*, u^*)$ and find $\bar{h}$ such that, for any $h \in (0, \bar{h}]$,

$$
E \left( \frac{1}{h} \int_s^{s+h} f(\tau, X^*(\tau), X^*(\tau) \in \mathbb{P}, u^*(\tau)) d\tau - f(s, X^*(s), X^*(s) \in \mathbb{P}, u^*(s)) \right)^p \leq 1.
$$

(16)

Using once more the assumption that $s$ is regular, we have that $\|u^*(s)\|_{L^p} < \infty$. This, (16) and assumption (H3) give that

$$
\|X^*(s + h) - X^*(s)\|_{L^p} = \left\| \int_s^{s+h} f(\tau, X^*(\tau), X^*(\tau) \in \mathbb{P}, u^*(\tau)) d\tau \right\|_{L^p}
\leq \left\| \int_s^{s+h} f(\tau, X^*(\tau), X^*(\tau) \in \mathbb{P}, u^*(\tau)) d\tau - h f(s, X^*(s), X^*(s) \in \mathbb{P}, u^*(s)) \right\|_{L^p}
+ h \|f(s, X^*(s), X^*(s) \in \mathbb{P}, u^*(s))\|_{L^p}
\leq h + C_\infty (1 + 2\|X^*(s)\|_{L^p} + \|u^*(s)\|_{L^p}) h = C_1 h.
$$

11
This and the second statement of the proposition imply that, if \( h \in (0, \bar{h}) \),
\[
\|Z^h_\nu(s + h) - X^\ast(s + h)\|_{L^p} \leq (C_1 + C'_1)h.
\]
Furthermore, \( u^h_\nu(t) = u^\ast(t) \) when \( t \in [s + h, T] \). This together with Lipschitz continuity of the function \( f \) (see (12)) yield the following inequality for \( t \in [s + h, T] \)
\[
\|Z^h_\nu(t) - X^\ast(t)\|_{L^p} \leq \|Z^h_\nu(s + h) - X^\ast(s + h)\|_{L^p} + \left\| \int_{s + h}^t \left[ f(\tau, Z^h_\nu(\tau), Z^h_\nu(\tau)\sharp \mathbb{P}, u^\ast(\tau)) - f(\tau, X^\ast(\tau), X^\ast(\tau)\sharp \mathbb{P}, u^\ast(\tau)) \right] d\tau \right\| \\
\leq (C_1 + C'_1)h + (C_x + C_m) \int_{s + h}^t \|Z^h_\nu(\tau) - X^\ast(\tau)\|_{L^p} d\tau.
\]
Using the Gronwall’s inequality, we obtain the third statement of the proposition.

\[\square\]

**Corollary 5.3.** There exists a sequence \( \{h_n\}_{n=1}^\infty \) such that

1. \( \{Z^{h_n}_\nu(t, \omega)\}_{n=1}^\infty \) converges to \( X^\ast(t, \omega) \) for \( \lambda \otimes \mathbb{P} \)-a.e. \((t, \omega) \in [s, T] \times \Omega; \)

2. \( \{Z^{h_n}_\nu(T, \omega)\}_{n=1}^\infty \) converges to \( X^\ast(T, \omega) \) for \( \mathbb{P} \)-a.e. \( \omega \in \Omega. \)

**Proof.** Proposition 5.2 implies that the family \( \{Z^h_\nu\}_{h \in (0, \bar{h})} \) converges to the function \( X^\ast \) in measure \( \lambda \otimes \mathbb{P} \) as \( h \to 0 \). By [20] Proposition 2.2.26, there exists a sequence \( \{h_n\}_{n=1}^\infty \) such that \( \{Z^{h_n}_\nu(t, \omega)\}_{n=1}^\infty \) converges to \( X^\ast(t, \omega) \) for \( \lambda \otimes \mathbb{P} \)-a.e. \((t, \omega) \). This proves the first statement of the corollary. To prove the second statement, it suffices to consider the sequence \( \{Z^{h_n}_\nu(T)\} \) that converges to \( X^\ast(T) \) in probability \( \mathbb{P} \) and find the subsequence still denoted by \( \{h_n\} \) such that \( Z^{h_n}_\nu(T) \to X^\ast(T) \) \( \mathbb{P} \)-a.s.

Below, we fix the sequence \( \{h_n\}_{n=1}^\infty \) satisfying the statements of Corollary 5.3.

Now consider the following backward system of ODEs on \([s, T] \):

\[
\frac{d}{dt} Y_\nu(t, \omega) = \nabla_x f(t, X^\ast(t, \omega), X^\ast(t)\sharp \mathbb{P}, u^\ast(t, \omega)) \cdot Y_\nu(t, \omega) + \int_\Omega \nabla_m f(t, X^\ast(t, \omega), X^\ast(t)\sharp \mathbb{P}, X^\ast(t, \omega'), u^\ast(t, \omega')) Y_\nu(t, \omega') \mathbb{P}(d\omega'),
\]

\[\text{(17)}\]

**Proposition 5.4.** System (17) admits a unique solution. Moreover, there exists a constant \( C_3 \) such that, for all \( t \in [s, T] \),
\[
\|Y_\nu(t)\|_{L^p} \leq C_3.
\]

**Proof.** The existence and uniqueness result for \( Y_\nu(\cdot) \) directly follows from Proposition C.1 and the boundness of \( \nabla_x f \) and \( \nabla_m f \). Furthermore, due to assumption [H5] we have that
\[
\|Y_\nu(t)\|_{L^p} \leq \|\Delta^\nu f^\ast\|_{L^p} + (C_x + C_m) \int_s^t \|Y_\nu(\tau)\|_{L^p} d\tau.
\]
Applying the Gronwall’s inequality we obtain that \( \|Y_\nu(t)\|_{L^p} \) is uniformly bounded.

\[\square\]

Below, we shorten the notation and denote, for \( t \in [0, T] \), \( \omega, \omega' \in \Omega, \)
\[
f^\ast_x(t, \omega) \triangleq \nabla_x f(t, X^\ast(t, \omega), X^\ast(t)\sharp \mathbb{P}, u^\ast(t, \omega)),
\]
\[\text{(18)}\]
\[
f^\ast_m(t, \omega, \omega') \triangleq \nabla_m f(t, X^\ast(t, \omega), X^\ast(t)\sharp \mathbb{P}, X^\ast(t, \omega'), u^\ast(t, \omega)) + \langle f^\ast_m, Y_\nu \rangle(\tau, \omega) \triangleq \int_\Omega f^\ast_m(t, \omega, \omega') Y_\nu(t, \omega') \mathbb{P}(d\omega').
\]
\[\text{(19)}\]
Proposition 5.5. Let \( \{h_n\}_{n=1}^{\infty} \) satisfy the statements of Corollary 5.3. Then,

\[
\frac{1}{h_n} \| Z_{h_n}(t) - X^*(t) - h_n Y_\nu(t) \|_{L^p} \to 0 \text{ as } n \to \infty
\]

uniformly for \( t \in (s, T) \).

Proof. For simplicity, put

\[
F(t, \omega) \triangleq f_s^*(t, \omega) Y_\nu(t, \omega) + (f_m^*, Y_\nu)(\tau, \omega).
\]

Notice that, due to Proposition 5.4 \( \| Y_\nu(t) \|_{L^p} \) is uniformly bounded. Therefore,

\[
\| F(t) \|_{L^p} \leq C_4,
\]

where \( C_4 \) is a constant (certainly, dependent on \( (X^*, u^*) \)).

Choose \( t \in (s, T) \). Let \( N \) be such that, for every \( n > N \), we have that \( t > s + h_n \).

We have that

\[
\begin{align*}
\frac{1}{h_n} \| Z_{h_n}(s+h_n) - X^*(s+h_n) - h_n Y_\nu(s+h_n) \|_{L^p} & \\
\leq \frac{1}{h_n} \left\| \int_s^{s+h_n} \left[ f(\tau, Z_{h_n}^\nu(\tau), Z_{h_n}^\nu(\tau) \|_{L^p}, \nu) - f(\tau, X^*(\tau), X^*(\tau) \|_{L^p}, u^*(\tau)) \right] d\tau \\
& \quad - h_n \Delta^\nu f^* \right\|_{L^p}.
\end{align*}
\]

Since \( f \) is Lipschitz continuous w.r.t. \( x \) and \( m \) with constants \( C_x \) and \( C_m \) respectively (see (22)), \( \| F(t) \|_{L^p} \) is bounded (see (21)), using assumption \( (H4) \) and Proposition 5.2 we derive the following

\[
\begin{align*}
\frac{1}{h_n} \| Z_{h_n}^\nu(s+h_n) - X^*(s+h_n) - h_n Y_\nu(s+h_n) \|_{L^p} & \\
\leq \frac{1}{h_n} \| f(s, X^*(s), X^*(s) \|_{L^p}, \nu) - f(s, X^*(s), X^*(s) \|_{L^p}, u^*(s)) \|_{L^p} \\
& \quad + (C_x C_1 + C_m C_1 + C_4) h_n + \zeta(h_n)(1 + 2 \| X^*(s) \|_{L^p} + \| \nu \|_{L^p}).
\end{align*}
\]

The first term is equal to 0 (see (23)). The second term

\[
\begin{align*}
\frac{1}{h_n} \left\| \int_s^{s+h_n} f(\tau, X^*(\tau), X^*(\tau) \|_{L^p}, u^*(\tau)) d\tau \right\|_{L^p}
\end{align*}
\]

tends to 0 due to the assumption that \( s \) is a regular point. Simultaneously, \( (C_x C_1 + C_m C_1 + C_4) h_n + \zeta(h_n)(1 + 2 \| X^*(s) \|_{L^p} + \| \nu \|_{L^p}) \to 0 \) as \( n \to \infty \). Thus,

\[
\begin{align*}
a_n^{(1)} & \triangleq \frac{1}{h_n} \left\| \int_s^{s+h_n} f(\tau, X^*(\tau), X^*(\tau) \|_{L^p}, u^*(\tau)) d\tau \right\|_{L^p} \\
& \quad + (C_x C_1 + C_m C_1 + C_4) h_n + \zeta(h_n)(1 + 2 \| X^*(s) \|_{L^p} + \| \nu \|_{L^p})
\end{align*}
\]

converge to 0 when \( n \to \infty \).

Further,

\[
\begin{align*}
\frac{1}{h_n} \| Z_{h_n}^\nu(t) - X^*(t) - h_n Y_\nu(t) \|_{L^p} & \\
\leq \frac{1}{h_n} \| Z_{h_n}^\nu(s+h_n) - X^*(s+h_n) - h_n Y_\nu(s+h_n) \|_{L^p} \\
& \quad + \frac{1}{h_n} \left\| \int_{s+h_n}^{t} \left[ f(\tau, Z_{h_n}^\nu(\tau), Z_{h_n}^\nu(\tau) \|_{L^p}, \nu) - f(\tau, X^*(\tau), X^*(\tau) \|_{L^p}, u^*(\tau)) \right] d\tau \right\|_{L^p}
\end{align*}
\]

\[
\leq a_n^{(1)} + \frac{1}{h_n} \left\| \int_{s+h_n}^{t} \left[ f(\tau, Z_{h_n}^\nu(\tau), Z_{h_n}^\nu(\tau) \|_{L^p}, \nu) - f(\tau, X^*(\tau), X^*(\tau) \|_{L^p}, u^*(\tau)) \right] d\tau \right\|_{L^p}.
\]
Now we evaluate the second term in the right-hand side of (24). First, notice that, by the Minkowski’s integral inequality \[\text{(9.12)}\],

\[
\left\| \int_{s+h_n}^{t} [f(\tau, Z^{h_n}_\nu(\tau), Z^{h_n}_\nu(\tau) \mathbb{P}, u^*(\tau)) - f(\tau, X^*(\tau), X^*(\tau) \mathbb{P}, u^*(\tau)) - h_n F(\tau)]d\tau \right\|_{L^p} \\
\leq \int_{s+h_n}^{t} \|f(\tau, Z^{h_n}_\nu(\tau), Z^{h_n}_\nu(\tau) \mathbb{P}, u^*(\tau)) - f(\tau, X^*(\tau), X^*(\tau) \mathbb{P}, u^*(\tau)) - h_n F(\tau)\|_{L^p} d\tau.
\]  

(25)

Simultaneously, the following equality holds \(\mathbb{P}\)-a.s.:

\[
f(\tau, Z^{h_n}_\nu(\tau), Z^{h_n}_\nu(\tau) \mathbb{P}, u^*(\tau)) - f(\tau, X^*(\tau), X^*(\tau) \mathbb{P}, u^*(\tau)) = (f(\tau, Z^{h_n}_\nu(\tau), Z^{h_n}_\nu(\tau) \mathbb{P}, u^*(\tau)) - f(\tau, X^*(\tau), Z^{h_n}_\nu(\tau) \mathbb{P}, u^*(\tau)) + (f(\tau, X^*(\tau), Z^{h_n}_\nu(\tau) \mathbb{P}, u^*(\tau)) - f(\tau, X^*(\tau), X^*(\tau) \mathbb{P}, u^*(\tau))).
\]

Therefore, by the triangle inequality and definition of the function \(F\) (see \(\text{(20)}\)), we have that

\[
\|f(\tau, Z^{h_n}_\nu(\tau), Z^{h_n}_\nu(\tau) \mathbb{P}, u^*(\tau)) - f(\tau, X^*(\tau), X^*(\tau) \mathbb{P}, u^*(\tau)) - h_n F(\tau)\|_{L^p} \\
\leq \|f(\tau, Z^{h_n}_\nu(\tau), Z^{h_n}_\nu(\tau) \mathbb{P}, u^*(\tau)) - f(\tau, X^*(\tau), Z^{h_n}_\nu(\tau) \mathbb{P}, u^*(\tau)) - h_n f^*_x(\tau) Y_\nu(\tau)\|_{L^p} + \|f(\tau, X^*(\tau), Z^{h_n}_\nu(\tau) \mathbb{P}, u^*(\tau)) - f(\tau, X^*(\tau), X^*(\tau) \mathbb{P}, u^*(\tau)) - h_n (f^*_m(\tau) Y_\nu(\tau) - F(\tau))\|_{L^p}.
\]

(26)

Since \(f\) is continuously differentiable w.r.t. \(x\), we conclude that

\[
f(\tau, Z^{h_n}_\nu(\tau), Z^{h_n}_\nu(\tau) \mathbb{P}, u^*(\tau)) - f(\tau, X^*(\tau), Z^{h_n}_\nu(\tau) \mathbb{P}, u^*(\tau)) = \int_0^1 \nabla_x f(\tau, X^*(\tau) + r(Z^{h_n}_\nu(\tau) - X^*(\tau)), Z^{h_n}_\nu(\tau) \mathbb{P}, u^*(\tau))(Z^{h_n}_\nu(\tau) - X^*(\tau))dr.
\]

Thus, taking into account the definition of \(f^*_x\) (see \(\text{(18)}\)), we have

\[
\|f(\tau, Z^{h_n}_\nu(\tau), Z^{h_n}_\nu(\tau) \mathbb{P}, u^*(\tau)) - f(\tau, X^*(\tau), Z^{h_n}_\nu(\tau) \mathbb{P}, u^*(\tau)) - h_n f^*_x(\tau) Y_\nu(\tau)\| \\
\leq \int_0^\tau \|\omega^x_x(\tau, \omega)\|d\tau \cdot \|Z^{h_n}_\nu(\tau) - X^*(\tau)\| + \|f^*_x(\tau)(Z^{h_n}_\nu(\tau) - X^*(\tau))\|.
\]

Here we denote

\[
\omega^x_x(\tau, \omega) \equiv \nabla_x f(\tau, X^*(\tau, \omega) + r(Z^{h}_\nu(\tau, \omega) - X^*(\tau, \omega)), Z^{h_n}_\nu(\tau) \mathbb{P}, u^*(\tau, \omega)) \\
- \nabla_x f(\tau, X^*(\tau, \omega), X^*(\tau) \mathbb{P}, u^*(\tau, \omega))
\]

and omit the dependence on \(\omega\). Therefore, using the Hölder inequality, one can estimate the integral over \([s+h, t]\) of the first term in the right-hand side of (26)

\[
\int_{s+h_n}^t \|f(\tau, Z^{h_n}_\nu(\tau), Z^{h_n}_\nu(\tau) \mathbb{P}, u^*(\tau)) - f(\tau, X^*(\tau), Z^{h_n}_\nu(\tau) \mathbb{P}, u^*(\tau)) - h_n f^*_x(\tau) Y_\nu(\tau)\|_{L^p} dt \\
\leq \left[ \int_s^T \int_0^1 \|\omega^x_x(\tau, \omega)\| q d\nu \mathbb{P}(d\omega) d\tau \right]^{1/q} \|Z^{h_n}_\nu - X^*\|_{L^p, s, T} + \int_{s+h_n}^t \|f^*_x(\tau)\|_{L^q} \|Z^{h_n}_\nu(\tau) - X^*(\tau) - h_n Y_\nu(\tau)\|_{L^p} d\tau.
\]

(27)

Recall that (see Proposition \(\text{(5.2)}\)) that, for every \(\tau \in [s, T]\), \(\|Z^{h_n}_\nu(\tau) - X^*(\tau)\|_{L^p} \leq C_2 h\). Therefore, by \(\text{(5)}\),

\[
\|Z^{h_n}_\nu - X^*\|_{L^p, s, T} \leq T^{1/p} C_2 h_n.
\]

Additionally, thanks to assumption \(\text{(H5)}\)

\[
\|f^*_x(\tau)\| \leq C_x.
\]

(28)
Since $f_x$ is continuous, while we choose $\{h_n\}$ such that $Z^{h_n}(\tau, \omega)$ converge to $X^*(\tau, \omega)$ for $\lambda \otimes \mathbb{P}$-a.e. $(t, \omega) \in [s, T] \times \Omega$, the sequence $\{\varpi^n_x(\tau, \omega)\}$ converges to zero $\lambda \otimes \mathbb{P}$-a.e. when $n \to \infty$. Moreover, due to assumption [(H5)]

$$\|\varpi^n_x(\tau, \omega)\| \leq 2C_x.$$ 

Therefore, by the dominating convergence theorem, the quantity

$$a_n^{(2)} \triangleq C_2 T^{1/p} \left[ \int_s^T \int_0^1 \left( \int_0^1 \|\varpi^m_n(\tau, \omega)\|^q d\tau \right)^{1/q} d\omega \right]$$

(29)

tends to zero as $n \to \infty$. Plugging this estimate and (28) into (27), we conclude that

$$\int_{s+h_n}^t \|f(\tau, Z^{h_n}(\tau), Z^{h_n}(\tau) \mathbb{P}, u^*(\tau)) - f(\tau, X^*(\tau), Z^{h_n}(\tau) \mathbb{P}, u^*(\tau)) - h_n f_x^*(\tau) Y^*_\nu \|_{L^p} dt$$

$$\leq a_n^{(2)} h + C_x \int_{s+h_n}^t \|Z^{h_n}(\tau) - X^*(\tau) - h_n Y^*_\nu(\tau)\|_{L^p} d\tau.$$

(30)

Now let us evaluate the integral over $[s + h, t]$ of the second term in the right-hand side of (26). Since the function $m \mapsto f(\tau, x, m, u)$ is continuously differentiable w.r.t. $m$, letting, for the given $\tau \in [s, T]$ and $\theta \in [0, 1]$,

$$m^n(\theta, \tau) \triangleq \theta Z^{h_n}(\tau) \mathbb{P} + (1 - \theta) X^*(\tau) \mathbb{P},$$

we obtain

$$f(\tau, X^*(\tau), Z^{h_n}(\tau) \mathbb{P}, u^*(\tau)) - f(\tau, X^*(\tau), X^*(\tau) \mathbb{P}, u^*(\tau))$$

$$= \int_0^1 \int_\Omega \left[ \delta f_m (t, X^*(\tau), m^n(\theta, \tau), y, u)(Z^{h_n}(\tau) \mathbb{P})(dy) - (X^*(\tau) \mathbb{P})(dy) \right] d\theta$$

$$= \int_0^1 \int_\Omega \left[ \delta f_m (t, X^*(\tau), m^n(\theta, \tau), Z^{h_n}(\tau, \omega'), u^*(\tau))$$

$$- \delta f_m (t, X^*(\tau), m^n(\theta, \tau), X^*(\tau, \omega'), u^*(\tau)) \right] \mathbb{P}(d\omega') d\theta$$

$$= \int_0^1 \int_\Omega \nabla_m f(t, X^*(\tau), m^n(\theta, \tau), X^*(\tau, \omega') + r(Z^{h_n}(\tau, \omega') - X^*(\tau, \omega'), u^*(\tau))$$

$$(Z^{h_n}(\tau, \omega') - X^*(\tau, \omega')) d\tau.$$

Denote

$$\varpi^m_n(\theta, \tau, \omega, \omega')$$

$$\triangleq \nabla_m f(t, X^*(\tau, \omega), \theta Z^{h_n}(\tau) \mathbb{P} + (1 - \theta) X^*(\tau) \mathbb{P}, X^*(\tau, \omega') + r(Z^{h_n}(\tau, \omega') - X^*(\tau, \omega'), u^*(\tau))$$

$$- \nabla_m f(t, X^*(\tau, \omega), X^*(\tau, \omega') \mathbb{P}, X^*(\tau, \omega'), u^*(\tau, \omega)).$$

Therefore, using the definitions of $\nabla f_m, Y^*_\nu$ (see [19]), we have that

$$\|f(\tau, X^*(\tau), Z^{h_n}(\tau) \mathbb{P}, u^*(\tau)) - f(\tau, X^*(\tau), X^*(\tau) \mathbb{P}, u^*(\tau)) - h_n f_x^*(\tau) Y^*_\nu(\tau, \omega)\|$$

$$\leq \int_s^T \int_0^1 \int_\Omega \left( \|\varpi^m_n(\theta, \tau, \omega, \omega')\| Z^{h_n}(\tau, \omega') - X^*(\tau, \omega') \| d\tau \right)^{1/q} d\theta$$

$$+ \int_\Omega \left( \|f_m^n(\tau, \omega, \omega')\| Z^{h_n}(\tau, \omega') - X^*(\tau, \omega') - h_n Y^*_\nu(\tau, \omega') \| d\omega' \right)^{1/q}.$$

This and the Hölder’s inequality and assumption [(H5)] give that

$$\int_{s+h_n}^t \|f(\tau, X^*(\tau), Z^{h_n}(\tau) \mathbb{P}, u^*(\tau)) - f(\tau, X^*(\tau), X^*(\tau) \mathbb{P}, u^*(\tau)) - h_n f_x^*(\tau) Y^*_\nu(\tau, \omega)\|_{L^p} d\tau$$

$$\leq \left[ \int_s^T \int_0^1 \int_\Omega \left( \|\varpi^m_n(\theta, \tau, \omega, \omega')\| d\omega' \right)^{1/q} \| Z^{h_n}(\tau, \omega') - X^*(\tau, \omega') \|_{L^p} d\tau \right]^{1/q}$$

$$+ C_m \int_{s+h_n}^t \|Z^{h_n}(\tau) - X^*(\tau) - h_n Y^*_\nu(\tau)\|_{L^p} d\tau.$$
Denote
\[ a_n^{(3)} = C_4 T^{1/p} \left[ \int_{s+h_n}^{T} \int_{\Omega} \int_{0}^{r} \int_{0}^{r} \int_{0}^{r} \left\| w_m(\theta, r, \tau, \omega, \omega') \right\|^q d\theta d\omega' d\theta d\omega d\tau \right]^{1/q}. \] (32)

Notice that
\[ \left\| w_m^p(\theta, r, \tau, \omega, \omega') \right\| \leq 2C_m, \]
while the sequence \( \{h_n\} \) is such that \( Z^{h_n}(\tau, \omega) \to X^*(\tau, \omega) \) \( \lambda \otimes \mathbb{P} \)-a.e. as \( n \to \infty \). This, continuity of the function \( \nabla_m f \), the fact that \( \|Z^{h_n}(t) - X^*(t)\|_{L^p} \) converges to zero uniformly w.r.t. \( t \) and the dominated convergence theorem imply that
\[ a_n^{(3)} \to 0 \text{ as } n \to \infty. \] (33)

Applying the Hölder inequality for the first term in the right-hand side of (31), using definition (32) and the third statement of Proposition 5.2, we have that
\[ \int_{s+h_n}^{t} \left\| f(\tau, X^*(\tau), Z^{h_n}(\tau) \mathbb{P}, u^*(\tau)) - f(\tau, X^*(\tau), X^*(\tau) \mathbb{P}, u^*(\tau)) - h_n(f_m, Y_\nu)(\tau, \omega) \right\|_{L^p} d\tau \]
\[ \leq a_n^{(3)} h + C_m \int_{s+h_n}^{t} \|Z^{h_n}(\tau) - X^*(\tau) - h_n Y_\nu(\tau)\|_{L^p} d\tau. \] (34)

Combining (24), (25), (26), (30), (34), we obtain the following
\[ \frac{1}{h_n} \|Z^{h_n}(t) - X^*(t) - h_n Y_\nu(t)\|_{L^p} \leq \Delta_n^{(1)} + \Delta_n^{(2)} + \Delta_n^{(3)} + (C_x + C_m) \int_{s+h_n}^{t} \frac{1}{h_n} \|Z^{h_n}(\tau) - X^*(\tau) - h_n Y_\nu(\tau)\|_{L^p} d\tau. \]

Applying to this estimate the Gronwall’s inequality, we obtain
\[ \frac{1}{h} \|Z^{h_n}(t) - X^*(t) - h Y_\nu(t)\|_{L^p} \leq \Delta_n^{(1)} + \Delta_n^{(2)} + \Delta_n^{(3)} e^{(C_x + C_m)(t-s)}. \]

This and the fact that the sequences \( \{a_n^{(1)}\}, \{a_n^{(2)}\}, \{a_n^{(3)}\} \) converge to zero (see 23, 29, 33) give the statement of the proposition.

Below we evaluate the variation of the running cost. For shortness, we will use the following notation:
\[ f^x_0(t, \omega) \triangleq \nabla_x f_0(t, X^*(t, \omega), X^*(t) \mathbb{P}, u^*(t, \omega)), \]
\[ f^x_{0,m}(t, \omega, \omega') \triangleq \nabla_m f_0(t, X^*(t, \omega), X^*(t) \mathbb{P}, X^*(t, \omega'), u^*(t, \omega)), \]
\[ (f^x_{0,m}, Y_\nu)(t, \omega) \triangleq \int_{\Omega} f^x_{0,m}(t, \omega, \omega') Y_\nu(t, \omega') d\omega'. \] (36)

**Lemma 5.6.** For every \( r_1, r_2 \in [s, T] \), \( r_1 < r_2 \), \( f^x_{0,x} \in L^q([r_1, r_2] \times \Omega; \mathcal{B}([r_1, r_2]) \otimes \mathcal{F}, \lambda \otimes \mathbb{P}; \mathbb{R}^d_{t, \omega}), \)
\( f^x_{0,m} \in L^q([r_1, r_2] \times \Omega; \mathcal{B}([r_1, r_2]) \otimes \mathcal{F} \otimes \mathcal{F}, \lambda \otimes \mathbb{P} \otimes \mathbb{P}; \mathbb{R}^{d,t}). \) Moreover, \( \|f^x_{0,x}\|_{L^q_{t,\omega}} \) and \( \|f^x_{0,m}\|_{L^q_{t,\omega}} \) are bounded uniformly w.r.t. \( r_1 \) and \( r_2 \).

**Proof.** We consider only \( f^x_{0,x} \). The case of \( f^x_{0,m} \) is the same.

Due to definition of \( f^x_{0,x} \) (see (33)), and assumption [H6]
\[ \|f^x_{0,x}(t, \omega)\|^q \leq C_0^q (1 + \|X^*(t, \omega)\|)^p + \|X^*(t)\|_{L^p} + \|u^*(t, \omega)\|^p. \]

Using the first statement of Proposition 5.2 for \( h = 0 \) and the fact that \( u^* \in \mathcal{U}^p \), we obtain that
\[ \|f^x_{0,x}\|_{L^q_{t,\omega}}^q = \int_{s}^{s+h_0} \int_{\Omega} \left\| f^x_{0,x}(t, \omega) \right\|^q d\omega d\tau \]
\[ \leq \int_{r_1}^{r_2} \int_{\Omega} C_0^q (1 + \|X^*(t, \omega)\|)^p + \|X^*(t)\|_{L^p} + \|u^*(t, \omega)\|^p) d\omega d\tau \]
\[ \leq C_0^q (T + 2TC_0^p + \|u^*\|_{L^p}) < \infty. \]

\[ \square \]
Proposition 5.7. The following equality holds true:

\[
\lim_{n \to \infty} \frac{1}{h_n} \left[ \int_0^T \mathbb{E}f_0(t, Z_{\nu_n}(t), Z_{\nu_n}(t)\mathbb{P}, u_{\nu_n}(t))dt - \int_0^T \mathbb{E}f_0(t, X^*(t), X^*(t)\mathbb{P}, u^*(t))dt \right]
\]

\[= \mathbb{E}\Delta^*_\nu f_0^* + \int_s^T \mathbb{E}[f_{0,x}^*(t)Y_\nu(t) + (f_{0,m}^*, Y_\nu)(t)]dt.\]

Proof. We split the proof into the five steps.

Step 1. Notice that, \(u_{\nu_n}(t) = u^*(t)\) when \(t \notin [s, s + h_n]\), and \(u_{\nu_n}(t) = \nu\) for \(t \in [s, s + h_n]\). Moreover, \(Z_{\nu_n}(t) = X^*(t)\) on \([0, s]\). Therefore,

\[
\left[ \int_0^T \mathbb{E}[f_0(t, Z_{\nu_n}(t), Z_{\nu_n}(t)\mathbb{P}, u_{\nu_n}(t))dt - \int_0^T \mathbb{E}f_0(t, X^*(t), X^*(t)\mathbb{P}, u^*(t))dt \right]
\]

\[\leq G_n^{(1)} + G_n^{(2)} + G_n^{(3)} + G_n^{(4)},\]

where we denote

\[
G_n^{(1)} \triangleq \int_s^{s+h_n} \mathbb{E}[f_0(t, Z_{\nu_n}(t), Z_{\nu_n}(t)\mathbb{P}, \nu) - f_0(t, X^*(t), X^*(t)\mathbb{P}, u^*(t))]dt - h_n\mathbb{E}\Delta^*_\nu f_0^*,
\]

\[
G_n^{(2)} \triangleq h_n \int_s^{s+h_n} \mathbb{E}[f_{0,x}^*(t)Y_\nu(t) + (f_{0,m}^*, Y_\nu)(t)]dt,
\]

\[
G_n^{(3)} \triangleq \int_{s+h_n}^T \mathbb{E}[f_0(t, Z_{\nu_n}(t), Z_{\nu_n}(t)\mathbb{P}, u^*(t)) - f_0(t, X^*(t), Z_{\nu_n}(t)\mathbb{P}, u^*(t)) - h_n f_{0,x}^*(t)Y_\nu(t)]dt
\]

\[
G_n^{(4)} \triangleq \int_{s+h_n}^T \mathbb{E}[f_0(t, X^*(t), Z_{\nu_n}(t)\mathbb{P}, u^*(t)) - f_0(t, X^*(t), X^*(t)\mathbb{P}, u^*(t)) - h_n (f_{0,m}^*, Y_\nu)(t)]dt.
\]

In the following, we will show that \(G_n^{(i)}/h_n \to 0\) as \(n \to \infty\).

Step 2. Now choose \(t \in [s, s + h]\). Notice that

\[
|\mathbb{E}[f_0(t, Z_{\nu_n}(t), Z_{\nu_n}(t)\mathbb{P}, \nu) - f_0(s, X^*(s), X^*(s)\mathbb{P}, \nu)]|
\]

\[\leq |\mathbb{E}[f_0(t, Z_{\nu_n}(t), Z_{\nu_n}(t)\mathbb{P}, \nu) - f_0(s, Z_{\nu_n}(t), Z_{\nu_n}(t)\mathbb{P}, \nu)]|
\]

\[+ |\mathbb{E}[f_0(s, Z_{\nu_n}(t), Z_{\nu_n}(t)\mathbb{P}, \nu) - f_0(s, X^*(s), Z_{\nu_n}(t)\mathbb{P}, \nu)]|
\]

\[+ |\mathbb{E}[f_0(s, X^*(s), Z_{\nu_n}(t)\mathbb{P}, \nu) - f_0(s, X^*(s), X^*(s)\mathbb{P}, \nu)]|.\]

Thanks to assumption \([114]\) we have that

\[
|f_0(t, Z_{\nu_n}(t), Z_{\nu_n}(t)\mathbb{P}, \nu) - f_0(s, Z_{\nu_n}(t), Z_{\nu_n}(t)\mathbb{P}, \nu)|
\]

\[\leq \varsigma(t-s)(1 + \|Z_{\nu_n}(t)\|^p + \|Z_{\nu_n}(t)\|_L^p + \|\nu\|^p).
\]

Therefore, using the first statement of Proposition \([5.2]\) we conclude that

\[
\mathbb{E}[f_0(t, Z_{\nu_n}(t), Z_{\nu_n}(t)\mathbb{P}, \nu) - f_0(s, Z_{\nu_n}(t), Z_{\nu_n}(t)\mathbb{P}, \nu)]
\]

\[\leq \varsigma(t-s)(1 + 2\|Z_{\nu_n}(t)\|_L^p + \|\nu\|_L^p) \leq \varsigma(t-s)(1 + 2C_0^p + \|\nu\|_L^p).
\]
Since \( f_0 \) is continuously differentiable w.r.t. \( x \), we have that
\[
|f_0(s, Z^{h_n}_{\nu}(t), Z^{h_n}_{\nu}(t)\mathbb{P}, \nu(\omega)) - f_0(s, X^*(s, \omega), Z^{h_n}_{\nu}(t)\mathbb{P}, \nu(\omega))| \\
\leq \int_0^1 \| \nabla_x f_0(s, X^*(s) + r(Z^{h_n}_{\nu}(t) - X^*(s)), Z^{h_n}_{\nu}(t)\mathbb{P}, \nu) \| \cdot \| Z^{h_n}_{\nu}(t) - X^*(s) \| dr.
\]

Using the Hölder’s inequality, we obtain
\[
\mathbb{E}|f_0(s, Z^{h_n}_{\nu}(t), Z^{h_n}_{\nu}(t)\mathbb{P}, \nu) - f_0(s, X^*(s), Z^{h_n}_{\nu}(t)\mathbb{P}, \nu)| \\
\leq \mathbb{E} \left[ \int_0^1 \| \nabla_x f_0(s, X^*(s) + r(Z^{h_n}_{\nu}(t) - X^*(s)), Z^{h_n}_{\nu}(t)\mathbb{P}, \nu) \| dr \cdot \| Z^{h_n}_{\nu}(t) - X^*(s) \| \right]^{1/\alpha} \| Z^{h_n}_{\nu}(t) - X^*(s) \|_{L^p}.
\]

Thanks to assumption [H6] and Proposition 5.2, we conclude
\[
\mathbb{E}|f_0(s, Z^{h_n}_{\nu}(t), Z^{h_n}_{\nu}(t)\mathbb{P}, \nu) - f_0(s, X^*(s), Z^{h_n}_{\nu}(t)\mathbb{P}, \nu)| \leq \left( C^0_0(1 + 2C^0_0 + \| \nu \|_{L^p})^{1/\alpha} C_1 h_n. \right)
\]

Since \( f_0 \) is continuously differentiable w.r.t. \( m \), the following estimate holds true \( \mathbb{P} \)-a.s.:
\[
|f_0(s, X^*(s, \omega), Z^{h_n}_{\nu}(t)\mathbb{P}, \nu(\omega)) - f_0(s, X^*(s, \omega), X^*(s)\mathbb{P}, \nu(\omega))| \\
= \left| \int_0^1 \left[ \frac{\delta f_0}{\delta m}(s, X^*(s, \omega), m^n(\theta, s), Z^{h_n}_{\nu}(t, \omega'), \nu(\omega)) \\
- \frac{\delta f_0}{\delta m}(s, X^*(s, \omega), m^n(\theta, s), X^*(s, \omega'), \nu(\omega)) \right] \mathbb{P}(d\omega') d\theta \right| \\
\leq \int_0^1 \int_0^1 \| \nabla f_0(s, X^*(s, \omega), m^n(\theta, s), y^n(r, s, \omega'), \nu(\omega)) \| \cdot \| Z^{h_n}_{\nu}(t, \omega') - X^*(s, \omega') \| dr \mathbb{P}(d\omega') d\theta.
\]

Above we denoted
\[
y^n(r, s, \omega') \triangleq X^*(s, \omega') + r(Z^{h_n}_{\nu}(t, \omega') - X^*(s, \omega')) \\
m^n(\theta, s) \triangleq \theta Z^{h_n}_{\nu}(\tau)\mathbb{P} + (1 - \theta)X^*(\tau)\mathbb{P}.
\]

Applying the Hölder inequality, we obtain that
\[
\mathbb{E}|f_0(s, X^*(s, \omega), Z^{h_n}_{\nu}(t)\mathbb{P}, \nu(\omega)) - f_0(s, X^*(s, \omega), X^*(t)\mathbb{P}, \nu(\omega))| \\
\leq \left[ \int_0^1 \int_0^1 \int_0^1 \| \nabla f_0(s, X^*(s, \omega), m^n(\theta, s), y^n(r, s, \omega'), \nu(\omega)) \| dr \mathbb{P}(d\omega') d\theta \mathbb{P}(d\omega) \right]^{1/\alpha} \| Z^{h_n}_{\nu}(t) - X^*(s) \|_{L^p}.
\]

Using estimates from assumption [H6] Proposition 5.2, we deduce the following estimate
\[
\mathbb{E}|f_0(s, X^*(s, \omega), Z^{h_n}_{\nu}(t)\mathbb{P}, \nu(\omega)) - f_0(s, X^*(s, \omega), X^*(t)\mathbb{P}, \nu(\omega))| \\
\leq \left( C^0_0(1 + 3C^0_0 + \| \nu \|_{L^p})^{1/\alpha} C_1 h_n. \right)
\]

Combining (42)–(46), we have that, for \( t \in [s, s + h_n] \),
\[
\left| \mathbb{E}|f_0(t, Z^{h_n}_{\nu}(t), Z^{h_n}_{\nu}(t)\mathbb{P}, \nu) - f_0(s, X^*(s), X^*(t)\mathbb{P}, \nu)| \right| \leq a_n^{(4)}.
\]

where
\[
a_n^{(4)} \triangleq \zeta(h_n)(1 + 2C^0_0 + \| \nu \|_{L^p}) + \left( C^0_0(1 + 3C^0_0 + \| \nu \|_{L^p})^{1/\alpha} C_1 h_n + C^0_0(1 + 3C^0_0 + \| \nu \|_{L^p})^{1/\alpha} C_1 h_n. \right)
\]
Notice that the sequence \( \{a_n^{(4)}\}_{n=1}^{\infty} \) converges to 0.

Therefore, recalling the definition of \( \Delta_{q}f_0^* \), we obtain the estimate

\[
G_n^{(1)} \leq \left| \int_{\Omega} \int_s^{s+h_n} \left[f_0(s, X^*(s, \omega), X^*(s, \omega))^2 \mathbb{P}, u^*(s, \omega)\right] - f_0(s, X^*(s, \omega), X^*(s, \omega))^2 \mathbb{P} (d\omega) dt \right| + a_n^{(4)} \cdot h_n
\]  

(48)

This and the assumption that \( s \) is regular (see Definition 4.1) imply that \( G_n^{(1)}/h_n \to 0 \) as \( n \to \infty \).

Step 3. Let us estimate

\[
\int_s^{s+h_n} \mathbb{E} \left[ f_0^*(t) Y_\nu(t) + \langle f_0^*, Y_\nu(t) \rangle \right] dt.
\]

We have that

\[
\int_s^{s+h_n} \mathbb{E} \left| f_0^*(t) Y_\nu(t) \right| dt \leq \left\| f_0^* \right\|_{L^q,s,s+h_n} \cdot \left\| Y_\nu \right\|_{L^p,s,s+h_n}.
\]  

(49)

By Lemma 5.6, the functions \( f_0^* \) are uniformly bounded. Further, due to Proposition 5.4 and \( (5) \),

\[
\left\| Y_\nu \right\|_{L^p,s,s+h_n} \leq (h_n)^{1/p} C_3.
\]  

(50)

Combining this, (49) and (50) we arrive at the estimate

\[
\int_s^{s+h_n} \mathbb{E} \left[ f_0^*(t) Y_\nu(t) \right] \mathbb{P} (d\omega) dt \leq C_5 h_n^{1/p},
\]  

(51)

where \( C_5 \) is a constant.

Analogously, we have

\[
\int_s^{s+h_n} \mathbb{E} \left[ \langle f_0^*, Y_\nu \rangle (t) \right] dt \leq \int_s^{s+h_n} \int_{\Omega} \int_s^{s+h_n} \left| f_0^* (t, \omega', \omega') Y_\nu(t, \omega') \mathbb{P} (d\omega') \mathbb{P} (d\omega) dt \right|
\]

\[
\leq \left\| f_0^* \right\|_{L^q,s,s+h_n} \cdot \left\| Y_\nu \right\|_{L^p,s,s+h_n}.
\]

Using Lemma 5.6 we obtain that \( f_0^*/m \) are uniformly bounded. This and (49) give the estimate

\[
\int_s^{s+h_n} \int_{\Omega} \left| f_0^* (t, \omega) \right| \mathbb{P} (d\omega) dt \leq C_6 h_n^{1/p},
\]  

where \( C_6 \) is a constant (certainly dependent on \( (X^*, u^*) \)). Using this inequality and (51), we conclude that

\[
\int_s^{s+h_n} \mathbb{E} \left[ f_0^*(t) Y_\nu(t) + \langle f_0^*, Y_\nu(t) \rangle \right] dt \leq (C_5 + C_6) h_n^{1/p}.
\]

Therefore, \( G_n^{(2)} \) defined by (38) is such that

\[
G_n^{(2)}/h_n \to 0 \text{ as } n \to \infty.
\]

Step 4. We have that

\[
f_0(t, Z^h_{\nu} (t), Z^h_{\nu} (t)) \mathbb{P} (u^*(t)) - f_0(t, X^*(t), Z^h_{\nu} (t)) \mathbb{P} (u^*(t)) = \int_0^1 \nabla_x f_0(t, X^*(t) + r(Z^h_{\nu} (t) - X^*(t)), Z^h_{\nu} (t)) \mathbb{P} (u^*(t))(Z^h_{\nu} (t) - X^*(t)) dr.
\]

Denote

\[
\varphi^h_{0, \nu}(r, t, \omega) \triangleq \nabla_x f_0(t, X^*(t, \omega) + r(Z^h_{\nu} (t, \omega) - X^*(t, \omega)), Z^h_{\nu} (t, \omega) \mathbb{P} (u^*(t, \omega))
\]

\[-\nabla_x f_0(t, X^*(t, \omega), Z^h_{\nu} (t) \mathbb{P} (u^*(t, \omega)).
\]
Therefore,
\[
G^{(3)}_n \leq \int_s^T \int_0^1 \int_0^1 \| \varpi^n_{0,x}(r,t,\omega) \| ds \| dr \mathbb{P}(d\omega) dt \\
+ \int_s^T \int_0^1 \| f^n_{0,x}(t,\omega) \| ds \| dr \mathbb{P}(d\omega) dt \\
\leq \left[ \int_s^T \int_0^1 \int_0^1 \| \varpi^n_{0,x}(r,t,\omega) \|^q dr \mathbb{P}(d\omega) dt \right]^{1/q} \| Z^{h_n}_t - X^* \|_{L^p,s,T} \\
+ \| f^n_{0,x} \|_{L^q,s,T} \| Z^{h_n}_t - X^* - h_n Y^*_t \|_{L^p,s,T}.
\]

(52)

Notice that, due to the choice of the sequence \( \{ h_n \}_{n=1}^\infty \), \( Z^{h_n}_t \to X^* \) \( \lambda \otimes \mathbb{P} \)-a.e. Therefore, \( \varpi^n_{0,x} \) converges to zero \( \lambda \otimes \mathbb{P} \)-a.e. as \( n \to \infty \). Moreover,
\[
\| \varpi^n_{0,x}(r,t,\omega) \| \leq \| \nabla_x f_0(t, X^*(t, \omega) + r(Z^{h_n}_t, \omega) - X^*(t, \omega)), Z^{h_n}_t(t) \|_{\mathbb{P}}, u^*(t, \omega)) \| \\
+ \| \nabla_x f_0(t, X^*(t, \omega), Z^{h_n}_t(t) \|_{\mathbb{P}}, u^*(t, \omega)) \|.
\]

Using assumption (H6), Jensen's inequality, Proposition 5.2 and the fact that \( u^* \in U^\mathbb{P} \), we obtain that
\[
\int_s^T \int_0^1 \int_0^1 \| \varpi^n_{0,x}(r,t,\omega) \|^q dr \mathbb{P}(d\omega) dt \leq 2C_2^1(1 + 2C_0 + \| u^* \|_{U^\mathbb{P}} < \infty.
\]

Therefore, by the dominated convergence theorem,
\[
\int_s^T \int_0^1 \int_0^1 \| \varpi^n_{0,x}(r,t,\omega) \|^q dr \mathbb{P}(d\omega) dt \to 0 \text{ as } n \to \infty.
\]

Furthermore, by the third statement of Proposition 5.2 and (8),
\[
\| Z^{h_n}_t - X^*(t) \|_{L^p} \leq T^{1/p} C_1 h_n.
\]

Additionally, by Lemma 5.6, \( \| f^n_{0,x} \|_{L^q,s,T} < \infty \). Finally, thanks to Proposition 5.5 and (8), \( \| Z^{h_n}_t(t) - X^*(t) - h_n Y^*_t(t) \|_{L^p,s,T} / h_n \to 0 \) as \( n \to \infty \).

Combining the above estimates of the right-hand side of estimate (52), we conclude that
\[
\frac{G^{(3)}_n}{h_n} \to 0 \text{ as } n \to \infty.
\]

Step 5. As above, we have that
\[
f_0(t, X^*(t, \omega), Z^{h_n}_t(t) \|_{\mathbb{P}}, u^*(t, \omega)) - f_0(t, X^*(t, \omega), X^*(t) \|_{\mathbb{P}}, u^*(t, \omega)) \\
= \int_0^1 \int_0^1 \nabla_m f_0(t, X^*(t, \omega), m^n(t, \theta), y^n(r, t, \omega'), u^*(t, \omega')) dr \mathbb{P}(d\omega') d\theta,
\]

where we denote
\[
y^n(r, t, \omega') = X^*(t, \omega') + r(Z^{h_n}_t(t, \omega') - X^*(t, \omega')), \\
m^n(t, \theta) \triangleq \theta Z^{h_n}_t(t) \|_{\mathbb{P}} + (1 - \theta) X^*(t) \|_{\mathbb{P}}.
\]

Arguing as in the proof of estimate (51), we have
\[
\int_s^T E \| f_0(t, X^*(t, \omega), Z^{h_n}_t(t) \|_{\mathbb{P}}, u^*(t)) - f_0(t, X^*(t, \omega), X^*(t) \|_{\mathbb{P}}, u^*(t)) - h_n \| f^n_{0,m}(Y^*_t(t)) \| dt \\
\leq \left[ \int_s^T \int_0^1 \int_0^1 \int_0^1 \| \varpi^n_{0,m}(r,t,\omega') \|^q dr \mathbb{P}(d\omega') d\theta \mathbb{P}(d\omega) dt \right]^{1/q} \| Z^{h_n}_t - X^* \|_{L^p,s,T} \\
+ \| f^n_{0,m} \|_{L^q,s,T} \| Z^{h_n}_t(t) - X^*(t) - h_n Y^*_t(t) \|_{L^p,s,T},
\]

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Here we put
\[ \varpi_{n,m}^n(\theta, r, t, \omega, \omega') \triangleq \nabla_{m} f_0(t, X^*(t, \omega), m^n(t, \theta), y^n(r, t, \omega'), u^n(t, \omega)) - \nabla_{m} f_0(t, X^*(t, \omega), X^*(t) \mu^P, X^*(t, \omega'), u^n(t, \omega)). \]

Further, \( \{ \varpi_{n,m}^n \} \) converges to zero \( \lambda \otimes \lambda \otimes P \otimes P \)–a.e., and the functions \( \| \varpi_{0,m}^n \| \) are uniformly integrable (here we use the same arguments as in Step 3). Thus, due to the dominated convergence theorem,
\[
\int_{s+h_n}^{T} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \| \varpi_{0,m}^n(\theta, r, t, \omega, \omega') \|^{q} d\omega (d\omega') d\theta P(d\omega) dt \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

Recall that the third statement of Proposition \ref{secondboundary} says that \( \| Z^n_{\nu}(t) - X^*(t) \|_{L^p} \leq C_2 h_n \). Moreover, by Lemma \ref{LemmaSecond}
\( \| f_{0,m}^n \|_{L^{q,s,T}} < \infty \). Finally, Proposition \ref{SecondP} states that \( \| Z^n_{\nu}(t) - X^*(t) - h_n Y_{\nu}(t) \|_{L^p} \rightarrow 0 \) uniformly w.r.t time variable. This and \ref{LemmaSecond} give that
\[
\frac{1}{h_n} \| Z^n_{\nu}(t) - X^*(t) - h_n Y_{\nu}(t) \|_{L^{p,s,T}} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

Therefore, \( G^{(4)}_{n}/h_n \) tends to 0 when \( n \rightarrow \infty \).

Steps 1–5 imply that
\[
\lim_{n \rightarrow \infty} \left[ \frac{1}{h_n} \int_{0}^{T} \mathbb{E} [ f_0(t, Z^n_{\nu}(t), Z^n_{\nu}(t) \mu^P, u^n_{\nu}(t)) dt - \int_{0}^{T} \mathbb{E} [ f_0(t, X^*(t), X^*(t) \mu^P, u^*(t)) dt ] \right]
\]
\[
- \mathbb{E} \Delta f_0^\nu(t) \mu^P + \int_{0}^{T} \mathbb{E} [ f_0^\nu(t, \omega) Y_{\nu}(t, \omega) + \langle f_{0,m}^\nu, Y_{\nu} \rangle(t) dt ]
\]
\[
\leq \frac{1}{h_n} (G^{(1)}_{n} + G^{(2)}_{n} + G^{(3)}_{n} + G^{(4)}_{n}) \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

Now let us examine the limit behavior of the terminal payoff. To simplify notation, put
\[
\sigma^*_x(\omega) \triangleq \nabla \sigma(X^*(T, \omega), X^*(T) \mu^P),
\]
\[
\sigma^*_m(\omega, \omega') \triangleq \nabla \sigma(X^*(T, \omega), X^*(T) \mu^P, X^*(T, \omega')).
\]

Additionally, it is convenient to define
\[
\langle \sigma^*_m, Y_{\nu} \rangle(\omega) \triangleq \int_{0}^{1} \sigma^*_m(\omega, \omega') Y_{\nu}(\omega') d\omega'.
\]

Proposition 5.8.
\[
\lim_{n \rightarrow \infty} \frac{1}{h_n} \mathbb{E} [ \sigma(Z^n_{\nu}(T), Z^n_{\nu}(T) \mu^P) - \mathbb{E} [\sigma(X^*(T), X^*(T) \mu^P) - h_n [\sigma^*_x + \langle \sigma^*_m, Y_{\nu} \rangle]] = 0.
\]

We omit the proof of this proposition since it mimics Steps 4 and 5 in the proof Proposition \ref{SecondP} and relies on the fact that \( \{ h_n \} \) is such that \( \{ Z^n_{\nu}(T) \} \) converges to \( X^*(T) \) \( P \)–a.e. (see Corollary \ref{LemmaSecond}).
6 Proof the Pontryagin maximum principle in the Lagrangian form

Proof Theorem 4.3. In the proof we use notation introduced in (18)–(19), (35)–(36), (53)–(55).

By the third statement of Proposition 5.2,
\[ \|Z^h_n(t) - X^*(t)\|_{L^p} \leq C_1 h_n, \]
while \( Z^h_n(0) = X^*(0) \). Moreover,
\[ (\lambda \otimes \mathbb{P})(u^h_n \neq u^*) = (\lambda \otimes \mathbb{P})([s, s + h_n] \times \Omega) = h_n. \]
Thus, the assumption that \((X^*, u^*)\) is a Pontryagin local \( L^p \)-minimizer at \( X_0 \) implies that, for sufficiently large \( n \),
\[ J_L(X^*, u^*) \leq J_L(Z^h_n, u^h_n). \]
This yields the inequality
\[ 0 \leq \lim_{n \to \infty} \frac{1}{h_n} [J_L(Z^h_n, u^h_n) - J_L(X^*, u^*)]. \]
(56)

The existence of the limit is due to Propositions 5.7, 5.8. Using them and definition of the functional \( J_L \)
(see (41)), we compute
\[
\lim_{n \to \infty} \frac{1}{h_n} [J_L(Z^h_n, u^h_n) - J_L(X^*, u^*)]
= \int_{\Omega} \Delta^*_p f_0^*(\omega) \mathbb{P}(d\omega) + \int_s^T \int_{\Omega} f^*_0, x(t, \omega) Y_\nu(\omega) \mathbb{P}(d\omega) \, dt \\
+ \int_s^T \int_{\Omega} f^*_0, m(t, \omega', \omega) Y_\nu(t, \omega') \mathbb{P}(d\omega') \mathbb{P}(d\omega) \\
+ \int_{\Omega} \sigma_x(\omega) Y_\nu(T, \omega) \mathbb{P}(d\omega) + \int_{\Omega} \int_{\Omega} \sigma_m^*(\omega, \omega') Y_\nu(T, \omega) \mathbb{P}(d\omega') \mathbb{P}(d\omega).
\]
Changing the order of integration, we obtain
\[
\lim_{n \to \infty} \frac{1}{h_n} [J_L(Z^h_n, u^h_n) - J_L(X^*, u^*)]
= \int_{\Omega} \Delta^*_p f_0^*(\omega) \mathbb{P}(d\omega) + \int_s^T \int_{\Omega} \left[ f^*_0, x(t, \omega) + \int_{\Omega} f^*_0, m(t, \omega, \omega') \mathbb{P}(d\omega') \right] Y_\nu(t, \omega) \mathbb{P}(d\omega) \, dt \\
+ \int_{\Omega} \left[ \sigma_x(\omega) + \int_{\Omega} \sigma_m^*(\omega, \omega') \mathbb{P}(d\omega') \right] Y_\nu(T, \omega) \mathbb{P}(d\omega).
\]
We put \( \Psi \) to satisfy the following system:
\[
\frac{d}{dt} \Psi(t, \omega) = -\Psi(t, \omega) f_m^*(t, \omega) \mathbb{P}(d\omega') \\
+ f^*_0, x(t, \omega) + \int_{\Omega} f^*_0, m(t, \omega, \omega') \mathbb{P}(d\omega'), \quad (57)
\]
\[ \Psi(T, \omega) = \sigma_x(\omega) + \int_{\Omega} \sigma_m^*(\omega, \omega') \mathbb{P}(d\omega'). \]
The existence and uniqueness of such \( \Psi \) can be obtained from Proposition C.1. Notice that due to this choice of \( \Psi \), the costate equation and the transversality condition are fulfilled.
Now, let us consider the maximization of the Hamiltonian condition. Expressing \( f_{n,m}^*(t, \omega) + \int_\Omega f_{n,m}^*(t, \omega') \mathbb{P}(d\omega') \) from \([57]\) and changing the order of integration once more, we have

\[
\lim_{n \to \infty} \frac{1}{h_n} [J_L(Z_{\nu}^h, u_{\nu}^h) - J_L(X^*, u^*)] = \int_\Omega \Delta^*_n f_0^* (\omega) \mathbb{P}(d\omega) + \int_\Omega \int_s^T \frac{d}{dt} \Psi(t, \omega) Y_\nu(t, \omega) dt \mathbb{P}(d\omega)
+ \int_\Omega \int_s^T \Psi(t, \omega) f_\nu^*(t, \omega) Y_\nu(t, \omega) dt \mathbb{P}(d\omega)
+ \int_\Omega \int_s^T \Psi(t, \omega) \int_\Omega f_m^*(t, \omega', \omega') Y_\nu(t, \omega') \mathbb{P}(d\omega') dt \mathbb{P}(d\omega)
+ \int_\Omega \left[ \sigma_x(\omega) + \int_\Omega \sigma_m^*(\omega', \omega) \mathbb{P}(d\omega') \right] Y_\nu(T, \omega) \mathbb{P}(d\omega).
\]

Taking into account the fact that

\[
\frac{d}{dt} Y_\nu(t, \omega) = f_x^*(t, \omega) Y_\nu(t, \omega) + \int_\Omega f_m^*(t, \omega', \omega') Y_\nu(t, \omega') \mathbb{P}(d\omega'),
\]

we arrive at the following equality

\[
\lim_{n \to \infty} \frac{1}{h_n} [J_L(Z_{\nu}^h, u_{\nu}^h) - J_L(X^*, u^*)] = \int_\Omega \Delta^*_n f_0^* (\omega) \mathbb{P}(d\omega) + \int_\Omega \int_s^T \frac{d}{dt} \Psi(t, \omega) Y_\nu(t, \omega) dt \mathbb{P}(d\omega)
+ \int_\Omega \int_s^T \Psi(t, \omega) \frac{d}{dt} Y_\nu(t, \omega) dt \mathbb{P}(d\omega)
+ \int_\Omega \left[ \sigma_x(\omega) + \int_\Omega \sigma_m^*(\omega', \omega) \mathbb{P}(d\omega') \right] Y_\nu(T, \omega) \mathbb{P}(d\omega).
\]

Since \( \Psi(T, \omega) = -\sigma_x(\omega) - \int_\Omega \sigma_m^*(\omega', \omega) \mathbb{P}(d\omega') \), \( Y_\nu(s, \omega) = \Delta^*_n f^* (\omega) \), the integration by part formula yields that

\[
\lim_{n \to \infty} \frac{1}{h_n} [J_L(Z_{\nu}^h, u_{\nu}^h) - J_L(X^*, u^*)] = \mathbb{E}[\Delta^*_n f_0^* - \Psi(s) \Delta^*_n f^*(s)].
\]

Recall that (see \([50]\)) this limit is nonnegative, while (see \([59]\), \([60]\), \([61]\))

\[
\Delta^*_n f_0^* (\omega) - \Psi(s, \omega) \Delta^*_n f^*(\omega)
= H(s, X^*(s, \omega), \Psi(s, \omega), X^*(s) \mathbb{P}, u^*(s, \omega)) - H(s, X^*(s, \omega), \Psi(s, \omega), X^*(s) \mathbb{P}, \nu(\omega))
\]

Hence,

\[
\mathbb{E}H(s, X^*(s), \Psi(s), X^*(s) \mathbb{P}, u^*(s)) \geq \mathbb{E}H(s, X^*(s), \Psi(s), X^*(s) \mathbb{P}, \nu).
\]

This is the integral form of maximization condition \([60]\).

It remains to show that this condition is equivalent to local maximization condition \([7]\). First notice that \([7]\) obviously implies \([6]\). To prove the implication \([6] \Rightarrow [7]\), we assume that \([6]\) is fulfilled, while \([7]\) violates. Thus, there exist a number \( \epsilon > 0 \) and a set \( \Xi \in \mathcal{F} \) such that, for each \( \omega \in \Xi \),

\[
H(s, X^*(s, \omega), \Psi(s, \omega), X^*(s) \mathbb{P}, u^*(s, \omega)) + \epsilon \leq \max_{u \in U} H(s, X^*(s, \omega), \Psi(s, \omega), X^*(s) \mathbb{P}, u)
\]

and \( \mathbb{P}(\Xi) > 0 \). Since \( (\Omega, \mathcal{F}, \mathbb{P}) \) is a standard probability space, one can assume that \( \Xi \) is compact, while the restriction of functions \( X^*(s, \cdot), \Psi(s, \cdot) \) and \( u^*(s, \cdot) \) on \( \Xi \) are continuous (the latter is due to the Lusin’s theorem \([2] \text{Theorem 12.8}\)). Notice that, in this case, the mappings \( \Xi \times U \ni (\omega, u) \mapsto\)
\[ H(s, X^*(s, \omega), X^*(s)\mathbb{P}, \Psi(s, \omega), u) \] and \( \Xi \ni \omega \mapsto H(s, X^*(s, \omega), X^*(s)\mathbb{P}, \Psi(s, \omega), u^*(s, \omega)) \) are continuous. Let the multivalued mapping \( \mathcal{G} : \Xi \ni U \) assign to each \( \omega \in \Xi \) the set
\[
\mathcal{G}(\omega) \triangleq \left\{ u \in U : H(s, X^*(s, \omega), X^*(s)\mathbb{P}, \Psi(s, \omega), u^*(s, \omega)) + \epsilon \leq H(s, X^*(s, \omega), X^*(s)\mathbb{P}, \Psi(s, \omega), u) \right\}.
\]
Due to (59) the sets \( \mathcal{G}(\omega) \) are nonempty. The continuity of mappings \( \Xi \times U \ni (\omega, u) \mapsto H(s, X^*(s, \omega), X^*(s)\mathbb{P}, \Psi(s, \omega), u) \) and \( \Xi \ni \omega \mapsto H(s, X^*(s, \omega), X^*(s)\mathbb{P}, \Psi(s, \omega), u^*(s, \omega)) \) implies that the multivalued mapping \( \mathcal{G} \) has a closed graph. Moreover, the values of \( \mathcal{G} \) are nonempty. By [2, Corollary 18.27], \( \mathcal{G} \) admits a measurable selector \( \hat{u} : \Xi \mapsto U \). Choose
\[
\hat{\nu} \triangleq \left\{ u^*(s, \omega), \omega \in \Omega \setminus \Xi, \hat{u}(\omega), \omega \in \Xi. \right\}
\]
By construction, we have that
\[ H(s, X^*(s, \omega), \Psi(s, \omega), X^*(s)\mathbb{P}, \hat{\nu}(\omega)) = H(s, X^*(s, \omega), \Psi(s, \omega), X^*(s)\mathbb{P}, u^*(s, \omega)) \]
for \( \omega \in \Omega \setminus \Xi \), and
\[ H(s, X^*(s, \omega), \Psi(s, \omega), X^*(s)\mathbb{P}, \hat{\nu}(\omega)) \geq H(s, X^*(s, \omega), \Psi(s, \omega), X^*(s)\mathbb{P}, u^*(s, \omega)) + \epsilon \]
if \( \omega \in \Xi \). Hence,
\[
\int_{\Omega} \left( H(s, X^*(s, \omega), \Psi(s, \omega), X^*(s)\mathbb{P}, u^*(s, \omega)) \mathbb{P}(d\omega) + \epsilon \mathbb{P}(\Xi) \right) \leq \int_{\Omega} H(s, X^*(s, \omega), \Psi(s, \omega), X^*(s)\mathbb{P}, \hat{\nu}(\omega)) \mathbb{P}(d\omega).
\]
Since \( \epsilon > 0 \) and \( \mathbb{P}(\Xi) > 0 \), this contradicts (58).
Therefore, (5) yields that (7) holds true \( \mathbb{P} \)-a.s. This completes the proof.

\[ \square \]

7 Kantorovich approach

In this section we introduce the concept of local minima within the Kantorovich formulation of the mean field type control problem, examines its link with the Lagrangian approach and derive the Pontryagin maximum principle for the Kantorovich formalization. Certainly, within this section we assume that conditions [H1]–[H7] are in force.

7.1 Kantorovich admissible processes and statement of PMP in the Kantorovich form

Definition 7.1. We say that a pair \((\eta, u_K)\), where \( \eta \in \mathcal{P}^p(\Gamma) \), \( u_K \in L^p([0, T] \times \Gamma, \mathcal{B}([0, T] \times \Gamma), \lambda \otimes \eta; L^p([0, T]; U)) \), is a Kantorovich control process if

- \( \eta \) is concentrated on the set of absolutely continuously curves;
- \( \eta \)-a.e. \( \gamma \in \Gamma \) satisfies the differential equation
\[
\frac{d}{dt} \gamma(t) = f(t, \gamma(t), e_t \sharp \eta, u_K(t, \gamma)).
\]

The outcome of the Kantorovich process \((\eta, u_K)\) is evaluated by the quantity
\[
J_K(\eta, u_K) \triangleq \int_{\Gamma} \sigma(\gamma(T), e_T \sharp \eta) \eta(d\gamma) + \int_{\Gamma} \int_{0}^{T} f_0(\gamma(t), e_t \sharp \eta, u_K(t, \gamma)) dt \eta(d\gamma) < +\infty.
\]
Definition 7.2. Given an initial distribution $m_0 \in \mathcal{P}_p(\mathbb{R}^d)$, we denote the set of Kantorovich control processes $(\eta, u_K)$ satisfying the initial condition $e_0 \sharp \eta = m_0$ by $\text{Adm}_K(m_0)$.

Let us formulate the following concept that provides the link between Kantorovich and Lagrangian approaches. It will play the crucial role in the derivation of the Pontryagin maximum principle in the Kantorovich framework. To introduce it, denote, for given a stochastic process $X$ with values in $\mathbb{R}^d$ defined on some standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$, by $\widehat{X}$ the operator that assigns to $\omega \in \Omega$ the whole realization $X(\cdot, \omega)$.

Definition 7.3. Let $(\eta, u_K)$ be an admissible Kantorovich control process and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a standard probability space. We say that a Lagrangian control process $(X, u_L)$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ realizes $(\eta, u_K)$ if

$$\eta = \widehat{X} \sharp \mathbb{P},$$

and, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and a.e. $t \in [0, T]$,

$$u_L(t, \omega) = u_K(t, \widehat{X}(\omega)).$$

Proposition 7.4. Let $(\eta, u_K)$ be Kantorovich control process. Assume also that the standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is such that either the probability $\mathbb{P}$ has no atoms or $(\Omega, \mathcal{F}, \mathbb{P})$ is $(\Gamma, \mathcal{B}(\Gamma), \eta)$.

Then, there exists a Lagrangian process $(X, u_L)$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ that realizes $(\eta, u_K)$. Furthermore, if $(\Omega, \mathcal{F}, \mathbb{P}) = (\Gamma, \mathcal{B}(\Gamma), \eta)$, one can put $X = \text{id}_\Omega$ and $u_K = u_L$.

This statement is proved in §10.2.

We consider only strong local minimizers for the Kantorovich case. This concept is defined as follows.

Definition 7.5. A Kantorovich control process $(\eta^*, u_K^*) \in \text{Adm}_K(m_0)$ is called a strong local minimizer at $m_0$ within the Kantorovich approach if there exists a positive $\epsilon$ such that $J_K(\eta, u_K) \geq J_K(\eta^*, u_K^*)$ for all admissible $(\eta, u_K) \in \text{Adm}_K(m_0)$ satisfying $W_2(e_\sharp \eta, e_\sharp \eta^*) < \epsilon$ when $t \in [0, T]$.

The next theorem states the link between local minimizers in the Kantorovich and Lagrangian approaches.

Theorem 7.6. Assume that $(\eta^*, u_K^*)$ is a strong local minimizer in the framework of the Kantorovich approach at $m_0 = e_\sharp \eta^*$. Let $(X^*, u_L^*)$ be an admissible Lagrangian process that realizes the Kantorovich process $(\eta^*, u_K^*)$. Then, $(X^*, u_L^*)$ is a strong local $W_p$-minimizer at $m_0$ in the framework of the Lagrangian approach.

This theorem is also proved in §10.2.

Finally, we formulate the Pontryagin maximum principle for the Kantorovich optimal control process.

Theorem 7.7. Let $(\eta^*, u_K^*) \in \text{Adm}_K(m_0)$ be a strong local minimizer in the framework of the Kantorovich approach.

Then, there exists a function $\psi \in L^q(\Gamma, \mathcal{B}(\Gamma), \eta^*; \Gamma^*)$ such that the following conditions holds true:

- **costate equation:** for $\eta^*$-a.e. $\gamma \in \Gamma$, $\psi(\cdot, \gamma)$ solves

$$\frac{d}{dt} \psi(t, \gamma) = -\psi(t, \gamma)\nabla_x f(t, \gamma(t), e_t \sharp \eta^*, u_K^*(t, \gamma))$$

$$+ \nabla_x f_0(t, \gamma(t), e_t \sharp \eta^*, u_K^*(t, \gamma))$$

$$- \int_\Gamma \psi(t, \gamma')\nabla_m f(t, \gamma'(t), e_t \sharp \eta^*, \gamma(t), u_K^*(t, \gamma'))\eta^*(d\gamma')$$

$$+ \int_\Gamma \nabla_m f_0(t, \gamma'(t), e_t \sharp \eta^*, \gamma(t), u_K^*(t, \gamma'))\eta^*(d\gamma');$$

- **transversality condition:**

$$\psi(T, \gamma)\sharp \eta^* = -\nabla_x \sigma(\gamma(T), e_T \sharp \eta^*) - \int_\Gamma \nabla_m \sigma(\gamma'(t), e_t \sharp \eta^*, \gamma(T))\eta^*(d\gamma')$$

for $\eta^*$-a.e. $\gamma \in \Gamma$.
maximization of the Hamiltonian condition: for a.e. $s \in [0,T]$, and $\eta^* -a.e. \gamma \in \Gamma$,

$$H(s, \gamma(s), \psi(s, \gamma), e_s \hat{\eta}^*, u^*(s, \gamma)) = \max_{u \in U} H(s, \gamma(s), \psi(s, \gamma), e_s \hat{\eta}^*, u).$$  \hspace{1cm} (65)

We postpone the proof of this Pontryagin maximum principle till § 8.1.

\subsection*{7.2 Link between Kantorovich and Lagrangian approaches}

\textit{Proof Proposition 7.4.} In the case where $(\Omega, \mathcal{F}, \mathbb{P}) = (\Gamma, \mathcal{B}(\Gamma), \eta)$, set $X = \text{id}_\Omega$. In the other case, i.e., when the probability $\mathbb{P}$ has no atoms, we use the Skorokhod representation theorem \cite[Theorem 6.7]{11} and the fact that $\Gamma$ is a Polish space. They give the existence of the measurable map $X \in \mathcal{B}((\Omega, \mathcal{F}, \Gamma)$ such that $\hat{X} \sharp \mathbb{P} = \eta$. In both cases, we obtain $e_s \hat{\eta} = e_s \sharp (\hat{X} \sharp \mathbb{P})$ and equality (61) holds. Further, by construction, we have that $\|X(\cdot, \omega)\|_{L^\infty}$ is finite for $\mathbb{P}$-a.e. $\omega$. Moreover,

$$\int_{\Omega} \|X(\cdot, \omega)\|_{L^\infty}^p \mathbb{P}(d\omega) = \int_{\Gamma} \|\gamma\|_{L^\infty}^p \eta(d\gamma) < +\infty. \hspace{1cm} (66)$$

Here the last inequality is due to assumption that $\eta \in \mathcal{P}^p(\Gamma)$. Thus,

$$X \in \mathcal{X}^p.$$

Now, for every $t \in [0, T]$ and $\omega \in \Omega$, set

$$u_L(t, \omega) \triangleq u_K(t, \hat{X}(\omega)).$$

Obviously, this control satisfies equality (62). Furthermore, from the inclusion $u_K \in L^p([0, T] \times \Gamma, \mathcal{B}([0, T] \times \Gamma), \lambda \otimes \eta; U)$ and the equality $\eta = \hat{X} \sharp \mathbb{P}$, it follows that

$$\|u_L\|_{L^p} = \int_0^T \int_{\Gamma} \|u_K(\cdot, \gamma)\|_{L^p([0, T]; U)} \eta(d\gamma) dt < +\infty.$$

Therefore, $u_L$ lies in $U^p$.

Finally, let us show that, for $\mathbb{P}$-a.e. $\omega$, $X(\cdot, \omega)$ satisfies the equation

$$\frac{d}{dt} X(t, \omega) = f(t, X(t, \omega), X(t) \sharp \mathbb{P}, u_L(t, \omega)),$$

or, equivalently,

$$X(t, \omega) = X(0, \omega) + \int_0^t f(\tau, X(\tau, \omega), e_s \hat{\eta}, u_L(\tau, \omega)) d\tau. \hspace{1cm} (67)$$

The latter follows from the assumption that, for $\eta$-a.e. $\gamma \in \Gamma$ and a.e. $t \in [0, T]$,

$$\gamma(t) = \gamma(0) + \int_0^t f(\tau, \gamma(\tau), e_s \hat{\eta}, u_K(\tau, \gamma)) d\tau.$$

Inclusions $X \in \mathcal{X}^p$, $u_L \in U^p$ and the fact that (67) is fulfilled for $\mathbb{P}$-a.e. $\omega$ imply that $(X, u_L)$ is an admissible Lagrangian process. By construction, it realizes $(\eta, u_K)$.

To prove the fact that each Kantorovich local minimizer is a minimizer in the Lagrangian framework, we will use the following.

\textbf{Definition 7.8.} Let $(X, u_L)$ be a Lagrangian control process defined on some standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that a Kantorovich control process $(\eta, u_K)$ improves $(X, u_L)$ if they satisfy (61) and $J_L(X, u_L) \geq J_K(\eta, u_K)$.

\textbf{Lemma 7.9.} Let $(X, u_L)$ be a Lagrangian control process defined on some standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, there exists a Kantorovich control process $(\eta, u_K)$ that improves $(X, u_L)$. 

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Proof. Define the probability $\eta \in \mathcal{P}(\Gamma)$ by the rule
\[
\eta \triangleq \hat{X}_*^{\mathbb{P}}.
\]
Since $X \in \mathcal{X}^p$, we have that
\[
\eta \in \mathcal{P}^p(\Gamma).
\]
Further, for $t \in [0,T]$, set
\[
m(t) \triangleq X(t)^{\mathbb{P}}.
\]
By construction, (61) holds true. Since the process $(X,u_L)$ is admissible, there exists a full measure set $\Omega_+ \in \mathcal{F}$ (i.e., $\mathbb{P}(\Omega_+) = 1$) such that, for every $\omega \in \Omega_+$, $\|u_L(\cdot,\omega)\|_{L^p} < +\infty$ and the function $X(\cdot,\omega)$ solves the differential equation
\[
\frac{d}{dt}X(t,\omega) = f(t,X(t,\omega),m(t),u_L(t,\omega))
\] (68)
on $[0,T]$. Let $\Gamma_+$ be equal to $\hat{X}(\Omega_+)$. We have that $\eta$ is concentrated on $\Gamma_+$, i.e., $\eta(\Gamma_+) = 1$.

By the disintegration theorem [25, III-70], there exists a system of probability measures $\{\mathbb{P}_\gamma\}_{\gamma \in \Gamma_+}$ such that, for every Borel function $\phi : \Omega \to \mathbb{R}$,
\[
\mathbb{E}\phi = \int_{\Gamma_+} \int_{\hat{X}^{-1}(\gamma)} \phi(\omega)\mathbb{P}_\gamma(d\omega)\eta(d\gamma).
\] (69)

Now, for each $\omega \in \Omega$, denote
\[
g(\omega) \triangleq \int_0^T f_0(t,X(t,\omega),m(t),u_L(t,\omega))dt.
\]
Moreover, put, for $\gamma \in \Gamma_+$,
\[
\bar{g}(\gamma) \triangleq \int_{\hat{X}^{-1}(\gamma)} g(\omega)\mathbb{P}_\gamma(d\omega), \quad \ell(\gamma) \triangleq \int_{\hat{X}^{-1}(\gamma)} \|u_L(\cdot,\omega)\|_{L^p}^p\mathbb{P}_\gamma(d\omega).
\]
Obviously, for $\eta$-a.e. $\gamma \in \Gamma_+$,
\[
\bar{g}(\gamma) \geq \inf_{\omega \in \hat{X}^{-1}(\gamma)} g(\omega).
\] (70)

Let $\Gamma_0 \subset \Gamma_+$ be such that $\eta(\Gamma_0) = 1$ and on $\Gamma_0$ (70) holds true. Notice that (70), in particular, implies that either $\bar{g}(\gamma) \geq g(\omega')$ for some $\omega' \in \hat{X}^{-1}(\gamma)$ or, for every $\omega \in \hat{X}^{-1}(\gamma)$, $\bar{g}(\gamma) = g(\omega)$. In the latter case, one can find $\omega'' \in \hat{X}^{-1}(\gamma)$ such that $\ell(\gamma) \geq \|u_L(\cdot,\omega'')\|_{L^p}^p$. For the sake of shortness, put
\[
\Gamma_0 \triangleq \{\gamma \in \Gamma_0 : \bar{g}(\gamma) \geq g(\omega')\text{ for some }\omega' \in \hat{X}^{-1}(\gamma)\}.
\]

Define the multivalued mapping $S : \Gamma_0 \rightrightarrows \Omega$ by the rule:

• if $\gamma \in \Gamma_0'$, we put
\[
S(\gamma) \triangleq \{\omega \in \hat{X}^{-1}(\gamma) : \bar{g}(\gamma) \geq g(\omega)\};
\]

• in the other case, we set
\[
S(\gamma) \triangleq \{\omega \in \hat{X}^{-1}(\gamma) : \ell(\gamma) \geq \|u_L(\cdot,\omega)\|_{L^p}^p\}.
\]

By the choice of $\Gamma_0$, $S(\gamma)$ is nonempty for every $\gamma \in \Gamma_0$. Furthermore, $S$ has a measurable graph. Thus, by the Aumann selection theorem [6] there exists a measurable selector $\Gamma_0 \ni \gamma \mapsto s(\gamma) \in S(\gamma) \subset \Omega_+$.

Similarly, the set
\[
V(\gamma) \triangleq \{\omega \in \hat{X}^{-1}(\gamma) : \ell(\gamma) \geq \|u_L(\cdot,\omega)\|_{L^p}^p\}
\]
is nonempty for $\eta$-a.e. $\gamma \in \Gamma_+$. Cutting, if necessary, the set $\Gamma_0$, one can assume that $V(\gamma)$ is nonempty for every $\gamma \in \Gamma_0$. Certainly, $\eta(\Gamma_0) = 1$. Furthermore, the graph of $V$ is measurable. Applying the
Aumann selection theorem to the restriction of $V$ on the set $\Gamma'_0$, we construct a measurable selector $\Gamma'_0 \ni \gamma \mapsto v(\gamma) \in V(\gamma) \subset \Omega_\gamma$. On the set $\Gamma_0 \setminus \Gamma'_0$, we put $v(\gamma) \equiv s(\gamma)$. Further, we extend the mappings $s(\cdot)$ and $v(\cdot)$ to $\Gamma \setminus \Gamma_0$. Without loss of generality, we assume these extensions still denoted by $s(\cdot)$ and $v(\cdot)$ are measurable.

Now, for each natural $k$, let a set $\mathcal{A}_k \subset \Gamma_0$ be equal to
$$\mathcal{A}_k \triangleq \{ \gamma \in \Gamma_0 : \|u_L(\cdot, s(\gamma))\|^p_{L^p} \leq k\|u_L(\cdot, v(\gamma))\|^p_{L^p} \}.$$ We have that the sequence $\{\mathcal{A}_k\}_{k=1}^\infty$ converges to $\Gamma_0$, i.e.,
$$\Gamma_0 = \bigcup_{k=1}^\infty \mathcal{A}_k.$$ For each natural $k$, define the strategy $u_k$ on $\Gamma$ by the following rule:
$$u_k(\cdot, \gamma) \triangleq \begin{cases} u_L(\cdot, s(\gamma)), & \gamma \in \mathcal{A}_k, \\ u_L(\cdot, v(\gamma)), & \gamma \notin \mathcal{A}_k. \end{cases} \quad (71)$$

Now let us show that $(\eta, u_k)$ is an admissible Kantorovich process for each natural $k$. By construction, we have that $\eta = \hat{X}_P \in P^P(\Gamma)$. To see that $\eta$ is $P$-a.e. $\gamma$ satisfies \( (69) \), it suffices to recall the definition of $\Omega_\eta$ (see \( (68) \)) and the fact that $\Gamma_0 \subset \hat{X}^{-1}(\Omega_\eta)$ while $\eta(\Gamma_0) = 1$. Finally, by definitions of $u_k$ and $s$ and the choice of the selector $\gamma \mapsto v(\gamma) \in V(\gamma)$, we also obtain
$$\int_{\Gamma} \|u_k(\cdot, \gamma)\|^p_{L^p} \eta(d\gamma) \leq k \int_{\Gamma_0} \|u_L(\cdot, v(\gamma))\|^p_{L^p} \eta(d\gamma) \leq k \int_{\Gamma_0} \ell(\gamma) \eta(d\gamma) \leq k \int_{\Gamma_0} \int_{\hat{X}^{-1}(\cdot)} \|u_L(\cdot, u)\|^p_{L^p} \eta(du) \eta(d\gamma) = k\|u_L\|^p_{L^p} < +\infty.$$ So, each $(\eta, u_k)$ is an admissible Kantorovich process.

Finally, we shall prove that $J_L(X, u_L) \geq J_k(\eta, u_k)$ for some natural $k$. First, notice that
$$\int_{\Omega} \sigma(X(T, \omega), m(T)) P(d\omega) = \int_{\Gamma} \sigma(\gamma(T), m(T)) \eta(d\gamma). \quad (72)$$

Now, let us show that
$$\int_{\Omega} \int_0^T f_0(t, X(t, \omega), m(t), u_L(t, \omega)) dt P(d\omega) \geq \int_{\Gamma} \int_0^T f_0(t, \gamma(t), m(t), u_k(t, \gamma)) dt \eta(d\gamma) \quad (73)$$
for some $k$.

The proof this fact relies on the following property. Given a measurable set $\mathcal{A} \subset \Gamma_0$, by definitions of the functions $g$, $\bar{g}$, $s$, the system of measures $\{P_\gamma\}_{\gamma \in \Gamma_0}$, the equality $\bar{g}(\gamma) = \int_{\hat{X}^{-1}(\gamma)} g(\omega) P_\gamma(\omega) d\omega$, and inequality \( (70) \), we have that
$$\int_{\hat{X}^{-1}(\mathcal{A})} \int_0^T f_0(t, X(t, \omega), m(t), u_L(t, \omega)) dt P(d\omega) = \int_{\mathcal{A}} \int_{\hat{X}^{-1}(\gamma)} g(\omega) P_\gamma(\omega) \eta(d\gamma) = \int_{\mathcal{A}} \bar{g}(\gamma) \eta(d\gamma) \geq \int_{\mathcal{A}} g(s(\gamma)) \eta(d\gamma) = \int_{\mathcal{A}} \int_0^T f_0(t, \gamma(t), m(t), u_L(t, s(\gamma))) dt \eta(d\gamma).$$
In particular, for $\mathcal{A} = \Gamma_0$, we obtain that
$$\int_{\Omega_\gamma} \int_0^T f_0(t, X(t, \omega), m(t), u_L(t, \omega)) dt P(d\omega) \geq \int_{\Gamma_0} \int_0^T f_0(t, \gamma(t), m(t), u_L(t, s(\gamma))) dt \eta(d\gamma).$$
First, let us consider the case when
$$\int_{\Omega_\gamma} \int_0^T f_0(t, X(t, \omega), m(t), u_L(t, \omega)) dt P(d\omega) = \int_{\Gamma_0} \int_0^T f_0(t, \gamma(t), m(t), u_L(t, s(\gamma))) dt \eta(d\gamma). \quad (74)$$
Since $\bar{g}(\gamma) \geq g(s(\gamma))$ (this is due to the fact that $s(\gamma) \in S(\gamma)$) we have that, (74) implies the equality $\bar{g}(\gamma) = g(s(\gamma))$ for $\eta$-a.e. $\gamma \in \Gamma_0$. Moreover, the definition of $v$ gives that, in this case, $v(\gamma) = s(\gamma)$ for $\eta$-a.e. $\gamma \in \Gamma_0$. Thus, $\mathcal{A}_1 = \Gamma_0$, and we obtain (73) with $k = 1$.

Now, let
\[
\int_{\Omega_+} \int_{0}^{T} f_0(t, X(t, \omega), m(t), u_L(t, \omega))d\mathbb{P}(d\omega) > \int_{\Gamma_0} \int_{0}^{T} f_0(t, \gamma(t), m(t), u_L(t, s(\gamma)))d\eta(d\gamma). \tag{75}
\]

In this case, one can find a positive number $\varepsilon$ such that
\[
\int_{\Omega_+} \int_{0}^{T} f_0(t, X(t, \omega), m(t), u_L(t, \omega))d\mathbb{P}(d\omega) > \int_{\Gamma_0} \int_{0}^{T} f_0(t, \gamma(t), m(t), u_L(t, s(\gamma)))d\eta(d\gamma) + 2\varepsilon. \tag{76}
\]

Since the sequence of $\mathcal{A}_k$ converges to $\Gamma_0$, we also obtain
\[
\int_{\Omega_+} \int_{0}^{T} f_0(t, X(t, \omega), m(t), u_L(t, \omega))d\mathbb{P}(d\omega) > \int_{\mathcal{A}_k} \int_{0}^{T} f_0(t, \gamma(t), m(t), u_L(t, s(\gamma)))d\eta(d\gamma) + \varepsilon \tag{77}
\]
for all sufficiently large $k$. Furthermore, by (43)
\[
\int_{\Gamma_0 \setminus \mathcal{A}_k} \int_{0}^{T} f_0(t, \gamma(t), m(t), u_L(t, v(\gamma)))d\eta(d\gamma)
\]
\[
\leq \int_{\Gamma_0 \setminus \mathcal{A}_k} \int_{0}^{T} C_\infty(1 + \|\gamma(t)\|_p + M^p_p(m(t)) + \|u_L(t, v(\gamma))\|_p)\,d\eta(d\gamma)
\]
\[
\leq C_\infty \int_{\Gamma_0 \setminus \mathcal{A}_k} (T + T\|\gamma\|_\infty + M^p_p(\eta) + \ell(\gamma))\,d\eta(d\gamma)
\]
\[
= C_\infty(T + M^p_p(\eta))\eta(\Gamma_0 \setminus \mathcal{A}_k) + C_\infty T \int_{\Omega \setminus \hat{X}^{-1}(\mathcal{A}_k)} \|X(\cdot, \omega)\|_p \,d\mathbb{P}(d\omega)
\]
\[
+ C_\infty \int_{\Omega \setminus \hat{X}^{-1}(\mathcal{A}_k)} \|u_L(\cdot, \omega)\|_{L^p_p} \,d\mathbb{P}(d\omega).
\]

Here the latter inequality is due to the facts that $\eta = \hat{X}\mathbb{P}$, definitions of $\{\mathbb{P}_\gamma\}$ and $\ell$. Recall that the sequence $\{\mathcal{A}_k\}$ converges to $\Gamma_0$, therefore $\eta(\Gamma_0 \setminus \mathcal{A}_k) \to 0$ as $k \to \infty$. Moreover, $\mathbb{P}(\Omega \setminus \mathcal{A}_k)$ tends to zero when $k \to \infty$. This and the facts that $\|X\|_{L^p_p} = \int_{\Omega} \|X(\cdot, \omega)\|_p^p \,d\mathbb{P}(d\omega)$ and $\|u_L\|_{L^p_p} = \int_{\Omega} \|u(\cdot, \omega)\|_{L^p_p} \,d\mathbb{P}(d\omega)$ are finite implies that two last terms also converge to zero. Therefore, for sufficiently large $k$,
\[
\int_{\Omega_+ \setminus \mathcal{A}_k} \int_{0}^{T} f_0(t, \gamma(t), m(t), u_L(t, v(\gamma)))d\eta(d\gamma) \leq \varepsilon.
\]

Summing up this inequality and (74), by the definition of the strategy $u_k$ (see (71)), we obtain
\[
\int_{\Omega_+} \int_{0}^{T} f_0(t, X(t, \omega), m(t), u_L(t, \omega))d\mathbb{P}(d\omega) > \int_{\Gamma_0} \int_{0}^{T} f_0(t, \gamma(t), m(t), u_k(t, \gamma))d\eta(d\gamma).
\]

This and (69) yield that $\mathcal{J}_3$ for some natural $k$ in the case when
\[
\int_{\Omega_+} \int_{0}^{T} f_0(t, X(t, \omega), m(t), u_L(t, \omega))d\mathbb{P}(d\omega) > \int_{\Gamma_0} \int_{0}^{T} f_0(t, \gamma(t), m(t), u_L(t, s(\gamma)))d\eta(d\gamma).
\]

The case of equality was considered above. This and (72) imply that
\[
J_L(X, u_L) \geq J_K(\eta, u_k)
\]
for some $k$. Therefore, the admissible Kantorovich process $(\eta, u_k)$ improves $(X, u_L)$. \qed

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Now let us prove Theorem 7.6 which state that if a Lagrangian process \((X^*, u^*_K)\) realizes a local minimizer within the Kantorovich framework \((\eta^*, u^*_K)\), then it is a strong local \(W_p\)-minimizer.

**Proof Theorem 7.6.** Let \(\varepsilon \) be such that \(J_K(\eta, u_K) \geq J_K(\eta^*, u^*_K)\) for every Kantorovich control process \((\eta, u_K) \in \text{Adm}_K(m_0)\) satisfying \(W_p(e\varepsilon \eta, e\varepsilon u_K) \leq \varepsilon\).

Further, let an admissible Lagrangian process \((X, u_L) \in \mathcal{A}_L(X_0)\) such that \(\|X - X^*\|_{X^P} < \varepsilon\). This implies that \(W_p(X(t)\mathbb{P}, X^*(t)\mathbb{P}) \leq \varepsilon\) for every \(t \in [0, T]\). By Lemma 7.9 there exists a Kantorovich process \((\eta, u_K) \in \text{Adm}_K(m_0)\) defined on \((\Omega, \mathcal{F}, \mathbb{P})\) that improves \((X, u_L)\). In particular, \(\eta = \tilde{X}\mathbb{P}\).

Moreover, since \((X^*, u^*_L)\) realizes \((\eta^*, u^*_K)\) we have that \(e\varepsilon \eta^* = \tilde{X}\mathbb{P}\). Therefore, \(m_0 = e_0 \varepsilon \eta^*\) and \(W_p(e\varepsilon \eta, e\varepsilon u_K) \leq \varepsilon\). By the definition of the local minimizer in the framework of the Kantorovich approach, we have \(J_K(\eta, u_K) \geq J_K(\eta^*, u^*_K)\). On the other hand, since \((X^*, u^*_L)\) realizes \((\eta^*, u^*_K)\) and \((\eta, u_K)\) improves \((X, u_L)\), we also obtain

\[J_L(X, u_L) \geq J_K(\eta, u_K) \geq J_K(\eta^*, u^*_K) = J_L(X^*, u^*_L).\]

Thus, \((\eta^*, u^*_K)\) is a strong local \(W_p\)-minimizer at \(m_0\) in the framework of the Kantorovich approach. \(\square\)

### 7.3 Proof PMP for the Kantorovich approach

**Proof Theorem 3.4.** Choose \((\Omega, \mathcal{F}, \mathbb{P}) \triangleq (\Gamma, \mathcal{B}(\Gamma), \eta^*)\). By Proposition 3.4 the Lagrangian process \((\text{id}_\Omega, u^*_K)\) realizes \((\eta^*, u^*_K)\). Theorem 7.6 gives that the process \((\text{id}_\Omega, u^*_K)\) is a strong local \(W_p\)-minimizer at \(m_0\) in the framework of the Kantorovich approach. Applying Theorem 4.3 for the Lagrangian control process \((\text{id}_\Omega, u^*_K)\), and \(\mathbb{P} = \eta^*\), we have that now equation (41) is (63) while conditions (3), (7) take the forms of conditions (11), (13) respectively. \(\square\)

### 8 Eulerian approach

This section is concerned with the Eulerian formalization of the mean field type control problems. Below, we study the links between local minimizers within the Eulerian and Lagrangian approaches. Using this, we deduce the Pontryagin maximum principle for the Eulerian formulation of the mean field type control problem.

In this section, we assume condition (H1)–(H7) and, additionally, we impose the following convexity assumption borrowed from [19]:

(C1) the set \(U\) is a closed convex subset of a Banach space;

(C2) the mapping \(U \ni u \mapsto f(t, x, m, u)\) is affine in \(u\), i.e., for \(t \in [0, T]\), \(x \in \mathbb{R}^d\), \(m \in \mathcal{P}(\mathbb{R}^d)\), \(u_1, u_2 \in U\), \(\alpha \in [0, 1]\),

\[f(t, x, m, \alpha u_1 + (1 - \alpha) u_2) = \alpha f(t, x, m, u_1) + (1 - \alpha) f(t, x, m, u_2);\]

(C3) the function \(f_0\) is convex in \(u\), i.e., for every \(t \in [0, T]\), \(x \in \mathbb{R}^d\), \(m \in \mathcal{P}(\mathbb{R}^d)\), \(u_1, u_2 \in U\), \(\alpha \in [0, 1]\),

\[f_0(t, x, m, \alpha u_1 + (1 - \alpha) u_2) \leq \alpha f_0(t, x, m, u_1) + (1 - \alpha) f_0(t, x, m, u_2).\]

Notice that this condition is always fulfilled if one uses the relaxed controls [19].

### 8.1 Optimal control processes in the Eulerian formulation

Below, we give the definitions of admissible and optimal processes in the framework of the Eulerian approach. Furthermore, we formulate the statement that provides the link with the Lagrangian approach as well as the Pontryagin maximum principle for the Eulerian framework.

**Definition 8.1.** We say that a pair \((m(\cdot), u_E)\), where \(m(\cdot) \in AC^p([0, T]; \mathcal{P}(\mathbb{R}^d))\), \(u_E \in L^p([0, T] \times \mathbb{R}^d, B([0, T] \times \mathbb{R}^d, \lambda; m; U))\), is an Eulerian control process if \(m(\cdot)\) and the velocity field \(v_E : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d\) defined by the rule

\[v_E(t, x) \triangleq f(t, x, m(t), u_E(t, x))\] (79)
satisfy the following continuity equation:

$$\partial_t m(t) + \text{div}(v_E(t,x)m(t)) = 0$$

in the sense of distribution, i.e., for every \( \varphi \in C_c^\infty((0,T) \times \mathbb{R}^d) \),

$$\int_0^T \int_{\mathbb{R}^d} [\partial_t \varphi(t,x) + \nabla \varphi(t,x)v_E(t,x)]m(t, dx) dt = 0.$$ 

Notice that due to assumption (H2) the vector field \( v_E \) defined by (79) for each Eulerian control process \((m(\cdot), u_E)\) lies in \( L^p([0,T] \times \mathbb{R}^d, \mathcal{B}([0,T] \times \mathbb{R}^d), \lambda \times m; \mathbb{R}^d) \).

The outcome of the Eulerian control \((m(\cdot), u_E)\) process is evaluated by the formula:

$$J_E(m(\cdot), u_E) \triangleq \int_{\mathbb{R}^d} \sigma(x, m(T))m(T, dx) + \int_0^T \int_{\mathbb{R}^d} f_0(t, x, m(t), u_E(t,x))m(t, dx) dt.$$ 

**Definition 8.2.** Let \( m_0 \in \mathcal{P}^p(\mathbb{R}^d) \). We say that a control process \((m(\cdot), u)\) satisfies the initial condition \( m_0 \) if \( m(0) = m_0 \). The set of Eulerian control processes satisfying the initial condition \( m_0 \) is denoted by \( \text{Adm}_E(m_0) \).

To study the link between the Eulerian and Langrangian approaches, let us introduce the following notions.

**Definition 8.3.** Let \((m(\cdot), u_E)\) be an Eulerian control process. A Lagrangian control process \((X, u_L)\) defined on a standard probability space \((\Omega, \mathcal{F}, \mathbb{P})\) realizes \((m(\cdot), u_E)\) provided that

- for every \( t \in [0,T] \),
  
  $$m(t) = X(t)\sharp \mathbb{P};$$
  
  (80)

- for a.e. \( t \in [0,T] \) and \( \mathbb{P}\text{-a.e. } \omega \in \Omega \),
  
  $$u_L(t, \omega) = u_E(t, X(t, \omega)).$$
  
  (81)

Notice that these conditions yield the equality

$$J_E(\mu, u_E) = J_L(X, u_L).$$

The next proposition states that each Eulerian process can be realized by a Lagrangian one.

**Proposition 8.4.** Assume that \((m(\cdot), u_E)\) is an Eulerian control process. Furthermore, let a standard probability space \((\Omega, \mathcal{F}, \mathbb{P})\) be such that at least one the following conditions satisfies:

1. **(Ω1)** the probability \( \mathbb{P} \) has no atoms,
2. **(Ω2)** \( \Omega = \Gamma, \mathcal{F} = \mathcal{B}(\Gamma), \mathbb{P} = \eta \in \mathcal{P}(\Gamma) \), while \( \eta\text{-a.e. } \gamma \) solves the equation

$$\frac{d}{dt} \gamma(t) = f(t, \gamma(t), m(t), u_E(t, \gamma(t)))$$

(82)

and \( m(t) = e_t \sharp \eta \).

Then, there exists a Lagrangian process \((X, u_L)\) defined on \((\Omega, \mathcal{F}, \mathbb{P})\) that realizes \((m(\cdot), u_E)\). Furthermore, in case **(Ω2)** we can put \( X = \text{id}_\Gamma \) and \( u_E(t, \gamma(t)) = u_L(t, \gamma) \).

This proposition is proved in §8.2.

For the Eulerian approach, we will consider only strong minima.

**Definition 8.5.** An Eulerian control process \((m^*(\cdot), u^*_E)\) \( \in \text{Adm}_E(m_0) \) is called a strong local minimizer if there exists a positive number \( \varepsilon \) such that \( J_E(m(\cdot), u) \geq J_E(m^*(\cdot), u^*) \) for all admissible Eulerian processes \((m(\cdot), u)\) \( \in \text{Adm}_E(m_0) \) satisfying \( W_p(m(t), m^*(t)) < \varepsilon \) when \( t \in [0,T] \).
The following theorem states that every Eulerian strong minimizer corresponds the minimizer within the Lagrangian approach.

**Theorem 8.6.** Let \((m(\cdot), u^*_E)\) be a strong local minimizer in the Euler framework and let \((X^*, u^*_E)\) be an admissible Lagrangian process defined on some standard probability space \((\Omega, \mathcal{F}, \mathbb{P})\) that realizes \((m(\cdot), u^*_E)\). Then, \((X^*, u^*_E)\) is a strong local \(W_p\)-minimizer at \(m_0\) within the Lagrangian framework.

We give the proof this theorem also in § 8.2.

Now let us formulate the Pontryagin maximum principle within the Eulerian approach. It will rely on the continuity equation for probabilities defined on \(\mathbb{R}^d \times \mathbb{R}^{d,*}\). As above, we consider the solutions in the distributional sense, i.e., if \(w\) is a velocity field defined on \([0, T] \times \mathbb{R}^d \times \mathbb{R}^{d,*}\) with values in \(\mathbb{R}^d \times \mathbb{R}^{d,*}\), we say that \([0, T] \mapsto \nu(t) \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^{d,*})\) solves the continuity equation

\[
\partial_t \nu + \text{div}(w(t, x, \psi) \nu) = 0
\]

if, for every \(\varphi \in C^\infty_c([0, T] \times \mathbb{R}^d \times \mathbb{R}^{d,*})\),

\[
\int_0^T \int_{\mathbb{R}^d \times \mathbb{R}^{d,*}} \left[ \partial_t \varphi(t, x, \psi) + \nabla_x \varphi(t, x, \psi) w_x(t, x, \psi) + w_\psi(t, x, \psi) \nabla_\psi \varphi(t, x, \psi) \right] \nu(t, d(x, \psi)) dt = 0.
\]

Here \(\nabla_x \varphi\) (respectively, \(\nabla_\psi \varphi\)) stands for the derivative of the function \(\varphi\) w.r.t. \(x\) (respectively, w.r.t. \(\psi\)), while \(w_x(t, x, \psi) \in \mathbb{R}^d\) and \(w_\psi(t, x, \psi) \in \mathbb{R}^{d,*}\) are components of the vector field \(w\): \(w(t, x, \psi) = (w_x(t, x, \psi), w_\psi(t, x, \psi))\).

**Theorem 8.7.** Let an Eulerian control process \((m^*, u^*_E) \in \text{Adm}_E(m_0)\) be a strong local minimizer at \(m_0\). Then, there exists a flow of probabilities \(\nu^*(\cdot) \in AC^{\text{loc}}([0, T]; \mathcal{P}^{\text{p}^\land q}(\mathbb{R}^d \times \mathbb{R}^{d,*}))\) satisfying the following conditions:

- **consistency with \(m^*(\cdot)\):**
  \[
p^1 \sharp \nu^*(t) = m^*(t) \quad \forall t \in [0, T];
  \]

- **joint state and costate continuity equation:** \(\nu^*(\cdot)\) is a distributional solution of the continuity equation
  \[
  \partial_t \nu^* + \text{div}(\mathcal{J}(t, x, \psi) \nu^*) = 0,
  \]

  where the vector field \(\mathcal{J}(t, x, \psi) = (\mathcal{J}_x(t, x, \psi), \mathcal{J}_\psi(t, x, \psi))\) is given by

  \[
  \mathcal{J}_x(t, x, \psi) \triangleq f(t, x, m^*(t), u^*_E(t, x)), \quad \mathcal{J}_\psi(t, x, \psi) \triangleq -\psi \nabla_x f(t, x, m^*(t), u^*_E(t, x)) + \nabla_x f_0(t, x, m^*(t), u^*_E(t, x)) - \int_{\mathbb{R}^d \times \mathbb{R}^{d,*}} \zeta \nabla_m f(t, y, m^*(t), u^*_E(t, y)) \nu^*(t, d(y, \zeta)) + \int_{\mathbb{R}^d} \nabla_m f_0(t, y, m^*(t), u^*_E(t, y)) m^*(t, dy);
  \]

- **transversality condition:**
  \[
  p^2 \sharp \nu^*(T) = \left[ -\nabla_x \sigma(\cdot, m^*(T)) - \int_{\mathbb{R}^d} \nabla_m \sigma(y, m^*(T), \cdot) m^*(T, dy) \right] \sharp m^*(T);
  \]

- **maximization condition:** for almost every \(s \in [0, T]\) and \(\nu^*(s)\)-a.e. \((x, \psi) \in \mathbb{R}^d \times \mathbb{R}^{d,*}\),
  \[
  H(s, x, \psi, m^*(s), u^*_E(s, x)) = \max_{u \in U} H(s, x, \psi, m^*(s), u).
  \]

This theorem is proved in § 8.3.
Remark 8.8. Notice that, for a constant $u \in U$, one can obtain the vector field $(\mathcal{J}_x, \mathcal{J}_\psi)$ as a Hamiltonian flow. Indeed, let
\[
\mathcal{H}(t, x, \psi, \nu, u) \triangleq \int_{\mathbb{R}^d \times \mathbb{R}^d} H(t, x, \psi, p^1 \sharp \nu, u) \nu(d(x, \psi)).
\]
Direct calculation (see Proposition 8.3) gives that
\[
\nabla_\nu \mathcal{H}(t, s, \psi, \nu, u) = (\nabla_x H(t, x, \psi, p^1 \sharp \nu, u), \nabla_\psi H(t, x, \psi, p^1 \sharp \nu, u))
\]
\[+ \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_m H(t, y, \zeta, p^1 \sharp \nu, x, u) \nu(dy, \zeta), 0 \right), \tag{87}
\]
where
\[
\nabla_x H(t, x, \psi, p^1 \sharp \nu, u) = \psi \nabla_x f(t, x, p^1 \sharp \nu, u) - \nabla_x f_0(t, x, p^1 \sharp \nu, u),
\]
\[
\nabla_\psi H(t, x, \psi, p^1 \sharp \nu, u) = f(t, x, p^1 \sharp \nu, u), \tag{88}
\]
while
\[
\nabla_m H(t, y, \zeta, p^1 \sharp \nu, x, u) = \zeta \nabla_m f(t, y, p^1 \sharp \nu, x, u) - \nabla_m f_0(t, y, p^1 \sharp \nu, x, u) \in \mathbb{R}^{d, *}. \tag{89}
\]
Further, let a linear function $\mathcal{J} : \mathbb{R}^{d, *} \times \mathbb{R}^d \to \mathbb{R}^{d} \times \mathbb{R}^{d, *}$ be defined by the rule:
\[
\mathcal{J}(\zeta, y) \triangleq (y, -\zeta).
\]
One can regard $\mathcal{J}$ as a unit symplectic matrix. Comparing the formulae for $\mathcal{J}_x$ and $\mathcal{J}_\psi$ with (87)–(89), we conclude that
\[
\mathcal{J}(t, x, \psi) = \mathcal{J}_\nu \mathcal{H}(t, x, \psi, \nu(t), u^*_E(t, x)).
\]

8.2 Link between Eulerian and Lagrangian approaches

In this section we prove Proposition 8.4 and Theorem 8.6.

Proof Proposition 8.4. First, let us construct a process $X$. If $(\Omega, F, \mathbb{P})$ satisfies condition (102) we simply put $X(\cdot, \gamma) \triangleq \gamma$. The case when $(\Omega, F, \mathbb{P})$ satisfies condition (111) is reduced to the previous one in the following way. Since $v_E$ defined by (79) lies in $L^p([0, T] \times \mathbb{R}^d, \mathcal{B}(0, T] \times \mathbb{R}^d), \lambda \ast m; \mathbb{R}^d)$, one can apply [3 Theorem 8.2.1] and construct a probability measure $\eta' \in \mathcal{P}^p(\Gamma)$ such that $m(t) = e_t \eta'$ and $\eta'$-a.e. $\gamma \in \Gamma$ satisfy (82). Further, since $\Gamma$ is Polish space, by the Skorokhod representation theorem [11 Theorem 6.7], there exists a measurable map $X$ from $\Omega$ to $\Gamma$ such that $X^*_E = \eta'$. This gives the desired process $X$ for the case when $(\Omega, F, \mathbb{P})$ satisfies condition (111).

Notice that $X$ satisfies condition (80). In both cases
\[
u_L(t, \omega) \triangleq u_E(t, X(t, \omega)).
\]
Therefore, (81) is fulfilled.

Now, let us show that the process $(X, u_L)$ is admissible. The construction of process $X$ implies that, for $\mathbb{P}$-a.e. $\omega \in \Omega$, $X(\cdot, \omega)$ solves
\[
\frac{d}{dt} X(t, \omega) = v_E(t, X(t, \omega)). \tag{90}
\]
Here $v_E(t, x)$ is defined by (79). Indeed, if $(\Omega, F, \mathbb{P})$ satisfies condition (102) this follows from the equality $X(\cdot, \gamma) \triangleq \gamma$. In the case (111) we use the construction of the probability $\eta' \in \mathcal{P}^p(\Gamma)$ that is concentrated on curves satisfying (82). Equality (79) and construction of $u_L$ implies that
\[
v_E(t, X(t, \omega)) = f(t, X(t, \omega), m(t), u_L(t, \omega)).
\]
This and (90) yield that, for $\mathbb{P}$-a.e. $\omega \in \Omega$, $X(\cdot, \omega)$ is a solution of the ODE
\[
\frac{d}{dt} X(t, \omega) = f(t, X(t, \omega), m(t), u_L(t, \omega)).
\]
Moreover, we have that
\[
\|X(\cdot,\omega)\|_\infty \leq \|X(0,\omega)\| + \int_0^T \|v_E(t,X(t,\omega))\| dt.
\]
This and construction of \(X\) imply that
\[
\|X\|_{\mathcal{X}^p} \leq \mathcal{M}_p(m(0)) + \int_0^T \int_\Omega \|v_E(t,X(t,\omega))\| d\mathbb{P}(d\omega) = \mathcal{M}_p(m(0)) + \int_0^T \int_{\mathbb{R}^d} \|v_E(t,x)\| m(t, dx) dt. \tag{91}
\]
Notice that
\[
\int_0^T \int_\Omega \|v_E(t,X(t,\omega))\| d\mathbb{P}(d\omega) = \int_0^T \int_{\mathbb{R}^d} \|v_E(t,x)\| m(t, dx) dt
\]
Due to assumption [\(\mathcal{H}3\)] and inclusion \(u_E \in L^p([0,T] \times \mathbb{R}^d, \mathcal{B}([0,T] \times \mathbb{R}^d), \lambda \ast m; U)\), we have that \(v_E \in L^p([0,T] \times \mathbb{R}^d, \mathcal{B}([0,T] \times \mathbb{R}^d), \lambda \ast m; \mathbb{R}^d)\). Using this, (91) and the Hölder’s inequality, we conclude that \(X\) belongs to \(\mathcal{X}^p\).

To complete the proof, let us show that \(u_L \in \mathcal{U}^p\). Indeed,
\[
\|u_L\|_{\mathcal{U}^p} = \int_0^T \int_\Omega \|u_L(t,\omega)\|^{p} d\mathbb{P}(d\omega) dt = \int_0^T \int_{\mathbb{R}^d} \|u_L(t,x)\|^{p} m(t, dx) dt < \infty.
\]
The latter inequality is due to the fact that any Eulerian process \((m(\cdot), u_E)\) satisfies \(u_E \in L^p([0,T] \times \mathbb{R}^d, \mathcal{B}([0,T] \times \mathbb{R}^d), \lambda \ast m; U)\).

**Remark 8.9.** Notice that the previous proposition does not rely on the convexity assumption.

Definition 8.3 provides the embedding of the set of Eulerian processes into the set of Lagrangian processes. The following concept plays the crucial role in the proof of Theorem 8.6. It can be regarded as an embedding converse to one given by Definition 8.3.

**Definition 8.10.** Let \((X,u_L)\) be a Lagrangian control process defined on a standard probability space \((\Omega, \mathcal{F}, \mathbb{P})\). We say that an Eulerian control process \((m(\cdot), u_E)\) improves \((X,u_L)\) if they satisfy (90) and \(J_L(X,u_L) \geq J_E(m(\cdot),u_E)\).

**Lemma 8.11.** Let \((X,u_L)\) be a Lagrangian control process defined on a standard probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Then, there exists an Eulerian process \((m(\cdot), u_E)\) that improves \((X,u_L)\).

**Proof.** Define the flow of probabilities \(m(\cdot) \in C([0,T]; \mathcal{P}(\mathbb{R}^d))\) and the velocity field \(v_L \in B([0,T] \times \Omega; \mathcal{B}([0,T]) \otimes \mathcal{F}; \mathbb{R}^d)\) by the following rules: for all \(t \in [0,T]\) and \(\omega \in \Omega\),
\[
m(t) \triangleq X(t)|\mathbb{P}, \quad v_L(t,\omega) \triangleq f(t,X(t,\omega), m(t), u_L(t,\omega)).
\]
So, (90) holds true. Moreover, since \((X,u_L)\) is an admissible process, using assumption [\(\mathcal{H}2\)] we conclude that
\[
v_L \in L^p([0,T] \times \Omega, \mathcal{B}([0,T]) \otimes \mathcal{F}; \lambda \otimes \mathbb{P}; \mathbb{R}^d). \tag{92}
\]
Now let us define the velocity field \(v_E \in B([0,T] \times \mathbb{R}^d, \mathcal{B}([0,T] \times \mathbb{R}^d); \mathbb{R}^d)\). To this end, we use the disintegration theorem and, given \(t \in [0,T]\), find a system of probability measures \(\{\mathbb{P}_x^t\}_{x \in \mathbb{R}^d}\) such that, for each Borel \(\phi\),
\[
\mathbb{E}\phi = \int_{\mathbb{R}^d} \int_{\Omega} \phi(\omega) \mathbb{P}_x^t(d\omega) m(t, dx).
\]
Define the control \(u_E \in B([0,T] \times \mathbb{R}^d, \mathcal{B}([0,T] \times \mathbb{R}^d); U)\) and the velocity field \(v_E\) by the rules
\[
u_E(t,x) \triangleq \int_{\Omega} u_L(t,\omega) \mathbb{P}_x^t(d\omega), \tag{93}
\]
Further, Assumption \([C1]\) gives that \(u_E(t,x) \in U\). By the Jensen’s inequality, we have that
\[
\|u_E\|_{L^p}^p = \int_0^T \int_{\mathbb{R}^d} \left\| u_L(t,\omega) \mathbb{P}_x^t(d\omega) \right\|^p m(t,dx)dt \leq \int_0^T \int_{\Omega} \left\| u_L(t,\omega) \right\|^p d\mathbb{P}(d\omega).
\]
This and the fact that \(u_L \in L^p\) yield that the Eulerian control \(u_E\) lies in \(L([0,T] \times \mathbb{R}^d,\mathcal{B}([0,T] \times \mathbb{R}^d),\lambda \ast m;U)\).

Due to \([C2]\) we have that
\[
f(t,x,m(t),u_E(t,x)) = \int_{\Omega} f(t,X(t,\omega),m(t),u_L(t,\omega))\mathbb{P}_x^t(d\omega)
\]
\[
= \int_{\Omega} v_L(t,\omega)\mathbb{P}_x^t(d\omega) = v_E(t,x)
\]
for each \(t \in [0,T]\) and \(m(t)\text{-a.e. } x\).

We claim that \(m(\cdot)\) is a distributional solution of the equation
\[
\partial_t m(t) + \text{div}(v_E(t,x)m(t)) = 0.\tag{94}
\]
on \([0,T] \times \mathbb{R}^d\). Indeed, choose a smooth function \(\varphi \in C_c^\infty((0,T) \times \mathbb{R}^d)\). Since \((X,u_L)\) is an admissible Lagrangian process, we have that, for \(\mathbb{P}\text{-a.e. } \omega \in \Omega,\)
\[
\int_0^T \left[ \partial_t \varphi(t,X(t,\omega)) + \nabla \varphi(t,X(t,\omega)) v_L(t,\omega) \right] dt = 0.
\]
Integrating this equality w.r.t. \(\mathbb{P}\) and using the equality \(v_L(t,\omega) = f(t,X(t,\omega),m(t),u_L(t,\omega))\), we obtain
\[
\int_{\Omega} \int_0^T \left[ \partial_t \varphi(t,X(t,\omega)) + \nabla \varphi(t,X(t,\omega)) v_L(t,\omega) \right] dt \mathbb{P}(d\omega) = 0.
\]
Notice that, for any \(t \in [0,T]\)
\[
\int_{\Omega} \nabla \varphi(t,X(t,\omega)) v_L(t,\omega) \mathbb{P}(d\omega) = \int_{\mathbb{R}^d} \nabla \varphi(t,x) \int_{\Omega} v_L(t,\omega) \mathbb{P}_x^t(d\omega)m(t,dx)
\]
\[
= \int_{\mathbb{R}^d} \nabla \varphi(t,x) v_E(t,x) m(t,dx).
\]
Analogously,
\[
\int_{\Omega} \partial_t \varphi(t,X(t,\omega)) \mathbb{P}(d\omega) = \int_{\mathbb{R}^d} \partial_t \varphi(t,x) m(t,dx).
\]
Therefore, \(m(\cdot)\) satisfies \((94)\) in the distributional sense. Further, since \(v_L \in L^p([0,T] \times \mathbb{R}^d,\mathcal{B}([0,T] \times \mathbb{R}^d),\lambda \ast m;\mathbb{R}^d)\), we have that the mapping \(t \mapsto \|v_L(t,\cdot)\|_{L^p}\) lies in \(L^p([0,T],\mathcal{B}([0,T]),\lambda;\mathbb{R})\). Hence, using the fact that \(m(\cdot)\) is the distributional solution of \((94)\), by \([3\text{, Theorem 8.3.1}]\), we conclude that \(m(\cdot) \in ACP([0,T];L^p(\mathbb{R}^d))\).

Thus, we have proved that \((m(\cdot),u_E)\) is an Eulerian control process.

Finally, let us show that \(J_L(X,u_L) \geq J_E(m(\cdot),u_E)\). By construction, we have that
\[
\int_{\Omega} \sigma(X(T,\omega),m(T))\mathbb{P}(d\omega) = \int_{\mathbb{R}^d} \sigma(x,m(T))m(T,dx).\tag{95}
\]
Further,
\[
\int_0^T \int_{\Omega} f_0(t,X(t,\omega),m(t),u_L(t,\omega)) \mathbb{P}(d\omega)dt
\]
\[
= \int_0^T \int_{\mathbb{R}^d} \int_{\Omega} f_0(t,x,m(t),u_L(t,\omega)) \mathbb{P}_x^t(d\omega)m(t,dx)dt.
\]
The definition of the control \( u_E \) (see (B.3)) and assumption (C3) give that

\[
\int_{\Omega} f_0(t, x, m(t), u_L(t, \omega))P(d\omega) \geq f_0(t, x, m(t), u_E(t, x)).
\]

Therefore,

\[
\int_0^T \int_{\Omega} f_0(t, X(t, \omega), m(t), u_L(t, \omega))P(d\omega) dt \geq \int_0^T \int_{\mathbb{R}^d} f_0(t, x, m(t), u_E(t, x))m(t, dx)dt.
\]

Combining this with (93), we arrive at the inequality \( J_L(X, u_L) \geq J_E(m(\cdot), u_E) \).

**Proof Theorem 8.6.** Since \( (m(\cdot), u_E^*) \) is a strong local minimizer within the Eulerian approach, there exists \( \varepsilon > 0 \) such that, for each \( (m(\cdot), u_E) \in \text{Adm}_E(m_0) \),

\[
J_E(m(\cdot), u_E^*) \leq J_E(m(\cdot), u_E)
\]

provided that \( W_p(m^*(t), m(t)) \leq \varepsilon, t \in [0, T] \).

Let \( (X, u_L) \in \text{Adm}_L(m_0) \) satisfy \( W_p(X(t)^*P, X^*(t)^*P) \leq \varepsilon \) for all \( t \in [0, T] \). By Lemma 8.11 there exists an Eulerian process \( (m(\cdot), u)^* \) defined on \( (\Omega, F, \mathbb{P}) \) that improves \( (X, u_L) \). Furthermore, from (80) it follows \( m(0) = X(0)^*P = X^*(0)^*P = m_0 \), \( (m(\cdot), u_E) \in \text{Adm}_E(m_0) \), and \( W_p(m(t), m^*(t)) \leq \varepsilon \) on \( [0, T] \), while \( J_E(m(\cdot), u_E) \geq J_E(m^*(\cdot), u_E^*) \).

Since \( (X^*, u^*_L) \) realizes \( (m(\cdot), u^*_E) \) and \( (m(\cdot), u_E) \) improves \( (X, u_L) \) we obtain

\[
J_L(X, u_L) \geq J_E(m(\cdot), u_E) \geq J_E(m^*(\cdot), u^*_E) = J_L(X^*, u^*_L).
\]

Thus, \( (X^*, u^*_L) \) is a strong local \( W_p \)-minimizer at \( m_0 \) in the framework of the Lagrangian approach.

**8.3 PMP in the Eulerian framework**

The purpose of this section is to prove Theorem 8.7. We rely on the fact that every Eulerian strong local minimizer is the strong local \( W_p \)-minimizer within the Lagrangian framework.

**Proof Theorem 8.7.** Since \( (m^*, u^*_L) \in \text{Adm}_E(m_0) \) is a strong local minimizer at \( m_0 \), by Proposition 8.4 there exist a probability measure \( \eta^* \in \mathcal{P}(\Gamma) \) and a probability space \( (\Omega, F, \mathbb{P}) = (\Gamma, B(\Gamma), \eta^*) \) such that Lagrangian process \( (\text{id}_\Omega, u^*_L) \) realizes \( (m^*(\cdot), u^*_L) \). Moreover, \( u^*_L(t, \gamma) = u^*_E(t, \gamma(t)) \). By Theorem 8.6 the process \( (\text{id}_\Omega, u^*_L) \) is a strong local \( W_p \)-minimizer at \( m_0 \) in the framework of the Lagrangian approach. Applying Corollary 1.5 to this process and taking into account the equalities \( \mathbb{P} = \eta^* \), \( m^*(t) = \eta^*P \), \( X^*(t, \gamma) = \gamma(t) \), and \( u^*_L(t, \gamma) = u^*_E(t, \gamma(t)) \), we find a function \( \Psi_L \in L^q(\Gamma, B(\Gamma), \mathbb{P}; \Gamma^*) \) satisfying \( \eta^* \)-a.s. the costate equation:

\[
\frac{d}{dt} \Psi_L(t, \gamma) = -\nabla_x f(t, \gamma(t), m^*(t), u^*_E(t, \gamma(t)))
+ \nabla f_0(t, \gamma(t), m^*(t), u^*_E(t, \gamma(t)))
- \int_{\Omega} \Psi_L(t, \gamma') \nabla m f(t, \gamma'(t), m^*(t), \gamma(t), u^*_E(t, \gamma'(t))) \eta^*(d\gamma')
+ \int_{\Omega} \nabla m f_0(t, \gamma'(t), m^*(t), \gamma(t), u^*_E(t, \gamma'(t))) \eta^*(d\gamma'),
\]

the transversality condition:

\[
\Psi_L(T, \gamma) = -\nabla_x \sigma(\gamma(T), m^*(T)) - \int_{\Omega} \nabla m \sigma(\gamma'(T), m^*(T), \gamma(T)) \eta^*(d\gamma'),
\]

and the maximization of the Hamiltonian condition in the local form which states that, for a.e. \( s \in [0, T] \) and \( \eta^*-\text{a.e.} \gamma \in \Gamma \):

\[
H(s, \gamma(s), \Psi_L(s, \gamma), m^*(s), u^*_E(s, \gamma(s))) = \max_{u \in U} H(s, \gamma(s), \Psi_L(s, \gamma), m^*(s), u).
\]
Since the mapping $\gamma \mapsto \Psi(\cdot, \gamma)$ belongs to $L^q(\Omega, \mathcal{F}, \mathbb{P}; \Gamma^*)$ and the measure $\mathbb{P}$ is concentrated on the set of curves $\gamma \in \mathcal{X}^p = L^p(\Omega, \mathcal{F}, \mathbb{P}; \Gamma)$, the mapping $\gamma \mapsto (\gamma, \Psi(\cdot, \gamma))$ lies in $L^{p,q}(\Omega, \mathcal{F}, \mathbb{P}; \Gamma \times \Gamma^*)$. Thus, the measure $\chi^* \in \mathcal{P}(\Gamma \times \Gamma^*)$ defined by the rule $\chi^* = (\mathbb{1}_0, \psi)\mathbb{P}$ lies in $\mathcal{P}^{p,q}(\Gamma \times \Gamma^*)$. In particular, we have that $p^1 \cdot \chi^* = \mathbb{P} = \eta^*$ and $(p^1 \circ e_t)\cdot \chi^* = \eta^*$, where

Furthermore, we claim that the measure $\chi^*$ is supported on $AC^{p,q}([0, T]; \mathbb{R}^d \times \mathbb{R}^{d,e})$. Indeed, the probability $p^1 \cdot \chi^* = \eta^*$ is concentrated on $AC^p([0, T]; \mathbb{R}^d) \subset AC^{p,q}([0, T]; \mathbb{R}^d)$. Furthermore, due to the fact that $\Psi(\cdot, \gamma)$ satisfies the costate equation [96], $\eta^*$-a.s., while $u_E \in L^p([0, T] \times \mathbb{R}^d, B([0, T] \times \mathbb{R}^d)$, $\lambda \neq m^*; U$, assuming assumption [16] we conclude that $\Psi(\cdot, \gamma) \in AC^q([0, T]; \mathbb{R}^{d,e}) \subset AC^{p,q}([0, T]; \mathbb{R}^{d,e})$

Hence, by [3, Theorem 8.2.1], the flow of probabilities $\nu^*(\cdot)$ defined by the rule $\nu^*(t) \triangleq e_t \cdot \chi^*$ is a solution of the continuity equation

$$
\partial_t \nu^*(t) + \nabla \cdot (w(t, x, \psi) {\nu^*(t)}) = 0
$$

with the vector field $w(t, x, \psi) = (w_x(t, x, \psi), w_y(t, x, \psi))$, where

$$
w_x(t, x, \psi) = f(t, x, m^*(t), u^*_E(t, x)),
$$

$$
w_y(t, x, \psi) = -\nabla f(t, x, m^*(t), u^*_E(t, x)) + \nabla f(t, x, m^*(t), u^*_E(t, x)) - \int_\Omega \beta'(t) \nabla f(t, \gamma'(t), m^*(t), u^*_E(t, \gamma'(t))) \chi^* (d\gamma', \beta') + \int_\Omega \nabla f(t, \gamma'(t), m^*(t), u^*_E(t, \gamma'(t))) \chi^* (d\gamma', \beta').
$$

We have that $w_x(t, x, \psi) = \mathcal{J}_x(t, x, \psi)$. The equalities $\nu^*(t) = e_t \cdot \eta^*$, $p^1 \cdot \nu^*(t) = m^*(t) = e_t \cdot \eta^*$ imply that $w_y(t, x, \psi) = \mathcal{J}_y(t, x, \psi)$. Thus, $\nu^*(\cdot)$ satisfies [55]. By the same arguments, we deduce from [97] the fact that $\nu^*$ satisfies the transversality condition [55] in the Eulerian form.

Furthermore, the equalities $e_t \cdot \mathbb{1}_0(\cdot, \psi) \mathbb{P}^\mathcal{X} = \nu^*(t)$, $u^*_E(s, \gamma(s)) = u^*_E(s, \gamma(s)) \eta^*$-a.s., and [55] lead maximization condition [55].

## 9 Mean field type linear-quadratic regulator

In this section, we come back to the Lagrangian approach and consider the model problem of linear-quadratic regulator with the additional terms describing the variance of the distribution of agents. We put $p = 2$. Moreover, we fix a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an initial assignment $X_0 \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$. The dynamics of each agent is given by the equation

$$
dt X(t, \omega) = A(t)X(t, \omega) + B(t)u(t, \omega),
$$

while the payoff functional is equal to

$$
\frac{1}{2} \mathbb{E} \left( \int_0^T [X^T(t)Q_x(t)X(t) + u^T(t)R(t)u(t)]dt + X^T(T)K_x X(T) \right)
$$

$$
+ \frac{1}{2} \mathbb{E} \left( \int_0^T [(X(t) - \mathbb{E}X(t))^T Q_m(t)(X(t) - \mathbb{E}X(t))]dt
\right) + \frac{1}{2} \mathbb{E} \left( (X(T) - \mathbb{E}X(T))^T K_m(X(T) - \mathbb{E}X(T)) \right).
$$

Here $X(t, \omega) \in \mathbb{R}^d$, $U = \mathbb{R}^{d'}$, $A(t), Q_x(t), Q_m(t), K_x, K_m$ are $(d \times d)$-matrices, $d'$ is natural, $B(t)$ is a $(d \times d')$-matrix, $R(t) \in \mathbb{R}^{d' \times d'}$. Additionally, the matrices $Q_x(t), Q_m(t), K_x, K_m$ and $R(t)$ are symmetric, while $R(t) > 0$. Finally, we assume that the matrix-valued functions $A(\cdot), B(\cdot), R(\cdot), Q_x(\cdot)$ and $Q_m(\cdot)$ are continuous on $[0, T]$.

Notice that the first term in (100) refers to the individual controls of the agents. The second term is the integral variance of the random variable $D(t)X(t)$, where $Q_m(t) = D^T(t)D(t)$. Finally, the third term is equal to the variance of the random variable $\Theta X(T)$ with $\Theta^T \Theta = K_m$. The last two terms evaluates
the cooperative behavior of the agents. Since, for each symmetric matrix $Q$ and every random variable $\xi$ with values in $\mathbb{R}^d$, $E[(\xi - E\xi)^T Q (\xi - E\xi)] = E(\xi^T Q \xi) - (E\xi^T)Q(E\xi)$, we may set

$$f_0(t, x, m, u) \triangleq \frac{1}{2} \left[ x^T Q_x(t)x + u^T R(t)u + x^T Q_m(t)x - \left( \int_{\mathbb{R}^d} y^T m(dy) \right) Q_m(t) \left( \int_{\mathbb{R}^d} y m(dy) \right) \right],$$

$$\sigma(x, m) \triangleq \frac{1}{2} \left[ x^T K_x x + x^T K_m x - \left( \int_{\mathbb{R}^d} y^T m(dy) \right) K_m \left( \int_{\mathbb{R}^d} y m(dy) \right) \right].$$

Below, to simplify notation, given a random variable $\xi$, we denote

$$\bar{\xi} \triangleq E\xi.$$

**Theorem 9.1.** If $(X^*, u^*)$ is a Pontryagin local $L^2$-minimizer at some initial assignment $X_0$ for problem (99), (100) with an initial assignment $X_0$. Then,

$$u^*(t, \omega) = -R^{-1}(t) B^T(t) \left[ P_1(t)(X^*(t, \omega) - \bar{X}^*(t)) + P_2(t) \bar{X}^*(t) \right],$$

where $P_1(\cdot)$ is the matrix-valued function solving the Ricatti differential equation

$$\frac{d}{dt} P_1(t) = -P_1(t) A(t) - A^T(t) P_1(t) + P_1(t) B(t) R^{-1}(t) B(t) P_1(t) - (Q_x(t) + Q_m(t))$$

with the boundary condition

$$P_1(T) = K_x + K_m,$$

while $P_2(\cdot)$ satisfies the Ricatti differential equation

$$\frac{d}{dt} P_2(t) = -P_2(t) A(t) - A^T(t) P_2(t) + P_2(t) B(t) R^{-1}(t) B(t) P_2(t) - Q_x(t)$$

and the boundary condition

$$P_2(T) = K_x.$$

**Proof.** We will use Theorem 4.3 to determine the optimal control. Notice that the Hamiltonian $H(t, x, \psi, m, u)$ for problem (99), (100) is equal to

$$H(t, x, \psi, m, u) \triangleq \psi A(t)x + \psi B(t)u - \frac{1}{2} \left[ x^T Q_x(t)x + u^T R(t)u + x^T Q_m(t)x - \left( \int_{\mathbb{R}^d} y^T m(dy) \right) Q_m(t) \left( \int_{\mathbb{R}^d} y m(dy) \right) \right].$$

Below, to use the matrix notation, we work with the vector $\Upsilon(t) = \Psi^T(t)$.

The maximization condition implies that

$$R(t) u^*(t, \omega) = B^T(t) \Upsilon(t, \omega).$$

Since $R(t) > 0$, we have that

$$u^*(t, \omega) = R^{-1}(t) B^T(t) \Upsilon(t, \omega).$$

Plugging this control to equation (99), we obtain

$$\frac{d}{dt} X^*(t, \omega) = A(t) X^*(t, \omega) + B(t) R^{-1}(t) B^T(t) \Upsilon(t, \omega).$$

Recall that $X^*$ satisfies the initial condition

$$X^*(0, \omega) = X_0(\omega).$$

Using the formula for the derivative of the function depending on mean (see Proposition A.2), we conclude that the transposed costate variable $\Upsilon(\cdot, \omega)$ satisfies the equation

$$\frac{d}{dt} \Upsilon(t, \omega) = (Q_x(t) + Q_m(t)) X^*(t, \omega) - A^T(t) \Upsilon(t, \omega) - Q_m(t) \bar{X}^*(t).$$

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and the boundary condition

\[ \Upsilon(T, \omega) = -(K_x + K_m)X^*(T, \omega) + K_m \tilde{X}^*(T). \]  \hfill (110)

For each \( \omega \in \Omega \), system (107), (109) is a nonhomogeneous system of linear equations. To analyze it, we take expectation in equations (107), (109) and in boundary conditions (108), (110). This leads to the following system on \( \bar{X}^* \) and \( \bar{Y} \):

\[
\frac{d}{dt} \bar{X}^*(t) = A(t)\bar{X}^*(t) + B(t)R^{-1}(t)B(t)\bar{Y}(t),
\]

\[
\frac{d}{dt} \bar{Y}(t) = Q_x(t)\bar{X}^*(t) - A^T(t)\bar{Y}(t)
\]

equipped with the boundary conditions

\[ \bar{X}^*(0) = X_0, \quad \bar{Y}(T) = -K_x \bar{X}^*(T). \]  \hfill (113)

Subtracting (111) from (107) and (112) from (109), we obtain that the differences \( X^*(t, \omega) - \bar{X}^*(t) \) and \( \Upsilon(t, \omega) - \bar{Y}(t) \) satisfies the following system of ODEs

\[
\frac{d}{dt}[X^*(t, \omega) - \bar{X}^*(t)] = A(t)[X^*(t, \omega) - \bar{X}^*(t)] + B(t)R^{-1}(t)B(t)[\Upsilon(t, \omega) - \bar{Y}(t)],
\]

\[
\frac{d}{dt}[\Upsilon(t, \omega) - \bar{Y}(t)] = (Q_x(t) + Q_m(t))[X^*(t, \omega) - \bar{X}^*(t)] - A^T(t)[\Upsilon(t, \omega) - \bar{Y}(t)].
\]

Furthermore,

\[
X^*(0, \omega) - \bar{X}^*(0) = X_0(\omega) - X_0,
\]

\[
\Upsilon(T, \omega) - \bar{Y}(T) = -(K_x + K_m)[X^*(T, \omega) - \bar{X}^*(T)].
\]

From the standard theory (see [41, §6.1.1, 6.1.2]) we have that

\[
\Upsilon(t, \omega) - \bar{Y}(t) = -P_1(t)[X^*(t, \omega) - \bar{X}^*(t)],
\]

where \( P_1(\cdot) \) satisfies (102) and (103).

Additionally, (111)–(113) and the standard theory of linear quadratic regulator yield that

\[
\bar{Y}(t) = -P_2(t)\bar{X}^*(t),
\]

where \( P_2(\cdot) \) satisfies (104), (105). Plugging \( \bar{Y}(t) \) into (110), we conclude that

\[
\Upsilon(t, \omega) = -P_1(t)[X^*(t, \omega) - \bar{X}^*(t)] - P_2(t)\bar{X}^*(t).
\]

This and (106) imply (101). \( \square \)

\section*{Appendices}

\section*{A Some properties of intrinsic derivative}

\textbf{Proposition A.1.} Let \( p > 1 \). Assume that \( \Phi : \mathcal{P}^p(\mathbb{R}^d) \to \mathbb{R} \) has the intrinsic derivative that is continuous and bounded by a constant \( \hat{C} \). Then \( \Phi \) is Lipschitz continuous with the constant equal to \( \hat{C} \).

\textbf{Proof.} Let \( m, m' \in \mathcal{P}^p(\mathbb{R}^d) \), and let \( \pi_0 \in \Pi(m', m) \) be an optimal plan between \( m \) and \( m' \). The existence of the optimal plan is due to [48, Theorem 4.1]. We have that

\[
\Phi(m') - \Phi(m) = \int_0^1 \int_{\mathbb{R}^d \times \mathbb{R}^d} \left[ \frac{\delta \Phi}{\delta m}((1 - s)m + sm', y') - \frac{\delta \Phi}{\delta m}((1 - s)m + sm', y) \right] \pi_0(dy', y)ds.
\]

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Further, notice that
\[
\left[ \frac{\delta \Phi}{\delta m}((1-s)m + sm', y') - \frac{\delta \Phi}{\delta m}((1-s)m + sm', y) \right] = \nabla_m \Phi((1-s)m + sm', y + \hat{r}(y' - y)) \cdot (y' - y),
\]
where $\hat{r} \in (0, 1)$ depends on $y'$ and $y$. By assumption $\nabla_m \Phi$ is bounded by some constant $\hat{C}$. Since $\pi$ is the optimal plan between $m'$ and $m$, using the Jensen’s inequality, we obtain
\[
\Phi(m') - \Phi(m) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \hat{C}||y' - y||\pi_0(d(y', y)) \leq \hat{C} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} ||y' - y||^p \pi_0(d(y', y)) \right)^{1/p} = \hat{C} W_p(m', m).
\]
\[
\square
\]

Now, let us compute the intrinsic derivative for the two important cases.

**Proposition A.2.** Assume that the function $\phi_1 : \mathbb{R}^d \rightarrow \mathbb{R}$ is differentiable,
\[
\Phi_1(m) \triangleq \phi_1 \left( \int_{\mathbb{R}^d} zm(dz) \right).
\]
Then,
\[
\nabla_m \Phi_1(m, y) = \nabla \phi_1 \left( \int_{\mathbb{R}^d} zm(dz) \right),
\]
where $\nabla \phi_1$ stands for the derivative of $\phi_1$.

**Proof.** Indeed, we have that, for any probabilities $m, m' \in \mathcal{P}^p(\mathbb{R}^d)$,
\[
\lim_{s \downarrow 0} \frac{\Phi(m + s(m' - m)) - \Phi(m)}{s} = \lim_{s \downarrow 0} \frac{1}{s} \left[ \phi_1 \left( \int_{\mathbb{R}^d} z((1-s)m + sm')(dz) \right) - \phi_1 \left( \int_{\mathbb{R}^d} zm(dz) \right) \right] = \int_{\mathbb{R}^d} \nabla \phi_1 \left( \int_{\mathbb{R}^d} zm(dz) \right) y[m'(dy) - m(dy)].
\]
Thus,
\[
\frac{\delta \Phi_1}{\delta m}(m, y) = \nabla \phi_1 \left( \int_{\mathbb{R}^d} zm(dz) \right) y.
\]
This yields the statement of the proposition.
\[
\square
\]

Now, let us compute the intrinsic derivative of the mean of the function depending also on a probability.

**Proposition A.3.** Let $\phi_2 : \mathbb{R}^d \times \mathcal{P}^p(\mathbb{R}^d) \rightarrow \mathbb{R}$ be differentiable w.r.t. $x$ and $m$,
\[
\Phi_2(m) \triangleq \int_{\mathbb{R}^d} \phi_2(x, m)m(dx).
\]
Then,
\[
\nabla_m \Phi_2(m, y) = \nabla_x \phi_2(y, m) + \int_{\mathbb{R}^d} \nabla_m \phi_2(x, m, y)m(dx),
\]
where $\nabla_x \phi_2$ stands for the derivative of $\phi_2$ w.r.t. finite dimensional variable, while $\nabla_m \phi_2$ denotes the derivative w.r.t. measure.

**Proof.** First, let us compute $\frac{\delta \Phi_2}{\delta m}$. We have that, given a probability $m'$,
\[
\lim_{s \downarrow 0} \frac{\Phi_2(m + s(m' - m)) - \Phi_2(m)}{s} = \lim_{s \downarrow 0} \int_{\mathbb{R}^d} \phi_2(x, m + s(m' - m))[m'(dx) - m(dx)] + \lim_{s \downarrow 0} \frac{1}{s} \left[ \int_{\mathbb{R}^d} [\phi_2(x, m + s(m' - m)) - \phi_2(x, m)]m(dx) \right].
\]

\[
40
\]
Therefore,
\[
\frac{\delta \Phi_2}{\delta m}(m, y) = \phi_2(y, m) + \int_{\mathbb{R}^d} \frac{\delta \phi_2}{\delta m}(x, m, y)m(dx).
\]

Taking the derivative w.r.t. \( y \), we obtain the statement of the proposition.

We complete this section with the formula of derivative of function depending on push-forward measure.

**Proposition A.4.** Let

- \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space;
- \(p > 1 \), \(q\) be conjugate to \(p\);
- \(\Phi : \mathcal{P}^p(\mathbb{R}^d) \to \mathbb{R}\) be such that \(\nabla_m \Phi\) is continuous and, for each \(m \in \mathcal{P}^p(\mathbb{R}^d)\), \(y \in \mathbb{R}^d\),
  \[
  \|\nabla_m \Phi(m, y)\|^q \leq C(1 + \mathcal{M}^p_p(m) + \|y\|^p);
  \]

Then, there exists a Gateaux derivative of the mapping \(L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \ni X \mapsto \Phi(X\sharp \mathbb{P})\) and

\[
\nabla_X \Phi(X\sharp \mathbb{P}) = \nabla_m \Phi(X\sharp \mathbb{P}, X).
\]

This proposition is a slight extension of [18, Proposition 2.2.3.] where only the case of bounded derivative is considered. Certainly, the proof follows the method used in [18].

**Proof.** Let \(X, Y \in L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)\). We shall prove that

\[
\lim_{h \downarrow 0} \frac{\Phi((X + hY)\sharp \mathbb{P}) - \Phi(X\sharp \mathbb{P})}{h} = \mathbb{E}[\nabla_m \Phi(X\sharp \mathbb{P}, X) \cdot Y].
\]  

(117)

Denote \(m \triangleq X\sharp \mathbb{P}, m^h \triangleq (X + hY)\sharp \mathbb{P}\). Due to (2.2) and change of variable formula, we have that

\[
\Phi((X + hY)\sharp \mathbb{P}) - \Phi(X\sharp \mathbb{P}) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta \Phi}{\delta m}(m + s(m^h - m), y)[m^h(dy) - m(dy)]ds
\]

\[
= \int_0^1 \mathbb{E}\left[\frac{\delta \Phi}{\delta m}(m + s(m^h - m), X + hY) - \frac{\delta \Phi}{\delta m}(m + s(m^h - m), X)\right]ds.
\]

Further, using the definition of \(\nabla_m \Phi\), we obtain

\[
\Phi((X + hY)\sharp \mathbb{P}) - \Phi(X\sharp \mathbb{P}) = h \int_0^1 \mathbb{E}[\nabla_m \Phi(m + s(m^h - m), X + h\xi_h Y)Y]ds,
\]

where, for each \(h\), \(\xi_h\) is a random variable taking values in \([0, 1]\). This implies that

\[
\left|\frac{\Phi((X + hY)\sharp \mathbb{P}) - \Phi(X\sharp \mathbb{P})}{h} - \mathbb{E}[\nabla_m \Phi(X\sharp \mathbb{P}, X) \cdot Y]\right|
\]

\[
\leq \int_0^1 \int_{\Omega} \|\nabla_m \Phi(m + s(m^h - m), X(\omega) + h\xi_h(\omega)Y(\omega)) - \nabla_m \Phi(m, X(\omega))\|\|Y(\omega)\|\mathbb{P}(d\omega)ds
\]

\[
\leq \left[\int_0^1 \int_{\Omega} \|\nabla_m \Phi(m + s(m^h - m), X(\omega) + h\xi_h(\omega)Y(\omega)) - \nabla_m \Phi(m, X(\omega))\|^q\mathbb{P}(d\omega)ds\right]^{1/q} \|Y\|_{L^p}.
\]  

(118)

Now assume that \(h \in [0, 1]\). Thanks to the assumption of the proposition, we have that

\[
\|\nabla_m \Phi(m + s(m^h - m), X(\omega) + h\xi_h(\omega)Y(\omega)) - \nabla_m \Phi(m, X(\omega))\|^q
\]

\[
\leq 2^{q-1}\|\nabla_m \Phi(m + s(m^h - m), X(\omega) + h\xi_h(\omega)Y(\omega))\|^q + 2^{q-1}\|\nabla_m \Phi(m, X(\omega))\|^q
\]

\[
\leq 2^{q-1}C(2 + \mathcal{M}^p_p(m + s(m^h - m)) + \mathcal{M}^p_p(m) + \|X(\omega) + h\xi_h(\omega)Y(\omega)\|^p + \|X(\omega)\|^p).
\]  

(119)
Further, we have that \( \mathcal{M}_p^\phi(m) = \mathcal{M}_p^\phi(X^2P) = \|X\|_{L^p}^p \), while \( \mathcal{M}_p^\phi(m + s(m^h - m)) \leq \mathcal{M}_p^\phi(X + hY) \leq (1 + 2^{p-1})\|X\|_{L^p}^{p-1} \|Y\|_{L^p}^p \). Plugging this estimates into right-hand side of (119), we obtain that

\[
\|\nabla_m \Phi(m + s(m^h - m), X(\omega) + h\xi_h(\omega)Y(\omega)) - \nabla_m \Phi(m, X(\omega))\|^q \\
\leq C_1(1 + \|X\|_{L^p}^p + \|Y\|_{L^p}^p + \|X(\omega)\|^p + \|Y(\omega)\|^p),
\]

where \( C_1 \) is a constant dependent only on \( C \) and \( p \). Thus, the functions

\[
\|\nabla_m \Phi(m + s(m^h - m), X(\omega) + h\xi_h(\omega)Y(\omega)) - \nabla_m \Phi(m, X(\omega))\|^q
\]

is bounded by the integrable function. Further, the assumption that \( \nabla_m \Phi \) is continuous yields that, for \( \lambda \otimes \mathbb{P}\)-a.e. \( (s, \omega) \in [0, 1] \times \Omega \),

\[
\nabla_m \Phi(m + s(m^h - m), X(\omega) + h\xi_h(\omega)Y(\omega)) \to \nabla_m \Phi(m, X(\omega)) \text{ as } h \to 0.
\]

Therefore, due to the dominated convergence theorem, we obtain that

\[
\int_0^1 \int_{\Omega} \|\nabla_m \Phi(m + s(m^h - m), X(\omega) + h\xi_h(\omega)Y(\omega)) - \nabla_m \Phi(m, X(\omega))\|^q \mathbb{P}(d\omega) ds
\]

tends to 0 while \( h \to 0 \). This means that the right-hand side of (118) tends to 0 and yields (117). \( \square \)

### B An integral Lebesgue differentiation theorem

Here we prove an extended version of the famous Lebesgue differentiation theorem.

**Theorem B.1.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a standard probability space, \(T > 0\), \(p \geq 1\), and let \(\phi : [0, T] \times \Omega \in L^p([0, T] \times \Omega, \mathcal{B}([0, T]) \otimes \mathcal{F}, \lambda \otimes \mathbb{P}; \mathbb{R})\). Then, for a.e. \( t \in [0, T] \),

\[
\lim_{h \downarrow 0} \frac{1}{h} \int_t^{t+h} \int_{\Omega} |\phi(\tau, \omega) - \phi(t, \omega)|^p \mathbb{P}(d\omega) d\tau = 0.
\]

**Proof.** First, if \( C > 0 \), then denote

\[
[\phi]_C \equiv (\phi \wedge C) \vee (-C).
\]

Since \((\Omega, \mathcal{F}, \mathbb{P})\) is a standard probability space, we can assume that \(\Omega\) is endowed by the distance \(d_{\Omega}\). Let \( N \) be a natural number. Since \(\phi \in L^p([0, T] \times \Omega, \mathcal{B}([0, T]) \otimes \mathcal{F}, \lambda \otimes \mathbb{P}; \mathbb{R})\), we have that there exists a number \(c_N\) such that

\[
\int_0^T \int_{\Omega} |\phi(t, \omega) - [\phi]_{c_N}(t, \omega)|^p \mathbb{P}(d\omega) dt \leq 2/N^3.
\]

Further, by the Lusin’s theorem, there exists a compact \( K_N \subset \Omega \) such that the restriction of the function \( [\phi]_{c_N} \) on \( K_N \) is continuous and \((\lambda \otimes \mathbb{P})(K_N) < 1/(N^3c_N^3)\). Let \(\varpi_N(\cdot)\) be a modulus of continuity of the function \( [\phi]_{c_N} \) on \( K_N \). One define the function \(\hat{\phi}_N\) by the rule:

\[
\hat{\phi}_N(t, \omega) \equiv \inf_{(t', \omega') \in [0, T] \times \Omega} ([\phi]_{c_N}(t', \omega') + \varpi_N(|t' - t| + d_{\Omega}(\omega', \omega))).
\]

By construction, we have that

- \(|\hat{\phi}_N| \leq c_N\);
- \(\hat{\phi}_N = [\phi]_{c_N}\) on \( K_N \).
Additionally,
\[
\int_0^T \int_\Omega |\phi(t, \omega) - \hat{\phi}_N(t, \omega)|^p \mathbb{P}(d\omega) dt \leq 2^{p-1} \int_0^T \int_\Omega |\phi(t, \omega) - [\phi]_{C_n}(t, \omega)|^p \mathbb{P}(d\omega) dt + 2^{p-1} \int_0^T \int_\Omega |[\phi]_{C_n}(t, \omega) - \hat{\phi}_N(t, \omega)|^p \mathbb{P}(d\omega) dt \leq 2^p / N^3 + 2^p c^p / (N^3 c^p_N) = 2^{p+1} / N^3.
\]

For each \( N \), let \( T_N \) be the set of \( t \in [0, T] \) such that
\[
\int_\Omega |\phi(t, \omega) - \hat{\phi}_N(t, \omega)|^p \mathbb{P}(d\omega) \geq 1 / N.
\]

By the Markov inequality and (121),
\[
\lambda(T_N) \leq 2^{p+1} / N^2.
\]

Denote,
\[
T_\infty \triangleq \limsup_{N \to \infty} T_N = \bigcap_{N=1}^\infty \bigcup_{n=N}^\infty T_n.
\]

Normalizing the Lebesgue measure on \([0, T]\) and using the Borel–Cantelli lemma for the sequence of events \( \{T_N\} \), we conclude that \( \lambda(T_\infty) = 0 \). Further, if \( t \in [0, T] \setminus T_\infty \), given \( \varepsilon > 0 \), there exists \( N \) such that, for every \( n \geq N \),
\[
\int_\Omega |\phi(t, \omega) - \hat{\phi}_n(t, \omega)|^p \mathbb{P}(d\omega) < \varepsilon.
\]

Indeed, in this case, we have that
\[
t \in \bigcup_{N=1}^\infty \bigcap_{n=N}^\infty ([0, T] \setminus T_n).
\]

Using the definition of \( T_N \) (see (122)), we arrive at (123).

Now, let \( L \) be a natural number. For \( l = 0, \ldots, L \), set
\[
\theta^l_L \triangleq Tl / L.
\]

We have that
\[
|\hat{\phi}_n(t, \omega) - \hat{\phi}_n(\theta^l_L, \omega)| \leq \varpi_n(T / L) \text{ when } t \in [\theta^l_L, \theta^{l+1}_L].
\]

Denote
\[
g^l_{n,L}(\omega) \triangleq \hat{\phi}_n(\theta^l_L, \omega).
\]

Given natural \( n \), \( L \), and \( l = 0, \ldots, L \), by the Lebesgue differentiation theorem, there exists a set \( A^l_{n,L} \subset [0, T] \) such that \( \lambda(A^l_{n,L}) = 0 \) and, for every \( t \in [0, T] \setminus A^l_{n,L} \),
\[
\frac{1}{h} \int_t^{t+h} \int_\Omega |\phi(\tau, \omega) - g^l_{n,L}(\omega)|^p \mathbb{P}(d\omega) d\tau \to \int_\Omega |\phi(t, \omega) - g^l_{n,L}(\omega)|^p \mathbb{P}(d\omega).
\]

Now, let
\[
A \triangleq \left( \bigcup_{n=1}^\infty \bigcup_{L=1}^L \bigcup_{l=1}^L A^l_{n,L} \right) \bigcup T_\infty.
\]

Notice that \( \lambda(A) = 0 \). Now, let \( t \in [0, T] \setminus A \), \( \varepsilon > 0 \). One can find \( n \) such that \( 1/n < \varepsilon \) and (123) holds true. Further, choose \( L \) satisfying \( \varpi_n(T / L) \leq \varepsilon^{1/p} \). Finally, let \( \theta^l_L \) be such that \( t \in [\theta^l_L, \theta^{l+1}_L] \). We have
that
\[
\frac{1}{h} \int_t^{t+h} \int_\Omega |\phi(\tau, \omega) - \phi(t, \omega)|^p \mathbb{P}(d\omega) d\tau \\
\leq 2^{p-1} \frac{1}{h} \int_t^{t+h} \int_\Omega |\phi(\tau, \omega) - g_{n,L}(\omega)|^p \mathbb{P}(d\omega) d\tau - 2^{p-1} \int_\Omega |\phi(t, \omega) - g_{n,L}(\omega)|^p \mathbb{P}(d\omega) \\
+ 2^p \int_\Omega |\phi(t, \omega) - g_{n,L}(\omega)|^p \mathbb{P}(d\omega)
\]  
(126)

By (125), there exists \( \hat{h}(\varepsilon) \) such that, for \( h < \hat{h}(\varepsilon) \),
\[
\frac{1}{h} \int_t^{t+h} \int_\Omega |\phi(\tau, \omega) - g_{n,L}(\omega)|^p \mathbb{P}(d\omega) d\tau - \int_\Omega |\phi(t, \omega) - g_{n,L}(\omega)|^p \mathbb{P}(d\omega) \leq \varepsilon.
\]  
(127)

Then, since we choose \( n \) such that \( 1/n < \varepsilon \) and (128) holds true, we obtain that
\[
\int_\Omega |\phi(t, \omega) - \hat{\phi}(t, \omega)|^p \mathbb{P}(d\omega) \leq \varepsilon.
\]  
(128)

Finally, since \( L \) is such that \( \varpi_n(T/L) < \varepsilon, |t - \theta'_L| < T/L, \) using (124), we conclude that
\[
\int_\Omega |\hat{\phi}(t, \omega) - g_{n,L}(\omega)|^p \mathbb{P}(d\omega) \leq \varepsilon.
\]

Combining this with (127) and (128), we estimate the right-hand side of (126) for every \( t \in \mathcal{A} \) by \( (2^{p-1} + 2^p)\varepsilon \) while \( h < \hat{h}(\varepsilon) \). This gives the statement of theorem. \( \square \)

Theorem B.1, the Fubini’s theorem and the Jensen’s inequality imply the following.

**Corollary B.2.** If \((\Omega, \mathcal{F}, \mathbb{P})\) is a standard probability space, \( T > 0, p \geq 1, \) and \( \phi \in L^p([0, T] \times \Omega, \mathcal{B}([0, T]) \otimes \mathcal{F}, \lambda \otimes \mathbb{P}; \mathbb{R}) \), then, for a.e. \( t \in [0, T] \),
\[
\lim_{h \downarrow 0} \int_\Omega \left\| \frac{1}{h} \int_t^{t+h} \phi(\tau, \omega) d\tau - \phi(t, \omega) \right\|^p \mathbb{P}(d\omega) = 0.
\]

C Mean field type differential equation

The aim of this appendix is to show that there exists a unique solution of the following mean field type system defined on a standard probability space \((\Omega, \mathcal{F}, \mathbb{P})\)
\[
\frac{d}{dt} X(t, \omega) = b(t, X(t, \omega), X(t)\mu \mathbb{P}, \omega), \quad X(0, \omega) = X_0(\omega).
\]  
(129)

Here \( b \) is a measurable function from \([0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times \Omega \to \mathbb{R} \). We say that \( X \in \mathcal{X}^p \) solves (129), if this equality is satisfied for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \).

**Proposition C.1.** Assume that
- \( X_0 \in L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d); \)
- for every \( x', x'' \in \mathbb{R}^d, m', m'' \in \mathcal{P}(\mathbb{R}^d), \)
\[
\|b(t, x', m', \omega) - b(t, x'', m'', \omega)\| \leq L\|x' - x''\| + W_p(m', m'');
\]
• there exist a constant \( \tilde{C} \) and a function \( a \in L^p([0, T] \times \Omega, \mathcal{B}([0, T]) \times \mathcal{F}, \lambda \otimes \mathbb{P}, \mathbb{R}) \) such that
\[
\|b(t, x, m, \omega)\| \leq \tilde{C}(\|x\| + \mathcal{M}_p(m)) + a(t, \omega).
\]

Then, there exists a solution of (129). If \( X', X'' \) solve (129), then \( X'(\cdot, \omega) = X''(\cdot, \omega) \) for \( \mathbb{P}\text{-a.e.} \ \omega \in \Omega \).

Proof. First, we claim that, for a.e. \( \omega \in \Omega \), the function \( a(\cdot, \omega) \) lies in \( L^p([0, T], \mathcal{B}([0, T]); \lambda) \). This follows from the Fubini theorem and the assumption that \( a \in L^p([0, T] \times \Omega, \mathcal{B}([0, T]) \times \mathcal{F}; \lambda \otimes \mathbb{P}) \). Now, let us consider equation (129) in the \( L^1 \) sense, i.e., we try to find a function \([0, T] \mapsto \Xi(t) \in L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)\) such that for every \( t \in [0, T], \)
\[
\int_{\Omega} \left\| \Xi(t, \omega) - \int_{0}^{t} b(\tau, \Xi(\tau, \omega), \Xi(\tau, \omega))d\tau \right\| \mathbb{P}(d\omega) = 0. \tag{130}
\]
Using the same technique as in the Picard’s existence and uniqueness theorem, one can prove that there exists a unique process \( \Xi \) satisfying (130). Additionally, using the sublinear growth assumption, we conclude that
\[
\|\Xi(t)\|_{L^1} \leq 2\tilde{C} \int_{0}^{t} \|\Xi(\tau)\|_{L^1} d\tau + \int_{0}^{t} \int_{\Omega} |a(\tau, \omega)| \mathbb{P}(d\omega) d\tau \leq 2\tilde{C} \int_{0}^{t} \|\Xi(\tau)\|_{L^1} d\tau + T^{1/q} \|a\|_{L^p}.
\]
This and the Gronwall’s inequality yield the estimate
\[
\|\Xi(t)\|_{L^1} \leq \tilde{C}_1
\]
for some constant \( \tilde{C}_1 \).

Further, for each \( \omega \), let \( X(\cdot, \omega) \) satisfy
\[
\frac{d}{dt} X(t, \omega) = b(t, X(t, \omega), \Xi(t, \omega)), \quad X(0, \omega) = X_0(\omega). \tag{131}
\]
We will prove that \( X(t) = \Xi(t) \). This will give the existence of pathwise solution. To this end, first notice that
\[
\|X(t)\|_{L^1} \leq \tilde{C} \int_{0}^{t} \|X(\tau)\|_{L^1} d\tau + \int_{0}^{t} \int_{\Omega} |a(\tau, \omega)| \mathbb{P}(d\omega) d\tau.
\]
Thus, the function \( t \mapsto \|X(t)\|_{L^1} \) is bounded. This estimate and the fact that \( X(\cdot, \omega) \) solves (131) imply that, for every \( t \in [0, T], \)
\[
\int_{\Omega} \left\| X(t, \omega) - \int_{0}^{t} b(\tau, X(\tau, \omega), \Xi(\tau, \omega))d\tau \right\| \mathbb{P}(d\omega) = 0.
\]
This means that \( X(\cdot) \) is the \( L^1 \)-solution to the equation
\[
\frac{d}{dt} X(t, \omega) = b(t, X(t, \omega), \Xi(t, \omega)), \quad X(0) = X_0. \tag{132}
\]
Simultaneously, due to (130), \( \Xi \) is the \( L^1 \)-solution of (132). Applying uniqueness result for the \( L^1 \)-solutions, we conclude that \( X(t) = \Xi(t) \).

The uniqueness of pathwise solution can be obtained using the standard arguments relying on the Gronwall’s inequality.

To complete the proof, we should show that \( X \in \mathcal{X}^p \). Indeed, for \( \mathbb{P}\text{-a.e.} \ \omega \in \Omega \), the function \( t \mapsto a(t, \omega) \) lies in \( L^p([0, T], \mathcal{B}([0, T]); \lambda; \mathbb{R}^d) \). Further,
\[
\|X(t, \omega)\| \leq \tilde{C} \int_{0}^{t} \|X(\tau, \omega)\| d\tau + \tilde{C} \int_{0}^{t} \|X(\tau)\|_{L^1} d\tau + \int_{0}^{t} |a(t, \omega)| dt.
\]
Since \( \|X(\tau)\|_{L^1} \) is uniformly bounded by a constant \( \tilde{C}_1 \), by the Gronwall’s inequality, we have that
\[
\|X(\cdot, \omega)\|_{\mathcal{X}^p} \leq \tilde{C}_2 \left[ \int_{0}^{T} |a(t, \omega)|^p dt \right]^{1/p}.
\]
Here \( \tilde{C}_2 \) is a constant. This and assumption that \( \int_{0}^{T} \int_{\Omega} |a(t, \omega)|^p dt \mathbb{P}(d\omega) < \infty \) give the inclusion \( X \in \mathcal{X}^p. \)
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