On the cohomology of the Lubin-Tate curve of level 2 and the Lusztig theory over finite rings

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Abstract

In [IMT], we study the étale cohomology group $W$ of irreducible components, with each having an affine model $X^q + X = \xi Y^{q+1} + c$ with $\xi \in \mathbb{F}_q^*$ and $c \in \mathbb{F}_q$, in the stable reduction of the Lubin-Tate curve $\mathcal{X}(\pi^q)$. Then, $W$ is considered as a $\text{GL}_2(\mathcal{O}_F/\pi^2)$-representation and relates to unramified cuspidal representations of $\text{GL}_2(F)$ of level 1. In this paper, we study a relationship between $W$ and the étale cohomology groups of some Lusztig varieties constructed in [Lus] and [Lus2].

1 Introduction

Let $F$ be a non-archimedean local field with ring of integers $\mathcal{O}_F$, uniformizer $\pi$ and residue field $\mathbb{F}_q$ of characteristic $p > 0$. We write $\mathbf{F}$ for an algebraic closure of $\mathbb{F}_q$. Let $\overline{\mathbf{F}}$ denote an algebraic closure of $\mathbf{F}$ and $C$ its completion. Roughly speaking, the Lubin-Tate curve $X(\pi^2)$ is the generic fiber of the deformation space of a formal $\mathcal{O}_F$-module of height 2 over $\mathbf{F}$ equipped with the Drinfeld level $\pi$-structure. Namely, $X(\pi^2)$ is a rigid analytic curve. In [IMT], we compute defining equations of irreducible components in the stable reduction of the Lubin-Tate curve $X(\pi^2)$, and study the étale cohomology group of some components as a representation of a product of $G^\times_2 := \text{GL}_2(\mathcal{O}_F/\pi^2)$, the central division algebra $\mathcal{O}_D^\times$ over $\mathbf{F}$ of invariant $1/2$, and the inertia group $I_F$. For $n \geq 1$, let $U^n_D$ be open compact subgroups of $\mathcal{O}_D^\times$. We put $\mathcal{O}^\times_n := \mathcal{O}_D^\times / U^n_D$ and $G := G^\times_2 \times \mathcal{O}^\times_3 \times I_F$.

We set

$$G_{00}^{\times} := \{(x_0, y_0) \in \mathbf{F}^2 \mid \xi := x_0^q y_0 - x_0 y_0^q \in \mathbb{F}_q^*, x_0^{q-1} = y_0^{q-1} = -1\}.$$ 

Then, we have $|G_{00}^{\times}| = |\text{GL}_2(\mathbb{F}_q)| = q(q-1)(q^2-1)$. For $i = (x_0, y_0) \in G_{00}^{\times}$, let $X_i$ denote the smooth compactification of an affine curve $X^\times - X = \xi(Y^{q+1} - Y^{q+1})$. Each curve $X_i$ has $q$ connected components and each component with genus $q(q-1)/2$ has an affine model $X^q + X = \xi Y^{q+1} + c$ with some $c \in \mathbb{F}_q$. In the stable reduction of $\mathcal{X}(\pi^2)$, the components $\{X_i\}_{i \in G_{00}^{\times}}$ appear, which is proved in [IMT]. We set

$$W := \bigoplus_{i \in G_{00}^{\times}} H^1(X_i, \mathbb{Q}_l) \subset H^1(\mathcal{X}(\pi^2)_C, \mathbb{Q}_l)$$
with a prime number $l \neq p$. Apriori, the product group $G$ acts on the curve $X(\pi^2)_C$ on the right, and hence on the étale cohomology group $H^1(X(\pi^2)_C, \mathbb{Q}_l)$ on the left. See [Ca] Section 1 for more details. Then, the subspace $W$ is stable under the $G$-action. In [IMT, 7.2], we analyze $W$ as a $G$-representation. Let $E/F$ denote the unramified quadratic extension. Canonically, the group $\mathcal{O}_F^\times$ contains $\Gamma := (\mathcal{O}_F/\pi^2)^\times$ as a subgroup. For any finite extension $L/F$, let $W_L$ denote the Weil group of $L$ and $I_L$ the inertia subgroup of $W_L$. Then, we have the Artin reciprocity map $\alpha_L : W_L^\mathrm{ab} \to L^\times$ normalized such that the geometric Frobenius goes to a prime element under this map. We have the restriction $\alpha_L : I_L^\mathrm{ab} \to \mathcal{O}_L^\times$. Hence, in particular, the map $\alpha_E$ induces the following map $I_F \to I_F^\mathrm{ab} \simeq I_F^\mathrm{ab} \to \mathcal{O}_E^\times \to \Gamma$, which we denote by $\alpha_E$. Then, the inertia group $I_F$ acts on $W$ by factoring through the map $\alpha_E : I_F \to \Gamma$. Hence, the restriction $W|_{G_E^F \times \{1\} \times I_F}$ can be considered as a $G_E^F \times \Gamma$-representation, which we denote by $W_1$. Let $W_2$ denote the restriction $W|_{G_E^F \times \Gamma \times \{1\}}$.

Let $G$ be a connected reductive algebraic group over $F$ with a given $\mathbb{F}_q$-rational structure with associated Frobenius morphism $F : G \to G$. For $n \geq 1$, in [Lus] and [Lus2], G. Lusztig constructs an affine algebraic variety over $\mathbb{F}_q$ with $G(\mathcal{O}_F/\pi^n)$-action and some torus $T(\mathcal{O}_F/\pi^n)$-action, whose étale cohomology group realize some irreducible representations of $G(\mathcal{O}_F/\pi^n)$.

These representations are attached to characters of $T(\mathcal{O}_F/\pi^n)$ in general position. The works [Lus] and [Lus2] are generalizations of the Deligne-Lusztig theory [DL] over finite fields to infinite rings $\mathcal{O}_F/\pi^n$. In this paper, we call the variety over $\mathbb{F}_q$ the Lusztig variety for $G(\mathcal{O}_F/\pi^n)$. Specialized to a case $S^F_2 := SL_2(\mathcal{O}_F/\pi^2)$, Lusztig gives an explicit description of the Lusztig variety of dimension 2 for $S^F_2$ and studies its cohomology groups in [Lus, Section 3]. In the cohomology groups, all unramified representations, in a sense of [Sha, p.37], occur each one with multiplicity 2. Hence, we call the variety the unramified Lusztig surface for $S^F_2$.

In this paper, we will study the affine unramified Lusztig surface $\tilde{X}$ for $G_E^F$ in the same way as in [Lus, Section 3]. The surface $\tilde{X}$ admits an action of a product group $G_E^F \times \Gamma$. Then, we investigate the étale cohomology group $H^2_E(\tilde{X}, \mathbb{Q}_l)$ as a $G_E^F \times \Gamma$-representation in the same way as in loc. cit. As a result, we show that a direct sum $\rho_{DL}$ of all cuspidal or unramified representations of $G_E^F$, in a sense of [Sta], appears as a $G_E^F$-subrepresentation of $H^2_E(\tilde{X}, \mathbb{Q}_l)$. By using the explicit description of $H^2_E(\tilde{X}, \mathbb{Q}_l)$ as a $G_E^F$-representation, we can compare $\rho_{DL}$ with $\rho_{DL}$.

For $S^F_2$, we obtain the same things as $G_E^F$. See Remarks 3.3 and 4.3 for more details. In Section 5, we write down the Lusztig curve $X_D$ for $G^F_2$ and study its cohomology group $H^2_D$ as a $G^F_2 \times \Gamma$-representation. The Lusztig variety for the group of reduced norm one in a central division algebra of invariant $1/n$ over $F$ is explicitly described in [Lus, Section 2]. We consider the product $X := \tilde{X} \times X_D$. Then, this variety $X$ admits an action of $G^F_2 \times \mathcal{O}^\times_2 \times \Gamma$. Hence, the étale cohomology group $H^3_D(\tilde{X}, \mathbb{Q}_l)$ is considered as a $G$-representation through a surjective map $G \to G^F_2 \times \mathcal{O}^\times_2 \times \Gamma$. As a result of the analyses of $H^1_D(X_D, \mathbb{Q}_l)$ as a $\mathcal{O}^\times_2 \times \Gamma$-representation in Lemma 5.1 and $H^2_D(\tilde{X}, \mathbb{Q}_l)$ as a $G^F_2 \times \Gamma$-representation in Proposition 5.2, we show that $W$ is contained in $H^3_D(X, \mathbb{Q}_l)$ as a $G$-subrepresentation in Theorem 5.2. See also [Y] for a connection between the non-abelian Lubin-Tate theory of level 1 and the Deligne-Lusztig theory over finite fields.

**Notation.** We fix some notations used throughout this paper. Let $F$ be a non-archimedean local field with ring of integers $\mathcal{O}_F$, uniformizer $\pi$ and residue field $\mathbb{F}_q$ of characteristic $p > 0$. We identify $(\mathcal{O}_F/\pi^2)^\times$ with $\mathbb{F}_q^\times \times \mathbb{F}_q$ by $a_0 + a_1 \pi \mapsto (a_0, (a_1/a_0))$. Let $E/F$ be the unramified quadratic extension and $\mathcal{O}_E$ the ring of integers. Of course, the residue field of $E$ is equal to $\mathbb{F}_q$. Let $\tau \in \text{Gal}(E/F)$ be a non-trivial element. Let $\mathbb{F}$ denote an algebraic closure of $\mathbb{F}_q$. We set $G^F_n := \text{GL}_2(\mathcal{O}_F/\pi^n)$ and $S^F_n := \text{SL}_2(\mathcal{O}_F/\pi^n)$ for $n \geq 1$. Let $D$ be the central division algebra over $F$ of invariant $1/2$. Let $\mathcal{O}_D$ denote the ring of integers of $D$, and $\varphi$ a prime element of $D$.
such that \( \varphi^2 = \pi \). Then, we have \( \mathcal{O}_D = \mathcal{O}_E \oplus \varphi \mathcal{O}_E \) with \( \varphi a = a^\varphi \) for \( a \in \mathcal{O}_E \). Furthermore, for \( n \geq 1 \), let \( U_n^b \) denote the open compact subgroups \( 1 + (\varphi^n) \) of \( D^\times \). We set \( \mathcal{O}_n^X := \mathcal{O}_D^n / U_n^b \).

Let \( \text{Nrd}_{D/F} : D^\times \to F^\times \) denote the reduced norm of \( D^\times \). For any finite extension \( L/F \), let \( W_L \) denote the Weil group of \( L \) and \( I_L \) the inertia subgroup of \( W_L \). Then, we have the Artin reciprocity map \( a_L : W_L \sim L^\times \) normalized such that the geometric Frobenius goes to a prime element under this map. Then, we have the restriction \( L/K \)

\[
\text{Nrd}_{D/F} : D^\times \to F^\times
\]

denote the Weil group of \( L \) and \( I_L \) the inertia subgroup of \( W_L \). Then, we have the Artin reciprocity map \( a_L : W_L \sim L^\times \) normalized such that the geometric Frobenius goes to a prime element under this map.

Then, we have the restriction \( a_L : I_L^b \to O_L^\times \). For a finite extension \( L/K \), we have \( a_K = \text{Nrd}_{L/K} \circ a_L \). If \( X \) is an affine algebraic variety over \( F \) and \( r \geq 1 \), we set \( X_r := X(F[[\pi]] / (\pi^r)) \). For a prime number \( l \neq p \) and a finite abelian group \( M \), we write \( M^\vee \) for a character group \( \text{Hom}(M, \hat{\mathbb{Q}}_l^\times) \). For a representation \( \pi \) of a finite group \( G \) over \( \hat{\mathbb{Q}}_l \), we write \( \pi^\vee \) for its dual. For any variety \( Y/F \) with an action of a finite abelian group and any character \( w \) of that finite group, let \( H^q_t(Y, \hat{\mathbb{Q}}_l)_w \) denote the subspace of \( H^q_t(Y, \hat{\mathbb{Q}}_l) \) on which the finite group acts according to \( w \).

### 2 Cuspidal representation of \( G_2^F \)

In this section, we briefly recall definitions of cuspidal representation of \( G_2^F \), and strongly primitive character of a torus \( \Gamma := (\mathcal{O}_E / \pi^2)^\times \) in Definitions 2.1 and 2.2 which are due to [AOPS 5.1 and 5.2], [Oroli 6.2], [Sta p.2835] and [Sta2]. A cuspidal representation of \( G_2^F \) has degree \( q(q-1) \), and there exist \( q(q-1)(q^2-1)/2 \) cuspidal representations up to equivalence. These cuspidal representations are parametrized by strongly primitive characters of a maximal torus \( \Gamma \subset G_2^F \). See Remark 2.3.3 for more details.

First, we prepare some notations. Let \( m : G_2^F \to G_1^F \) be the canonical surjection. We write \( \tilde{g} \) for the image of \( g \in G_2^F \) by the map \( m \). We set \( N := \text{Ker} \ m \). Then, the group \( N \) is isomorphic to an abelian group \( M_2(F_q) \). All irreducible representations of \( G_2^F \) occur as an irreducible component of the induced representation \( \text{Ind}_{G_1^F}^{G_2^F}(\Phi) \) with some character \( \Phi \in N^\vee \). We fix a non-trivial character \( \eta \in F_q^\times \). Then, we have the following isomorphism

\[
M_2(F_q) \sim N^\vee : \beta \mapsto (\eta_\beta : h \mapsto \eta(\text{Tr}_{M_2(F_q)/F_q}(\beta \pi^{-1}(h - 1)))) \quad (2.1)
\]

For example, see [Sta2 Section 2] for more details. The group \( G_2^F \) acts on \( N^\vee \) by conjugation. For \( \Phi \in N^\vee \), let \( T(\Phi) \) denote the centralizer of \( \Phi \) in \( G_2^F \). Then, \( T(\Phi) \) is a torus of \( G_2^F \). Let \( N(\Phi) := T(\Phi) \cap N \). Then, \( N(\Phi) \) is the stabilizer of \( \Phi \) in \( G_2^F \). By the Clifford theory, any irreducible representation \( U \) of \( N(\Phi) \) such that \( \Phi \in U \cap N \) is of the form \( \Psi_{\psi, \Phi} : \Psi_{\psi, \Phi}(n) = \psi(t) \Phi(n) \) for \( t \in T(\Phi) \), \( n \in N \) where \( \psi \in T(\Phi)^\vee \) and satisfies \( \psi |_{T(\Phi) \cap N} = \Phi |_{T(\Phi) \cap N} \). See [Sta2 Theorem 2.1] for more details.

**Definition 2.1.** Let \( \rho \) be an irreducible representation of \( G_2^F \). We assume that \( \rho \) does not factor through \( G_1^F \). We call \( \rho \) cuspidal if \( \rho \) is equivalent to \( \text{Ind}_{N(\Phi)}^{G_2^F}(\Phi) \) with \( \Phi = \eta_\beta \in \mathcal{O}_E / \pi^2 \), and \( \beta \) is conjugate to the following

\[
\begin{pmatrix}
0 & 1 \\
-\Delta & s
\end{pmatrix}
\]

where \( X^2 - sX + \Delta \) is an irreducible polynomial in \( F_q[X] \).

We also call cuspidal representation of \( G_2^F \) unramified.

**Definition 2.2.** ([AOPS 5.1]) We call a character \( \psi \) of \( \Gamma : = (\mathcal{O}_E / \pi^2)^\times \) strongly primitive if the restriction of \( \psi \) to a subgroup \( F_q^\times \simeq \text{Ker} (\Gamma \to (\mathcal{O}_E / \pi)^\times) \) does not factor through the trace map \( \text{Tr}_{F_q^\times / F_q} : F_q^\times \to F_q \).
Remark 2.3. 1. For $SL_2$, a similar definition to Definition 2.1 is found in [Sha p.37].
2. The dimension of the cuspidal representation of $G^F_2$ is equal to $q(q-1)$, because we have $[G^F_2 : N(\Phi)] = q(q-1)$ if $\Phi$ is conjugate to $[G^F_2 : \Phi]$. 3. The definition 2.1 of cuspidal representation in [Lus] is not equal to the one in [Lus, 3.2]. To obtain a left $x$ with some $g$ fixed rank 2. Let $T$ be an unramified Lusztig surface for $G$, which we call the group of $\tilde{\Gamma}$.

3 Unramified Lusztig surface for $G^F_2$ and its cohomology $H^2_c$

In this section, we recall the Lusztig theory in [Lus] and [Lus2], specialized to the case $G^F_2$. The papers [Lus] and [Lus2] are generalizations of the Deligne-Lusztig theory over a finite field to finite rings $O_F/\pi^r$ with $r \geq 2$. Let $G$ be a reductive connected algebraic group over $F$ with associated Frobenius map $F : G \to G$ and $T$ is a maximal torus of $G$. Roughly speaking, the Lusztig variety for $G(O_F/\pi^r)$ means some variety over $F$ with $G(O_F/\pi^r) \times T(O_F/\pi^r)$-action, and its $\ell$-adic cohomology groups give a correspondence between characters of $T(O_F/\pi^r)$ in general position and some irreducible representations of $G(O_F/\pi^r)$. A concrete description of the Lusztig surface for $S^2_F$ is given in [Lus, section 3]. In this section, in the same way as in loc. cit., we explicitly describe the Lusztig surface for $G^F_2$ and the étale cohomology group $H^2_c$ of it. The cohomology group realizes all unramified or cuspidal representations of $G^F_2$ introduced in Definition 2.1. See Proposition 6.1 and Corollary 3.2 for more details. Arguments in this section are almost same as the ones in loc. cit.

In the following, we assume that $p$ is odd and

$$\text{char } F = p > 0, \text{ or, char } F = 0 \text{ and } e_{F/q_F} \geq 2. \quad (3.1)$$

Let $F''_F$ be the maximal unramified extension of $F$ in a fixed algebraic closure of $F$. $A := O_F/F''_F$. Then, by the assumption $(3.1)$, we have $A \simeq F[[\pi]]/(\pi^2)$. Define $F : A \to A$ by $F(a_0 + a_1 \pi) = a_0^q + a_1^q \pi$ where $a_0, a_1 \in F$. Let $V$ be a 2-dimensional $F$-vector space with a fixed $F_q$-rational structure with the Frobenius map $F : V \to V$. Let $V'$ be a basis of $V$. Let $G := GL(V)$. We identify $G \simeq GL_2(F)$ by $g \mapsto \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ with $eg = ae + ce' , e'g = be + de'$. Then, $G$ also has a $F_q$-rational structure with the Frobenius map $F : G \to G$ such that $F(vg) = F(v)F(g)$ for $g \in G, v \in V$. We put $V_2 := V \otimes_F A$. Clearly, $V_2$ is a free $A$-module of rank 2. Let $T := \{ \left( \begin{array}{cc} F(a) & 0 \\ 0 & a \end{array} \right) \} \subset G$ and $U := \{ \left( \begin{array}{cc} 1 & c \\ 0 & 1 \end{array} \right) \} \subset G$. Let $v := \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$.

We define as follows

$$\tilde{X} := \{ g \in G_2 \ | \ F(g)g^{-1} \in U_2v \}, \quad (3.2)$$

which we call the unramified Lusztig surface for $G^F_2$. Then, $G^F_2$ acts on $\tilde{X}$ on the right. This definition is not equal to the one in [Lus 3.2]. To obtain a left $G^F_2$-action on the cohomology group of $\tilde{X}$, in (3.2), we slightly modify the definition in loc. cit. By a direct computation, we check that, for $g \in G^F_2$, the condition $F(g)g^{-1} \in U_2v$ is equivalent to $g = \left( \begin{array}{cc} -F(x) & -F(y) \\ x & y \end{array} \right)$ with some $x, y \in A$ and $\det(g) \in (O_F/\pi^2)^\times$. We set

$$\Gamma := \{ t \in T_2 \ | \ tF(t) \in T_2^F \}. \quad (3.3)$$
Let $\Gamma \ni t$ act on $\tilde{X}$ by $g \mapsto t^{-1}g$. Then, the group $\Gamma$ is isomorphic to the following $\{(a_0, a_1) \in \mathbb{F}^2 \mid a_0 \in \mathbb{F}^\times_q, a_0^q a_1 + a_0 d_1^q \in \mathbb{F}_q\}$. Hence, $\Gamma$ is isomorphic to a group $\left(\mathcal{O}_E/\pi^2\right)^\times \ni a_0 + a_1 \pi$ of order $q^2(q^2 - 1)$. In the following, we fix the identification $\Gamma \simeq \left(\mathcal{O}_E/\pi^2\right)^\times$. We define a subgroup $A' := \{1 + \pi a_1 \mid a_1^q + a_1 = 0\} \subset \Gamma$. We put 

$$\mathcal{S} := \{x \in V_2 \mid x \wedge F(x) \in \left(\mathcal{O}_E/\pi^2\right)^\times (e \wedge e')\}$$

with a right $G_2^F$-action $x \mapsto xg$. The group $\Gamma$ acts on $\mathcal{S}$ by the inverse of scalar multiplication. Then, as in [Lus 3.4], we have an isomorphism $\iota : \tilde{X} \cong \mathcal{S} : g \mapsto \tilde{g}$, which is compatible with the $G_2^F \times \Gamma$-actions. We write $x = xe + ye' \in \mathcal{S}$ with $x = x_0 + x_1 \pi$ and $y = y_0 + y_1 \pi$. Then, we have the following isomorphism by a direct computation

$$\mathcal{S} \simeq \{(x_0, x_1, y_0, y_1) \in \mathbb{F}^4 \mid x_0 y_0^q - x_0^q y_0 \in \mathbb{F}_q^\times, x_1 y_0^q + x_0 y_1^q - x_0^q y_1 - x_0^q y_0 \in \mathbb{F}_q\}. \quad (3.4)$$

Let $\lambda = a_0 + \pi a_1 \in \Gamma$. Then, $\lambda^{-1}$ acts on $\mathcal{S}$ as follows

$$\lambda^{-1} : (x_0, x_1, y_0, y_1) \mapsto (a_0 x_0, a_1 x_0 + a_0 x_1, a_0 y_0, a_1 y_0 + a_0 y_1). \quad (3.5)$$

Let $g = \left(\begin{array}{cc} a_0 + a_1 \pi & b_0 + b_1 \pi \\ c_0 + c_1 \pi & d_0 + d_1 \pi \end{array}\right) \in G_2^F$. Then, the element $g$ acts on $\mathcal{S}$ as follows

$$g : (x_0, x_1, y_0, y_1) \mapsto (a_0 x_0 + c_0 y_0, a_1 x_0 + a_0 x_1 + c_1 y_0 + c_0 y_1, b_0 x_0 + d_0 y_0, b_1 x_0 + b_0 x_1 + d_0 y_1 + d_1 y_0). \quad (3.6)$$

Let $l \neq p$ be a prime number. In the following, we analyze the étale cohomology group $H^2_\text{ét}(\tilde{X}, \mathbb{Q}_l)$ on the basis of arguments in [Lus Section 3]. To do so, as in loc. cit., we consider the following fibration to the Deligne-Lusztig curve $\mathcal{S}_0 := \{(x_0, y_0) \in \mathbb{F}^2 \mid x_0 y_0^q - x_0^q y_0 \in \mathbb{F}_q^\times\}$ for $G_1^F$

$$\kappa : \mathcal{S} \longrightarrow \mathcal{S}_0 : (x_0, x_1, y_0, y_1) \mapsto (x_0, y_0). \quad (3.7)$$

Then, we have $\kappa(xg) = \kappa(x)\tilde{g}$ for all $x \in \mathcal{S}$ and $g \in G_2^F$. The fiber of $\kappa$ is stable under the action of $A'$ by [3.5]. Let $\mathcal{S}_0^\times \subset \mathcal{S}_0$ be the closed subset defined by $x_0 y_0^q = x_0^q y_0 = 1$. The set $\mathcal{S}_0^\times$ is stable under the action of $G_1^F$, and its action is simply transitive. Hence, the set $\mathcal{S}_0^\times$ consists of $|G_1^F| = q(q - 1)(q^2 - 1)$ closed points. We let $\xi := x_0^q y_0 - x_0 y_0^q$. Furthermore, we put as follows

$$s_0 := y_0^q x_1 - x_0^q y_1, \quad s_1 := -\xi^{-1}(x_0 y_1 - x_1 y_0), \quad t_0 := x_0 y_0^q - x_0^q y_0.$$ 

Then, the fiber $\kappa^{-1}((x_0, y_0))$ for $(x_0, y_0) \in \mathcal{S}_0$ is identified with the following by a direct computation,

$$\{(s_0, s_1) \in \mathbb{F}^2 \mid s_0^q + s_1^q t_0 \in \mathbb{F}_q\}. \quad (3.8)$$

For $(x_0, y_0) \in \mathcal{S}_0, x_0 y_0^q = x_0^q y_0 = -1$ is equivalent to $t_0 = 0$. First, let $(x_0, y_0) \in \mathcal{S}_0 := \mathcal{S}_0 \setminus \mathcal{S}_0^\times$. Then, we have $t_0 \neq 0$. Hence, by (3.8), the fiber $\kappa^{-1}((x_0, y_0))$ is a disjoint union of $q$ affine lines. Secondly, we have the following identification again by (3.8)

$$\mathcal{S}_* := \kappa^{-1}(\mathcal{S}_0^\times) \simeq \{(x_1, y_1) \in \mathbb{F}^2 \mid x_1 y_0^q - x_0^q y_1 = s_0 \} \simeq \mathbb{A}^1 \times \mathbb{F}_q^\times \times \mathcal{S}_0^\times \simeq \mathbb{A}^1 \times \mathbb{F}_q^2 \times \mathcal{S}_0^\times. \quad (3.9)$$

Furthermore, $\mathcal{S}_*$ is stable under the actions of $G_2^F$ and $\Gamma$. We write down the action of $G_2^F$ and $\Gamma$ on the fiber space $\mathcal{S}_*$ under the identification (3.9). For $g = \left(\begin{array}{cc} a_0 + a_1 \pi & b_0 + b_1 \pi \\ c_0 + c_1 \pi & d_0 + d_1 \pi \end{array}\right) \in G_2^F$, we set

$$f(i, g) := \{(a_1 x_0 + c_1 y_0)(b_0 x_0 + d_0 y_0)^q - (a_0 x_0 + c_0 y_0)^q(b_1 x_0 + d_1 y_0) \in \mathbb{F}_q^2\}. \quad (3.10)$$
We have the following long exact sequence

On the other hand, by (3.5), an element \( \lambda = a_0 + a_1 \pi \in \Gamma \) acts on \( s_0 \in \mathbb{F}_q^2 \) as follows

\[
\lambda : s_0 \mapsto a_0^{-(q+1)}(s_0 + (a_1/a_0) \xi).
\]

We have the following long exact sequence

\[
\cdots \to H^j_c(\kappa^{-1}(\mathcal{S}_{01}), \mathbb{T}_{\ell}) \to H^j_c(\mathcal{S}, \mathbb{T}_{\ell}) \to H^j_c(\mathcal{S}, \mathbb{T}_{\ell}) \to \cdots.
\]

The action of \( A' \) on \( \kappa^{-1}(\mathcal{S}_{01}) \) preserves each connected component of each fiber of the map \( \kappa : \kappa^{-1}(\mathcal{S}_{01}) \to \mathcal{S}_{01} \). Hence, \( A' \) acts trivially on \( H^j_c(\kappa^{-1}(\mathcal{S}_{01}), \mathbb{T}_{\ell}) \) for all \( j \). Therefore, by (3.13), the restriction \( H^j_c(\mathcal{S}, \mathbb{T}_{\ell}) \to H^j_c(\mathcal{S}, \mathbb{T}_{\ell}) \) is an isomorphism on the part where we have \( \sum_{\lambda \in A'} \lambda = 0 \). Since \( \mathcal{S} \) is an affine line bundle over \( \mathcal{S}_{00} \times \mathbb{F}_q^2 \), we acquire an isomorphism

\[
H^2_c(\mathcal{S}, \mathbb{T}_{\ell}) \cong H^0(\mathcal{S}_{00} \times \mathbb{F}_q^2, \mathbb{T}_{\ell})
\]

Note that we have \( \dim H^2_c(\mathcal{S}, \mathbb{T}_{\ell}) \sum_{\lambda \in A'} \lambda = 0 = q^2(q - 1)^2(q^2 - 1) \).

In the following, we will prove that the \( \text{étale cohomology group} \ H^2_c(\tilde{X}, \mathbb{T}_{\ell}) \) contains all cuspidal representations of \( G^\rho \) each one with multiplicity 2 in Corollary 3.2. More precisely, the part \( H^2_c(\tilde{X}, \mathbb{T}_{\ell}) \sum_{\lambda \in A'} \lambda = 0 \) is the direct sum of all cuspidal representations each with multiplicity 2 similarly as in [Lus, subsection 3.4]. The part \( H^2_c(\tilde{X}, \mathbb{T}_{\ell}) \) is a cuspidal representation if \( w|_{A'} \neq 1 \), which we will give a proof in Proposition 3.1 later. Note that \( w \in \Gamma^\vee \) such that \( w|_{A'} \neq 1 \) is a strongly primitive character in Definition 2.2.

To describe \( H^2_c(\tilde{X}, \mathbb{T}_{\ell}) \sum_{\lambda \in A'} \lambda = 0 \), we define a \( G^\rho \times \Gamma \)-representation \( \rho_{DL} \). We consider \( \mathbb{F}_q^\vee \) as a subgroup of \( \mathbb{F}_q^\vee \) by \( \text{Tr}_{\mathbb{F}_q^2/\mathbb{F}_q} \). For a character \( \psi \in \mathbb{F}_q^\vee \) and \( a \in \mathbb{F}_q^2 \), we write \( \psi(a) \) for the character \( x \mapsto \psi(ax) \). We set as follows

\[
V_\rho := \bigoplus_{i \in \mathbb{F}_q^\vee} \bigoplus_{\psi \in \mathbb{F}_q^\vee/\mathbb{F}_q^2} \mathbb{T}_{\ell}e_{i,\psi}.
\]

Then, clearly we have \( \dim V_\rho = q^2(q - 1)^2(q^2 - 1) \). We define an action of \( G^\rho \times \Gamma \) on the space \( V_\rho \). Let \( g = \begin{pmatrix} a_0 + a_1 \pi & b_0 + b_1 \pi \\ c_0 + c_1 \pi & d_0 + d_1 \pi \end{pmatrix} \in G^\rho \). Let \( i = (x_0, y_0) \in \mathcal{S}_{00} \). We set \( ig := (a_0x_0 + c_0y_0, b_0x_0 + d_0y_0) \in \mathcal{S}_{00}^\times \). Recall \( f(i, g) \) in (3.10). Then, we define a \( G^\rho \)-action as follows

\[
G^\rho \ni g : e_{i,\psi} \mapsto \psi_{\det(g)}(f(i, g^{-1}))e_{i^{-1},\psi_{\det(g)}}.
\]

Note that the \( G^\rho \)-action (3.16) on \( V_\rho \) is a left action. Let \( t = a_0 + a_1 \pi \in \Gamma \). We set \( t^{-1}i := (a_0x_0, a_0y_0) \in \mathcal{S}_{00}^\times \). Then, we define a \( \Gamma \)-action as follows

\[
\Gamma \ni t : e_{i,\psi} \mapsto \psi(-a_1/a_0)\xi e_{i^{-1},\psi_{a_0q^{-1}}(a_0)\xi}.
\]

We denote this representation by \( \rho_{DL} \). Let \( \Gamma^\vee_{\text{stp}} \subset \Gamma^\vee \) be the subset of strongly primitive characters in Definition 2.2. Then, by (3.11), (3.12) and (3.14), we acquire an isomorphism

\[
\bigoplus_{w \in \Gamma^\vee_{\text{stp}}} (H^2_c(\tilde{X}, \mathbb{T}_{\ell}) \otimes w) = H^2_c(\tilde{X}, \mathbb{T}_{\ell}) \sum_{\lambda \in A'} \lambda = 0 \cong \rho_{DL}
\]
as a $G^F_2 \times \Gamma$-representation.

In the following, we give a more concrete description of the part $H^F_2(X, \widetilde{Q})_w$ for each $w \in \Gamma^\vee_{\text{stp}}$. To do so, for each $w \in \Gamma^\vee_{\text{stp}}$, we will define a cuspidal representation $\pi_w$ as in [AOPS 5.2]. Now, we fix an element $\zeta_0 \in F/q \setminus F_q$ and an embedding

$$\iota_{\zeta_0} : \Gamma = (O_E/\pi^2)^{\times} \hookrightarrow G^F_2$$

$$a + b\zeta_0 \mapsto a1_2 + b\left( \begin{array}{cc} \zeta_0^2 + \zeta_0 & 1 \\ -\zeta_0^{q+1} & 0 \end{array} \right)$$

with $a, b \in (O_E/\pi^2)$. We identify $\Gamma \simeq F_{q^2}^\times \times F_{q^2}^\times$ by $a_0 + a_1 \pi \mapsto (a_0, (a_1/a_0))$. For a character $\psi \in F_{q^2}^\times \setminus F_{q^2}^\times$ and an element $\zeta \in F_{q^2} \setminus F_q$, we define a character $\psi_\zeta$ of $N$ by the following

$$\psi_\zeta : N \ni \left( \begin{array}{cc} 1 + \pi a_1 & \pi b_1 \\ \pi c_1 & 1 + \pi d_1 \end{array} \right) \mapsto \psi \left( \frac{-a_1\zeta + c_1 - \zeta^q(b_1\zeta + d_1)}{\zeta - \zeta^q} \right).$$

Then, by (3.19), the restriction of $\psi_{\zeta_0}$ to a subgroup $F_{q^2} = \Gamma \cap N \subset N$ is equal to $\psi$. For a strongly primitive character $w = (\chi, \psi) \in \Gamma^\vee \simeq (F_{q^2}^\times)^\vee \times F_{q^2}^\times$, i.e. $\psi \notin F_q^\times$, we define a character $w$ of $\Gamma N = F_{q^2}^\times N$ by

$$w(xu) = \chi(x)\psi_{\zeta_0}(u)$$

(3.20)

for all $x \in F_{q^2}^\times$ and $u \in N$. We set

$$\pi_w := \text{Ind}_{\Gamma_{\text{stp}}}^{G^F_2}(w).$$

Then, $\pi_w$ is a cuspidal representation of $G^F_2$ by Definition 2.1. Recall that we have $\dim \pi_w = q(q-1)$. See [AOPS subsection 5.2] for more details.

Again, we consider the $G^F_2 \times \Gamma$-representation $\rho_{\text{DL}}$. In the following, we give a decomposition of $V_\rho$ to irreducible components. To do so, we define several subspaces $\{W_w\}_{w \in \Gamma^\vee_{\text{stp}}}$ of $V_\rho$, and prove $W_w \simeq \pi_w \times w$ as a $G^F_2 \times \Gamma$-representation. We choose $w \in \Gamma^\vee_{\text{stp}}$ and write $w = (\chi, \psi)$ with $\chi \in (F_{q^2}^\times)^\vee$ and $\psi \in F_{q^2}^\times \setminus F_q^\times$. We fix an element $y_0 \in F$ such that $y_0^{q-1} = -1$. For each $\zeta \in F_{q^2} \setminus F_q$, we define a vector of $V_\rho$ in (3.15)

$$e^w_\zeta := \sum_{\mu \in F_{q^2}} \chi^{-1}(\mu)e_{((\chi y_0, \mu y_0), \psi_{-((\mu y_0)^q + 1)(\zeta^q - \zeta)^{-1}})} \in V_\rho$$

and set $W^w_\zeta := \bigoplus_{\zeta \in F_{q^2} \setminus F_q} W^w_\zeta \subset V_\rho$. Furthermore, we put

$$W_w := \bigoplus_{\zeta \in F_{q^2} \setminus F_q} W^w_\zeta \subset V_\rho.$$
other hand, by (3.17), we can easily check that \( t e^w \equiv w(t)e^w \) for \( t \in \Gamma \). Hence, by (3.22), \( W_w \) is a \( G^F_2 \times \Gamma \)-representation. By (3.22), the stabilizer of the subspace \( W^\infty_{\zeta} \) in \( G^F_2 \) is equal to \( \Gamma N \). Moreover, we acquire \( W^\infty_w \simeq \tau w^{-1} \) as a \( \Gamma N \)-representation. Since \( G^F_2 \) permutes the subspaces \( \{ W^\infty_{\zeta}, \zeta \in \mathbb{F}_q \setminus \mathbb{F}_q \} \) transitively again by (3.22), we obtain \( W_w \simeq \pi^\gamma_w \otimes w \) as a \( G^F_2 \times \Gamma \)-representation.

Now, we have the following proposition.

**Proposition 3.1.** Let the notation be as above. Then, we have the followings:
1. The following isomorphism as a \( G^F_2 \)-representation holds
   \[
   H^1_c(\bar{X}, \mathcal{Q}_2) \simeq \pi^\gamma_w.
   \]
2. We have the following isomorphism
   \[
   \rho_{DL} \simeq \bigoplus_{w \in \Gamma_{st}} (\pi^\gamma_w \otimes w)
   \]
as a \( G^F_2 \times \Gamma \)-representation.

**Proof.** The required assertion 2 follows from (3.21) and the isomorphism \( W_w \simeq \pi^\gamma_w \otimes w \) as a \( G^F_2 \times \Gamma \)-representation. Therefore, we conclude that \( H^1_c(\bar{X}, \mathcal{Q}_2) \simeq \pi^\gamma_w \) by (3.18). Hence, the assertion 1 is proved.

As a direct consequence of Proposition 3.1, we obtain the following corollary.

**Corollary 3.2.** The representation \( \rho_{DL} \) contains all cuspidal representations each with multiplicity 2.

**Proof.** Let \( \tau \neq 1 \in \text{Gal}(E/F) \). Then, we have \( \pi_w \simeq \pi_{w^\tau} \) as a \( G^F_2 \)-representation as mentioned in Remark 2.3. Hence, the required assertion follows from Proposition 3.1.2.

We call \( \rho_{DL} \) the unramified Lusztig representation for \( G^F_2 \).

**Remark 3.3.** We study the part of \( H^1_c(\bar{X}, \mathcal{Q}_2) \) where \( A' \) acts trivially. Clearly, we have \( H^1_c(\bar{X}, \mathcal{Q}_2)_{A'=1} \simeq H^1_c(A' \setminus \mathcal{G}, \mathcal{Q}_2) \) for any \( A' \). By (3.18), \( \mathcal{G} \) is identified with \( \{ (x_0, y_0, s_0, s_1) \in \mathcal{S}_0 \times \mathbb{F}_q^2 | s_0^2 + s_1 s_0 t_0 = 0 \} \). Then, \( A' \ni 1 + a_1 \pi \) acts on \( \mathcal{G} \) by \( (x_0, y_0, s_0, s_1) \mapsto (x_0, y_0, s_0 + a_1, s_1) \). Hence, the quotient \( A' \setminus \mathcal{G} \) is isomorphic to a disjoint union of \( q \) copies of \( \mathcal{S}_0 \times \mathcal{A}' \ni ((x_0, y_0), s_1) \). We denote by \( d_2 \) the composite \( \mathcal{G}_2^F \to (\mathcal{G}_2^F / \mathcal{P}^F)^{\mathbb{F}_q} \to \mathbb{F}_q \). Then, on the set \( \mathcal{G}_2^F \ni i \) of connected components of \( A' \setminus \mathcal{G} \), \( G^F_2 \times \Gamma \ni (g, t) \) acts by \( i \mapsto d_2(g)t_0^{-1}(t) \). We put \( M^j := H^j_c(\mathcal{S}_0, \mathcal{Q}_2) \). On \( M^j \), the group \( G^F_2 \times \Gamma \) acts by factoring through \( G^F_2 \times \Gamma \to G^F_2 \times \mathbb{F}^\times_q \). Recall that the group \( G^F_2 \times \mathbb{F}^\times_q \ni (g, \zeta) \) acts on \( \mathcal{S}_0 \ni (x_0, y_0) \) by \( g : (x_0, y_0) \mapsto (a_0 x_0 + c_0 y_0, b_0 x_0 + d_0 y_0) \) and \( \zeta : (x_0, y_0) \mapsto (\zeta^{-1} x_0, \zeta^{-1} y_0) \). Then, we obtain an isomorphism \( \rho_c : (\mathcal{S}_0, \mathcal{Q}_2) \simeq \bigoplus_{\chi \in \mathbb{F}^\times_q} (M^j \otimes \chi) \otimes d_2 \chi^{-1} \otimes t_2 \) as a \( G^F_2 \times \Gamma \)-representation. Hence, in particular, on the part \( H^1_c(\bar{X}, \mathcal{Q}_2)_{A'=1} \), the classical Deligne-Lusztig correspondence for \( G^F_2 \) realizes.

**Remark 3.4.** Let \( S := \text{SL}_2(\mathbb{F}_q) \) and \( N_0 := \text{Ker}(S^F_2 \to S_2^F) \). Clearly, we have an isomorphism \( N_0 \simeq \{ g \in M_2(\mathbb{F}_q) \mid \text{Tr}_{M_2(\mathbb{F}_q)/\mathbb{F}_q}(g) = 0 \} \) as a group, and hence \( N_0 \) is an abelian group of order \( q^3 \). As in [LM 3.2], we consider the Lusztig surface \( \tilde{X}_0 := \{ g \in S_2 | g F(g)^{-1} \in U_2 v \} \) for \( S^F_2 \).

More explicitly, \( \tilde{X}_0 \) is isomorphic to the following
\[
\mathcal{S}^0 := \{ (x_0, x_1, y_0, y_1) \in \mathbb{F}_q^4 \mid x_0 y_0^q - x_1 y_0 q y_0 = 1, x_1 y_0 + x_0 y_1 - x_0^q y_1 - x_1^q y_0 = 0 \}
\]
as in \cite{Sha}. In this case, the torus \( \Gamma \) is replaced by \( \Gamma_0 := \{(a_0, a_1) \in \mathbb{F}^2 \mid a_0^{q+1} = 1, \ a_0^q a_1 + a_0 a_1^q = 0 \} \) of order \( q(q+1) \) as in \cite{Lus} 3.2. We put \( \mathcal{I} := \text{Ker} (\text{Tr}_{\mathbb{F}^2/\mathbb{F}} : \mathbb{F}^2 \to \mathbb{F}) \). Then, obviously, we have isomorphisms \( \Gamma_0 \cong \text{Ker} (N_{E/F} : \Gamma \to (O_F/\mathfrak{p}^2)^\times) \) and \( \Gamma' \simeq \mathcal{I} \) as an abelian group. Furthermore, \( \Gamma_0 \) contains \( \Gamma' \) as a subgroup. For a character \( \chi \in \Gamma_0 \), we say that \( \chi \) is primitive if \( \chi|_{\Gamma'} \neq 1 \). Let \( \Gamma_0', \Gamma_0'' \subset \Gamma_0' \) be the subset of primitive characters. Obviously, we have a natural surjection \( \Gamma_{\text{stp}}' \to \Gamma_0' \). Then, the variety \( \tilde{X}_0 \) admits an action of \( S^2_2 \times \Gamma_0 \) and the étale cohomology group \( H^2_c(X_0, \mathcal{O}^\times_\ell) \) does so. Let \( \rho_{DL}^0 \) denote the \( S^2_2 \)-representation \( H^2_c(X_0, \mathcal{O}^\times_\ell)_{\sum \lambda \in \Lambda'(\lambda = 0)} \). Note that we have \( \dim \rho_{DL}^0 = q(q-1)(q^2-1) \). The restriction of \( \iota_{\mathcal{O}_E} \) in (3.19) to a subgroup \( \Gamma_0 \) induces an embedding \( \Gamma_0 \hookrightarrow S^2_2 \) by \( \text{det} \circ \iota_{\mathcal{O}_E} = N_{E/F} \). Any character \( \chi \in \Gamma_0' \) extends to the character of \( \Gamma_0 N_0 \) uniquely as in (3.20), which we denote by the same letter \( \chi \). Then, we define \( \pi_w^0 := \text{Ind}_{\Gamma_0 N_0}^{S^2_2}(w) \). This representation is also called a cuspidal representation of \( S^2_2 \) in \cite{Sha}, and all cuspidal representations arise in this way. See also \cite{Sha} 4.2. In \cite{Sha} p.37, a cuspidal representation of \( S^2_2 \) is called an unramified representation. All cuspidal representations have degree \( q(q-1) \) and the number of them is \( (q^2-1)/2 \) by loc. cit. Similarly as in Proposition 3.1, we have the following isomorphism

\[
\rho_{DL}^0 \simeq \bigoplus_{\chi \in \Gamma_0'} (\pi_w^0 \otimes \chi).
\]

Hence, as proved in \cite{Lus} 3.4, the representation \( \rho_{DL}^0 \) contains all cuspidal representations of \( S^2_2 \) each with multiplicity 2.

### 4 Review of [IMT, 7.2]

In this section, we recall the results in [IMT, 7.2] and prove Proposition 4.1 by using the results in the previous section. Let \( \mathcal{S}^{\times}_0 \) be as in the previous section. For \( \chi = (x_0, y_0) \in \mathcal{S}^{\times}_0 \), we set \( \xi := x_0^q y_0 - x_0 y_0^q \). Then for each \( i \in \mathcal{S}^{\times}_0 \), let \( X_i \) denote the smooth compactification of the following affine curve \( \mathcal{X}^2 - X = \xi(Y^{q+1} - Y^{-q+1}) \). Then, \( X_i \) has \( q \) connected components and each component has genus \( q(q-1)/2 \) in loc. cit., we prove that the components \( X_i \) for \( i \in \mathcal{S}^{\times}_0 \) appear in the stable reduction of the Lubin-Tate curve \( \mathcal{X}(\pi^2) \). Furthermore, in loc. cit., we analyze the following étale cohomology group

\[
W := \bigoplus_{i \in \mathcal{S}^{\times}_0} H^1(X_i, \mathcal{O}^\times_\ell).
\]

Then, \( W \) has a left action of \( \mathcal{G} := G^F_2 \times O^\times_3 \times I_F \). We consider \( \Gamma \) as a subgroup of \( O^\times_3 \). We will observe that the restriction of \( W \) to a subgroup \( G^F_2 \times \Gamma \times \{1\} \subset \mathcal{G} \) is related to the unramified Lusztig representation \( \rho_{DL} \) computed in the previous section. The restriction \( W|_{G^F_2 \times \{1\} \times I_F} \) is also written with respect to \( \rho_{DL} \). See Proposition 4.1 for precise statements.

As in the previous section, we assume (3.10). Since we have \( |\mathcal{S}^{\times}_0| = |G^F_1| = q(q^2-1)(q-1) \) and \( \dim H^1(X_i, \mathcal{O}^\times_\ell) = q^2(q-1) \) for each \( i \), we have \( \dim W = q^3(q-1)^2(q^2-1) \). In the following, we write down the right action of \( \mathcal{G} \) on the components \( \{X_i\} \) given in loc. cit.

This action induces the left \( \mathcal{G} \)-action on \( W \) as mentioned above. First, we recall the action of \( G^F_1 \). Let \( g = \begin{pmatrix} a_0 + a_1 \pi & b_0 + b_1 \pi \\ c_0 + c_1 \pi & d_0 + d_1 \pi \end{pmatrix} \in G^F_1 \). Then, \( g \) acts on \( \mathcal{S}^{\times}_0 \) as follows, factoring through \( G^F_1 \).

\[
g : i := (x_0, y_0) \mapsto i \hat{g} := (a_0 x_0 + c_0 y_0, b_0 y_0 + d_0 x_0).
\]
Moreover, $g$ induces the following morphism
\begin{equation}
    g : X_i \to X_{ig} : (X, Y) \mapsto (\det(g)X + f(i, g), Y)
\end{equation}
where $f(i, g)$ is defined in (4.10). Compare (4.3) with (4.11). Secondly, we recall the action of $\mathcal{O}_3^\times$. Let $b = a_0 + \varphi b_0 + \pi a_1 \in \mathcal{O}_3^\times$ with $a_0 \in \mathbb{F}_q^\times$ and $a_1, b_0 \in \mathbb{F}_q$. Then, $b$ acts on $\mathbb{S}^\times_{00}$ as follows
\begin{equation}
    i = (x_0, y_0) \mapsto ib := (a_0^{-1}x_0, a_0^{-1}y_0).
\end{equation}
Moreover, $b$ induces a morphism
\begin{equation}
    X_i \to X_{ib} : (X, Y) \mapsto \left( a_0^{-(q+1)} \left( X - (b_0a_0^{-1})\xi Y + (a_1a_0^{-1})\xi \right), a_0^{-1} (Y - (b_0a_0^{-1})^q) \right).
\end{equation}
In particular, the element $t := a_0 + \pi a_1 \in \Gamma \subset \mathcal{O}_3^\times$ acts as follows
\begin{equation}
    X_i \to X_t : (X, Y) \mapsto \left( a_0^{-(q+1)} \left( X + (a_1a_0^{-1})\xi \right), a_0^{-1}Y \right).
\end{equation}
Compare this action with (4.12). Thirdly, we recall the inertia action $I_F$. We define a homomorphism
\begin{equation}
    a_{E} : I_F \to I_{F}^b \simeq I_{E}^{ab} \xrightarrow{\alpha |}_{gb} \mathcal{O}_E^\times \xrightarrow{\text{can.}} \Gamma \simeq \mathbb{F}_q^\times \times \mathbb{F}_q^\times : \sigma \mapsto (\zeta(\sigma), \lambda(\sigma)).
\end{equation}
Then, $\sigma \in I_F$ acts on $\mathcal{S}_{00}^\times$ as follows
\begin{equation}
    (x_0, y_0) \mapsto i\sigma := (\zeta(\sigma)^{-1}x_0, \zeta(\sigma)^{-1}y_0).
\end{equation}
Moreover, $\sigma$ induces a morphism
\begin{equation}
    X_i \to X_{i\sigma} : (X, Y) \mapsto (\zeta(\sigma)^{(q+1)}(X + \lambda(\sigma)\xi), Y).
\end{equation}
The product group $\mathbb{F}_q^\times \times \mu_{q+1} \supset (a, \zeta)$ acts on $X_i$ with $i \in \mathcal{S}_{00}^\times$ by $(X, Y) \mapsto (X + a, \zeta Y)$. Then, we can decompose $H^1(X_i, \mathcal{Q}_I)$ to a direct sum of some characters of $\mathbb{F}_q^\times \times \mu_{q+1}$ as follows
\begin{equation}
    H^1(X_i, \mathcal{Q}_I) \simeq \bigoplus_{x_0 \in \mu_{q+1}\{1\}} \bigoplus_{\psi \in \mathcal{F}_{q}^\times \setminus \mathcal{F}_q^\times} \mathbb{Q} e_{i, \psi, x_0}.
\end{equation}
See [IMT] Corollary 7.5 for a proof of (4.9). By (4.9), we acquire the following isomorphism
\begin{equation}
    W \simeq \bigoplus_{i \in \mathcal{S}_{00}^\times \ x_0 \in \mu_{q+1}\{1\}} \bigoplus_{\psi \in \mathcal{F}_{q}^\times \setminus \mathcal{F}_q^\times} \mathbb{Q} e_{i, \psi, x_0}
\end{equation}
as a $\mathcal{Q}_I$-vector space. Explicit descriptions of the $G$-action on the right hand side of (4.10), which is induced by the $G$-action on $W$, are given in [IMT] (7.17)-(7.19). Now, we write down them. We have the following $G^F_2$-action
\begin{equation}
    G^F_2 \ni g : e_{i, \psi, \chi_0} \mapsto \psi_{\det(g)}(f(i, g^{-1}))e_{i(g^{-1}), \psi_{\det(g)}, \chi_0}
\end{equation}
and the following action of $\Gamma \subset \mathcal{O}_3^\times$
\begin{equation}
    \Gamma \ni t = a_0 + a_1 \pi : e_{i, \psi, \chi_0} \mapsto \psi(-(a_1/a_0)\xi)\chi_0^{-1}(t)e_{i, \psi_{\det(g)}, \chi_0}^{-1}e_{i^{-1}, \psi_{\chi_0}^{-1}(a_1), \chi_0}.
\end{equation}
Furthermore, we have the following $I_F$-action
\begin{equation}
    I_F \ni \sigma : e_{i, \psi, \chi_0} \mapsto \psi(-\lambda(\sigma)\xi)e_{i, \psi_{\chi_0}^{-1}(a_1), \chi_0}.
\end{equation}
These descriptions of the $G$-action are almost direct consequences of the $G$-action given in (4.12)-(4.18). Now, we have the following proposition.
Proposition 4.1. Let \( \nu : \Gamma \to \mu_{q+1} \) be a surjective homomorphism defined by \( a_0 + a_1 \pi \mapsto a_0^q - 1 \). For a character \( \chi_0 \in \mu_{q+1}^\vee \), we denote by the same letter \( \chi_0 \) for the composite \( \chi_0 \circ \nu \in \Gamma^\vee \).

1. Then, we have the following isomorphism

\[
W|_{G^F_2 \times \Gamma \times \{1\}} \simeq \bigoplus_{\chi_0 \in \mu_{q+1}^\vee \setminus \{1\}} (\rho_{DL} \otimes \chi_0^{-1})
\]

as a \( G^F_2 \times \Gamma \)-representation.

2. Let \( \hat{\rho}_{DL} \) denote the inflation to \( G^F_2 \times I_F \) of the \( G^F_2 \times \Gamma \)-representation \( \rho_{DL} \) by a map \( \text{id} \times a_E : G^F_2 \times I_F \to G^F_2 \times \Gamma \). Then, we have the following isomorphism

\[
W|_{G^F_2 \times \Gamma \times \{1\} \times I_F} \simeq \psi (\hat{\rho}_{DL})^{\otimes q}
\]

as a \( G^F_2 \times I_F \)-representation.

Proof. The required assertions follow by comparing the \( G^F_2 \times \Gamma \)-action (3.16) and (3.17) on \( V_{\nu} \) with the \( G \)-action on \( W \) given in (4.11)-(4.13). □

By Proposition 4.1, we acquire the following corollary, which is proved in [IMT Proposition 7.6].

Corollary 4.2. ([IMT Proposition 7.6]) We set \( H := G^F_2 \times \Gamma \times I_F \subset G \). Then, we have the following isomorphism

\[
W|_H \simeq \bigoplus_{w \in \Gamma_{\nu}^{x_0}} \left( \pi_{w} \otimes \sum_{\chi_0 \in \mu_{q+1}^\vee \setminus \{1\}} w \chi_0^{-1} \otimes w \circ a_E \right)
\]

as a \( H \)-representation.

Proof. The required assertion follows from Proposition 3.1 and Theorem 4.1 immediately. □

Remark 4.3. Let the notation be as in Remark 3.3. We briefly remark that similar things to Theorem 4.1 hold for \( S^F_2 \). We set \( \cal O_{3}^{1,\times} := \text{Ker} (\text{Nrd}_{D/F} : \cal O_{3}^{\times} \to (\cal O_{F}/\pi^{2})^{\times}) \). Then, obviously, we have a natural inclusion \( \Gamma_0 \hookrightarrow \cal O_{3}^{1,\times} \) by \( \text{Nrd}_{D/F} | \Gamma = \text{Nrd}_{E/F} \). We write \( a_F \) for the composite \( I_F \to I_{F_{a}}^{\text{ab}} \to \cal O_{F}^{\times} \to (\cal O_{F}/\pi^{2})^{\times} \). Then, we set \( I_{F,3}^{1} := \text{Ker} (a_{F} : I_{F} \to (\cal O_{F}/\pi^{2})^{\times}) \). Moreover, the restriction of the map (4.10) to a subgroup \( I_{F,3}^{1} \) induces a surjection \( a_{F}^{0} : I_{F,3}^{1} \to \Gamma_0 \) by \( a_{F} = \text{Nrd}_{E/F} \circ a_{E} \). Let \( \cal G_{0} := \{ (x_{0}, y_{0}) \in \cal F^{2} \mid x_{0}y_{0}^{q} - x_{0}y_{0} = 1, y_{0}^{q-1} - 1 \} \). Then, we have \( |\cal G_{0}| = |S_{1}^{\times} | = q(q^{2}-1) \). For each \( j \in \cal G_{0} \), let \( X_{j} \) be the smooth compactification of an affine curve \( X^{q} + X + Y^{q+1} = 0 \) with genus \( q(q-1)/2 \). Then, a product group \( \cal I \times \mu_{q+1} \cong (a, \zeta) \) acts on \( X_{j} \) by \( (X, Y) \mapsto (X + a, \zeta Y) \). Then, we consider

\[
W_{0} := \bigoplus_{j \in \cal G_{0}} H_{c}^{1}(X_{j}, \mathcal{Q}_{\ell}) \simeq \bigoplus_{j \in \cal G_{0}} \bigoplus_{\chi_{0} \in \mu_{q+1} \setminus \{1\}} \bigoplus_{\psi \in \cal I^{\times} \setminus \{0\}} \mathcal{Q}_{\ell}^{j}_{\psi, \chi_{0}}
\]

where the second isomorphism follows from [IMT Lemma 7.4]. Note that we have \( \dim W_{0} = q^{2}(q^{2}-1) \). As mentioned in section 3, \( f(i, g) \) in (3.10) is contained in \( \cal I \) if \( g \in S^F_2 \). Similarly, for \( \sigma \in I_{F,3}^{1} \), we have \( \zeta(\sigma)q^{2}+1 \) and \( \lambda(\sigma) \in \cal I \). For \( b = a_{0} + a_{1} \pi \in \cal O_{F}^{1,\times} \), we have \( a_{0}^{q^{2}+1} = 1 \) and \( b_{0}/a_{0}q^{2} = (a_{1}/a_{0})q + (a_{1}/a_{0}) \). Hence, a product group \( \cal G_{0} := S^F_2 \times \cal O_{F}^{1,\times} \times I_{F,3}^{1} \) acts on the components \( \{ X_{j} \}_{j \in \cal G_{0}} \) in the same way as (4.2)-(4.3). Therefore, \( \cal G_{0} \) also acts on \( W_{0} \) in the same way as (4.11)-(4.13). Then, for \( W_{0} \), the same statements as in Theorem 4.1 hold by replacing \( \rho_{DL} \) by \( \rho_{DL}^{0} \). Namely, we have the following isomorphism \( W_{0}|_{S^F_2 \times \Gamma_0 \times \{1\}} \simeq \bigoplus_{\chi_{0} \in \mu_{q+1}^\vee \setminus \{1\}} (\rho_{DL}^{0}) \otimes \chi_{0}^{-1} \).
\(\chi^{-1}\) as a \(S_F^G \times \Gamma_0\)-representation. Let \(\rho_{DL}\) denote the inflation of \(\rho_{DL}^0\) to \(S_F^G \times I\) by the map \(\text{id} \times a_E^0 : S_F^G \times I \to S_F^G \times \Gamma_0\). Then, we have an isomorphism \(W_0|_{S_F^G \times \{1\} \times I} \simeq (\rho_{DL}^0)_{x_0}q\) as a \(S_F^G \times I\)-representation. Let \(H_0 := S_F^G \times \Gamma_0 \times I_{1,2} \subset G_0\). Then, similarly as Corollary 4.2, we acquire an isomorphism \(W_0|_{H_0} \simeq \bigoplus_{w \in \Gamma_0^\vee} (\pi_w \otimes (\bigoplus_{\chi \in \rho_{DL}^0 \setminus \{1\}} w \chi^{-1}) \otimes w \circ a_E^0)\) as a \(H_0\)-representation.

### 5 Main theorem and its proof

In this section, we write down the Lusztig curve \(X_D\) for \(O_3^+\). The Lusztig curve \(X_D\) has an \(O_3^+ \times \Gamma\)-action. We study the cohomology group \(H^1(X_D, \mathbb{Q}_l)\) as a \(O_3^+ \times \Gamma\)-representation in Lemma 5.1. We consider the product \(X := X \times X_D\) with an action of \(G_F^G \times O_3^+ \times \Gamma\). The étale cohomology group \(H^3_\text{ét}(X, \mathbb{Q}_l)\) is considered as a \(G\)-representation via the map \(\text{id} \times \text{id} \times a_E : G \to G_F^G \times O_3^+ \times \Gamma\). Then, in Theorem 5.2, we show that \(W\) is contained in the étale cohomology group \(H^3_\text{ét}(X, \mathbb{Q}_l)\) as a \(G\)-subrepresentation.

As a direct consequence of Corollary 4.2, we prove that the following isomorphism holds

\[
W \simeq \bigoplus_{w \in \Gamma_0^\vee} (\pi_w \otimes \rho_w \otimes w \circ a_E^0)
\]

as a \(G\)-representation in [Lus2 Corollary 7.7]. Here, \(\rho_w\) with \(w = (\chi, \psi) \in \Gamma_0^\vee \simeq (\mathbb{F}_q^\times)^\vee \times (\mathbb{F}_q^\times)^\vee\) is an irreducible representation of \(O_3^+\) of degree \(q\), which is uniquely characterized by the following \(\rho_w\) for \(U = \psi^{\otimes q}\). We set \(\rho_w(\zeta) = -\chi(\zeta)\) for \(\zeta \in \mathbb{F}_q^\times\), where we set \(U := U_{2,1}^2/U_0^2 \simeq \mathbb{F}_q^\times \subset O_3^+\). See [Lus2 Corollary 7.4] and [Hum Lemma 16.2] for more details on \(\rho_w\).

In the following, we recall the Lusztig curve for \(O_3^+\) similarly as in [Lus2 Section 2]. Let \(\varphi' := \left( \begin{array}{cc} 0 & 1 \\ \pi & 0 \end{array} \right) \in G_2^G\). We define a morphism \(F' : G_2 \to G_2\) by \(F'(g) = \varphi F(g) \varphi^{-1}\). We define as follows

\[
X_D := \{ g \in G_2 \mid F'(g) g^{-1} \in U_2 \},
\]

which we call the Lusztig curve for \(O_3^+\). The condition \(F'(g) g^{-1} \in U_2\) is equivalent to \(g = \left( \begin{array}{cc} x & y \\ \pi F(y) & F(x) \end{array} \right) \) with \(x \in A, y \in \mathbb{F}\) and \(\det(g) \in (\mathbb{O}_F/\pi)^\times\). The fixed part \(G_2^c\) is equal to the following \([a, b, 0] := \left( \begin{array}{c} a \\ \pi b_0 \\ F(a) \end{array} \right) \) \(a \in (\mathbb{F}_q[[\pi]]/\pi)^\times, b \in \mathbb{F}_q\)\). Hence, we fix the following isomorphism \(G_2^c \simeq O_3^+\) : \([a, b, 0] \mapsto a + b\varphi_0\). Then, the group \(G_2^c \times \Gamma \ni (b, t)\) acts on \(X_D \ni g\) by \(g \mapsto \left( \begin{array}{cc} t & 0 \\ 0 & F(t) \end{array} \right)^{-1} gb\). We set as follows

\[
\mathcal{G}_D := \{ x \in (O_3 \otimes \mathbb{F}_q) \times (\mathbb{O}_F/\pi)^\times \mid \text{Nrd}_{D/F}(x) \in (\mathbb{O}_F/\pi)^\times \}.
\]

The group \(O_3^+ \times \Gamma \ni (b, t)\) acts on \(\mathcal{G}_D\) by \(x \mapsto t^{-1}xb\). We acquire an isomorphism \(\mathcal{G}_D \simeq X_D : x_0 + y_0 \varphi + x_1 \pi \mapsto \left( x_0 + \pi x_1 + y_0 \pi F(y_0) \right) \), which is compatible with the \(O_3^+ \times \Gamma\)-actions. Let \(X = x_0 + y_0 \varphi + x_1 \pi \in \mathcal{G}_D\). Then, by the \(\text{Nrd}_{D/F}(x) \in (\mathbb{O}_F/\pi)^\times\), we acquire \(x_0 \in \mathbb{F}_q^{x_0}\) and \(x_0^2 x_1 + x_0 x_1^2 - y_0 y_1^2 \in \mathbb{F}_q\). We fix \(x_0^2 \in \mathbb{F}_q\). By changing variables as follows \(X = x_0^2 x_1\) and \(Y = y_0 / x_0\), we obtain \(X^q + X - Y^{q+1} \in \mathbb{F}_q\). For each \(x_0 \in \mathbb{F}_q\), let \(X_{x_0}\) denote the affine curve \(X^q - X = x_0^{q+1}(Y^{q(1+2q)} - Y^{q+1})\). Then, \(X_D\) is isomorphic to a disjoint union of \(q^2 - 1\) affine curves \(\{X_{x_0}\}_{x_0 \in \mathbb{F}_q}\). Furthermore, by a direct computation, \(b = a_0 + \varphi b_0 + \pi a_1 \in O_3^+\) induces the following morphism

\[
X_{x_0} \to X_{a_0 x_0} : (X, Y) \mapsto (a_0^{q+1}(X + (b_0/a_0))Y + (a_1/a_0)\zeta) + a_0^{-1}(Y + (b_0/a_0)\zeta)) \quad (5.1)
\]
On the other hand, $\Gamma \ni t = a_0 + a_1 \pi$ induces the following map

$$X_{x_0} \to X_{x_0}^{-1} : (X,Y) \mapsto (a_0^{(q+1)}(X - (a_1/a_0)\xi), Y).$$  \hspace{1cm} (5.2)

**Lemma 5.1.** Let the notation be as above. Then, we have the following isomorphism $H^1_c(X_D, \overline{Q}_l) \simeq \bigoplus_{w \in \Gamma_{\text{stp}}} (\rho_w \otimes w^{-1})$ as an $O^\wedge \times \Gamma$-representation.

**Proof.** Similarly as in section 3, we prove the following isomorphism by using (4.9)

$$H^1_c((X_D, \overline{Q}_l))_{|\Gamma\times \Gamma} \simeq \bigoplus_{w \in \Gamma_{\text{stp}}} \left( \bigoplus_{\chi_0 \in \mu_{q+1}\setminus \{1\}} (w\chi_0 \otimes w^{-1}) \right)$$

as a $\Gamma \times \Gamma$-representation. Thereby, for $w = (\chi, \psi) \in \Gamma_{\text{stp}}$, $\rho_w := H^1_c((X_D, \overline{Q}_l)_{|w^{-1}}$ satisfies $\rho_w|_U = \psi \otimes q$ and $\text{Tr} \rho_w(\zeta) = -\chi(\zeta)$ for $\zeta \in F_\gamma \setminus F_q$. Hence, we acquire $\rho_w \simeq \rho_{w'}$. \hfill $\square$

Let $\tilde{X}$ be as in section 3. Now, we consider the fiber product $X := \tilde{X} \times X_D$. On this variety, $G^F \times O^\wedge \times \Gamma \times \Gamma$ acts. We restrict the $\Gamma \times \Gamma$-action on $X$ to a subgroup $\Gamma \to \Gamma \times \Gamma : t \mapsto (t, t^{-1})$. Then, the product $G^F \times O^\wedge \times \Gamma$ acts on the variety $X$. Hence, let $G$ act on $X$ according to $\text{id} \times \text{id} : G \to G^F \times O^\wedge \times \Gamma$. Now, we consider $H^3_c(X, \overline{Q}_l)$ as a $G$-representation. For each $w \in \Gamma_{\text{stp}}$, let $H^3_c(X, \overline{Q}_l)_w$ be the subspace of $H^3_c(X, \overline{Q}_l)$ on which $I_F$ acts according to $w \circ \rho_E$.

**Theorem 5.2.** Let the notation be as above. Then, we have the following:
1. For each $w \in \Gamma_{\text{stp}}^\vee$, the following isomorphism holds $H^3_c(X, \overline{Q}_l)_w \simeq \pi_w \otimes \rho_w \circ w \circ \rho_E$ as a $G$-representation.
2. We have the following isomorphism

$$W \simeq \bigoplus_{w \in \Gamma_{\text{stp}}} H^3_c(X, \overline{Q}_l)_w$$

as a $G$-representation.

**Proof.** The assertion 2 follows from 1 immediately. We prove the assertion 1. Since $\mathcal{G}_s$ is an affine line bundle over a finite set, we obtain $H^3_c(X, \overline{Q}_l)_{|\Sigma \leq \Sigma'} \lambda \otimes \lambda' \otimes \lambda'' = 0$. On the other hand, we have $H^3_c(X, \overline{Q}_l)_{|\Sigma = \Sigma'} = 0$ by Remark 3.3. Hence, we acquire $H^3_c(X, \overline{Q}_l) = 0$. Therefore, the following holds $H^3_c(X, \overline{Q}_l) \simeq H^3_c(X, \overline{Q}_l) \otimes H^3_c(X_D, \overline{Q}_l)$ by the Kunneth formula. Hence, for each $w \in \Gamma_{\text{stp}}$, the part $H^3_c(X, \overline{Q}_l)_w$ is isomorphic to $\pi_w \otimes \rho_w$ as a $G^F \times O^\wedge$-representation by Proposition 5.1 and Lemma 5.1. Therefore, the required assertion follows. \hfill $\square$

**Remark 5.3.** Let the notation be as in Remark 4.3. Let $X_{0,D} := \{ g \in S_2 \mid F^*(g)g^{-1} \in U_2 \}$. This is the Lusztig curve for $O^{1,X}$, which is computed in [Lus2, Section 2]. Then, $X_{0,D}$ is isomorphic to a disjoint union of $d+1$ copies of an affine curve $X^g + X = Y^g + Y$, for which we write $\{X_{x_0}\}_{x_0 \in \mu_{q+1}}$, as well as in (4.5) and (5.2). We set $U_0 := U \cap O^{1,X} \times \tilde{T}$. Then, as in Lemma 5.1, we acquire $H^1_c((X_0, \overline{Q}_l)) \simeq \bigoplus_{w \in \Gamma_{\text{stp}}} ( \rho_w^0 \otimes w^{-1})$ as an $O^{1,X} \times \Gamma_0$-representation. Here, for $w = (\chi, \psi) \in \Gamma_{\text{stp}}$, $\rho_w^0|_{U_0} = \psi \otimes q$ and $\text{Tr} \rho_w^0(\zeta) = -\chi(\zeta)$ for $\zeta \in \mu_{q+1}\setminus \{1\}$. We consider $X_0 : \tilde{X}_0 \times X_{0,D}$ with a right $S^F \times O^{1,X} \times \Gamma_0 \times \Gamma_0$. We restrict the $\Gamma_0 \times \Gamma_0$-action on $X_0$ to a subgroup $\Gamma_0 \to \Gamma_0 \times \Gamma_0 : t \mapsto (t, t^{-1})$. By a surjective map $\text{id} \times \text{id} : \rho_E : \Gamma_0 \to \Gamma_0 \times \Gamma_0$, which is computed in [Lus2, Section 2]. We restrict the $\Gamma_0 \times \Gamma_0$-action on $X_0$ to a subgroup $\Gamma_0 \to \Gamma_0 \times \Gamma_0 : t \mapsto (t, t^{-1})$. By a surjective map $\text{id} \times \text{id} : \rho_E : \Gamma_0 \to \Gamma_0 \times \Gamma_0$, which is computed in [Lus2, Section 2]. We restrict the $\Gamma_0 \times \Gamma_0$-action on $X_0$ to a subgroup $\Gamma_0 \to \Gamma_0 \times \Gamma_0 : t \mapsto (t, t^{-1})$. By a surjective map $\text{id} \times \text{id} : \rho_E : \Gamma_0 \to \Gamma_0 \times \Gamma_0$, which is computed in [Lus2, Section 2]. We restrict the $\Gamma_0 \times \Gamma_0$-action on $X_0$ to a subgroup $\Gamma_0 \to \Gamma_0 \times \Gamma_0 : t \mapsto (t, t^{-1})$. By a surjective map $\text{id} \times \text{id} : \rho_E : \Gamma_0 \to \Gamma_0 \times \Gamma_0$, which is computed in [Lus2, Section 2]. We restrict the $\Gamma_0 \times \Gamma_0$-action on $X_0$ to a subgroup $\Gamma_0 \to \Gamma_0 \times \Gamma_0 : t \mapsto (t, t^{-1})$. By a surjective map $\text{id} \times \text{id} : \rho_E : \Gamma_0 \to \Gamma_0 \times \Gamma_0$, which is computed in [Lus2, Section 2]. We restrict the $\Gamma_0 \times \Gamma_0$-action on $X_0$ to a subgroup $\Gamma_0 \to \Gamma_0 \times \Gamma_0 : t \mapsto (t, t^{-1})$. By a surjective map $\text{id} \times \text{id} : \rho_E : \Gamma_0 \to \Gamma_0 \times \Gamma_0$, which is computed in [Lus2, Section 2]. We restrict the $\Gamma_0 \times \Gamma_0$-action on $X_0$ to a subgroup $\Gamma_0 \to \Gamma_0 \times \Gamma_0 : t \mapsto (t, t^{-1})$. By a surjective map $\text{id} \times \text{id} : \rho_E : \Gamma_0 \to \Gamma_0 \times \Gamma_0$, which is computed in [Lus2, Section 2]. We restrict the $\Gamma_0 \times \Gamma_0$-action on $X_0$ to a subgroup $\Gamma_0 \to \Gamma_0 \times \Gamma_0 : t \mapsto (t, t^{-1})$. By a surjective map $\text{id} \times \text{id} : \rho_E : \Gamma_0 \to \Gamma_0 \times \Gamma_0$, which is computed in [Lus2, Section 2]. We restrict the $\Gamma_0 \times \Gamma_0$-action on $X_0$ to a subgroup $\Gamma_0 \to \Gamma_0 \times \Gamma_0 : t \mapsto (t, t^{-1})$. By a surjective map $\text{id} \times \text{id} : \rho_E : \Gamma_0 \to \Gamma_0 \times \Gamma_0$, which is computed in [Lus2, Section 2]. We restrict the $\Gamma_0 \times \Gamma_0$-action on $X_0$ to a subgroup $\Gamma_0 \to \Gamma_0 \times \Gamma_0 : t \mapsto (t, t^{-1})$. By a surjective map $\text{id} \times \text{id} : \rho_E : \Gamma_0 \to \Gamma_0 \times \Gamma_0$, which is computed in [Lus2, Section 2]. We restrict the $\Gamma_0 \times \Gamma_0$-action on $X_0$ to a subgroup $\Gamma_0 \to \Gamma_0 \times \Gamma_0 : t \mapsto (t, t^{-1})$. By a surjective map $\text{id} \times \text{id} : \rho_E : \Gamma_0 \to \Gamma_0 \times \Gamma_0
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