Random-Unitary Depolarization and the Reversibility of All Quantum Channels

Samuel R. Hedemann
Dept. of Physics and Engineering Physics, Stevens Institute of Technology, Hoboken, NJ 07030, USA
(Dated: February 18, 2014)

A simple method for finding random-unitary (RU) state-independent Kraus decompositions of the depolarization channel for any n-level quantum system is presented. Furthermore, these RU Kraus operators are shown to be Hilbert-Schmidt (HS) complete or overcomplete in all dimensions, which enables the proof that all quantum channels are reversible, by virtue of the reversibility of any RU decomposition and the HS completeness of these Kraus operators. The major result of this work is that all quantum channels are reversible, and a universal reversal operation is presented along with several examples showing the full reversibility of some important n-level quantum channels.

PACS numbers: 03.67.Pp, 03.67.Ac, 03.65.Yz, 03.65.Aa

I. INTRODUCTION

As the technological demand for the development of quantum computers increases, the need for efficient quantum error correction becomes more important, as well [1, 2]. Many quantum error-correction schemes have been proposed, such as the Shor code [3], the Steane code [4], and environment-assisted methods [5], such as random-unitary (RU) decomposition [6–13], with a wide range of pioneering work in general quantum error correction becoming more important, as well [14–18].

Here, we will focus on RU Kraus decompositions [19–22] of quantum channels for discrete quantum systems, first deriving an RU decomposition of all depolarization channels, and then showing that the Hilbert-Schmidt (HS) completeness of this set of RU Kraus operators guarantees the reversibility of all quantum channels.

To review, a quantum operation \( \mathcal{E} \) is a map between physical states given by density operators \( \rho \) and \( \rho' \), as

\[
\rho' = \mathcal{E}(\rho).
\]

Mathematically, \( \mathcal{E} \) can have Kraus representation as

\[
\mathcal{E}(\rho) = \sum_m K_m \rho K_m^\dagger,
\]

where the \( K_m \) are Kraus operators, with completeness

\[
\sum_m K_m^\dagger K_m = I.
\]

Quantum operations are important because they can describe both closed and open systems, meaning systems of interest that interact with other systems which are either ignored or not fully known.

In the subject of quantum noise, a quantum operation is often referred to as a quantum channel. A good example of a quantum channel is the depolarizing channel,

\[
\mathcal{D}(\rho) = p \frac{I}{n} + (1 - p) \rho,
\]

where \( n \) is the dimension of the Hilbert space, \( I \) is the identity, and \( p \) is a probability \( p \in [0, 1] \). Physically, \( \mathcal{D}(\rho) \) transmits \( \rho \) exactly with probability \( 1 - p \) and replaces it with the maximally mixed state \( \frac{I}{n} \) with probability \( p \).
HS-completeness of a Kraus set guarantees the reversibility of any quantum channel. The fact that the n-level RU decomposition of the depolarization channel from Sec. II possesses these properties then proves the reversibility of all quantum channels. In Sec. IV, we demonstrate the main application of this discovery by showing how to find a universal reversal channel, with examples for environment-assisted correction of both the n-level dephasing and n-level depolarization channels.

II. RU DECOMPOSITIONS OF ALL DEPOLARIZATION CHANNELS

We will use two methods to decompose depolarization channels. One uses a “top-down” approach looking at behaviors of groups of unitary operators on an input state, and the other uses a more direct approach to build the Kraus operators directly. Since the top-down approach is more intuitive we will look at that method now.

A. Recursive Definition of RU Depolarization

To motivate this discussion as naturally as possible, we will break it up into several smaller tasks which will serve to illustrate why the RU decomposition of depolarization channels is always possible with this method.

1. n-Level RU Maximal-Dephasing Channel

First, let the RU maximal-dephasing channel $\Delta(\rho)$ be the channel that sends all off-diagonal elements of $\rho$ to 0, while preserving its diagonal elements, doing so with RU Kraus operators. This is given by the recursive formula

$$\Delta(\rho) \equiv \Lambda_{n-1} \circ \cdots \circ \Lambda_1(\rho) \equiv \Lambda_{n-1}((\cdots(\Lambda_1(\rho))\cdots),$$

where $\Lambda_k$ is defined by

$$\Lambda_k(\rho) \equiv \frac{1}{2}\rho + \frac{1}{2}N_k\rho N_k^\dagger,$$

and $N_k$ is the diagonal unitary matrix

$$N_k \equiv I - 2E_{(k,k)}^{[n]},$$

where $E_{(a,b)}^{[n]}$ is the $n \times n$ elementary matrix with a 1 in its row-$a$, column-$b$ entry and zeros elsewhere.

As an example of how $\Delta(\rho)$ works, we will apply $\Lambda_k(\rho)$ recursively to an arbitrary 4-level system. First, note that $N_k$ are just diagonal matrices of ones except that the row-$k$, column-$k$ entry is $-1$, so for example, $N_1 = \text{diag}\{-1,1,1,1\}$. Then, the effect of these on $\rho$ is to flip the sign of the row-$k$, column-$k$ entries, such as

$$N_1\rho N_1^\dagger = \begin{pmatrix} \rho_{1,1} & -\rho_{1,2} & -\rho_{1,3} & -\rho_{1,4} \\ -\rho_{2,1} & \rho_{2,2} & \rho_{2,3} & \rho_{2,4} \\ -\rho_{3,1} & \rho_{3,2} & \rho_{3,3} & \rho_{3,4} \\ -\rho_{4,1} & \rho_{4,2} & \rho_{4,3} & \rho_{4,4} \end{pmatrix}.$$ 

Thus, adding $\rho$ to this and dividing by 2 gets rid of all the sign-flipped entries in (11). Performing this operation repeatedly for $k$ up to $n - 1$ decoheres the input state as

$$\Lambda_1(\rho) = \begin{pmatrix} \rho_{1,1} & 0 & 0 & 0 \\ 0 & \rho_{2,2} & \rho_{2,3} & \rho_{2,4} \\ 0 & \rho_{3,2} & \rho_{3,3} & \rho_{3,4} \\ 0 & \rho_{4,2} & \rho_{4,3} & \rho_{4,4} \end{pmatrix},$$

$$\Lambda_2(\Lambda_1(\rho)) = \begin{pmatrix} \rho_{1,1} & 0 & 0 & 0 \\ 0 & \rho_{2,2} & 0 & 0 \\ 0 & 0 & \rho_{3,3} & \rho_{3,4} \\ 0 & 0 & \rho_{4,3} & \rho_{4,4} \end{pmatrix},$$

$$\Delta(\rho) \equiv \Lambda_3(\Lambda_2(\Lambda_1(\rho))) = \begin{pmatrix} \rho_{1,1} & 0 & 0 & 0 \\ 0 & \rho_{2,2} & 0 & 0 \\ 0 & 0 & \rho_{3,3} & 0 \\ 0 & 0 & 0 & \rho_{4,4} \end{pmatrix},$$

thus removing coherence while preserving probabilities. This channel was symbolically verified for all dimensions up to $n = 13$, and we can claim that it works for all $n$.

2. n-Level RU Diagonal-Input Randomizer Channel

Now, the next step towards an RU depolarization channel is a method of equalizing all the probabilities of a diagonal state using RU Kraus operators. This can be accomplished using an RU permutation channel,

$$\Pi(\rho) \equiv \frac{1}{n} \sum_{m=1}^{n} \Pi_m \rho \Pi_m^\dagger,$$

where we use unitary permutation matrices,

$$\Pi_m \equiv \begin{pmatrix} R_m \\ R_{m+1} \\ \vdots \\ R_{m-1} \end{pmatrix},$$

where $R_m$ is the $m$th elemental row vector,

$$R_m = (0, \cdots, 0, 1_m, 0_{m+1}, \cdots, 0_n),$$

so the rows of $\Pi_m$ start with $R_m$ and cycle through all indices $m$ in increasing order, starting over at 1 after $n$. For example, for $n = 4$,

$$\Pi_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Pi_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\Pi_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \Pi_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$ 

Thus, any diagonal input to $\Pi(\rho)$ is equalized, such as

$$\Pi(\text{diag}\{a,b,c,d\}) = \frac{a+b+c+d}{4} I.$$  

(17)
3. n-Level RU Maximal-Mixing Channel

Immediately, we see that if the trace of the diagonal input to $\Pi(\rho)$ is 1, then $\Pi(\rho)$ will produce the maximally mixed state $\frac{I}{n}$. Then, since $\Delta(\rho)$ always produces a diagonal state for which $\text{tr}(\rho) = 1$, we can define an RU maximal-mixing channel as

$$\mathcal{M}(\rho) \equiv \Pi(\Delta(\rho)) = \frac{I}{n},$$

(18)

where $\Pi(\rho)$ is given in (13), and $\Delta(\rho)$ is given in (8). For example, putting the result of (12) into (18) produces

$$\mathcal{M}(\Delta(\rho)) = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix},$$

(19)

for all possible input states $\rho$, whether they are mixed or pure. Thus, (18) produces the maximally mixed state for all $n$-level systems using only RU Kraus operators.

4. The n-Level RU Depolarization Channel

Essentially our work is done by (18), but to finish the job, note that $\Delta(\rho)$ in (4) involves another term past the maximally mixed state. However, this feature is trivial because it merely increases the weight of the identity operator term in an RU Kraus decomposition while scaling the others by $\rho$, which preserves the RU property.

Thus, the full RU Kraus decomposition of the $n$-level depolarizing channel is

$$\mathcal{D}(\rho) = p\mathcal{M}(\rho) + (1-p)\rho,$$

(20)

where $\mathcal{M}(\rho)$ is given in (18), and the actual RU Kraus operators of $\mathcal{D}(\rho)$ are not immediately visible in (20) because of the recursive nature of the functions used to define it. However, that is acceptable here since the purpose of this section was merely to motivate the procedure. Figure 1 demonstrates that the RU depolarization channel $\mathcal{D}(\rho)$ defined in (20) truly produces the correct output which is $p\frac{I}{n} + (1-p)\rho$ as defined in (4).

Next, we will determine the RU Kraus operators directly, which will be more abstract, but more practically useful in many cases.

Also, hereafter we will only focus on the maximal-mixing channel $\mathcal{M}(\rho)$, since we have just shown that if $\mathcal{M}(\rho)$ has an RU decomposition, then $\mathcal{D}(\rho)$ does as well.
operators are merely different from the others by a global matrix factor of \(-1\), which cannot affect the decomposition. For example, the unitary parts of the Kraus operators of \(\Delta(\rho)\) for \(n = 4\) are

\[
\begin{align*}
N_0 &= \text{diag}\{1, 1, 1, 1\} \\
N_1 &= \text{diag}\{-1, 1, 1, 1\} \\
N_2 &= \text{diag}\{-1, -1, 1, 1\} \\
N_3 &= \text{diag}\{1, -1, 1, 1\} \\
N_4 &= \text{diag}\{1, 1, -1, 1\} \\
N_5 &= \text{diag}\{-1, 1, -1, 1\} \\
N_6 &= \text{diag}\{-1, -1, -1, 1\} \\
N_7 &= \text{diag}\{-1, -1, -1, 1\} \\
N_8 &= \text{diag}\{-1, -1, -1, 1\}
\end{align*}
\]  

(21)

which include only \(2^{n-1} = 8\) unique operators.

Then, for the full channel \(\mathcal{M}(\rho)\), there are \(n\) permutation families of Kraus operators, where the \(m\)th family is simply \(\Pi_m\) left-multiplied to the set of unique \(N\)-operators, such as the first half of (21), resulting in a total of \(n2^{n-1}\) unitary operators. The set is made properly RU by then normalizing with a factor of \(\frac{1}{\sqrt{n2^{n-1}}}\). Thus, if we call the half-set of \(N\)-operators \(\{N\}\), then the total set of unique RU Kraus operators for \(\mathcal{M}(\rho)\) is

\[
\{M_k\} = \left\{ \frac{1}{\sqrt{n2^{n-1}}} \Pi_1\{N\}, \ldots, \frac{1}{\sqrt{n2^{n-1}}} \Pi_n\{N\} \right\}.
\]  

(22)

2. Obtaining RU Kraus Operators from Qualitative Observation

A much simpler qualitative way to generate the RU Kraus operators for \(\mathcal{M}(\rho)\) is to start with the permutation matrices \(\Pi_m\), such as in (16), and then simply consider all possible sign distributions among their nonzero elements. Then, normalizing those by \(\frac{1}{\sqrt{n2^{n-1}}}\) and weed out the half that are just negatives of the others, one obtains the RU Kraus set for \(\mathcal{M}(\rho)\).

3. Explicit Construction of RU Kraus Operators

Finally, we have the information we need to explicitly construct the RU Kraus operators of \(\mathcal{M}(\rho)\). First, define the vector-index version of the \(N\)-operators as

\[
N_x \equiv N_x[n] \equiv I[n] - 2 \sum_{k=1}^{\text{dim}(x)} \text{sgn}(x_k) E_{(x_k,x_k)},
\]  

(23)

where \(x_k\) are nonnegative integer components of vector \(x\), and note that \(N_0 = I\).

Next, define the vectorized \(n\)-choose-\(k\) function as

\[
\text{nCk}(x, k) \equiv \text{each unique combinations of the elements of } x \text{ taken } k \text{ at a time}.
\]

(24)

For example, if \(n = 3\), then

\[
\text{nCk}([1, 2, 3], 1) = \begin{pmatrix} 1 \end{pmatrix}, \quad \text{nCk}([1, 2, 3], 2) = \begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix}, \\
\text{nCk}([1, 2, 3], 3) = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix},
\]

(25)

where, notice that by convention, the order of the combinations is taken to be increasing left to right, and counting in the standard format where the right-most digit counts upward to \(n\) and then resets as the digit to its left increases by one and so-on.

Then, the unique set of \(N\)-operators is

\[
\{N_x\} : x \in \{0, \text{nCk}([1, \ldots, n], k)\}_{k=1}^{k=n}\} = \{n=2^{n-1}\},
\]  

(26)

where we use set-building notation such that \(\{a, \{b, c\}\} \equiv \{a, b, c\}\), and \(\{a_k\}_{k=1}^{k=n} \equiv \{a_1, \ldots, a_n\}\), and \(\{\ldots\}_{\#=z}\) means to stop adding elements to the set after \(z\) elements have been included. Thus, (26) instructs us to make all the vectors \(x\) in the set it defines, and then use that set of \(2^{n-1}\) vector-indices to populate the set \(\{N_x\}\).

Finally, the full set of RU Kraus operators for the maximal-mixing channel \(\mathcal{M}(\rho)\) is given explicitly by

\[
\{M_{(m,x)}\} \equiv \left\{ \frac{1}{\sqrt{n2^{n-1}}} \Pi_1\{N_x\}, \ldots, \frac{1}{\sqrt{n2^{n-1}}} \Pi_n\{N_x\} \right\},
\]

(27)

meaning that we simply left-multiply each member of \(\{N_x\}\) by each of the permutation matrices \(\Pi_m\), and then their union, with each member normalized by \(\frac{1}{\sqrt{n2^{n-1}}}\), constitutes the RU Kraus decomposition of \(\mathcal{M}(\rho)\), and thus can be used to RU-decompose \(\mathcal{D}(\rho)\) as well.

For \(n = 4\), the unitary parts of the RU Kraus operators of \(\mathcal{M}(\rho)\) for the first permutation set are

\[
\begin{align*}
\Pi_1 N_0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\Pi_1 N_4 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\Pi_1 N_1 &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\Pi_1 N_5 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\Pi_1 N_2 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\Pi_1 N_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\
\Pi_1 N_6 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\Pi_1 N_7 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\
\end{align*}
\]

(28)
and for the second permutation set,

$$\Pi_2 N_0 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}, \quad \Pi_2 N_4 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix},$$

$$\Pi_2 N_1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0
\end{pmatrix}, \quad \Pi_2 N_{(1,2)} = \begin{pmatrix}
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0
\end{pmatrix},$$

$$\Pi_2 N_2 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}, \quad \Pi_2 N_{(1,3)} = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0
\end{pmatrix},$$

$$\Pi_2 N_3 = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}, \quad \Pi_2 N_{(1,4)} = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0
\end{pmatrix},$$

and for the third permutation set,

$$\Pi_3 N_0 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}, \quad \Pi_3 N_4 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix},$$

$$\Pi_3 N_1 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}, \quad \Pi_3 N_{(1,2)} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix},$$

$$\Pi_3 N_2 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{pmatrix}, \quad \Pi_3 N_{(1,3)} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix},$$

$$\Pi_3 N_3 = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}, \quad \Pi_3 N_{(1,4)} = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{pmatrix},$$

and for the fourth permutation set,

$$\Pi_4 N_0 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad \Pi_4 N_4 = \begin{pmatrix}
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix},$$

$$\Pi_4 N_1 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad \Pi_4 N_{(1,2)} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix},$$

$$\Pi_4 N_2 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad \Pi_4 N_{(1,3)} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0
\end{pmatrix},$$

$$\Pi_4 N_3 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \Pi_4 N_{(1,4)} = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix},$$

all 32 of which are visibly unitary. The sum of each of these operations applied to any input \( \rho \) as

$$\frac{1}{\sqrt{n^2}} \sum_{x=1}^{n^2} \Pi_m N_x \rho (\Pi_m N_x)$$

then constitutes the RU decomposition of \( M(\rho) \), and the output will always be the maximally mixed state \( \frac{1}{n} I \) for all possible input states.

4. Hilbert-Schmidt Completeness of the RU Kraus Operators

Since it will be useful to us in the next section, we will show here that these sets of RU Kraus operators always contain at least enough operators to be HS complete.

First, given the total RU set in (27), consider only the first \( n \) operators from each permutation set. Then, expanding each in terms of the elementary matrices, we find that all are related by a single transformation matrix,

$$T \equiv T^{[n]} = \frac{1}{\sqrt{n^2}} (\Omega^{[n]} - 2 \sum_{k=1}^{n-1} E_{(k+1,k)}^{[n]})$$

where \( \Omega^{[n]} \) is the matrix of all-ones, defined as

$$\Omega^{[n]} = \sum_{j=1}^{n} \sum_{k=1}^{n} E_{(j,k)}^{[n]}.$$ (33)

One finds that each group of elementary matrices that supports the non-zero elements in a given permutation matrix \( \Pi_m \) is linearly combined by \( T \) to form each of the first \( n \) RU Kraus operators of that permutation group.

Now here are the crucial two points. First, since \( T \) has a non-vanishing determinant for all dimensions,

$$\det(T^{[n]}) = \frac{2^{n(n-1)}}{(3n-1)!} \neq 0 \ \forall n,$$ (34)

then \( T \) is always invertible, allowing us to express the elementary matrices as linear combinations of the RU operators. Second, since the union of each of the first \( n \) operators of each of the \( n \) permutation sets involves all \( n^2 \) of the unique elementary matrices \( E_{(j,k)}^{[n]} \), then this plus the invertibility of each group ensures that all of the elementary matrices can be expressed using this subset of the RU Kraus operators defined in (27).

Since the set of all \( E_{(j,k)}^{[n]} \) are HS complete, and since they can be expanded with a subset of the RU Kraus operators of (27), then this means that these RU Kraus operators are HS overcomplete, meaning that they provide more than enough operators to expand any matrix as a linear combination of them.

For example, for \( n = 4 \) the transformation matrix is

$$T = \frac{1}{\sqrt{52}} \begin{pmatrix}
1 & 1 & 1 & 1 \\
-1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1
\end{pmatrix},$$ (35)

and the four permutation sets of RU Kraus operators are
where $E_1$, we could produce infinite expansions of $\alpha, \beta$

Then, given any complex numbers $Kraus$ expansions of a given elementary matrix, $E$.

Note that the overcompleteness of our RU Kraus operators can produce a particular $\sum_i R_i$.

Moreover, Kraus operators can be used to expand any operator.

Note that the overcompleteness of our RU Kraus operators is not a problem. In general, many different combinations of the RU Kraus operators can produce a particular $E[k]$. For example, suppose we have two different Kraus expansions of a given elementary matrix,

$$E_{(j,k)}^{[n]} = aM_1 + bM_2, \quad \text{and} \quad E_{(j,k)}^{[n]} = cM_3 + dM_4. \quad (37)$$

Then, given any complex numbers $\alpha, \beta$ such that $\alpha + \beta = 1$, we could produce infinite expansions of $E_{(j,k)}^{[n]}$ as

$$E_{(j,k)}^{[n]} = (\alpha + \beta)E_{(j,k)}^{[n]} = \alpha(aM_1 + bM_2) + \beta(cM_3 + dM_4). \quad (38)$$

Thus, we have a great deal of freedom as to how to involve the RU Kraus operators of (27) in the expansion of any given operator.

The importance of all of this will become clearer in the discussion of the reversibility of quantum channels.

III. REVERSIBILITY OF ALL QUANTUM CHANNELS

Now that we have shown that it is always possible to find an HS-complete set of RU Kraus operators for the maximal-mixing and depolarization channels in all dimensions, here we will show that this is sufficient to guarantee the reversibility of all quantum channels.

To simplify the argument, we will start by observing several essential facts, and culminate by showing that they prove the above claim.

A. Supporting Facts

1. General Conditions for Quantum Channel Reversibility Operations

The essential goal of quantum error correction is to correct for the effects of any given quantum channel on some unknown input state, restoring it to its state before the noise channel acted. Thus, we will refer to this ability of correction or restoration as the reversibility of a channel or noise.

As in [23], we will not assume any particular error-correction scheme, but rather follow only the bare-minimum of requirements. Thus, we suppose that if a restoration procedure exists to correct noise from channel $E$, that it can be represented by a single trace-preserving quantum operation $R(\rho)$, such that

$$R(\rho) \propto \rho. \quad (39)$$

where $\rho$ is the generally mixed unknown input state to $E(\rho)$. By trace-preserving, we mean that Kraus operators $R_k$ of $R$ have Kraus completeness $\sum_k R_k I_k = 1$.

Note that since the input is unknown, the reversal operation $R$ should not be dependent on the input state. However, later we will see that it is always possible to find an RU decomposition for any quantum channel if one allows state-dependent Kraus decompositions, though that is not necessary for state-independent correction.

2. Existence of Reversibility Operation for a Set of Errors

From [23], a necessary and sufficient condition for the existence of a quantum error correction operation $R$ correcting channel $E$ with Kraus operators $\{E_k\}$ on quantum code $C$ is that

$$P^j E^j_k P = \alpha, \quad (40)$$

where $\alpha$ is a Hermitian matrix, and $P$ is a projector onto the code space $C$. We can think of the $\{E_k\}$ as errors, and if $R$ exists for channel $E$, then $\{E_k\}$ is said to be a correctable set of errors.

3. Generality of a Reversibility Operation

Also from [23], suppose that $R$ exists and corrects errors $\{E_k\}$ for channel $E$. Then, given any other quantum operation $F$ with errors $\{F_j\}$, if each $F_j$ is a linear combination of the $\{E_k\}$ such that

$$F_j = \sum_k m_{j,k} E_k, \quad (41)$$

where the $m_{j,k}$ are complex elements of a generally nonunitary matrix $m$, then $R$ also corrects the effects of noise $F$ on code $C$. 

b
4. Kraus Sets of Same Channel Are Unitarily Related

Another well-known fact is that given any two different Kraus sets \( \{ E_k \} \) and \( \{ G_l \} \) that decompose the same quantum channel \( \mathcal{E} \), if the set with fewer elements is “padded” with the necessary number of zero-operators then both sets are unitarily related as

\[
E_k = \sum_l U_{k,l} G_l, \quad \text{or} \quad G_l = \sum_m U_{m,l}^* E_m, 
\]

(42)

where \( U \) is a unitary matrix. Note that Kraus sets from different channels are not unitarily related, and in general, one channel may not even have the completeness necessary to decompose all or any members of a Kraus set belonging to a different channel.

5. Corollary: Unitarily-Related Kraus Sets Share Reversibility

By using the fact that the rows (or columns) of a unitary matrix are orthonormal, one can show that two sets of unitarily-related Kraus operators both decompose the same quantum channel. For example, if \( \{ G_l \} \) is unitarily related to \( \{ E_k \} \), then

\[
\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger = \sum_l G_l \rho G_l^\dagger. 
\]

(43)

Then, if \( \{ E_k \} \) is also a correctable set of errors, then channel \( \mathcal{E}(\rho) \) is reversible, and thus from (43), \( \{ G_l \} \) is also a set of correctable errors. More rigorously, testing the reversibility conditions for \( \{ G_l \} \) and using (42) gives

\[
P^l G^l_k P = \sum_l \sum_m U_{m,j} U_{m,k}^* P^l E^l_m P \\
= \sum_l \sum_m U_{m,j} U_{m,k}^* \alpha_{m,l} P 
\]

(44)

where \( \beta_{j,k} \equiv \sum_m U_{m,j} U_{m,k}^* \alpha_{m,l} \), so that \( \beta \) is Hermitian, since \( \alpha \) is Hermitian from (40). Thus, since \( \beta \) is Hermitian, it follows that \( \{ G_l \} \) is a correctable set of errors, and this result was guaranteed by both the correctability of \( \{ E_k \} \) and its unitary relation to \( \{ G_l \} \).

B. Proof that All Quantum Channels are Reversible

Now that we have amassed several useful facts, we are ready to put them together to prove the claim that all quantum channels are reversible.

1. Proof of Reversibility of All Quantum Channels

The motivation for this proof is actually a corollary to Sec. III A 3. Simply put, if a correctable set of errors \( \{ E_k \} \) is also HS complete, then any operator can be expanded as a linear combination of the \( E_k \), as in (41). This means that any other Kraus set \( \{ F_k \} \) from any channel \( \mathcal{F} \) can be expanded using correctable set \( \{ E_k \} \), and therefore \( \mathcal{F} \) is also correctable. Then, since RU decompositions are always correctable, and since (27) proves that we can always find an RU decomposition that is also HS complete, then we can always find a correctable set of HS-complete Kraus operators, and therefore any Kraus set is correctable, so any channel is reversible.

Now we shall work this out more rigorously. Suppose we have a quantum channel

\[
\mathcal{F}(\rho) = \sum_r F_r \rho F_r^\dagger, 
\]

(45)

with Kraus operators \( \{ F_r \} \) for which no reversal operation is known. Now, suppose that for some different channel \( \mathcal{E} \), its Kraus operator set \( \{ E_k \} \) is both RU and HS complete. Then, by their HS completeness, we can expand as \( F_r = \sum_k m_{r,k} E_k \), so that (45) becomes

\[
\mathcal{F}(\rho) = \sum_k \sum_r H_{k,l} E_k \rho E_l^\dagger, 
\]

(46)

where \( H_{k,l} \equiv \sum_j m_{r,j} m_{r,k}^* \) are the elements of a Hermitian matrix \( H \) whose Hermiticity is a consequence of the form of the sum that defines it.

Now let \( \{ G_j \} \) be a different set of Kraus operators for reversible channel \( \mathcal{E} \), so that they are unitarily related to correctable set \( \{ E_k \} \), as in (42). Then for some unitary matrix \( U \), (46) becomes

\[
\mathcal{F}(\rho) = \sum_r \sum_s \left( \sum_k \sum_l U_{k,r} H_{k,l} U_{l,s}^\dagger \right) G_r \rho G_s^\dagger. 
\]

(47)

Then, if we let \( U \equiv \epsilon_H \), where \( \epsilon_H \) is the eigenvector matrix of \( H \) so that \( \epsilon_H^\dagger H \epsilon_H \) is diagonal, then \( U_{k,r} = (\epsilon_H^\dagger)_{k,r} \) and \( U_{l,s}^\dagger = (\epsilon_H)_{l,s}^\dagger \), so then

\[
\sum_k \sum_l U_{k,r} H_{k,l} U_{l,s}^\dagger = \sum_k (\epsilon_H^\dagger)_{r,k} H_{k,l} (\epsilon_H)_{l,s}^\dagger \\
= (\epsilon_H^\dagger H \epsilon_H)_{r,s} \\
= \lambda_{r,s} \delta_{r,s} 
\]

(48)

where \( \lambda_{r,s} \) are the (real) eigenvalues of Hermitian matrix \( H \). Then, putting (48) into (47) yields

\[
\mathcal{F}(\rho) = \sum_r \lambda_r G_r \rho G_r^\dagger = \sum_r \tilde{F}_r \rho \tilde{F}_r^\dagger. 
\]

(49)

Now, from the fact proved in Sec. III A 5 that unitarily-related Kraus sets share reversibility, since the Kraus operators \( \tilde{F}_r = \sqrt{\lambda_r} G_r \) of (49) are proportional to operators of correctable set \( \{ G_r \} \), then \( \tilde{F}_r \) is also a correctable set and therefore we have proven that arbitrary quantum channels \( \mathcal{F}(\rho) \) are reversible.

To be more rigorous, applying the correctability conditions to the Kraus operators of (49) gives

\[
P^l \tilde{F}_j^\dagger P = P^l \sqrt{\lambda_j} G_j P = \sqrt{\lambda_j} \sum_k \sum_r \epsilon_H^* (\epsilon_H)_{r,k} P^l E_r E_l P \\
= \sqrt{\lambda_j} \sum_k \sum_r \epsilon_H^* (\epsilon_H)_{r,k} \alpha_{s,r} P \\
= \gamma_{j,k} P, 
\]

(50)
where $\gamma$ is a Hermitian matrix with elements
\[
\gamma_{j,k} = \sqrt{\lambda_r \lambda_k} \sum_r \epsilon_H^* (\epsilon_H)^r_{j,k} \alpha_{s,r},
\]
and again $\alpha$ is Hermitian because $E$ is correctable. Since $\{F_r\}$ satisfies the correctability conditions, then all quantum channels $\mathcal{F}(\rho)$ are truly reversible.

Thus we have proved the central claim of this paper. Moreover, applying Sec. III A 3 to this result means that any restoration operation $\mathcal{R}$ that corrects the effects of $E$ will also correct the effects of all other quantum operations, where we must keep in mind that this requires that $\{E_k\}$ be both correctable and HS complete.

Then, since (27) proves-by-demonstration that there always exists an RU (and therefore correctable) set of HS complete operators in all dimensions, this guarantees that the restoration $\mathcal{R}$ that corrects the Kraus operators of (27) will also correct errors from all other channels.

2. Conversion of Any Kraus Set to a Correctable Set of Errors

Suppose that $\{F_j\}$ is some arbitrary set of Kraus operators belonging to channel $\mathcal{F}$ for which a recovery operation is not known.

To find a set of correctable errors for $\mathcal{F}$, we first use (27) to identify an HS-complete set of RU Kraus operators $\{E_k\}$. Then, expand the $F_j$ as
\[
F_j = \sum_k m_{j,k} E_k,
\]
and then form Hermitian matrix $H$ with elements
\[
H_{k,l} = \sum_j m_{j,k} m^*_{j,l}.
\]
Next, define a new set of Kraus operators as
\[
G_k = \sum_r (\epsilon_H)^r_{k,r} E_r.
\]
where $\epsilon_H$ is the eigenvector matrix of $H$. Finally, a set of correctable Kraus operators for $\mathcal{F}$ is given by
\[
\{\tilde{F}_r\} \equiv \{\sqrt{\lambda_r} G_r\},
\]
where $\lambda_r$ are the eigenvalues of $H$. $G_r$ are defined in (54), and the corresponding correctable decomposition of $\mathcal{F}$ is given in (49).

3. Proof That the Converted Kraus Set is Reversible

Now, given that (52-55) shows how to find a correctable set of Kraus operators for any channel $\mathcal{F}$, here we will verify that it is actually correctable by finding the reversal operation $\mathcal{R}$ that restores the input state. This section relies heavily on a less-detailed derivation in [23], but is shown here as a convenient illustration of why the reversal operation works.

Thus, starting with an arbitrary channel $\mathcal{F}$, if its correctable set is $\{\tilde{F}_r\} \equiv \{\sqrt{\lambda_r} G_r\}$ as in (55), then the correctability conditions for $\{\tilde{F}_r\}$ are $P^\dagger \tilde{F}_r \tilde{F}_i P = \gamma_{i,j}$ as given in (50), where $\gamma$ is Hermitian, with elements given in (51). Then, to simplify (50), define the unitarily related Kraus set,
\[
\tilde{F}_a = \sum_k (\epsilon^*_\gamma)_{a,k} \tilde{F}_k,
\]
where $\epsilon_\gamma$ is the eigenvector matrix of $\gamma$ so that $\epsilon^*_\gamma \epsilon_\gamma$ is diagonal. Then the correctability conditions for $\{\tilde{F}_a\}$ are
\[
P^\dagger \tilde{F}_a^\dagger \tilde{F}_b P = \sum_k \sum_{l} (\epsilon^*_\gamma)_{a,k}(\epsilon^*_\gamma)_{b,l} P^\dagger \tilde{F}_k \tilde{F}_l P
\]
(57)
\[
\equiv (\sum_k \sum_{l} (\epsilon^*_\gamma)_{a,k} \gamma_{l,b} (\epsilon^*_\gamma)_{l,b}) P
\]
(58)
\[
= \delta_{a,b} d_k P,
\]
where we used (50), and $d_k$ are the eigenvalues of $\gamma$. Then using a polar decomposition of $\tilde{F}_k P$ as
\[
\tilde{F}_k P = U_k \sqrt{(\bar{F}_k P)^\dagger(\bar{F}_k P)}
\]
(59)
\[
= U_k \sqrt{P^\dagger \tilde{F}_k^\dagger \bar{F}_k P}
\]
(60)
\[
= U_k \sqrt{\frac{P^\dagger \bar{F}_k P}{d_k} P}
\]
where right-multiplying (58) by $U_k^\dagger$, we obtain $\tilde{F}_k P U_k^\dagger = \sqrt{d_k} P U_k$, with projectors $P_k \equiv U_k P U_k^\dagger$, from which we find that
\[
P_k \equiv \frac{\tilde{F}_k P U_k^\dagger}{\sqrt{d_k}}, \quad P_k^\dagger \equiv \frac{U_k P \tilde{F}_k^\dagger}{\sqrt{d_k}}.
\]
(61)
If the syndrome measurement is defined by projectors $P_k$, possibly supplemented by another for completeness so that $\sum_k P_k = I$, then reversal is accomplished by projective measurement $P_k$ followed by unitary operation $U_k^\dagger$. Thus, the general reversal channel is
\[
\mathcal{R}(\sigma) = \sum_k U_k^\dagger P_k \sigma P_k U_k
\]
(62)
for input state $\sigma$. Then, if this reversal operation is for a state $P \rho P$ in the codespace that has experienced channel $\mathcal{F}$ so that the input to the reversal channel is $\sigma = \mathcal{F}(P \rho P) = \sum_l \tilde{F}_l P \tilde{F}_l^\dagger$, then the reversal channel acting on this input can be written as
\[
\mathcal{R}(\mathcal{F}(P \rho P)) = \sum_k \sum_l U_k^\dagger P_k \tilde{F}_l P \sqrt{\rho} \sqrt{\rho} \tilde{F}_l^\dagger P_k U_k.
\]
(63)
Then, in the left half of (61), using (59) and (57) produces
\[
U_k^\dagger P_k \tilde{F}_l P \sqrt{\rho} = U_k^\dagger U_k \frac{\tilde{F}_l P \sqrt{\rho}}{\sqrt{d_k}} = U_k^\dagger \sqrt{d_k} P \tilde{F}_l P \sqrt{\rho}
\]
(64)
\[
= \frac{d_k}{\sqrt{d_k}} \sqrt{\rho}
\]
(65)
\[
\delta_{l,i} \sqrt{d_k} P \sqrt{\rho}
\]
(66)
and then putting (62) into (61) reveals that

\[ \mathcal{R}(\mathcal{F}(P\rho P)) = \sum_k \sum_l \delta_{k,l} \sqrt{d_k} P \sqrt{\rho} \sqrt{d_k} P \sqrt{\rho} = \sum_k \sum_l \delta_{k,l} \sqrt{d_k} P \rho P \sum_k \sum_l \delta_{k,l} P \rho P \mathcal{R}(\mathcal{F}(P\rho P)) \propto P\rho P, \]

(63)

which shows that the coded input state \( P\rho P \) is restored by \( \mathcal{R} \), despite the effects of noise channel \( \mathcal{F} \), where we note that the proportionality constant \( \sum_k d_k = \text{tr}(\gamma) \) is eliminated through normalization.

Thus, we have shown that, starting from a completely arbitrary channel \( \mathcal{F} \) and its action on an arbitrary input state \( \rho \), that there exists a reversal operation \( \mathcal{R} \) that reverses the effects of \( \mathcal{F} \) on \( \rho \). The key step in proving that this is possible for arbitrary channels \( \mathcal{F} \) was actually in the last section, in which we outlined how to find a set of correctable Kraus operators for any arbitrary channel. The existence of the reversal operation \( \mathcal{R} \) is then guaranteed as well, as demonstrated explicitly in this section.

It is also possible to show that the projectors \( P_k \) are orthonormal, and that can then be used to prove that all RU Kraus decompositions belong to reversible channels, since their correctability conditions are always expressible with elements of a diagonal nonnegative unit-trace Hermitian matrix, specifically the matrix of the probabilities for the RU decomposition. It was not necessary for us to use an RU decomposition as in (27), but since such decompositions are always reversible, that was sufficient to prove that such HS-complete Kraus sets guarantee the reversibility of all quantum channels.

The final and most impressive step is now to use Sec. III A 3 and all subsequent results to develop a single reversal channel \( \mathcal{R} \) that will correct the effects of any other quantum channel.

IV. PHYSICAL APPLICATIONS

The purpose of this paper is in fact the application of the ideas presented here to accomplish the reversal of the effects of any quantum channel acting on a discrete system. Since the proof given in the last section implies that it is always possible to reverse the effects of any quantum channel, then the most general application is the construction of a universal reversal channel \( \mathcal{R} \). This channel is not unique and depends both upon the quantum code used and the HS-complete correctable Kraus set used. Nevertheless, we will see an example of such a universal reversal channel here. Then, this section will conclude with a few examples of how to accomplish RU-type corrections for the dephasing channel and depolarization channel, both in all dimensions \( n \). Since these are RU corrections, they can be accomplished with environment-assisted correction techniques, and we do not need to focus on a particular code.

A. Universal Reversal Channels

Since the RU Kraus set of (27) is reversible by virtue of being RU, and HS complete as proven in (34), then any Kraus operators of any quantum channel can be expanded in a linear combination of them, and from Sec. III A 3 and the proof in (56-63), this means that all quantum channels are reversible using the same reversal channel \( \mathcal{R} \) that reverses the maximal-mixing channel \( \mathcal{M} \) to which the RU Kraus set of (27) belongs. Therefore, to get a universal reversal channel \( \mathcal{R} \), we simply need to find the \( \mathcal{R} \) that reverses the effects of \( \mathcal{M} \).

Thus, from (27), our HS-complete RU Kraus set is

\[ \{E_k \} \equiv \{ M_{(m,x)} \} \equiv \{ \frac{1}{\sqrt{n^2 + 1}} \Pi_{m} N_{x} \}, \]

(64)

where the notation is explained in Sec. II B 3. Since this set is RU, meaning that \( E_k = \sqrt{p_k} U_k \) for some probability \( p_k \) and unitary matrix \( U_k \), then its correctability conditions are

\[ PE_j^\dagger E_k P = \delta_{j,k} \frac{1}{n^2 + 1}, \]

(65)

where \( P \) is the projector for the code, and we defined orthonormal projectors within the codespace as

\[ P_k \equiv P_{(m,x)} \equiv (\Pi_{m} N_{x}) P (\Pi_{m} N_{x})^\dagger. \]

(66)

Then, following the argument in Sec. II B 3, we find that the universal reversal operation on states \( \sigma \equiv P\rho P \) in the codespace is

\[ \mathcal{R}(\sigma) = \frac{1}{n^2 + 1} \sum_{m,x} N_{x}^{\dagger} \Pi_{m} P_{(m,x)} \sigma P_{(m,x)} \Pi_{m} N_{x}. \]

(67)

Thus, provided that a code has been defined and code-projector \( P \) has been chosen, then this \( \mathcal{R} \) will correct the effects of any quantum channel acting on a state \( \sigma \equiv P\rho P \) within the codespace. Using (66), an alternative way to express this is

\[ \mathcal{R}(\sigma) = P \left( \frac{1}{n^2 + 1} \sum_{m,x} N_{x}^{\dagger} \Pi_{m} \sigma \Pi_{m} N_{x} \right) P, \]

(68)

which shows that \( \mathcal{R} \) can be thought of as the application of an RU channel on states in the codespace followed by projection into the codespace.

Please note that the use of \( \mathcal{R} \) requires the definition of the quantum code to be established first; we cannot simply plug any transformed state into it and expect a correction to occur. The input state must be within the codespace. Thus, (67) and (68) merely provide an example of a general form of a universal reversal operation. To implement it, one must make syndrome measurements on \( P\rho P \) using projectors \( P_{(m,x)} \), followed by the application of the corresponding unitary \( N_{x}^{\dagger} \Pi_{m} \). See [3, 23] for more details about how to implement such codes.

Unfortunately, while we have shown that reversal of any quantum channel is possible, quantum codes can require an inconveniently large number of subsystems which must all be controlled and connected. For example, the fewest number of qubits needed to implement a
universal correction code on a single qubit is five, requiring a Hilbert space of 32 levels [14, 16, 23]. However, environment-assisted correction schemes can offer a conceptually cleaner alternative, and we shall now look at a few examples of these, showing how to reverse the effects of some channels of interest.

B. Some Important RU-Correctable Channels

Here, we look at RU-type correction schemes for some important quantum channels, starting with a review of environment-assisted RU correction.

1. Review of RU Environment-Assisted Error Correction

Consider a bipartite joint system consisting of a primary system $S$ of $n(S)$ levels and an environment $E$ of $n(E)$ levels, so that the joint system has Hilbert space $\mathcal{H}(S) \otimes \mathcal{H}(E)$ of $n = n(S)n(E)$ levels.

Next, suppose that the initial joint state is the product $\rho(S) \otimes \rho(E)$ where $\rho(S)$ and $\rho(E)$ are the initial states of the system and environment. Then, if the joint system experiences a unitary transformation $U$, the transformed reduced system is

$$ \rho(S)'' = \text{tr}_E(U(\rho(S) \otimes \rho(E))U^\dagger), \quad (69) $$

where $\text{tr}_E(\rho)$ is the partial trace over the environment,

$$ \text{tr}_E(\rho) \equiv \sum_{k=1}^{n(E)} \langle k(E) | \rho | k(E) \rangle, \quad (70) $$

where $\{ | k(E) \rangle \}$ is a complete basis for $E$. Then, assuming that the initial environment state is the pure state $| \psi_1 \rangle \equiv | \psi_1(E) \rangle$ (which is reasonable because a mixed state can always be purified), then $\rho(E) \equiv | \psi_1 \rangle \langle \psi_1 |$, and using

$$ \rho(S) \otimes | \psi_1 \rangle \langle \psi_1 | = (I(S) \otimes | \psi_1 \rangle \langle \psi_1 |)(\rho(S) \otimes 1(E))(I(S) \otimes | \psi_1 \rangle \langle \psi_1 |) = (I(S) \otimes | \psi_1 \rangle \langle \psi_1 |)\rho(S)(I(S) \otimes | \psi_1 \rangle \langle \psi_1 |), \quad (71) $$

we then find that the transformed state has Kraus form,

$$ \rho(S)'' \equiv E(\rho(S)) = \sum_{k=1}^{n(E)} E^{(S)}_{(k,1)} \rho(S) E^{(S)\dagger}_{(k,1)} = \sum_k E_k \rho(S) E_k^\dagger, \quad (72) $$

where the Kraus operators are defined by

$$ E^{(S)}_{(k,1)} = (I(S) \otimes | k(E) \rangle \langle k(E) |)U(I(S) \otimes | \psi_1 \rangle \langle \psi_1 |) \equiv | k \rangle \langle U | \psi_1 \rangle \langle \psi_1 \rangle \equiv E_k. \quad (73) $$

This last fact provides a link between the outcomes of the environment and particular Kraus operation that gives rise to that pure state member of the mixed state $E(\rho(S))$. Specifically, if $U$ includes a transformation that leaves the environment in its eigenbasis, then if the environment is measured and found to produce outcome $m$, then after we look at the measurement, the unnormalized post-measurement state of the system is

$$ \tilde{\rho}_m(S)' = E_m \rho(S) E_m^\dagger. \quad (74) $$

If the Kraus set $\{ E_k \} \equiv \{ \sqrt{p_k} U_k \}$ is RU, then since the measurement of $E$ tells us the present state of $S$, we merely need to apply the inverse of the unitary part of $E_m$ as $U_m^\dagger$ to reverse its effect on the state. Since this works for every outcome of the environment, provided that we have arranged an environment for which the levels each correspond to a particular Kraus operator, then we can always correct the effects of any RU channel.

This procedure is called environment-assisted RU error correction, as illustrated by Fig. 2 below.

FIG. 2. Schematic of environment-assisted RU error correction. Starting with a product state input $\rho(S) \otimes \rho(E)$ where $\rho(E)$ is pure, some unitary $T$ acts on the joint system, and then the environment is rotated to its eigenbasis with $V$ so the total unitary matrix acting on the joint system is $U \equiv (I(S) \otimes V)T$.

Then outcomes of measurements on $E$ correspond to particular terms $E_m \rho(S) E_m^\dagger$ in the system $S$, so that we know to conditionally apply reversal operation $R_m \propto E_m^{-1}$ on $S$ to restore the system state to $\rho(S)$. For RU Kraus decompositions, $R_m$ is the adjoint of the unitary part of $E_m$.

Now that we have covered the basics of correcting RU channels, we can apply this technique along with the HS-complete RU Kraus set presented in (27) to show how to explicitly correct the effects of several important channels in all dimensions.

2. Reversal of Dephasing Channel in all Dimensions

The dephasing channel, also called the phase-damping channel, is defined by the complete loss of coherence throughout the state, without changing the populations. In terms of a density matrix, it causes all off-diagonal elements to become zero, while the diagonal elements remain unchanged. A good example of a common system in which dephasing happens is the state of the field of a
laser. It begins as a pure coherent state, but then over time, random phase fluctuations cause the off-diagonal elements to vanish, leaving the beam in a diagonal phase-mixed coherent state, a classical Poisson distribution.

Though we have already seen a recursive definition for the $n$-level dephasing channel in (8), we can now use (27) to give it explicitly in terms of RU Kraus operators for any dimension. Note that here, we use $n$ as an abbreviation for $n_{(S)}$, the number of outcomes of the system $S$, and likewise we use $\rho$ to abbreviate $\rho_{(S)}$.

Thus, the $n$-level dephasing channel can be defined by the RU Kraus set,

$$\{\Delta_x\} \equiv \frac{1}{\sqrt{2^n}}\{N_x\},$$

(75)

where $N_x$ is defined in (23), and $\{N_x\}$ is the unique set of $N$-operators as defined in (26). The $n$-level dephasing channel then has Kraus representation as

$$\Delta(\rho) = \sum_x \Delta_x \rho \Delta_{x}^\dagger.$$

(76)

For example, if $n = 4$, the RU dephasing channel is

$$\Delta(\rho) = \frac{1}{8} \left( \rho + N_1 \rho N_1 + N_2 \rho N_2 + N_3 \rho N_3 + N_4 \rho N_4 + N_{12} \rho N_{12} + N_{13} \rho N_{13} + N_{14} \rho N_{14} \right),$$

(77)

where the unitary $N$-operators shown here are given in (28), using the fact that there, $\Pi_1 \equiv I$, and we used the fact that these operators are self-adjoint.

Then, the correction procedure is simply to prepare the system in product with a pure environment of dimension $n_{(E)} \equiv 2^{n-1}$ such that the outcomes in its eigenbasis each correspond to a different Kraus operator $\Delta_x$. Thus, for the dephasing channel, since all of the RU Kraus operators are self-adjoint, the correction is simply the unitary part of the Kraus operator itself,

$$R_x \equiv \sqrt{2^{n-1}} \Delta_{x}^\dagger = N_x.$$

(78)

For example, in the $n = 4$ dephasing channel in (77), if the $(2^{n-1} = 8)$-level environment is measured and the outcome is found to be $x = (1, 3)$, then to restore the state, we simply apply

$$R_{(1, 3)} = \sqrt{8} \Delta_{(1, 3)}^\dagger = N_{(1, 3)},$$

and after normalization, the state is restored to $\rho$. Since this procedure works for all $x$, then the restoration operation can occur with perfect success.

Prior to this paper, restoration of the dephasing channel was only shown to be possible in RU form for $n = 2$ and $n = 3$, in [6, 7]. Thus, to this author’s knowledge, this is the first presentation of an RU environment-assisted correction for $n$-level dephasing channels.

3. **Reversal of Depolarization Channel in all Dimensions**

The depolarization channel is given in (20) and is simply a mixture of the maximal mixing channel $\mathcal{M}$ and the identity channel. Focusing first on $\mathcal{M}$, we can write it in explicit Kraus form as

$$\mathcal{M}(\rho) \equiv \frac{1}{n^{2^{n-1}}} \sum_{m, x} (\Pi_m N_x \rho (\Pi_m N_x)^\dagger),$$

(79)

where $\Pi_m$ and $N_x$ are defined in Sec. II A 2 and Sec. III B 3, and again, we only use the unique set $\{N_x\}$.

For example, the maximal mixing channel for $n = 4$ is

$$\mathcal{M}(\rho) = \frac{1}{32} \sum_{m=1}^{4} \Pi_m \left( \rho + N_1 \rho N_1 + N_2 \rho N_2 + N_3 \rho N_3 + N_{12} \rho N_{12} + N_{13} \rho N_{13} + N_{14} \rho N_{14} \right) \Pi_m^\dagger,$$

(80)

and explicit examples of each of the RU Kraus operators appearing in (80) are given in (28-31).

Now, since the maximal-mixing channel is merely a special case of the depolarization channel $\mathcal{D}(\rho) \equiv p \mathcal{M}(\rho) + (1 - p) \rho$, then putting (79) into this gives its explicit RU Kraus form as

$$\mathcal{D}(\rho) \equiv (1 - p) \rho + p \frac{1}{n^{2^{n-1}}} \sum_{m, x} (\Pi_m N_x \rho (\Pi_m N_x)^\dagger).$$

(81)

However, since both main terms include identity operations, it is more convenient to collect them as

$$\mathcal{D}(\rho) = \left( 1 - \frac{n^{2^{n-1}-1}}{n^{2^{n-1}}} p \right) \rho + \frac{1}{n^{2^{n-1}}} p \sum_{l=2}^{n} \Pi_l \rho \Pi_l^\dagger.$$

(82)

Thus, (82) gives an explicit RU decomposition of the depolarization channel in all dimensions $n$. In an environment-assisted correction scheme, for a measurement of the environment that corresponds to outcome $(m, x)$ then requires the unitary reversal operation $N_x \Pi_m^\dagger$. This allows perfect reversal of the effects of the depolarization channel in all dimensions, including the extreme case of $p = 1$, when $\mathcal{D}$ becomes the maximal mixing channel $\mathcal{M}$.

For example, the depolarization channel for $n = 4$ is

$$\mathcal{D}(\rho) = \left( 1 - \frac{41}{52} p \right) \rho + \frac{1}{52} p \sum_{l=2}^{4} \Pi_l \rho \Pi_l^\dagger + \frac{1}{52} p \sum_{m=1}^{4} \Pi_m \left( N_1 \rho N_1 + N_2 \rho N_2 + N_3 \rho N_3 + N_{12} \rho N_{12} + N_{13} \rho N_{13} + N_{14} \rho N_{14} \right) \Pi_m^\dagger,$$

(83)

and in an environment-assisted correction scheme, the environment would need $n 2^{n-1} = 32$ levels. Then, for instance, if measurement of the environment produced the outcome corresponding to $(m, x) = (2, (1, 4))$, then the recovery operation would be $N_{(1, 4)} \Pi_2^\dagger$.

Since the environment for the two-qubit RU-correction scheme requires 32 levels, just as the simplest five-qubit realization of the Shor code [14, 16, 23] for one qubit,
then the RU method is more compact than the Shor code to implement. However, the main advantage of the RU method is its conceptually simple interpretation as the application of unitary gates contingent upon measured outcomes of the environment.

Thus, we have seen two examples of general n-level channels that can be perfectly reversed using environment-assisted RU correction. Such methods have the virtue of not requiring the complicated setups needed by traditional quantum codes such as the Shor code. However, those traditional codes offer a more general framework for quantum error correction, and as we proved in Sec. IIIIB, the general quantum code format allows any quantum channel to be corrected perfectly.

C. How to Verify Correctability of an Arbitrary Set of Errors

Here we illustrate the steps in (52-55) for a specific channel and a set of Kraus operators that we do not know are correctable simply by looking at them. This actually works for any set of Kraus operators for any channel, but we use a specific example here as a demonstration.

Consider the channel \( F \) with non-RU Kraus operators

\[
F_1 = \text{diag}\{p^2, p, p, 1\}, \quad F_2 = \text{diag}\{pq, 0, q, 0\}, \quad F_3 = \text{diag}\{qp, q, 0, 0\}, \quad F_4 = \text{diag}\{q^2, 0, 0, 0\},
\]

where \( q \equiv \sqrt{1-p^2} \), and \( p \in [0, 1] \) is time-dependent, and this channel models Ornstein-Ulenbeck phase noise, the details of which can be found in [28], and the output of the channel acting on an arbitrary initial state \( \rho(0) \) is

\[
F(\rho(0)) = \begin{pmatrix}
\rho_{1,1}(0) & \rho_{1,2}(0)p & \rho_{1,3}(0)p & \rho_{1,4}(0)p^2 \\
\rho_{2,1}(0)p & \rho_{2,2}(0)p & \rho_{2,3}(0)p^2 & \rho_{2,4}(0)p^2 \\
\rho_{3,1}(0)p^2 & \rho_{3,2}(0)p & \rho_{3,3}(0) & \rho_{3,4}(0)p \\
\rho_{4,1}(0)p^3 & \rho_{4,2}(0)p^2 & \rho_{4,3}(0)p & \rho_{4,4}(0)
\end{pmatrix},
\]

showing that this a kind of phase damping.

Observation of (84) shows that the \( F_j \) are not RU. Beyond that, to see if this set of errors is correctable, we might try testing the correctability conditions of (40). However, for that we need some way to choose the code projector \( P \), otherwise that test is not very useful.

This is where the steps of (52-55) become useful. They will allow us to find a different set of Kraus operators \( \{\tilde{F}_j\} \) for the same channel \( F \), where the new set has the desired property of being able to satisfy the correctability conditions of (40), thus guaranteeing that \( F \) is reversible.

First, applying (52), if we define the known-correctable errors as the RU set from (27), then we find we only need five of them, which we label as \( E_1 \equiv \frac{1}{\sqrt{32}}\Pi_1N_0, E_2 \equiv \frac{1}{\sqrt{32}}\Pi_1N_1, E_3 \equiv \frac{1}{\sqrt{32}}\Pi_1N_2, E_4 \equiv \frac{1}{\sqrt{32}}\Pi_1N_3, \) and \( E_5 \equiv \frac{1}{\sqrt{32}}\Pi_1N_4. \) Then the given operators can be expanded as

\[
F_1 = \frac{a}{2}(p + 1)^2E_1 - \frac{a}{2}pqE_2 - \frac{a}{2}pqE_3 - \frac{a}{2}pqE_4 - \frac{a}{2}pqE_5 \\
F_2 = \frac{a}{2}(p + 1)qE_1 - \frac{a}{2}pqE_2 + 0E_3 - \frac{a}{2}pqE_4 + 0E_5 \\
F_3 = \frac{a}{2}(p + 1)qE_1 + \frac{a}{2}pqE_2 - \frac{a}{2}pqE_3 + 0E_4 + 0E_5 \\
F_4 = \frac{a}{2}q^2E_1 - \frac{a}{2}q^2E_2 + 0E_3 + 0E_4 + 0E_5,
\]

where \( a \equiv \sqrt{32} \), which produces the coefficient matrix,

\[
m = aq \begin{pmatrix}
(p + 1)^2 & -p^2 & -p & -p & -1 \\
(p + 1)q & -pq & 0 & -q & 0 \\
(p + 1)q & -pq & -q & 0 & 0 \\
q^2 & -q^2 & -q^2 & -q^2
\end{pmatrix}.
\]

Then, putting (87) into (53) and eliminating \( q \) using its definition produces the Hermitian matrix,

\[
H = 8 \begin{pmatrix}
4s^2 & -s^2 & -s^2 & -s^2 & -s^2 \\
-s^2 & 1 & p & p & p^2 \\
-s^2 & p & 1 & p^2 & p \\
-s^2 & p^2 & p & 1 & p \\
-s^2 & p^2 & p & p & 1
\end{pmatrix},
\]

where \( s \equiv 1 + p. \)

![FIG. 3. (color online) Plot of the two-qubit channel F computed two ways for 1000 different values of p ∈ [0, 1]. First, for visual evidence that different mixed input states ρ ∈ [0, 1] and probabilities p are used, the dark blue dots are the Bloch purities P_R (defined in Fig. 1) of the output of F using the \{F_j\} of (84). The light blue dots are the Bloch purities of the quantum operation using \{tilde{F}_j\} obtained by putting (88) into (54) and (55). However, the true necessary and sufficient test is the red dots, which are the square magnitudes of the difference of the Bloch vectors of two differently-computed output states, computed as \( \frac{1}{n} \text{tr}((\sum_j F_j \rho F_j^†) - \sum_j \tilde{F}_j \rho \tilde{F}_j^†)^2)\). The two decompositions of \( F \) produce the same output iff the red dot has a height of zero. Since all pairs of decompositions produce the exact same output, this gives us good confidence that \{tilde{F}_j\} is a valid Kraus decomposition of \( F \).](image-url)
$H$ and the eigenvalues $\lambda_i$ of $H$. At this point, we must abandon symbolic representation and test numerical values. Therefore, by choosing random values for $p \in [0, 1]$, and choosing arbitrary input states $\rho(0)$, we can test that the output states are the same, for both $\{F_j\}$ and $\{\tilde{F}_j\}$, which is tested by Fig. 3.

Now that Fig. 3 has given us confidence that the new Kraus set $\{\tilde{F}_j\}$ is a valid decomposition of $\mathcal{F}$, it is easy to prove that $\mathcal{F}$ is reversible. The correctability conditions of $\{\tilde{F}_j\}$ are given in (50), given a known-correctable set $\{E_k\}$, as $P^\dagger \tilde{F}_k P = \gamma_{j,k} P$, where the necessary and sufficient condition for correctability is satisfied because $\gamma$ is Hermitian, since its elements are

$$\gamma_{j,k} = \sqrt{\lambda_j \lambda_k} \sum_s \sum_r (\epsilon_H)_{s,j}^* (\epsilon_H)_{r,k} \alpha_{s,r}. \quad (89)$$

However, to derive this, we assumed that we already knew the correctability conditions for the known-correctable set $\{E_k\}$. In this case, since $\{E_k\}$ is a subset of the HS-complete RU set from (27), we already know its correctability conditions from (65), which tell us that

$$\alpha_{s,r} = \delta_{s,r} \frac{1}{\sqrt{n^2 - 1}}. \quad (90)$$

Then, putting (90) into (89) yields

$$\gamma_{j,k} = \frac{1}{\sqrt{n^2 - 1}} \sqrt{\lambda_j \lambda_k} \sum_r (\epsilon_H)_{r,j}^* (\epsilon_H)_{r,k} \delta_{j,k}, \quad (91)$$

which shows that $\gamma$ in this case is definitely Hermitian, since it is diagonal and real. Therefore the $\{\tilde{F}_j\}$ are correctable and thus $\mathcal{F}$ is reversible.

To summarize what we just did, we started with a set of Kraus operators that we were not sure was correctable just by looking at them. Then, we used the steps of (52-55) to find a different set of Kraus operators for the same channel. This new set is unitarily related to the set of HS-complete RU Kraus operators from (27), and therefore, they inherit the correctability of that set, according to Sec. III A 5, which we verified in (91). The fact that we used a specific example here is just for illustration purposes. This proof is possible for any quantum channel, as we saw in Sec. III B.

D. Special Topic: State-Dependent RU Decomposition for All Channels

Here, we address a related, but different topic. Note that this section is not necessary for anything we have discussed so far. However, its relevance arises from the fact that it proves that all quantum channels can be expressed as RU Kraus expansions, and are therefore reversible, in principle. The catch is that such RU expansions are generally state-dependent in terms of both the input state and the output state of a given quantum channel.

Despite the state-dependence of this method, the fact that all possible mixed output states can be expressed as RU decompositions of all possible pure input states is incredibly powerful in that it suggests, at least theoretically, that the action of any quantum channel is reversible since we can always obtain an RU decomposition.

However, the state-dependence of this decomposition prevents it from being useful for general correction schemes in which we do not know the input state. Nevertheless, we will propose an application for it beyond its theoretical value. Now, we consider this universal RU decomposition method.

First, note that any quantum channel $\mathcal{E}$ is just a mapping of some input state $\rho$ to an output state $\rho'$, as

$$\mathcal{E}(\rho) = \rho'.$$  \quad (92)

Since both states are physical, both have spectral decompositions, and thus, given a particular input-output pair $\{\rho, \rho'\}$ of $n$-level states, we can define the spectral decomposition of the channel $\mathcal{E}$ as being that of $\rho'$,

$$\mathcal{E}(\rho) = \rho' = \sum_{j=1}^R \lambda_j \rho_j', \quad (93)$$

where $\rho_j' = |e_j\rangle \langle e_j|$ are the pure eigenstates of $\rho'$, with corresponding eigenvalues $\lambda_j$ such that $\sum_{j=1}^n \lambda_j = 1$ and $\lambda_j \in [0, 1]$, and $R = \text{rank}(\rho')$, where we use the descending order convention for the eigenvalues, $\lambda_1 \geq \cdots \geq \lambda_n$.

Then, given pure state input $\rho \equiv |\psi\rangle \langle \psi|$, since all pure states have the same diagonal matrix of eigenvalues $D$, we can eliminate this using

$$e_j^\dagger \rho_j' e_j = D = e_j^\dagger \rho e_j, \quad (94)$$

where $e_A$ is the unitary eigenvector matrix of $A$, such that $e_A^\dagger A e_A$ is diagonal. Then, solving (94) for the channel eigenstates $\rho_j'$ yields

$$\rho_j' = \epsilon_{\rho_j'} \epsilon_{\rho_j} e_{\rho_j}^\dagger = U_j \rho U_j^\dagger, \quad (95)$$

where the unitary matrix $U_j$ that converts $\rho$ to $\rho_j'$ is

$$U_j \equiv \epsilon_{\rho_j} e_{\rho_j}^\dagger, \quad (96)$$

Thus, putting (95) into (93) yields the RU Kraus decomposition of $\mathcal{E}(\rho)$ as

$$\mathcal{E}(\rho) = \sum_{j=1}^R \lambda_j U_j \rho U_j^\dagger = \sum_{j=1}^K K_j \rho K_j^\dagger, \quad (97)$$

where the RU Kraus operators are

$$K_j \equiv \sqrt{\lambda_j} U_j = \sqrt{\lambda_j} \epsilon_{\rho_j} e_{\rho_j}^\dagger. \quad (98)$$

Thus, we have proven that given a known pure state input $\rho \equiv |\psi\rangle \langle \psi|$, and a quantum channel $\mathcal{E}(\rho)$ producing
state \( \rho' \) from input \( \rho \), it is always possible to obtain an RU Kraus decomposition of \( \mathcal{E}(\rho) \).

Physically, this means that the action of any quantum channel on a pure-state input can be thought of as applying each of a set of \( R \) unitary operations \( U_j \) on input \( \rho \equiv |\psi\rangle\langle\psi| \) with probability \( \lambda_j \), which are the descending-order eigenvalues of the generally mixed output state \( \rho' \).

As a demonstration that this works, Fig. 4 plots 1000 random input-output state pairs for various quantum systems of different sizes and compositions.

![Figure 4](image)

**FIG. 4.** (color online) Plot of 1000 random two-qubit input-output state pairs represented two ways each. First, for visual evidence that different states are used, the height of the dark blue dots is the Bloch purity of the arbitrarily chosen output state \( P_B(\rho') \), as defined in Fig. 1. The height of the light blue dots is the Bloch purity of the RU-reconstructed output state \( P_B(\sum_{j=1}^{R} K_j \rho K_j^\dagger) \), where the construction is based on the state-dependent Kraus operators of (98), and \( \rho \) is a randomly chosen pure input state. However, the true necessary and sufficient test is the height of the red dots, which is the square magnitude of the difference of the Bloch vectors of the correct output state and the reconstructed output state, computed as \( \frac{n}{n-1} \text{tr}((\rho' - \sum_{j=1}^{R} K_j \rho K_j^\dagger)^2) \). The reconstruction is successful iff the red dot has a height of zero. Thus, this method works for all states tested. Note that it is not limited to two qubits; that is merely an arbitrary choice here.

Note that the restriction to pure input states is not limiting, because it is always possible to purify a mixed input, thus enabling this method.

Again, the application of this method is not necessarily useful for general error correction because it requires us to know the input and output states.

However, suppose that we had some device that produced a particular desired quantum state, but that due to internal imperfections it produces a somewhat corrupted version of the state. For example, this device could be a laser, inside of which we are certain that it produces an excellent approximation to a pure coherent state, yet random phase kicks cause dephasing, as mentioned earlier. Since this dephasing is generally time-dependent, that means that we know the output state of this dephasing channel at any time, and of course the input is a pure coherent state.

Thus, in principle, one could reverse the dephasing effects of a laser field by using the input and output states to form its state-dependent RU-decomposition.

Of course, the method presented in Sec. IV B 2 shows us how to accomplish a state-independent RU decomposition of the dephasing channel. However, the state-dependent method here offers a reduction in the number of outcomes needed, requiring only the number of terms equal to the rank \( R \) of the output state. In the case of a laser in a pure coherent state, for a weak enough field, we can truncate the number of levels to some finite value \( n \), retaining an excellent approximation to the state. Thus, for the state-independent method, we would need \( 2^{n-1} \) terms, which is generally larger than \( n \), the maximum number needed with the state-dependent method.

Thus, we see that the state-dependent RU decomposition enables greater efficiency in correction of certain quantum channels under conditions where we know both the input and output state of the channel to be corrected.

However, the greater worth of the state-dependent RU decomposition is its theoretical suggestion that, in principle, all quantum channels are reversible. This lends further support to the proof we gave of this using state-independent decompositions in Sec. III B.

V. CONCLUSIONS

The main goal of this paper is the proof of the reversibility of all quantum channels for discrete systems. To accomplish this, Sec. II A proposed an organic, recursive method to obtain an RU decomposition for the \( n \)-level dephasing channel \( \Delta \) and the depolarization channel \( \mathcal{D} \) and its extreme form, the maximal-mixing channel \( \mathcal{M} \). This established the fact that RU decompositions exist for the depolarization channel in all dimensions \( n \).

Then, in Sec. II B, we developed a method for explicit construction of the RU Kraus operators \( \{M_{\text{(m,x)}}\} \) of (27) for the maximal-mixing-channel \( \mathcal{M} \), used instead of the depolarization channel because it is more symmetrical in form. The explicit RU Kraus operators then allowed us to prove that this set always contains at least enough operators to form linear combinations of all the elementary matrices \( E_{\text{(a,b)}}^{\text{(n)}} \), which are Hilbert-Schmidt (HS) complete on the set of all matrices, meaning that any matrix can be expressed as a linear combination of the \( E_{\text{(a,b)}}^{\text{(n)}} \). The invertibility of the RU Kraus set \( \{M_{\text{(m,x)}}\} \) with \( \{E_{\text{(a,b)}}^{\text{(n)}}\} \) then guarantees that \( \{M_{\text{(m,x)}}\} \) can be used to expand any other operator, and is thus HS complete.

In Sec. III A, we reviewed various facts about quantum error-correction codes, including the correctability conditions, and in particular, Sec. III A 3 stated that any set of Kraus operators expandable as a linear combination of correctable Kraus operators is also correctable on the same code. This fact is the basis for the univer-
of HS-complete correctable Kraus operators. Then, by virtue of the correctability conditions for the code with projector \( P \), which satisfies the correctability conditions for the code with projector \( P \) for which \( \{ E_k \} \) is also HS complete. Thus, arbitrary channels can be corrected, given the existence of an HS-complete correctable set of Kraus operators.

The fact that we had already found an \( n \)-level set of HS-complete correctable Kraus operators \( \{ M_{(m,x)} \} \) proved the existence of the set \( \{ E_k \} \) used in the proof for correctability of all quantum channels \( F \), and therefore this proved the central claim of this paper. We then demonstrated that the reversal operation \( \mathcal{R} \) does in fact work, restoring states in the codespace \( P\rho P \) up to a proportionality factor, just as a reversal operation should.

The remainder of the paper focused on applications. First, we used the HS-complete RU Kraus set \( \{ M_{(m,x)} \} \) to construct a universal reversal operation \( \mathcal{R} \) for a code with projector \( P \). However, due to the complicated nature of such codes, we did not elaborate further. For example, even for a single qubit, the general error-correction scheme can have many different implementations, such as five qubits, seven qubits, etc. Thus, we leave the details of such schemes to be adapted by those using them, and the universal reversal channel \( \mathcal{R} \) in Sec. IV A can be used as a general guideline.

For more tangible examples, we then highlighted the most obvious application of the HS-complete RU Kraus set \( \{ M_{(m,x)} \} \), which is their use in environment-assisted RU correction schemes. Specifically, we showed how to perfectly reverse the effects of both the dephasing channel and the depolarization channel in all dimensions \( n \).

Finally, as food for thought, we considered a special topic that showed how to obtain state-dependent RU decompositions of all quantum channels. While this state-dependence limits its application for general error-correction, we showed that there do exist cases such as the state of a laser field that would benefit from such a state-dependent environmental error correction scheme. However, more importantly, the fact that this proves that RU decompositions exist for all possible mixed output states resulting from all possible quantum operations acting on all possible pure input states means that in principle, RU decompositions always exist for all quantum channels, even though they may generally need to be state-dependent. Since RU-decomposable channels are always correctable, the fact that RU decompositions can always be found suggests that all channels should be correctable, which further supports the earlier proof of this fact in Sec. III B for state-independent methods. Furthermore, the fact that RU decompositions are always possible means that we can always use the more general quantum-operations perspective to describe master equations for open systems, rather than the Lindblad equation [29] which is only valid under certain conditions.

Thus, we have proven that all discrete-system quantum channels are reversible with perfect success. There are undoubtedly many practical difficulties associated with implementation of the general correction schemes needed to realize such perfect channel reversal, however this newly proven fact that it is always possible to do so is enormously encouraging news for the field of quantum computation. It also has profound implications for physics in general because it shows that at least for discrete systems, there is no quantum operation that cannot be reversed completely with perfect success. Thus, at least in some local subsystem of interest, it is always possible to reverse any quantum process, even if the state of such a system starts in the maximally mixed state.

It is likely that there are a large number of applications and explorations possible as a result of these findings, and it is hoped that this paper will provide a useful starting point for further research in quantum error correction and other fields, as well.

**ACKNOWLEDGMENTS**

Many thanks to Ting Yu for his insightful comments. This project was supported by the I&E Fellowship at Stevens Institute of Technology.

[1] R. P. Feynman, Found. Phys. 16, 507 (1986).
[2] D. P. DiVincenzo, *The Physical Implementation of Quantum Computation* (Unpublished Article, arXiv:quant-ph/0002077, 2000).
[3] P. W. Shor, Phys. Rev. A 52, R2493 (1995).
[4] A. M. Steane, Phys. Rev. Lett. 77, 794 (1996).
[5] M. Gregoratti and R. F. Werner, J. Mod. Opt. 50, 915 (2003).
[6] L. J. Landau and R. F. Streater, Linear Algebra Appl. 193, 107 (1993).
[7] F. Buscemi, G. Chiribella, and G. M. D' Ariano, Phys. Rev. Lett. 95, 090501 (2005).
[8] C. B. Mendl, *Unital Quantum Channels* (Thesis, 2008).
[9] K. M. R. Audenaert and S. Scheel, New J. Phys. 10, 023011 (2008).
[10] B. Trendelkamp-Schroer, J. Helm, and W. T. Strunz, Phys. Rev. A 84, 062314 (2011).
[11] T. Heinosaarai, M. A. Jivulescu, D. Reeb, and M. M.
Wolf, J. Math. Phys. 53, 102208 (2012).
[12] X. Zhao, S. R. Hedemann, and T. Yu, Phys. Rev. A 88, 022321 (2013).
[13] M. E. Shirokov, Sb. Math. 204, 1215 (2013).
[14] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A 54, 3824 (1996).
[15] A. Ekert and C. Macchiavello, Phys. Rev. Lett. 77, 2585 (1996).
[16] R. Laflamme, C. Miquel, J. P. Paz, and W. H. Zurek, Phys. Rev. Lett. 77, 198 (1996).
[17] E. Knill and R. Laflamme, Phys. Rev. A 55, 900 (1997).
[18] E. Knill and R. Laflamme, Quantum Computation and Quadratically Signed Weight Enumerators (Unpublished Article, arXiv:quant-ph/9909094, 1999).
[19] K. E. Hellwig and K. Kraus, Commun. Math. Phys. 11, 214 (1969).
[20] K. E. Hellwig and K. Kraus, Commun. Math. Phys. 16, 142 (1970).
[21] M. D. Choi, Linear Algebra Appl. 10, 285 (1975).
[22] K. Kraus, States, Effects, and Operations: Fundamental Notions of Quantum Theory, Lecture Notes in Physics 190 (Springer-Verlag, Berlin, 1983).
[23] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, 2010).
[24] M. Gell-Mann, Phys. Rev. 125, 1067 (1962).
[25] Y. Ne’eman, Nucl. Phys. 26, 222 (1961).
[26] D. J. Griffiths, Introduction to Elementary Particles (Wiley-VCH, 2004).
[27] S. R. Hedemann, Hyperspherical Bloch Vectors with Applications to Entanglement and Quantum State Tomography, Ph.D. thesis, Stevens Institute of Technology (2014).
[28] T. Yu and J. H. Eberly, Optics Communications 283, 676 (2010).
[29] G. Lindblad, Commun. Math. Phys. 48, 119 (1976).