Dominoins in varieties generated by simple groups
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Abstract. Let $S$ be a finite nonabelian simple group, and let $H$ be a subgroup of $S$.
In this work, the dominion (in the sense of Isbell) of $H$ in $S$ in $\text{Var}(S)$ is determined,
generalizing an example of B.H. Neumann. A necessary and sufficient condition for $H$
to be epimorphically embedded in $S$ is obtained. These results are then extended to a
variety generated by a family of finite nonabelian simple groups.

Section 1. Introduction

An epimorphism in a given category $\mathcal{C}$ is defined to be a right cancellable function.
That is, given $\mathcal{C}$, a map $f: A \to B$ in $\mathcal{C}$ is an epimorphisms if and only if for every
object $C \in \mathcal{C}$, and every pair of maps $g, h: B \to C$, if $g \circ f = h \circ f$ then $g = h$.
In many familiar categories, such as $\text{Group}$, being an epimorphism is equivalent
to being a surjective map (for a proof of this, see [9]). On the other hand, this
is not the case in other familiar categories. For example, in the category of rings,
the embedding $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphisms, even though it is not surjective.

Isbell [4] has introduced the concept of dominion to study epimorphisms in cat-
egories of algebras (in the sense of Universal Algebra). Recall that given a full
subcategory $\mathcal{C}$ of the category of all algebras of a given type, and $A \in \mathcal{C}$ with a
subalgebra $B$ of $A$, we define the dominion of $B$ in $A$ in the category $\mathcal{C}$ to be
the intersection of all equalizer subalgebras of $A$ containing $B$. Explicitly,

$$\text{dom}_{\mathcal{C}}^A(B) = \left\{ a \in A \mid \forall C \in \mathcal{C}, \forall f, g: A \to C, \text{ if } f|_B = g|_B \text{ then } f(a) = g(a) \right\}.$$ 

It is clear that $B$ is epimorphically embedded in $A$ (in the category $\mathcal{C}$) if and
only if $\text{dom}_{\mathcal{C}}^A(B) = A$.

If $\text{dom}_{\mathcal{C}}^A(B) = B$, we will say that the dominion of $B$ in $A$ is trivial (meaning it
is as small as possible), and we will say it is nontrivial otherwise.

As Isbell notes, an arbitrary morphism $f: A' \to A$ of algebras may be factored as a
surjection onto $f(A')$ followed by the embedding of $f(A')$ into $A$. The surjection
$A' \hookrightarrow f(A')$ is well understood in terms of congruences, and so we may reduce the
study of epimorphisms to the study of dominions.

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For the rest of the present work we will restrict ourselves to groups and group morphisms, unless otherwise specified. Recall that a variety of groups is a full subcategory of Group which is closed under taking quotients, subgroups and arbitrary direct products. For basic properties of varieties, we direct the reader to Hanna Neumann’s book \[11\].

Although, as we mentioned above, all epimorphisms in Group are surjective (in fact, all epimorphisms in any variety of solvable groups are surjective by a theorem of P.M. Neumann \[12\], this is not true of other varieties of groups. For example, B.H. Neumann \[12\] has exhibited a nonsurjective epimorphism in Var(\(A_5\)), namely the embedding \(A_4 \hookrightarrow A_5\). In the present work we will generalize his example, by studying the variety of groups generated by a single finite nonabelian simple group.

Groups will be written multiplicatively. The identity element of a group \(G\) will be denoted by \(e_G\), with the subscript omitted if it is understood from context.

We quickly recall the basic properties of dominions in the context of varieties of groups: \(\text{dom}^V_G(-)\) is a closure operator on the lattice of subgroups of \(G\); the dominion construction respect finite direct products, and respects quotients. That is, if \(V\) is a variety of groups, \(G \in V\), and \(H\) is a subgroup of \(G\), \(N\) a normal subgroup of \(G\) contained in \(H\), then

\[
\text{dom}^V_{G/N}(H/N) = \text{dom}^V_G(H) / N.
\]

In Section 2 we will generalize a result of S. Oates to describe the structure of finitely generated groups in the variety generated by \(S\), a fixed finite nonabelian simple group. In Section 3 we will use this to generalize the example of B.H. Neumann mentioned above, and give a complete description of dominions of subgroups of \(S\). Finally, in Section 4 we will extend the results to a variety generated by a family of finite nonabelian simple groups.

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**Section 2. Finitely generated groups in Var(\(S\))**

Let \(S\) be fixed finite nonabelian simple group, and let \(V = \text{Var}(S)\) be the variety generated by \(S\). Recall that by Birkhoff’s HSP theorem \[1\], \(\text{Var}(S)\) is the collection of all homomorphic images of subgroups of direct powers of \(S\).

**Definition 2.1.** Given a group \(G\), by a *subfactor* of \(G\) we mean any group of the form \(H/N\), where \(N \triangleleft H\), and \(H\) is a subgroup of \(G\). A subfactor of \(G\) is called *proper*, except in the case where \(H = G\) and \(N = \{e\}\).
The name “subfactor” is somewhat troublesome. In Hanna Neumann’s book [11], they are called “factors”, but this is no longer standard terminology. Many authors say that a group \( H/N \) as above is involved in \( G \), but this seems like a bad choice, since it is a phrase that we ought to be free to use in its non-technical sense. Another common way to refer to such a group is as a “section” of \( G \), but again this may lead to confusion since a section is usually used to denote the image of a right inverse of a surjective map, especially when dealing with cohomology of groups and extensions (see for example [2]). And even “subfactor” is not an entirely satisfactory choice, since it has a different meaning in operator algebra theory. I have settled on subfactor as the terminology least likely to lead to confusion in the present work, and most suggestive of the situation we are dealing with.

**Definition 2.2.** A group \( G \) is called critical if \( G \) is finite and is not in the variety generated by the proper subfactors of \( G \).

The following notation was suggested by B.H. Neumann, and is used in Hanna Neumann’s book [11].

**Definition 2.3.** Given a group \( G \), we denote by \((\text{HS} - 1)(G)\) the collection of all proper subfactors of \( G \).

This follows the notations of Birkhoff’s \( H \), \( S \) and \( P \) operators. Like these three, the operator \( \text{HS} - 1 \) can easily be interpreted in the more general setting of universal algebras. The notation is suggestive, in that \( \text{HS} \) denotes the collection of subfactors, so \( \text{HS} - 1 \) is to be interpreted as the difference between the operator \( \text{HS} \) and the identity operator; so it yields all subfactors except for \( G \).

In this notation, the variety generated by all proper subfactors of a group \( G \) is denoted \( \text{Var}(\text{HS} - 1)(G) \). We can now express the fact that a finite group \( G \) is critical by writing

\[
G \notin \text{Var}(\text{HS} - 1)(G).
\]

We record the following fact for future use:

**Lemma 2.4.** (See 51.34 in [11]) Let \( S \) be a finite nonabelian simple group. Then \( S \) is critical.

We will later wish to study pairs of maps \( f, g : S \to K \) in \( \text{Var}(S) \). Therefore, we would like to describe the structure of finitely generated groups in \( \text{Var}(S) \). We do this now as a series of lemmas. The first one is easy to prove:

**Lemma 2.5.** (Remak, [14]) Let \( G = A_1 \times \cdots \times A_n \), where each \( A_i \) is a nonabelian simple group, and let \( N \triangleleft G \). Then \( N \) is equal to a direct product of some of the \( A_i \)'s.

The proof of the following lemma is based on a result of S. Oates (Lemma 3.2 in [13]).

**Lemma 2.6.** Let \( S \) be a finite nonabelian simple group, and let \( G \) be a finitely generated group in \( V = \text{Var}(S) \). Then

\[
G \cong S^n \times K
\]
where \( K \in \mathcal{U} = \text{Var}\left( (HS - 1)(S) \right) \).

**Proof:** By the finite HSP theorem, due to Higman [3], a finitely generated group in \( \mathcal{V} \) is a factor of a subgroup of a direct product of finitely many copies of \( S \). Since \( S \) itself can be written in the form of (2.7), it will suffice to show that the direct products of two groups of the given form can also be written as in (2.7), and that subgroups and homomorphic images of groups of the given form can also be written as in (2.7).

That the direct products of two groups of the form described in (2.7) is also of that form is obvious. We prove the subgroup clause by induction on \( n \). Consider \( H \), where

\[
H \subseteq S^{(1)} \times S^{(2)} \times \cdots \times S^{(n)} \times K
\]

is a subgroup.

If \( n = 0 \), then \( H \subseteq K \), and therefore, \( H \in \mathcal{U} \), so we are done. Assume inductively the result for any \( k < n \). Let \( \pi_i : H \rightarrow S^{(i)} \) be the canonical projection of \( H \) onto \( S^{(i)} \), and \( \pi_0 : H \rightarrow K \) be the canonical projection onto \( K \). If for any \( i \) we have \( \pi_i(H) \not\subseteq S^{(i)} \), then we can realize \( H \) as a subgroup of \( S^{(1)} \times \cdots \times S^{(i-1)} \times \pi_i(H) \times S^{(i+1)} \times \cdots \times S^{(n)} \times K \) and since \( \pi_i(H) \times K \in \text{Var}\left( (HS - 1)(S) \right) \) we can apply induction. So we may assume that \( \pi_1, \pi_2, \ldots, \pi_n \) are surjective; thus, replacing \( K \) with \( \pi_0(H) \) if necessary, we may assume that \( H \) is a subdirect product.

Now consider \( H \cap S^{(1)} \). This subgroup of \( S^{(1)} \) is normal in \( S^{(1)} \), and since \( S^{(1)} \) is simple, it follows that \( H \cap S^{(1)} = \{e\} \), or \( H \cap S^{(1)} = S^{(1)} \). In the former case it means that the canonical projection from \( H \) onto the group \( S^{(2)} \times \cdots \times S^{(n)} \times K \) is injective, so we can identify \( H \) with a subgroup of \( S^{n-1} \times K \) and apply the induction hypothesis. If, on the other hand, \( H \cap S^{(1)} = S^{(1)} \), then we can write \( H = S^{(1)} \times H' \), where \( H' \) is a subgroup of \( S^{(2)} \times \cdots \times S^{(n)} \times K \). We apply the induction hypothesis to \( H' \), and then note that \( H \) is a product of two groups of the form shown in (2.7), hence also of that form. This proves the subgroup clause.

Finally, to consider the homomorphic images of a group as shown in (2.7), we first need to tabulate its normal subgroups. Let \( N \) be a normal subgroup of \( S^n \times K \) having projections \( X \) and \( Y \) in \( S^n \) and \( K \), respectively. Then \( X \trianglelefteq S^n \), and \( Y \trianglelefteq K \), and by Goursat’s Lemma, \( X/X \cap N \cong Y/Y \cap N \).

By Lemma 2.5, \( X \) is necessarily a direct product of some copies of \( S \), say \( S^r \), and hence \( X/X \cap N \) and \( Y/Y \cap N \) have composition factors in disjoint sets, therefore they both must be the trivial group. In particular, \( N = X \times Y \), so

\[
(S^n \times K)/N \cong (S^{n-r}) \times (K/Y)
\]

which is of the required form. This proves the lemma. \qed
Theorem 2.8. Let $S$ be a finite nonabelian simple group, $H$ a subgroup of $S$, and let $\mathcal{V} = \text{Var}(S)$. Then

$$\text{dom}_{S}^{\mathcal{V}}(H) = \{ s \in S \mid \forall \phi \in \text{Aut}(S) \text{ if } \phi|_{H} = \text{id}_{H} \text{ then } \phi(s) = s \}.$$ 

Proof: Let $\mathcal{U}$ be the variety generated by all proper subfactors of $S$. If $H$ is trivial, then the dominion is also the trivial subgroup by definition. Now note that every inner automorphism of $S$ fixes the trivial group, so the right hand side of the display is contained in the fixed subgroup of all inner automorphisms, that is, the center of $S$. But since $S$ is nonabelian and simple, it has trivial center, so we get equality. We may, therefore, assume that $H$ is nontrivial. Denote the set described in the statement by $D$. Clearly, we must have $\text{dom}_{S}^{\mathcal{V}}(H) \subseteq D$.

Let $G \in \mathcal{V}$, and let $\theta, \psi : S \to G$ be two homomorphisms such that $\theta|_{H} = \psi|_{H}$. We want to show that they must also agree on $D$. Obviously we may assume that $G$ is generated by the images of $\theta$ and $\psi$, and in particular we may take $G$ to be finitely generated. Consequently, by Lemma 2.6 we may write $G = S^{n} \times K$, where $K \in \mathcal{U}$. Let $\pi_{0}$ be the projection of $G$ onto $K$, and let $\pi_{i}$ be the projection onto the $i$-th copy of $S$, for each $i$. Let $\theta_{i} = \pi_{i} \circ \theta$ and let $\psi_{i} = \pi_{i} \circ \psi$ for all $0 \leq i \leq n$.

Since $K \in \mathcal{U}$, and $S$ is simple and critical, $\theta_{0}$ and $\psi_{0}$ are both the trivial map, so $\theta_{0} = \psi_{0}$, and they agree on $D$. Moreover, since $\theta_{i}$ and $\psi_{i}$ agree on $H$, they are either both monomorphisms, or they are both trivial. In the latter case, $\theta_{i} = \psi_{i}$. If they are both monomorphisms, then necessarily they are isomorphisms. In that case, let $\phi_{i} = \psi_{i}^{-1} \circ \theta_{i}$. Then $\phi_{i}$ is an automorphism of $S$, which acts as the identity map on $H$. Therefore, $\phi_{i}$ fixes $D$ pointwise, so $\theta_{i}$ and $\psi_{i}$ must agree on $D$. Thus $\theta_{i}$ and $\psi_{i}$ agree on $D$ for all $i$, so $\theta$ and $\psi$ agree on $D$. This proves that $D \subseteq \text{dom}_{S}^{\mathcal{V}}(H)$, and we are done.

Remark 2.9. I remark that I cannot at present give a complete classification of the dominions in $\mathcal{V}$, although as Theorem 2.8 shows we can describe all dominions in the generating object $S$.

We obtain the following consequence of Theorem 2.8:

Corollary 2.10. Let $S$ be a finite nonabelian simple group. A subgroup $H$ of $S$ is epimorphically embedded in $S$ (in the variety $\text{Var}(S)$) if and only if

$$\left\{ \psi \in \text{Aut}(S) \mid \psi(h) = h \text{ for all } h \in H \right\} = \{ \text{id}_{S} \}.$$ 

In particular, if all automorphisms of $S$ are inner and $H$ is maximal, this will hold if and only if $\text{Z}(H) = \{ e \}$.

Proof: Note that if $S$ is a simple group and $H$ is a maximal subgroup of $S$, any element of $S$ which centralizes $H$ must lie in $H$ (otherwise, $H$ would be a normal subgroup of $S$). Therefore, the centralizer of $H$ in $S$ equals the center of $H$, which yields the last statement. The rest of the result is clear.
Section 3. Applications and examples

Theorem 2.8 provides us with a family of examples of nonsurjective epimorphisms. We start with a special case, and then generalize.

Lemma 3.11. (B.H. Neumann, Example A in [12]) Let $V = \text{Var}(A_5)$. Let $A_4$ be identified with the subgroup of $A_5$ fixing 5. Then $\text{dom}_V(A_4) = A_5$. Equivalently, the embedding $A_4 \hookrightarrow A_5$ is a nonsurjective epimorphism in $\text{Var}(A_5)$.

Proof: By Theorem 2.8 we need to look at

$$D = \{ g \in A_5 \mid \forall \phi \in \text{Aut}(A_5) \text{ if } \phi|_{A_4} = \text{id}_{A_4} \text{ then } \phi(g) = g \}.$$

Since $\text{Aut}(A_5)$ may be identified with $S_5$ acting by conjugation, we need to find the centralizer of $A_4$ in $S_5$. This is easily seen to be trivial. Therefore, the dominion of $A_4$ is equal to the even permutations of 5 elements which commute with the identity, that is, $\text{dom}_V(A_4) = A_5$, as claimed.

Remark 3.12. Note that $A_4$ lies in a variety of solvable groups, whereas $A_5$ does not. Compare with Corollary 2.18 in [10], which says that the dominion of an abelian group must be abelian. Also, $A_4$ lies in $B_6$, the variety determined by the identity $x^6 = e$, but $A_5$ does not, since it contains $\mathbb{Z}/5\mathbb{Z}$.

Theorem 3.13. Let $n \geq 4$, and let $V = \text{Var}(A_{n+1})$. If we identify $A_n$ with the subgroup of $A_{n+1}$ fixing $n+1$, then $\text{dom}_V(A_n) = A_{n+1}$, so the embedding $A_n \hookrightarrow A_{n+1}$ is a nonsurjective epimorphism in $\text{Var}(A_{n+1})$.

Proof: It is easy to verify that the group of automorphisms of $A_{n+1}$ having $A_n$ in their fixed subgroups is just the identity: the only possible difficulty is in the case $n = 5$ (as then the automorphism group is strictly bigger than $S_{n+1}$), but every automorphism of $A_6$ which does not come from conjugation by an element of $S_6$ fixes no group elements of exponent 3 other than the identity (see [7]).

So by Theorem 2.8,

$$\text{dom}_V(A_{n+1}) = \{ x \in A_{n+1} \mid \text{Id}_{A_{n+1}}(x) = x \} = A_{n+1}$$

as claimed.

Remark 3.14. It might appear, since the inclusion $A_4 \to A_5$ is an epimorphism, and the inclusion $A_5 \to A_6$ is also an epimorphism, that the composite inclusion $A_4 \to A_6$ is an epimorphism. But, the precise statements are that the first inclusion is an epimorphism in $\text{Var}(A_5)$, and the second in $\text{Var}(A_6)$. There is in fact no variety in which they are both epimorphisms, since one can easily verify, using Theorem 2.8, that the inclusion $A_4 \to A_6$ is not an epimorphism in the smallest variety in which the question makes sense, namely $\text{Var}(A_6)$. 

We can prove that a larger family of subgroups of $A_n$ are epimorphically embedded in $A_n$. But before doing this, we should recall the definition of a primitive permutation group.

Recall that given a group $G$ acting on a set $\Omega$, a subset $\Delta \subseteq \Omega$ and $g \in G$, we write

$$\Delta^g = \{g(d) \mid d \in \Delta\}.$$  

If $G$ acts transitively on $\Omega$, we say that a non-empty subset $\Delta \subseteq \Omega$ is a block if for every $g$ in $G$, $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \emptyset$. Given a group $G$ acting transitively on a set $\Omega$, we say that $G$ is primitive if $G$ admits no nontrivial blocks in $\Omega$.

We can now give a rough classification for the maximal subgroups of $A_n$:

**Theorem 3.15.** (Theorem 1 in [8]) Every maximal subgroups of $A_n$ is of one of the following three sorts:

(i) Primitive.

(ii) (Intransitive) The set stabilizer of some $\Delta \subseteq \{1, \ldots, n\}$, $1 \leq |\Delta| \leq \frac{n}{2}$, that is $H = (S_m \times S_k) \cap A_n$, with $n = m + k$ and $m \neq k$.

(iii) (Imprimitive) A subgroup $S(\Pi)$ of all even permutations which preserve a partition

$$\Pi = \{\Delta_1, \Delta_2, \ldots, \Delta_m\}$$

of $\{1, \ldots, n\}$ into parts of size $\frac{n}{m}$, $1 < m < n$, that is $H = (S_m \wr S_k) \cap A_n$ with $n = mk$, $m > 1$ and $k > 1$.

**Remark 3.16.** The notation $S_m \wr S_k$ denotes the standard wreath product of $S_m$ by $S_k$.

We shall now determine, for two of the above three classes of maximal subgroups, precisely which members of the class are epimorphically embedded in $A_n$.

**Theorem 3.17.** Let $H$ be an intransitive maximal subgroup of $A_n$, $n \geq 5$; that is, $H$ is of the form

$$H = (S_m \times S_k) \cap A_n,$$

where $n = m + k$, and $1 \leq m \leq \frac{n}{2}$, $m \neq k$. Then the inclusion of $H$ into $A_n$ is an epimorphism in $\text{Var}(A_n)$ if and only if $m \neq 2$.

**Proof:** If $m = 1$, then $H \cong A_{n-1}$, and the result is just Theorem 3.13. Suppose first that $m > 2$; then $S_m$ and $S_k$ are both centerless, any element of $S_n$ that centralizes $S_m \times S_n$ must stabilize the partition given by $\Delta$, and each component must lie in the center of $S_m$ and $S_k$, respectively; therefore, no nontrivial element of $S_n$ can centralize $S_m \times S_k$. By the description of dominions given in Theorem 2.8, $\text{dom}_{A_n}^\text{Var}(A_n)(H) = A_n$, as claimed.

Finally, if $m = 2$, then $H \cong S_{n-2}$, but this is equal to its own dominion; if $\Delta = \{i, j\}$, then the permutation $(i, j)$ lies in the centralizer of $H$ in $S_n$, and in turn the centralizer of $(i, j)$ in $A_n$ is none other than $H$ itself. This establishes the theorem.
Theorem 3.18. Let $H$ be an imprimitive maximal subgroup of $A_n$, $n \geq 5$; that is, $H$ is of the form

$$H = (S_m \wr S_k) \cap A_n,$$

where $n = mk$ and $1 < m < n$. Then the embedding of $H$ into $A_n$ is an epimorphism in $\text{Var}(A_n)$ if and only if $m > 2$.

Proof: Again, all dominions are being considered inside of $\text{Var}(A_n)$. Assume first that $m > 2$. Let $x \in S_n$ be an element in the centralizer of $H$. Suppose that $\Delta_1$ contains the letters $i$, $j$ and $k$, and that $\Delta_2$ contains $p$ and $q$. Then $H$ contains the element $(i,j)(p,q)$. Since $x$ centralizes $H$, it must either fix $\{i,j\}$ as a set, or exchange it with $\{p,q\}$. By the same token, $H$ contains the elements $(i,k)(p,q)$, and so $x$ must fix $\{i,k\}$ or exchange it with $\{p,q\}$. Hence $x$ cannot exchange $\{p,q\}$ with both $\{i,j\}$ and $\{i,k\}$, it follows that it must fix both $\{i,j\}$ and $\{i,k\}$ as sets; in particular, it must fix $i$. Since no special properties of $\Delta_1$ or its element $i$ were used, we conclude that $x$ must fix all points, that is $x = e$. Therefore,

$$\text{dom}_{A_n}(H) = C_{A_n}(e) = A_n$$

so $H$ is epimorphically embedded in $A_n$, as claimed.

In the case where $m = 2$, the embedding of $H$ into $A_n$ is not an epimorphism; say $H$ is the stabilizer of the partition

$$\Pi = \left\{ \{1,2\}, \{3,4\}, \ldots, \{2k-1,2k\} \right\}.$$ 

Then the element $(1,2)(3,4) \cdots (2k-1,2k) \in S_n$ is in the centralizer of $H$ (in $S_n$), and so the dominion of $H$ cannot be all of $A_n$. \qed

The method used above to study when maximal subgroups are epimorphically embedded into the group may also be applied to some nonmaximal subgroups of $A_n$. Note that Theorem 2.8, together with our observation on outer automorphisms of $S_6$, shows that every subgroup of $A_n$, with $n \geq 5$ which has trivial centralizer in $S_n$ (and has elements of order 3, if $n = 6$) is epimorphically embedded in $A_n$ (in the variety $\text{Var}(A_n)$). In particular, using arguments like those in Theorem 3.17 and Theorem 3.18, we see that this includes every subgroup of the form

$$A_n \cap (S_{m_1} \times S_{m_2} \times \cdots \times S_{m_r})$$

where $m_1 + m_2 + \cdots + m_r = n$, none of the $m_i$ equals 2, and at most one of them equals 1, with $S_{m_1} \times \cdots \times S_{m_r}$ embedded in $S_n$ in the natural way.

Also, given any partition

$$\Pi = \{\Delta_1, \Delta_2, \ldots, \Delta_m\}$$
of \{1, \ldots, n\} such that no \(\Delta_i\) has cardinality two, and at most one of them has cardinality 1, the group of even permutations preserving \(\Pi\) contains (after possible relabeling of \{1, \ldots, n\}) a subgroup of the form mentioned in the preceding paragraph, hence it will also be epimorphically embedded into \(A_n\).

We also note in passing that Theorem 2.8 may also be used to establish other instances of nonsurjective epimorphisms. For example, let \(M_{11}\) denote the Mathieu group on 11 letters, a sharply 4-transitive group acting on \{1, 2, \ldots, 11\}, which is the smallest sporadic simple group. It is known that \(\text{Aut}(M_{11}) = M_{11}\). From this fact, it is not too hard to verify that the dominion of \(M_{10}\), the stabilizer of a point in \(M_{11}\), is all of \(M_{11}\), so that the embedding \(M_{10} \hookrightarrow M_{11}\) is a nonsurjective epimorphism in \(\text{Var}(M_{11})\).

Section 4. Dominions in \(\text{Var}(\{S_i\}_{i \in I})\)

We would like to extend the results in the previous sections to varieties generated by families of finite nonabelian simple groups. First, we note that we can restrict to the case of varieties generated by finitely many finite nonabelian simple groups:

Theorem 4.19. (Jones [6], Weigel [15], [16], [17]) If a variety \(V\) contains infinitely many isomorphism classes of finite nonabelian simple groups, then \(V\) is the variety of all groups.

Remark 4.20. Theorem 4.19 depends on the classification of finite simple groups, or at least on the fact that there are at most finitely many exceptions to the classification. Weigel proved that the absolutely free group of rank 2 is residually in any infinite collection of nonisomorphic known finite nonabelian simple groups, while Jones proved that any proper subvariety of \(\text{Groups}\) contains at most finitely many nonabelian simple groups of the known types.

In the proof of the following lemma, we shall use the concept of a product variety. Recall that given two varieties of groups \(N\) and \(Q\), the class of all groups which are extensions of an \(N\)-group by a \(Q\)-group forms a variety, denoted by \(N Q\); this variety clearly contains both \(N\) and \(Q\).

Lemma 4.21. Let \(S\) be a simple group, and let \(V_1, \ldots, V_n\) be a finite collection of varieties. Let \(V\) be the join of the \(V_i\), that is, the least variety that contains \(V_i\) for \(i = 1, \ldots, n\). If \(S \in V\) then there exists \(i_0\), \(1 \leq i_0 \leq n\) such that \(S \in V_{i_0}\).

Proof: First note that if a simple group \(S\) lies in a product variety \(N Q\), then either \(S \in N\) or \(S \in Q\), by definition of \(N Q\).

Next note that \(V \subseteq V_1 \cdots V_n\), since a product variety contains each of the factors. Therefore, if \(S \in V\), it follows that there exist \(i_0\) such that \(S \in V_{i_0}\), as claimed.

Lemma 4.22. (See Theorem 51.2 in [11]) Let \(X\) be a class of finite groups such that \(\text{Var}(X)\) is locally finite, and let \(A\) be a finite group in \(\text{Var}(X)\). Then the composition factors of \(A\) are subfactors of groups in \(X\).
It is not hard to verify that if $\mathcal{X}$ is a finite class of finite groups, then $\text{Var}(\mathcal{X})$ is indeed locally finite. From this we deduce:

**Lemma 4.23.** If $\mathcal{X}$ is a family of finite (not necessarily simple) groups such that $\text{Var}(\mathcal{X})$ is locally finite, then the only finite simple groups in the variety $\text{Var}(\mathcal{X})$ are the simple subfactors of the members of $\mathcal{X}$.

**Proof:** By Lemma 4.22, if $A$ is a finite group in $\text{Var}(\mathcal{X})$, then the composition factors of $A$ are subfactors of the members of $\mathcal{X}$. If $S$ is a finite simple group in $\text{Var}(\mathcal{X})$, since the only composition factor of $S$ is $S$ itself, it follows that $S$ must be a subfactor of a member of $\mathcal{X}$, as claimed.

**Theorem 4.24.** Let $\{S_i\}_{i \in I}$ be a finite collection of pairwise non-isomorphic finite nonabelian simple groups. Let $\{S_i\}_{i \in I'}$ be the subfamily consisting of those members of $\{S_i\}_{i \in I}$ which are not isomorphic to subfactors of other members of this family. Then

$$\text{Var}(\{S_i\}_{i \in I}) = \text{Var}(\{S_i\}_{i \in I'}),$$

and for each $i \in I'$, $S_i \notin \text{Var}(\{S_j\}_{j \in I \setminus \{i\}}, (HS - 1)(S_i))$.

**Proof:** Clearly, if $i \in I$, then $S_i \in \text{Var}(\{S_i\}_{i \in I'})$ (since the $S_i$ are pairwise non-isomorphic). The reverse inclusion is obvious, giving equality. The final statement follows from the definition of $I'$ and the fact that $S_i$ is simple, hence critical.

Therefore, when talking about $\text{Var}(\{S_i\})$ we may assume without loss of generality the $S_i$ are pairwise non-isomorphic, and that $\text{Var}(\{S_i\}) = \text{Group}$, or else that $\{S_i\}_{i \in I}$ is a finite collection finite nonabelian simple groups such that no $S_i$ is a subfactor of any other.

**Theorem 4.25.** Let $\mathcal{V} = \text{Var}(\{S_i\}_{i \in I})$ be a variety generated by finitely many finite nonabelian simple groups, such that no $S_i$ is a subfactor of any other. Let $G$ be a finitely generated group in $\mathcal{V}$. Then

$$G \cong S_1^{m_1} \times \cdots \times S_n^{m_n} \times K$$

where $K$ is in the variety generated by the proper subfactors of all the $S_i$.

**Proof:** The argument in the proof of Lemma 2.6 will establish this result, once we note that for each $S_i$, a map into $K$ must be trivial, and by the hypothesis on the set $\{S_i\}$, a mapping $S_i \to S_j$, with $i \neq j$ must also be trivial.

**Theorem 4.26.** Let $\{S_i\}_{i=1}^n$ be a finite set of finite nonabelian simple groups, none of which is a subfactor of any other, and let $\mathcal{V} = \text{Var}(\{S_i\})$. Fix some $i_0 \in \{1, 2, \ldots, n\}$, and let $H$ be a subgroup of $S_{i_0}$. Then

$$\text{dom}_{S_{i_0}}^\mathcal{V}(H) = \left\{ s \in S_{i_0} \mid \forall \phi \in \text{Aut}(S_{i_0}) \text{ if } \phi|_H = \text{id}|_H, \text{ then } \phi(s) = s \right\}.$$ 

**Proof:** This follows from Theorem 4.25, noting as in the proof of the latter that maps between distinct $S_i$ are trivial.
Remark 4.27. We remark again that we cannot at present give a complete classification of the dominions in $\mathcal{V}$, although as Theorem 4.26 shows, we can describe the dominions in any of the generating objects $S_i$. This together with Theorem 4.19 gives us a precise description of a large number of dominions in any variety which is generated by finite nonabelian simple groups.

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