Statistical mechanics of gravitating systems: An Overview

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Abstract

I review several issues related to statistical description of gravitating systems in both static and expanding backgrounds. After briefly reviewing the results for the static background, I concentrate on gravitational clustering of collision-less particles in an expanding universe. In particular, I describe (a) how the nonlinear mode-mode coupling transfers power from one scale to another in the Fourier space if the initial power spectrum is sharply peaked at a given scale and (b) what are the asymptotic characteristics of gravitational clustering that are independent of the initial conditions. Numerical simulations as well as analytic work shows that power transfer leads to a universal power spectrum at late times, somewhat reminiscent of the existence of Kolmogorov spectrum in fluid turbulence.

1 Overview of the key issues and results

The statistical mechanics of systems dominated by gravity has close connections with areas of condensed matter physics, fluid mechanics, renormalization group etc. and poses an incredible challenge as regards the basic formulation. The ideas also find application in many different areas of astrophysics and cosmology, especially in the study of globular clusters, galaxies and gravitational clustering in the expanding universe. (For an overall review of statistical mechanics of gravitating systems, see [1], [2], [3]; review of gravitational clustering in expanding background is available in [4] and in several textbooks in cosmology [5]; for a sample of different attempts to understand these phenomena by different groups; see [6], [7], [8], [9], [10], [11] and the references cited therein.) It will be useful to begin with a broad overview and a description of the issues which will be addressed in this article.

In Newtonian theory, the gravitational force can be described as a gradient of a scalar potential and the evolution of a set of particles under the action of gravitational forces can be described the equations

\[ \dot{x}_i = -\nabla \phi(x_i, t); \quad \nabla^2 \phi = 4\pi G \sum_i m_i \delta_D(x - x_i) \] (1)
where $x_i$ is the position of the $i$-th particle, $m_i$ is its mass. For an isolated system with sufficiently large number of particles, it is useful to investigate whether some kind of statistical description of such a system is possible. Such a description, however, is complicated by the long range, unscreened, nature of gravitational force. If a self gravitating system is divided into two parts, the total energy of the system cannot be expressed as the sum of the gravitational energy of the components. The conventional results in statistical physics are based on the extensivity of the energy which is clearly invalid for gravitating systems. To construct the statistical description of such a system, one must begin with the construction of the micro-canonical ensemble describing such a system. If the Hamiltonian of the system is $H(p_i, q_i)$ then the volume $g(E)$ of the constant energy surface $H(p_i, q_i) = E$ will be of primary importance in the micro-canonical ensemble. The logarithm of this function will give the entropy $S(E) = \ln g(E)$ and the temperature of the system will be $T(E) \equiv \beta(E)^{-1} = \left(\frac{\partial S}{\partial E}\right)^{-1}$.

Systems for which a description based on canonical ensemble is possible, the Laplace transform of $g(E)$ with respect to a variable $\beta$ will give the partition function $Z(\beta)$. It is, however, trivial to show that gravitating systems of interest in astrophysics cannot be described by a canonical ensemble [1], [12], [13]. Virial theorem holds for such systems and we have $(2K + U) = 0$ where $K$ and $U$ are the total kinetic and potential energies of the system. This leads to $E = K + U = -K$; since the temperature of the system is proportional to the total kinetic energy, the specific heat will be negative: $C_V \equiv \langle \frac{\partial E}{\partial T} \rangle_V \propto \langle \frac{\partial E}{\partial K} \rangle < 0$. On the other hand, the specific heat of any system described by a canonical ensemble $C_V = \beta^2 < (\Delta E)^2 >$ will be positive definite. Thus one cannot describe self gravitating systems of the kind we are interested in by canonical ensemble.

One may attempt to find the equilibrium configuration for self gravitating systems by maximizing the entropy $S(E)$ or the phase volume $g(E)$. It is again easy to show that no global maximum for the entropy exists for classical point particles interacting via Newtonian gravity. To prove this, we only need to construct a configuration with arbitrarily high entropy which can be achieved as follows: Consider a system of $N$ particles initially occupying a region of finite volume in phase space and total energy $E$. We now move a small number of these particles (in fact, a pair of them, say, particles 1 and 2 will do) arbitrarily close to each other. The potential energy of interaction of these two particles, $-Gm_1m_2/r_{12}$, will become arbitrarily high as $r_{12} \to 0$. Transferring some of this energy to the rest of the particles, we can increase their kinetic energy without limit. This will clearly increase the phase volume occupied by the system without bound. This argument can be made more formal by dividing the original system into a small, compact core and a large diffuse halo and allowing the core to collapse and transfer the energy to the halo.

The absence of the global maximum for entropy — as argued above — depends on the idealization that there is no short distance cut-off in the interaction of the particles, so that we could take the limit $r_{12} \to 0$. If we assume, instead, that each particle has a minimum radius $a$, then the typical lower bound to the gravitational potential energy contributed by a pair of particles will be $-Gm_1m_2/2a$. This will put an upper bound on the amount of energy that can be made available to the rest of the system.

We have also assumed that part of the system can expand without limit — in
the sense that any particle with sufficiently large energy can move to arbitrarily large distances. In real life, no system is completely isolated and eventually one has to assume that the meandering particle is better treated as a member of another system. One way of obtaining a truly isolated system is to confine the system inside a spherical region of radius \( R \) with, say, reflecting wall. (Most of our discussion is confined to 3-dimensions and the situation is different in 2-dimensions; see e.g [14])

The two cut-offs \( a \) and \( R \) will make the upper bound on the entropy finite, but even with the two cut-offs, the primary nature of gravitational instability cannot be avoided. The basic phenomenon described above (namely, the formation of a compact core and a diffuse halo) will still occur since this is the direction of increasing entropy. Particles in the hot diffuse component will permeate the entire spherical cavity, bouncing off the walls and having a kinetic energy which is significantly larger than the potential energy. The compact core will exist as a gravitationally bound system with very little kinetic energy. A more formal way of understanding this phenomena is based on the virial theorem for a system with a short distance cut-off confined to a sphere of volume \( V \). In this case, the virial theorem will read as [2]

\[
2T + U = 3PV + \Phi
\]  

(2)

where \( P \) is the pressure on the walls and \( \Phi \) is the correction to the potential energy arising from the short distance cut-off. This equation can be satisfied in essentially three different ways. If \( T \) and \( U \) are significantly higher than \( 3PV \), then we have \( 2T + U \approx 0 \) which describes a self gravitating systems in standard virial equilibrium but not in the state of maximum entropy. If \( T \gg U \) and \( 3PV \gg \Phi \), one can have \( 2T \approx 3PV \) which describes an ideal gas with no potential energy confined to a container of volume \( V \); this will describe the hot diffuse component at late times. If \( T \ll U \) and \( 3PV \ll \Phi \), then one can have \( U \approx \Phi \) describing the compact potential energy dominated core at late times. In general, the evolution of the system will lead to the production of the core and the halo and each component will satisfy the virial theorem in the form (2). Such an asymptotic state with two distinct phases [15] is quite different from what would have been expected for systems with only short range interaction. Considering its importance, I shall briefly describe in section 2 a toy model which captures the essential physics of the above system in an exactly solvable context.

The above discussion focussed on the existence of global maximum to the entropy and we proved that it does not exist in the absence of two cut-offs. It is, however, possible to have local extrema of entropy which are not global maxima. Intuitively, one would have expected the distribution of matter in the configuration which is a local extrema of entropy to be described by a Boltzmann distribution, with the density given by \( \rho(x) \propto \exp[-\beta\phi(x)] \) where \( \phi \) is the gravitational potential related to \( \rho \) by Poisson equation. This is indeed true; for a formal proof see [1]. This configuration is usually called the isothermal sphere (because it can be shown that, among all solutions to this equation, the one with spherical symmetry maximizes the entropy) and it is a local maximum of entropy. The second (functional) derivative of the entropy with respect to the configuration variables will determine whether the local extremum of entropy is a local maximum or a saddle point [16], [17].
The relevance of the long range of gravity in all the above phenomena can be understood by studying model systems with an attractive potential varying as \( r^{-\alpha} \) with different values for \( \alpha \). Such studies confirm the results and interpretation given above; (see [18] and references cited therein).

Let us now consider the situation in the context of an expanding background. There is considerable amount of observational evidence to suggest that one of the dominant energy densities in the universe is contributed by self-gravitating point particles. The smooth average energy density of these particles drive the expansion of the universe while any small deviation from the homogeneous energy density will cluster gravitationally. [For a review of cosmology from a contemporary perspective, see e.g., [19]] One of the central problems in cosmology is to describe the non linear phases of this gravitational clustering starting from a initial spectrum of density fluctuations. It is often enough (and necessary) to use a statistical description and relate different statistical indicators (like the power spectra, \( n^{th} \) order correlation functions ....) of the resulting density distribution to the statistical parameters (usually the power spectrum) of the initial distribution. The relevant scales at which gravitational clustering is non linear are less than about 10 Mpc (where 1 Mpc = \( 3 \times 10^{24} \) cm is the typical separation between galaxies in the universe) while the expansion of the universe has a characteristic scale of about few thousand Mpc. Hence, non linear gravitational clustering in an expanding universe can be adequately described by Newtonian gravity provided the rescaling of lengths due to the background expansion is taken into account. This is easily done by introducing a proper coordinate for the \( i^{th} \) particle \( r_i \), related to the comoving coordinate \( x_i \), by \( r_i = a(t)x_i \) with \( a(t) \) describing the stretching of length scales due to cosmic expansion. The Newtonian dynamics works with the proper coordinates \( r_i \) which can be translated to the behaviour of the comoving coordinate \( x_i \) by this rescaling. [This implies that, for all practical purposes, we are still in the domain of Newtonian gravity. There is a far deeper connection between thermodynamics and gravity [20] in the general relativistic domain which we will not discuss in these lectures.]

As to be expected, cosmological expansion completely changes the nature of the problem because of several new factors which come in: (a) The problem has now become time dependent and it will be pointless to look for equilibrium solutions in the conventional sense of the word. (b) On the other hand, the expansion of the universe has a civilizing influence on the particles and acts counter to the tendency of gravity to make systems unstable. (c) In any small local region of the universe, one would assume that the conclusions describing a finite gravitating system will still hold true approximately. In that case, particles in any small sub region will be driven towards configurations of local extrema of entropy (say, isothermal spheres) and towards global maxima of entropy (say, core-halo configurations).

An extra feature comes into play as regards the expanding halo from any sub region. The expansion of the universe acts as a damping term in the equations of motion and drains the particles of their kinetic energy — which is essentially the lowering of temperature of any system participating in cosmic expansion. This, in turn, helps gravitational clustering since the potential wells of nearby sub regions can capture particles in the expanding halo of one region when the kinetic energy of the expanding halo has been sufficiently reduced.

The actual behaviour of the system will, of course, depend on the form of
However, for understanding the nature of clustering, one can take $a(t) \propto t^{2/3}$ which describes a matter dominated universe with critical density. Such a power law has the advantage that there is no intrinsic scale in the problem. Since Newtonian gravitational force is also scale free, one would expect some scaling relations to exist in the pattern of gravitational clustering. Converting this intuitive idea into a concrete mathematical statement turns out to be non trivial and difficult.

It is clear that cosmological expansion introduces several new factors into the problem when compared with the study of statistical mechanics of isolated gravitating systems. (For a general review of statistical mechanics of gravitating systems, see [1]. For a sample of different approaches, see [21] and the references cited therein. Review of gravitational clustering in expanding background is also available in several textbooks in cosmology [5, 22].) Though this problem can be tackled in a ‘practical’ manner using high resolution numerical simulations (for a review, see [23]), such an approach hides the physical principles which govern the behaviour of the system. To understand the physics, it is necessary to attack the problem from several directions using analytic and semi analytic methods. Several such attempts exist in the literature based on Zeldovich(like) approximations [24], path integral and perturbative techniques [25], nonlinear scaling relations [26] and many others. In spite of all these it is probably fair to say that we still do not have a clear analytic grasp of this problem, mainly because each of these approximations have different domains of validity and do not arise from a central paradigm.

I propose to attack the problem from a different angle, which has not received much attention in the past. The approach begins from the dynamical equation for the density contrast in the Fourier space and casts it as an integro-differential equation. Though this equation is known in the literature (see, e.g. [22]), it has received very little attention because it is not ‘closed’ mathematically; that is, it involves variables which are not natural to the formalism and thus further progress is difficult. I will, however, argue that there exists a natural closure condition for this equation based on Zeldovich approximation thereby allowing us to write down a closed integro-differential equation for the gravitational potential in the Fourier space.

It turns out that this equation can form the basis for several further investigations some of which are described in ref. [27] and in the second reference in [1]. Here I will concentrate on just two specific features, centered around the following issues:

- If the initial power spectrum is sharply peaked in a narrow band of wavelengths, how does the evolution transfer the power to other scales? In particular, does the non linear evolution in the case of gravitational interactions lead to a universal power spectrum (like the Kolmogorov spectrum in fluid turbulence)?
- What is the nature of the time evolution at late stages? Does the gravitational clustering at late stages wipe out the memory of initial conditions and evolve in a universal manner?

Fair amount of progress can be made as regards these questions using the integro-differential equation mentioned above and some of these aspects will be discussed in detail.
2 Phases of the self gravitating system

As described in section 1 the statistical mechanics of finite, self gravitating, systems have the following characteristic features: (a) They exhibit negative specific heat while in virial equilibrium. (b) They are inherently unstable to the formation of a core-halo structure and global maximum for entropy does not exist without cut-offs at short and large distances. (c) They can be broadly characterized by two phases — one of which is compact and dominated by potential energy while the other is diffuse and behaves more or less like an ideal gas. The purpose of this section is to describe a simple toy model which exhibits all these features and mimics a self gravitating system [1].

Consider a system with two particles described by a Hamiltonian of the form

$$H(P, Q, p, r) = \frac{P^2}{2M} + \frac{p^2}{2\mu} - \frac{Gm^2}{r}$$  \hspace{1cm} (3)

where \((Q, P)\) are coordinates and momenta of the center of mass, \((r, p)\) are the relative coordinates and momenta, \(M = 2m\) is the total mass, \(\mu = m/2\) is the reduced mass and \(m\) is the mass of the individual particles. This system may be thought of as consisting of two particles (each of mass \(m\)) interacting via gravity. We shall assume that the quantity \(r\) varies in the interval \((a, R)\). This is equivalent to assuming that the particles are hard spheres of radius \(a/2\) and that the system is confined to a spherical box of radius \(R\). We will study the “statistical mechanics” of this simple toy model.

To do this, we shall start with the volume \(g(E)\) of the constant energy surface \(H = E\). Straightforward calculation gives

$$g(E) = AR^3 \int_a^{r_{\max}} r^2 dr \left[ E + \frac{Gm^2}{r} \right]^2.$$  \hspace{1cm} (4)

where \(A = (64\pi^5m^3/3)\). The range of integration in (4) should be limited to the region in which the expression in the square brackets is positive. So we should use \(r_{\max} = (Gm^2/|E|)\) if \((-Gm^2/a) < E < (-Gm^2/R)\), and use \(r_{\max} = R\) if \((-Gm^2/R) < E < +\infty\). Since \(H \geq (-Gm^2/a)\), we trivially have \(g(E) = 0\) for \(E < (-Gm^2/a)\). The constant \(A\) is unimportant for our discussions and hence will be omitted from the formulas hereafter. The integration in (4) gives the following result:

$$\frac{g(E)}{(Gm^2)^3} = \begin{cases} \frac{R^3}{3}(-E)^{-1} \left( 1 + \frac{RE}{Gm^2} \right)^3, & (-Gm^2/a) < E < (-Gm^2/R) \\ \frac{R^3}{3}(-E)^{-1} \left[ (1 + \frac{RE}{Gm^2})^3 - (1 + \frac{aE}{Gm^2})^3 \right], & (-Gm^2/R) < E < \infty. \end{cases}$$  \hspace{1cm} (5)

This function \(g(E)\) is continuous and smooth at \(E = (-Gm^2/R)\). We define the entropy \(S(E)\) and the temperature \(T(E)\) of the system by the relations

$$S(E) = \ln g(E); \quad T^{-1}(E) = \beta(E) = \frac{\partial S(E)}{\partial E}.$$  \hspace{1cm} (6)

All the interesting thermodynamic properties of the system can be understood from the \(T(E)\) curve.
Consider first the case of low energies with \((-Gm^2/a) < E < (-Gm^2/R)\). Using (5) and (6) one can easily obtain \(T(E)\) and write it in the dimensionless form as

\[
t(\epsilon) = \left[ \frac{3}{1 + \epsilon} - \frac{1}{\epsilon} \right]^{-1}
\]

(7)

where we have defined \(t = (aT/Gm^2)\) and \(\epsilon = (aE/Gm^2)\).

This function exhibits the peculiarities characteristic of gravitating systems. At the lowest energy admissible for our system, which corresponds to \(\epsilon = -1\), the temperature \(t\) vanishes. This describes a tightly bound low temperature phase of the system with negligible random motion. The \(t(\epsilon)\) is clearly dominated by the first term of (7) for \(\epsilon \approx -1\). As we increase the energy of the system, the temperature increases, which is the normal behaviour for a system. This trend continues up to

\[
\epsilon = \epsilon_1 = -\frac{1}{2}(\sqrt{3} - 1) \simeq -0.36
\]

(8)
at which point the \(t(\epsilon)\) curve reaches a maximum and turns around. As we increase the energy further the temperature decreases. The system exhibits negative specific heat in this range.

Equation (7) is valid from the minimum energy \((-Gm^2/a)\) all the way up to the energy \((-Gm^2/R)\). For realistic systems, \(R \gg a\) and hence this range is quite wide. For a small region in this range, [from \((-Gm^2/a)\) to \((-0.36Gm^2/a)\)] we have positive specific heat; for the rest of the region the specific heat is negative. The positive specific heat region owes its existence to the nonzero short distance cutoff. If we set \(a = 0\), the first term in (7) will vanish; we will have \(t \propto (-\epsilon^{-1})\) and negative specific heat in this entire domain.

For \(E \geq (-Gm^2/R)\), we have to use the second expression in (5) for \(g(E)\). In this case, we get:

\[
t(\epsilon) = \left[ \frac{3}{(1 + \epsilon)^3} \frac{[(1 + \epsilon)^2 - \frac{K}{a}(1 + \frac{K\epsilon}{a})^2]}{(1 + \epsilon)^3 - (1 + \frac{K\epsilon}{a})^3} - \frac{1}{\epsilon} \right]^{-1}
\]

(9)

This function, of course, matches smoothly with (7) at \(\epsilon = -(a/R)\). As we increase the energy, the temperature continues to decrease for a little while, exhibiting negative specific heat. However, this behaviour is soon halted at some \(\epsilon = \epsilon_2\), say. The \(t(\epsilon)\) curve reaches a minimum at this point, turns around, and starts increasing with increasing \(\epsilon\). We thus enter another (high-temperature) phase with positive specific heat. From (11) it is clear that \(t \simeq (1/2)\epsilon\) for large \(\epsilon\). (Since \(E = (3/2)NkT\) for an ideal gas, we might have expected to find \(t \simeq (1/3)\epsilon\) for our system with \(N = 2\) at high temperatures. This is indeed what we would have found if we had defined our entropy as the logarithm of the volume of the phase space with \(H \leq E\). With our definition, the energy of the ideal gas is actually \(E = [(3/2)N - 1]kT\); hence we get \(t = (1/2)\epsilon\) when \(N = 2\). The form of the \(t(\epsilon)\) for \((a/R) = 10^{-4}\) is shown in figure 1 by the dashed curve. The specific heat is positive along the portions AB and CD and is negative along BC.

The overall picture is now clear. Our system has two natural energy scales: \(E_1 = (-Gm^2/a)\) and \(E_2 = (-Gm^2/R)\). For \(E \gg E_2\), gravity is not strong enough to keep \(r < R\) and the system behaves like a gas confined by the container; we have a high temperature phase with positive specific heat. As we lower
the energy to \( E \simeq E_2 \), the effects of gravity begin to be felt. For \( E_1 < E < E_2 \),
the system is unaffected by either the box or the short distance cutoff; this is
the domain dominated entirely by gravity and we have negative specific heat.
As we go to \( E \simeq E_1 \), the hard core nature of the particles begins to be felt and
the gravity is again resisted. This gives rise to a low temperature phase with
positive specific heat.

We can also consider the effect of increasing \( R \), keeping \( a \) and \( E \) fixed. Since
we imagine the particles to be hard spheres of radius \((a/2)\), we should only
consider \( R > 2a \). It is amusing to note that, if \( 2 < (R/a) < (\sqrt{3} + 1) \), there is
no region of negative specific heat. As we increase \( R \), this negative specific heat
region appears and increasing \( R \) increases the range over which the specific heat
is negative. Suppose a system is originally prepared with some \( E \) and \( R \) values
such that the specific heat is positive. If we now increase \( R \), the system may
find itself in a region of negative specific heat. This suggests the possibility that
an instability may be triggered in a constant energy system if its radius increases
beyond a critical value. We will see later that this is indeed true.
Since systems described by canonical distribution cannot exhibit negative specific heat, it follows that canonical distribution will lead to a very different physical picture for this range of (mean) energies \( E_1 < E < E_2 \). It is, therefore, of interest to look at our system from the point of view of canonical distribution by computing the partition function. In the partition function

\[
Z(\beta) = \int d^3Pd^3pd^3Qd^3r \exp(-\beta H)
\]  

(10)

the integrations over \( P, p \) and \( Q \) can be performed trivially. Omitting an overall constant which is unimportant, we can write the answer in the dimensionless form as

\[
Z(t) = t^3 \frac{R}{a}^3 \int_1^{R/a} dx^2 \exp \left( \frac{1}{x t} \right)
\]  

(11)

where \( t \) is the dimensionless temperature defined in (7). Though this integral cannot be evaluated in closed form, all the limiting properties of \( Z(\beta) \) can be easily obtained from (11).

The integrand in (11) is large for both large and small \( x \) and reaches a minimum for \( x = x_m = (1/2t) \). At high temperatures, \( x_m < 1 \) and hence the minimum falls outside the domain of integration. The exponential contributes very little to the integral and we can approximate \( Z \) adequately by

\[
Z \approx t^3 \left( \frac{R}{a} \right)^3 \int_1^{R/a} dx^2 \left[ 1 + \frac{2x_m}{x} \right] = t^3 \left( \frac{R}{a} \right)^6 \left( 1 + \frac{3a}{2Rt} \right).
\]

(12)

On the other hand, if \( x_m > 1 \) the minimum lies between the limits of the integration and the exponential part of the curve dominates the integral. We can easily evaluate this contribution by a saddle point approach, and obtain

\[
Z \approx \left( \frac{R}{a} \right)^3 t^4 (1 - 2t)^{-1} \exp \left( \frac{1}{t} \right).
\]

(13)

As we lower the temperature, making \( x_m \) cross 1 from below, the contribution switches over from (12) to (13). The transition is exponentially sharp. The critical temperature at which the transition occurs can be estimated by finding the temperature at which the two contributions are equal. This occurs at

\[
t_c = \frac{1}{3} \ln(R/a).
\]

(14)

For \( t < t_c \), we should use (13) and for \( t > t_c \), we should use (12).

Given \( Z(\beta) \) all thermodynamic functions can be computed. In particular, the mean energy of the system is given by \( E(\beta) = -\langle \partial \ln Z/\partial \beta \rangle \). This relation can be inverted to give the \( T(E) \) which can be compared with the \( T(E) \) obtained earlier using the micro-canonical distribution. From (12) and (13) we get,

\[
\epsilon(t) = \frac{aE}{Gm^2} = 4t - 1
\]

for \( t < t_c \) and

\[
\epsilon(t) = 3t - \frac{3a}{2R}
\]

for \( t > t_c \).

(15)

(16)
for \( t > t_c \). Near \( t \approx t_c \), there is a rapid variation of the energy and we cannot use either asymptotic form. The system undergoes a phase transition at \( t = t_c \) absorbing a large amount of energy

\[
\Delta \epsilon \approx \left(1 - \frac{1}{3 \ln(R/a)}\right). \tag{17}
\]

The specific heat is, of course, positive throughout the range. This is to be expected because canonical ensemble cannot lead to negative specific heats.

The \( T - E \) curves obtained from the canonical (unbroken line) and micro-canonical (dashed line) distributions are shown in figure [1]. For convenience, we have rescaled the \( T - E \) curve of the micro-canonical distribution so that \( \epsilon \approx 3t \) asymptotically.) At both very low and very high temperatures, the canonical and micro-canonical descriptions match. The crucial difference occurs at the intermediate energies and temperatures. Micro-canonical description predicts negative specific heat and a reasonably slow variation of energy with temperature. Canonical description, on the other hand, predicts a phase transition with rapid variation of energy with temperature. Such phase transitions are accompanied by large fluctuations in the energy, which is the main reason for the disagreement between the two descriptions [1], [12], [13].

Numerical analysis of more realistic systems confirm all these features. Such systems exhibit a phase transition from the diffuse virialized phase to a core dominated phase when the temperature is lowered below a critical value [15]. The transition is very sharp and occurs at nearly constant temperature. The energy released by the formation of the compact core heats up the diffuse halo component.

### 3 Isothermal sphere

While a global maximum to the entropy does not exist in the absence of two cut-offs, it is, however, possible to have local extrema of entropy which are not global maxima. Such a configuration is described by a Boltzmann distribution, with the density given by \( \rho(x) \propto \exp[-\beta \phi(x)] \) where \( \phi \) is the gravitational potential related to \( \rho \) by Poisson equation (for a formal proof see [1]). Among all solutions to this equation, since the one with spherical symmetry maximizes the entropy this configuration is usually called the isothermal sphere. The extremum condition for the entropy, is equivalent to the differential equation for the gravitational potential:

\[
\nabla^2 \phi = 4\pi G \rho_c e^{-\beta [\phi(x) - \phi(0)]} \tag{18}
\]

Given the solution to this equation, all other quantities can be determined. As we shall see, this system shows several peculiarities.

It is convenient to introduce the length, mass and energy scale by the definitions

\[
L_0 \equiv (4\pi G \rho_c \beta)^{1/2}, \quad M_0 = 4\pi \rho_c L_0^3, \quad \phi_0 \equiv \beta^{-1} = \frac{GM_0}{L_0} \tag{19}
\]

where \( \rho_c = \rho(0) \). All other physical variables can be expressed in terms of the dimensionless quantities

\[
x \equiv \frac{r}{L_0}, \quad n \equiv \frac{\rho}{\rho_c}, \quad m = \frac{M(r)}{M_0}, \quad y \equiv \beta [\phi - \phi(0)]. \tag{20}
\]
In terms of $y(x)$ the isothermal equation (18) becomes

$$\frac{1}{x^2} \frac{d}{dx} \left( x^2 \frac{dy}{dx} \right) = e^{-y}$$

(21)

with the boundary condition $y(0) = y'(0) = 0$. Let us consider the nature of solutions to this equation.

By direct substitution, we see that

$$n = \frac{2}{x^2}, \quad m = 2x, \quad y = 2\ln x$$

satisfies these equations. This solution, however, is singular at the origin and hence is not physically admissible. The importance of this solution lies in the fact that other (physically admissible) solutions tend to this solution [1], [28] for large values of $x$. This asymptotic behavior of all solutions shows that the density decreases as $(1/r^2)$ for large $r$ implying that the mass contained inside a sphere of radius $r$ increases as $M(r) \propto r$ at large $r$. To find physically useful solutions, it is necessary to assume that the solution is cutoff at some radius $R$. For example, one may assume that the system is enclosed in a spherical box of radius $R$. In what follows, it will be assumed that the system has some cutoff radius $R$.

The equation (21) is invariant under the transformation $y \rightarrow y + a; \quad x \rightarrow kx$ with $k^2 = e^a$. This invariance implies that, given a solution with some value of $y(0)$, we can obtain the solution with any other value of $y(0)$ by simple rescaling. Therefore, only one of the two integration constants in (21) is really non-trivial. Hence it must be possible to reduce the degree of the equation from two to one by a judicious choice of variables [28]. One such set of variables are:

$$v \equiv \frac{m}{x}; \quad u \equiv \frac{nx^3}{m} = \frac{nx^2}{v}.$$  

(22)

In terms of $v$ and $u$, equation (18) becomes

$$\frac{u}{v} \frac{dv}{du} = -\frac{(u - 1)}{(u + v - 3)}.$$  

(23)

The boundary conditions $y(0) = y'(0) = 0$ translate into the following: $v$ is zero at $u = 3$, and $(dv/du) = -5/3$ at $(3,0)$. The solution $v(u)$ has to be obtained numerically: it is plotted in figure 2 as the spiraling curve. The singular points of this differential equation are given by the intersection of the straight lines $u = 1$ and $u + v = 3$ on which, the numerator and denominator of the right hand side of (23) vanishes; that is, the singular point is at $u_s = 1, v_s = 2$ corresponding to the solution $n = (2/x^2), m = 2x$. It is obvious from the nature of the equations that the solutions will spiral around the singular point.

The nature of the solution shown in figure 2 allows us to put an interesting bounds on physical quantities including energy. To see this, we shall compute the total energy $E$ of the isothermal sphere. The potential and kinetic energies are

$$U = -\int_0^R \frac{GM(r)}{r} \frac{dM}{dr} dr = -\frac{GM^2}{L_0} \int_0^{x_0} mn xd x$$

$$K = \frac{3M}{2} \beta = \frac{3GM^2}{2L_0} m(x_0) = \frac{GM^2}{L_0} \frac{3}{2} \int_0^{x_0} n x^2 dx$$  

(24)

where $x_0 = R/L_0$. The total energy is, therefore,

$$E = K + U = \frac{GM^2}{2L_0} \int_0^{x_0} dx (3nx^2 - 2mn x)$$

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\[= \frac{GM^2}{2L_0} \int_0^{x_0} dx \frac{d}{dx}(2nx^3 - 3m) = \frac{GM^2}{L_0} (n_0x_0^3 - \frac{3}{2}m_0) \tag{25}\]

where \(n_0 = n(x_0)\) and \(m_0 = m(x_0)\). The dimensionless quantity \((RE/GM^2)\) is given by

\[\lambda = \frac{RE}{GM^2} = \frac{1}{v_0} \left( u_0 - \frac{3}{2} \right). \tag{26}\]

Note that the combination \((RE/GM^2)\) is a function of \((u, v)\) alone. Let us now consider the constraints on \(\lambda\). Suppose we specify some value for \(\lambda\) by specifying \(R, E\) and \(M\). Then such an isothermal sphere must lie on the curve

\[v = \frac{1}{\lambda} \left( u - \frac{3}{2} \right); \quad \lambda \equiv \frac{RE}{GM^2} \tag{27}\]

which is a straight line through the point \((1.5, 0)\) with the slope \(\lambda^{-1}\). On the other hand, since all isothermal spheres must lie on the \(u-v\) curve, an isothermal sphere can exist only if the line in (27) intersects the \(u-v\) curve.

Figure 2: Bound on \(RE/GM^2\) for the isothermal sphere

For large positive \(\lambda\) (positive \(E\)) there is just one intersection. When \(\lambda = 0\), (zero energy) we still have a unique isothermal sphere. (For \(\lambda = 0\), equation (27) is a vertical line through \(u = 3/2\).) When \(\lambda\) is negative (negative \(E\)), the line can cut the \(u-v\) curve at more than one point; thus more than one isothermal sphere can exist with a given value of \(\lambda\). [Of course, specifying \(M, R, E\) individually will remove this non-uniqueness]. But as we decrease \(\lambda\) (more and more negative \(E\)) the line in (27) will slope more and more to the left; and when \(\lambda\) is smaller than a critical value \(\lambda_c\), the intersection will cease to exist. Thus no isothermal sphere can exist if \((RE/GM^2)\) is below a critical value \(\lambda_c\).\(^1\) This fact follows immediately from the nature of \(u-v\) curve and

\(^1\)This derivation is due to the author [17]. It is surprising that Chandrasekhar, who has
equation (27). The value of $\lambda_c$ can be found from the numerical solution in figure. It turns out to be about $(-0.335)$.

The isothermal sphere has a special status as a solution to the mean field equations. Isothermal spheres, however, cannot exist if $(RE/GM^2) < -0.335$. Even when $(RE/GM^2) > -0.335$, the isothermal solution need not be stable. The stability of this solution can be investigated by studying the second variation of the entropy. Such a detailed analysis shows that the following results are true [16], [29], [17]. (i) Systems with $(RE/GM^2) < -0.335$ cannot evolve into isothermal spheres. Entropy has no extremum for such systems. (ii) Systems with $(RE/GM^2) > -0.335$ and $(\rho(0) > 709 \rho(R))$ can exist in a meta-stable (saddle point state) isothermal sphere configuration. Here $\rho(0)$ and $\rho(R)$ denote the densities at the center and edge respectively. The entropy extrema exist but they are not local maxima. (iii) Systems with $(RE/GM^2) > -0.335$ and $(\rho(0) < 709 \rho(R))$ can form isothermal spheres which are local maximum of entropy.

4 An integral equation to describe nonlinear gravitational clustering

Let us next consider the gravitational clustering of a system of collision-less point particles in an expanding universe which poses several challenging theoretical questions. Though the problem can be tackled in a ‘practical’ manner using high resolution numerical simulations, such an approach hides the physical principles which govern the behaviour of the system. To understand the physics, it is necessary that we attack the problem from several directions using analytic and semi analytic methods. These sections will describe such attempts and will emphasize the semi analytic approach and outstanding issues, rather than more well established results.

The expansion of the universe sets a natural length scale (called the Hubble radius) $d_H = c(\dot{a}/a)^{-1}$ which is about 4000 Mpc in the current universe. In any region which is small compared to $d_H$ one can set up an unambiguous coordinate system in which the proper coordinate of a particle $\mathbf{r}(t) = a(t)\mathbf{x}(t)$ satisfies the Newtonian equation $\ddot{\mathbf{r}} = -\nabla\Phi$ where $\Phi$ is the gravitational potential. The Lagrangian for such a system of particles is given by

$$L = \sum_i \left[ \frac{1}{2} m_i \dot{r}_i^2 + \frac{G}{2} \sum_j \frac{m_i m_j}{|r_i - r_j|} \right]$$

(28)

In the term

$$\frac{1}{2} \dot{r}_i^2 = \frac{1}{2} \left[ a^2 x_i^2 + \dot{a}^2 x_i^2 + a\ddot{a} \frac{dx_i^2}{dt} \right] = \frac{1}{2} \left[ a^2 \dot{x}_i^2 - a\ddot{a} \dot{x}_i^2 + \frac{da}{dt} \frac{dx_i}{dt} \right]$$

(29)

we note that: (i) the total time derivative can be ignored; (ii) using $\dot{a} = -(4\pi G/3)\rho_0$, the term $\Phi_{FRW} = -(1/2)a\ddot{a} x_i^2 = (2\pi G/3)\rho_0(x_i^2/a)$ can be identified as the gravitational potential due to the uniform Friedman background of worked out the isothermal sphere in uv coordinates as early as 1939, missed discovering the energy bound shown in figure 2. Chandrasekhar [28] has the uv curve but does not over-plot lines of constant $\lambda$. If he had done that, he would have discovered Antonov instability decades before Antonov did [16].

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density $\rho_b = \rho_0 / a^3$. Hence the Lagrangian can be expressed as

$$L = \sum_i m_i \left[ \frac{1}{2} a^2 \dot{x}_i^2 + \phi(t, x_i) \right] = T - U$$

(30)

where

$$\phi = -\frac{G}{2a} \sum_j \frac{m_j}{|x_i - x_j|} - \frac{2\pi G \rho_0 x_i^2}{3a}$$

(31)

is the difference between the total potential and the potential for the background Friedmann universe $\Phi_{FRW}$. Varying the Lagrangian in Eq. (30) with respect to $x_i$, we get the equation of motion to be:

$$\ddot{x} + 2\frac{\dot{a}}{a} \dot{x} = -\frac{1}{a^2} \nabla_x \phi$$

(32)

Since $\phi$ is the gravitational potential generated by the perturbed mass density, it satisfies the equation with the source $(\rho - \rho_b) \equiv \rho_b \delta$:

$$\nabla_x^2 \phi = 4\pi G \rho_b a^2 \delta$$

(33)

Equation (32) and Eq. (33) govern the nonlinear gravitational clustering in an expanding background.

Usually one is interested in the evolution of the density contrast $\delta(t, x)$ rather than in the trajectories. Since the density contrast can be expressed in terms of the trajectories of the particles, it should be possible to write down a differential equation for $\delta(t, x)$ based on the equations for the trajectories $x(t)$ derived above. It is, however, somewhat easier to write down an equation for $\delta_k(t)$ which is the spatial Fourier transform of $\delta(t, x)$. To do this, we begin with the fact that the density $\rho(x, t)$ due to a set of point particles, each of mass $m$, is given by

$$\rho(x, t) = m \frac{a}{a^3(t)} \sum_i \delta_D[x - x_i(t)]$$

(34)

where $x_i(t)$ is the trajectory of the $i$th particle and $\delta_D$ is the Dirac delta function.

The density contrast $\delta(x, t)$ is related to $\rho(x, t)$ by

$$1 + \delta(x, t) \equiv \frac{\rho(x, t)}{\rho_b} = \frac{V}{N} \sum_i \delta_D[x - x_i(t)] = \int d^3q \delta_D[x - x_T(t, q)].$$

(35)

In arriving at the last equality we have taken the continuum limit by: (i) replacing $x_i(t)$ by $x_T(t, q)$ where $q$ stands for a set of parameters (like the initial position, velocity etc.) of a particle; for simplicity, we shall take this to be initial position. The subscript ‘$T$’ is just to remind ourselves that $x_T(t, q)$ is the trajectory of the particle. (ii) replacing $(V/N)$ by $d^3q$ since both represent volume per particle. Fourier transforming both sides we get

$$\delta_k(t) \equiv \int d^3x e^{-ik\cdot x} \delta(x, t) = \int d^3q \exp[-ik\cdot x_T(t, q)] - (2\pi)^3 \delta_D(k)$$

(36)

Differentiating this expression, and using Eq. (32) for the trajectories give one can obtain an equation for $\delta_k$ (see e.g. ref. [27]; Eq.10). The structure of this
equation can be simplified if we use the perturbed gravitational potential (in Fourier space) $\phi_k$ related to $\delta_k$ by

$$\delta_k = -\frac{k^2 \phi_k}{4\pi G \rho_b a^2} = -\left(\frac{k^2 a}{4\pi G \rho_0}\right) \phi_k = -\left(\frac{2}{3H_0^2}\right) k^2 a \phi_k$$  \hfill (37)

In terms of $\phi_k$ the exact evolution equation reads as:

$$\ddot{\phi}_k + 4 \frac{\dot{a}}{a} \dot{\phi}_k = -\frac{1}{2a^2} \int \frac{d^3p}{(2\pi)^3} \phi_{\frac{1}{2}k+p} \phi_{\frac{1}{2}k-p} \left[ \left(\frac{k}{2}\right)^2 + p^2 - 2 \left(\frac{k}{2}\right)^2 \right] + \left(\frac{3H_0^2}{2}\right) \int \frac{d^3q}{a} \left(\frac{k}{k}\right)^2 e^{ik \cdot x}$$  \hfill (38)

where $x = x_T(t, q)$. Of course, this equation is not ‘closed’. It contains the velocities of the particles $\dot{x}_T$ and their positions explicitly in the second term on the right and one cannot — in general — express them in simple form in terms of $\phi_k$. As a result, it might seem that we are in no better position than when we started. I will now motivate a strategy to tame this term in order to close this equation. This strategy depends on two features:

(1) First, extremely nonlinear structures do not contribute to the right hand side of Eq.(38) though, of course, they contribute individually to the second term. More precisely, the right hand side of Eq.(38) will lead to a density contrast that will scale as $\delta_k \propto k^2$ if originally — in linear theory — $\delta_k \propto k^n$ with $n > 2$ as $k \to 0$. This leads to a $P \propto k^2$ tail in the power spectrum [22, 27]. (We will provide a derivation of this result in the next section.)

(ii) Second, we can use Zeldovich approximation to evaluate this term, once the above fact is realized. It is well known that, when the density contrasts are small, it grows as $\delta_k \propto a$ in the linear limit. One can easily show that such a growth corresponds to particle displacements of the form

$$x_T(a, q) = q - a \nabla \psi(q); \quad \psi \equiv (4\pi G \rho_0)^{-1} \phi$$  \hfill (39)

A useful approximation to describe the quasi linear stages of clustering is obtained by using the trajectory in Eq.(39) as an ansatz valid even at quasi linear epochs. In this approximation, (called Zeldovich approximation), the velocities $\dot{x}$ can be expressed in terms of the initial gravitational potential.

We now combine the two results mentioned above to obtain a closure condition for our dynamical equation. At any given moment of time we can divide the particles in the system into three different sets. First, there are particles which are already a part of virialized cluster in the nonlinear regime. Second set is made of particles which are completely unbound and are essentially contributing to power spectrum at the linear scales. The third set is made of all particles which cannot be put into either of these two baskets. Of these three, we know that the first two sets of particles do not contribute significantly to the right hand side of Eq.(38) so we will not incur any serious error in ignoring these particles in computing the right hand side. For the description of particles in the third set, the Zeldovich approximation should be fairly good. In fact, we can do slightly better than the standard Zeldovich approximation. We note that in Eq. (39) the velocities were taken to be proportional to the gradient of the initial gravitational potential. We can improve on this ansatz by taking the velocities to be given by the gradient of the instantaneous gravitational potential.
which has the effect of incorporating the influence of particles in bound clusters on the rest of the particles to certain extent.

Given this ansatz, it is straightforward to obtain a closed integro-differential equation for the gravitational potential. Direct calculation shows that the gravitational potential is described by the closed integral equation: (The details of this derivation can be found in ref. [27] and will not be repeated here.)

\[
\ddot{\phi}_k + 4\frac{\dot{a}}{a}\dot{\phi}_k = -\frac{1}{3a^2} \int \frac{d^3p}{(2\pi)^3} \phi_{k+p}\phi_{k-p} \left[ \frac{7}{8} k^2 + \frac{3}{2} p^2 - 5 \left( \frac{k \cdot p}{k} \right)^2 \right] \quad (40)
\]

This equation provides a powerful method for analyzing non-linear clustering since estimating Eq.(38) by Zeldovich approximation has a very large domain of applicability. In the next two sections, I will use this equation to study the transfer of power in gravitational clustering.

5 Inverse cascade in non-linear gravitational clustering: The \(k^4\) tail

There is an interesting and curious result which is characteristic of gravitational clustering that can be obtained directly from our Eq. (40). Consider an initial power spectrum which has very little power at large scales; more precisely, we shall assume that \(P(k) \propto k^n\) with \(n > 4\) for small \(k\) (i.e., the power dies faster than \(k^4\) for small \(k\)). If these large spatial scales are described by linear theory — as one would have normally expected — then the power at these scales can only grow as \(P \propto a^2 k^n\) and it will always be sub-dominant to \(k^4\). It turns out that this conclusion is incorrect. As the system evolves, small scale nonlinearities will develop in the system and — if the large scales have too little power intrinsically — then the long wavelength power will soon be dominated by the “\(k^4\)-tail” of the short wavelength power arising from the nonlinear clustering. This is a purely non linear effect which we shall now describe. (This result is known in literature [22, 27] but we will derive it from the formalism developed in the last section which adds fresh insight.)

A formal way of obtaining the \(k^4\) tail is to solve Eq. (40) for long wavelengths; i.e. near \(k = 0\). Writing \(\phi_k = \phi_k^{(1)} + \phi_k^{(2)} + \ldots\) where \(\phi_k^{(1)} = \phi_k^{(L)}\) is the time independent gravitational potential in the linear theory and \(\phi_k^{(2)}\) is the next order correction, we get from Eq. (40), the equation

\[
\ddot{\phi}_k^{(2)} + 4\frac{\dot{a}}{a}\dot{\phi}_k^{(2)} \cong -\frac{1}{3a^2} \int \frac{d^3p}{(2\pi)^3} \phi_{k+p}\phi_{k-p} G(k, p) \quad (41)
\]

where \(G(k, p) \equiv [(7/8)k^2 + (3/2)p^2 - 5(k \cdot p/k)^2].\) The solution to this equation is the sum of a solution to the homogeneous part [which decays as \(\phi \propto a^{-4} \propto t^{-8/3}\) giving \(\phi \propto t^{-5/3}\)] and a particular solution which grows as \(a\). Ignoring the decaying mode at late times and taking \(\phi_k^{(2)} = aC_k\) one can determine \(C_k\) from the above equation. Plugging it back, we find the lowest order correction to be,

\[
\phi_k^{(2)} \cong -\left( \frac{2a}{21H_0^2} \right) \int \frac{d^3p}{(2\pi)^3} \phi_{k+p}\phi_{k-p} G(k, p) \quad (42)
\]
Figure 3: The transfer of power to long wavelengths forming a $k^4$ tail is illustrated using simulation results. Power is injected in the form of a narrow peak at $L = 8$. Note that the $y$-axis is $(\Delta/a)$ so that there will be no change of shape of the power spectrum under linear evolution with $\Delta \propto a$. As time goes on a $k^4$ tail is generated purely due to nonlinear coupling between the modes. (Figure adapted from ref. [30].)

Near $k \simeq 0$, we have

$$\phi_{k=0}^{(2)} \simeq \frac{2a}{21H^2_0} \int \frac{d^3p}{(2\pi)^3} |\phi^{(L)}_p|^2 \left[ \frac{7}{8} k^2 + \frac{3}{2} p^2 - \frac{5(k \cdot p)^2}{k^2} \right]$$

$$= \frac{a}{126\pi^2 H^2_0} \int_0^\infty dp \int |\phi^{(L)}_p|^2$$

where $\phi^{(L)}_p$ is the power per logarithmic band in $k$. In linear theory $\Delta \propto a$ and this quantity should not change. The curves labelled by $a = 0.12$ to $a = 20.0$ show the effects of nonlinear evolution, especially the development of $k^4$ tail. (Actually one can do better. The formation can also reproduce the sub-harmonic at $L \simeq 4$ seen in Fig. 3 and other details; see ref. [27].)
6 Analogue of Kolmogorov spectrum for gravitational clustering

If power is injected at some scale $L$ into an ordinary viscous fluid, it cascades down to smaller scales because of the non linear coupling between different modes. The resulting power spectrum, for a wide range of scales, is well approximated by the Kolmogorov spectrum which plays a key, useful, role in the study of fluid turbulence. It is possible to obtain the form of this spectrum from fairly simple and general considerations though the actual equations of fluid turbulence are intractably complicated. Let us now consider the corresponding question for non linear gravitational clustering. If power is injected at a given length scale very early on, how does the dynamical evolution transfers power to other scales at late times? In particular, does the non linear evolution lead to an analogue of Kolmogorov spectrum with some level of universality, in the case of gravitational interactions?

Surprisingly, the answer is “yes”, even though normal fluids and collisionless self gravitating particles constitute very different physical systems. If power is injected at a given scale $L = 2\pi/k_0$ then the gravitational clustering transfers the power to both larger and smaller spatial scales. At large spatial scales the power spectrum goes as $P(k) \propto k^4$ as soon as non linear coupling becomes important. We have already seen this result in the previous section. More interestingly, the cascading of power to smaller scales leads to a universal pattern at late times just as in the case of fluid turbulence. This is because, Eq.(40) admits solutions for the gravitational potential of the form $\phi_k(t) = F(t)D(k)$ at late times when the initial condition is irrelevant; here $F(t)$ satisfies a non linear differential equation and $D(k)$ satisfies an integral equation. One can analyze the relevant equations analytically as well as verified the conclusions by numerical simulations. This study (the details of which can be found in ref.[31]) confirms that non linear gravitational clustering does lead to a universal power spectrum at late times if the power is injected at a given scale initially. (In cosmology there is very little motivation to study the transfer of power by itself and most of the numerical simulations in the past concentrated on evolving broad band initial power spectrum. So this result was missed out.)

Our aim is to look for late time scale free evolution of the system exploiting the fact Eq.(40) allows self similar solutions of the form $\phi_k(t) = F(t)D(k)$. Substituting this ansatz into Eq. (40) we obtain two separate equations for $F(t)$ and $D(k)$. It is also convenient at this stage to use the expansion factor $a(t) = (t/t_0)^{2/3}$ of the matter dominated universe as the independent variable rather than the cosmic time $t$. Then simple algebra shows that the governing equations are

$$a \frac{d^2F}{da^2} + \frac{7}{2} \frac{dF}{da} = -F^2$$

(44)

and

$$H_0^2 D_k = \frac{1}{3} \int \frac{d^3p}{(2\pi)^3} D_{2k+p}D_{2k-p}G(k,p)$$

(45)

Equation (44) governs the time evolution while Eq. (45) governs the shape of the power spectrum. (The separation ansatz, of course, has the scaling freedom $F \rightarrow \mu F, D \rightarrow (1/\mu)D$ which will change the right hand side of Eq. (44) to $-\mu F^2$ and the left hand side of Eq. (45) to $\mu H_0^2 D_k$. But, as to be expected, our
results will be independent of $\mu$; so we have set it to unity). Our interest lies in analyzing the solutions of Eq. (44) subject to the initial conditions $F(a_i) = \text{constant}$, $(dF/da)_i = 0$ at some small enough $a = a_i$.

Inspection shows that Eq. (44) has the exact solution $F(a) = (3/2)a^{-1}$. This, of course, is a special solution and will not satisfy the relevant initial conditions. However, Eq. (44) fortunately belongs to a class of non-linear equations which can be mapped to a homologous system. In such cases, the special power law solutions will arise as the asymptotic limit. (The example well known to astronomers is that of isothermal sphere [28]. Our analysis below has a close parallel.) To find the general behaviour of the solutions to Eq. (44), we will make the substitution $F(a) = (3/2)a^{-1}g(a)$ and change the independent variable from $a$ to $q = \log a$. Then Eq. (44) reduces to the form

$$\frac{d^2g}{dq^2} + \frac{1}{2} \frac{dg}{dq} + \frac{3}{2}g(g - 1) = 0 \quad (46)$$

This represents a particle moving in a potential $V(g) = (1/2)g^3 - (3/4)g^2$ under friction. For our initial conditions the motion will lead the “particle” to asymptotically come to rest at the stable minimum at $g = 1$ with damped oscillations. In other words, $F(a) \rightarrow (3/2)a^{-1}$ for large $a$ showing this is indeed the asymptotic solution. From the Poisson equation, it follows that $k^2\phi_k \propto (\delta_k/a)$ so that $\delta_k(a) \propto g(a)k^2D(k)$ giving a direct physical meaning to the function $g(a)$ as

![Figure 4: The solution to Eq. (46) plotted in $g-a$ plane. The function $g(a)$ asymptotically approaches unity with oscillations which are represented by the spiral in the right panel. The different curves in the left panel corresponds to the rescaling freedom in the initial conditions. One fiducial curve which was used to model the simulation is shown in the red. For more details, see ref. [31].](image-url)
Figure 5: Left panel: The results of the numerical simulation with an initial power spectrum which is a Gaussian peaked at $L = 24$. The y-axis gives $\Delta_k^2$, where $\Delta_k^2 = k^3 P/2\pi^2$ is the power per logarithmic band. The evolution generates a well known $k^4$ tail at large scales (see for example, [30]) and leads to cascading of power to small scales. Right panel: The simulation data is re-expressed by factoring out the time evolution function $g(a)$ obtained by integrating Eq. (46). The fact that the curves fall nearly on top of each other shows that the late time evolution is scale free and described by the ansatz discussed in the text. The rescaled spectrum is very well described by $P(k) \propto k^{-0.4}/(1 + (k/k_0)^{2.6})$ which is shown by the, completely overlapping, broken blue curve. For more details, see ref. [31].
the growth factor for the density contrast. The asymptotic limit \( g \approx 1 \) corresponds to a rather trivial case of \( \delta_k \) becoming independent of time. What will be more interesting — and accessible in simulations — will be the approach to this asymptotic solution. To obtain this, we introduce the variable

\[
    u = g + 2 \left( \frac{dg}{dq} \right)
\]

so that our system becomes homologous. It can be easily shown that we now get the first order form of the autonomous system to be

\[
    \frac{du}{dg} = -\frac{6g(g-1)}{u-g}
\]

The critical points of the system are at \((0, 0)\) and \((1, 1)\). Standard analysis shows that: (i) the \((0, 0)\) is an unstable critical point and the second one \((1, 1)\) is the stable critical point; (ii) for our initial conditions the solution spirals around the stable critical point.

Figure 4 (from ref. \[31\]) describes the solution in the \( g - a \) plane. The \( g(a) \) curves clearly approach the asymptotic value of \( g \approx 1 \) with superposed oscillations. The different curves in Fig. 4 are for different initial values which arise from the scaling freedom mentioned earlier. (The thick red line correspond to the initial conditions used in the simulations described below.). The solution \( g(a) \) describes the time evolution and solves the problem of determining asymptotic time evolution.

To test the correctness of these conclusions, we performed a high resolution simulation using the TreePM method \[32, 33\] and its parallel version \[34\] with \( 128^3 \) particles on a \( 128^3 \) grid. Details about the code parameters can be found in \[33\]. The initial power spectrum \( P(k) \) was chosen to be a Gaussian peaked at the scale of \( k_p = 2\pi/L_p \) with \( L_p = 24 \) grid lengths and with a standard deviation \( \Delta k = 2\pi/L_{box} \), where \( L_{box} = 128 \) is the size of one side of the simulation volume. The amplitude of the peak was taken such that \( \Delta^2 k \equiv k^3 P(k)/2\pi^2 \) where \( P = |\delta_k|^2 \) is the power spectrum of density fluctuations obtained from the simulations is shown in Fig. 6 (left panel). In the right panel, we have rescaled the \( \Delta_k \), using the appropriate solution \( g(a) \). The fact that the curves fall on top of each other shows that the late time evolution indeed scales as \( g(a) \) within numerical accuracy. A reasonably accurate fit for \( g(a) \) at late times used in this figure is given by \( g(a) \propto a(1-0.3 \ln a) \). The key point to note is that the asymptotic time evolution is essentially \( \delta(a) \propto a \) except for a logarithmic correction, even in highly nonlinear scales. (This was first noticed from somewhat lower resolution simulations in \[30\].) Since the evolution at linear scales is always \( \delta \propto a \), this allows for a form invariant evolution of power spectrum at all scales. Gravitational clustering evolves towards this asymptotic state.

To the lowest order of accuracy, the power spectrum at this range of scales is approximated by the mean index \( n \approx -1 \) with \( P(k) \propto k^{-1} \). A better fit to the power spectrum in Fig. 5 is given by

\[
    P(k) \propto \frac{k^{-0.4}}{1 + (k/k_0)^{2.6}}, \quad \frac{2\pi}{k_0} \approx 4.5
\]

This fit is shown by the broken blue line in the figure which completely overlaps with the data and is barely visible. (Note that this fit is applicable only at
$L < L_p$ since the $k^4$ tail will dominate scales to the right of the initial peak; see the discussion in [30]. At nonlinear scales $P(k) \propto k^{-3}$ making $\Delta_k$ flat, as seen in Fig. [3] (This is not a numerical artifact and we have sufficient dynamic range in the simulation to ascertain this.) At quasi linear scales $P(k) \propto k^{-0.4}$. The effective index of the power spectrum varies between $-3$ and $-0.4$ in this range of scales.

What could be a possible interpretation for this behaviour? It is difficult to provide a simple but precise answer but one possible line of reasoning is as follows: In the case of viscous fluids, the energy is dissipated at the smallest scales as heat. In steady state, energy cannot accumulate at any intermediate scale and hence the rate of flow of energy from one scale to the next (lower) scale must be a constant. This constancy immediately leads to Kolmogorov spectrum. In the case of gravitating particles, there is no dissipation and each scale will evolve towards virial equilibrium. At any given time, the power would have cascaded down only up to some scale $l_{min}(t)$ which it, of course, is a decreasing function of time. So, at time $t$ we expect very little power for $1 < kl_{min}(t)$ and a $k^4$ tail for $kL_p < 1$, say. The really interesting band is between $l_{min}$ and $L_p$.

To understand this band, let us recall that the Lagrangian in Eq. (30) leads to the time dependent Hamiltonian is $H(p,x,t) = \frac{1}{2}p^2/2ma^2 + U$. The evolution of the energy in the system is governed by the equation $dH/da = (\partial H/\partial a)p_x$. It is clear from Eq. (31) that $\langle \partial U/\partial a \rangle p_x = -U/a$ while $\langle \partial T/\partial a \rangle p_x = -2T/a$. Hence the time evolution of the total energy $H = E$ of the system is described by

$$\frac{dE}{da} = -\frac{1}{a}(2T + U) = \frac{2E}{a} - \frac{U}{a} = -\frac{E}{a} + \frac{T}{a} \quad (50)$$

In the continuum limit, ignoring the infinite self-energy term, the potential energy can be written as:

$$U = -\frac{G\rho_0^2}{2a} \int d^3x \int d^3y \frac{\delta(x,a)\delta(y,a)}{|x - y|} \quad (51)$$

Hence

$$\frac{d(a^2E)}{da} = -a^2U = \frac{G\rho_0^2}{2} \int d^3x \int d^3y \frac{\delta(x,a)\delta(y,a)}{|x - y|} \quad (52)$$

The ensemble average of the right hand side, per unit proper volume will be

$$\mathcal{E} = -\frac{G\rho_0^2}{2V a} \int d^3x d^3y \frac{\langle \delta(x,a)\delta(y,a) \rangle}{|x - y|} \propto \int d^3k \frac{\langle \delta_k \rangle^2}{a^2k^2} \propto \int_0^\infty \frac{dk}{k} \frac{P(k)}{a^2} \quad (53)$$

where $V$ is the comoving volume.

When a particular scale is virialized, we expect $\mathcal{E} \approx$ constant at that scale in comoving coordinates. That is, we would expect the energy density in Eq. (53) would have reached equipartition and contribute same amount per logarithmic band of scales in the intermediate scales between $l_{min}$ and $L_{peak}$. This requires $P(k) \propto a^2/k$ which is essentially what we found from simulations. The time dependence of $P$ is essentially $P \propto a$ except for a logarithmic correction. Similarly the scale dependence is $P \propto k^{-1}$ which is indeed a good fit to the simulation results. The flattening of the power at small scales, modeled by the more precise fitting function in Eq. (49), can be understood from the fact that, equipartition
is not yet achieved at smaller scales. The same result holds for kinetic energy if the motion is dominated by scale invariant radial flows \[30, 35\]. Our result suggests that gravitational power transfer evolves towards this equipartition.

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