FINITE INDEX SUBGROUPS IN CHEVALLEY GROUPS ARE BOUNDED:
AN ADDENDUM TO “ON BI-ININVARIANT WORD METRICS”

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Abstract. We prove that finite index subgroups in $S$-arithmetic Chevalley groups are bounded.

1. Introduction

A group $G$ is called bounded if every conjugation invariant norm on $G$ has finite diameter. Examples of bounded groups include $\text{SL}(n, \mathbb{Z})$ for $n \geq 3$ [3, 6], $\text{Diff}_0(M)$, where $M$ is a manifold of dimension different from 2 and 4 [9], the commutator subgroup of Thompson’s group $F$ [5] and many others. A finite index subgroup of a bounded group does not have to be bounded. The simplest example is the infinite cyclic subgroup of the infinite dihedral group.

Theorem. Let $G$ be a Chevalley group of rank at least 2 over the ring of $S$-integers in a number field $k$. If $H \leq G$ is a subgroup of finite index then it is bounded.

The above theorem generalises the main result of the paper [6]. The proof is similar with an additional ingredient being an explicit form of bounded generation of a finite index subgroup of a boundedly generated group.

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2. Definitions and known facts

2.1. Norm. A conjugation invariant norm of a group $G$ is a nonnegative function $\nu: G \to \mathbb{R}$ such that the following conditions hold for every $g, h \in G$.

1. $\nu(g) = 1$ if and only if $g = 1$,
2. $\nu(g^{-1}) = \nu(g)$,
3. $\nu(gh) \leq \nu(g) + \nu(h)$,
4. $\nu(h^{-1}gh) = \nu(g)$

2.2. Bounded group. A group $G$ is called bounded if the diameter of every conjugation invariant norm $\nu$ is finite:

\[ \text{diam}(G, \nu) = \sup\{\nu(g) \mid g \in G\} < \infty. \]

2.3. Bounded generation. A group $G$ has bounded generation property if there exist $E_1, \ldots, E_m \in G$ such that every element $g \in G$ can be written as $g = E_{1}^{k_1} \cdots E_{m}^{k_m}$ for some $k_i \in \mathbb{Z}$ [2, Chapter 4].

2.4. Arithmetic group. Let $k$ be a number field (i.e. a finite extension of $\mathbb{Q}$) and let $V_k$ denote the set of equivalence classes of valuations of $k$. Let $S \subset V_k$ be a finite set containing all non-archimedean valuations. The ring of $S$-integers is defined by \( \mathcal{O}_S = \{x \in k \mid \nu(x) \geq 0 \text{ for all } \nu \in S\} \). Let $G$ be a connected algebraic group defined over $k$ with a fixed embedding $G \to \text{GL}(r)$. A subgroup of $G$ that is commensurable with $G(\mathcal{O}_S) = G \cap \text{GL}(r, \mathcal{O}_S)$ is called an $S$-arithmetic group [7, page 61]. More generally, an $S$-arithmetic group can be defined over any global field.

2.5. Chevalley group [1]. Let $\mathfrak{g}$ be a semi-simple complex Lie algebra over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$. Let $\Phi$ be a root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ with simple roots $\{\alpha_1, \ldots, \alpha_k\} \subset \Phi$. Let

\[ \{H_{\alpha_i}(1 \leq i \leq k); X_\alpha(\alpha \in \Phi)\} \]

be a Chevalley basis of the algebra $\mathfrak{g}$, and let $\mathfrak{gl}$ be its linear envelope over $\mathbb{Z}$. Let $\varphi: \mathfrak{g} \to \mathfrak{gl}(r, \mathbb{C})$ be a faithful representation. There is a lattice $L \subset \mathbb{C}^r$ which is invariant with respect to all operators of the form $\varphi(X_\alpha)^m/m!$, where $m \in \mathbb{N}$. If $k$ is an arbitrary field then
homomorphisms $x_\alpha : (k, +) \to \text{GL}(L \otimes k)$ of the additive group of $k$ into $\text{GL}(L \otimes k)$ are defined and given by the formulas
\[ x_\alpha(t) = \sum_{m=0}^{\infty} t^m \varphi(X_\alpha)^{m/m!}. \]

The subgroup $G(\Phi, k) \subset \text{GL}(L \otimes k)$ generated by $\{x_\alpha(t) : \alpha \in \Phi, t \in k\}$, is called the Chevalley group associated with the root system $\Phi$, the representation $\varphi$ and the field $k$.

2.6. Chevalley’s commutator formula. The root elements of the Chevalley group $G(\Phi, k)$ satisfy the following relations:
\begin{align*}
(2.1) & \quad x_\alpha(s)x_\alpha(t) = x_\alpha(s + t) \\
(2.2) & \quad [x_\alpha(s), x_\beta(t)] = \prod_{i,j>0} x_{i\alpha + j\beta}(C_{ij\alpha\beta}(-t)^i s^j)
\end{align*}
where the product is taken in the increasing order of $i + j > 0$ and $C_{ij\alpha\beta} \in \{\pm 1, \pm 2, \pm 3\}$ [4, Theorem 5.2.2].

2.7. $S$-arithmetic Chevalley groups. Let $G(\Phi, k)$ be a Chevalley group over a number field $k$. We consider the $S$-arithmetic group $G(\Phi, \mathcal{O}_S)$ over the ring of $S$-integers $\mathcal{O}_S \subset k$. Let $E(\Phi, \mathcal{O}_S) \subset G(\Phi, \mathcal{O}_S)$ be the subgroup generated by the root elements $x_\alpha(t)$, where $\alpha \in \Phi$ and $t \in \mathcal{O}_S$. It is known that it is a subgroup of finite index. We shall call it an elementary Chevalley group.

2.8. Tavgen’s theorem. Let $\Phi$ be a root system of rank at least 2. If $G = E(\Phi, \mathcal{O}_S)$ is an elementary $S$-arithmetic Chevalley group then there exists a number $m \in \mathbb{N}$ such that every element $g \in G$ can be written as $g = x_{\alpha_1}(t_1) \cdots x_{\alpha_m}(t_m)$, for some $t_i \in \mathcal{O}_S$ [8, Theorem A]. This means that $G$ has bounded generation with respect to root elements.

Let $\xi_1, \ldots, \xi_r \in \mathcal{O}_S$ be elements such that there is an isomorphism of abelian groups $\mathcal{O}_S \cong \mathbb{Z} \oplus \mathbb{Z}\xi_1 \oplus \cdots \oplus \mathbb{Z}\xi_r$. It follows from Tavgen’s theorem that $G$ is boundedly generated by the elements $x_\alpha(1), x_\alpha(\xi_i)$, where $\alpha \in \Phi$.

3. Proof of Theorem

**Lemma 3.1.** Let $G$ be a group with bounded generation. If $H \leq G$ is a finite index subgroup then it also have bounded generation. (cf. [2, Exercise 4.4.3].)
Proof. Assume first that $H \leq G$ is a normal subgroup.

Let $E_1, \ldots, E_m \in G$ be elements boundedly generating $G$. Let $p \in \mathbb{N}$ be the smallest positive integer such that $E_i^p \in H$, for all $i \in \{1, \ldots, m\}$. Consider (finitely many) $m$-tuples $K = (k_1, \ldots, k_m)$, where $0 \leq k_i < p$. Let $E_K = E_1^{k_1} \cdots E_m^{k_m}$. We claim that $H$ is boundedly generated by elements $E_K E_i^p E_K^{-1}$ and those $E_K$’s which are elements of $H$. Indeed, if $h \in H$ then

$$h = E_1^{n_1 p + k_1} \cdots E_m^{n_m p + k_m}$$

$$= \left( \prod_{i=1}^m \left( E_1^{k_1} \cdots E_{i-1}^{k_{i-1}} \right) E_i^{n_i p} \left( E_1^{k_1} \cdots E_{i-1}^{k_{i-1}} \right)^{-1} \right) \cdot \left( E_1^{k_1} \cdots E_m^{k_m} \right)$$

The case of general $H \leq G$ follows from the facts that bounded generation of a finite index subgroup implies bounded generation of the ambient group and that every finite index subgroup contain a finite index normal subgroup.

Remark 3.1. The above proof shows more than the statement. Namely, if $H \leq G$ is a normal subgroup then each $h \in H$ can be expressed as a product of powers of conjugates of $E_i$ in $G$ and elements of the form $E_K$ (in powers either zero or one).

Proof of Theorem. Let $G = G(\Phi, \mathcal{O}_S)$ be a Chevalley group and let $H \leq G$ be a subgroup of finite index. Let $\xi_0 = 1, \xi_1, \ldots, \xi_r \in \mathcal{O}_S$ be a basis of $\mathcal{O}_S$ over $\mathbb{Z}$, i.e. $\mathcal{O}_S$ splits as a direct product $\bigoplus_{i=0}^r \mathbb{Z} \xi_i$. Recall that $E(\Phi, \mathcal{O}_S)$ is boundedly generated by the elements $x_\alpha(\xi_i)$, where $\alpha \in \Phi$. Consider the subgroup $H \cap E(\Phi, \mathcal{O}_S)$. It is of finite index in $G$ and it follows from Lemma 3.1 that it is boundedly generated.

Let $H_E \leq H \cap E(\Phi, \mathcal{O}_S)$ be a subgroup which is of finite index which is normal in $G$. Let $p \in \mathbb{N}$ be the smallest positive integer such that the root element $x_\alpha(p\xi_i) \in H_E$ for each $\alpha \in \Phi$ and $\xi_i, i = 0, 1, \ldots, r$. It follows from the proof of Lemma 3.1 than every element $h \in H_E$ can be written as

$$h = \left( \prod_{\alpha \in \Phi} \prod_{i=0}^r g_{\alpha i} x_\alpha(p\xi_i)^{n_i} g_{\alpha i}^{-1} \right) \cdot E_K,$$

where $E_K$ is as in the proof of Lemma 3.1.
The boundedness in $H_E$ with respect to any conjugation invariant norm of the cyclic subgroup generated by $h$ is equivalent to boundedness of the cyclic subgroup generated by $ghg^{-1}$ for any $g \in G$. It thus follows that in order to prove boundedness of $H_E$ it is enough to show boundedness of cyclic subgroups generated by $x_\alpha(p\xi_i)$ with respect to any conjugation invariant norm.

Consider a conjugation invariant norm $\nu: H_E \to \mathbb{R}$. Define

$$M = \max\{\nu(x_\alpha(p\xi_i)), \nu(E_K) \mid \alpha \in \Phi, \ i = 0, \ldots, r, E_K \in H_E\}.$$

This is a maximum over a finite set.

Let $\alpha \in \Phi$ be a root. There exists a subsystem $\Psi \subset \Phi$ isomorphic to one of $A_2$, $B_2$ or $G_2$ such that $\alpha \in \Psi$. Moreover, if $\alpha \in A_2$ there exist $\beta, \gamma \in A_2$ such that $\alpha = \beta + \gamma$ and that no other positive combination of $\beta$ and $\gamma$ is a root. The same holds if $\alpha$ is a long root in either $B_2$ or $G_2$. It then follows from (2.2) that

$$x_\alpha(p\xi)^{Cn} = [x_\beta(q\xi), x_\gamma(pn)]$$

for any $n \in \mathbb{Z}$ and some $C \in \{\pm 1, \pm 2, \pm 3\}$, where $p = p(\alpha, \xi)$ and $q = p(\beta, \xi)$. In particular, we have that

$$\nu\left(x_\alpha(p\xi)^{Cn}\right) \leq 2M.$$

Given $m = Cn + \epsilon$, where $0 \leq \epsilon \leq C$ we have

$$\nu(x_\alpha(p\xi)^m) = \nu(x_\alpha(p\xi)^{Cn})\nu(x_\alpha(p\xi)^\epsilon) \leq 4M$$

if $\alpha$ is either a root contained in $A_2$ or a long root in either $B_2$ or $G_2$. If $\alpha$ is a short root in either $B_2$ or $G_2$ then there exist $\beta, \gamma \in B_2$ (respectively $\beta, \gamma \in G_2$) such that $\alpha = \beta + \gamma$ and $\beta + 2\gamma$ is a long root and no other positive combination of $\beta$ and $\gamma$ is a root. Applying (2.6) again, we get that

$$[x_\beta(p\xi), x_\gamma(-pn)] = x_\alpha(p^2\xi Cn) x_{\beta+2\gamma}(p^2\xi C'n)$$

$$x_\alpha(p\xi)^{Cpn} = [x_\beta(p\xi), x_\gamma(-pn)] x_{\beta+2\gamma}(-p\xi)^{C'n}$$

which implies that $\nu(x_\alpha(p\xi)^n) \leq 6M$ is bounded. This finishes the proof for $H_E$. Since it is of finite index in $H$ it follows that $H$ is bounded as well which finishes the proof. \qed

Remark 3.2. The last part of the proof of boundedness of Chevalley groups in [6] is incorrect and the above argument fixes it.
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