Real-Time Thermal Propagators
and the QED Effective Action
for an External Magnetic Field

Per Elmfors,a David Perssonb and Bo-Sture Skagerstamb,c

aNORDITA, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark
bInstitute of Theoretical Physics, Chalmers University of Technology and
University of Göteborg, S-412 96 Göteborg, Sweden
cUniversity of Kalmar, Box 905, S-391 29 Kalmar, Sweden

Abstract

The thermal averaged real-time propagator of a Dirac fermion in a static uniform magnetic field $B$ is derived. At non-zero chemical potential and temperature we find explicitly the effective action for the magnetic field, which is shown to be closely related to the Helmholtz free energy of a relativistic fermion gas, and it exhibits the expected de Haas – van Alphen oscillations. An effective QED coupling constant at finite temperature and density is derived, and compared with renormalization group results. We discuss some astrophysical implications of our results.

1Email address: elmfors@nordita.dk.
2Email address: tfedp@fy.chalmers.se.
3Email address: tfebss@fy.chalmers.se. Research supported by the Swedish National Research Council under contract no. 8244-311.
Large magnetic fields $B$ can be associated with certain compact astrophysical objects like supernovae \cite{1,2} where $B = \mathcal{O}(10^{10}) \text{T}$, neutron stars \cite{3,4} where $B = \mathcal{O}(10^{8}) \text{T}$, or white magnetic dwarfs \cite{4,5} in which case $B = \mathcal{O}(10^{4}) \text{T}$. (As a reference the electron mass in units of Tesla is $m_e^2/e = 4.414 \cdot 10^9 \text{T}$.) It has recently been argued that a plasma at thermal equilibrium can sustain fluctuations of the electromagnetic fields. In particular, for the primordial Big-Bang plasma the amplitude of magnetic field fluctuations at the time of the primordial nucleosynthesis can be as large as $B = \mathcal{O}(10^{10}) \text{T}$ \cite{6}. Furthermore, a model for extragalactic gamma bursts in terms of mergers of massive binary stars suggests magnetic fields up to the order $B = \mathcal{O}(10^{13}) \text{T}$ \cite{7}. A more speculative system where even larger macroscopic magnetic fields can be contemplated are superconducting strings \cite{8}. Here one may conceive fields as large as $B \gtrsim \mathcal{O}(10^{14}) \text{T}$. It has also recently been suggested that due to gradients in the Higgs field during the electroweak phase transition in the early universe very large magnetic fields, $B = \mathcal{O}(10^{19}) \text{T}$, may be generated \cite{9}. If one encounters magnetic fields of this order of magnitude the complete electroweak model has to be considered and the concept of electroweak magnetism becomes important (for a recent account see e.g. Ref.\cite{10}). In the present paper we consider, however, magnetic fields such that calculations within QED are sufficient. A shorter version of this report has been published elsewhere \cite{11}.

In many of these systems one has to consider the effects of a thermal environment and a finite chemical potential. In this paper we derive the appropriate effective fermion propagator and the effective action in QED for a thermal environment treated exactly in the external constant magnetic field but with no virtual photons present, i.e. we consider the weak coupling limit.

Calculations of the QED effective Lagrangian density in an external field have been attempted before either at finite temperature \cite{12,13} or at finite chemical potential \cite{14}. In the latter case \cite{14} the effective action is not complete but the correct form is presented here. At finite chemical potential and for sufficiently small temperatures, the QED effective action should exhibit a certain periodic dependence of the external field, i.e. the well-known de Haas – van Alphen oscillations in condensed matter physics. This was not obtained in Ref.\cite{14}. Elsewhere, the radiative corrections to the anomalous magnetic moment has been estimated in the presence of large magnetic fields and it was argued that they are extremely small \cite{13,16}.
By making use of the effective action we derive the effective QED coupling as a function of the external field, the chemical potential and temperature. In a future publication we will discuss the fermion self-energy and radiative corrections to the electrons anomalous magnetic moment, in terms of the formalism derived here [17].

2 Thermal propagators in the Furry picture

We consider Dirac fermions in the presence of an external static field as described by the vector potential $A_\mu$. Using static energy solutions we may represent the second quantized fermion field in the Furry picture [18]. It is given by

$$\Psi(x, t) = \sum_{\lambda, \kappa} b_{\lambda\kappa} \psi^{(+)}_{\lambda\kappa}(x, t) + d^\dagger_{\lambda\kappa} \psi^{(-)}_{\lambda\kappa}(x, t) ,$$  \hspace{1cm} (2.1)

where $\lambda$ is a polarization index, $\kappa$ denotes the energy and momentum (or other) quantum numbers (discrete and/or continuous) needed in order to completely characterize the solutions, and $(\pm)$ denotes positive and negative energy solutions of the corresponding Dirac equation,

$$(i \not{D} - m)\psi^{(\pm)}_{\lambda\kappa}(x, t) = 0 ,$$  \hspace{1cm} (2.2)

where $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative. The creation and annihilation operators satisfy the canonical anti-commutation relations

$$\{d_{\lambda\kappa'}, d^\dagger_{\lambda\kappa}\} = \delta_{\lambda\lambda'}\delta_{\kappa\kappa'} = \{b_{\lambda\kappa'}, b^\dagger_{\lambda\kappa}\} ,$$  \hspace{1cm} (2.3)

while other anti-commutators are zero. The completeness relation

$$\sum_{\lambda, \kappa} \psi^{(+)}_{\lambda\kappa,a}(x', t)\psi^{(+)}_{\lambda\kappa,b}(x, t) + \psi^{(-)}_{\lambda\kappa,a}(x', t)\psi^{(-)}_{\lambda\kappa,b}(x, t) = \delta_{ab}\delta^3(x' - x) ,$$  \hspace{1cm} (2.4)

where $\psi^{(\pm)}_{\lambda\kappa,a}$ denotes the $a$-component of the Dirac spinor $\psi^{(\pm)}_{\lambda\kappa}$, leads to the canonical anti-commutation relations for the fields

$$\{\Psi_a(x', t), \Psi^\dagger_b(x, t)\} = \delta_{ab}\delta^3(x' - x) .$$  \hspace{1cm} (2.5)

In vacuum, the fermion propagator $iS_F(x'; x|m)$ is defined by

$$iS_F(x'; x|m) = \langle 0| T \left( \psi(x', t')\overline{\psi}(x, t) \right) |0\rangle = \theta(t' - t) \sum_{\lambda, \kappa} \psi^{(+)}_{\lambda\kappa}(x', t')\overline{\psi}^{(+)}_{\lambda\kappa}(x, t) - \theta(t - t') \sum_{\lambda, \kappa} \psi^{(-)}_{\lambda\kappa}(x', t')\overline{\psi}^{(-)}_{\lambda\kappa}(x, t) ,$$  \hspace{1cm} (2.6)
where the conjugated spinor $\overline{\psi}_{\lambda\kappa}^{(\pm)}$ is given by $\overline{\psi}_{\lambda\kappa}^{(\pm)} = (\psi_{\lambda\kappa}^{(\pm)})^\dagger \gamma_0$. Since $\psi_{\lambda\kappa}^{(\pm)}(x, t)$ satisfies the Dirac equation, only the time derivative acting on the step functions gives a non-zero contribution, so one finds that

$$
(i \Slash{D} - m)S_F(x'; x|m) = 1 \cdot \delta^4(x' - x) .
$$

(2.7)

The real-time propagator at finite temperature $T$ and chemical potential $\mu$, denoted by $\langle iS_F(x'; x|m) \rangle_{\beta, \mu}$, can now be obtained by the following reasoning. Let $f^+_F(\omega)$ denote the Fermi-Dirac thermodynamical distribution function

$$
f^+_F(\omega) = \frac{1}{\exp(\beta(\omega - \mu)) + 1} ,
$$

(2.8)

where $\beta$ is the inverse temperature, $\mu$ the chemical potential and $\omega$ is the energy of the quantum state under consideration. A particle can propagate forward in time in a state which is unoccupied by thermal particles, whereas a hole in the occupied states can propagate backwards in time. We can therefore write

$$
\langle iS_F(x'; x|m) \rangle_{\beta, \mu} = \sum_{\lambda, \kappa}
\left[ \theta(t' - t) \left( [1 - f^+_F(E_\kappa)]\psi^+_{\lambda\kappa}(x', t')\overline{\psi}^+_{\lambda\kappa}(x, t) + [1 - f^+_F(-E_\kappa)]\psi^-_{\lambda\kappa}(x', t')\overline{\psi}^-_{\lambda\kappa}(x, t) \right)
- \theta(t - t') \left( f^+_F(-E_\kappa)\psi^-_{\lambda\kappa}(x', t')\overline{\psi}^-_{\lambda\kappa}(x, t) + f^+_F(E_\kappa)\psi^+_{\lambda\kappa}(x', t')\overline{\psi}^+_{\lambda\kappa}(x, t) \right) \right] .
$$

(2.9)

We can now extract the vacuum part of the propagator Eq.(2.6) and write

$$
\langle iS_F(x'; x|m) \rangle_{\beta, \mu} = iS_F(x'; x|m) + iS^{\beta, \mu}_F(x'; x|m) ,
$$

(2.10)

where the thermal part $iS^{\beta, \mu}_F(x'; x|m)$ is defined by

$$
S^{\beta, \mu}_F(x'; x|m) = i \sum_{\lambda, \kappa}
\left( f^+_F(E_\kappa)\psi^+_{\lambda\kappa}(x', t')\overline{\psi}^+_{\lambda\kappa}(x, t) - f^-_F(E_\kappa)\psi^-_{\lambda\kappa}(x', t')\overline{\psi}^-_{\lambda\kappa}(x, t) \right) ,
$$

(2.11)

and where we have defined the distribution

$$
f^-_F(E_\kappa) = 1 - f^+_F(-E_\kappa) .
$$

(2.12)

Notice that there is no time-ordering in $S^{\beta, \mu}_F(x'; x|m)$ despite the fact that the time-ordering in Eq.(2.9) is non-trivial. The thermal propagator Eq.(2.10) therefore also trivially satisfies Eq.(2.7). These considerations can, of course, easily be extended to treat particles with Bose-Einstein statistics as well.
The result Eq. (2.11) can also be derived from an explicit calculation using the second-quantized field operators and appropriate thermal averages, i.e. we use Wicks theorem

\[ \mathcal{T} \left( \Psi(x', t') \bar{\Psi}(x, t) \right) = i S_F(x'; x|m) + \bar{\Psi}(x', t') \bar{\Psi}(x, t) : , \] (2.13)

where the last term corresponds to a normal ordering. We then obtain

\[ \langle \mathcal{T} \left( \Psi(x', t') \bar{\Psi}(x, t) \right) \rangle_{\beta,\mu} = i S_F(x'; x|m) + i S_F^{\beta\mu}(x'; x|m) , \] (2.14)

where we have used the only non-zero bilinear thermal averages

\[ \langle b^\dagger_{\lambda\kappa} b_{\lambda'\kappa'} \rangle_{\beta,\mu} = f^+_{F}(E_\kappa) \delta_{\lambda\lambda'} \delta_{\kappa\kappa'} , \]

\[ \langle d^\dagger_{\lambda\kappa} d_{\lambda'\kappa'} \rangle_{\beta,\mu} = f^-_{F}(E_\kappa) \delta_{\lambda\lambda'} \delta_{\kappa\kappa'} . \] (2.15)

In principle we do not have to restrict ourselves to thermal distributions as given by Eq. (2.8). In fact, we can allow for any such one-particle distribution function \( f^{\pm}_{F}(\omega) \) and the definition Eq. (2.12).

### 3 External Uniform and Static Magnetic Field

For the convenience of the reader, we summarize some of the relevant expressions in the case of a constant magnetic field \( B \) parallel to the \( z \)-direction in the gauge \( A_\mu = (0, 0, -Bx, 0) \). Using \( \kappa \) as a collective index for \( (n, k_y, k_z) \), where \( n = 0, 1, 2, ... \); \( k_y, k_z \) are continuous, and the \( \gamma \) matrices in the chiral representation, we can write the solutions in the form

\[ \psi_{\lambda,\kappa}(x, t) = \frac{1}{2\pi} \exp[\pm(-iE_\kappa t + ik_y y + ik_z z)] \frac{\sqrt{2E}}{E} \Phi_{\lambda,\kappa}^{(\pm)}(x) , \] (3.1)

where

\[ \Phi_{1,\kappa}^{(\pm)}(x) = \frac{1}{\sqrt{E_\kappa + k_z}} \begin{pmatrix} (E_\kappa + k_z) I_{n;k_y}(x) \\ \pm i \sqrt{2eBn} I_{n-1;k_y}(x) \\ -m I_{n;k_y}(x) \\ 0 \end{pmatrix} , \] (3.2)

\[ \Phi_{2,\kappa}^{(\pm)}(x) = \frac{1}{\sqrt{E_\kappa + k_z}} \begin{pmatrix} 0 \\ -m I_{n-1;k_y}(x) \\ \pm i \sqrt{2eBn} I_{n;k_y}(x) \\ (E_\kappa + k_z) I_{n-1;k_y}(x) \end{pmatrix} , \] (3.3)
\[ \Phi_{1,\kappa}^{(-)}(x) = \frac{1}{\sqrt{E_\kappa - k_y}} \begin{pmatrix} -mI_{n; -k_y}(x) \\ 0 \\ (-E_\kappa + k_z)I_{n; -k_y}(x) \\ i\sqrt{2eBn}I_{n-1; -k_y}(x) \end{pmatrix} , \] (3.4)

\[ \Phi_{2,\kappa}^{(-)}(x) = \frac{1}{\sqrt{E_\kappa - k_y}} \begin{pmatrix} i\sqrt{2eBn}I_{n; -k_y}(x) \\ (-E_\kappa + k_z)I_{n-1; -k_y}(x) \\ 0 \\ -mI_{n-1; -k_y}(x) \end{pmatrix} . \] (3.5)

In these expressions the energy \( E_\kappa \) is given by

\[ E_\kappa = \sqrt{m^2 + k_z^2 + 2eBn} \] ,

(3.6)

and the \( I_{n;k_y}(x) \) functions are explicitly written

\[ I_{n;k_y}(x) \equiv \left( \frac{eB}{\pi} \right)^{1/4} \exp \left[ -\frac{1}{2} \frac{eB}{\pi} \left( x - \frac{k_y}{eB} \right)^2 \right] \frac{1}{\sqrt{n!}} H_n \left[ \sqrt{2eB} \left( x - \frac{k_y}{eB} \right) \right] . \] (3.7)

Here \( H_n \) is the Hermite polynomial given by Rodrigues’ formula as

\[ H_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n} e^{-\frac{1}{2}x^2} , \] (3.8)

and we define \( I_{-1;k_y} = 0 \). The functions \( I_{n;k_y}(x) \) are normalized as

\[ \int dx I_{n;k_y}(x) I_{m;k_y}(x) = \delta_{n,m} \] (3.9)

if \( n, m \geq 0 \), so that it is easily shown that the collection of all \( \Psi \)’s form a complete orthonormal set. The vacuum part of the propagator Eq.(2.6) is then given by (see e.g. Ref.[19])

\[ S_F(x'; x|m)_{ab} = \sum_{n=0}^{\infty} \int {d\omega \frac{dk_y}{(2\pi)^3}} \exp\left[ -i\omega(t' - t) + ik_y(y' - y) + ik_z(z' - z) \right] \]

\[ \times \frac{1}{\omega^2 - k_y^2 - m^2 - 2eBn + i\epsilon} S_{ab}(n; \omega, k_y, k_z) . \] (3.10)

The matrix \( S(n; \omega, k_y, k_z) \) entering above is given by
\[ S(n; \omega, k_y, k_z) \equiv \]
\[
\begin{pmatrix}
mI_{n,n} & 0 & -(\omega+k_z)I_{n,n} & -i\sqrt{2eB}nI_{n,n-1} \\
0 & mI_{n-1,n-1} & i\sqrt{2eB}nI_{n-1,n} & -(\omega-k_z)I_{n-1,n-1} \\
-(\omega-k_z)I_{n,n} & i\sqrt{2eB}nI_{n,n-1} & mI_{n,n} & 0 \\
-i\sqrt{2eB}nI_{n-1,n} & -(\omega+k_z)I_{n-1,n-1} & 0 & mI_{n-1,n-1}
\end{pmatrix},
\]
(3.11)

where we have defined
\[ I_{n,n'} \equiv I_{n;k_y}(x)I_{n';k_y}(x') \]  
(3.12)

Similarly we find the thermal part of the fermion propagator
\[
S_F^{\beta\mu}(x'; x|m)_{ab} = \sum_{n=0}^{\infty} \int \frac{d\omega dk_y dk_z}{(2\pi)^3} \exp[-i\omega(t' - t) + ik_y(y' - y) + ik_z(z' - z)] 
\times 2\pi i \delta(\omega^2 - k_z^2 - m^2 - 2eBn)f_F(\omega)S_{ab}(n; \omega, k_y, k_z),
\]
(3.13)

where \( f_F(\omega) \) is the thermal distribution
\[
f_F(\omega) = \theta(\omega)f_F^+(\omega) + \theta(-\omega)f_F^-(\omega).
\]
(3.14)

By making use of the completeness relation
\[
\sum_{n=0}^{\infty} I_{n;k_y}(x)I_{n;k_y}(x') = \delta(x - x'),
\]
(3.15)

one can show [17] that the propagators Eq.(3.10) and Eq.(3.13) reduce to the free-field propagators in the limit when the magnetic field \( B \) tends to zero.

4 Propagators in thermo-field dynamics

The thermal propagators in Eqs.(3.10) and (3.13) cannot be used for a perturbative expansion in a naive way. The reason is that the \( \delta \)-functions can occur on several internal legs with coinciding arguments and that such expressions are not well-defined. It is known that such problems can be avoided by means of a correctly derived real-time finite temperature formalism where one must invoke a doubling of the degrees of freedom. There are several formalisms for doing that and we shall use thermo field dynamics (TFD) since it is very easy in the operator formalism \[20\]. In TFD the propagator is obtained as the expectation value of the time-ordered product in the thermal vacuum \( |O_\beta\rangle \) which is annihilated by the thermal operators \( (b_{\lambda\kappa}(\beta), d_{\lambda\kappa}(\beta)) \) and their tilde partners \( (\tilde{b}_{\lambda\kappa}(\beta), \tilde{d}_{\lambda\kappa}(\beta)) \).
The TFD propagator can be given by a simple expression for independent harmonic oscillators. We have solved the Dirac equation exactly in the external field, but in the free propagator the interaction between the particles is neglected. Each mode is therefore still an independent harmonic oscillator, but with a different frequency labeled by the quantum numbers \((n, k_y, k_z)\) and corresponding to a definite Landau level. Thus, in the derivation of the propagator we can copy the usual procedure for free particles.

The Bogoliubov transformation between the zero temperature and thermal operators is given by

\[
\begin{pmatrix}
    b_{\lambda\kappa} \\
    i\tilde{b}_{\lambda\kappa}^\dagger
\end{pmatrix}
= \begin{pmatrix}
    \cos \vartheta_{\lambda\kappa}^{(+)} & -\sin \vartheta_{\lambda\kappa}^{(+)} \\
    \sin \vartheta_{\lambda\kappa}^{(+)} & \cos \vartheta_{\lambda\kappa}^{(+)}
\end{pmatrix}
\begin{pmatrix}
    b_{\lambda\kappa}(\beta) \\
    i\tilde{b}_{\lambda\kappa}^\dagger(\beta)
\end{pmatrix},
\]

and

\[
\begin{pmatrix}
    d_{\lambda\kappa} \\
    i\tilde{d}_{\lambda\kappa}^\dagger
\end{pmatrix}
= \begin{pmatrix}
    \cos \vartheta_{\lambda\kappa}^{(-)} & -\sin \vartheta_{\lambda\kappa}^{(-)} \\
    \sin \vartheta_{\lambda\kappa}^{(-)} & \cos \vartheta_{\lambda\kappa}^{(-)}
\end{pmatrix}
\begin{pmatrix}
    d_{\lambda\kappa}(\beta) \\
    i\tilde{d}_{\lambda\kappa}^\dagger(\beta)
\end{pmatrix}.
\]

The number expectation values in Eq.(2.15) imply that the coefficients in the Bogoliubov matrices must satisfy

\[
\sin^2 \vartheta_{\lambda\kappa}^{(\pm)} = f_\pm(E_\kappa).
\]

We use the convention that \(b\) and \(\tilde{b}\) anti-commute. The Bogoliubov matrices in Eqs.(4.1) and (4.2), and the factors of \(i\), are carefully explained in Ref.[21]. The definition of the fermion propagator varies slightly in the literature and we shall for definiteness follow Ref.[21]. Other propagators correspond to other definitions of the thermal doublet and they may be computed in a similar way. Since we are not doing any higher loop calculations these conventions do not matter. We compute the TFD propagator matrix as

\[
iS_F^{TFD}(x'; x|m)_{ab} = \langle \mathcal{O}_\beta | T \left[ \left( \Psi_a(x') \right)^\dagger \left( \tilde{\Psi}_b(x), -i\tilde{\Psi}_b^\dagger(x) \right) \right] | \mathcal{O}_\beta \rangle.
\]

The structure of the propagator is the same as in absence of the external field except that we now expand in another basis corresponding to the new energy eigenvalues. We obtain

\[
iS_F^{TFD}(x'; x|m)_{ab} = \sum_{n=0}^{\infty} \int \frac{d\omega}{(2\pi)^3} \frac{dk_y dk_z}{(2\pi)^3} \exp[-i\omega(t' - t) + ik_y(y' - y) + ik_z(z' - z)]
\times S_{ab}(n; \omega, k_y, k_z)U_F(\omega) \left( \begin{array}{cc}
    1 & 0 \\
    \frac{\omega^2 - E_\kappa^2 + i\epsilon}{\omega^2 - E_\kappa^2 - i\epsilon} & 0
\end{array} \right) U_F^T(\omega),
\]

7
where
\[ U_F(\omega) = \begin{pmatrix} \cos \vartheta(\omega) & -\sin \vartheta(\omega) \\ \sin \vartheta(\omega) & \cos \vartheta(\omega) \end{pmatrix}, \] (4.6)
and
\[ \sin \vartheta(\omega) = \theta(\omega) \sqrt{f_F^+(\omega) - \theta(-\omega) \sqrt{f_F^-(\omega)}} , \]
\[ \cos \vartheta(\omega) = \theta(\omega) \sqrt{1 - f_F^+(\omega) + \theta(-\omega) \sqrt{1 - f_F^-(\omega)}} . \] (4.7)

Here, \( U_F^T(\omega) \) is the transpose of the matrix \( U_F(\omega) \). The \( S_{TFD}^{TFD}(x'; x|m)^{11}_{ab} \) component is, of course, the same as the propagator in Eqs.(3.10) and (3.13) and the other components are only needed in higher loop calculations.

The derivation of the propagator in this Section can be repeated for a non-equilibrium distribution if only we assume certain factorization properties of the density matrix. The essential assumption is that there are no non-trivial multiparticle correlations so that everything is determined in terms of the single particle distribution. This freedom amounts to replacing the functions \( f_K^\pm(E_\kappa) \) with some other positive functions that describes the distribution. Some applications of such a formalism in absence of the external field can be found in Ref.[22].

## 5 The Effective Action

As was shown by Schwinger a long time ago [23], an external electromagnetic field, slowly varying in space and time, can be treated to all orders in the external field in the weak-coupling limit. Here we make use of a technique similar to that of Schwinger’s in order to evaluate the thermodynamical partition function in a static uniform magnetic field \( B \) for charged fermions as well as for charged bosons.

### 5.1 QED and Charged Fermions

The generating functional of fermionic Green’s functions in an external field \( Z[\bar{\eta}, \eta, A_\mu] \), formally defined by
\[ Z[\bar{\eta}, \eta, A_\mu] = \int d[\bar{\psi}] d[\psi] \exp[i \int d^4x (-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i \mathcal{D} - m) \psi - \bar{\eta} \psi + \bar{\psi} \eta)] \], (5.1)
describes second-quantized electrons and positrons interacting with a classical electromagnetic field expressed in terms of the vector potential $A_\mu$. The expectation value of $\psi$ (and $\bar{\psi}$) can formally be fixed by choosing appropriate $\bar{\eta}$ (and $\eta$), i.e. $\varphi(x) \equiv \langle \psi(x) \rangle = -i\delta/\delta \bar{\eta}(x) \log Z$ (and similarly for $\bar{\psi}$). The equation of motion for $\varphi(x)$ tells us how the electrons interact with the electromagnetic field which includes effects due to all virtual $e^+e^-$-pairs.

The fermionic Gaussian functional integral in Eq.(5.1) can formally be performed with the result that

$$Z[\bar{\eta}, \eta, A_\mu] = \text{Det} \left[ i \left( i\not{D} - m \right) \right] \exp \left[ i \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \int d^4y \bar{\eta}(x) S_F(x;y|m) \eta(y) \right) \right] ,$$

where $S_F(x;y|m)$ is the external field vacuum propagator as given by Eq.(3.10). It follows that $\varphi(x)$ satisfies the Dirac equation in the external field, i.e. $(i\not{D} - m) \varphi(x) = 0$.

The functional determinant $\text{Det}(i\not{D} - m)$ gives rise to a contribution to the effective Lagrangian density $L_{\text{eff}}$. Using a complete orthogonal basis to rewrite $\log \text{Det}$ as $\text{Tr} \log$, the effective action can thus be written

$$S_{\text{eff}} = \int d^4x L_{\text{eff}} = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] - i \text{Tr} \log \left[ i \left( i\not{D} - m \right) \right] .$$

We now write the effective Lagrangian density as

$$L_{\text{eff}} = L_0 + L_1 ,$$

where the tree level part in the case of a pure magnetic field is

$$L_0 = -\frac{1}{2} B^2 ,$$

and $L_1$ corresponds to the functional determinant. Differentiating Eq.(5.3) with respect to the fermion mass we now find the one-loop correction according to

$$\frac{\partial L_1}{\partial m} = i \text{tr} S_F(x;x|m) ,$$

where the trace now only is over spinor indices. After a straightforward calculation of the trace using Eq.(3.10), we obtain in terms of renormalized quantities the well-known result [23] that

$$L_1 = -\frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^3} \left[ esB \coth(esB) - 1 - \frac{1}{3} (esB)^2 \right] \exp(-m^2 s) .$$
We have here performed the standard renormalizations leaving \( eB \) invariant, i.e.

\[
A_\mu \rightarrow (1 + C e^2)^{-1/2} A_\mu ,
\]

\[
e^2 \rightarrow e^2 \left( 1 + C e^2 \right) ,
\]

where the divergent constant \( C \) is given by

\[
C = \frac{1}{12\pi^2} \int_0^\infty \frac{ds}{s} \exp(-m^2 s) .
\]

We shall now find the corresponding correction \( S^\beta\mu_{\text{eff}} = \int d^4x L^\beta\mu_{\text{eff}} \), to the effective action \( S_{\text{eff}} \) at finite chemical potential and temperature such that

\[
L_{\text{eff}} = L_0 + L_1 + L^\beta\mu_{\text{eff}} .
\]

We notice that the correction \( L^\beta\mu_{\text{eff}} \), due to the presence of thermal fermions, can be written in the form

\[
\frac{\partial L^{\beta\mu}_{\text{eff}}}{\partial m} = i \text{Tr} S^{\beta\mu}_{F}(x; x|m) .
\]

By performing the trace operation in Eq.(5.11), using the thermal propagator Eq.(3.13), we obtain

\[
L^\beta\mu_{\text{eff}} = \frac{4eB}{(2\pi)^2} \sum_{n=0}^{\infty} \sum_{\lambda=1}^{2} \int_0^\infty d\omega f_F(\omega) \times \int_0^\infty dk k^2 \delta(\omega^2 - k^2 - 2eB(n + \lambda - 1) - m^2) ,
\]

where we have integrated by parts with respect to \( k \). We, therefore, see that \( L^\beta\mu_{\text{eff}} \) is directly related to the partition function \( Z(B, T, \mu) \) of the relativistic fermion gas in the presence of an external magnetic field \( B \) in a sufficiently large quantization volume \( V \), as given in for example Ref.[24], according to

\[
L^\beta\mu_{\text{eff}} = \log \frac{Z(B, T, \mu)}{\beta V} = \frac{eB}{(2\pi)^2} \sum_{n=0}^{\infty} \sum_{\lambda=1}^{2} \int_{-\infty}^{\infty} dk \frac{k^2}{E_{\lambda,n}} \times \left( \frac{1}{1 + \exp[\beta(E_{\lambda,n} - \mu) - 1 + \exp[\beta(E_{\lambda,n} + \mu)]]} \right) ,
\]

where

\[
E_{\lambda,n} = \sqrt{k^2 + 2eB(n + \lambda - 1) + m^2} .
\]
Separating the field independent part we write

\[ \mathcal{L}_{\text{eff}}^{\beta,\mu} = \mathcal{L}^{\beta,\mu}_0 + \mathcal{L}^{\beta,\mu}_1, \tag{5.15} \]

where

\[ \mathcal{L}^{\beta,\mu}_0 = \frac{1}{3\pi^2} \int_{-\infty}^{\infty} d\omega \theta(\omega^2 - m^2) f_F(\omega) \left( \omega^2 - m^2 \right)^{3/2}. \tag{5.16} \]

We, therefore, conclude that the field independent thermal correction to the Lagrangian density \( \mathcal{L}^{\beta,\mu}_0 \) can be identified as

\[ \mathcal{L}^{\beta,\mu}_0 = \log \frac{Z(T,\mu)}{\beta V} = -F(T,\mu)/V, \tag{5.17} \]

where \( Z(T,\mu) \) is the partition function, and \( F(T,\mu) \) the free energy, for an ideal \( e^+e^- \)-gas with particle energy \( E = \sqrt{k^2 + m^2} \), i.e.

\[ \log \frac{Z(T,\mu)}{V} = 2 \int_{0}^{\infty} \frac{d^3k}{(2\pi)^3} \left[ \log[1 + e^{-\beta(E-\mu)}] + \log[1 + e^{-\beta(E+\mu)}] \right], \tag{5.18} \]

consistent with the general identification above. Using the identity

\[ \frac{\exp(-|x|)}{|x|} = \int_{0}^{\infty} \frac{dt}{\sqrt{2\pi t}} \exp \left( -\frac{1}{2} (x^2 t + \frac{1}{t}) \right), \tag{5.19} \]

the following representation of \( \mathcal{L}^{\beta,\mu}_1 \), valid for \( |\mu| < m \), can be derived in a straightforward manner

\[ \mathcal{L}^{\beta,\mu}_1 = \frac{1}{4\pi^2} \sum_{l=1}^{\infty} (-1)^{l+1} \int_{0}^{\infty} \frac{ds}{s^3} \exp \left( -\frac{\beta^2 l^2}{4s} - m^2 s \right) \frac{\cosh(\beta l \mu)}{2} [eBs \coth(eBs) - 1]. \tag{5.20} \]

In the case \( \mu = 0 \), Eq.(5.20) agrees with the result obtained in Refs.[12, 13]. However, it is not always obvious, when written in this form, to see how to extract the physical content, and particularly not obvious how to generalize \( \mathcal{L}_{\text{eff}}^{\beta,\mu} \) to \( |\mu| \geq m \), since then it appears to be divergent. In particular we notice that the high \( T \) behaviour given in Ref.[12] is not correct. As explained in Appendix A it is, however, possible to show that Eq.(5.20) is equal to Eq.(5.21) given below, which is valid for all \( T \) and \( \mu \).

In order to calculate the thermal part \( \mathcal{L}_{\text{eff}}^{\beta,\mu} \) of the effective action in a more useful form, we have to be careful with the convergence and the analytical structure. Some details of the calculation are given in Appendix A. We get \( \mathcal{L}_{\text{eff}}^{\beta,\mu} = \mathcal{L}^{\beta,\mu}_0 + \mathcal{L}^{\beta,\mu}_1 \), where \( \mathcal{L}^{\beta,\mu}_0 \), the ideal gas contribution in absence of the external field \( B \), is given in Eq.(5.17), and

\[ \mathcal{L}^{\beta,\mu}_1 = \mathcal{L}^{\beta,\mu}_{1,\text{reg}} + \mathcal{L}^{\beta,\mu}_{1,\text{osc}}. \]
\[
\begin{align*}
&= \int_{-\infty}^{\infty} d\omega (\omega^2 - m^2) f_F(\omega) \left[ \frac{1}{4\pi^{5/2}} \int_0^{\infty} \frac{ds}{s^{5/2}} e^{-s(\omega^2 - m^2)} [seB \coth(seB) - 1] \right] \\
&\quad - \int_{-\infty}^{\infty} d\omega (\omega^2 - m^2) f_F(\omega) \left[ \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \left( \frac{eB}{n} \right)^{3/2} \sin \left( \frac{\pi}{4} - \frac{\pi n}{eB} (\omega^2 - m^2) \right) \right].
\end{align*}
\]

The term with the sum over \( n \), \( \mathcal{L}_{1,\text{osc}}^{\beta,\mu} \), was neglected in Ref.\[14\] and we show in Section 8 that it is essential to keep this term in order to get the correct physical result. We may also use the generalized \( \zeta \)-function to rewrite \( \mathcal{L}_{1,\text{osc}}^{\beta,\mu} \) in a different form, sometimes more suited for numerical calculations
\[
\mathcal{L}_{1,\text{osc}}^{\beta,\mu} = \int_{-\infty}^{\infty} d\omega (\omega^2 - m^2) f_F(\omega)(eB)^{3/2} \frac{\sqrt{2}}{\pi^2} \zeta \left( \frac{1}{2}, \text{mod} \left[ \frac{\omega^2 - m^2}{2eB} \right] \right),
\]
where \( \text{mod}[A] \) is a shorthand notation for \( A \) modulo 1, i.e.
\[
\text{mod}[A] = A - \text{int}[A].
\]

An alternative way to write Eq. (5.22) is
\[
\mathcal{L}_{1,\text{osc}}^{\beta,\mu} = \sum_{n=0}^{\infty} \int_0^1 ds \sqrt{m^2 + 2eB(n + s)}
\times \left( f_F^+(\sqrt{m^2 + 2eB(n + s)}) + f_F^-(\sqrt{m^2 + 2eB(n + s)}) \right) \zeta(-\frac{1}{2}, s),
\]
where the various Landau-level contributions are made explicit. In addition to \( \mathcal{L}_{\text{eff}}^{\beta,\mu} \) the free energy has a contribution from the thermal photons, i.e.
\[
\frac{F_\gamma(T)}{V} = -\frac{T^4 \pi^2}{45},
\]
which is background field independent since there is no self-interaction among abelian gauge fields.

### 5.2 QED and Charged Scalars

The formalism used so far applies also to scalar QED. We give some of the corresponding results here for completeness. Equation (5.6) becomes in this case
\[
\frac{\partial \mathcal{L}_1}{\partial m^2} = -iG_F(x; x|m^2),
\]
and the thermal propagator is
\[
\langle G_F(x; x|m^2) \rangle_{\beta,\mu} = \sum_{n=0}^{\infty} \int \frac{d\omega dk_y dk_z}{(2\pi)^3} \left( I_{n;k_y}(x) \right)^2
\times \left[ \frac{1}{\omega^2 - E_n^2 + i\epsilon} - 2\pi i \delta(\omega^2 - E_n^2) f_B(\omega) \right],
\]

12
where
\[ E_n^2 = k_x^2 + (2n + 1)eB + m^2, \tag{5.28} \]
and
\[ f_B(\omega) = \frac{\theta(\omega)}{e^{\beta(\omega - \mu)} - 1} + \frac{\theta(-\omega)}{e^{\beta(-\omega + \mu)} - 1}. \tag{5.29} \]

It is rather straightforward to obtain the correction
\[
\mathcal{L}_1 = \frac{1}{16\pi^2} \int_0^\infty \frac{ds}{s^3} \exp(-m^2s) \left( \frac{eBs}{\sinh(eBs)} - 1 + \frac{(eBs)^2}{6} \right), \tag{5.30}
\]
to the effective action in the vacuum sector. At finite chemical potential and temperature we similarly find the following contribution to the effective action
\[
\mathcal{L}_{\beta,\mu}^{\text{eff}} = \frac{1}{6\pi^2} \int d\omega \theta(\omega^2 - m^2 - eB) f_B(\omega) (\omega^2 - m^2)^{3/2} \]
\[ + \int d\omega \theta(\omega^2 - m^2 - eB) f_B(\omega) \left[ \frac{1}{8\pi^{5/2}} \int \frac{ds}{s^{5/2}} e^{-s(\omega^2 - m^2)} \left( \frac{eBs}{\sinh(eBs)} - 1 \right) \right] \]
\[- \int d\omega \theta(\omega^2 - m^2 - eB) f_B(\omega) \left[ \frac{1}{4\pi^2} \sum_{k=1}^{\infty} \left( \frac{eB}{k} \right)^{3/2} \sin \left( \frac{\pi}{4} - \frac{\pi k}{eB} (\omega^2 - m^2 - eB) \right) \right]. \tag{5.31}
\]

The zero temperature part \( \mathcal{L}_1 \) was derived in Ref.\[23\]. Physically this effective action is quite different from the fermionic one. We shall not pursue this investigation here but only make a few remarks. Since for charged bosons there is no sharp Fermi surface, there are no de Haas – van Alphen oscillations either. Furthermore, even the energy of the lowest Landau level \((n = 0)\) depends on \(B\), so that, for example, in the case of a vanishing chemical potential, the number density is Boltzmann suppressed for large fields.

6 The Physical Content of \( \mathcal{L}_{\text{eff}} \)

There are several dimensionful parameters related to \( \mathcal{L}_{\text{eff}}, \) i.e. \(T, \mu, m,\) and \(B,\) that can be large or small compared to each other. We shall discuss some of these limits which we think are particularly interesting. A central feature of a fermion gas is whether it is degenerate or not, i.e. whether or not the Fermi surface is sharp on the scale of the Fermi energy. With an external magnetic field it is also important to compare the smoothness of the Fermi surface with the spacing of the Landau levels. A criteria for the de Haas – van Alphen effect is that the distance between the Landau levels close to the Fermi...
surface is considerably larger than the diffuseness or fluctuations in the Fermi surface due to finite temperature, electron–electron interactions, impurities etc. This can sometimes be achieved even at high $T$ by having large $\mu$ and $B$.

The effective Lagrangian is here given as a function of the chemical potential $\mu$. In many situations it is more natural to consider the expectation value of charge density $Q/V$ as given, where $Q$ is the total conserved charge. It is calculated from $Q/V = -e\rho(\mu)$, where

$$\rho(\mu) = -\frac{1}{V} \frac{\partial F}{\partial \mu} = \frac{\partial \mathcal{L}_{\text{eff}}^{3,\mu}}{\partial \mu},$$

which in the case of vanishing magnetic field and temperature reduces to

$$\sqrt{\mu^2 - m^2} = (3\pi^2 |\rho|)^{1/3},$$

and $\mu$ has the same sign as $\rho$. For large $B$ field this relation gets substantial correction, see e.g. Section 5.2. We notice that $\rho$ is equal to the difference between the electron and positron number densities, that may be useful on comparison with condensed matter physics calculations. In other situations one may consider adiabatic changes of $B$, and then keep the entropy fixed, or the pressure. All these different cases are described by suitable Legendre transformations of the thermodynamical potential $F$.

### 6.1 The de Haas–van Alphen Effect

At low temperature one may attempt an expansion in $T$ using Sommerfeld’s method [25]. We assume that $\mu > m$ since for $|\mu| < m$ the thermal contribution is exponentially suppressed. The Sommerfeld expansion for a function $H(\omega)$ is

$$\int_{m}^{\infty} d\omega f_F^+(\omega)H(\omega) = f_F^+(m) \int_{m}^{\infty} d\omega H(\omega) + \sum_{n=1}^{\infty} T^n a_n \frac{d^{n-1}H(\omega)}{d\omega^{n-1}} \bigg|_{\omega=\mu},$$

where

$$a_n = \int_{-\mu-m}^{\infty} \frac{dx}{n!} x^n \left( -\frac{\partial}{\partial x} \frac{1}{e^x + 1} \right),$$

but the odd powers of $T$ are exponentially suppressed. This formula can be applied to $\mathcal{L}_{0,\mu}^{3,\mu}$ and $\mathcal{L}_{1,\text{reg}}^{3,\mu}$, but in $\mathcal{L}_{1,\text{osc}}^{3,\mu}$ performing the derivative inside the summation sign is not allowed since the sum is not uniformly convergent, and when acting on the form containing the $\zeta$-function there will obviously be divergences at discrete points. This indicates that an expansion in $mT/eB$ is not possible. Anyway, the $T=0$ part of $\mathcal{L}_{1,\text{osc}}^{3,\mu}$ can be calculated,
and if we in particular assume \( \{ T = 0, eB \ll \mu^2 - m^2 \ll m^2 \} \) we get
\[
\mathcal{L}_{1}^{\beta,\mu} \approx \frac{(eB)^2}{12\pi^2} \frac{\sqrt{\mu^2 - m^2}}{m} - \frac{(eB)^{5/2}}{4\pi^3 m} \sum_{n=1}^{\infty} \frac{1}{n^{5/2}} \left[ \cos \left( \frac{\pi}{4} - n\pi \frac{\mu^2 - m^2}{eB} \right) - \frac{1}{\sqrt{2}} \right] . \tag{6.5}
\]
This is a non-relativistic limit (in the sense that the kinetic energy is much smaller than \( m \)) with a degenerate Fermi sea and a weak external field.

The vacuum correction is in this limit given by
\[
\mathcal{L}_{1} \approx \frac{(eB)^2}{360\pi^2} \left( \frac{eB}{m^2} \right)^2 , \tag{6.6}
\]
so that the finite density correction
\[
\mathcal{L}_{1}^{\beta,\mu} \approx \frac{(eB)^2}{12\pi^2} \left( \frac{3\pi^2 \rho}{m^3} \right)^{1/3} , \tag{6.7}
\]
therefore dominates over \( \mathcal{L}_{1} \) when
\[
\left( \frac{\rho}{m^3} \right)^{1/3} \gg \frac{1}{30(3\pi^2)^{1/3}} \left( \frac{eB}{m^2} \right)^2 , \tag{6.8}
\]
or equivalently, in terms of the chemical potential
\[
\frac{\sqrt{\mu^2 - m^2}}{m} \gg \frac{1}{30} \left( \frac{eB}{m^2} \right)^2 . \tag{6.9}
\]
This is always satisfied in the limit \( \{ eB \ll \mu^2 - m^2 \ll m^2 \} \).

Even though the \( B^2 \) dominates over \( B^{5/2} \) for small \( B \), the magnetization of the heat bath\(^4\) gets a larger contribution from \( \mathcal{L}_{1,\text{osc}}^{\beta,\mu} \)
\[
M = M_{\text{reg}} + M_{\text{osc}} = -\frac{1}{V} \frac{\partial F}{\partial B} = \frac{\partial \mathcal{L}_{\text{eff}}^{\beta,\mu}}{\partial B} , \tag{6.10}
\]
where to the lowest order in the magnetic field
\[
M_{\text{osc}} = \frac{e\sqrt{eB}(\mu^2 - m^2)}{4\pi^3 m} \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \sin \left( \frac{\pi}{4} - n\pi \frac{\mu^2 - m^2}{eB} \right) = -\left( \frac{e}{2m} \right) \sqrt{2eB} (\mu^2 - m^2) \zeta \left( -\frac{1}{2}, \text{mod} \left[ \frac{\mu^2 - m^2}{2eB} \right] \right) , \tag{6.11}
\]
and
\[
M_{\text{reg}} = \left( \frac{e}{2m} \right) \frac{eB}{3\pi^2} \sqrt{\mu^2 - m^2} . \tag{6.12}
\]
\(^4\)The vacuum contribution to the magnetization is not included in Eq.(6.10) since it is very small for small \( B \).
The \( \zeta \)-function has its maximal modulus at \( \zeta \left( -\frac{1}{2}, 1 \right) = \zeta \left( -\frac{1}{2}, 0 \right) \approx -0.208 \), which implies that the peak magnetization from the oscillating term is larger than that from the regular term for \( \{ eB \lesssim 0.78(\mu^2 - m^2) \} \), i.e. when the approximations used here are valid. Defining the magnetic susceptibility as the response in the magnetization due to a magnetic field, i.e. \( M = \chi B \), as in Ref.\[26\], we get exact agreement with this reference, but not with Refs.\[27, 28\], which have an extra factor \( (-1)^n \) in the sum over \( n \), that we find only should be present in the case of spinless bosons. In Section 6.2 we give an argument why our result has to be correct.

The oscillatory behaviour as a function of \( B \) is well-known as the de Haas – van Alphen effect. The frequency of this periodic function agrees with the one derived by Onsager \[29\]. Equation (5.21) describes the full relativistic generalization of this effect, and in Section 7 we consider some astrophysical applications where the non-relativistic approximation is not valid. The distance between the magnetic field of two adjacent minima of the magnetization is determined by

\[
\left| \frac{1}{eB_i} - \frac{1}{eB_{i+1}} \right| = \frac{2\pi}{A},
\]

(6.13)

where \( A \) is the area of an extremal cross section of the Fermi sea.

Sometimes (e.g. in Ref.\[25\]) the magnetic susceptibility is defined by

\[
\chi = \frac{\partial M}{\partial B},
\]

(6.14)

but again we find that the sum over \( n \) does not converge, and that the form containing the \( \zeta \)-function contains divergences at discrete values of \( B \), and is poorly illuminating.

### 6.2 Strong B-field

In the limit of strong field, \( \{ eB \gg T^2, m^2, |\mu^2 - m^2| \} \), we can see from Eq.(5.13) that only the lowest Landau level contribute and \( \mathcal{L}_{\text{eff}}^{\beta, \mu} \) goes like a linear function of \( eB \). We shall now reproduce this result from Eq.(5.21) and it turns out to be rather non–trivial. The leading \( B \) dependence in the first term in Eq.(5.21) is obtained by scaling out \( eB \) and taking \( eB \to \infty \) in the remainder. The total contribution is, apart from the thermal integration (see Appendix B)

\[
\frac{(eB)^{3/2}}{4\pi^{5/2}} \left[ \int_0^\infty \frac{dx}{x^{5/2}} (x \coth x - 1) - \sqrt{\frac{2}{\pi}} \sum_{n=1}^\infty \frac{1}{n^{3/2}} \right],
\]

(6.15)
but this is actually identically zero. The next subleading term can be shown to be

\[ \mathcal{L}_1^{\beta,\mu} = \frac{eB}{2\pi^2} \int_{-\infty}^{\infty} d\omega \theta(\omega^2 - m^2) f_F(\omega) \sqrt{\omega^2 - m^2}, \quad (6.16) \]

which is exactly the leading term from Eq.(5.13). This calculation shows that the oscillatory term in Eq.(5.21) is absolutely necessary to cancel the $B^{3/2}$ term and to give the correct linear term. Also, notice that the expression presented here for this term has to be correct, without the extra factor $(-1)^n$ of Refs.[28, 27], for the $B^{3/2}$ terms to cancel.

In this limit of strong magnetic field the thermal and density corrections given above are small compared to

\[ \mathcal{L}_1 \approx \frac{(eB)^2}{24\pi^2} \log \left( \frac{eB}{m^2} \right). \quad (6.17) \]

The vacuum polarization effects are dominating here, which comes quite naturally, since the magnetization from real thermal particles becomes saturated when all spins are aligned, whereas the magnetization from vacuum polarization increases like $B \log B$. This has not always been recognized in the literature [24].

Another issue when the $B$ field is strong compared to $\mu^2 - m^2$ is that the relation between $\rho$ and $\mu$ is changed. In fact, we have from Eq.(6.16) at $T = 0$

\[ \rho(\mu) \approx \frac{eB}{2\pi^2} \sqrt{\mu^2 - m^2}. \quad (6.18) \]

The linear dependence on the Fermi momentum $k_F = \sqrt{\mu^2 - m^2}$ can be understood from the fact that only the lowest Landau level is filled and therefore the phase space is essentially one-dimensional.

### 6.3 Weak B-field

In Section 6.1 we had an expression for $\mathcal{L}_{eff}^{\beta,\mu}$ in a weak ($\ll \mu^2 - m^2$) field but $T^2$ still smaller than $eB$. An expansion for $B$ smaller than all other scales would be desirable but there are some subtleties involved in such an expansion. The vacuum part can be expanded in a naive way and we get

\[ \mathcal{L}_1 = -\frac{m^4}{4\pi^2} \sum_{k=1}^{\infty} \left( \frac{eB}{\pi m^2} \right)^{2k+2} (-1)^k \zeta(2k+2)\Gamma(2k). \quad (6.19) \]

This series is not convergent but Borel summable for small $eB/m^2$ so we expect the first few terms to be a good approximation for weak fields. Expanding the integrand of $\mathcal{L}_{1,reg}^{\beta,\mu}$ (see Eq.(5.21)) in powers of $B$ leads to the same problem after the $s$-integration. Moreover,
the $\omega$-integration becomes infra-red divergent, for higher order terms. We cannot even expand the integrand of $L_{1,osc}^{\beta,\mu}$ in powers of $B$, but after repeated partial integrations with respect to $\omega$ we obtain

$$L_{1,osc}^{\beta,\mu} = \frac{m^4}{4\sqrt{2}\pi^{3/2}} \sum_{k=0}^{\infty} \left( \frac{eB}{\pi m^2} \right)^{5/2+k} \zeta(5/2+k)(-1)^{k/2}$$

$$\times \left( m^2 \frac{d}{d\omega^2} \right)^k \frac{m}{\omega} \left( f_+^\omega(\omega) + f_-^\omega(\omega) \right) \bigg|_{\omega=m},$$

(6.20)

where $[k/2]$ is the integral part of $k/2$. When $|\mu| > m$ the factor with derivatives of $f_+^\omega(\omega)$ at $\omega = m$ contains powers of $m/T$. These factors, combined with the $B/m^2$ factors, show that we must have $\{B \ll m^2, T^2\}$ in order for the expansion to be valid. For $|\mu| < m$ these terms are exponentially suppressed at small $T$. We thus see that there is an intricate interplay between $B$ and $T$ in such a way that when $\{eB \ll T^2\}$ $L_{1,osc}^{\beta,\mu}$ is smaller than $L_{1,reg}^{\beta,\mu}$, as well as their derivatives. However, when $\{T^2 \ll eB\}$, even though $\{eB \ll \mu^2 - m^2, m^2\}$, the $B$ derivatives of $L_{1,osc}^{\beta,\mu}$ are large and show a periodic behaviour as shown in Section 6.1. Also the expansion from Eq.(5.20) is only asymptotic. In view of the observations above, especially the half-integer powers of $B$ in Eq.(6.20), it seems unlikely that the same result can be obtained in ordinary perturbation theory using diagrammatic techniques. The vanishing radius of convergence for the expansion of $L_1$, and the same for $L_{1,reg}^{\beta,\mu}$, also including the infra-red divergences, arise due to the fact that we get substantial contributions to the parameter integrals when we are outside the radius of convergence for the series expansion of the $\coth(eBs)$, i.e. for large $s$, for high order terms. We will investigate this, the non-analyticity in $B$, and the connection to ordinary perturbation theory more carefully in a future project. Some weak-field results can nevertheless be obtained and, for instance, the magnetic susceptibility can be computed in the limit $\{B \to 0, T \ll \mu^2 - m^2\}$. It gets contribution only from $L_{1,reg}^{\beta,\mu}$,

$$\chi = \lim_{B \to 0} \frac{\partial^2 L_{1,reg}^{\beta,\mu}}{\partial B^2} = \frac{e^2}{6\pi^2} \log \left( \frac{|\mu|}{m} + \frac{\sqrt{\mu^2 - m^2}}{m} \right).$$

(6.21)

If we further assume that $\{\mu^2 - m^2 \ll m^2\}$ and write it in terms of the Bohr magneton $\mu_B = e/2m$ and the density of states at the Fermi surface $g(\mu) = m\sqrt{\mu^2 - m^2}/\pi^2$, we find

$$\chi = \frac{2}{3} \mu_B^2 g(\mu) = \chi_{Pauli} + \chi_{Landau}.$$

(6.22)

It coincides with the well-known result \cite{25} where $\chi_{Pauli}$ is the Pauli paramagnetic spin contribution and $\chi_{Landau} = -\frac{1}{3}\chi_{Pauli}$ is the Landau diamagnetic orbital contribution.
Notice that in this weak field limit the thermal corrections dominate, i.e.

\[ \mathcal{L}^{\beta,\mu}_1 \approx \frac{(eB)^2}{12\pi^2} \sqrt{\frac{\mu^2 - m^2}{m}} \gg \mathcal{L}_1, \]  

(6.23)

where \( \mathcal{L}_1 \) is given by Eq. (6.6).

### 6.4 High Temperatures

At high temperatures one may find an analytical approximation in the limit \( \{ T^2 \gg m^2 \gg eB, \mu = 0 \} \), where we have that

\[ \mathcal{L}^{\beta,\mu}_1 \approx \frac{(eB)^2}{24\pi^2} \log \left( \frac{T^2}{m^2} \right), \]  

(6.24)

and we do not agree with the high temperature and weak field limit in Ref. [12]. (We notice the similarity between Eq. (6.24) and \( \mathcal{L}_1 \) for \( eB \gg m^2 \) in Eq. (5.17).) The thermal contribution \( \mathcal{L}^{\beta,\mu}_1 \) thus dominates over \( \mathcal{L}_1 \) as given by Eq. (6.6) when

\[ \frac{T}{m} \gg \exp \left[ \frac{1}{30} \left( \frac{eB}{m^2} \right)^2 \right] \approx 1, \]  

(6.25)

i.e. when the approximations used here are valid.

### 7 Some Astrophysical Applications

As mentioned in the Introduction, strong magnetic fields at finite temperature and density are situations that are frequently encountered in astrophysical contexts. We have investigated the possibility of some interesting behaviour mainly for white dwarfs, neutron stars and supernovae since they present the most extreme conditions while still being directly observable, in contrast to e.g. cosmic strings, the existence of which has yet to be confirmed. We can use the effective action in two ways. Either we consider the response of the system to a given external magnetic field \( H \), or we study the properties of an isolated system with only the induced magnetic field. In the first case the free energy is given by

\[ F = -\mathcal{L}_1(B) - \mathcal{L}_{eff}^{\beta,\mu}(B), \]  

(7.1)

where \( B \) is determined by the mean field equation

\[ B = H + M(B) = H + \frac{\partial \mathcal{L}_1}{\partial B} + \frac{\partial \mathcal{L}_{eff}^{\beta,\mu}}{\partial B}. \]  

(7.2)
The magnetization $M(B)$ is thus calculated in the presence of the microscopic magnetic field $B$. Note that we include both the contribution from real electrons in the heat bath and virtual electrons from vacuum polarization. If we consider the dynamics of the system without any external field we should add $\mathcal{L}_0$ to the effective action and determine stationary values of the field by

$$\frac{\partial \mathcal{L}_{\text{eff}}}{\partial B} = -B + \frac{\partial \mathcal{L}_1}{\partial B} + \frac{\partial \mathcal{L}_\beta\mu_{\text{eff}}}{\partial B} = 0 ,$$

(7.3)

which, of course, is the same as putting $H = 0$ in Eq.(7.2). As discussed in Section 6.2 the vacuum contribution is dominant for large fields. Using the result from \cite{24} we see that for $T = m$ the thermal contribution saturates at about $eB = 10 m^2$. At that value of the magnetic field, the vacuum contribution is about twice as large as the thermal and cannot be ignored.

It would be most interesting if we could find astrophysical objects showing the de Haas – van Alphen oscillations. The magnitude of the oscillations might then be large enough to effectively trap the magnetic field in a local minimum satisfying Eq.(7.3). A candidate for such a system is a neutron star with a strong $B$ field and a degenerate electron gas. In order to get de Haas – van Alphen oscillations as a function of $B$ the spacing of Landau levels near the Fermi surface need to be larger than the spreading of the Fermi surface due to finite temperature. If the $n$-th Landau level is at the Fermi surface, $E_n = \mu$, then we require $E_{n+1} - E_n \gtrsim T$. For $\mu^2 \gg eB$, which is the case for neutron stars, we get the condition

$$eB \gtrsim \mu T .$$

(7.4)

According to Appendix C we can even get a more stringent condition in the case of large chemical potential

$$eB \gtrsim 2\pi^2 \mu T .$$

(7.5)

As a comparison, we find in the non-relativistic case, instead of Eq.(7.4) that

$$eB \gtrsim mT , \quad \{eB, \mu^2 - m^2 \ll m^2\} .$$

(7.6)

In order to see any oscillations the field must not be so high that all fermions are in the lowest Landau level, i.e. integer $n$ above must be greater than unity, that gives

$$(\mu^2 - m^2)/2 > eB .$$

(7.7)
Approximate values for \( eB, T \) and \( \mu \) for what we find the most interesting astrophysical objects in this context, a supernova; a neutron star; and a white dwarf, are given in Table 1. According to above, the number in the last two rows of this table should be greater than unity for de Haas – van Alphen oscillations to appear. That is not the case in either of the situations. For a neutron star we have numerically computed the different parts of the effective Lagrangian, and the corresponding magnetization. The results are given in Table 2 and Table 3, respectively. The effective Lagrangian is totally dominated by the thermal contribution in absence of a magnetic field, \( \mathcal{L}_0^{\beta,\mu} \), due to the extreme chemical potential. We would like to stress that there are no oscillations in the so called oscillating part of the magnetization, \( M_{osc} \), in this region of parameters. Obviously we do not expect to see any de Haas – van Alphen oscillations unless the neutron star is very cold \( (T = \mathcal{O}(1) \text{ eV}) \), or if the electron density is very low in some region, for example close to the surface, where the field still is strong.

### Table 1: Typical values of \( eB, T \) and \( \mu \) for some astrophysical objects, and an indication of the possibility for oscillations in the magnetization. The references are given in brackets.

|            | White Dwarf | Neutron Star | Supernova |
|------------|-------------|--------------|-----------|
| \( \mu/m \) | 1.02 \( \times 10^3 \) | 6 \( \times 10^2 \) | 6 \( \times 10^2 \) |
| \( T/m \)  | 2 \( \times 10^{-3} \) | 1 | 1 \( \times 10^2 \) |
| \( eB/m^2 \) | 2 \( \times 10^{-6} \) | 2 \( \times 10^{-1} \) | 2 |
| \( (\mu^2 - m^2)/(2eB) \) | 1 \( \times 10^3 \) | 2 \( \times 10^6 \) | 2 \( \times 10^5 \) |
| \( eB/(\mu T) \) | 1 \( \times 10^{-3} \) | 3 \( \times 10^{-4} \) | 3 \( \times 10^{-5} \) |

### Table 2: The different parts of the effective Lagrangian for a typical neutron star, in natural units.

| \( \mathcal{L}_0 \) \( (m^4) \) | \( \mathcal{L}_1 \) \( (m^4) \) | \( \mathcal{L}_0^{\beta,\mu} \) \( (m^4) \) | \( \mathcal{L}_{1, \text{reg}}^{\beta,\mu} \) \( (m^4) \) | \( \mathcal{L}_{1, \text{osc}}^{\beta,\mu} \) \( (m^4) \) |
|----------------|----------------|----------------|----------------|----------------|
| \( -2 \times 10^{-2} \) | 2 \( \times 10^{-6} \) | 1 \( \times 10^9 \) | 6 \( \times 10^{-2} \) | 1 \( \times 10^{-3} \) |

### Table 3: The different parts of the magnetization for a typical neutron star, in natural units.

| \( M_1 \) \( (em^2) \) | \( M_{\text{reg}} \) \( (em^2) \) | \( M_{\text{osc}} \) \( (em^2) \) |
|----------------|----------------|----------------|
| 2 \( \times 10^{-5} \) | 4 \( \times 10^{-2} \) | 1 \( \times 10^{-2} \) |
In order to investigate the behaviour of a relativistic gas of fermions showing de Haas – van Alphen oscillations, we have numerically calculated the effective action, and the magnetization for \( \{ \mu/m = 4; \ T/m = 0.01, 0.1, 1.0 \} \). The latter is shown in Fig. 1. We see clearly how the oscillations disappear as the temperature is raised. There is also a last oscillation at about \( eB \simeq 7 m^2 \) which occurs when the second Landau level leaves the Fermi surface. For the values above we do not find any non–trivial solution to Eq.(7.3) because the tree level \(-B\) dominates. It is, in fact, only for a rather limited range of parameters that \( \mathcal{L}^\beta_{1,osc} \) can give local maxima for the total effective action. As an example, let us first put \( T = 0 \) since that only enhances the oscillations. Then we look at small \( B \) so that the tree part is small. There is a chance that the \( \sqrt{B} \) term in Eq.(6.11) can dominate. Using \( \mu \simeq m \) and making the approximation \( |\zeta(-\frac{1}{2}, x)| \leq 0.2 \), we get

\[
|M_{osc}| \lesssim 0.2 \frac{\sqrt{2} e^{3/2}}{2 m \pi^2} \sqrt{B}(\mu^2 - m^2) \simeq 0.005 \sqrt{B}(\mu - m). \tag{7.8}
\]

For this term to dominate over \( |M_{tree}| = B \) we need \( B \lesssim 10^{-5}(\mu - m)^2 \) which complicates numerical calculations. Also, since the field is small the probability of tunneling through
the barrier between the maxima is not very suppressed and it is probably not an efficient way of trapping magnetic fields. At very large values of $B$ the vacuum part eventually dominates over the tree level, but this is just the Landau ghost and we cannot draw any conclusion about any instability.

Even if there are no local minima in $-B^2/2 - \mathcal{L}_1 - \mathcal{L}_\text{eff}^\beta\mu$, there may be intervals in $B$ where $-\mathcal{L}_1 - \mathcal{L}_\text{eff}^\beta\mu$ is concave, i.e. where the susceptibility is positive. Domains with different magnetization could then be formed in presence of an external field, just like in some solid state materials [27].

8 THE EFFECTIVE QED COUPLING

The charge renormalization given by Eq.(5.8) also leads to the weak coupling expansion of the QED $\beta$-function, i.e.

$$\lambda \frac{d}{d\lambda} \alpha(\lambda) = \beta(\alpha(\lambda)) = \frac{2}{3\pi} \alpha^2(\lambda) + \mathcal{O}(\alpha^3(\lambda)) \quad , \quad (8.1)$$

where $\lambda$ is a momentum scale factor. We notice that due to the scale invariance of $eB$, we can also define an effective coupling constant from $\mathcal{L}_\text{eff}$ as [23, 31]

$$-\frac{1}{e^2(eB, \mu, T)} = \frac{1}{eB} \frac{\partial \mathcal{L}_\text{eff}}{\partial (eB)} \quad , \quad (8.2)$$

that gives for the electromagnetic fine structure constant $\alpha(eB, \mu, T) \equiv e^2(eB, \mu, T)/4\pi$

$$\frac{1}{\alpha(T, \mu, B)} = \frac{1}{\alpha} - \frac{1}{\alpha B} \frac{\partial (\mathcal{L}_1 + \mathcal{L}_\text{eff}^\beta\mu)}{\partial B} \quad , \quad (8.3)$$

in analogy with the definition of the renormalized coupling in the vacuum sector in connection with Eq.(8.1). Special care has to be taken when evaluating the derivative of the oscillating term in Eq.(5.21). In the limit when $eB = 0$, we obtain the effective coupling $\alpha(T, \mu) = \alpha(T, \mu, B = 0)$ given by

$$\frac{1}{\alpha(T, \mu)} = \frac{1}{\alpha} - \frac{2}{3\pi} \int_{-\infty}^{\infty} d\omega \frac{\theta(\omega^2 - m^2)}{\sqrt{\omega^2 - m^2}} f_F(\omega) \quad . \quad (8.4)$$

When $T = 0$, we therefore get an effective coupling $\alpha(\mu) = \alpha(T = 0, \mu)$ such that

$$\frac{1}{\alpha(\mu)} = \frac{1}{\alpha} - \frac{2}{3\pi} \log \left( \frac{|\mu|}{m} + \sqrt{\frac{\mu^2}{m^2} - 1} \right) \quad . \quad (8.5)$$
In the limit \( \mu = 0 \), we find the following asymptotic behaviour of the corresponding effective coupling \( \alpha(T) = \alpha(T, \mu = 0) \),

\[
\frac{1}{\alpha(T)} = \frac{1}{\alpha} - \frac{4}{3\pi} \int_{\beta m}^\infty \frac{dx}{\sqrt{x^2 - (\beta m)^2}} e^{x^2} 1 + 1 \approx \frac{1}{\alpha} - \frac{2}{3\pi} \log \left( \frac{T}{m} \right),
\]

(8.6)

for \( T \gg m \). It is now clear that (only) for \( \mu \gg m \) and \( T \gg m \) the effective couplings \( \alpha(\mu) \) and \( \alpha(T) \) are solutions to the renormalization group equation (8.1) when \( \lambda \) is identified with \( \mu \) and \( T \) respectively (see in this context e.g. Refs. [32, 13]). We also note that Eq.(6.17) leads to an effective coupling \( \alpha(B) = \alpha(T = 0, \mu = 0, B) \) with an asymptotic behaviour

\[
\frac{1}{\alpha(B)} \approx \frac{1}{\alpha} - \frac{2}{3\pi} \log \left( \frac{\sqrt{eB}}{m} \right),
\]

(8.7)

that also satisfies the renormalization group equation Eq.(8.1). The effective coupling defined in Eq.(8.3) can also be extracted from the residue of the thermal Debye-screened photon propagator (see Ref.[32]).

The effective couplings as given in Eqs.(8.5), (8.6) and (8.7) can be interpreted as follows. If we use the lowest order \( \beta \)-function in Eq.(8.1), then the scale dependent coupling is given by

\[
\frac{1}{\alpha(\lambda)} = \frac{1}{\alpha} - \frac{1}{3\pi} \log \left( \frac{\lambda^2}{m^2} \right).
\]

(8.8)

Then we can write

\[
\frac{1}{\alpha(x)} \approx \frac{1}{\alpha(\lambda)} - \frac{2}{3\pi} \log \left( \frac{x}{\lambda} \right),
\]

(8.9)

where \( x = \mu, T \) or \( \sqrt{eB} \). If \( \lambda \) is identified with any of these scales, we can in each such case write

\[
\mathcal{L}_{\text{eff}} = -\frac{1}{2} \frac{(eB)^2}{e^2(x)} + \mathcal{L}_{0}^{\beta,\mu} + \mathcal{L}_{0}^{\beta,\mu},
\]

(8.10)

when \( x \gg (m \text{ and any other scale of dimension energy}) \).

In terms of the effective fine-structure constant, and in the case of small chemical potentials, so that \( |\mu| < m \), we obtain

\[
\alpha(eB, T, \mu) = \frac{\alpha}{1 - \alpha X(eB, T, \mu)},
\]

(8.11)

where we have defined the functions \( X(eB, T, \mu) = X_1(eB) + X_2(eB, T, \mu) \),

\[
X_1(eB) = \frac{1}{2\pi} \int_0^\infty \frac{dx}{x} \exp\left( -\frac{m^2}{eB} \right) \left[ \frac{1}{\sinh^2(x)} - \frac{\coth(x)}{x} + \frac{2}{3} \right],
\]

(8.12)
and

\[ X_2(eB, T, \mu) = \frac{1}{2\pi} \sum_{l=1}^{\infty} (-1)^l \int_0^\infty \frac{dx}{x} \exp \left( -\frac{\beta^2 l^2}{4x} - m^2 x \right) \]
\[ \times \left[ \frac{1}{\sinh^2(eBx)} - \frac{\coth(eBx)}{eBx} \right] \cosh(\beta l \mu) \ . \tag{8.13} \]

The function \(X_1(eB)\) has the following expansions

\[ X_1(eB) = 2 \frac{45}{45\pi} \left( \frac{eB}{m^2} \right)^2 + O \left( \left( \frac{eB}{m^2} \right)^4 \right) \] (8.14)

if \(eB \ll m^2\) and

\[ X_1(eB) = \frac{1}{3\pi} \log \left( \frac{eB}{m^2} \right) \left( 1 + \frac{3m^2}{2eB} \right) + O \left( \frac{m^2}{eB} \right) \] (8.15)

if \(eB \gg m^2\). In the case of a vanishing chemical potential we can in Eq.(8.13) identify a \(\vartheta_4\)-function, given as

\[ \vartheta_4[z, q] = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^n \cos(2nz) \ , \tag{8.16} \]

and write

\[ X_2(eB, T) = \frac{1}{2\pi} \int_0^\infty \frac{dx}{x} \exp(-x \frac{m^2}{eB}) \]
\[ \times \left\{ 1 - \vartheta_4 \left[ 0, \exp \left( -\frac{eB\beta^2}{4x} \right) \right] \right\} \left[ \frac{\coth(x) - \frac{1}{\sinh^2(x)}}{x} \right] \ . \tag{8.17} \]

If \(eB \ll m^2\), we can write

\[ X_2(eB, T) = 4 \frac{3\pi}{3\pi} \sum_{l=1}^{\infty} (-1)^{l+1} \left( K_0(\beta ml) - \frac{(\beta ml)^2}{2} K_2(\beta ml) O \left( \frac{eB^2}{m^2} \right) \right) \] (8.18)

For \(T \gg m\) we can use

\[ \sum_{l=1}^{\infty} K_0(xl)(-1)^{l+1} \rightarrow -\frac{1}{2} \log x \ ; \ x \rightarrow 0 \ , \tag{8.19} \]

to find that it leads to a \(\log(T/m)\) dependence with the correct prefactor in accordance with Eq.(8.9). (The approximation of keeping only the \(l = 1\), as in Ref.[13], excludes the factor \(1/2\), and thus is not correct.) In general we have that

\[ X_1(eB) + X_2(eB, T) = \frac{1}{2\pi} \int_0^\infty \frac{dx}{x} \exp(-x \frac{m^2}{eB}) \]
\[ \times \left\{ 2 - \vartheta_4 \left[ 0, \exp \left( -\frac{eB\beta^2}{4x} \right) \right] \right\} \left[ \frac{\coth(x) - \frac{1}{\sinh^2(x)}}{x} \right] \ . \tag{8.20} \]
9 Discussion and final remarks

9.1 Inclusion of interparticle interactions

In our one-loop treatment of the effective action we have not included interactions between electrons. The interaction energy between the particles increases with $T$ and $\mu$ since the density increases, but so does kinetic energy. For a degenerate electron gas with large chemical potential the kinetic energy dominates over the potential energy for electrons close to the Fermi surface. However, not all electrons have large kinetic energy and corrections from interactions have to be considered for electrons with low momenta. The self-energy correction for fermions at high temperature and density, but zero external field, has been computed in e.g. Refs.\[33, 34\] (in Ref.\[34\] only massless fermions were considered but it gives an indication of the correction, especially in view of the result in Ref.\[33\]). There appear some completely new collective phenomena, such as hole excitations \[35\], which are not taken into account in this paper. For the particle excitations the dispersion relation can be approximated by an ordinary massive particle provided the mass is replaced by an effective $T$ and $\mu$ dependent mass \[33\]

$$m_p = \sqrt{m_e^2 + 4M^2 + m_e^2}, \quad (9.1)$$

where $M$ is the thermally induced mass which in the case of QED is

$$M^2 = \frac{e^2\mu^2}{8\pi^2}; \quad T = 0, \mu \neq 0,$$

$$M^2 = \frac{e^2T^2}{8}; \quad T \neq 0, \mu = 0, \quad (9.2)$$

at least if $T \gg m$ and $\mu \gg m$. The hole excitation has a more peculiar dispersion relation but its spectral weight is on the other hand lower. It is difficult to make any quantitative estimates of the importance of self-energy corrections. We do not, however, expect that phenomena like the de Haas – van Alphen oscillations should be altered since it depends on the electrons at the Fermi surface.

9.2 Further developments

There are some extensions of our work that may be of physical importance. First, we can consider the self-energy correction of an electron in presence of an external $B$ field.
From that the anomalous magnetic moment can be extracted and compared with previous
calculations for small $B$ field, where there appears some problems of analyticity in the
external photon momentum at finite density [15]. The self-energy is also important for
the higher loop corrections of the effective action as discussed above. The QED radiative
corrections could effect the electroweak transition rates, relevant for the Big-Bang primor-
dial nucleosynthesis [30]. The photon polarization tensor should also be calculated, and
in particular its imaginary part which is related to the decay into an $e^+e^-$-pair. Also the
three-photon vertex is interesting since it does not exist in absence of the external field.
Such photon splitting processes have been considered earlier in vacuum [37, 38, 39, 40]
and it would be interesting to study the correction from a thermal environment.

The physically more complicated case of a constant (or slowly varying) $E$ field is
equally interesting. A plasma does not stay in equilibrium since the $E$ field gets screened
and the physical picture is very different from the one discussed in this paper. Yet another
generalization would be to expand a non-constant field in powers of the derivative. Such
an expansion has been studied in Ref.[41] at zero temperature.

9.3 Conclusion

The main objective of this paper has been to establish the correct form of the one-
loop QED effective action at finite temperature and density to all orders in a constant
external magnetic field, and the result differs from earlier attempts. From the form of
$L_{\epsilon}^{\beta, \mu}$ presented in Eq.(5.21) we have checked several limits that can be understood from
a physical point of view. A great advantage with our expression for $L_{\epsilon}^{\beta, \mu}$ is that the
thermal distribution function $f_F(\omega)$ occurs explicitly. This means that it is easy to study
other thermal situations by simply replacing $f_F(\omega)$ with some other (non-equilibrium)
distribution (see e.g. Ref.[22]). The importance of the thermal correction depends on the
value of $B$, $T$ and $\mu$. In some physically interesting cases they may be large compared to
$m$ but often of the same order of magnitude, which makes it difficult to obtain analytical
approximations. It is, however, possible to use Eq.(5.21), or the expressions in Appendix
C, for numerical calculations.

Even though the correction to the free energy may be small compared to the value
without the external field there are other quantities that are effected by the presence of the
heat bath. For instance, the magnetization of a degenerate Fermi sea shows the de Haas –
van Alphen effect. We found, however, that for a neutron star this effect does not show up
in spite of the extreme degeneracy and magnetic field. The reason is the relativistic form
of the energy spectrum which suppresses the oscillations at a large chemical potential. We also briefly discussed the importance of including the vacuum contribution to the magnetization when the $B$ field is comparable to $m^2/e$.

We have, furthermore, calculated an effective coupling constant defined from the derivative of $L_{\text{eff}}^{\beta,\mu}$ with respect to $B$. It satisfies asymptotically a naive zero temperature renormalization group equation where the renormalization scale is replaced by $T, \mu$ or $\sqrt{eB}$.

**ACKNOWLEDGMENT**

One of the authors (B.-S. S.) would like to thank John Ellis for the hospitality of the Theory Division at CERN where some of this work was initiated and NFR for providing the financial support. P. E. wants to thank C. Pethick for discussions about neutron stars. It is a pleasure to thank the organizers of the 3rd Workshop on Thermal Field Theories, 1993, and in particular R. Kobes and G. Kunstatter, for providing a stimulating atmosphere during which parts of the present work were finalized.

**APPENDIX A**

In this appendix we give some details of how to calculate the effective action in Eq.(5.21). First we show that Eq.(5.20) is equal to Eq.(5.21). To do that we start with a Poisson resummation in $l$ using

$$
\sum_{l=1}^{\infty} (-1)^l \exp(-\frac{l^2}{4a}) = \sqrt{4\pi a} \sum_{l=0}^{\infty} \exp(-a\pi^2(2l+1)^2) - \frac{1}{2},
$$

(A.1)

and rewrite the sum over $l$ as a contour integral by means of the formula

$$
\sum_{l=0}^{\infty} f\left(\frac{n}{\beta}(2n + l)\right) = \frac{\beta}{2\pi} \int_{C} \frac{d\omega f(\omega)}{e^{\beta \omega} + 1}.
$$

(A.2)

The integration contour $C$ is chosen to go from $\infty + i\epsilon$ to $\epsilon$ in the upper half plane and back to $\infty - i\epsilon$ in the lower half plane (i.e. $\omega \in \{\infty + i\epsilon \rightarrow \epsilon \rightarrow \infty - i\epsilon\}$), without encircling the origin. In this way all the poles on the positive real axis are encircled.

We would now like to deform the $\omega$-integral to the imaginary axis and the $s$-integral to the negative real axis. This is not straightforward since there are poles on the imaginary $s$-axis and the section at infinity has to be chosen to give a vanishing contribution. It is, therefore, necessary to divide the integral into several pieces and to do the deformation
for each piece separately. Let us start with the part where \( \omega \) is in the upper half plane. Then the \( s \)-contour can be deformed to the negative imaginary axis, but to the right of the poles. After that the \( \omega \) contour is deformed to the positive imaginary axis. Finally, for \( |\omega| > m \) we further continue the \( s \)-integral to the negative real axis and pick up the poles on the negative imaginary axis, while for \( |\omega| < m \) we deform the \( s \)-contour back to the positive real axis.

The whole procedure can be repeated for \( \omega \) below the real axis, reflecting all deformations around the real axis. To get the correct convergence for the \( \omega \)-contour deformation, the constant \(-\frac{1}{2}\) in Eq.(A.1) should be associated with the \( \omega \) in the lower half plane. After summing the pieces there is only a contribution from \( |\omega| > m \), as expected, and it consists of an \( s \)-integral and a sum over the residues of the poles.

In the deformations above we have been careful with the convergence for large \( |s| \) and \( |\omega| \), but we have said nothing about the possible singularity at \( s = 0 \). One way of dealing with that is to multiply the expression with \( s^\nu \) and perform the integration for such a \( \nu \) that there is no divergence at \( s = 0 \), and to do the analytic continuation at the end.

Equation (5.21) can also be obtained from the thermal propagator in Eq.(3.13) by representing the \( \delta \)-function as

\[
2\pi i \delta(x) = i \text{Im} \frac{1}{x - i\epsilon} = i \text{Im} \int_0^\infty ds \, e^{-i(x - i\epsilon)} .
\]  

(A.3)

Then the \( k_y \) and \( k_z \) integrations can be carried out (using Eq.(3.9) as well). The summation over \( n \) is just a geometric series but it is not absolutely convergent so we sum only to a finite \( N \) and take the limit \( N \to \infty \) at the end. This gives

\[
\text{Tr} \, S_F^{\beta\mu}(x; x|m) = \lim_{N \to \infty} i \frac{mB}{\pi^{3/2}} \text{Im} \int_{-\infty}^\infty \frac{d\omega}{2\pi f_F(\omega)} \int_0^\infty \frac{ds}{s^{1/2-\nu}} e^{i\frac{\nu}{2} + i \frac{\nu}{2} e^{-i\frac{3\pi}{4}}} e^{-i\nu(\omega^2 - m^2 - 2i\epsilon)}
\]

\[
\times \left[ \frac{1 + e^{i2sB}}{1 - e^{i2sB}} - \frac{2e^{i2\nu sB}}{1 - e^{i2sB}} \right] ,
\]  

(A.4)

where we also have introduced the dimensional regularization \( \nu \) in \( 4 - 2\nu \) dimensions, and we are to analytically continue to \( \nu = 0 \) in the end. Keeping \( \nu \) large enough that the integral is absolutely convergent, the expression above can easily be integrated with respect to \( m \) to yield \( \mathcal{L}_{\text{eff}}^{\beta\mu} \). To be more precise, there is an integration constant from the lower limit in

\[
i \int_{m_0}^m dm' \text{Tr} \, S_F^{\beta\mu}(x; x|m') = \mathcal{L}_{\text{eff}}^{\beta\mu}(m) - \mathcal{L}_{\text{eff}}^{\beta\mu}(m_0) .
\]  

(A.5)

We expect that in the limit \( m \to \infty \) the thermal part of the effective action is zero since an infinitely massive particle has zero Boltzmann weight. Therefore we let \( m_0 \to \infty \) and thereby put the integration constant \( \mathcal{L}_{\text{eff}}^{\beta\mu}(m_0) \) to zero.
The poles in the last factor in Eq. (A.4) cancel for finite \( N \), and we cannot let \( N \to \infty \) in a naive way before deforming the \( s \) integration contour to the imaginary axis. The two terms have to be treated separately so we must choose an integration contour for \( s \) slightly above or below the real axis. Since, according to the discussion above, \( \mathcal{L}_{\text{eff}}^\beta(\mu, m\to\infty) = 0 \) we see that the the original contour must be chosen slightly above the real axis. Depending on the sign of \( \omega^2 - m^2 \) (or \( \omega^2 - m^2 - 2eB(N-1) \) in the second term) we deform the \( s \)-contour to either the positive or negative imaginary axis. In one of the cases we get a contribution from the poles. After deforming the contours we take the \( N \to \infty \) limit and also take the limit \( \nu \to 0 \) what concerns taking the imaginary part, in order to get a more apparent expression, but we still need to keep \( \nu > 0 \) to have the integration over \( s \) finite, with the result

\[
\mathcal{L}_{\text{eff}}^\beta = \int_{-\infty}^{\infty} d\omega \theta(\omega^2 - m^2) f_F(\omega) \left[ \frac{1}{4\pi^{5/2}} \int_0^\infty \frac{ds}{s^{5/2}} e^{-s(\omega^2 - m^2)} seB \coth(seB) \right] - \int_{-\infty}^{\infty} d\omega \theta(\omega^2 - m^2) f_F(\omega) \left[ \frac{1}{2\pi^3} \sum_{n=1}^{\infty} \left(\frac{eB}{n}\right)^{3/2} \sin\left(\frac{\pi}{4} - \frac{\pi n}{eB(\omega^2 - m^2)}\right) \right]. \tag{A.6}
\]

Actually we must have \( \nu > 3/2 \), i.e. less than one dimension, but we may just consider it as an analytical continuation in \( \nu \), in order to be able to change the order of integration. If we now take the limit \( B \to 0 \), we get

\[
\mathcal{L}_0^\beta = \frac{1}{4\pi^{5/2}} \int_{-\infty}^{\infty} d\omega \theta(\omega^2 - m^2) f_F(\omega) \int_0^\infty ds s^{-5/2} e^{-s(\omega^2 - m^2)} = \frac{1}{4\pi^{5/2}} \int_{-\infty}^{\infty} d\omega \theta(\omega^2 - m^2) f_F(\omega)(\omega^2 - m^2)^{3/2-\nu} \Gamma(\nu - 3/2). \tag{A.7}
\]

We may now take the limit \( \nu \to 0 \) to get Eq. (5.17), and after subtraction of this term we may also let \( \nu \) vanish in Eq. (A.6) and get Eq. (5.21).

**Appendix B**

In the limit of very strong fields \( \{eB \gg T^2, m^2, |\mu^2 - m^2|\} \), the first term in Eq. (5.21) can be written as

\[
\mathcal{L}_1^{\beta,\mu}_{\text{reg}} = \int_{-\infty}^{\infty} d\omega \theta(\omega^2 - m^2) f_F(\omega) \left(\frac{eB}{4\pi^{5/2}}\right)^{3/2} \int_0^\infty ds \frac{s^{3/2}}{s^{3/2}} (s \coth s - 1). \tag{B.1}
\]

Similarly we find in this limit

\[
\mathcal{L}_1^{\beta,\mu}_{\text{osc}} = -\int_{-\infty}^{\infty} d\omega \theta(\omega^2 - m^2) f_F(\omega) \left(\frac{eB}{2\sqrt{2\pi^3}}\right)^{3/2} \zeta(3/2). \tag{B.2}
\]
where $\zeta$ is the Riemann $\zeta$-function. It can be shown by residue calculations that

$$\int_0^\infty \frac{ds}{s^{5/2}} (s \coth s - 1) = \sqrt{\frac{2}{\pi}} \zeta(3/2),$$

so that the $O(B^{3/2})$ terms cancel in this limit. In order to extract the next term in the strong field expansion of $L_1^\beta,\mu$, we consider the expression entering in the $\omega$ integral in $L_1^\beta,\mu$, expanded for large $B$, i.e.

$$\frac{(eB)^{3/2}}{4\pi^{5/2}} \left\{ \int_0^\infty \frac{ds}{s^{5/2}} \left[ \exp\left(-\frac{s}{eB}(\omega^2 - m^2)\right) - 1\right] (s \coth s - 1) - \frac{2}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \left( \sin\left(\frac{\pi}{4} - \frac{n\pi}{eB}(\omega^2 - m^2)\right) - \sin\frac{\pi}{4} \right) \right\}. \quad (B.4)$$

If we now use the cancellations depicted above, and the fact that the sum converges towards an integral in the limit $B \to \infty$, the expression above may be written as

$$\frac{(eB)^{3/2}}{4\pi^{5/2}} \left\{ \int_0^\infty \frac{ds}{s^{3/2}} \left[ \exp\left(-\frac{s}{eB}(\omega^2 - m^2)\right) \right] \right. \left. - \frac{1}{\sqrt{\pi B}} \int_0^\infty \frac{dx}{x^{3/2}} \left( \sin\left(\frac{\pi}{4} - x\pi(\omega^2 - m^2)\right) - \frac{1}{\sqrt{2}} \right) \right\}. \quad (B.5)$$

By performing the integrals in this expression, we find the following leading contribution

$$L_1^\beta,\mu = \frac{eB}{2\pi^2} \int_{-\infty}^\infty d\omega \theta(\omega^2 - m^2) f_F(\omega) \sqrt{\omega^2 - m^2}. \quad (B.6)$$
Appendix C

In the case of large chemical potentials, e.g. in a neutron star, when \((\mu^2 - m^2)/2eB \gg 1\), the form for \(L_{1,\text{osc}}^{\beta,\mu}\) given in Eq.\(5.22\) is difficult to handle due to the rapid oscillations in the \(\zeta\)-function. Let us instead start from Eq.\(5.21\), and rewrite it as

\[
L_{1,\text{osc}}^{\beta,\mu} = \frac{m^4}{2\pi^3} \left(\frac{eB}{m^2}\right)^{3/2} \sum_{n=1}^{\infty} n^{-3/2} \text{Im} \left\{ \exp \left[ -i \left( \frac{\pi}{4} + \frac{\pi n}{eB/m^2} \right) \right] I_n \right\},
\]

where we have defined

\[
I_n \equiv \int_1^\infty dx \frac{\exp \left( i \frac{\pi n}{eB/m^2} x^2 \right)}{1 + \exp[m\beta(x - \mu/m)]}.
\]

Since the exponential function here is oscillating rapidly and we desire a rapidly decreasing function instead, we close the contour with a circular section at infinity, a straight line from the origin to infinity with complex argument \(\pi/4\), and the small section from the origin to \(x = 1\), and use Cauchy’s theorem to get

\[
I_n = e^{i\pi/4} \int_0^\infty dx \frac{\exp \left( -\frac{\pi n}{eB/m^2} x^2 \right)}{1 + \exp[m\beta(x - \mu/m)]} - \int_0^1 dx \frac{\exp \left( i \frac{\pi n}{eB/m^2} x^2 \right)}{1 + \exp[m\beta(x - \mu/m)]} + I_n^{\text{poles}}.
\]

The contribution from the residues at the poles is

\[
I_n^{\text{poles}} = -2\pi i \frac{T}{m} \exp \left[ -2\pi^2 \frac{\mu T}{eB} + i\pi n \frac{\mu^2}{eB} \right] \sum_{\nu=0}^{\nu_{\text{max}}} \exp \left[ -2\pi^2 n \frac{\mu T}{eB} 2\nu - i\pi T^2 (2\nu + 1) \right],
\]

where we have defined \(\nu_{\text{max}}\) as the number of poles encircled by the contour

\[
\nu_{\text{max}} = \text{int} \left[ \frac{\mu}{2\pi T} - \frac{1}{2} \right].
\]

In the case of large chemical potential compared to the temperature and the square root of the magnetic field, we may assume the thermal distributions to be unity, and perform the integrals with the result

\[
I_n = e^{i\pi/4} \frac{1}{2} \sqrt{\frac{eB/m^2}{n}} \left( 1 - \text{erf} \left[ \frac{n}{eB/m^2} e^{-i\pi/4} \right] \right) + I_n^{\text{poles}} + O[e^{-\beta(\mu - \sqrt{\frac{\pi eB}{m^2}})}].
\]

It turns out that the phase from one minus the error function in Eq.\(C.6\) cancels the phase from \(\exp \left[ -i \left( \frac{\pi}{4} + \frac{\pi n}{eB/m^2} \right) \right]\) in Eq.\(C.1\), when taking the imaginary part. The oscillations are thus only originating from the residues at the poles, that all have \(\text{Re}[\omega] = \mu\), i.e. they are lying at the Fermi surface. Also, notice that the contribution from these poles is exponentially suppressed as \(\exp \left[ -2\pi^2 \frac{\mu T}{eB} \right]\), in agreement with the general discussion on de Haas – van Alphen oscillations in Section 7.
References

[1] V. Ginzburg, “High Energy Gamma Ray Astrophysics” (North Holland, Amsterdam, 1991).

[2] E. Müller , “Supernova Theory and the Nuclear Equation of State”, J. Phys. G: Nucl. Part. Phys. 16 (1990) 1571.

[3] S. L. Shapiro and S. A. Teukolsky, “Black Holes, White Dwarfs and Neutron Stars, The Physics of Compact Objects ” (Wiley, New York, 1983).

[4] G. Chanmugam, “Magnetic Fields of Degenerate Stars”, Ann. Rev. Astron. Astrophys. 30 (1992) 143.

[5] J. R. P. Angel, “Magnetic White Dwarfs”, Ann. Rev. Astron. Astrophys. 16 (1978) 487. Also see J. D. Landstreet, “Magnetic Fields at the Surfaces of Stars”, The Astron. Astrophys. Rev. 4 (1992) 35.

[6] T. Tajima, S. Cable, K. Shibita and R. M. Kulsrud, “On the Origin of Cosmological Magnetic Fields”, Ap. J. 390 (1992) 309.

[7] R. Narayan, P. Paczyński and T. Piran, “Gamma-Ray Bursts as the Death Throes of Massive Binary Stars”, Ap. J. 395 (1992) L83.

[8] E. Witten, “Superconducting Strings”, Nucl. Phys. B249 (1985) 557; J. P. Ostriker, C. Thompson and E. Witten, “Cosmological Effects of Superconducting Strings”, Phys. Lett. 180B (1986) 231; V. Berezinsky and H. R. Rubinstein, “Evolution and Radiation From Superconducting Cosmic Strings”, Nucl. Phys. B323 (1989) 95.

[9] T. Vachaspati, “Magnetic Fields from Cosmological Phase Transitions”, Phys. Lett. B265 (1991) 258; K. Enqvist and P. Olesen, “On Primordial Magnetic Fields of Electroweak Origin”, preprint (NBI-HE-93-33, The Niels Bohr Institute, 1993).

[10] J. Ambjørn and P. Olesen, “Electroweak Magnetism, W-Condensation and Anti-Screening”, preprint (NBI-HE-93-17, The Niels Bohr Institute, 1993).

[11] P. Elmfors, D. Persson and B.-S. Skagerstam, “The QED Effective Action at Finite Temperature and Density”, Phys. Rev. Lett. 71 (1993) 480.

33
[12] W. Dittrich, “Effective Lagrangians at Finite Temperature”, Phys. Rev. D19 (1979) 2385.

[13] M. Loewe and J. C. Rojas, “Thermal Effects and the Effective Action of Quantum Electrodynamics”, Phys. Rev. D46 (1992) 2689.

[14] A. Chodos, K. Everding and D. A. Owen, “QED With a Chemical Potential: The Case of a Constant Magnetic Field”, Phys. Rev. D42 (1990) 2881.

[15] B.-S. Skagerstam, “Thermal Effects in Particle and String Theories”, in Proceedings of the the 1989 Workshop on Superstrings and Particle Theory, Eds. L. Clavelli and B. Harms (World Scientific, Singapore, 1990); P. Elmfors and B.-S. Skagerstam “Anomalous Magnetic Moment at Finite Chemical Potential and External Field Effects”, Z. Phys. C49 (1991) 251 and “Anomalous QED Magnetic Moment at Finite Chemical Potential”, in Proceedings of the 2nd Workshop on Thermal Field Theories and Their Applications, Eds. H. Ezawa, T. Arimitsu and Y. Hashimoto (Elsevier, 1991)

[16] A. I. Studenikin, “Anomalous Magnetic Moments of Charged Leptons and Problems of Elementary-Particle Physics”, Sov. J. Part. Nucl. 21 (1990) 259.

[17] P. Elmfors, D. Persson and B.-S. Skagerstam, “Thermal Radiative Corrections and External Fields”, in preparation.

[18] W. H. Furry, “On Bound States and Scattering in Positron Theory, Phys. Rev. 81 (1951) 115.

[19] M. Kobayashi and M. Sakamoto, “Radiative Corrections in a Strong Magnetic Field”, Prog. Theor. Phys. 70 (1983) 1375.

[20] H. Umezawa, H. Matsumoto and M. Tachiki, “Thermo Field Dynamics and Condensed States”, (North Holland, 1982).

[21] I. Ojima, “Gauge Fields at Finite Temperatures - ‘Thermo Field Dynamics’ and the KMS Condition and Their Extensions to Gauge Theories”, Ann. Phys. (N.Y.) 137 (1981) 1.

[22] P. Elmfors, K. Enqvist and I. Vilja, “On the Non–equilibrium Early Universe”, preprint (Nordita, December 1993).
[23] J. Schwinger, “On Gauge Invariance and Vacuum Polarization”, Phys. Rev. 82 (1951) 664 and “Particles, Sources and Fields”, Vol.3 (Addison-Wesley Pub. Co., 1988).

[24] D. Miller and P. S. Ray, “Thermodynamics of a Relativistic Fermi Gas in a Strong Magnetic Field”, Helv. Phys. Acta 57 (1984) 96.

[25] N. W. Ashcroft and N. D. Mermin, “Solid State Physics”, (Holt-Saunders, 1976).

[26] A. Isihara, “Condensed Matter Physics”, (Oxford University Press 1991).

[27] A. A. Abrikosov, “Fundamentals of the Theory of Metals”, (North-Holland, 1988).

[28] C. Kittel, “Quantum Theory of Solids”, (Wiley, 1963).

[29] L. Onsager, “Interpretation of the de Haas – van Alphen effect”, Phil. Mag. 43 (1952) 1006.

[30] K. Freese, R. Canal and R. G. Kron, “Do Monopoles Keep White dwarfs Hot?”, preprint (Chicago University, PRINT-84-0573).

[31] A. Chodos, D. A. Owen and C. M. Sommerfield, “Strong Field Dependence of the Fine Structure Constant”, Phys. Lett. 212B (1988) 491.

[32] P. D. Morley and M. B. Kislinger, “Relativistic Many-Body Theory, Quantum Chromodynamics and Neutron Stars/Supernova”, Phys. Rep. 51 (1979) 63 and J. F. Donoghue, B. Holstein and R. W. Robinett, “Quantum Electrodynamics at Finite Temperature”, Ann. Phys. (N.Y.) 164 (1985) 233.

[33] E. Petitgirard, “Massive Fermion Dispersion Relation at Finite Temperature”, Z. Phys. C54 (1992) 673.

[34] T. Altherr and U. Kraemmer, “Gauge Field Theory Methods for Ultra-degenerate and Ultra-relativistic Plasmas”, Astropart. Phys. 1 (1992) 133.

[35] H. A. Weldon, “Dynamical Holes in the Quark-Gluon Plasma”, Phys. Rev. D40 (1989) 2410.

[36] B. Cheng, D. N. Schramm and J. W. Truran, “Interaction Rates at High Magnetic Field Strengths and High Degeneracy”, Phys. Lett. B316 (1993) 521.
[37] S. L. Adler, J. N. Bahcall, C. G. Callan and M. N. Rosenbluth, “Photon Splitting in a Strong Magnetic Field”, Phys. Rev. Lett. 25 (1970) 1061.

[38] Z. Bialynicka-Birula and I. Bialynicka-Birula, “Nonlinear Effects in Quantum Electrodynamics. Photon Propagation and Photon Splitting in an External Field”, Phys. Rev. D2 (1970) 2341.

[39] E. Brézin and C. Itzykson, “Polarization Phenomena in Vacuum Nonlinear Electrodynamics”, Phys. Rev. D3 (1971) 618.

[40] V. O. Papayan and V. I. Ritus, “Vacuum Polarization and Photon Splitting in an Intense Field”, Sov. Phys. JETP 34 (1972) 1195.

[41] J. Hauknes, “An Effective Action for a Variable Electromagnetic Field”, Ann. Phys. (N.Y.) 156 (1984) 303.