Atomic spin squeezing in a $\Lambda$ system

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Using a quantum theory for an ensemble of three-level atoms ($\Lambda$) placed in an optical cavity and driven by electromagnetic fields, we show that the long-lived spin associated with the ground state sublevels can be squeezed. Two kinds of squeezing are obtained: self-spin squeezing, when the input fields are coherent states and the atomic ensemble exhibits a large non-linearity; squeezing transfer, when one of the incoming fields is squeezed.

I. INTRODUCTION

In this paper, we present calculations performed on a model system consisting of three-level atoms that interact with two fields. The atoms are placed in an optical cavity which ensures a sizeable level of interaction, in contrast with single pass schemes. Under such conditions, the validity of quantum fluctuation calculations based on quantum Langevin equations is well established. We will consider a set of three-level atoms in the so-called $\Lambda$ configuration, consisting of two ground state sublevels and one excited state.

It is well-known that, in the case of large one-photon detuning (Raman scheme), terms involving the ground - excited state coherence and excited state population can be adiabatically eliminated. The three-level system is then equivalent to an effective two-level system in which the spin associated with the ground state sublevels may be squeezed. Because of its simplicity the effective two-level system provides a good understanding of the physical
phenomena responsible for the creation of atomic squeezing and allows one to carry out analytical calculations of optimal squeezing.

We study two regimes. In the first, the atoms interact with coherent light. One can achieve atomic self-spin squeezing if the non-linearity of the medium is sufficiently high. This atomic squeezing originates from the same physical process that gives rise to the squeezing of the field exiting the cavity, predicted theoretically in Refs. [1,2,3] and observed experimentally a little later [4,5,6]. In the early 1980’s, it was conjectured that atomic spin squeezing appears as a counterpart of squeezing of the electromagnetic field [7,8]. However, the quantum noise reduction on atomic variables computed in several papers can be obtained by a rotation of the atomic variables of a two-level system interacting with a coherent field. As shown in [9,10,11], in order to be relevant, spin squeezing should be computed in the plane orthogonal to the direction of the mean spin. We have performed full quantum calculations in the relevant basis. Here we generalize the results obtained previously on two-level systems [12] and three-level (V) [13] systems to the Λ system.

In the second regime, the atoms interact with a squeezed incoming field. This scheme was proposed recently and experimentally tested in a single pass configuration to produce spin squeezing between two excited levels [14,15]. In [16], it was shown that the cavity configuration should allow one to enhance the squeezing transfer between two-level atoms and field. In both cases the transfer was limited by the vacuum field fluctuations. We find here that the Λ scheme allows for a quasi-perfect squeezing transfer from the field to the atoms. Unlike configurations in which the squeezing occurs between two excited states or between a ground and excited state, the squeezing is created between two long-lived ground states. Such a system offers significant advantages from an experimental perspective.
II. EQUATIONS FOR ATOMIC FLUCTUATIONS

A. Three-level atoms interacting with two fields

We consider an ensemble of three-level atoms, the three levels being labelled 1, 2, 3 in a Λ configuration (Fig. 1). The atoms interact with two light fields: an intense classical pump field $A_1$, having frequency $\omega_1$ and a quantum probe field $A_2$, having frequency $\omega_2$. Field $A_1$ transverses the medium in a single pass and drives the $1 \rightarrow 3$ transition, while field $A_2$ is confined to an optical cavity and drives the $2 \rightarrow 3$ transition. The detunings from atomic resonance are $\Delta_1 = \omega_{31} - \omega_1$ for the pump, and $\Delta_2 = \omega_{32} - \omega_2$ for the probe. The cavity resonance frequency closest to the probe frequency is $\omega_c$. The cavity detuning for the probe field is $\Delta_c = \omega_2 - \omega_c$. The incoming quantum field that is coupled into the cavity is denoted by $A_{2n}^\text{in}$. The intensity of field $A_1$ is assumed to be much greater than that of the quantum field.

The three-level system is described using nine collective operators for the $N$ atoms of the ensemble, the populations of levels $|1\rangle$, $|2\rangle$ and $|3\rangle$,

$$
\Pi_1 = \sum_{i=1}^{N} |1\rangle_i \langle 1|_i, \quad \Pi_2 = \sum_{i=1}^{N} |2\rangle_i \langle 2|_i, \quad \Pi_3 = \sum_{i=1}^{N} |3\rangle_i \langle 3|_i, \tag{1}
$$

the components of the optical dipoles in frames rotating at the laser frequencies,

$$
P_1(t) = \sum_{i=1}^{N} |1\rangle_i \langle 3|_i e^{i\omega_1 t}, \quad P_2(t) = \sum_{i=1}^{N} |2\rangle_i \langle 3|_i e^{i\omega_2 t} \tag{2}
$$

(and their hermitian conjugates), and the operator associated with the coherence between levels $|1\rangle$ and $|2\rangle$,

$$
P_r(t) = \sum_{i=1}^{N} |2\rangle_i \langle 1|_i e^{i(\omega_2 - \omega_1)t} \tag{3}
$$

(and its hermitian conjugate).

The Hamiltonian for the atom-field system is given by

$$
H = \hbar \omega_{21}\Pi_2 + \hbar \omega_{31}\Pi_3 + \hbar \omega_2 a_2^\dagger a_2 + \hbar g(P_2^\dagger A_2 + A_2^\dagger P_2) + \hbar(\Omega_1^* P_1 + \Omega_1 P_1^\dagger),
$$
where \( g = \mathcal{E}_0 d_{23} / \hbar \) is a coupling constant, assumed real, associated with the (quantum) cavity field, \( \mathcal{E}_0 = \sqrt{\hbar \omega_2 / 2 \epsilon_0 Sc} \) is an amplitude appearing in the equation for the cavity field operator \( E_2 = \mathcal{E}_0 \left( e^{-i\omega_2 t} A_2 + e^{i\omega_2 t} A_2^\dagger \right) \), \( A_2 \) and \( A_2^\dagger \) are field creation and annihilation operators of having commutator
\[
\left[ A_2(t), A_2^\dagger(t') \right] = e^{-\kappa |t-t'|} / \tau,
\]
\( \kappa \) is the cavity decay rate, \( \tau = L/c \), where \( L \) is the cavity length, \( S \) is the cross sectional area of the cavity field, \( \Omega_1 = E_1 d_{13} / \hbar \) is a coupling constant associated with the classical pump field, \( E_1(t) = (e^{-i\omega_1 t} E_1 + e^{i\omega_1 t} E_1^*) \), \( d_{23} \) and \( d_{13} \) are dipole moment matrix elements.

Note that \( \Omega_1 \) has units of frequency, while \( g \) and \( A_2 \) have units of \((\text{frequency})^{1/2}\). The decay constants associated with dipole operators \( P_1 \) and \( P_2 \) are taken to be equal and denoted by \( \gamma \). In order to account for the finite lifetime of the two ground state sublevels 1 and 2, we include in the model another decay rate \( \gamma_0 \), which is much smaller than \( \gamma \). For example, \( \gamma_0 \) can reflect transit-time broadening in the system; as such, it affects level 3 as well as levels 1 and 2. Typically, the atoms fall out of the interaction area with the light beam in a time of the order of tens of milliseconds, whereas \( \gamma \) is of the order of the MHz for the excited state. We also include incoherent pumping of levels 1 and 2 with rates \( \Lambda_1 \) and \( \Lambda_2 \), respectively, to allow for ground state atoms entering the interaction volume. We will neglect any fluctuations in the total number of atoms in the interaction volume and assume that \( \Pi_T = \Pi_1 + \Pi_2 + \Pi_3 = (\Lambda_1 + \Lambda_2) / \gamma_0 = N \).

The time evolution of atomic polarizations and populations is obtained by adding to the Heisenberg equations of motion, \( i\hbar \dot{\mathcal{O}} = [O, H] \), terms corresponding to the Langevin forces associated with the decay of these quantities,

\[
\frac{d\Pi_1}{dt} = i\Omega_1^* P_1 - i\Omega_1 P_1^\dagger + \gamma \Pi_3 - \gamma_0 \Pi_1 + \Lambda_1 + F_{\Pi_1} \tag{4}
\]

\[
\frac{d\Pi_2}{dt} = ig A_2^\dagger P_2 - ig A_2 P_2^\dagger + \gamma \Pi_3 - \gamma_0 \Pi_2 + \Lambda_2 + F_{\Pi_2} \tag{5}
\]

\[
\frac{d\Pi_3}{dt} = -(i\Omega_1^* P_1 - i\Omega_1 P_1^\dagger) - (ig A_2^\dagger P_2 - ig A_2 P_2^\dagger) - 2\gamma \Pi_3 - \gamma_0 \Pi_3 + F_{\Pi_3} \tag{6}
\]
\[
\frac{dP_1}{dt} = -(i\Delta_1 + \gamma)P_1 + i\Omega_1(\Pi_1 - \Pi_3) + igA_2P^\dagger_r + F_{P_1} \tag{7}
\]

\[
\frac{dP_2}{dt} = -(i\Delta_2 + \gamma)P_2 + igA_2(\Pi_2 - \Pi_3) + i\Omega_1P_r + F_{P_2} \tag{8}
\]

\[
\frac{dP_r}{dt} = -(\gamma_0 - i\delta)P_r + i\Omega_1^*P_2 - igA_2P^\dagger_1 + F_{P_r} \tag{9}
\]

where

\[\delta = \Delta_1 - \Delta_2\]

is the detuning between the ground state sublevels. "In" terms for \(P_r\) resulting from spontaneous emission have not been included in Eq. (9), owing either to polarization selection rules or a hyperfine separation of levels 1 and 2 that is much greater than \(\gamma\). To these equations must be added the evolution equation for the cavity field.

Changes of the intracavity field result from the incoming field \(A_{2}^{\text{in}}\), losses through the coupling mirror, and the source field associated with the atomic polarization. The evolution equation for the intracavity field is [12]

\[
\frac{dA_2}{dt} = -(\kappa + i\Delta_c)A_2 + \frac{ig}{\tau}P_2 + \sqrt{\frac{2\kappa}{\tau}}A_{2}^{\text{in}} \tag{10}
\]

As is evident from this equation, fluctuations of the incoming field give rise to a Langevin force for the cavity field.

Our aim is to obtain the fluctuations of the spin operators associated with levels 1 and 2. For this, we need to determine the fluctuations of operators \(P_r, P^\dagger_r, \Pi_1\) and \(\Pi_2\). To obtain equations for the fluctuations, we linearize equations (7-9, 10), and (7-9, 10) and their hermitian conjugates. We define a 10-dimensional vector \(|\xi(t)\rangle\) as

\[|\xi(t)\rangle = |A_2(t), A_{2}^{\dagger}(t), P_1(t), P_{1}^{\dagger}(t), P_2(t), P_{2}^{\dagger}(t), P_r(t), P_{r}^{\dagger}(t), S_{z1}(t), S_{z2}(t)\rangle^T\]

where

\[S_{z1} = (\Pi_1 - \Pi_3) / 2; \quad S_{z2} = (\Pi_2 - \Pi_3) / 2\]
and then set

\[ |\xi(t)| = |\langle \xi(t) \rangle| + |\delta\xi(t)| \]

The steady state values, \( \langle \xi(t) \rangle \), are obtained from Eqs. (4-9), (10) in steady state, using the fact that the mean value of the Langevin operators are zero; however, these analytical solutions are of marginal use.

The fluctuation vector \( |\delta\xi(t)| \) obeys an equation of motion

\[ \frac{d|\delta\xi(t)|}{dt} = -[B] |\delta\xi(t)| + |F_\xi(t)| \]  

(11)

where \( [B] \) is the linearized evolution matrix of the atom-field system and the column vector \( |F_\xi(t)| \) contains the Langevin forces:

\[ |F_\xi(t)| = \left[ \sqrt{2\kappa/\tau} \delta A_2^\text{in}(t), \sqrt{2\kappa/\tau} \delta A_2^\text{int}(t), F_{P_1}(t), F_{P_1^\dagger}(t), F_{P_2}(t), F_{P_2^\dagger}(t), F_{P_r}(t), F_{P_r^\dagger}(t), F_{S_{z1}}(t), F_{S_{z2}}(t) \right]^T \]

(12)

The correlation matrix \([G(t)]\) of the fluctuations is:

\[ [G(t)] = [\delta\xi(t)][\delta\xi(0)] \]  

(13)

The variances are obtained from the zero time correlation functions, contained in the matrix \([G(0)]\) which satisfies:

\[ [B] [G(0)] + [G(0)] [B]^\dagger = [D] \]  

(14)

where \([D]\) is the correlation matrix of the Langevin forces:

\[ \langle |F_\xi(t)| |F_\xi(t')\rangle = [D] \delta(t - t'). \]  

(15)

The sub-matrix \([G_c(0)]\) is defined as the 4 x 4 lower diagonal block of \([G(0)]\) and contains the variances of \(P_r, P_r^\dagger, S_{z1}, \) and \(S_{z2}.\)
B. Spin squeezing

1. Definition

In the same way as a squeezed state of the electromagnetic field is defined by comparison with a coherent state, a squeezed spin state will be defined as having fluctuations in one component lower than that of a coherent spin state [11]. A coherent spin state for \(N\) atoms is defined as a product state of \(N\) uncorrelated spins, in which the state vector for the \(i\)-th spin is an eigenstate the individual spin operator in the \((\theta, \phi)\) direction:

\[
\sigma_{\theta,\phi} i = \sigma_{x i} \sin \theta \cos \phi + \sigma_{y i} \sin \theta \sin \phi + \sigma_{z i} \cos \theta
\]

having eigenvalue +1/2, with \(\sigma_{x i} = (\sigma_{+ i} + \sigma_{- i}^\dagger)/2\), \(\sigma_{y i} = (\sigma_{+ i} - \sigma_{- i}^\dagger)/2i\). The coherent spin state can be obtained as a rotation from the state having all spins aligned along the \(z\) axis. This coherent spin state is an eigenvalue of the collective spin operator \(S_{\theta,\phi} = \sum_{i=1,N} \sigma_{\theta,\phi} i\), with eigenvalue \(S = N/2\) [18]. It satisfies the minimum uncertainty relationship with fluctuations equally distributed over any two orthogonal components normal to the \((\theta, \phi)\) direction, the variance of which is equal to \(S/2 = N/4\). If one can squeeze the fluctuations of the total spin within the plane orthogonal to the mean value, it will result in noise reduction in spin measurements. The condition for spin squeezing is then [19]

\[
\Delta S_\alpha \leq |\langle S_Z \rangle|/2
\]

where the axes have been rotated in such a way that the \(Z\) axis is in the direction of the mean spin and \(\alpha\) represents a direction in the \(X, Y\) plane. The quantity \(\langle S_Z \rangle\) is then the mean value of the spin and \(S_X\) and \(S_Y\) have zero mean values. Spin squeezing can occur only for a spin ensemble with \(N > 1\) since it implies the emergence of quantum correlations within the spin ensemble.
2. Minimum variance calculation

We calculate the variances $\Delta S_X$ and $\Delta S_Y$ of the spin variables in the new reference frame. For this, we perform a rotation defined by angles $\phi$ around the $z$ axis and $\theta$ around the $Y$ axis (defined by the first rotation) such that

$$\cos \theta = \frac{\langle S_z \rangle}{|\langle \vec{S} \rangle|} \quad \cos \phi = \frac{\langle S_x \rangle}{S_\phi}$$

with $|\langle \vec{S} \rangle| = \sqrt{\langle S_x \rangle^2 + \langle S_y \rangle^2 + \langle S_z \rangle^2}$ and $S_\phi = \sqrt{\langle S_x \rangle^2 + \langle S_y \rangle^2}$, where the mean values are the solutions of the steady state equations. A spin component in the $X,Y$ plane making an angle $\alpha$ with the $X$ axis has a variance given by:

$$\Delta S_\alpha = \cos^2 \alpha \, \Delta S_X + \sin^2 \alpha \, \Delta S_Y + \sin 2\alpha \, \text{Re} (\Delta S_{XY})$$

with $\Delta S_{\mu\nu} = <\delta S_\mu(0) \, \delta S_\nu(0)>$ ($\mu, \nu = X,Y$). As a consequence the values $\alpha_0$ of $\alpha$ for the spin components having maximal and minimal variances satisfy:

$$\tan 2\alpha_0 = \frac{2 \, \text{Re} (\Delta S_{XY})}{\Delta S_X - \Delta S_Y}$$

In order to investigate squeezing, we compare the minimal variance to $|\langle S_z \rangle|/2$. The corresponding normalized variance thus obtained is called $\Delta S_{\text{min}}$. Spin squeezing is achieved when $\Delta S_{\text{min}} < 1$.

For the three-level system, the spin fluctuations corresponding to the atomic coherence between levels 1 and 2 are given by lower diagonal block $[G_p(0)]$ of the correlation matrix. To go into the relevant basis, one must first perform the transformation from $P_r$, $P_r^\dagger$, $S_{z1}$, $S_{z2}$ to $S_{cx}$, $S_{cy}$, $S_{cz}$ with:

$$S_{cx} = \frac{P_r + P_r^\dagger}{2}, \quad S_{cy} = \frac{P_r - P_r^\dagger}{2i}, \quad S_{cz} = \frac{\Pi_2 - \Pi_1}{2}$$

For this we define matrix $[R_1]$:

$$[R_1] = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -i & i & 0 & 0 \\ 0 & 0 & -2 & 2 \end{pmatrix}$$
Then one goes into the basis \( S_{cX}, S_{cY} \), where the mean spin direction is along \( OZ \), by using the \([R_2]\) matrix:

\[
[R_2] = \begin{pmatrix}
\cos \theta \cos \varphi & \cos \theta \sin \varphi & -\sin \theta \\
-\sin \varphi & \cos \varphi & 0
\end{pmatrix}
\] (23)

with angles \( \theta \) and \( \varphi \) given by Eq.(18).

The atomic correlation matrix \([G^\perp_1(0)]\) in the \( S_{cX}, S_{cY} \) basis is

\[
[G^\perp_1(0)] = [R_2] [R_1] [G_1(0)] [R_1]_{hc} [R_2]_{hc}
\] (24)

The variances in the XY plane are then

\[
\Delta S_{cX} = [G^\perp_1(0)]_{1,1}, \quad \Delta S_{cY} = [G^\perp_1(0)]_{2,2}, \quad \Delta S_{cXY} = [G^\perp_1(0)]_{1,2}
\] (25)

The minimal variance \( \Delta S_{c\min} \) corresponds to the angle \( \alpha_0 \) given by Eq.(20).

**III. ADIABATIC ELIMINATION IN THE RAMAN SCHEME**

**A. Effective two-level system**

In the limit that \( |\Delta_{1,2}| \gg \gamma, |\Omega_1| \), one can adiabatically eliminate the ground-excited state polarizations. In this limit, the excited state population is much less than unity and can be neglected. We assume also that \( |\delta| = |\Delta_1 - \Delta_2| \ll |\Delta| \equiv (\Delta_1 + \Delta_2)/2 \), since the values of \( |\delta| \) to be considered are of order of \( \gamma_0 \ll |\Delta| \). Performing this adiabatical elimination and neglecting terms of order \( \gamma/\Delta^2 \), one obtains simplified equations for the sublevels variables \( S_+ = P_r, S_- = P_r^\dagger, S_z = (\Pi_2 - \Pi_1)/2 \) and \( \Pi = \Pi_2 + \Pi_1 = N \):

\[
\frac{dS_+}{dt} = -(\gamma_0 - i\tilde{\delta})S_+ + 2i\tilde{g}A_2S_z + F_{S_+}
\] (26)

\[
\frac{dS_-}{dt} = -(\gamma_0 + i\tilde{\delta})S_- - 2i\tilde{g}A_2^\dagger S_z + F_{S_-}
\] (27)

\[
\frac{dS_z}{dt} = -\gamma_0 S_z + \frac{\Lambda_2 - \Lambda_1}{2} + i\tilde{g}(A_2^\dagger S_+ - S_- A_2) + F_{S_z}
\] (28)
\[ \frac{d\Pi}{dt} = -\gamma_0 \Pi + \Lambda \]  
where \( \Lambda = \Lambda_1 + \Lambda_2 \),

\[ \tilde{\delta} = \delta + \frac{|\Omega_1|^2}{\Delta} - \frac{g^2 A_2^1 A_2}{\Delta} \]
is the effective atomic detuning corrected for light-shifts, and

\[ \tilde{g} = g \frac{\Omega_1}{\Delta} \]
is an effective coupling constant. The evolution equation (30) for the quantum field becomes

\[ \frac{dA_2}{dt} = -(\kappa + i\Delta_c)A_2 + i\tilde{g} S_+ + \sqrt{2\kappa \tau} A_2^{in} \]  

In principle, the atomic fluctuation operators contain terms related to repopulation of the ground states via spontaneous emission; however these contributions are down by \((\gamma/\Delta)^2\) and are neglected. As a consequence, the fluctuation operators are associated only with the decay rate \(\gamma_0\) and the repopulating processes; we shall see that the diffusion matrix formed from these operators has a fairly simple form.

The problem has been reduced to that of an ensemble of two-level atoms interacting with one field in an optical cavity whose effective Rabi frequency is \(\Omega_{eff} = \tilde{g} \langle A_2 \rangle = g \frac{\Omega_1}{\Delta} \langle A_2 \rangle \).

We will use these simplified equations for the spin squeezing calculations and give a physical interpretation to our results. Note that, beside the incoherent pumping terms, our two-level system differs slightly from a classical "excited-ground state" two-level system. The decay terms in (26-27-28) are all \(\gamma_0\); the coherence relaxes at the same rate as the populations. This remark, as simple as it may seem, critically affects the steady states that are allowed by the system (see Sec. IV C). Note also that the atomic noise scale is no longer \(\gamma\), but \(\gamma_0\), hence much smaller. We thus expect the atomic noise not to destroy the squeezing too much.
B. Steady state and bistability

The steady state values are obtained solving Eq. (26-27-28-30) in steady state, using the fact that the mean value of the Langevin operators are zero. Explicitly, one finds

\[
\langle S_+ \rangle = \frac{\Lambda_2 - \Lambda_1}{\gamma_0} \frac{i - \bar{\delta}}{1 + \bar{\delta}^2 + 4I_2}
\]

(31)

\[
\langle S_- \rangle = -\frac{\Lambda_2 - \Lambda_1}{\gamma_0} \frac{i + \bar{\delta}}{1 + \bar{\delta}^2 + 4I_2}
\]

(32)

\[
\langle S_z \rangle = \frac{\Lambda_2 - \Lambda_1}{2\gamma_0} \frac{1 + \bar{\delta}^2}{1 + \bar{\delta}^2 + 4I_2}
\]

(33)

with the dimensionless variables (in units of \(\gamma_0\))

\[
\bar{\delta} = \frac{\bar{\delta}}{\gamma_0}, \quad \beta_2 = \frac{\bar{g}\langle A_2 \rangle}{\gamma_0}, \quad I_2 = \beta_2^2.
\]

(34)

For certain values of the parameters, the system exhibits bistability. From (30) in steady state, one derives the mean value of the intracavity intensity

\[
I_2^{in} = I_2 \left[ \left( 1 + \frac{2\bar{C}}{1 + \bar{\delta}^2 + 4I_2} \right)^2 + \left( \delta_c + \frac{2\bar{C}\bar{\delta}}{1 + \bar{\delta}^2 + 4I_2} \right)^2 \right]
\]

(35)

where \(\delta_c = \Delta_c/\kappa\) is the dimensionless cavity detuning, and \(\bar{C}\) is an effective cooperativity or strong-coupling parameter

\[
\bar{C} = \frac{\bar{g}^2}{2\kappa^2\gamma_0} \frac{\Lambda_2 - \Lambda_1}{\gamma_0}
\]

(36)

We will limit our discussion to an incoherent pumping scheme in which \(\Lambda_1 = 0\) and \(\Lambda_2 = N\gamma_0\), corresponding to absorption for the probe field. We will show in the next sections that it is indeed the most favorable case for squeezing. The effective cooperativity is then related to the usual cooperativity parameter \(C = g^2N/2\kappa^2\gamma\) by

\[
\bar{C} = C \left| \frac{\Omega_1}{\Delta} \right|^2 \frac{\gamma}{\gamma_0}
\]

(37)

The cooperativity parameter determines the efficiency of the non-linear effect. If \(C\) - or \(\bar{C}\) - and \(I_2^{in}\) are large enough, the intracavity field exhibits a bistable behavior. It has been
shown that the field quantum fluctuations are maximal around the lower turning point of the bistability curve, allowing for the outgoing field to be squeezed. In Ref. [12], it has been demonstrated that this point is also interesting for atomic squeezing. Therefore, we will study the fluctuations in the vicinity of this turning point.

C. Linearization and diffusion matrix

The linearized equations obtained from (26-30) are

$$\frac{d\delta S_+}{dt} = -(\gamma_0 - i\bar{\delta})\delta S_+ + 2i\bar{g}\langle S_z\rangle\delta A_2 + 2i\bar{g}\langle A_2\rangle\delta S_z + F_{S_+}$$  \hspace{1cm} (38)

$$\frac{d\delta S_-}{dt} = -(\gamma_0 + i\bar{\delta})\delta S_- - 2i\bar{g}\langle S_z\rangle\delta A_2^\dagger - 2i\bar{g}\langle A_2\rangle^*\delta S_z + F_{S_-}$$  \hspace{1cm} (39)

$$\frac{d\delta S_z}{dt} = -\gamma_0\delta S_z - i\bar{g}(\langle S_-\rangle\delta A_2 - \langle S_+\rangle\delta A_2^\dagger - \langle A_2\rangle^*\delta S_+ + \langle A_2\rangle\delta S_-) + F_{S_z}$$  \hspace{1cm} (40)

$$\frac{d\delta A_2}{dt} = -(\kappa + i\Delta_c)\delta A_2 + i\bar{g}/\tau\delta S_+ + \sqrt{2\kappa/\tau}\delta A_2^{in}$$  \hspace{1cm} (41)

We can write Eqs. (38-41) and the hermitian conjugate of (41) in matrix form as

$$\frac{d}{dt} [\delta\xi(t)] = -[B] [\delta\xi(t)] + [F\xi]$$  \hspace{1cm} (42)

where

$$[\delta\xi(t)] = [\delta A_2(t), \delta A_2^\dagger(t), \delta S_+(t), \delta S_-(t), \delta S_z(t)]^T,$$  \hspace{1cm} (43)

and

$$[B] = \begin{pmatrix}
\kappa + i\Delta_c & 0 & -i\bar{g}/\tau & 0 & 0 \\
0 & \kappa - i\Delta_c & 0 & i\bar{g}/\tau & 0 \\
-2i\bar{g}\langle S_z\rangle & 0 & (\gamma_0 - i\bar{\delta}) & 0 & -2i\bar{g}\langle A_2\rangle \\
0 & 2i\bar{g}\langle S_z\rangle & 0 & (\gamma_0 + i\bar{\delta}) & 2i\bar{g}\langle A_2\rangle^* \\
i\bar{g}\langle S_-\rangle & -i\bar{g}\langle S_+\rangle & -i\bar{g}\langle A_2\rangle^* & i\bar{g}\langle A_2\rangle & \gamma_0 \\
\end{pmatrix}$$
\[ |F_\xi(t)| = \left[ \sqrt{2\kappa/\tau} \delta A_2^m(t), \sqrt{2\kappa/\tau} \delta A_2^{m\dagger}(t), F_{S_+}(t), F_{S_-}(t), F_{S_z}(t) \right]^T \]  \hspace{1cm} (44)

The covariance matrix \([G(t)]\) is defined by

\[ [G(t)] = |\delta \xi(t)| [\delta \xi(0)] \]  \hspace{1cm} (45)

and the diffusion matrix by

\[ \langle |F_\xi(t)| [F_\xi(t')] \rangle = [D] \delta(t - t') \]  \hspace{1cm} (46)

The values of the atomic diffusion coefficients can be derived from the quantum regression theorem or operator identities. The complete diffusion matrix is given in the Appendix. The variances of the spin components and their correlation functions are the elements of the zero time correlation matrix \([G(0)]\), which satisfies

\[ [B] [G(0)] + [G(0)] [B]^\dagger = [D] \]  \hspace{1cm} (47)

Inverting Eq. (47), one obtains \([G(0)]\), and, consequently, the spin variances. We proceed with the calculation of the minimal variance as in the first Section.

**IV. SELF SPIN SQUEEZING**

**A. Minimal variance calculation**

We first study the case of a coherent input field \(A_2^m\). We have studied a wide range of parameters. First, in Fig. 2, we have plotted the minimum variance \(\Delta S_{\min}\) as a function of the cooperativity \(\tilde{C}\) for a typical steady state point (\(\tilde{\delta} = 10, \tilde{\delta}_c = 0, I_2 = 25.2\)).

We see that the squeezing naturally increases with the cooperativity, the non-linear interaction increasing, but saturates a little under 30 \% when \(\tilde{C}\) exceeds 100. As it is a relatively accessible value for experiments, we will often choose this value for \(\tilde{C}\) in the following.
Then, for a fixed cooperativity $\tilde{C}$, we have optimized, for each value of $\tilde{\delta}$, the values of $\delta_c$ and $I_2$ yielding the best atomic squeezing, that is the lowest $\Delta S_{\text{min}}$. The results are reported in Table I for $\tilde{C} = 100$.

| $\tilde{\delta}$ | 0  | 5  | 10 | 15 | 20 |
|------------------|----|----|----|----|----|
| $\delta_c$       | 0  | -0.2 | 0  | -0.4 | -0.2 |
| $I_2$            | 0.25 | 6.5  | 25.2 | 56.5 | 100 |
| $\Delta S_{\text{min}}$ | 0.713 | 0.716 | 0.72 | 0.72 | 0.728 |

*Table I. Minimal spin variances for the effective two-level system as a function of $\tilde{\delta}$ ($\tilde{C}$ is equal to 100, the other parameters are adjusted for optimal noise reduction).*

We see that the squeezing does not vary much with $\tilde{\delta}$, and is always close to 28-29%. Increasing the cooperativity does not improve the squeezing much: one saturates at a squeezing value of almost 30%. The fact that the minimum variance is rather independent from the triplet $(\tilde{\delta}, \delta_c, I_2)$ when the cooperativity increases shows that there exists an optimal turning point for the system that reaches a steady state independent from the above mentioned triplet as soon as the cooperativity is large enough. Furthermore, the minimal variance seems to be rather indifferent to the atomic noise for large values of $\tilde{C}$. The following sections aim to explain these *a priori* surprising results.

**B. Outgoing field spectrum**

As mentioned above, with the existence of atomic squeezing, we can expect squeezing for the outgoing field $A_2^{\text{out}}$. To analyze field squeezing, a good variable is the noise spectrum in the frequency domain obtained from the variance matrix $[V_{\text{out}}(\omega)]$ defined by

$$\langle \left[ \delta A_2^{\text{out}}(\omega) \right] \left[ \delta A_2^{\text{out}}(\omega') \right] \rangle = 2\pi \delta(\omega - \omega') [V_{\text{out}}(\omega)]$$

(48)

where

$$\left[ \delta A_2^{\text{out}}(\omega) \right] = \begin{bmatrix} \delta A_2^{\text{out}}(\omega) \\ \delta A_2^{\text{out}\dagger}(\omega) \end{bmatrix}$$

(49)
is the vector Fourier transform of $|\delta A_2^{\text{out}}(t)|$. The minimal and maximal component of the noise spectrum are then

$$S_{\text{out min}}(\omega) = [V_{\text{out}}(\omega)]_{1,1} + [V_{\text{out}}(\omega)]_{2,2} - 2 ||V_{\text{out}}(\omega)||_{1,2}$$  \hspace{1cm} (50)$$

$$S_{\text{out max}}(\omega) = [V_{\text{out}}(\omega)]_{1,1} + [V_{\text{out}}(\omega)]_{2,2} + 2 ||V_{\text{out}}(\omega)||_{1,2}$$  \hspace{1cm} (51)$$

We have plotted in Fig. 3 these two components for a given set of parameters of Table I. As expected, the field is squeezed (maximum squeezing around 12%), but the squeezing occurs over a frequency range having width $100\gamma_0$, much larger than the value of order $\gamma_0$ one might have expected from Eq. (38-41) where the atomic time constant is $1/\gamma_0$. This is a consequence of the field-atom interaction in the "bad cavity limit", as will be explained in the next section.

C. Adiabatic elimination of the intracavity field

Unlike the three-level system, the effective two-level system we consider is in what is called the "bad cavity limit": $\kappa \gg \gamma_0$, the field evolves much faster than the atoms. We can therefore adiabatically eliminate the field operators in the linearized Eqs. (38-41). One obtains a new set of equations

$$\frac{d|\delta S|}{dt} = -\gamma_0 [B_3]|\delta S| + |F'_S|$$  \hspace{1cm} (52)$$

with $|\delta S| = [\delta S_+ , \delta S_- , \delta S_z]^T$, $|F'_S| = [F'_S^+, F'_S^-, F'_S^z]^T$,

$$[B_3] = \begin{pmatrix} 1 - i\tilde{\delta} + 4\tilde{C}\frac{s^z_+}{1+i\delta_c} & 0 & -2i\beta_2 \\ 0 & 1 + i\tilde{\delta} + 4\tilde{C}\frac{s^z_-}{1-i\delta_c} & 2i\beta_2 \\ -i\beta_2 - 2\tilde{C}\frac{s_-}{1-i\delta_c} & i\beta_2 - 2\tilde{C}\frac{s^+_z}{1+i\delta_c} & 1 \end{pmatrix}$$  \hspace{1cm} (53)$$

where

$$s_i = \langle S_i \rangle / N \quad (i = +, -, z)$$
In these equations there appears a new decay constant

\[ \gamma' = \gamma_0 \left[ 1 + 4\tilde{C} \frac{s_z}{1 + \delta_z^2} \right] \]

which is approximately equal to \( \tilde{C}\gamma_0 \) if \( \tilde{C} \gg 1 \). If we take the values of the parameters of Fig. 3, we find \( \gamma' = 100\gamma_0 \) which corresponds exactly to the frequency bandwidth over which the outgoing field is squeezed. The atomic spectrum is broadened as a result of the interaction with the field. The new diffusion matrix defined by

\[ \langle |F'(t)| [F'(t')] \rangle = [D'] \delta(t - t') \]

is given by

\[
[D'] = \gamma_0 N \begin{pmatrix}
4s_z^2 & 0 & -2s_z s_+ \\
0 & 0 & 0 \\
-2s_z s_- & 0 & s_+ s_- \\
\end{pmatrix} + \tilde{C} [D_{at}]
\]

where \([D_{at}]\) is the atomic diffusion matrix calculated from the quantum regression theorem (see Appendix).

Equation (54) shows that \([D']\) is the sum of the usual atomic diffusion matrix \(\gamma_0 N D_{at}\), which is proportional to the atomic decay rate \(\gamma_0\) and the number of atoms \(N\), and another matrix, which represents the contribution of the field fluctuations, proportional to \(N^2\), since \(\tilde{C} \propto N\). For large values of the cooperativity, the main contribution to the noise originates from the field. This conclusion is valid only if

\[ \gamma_0 \ll \tilde{C}\gamma_0 \ll \kappa; \]

if \(\tilde{C}\) becomes too large, the atoms evolve as fast as the field, and the adiabatic elimination is no longer justified. When inequalities (53) hold, one is able to perform analytical calculations for the minimum variance, considering only the larger terms in (54) and (53), and using an optimal bistable point as in Table I. It is possible to show that the lowest minimum variance \(\Delta S_{min}\) that can be obtained is \(1/\sqrt{2} = 0.707\), in good agreement with the results announced in Sec. IV A. The optimal bistable point is obtained considering the most favorable steady state for squeezing. Since the squeezing originates from atom correlations we expect it to be maximum when the modulus of coherence \(S_+\) is a maximum. From the steady state equations
(31-32-33), it is straightforward to show that the maximum value of $|S_+|^2$ is $N^2/16$, when $4I_2 = 1 + \delta^2$. In this case, $S_z = N/4$ and, integrating (52), one finds, for $\tilde{C} \gg 1$,

$$\Delta S_{\text{min}} = \frac{1}{\sqrt{2}} + O\left(\frac{1}{\tilde{C}}\right)$$

(56)

We see that the self squeezing limit in this system is approximately 30 %, as was found in the numerical simulations.

D. Noise contributions to the atomic variance

To confirm our statement that the squeezing can be traced to fluctuations in the cavity field, we have computed the different contributions to the atomic noise. Indeed, instead of calculating directly the atomic variances from the time domain equations, it is possible to derive the atomic noise spectra in the Fourier domain and then integrate the various contributions over the frequency to get the variances. The linear response theory allows for calculating the response of the atoms to the field excitation. The fluctuations of the incident field cause the intracavity field to fluctuate, which induces fluctuations for the atomic variables. These fluctuations are coupled back to the field by polarization fluctuations, hence the non-linear coupling between the intracavity field $|\delta A_2|$ and the spin fluctuations $|\delta S|$. To these sources of fluctuations, we must add the coupling with the vacuum surrounding modes of the cavity and the noise due to the decay and pumping of the sublevels. Taking all these fluctuations into account, the atomic variance matrix $[V_{at}(\omega)]$ is the sum of four contributions [12]:

$$[V_{at}(\omega)] = [V_f(\omega)] + [V_v(\omega)] + [V_{dip}(\omega)] + [V_{int}(\omega)]$$

(57)

in which

1. $[V_f(\omega)]$ represents the contribution of the incident field whose fluctuations are modified by the atoms

2. $[V_v(\omega)]$ is the contribution of the vacuum fluctuations
3. \([V_{dip}(\omega)]\) corresponds to the fluorescence emitted in the cavity mode

4. \([V_{int}(\omega)]\) is the interference term between the vacuum fluctuations and the dipole fluctuations.

In Fig. 4, are plotted the various contributions to the minimal component of the spin noise spectrum. It is clear that the atomic noise is due entirely to the fluctuations of the incident field, whereas the sum of the contributions from the other terms is negligible. We have integrated the different spectra to yield the contributions to the variances \(\Delta S_{\text{min}}\) and present the results in Table II.

| \(\Delta S_f\) | \(\Delta S_v\) | \(\Delta S_{dip}\) | \(\Delta S_{int}\) | \(\Delta S_{\text{min}}\) | \(\Delta S_f/\Delta S_{\text{min}}\) |
|---------|---------|---------|---------|---------|----------------|
| 0.701   | 1.414   | 1.373   | -2.772  | 0.716   | 97.9 %         |

Table II: Contributions to the atomic variance (value of the parameters: \(\bar{C} = 100, \bar{\delta} = 12, \delta_c = -0.2, I_2 = 40, \gamma_0 = \gamma/1000, \kappa = 2\gamma = 5.2 \text{MHz}\))

We see that the contribution of the field represents about 98 % of the total variance for the minimal component, whereas all the other contributions amount to only 2% of the global noise. Consequently, the \(\Lambda\) scheme provides a non-linear regime in which the field imprints its fluctuations on the atoms and in which all the other causes of noise are negligible compared to the non-linear interaction. We can thus easily reach the squeezing limit predicted in the previous section, for much smaller values of the cooperativity (50-100). In [13], for a classical ground-excited state two-level system, the self-squeezing limit was 50 %, but, in order to approach that limit, much larger values of the cooperativity were required (1000 to 10000).

V. SQUEEZING TRANSFER

The second configuration that we have studied is the case where the input light is a broadband squeezed vacuum. In this case, \(\langle A_{2}^{\text{in}}\rangle = 0\). Note that we are no longer in a non-linear regime as in the previous section. We have shown that, for the 3-level Raman scheme considered in this paper and for the optimized transfer conditions given in Ref. [13],
the squeezing transfer from the field to the atoms is almost 100%, even though we are not in the strong coupling regime as in [16]. The only approximation made is the same as in Ref. [16]: the intracavity intensity stays negligible and the steady state mean spin is then 
\[ \langle S_z \rangle = \frac{N}{2} \] ("small angle approximation" [11]). Of course, even though \( \langle A_2^{in} \rangle = 0 \), the average number of intracavity photons is not 0, but \( \sinh^2 r \) for a squeezed vacuum, where \( r \) is a squeezing parameter. If the average photon number is much less than \( N \), however, one can set

\[ \langle S_+ \rangle = \langle S_- \rangle = 0, \quad \langle S_z \rangle = \frac{N}{2}. \quad (58) \]

Under these assumptions, the calculation of the minimum variance can be done analytically [16], and one can deduce the optimal transfer condition

\[ \bar{\delta} = \delta_c = 0. \quad (59) \]

In this case, the minimal variance can be expressed as a function of the effective cooperativity \( \tilde{C} \), the ratio \( \rho = \gamma_0/\kappa \) and the squeezing rate \( e^{-2r} \) as

\[ \Delta S_{min} = 1 - \frac{2\tilde{C}}{(1 + \rho)(1 + 2\tilde{C})}(1 - e^{-2r}). \quad (60) \]

As we pointed out, we are not in a strong coupling regime, which corresponds to \( \rho \tilde{C} \gg 1 \). In our case, \( \tilde{C} \sim 100 \) and \( \rho = \gamma_0/\kappa \sim 1/2000 \ll \gamma/\kappa \) (for a ground-excited state (g-e) system with \( \kappa = 2\gamma, \rho = 1/2 \) and \( \rho C \gg 1 \)). For large values of \( \tilde{C} \), the factor in front of \( (1 - e^{-2r}) \) is close to unity. If we define the transfer efficiency \( \eta \) by

\[ \eta = \frac{1 - \Delta S_{min}}{1 - e^{-2r}}, \quad (61) \]

we find, for large \( \tilde{C} \),

\[ \eta = \frac{1}{(1 + \rho)(1 + \frac{1}{2\tilde{C}})} \approx \frac{1}{(1 + \rho)} \left(1 - \frac{1}{2\tilde{C}} \right). \quad (62) \]

In particular, for infinite cooperativity, we find the same limit of maximal squeezing as in [16], but in a regime for which \( \gamma_0 \ll \kappa \) (\( \rho \ll 1, \eta \sim 1 - \rho \)).
the minimum variance versus the squeezing of the input field $R_{in} = 1 - e^{-2r}$, for $\tilde{C} = 100$
and two different values of $\rho$ corresponding to the case of our effective two-level system
($\rho = 1/2000$), and the case of an g-e system ($\rho = 1/2$). In particular, the $\Lambda$ system enables
one to reach much higher squeezing values than in a classical g-e two-level scheme. To
understand the influence of the cooperativity on the transfer, the efficiency versus $\tilde{C}$ is
plotted in Fig. 6 in the two cases discussed previously. The first conclusion is that the
transfer is close to 100 % in the regime considered ($\rho = 1/2000$) and is not limited by the
vacuum noise as in a g-e transition [16]. Second, the efficiency quickly saturates with $\tilde{C}$; a
cooperativity of 100 enables a transfer of 99.5 % of the squeezing from the field to the atoms
($\rho = 1/2000$). We find again that the transfer efficiency is excellent for very reasonable
values of the cooperativity ($\sim 100$), which is usually the limiting experimental factor.

VI. VALIDITY OF THE TWO-LEVEL MODEL

In this section, we discuss the validity conditions for our effective two-level model and
see how the model can be modified slightly to extend its range of validity. In carrying out
the adiabatic elimination that took us from the full, three-level equations to the two-level
equations, two basic assumptions were made. First, it was assumed that optical pumping
terms of order

$$\Gamma_p = \gamma |\Omega_1/\Delta|^2$$

could be neglected. Second, it was assumed implicitly that the probe field was sufficiently
weak to neglect terms of order $|\Omega_2/\Omega_1|$, where

$$\hat{\Omega}_2 = gA_2; \quad \Omega_2 = \langle \hat{\Omega}_2 \rangle$$

If one adiabatically eliminates both the excited state - ground state coherence and the
excited state population operators from Eqs. (4-9), the resulting contributions to the time
evolution of the ground state operators of optical pumping, transit time decay, and incoherent
pumping is
\[ \dot{S}_z = -(\gamma_0 + \Gamma_p) S_z + N\Gamma_p/2 + (\Lambda_2 - \Lambda_1)/2 \]  
\[ \dot{S}_+ = -(\gamma_0 + \Gamma_p) S_+ - \Gamma_p\hat{\Omega}_2/\Omega_1 \]  
(63a)  
(63b)

where terms of order $\Gamma_p |\Omega_2/\Omega_1|^2$ have been neglected. Aside from the term of order $\Gamma_p |\Omega_2/\Omega_1|$, which we neglect for the moment but return to shortly, these equations have basically the same structure that we used for our effective two-state model. If $\Gamma_p \ll \gamma_0$, the two models coincide. For arbitrary ratios of $\Gamma_p/\gamma_0$, one can still use an effective two-level model if the assignments

\[ \gamma_0\text{(two-level)} = \gamma_0\text{(three-level)} + \Gamma_p \]  
(64a)  
\[ \Lambda_1\text{(two-level)} = \Lambda_1\text{(three-level)} \]  
(64b)  
\[ \Lambda_2\text{(two-level)} = \Lambda_2\text{(three-level)} + N\Gamma_p \]  
(64c)

are made [recall that $\Lambda_1 + \Lambda_2 = N\gamma_0$]. To check this hypothesis, we set $\gamma_0\text{(three-level)} = \Gamma_p = \gamma/1000$, so that $\Gamma_p$ is no longer negligible compared to $\gamma_0$, and we compare the two- and three-level calculations for self-squeezing when $\Delta = 100\gamma$, $\Omega_1 = \sqrt{10}\gamma$. For these parameters, $C = \tilde{C} = 100$ and $|\Omega_2/\Omega_1|$ is typically of order 0.01 at the point of optimal squeezing. The results, displayed in Fig. 7, are in good agreement for both models. We have generalized the results obtained in Sec. IV A for $\Gamma_p/\gamma_0 \ll 1$ to arbitrary ratios $\Gamma_p/\gamma_0$. We would like to point out that the steady state values (31), (32), (33) are modified by the presence of $\Gamma_p$, and the relation $4I_2 = (1 + \Gamma_p/\gamma_0)^2 + \delta^2$ must be satisfied in order to have the same optimal value of $|\langle S_+ \rangle| = N/4$.

To achieve better agreement between the effective two-level and full three-level calculations, it may also necessary to include the $|\Omega_2/\Omega_1|$ term in Eq. (63b). In the previous cases, this term can be neglected, but, for closed systems [$\Gamma_p \gg \gamma_0\text{(three-level)}$], the term linear in $|\Omega_2/\Omega_1|$ in Eq. (63b) can slightly increase the value of $|\langle S_+ \rangle|$ above the maximum value of $N/4$ it has for open systems [$\Gamma_p \ll \gamma_0\text{(three-level)}$]. As such the value of $\Delta S_{\min}$ can be reduced below $1/\sqrt{2}$. This feature is seen in Fig. 8, where inclusion of the linear term in a corrected two-level model brings the results into good agreement with the full three-level
calculation. Squeezing values of 35% can be reached in such a regime. The corrected two-level model also includes a slight correction arising from a term linear in $|\Omega_2/\Omega_1|$ that we neglected in going from Eq. (10) to (30) for the field evolution. When this linear term is included, Eq. (30) is modified as

$$\frac{dA_2}{dt} = - (\kappa + i \Delta_c) A_2 + \frac{i \tilde{g}}{\tau} \left[ S_+ \left( \hat{\Omega}_2/\Omega_1 \right) \Pi_2 \right] + \sqrt{\frac{2\kappa}{\tau}} A_2^\text{in}$$ (65)\)

The simplified two-level model considered in the earlier sections provides a good understanding of squeezing in a three-level system and allows one to optimize the parameters in a very simple way. It can be brought into full agreement with the three-level model when $|\Delta_{1,2}| \gg \gamma, |\Omega_1|$ and $|\Omega_2/\Omega_1| \ll 1$, provided one uses the prescription (64) for the decay and incoherent pumping and Eq. (65) for the field evolution. Finally, we would like to point out that, although the value of $\gamma_0$ itself is not critical, the ratio of the decay constants of the coherence $S_\pm$ to the population difference $S_z$ is important, since it determines the maximal coherence that can be obtained from the system, hence the maximal squeezing. Changing this ratio from 1 (as in our simple two-level system) to 1/2 (as in a classical e-g system) would enable one to achieve a 50% squeezing limit.

VII. CONCLUSION

Using a quantum model for an ensemble of three-level atoms in a Raman (\Lambda) type configuration, we have derived the atomic spin fluctuation spectra and variances and have shown rigorously the occurrence of spin squeezing between the two ground state sublevels. Unlike configurations in which the squeezing occurs between two excited states or between a ground state and an excited state, the squeezing obtained here has a long lifetime, does not require a very strong coupling and, as a consequence, presents substantial advantages for experimental realization.

Spin squeezing may occur in two different regimes. In the first one, the non-linearity of the atomic ensemble is exploited to squeeze the intracavity field, which in turn imprints
squeezing on the atomic ensemble, yielding self-spin squeezing. In the second one, the atomic ensemble has a linear behavior. It cannot create squeezing in the intracavity field. However, if one of the incoming fields is squeezed, the atom-field coupling in the cavity enables a transfer of squeezing of almost 100%.

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IX. APPENDIX

We give here the expression of the diffusion matrix appearing in (16) for the two-level system. The matrix elements of the upper $2 \times 2$ diagonal sub-block of $[D]$ are the correlation functions of a broadband squeezed field, equal to the one of a single mode field squeezed by a factor $e^{-2r}$ [16]

$$<\delta A^{in}(t)\delta A^{in\dagger}(t)> = \cosh^2(r) \delta(t-t')$$  \hspace{1cm} (66)

$$<\delta A^{in}(t)\delta A^{in}(t)> = 1/2 \sinh(r) e^{i\theta} \delta(t-t')$$  \hspace{1cm} (67)

$$<\delta A^{in\dagger}(t)\delta A^{in\dagger}(t)> = 1/2 \sinh(r) e^{-i\theta} \delta(t-t')$$  \hspace{1cm} (68)

$$<\delta A^{in\dagger}(t)\delta A^{in}(t)> = \sinh^2(r) \delta(t-t')$$  \hspace{1cm} (69)

leading to a field diffusion matrix

$$[D_f] = \frac{2\kappa}{\tau} \begin{bmatrix} \cosh^2(r) & 1/2 \sinh(r) e^{-i\theta} \\ 1/2 \sinh(r) e^{i\theta} & \sinh^2(r) \end{bmatrix}$$  \hspace{1cm} (70)
The matrix elements of the lower $3 \times 3$ diagonal sub-block of $[D]$ are the correlation functions of the atomic noise operators appearing in Eqs. (38)-(40). They were evaluated with the Einstein generalized relations [17], and grouped in the atomic diffusion matrix $[D_{at}]$:

$$
[D_{at}] = N\gamma_0 \begin{pmatrix}
1 + \frac{\lambda_1 - \lambda_2}{2} - s_z & 0 & (1 + \lambda_1 - \lambda_2) \frac{s_+}{2} \\
0 & 1 - \frac{\lambda_1 - \lambda_2}{2} + s_z & (-1 + \lambda_1 - \lambda_2) \frac{s_+}{2} \\
(1 + \lambda_1 - \lambda_2) \frac{s_+}{2} & (-1 + \lambda_1 - \lambda_2) \frac{s_-}{2} & \frac{1}{2} + (\lambda_1 - \lambda_2)s_z
\end{pmatrix}
$$

(71)

where $\lambda_{1,2} = \Lambda_{1,2}/N\gamma_0$ are the dimensionless incoherent pumping terms for one atom and $s_{\pm,z} = \langle S_{\pm,z} \rangle / N$ are the steady state values for one atom. The other elements of $[D]$ are equal to zero since there is no correlation between atomic and field fluctuations at the same time. Therefore,

$$
[D] = \begin{pmatrix}
[D_f] & 0 \\
0 & [D_{at}]
\end{pmatrix}
$$

(72)

where $[D_{at}]$ and $[D_f]$ are defined above (70)-(71).

Fig 1. Three-level system in a $\Lambda$ configuration.

Fig 2. Minimal variance versus effective cooperativity $\tilde{C}$ for a typical steady state point ($\tilde{\delta} = 10$, $\delta_c = 0$, $I_2 = 25.2$).

Fig 3. Minimal (below) and maximal (above) spectra for the outgoing field for the same bistable point as in Fig. 1 (the frequency unit is $\gamma_0$).

Fig 4. Contributions to the atomic noise spectrum (black): the field contribution (light grey) is predominant over the sum of the three other contributions, (dark grey). The bistable parameters are $\tilde{C} = 100$, $\tilde{\delta} = 12$, $\delta_c = -0.2$, $I_2 = 40$ as in Table II.

Fig 5. Minimal variance $\Delta S_{\text{min}}$ as a function of the input field squeezing $R_{in}$ for $\rho = 1/2000$ [$\Lambda$ system] and $\rho = 1/2$ [classical 2-level system].

Fig 6. Transfer efficiency $\eta$ versus effective cooperativity $\tilde{C}$ for the same systems as in Fig. 5.

Fig 7. Minimal variance versus effective detuning $\tilde{\delta}$ in the case when the ratio $\Gamma_p/\gamma_0 = 1$, calculated using the 2-level model with prescription (54) and the full 3-level model.
Fig 8. Minimal variance versus effective detuning $\tilde{\delta}$ in the case of a closed system $[\Gamma_p \gg \gamma_0]$, calculated using the corrected 2-level model and the 3-level model. The small field term in Eq. (63b) allows for going below the $1/\sqrt{2}$ minimal variance limit and improving the squeezing (up to 35% in this case).

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