AN EXPLICIT INTEGRAL REPRESENTATION OF
SIEGEL-WHITTAKER FUNCTIONS ON \( \text{Sp}(2, \mathbb{R}) \) FOR THE LARGE
DISCRETE SERIES REPRESENTATIONS

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Abstract. We obtain an explicit integral representation of Siegel-Whittaker functions on \( \text{Sp}(2, \mathbb{R}) \) for the large discrete series representations. We have another integral expression different from that of Miyazaki [7].

1. Introduction

In this article, we study Siegel-Whittaker functions on \( G = \text{Sp}(2, \mathbb{R}) \), the real symplectic group of degree two for the large discrete series representations. Let \( P_S \) be the Siegel parabolic subgroup of \( G \), which is a maximal parabolic subgroup with abelian unipotent radical \( N_S \). Let \( \pi \) be an admissible representation of \( G \) and \( \xi \) be a definite unitary character of \( N_S \). Siegel-Whittaker model for an admissible \( \pi \) is a realization of \( \pi \) in the induced module from a certain closed subgroup \( R \) which contains \( N_S \). (See (2.2) for definition of \( R \).) We consider the intertwining space

\[
\text{SW}(\pi; \eta) = \text{Hom}_{(g_{\mathbb{C}}, K)}(\pi, C^\infty(R\backslash G)),
\]

where \( \eta \) is an irreducible \( R \)-module such that \( \eta|_{N_S} \) contains \( \xi \), \( K \) is a maximal compact subgroup of \( G \) and \( g_{\mathbb{C}} \) is the complexification of the Lie algebra of \( G \). A function in this realization is called a Siegel-Whittaker function for \( \pi \).

Takuya Miyazaki obtained a system of partial differential equations satisfied by Siegel-Whittaker functions for the large discrete representations in [7]. He also obtained multiplicity one property and its formal power series solutions. In this article, we investigate further the system obtained by Miyazaki (Proposition 3.1) and give an explicit integral representation of the Siegel-Whittaker functions, which is of rapid decay, for a large discrete series representation \( \pi \). (Theorem 5.1) In other words, we show that the rapidly decreasing Siegel-Whittaker function for the large discrete series representation \( \pi \) is uniquely determined and (up to polynomials) described by the partially confluent hypergeometric

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functions in the $A$-radial part $(a_1, a_2) \in (\mathbb{R}_{>0})^2$:

\begin{equation}
\Psi_{\alpha, \beta}(a_1, a_2) = e^{-2\pi(h_1a_1^2 + h_2a_2^2)} \int_0^1 F_{SW}(2\pi h_1a_1^2 t + 2\pi h_2a_2^2(1-t)) t^{\alpha-1}(1-t)^{\beta-1} dt,
\end{equation}

with

\[ F_{SW}(x) = F(x) = e^{x}x^{-\gamma}W_{\kappa, \mu}(2x). \]

Here, $W_{\kappa, \mu}(x)$ is the Whittaker’s classical confluent hypergeometric function, $\alpha, \beta, \gamma > 0$ and $\kappa, \mu$ are half integers depending on parameters $\pi$ and $\eta$. This kind of integral already appeared in Gon [1], treated the cases for the large discrete series representations and $P_J$-principal series representations of $SU(2, 2)$.

The reason we begin to believe the validity of similar formula for $Sp(2, \mathbb{R})$, is the paper Hirano-Ishii-Oda [2]. By this new integral expression (1.1), it seems possible to extend the former argument in [2] of the confluence from Siegel-Whittaker functions to Whittaker functions for $P_J$-principal series, to the confluence from Siegel-Whittaker functions to Whittaker functions [3] for the large discrete series of $Sp(2, \mathbb{R})$. Here we recall the template of the integral formulas of Whittaker functions in $(a_1, a_2) \in (\mathbb{R}_{>0})^2$:

\begin{equation}
\mu_W(a_1, a_2) \int_0^\infty F_W(t) t^{-C} e^{-A t^2/a_1^2 - B a_2^2 t^2} dt,
\end{equation}

where $F_W(t) = W_{0, \gamma}(t)$ with a positive integral parameter $\gamma$, $\mu_W(a_1, a_2)$ is a monomial in $a_1, a_2$ times $e^{\delta a_2^2}$ ($\delta > 0$) and $A, B, C$ are positive real parameters.

Probably we can proceed a bit more. Since Iida [3], Theorems 8.7 and 8.9 (Formulas (8.8) and (8.10), resp.) give analogous integral expressions for the matrix coefficient of $P_J$-principal series, and also Oda [9] (p.247), Theorem 6.2 similarly gives analogous integral expression for matrix coefficients of the large discrete series. We notice that the template of formulas in Theorem 6.2 of [9] is given by

\begin{equation}
\mu_{MC}(a_1, a_2) \int_0^1 F_{MC}(tx_1 + (1-t)x_2) t^{\alpha-1}(1-t)^{\beta-1} dt,
\end{equation}

with $x_i = -(a_i - a_i^{-1})^2/4$ ($i = 1, 2$). Here $F_{MC}(x) = {}_2F_1(A, B; C : x)$ with $A, B, C, \alpha, \beta$ determined by Harish-Chandra parameters and the components of the minimal $K$-type. The elementary factor $\mu_{MC}(a_1, a_2)$ is a monomial of $((a_i \pm a_i^{-1})/2)^{\pm 1/2}$ ($i = 1, 2$).

These facts suggests that there will be a deformation of confluence [1] $\mapsto$ [1] from the matrix coefficients realization to Siegel-Whittaker realization in a very simple natural way. We believe the similar argument is possible for the principal series: but in this case the natures of integrands of Ishii [2], Theorem 3.2 and Siegel-Whittaker case [3], Theorem 10.1 are still difficult to deform.

For the $P_J$-principal series of the other group $SU(2, 2)$, we have explicit integral expression of the matrix coefficients (Theorem 5.4 of [6]) similar to the above formula (1.3). We can expect the confluence from this formula to the integral expression of the Siegel-Whittaker realization of [1] analogous to (1.1), corresponding to the limit of one parameter
conjugations of a compact subgroup $K$ of SU(2, 2):
\[
\lim_{t \to 0} h_tKh_t^{-1} = R.
\]
The deformation of the Siegel-Whittaker function of the $P_J$-principal series to the Whittaker function could be handled in a similar way as in [2].

In view of this observation, we may hope that similar phenomenon occurs for more general groups SO(2, q) ($q > 4$).

2. Preliminaries

2.1. Basic notations. Let $G$ be the real symplectic group of degree two:
\[
\text{Sp}(2, \mathbb{R}) = \left\{ g \in \text{SL}(4, \mathbb{R}) \mid ^t gJ_2 g = J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\},
\]
with $1_2$ the unit matrix of degree two and $0_2$ the zero matrix of degree two.

Fix a maximal compact subgroup $K$ of $G$ by
\[
K = \left\{ k(A, B) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in G \mid A, B \in M(2, \mathbb{R}) \right\}.
\]
It is isomorphic to the unitary group $U(2)$ via the homomorphism
\[
K \ni k(A, B) \mapsto A + \sqrt{-1}B \in U(2).
\]

We define a certain spherical subgroup $R$ of $G$ as follows. Let $P_S = L_S \ltimes N_S$ be the Siegel parabolic subgroup with the Levi part $L_S$ and the abelian unipotent radical $N_S$ given by
\[
L_S = \left\{ \begin{pmatrix} A & 0_2 \\ 0_2 & ^t A^{-1} \end{pmatrix} \mid A \in \text{GL}(2, \mathbb{R}) \right\},
\]
\[
N_S = \left\{ n(T) = \begin{pmatrix} 1_2 \\ 0_2 \end{pmatrix} \mid ^t T = T \in M(2, \mathbb{R}) \right\}.
\]

Fix a non-degenerate unitary character $\xi$ of $N_S$ by
\[
\xi(n(T)) = \exp(2\pi \sqrt{-1} \text{tr}(H_\xi T))
\]
with $H_\xi = \begin{pmatrix} h_1 & h_3/2 \\ h_3/2 & h_2 \end{pmatrix} \in M(2, \mathbb{R})$ and det $H_\xi \neq 0$. Consider the action of $L_S$ on $N_S$ by conjugation and the induced action on the character group $\hat{N}_S$. Define $SO(\xi)$ to the identity component of the subgroup of $L_S$ which stabilize $\xi$:
\[
SO(\xi) := \text{Stab}_{L_S}(\xi)^0 = \left\{ \begin{pmatrix} A & 0_2 \\ 0_2 & ^t A^{-1} \end{pmatrix} \in L_S \mid ^t A H_\xi A = H_\xi \right\}.
\]
Then $SO(\xi)$ is isomorphic to $SO(2)$ if det $H_\xi > 0$ and to $SO_c(1, 1)$ if det $H_\xi < 0$. In this article we treat the case that $\xi$ is a ‘definite’ character, that is det $H_\xi > 0$. So we may assume $h_1, h_2 > 0$ and $h_3 = 0$ without loss of generality. We sometimes identify the element
of \( \text{SO}(\xi) \) with its upper left \( 2 \times 2 \) component. Fix a unitary character \( \chi_{m_0} \) \( (m_0 \in \mathbb{Z}) \) of \( \text{SO}(\xi) \cong \text{SO}(2) \) by

\[
(2.1) \quad \chi_{m_0} = \begin{pmatrix} \sqrt{t_1} & \cos \theta & \sin \theta \\ \sqrt{h_2} & -\sin \theta & \cos \theta \end{pmatrix} = \exp(\sqrt{-1}m_0 \theta).
\]

We define \( R \) by

\[
(2.2) \quad R = \text{SO}(\xi) \ltimes N_S \quad \text{and} \quad \eta = \chi_{m_0} \rtimes \xi.
\]

Taking a maximal split torus \( A \) of \( G \) by

\[
A = \{ a = (a_1, a_2) = \text{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \mid a_1, a_2 > 0 \},
\]

we have the decomposition \( G = RAK \).

### 2.2. Siegel-Whittaker functions.

We consider the space \( C^\infty_\eta(R \backslash G) \) of complex valued \( C^\infty \) functions \( f \) on \( G \) satisfying

\[
f(rg) = \eta(r) f(g) \quad \forall (r, g) \in R \times G.
\]

By the right translation, \( C^\infty_\eta(R \backslash G) \) is a smooth \( G \)-module and we denote the same symbol its underlying \( (g_C, K) \)-module. For an irreducible admissible representation \( (\pi, H_\pi) \) of \( G \) and the subspace \( H_{\pi, K} \) of \( K \)-finite vectors, the intertwining space

\[
I_{\eta, \pi} = \text{Hom}_{(g_C, K)}(H_{\pi, K}, C^\infty_\eta(R \backslash G))
\]

between the \( (g_C, K) \)-modules is called the space of algebraic Siegel-Whittaker functionals. For a finite-dimensional \( K \)-module \( (\tau, V_\tau) \), denote by \( C^\infty_{\eta, \tau}(R \backslash G/K) \) the space

\[
\{ \phi: G \rightarrow V_\tau, C^\infty \mid \phi(r \eta \tau(k^{-1}) \phi(g) \quad \forall (r, g, k) \in R \times G \times K \}.
\]

Let \( (\tau^*, V_{\tau^*}) \) be a \( K \)-type of \( \pi \) and \( \iota: V_{\tau^*} \rightarrow H_\pi \) be an injection. Here, \( \tau^* \) means the contragredient representation of \( \tau \). Then for \( \Phi \in I_{\eta, \pi} \), we can find an element \( \phi_i \) in

\[
C^\infty_{\eta, \tau}(R \backslash G/K) = C^\infty_{\eta, \tau}(R \backslash G) \otimes V_{\tau^*} \cong \text{Hom}_K(V_{\tau^*}, C^\infty_\eta(R \backslash G))
\]

via \( \Phi(\iota(v^*)))(g) = \langle v^*, \phi_i(g) \rangle \) with \( \langle , \rangle \) the canonical pairing on \( V_{\tau^*} \times V_\tau \).

Since there is the decomposition \( G = RAK \), our generalized spherical function \( \phi_i \) is determined by its restriction \( \phi_i|_A \), which we call the radial part of \( \phi_i \). For a subspace \( X \) of \( C^\infty_{\eta, \tau}(R \backslash G/K) \), we denote \( X|_A = \{ \phi|_A \in C^\infty(A) \mid \phi \in X \} \).

Let us define the space \( \text{SW}(\pi, \eta, \tau) \) of Siegel-Whittaker functions and its subspace \( \text{SW}(\pi, \eta, \tau)^{\text{rap}} \) as follows:

\[
\text{SW}(\pi, \eta, \tau)^{\text{rap}} = \bigcup_{\iota \in \text{Hom}_K(\tau^*, \pi)} \{ \phi_i \mid \Phi \in I_{\eta, \pi} \}
\]

and

\[
\text{SW}(\pi, \eta, \tau) = \{ \phi_i \in \text{SW}(\pi, \eta, \tau) \mid \phi_i|_A \text{ decays rapidly as } a_1, a_2 \rightarrow \infty \}.
\]

We call an element in \( \text{SW}(\pi, \eta, \tau) \) a Siegel-Whittaker function for \( (\pi, \eta, \tau) \).
2.3. **Parametrization of the discrete series representations.** Let $E_{ij} \in M_d(\mathbb{R})$ be the matrix unit with 1 as its $(i, j)$-component and 0 at the other entries. The root system of $G$ with respect to a compact Cartan subgroup

$$T = \exp(\mathbb{R}(E_{13} - E_{31}) + \mathbb{R}(E_{24} - E_{42}))$$

is given by a set of vectors in the Euclidean plane:

$$\{\pm 2\varepsilon_1, \pm 2\varepsilon_2, \pm \varepsilon_1 \pm \varepsilon_2\}.$$ 

Here,

$$\varepsilon_1(r_1(E_{13} - E_{31}) + r_2(E_{24} - E_{42})) = \sqrt{-1}r_1,$$

$$\varepsilon_2(r_1(E_{13} - E_{31}) + r_2(E_{24} - E_{42})) = \sqrt{-1}r_2.$$ 

We fix a subset of simple roots and the associated positive roots by

$$\hat{\varepsilon}_1$$

and $\hat{\varepsilon}_2$ respectively.

Then the set of the unitary characters of $T$ (or their derivatives) is identified naturally with $\mathbb{Z} \oplus \mathbb{Z}$, and the subset consisting of dominant integral weight is

$$\Xi = \{(n_1, n_2) \in \mathbb{Z} \oplus \mathbb{Z} \mid n_1 \geq n_2\}.$$ 

There is a bijection between $\hat{\mathbb{K}}$ and $\Xi$ by highest weight theory. Because the half-sum of the positive root is integral, the discrete series representations of $G = \operatorname{Sp}(2, \mathbb{R})$ are parametrized by the subset of regular elements in $\Xi$:

$$\Xi' = \{(n_1, n_2) \in \mathbb{Z} \oplus \mathbb{Z} \mid n_1 > n_2, n_1 \neq 0, n_2 \neq 0, n_1 + n_2 \neq 0\}.$$ 

Here the condition $n_1 > n_2$ means the positivity of weight $(n_1, n_2)$ with respect to the compact root $\varepsilon_1 - \varepsilon_2 = (1, -1)$.

The subsets $\Xi_I = \{(n_1, n_2) \mid n_1 > n_2 > 0\}$ and $\Xi_{IV} = \{(n_1, n_2) \mid 0 > n_1 > n_2\}$ parametrize the holomorphic discrete series and the anti-holomorphic discrete series representations, respectively. Set

$$\Xi_{II} = \{(n_1, n_2) \mid n_1 > 0 > n_2, n_1 + n_2 > 0\},$$ 

and

$$\Xi_{III} = \{(n_1, n_2) \mid n_1 > 0 > n_2, 0 > n_1 + n_2\}.$$ 

Then the union $\Xi_{II} \cup \Xi_{III}$ parametrize the large discrete series representations of $G$.

3. Miyazaki’s results

Miyazaki derived a system of partial differential equations satisfied by Siegel-Whittaker functions for the large discrete representations in [7]. He also obtained multiplicity one property. We recall his results in this section.

Let $\tau = \tau(\lambda_1, \lambda_2) = \operatorname{Sym}^{\lambda_1 - \lambda_2} \otimes \det^{\lambda_2}$ be the irreducible $K$-module with the highest weight $(\lambda_1, \lambda_2)$, then the dimension of $\tau$ is $d + 1$ with $d = \lambda_1 - \lambda_2$. We take the basis $\{v_j\}_{j=0}^d$ of $V_\tau$ with $\tau^* = \tau(-\lambda_2, -\lambda_1)$ as in [7] Lemma 3.1].
We remark on a compatibility condition. For a non-zero function \( \phi \) in \( C_{\eta^*}(R \setminus G/K) \), we have

\[
\phi(a) = \phi(mam^{-1}) = (\chi_{m_0} \otimes \xi)(m) \tau_{(-\lambda_2,-\lambda_1)}(m) \phi(a),
\]

where, \( a \in A \) and \( m \in SO(\xi) \cap Z_K(A) = \{ \pm 1 \} \). If we take \( m = -1 \), \( (\chi_{m_0} \otimes \xi)(m) = \chi_{m_0}(m) = (-1)^{m_0} \) and \( \tau_{(-\lambda_2,-\lambda_1)}(m) = (-1)^d \) imply that \( (m_0 + d)/2 \) is an integer.

**Proposition 3.1** (Miyazaki [7]). Let \( \pi = \pi_\Lambda \) be a large discrete series representation of \( G \) with the Harish-Chandra parameter \( \Lambda = (\lambda_1 - 1, \lambda_2) \in \Xi_H \) and its minimal K-type \( \tau = \tau_{(\lambda_1, \lambda_2)} \). Let \( \xi \) be a unitary character of \( N_S \) associated with a positive definite matrix \( H_\xi = (h_{ij}) \). Put \( \eta = \chi_{m_0} \otimes \xi \), as in (2.7). Then we have the following:

(i) we have \( \dim_{\mathbb{C}} \text{SW}(\pi, \eta, \tau) \leq 4 \) and a function

\[
\phi_{SW}(a) = \sum_{j=0}^{d} (\sqrt{h_1 a_1})^{\lambda_1-j}(\sqrt{h_2 a_2})^{\lambda_2+j}e^{-2\pi(h_1 a_1^2 + h_2 a_2^2)c_j(a)} v_j
\]

is in the space \( \text{SW}(\pi_\Lambda, \eta, \tau)|_A \) if and only if \( \{c_j(a)\}_{j=0}^{d} \) is a smooth solution of the following system:

\[
(3.1) \left[ \partial_1 + j \frac{h_2 a_2^2}{\Delta} \right] c_{j-1}(a) + \sqrt{-1}m_0 \frac{h_2 a_2^2}{\Delta} c_j(a) - (d - j) \frac{h_2 a_2^2}{\Delta} c_{j+1}(a) = 0 \quad (1 \leq j \leq d),
\]

\[
(3.2) j \frac{h_1 a_1^2}{\Delta} c_{j-1}(a) + \sqrt{-1}m_0 \frac{h_1 a_1^2}{\Delta} c_j(a) + \left[ \partial_2 - (d - j) \frac{h_1 a_1^2}{\Delta} \right] c_{j+1}(a) = 0 \quad (0 \leq j \leq d - 1),
\]

\[
(3.3) h_1 a_1^2 \left[ \partial_1 - 8\pi h_2 a_2^2 - 2j \frac{h_2 a_2^2}{\Delta} + 2\lambda_2 - 2 \right] c_{j-1}(a) - 2\sqrt{-1}m_0 \frac{h_1 a_1^2 h_2 a_2^2}{\Delta} c_j(a) + h_2 a_2^2 \left[ \partial_1 - 8\pi h_1 a_1^2 + 2(d - j) \frac{h_1 a_1^2}{\Delta} + 2\lambda_2 - 2 \right] c_{j+1}(a) = 0 \quad (1 \leq j \leq d - 1),
\]

with \( \partial_i = a_i(\partial/\partial a_i) \) (i = 1, 2) and \( \Delta = h_1 a_1^2 - h_2 a_2^2 \).

(ii) \( \dim_{\mathbb{C}} \text{SW}(\pi, \eta, \tau)^{\text{rap}} \leq 1 \).

This is a paraphrase of Propositions 10.2, 10.7 and Theorem 11.5 of [7]. Here (3.1), (3.2), (3.3) are essentially identical equations to (10.4), (10.5), (10.6) of [7] deduced from Proposition 10.2. However we replace the symbol \( \chi(Y_\eta) \) of [7] by its explicit value \( \sqrt{-1}m_0/\sqrt{h_1 h_2} \), and the symbol \( D \) of [7] by \( \Delta \).

4. Partially Confluent hypergeometric functions in two variables

We introduce and study certain partially confluent hypergeometric functions in two variables on \( A \simeq (\mathbb{R}_{>0})^2 \). These functions play a key role in describing Siegel-Whittaker functions \( \phi_{SW}(a) \). We remark that these types of confluent hypergeometric functions have also appeared in [1], for the large discrete series representations and \( P_J \)-principal series representations of \( SU(2,2) \), and [2], for the \( P_J \)-principal series representations of \( \text{Sp}(2, \mathbb{R}) \).
Definition 4.1 (Partially confluent hypergeometric functions). Let \( \pi = \pi_\Lambda \) be a large discrete series representation of \( G \) with the Harish-Chandra parameter \( \Lambda = (\lambda_1 - 1, \lambda_2) \in \Xi_\Pi \) and its minimal \( K \)-type \( \tau = \tau(\lambda_1, \lambda_2) \). Let \( \xi \) be a unitary character of \( N_S \) associated with a positive definite matrix \( H_\xi = (h_{ij}) \). Put \( \eta = \chi_{m_0} \otimes \xi \), as in (2.1).

We assume that
\[
|m_0| \geq d \quad \text{and} \quad |m_0| \equiv d \pmod{2}.
\]
For \( 0 \leq k \leq d \), define
\[
f_k(a) = f_k(a; [\pi, \eta, \tau]) \quad (4.1)
\]
\[
= (\sqrt{h_1 a_1})^{2k+1} (\sqrt{h_2 a_2})^{2d+1-2k} (h_1 a_1^2 - h_2 a_2^2)^{\frac{m_0|-d}{2}} \times \int_0^1 F(2\pi h_1 a_1^2 t + 2\pi h_2 a_2^2 (1-t)) t^{\frac{|m_0|-d-1}{2} + k} (1-t)^{\frac{|m_0|+d-1}{2} - k} dt,
\]
with
\[
F(x) = e^{\pi x} x^{-\frac{|m_0|+\lambda_1+1}{2}} W_{\lambda_1-|m_0|-\lambda_2}(2x).
\]
Here, \( W_{\kappa, \mu}(z) \) is Whittaker’s confluent hypergeometric function. (See [10] Chapter 16 for definition.)

Since the indices \( \alpha_k - 1 = \frac{|m_0|-d-1}{2} + k, \beta_k - 1 = \frac{|m_0|+d-1}{2} - k \) in the integrand of \( f_k \) satisfy \( \alpha_k, \beta_k > 0 (0 \leq k \leq d) \), we see that \( f_k(a) \) is a smooth function on \( A \) and of moderate growth when each \( a_1, a_2 \) tends to infinity. We have further more,

Proposition 4.2. Partially confluent hypergeometric functions \( \{f_k(a)\}_{k=0}^d \) satisfy the following system of the difference-differential equations.

\[
(4.2) \quad \partial_1 f_k = -(2k+1)^2 h_2 a_2^2 \frac{\Delta}{\triangle} f_k + (|m_0| + d - 1 - 2k) h_2 a_2^2 \frac{\Delta}{\triangle} f_{k+1} \quad (0 \leq k \leq d - 1),
\]

\[
(4.3) \quad \partial_2 f_k = (2d - 2k + 1)^2 h_1 a_1^2 \frac{\Delta}{\triangle} f_k - (|m_0| - d - 1 + 2k) h_1 a_1^2 \frac{\Delta}{\triangle} f_{k-1} \quad (1 \leq k \leq d),
\]

\[
(4.4) \quad [(\partial_1 + \partial_2)^2 + 2(\lambda_2 - 2)(\partial_1 + \partial_2) - 8\pi(h_1 a_1^2 \partial_1 + h_2 a_2^2 \partial_2) - 4(\lambda_2 - 1)] f_k(a) = 0 \quad (0 \leq k \leq d).
\]

Here, \( \triangle = h_1 a_1^2 - h_2 a_2^2 \).

Proof. For \( 0 \leq k \leq d \), put
\[
\tilde{f}_k(a) = \int_0^1 F(2\pi h_1 a_1^2 t + 2\pi h_2 a_2^2 (1-t)) t^{\frac{|m_0|-d-1}{2} + k} (1-t)^{\frac{|m_0|+d-1}{2} - k} dt.
\]
Then we can verify that
\[
\partial_1 \tilde{f}_k = (\sqrt{h_1 a_1})^{2k+1} (\sqrt{h_2 a_2})^{2d+1-2k} \Delta \frac{|m_0|-d}{2} \left[ \partial_1 + (2k+1) + (|m_0| - d) \frac{h_1 a_1^2}{\Delta} \right] \tilde{f}_k,
\]
\[
\partial_2 \tilde{f}_k = (\sqrt{h_1 a_1})^{2k+1} (\sqrt{h_2 a_2})^{2d+1-2k} \Delta \frac{|m_0|-d}{2} \left[ \partial_2 - (2d-2k+1) + (|m_0| - d) \frac{h_2 a_2^2}{\Delta} \right] \tilde{f}_k.
\]
\[
\partial_1 \dot{f}_k = -(|m_0| - d + 1 + 2k) \frac{h_1 a_1^2}{\Delta} \dot{f}_k + (|m_0| + d - 1 - 2k) \frac{h_1 a_1^2}{\Delta} f_{k+1}
\]
for \(0 \leq k \leq d - 1\). Therefore, we obtain the formula (1.2). Similarly, we have
\[
\partial_2 f_k = (\sqrt{h_1 a_1})^{2k+1} (\sqrt{h_2 a_2})^{2d+1-2k} \frac{|m_0| - d}{\Delta} \left[ \partial_2 + (2d + 1 - 2k) \right] \frac{h_2 a_2^2}{\Delta} \dot{f}_k
\]
and
\[
\partial_2 \dot{f}_k = -(|m_0| - d + 1 + 2k) \frac{h_2 a_2^2}{\Delta} \dot{f}_{k-1} + (|m_0| + d + 1 - 2k) \frac{h_1 a_1^2}{\Delta} \dot{f}_k
\]
for \(1 \leq k \leq d\). Thus, we have the formula (4.3).

Let us prove (4.4). Put \(\Omega\) and \(\check{\Omega}\) be the partial differential operators defined by
\[
\Omega = (\partial_1 + \partial_2)^2 + 2(\lambda_2 - 2)(\partial_1 + \partial_2) - 8\pi(h_1 a_1^2 \partial_1 + h_2 a_2^2 \partial_2) - 4(\lambda_2 - 1)
\]
and
\[
\check{\Omega} = (\partial_1 + \partial_2)^2 + 2(|m_0| + d + \lambda_2)(\partial_1 + \partial_2) - 8\pi(h_1 a_1^2 \partial_1 + h_2 a_2^2 \partial_2)
\]
\[
- 8\pi(h_1 a_1^2 + h_2 a_2^2)(|m_0| + 1) - 8\pi(h_1 a_1^2 - h_2 a_2^2)(2k - d) + (|m_0| + d)(|m_0| + d + 2\lambda_2).
\]
Then we can check that
\[
\Omega f_k = (\sqrt{h_1 a_1})^{2k+1} (\sqrt{h_2 a_2})^{2d+1-2k} \frac{|m_0| - d}{\Delta} \frac{h_2 a_2^2}{\Delta} \check{\Omega} \dot{f}_k.
\]
By interchanging differentiation and integration, we have
\[
\check{\Omega} \dot{f}_k(t) = \int_0^1 G(2\pi h_1 a_1^2 t + 2\pi h_2 a_2^2 (1 - t)) t^{\frac{|m_0| - d - 1}{2} + k (1 - t)} \frac{|m_0| + d - 1}{2} \lambda_2^2 dt
\]
with
\[
G(x) = 4 \left[ x^2 \frac{d^2}{dx^2} - \left( 2x - (|m_0| + \lambda_1 + 1) \right) x \frac{d}{dx} - 2(|m_0| + 1) x + \frac{(|m_0| + \lambda_1)^2 - \lambda_2^2}{4} \right] F(x).
\]
Put \(F(x) = e^x x^{-|m_0|+\lambda_1+1/2} H(x)\). Then we have
\[
G(x) = 4 e^x x^{-|m_0|+\lambda_1+1/2} \cdot x^2 \left[ \frac{d^2}{dx^2} + \left\{ -1 + \frac{\lambda_1 - |m_0| - 1}{x} + \frac{1 - \lambda_2^2}{4x^2} \right\} \right] H(x).
\]
It is known that the differential equation
\[
\left[ \frac{d^2}{dx^2} + \left\{ -1 + \frac{\lambda_1 - |m_0| - 1}{x} + \frac{1 - \lambda_2^2}{4x^2} \right\} \right] H(x) = 0
\]
has two linearly independent solutions:
\[
W_{\lambda_1-|m_0|+1/2,\lambda_2}(2x), \quad M_{\lambda_1-|m_0|+1/2,\lambda_2}(2x).
\]
Here, \(W_{\kappa,\mu}(z)\) and \(M_{\kappa,\mu}(z)\) are Whittaker’s confluent hypergeometric functions. (See [10] Chapter 16 for definition.) Therefore, we have (4.4). It completes the proof. \(\square\)
5. Main results

We state our main results on an explicit integral formula of Siegel-Whittaker functions which are of rapid decay for the large discrete series representations of $\text{Sp}(2, \mathbb{R})$.

**Theorem 5.1.** Let $\pi = \pi_\Lambda$ be a large discrete series representation of $G$ with the Harish-Chandra parameter $\Lambda = (\lambda_1 - 1, \lambda_2) \in \Xi_{II}$ and its minimal $K$-type $\tau = \tau(\lambda_1, \lambda_2)$. Let $\xi$ be a unitary character of $N_S$ associated with a positive definite matrix $H_\xi = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$. Put $\eta = \chi_{m_0} \boxtimes \xi$, as in (2.1).

We assume that $|m_0| \geq d$ and $|m_0| \equiv d \pmod{2}$.

Then we have the following:

(i) $\dim C_{\text{SW}}(\pi, \eta, \tau)_{\text{rap}} = 1$.

(ii) Let $\{f_k(a)\}_{k=0}^d$ be the partially confluent hypergeometric functions, defined in Definition 4.1. We consider the following $C$-linear combinations $\{g_j(a)\}_{j=0}^d$ of elements of $\{f_k(a)\}_{k=0}^d$, given by

$$g_j(a) = \sum_{k=0}^d x_{jk} f_k(a),$$

where the complex numbers $\{x_{jk}\}_{0 \leq j, k \leq d}$ are given by

$$x_{jk} = (-1)^{j+k} (\text{sgn}(m_0) \sqrt{-1})^j \binom{d}{2k-j} |m_0|^j \prod_{l=1}^k \frac{2l-1}{|m_0| - d + 2l - 1}$$

$$\times \sum_{r=0}^{[j/2]} \binom{[j/2]}{r} (d - 2r - \delta(j))! \prod_{l=1}^r \left( \frac{2k-j-2l+2-\delta(j)}{2k-2l+1} \right) (|m_0|^2 - (d - 2l + 2)^2)$$

$(\delta(j) = 1$ if $j$ is odd, otherwise 0). Then the function

$$\phi_{\text{SW}}(a) = \sum_{j=0}^d \left( (\sqrt{h_1} a_1)^{\lambda_1-j} (\sqrt{h_2} a_2)^{\lambda_2+j} e^{-2\pi(h_1 a_1^2 + h_2 a_2^2)} g_j(a) \right) v_j$$

gives a non-zero element in $\text{SW}(\pi, \eta, \tau)_{\text{rap}}|_A$ which is unique up to constant multiple.

6. Proof of main results

We claim that the coefficient functions $c_0(a), c_1(a), \ldots, c_d(a)$ appearing in the Siegel-Whittaker function $\phi_{\text{SW}}(a)$, in Proposition 3.1, are $C$-linear combination of the confluent hypergeometric functions $f_0(a), f_1(a), \ldots, f_d(a)$ defined in Definition 4.1.

**Proposition 6.1.** We assume that $|m_0| \geq d$ and $|m_0| \equiv d \pmod{2}$. 
Let \( \{x_{jk}\}_{0 \leq j, k \leq d} \) be a sequence of complex numbers, which satisfy
\[
\begin{align*}
(6.1) & \quad j x_{j-1,k} + \sqrt{-1} m_0 x_{j,k} + (d - 2k + j + 1) x_{j+1,k} - (|m_0| - d + 2k + 1) x_{j+1,k+1} = 0 \\
& \quad (0 \leq j \leq d - 1, 0 \leq k \leq d), \\
(6.2) & \quad (j - 2k - 1) x_{j-1,k} + (|m_0| + d - 2k + 1) x_{j-1,k-1} + \sqrt{-1} m_0 x_{j,k} - (d - j) x_{j+1,k} = 0 \\
& \quad (1 \leq j \leq d, 0 \leq k \leq d), \\
(6.3) & \quad x_{1,0} = x_{2,0} = \cdots = x_{d,0} = 0, \\
(6.4) & \quad x_{0,d} = x_{1,d} = \cdots = x_{d-1,d} = 0.
\end{align*}
\]

For \( 0 \leq j \leq d \), define
\[
g_j(a) = \sum_{k=0}^{d} x_{jk} f_k(a),
\]
then \( \{g_j(a)\}_{j=0}^{d} \) is a smooth solution of the system of the difference-differential equations \((6.1), (6.2)\) and \((6.3)\) in Proposition 3.1.

Proof. By using Proposition 4.2, we see that
\[
\begin{align*}
\nabla h_2 a_2^2 \left[ \partial_1 + j \frac{h_2 a_2^2}{\nabla} \right] g_{j-1}(a) & = \sum_{k=0}^{d} (j - 2k - 1) x_{j-1,k} f_k + \sum_{k=0}^{d} (|m_0| + d - 2k - 1) x_{j-1,k} f_{k+1}.
\end{align*}
\]

Therefore, we have
\[
\begin{align*}
\nabla h_2 a_2^2 \left[ \partial_1 + j \frac{h_2 a_2^2}{\nabla} \right] g_{j-1}(a) & + \sqrt{-1} m_0 g_j(a) - (d - j) g_{j+1}(a) \\
& = \sum_{k=0}^{d} (j - 2k - 1) x_{j-1,k} f_k + \sum_{k=1}^{d} (|m_0| + d - 2k + 1) x_{j-1,k-1} f_k \\
& \quad + \sqrt{-1} m_0 \sum_{k=0}^{d} x_{j,k} f_k - (d - j) \sum_{k=0}^{d} x_{j+1,k} f_k \\
& = 0 \quad (1 \leq j \leq d).
\end{align*}
\]

This is the desired \((3.1)\) for \( \{g_j(a)\}_{j=0}^{d} \). In the above calculation, we used the relations \((6.2)\) and \((6.4)\) on \( \{x_{jk}\}_{0 \leq j, k \leq d} \). Again, by using Proposition 4.2, we see that
\[
\begin{align*}
\nabla h_1 a_1^2 \left[ \partial_2 - (d - j) \frac{h_1 a_1^2}{\nabla} \right] g_{j+1}(a) & = \sum_{k=0}^{d} (d - 2k + j + 1) x_{j+1,k} f_k - \sum_{k=0}^{d} (|m_0| - d + 2k - 1) x_{j+1,k} f_{k-1}.
\end{align*}
\]
where we put

\[ \beta (6.1) \text{ and (6.3) on } \{ \}

Therefore, we have

This is the desired (3.2) for (6.6)

Combining (6.7) and (6.6), we have, for 0

By considering (3

Lastly we prove that \( \{ g_j(a) \} \}_{j=0}^d \) for \( \{ x_{jk} \}_{0 \leq j, k \leq d} \).

By considering (3

Operating \( \partial_2(h_2a_2)^{-1} \) on both hand sides of (6.5), we have

(6.7)

Combining (6.7) and (6.6), we have, for 0 \( \leq j \leq d - 2 \),

(6.8)

Operating \( \partial_1(h_1a_1)^{-1} \) on both hand sides of (6.5) and combining with (6.6), we have (6.8) for 2 \( \leq j \leq d \). As a result, we have (6.8) for 0 \( \leq j \leq d \).

Lastly we prove that \( \{ g_j(a) \} \}_{j=0}^d \) satisfy (3.3). By Proposition 4.2 \( f_0(a), f_1(a), \ldots, f_d(a) \) satisfy the same differential equation (6.8), therefore their \( \mathbb{C} \)-linear combinations \( \{ g_j(a) \} \}_{j=0}^d \) also satisfy (6.8). Since \( \{ g_j(a) \} \}_{j=0}^d \) satisfy (6.6) and (6.8), we see that

(6.9)

where we put

\[ F_j(a) = h_1a_1^2 \left[ (\partial_1 + \partial_2) - 8\pi h_2a_2^2 + 2\lambda_2 - 2 \right] g_{j-1}(a) + h_2a_2^2 \left[ (\partial_1 + \partial_2) - 8\pi h_1a_1^2 + 2\lambda_2 - 2 \right] g_{j+1}(a). \]

By (6.9), there exist constants \( \beta_j \) (1 \( \leq j \leq d - 1 \)) such that

\[ F_j(a) = \beta_j h_1a_1^2 h_2a_2^2. \]
Let us determine \( \beta_j \). For \( y > 0 \), define \( L = \{(a_1, a_2) \in A \mid h_1a_1^2 = h_2a_2^2 = y\} \). We show that \( (F_j|_L)(y) = 0 \) to deduce \( \beta_j = 0 \). We can verify that
\[
(\partial_1 + \partial_2) f_k = (\sqrt{h_1a_1})^{2k+1}(\sqrt{h_2a_2})^{2d+1-2k} \Delta [\partial_1 + \partial_2] + (d + |m_0| + 2) \]
\[
\times \int_0^1 F(2\pi h_1a_1^2 t + 2\pi h_2a_2^2(1-t))^t \frac{|m_0-d-1|}{2} k(1-t) \frac{|m_0+d-1|}{2} k dt
\]
\[
= (\sqrt{h_1a_1})^{2k+1}(\sqrt{h_2a_2})^{2d+1-2k} \Delta
\times \int_0^1 F^*(2\pi h_1a_1^2 t + 2\pi h_2a_2^2(1-t))^t \frac{|m_0-d-1|}{2} k(1-t) \frac{|m_0+d-1|}{2} k dt.
\]
Here,
\[
F^*(x) = \left( (2x \frac{d}{dx} + (d + |m_0| + 2)) F \right)(x).
\]
There are two cases:
(i) If \(|m_0| - d \geq 2\), then both \( (\partial_1 + \partial_2) f_k \) and \( f_k \) have zeros at \( \{h_1a_1^2 - h_2a_2^2 = 0\} \). Hence, both \( (\partial_1 + \partial_2) g_j \) and \( g_j \) have zeros at there. Therefore we have \( (F_j|_L)(y) = 0 \).
(ii) Suppose that \(|m_0| = d\), then we have
\[
((\partial_1 + \partial_2) f_k)|_L(y) = y^{d+1} F^*(2\pi y) B\left(k + \frac{1}{2}, d - k + \frac{1}{2} \right).
\]
Here,
\[
B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)}
\]
is the Beta function. Then we can verify that
\[
((\partial_1 + \partial_2) g_j)|_L(y) = \left( \sum_{k=0}^d x_{jk} B\left(k + \frac{1}{2}, d - k + \frac{1}{2} \right) \right) y^{d+1} F^*(2\pi y).
\]
To derive (6.10), we used (6.16) in Proposition 6.2. (We will prove later.)
\[
x_{jk} = (-1)^{j+k} (\text{sgn}(m_0) \sqrt{-1})^j \left( \frac{d}{2k-j} \right) \text{ (when } |m_0| = d),
\]
under the condition \( x_{0,0} = 1 \), and the equality:
\[
\sum_{k=0}^d (-1)^k \left( \frac{2k}{j} \right) \left( \frac{2d-2k}{d-j} \right) \left( \frac{d}{k} \right) = (-1)^j 2^d \left( \frac{d}{j} \right).
\]
Let \( S_{d,j} \) be the left hand side of (6.11). We remark that the above equality (6.11) is proved by showing that
\[
S_{d,j} = \frac{2d}{j} S_{d-1,j-1} \quad (j \geq 1) \text{ and } S_{d,0} = 2^d.
\]
See p.620 no. 63 in [11] for the equality \( S_{d,0} = 2^d \).
Similarly, we also obtain
\[ g_j \big|_L(y) = \left( \text{sgn}(m_0) \sqrt{-1} \right)^j 2^{-d} \pi^d y^{d+1} F(2\pi y). \]

Therefore, we have
\[ ((\partial_1 + \partial_2) (g_{j-1} + g_{j+1}))\big|_L(y) = (g_{j-1} + g_{j+1})\big|_L(y) = 0, \]
for \(1 \leq j \leq d - 1\). Therefore we have \((F_j|_L)(y) = 0\).

In any cases, \(\{g_j(a)\}_{j=0}^d\) satisfy (6.5). Hence, \(\{g_j(a)\}_{j=0}^d\) satisfy (3.3) and it completes the proof. □

**Proposition 6.2.** We assume that
\[ |m_0| \geq \delta \quad \text{and} \quad |m_0| \equiv \delta \mod 2. \]

Let \(\{x_{jk}\}_{0 \leq j,k \leq \delta}\) be a sequence of complex numbers, which satisfy
\[ \begin{align*}
(6.12) \quad & j x_{j-1,k} + \sqrt{-1} m_0 x_{j,k} + (d - 2k + j + 1) x_{j+1,k} - (|m_0| - d + 2k + 1) x_{j+1,k+1} = 0 \\
& (0 \leq j \leq d - 1, 0 \leq k \leq \delta), \\
(6.13) \quad & (j - 2k - 1) x_{j-1,k} + (|m_0| + d - 2k + 1) x_{j-1,k-1} + \sqrt{-1} m_0 x_{j,k} - (d - j) x_{j+1,k} = 0 \\
& (1 \leq j \leq d, 0 \leq k \leq \delta), \\
(6.14) \quad & x_{1,0} = x_{2,0} = \cdots = x_{d,0} = 0,
\end{align*} \]
and
\[ x_{0,d} = x_{1,d} = \cdots = x_{d-1,d} = 0. \]

Then the sequence \(\{x_{jk}\}_{0 \leq j,k \leq \delta}\) is uniquely determined and given by, up to a constant multiple,
\[ \begin{align*}
(6.16) \quad & x_{jk} = (-1)^{j+k} (\text{sgn}(m_0)\sqrt{-1})^j \left( \frac{d}{2k-j} \right)^{|m_0|\delta / j} \prod_{l=1}^{k} \frac{2l-1}{|m_0| - d + 2l - 1} \\
& \times \sum_{r=0}^{[j/2]} \binom{[j/2]}{r} \frac{(d - 2r - \delta(j))!}{d!} \prod_{l=1}^{r} \frac{2k - j - 2l + 2 - \delta(j)}{2k - 2l + 1} \left( |m_0|^2 - (d - 2l + 2)^2 \right),
\end{align*} \]
where, \(\delta(j) = 1\) if \(j\) is odd, otherwise 0.

**Proof.** We may assume that \(x_{0,0} = 1\). Let us write down (6.12) with \(j = 0, 1\) and \(k\) replaced by \(k - 1, k\); and (6.13) with \(j = 1\) and \(k\) replaced by \(k - 1, k, k + 1\). Then we have the
We eliminate \( \{ \) equation for \( (6.20) \) with \( (6.19) \) By solving the difference equation \( (6.18) \) for \( (m_0) \) \( (6.17) \) \( \sqrt{-1}m_0 x_{0,k-1} + (d - 2k + 3) x_{1,k-1} - (|m_0| - d + 2k - 1) x_{1,k} = 0, \)
\( \sqrt{-1}m_0 x_{0,k} + (d - 2k + 1) x_{1,k} - (|m_0| - d + 2k + 1) x_{1,k+1} = 0, \)
\( x_{0,k-1} + \sqrt{-1}m_0 x_{1,k-1} + (d - 2k + 4) x_{2,k-1} - (|m_0| - d + 2k - 1) x_{2,k} = 0, \)
\( x_{0,k} + \sqrt{-1}m_0 x_{1,k} + (d - 2k + 2) x_{2,k} - (|m_0| - d + 2k + 1) x_{2,k+1} = 0, \)
\( - (2k - 2) x_{0,k-1} + (|m_0| + d - 2k + 3) x_{0,k-2} + \sqrt{-1m_0 x_{1,k-1} - (d - 1) x_{2,k-1} = 0, \)
\( - 2k x_{0,k} + (|m_0| + d - 2k + 1) x_{0,k-1} + \sqrt{-1m_0 x_{1,k} - (d - 1) x_{2,k} = 0, \)
\( - (2k + 2) x_{0,k+1} + (|m_0| + d - 2k - 1) x_{0,k} + \sqrt{-1m_0 x_{1,k+1} - (d - 1) x_{2,k+1} = 0. \)

We eliminate \( x_{j,k}, x_{j,k \pm 1} (j = 1, 2) \) in the above system. Then we have
\[
- 2(k + 1) (|m_0| - d + 2k + 1) (|m_0| - d + 2k - 1) x_{0,k+1} \\
+ \left\{ 2k (4d - 6k + 3) - d(d - 1) \right\} (|m_0| - d + 2k - 1) x_{0,k} \\
+ \left[ 2(d - 2k + 2) \left\{ d(d - 1) - (2k - 2)(2d - 2k + 3) \right\} \\
+ (d - 2k)(d - 2k + 1) (|m_0| - d + 2k - 1) \right] x_{0,k-1} \\
+ (d - 2k + 4)(d - 2k + 3)(|m_0| + d - 2k + 3) x_{0,k-2} \\
= 0.
\]

By solving the difference equation \( (6.18) \) for \( \{ x_{0,k} \} \) with \( x_{0,0} = 1, \) we obtain
\[
(6.19) \quad x_{0,k} = (-1)^k \left( \frac{d}{2k} \right) \prod_{l=1}^{k} \frac{2l - 1}{|m_0| - d + 2l - 1}.
\]

From the second equation of \( (6.17), (6.19) \) and \( (6.14), \) we have the following difference equation for \( \{ x_{1,k} \}: \)
\[
(|m_0| - d + 2k + 1) x_{1,k+1} = (d - 2k + 1) x_{1,k} + \sqrt{-1m_0} (-1)^k \left( \frac{d}{2k} \right) \prod_{l=1}^{k} \frac{2l - 1}{|m_0| - d + 2l - 1}
\]
with \( x_{1,0} = 0. \) Thus, we obtain
\[
(6.20) \quad x_{1,k} = (-1)^{k+1} (\text{sgn} (m_0) \sqrt{-1}) \left( \frac{d}{2k - 1} \right) \frac{|m_0|}{d} \prod_{l=1}^{k} \frac{2l - 1}{|m_0| - d + 2l - 1}.
\]
We prove the formula (6.16) for general $x_{jk}$ by induction on the index $j$. For $0 \leq 2p, 2p+1, k \leq d$, let us prove

$$x_{2p,k} = (-1)^{k+p} \left( \frac{d}{2k-2p} \right) \prod_{l=1}^{k} \frac{2l-1}{|m_0| - d + 2l - 1} \times \sum_{r=0}^{p} \left( \frac{p}{r} \right) \frac{(d-2r)!}{d!} \prod_{l=1}^{r} \left\{ \frac{2k-2p-2l+2}{2k-2l+1} \left( |m_0|^2 - (d-2l+2)^2 \right) \right\},$$

(6.21)

$$x_{2p+1,k} = (-1)^{k+p+1} \left( \text{sgn}(m_0) \sqrt{-1} \right) \left( \frac{d}{2k-2p} \right) \prod_{l=1}^{k} \frac{2l-1}{|m_0| - d + 2l - 1} \times \sum_{r=0}^{p} \left( \frac{p}{r} \right) \frac{(d-2r-1)!}{d!} \prod_{l=1}^{r} \left\{ \frac{2k-2p-2l}{2k-2l+1} \left( |m_0|^2 - (d-2l+2)^2 \right) \right\}.$$

(6.22)

Suppose that $2p+1 \leq d$ and the above formulas are true for $x_{2p-1,k}, x_{2p,k}$ ($0 \leq k \leq d$), then substitute them into (6.13):

$$x_{2p+1,k} = \frac{2p - 2k - 1}{d - 2p} x_{2p-1,k} + \frac{|m_0| + d - 2k + 1}{d - 2p} x_{2p-1,k-1} + \frac{\sqrt{-1} m_0}{d - 2p} x_{2p,k}.$$

(6.23)

Put

$$a(k) = \prod_{l=1}^{k} \frac{2l-1}{|m_0| - d + 2l - 1}, \quad b(r; k, p) = \prod_{l=1}^{r} \left\{ \frac{2k-2p-2l}{2k-2l+1} \left( |m_0|^2 - (d-2l+2)^2 \right) \right\}.$$

Then, we have

$$\frac{(-1)^{k+p+1}(2p - 2k - 1)}{\sqrt{-1} m_0(d - 2p)} x_{2p-1,k}$$

$$= \frac{2p - 2k - 1}{d - 2p} \left( \frac{d}{2k-2p+1} \right) a(k) \sum_{r=0}^{p-1} \left( \frac{p-1}{r} \right) \frac{(d-2r)!}{d!} b(r; k, p - 1)$$

$$= a(k) \frac{d}{d - 2p} \left( \frac{d}{2k-2p-1} \right) (d - 2k + 2p + 1)(d - 2k + 2p)$$

$$\times \sum_{r=0}^{p-1} \left( \frac{p-1}{r} \right) \frac{(d-2r-1)!}{d!} b(r; k, p) \frac{2k-2p-2r}{2k-2p-2r}.$$
and

\[
(-1)^{k+p+1} \frac{\sqrt{1} m_0 (d - 2p)}{d - 2p} x_{2p-1, k-1}
\]

\[
= \frac{|m_0| + d - 2k + 1}{d - 2p} \left( \frac{d}{2k - 2p - 1} \right) a(k - 1) \sum_{r=0}^{p-1} \binom{p-1}{r} \frac{(d - 2r - 1)!}{d!} b(r; k - 1, p - 1)
\]

\[
= \frac{a(k)}{d - 2p} \left( \frac{d}{2k - 2p - 1} \right) \sum_{r=0}^{p-1} \left( \binom{p-1}{r} \frac{(d - 2r - 1)!}{d!} b(r; k, p) \right) + \frac{a(k)}{d - 2p} \left( \frac{d}{2k - 2p - 1} \right) \sum_{r=1}^{p} \left( \binom{p}{r} \frac{(d - 2r + 1)!}{d!} b(r; k, p) \right)
\]

Here, we used the equality:

\[
\{ |m_0|^2 - (d - 2k + 1)^2 \} = \{ |m_0|^2 - (d - 2r)^2 \} + (2d - 2r - 2k + 1)(2k - 2r - 1)
\]

in the above calculation. By noting the equality:

\[
\frac{(d - 2k + 2p + 1)(d - 2k + 2p)}{2k - 2p - 2r} + (2d - 2r - 2k + 1) = \frac{(d - 2r)(d - 2r + 1)}{2k - 2p - 2r} - 2p,
\]

we have

\[(6.24) + (6.25)\]

\[
= \frac{a(k)}{d - 2p} \left( \frac{d}{2k - 2p - 1} \right) \sum_{r=0}^{p-1} \left( \binom{p-1}{r} \frac{(d - 2r - 1)!}{d!} \left( \frac{(d - 2r)(d - 2r + 1)}{2k - 2p - 2r} - 2p \right) b(r; k, p) \right)
\]

\[
+ \frac{a(k)}{d - 2p} \left( \frac{d}{2k - 2p - 1} \right) \sum_{r=1}^{p} \left( \binom{p}{r} \frac{(d - 2r + 1)!}{d!} \frac{b(r; k, p)}{2k - 2p - 2r} \right)
\]

\[
= \frac{a(k)}{d - 2p} \left( \frac{d}{2k - 2p - 1} \right) \sum_{r=0}^{p} \left( \binom{p}{r} \frac{(d - 2r + 1)!}{d!} \frac{b(r; k, p)}{2k - 2p - 2r} \right)
\]

\[- 2p \frac{a(k)}{d - 2p} \left( \frac{d}{2k - 2p - 1} \right) \sum_{r=0}^{p-1} \left( \binom{p-1}{r} \frac{(d - 2r - 1)!}{d!} b(r; k, p) \right).\]
On the other hand, we have

\[\frac{(-1)^{k+p+1}}{\sqrt{-1} m_0(d-2p)} x_{2p,k}\]

\[= -\frac{1}{d-2p} \left( \frac{d}{2k-2p} \right) a(k) \sum_{r=0}^{p} \frac{(d-2r)!}{d!} b(r; k, p - 1)\]

\[= -\frac{a(k)}{d-2p} \left( \frac{d}{2k-2p-1} \right) (d-2k+2p+1) \sum_{r=0}^{p} \frac{(d-2r)!}{d!} \frac{b(r; k, p)}{2k-2p-2r}\]

\[= a(k) \left( \frac{d}{2k-2p-1} \right) \sum_{r=0}^{p} \frac{(d-2r+1)!}{d!} b(r; k, p)\]

Therefore, we have

\[\text{(6.24)} + \text{(6.25)} + \text{(6.26)}\]

\[= a(k) \left( \frac{d}{2k-2p-1} \right) \sum_{r=0}^{p} \frac{(d-2r)!}{d!} b(r; k, p)\]

\[- 2p \frac{a(k)}{d-2p} \left( \frac{d}{2k-2p-1} \right) \sum_{r=0}^{p-1} \frac{(p-1)!}{d!} b(r; k, p)\]

\[= a(k) \left( \frac{d}{2k-2p-1} \right) \sum_{r=0}^{p} \frac{(d-2r-1)!}{d!} b(r; k, p).\]

By (6.23) and the above formula, we obtain

\[x_{2p+1,k} = (-1)^{k+p+1} (m_0 \sqrt{-1}) a(k) \left( \frac{d}{2k-2p-1} \right) \sum_{r=0}^{p} \frac{(d-2r)!}{d!} b(r; k, p).\]

Therefore, the formula is valid for \(x_{2p+1,k} (0 \leq k \leq d)\). Similarly, if we suppose that \(2p \leq d\) and the formulas (6.21) and (6.22) are true for \(x_{2p-1,k}, x_{2p-2,k} (0 \leq k \leq d)\), then we can prove the formula is also valid for \(x_{2p,k} (0 \leq k \leq d)\). It completes the proof. \(\square\)

Let us complete the proof of Theorem 5.1.

**Proof of 5.1.** By the condition \(|m_0| \geq d, |m_0| \equiv d \pmod{2}\), Propositions 6.1 and 6.2, \(\{g_j(a)\}_{j=0}^{d}\) is a non-zero smooth solution of the system (3.1), (3.2) and (3.3) in Proposition 3.1. Furthermore, we can check that all of \(\{e^{-2\pi(h_1a_1^2+h_2a_2^2)} g_j(a)\}_{j=0}^{d}\) are rapidly decreasing when each \(a_1, a_2\) tends to infinity, by Definition 4.1. We completes the proof. \(\square\)
7. Remarks on \( \{x_{jk}\} \)

We have some remarks on the sequence of complex numbers \( \{x_{jk}\}_{0 \leq j, k \leq d} \) appearing in Theorem 5.1 and Proposition 6.2. Let \( X_d = (x_{jk})_{0 \leq j, k \leq d} \in M_{d+1}(\mathbb{C}) \) be the square matrix of size \((d + 1)\), defined by the sequence \( \{x_{jk}\}_{0 \leq j, k \leq d} \).

Put \( \varepsilon = \text{sgn}(m_0)\sqrt{-1} \) and \( t = |m_0| \). Then \( x_{jk} \) is given by

\[
(7.1) \quad x_{jk} = (-1)^{j+k} \left( \frac{d}{2k-j} \right) t^{\delta(j)} \prod_{l=1}^{k} \frac{2l-1}{t-d+2l-1} \times \sum_{r=0}^{[j/2]} \left( \frac{d-2r-\delta(j)}{d!} \right) \prod_{l=1}^{r} \left\{ \frac{2k-j-2l+2-\delta(j)}{2k-2l+1} \left( t^2 - (d-2l+2)^2 \right) \right\},
\]

where, \( \delta(j) = 1 \) if \( j \) is odd, otherwise 0.

Furthermore, we set

\[
(7.2) \quad Z_d = (z_{jk})_{0 \leq j, k \leq d} \in M_{d+1}(\mathbb{C}) \quad \text{with} \quad z_{jk} = \varepsilon^{-j} x_{jk}.
\]

Though the Harish-Chandra parameter \( \Lambda = (\lambda_1 - 1, \lambda_2) \in \Sigma_{\Pi} \) implies that \( d = \lambda_1 - \lambda_2 \geq 4 \), we formally write down matrices \( Z_d \) for \( 1 \leq d \leq 7 \).

**Example 7.1 (\( Z_d \) for \( 1 \leq d \leq 7 \)).**

\[
Z_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 1 & 0 & -\frac{1}{t} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{t} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad Z_3 = \begin{bmatrix} 1 & -\frac{3}{t-2} & 0 & 0 \\ 0 & t-2 & -\frac{1}{t-2} & 0 \\ 0 & 0 & -\frac{t-3}{2} & 0 \\ 0 & 0 & 0 & t-2 \end{bmatrix},
\]

\[
Z_4 = \begin{bmatrix} 1 & -\frac{6}{t-3} & -\frac{3}{(t-3)(t-1)} & 0 & 0 \\ 0 & \frac{t-3}{2} & -\frac{(t-3)(t-1)}{2t} & 0 & 0 \\ 0 & -\frac{1}{t-3} & \frac{(t-3)(t-1)}{t^2+2} & -\frac{1}{t-3} & 0 \\ 0 & 0 & -\frac{(t-3)(t-1)}{3} & \frac{t-3}{3} & 0 \\ 0 & 0 & 0 & \frac{t-3}{6} & 1 \end{bmatrix},
\]

\[
Z_5 = \begin{bmatrix} 1 & -\frac{10}{t-4} & -\frac{15}{(t-4)(t-2)} & 0 & 0 & 0 \\ 0 & \frac{t-4}{6} & -\frac{(t-4)(t-2)}{6t} & 0 & 0 & 0 \\ 0 & -\frac{1}{t-4} & \frac{(t-4)(t-2)}{3t} & 0 & 0 & 0 \\ 0 & 0 & -\frac{(t-4)(t-2)}{3t} & \frac{3}{t^2+5} & -\frac{1}{t-4} & 0 \\ 0 & 0 & 0 & -\frac{(t-4)(t-2)}{3t} & \frac{1}{t-4} & 0 \\ 0 & 0 & 0 & 0 & -\frac{10}{t-4} & 1 \end{bmatrix},
\]
**Proposition 7.2** (det \( Z_d \) for \( 1 \leq d \leq 7 \)).

\[
\det Z_1 = 1, \quad \det Z_2 = \frac{t}{t-1}, \quad \det Z_3 = \frac{t^2-1}{(t-2)^2}, \quad \det Z_4 = \frac{t^2(t^2-4)}{(t-1)(t-3)^3}, \\
\det Z_5 = \frac{(t^2-1)^2(t^2-9)}{(t-2)(t-4)^4}, \quad \det Z_6 = \frac{t^3(t^2-4)^2(t^2-16)}{(t-1)(t-3)^3(t-5)^5}, \\
\det Z_7 = \frac{(t^2-1)^3(t^2-9)^2(t^2-25)}{(t-2)(t-4)(t-6)^6}.
\]

Since \( \det X_d = \varepsilon^{d(d+1)/2} \det Z_d \), we have the following conjecture on \( \det X_d \).

**Conjecture 7.3.** For a natural number \( q \), we conjecture that

\[
\det X_{2q-1} = \left( \text{sgn}(m_0) \sqrt{-1} \right)^{q(2q-1)} \prod_{l=1}^{q-1} \frac{(t^2 - (2l - 1)^2)^{q-l}}{(t-2l)^{2l}}, \\
\det X_{2q} = \left( \text{sgn}(m_0) \sqrt{-1} \right)^{q(2q+1)} \prod_{l=1}^{q-1} \frac{t^2 - (2l)^2)^{q-l}}{(t-2l+1)^{2l-1}}.
\]
In this article, we have shown that the Siegel-Whittaker functions of rapid decay for the large discrete series representations of $\text{Sp}(2, \mathbb{R})$ are described by the partially confluent hypergeometric functions $\{f_k(a)\}_{k=0}^{d}$. Conversely, if the above conjecture is true, we have,

**Corollary 7.4.** We assume that $|m_0| \geq d$ and $|m_0| \equiv d \pmod{2}$.

Let $\{c_j(a)\}_{j=0}^{d}$ be a smooth solution of the system (3.1), (3.2) and (3.3) in Proposition 3.1, and suppose that all of $\{e^{-2\pi(h_1a_1^2+h_2a_2^2)}c_j(a)\}_{j=0}^{d}$ are rapidly decreasing.

If Conjecture 7.3 is true, then all of $\{f_k(a)\}_{k=0}^{d}$ are $\mathbb{C}$-linear combinations of elements of $\{c_j(a)\}_{j=0}^{d}$.

**Proof.** Since $|m_0| \geq d$ and $|m_0| \equiv d \pmod{2}$, we can check that $\det X_d \neq 0,$

and $X_d^{-1}$ exists. It completes the proof. \hfill \Box

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