Second law of black hole mechanics for all 2d dilaton theories

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It is shown that all generalized two–dimensional dilaton theories with arbitrary matter content (with a curvature independent coupling to gravity) do not only obey a first law of black hole mechanics (which follows from Wald’s general considerations, if the entropy \( S \) is defined appropriately), but also a second law: \( \delta S \geq 0 \) provided only that the null energy condition holds and that, loosely speaking, for late times a stationary state is assumed. Also any two-dimensional \( f(R) \)–theory is covered. This generalizes a previous proof of Frolov [1] to a much wider class of theories.

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I. INTRODUCTION

Currently there is a noteworthy activity in finding a microscopic statistical mechanical explanation for the entropy of black holes (cf., e.g., [2–5]), where the most impressive results where obtained within the string theoretical approach. Underlying these attempts is the by now old analogy of black hole mechanics (BHM) in classical Einstein gravity [6] with standard thermodynamics. This originally purely mathematical analogy was further supported by Bekenstein’s Gedanken experiments [7] concerning the second law of thermodynamics for a system in a spacetime with a black hole followed by Hawking’s observation [8] that black holes should emit thermal radiation when quantum effects are taken into account.

The Einstein-Hilbert action is just the (bosonic) leading order term in the low energy effective action of string theory. Also other modifications of Einstein gravity can be envisaged yielding agreement with current experimental evidence. Thus the question arises [9], if, and in what sense, the three to four laws of BHM hold also in more general theories than Einstein gravity.

At least what concerns the entropy of stationary black holes, Wald [10] provided a very general definition of black hole entropy \( S \), which satisfies a (certain kind of) first law; in particular, his definition of \( S \) and the corresponding first law applies to any diffeomorphism invariant theory of gravity in \( D \) spacetime dimensions formulated in terms of a metric \( g \). In general, this entropy is no more proportional to the area of the \((D–2)\)–dimensional black hole horizon, while it certainly reduces to this result for the specific case of Einstein gravity. The upshot is that, up to a proportionality constant, the entropy (at least for stationary black holes) is defined as the integral over the black hole horizon \( \Sigma \) of the (functional) derivative of the diffeomorphism invariant action \( L \) with respect to the curvature tensor \( R_{\mu\nu\rho\sigma} \) (having expressed all dependences of \( L \) on derivatives of \( g \) in terms of the curvature of \( g \) and its symmetrized covariant derivatives; this is always possible as a consequence of the diffeomorphism invariance of \( L \)).

Wald’s first law of BHM takes the form

\[
\frac{\kappa}{2\pi} \delta S = \delta \mathcal{E} - \Omega_H^{(l)} \delta J_{(l)},
\]

where \( \kappa \) is the surface gravity of the original (unperturbed) stationary spacetime, \( \mathcal{E} \) is the “canonical energy”, defined as the evaluation of the Hamiltonian (for Killing time evolution of the original spacetime) on the classical solutions, and, similarly, the \( J_{(l)} \)’s, \( l = 1, \ldots, r \), are “canonical angular momenta” associated with \( r \) axial Killing vectors (\( \Omega_H^{(l)} \) being the appropriate angular velocities of the horizon; certainly \( r \leq 1 \) for spacetime dimension \( D = 4 \) and \( D = 3 \), while \( r = 0 \) for \( D = 2 \)). For pure Einstein gravity, \( \mathcal{E} \) coincides with the ADM mass, but, e.g., for Einstein-Maxwell theory it also yields the

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standard contribution proportional to the change in the charge $Q$ of the black hole \[1\].

While there thus exists a formulation of the first law of BHM that applies to shear any conceivable theory of gravity, formulated even in arbitrary spacetime dimension $D \geq 2$, the generality of the second law for the so defined entropy is an essential and in general very complicated open problem.

The second law of BHM states that during any dynamical process the entropy $S$ can only increase, or, more precisely,

$$\delta S \geq 0 \quad \text{provided} \quad T_{\mu\nu} u^\mu u^\nu \geq 0 \quad \forall \, u^\mu, \, u^\mu u_\mu \geq 0. \tag{2}$$

Here $T_{\mu\nu}$ is the energy momentum tensor of the matter content of the theory and the second half of (2) is known as the weak energy condition. This statement has been proven by Hawking \[12\] for Einstein gravity in four spacetime dimensions under the (unproven) assumption that cosmic censorship holds true. The proof, which recently has been put somewhat into question in \[13\] (but cf. also \[14\]), relies on the fact that in this case $S$ has the geometrical meaning of (a quarter of) the area of the black hole horizon, and the weak energy condition is brought in by the additional use of the specific form of the Einstein field equations. It is by no means clear, if (and under what conditions) the entropy $S$ defined by Frolov coincides, for more general gravitational theories. However, for an analogy of BHM with thermodynamics, the validity of a second law seems indispensable!

In a pioneering work, Frolov \[1\] showed that there is a first and a second law of black hole mechanics for two–dimensional dilaton theories with, more precisely, an arbitrary two–dimensional dilaton theories with, more generally, according to the formula

$$T_{\mu\nu} = -\frac{4\pi}{\sqrt{-\det g}} \frac{\delta L_{\text{matter}}}{\delta g^{\mu\nu}}, \tag{4}$$

or, more generally, according to the formula $T_{\mu\nu} = -\frac{4\pi}{\sqrt{-\det g}} \left( \frac{\delta L_{\text{matter}}}{\delta e^\nu_2} \right) / \det(e^\nu_2)$, if the matter part of the action $L_{\text{matter}}$ depends also on a vielbein $e^\mu_\nu$, as is the case for fermionic fields (cf., e.g., ref. \[13\] for further details).

Eq. (4) will be established under the assumption that for late “times” $S$ approaches zero sufficiently fast, i.e. there is a function $F$ within a certain class of allowed functions (the class depending on the given action functional) such that $\lim_{t \to \infty} \frac{S}{F} = 0$, where $F$ is a future directed affine parameter along the (null) horizon and $S$ is fully defined in terms of geometrical quantities defined on the horizon. For a more precise formulation of this assumption cf. Theorem \[4\] below, which is our main result. We remark that the assumption is, e.g., automatically satisfied, if timelike infinity $i^+$ has a neighborhood which is vacuum (all matter fields vanish there), cf. Corollary 2.

The organization of this note is as follows: In the next section we recall the action and field equations for the class of two–dimensional dilaton theories considered here and specify Wald’s formula for the entropy $S$ to these models. In the following section we then review (in own words) Frolov’s nice proof of the second law. At first sight, his proof seems applicable only to a very restricted subclass of models. However, by a rather simple change of variables in the action functional, this proof may be extended to the whole class of theories. With some patching and continuity considerations, global obstructions to the change of variables will be shown to not spoil the final statement. The significance of the result relies primarily in its generality as well as in its close analogy with the (much more involved) four–dimensional Einstein case.

Instead of a generalized dilaton theory we may also take $L_{\text{grav}} \propto \int d^2x \sqrt{-g} f(R)$ as for the gravitational part of the action, where $f$ is any nonlinear, twice differentiable function of the curvature scalar $R$. The matter part is again assumed to not depend on $R$ explicitly. This is shown in a separate section. A final section then contains a brief summary and outlook on possible further developments.

*This formulation reproduces the result in the particular case considered by Frolov. Note that the quantity $T_{\mu\nu}$ below does not coincide with what Frolov denoted by $T_{\mu\nu}$ and that, on the other hand, the condition formulated by him may be reexpressed as the condition below on the energy momentum tensor.

$$T_{\mu\nu} = -\frac{4\pi}{\sqrt{-\det g}} \frac{\delta L_{\text{matter}}}{\delta g^{\mu\nu}}, \tag{4}$$
II. THE CLASS OF THEORIES CONSIDERED

A. Generalized two–dimensional dilaton theories

In two spacetime dimensions the integrand of the Einstein-Hilbert action \( L_{EH} = \int d^2x \sqrt{-\det g} R \) becomes a total derivative (locally) and thus does not yield any field equations. Introducing an additional scalar field \( \Phi \), the “dilaton”, as a second “gravitational” variable beside the spacetime-metric \( g_{\mu\nu} \), one may consider the following class of actions \([13]\) for 2d gravity theories as a substitute of the Einstein-Hilbert action in higher dimensions:

\[
L_{\text{grav}}[g, \Phi] = \frac{1}{4\pi} \int d^2x \sqrt{-\det g} \left[ U(\Phi) R + V(\Phi) g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + W(\Phi) \right]. \tag{5}
\]

Here \( R \) denotes the Ricci scalar of the Levi-Civita connection of \( g \) and \( U, V, \) and \( W \) are some essentially arbitrary functions (“potentials”) of the dilaton, further specified below. Possible surface terms have been omitted in \([3]\). The total action results upon adding some matter part,

\[
L_{\text{tot}}[g, \Phi, \psi] = L_{\text{grav}}[g, \Phi] + L_{\text{matter}}[g, \Phi, \psi], \tag{6}
\]

where all the matter fields collectively have been denoted by \( \psi \). For further discussion of generalized dilaton theories cf., e.g., the review articles \([17, 13]\) and citations therein. We will assume in the following without further mention that all the \( g \)–dependence of \( L_{\text{matter}} \) may be expressed without using the curvature scalar or its derivatives, with eventual multiple covariant derivatives entering in symmetric combinations only. The \( \Phi \)–dependence of \( L_{\text{matter}} \) is unrestricted, however.

To give an example: In the case of a complex scalar field \( f \) and a U(1) gauge field \( A_\mu \) (with its curvature \( F = dA \)), the matter part of the action can take the form

\[
L_{\text{matter}} = \frac{1}{2\pi} \int d^2x \sqrt{-\det g} \left[ \alpha(\Phi) F^{\mu\nu} F_{\mu\nu} + \beta(\Phi) \left( \partial^\mu + iqA^\mu \right) f^* (\partial_\mu - iqA_\mu) f + \frac{\mu^2(\Phi)}{2} f^* f \right], \tag{7}
\]

where \( \alpha, \beta, \) and \( \mu \) are \( \Phi \)–dependent coupling constants and a “mass” parameter, respectively.

The interpretation of \( g \) and \( \Phi \) as the basic gravitational variables may be backed up further by the example of the spherically symmetric sector of \( D = 4 \) Einstein gravity (or likewise for some other dimension \( D \geq 3 \)): Implementing \([24]\) the spherical ansatz \( ds^2_{D=4} = g_{\mu\nu}(x^\rho) dx^\mu dx^\nu - \Phi^2(x^\rho) \left[ d\theta^2 + \sin^2 \theta \, d\phi^2 \right] \), \( \mu, \nu, \rho \in \{0,1\} \), into the \( D = 4 \) action yields a two–dimensional action of the form \([3]\) with an explicitly \( \Phi \)–dependent part \( L_{\text{matter}} \) and shows that in this case \( \Phi \) corresponds to a component of the metrical tensor in the higher dimension. In general, \( \Phi \) does not have such a clear interpretation; nevertheless, it seems advisable also for the general class of 2d toy models to treat \( g \) and \( \Phi \) on a similar footing and to regard the action \([3]\) as the 2d analogue of the 4d Einstein action. It is also this point of view that leads to the simple and “physical” condition in Eq. \([3]\).

The model considered by Frolov \([1]\) results from Eqs. \([3] - [7]\) upon the choice

\[
U \equiv V \equiv \alpha \equiv \beta = 2\pi \exp \Phi, \ W = 2\pi \lambda \exp \Phi, \tag{8}
\]

and \( \mu = \text{const.} \). The gravitational part of this action governs the s-wave modes of the low energy effective action for string theory in four dimensions. If the matter part of the action is taken to be a minimally coupled massless, real scalar field, the action is the well–known CGHS model \([21]\), which may be used as a toy model for Hawking radiation (cf. e.g. \([17]\) for a review as well as \([22]\) for a different approach to this issue within the same model). The significance of the CGHS model resides in the fact that it may be fully integrated, at least on the classical level. This is certainly no more true for the general action \( L_{\text{tot}} \) with arbitrary \( U, V, W \); it can be integrated only in relatively specific cases (cf. \([15]\) for a general discussion and a collection of integrable cases).

B. Entropy for 2d black holes

According to Wald the entropy \( S \) of a (stationary) black hole is determined by the functional derivative of the total action with respect to the curvature of the metric integrated over the event horizon at an instant of time. In \( D = 2 \), taking the integral reduces to mere evaluation at the given point on the horizon. The functional derivative of \( L_{\text{tot}} \) with respect to the curvature obviously yields just a term proportional to \( U(\Phi) \).

The explicit formula of Wald \([10]\) prescribes the prefactor. Let us remark, however, that clearly the resulting entropy can be rescaled if the total action of the theory under discussion is multiplied by a prefactor. Note that the thus rescaled entropy still satisfies an equation of the form \([3]\) with the same surface gravity \( \kappa \), since the latter quantity is determined by the geometry of the solutions, which clearly is not changed by a different overall factor to the action; this is possible, since then also \( E \) and \( J \) will be rescaled by the same factor. In the case of toy models with no underlying experiments it is hardly possible to fix prefactors in front of the action, and, on the same footing, having fixed the prefactor in the action, we think that one should not regard the prefactor of the entropy to be fixed at the same time. In the present case we decide for the simplest normalization,

\[
S = U(\Phi) \big|_{\text{event horizon}}. \tag{9}
\]
having chosen the prefactors in \( \mathfrak{r} \) so that this formula agrees with what one obtains also by naive application of the formula provided in \( \mathfrak{s} \). (We have chosen conventions in which \( R_{\mu\nu\rho}^\sigma \) is defined by \( 2[\nabla_\mu, \nabla_\nu]\omega_\rho = R_{\mu\nu\rho}^\sigma \omega_\sigma \) on 1-forms \( \omega_\mu \) and the norm of a spacelike vector is positive). Still, one should keep in mind the eventual possibility to reinterpret \( aS + bP \) for some constants \( a, b \), as the “true” entropy. This freedom may, e.g., be essential (although not sufficient in general) to ensure positivity of the entropy. Note also that adding a constant \( b \) amounts just to the addition of a total divergence to the Lagrangian \( \mathfrak{r} \).

The above result \( \mathfrak{r} \) for the entropy of a 2d black hole in a theory defined by the action \( L_{\text{tot}} \) may be obtained also by various other approaches (cf., e.g., \( \mathfrak{z}, \mathfrak{g} \)). The strength of Wald’s approach is its generality and simplicity, while in any known particular case it reproduces the result of other, generically more limited approaches.

Strictly speaking, Wald’s definition of the entropy applies only to stationary black holes. But this definition is extended in the second paper of ref. \( \mathfrak{I} \) to an entropy for nonstationary black holes \( S_{\text{dyn}} \) which happens to coincide with the original prescription in the case of 2d dilaton gravity theories; at an “instant of time” the entropy \( S \) is defined by \( U(\Phi) \) evaluated at the cross section of the event horizon with the spacelike hypersurface specifying the instant of time. It is this notion of entropy for which a second law will be established.

C. Technical Preliminaries

To establish a second law, one needs the dynamics of the system under consideration, i.e., one needs its field equations. In the present context we will mainly need the variation of \( L_{\text{tot}} \) with respect to \( g_{\mu\nu} \), which yields after some manipulations:

\[
- \nabla_\mu \partial_\nu U(\Phi) + \frac{1}{2} g_{\mu\nu} \left[ W(\Phi) - V(\Phi) (\nabla \Phi)^2 \right] - \\
+ \partial_\mu \Phi \partial_\nu \Phi V(\Phi) = T_{\mu\nu} - g_{\mu\nu} T,
\]

where \( T_{\mu\nu} \) is the energy momentum tensor, defined in Eq. \( \mathfrak{I} \), and \( T \) is its trace. The variation with respect to \( \Phi \) yields

\[
U' R - V' (\nabla \Phi)^2 - 2V \Box \Phi + W' = \frac{-4\pi}{\sqrt{-\det g}} \frac{\delta L_{\text{matter}}}{\delta \Phi}.
\]

It turns out that we will not need the remaining field equations, \( \delta L_{\text{matter}}/\delta \psi = 0 \). As remarked already above, the combined field equations cannot be solved in general; nevertheless, the validity of the second law, Eq. \( \mathfrak{r} \), still can be established, as we will demonstrate in the subsequent section.

We now specify some technical assumptions on the allowed class of potentials in the action, in part so as to exclude pathological cases, but also adopting a rather pragmatic point of view here: We want to present simple conditions on the potentials, still covering the generic cases, at least from a physicist point of view. (Although in part possible, we do not intend to relax the conditions as much as possible so that the main statements do still hold true, possibly in a refined version). Moreover, with respect to the restrictions on the potentials we focus just on the functions \( U, V, \) and \( W \) in \( \mathfrak{I} \); likewise restrictions will apply to dilaton–dependent coupling constants such as to the functions \( \alpha \) and \( \beta \) in Eq. \( \mathfrak{I} \).

First, we require the potentials to be analytic\(^1\) functions on some common, connected, open domain of definition \( D \subseteq \mathbb{R} \) and \( U \) to be nonconstant. In this way we ensure, e.g., that the action contains a kinetic term for the metric (and this for all values of the dilaton).

Next, we restrict ourselves to such (generic) choices of the potentials, for which eventual extrema of \( U \) do not coincide with zeros of \( W \). E.g., in the case of higher order zeros of \( W \), inspection of the field equations shows that there would be solutions with constant dilaton and vanishing matter fields for which the metric remains completely unspecified. Clearly, such an action functional would be no good toy model for Einstein gravity and should be excluded from our considerations. We furthermore require for some technical reasons that at any extremum \( \Phi_0 \) of \( U \), \( \frac{V(\Phi_0)}{\sqrt{\det g(\Phi_0)}} \neq 2i, i \in \mathbb{N} \) (cf. \( \mathfrak{z} \)).

We focus on smooth \((C^\infty)\) solutions \( g, \Phi, \psi \) only and always take spacetime as to lie on the boundary \( \partial M \) of spacetime and, provided such a boundary point exists, it is either infinitely far away in terms of the spacetime metric \( g \) (more precisely, at least one geodesic approaching this point is complete) or \( R \) diverges there.

This includes an implicit restriction on the potentials. To illustrate this, consider the simple action resulting from \( \mathfrak{I} \) upon the choice \( U := \Phi, V := 0, \) and \( W := 1 \). Then the general, maximally extended vacuum solution

\(^1\) In most cases considered below this may be replaced by “(sufficiently) smooth”, provided only eventual extrema of \( U(\Phi) \) are isolated.
is easily seen to be Minkowski space with \( \Phi \) taking its limiting values \( \pm \infty \) on the boundaries of the diamond–shaped Penrose–Walker diagram. (Note that \( \Phi(x) \) is always nonconstant on \( M \) and correspondingly only one of the Killing vectors provides an isometry vector field of the total solution; which one is determined by a (gauge–independent) integration constant present in the solution for \( \Phi(x) \).) Now let us replace \( \Phi \) by \( \exp \Phi \) in the action; in the present case (for a more general discussion cf. Eq. (12) below) this obviously changes only \( U \) to \( U = \exp \Phi \). By the above requirements, this choice of potentials would not be permitted: The part of Minkowski spacetime which previously was described by negative values of the dilaton is now no more covered; the line in Minkowski space where now \( \Phi \to -\infty \) was previously just the internal line of vanishing dilaton field. Clearly at this line the metric is still well–behaved (even flat in this simple example) and the line is by no means “infinitely far away”.

This example may be easily generalized. For any model with potentials satisfying the above conditions, there are infinitely many, which do not: Indeed, we only need to map any proper subset of \( D \) onto \( \mathbb{R} \) by a diffeomorphism \( \Phi \to \tilde{\Phi}(\tilde{\Phi}) \); the thus induced potentials will violate the (reasonable) condition that the maximal extension of the solutions is restricted solely by the metric. So, “artificial parametrizations” of the dilaton field \( \Phi \) are excluded by the above conditions.

Still within the allowed class of potentials there remains a freedom to reparametrize the dilaton in a globally well–defined manner, which we will make use of below. Note that while \( U \) and \( W \) transform like ordinary functions under such diffeomorphisms \( \tilde{\Phi} : D \to I \subset \mathbb{R} \), \( V \) is density of weight two:

\[
U \to U \circ f, \quad W \to W \circ f, \quad V \to f^2 \cdot V \circ f.
\]

Here \( \circ \) denotes the composition of maps, prime differentiation with respect to the argument, \( \cdot \) pointwise multiplication, and \( f \equiv \tilde{\Phi}^{-1} \) the inverse to the function \( \tilde{\Phi} \).

We conclude this section with some facts about the vacuum theories (cf. [23,24] for the details). First, for any vacuum solution there is a Killing vector field, the integral lines of which coincide with lines of constant dilaton. Thus, in the vacuum case the horizon is a Killing horizon. Under the assumptions specified above, it may be shown [24], furthermore, that on the Killing horizon the dilaton never assumes a critical value of \( U \).

III. PROOF OF THE SECOND LAW

A. Frolov’s proof for the case given in Eq. (8)

By definition of the event horizon as the boundary of the causal past of future null infinity, the event horizon is a null line. (The cautious addition “at its regular points” in [8] is not necessary, since in two spacetime dimensions there is room for any “cusps” [13] as in higher dimensions). Let \( l^\mu \) be a generator of this line, such that

\[
l^\mu \nabla_\mu l^\nu = 0.
\]

Such a choice of \( l \) is always possible, since clearly any null line in \( D = 2 \) coincides with a geodesic. Next we contract the field equations (11) with \( l^\mu l^\nu \):

\[
l^\mu l^\nu T_{\mu\nu} = -l^\mu \partial_\mu [l^\nu \partial_\nu U(\Phi)] + (l^\nu \partial_\nu \Phi)^2 V(\Phi) \equiv (V(\Phi) - U''(\Phi)) \dot{\Phi}^2 - U'(\Phi) \ddot{\Phi},
\]

where use of Eq. (11) has been made. In the second line of this equation a dot denotes differentiation with respect to the affine parameter of \( l \), which we will call \( \tau \) in the following. In view of Eq. (8) and our goal to obtain a condition on the increase of \( S \), the right-hand side of the still general Eq. (14) is somewhat demotivating.

In the case (8), considered by Frolov, now the following simplification occurs in Eq. (14): Clearly to show that \( U|_H = 2\pi \exp \Phi|_H \) is increasing, \( H \) denoting the event horizon here and in what follows, it is sufficient to show that \( V \Phi|_H\) itself is increasing. Since, moreover, \( U'' = V \) for the choice (8) Eq. (14) simplifies to

\[
\ddot{\Phi} = \frac{-1}{2\pi} \exp(-\Phi) l^\mu l^\nu T_{\mu\nu}.
\]

Assuming that at late times the black hole settles down to a stationary state, \( \Phi(+\infty) = 0 \), Eq. (15) may be integrated to yield

\[
\dot{\Phi}(\tau) = \frac{1}{2\pi} \int_{\tau}^{\infty} [\exp(-\Phi) l^\mu l^\nu T_{\mu\nu}] (\lambda) d\lambda.
\]

Since the right-hand side of this equation is positive upon the assumption \( l^\mu l^\nu T_{\mu\nu} \geq 0 \), the second law, Eq. (13), is established.

There still is a small loophole in the above proof, if we require only \( \lim_{\tau \to -\infty} \dot{S} = 0 \). Indeed, according to Eqs. (8) and (11), the above asymptotic condition implies \( \lim_{\tau \to -\infty} \dot{\Phi} = 0 \) only if \( \Phi \) does not approach \( -\infty \) for \( \tau \to -\infty \). At least for generic choices of \( L_{\text{matter}} \), including the case considered by Frolov (cf. Eq. (8)) as well as dilaton–independent and conformally invariant couplings to matter fields, it may be shown by means of the field equations that \( \Phi \to -\infty \) implies \( R \to \infty \). This to happen for \( \tau \to \infty \) is, however, in conflict with the notion of a horizon and thus may be excluded in the present case.

B. The general case

In the notation introduced above and making use of Eq. (8), Eq. (14) takes the form
\[ S = - \left[ t^\mu t^\nu T_{\mu\nu} - V(\Phi)\Phi^2 \right]_H. \] (17)

First of all, we observe that if \( V(\Phi) \) is a nonpositive function of \( \Phi \) or, more generally, if one can ensure that evaluated on the event horizon this function is always nonpositive, we can integrate (17) directly to establish (3) under the late-time assumption
\[ \lim_{t \to \infty} \dot{S} = 0. \] (18)

**Proposition 1** If the potential \( V \) in (3) is nonpositive and (18) holds true, the entropy-function \( S \) obeys the second law (3).

Note that for the string inspired dilaton theory considered in the previous subsection as well as for a spherically reduced gravity theory, the potential \( V \) is positive definite, and the assumption for the above reasoning is thus not satisfied in these cases. On the other hand, many authors consider the action functional (5) with \( U \equiv 0 \); then Proposition 1 may be applied. (This result should, however, not be confused with one for the general theory (3); we will come back to this issue below).

The situation improves if one knows that \( U \) is some strictly monotonic function of its variable \( \Phi \) evaluated along the horizon. Let us assume first that \( U = U^\mu - V \equiv 0 \) is monotonic and (analogue considerations apply to the decreasing case). Then we can make use of the relation \( \dot{S} > 0 \Leftrightarrow \Phi > 0 \). Employing Eq. (14), on the other hand, we obtain the integral equality
\[ \Phi(\tau) = \int_0^\tau \frac{t^\mu t^\nu T_{\mu\nu} + \Phi^2(U^\mu - V)}{U^\mu} (\lambda) d\lambda \] (19)

From this it is obvious that the second law is satisfied, if \( U^\mu - V \) is nonnegative, at least at the horizon. In summary:

**Proposition 2** If the potentials in (3) are such that \( U \) is monotonic and \( V - U^\mu \) is nonpositive and if \( \lim_{t \to \infty} \Phi = 0 \) holds true, the entropy-function \( S \) obeys the second law (3).

In string inspired dilaton gravity \( U \) is monotonic and the expression \( (U^\mu - V) \) vanishes identically; Proposition 2 is the straightforward generalization of the considerations in the previous section III A.

Still, in general we will not be able to ensure that either \( V|_H \) or \( [V - U^\mu]|_H \) in a given action functional is nonpositive and the applicability of Propositions 1 and 2 seems to be quite limited. However, as noted already at the end of the previous section, the potentials in the action functional change under a reparametrization of the coordinate \( \Phi \) (in a one-dimensional target space), cf. Eq. (13). While obviously the sign of \( V \) evaluated on the event horizon can not be changed by such a transformation, the sign of \( V - U^\mu \) can, and by these means we will now in fact be able to establish the second law under much more general conditions. Defining the following nonnegative function
\[ F_0(Z) := \exp \left[ \int Z \frac{V(z)}{U'(z)} dz \right], \] (20)

where a constant of integration is assumed to be fixed somehow within the integral, we have

**Proposition 3** If \( U \) in the action functional (3) as a function of the dilaton \( \Phi \) has no extrema on its domain of definition \( D \) and the late time assumption \( \lim_{t \to \infty} \dot{S}/F_0(\Phi(\tau)) = 0 \) holds, then the entropy \( S \), defined in Eq. (3), obeys the second law (3).

**Proof:** We can get rid of the term \( U^\mu - V \) in Eq. (14) by an appropriate reparametrization \( \Phi = f(\phi) \) of the dilaton. With (12) we see that \( f \) has to obey
\[ 0 \equiv \frac{d^2 f}{d\phi^2} U'(f(\phi)) + \left( \frac{df}{d\phi} \right)^2 [U''(f(\phi)) - V(f(\phi))] \] (21)

After one integration this may be verified to become
\[ \frac{d}{d\phi} \left[ U'(f(\phi)) \frac{df}{d\phi} \right] = F_0(f(\phi)). \] (22)

We choose \( f \) to be orientation preserving, i.e. \( f'(\phi) > 0 \). Our assumption \( U' \neq 0 \) and the regularity of \( V \) within \( D \) guarantee that the derivative of \( f \) with respect to \( \phi \) does not vanish or diverge. \( f \) is a (globally well-defined) diffeomorphism from some domain to the common domain of definition \( D \) of the potentials. Now our claim is easily established by noting that
\[ \dot{S} = \text{sgn}(U') F_0(\dot{\phi}) \] (23)

and that (using Eq. (19) for the redefined field \( \phi \) and, in a second step, the null energy condition)
\[ \text{sgn}(U') \dot{\phi}(\tau) = \int_0^\tau \frac{t^\mu t^\nu T_{\mu\nu}}{|U'(f(\phi)) f'(\phi)|} (\lambda) d\lambda \geq 0. \] (24)

\[ \Box \]
The condition in Proposition 3 on the late time behavior is implicitly one on the asymptotic behavior of \( \Phi(\tau) \), since \( S(\tau) \equiv U(\Phi(\tau)) \); here again \( \tau \) is an affine parameter along the event horizon. For any finite value of \( \tau \), the function \( F_0 \) is nonzero by its definition \( (20) \) upon our conditions on the potentials \( U \) and \( V \). However, \( U' \) can vanish at the boundary \( \partial D \) of its domain of definition (which does not belong to \( D \) itself, being an open interval in \( \mathbb{R} \)) and similarly \( V \) can diverge there. Thus in general the condition on the late time behavior is different from Eq. \((18)\); it is stronger (weaker) whenever \( \lim_{\Phi \to \partial Dz} F_0(\Phi) \) becomes zero (divergent). Whenever \( G_0(\Phi) := \int \Phi(V(z)/U'(z))dz \) does not approach minus infinity at one of its boundary points, we consequently may also drop the function \( F_0 \) in the condition of Proposition 3.

Note that in the vacuum theory always \( \Phi \equiv 0 \) along the horizon (cf. the remarks at the end of section II C). Thus even in the case that \( F_0(\Phi) \) vanishes on one of the two boundary points \( \partial D \), the above condition may be well motivated by the requirement of a sufficiently fast approach of the spacetime in the neighborhood of the horizon to the one of a vacuum black hole.

Mostly in the literature the potential \( U \) in the action functional \( (5) \) is put to the identity map right at the beginning. We thus specify the above Proposition to this particular case.

**Corollary 1** For \( U = \Phi \) and

\[
\lim_{\tau \to \infty} \exp \left( - \int \Phi(\tau) \right) \Phi(\tau) = 0
\]

the second law \((1)\) holds true.

There is also another possibility for establishing Proposition 3, which we intend to sketch as it provides an interesting alternative point of view. It makes use of a trick which is by now well-known in the context of the general dilaton theories \((5)\), although in the present context one has to be somewhat careful as we will emphasize below (in particular if we want to generalize the considerations to nonmonotonic potentials \( U \)): By a dilaton-dependent conformal redefinition of the metric we can effectively remove the potential \( V \), which was potentially disturbing when proceeding from Eq. \((13)\). In the notation introduced above, this transformation takes the form

\[
g_{\mu\nu} \to \tilde{g}_{\mu\nu} := F_0(\Phi(x)) g_{\mu\nu},
\]

where the function \( F_0 \) is the same function as the one encountered already above in quite a different context (reparametrization of the dilaton, with \( g \) held fixed). In terms of the rescaled variables the action functional \((5)\) reads

\[
L_{\text{grav}}[\tilde{g}, \Phi] = \frac{1}{4\pi} \int_M d^2x \sqrt{-\det \tilde{g}} \left[ U(\Phi) \tilde{R} + \frac{W(\Phi)}{F_0(\Phi)} \right],
\]

where \( \tilde{R} \) is the curvature scalar of the metric \( \tilde{g} \). Repeating the steps leading to Eq. \((14)\), we obtain

\[
\frac{d^2S}{d\tau^2}(\tau) = -\tilde{\nu}^{\mu\nu} T_{\mu\nu} \leq 0,
\]

where \( \tilde{r} \) denotes the affine parameter of the horizon in the auxiliary spacetime with metric \( \tilde{g} \), and \( \tilde{I} \) is the corresponding generator satisfying \( \tilde{\nu}^{\mu\nu} T_{\mu\nu} = 0 \). Thus while \( S \) will in general not be a convex function of \( \tau \), cf. Eq. \((17)\), as a function of \( \tilde{r} \) it is. We then may proceed as for Proposition 3 for \( S(\tilde{r}) \), taking care, however, of the relation between \( \tau \) and \( \tilde{r} \).

The affine parameter \( \tilde{r} \) along the (lightlike) image of the event horizon is connected to the respective affine parameter in the physical spacetime with metric \( g \) by (cf., e.g., \([27]\])

\[
\frac{d\tilde{r}}{d\tau} \propto F_0(\Phi(\tau)).
\]

Thus (by an appropriate choice of the proportionality constant)

\[
\frac{dS}{d\tau} = F_0(\Phi(\tau)) \frac{dS}{d\tilde{r}},
\]

Thus from Eq. \((27)\) with \( \lim dS/d\tilde{r} = 0 \) we reobtain Proposition 3.

Alternatively, but equivalently to this second approach, is the use of a different generator of the event horizon \( \tilde{I} \) than the one in \((13)\). Introducing \( \tilde{I} \) according to

\[
\tilde{\nu}^{\mu\nu} \tilde{I}_\nu = -\frac{V(\Phi)}{U'(\Phi)} \left( \tilde{\nu}^{\mu\nu} \partial_\nu \Phi \right),
\]

the term including \( V \) in Eq. \((14)\) cancels and, using \( \tilde{r} \) as the flow parameter of \( \tilde{I} \), we again obtain \((27)\).

In the context of potentials with nonvanishing \( U' \) both approaches, the one with a redefined dilaton field \( \phi \) and the one with a conformally rescaled metric \( \tilde{g} \) (respectively a new parameter \( \tilde{r} \)) are equally applicable. However, when we turn to the case of the general class of potentials, restricted only by the conditions presented in section II C, the former method turns out to be much better suited. In fact, we managed to establish Theorem 3 below only by means of the first method.

Before we turn to the case of nonmonotonic potentials \( U \), however, let us first unify the conditions in Propositions 1 and 3 (for the case of potentials with \( U' \neq 0 \) everywhere). As the propositions are formulated right now, the respective fall-off conditions for late times are in part different. E.g., if \( U'' \equiv V \), then the function \( F_0 \) in Eq. \((29)\) becomes (proportional to) \( U' \) and
\[ \lim_{\tau \to \infty} \dot{S}/F_P(\Phi(\tau)) = \lim_{\tau \to \infty} \dot{\Phi} \] but if, more generally, \( V - U'' \) is just a nonpositive function, \( \lim_{\tau \to \infty} \dot{\Phi} = 0 \) is a different condition than the one used in Proposition 3, which, however, is applicable as well.

To incorporate also Proposition 3 as a particular case into Proposition 2 we need to relax the conditions on the function \( F \). In fact, in order to proceed along the lines of the proof for Proposition 2, it is sufficient to replace the left hand side of Eq. (21) by any nonnegative function of \( \phi \). Making for this function the \( f \)-dependent choice \((f'(\phi))^2 P(f(\phi))\) with some nonnegative function \( P \), we merely need to replace \( F_0 \) in Eq. (30) by its generalization

\[ F_P(Z) := \exp \left[ \int_Z V(z) + P(z) \frac{U'(z)}{U''(z)} \, dz \right]. \tag{30} \]

All the remaining relations in the proof then remain the same (just with \( F_0 \) replaced by \( F_P \) and with now the correspondingly changed function \( f(\phi) \)). Thus, in Proposition 3 we are allowed to replace \( F_0 \) by \( F_P \) for any nonnegative function \( P \).

Now, if \( U'' - V \) is some nonnegative function \( P(\Phi) \), we can choose this function in Eq. (30) so as to obtain \( F_P = |U'| \) \((f \) then is seen to become just the identity map) and Proposition 3 becomes a particular case of Proposition 2 (with \( F_0 \) generalized to \( F_P \)). Similarly, if \( V \) is some nonpositive function, we can choose \( P := -V \) yielding \( F_P = 1 \) and thus, in the case of a theory with \( U' \neq 0 \), now also Proposition 3 becomes incorporated into (the above generalization of) Proposition 2.

The generalized assumption using \( F_P \) instead of \( F_0 \) may be obtained also in the second approach with a dilaton dependent conformal transformation of the metric. Indeed, in repeating the steps leading to Proposition 3 with redefined metric \( \tilde{g} \) and affine parameter \( \tau \), we need not cancel the potential \( V \) in (22), but it obviously is sufficient to transform it to some nonpositive function \(-P\). This, however, is achieved by just replacing \( F_0 \) in Eq. (22) by the function \( F_P \) defined in Eq. (30).

In the remainder of this section we want to extend our results to the completely general case (no restrictions on the potentials except those mentioned in section 11) and show:

**Theorem 1** Provided there is some nonnegative function \( P : D \to \mathbb{R} \) such that along the horizon \( \lim_{\tau \to \infty} \dot{S}/F_P(\Phi(\tau)) = 0 \), with \( F_P \) as defined in Eq. (28), the entropy \( S \) as defined by Wald, cf. Eq. (3), obeys the second law (3).

In particular we will now permit extrema of \( U \) within its domain of definition \( D \). This implies that the function \( F_P \) defined in Eq. (28) and evaluated along the event horizon can become divergent or zero within spacetime. Either of the two approaches provided to establish Proposition 3 can then be applied only within those sectors of the horizon where \( \ln F_P := G_P \) is nondivergent. To prove Theorem 1 we then need to glue together different sectors. As mentioned already above, in this context the first approach appears superior over the second one: In the latter the parameter \( \tilde{\tau} \) becomes ill-defined at the transition between two sectors and there one can thus hardly make statements about \( S \) as a function of \( \tilde{\tau} \). On the other hand, by introducing an auxiliary dilaton field \( \phi \) within each sector, the differential equations remain formulated in terms of an everywhere well-defined affine parameter \( \tau \) and continuity arguments may be applied.

In the following we will make use of a small lemma, the proof of which is obvious:

**Lemma 1** Let \( \varphi(\tau) \) and \( G(\tau) \) be smooth functions on some open interval \((a, b) \subset \mathbb{R}\), with possibly a divergent limiting value at \( a \) or \( b \), and \( \dot{S}(\tau) \) be a smooth function on \([a, b]\) with \( \dot{S}(b) \geq 0 \), satisfying

\[ \varphi = \exp(-G)\dot{S}. \tag{31} \]

Then we may conclude from

\[ \varphi \leq 0 \quad \text{and} \quad \lim_{\tau \to \tau^+} \varphi \quad \text{nonnegative} \tag{32} \]

that \( \dot{S} \geq 0 \) on all \([a, b]\).

In the above \( a \) may be replaced also by \(-\infty \) and likewise \( b \) by \( \infty \). Furthermore, \( \lim_{\tau \to \pm \infty} \varphi \) need not necessarily exist in the condition for the lemma; it is already sufficient that there is an \( \epsilon > 0 \) such that \( \varphi(\tau) \geq 0 \) for all \( \tau \in (b - \epsilon, b) \). We will interpret the respective condition in Eq. (32) in this manner.

**Proof** (of Theorem 1): Due to the analyticity required for \( U \), eventual zeros of \( U' \) are always isolated. Correspondingly, if \( G(\tau) := G_P(\Phi(\tau)) \equiv \ln F_P(\Phi(\tau)) \) diverges not just at an isolated point on the horizon but on a whole segment, then \( \Phi \) will be constant along this segment and therefore \( \dot{S} \) will vanish there. \( \dot{S}(\tau) \), on the other hand, is a well-defined smooth function of \( \tau \), since \( U \) and \( \Phi \) are smooth and thus differentiable by assumption (and \( \tau \) is just the flow parameter of a globally well-defined vector field \( l \)). Thus, different sectors may be glued together by continuity of \( \dot{S} \). If the sectors are separated by segments of diverging \( G_P \), then the gluing is even simplified, being achieved by \( \dot{S} = 0 \) in this case. For notational simplicity we thus restrict our attention to the case where different sectors are separated by points \( \tau_i \) along the horizon on which \( G \) diverges.

Within each sector we can again perform the reparametrization \( \Phi = f(\phi) \) with \( f \) defined implicitly through \( |U'(f(\phi))| f'(\phi)| = F_P(f(\phi)) \) (cf. Eq. (22)). Introducing the function \( \varphi := \text{sgn}(U'(f(\phi))) \), Eq. (28) and Eq. (24) (differentiated with respect to \( \tau \) and using the null energy condition again) take the form

\[ \dot{S} = F_P \varphi, \quad \varphi F_P = -T_{\mu \nu} l^\mu l^\nu \leq 0. \tag{33} \]
Since $F_P$ is nonnegative in each sector by construction, we only need to make sure that the second condition in Eq. (32) is satisfied so as to conclude from Lemma 1 above that $\dot{S} \geq 0$ within the respective sector.

Theorem 1 can now be proven by induction. Let $\tau_1 > \tau_2 > \ldots$ denote the values of the affine parameter along the horizon where $G$ happens to diverge. By our assumption we know that $\varphi \to 0$ for $\tau \to \infty$. Thus in the rightmost sector, $\tau \geq \tau_1$, $\dot{S} \geq 0$ is established.

For the induction let $\dot{S} \geq 0$ for $\tau \geq \tau_i$. The continuity of $\dot{S}$ and the positivity of $F_P = \exp G$ assures that $\lim_{\tau \to \tau_{i-}} \varphi$ is nonnegative. With Lemma 1 we obtain $\dot{S} \geq 0$ for $\tau_{i+1} \leq \tau \leq \tau_i$ and, by induction, thus for all $\tau$.

The condition on the late time behavior of the entropy of spacetime may be dropped altogether, if for late times spacetime becomes equal to the one of a vacuum black hole:

**Corollary 2** If for all $\tau > \tau_0$ for some $\tau_0$ the event horizon of the two-dimensional spacetime has a neighborhood which is vacuum ($T_{\mu \nu} = 0$ in this region), then the second law (1) is satisfied for all times.

This follows immediately from Theorem 1 and section III C (note in particular that according to section III C $\ln F_0$ is nondivergent for $\tau > \tau_0$).

**IV. SECOND LAW FOR $f(R)$-THEORIES**

Instead of taking recourse to a dilaton field $\Phi$ so as to define a nontrivial gravitational part of the action in two spacetime dimensions, we may also consider higher derivative theories, where the Ricci scalar $R$ in the Einstein action is replaced by a nonlinear function of it. This leads us to consider (cf. also [8, 9])

$$L_{geom} = \frac{1}{4\pi} \int_M \sqrt{-g} f(R)$$

(34)

as for the gravitational part of the action instead of (3). Correspondingly, the matter part of the action is then also assumed not to depend on a dilaton, or alternatively, the dilaton is understood as yet just another scalar scalar field among other matter fields described collectively by means of $w$ in $L_{matter}[g, \psi]$. Again $L_{matter}$ is assumed to not depend explicitly on $R$. Then according to Wald the entropy $S$ of a black hole in an $f(R)$-theory is nothing but $f'(R)$ evaluated on the horizon at a given instant of time (parametrized by the affine parameter $\tau$).

To have a nontrivial kinetic term for the metric, we should require that the function $f$ is locally nonlinear, i.e. that there is no interval (within the domain of definition of $f$) in which it is linear. The physical requirement of nonlinearity may also be dropped on a formal level, however, since a linear function $f$ yields a constant entropy according to Wald, which then clearly satisfies a second law, too. Finally, we require $f$ to be three times differentiable (cf. the field equation Eq. (5) below).

For convex functions $f$, the model of the present section may be considered a special case of the one in the previous sections. Indeed, choosing $U = \Phi$, $V = 0$, and $-W$ as the Legendre transform of $f$ within the action (4), and eliminating the dilaton $\Phi$, we just reproduce the action (3). However, even in the case of a general, non-convex choice for the function $f$ in the gravitational part of the action, we can establish a second law. We thus will not take recourse to our previous results (which then would have to be “glued together” appropriately along regions in spacetime where $R$ takes a value for which $f'$ becomes extremal), but rather show directly:

**Theorem 2** If the spacetime in an $f(R)$-theory satisfies

$$\lim_{\tau \to \infty} \dot{S} = 0 \text{ or, more generally, if there exists some strictly monotonically increasing function } H \text{ with nonnegative second derivative such that } \lim_{\tau \to \infty} H(S) = 0,$$

then the second law (1) holds, i.e. provided the matter fields satisfy the null energy condition, $\dot{S}(\tau)$ can at most increase in time.

*Proof:* The equations of motion of the class of models under consideration are given by

$$- \nabla_{\mu} \nabla_{\nu} f'(R) + \frac{1}{2} g_{\mu \nu} (f(R) - R f'(R)) = T_{\mu \nu} - g_{\mu \nu} T \ .$$

(35)

By contracting this equation with the generator $l$ of the horizon, we obtain

$$T_{\mu \nu} l^\mu l^\nu = - l^\mu \partial_\mu (l^\nu \partial_\nu f'(R)) = - \dot{S}(\tau) \ ,$$

(36)

where this equation is assumed to be evaluated on the horizon. By integrating Eq. (36) Theorem 2 is now established as in the previous section provided the late-time assumption $\lim_{\tau \to \infty} \dot{S} = 0$ holds.

Let us now consider the more general case of the late-time assumption $\lim_{\tau \to \infty} \dot{H}(S) = 0$. By introducing the function $\varphi = H(S)$ we easily obtain the relations

$$\dot{S}(\tau) = \frac{\dot{\varphi}(\tau)}{H'(S(\varphi(\tau)))}$$

(37)

as well as

$$\ddot{S}(\tau) = \frac{\ddot{\varphi}(\tau)}{H'(S(\varphi(\tau)))} - \frac{H''(S(\varphi(\tau))) \dot{\varphi}^2(\tau)}{H'^3(S(\varphi(\tau)))} \ .$$

(38)

From Eq. (37) and our assumptions concerning $H$ we see that nonnegativity of $\dot{S}$ is equivalent to nonnegativity of $\dot{\varphi}$. Eq. (38), on the other hand, can be integrated for $\dot{\varphi}$ using (36) and our late-time assumption to yield
\[ \dot{\phi}(\tau) = \int_{\tau}^{\infty} d\lambda \left( T_{\mu \nu} T^{\mu \nu} H'(S(\phi)) - \frac{H''(S(\phi)) \dot{\phi}^2}{H'(S(\phi))} \right) (\lambda). \]  

(39)

Using the positivity of \( H' \), the nonpositivity of \( H'' \) and the null energy condition, we conclude from this equation that \( \dot{\phi} \) is nonnegative. This is equivalent to the proposition of the theorem. \( \square \)

As mentioned above, in a region where the function \( f \) in the action is convex the geometric action may be reformulated in terms of a dilaton theory. By comparing the notations in the present section and in the previous one, it is easy to see that the function \( H'(S) \) is the analogue of \( 1/F \). This function is strictly positive and monotonically decreasing. These are exactly the requirements we posed on \( H' \) in Theorem 2.

One may ask whether the version of the theorem with the late-time assumption involving the function \( H \) is really a generalization of the statement using the assumption \( \lim_{\tau \to \infty} \dot{S} = 0 \). As can be seen from Eq. (37), this is the case only if \( H'(S) \) tends to zero and \( \dot{S} \) diverges for large \( \tau \) (without \( R \) diverging at the same time since this would be in conflict with the notion of a horizon). This, on the other hand, can happen only if the function \( f \) in the action (34) is defined on an interval with a divergent limit at a boundary value (such as e.g. \( f(R) = \tan R \)). For functions \( f \) which are continuous on all of \( \mathbb{R} \), this part of Theorem 2 may be dropped without loss of generality.

V. CONCLUSION AND OUTLOOK

Our results are contained in the Theorems 1 and 2 as well as in Corollary 2. We established the validity of the second law of black hole mechanics for all generalized 2d dilaton gravity theories. The class of theories was recapitulated in section 1. The gravitational part \( L_{\text{grav}} \) of the action replacing the standard Einstein term in four dimensions has the form (11) (satisfying conditions on the potentials such as analyticity as specified in detail in section 1C), while the matter part \( L_{\text{matter}} \) was left essentially arbitrary. For the notion of an entropy \( S \) of a 2d black hole we applied the general formalism of Wald [10], extrapolated to the dynamical situation, which yielded \( U(\Phi) \) evaluated at the event horizon at a given instant of time. (Here enters the only restriction on \( L_{\text{matter}} \): it should not depend explicitly on \( R \), since otherwise the formula for \( S \) would change). We then showed that during any dynamical process which finally settles down (sufficiently fast) to a stationary (e.g. vacuum) situation the entropy \( S \) can at most increase. Here we even need not require the weak energy condition (or assume anything like cosmic censorship): it is sufficient that the energy momentum tensor of matter satisfies the null energy condition. We also showed the second law for models in which the gravity part of the action is replaced by the geometrical one (34). For these theories the entropy turned out to be \( f'(R) \) evaluated on the horizon.

We remark here in parenthesis that the s-wave sector (i.e. the spherically symmetric sector) of Einstein gravity in \( D \geq 3 \) dimensions with arbitrary minimally coupled matter content is contained as a particular example to the present considerations. The type of theories considered here may be of interest also as the s-wave sector of more complicated dilaton gravity theories in higher dimensions, e.g. as they arise generically in (super) string theory at low energies.

It may be worthwhile to extend our results to even more general 2d gravity theories, such as theories with dynamical torsion:

\[ L = \frac{1}{4\pi} \int_{M} d^2 x \sqrt{-g} \mathcal{F}(R, \tau^a \tau_a) \]  

(40)

where \( \mathcal{F} \) is a sufficiently smooth function and \( \tau^a \) denotes the Hodge dual of the torsion 2-form \( De^a \). For functions \( F \) with a well-defined Legendre transform there is a formulation of these theories in terms of Poisson Sigma Models (cf., e.g., [13]), which, on the other hand, also cover the \( f(R) \)-theories and the general dilaton theories with monotonic potential \( U \) discussed within the present paper. It is thus plausible that a second law may be established also for the class of theories where the gravity part of the action has the form of Eq. (40).

Concerning the first law of black hole mechanics we relied on the general formalism of Wald guaranteeing the existence of such a law. Still it may be worthwhile, also in combination with the present second law, to discuss the resulting “2d black hole mechanics” in further detail and from different physical perspectives.

Our considerations remained on a purely classical and mathematically rigorous level. Still, it would be interesting, if and in what sense the results may be extended also into at least a semiclassical regime. Is there, e.g., a “generalized second law” of black hole mechanics when Hawking radiation and some approximate back reaction are taken into account? For specific 2d models such issues have been discussed already partially in the literature [29].

We finally come back to the issue of a microscopic explanation for the entropy of black holes, which certainly also arises within the two-dimensional setting. Here we mention two approaches which turned out successful in the case of 2+1 Einstein gravity with a cosmological term, giving rise to BTZ black holes [30]. First, in the spirit of the AdS/CFT correspondence [31], Strominger [4] obtained the statistical entropy by counting the degeneracies of the representations of the algebra of asymptotic symmetries of this spacetime. In the two-dimensional setting this idea was applied [12] to a very particular theory of the type (5), namely the so-called
Another approach \[\text{(1)}\] which turned out successful in the case of BTHZ black holes, ascribes the black hole entropy to tracing out the degrees of freedom of an effective theory living on the event horizon. In two space-time dimensions this corresponds to a quantum theory of point particles. Adapting Carlip’s approach to the general class of theories with action \[\text{(1)}\] did not reproduce the expected answer: In some cases it just produced an ill-defined infinity, in other cases (such as the above mentioned JT model) it produced the logarithm of the expected result. Let us remark here, however, that by means of some additional steps one can obtain the expected result, at least for cases such as the JT model or spherically reduced gravity and up to prefactors. Roughly speaking this works as follows:\[\text{(5)}\] The point particle phase space found in \[\text{(1)}\] was a copy of the two-dimensional spacetime manifold, equipped with a specific symplectic form \(\Omega\). Now one keeps only the part of phase space corresponding to the interior of the black hole, appropriately euclideanizes \(\Omega\), and then performs a second quantization with fermionic statistics; then \(S\) may be identified with the logarithm of the dimension of the resulting Hilbert space.

But even if correct numbers may be produced in such or another manner, there always remains the question for the underlying general principle and, all the more, a principle which would work in general dimensions (in analogy to Wald’s general formalism for obtaining a classical prediction for the entropy \(S\), satisfying a first law). The relatively large class of 2d gravity models discussed in this note may give us decisive hints to the right answers in this context.

The existence of a second law of black hole mechanics, despite still on a classical level, is already an important step in this direction. Combined with the first law and due to its close formal analogy to the much more involved four dimensional situation, it should provide enough motivation for further studies of such theories.

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