Destabilization effect of exchange dipole-dipole interaction on the spectrum of electric dipolar ultracold Fermi gas

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The self-consistent field approach for the electric dipolar ultracold spin-1/2 fermions is discussed. Contribution of the exchange part of the electric dipole interaction is found. Hence we obtain a model of dipolar fermions beyond the self-consistent field approximation. It is shown that the exchange interaction of electric dipolar fermions depends on the spin-polarization of the system. For instance the electric dipole exchange interaction equals to zero for spin-unpolarised systems, namely all low laying quantum states occupied by two-particles with opposite spins. In opposite limit of the full spin polarization of the degenerate fermions, then we have one particle in each quantum states, the exchange interaction has maximum value, which is comparable with the self-consistent field part of the dipole-dipole interaction. The self-consistent part of the electric dipole-dipole interaction gives a positive contribution into the spectrum of collective excitations, while the exchange part of the dipole-dipole interaction leads to a negative term in the spectrum. At angles between the equilibrium polarization and the direction of wave propagation close to $\pi/2$ the full dipolar part of the spectrum becomes negative. At the electric dipole moment of fermions of order of several Deby the dipolar part is large enough to exceed the Fermi pressure, that reveals in an instability. We also consider spectrum of the quasi two-dimensional cloud of fermions in the trap. We consider the regime of purely two dimensional structure of dipolar fermions with the exchange dipole-dipole interaction in the three dimensional space and calculate the spectrum in this regime. We assume that the equilibrium polarization is perpendicular to the two dimensional structure. Major picture of the spectrum behavior in low dimensional regimes is similar to the three-dimensional one. Since the two-dimensional perturbations propagate perpendicular to the equilibrium polarization we find that the dipolar part of the spectrum is negative in these regimes.

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I. INTRODUCTION

Dipolar Fermi molecules [1, 10] attract as much attention as electrically polarised molecules in the Bose-Einstein condensate (BEC) state [11, 31]. Majority of researches in the field of dipolar quantum gases are focused on the aligned dipoles [32, 38], but there are paper where evolution of the dipole directions is included as well [39, 42].

A lot of achievements in physics of ultracold fermions were reviewed in Ref. [13]. The non-linear Schrodinger equation and the hydrodynamic equations have been applied for description of ultracold fermions [13, 53]. Hydrodynamic model of dipolar ultracold fermions was derived in approximation of aligned dipoles applying the microscopic density matrix [54]. This model is in agreement with the similar model of dipolar BECs suggested earlier [55, 56]. Collective excitations of quasi-two-dimensional trapped dipolar fermions were considered in Ref. [59]. Different generalizations of the model of dipolar BECs were suggested. Contribution of the quantum fluctuations in the characteristics of dipolar BECs was considered in Ref. [60]. The dipole-dipole interaction, in terms of collisions, in the dipolar BECs, beyond the first Bohm approximation was developed in Ref. [61]. Spectrums of dipolar BECs at finite temperatures were calculated in Refs. [62, 63]. Exchange effects in temperature distributed quantum gases (bosons and fermions) were considered in Refs. [64, 65]. Non-integral Gross-Pitaevskii equations for dipolar (electric and magnetic) BECs were obtained in Refs. [66, 67]. Difference between the magnetic and electric dipolar BECs was discussed in Refs. [68, 69]. Dipole direction evolution and its influence on spectrum of electric dipolar BECs were described in Refs. [70, 71]. Model of ultracold dipolar fermions with the dipole direction evolution was developed in Refs. [72, 73].

Fisher theoretically demonstrated stability of quasi two-dimensional dipolar Bose-Einstein condensates [74]. This result extended the list of fundamental results on the trapped dipolar Bose-Einstein condensates [75, 76]. Further analysis of general properties of the flattened dipolar condensates was presented in Ref. [77]. The evolution of correlations in a quasi-two-dimensional dipolar gas driven out of equilibrium by a sudden ramp of the interactions was investigated in Ref. [78]. Anisotropic superfluidity in dipolar Bose-Einstein condensates in a quasi-two-dimensional geometry was considered [79]. Fedorov et al. [80] predicted the effect of the roton instability for a two-dimensional weakly interacting gas of
tilted dipoles in a single homogeneous quantum layer. Low dimensional dipolar fermions [54], [80] attract a lot of attention along with the quasi-two-dimensional dipolar Bose-Einstein condensates. The strong correlations on the phase diagram and collective modes of quasi-two-dimensional dipolar fermions were considered in Ref. [81]. The density-wave phase of a two-dimensional dipolar fermions was studied in Refs. [82], [83]. The BCS superfluid transition in a single-component fermionic gas of dipolar particles loaded in a tight bilayer trap was studied in Ref. [84]. The three-dimensional dipolar fermions [59], [86], two component systems of dipolar bosons [87], and dipolar boson-fermion mixtures [88] are also under consideration. Jona-Lasinio et al demonstrated that the density dependence of the roton minimum results in a spatial roton confinement, that roton confinement plays a crucial role in the dynamics after roton instability, and that arresting the instability may create a trapped roton gas revealed by confined density modulations [89]. The ground state of a system of bosons with aligned dipoles located in a plane is analyzed in Ref. [90]. The Feshbach resonances in dipolar quantum gases were described by Kotochigova [91].

In this paper we consider the fermi molecules with the aligned electric dipoles. We use the method of many-particle quantum hydrodynamics to derive the model. We consider dipolar fermions beyond the self-consistent field approximation, so we calculate contribution of the exchange electric dipole-dipole interaction. We obtain that the exchange part of the interaction is comparable with the self-consistent field part for fully spin polarised spin-1/2 fermions possessing the electric dipole moment. Contribution of the exchange interaction becomes smaller with decreasing of the spin polarisation. It equals to zero for the spin unpolarised systems of fermions with the aligned electric dipoles. The dynamic of spins and the spin-spin interaction are not considered in this paper, since we focused on the electric dipole contribution, and the electric dipoles have rather larger contribution than the contribution of the magnetic dipoles. However we mention spin of particles, since their spin states give influence on the exchange part of the electric dipole-dipole interaction. We should note that the exchange spin-spin interaction in systems of spin-1/2 particles was derived in Ref. [92].

The self-consistent field part and the exchange part of the electric dipole interaction have different signs. Thus the exchange electric dipole interaction can cause an instability of the three dimensional cloud of dipolar fermions. Therefore we have calculated the spectrum of collective excitations (an analog of the Bogoliubov spectrum of dipolar Bose-Einstein condensates) of the quasi-two-dimensional and quasi-one-dimensional Fermi gas. We also calculate spectrums for two-dimensional and one-dimensional cases to compare it with the quasi-two-dimensional and quasi-one-dimensional systems.

Dipolar Bose-Einstein condensates (BECs) in terms of the quantum hydrodynamic method beyond the self-consistent field approximation was considered in Ref. [93]. This consideration revealed a strange result. It was shown that the self-consistent field part equals to zero. Hence all the dipole-dipole interaction in dipolar BECs is related to the exchange dipole-dipole interaction. Moreover the explicit form of the exchange interaction of the electric dipoles coincides with the result of formal application of the self-consistent field approximation. It was shown that this conclusion is correct for both the aligned dipoles and systems of dipoles with the dipole-direction evolution. Formal application of the self-consistent field approximation in both described cases was performed in Refs. [93], [100]. The three-dimensional dipolar fermions [64], [65], two component systems of dipolar bosons [66], and dipolar boson-fermion mixtures [67] are also under consideration. Jona-Lasinio et al demonstrated that the density dependence of the roton minimum results in a spatial roton confinement, that roton confinement plays a crucial role in the dynamics after roton instability, and that arresting the instability may create a trapped roton gas revealed by confined density modulations [68]. The ground state of a system of bosons with aligned dipoles located in a plane is analyzed in Ref. [69].

Collective excitations of a harmonically trapped, two-dimensional, spin-polarized dipolar Fermi gas in the hydrodynamic regime was considered in Ref. [101], where authors applied the Thomas-Fermi von Weizsäcker energy functional for the neutral atoms with the magnetic moments. They obtained the force field $F$ of the magnetic dipole-dipole interaction proportional to $\nabla n_{2D}^{3/2}$ (in their paper they have $F \sim \nabla n_{2D}^{3/2}$ due to different definition of the force $F$). Their formalism is based on the two-dimensional of particles [$n_{2D} = cm^{-2}$]. The structure of the full force of the magnetic dipole-dipole interaction obtained in Ref. [102] corresponds to the exchange part of the exchange electric dipole interaction derived in our paper for purely two dimensional dipolar fermions (see formula (53) below). Let us also note that quantum hydrodynamic derivation of the force of magnetic dipole-dipole interaction (the spin-spin interaction) for three dimensional systems of particles with the exchange spin-spin interaction was presented in Ref. [92].

This paper is organized as follows. In Sec. II we describe the model of dipolar degenerate fermions in the self-consistent field approximation for align electric dipoles. In Sec. III we present generalisation of the model including the exchange part of the electric dipole-dipole interaction. In Sec. IV we consider excitation spectrum of three-dimensional plane waves. In Sec. V we consider quasi-two-dimensional cloud of trapped degenerate fermions. In Sec. VI we consider purely two-dimensional dipolar fermions to compare it with the quasi-two-dimensional case. In Sec. VII brief conclusions are presented.
II. SELF-CONSISTENT FIELD MODEL

The many-particle quantum hydrodynamic method [94], [93] for quantum gases with the short-range interaction was developed in Ref. [93]. The method of derivation of the quantum hydrodynamic equations for electric dipolar particles was presented in Ref. [106]. It was applied to dipolar ultracold bosons in Refs. [39], [66]. Similar derivation can be performed for ultracold dipolar fermions. Result of derivation for ultracold dipolar fermions with the evolution of dipole directions was presented in Ref. [106]. It was applied to dipolar ultracold bosons in Refs. [39], [66].

For the fully spin polarized systems of electric fermions with the evolution of dipole directions was presented in Ref. [106]. It was applied to dipolar ultracold bosons in Refs. [39], [66].

Let us note that the pressure of the fully polarised spins larger than the pressure of the unpolarised fermions: $p_{F,E}^{3D} = \sqrt{4}$ and $p_{F,E}^{3D} = 2$.

The kinetic theory shows that the small perturbations of pressure, in the linear regime, differ from the results of application of the linearisation of the Fermi pressure by the factor $\chi_K = 9/5$, for three dimensional mediums. To include this information we substitute coefficient $\chi_K$ in the pressure perturbations $p = p_0 + \delta p$, and

$$\delta p = \chi_K \delta p_{3D E}.$$  (4)

Polarisation $P$ simplifies in the case of the aligned dipoles. In this case evolution of polarisation is reduced to the concentration evolution $P = n\vec{d}$, where $\vec{d}$ is the unit vector in direction of the polarization formed by the external electric field. Approximations of the self-consistent field and the aligned dipoles allow to obtain a closed set of the QHD equations for dipolar ultracold fermions.

The self-consistent field approximation applied in equation (2) allows to introduce the intrinsic electric field created by dipoles

$$E_{int}^{\alpha} = d^{\beta} \int d\vec{r}' \mathcal{G}^{\alpha\beta}(\vec{r}, \vec{r}') n(\vec{r}', t).$$  (5)

Using notion of full electric field $E_{full} = E_{ext} + E_{int}$ we can represent the Euler equation (2) in more simple non-integral form

$$mn(\partial_t + \vec{v} \cdot \nabla) \vec{v} + \vec{v} \cdot \nabla p - \frac{\hbar^2}{4m} n \nabla \left( \frac{\nabla n}{n} \right) = P^{\beta} \nabla E_{ext}^{\beta} + P^{\beta} \nabla \int d\vec{r}' \mathcal{G}^{\beta\gamma}(\vec{r}, \vec{r}') P^{\gamma}(\vec{r}', t).$$  (6)

The non-integral form of the Euler equation contains an extra variable: the electric field $E = E_{full}$. To get closed set of equations we need to obtain equations for electric field $E$.

Acting by the operators div and curl on the explicit form of electric field [106] we show that the electric field satisfies the Maxwell equations

$$\nabla \cdot \vec{E}(\vec{r}, t) = -4\pi \nabla \cdot \vec{P}(\vec{r}, t) = -4\pi d(1 \cdot \nabla)n(\vec{r}, t).$$  (7)
\[ \nabla \times \mathbf{E}(\mathbf{r}, t) = 0. \quad (8) \]

In the limit of the aligned dipoles the force field of the dipole-dipole interaction and the interaction of the dipoles with the external electric field can be rewritten as \( n \hat{d} \nabla \phi(\mathbf{E}) \), where \( \hat{d} \) is the unit vector in direction of the polarization formed by the external electric field.

Under assumption of the potential velocity field \( \mathbf{v} = \nabla \phi \), the set of quantum hydrodynamic equations \( 11 \) and \( 12 \) can be presented in the form of the non-linear Schrödinger equation for the effective many-particle wave function \( \Phi = \sqrt{n} \exp(\imath m \phi / \hbar) \):

\[ \imath \hbar \partial_t \Phi(\mathbf{r}, t) = \left( -\frac{\hbar^2}{2m} \Delta + \imath \beta_3 \partial_t - \frac{(3 \pi^2)^{2/3} \hbar^2}{2m} \mathbf{d} \cdot \mathbf{E} \right) \Phi(\mathbf{r}, t), \]

\( n(\mathbf{r}, t) = \Phi^*(\mathbf{r}, t) \Phi(\mathbf{r}, t) \) is the concentration of particles, and \( \mathbf{d} = \mathbf{d}_ \mathbf{l} \). The first term on the right-hand side of the NLSE \( 9 \) is the kinetic energy operator. The second term presents the Fermi pressure of the partially spin-polarised systems of spin-1/2 fermions possessing the electric dipole moment. The last term describes the potential energy of the electric dipoles being in the electric field \( \mathbf{E} \) consisting of the external electric field and the internal electric field created by the dipoles. Formula \( 9 \) gives the NLSE for dipolar fermions in the self-consistent field approximation.

Definitions of the particle concentration \( n \), the velocity field \( \mathbf{v} \), and the polarisation \( \mathbf{P} \) are

\[ n(\mathbf{r}, t) = \int d\mathbf{R}_N \sum_i \delta(\mathbf{r} - \mathbf{r}_i) \psi^*(\mathbf{R}, t) \psi(\mathbf{R}, t), \]

\[ \mathbf{P}(\mathbf{r}, t) = \int d\mathbf{R}_N \sum_i \delta(\mathbf{r} - \mathbf{r}_i) \mathbf{d}_i \psi^*(\mathbf{R}, t) \psi(\mathbf{R}, t), \]

and

\[ \mathbf{v}(\mathbf{r}, t) = \frac{j(\mathbf{r}, t)}{n(\mathbf{r}, t)} = \int d\mathbf{R}_N \sum_i \delta(\mathbf{r} - \mathbf{r}_i) \times \left( \psi^*(\mathbf{R}, t) \mathbf{p}_i \psi(\mathbf{R}, t) + \text{c.c.} \right), \]

\( 10 \)
\( 11 \)
\( 12 \)
correspondingly.

Definitions \( 10 \)–\( 12 \) are presented in terms of the microscopic many-particle wave function \( \psi(\mathbf{R}, t) \). The wave function \( \psi(\mathbf{R}, t) \) obeys the many-particle Schrödinger equation. We do not consider the spin evolution assuming that all atoms or molecules in the same fine structure state, hence we do not need to use the Pauli equation. Consequently Schrödinger equation, corresponding to the systems under consideration, within the quasi-static approximation, has the following form

\[ \imath \hbar \partial_t \psi = \hat{H} \psi, \]

\( 13 \)

with

\[ \hat{H} \equiv \sum_i \left( \frac{1}{2 \hbar^2} \mathbf{p}_i^2 - \mathbf{d}_i \mathbf{E}_{i, \text{ext}} + V_{\text{trap}}(\mathbf{r}_i, t) \right) + \frac{1}{2} \sum_{i,j \neq i} \left( U_{ij} - d_i^a d_j^b G_{ij}^{a\beta} \right). \]

\( 14 \)

The first term in the Hamiltonian is the operator of the kinetic energy. The second term represents the interaction between the dipole moment \( d_i^a \) and the external electrical field. The subsequent terms represent the short-range \( U_{ij} \) and the dipole-dipole \( d_i^a d_j^b G_{ij}^{a\beta} \) interactions between particles. The Green function for the dipole-dipole interaction reads \( G_{ij}^{a\beta} = \nabla_i \nabla_j \phi(1/r_{ij}) \). This Schrödinger equation coincides with the Schrödinger equation applied for dipolar bosons dynamics in Refs. \( 39, 40, 41, 67 \).

Equation \( 13 \) shows that we do not apply the second quantization. The wave function \( \psi(\mathbf{R}, t) \) governs the microscopic evolution of the systems of interacting particles.

The explicit form of the Green function of the dipole-dipole interaction can be written in different forms:

\[ G^{a\beta}(\mathbf{r}, \mathbf{r}') = \partial^a \partial^\beta \frac{1}{|\mathbf{r} - \mathbf{r}'|} \]

\[ = -\frac{\delta^{a\beta}}{3 \pi^3} \left( \frac{\delta^{a\beta} \delta(\mathbf{r})}{r^3} + \frac{1}{3} \delta^{a\beta} \Delta \frac{1}{r} \right). \]

\( 15 \)

Explicit form of the Green function of the electric dipole-dipole interaction consists of two parts: the reduced part (the first term in the second line of formula \( 15 \)) and the delta function term. The full theory, which is in accordance with the Maxwell equations, requires the application of the full potential of the dipole-dipole interaction containing the delta-function term \( 15 \).

Below, at the consideration of the two dimensional dipolar fermions we will need to apply the explicit form of the \( zz \) element of the tensor of Green function of the electric dipole interaction \( G^{zz} \). Hence we find this element from the formula \( 15 \):

\[ G^{zz}(\xi) = G^{a\beta}(\xi) \delta^{z\alpha} \delta^{z\beta} \]

\[ = -\frac{\delta^{a\beta}}{3 \xi^3} + \frac{3 \xi^\alpha \xi^\beta}{\xi^5} \left( -\frac{4 \pi}{3} \delta^{a\beta} \delta(\xi) \right) \delta^{z\alpha} \delta^{z\beta} \]

\[ = -\frac{1}{\xi^3} + \frac{3(\xi^z)^2}{\xi^5} - \frac{4 \pi}{3} \delta(\xi). \]

\( 16 \)
III. Dipolar Fermions Beyond the Self-Consistent Field Approximation: Exchange Dipole-Dipole Interaction

General form of the Euler equation beyond the self-consistent field approximation, which arises at derivation [104], is

\[
mn(\partial_t + \mathbf{v} \cdot \nabla)v + \nabla p - \frac{\hbar^2}{4m} \nabla \left( \frac{\Delta n}{n} - \frac{(\nabla n)^2}{2n^2} \right) = -\Sigma + P^\beta \nabla \Pi^\beta + \int d\mathbf{r}' (\nabla G^\gamma)(\mathbf{r} - \mathbf{r}') P^\gamma_2(\mathbf{r}, \mathbf{r}', t),
\]

(17)

where the first term on the right-hand side presents the general form of the short-range interaction \(\Sigma = \partial^\alpha \sigma^{\alpha\beta} \), with the quantum stress tensor \(\sigma^{\alpha\beta} \) [96], [107], the last term on the right-hand side describes the dipole-dipole interaction. The quantum stress tensor appears as an expansion in the series on the interaction radius of the short-range interaction \(\sigma^{\alpha\beta} \) [96]. In systems of bosons, being in the Bose-Einstein condensates state, a contribution of the short range interaction arises in the first order by the interaction radius, if we consider spherically symmetric short-range potential. It leads to the Gross-Pitaevskii approximation [96]. Generalization of the Gross-Pitaevskii model in the third order by the interaction radius can be found in Ref. [96]. Influence of the short interaction considered up to the third order by the interaction radius on spectrum of collective excitations in dipolar Bose-Einstein condensates and dipolar fermions was considered in Refs. [49] and [42] correspondingly. If we consider fermions then the contribution of the short-range interaction equals to zero due to the antisymmetry of the wave function of fermions. However, a non-zero contribution was derived in the third order by the interaction radius [96]. Including the dependence on the spin polarisation, the result of Ref. [96] can be presented as

\[
\sigma^{\alpha\beta} = \frac{4\pi \hbar^2}{m^4} \sum \delta^{\alpha\beta} \delta \delta^3 \left( \frac{3}{2} \frac{1}{m} n^2 + \delta^{\alpha\beta} n \bar{T} \gamma \gamma + 2n \bar{T} \gamma \beta \right),
\]

where \(\bar{T} \gamma \beta = -\frac{1}{4m} (\partial^\alpha \partial^\beta n + \frac{1}{n} (\partial^\alpha n) (\partial^\beta n))\), and \(\bar{\delta} = \frac{4\pi}{m} \int d\mathbf{r}' (\partial^\alpha \partial^\beta \phi)(\mathbf{r})\).

This is the explicit definition of the two-particle concentration in the expression of the many-particle wave function occurring at derivation of the Euler equation

\[
P_{2}^{\alpha\beta}(\mathbf{r}, \mathbf{r}', t) = \int d\mathbf{r}_N \sum_{i,j \neq i} \delta_{i}(\mathbf{r} - \mathbf{r}_i) \delta_{j}(\mathbf{r}' - \mathbf{r}_j) \times
\]

\[
\times d^3 \psi^*(\mathbf{r}, t) \psi(\mathbf{r}, t).
\]

(18)

In this paper we focus our attention on dipolar fermions with aligned dipoles. Consequently polarisation of the system reduces to the concentration

\[
P = d\mathbf{n}.
\]

(19)

with \(\mathbf{l}\) direction of all dipoles parallel to the external electric field. Let us choose \(\mathbf{l} = \mathbf{e}_z\).

For the aligned dipoles the two-particle polarisation can be simplified as

\[
P_{2}^{\alpha\beta}(\mathbf{r}, \mathbf{r}', t) = d^2 \delta^{\alpha z} \delta^{\beta z} n_{2}(\mathbf{r}, \mathbf{r}', t),
\]

(20)

where the two-particle concentration appears to be

\[
n_{2}(\mathbf{r}, \mathbf{r}', t) = \int d\mathbf{r}_N \sum_{i,j \neq i} \delta_{i}(\mathbf{r} - \mathbf{r}_i) \delta_{j}(\mathbf{r}' - \mathbf{r}_j) \psi^*(\mathbf{R}, t) \psi(\mathbf{R}, t).
\]

(21)

General form of the two-particle concentration for fermions was obtained in Refs. [94], [96]

\[
n_{2}(\mathbf{r}, \mathbf{r}', t) = n(\mathbf{r}) n(\mathbf{r}', t) - |\rho(\mathbf{r}, \mathbf{r}', t)|^2,
\]

(22)

where,

\[
\rho(\mathbf{r}, \mathbf{r}', t) = \sum_{g} n_{g} \varphi_2^g(\mathbf{r}, t) \varphi_g(\mathbf{r}, t),
\]

(23)

and

\[
\varphi_g(\mathbf{r}, t) = \sum_{g} n_{g} \varphi_2^g(\mathbf{r}, t) \varphi_g(\mathbf{r}', t),
\]

(24)

with \(\varphi_g(\mathbf{r}, t)\) are the arbitrary single-particle wave functions.

Exchange electric dipole-dipole interaction in system of spin-1/2 fully spin polarised degenerate fermions of align electric dipoles is

\[
F_{SHF,3D,Exc} = \frac{8\pi}{3} d^2 n \nabla n.
\]

(25)

Formula (25) introduce an attractive interaction.

Including of the partial spin polarisation degenerate fermions of align electric dipoles gives

\[
F_{SHF,3D,Exc} = \frac{8\pi}{3} d^2 n \nabla n,
\]

(26)

where

\[
\eta = \frac{|n_+ - n_-|}{n_+ + n_-}.
\]

(27)

Let us present the explicit form of the Euler equation for 3D degenerate fully polarised dipolar fermions with the exchange dipole-dipole interaction

\[
\begin{align*}
    mn(\partial_t + \mathbf{v} \cdot \nabla)v + \sqrt{2} \nabla p_{Fe} - \frac{\hbar^2}{4m} n \nabla \left( \frac{\Delta n}{n} - \frac{(\nabla n)^2}{2n^2} \right) & = P^\beta \nabla \Pi^\beta + \frac{8\pi}{3} d^2 n \nabla n.
\end{align*}
\]

(28)
FIG. 1: (Color online) The figure shows the dimensionless sound velocity $U = U_{s,3D}/v_{Fe,3D}$ as the function of the electric dipole moment $d$, in Debay (D) units, and the angle $\theta$ between the direction of the external field and the direction of wave propagation at $n_0 = 10^{15}\text{cm}^{-3}$ and mass of particles $m = 127$ amu (the atomic mass unit).

Corresponding non-linear Schrödinger equation arises as

\[ i\hbar \partial_t \Phi(r, t) = \left( -\frac{\hbar^2}{2m} \Delta + \sqrt{4\pi^2} \frac{2}{3} \hbar^2 n_0^{2/3} \right) \Phi(r, t) - d \cdot E - \frac{8\pi}{3} d^2 n \Phi(r, t). \]  

(29)

Formula (29) presents the NLSE for the fully spin polarised fermions with the aligned electric dipoles with the account of the exchange dipole-dipole interaction. The exchange interaction is presented by the last term on the right-hand side of the NLSE.

IV. COLLECTIVE EXCITATIONS IN 3D UNLIMITED SAMPLE

Our calculation gives following spectrum of collective excitations

\[ \omega^2 = \frac{4\pi n_0 d^2 k^2}{m} \cos^2 \theta - \eta \frac{8\pi n_0 d^2 k^2}{3m} + \varrho_{3D} X_K \frac{(3\pi^2)^{2/3} \hbar^2 n_0^{2/3}}{3 m^2} k^2 + \frac{\hbar^2 k^4}{4m^2}, \]  

(30)

where $\cos \theta = k_z/k$.

At small wave vectors $k$ and the full spin polarisation the frequency can be written as

\[ \omega = \sqrt{\chi K \frac{(6\pi^2)^{2/3} \hbar^2 n_0^{2/3}}{3m^2} + \frac{4\pi n_0 d^2}{m} \left( \cos^2 \theta - \frac{2}{3} \right) k}. \]  

(31)

Without the dipole-dipole interaction formula (31) gives $\omega = U_{s,3D}k$, where $U_{s,3D}$ is the sound velocity of ideal Fermi gas caused by the Fermi pressure: $U_{s,3D} = \sqrt{\chi K \frac{4}{3}} v_{Fe,3D}$, with the traditional Fermi velocity $v_{Fe,3D} = (3\pi^2 n_0^{1/3})^{1/3} \hbar/m$.

For numerical analysis of the spectrum (31) we apply a fixed value of the equilibrium concentration $n_0 = 10^{15}\text{cm}^{-3}$ and mass of particles $m = 127$ amu. Fig. (1) shows spectrum (31), or the rate of the sound velocity to the Fermi velocity, at the small electric dipole moments $d < 1$ Debay (D). Fig. (2) shows the arising of the instability at the larger electric dipole moments. The area of instability is shown by the clipping of the surface.

Change of spectrum at the increase of the mass of particles from $m = 127$ amu (the green surface) to $m = 200$ amu (the blue surface) at $n_0 = 10^{15}\text{cm}^{-3}$.

FIG. 2: (Color online) The figure shows the appearance of instability for the dipolar fermions possessing the electric dipole moment more than 2 Debay at $n_0 = 10^{15}\text{cm}^{-3}$ and mass of particles $m = 127$ amu. Area of instability faster increases at $d \in (2, 4)$ Debay reaching angle $\theta_0 \approx 0.61$ radian.

FIG. 3: (Color online) The figure shows behavior of the sound velocity $U_{s,3D}(d, \theta)$ for particles of different mass ($m = 127$ amu (the green surface) and $m = 200$ amu (the blue surface)) at $n_0 = 10^{15}\text{cm}^{-3}$.
are perpendicular to the plane of confinement (parallel
So, two last terms have similar form. Combining the two
where \( \Delta \)
the electric dipole interaction \( 15 \)
Next we use the explicit form of the Green function of
we apply the NLSE for fully spin-polarised fermions pos-
were considered in Ref. \[69\].
In this section we follow Refs. \[69\], \[108\], where
We have considered homogeneous three dimensional
systems of dipolar degenerate fermions. In this section we
consider fermions being in a trap, we choose to consider
quasi-two dimensional trap with the strong confinement
along z-direction.
In this section we follow Refs. \[69\], \[108\], where
method of consideration of quasi two-dimensional clouds
of ultracold gases. Quasi two-dimensional dipolar BECs
were considered in Ref. \[69\].
Before we perform transformation of the NLSE in the
form corresponding to the quasi-two-dimensional trap, let
us represent the NLSE in the suitable form. Here
we apply the NLSE for fully spin-polarised fermions pos-
sessing the electric dipole moment, which includes the
exchange dipole-dipole interaction \( 20 \). First we rewrite
it in an integral form
\[
\hbar \partial_t \Phi(r, t) = \left(-\frac{\hbar^2}{2m} \Delta + \frac{(6\pi^2)^{2/3} \hbar^2 n^{2/3}}{2m} \right) \\
- d^3 d^3 \int d^3 G^3 \gamma(r, r') n(r', t) - \frac{8\pi}{3} d^2 n \Phi(r, t).
\]
Next we use the explicit form of the Green function of
the electric dipole interaction \( 15 \)
\[
\hbar \partial_t \Phi(r, t) = \left(-\frac{\hbar^2}{2m} \Delta + \frac{(6\pi^2)^{2/3} \hbar^2 n^{2/3}}{2m} \right) \\
+ d^3 d^3 \int d^3 G^3 \gamma(r, r') n(r', t) - \frac{8\pi}{3} d^2 n \Phi(r, t),
\]
where \( \Delta r = r - r' \), \( \Delta r = |\Delta r| \), and we have taken integral
with the term proportional to the Dirac delta function.
So, two last terms have similar form. Combining the two
last terms together and assuming that all electric dipoles
are perpendicular to the plane of confinement (parallel
to the z-direction) we find
\[
\hbar \partial_t \Phi(r, t) = \left(-\frac{\hbar^2}{2m} \Delta + \frac{(6\pi^2)^{2/3} \hbar^2 n^{2/3}}{2m} \right) \\
+ d^2 \int d^3 r' \frac{1 - 3 \cos^2 \psi r n(r', t) - \frac{4\pi}{3} d^2 n}{} \Phi(r, t),
\]
where \( \psi r \) is the angle between \( \Delta r \) and the z-direction.
The last term in equation \( 33 \) can be represented via
effective interaction constant \( \Delta r f f = -4\pi d^2/3 \).
Now we are going to find the NLSE for quasi-two-
dimensional electric dipolar fermions. We ready to apply
results of Ref. \[69\] to reach our goal.
}
we can use more simple procedure to perform the transition of the Hamiltonian to the quasi two-dimensional regime. We can simply integrate over the confined direction

\[
H_{SR1} = \frac{1}{2} \int dr g_{3D} n^2(r)
\]

\[
= \frac{1}{2} g_{3D} \frac{1}{\pi a_z^2} \int dr_2 dz e^{-\frac{r_2^2}{\pi a_z^2}} n^2(x, y)
\]

\[
= \frac{1}{2} g_{2D} \int dr_2 n^2(x, y),
\]

where \(dr_2 = dx dy\), and \(g_{2D} = \frac{9D}{2\pi a_z}\).

Corresponding NLSE arises as

\[
\iu \hbar \partial_t \Phi(x, y, t) = \left( -\frac{\hbar^2}{2m} \Delta_{2D} + g_{2D} n \right) \Phi(x, y, t).
\]

Similarly we can consider the Fermi pressure. Corresponding Hamiltonian and following calculations are

\[
H_F = \frac{3}{5} \int dr C_{3D} n^{5/3}(r)
\]

\[
= \frac{3}{5} C_{3D} \int dr_2 dzn^{5/3}(x, y) \left( \frac{1}{\sqrt{\pi a_z^2}} e^{-\frac{r_2^2}{\pi a_z^2}} \right)^{5/3}
\]

\[
= \frac{3}{5} C_{2D} \int dr_2 n^{5/3}(x, y),
\]

where \(C_{3D} = \vartheta_{3D}(3\pi^2/2)\xi/3\hbar^2/(2m)\), and \(C_{2D} = \sqrt{\frac{2}{3}} (\vartheta_{2D}/\vartheta_{3D}) \xi/3\hbar^2 \sqrt{\vartheta_{3D} \xi/2}\).

Corresponding the NLSE, for fermions with the Fermi pressure in absence of the interaction, in quasi-two-dimensional case appears as

\[
\iu \hbar \partial_t \Phi(x, y, t) = \left( -\frac{\hbar^2}{2m} \Delta_{2D} \right)
\]

\[
+ \vartheta_{3D} \sqrt{\frac{3}{5}} \frac{9^{1/3} \pi^2 \hbar^2}{2ma_z^2} \Phi(x, y, t)
\]

(46)

Quasi-two-dimensional Fermi pressure appears as

\[
p_{F\epsilon(q2D)} = \vartheta_{3D} \xi K \sqrt{\frac{3}{5}} \frac{3^{2/3} \pi^2 \hbar^2}{5ma_z^2} n_{q2D}^{5/3}
\]

(47)

Correspondingly, for the quasi-two-dimensional sound velocity of ideal fermi gas we have

\[
U_{s(q2D)} = \sqrt{\vartheta_{3D} \xi K \frac{\sqrt{3} \sqrt{\pi}}{\sqrt{15}} \frac{\hbar}{ma_z} n_{q2D,0}^{1/3}}
\]

(48)
C. Spectrum of collective excitations in quasi two-dimensional dipolar Fermi gas

Dipolar part of the spectrum of the linear collective excitations is proportional to $V_{tot}^2D(k_2D a_z)$.

The full spectrum also contains the contribution of the Fermi pressure and the quantum Bohm potential.

For small $ξ$, function $w(ξ/√2)$ behaves like $w(ξ/√2) = 1 − 2/πξ + O(ξ^2)$.

Spectrum in the small wave vector regime, containing terms up to $k_2D^3$ can be presented as follows

$$\omega^2 = \frac{\pi \sqrt{π n(0)} d^2}{m} k_2D^2 \left[-2k_2D a_z + 2\sqrt{\frac{3}{π}} k_2D^2 a_z^2\right]$$

$$+ U_{s(q2D)}^2 k_2D^4 + \frac{\hbar^2 k_2D^4}{4m^2} \tag{49}$$

The first group of terms in formula (49) describes the contribution of the dipole-dipole interaction in the small wave vector limit. It consists of two parts. The first of them is negative and proportional to $k^3$. Another term is positive and proportional to $k^4$ as the quantum Bohm potential presented by the last term in formula (49). The second term in formula (49) presents the Fermi pressure, and it is proportional to $k^2$.

We see that the Fermi pressure leads to the sound like spectrum at the small $k$. Moreover, the account of the exchange part of the electric dipole-dipole interaction in the fully spin polarised spin-1/2 fermions reveals in the fact that the dipole-dipole interaction does not show sound like spectrum. The contribution of the dipole-dipole interaction is proportional to the third degree of the module of wave vector and to higher orders of the module of wave vector.

Parameter $R$ introduced in Ref. 69, in our case, has the magnitude of

$$R = \frac{\sqrt{π/2}}{1 + 2gd/9g} = 2.51, \tag{50}$$

where $g_{3D} = g_{eff} = 4\pi d^2/3$, $g_d = 4\pi d^2/3$.

VI. TWO-DIMENSIONAL DIPOLAR FERMI GAS

In the previous section we have considered the quasi two-dimensional dipolar fermions, where we have explicitly considered confinement of particles by the harmonic trap. At consideration of the reduced potential of the dipole-dipole interaction we have followed Ref. 69, where the quasi two-dimensional dipolar Bose-Einstein condensates were considered for the first time. Analysis of quasi two-dimensional dipolar fermions was performed in Ref. 54. Two dimensional dipolar BECs were recently considered in Refs. 105 and 110, where authors apply the purely two-dimensional Fourier image for ultrathin plane of dipolar fermions with no trace of the delta function term in the potential of electric dipole interaction. Similarly to earlier works on three-dimensional dipolar Bose-Einstein condensates [53, 57, 58]. Purely two-dimensional models are widely spread in the condensed matter physics at description of the two-dimensional electron gas (2DEG) (see for instance 111–113).

In previous section we have considered three dimensional dipolar fermions in traps with the strong confinement in the z-direction and explicit account of the trapping in the z-direction. It have given us the quasi-two dimensional distribution of particles. However, there is an approach to consider ultrathin plane-like structures as purely two dimensional objects in the three dimensional space. Let us apply this approach to derive a model of the two-dimensional dipolar fermions with the exchange electric dipole-dipole interaction.

We can start our analysis with the Schrodinger equation in the plane-like two dimensional layer. As for three dimensional case we give definition of the two-dimensional concentration of particles

$$n_{2D}(r,t) = \int dR_{2N} \sum_i \delta(r_{2D} - r_{2D,i}) \psi^*(R_{2N},t) \psi(R_{2N},t), \tag{51}$$

where $[n_{2D}] = cm^{-2}$, $r_{2D} = \{x,y\}$ is the coordinate vector in the two dimensional physical space, $R_{2N} = \{r_{2D,1}, ..., r_{2D,i}, ..., r_{2D,N}\}$ set of all coordinates in 2N configurational space with $r_{2D,i} = \{x_i, y_i\}$, and $dR_{2N} = 2i d^2r_{2D,i}$ is the element of configurational space.

Definition 51 allows to derive the continuity equation and the Euler equation:

$$\partial_t n_{2D} + \nabla \cdot (n_{2D} \mathbf{v}) = 0, \tag{52}$$

and

$$mn_{2D}(\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p_{2D} - \frac{\hbar^2}{4m} n_{2D} \nabla \left(\frac{\Delta n_{2D}}{n_{2D}} - \frac{\nabla n_{2D}^2}{2n_{2D}^2}\right)$$

$$= P_{2D}^\beta \nabla P_{ext}^\beta + P_{2D}^\beta \nabla \int d\mathbf{r}'_2D G^{\beta\gamma}(\mathbf{r}, \mathbf{r}') P_{2D}^\gamma(\mathbf{r}', t)$$

$$+ \zeta_{2D} \sqrt{2\pi d^2} I_2 \nabla n_{2D}^\beta, \tag{53}$$

where $I_2 = 8.045$, in this section, $\Delta = \nabla^2_{2D} = i\partial_x + j\partial_y$, $\nabla_{2D} = \partial_x^2 + \partial_y^2$, $\mathbf{v} = \nabla_{2D} = \{v_x, v_y\}$, $\mathbf{P} = \{P_x, P_y, P_z\}$, $P_z$ is the projection of the polarisation on the direction perpendicular to the plane, with $\beta = \gamma = z$ in equation (53), $p_{2D}$ is the two dimensional pressure of degenerate partially spin polarised spin-1/2 fermions, its explicit form is

$$p_{2D} = p_{2D}^\beta = (1 + \eta^2)\pi h^2 n_{a,2D}^2 / (2m_0). \tag{54}$$
parameter $\zeta_{2D}$ describes dependence of the exchange electric dipole-dipole interaction on the equilibrium spin condition:

$$\zeta_{2D} = (1 + \eta)^{5/2} - (1 - \eta)^{5/2},$$  \hspace{1cm} (55)

and

$$G^{zz} = -\frac{2}{3} \frac{1}{\xi^3},$$  \hspace{1cm} (56)

is an explicit form of the $zz$ matrix element of the Green function of the electric dipole interaction, where we have applied formula (16) assuming that the two dimensional layer of fermions is located in plane $z = 0$. Next we can use $\Delta_{2D} \frac{1}{\xi} = \frac{1}{\xi}$ to obtain the final form of $G^{zz}$ for two dimensional systems

$$G^{zz} = -\frac{2}{3} \frac{1}{\xi^3}. \hspace{1cm} (57)$$

Applying formula (57) in the Euler equation we can calculate spectrum of the collective excitations of two dimensional dipolar fermions.

Three dimensional dipolar quantum gases have one preferable direction $E_{ext}$, while two dimensional dipolar quantum gases have two preferable directions $E_{ext}$ and $n$, where $n$ is the perpendicular to the plane. In this paper we do not use this degree of freedom and assume $E_{ext} \parallel n$. We focus our attention on the electric field perpendicular to the sample, so we have $P_{2D} = n_{2D}dI$, with $l \parallel e_2$.

The spectrum of collective excitations propagating as the plane waves in the two dimensional structure of dipolar ultracold fermions appears as follows

$$\omega^2 = \frac{4\pi d^2 n_0 k^3}{3m} - \zeta_{2D} \frac{5}{2} \sqrt{2\pi I_2} \frac{d^2 n_{2D} k^2}{m}$$

$$+ \vartheta_{2D} \frac{\pi \hbar^2 n_0 k^2}{m^2} + \frac{\hbar^2 k^4}{4m^2}. \hspace{1cm} (58)$$

As the three-dimensional and quasi two-dimensional spectrums, solution (58) consists of four parts. The first terms describes the contribution of the self-consistent field part of electric dipole-dipole interaction under assumption that the external electric field creating the equilibrium polarisation is perpendicular to the sample. The second term presents the contribution of the exchange part of the electric dipole-dipole interaction. Both of them are negative. The third term is the Fermi pressure contribution derived for system of particles moving inside the plane. The last term is the quantum Bohm potential.

Let us note that we have considered systems of two-dimensional aligned dipolar fermions with the exchange interaction. The spectrum of collective excitations of dipolar fermions with the account of the dipole direction evolution, with no account of the exchange dipole-dipole interaction, was obtained in Refs. 19, 20 for the three-dimensional systems, and 12 for the two-dimensional systems.

It is essential to note the differences between spectrum of the quasi two-dimensional and the two-dimensional dipolar fermions 19, 20. The quasi two-dimensional regime 19, 20 contains trace of the three dimensional equation of state, being proportional to $n_{2D}^{2/3}$, while the two-dimensional limit gives $\omega^2 \sim n_{2D}$. At the eddy-free motion $\omega_{2D} = \sqrt{2\pi I_2 n_{2D}}$, the quasi two-dimensional medium of dipolar fermions with the aligned electric dipoles the hydrodynamic equations (52), 53 can be rewritten in the form of the non-linear Schrodinger equation

$$ih\partial_t \Phi_{2D}(r, t) = \left( -\frac{\hbar^2}{2m} \Delta_{2D} + \vartheta_{2D} \frac{\pi \hbar^2}{m} n - dE_{ext}$$

$$-d^2 \int d^2 r' G_{2D}(r, r') n_{2D}(r', t) \zeta_{2D} \frac{5}{3} \sqrt{2\pi I_2 n_{2D}} \right) \Phi_{2D}(r, t). \hspace{1cm} (59)$$

The NLSE (59) is obtained for the effective macroscopic wave function $\Phi_{2D}(r, t)$ defined in terms of hydrodynamic variables: $\Phi_{2D}(r, t) = \sqrt{n_{2D}} \exp(i m_0 \phi / \hbar)$.

VII. DISCUSSIONS AND CONCLUSIONS

The quantum hydrodynamic method for ultracold dipolar fermions descriptions for the aligned dipoles has been developed. This method describes the electric dipole-dipole interaction as a long-range interaction. However, the self-consistent field approximation has not been applied in this paper. We have explicitly considered the two-particle hydrodynamic concentration appearing in the Euler equation in the force field of dipole-dipole interaction. We have considered this function in the limit of weak interparticle interaction. At approximate calculation of the two-particle concentration we have explicitly considered antisymmetric N-particle wave function as the Slater determinant. The final result for the force field arises as the sum of two parts: the self-consistent field part and the exchange part. Analysis of the force field shows that the exchange interaction can reach same magnitude as the self-consistent field part. However the exchange part of the force field strongly depend on distribution of fermions over quantum states.

Spin-1/2 fermions at temperatures considerably less than the Fermi temperature may occupy one of two spin states. In this case we can put one fermion in each quantum state in the momentum space, hence fermions occupy all states with the momentum smaller than $\sqrt{2} \nu_{F,e}$, where $\nu_{F,e} = \nu_{F,e,3D} = (3\pi^2 n_0)^{1/3} \hbar$ is the Fermi momentum, with the equilibrium particle concentration $n_0$ (or, in the case of two dimensional plane-like distribution of particles, they occupy all states with the momentum smaller than $\sqrt{2} \nu_{F,e,2D}$, where $\nu_{F,e,2D} = \sqrt{2\pi n_{0,2D}} \hbar$
the 2D Fermi momentum). In this case we have the fully spin polarised systems of spin-1/2 fermions possessing the electric dipole moments.

The opposite limit case, when pairs of fermions with the opposite spins occupy each quantum state in the momentum space. In this case fermions occupy all states with the momentum smaller than $p_{F,\perp 3D}$ (or $p_{F,\perp 2D}$ for plane like structure). In this case we have no spin polarisation. The exchange interaction force monotonically depends on the spin polarisation. Therefore, the exchange force field equals to zero at the zero spin polarisation. While it has maximum value at the full spin polarisation.

Considering spin-1/2 atoms and molecules, when all of them located in the same state of the fine structure, we have rather large contribution of the exchange electric dipole-dipole interaction in the Euler equation. This contribution is the same order as the self-consistent field part. Since they have opposite signs (repulsing self-consistent field part and attractive exchange part), they can cancel each other at some conditions.

Spectrum of the collective excitations of ultracold electric dipolar fermions contains the positive anisotropic contribution of the self-consistent field part of the dipole-dipole interaction ($\sim \cos^2 \theta$), and the isotropic negative contribution of the exchange part of the dipole-dipole interaction. Together they give the term proportional \( \cos^2 \theta - 2/3 \). If the dipole-dipole interaction dominate over the short-range interaction, then the obtained spectrum shows the famous roton instability.

Usually authors obtain the roton instability in dipolar BECs and dipolar fermions in the self-consistent field approximation due to consideration of the reduced potential energy of the dipole-dipole interaction, instead of the full potential considered in this paper. Here, the roton instability arises at another angle (we have \( \cos^2 \theta - 2/3 \) instead of \( \cos^2 \theta - 1/3 \)), and it has another physical mechanism, i.e. the exchange part of the electric dipole-dipole interaction. We should note that similar approximation for the dipolar BECs does not lead to any instability (see Ref. [93]).

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