Majorana - entanglement relation in topological quantum computation

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Weyl and Majorana bispinor solutions to the four-dimensional massless Dirac equation are considered and shown to be unitary equivalent. It is shown that Weyl bispinors are algebraically equivalent to two-qubit direct product states, and that the massless Majorana bispinors are algebraically equivalent to maximally entangled states (Bell states), with the transformations relating the two bispinors types acting as entangling gates in quantum computation. A new set of entangling gates is presented which fulfills the required properties for Majorana zero mode operators in a topological quantum computation setting. Based on this set a toy model with four Majorana operators is presented that admits entanglement of two logical qubits from braiding.

Keywords: Entanglement, braiding, Majorana zero modes, massless Majorana bispinors, Weyl bispinors.

I. INTRODUCTION

A Majorana fermion is a spin 1/2 particle that is its own antiparticle. They were first proposed in 1937 by E. Majorana[1] in the context of particle physics. As an elementary particle, the only fundamental candidate for a Majorana fermion is the massive neutrino. It could also be a Dirac particle, although the Majorana alternative is theoretically preferred[2, 3]. The experimental verification of the Majorana nature of the neutrino, through the observation of neutrinoless double beta decay processes, is still an open question.

Majorana fermions arise also in condensed matter systems[4–7]. Here they are not elementary particles and, in principle, not at all related to the ones in particle physics, but rather localized zero-energy bound states (Bogoliubov quasiparticles) of electrons and holes, better known as Majorana zero modes[8] (MZMs). In this case the Majorana condition is satisfied through the use of Hermitian operators to describe MZMs. The composite objects consisting of Majorana bound states coupled to topological defects, such as vortices, obey non-Abelian statistics and are known as Ising anyons[9, 10], which constitute a particular type of non-Abelian anyons. Examples of 2-d systems admitting Ising anyons are the \( \nu = 5/2 \) fractional quantum Hall state[9, 11], \( p + ip \) superconductors[12, 13], and the surface of topological insulators[14], among others.

The interest in Ising anyons, from the perspective of quantum computation, is because they provide a means for fault-tolerant quantum computation[7, 15–17]. In a system with localized anyons quantum information can be stored non-locally in pairs, or in general \( n \)-tuplets, with \( n \) even, of anyons. Computations are performed by adiabatically braiding the anyons worldlines. These braiding operations constitute the logical quantum gates acting on the states and, up to a phase, depend only on the topology of the trajectories, in turn classified by the braid group. A topological quantum computation (TQC) model is specified[10] by providing the Hilbert space, the initial state, the braid operators and the measurable observables.

It has been shown that the operators representing the MZMs can be given in terms of Dirac gamma matrices[11, 18, 19] and, in particular, in Refs. 18 and 19 it is shown that the Clifford algebra of the Majorana operators, for a 2-d system with four vortices, can be realized by elements of the 4-d spacetime Clifford algebra. This result opens up the question about the feasibility of employing four-component spinors (bispinors) to describe the relevant particle states, namely Weyl[20] and massless Majorana states.

In this paper we study massless Majorana bispinors, that is solutions to the 4-d massless Dirac equation satisfying the Majorana condition, in two different settings: relativistic quantum mechanics (RQM) and quantum computation (QC), providing an algebraic relation between the two. We show that massless Majorana and Weyl bispinors are unitary equivalent and that in QC they correspond, respectively, to maximally entangled and separable two-qubit states. The unitary transformations relating them constitute entangling gates, and we present a new set of these gates which are Hermitian, besides being unitary, and that satisfy a Clifford algebra relation. These characteristics allow for these matrices to be interpreted as Majorana operators in TQC. We present a toy model based on them, where we show that it is possible to obtain entanglement of two logical qubits solely from topological operations in a system with four MZMs. This is the main result.

The organization is as follows: In section II we obtain massless Majorana and Weyl bispinors and show that they are unitary equivalent. This fills a gap in the literature, where the known equivalence between massless Majorana and Weyl free field operators is shown to also hold for \( c \)-number spinors in RQM. In section III we establish an algebraic equivalence between Weyl bispinors and separable two-qubit states, and between maximally entan-
gled states (Bell states) and massless Majorana bispinors. We refer to the latter as the Majorana - entanglement relation[21]. Two types of entangling gates are discussed and we provide a set not previously found in the literature. In section IV we show that the new set of entangling gates fulfill all the requirements for MZMs operators. Based on these we construct a TQC toy model with four MZMs that admits entanglement from braiding, and we show that the Majorana - entanglement relation continues to hold. Finally, concluding remarks are given.

II. MASSLESS C-NUMBER BISPINORS

A. Weyl

Let us begin by considering four-component Weyl bispinors with four-momentum \( p^\mu = (\pm |p|, p) \), respectively for positive- and negative-energy \( p^0 = \pm E = \pm |p| \), which are solutions to the massless Dirac equation

\[
i \gamma^\mu \partial_\mu \Psi = 0. \tag{1}
\]

The gamma matrices \( \gamma^\mu = (\gamma^0, \gamma) \) obey the Clifford algebra relation

\[
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}, \tag{2}
\]

with \( g^{\mu\nu} \) the metric tensor with signature \( \text{diag}(1, -1, -1, -1) \), and the Weyl representation

\[
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}, \tag{3}
\]

with \( \sigma = (\sigma^1, \sigma^2, \sigma^3) \) the standard Pauli matrices will be used throughout. Using the plane waves

\[
\Psi = u(p) \exp \{i (\pm Et - \mathbf{x} \cdot \mathbf{p}) \}, \tag{4}
\]

and the matrices

\[
\gamma^5 \equiv i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Sigma \equiv \gamma^5 \gamma^0 \gamma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}, \tag{5}
\]

equation (1) is rewritten as

\[
\Sigma \cdot \hat{p} u(p) = \pm \gamma^5 u(p), \tag{6}
\]

with \( \hat{p} = p/|p| \). Thus, the bispinors \( u(p) \) are eigenvectors of both helicity \( \Sigma \cdot \hat{p} \) and chirality \( \gamma^5 \) operators, and Eq. (6) expresses the known result that chirality equals the helicity for massless, positive-energy bispinors, while it is opposite for negative-energy ones. Taking the direction of \( \mathbf{p} \) along \( \mathbf{z} \) (from now on called the canonical frame) in Eq. (6) one obtains the four independent solutions, with their eigenvalues given in Table 1.

| Energy | + | + | - | - |
|--------|---|---|---|---|
| Helicity | 1 | -1 | -1 | 1 |
| Chirality | 1 | -1 | 1 | -1 |

Table I. Eigenvalues of the canonical frame Weyl bispinors

\[
u^{(1)}(p_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad u^{(2)}(p_2) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},
\]

\[
u^{(3)}(p_2) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad u^{(4)}(p_2) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.
\]

To obtain solutions for general three-momentum we use spherical polar coordinates

\[
\hat{p} = \sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta, \tag{8}
\]

and the rotation

\[
\Lambda (\theta, \varphi) = \exp \left\{-\frac{\theta}{2} \left( \gamma^1 \cos \varphi + \gamma^2 \sin \varphi \right) \gamma^3 \right\}. \tag{9}
\]

Applying Eq. (9) to the bispinors in Eq. (7) we have

\[
u^{(i)}(p_2) = u^{(i)}(p), \quad i = 1, \ldots, 4, \tag{10}
\]

with the general momentum bispinors, in two-block notation, given by

\[
u^{(1)}(p) = \begin{pmatrix} 0 \\ \chi_+ (p) \end{pmatrix}, \quad u^{(2)}(p) = \begin{pmatrix} \chi_- (p) \\ 0 \end{pmatrix}, \tag{11}
\]

\[
u^{(3)}(p) = \begin{pmatrix} 0 \\ \chi_- (p) \end{pmatrix}, \quad u^{(4)}(p) = \begin{pmatrix} \chi_+ (p) \\ 0 \end{pmatrix},
\]

where \( \chi_{\pm} (p) \) are the two-component helicity eigenspinors

\[
\chi_+ (p) = \begin{pmatrix} \cos \left( \frac{\varphi}{2} \right) \\ e^{i \varphi} \sin \left( \frac{\varphi}{2} \right) \end{pmatrix}, \tag{12}
\]

\[
\chi_- (p) = \begin{pmatrix} -e^{-i \varphi} \sin \left( \frac{\varphi}{2} \right) \\ \cos \left( \frac{\varphi}{2} \right) \end{pmatrix},
\]

satisfying the equation
\[ \mathbf{\sigma} \cdot \hat{p} \chi_{\pm} (\mathbf{p}) = \pm \chi_{\pm} (\mathbf{p}). \]  

(13)

The bispinors in Eq. (11) are orthonormal

\[ u^{(1)} (\mathbf{p}) u^{(j)} (\mathbf{p}) = \delta_{ij}, \]  

(14)

with a normalization that is adequate for massless spinors, as the Dirac adjoint \( \Gamma \equiv u^\dagger \gamma^0 \) is not needed in this case. Another useful, Lorentz invariant normalization is to re-scale them to \( \sqrt{2E} \). These bispinors are also solutions to Eq. (6), which in Hamiltonian form reads

\[ i \alpha \cdot \hat{p} u^{(s)} (\mathbf{p}) = \pm u^{(s)} (\mathbf{p}), \]  

\[ \alpha \cdot \hat{p} u^{(s+2)} (\mathbf{p}) = - u^{(s+2)} (\mathbf{p}), \quad s = 1, 2 \]  

(15)

making explicit that \( u^{(1)} (\mathbf{p}) \) and \( u^{(2)} (\mathbf{p}) \) are positive-energy bispinors, while \( u^{(3)} (\mathbf{p}) \) and \( u^{(4)} (\mathbf{p}) \) are negative-energy ones. The helicity and chirality eigenvalues are the same as in Eq. (7).

B. Majorana

Using the canonical frame bispinors in Eq. (7) we define the following Majorana bispinors

\[ u^{(1)}_M (p_z) = \frac{1}{\sqrt{2}} \left( u^{(2)} (p_z) + i \gamma^2 u^{* (2)} (p_z) \right), \]  

(16)

\[ u^{(2)}_M (p_z) = \frac{1}{\sqrt{2}} \left( u^{(1)} (p_z) - i \gamma^2 u^{* (1)} (p_z) \right), \]

\[ u^{(3)}_M (p_z) = \frac{1}{\sqrt{2}} \left( u^{(3)} (p_z) - i \gamma^2 u^{* (3)} (p_z) \right), \]

\[ u^{(4)}_M (p_z) = \frac{1}{\sqrt{2}} \left( u^{(4)} (p_z) + i \gamma^2 u^{* (4)} (p_z) \right), \]

where the asterisk denotes complex conjugation, even though it is superfluous in this case because the \( u^{(3)} (p_z) \) are real. The bispinors in Eq. (16) are eigenstates of the standard charge conjugation operator\([22, 23]\]

\[ C \equiv CK \equiv i \gamma^2 K, \]  

(17)

where \( C = i \gamma^2 \) is the charge conjugation matrix, and \( K \) stands for the operation of complex conjugation to the right. We then have

\[ C u^{(1,4)}_M (p_z) = + u^{(1,4)}_M (p_z), \]

\[ C u^{(2,3)}_M (p_z) = - u^{(2,3)}_M (p_z), \]  

(18)

relating them to the Weyl bispinors in Eq. (7). Among several possibilities, to be discussed in the next section, we choose

\[ R_3 = \exp \left( \frac{i \pi}{4} \gamma^0 \gamma^3 \right), \]  

(19)

as the transformation matrix, which besides being unitary is also of unit determinant, therefore a rotation. Thus, we have the following equivalence between the bispinors in Eqs. (7) and (16)

\[ R_3 u^{(1)} (p_z) = - u^{(1)}_M (p_z), \quad R_3 u^{(2)} (p_z) = + u^{(2)}_M (p_z), \]

\[ R_3 u^{(3)} (p_z) = + u^{(4)}_M (p_z), \quad R_3 u^{(4)} (p_z) = - u^{(3)}_M (p_z). \]  

(20)

It is now straightforward to generalize this result for arbitrary momentum bispinors. Using the ones in Eq. (11) we obtain the generalization of Eq. (16)

\[ u^{(1)}_M (p) = \frac{1}{\sqrt{2}} \left( u^{(2)} (p) + i \gamma^2 u^{* (2)} (p) \right), \]

\[ u^{(2)}_M (p) = \frac{1}{\sqrt{2}} \left( u^{(1)} (p) - i \gamma^2 u^{* (1)} (p) \right), \]

\[ u^{(3)}_M (p) = \frac{1}{\sqrt{2}} \left( u^{(3)} (p) - i \gamma^2 u^{* (3)} (p) \right), \]

\[ u^{(4)}_M (p) = \frac{1}{\sqrt{2}} \left( u^{(4)} (p) + i \gamma^2 u^{* (4)} (p) \right). \]  

(21)

These Majorana bispinors are obtained from the canonical frame ones in Eq. (16) by the same rotation in Eq. (9)

\[ \Lambda (\theta, \varphi) u^{(i)}_M (p_z) = u^{(i)}_M (p), \quad i = 1, \ldots, 4. \]  

(22)

Then defining the rotation

\[ \Omega (\theta, \varphi) \equiv \Lambda (\theta, \varphi) R_3 \Lambda^\dagger (\theta, \varphi), \]  

(23)

equations (10) and (20) yield

\[ \Omega (\theta, \varphi) u^{(1)} (p) = - u^{(1)}_M (p), \quad \Omega (\theta, \varphi) u^{(2)} (p) = + u^{(2)}_M (p), \]

\[ \Omega (\theta, \varphi) u^{(3)} (p) = + u^{(4)}_M (p), \quad \Omega (\theta, \varphi) u^{(4)} (p) = - u^{(3)}_M (p). \]  

(24)

Observing that \( \Omega (\theta, \varphi) \) and \( \alpha \cdot \hat{p} \) commute, it is readily verified that the bispinors in Eq. (21) are solutions to the massless Dirac equation

\[ \alpha \cdot \hat{p} u^{(s)}_M (p) = + u^{(s)}_M (p), \quad s = 1, 2, \]  

(25)

They also satisfy the Majorana condition

\[ C u^{(1,4)}_M (p) = + u^{(1,4)} (p)_C, \quad C u^{(2,3)}_M (p) = - u^{(2,3)}_M (p)_C, \]  

\[ C u^{(1,4)}_M (p) = + u^{(1,4)} (p)_C, \quad C u^{(2,3)}_M (p) = - u^{(2,3)}_M (p)_C. \]  

(26)
Accordingly, Eq. (24) establishes an equivalence between Weyl and massless Majorana bispinors. This relation is the c-number analogue of the known equivalence between Weyl and massless Majorana field operators, related by a Pauli-Gursey transformation[24–26]. In this sense this result completes the equivalence between massless Majorana and Weyl fermions, which is now seen to hold for both quantum fields and c-number spinors.

III. MAJORANA - ENTANGLEMENT RELATION

A. Massless bispinors as bipartite qubits

Let us denote the computational basis states by |0⟩ and |1⟩. For spin-1/2 systems they can be chosen as the eigenstates of \( \sigma^3 \)

\[
|0⟩ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1⟩ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

(27)

In this basis, the helicity spinors in Eq. (12) are given by the general pure-state qubits

\[
|\chi_+⟩ = \cos \left( \frac{\theta}{2} \right) |0⟩ + e^{i\varphi} \sin \left( \frac{\theta}{2} \right) |1⟩,
\]

\[
|\chi_-⟩ = -e^{-i\varphi} \sin \left( \frac{\theta}{2} \right) |0⟩ + \cos \left( \frac{\theta}{2} \right) |1⟩,
\]

(28)

which are antipodal in the unit Bloch sphere representation[27, 28], with the three-momentum in Eq. (8) taken as the Bloch vector.

The computational basis for the space of two pure-state qubits is then given by the set \( \{|0⟩, |1⟩\} \otimes \{|0⟩, |1⟩\} \), whence, upon using Eq. (27) and the notation |00⟩ = |0⟩ ⊗ |0⟩ and so on, we obtain the explicit representation

\[
|00⟩ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |01⟩ = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |10⟩ = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad |11⟩ = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},
\]

(29)

and we see that the elements of the basis are just the canonical frame Weyl bispinors in Eq. (7), so we make the identification

\[
|00⟩ = u_M^{(1)}(p_z), \quad |01⟩ = u_M^{(2)}(p_z),
\]

\[
|10⟩ = u_M^{(3)}(p_z), \quad |11⟩ = u_M^{(4)}(p_z).
\]

(30)

Another basis for this space is provided by the Bell states, which are maximally entangled states

\[
|\Phi^+⟩ = \frac{1}{\sqrt{2}} \left( |00⟩ + |11⟩ \right),
\]

\[
|\Phi^−⟩ = \frac{1}{\sqrt{2}} \left( |00⟩ - |11⟩ \right),
\]

\[
|\Psi^+⟩ = \frac{1}{\sqrt{2}} \left( |01⟩ + |10⟩ \right),
\]

\[
|\Psi^−⟩ = \frac{1}{\sqrt{2}} \left( |01⟩ - |10⟩ \right).
\]

(31)

Using either of Eqs. (7) or (29), explicit representations of the Bell states, as well as the massless Majorana bispinors in Eq. (16), are directly obtained, and upon comparing the two sets we arrive at the interesting result that the Bell states are algebraically equivalent to the massless Majorana bispinors in the canonical frame

\[
u_M^{(1)}(p_z) = |\Psi^−⟩, \quad u_M^{(2)}(p_z) = |\Psi^+⟩,
\]

\[
u_M^{(3)}(p_z) = −|\Phi^−⟩, \quad u_M^{(4)}(p_z) = |\Phi^+⟩.
\]

(32)

This result is generalized to arbitrary momentum by defining the general-momentum Bell states

\[
|\Phi^+(p)⟩ = \frac{1}{\sqrt{2}} \left( u^{(4)}(p) + u^{(3)}(p) \right),
\]

\[
|\Phi^-(p)⟩ = \frac{1}{\sqrt{2}} \left( u^{(4)}(p) - u^{(3)}(p) \right),
\]

\[
|\Psi^+(p)⟩ = \frac{1}{\sqrt{2}} \left( u^{(2)}(p) + u^{(1)}(p) \right),
\]

\[
|\Psi^-(p)⟩ = \frac{1}{\sqrt{2}} \left( u^{(2)}(p) - u^{(1)}(p) \right),
\]

(33)

then, from Eqs. (11), (12), and (21) we get

\[
u_M^{(1)}(p) = |\Psi^-(p)⟩, \quad u_M^{(2)}(p) = |\Psi^+(p)⟩,
\]

\[
u_M^{(3)}(p) = −|\Phi^-(p)⟩, \quad u_M^{(4)}(p) = |\Phi^+(p)⟩.
\]

(34)

Thus, we conclude that for massless bispinors obeying the Dirac equation, the Majorana condition produces maximal entanglement.

B. Entangling gates

Operations on qubits are given by unitary quantum gates, and from Eqs. (20) and (32) we see that the rotation in Eq. (19) serves as a two-qubit gate that produces entanglement. For the two-qubit case it has been shown that entangling gates, together with suitable one-qubit gates, are universal for quantum computation[29]. The common procedure for producing entanglement for logical qubits is by a combination of a CNOT (controlled not) gate and a Hadamard gate[28]. We now present two
The gate

They have the interesting property of being solutions to these gates on the computational basis in Eq. (29). The action of the entangling gates in Eq. (37) on the computational basis in Eq. (29). The table is read so that the gates in the first column act on the basis states in the top first row and produce the given Bell state in the intersection.

They do not obey Eq. (36), but satisfy the Clifford algebra

a property not shared by the gates in Eq. (35). These matrices are all orthogonal to each other, as is verified with the inner product

hence, they are linearly independent. Using the 16 elements of the 4-d gamma matrices Clifford algebra it can be verified that no other matrix exists with these characteristics that fulfills Eq. (38) and also closes the algebra in Eq. (39). In this sense the set in Eq. (37) is complete. Their action on the computational basis is shown in Table 3.

Another set of entangling gates, denoted by \( \hat{R}_i \), \( i = 1, \ldots, 4 \), is given by the rotations,

\[
\begin{align*}
\hat{R}_1 &= i \frac{\sqrt{2}}{\sqrt{2}} (\mathbb{1} + \gamma^1), \\
\hat{R}_2 &= i \frac{\sqrt{2}}{\sqrt{2}} (\mathbb{1} + \gamma^2), \\
\hat{R}_3 &= i \frac{\sqrt{2}}{\sqrt{2}} (\mathbb{1} + \gamma^3), \\
\hat{R}_4 &= i \frac{\sqrt{2}}{\sqrt{2}} (\gamma^0 \gamma^2 \gamma^3 + i \gamma^5).
\end{align*}
\]

They are also Hermitian and therefore square to the identity matrix

\[ \hat{R}_i = \hat{R}_i^\dagger, \hat{R}_i^2 = \mathbb{1}, \ i = 1, \ldots, 4. \] (38)

| | [10] | [01] | [11] | [00] |
|---|---|---|---|---|
| \( R_1 \) | \( \Psi^+ \) | \( \Psi^- \) | \( \Phi^+ \) | \( \Phi^- \) |
| \( R_2 \) | \( \Psi^- \) | \( \Psi^+ \) | \( -\Phi^- \) | \( \Phi^+ \) |
| \( R_3 \) | \( \Psi^- \) | \( \Psi^+ \) | \( \Phi^+ \) | \( -\Phi^- \) |
| \( R_4 \) | \( \Psi^+ \) | \( \Psi^- \) | \( -\Phi^- \) | \( \Phi^+ \) |

Table II. Action of the entangling gates in Eq. (35) on the computational basis in Eq. (29). The table is read so that the gates in the first column act on the basis states in the top first row and produce the given Bell state in the intersection.

| | [10] | [01] | [11] | [00] |
|---|---|---|---|---|
| \( \hat{R}_1 \) | \( i \phi^+ - i \phi^- - i \phi^+ i \phi^- \) |
| \( \hat{R}_2 \) | \( \phi^+ \phi^- \phi^+ - \phi^- \) |
| \( \hat{R}_3 \) | \( \phi^+ \phi^- - \phi^- \phi^+ \) |
| \( \hat{R}_4 \) | \( \phi^- \phi^+ - \phi^+ \phi^- \) |

Table III. Action of the entangling gates in Eq. (37) on the computational basis in Eq. (29). The table is read so that the gates in the first column act on the basis states in the top first row and produce the given Bell state in the intersection.
To define braid operators we take the branch cuts of the Majorana zero modes, described by the Majorana operators, in the same direction, and order them in a way that exchanging $\hat{R}_i$ and $\hat{R}_{i+1}$ clockwise ensures that $\hat{R}_i$ crosses solely the branch cut of $\hat{R}_{i+1}$, with no other operator crossing any other branch cut. Then the local (nearest-neighbor) braid operators are given by

$$B_{12} = \exp \left( -\frac{\pi}{4} \hat{R}_1 \hat{R}_2 \right),$$
$$B_{23} = \exp \left( -\frac{\pi}{4} \hat{R}_2 \hat{R}_3 \right),$$
$$B_{34} = \exp \left( -\frac{\pi}{4} \hat{R}_3 \hat{R}_4 \right).$$

They are unitary by construction, and satisfy the required properties for braiding operators\[5, 10, 12\], namely the Yang-Baxter equations

$$B_{12}B_{23}B_{12} = B_{23}B_{12}B_{23},$$
$$B_{23}B_{34}B_{23} = B_{34}B_{23}B_{34},$$

and commutation relations

$$[B_{12}, B_{34}] = 0,$$
$$[B_{12}, B_{23}] = \hat{R}_1 \hat{R}_3,$$
$$[B_{23}, B_{34}] = \hat{R}_2 \hat{R}_4.$$

We also have the non-local braid operators

$$B_{13} = B_{23}B_{12}B_{23}^\dagger,$$
$$B_{14} = B_{34}B_{23}B_{12}B_{23}^\dagger B_{34}^\dagger,$$
$$B_{24} = B_{34}B_{23}B_{34}^\dagger.$$

The relevant result is that $B_{23}$ in Eq. (43) cannot be written as the tensor product of two $2 \times 2$ matrices, and therefore is an entangling gate. This also holds for all three operators in Eq. (47). $B_{12}$ and $B_{34}$, on the other hand, are separable

$$B_{12} = 1_2 \otimes R_x (\pi/2),$$
$$B_{34} = R_y (\pi/2) \otimes 1_2,$$

where $R_x (\pi/2)$ and $R_y (\pi/2)$ are the one-qubit gates (rotation matrices)

$$R_x (\pi/2) = \exp \left( \frac{\pi}{4} \sigma^x \right),$$
$$R_y (\pi/2) = \exp \left( \frac{\pi}{4} \sigma^y \right).$$

Thus, leaving out the identity, the braid gates of the model form the set

$$\{R_x (\pi/2), R_y (\pi/2), B_{23}\}.$$ (50)

Acting on the Majorana operators in Eq. (37), the braid operators in Eqs. (43) and (46) yield

$$B_{pq} \hat{R}_k B_{pq}^\dagger = \begin{cases} 
\hat{R}_k & \text{if } k \notin \{p, q\}, \\
-\hat{R}_p & \text{if } k = q.
\end{cases}$$ (51)

We also specify the observables $F_{pq}$

$$F_{pq} = -i \hat{R}_p \hat{R}_q, \quad p < q,$$ (52)

which are the fermion parity operators for the pair of Majoranas $pq$, and the total parity operator $Q$ (topological charge)

$$Q = F_{12}F_{34} = -\hat{R}_1 \hat{R}_3 \hat{R}_4 \hat{R}_4.$$ (53)

It can be verified that $Q$ commutes with all braid operators and observables, in compliance with the superselection rules for total topological charge conservation\[10\].

To complete the model a computational basis needs to be specified. We choose to fuse the anyons in the pairs 1,2 and 3,4, so we consider the fermionic operators

$$f_{12} = \frac{1}{2} \left( \hat{R}_1 + i \hat{R}_2 \right),$$
$$f_{34} = \frac{1}{2} \left( \hat{R}_3 + i \hat{R}_4 \right),$$

producing the elements

$$|00\rangle, \quad |10\rangle = f_{12}^\dagger |00\rangle,$$
$$|01\rangle = f_{34}^\dagger |00\rangle, \quad |11\rangle = f_{34}^\dagger f_{12}^\dagger |00\rangle,$$ (55)

where $|00\rangle$ is such that $f_{12} |00\rangle = f_{34} |00\rangle = 0$, and the over bar is used to distinguish them from the canonical states in Eq. (29). Explicitly
operator, produces the states
\[ |0\bar{0}⟩ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ -i \\ i \end{pmatrix}, \quad |\bar{1}\bar{1}⟩ = \frac{e^{-i\pi/2}}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ e^{-i\pi/2} \\ e^{-i\pi/2} \end{pmatrix}, \]
\[ |\bar{0}\bar{1}⟩ = \frac{i}{\sqrt{2}} \begin{pmatrix} e^{-i\pi/2} \\ 1 \\ 1 \\ -i \end{pmatrix}, \quad |\bar{1}\bar{1}⟩ = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ -i \\ 1 \\ 1 \end{pmatrix}. \] (56)

These states are separable as is readily checked. The first
digit in the kets corresponds to the occupation number of
the fermion operator \( f_{12} \), while the second digit to that
of the \( f_{34} \) operator. This is verified by acting on the basis
with the fermion parity operators in Eq. (52), giving
\[ F_{12} |0\bar{0}⟩ = |0\bar{0}⟩, \quad F_{12} |\bar{1}\bar{1}⟩ = |\bar{1}\bar{1}⟩, \quad F_{12} |\bar{0}\bar{1}⟩ = |\bar{0}\bar{1}⟩, \quad F_{12} |\bar{1}\bar{1}⟩ = |\bar{1}\bar{1}⟩, \] (57)
\[ F_{34} |0\bar{0}⟩ = |0\bar{0}⟩, \quad F_{34} |\bar{1}\bar{1}⟩ = |\bar{1}\bar{1}⟩, \quad F_{34} |\bar{0}\bar{1}⟩ = |\bar{0}\bar{1}⟩, \quad F_{34} |\bar{1}\bar{1}⟩ = |\bar{1}\bar{1}⟩, \] (58)
with the plus eigenvalue corresponding to the vacant slot
\( 0 \) and the minus sign to the occupied state \( 1 \). The total
parity operator gives
\[ Q |0\bar{0}⟩ = |0\bar{0}⟩, \quad Q |\bar{1}\bar{1}⟩ = |\bar{1}\bar{1}⟩, \quad Q |\bar{0}\bar{1}⟩ = |\bar{0}\bar{1}⟩, \quad Q |\bar{1}\bar{1}⟩ = |\bar{1}\bar{1}⟩. \] (59)

The model is now complete and the system can be
initiated in any pair of the basis states with the same
\( Q \) parity, due to total parity conservation. The last two
states in Eq. (59) correspond to the fusion rule \( \sigma × \sigma = \psi \),
while the first ones to \( \sigma × \sigma = \mathbf{1}_{\text{vac}} \) and \( \sigma × \sigma × \sigma × \sigma = \mathbf{1}_{\text{vac}} \),
respectively. Whatever the initial states are, braiding anyons two and three, with the \( B_{23} \)
operator, produces the states
\[ B_{23} |0\bar{0}⟩ = \frac{1}{\sqrt{2}} ( |0\bar{0}⟩ + i |\bar{1}\bar{1}⟩), \]
\[ B_{23} |\bar{0}\bar{1}⟩ = \frac{1}{\sqrt{2}} ( |\bar{0}\bar{1}⟩ - i |\bar{1}\bar{1}⟩), \]
\[ B_{23} |\bar{1}\bar{0}⟩ = \frac{1}{\sqrt{2}} ( -i |\bar{0}\bar{1}⟩ + |\bar{1}\bar{1}⟩), \]
\[ B_{23} |\bar{1}\bar{1}⟩ = \frac{1}{\sqrt{2}} ( i |0\bar{0}⟩ + |\bar{1}\bar{1}⟩), \] (60)
which conserve total parity and are maximally entan-
gled. The former is directly seen from Eq. (59),
while the latter can be established by their Schmidt
decomposition, e. g., for \( B_{23} |0\bar{0}⟩ \) we have \( B_{23} |0\bar{0}⟩ = \frac{1}{\sqrt{2}} ( |0⟩ \otimes |0⟩ + i |1⟩ \otimes |1⟩) \),
with \( |0⟩, |1⟩ \) given in Eq. (27).

Similar relations hold for the rest of the states in Eq.
(60). On the other hand, the braid operators \( B_{12} \) and
\( B_{34} \) produce the same state multiplied by a phase of the
type \( \exp (±i\pi/4) \) when acting on the basis in Eq. (56),
as expected from their Abelian nature expressed in the
first relation of Eq. (45). The states in Eq. (60) corre-
spond to the fusion rule \( \sigma × \psi = \sigma \). Finally, we also verify that these maximal entangled states satisfy the Majorana
condition
\[ i\gamma^2 (B_{23} |0\bar{0}⟩)^* = -iB_{23} |0\bar{0}⟩, \]
\[ i\gamma^2 (B_{23} |\bar{0}\bar{1}⟩)^* = -B_{23} |\bar{0}\bar{1}⟩, \]
\[ i\gamma^2 (B_{23} |\bar{1}\bar{1}⟩)^* = -iB_{23} |\bar{1}\bar{1}⟩, \] (61)
in accord with the Majorana-entanglement relation.

V. CONCLUDING REMARKS

We have shown that Weyl and massless Majorana
bispinors are related by unitary transformations and presented
an algebraic equivalence between the former and
two-qubit states, and the later and maximally entangled
states. The Majorana-entanglement relation, while interest-
ing on its own, provides yet another criterion for maximal
entanglement of two logical qubits. It also implies that an entangled state of spin one-half states cannot
carry any \( U(1) \) charge.

The unitary transformations connecting the two
bispinors types play the role of two-qubit entangling gates
as they are used in quantum computation. Some of the
matrices in Eq. (35) have been studied by Kauffman
et al[34] in connection with knot theory and topological
linking. The matrices in Eq. (37), on the other hand,
constitute a new set of entangling gates which also
serve as Majorana operators in TQC. Being entangling,
a feature not previously considered for Majorana opera-
tors, they permit entanglement of two logical qubits from
topological operations in a system with four quasiparti-
cles, contrary to what was previously thought[10], as is
shown in the TQC toy model presented.

The use of relativistic spinors and the Clifford algebra
of the Dirac gamma matrices in quantum computation
have the potential of providing physical insight as well as
facilitating calculations[19, 35]. As an outlook, it would
be interesting to study the the Majorana operators in the
context of quantum field theory, since in that setting
operators that are both unitary and Hermitian act
as symmetry transformations that are also observables
and they are associated with the discrete symmetries of
parity, time reversal and charge conjugation. Also, it would be interesting to determine whether or not the Majorana-entanglement relation remains valid for more than two-qubit states.

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