Online Bin Covering with Advice*

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Abstract. The bin covering problem asks for covering a maximum number of bins with an online sequence of \( n \) items of different sizes in the range \((0,1]\); a bin is said to be covered if it receives items of total size at least 1. We study this problem in the advice setting and provide tight bounds for the size of advice required to achieve optimal solutions. Moreover, we show that any algorithm with advice of size \( o(\log \log n) \) has a competitive ratio of at most 0.5. In other words, advice of size \( o(\log \log n) \) is useless for improving the competitive ratio of 0.5, attainable by an online algorithm without advice. This result highlights a difference between the bin covering and the bin packing problems in the advice model: for the bin packing problem, there are several algorithms with advice of constant size that outperform online algorithms without advice. Furthermore, we show that advice of size \( O(\log \log n) \) is sufficient to achieve a competitive ratio that is arbitrarily close to 0.533 and hence strictly better than the best ratio 0.5 attainable by purely online algorithms. The technicalities involved in introducing and analyzing this algorithm are quite different from the existing results for the bin packing problem and confirm the different nature of these two problems. Finally, we show that a linear number of bits of advice is necessary to achieve any competitive ratio better than \( \frac{15}{16} \) for the online bin covering problem.

1 Introduction

In the bin covering problem [3], the input is a multi-set of items of different sizes in the range \((0,1]\) which need to be placed into a set of bins. A bin is said to be covered if the total size of items in it is at least 1. The goal of the bin covering problem is to place items into bins so that a maximum number of bins is covered. In the online setting, items form a sequence which is revealed in a piece-by-piece manner; that is, at each given time, one item of the sequence is revealed and an online algorithm has to place the item into a bin without any information about the forthcoming items. The decisions of the algorithm are irrevocable.

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Bin covering is closely related to the classic bin packing problem and is sometimes called the dual bin packing problem. The input to both problems is the same. In the bin packing problem, however, the goal is to place items into a minimum number of bins so that the total size of items in each bin is at most 1. Online algorithms for bin packing can naturally be extended to bin covering. For example, Next-Fit is a bin packing algorithm which keeps one “open” bin at any time: To place an incoming item $x$, if the size of $x$ is smaller than the remaining capacity of the open bin, $x$ is placed in the open bin; otherwise, the bin is closed (never used again) and a new bin is opened. Dual-Next-Fit is a bin covering algorithm that behaves similarly, except that it closes the bin when the total size of items in it becomes at least 1.

**Offline algorithms**

In the offline setting, the bin packing and bin covering problems are NP-hard. There is an asymptotic fully polynomial-time approximation scheme (AFPTAS) for bin covering. There is also a bin packing algorithm which opens $\text{OPT}(\sigma) + o(\text{OPT}(\sigma))$ bins, where $\text{OPT}(\sigma)$ is the number of bins in the optimal packing. The additive term in was $O(\log^2 \text{OPT}(\sigma))$ which was improved to $O(\log \text{OPT}(\sigma) \log \log \text{OPT}(\sigma))$ in and further, to $O(\log \text{OPT}(\sigma))$, in.

**Online algorithms**

Online algorithms are often compared under the framework of competitive analysis. Roughly speaking, the competitive ratio of a bin covering (respectively bin packing) algorithm is the minimum (respectively maximum) ratio between the number of bins covered (respectively opened) by the algorithm and that of an optimal offline algorithm on the same input.

Despite similarities between bin covering and bin packing, the status of these problems are different in the online setting. In the case of bin covering, it is known that no online algorithm can achieve a competitive ratio better than $1/2$, while bin covering algorithms such as Dual-Next-Fit have the best possible competitive ratio of $1/2$. Hence, we have a clear picture of the complexity of deterministic bin covering under competitive analysis. The situation is more complicated for the bin packing problem. It is known that no deterministic algorithm can achieve a competitive ratio of $1.54278$ while the best existing deterministic algorithm has a competitive ratio of $1.5783$. Note there is a gap between the best known upper and lower bounds.

**Online algorithms with advice**

Advice complexity is a formalized way of measuring how much knowledge of the future is required for an online algorithm to obtain a certain level of performance, as measured by the competitive ratio. When such advice is available,
algorithms with advice could lead to semi-online algorithms. Unlike related approaches such as “lookahead” [13] (in which some forthcoming items are revealed to the algorithm) and “closed bin packing” [2] (where the length of the input is revealed), any information can be encoded and sent to the algorithm under the advice setting. This generality means that lower bound results under the advice model also imply strong lower bound results on semi-online algorithms, where one can infer impossibility results simply from the length of an encoding of the information a semi-online algorithm is provided with. Advice complexity is also closely related to randomization; complexity bounds from advice complexity can be transferred to the randomization case and vice versa [8,21,14].

We use the advice on tape model defined in [17,9]: The advice is generated by a benevolent oracle with unlimited computational power. The advice is written on a tape and the algorithm knows its meaning. This general approach has been studied for many problems (we refer the reader to a recent survey on advice complexity of online problems [11]). In particular, bin packing has been studied under the advice complexity [12,23,1].

For bin covering with advice, the decision of where to pack the $i$th item is based on the content of the advice tape and the sizes of the first $i$ items. A bin covering algorithm, $A$, is $c$-competitive with advice complexity $s(n)$ if there exists a constant $b$ such that, for all $n$ and for all input sequences $\sigma$ of length at most $n$, there exists some advice $\Phi$ such that $A(\sigma) \geq c \cdot \text{Opt} - b$ and at most $s(n)$ bits of $\Phi$ are accessed by the algorithm. If $c = 1$ and $b = 0$, the algorithm is optimal. For a given algorithm, $A$, with a given advice complexity, $s(n)$, the competitive ratio is $\sup\{c \mid A$ is $c$-competitive$\}$.

Note that bin covering is a maximization problem. For minimization problems, like bin packing, the competitive ratio is defined analogously, except that the inequality is $A(\sigma) \leq c \cdot \text{Opt} + b$. Similarly, the competitive ratio is the infimum over all $c$ such that algorithm is $c$-competitive.

For Bin Packing, it is known that $\Theta(n \log(\text{Opt}))$ advice bits are necessary and sufficient to produce optimal solutions [12], but a constant number of advice bits are sufficient to obtain a competitive ratio close to 1.47, beating the best possible online algorithm without advice [14]. Furthermore, $2n + o(n)$ advice bits suffice to get arbitrarily close to a competitive ratio of $4/3$ [12], and getting below 1.17 requires at least a linear number of bits [21]. In [23], $(1 + \varepsilon)$-competitive online algorithms using $O(n \cdot \frac{1}{1+\varepsilon} \log \frac{1}{\varepsilon})$ advice bits are designed based on round and group techniques known from offline algorithms.

**Contributions**

In this article, we provide the first results with respect to the advice complexity of the bin covering problem.

To obtain an optimal result, advice essentially corresponding to an encoding of an entire optimal solution is necessary and sufficient. Not surprisingly, this follows from a similar proof for bin packing, since for both problems, bins filled to size one in an optimal solution are at the core of the proof.
Unlike the bin packing problem, advice of constant size cannot help improve the competitive ratio of algorithms. We establish this result by showing that any algorithm with advice of size $o(\log \log n)$ has a competitive ratio of at most 0.5, which is the competitive ratio of online algorithms without advice.

We prove a tight result that advice of size $O(\log \log n)$ suffices to achieve a competitive ratio arbitrarily close to 0.533. Some techniques that we develop for this result are quite different from the existing results for bin packing and are likely helpful for future analysis of bin covering with advice. The idea is to let the advice communicate the number of bins of certain “types” in an optimal packing. However, to get down to $O(\log \log n)$ bits of advice, only approximate values are given. This idea is similar to that for bin packing in [1], except that in [1] only a constant number of bits are used, approximating the ratio of the number bins in two different sets.

Finally, using a reduction from the binary string guessing problem [7], we show that advice of linear size is necessary to achieve any competitive ratio larger than 15/16. This is similar to, but more intricate, than the corresponding result for bin packing.

Techniques

We provide an intuitive explanation of the difference between bin packing and bin covering under the advice model, and use this explanation to describe our techniques in designing the bin covering algorithm of Section 4. Online bin packing is relatively “easy” when items are relatively large or small. Online algorithms can place large and small items separately, and this gives relatively good competitive ratios because an optimal algorithm has to open a bin for every large item and the online algorithm can fill any bin almost completely with small items. As such, in the bin packing problem, the “tricky” items are those that are close to 1/2 (a bit more or less than 1/2), and other items can be handled without wasting too much space. For inputs formed by the tricky items, a bin packing algorithm acts like a “matching algorithm”, where items smaller than 1/2 can match with themselves or some items larger than 1/2. Advice can help by encoding the number of items slightly larger than 1/2. This is consistent with reserving some space for “critical” items, which is the main technique used in some results for bin packing [12,1]. The resulting bins reflect the matching of large and small items or small items with themselves. It turns out that if we know the ratio between these two “types” of bins (approximated by a constant number of bits), we can do better than any deterministic online algorithm (see [1] for details). For bin covering, however, the tricky items are those that are either very large (close to 1) or very small (close to 0). Inputs formed by such tricky items are used in [13] to derive a lower bound on the competitive ratio of purely online algorithms and we also use them in Section 3 to derive a lower bound when the advice size is $o(\log \log n)$. Similarly to bin packing, for inputs formed by the tricky items, a bin covering algorithm becomes a “matching algorithm”. The matching process is albeit a bit harder than it is for bin packing. This is because, unlike bin packing where matching two large items is not possible (as
they do not fit in the bin), in bin covering, matching two large items is possible and sometimes necessary. To be a bit more precise, in bin covering, we cannot afford having a bin with an unmatched large item; such a bin will be almost full, but not covered, which implies that the item placed in the bin is wasted. When designing algorithms for bin covering in Section 4 we exploit advice in order to form packings that ensure that large items are always matched with other items, preferably with a set of small items, and if not possible with other large items. This ensures that items are not wasted in our packings, that is, all bins that receive items will be covered (except potentially a constant number of them). This family of packings is formally defined in Section 4 as “(α, k)-desirable coverings”, in which large items are matched with each other, except for a fraction α of them that are placed in bins that are covered by small items. The value of k is a parameter to encode the “approximate value” of a few numbers passed to the algorithm in the form of advice.

2 Optimal covering and advice

It is not hard to see that advice of size $O(n \log(\text{Opt}(\sigma)))$ is sufficient to achieve an optimal covering for an input $\sigma$ of length $n$; note that Opt$(\sigma)$ denotes the number of bins in an optimal covering of $\sigma$. Provided with $O(\log(\text{Opt}(\sigma)))$ bits of advice for each item, the offline oracle can indicate in which bin the item is placed in the optimal packing. Provided with this advice, the online algorithm just needs to pack each item in the bin indicated by the advice. Clearly, the size of the advice is $O(n \log(\text{Opt}(\sigma)))$ and the outcome is an optimal packing. Note that it is always assumed that the oracle that generates the advice has unbounded computational power. However, if the time complexity of the oracle is a concern, we can use the AFPTAS of [18] to generate an almost-optimal packing and encode it in the advice. Similarly, if the input is assumed to have only $m$ distinct known sizes, one can encode the entire request sequence, specifying for each distinct size how many of that size occur in the sequence. This only requires $O(m \log(n))$ bits of advice. The following theorem shows that the above naive solutions are asymptotically tight.

**Theorem 1.** For online bin covering on sequences $\sigma$ of length $n$, advice of size $\Theta(n \log \text{Opt}(\sigma))$ is required and sufficient to achieve an optimal solution, assuming $2 \text{Opt}(\sigma) \leq (1 - \varepsilon)n$ for some positive value of $\varepsilon$. When the input is formed by $n$ items with $m \in o(n)$ distinct, known item sizes, advice of size $\Theta(m \log n)$ is required and sufficient to achieve an optimal solution.

**Proof.** The lower bounds follow immediately from the corresponding results for bin packing [12 Theorems 1, 3]. Since the optimal result in those proofs have all bins filled to size 1, any non-optimal bin packing would also lead to a non-optimal bin covering. \qed
3 Advice of size $o(\log \log n)$ is not helpful

In this section, we show that advice of size $o(\log \log n)$ does not help for improving the competitive ratio of bin covering algorithms. This result is in contrast to bin packing where advice of constant size can improve the competitive ratio. Our lower bound sequence is similar to the one in [13] where the authors proved a lower bound on the competitive ratio of purely online algorithms.

**Theorem 2.** There is no algorithm with advice of size $o(\log \log n)$ and competitive ratio better than $1/2$.

**Proof.** Consider a family of sequences formed as follows:

$$\sigma_j = (\varepsilon, \varepsilon, \ldots, \varepsilon, 1 - j\varepsilon, 1 - j\varepsilon, \ldots, 1 - j\varepsilon)$$

Here, $j$ takes a value between 1 and $n$ and hence there are $n$ sequences in the family. All sequences start with the same prefix of $n$ items of size $\varepsilon$. We assume that $\varepsilon < \frac{1}{2n}$ to ensure that, even if all these items are placed in the same bin, the level of that bin is still less than $1/2$. Note that the suffix, formed by items of size $1 - j\varepsilon$ has length $O(n)$, and hence the length of all sequences is $\Theta(n)$.

Clearly, for packing $\sigma_j$, an optimal algorithm places $j$ items of size $\varepsilon$ in each bin and covers $n/j$ bins. So we have $\text{Opt}(\sigma_j) = n/j$.

The proof is by contradiction, so assume there is an algorithm, $A$, using $o(\log \log n)$ advice bits and having competitive ratio $1/2 + \mu$ for some constant $\mu > 0$. Thus, there exists a fixed constant $d$ such that for any sequence $\sigma_j$ we have

$$A(\sigma_j) \geq (1/2 + \mu) \text{Opt}(\sigma_j) - d = \frac{n}{2j} + \frac{\mu n}{j} - d$$

We say two sequences belong to the same sub-family if they receive the same advice string. Since the advice has size $o(\log \log n)$, there are $o(\log n)$ sub-families. Let $\sigma_{a_1}, \ldots, \sigma_{a_w}$ be the sequences in one sub-family. Since the advice and the first $n$ items (of size $\varepsilon$) are the same for any two members of this sub-family, $A$ will place these $n$ items identically. Let $m_i$ denote the number of bins receiving at least $i$ items in such a placement. So, we have $\sum_{i=1}^n m_i = n$ (a bin with exactly $x$ items is counted $x$ times). Moreover, for any $\sigma_j$, we have

$$A(\sigma_j) \leq m_j + (n/j - m_j)/2 = \frac{n}{2j} + \frac{m_j}{2}$$

This follows since any bin with at least $j$ items of size $\varepsilon$ can be covered using only one item of size $1 - j\varepsilon$, while the other bins require two such items.

From Equations [1] and [2] we get $\mu \frac{n}{j} \leq \frac{m_j}{2} + d$. Summing over $j \in \{a_1, \ldots, a_w\}$, we get that

$$\mu n \left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_w}\right) \leq \frac{1}{2}(m_{a_1} + m_{a_2} + \ldots + m_{a_w}) + wd$$
Since $\frac{1}{2}(m_{a_1} + m_{a_2} + \ldots + m_{a_w}) + dw \leq \frac{1}{2} \cdot \sum_{i=1}^{n} m_i + d n = (d + \frac{1}{2})n$, we have

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_w} \in O(1)$$

Summing the left-hand side over all families, we include every sequence once and obtain $\sum_{i=1}^{n} \frac{1}{a_i}$. Since there are $o(\log n)$ sub-families, it follows that $\sum_{i=1}^{n} \frac{1}{a_i} \in o(\log n)$. This is a contradiction since the Harmonic number $\sum_{i=1}^{n} \frac{1}{i} \in \Theta(\log n)$.

Thus, our initial assumption is wrong and with advice of size $o(\log \log n)$, no algorithm with competitive ratio strictly better than $1/2$ can exist.

\[\square\]

4 An algorithm with advice of size $O(\log \log n)$

In this section, we show that advice of size $O(\log \log n)$ is sufficient to achieve a competitive ratio arbitrarily close to 0.533. Throughout this section, we call an item small if it has size less than $1/2$ and large otherwise.

Consider a packing of the input sequence $\sigma$. We partition the bins in this packing into three groups. A large-small (LS) bin includes one large item and some small items, a large-large (LL) bin includes only two large items, and a small (S) bin includes only small items. We assume there is no small item in the LL bins of Opt (such small items can be moved to another bin without decreasing the number of covered bins). We also assume that, in the optimal packing, large items in LS bins are larger than those in LL bins (otherwise, we can move them around and LL bins will be still covered while the level of LS bins will increase). We use $m$ and $m'$, respectively, to denote the number of LS and LL bins in the optimal packing. For $m \geq 1$, we let $\beta \geq 1$ satisfy

$$m + m' = \beta m.$$  

See Table 1 for a summary of notation used in this section.

| Notation | Meaning |
|----------|---------|
| $n$      | The length of the input |
| $m$      | The number of LS bins in the optimal packing |
| $m'$     | The number of LL bins in the optimal packing |
| $S_l$    | An integer representing the total size of small items in the LS bins of the optimal packing (rounded down). |
| $S_s$    | An integer representing the total size of small items in the S bins of the optimal packing (rounded down). |
| $\beta$  | The value of $m/m'$. The algorithm behaves differently when $\beta \geq 15/14$ compared to when $\beta < 15/14$. |
| $\alpha$ | A parameter of the algorithm when $\beta < 15/14$. Approximately $\lfloor \alpha m \rfloor$ of covered bins include exactly one large item. We assume $\alpha < \frac{7-6\beta}{15} < \frac{1}{105}$. |
| $k$      | An integer representing the precision of approximate encodings in $O(\log \log n)$ bits. We assume $k$ is a large constant and we have $k \geq 6$. |
In the following lemma and later, we use the algorithm Dual-Worst-Fit, which, given a fixed number of bins, places an item in a least full bin.

**Lemma 1.** Given an integer $q$, assume we apply Dual-Worst-Fit to cover $q$ bins. Let $S$ denote the total size of packed items and $d$ denote the maximum size of any item in the sequence. The level of any bin is at least $\frac{S}{q} - d$.

**Proof.** The level of any two bins cannot differ by more than $d$; otherwise the last item placed in the bin with the larger level had to be placed in the bin with the smaller level. Let $B_{\text{min}}$ and $B_{\text{max}}$ be the two bins with minimum and maximum levels, respectively. From the above observation, we have $\text{level}(B_{\text{min}}) \geq \text{level}(B_{\text{max}}) - d$. The maximum level of any bin is no less than the average level of all bins. That is, $\text{level}(B_{\text{max}}) \geq \frac{S}{q}$ which gives $\text{level}(B_{\text{min}}) \geq \frac{S}{q} - d$. $\Box$

The following lemma shows that sequences with relatively few LS bins in an optimal packing are “easy” instances. The lemma is used in the case where $\beta \geq \frac{15}{14}$, i.e., when there are at most 14 times as many LS bins as LL bins in the optimal packing. Note that, in this case, $\frac{2(\beta - 1)}{2\beta} \geq \frac{8}{15}$.

**Lemma 2.** There is an online bin covering algorithm with competitive ratio at least $\min\{\frac{2}{3}, \frac{2\beta - 1}{2\beta}\}$.

**Proof.** Consider a simple algorithm, $A$, that places large and small items separately. Each pair of large items cover one bin and small items are placed using the Dual-Next-Fit strategy, that is, they are placed in the same bin until the bin is covered (and then a new bin is started). Let $S$ denote the total size of small items. Note that the number of large items is $m + 2m'$. The number of bins covered by the algorithm is at least $\left\lfloor \frac{m + 2m'}{2} + \frac{2S}{3} \right\rfloor$. The number of bins covered by $\text{Opt}$ is at most $m + m' + \lfloor S \rfloor$. Thus, for any input sequence, $\sigma$,

$$A(\sigma) \geq \frac{\left\lfloor (m + 2m')/2 \right\rfloor + \left\lfloor 2S/3 \right\rfloor}{m + m' + \lfloor S \rfloor} \cdot \text{Opt}$$

$$= \frac{(m + 2m')/2 + 2S/3}{m + m' + S} \cdot \text{Opt} - O(1)$$

This proves a competitive ratio of at least $\min\{\frac{2}{3}, \frac{2\beta - 1}{2\beta}\}$, since

$$\frac{(m + 2m')/2}{m + m'} = \frac{2m + 2m' - m}{2m + 2m'} = \frac{\frac{2m + m'}{m} - 1}{\frac{2m + m'}{m}} = \frac{2\beta - 1}{2\beta}.$$  

$\Box$

Recall that among the large items, we assume that the largest $m$ items form LS bins in the optimal packing. Let $S_l$ and $S_s$ be two integers that denote the floor of the total size of small items placed in respectively the LS and SS bins. So, the number of bins covered by $\text{Opt}$ is at most $\beta m + S_s$. In what follows, we define $(\alpha, k)$-desirable packings, which act as reference packings for our algorithm. Here $\alpha$ and $k$ are two parameters of the algorithm that we will introduce later.

For the following definition, it may be helpful to confer with Figure 1.
Fig. 1: (top) an optimal packing with $m = 4$, $m' = 2$, $S_l = \lfloor 1.18 \rfloor$ and $S_s = \lfloor 3.08 \rfloor$ (bottom) an $(\alpha, k)$-desirable packing with $\alpha = 1/2$ and $k = 6$.

**Definition 1.** A covering is $(\alpha, k)$-desirable, where $\alpha$ is a real number in the range $(0, 1]$ and $k$ is a positive integer, if and only if all the following hold:

I The covering has at least $\lfloor \alpha m \rfloor$ LS bins. All LS bins, except possibly a constant number of them, are covered.

II The large items that are not packed in LS bins appear in pairs, with each pair covering one bin (except one item when there are an odd number of such large items).

III The small items that are not packed in LS bins cover at least $\left\lfloor (1 - \frac{1}{2k}) \frac{2S_s}{3} \right\rfloor - 1$ bins.

**Lemma 3.** For any input sequence, $\sigma$, the number of bins covered in an $(\alpha, k)$-desirable packing is at least $\min\{\frac{\alpha + 2\beta - 1}{2}, (1 - \frac{1}{2k}) \frac{2S_s}{3}\} \cdot \text{OPT}(\sigma) - O(1)$.

Proof. The number of bins covered in the optimal packing is at most $m' + m + S_s = \beta m + S_s$. The number of bins covered by an $(\alpha, k)$-desirable packing is at
least \( \lfloor \alpha m \rfloor - c \) (for covered LS bins; \( c \) is a constant) plus \( \lfloor (2\beta - 1 - \alpha)m/2 \rfloor \) (for bins covered by pairs of large items) plus at least \( \lfloor (1 - \frac{1}{2^k}) \frac{2S_s}{3} \rfloor - 1 \) bins (covered by small items). So, the number, \( d \), of bins covered in the \((\alpha, k)\)-desirable packing of \( \sigma \) will be

\[
d \geq \frac{\alpha m/2 + (2\beta - 1)m/2 + (1 - \frac{1}{2^k})2S_s/3}{\beta m + S_s} \cdot \text{Opt}(\sigma) - O(1)
\]

\[
\geq \min \left\{ \frac{\alpha + 2\beta - 1}{2\beta}, \left(1 - \frac{1}{2^k}\right) \frac{2}{3} \right\} \cdot \text{Opt}(\sigma) - O(1).
\]

\( \square \)

In the remainder of this section, we describe an algorithm that achieves an \((\alpha, k)\)-desirable covering for certain values of \( \alpha \) and \( k \). Here, \( k \) is used as a parameter to encode approximate values of a few numbers passed to the algorithm. Before describing these numbers, we explain how the approximate encodings work. Given a positive integer \( x \), we can write the length of the binary encoding of \( x \) in \( O(\log \log x) \) bits, using self-delimited encoding as in [20]. The approximate value of \( x \) will be represented by the binary encoding of the length of \( x \), plus the \( k \) most significant bits of \( x \) after the high-order 1. Setting the unknown lower order bits to zero gives an approximation to \( x \) which we denote by \( \bar{x} \). We can bound \( \bar{x} \) as follows: If \( \bar{x} = y \cdot 2^f \) for some \( y \) represented by \( k + 1 \) bits, where the high-order bit is a one, then \( 2^k \leq y < 2^{k+1} \). Given \( \bar{x} \), the largest \( x \) could be is \( y \cdot 2^f + (2^f - 1) \). Thus, \( (1 - \frac{1}{2^k})x < \bar{x} \leq x \).

In the remaining more technical part of the section, it may be beneficial to consider that if we had had \( O(\log n) \) bits of advice instead of \( O(\log \log n) \), many arguments would be simplified, and it could be helpful on a first reading to ignore multiplicative terms such as \( 1 - \frac{1}{2^k} \) that are there because we know only approximate as opposed to exact values of the parameters we receive information about in the advice.

First, we describe how the algorithm treats the small items and then discuss the large items. The algorithm receives \( \bar{S}_s \) and \( \bar{m} \), i.e., the approximate values of \( S_s \) and \( m \), in \( O(\log \log n + k) \) bits of advice. It places small items using the Dual-Next-Fit strategy until a point at which the sum of small items observed so far becomes larger than \( \bar{S}_s \). Let \( p \) be the small item that causes the sum to exceed \( \bar{S}_s \). The algorithm places \( p \) and any other small item that follows it using the Dual-Worst-Fit strategy in \( \lceil \bar{m}/3 \rceil \) bins, ignoring any large items when calculating the levels of the bins. In what follows, we refer to these \( \lceil \bar{m}/3 \rceil \) bins as reserved bins. The items before \( p \) have a total size of more than \( \bar{S}_s - 1 \) and hence cover at least \( \lceil (2/3)(\bar{S}_s - 1) \rceil \geq 2\bar{S}_s/3 - 1 \geq (2/3)(1 - 1/2^k)\bar{S}_s - 1 \). So, Property III of an \((\alpha, k)\)-desirable covering holds.

Next, we describe how the algorithm places large items so that properties I and II also hold. For that, the algorithm will need the approximate value of \( m \) (which was also required for small items) and \( m' \). As before, these values can be encoded in \( O(\log \log n + k) \) bits of advice. We call the largest \( \lceil m/3 \rceil \) items in the input sequence good items. The algorithm aims at placing \( \lfloor \alpha m \rfloor \) of the good items in the reserved bins. Before describing how the algorithm detects good
items, we prove the following lemma, showing that the reserved bins with one good item will be covered.

**Lemma 4.** A reserved bin that includes any good item will be covered in the final solution (covering) of the algorithm.

**Proof.** Define the desired level to be \( d = 1 - \text{size}(x) \) where \( x \) is the smallest good item. Consider an LS-bin \( B \) in the optimal packing that does not include a good item (that is, it has one large item smaller than any good item). The total size of small items in \( B \) will be at least \( d \). As there are at least \( \lceil \frac{2m}{3} \rceil \) such bins, we have that \( S_l \geq \frac{2S_l}{2m} \). On the other hand, the total size of small items placed in the reserved bins is at least \( S_l + S_l - \bar{S}_s \geq S_l \). Since we use the Dual-Worst-Fit strategy to place these items into \( \frac{m}{3} \) reserved bins, by Lemma 1, the total size of small items in any reserved bin is at least \( S_l \frac{m}{3} - y \) where \( y \) is the largest small item in those bins. Now, if a reserved bin includes a small item of size at least \( d \), its level is already at least \( d \); otherwise, \( S_l \frac{m}{3} - x \) will be at least \( 3S_l/m - d \) and since \( d \leq \frac{3S_l}{2m} \), the level of the bins is at least \( d \). \( \square \)

So, in order to achieve an \((\alpha, k)\)-desirable packing, our algorithm needs to select \( \lfloor \alpha m \rfloor \) good items and place them in the reserved bins; the above lemma indicates that these bins will be covered (Property I holds). Meanwhile, the algorithm ensures that other large items are paired and hence each pair of them covers a bin (Property II holds). In order to provide the above guarantees, the algorithm considers three cases depending on the location of good items (advice will be used to select the correct case).

**Lemma 5.** When \( \beta < \frac{15}{14} \), there exists an \((\alpha, k)\)-desirable packing for \( \alpha \leq \frac{7 - 6\beta}{15} \) and \( k \) sufficiently large.

**Proof.** Throughout the proof, \( 1 \leq \beta < \frac{15}{14} \) and \( \alpha < \frac{7 - 6\beta}{15} - \frac{176 - 18\beta}{75 - 2\beta + 120} \). Note that under the description of the algorithm, we established Property III, and just prior to the statement of the lemma, we established Property I. The proof to establish Property II is a case analysis on where good items appear in the request sequence.

**Case 1:** Assume there are \( \lfloor \alpha \bar{m}/(1 - 1/2^k) \rfloor \) good items among the first \( A = \lfloor \bar{m}/3 \rfloor \) large items in the sequence. In this case, the algorithm places the first \( A \) large items into the reserved bins. After seeing all these \( A \) items, the algorithm chooses the largest \( \lfloor \alpha \bar{m}/(1 - 1/2^k) \rfloor \) of them and declares them to be good items, which by Lemma 4 are guaranteed to be covered. The remaining \( A - \lfloor \alpha m/(1 - 1/2^k) \rfloor \) large items in the reserved bins will be paired with forthcoming large items. Since there are at least \( m - A \geq 2m/3 \) forthcoming large items and fewer than \( m/3 \) large items in the reserved bins waiting to be paired, all these large items (except possibly one) can be paired (Property II holds). In summary, in the final covering, there are \( \lfloor \alpha \bar{m}/(1 - 1/2^k) \rfloor \) bins covered by a large item (and some small items) while the remaining large items are paired (except possibly one). Hence, the result will be an \((\alpha, k)\)-desirable packing.
Case 2: Assume there are fewer than \( \lfloor \alpha \bar{m} / (1 - 1/2^k) \rfloor \) good items among the first \( A \) large items in the sequence (Case 1 does not apply). Furthermore, assume there are \( \lfloor \alpha \bar{m} / (1 - 1/2^k) \rfloor \) good items among the \( A = \lceil \bar{m} / 3 \rceil \) large items that follow the first \( A \) large items.

In this case, the algorithm places the first \( A \) large items pairwise in \( \lceil A/2 \rceil \) bins. The \( A \) large items that follow are placed in the reserved bins. After placing the last of these items in the reserved bins, the algorithm considers these \( A \) items and declares the \( \lfloor \alpha \bar{m} / (1 - 1/2^k) \rfloor \geq \lfloor \alpha m \rfloor \) largest to be good items, which by Lemma 4 are guaranteed to be covered. The remaining \( A - \lfloor \alpha \bar{m} \rfloor \) reserved bins (with large items) will need to be covered by forthcoming large items. We know there are at least \( m - 2A \geq \lfloor m / 3 \rfloor \) forthcoming large items and fewer than \( \lfloor m / 3 \rfloor \) large items in reserved bins waiting to be paired, so all these large items (except possibly one) can be paired (Property II holds). Thus, the result is an \((\alpha, k)\)-desirable packing.

Case 3: Assume there are fewer than \( \lfloor \alpha \bar{m} / (1 - 1/2^k) \rfloor \) good items among the first \( A = \lceil \bar{m} / 3 \rceil \) large items and also fewer than \( \lfloor \alpha \bar{m} / (1 - 1/2^k) \rfloor \) good items among the following \( A \) items (Cases 1 and 2 do not apply). In what follows, we assume \( \beta < 15/14 \) and let \( \alpha \) be some positive value such that \( \alpha \leq \frac{7 - 6 \beta}{15} - \frac{176}{15} \cdot \frac{18 \beta}{75} \). We will later choose \( k \) sufficiently large; here \( k \geq 6 \) will ensure that \( \alpha \) is positive. Note also that we have \( \alpha < \frac{7 - 6 \beta}{15} \).

In this case, the algorithm places the first \( 2A \) large items in pairs. There are \( C = 2m' + m - 2A = 2m' + m - 2\lceil \bar{m} / 3 \rceil \) remaining large items. The algorithm places the first \( F = m' + \lceil \bar{m} / 6 \rceil + \lfloor \alpha \bar{m} / 2 \rfloor - 1 - \lceil \bar{m} / 3 \rceil \) of the last \( C \) large items in the reserved bins (note that this is roughly half of the last \( C \) large items when \( \alpha \) is small). For this to be possible, we show first that the number of reserved bins is at least \( F \). Since there are \( \lceil \bar{m} / 3 \rceil \) reserved bins, as long as we have \( F \leq \lceil \bar{m} / 3 \rceil \), there are enough reserved bins. Since \( m' = (\beta - 1)m, \ 1 \leq \beta < 15/14, \) and \( \alpha < \frac{7 - 6 \beta}{15}, \) we get:

\[
\begin{align*}
F - \lfloor \frac{\bar{m}}{3} \rfloor &= \bar{m}' + \lfloor \frac{\bar{m}}{6} \rfloor + \lfloor \frac{\alpha \bar{m}}{2} \rfloor - 1 - \lfloor \frac{\bar{m}}{3} \rfloor \\
&< \bar{m}' + \frac{\bar{m}}{6} + \frac{\alpha \bar{m}}{2} - \frac{\bar{m}}{3} \\
&= \bar{m}' + \left( \frac{\alpha}{2} - \frac{1}{6} \right) \bar{m} \\
&< m' + \left( \frac{\alpha}{2} - \frac{1}{6} \right) \left( 1 - \frac{1}{2^k} \right) m, \text{ since } \frac{\alpha}{2} - \frac{1}{6} < 0 \text{ and } \bar{m} > \left( 1 - \frac{1}{2^k} \right) m \\
&= (\beta - 1)m + \left( \frac{\alpha}{2} - \frac{1}{6} \right) \left( 1 - \frac{1}{2^k} \right) m \\
&< \frac{1}{14} m - \frac{2}{15} \left( 1 - \frac{1}{2^k} \right) m \\
&= \left( -\frac{13}{210} + \frac{2}{15} \cdot \frac{1}{2^k} \right) m \\
&< 0, \text{ for } k \geq 2,
\end{align*}
\]
proving that there are enough reserved bins.

Next, we show that at least $\alpha m - 6$ of the $F$ items placed in the reserved bins are good items. There are fewer than $2\lfloor \alpha m/(1 - 1/2^k) \rfloor$ good items among the first $2A$ large items. Thus, the number of good items among the last $C$ large items is more than $|m/3| - 2\lfloor \alpha m/(1 - 1/2^k) \rfloor$. Even if all items among the last $C - F$ large items are good, there are still more than $X = |m/3| - 2\lfloor \alpha m/(1 - 1/2^k) \rfloor + C + F$ good items that are placed in the reserved bins. In the following we deduce that $X > \alpha m - 6$:

\[
X = \left(\frac{m}{3} - 2\lfloor \alpha m/(1 - 1/2^k) \rfloor\right) - \left(2m' + m - 2\lfloor \frac{m}{3} \rfloor\right) + \left(\frac{m'}{6} + \frac{\bar{m}}{6} + \left\lceil \frac{\alpha m}{2} \right\rceil - 1\right)
\]

\[
= \left(\frac{m}{3} - 2\lfloor \alpha m/(1 - 1/2^k) \rfloor\right) - \left(2m' + m - \frac{2m}{3} + 2\right) + \left(\frac{m'}{6} + \frac{\bar{m}}{6} + \frac{\alpha m}{2} - 3\right)
\]

\[
= -\frac{2m}{3} - \frac{3 + \frac{1}{2^k}}{2(1 - \frac{1}{2^k})} \alpha m - 2m' + \frac{\bar{m}}{6} + 5\bar{m} - 6
\]

\[
\geq -\frac{2m}{3} - \frac{3 + \frac{1}{2^k}}{2(1 - \frac{1}{2^k})} \alpha m - \left(1 + \frac{1}{2^k}\right) m' + \left(1 + \frac{1}{2^k}\right) 5m - \frac{6}{2^k} - 6
\]

Since we assumed $\alpha \leq \frac{7 - 6\beta}{15} - \frac{176 - 18\beta}{75 + 2\beta + 120} = \frac{(7 - 6\beta) - (6\beta + 24)/2^k}{15 + 24/2^k}$, we can write $\frac{7 - 6\beta}{15} \geq \frac{\alpha m}{2} + \frac{\alpha m + \beta + 1}{4}$. Recall that $m' = (\beta - 1)m$. So,

\[
X = \left(\frac{m}{6} - \frac{3\alpha m}{2} - \frac{(5 + 4\alpha)m}{2^k}\right) - (1 + \frac{1}{2^k})(\beta - 1)m - 6
\]

\[
= \left(\frac{7 - 6\beta}{15}\right) m - \frac{5\alpha m}{2} - \frac{(\beta + 4\alpha + 4)m}{2^k} + \alpha m - 6
\]

\[
\geq \frac{5\alpha m}{2} - \frac{(\beta + 4\alpha + 4)m}{2^k} - \frac{\alpha m}{2} - \frac{(\beta + 4\alpha + 4)m}{2^k} + \alpha m - 6
\]

Hence, at least $|\alpha m| - 6$ of the $F$ items placed in the reserved bins are good items. After placing these $F$ items, the algorithm declares the largest $|\alpha m| - 6$ among them to be good items. By Lemma 4 these items (along with small items
in the reserved bins) will cover their respective bins. There are $F - |\alpha m| + 6$ (positive for $k \geq 1$ and $\alpha \leq \frac{15}{14}$) large items in reserved bins which have not been declared good, and the $C - F$ large items which have not arrived at this point will be paired with them. Thus,

$$F - |\alpha m| + 6 = \tilde{m}' + \lceil \tilde{m}/6 \rceil + \left\lfloor \frac{\alpha \tilde{m}}{2(1 - 1/2^k)} \right\rfloor - 1 - |\alpha m| + 6$$

$$< (2m' + m - 2\lceil \tilde{m}/3 \rceil) - (\tilde{m}' + \lceil \tilde{m}/6 \rceil + \left\lfloor \frac{\alpha \tilde{m}}{2(1 - 1/2^k)} \right\rfloor - 1) + 6$$

$$= C - F + 6$$

So, the number of large items in the reserved bins which are not paired will be at most 6. The bins in which these items are placed, along with the $|\alpha m| - 6$ bins that include good items, will be the LS bins in the final $(\alpha, k)$-desirable packing. Note that the number of these bins is $|\alpha m|$ minus an additive constant which is allowed in desirable packings. All large items placed in bins other than LS bins are paired and hence, Property II also holds.

**Theorem 3.** There is an algorithm that, provided with $O(\log \log n)$ bits of advice, achieves a competitive ratio of at least $\frac{12\beta - 4}{25} - \frac{88 - 9\beta}{75\beta^2 + 120\beta}$, where $k$ is a large but constant parameter of the algorithm. Since $\beta \geq 1$, for any $\varepsilon > 0$, there exists an algorithm using a sufficiently large $k$ with competitive ratio at least $\frac{8}{15} - \varepsilon$.

**Proof.** The advice indicates the values of $\tilde{m}$, $\tilde{m}'$, and $\tilde{S}$. These values can all be encoded in $O(\log \log n)$ bits of advice. Note that one cannot calculate $\beta$ exactly, since $m$ and $m'$ are not known exactly. Thus, the advice also includes 1 bit to indicate if Lemma 2 should be used because $\beta$ is larger than 15/14. If not, the advice also indicates one of the three cases described above; this requires two more bits. Thus, the size of advice is $O(\log \log n)$.

If Lemma 2 is used, the competitive ratio is at least $\min\{\frac{2}{3}, \frac{2\beta - 1}{2\beta}\}$, which for $\beta \geq 15/14$ is at least 8/15. Otherwise, provided with this advice and a sufficiently large integer parameter $k$, the algorithm can create an $(\alpha, k)$-packing of the input sequence for any $\alpha \leq \frac{7 - 6\beta}{15} - \frac{176 - 18\beta}{75\beta^2 + 120\beta}$. By Lemma 2 the resulting packing has a competitive ratio of at least $\frac{2\beta - 1}{2\beta}$, choosing $\alpha = \frac{7 - 6\beta}{15} - \frac{176 - 18\beta}{75\beta^2 + 120\beta}$ gives a scheme with competitive ratio at least $\frac{12\beta - 4}{25\beta^2} - \frac{88 - 9\beta}{75\beta^2 + 120\beta}$. Since this is an increasing function of $\beta$ and $\beta \geq 1$, the competitive ratio approaches $\frac{8}{15}$ for large values of $k$.

5 **Impossibility result for advice of sub-linear size**

This section uses what is normally referred to as lower bound techniques, but since our ratios are smaller than 1, an upper bound is a negative result, and we refer to such results as negative or impossibility results. In what follows, we show that, in order to achieve any competitive ratio larger than 15/16, advice of linear size is necessary. We use a reduction from the binary separation problem:
Definition 2. The Binary Separation Problem is the following online problem. The input $I = (n_1, \sigma = (y_1, y_2, \ldots, y_n))$ consists of $n = n_1 + n_2$ positive values which are revealed one by one. There is a fixed partitioning of the set of items into a subset of $n_1$ large items and a subset of $n_2$ small items, so that all large items are larger than all small items. Upon receiving an item $y_i$, an online algorithm must guess if $y$ belongs to the set of small or large items. After the algorithm has made a guess, it is revealed to the algorithm which class $y_i$ belongs to.

A reduction from a closely related problem named “binary string guessing with known history” shows that, in order to guess more than half of the items correctly, advice of linear size is required:

Lemma 6. [12] For any fixed $\beta > 0$, any deterministic algorithm for the Binary Separation Problem that is guaranteed to guess correctly on more than $(1/2 + \beta)n$ input items on an input of length $n$ needs at least $\Omega(n)$ bits of advice.

The following lemma provides the actual reduction from the Binary Separation Problem to bin covering.

Lemma 7. Consider the bin covering problem on sequences of length $2n$ for which $\text{OPT}$ covers $n$ bins. Assume that there is an online algorithm $A$ that solves the problem on these instances using $b(n)$ bits of advice and covers at least $n - r(n)/8$ bins. Then there is also an algorithm $B_{sa}$ that solves the Binary Separation Problem on sequences of length $n$ using $b(n)$ bits of advice and guessing incorrectly at most $r(n)$ times.

Proof. In the reduction, we encode requests for the algorithm $B_{sa}$ as items for bin covering, which will be given to the algorithm $A$. Assume we are given an instance $I = (n_1, \sigma = (y_1, y_2, \ldots, y_n))$ of the Binary Separation Problem, in which $n_1$ is the number of large items ($n_1 + n_2 = n$), and the values of $y_i$s are revealed in an online manner ($1 \leq t \leq n$). We create an instance of the bin covering problem for $A$ which has length $2n$.

The bin covering sequence starts with $n_1$ “huge” items of size $1 - \varepsilon$ for some $\varepsilon < \frac{1}{2n}$ (this will ensure that at least one item of size at least $1/2$ is in every covered bin). In the optimal covering, each of these huge items will be placed in a different bin. The next $n$ items are created in an online manner, so that we can use the result of their packing to guess the requests for the Binary Separation Problem. Let $\tau = y_t$ ($1 \leq t \leq n$) be a requested item from the Binary Separation Problem, and choose an increasing function $f : \mathbb{R} \to (\varepsilon, 2\varepsilon)$. When $\tau$ is presented to $B_{sa}$, $f(\tau)$ is presented to $A$, and the item $f(\tau)$ is said to be associated with the request $\tau$. If the algorithm places the item $f(\tau)$ in one of the bins opened by the huge items, $B_{sa}$ will answer that $\tau$ is small; otherwise, $B_{sa}$ will answer that $\tau$ is large.

The last $n_2$ items of the bin covering instance are defined as complements of the items in the bin covering instance associated with large items in the binary separation instance (the complement of item $x$ is $1 - x$). We do not need to give
the last items complementing the small items in order to implement the algorithm $B_{SA}$, but we need them for the proof of the quality of the correspondence that we are proving.

Call an item in the bin covering sequence “large” if it is associated with large items in the Binary Separation Problem, and “small” if associated with a small item. For the bin covering sequence produced by the reduction, an optimal algorithm pairs each of the small items with a huge item, one of the first $n_1$ items, placing it in one of the first $n_1$ bins. $OPT$ pairs the large items with their complements, starting one of the next $n_2$ bins with each of these large items. Hence, the number of bins in an optimal covering is $n_1 + n_2 = n$.

Let $a_1$ and $a_2$ denote the number of the two types of mistakes that the bin covering algorithm $A$ makes, causing incorrect answers to be given by $B_{SA}$. Thus, we let $a_1$ denote the number of small items which do not get placed with a huge item, and $a_2$ denote the number of large items which are placed with a huge item. Clearly, the number of errors in the answers provided by the binary separation algorithm $B_{SA}$ is $a_1 + a_2$. We claim that the number of bins covered by the bin covering algorithm $A$ is at most $n_1 + n_2 - (a_1 + a_2)/8$. $A$ covers at most $(n_1 - a_1) + (n_2 - a_2)$ bins in the same way that $OPT$ does; these bins include either a huge item with a small item or a large item and its complement. Moreover, at most $\min\{a_1, a_2\}$ bins are covered by huge items that are placed with large items, but no small item. Similarly, at most $\min\{a_1/2, a_2\}$ bins are covered with two small items and a complement of a large item. Finally, there are $p = \max\{a_1 - a_2, 0\}$ bins with huge items that are not covered with any small or large items; similarly, there are $q = \max\{a_2 - a_1/2, 0\}$ complements of large items that are not covered by large or pairs-of-small items. These $p + q$ items can be paired to cover at most $(p + q)/2$ bins. In total, the number of bins that are covered by $A$ is at most

\[
(n_1-a_1) + (n_2-a_2) + \min\{a_1, a_2\} + \min\{a_1/2, a_2\} + \frac{\max\{a_1 - a_2, 0\} + \max\{a_2 - a_1/2, 0\}}{2}
\]

If $a_1 \leq a_2$, the above value becomes $n_1 + n_2 - a_2/2 + a_1/4$ which is indeed at most $n_1 + n_2 - (a_1 + a_2)/8$. If $a_1/2 \leq a_2 \leq a_1$, the above value becomes $n_1 + n_2 - a_1/4$ which is at most $n_1 + n_2 - (a_1 + a_2)/8$. Finally, if $a_2 < a_1/2$, the above value becomes $n_1 + n_2 - (a_1 + a_2)/2$ which is less than $n_1 + n_2 - (a_1 + a_2)/8$. In summary, when the algorithm $B_{SA}$ makes $a_1 + a_2$ errors in partitioning small and large items, $A$ covers at most $n - (a_1 + a_2)/8$ bins (intuitively speaking, each eight mistakes in binary guessing causes at least one fewer bin to be covered; see Figure 2). In other words, if the number of covered bins is at least $n - r(n)/8$, then the number of binary separation errors must be at most $r(n)$. $ \square$

It turns out that reducing the Binary Separation Problem to bin covering (the above lemma) is more involved than a similar reduction to the bin packing problem [12]. The difference roots in the fact that there are more ways to place items into bins in the bin covering problem compared to bin packing; this is because many arrangements of items are not allowed in bin packing due to the capacity constraint.
Fig. 2: Two coverings for the sequences used in reduction from binary separation to bin covering. Grey items show huge items that arrive first. Red and green items are respectively large and small items that need to be separated. The sequence ends with complements of large items. (a) shows an optimal packing and (b) shows a covering with eight mistakes; as a result of the eight mistakes, one fewer bin is covered.

Theorem 4. Consider the bin covering problem on sequences of length $n$. To achieve a competitive ratio of $15/16 + \delta$, in which $\delta$ is a small, but fixed positive constant, an online algorithm needs to receive $\Omega(n)$ bits of advice.

Proof. Suppose for the sake of contradiction that there is a bin covering algorithm $A$ with competitive ratio $15/16 + \delta$ using $o(n)$ bits of advice. Consider sequences of length $2n$ for which $\text{OPT}$ covers $n$ bins. $A$ covers $(15/16 + \delta)n = n - r(n)/8$ bins for $r(n) = (1/2 - 8\delta)n$. Applying Lemma 7, we conclude that there is an algorithm that solves the Binary Separation Problem on sequences of length $n$ using $o(n)$ bits of advice, while making at most $(1/2 - 8\delta)n$ errors.
By Lemma 6, we know that such an algorithm requires $\Omega(n)$ bits of advice. So, our initial assumption that $A$ required only $o(n)$ bits of advice is wrong. 

6 Concluding remarks

We have established that $\Theta(\log \log n)$ bits of advice are necessary and sufficient to improve the competitive ratio obtainable by purely online algorithms.

Obvious questions are: How much better than our bound of $8/15 = 0.533$ can one do with $O(\log \log n)$ bits of advice? Can one do better with $O(\log n)$ bits of advice?

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