The Role of the Fifth Postulate in the Euclidean Construction of
Parallels

Iosif Petrakis
Ludwig-Maximilians Universität München, Germany
petrakis@mathematik.uni-muenchen.de

Abstract
We ascribe to the Euclidean Fifth Postulate a genuine constructive role, which makes it absolutely necessary in the parallel construction. For that, we present a reconstruction of the general principles underlying the Euclidean construction of a geometric property. As a consequence, the epistemological role of Euclidean constructions is revealed. We also examine some first implications of our interpretation to the relation between Euclidean and non-Euclidean geometries. The Bolyai construction of limiting parallels is also discussed from the Euclidean point of view, as this is reconstructed here.

1 The Standard Interpretation of the Fifth Postulate

From Proclus up to our days a hermeneutic tradition regarding the Fifth Postulate (FP) has been developed, which we call the Standard Interpretation (SI). According to it, the Euclidean FP, though originally is formulated differently, actually asserts that through a given point outside a given straight line at most a unique parallel straight line can be drawn to it. This formulation, commonly known as Playfair’s Axiom (PA), is logically equivalent to the original FP. As the existence of a parallel line is independent from PA, the addition of PA establishes the existence of exactly one such parallel. A clear indication of the predominance of SI is that PA was made the standard form of presenting FP in the axiomatic formulations of Euclidean geometry. In order to describe the shortcomings of SI, we give briefly the Euclidean line of presentation of the parallel construction in a more formal scheme compatible to our later reconstruction.

Definition 1.1. If \(a, b\) and \(c\) are Euclidean coplanar straight lines, we define their following properties:
\[
T(a, b, c) :\iff c \text{ falls on } a \text{ and } b,
\]
\[
Q_b(a) :\iff a \text{ is parallel to } b,
\]
\[
P_{b,c}(a) :\iff T(a, b, c) \text{ and } c \text{ makes the alternate angles equal to one another.}
\]

The path to the Euclidean parallel construction can be described as follows:

- Proposition 27 of Book I of the Elements, or Proposition I.27, is a criterion of parallelism, which, according to Definition 1.1 is the implication \(P_{b,c}(a) \Rightarrow Q_b(a)\).

- Proposition I.28 contains two more criteria of parallelism, reducible to that of Proposition I.27.

- Proposition I.29: Let \(a, b, c\), such that \(T(a, b, c)\), then \(Q_b(a) \Rightarrow P_{b,c}(a)\).
  Hence, in Proposition I.29 the converse to the implication of Proposition I.27 is established. In its proof Euclid uses FP\(^1\) for the first time.

- Proposition I.30: If \(Q_b(a)\) and \(Q_b(c)\), then \(Q_c(a)\).
  This proposition, the proof of which uses Proposition I.29, is crucial to SI, as it proves the uniqueness of the parallel line. This uniqueness result though, is not included in the Elements.

\(^1\)The original formulation of FP is the following: if \(T(a, b, c)\) and \(c\) makes the interior angles less than two right angles (\(2c\)), then \(a, b\), if produced indefinitely, meet on that side on which are the angles less than \(2c\).
• **Proposition I.31**: Construction of a straight line \( a \), through a given point \( A \) outside line \( b \), such that \( Q_b(a) \). The construction consists in the construction of lines \( c \) and \( a \), such that \( P_{b,c}(a) \). Then, by Proposition I.27, we also get \( Q_b(a) \).

The core of SI can be summarised as follows:

- The construction in Proposition I.31 requires only Proposition I.27 and hence it is independent from FP. Consequently, it could be placed after Proposition I.27 and before Proposition I.29.
- Within SI the place of the parallel construction after the first use of FP is explained, although not with absolute certainty, as an expression of Euclid’s need, before giving the construction, to place beyond all doubt the fact that only one such parallel can be drawn (see [11], p. 316). Actually, this is an argument of Proclus, as expressed in his Commentary [19], pp. 295–296. If the parallel construction was placed right after Proposition I.27, then only the existence of the parallel line would be established.

The independence of FP from the parallel construction within SI is one of the main reasons why mathematicians, before the emergence of non-Euclidean geometries, used to consider FP as a theorem rather than as a Postulate. According to SI, the Euclidean line of presentation certifies the existence and the uniqueness of the parallel line, hence the “true” meaning of FP is the uniqueness formulation of the parallel line. It is this emphasis of SI on the uniqueness of the parallel which pushed it forward as a central characteristic of Euclidean geometry. Gradually, the difference between Euclidean geometry and non-Euclidean geometries was identified with the different number of parallels they permit.

The uniqueness interpretation though, is, in our view, inadequate. In the first place, there is no explanation within SI why Euclid preferred his formulation of FP rather than the uniqueness assumption. Furthermore, the examination of the Elements shows that Euclid seems indifferent to questions of uniqueness. In the First Postulate (construction of a line segment between two points) there is no reference to the uniqueness of the corresponding segment, though it is used in Proposition I.4 in the form “two straight lines cannot enclose a space”. The circle under construction in the Third Postulate (construction of a circle of any center and radius) is not mentioned to be unique either. The investigation of the perpendicular constructions in Propositions I.11 and I.12 reveals the same Euclidean indifference to the uniqueness of a constructed object.

## 2 The basic principles of a Euclidean construction and the constructive role of the Fifth Postulate

In this section we present a reconstruction of the general principles underlying the Euclidean construction of a geometric property. As a consequence, FP plays a crucial role to the parallel construction of Proposition I.31. The first three Euclidean postulates have a direct constructive role, as they provide the fundamental elements for the subsequent line and circle constructions. In our view, the Fourth and the Fifth Postulate have an indirect, though genuine, constructive role. Both are less elementary and they participate in the less elementary parallel construction.

**The constructive role of the Fourth Postulate**: It is used in Proposition I.16 (through Proposition I.15), which is necessary in the proof of Proposition I.27. By this line of thought, it participates in the construction of Proposition I.31. Also, by the Fourth Postulate, the right angle is a fixed and universal standard, to which other angles can be compared. In this way FP, treating the \( 2 \) as a fixed quantity, “depends” on the Fourth Postulate.

To reveal the constructive character of FP, we need to understand the conceptual requirements of ancient Greek mathematics regarding the nature of geometric constructions as these are embodied in the Euclidean Elements. These requirements are not explicitly found in Euclid, but we propose them as an accurate reconstruction of the Euclidean constructive spirit.

**Basic principles of the Euclidean construction \( K(a, P) \) of an object \( a \) satisfying a geometric property \( P \):

---

2A first presentation of our reconstruction of the principles underlying the Euclidean constructions is found in [17].
K1: In $K(a, P)$ an object $a$ is constructed, satisfying, as accurately as possible, the definition of $P^3$.

K2: If an object $b$ satisfying a geometric property $R$ is used in construction $K(a, P)$, then the construction $K(b, R)$ must have already been completed.

K3: If $a$ is a geometric object satisfying $P$ and $Q$ is another geometric property, such that whenever $a$ satisfies $P$ it satisfies $Q$, but not the converse i.e., $P(a) \Rightarrow Q(a)$, but not $Q(a) \Rightarrow P(a)$, then $K(a, Q)$ cannot be established through $K(a, P)$.

K4: If $a$ is a geometric object satisfying $P$ and $Q$ is another geometric property, such that whenever $a$ satisfies $P$ it satisfies $Q$, and the converse i.e., $P(a) \Leftrightarrow Q(a)$, then $K(a, Q)$ can be established through $K(a, P)$, and conversely.

Principle K2 guarantees that $K(a, P)$ does not have constructive gaps, i.e., all geometric concepts used in construction $K(a, P)$ are already constructed$^3$. Principle K3 is the most crucial to our reconstruction of the role of FP in the Euclidean parallel construction. It guarantees that the construction of the abstract object $a$ satisfying property $Q$ cannot be established through the construction of the less general property $P$ i.e., construction $K(a, P)$ respects the generality hierarchy of geometric concepts. For example, the construction of an isosceles triangle cannot be established through the construction of an equilateral triangle, since there are isosceles triangles which are not equilateral$^5$. Principle K4 guarantees that whenever properties $P$ and $Q$ are logically equivalent, having the same generality, they do not differ with respect to construction. K4 is the natural complement to K3 and together they form the core of the Euclidean constructive method. In order to explain the use of the above set of principles in the parallel construction and their relation to FP, we define the following notions of construction.

Definition 2.1. We call a construction $K(a, P)$ direct, if it establishes an object $a$ reproducing exactly the definition of $P$. In this case we call $P$ a finite property. A geometric property $Q$ is called infinite, if it is impossible to give a direct construction of $Q$. A construction $K(a, Q)$ is called indirect, if $K(a, Q)$ establishes an object $a$, which satisfies the definition of $Q$ indirectly, i.e., through a provably equivalent, finite property $P$.

Most of Euclidean constructions are direct. For example, at the end of the perpendicular construction of Proposition I.12 Euclid restates the definition of the perpendicular line, showing that he has constructed an object which satisfies completely that very definition. In our terminology, the property of perpendicularity is finite. On the other hand, the parallel property is infinite. Euclid defined parallel lines (Definition 23 of Book 1) as straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction. It is impossible to give a direct construction of a line parallel to a given one, since we cannot reproduce the above definition. The infinite character of this definition lies in our mental inability to produce a line indefinitely and act as if this ever extended object exists as a whole. Each moment we only know a finite part of the on going line, from which we cannot infer that every extension of it does not meet the given line. The formation of the parallel line never ends.

Euclidean construction of the infinite parallel property: Euclid gradually constructed (mainly through the Fourth Postulate and Propositions I.16 and I.27) the geometric property $P_{b,c}(a)$, which is a finite property. Given a line $b$, we can construct directly lines $c$ and $a$ such that $P_{b,c}(a)$ (actually this is the construction of Proposition I.31), using only the direct construction of Proposition I.23 (construction of a rectilinear angle equal to a given one, on a given straight line and at a point on it). The implication $P_{b,c}(a) \Rightarrow Q_{b}(a)$ is established by Proposition I.27, but it would be a violation of K3 if $P_{b,c}(a)$ is used in construction $K(a, P)$.

---

3The expression “as accurately as possible” in K1 will be evident in section 4. K1 can also be found, though not as explicitly as here, in the intuitionistic literature on the concept of species, or an intuitionistic property. A constructive principle such as K1 can be detected in Brouwer’s notes. Furthermore, for Griss, a species is defined by a property of mathematical objects, but such a property can only have a clear sense after we have constructed an object which satisfies it (see [12], p. 126). For the construction of species see also [14, 15].

4Though K2 is very natural to accept, it is not trivial. In a sense described in section 4 Bolyai’s construction of limiting parallels violates it.

5Euclid uses the concept of an isosceles triangle in Proposition I.5, without providing first a construction of it, because this construction is a simple generalization of the equilateral one (Proposition I.1). Evidently, Euclid found no reason to include this, strictly speaking, different, but expected construction.

6This impossibility is not a logical one, but just a result of the definition of $Q$. 

3

4

5

6
construction \( K(a, P_{bc}(a)) \) was considered as construction \( K(a, Q_k(a)) \). Construction \( K(a, P_{bc}(a)) \) can be considered as construction \( K(a, Q_k(a)) \) only if the converse implication \( Q_k(a) \Rightarrow P_{bc}(a) \) is proved. Then, \( P \) and \( Q \) will have the same generality and then \( K4 \) can be applied. In our view, this is why Euclid “postponed” the parallel construction, placing it after Proposition I.29, which establishes the converse implication.

The constructive role of FP: The fifth postulate is this (intuitively true) proposition, through which the implication \( Q_k(a) \Rightarrow P_{bc}(a) \) is shown, and then by \( K4 \), construction \( K(a, P_{bc}(a)) \) in Proposition I.31 is also construction \( K(a, Q_k(a)) \) of parallels.

Euclid used his specific formulation of FP so that the proof of Proposition I.29 requires one only conceptual step, reaching his goal in the simplest way. According to our interpretation, Euclid does not postpone the use of FP as long as possible, recognizing its “problematic” nature. On the contrary, he uses it exactly the moment he needs it, revealing in this way its function. In Euclid’s Elements, if \( P \) is a finite property then \( K(a, P) \) is always given through \( P \) itself and not through an equivalent property \( Q \) i.e., \( K4 \) is not used in constructions of finite properties. It is used only when an infinite property \( Q \) is to be constructed. Otherwise, its function wouldn’t be clear.

The indirect construction of an infinite geometric property is not the only way ancient Greeks used to handle an infinite property. If an infinite property \( Q \) has no finite equivalent, it may have a special case \( F \) with a strong finite character accompanying the infinite one. Infinite anthyphairesis i.e., infinite continued fraction \( Q \) is an infinite property studied in Book X of the Elements, which does not have a finite equivalent. Periodic anthyphairesis (periodic continued fraction) \( F \) is a special case of \( Q \), which possesses a strong finite character beside its infinity. Although the sequence of the quotients forming the periodic continued fraction never ends (infinity of \( F \)), its finite period expresses our knowledge of this sequence (finite character of \( F \)). In intuitionistic terms, periodic anthyphairesis is a strong form of law-likeness.

3 The epistemological role of Euclidean constructions

Our description of the Euclidean constructive principles also reveals the difference between “Euclidean construction and “Euclidean existence”. Let the existential formula

\[ \exists a Q(a) \iff \text{there exists a geometric object } a \text{ satisfying geometric property } Q. \]

In Euclid’s Elements \( \exists a Q(a) \) is established either by \( K(a, Q) \) or by \( K(a, P) \), where \( P(a) \Rightarrow Q(a) \) (but not the converse). Euclidean geometry (except Eudoxus’ theory of ratios) is a basic paradigm of a constructive mathematical theory, since existence of a mathematical object or concept is constructively established. For example, if the construction of Proposition I.31 was placed right after Proposition I.27, that would only show the existence of a parallel line. This proof of existence though, does not constitute a construction of the concept of parallelism.

The traditionally accepted independence between FP and the construction of Proposition I.31 is based on the identification between \( \exists a Q(a) \) and \( K(a, Q) \). For Euclid though, the construction of property \( Q \) is generally an enterprise larger than the exhibition-construction of a single object satisfying \( Q \). The parallel construction reflects this fact very clearly. According to our reconstruction, \( \exists a Q(a) \) shows that property \( Q \) is not void, that it possesses, in modern terms, an extension. On the other hand, \( K(a, Q) \) shows that we have found a way to mentally grasp property \( Q \), fully if \( Q \) is finite, as much as possible if \( Q \) is infinite.

Traditionally, the Elements are considered as the original model of the axiomatic method and logical deduction. In our view, they also constitute a first model of the constructive method, quite different though, from modern constructivism in mathematics (see e.g., [4]). It is this combination of

---

\(^7\)For a recent reference to this long repeated view see [10].

\(^8\)Ancient Greeks had also found a necessary and sufficient condition for an infinite anthyphairesis to be periodic (logos criterion). Its knowledge and its importance in Plato’s system have been developd in recent times in Negrepontis’ program on reconstructing Plato (see e.g., [5]).

\(^9\)According to Zeuthen [22], the main purpose of a geometric construction is to provide a proof of existence, so the purpose of the FP is to ensure the existence of the intersection point of the non parallel lines. This approach fails to see, in our view, the difference between existence and construction.
the axiomatic and the constructive method that reflects the philosophical importance of the Elements. For the first time in the history of mathematics a mathematical theory answers simultaneously the ontological and the epistemological problem of the mathematical concepts involved. The ontology of Euclidean geometric objects and concepts is of mental (and not empirical) nature. Almost certainly Euclidean ontology is very close to Platonic ontology. This mental ontology of mathematical concepts imposes the constructive method. It is the construction of mathematical concepts which provides their study with a firm epistemology. Euclid does not only care about the logical relations between geometric concepts and objects. He also needs to answer the main epistemological question: *how do we understand the concepts that we employ in our deductions?* And his answer, in our view, is: *we understand them because we construct them.* Thus, geometric constructions form the indispensable epistemology of Euclidean geometry.

### 4 On the relation between Euclidean and non-Euclidean Geometry

It is impossible here to study fully the relation between Euclidean geometry and non-Euclidean geometries. We shall only stress some points that can be derived from our previous analysis. There is a traditional view regarding the above relation too. According to it, the two geometries can be seen as mathematical structures of the same kind, differing only in the number of parallels they allow. One such common mathematical framework is the concept of Hilbert plane. A Hilbert plane is a system of points, lines and planes satisfying the well known Hilbert axioms of incidence, betweenness and congruence. In a Hilbert plane the parallel line (as any other geometric property) is not constructed, only its existence is established. A general Hilbert plane is neutral with respect to the uniqueness of the parallel line. A Euclidean plane is a Hilbert plane allowing only one parallel, while a hyperbolic plane is a Hilbert plane allowing more than one parallels. The consequences of this “coexistence” of Euclidean and non-Euclidean geometry were very serious. The foundations of mathematics and mathematics itself were influenced immensely from the loss of the a priori character of Euclidean geometry. As Euclidean geometry became just one possible geometry, the Kantian a priori suffered a serious blow and especially the a priori of space. As a result, all major foundational programs if mathematics in the twentieth century rest either on a Kantian a priori of discrete nature (e.g., Brouwer’s intuitionism and Bishop’s constructivism), or on a purely logical substratum (e.g., Frege’s logicism).

Our reconstruction of the parallel construction suggests a class with this traditional view. In our opinion, Euclidean geometry has a constructive character, of a specific type, which non-Euclidean geometry lacks. Of course, this opinion echoes Kant. In [21], p. 1, Webb remarks the following:

> [It was a commonplace of older Kantian scholarship that the discovery of non-euclidean geometry undermined his theory of the synthetic a priori status of geometry. It is commonplace of newer Kant scholarship that he already knew about non-euclidean geometry from his friend Lambert, one of the early pioneers of this geometry, and that in fact its very possibility only reinforces Kant’s doctrine that euclidean geometry is synthetic a priori because only its concepts are constructible in intuition.]

The common language of Hilbert planes (or any such common mathematical framework) ignores the role and the necessity of FP in the parallel construction, just as the epistemological role of constructions. Modern geometry, developed within classical logic, seems quite indifferent to epistemological questions. We can only indicate here that the two geometries are not directly comparable, from the constructive point of view. Consequently, Euclidean geometry has not lost its a priori character. Next we explain

---

10Euclid was a Platonist and his definitions are closely related to the Platonic ones (see [12], p. 168).
11For a more recent discussion on the role of Euclidean constructions see [14] and [9]. In our opinion, the interpretations proposed there are not satisfactory.
12This framework is not as absolute as it is often named, since it does not include the elliptic plane, in which there exist no parallels at all, and every line through the pole of a given line is perpendicular to it. Hilbert’s classic work is still the best introduction to Hilbert planes. A more absolute framework that includes elliptic geometry, is the concept of a Bachmann plane, or metric plane (see [1]).
13Putnam’s assessment in [20], p. x, is characteristic: “…the overthrow of Euclidean geometry is the most important event in the history of science for the epistemologist”.

---

5
why Bolyai’s construction of limiting parallels is problematic from the Euclidean point of view, as this is reconstructed above.

The path to the Bolyai construction of limiting parallels can be described as follows:

- **A hyperbolic plane** is a Hilbert plane satisfying Lobachevsky’s axiom (L): If $a$ is a line and $A$ is a point outside $a$, there exist rays $Ab, Ac$, not on the same line, which do not intersect $a$, and each ray $Ad$ in the angle $bAc$ intersects $a$.

- **Proposition 4.1**: A triangle in a hyperbolic plane has angle sum less than $2\pi$.

- A quadrilateral $PQRS$ is a Lambert quadrilateral, if it has right angles at $P, Q$ and $S$.

- **Proposition 4.2**: In a hyperbolic plane the fourth angle (the angle at $R$) of a Lambert quadrilateral $PQRS$ is acute, and a side adjacent to it is greater than its opposite side ($QR > PS$ and $SR > PQ$).

- **Proposition 4.3**: Suppose we are given a line $a$ and a point $P$ not on $a$, in a hyperbolic plane. Let $PQ$ be the perpendicular to $a$. Let $m$ be a line through $P$, perpendicular to $PQ$. Choose any point $R$ on $a$, and let $RS$ be the perpendicular to $m$. If $Pc$ is a limiting parallel ray intersecting $RS$ at $X$, then $PX = QR$.

- **Elementary Continuity Principle** (ECP): If one endpoint of a line segment is inside a circle and the other outside, then the segment intersects the circle.

- **Bolyai’s construction**: Consider a hyperbolic plane satisfying ECP. Suppose we are given a line $a$ and a point $P$ not on $a$. Let $PQ$ be the perpendicular to $a$. Let $m$ be a line through $P$, perpendicular to $PQ$. Choose any point $R$ on $a$, and let $RS$ be the perpendicular to $m$. Then the circle of radius $QR$ around $P$ will meet the segment $RS$ at a point $X$, and the ray $PX$ will be the limiting parallel ray to $a$ through $P$.

The proof of Bolyai’s construction goes as follows: as $Q = \pi$ and $PR > QR$, by Proposition 4.1 the angle at $Q$ is the largest angle in the triangle $PQR$. Moreover, $PS < QR$, since $PQRS$ is a Lambert quadrilateral satisfying the hypothesis of Proposition 4.2. Consequently, the endpoints $R$ and $S$ of the segment $RS$ are outside and inside the circle $(P, QR)$. By ECP the segment $RS$ intersects $(P, QR)$ at a (unique) point $X$, and $PX$ is the limiting parallel ray to $a$ through $P$, since L guarantees its existence and by Proposition 4.3 it satisfies $PX = QR$.

The curious feature of this proof is that we prove that this construction works by assuming first (via axiom L) that the object we wish to construct already exists. This curiosity is stressed by Hartshorne in [10] p. 398. As this presupposed existence of the limiting parallel is axiomatic and not constructive, Bolyai’s construction violates the Euclidean Principle K2. Another aspect of the problematic character of Bolyai’s construction is related to the constructive principles K3 and K4. Proposition 4.3 is in analogy to Proposition I.29, since it can be written in the form

$$L \Rightarrow PX = QR.$$ 

In our language, L is an infinite property and $PX = QR$ is a finite one. In order to consider, from the Euclidean point of view, the direct construction of $X$ as the construction of the limiting ray, we have to prove directly, in a hyperbolic plane satisfying ECP, the analogue to Proposition I.27:

$$PX = QR \Rightarrow L.$$ 

Such a direct proof has not yet been found. Although the above line and circle construction of the most important concept of hyperbolic geometry shows Bolyai’s constructive sensitivity, it does not satisfy the constructive principles of the Euclidean parallel construction. The usual proof of the existence of limiting parallel is based on Dedekind’s continuity axiom (D) (see e.g., [3] p. 156). According to it, any (set-theoretic) separation of points on a line i.e., a Dedekind cut, is produced by a unique point. Axiom D seems unfathomable from the Euclidean point of view, maybe because of its set-theoretic
nature. The question whether Bolyai’s construction could be used to prove the existence of a limiting parallel for a system of axioms that includes ECP but not D, was naturally raised by Greenberg in [6]. Pejas, working in the framework of Bachmann plane geometry, a geometry without betweenness and continuity axioms, succeeded to classify all Hilbert planes with Greenberg, using Pejas’ classification of Hilbert planes, managed in [6] to answer his question positively.

**Proposition 4.4** (Pejas-Greenberg): If ECP holds and the fourth angle of a Lambert quadrilateral is acute, then Bolyai’s construction gives the two lines through \( P \) that have a “common perpendicular at infinity” with \( a \) through the ideal points at which they meet \( a \). Among Hilbert planes satisfying ECP, the Klein models are the only which are hyperbolic, and Bolyai’s construction gives the asymptotic parallels for them.

An important corollary of Proposition 4.4 is the following proposition.

**Proposition 4.5**: Every Archimedean, non-Euclidean Hilbert plane in which ECP holds is hyperbolic.

Though Pejas-Greenberg managed to show that the Bolyai construction does yield the limiting parallel replacing D with more elementary axioms, their proof is indirect, since it is based on a classification theorem. Hence, from the Euclidean, constructive point of view, there is still no direct constructive proof of the concept of limiting parallel. We conjecture that such a proof cannot be found.

**References**

[1] F. Bachmann: *Aufbau der Geometrie aus dem Spiegelungsbegriff*, New York: Springer, 1971.

[2] E. Bishop: *Foundations of Constructive Analysis*, McGraw-Hill, 1967.

[3] E. Bishop, D. S. Bridges: *Constructive Analysis*, Grundlehren der math. Wissenschaften 279, Springer-Verlag, Heidelberg-Berlin-New York, 1985.

[4] D. S. Bridges, H. Ishihara, M. Rathjen, H. Schwichtenberg: *Handbook of Constructive Mathematics*, Cambridge University Press, to appear, 2023.

[5] V. Farmaki, S. Negrepontis: The paradoxical nature of mathematics, in A. Papadopoulos (Ed.) *Topology and Geometry*, EMS, 2021, 599–642 DOI 10.4171/IRMA/33-1/27

[6] M. J. Greenberg: On J. Bolyai’s Parallel Construction, Journal of Geometry 12, 45-64, 1979.

[7] M. J. Greenberg: Euclidean and Non-Euclidean geometries without continuity, American Mathematical Monthly 86 (9), 757-764, 1979.

[8] M. J. Greenberg: *Euclidean and Non-Euclidean Geometries*, Freeman (2nd ed.), 1980.

[9] O. Harari: The Concept of Existence and the Role of Constructions in Euclid’s Elements, Archive for History of Exact Sciences 57, 1-23, 2003.

[10] R. Hartshorne: *Geometry: Euclid and Beyond*, Springer, 2000.

[11] T. L. Heath: *The Thirteen Books of Euclid’s Elements*, Dover, vol.1, 1956.

[12] A. Heyting: *Intuitionism*, North-Holland, 1971.

[13] D. Hilbert: *Foundations of Geometry*, Open Court, La Salle, 1971.

[14] W. R. Knorr: Constructions as Existence Proofs in Ancient Geometry, Ancient Philosophy 3, 125-148, 1983.

---

14A Hilbert plane corresponds to an ordered Bachmann plane with free mobility. As Greenberg puts it in [7], Hilbert’s approach is thus incorporated into Klein’s Erlangen program, whereby the group of motions becomes the primordial object of interest. For Pejas’ classification theorem see [15].

15A Hilbert plane \( P \) is called non-Euclidean if PA fails in \( P \).
[15] W. Pejas: Die Modelle des Hilbertschen Axiomensystems der absoluten Geometrie, Mathematische Annalen 143, 212-235, 1961.

[16] I. Petrakis: The role of the fifth postulate in the Euclidean construction of parallels, in *History and Epistemology in Mathematics Education*, Proceedings of the 5th European Summer University, E. Barbin, N. Stehlíková and C. Tzanakis (eds.), Vydatelský servis, Plzeň, 2008, 595-604.

[17] I. Petrakis: Brouwer’s intuitionism as a self-interpreted mathematical theory, in R. Cantoral et al (Eds.) Electronic Proceedings of the HPM 2008, History and Pedagogy of Mathematics, Mexico City, 14-18 July 2008.

[18] I. Petrakis: *Brouwer’s Fan Theorem*, Master Thesis, Aristotle University of Thessaloniki, 2010.

[19] Proclus: *A Commentary on the First Book of Euclid’s Elements*, translated by G. R. Morrow, Princeton University Press, 1992.

[20] H. Putnam: *Mathematics, Matter and Method*, Cambridge University Press, 1975.

[21] J. Webb: Tracking Contradictions in Geometry: The Idea of a Model from Kant to Hilbert, in *From Dedekind to Gödel*, J. Hintikka (ed.), Kluwer Academic Publishers, pp. 1-20, 1995.

[22] H. G. Zeuthen: Die geometrische Construction als “Existenzbeweis” in der antiken Geometrie, Mathematische Annalen 47, 222-228, 1896.