Large–$N_c$ QCD and Low Energy Interactions

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Abstract. This talk reviews recent progress in formulating the dynamics of the electroweak interactions of hadrons at low energies, within the framework of the $1/N_c$–expansion in QCD. The emphasis is put on the basic issues of the approach.

INTRODUCTION

In the Standard Model, the electroweak interactions of hadrons at very low energies can be described by an effective Lagrangian which only has as active degrees of freedom the flavour $SU(3)$ octet of the low–lying pseudoscalar particles. The underlying theory, however, is the gauge theory $SU(3)_C \times SU(2)_L \times U(1)_Y$ which has as dynamical degrees of freedom quarks and gauge fields. Going from these degrees of freedom at high energies to an effective description in terms of mesons at low energies is, in principle, a problem which should be understood in terms of the evolution of the renormalization group from short–distances to long–distances. Unfortunately, it is difficult to carry out explicitly this evolution because at energies, typically of a few GeV, non–perturbative dynamics like spontaneous chiral symmetry breaking and color confinement sets in.

It has been possible, however, to integrate out the heavy degrees of freedom of the Standard Model gauge theory, in the presence of the strong interactions, perturbatively, thanks to the asymptotic freedom property of QCD at short–distances. This procedure results in an effective Lagrangian which consists of the usual QCD Lagrangian with the light quark flavours $u$, $d$, and $s$ still active, plus a sum of effective four–quark operators of the light quarks, modulated by c–number coefficients (the Wilson coefficients,) which are functions of the masses of the heavy particles which have been integrated out, and of the renormalization scale. We are still left with the evolution from this effective field theory, appropriate at intermediate scales higher than a few GeV, to an effective Lagrangian description in terms of the low–lying pseudoscalar particles which are the Goldstone modes associated to the spontaneous chiral symmetry breaking of the Standard Gauge Theory in the light quark sector. In this talk, I shall review recent progress which has been made in approaching this last step, when the problem is formulated within the framework of QCD in the limit of a large number of colours $N_c$. The emphasis is put on basic issues. Details of the applications reviewed here can be found in the original publications.

The suggestion to keep $N_c$ as a free parameter was first made by G. ’t Hooft [1] as a possible way to approach the study of non–perturbative aspects of QCD. The limit $N_c \to \infty$ is taken with the product $\alpha_s N_c$ kept fixed and it is highly non–trivial. In spite
of the efforts of many illustrious theorists who have worked on the subject, QCD in the large–$N_c$ limit still remains unsolved; but many interesting properties have been proved, which suggest that, indeed, the theory in this limit has the bulk of the non–perturbative properties of the full QCD. In particular, it has been shown that, if confinement persists in this limit, there is spontaneous chiral symmetry breaking [2].

The spectrum of the theory in the large–$N_c$ limit consists of an infinite number of narrow stable meson states which are flavour nonets [3]. This spectrum looks \textit{a priori} rather different to the real world. The vector and axial–vector spectral functions measured in $e^+e^- \rightarrow$ hadrons and in the hadronic $\tau$–decay show indeed a richer structure than just a sum of narrow states. There are, however, many instances where one is only interested in observables which are given by weighted integrals of some hadronic spectral functions. In these cases, it may be enough to know a few \textit{global} properties of the hadronic spectrum, and to have a good interpolation. Typical examples of that are, as we shall see, the coupling constants of the effective chiral Lagrangian of QCD at low energies, as well as the coupling constants of the effective chiral Lagrangian of the electroweak interactions of pseudoscalar particles in the Standard Model. Some of these couplings are needed in order to understand $K$–Physics quantitatively. In these examples the \textit{hadronic world} predicted by large–$N_c$ QCD provides already a good approximation to the real hadronic spectrum. It is in this sense that I shall show that large–$N_c$ QCD is a very useful phenomenological approach for understanding non–perturbative QCD physics at low energies.

There are a number of good articles and lecture notes on large–$N_c$ QCD in the literature \textsuperscript{1}. Here I shall limit myself to make a couple of comments on prejudices one often encounters concerning the QCD large–$N_c$ limit.

- The first prejudice has to do with the “extrapolation” from $N_c = 3$ to $N_c = \infty$. In fact, $N_c$ is really used as a label to select specific topologies among Feynman diagrams. The topology which corresponds to the highest power in the $N_c$–label is the one which selects \textit{planar} diagrams only; and the claim is that it is this class that already provides a good approximation to the full theory.
- The second prejudice has to do with the fact that some physical quantities are absent in the large–$N_c$ limit; the $\eta'$–mass e.g. is zero in that limit. That does not mean that the $1/N_c$–expansion fails, as sometimes it is argued, but rather that some observables only appear at subleading topologies, (\textit{planar} diagrams with \textit{one handle} in this case,) much the same as in QED, there is no light–by–light scattering at the Born approximation and one has to go to \textit{one loop} diagrams to evaluate its leading behaviour.

\textbf{THE CHIRAL LAGRANGIAN AT LOW ENERGIES}

The strong and electroweak interactions of the Goldstone modes at very low energies are described by an effective Lagrangian which has terms with an increasing number of

\textsuperscript{1} See e.g., the book in ref. [4] and the lectures in [5]
derivatives (and quark masses if explicit chiral symmetry breaking is taken into account.) These terms are modulated by couplings which encode the dynamics of the underlying theory. The evaluation of these couplings from the underlying theory is the question we are interested in. Typical terms of the chiral Lagrangian are

\[
L_{\text{eff}} = \frac{1}{4} F_0^2 \frac{\text{tr} \left( D_\mu U D^\mu U^\dagger \right)}{\pi \pi \to \pi \pi}, \quad K \to \pi \nu \nu \\
+ L_{10} \frac{\text{tr} \left( U^\dagger F_{R\nu} U T^{\mu \nu}_{L} \right)}{\pi \to \epsilon \gamma f} + \cdots
\]

where \( U \) is a 3 \times 3 unitary matrix in flavour space which collects the Goldstone fields and which under chiral rotations transforms as \( U \to V_R U V_L^\dagger \). The first line shows typical terms of the strong interactions in the presence of external currents [6],[7, 8]; the second line shows typical terms which appear when photons, \( W' \)s and \( Z' \)s have been integrated out in the presence of the strong interactions. We show under the braces the typical physical processes to which each term contributes. Each term is modulated by a constant: \( F_0^2, L_{10}, \ldots \), which encode the underlying dynamics responsible for the appearance of the corresponding effective term. Knowing \( g_8 \) for example, means that we can calculate from first principles the dominant \( \Delta I = 1/2 \) transitions for \( K \)–decays to leading order in the chiral expansion.

There are two crucial observations concerning the relation of these low energy constants to the underlying theory, that I want to discuss.

- The low–energy constants of the Strong Lagrangian, like \( F_0^2 \) and \( L_{10} \), are the coefficients of the Taylor expansion of appropriate QCD Green’s Functions. For example, with \( \Pi_{LR}(Q^2) \) the correlation function of a left–current with a right–current in the chiral limit, (where the light quark masses are neglected,) i.e.,

\[
2i \int d^4 x \ e^{iq \cdot x} \langle 0 | T \left( \bar{u}_L \gamma^\mu d_L(x) \bar{u}_R \gamma^\nu d_R(0) \right) | 0 \rangle = (q^\mu q^\nu - g^{\mu \nu} q^2) \Pi_{LR}(Q^2),
\]

the Taylor expansion

\[
-Q^2 \Pi_{LR}(Q^2) \big|_{Q^2 \to 0} = F_0^2 - 4 L_{10} Q^2 + O(Q^4),
\]

defines the constants \( F_0^2 \) and \( L_{10} \).

- By contrast, the low–energy constants of the ElectroWeak Lagrangian, like e.g. \( C \) and \( g_8 \), are integrals of appropriate QCD Green’s Functions. For example, including the effect of weak neutral currents [9],

\[
C = \frac{3}{32 \pi^2} \int_0^\infty dQ^2 \left( 1 - \frac{Q^2}{Q^2 + M_Z^2} \right) \left( -Q^2 \Pi_{LR}(Q^2) \right).
\]

Their evaluation appears to be, a priori, quite a formidable task because they require the knowledge of Green’s functions at all values of the euclidean momenta;
i.e. they require a precise matching of the short–distance and the long–distance contributions of the underlying Green’s functions.

The observations above are completely generic to the Standard Model independently of the \(1/N_c\)–expansion. The large–\(N_c\) approximation helps, however, because it restricts the analytic structure of the Green’s functions in general, and \(\Pi_{LR}(Q^2)\) in particular, to be meromorphic functions: they only have poles as singularities; e.g., in large–\(N_c\) QCD,

\[
\Pi_{LR}(Q^2) = \sum_V \frac{f_V^2 M_V^2}{Q^2 + M_V^2} - \sum_A \frac{f_A^2 M_A^2}{Q^2 + M_A^2} - \frac{F_0^2}{Q^2},
\]

where the sums are, in principle, extended to an infinite number of states.

There are two types of important restrictions on Green’s functions like \(\Pi_{LR}(Q^2)\). One type follows from the fact that, as already stated above, the Taylor expansion at low euclidean momenta must match the low energy constants of the strong chiral Lagrangian. This results in a series of long–distance sum rules like e.g.

\[
\sum_V f_V^2 - \sum_A f_A^2 = -4L_{10}. \tag{6}
\]

Another type of constraints follows from the short–distance properties of the underlying Green’s functions. The behaviour at large euclidean momenta of the Green’s functions which govern the low energy constants of the chiral Lagrangian in Eq. (1) can be obtained from the operator product expansion (OPE) of local currents in QCD. In the large–\(N_c\) limit, this results in a series of algebraic sum rules [10] which restrict the coupling constants and masses of the hadronic poles. In the case of the LR–correlation function in Eq. (5) one has,

\[
\begin{align*}
\text{No } \frac{1}{Q^2} \text{ term in OPE } & \Rightarrow \sum_V f_V^2 M_V^2 - \sum_A f_A^2 M_A^2 - F_0^2 = 0, \quad \text{1st Weinberg sum rule.} \tag{7} \\
\text{No } \frac{1}{Q^4} \text{ term in OPE } & \Rightarrow \sum_V f_V^2 M_V^4 - \sum_A f_A^2 M_A^4 = 0, \quad \text{2nd Weinberg sum rule.} \tag{8} \\
\text{Matching } \frac{1}{Q^6} \text{ terms in the OPE } & \Rightarrow \sum_V f_V^2 M_V^6 - \sum_A f_A^2 M_A^6 = \langle O^{(6)} \rangle, \tag{9}
\end{align*}
\]

where [10, 11], in large–\(N_c\) QCD,

\[
\langle O^{(6)} \rangle = \left[-4\pi\alpha_s + O(\alpha_s^2)\right] \left\langle \bar{\psi}\psi \right\rangle^2.
\]

The Minimal Hadronic Ansatz Approximation to Large–\(N_c\) QCD

In most cases of interest, the Green’s functions which govern the low–energy constants of the chiral Lagrangian are order parameters of spontaneous chiral symmetry breaking; i.e. they vanish, in the chiral limit, order by order in the perturbative vacuum of QCD. That implies that they have a power fall–out in \(1/Q^2\) at large–\(Q^2\); like e.g., the function \(\Pi_{LR}(Q^2)\), which as explicitly shown in Eq. (9), falls as \(1/Q^6\). That also implies that within a finite radius in the complex \(Q^2\)–plane, these Green’s functions in large–\(N_c\)
QCD, only have a finite number of poles. The natural question which arises is: WHAT IS THE MINIMAL NUMBER OF POLES REQUIRED TO SATISFY THE OPE CONSTRAINTS? The answer to that follows from a well known theorem in analysis [12] which we illustrate with the example of the Green’s function

\[-Q^2 \Pi_{LR}(Q^2) \equiv \Delta[z] \text{ with } z = \frac{Q^2}{M_V^2},\]  

(11)

where for convenience we normalize \(Q^2\) to the mass of the lowest vector state. The function \(\Delta[z]\) has the property that

\[N - P = \frac{1}{2\pi i} \oint \frac{\Delta'[z]}{\Delta[z]} dz,\]  

(12)

where \(N\) is the number of zeros and \(P\) is the number of poles inside the integration contour (a zero and/or a pole of order \(m\) is counted \(m\) times.) For a contour of radius sufficiently large so that the OPE applies, we simply have that \(N - P = -p_{OPE}\) where \(p_{OPE}\) denotes the leading power fall–out in \(1/z\) predicted by the OPE. Since \(N \geq 0\), it follows that \(P \geq p_{OPE}\). In our case \(p_{OPE} = 2 \Rightarrow P \geq 2\) and the minimal hadronic ansatz (MHA) compatible with the OPE requires two poles: one vector state and one axial–vector state. The MHA approximation to the large–\(N_c\) expression in Eq. (5) is then the simple function

\[-Q^2 \Pi_{LR}(Q^2) = F^2_0 \frac{M_V^2M_A^2}{(Q^2 + M_V^2)(Q^2 + M_A^2)}.\]  

(13)

Inserting this function in Eq. (4) gives a prediction to the low–energy constant \(C\) which governs the electromagnetic \(\pi^+ - \pi^0 \equiv \Delta m_\pi\) mass difference, with the result

\[\Delta m_\pi = (4.9 \pm 0.4) \text{ MeV}, \quad \text{MHA to Large–}\(N_c\) QCD,\]  

(14)

to be compared with the experimental value

\[\Delta m_\pi = (4.5936 \pm 0.0005) \text{ MeV}, \quad \text{Particle Data Book [15].}\]  

(15)

The shape of the function in Eq. (13), normalized to its value at \(Q^2 = 0\), is shown in Fig. 1 below, (the continuous red curve.) It provides a good interpolation between the low–\(Q^2\) regime where \(\chi PT\) applies and the high–\(Q^2\) regime where the OPE applies. Also shown in the same plot is the experimental curve, (the green dotted curve) obtained from the ALEPH collaboration data [16]. Except for the intermediate energy region, where the MHA overestimates slightly the experimental curve, the overall agreement is quite remarkable.

\[\text{Notice that with the definition of } \Delta[z]\text{ in Eq. (11) the pion pole is removed.}\]

\[\text{This is the result for } M_V = (748 \pm 29) \text{ MeV and } g_A = \frac{M_V^2}{M_A^2} = 0.50 \pm 0.06. \text{ These values follow from an overall fit to predictions of the low energy constants [13, 14].}\]
The MHA approximation to large–$N_c$ QCD is a starting point to do well defined approximations in nonperturbative QCD. The approximation has been tested with the ALEPH data [17]. In principle it is improvable: inserting more terms in the OPE provides extra sum rules which can be used to fix the extra hadronic parameters. We have made tests with models of large–$N_c$ QCD [18, 19]. It has also been shown [20] that in the case of $\Pi_{LR}$, inserting an extra $\rho'$–like vector state, improves the overall picture; the $\Delta m_\pi$ prediction in particular.

At this stage, it is perhaps illustrative to compare the large–$N_c$ approach we have discussed so far with other analytic approaches which exist in the literature. The $\Pi_{LR}$ correlation function provides us with an excellent theoretical laboratory to do that. The different shapes of this correlation function predicted by other analytic approaches are also collected in Fig. 1. Let us comment on them individually.

- The suggestion to use large–$N_c$ QCD combined with lowest order $\chi$PT loops, was first proposed by Bardeen, Buras and Gérard in a series of seminal papers [4]. The same approach has been applied by the Dortmund group [24], in particular to the evaluation of $\epsilon'/\epsilon$. In this approach the hadronic ansatz to the Green’s functions consists of Goldstone poles only and their integrals, (which become of course UV–divergent, often quadratically divergent since the correct QCD short–distance behaviour is not implemented,) are cut–off sharply. In this case, the predicted hadronic shape of the LR–correlation function normalized to its value at $Q^2 = 0$ is constant, as shown by the BBG, HKPSB line (black dotted) in Fig. 1.

- The Trieste group evaluate the relevant Green’s functions using the constituent chiral quark model ($C\chi$QM) proposed in refs. [25] and [26, 27]. They have obtained

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4 See refs. [21, 22, 23] and references therein.
a long list of predictions [28], in particular $\epsilon'/\epsilon$. The model gives an educated first guess of the low--$Q^2$ behaviour of the Green's functions, as one can judge from the $\chi$QM--curve (green dashed) in Fig. 1, but it fails to reproduce the short--distance QCD--behaviour. Another objection to this approach is that the “natural matching scale” to the short--distance behaviour in this model should be $\sim 4M_Q^2$, ($M_Q$ the constituent quark mass,) too low to be trusted.

- The extended Nambu–Jona-Lasinio (ENJL) model was developed as an improvement on the $\chi$QM, since in a certain way it incorporates the vector--like fluctuations of the underlying QCD theory, which are known to be phenomenologically important. The model is, indeed, rather successful in predicting the low--energy $O(p^4)$ constants of the chiral Lagrangian. It has, indeed, a better low--energy behaviour than the $\chi$QM, as the ENJL--curve (blue dot--dashed) in Fig. 1 shows; but it fails to reproduce the short--distance behaviour of the OPE in QCD. Arguments to do the matching to short--distance QCD have been forcefully elaborated in refs. [30], which also have made a lot of predictions; a large value for $\epsilon'/\epsilon$ in particular.

- The problem with the ENJL model as a plausible model of large--$N_c$ QCD, is that the on--shell production of unconfined constituent quark $Q\bar{Q}$ pairs that it predicts violates the large--$N_c$ QCD counting rules. In fact, as shown in ref. [13], when the unconfining pieces in the ENJL spectral functions are removed by adding an appropriate series of local counterterms, the resulting theory is entirely equivalent to an effective chiral meson theory with three narrow states $V$, $A$ and $S$; very similar to the phenomenological Resonance Dominance Lagrangians proposed in refs. [31, 32]. These Resonance Dominance Lagrangians can be viewed as particular models of large--$N_c$ QCD. They predict the same Green’s functions as the MHA approximation to large--$N_c$ QCD discussed above, in some particular cases but not in general.

In view of the difficulties that these analytic approaches have in reproducing the shape of the simplest Green’s function one can think of, it is difficult to attribute more than a qualitative significance to their “predictions”; $\epsilon'/\epsilon$ in particular, which requires the interplay of several other Green’s functions much more complex than $\Pi_{LR}(Q^2)$.

**METHODODOLOGY AND APPLICATIONS**

The approach that we propose in order to compute a specific coupling of the chiral electroweak Lagrangian consists of the following steps:

1. **Identify the underlying QCD Green’s functions.**
   In most cases of interest, the Green’s functions in question are two–point functions with zero momentum insertions of vector, axial vector, scalar and pseudoscalar currents. The higher the power in the chiral expansion, the higher will be the

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5 For a review, see e.g. ref. [29] where earlier references can be found.

6 See e.g. the three–point functions discussed in ref. [33].
number of insertions. This step is totally general and does not invoke any large–$N_c$ approximation.

2. Work out the short–distance behaviour and the long–distance behaviour of the relevant Green’s functions.

The long–distance behaviour is governed by the Goldstone singularities and can be obtained from $\chi$PT. The short–distance behaviour is governed by the OPE of the currents through which the hard momenta flows. Again, this step is well defined independently of the large–$N_c$ expansion; in practice, however, the calculations simplify a lot when restricted to the appropriate order in the $1/N_c$–expansion one is interested in.

3. Large–$N_c$ ansatz for the underlying Green’s functions.

As already mentioned, the large–$N_c$ ansatz involves only sums of poles; the minimal hadronic ansatz consists in limiting these sums to the minimum number required to satisfy the leading power fall–out at short–distances, as well as the appropriate $\chi$PT long–distance constraints.

All the three steps can be done analytically which helps to show the crucial points of the underlying dynamics. The method is, in principle, improvable by adding constraints from the next–to–leading short–distance inverse power behaviour and/or higher orders in the chiral expansion.

We have tested this approach with the calculation of a few low–energy observables:

• The electroweak $\Delta m_\pi$ mass difference [9] which we have already discussed.

• The hadronic vacuum polarization contribution to the anomalous magnetic moment of the muon $a_\mu$ [34]. The MHA in this case requires one vector–state pole and a pQCD continuum. The absence of dimension two operators in QCD in the chiral limit, constrains the threshold of the continuum. The result, which includes an estimate of the systematic error of the approach, is

$$a_\mu|_{\text{HVP}} = (5.7 \pm 1.7) \times 10^{-8},$$

(16)

to be compared with an average of recent phenomenological determinations

$$a_\mu|_{\text{HVP}} = (6.949 \pm 0.064) \times 10^{-8}.$$  

(17)

• The $\pi^0 \to e^+e^-$ and $\eta \to \mu^+\mu^-$ decay rates [36]. These processes are governed by a $\langle PVV \rangle$ three–point function, with the $Q^2$–momentum flowing through the two $V$ insertions. The MHA in this case requires a vector–pole and a double vector–pole. The predictions of the branching ratios

$$R(P \to l^+l^-) = \frac{\Gamma(P \to l^+l^-)}{\Gamma(P \to \gamma\gamma)},$$

(18)

compared to the experimental determinations are shown in Table 1 below.

7 Unlike the other analytic methods discussed above.

8 See e.g. Prades’s talk at KAON2001 [35] and references therein.
TABLE 1. Summary of branching ratios results

| Branching Ratio (18) | Large–Nc Approach | Experiment [15] |
|----------------------|--------------------|-----------------|
| \( R(\pi^0 \rightarrow e^+ e^-) \times 10^8 \) | 6.2 ± 0.3          | 6.28 ± 0.55     |
| \( R(\eta \rightarrow \mu^+ \mu^-) \times 10^3 \) | 1.4 ± 0.2          | 1.47 ± 0.20     |
| \( R(\eta \rightarrow e^+ e^-) \times 10^8 \) | 1.15 ± 0.05        | < 1.8 × 10^4    |

These successful predictions have encouraged us to start a project of a systematic analysis of non–leptonic \( K \)–decays within this large–\( N_c \) approach. We have first used the example of the neutral current contribution to the \( \Delta m_\pi \) mass difference, (see Eq. (4),) as a theoretical laboratory to show explicitly the cancellation between the renormalization scale in the Wilson coefficient of four–quark operators and in the hadronic matrix elements evaluated in our approach. We have later shown that this cancellation can also be made renormalization scheme independent [37, 38].

So far we have completed two calculations of \( K \)–matrix elements within this large–\( N_c \) approach, which we next discuss.

The \( B_K \)–Factor of \( K^0 - \bar{K}^0 \) Mixing

The factor in question is conventionally defined by the matrix element of the four–quark operator \( Q_{\Delta S=2}(x) = (\bar{s}_L \gamma^\mu d_L)(\bar{s}_L \gamma_\mu d_L)(x) \):

\[
\langle \bar{K}^0 | Q_{\Delta S=2}(0) | K^0 \rangle = \frac{4}{3} f_K^2 M_K^2 B_K(\mu) .
\]  

(19)

To lowest order in the chiral expansion the operator \( Q_{\Delta S=2}(x) \) bosonizes into a term of \( O(p^2) \)

\[
Q_{\Delta S=2}(x) \Rightarrow -\frac{F_0^4}{4} g_{\Delta S=2}(\mu) \left[ (D^\mu U) U \right]_{23} \left[ (D_\mu U^\dagger U) \right]_{23} ,
\]  

(20)

with \( g_{\Delta S=2}(\mu) \) a low energy constant, to be determined, which is a function of the renormalization scale \( \mu \) of the Wilson coefficient \( C(\mu) \) which modulates the operator \( Q_{\Delta S=2}(x) \) in the four–quark effective Lagrangian. A convenient choice of the underlying Green’s function here is the four–point function \( W_{LRLR}(Q^2) \) of two left–currents which carry the \( Q^2 \)–momentum one has to integrate over, and two right–currents with zero momentum insertions. The coupling constant \( g_{\Delta S=2}(\mu) \), which has to be evaluated in the same renormalization scheme as the Wilson coefficient \( C(\mu) \) has been calculated, is then given by an integral [37]

\[
g_{\Delta S=2}(\mu) = 1 - \frac{1}{32 \pi^2 F_0^2} \int_0^\infty dQ^2 \left( \frac{4 \pi \mu^2}{Q^2} \right)^{\epsilon/2} W_{LRLR}(Q^2) ,
\]  

(21)

conceptually similar to the one which determines the electroweak constant \( C \) in Eq. (4). The hadronic ansatz, in the \( 1/N_c \)–expansion, of the Green’s function \( W_{LRLR}(Q^2) \), which
fulfills the leading short–distance constraint and the long–distance constraints which fix $W_{LRLR}(0)$ and $W'_{LRLR}(0)$, requires one vector–pole, a double vector–pole and a triple vector–pole. The invariant $\hat{B}_K$ defined as

$$\hat{B}_K = \frac{3}{4} C(\mu) \times g_{\Delta S=2}(\mu),$$

can then be evaluated, with no free parameters, with the result [37]

$$\hat{B}_K = 0.38 \pm 0.11.$$ (23)

When comparing this result to other determinations, specially in lattice QCD, it should be realized that the unfactorized contribution in Eq. (21) is the one in the chiral limit. It is possible, in principle, to calculate chiral corrections within the same large–$N_c$ approach, but this has not yet been done.

The result in Eq. (23) is compatible with the old current algebra prediction [39] which, to lowest order in chiral perturbation theory, relates the $B_K$ factor to the $K^+ \to \pi^+ \pi^0$ decay rate. In fact, our calculation of the $B_K$ factor can be viewed as a successful prediction of the $K^+ \to \pi^+ \pi^0$ decay rate!

As discussed in ref. [40] the bosonization of the four–quark operator $Q_{\Delta S=2}$ and the bosonization of the operator $Q_2 - Q_1$ which generates $\Delta I = 1/2$ transitions are related to each other in the combined chiral limit and next–to–leading order in the $1/N_c$–expansion. Lowering the value of $\hat{B}_K$ from the leading large–$N_c$ prediction $\bar{B}_K = 3/4$ to the result in Eq. (23) is correlated with an increase of the coupling constant $g_8$ in the lowest order effective chiral Lagrangian, (see Eq. (1),) which generates $\Delta I = 1/2$ transitions, and provides a first step towards a quantitative understanding of the dynamical origin of the $\Delta I = 1/2$ rule.

**Electroweak Four–Quark Operators**

These are the four–quark operators generated by the so called electroweak Penguin like diagrams

$$L \Rightarrow \cdots C_7(\mu)Q_7 + C_8(\mu)Q_8,$$ (24)

with

$$Q_7 = 6(\bar{s}_L\gamma^\mu d_L) \sum_{q=u,d,s} e_q(\bar{q}_R \gamma_\mu q_R) \quad \text{and} \quad Q_8 = -12 \sum_{q=u,d,s} e_q(\bar{s}_L q_R)(\bar{q}_R d_L),$$ (25)

and $C_7(\mu), C_8(\mu)$ their corresponding Wilson coefficients. They generate a term of $O(\mu^0)$ in the effective chiral Lagrangian[42]; therefore, the matrix elements of these operators,

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9 This goes beyond the strict MHA which, in this case, only requires a vector–pole. It is the extra information of knowing $W_{LRLR}(0)$ and $W'_{LRLR}(0)$ in $\chi$PT which forces the presence of the double and triple poles.

10 See e.g., Buras lectures[41]
although suppressed by an $e^2$ factor, are chirally enhanced. Furthermore, the Wilson coefficient $C_8$ has a large imaginary part, which makes the matrix elements of the $Q_8$ operator to be particularly important in the evaluation of $\epsilon'/\epsilon$.

Within the large–$N_c$ framework, the bosonization of these operators produce matrix elements with the following counting

$$\langle Q_7 \rangle |_{O(p^0)} = O(N_c) + O(N_c^0) \quad \text{and} \quad \langle Q_8 \rangle |_{O(p^0)} = O(N_c^2) + \frac{O(N_c^0)}{N_c}$$

Zweig suppressed

A first estimate of the underlined contributions was made in ref. [43]\(^{11}\). The bosonization of the $Q_7$ operator to $O(p^0)$ in the chiral expansion and to $O(N_c)$ is very similar to the calculation of the $Z$–contribution to the coupling constant $C$ in Eq. (4). An evaluation which also takes into account the renormalization scheme dependence has been recently made in ref. [38].

The contribution of $O(N_c^0)$ to $\langle Q_8 \rangle |_{O(p^0)}$ is Zweig suppressed. It involves the sector of scalar (pseudoscalar) Green’s functions where it is hinted from various phenomenological sources that the restriction to just the leading large–$N_c$ contribution may not always be a good approximation. Fortunately, as first pointed out by the authors of ref. [47], independently of large–$N_c$ considerations, the bosonization of the $Q_8$ operator to $O(p^0)$ in the chiral expansion can be related to the four–quark condensate $\langle O_2 \rangle \equiv \langle 0 | (\bar{s}_L \gamma_\mu d_L) (\bar{d}_R \gamma_\nu s_R) | 0 \rangle$ by current algebra Ward identities, the same four–quark condensate which also appears in the OPE of the $\Pi_{LR}(Q^2)$ function discussed above. More precisely

$$\lim_{Q^2 \to \infty} \left( -Q^2 \Pi_{LR}(Q^2) \right) Q^4 = 4\pi^2 \frac{\alpha_s}{\pi} \left( 4\langle O_2 \rangle + \frac{2}{N_c}\langle O_1 \rangle \right) + O\left( \frac{\alpha_s}{\pi} \right)^2,$$

where $\langle O_1 \rangle \equiv \langle 0 | (\bar{s}_L \gamma_\mu d_L) (\bar{d}_R \gamma_\nu s_R) | 0 \rangle$ is the vev which governs $\langle Q_7 \rangle |_{O(p^0)}$. In fact, in the $1/N_c$–expansion [38]

$$\langle O_1 \rangle = \left( -\frac{1}{2}g_{\mu\nu} \int \frac{d^4 q}{(2\pi)^4} \Pi_{LR}^{\mu\nu}(q) \right)_{\text{ren.}} \ms$$

$$= -\frac{3}{32\pi^2} \left[ \sum_A f_A^2 M_A^4 \log \frac{\Lambda^2}{M_A^2} - \sum_V f_V^2 M_V^4 \log \frac{\Lambda^2}{M_V^2} \right],$$

with (NDR means naive dimensional renormalization scheme; HV means ’t Hooft–Veltman scheme,)

$$\Lambda^2 = \mu^2 \exp(1/3 + \kappa); \quad \kappa = -1/2 \quad \text{in NDR}, \quad \text{and} \quad \kappa = +3/2 \quad \text{in HV.} \quad (30)$$

\(^{11}\) The inclusion of final state interaction effects based on the leading large–$N_c$ determination of $\langle Q_8 \rangle$ (and $\langle Q_6 \rangle$) in connection with a phenomenological determination of $\epsilon'/\epsilon$, has been recently discussed in ref. [44].
The crucial observation is that large–$N_c$ QCD gives a rather good description of the $\Pi_{LR}(Q^2)$–function, as we have seen earlier; in particular it implies that, (see Eq. (9),)

$$\lim_{Q^2 \to \infty} (-Q^2 \Pi_{LR}(Q^2)) \propto \sum_V f^2_V M^0_V - \sum_A f^2_A M^0_A.$$  \hspace{1cm} (31)

Solving these equations in the MHA approximation, results in a determination of the matrix elements

$$M_{7,8} \equiv \langle (\pi\pi)_{I=2} | Q_{7,8} | K^0 \rangle_{2\text{GeV}}$$  \hspace{1cm} (32)

at the renormalization scale $\mu = 2$ GeV in the two schemes NDR and HV and in units of GeV$^3$ are shown in Table 2 below, (the first line.)

| TABLE 2. Matrix elements results, (see Eq. (32)) |
|--------------------------------------------------|
| METHOD               | $M_7$(NDR) | $M_7$(HV) | $M_8$(NDR) | $M_8$(HV) |
| Large–$N_c$ Approach |            |           |            |           |
| Ref. [38]            | 0.11 ± 0.03 | 0.67 ± 0.20 | 3.5 ± 1.1 | 3.5 ± 1.1 |
| Lattice QCD          |            |           |            |           |
| Ref. [46]            | 0.11 ± 0.04 | 0.18 ± 0.06 | 0.51 ± 0.10 | 0.62 ± 0.12 |
| Dispersive Approach  |            |           |            |           |
| Ref. [47]            | 0.22 ± 0.05 |          | 1.3 ± 0.3  |           |
| Ref. [48]            | 0.35 ± 0.10 |          | 2.7 ± 0.6  |           |
| Ref. [49]            | 0.18 ± 0.12 | 0.50 ± 0.06 | 2.13 ± 0.85 | 2.44 ± 0.86 |
| Ref. [51]            | 0.16 ± 0.10 | 0.49 ± 0.07 | 2.22 ± 0.67 | 2.46 ± 0.70 |

Also shown in the same table are other evaluations of matrix elements with which we can compare scheme dependences explicitly $^{12}$. Several remarks are in order

- Our evaluations of $M_8$ do not include the terms of $O(\alpha_s^2)$ in Eq. (27) because, as pointed out in Ref. [38], the available results in the literature [45] were not calculated in the right basis of four–quark operators needed here.
- We find that our results for $M_7$ are in very good agreement with the lattice results in the NDR scheme, but not in the HV scheme. This disagreement is, very likely, correlated with the strong discrepancy we have with the lattice result for $M_8$(NDR).
- The recent revised dispersive approach results [49, 51], which now include the effect of higher terms in the OPE, are in agreement, within errors, with the large–$N_c$ approach results. In fact, the agreement improves further if the new $O(\alpha_s^2)$ corrections, which have now been calculated in the right basis [51], are also incorporated in the large–$N_c$ approach.

$^{12}$ There is another “dispersive determination” in the literature [50] since the HEP-2001 conference, but it is controversial as yet; this is why we do not include it in the Table.
Both the revised dispersive approach results and the large–$N_c$ approach results for $M_8$ are higher than the lattice results. The discrepancy may originate in the fact that, for reasonable values of $\langle \bar{\psi} \psi \rangle$, most of the contribution to $M_8$ appears to come from an OZI–violating Green’s function which is something inaccessible in the quenched approximation at which the lattice results, so far, have been obtained.

Conclusion and Outlook

We hope to have shown with these examples that the large–$N_c$ approach that we have discussed, provides a very useful framework to formulate calculations of genuinely non–perturbative nature, like the low–energy constants of the effective chiral Lagrangian, both in the strong and the electroweak sector.

Other calculations in progress, by various groups of people and in order of advancement, are

- *The electroweak hadronic contributions to $g_\mu - 2$.*
- *The matrix elements of the strong Penguin operator $Q_6$, relevant for $\epsilon'/\epsilon$.*
- *The light–by–light hadronic contributions to $g_\mu - 2$; in particular the one generated by the convolution of two $\langle PVV \rangle$ three–point functions.*
- *The chiral corrections to the $B_K$–factor.*

We hope to have the results in the near future.

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