AFFINE LINES ON $\mathbb{Q}$-HOMOLOGY PLANES AND GROUP ACTIONS

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Abstract. This note is a supplement to the papers [KiKo] and [GMMR]. We show the role of group actions in classification of affine lines on $\mathbb{Q}$-homology planes.

INTRODUCTION

This note is a supplement to the papers [KiKo] and [GMMR]. Our aim is to shed light on the role of group actions in classification of affine lines on $\mathbb{Q}$-homology planes with logarithmic Kodaira dimension $-\infty$. This enables us to strengthen certain results in loc. cit. (see Section 1).

Let us fix terminology. It is usual [Mi] Ch. 3, §4 to call a smooth $\mathbb{Q}$-acyclic ($\mathbb{Z}$-acyclic, respectively) surface over $\mathbb{C}$ a $\mathbb{Q}$-homology plane (a homology plane, respectively). By Fujita’s Lemma [Fu, 2.5] such a surface is necessarily affine. Likewise we call a homology line an irreducible affine curve $\Gamma$ with Euler characteristic $e(\Gamma) = 1$. So $\Gamma$ is homeomorphic to $\mathbb{R}^2$ and its normalization is isomorphic to $\mathbb{A}^1 = \mathbb{A}^1_{\mathbb{C}}$. A smooth curve isomorphic to $\mathbb{A}^1$ will be called an affine line. Following [Mi] we let $\mathbb{A}^1_{k} = \mathbb{A}^1 \setminus \{0\}$. As usual $k$ stands for logarithmic Kodaira dimension.

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1. Main results

**Theorem 1.** Let $X$ be a $\mathbb{Q}$-homology plane and $\Gamma$ a homology line on $X$. Then the following hold.

(a) Suppose that $\bar{k}(X \setminus \Gamma) = -\infty$. Then $\Gamma$ is either an orbit of an effective $\mathbb{C}^+$-action on $X$ or a connected component of the fixed point set of such an action. Anyhow $\Gamma \simeq \mathbb{A}^1$ is a fiber component of the corresponding orbit map (an $\mathbb{A}^1$-ruling) $\pi : X \to \mathbb{A}^1$.

(b) Suppose that $\bar{k}(X \setminus \Gamma) \geq 0$. Suppose further that $\Gamma \simeq \mathbb{A}^1$ and $\bar{k}(X) = -\infty$. Then $\Gamma$ is an orbit closure of an effective hyperbolic $\mathbb{C}^*$-action on $X$. Moreover $X$ admits an effective action of a semidirect product $G = \mathbb{C}^* \rtimes \mathbb{C}^+$ with an open orbit $U$. The orbit map $X \to \mathbb{A}^1$ of the induced $\mathbb{C}^+$-action defines an $\mathbb{A}^1$-ruling on $X$ with a unique multiple fiber say $\Gamma' \simeq \mathbb{A}^1$ such that $\Gamma$ and $\Gamma'$ meet at one point transversally and $U = X \setminus \Gamma' \simeq \mathbb{A}^1 \times \mathbb{A}^1$. Furthermore this $\mathbb{C}^+$-action moves $\Gamma$. Consequently there exists a continuous family of affine lines $\Gamma_t$ on $X$ with the same properties as $\Gamma$.

(c) Suppose that $\Gamma$ is singular. Then $X \simeq \mathbb{A}^2$ and $\bar{k}(X \setminus \Gamma) = 1$. Moreover there is an isomorphism $X \simeq \mathbb{A}^2$ sending $\Gamma$ to a curve $V(x^k - y^l)$ with coprime $k, l \geq 2$. Consequently $\Gamma$ is an orbit closure of an elliptic $\mathbb{C}^*$-action on $X$.

We indicate below a proof of the theorem. The cases (a), (b) and (c) are proven in Sections 2, 3 and 4, respectively. Besides, in cases (a) and (b) we provide in Lemmas 3 and 7, respectively, a description of the pairs $(X, \Gamma)$ satisfying their assumptions. The assertion of (b) follows from Theorem 1.1 in [KiKo], cf. also Theorem 3.10 in [GMMR]. In the case of a $\mathbb{Z}$-homology plane (c) was established in [Za]; the proof for a $\mathbb{Q}$-homology plane is similar. This gives a strengthening of Theorem 1.3 in [KiKo].

The cases (a)-(c) of Theorem 1 do not exhaust all the possibilities for the pair $(X, \Gamma)$ as above. To complete the picture let us summarize some known facts, see e.g. [Za, GuPa, Mi, Ch. 3, §4] and the references therein.

**Theorem 2.** We let as before $X$ be a $\mathbb{Q}$-homology plane and $\Gamma \subseteq X$ a homology line. If $\Gamma$ is singular then $(X, \Gamma)$ is as in Theorem 1(c). Suppose further that $\Gamma$ is smooth i.e. is an affine line. Then $\bar{k}(X) \leq \bar{k}(X \setminus \Gamma) \leq 1$ and one of the following cases occurs.

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2 This is due to the Lin-Zaidenberg Theorem [LiZa, Mi, Ch. 3, §3].

3 See [Mi, Ch.2, Theorem 6.7.1].
(a) \((\bar{k}(X), \bar{k}(X \setminus \Gamma)) = (-\infty, -\infty)\) and \((X, \Gamma)\) is as in Theorem 1(a) that is, \(\Gamma\) is of fiber type and \(X \setminus \Gamma\) carries a family of disjoint affine lines.

(b) \((\bar{k}(X), \bar{k}(X \setminus \Gamma)) = (-\infty, 0)\) or \((-\infty, 1)\) and \((X, \Gamma)\) is as in Theorem 1(b).

(c) \((\bar{k}(X), \bar{k}(X \setminus \Gamma)) = (0, 0)\) and either \(X\) is not NC minimal or \(X\) is one of the Fujita’s surfaces \(H[-k, k]\) \((k \geq 1)\). Anyhow \(\Gamma\) is a unique affine line on \(X\) unless \(X = H[-1, 1]\).

(d) \((\bar{k}(X), \bar{k}(X \setminus \Gamma)) = (0, 1)\), \(X = H[-1, 1]\) and there are exactly two affine lines, say, \(\Gamma_0\) and \(\Gamma_1 = \Gamma\) on \(X\). These lines meet transversally in two distinct points, moreover \(\bar{k}(X \setminus \Gamma_0) = 0\) and \(\bar{k}(X \setminus \Gamma_1) = 1\).

(e) \((\bar{k}(X), \bar{k}(X \setminus \Gamma)) = (1, 1)\), there is a unique \(\mathbb{A}^1\)-fibration on \(X\) and \(\Gamma\) is a fiber component of its degenerate fiber. There can be at most one further affine line on \(X\), which is then another component of this same degenerate fiber, and these components meet transversally in one point.

Remark 1. Let \(X\) be a \(\mathbb{Z}\)-homology plane. By [Fu] then \(\bar{k}(X) \neq 0\). By [Za] (supplement) \(\bar{k}(X) = 1\) if and only if there exists a unique homology (in fact, affine) line on \(X\).

2. \(\mathbb{Q}\)-HOmology PlanEs wIth an \(\mathbb{A}^1\)-ruling

These occur to be smooth affine surfaces with \(\mathbb{A}^1\)-rulings \(X \to \mathbb{A}^1\) which possess only irreducible degenerate fibers. They were studied in details e.g. in [Fu 4.14], [Ba], [Fi], [FLZa], [4]. See also [Mi] Ch. 3, 4.3.1 for a brief summary.\(^5\) In Lemma 3 below we show that every \(\mathbb{A}^1\)-ruling \(\pi : X \to \mathbb{A}^1\) on a \(\mathbb{Q}\)-homology plane \(X\) can be obtained starting from a standard linear \(\mathbb{A}^1\)-ruling \(\mathbb{A}^2 \to \mathbb{A}^1\) and replacing several fibers by multiple fibers via a procedure called in [FLZa] a comb attachment. More precisely, this replacement goes as follows.

Attaching combs. On the quadric \(\mathbb{P}^1 \times \mathbb{P}^1\) with a \(\mathbb{P}^1\)-ruling \(\pi_0 = \text{pr}_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1\) we fix a finite set of points \(\{A_j\}, j = 1, \ldots, n\) \((n \geq 0)\) in different fibers \(F_j = \{t_j\} \times \mathbb{P}^1\) of \(\pi_0\). We fix further a

\(^4\)See also Lemma 3 below.

\(^5\)The both possibilities actually occur, see the Construction in Section 3 and also Lemma 7.

\(^6\)We refer e.g. to [Fu, GuPa, Mi] Ch. 3, 4.4.1-4.4.2 for definitions.

\(^7\)The same conclusions hold also in case (c) if \(X\) is not NC-minimal [GuPa].

\(^8\)We note [Ba] that \(\pi_1(X)\) is a free product of cyclic groups, namely, \(\pi_1(X) \cong \ast_j \mathbb{Z}/m_j \mathbb{Z}\), where \((m_j)_{j}\) is the sequence of multiplicities of degenerate fibers, and so \(H_1(X; \mathbb{Z}) \cong \bigoplus_j \mathbb{Z}/m_j \mathbb{Z}\).
sequence \( \sigma : V \to \mathbb{P}^1 \times \mathbb{P}^1 \) of blowups with centers at the points \( A_j \) and infinitesimally near points. Letting \( \bar{\pi} : V \to \mathbb{P}^1 \) be the induced \( \mathbb{P}^1 \)-ruling, we suppose that \( \bar{\pi} \) enjoys the following properties:

- the center of every blowup over \( A_j \) except for the first one belongs to the exceptional \((-1)\)-curve of the previous blowup;
- \( D_\infty \cdot E_j = 0 \) \( \forall j = 1, \ldots, n \), where \( D_\infty \) is the proper transform in \( V \) of the section \( \mathbb{P}^1 \times \{\infty\} \) of \( \text{pr}_1 \) and \( E_j \) is the last \((-1)\)-curve in the fiber \( \bar{\pi}^{-1}(t_j) \).
- \( E_j \) is a tip of the dual graph of the fiber \( \bar{\pi}^{-1}(t_j) \).

Under these assumptions the dual graph as above is a comb, with all vertices of degree \( \leq 3 \). Let \( F_\infty = \bar{\pi}^{-1}(t_\infty) \subset V \) be a fiber over an extra point \( t_\infty = t_1, \ldots, t_n \) and \( E \subset V \) be the reduced exceptional divisor of \( \sigma : V \to \mathbb{P}^1 \times \mathbb{P}^1 \). We consider the open surface \( X = V \setminus D \), where \( D = F_\infty + D_\infty + E + \sum_{j=1}^n (F_j' - E_j) \) and \( F_j' \) is the proper transform of \( F_j \) in \( V \). Then \( \bar{\pi} : V \to \mathbb{P}^1 \) restricts to an \( \mathbb{A}^1 \)-ruling \( \pi : X \to \mathbb{A}^1 \) with only irreducible fibers; all fibers of \( \pi \) are reduced except possibly the fibers \( \pi^{-1}(t_j) = E_j \cap X \).

The following lemma is well known, see e.g. [FlZa] Proposition 4.9.

**Lemma 3.** Under the notation as above the surface \( X \) is a \( \mathbb{Q} \)-homology plane. Moreover, every \( \mathbb{Q} \)-homology plane \( X \) with an \( \mathbb{A}^1 \)-ruling \( \pi : X \to \mathbb{A}^1 \) arises in this way.

**Proof.** Let \( X \) be constructed as above. By the Suzuki formula [Suz, Za, Gu], \( e(X) = 1 \) and so the equality \( b_2 = b_1 + b_3 \) for the Betti numbers of \( X \) holds. Thus \( X \) is \( \mathbb{Q} \)-acyclic if and only if \( b_2 = 0 \) or equivalently, if \( \text{Pic}(D) \otimes \mathbb{Q} \) generates \( \text{Pic}(V) \otimes \mathbb{Q} \), see [Mi] Ch. 3, 4.2.1. The latter is easily seen to be the case in our construction. The first assertion follows now by Fujita’s Lemma [Fu, 2.5].

As for the second one, given an \( \mathbb{A}^1 \)-ruling \( \pi : X \to \mathbb{A}^1 \) on a \( \mathbb{Q} \)-homology plane \( X \) it extends to a pseudominimal \( \mathbb{P}^1 \)-ruling \( \pi : V \to \mathbb{P}^1 \) on a smooth completion \( V \) of \( X \) with an SNC boundary divisor \( D \). The pseudominimality means that none of the \((-1)\)-curves in \( D - D_\infty \), where \( D_\infty \) is the horizontal component of \( D \), can be contracted without losing the SNC property, see [Za, 3.4]. Since \( e(X) = 1 \) all fibers of \( \pi : X \to \mathbb{A}^1 \) are irreducible. We let \( \bar{\pi}^{-1}(t_j), j = 1, \ldots, n \) be the degenerate fibers of \( \bar{\pi} \) and \( E_j \) be the component of the fiber \( \bar{\pi}^{-1}(t_j) \) such that \( E_j \cap X = \pi^{-1}(t_j) \cong \mathbb{A}^1 \). By the pseudominimality assumption, \( E_j \) is the only \((-1)\)-curve in the fiber \( \bar{\pi}^{-1}(t_j) \). Therefore \( V \) is obtained from a Hirzebruch surface \( \Sigma_m \) by blowing up process which enjoys the properties of a comb attachment. Performing, if necessary, elementary
transformations in the fiber $F_\infty$ we may assume that $D_\infty^2 = 0$, where $D_\infty$ is the image of $D_\infty$ in $\Sigma_m$ and so, $\Sigma_m = \Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$. \hfill \qed

**Remark 2.** Every surface $X$ as considered above can actually be obtained from affine plane $\mathbb{A}^2$ via a suitable affine modification that is [KaZa, §1], by blowing up with center in a zero dimensional subscheme $V(I)$ of $\mathbb{A}^2$ located on a principal divisor $D$ and deleting the proper transform of $D$. Indeed $X$ contains a cylinder $U \times \mathbb{A}^1$, where $U \subseteq \mathbb{A}^1$ is a Zariski open subset, see [MiSu] or [Mi, Ch. 3, 1.3.2]. The canonical projections of $U \times \mathbb{A}^1$ to the factors regarded as rational functions on $X$, say, $f, h$, can be made regular by multiplying $h$ by an appropriate polynomial $q \in \mathbb{C}[t]$. Then $\varphi = (f, g) : X \to \mathbb{A}^2$, where $g = qh$, yields a birational morphism. Since every birational morphism between affine varieties is an affine modification [KaZa, Prop. 1.1] the claim follows.

For instance the following example from [Be] can be treated in terms of affine modifications.

**Example 1.** ([Be, Ex. 2.6.1], [KaZa, 7.1]) The Bertin surfaces are surfaces in $\mathbb{A}^3$ with equations

$$x^e z = x + y^d.$$  

Every such surface $X$ appears as affine modification of the plane $\mathbb{A}^2 = \text{Spec} \mathbb{C}[x, y]$ with center $(I, (x^e))$, where $I = (x^e, x + y^d) \subseteq \mathbb{C}[x, y]$. Actually $X$ is a $\mathbb{Q}$-homology plane with $\text{Pic}(X) \cong H_1(X; \mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$, and $\pi = x|X : X \to \mathbb{A}^1$ gives an $\mathbb{A}^1$-ruling on $X$ with a unique multiple fiber of multiplicity $d$ over $x = 0$. Whereas the $\mathbb{A}^1$-fibration $f = x^{e-1}z : X \to \mathbb{A}^1$ appears as the orbit map of a $\mathbb{C}^*$-action on $X$ (cf. Remark 1.1 below).

**Proposition 4.** Any two disjoint homology lines $\Gamma_0$ and $\Gamma_1$ on a $\mathbb{Q}$-homology plane $X$ appear as two different fibers of an $\mathbb{A}^1$-ruling $\pi : X \to \mathbb{A}^1$. In particular, $X$ arises as in the above construction. If, moreover, $X$ is a $\mathbb{Z}$-homology plane then there exists an isomorphism $X \cong \mathbb{A}^2$ sending $\Gamma_0, \Gamma_1$ to two parallel lines.

**Proof.** The second assertion is proven in [Za §9]. The first one can be deduced by a similar argument. Namely, $\text{Pic}(X)$ being a torsion group, $m\Gamma_0$ is a principal divisor and so $m\Gamma_0 = q^*(0)$ for some $m \in \mathbb{N}, q \in \mathcal{O}[X]$. Applying Stein factorisation we may assume that the general fibers of $q$ are irreducible. Since $\Gamma_0$ and $\Gamma_1$ are disjoint, $\Gamma_1$ is a component of a fiber, say, $F_1 = q^{-1}(1)$. Let the degenerate fibers of $q$ be among the fibers $F_0 = \Gamma_0, F_1, \ldots, F_n$, and denote $F$ a general fiber.
of $q$. By the Suzuki formula \textit{loc. cit.}

$$\sum_{j=0}^{n} (e(F_j) - e(F)) = 1 - e(F),$$

where all summands are non-negative by [Za, 3.2]. Since $e(F_0) = e(\Gamma_0) = 1$ we have $e(F_j) = e(F) \forall j = 1, \ldots, n$. It follows by [Za, 3.2] or [Gu] that either

(i) the fibers $F_j$ are general \(\forall j = 1, \ldots, n\), or

(ii) $F \cong F_j \cong \mathbb{A}^1 \forall j = 1, \ldots, n$.

The case (ii) must be excluded since $\Gamma_1 \subseteq F_1$ and $e(\Gamma_1) = 1$. If (i) holds then $F_1 \cong \pi^{-1}(1)$ is a general fiber of $q$, hence $F \cong F_1 \cong \mathbb{A}^1$. Thus in any case all fibers of $\pi$ are isomorphic to $\mathbb{A}^1$ and so, $\Gamma_0, \Gamma_1$ are fibers of the $\mathbb{A}^1$-ruling $\pi = q : X \to \mathbb{A}^1$.

\[\square\]

Now one can easily deduce Theorem 1(a).

\textbf{Proof of Theorem 1(a).} $X \setminus \Gamma$ being a smooth affine surface with $\bar{k}(X \setminus \Gamma) = -\infty$, there exists an $\mathbb{A}^1$-ruling on $X \setminus \Gamma$ [Mi, 2.1.1]. The curve $\Gamma_0 = \Gamma$ and a general fiber, say, $\Gamma_1$ of this ruling provide two disjoint homology lines on $X$. By Proposition 4 $\Gamma_0$ and $\Gamma_1$ are two different fibers of an $\mathbb{A}^1$-ruling $\pi : X \to \mathbb{A}^1$ on $X$ and so, $\Gamma = \Gamma_0$ is an affine line stable under an effective $\mathbb{C}^*$-action on $X$ along this ruling, see e.g. [FlZa3, 1.6]. Now the assertion follows easily. \[\square\]

\textbf{3. $\mathbb{Q}$-HOMOLOGY PLANES WITH A $\mathbb{C}^*$-ACTION}

To deduce Theorem 1(b) we recall first Example 1 in [KiKo], cf. also Examples 3.8, 3.9 in [GMMR].

\textbf{Construction.} The construction begins with a divisor $D_0 = M_a + F_0 + F_1 + F_\infty$ on a Hirzebruch surface $\Sigma_a$, where $F_0, F_1, F_\infty$ are 3 distinct fibers of the standard projection $\pi_0 : \Sigma_a \to \mathbb{P}^1$ and $M_a$ are two disjoint sections with $M_a^2 = -a$. We may suppose that $F_j = \pi_0^{-1}(j)$ ($j = 0, 1, \infty \in \mathbb{P}^1$). Besides, the construction involves a sequence of \textit{inner} blowups $\mu : V \to \Sigma_a$ over $D_0$ i.e., successive blowups with centers at double points on $D_0$ or on its total transforms. This results in a $\mathbb{Q}$-homology plane $X = V \setminus D$, where $D \subseteq \mu^{-1}(D_0)$ is a suitable SNC tree of rational curves on $V$ (see a description below). The induced $\mathbb{P}^1$-ruling $\bar{\pi} : V \to \mathbb{P}^1$ restricts to an untwisted $\mathbb{A}^1$-fibration $\pi : X \to \mathbb{A}^1$.

More precisely $\mu$ replaces the fiber $F_j$ ($j = 0, 1$) by a linear chain of smooth rational curves with a unique $(-1)$-curve $E_j$, which is a multiple component of the corresponding divisor $\bar{\pi}^*(j)$. The dual graph...
of the chain $\bar{\pi}^{-1}(0)$ has a sequence of weights $[-n, -1, -2, \ldots, -2]$, where for the strict transform $F'_0$ of $F_0$ on $V$ one has $F'_0 = -n \leq -2$. The boundary divisor $D$ appears as the total transform of $D_0$ in $V$ with the components $E_0, E_1$ and $F'_0$ being deleted. In the affine part $X = V \setminus D$ these deleted components form the only degenerate fibers of $\pi$, namely $\pi^*(0) = \Gamma + n(E_0 \cap X)$ and $\pi^*(1) = m(E_1 \cap X)$, where $\Gamma = F'_0 \cap X$ and $m, n \geq 2$. Thus the only reducible affine fiber $\bar{\pi}^{-1}(0) = (F'_0 + E_0) \cap X$ is isomorphic to the cross $A^1 \times A^1 = \{(xy = 0) \subset A^2\}$. Furthermore $\bar{\pi}^{-1}(1) = E_1 \cap X \simeq A^1_U$ is an irreducible multiple fiber. Finally $\bar{\pi}^{-1}(F_\infty) = F'_\infty \subseteq D$. A computation in [KiKo, Example 1] shows that $\bar{k}(X \setminus \Gamma) = 0$ if $m = n = 2$ and $\bar{k}(X \setminus \Gamma) = 1$ otherwise.

Remark 3. One could consult e.g. [FlZa1] for a construction giving all $\mathbb{Q}$-homology planes $X$ with an $A^1_U$-fibration $\pi : X \to B$. In the terminology of [FlZa1], such a surface with a twisted (untwisted) $A^1_U$-fibration over $B = A^1$, $B = \mathbb{P}^1$, respectively, is said to be of type $A1$, $A2$ ($B1$, $B2$), respectively. Thus the surface $X$ as in the Construction above is of type $B1$, with a comb attachment applied at $F_0$ and with $F_1$ replaced by a fiber of a broken chain type in the terminology of [FlZa1].

In the following lemma we prove the first assertion of Theorem 1(b).

We recall that a $\mathbb{C}^*$-action on $X$ is hyperbolic if its general orbits are closed, elliptic if it possesses an attractive or repelling fixed point in $X$, and parabolic if its fixed point set is one-dimensional.

Lemma 5. If $(X, \Gamma)$ satisfies the assumptions of Theorem 1(b) then $\Gamma$ is an orbit closure of an effective hyperbolic $\mathbb{C}^*$-action on $X$.

Proof. According to Theorem 1.1 in [KiKo] (cf. Theorem 3.10 in [GMNR]), under our assumptions $(X, \Gamma)$ is one of the pairs as in the above Construction. There exists an effective $\mathbb{C}^*$-action on $\Sigma_a$ along the fibers of $\pi_0$ with the fixed point set equal to $M_a \cup \bar{M}_a$. By induction on the number of blowups this $\mathbb{C}^*$-action lifts to $V$ stabilizing the total transform $\mu^{-1}(D_0)$. Indeed the centers of successive inner blowups in $\mu$ are fixed under the $\mathbb{C}^*$-action constructed on the previous step, and so Lemma 2.2(b) in [FKZ] applies. It follows that the curve $D \subseteq \mu^{-1}(D_0)$ as in the Construction above is stable under the lifted $\mathbb{C}^*$-action, so this action restricts to a hyperbolic $\mathbb{C}^*$-action on $X = V \setminus D$. In turn, the affine line $\Gamma$ on $X$ as in the Construction is an orbit closure for this restricted action, as required.  \[\square\]
The resulting surface $X$ with a hyperbolic $\mathbb{C}^*$-action admits the following description in terms of the DPD orbifold presentation as elaborated in \[\text{[FlZa2]}\].

**DPD presentation.** Let $C = \text{Spec } A_0$ be a smooth affine curve and $(D_+, D_-)$ be a pair of $\mathbb{Q}$-divisors on $C$ with $D_+ + D_- \leq 0$. Letting $A_{\pm k} = H^0(C, \mathcal{O}(kD_\pm))$, $k \geq 0$ we consider the graded $A_0$-algebra $A = A_0[D_+, D_-] = \bigoplus_{k \in \mathbb{Z}} A_k$ and the associated normal affine surface $X = \text{Spec } A$. The grading determines, in a usual way, a graded semisimple Euler derivation $\delta$ on $A$, where $\delta(a_k) = ka_k \forall a_k \in A_k$, and, in turn, an effective hyperbolic $\mathbb{C}^*$-action on $X$. Vice versa, any effective hyperbolic $\mathbb{C}^*$-action on a normal affine surface with the orbit space $C$ arises in this way \[\text{[FlZa2]}\ 4.3\].

Let $\pi : X \to C$ be the orbit map. Given a point $p \in C$ we let $m_{\pm}(p)$ denote the minimal positive integer such that $m_{\pm}(p)D_{\pm}(p) \in \mathbb{Z}$. In case where $(D_+ + D_-)(p) = 0$ we set $m(p) = m_{\pm}(p)$. If $(D_+ + D_-)(p) < 0$ then the fiber $\pi^{-1}(p)$ is reducible, isomorphic to the cross $\mathbb{A}^1 \setminus \mathbb{A}^1$ in $\mathbb{A}^2$ and consists of two orbit closures $\bar{O}_p^\pm$. Its unique double point $p'$ is a fixed point; $p'$ is smooth on $X$ if and only if $(D_+ + D_-)(p) = -1/m_+m_- \text{ [FlZa2] \ 4.15\}$. Actually $m_+(p)$ are the multiplicities of the curves $\bar{O}_p^\pm$, respectively, in the divisor $\pi^*(p)$.

In case where $(D_+ + D_-)(p) = 0$ the fiber $O_p = \pi^{-1}(p) \simeq \mathbb{A}^1_+$ is irreducible of multiplicity $m(p)$ in $\pi^*(p)$.

The inversion $\lambda \mapsto \lambda^{-1}$ in $\mathbb{C}^*$ results in interchanging $D_+$ and $D_-$, respectively, $\bar{O}_p^+$ and $\bar{O}_p^-$. Passing from the pair $(D_+, D_-)$ to another one $(D'_+, D'_-) = (D_+ + D_0, D_- - D_0)$ with a principal divisor $D_0$ on $C$ results in passing from $A$ to an isomorphic graded $A_0$-algebra $A'$, so the corresponding $\mathbb{C}^*$-surfaces are equivariantly isomorphic over $C$.

**Lemma 6.** Given a normal affine surface $X = \text{Spec } A$ with a hyperbolic $\mathbb{C}^*$-action determined by a pair $(D_+, D_-)$ of $\mathbb{Q}$-divisors on the affine curve $C = \text{Spec } A_0$ with $D_+ + D_- \leq 0$, we denote by $p_1, \ldots, p_l, q_1, \ldots, q_k$ the points of $C$ with $(D_+ + D_-)(p_j) < 0$, $(D_+ + D_-)(q_i) = 0$ and $m(q_j) \geq 2$, respectively. Letting $\pi : X \to C$ be the orbit map we assume that $C \simeq \mathbb{A}^1$ and that $X$ is smooth that is, $(D_+ + D_-)(p_j) = -1/m_+(p_j)m_-(p_j) \forall j = 1, \ldots, l$. Then the following hold.

(a) $e(X) = l$.
(b) $\text{Pic}(X) \otimes \mathbb{Q} = 0$ if and only if $l \leq 1$ that is, $\pi$ has at most one reducible fiber.

\[\text{i.e. the Dolgachev-Pinkham-Demazure presentation.}\]
(c) Moreover $X$ is $\mathbb{Q}$-acyclic if and only if $l = 1$. In the latter case
\[ \pi : X \to \mathbb{A}^1 \] is an untwisted $\mathbb{A}^1_\ast$-fibration.$^{10}$

Proof. (a) holds by the additivity of the Euler characteristic, and (b) follows from the description of the Picard group $\text{Pic}(X)$ in \cite[4.24]{FlZa}. For a smooth rational affine surface $X$ we have $b_3 = b_4 = 0$ and $b_1 = \rho(X)$, where $\rho(X)$ is the Picard number of $X$ \cite[Ch. 3, 4.2.1]{Mi}. Thus $X$ is $\mathbb{Q}$-acyclic if and only if $e(X) = 1$ and $\text{Pic}(X) \otimes \mathbb{Q} = 0$, whence (c) follows. □

Lemma 7. Every $\mathbb{Q}$-homology plane $X$ as in the above Construction is isomorphic to a $\mathbb{C}^\ast$-surface $\text{Spec} \, A_0[D_+, D_-]$, where $A_0 = \mathbb{C}[t]$ and
\[ D_+ = \frac{e}{m}[1], \quad D_- = -\frac{1}{n}[0] - \frac{e}{m}[1] \] with $0 < e < m$, $\gcd(e, m) = 1$, $m, n \geq 2$.

Conversely, every $\mathbb{C}^\ast$-surface with such a DPD-presentation appears via the above Construction.

Proof. By our Construction, the degenerate fibers of the induced $\mathbb{A}^1_\ast$-family $\pi : X \to \mathbb{A}^1$ are $\pi^*(0) = n(E_0 \cap X) + \Gamma$ and $\pi^*(1) = m(E_1 \cap X)$, where $\Gamma = F'_0 \cap X$, $E_0 \cdot \Gamma = 1$ and $E_j (j=0,1)$ is the unique $(-1)$-curve in the fiber $\tilde{\pi}^{-1}(j)$ of the induced $\mathbb{P}^1$-ruling $\tilde{\pi} : V \to \mathbb{P}^1$. Clearly, all these curves are orbit closures for the $\mathbb{C}^\ast$-action on $X$ as in Lemma 5.

We may suppose that, in the notation as above, $\Gamma = O_0^+$, $E_0 \cap X = O_0^-$ and $E_1 \cap X = O_1$ so that
\[ k = l = 1, \quad p_1 = 0, \quad q_1 = 1, \quad m_+(0) = 1, \quad m_-(0) = n \] and $m(1) = m$.

Since every integral divisor on $C = \mathbb{A}^1$ is principal, passing to an equivalent pair of $\mathbb{Q}$-divisors we may achieve that $(D_+, D_-)$ is a pair as in the lemma. This proves the first assertion. The converse easily follows by virtue of Lemma 6. □

Remarks 1. 1. According to \cite[5.5]{FlZa}, for $e = 1$ and $m | n$ the above surfaces actually coincide with the Bertin surfaces from Example 1.

2. The formula for the canonical divisor in \cite[4.25]{FlZa} gives in our case
\[ K_X = -(e(n-1) + 1)[O_1], \quad \text{where} \quad m[O_1] = 0. \]

Therefore $K_X = 0$ if and only if $e(n-1) \equiv -1 \mod m$. The question arises whether, among the $\mathbb{C}^\ast$-surfaces from Lemma 7 satisfying the latter condition, the Bertin surfaces are the only hypersurfaces.

3. For $n > 1$ the fractional part $\{D_-\}$ in Lemma 7 is supported on 2 points, hence by \cite[4.5]{FlZa} the surface $X$ as in Lemma 7 admits a

\[ ^{10}\text{It is of type B1 in the classification of \cite{FlZa}.} \]
unique $A^1$-ruling $X \to A^1$ (i.e., $X$ is of class $ML_1$ in the terminology of [GMMR]).

In contrast, for $n = 1$ there is a second $A^1$-ruling $X \to A^1$, so $X$ has trivial Makar-Limanov invariant. In particular for $e/m = 1/2$ and $n = 1$ by virtue of [FlZa3, 5.1], $X \simeq \mathbb{P}^2 \setminus \Delta$, where $\Delta$ is a smooth conic in $\mathbb{P}^2$.

4. Following [FlZa2, 4.8] it is possible to define, by explicit equations, a family of surfaces in $\mathbb{A}^4$, not necessarily complete intersections, whose normalizations are the $\mathbb{Q}$-homology planes in the above Construction.

Now we are ready to complete the proof of Theorem 1(b).

Proof of Theorem 1(b). Since the fractional part $\{D_+\}$ of the divisor $D_+$ as in Lemma 7 is supported on one point, there exists a graded locally nilpotent derivation on $A$ of positive degree, see [FlZa3, 2.2, 3.23]. This derivation generates an effective $\mathbb{C}^+$-action on $X$, and also an action of a semidirect product $G = \mathbb{C}^* \rtimes \mathbb{C}^+$ with an open orbit $U \simeq \mathbb{A}^1 \times \mathbb{A}^1$. Moreover by [FlZa3, 3.25], the orbit map $X \to \mathbb{A}^1$ of the associate $\mathbb{C}^+$-action has a unique irreducible multiple fiber $\Gamma' = \mathbb{O}_0 \setminus X$ of multiplicity $m \geq 2$. General orbits of this $\mathbb{C}^+$-action on $X$ being transversal to $\Gamma$, the action moves $\Gamma$, as stated. □

4. ISOTRIVIAL FAMILIES OF CURVES AND $\mathbb{C}^*$-ACTIONS

To indicate a proof of Theorem 1(c) let us recall first a necessary result from [LiZa, Za]. For the sake of completeness we sketch the proof.

Lemma 8. ([LiZa, Lemma 5]) Let $X^*$ be a smooth affine surface and $\pi : X^* \to A^1$ be a family of curves without degenerate fibers which is not a twisted $A^1_+$-family. Then $\pi$ is equivariant with respect to a suitable effective $\mathbb{C}^*$-action on $X^*$ and a nontrivial $\mathbb{C}^*$-action on $A^1_+$.

Proof. Let $F$ denote a general fiber of $\pi$. In the case where $F \simeq A^1$ the surface $X^*$ admits a completion which is a Hirzebruch surface $\Sigma_a$ with the boundary divisor $D = \Sigma_a \setminus X^*$ consisting of a section and two fibers. It follows that $\pi$ is a trivial family, which implies the assertion. The same argument applies if $F \simeq A^1_+$ since in this case by our assumption $\pi$ is untwisted.

Suppose further that $e(F) < 0$ i.e. that $F$ is a hyperbolic curve. By Bers’ Theorem the Teichmuller space corresponding to $F$, with its natural complex structure, is biholomorphic to a bounded domain in $\mathbb{C}^M$ for some $M > 0$, hence is as well hyperbolic. Therefore the family $\pi$ over a non-hyperbolic base $A^1_+$ is isotrivial i.e., its fibers are all pairwise isomorphic. Since $\text{Aut}(F)$ is a finite group the monodromy $\mu \in \text{Aut}(F)$
of the family $\pi$ has finite order, say, $N$. After a cyclic étale base change $z \mapsto z^N$ we obtain a trivial family $F \times \mathbb{A}^1_\mathbb{C} \to \mathbb{A}^1_\mathbb{C}$, which is a cyclic étale covering of the given family $\pi$. The standard $\mathbb{C}^*$-action on its base lifts to a free $\mathbb{C}^*$-action on $F \times \mathbb{A}^1_\mathbb{C}$ commuting with the monodromy $\mathbb{Z}/N\mathbb{Z}$-action. Therefore the lifted $\mathbb{C}^*$-action descends to $X^*$ so that $\pi$ becomes equivariant with respect to the $\mathbb{C}^*$-action $\lambda z = \lambda^N z$ on $\mathbb{A}^1_\mathbb{C}$, as needed.

**Remark 4.** The $\mathbb{A}^1_\mathbb{C}$-family of orbits of a hyperbolic $\mathbb{C}^*$-action on an affine surface is always untwisted [FKZ]. Hence the conclusion of Lemma 8 does not hold for twisted $\mathbb{A}^1_\mathbb{C}$-families.

**Proof of Theorem 1(c).** Let $\Gamma$ be a non-smooth homology line on a $\mathbb{Q}$-homology plane $X$, and let $m\Gamma = f^*(0)$ for a suitable $m \in \mathbb{N}$ and a primitive regular function $f \in \mathcal{O}(X)$ with irreducible general fiber $F$ (cf. the proof of Proposition 4). Let $p' \in \Gamma$ be a singular point of $\Gamma$ with Milnor number $\mu > 0$. In a suitable small spherical neighborhood $B$ of $p'$, the function $f^{1/m}$ is holomorphic and its general fiber say $R$ (which is the Milnor fiber of $(\Gamma, p')$) is a Riemann surface with boundary of positive genus $g = \mu/2$ [Mil 10.2]. For a fixed general fiber $F$ of $f$ sufficiently close to $\Gamma$, the intersection $F \cap B$ is a disjoint union of $m$ copies of the Milnor fiber $R$, hence $F$ as well is of positive genus.

Therefore $e(F) < 0$. By Lemma 3.2 in [Za], since $e(X \setminus \Gamma) = 0$ the family $\pi = f|(X \setminus \Gamma) : X \setminus \Gamma \to \mathbb{A}^1_\mathbb{C}$ has no degenerate fiber, and so Lemma 8 applies.

As a matter of fact, the $\mathbb{C}^*$-action on $X \setminus \Gamma$ as in Lemma 8 extends to an elliptic $\mathbb{C}^*$-action on $X$ making $f$ equivariant and $p'$ an attractive or repelling fixed point. For a $\mathbb{Z}$-homology plane $X$, the existence of such an extension was shown in [LiZa] and in [Za] in two different ways. The both proofs work *mutatis mutandis* in our more general setting. We choose below to follow the lines of the proof of Lemma 6 in [LiZa].

Let $\bar{F}$ be a smooth projective model of $F$. The cyclic étale covering $\rho : F \times \mathbb{A}^1_\mathbb{C} \to X \setminus \Gamma$ as in the proof of Lemma 8 extends to an equivariant rational map which fits into the commutative diagram

$$
\begin{array}{ccc}
F \times \mathbb{P}^1 & \xrightarrow{\rho} & V \\
\downarrow \text{pr}_2 & & \downarrow \pi \\
\mathbb{P}^1 & \xrightarrow{z \mapsto z^N} & \mathbb{P}^1
\end{array}
$$

where $V$ is a smooth equivariant SNC completion of $X \setminus \Gamma$. If the $\mathbb{C}^*$-action on $X \setminus \Gamma$ possesses an orbit $O$ which is not closed in $X$ then
so are all orbits and the action extends to $X$. Indeed the closure $\bar{O}$ meets $\Gamma$ in one point say $q$, and $n\bar{O} = h^*(0)$ for some regular function $h$ on $X$ and for some $n \in \mathbb{N}$. Let $P$ be the connected component of the polyhedron $|f| \leq \varepsilon$, $|h| \leq \varepsilon$ which contains $q$. Since $\lambda \cdot h = \lambda^k \cdot h$ for some $k \in \mathbb{Z}$, for every $\lambda \in \mathbb{C}^*$ with $|\lambda| = 1$ the complement $P \setminus \Gamma$ is stable under the action of $\lambda$. The polyhedron $P$ being compact for a sufficiently small $\varepsilon > 0$, such an action on $P \setminus \Gamma$ extends across $\Gamma$. Hence it also extends through $\Gamma$ to an action on the whole $X$ for all $\lambda \in \mathbb{C}^*$ with $|\lambda| = 1$, and then also for all $\lambda \in \mathbb{C}^*$.

The indeterminacy set of $\rho$ being at most 0-dimensional, $\rho$ restricts to the fiber over $0 \in \mathbb{P}^1$ yielding a morphism $\rho : \bar{F} \to \pi^{-1}(0)$. The following alternative holds: either $\rho(\bar{F}) = \rho(p) \in \Gamma$, or $\rho(\bar{F}) = \bar{\Gamma}$, or finally $\rho(\bar{F}) \cap \Gamma = \emptyset$. Let us show that the last two possibilities cannot occur.

Indeed supposing that $\rho(\bar{F}) = \rho(p) \in \Gamma$ ($\rho(\bar{F}) = \bar{\Gamma}$, respectively) the general orbits of the $\mathbb{C}^*$-action on $X \setminus \Gamma$ are not closed in $X$, and the action extends to an elliptic (parabolic, respectively) $\mathbb{C}^*$-action on $X$. Since the fixed point set of a parabolic $\mathbb{C}^*$-action on a normal affine surface is smooth (see [FLZa 3]), the latter case must be excluded.

To exclude the last possibility, suppose on the contrary that $\rho(\bar{F}) \cap \Gamma = \emptyset$. Letting $p_1, \ldots, p_k \in \bar{F} \times \{0\}$ be the indeterminacy points of $\rho$ on the central fiber, we observe that under our assumption, all $\mathbb{C}^*$-orbits $\rho(\{p_i\} \times \mathbb{A}^1)$ are closed in $X$, because general orbits are. The orbits $\rho(\{p_i\} \times \mathbb{A}^1)$, $i = 1, \ldots, k$, meet any fiber $F_\xi = f^{-1}(\xi)$, $\xi \in \mathbb{A}^1$, in a finite set, say, $T$.

Fixing further a general fiber $F = F_\xi$ sufficiently close to $\Gamma$ and a sufficiently small neighborhood $\omega$ of the finite set $T \cup (F \setminus F)$ in $\bar{F}$, we let $K = \bar{F} \setminus \omega \subseteq F$. Under our assumptions $K$ is a compact Riemann surface of positive genus with boundary, and $B \cap \lambda . K = \emptyset$ for all sufficiently small $\lambda \in \mathbb{C}^*$. Hence $F \setminus \lambda^{-1}.B \subseteq \omega$ is a disjoint union of Riemann surfaces of genus 0. On the other hand

$$F \setminus \lambda^{-1}.B \cong B \cap \lambda . F = B \cap F_{\lambda^{-1} \xi}$$

is a disjoint union of $m$ copies of the Milnor fiber $R$ of the analytic plane curve singularity $(\Gamma, \rho')$. This is a contradiction because $R$ is of positive genus.

Thus $\Gamma$ is stable under the extended elliptic $\mathbb{C}^*$-action on $X$. So $\Gamma$ is an orbit closure of this action and the singular point $p' \in \Gamma$ is a fixed point of the action. Consider an equivariant embedding $X \hookrightarrow \mathbb{A}^N$ which sends $p'$ to the origin, where $\mathbb{A}^N$ is equipped with a linear $\mathbb{C}^*$-action, and fix an equivariant linear projection $\mathbb{A}^N \to T$, where $T \simeq \mathbb{A}^2$ is the tangent plane of $X$ at $p' = 0$. This projection restricted to $X$
gives an equivariant isomorphism $X \simeq A^2$, where the $\mathbb{C}^*$-action on $A^2$ is linear (indeed, both actions have the origin as an attractive fixed point).

In appropriate linear coordinates the latter linear action is diagonal: $\lambda. (x, y) \mapsto (\lambda^k x, \lambda^l y)$ with $\gcd(k, l) = 1$. So either the image of $\Gamma$ is one of the axes, which contradicts the assumption that $\Gamma$ is singular, or it is a curve $\alpha x^k - \beta y^l = 0$ for some $\alpha, \beta \in \mathbb{C}^*$. This proves (c) of Theorem 1. Now the proof of Theorem 1 is completed. \qed

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