Generalized Hermite process: tempering, properties and applications

Héctor Araya$^{1,2}$

1 Facultad de Ingeniería y Ciencias, Universidad Adolfo Ibañez, hector.araya@uai.cl
2 Data Observatory Foundation, Chile

October 7, 2022

Abstract

In this work, we introduce a new process by modifying the kernel in the time domain representation of the generalized Hermite process. This modification is constructed by means of multiplication of the kernel in the time definition of the process by an exponential tempering factor $\lambda > 0$ such that this new process is well defined. Several properties of the process are studied and an application to non-parametric regression is also given.

2010 AMS Classification Numbers:

Key Words and Phrases: Non Gaussian; Generalized Hermite process; Tempering; Wiener chaos; Limit theorem.

1 Introduction

An interesting, non-Gaussian, extension of fractional Brownian motion (fBm) is the so called Hermite process [21, 27]. This process can be defined as an iterated Wiener-Itô integral [13], and share many of the properties of fBm, such as: self-similarity, stationarity of the increments, regularity of the paths, covariance structure, among others. Since the Hermite process is a non-Gaussian, self-similar with stationary increments process, it can be a good candidate to be an input in models where self-similarity is observed in empirical data which appears to be non-Gaussian [9, 11].

Recently, a new process has been introduced in [2]. This process, called generalized Hermite process, replace the kernel of the Hermite process by some general kernel. In fact, the process is defined as $Z(t) = I_k(h_t)$, where $I_k(\cdot)$ denotes de k-tuple Wiener-Itô integral, and

$$h_t(x_1, \ldots, x_k) := \int_0^t g(s - x_1, \ldots, s - x_k)1_{\{s > x_1, \ldots, s > x_k\}}ds$$
with a suitable homogeneous function $g$ called the generalized Hermite kernel (see Section 2 below for details).

In this work, using the generalized Hermite process introduced above, we construct a new process by the modification of the kernel in the time representation of the generalized Hermite process introduced in [2]. For this process we study several properties, such as, stationarity of the increments, covariance structure, scaling properties, among others. Also, we study the special case of the Hermite kernel introduced in [22] and the filtered version of this same kernel. For these representations, in addition to the properties mentioned above, in the case of the second chaos (Rosenblatt case), we provide a formula for the cumulants of the processes and we show their behavior in terms of the cumulants, as the tempering factor goes to the critical value zero. Finally, a non-parametric estimation application is performed. Precisely, we study the problem of non parametric estimation in a co-integrated model where the noise is a special type of tempered generalized Hermite process.

Tempered processes are defined, in general, by exponentially tempering the power law kernel in the moving average representation of different types of processes. One of the first cases of this kind of process is the tempered fractional Brownian motion (TFBM) introduced in [14]. In this paper, the authors defined the TFBM and study different properties, such as; stationarity of the increments, scaling, the definition of the tempered fractional Gaussian noise (TFGN), among others. Since then, a numerous amount of references related to the TFBM can be found in the literature, and although we do not attempt to do a complete review on the topic, we only mention a few references (see [5, 14, 15, 17, 23, 24] and the reference therein).

With respect to the case of tempered non-Gaussian processes related to long memory there are significantly less references. In [22] the author analyze the case of the tempered Hermite process, give the properties of the process and construct a weak approximation by means of a certain discrete chaos process. Related to the aforementioned work, the author in [10] develops the theory related to the two-parameter tempered Hermite field, including moving average, sample path properties, spectral representations and the theory of Wiener stochastic integration with respect to the two-parameter tempered Hermite field of order one. In [3] the authors define two new classes of stochastic processes, called tempered fractional Lévy processes of the first and second kinds. These processes, as usual, are constructed by exponentially tempering the power law kernel in the moving average representation of a fractional Lévy process. Using the framework of tempered fractional integrals and derivatives the authors develop the theory of stochastic integration with respect to both processes. In this same direction we can mention [8, 12, 16].

The remainder of this paper is organized as follows. In Section 2, we briefly recall some relevant aspects and notations related to generalized kernels. Section 3 is devoted to the construction of the process and the study of the properties of stationary increments, scaling, regularity of the paths and, under some special assumptions, its covariance structure. In Section 4, a filtered version of the same process is constructed, and as before, the same properties are established. Section 5, is concerned with the analysis of the process in the case of the Hermite kernel (regular kernel and filtered). In Section 6, we consider the problem
of non-parametric estimation. Precisely, we consider a co-integrated regressor model where the regressor is a fractional Brownian motion with Hurst parameter $H_1 \in (0, 1)$ and $Z^\lambda$ is a generalized tempered Hermite process. Finally, Section 7 contains brief appendix related to elementary topics of Malliavin calculus.

2 Preliminaries

We briefly recall some relevant aspects and notations related to generalized kernels, our main reference is [2].

2.1 Generalized kernels

In this part, we introduce the definition and properties of a general type of kernel that we will use in order to construct our tempered process.

We need to introduce the following notation, let $z = (z_1, \ldots, z_k) \in \mathbb{R}^k$, $i = (i_1, \ldots, i_k)$, $0 = (0, \ldots, 0)$; $1 = (1, \ldots, 1)$, where the dimensions of $0$ and $1$ will be determined by the context of the computations. Also, for any real number $z$, $[z] = \sup\{n \in \mathbb{Z}, n \leq z\}$, and $[z] = ([z_1], \ldots, [z_k])$. Furthermore, we write $x > y$ for $x_j > y_j$ with $j = 1, \ldots, k$ (the definition for $\geq$ is exactly the same). $\langle y, z \rangle = \sum_{j=1}^k y_j z_j$, $\|z\| = \langle z, z \rangle$. We use $\| \cdot \|$ with a subscript to denote the norm of some other space (the space will be determined by the subscript). For a set $B \subset \mathbb{R}$, $B^k$ is the $k$-fold cartesian product. To simplify the computations we establish the following notation for the tempering factor

$$e^{-\lambda(s^1-x)} := \prod_{i=1}^k e^{-\lambda(s-x_i)}.$$

Now, we recall a proposition from the reference [2]. This proposition allow us to construct general H-ssi process living in a Wiener chaos

Proposition 1. Let $H \in (0, 1)$. Suppose that $\{h_t(\cdot), t > 0\}$ is a family of functions defined on $\mathbb{R}^k$ satisfying

1. $h_t \in L^2(\mathbb{R}^k)$;
2. $\forall \lambda > 0, \exists \beta \neq 0$, such that $h_M(x) = \lambda^{H+k\beta/2} h_t(\lambda^{\beta} x)$ for a.e. $x \in \mathbb{R}^k$ and all $t > 0$;
3. $\forall s, \exists a \in \mathbb{R}^k$, such that $h_{t+s}(x) - h_t(x) = h_s(x + ta)$ for a.e. $x \in \mathbb{R}^k$ and all $t > 0$.

Then

$$Z(t) := I_k(h_t) = \int_{\mathbb{R}^k} \int_0^t g(s-x_1, \ldots, s-x_k) 1_{\{s \geq x_1, \ldots, s \geq x_k\}} ds W(dx_1) \ldots W(dx_k)$$

is an H-ssi process with

$$h_t(x) := \int_0^t g(s^1-x) 1_{\{s^1 > x\}} ds.$$
Now, the idea is to give the conditions over a general function $g$, such that, we can verify the conditions of Proposition 1.

**Definition 1.** A nonzero measurable function $g$ defined on $\mathbb{R}^k_+$ is called generalized tempered Hermite kernel, if it satisfies the following conditions:

1. **(H1)** $g(cx) = c^\alpha g(x)$, $\forall c > 0$, where $\alpha \in \left(-\frac{k+1}{2}, -\frac{k}{2}\right)$.
2. **(H2)** $\int_{\mathbb{R}^k_+} |g(x)g(1+x)|e^{-2\lambda u x} < \infty$.

### 3 The Process

In this part, we introduce the tempered generalized Hermite process. Then, we study some sample properties of the process: scaling, stationarity of the increments, covariance and sample path regularity.

Let $\lambda > 0$, the tempered generalized Hermite process is given by

$$Z^\lambda(t) := \int_{\mathbb{R}^k_+} \int_0^t g(s-x_1, \ldots, s-x_k) \prod_{i=1}^k e^{-\lambda(s-x_i)} 1_{\{s>x_i\}} ds W(dx_1) \ldots W(dx_k)$$

$$= I_k(h^\lambda_t),$$

where

$$h^\lambda_t(x) := \int_0^t g(s-1-x)e^{-\lambda(s-1-x)} 1_{\{s>x\}} ds,$$

in here, to simplify the computations we establish the following notation for the tempering factor

$$e^{-\lambda(s-x)} := \prod_{i=1}^k e^{-\lambda(s-x_i)}.$$  

Next, we will prove that $Z^\lambda$ is well defined. In fact, we have the following result:

**Proposition 2.** Let $g(x)$ be a generalized Hermite kernel presented in Definition 1. Then,

$$h^\lambda_t(x) := \int_0^t g(s-1-x)e^{-\lambda(s-1-x)} 1_{\{s>x\}} ds$$

is well defined in $L^2(\mathbb{R}^k)$.

**Proof:** To check that $h^\lambda_t$ is well defined, we write

$$\int_{\mathbb{R}^k} h^\lambda_t(x)^2 \, dx = \int_{\mathbb{R}^k} dx \int_0^t ds_1 \int_0^t ds_2 g(s_1-1-x)g(s_2-1-x) \times e^{-\lambda(s_1-1-x)} e^{-\lambda(s_2-1-x)} 1_{\{s_1>x\}} 1_{\{s_2>x\}}.$$
now, we want to use Fubini theorem. Therefore, we need to check that the absolute value of the integrand is integrable, i.e.

\[
\int_{\mathbb{R}^k} h_1^\lambda(x)^2 \, dx = 2 \int_0^t ds_1 \int_0^{t-s_1} ds_2 \int_{\mathbb{R}^k} dx |g(s_11 - x)g(s_21 - x)|
\times e^{-\lambda(s_11-x)}e^{-\lambda(s_21-x)}1_{\{s_11>x\}}1_{\{s_21>x\}},
\]

also, we have used the symmetry in \(s_1\) and \(s_2\). Now, making the change of variables \(s = s_1\), \(u = s_2 - s_1\), \(w = s_11 - x\), we get

\[
\int_{\mathbb{R}^k} h_1^\lambda(x)^2 \, dx = 2 \int_0^t ds \int_0^{t-s} du \int_{\mathbb{R}^k} dw |g(w)g(u1 + w)|e^{-\lambda(w)}e^{-\lambda(u1+w)}
\times dy |g(y)g(1 + y)|e^{-2\lambda uy},
\]

here, in the last equality, we have used the change of variables \(y = w\), the condition (H1), and condition (H2). We can see that the last expression is finite by \(2\alpha + k + 1 > 0\) and (H2).

If we compute the same without the absolute value we arrive to

\[
\int_{\mathbb{R}^k} h_1^\lambda(x)^2 \, dx \leq \frac{t^{2\alpha+k+2}}{(\alpha + k/2 + 1)(2\alpha + k + 2)} \int_{\mathbb{R}^k} dy |g(y)g(1 + y)|e^{-2\lambda uy},
\]

which is finite for the same reason as before.

**Remark 1.** If we consider necessary, in some of the proofs, to be more clear in the computations, we will use the product notation, i.e., we will not use the notation defined in (2) and (3).

**Lemma 1.** Let \(H > 1/2\) and \(\lambda > 0\). Then, the tempered generalized Hermite process \(Z^\lambda\) is a stationary increments process with the scaling property

\[
\{Z^\lambda(ct)\}_{t \in \mathbb{R}} \overset{d}{=} \{cH^zZ^{\lambda}(t)\}_{t \in \mathbb{R}},
\]

where \(c > 0\) and \(\overset{d}{=}\) means equality in sense of finite dimensional distributions.

**Proof:** To check the stationarity of the increments, we write for \(t, h > 0\)
Making the change of variable $u = s - t$, we obtain
\[
Z^\lambda(t + h) - Z^\lambda(t) = \int_{\mathbb{R}^k} g(u + t - x_1, \ldots, u + t - x_k) \prod_{i=1}^{k} e^{-\lambda(u-t-x_i)} 1_{\{u+x_1>0, \ldots, u+x_k>0\}} du W(dx_1) \ldots W(dx_k).
\]

Now, by the change of variable $y_i = x_i + t$, we get
\[
Z^\lambda(t + h) - Z^\lambda(t) = \int_{\mathbb{R}^k} g(u - y_1, \ldots, u - y_k) \prod_{i=1}^{k} e^{-\lambda(u-y_i)} 1_{\{u+y_1>0, \ldots, u+y_k>0\}} du W(dy_1 + t) \ldots W(dy_k + t)
\]
\[
\overset{d}{=} Z^\lambda(h),
\]

where the last equality comes from the stationarity of Brownian motion. Now, to prove the scaling property, we have for $c > 0$
\[
Z^\lambda(ct) = \int_{\mathbb{R}^k} g(s - x_1, \ldots, s - x_k) \prod_{i=1}^{k} e^{-\lambda(s-x_i)} 1_{\{s+x_1>0, \ldots, s+x_k>0\}} ds W(dx_1) \ldots W(dx_k)
\]
\[
= c \int_{\mathbb{R}^k} g(u c - x_1, \ldots, u c - x_k) \prod_{i=1}^{k} e^{-\lambda(u c-x_i)} 1_{\{u c+x_1>0, \ldots, u c+x_k>0\}} du W(dx_1) \ldots W(dx_k),
\]

here, we used the change of variable $u = s/c$. Continuing, we use the change of variable $y_i = x_i/c$
\[
Z^\lambda(ct) = c \int_{\mathbb{R}^k} g(u c - y_1, \ldots, u c - y_k) \prod_{i=1}^{k} e^{-\lambda(u c-y_i)} 1_{\{u c+y_1>0, \ldots, u c+y_k>0\}} du W(dy_1) \ldots W(dy_k)
\]
\[
\overset{d}{=} c^{\alpha+1+k/2} Z^\lambda(t),
\]

where, in the last equality we have used the condition $(H1)$ and the self-similarity of Brownian motion. Finally, the result is achieved by taking $\alpha + 1 = H - k/2$. 
Lemma 2. The stochastic process $Z^\lambda$ has a continuous version.

Proof: By Proposition [2] and Lemma [1] we can get

$$E([Z^\lambda(t) - Z^\lambda(s)]^2) \leq C|t - s|^{2\alpha + k + 2},$$

noticing that taking $\alpha + 1 = H + k/2$, we have

$$E([Z^\lambda(t) - Z^\lambda(s)]^2) \leq C|t - s|^{2H},$$

where $H \in (1/2, 1)$. Then, the result is achieved by means of Kolmogorov Chensov theorem.

Lemma 3. Let us assume that $g$ given by Definition [4] with

$$g(x_1, \ldots, x_k) = \prod_{i=1}^{k} g(x_i)$$

and $g(0) = 0$ for $i = 1, \ldots, k$. Then, the tempered generalized Hermite process $Z^\lambda$ has the covariance function

$$E(Z^\lambda(t)Z^\lambda(s)) = k! \int_0^t \int_0^s e^{-\lambda k|u-v|} |u-v|^{k(2\alpha-1)} \left[ \int_0^\infty g(x)[1 + x]e^{-2\lambda|u-v|x} dx \right]^k du dv,$$

where $\lambda > 0$, and $\alpha \in \left(-\frac{k+1}{2}, -\frac{k}{2}\right)$ (equivalently $H > 1/2$).

Proof: By the definition of $Z^\lambda$, the fact that $g(0) = 0$, Fubini theorem and the isometry of multiple Wiener-Itô integrals, we get

$$E(Z^\lambda(t)Z^\lambda(s)) = E[I_k(h^\lambda_k)I_k(h^\lambda_k)]$$

$$= k! \int_{\mathbb{R}^k} h^\lambda_k(x)h^\lambda_k(x) dx$$

$$= k! \int_0^t \int_0^s \left[ \int_{\mathbb{R}^k} g((u1 - x)_+)|g((v1 - x)_+)|e^{-\lambda(u1 - x)_+}e^{-\lambda(v1 - x)_+} dx \right] du dv,$$

Now, using (4) we can obtain

$$E(Z^\lambda(t)Z^\lambda(s)) = k! \int_0^t \int_0^s \left[ \int_{\mathbb{R}^k} \prod_{i=1}^{k} g((u - x_i)_+)|g((v - x_i)_+)|e^{-\lambda(u - x_i)_+}e^{-\lambda(v - x_i)_+} dx_1 \cdots dx_k \right] du dv$$

$$= k! \int_0^t \int_0^s \left[ \int_{\mathbb{R}^k} g((u - x)_+)|g((v - x)_+)|e^{-\lambda(u - x)_+}e^{-\lambda(v - x)_+} dx \right]^k du dv$$

$$= k! \int_0^t \int_0^s \left[ \int_{-\infty}^{u\wedge v} g[u - x]g[v - x]e^{-\lambda(u - x)}e^{-\lambda(v - x)} dx \right]^k du dv$$

$$= k! \int_0^t \int_0^s e^{-\lambda k|u - v|} \left[ \int_0^\infty g[w]g[|u - v| + w]e^{-2\lambda w} dw \right]^k du dv.$$
By the properties of $g$

$$E(Z^\lambda(t)Z^\lambda(s)) = k! \int_0^t \int_0^s e^{-\lambda |u-v|} |u-v|^{k(2\alpha-1)} \left[ \int_0^\infty g[x]g[1+x]e^{-2\lambda |u-v|x} dx \right]^k dudv.$$ 

4 The process: Fractionally filtered kernels

In this part, we introduce the tempered generalized Hermite process. Then, we study some sample properties of the process: scaling, stationarity of the increments, covariance and sample path regularity.

Let $\lambda > 0$, we define the tempered generalized Hermite process with filtered kernel by

$$Z^\lambda,\beta(t) := \int_{\mathbb{R}^k} \int_{\mathbb{R}} g(s - x_1, \ldots, s - x_k) \frac{1}{\beta}[(t-s)_+^\beta - (-s)_+^\beta]$$

$$\times \prod_{i=1}^k e^{-\lambda (s-x_i)} 1_{\{s>x_i, \ldots, s>x_k\}} ds W(dx_1) \ldots W(dx_k)$$

$$= I_k(h^\lambda,\beta), \quad (5)$$

where

$$h^\lambda,\beta := \int_{\mathbb{R}} l^\beta_t(s)g(s1 - x)e^{-\lambda(s1-x)} 1_{\{s1>x\}} ds,$$

with

$$l^\beta_t(s) = \frac{1}{\beta}[(t-s)_+^\beta - (-s)_+^\beta]$$

and $e^{-\lambda(s1-x)}$ given by (3).

Next, we will prove that $Z^\lambda,\beta$ is well defined. In fact, we have the following result:

**Proposition 3.** Let $g(x)$ be a generalized Hermite kernel given in Definition 1. If

$$-1 < -\alpha - \frac{k}{2} - 1 < \beta < -\alpha - \frac{k}{2} < \frac{1}{2}, \quad \beta \neq 0.$$

Then,

$$h^\lambda,\beta := \int_{\mathbb{R}} l^\beta_t(s)g(s1 - x)e^{-\lambda(s1-x)} 1_{\{s1>x\}} ds$$

is well defined in $L^2(\mathbb{R}^k)$.

**Proof:** By the defintion of $h^\lambda,\beta$, we get

$$\int_{\mathbb{R}^k} h^\lambda,\beta(x)^2 dx \leq 2 \int_{\mathbb{R}} ds_1 \int_{s_1}^\infty ds_2 \int_{\mathbb{R}^k} d\mathbf{x} l^\beta_t(s_1) l^\beta_t(s_2)$$

$$\times |g(s_11-x)g(s_21-x)|e^{-\lambda(s_11-x)} e^{-\lambda(s_21-x)} 1_{\{s11>x\}}.$$
Making the change of variables $s = s_1$, $u = s_2 - s_1$, $w = s_1 t - x$, we get

$$\int_{\mathbb{R}^k} h_{t}^{\lambda,\beta}(x)^2 \, dx \leq 2 \int_{\mathbb{R}} ds \int_{0}^{\infty} du \int_{\mathbb{R}^k_+} dx \, \left| g(w)g(u1 + w) \right| e^{-\lambda(w)} e^{-\lambda(u1 + w)}$$

$$\leq 2 \int_{\mathbb{R}} ds \int_{0}^{\infty} du \int_{\mathbb{R}^k_+} dx \, |g(y)|g(1 + y)|e^{-2\lambda\alpha}y|.$$  

By condition $(H2)$ and the fact $\lambda > 0$, we need to prove that

$$\int_{\mathbb{R}} ds \int_{0}^{\infty} du \int_{\mathbb{R}^k_+} dx \, |g(y)|g(1 + y)|e^{-2\lambda\alpha}y| < \infty.$$  

However, since $u \in (0, \infty)$, we have

$$\int_{\mathbb{R}} ds \int_{0}^{\infty} du \int_{0}^{\infty} dx \, u^{2\alpha + k} e^{-\lambda uk} < \int_{\mathbb{R}} ds \int_{0}^{\infty} du \int_{0}^{\infty} dx \, u^{2\alpha + k}$$

and this last term if finite for

$$-1 < -\alpha - \frac{k}{2} - 1 < \beta < -\alpha - \frac{k}{2} < \frac{1}{2}, \quad \beta \neq 0$$

due to Proposition 3.25 in [2].

**Lemma 4.** Let $\lambda > 0$, and

$$-1 < -\alpha - \frac{k}{2} - 1 < \beta < -\alpha - \frac{k}{2} < \frac{1}{2}, \quad \text{with} \ \beta \neq 0.$$  

Then, the tempered generalized Hermite process $Z^{\lambda,\beta}$ is a stationary increments process with the scaling property

$$\{Z^{\lambda,\beta}(ct)\}_{ct \in \mathbb{R}} = \{c^{\beta + 1 + \alpha + k/2}Z^{c\lambda,\beta}(t)\}_{t \in \mathbb{R}},$$

where $c > 0$ and $\overset{d}{=} \text{means equality in sense of finite dimensional distributions.}$
Proof: To check the stationarity of the increments, we write for $t, h > 0$

$$Z^{\lambda, \beta}(t + h) - Z^{\lambda, \beta}(t) = \int_{\mathbb{R}}^{'} \frac{1}{\beta} [(t + h - s)^{\beta} - (s)^{\beta} \mathbb{1}_{s > x_1, \ldots, s > x_k}] g(s - x_1, \ldots, s - x_k)$$

$$\times \prod_{i=1}^{k} e^{-\lambda(s-x_i)} \mathbb{1}_{s > x_1, \ldots, s > x_k} ds W(dx_1) \ldots W(dx_k) - \int_{\mathbb{R}}^{'} \frac{1}{\beta} [(t - s)^{\beta} - (s)^{\beta} \mathbb{1}_{s > x_1, \ldots, s > x_k}] g(s - x_1, \ldots, s - x_k)$$

$$\times \prod_{i=1}^{k} e^{-\lambda(s-x_i)} \mathbb{1}_{s > x_1, \ldots, s > x_k} ds W(dx_1) \ldots W(dx_k).$$

Making the change of variable $t - s = -v$, we get

$$Z^{\lambda, \beta}(t + h) - Z^{\lambda, \beta}(t) = \int_{\mathbb{R}}^{'} \frac{1}{\beta} [(h - v)^{\beta} - (v)^{\beta} \mathbb{1}_{v > x_1, \ldots, v > x_k}] g(t + v - x_1, \ldots, t + v - x_k)$$

$$\times \prod_{i=1}^{k} e^{-\lambda(t+v-x_i)} \mathbb{1}_{t+v > x_1, \ldots, t+v > x_k} dv W(dx_1) \ldots W(dx_k).$$

Now, by the change of variable $y_i = x_i - t$ for $i = 1, \ldots, k$, we obtain

$$Z^{\lambda, \beta}(t + h) - Z^{\lambda, \beta}(t) = \int_{\mathbb{R}}^{'} \frac{1}{\beta} [(h - v)^{\beta} - (v)^{\beta} \mathbb{1}_{v > y_1, \ldots, v > y_k}] g(v - y_1, \ldots, v - y_k)$$

$$\times \prod_{i=1}^{k} e^{-\lambda(v-y_i)} \mathbb{1}_{v > y_1, \ldots, v > y_k} dv W(d(y_1 + t)) \ldots W(d(y_k + t)).$$

This last equality is due to the stationarity of Brownian motion. With respecto to the scaling property, we have for $c > 0$

$$Z^{\lambda, \beta}(ct) = \int_{\mathbb{R}}^{'} \frac{1}{\beta} [(ct - s)^{\beta} - (s)^{\beta} \mathbb{1}_{s > x_1, \ldots, s > x_k}] g(s - x_1, \ldots, s - x_k)$$

$$\times \prod_{i=1}^{k} e^{-\lambda(s-x_i)} \mathbb{1}_{s > x_1, \ldots, s > x_k} ds W(dx_1) \ldots W(dx_k),$$

10
by the change of variable $v = s/c$, we get

$$Z^{\lambda, \beta}(ct) = c^{\beta+1} \int_{\mathbb{R}} \frac{1}{\beta} [(t - v)^{\beta} - (-v)^{\beta}] g(cv - x_1, \ldots, cv - x_k)$$

$$\times \prod_{i=1}^{k} e^{-\lambda(cv - x_i)} 1_{\{cv > x_1, \ldots, cv > x_k\}} \, dv W(dx_1) \ldots W(dx_k),$$

in a similar way, we use the change of variable $y_i = x_i/c$, this allow us to obtain

$$Z^{\lambda, \beta}(ct) = c^{\beta+1+\alpha} \int_{\mathbb{R}} \frac{1}{\beta} [(t - v)^{\beta} - (-v)^{\beta}] g(v - y_1, \ldots, v - y_k)$$

$$\times \prod_{i=1}^{k} e^{-\lambda(v - x_i)} 1_{\{v > y_1, \ldots, v > y_k\}} \, dv W(dy_1) \ldots W(dy_k)$$

$$= c^{\beta+1+\alpha+k/2} Z^{\lambda, \beta}(t).$$

**Remark 2.** We can take $H = \beta + 1 + \alpha + k/2 \in (0, 1)$, then

$$\{Z^{\lambda, \beta}(ct)\}_{t \in \mathbb{R}} \overset{d}{=} \{c^H Z^{\lambda, \beta}(t)\}_{t \in \mathbb{R}},$$

if we want to consider the anti-persistent case $H < 1/2$, then we have to take

$$\beta \in \left(-\alpha - \frac{k}{2}, -\alpha - \frac{k}{2} - \frac{1}{2}\right).$$

**Lemma 5.** Let us assume that $g$ given by Definition (\ref{def:g}) with

$$g(x_1, \ldots, x_k) = \prod_{i=1}^{k} g(x_i)$$

and $g(0) = 0$ for $i = 1, \ldots, k$. Then, the tempered generalized Hermite process $Z^{\lambda}$ has the covariance function

$$E(Z^{\lambda}(t)Z^{\lambda}(s)) = k! \int_{0}^{t} \int_{0}^{s} e^{-\lambda|u-v|} (u-v)^{k(2\alpha-1)} i^3(u) j^3(v) \left[ \int_{0}^{\infty} g(x) [1 + x] e^{-2\lambda|u-v| x} \, dx \right]^{k} \, du dv.$$
Proof: By the definition of $Z^\lambda$, the fact that $g(0) = 0$, Fubini theorem and the isometry of multiple Wiener - Itô integrals, we get

$$E(Z^\lambda(t)Z^\lambda(s)) = E[I_k(h^\lambda_k)I_k(h^\lambda_s)]$$

$$= k! \int_{\mathbb{R}^k} h^\lambda_k(x)h^\lambda_s(x)dx$$

$$= k! \int_{\mathbb{R}} \int_{\mathbb{R}} l^\beta_s(u)l^\beta_t(v) \left[ \int_{\mathbb{R}^k} g((u1 - x)_+)g((v1 - x)_+)e^{-\lambda(u1 - x)}e^{-\lambda(v1 - x)}dx \right] dudv,$$

Now, using (6) we can obtain

$$E(Z^\lambda(t)Z^\lambda(s)) = k! \int_{\mathbb{R}} \int_{\mathbb{R}} l^\beta_s(u)l^\beta_t(v) \left[ \int_{\mathbb{R}} \prod_{i=1}^k g((u - x_i)_+)g((v - x_i)_+)e^{-\lambda(u - x_i)}e^{-\lambda(v - x_i)}dx_1 \cdots dx_k \right] dudv$$

$$= k! \int_{\mathbb{R}} \int_{\mathbb{R}} l^\beta_s(u)l^\beta_t(v) \left[ \int_{\mathbb{R}} g((u - x)_+)g((v - x)_+)e^{-\lambda(u - x)}e^{-\lambda(v - x)}dx_1 \cdots dx_k \right]^k dudv$$

$$= k! \int_{\mathbb{R}} \int_{\mathbb{R}} l^\beta_s(u)l^\beta_t(v) \left[ \int_{-\infty}^{u/v} g[u - x]g[v - x]e^{-\lambda(u - x)}e^{-\lambda(v - x)}dx \right]^k dudv$$

$$= k! \int_{\mathbb{R}} \int_{\mathbb{R}} l^\beta_s(u)l^\beta_t(v) e^{-\lambda|u - v|} \left[ \int_{0}^{\infty} g[w]g[|u - v| + w]e^{-2\lambda w}dw \right]^k dudv.$$

By the properties of $g$

$$E(Z^\lambda(t)Z^\lambda(s)) = k! \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\lambda|u - v||u - v|^{(2\alpha-1)}} l^\beta_s(u)l^\beta_t(v) \left[ \int_{0}^{\infty} g[x]g[1 + x]e^{-2\lambda|u - v|x}dx \right]^k dudv.$$
5 The Hermite case

Here, we consider and study a special case of kernel that fulfill the conditions of Proposition 1 and Definition 1. We will consider the Hermite kernel and then, the filtered version of the same kernel.

5.1 Hermite kernel

Let us recall that the Hermite kernel is given by

\[ g(x) = \prod_{j=1}^{k} x_j^{d-1}, \quad x_j > 0. \]

Clearly, this kernel meets all the conditions of Definition 1. In fact, in [22] the author study the following process

\[ Z_\lambda(t) = \int_{\mathbb{R}^k} \int_0^t \prod_{i=1}^{k} (s-x_i)^{d-1}e^{-\lambda(s-x_i)+ds}W(dx_1)\ldots W(dx_k), \quad (7) \]

where \( \lambda > 0 \), \( (x)_+ = x1_{x>0} \) and \( d = \frac{1}{2} - \frac{1-H}{k} \in \left( \frac{1}{2} - \frac{1}{2k}, \infty \right) \), and \( H > 1/2 \). By Proposition 2 and Lemma 1 we know that \( Z_\lambda \) is a stationary increment process with a scaling property. Furthermore, the author in [22] obtains the following properties given in Proposition 4 and Proposition 5 (for all the proofs the reader can refer to [22]).

**Proposition 4.** Let \( Z_\lambda \) be the process given by (7) has the covariance function

\[ E(Z_\lambda(t)Z_\lambda(s)) = 2 \left[ \frac{\Gamma(d)}{\sqrt{\pi}2\lambda^{d-1/2}} \right]^k \int_{0}^{t} \int_{0}^{s} |u-v|^{d-1/2} K_{1/2-d}\left(\lambda|u-v|\right) \left[ K_v(\lambda|x-x_1|) \right] \ldots \left[ K_v(\lambda|x-x_k|) \right] \]

where \( \lambda > 0 \), \( d > \frac{1}{2} - \frac{1}{2k} \) (equivalently \( H > 1/2 \)) and \( K_v(x) \) is a modified Bessel function of the second kind (see [4, 6] for details).

**Proposition 5.** Let \( Z_\lambda \) be the process given by (7) has the spectral domain representation

\[ Z_\lambda(t) = C_{d,k} \int_{\mathbb{R}^k} e^{i(\omega_1+\ldots+\omega_k)} \frac{1}{i(\omega_1+\ldots+\omega_k)} \prod_{j=1}^{k} (\lambda+i\omega_j)^{-d}W(dx_1)\ldots W(dx_k), \]

Now, we give the expression for the cumulants for the process when \( k = 2 \). It is known that, for a second chaos process the law of the process is completely determined by their cumulants (see [7]). In fact, if we consider a multiple integral \( I_2(f) \) of order two with
\( f \in L^2(\mathbb{R}^2) \) symmetric. Then the m-th cumulant of the random variable \( I_2(f) \) are given by (see [18])

\[
C_m(I_2(f)) = 2^{m-1}(m-1)! \int_{\mathbb{R}^m} f(x_1, x_2)f(x_2, x_3) \cdots f(x_{m-1}, x_m)f(x_m, x_1)dx_1 \cdots dx_m. \tag{8}
\]

Also, we will need the following formula (see [22] for the details)

**Lemma 6.** Let \( \tau \in (0, 1/2) \) and \( \lambda > 0 \). Then,

\[
\int_{\mathbb{R}} e^{-\lambda(u-x)} + e^{-\lambda(v-x)} + (u-x)^{\tau-1} e^{-\lambda u} + (v-x)^{\tau-1} e^{-\lambda v} dx = (2\lambda)^{1/2} \sqrt{\pi} \frac{\Gamma(\tau)}{2^{\tau}} K_{\frac{1}{2}-\tau}(|u-v|) u-v|^{\tau-1/2},
\]

where \( K_{\nu}(x) \) is a modified Bessel function of the second kind.

Using formula (8) and Lemma 6, we can obtain the following result concerning the cumulants of the process \( Z^\lambda \) with \( k = 2 \).

**Lemma 7.** Let \( Z^\lambda \) the process given by (1) with \( k = 2, H > 1/2 \) and \( \lambda > 0 \), and

\[
h^\lambda_t(x_1, x_2) = \int_0^t e^{-\lambda(s-x_1)} + e^{-\lambda(s-x_2)} + (s-x_1)^{d-1} e^{-\lambda (s-x_2)d-1} ds,
\tag{9}
\]

then

\[
C_m(Z^\lambda(t)) = 2^{m-1+\left(\frac{1}{2}-d\right)m}(m-1)! \lambda^{\left(\frac{1}{2}-d\right)m} \left( \frac{\Gamma(d)}{\sqrt{\pi}} \right)^m \int_0^t \cdots \int_0^t ds_1 \cdots ds_m
\times K_{\frac{1}{2}-d}(\lambda |s_1 - s_2|) |s_1 - s_2|^{d-1/2} K_{\frac{1}{2}-d}(\lambda |s_2 - s_3|) |s_2 - s_3|^{d-1/2}
\times \cdots \times K_{\frac{1}{2}-d}(\lambda |s_m - s_1|) |s_m - s_1|^{d-1/2}. \tag{10}
\]

**Proof:** By (11) and (8) we have that, for \( k = 2 \), we can write

\[
Z^\lambda(t) = \int_{\mathbb{R}^2} \int_0^t \prod_{i=1}^2 (s-x_i)^{d-1} e^{-\lambda(s-x_i)} ds W(dx_1) \cdots W(dx_2)
\]

\[
= \int_{\mathbb{R}^2} h^\lambda_t(x_1, x_2) W(dx_1) \cdots W(dx_2).
\]

14
Using Lemma 6, we can obtain

\[ C_m(Z^\lambda(t)) = 2^{m-1}(m-1)! \int_{\mathbb{R}^m}^t h^{(x_1, x_2)} h^{(x_2, x_3)} \cdots h^{(x_{m-1}, x_m)} h^{(x_m, x_1)} \, dx_1 \cdots dx_m \]

\[ = 2^{m-1}(m-1)! \int_{\mathbb{R}^m} dx_1 \cdots dx_m \]

\[ \times \left( \int_0^t e^{-\lambda(s_1-x_1)+e^{-\lambda(s_1-x_2)}} (s_1-x_1)^{d-1} + (s_1-x_2)^{d-1} \, ds_1 \right) \]

\[ \times \left( \int_0^t e^{-\lambda(s_2-x_2)+e^{-\lambda(s_2-x_3)}} (s_2-x_2)^{d-1} + (s_2-x_3)^{d-1} \, ds_2 \right) \]

\[ \vdots \]

\[ \times \left( \int_0^t e^{-\lambda(s_m-x_m)+e^{-\lambda(s_m-x_1)}} (s_m-x_m)^{d-1} + (s_m-x_1)^{d-1} \, ds_m \right). \]

Then, by Fubini theorem, we can get

\[ C_m(Z^\lambda(t)) = 2^{m-1}(m-1)! \int_0^t \cdots \int_0^t ds_1 \cdots ds_m \]

\[ \times \left( \int_\mathbb{R} e^{-\lambda(s_1-x_1)} + e^{-\lambda(s_m-x_1)} (s_1-x_1)^{d-1} + (s_m-x_1)^{d-1} \, dx_1 \right) \]

\[ \vdots \]

\[ \times \left( \int_\mathbb{R} e^{-\lambda(s_{m-1}-x_m)} + e^{-\lambda(s_m-x_m)} (s_{m-1}-x_m)^{d-1} + (s_m-x_m)^{d-1} \, dx_m \right). \]

Using Lemma 3 we can obtain

\[ C_m(Z^\lambda(t)) = 2^{m-1+(\frac{1}{2}-d)m}(m-1)! \lambda^{(\frac{1}{2}-d)m} \left( \frac{\Gamma(d)}{\sqrt{\pi}} \right)^m \int_0^t \cdots \int_0^t ds_1 \cdots ds_m \]

\[ \times K_{(\frac{1}{2}-d)}(\lambda|s_1-s_2|)|s_1-s_2|^{d-1/2} K_{(\frac{1}{2}-d)}(\lambda|s_2-s_3|)|s_2-s_3|^{d-1/2} \]

\[ \cdots K_{(\frac{1}{2}-d)}(\lambda|s_m-s_1|)|s_m-s_1|^{d-1/2}. \]

**Remark 3.** By taking \( m=2 \) in Formula (10), we can recover the formula for the variance of the tempered Hermite process with \( k = 2 \).

Now, we present an interesting result related to the behavior of the process as \( \lambda \to 0^+ \)

**Lemma 8.** Let \( Z^\lambda \) be the process given by (11) with \( k = 2, H > 1/2 \) and \( \lambda > 0 \), and

\[ h^{(x_1, x_2)} = \int_0^t e^{-\lambda(s-x_1)+e^{-\lambda(s-x_2)}} (s-x_1)^{d-1} + (s-x_2)^{d-1} \, ds, \]

then

\[ \lim_{\lambda \to 0^+} Z^\lambda_t = Z_t, \]

where \( Z \) is the Rosenblatt process.
\textbf{Proof:} Let us consider \(b_1, \ldots, b_n \in \mathbb{R}\) and \(t_1, \ldots, t_n \in (0, \infty)\). We need to show that the random variables
\[
\lim_{\lambda \to 0^+} \sum_{l=1}^n b_l Z_{t_l}^\lambda ; \quad \sum_{l=1}^n b_l Z_{t_l}
\]
have the same distribution. To do this, we will use the cumulant criterium (see Formula \((5)\)). Also, to simplified computations we will study the limit when \(\lambda \to 0^+\) of the cumulants of \(Z_t^\lambda + Z_s^\lambda\); the general case follows by similar arguments.

We have that
\[
Z_t^\lambda + Z_s^\lambda = I_2(h_{t,s}^\lambda),
\]
where
\[
h_{t,s}^\lambda = \int_0^t e^{-\lambda(u-x_1)+e^{-\lambda(u-x_2)}}(u-x_1)_{+}^{d-1}(u-x_2)_{+}^{d-1}du + \int_0^s e^{-\lambda(u-x_1)+e^{-\lambda(u-x_2)}}(u-x_1)_{+}^{d-1}(u-x_2)_{+}^{d-1}du.
\]

By Formula \((5)\)
\[
C_m(Z_t^\lambda + Z_s^\lambda) = 2^{m-1}(m-1)! \int_{\mathbb{R}^m} h_{t,s}^\lambda(x_1, x_2)h_{t,s}^\lambda(x_2, x_3) \cdots h_{t,s}^\lambda(x_{m-1}, x_m)h_{t,s}^\lambda(x_m, x_1)dx_1 \cdots dx_m
\]
\[
= 2^{m-1}(m-1)! \int_{\mathbb{R}^m} dx_1 \cdots dx_m
\]
\[
\times \left( \int_0^t e^{-\lambda(u_1-x_1)+e^{-\lambda(u_1-x_2)}}(u_1-x_1)_{+}^{d-1}(u_1-x_2)_{+}^{d-1}du_1 + \int_0^s e^{-\lambda(u_1-x_1)+e^{-\lambda(u_1-x_2)}}(u_1-x_1)_{+}^{d-1}(u_1-x_2)_{+}^{d-1}du_1 \right)
\]
\[
\times \left( \int_0^t e^{-\lambda(u_2-x_2)+e^{-\lambda(u_2-x_3)}}(u_2-x_2)_{+}^{d-1}(u_2-x_3)_{+}^{d-1}du_2 + \int_0^s e^{-\lambda(u_2-x_2)+e^{-\lambda(u_2-x_3)}}(u_2-x_2)_{+}^{d-1}(u_2-x_3)_{+}^{d-1}du_2 \right)
\]
\[
\vdots
\]
\[
\times \left( \int_0^t e^{-\lambda(u_m-x_m)+e^{-\lambda(u_m-x_1)}}(u_m-x_m)_{+}^{d-1}(u_m-x_1)_{+}^{d-1}du_m + \int_0^s e^{-\lambda(u_m-x_m)+e^{-\lambda(u_m-x_1)}}(u_m-x_m)_{+}^{d-1}(u_m-x_1)_{+}^{d-1}du_m \right).
By Fubini theorem, we can get
\[ C_m(Z^\lambda(t)) = 2^{m-1}(m-1)! \sum_{t_j \in \{t,s\}} \int_0^{t_1} \cdots \int_0^{t_m} du_1 \cdots du_m \times \left( \int_\mathbb{R} e^{-\lambda(u_1-x_1)} + e^{-\lambda(u_m-x_1)} (u_1-x_1)^{d-1} dx_1 \right) \]
\[ \vdots \]
\[ \times \left( \int_\mathbb{R} e^{-\lambda(u_{m-1}-x_m)} + e^{-\lambda(u_m-x_m)} (u_m-x_m)^{d-1} dx_m \right). \]

As before, using Lemma 6, we can obtain
\[ C_m(Z^\lambda_t + Z^\lambda_s) = 2^{m-1+(\frac{d}{2})m}(m-1)! \lambda^{(\frac{d}{2}-d)m} \left( \frac{\Gamma(d)}{\sqrt{\pi}} \right)^m \sum_{t_j \in \{t,s\}} \int_0^{t_1} \cdots \int_0^{t_m} du_1 \cdots du_m \times K_{\frac{d}{2}-d}(\lambda|u_1-u_2|)|u_1-u_2|^{d-1/2} \]
\[ \cdots K_{\frac{d}{2}-d}(\lambda|u_m-u_1|)|u_m-u_1|^{d-1/2}. \]

Now, we use that the function \( K_v \) is continuous and, for any \( v \in \mathbb{R} \), it satisfies as \( u \to 0^+ \)
\[ K_v = \begin{cases} 2^{\frac{1}{|v|}} \Gamma(|v|)u^{-|v|} & \text{if } v \neq 0 \\ -\log u & \text{if } v = 0, \end{cases} \]

with this, we can obtain
\[ \lim_{\lambda \to 0^+} C_m(Z^\lambda_t + Z^\lambda_s) = a(m) \sum_{t_j \in \{t,s\}} \int_0^{t_1} \cdots \int_0^{t_m} du_1 \cdots du_m \times |s_1 - s_2|^{2d-1} |s_2 - s_3|^{2d-1} \cdots |s_m - s_1|^{2d-1}, \quad (11) \]
where \( a(m) = 2^{-1}(m-1)! \left( \frac{\Gamma(d)\Gamma(d-1/2)}{\sqrt{\pi}} \right)^m \). To conclude, we compare Formula (11) with the Formula (54) in [26].

**Remark 4.** Here, we assumed that, if two processes differs by a constant, they have the same distribution.

### 5.2 Hermite kernel: filtered version

In this part, we consider the filtered Hermite kernel given by
\[ g(x) = t^\beta_i u \prod_{j=1}^k x_j^{d-1}, \quad x_j > 0 \quad \text{and } u \in \mathbb{R}. \]
Proof: By Lemma 5 this comes by following the lines of Proposition 3 in Section 3 and Lemma 1 in [22]. Using this kernel, we define the following process

\[ Z^\lambda(t) = \int \int_{\mathbb{R}^k} l^\beta(u) \prod_{i=1}^{k} (u - x_i)_{+}^{d-1} e^{-\lambda(u - x_i)} du W(dx_1) \ldots W(dx_k), \]

(12)

where \( \lambda > 0 \), \((x)_+ = x1_{\{x>0\}}\) and \( d \in \left( \frac{1}{2} - \frac{1}{2k}, \infty \right) \), and

\[-1 < H < \beta < 1 - H < \frac{1}{2} \quad \beta \neq 0.\]

Recall that, the function \( l^\beta \) is given by

\[ l^\beta_t(u) = (t - u)^\beta - (-u)^\beta \quad \text{if} \quad \beta \neq 0. \]

and \( l^\beta_t(u) = 1_{[0,t]}(u) \) if \( \beta = 0 \).

**Remark 5.** If \( \beta = 0 \) we recover the definition of the tempered Hermite process, and if the tempering factor is one with \( \beta \neq 0 \) we have a special case of generalized Hermite process introduced in [2] and further studied in [1].

By Proposition 3 and Lemma 1 we know that \( Z^\lambda \) is a stationary increment process with a scaling property. Now, we consider the computation of the explicit expression for the covariance of the process

**Proposition 6.** Let \( Z^\lambda \) be the process given by (3) with \( g(x) = (x)^{d-1} \). Then, \( Z^\lambda \) has the covariance function

\[ E(Z^\lambda(t)Z^\lambda(s)) = \frac{1}{2} \left[ C_{[t]} |t|^{2\beta+k(d-1)/2} + C_{[s]} |s|^{2\beta+k(d-1)/2} - C_{[t-s]} |t-s|^{2\beta+k(d-1)/2} \right], \]

\[ C_{[t]} := k! \left( \frac{\Gamma(d)}{\sqrt{\pi}(2\lambda)^{d-1/2}} \right)^{k} \times \int_{\mathbb{R}} \int_{\mathbb{R}} |(1-u)^\beta - (-u)^\beta| |(1-v)^\beta - (-v)^\beta| |u-v|^{k(d-1)/2} K_{1/2-a}^k(\lambda |u-v|) dv du. \]

**Proof:** By Lemma 5

\[ E(Z^\lambda(t)Z^\lambda(s)) = k! \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\lambda |u-v|} |u-v|^{k(2d-1)} l^\beta_s(u) l^\beta_t(v) \left[ \int_0^\infty g(x) g[1+x] e^{-2\lambda |u-v|} dx \right]^k dv du, \]

now, taking \( g(x) = (x)^{d-1} \) and the fact that

\[ \int_0^\infty x^{d-1} e^{-2\lambda |u-v|} dx = \frac{\Gamma(d)}{\sqrt{\pi}(2\lambda)^{d-1/2}} |u-v|^{1/2-d} e^{\lambda |u-v|} K_{1/2-a}^k(\lambda |u-v|). \]

18
With this, we have
\[ E(Z^\lambda(t)Z^\lambda(s)) = k! \left( \frac{\Gamma(d)}{\sqrt{\pi}(2\lambda)^{d-1/2}} \right)^k \int_{\mathbb{R}} \int_{\mathbb{R}} l_\lambda^\beta(u)l_\lambda^\beta(v)|u - v|^{k(d-1/2)} K_{1/2-d}(\lambda|u - v|)dudv, \]

let us recall that \( l_\lambda^\beta(u) = (t - u)_+^\beta - (-u)_+^\beta \) if \( \beta \neq 0 \). Therefore,
\[ E(Z^\lambda(t)Z^\lambda(s)) = k! \left( \frac{\Gamma(d)}{\sqrt{\pi}(2\lambda)^{d-1/2}} \right)^k \times \int_{\mathbb{R}} \int_{\mathbb{R}} [(s - u)_+^\beta - (-u)_+^\beta][(t - v)_+^\beta - (-v)_+^\beta]|u - v|^{k(d-1/2)} K_{1/2-d}(\lambda|u - v|)dudv. \]  \tag{13}

To take advantage of the expression (13), we will use the fact that
\[ E(Z^\lambda(t)Z^\lambda(s)) = \frac{1}{2} \left[ E([Z^\lambda(t)]^2) + E([Z^\lambda(s)]^2) - E([Z^\lambda(t) - Z^\lambda(s)]^2) \right]. \]

Now, we concentrate in the computation of \( E([Z^\lambda(t)]^2) \). To do this, we make the successive change of variables \( \tilde{u} = u/t \) and \( \tilde{v} = v/t \)
\[ E(Z^\lambda(t)^2) = k! \left( \frac{\Gamma(d)}{\sqrt{\pi}(2\lambda)^{d-1/2}} \right)^k |t|^{2\beta + k(d-1/2)} \times \int_{\mathbb{R}} \int_{\mathbb{R}} [(1 - u)_+^\beta - (-u)_+^\beta][(1 - v)_+^\beta - (-v)_+^\beta]|u - v|^{k(d-1/2)} K_{1/2-d}(\lambda t|u - v|)dudv. \]

If we define
\[ C_{|t|} := k! \left( \frac{\Gamma(d)}{\sqrt{\pi}(2\lambda)^{d-1/2}} \right)^k \times \int_{\mathbb{R}} \int_{\mathbb{R}} [(1 - u)_+^\beta - (-u)_+^\beta][(1 - v)_+^\beta - (-v)_+^\beta]|u - v|^{k(d-1/2)} K_{1/2-d}(\lambda t|u - v|)dudv. \]

Then,
\[ E(Z^\lambda(t)^2) = C_{|t|}|t|^{2\beta + k(d-1/2)}. \]

With this and using the stationarity of the process \( Z^\lambda \) we can get
\[ E(Z^\lambda(t)Z^\lambda(s)) = \frac{1}{2} \left[ C_{|t|}|t|^{2\beta + k(d-1/2)} + C_{|s|}|s|^{2\beta + k(d-1/2)} - C_{|t-s|}|t - s|^{2\beta + k(d-1/2)} \right]. \]

As before, using Formula \( \mathcal{F} \), we can obtain the following result concerning the cumulants of the process \( Z^\lambda \) with \( k = 2 \) in the filtered case.
Lemma 9. Let $Z^\lambda$ the process given by (9) with $g(x) = (x)^{d-1}_+$, $k = 2; \lambda > 0$, and

$$h_t^\lambda(x_1, x_2) = \int_R t^3 u(u - x_1)^{d-1}_+ e^{-\lambda(u-x_1)}(u - x_2)^{d-1}_+ e^{-\lambda(u-x_2)}du,$$

then

$$C_m(Z^\lambda(t)) = 2^{m-1}(\frac{1}{2}-d)m(m-1)!\lambda^{(\frac{1}{2}-d)m}(\frac{\Gamma(d)}{\sqrt{\pi}})^m \int_R \cdots \int_R du_1 \cdots du_m \prod_{j=1}^m l_t^\beta(u_j)$$

$$\times K_{\frac{1}{2}-d}(\lambda|u_1 - u_2|)|u_1 - u_2|^{d-1/2}K_{\frac{1}{2}-d}(\lambda|u_2 - u_3|)|u_2 - u_3|^{d-1/2}$$

$$\cdots K_{\frac{1}{2}-d}(\lambda|u_m - u_1|)|u_m - u_1|^{d-1/2}.$$

Proof: By (12) we have that, for $k = 2$ we can write

$$Z^\lambda(t) = \int_{R^2} \int_R t^3 u \prod_{i=1}^2 (u - x_i)^{d-1}_+ e^{-\lambda(u-x_i)}duW(dx_1)W(dx_2)$$

$$= \int_{R^2} h_t^\lambda(x_1, x_2)W(dx_1)\cdots W(dx_2).$$

Now, using the Formula (13) we can obtain

$$C_m(Z^\lambda(t)) = 2^{m-1}(m-1) \int_{R^m} h_t^\lambda(x_1, x_2)h_t^\lambda(x_2, x_3)\cdots h_t^\lambda(x_{m-1}, x_m)h_t^\lambda(x_m, x_1)dx_1\cdots dx_m$$

$$= 2^{m-1}(m-1) \int_{R^m} dx_1 \cdots dx_m$$

$$\times \left( \int_R t^3 u_1 e^{-\lambda(u_1 - x_1)}e^{-\lambda(u_1 - x_2)}(u_1 - x_1)^{d-1}_+(u_1 - x_2)^{d-1}_+ du_1 \right)$$

$$\times \left( \int_R t^3 u_2 e^{-\lambda(u_2 - x_2)}e^{-\lambda(u_2 - x_3)}(u_2 - x_2)^{d-1}_+(u_2 - x_3)^{d-1}_+ du_2 \right)$$

$$\vdots$$

$$\times \left( \int_R t^3 u_m e^{-\lambda(u_m - x_m)}e^{-\lambda(u_m - x_1)}(u_m - x_m)^{d-1}_+(u_m - x_1)^{d-1}_+ ds_m \right).$$

Then, by Fubbbini theorem, we can obtain

$$C_m(Z^\lambda(t)) = 2^{m-1}(m-1)! \int_{R} \cdots \int_{R} du_1 \cdots du_m \prod_{j=1}^m l_t^\beta(u_j)$$

$$\times \left( \int_R e^{-\lambda(u_1 - x_1)}e^{-\lambda(u_m - x_1)}(u_1 - x_1)^{d-1}_+(u_m - x_1)^{d-1}_+ dx_1 \right)$$

$$\vdots$$

$$\times \left( \int_R e^{-\lambda(u_m - x_m)}e^{-\lambda(u_m - x_m)}(u_m - x_m)^{d-1}_+(u_m - x_m)^{d-1}_+ dx_m \right).$$
Using Lemma 6 again, we can obtain

\[ C_m(Z^\lambda(t)) = 2^{m-1+(\frac{1}{2}-d)m}(m-1)!\lambda^{(\frac{1}{2}-d)m} \left( \frac{\Gamma(d)}{\sqrt{\pi}} \right)^m \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} du_1 \cdots du_m \prod_{j=1}^m l^\beta_j(u_j) \times K_{(\frac{1}{2}-d)}(\lambda|u_1-u_2|)|u_1-u_2|^{d-1/2}K_{(\frac{1}{2}-d)}(\lambda|u_2-u_3|)|u_2-u_3|^{d-1/2} \cdots K_{(\frac{1}{2}-d)}(\lambda|u_m-u_1|)|u_m-u_1|^{d-1/2}. \]  

(14)

Remark 6. As in the tempered Hermite case, taking \(m=2\) in the formula (14), we can recover the formula for the variance of the tempered filtered Hermite process with \(k=2\) (see Equality (13)).

In a similar manner to the tempered Rosenblatt process, we have the following result concerning the behavior of the tempered Rosenblatt process with filtered kernel when \(\lambda \to 0^+\)

Lemma 10. Let \(Z^\lambda\) be given by \(\Phi\) with \(g(x) = (x)^{d-1}_+, k = 2, \) and \(\lambda > 0, \) and

\[ h^\lambda_t(x_1, x_2) = \int_{\mathbb{R}} l^\beta_t(u)(u - x_1)^{d-1}_+e^{-\lambda(u-x_1)}(u - x_2)^{d-1}_+e^{-\lambda(u-x_2)}du, \]

then

\[ \lim_{\lambda \to 0^+} Z^\lambda_t \overset{d}{=} Z_t, \]

where \(Z\) is the Rosenblatt process with filtered kernel.

Proof: The proof follows the same lines of the proof of Lemma \(\Phi\) so we will omit it.

6 An application: non parametric estimation

Here, we consider the problem of non parametric estimation. Precisely, we consider a cointegrated regressor model where the regressor is a fractional Brownian motion with Hurst parameter \(H_1 \in (0, 1)\) and \(Z^\lambda\) is a generalized tempered Hermite process.

Let us consider the following model

\[ Y_{i/n} = r(B_{i/n}^{H_1}) + S_n(Z_{i+1/n}^\lambda - Z_{i/n}^\lambda), \quad 0 \leq i \leq n-1 \quad \text{and} \quad n \geq 1, \]  

(15)

with, as mentioned before, \(B^{H_1}\) is a fractional Brownian motion and \(Z^\lambda\) is a generalized tempered Hermite process. Here,

\[ S_n = \sqrt{\text{Var}(Z_{i+1/n}^\lambda - Z_{i/n}^\lambda)}. \]
As usual, the estimator of the function $r$ can be written as

$$\hat{r}_n(x) = \frac{\sum_{i=0}^{n-1} Y_i/n K \left( \frac{x - B_{i/n}^{H_1}}{h} \right)}{\sum_{i=0}^{n-1} K \left( \frac{x - B_{i/n}^{H_1}}{h} \right)}, \quad (16)$$

by means of the expression (15) we can decomposed $\hat{r}_n$ as

$$\hat{r}_n(x) = \frac{\sum_{i=0}^{n-1} K \left( \frac{x - B_{i/n}^{H_1}}{h} \right) r(B_{i/n}^{H_1})}{\sum_{i=0}^{n-1} K \left( \frac{x - B_{i/n}^{H_1}}{h} \right)} + \frac{\sum_{i=0}^{n-1} K \left( \frac{x - B_{i/n}^{H_1}}{h} \right) S_n(Z_{i+1/n}^\lambda - Z_{i/n}^\lambda)}{\sum_{i=0}^{n-1} K \left( \frac{x - B_{i/n}^{H_1}}{h} \right)} := M_1^{(n)} + M_2^{(n)} \quad (17)$$

for every $x \in \mathbb{R}$. Where, $K$ is a non-negative real kernel function satisfying $\int_{\mathbb{R}} K(y)dy = 1, \int_{\mathbb{R}} yK(y)dy = 0$ and $\int_{\mathbb{R}} K(y)^2dy < \infty$. The bandwith $h \equiv h_n$ satisfies $h_n \to 0$ as $n \to \infty$ and

$$h_n := h = n^{-\kappa} \quad \text{with} \quad 0 < \kappa < 1.$$

Now, the idea (as in [25]) is to prove that the estimator $\hat{r}_n$ is consistent, that is, $\hat{r}_n$ converges in probability to $r(x) \ \forall x \in \mathbb{R}$. To prove this, we will handle the terms $M_1$ and $M_2$ by separate. In fact, we will prove that $M_1^{(n)}(x) \to r(x)$ and $M_2^{(n)}(x) \to 0$ as $n \to \infty$, where both limits are a.s. Then, the final result comes by an application of the continuous mapping theorem. As mentioned before, we will study the terms $M_1$ and $M_2$ by separate. In fact, we have the following result for $M_1$.

**Proposition 7.** Let assume that $r$ is H"older continuous with exponent $\gamma_r$. Take $\kappa < \min{(H_1/2, H_1\gamma_r)}$ and for $h = n^{-\kappa}$, let $M_1^{(n)}$ be given by (17), with Lipschitz kernel $K$ satisfying conditions (3.7) and (3.9) in [25]. Then, for every $x \in \mathbb{R}$

$$M_1^{(n)}(x) \to r(x)$$

as $n \to \infty$.

**Proof:** The proof is a consequence of Lemmas 3.1, 3.3, 3.4 and 3.5 in [25].
Remark 7. The kernels that satisfies the Lipschitz condition are:

- The Gaussian kernel
  \[ K(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}. \]

- The triangle kernel
  \[ K(x) = (1 - |x|)1_{[-1,1]}(x). \]

- The Epanechnikov kernel
  \[ K(x) = \frac{3}{4}(1 - x^2)1_{[-1,1]}(x). \]

- The quartic kernel
  \[ K(x) = \frac{15}{16}(1 - x^2)^21_{[-1,1]}(x). \]

Now, we need to prove that \( M(n)_{2}(x) \rightarrow 0 \) as \( n \rightarrow \infty \). To do this we need to study the asymptotic behavior of the term \( M_2 \). Let us recall that we can write

\[
M_2^{(n)}(x) = \frac{n^{\kappa-1} \sum_{i=0}^{n-1} K \left( \frac{x - B_{i/n}^{H_1}}{h} \right) S_n(Z_{i+1/n}^{\lambda} - Z_{i/n}^{\lambda})}{n^{\kappa-1} \sum_{i=0}^{n-1} K \left( \frac{x - B_{i/n}^{H_1}}{h} \right)} = M_2^{(n)}(x). \tag{18}
\]

First, we will handle the numerator of the term \( M_2 \). In fact, by Lemma 3 and considering for \( i = 0 \) is easy to see that the convergence holds for \( \kappa < 1 \). We can obtain

\[
E \left[ \left( M_{2,1}^{(n)}(x) \right)^2 \right] \leq n^{2(\kappa-1)+2H} \sum_{i,j=1}^{n-1} E K \left( \frac{x - B_{i/n}^{H_1}}{h} \right) K \left( \frac{x - B_{j/n}^{H_1}}{h} \right) \times E(Z_{i+1/n}^{\lambda} - Z_{i/n}^{\lambda})(Z_{i+1/n}^{\lambda} - Z_{j/n}^{\lambda})
\]

Now, we will consider, only, the Gaussian kernel. Although, the others examples given in Remark can be used without any problem.

\[
E \left[ \left( M_{2,1}^{(n)}(x) \right)^2 \right] \leq Cn^{2(\kappa-1)} \sum_{i=1}^{n-1} E K^2 \left( \frac{x - B_{i/n}^{H_1}}{h} \right) + Cn^{2(\kappa-1)} \sum_{i,j=1}^{n-1} E K \left( \frac{x - B_{i/n}^{H_1}}{h} \right) K \left( \frac{x - B_{j/n}^{H_1}}{h} \right) R(i,j)
\]

\[= m_{2,1.1}^{(n)}(x) + m_{2,1.2}^{(n)}(x). \]
For the first term, we can use the Results from Section 3.2.1 in [25] to get

\[ m_{2,1.1}^{(n)}(x) \leq Cn^{\kappa-1} \]  \hspace{1cm} (19)

and this always goes to zero for \( \kappa < 1 \). With respect to the second term we must impose the condition that \( |R(i,j)| \leq C|i-j|^{l_R} \), where \(-1 < l_R < 0\), with this and using, again, the results from Section 3.2.1 in [25], it allow us to obtain

\[ m_{2,1.2}^{(n)}(x) \leq Cn^{2(\kappa-1)+\frac{4-3\kappa}{2}+l_R} \]  \hspace{1cm} (20)

which converges to zero under the condition \( \kappa < -2l_R \). Taking into account the previous computations we can have

**Lemma 11.** Let us assume that \( K \) is the Gaussian kernel and \( \kappa < -2l_R \)

with \(-1 < l_R < 0\) and \( M_{2,1}^{(n)}(x) \) given by [18]. Then,

\[ M_{2,1}^{(n)}(x) \rightarrow 0 \]

as \( n \rightarrow \infty \) in \( L^2(\Omega) \).

Now we consider the behavior of \( M_2 \).

**Lemma 12.** Let us assume that \( K \) is the Gaussian kernel and

\[ \kappa < \min \{ H_1/2, -2l_R \} \]

with \(-1 < l_R < 0\) and \( M_2^{(n)}(x) \) given by [18]. Then,

\[ M_2^{(n)}(x) \rightarrow 0 \]

as \( n \rightarrow \infty \) in probability.

**Proof:** The proof follows by Lemma 3.1 in [25] and Lemma 11.

Finally, we can obtain the following result concerning the convergence of \( \hat{r}_n \)

**Theorem 1.** Let us assume that the function \( r \) is Hölder continuous with exponent \( \gamma_r \) and \( K \) is the Gaussian kernel. Also assume that

\[ \kappa < \min \{ H_1/2, l_R, H_1 \gamma_r \} \]

Then, for every \( x \in \mathbb{R} \)

\[ \hat{r}(x) \rightarrow r(x) \]

as \( n \rightarrow \infty \) in probability.
Proof: The proof follows by Proposition 7, Lemma 11 and Lemma 12.

Remark 8. The condition $|R(i,j)| \leq C|i-j|^l_R$, where $-1 < l_R < 0$ is not easy to fulfill, even it seems hard to prove for the simplest non-Gaussian case. At least, we have that this condition is satisfied in the Gaussian case. However, for the non-Gaussian cases there is the tempered fractional levy [19] and the fractional stable [16] processes that fulfill this condition. Nonetheless, this process is not in the space of generalized Hermite process. Although, is a proper non-Gaussian process for our application. The long time behavior of covariance of the generalized Hermite process remain as an open question and is a topic of future research.

7 Appendix: Some elements from Malliavin calculus

Here, we briefly recall some elements from stochastic analysis; for an in-depth introduction we refer the reader to [20]. Consider $(\mathcal{H}, \langle \cdot, \cdot \rangle_\mathcal{H})$ a real separable Hilbert space and $(B(\varphi), \varphi \in \mathcal{H})$ an isonormal Gaussian process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is a centered Gaussian family of random variables such that $\mathbb{E} (B(\varphi) B(\psi)) = \langle \varphi, \psi \rangle_\mathcal{H}$, for every $\varphi, \psi \in \mathcal{H}$. Denote $I_q$ the qth multiple stochastic integral with respect to $B$. This $I_q$ is actually an isometry between the Hilbert space $\mathcal{H}^\otimes q$ (symmetric tensor product) equipped with the scaled norm $\frac{1}{\sqrt{q!}} \cdot \| \cdot \|_{\mathcal{H}^\otimes q}$ and the Wiener chaos of order $q$, which is defined as the closed linear span of the random variables $H_q(B(\varphi))$ where $\varphi \in \mathcal{H}$, $\| \varphi \|_{\mathcal{H}} = 1$ and $H_q$ is the Hermite polynomial of degree $q \geq 1$ defined by:

$$H_q(x) = (-1)^q \exp \left( \frac{x^2}{2} \right) \frac{d^q}{dx^q} \left( \exp \left( -\frac{x^2}{2} \right) \right), \quad x \in \mathbb{R}. \quad (21)$$

The isometry of multiple integrals can be written as: for $p, q \geq 1$, $f \in \mathcal{H}^\otimes p$ and $g \in \mathcal{H}^\otimes q$,

$$\mathbb{E} \left( I_p(f) I_q(g) \right) = \begin{cases} q! \langle \hat{f}, \hat{g} \rangle_{\mathcal{H}^\otimes q} & \text{if } p = q \\ 0 & \text{otherwise.} \end{cases} \quad (22)$$

It also holds that:

$$I_q(f) = I_q(\hat{f})$$

where $\hat{f}$ denotes the symmetrization of $f$ defined by $\hat{f}(x_1, \ldots, x_q) = \frac{1}{q!} \sum_{\sigma \in S_q} f(x_{\sigma(1)}, \ldots, x_{\sigma(q)})$.

Acknowledgements

The author was partially supported by Proyecto Fondecyt PostDoctorado 3190465, Project ECOS210037, MEC 80190045 and Mathamsud AMSUD210023.
References

[1] Assaad, O., Diez, C. P., and Tudor C. A. (2022). Generalized Wiener-Hermite integrals and rough non-Gaussian Ornstein–Uhlenbeck process. Stochastics, DOI: 10.1080/17442508.2022.2068955.

[2] Bai, S., and M. Taqqu. (2014). Generalized Hermite processes, discrete chaos and limit theorems. Stochastic Processes and Their Applications 124:1710-1739.

[3] Boniece, B.C., Didier, G., and Sabzikar, F. (2020). On Fractional Lévy Processes: Tempering, Sample Path Properties and Stochastic Integration. J Stat Phys 178, 954-985.

[4] Bowman, F. (1958). Introduction to Bessel Functions, Dover New York.

[5] Chen, Y., Wang, X., and Deng, W. (2017). Localization and Ballistic Diffusion for the Tempered Fractional Brownian–Langevin Motion. J Stat Phys 169, 18-37.

[6] Embrechts, P., and Maejima, M. (2002). Selfsimilar Processes. Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ.

[7] Fox, R., and Taqqu, M.S. (1987). Multiple stochastic integrals with dependent integrators. J. Multivariate Analysis 21, 105-127.

[8] Fan, X., and Lévy Véhel, J. (2019). Tempered fractional multistable motion and tempered multifractional stable motion. ESAIM: PS 23 37-67.

[9] Lawrance, A.J., and Kottegoda, N.T. (1977). Stochastic modelling of riverflow time series. J. Roy. Statist. Soc. Ser. A, 140(1), 1-47.

[10] Lechiheb, A. (2021). Wiener integrals with respect to the two-parameter tempered Hermite random fields. Arxiv.

[11] Lupascu-Stamate, O., and Tudor, C.A. (2019). Rosenblatt Laplace Motion. Mediterr. J. Math. 16, 15.

[12] Madan, D.B., and Wang, K. (2022). Stationary increments reverting to a Tempered Fractional Lévy Process (TFLP), Quantitative Finance, 22:7, 1391-1404.

[13] Major, P. (2014). Multiple Wiener-Itô Integrals: With Applications to Limit Theorems, Springer Cham.

[14] Meerschaert, M.M., and Sabzikar, F. (2013). Tempered fractional Brownian motion. Statistics & Probability Letters 83(10), 2269-2275.

[15] Meerschaert, M.M., and Sabzikar, F. (2014). Stochastic Processes and their Applications, 124(7), 2363-2387.
[16] Meerschaert, M.M., and Sabzikar, F. (2016). Tempered Fractional Stable Motion. J Theor Probab 29, 681-706.

[17] Meerschaert, M.M., Sabzikar, F., Phanikumar M. S., and Zeleke A. (2014). Tempered fractional time series model for turbulence in geophysical flows. Journal of Statistical Mechanics: Theory and Experiment. 2014(9), 09-023.

[18] Nourdin., I., and Peccati. G. (2010): Cumulants onWiener space. Journal of Functional Analysis 258, 3775-3791

[19] Nourdin, I., and T. T. Tran. (2019).Statistical inference for vasicek-type model driven by hermite processes. Stochastic Processes and Their Applications 129(10):3374-3791. dpi:https://doi.org/10.1016/j.spa.2018.10.005.

[20] Nualart, D. (2006). Malliavin Calculus and Related Topics, 2nd ed., Springer.

[21] Pipiras, V., and Taqqu, M. (2017). Long-Range Dependence and Self-Similarity (Cambridge Series in Statistical and Probabilistic Mathematics). Cambridge: Cambridge University Press.

[22] Sabzikar, F. (2015). Tempered Hermite process. Modern Stochastic: Theory and Applications 2: 327-341.

[23] Sabzikar, F., Meerschaert, M.M., and Chen, J. (2015). Tempered fractional calculus, Journal of Computational Physics, 293, 14-28.

[24] Sabzikar, F., and Surgailis, D. (2018). Tempered fractional Brownian and stable motions of second kind, Statistics & Probability Letters, 132, 17-27.

[25] Sued, M., Torres, S., and Tudor., C.A. (2013). Nonparametric regression with non-Gaussian long memory, Communications on Stochastic Analysis: Vol. 7 : No. 2 , Article 6.

[26] Tudor., C.A. (2008). Analysis of the Rosenblatt process. ESAIM: Probability and Statistics, Tome 12, pp. 230-257.

[27] Tudor., C.A. (2013). Analysis of Variations for Self-Similar Processes. A Stochastic Calculus Approach, Probability and its Applications, Springer, Cham, New York.