CROSS-POWER SPECTRUM AND ITS APPLICATION ON WINDOW FUNCTIONS IN THE WILKINSON MICROWAVE ANISOTROPY PROBE DATA

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ABSTRACT

The cross-power spectrum is a quadratic estimator between two maps that can provide unbiased estimate of the underlying power spectrum of the correlated signals, which is therefore used for extracting the power spectrum in the Wilkinson Microwave Anisotropy Probe (WMAP) data. In this paper, we discuss the limit of the cross-power spectrum and derive the residual from the uncorrelated signal, which is the source of error in power spectrum extraction. We employ the estimator to extract window functions by crossing pairs of extragalactic point sources. We demonstrate its usefulness in WMAP difference assembly maps where the window functions are measured via Jupiter and then extract the window functions of the five WMAP frequency band maps.

Key words: cosmic background radiation – cosmology: observations – methods: data analysis

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1. INTRODUCTION

Variables and parameters in theories of cosmology are often derived based on a statistical ensemble. In practice, particularly in data analysis on the cosmic microwave background (CMB), however, one often encounters the well known and rather unique issue of being able to conduct statistical analysis on only one version of the celestial sphere using an ergodic hypothesis. For example, the angular power spectrum of the temperature anisotropies is $C_\ell \equiv \langle a_{\ell m} a_{\ell m}^* \rangle$, but with only one version of the sphere it can be estimated with only two $\ell + 1$ modes: $C_\ell^\text{est} = (2\ell + 1)^{-1} \sum_{m=-\ell}^{\ell} a_{\ell m} a_{\ell m}^*$. Lack of a statistical ensemble thus induces in power spectrum analysis the uncertainty called “cosmic variance” (White et al. 1993). Another issue arising from lack of an ensemble is coined “cosmic covariance,” which refers to a non-zero chance correlation between the CMB and the foregrounds (Chiang et al. 2009). The cosmic covariance is the source of error in employing quadratic minimization for separating the CMB and the foregrounds (Bennett et al. 2003; Hinshaw et al. 2007).

In this paper, we address another issue in the cross-power spectrum arising from lack of an ensemble. The cross-power spectrum is useful in eliminating uncorrelated signal and is thus commonly used in power spectrum estimation from multifrequency measurement of CMB. It is required, however, to have an ensemble of modes in the summation to completely eliminate the uncorrelated signal. Therefore, lack of an ensemble of modes results in residual (hence error) in the cross-power spectrum.

We then employ the cross-power spectrum to estimate window functions. An important issue related to extraction of the power spectrum of CMB temperature anisotropies is the calibration of the window function convolving the signal, which is usually obtained via measuring Jupiter (Page et al. 2003). It can also be obtained by measuring bright extragalactic point sources because they are manifestation of the window function as well. Therefore, in employing the cross-power spectrum on pairs of extragalactic point sources from different patches of the sky, the background CMB and noise are uncorrelated and thus can be largely eliminated.

This paper is arranged as follows. In Section 2, we examine the limit of the cross-power spectrum as a quadratic estimator. We in Section 3 employ the estimator on a pair of simulated point sources to obtain the window function. In Section 4, we put this method to the test on Wilkinson Microwave Anisotropy Probe (WMAP) Difference Assembly (DA) maps, whose window functions are already carefully calibrated via measuring Jupiter and we extract the window functions of WMAP frequency band maps. Conclusion and discussion are in Section 5.

2. CROSS-POWER SPECTRUM AND ITS LIMITATION

The cross-power spectrum (XPS) is a quadratic estimator between two maps $j$ and $j'$, and on a sphere it is written as

$$X_{jj'} = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} j_{\ell m}^* j'_{\ell m},$$

where $j_{\ell m}$ and $j'_{\ell m}$ are the spherical harmonic coefficients of the two maps, and $*$ denotes the complex conjugate (Hinshaw et al. 2003). In order for the XPS to be real, one usually replaces $j_{\ell m}^* j'_{\ell m}$ in Equation (1) with

$$\langle j_{\ell m} j'_{\ell m} + j'_{\ell m}^* j_{\ell m} \rangle/2 \equiv |j_{\ell m}| |j'_{\ell m}| \cos \Delta \phi_{\ell m},$$

where $\Delta \phi_{\ell m}$ is the phase difference between $j_{\ell m}$ and $j'_{\ell m}$. The advantage of XPS as an unbiased quadratic estimator for the power spectrum estimation lies in the fact that XPS returns with its usual power spectrum $\sum_m |j_{\ell m}|^2$ if $j$ and $j'$ are of the same signal, and if, on the other hand, $j$ and $j'$ are uncorrelated then $\Delta \phi_{\ell m}$ is distributed uniformly random in $[0, 2\pi]$ and thus $|j_{\ell m}| |j'_{\ell m}| \cos \Delta \phi_{\ell m} = 0$, where the angle brackets denote ensemble average. Hence XPS is useful in eliminating uncorrelated signals while preserving the correlated one and has been employed in WMAP analysis to extract CMB spectrum by crossing the differencing assemblies (DAs; Hinshaw et al. 2003, 2007; Nolta et al. 2009; Larson et al. 2010). In practice, however, one does not have a statistical ensemble. The capability of XPS to extract the underlying power spectrum from the correlated signal thus depends on how much
the uncorrelated signal is decreased with finite realizations and a finite number of multipole modes involved in the summation at $\ell$, i.e., $\sum_{m=-\ell}^{\ell} |j_{\ell m}| |j'_{\ell m}| \cos \Delta \phi_{\ell m} \neq 0$.

Take as an example the extraction of the CMB signal from the WMAP data. The W1 and W2 DA have the same CMB but uncorrelated noise, which can be written as $a_{\ell m}^{W1} = a_{\ell m}^{c} + n_{\ell m}$ and $a_{\ell m}^{W2} = a_{\ell m}^{c} + n'_{\ell m}$. In XPS the correlated signal $\sum_{m} |a_{\ell m}^{c}|^2$ is what we are after whereas those uncorrelated terms between CMB and noises $X_{\ell}^{cn}$, $X_{\ell}'^{cn}$ and between noises $X_{\ell}^{nn}$ shall be decreased but not to zero, which is then the source of error in the XPS.

In order to examine the level of residual in XPS, we consider two Gaussian and uncorrelated maps $j$ and $j'$, whose spherical harmonic coefficient $j_{\ell m}$ and $j'_{\ell m}$ have zero mean and variance $J_{\ell}$ and $J'_{\ell}$, respectively, and the corresponding phase difference $\Delta \phi_{\ell m}$ is uniformly random in $[0, 2\pi]$. One should note that the distribution of Equation (2) under this condition is a normal product distribution, which is itself not Gaussian but can be expressed as

$$p_{j,j'}(x) = \frac{4K_0 \left( \sqrt{x J J'} \right)}{\pi \sqrt{J J'}} ,$$

where $K_0$ is the zeroth-ordered modified Bessel function of the second kind (Springer 1979). Although the distribution of the variable is not Gaussian, what we need to examine is its summation. Since the XPS defined in Equation (2) can be either positive or negative, we look instead at its root mean square $\sqrt{\langle (X_{\ell}^{ij})^2 \rangle}$. Due to equal partition on both perpendicular directions of complex plane, it can be written as

$$\sqrt{\langle (X_{\ell}^{ij})^2 \rangle} = \frac{1}{2\ell + 1} \sqrt{\left( \sum_{m=-\ell}^{\ell} |j_{\ell m}| |j'_{\ell m}| \exp(i \Delta \phi_{\ell m}) \right)^2} ,$$

the right-hand side of which is closely related to a two-dimensional random walk. Random walk statistics is employed as a Gaussianity test on CMB (Naselsky et al. 2005; Stannard & Coles 2005). Recall that the celebrated Pearson’s random walk is a two-dimensional isotropic random walk with equal step length, expressed in the complex plane $r = \sum_{k} |d_k| \exp(i \phi_k)$ where $|d_k| = d$. The probability density function (pdf) of the resultant displacement $r \equiv |r|$ after $\ell$ steps has the form

$$p(r) = \frac{1}{\pi \ell d^2} \exp \left( - \frac{r^2}{\ell d^2} \right) ,$$

from which $\langle r^2 \rangle = \ell d^2$. Pearson’s walk can be generalized to unequal step length and it can be shown (Pearson 1905; Rayleigh 1905) that when $\ell$ is large and the mean-square step length is known: $(\ell)^{-1} \sum_{k} |d_k| = d^2$, the pdf for the displacement has the same form as Equation (5) (Hughes 1995). For unequal step length $|d_k| = |j_{\ell m}| |j'_{\ell m}|$, the ensemble-average square step length is then $\langle |j_{\ell m}|^2 |j'_{\ell m}|^2 \rangle = J_{\ell} J'_{\ell}$, so the mean-square displacement of the random walk $\langle |\sum_{m=-\ell}^{\ell} |j_{\ell m}| |j'_{\ell m}| \exp(i \Delta \phi_{\ell m})|^2 \rangle = (2\ell + 1)J_{\ell} J'_{\ell}$. Note that in deriving the result we only assume $j$ and $j'$ are Gaussian, so they can be either white noise (with a flat power spectrum) or others whose power spectrum has scale dependence. In any case, we can write

$$\sqrt{\langle (X_{\ell}^{ij})^2 \rangle} = \frac{1}{2\ell + 1} .$$

The decreasing of the “junk” is inversely proportional to the square root of the number of random walk steps, and it can be further decreased by $1/\sqrt{\ell N}$ with binning interval $L \equiv \Delta \ell$ multipole numbers and averaging from $N$ sets of XPS. We demonstrate Equation (6) in Figure 1 by plotting $X_{\ell}$ from two white-noise maps. One can see that the decreasing of the power spectrum is proportional to $(2\sqrt{\ell + 1/2})^{-1}$. Note that $X_{\ell}$ is jumping either positively or negatively but only positive points are plotted.

Knowing the residual of the “junk” from XPS, we can estimate its capability in extracting CMB. In crossing $N$ pairs of maps containing CMB with the same noise level, we assume that the CMB power spectrum is $C_{\ell} \simeq A \ell^{-2}$ and the instrument noise level is $N_{\ell} = RA$, where $R$ indicates the noise level.
relative to the CMB and $R \ll 1$.\footnote{The noise level $N_e = RA$ is related to noise fluctuation $\sigma^2$ through $\sigma^2 = N_{\text{pix}} RA / 4\pi$, where $N_{\text{pix}}$ is the total pixel number of the map.}\ The residual level from between CMB and noise is $2X_\ell^{\text{cm}} \simeq A\sqrt{\ell} (\ell^3 LN)^{-1/2}$ and between the noises is $X_\ell^{\text{mn}} \simeq RA (4\ell LN)^{-1/2}$. They intersect at $\ell \simeq 2/\sqrt{R}$. For $\ell < 2/\sqrt{R}$, $X_\ell^{\text{cm}}$ is the main residual with contribution $2X_\ell^{\text{cm}}/C_\ell^c \simeq \sqrt{R}/LN$, and $LN > \ell^2 R$ in order for the residual contribution to be within cosmic variance, which shall be the case as long as $LN \gg 4$. For $\ell \geq 2/\sqrt{R}$, on the other hand, the residual in the CMB extraction is $R\ell\sqrt{\ell^3/4LN}$, and beyond $\ell = 4LN/R^2$ the residual level is higher than the CMB, thereby the XPS fails.

For a more realistic situation, where the XPS is employed on signals that are convolved with the window function: $A\ell^{-2}\exp(-B^2\ell^2)$, we plot in Figure 2 the upper limit of the noise level $R$ if the XPS is to retrieve the CMB at the corresponding multipole number. Various beam sizes are chosen according to those from channels of ESA Planck Surveyor (Tauber et al. 2010). As expected, more binning and more sets of XPS shall retrieve CMB up to higher modes for the same noise level.

In Figure 3, we show the XPS from two simulated maps. The left panel shows Gaussian random signal with power spectrum $C_\ell = 10^4\ell^{-2}(\mu K^2)$ convolved with beam FWHM 30 arcmin in both maps, the theoretical power spectrum of which is shown in dash-dotted curve. They are added with pixel (white) noise signals with different levels $N_\ell = 10^{-3}$ and $N_\ell' = 10^{-3}(\mu K^2)$. The predicted residual from XPS (i.e., Equation (6)) is plotted in dash line. Note that the residual in employing XPS on maps with different noise levels is proportional to $\sqrt{N_\ell N_\ell'}$, which reflects the real situation of crossing between different frequency bands such as the WMAP V and W bands. On the right panel, we simulate four WMAP W channel DA maps, convolving with beam FWHM 12 arcmin before adding WMAP simulated noise. There are six XPS and the binned ($\Delta\ell = 10$) retrieved power spectrum is denoted with the plus sign. One can see the residual from the noise is the main error from XPS, even if it is not white noise.

\section{3. Application of Cross-Power Spectrum}

One of the applications of XPS is to retrieve the window function from bright extragalactic point sources. The standard way of measuring the window function in CMB experiments is via measuring planets such as Jupiter (Page et al. 2003; Hill et al. 2009).
The importance of window functions can be illustrated in the following simple example. The final step of CMB power spectrum estimate often involves deconvolution because the retrieved signal (either via ILC method or foreground template fitting) is also convolved with a common window function. The final power spectrum, after deconvolution, can be written as \( \delta T^2 = (2\pi)^{-4} \ell(\ell + 1)C_\ell \exp[b^2 \ell(\ell + 1)] \), where \( C_\ell \) is the (retrieved) CMB power spectrum and \( b = (\sqrt{8 \ln 2})^{-1} \) FWHM. An error in deconvolution scale \( \Delta b \) produces an error in the power \( \Delta(\delta T^2) = 2b \Delta b \ell(\ell + 1)\delta T^2 \), so, for example, deconvolution with 1\(^\circ\) FWHM on an originally smoothed map by 59\(^\circ\) (i.e., overestimated by a mere 1 arcmin) will overestimate the power at the first Doppler peak by \( \Delta(\delta T^2_{220})/\delta T^2_{220} \approx 8.9\% \) and the error goes with \( -\ell^2 \) for high \( \ell \). Although in practice deconvolution involves estimating the inverse covariance matrix, it demonstrates that the window function is one of the most crucial issues in CMB power spectrum estimation. The sensitivity of window function is also investigated in Sawangwit & Shanks (2010).

Based on the flat-sky approximation and Fourier transform, we can estimate the window function from bright point sources by taking two square patches of sky where there is a bright point source in the center: \( T_1(k) = ab(k) + c(k) + n(k), \) \( T_2(k) = b b(k) + c(k) + n'(k), \) where \( k \) is the Fourier wavenumber, \( a, c, c' \) and \( n, n' \) represent different CMB (+ foreground) and noise, respectively. The CMB and noise are not correlated in different parts of the sky so they are now the “junk” in XPS whereas the bright point sources each in the center of the patches are manifestation of the inflight beam profile and thus are a correlated signal. The window function is then \( W_0 = \sum b^2(k) \) with a rescaling relation \( \ell = 2\pi k/L \), where \( L \) is the size of the patch.

We simulate the full-sky CMB signal with the WMAP best-fit \( \Lambda \)CDM model on the Planck 30 GHz channel and add two bright point sources on different parts of the sky, which is then convolved with beam 33 arcmin FWHM before adding noise with \( N_0 = 10^{-2} \mu K^2 \). We extract \( 10^5 \times 10^6 \) with the point source at the center. The point sources have amplitudes 40\( \delta T \) and 46\( \delta T \), where \( \sigma^2 \) is the variance of the CMB. In Figure 4, we present the simulation and the retrieving capability of XPS on the window function from bright point sources. One can see that XPS extended the estimate by more than 10 dB.

### 4. WINDOW FUNCTIONS OF THE WMAP DIFFERENCE ASSEMBLY AND FREQUENCY BAND MAPS

Having demonstrated that the window function can be retrieved with XPS from simulated bright point sources, we now employ the same method on WMAP DA maps: K1, Ka1, Q1, Q2, V1, V2, and W1–W4. For each DA map, we extract the square patches where the top two brightest point sources are in the center of the patch.

Care has to be taken when one chooses the patch size. Fourier transform spreads the variance of the map into the power at different wavelengths. For stationary noise, a change in the choice of patch area from \( L_0^2 \) to \( L^2 \) (with the same pixel size) will shift the level of white spectrum by \( (L_0/L)^2 \) as the variance is fixed. For a point source, on the other hand, changing the patch size has an extra effect on the variance by \( (L_0/L)^2 \), resulting in the large-scale power level of a point source being shifted by \( (L_0/L)^2 \). Hence, choosing a smaller patch \( L < L_0 \) enhances the relative power level of the point source to noise by \( (L_0/L)^2 \), rendering a better profile of the window function. One cannot, however, choose as small as sizes as possible. The first multipole number to display the profile of the window function is decided by the size of the patch \( L \) via \( \ell_{\text{ext}} = 2\pi/L \), so one loses the large-scale profile if the patch size is chosen too small. Furthermore, a fixed pixel size sets the maximum multipole number regardless of the patch size, thus a smaller patch with a fixed pixel size has fewer Fourier modes with spacing \( \Delta \ell = 2\pi/L \).

We take the patch size \( L = 8^\circ, 5^\circ, 4^\circ, 3^\circ, \text{and } 3^\circ \) for the K, Ka, Q, V, and W bands, respectively, and show the XPS results of all 10 DA maps in Figure 6: solid curves are the window functions from WMAP DA maps and diamond sign denotes the
retrieved window functions from XPS. One can see in Figure 5 that the XPS retrieves the window functions nicely, except for the four W band DA maps. The reason is that W band not only has the smallest beam size, but also the highest noise level than the other bands.

According to the previous XPS DA results, our method for retrieving the window function from bright point sources is demonstrated useful. We can now retrieve the window functions of the WMAP frequency band maps. The WMAP frequency band maps, together with the corresponding window functions, can be as useful as the DA maps in extracting the CMB power spectrum. The WMAP K, Ka, Q, V, and W bands have 1, 1, 2, 2, and 4 DA maps, respectively, and the frequency band maps are produced from combining the DA maps, thus there is no direct measurement of the corresponding window functions for each map. One should note that the window functions of the DA maps at the same frequency band do not necessarily have the same profile, which is particularly true for W band (e.g., see W1 and W2 DA window functions in Figure 5).

In order to retrieve the window functions of the frequency band maps, we perform the same procedure as that on the DA maps, choosing the same patches of the sky and patch size for the corresponding band. The retrieved window functions are showed in Figure 6, and for comparison we plot the window function profiles of the DA maps for the corresponding frequency band. Though there are two DA maps for Q and V bands, the profiles are close to each other, and one can see that our retrieved window function for the frequency band maps is close to that of DA maps. For W band, we plot the W1 DA map (solid curve) and W2 DA map (long dashed line). The profiles of W3 and W4 are not plotted as they are close to those of W2 and W1, respectively. One can see that the pronounced difference in the W1 and W2 DA window function profiles and the retrieved W band window function is close to that of W1 (and W4), not W2 (and W3).

5. CONCLUSION AND DISCUSSION

In the first part of this paper, we discuss the widely used XPS and its limitation. We demonstrate that it has the ability to decrease uncorrelated “junk” signal by a factor inversely proportional to the square root of the number of modes involved in the summation. It indeed can decrease the uncorrelated signal on small scales in the power spectrum, but has limited ability in eliminating that of large scales. Understanding this limitation, in the second part of the paper we then employ XPS to retrieve the window functions via extragalactic point sources. One of the crucial steps for estimating CMB power spectrum is to know about the window function that convolves the observed signal. In this case the “junk” signal is composed of CMB and pixel noise. The WMAP DA maps have well-calibrated window functions obtained by measuring Jupiter. We thus use WMAP DA maps as test samples first to check the usefulness of the method and then conduct the same procedures for the window functions of the WMAP frequency band maps, which are unknown prior to our investigation. Our method will be useful for the upcoming Planck data as an auxiliary tool for estimation of the Planck window functions.

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