HOCHSCHILD COHOMOLOGY OF FROBENIUS ALGEBRAS

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Abstract.
Let $k$ be a field, $A$ a finite dimensional Frobenius $k$-algebra and $\rho: A \to A$, the Nakayama automorphism of $A$ with respect to a Frobenius homomorphism $\varphi: A \to k$. Assume that $\rho$ has finite order $m$ and that $k$ has a primitive $m$-th root of unity $w$. Consider the decomposition $A = A_0 \oplus \cdots \oplus A_{m-1}$ of $A$, obtained defining $A_i = \{a \in A : \rho(a) = w^i a\}$, and the decomposition $\text{HH}^*(A) = \bigoplus_{i=0}^{m-1} \text{HH}^i(A)$ of the Hochschild cohomology of $A$, obtained from the decomposition of $A$. In this paper we prove that $\text{HH}^*(A) = \text{HH}^*_{0}(A)$ and that if decomposition of $A$ is strongly $\mathbb{Z}/m\mathbb{Z}$-graded, then $\mathbb{Z}/m\mathbb{Z}$ acts on $\text{HH}^*(A_0)$ and $\text{HH}^*(A) = \text{HH}^*_{0}(A) = \text{HH}^*(A_0)\mathbb{Z}/m\mathbb{Z}$.

Introduction

Let $k$ be a field, $A$ a finite dimensional $k$-algebra and $DA = \text{Hom}_k(A, k)$ endowed with the usual $A$-bimodule structure. Recall that $A$ is said to be a Frobenius algebra if there exists a linear form $\varphi: A \to k$, such that the map $A \to DA$, defined by $x \mapsto x\varphi$ is a left $A$-module isomorphism. This linear form $\varphi: A \to k$ is called a Frobenius homomorphism. It is well known that this is equivalent to say that the map $x \mapsto \varphi x$, from $A$ to $DA$, is an isomorphism of right $A$-modules. From this it follows easily that there exists an automorphism $\rho$ of $A$, called the Nakayama automorphism of $A$ with respect to $\varphi$, such that $x\varphi = \varphi \rho(x)$, for all $x \in A$. It is easy to check that a linear form $\tilde{\varphi}: A \to k$ is another Frobenius homomorphism if and only if there exists $x \in A$ invertible, such that $\tilde{\varphi} = x\varphi$. It is also easy to check that the Nakayama automorphism of $A$ with respect to $\tilde{\varphi}$ is the map given by $a \mapsto \rho(x)^{-1}\rho(a)\rho(x)$.

Let $A$ be a Frobenius $k$-algebra, $\varphi: A \to k$ a Frobenius homomorphism and $\rho: A \to A$ the Nakayama automorphism of $A$ with respect to $\varphi$.

Definition. We say that $\rho$ has order $m \in \mathbb{N}$ and we write $\text{ord}_\rho = m$ if $\rho^m = id_A$ and $\rho^r \neq id_A$, for all $r < m$.

Assume that $\rho$ has finite order and that $k$ has a primitive $\text{ord}_\rho$-th root of unity $w$. Since the minimal polynomial $X^{\text{ord}_\rho} - 1$ of $\rho$ has distinct roots $w^i$ $(0 \leq i < \text{ord}_\rho)$, the algebra $A$ becomes a $\mathbb{Z}/\text{ord}_\rho\mathbb{Z}$-graded algebra

$$A = A_0 \oplus \cdots \oplus A_{\text{ord}_\rho - 1},$$

where $A_i = \{a \in A : \rho(a) = w^i a\}$.

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Let \((\text{Hom}_k(A^\otimes \ast, A), b^\ast)\) be the cochain Hochschild complex of \(A\) with coefficients in \(A\). For each \(0 \leq i < \text{ord}_\rho\), we let \((\text{Hom}_k(A^\otimes \ast, A)_i, b^\ast)\) denote the subcomplex of \((\text{Hom}_k(A^\otimes \ast, A), b^\ast)\), defined by

\[
\text{Hom}_k(A^\otimes \ast, A)_i = \bigoplus_{\tilde{B}_{i,n}} \text{Hom}_k(A_{u_1} \otimes \cdots \otimes A_{u_n}, A_v),
\]

where \(\tilde{B}_{i,n} = \{(u_1, \ldots, u_n, v)\text{ such that } v - u_1 - \cdots - u_n \equiv i \pmod{\text{ord}_\rho}\}\). The cochain Hochschild complex \((\text{Hom}_k(A^\otimes \ast, A), b^\ast)\) decomposes as the direct sum

\[
(\text{Hom}_k(A^\otimes \ast, A), b^\ast) = \bigoplus_{i=0}^{\text{ord}_\rho-1} (\text{Hom}_k(A^\otimes \ast, A)_i, b^\ast).
\]

Thus, the Hochschild cohomology \(\text{HH}^n(A)\), of \(A\) with coefficients in \(A\), decomposes as the direct sum

\[
\text{HH}^n(A) = \bigoplus_{i=0}^{\text{ord}_\rho-1} \text{HH}_i^n(A),
\]

where \(\text{HH}_i^n(A) = H^n(\text{Hom}_k(A^\otimes \ast, A)_i, b^\ast)\).

The aim of this paper is to prove the following results:

**Theorem A.** Let \(A\) be a Frobenius \(k\)-algebra, \(\varphi: A \rightarrow k\) a Frobenius homomorphism and \(\rho: A \rightarrow A\) the Nakayama automorphism of \(A\) with respect to \(\varphi\). If \(\rho\) has finite order and \(k\) has a primitive \(\text{ord}_\rho\)-th root of unity \(w\), then

\[
\text{HH}^n(A) = \text{HH}^n_0(A), \quad \text{for all } n \geq 0.
\]

Recall that \(A = A_0 \oplus \cdots \oplus A_{\text{ord}_\rho-1}\) is said to be strongly \(\mathbb{Z}/\text{ord}_\rho\mathbb{Z}\)-graded if \(A_iA_j = A_{i+j}\), for all \(i, j \in \{0, \ldots, \text{ord}_\rho-1\}\), where \(i+j\) denotes the sum of \(i\) and \(j\) in \(\mathbb{Z}/\text{ord}_\rho\mathbb{Z}\).

**Theorem B.** Let \(A\) be a Frobenius \(k\)-algebra, \(\varphi: A \rightarrow k\) a Frobenius homomorphism and \(\rho: A \rightarrow A\) the Nakayama automorphism of \(A\) with respect to \(\varphi\). If \(\rho\) has finite order, \(k\) has a primitive \(\text{ord}_\rho\)-th root of unity \(w\) and \(A = A_0 \oplus \cdots \oplus A_{\text{ord}_\rho-1}\) is strongly \(\mathbb{Z}/\text{ord}_\rho\mathbb{Z}\)-graded, then

\[
\text{HH}^n(A) = \text{HH}^n(A_0)^{\mathbb{Z}/\text{ord}_\rho\mathbb{Z}}, \quad \text{for all } n \geq 0.
\]

**Corollary.** Assume that the hypothesis of Theorem B are verified. If \(\text{HH}^2(A_0) = 0\), then \(A\) is rigid.

**Remark.** As it is well known, every finite dimensional Hopf algebra \(H\) is Frobenius, being a Frobenius homomorphism any right integral \(\varphi \in H^* \setminus \{0\}\). Moreover, by [S, Proposition 3.6], the compositional inverse of the Nakayama map \(\rho\) with respect to \(\varphi\), is given by

\[
\rho^{-1}(h) = \alpha(h_{(1)})\overline{S}(h_{(2)}),
\]

where \(\alpha \in H^*\) is the modular element of \(H^*\) and \(\overline{S}\) is the compositional inverse of \(S\) (note that the automorphism of Nakayama considered in [S] is the compositional inverse of \(S\) with respect to \(\varphi\)).
inverse of the one considered by us). Using this formula and that \( \alpha \circ S^{2} = \alpha \), it is easy to check that \( \rho(h) = \alpha(S(h_{(1)}))S^{2}(h_{(2)}) \), and more generality, that

\[
\rho^{l}(h) = \alpha^{*l}(S(h_{(1)}))S^{2l}(h_{(2)}),
\]

where \( \alpha^{*l} \) denotes the \( l \)-fold convolution product of \( \alpha \). Since \( \alpha \) has finite order respect to the convolution product and, by the Radford formula for \( S^{4} \) (see [S, Theorem 3.8]), the antipode \( S \) has finite order respect to the composition, we have that \( \rho \) has finite order. So, the above theorems apply to finite dimensional Hopf algebras.

We think that the decomposition of \( H \) associated with \( \rho \) can be useful to study the structure of finite dimensional Hopf algebras. In this paper we exploit it in a cohomological level. Recently has been considered another decomposition of \( H \), similar to this one, but distinct. Namely the one associated to \( S^{2} \) (see [R-S]).

**Example.** Let \( k \) a field and \( N \) a natural number. Assume that \( k \) has a primitive \( N \)-th root of unity \( w \). Let \( H \) be the Taft algebra of order \( N \). That is, \( H \) is the algebra generated over \( k \) by two elements \( g \) and \( x \) subject to the relations \( g^{N} = 1 \), \( x^{N} = 0 \) and \( xg = wxg \). The Taft algebra \( H \) is a Hopf algebra with comultiplication \( \Delta \), counity \( \epsilon \) and antipode \( S \) given by

\[
\Delta(g) = g \otimes g, \quad \Delta(x) = 1 \otimes x + x \otimes g,
\]
\[
\epsilon(g) = 1, \quad \epsilon(x) = 0,
\]
\[
S(g) = g^{-1}, \quad S(x) = -xg^{-1}.
\]

Using that \( t = \sum_{j=0}^{N-1} w^{j}g^{j}x^{N-1} \) is a right integral of \( H \), it is easy to see that the modular element \( \alpha \in H^{*} \) verifies \( \alpha(g) = w^{-1} \) and \( \alpha(x) = 0 \). By the remark above, the Nakayama map \( \rho \colon H \to H \) is given by \( \rho(g) = wg \) and \( \rho(x) = w^{-1}x \). Hence, \( H = H_{0} \oplus \cdots \oplus H_{N-1} \), where

\[
H_{i} = \{ a \in H : \rho(a) = w^{-i}a \}
= \langle x^{i}, x^{i+1}g, \ldots, x^{N-1}g^{N-i-1}, g^{N-i}, xg^{N-i+1}, \ldots, x^{i-1}g^{N-1} \rangle.
\]

Let \( C_{N} = \{ 1, t, \ldots, t^{N-1} \} \) be the cyclic group of order \( N \). It is easy to see that \( C_{N} \) acts on \( H_{0} \) via \( t \cdot x^{i}g^{j} = w^{i}x^{i}g^{j} \) and that \( H \) is isomorphic to the skew product of \( H_{0} \# C_{N} \). By Theorem B

\[
\text{HH}^{n}(H) = \text{HH}^{n}(H_{0})^{C_{N}} \quad \text{for all } n \geq 0,
\]

where the action of \( C_{N} \) on \( \text{HH}^{n}(H_{0}) \) is induced by the one of \( C_{N} \) on \( \text{Hom}_{k}(H_{0}^{\otimes n}, H_{0}) \), given by

\[
t \cdot \varphi(x^{i_{1}}g^{j_{1}} \otimes \cdots \otimes x^{i_{n}}g^{i_{n}}) = g^{N-1} \varphi(t \cdot x^{i_{1}}g^{j_{1}} \otimes \cdots \otimes t \cdot x^{i_{n}}g^{i_{n}})g.
\]
1. PROOF OF THEOREMS A AND B

Let $k$ be a field and $A$ a $k$-algebra. To begin, we fix some notations:

1) As in the introduction, we let $DA$ denote $\text{Hom}_k(A, k)$, endowed with the usual $A$-bimodule structure.

2) For each $k$-module $V$, we let $V^\otimes n$ denote the $n$-fold tensor product $V \otimes \cdots \otimes V$.

3) Given $x \in A \cup DA$, we write

$$\pi_A(x) = \begin{cases} x & \text{if } x \in A, \\ 0 & \text{if } x \in DA, \end{cases} \quad \text{and} \quad \pi_{DA}(x) = \begin{cases} x & \text{if } x \in DA, \\ 0 & \text{if } x \in A, \end{cases}$$

4) For $n \geq 1$, we let $B^n \subseteq (A \oplus DA)^\otimes n$ denote the vector subspace spanned by $n$-tensors $x_1 \otimes \cdots \otimes x_n$ such that exactly 1 of the $x_i$’s belong to $DA$, while the other $x_i$’s belong to $A$.

5) Given $i < j$ and $x_i, x_{i+1}, \ldots, x_j \in A \cup DA$, we write $x_{i,j} = x_i \otimes \cdots \otimes x_j$.

6) For each map $f : X \to Y$ and each element $x \in X$, we let $\langle f, x \rangle$ denote the evaluation of $f$ in $x$.

The complex $X^{*,*}(A)$. For each $k$-algebra $A$, we consider the double complex

$$\begin{array}{ccc}
\cdots & \downarrow b^{0,2} & \cdots \\
\downarrow b^{0,3} & \text{Hom}_k(A^{\otimes 3}, A) \rightarrow & \text{Hom}_k(B^4, DA) \\
\downarrow b^{0,4} & \delta^{3,3} & \downarrow \delta^{3,2} \\
\text{Hom}_k(A^{\otimes 2}, A) \rightarrow & \text{Hom}_k(B^3, DA) \\
\downarrow b^{0,1} & \delta^{3,1} & \downarrow b^{1,1} \\
\text{Hom}_k(A, A) \rightarrow & \text{Hom}_k(B^2, DA) \\
\downarrow b^{0,1} & \delta^{3,0} & \downarrow b^{1,0} \\
\text{Hom}_k(k, A) \rightarrow & \text{Hom}_k(B^1, DA), \\
\end{array}$$

where

$$\langle \langle b^{0,n+1}, f \rangle, x_{1,n+1} \rangle = x_1 \langle f, x_{2,n+1} \rangle + \sum_{i=1}^{n} (-1)^i \langle f, x_{1,i-1} \otimes x_i x_{i+1} \otimes x_{i+1,n+1} \rangle$$

$$+ (-1)^{n+1} \langle f, x_{1,n} \rangle x_{n+1},$$

$$\langle \langle b^{1,n}, g \rangle, y_{1,n+1} \rangle = \langle \pi_A, y_1 \rangle \langle g, y_{2,n+1} \rangle + \sum_{i=1}^{n} (-1)^i \langle g, y_{1,i-1} \otimes y_i y_{i+1} \otimes y_{i+1,n+1} \rangle$$

$$+ (-1)^{n+1} \langle g, y_{1,n} \rangle \langle \pi_A, y_{n+1} \rangle,$$

$$\langle \langle \delta^{1,n}, f \rangle, y_{1,n+1} \rangle = \langle \pi_{DA}, y_1 \rangle \langle f, y_{2,n+1} \rangle + (-1)^{n+1} \langle f, y_{1,n} \rangle \langle \pi_{DA}, y_{n+1} \rangle,$$

for $f \in \text{Hom}_k(A^{\otimes n}, A), g \in \text{Hom}_k(B^n, DA), x_{1,n+1} = x_1 \otimes \cdots \otimes x_{n+1} \in A^{\otimes n+1}$ and $y_{1,n+1} = y_1 \otimes \cdots \otimes y_{n+1} \in B^n$.
Proposition 1.1. Let \( X^*(A) \) be the total complex of \( X^{*,*}(A) \). It is true that

\[
H^n(X^*(A)) = \begin{cases} 
H^0(X^{0,*}(A)) & \text{if } n = 0, \\
H^n(X^{0,*}(A)) \oplus H^{n-1}(X^{0,*}(A)) & \text{if } n \geq 1.
\end{cases}
\]

Proof. Let

\[
\delta^{1,*}: (\text{Hom}_k(A^{\otimes*}, A), -b^{0,*+1}) \to (\text{Hom}_k(B^{*,1}, DA), b^{1,*+1})
\]

be the map defined by

\[
\langle \langle \delta^{1,n}, f \rangle, x_{1,n+1} \rangle = \langle \pi_{DA}, x_1 \rangle \langle f, x_{2,n+1} \rangle + (-1)^{n+1} \langle f, x_{1,n} \rangle \langle \pi_{DA}, x_{n+1} \rangle.
\]

Since \( X^*(A) \) is the mapping cone of \( \delta^{1,*} \), in order to obtain the result it suffices to check that \( \delta^{1,*} \) is null homotopic. Let \( \sigma_+: \text{Hom}_k(A^{\otimes*}, A) \to \text{Hom}_k(B^*, DA) \) be the family of maps defined by

\[
\langle \langle \sigma_n, f \rangle, x_{1,n} \rangle, a \rangle = (-1)^{j+1} \langle x_j, \langle f, x_{j+1,n} \otimes a \otimes x_{1,j-1} \rangle \rangle \quad \text{if } j = 1.
\]

We assert that \( \sigma_* \) is an homotopy from \( \delta^{1,*} \) to 0. By definition,

\[
\langle \langle b^{1,n}, \langle \sigma_n, f \rangle \rangle, x_{1,n+1} \rangle = \langle \pi_A, x_1 \rangle \langle \sigma_n, f \rangle, x_{2,n+1} \rangle + \sum_{i=1}^{n} (-1)^i \langle \sigma_n, f \rangle, x_{1,i-1} \otimes x_i x_{i+1} \otimes x_{i+2,n+1} \rangle + (-1)^{n+1} \langle \sigma_n, f \rangle, x_{1,n} \rangle \langle \pi_A, x_{n+1} \rangle.
\]

Hence, if \( x_1 \in DA \), then

\[
\langle \langle b^{1,n}, \langle \sigma_n, f \rangle \rangle, x_{1,n+1} \rangle, x_{n+2} \rangle = (-1)^{n+2} \langle x_1, x_2 \langle f, x_{3,n+2} \rangle \rangle - \sum_{i=2}^{n+1} (-1)^{n+i} \langle x_1, \langle f, x_{2,i-1} \otimes x_i x_{i+1} \otimes x_{i+2,n+2} \rangle \rangle;
\]

if \( x_j \in DA \) for \( 1 < j \leq n \), then

\[
\langle \langle b^{1,n}, \langle \sigma_n, f \rangle \rangle, x_{1,n+1} \rangle, x_0 \rangle = (-1)^{(j-1)n+j} \langle x_j, \langle f, x_{j+1,n+1} \otimes x_{0,j-2} \rangle x_{j-1} \rangle - \sum_{i=0}^{j-2} (-1)^{(j-1)n+i} \langle x_j, \langle f, x_{j+1,n+1} \otimes x_{0,i-1} \otimes x_i x_{i+1} \otimes x_{i+2,j-1} \rangle \rangle - (-1)^{jn+j} \langle x_j, x_{j+1} \langle f, x_{j+2,n+1} \otimes x_{0,j-1} \rangle \rangle
\]

\[
- \sum_{i=j+1}^{n} (-1)^{jn+i} \langle x_j, \langle f, x_{j+1,i-1} \otimes x_i x_{i+1} \otimes x_{i+2,n+1} \otimes x_{0,j-1} \rangle \rangle + (-1)^{jn+n} \langle x_j, \langle f, x_{j+1,n} \otimes x_{n+1} x_0 \otimes x_{1,j-1} \rangle \rangle;
\]

and if \( x_{n+1} \in DA \), then

\[
\langle \langle b^{1,n}, \langle \sigma_n, f \rangle \rangle, x_{1,n+1} \rangle, x_0 \rangle = \sum_{i=0}^{n-1} (-1)^{n+i+1} \langle x_{n+1}, \langle f, x_{0,i-1} \otimes x_i x_{i+1} \otimes x_{i+2,n} \rangle \rangle
\]
On the other hand, if \( x_1 \in DA \), then
\[
\langle\langle \sigma_{n+1}, -\langle b^{0,n+1}, f \rangle, x_{1,n+1}, x_{n+2} \rangle \rangle = (-1)^{n+1} x_1 (b^{0,n+1}(f)(x_{2,n+2})) \\
= (-1)^{n+1} \langle x_1, x_2 (f, x_{3,n+2}) \rangle + \langle x_1, \langle f, x_{2,n+1} \rangle x_{n+2} \rangle \\
+ \sum_{i=2}^{n} (-1)^{n+i} \langle x_1, \langle f, x_{2,i-1} \otimes x_i x_{i+1} \otimes x_{i+2,n+2} \rangle \rangle;
\]
if \( x_j \in DA \) for \( 1 < j \leq n \), then
\[
\langle\langle \sigma_{n+1}, -\langle b^{0,n+1}, f \rangle, x_{1,n+1}, x_0 \rangle \rangle \\
= (-1)^{j(n+1)} \langle x_j, \langle b^{0,n+1}, f \rangle, x_{j+1,n+1} \otimes x_{0,j-1} \rangle \rangle \\
= (-1)^{j(n+1)} \langle x_j, x_{j+1} (f, x_{j+2,n+1} \otimes x_{0,j-1}) \rangle \rangle \\
+ \sum_{i=j+1}^{n} (-1)^{j(n+1)+i-j} \langle x_j, \langle f, x_{j+1,i-1} \otimes x_i x_{i+1} \otimes x_{i+2,n+1} \otimes x_{0,j-1} \rangle \rangle \\
+ (-1)^{j(n+1)+n-j+1} \langle x_j, \langle f, x_{j+1,n} \otimes x_{n+1} x_0 \otimes x_{1,j-1} \rangle \rangle \\
+ \sum_{i=0}^{j-2} (-1)^{j(n+1)+i+n-j} \langle x_j, \langle f, x_{j+1,n+1} \otimes x_{0,i-1} \otimes x_i x_{i+1} \otimes x_{i+2,j-1} \rangle \rangle \\
+ (-1)^{j(n+1)+n+1} \langle x_j, \langle f, x_{j+1,n+1} \otimes x_{0,j-2} \rangle x_{j-1} \rangle;
\]
and if \( x_{n+1} \in DA \), then
\[
\langle\langle \sigma_{n+1}, -\langle b^{0,n+1}, f \rangle, x_{1,n+1}, x_0 \rangle \rangle = (-1)^{n+1} \langle x_{n+1}, \langle b^{0,n+1}, f \rangle, x_{0,n} \rangle \rangle \\
= (-1)^{n+1} \langle x_{n+1}, x_0 (f, x_{1,n}) \rangle + \langle x_{n+1}, \langle f, x_{0,n-1} \rangle x_{n} \rangle \\
+ \sum_{i=0}^{n-1} (-1)^{n+i} \langle x_{n+1}, \langle f, x_{0,i-1} \otimes x_i x_{i+1} \otimes x_{i+2,n} \rangle \rangle;\]
The assertion follows immediately from these equalities.  

The complex \( Y^{*,*}(A) \). From now on we fix a Frobenius algebra \( A \), a Frobenius homomorphism \( \varphi : A \to k \) of \( A \) and we let denote \( \rho \) the Nakayama automorphism of \( A \) with respect to \( \varphi \). Let \( A_\rho \) be \( A \), endowed with the \( A \)-bimodule structure given by \( a \cdot x \cdot b := \rho(a) xb \). Let \( \Theta : DA \to A_\rho \) be the \( A \)-bimodule isomorphism given by \( \Theta(\varphi x) = x \) and let
\[
A_\rho \xleftarrow{\mu} A \otimes A_\rho \xleftarrow{b_1^\prime} A^\otimes 2 \otimes A_\rho \xleftarrow{b_2^\prime} A^\otimes 3 \otimes A_\rho \xleftarrow{b_3^\prime} A^\otimes 4 \otimes A_\rho \xleftarrow{b_4^\prime} \ldots,
\]
be the bar resolution of \( A_\rho \).

**Proposition 1.2.** The following assertions hold:

1) The complex

\[
(*), \quad DA \xleftarrow{d''} A \otimes B^1 \otimes A \xleftarrow{b_1''} A \otimes B^2 \otimes A \xleftarrow{b_2''} A \otimes B^3 \otimes A \xleftarrow{b_3''} \ldots,
\]

be the bar resolution of \( A_\rho \).
where \((\mu', x_0 \otimes x_1 \otimes x_2) = x_0x_1x_2\) and

\[
\langle b''_n, x_{0,n+2} \rangle = x_0 \langle \pi_A, x_1 \rangle \otimes x_{2,n+2} + \sum_{i=1}^{n} (-1)^i x_{0,i-1} \otimes x_i x_{i+1} \otimes x_{i+2,n+2} + (-1)^{n+1} x_{0,n} \otimes \langle \pi_A, x_{n+1} \rangle x_{n+2},
\]

is a projective resolution of \(DA\).

2) There is a chain map \(\Psi_*: (A \otimes^{*+1} \otimes A, b'_*) \to (A \otimes B^{*+1} \otimes A, b''_*),\) given by

\[
\langle \Psi'_n, x_{0,n+1} \rangle = \sum_{i=0}^{n} (-1)^{i+n} x_{0,i} \otimes \varphi \otimes \langle \rho, x_{i+1} \rangle \otimes \cdots \otimes \rho, x_{n} \otimes x_{n+1}.
\]

3) \(\Theta \circ \mu' \circ \Psi'_0 = \mu.\)

Proof. Items 2) and 3) follow by a direct computation and item 1) is well known. For instance, the family of maps

\[\sigma_0: DA \to A \otimes B^1 \otimes A\] and \[\sigma_n: A \otimes B^n \otimes A \to A \otimes B^{n+1} \otimes A (n \geq 1),\]

given by

\[
\langle \sigma_0, x \rangle = 1 \otimes x \otimes 1,
\]

\[
\langle \sigma_{n+1}, x_{0,n+1} \rangle = \begin{cases} 1 \otimes x_{0,n+1} + (-1)^{n+1} \otimes x_0 x_1 \otimes x_{2,n+1} \otimes 1 & \text{if } x_1 \in DA, \\ 1 \otimes x_{0,n+1} & \text{if } x_1 \notin DA,
\end{cases}
\]

where \(x_{0,n+1} = x_0 \otimes \cdots \otimes x_{n+1} \in A \otimes B^n \otimes A,\) is a contracting homotopy of \((*)\) as a \(k\)-module complex. \(\square\)

Let \(Y^{*,*}(A)\) be the double complex

\[
\begin{array}{ccc}
\vdots & \vdots & \\
\tilde{b}^{0,4} & \tilde{b}^{1,4} & \\
\Hom_k(A \otimes^3, A) & \rightarrow & \Hom_k(A \otimes^3, A) \\
\uparrow & \uparrow & \\
\tilde{b}^{0,3} & \tilde{b}^{1,3} & \\
\Hom_k(A \otimes^2, A) & \rightarrow & \Hom_k(A \otimes^2, A) \\
\uparrow & \uparrow & \\
\tilde{b}^{0,2} & \tilde{b}^{1,2} & \\
\Hom_k(A, A) & \rightarrow & \Hom_k(A, A) \\
\uparrow & \uparrow & \\
\tilde{b}^{0,1} & \tilde{b}^{1,1} & \\
\Hom_k(k, A) & \rightarrow & \Hom_k(k, A),
\end{array}
\]

with boundary maps

\[
\langle \langle \tilde{b}^{n,n}, f \rangle, x_{1,n} \rangle = x_1 f(x_{2,n}) + \sum_{i=1}^{n-1} (-1)^i \langle f, x_{1,i-1} \otimes x_i x_{i+1} \otimes x_{i+2,n} \rangle + (-1)^{n} \langle f, x_{1,n-1} \rangle x_n,
\]

\[
\langle \langle \tilde{d}^{1,n-1}, f \rangle, x_{1,n-1} \rangle = (-1)^n \langle f, x_{1,n-1} \rangle + \sum_{i=1}^{n-1} (-1)^{n-1} \langle f, x_{1,i-1} \otimes x_i x_{i+1} \otimes x_{i+2,n} \rangle.
\]
where \( u = 0, 1, f \in \text{Hom}_k(A^{n-1}, A) \) and \( x_{1,n} = x_1 \otimes \cdots \otimes x_n \in A^{\otimes n} \).

**Proposition 1.3.** The double complexes \( X^{*,*}(A) \) and \( Y^{*,*}(A) \) are quasiisomorphic.

**Proof.** It is immediate that \( X^{1,*}(A) \simeq \text{Hom}_{A^e}((A \otimes B^{*+1} \otimes A, b'_*^0), DA) \). Moreover, by Proposition 1.2, the map \( \Psi^* := \text{Hom}_{A^e}(\Psi'_*, DA) \) is a quasiisomorphism from \( \text{Hom}_{A^e}((A \otimes B^{*+1} \otimes A, b'_*^0), DA) \) to \( \text{Hom}_{A^e}((A^{\otimes *+1} \otimes A, b'_*), DA) \). On the other hand the family of bijective maps

\[
\Upsilon^n : Y^{1,n}(A) \to \text{Hom}_{A^e}(A^{\otimes n+1} \otimes A_\rho, DA) \quad (n \geq 0),
\]

defined by \( \langle \langle \Upsilon^n, f \rangle, x_{0,n+1} \rangle = x_0 f(x_{1,n}) \varphi x_{n+1} \), is an isomorphism of complexes from \( Y^{1,*}(A) \) to \( \text{Hom}_{A^e}((A^{\otimes *+1} \otimes A, b'_*), DA) \). In fact, we have

\[
\langle \langle \Upsilon^{n+1}, \langle \langle \delta^{1,n+1}, f \rangle \rangle, x_{0,n+2} \rangle = x_0 \langle \langle \delta^{1,n+1}, f \rangle, x_{1,n+1} \rangle \varphi x_{n+2} = x_0 x_1 \langle f, x_{2,n+1} \rangle \varphi x_{n+2} + \sum_{i=1}^{n} (-1)^i x_0 \langle f, x_{1,i-1} \otimes x_i x_{i+1} \otimes x_{i+2,n+1} \rangle \varphi x_{n+2} + (-1)^{n+1} x_0 \langle f, x_{1,n} \rangle \varphi x_{n+2} = x_0 x_1 \langle f, x_{2,n+1} \rangle \varphi x_{n+2} + \sum_{i=1}^{n} (-1)^i x_0 \langle f, x_{1,i-1} \otimes x_i x_{i+1} \otimes x_{i+2,n+1} \rangle \varphi x_{n+2} + (-1)^{n+1} x_0 \langle f, x_{1,n} \rangle \varphi \langle \rho, x_{n+1} \rangle x_{n+2} = \sum_{i=0}^{n} (-1)^i \langle \langle \Upsilon^n, f \rangle, x_{0,i-1} \otimes x_i x_{i+1} \otimes x_{i+2,n+2} \rangle + (-1)^{n+1} \langle \langle \Upsilon^n, f \rangle, x_{0,n} \otimes \langle \rho, x_{n+1} \rangle x_{n+2} \rangle = \langle \langle \Upsilon^n, f \rangle, \langle b'_n, x_{0,n+2} \rangle \rangle.
\]

Hence, to finish the proof it suffices to check that \( \Psi^* \circ \delta^{1,*} = \Psi^* \circ \delta^{1,*} \). But,

\[
\langle \langle \Psi^n, \langle \delta^{1,n}, f \rangle \rangle, x_{0,n+1} \rangle
= \sum_{i=0}^{n} (-1)^{i+n} x_0 \langle \langle \delta^{1,n}, f \rangle, x_{1,i} \otimes \varphi \otimes \langle \rho, x_{i+1} \rangle \otimes \cdots \otimes \langle \rho, x_{n} \rangle \rangle x_{n+1}
= (-1)^n x_0 \varphi \langle f, \langle \rho, x_1 \rangle \otimes \cdots \otimes \langle \rho, x_n \rangle \rangle x_{n+1} + (-1)^{n+1} x_0 \langle f, x_{1,n} \rangle \varphi x_{n+1}
= (-1)^n x_0 \langle \rho^{-1}, f, \langle \rho, x_1 \rangle \otimes \cdots \otimes \langle \rho, x_n \rangle \rangle \varphi x_{n+1} + (-1)^{n+1} x_0 \langle f, x_{1,n} \rangle \varphi x_{n+1}
= x_0 \langle \langle \delta^{1,n}, f \rangle, x_{1,n} \rangle \varphi x_{n+1}
= \langle \langle \langle \Upsilon^n, \delta^{1,n}, f \rangle \rangle, x_{0,n+1} \rangle,
\]

as desired. \( \square \)
Proposition 1.4. Let $Y^*(A)$ denote the total complex of $Y^{*,*}(A)$. If the Nakayama automorphism $\rho$ has finite order and $k$ has a primitive $\text{ord}_{\rho}$-th root of unity $w$, then

$$H^n(Y^*(A)) = \begin{cases} HH^0_0(A) & \text{if } n = 0, \\ HH^0_n(A) \oplus HH^{n-1}_0(A) & \text{if } n \geq 1. \end{cases}$$

Proof. For each $0 \leq i < \text{ord}_{\rho}$, let $Y^i_*(A)$ be the subcomplex of $Y^{*,*}(A)$ defined by

$$Y^i_{u,n} = \bigoplus_{B_{i,n}} \text{Hom}(A_{u_1} \otimes \cdots \otimes A_{u_n}, A_v),$$

where $B_{i,n} = \{(u_1, \ldots, u_n, v) \mid v - u_1 - \cdots - u_n \equiv i \pmod{\text{ord}_{\rho}}\}$. It is clear that $Y^{*,*}(A) = \bigoplus_{i=0}^{\text{ord}_{\rho}} Y^i_*(A)$. Let $f \in Y^0_{0,n}(A)$. A direct computation shows that

$$\langle (\delta_{1,n}, f), x_{1,n} \rangle = (-1)^{n+1}(1 - w^{-i})\langle f, x_{1,n} \rangle.$$

Hence the horizontal boundary maps of $Y^i_*(A)$ are isomorphisms if $i \neq 0$, and they are zero maps if $i = 0$. So,

$$H^n(Y^i_*(A)) = \begin{cases} 0 & \text{if } i \neq 0, \\ H^0(Y^0_0(A)) & \text{if } i = 0 \text{ and } n = 0, \\ H^n(Y^0_0(A)) \oplus H^{n-1}(Y^1_1(A)) & \text{if } i = 0 \text{ and } n > 0, \end{cases}$$

where $Y^i_*(A)$ is the total complex of $Y^i_*(A)$. The result follows easily from this fact, since $Y^0_0(A) = Y^1_0(A) \simeq (\text{Hom}_k(A^{\otimes n}, A), b^*)$. □

Proof of Theorem A. By Proposition 1.3,

$$H^n(Y^*(A)) = H^n(X^*(A)) \quad \text{and} \quad H^n(Y^u_*, A) = H^n(X^{u,*}(A)), \text{ for } u = 0, 1.$$

Hence, by Propositions 1.1 and 1.4,

$$HH^0_0(A) = H^0(Y^*(A)) = H^0(X^*(A)) = H^0(X^{0,*}(A)) = H^0(Y^{0,*}(A)) = HH^0_0(A)$$

and

$$HH^0_0(A) \oplus HH^{n-1}_0(A) = H^n(Y^*(A)) = H^n(X^*(A)) = H^n(X^{0,*}(A)) = H^n(Y^{0,*}(A)) = HH^0_0(A) \oplus HH^{n-1}_0(A),$$

for all $n \geq 1$. From this it follows easily that $HH^n(A) = HH^n_0(A)$, for all $n \geq 0$, as desired. □

Proof of Theorem B. By [St] or the cohomological version of [L], $\mathbb{Z}/\text{ord}_{\rho}\mathbb{Z}$ acts on $H^*(A_0, A)$ and there is a converging spectral sequence

$$E_2^{p,q} = H^p(\mathbb{Z}/\text{ord}_{\rho}\mathbb{Z}, H^q(A_0, A)) \Rightarrow HH_{p+q}(A).$$
Since $k$ has a primitive ord$_{\rho}$-th root of unity, ord$_{\rho}$ is invertible in $k$. Hence, the above spectral sequence gives isomorphisms

$$\text{HH}^n(A) = H^n(A_0, A)^{\mathbb{Z}/\text{ord}_\rho \mathbb{Z}} \quad (n \geq 0).$$

These maps are induced by the canonical inclusion of $A_0$ in $A$, and the action of $i \in \mathbb{Z}/\text{ord}_\rho \mathbb{Z}$ on $H^n(A_0, A)$ is induced by the map of complexes

$$\theta_i^*: (\text{Hom}_k(A_0^{\otimes *}, A), b^*) \to (\text{Hom}_k(A_0^{\otimes *}, A), b^*),$$

defined by

$$\theta_i^n(\varphi)(a_1 \otimes \cdots \otimes a_n) = \sum_{j_1, \ldots, j_{n+1} \in J_i} s'_{i,j_1} \varphi(s_{i,j_1}a_1s'_{i,j_2} \otimes s_{i,j_2}a_2s'_{i,j_3} \otimes \cdots \otimes s_{i,j_n}a_ns'_{i,j_{n+1}})s_{i,j_{n+1}},$$

where $(s_{i,j})_{j \in J_i}$ and $(s'_{i,j})_{j \in J_i}$ are families of elements of $A_i$ and $A_{n-i}$ respectively, that satisfy $\sum_{j \in J_i} s'_{i,j} s_{i,j} = 1$. From this it follows easily that we have isomorphisms

$$\text{HH}_i^n(A) = H^n(A_0, A_i)^{\mathbb{Z}/\text{ord}_\rho \mathbb{Z}} \quad (n \geq 0, 0 \leq i < \text{ord}_\rho).$$

By combining this result with Theorem A, we obtain the desired result. □

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