Infinite-order Differential Operators Acting on Entire Hyperholomorphic Functions

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Abstract

Infinite-order differential operators appear in different fields of mathematics and physics and in the past decade they turned out to be of fundamental importance in the study of the evolution of superoscillations as initial datum for Schrödinger equation. Inspired by the operators arising in quantum mechanics, in this paper, we investigate the continuity of a class of infinite-order differential operators acting on spaces of entire hyperholomorphic functions. We will consider two classes of hyperholomorphic functions, both being natural extensions of holomorphic functions of one complex variable. We show that, even though these two notions of hyperholomorphic functions are quite different from each other, in both cases, entire hyperholomorphic functions with exponential bounds play a crucial role in the continuity of infinite-order differential operators acting on these two classes of functions. This is particularly remarkable since the exponential function is not in the kernel of the Dirac operator, but it plays an important role in the theory of entire monogenic functions with growth conditions.

Keywords

Infinite-order differential operators · Slice hyperholomorphic functions · Dirac operator · Entire functions with growth conditions · Spaces of entire functions

Mathematics Subject Classification

32A15 · 32A10 · 47B38

1 Introduction

Infinite-order differential operators are of fundamental importance in the study of the evolution of superoscillations as initial datum for Schrödinger equation.
the evolution of superoscillatory functions under the Schrödinger equation is highly nontrivial, and a natural functional setting is the space of entire functions with growth conditions, for more details see, e.g., the monograph [8] and [22]. In fact, the Cauchy problem for the Schrödinger equation with superoscillatory initial datum leads to infinite-order differential operators of the type

$$U(t, x; D_x) = \sum_{m=1}^{\infty} u_m(t, x) \partial_x^m,$$

where the coefficients $u_m(t, x)$ depend on the Green function of the Schrödinger equation with the potential $V$, and $t$ and $x$ are the time and the space variable, respectively. According to the structure of the Green function, the coefficients $u_m(t, x)$ satisfy given growth conditions. For some potentials $V$, we are forced to consider infinite-order differential operators $P(t, x; \partial_\xi)$ depending on an auxiliary complex variable $\xi$

$$P(t, x; \partial_\xi) = \sum_{m=1}^{\infty} u_m(t, x) \partial_\xi^m,$$

with coefficients $u_m(t, x)$ that depend on $V$. The continuity properties of the operators $U(t, x; \partial_x)$ or $P(t, x; \partial_\xi)$ acting on the spaces of entire functions with exponential bounds are the heart of the study of the evolution of superoscillatory initial data in quantum mechanics. For $p \geq 1$, the natural spaces on which such operators $U(t, x; \partial_x)$ and $P(t, x; \partial_\xi)$ act are the spaces of entire functions with either order lower than $p$ or order equal to $p$ and finite type. In other words, they consist of entire functions $f$ for which there exist constants $B, C > 0$ such that $|f(z)| \leq Ce^{B|z|^p}$.

In this paper, we investigate the continuity of a class of infinite-order differential operators acting on spaces of entire hyperholomorphic functions. We consider two classes of such functions, both extending holomorphic functions of one complex variable to the case of functions of a paravector variable: the slice monogenic and the monogenic functions. These two classes are related by the celebrated Fueter–Sce–Qian mapping theorem, rephrased in a modern way.

The Fueter–Sce–Qian result generalizes a theorem of Fueter in a nontrivial and original way from the quaternionic setting to the Clifford algebra setting. Here, we recall some basic facts, and we refer to the book [34] for an overview of this profound theorem. Let $f(z) = f_0(u, v) + i f_1(u, v)$ be a holomorphic function of the variable $z = u + iv$, defined in a domain (open and connected) $D \subseteq \mathbb{C} \cong \mathbb{R}^2$ such that $f_0(u, v) + i f_1(u, v)$ satisfies the conditions:

$$f_0(u, -v) = f_0(u, v), \quad f_1(u, -v) = -f_1(u, v).$$  \hspace{1cm} (1)

Let

$$\Omega_D = \{x = x_0 + \bar{x} \mid (x_0, |\bar{x}|) \in D\}$$
be the open set induced by $D$ in $\mathbb{R}^{n+1}$.

(Step I) The map $T_1 : \tilde{f} \mapsto f$ defined by:

$$f(x) = T_1(\tilde{f}) := f_0(x_0, |x|) + \frac{x}{|x|} f_1(x_0, |x|)$$

is such that the Clifford-valued function $f(x)$ is slice hyperholomorphic.

(Step II) The map $T_2 : f \mapsto \Delta_{n+1}^{\frac{n-1}{2}} f$, where $\Delta_{n+1}$ denotes the Laplacian in $n+1$ dimensions, defines a function $\tilde{f}(x) := T_2(f(x))$ such that $D\tilde{f}(x) = 0$ on $\Omega_D$, $D$ being the generalized Cauchy–Riemann operator. The function $\tilde{f}$ is said to be monogenic. If we consider $f_0(u, v)$ and $f_1(u, v)$ Clifford algebra-valued functions, we obtain the class of the so-called slice monogenic functions, and applying the map $T_2 := \Delta_{n+1}^{\frac{n-1}{2}}$, in Step II, we still get a monogenic function. The case when $\frac{n-1}{2}$ is fractional has been studied by Qian, see the notes of the book [34].

We show that, even though the two notions of hyperholomorphic functions are quite different from each other and the exponential function is not in the kernel of the Dirac operator, hyperholomorphic functions with exponential bounds play a crucial role in the continuity of a class of infinite-order differential operators in the hypercomplex settings. The complex version of these results was studied in the papers [16,18].

The two function theories have several differences and also some analogies under the action of infinite-order differential operators.

(A) The pointwise product of two hyperholomorphic functions, in general, is not hyperholomorphic, so we need to define the product in a way that preserves the hyperholomorphicity. Given two entire left-slice monogenic functions $f$ and $g$, then their star product (or slice hyperholomorphic product) is defined by

$$(f \star L g)(x) = \sum_{\ell=0}^{+\infty} x^{\ell} \sum_{k=0}^{\ell} a_k b_{\ell-k},$$

where $f(x) = \sum_{k=0}^{+\infty} x^k a_k$ and $g(x) = \sum_{k=0}^{+\infty} x^k b_k$.

When we deal with monogenic functions the Fueter’s polynomials $V_k(x)$ defined by

$$V_k(x) := \frac{k!}{|k|!} \sum_{\sigma \in \text{perm}(k)} z_{j_\sigma(1)} z_{j_\sigma(2)} \cdots z_{j_\sigma(|k|)},$$

where $k$ is a multi-index, play the same role as the monomials $x^k$, for $k \in \mathbb{N}_0$, of the paravector variable $x$ for slice monogenic functions (see Definition 4.2 for the notation needed here). The C-K product of two left entire monogenic $f$ and $g$ is defined by

$$f \odot_L g := \sum_{|k|=0}^{+\infty} \sum_{|j|=0}^{+\infty} V_{k+j}(x) f_k g_j,$$
where \( f(x) = \sum_{|k|=0}^{\infty} V_k(x) f_k \) and \( g(x) = \sum_{|k|=0}^{\infty} V_k(x) g_k \) are given in terms of \( V_k(x) \).

(B) The contour integral, in the Cauchy formula of slice monogenic functions and their derivatives, is computed on the complex plane \( \mathbb{C}_j \) in \( \mathbb{R}^{n+1} \) (for \( j \in S \)). Such contour is the boundary of \( U \cap \mathbb{C}_j \), where the regular domain \( U \) is contained in \( \mathbb{R}^{n+1} \) and is contained in a set where \( f \) is slice monogenic. For the monogenic case, the integral, in the Cauchy formula for monogenic functions and their derivatives, is computed on the boundary of \( U \subset \mathbb{R}^{n+1} \) where \( \overline{U} \) is contained in the set of monogenicity of \( f \).

(C) For slice monogenic functions, there exists two different Cauchy kernels according to left- and right-slice hyperholomorphicity, while left and right monogenic functions have same the Cauchy kernel.

Finally, we point out that it is possible to define slice hyperholomorphic functions, as functions in the kernel of the first-order linear differential operator, introduced in [28], and defined by

\[
G f = \left( |x|^2 \frac{\partial}{\partial x_0} + x \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} \right) f = 0,
\]

where \( x = x_1 e_1 + \cdots + x_n e_n \). The operator \( G \) is a linear nonconstant coefficients differential operator while the generalized Cauchy–Riemann operator \( \mathcal{D} \) in which kernel gives the monogenic functions is linear, with constant coefficients.

We now provide an overview of some of the main results.

(I) Consider the formal infinite-order differential operator

\[
U_L(x, \partial_{x_0}) f(x) := \sum_{m=0}^{\infty} u_m(x) \ast_L \partial_{x_0}^m f(x),
\]

defined on entire left-slice monogenic functions \( f \), where \( \ast_L \) denotes the hyperholomorphic product. Suppose that \( (u_m)_{m \in \mathbb{N}_0} : \mathbb{R}^{n+1} \to \mathbb{R}^n \) is a sequence of entire left-slice monogenic functions. Assume, see [18], that \( (u_m)_{m \in \mathbb{N}_0} \) satisfy the condition that for every \( \varepsilon > 0 \), there exist \( B_\varepsilon > 0, C_\varepsilon > 0 \) for which

\[
|u_m(x)| \leq C_\varepsilon \frac{e^m}{(m!)^{1/q}} \exp(B_\varepsilon |x|^p), \quad \text{for all } m \in \mathbb{N}_0,
\]

where \( 1/p + 1/q = 1 \) and \( 1/q = 0 \) when \( p = 1 \). Then in Theorem 3.2, we show that for \( p \geq 1 \) the operator \( U_L(x, \partial_{x_0}) \) acts continuously on the space of entire left-slice monogenic functions with growth condition of the form \( |f(x)| \leq C e^{B|x|^p} \). In the same theorem, we also considered right-slice monogenic functions.

(II) For monogenic functions we let \( p \geq 1 \) and set \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Let \( (u_m)_{m \in \mathbb{N}_0^n} : \mathbb{R}^{n+1} \to \mathbb{R}^n \) be left entire monogenic functions such that for every \( \varepsilon > 0 \) there exist \( B_\varepsilon > 0, C_\varepsilon > 0 \) for which

\[
|u_m(x)| \leq C_\varepsilon \frac{e^{|m|}}{(|m|!)^{1/q}} \exp(B_\varepsilon |x|^p), \quad \text{for all } m \in (\mathbb{N}_0)^n,
\]
where \(1/p + 1/q = 1\) and \(1/q = 0\) when \(p = 1\), and \(m\) is a multi-index. We define the formal infinite-order differential operator

\[ U_L(x, \partial_x) f(x) := \sum_{|m|=0}^{\infty} u_m(x) \odot_L \partial^m_x f(x), \]

for left entire monogenic functions \(f\) where \(\partial^m_x := \partial_{x_1}^{m_1} \ldots \partial_{x_n}^{m_n}\) and \(\odot_L\) denotes the C-K product. Then for \(p \geq 1\), in Theorem 5.3, we prove that the operator \(U_L(x, \partial_x)\) acts continuously on the space of left monogenic functions with the condition \(|f(x)| \leq Ce^{B|x|^p}\).

Even though the two classes of hyperholomorphic functions have very different Taylor series expansions, they have strong similarities with respect to the action of infinite-order differential operators when we assume similar growth conditions on the coefficients of the operators. The results are even more surprising because of the exponential bounds, \(|f(x)| \leq Ce^{B|x|^p}\) is used for both classes of functions even though the function \(f(x) = e^{Bx}\), for \(B \in \mathbb{R}\), is slice monogenic but it is not monogenic.

The two hyperholomorphic function theories have several applications both in Mathematics and in Physics. Precisely, associated with slice hyperholomorphic functions [27,29,40] it is possible to define the spectral theory on the \(S\)-spectrum [27,30] that has applications in quaternionic quantum mechanics [1,39], in fractional diffusion processes [25,26,31,35], characteristic operator functions [14], the spectral theorem for quaternionic normal operators [12], the perturbations theory [24], Schur analysis [13], and other fields are under investigation. The monogenic function theory is associated with harmonic analysis in higher dimension. Moreover, there exists a functional calculus based on the Cauchy formula from which the monogenic spectrum was defined, see [43], the function theory has applications to boundary value problems, see [41].

The notion of superoscillatory functions first appears in a series of works of Aharonov, Berry, and co-authors, see [2,3,20]. In this context, there are good physical reasons for such a behavior, but the discoverers pointed out the apparently paradoxical nature of such functions, thus, opening the way for a more thorough mathematical analysis of the phenomenon. In a series of recent papers, there are some systematic study of superoscillations from the mathematical point of view, see [4–7,9–11,17,19,33] and see also [21,32,44].

The theory of superoscillations appears in various questions, namely extension of positive definite functions, interpolation of polynomials, and also of \(R\)-functions and have applications to signal theory and prediction theory of stationary stochastic processes. Thing in some of this area is still under investigation as one can see in the paper [15].

This paper is addressed to a double audience: to researchers working in complex and hypercomplex analysis and to experts working in the area of infinite-order differential operators. In Sects. 2 and 4, we collect the preliminary results on function spaces of entire slice monogenic functions and of entire monogenic functions with growth conditions, respectively. These results are of crucial importance in order to study the continuity properties of a class of infinite-order differential operators acting on...
entire slice monogenic and monogenic functions that are treated in Sects. 3 and 5, respectively.

2 Function Spaces of Entire Slice Monogenic Functions

In this section, we recall some results on slice monogenic functions (see Chapter 2 in [27]), and we prove some important properties of entire slice monogenic functions that appear here for the first time. We recall that $\mathbb{R}_n$ is the real Clifford algebra over $n$ imaginary units $e_1, \ldots, e_n$. The element $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ will be identified with the paravector $x = x_0 + x = x_0 + \sum_{\ell=1}^n x_\ell e_\ell$, and the real part $x_0$ of $x$ will also be denoted by $\text{Re}(x)$. An element in $\mathbb{R}_n$, called a Clifford number, can be written as follows:

$$a = a_0 + a_1 e_1 + \cdots + a_n e_n + a_{12} e_1 e_2 + \cdots + a_{123} e_1 e_2 e_3 + \cdots + a_{12\ldots n} e_1 e_2 \ldots e_n.$$ 

Denote by $A$ an element in the power set $P(1, \ldots, n)$. If $A = \{i_1 \ldots i_r\}$, then the element $e_{i_1} \cdots e_{i_r}$ can be written as $e_{i_1 \ldots i_r}$ or, in short, $e_A$. Thus, in a more compact form, we can write a Clifford number as follows:

$$a = \sum_A a_A e_A.$$

Possibly using the defining relations $e_i^2 = -1, e_i e_j + e_j e_i = 0, i, j \in \{1, \ldots, n\}, i \neq j$, we will order the indices in $A$ as $i_1 < \ldots < i_r$. When $A = \emptyset$, we set $e_\emptyset = 1$. The Euclidean norm of an element $y \in \mathbb{R}_n$ is given by $|y|^2 = \sum_A |y_A|^2$, in particular the norm of the paravector $x \in \mathbb{R}^{n+1}$ is $|x|^2 = x_0^2 + x_1^2 + \cdots + x_n^2$. The conjugate of $x$ is given by $\bar{x} = x_0 - x = x_0 - \sum_{\ell=1}^n x_\ell e_\ell$. Recall that $\mathbb{S}$ is the sphere

$$\mathbb{S} = \{x = e_1 x_1 + \cdots + e_n x_n \mid x_1^2 + \cdots + x_n^2 = 1\};$$

so for $j \in \mathbb{S}$, we have $j^2 = -1$. Given an element $x = x_0 + x \in \mathbb{R}^{n+1}$, let us define $j_x = x/|x|$ if $x \neq 0$, and given an element $x \in \mathbb{R}^{n+1}$, the set

$$[x] := \{y \in \mathbb{R}^{n+1} : y = x_0 + j_x y, \ j \in \mathbb{S}\}$$

is an $(n-1)$-dimensional sphere in $\mathbb{R}^{n+1}$. The vector space $\mathbb{R} + j\mathbb{R}$ passing through 1 and $j \in \mathbb{S}$ will be denoted by $\mathbb{C}_j$, and an element belonging to $\mathbb{C}_j$ will be indicated by $u + j v$, for $u, v \in \mathbb{R}$. With an abuse of notation, we will write $x \in \mathbb{R}^{n+1}$. Thus, if $U \subseteq \mathbb{R}^{n+1}$ is an open set, a function $f : U \subseteq \mathbb{R}^{n+1} \rightarrow \mathbb{R}_n$ can be interpreted as a function of the paravector $x$. We say that $U \subseteq \mathbb{R}^{n+1}$ is axially symmetric if $[x] \subseteq U$ for any $x \in U$.

**Definition 2.1** (Slice hyperholomorphic functions with values in $\mathbb{R}_n$ (or slice monogenic functions)) Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric open set and set $\mathcal{U} =$
\(\{(u, v) \in \mathbb{R}^2 : u + Sv \subset U\}\). A function \(f : U \to \mathbb{R}_n\) is called a left-slice function, if it is of the form:

\[
f(q) = f_0(u, v) + jf_1(u, v) \quad \text{for } q = u + jv \in U
\]

where the two functions \(f_0, f_1 : U \to \mathbb{R}_n\) satisfy the compatibility conditions:

\[
f_0(u, -v) = f_0(u, v), \quad f_1(u, -v) = -f_1(u, v).
\]

If in addition, \(f_0\) and \(f_1\) satisfy the Cauchy–Riemann-equations

\[
\frac{\partial}{\partial u} f_0(u, v) - \frac{\partial}{\partial v} f_1(u, v) = 0
\]

\[
\frac{\partial}{\partial v} f_0(u, v) + \frac{\partial}{\partial u} f_1(u, v) = 0,
\]

then \(f\) is called left-slice hyperholomorphic (or left-slice monogenic). A function \(f : U \to \mathbb{R}_n\) is called a right-slice function if it is of the form:

\[
f(q) = f_0(u, v) + f_1(u, v)j \quad \text{for } q = u + jv \in U
\]

with two functions \(f_0, f_1 : U \to \mathbb{R}_n\) that satisfy (5). If in addition, \(f_0\) and \(f_1\) satisfy the Cauchy–Riemann equation, then \(f\) is called right-slice hyperholomorphic (or right slice monogenic).

If \(f\) is a left (or right) slice function such that \(f_0\) and \(f_1\) are real valued, then \(f\) is called intrinsic. We denote the sets of left- and right-slice hyperholomorphic functions on \(U\) by \(\mathcal{SM}_L(U)\) and \(\mathcal{SM}_R(U)\), respectively. When we do not distinguish between left of right, we indicate the space \(\mathcal{SM}(U)\).

**Definition 2.2** Let \(f : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n\) and let \(x = u + jv \in U\). If \(x\) is not real, then we say that \(f\) admits left-slice derivative in \(x\) if (denoting by \(f_j\) the restriction of \(f\) to the complex plane \(\mathbb{C}_j\))

\[
\partial_S f(x) := \lim_{p \to x, p \in \mathbb{C}_j} (p - x)^{-1}(f_j(p) - f_j(x))
\]

exists and is finite. If \(x\) is real, then we say that \(f\) admits left-slice derivative in \(x\) if (8) exists for any \(j \in \mathbb{S}\). Similarly, we say that \(f\) admits right-slice derivative in a nonreal point \(x = u + jv \in U\) if

\[
\partial_S f(x) := \lim_{p \to x, p \in \mathbb{C}_j} (f_j(p) - f_j(x))(p - x)^{-1}
\]

exists and is finite, and we say that \(f\) admits right-slice derivative in a real point \(x \in U\) if (9) exists and is finite, for any \(j \in \mathbb{S}\).
Remark 2.3 Observe that $\partial_{\mathcal{S}} f(x)$ is uniquely defined and independent of the choice of $j \in \mathbb{S}$ even if $x$ is real. If $f$ admits slice derivative, then $f_j$ is $\mathbb{C}_j$-complex left, resp. right, differentiable, and we find
\[
\partial_{\mathcal{S}} f(x) = f'_j(x) = \frac{\partial}{\partial u} f_j(x) = \frac{\partial}{\partial u} f(x), \quad x = u + jv. \tag{10}
\]

Theorem 2.4 [27, Proposition 2.3.1] Let $a \in \mathbb{R}$, let $r > 0$ and let $B_r(a) = \{x \in \mathbb{R}^{n+1} : |x - a| < r\}$. If $f \in \mathcal{S}M_L(B_r(a))$, then
\[
f(x) = \sum_{k=0}^{+\infty} (x - a)^k \frac{1}{k!} \partial_{\mathcal{S}}^k f(a) \quad \forall x = u + jv \in B_r(a). \tag{11}
\]

If on the other hand, $f \in \mathcal{S}M_R(B_r(a))$, then
\[
f(x) = \sum_{k=0}^{+\infty} \frac{1}{k!} \left( \partial_{\mathcal{S}}^k f(a) \right) (x - a)^k \quad \forall x = u + jv \in B_r(a).
\]

We now recall the natural product that preserves slice monogenicity of functions admitting power series expansion as shown by Theorem 2.4.

Definition 2.5 Let $f(x) = \sum_{k=0}^{+\infty} x^k a_k$ and $g(x) = \sum_{k=0}^{+\infty} x^k b_k$ be two left-slice monogenic power series, the left-star product, denoted by $\star_L$, is defined by
\[
(f \star_L g)(x) = \sum_{\ell=0}^{+\infty} x^\ell \left( \sum_{k=0}^{\ell} a_k b_{\ell-k} \right). \tag{12}
\]

Similarly, for right-slice monogenic power series $f(x) = \sum_{k=0}^{+\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{+\infty} b_k x^k$ the right-star product, denoted by $\star_R$, is defined by
\[
(f \star_R g)(x) = \sum_{\ell=0}^{+\infty} \left( \sum_{k=0}^{\ell} a_k b_{\ell-k} \right) x^\ell. \tag{13}
\]

The Cauchy formula of slice monogenic functions has two different Cauchy kernels according to left- or right-slice monogenicity. Let $x, s \in \mathbb{R}^{n+1}$, with $x \notin [s]$, be paravectors then the slice monogenic Cauchy kernels are defined by

\[
S_{\mathcal{L}}^{-1}(s, x) := -(x^2 - 2\text{Re}(s)x + |s|^2)^{-1}(x - \bar{s}),
\]

and
\[
S_{\mathcal{R}}^{-1}(s, x) := -(x - \bar{s})(x^2 - 2\text{Re}(s)x + |s|^2)^{-1}.
\]

The two results below can be found in [27], Sect. 2.8.
**Theorem 2.6** (The Cauchy formulas for slice monogenic functions) Let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric domain. Suppose that $\partial (U \cap \mathbb{C}_j)$ is a finite union of continuously differentiable Jordan curves for every $j \in S$ and set $ds_j = -ds_j$ for $j \in S$. Let $f$ be a slice monogenic function on an open set that contains $\overline{U}$ and set $x = x_0 + x$, $s = s_0 + s$. Then

$$f(x) = \frac{1}{2\pi} \int_{\partial (U \cap \mathbb{C}_j)} S_{L}^{-1}(s, x) \, ds_j f(s), \quad \text{for any } x \in U. \quad (14)$$

If $f$ is a right-slice monogenic function on a set that contains $\overline{U}$, then

$$f(x) = \frac{1}{2\pi} \int_{\partial (U \cap \mathbb{C}_j)} f(s) \, ds_j S_{R}^{-1}(s, x), \quad \text{for any } x \in U. \quad (15)$$

Moreover, the integrals depend neither on $U$ nor on the imaginary unit $j \in S$.

**Theorem 2.7** (Derivatives of slice monogenic functions) Let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric domain. Suppose that $\partial (U \cap \mathbb{C}_j)$ is a finite union of continuously differentiable Jordan curves for every $j \in S$ and set $ds_j = -ds_j$ for $j \in S$. Let $f$ be a left-slice monogenic function on an open set that contains $\overline{U}$ and set $x = x_0 + x$, $s = s_0 + s$. Then the slice derivatives $\partial_S^k f(x)$ are given by

$$\partial_S^k f(x) = \frac{k!}{2\pi} \int_{\partial (U \cap \mathbb{C}_j)} (x^2 - 2s_0 x + |s|^2)^{-k-1}(x - \overline{s})^{*(k+1)} \, ds_j f(s), \quad (16)$$

where

$$(x - \overline{s})^{*k} = \sum_{m=0}^{k} \frac{k!}{(k-m)!m!} x^{k-m} \overline{s}^m. \quad (17)$$

Moreover, the integral depends neither on $U$ nor on the imaginary unit $j \in S$.

A similar formula holds also for right-slice monogenic functions.

After the basic facts on slice monogenic functions, we can introduce some function spaces of entire slice monogenic functions in the spirit of their quaternionic counterpart studied in the book [29]. Let $f$ be a nonconstant entire monogenic function. We define

$$M_{f|\mathbb{C}_j}(r) = \max_{|z|=r, z \in \mathbb{C}_j} |f(z)|, \quad \text{for } r \geq 0$$

and

$$M_f(r) = \max_{|x|=r} |f(x)|, \quad \text{for } r \geq 0.$$
Definition 2.8 Let \( f \) be an entire slice monogenic function. Then we say that \( f \) is of finite order if there exists \( \kappa > 0 \) such that

\[
M_f(r) < e^{r^\kappa}
\]

for sufficiently large \( r \). The greatest lower bound \( \rho \) of such numbers \( \kappa \) is called order of \( f \). Equivalently, we can define the order as

\[
\rho = \limsup_{r \to \infty} \frac{\ln \ln M_f(r)}{\ln r}.
\]

Definition 2.9 Let \( f \) be an entire slice monogenic function of order \( \rho \) and let \( A > 0 \) be such that for sufficiently large values of \( r \), we have

\[
M_f(r) < e^{Ar^\rho}.
\]

We say that \( f \) of order \( \rho \) is of type \( \sigma \) if \( \sigma \) is the greatest lower bound of such numbers and we have

\[
\sigma = \limsup_{r \to \infty} \frac{\ln M_f(r)}{r^\rho}.
\]

Moreover

- When \( \sigma = 0 \), we say that \( f \) is of minimal type.
- When \( \sigma = \infty \), we say that \( f \) is of maximal type.
- When \( \sigma \in (0, \infty) \), we say that \( f \) is of normal type.

The constant functions are said to be of minimal type of order zero.

Definition 2.10 Let \( p \geq 1 \). We denote by \( \mathcal{SM}^p \) the space of entire slice monogenic functions with either order lower than \( p \) or order equal to \( p \) and finite type. It consists of those functions \( f : \mathbb{R}^{n+1} \to \mathbb{R}_n \), for which there exist constants \( B, C > 0 \) such that

\[
|f(x)| \leq Ce^{B|x|^p}, \forall x \in \mathbb{R}^{n+1}.
\] (18)

Let \( (f_m)_{m \in \mathbb{N}}, f_0 \in \mathcal{SM}^p \). Then \( f_m \to f_0 \) in \( \mathcal{SM}^p \) if there exists some \( B > 0 \) such that

\[
\lim_{m \to \infty} \sup_{x \in \mathbb{R}^{n+1}} \left| (f_m(x) - f_0(x))e^{-B|x|^p} \right| = 0.
\] (19)

Functions in \( \mathcal{SM}^p \) that are left-slice monogenic will be denoted by \( \mathcal{SM}_L^p \), while right-slice monogenic will be denoted by \( \mathcal{SM}_R^p \).

We now give a characterization of functions in \( \mathcal{SM}^p \) in terms of their Taylor coefficients. To prove our results, we need some very well-known estimates on the Gamma function \( \Gamma \) and on the binomial. We collect them in the following lemma.
Lemma 2.11 We have the following estimates:

(I) For \( j, k \in \mathbb{N} \), \((j + k)! \leq 2^{j+k} j! k! \).

(II) For \( n, k \in \mathbb{N} \), \( \Gamma(n+1)\Gamma(k+1) \leq \Gamma(n+k+2) \).

(III) For \( q \in [1, \infty) \) and \( n \in \mathbb{N} \), we have \( \Gamma\left(\frac{n}{q} + 1\right) \leq (n!)^{1/q} \).

(IV) \((a+b)^p \leq 2^p (a^p + b^p), \ a > 0, \ b > 0, \ p > 0. \)

Lemma 2.12 Let \( x \in \mathbb{R}^{n+1} \), then the Mittag–Leffler function

\[
E_{\alpha, \beta}(x) = \sum_{k=1}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}
\]

is an entire slice monogenic function of order \( 1/\alpha \) (and of type 1) for \( \alpha > 0 \) and \( \text{Re}(\beta) > 0 \).

Proof The proof follows the one in the complex case.

We are now ready to prove a crucial result which is the slice monogenic version of the complex version proved in [16]. We consider the case of left-slice monogenic functions, but the results can be stated and proved for right-slice monogenic functions along similar lines.

Lemma 2.13 Let \( p \geq 1 \). A function

\[
f(x) = \sum_{k=0}^{\infty} x^k \alpha_k
\]

belongs to \( \mathcal{SM}_L^p \) if and only if there exist constants \( C_f, b_f > 0 \) such that

\[
|\alpha_k| \leq C_f \frac{b_f^k}{\Gamma\left(\frac{k}{p} + 1\right)}. \tag{20}
\]

Furthermore, let \( f_m \) be a sequence in \( \mathcal{SM}_L^p \); then \( f_m \to 0 \) for \( m \to +\infty \) if and only if we can take constants \( C_{f_m} \) and \( b_m, m \in \mathbb{N} \) such that the sequence \( \{b_{f_m}\}_{m \in \mathbb{N}} \) is bounded and \( C_{f_m} \to 0 \) as \( m \to +\infty \).

Proof We first prove that if \( f \in \mathcal{SM}_L^p \), we have the estimates (20) on the coefficients \( \alpha_k \), for \( k \in \mathbb{N}_0 \). Observe that the kernel

\[
(x, s) \mapsto (x^2 - 2s_0 x + |s|^2)^{-k-1}(x - \overline{s})^{*(k+1)}
\]

in formula (16) can be decomposed by the representation formula, see [27], Theorem 2.2.18, as

\[
(x^2 - 2s_0 x + |s|^2)^{-k-1}(x - \overline{s})^{*(k+1)} = \frac{1 - ij}{2} \frac{1}{(s - \overline{w})^{k+1}} + \frac{1 + ij}{2} \frac{1}{(s - \overline{w})^{k+1}} \tag{21}
\]
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for \( x = u + iv \), \( w = u + jv \), and \( s \in \mathbb{C} \). Moreover, the zeros of the function
\( x \mapsto x^2 - 2s_0x + |s|^2 \) consist of a real point or of a 2-sphere. In fact, on \( \mathbb{C}_j \), we find
only the point \( x \) as a singularity and the result follows from the Cauchy formula on
the plane \( \mathbb{C}_j \). In the complex plane \( \mathbb{C}_j \) for \( j \neq j_x \), if the singularity is real, we obtain
again the Cauchy formula of complex analysis. If the zeros are not real and \( j \neq j_x \),
then on any complex plane \( \mathbb{C}_j \), we find the two conjugate zeros \( s_{1,2} = x_0 \pm j|x| \) in
this case using the representation formula (21). So, the the slice derivatives can be written
in integral form as follows:

\[
\partial^k_j f(x) = \frac{k!}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \frac{1 - ij}{2(s - w)^{k+1}} + \frac{1 + ij}{2(s - w)^{k+1}} ds_j f(s)
\]

and also

\[
\partial^k_j f(x) = \frac{1 - ij}{2\pi} \frac{k!}{\partial(U_1 \cap \mathbb{C}_j)} \frac{1}{(s - w)^{k+1}} ds_j f(s)
\]

\[
+ \frac{1 + ij}{2\pi} \frac{k!}{\partial(U_2 \cap \mathbb{C}_j)} \frac{1}{(s - w)^{k+1}} ds_j f(s),
\]

where \( \partial(U_1 \cap \mathbb{C}_j) \) is the path of integration in \( \mathbb{C}_j \) that contains the point \( w = x_0 + j|x| \)
and \( \partial(U_2 \cap \mathbb{C}_j) \) is the path of integration in \( \mathbb{C}_j \) that contains the point \( \bar{w} = x_0 - j|x| \).
Now, we suppose that the above paths of integration are the two circles \( |s - w| = \tau |w| \)
and \( |s - \bar{w}| = \tau |ar{w}| \) where \( \tau > 0 \) is a parameter. As it is well known, the slice derivative
satisfies

\[
\partial^k_j f(x) = \partial^k_{x_0} f(x) = f^k(x),
\]

so that, we now estimate the two terms

\[
f^{(k)}(w) := \frac{k!}{2\pi} \int_{\partial(U_1 \cap \mathbb{C}_j)} \frac{1}{(s - w)^{k+1}} ds_j f(s)
\]

and

\[
f^{(k)}(\bar{w}) := \frac{k!}{2\pi} \int_{\partial(U_2 \cap \mathbb{C}_j)} \frac{1}{(s - \bar{w})^{k+1}} ds_j f(s)
\]

using the Cauchy formula for the derivatives in the complex plane \( \mathbb{C}_j \). Recalling that
we assume the growth condition \( |f(x)| \leq C_f e^{B|x|^p} \), we obtain

\[
|f^{(k)}(w)| \leq \frac{k!}{(\tau |w|)^j} \max_{|s - w| = \tau |w|} |f(s)| \leq \frac{C_k k!}{(\tau |w|)^j} \exp(B(1 + \tau)^p |w|^p)
\]

\( \mathbb{C} \) Springer
and similarly for $f^{(k)}(\bar{w})$

$$|f^{(k)}(\bar{w})| \leq \frac{C_f k!}{(\tau |\bar{w}|)^k} \exp(B(1 + \tau)^p |\bar{w}|^p).$$

We then conclude that

$$|f^{(k)}(w)| + |f^{(k)}(\bar{w})| \leq 2 \frac{C_f k!}{(\tau |w|)^k} \exp(B(1 + \tau)^p |w|^p)$$

(23)

for all $\tau > 0$. Now we can estimate the slice derivative from the formula (22), precisely

$$|\partial^k_S f(x)| \leq \left| \frac{1 - ij}{2} \right| |f^{(k)}(w)| + \left| \frac{1 + ij}{2} \right| |f^{(k)}(\bar{w})| \leq |f^{(k)}(w)| + |f^{(k)}(\bar{w})|.$$

The well-known estimate (IV) in Lemma 2.11 gives $(1 + \tau)^p \leq 2^{p} (\tau^p + 1)$ for all $\tau > 0$. Hence, from (23) we have

$$|f^{(k)}(w)| + |f^{(k)}(\bar{w})| \leq 2C_f \frac{k!}{(\tau |w|)^k} \exp(B \cdot 2^p \tau^p |w|^p) \exp(B \cdot 2^p |w|^p)$$

(24)

for all $w \in \mathbb{C}_j$ and $\tau > 0$. Now, we observe that the point $\tau_{\text{min}} = \left( \frac{k}{2^p B^p} \right)^{1/p} \frac{1}{|w|}$ for $w \neq 0$ is the minimum of the function:

$$\tau \mapsto \frac{1}{(\tau |w|)^k} \exp(B \cdot 2^p \tau^p |w|^p)$$

which is the right-hand side of (24), so that we obtain

$$|\partial^k_S f(x)| \leq |f^{(k)}(w)| + |f^{(k)}(\bar{w})| \leq 2C_f k! \left( \frac{2^p Bp}{k} \right)^{k/p} e^{k/p} \exp(B \cdot 2^p |w|^p).$$

If we set

$$b := (2^p Bpe)^{1/p}$$

we deduce the estimate

$$|\partial^k_S f(x)| \leq |f^{(k)}(w)| + |f^{(k)}(\bar{w})| \leq 2C_f k! \frac{b^k}{k^{k/p}} \exp(B \cdot 2^p |w|^p)$$

for all $w \in \mathbb{C}_j$. The maximum modulus principle applied in a disc centered at the origin and with radius $\epsilon > 0$ sufficiently small, in the complex plane $\mathbb{C}_j$ and the fact that

$$\alpha_k = \frac{\partial^k_S f(0)}{k!}.$$
finally give

\[ |\alpha_k| \leq 2Cf \frac{b^k}{k^{k/p}} \exp(B \cdot 2^p \epsilon^p) \leq 2Cf \frac{b^k}{k^{k/p}} \leq C'f \frac{b^k}{(k!)^{1/p}} \leq C'f \frac{b^k}{\Gamma(\frac{k}{p} + 1)}. \]

The other direction follows from the properties of the Mittag–Leffler function because it is of order $1/\alpha$ (and of type 1) for $\alpha > 0$ and $\text{Re}(\beta) > 0$, so, in our case, $f$ is entire of order $p$. The fact that $f_m \to 0$ in $SM^p$ if and only if $C f_m \to 0$ and $b f_m < b$ for some $b > 0$ is a consequence of the estimate on the $\alpha_k$. □

3 Infinite-order Differential Operators on Slice Monogenic Functions

In this section, we study a class of infinite-order differential operators acting on spaces of entire slice monogenic functions. The definition of these infinite-order differential operators is designed to preserve the slice monogenicity. In fact, we consider the $\star$-product of the coefficients $(u_m)_{m \in \mathbb{N}_0}$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, of the operator and the slice derivative $\partial^{m}_{x_0} f(x)$ of the slice monogenic function $f : \mathbb{R}^{n+1} \to \mathbb{R}_n$. Precisely, we have

**Definition 3.1** Let $(u_m)_{m \in \mathbb{N}_0} : \mathbb{R}^{n+1} \to \mathbb{R}_n$ be entire functions in $SM_L$ (resp. $SM_R$) and $p \geq 1$. Assume that for every $\epsilon > 0$, there exist $B_\epsilon > 0, C_\epsilon > 0$ for which

\[ |u_m(x)| \leq C_\epsilon \frac{\epsilon^m}{(m!)^{1/q}} \exp(B_\epsilon |x|^p), \quad \text{for all} \quad m \in \mathbb{N}_0, \quad (25) \]

where $1/p + 1/q = 1$ and $1/q = 0$ when $p = 1$.

- For $(u_m)_{m \in \mathbb{N}_0}$ in $SM_L$ entire functions as above, $D^L_{p,0}$ denotes the set of formal operators defined by

\[ U_L(x, \partial_{x_0}) f(x) := \sum_{m=0}^{\infty} u_m(x) \partial^m_{x_0} f(x), \]

and acting on entire functions in $SM_L$.

- For $(u_m)_{m \in \mathbb{N}_0}$ in $SM_R$ entire functions as above, $D^R_{p,0}$ denotes the set of formal operators defined by

\[ U_R(x, \partial_{x_0}) f(x) := \sum_{m=0}^{\infty} \partial^m_{x_0} f(x) \partial^m_{x_0} u_m(x) \]

and acting on entire functions in $SM_R$.

We are now in the position to state and prove the main result of this section.

**Theorem 3.2** Let $p \geq 1$ and let $D^L_{p,0}$ and $D^R_{p,0}$ be the sets of formal operators as in Definition 3.1.
(I) Let $U_L(x, \partial_{x_0}) \in D^L_{p, 0}$ and let $f \in SM^p_L$, then $U_L(x, \partial_{x_0})f \in SM^p_L$ and the operator $U_L(x, \partial_{x_0})$ acts continuously on $SM^p_L$, i.e., if $(f_m) \subset SM^p_L$ and $f_m \to 0$ in $SM^p_L$, then $U_L(x, \partial_{x_0})f_m \to 0$ in $SM^p_L$.

(II) Let $U_L(x, \partial_{x_0}) \in D^R_{p, 0}$ and let $f \in SM^p_R$, then $U_R(x, \partial_{x_0})f \in SM^p_R$ and the operator $U_R(x, \partial_{x_0})$ acts continuously on $SM^p_R$, i.e., if $(f_m) \subset SM^p_R$ and $f_m \to 0$ in $SM^p_R$, then $U_R(x, \partial_{x_0})f_m \to 0$ in $SM^p_R$.

**Proof** Let us prove case (I), since case (II) follows with similar computations. We apply the operator $U_L(x, \partial_{x_0}) \in D^L_{p, 0}$ (see Definition 3.1) to a function $f \in SM^p_L$, and get

$$U_L(x, \partial_{x_0})f(x) = \sum_{m=0}^{\infty} u_m(x) \partial_x^m \left( \sum_{j=0}^{\infty} x^j \alpha_j \right)$$

$$= \sum_{m=0}^{\infty} u_m(x) \partial_x^m \sum_{j=0}^{\infty} x^j \alpha_j$$

$$= \sum_{m=0}^{\infty} u_m(x) \partial_x^m \sum_{j=m}^{\infty} \frac{j!}{(j-m)!} x^{j-m} \alpha_j$$

$$= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} x^k u_m(x) \alpha_{m+k}.$$  

Now we observe that

$$|U_L(x, \partial_{x_0})f(x)| \leq \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} |x|^k |u_m(x)||\alpha_{m+k}|,$$

and we recall that since $U_L(x, \partial_{x_0}) \in D^L_{p, 0}$ the coefficients $u_m(x)$ satisfy the estimate (25) and since $f \in SM^p_L$, the coefficients $|\alpha_k|$ of $f$ satisfy (20), so we deduce

$$|U_L(x, \partial_{x_0})f(x)| \leq C_f C \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{\epsilon^m}{(m!)^{1/p}} \exp(B \epsilon |x|^p) \frac{b^{m+k}}{\Gamma\left(\frac{m+k}{p}+1\right)} \frac{(k+m)!}{k!} |x|^k.$$
We now use estimates (I) and (III) in Lemma 2.11 to get

\[
|U_L(x, \partial_{x_0})f(x)| \leq C_\epsilon \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{\epsilon^m}{\Gamma\left(\frac{m}{q} + 1\right)} \frac{k^{m+k}}{\Gamma\left(\frac{m+k}{p} + 1\right)} \frac{2^{k+m}k!m!}{k!} |x|^k \exp(B_\epsilon |x|^p)
\]

\[
\leq C_\epsilon \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} (2b)^k (2\epsilon b)^m \frac{1}{\Gamma\left(\frac{m}{q} + 1\right)} \frac{m!}{\Gamma\left(\frac{m+k}{p} + 1\right)} |x|^k \exp(B_\epsilon |x|^p).
\]

From (III) in Lemma 2.11 it follows that \(\pi \Gamma\left(\frac{k+m}{p} + 1\right) \geq \Gamma\left(\frac{k}{p} + \frac{1}{2}\right) \Gamma\left(\frac{m}{q} + \frac{1}{2}\right)\), and so we can write (26) as

\[
|U_L(x, \partial_{x_0})f(x)| \leq \beta(p, q, b, \epsilon) C_\epsilon \sum_{k=0}^{\infty} \frac{(2b)^k}{\Gamma\left(\frac{k}{p} + \frac{1}{2}\right)} |x|^k \exp(B_\epsilon |x|^p)
\]

where we set

\[
\beta(p, q, b, \epsilon) := \pi \sum_{m=0}^{\infty} (2\epsilon b)^m \frac{m!}{\Gamma\left(\frac{m}{p} + \frac{1}{2}\right) \Gamma\left(\frac{m}{q} + 1\right)}.
\]

We note that the series (27) is convergent, for \(\epsilon\) arbitrary small, in fact using the asymptotic expansion of the Gamma function, we have

\[
(2\epsilon b)^m \frac{m!}{\Gamma\left(\frac{m}{p} + \frac{1}{2}\right) \Gamma\left(\frac{m}{q} + 1\right)} \sim \frac{m^n (2\epsilon b)^m}{\left(\frac{m}{p}\right)^{m/p} \left(\frac{m}{q}\right)^{m/q}} = (2\epsilon b)^m [p^{1/p} q^{1/q}]^m.
\]

We finally obtain

\[
|U_L(x, \partial_{x_0})f(x)| \leq \beta(p, q, b, \epsilon) C_\epsilon \sum_{k=0}^{\infty} \frac{(2b)^k}{\Gamma\left(\frac{k}{p} + \frac{1}{2}\right)} |x|^k \exp(B_\epsilon |x|^p)
\]

and, by the properties of the Mittag–Leffler function, we have

\[
\sum_{k=0}^{\infty} \frac{(2b)^k}{\Gamma\left(\frac{k}{p} + \frac{1}{2}\right)} |x|^k \leq C' \exp(B' |x|^p).
\]
We conclude that there exists $B''_e > 0$ such that

$$|U_L(x, \partial_{x_0}) f(x)| \leq \beta(p, q, b, \varepsilon) C_f C_e C' \exp(B''_e |x|^p)$$

that is $U_L(x, \partial_{x_0}) f(x) \in SM^p_L$. The same estimate proves the continuity, using Lemma 2.13. In fact, given a sequence $(f_m)$, if $f_m \to 0$ then $C_{f_m} \to 0$ and so $|U_L(x, \partial_{x_0}) f_m(x)| \to 0$. □

4 Function Spaces of Entire Monogenic Functions

In this section, we deal with monogenic functions. We provide only the basic notions, and for the notations, we refer the reader to Sect. 2, and for more details on this class of functions, we refer to the book [23]. We recall that the generalized Cauchy–Riemann operator in $\mathbb{R}^{n+1}$ is defined by

$$\mathcal{D} := \frac{\partial}{\partial x_0} + \sum_{i=1}^n e_i \frac{\partial}{\partial x_i}.$$ 

**Definition 4.1** (Left and right Monogenic Functions) Let $U \subseteq \mathbb{R}^{n+1}$ be an open subset. A function $f : U \to \mathbb{R}^n$, of class $C^1$, is called left monogenic if

$$\mathcal{D} f = \frac{\partial}{\partial x_0} f + \sum_{i=1}^n e_i \frac{\partial}{\partial x_i} f = 0.$$ 

A function $g : U \to \mathbb{R}^n$, of class $C^1$, is called right monogenic if

$$f \mathcal{D} = \frac{\partial}{\partial x_0} f + \sum_{i=1}^n \frac{\partial}{\partial x_i} f e_i = 0.$$ 

The set of left monogenic functions (resp. right monogenic functions) will be denoted by $\mathcal{M}_L(U)$ (resp. $\mathcal{M}_R(U)$); if $U = \mathbb{R}^{n+1}$ we simply denote it by $\mathcal{M}_L$ (resp. $\mathcal{M}_R$)

**Definition 4.2** (Fueter’s homogeneous polynomials) Let $x \in \mathbb{R}^{n+1}$. Given a multi-index $k = (k_1, \ldots, k_n)$ where $k_i$ are integers, we set $|k| = \sum_{i=1}^n k_i$ and $k! = \prod_{i=1}^n k_i!$. We define the homogeneous polynomials $P_k(x)$ as follows:

(I) For a multi-index $k$ with at least one negative component, we set

$$P_k(x) := 0$$

for $0 = (0, \ldots, 0)$, we set

$$P_0(x) := 1.$$
(II) For a multi-index \( k \) with \( |k| > 0 \) and the integers \( k_j \) nonnegative, we define \( P_k(x) \) as follows: for each \( k \) consider the sequence of indices \( j_1, j_2, \ldots, j_{|k|} \) be given such that 1 appears in the sequence \( k_1 \) times, 2 appears \( k_2 \) times, etc., and, finally, \( n \) appears \( k_n \) times. We define \( z_i = x_i - x_0 e_i \) for any \( i = 1, \ldots, n \) and \( z = (z_1, \ldots, z_n) \). We set

\[
z^k := z_j_1 z_j_2 \cdots z_{j_{|k|}}
\]

and

\[
|z|^k = |z_1|^{k_1} \cdots |z_n|^{k_n}
\]

these products contain \( z_1 \) exactly \( k_1 \)-times, \( z_2 \) exactly \( k_2 \)-times, and so on. We define

\[
P_k(x) = \frac{1}{|k|!} \sum_{\sigma \in \text{perm}(k)} \sigma(z^k) := \frac{1}{|k|!} \sum_{\sigma \in \text{perm}(k)} z_{\sigma(1)} z_{\sigma(2)} \cdots z_{\sigma(|k|)},
\]

where \( \text{perm}(k) \) is the permutation group with \( |k| \) elements.

We note that it can be useful to consider the polynomials:

\[
V_k(x) := k! P_k(x) = \frac{k!}{|k|!} \sum_{\sigma \in \text{perm}(k)} z_{\sigma(1)} z_{\sigma(2)} \cdots z_{\sigma(|k|)},
\]

The polynomials \( P_k(x) \) play an important role in the monogenic function theory, and we collect some of their properties in the next proposition (see Theorem 6.2 in [42]):

**Theorem 4.3** Consider the Fueter polynomials \( P_k(x) \) defined above. Then the following facts hold:

(I) the recursion formula

\[
k P_k(x) = \sum_{i=1}^{n} k_i P_{k-e_i}(x) z_i = \sum_{i=1}^{n} k_i z_i P_{k-e_i}(x),
\]

and also

\[
\sum_{i=1}^{n} k_i P_{k-e_i}(x) e_i = \sum_{i=1}^{n} k_i e_i P_{k-e_i}(x),
\]

where \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \) with 1 in the position \( i \).

(II) The derivatives \( \partial x_j \) for \( j = 1, \ldots, n \), are given by

\[
\partial x_j P_k(x) = k_j P_{k-e_j}(x).
\]
(III) The Fueter polynomials \( P_k(x) \) are both left and right monogenic.

(IV) The following estimates holds

\[
|P_k(x)| \leq |x|^{|k|}.
\]

(V) (Binomial formula) For all paravectors \( x \) and \( y \), and for the multi-index \( k, j \) and \( i \)

\[
P_k(x + y) = \sum_{i+j=k} \frac{k!}{i!j!} P_i(x) P_j(y).
\]

We now introduce the Cauchy kernel function.

**Definition 4.4** The Cauchy kernel \( G(x) \) is defined by

\[
G(x) = \frac{1}{\sigma_n} \frac{x}{|x|^{n+1}}, \quad x \in \mathbb{R}^{n+1} \setminus \{0\}, \quad \sigma_n := 2 \frac{2\pi^{(n+1)/2}}{\Gamma((n + 1)/2)}.
\]

Moreover, for any multi-index \( k = (k_1, \ldots, k_n) \), we define

\[
G_k(x) = \frac{\partial^{|k|}}{\partial x^k} G(x).
\]

Monogenic functions satisfy a generalized integral Cauchy formula (see Theorem 7.12 in [42]).

**Theorem 4.5** (The Cauchy formula) Let \( U \) be a bounded domain in \( \mathbb{R}^{n+1} \) with smooth boundary \( \partial G \) so that the normal unit vector is orientated outwards. For the left monogenic functions \( f \), defined on an open set that contains \( \overline{U} \), we have

\[
f(x) = \int_{\partial G} G(y - x) Dy f(y), \quad x \in G.
\]

where

\[
Dy = \sum_{j=0}^{n} (-1)^j e_j dy_0 \wedge \cdots \wedge dy_{j-1} \wedge dy_{j+1} \wedge \cdots \wedge dy_n.
\]

Moreover, for any \( x \in G \) and for any multi-index \( k = (k_1, \ldots, k_n) \), we also have

\[
\frac{\partial^{|k|}}{\partial x^k} f(x) = (-1)^{|k|} \int_{\partial G} G_k(y - x) Dy f(y).
\]
If $f$ is left monogenic in a ball centered at the origin and of radius $R$ then for any $|x| < r$ with $0 < r < R$, we have

$$f(x) = \sum_{|k|=0}^{+\infty} V_k(x) a_k,$$

where the $a_k$'s are Clifford numbers defined by

$$a_k := \frac{(-1)^{|k|}}{k!} \int_{|y|=r} G_k(y) Dyf(y).$$

From

$$|G_k(x)| \leq \frac{n(n+1) \cdots (n + |k| - 1)}{|x|^{n+|k|}}$$ (28)

we obtain the following sharp estimate (see [36]):

$$|a_k| \leq M_g(r) \frac{c(n,k)}{r^{|k|}}.$$ (29)

where

$$c(n,k) := \frac{n(n+1) \cdots (n + |k| - 1)}{k!} = \frac{(n + |k| - 1)!}{(n-1)! k!}.$$ (30)

The first important property that concerns the number $c(n,k)$ is contained in the following (see Lemma 1 in [38]).

**Lemma 4.6** Let $n \in \mathbb{N}$ be nonzero. For all multi-indexes $k \in (\mathbb{N}_0^n \setminus \{0\}$, we have

$$\limsup_{p \to +\infty} \left( \sum_{|k|=p} c(n,k) \right)^{\frac{1}{p}} = n.$$

**Definition 4.7** Let $f$ be an entire left monogenic function. Then we say that $f$ is of finite order if there exists $\kappa > 0$ such that

$$M_f(r) < e^{\kappa r}$$

for sufficiently large $r$. The greatest lower bound $\rho$ of such numbers $\kappa$ is called order of $f$. Equivalently, we can define the order as follows:

$$\rho = \limsup_{r \to \infty} \frac{\ln M_f(r)}{\ln r}.$$
Definition 4.8 Let \( f \) be an entire left monogenic function of order \( \rho \) and let \( A > 0 \) be such that for sufficiently large values of \( r \), we have

\[
M_f(r) < e^{Ar^\rho}.
\]

We say that \( f \) of order \( \rho \) is of type \( \sigma \) if \( \sigma \) is the greatest lower bound of such numbers and we have

\[
\sigma = \limsup_{r \to \infty} \frac{\ln M_f(r)}{r^\rho}.
\]

Moreover, we have

- When \( \sigma = 0 \), we say that \( f \) is of minimal type.
- When \( \sigma = \infty \), we say that \( f \) is of maximal type.
- When \( \sigma \in (0, \infty) \), we say that \( f \) is of normal type.

The constant functions are said to be of minimal type of order zero. The next two theorems are the generalizations of the Lindelöf and Pringsheim theorems on the growth of the Taylor coefficients of an entire holomorphic function to the case of entire monogenic functions (see Theorem 1 in [38] and Theorem 1 in [37]).

Theorem 4.9 For an entire monogenic function, \( f : \mathbb{R}^{n+1} \to \mathbb{R}_n \) with a Taylor series representation of the form \( f(x) = \sum_{|k|=0}^{+\infty} V_k(x) a_k \) set

\[
\Pi = \limsup_{|k| \to +\infty} \frac{|k| \log |k|}{-\log \left| \frac{1}{c(n,k)} a_k \right|},
\]

then \( \rho(f) = \Pi \) where \( c(n,k) \) is the number defined in (30).

Theorem 4.10 For an entire monogenic function \( f : \mathbb{R}^{n+1} \to \mathbb{R}_n \) with a Taylor series representation of the form \( f(x) = \sum_{|k|=0}^{+\infty} V_k(x) a_k \) with order \( \rho \) \((0 < \rho < +\infty)\) and set

\[
\Pi_0 = \limsup_{|k| \to +\infty} |k| \left( |a_k| \right)^{\rho / |k|},
\]

then

\[
\sigma(f) = \frac{\Pi_0}{e^{\rho(f)}}.
\]

The next lemma will be useful in the proof of Lemma 4.12, and it is a generalization of Lemma 2.11 (III).

Lemma 4.11 Given a multi-index \( k \in (\mathbb{N}_0)^n \), then for any \( q \geq 1 \), we have

\[
\Gamma\left( \frac{|k|}{nq} + 1 \right)^{nq} \leq k!.
\]
Proof. It is sufficient to observe that

\[
\Gamma\left(\frac{|k|}{nq} + 1\right) = \int_0^{+\infty} e^{-t} t^{\frac{|k|}{nq}} dt = \int_0^{+\infty} \prod_{i=1}^{n} e^{-\frac{t}{nq} t^{\frac{k_i}{nq}}} dt \leq \prod_{i=1}^{n} \Gamma\left(\frac{k_i}{q} + 1\right) \quad \text{(Hölder inequality)}
\]

so we get the statement. \(\Box\)

In the next lemma, we introduce the counterpart of the entire Mittag–Leffler functions in one complex variable in the context of monogenic functions. We follow the way presented at p. 159 in [37] and at p. 772 in [38].

Lemma 4.12 Let \(x \in \mathbb{R}^{n+1}\) then for any \(\alpha, \beta \in \mathbb{R}_{>0}\) the Mittag–Leffler function

\[
E_{\alpha,\beta}(x) = \sum_{|k|=0}^{\infty} c(n, k) V_k(x) \Gamma(\alpha |k| + \beta)
\]

is an entire monogenic function of order \(1/\alpha\) and of type \(n^{1/\alpha}\) where \(c(n, k)\) is defined in (30).

Proof. The radius of convergence of \(E_{\alpha,\beta}(x)\) is \(+\infty\) so that \(E_{\alpha,\beta}(x)\) is entire. For it is sufficient to observe that

\[
\sum_{|k|=0}^{\infty} c(n, k) |V_k(x)|/\Gamma(\alpha |k| + \beta) \leq \sum_{p=0}^{\infty} \left( \sum_{|k|=p}^{\infty} c(n, k) \right) |x|^p /\Gamma(\alpha p + \beta)
\]

and the conclusion follows by the Cauchy–Hadamard Theorem for the power series once we note that

\[
\limsup_{p \to +\infty} \left( \left( \sum_{|k|=p}^{\infty} c(n, k) \right) \frac{1}{\Gamma(\alpha p + \beta)} \right)^{\frac{1}{p}} \overset{\text{Lemma 4.6}}{=} 0.
\]

We prove the remaining part of the lemma for \(\beta = 1\) since we can deduce the general case by observing that

\[
\Gamma(\alpha x + \beta) = \Gamma\left(\alpha \left( x + \frac{\beta - 1}{\alpha} \right) + 1\right).
\]
To prove that the order of $E_{\alpha,1}$ is equal to $\frac{1}{\alpha}$, we apply Theorem 4.9 with $a_k = \frac{c(n,k)}{\Gamma(\alpha|k|+1)}$ and the Stirling–De Moivre formula to obtain

$$\rho (E_{\alpha,1}) = \limsup_{|k| \to +\infty} \frac{|k| \log |k|}{|k|} - \log \left| \frac{1}{c(n,k)} a_k \right|$$

$$= \limsup_{|k| \to +\infty} \frac{|k| \log |k|}{\log \Gamma(\alpha|k|+1)} \text{ Stirling–De Moivre} = \frac{1}{\alpha}.$$

To prove that the type of $E_{\alpha,1}$ is equal to $n^{\frac{1}{\alpha}}$, we apply Theorem 4.10 with $a_k = \frac{c(n,k)}{\Gamma(\alpha|k|+1)}$ to obtain

$$\sigma (E_{\alpha,1}) = \frac{\alpha}{e} \limsup_{|k| \to +\infty} |k| \left( |a_k| \right)^{\frac{1}{|a_k|}}$$

$$= \frac{\alpha}{e} \limsup_{|k| \to +\infty} |k| \left( \frac{c(n,k)}{\Gamma(\alpha|k|+1)} \right)^{\frac{1}{|a_k|}}.$$

Since by the Lemma 4.11, we have

$$\sup_{|k|=p} \frac{c(n,k)}{\Gamma(\alpha|k|+1)} \leq \sup_{|k|=p} \frac{(n + |k| - 1)!}{(n - 1)! k! \Gamma(\alpha|k|+1)}$$

$$\leq \sup_{|k|=p} \frac{(n + |k| - 1)!}{(n - 1)! \Gamma \left( \frac{|k|}{n} + 1 \right)^n \Gamma(\alpha|k|+1)}$$

with the equality when $|k|$ is a multiple of $n$ and $k = \left( \frac{|k|}{n}, \ldots, \frac{|k|}{n} \right)$, we can conclude

$$\sigma (E_{\alpha,1}) = \frac{\alpha}{e} \limsup_{|k| \to +\infty} |k| \left( \frac{(n + |k| - 1)!}{(n - 1)! \Gamma \left( \frac{|k|}{n} + 1 \right)^n \Gamma(\alpha|k|+1)} \right)^{\frac{1}{|a_k|}}$$

$$= \frac{\alpha}{e} \limsup_{|k| \to +\infty} |k| \left( \frac{(n + |k| - 1)^{n+|k|-1}}{\left( \frac{|k|}{n} \right)^{|k|} (\alpha|k|)^{\alpha|k|} \exp(-\alpha|k|)} \right)^{\frac{1}{|a_k|}} = n^{\frac{1}{\alpha}},$$

where in the second equality, we deleted the terms that do not affect the lim sup. □

**Definition 4.13** Let $p \geq 1$. We denote by $\mathcal{M}^p$ the space of entire monogenic functions with either order lower than $p$ or order equal to $p$ and finite type. It consists of functions $f$, for which there exist constants $B, C > 0$ such that

$$|f(x)| \leq Ce^{B|x|^p}.$$  \hspace{1cm} (31)
Let \((f_m)_{m \in \mathbb{N}}, f_0 \in \mathcal{M}^p\). Then \(f_m \to f_0\) in \(\mathcal{M}^p\) if there exists some \(B > 0\) such that

\[
\lim_{m \to \infty} \sup_{x \in \mathbb{R}^{n+1}} \left| (f_m(x) - f_0(x))e^{-B|x|^p} \right| = 0. \tag{32}
\]

Functions in \(\mathcal{M}^p\) that are left monogenic will be denoted by \(\mathcal{M}^p_L\), while right monogenic will be denoted by \(\mathcal{M}^p_R\).

We extend the Lemma 2.2 in [16] to the case of the monogenic entire function.

**Lemma 4.14** Let \(p \geq 1\). A function

\[
f(x) = \sum_{|k|=0}^{\infty} V_k(x)a_k
\]

belongs to \(\mathcal{M}^p\) if and only if there exist constants \(C_f, b_f, \beta > 0\) such that

\[
|a_k| \leq C_f \frac{b_f^{|k|}c(n,k)}{\Gamma \left( \frac{|k|}{p} + \beta \right)} . \tag{33}
\]

Furthermore, a sequence \(f_m\) in \(\mathcal{M}^p\) tends to zero if and only if \(C_{f_m} \to 0\) and \(b_{f_m} < b\) for some \(b > 0\) where \(c(n,k)\) is the number defined in (30).

**Proof** \((\Rightarrow)\) By the Theorem 4.5, we have

\[
\partial^k_s f(x) = (-1)^{|k|} \int_{\partial B(x,s|x|)} G_k(y-x)Dyf(y)
\]

where \(s > 0\) is a constant to be determined later. In view of the estimate (28) and since \(f \in \mathcal{M}^p\), we have

\[
|\partial^k_s f(x)| \leq k! \frac{c(n,k)}{(s|x|)^{|k|}} \sup_{|\xi-x|=s|x|} |f(\xi)|
\]

\[
\leq k! \frac{c(n,k)}{(s|x|)^{|k|}} M_f((1+s)|x|)
\]

\[
\leq C_f k! \frac{c(n,k)}{(s|x|)^{|k|}} \exp(1+s)^p|x|^p)
\]

\[
\leq C_f k! \frac{c(n,k)}{(s|x|)^{|k|}} \exp(2^p s^p|x|^p) \exp(B^2 s^p|x|^p)
\]

where the last inequality is due to the estimate: \((1+s)^p \leq 2^p (1+s^p)\) which follows by Lemma 2.11, (IV). We define

\[
g(s) := \frac{\exp(2^p s^p|x|^p)}{(s|x|)^{|k|}}
\]

\[
\square \ Springer
\]
and we note that this function gets its minimum at

\[ s_0 := \frac{1}{2} \left( \frac{|k|}{pB} \right)^{\frac{1}{p}} \frac{1}{|x|}. \]

Thus, we have

\[ g(s_0) = \exp \left( \frac{|k|}{p} \right) \left( \frac{2pB}{|m|} \right)^{|m|/p}, \]

and

\[ |\partial_x^k f(x)| \leq C_f c(n, k) \left[ (2epB)^{\frac{1}{p}} \right] |k|^{-\frac{|k|}{p}} \exp(B2^p|x|^p). \]

We set \( b = (2epB)^{\frac{1}{p}} \), and by the maximum modulus principle, we have

\[
|a_k| = \frac{|\partial_x^k f(0)|}{k!} \leq \sup_{|x|=r} |\partial_x^k f(x)| \leq C_f c(n, k) b^{|k|} |k|^{-\frac{|k|}{p}} \exp(B2^pR^p) \\
\leq 2C_f c(n, k) b^{|k|} |k|^{-\frac{|k|}{p}} \\
\leq 2C'_f c(n, k) b^{|k|} (|k|!)^{-\frac{1}{p}} \\
\leq 2C'_f c(n, k) \frac{b^{|k|}}{\Gamma \left( \frac{|k|}{p} + 1 \right)}. \]

\((\Leftarrow)\) The other direction is a consequence of the properties of the Mittag–Leffler function described in Lemma 4.12 \(\blacksquare\)

To define a class of operators that act over \( \mathcal{M}_L^p \) with image in the same space, it is useful to introduce the left (resp. right) C-K product between left (resp. right) monogenic entire functions (see p. 114 in [23]).

**Definition 4.15** Let \( f, g \in \mathcal{M}_L \) be entire functions (resp. \( f, g \in \mathcal{M}_R \)). Using their Taylor series representation

\[
f(x) = \sum_{|k|=0}^{+\infty} V_k(x) f_k \quad \text{(resp. } f(x) = \sum_{|k|=0}^{+\infty} f_k V_k(x) \text{)}
\]

and

\[
g(x) = \sum_{|k|=0}^{+\infty} V_k(x) g_k \quad \text{(resp. } g(x) = \sum_{|k|=0}^{+\infty} g_k V_k(x) \text{)}
\]
we define
\[
 f \odot_L g := \sum_{|k|=0}^{+\infty} \sum_{|j|=0}^{+\infty} V_{k+j}(x) f_k g_j \quad \text{(resp. } f \odot_R g := \sum_{|k|=0}^{+\infty} \sum_{|j|=0}^{+\infty} f_k g_j V_{k+j}(x)).
\]

**Remark 4.16** At p. 114 in [23], the definition of C-K product is given between the Fueter polynomials. Here, we adapt that definition for the polynomial \( V_k(x) \) introduced in Definition 4.2.

5 Infinite-order Differential Operators on Monogenic Functions

In this section, using the definition of the C-K product of two monogenic entire functions, we define suitable classes of infinite-order differential operators in the monogenic setting.

**Definition 5.1** Let \((u_m)_{m \in (\mathbb{N}_0)^n} : \mathbb{R}^{n+1} \to \mathbb{R}^n\) be entire functions in \(\mathcal{M}_L\) (resp. \(\mathcal{M}_R\)), and let \(p \geq 1\). Assume that for every \(\varepsilon > 0\), there exist \(B_\varepsilon > 0\), \(C_\varepsilon > 0\) for which
\[
|u_m(x)| \leq C_\varepsilon e^{(|m|)!^{1/q}} \exp(B_\varepsilon |x|^p), \quad \text{for all } m \in (\mathbb{N}_0)^n,
\]
where \(1/p + 1/q = 1\) and \(1/q = 0\) when \(p = 1\).

- Let \((u_m)_{m \in (\mathbb{N}_0)^n}\) be in \(\mathcal{M}_L\) be as above. We define the set \(\mathcal{D}_L^{p,0}\) of formal operators by
\[
U_L(x, \partial_x) f(x) := \sum_{|m|=0}^{+\infty} u_m(x) \odot_L \partial_x^m f(x),
\]
for entire functions \(f\) in \(\mathcal{M}_L\) where \(\partial_x^m := \partial_{x_1}^{m_1} \ldots \partial_{x_n}^{m_n}\) (in particular no derivatives along the \(x_0\)-direction appear).

- Let \((u_m)_{m \in (\mathbb{N}_0)^n}\) be entire functions in \(\mathcal{M}_R\) be as above. We define the set \(\mathcal{D}_R^{p,0}\) of formal operators by
\[
U_R(x, \partial_x) f(x) := \sum_{|m|=0}^{+\infty} \partial_x^m f(x) \odot_R u_m(x)
\]
for entire functions \(f\) in \(\mathcal{M}_R\) where \(\partial_x^m := \partial_{x_1}^{m_1} \ldots \partial_{x_n}^{m_n}\) (in particular no derivatives along the \(x_0\)-direction appear).

**Remark 5.2** If we consider the Taylor expansion of the \(u_m\)s, then we can write them as
\[
u_m(x) = \sum_{|m|=0}^{+\infty} V_j(x) a_j^m
\]
and, thanks to the Lemma 4.14, the coefficients $a^m_j$ satisfy the estimate

$$|a^m_j| \leq C \epsilon e^{\frac{|m|}{2} \frac{|j| c(n, j)}{(m!)^{\frac{1}{q}} \Gamma \left( \frac{1}{p} + 1 \right)}},$$

where $c(n, j)$ is the number defined in (30).

We are now in the position to state and proof the main result of this section (the analogue of Theorem 2.4 in [16]).

**Theorem 5.3** Let $p \geq 1$ and let $\mathcal{DM}^L_{p,0}$ and $\mathcal{DM}^R_{p,0}$ be the sets of formal operators in Definition 5.1.

(I) Let $u_m(x, \partial_x) \in \mathcal{DM}^L_{p,0}$ and let $f \in \mathcal{M}_L^p$, then $U_L(x, \partial_x) f \in \mathcal{M}_L^p$ and the operator $U_L(x, \partial_x)$ acts continuously on $\mathcal{M}_L^p$, i.e., if $(f_m) \subset \mathcal{M}_L^p$ and $f_m \to 0$ in $\mathcal{M}_L^p$, then we have $U_L(x, \partial_x) f_m \to 0$ in $\mathcal{M}_L^p$.

(II) Let $U_R(x, \partial_x) \in \mathcal{DM}^R_{p,0}$ and let $f \in \mathcal{M}_R^p$, then $U_R(x, \partial_x) f \in \mathcal{M}_R^p$ and the operator $U_R(x, \partial_x)$ acts continuously on $\mathcal{M}_R^p$, i.e., if $(f_m) \subset \mathcal{M}_R^p$ and $f_m \to 0$ in $\mathcal{M}_R^p$, then we have $U_R(x, \partial_x) f_m \to 0$ in $\mathcal{M}_R^p$.

**Proof** We write in details only the proof of the statement (I) since the proof of the statement (II) follows from minor changes. Since $u_m, f \in \mathcal{M}_L^p$ are entire functions, we can rewrite them using their Taylor expansion series:

$$f(x) = \sum_{|k| = 0}^{\infty} V_k(x) f_k \quad \text{and} \quad u_m(x) = \sum_{|k| = 0}^{\infty} V_k(x) a^m_k$$

where $a^m_k, f_k \in \mathbb{R}_n$ for any multi-indexes $m, k \in (\mathbb{N}_0)^n$. According to the Definition 5.1, we have

$$U_L(x, \partial_x) f(x) = \sum_{|m| = 0}^{\infty} u_m(x) \circ L \partial^m x f(x)$$

$$= \sum_{|m| = 0}^{\infty} u_m(x) \circ L \sum_{|k| = 0}^{\infty} \partial^m x (V_k(x)) f_k$$

$$= \sum_{|m| = 0}^{\infty} u_m(x) \circ L \sum_{|k| = 0}^{\infty} \frac{(m + k)!}{k!} V_k(x) f_{m+k}$$

$$= \sum_{|m| = 0}^{\infty} \sum_{|k| = 0}^{\infty} \sum_{|j| = 0}^{\infty} \frac{(m + k)!}{k!} V_{k+j}(x) a^m_j f_{m+k}.$$
Since the $V_k(x)$s are paravectors, using the classical inequality: $|xy| \leq 2 \frac{n}{2} |x||y|$ for any $x, y \in \mathbb{R}_n$, we have that

$$|U_L(x, \partial_x) f(x)| \leq 2^\frac{n}{2} \sum_{|m|=0}^\infty \sum_{|k|=0}^\infty \sum_{|j|=0}^\infty \frac{(m+k)!}{k!} |a_j^m| |f_{m+k}| |V_{k+j}(x)|.$$ 

We observe that if we define

$$\underline{y} = y_1 e_1 + \cdots + y_n e_n := \left(\sqrt{x_1^2 + x_0^2}\right) e_1 + \cdots + \left(\sqrt{x_n^2 + x_0^2}\right) e_n$$

then we have

$$|V_k(x)| \leq V_k(\underline{y}).$$

Since

$$V_k(\underline{y}) = k! \Pi_{i=1}^n (y_i)^{k_i}$$

we get

$$|V_{k+j}(x)| \leq V_{k+j}(\underline{y}) = \frac{(k+j)!}{k! j!} V_k(\underline{y}) V_j(\underline{y}) \overset{\text{Lemma 2.11 (I)}}{\leq} 2^{k+j} |V_k(\underline{y}) V_j(\underline{y})|. \tag{35}$$

Moreover, since $|\underline{y}| \leq \sqrt{n} |x|$ and using the Remark 5.2 in combination with the Lemma 4.14, we have

$$|U_L(x, \partial_x) f(x)| \leq 2^\frac{n}{2} \sum_{|m|=0}^\infty \sum_{|k|=0}^\infty \left( \sum_{|j|=0}^\infty 2^{|j|} |a_j^m| V_j(\underline{y}) \right) \frac{2^{|k|}(m+k)!}{k!} |f_{m+k}| |V_k(\underline{y})|$$

$$\leq 2^\frac{n}{2} C_\epsilon \sum_{|m|=0}^\infty \sum_{|k|=0}^\infty \epsilon^{|m|} \frac{e^{(|m|)!/q}}{|m|!^{1/q}} \exp(B_\epsilon |x|^p) \frac{2^{|k|}(m+k)!}{k!} |f_{m+k}| |V_k(\underline{y})|. \quad \Box$$
Using Lemma 4.14 and the estimates (34), we obtain

\[ |U_L(x, \partial_x) f(x)| \leq 2^{2n} C_f C_\varepsilon \sum_{|m|=0}^{\infty} \sum_{|k|=0}^{\infty} \frac{(m + k)! \cdot |b_f|^{|m|}}{k! \cdot (m!)^{1/q}} \]

\[ \frac{(2b_f)^{|k|}c(n, m + k)}{\Gamma \left( \frac{|m + k|}{p} + 1 \right)} \exp(B_\varepsilon |x|^p) V_k(y) \]

\[ \leq 2^{2n} C_f C_\varepsilon \pi \sum_{|m|=0}^{\infty} \frac{(\varepsilon 2b_f)^{|m|} c(n, m) m! \cdot \exp(B_\varepsilon |x|^p)}{\Gamma \left( \frac{|m|}{q} + 1 \right) \Gamma \left( \frac{|m|}{p} + \frac{1}{2} \right)} \]

\[ \sum_{|k|=0}^{\infty} \frac{c(n, m + k)}{c(n, m) c(n, k)} \frac{(4b_f)^{|k|} c(n, k) V_k(y)}{\Gamma \left( \frac{|k|}{p} + \frac{1}{2} \right)} \]

\[ \leq 2^{3n - 1} (n - 1)! C_f C_\varepsilon \pi \sum_{|m|=0}^{\infty} \frac{(\varepsilon 4b_f)^{|m|} c(n, m) m! \cdot \exp(B_\varepsilon |x|^p)}{\Gamma \left( \frac{|m|}{q} + 1 \right) \Gamma \left( \frac{|m|}{p} + \frac{1}{2} \right)} \]

\[ \sum_{|k|=0}^{\infty} \frac{(8b_f)^{|k|} c(n, k) V_k(y)}{\Gamma \left( \frac{|k|}{p} + \frac{1}{2} \right)} , \]

where the last inequality is due to the following estimate:

\[ \frac{c(n, m + k)}{c(n, m) c(n, k)} = \frac{(n + |m| + |k| - 1)! ((n - 1)!)^{2k} m!}{(n - 1)! (m + k)! (n + |m| - 1)! (n + |k| - 1)!} \leq (n - 1)! 2^{n + |m| + |k| - 1}. \]

We observe that

\[ \frac{m!}{\Gamma \left( \frac{|m|}{q} + 1 \right) \Gamma \left( \frac{|m|}{p} + \frac{1}{2} \right)} \leq \frac{|m|!}{\Gamma \left( \frac{|m|}{q} + 1 \right) \Gamma \left( \frac{|m|}{p} + \frac{1}{2} \right)} \]

\[ \text{Stirling–De Moivre} \quad \left( \frac{|m|!}{\left( \frac{|m|}{q} \right) \left( \frac{|m|}{p} - \frac{1}{2} \right)^{\frac{|m|}{p} - \frac{1}{2}} |m| \cdot \exp(-|m|)} \right) \approx \left( \frac{\frac{1}{p} \cdot \frac{1}{q} \cdot \frac{1}{q}}{|m|!} \right). \]

In particular, the previous inequality implies that

\[ \sum_{|m|=0}^{\infty} \frac{(\varepsilon 4b_f)^{|m|} c(n, m) m!}{\Gamma \left( \frac{|m|}{q} + 1 \right) \Gamma \left( \frac{|m|}{p} + \frac{1}{2} \right)} \lesssim \sum_{|m|=0}^{\infty} \left( \frac{\frac{1}{p} \cdot \frac{1}{q} \cdot \frac{1}{q}}{|m|!} \varepsilon 4b_f \right)^{|m|} c(n, m). \]
By the Lemma 4.6 and since \( \epsilon > 0 \) can be chosen small enough, using the Cauchy–Hadamard Theorem for the power series, we have that the previous series converges. Thus, there exists a constant \( C' > 0 \) such that

\[
\sum_{|m|=0}^{\infty} \frac{(\epsilon 4 b f)^{|m|} c(n, m) m!}{\Gamma\left(\frac{|m|}{q} + 1\right) \Gamma\left(\frac{|m|}{p} + \frac{1}{2}\right)} \leq C'.
\]  

(38)

By the Lemma 4.14, there exist two constants: \( B' > 0 \) and \( C'' > 0 \) such that

\[
\sum_{|k|=0}^{\infty} \frac{(8 b f)^{|k|} c(n, k)}{\Gamma\left(\frac{|k|}{p} + \frac{1}{2}\right)} V_k(y) \leq C'' \exp\left((B' + B')|x|^p\right).
\]  

(39)

In conclusion by the estimates (38) and (39), we have proved that

\[
|U_L(x, \partial_x) f(x)| \leq 2^{2n-1} (n - 1)! C' C_f C_c C' \exp\left((B_e + B')|x|^p\right)
\]

which means that \( U_L(x, \partial_x) f(x) \in \mathcal{M}^p_L \) and also that \( U_L(x, \partial_x) f(x) \to 0 \) as \( f \to 0 \) or, equivalently, \( C_f \to 0 \).

\[\square\]

**Remark 5.4** If in the Definition 5.1, we use the pointwise product instead of the C-K product to define \( U_L(x, \partial_x) \) (or \( U_R(x, \partial_x) \)), the resulting operator does not preserve the monogenicity, although it remains a continuous operator from \( \mathcal{M}^p_L \) (or \( \mathcal{M}^p_R \)) to \( C^0(\mathbb{R}^{n+1}, \mathbb{R}_n) \). This can be seen starting from the inequality

\[
\left| \sum_{|m|=0}^{\infty} u_m(x) \partial_x^m f(x) \right| \leq 2^n \sum_{|m|=0}^{\infty} \sum_{|k|=0}^{\infty} \frac{(m + k)!}{k!} |u_m(x)||f_{m+k}|V_k(y)
\]

\[
\leq 2^n \sum_{|m|=0}^{\infty} \sum_{|k|=0}^{\infty} \frac{(m + k)!}{k!} \frac{\epsilon^q}{(|m|)!^q} \exp(B|x|^p)|f_{m+k}|V_k(y)
\]

and applying to the last term in the right the same estimates used in (36), (38), and (39).

**Acknowledgements** The authors would like to thank the referees for carefully reading the manuscript and for the useful comments.

**References**

1. Adler, S.: Quaternionic Quantum Mechanics and Quaternionic Quantum Fields. International Series of Monographs on Physics. Oxford University Press, New York (1995)
2. Aharonov, Y., Rohrlich, D.: Quantum Paradoxes: Quantum Theory for the Perplexed. Wiley-VCH Verlag, Weinheim (2005)
3. Aharonov, Y., Albert, D., Vaidman, L.: How the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100. Phys. Rev. Lett. 60, 1351–1354 (1988)
| 4. | Aharonov, Y., Colombo, F., Sabadini, I., Struppa, D.C., Tollaksen, J.: Some mathematical properties of superoscillations. J. Phys. A 44, 365304 (2011) |
| 5. | Aharonov, Y., Colombo, F., Sabadini, I., Struppa, D.C., Tollaksen, J.: On the Cauchy problem for the Schrödinger equation with superoscillatory initial data. J. Math. Pures Appl. 99, 165–173 (2013) |
| 6. | Aharonov, Y., Colombo, F., Sabadini, I., Struppa, D.C., Tollaksen, J.: Superoscillating sequences as solutions of generalized Schrödinger equations. J. Math. Pures Appl. 103, 522–534 (2015) |
| 7. | Aharonov, Y., Colombo, F., Sabadini, I., Struppa, D.C., Tollaksen, J.: Superoscillating sequences in several variables. J. Fourier Anal. Appl. 22, 751–767 (2016) |
| 8. | Aharonov, Y., Colombo, F., Sabadini, I., Struppa, D.C., Tollaksen, J.: The mathematics of superoscillations. Mem. Am. Math. Soc. 247, 107 (2017) |
| 9. | Aharonov, Y., Colombo, F., Sabadini, I., Struppa, D.C., Tollaksen, J.: Evolution of superoscillations in the Klein–Gordon Field. Milan J. Math. 88, 171–189 (2020) |
| 10. | Aharonov, Y., Colombo, F., Sabadini, I., Struppa, D.C., Tollaksen, J.: How superoscillating tunneling waves can overcome the step potential. Ann. Phys. 414, 168088 (2020) |
| 11. | Aharonov, Y., Behrndt, J., Colombo, F., Schlosser, P.: Schrödinger evolution of superoscillations with δ- and δ’-potentials. Quantum Stud. Math. Found. https://doi.org/10.1007/s40509-019-00215-4 |
| 12. | Alpay, D., Colombo, F., Kimsey, D.P.: The spectral theorem for quaternionic unbounded normal operators based on the S-spectrum. J. Math. Phys. 57, 023503 (2016) |
| 13. | Alpay, D., Colombo, F., Sabadini, I.: Slice Hyperholomorphic Schur Analysis, Operator Theory: Advances and Applications, 256. Birkhäuser/Springer, Cham (2016) |
| 14. | Alpay, D., Colombo, F., Sabadini, I.: Quaternionic de Branges spaces and characteristic operator function. Springer Briefs in Mathematics, Springer, Cham (2020) |
| 15. | Alpay, D., Colombo, F., Sabadini, I.: Superoscillations and analytic extension. J. Fourier Anal. Appl |
| 16. | Aoki, T., Colombo, F., Sabadini, I., Struppa, D.C.: Continuity theorems for a class of convolution operators and applications to superoscillations. Ann. Mat. Pura Appl. 197(5), 1533–1545 (2018) |
| 17. | Aoki, T., Colombo, F., Sabadini, I., Struppa, D.C.: Continuity of some operators arising in the theory of superoscillations. Quantum Stud. Math. Found. 5, 463–476 (2018) |
| 18. | Aoki, T., Ishimura, R., Okada, Y., Struppa, D.C., Uchida, S.: Characterisation of continuous endomorphisms of the space of entire functions of a given order. Complex Var. Elliptic Equ. https://doi.org/10.1080/17476933.2020.1767086 |
| 19. | Behrndt, J., Colombo, F., Schlosser, P.: Evolution of Aharonov–Berry superoscillations in Dirac δ-potential. Quantum Stud. Math. Found. 6, 279–293 (2019) |
| 20. | Berry, M.V.: Faster than Fourier, in Quantum Coherence and Reality; in celebration of the 60th Birthday of Yakir, Aharonov, pp. 55–65. World Scientific, Singapore (1994) |
| 21. | Berry, M.V.: Representing superoscillations and narrow Gaussians with elementary functions. Milan J. Math. 84, 217–230 (2016) |
| 22. | Berry, M.V., et al.: Roadmap on superoscillations. J. Optics 21, 053002 (2019) |
| 23. | Brackx, F., Delanghe, R., Sommen, F.: Clifford Analysis, Research Notes in Mathematics, 76. Pitman (Advanced Publishing Program), Boston, MA. x+308 pp (1982) |
| 24. | Cerejeiras, P., Colombo, F., Kähler, U., Sabadini, I.: Perturbation of normal quaternionic operators. Trans. Am. Math. Soc. 372, 3257–3281 (2019) |
| 25. | Colombo, F., Gantner, J.: An application of the S-functional calculus to fractional diffusion processes. Milan J. Math. 86, 225–303 (2018) |
| 26. | Colombo, F., Gantner, J.: Quaternionic closed operators, fractional powers and fractional diffusion process, Operator Theory: Advances and Applications, vol 274. ISBN 978-3-030-16409-6 (2019) |
| 27. | Colombo, F., Sabadini, I., Struppa, D.C.: Noncommutative functional calculus. Theory and applications of slice hyperholomorphic functions, Volume 289 of Progress in Mathematics. Birkhäuser/Springer Basel AG, Basel (2011) |
| 28. | Colombo, F., Gonzalez-Cervantes, J.O., Sabadini, I.: A nonconstant coefficients differential operator associated to slice monogenic functions. Trans. Am. Math. Soc. 365, 303–318 (2013) |
| 29. | Colombo, F., Sabadini, I., Struppa, D.C.: Entire Slice Regular Functions, SpringerBriefs in Mathematics. Springer, Cham. v+118 pp (2016) |
| 30. | Colombo, F., Gantner, J., Kimsey, D.P.: Spectral theory on the S-spectrum for quaternionic operators, Operator Theory: Advances and Applications, 270. Birkhäuser/Springer, Cham. ix+356 pp. ISBN: 978-3-030-03073-5; 978-3-030-03074-2 47-02 (2018) |
| 31. | Colombo, F., Peloso, M., Pinton, S.: The structure of the fractional powers of the noncommutative Fourier law. Math. Methods Appl. Sci. 42, 6259–6276 (2019) |
32. Colombo, F., Sabadini, I., Struppa, D.C., Yger, A.: Superoscillating sequences and hyperfunctions. Publ. Res. Inst. Math. Sci. 55, 665–688 (2019)
33. Colombo, F., Gantner, J., Struppa, D.C.: Evolution by Schrödinger equation of Aharonov–Berry superoscillations in centrifugal potential. Proc. A. 475, 20180390 (2019)
34. Colombo, F., Sabadini, I., Struppa, D.C.: Michele Sce’s Works in Hypercomplex Analysis. A Translation with Commentaries, Birkhäuser, Hardcover ISBN 978-3-030-50215-7 (2020)
35. Colombo, F., Gonzalez, D.D., Pinton, S.: Fractional powers of vector operators with first order boundary conditions. J. Geom. Phys. 151, 103618 (2020)
36. Constales, D., Krausshar, R.S.: Representation formulas for the general derivatives of the fundamental solution of the Cauchy–Riemann operator in Clifford analysis and applications. Z. Anal. Anwendungen 21, 579–597 (2002)
37. Constales, D., De Almeida, R., Krausshar, R.S.: On the growth type of entire monogenic functions. Arch. Math. 88, 153–163 (2007)
38. Constales, D., De Almeida, R., Krausshar, R.S.: On the relation between the growth and the Taylor coefficients of entire solutions to the higher-dimensional Cauchy–Riemann system in $\mathbb{R}^{n+1}$. J. Math. Anal. Appl. 327, 763–775 (2007)
39. Gantner, J.: On the equivalence of complex and quaternionic quantum mechanics. Quantum Stud. Math. Found. 5, 357–390 (2018)
40. Gentili, G., Stoppato, C., Struppa, D.C.: Regular functions of a quaternionic variable, Springer Monographs in Mathematics. Springer, Heidelberg. x+185 pp (2013)
41. Gürlbeck, K., Sprössig, W.: Quaternionic Analysis and Elliptic Boundary Value Problems, International Series of Numerical Mathematics, 89, p. 253. Birkhäuser Verlag, Basel (1990)
42. Gürlbeck, K., Habetha, K., Sprössig, W.: Holomorphic Functions in the Plane and n-dimensional Space, Birkhäuser (2008)
43. Jefferies, B.: Spectral Properties of Noncommuting Operators. Lecture Notes in Mathematics, vol. 1843. Springer-Verlag, Berlin (2004)
44. Kempf, A.: Four aspects of superoscillations. Quantum Stud. Math. Found. 5, 477–484 (2018)

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