PIECEWISE INTERLACING ZEROS OF POLYNOMIALS

DAVID G.L. WANG†‡ AND JERRY J.R. ZHANG

ABSTRACT. We introduce the concept of piecewise interlacing zeros for studying the relation of root distribution of two polynomials. The concept is pregnant with an idea of confirming the real-rootedness of polynomials in a sequence. Roughly speaking, one constructs a collection of disjoint intervals such that one may show by induction that consecutive polynomials have interlacing zeros over each of the intervals. We confirm the real-rootedness of some polynomials satisfying a recurrence with linear polynomial coefficients. This extends Gross et al.’s work where one of the polynomial coefficients is a constant.

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1. INTRODUCTION

The root distribution of a single polynomial is a long-standing topic all along the history of mathematics; see Rahman and Schmeisser’s book [18]. For instance,

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the significance of real-rootedness and stability of polynomials can be found from Stanley [21, §4] and Borcea and Bränden [2].

Motivated by the LCGD conjecture from topological graph theory, Gross, Mansour, Tucker and the first author [9,10] studied the root distribution of polynomials satisfying some recurrences of order two, with one of the polynomial coefficients in the recurrence linear and the other constant. In this paper, we continue to study the root distribution of polynomials defined by a recurrence of order two, but with both the polynomial coefficients linear. It turns out that this change brings much richer root geometry. Orthogonal polynomials have a closed relation with such recurrences; see Szegő [24], Andrews, Richard and Ranjan [1], and Stahl and Totik [19]. Equally closed connection with quasi-orthogonal polynomials was pointed by Brezinski, Driver and Redivo-Zaglia [3].

A popular way to show the real-rootedness of all polynomials in a sequence is to show that consecutive polynomials have interlacing real zeros by induction; see Liu and Wang [16]. The classical notion of interlacing zeros can be found from Rahman and Schmeisser’s book [18]. Unfortunately and commonly encountered, consecutive polynomials in a sequence, such as those in Section 3, do not have interlacing zeros over the whole real line.

We introduce the concept of piecewise interlacing zeros for studying the relation of root distribution of two polynomials. The concept is pregnant with an idea of confirming the real-rootedness of polynomials in a sequence. Roughly speaking, one constructs a collection of disjoint intervals such that one may show by induction that consecutive polynomials have interlacing zeros over each of the intervals.

We should mention that He and Saff [12] considered the root distributions of sequences of Faber polynomials associated with a compact set on the complex plane. By virtue of choosing the compact set to be the $m$-cusped hypocycloid with the parametric equation

$$z = e^{i\theta} + \frac{1}{m-1}e^{(1-m)i\theta},$$

the roots of corresponding Faber polynomials are located on the $m$ line segments each of which connects the origin and a complex number $\frac{m}{m-1}e^{2k\pi i/m}$ for some $k = 0,1,\ldots,m-1$. They proved that consecutive Faber polynomials for any given $m$ interlace on each line segment.

The idea of piecewise interlacing zeros presented in this paper has an essential difference from the one of He and Saff’s. Our idea needs a discovery of disjoint intervals from a single line to form the pieces, such that the real-rootedness of each polynomial on each resulting interval can be shown by induction. In comparison, He and Saff’s way of forming the pieces is somehow straightforward, for at least
zeros lying on distinct lines are not said to be interlacing, although it is literally true to say that the zeros are “piecewise” interlacing for the above Faber polynomials.

Furthermore, the extensive concept of piecewise interlacing inherits some attributes of the classical notion of interlacing property. For example, Jordaan and Toókos [14] call the interlacing properties among any two of the zero sets of three consecutive polynomials in the sequence the “triple interlacing property”. In Theorem 4.2, the piecewise interlacing properties is shown to have such triple behaviour either.

This paper is organised as follows. Section 2 explains the idea of confirming the real-rootedness of polynomials in a sequence based on the concept of piecewise interlacing zeros. In §3, we give an application of the idea as Theorem 3.3 with some numerical examples. A proof of Theorem 3.3 lies in §4. We end this paper with a conjecture in §5.

2. Piecewise interlacing zeros

The notion of interlacing zeros can be found from Rahman and Schmeisser’s book [18, Definition 6.3.1].

Definition 2.1. We say that an unordered pair \((X, Y)\) of sets of real numbers interlaces if (i) the number of elements in \(X\) and \(Y\) are equal or differ by one, and (ii) there exists an ordering \(\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \cdots \leq \alpha_\nu \leq \beta_\nu \leq \cdots\), where \(\alpha_1, \alpha_2, \ldots\) are the elements of one of the sets and \(\beta_1, \beta_2, \ldots\) are those of the other; and strictly interlaces if no equality in the foregoing ordering holds. Two real-rooted polynomials \(P\) and \(Q\) are said to have interlacing zeros if the pair of their zero sets interlaces. A degenerated case is that the pair of any singleton and the empty set \(\emptyset\) interlaces, and so does the pair \((\emptyset, \emptyset)\).

We introduce the concept of piecewise interlacing pairs.

Definition 2.2. We say a pair \(X\) and \(Y\) of sets of real numbers piecewise interlaces (resp. piecewise strictly interlaces) on the disjoint union \(\sqcup_{\lambda \in \Lambda} I_\lambda\) of intervals, if the subset pair \((X \cap I_\lambda, Y \cap I_\lambda)\) for each \(\lambda \in \Lambda\) interlaces (resp. strictly interlaces).

Suppose that we are going to show the real-rootedness of every polynomial in a polynomial sequence \(\{W_n(z)\}\), or to study the relation of root distribution of consecutive polynomials \(W_n(z)\) and \(W_{n+1}(z)\). With the aid of the concept of piecewise interlacing zeros, we can do this in two steps. First, we deal with repeated zeros of each \(W_n(z)\), and with common zeros of different polynomials \(W_n(z)\), individually. This step may be done in specific ways; see [13] for instance. Then, secondly, we can suppose that the polynomials \(W_n(z)\) have neither common zeros nor repeated zeros.
The basic idea of confirming the real-rootedness is to show that consecutive polynomials have piecewise interlacing zeros by induction using the intermediate value theorem. The difficulty lies in the division of the real line into disjoint intervals so that an induction proof works. More precisely, the points that one would like to assign to be an end of the desired intervals often obey a regular law of sign changing in the subscript $n$. For example, traditionally one considers the whole real line on which polynomials are expected to have interlacing zeros, that is, to take the interval ends to be $-\infty$ and $+\infty$. That is partially because that the signs of $\lim_{x \to -\infty} W_n(x)$ and $\lim_{x \to \infty} W_n(x)$ for each polynomial $W_n(x)$ in real $x$ can be simply determined by the sign of leading coefficient of $W_n(x)$ and the parity of the degree of $W_n(x)$. It is usually easy to figure out the sign of leading coefficient and the parity of the degree for each $W_n(x)$ from a recursive definition.

Of course it is possible that two polynomials may not have interlacing zeros over the whole real line. It is not uncommon that the intervals are well hidden behind the appearance of the given polynomial sequence. Besides the points $\pm \infty$, one may also select “isolated limits of zeros” to be interval ends; see Beraha, Kahane and Weiss [4, 5] and Sokal [22] for definition. We will illustrate this idea in the next section.

In fact, there has been a number of study on the limiting root distribution of polynomials in a sequence; see Szegö [23] for instance. Stanley on his website [20] provides some figures for the root distribution of some polynomials in a sequence arising from combinatorics. Bleher and Mallison [6] studied the asymptotics and zeros of Taylor polynomials for linear combinations of exponentials. In answering a problem posed by Herbert Wilf, Boyer and Goh [7] figured out that the set of limit points of zeros of Euler polynomials, called its “zero attractor”, to be the union of an interval and a curve closely related to the Szegö’s curve; See also [8, 11].

3. AN APPLICATION OF THE PIECEWISE INTERLACING ZEROS

Let $a, b, c, d \in \mathbb{R}$ and $ac \neq 0$. In this paper, we consider polynomials $W_n(z)$ satisfying the recurrence

\begin{equation}
W_n(z) = (az + b)W_{n-1}(z) + (cz + d)W_{n-2}(z).
\end{equation}

We call the sequence $\{W_n(z)\}_{n \geq 0}$ normalised if $W_0(z) = 1$ and $W_1(z) = z$. For any complex number $z = re^{i\theta}$ with $\theta \in (-\pi, \pi]$, we use the square root notation $\sqrt{z}$ to denote the number $\sqrt{r}e^{i\theta/2}$. Lemma 3.1 is the base of our study, which can be found in [9, 10, 15].
Lemma 3.1. Let $A, B \in \mathbb{C}$. Suppose that $W_0 = 1$ and $W_n = AW_{n-1} + BW_{n-2}$ for $n \geq 2$. Then for $n \geq 0$, we have

$$W_n = \begin{cases} 
\alpha_+ \lambda_+^n + \alpha_- \lambda_-^n, & \text{if } \Delta \neq 0, \\
\frac{A + n(2W_1 - A)}{2} \left(\frac{A}{2}\right)^{n-1}, & \text{if } \Delta = 0,
\end{cases}$$

where $\lambda_{\pm} = (A \pm \sqrt{\Delta})/2$ with $\Delta = A^2 + 4B$ are the eigenvalues, and

$$\alpha_\pm = \frac{\sqrt{\Delta} \pm (2W_1 - A)}{2\sqrt{\Delta}}.$$

We employ the notations

$$\Delta(z) = A(z)^2 + 4B(z) = a^2 z^2 + (2ab + 4c)z + (b^2 + 4d),$$

$$\alpha_\pm(z) = \frac{\sqrt{\Delta(z)} \pm (2W_1(z) - A(z))}{2\sqrt{\Delta(z)}},$$

$$g(z) = -\alpha_+(z)\alpha_-(z)\Delta(z) = (1 - a)z^2 - (b + c)z - d,$$

whose zeros are respectively

$$(3.2) \quad x_\Delta^\pm = \frac{-ab - 2c \pm 2\sqrt{\Delta_\Delta}}{a^2}$$

where $\Delta_\Delta = -a^2d + abc + c^2$, and

$$(3.3) \quad x_g^\pm = \begin{cases} 
\frac{b + c}{2(1 - a)} \pm \frac{\sqrt{\Delta_g}}{2|1 - a|}, & \text{if } a \neq 1, \\
\frac{d}{b + c}, & \text{if } a = 1 \text{ and } b + c \neq 0,
\end{cases}$$

where $\Delta_g = (b + c)^2 + 4d(1 - a)$. In fact, the function $g(z) = -d$ is a constant when $a = 1$ and $b + c = 0$. For convenience, we also use

$$x_A = -b/a \quad \text{and} \quad x_B = -d/c.$$

From Recurrence (3.1), it is routine to check that $W_n(z)$ has leading coefficient $a^{n-1}z^n$, and that

$$(3.4) \quad W_n(x_g^\pm) = (x_g^\pm)^n,$$

$$(3.5) \quad W_n(x_\Delta^\pm) = \frac{A(x_\Delta^\pm) + nh(x_\Delta^\pm)}{2} \cdot \left(\frac{A(x_\Delta^\pm)}{2}\right)^{n-1},$$

where $h(z) = 2W_1(z) - A(z) = (2 - a)z - b$.

The polynomials $W_n(z)$ in Lemma 3.1 satisfy another recurrence as presented in Lemma 3.2, which helps in proving Theorems 3.3 and 4.2.
Lemma 3.2. Let $A, B \in \mathbb{R} \setminus \{0\}$. Suppose that $W_0 = 1$ and $W_n = AW_{n-1} + BW_{n-2}$ for $n \geq 2$. Then $W_n = (A^2 + 2B)W_{n-2} - B^2W_{n-4}$ for $n \geq 4$.

To the end of this paper, let $\{W_n(z)\}_n$ be a normalised polynomial sequence satisfying Recurrence (3.1) with $a, b, d < 0$ and $c > 0$. We introduce the notation

$$c^\pm = \pm 2\sqrt{d(a - 1)} - b,$$

as the zeros of the discriminant $\Delta_g$ in $c$. It is clear that

$$\max(0, c^-) < c^+.$$

From definition, we have

$$x_g^+ \in \mathbb{R} \iff c \leq c^- \text{ or } c \geq c^+.$$

In this case, we denote

$$J_g = (x_g^-, x_g^+).$$

Let $R_n$ be the zero set of $W_n(z)$. It is a multi-set when $W_n(z)$ has repeated zeros, though all zeros considered in this paper turn out to be simple eventually. Denote

$$R_n \cap J = R_n^J \text{ for any interval } J.$$

For the statement of Theorem 3.3, we denote

$$n^+ = -\frac{A(x_\Delta^+)}{h(x_\Delta^+)} \text{ whenever } h(x_\Delta^+) \neq 0.$$

Theorem 3.3. Let $\{W_n(z)\}_n$ be a normalised polynomial sequence satisfying Recurrence (3.1) with $a, b, d < 0$ and $c > 0$. Define

$$J_1 = (x_A^-, x_A), \quad J_2 = (x_A, x_\Delta^+), \quad J_3 = [x_\Delta^-, x_g^-), \quad J_4 = (x_g^+, 0].$$

(i) If $c \leq c^-$, then every polynomial $W_n(z)$ is real-rooted,

$$|R_n^{J_1}| = \lfloor (n - 1)/2 \rfloor, \quad |R_n^{J_1 \cup J_3}| = \lfloor n/2 \rfloor, \quad \text{and} \quad |R_n^{J_4}| = 1.$$

Moreover, we have

$$|R_n^{J_3}| = \begin{cases} 1, & \text{if } \Delta_A > \Delta_g \text{ and } n \geq n^+; \\ 0, & \text{otherwise.} \end{cases}$$

(ii) If $c \geq c^+$, then every polynomial $W_{2n}(z)$ is real-rooted, and

$$|R_{2n}^{J_1}| = |R_{2n}^{J_2}| = n - 1 \quad \text{and} \quad |R_{2n}^{J_4}| = |R_{2n}^{J_3}| = 1.$$

(iii) If $c > c^+$ and the polynomial $W_3(z)$ has zeros $\xi_{3,2} \leq \xi_{3,3}$ lying in the interval $J_g$, then every polynomial $W_{2n+1}(z)$ is real-rooted, and

$$|R_{2n+1}^{J_1}| = n - 1, \quad |R_{2n+1}^{J_2}| = n, \quad \text{and} \quad |R_{2n+1}^{J_4}| = |R_{2n+1}^{J_3}| = 1,$$

where $I_3 = (x_A^-, \xi_{3,2})$ and $I_4 = (\xi_{3,3}, x_g^+)$ for $n \geq 2$. 


We illustrate Theorem 3.3 as in Figs. 3.1 to 3.4 respectively. In each figure, we thicken the line segment $J_{\Delta}$ and the points $x_y^{\pm}$. The numbers of zeros are indicated above corresponding intervals.

\[ \begin{array}{cccccccc}
& \left\lfloor \frac{n-1}{2} \right\rfloor & & & & & & \\
x_{\Delta} & J_1 & x_A & J_2 & x_{\Delta}^- & J_3 & x_y^- & J_4 & 0 \\
\end{array} \]

**Figure 3.1.** Illustration of the root distribution in Theorem 3.3 (i) when $\Delta_{\Delta} > \Delta_g$ and $n \geq n^+$.

\[ \begin{array}{cccccccc}
& & \left\lfloor \frac{n}{2} \right\rfloor & & & & & \\
x_{\Delta} & J_1 & x_A & J_2 & x_{\Delta}^- & J_3 & x_y^- & x_y^+ & J_4 & 0 \\
\end{array} \]

**Figure 3.2.** Illustration of the root distribution in Theorem 3.3 (i) when either $\Delta_{\Delta} > \Delta_g$ and $n < n^+$, or $\Delta_{\Delta} \leq \Delta_g$.

\[ \begin{array}{cccccccc}
& n-1 & & n-1 & & 1 & & 1 & \\
x_{\Delta} & J_1 & x_A & J_2 & x_{\Delta}^- & x_B & J_3 & x_y^- & x_y^+ & J_4 & 0 \\
\end{array} \]

**Figure 3.3.** Illustration of the root distribution in Theorem 3.3 (ii).

\[ \begin{array}{cccccccc}
& n-1 & & n & & 1 & & 1 & \\
x_{\Delta} & J_1 & x_A & J_2 & x_{\Delta}^- & x_y^- & J_3 & \xi_{3,2} & \xi_{3,3} & J_4 & x_y^+ \\
\end{array} \]

**Figure 3.4.** Illustration of the root distribution in Theorem 3.3 (iii).

Below are some examples as an application of Theorem 3.3.

**Example 3.4.** Let $c \in \mathbb{R}$. Let $W_n(z)$ be polynomials defined by the recurrence

\[ W_n(z) = (-3z - 5)W_{n-1}(z) + (cz - 1)W_{n-2}(z), \]

with $W_0(z) = 1$ and $W_1(z) = z$. It is routine to compute that

\[ W_2(z) = -3z^2 + (c - 5)z - 1, \]

\[ W_3(z) = 9z^3 + (30 - 2c)z^2 + (27 - 5c)z + 5, \]
\[ W_4(z) = -27z^4 - (3c - 135)z^3 + (c^2 + 20c - 228)z^2 + (23c - 145)z - 24, \]
\[ W_5(z) = 81z^5 - (5c^2 + 45c - 1350)z^4 + (212 - 3c^2)z^3 \]
\[- (10c^2 + 140c - 1545)z^2 + (770 - 105c)z + 115. \]

From definition, one may compute that \( c^- = 1 \) and \( c^+ = 9 \).

- If \( c = 0.8 \), then Theorem 3.3 (i) implies the real-rootedness of every polynomial \( W_n(z) \). For example, \( W_4(z) \) has the real-zeros
  \[ \xi_{4,1} \approx -2.396, \quad \xi_{4,2} \approx -1.446, \quad \xi_{4,3} \approx -0.704, \quad \xi_{4,4} \approx -0.364. \]

- If \( c = 10 \), then every polynomial \( W_{2n}(z) \) is real-rooted by Theorem 3.3 (ii). In this case, we can compute that
  \[ x_g^- = 0.25, \quad \xi_{3.2} \approx 0.251, \quad \xi_{3.3} \approx 0.955, \quad x_g^+ = 1. \]

Thus \( x_g^- < \xi_{3.2} < \xi_{3.3} < x_g^+ \). By Theorem 3.3 (iii), every polynomial \( W_{2n-1}(z) \) is also real-rooted. For example, the zeros of \( W_5(z) \) have the following approximations.

\[ \xi_{5,1} \approx -15.70, \quad \xi_{5,2} \approx -1.962, \quad \xi_{5,3} \approx -0.534, \quad \xi_{5,4} \approx 0.250, \quad \xi_{5,5} \approx 0.999. \]

In fact, when \( c^- > 0 \), then the real-rootedness of the polynomial \( W_3(z) \) implies that \( |R_{3g}^i| = 2 \). This can be seen from Theorem 4.3. When \( c^- < 0 \), nevertheless, it is possible that \( |R_{3g}^i| = 2 \). Below is such an example.

**Example 3.5.** Let \( c \in \mathbb{R} \). Let \( W_n(z) \) be polynomials defined by the recurrence

\[ W_n(z) = (-0.3z - 1)W_{n-1}(z) + (cz - 60)W_{n-2}(z), \]

with \( W_0(z) = 1 \) and \( W_1(z) = z \). From definition, one may compute that \( c^- \approx -16.6 \) and \( c^+ \approx 18.6 \). If \( c = 65 \), then

\[ x_g^- \approx 0.956, \quad \xi_{3.2} \approx 1.014, \quad \xi_{3.3} \approx 1.276, \quad x_g^+ \approx 48.274. \]

Thus \( x_g^- < \xi_{3.2} < \xi_{3.3} < x_g^+ \). By Theorem 3.3, every polynomial \( W_n(z) \) is real-rooted. For example, the zeros of \( W_5(z) \) have the following approximations.

\[ \xi_{5.1} \approx -1844.053, \quad \xi_{5.2} \approx -1255.040, \quad \xi_{5.3} \approx 0.912, \quad \xi_{5.4} \approx 0.958, \quad \xi_{5.5} \approx 4.352. \]

Before ending this section, we explain how one obtains the intervals \( J_i \) in Theorem 3.3. According to Beraha et al.’s result [5], one may calculate all limits of zeros of the polynomials. **Non-isolated limits of zeros** depend only on Recurrence (3.1), while those **isolated** relies on the initial polynomials additionally, that is \( W_0(z) = 1 \) and \( W_1(z) = z \) in the normalised case. In fact, the non-isolated limits have bounds \( x_\Delta^- \) and \( x_\Delta^+ \), and the set of isolated limits consists of the points \( x_g^\pm \).
Now we have the four cutting points $x_A^+ \pm x_g$ in hand. They work already well when the coefficients $A(z)$ and $B(z)$ have lower degrees, see [9, 10] for when one of them is a constant. The coefficients in this paper have both degree one, and we adopt a fifth point $x_A$ to be an end of the desired intervals. In fact, the point $x_A$ is a non-isolated limit of zeros as long as the coefficient $A(z)$ is of degree one, and it is easy to determine the sign of $W_n(x_A)$ for each $n$ by Recurrence (3.1). In general, the increasing order prompts us to introduce more cutting points to divide the real line.

4. PROOFS OF THE MAIN RESULT

The section contains a proof of Theorem 3.3, a result on the root distribution of the polynomial $W_3(z)$ with a proof.

In Lemma 4.1, we list some signs that will be used in the sequel.

**Lemma 4.1.** Let $\{W_n(z)\}_{n \geq 0}$ be a normalised polynomial sequence satisfying Recurrence (3.1) with $a, b, d < 0$ and $c > 0$. Then we have

$$x_\Delta < x_A < 0 < x_B \quad \text{and} \quad x_A < x^+_\Delta < x_B.$$  

It follows that $A(x^-_\Delta) > 0 > A(x^+_\Delta)$. For $n \geq 1$, we have $(-1)^{[n/2]} W_n(x_A) > 0$, $(-1)^n W_n(x_B) < 0$, $W_n(x^-_\Delta) < 0$, and the following.

(i) If $c \leq c^-$, then $x^+_\Delta \leq x^-_g \leq x^+_g < 0$, $W_n(x^+_g)(-1)^n > 0$, and

$$(-1)^n W_n(x^+_\Delta) = \begin{cases} < 0, & \text{if } \Delta_\Delta > \Delta_g \text{ and } n > n^+; \\ = 0, & \text{if } \Delta_\Delta > \Delta_g \text{ and } n = n^+; \\ > 0, & \text{otherwise}. \end{cases}$$

(ii) If $c \geq c^+$, then $x_B < x^-_g$, $W_n(x^+_g) > 0$, and $(-1)^n W_n(x^+_\Delta) < 0$ for $n \geq 2$.

**Proof.** The order relations among the numbers $x^+_\Delta$, $x_A$ and $x_B$ can be derived directly from definition. From the linearity of the function $A(z)$ and that of $h(z)$, we find $A(x^-_\Delta) > 0 > A(x^+_\Delta)$ and $h(x^-_\Delta) < 0$. From Recurrence (3.1), we can deduce that

$$W_n(x_A) = \begin{cases} (c(x_A - x_B))^{n/2}, & \text{if } n \text{ is even}, \\ (c(x_A - x_B))^{(n-1)/2} x_A, & \text{if } n \text{ is odd}, \end{cases}$$

$$(-1)^n W_n(x_B) = (-1)^n (a(x_B - x_A))^{n-1} x_B < 0.$$  

On the other hand, we note that

$$A(x^-_\Delta) + n \cdot h(x^-_\Delta) = 2x^-_\Delta + (n - 1)h(x^-_\Delta) < 0.$$
By Eq. (3.5), we infer that $W_n(x^-) < 0$. We proceed according to the range of $c$.

**Case 4.1.1** ($c \leq c^-$). In this case, the negativities of $x^\pm_g$ can be shown by using Vièta’s formula to the polynomial $g(z)$. It follows from Eq. (3.4) that $W_n(x^\pm_g)(-1)^n > 0$. By Eqs. (3.2) and (3.3), it is routine to verify that

$$x^-_g - x^+_g = \frac{2(\sqrt{\Delta_g} - \sqrt{\Delta_g})^2}{4(1-a)^2 - a^2 + 2ab - 4ac + 4c} \geq 0.$$  

Equally routine can we check that

$$h(x^\pm_g) = \frac{2(\Delta_g - \Delta_g)}{ab - ac + 2c + (2-a)\sqrt{\Delta_g}},$$

which implies that the values $h(x^\pm_g)$ and $\Delta_g - \Delta_g$ have the same sign. The sign of $W_n(x^\pm_g)$ can be determined by Eq. (3.5).

**Case 4.1.2** ($c \geq c^+$). Vièta’s formula gives the positivities of $x^\pm_g$. Then $W_n(x^\pm_g) > 0$. In this case, we have

$$c(b + c) + 2d(1 - a) \geq 2\sqrt{d(a - 1)(c - \sqrt{d(a - 1)})} \geq 2\sqrt{d(a - 1)(\sqrt{d(a - 1)} - b)} > 0.$$  

Thus

$$x^-_g - x_B = \frac{2x_B((a - 1)(d - bc))}{c\sqrt{\Delta_g} + c(b + c) + 2d(1 - a)} > 0.$$  

By using the condition $c \geq c^+$, it is routine to verify that $\Delta_g < \Delta_g$. Denote by $z_h = b/(a - 2)$ the zero of $h(z)$. Then $\Delta(z_h) = 4(\Delta_g - \Delta_g)/(2-a)^2 < 0$. It follows that $z_h < x^+_g$, $h(x^+_g) > 0$ and thus $n^+ > 0$. By Eq. (3.5), we can deduce that

$$(-1)^nW_n(x^+_g) < 0 \iff n > n^+.$$  

It remains to show $n^+ < 2$. It is routine to check that $n^+ = N^+/D^+$, where

$$N^+ = a(\sqrt{\Delta_g} - c) < 0 \quad \text{and} \quad D^+ = (a - 2)(\sqrt{\Delta_g} - c) + ab.$$  

Thus $D^+ < 0$, and the desired inequality $n^+ < 2$ holds if and only if $N^+ > 2D^+$, which simplifies to $R > L$, where

$$L = 2ab - ac + 4c > 0 \quad \text{and} \quad R = (4-a)\sqrt{\Delta_g} > 0.$$  

The remaining proof follows immediately from the fact $R^2 > L^2$.

This completes the proof. \qed

Theorem 4.2 is the aforementioned result that the polynomials $W_n(z)$ have triple piecewise interlacing zeros.

**Theorem 4.2** (Piecewise interlacing zeros). Let $\{W_n(z)\}_n$ be a normalised polynomial sequence satisfying Recurrence (3.1) with $a, b, d < 0$ and $c > 0$. 


(i) If $0 < c \leq c^-$, then the pair $(R_{n+1}, R_n)$ piecewise strictly interlaces on the union $\bigcup_{j=1}^{4}J_j$, and so do the pairs $(R_{2n+2}, R_{2n})$ and $(R_{2n+1}, R_{2n-1})$. As $n \to \infty$, the possible zero in $R_n^{J_4}$ increases, and the zero in $R_n^{J_4}$ decreases.

(ii) If $c \geq c^+$, then the pair $(R_{2n+2}, R_{2n})$ piecewise strictly interlaces on the union $\bigcup_{j=1}^{4}J_j$. As $n \to \infty$, the zero in $R_n^{J_4}$ increases and the zero in $R_n^{J_4}$ decreases.

(iii) If $c > c^+$ and $W_3(z)$ has zeros $\xi_{3,2} \leq \xi_{3,3}$ lying in $J_g$, then the pair $(R_{n+1}, R_n)$ piecewise strictly interlaces on the union $\bigcup_{j=1}^{4}J_j$, and so does the pair $(R_{2n+1}, R_{2n-1})$. As $n \to \infty$, the zero in $R_n^{J_4}$ decreases and the zero in $R_n^{J_4}$ increases.

We shall prove Theorems 3.3 and 4.2 together. From Lemma 4.1, we see that $x_A < x_A < x_g < x_g < 0$ when $c \leq c^-$.

**Proof of Theorem 3.3 (i) and Theorem 4.2 (i).** First of all, we show the results in Theorem 3.3 (i) and Theorem 4.2 (i) except that for the cardinality $|R_n^{J_4}|$.

From Lemma 4.1, we see that $B(x) < 0$ for all negative $x$. Before proceeding by induction, we check the desired results for $n \leq 4$.

- **$n = 1$.** The polynomial $W_1(z) = z$ has the unique zero $0 \in J_4$.
- **$n = 2$.** It is direct to compute that $W_2(0) = d < 0$. By the intermediate value theorem, with aid of Lemma 4.1, the polynomial $W_2(x)$ has zeros
  $$\xi_{2,1} \in J_2, J_3 \quad \text{and} \quad \xi_{2,2} \in J_4.$$

- **$n = 3$.** Since $\xi_{2,j} < 0$ for $j \in [2]$, we infer that $B(\xi_{2,j}) < 0$. Since $W_2(\xi_{2,3}) = 0$, we deduce from Recurrence (3.1) that
  $$W_3(\xi_{2,j}) = B(\xi_{2,j})W_1(\xi_{2,j}) = B(\xi_{2,j})\xi_{2,j} > 0.$$  

  For the same reason, the polynomial $W_3(x)$ has zeros
  $$\xi_{3,1} \in J_1, \quad \xi_{3,2} \in (\xi_{2,1}, x_g) \subset J_2 \cup J_3, \quad \text{and} \quad \xi_{3,3} \in (x_g, \xi_{2,2}) \subset J_4.$$

- **$n = 4$.** By Lemma 3.2 and the facts $W_2(\xi_{2,1}) = 0$ and $W_0(z) = 1$, we infer that
  $$W_4(\xi_{2,1}) = -B^2(\xi_{2,1})W_0(\xi_{2,1}) = -B^2(\xi_{2,1}) < 0.$$  

  Since $\xi_{3,j} < 0$ for $j \in [3]$, we obtain $B(\xi_{3,j}) < 0$. By Recurrence (3.1) and together with the facts $W_3(\xi_{3,j}) = 0$ and
  $$\text{sgn}(W_2(\xi_{3,j})) = \begin{cases} 
  -1, & \text{if } j = 1; \\
  1, & \text{if } j = 2, 3,
  \end{cases}$$

  we infer that
  $$\text{sgn}(W_4(\xi_{3,j})) = \text{sgn}(B(\xi_{3,j})W_2(\xi_{3,j})) = \begin{cases} 
  1, & \text{if } j = 1; \\
  -1, & \text{if } j = 2, 3.
  \end{cases}$$
Again, we can deduce that $W_4(z)$ has a zero each of the following intervals: 
\((x^-_\Delta, \xi_{3,1}) \subset J_1, \ (x_A, \xi_{2,1}) \subset J_2 \cup J_3, \ (\xi_{3,2}, x^-_g) \subset J_2 \cup J_3, \ and \ (x^+_g, \xi_{3,3}) \subset J_4.\)

Let \(n \geq 5.\) We proceed by induction. Suppose that the polynomial $W_{n-2}(z)$ has zeros \(x_1 < \cdots < x_{n-2},\) and that the polynomial $W_{n-1}(z)$ has zeros \(y_1 < \cdots < y_{n-1}.\) The induction hypotheses on the interlacing zeros give us immediately
\[
\text{sgn}(W_{n-4}(x_j)) = \begin{cases} 
(-1)^j, & \text{if } j \in [(n-3)/2]; \\
(-1)^{j+1}, & \text{if } [(n-1)/2] \leq j \leq n-3.
\end{cases}
\]
Note that \(B(x_j) \neq 0.\) By Lemma 3.2 and the fact $W_{n-2}(x_j) = 0$, we infer that
\[
\text{sgn}(W_n(x_j)) = \text{sgn}(-B^2(x_j)W_{n-4}(x_j)) = \begin{cases} 
(-1)^{j+1}, & \text{if } j \in [(n-3)/2]; \\
(-1)^j, & \text{if } [(n-1)/2] \leq j \leq n-3.
\end{cases}
\]
On the other hand, by Recurrence (3.1), the facts $W_{n-1}(y_{n-1}) = 0$ and $B(y_{n-1}) < 0$, and the interlacing zeros by induction, we deduce that
\[
W_{n-2}(x^+_g)W_{n-2}(y_{n-1}) = W_{n-2}(x^+_g) \cdot (B(y_{n-1})W_{n-2}(y_{n-1})) = B(y_{n-1}) \cdot (W_{n-2}(x^+_g)W_{n-2}(y_{n-1})) < 0.
\]
By the intermediate value theorem and with the aid of Lemma 4.1, the polynomial $W_n(z)$ has zeros $z_j (j \in [n])$ such that
\[
x^-_\Delta < z_1 < x_1 < z_2 < x_2 < \cdots < x_{[(n-3)/2]} < z_{[(n-1)/2]} < x_A < z_{[(n+1)/2]} < x_{[(n+3)/2]} < z_{[(n+1)/2]} < \cdots < x_{n-1} < z_{n-1} < x^-_g < z_n < y_{n-1} < x_{n-2} < \cdots \leq 0.
\]
Since \(\deg W_n(z) = n,\) we conclude that the zeros $z_j (j \in [n])$ constitute the zero set of $W_n(z)$. This proves the real-rootedness of $W_n(z)$, verifies the cardinalities $|R^L_n|$, $|R^{L \cup J_1}_n|$, and $|R^{J_1}_n|$, and establishes the desired interlacing properties of the sequences. By induction, we have $W_{n-2}(y_j)(-1)^j > 0$ for $j \in [n-2]$. By Recurrence (3.1) and the facts $W_{n-1}(y_j) = 0$ and $B(y_j) < 0$, we derive
\[
W_n(y_j)(-1)^j = B(y_j)W_{n-2}(y_j)(-1)^j < 0.
\]
For the same reason, we obtain that $W_n(z)$ and $W_{n-1}(z)$ have interlacing zeros over each of the intervals $J_j$.

Now we are going to determine the cardinality $|R^{J_1}_n|$.

When $\Delta_\Delta \leq \Delta_g$, from Lemma 4.1, we see that the numbers $W_n(x^+_\Delta)$ and $W_n(x^-_g)$ have the same sign. This leads us to replace the number $x^-_g$ by $x^+_\Delta$ in the foregoing proof, and the interval $J_2 \cup J_3$ becomes $J_2$. The whole deduction remains true,
which changes the conclusion \( |R_{n}^{J_{2} \cup J_{3}}| = \lfloor n/2 \rfloor \) to \( |R_{n}^{J_{2}}| = \lfloor n/2 \rfloor \). Thus \( R_{n}^{J_{4}} = \emptyset \) as desired.

When \( \Delta_{\Delta} > \Delta_{g} \), the desired cardinality can be determined in a way highly similar to the proof of Theorem 4.5 in [10]. We provide a proof sketch for completeness. In fact, we can deduce that \( h(x_{\Delta}^{+}) > 0, n^{+} > 0, \) and \( W_{n}(x_{\Delta}^{+})W_{n}(x_{g}^{+})(n - n^{+}) \leq 0 \), with the equality holds if and only if \( n = n^{+} \). The interlacing property over the interval \( J_{2} \cup J_{3} \) implies \( |R_{n+1}^{J_{3}}| - |R_{n}^{J_{3}}| \leq 1 \). By bootstrapping and with the aid of the intermediate value theorem, one may show that \( R_{n}^{J_{3}} = \emptyset \) when \( n < n^{+} \); the number \( x_{\Delta}^{+} \) is the unique zero of \( W_{n}(z) \) in \( J_{3} \) in case \( n = n^{+} \); and \( |R_{n}^{J_{3}}| = 1 \) for \( n > n^{+} \). This completes the proofs of Theorem 3.3 (i) and Theorem 4.2 (i).

When \( c \geq c^{+} \), recall from Lemma 4.1 that \( x_{\Delta}^{-} < x_{A} < x_{\Delta}^{+} < x_{B} < x_{g}^{-} \leq x_{g}^{+} \). The idea of piecewise interlacing still works for proving Theorem 3.3 (ii) and Theorem 4.2 (ii).

**Proof of Theorem 3.3 (ii) and Theorem 4.2 (ii).** By Lemma 4.1, we have \( W_{2}(x_{\Delta}^{+}) < 0 \). By using the facts \( W_{2}(z) = z^{2} - g(z) \), it is elementary to verify the truth for \( n = 1 \) that \( W_{2}(z) \) has a zero in the interval \( J_{3} \) and another zero in \( J_{4} \). Let \( n \geq 2 \). By induction, we can suppose that the polynomial \( W_{2n-2}(z) \) has zeros \( \xi_{1}, \ldots, \xi_{2n-2} \) such that

\[
x_{\Delta}^{-} < \xi_{1} < \cdots < \xi_{n-2} < x_{A} < \xi_{n-1} < \cdots < \xi_{2n-4} < x_{\Delta}^{+} < \xi_{2n-3} < x_{g}^{-} \leq x_{g}^{+} < \xi_{2n-2}.
\]

Similar to the proof of Theorem 3.3 (i), we can deduce that

\[
\text{sgn}(W_{2n}(\xi_{j})) = \begin{cases} 
(-1)^{j+1}, & \text{if } j \in [n-2]; \\
(-1)^{j}, & \text{if } n-1 \leq j \leq 2n-4; \\
-1, & \text{if } j = 2n-3, 2n-4.
\end{cases}
\]

From Lemma 4.1, we see that

\[
W_{2n}(x_{\Delta}^{-}) < 0, \quad W_{2n}(x_{A})(-1)^{n} > 0, \quad W_{2n}(x_{\Delta}^{+}) < 0 \quad \text{and} \quad W_{2n}(x_{g}^{+}) > 0.
\]

The remaining of the proof follows from the intermediate value theorem. \( \square \)

Theorem 3.3 (iii) and Theorem 4.2 (iii) can be shown along the same vein, and we omit their proofs.

Since \( W_{3}(x_{\Delta}^{-}) < 0 < W_{3}(x_{A}) \), we infer that the polynomial \( W_{3}(z) \) has a zero in the interval \( (x_{\Delta}^{-}, x_{A}) \subset \mathbb{R}^{-} \). It is direct to compute that \( W_{3}(0) = bd \neq 0 \). Theorem 4.3 explains the extent to which the condition on the root distribution of \( W_{3}(z) \) in Theorem 3.3 (iii) is acceptable. Recall that \( J_{g} = (x_{g}^{-}, x_{g}^{+}) \).

**Theorem 4.3.** Let \( \{W_{n}(z)\}_{n} \) be a normalised polynomial sequence satisfying Recurrence (3.1) with \( a, b, d < 0 \) and \( c \geq c^{+} \). Then the polynomial \( W_{3}(z) \) has a
unique negative zero. If it has a positive zero lying outside the interval $J_g$, then it has two zeros lying in the interval $(-b/(a+1), x_0)$, where $x_0$ is the positive zero of the quadratic polynomial $A^2(z) + B(z)$, and we have

$$x_0 < x^+_{\Delta}, \quad a > -1, \quad c < \frac{b}{a+1} + \frac{a+1}{b} \cdot d \quad \text{and} \quad c^- < 0.$$ 

When the polynomial $W_3(z)$ is real-rooted, its root distribution is illustrated in Figs. 4.1 and 4.2.

![Figure 4.1](image1)

**Figure 4.1.** Illustration for the root distribution of $W_3(z)$ when two of its zeros lie in the interval $J_g$.

![Figure 4.2](image2)

**Figure 4.2.** Illustration for the root distribution of $W_3(z)$, when $W_3(z)$ has a positive zero lying outside the interval $J_g$. It turns out that it has two positive zeros lying in the interval $(-b/(a+1), x_0)$.

**Proof.** By Recurrence (3.1), one may compute that $W_3(z) = a^2z^3 + Uz^2 + Vz + bd$, where $U = 2ab + ac + c$ and $V = ad + bc + b^2 + d$. Suppose that $c \geq c^+$. Let $\{x_1, x_2, x_3\}$ be the zero set of $W_3(z)$ such that $x_1 \in (x_{\Delta}, x_A)$. By Vieta’s formula, we have

$$x_1 + x_2 + x_3 = -U/a^2, \quad (4.1)$$

$$x_1 x_2 + x_1 x_3 + x_2 x_3 = V/a^2, \quad (4.2)$$

$$x_1 x_2 x_3 = -bd/a^2 > 0. \quad (4.3)$$

From Eq. (4.3) and the fact $x_1 < 0$, we infer that the zeros $x_2$ and $x_3$ have the same sign. We shall show that both $x_2$ and $x_3$ are positive.

Assume, to the contrary, that $x_2, x_3 < 0$. Then Eq. (4.1) implies $U > 0$, and Eq. (4.2) implies $V > 0$. On one hand, since

$$ad + b^2 + d = V - bc > -bc \geq -bc^+ = -b(2\sqrt{d(a-1)} - b),$$

that is,

$$\quad (a+1)\sqrt{-d} < 2b\sqrt{1-a}, \quad (4.4)$$
we infer that \( a + 1 < 0 \). On the other hand,
\[
2ab = U - (a + 1)c > -(a + 1)c \geq -(a + 1)c^+ = -(a + 1)(2\sqrt{d(a - 1)} - b),
\]
that is, \( b\sqrt{1-a} < 2(a+1)\sqrt{-d} \). Combining it with Ineq. (4.4), we find \( a + 1 > 0 \), a contradiction. Therefore, both the zeros \( x_2 \) and \( x_3 \) are positive. This proves the uniqueness of the negative zero \( x_1 \).

Suppose that \( \{x_2, x_3\} \setminus J_g \neq \emptyset \). From Lemma 4.1, we see that \( W_n(x_g^+) > 0 \). By the intermediate value theorem, the number of zeros of \( W_3(z) \) in \( J_g \) is even. Thus \( \{x_2, x_3\} \cap J_g = \emptyset \). Let \( \xi \in \{x_2, x_3\} \). Then \( \xi > 0 \). Since the polynomial \( g(z) \) is quadratic with leading coefficient \( 1 - a > 0 \), and since \( \xi \notin J_g \), we infer that \( g(\xi) > 0 \).

Let \( F(z) = A^2(z) + B(z) \). By Recurrence (3.1), one may deduce that
\[
0 = W_3(\xi) = A(\xi)W_2(\xi) + B(\xi) \cdot \xi = A(\xi) \cdot (A(\xi) \cdot \xi + B(\xi)) + B(\xi) \cdot \xi \tag{4.5}
\]
\[
= A^2(\xi) \cdot \xi + B(\xi) \cdot (A(\xi) + \xi) \tag{4.6}
\]
It follows that \( F(\xi) \neq 0 \) and \( \xi = -A(\xi)B(\xi)/F(\xi) \). Thus \( g(\xi) = -B^2(\xi)/F(\xi) \). Since \( g(\xi) > 0 \), we deduce that \( B(\xi) < 0 \). By Eq. (4.5), we infer that \( A(\xi) + \xi > 0 \), which implies that \( a > -1 \) and \( \xi > -b/(a + 1) \).

Since \( x_A < 0 < \xi \), we have \( A(\xi) < 0 \). By Eq. (4.6), we infer that \( F(\xi) < 0 \). Note that \( F(z) = a^2z^2 + (2ab + c)z + (b^2 + d) \) is quadratic with positive leading coefficient. The fact \( F(\xi) < 0 \) implies that \( F(z) \) is real-rooted and it has a zero larger than \( \xi > 0 \). Applying Viéta’s formula on \( F(z) \), we obtain that \( F(z) \) has a negative zero. Thus \( F(z) \) has a unique positive zero, say, \( x_0 \), and \( x_0 > \xi \). From Lemma 4.1, we see that \( x_A^+ < x_B \). Thus
\[
F(x_A^+) = \Delta(x_A^+) - 3B(x_A^+) = -3B(x_A^+) > 0.
\]
Thus \( x_0 < x_A^+ \). Since the polynomial \( F(z) \) is increasing in the interval \((0, +\infty)\) and \( F(\xi) < 0 \), we infer that \( F(-b/(a + 1)) < 0 \), that is,
\[
c < \frac{b}{a+1} + \frac{a+1}{b} \cdot d, \quad \text{or equivalently,} \quad d < \frac{bc(a + 1) - b^2}{(a + 1)^2}.
\]
It follows that \( d < -b^2 \), and \( c^- < 0 \) from definition. This completes the proof. \( \square \)

Continuing Example 3.5, we illustrate Theorem 4.3 by presenting that \( W_3(z) \) may have two positive zeros in the interval \((-b/(a + 1), x_0) \) and that \( W_3(z) \) may have non-real zeros.

**Example 4.4.** Let \( c \in \mathbb{R} \). Let \( W_n(z) \) be polynomials defined by the recurrence
\[
W_n(z) = (-0.3z - 1)W_{n-1}(z) + (cz - 60)W_{n-2}(z),
\]
with $W_0(z) = 1$ and $W_1(z) = z$. Then $c^- \approx -16.6$ and $c^+ \approx 18.6$.

- If $c = 20$, then $c > c^+$, and every polynomial $W_{2n}(z)$ is real-rooted by Theorem 3.3 (ii). For example, $W_4(z)$ has the real zeros 
  
  $\xi_{4,1} \approx -423.39, \quad \xi_{4,2} \approx 2.89, \quad \xi_{4,3} \approx 3.67, \quad \xi_{4,4} \approx 29.03.

  Moreover, we can compute that
  
  $-b/(a + 1) \approx 1.42, \quad \xi_{3,2} \approx 1.61, \quad \xi_{3,3} \approx 2.48, \quad x_0 \approx 2.82.

  Thus $-b/(a + 1) < \xi_{3,2} < \xi_{3,3} < x_0$. In this case, the polynomial $W_5(z)$ has real zeros approximately $2.93, -47.44, \text{ and } -574.73$, and non-real zeros approximately $2.95 \pm 1.52i$.

- If $c = 30$, then $W_3(z)$ has non-real zeros approximately $1.63 \pm 0.30i$.

5. Concluding remarks

With a bit more work on the root distribution of the polynomial $W_3(z)$ in Theorem 4.3, one may obtain Theorem 5.1.

**Theorem 5.1.** Let $\{W_n(z)\}_n$ be a normalised polynomial sequence satisfying Recurrence (3.1) with $a, b, d < 0$ and $c \in \mathbb{R}$. Then every polynomial $W_n(z)$ is real-rooted if the number $c$ is sufficiently large.

**Proof.** By routine computation, one sees that the discriminant of the polynomial $W_3(z)$ is a quartic polynomial in $c$ with a positive leading coefficient. Thus there exists $N$ such that the discriminant is positive as if $c > N$. Let

$$c^* = \max\left(c^+, \frac{b}{a + 1} + \frac{a + 1}{b} \cdot d\right).$$

Suppose that $c > c^*$. Then $W_3(z)$ is real-rooted, because of the positivity of its discriminant. By Theorem 4.3, $W_3(z)$ has two positive zeros in the interval $J_g$. By Theorem 3.3 (ii) and (iii), every polynomial $W_n(z)$ is real-rooted. \(\square\)

We end this paper by proposing Conjecture 5.2, which is partially supported by Theorem 5.1.

**Conjecture 5.2.** Let $\{W_n(z)\}_n$ be a normalised polynomial sequence satisfying Recurrence (3.1) with $a, b, d < 0$ and $c > 0$. Then every polynomial $W_n(z)$ has at most two non-real zeros.

We notice that the polynomial sequence $\{W_n(z)\}_n$ satisfying Recurrence (3.1) can be written as $W_n(z) = f_n(z) + g_n(z)$, where $\{f_n(z)\}_n$ is a polynomial sequence whose root distribution has been studied by Tran [25] and $\{g_n(z)\}_n$ is a polynomials
sequence whose root distribution has been studied by Mai [17]. We did not find a modification of any proof of theirs which may show Theorem 3.3.

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†School of Mathematics and Statistics, Beijing Institute of Technology, 102488 Beijing, P. R. China, ‡Beijing Key Laboratory on MCAACI, Beijing Institute of Technology, 102488 Beijing, P. R. China

E-mail address: david.combin@gmail.com

E-mail address: jrzhang.combin@gamil.com