Equidistribution in Shrinking Sets and 

$L^4$-Norm Bounds for Automorphic Forms

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Abstract

This thesis deals with two closely related problems stemming from the random wave conjecture for Maass forms. The first problem is quantum unique ergodicity in shrinking sets; we show that by averaging over the centre of hyperbolic balls in $\Gamma \backslash \mathbb{H}$, quantum unique ergodicity holds for almost every shrinking ball whose radius is larger than the Planck scale. This result is conditional on the generalised Lindelöf hypothesis for Maass eigenforms but is unconditional for Eisenstein series. We also show that equidistribution for Maass eigenforms need not hold at or below the Planck scale. The second problem is bounding the $L^4$-norm of a Maass form in the large eigenvalue limit; we complete the work of Spinu [Spi03] to show that the $L^4$-norm of an Eisenstein series $E(z, 1/2 + it_g)$ restricted to compact sets is bounded by $\sqrt{\log t_g}$. 
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Chapter 1

Introduction

1.1 Automorphic Forms

Let $\kappa \in \{0, 1\}$, let $q$ be a positive integer, and let $\chi$ be a Dirichlet character modulo $q$ of conductor $q_\chi$, where $q_\chi$ divides $q$, satisfying $\chi(-1) = (-1)^\kappa$. We let $\Gamma = \text{SL}_2(\mathbb{Z})$ and let

$$\Gamma_0(q) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{q} \right\}$$

denote the Hecke congruence subgroup of level $q$. Consider the space of smooth functions $f : \mathbb{H} \rightarrow \mathbb{C}$ from the upper half-plane to the complex numbers that are of moderate growth and satisfy $f(\gamma z) = \chi(\gamma) j_\gamma(z)^\kappa f(z)$ for all $\gamma \in \Gamma_0(q)$ and $z = x + iy \in \mathbb{H}$, where for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q)$,

$$\gamma z := \frac{az + b}{cz + d},$$

$$\chi(\gamma) := \chi(d),$$

$$j_\gamma(z) := \frac{cz + d}{|cz + d|}.$$
We denote by $L^2(\Gamma_0(q) \backslash \mathbb{H}, \kappa, \chi)$ the $L^2$-completion of this space with respect to the inner product

$$\langle f, g \rangle_q := \int_{\Gamma_0(q) \backslash \mathbb{H}} f(z) \overline{g(z)} \, d\mu(z),$$

where $d\mu(z)$ is equal to $\frac{dx \, dy}{y^2}$ on any fundamental domain of $\Gamma_0(q) \backslash \mathbb{H}$. When $q = 1$, we simply write $\langle \cdot, \cdot \rangle$. Note that

$$\text{vol} \left( \Gamma_0(q) \backslash \mathbb{H} \right):= \int_{\Gamma_0(q) \backslash \mathbb{H}} d\mu(z) = \text{vol} \left( \Gamma \backslash \mathbb{H} \right) \left[ \Gamma: \Gamma_0(q) \right] = \frac{\pi}{3} q \prod_{p | q} \left( 1 + \frac{1}{q} \right),$$

so that $\Gamma_0(q) \backslash \mathbb{H}$, though noncompact, has finite volume.

As detailed in [DFI02, Section 4], the space $L^2(\Gamma_0(q) \backslash \mathbb{H}, \kappa, \chi)$ has the spectral decomposition

$$L^2(\Gamma_0(q) \backslash \mathbb{H}, \kappa, \chi) = \mathcal{A}_\kappa(q, \chi) \oplus \mathcal{E}_\kappa(q, \chi)$$

with respect to the weight $\kappa$ Laplacian

$$\Delta_\kappa = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + i\kappa y \frac{\partial}{\partial x}.$$

To describe these spaces, we first define the stabiliser

$$\Gamma_a := \{ \gamma \in \Gamma_0(q) : \gamma a = a \}$$

of a cusp $a$ of $\Gamma_0(q) \backslash \mathbb{H}$. There exists a scaling matrix $\sigma_a \in \text{SL}_2(\mathbb{R})$, unique up to translation on the right, such that

$$\sigma_a \infty = a, \quad \sigma_a^{-1} \Gamma_\infty \sigma_a = \Gamma_a.$$

Then $\gamma_a := \sigma_a \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right) \sigma_a^{-1}$ is a parabolic element of $\Gamma_0(q)$ that generates $\Gamma_a$. A cusp $a$ of $\Gamma_0(q) \backslash \mathbb{H}$ is said to be singular with respect to $\chi$ if $\chi(\gamma_a) = 1$. 

2
The subspace $A_\kappa(q, \chi)$ of $L^2(\Gamma_0(q) \backslash \mathbb{H}, \kappa, \chi)$ is the $L^2$-closure of the span of the set of all Maaß cusp forms of weight $\kappa$, level $q$, and nebentypus $\chi$. These are smooth functions $f$ in $L^2(\Gamma_0(q) \backslash \mathbb{H}, \kappa, \chi)$ that are eigenfunctions of $\Delta_\kappa$ with eigenvalue $\lambda_f = 1/4 + t_f^2$, where $t_f$ is the spectral parameter of $f$, and satisfy

$$\int_0^1 j_\sigma(z)^{-\kappa} f(z) \, dx = 0$$

for every singular cusp $a$. The Fourier expansion of a Maaß cusp form is

$$f(z) = \sum_{n=-\infty}^{\infty} \rho_f(n)W_{\text{sgn}(n)}(4\pi|n|y)e(nx),$$

where

$$\rho_f(n)W_{\text{sgn}(n)}(4\pi|n|y) = \left(\int_0^1 f(z)e(-nx) \, dx\right)$$

and $W_{\alpha,\beta}$ is the Whittaker function. The Laplacian eigenvalue $\lambda_f = 1/4 + t_f^2$ of a Maaß cusp form is positive, so either $t_f \in [0, \infty)$ or $it_f \in (0, 1/2)$. In the latter case, $\lambda_f$ is said to be an exceptional eigenvalue, and Selberg’s eigenvalue conjecture states that this can never occur. When $q = 1$, this conjecture is known to hold.

For a singular cusp $a$, we define the Eisenstein series

$$E_a(z, s, \chi) := \sum_{\gamma \in \Gamma_a \setminus \Gamma_0(q)} \chi(\gamma)j_{\sigma_a^{-1} \gamma}(z)^{-\kappa} \Im(\sigma_a^{-1} \gamma z)^s,$$

which is absolutely convergent for $\Re(s) > 1$ and extends meromorphically to $\mathbb{C}$, with the Fourier expansion

$$\delta_{a,\infty}y^{1/2+it} + \varphi_{a,\infty}\left(\frac{1}{2} + it, \chi\right)y^{1/2-it} + \sum_{n=-\infty}^{\infty} \sum_{n \neq 0} \rho_a(n, t, \chi)W_{\text{sgn}(n)}(4\pi|n|y)e(nx)$$
for \( s = 1/2 + it \) with \( t \in \mathbb{R} \setminus \{0\} \), where

\[
\delta_{\alpha, \infty} y^{1/2 + it} + \varphi_{\alpha, \infty} \left( \frac{1}{2} + it, \chi \right) y^{1/2 - it} = \int_{0}^{1} E_{\alpha} \left( z, \frac{1}{2} + it, \chi \right) \, dx,
\]

\[
\rho_{\alpha}(n, t, \chi) W_{\text{sgn}(n)} \frac{\pi}{2} e(it) = \int_{0}^{1} E_{\alpha} \left( z, \frac{1}{2} + it, \chi \right) e(-nx) \, dx.
\]

The subspace \( E_{\kappa}(q, \chi) \) consists of functions \( g \in L^{2}(\Gamma_{0}(q) \setminus \mathbb{H}, \kappa, \chi) \) that are orthogonal to every Maaß cusp form \( f \in A_{\kappa}(q, \chi) \); it is the \( L^{2} \)-closure of the space spanned by incomplete Eisenstein series, which are functions of the form

\[
E_{\alpha}(z, \psi, \chi) := \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} E_{\alpha}(z, s, \chi) \hat{\psi}(s) \, ds
\]

for some singular cusp \( \alpha \) and some smooth function of compact support \( \psi: \mathbb{R}^{+} \to \mathbb{C} \), where \( \sigma > 1 \) and

\[
\hat{\psi}(s) := \int_{0}^{\infty} \psi(x) x^{-s} \frac{dx}{x}.
\]

**Lemma 1.1.1** ([IK04, Theorem 15.5]). Let

\[
f_{0}(z) := \frac{1}{\sqrt{\text{vol}(\Gamma_{0}(q) \setminus \mathbb{H})}},
\]

so that \( \langle f_{0}, f_{0} \rangle_{q} = 1 \), and let \( B_{\kappa}(q, \chi) \) be an orthonormal basis of Maaß cusp forms of \( A_{\kappa}(q, \chi) \). Then a function \( g \in L^{2}(\Gamma_{0}(q) \setminus \mathbb{H}, \kappa, \chi) \) has the spectral expansion, valid in the \( L^{2} \)-sense, of the form

\[
g(z) = \langle g, f_{0} \rangle_{q} \delta_{\chi, \chi_{0}(q)} + \sum_{f \in B_{\kappa}(q, \chi)} \langle g, f \rangle_{q} f(z)
\]

\[
+ \sum_{\alpha} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\langle g, E_{\alpha} \left( \frac{1}{2} + it, \chi \right) \right\rangle \, E_{\alpha} \left( z, \frac{1}{2} + it, \chi \right) \, dt,
\]

where \( \delta_{\chi, \chi_{0}(q)} \) is equal to 1 if \( \chi \) is the principal character modulo \( q \) and is equal to 0
otherwise. Moreover, we have Parseval’s identity
\[
\langle g_1, g_2 \rangle_q = \langle g_1, f_0 \rangle_q \langle f_0, g_2 \rangle_q \delta_{\chi, \chi_0(q)} + \sum_{f \in B_\kappa(q, \chi)} \langle g_1, f \rangle_q \langle f, g_2 \rangle_q \\
+ \sum_a \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\langle g_1, E_a \left( \cdot, \frac{1}{2} + it, \chi \right) \right\rangle_q \left\langle E_a \left( \cdot, \frac{1}{2} + it, \chi \right), g_2 \right\rangle_q \, dt
\]
for \( g_1, g_2 \in L^2 \left( \Gamma_0(q) \backslash \mathbb{H}, \kappa, \chi \right) \).

We let \( A^*_\kappa(q, \chi) \) denote the subspace of \( A_\kappa(q, \chi) \) spanned by \( B^*_\kappa(q, \chi) \), where \( B^*_\kappa(q, \chi) \) denotes the set of Maaß newforms \( f \in A_\kappa(q, \chi) \) normalised such that
\[
\|f\|^2_{L^2(\Gamma_0(q) \backslash \mathbb{H})} = \langle f, f \rangle_q = 1.
\]
Maaß newforms \( f \) are eigenfunctions of the weight \( \kappa \) Laplacian \( \Delta_\kappa \) with eigenvalue \( \lambda_f = 1/4 + t_f^2 \). Maaß newforms are additionally eigenfunctions of every Hecke operator \( T_n \) with eigenvalue \( \lambda_f(n) \), where for \( n \geq 1 \),
\[
(T_n f)(z) := \frac{1}{\sqrt{n}} \sum_{a \text{ mod } n} \chi(a) \sum_{b \text{ mod } d} f \left( \frac{a z + b}{d} \right).
\]
Moreover, a Maaß newform is an eigenfunction of the operator \( Q_{1/2+i t_f, \kappa} \) as defined in [DFI02, Section 4], with eigenvalue \( \epsilon_f \in \{-1, 1\} \); we say that \( f \) is even if \( \epsilon_f = 1 \) and \( f \) is odd if \( \epsilon_f = -1 \). When \( \kappa = 0 \), the Fourier coefficients of a Maaß newform \( f \) satisfy \( \rho_f(n) = \epsilon_f \rho_f(-n) \) for every integer \( n \).

When \( \kappa = 0 \) and \( \chi = \chi_0 \) is the principal character, we write \( E_a(z, s) \) in place of \( E_a(z, s, \chi_0) \), and write all these spaces in the form \( L^2 \left( \Gamma_0(q) \backslash \mathbb{H} \right), A_0 \left( \Gamma_0(q) \right), E_0 \left( \Gamma_0(q) \right), A_0^* \left( \Gamma_0(q) \right), \) and \( B_0^* \left( \Gamma_0(q) \right) \) for brevity’s sake.
Randomness of Maaß Newforms

1.2.1 Random Wave Conjecture

A well-known conjecture of Berry [Ber77] and Hejhal and Rackner [HejRa92] states that a Maaß newform \( g \in \mathcal{B}_\kappa^*(q, \chi) \) of large Laplacian eigenvalue \( \lambda_g \) ought to behave like a random wave. Here by a random wave, we mean a function of the form

\[
g_{\lambda}(z) = \sum_{\lambda \leq \lambda_f \leq \lambda + \eta(\lambda)} c_f f(z),
\]

where \( \eta(\lambda) \to \infty \) as \( \lambda \to \infty \) and \( \eta(\lambda) = o(\lambda) \), each \( f \) is a normalised Maaß newform, and the coefficients \( c_f \) are independent Gaussian random variables of mean 0 and variance 1. These are a randomised model of eigenfunctions of the Laplacian in the large eigenvalue limit \( \lambda \to \infty \), and it is easier to prove (almost surely) results for random waves than for true eigenfunctions.

For \( \Gamma_0(q) \backslash \mathbb{H} \), there are situations in which random waves do not behave precisely like Laplacian eigenfunctions: random waves satisfy \( \sup_{w \in K} |g_{\lambda}(w)| \asymp_K \sqrt{\log \log \lambda} \) almost surely for every compact subset \( K \), whereas Milićević [Mil10, Theorem 1] proved the existence of a dense subset of points \( w \in \Gamma_0(q) \backslash \mathbb{H} \) for which Maaß newforms \( g \in \mathcal{B}_0^*(\Gamma_0(q)) \) may be much larger than \( \sqrt{\log \log \lambda_g} \). Nonetheless, it is conjectured that for Laplacian eigenfunctions should, on the whole, be well-modelled by random waves. This (admittedly quite vague) conjecture is known as the random wave conjecture.

1.2.2 Gaussian Distribution Conjecture

A particular manifestation of the random wave conjecture states that the distribution of a Maaß newform \( g \in \mathcal{B}_\kappa^*(q, \chi) \) should be that of a Gaussian random variable in the large eigenvalue limit. More precisely, when the nebentypus \( \chi \) of \( g \) is principal, so that \( g \) is real-valued, the distribution of \( g \) should converge to that of a real Gaussian
random variable as $\lambda_g$ tends to infinity, while when $\chi$ is nonprincipal, so that $g$ is necessarily complex-valued, the distribution of $g$ should converge to that of a complex Gaussian random variable.

We recall that a sequence of probability measures $\{\nu_n\}$ on a topological space $X$ converges in distribution to a probability measure $\nu$ on $X$ if

$$\lim_{n \to \infty} \int_X f(x) \, d\nu_n(x) = \int_X f(x) \, d\nu(x)$$

for every bounded continuous function $f : X \to \mathbb{C}$. By the Portmanteau theorem, this is equivalent to

$$\lim_{n \to \infty} \nu_n(B) = \nu(B)$$

for every continuity set $B \subset X$; recall that a continuity set $B$ is a Borel set whose boundary has $\nu$-measure zero.

In [HejRa92, Section 5], the conjectural randomness of newforms is quantified as follows. Let $K$ be any fixed compact continuity set of $\Gamma_0(q) \backslash \mathbb{H}$, and for each $g \in \mathcal{B}_0^* (\Gamma_0(q))$ normalised such that $\langle g, g \rangle_q = 1$, define the probability measure $\nu_{g,K}$ by

$$\nu_{g,K}(B) := \frac{\text{vol}(\{z \in K : \text{Var}_K(g)^{-1/2} g(z) \in B\})}{\text{vol}(K)}$$

for every Lebesgue-measurable subset $B$ of $\mathbb{R}$, where

$$\text{Var}_K(g) := \frac{1}{\text{vol}(K)} \int_K |g(z)|^2 \, d\mu(z).$$

Similarly, if $\chi$ is nonprincipal, then for each $g \in \mathcal{B}_*^* (q, \chi)$ normalised such that $\langle g, g \rangle_q = 1$, $\nu_{g,A}$ is defined analogously on $\mathbb{C}$.

**Conjecture 1.2.1** (Gaussian Distribution Conjecture). For any fixed compact continuity set $K$ of $\Gamma_0(q) \backslash \mathbb{H}$, and for $g \in \mathcal{B}_0^* (\Gamma_0(q))$ normalised such that $\langle g, g \rangle_q = 1$, the sequence of probability measures $\nu_{g,K}$ on $\mathbb{R}$ converges in distribution as the spectral
parameter $t_g$ of $g$ tends to infinity to the probability measure of a real Gaussian random variable with mean 0 and variance 1.

Similarly, for any fixed compact continuity set $K$ of $\Gamma_0(q) \backslash \mathbb{H}$, and $\chi$ nonprincipal and $g \in \mathcal{B}_k^\chi(q,\chi)$ normalised such that $\langle g, g \rangle_q = 1$, the sequence of probability measures $\nu_{g,K}$ on $\mathbb{C}$ converges in distribution as $t_g$ tends to infinity to the probability measure of a complex Gaussian random variable with mean 0 and variance 1.

In particular, Hejhal and Rackner conjecture that $\nu_{g,K}$ converges in distribution to a probability measure that is independent of $A$. Note that the spectral parameter $t_g$ of $g$ tending to infinity is equivalent to the Laplacian eigenvalue $\lambda_g = 1/4 + t_g^2$ of $g$ tending to infinity.

1.2.3 Gaussian Moments Conjecture

Another way to quantify the random wave conjecture is to study the moments of a newform.

**Conjecture 1.2.2 (Gaussian Moments Conjecture).** Let $K$ be any fixed compact continuity set of $\Gamma_0(q) \backslash \mathbb{H}$, and let $g \in \mathcal{B}_0^\chi(\Gamma_0(q))$ be normalised such that $\langle g, g \rangle_q = 1$. Then for every nonnegative integer $n$,

$$\frac{1}{\text{Var}_K(g)^{n/2} \text{vol}(K)} \int_K g(z)^n \, d\mu(z)$$

converges to

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}} \, dx = \begin{cases} \frac{2^{n/2}}{\sqrt{\pi}} \Gamma \left( \frac{n+1}{2} \right) & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}, \end{cases}$$

as $t_g$ tends to infinity.

Similarly, let $K$ be any fixed compact continuity set of $\Gamma_0(q) \backslash \mathbb{H}$, and for $\chi$ non-
principal, let $g \in \mathcal{B}_k^*(q, \chi)$ be normalised such that $\langle g, g \rangle_q = 1$. Then for every pair of nonnegative integers $n_1, n_2$,

$$\frac{1}{\text{Var}_K(g)^{n_1+n_2/2} \text{vol}(K)} \int_K g(z)^{n_1} \overline{g(z)}^{n_2} \, d\mu(z)$$

(1.2.4)

converges to

$$\frac{1}{\pi} \int_{\mathbb{C}} z^{n_1} \overline{z}^{n_2} e^{-|z|^2} \, dz = \begin{cases} 2^{n_1+n_2+1} \Gamma \left( \frac{n_1+n_2+1}{2} \right) & \text{if } n_1 = n_2, \\ 0 & \text{otherwise}, \end{cases}$$

as $t_g$ tends to infinity.

When $K$ is replaced by a noncompact set, the Gaussian moments conjecture ought not necessarily to hold for high moments. As explained in [HeSt01, Section 4], using a heuristic appearing in [Hej99, Section 7], the transition range of the Whittaker function leads to a “tidal pulse” phenomenon near the cusps of $\Gamma_0(q) \backslash \mathbb{H}$; when $K$ is replaced by $\Gamma_0(q) \backslash \mathbb{H}$, so that $\text{Var}_{\Gamma_0(q) \backslash \mathbb{H}}(g) = \text{vol}(\Gamma_0(q) \backslash \mathbb{H})^{-1}$, one can thereby show that (1.2.3) grows like a power of $t_g$ whenever $n \geq 12$ is even, and similarly that (1.2.4) grows like a power of $t_g$ whenever $n_1 = n_2 \geq 6$. This is closely related to the fact that for $g \in \mathcal{B}_k^*(q, \chi)$ normalised such that $\langle g, g \rangle_q = 1$,

$$\|g\|_\infty \gg_{q, \varepsilon} t_g^{1/6 - \varepsilon}.$$

Nonetheless, it is not unreasonable to conjecture that the Gaussian moments conjecture holds for smaller moments when $K$ is replaced by $\Gamma_0(q) \backslash \mathbb{H}$. Indeed, when $g$ is real-valued, the conjecture holds by definition for $n \in \{0, 2\}$ and is easily shown to also be true when $n = 1$, as both sides vanish, while for $n = 3$, this can be shown to hold via the work of Watson [Wat08]. When $g$ is complex-valued, the conjecture again the conjecture holds by definition when $(n_1, n_2) \in \{(0, 0), (1, 1)\}$, is easily proven
whenever \( n_1 - n_2 \) is not a multiple of the order of \( \chi \), for then both sides vanish, and can also be shown to hold for \((n_1, n_2) \in \{(2, 1), (1, 2)\}\) via the work of Watson [Wat08].

### 1.2.4 Quantum Unique Ergodicity

Another manifestation of the randomness of Maaß newforms is quantum unique ergodicity.

**Conjecture 1.2.5** (Quantum Unique Ergodicity in Configuration Space). Let \( g \in B_\kappa^*(q, \chi) \) be a newform normalised such that \( \langle g, g \rangle_q = 1 \). Then the probability measure \( |g(z)|^2 \, d\mu(z) \) converges in distribution as \( g \) tends to infinity to the uniform probability measure on \( \Gamma_0(q) \setminus \mathbb{H} \), so that for every continuity set \( B \subset \Gamma_0(q) \setminus \mathbb{H} \),

\[
\int_B |g(z)|^2 \, d\mu(z) = \frac{\text{vol}(B)}{\text{vol}(\Gamma_0(q) \setminus \mathbb{H})} + o_{q,B}(1)
\]

as \( g \) tends to infinity.

By the Portmanteau theorem, this conjecture is equivalent to

\[
\int_{\Gamma_0(q) \setminus \mathbb{H}} f(z) |g(z)|^2 \, d\mu(z) = \frac{1}{\text{vol}(\Gamma_0(q) \setminus \mathbb{H})} \int_{\Gamma_0(q) \setminus \mathbb{H}} f(z) \, d\mu(z) + o_{q,f}(1)
\]

(1.2.6)

for every bounded continuous function on \( \Gamma_0(q) \setminus \mathbb{H} \). By the argument presented in [NPS14, Section 3.6], it is sufficient to prove this for all compactly supported continuous functions on \( \Gamma_0(q) \setminus \mathbb{H} \), and the space of all such functions is contained in the span of Maaß cusp forms and incomplete Eisenstein series, so it suffices to prove (1.2.6) for all \( f \in B_0(\Gamma_0(q)) \) and all \( E_a(z, \psi) \) with \( \psi \in C^\infty_c(\mathbb{R}^+) \).

It behooves us to mention that there is a stronger formulation of quantum unique ergodicity, namely quantum unique ergodicity in phase space, which is the cosphere bundle \( S^*(\Gamma_0(q) \setminus \mathbb{H}) \cong \Gamma_0(q) \setminus \text{SL}_2(\mathbb{R}) \): not only should the sequence of probability measures \( |g(z)|^2 \, d\mu(z) \) equidistribute on the configuration space \( \Gamma_0(q) \setminus \mathbb{H} \), but that
a microlocal lift of these measures to Wigner distributions on phase space should equidistribute with respect to the uniform probability measure on this space, namely the Liouville measure. Should one only ask that equidistribution holds for a subsequence of eigenfunctions of full density, then this is called quantum ergodicity; for $\Gamma \backslash \mathbb{H}$, this is a result of Zelditch [Zel91].

For $q = 1$, quantum unique ergodicity in phase space, and hence also in configuration space, is known to be true via the work of Lindenstrauss [Lind06] and Soundararajan [Sou10]. However, this proof does not quantify the rate of convergence; in particular, it does not give explicit rates of decay for the terms

$$\int_{\Gamma \backslash \mathbb{H}} f(z)|g(z)|^2 \, d\mu(z), \quad \int_{\Gamma \backslash \mathbb{H}} E(z, \psi)|g(z)|^2 \, d\mu(z)$$

for fixed $f \in B_0(\Gamma)$ and $\psi \in C^\infty_c(\mathbb{R}^+)$ as $t_g$ tends to infinity. Watson [Wat08, Corollary 1] has shown that optimal decay rates for these integrals follow directly from the generalised Lindelöf hypothesis.

The $n = 2$ case of the Gaussian moments conjecture for the set $K = \Gamma_0(q) \backslash \mathbb{H}$ — namely the $L^4$-norm of $g$ — shares many similarities with quantum unique ergodicity in configuration space. In fact, it is extremely closely related to a more refined version of quantum unique ergodicity, namely equidistribution on shrinking sets. We discuss this similarity in Sections 1.4 and 1.5.

### 1.3 Randomness of Eisenstein Series

The random wave conjecture, Gaussian moments conjecture, and quantum unique ergodicity ought to be true, once suitably modified, when $g(z) = E(z, 1/2 + it_g)$ is an Eisenstein series; here we set $q = 1$ for simplicity. Although Eisenstein series are complex-valued, the real and imaginary parts of $g$ do not seem to be independent
because of the functional equation

\[ E(z, s) = \frac{\Lambda(2 - 2s)}{\Lambda(2s)} E(z, 1 - s), \]

where \( \Lambda(s) \) is the completed Riemann zeta function; we define and discuss this function in Section 2.1.1. In particular,

\[ \Re\left( E\left(z, \frac{1}{2} + it_g\right)\right) = \frac{1}{2} \left( 1 + \frac{\Lambda(1 + 2it_g)}{\Lambda(1 - 2it_g)} \right) E\left(z, \frac{1}{2} - it_g\right), \]
\[ \Im\left( E\left(z, \frac{1}{2} + it_g\right)\right) = \frac{i}{2} \left( 1 - \frac{\Lambda(1 + 2it_g)}{\Lambda(1 - 2it_g)} \right) E\left(z, \frac{1}{2} - it_g\right). \]

Moreover, Eisenstein series are not square-integrable, so one must use some sort of regularisation. One method is to use the Zagier’s regularisation of divergent integrals [Zag82]. Another method is to replace \( g \) with the truncated Eisenstein series

\[ \Lambda^T E(z, s) := E(z, s) - \sum_{\gamma \in \Gamma_\infty \setminus \Gamma \atop \Im(\gamma z) > T} \left( \Im(\gamma z)^s + \frac{\Lambda(2 - 2s)}{\Lambda(2s)} \Im(\gamma z)^{-s} \right) \]

for some \( T \geq 1 \). In [HejRa92, Section 7], Hejhal and Rackner define

\[ F\left(z, \frac{1}{2} + it_g\right) := \frac{\Lambda(1 + 2it_g)}{\Lambda(1 + 2it_g)} \Lambda^T E\left(z, \frac{1}{2} + it_g\right), \]

which is real-valued. They again define the probability measure \( \nu_{F,K} \) for fixed compact continuity sets of \( \Gamma \setminus \mathbb{H} \) by

\[ \nu_{F,K}(B) := \frac{\text{vol}\left( \left\{ z \in K : \text{Var}_K(F)^{-1/2} F(z) \in B \right\} \right)}{\text{vol}(K)} \]

for every Lebesgue-measurable subset \( B \) of \( \mathbb{R} \), where

\[ \text{Var}_K(F) := \frac{1}{\text{vol}(K)} \int_K F(z)^2 \, d\mu(z). \]
They then give numerical evidence towards the following.

**Conjecture 1.3.1** (Gaussian Distribution Conjecture for Eisenstein Series). For any fixed compact continuity set $K$ of $\Gamma \backslash \mathbb{H}$, the sequence of probability measures $\nu_{F,K}$ on $\mathbb{R}$ converges in distribution as the spectral parameter $t_g$ of $g$ tends to infinity to the probability measure of a real Gaussian random variable with mean 0 and variance 1.

Similarly, we can formulate the Gaussian moments conjecture for Eisenstein series. For quantum unique ergodicity, we need not deal with the truncated version of the Eisenstein series provided that we take into account the growth of the $L^2$-norm of an Eisenstein series on compact sets. When $q = 1$, Luo and Sarnak have shown the following.

**Theorem 1.3.2** ([LS95, Theorem 1.1]). For any compact continuity set $K \subset \Gamma \backslash \mathbb{H}$ and for $g(z) = E(z, 1/2 + it_g)$,

$$\int_K |g(z)|^2 d\mu(z) = \frac{\log \left( \frac{1}{4} + t_g^2 \right) \text{vol}(K)}{\text{vol}(\Gamma \backslash \mathbb{H})} + o_K(\log t_g)$$

as $t_g$ tends to infinity.

Since $K$ is compact, one can replace $g(z)$ with $\Lambda^T E(z, 1/2 + it_g)$ for some $T$ sufficiently large dependent on $K$. The presence of $\log \left( \frac{1}{4} + t_g^2 \right)$ stems from the fact that the $L^2$-norm of $g(z)$ restricted to compact subsets grows like $\sqrt{\log \left( \frac{1}{4} + t_g^2 \right)}$; we expand upon this point when we prove the Maaß–Selberg relation in Corollary 2.2.5.

Quantum unique ergodicity in phase space is also known for Eisenstein series; this is a result of Jakobsen [Jak94, Theorem 1].

### 1.4 The $L^4$-Norm Problem

The $L^4$-norm problem for a Maaß newform $g$ is the second nontrivial case of the Gaussian moments conjecture.
Conjecture 1.4.1 ($L^4$-Norm Problem). Let $g \in \mathcal{B}_\kappa^\ast(q, \chi)$ be a newform normalised such that $\langle g, g \rangle_q = 1$. As $t_g$ tends to infinity,

$$
\int_{\Gamma_0(q) \backslash \mathbb{H}} |g(z)|^4 \, d\mu(z) = \begin{cases} 
3 \frac{\text{vol} \left( \Gamma_0(q) \backslash \mathbb{H} \right)}{\text{vol} \left( \Gamma_0(q) \backslash \mathbb{H} \right)} + o_q(1) & \text{if } \chi \text{ is principal,} \\
2 \frac{\text{vol} \left( \Gamma_0(q) \backslash \mathbb{H} \right)}{\text{vol} \left( \Gamma_0(q) \backslash \mathbb{H} \right)} + o_q(1) & \text{if } \chi \text{ is nonprincipal.}
\end{cases}
$$

A similar statement can be formulated when $g$ is an Eisenstein series, though some care must be taken, since Eisenstein series are not square-integrable.

The key to proving $L^4$-norm bounds is the spectral decomposition of $L^2(\Gamma_0(q) \backslash \mathbb{H})$ given in Lemma 1.1.1. In particular, the following is simply Parseval’s identity with $g_1 = g_2 = |g|^2$.

Corollary 1.4.2. Let $g \in L^2(\Gamma_0(q) \backslash \mathbb{H}, \kappa, q)$ be of rapid decay. Then $\|g\|_{L^4(\Gamma_0(q) \backslash \mathbb{H})}^4$ is equal to

$$
\left| \left| \left| \langle |g|^2, f_0 \rangle_q \right| \right|^2 + \sum_{f \in \mathcal{B}_0(\Gamma_0(q))} \left| \left| \langle |g|^2, f \rangle_q \right| \right|^2 + \sum_a \frac{1}{4\pi} \int_{-\infty}^{\infty} \left| \left| \left\langle |g|^2, E_a \left( \cdot, \frac{1}{2} + it \right) \right\rangle_q \right| \right|^2 \, dt.
$$

We then use the Watson–Ichino formula to write $\left| \left| \langle |g|^2, f \rangle_q \right| \right|^2$ as a product of $L$-functions; we will expand upon this point in Chapter 2.

In general, an unconditional proof of the $L^4$-norm problem seems quite difficult. A weaker conjecture is that

$$
\|g\|_{L^4(\Gamma_0(q) \backslash \mathbb{H})}^4 \ll_{q, \varepsilon} t_g^\varepsilon.
$$

In certain special cases, this has been shown: when $g$ is a dihedral Maaß cusp form, this is a result of Luo [Luo14], while when $g$ is a truncated Eisenstein series, this is a result of Spinu [Spi03] (with the implicit constant of course dependent on the truncation parameter $T$).
Recently, Buttcane and Khan [BK17b, Theorem 1.1] have given a conditional proof of the $L^4$-norm problem for a Hecke eigenform $g \in B_0(\Gamma)$. When $q = 1$ and $g(z) = \Lambda^T E(z, 1/2 + it_g)$, Spinu [Spi03, Chapter 6] sketches an unconditional proof of the bound

$$\|g\|_{L^4(\Gamma \setminus H)}^4 \ll_T (\log t_g)^2.$$ 

His proof, however, only treats the spectral sum in Corollary 1.4.2 in the range $\alpha t_g < t_f < 2(1 - \alpha)t_g$ for any $\alpha > 0$. He does not address the remaining ranges, which all ought to contribute a negligible amount.

We will show how to prove that these ranges are indeed negligible, thereby completing Spinu’s proof. The key is to use results of Jutila [Jut04], Ivić [Ivi01], and Jutila and Motohashi [JM05] on sums in dyadic spectral ranges of products of certain $L$-functions.

**Theorem 1.4.3.** Let $g(z) = \Lambda^T E(z, 1/2 + it_g)$. We have that

$$\|g\|_{L^4(\Gamma \setminus H)}^4 \ll_T (\log t_g)^2.$$ 

Up to the implicit constant, this result should be sharp, for the Maaß–Selberg relation implies that

$$\|g\|_{L^2(\Gamma \setminus H)}^4 = \left( \log \left( \left( \frac{1}{4} + t_g^2 \right) T^2 \right) + O \left( (\log t_g)^{2/3} (\log \log t_g)^{1/3} \right) \right)^2,$$

as we show in Corollary 2.2.5.

### 1.5 Quantum Unique Ergodicity in Shrinking Sets

A natural strengthening of quantum unique ergodicity is to determine whether equidistribution still occurs if we vary the set $B$ with $t_g$; in particular, if the size of $B$ shrinks
as \( t_g \) increases. This is closely related to determining the rate of equidistribution. Proving equidistribution in shrinking sets has applications towards bounds for the \( L^p \)-norms and size of nodal domains of eigenfunctions of the Laplacian; see [HezRi16].

We denote by \( B = B_R(w) \) the hyperbolic ball of radius \( R \) centred at \( w \in \Gamma_0(q)\backslash \mathbb{H} \): its hyperbolic volume is

\[
\text{vol} (B_R) = 4\pi \sinh^2 \frac{R}{2},
\]

which is independent of the centre \( w \).

**Question 1.5.1.** Let \( g \in \mathcal{B}^\kappa(q, \chi) \) be a newform normalised such that \( \langle g, g \rangle_q = 1 \).

For what conditions on \( R \), with regards to \( t_g \), is it still true that

\[
\frac{1}{\text{vol} (B_R)} \int_{B_R(w)} |g(z)|^2 d\mu(z) = \frac{1}{\text{vol} (\Gamma_0(q)\backslash \mathbb{H})} + o_{w,R}(1) \tag{1.5.2}
\]

as \( t_g \) tends to infinity?

We should not expect equidistribution to hold when \( R \ll t_g^{-1} \); indeed, Hejhal and Rackner [HejRa92, Section 5], writing \( \Psi_n \) in place of \( g \), \( \lambda_n \) in place of \( \lambda_g = 1/4 + t_g^2 \), and \( A \) in place of \( R \), state that

\[
\ldots \text{in the physics literature, } c/\sqrt{\lambda_n} \text{ is commonly referred to as the de Broglie wavelength. At length scales below } c/\sqrt{\lambda_n}, \text{ one expects the topography of } \Psi_n \text{ to look “essentially sinusoidal”, that is, regular. It is only when } A \text{ is substantially bigger than the de Broglie wavelength that one stands any chance of seeing any type of Gaussian distribution.}
\]

We confirm this statement, by showing that if \( R \ll_A t_g^{-1}(\log t_g)^A \) for any \( A > 0 \), then there exist infinitely many points \( w \in \Gamma_0(q)\backslash \mathbb{H} \) for which (1.5.2) does not hold, so that the sequence of probability measures \( |g(z)|^2 d\mu(z) \) does not equidistribute on the shrinking balls of radius \( t_g^{-1}(\log t_g)^A \) centred at these points. We think of \( R \asymp t_g^{-1} \) as...
being the Planck scale, so that equidistribution need not occur within a logarithmic window of the Planck scale.

**Theorem 1.5.3.** Let \( g \in \mathcal{B}_0^* (\Gamma_0(q)) \) be a newform normalised such that \( \langle g, g \rangle_q = 1 \). For every fixed Heegner point \( w \in \mathbb{H} \), we have that

\[
\frac{1}{\text{vol}(B_R)} \int_{B_R(w)} |g(z)|^2 d\mu(z) = \Omega \left( \exp \left( 2 \sqrt{\frac{\log t_g}{\log \log t_g}} \left( 1 + O \left( \frac{\log \log \log t_g}{\log \log t_g} \right) \right) \right) \right)
\]

for \( R \ll_A t_g^{-1} (\log t_g)^A \) for any \( A > 0 \) as \( t_g \) tends to infinity.

Nevertheless, we should expect equidistribution to occur at every scale larger than the Planck scale, namely \( R \gg t_g^{-\delta} \) for any \( \delta < 1 \). Towards this, Young [You16] has proved the following.

**Theorem 1.5.4** (Young [You16, Proposition 1.5]). Let \( g \in \mathcal{B}_0 (\Gamma) \) be a Hecke eigenform normalised such that \( \langle g, g \rangle = 1 \). Assume the generalised Lindelöf hypothesis, and suppose that \( R \asymp t_g^{-\delta} \) with \( \delta < 1/3 \). Then

\[
\frac{1}{\text{vol}(B_R)} \int_{B_R(w)} |g(z)|^2 d\mu(z) = \frac{1}{\text{vol}(\Gamma \backslash \mathbb{H})} + o_{w,\delta}(1)
\]

for every fixed point \( w \in \Gamma \backslash \mathbb{H} \).

Similarly, let \( g(z) = E(z, 1/2 + it_g) \), and suppose that \( R \asymp t_g^{-\delta} \) with \( \delta < 1/9 \). Then unconditionally

\[
\frac{1}{\log \left( \frac{1}{4} + t_g^2 \right) \text{vol}(B_R)} \int_{B_R(w)} |g(z)|^2 d\mu(z) = \frac{1}{\text{vol}(\Gamma \backslash \mathbb{H})} + o_{w,\delta}(1)
\]

for every fixed point \( w \in \Gamma \backslash \mathbb{H} \).

In fact, with little work, we can improve the range in Young’s result for Eisenstein series.
Theorem 1.5.5. Let \( g(z) = E(z, 1/2 + it) \), and suppose that \( R \asymp t_g^{-\delta} \) with \( \delta < 1/3 \). Then unconditionally

\[
\frac{1}{\log \left( \frac{1}{4} + t_g^2 \right)} \frac{\text{vol}(B_R)}{\text{vol}(\mathbb{H})} \int_{B_R(w)} |g(z)|^2 d\mu(z) = \frac{1}{\text{vol}(\Gamma\setminus\mathbb{H})} + o_{w,\delta}(1)
\]

for every fixed point \( w \in \Gamma\setminus\mathbb{H} \).

A simpler version of Question 1.5.1 is to instead consider eigenfunctions of the Laplacian on the \( d \)-torus \( \mathbb{T}^d \) for any \( d \geq 2 \). Hezari and Rivièrie [HezRi15, Corollary 1.5] give strong bounds for equidistribution in shrinking balls along a full density subsequence of eigenfunctions of the Laplacian on \( \mathbb{T}^d \) with eigenvalue \( \lambda \), namely equidistribution on all balls of radius \( R \gg \lambda^{-\frac{1}{2}d^{-1}} \). Lester and Rudnick [LR17, Theorem 1.1] improve this to \( R \gg_{\varepsilon} \lambda^{-\frac{1}{2}d^{-1}+\varepsilon} \). Moreover, they prove [LR17, Theorems 3.1 and 4.1] that this is essentially sharp, in that there exists a subsequence of eigenfunctions for which equidistribution does not occur on shrinking balls of radius \( R \ll_{\varepsilon} \lambda^{-\frac{1}{2}d^{-1}-\varepsilon} \). For \( d = 2 \), Granville and Wigman [GW16, Corollary 3.2] have subsequently sharpened Lester and Rudnick’s results to show there exists \( A > 0 \) such that equidistribution may not occur on shrinking balls of radius \( R \ll A \lambda^{-1/2}(\log \lambda)^A \).

We study a related question: instead of demanding that equidistribution hold in shrinking balls of radius \( R > 0 \) centred at \( w \) for every point \( w \in \Gamma_0(q)\setminus\mathbb{H} \), we relax this requirement by instead asking whether equidistribution holds in shrinking balls \( B_R(w) \) for almost every \( w \in \Gamma_0(q)\setminus\mathbb{H} \).

1.5.1 Conditional Results

We are able to give a conditional proof of equidistribution in almost every shrinking ball when \( g \in B_0(\Gamma) \) and \( R \gg t_g^{-\delta} \) for any \( 0 < \delta < 1 \), that is, at all scales above the Planck scale. This is a simple consequence of the following variance bound.
Theorem 1.5.6. Let \( g \in \mathcal{B}_0(\Gamma) \) be a Hecke eigenform normalised such that \( (g, g) = 1 \).

For \( R > 0 \), let

\[
\text{Var}(g; R) := \int_{\Gamma_0(q) \backslash \mathbb{H}} \left( \frac{1}{\text{vol}(B_R)} \int_{B_R(w)} |g(z)|^2 \, d\mu(z) - \frac{1}{\text{vol}(\Gamma_0(q) \backslash \mathbb{H})} \right)^2 \, d\mu(w).
\]

Assume the generalised Lindelöf hypothesis, and suppose that \( R \asymp t_g^{-\delta} \) for some \( \delta > 0 \). Then for \( 0 < \delta < 1 \),

\[
\text{Var}(g; R) \ll \epsilon t_g^{-(1-\delta)+\epsilon}
\]

as \( t_g \) tends to infinity, while for \( \delta > 1 \),

\[
\text{Var}(g; R) \sim \frac{2}{\text{vol}(\Gamma \backslash \mathbb{H})} = \frac{6}{\pi}
\]

as \( t_g \) tends to infinity.

By Chebyshev’s inequality,

\[
\text{vol} \left( \left\{ w \in \Gamma_0(q) \backslash \mathbb{H} : \left| \frac{1}{\text{vol}(B_R)} \int_{B_R(w)} |g(z)|^2 \, d\mu(z) - \frac{1}{\text{vol}(\Gamma_0(q) \backslash \mathbb{H})} \right| > c \right\} \right) \\
\leq \frac{1}{c^2} \text{Var}(g; R)
\]

for any \( c > 0 \). This yields the following.

Theorem 1.5.7. Let \( g \in \mathcal{B}_0(\Gamma) \) be a Hecke eigenform normalised such that \( (g, g) = 1 \).

Assume the generalised Lindelöf hypothesis, and suppose that \( R \asymp t_g^{-\delta} \) for some \( 0 < \delta < 1 \). Then for any \( c \gg \epsilon t_g^{-1+\frac{1}{2}+\epsilon} \),

\[
\text{vol} \left( \left\{ w \in \Gamma \backslash \mathbb{H} : \left| \frac{1}{\text{vol}(B_R)} \int_{B_R(w)} |g(z)|^2 \, d\mu(z) - \frac{1}{\text{vol}(\Gamma \backslash \mathbb{H})} \right| > c \right\} \right)
\]

converges to zero as \( t_g \) tends to infinity.
We emphasise that as a consequence, if $R \gg \varepsilon t^{-1+\varepsilon}$, then for any fixed $c > 0$, 

$$\lim_{t \to \infty} \mathrm{vol} \left( \left\{ w \in \Gamma \setminus \mathbb{H} : \left| \frac{1}{\mathrm{vol}(B_R)} \int_{B_R(w)} |g(z)|^2 \, d\mu(z) - \frac{1}{\mathrm{vol}(\Gamma \setminus \mathbb{H})} \right| > c \right\} \right) = 0,$$

so that assuming the generalised Lindelöf hypothesis, equidistribution holds for almost every shrinking ball at any power larger than the Planck scale.

The method of calculating the variance in order to show equidistribution in almost every shrinking ball is used in [GW16, Theorem 1.6] for eigenvalues of the Laplacian on $T^2$, as well as in both [EMV13, Theorem 1.3] and [BRS16, Theorem 1.7], where the problem investigated is not quantum unique ergodicity, but rather the equidistribution of lattice points on the sphere.

The key to calculating the variance in Theorem 1.5.6 is the following spectral expansion.

**Proposition 1.5.8.** Let $g \in \mathcal{B}_k^*(q, \chi)$ be a newform normalised such that $\langle g, g \rangle_q = 1$. Then $\text{Var}(g; R)$ is equal to

$$\sum_{f \in \mathcal{B}_0(\Gamma_0(q))} |h_R(t_f)|^2 \left| \langle |g|^2, f \rangle_q \right|^2 + \sum_{a} \frac{1}{4\pi} \int_{-\infty}^{\infty} |h_R(t)|^2 \left| \left\langle |g|^2, E_a \left( \cdot, \frac{1}{2} + it \right) \right\rangle_q \right|^2 dt,$$

where

$$h_R(t) := \frac{R}{\pi \sinh \frac{R}{2}} \int_{-1}^{1} \sqrt{1 - \left( \frac{\sinh \frac{Rr}{2}}{\sinh \frac{R}{2}} \right)^2} e^{iRrt} \, dr.$$

Again, the terms $\left| \langle |g|^2, f \rangle_q \right|^2$ will be shown in Chapter 2 to be equal to a product of $L$-functions. We also require the following asymptotics for $h_R(t)$, which are be extremely similar to the analogous result for $T^2$; see [GW16, Lemma 2.1].
Lemma 1.5.9. As $R$ tends to zero, we have that

$$h_R(t) \sim\begin{cases} 1 & \text{if } R \text{ tends to zero,} \\ \frac{2J_1(Rt)}{Rt} & \text{if } R \in (0, \infty), \\ \frac{1}{\sqrt{\pi}} \left( \frac{2}{Rt} \right)^{3/2} \sin \left( Rt - \frac{\pi}{4} \right) & \text{if } R \text{ tends to infinity.} \end{cases}$$

In particular, this shows that the spectral sum for $\text{Var}(g; R)$ is essentially the same as the spectral sum for the $L^4$-norm as in Corollary 1.4.2 in the range $0 < t_f \ll R^{-1+\epsilon}$, whereas for $t_f \gg 1/R$, it is much smaller.

1.5.2 Unconditional Results

Proving unconditional results seems to be much more difficult. Nevertheless, we are able to do so when $g(z) = E(z, 1/2 + it_g)$ is an Eisenstein series. Young [You16, Theorem 1.4] has shown that for Eisenstein series, equidistribution occurs on every shrinking ball of radius $R \approx t_g^{-\delta}$ whenever $\delta < 1/9$. In fact, this result can be extended to the range $\delta < 1/3$ by combining his method of proof together with Lemma 3.4.1. By relaxing the condition of equidistribution to almost every shrinking ball, we can unconditionally prove equidistribution for Eisenstein series at all scales above the Planck scale.

Theorem 1.5.10. Let $g(z) = E(z, 1/2 + it_g)$. For $R > 0$, let

$$\text{Var}(g; R) := \int_{\Gamma \setminus \mathbb{H}} \left( \frac{1}{\text{vol}(B_R)} \int_{B_R(w)} |g(z)|^2 \, d\mu(z) - C(g; R; w) \right)^2 \, d\mu(w),$$

where $C(g; R; w)$ is given by (5.2.3). Suppose that $R \approx t_g^{-\delta}$ for some $0 < \delta < 1$. Then

$$\text{Var}(g; R) \ll_{\epsilon} t_g^{-\min \left\{ \frac{\delta}{2}, \frac{1}{2}\epsilon \right\} + \epsilon}.$$
Using this, we are able to prove the following.

**Theorem 1.5.11.** Let \( g(z) = E(z, 1/2 + it) \). Suppose that \( R \gg t_g^{-\delta} \) for some \( 0 < \delta < 1 \). Then for any \( c \gg \varepsilon t_g \),

\[
\lim_{t_g \to \infty} \text{vol} \left( \left\{ w \in \Gamma \backslash \mathbb{H} : \left| \frac{1}{\text{vol} (B_R)} \int_{B_R(w)} |g(z)|^2 d\mu(z) - D(g; w) \right| > c \right\} \right) = 0,
\]

converges to zero as \( t_g \) tends to infinity, where \( D(g; w) \) is given by (5.2.2).

This result is consistent with Theorem 1.3.2 due to the following.

**Lemma 1.5.12.** In any compact subset \( K \) of \( \Gamma \backslash \mathbb{H} \), we have that for all \( w \in K \),

\[
D(g; w) = \frac{\log \left( \frac{1}{4} + t_g^2 \right)}{\text{vol} (\Gamma \backslash \mathbb{H})} + O_K \left( (\log t_g)^{2/3} (\log \log t_g)^{1/3} \right).
\]

In particular, we may rephrase Theorem 1.5.11 in the following way.

**Corollary 1.5.13.** Let \( g(z) = E(z, 1/2 + it) \), and let \( K \) be a fixed compact subset of \( \Gamma \backslash \mathbb{H} \). Suppose that \( R \gg t_g^{-1+\varepsilon} \). Then for any fixed \( c > 0 \),

\[
\lim_{t_g \to \infty} \text{vol} \left( \left\{ w \in K : \frac{1}{\log \left( \frac{1}{4} + t_g^2 \right) \text{vol} (B_R)} \int_{B_R(w)} |g(z)|^2 d\mu(z) - \frac{1}{\text{vol} (\Gamma \backslash \mathbb{H})} \right| > c \right\} \right) = 0.
\]

Once again, our starting point is a spectral expansion for \( \text{Var}(g; R) \).

**Proposition 1.5.14.** Let \( g(z) = E(z, 1/2 + it) \). Then \( \text{Var}(g; R) \) is equal to

\[
\sum_{f \in \mathcal{B}_0(\Gamma)} |h_R(t_f)|^2 \left| \langle |g|^2, f \rangle \right|^2 + \frac{1}{4\pi} \int_{-\infty}^{\infty} |h_R(t)|^2 \left| \left\langle |g|^2, E \left( \cdot, \frac{1}{2} + it \right) \right|_{\text{reg}} \right|^2 dt,
\]

where the regularised inner product of Eisenstein series is defined in Proposition 2.3.8.
1.6 Equidistribution of Geometric Invariants of Quadratic Fields in Shrinking Sets

Finally, in Chapter 6, we study a similar equidistribution problem in shrinking sets. Associated to each narrow ideal class $A$ of the narrow class group $\text{Cl}_K^+$ of a quadratic number field $K = \mathbb{Q}(\sqrt{D})$ is a geometric invariant. For $D < 0$, this is a Heegner point $z_A$, while for $D > 0$, this is a closed geodesic $C_A$ or a hyperbolic orbifold $\Gamma_A \backslash \mathcal{N}_A$ having this closed geodesic as its boundary; we explain these geometric invariants in more detail in Section 6.1.

For each fundamental discriminant $D$, we choose a genus $G_K \subset \text{Cl}_K^+$ in the group of genera $\text{Gen}_K = \text{Cl}_K^+ / (\text{Cl}_K^+)^2$, so that $G_K$ is a coset $A(\text{Cl}_K^+)^2$ of narrow ideal classes in $\text{Cl}_K^+$. We have that $\text{Gen}_K \cong (\mathbb{Z}/2\mathbb{Z})^{\omega(|D|)-1}$, where $\omega(|D|)$ is the number of distinct prime factors of $|D|$, so that $\#G_K = \#(\text{Cl}_K^+)^2 = 2^{1-\omega(|D|)}{h_K^+}$, where $h_K^+ := \#\text{Cl}_K^+$ denotes the narrow class number of $K$. Then Duke, Imamoglu, and Tóth have proved the following equidistribution theorem.

**Theorem 1.6.1 ([DIT16, Theorem 2]).** For every continuity set $B \subset \Gamma \backslash \mathbb{H}$,

$$\frac{\# \{ A \in G_K : z_A \in B \}}{\# G_K} = \frac{\text{vol}(B)}{\text{vol}(\Gamma \backslash \mathbb{H})} + o_B(1)$$

as $D \to -\infty$ through fundamental discriminants, and

$$\frac{\sum_{A \in G_K} \ell(C_A \cap B)}{\sum_{A \in G_K} \ell(C_A)} = \frac{\text{vol}(B)}{\text{vol}(\Gamma \backslash \mathbb{H})} + o_B(1),$$

$$\frac{\sum_{A \in G_K} \text{vol}(\Gamma_A \backslash \mathcal{N}_A \cap B)}{\sum_{A \in G_K} \text{vol}(\Gamma_A \backslash \mathcal{N}_A)} = \frac{\text{vol}(B)}{\text{vol}(\Gamma \backslash \mathbb{H})} + o_B(1)$$

as $D \to \infty$ through fundamental discriminants, where $\ell(C_A) := \int_{C_A} ds$, with $ds^2 = \frac{dx^2 + dy^2}{y^2}$.

If we sum over all genera, so that we are studying equidistribution associated to
the full narrow class group, then this result is due to Duke [Duk88, Theorem 1] for
Heegner points and closed geodesics, while this result becomes trivial for hyperbolic
orbifolds, for there is no error term whatsoever in this case. Moreover, the equidis-
tribution of closed geodesics has a stronger realisation: instead of merely asking for
the equidistribution of closed geodesics on $\Gamma \backslash \mathbb{H}$, we may lift these geodesics to phase
space $S^*(\Gamma \backslash \mathbb{H}) \cong \Gamma \backslash \text{SL}_2(\mathbb{R})$ and demand equidistribution with respect to the Liouville
measure. This has been proved by Chelluri [Che04].

It is natural to ask whether equidistribution still occurs if $B$ shrinks as $|D|$ grows.
Towards this, Young [You17] has proved the following.

**Theorem 1.6.2** (Young [You17, Theorem 2.1]). Fix $w \in \Gamma \backslash \mathbb{H}$, and suppose that
$R \asymp (-D)^{-\delta}$. Unconditionally, as $D \to -\infty$ through odd fundamental discriminants,
\[
\frac{\# \{ A \in \text{Cl}_K : z_A \in B_R(w) \}}{\text{vol}(B_R) h_K} = \frac{1}{\text{vol}(\Gamma \backslash \mathbb{H})} + o_{w,\delta}(1) \quad \text{for} \ \delta < 1/24,
\]
where $\text{Cl}_K$ denotes the class group of $K$ and $h_K := \# \text{Cl}_K$ denotes the class number.
Assuming the generalised Lindel"of hypothesis, as $D \to -\infty$ through fundamental
discriminants,
\[
\frac{\# \{ A \in \text{Cl}_K : z_A \in B_R(w) \}}{\text{vol}(B_R) h_K} = \frac{1}{\text{vol}(\Gamma \backslash \mathbb{H})} + o_{w,\delta}(1) \quad \text{for} \ \delta < 1/8.
\]

In fact, from the method of proof, it is clear that Young’s theorem applies to
genera mutatis mutandis, and proves equidistribution not only of Heegner points, but
also of closed geodesics and hyperbolic orbifolds.

**Theorem 1.6.3.** Fix $w \in \Gamma \backslash \mathbb{H}$, and suppose that $R \asymp D^{-\delta}$. Unconditionally, as
$D \to \infty$ through odd fundamental discriminants,
\[
\frac{\sum_{A \in G_K} \ell(C_A \cap B_R(w))}{\text{vol}(B_R) \sum_{A \in G_K} \ell(C_A)} = \frac{1}{\text{vol}(\Gamma \backslash \mathbb{H})} + o_{w,\delta}(1) \quad \text{for} \ \delta < 1/18,
\]
\[ \sum_{A \in G_K} \frac{\text{vol} (\Gamma_A \setminus \mathcal{N}_A \cap B_R(w))}{\text{vol} (B_R) \sum_{A \in G_K} \text{vol} (\Gamma_A \setminus \mathcal{N}_A)} = \frac{1}{\text{vol} (\Gamma \setminus \mathbb{H})} + o_{w,\delta}(1) \quad \text{for } \delta < 1/12. \]

Assuming the generalised Lindelöf hypothesis, as \( D \to \infty \) through fundamental discriminants,

\[ \sum_{A \in G_K} \frac{\ell (\mathcal{C}_A \cap B_R(w))}{\text{vol} (B_R) \sum_{A \in G_K} \ell (\mathcal{C}_A)} = \frac{1}{\text{vol} (\Gamma \setminus \mathbb{H})} + o_{w,\delta}(1) \quad \text{for } \delta < 1/6, \]

\[ \sum_{A \in G_K} \frac{\text{vol} (\Gamma_A \setminus \mathcal{N}_A \cap B_R(w))}{\text{vol} (B_R) \sum_{A \in G_K} \text{vol} (\Gamma_A \setminus \mathcal{N}_A)} = \frac{1}{\text{vol} (\Gamma \setminus \mathbb{H})} + o_{w,\delta}(1) \quad \text{for } \delta < 1/4. \]

Once again, we may weaken the demand that equidistribution hold in shrinking balls of radius \( R > 0 \) centred at \( w \) for every point \( w \in \Gamma \setminus \mathbb{H} \) and instead study whether equidistribution holds in shrinking balls \( B_R(w) \) for almost every \( w \in \Gamma \setminus \mathbb{H} \). To this end, we define

\[ \text{Var} (G_K (z_A); R) := \int_{\Gamma \setminus \mathbb{H}} \left( \frac{\# \{ A \in G_K : z_A \in B_R(w) \}}{\text{vol} (B_R) \# G_K} - \frac{1}{\text{vol} (\Gamma \setminus \mathbb{H})} \right)^2 d\mu(w), \]

\[ \text{Var} (G_K (\mathcal{C}_A); R) := \int_{\Gamma \setminus \mathbb{H}} \left( \frac{\sum_{A \in G_K} \ell (\mathcal{C}_A \cap B_R(w))}{\text{vol} (B_R) \sum_{A \in G_K} \ell (\mathcal{C}_A)} - \frac{1}{\text{vol} (\Gamma \setminus \mathbb{H})} \right)^2 d\mu(w), \]

\[ \text{Var} (G_K (\Gamma_A \setminus \mathcal{N}_A); R) := \int_{\Gamma \setminus \mathbb{H}} \left( \frac{\sum_{A \in G_K} \text{vol} (\Gamma_A \setminus \mathcal{N}_A \cap B_R(w))}{\text{vol} (B_R) \sum_{A \in G_K} \text{vol} (\Gamma_A \setminus \mathcal{N}_A)} - \frac{1}{\text{vol} (\Gamma \setminus \mathbb{H})} \right)^2 d\mu(w). \]

We prove the following conditional result.

**Theorem 1.6.4.** Suppose that \( R \asymp |D|^{-\delta} \). Assuming the generalised Lindelöf hypothesis, we have that as \( D \to -\infty \) along fundamental discriminants

\[ \text{Var} (G_K (z_A); R) \ll_{\varepsilon} (-D)^{-\left(\frac{1}{4} - \delta\right) + \varepsilon} \quad \text{for } 0 < \delta < 1/4, \]

\[ \sum_{A \in G_K} \frac{\text{vol} (\Gamma_A \setminus \mathcal{N}_A \cap B_R(w))}{\text{vol} (B_R) \sum_{A \in G_K} \text{vol} (\Gamma_A \setminus \mathcal{N}_A)} = 1 \quad \text{vol} (\Gamma \setminus \mathcal{H}) + o_{w,\delta}(1) \]

for \( \delta < 1/12 \) and

\[ \sum_{A \in G_K} \frac{\ell (\mathcal{C}_A \cap B_R(w))}{\text{vol} (B_R) \sum_{A \in G_K} \ell (\mathcal{C}_A)} = 1 \quad \text{vol} (\Gamma \setminus \mathcal{H}) + o_{w,\delta}(1) \quad \text{for } \delta < 1/6, \]

\[ \sum_{A \in G_K} \frac{\text{vol} (\Gamma_A \setminus \mathcal{N}_A \cap B_R(w))}{\text{vol} (B_R) \sum_{A \in G_K} \text{vol} (\Gamma_A \setminus \mathcal{N}_A)} = 1 \quad \text{vol} (\Gamma \setminus \mathcal{H}) + o_{w,\delta}(1) \quad \text{for } \delta < 1/4. \]
while as $D \to \infty$ along fundamental discriminants,

$$\text{Var} (G_K (C_A); R) \ll \varepsilon D^{-\frac{1}{2}(\frac{1}{6}-\delta)} + \varepsilon \quad \text{for } 0 < \delta < 1/2.$$  

By Chebyshev’s inequality, we obtain the following.

**Theorem 1.6.5.** Suppose that $R \asymp |D|^{-\delta}$. Assuming the generalised Lindelöf hypothesis, we have that for $0 < \delta < 1/4$ and $c \gg \varepsilon (-D)^{-\frac{1}{2}(\frac{1}{12} - \delta)} + \varepsilon$,

$$\text{vol} \left( \left\{ w \in \Gamma \backslash \mathbb{H} : \left| \frac{\# \{ A \in G_K : z_A \in B_R (w) \}}{\text{vol} (B_R) \# G_K} - \frac{1}{\text{vol} (\Gamma \backslash \mathbb{H})} \right| > c \right\} \right)$$

converges to zero as $D \to -\infty$ along fundamental discriminants, while for $0 < \delta < 1/2$ and $c \gg \varepsilon (-D)^{-\frac{1}{2}(\frac{1}{12} - \delta)} + \varepsilon$,

$$\text{vol} \left( \left\{ w \in \Gamma \backslash \mathbb{H} : \left| \frac{\sum_{A \in G_K} \ell (C_A \cap B_R (w))}{\text{vol} (B_R) \sum_{A \in G_K} \ell (C_A)} - \frac{1}{\text{vol} (\Gamma \backslash \mathbb{H})} \right| > c \right\} \right)$$

converges to zero as $D \to \infty$ along fundamental discriminants.

Unconditionally, we obtain the following weaker results.

**Theorem 1.6.6.** Suppose that $R \asymp |D|^{-\delta}$. Then as $D \to -\infty$ along odd fundamental discriminants,

$$\text{Var} (G_K (z_A); R) \ll \varepsilon (-D)^{-\frac{1}{12}(\frac{1}{12} - \delta)} + \varepsilon \quad \text{for } 0 < \delta < 1/12,$$

while as $D \to \infty$ along odd fundamental discriminants,

$$\text{Var} (G_K (C_A); R) \ll \varepsilon D^{-\frac{1}{6}(\frac{1}{6} - \delta)} + \varepsilon \quad \text{for } 0 < \delta < 1/6,$$

$$\text{Var} (G_K (\Gamma_A \backslash \mathcal{N}_A); R) \ll \varepsilon D^{-1/2+\varepsilon} \quad \text{for all } \delta > 0.$$

**Theorem 1.6.7.** Suppose that $R \asymp |D|^{-\delta}$. We have that for $0 < \delta < 1/12$ and
\[ c \gg \varepsilon (-D)^{-\frac{3}{2}\left(\frac{1}{6} - \delta\right) + \varepsilon}, \]

\[
\text{vol}\left( \left\{ w \in \Gamma \backslash \mathbb{H} : \left| \frac{\# \{ A \in G_K : z_A \in B_R(w) \}}{\text{vol}(B_R) \# G_K} - \frac{1}{\text{vol}(\Gamma \backslash \mathbb{H})} \right| > c \right\} \right)
\]

converges to zero as \( D \to -\infty \) along odd fundamental discriminants, while for \( 0 < \delta < 1/6 \) and \( c \gg \varepsilon D^{-\frac{1}{2}\left(\frac{1}{6} - \delta\right) + \varepsilon} \),

\[
\text{vol}\left( \left\{ w \in \Gamma \backslash \mathbb{H} : \left| \frac{\sum_{A \in G_K} \ell(C_A \cap B_R(w))}{\text{vol}(B_R) \sum_{A \in G_K} \ell(C_A)} - \frac{1}{\text{vol}(\Gamma \backslash \mathbb{H})} \right| > c \right\} \right)
\]

converges to zero as \( D \to \infty \) along odd fundamental discriminants, and for all \( \delta > 0 \) and \( c \gg \varepsilon D^{-1/4 + \varepsilon} \),

\[
\text{vol}\left( \left\{ w \in \Gamma \backslash \mathbb{H} : \left| \frac{\sum_{A \in G_K} \text{vol}(\Gamma_A \backslash \mathcal{N}_A \cap B_R(w))}{\text{vol}(B_R) \sum_{A \in G_K} \text{vol}(\Gamma_A \backslash \mathcal{N}_A)} - \frac{1}{\text{vol}(\Gamma \backslash \mathbb{H})} \right| > c \right\} \right)
\]

converges to zero as \( D \to \infty \) along odd fundamental discriminants.

The fact that these geometric invariants equidistribute on almost every ball of different scales should not come as a surprise, and essentially boils down to the fact that a Heegner point has dimension 0, a closed geodesic has dimension 1, and a hyperbolic orbifold has dimension 2. For Heegner points, we need roughly \( R^2 \) balls to cover \( \Gamma \backslash \mathbb{H} \), so we require the number of Heegner points \#\( G_K \) corresponding to the genus \( G_K \) to be at least \( R^2 \) in order to expect equidistribution; this is the scale \( R \approx (-D)^{-1/4} \). For closed geodesics, on the other hand, \( R \) balls will cover roughly \( 1/R \) of \( \Gamma \backslash \mathbb{H} \), but a closed geodesic may intersect more than one ball, so we only require the total length \( \sum_{A \in G_K} \ell(C_A) \) of closed geodesics corresponding to the genus \( G_K \) to be at least \( R \); this is the scale \( R \approx D^{-1/2} \). Finally, we should expect equidistribution at all scales for hyperbolic orbifolds, since these are just (possibly uneven) coverings of \( \Gamma \backslash \mathbb{H} \).
Chapter 2

Integrals of Automorphic Forms and $L$-Functions

2.1 $L$-Functions

Recall that an $L$-function associated to an automorphic representation $\pi$ of $\text{GL}_n(\mathbb{A}_\mathbb{Q})$ is an Euler product of local $L$-functions,

$$L(s, \pi) := \prod_p L_p(s, \pi).$$

Each nonarchimedean local $L$-function $L_p(s, \pi)$ is of the form

$$L_p(s, \pi) = \prod_{j=1}^n \frac{1}{(1 - \alpha_{\pi,j}(p)p^{-s})}.$$

The completed $L$-function associated to $\pi$ is

$$\Lambda(s, \pi) = q(\pi)^{s/2}L_\infty(s, \pi)L(s, \pi),$$
with $q(\pi)$ a positive integer; it is called the conductor of $\pi$, and is such that $\alpha_{\pi,j}(p) \neq 0$ for all $j \in \{1, \ldots, n\}$ whenever $p \nmid q(\pi)$. The archimedean local $L$-function $L_\infty(s, \pi)$ is of the form
\[
L_\infty(s, \pi) = \pi^{-\frac{ns}{2}} \prod_{j=1}^{n} \Gamma \left( \frac{s + \kappa_{\pi,j}}{2} \right).
\]
A completed $L$-function satisfies a functional equation
\[
\Lambda(s, \pi) = \epsilon(\pi) \Lambda \left( 1 - s, \tilde{\pi} \right),
\]
where $\epsilon(\pi)$ is the root number of $\pi$, which has absolute value 1, and $\tilde{\pi}$ is the contragredient of $\pi$. For a squarefree integer $q$, we define $L_q(s, \pi) := \prod_{p|q} L_p(s, \pi)$ and $\Lambda^q(s, \pi) := \Lambda(s, \pi)L_q(s, \pi)^{-1}$.

The analytic conductor of $L(s, \pi)$ is defined to be
\[
q(\pi, s) := q(\pi) \prod_{j=1}^{n} (|s + \kappa_{\pi,j}| + 3).
\]
The $L$-function associated to $\pi$ satisfies the convexity bound
\[
L(s, \pi) \ll_{\varepsilon} q(\pi, s)^{\frac{1}{4} + \varepsilon}
\]
whenever $\Re(s) = 1/2$. Any improvement in lowering the exponent 1/4 is known as a subconvexity bound. The generalized Lindelöf hypothesis is the conjecture that the exponent 1/4 can be lowered to 0.

### 2.1.1 Riemann Zeta Function

The completed $L$-function of the Riemann zeta function is
\[
\Lambda(s) = \pi^{-s/2} \Gamma \left( \frac{s}{2} \right) \zeta(s),
\]
where for $\Re(s) > 1$,
\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \zeta_p(s)
\]
with
\[
\zeta_p(s) = \frac{1}{1 - p^{-s}}.
\]
The completed $L$-function satisfies the functional equation
\[
\Lambda(s) = \Lambda(1 - s).
\]
The analytic conductor of $\zeta(s)$ is
\[
q(1, s) = |s| + 3.
\]

### 2.1.2 Dirichlet $L$-Functions

For a primitive Dirichlet character $\chi$ modulo $q$,
\[
\Lambda(s, \chi) := \left(\frac{q}{\pi}\right)^{s/2} \Gamma\left(\frac{s + \kappa_\chi}{2}\right) L(s, \chi),
\]
where
\[
\kappa_\chi := \frac{1 - \chi(-1)}{2},
\]
and for $\Re(s) > 1$,
\[
L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p L_p(s, \chi)
\]
with
\[
L_p(s, \chi) := \frac{1}{1 - \chi(p)p^{-s}}.
\]
The completed $L$-function satisfies the functional equation

$$\Lambda(s, \chi) = \epsilon(\chi) \Lambda(1 - s, \overline{\chi})$$

with

$$\epsilon(\chi) := \frac{\tau(\chi)}{i^k \sqrt{q}},$$

where $\tau(\chi)$ is the Gauss sum of $\chi$. The analytic conductor of $L(s, \chi)$ is

$$q(\chi, s) = q(|s + \kappa| + \chi).$$

### 2.1.3 Hecke $L$-Functions

For $f \in B^*_\kappa(q, \chi)$,

$$\Lambda(s, f) := \left(\frac{\sqrt{q}}{\pi}\right)^s \Gamma\left(\frac{s + \kappa_f}{2} + \frac{it_f}{2}\right) \Gamma\left(\frac{s + \kappa'_f}{2} - \frac{it_f}{2}\right) L(s, f),$$

where

$$\kappa_f := \frac{1 - \epsilon_f}{2}, \quad \kappa'_f := \frac{1 - (-1)^{\kappa_f} \epsilon_f}{2},$$

and for $\Re(s) > 1$,

$$L(s, f) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p L_p(s, f)$$

with

$$L_p(s, f) := \frac{1}{1 - \lambda_f(p)p^{-s} + \chi(p)p^{-2s}}.$$

The completed $L$-function satisfies the functional equation

$$\Lambda(s, f) = \epsilon(f) \Lambda\left(1 - s, \tilde{f}\right)$$
with

\[
\epsilon(f) := i^{\kappa + 2\left\lfloor \frac{\kappa - \kappa_f}{2} \right\rfloor} \eta_f \frac{\rho_f(1)}{\rho_f(1)},
\]

where \( \eta_f \) is the eigenvalue of \( f \) of the involution

\[
(W_\kappa f)(z) := \left( \frac{-z}{|z|} \right)^{-\kappa} \overline{f} \left( \frac{1}{qz} \right).
\]

Note that \( \eta_f = 1 \) if \( \chi \) is the principal character, and so \( L(1/2, f) = 0 \) if \( \chi \) is the principal character and \( f \) is odd. The analytic conductor of \( L(s, f) \) is

\[
q(f, s) = q(\left| s + \kappa_f + it_f \right| + 3) \left( \left| s + \kappa'_f - it_f \right| + 3 \right).
\]

### 2.1.4 Hecke L-Functions of Twists of Newforms

When \( f \in \mathcal{B}_\kappa(q, \chi) \) and \( \psi \) is a primitive Dirichlet character modulo \( r \) with \( (q, r) = 1 \), then \( f \times \psi \in \mathcal{B}_\kappa(qr^2, \chi \psi^2) \), where

\[
(f \times \psi)(z) := \frac{1}{\tau(\psi)} \sum_{a \mod r} \overline{\psi}(a) f \left( z + \frac{a}{r} \right).
\]

We have that \( \epsilon_{f \times \psi} = \psi(-1) \epsilon_f \), so that

\[
\Lambda(s, f \times \psi) = \left( \frac{\sqrt{qr}}{\pi} \right)^s \Gamma \left( \frac{s + \kappa_{f \times \psi}}{2} + \frac{it_f}{2} \right) \Gamma \left( \frac{s + \kappa'_{f \times \psi}}{2} - \frac{it_f}{2} \right) L(s, f \times \psi),
\]

where

\[
\kappa_{f \times \psi} = \frac{1 - \psi(-1) \epsilon_f}{2}, \quad \kappa'_{f \times \psi} = \frac{1 - (1)^s \psi(-1) \epsilon_f}{2},
\]

and for \( \Re(s) > 1 \),

\[
L(s, f \times \psi) = \sum_{n=1}^{\infty} \frac{\lambda_f(n) \psi(n)}{n^s} = \prod_p L_p(s, f \times \psi)
\]
with
\[ L_p(s, f \times \psi) = \frac{1}{1 - \lambda_f(p)\psi(p)p^{-s} + \chi(p)\psi(p)^2p^{-2s}}. \]

The completed \( L \)-function satisfies the functional equation
\[ \Lambda(s, f \times \psi) = \epsilon(f \times \psi) \Lambda(1 - s, \tilde{f} \times \tilde{\psi}) \]
with
\[ \epsilon(f \times \psi) = \chi(r)\psi(q)\frac{\tau(\psi)^2}{r'} q^{\kappa + 2|\kappa - \kappa_{f \times \psi}|} j_{f \times \psi}. \]

In particular, \( L(1/2, f \times \psi) = 0 \) if \( \chi \) is the principal character, \( \psi \) is a quadratic character, and \( f \) is odd. The analytic conductor of \( L(s, f \times \psi) \) is
\[ q(f \times \psi, s) = qr^2 (|s + \kappa_{f \times \psi} + it_f| + 3) (|s + \kappa'_{f \times \psi} - it_f| + 3). \]

### 2.2 The Maaß–Selberg Relation

Let \( q = 1 \), so that there is only one cusp of \( \Gamma \backslash \mathbb{H} \), with the associated Eisenstein series being
\[ E(z, s) = \sum_{\gamma \in \Gamma} \Im(\gamma z)^s. \]

The function \( E(z, 1/2 + it) \) is not square-integrable for any \( t \in \mathbb{R} \). However, this is no longer the case when we replace the Eisenstein series with the truncated Eisenstein series
\[ g(z) = \Lambda^T E \left( z, \frac{1}{2} + it \right), \]
where
\[ \Lambda^T E(z, s) := E(z, s) - \sum_{\gamma \in \Gamma \backslash \Gamma} c(\gamma z, s), \]

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and \( c(z, s) = \Im(z)^s + \varphi(s) \Im(z)^{1-s} \) denotes the constant term in the Fourier expansion of \( E(z, s) \), with

\[
\varphi(s) = \frac{\Lambda(2 - 2s)}{\Lambda(2s)}.
\]

The function \( \Lambda^T E(z, s) \) is of rapid decay at the cusp of \( \Gamma \backslash \mathbb{H} \), and so is square-integrable.

To deal with the term \( |\langle |g|, f_0 \rangle|^2 \) appearing in Corollary 1.4.2, we must explicitly calculate the \( L^2 \)-norm of \( g \). This is known as the Maaß–Selberg relation.

**Lemma 2.2.1.** For any \( z = x + iy \in \mathbb{H} \), we have that \( \Im(z) \Im(\gamma z) \leq 1 \) if \( \gamma \notin \Gamma_\infty \). If \( \gamma \in \Gamma_\infty \), then \( \Im(\gamma z) = \Im(z) \).

**Proof.** For \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \), we have that

\[
\Im(\gamma z) = \frac{y}{(cx+d)^2 + c^2 y^2}.
\]

By the Bruhat decomposition, if \( \gamma \notin \Gamma_\infty \), then \( c \) must be nonzero, and consequently \( \Im(z) \Im(\gamma z) \leq 1 \). If \( \gamma \in \Gamma_\infty \), then it is clear that \( \Im(\gamma z) = \Im(z) \). \( \square \)

**Corollary 2.2.2.** If \( T \geq 1 \), then

\[
\Lambda^T E(z, s) = \begin{cases} 
E(z, s) - c(z, s) & \text{if } \Im(z) > T, \\
E(z, s) & \text{if } 1/T \leq \Im(z) \leq T.
\end{cases}
\]

With these results in hand, we can prove the following Maaß–Selberg relation.

**Proposition 2.2.3** ([Iwa02, Proposition 6.8]). For \( T \geq 1 \), and \( s \neq \overline{r}, s + \overline{r} \neq 1 \),

\[
\int_{\Gamma \backslash \mathbb{H}} \Lambda^T E(z, s) \Lambda^T E(z, r) \, d\mu(z)
= \frac{T^{s+\overline{r}} - 1}{s + \overline{r} - 1} + \frac{T^{\overline{r}}}{s - \overline{r}} + \varphi(s) \frac{T^{r-s}}{r - s} + \varphi(s) \varphi(r) \frac{T^{1-s-\overline{r}}}{1 - s - \overline{r}}. \tag{2.2.4}
\]

**Proof.** We initially assume that \( \Re(s), \Re(r) > 1 \) with \( \Re(s) - \Re(r) > 1 \); the identity then extends to all \( s, r \in \mathbb{C} \) with \( s \neq \overline{r} \) and \( s + \overline{r} \neq 1 \) by analytic continuation. We
first show that

\[ \int_{\Gamma \setminus \mathbb{H}} \Lambda^T E(z, s) \left( \Lambda^T E(z, r) - E(z, r) \right) d\mu(z) = 0. \]

Indeed, the left-hand side is equal to

\[ - \int_{\Gamma \setminus \mathbb{H}} \sum_{\gamma \in \Gamma' \setminus \Gamma \atop \Im(\gamma z) > T} \frac{c(\gamma z, r) \Lambda^T E(\gamma z, s) d\mu(z)}{\gamma z, r} = - \int_T^\infty \int_0^1 \frac{c(z, r) \Lambda^T E(z, s)}{y^2} dy \]

upon unfolding the integral. But \( c(z, r) \) is independent of \( x \), while for \( \Im(z) > T \geq 1 \), the zeroth Fourier coefficient of the function \( \Lambda^T E(z, s) \) vanishes via Corollary 2.2.2, and so this vanishes. Consequently,

\[ \int_{\Gamma \setminus \mathbb{H}} \Lambda^T E(z, s) \Lambda^T E(z, r) d\mu(z) = \int_{\Gamma \setminus \mathbb{H}} \Lambda^T E(z, s) E(z, r) d\mu(z). \]

The right-hand side can be written as

\[ \int_{\Gamma \setminus \mathbb{H}} \sum_{\gamma \in \Gamma' \setminus \Gamma \atop \Im(\gamma z) \leq T} \Im(\gamma z)^s E(\gamma z, r) d\mu(z) + \int_{\Gamma \setminus \mathbb{H}} \sum_{\gamma \in \Gamma' \setminus \Gamma \atop \Im(\gamma z) > T} \varphi(s) \Im(\gamma z)^{1-s} E(\gamma z, r) d\mu(z). \]

The first term unfolds to

\[ \int_T^\infty \int_0^1 y^s E(z, r) \frac{dy}{y^2} = \int_T^\infty \int_0^1 y^s c(z, r) \frac{dy}{y^2} = \frac{T^{s+\tau-1}}{s+\tau-1} + \varphi(r) \frac{T^s - \tau}{s - \tau} \]

and similarly the second term unfolds to

\[ \int_T^\infty \varphi(s) y^{1-s} c(z, r) \frac{dy}{y^2} = \varphi(s) \frac{T^{\tau-s}}{\tau - s} + \varphi(s) \varphi(r) \frac{T^{1-s-\tau}}{1 - s - \tau}. \]

This yields the result. \( \square \)
Corollary 2.2.5. We have that
\[ \int_{\Gamma \setminus \mathbb{H}} \bigg| \Lambda T E \left( z, \frac{1}{2} + it_g \right) \bigg|^2 d\mu(z) = \log \left( \left( \frac{1}{4} + t_g^2 \right) T^2 \right) + O \left( (\log t_g)^{2/3} (\log \log t_g)^{1/3} \right). \]

Proof. We take \( s = r = 1/2 + it_g + \varepsilon \) with \( \varepsilon > 0 \) in the Maaß–Selberg relation (2.2.4) to obtain
\[ \int_{\Gamma \setminus \mathbb{H}} \bigg| \Lambda T E \left( z, \frac{1}{2} + it_g + \varepsilon \right) \bigg|^2 d\mu(z) = \frac{T^{2\varepsilon}}{2\varepsilon} - \left| \varphi \left( \frac{1}{2} + it_g + \varepsilon \right) \right|^2 \frac{T^{-2\varepsilon}}{2\varepsilon}. \]

Using the Taylor expansions
\[ T^{2\varepsilon} = 1 + 2\varepsilon \log T + O \left( \varepsilon^2 \right), \]
\[ \varphi \left( \frac{1}{2} + it_g + \varepsilon \right) = \varphi \left( \frac{1}{2} + it_g \right) + \varepsilon \varphi' \left( \frac{1}{2} + it_g \right) + O \left( \varepsilon^2 \right), \]

together with the fact that \( |\varphi(1/2 + it_g)| = 1 \) and that
\[ \frac{\varphi'}{\varphi} \left( \frac{1}{2} + it_g \right) = -4 \Re \left( \frac{\Lambda'}{\Lambda} \left( 1 + 2it_g \right) \right) \]
\[ = 2 \log \pi - 2 \Re \left( \frac{\Gamma'}{\Gamma} \left( \frac{1}{2} + it_g \right) \right) - 4 \Re \left( \frac{\zeta'}{\zeta} \left( 1 + 2it_g \right) \right), \]
\[ \text{(2.2.6)} \]

we find that
\[ \int_{\Gamma \setminus \mathbb{H}} |g(z)|^2 d\mu(z) = 2 \log T - 2 \log \pi + 2 \Re \left( \frac{\Gamma'}{\Gamma} \left( \frac{1}{2} + it_g \right) \right) + 4 \Re \left( \frac{\zeta'}{\zeta} \left( 1 + 2it_g \right) \right). \]

It remains to use Stirling’s formula to find that
\[ 2 \Re \left( \frac{\Gamma'}{\Gamma} \left( \frac{1}{2} + it_g \right) \right) = \log \left( \frac{1}{4} + t_g^2 \right) + O \left( \frac{1}{t_g} \right), \]
\[ \text{(2.2.7)} \]
and [IK04, Theorem 8.29] to give the bound

$$\zeta'(1 + 2it_g) \ll (\log t_g)^{2/3} (\log \log t_g)^{1/3}. \quad (2.2.8)$$

## 2.3 The Watson–Ichino Formula

To deal with spectral sums involving terms of the form $|\langle |g|^2, f \rangle_q|^2$, one can use the Watson–Ichino formula, which essentially states that the square of the integral over $\Gamma_0(q) \backslash \mathbb{H}$ of the product of three automorphic forms is equal to a product of completed $L$-functions involving those automorphic forms. In particular, if $g \in B^*_\kappa(q, \chi)$ and $f \in B^*_0(\Gamma_0(r))$ for some $r \mid q$, then from [Ich08, Theorem 1.1] and [Wat08, Theorem 3],

$$\left| \langle |g|^2, f \rangle_q \right|^2 = \frac{I'(\langle |g|^2, f \rangle)}{8} \frac{\Lambda \left( \frac{1}{2}, g \times \tilde{g} \times f \right)}{\Lambda(1, \text{ad} g)^2 \Lambda(1, \text{sym}^2 f)} ,$$

where

$$I'(\langle |g|^2, f \rangle) = \prod_{p \mid q} I'_p(\langle |g|^2, f \rangle)$$

is a product of local constants $I'_p(\langle |g|^2, f \rangle)$ that depend on the local components at $p$ of the automorphic representations of $\text{GL}_2(\mathbb{A}_q)$ corresponding to $f$ and $g$. In certain cases, the constants $I'_p(\langle |g|^2, f \rangle)$ have been calculated explicitly. Note that the numerator in the Watson–Ichino formula factorises:

$$\Lambda \left( s, g \times \tilde{g} \times f \right) = \Lambda(s, f) \Lambda(s, \text{ad} g \times f) .$$

Similar results also hold when either $f$ or $g$ is replaced with an Eisenstein series.

**Proposition 2.3.1** ([BK17b, Equations (2.2) and (4.2)]). For $f, g \in B_0(\Gamma)$,

$$\left| \langle |g|^2, f \rangle \right|^2 = \frac{1}{8} \frac{\Lambda \left( \frac{1}{2}, f \right) \Lambda \left( \frac{1}{2}, \text{sym}^2 g \times f \right)}{\Lambda(1, \text{sym}^2 g)^2 \Lambda(1, \text{sym}^2 f)} .$$
\[ \left| \left\langle |g|^2, E\left( 1, \frac{1}{2} + it \right) \right\rangle \right|^2 = \frac{1}{4} \frac{\Lambda \left( \frac{1}{2} + it \right) \Lambda \left( \frac{1}{2} - it \right) \Lambda \left( \frac{1}{2} + it, \text{sym}^2 g \right) \Lambda \left( \frac{1}{2} - it, \text{sym}^2 g \right)}{\Lambda \left( 1, \text{sym}^2 g \right)^2 \Lambda \left( 1 + 2it \right) \Lambda \left( 1 - 2it \right)} \].

**Proof.** The first identity follows from [Wat08, Theorem 3]. For the second identity, we recall that the Hecke eigenvalues \( \lambda_g(n) \) of \( g \) satisfy \( \rho_g(1) \lambda_g(n) = \sqrt{n} \rho_g(n) \) for all \( n \geq 1 \), and that \( \rho_g(-n) = \pm \rho_g(n) \). So for \( \Re(s) \) sufficiently large,

\[ \int_{\Gamma \setminus \mathbb{H}} |g(z)|^2 E(z, s) d\mu(z) \] (2.3.2)

is equal to

\[ \frac{2|\rho_g(1)|^2}{(4\pi)^{s-1}} \sum_{n=1}^{\infty} \frac{\lambda_g(n)^2}{n^s} \int_{0}^{\infty} y^{s-1} W_{0, it_g}(y)^2 \frac{dy}{y} \]

upon unfolding the integral. Using the identity [GR07, 9.235.2]

\[ W_{0, it_g}(y) = \sqrt{\frac{y}{\pi}} K_{it_g} \left( \frac{y}{2} \right) \] (2.3.3)

and the Mellin–Barnes formula [GR07, 6.576.4]

\[ \int_{0}^{\infty} y^s K_{it_g}(y) K_{it_f}(y) \frac{dy}{y} = 2^{s-3} \frac{\Gamma \left( \frac{s}{2} + \frac{i(t_g + t_f)}{2} \right) \Gamma \left( \frac{s}{2} + \frac{i(t_g - t_f)}{2} \right) \Gamma \left( \frac{s}{2} - \frac{i(t_g - t_f)}{2} \right) \Gamma \left( \frac{s}{2} - \frac{i(t_g + t_f)}{2} \right)}{\Gamma(s)} \] (2.3.4)

we find that (2.3.2) is equal to

\[ \frac{|\rho_g(1)|^2}{\pi^s} \frac{\Gamma \left( \frac{s}{2} + it_g \right) \Gamma \left( \frac{s}{2} \right)^2 \Gamma \left( \frac{s}{2} - it_g \right)}{\Gamma(s)} \sum_{n=1}^{\infty} \frac{\lambda_g(n)^2}{n^s} \]

We have that

\[ \sum_{n=1}^{\infty} \frac{|\lambda_g(n)|^2}{n^s} = \frac{\zeta(s) L(s, \text{sym}^2 g)}{\zeta(2s)} \]

Taking the residue of (2.3.2) at \( s = 1 \) and using the fact that \( \langle g, g \rangle = 1 \) and that the
residue of $E(z, s)$ at $s = 1$ is $\text{vol} (\Gamma \backslash \mathbb{H})^{-1} = 3/\pi$, we find that

$$|\rho_g(1)|^{-2} = \frac{2}{\pi} \Gamma \left( \frac{1}{2} + it_g \right) \Gamma \left( \frac{1}{2} - it_g \right) L \left( 1, \text{sym}^2 g \right) = 2 \Lambda \left( 1, \text{sym}^2 g \right). \quad (2.3.5)$$

It follows that (2.3.2) is equal to

$$\frac{1}{2\pi^{s-1}} \frac{\Gamma \left( \frac{s}{2} + it_g \right) \Gamma \left( \frac{s}{2} - it_g \right) \zeta(s)L \left( s, \text{sym}^2 g \right)}{\Gamma \left( \frac{1}{2} + it_g \right) \Gamma \left( \frac{1}{2} - it_g \right) \zeta(2s)L \left( 1, \text{sym}^2 g \right)} = \frac{1}{2} \frac{\Lambda(s) \Lambda \left( s, \text{sym}^2 g \right)}{\Lambda \left( 1, \text{sym}^2 g \right) \Lambda(2s)}. \quad (2.3.6)$$

By analytic continuation, we may take $s = 1/2 - it$, and the result again follows from Stirling’s approximation.

We may also prove a similar result when $g$ is an Eisenstein series.

**Proposition 2.3.6** ([LS95, Equation (17)], [Spi03, Theorem 4.1]). For $f \in \mathcal{B}_0(\Gamma)$,

$$\left\| \left\langle \left\| E \left( z, \frac{1}{2} + it \right) \right\|^2, f \right\rangle \right\|^2 = \frac{1}{2} \frac{\Lambda \left( \frac{1}{2}, f \right)^2 \Lambda \left( \frac{1}{2} + 2it_g, f \right) \Lambda \left( \frac{1}{2} - 2it_g, f \right)}{\Lambda(1 + 2it_g) \Lambda(1 - 2it_g) \Lambda(1, \text{sym}^2 f)}. \quad (2.3.7)$$

**Proof.** We recall that

$$E(z, s) = y^s + \frac{\Lambda(2 - 2s)}{\Lambda(2s)} y^{1-s} + \sum_{n=-\infty}^{\infty} \rho(n, s) W_{0,s-1/2}(4\pi |n| y) e(nx)$$

with

$$\rho(n, s) = \frac{|n|^{s-1} \sigma_{1-2s}(|n|)}{\Lambda(2s)},$$

where

$$\sigma_{1-2s}(|n|) := \sum_{d||n|} d^{1-2s}.$$ 

It follows by unfolding that for $\Re(s)$ sufficiently large,

$$\int_{\Gamma \backslash \mathbb{H}} E \left( z, \frac{1}{2} + it_g \right) E(z, s) f(z) d\mu(z) \quad (2.3.7)$$
vanishes if \( f \) is odd, while if \( f \) is even, this is equal to

\[
\frac{2\rho_f(1)}{\Lambda(1 + 2it_g)(4\pi)^{s-1}} \sum_{n=1}^{\infty} \frac{n^{it_g}\sigma-2it_g(n)\lambda_f(n)}{n^s} \int_0^\infty y^{s-1}W_{0,it_g}(y)W_{0,it_f}(y) \frac{dy}{y}.
\]

Using (2.3.3) and (2.3.4) and the Ramanujan identity [LS95, Equation (15)]

\[
\sum_{n=1}^{\infty} \frac{n^{it_g}\sigma_{-2it_g}(n)\lambda_f(n)}{n^s} = \frac{L(s + it_g, f) L(s - it_g, f)}{\zeta(2s)},
\]

we find that (2.3.7) is equal to

\[
\frac{\rho_f(1)}{\pi^{s-1/2-it_g}} \frac{\Gamma\left(\frac{s}{2} + \frac{i(t_g+t_f)}{2}\right)\Gamma\left(\frac{s}{2} + \frac{i(t_g-t_f)}{2}\right)\Gamma\left(\frac{s}{2} - \frac{i(t_g-t_f)}{2}\right)\Gamma\left(\frac{s}{2} - \frac{i(t_g+t_f)}{2}\right)}{\Gamma\left(\frac{1}{2} + it_g\right)\Gamma(s)}
\times \frac{L(s + it_g, f) L(s - it_g, f)}{\zeta(1 + 2it_g)\zeta(2s)} = \frac{\rho_f(1)\Lambda(s + it_g, f)\Lambda(s - it_g, f)}{\Lambda(1 + 2it_g)\Lambda(2s)}.
\]

By analytic continuation, we may take \( s = 1/2 - it_g \), from which the result for \( f \) even follows via (2.3.5). Finally, since \( L\left(\frac{1}{2}, f\right) \) vanishes when \( f \) is odd, this identity is valid for all \( f \in \mathcal{B}_0(\Gamma) \).

Finally, a similar result holds when \( f \) is also an Eisenstein series. In this case, the integral is no longer convergent. One can work around this issue by replacing this integral with a regularised integral. This is defined by Zagier [Zag82]. For a continuous function \( F: \Gamma\backslash \mathbb{H} \to \mathbb{C} \) of moderate growth, so that there exists \( c_j, \alpha_j \in \mathbb{C} \) and nonnegative integers \( n_j \) such that

\[
F(z) = \sum_{j=1}^{\ell} \frac{c_j}{n_j!} y^{\alpha_j} (\log y)^{n_j} + O_N \left( y^{-N} \right)
\]

for all \( N \geq 0 \) at the cusp at infinity, with no \( \alpha_j \) equal to 0 or 1, there exists a function \( \mathcal{E}(z) \) that is a linear combination of Eisenstein series and derivatives of Eisenstein series.
series $E(z, \alpha)$, each satisfying $\Re(\alpha) > 1/2$, such that for some $\delta > 0$,

$$F(z) - E(z) = O\left(y^{1/2-\delta}\right)$$

at the cusp at infinity. The regularised inner product of two functions $f, g$ such that $f\bar{g} = F$ is continuous and of moderate growth is defined to be

$$(f, g)_{\text{reg}} := \int_{\Gamma \backslash \mathbb{H}} (F(z) - E(z)) \, d\mu(z).$$

Moreover, if $f$ and $g$ depend on complex parameters, then we may extend both sides via analytic continuation where possible.

**Proposition 2.3.8 ([Zag82, Equation (44)])**. We have that

$$(E(\cdot, s_1)E(\cdot, s_2), E(\cdot, s))_{\text{reg}} = \frac{\Lambda(\overline{s} + s_1 + s_2 - 1) \Lambda(\overline{s} + s_1 - s_2) \Lambda(\overline{s} - s_1 + s_2) \Lambda(\overline{s} - s_1 - s_2 + 1)}{\Lambda(2\overline{s}) \Lambda(2s_1) \Lambda(2s_2)}. \quad (2.3.9)$$

In practice, it is the finite part $L(s, \pi)$ of a completed $L$-function $\Lambda(s, \pi)$ that is difficult to deal with; this is because the asymptotic behaviour of the archimedean components of a completed $L$-function can be inferred via Stirling’s approximation.

**Lemma 2.3.10.** The product of the archimedean components of the completed $L$-functions in Propositions 2.3.1, 2.3.6 (with $t = t_f$), and 2.3.8 (with $s_1 = s_2 = 1/2 + it_g$ and $s = 1/2 + it_f$) is equal to

$$\frac{8\pi^2 \exp(-\pi\Omega(t_f, t_g))}{(1 + t_f)(1 + 2t_g + t_f)^{1/2}(1 + |2t_g - t_f|)^{1/2}} \times \left(1 + O\left(\frac{1}{1 + t_f} + \frac{1}{1 + 2t_g + t_f} + \frac{1}{1 + |2t_g - t_f|}\right)\right), \quad (2.3.11)$$
where
\[
\Omega(t_f, t_g) := \begin{cases} 
0 & \text{if } 0 < t_f \leq 2t_g, \\
t_f - 2t_g & \text{if } t_f > 2t_g.
\end{cases}
\]

Proof. The product of the archimedean components of the completed $L$-functions is equal to
\[
\pi \left( \frac{1}{4} + \frac{i(t_g + t_f)}{2} \right) \Gamma \left( \frac{1}{4} + \frac{i(t_g - t_f)}{2} \right) \Gamma \left( \frac{1}{4} - \frac{i(t_g + 2t_f)}{2} \right) \\
\Gamma \left( \frac{1}{2} + it_g \right)^2 \Gamma \left( \frac{1}{2} - it_f \right)^2
\times \\
\pi \left( \frac{1}{4} - \frac{it_f}{2} \right)^2 \Gamma \left( \frac{1}{4} - \frac{it_f}{2} \right)^2
\]
\[
\frac{\Gamma \left( \frac{1}{2} + it_f \right) \Gamma \left( \frac{1}{2} - it_f \right)}{\Gamma \left( \frac{1}{2} + it_f \right) \Gamma \left( \frac{1}{2} - it_f \right)}.
\]

The result then follows directly from Stirling’s approximation. ∎

On occasion, we also need to deal with lower bounds for $L(1, \text{sym}^2 f)$. This is less complex than values of $L$-functions within the critical strip $0 < \Re(s) < 1$; indeed, the following is known.

**Lemma 2.3.12** (Hoffstein–Lockhart [HL94]). For $f \in B_0(\Gamma)$,
\[
L(1, \text{sym}^2 f) \gg \frac{1}{\log(t_f + 3)}.
\]

### 2.4 Ranges of the Spectral Decomposition

#### 2.4.1 Ranges of the Spectral Decomposition for the $L^4$-Norm

We divide the spectral decomposition of the $L^4$-norm of $g(z) = \Lambda^T E(z, 1/2 + it_g)$ given in Corollary 1.4.2 into different parts, then analyse each part individually.

There are two main ranges of the continuous spectrum to consider, which depend on a small fixed parameter $\delta > 0$:

- the initial range $0 \leq |t| < 2t_g - t_g^{1-\delta}$, and
the tail range $|t| > 2t_g + t_g^{1-\delta}$.

Both of these ranges should contribute a negligible amount via subconvexity estimates for the $L$-functions appearing in the integral.

For the contribution from the cuspidal spectrum, one can break up the summation over $B_0(\Gamma)$ into different ranges depending on $t_f$. There are four main ranges of the cuspidal spectrum left to consider, which depend on a fixed small parameter $\delta > 0$:

- the short initial range $0 \leq t_f \leq t_g^{1-\delta}$,
- the bulk range $t_g^{1-\delta} < t_f < 2t_g - t_g^{1-\delta}$,
- the short transition range $2t_g - t_g^{1-\delta} \leq t_f \leq 2t_g + t_g^{1-\delta}$, and
- the tail range $t_f > 2t_g + t_g^{1-\delta}$.

We divide the spectral sum into these particular ranges due to the size of the product of analytic conductors of $L$-functions. The analytic conductor of

$$L \left( \frac{1}{2}, f \right)^2 L \left( \frac{1}{2} + 2it_g, f \right) L \left( \frac{1}{2} - 2it_g, f \right)$$

is approximately

$$\left( \frac{1}{4} + t_f^2 \right)^2 \left( \frac{1}{4} + (2t_g^2 - t_f^2) \right) \left( \frac{1}{4} - (2t_g^2 - t_f^2) \right),$$

which is large when $t_f$ lies in the bulk range, but is small in the short initial range, and drops in the short transition range. For this reason, we expect the main contribution to be from the bulk range, while the contribution from the two short ranges ought to be negligible. Assuming the generalised Lindelöf hypothesis, this can be proven this directly; see [BK17b, Section 5]. Finally, the exponential decay in (2.3.11) arising from the archimedean components of the completed $L$-functions indicates that the tail range should contribute a negligible amount.
2.4.2 Ranges of the Spectral Decomposition for the Variance

Let \( g \in \mathcal{B}_0(\Gamma) \) be a Hecke eigenform, and let \( R \asymp t_g^{-\delta} \) for some \( \delta > 0 \). We again divide the spectral decomposition of the variance \( \text{Var}(g; R) \) given in Proposition 1.5.8 into different parts.

There are again two main ranges of the continuous spectrum to consider, which depend on a small fixed parameter \( \delta' > 0 \):

- the initial range \( 0 \leq |t| < 2t_g - t_g^{1-\delta'} \), and
- the tail range \( |t| > 2t_g + t_g^{1-\delta'} \).

Both of these ranges should contribute a negligible amount via subconvexity estimates for the \( L \)-functions appearing in the integral, regardless of the size of \( \delta \).

The division of the cuspidal spectrum into parts depends on \( \delta \). When \( R \asymp t_g^{-\delta} \) with \( 0 < \delta < 1 \), the ranges are:

- the short initial range \( 0 < t_f \leq t_g^\delta \),
- the polynomial decay range \( t_g^\delta < t_f < 2t_g + t_g^{1-\delta} \),
- the tail range \( t_f \geq 2t_g + t_g^{1-\delta} \).

When \( R \asymp t_g^{-\delta} \) with \( \delta > 1 \), there is an additional range, and these ranges depend on a small fixed parameter \( \delta' > 0 \):

- the short initial range \( 0 < t_f \leq t_g^{1-\delta'} \),
- the bulk range \( t_g^{1-\delta'} < t_f < 2t_g - t_g^{1-\delta'} \),
- the short transition range \( 2t_g - t_g^{1-\delta'} \leq t_f \leq 2t_g + t_g^{1-\delta'} \), and
- the tail range \( t_f > 2t_g + t_g^{1-\delta'} \).
We expect the short and tail ranges to be negligible for the same reasons as for the \( L^4 \)-norm of \( g \). When \( 0 < \delta < 1 \), the decay of \( h_R(t) \) given in Lemma 1.5.9 suggests that the polynomial decay range should be negligible, whereas when \( 0 < \delta < 1 \), Lemma 1.5.9 shows that the bulk range will give the same contribution as the bulk range for the \( L^4 \)-norm of \( g \).

Similarly, when \( g(z) = E(z, 1/2 + it_g) \), the spectral decomposition of the variance \( \text{Var}(g; R) \) given in Proposition 1.5.14 can be broken up into different ranges.

The two ranges of the continuous spectrum, which depend on a small fixed parameter \( \delta' > 0 \), are:

- the initial range \( 0 \leq |t| < 2t_g - t_g^{1 - \delta'} \), and
- the tail range \( |t| > 2t_g + t_g^{1 - \delta'} \).

Again, these should contribute a negligible amount.

The cuspidal spectrum can be broken into five ranges, which depend on a small fixed parameter \( 0 < \delta' < 1 - \delta \):

- the short initial range \( 0 < t_f \leq t_g^\delta \),
- the short initial polynomial decay range \( t_g^\delta < t_f < t_g^{1 - \delta'} \),
- the bulk polynomial decay range \( t_g^{1 - \delta'} \leq t_f \leq 2t_g - t_g^\delta \),
- the short transition polynomial decay range \( 2t_g - t_g^\delta < t_f < 2t_g + t_g^\delta \),
- the tail range \( t_f \geq 2t_g + t_g^\delta \).

As \( 0 < \delta < 1 \), Lemma 1.5.9 shows that each range should be negligible.
Chapter 3

Sharp Bounds for the $L^4$-Norm of a Truncated Eisenstein Series

We wish to determine sharp bounds for

$$\|g\|_{L^4(\Gamma\backslash \mathbb{H})}^4 = \int_{\Gamma\backslash \mathbb{H}} |g(z)|^4 d\mu(z)$$

with $g(z) = \Lambda^T E(z, 1/2 + it_g)$ in terms of $t_g$. Via Corollary 1.4.2, this is reduced to understanding bounds for the inner product of $|g|^2$ with eigenfunctions of the Laplacian. The first term in this decomposition is the inner product of $|g|^2$ with the constant function

$$f_0(z) = \frac{1}{\sqrt{\text{vol}(\Gamma\backslash \mathbb{H})}},$$

and Corollary 2.2.5 shows that

$$|\langle |g|^2, f_0 \rangle|^2 = \left( \frac{\log \left( \frac{1}{4} + t_g^2 \right)}{\text{vol}(\Gamma\backslash \mathbb{H})} \right)^2 + O_T \left( (\log t_g)^{5/3} (\log \log t_g)^{1/3} \right).$$

For the remaining terms, we divide into ranges as discussed in Section 2.4.1 and treat each range individually.
3.1 Spectral Methods to Bound the Continuous Spectrum

From Corollary 1.4.2, we must bound

\[
\frac{1}{4\pi} \int_{-\infty}^{\infty} \left| \left\langle |g|^2, E \left( \cdot, \frac{1}{2} + it \right) \right\rangle \right|^2 \, dt.
\]

Lemma 3.1.1 ([Spi03, Theorem 3.3]). There exists a positive constant \( c > 0 \) such that

\[
\frac{1}{4\pi} \int_{-\infty}^{\infty} \left| \left\langle |g|^2, E \left( \cdot, \frac{1}{2} + it \right) \right\rangle \right|^2 \, dt \leq 108T + O \left( t_g^{-c} \right).
\]

Here \( c \) is any constant less than \( 1/2 - 2\theta \), where \( \theta \) is a positive constant such that

\[
\zeta \left( \frac{1}{2} + it \right) \ll \varepsilon (|t| + 1)^{\theta + \varepsilon}.
\]

The best bound known is \( \theta = 13/84 \), due to Bourgain [Bou17, Theorem 5].

3.2 Reduction to Untruncated Eisenstein Series for the Cuspidal Spectrum

From Corollary 1.4.2, we must bound

\[
\sum_{f \in \mathcal{B}_0(\Gamma)} \left| \left\langle |g|^2, f \right\rangle \right|^2.
\]

First, we observe that \( g(z) = \Lambda^T E \left( z, 1/2 + it_g \right) \) can be replaced by \( E \left( z, 1/2 + it_g \right) \).

Lemma 3.2.1 ([Spi03, Theorem 4.2]). We have that

\[
\sum_{f \in \mathcal{B}_0(\Gamma)} \left| \left\langle |g|^2, f \right\rangle \right|^2 \leq \sum_{f \in \mathcal{B}_0(\Gamma)} \left| \left\langle E \left( \cdot, \frac{1}{2} + it_g \right), f \right\rangle \right|^2 + O_T \left( (\log t_g)^2 \right).
\]
This allows us to use Proposition 2.3.6 and Lemma 2.3.10. We divide the cuspidal spectrum into four ranges, as discussed in Section 2.4.1. The convexity bound for the associated $L$-functions together with the Weyl law shows that the tail range is negligible. So it remains to bound the first three ranges. In [Spi03, Chapter 5], Spinu uses the large sieve to bound these three ranges and obtains the weaker bound

$$\|g\|_{L^4(\Gamma \backslash \mathbb{H})} \ll T^{\varepsilon} t_g.$$

In [Spi03, Chapter 6], Spinu sketches a proof of the stronger result

$$\|g\|_{L^4(\Gamma \backslash \mathbb{H})} \ll T \left( \log \left( \frac{1}{4} + t_g^2 \right) \right)^2.$$

However, the only range dealt with in this proof sketch is bulk range, namely $2\alpha t_g \leq t_f \leq 2(1 - \alpha)t_g$ for any $\alpha > 0$; it is not shown how to deal with the short initial and transition ranges.

Nevertheless, we will show how to deal with these remaining ranges and thereby complete Spinu’s work; the key estimates required are due to Jutila [Jut04], Ivić [Ivi01], and Jutila and Motohashi [JM05].

### 3.3 Weaker Bounds via the Large Sieve

In [Spi03, Chapter 5], Spinu uses the large sieve to bound the initial range $0 < t_f < 2t_g - t_g^{1-\delta}$, which is the union of the short initial and bulk ranges. For this range, the idea is to divide into dyadic intervals $H \leq t_f \leq 2H$, upon which (2.3.11) is bounded by $H^{-1}t_g^{1+\frac{\delta}{2}}$, and show the following.
Lemma 3.3.1 ([Spi03, Proposition 5.4]). We have that

$$\sum_{H \leq t_{f} \leq 2H} L \left( \frac{1}{2}, f \right)^2 \left| L \left( \frac{1}{2} + 2it_{g}, f \right) \right|^2 \ll \varepsilon H_{g}^{1+\varepsilon}$$

uniformly in $H \leq 2t_{g} - t_{g}^{1-\delta}$.

Corollary 3.3.2. We have that

$$\sum_{0 < t_{f} \leq 2t_{g} - t_{g}^{1-\delta}} \frac{\Lambda \left( \frac{1}{2}, f \right)^2 \Lambda \left( \frac{1}{2} + 2it_{g}, f \right) \Lambda \left( \frac{1}{2} - 2it_{g}, f \right)}{\Lambda (1 + 2it_{g})^2 \Lambda (1 - 2it_{g})^2 \Lambda (1, \text{sym}^2 f)} \ll \varepsilon t_{g}^{\frac{2}{3} + \varepsilon}.$$

To prove this, Spinu first uses the Cauchy–Schwarz inequality, so that

$$\sum_{H \leq t_{f} \leq H} L \left( \frac{1}{2}, f \right)^2 \left| L \left( \frac{1}{2} + 2it_{g}, f \right) \right|^2 \leq \left( \sum_{H \leq t_{f} \leq 2H} L \left( \frac{1}{2}, f \right)^4 \right)^{1/2} \left( \sum_{H \leq t_{f} \leq 2H} \left| L \left( \frac{1}{2} + 2it_{g}, f \right) \right|^4 \right)^{1/2},$$

and then use the large sieve to bound each of these terms, obtaining

$$\sum_{H \leq t_{f} \leq 2H} L \left( \frac{1}{2}, f \right)^4 \ll \varepsilon H^{2+\varepsilon}$$

and

$$\sum_{H \leq t_{f} \leq 2H} \left| L \left( \frac{1}{2} + 2it_{g}, f \right) \right|^4 \ll \varepsilon H_{g}^{1+\varepsilon}.$$
$|2t_g - t_f| \sim H$, so that (2.3.11) bounded by a constant multiple of $H^{-1/2}t_g^{-3/2}$. It therefore suffices to show the following.

**Lemma 3.3.3** ([Spi03, Proposition 5.5]). We have that

$$
\sum_{H < |t_f - 2t_g| < 2H} \left| L \left( \frac{1}{2}, f \right) \right|^2 \left| L \left( \frac{1}{2} + 2it_g, f \right) \right|^2 \ll H^{1/2}t_g^{-\frac{3+\delta}{2}}
$$

uniformly in $1 \leq H \ll t_g^{1-\delta}$.

**Corollary 3.3.4.** We have that

$$
\sum_{2t_g - t_g^{-1+\delta} < t_f < 2t_g + t_g^{-1+\delta}} \frac{\Lambda \left( \frac{1}{2}, f \right) \Lambda \left( \frac{1}{2} + 2it_g, f \right) \Lambda \left( \frac{1}{2} - 2it_g, f \right)}{\Lambda(1 + 2it_g)^2\Lambda(1 - 2it_g)^2\Lambda(1, \text{sym}^2 f)} \ll \epsilon t_g^{\frac{3}{2} + \epsilon}.
$$

Spinu proves this by using the approximate functional equation to replace $L(1/2, f)$ and $L(1/2 + 2it_g, f)$ by Dirichlet polynomials of length $t_g^2$ and $H^{1/2}t_g^{1/2}$ respectively. One then extends the summation to $|t_f - 2t_g| \ll H^{1/2}t_g^{1+\delta}$ by introducing a smooth weight, which is advantageous in the ensuing application of the Kuznetsov formula. The delta term is easily estimated. To analyse the Kloosterman term, some work is required to bound the transform of the test function.

**Remark 3.3.5.** In fact, Spinu’s treatment of the short transition range is in some sense overkill, for one can in fact use the local large sieve, as stated in [Luo14, Lemma], to bound this range; see [Luo14, Proof of Theorem].

### 3.4 Spectral Methods to Bound the Short Initial Range

From [IK04, Theorem 8.29], we have that bound

$$
\frac{1}{\zeta(1 + it)} \ll (\log t)^{2/3}(\log \log t)^{1/3}.
$$
It therefore suffices to show that
\[
\sum_{0 < t_f < t_g^{-\delta}} \frac{L \left( \frac{1}{2}, f \right)^2 \left| L \left( \frac{1}{2} + 2it_g, f \right) \right|^2}{(1 + t_f)(1 + 2t_g + t_f)^{1/2}(1 + 2t_g - t_f)^{1/2} L(1, \text{sym}^2 f)} \ll t_g^{-\delta'}
\]
for some $\delta' > 0$. We divide the short transition range $0 < t_f < t_g^{1-\delta}$ into dyadic intervals $H \leq t_f < 2H$, of which there are roughly $\log t_g$ intervals, on which
\[
(1 + t_f)(1 + 2t_g + t_f)^{1/2}(1 + 2t_g - t_f)^{1/2} \approx Ht_g.
\]
It then suffices to show that for $H \ll t_g^{1-\delta}$,
\[
\sum_{H \leq t_f \leq 2H} \frac{L \left( \frac{1}{2}, f \right)^2 \left| L \left( \frac{1}{2} + 2it_g, f \right) \right|^2}{L(1, \text{sym}^2 f)} \ll Ht_g^{1-\delta'}.
\]
This bound follows from the work of Jutila [Jut04], Ivić [Ivi01], and Jutila and Motohashi [JM05]. It is worth noting that the purpose of these works is to obtain Weyl-type subconvexity bounds
\[
L \left( \frac{1}{2} + it, f \right) \ll \varepsilon \left( L \left( \frac{1}{2} + it, f \right) \right)^{\frac{1}{6} + \varepsilon}
\]
for Hecke eigenforms $f \in \mathcal{B}_0(\Gamma)$, so long as $|t|$ is not too close to $t_f$. Conveniently, their methods to obtain such bounds involve obtaining bounds for the exact type of spectral sum that we are studying.

**Lemma 3.4.1.** For $t \geq 0$ and $H \gg 1$, we have that
\[
\sum_{H \leq t_f \leq 2H} \frac{L \left( \frac{1}{2}, f \right)^2 \left| L \left( \frac{1}{2} + it, f \right) \right|^2}{L(1, \text{sym}^2 f)} \ll_{\varepsilon} \begin{cases} H^{2+\varepsilon} & \text{if } H \geq t^{2/3}, \\ t^{4/3+\varepsilon} & \text{if } t^{1/3} \leq H \leq t^{2/3}, \\ H^{8/3+\varepsilon} & \text{if } t^{1/3} \leq H \leq t^{1/2}, \\ H^{2/3+\varepsilon}t^{2/3+\varepsilon} & \text{if } H \leq t^{1/3}. \end{cases}
\]
Proof. For $H \geq t^{1/2}$, this follows from [JM05, Theorem 2], which states that for $t \geq 0$ and $H \gg 1$,

$$\sum_{H \leq t_f \leq 2H} \frac{L\left(\frac{1}{2}, f\right)^2 \left|L\left(\frac{1}{2} + it, f\right)\right|^2}{L(1, \text{sym}^2 f)} \ll_{\varepsilon} (H^2 + t^{4/3})^{1+\varepsilon}.$$ 

For $H \leq t^{1/2}$, this follows from the subconvexity bound

$$L\left(\frac{1}{2}, f\right) \ll_{\varepsilon} t_f^{1/3+\varepsilon}$$

of Ivić [Ivi01, Corollary 2], and from [Jut04, Theorem], which states that for $t \geq 0$ and $1 \ll G \ll H$,

$$\sum_{H \leq t_f \leq H+G} \frac{|L\left(\frac{1}{2} + it, f\right)|^2}{L(1, \text{sym}^2 f)} \ll_{\varepsilon} (GH + t^{2/3})^{1+\varepsilon}.$$ 

\[ \square \]

Corollary 3.4.2. For any $\delta > 0$, we have that

$$\sum_{0 < t_f < t_g^{1-\delta}} \frac{\Lambda\left(\frac{1}{2}, f\right)^2 \Lambda\left(\frac{1}{2} + 2it_g, f\right) \Lambda\left(\frac{1}{2} - 2it_g, f\right) \Lambda(1 + 2it_g)^2 \Lambda(1 - 2it_g)^2 \Lambda(1, \text{sym}^2 f)}{\Lambda(1 + 2it_g)^2 \Lambda(1 - 2it_g)^2 \Lambda(1, \text{sym}^2 f)} \ll_{\varepsilon} t_g^{-\min\{\frac{\delta}{2}, \frac{1}{6}\} + \varepsilon}.$$ 

3.5 Spectral Methods to Bound the Short Transition Range

In [BK17a, Section 1], Buttcane and Khan state for a dihedral Maaß newform $g$,

\[ \ldots \]

the range $[2t_g - t_g^{1-\delta} < t_f < 2t_g]$ can be handled by applying Hölder’s inequality as Luo does and then applying Jutila’s [Jut01] and Ivić’s [Ivi01] bounds for moments of $L(1/2, f)$ in short intervals of $t_f$ close to $2t_g$. 

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A similar idea works when $g$ is a truncated Eisenstein series. We must show that
\[
\sum_{2t_g-t_g^{1-\delta} \leq t_f \leq 2t_g+t_g^{1-\delta}} \frac{L \left( \frac{1}{2}, f \right)^2 |L \left( \frac{1}{2} + 2it_g, f \right)|^2}{(1 + t_f) (1 + 2t_g + t_f)^{1/2} (1 + |2t_g - t_f|)^{1/2} L \left( 1, \text{sym}^2 f \right)} \ll t_g^{-\delta'}
\]
for some $\delta' > 0$. We use the Cauchy–Schwarz inequality to see that this spectral sum is bounded by $t_g^{-3/2}$ times the square root of the product of
\[
\sum_{2t_g-t_g^{1-\delta} \leq t_f \leq 2t_g+t_g^{1-\delta}} \frac{L \left( \frac{1}{2}, f \right)^4}{(1 + |2t_g - t_f|)^{1/2} L \left( 1, \text{sym}^2 f \right)}
\]
and
\[
\sum_{2t_g-t_g^{1-\delta} \leq t_f \leq 2t_g+t_g^{1-\delta}} \frac{|L \left( \frac{1}{2} + 2it_g, f \right)|^4}{(1 + |2t_g - t_f|)^{1/2} L \left( 1, \text{sym}^2 f \right)}.
\]
The first sum is bounded by
\[
\sum_{k=0}^{|t_g^{2/3-\delta}|} \frac{1}{(1 + kt_g^{1/3})^{1/2}} \sum_{2t_g-(k+1)t_g^{1/3} \leq t_f < 2t_g-kt_g^{1/3}} \frac{L \left( \frac{1}{2}, f \right)^4}{L \left( 1, \text{sym}^2 f \right)}
\]
\[
+ \sum_{k=0}^{|t_g^{2/3-\delta}|} \frac{1}{(1 + kt_g^{1/3})^{1/2}} \sum_{2t_g+kt_g^{1/3} \leq t_f < 2t_g+(k+1)t_g^{1/3}} \frac{L \left( \frac{1}{2}, f \right)^4}{L \left( 1, \text{sym}^2 f \right)},
\]
and a similar expression holds for the second sum. We then apply the following lemma to show that each sum is bounded by a constant multiple dependent on $\varepsilon$ of $t_g^{\frac{1}{3}+\varepsilon}$, from which the result follows.

**Lemma 3.5.1** ([Jut01, Theorem], [JM05, Theorem 1]). For $H \gg 1$ and $1 \ll G \ll H$, we have that
\[
\sum_{H \leq t_f \leq H+G} \frac{L \left( \frac{1}{2}, f \right)^4}{L \left( 1, \text{sym}^2 f \right)} \ll_{\varepsilon} \left( H^{1/3} + G \right) H^{1+\varepsilon}.
\]
Similarly, for $H \gg 1$, $0 \leq t \ll H^{3/2-\varepsilon}$, and $0 \leq G \leq (H + t)^{4/3}H^{-1+\varepsilon}$, we have that

$$
\sum_{H \leq t_f \leq H+G} \frac{|L(\frac{1}{2} + it, f)|^4}{L(1, \text{sym}^2 f)} \ll \varepsilon (H + t)^{4/3}H^\varepsilon.
$$

**Corollary 3.5.2.** For any $0 < \delta < 2/3$, we have that

$$
\sum_{2t_g - t_g^{1-\delta} \leq t_f \leq 2t_g + t_g^{1-\delta}} \frac{\Lambda\left(\frac{1}{2}, f\right)^2 \Lambda\left(\frac{1}{2} + 2it_g, f\right) \Lambda\left(\frac{1}{2} - 2it_g, f\right)}{\Lambda(1 + 2it_g)^2 \Lambda(1 - 2it_g)^2 \Lambda(1, \text{sym}^2 f)} \ll \varepsilon t_g^{\frac{1}{2}+\varepsilon}.
$$

### 3.6 Spectral Methods to Bound the Bulk Range

In [Spi03, Chapter 6], Spinu sketches a proof of the bound

$$
\sum_{2\alpha t_g \leq t_f \leq 2(1-\alpha)t_g} \frac{\Lambda\left(\frac{1}{2}, f\right)^2 \Lambda\left(\frac{1}{2} + 2it_g, f\right) \Lambda\left(\frac{1}{2} - 2it_g, f\right)}{\Lambda(1 + 2it_g)^2 \Lambda(1 - 2it_g)^2 \Lambda(1, \text{sym}^2 f)} \ll \alpha \left(\log \left(\frac{1}{4} + t_g^2\right)\right)^2
$$

for any small $\alpha > 0$. Via the methods of Buttcane and Khan [BK17a, BK17b], one can extend this to the full bulk range $t_g^{1-\delta} < t_f < 2t_g - t_g^{1-\delta}$, which thereby completes the unconditional proof of Theorem 1.4.3.
Chapter 4

Failure of Equidistribution at the Planck Scale

4.1 The Selberg–Harish-Chandra Transform

For $z, w \in \mathbb{H}$, set
\[
u(z, w) := \frac{|z-w|^2}{4 \Im(z) \Im(w)} = \sinh^2 \frac{\rho(z, w)}{2},
\]
where
\[
\rho(z, w) := \log \frac{|z - \overline{w}| + |z - w|}{|z - \overline{w}| - |z - w|}
\]
denotes the hyperbolic distance on $\mathbb{H}$. The function $u: \mathbb{H} \times \mathbb{H} \rightarrow [0, \infty)$ is a point-pair invariant. From this, a function $k: [0, \infty) \rightarrow \mathbb{C}$ gives rise to a point-pair invariant $k(z, w) := k(u(z, w))$ on $\mathbb{H}$. The Selberg–Harish-Chandra transform maps sufficiently well-behaved functions $k: [0, \infty) \rightarrow \mathbb{C}$ to functions $h: \mathbb{R} \rightarrow \mathbb{C}$. This transform is given in three steps as follows:

\[
q(v) := \int_v^\infty \frac{k(u)}{\sqrt{u-v}} \, du,
\]
\[
g(r) := 2q \left( \sinh^2 \frac{r}{2} \right).
\]
\[ h(t) := \int_{-\infty}^{\infty} g(r) e^{irt} \, dr. \]

Note that \( h(t) \) is real whenever \( t \) is real.

We shall take \( k(z, w) = k_R(z, w) \) equal to the indicator function of a small ball of radius \( R \) centred at a point \( w \),

\[ B_R(w) := \{ z \in \mathbb{H} : \rho(z, w) \leq R \} = \left\{ z \in \mathbb{H} : u(z, w) \leq \sinh^2 \frac{R}{2} \right\}, \]

normalised by the volume of this ball,

\[ \text{vol} (B_R) = \text{vol} (B_R(w)) = 4\pi \sinh^2 \frac{R}{2}. \]

So

\[ k(u) = k_R(u) := \begin{cases} 
\frac{1}{4\pi \sinh^2 \frac{R}{2}} & \text{if } u \leq \sinh^2 \frac{R}{2}, \\
0 & \text{otherwise},
\end{cases} \quad (4.1.1) \]

and consequently

\[ h(t) = h_R(t) := \frac{R}{\pi \sinh \frac{R}{2}} \int_{-1}^{1} \sqrt{1 - \left( \frac{\sinh \frac{R r}{2}}{\sinh \frac{R}{2}} \right)^2} e^{iRrt} \, dr. \]

**Proof of Lemma 1.5.9.** If \( R \) and \( Rt \) both converge to zero, then the dominated convergence theorem implies that

\[ h_R(t) \sim \frac{2}{\pi} \int_{-1}^{1} \sqrt{1 - r^2} \, dr = 1. \]

If \( R \) converges to 0 and \( Rt \) converges to some value in \((0, \infty)\), then similarly

\[ h_R(t) \sim \frac{2}{\pi} \int_{-1}^{1} \sqrt{1 - r^2} e^{iRrt} \, dr = \frac{2 J_1(Rt)}{Rt}. \]
via [GR07, 8.411.10]. So it remains to prove the case that \( R \) converges to 0 and \( Rt \) tends to infinity. To do this, we let

\[
h(R, x) := \frac{R}{\pi \sinh \frac{R}{2}} \int_{-1}^{1} \sqrt{1 - \left( \frac{\sinh \frac{Rr}{2}}{\sinh \frac{R}{2}} \right)^2} e^{irx} \, dr.
\]

We show that

\[
x^{3/2}h(R, x) - 2\sqrt{\frac{2}{\pi \sinh R}} \sin \left( x - \frac{\pi}{4} \right)
\]

is pointwise convergent as \( R \) tends to zero and is uniformly convergent to 0 as \( x \) tends to infinity, from which the Moore–Osgood theorem allows us to interchange the order of limits taken in order to obtain the desired asymptotic. Indeed, the dominated convergence theorem once again shows that \( h(R, x) \) converges to

\[
\frac{2}{\pi} \int_{-1}^{1} \sqrt{1 - r^2} e^{irx} \, dr = \frac{2J_1(x)}{x}
\]

as \( R \) tends to zero. For the uniform convergence as \( x \) tends to infinity, we integrate by parts and make the substitution \( r = \frac{2}{R} \arcsinh \left( \sin v \sinh \frac{R}{2} \right) \), yielding

\[
h(R, x) = \frac{R}{2 \sinh \frac{R}{2}} \frac{2}{\pi ix} \int_{-\pi/2}^{\pi/2} \sin ve^{ivx} \frac{2}{\pi} \arcsinh \left( \sin v \sinh \frac{R}{2} \right) \, dv.
\]

Using stationary phase, with the two critical points being the endpoints \( \pm \pi/2 \), we find that there exists some \( R_0 > 0 \) such that

\[
\sup_{R \in (0, R_0)} \left| x^{3/2}h(R, x) - 2\sqrt{\frac{2}{\pi \sinh R}} \sin \left( x - \frac{\pi}{4} \right) \right| \ll \frac{1}{x}.
\]

For a function \( k: [0, \infty) \to \mathbb{C} \), we may form the automorphic kernel

\[
K(z, w) := \sum_{\gamma \in \Gamma_0(q)} k(\gamma z, w),
\]
which is $\Gamma_0(q)$-invariant in both variables. When \( k(u) = k_R(u) \), we write \( K(z, w) = K_R(z, w) \).

**Lemma 4.1.2.** If \( f : \Gamma_0(q) \backslash \mathbb{H} \to \mathbb{C} \) is an eigenfunction of the Laplacian with eigenvalue \( 1/4 + t_f^2 \), then

\[
\frac{1}{\text{vol}(B_R)} \int_{B_R(w)} f(z) \, d\mu(z) = \langle f, K_R(\cdot, w) \rangle_q = h_R(t_f)f(w).
\]

**Proof.** This follows from [Iwa02, Theorem 1.14]. Note that there it is assumed that not only is \( k(u) \) compactly supported, but that it is smooth; this, however, is not essential to the proof. Instead, we merely require that \( k(z, w) \) be twice differentiable in both variables \( \mu \)-almost everywhere. \( \square \)

**Proof of Proposition 1.5.8.** Via Parseval’s identity,

\[
\frac{1}{\text{vol}(B_R)} \int_{B_R(w)} |g(z)|^2 \, d\mu(z) = \frac{\langle g, g \rangle_q}{\text{vol}(\Gamma_0(q) \backslash \mathbb{H})} + \sum_{f \in \mathcal{B}_0(\Gamma_0(q))} h_R(t_f)f(w) \langle |g|^2, f \rangle_q
\]

\[
+ \sum_a \frac{1}{4\pi} \int_{-\infty}^{\infty} h_R(t) E_a\left(w, \frac{1}{2} + it\right) \left\langle |g|^2, E_a\left(\cdot, \frac{1}{2} + it\right) \right\rangle_q \, dt.
\]

Upon squaring and integrating over \( w \), we obtain Proposition 1.5.8. \( \square \)

### 4.2 Proof of Theorem 1.5.3

**Proposition 4.2.1 ([Mil10, Theorem 1]).** For every fixed Heegner point \( w \in \mathbb{H} \),

\[
|g(w)| = \Omega \left( \exp \left( \sqrt{\frac{\log t_g}{\log \log t_g}} \left( 1 + O \left( \frac{\log \log \log t_g}{\log \log t_g} \right) \right) \right) \right)
\]

as \( t_g \) tends to infinity.
Proof of Theorem 1.5.3. For \( g \in B^*_0(\Gamma_0(g)) \),

\[
\frac{1}{\text{vol}(B_R)} \int_{B_R(w)} g(z) d\mu(z) = \int_{\Gamma_0(q) \setminus \mathbb{H}} K_R(z, w)g(z) d\mu(z) = h_R(t_g)g(w).
\]

It follows by the Cauchy–Schwarz inequality that

\[
|h_R(t_g)|^2 |g(w)|^2 = \left| \left\langle g\sqrt{K_R(\cdot, w)}, \sqrt{K_R(\cdot, w)} \right\rangle_q \right|^2 
\leq \left\langle g\sqrt{K_R(\cdot, w)}, g\sqrt{K_R(\cdot, w)} \right\rangle_q \left\langle \sqrt{K_R(\cdot, w)}, \sqrt{K_R(\cdot, w)} \right\rangle_q 
= \frac{1}{\text{vol}(B_R)} \int_{B_R(w)} |g(z)|^2 d\mu(z).
\]

Theorem 1.5.3 then follows from Lemma 1.5.9 and Proposition 4.2.1. \( \square \)

Remark 4.2.2. Since it is conjectured that \( \max_{w \in K} |g(w)| \ll_{K, \varepsilon} t^\varepsilon_g \) for every compact subset \( K \) of \( \Gamma_0(q) \setminus \mathbb{H} \), we cannot expect any significant improvement to Theorem 1.5.3 via this line of reasoning.
Chapter 5

Equidistribution in Almost Every Shrinking Ball

5.1 Proof of Conditional Results

*Proof of Theorem 1.5.6 for $0 < \delta < 1$. We use Propositions 1.5.8 and 2.3.1 and Lemmata 1.5.9 and 2.3.10. We then divide the spectral decomposition in Proposition 1.5.8 into the ranges discussed in Section 2.4.2, which shows that $\text{Var}(g; R)$ is bounded by a constant multiple dependent on $\varepsilon$ of

$$
t^{-1+\varepsilon} \sum_{0 < t_f \leq t_g} \frac{L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \text{sym}^2 g \times f \right)}{t_f L \left( 1, \text{sym}^2 f \right)}
+ t_g^{3\delta - \frac{1}{2} + \varepsilon} \sum_{t_f^2 < 2t_g + t_g^{-\delta}} \frac{L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \text{sym}^2 g \times f \right)}{t_f^4 (1 + |2t_g - t_f|)^{1/2} L \left( 1, \text{sym}^2 f \right)}
+ t_g^{3\delta + \varepsilon} \sum_{t_f \geq 2t_g + t_g^{-\delta}} \exp(-\pi(t_f - 2t_g)) \frac{L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \text{sym}^2 g \times f \right)}{t_f^2 (1 + t_f - 2t_g)^{1/2} L \left( 1, \text{sym}^2 f \right)}
+ t_g^{-1/2 + \varepsilon} \int_{0}^{2t_g + t_g^{-\delta}} \frac{|L \left( \frac{1}{2} + it \right) L \left( \frac{1}{2} + it, \text{sym}^2 g \right)|^2}{(1 + t)(1 + |2t_g - t|)^{1/2} |\zeta(1 + 2it)|^2} dt
+ t_g^{-1/2 + \varepsilon} \int_{2t_g + t_g^{-\delta}}^{\infty} \exp(-\pi(t - 2t_g)) \frac{|L \left( \frac{1}{2} + it \right) L \left( \frac{1}{2} + it, \text{sym}^2 g \right)|^2}{(1 + t)(1 + |2t_g - t|)^{1/2} |\zeta(1 + 2it)|^2} dt.

$$
• From [BK17b, Lemma 2.1], the initial and tail ranges of the continuous spectrum are bounded by $t_g^{-1+\varepsilon}$.

• The convexity bounds for $L(1/2, f)$ and $L(1/2, \text{sym}^2 g \times f)$ show that the tail range of the cuspidal spectrum is rapidly decaying.

• For the other two ranges, the generalised Lindelöf hypothesis implies that the product of these two $L$-functions is bounded by a constant multiple dependent on $\varepsilon$ of $t_g^{\varepsilon}$, and then the Weyl law for $\Gamma \backslash \mathbb{H}$ and partial summation imply that the contribution of the cuspidal spectrum is bounded by $t_g^{\delta - 1+\varepsilon}$.

\[ \square \]

**Proof of Theorem 1.5.6 for $\delta > 1$.**  

• Once again, [BK17b, Lemma 2.1] shows that the contribution from the continuous spectrum is bounded by $t_g^{-1+\varepsilon}$.

• The contribution from the short initial and transition ranges is bounded by $t_g^{-\delta'/2+\varepsilon}$ via the generalised Lindelöf hypothesis.

• The tail range is easily seen to be negligible.

• Finally, the bulk range is asymptotic to $6/\pi$ from the proof of [BK17b, Proposition 2.2].

\[ \square \]

**Remark 5.1.1.** In fact, the method of proof of [BK17b, Proposition 2.2] together with Lemma 1.5.9 show that if $R \sim (Ct_g)^{-1}$ for some positive constant $C$, then

\[
\Var(g; R) \sim \frac{12C}{\pi^2} \int_0^1 \frac{J_1 \left( \frac{2t}{C} \right)}{t\sqrt{1-t^2}} \, dt = \frac{6}{\pi} \left( J_0 \left( \frac{1}{C} \right)^2 + J_1 \left( \frac{1}{C} \right)^2 \right)
\]

by [GR07, (8.473.1) and (6.552.4)], which converges to $6/\pi$ as $C$ tends to infinity.
5.2 Proof of Unconditional Results

We first sketch how to prove Theorem 1.5.5.

**Proof of Theorem 1.5.5.** In [You16], after [You16, (4.24)], we use Lemma 3.4.1 instead of the subconvexity bound $L(1/2 + it, f) \ll \varepsilon (t f + t)^{1/3 + \varepsilon}$. Using this, the right-hand side of [You16, (4.26)] is improved to $T^{-1/2 + \varepsilon} A^{1/2} \| \phi \|_2$, which yields the result. 

Next, we cover the proof of Theorem 1.5.10. To begin, we wish to calculate

$$\frac{1}{\text{vol}(B_R)} \int_{B_R(w)} |g(z)|^2 d\mu(z),$$

where $g(z) = E(z, 1/2 + it_g)$. However, we cannot use Parseval’s identity because $|g|^2 \notin L^2 (\Gamma \backslash \mathbb{H})$. Instead, we replace $|g(z)|^2$ with $E(z, s_1) E(z, s_2)$ and subtract away a linear combination of Eisenstein series $\mathcal{E}$ such that the resulting function is square-integrable. After applying Parseval’s identity, we finally send $s_1$ to $1/2 + it_g$ and $s_2$ to $1/2 - it_g$.

**Lemma 5.2.1** (cf. [You16, Lemma 4.1]). For $1/2 < \Re(s_1), \Re(s_2) < 3/4$,

$$\frac{1}{\text{vol}(B_R)} \int_{B_R(w)} E(z, s_1) E(z, s_2) d\mu(z)$$

is equal to

$$h_R \left( i \left( s_1 + s_2 - \frac{1}{2} \right) \right) E(w, s_1 + s_2)$$

$$+ h_R \left( i \left( \frac{1}{2} - s_1 + s_2 \right) \right) \frac{\Lambda(2 - 2s_1)}{\Lambda(2s_1)} E(w, 1 - s_1 + s_2)$$

$$+ h_R \left( i \left( \frac{1}{2} + s_1 - s_2 \right) \right) \frac{\Lambda(2 - 2s_2)}{\Lambda(2s_2)} E(w, 1 + s_1 - s_2)$$

$$+ h_R \left( i \left( \frac{3}{2} - s_1 - s_2 \right) \right) \frac{\Lambda(2 - 2s_1) \Lambda(2 - 2s_2)}{\Lambda(2s_1) \Lambda(2s_2)} E(w, 2 - s_1 - s_2)$$

$$+ \sum_{f \in B_0(\Gamma)} h_R(t_f) f(w) \langle E(\cdot, s_1) E(\cdot, s_2), f \rangle$$
\begin{align*}
+ \frac{1}{4\pi} \int_{-\infty}^{\infty} h_R(t) E \left( w, \frac{1}{2} + it \right) \left< E(\cdot, s_1) E(\cdot, s_2), E \left( \cdot, \frac{1}{2} + it \right) \right>_{\text{reg}} dt.
\end{align*}

Proof. Let \( F(z) := E(z, s_1) E(z, s_2) \) and let

\begin{align*}
\mathcal{E}(z) := E(z, s_1 + s_2) + \frac{\Lambda(2 - 2s_1)}{\Lambda(2s_1)} E(z, 1 - s_1 + s_2) \\
+ \frac{\Lambda(2 - 2s_2)}{\Lambda(2s_2)} E(z, 1 + s_1 - s_2) + \frac{\Lambda(2 - 2s_1) \Lambda(2 - 2s_2)}{\Lambda(2s_1) \Lambda(2s_2)} E(z, 2 - s_1 - s_2).
\end{align*}

Since the constant term of \( F(z) \) is

\begin{align*}
y^{s_1+s_2} + \frac{\Lambda(2 - 2s_1)}{\Lambda(2s_1)} y^{1-s_1+s_2} + \frac{\Lambda(2 - 2s_2)}{\Lambda(2s_2)} y^{1+s_1-s_2} + \frac{\Lambda(2 - 2s_1) \Lambda(2 - 2s_2)}{\Lambda(2s_1) \Lambda(2s_2)} y^{2-s_1-s_2},
\end{align*}

we have that \( F(z) - \mathcal{E}(z) = O(y^{1/2-\delta}) \) for some \( \delta > 0 \) as \( y \) at the cusp at infinity, and consequently \( F - \mathcal{E} \in L^2(\Gamma \setminus \mathbb{H}) \). Parseval’s identity and Lemma 4.1.2 then imply that

\begin{align*}
\left< F - \mathcal{E}, K_R(\cdot, w) \right> &= \frac{\langle F - \mathcal{E}, 1 \rangle}{\text{vol}(\Gamma \setminus \mathbb{H})} + \sum_{f \in B_0(\Gamma)} h_R(t_f) f(w) \langle F - \mathcal{E}, f \rangle \\
&\quad + \frac{1}{4\pi} \int_{-\infty}^{\infty} h_R(t) E \left( w, \frac{1}{2} + it \right) \left< F - \mathcal{E}, E \left( \cdot, \frac{1}{2} + it \right) \right> dt.
\end{align*}

The left-hand side is equal to \( \langle F, K_R(\cdot, w) \rangle - \langle \mathcal{E}, K_R(\cdot, w) \rangle \), and Lemma 4.1.2 allows us to calculate \( \langle \mathcal{E}, K_R(\cdot, w) \rangle \) explicitly. On the right-hand side, the inner product \( \langle \mathcal{E}, f \rangle \) vanishes whenever \( f \in B_0(\Gamma) \), being the linear combination of inner products of Eisenstein series with a cusp form, and similarly \( \langle F - \mathcal{E}, 1 \rangle \) vanishes via [Zag82, Equation (36) and Section 2]. Finally, we claim that the inner product \( \langle F - \mathcal{E}, E \left( \cdot, \frac{1}{2} + it \right) \rangle \) is equal to

\begin{align*}
\frac{\Lambda \left( s_2 - s_1 + \frac{1}{2} + it \right) \Lambda \left( s_1 + s_2 - \frac{1}{2} + it \right) \Lambda \left( s_2 - s_1 + \frac{1}{2} - it \right) \Lambda \left( s_1 + s_2 - \frac{1}{2} - it \right)}{\Lambda (2s_1) \Lambda (2s_2) \Lambda (1 - 2it)}.
\end{align*}

Indeed, we may add and subtract a linear combination of Eisenstein series \( \mathcal{E}' \) such
that both $FE(\cdot, 1/2 - it) - \mathcal{E}'$ and $\mathcal{E}E(\cdot, 1/2 - it) - \mathcal{E}'$ are integrable. Then the integral of $\mathcal{E}E(\cdot, 1/2 + it) - \mathcal{E}'$ vanishes via [Zag82, Equation (36) and Section 2], and the integral of $FE(\cdot, 1/2 + it) - \mathcal{E}'$ is equal to the desired product of completed zeta functions via [Zag82, Equation (44)].

We now define

$$D(g; w) := \frac{2}{\text{vol}(\Gamma \backslash \mathbb{H})} \left( 2\Re \left( \frac{\Lambda'}{\Lambda} (1 + 2it_g) \right) + 2\gamma_0 - \frac{12\zeta'(2)}{\pi^2} - \log |4\Im(w)\eta(w)^4| \right).$$  

(5.2.2)

Here $\gamma_0$ is the Euler–Mascheroni constant and

$$\eta(w) := e\left( \frac{w}{24} \right) \prod_{m=1}^{\infty} (1 - e(mw))$$

denotes the Dedekind eta function; note that $\Im(w)^6\eta(w)^{24}$ is a Maaß cusp form of weight 12 and level 1 (in the sense of [DFI02, Section 4]) that is nonvanishing outside the single cusp of $\Gamma \backslash \mathbb{H}$. That $D(g; w)$ is, in some sense, the “true” average of $|E(z, 1/2 + it_g)|^2$ on compact sets, rather than

$$\log \left( \frac{1}{4} + t_g^2 \right) / \text{vol}(\Gamma \backslash \mathbb{H}),$$

has previously been observed by Young [You16, Section 4.2] and also Hejhal and Rackner [HejRa92, p. 300], though in the latter case, their expression does not include the Dedekind eta function.

**Proof of Lemma 1.5.12.** This follows from (2.2.6), (2.2.7), and (2.2.8), together with the fact that $\Im(w)^6\eta(w)^{24}$ is nonvanishing in $K$. 

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We define
\[ C(g; R; w) := D(g; w) + \frac{2ih'_R (\frac{i}{2})}{\text{vol} \left( \Gamma \backslash \mathbb{H} \right)} + 2\Re \left( h_R \left( \frac{2t_g + i}{2} \right) \frac{\Lambda(1 - 2it_g)}{\Lambda(1 + 2it_g)} E(w, 1 - 2it_g) \right). \]

(5.2.3)

**Corollary 5.2.4.** Let \( g(z) = E(z, 1/2 + it_g) \). Then
\[
\frac{1}{\text{vol}(B_R)} \int_{B_R(w)} |g(z)|^2 d\mu(z) = C(g; R; w) + \sum_{f \in \mathcal{B}_0(\Gamma)} h_R(t_f) f(w) \langle |g|^2, f \rangle + \frac{1}{4\pi} \int_{-\infty}^{\infty} h_R(t) E \left( w, \frac{1}{2} + it \right) \left\langle |g|^2, E \left( \cdot, \frac{1}{2} + it \right) \right\rangle_{\text{reg}} \, dt.
\]

Proposition 1.5.14 is then a direct consequence via Parseval’s identity.

**Proof.** This follows from Lemma 5.2.1 upon taking \( s_1 = 1/2 + it_g + \varepsilon \) and \( s_2 = 1/2 - it_g + \varepsilon \) and using the expansions
\[
h_R \left( i \left( \frac{1}{2} + 2\varepsilon \right) \right) = 1 + 2ih'_R \left( \frac{i}{2} \right) \varepsilon + O(\varepsilon^2),
\]
\[
\frac{\Lambda(1 - 2it_g - 2\varepsilon)\Lambda(1 + 2it_g - 2\varepsilon)}{\Lambda(1 + 2it_g + 2\varepsilon)\Lambda(1 - 2it_g + 2\varepsilon)} = 1 - 8\Re \left( \frac{\Lambda' \Lambda}{\Lambda} (1 + 2it_g) \right) \varepsilon + O(\varepsilon^2),
\]
\[
\text{vol} \left( \Gamma \backslash \mathbb{H} \right) E(w, 1 + 2\varepsilon) = \frac{1}{2\varepsilon} + 2\gamma_0 - \log |4\Im(w)\eta(w)|^4 - \frac{12\zeta'(2)}{\pi^2} + O(\varepsilon),
\]

where the last line is the Kronecker limit formula. \( \square \)

**Proof of Theorem 1.5.10.** We use Propositions 1.5.14 and 2.3.6 and Lemmata 1.5.9 and 2.3.10. We then divide the spectral decomposition in Proposition 1.5.14 into the ranges discussed in Section 2.4.2, so that \( \text{Var}(g; R) \) is bounded by a constant multiple dependent on \( \varepsilon \) of
\[
t_g^{1+\varepsilon} \sum_{0 < t_f \leq t_g^0} \frac{L \left( \frac{1}{2}, f \right)^2 |L \left( \frac{1}{2} + 2it_g, f \right)|^2}{t_f L \left( 1, \text{sym}^2 f \right)}
\]

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For the cuspidal spectrum, we have the following.

\[ + t_g^{3\delta - 1 + \epsilon} \sum_{t_g^2 < t < t_g^{1 - \delta'}} \frac{L \left( \frac{1}{2}, f \right)^2 |L \left( \frac{1}{2} + 2it_g, f \right)|^2}{t_g^4 L(1, sym^2 f)} \]

\[ + t_g^{3\delta - \frac{3}{2} + \epsilon} \sum_{t_g^{1 - \delta'} \leq t \leq 2t_g - t_g^3} \frac{L \left( \frac{1}{2}, f \right)^2 |L \left( \frac{1}{2} + 2it_g, f \right)|^2}{(1 + |2t_g - t_f|)^{1/2} L(1, sym^2 f)} \]

\[ + t_g^{3\delta + \epsilon} \sum_{t_f \geq 2t_g + t_g^3} \exp(-\pi(t_f - 2t_g)) \frac{L \left( \frac{1}{2}, f \right)^2 |L \left( \frac{1}{2} + 2it_g, f \right)|^2}{t_g^7 (1 + t_f - 2t_g)^{1/2} L(1, sym^2 f)} \]

\[ + t_g^{-1/2 + \epsilon} \int_0^{2t_g + t_g^3} \left| \zeta \left( \frac{1}{2} + i(2t_g + t) \right) \zeta \left( \frac{1}{2} + it \right)^2 \zeta \left( \frac{1}{2} + i(2t_g - t) \right) \right|^2 \frac{dt}{(1 + t)(1 + |2t_g - t|)^{1/2} |\zeta(1 - 2it)|^2} \]

\[ + t_g^{1/2 + \epsilon} \int_{2t_g + t_g^3}^{\infty} \exp(-\pi(t_2 - 2t_g)) \frac{\left| \zeta \left( \frac{1}{2} + i(2t_g + t) \right) \zeta \left( \frac{1}{2} + it \right)^2 \zeta \left( \frac{1}{2} + i(2t_g - t) \right) \right|^2}{(1 + t)(1 + |2t_g - t|)^{1/2} |\zeta(1 - 2it)|^2} \ dt. \]

- From [Spi03, Proposition 3.4] and [Bou17, Theorem 5], the initial and tail ranges of the continuous spectrum are bounded by a constant multiple dependent on \( \epsilon \) of \( t_g^{-1/13 + \epsilon} \).

For the cuspidal spectrum, we have the following.

- The convexity bounds for \( L(1/2, f) \) and \( L(1/2 + 2it_g, f) \) show that the tail range is rapidly decaying.

- The short initial range is bounded by a constant multiple dependent on \( \epsilon \) of \( t_g^{-\min\{1-\delta,1/6\} + \epsilon} \) upon dividing into dyadic intervals and applying Lemma 3.4.1.

- The same method bounds the short initial polynomial decay range by \( t_g^{-\min\{\delta',1/6\} + \epsilon} \).

- For the bulk polynomial decay range, we divide into dyadic intervals and use Lemma 3.3.1, which shows that this range is bounded by \( t_g^{-1/2(1-\delta-\delta') + \epsilon} \).

- We divide the short transition polynomial decay range into intervals of length \( t_g^{1/3} \), use the Cauchy–Schwarz inequality, and apply Lemma 3.5.1, which gives
the bound $t_g^{-\frac{2}{3}(1-\delta)+\varepsilon}$.

Theorem 1.5.10 is proven upon taking $\delta' = \frac{5}{7}(1-\delta)$. 

**Proof of Theorem 1.5.11.** By Chebyshev’s inequality and Theorem 1.5.10,

$$\text{vol}\left(\left\{w \in \Gamma \setminus \mathbb{H} : \left| \frac{1}{\text{vol}(B_R)} \int_{B_R(w)} |g(z)|^2 \, d\mu(z) - C(g; R; w) \right| > c \right\}\right) \leq \frac{1}{t_g^{\varepsilon}} \frac{t_g^{-\min\left\{ \frac{2}{3}(1-\delta), \frac{1}{2}\right\} + \varepsilon}}{c^2}.$$

Again by Chebyshev’s inequality together with Corollary 2.2.2,

$$\text{vol}\left(\left\{w \in \Gamma \setminus \mathbb{H} : \right| 1 - \frac{1}{\text{vol}(B_R)} \int_{B_R(w)} |g(z)|^2 \, d\mu(z) > c \right\}\right) \leq \text{vol}\left(\left\{w \in \Gamma \setminus \mathbb{H} : \Im(w) > T \right\}\right) + \frac{1}{t_g^{\varepsilon}} \frac{t_g^{-\min\left\{ \frac{2}{3}(1-\delta), \frac{1}{2}\right\} + \varepsilon}}{c^2} \int_{\Gamma \setminus \mathbb{H}} |\Lambda^T E(w, 1 - 2it_g)|^2 \, d\mu(z)$$

for any $T \geq 1$, which, by the Maass–Selberg relation (2.2.4), is equal to

$$\frac{1}{T} + \frac{|h_R(2t_g + \frac{i}{2})|^2}{c^2} \left( T + 2\Re\left( \frac{\Lambda(1 - 4it_g)}{\Lambda(2 - 4it_g)} \frac{T^{4it_g}}{4it_g} \right) + \left| \frac{\Lambda(1 - 4it_g)}{\Lambda(2 - 4it_g)} \right|^2 \frac{1}{T} \right).$$

We have that $|h_R(2t_g + \frac{i}{2})|^2 \asymp t_g^{-3(1-\delta)}$ and that

$$\frac{\Lambda(1 - 2it_g)}{\Lambda(2 - 4it_g)} \ll t_g^{-1/2+\varepsilon}.$$

Next, we note that

$$ih'_R\left(\frac{i}{2}\right) = \frac{R^2}{\pi} \int_{-1}^{1} r \left[ \sqrt{1 - \left( \frac{\sinh \frac{Rr}{2}}{\sinh \frac{R}{2}} \right)^2} \frac{\sinh \frac{Rr}{2}}{\sinh \frac{R}{2}} \right] \, dr \sim \frac{R^2}{8} \asymp t_g^{-2\delta},$$

for any $T \geq 1$. This completes the proof.
so if \( c \gg \varepsilon t_g^{-2\delta + \varepsilon} \), then for all sufficiently large \( t_g \),

\[
\left| \frac{2ih_R'(i)}{\text{vol}(\Gamma \backslash \mathbb{H})} \right| < c.
\]

So piecing everything together, we find that if \( c \gg \varepsilon t_g^{-2\delta + \varepsilon} \),

\[
\text{vol} \left( \left\{ w \in \Gamma \backslash \mathbb{H} : \left| \frac{1}{\text{vol}(B_R)} \int_{B_R(w)} |g(z)|^2 d\mu(z) - D(g; w) \right| > c \right\} \right) \\
\ll \varepsilon \left( t_g^{-\frac{2}{7}(1-\delta) + \varepsilon} + t_g^{-\frac{1}{6} + \varepsilon} + \frac{1}{T} + \frac{t_g^{-3(1-\delta)} T}{c^2} \right).
\]

Taking \( T = ct_g^{\frac{3}{7}(1-\delta)} \) yields the result. \( \square \)
Chapter 6

Equidistribution of Geometric Invariants of Quadratic Fields

6.1 Geometric Invariants of Quadratic Fields

An integer $D$ is said to be a fundamental discriminant if either $D$ is squarefree with $D \equiv 1 \pmod{4}$ or $D = 4D'$ for some squarefree $D' \in \mathbb{Z}$ satisfying either $D' \equiv 2 \pmod{4}$ or $D' \equiv 3 \pmod{4}$. The set of fundamental discriminants is precisely the set of discriminants of $\mathbb{Q}$ and all of its quadratic extensions.

Let $K = \mathbb{Q}(\sqrt{D})$ be a quadratic field of discriminant $D$. We denote by $\text{Cl}_K^+$ the narrow class group of $K$: this is the quotient group $I_K/P_K^+$, where $I_K$ denotes the group of fractional ideals of $K$ and $P_K^+$ denotes the subgroup of totally positive principal fractional ideals, so that $a \in P_K^+$ if and only if $a = \alpha \mathcal{O}_K$ for some $\alpha \in K^\times$ satisfying $\sigma(\alpha) > 0$ for every embedding $\sigma: K \to \mathbb{R}$. When $D < 0$, $P_K^+$ is equal to $P_K$, the subgroup of principal fractional ideals, so that $\text{Cl}_K^+$ is the (wide) class group $\text{Cl}_K := I_K/P_K$ of $K$. We let $J_K \in \text{Cl}_K^+$ denote the narrow ideal class containing the different $\mathfrak{d}_K = (\sqrt{D})$ of $K$. Then the class group $\text{Cl}_K$ of $K$ is isomorphic to $\text{Cl}_K^+ / J_K$. We denote by $h_K^+ := \# \text{Cl}_K^+$ the narrow class number of $K$ and $h_K := \# \text{Cl}_K$ the
(wide) class number of $K$. We have that $\text{Cl}_K^+ = \text{Cl}_K$, so that $h_K^+ = h_K$, except when $D > 1$ and $\mathcal{O}_K^\times$ contains no elements of norm $-1$, in which case $h_K^+ = 2h_K$.

Suppose that $D < 0$, so that $K$ is an imaginary quadratic field. Then each ideal class $A \in \text{Cl}_K$ can be associated to a geometric invariant, namely a Heegner point $z_A \in \Gamma \backslash \mathbb{H}$. Similarly, let $K = \mathbb{Q}(\sqrt{D})$ be a real quadratic field of discriminant $D > 0$. Then a narrow ideal class $A \in \text{Cl}_K^+$ can be associated to a geometric invariant, namely a closed geodesic $C_A \subset \Gamma \backslash \mathbb{H}$. In [DIT16], Duke, Imamoğlu, and Tóth introduce another geometric invariant for real quadratic fields: associated to each narrow ideal class $A \in \text{Cl}_K^+$ is a hyperbolic orbifold $\Gamma_A \backslash \mathcal{N}_A$ whose boundary is $C_A$.

To describe these invariants, we first discuss the relation between binary quadratic forms and narrow ideal classes.

### 6.1.1 Binary Quadratic Forms and Narrow Ideal Classes

Let $Q(x, y) = ax^2 + bxy + cy^2$ be a binary quadratic form with $a, b, c \in \mathbb{Z}$. We denote by $D = b^2 - 4ac$ the discriminant of $Q$. If $D$ is a fundamental discriminant, then every binary quadratic form of discriminant $D$ is primitive, in the sense that $(a, b, c) = 1$. Two binary quadratic forms $Q(x, y), Q'(x, y)$ are said to be equivalent if there exists some $(\alpha \beta \gamma \delta) \in \Gamma$ such that $Q(\alpha x + \beta y, \gamma x + \delta y) = Q'(x, y)$. This equivalence preserves discriminants and primitivity.

Gauss showed that there are only finitely many equivalence classes of binary quadratic forms with a given fundamental discriminant. Indeed, there is a bijective correspondence between narrow ideal classes $A$ of $K = \mathbb{Q}(\sqrt{D})$ and equivalence classes of binary quadratic forms of fundamental discriminant $D$. For $D < 0$, this correspondence is induced by the map

$$a = w\mathbb{Z} + \mathbb{Z} \mapsto Q(x, y) = \frac{N(x - wy)}{N(a)}$$
where \( a \in I_K \) is a fractional ideal with \( w = w_1 + w_2\sqrt{D} \in K \). The map takes the ideal class of \( a \) to the \( \Gamma \)-equivalence class of \( Q \). The inverse map is given by \( Q(x, y) \mapsto w\mathbb{Z} + \mathbb{Z} \), where \( w = \frac{-b + \sqrt{D}}{2a} \). When \( D > 0 \), this correspondence of narrow ideal classes and equivalence classes of binary quadratic forms is essentially the same, except that the representative \( a = w\mathbb{Z} + \mathbb{Z} \) must be chosen such that \( w_2 > 0 \) and the representative \( Q(x, y) = ax^2 + bxy + cy^2 \) must be chosen such that \( a > 0 \).

### 6.1.2 Heegner Points \( z_A \)

Let \( K = \mathbb{Q}(\sqrt{D}) \) be an imaginary quadratic field of discriminant \( D < 0 \). Given a binary quadratic form \( Q(x, y) = ax^2 + bxy + cy^2 \) of discriminant \( b^2 - 4ac = D \), the point

\[
  z = \frac{-b + i\sqrt{-D}}{2a}
\]

lies in \( \mathbb{H} \). The equivalence class of binary quadratic forms containing \( Q(x, y) \) thereby corresponds to a point \( z \) in \( \Gamma \backslash \mathbb{H} \); equivalently, the corresponding ideal class \( A \in \text{Cl}_K \) corresponds to \( z = z_A \). We call such a point \( z_A \in \Gamma \backslash \mathbb{H} \) a Heegner point.

### 6.1.3 Closed Geodesics \( C_A \)

Let \( K = \mathbb{Q}(\sqrt{D}) \) be a real quadratic field of discriminant \( D > 1 \). Given a binary quadratic forms \( Q(x, y) = ax^2 + bxy + cy^2 \) of discriminant \( b^2 - 4ac = D \), the points

\[
  \frac{-b \pm \sqrt{D}}{2a}
\]

determine the endpoints of a closed geodesic in \( \mathbb{H} \). The equivalence class of binary quadratic forms containing \( Q(x, y) \) thereby corresponds to a closed geodesic \( C \) in \( \Gamma \backslash \mathbb{H} \); equivalently, the corresponding narrow ideal class \( A \in \text{Cl}_K \) corresponds to \( C = C_A \).
The length
\[ \ell(C_A) := \int_{C_A} ds \]
of \(C_A\), with \(ds^2 = \frac{dx^2 + dy^2}{y^2}\), is equal to \(2 \log \epsilon^+_K\), where \(\epsilon^+_K > 1\) is the smallest unit with positive norm in the ring of integers \(\mathcal{O}_K\) of \(K\), so that \(\epsilon^+_K = \frac{x + y\sqrt{D}}{2}\) with \((x, y) \in \mathbb{R}^2_+\) the fundamental solution to the Pell equation \(x^2 - Dy^2 = 4\). Note that \(\epsilon^+_K\) is equal to \(\epsilon_K\), the fundamental unit of \(K\), if \(\mathcal{O}_K^\times\) contains no elements of norm \(-1\), whereas \(\epsilon^+_K = \epsilon^2_K\) if \(\mathcal{O}_K^\times\) does contain elements of norm \(-1\).

### 6.1.4 Hyperbolic Orbifolds \(\Gamma_A \backslash \mathcal{N}_A\)

Let \(K = \mathbb{Q}(\sqrt{D})\) be a real quadratic field of discriminant \(D > 1\). Associated to a narrow ideal class \(A \in \text{Cl}^+_K\) is an invariant \(((n_1, \ldots, n_{\ell_A}))\), where \(\ell_A\) is a positive integer and \(n_1, \ldots, n_{\ell_A}\) are integers; this is the primitive cycle, unique up to cyclic permutations, occurring in the minus continued fraction expansion of each point \(w \in K\) for which \(1 > w > \sigma(w) > 0\) and \(w \mathbb{Z} + \mathbb{Z} \in A\). We define the elements

\[
S := \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T := \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]

of \(\text{PSL}_2(\mathbb{Z})\), which generate \(\text{PSL}_2(\mathbb{Z})\) as the free product of \(S\) and \(T\). For each \(k \in \{1, \ldots, \ell_A\}\), define

\[
S_k := T^{n_1 + \cdots + n_k} ST^{-n_1 - \cdots - n_k}.
\]

This is an elliptic element of order 2 in \(\text{PSL}_2(\mathbb{Z})\). We set

\[
\Gamma_A := \langle S_1, \ldots, S_{\ell_A}, T^{n_1 + \cdots + n_{\ell_A}} \rangle,
\]

which is a thin subgroup of \(\text{PSL}_2(\mathbb{Z})\). The Nielsen region \(\mathcal{N}_A\) of \(\Gamma_A\) is defined to be the smallest nonempty \(\text{PSL}_2(\mathbb{Z})\)-invariant open convex subset of \(\mathbb{H}\). Then \(\Gamma_A \backslash \mathcal{N}_A\) is
a hyperbolic orbifold, which naturally projects onto $\Gamma \setminus \mathbb{H}$. The boundary of $\Gamma_A \setminus \mathcal{N}_A$ is a simple closed geodesic whose image in $\Gamma \setminus \mathbb{H}$ is $\mathcal{C}_A$, and the volume of $\Gamma_A \setminus \mathcal{N}_A$ is $\pi \ell_A$.

Remark 6.1.1. In fact, $\Gamma_A$ depends on the choice of $w$. The resulting hyperbolic orbifold $\Gamma_A \setminus \mathcal{N}_A$ ends up being only unique up to translation; however, the projection of $\Gamma_A \setminus \mathcal{N}_A$ onto $\Gamma \setminus \mathbb{H}$ is independent of the choice of $w$.

6.2 Weyl Sums

6.2.1 Variance and Weyl Sums

We define the Weyl sums

$$W_{G_K(z_A),f} := \sum_{A \in G_K} f(z_A),$$

$$W_{G_K(z_A),\infty}(t) := \sum_{A \in G_K} E\left(z_A, \frac{1}{2} + it\right),$$

$$W_{G_K(c_A),f} := \sum_{A \in G_K} \int_{C_A} f(z) \, ds,$$

$$W_{G_K(c_A),\infty}(t) := \sum_{A \in G_K} \int_{C_A} E\left(z, \frac{1}{2} + it\right) \, ds,$$

$$W_{G_K(\Gamma_A \setminus \mathcal{N}_A),f} := \sum_{A \in G_K} \int_{\Gamma_A \setminus \mathcal{N}_A} f(z) \, d\mu(z),$$

$$W_{G_K(\Gamma_A \setminus \mathcal{N}_A),\infty}(t) := \sum_{A \in G_K} \int_{\Gamma_A \setminus \mathcal{N}_A} E\left(z, \frac{1}{2} + it\right) \, d\mu(z).$$

Proposition 6.2.1. We have that

$$\text{Var}(G_K(z_A); R) = \sum_{f \in B_0(\Gamma)} |h_R(t_f)|^2 \frac{|W_{G_K(z_A),f}|^2}{(#G_K)^2}$$

$$+ \frac{1}{4\pi} \int_{-\infty}^{\infty} |h_R(t)|^2 \frac{|W_{G_K(z_A),\infty}(t)|^2}{(#G_K)^2} \, dt,$$
\[
\text{Var} \left( G_K (C_A) ; R \right) = \sum_{f \in B_0(\Gamma)} |h_R(t_f)|^2 \frac{|W_{G_K(C_A), f}|^2}{(\sum_{A \in G_K} \ell (C_A))^2} \\
+ \frac{1}{4\pi} \int_{-\infty}^{\infty} |h_R(t)|^2 \frac{|W_{G_K(C_A), \infty}(t)|^2}{(\sum_{A \in G_K} \ell (C_A))^2} \, dt,
\]

\[
\text{Var} \left( G_K (\Gamma_A \setminus \mathcal{N}_A) ; R \right) = \sum_{f \in B_0(\Gamma)} |h_R(t_f)|^2 \frac{|W_{G_K(\Gamma_A \setminus \mathcal{N}_A), f}|^2}{(\sum_{A \in G_K} \text{vol}(\Gamma_A \setminus \mathcal{N}_A))^2} \\
+ \frac{1}{4\pi} \int_{-\infty}^{\infty} |h_R(t)|^2 \frac{|W_{G_K(\Gamma_A \setminus \mathcal{N}_A), \infty}(t)|^2}{(\sum_{A \in G_K} \text{vol}(\Gamma_A \setminus \mathcal{N}_A))^2} \, dt.
\]

**Proof.** This follows from the spectral expansion of \( K_R \) and Parseval's identity. \( \square \)

To bound the variance, we require upper bounds for the Weyl sums as well as lower bounds for \#G_K, \( \sum_{A \in G_K} \ell (C_A) \), and \( \sum_{A \in G_K} \text{vol}(\Gamma_A \setminus \mathcal{N}_A) \).

**Lemma 6.2.2.** We have that

\[
(-D)^{1/2-\varepsilon} \ll \#G_K \ll \sqrt{-D} \log(-D),
\]

\[
D^{1/2-\varepsilon} \ll \sum_{A \in G_K} \ell (C_A) \ll \sqrt{D} \log D,
\]

\[
D^{1/2-\varepsilon} \ll \sum_{A \in G_K} \text{vol}(\Gamma_A \setminus \mathcal{N}_A) \ll \sqrt{D} \log D.
\]

**Proof.** We have that \#G_K = 2^{1-\omega(|D|)} h_K^+ \quad \text{and} \quad \ell(C_A) = 2 \log \epsilon_K^+, \quad \text{while it is shown in} \quad \text{[DIT16, Proposition 1]} \quad \text{that}

\[
\frac{\#G_K \log \epsilon_K^+}{\log D} \ll \sum_{A \in G_K} \text{vol}(\Gamma_A \setminus \mathcal{N}_A) \ll \#G_K \log \epsilon_K^+.
\]

The class number formula states that

\[
h_K^+ = \begin{cases} \\
\frac{\sqrt{D} L(1, \chi_D)}{\log \epsilon_K^+} & \text{if } D > 0, \\
\frac{w_K \sqrt{-D} L(1, \chi_D)}{2\pi} & \text{if } D < 0,
\end{cases}
\]

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where

\[ w_K := \#O_{K,\text{tors}}^K = \begin{cases} 
4 & \text{if } D = -4, \\
6 & \text{if } D = -3, \\
2 & \text{otherwise.}
\end{cases} \]

The result then follows from the Landau–Siegel theorem and the bound \( L(1, \chi_D) \ll \log |D| \).

\[ \square \]

### 6.2.2 Genus Characters

The character group \( \widehat{\text{Gen}}_K \) of \( \text{Gen}_K \) is the group of real characters of \( \text{Cl}_K^+ \). These genus characters are indexed by unordered pairs of coprime fundamental discriminants \( d_1, d_2 \in \mathbb{Z} \) satisfying \( d_1 d_2 = D \). To each pair \( d_1, d_2 \), we let \( \chi = \chi_{d_1, d_2} \) denote the genus character corresponding to \( d_1, d_2 \): this is a real character of the narrow class group \( \text{Cl}_K^+ \) that extends multiplicatively to all nonzero fractional ideals via

\[ \chi(p) := \begin{cases} 
\chi_{d_1}(N(p)) & \text{if } (N(p), d_1) = 1, \\
\chi_{d_2}(N(p)) & \text{if } (N(p), d_2) = 1,
\end{cases} \]

for any prime ideals \( p \nmid \mathfrak{d}_K \), where \( \chi_{d_1}, \chi_{d_2} \) are the primitive real Dirichlet characters modulo \( d_1, d_2 \) respectively. In particular, \( \chi \) is a quadratic character unless either \( d_1 \) or \( d_2 \) is 1, in which case it is the trivial character.

**Lemma 6.2.3.** For any narrow ideal classes \( A_1, A_2 \in \text{Cl}_K^+ \), we have that

\[ \frac{1}{2\omega(|D|)-1} \sum_{\chi \in \widehat{\text{Gen}}_K} \chi(A_1 A_2) = \begin{cases} 
1 & \text{if } A_2 \in A_1 (\text{Cl}_K^+)^2, \\
0 & \text{otherwise.}
\end{cases} \]

**Proof.** This is character orthogonality for finite abelian groups. \( \square \)

We abuse notation and write \( G_K \) for an element in the coset of \( \text{Cl}_K^+ \) corresponding
to the genus $G_K$. This allows us to write

$$W_{G_K(z_A),f} = \frac{1}{2\omega(-D)-1} \sum_{\chi \in \text{Gen}_K} \chi(G_K) \sum_{A \in \text{Cl}_K} \chi(A)f(z_A),$$

$$W_{G_K(z_A),\infty}(t) = \frac{1}{2\omega(-D)-1} \sum_{\chi \in \text{Gen}_K} \chi(G_K) \sum_{A \in \text{Cl}_K} \chi(A)E\left(z_A, \frac{1}{2} + it\right),$$

$$W_{G_K(c_A),f} = \frac{1}{2\omega(D)-1} \sum_{\chi \in \text{Gen}_K} \chi(G_K) \sum_{A \in \text{Cl}_K^+} \chi(A) \int_{c_A} f(z) \, ds,$$

$$W_{G_K(c_A),\infty}(t) = \frac{1}{2\omega(D)-1} \sum_{\chi \in \text{Gen}_K} \chi(G_K) \sum_{A \in \text{Cl}_K^+} \chi(A) \int_{c_A} E\left(z, \frac{1}{2} + it\right) \, d\mu(z),$$

$$W_{G_K(\Gamma_A \backslash \mathcal{N}_A),f} = \frac{1}{2\omega(D)-1} \sum_{\chi \in \text{Gen}_K} \chi(G_K) \sum_{A \in \text{Cl}_K^+} \chi(A) \int_{\Gamma_A \backslash \mathcal{N}_A} f(z) \, d\mu(z),$$

$$W_{G_K(\Gamma_A \backslash \mathcal{N}_A),\infty}(t) = \frac{1}{2\omega(D)-1} \sum_{\chi \in \text{Gen}_K} \chi(G_K) \sum_{A \in \text{Cl}_K^+} \chi(A) \int_{\Gamma_A \backslash \mathcal{N}_A} E\left(z, \frac{1}{2} + it\right) \, d\mu(z).$$

This has the advantage that we are able to show in each case that the square of the sum over $A \in \text{Cl}_K^+$ is essentially equal to a product of $L$-functions.

### 6.2.3 Maaß Form Weyl Sums

**Lemma 6.2.4.** We have that

$$|W_{G_K(z_A),f}|^2 \ll \sqrt{D} \sum_{\chi \in \text{Gen}_K} \frac{L\left(\frac{1}{2}, f \times \chi_{d_1}\right) L\left(\frac{1}{2}, f \times \chi_{d_2}\right)}{L(1, \text{sym}^2 f)}.$$

$$|W_{G_K(c_A),f}|^2 \ll \sqrt{\frac{1}{4} + t_j^2} \sum_{\chi \in \text{Gen}_K} \frac{L\left(\frac{1}{2}, f \times \chi_{d_1}\right) L\left(\frac{1}{2}, f \times \chi_{d_2}\right)}{L(1, \text{sym}^2 f)},$$

$$|W_{G_K(\Gamma_A \backslash \mathcal{N}_A),f}|^2 \ll \sqrt{\frac{D}{\left(\frac{1}{4} + t_j^2\right)^3}} \sum_{\chi \in \text{Gen}_K} \frac{L\left(\frac{1}{2}, f \times \chi_{d_1}\right) L\left(\frac{1}{2}, f \times \chi_{d_2}\right)}{L(1, \text{sym}^2 f)}.$$

**Proof.** For $\chi = \chi_{d_1,d_2}$ and even $f \in \mathcal{B}_0(\Gamma)$, it is shown in [DIT16, Theorem 4 and
Equation (5.17)] that the quantity

\[
\left\{ \begin{array}{l}
\sum_{A \in \text{Cl}_K} \chi(A) \left| \frac{4\sqrt{\pi}}{w_K} f(z_A) \right|^2 & \text{if } d_1 d_2 < 0, \\
\sum_{A \in \text{Cl}_K^+} \chi(A) \left( \int_{C_A} f(z) \, ds \right)^2 & \text{if } d_1, d_2 > 0, \\
\sum_{A \in \text{Cl}_K^+} \chi(A) \left( \frac{\frac{1}{4} + t_f^2}{2} \int_{\Gamma_A \setminus \mathbb{N}_A} f(z) \, d\mu(z) \right)^2 & \text{if } d_1, d_2 < 0
\end{array} \right.
\]

is equal to

\[
\frac{|\rho_f(1)|^2}{\pi} \sqrt{|D|} \left| \Gamma \left( \frac{1}{2} - \frac{\text{sgn}(d_1)}{4} + \frac{it_f}{2} \right) \Gamma \left( \frac{1}{2} - \frac{\text{sgn}(d_2)}{4} + \frac{it_f}{2} \right) \right|^2 \\
\times L \left( \frac{1}{2}, f \times \chi_{d_1} \right) L \left( \frac{1}{2}, f \times \chi_{d_2} \right) \\
= \frac{\Lambda \left( \frac{1}{2}, f \times \chi_{d_1} \right) \Lambda \left( \frac{1}{2}, f \times \chi_{d_2} \right)}{2 \Lambda (1, \text{sym}^2 f)}.
\]

This also holds when \( f \) is odd, because in this case \( L(1/2, f \times \chi_d) = 0 \). Finally, it is also shown that

\[
\sum_{A \in \text{Cl}_K^+} \chi(A) \int_{\Gamma_A \setminus \mathbb{N}_A} f(z) \, d\mu(z)
\]

vanishes if \( d_1, d_2 > 0 \), and similarly

\[
\sum_{A \in \text{Cl}_K^+} \chi(A) \int_{C_A} f(z) \, ds
\]

vanishes if \( d_1, d_2 < 0 \). The result then follows from the Cauchy-Schwarz inequality and Stirling’s approximation.

Remark 6.2.6. The terms (6.2.5) can be viewed as toric integrals in the sense of [MV06, Section 2.2.1], and these can be generalised to involve Hecke characters \( \chi \) of \( K \) that are not necessarily genus characters. The resulting toric integral in this generalised setting
will essentially be equal to the completed Rankin–Selberg $L$-function $\Lambda\left(\frac{1}{2}, f \times g_\chi\right)$, where $g_\chi$ denotes the automorphic induction of the Hecke character $\chi$ to a newform $g_\chi$. When $\chi$ is a genus character $\chi_{d_1,d_2}$, this Rankin–Selberg $L$-function factorises as $\Lambda\left(\frac{1}{2}, f \times \chi_{d_1}\right) \Lambda\left(\frac{1}{2}, f \times \chi_{d_2}\right)$, while the case of $\chi$ being an ideal class character of an imaginary quadratic field $K$ and its applications towards equidistribution of Heegner points in conjugates of $\Gamma \backslash \mathbb{H}$ in $\Gamma_0(q) \backslash \mathbb{H}$ is investigated in [LMY13].

6.2.4 Eisenstein Series Weyl Sums

**Lemma 6.2.7.** We have that

$$\left|W_{G_K(z_A),\infty}(t)\right|^2 \ll \sqrt{-D} \sum_{\chi \in \hat{\text{Gen}_K}} \left| L\left(\frac{1}{2} + it, \chi_{d_1}\right) L\left(\frac{1}{2} + it, \chi_{d_2}\right) \frac{\zeta(1 + 2it)}{\zeta(1 + 2it)} \right|^2,$$

$$\left|W_{G_K(c_A),\infty}(t)\right|^2 \ll \sqrt{\frac{D}{\frac{1}{4} + t^2}} \sum_{\chi \in \hat{\text{Gen}_K}} \left| L\left(\frac{1}{2} + it, \chi_{d_1}\right) L\left(\frac{1}{2} + it, \chi_{d_2}\right) \frac{\zeta(1 + 2it)}{\zeta(1 + 2it)} \right|^2,$$

$$\left|W_{G_K(\Gamma_A \backslash \mathcal{N}_A),\infty}(t)\right|^2 \ll \sqrt{\frac{D}{(\frac{1}{4} + t^2)^3}} \sum_{\chi \in \hat{\text{Gen}_K}} \left| L\left(\frac{1}{2} + it, \chi_{d_1}\right) L\left(\frac{1}{2} + it, \chi_{d_2}\right) \frac{\zeta(1 + 2it)}{\zeta(1 + 2it)} \right|^2.$$

**Proof.** Again,

$$\sum_{A \in \text{Cl}_K^+} \chi(A) \int_{\Gamma_A \backslash \mathcal{N}_A} E\left(z, \frac{1}{2} + it\right) d\mu(z)$$

vanishes if $d_1, d_2 > 0$, as does

$$\sum_{A \in \text{Cl}_K^+} \chi(A) \int_{c_A} E\left(z, \frac{1}{2} + it\right) ds$$

if $d_1, d_2 < 0$. It is shown in [DIT16, Theorem 3] that

$$\frac{\Lambda\left(\frac{1}{2} + it, \chi_{d_1}\right) \Lambda\left(\frac{1}{2} - it, \chi_{d_1}\right) \Lambda\left(\frac{1}{2} + it, \chi_{d_2}\right) \Lambda\left(\frac{1}{2} - it, \chi_{d_2}\right)}{\Lambda(1 + 2it) \Lambda(1 - 2it)}$$
\[
\begin{aligned}
&= \begin{cases}
\sum_{A \in \text{Cl}_K} \chi(A) \frac{4\sqrt{\pi}}{w_K} E \left( z_A, \frac{1}{2} + it \right) \left| \left( z_A, \frac{1}{2} + it \right) \right|^2 & \text{if } d_1 d_2 < 0, \\
\sum_{A \in \text{Cl}_K^+} \chi(A) \int_{c_{A}} E \left( z, \frac{1}{2} + it \right) \, ds \left| \left( z, \frac{1}{2} + it \right) \right|^2 & \text{if } d_1, d_2 > 0, \\
\sum_{A \in \text{Cl}_K^{1+}} \chi(A) \frac{1}{2} + t^2 \int_{\Gamma_A \setminus \text{V}_{A}} E \left( z, \frac{1}{2} + it \right) \, d\mu(z) \left| \left( z, \frac{1}{2} + it \right) \right|^2 & \text{if } d_1, d_2 < 0,
\end{cases}
\end{aligned}
\]

from which the result again follows from the Cauchy-Schwarz inequality and Stirling’s approximation. \(\square\)

### 6.3 Bounds for the Variance

**Proof of Theorem 1.6.4.** For \(R \asymp (-D)^{-\delta}\), \(\text{Var} \left( G_K (z_A); R \right)\) is bounded by a constant multiple dependent on \(\varepsilon\) of

\[
(-D)^{-1/2+\varepsilon} \sum_{\chi \in \text{Gen}_K} \sum_{0 < t_f < 2(-D)^{\delta}} \frac{L \left( \frac{1}{2}, f \times \chi_{d_1} \right) L \left( \frac{1}{2}, f \times \chi_{d_2} \right)}{L \left( 1, \text{sym}^2 f \right)}
\]

\[
+ (-D)^{-1/2+3\delta+\varepsilon} \sum_{\chi \in \text{Gen}_K} \sum_{t_f \geq 2(-D)^{\delta}} \frac{L \left( \frac{1}{2}, f \times \chi_{d_1} \right) L \left( \frac{1}{2}, f \times \chi_{d_2} \right)}{t_f^3 L \left( 1, \text{sym}^2 f \right)}
\]

\[
+ (-D)^{-1/2+\varepsilon} \sum_{\chi \in \text{Gen}_K} \int_0^{2(-D)^{\delta}} \frac{\left| L \left( \frac{1}{2} + it, \chi_{d_1} \right) \right|^2 \left| L \left( \frac{1}{2} + it, \chi_{d_2} \right) \right|^2}{\left| \zeta(1+2it) \right|^2} \, dt
\]

\[
+ (-D)^{-1/2+3\delta+\varepsilon} \sum_{\chi \in \text{Gen}_K} \int_0^{\infty} \frac{\left| L \left( \frac{1}{2} + it, \chi_{d_1} \right) \right|^2 \left| L \left( \frac{1}{2} + it, \chi_{d_2} \right) \right|^2}{t^3 \left| \zeta(1+2it) \right|^2} \, dt.
\]

Similarly, when \(R \asymp D^{-\delta}\), \(\text{Var} \left( G_K (C_A); R \right)\) is bounded by a constant multiple dependent on \(\varepsilon\) of

\[
D^{-1/2+\varepsilon} \sum_{\chi \in \text{Gen}_K} \sum_{0 < t_f < 2D^\delta} \frac{L \left( \frac{1}{2}, f \times \chi_{d_1} \right) L \left( \frac{1}{2}, f \times \chi_{d_2} \right)}{t_f L \left( 1, \text{sym}^2 f \right)}
\]

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\[ + D^{-1/2+3\delta+\varepsilon} \sum_{\substack{\chi \in \hat{\text{Gen}}_{d_1,d_2} > 0}} \sum_{t_f \geq 2D^\delta} \frac{L \left( \frac{1}{2}, f \times \chi d_1 \right) L \left( \frac{1}{2}, f \times \chi d_2 \right)}{t_f^1 L \left( 1, \text{sym}^2 f \right)} \]

\[ + D^{-1/2+\varepsilon} \sum_{\chi \in \hat{\text{Gen}}_{d_1,d_2} > 0} \int_0^{2D^\delta} \frac{\left| L \left( \frac{1}{2} + it, \chi d_1 \right) \right|^2 \left| L \left( \frac{1}{2} + it, \chi d_2 \right) \right|^2}{\sqrt{\frac{1}{4} + t^2} |\zeta(1+2it)|^2} dt \]

\[ + D^{-1/2+3\delta+\varepsilon} \sum_{\chi \in \hat{\text{Gen}}_{d_1,d_2} < 0} \int_{2D^\delta}^{\infty} \frac{\left| L \left( \frac{1}{2} + it, \chi d_1 \right) \right|^2 \left| L \left( \frac{1}{2} + it, \chi d_2 \right) \right|^2}{t^4 |\zeta(1+2it)|^2} dt, \]

Upon applying the generalised Lindelöf hypothesis to each expression and using the Weyl law, Theorem 1.6.4 follows immediately. □

For unconditional results, we make use of the following bounds.

**Lemma 6.3.1 ([Ivi01, Theorem]).** For \( T \gg 1 \),

\[ \sum_{T \leq t_f \leq T+1} \frac{L \left( \frac{1}{2}, f \right)^3}{L(1, \text{sym}^2 f)} \ll_{\varepsilon} T^{1+\varepsilon}, \]

\[ \int_T^{T+1} \frac{\left| \zeta \left( \frac{1}{2} + it \right) \right|^6}{\left| \zeta(1+2it) \right|^2} dt \ll_{\varepsilon} T^{1+\varepsilon}. \]

**Lemma 6.3.2 ([You17, Theorem 1.1]).** For odd fundamental discriminants \( D \neq 1 \) and \( T \gg 1 \),

\[ \sum_{T \leq t_f \leq T+1} \frac{L \left( \frac{1}{2}, f \times \chi D \right)^3}{L(1, \text{sym}^2 f)} \ll_{\varepsilon} (|D|T)^{1+\varepsilon}, \]

\[ \int_T^{T+1} \frac{\left| L \left( \frac{1}{2} + it, \chi D \right) \right|^6 dt}{\left| \zeta(1+2it) \right|^2} \ll_{\varepsilon} (|D|T)^{1+\varepsilon}. \]

**Proof of Theorem 1.6.6.** The proof begins in the same way as the proof of Theorem 1.6.6, noting additionally that \( \text{Var} \left( G_K \ (\Gamma_A \backslash \mathcal{N}_A \ ; R) \right) \) is bounded by a constant multiple dependent on \( \varepsilon \) of
\[ D^{-1/2+\varepsilon} \sum_{\chi \in \text{Gen}_K \atop d_1, d_2 < 0} \sum_{0 < t_f < 2D^\delta} \frac{L \left( \frac{1}{2}, f \times \chi_{d_1} \right) L \left( \frac{1}{2}, f \times \chi_{d_2} \right)}{t_f^3 L (1, \text{sym}^2 f)} \]
\[ + D^{-1/2+3\delta+\varepsilon} \sum_{\chi \in \text{Gen}_K \atop d_1, d_2 < 0} \sum_{t_f \geq 2D^\delta} \frac{L \left( \frac{1}{2}, f \times \chi_{d_1} \right) L \left( \frac{1}{2}, f \times \chi_{d_2} \right)}{t_f^3 L (1, \text{sym}^2 f)} \]
\[ + D^{-1/2+\varepsilon} \sum_{\chi \in \text{Gen}_K \atop d_1, d_2 < 0} \int_0^{2D^\delta} \frac{\left| L \left( \frac{1}{2} + it, \chi_{d_1} \right) \right|^2 \left| L \left( \frac{1}{2} + it, \chi_{d_2} \right) \right|^2}{(\frac{1}{4} + t^2)^{3/2} |\zeta(1 + 2it)|^2} dt \]
\[ + D^{-1/2+3\delta+\varepsilon} \sum_{\chi \in \text{Gen}_K \atop d_1, d_2 < 0} \int_{2D^\delta}^\infty \frac{\left| L \left( \frac{1}{2} + it, \chi_{d_1} \right) \right|^2 \left| L \left( \frac{1}{2} + it, \chi_{d_2} \right) \right|^2}{t^6 |\zeta(1 + 2it)|^2} dt. \]

For each term, we use the generalised Hölder inequality with exponents \((3, 3, 3)\). Via the bounds in Lemmata 6.3.1 and 6.3.2, together with the Weyl law, we obtain the result.

\section{6.4 Representations of Integers by Indefinite Ternary Quadratic Forms}

We briefly describe how the results in this chapter can be interpreted in terms of indefinite ternary quadratic forms. For simplicity, we only discuss the case of negative discriminant and summing over all genera; for positive discriminant, a detailed presentation can be found in [ELMV12, Section 2].

Consider the indefinite ternary quadratic form

\[ Q(a, b, c) = b^2 - 4ac. \]

We are interested in the level sets

\[ V_{Q,D}(\mathbb{Z}) := \{(a, b, c) \in \mathbb{Z}^3 : b^2 - 4ac = D\}, \]
where $D < 0$ is a fundamental discriminant; these sets parametrise the different ways that the integer $D$ can be represented by the ternary quadratic form $Q$. The normalised level set $\mathcal{G}_D := (-D)^{-1/2}V_{Q,D}(\mathbb{Z})$ lies inside the two-sheeted hyperboloid

$$V_-(\mathbb{R}) := \{(a, b, c) \in \mathbb{R}^3 : b^2 - 4ac = -1\}.$$ 

It is natural to ask whether the normalised level sets $\mathcal{G}_D$ cover $V_{-1}(\mathbb{R})$ randomly as $D$ tends to $-\infty$ along fundamental discriminants. Of course, each level set $V_{Q,D}(\mathbb{Z})$ is countably infinite, and $V_-(\mathbb{R})$ is isomorphic to $\mathbb{C} \setminus \mathbb{R}$, the union of the upper and lower half-planes, which is not of finite volume; as such, one cannot immediately rephrase this random covering as equidistribution. On the other hand, the group

$$\text{SO}_Q(\mathbb{Z}) := \{A \in \text{SL}_3(\mathbb{Z}) : Q(Ax) = Q(x) \text{ for all } x = (a, b, c) \in \mathbb{Z}^3\}$$

acts transitively on $V_{Q,D}(\mathbb{Z})$, and the quotient space $\text{SO}_Q(\mathbb{Z}) \backslash \mathcal{G}_D$ is finite for all fundamental discriminants $D$, with cardinality equal to $h_K$. Moreover, $\text{SO}_Q(\mathbb{Z})$ is a discrete subgroup of $\text{SO}_Q(\mathbb{R})$ of finite covolume, where

$$\text{SO}_Q(\mathbb{R}) := \{A \in \text{SL}_3(\mathbb{R}) : Q(Ax) = Q(x) \text{ for all } x = (a, b, c) \in \mathbb{R}^3\},$$

and $V_-(\mathbb{R}) \cong \text{SO}_Q(\mathbb{R})/K$ with $K$ equal to the maximal compact subgroup of $\text{SO}_Q(\mathbb{R})$, and so the space $\text{SO}_Q(\mathbb{Z}) \backslash V_-(\mathbb{R})$ is of finite volume.

So to ask whether the normalised level sets $\mathcal{G}_D$ randomly cover $V_{-1}(\mathbb{R})$ can be rephrased as asking whether the finite sets $\text{SO}_Q(\mathbb{Z}) \backslash \mathcal{G}_D$ equidistribute in the finite volume space $\text{SO}_Q(\mathbb{Z}) \backslash V_-(\mathbb{R})$. This has a positive answer by naturally realising this result in terms of the equidistribution of Heegner points on $\Gamma \backslash \mathbb{H}$, as proved by Duke [Duk88, Theorem 1]. Indeed, the fact that $Q$ is indefinite implies that $\text{SO}_Q$ is isomorphic to the split special orthogonal group $\text{SO}_{1,2}$, and we have the
accidental isomorphism $\text{SO}_{1,2} \cong \text{PGL}_2$, while $K \cong \text{SO}_2(\mathbb{R})$. From this, we see that $\text{SO}_Q(\mathbb{Z}) \backslash V_-(\mathbb{R}) \cong \text{PGL}_2(\mathbb{Z}) \backslash \text{PGL}_2(\mathbb{R}) / \text{SO}_2(\mathbb{R}) \cong \Gamma \backslash \mathbb{H}$, while $\text{SO}_Q(\mathbb{Z}) \backslash \mathcal{G}_D$ is naturally identified with the set of Heegner points $\{ z_A \in \Gamma \backslash \mathbb{H} : A \in \text{Cl}_K \}$.

With this reinterpretation in mind, we now see that Theorem 1.6.4 implies that under the assumption of the generalised Lindelöf hypothesis, almost every shrinking ball of radius $R \asymp (-D)^{-\delta}$ with $0 < \delta < 1/4$ in $\text{SO}_Q(\mathbb{Z}) \backslash V_-(\mathbb{R})$ contains a normalised equivalence class of points $(a, b, c) \in \mathbb{Z}^3$ that represent the integer $D$ by the indefinite ternary quadratic form $Q(a, b, c) = b^2 - 4ac$. This complements [BRS16, Theorem 1.7], where the analogous result is proved for the definite ternary quadratic form $Q(a, b, c) = a^2 + b^2 + c^2$. 
Chapter 7

Further Problems

7.1 Quantum Unique Ergodicity in Shrinking Sets for Dihedral Maaß Newforms

The main reason that we are able to prove quantum unique ergodicity in shrinking sets for Eisenstein series is that the Watson–Ichino formula, as derived in Proposition 2.3.6, is particularly simple in this case; the numerator involves the product of four degree 2 $L$-functions, as opposed the general case given in Proposition 2.3.1, whose numerator involves the product of a degree 2 $L$-function and a degree 6 $L$-function.

Another case where the $L$-functions in the Watson–Ichino formula are less complicated than the general case is when $g_\psi \in B_0^\ast(q, \chi)$ is a dihedral Maaß newform, so that $q > 1$ is the fundamental discriminant of a real quadratic field $K = \mathbb{Q}(\sqrt{q})$ and $\chi$ is the primitive quadratic character modulo $q$, and $g_\psi$ is automorphically induced from a Hecke Größencharakter $\psi$ of $K$ of conductor $\mathcal{O}_K$. As we shall state in Proposition 7.1.1, the numerator in the Watson–Ichino formula involves the product of two degree 2 $L$-functions and one degree 4 $L$-function; for this reason, one might hope that proving quantum unique ergodicity in shrinking sets is attainable in this case.

There are, however, new issues that arise. The spectral expansion of the variance
and the Watson–Ichino formula become slightly more complex due to ramification at nonarchimedean places. Moreover, the discrete spectrum decomposes into newforms and oldforms, which is not an orthonormal decomposition, whereas the Watson–Ichino formula is best understood for newforms, so we need to take care in understanding this decomposition.

For the Watson–Ichino formula, we are able to prove the following.

**Proposition 7.1.1.** Suppose that \( q \equiv 1 \pmod{4} \) is squarefree and that \( \chi \) is the primitive quadratic character modulo \( q \). Let \( g_\psi \in \mathcal{B}_0^*(q, \chi) \) be dihedral, and let \( f \in \mathcal{B}_0^*(\Gamma_0(q_2)) \) with \( q_2 \mid q \), with \( f, g \) normalised such that \( \langle g_\psi, g_\psi \rangle_q = \langle f, f \rangle_q = 1 \). Then

\[
\left| \langle |g_\psi|^2, f \rangle_q \right|^2 = \frac{1}{8q^2} \frac{q^2}{\sigma_{-1}(q_1)\varphi(q_1)^2} \frac{\Lambda(\frac{1}{2}, f) \Lambda(\frac{1}{2}, f \otimes \chi) \Lambda(\frac{1}{2}, f \otimes g_\psi^2)}{\Lambda(1, \chi)^2 \Lambda(1, g_\psi^2)^2 \Lambda(1, \text{sym}^2 f)},
\]

where \( g_\psi^2 \in \mathcal{B}_0^*(q, \chi) \) denotes the automorphic induction from \( \text{GL}_1(\mathbb{A}_K) \) to \( \text{GL}_2(\mathbb{A}_Q) \) of the Hecke Größencharakter \( \psi^2 \) of \( K \).

Similarly, let \( E_a(z, s) \) denote the Eisenstein series on \( \Gamma_0(q) \backslash \mathbb{H} \) associated to the cusp \( a \) of \( \Gamma_0(q) \backslash \mathbb{H} \). Then

\[
\left| \langle |g_\psi|^2, E_a \left( \cdot, \frac{1}{2} + it \right) \rangle_q \right|^2 = \left( \frac{1}{4q^2} \right)^2 \frac{\Lambda^q(\frac{1}{2} + it) \Lambda^q(\frac{1}{2} - it, \chi) \Lambda(\frac{1}{2} - it, \chi) \Lambda(\frac{1}{2} + it, g_\psi^2) \Lambda(\frac{1}{2} - it, g_\psi^2)}{\Lambda(1, \chi)^2 \Lambda(1, g_\psi^2)^2 \Lambda^q(1 + 2it) \Lambda^q(1 - 2it)}.
\]

This makes use of recent work of Collins [Col16, Proposition 3.2.3], as well as unpublished work of the author, in determining local constants in the Watson–Ichino theorem.

For the spectral decomposition of the variance, we are able to make use of the orthonormal basis \( \mathcal{B}_0(\Gamma_0(q)) \) in terms of linear combinations of newforms and oldforms given by Iwaniec, Luo, and Sarnak [ILS00, Proposition 2.6], together with results of Asai [Asa76, Theorems 1 and 2] on the relation between Atkin–Lehner operators and
Proposition 7.1.2. Suppose that \( q \equiv 1 \pmod{4} \) is squarefree and that \( \chi \) is the primitive quadratic character modulo \( q \). Let \( g_\psi \in \mathcal{B}_0^* (q, \chi) \) be dihedral and normalised such that \( \langle g_\psi, g_\psi \rangle_q = 1 \). Then

\[
\text{Var}(g_\psi; R) := \int_{\Gamma_0(q) \backslash \mathbb{H}} \left( \frac{1}{\text{vol}(B_R)} \int_{B_R(w)} |g_\psi(z)|^2 \, d\mu(z) - \frac{1}{\text{vol}(\Gamma_0(q) \backslash \mathbb{H})} \right)^2 \, d\mu(w)
\]

is equal to

\[
\sum_{q_1, q_2 = q} \frac{2^{\omega(q_1)} \sigma_0(q_1) \varphi(q_1)}{q_1} \sum_{f \in \mathcal{B}_0^* (\Gamma_0(q_2))} \frac{L_{q_1}(1, \text{sym}^2 f)}{L_{q_1} \left( \frac{1}{2}, f \right)} |h_R(t_f)|^2 \left| \left\langle |g_\psi|^2, f \right\rangle_q \right|^2
\]

\[
\quad + \frac{2^{\omega(q)}}{4\pi} \int_{-\infty}^{\infty} |h_R(t)|^2 \left| \left\langle |g_\psi|^2, E_\infty \left( \cdot, \frac{1}{2} + it \right) \right\rangle_q \right|^2 \, dt,
\]

where each \( f \in \mathcal{B}_0^* (\Gamma_0(q_2)) \) is normalised such that \( \langle f, f \rangle_q = 1 \).

With these results in hand, proving unconditionally that \( \text{Var}(g_\psi; R) = o(1) \) for \( R \asymp_q t_g^{-\delta} \) with \( 0 < \delta < 1 \) essentially reduces to proving an analogue of Lemma 3.4.1, namely proving that for \( t_g \geq 0 \) and \( 1 \ll_q H \ll_q t_g^{1-\delta} \) with \( \delta > 0 \) fixed, we have that

\[
\sum_{\substack{f \in \mathcal{B}_0^* (\Gamma_0(q_2)) \\ H \leq t_f \leq 2H}} \frac{L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, f \times \chi \right) L \left( \frac{1}{2}, f \times g_\psi^2 \right)}{L \left( 1, \text{sym}^2 f \right)} \ll_q H t_g^{1-\delta'}
\]

for some \( \delta' > 0 \). A result of this form would deal with the short initial and short polynomial decay ranges; the remaining ranges can then be dealt with via the large sieve.

To prove such an estimate, a natural approach would be to mimic the proof of Lemma 3.4.1.

- For the range \( 0 \leq t_f \ll_q t_g^{\delta} \) with \( \delta > 0 \) fixed, we can use the use the subconvexity
bounds of Michel and Venkatesh [MV10, Theorem 1.2].

• For the range \( t^\delta_g \ll q \ t_f \ll_q t_g^{2/3-\delta} \), we may mimic the proof of Jutila [Jut04, Section 2], in which it is proved that

\[
\sum_{H \leq t_f \leq H+G} \frac{|L(\frac{1}{2} + it, f)|^2}{L(1, \text{sym}^2 f)} \ll_{\varepsilon} \left( GH + t^{2/3} + \frac{t}{H} \right)^{1+\varepsilon}
\]

for \( t \geq 0 \) and \( 1 \ll G \ll H \). The method of proof ought to be able to be generalised to prove that

\[
\sum_{f \in \mathcal{B}_0^*(\Gamma_0(q_2)) \atop H \leq t_f \leq H+G} \frac{L(\frac{1}{2}, f \times g_{\psi^2})}{L(1, \text{sym}^2 f)} \ll_{q,\varepsilon} \left( GH + t_g^{2/3} + \frac{t_g}{H} \right)^{1+\varepsilon}.
\]

The chief new idea is to use the fact that \( g_{\psi^2} \) is the automorphic induction of the Hecke Größencharakter \( \psi^2 \) to replace [Jut04, Equation (2.7)] with a sum over integral ideals of the real quadratic number field \( K \), and then generalise the partial summation formula [Jut04, Lemma 2] to this setting.

• For the range \( t_g^{2/3-\delta} \ll_q t_f \ll_q t_g^{1-\delta} \), we need to mimic the proof of Jutila and Motohashi [JM05, Theorem 2], which is more challenging; for example, we need an explicit spectral decomposition of certain shifted convolution sums. Nevertheless, it seems that the machinery for such a proof is in place.

Remark 7.1.3. In [BK17a, Section 1], Buttcane and Khan mention that they do not know how to deal with the short initial range for the \( L^4 \)-norm problem for dihedral Maaß newforms; the above discussion suggests a method of dealing with this range.
7.2 Random Wave Conjecture for Holomorphic Newforms

Much of the discussion of Section 1.2 can also be applied to \( g(z) = y^{k/2}G(z) \) with \( G \in B_k(\Gamma) \) a holomorphic Hecke eigenform of even weight \( k \geq 2 \) normalised such that \( \langle g, g \rangle = 1 \). In this case, we expect holomorphic Hecke eigenforms of large weight to behave like complex random waves, so that the distribution of \( g \) should behave like the distribution of a complex Gaussian random variable as \( k \) tends to infinity. Similarly, the moments of \( g \) should converge to the moments of a complex Gaussian random variable. Finally, one can study quantum unique ergodicity for holomorphic Hecke eigenforms.

7.2.1 Quantum Unique Ergodicity in Shrinking Sets for Holomorphic Eigenforms

Quantum unique ergodicity for holomorphic Hecke eigenforms is the statement that for \( G \in B_k(\Gamma) \), \( y^k|G(z)|^2 \, d\mu(z) \) converges in distribution as \( k \) tends to infinity to the uniform probability measure on \( \Gamma \setminus \mathbb{H} \); this is a result of Holowinsky and Soundararajan [HoSo10, Theorem 1]. As for Hecke–Maaß eigenforms, one can study quantum unique ergodicity for holomorphic Hecke eigenforms in shrinking sets. The same method of proof of Theorem 1.5.6 shows the following.

**Theorem 7.2.1.** Let \( G \in B_k(\Gamma) \) be a holomorphic Hecke eigenform of even weight \( k \geq 2 \) normalised such that \( \langle g, g \rangle = 1 \), where \( g(z) = y^{k/2}G(z) \). For \( R > 0 \), let

\[
\text{Var}(G; R) := \int_{\Gamma \setminus \mathbb{H}} \left( \frac{1}{\text{vol}(B_R)} \int_{B_R(w)} y^k|G(z)|^2 \, d\mu(z) - \frac{1}{\text{vol}(\Gamma \setminus \mathbb{H})} \right)^2 \, d\mu(w).
\]

Assume the generalised Lindelöf hypothesis, and suppose that \( R \asymp k^{-\delta} \) for some
\( 0 < \delta < 1. \) Then

\[
\text{Var}(G; R) \ll_{\varepsilon} k^{-(1-\delta)+\varepsilon}
\]

as \( k \) tends to infinity.

A classical result states that the (properly weighted) number of zeroes of a holomorphic cusp form is asymptotic to \( k/12 \). A nontrivial consequence of quantum unique ergodicity for holomorphic Hecke eigenforms due to Rudnick [Rud05, Theorem 1.1] is that the zeroes of a holomorphic Hecke eigenform equidistribute in \( \Gamma \setminus \mathbb{H} \) as \( k \) tends to infinity. Subsequently, Lester, Matomäki, and Radziwiłł have given an unconditional proof that this equidistribution still holds in shrinking ball centred about a fixed point of radius \( R \gg_{\varepsilon} (\log k)^{-\delta+\varepsilon} \) for some explicit constant \( -\delta > 0 \) [LMR15, Theorem 1.1], while conditionally on the generalised Lindelöf hypothesis, the same result holds for \( R \gg_{\varepsilon} k^{-1/8+\varepsilon} \) [LMR15, Theorem 1.2]. It is natural to wonder, though currently remains unclear, whether Theorem 7.2.1 implies an analogous result, contingent on the generalised Lindelöf hypothesis, of the equidistribution of the zeroes of holomorphic Hecke eigenforms in almost every shrinking ball of radius \( R \gg_{\varepsilon} k^{-1/2+\varepsilon} \).

### 7.2.2 \( L^4 \)-Norm Bounds for Holomorphic Hecke Eigenforms

Similarly, one can study the \( L^4 \)-norm of a holomorphic Hecke eigenform \( G \in \mathcal{B}_k(\Gamma) \) of even weight \( k \geq 2 \). Towards this, Blomer, Khan, and Young [BKY13, Theorem 1.1] have shown that for \( g(z) = y^{k/2}G(z) \) normalised such that \( \langle g, g \rangle = 1 \),

\[
\|g\|_{L^4(\Gamma \setminus \mathbb{H})}^4 \ll_{\varepsilon} k^{\frac{1}{4}+\varepsilon}
\]

as \( k \) tends to infinity. Moreover, they conjecture that these holomorphic Hecke eigenforms are modelled by a complex Gaussian random variable [BKY13, Conjecture]
1.2), so that we should expect that

$$\|g\|_{L^4(\Gamma \backslash \mathbb{H})}^4 = \frac{2}{\text{vol}(\Gamma \backslash \mathbb{H})} + o(1)$$

as $k$ tends to infinity.

The approach of the proof of [BKY13, Theorem 1.1] is to use the fact that $G(z)^2$ is a holomorphic cusp form of weight $2k$, which allows one to apply Parseval’s identity to express $\|g\|_{L^4(\Gamma \backslash \mathbb{H})}^4$ as a spectral sum only over holomorphic Hecke eigenforms of weight $2k$. Another approach is to use the fact that $|g|^2 \in L^2(\Gamma \backslash \mathbb{H})$ and then apply Parseval’s identity in order to express $\|g\|_{L^4(\Gamma \backslash \mathbb{H})}^4$ as a spectral sum over Hecke–Maaß eigenforms of weight zero, together with the contributions from the residual and continuous spectra. With this spectral decomposition at hand, one ought to be able to mimic the result of Buttcane and Khan [BK17b, Theorem 1.1], where it is shown that for $g \in B_0(\Gamma)$,

$$\|g\|_{L^4(\Gamma \backslash \mathbb{H})}^4 = \frac{3}{\text{vol}(\Gamma \backslash \mathbb{H})} + o(1),$$

conditional on the generalised Lindelöf hypothesis.

### 7.3 Equidistribution of Closed Geodesics in Shrinking Sets via Ergodic Methods

Ellenberg, Michel, and Venkatesh [EMV13, Theorem 1.3] have given an unconditional proof of a result akin to Theorem 1.6.4; more precisely, they prove a modular analogue of equidistribution of lattice points on the sphere in almost every shrinking ball, subject to a congruence requirement on the discriminant $D$, which is equivalent to requiring a given prime split in $\mathbb{Q}(\sqrt{D})$ with $D < 0$. The proof does not involve spectral decompositions and Weyl sums; instead, it is by ergodic methods that go back to the seminal work of Linnik [Linn68].
These ergodic methods have also been used by Einsiedler, Lindenstrauss, Michel, and Venkatesh [ELMV12, Theorem 1.3] to reprove Duke’s theorem [Duk88, Theorem 1] on the equidistribution of closed geodesics on $\Gamma \backslash \mathbb{H}$; this makes use of the fact that no congruence restriction on the discriminant is required due, for the archimedean place of $\mathbb{Q}$ splits over $\mathbb{Q}(\sqrt{D})$ with $D > 0$. With this in mind, it would be of interest whether one could combine the methods of [EMV13] and [ELMV12] to give an unconditional proof via ergodic methods of Theorem 1.6.4 for closed geodesics.

7.4 Hybrid Random Wave Conjecture

It is natural to wonder whether Maas newforms $g \in \mathcal{B}_0^\ast(q, \chi)$ and holomorphic newforms $G \in \mathcal{B}_k^\ast(q, \chi)$ exhibit the behaviour of random waves if one allows the level $q$ to tend to infinity. In certain cases, this problem has previously been studied. Quantum unique ergodicity for holomorphic newforms as $qk^2$ tends to infinity is known via the work of Nelson, Pitale, and Saha [NPS14, Theorem 1.1]; equidistribution occurs in the sense that

$$\int_B y^k |G(z)|^2 f(z) \, d\mu(z) = \frac{\text{vol}(B)}{\text{vol}(\Gamma \backslash \mathbb{H})} + o_B(1)$$

as $qk^2$ tends to infinity for every continuity set $B \subset \Gamma \backslash \mathbb{H}$, with such a set extending naturally to a continuity set contained in $\Gamma_0(q) \backslash \mathbb{H}$.

Due to the fact that the underlying space $\Gamma_0(q) \backslash \mathbb{H}$ is varying, it is less clear how one might prove quantum unique ergodicity in almost every shrinking ball as $q(g) \asymp qt_g^2$ tends to infinity, for the proof of Theorem 1.5.6 makes use of an additional averaging over the centre $w$ of the ball $B_R(w)$ on the full space upon which $g$ lives. Nonetheless, there remain natural problems to study in the hybrid limit $qt_g^2 \to \infty$.  

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Given a Maaß newform $g \in \mathcal{B}_0^*(q, \chi)$ or a holomorphic newform $G \in \mathcal{B}_k^*(q, \chi)$ normalised such that $\langle g, g \rangle_q = 1$, where in the latter case we set $g(z) = y^{k/2}G(z)$, it is natural to ask what bounds one can achieve for the $L^4$-norm $\|g\|_{L^4(\Gamma_0(q) \backslash \mathbb{H})}$ that are uniform in both the archimedean aspect, namely $t_g$ or $k$, and the level aspect, namely $q$. It is natural to conjecture the following.

**Conjecture 7.4.1.** We have that

$$\text{vol} \left( \Gamma_0(q) \backslash \mathbb{H} \right) \int_{\Gamma_0(q) \backslash \mathbb{H}} |g(z)|^4 d\mu(z) = \begin{cases} 3 + o(1) & \text{as } qt_g^2 \to \infty \text{ if } g \in \mathcal{B}_0^*(\Gamma_0(q)), \\ 2 + o(1) & \text{as } qt_g^2 \to \infty \text{ if } g \in \mathcal{B}_0^*(q, \chi) \text{ with } \chi \text{ nonprincipal}, \\ 2 + o(1) & \text{as } qk^2 \to \infty \text{ if } g(z) = y^{k/2}G(z) \text{ with } G \in \mathcal{B}_k^*(q, \chi). \end{cases}$$

Towards this conjecture, little is known, though this problem has been studied in the level aspect. The strongest result is due to Buttcane and Khan.

**Theorem 7.4.2** (Buttcane–Khan [BK15, Theorem 1.1]). For $q$ prime and $g(z) = y^{k/2}G(z)$ with $G \in \mathcal{B}_k^*(q, \chi)$,

$$\|g\|_{L^4(\Gamma_0(q) \backslash \mathbb{H})} \ll_{k, \varepsilon} q^{-\frac{3}{4}-\delta+\varepsilon},$$

where $\delta > 0$ is such that the subconvexity estimate

$$L \left( \frac{1}{2}, F \right) \ll_{k, \varepsilon} q^{\frac{1}{2}-\delta+\varepsilon}$$

holds for all $F \in \mathcal{B}_{2k}^*(\Gamma_0(q))$.

When $G_\psi \in \mathcal{B}_k^*(q, \chi)$ is a dihedral holomorphic newform with $q$ prime, Liu [Liu15,
Theorem 1.1] gives a proof of the stronger bound
\[ \|g_\psi\|_{L^4(G_0(q)\backslash \mathbb{H})} \ll_{k,\varepsilon} q^{-\frac{11}{12} - \frac{4}{3} + \varepsilon}, \]
where \( g_\psi(z) = y^{k/2}G_\psi(z) \). Unfortunately, the proof is flawed, and instead only the weaker result
\[ \|g_\psi\|_{L^4(G_0(q)\backslash \mathbb{H})} \ll_{k,\varepsilon} q^{-\frac{2}{3} - \frac{4}{3} + \varepsilon} \]
can be shown to hold. Indeed, the application of the Watson–Ichino formula and factorisation of the triple product \( L \)-function in [Liu15, Equations (2.2) and (2.4)] is invalid; the latter is trivially rectified, while the former can be fixed via the recent work of Collins [Col16, Proposition 3.2.3], as well as unpublished work of the author, in determining local constants in Ichino’s theorem. However, there is an error in the proof of [Liu15, Lemma 3.2] that cannot be corrected without weakening the result; the conductor of \( L(1/2, F \times g) \) is not \( \asymp_k q^2 \) but instead \( \asymp_k q^3 \), so the application of the large sieve here leads only to the bound \( M_2 \ll_{k,\varepsilon} q^{3/2+\varepsilon} \).

In fact, proving \( L^4 \)-norm bounds for dihedral newforms is more difficult than for newforms of principal nebentypus, because the conductor of the \( L \)-functions arising from the Watson–Ichino formula is larger in the \( q \) aspect. In spite of this, we are able to prove the following hybrid \( L^4 \)-norm bound for dihedral Maaß newforms.

**Theorem 7.4.3.** Suppose that \( q \equiv 1 \pmod{4} \) is squarefree and that \( \chi \) is the primitive quadratic character modulo \( q \). Let \( g_\psi \in \mathcal{B}_0^*(q, \chi) \) be dihedral and normalised such that \( \langle g_\psi, g_\psi \rangle_q = 1 \). Then
\[ \|g_\psi\|_{L^4(G_0(q)\backslash \mathbb{H})} \ll_{\varepsilon} q^{-\frac{1}{2} + \varepsilon} t_g^\varepsilon, \]

**Sketch of Proof.** The proof begins via Proposition 7.1.1 and (the method of proof of) Proposition 7.1.2. Then one proceeds by the same method as the proof of [Luo14,
Theorem]; via Hölder’s inequality, it suffices to determine bounds for

\[ \sum_{f \in \mathcal{B}_0^r(\Gamma_0(q_2))} \frac{L\left(\frac{1}{2}, f\right)^4}{L(1, \text{sym}^2 f)}, \]

\[ \sum_{f \in \mathcal{B}_0^r(\Gamma_0(q_2))} \frac{L\left(\frac{1}{2}, f \times \chi\right)^4}{L(1, \text{sym}^2 f)}, \]

\[ \sum_{f \in \mathcal{B}_0^r(\Gamma_0(q_2))} \frac{L\left(\frac{1}{2}, f \times g_{\psi^2}\right)^2}{L(1, \text{sym}^2 f)}. \]

We then use the large sieve to bound each these spectral sums, taking care to keep track of both the $q$ and $t_g$ dependence.

With more effort, it is likely that one can lower the exponent of $q$, though it is probably currently out of reach to prove

\[ \|g_{\psi}\|_{L^4(\Gamma_0(q)\setminus \mathbb{H})}^4 \ll q^{-1+\varepsilon} t_g^\varepsilon, \]

let alone the asymptotic expression in Conjecture 7.4.1.
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