Quantum coadjoint orbits of $GL(n)$ and generalized Verma modules

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Abstract

In our previous paper, we constructed an explicit $GL(n)$-equivariant quantization of the Kirillov–Kostant-Souriau bracket on a semisimple coadjoint orbit. In the present paper, we realize that quantization as a subalgebra of endomorphisms of a generalized Verma module. As a corollary, we obtain an explicit description of the annihilators of generalized Verma modules over $U(gl(n))$. As an application, we construct real forms of the quantum orbits and classify finite dimensional representations. We compute the non-commutative Connes index for basic homogeneous vector bundles over the quantum orbits.

Key words: Kirillov-Kostant-Souriau bracket, equivariant quantization, generalized Verma modules.

AMS classification codes: 53D55, 53D05, 22E47

1 Introduction

Amongst Poisson manifolds with symmetries the simplest and the most natural one is the dual space $\mathfrak{g}^*$ to a Lie algebra $\mathfrak{g}$. The Poisson structure on $\mathfrak{g}^*$ is induced by the Lie bracket of $\mathfrak{g}$ considered as a bivector field on $\mathfrak{g}^*$. The symplectic leaves of $\mathfrak{g}^*$ are precisely the coadjoint orbits of the corresponding Lie group $G$. The restriction of the Poisson-Lie structure from $\mathfrak{g}^*$

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to an orbit is called the Kirillov-Kostant-Souriau (KKS) bracket. Importance of this bracket is accounted for the fact that every $G$-homogeneous symplectic manifold is locally isomorphic to a $G$-coadjoint orbit via the moment map. Construction of $G$-equivariant quantization of the KKS bracket is a classic problem of the deformation quantization theory, [BFlFrLSt].

By a quantization of a manifold $M$ we mean a flat $\mathbb{C}[[t]]$-algebra $A_t(M)$ with an isomorphism $A_t(M)/tA_t(M) \rightarrow \mathcal{A}(M)$, where $\mathcal{A}(M)$ is the function algebra on $M$. A quantization of an $G$-space $M$ is called equivariant if $G$ acts on $A_t(M)$ by algebra automorphisms and that action extends the original action of $G$ on $\mathcal{A}(M)$. There are various approaches to quantization of Poisson manifolds. One of them, the $\ast$-product, presents a deformed multiplication in $\mathcal{A}(M) \otimes_{\mathbb{C}} \mathbb{C}[[t]]$ (the tensor product is completed in the $t$-adic topology) as a formal $t$-series with coefficients being bi-differential operators. Famous Fedosov’s construction guarantees existence of an equivariant $\ast$-product on a symplectic manifold with a $G$-invariant connection. However, Fedosov’s quantization is not given by an explicit formula for a particular manifold. Another approach to quantization is to describe the algebra $A_t(M)$ in terms of generators and relations. While associativity holds by the very construction and $G$-equivariance can be easily guaranteed, the principal difficulty is to ensure flatness of an algebra built in such a way.

There is a universal approach to equivariant quantization of semisimple coadjoint orbits of complex reductive Lie groups which is based on generalized Verma modules, [DGS]. By a generalized Verma module $V_{p,\lambda}$ we mean the left $\mathcal{U}(\mathfrak{g})$-module $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \mathbb{C}_\lambda$, where $\mathfrak{p} \subset \mathfrak{g}$ is a parabolic subalgebra and $\mathbb{C}_\lambda$ is a one-dimensional representation of $\mathcal{U}(\mathfrak{p})$. It is generated by a $\mathfrak{p}$-character $\lambda$, which can be identified with a certain element of $\mathfrak{g}^*$. The approach of [DGS] presents a quantized orbit as a subalgebra of endomorphisms of a generalized Verma module. Consider the Lie algebra $\mathfrak{g}_t$ over $\mathbb{C}[t]$ which coincides with $\mathfrak{g}[t]$ as a $\mathbb{C}[t]$-module and whose Lie bracket $[\ldots]_t$ extends from the bracket of $\mathfrak{g}$: $[x, y]_t = t[x, y]$ for all $x, y \in \mathfrak{g}$. The universal enveloping algebra $\mathcal{U}(\mathfrak{g}_t)$ is a $G$-equivariant quantization of the polynomial algebra on $\mathfrak{g}^*$. There is a family of isomorphisms $\mathcal{U}(\mathfrak{g}_t) \rightarrow \mathcal{U}(\mathfrak{g})$ at $t \neq 0$ (this extends to a natural embedding of $\mathbb{C}[t]$-algebras) such that the image of the composite map $\mathcal{U}(\mathfrak{g}_t) \rightarrow \mathcal{U}(\mathfrak{g}) \rightarrow \text{End}(V_{p,\lambda/t})$ gives an equivariant quantization of the orbit passing through $\lambda$.

The approach of [DGS] also presents quantized orbits as quotients of $\mathcal{U}(\mathfrak{g}_t)$. This fact was used in [DM2] for explicit description of quantized semisimple $GL(n)$-orbits in terms of generators and relations. There was built a $\mathbb{C}[t]$-algebra $\mathcal{A}_{m,\mu,\lambda}$ giving quantization of the polynomial algebra on the orbit of matrices with eigenvalues $\mu = (\mu_1, \ldots, \mu_k)$ of multiplicities $m = (m_1, \ldots, m_k)$. The algebra $\mathcal{A}_{m,\mu,\lambda}$ is a quotient of $\mathcal{U}(\mathfrak{g}_t)$ by a certain ideal whose
generators are given explicitly. This ideal can be realized (not uniquely) as the annihilator of a generalized Verma module $V_{p,\lambda}/t$, where $p$ and $\lambda$ depend on $m$, $\mu$, and $t$. In the present paper, we find this dependence explicitly.

Our first result is the following. We relate the two approaches to equivariant quantization of semisimple orbits: the quantization via generalized Verma modules and the one in terms of generators and relations.

As our second result, we describe the annihilator of a generalized Verma module $V_{p,\lambda}$ for generic $\lambda$ as an ideal in $\mathcal{U}(g)$ by presenting its set of generators.

As an application, we classify all finite dimensional representations of the algebras $A_{m,\mu,t}$ for fixed $m$, $\mu$, and $t$. We show that they all are factored through generalized Verma modules. For a finite dimensional representation to exist, the numbers $(\mu_i - \mu_j)/t$ should satisfy certain integral conditions. When $\mu$ is fixed, the dimension of representations grows as the deformation parameter $t$ tends to zero. This process yields "fuzzyfication" for semisimple orbits of $GL(n)$. Let us note that a reversed approach from fuzzy geometry to $*$-product quantization was used for quantization of grassmanian spaces in [DolJ]. For applications of fuzzy geometry to theoretical physics see e.g. [ARSch].

As another application, we construct real forms $A_{m,\mu,t}$ that are compatible with the standard real forms of $gl(n)$.

Finally, we show that special rational functions defining the central character of a quantum orbit, [DM2], give the Connes index for the basic quantum homogeneous vector bundles on the orbit.

The paper is organized as follows. In Sections 2 and 3 we recall respectively the quantization via generalized Verma modules and the quantization in terms of generators and relations. The correspondence between these two approaches is established in Section 4. Subsection 4.6 is devoted to finite-dimensional representations of the algebras $A_{m,\mu,t}$. In Section 5 we construct real forms on $A_{m,\mu,t}$. In Subsection 6.2, we compute the non-commutative Connes index for the case of two-parameter quantization of [DM2]. In Appendix we deduce an explicit polynomial expression for the central characters of quantum orbits.

2 Quantization via generalized Verma modules.

2.1. Consider a complex reductive Lie algebra $g$ and let $G$ be the corresponding connected Lie group. The dual space $g^*$ is a Poisson $G$-manifold endowed with the standard Poisson-Lie structure induced by the Lie bracket of $g$. This bracket can be restricted to every orbit of
$G$ in $\mathfrak{g}^*$ making it a homogeneous symplectic manifold. That symplectic structure is called Kirillov-Kostant-Souriau bracket. An element from $\mathfrak{g}^*$ is called semisimple if it is the image of a semisimple element under the $G$-equivariant isomorphism $\mathfrak{g} \to \mathfrak{g}^*$ via a non-degenerate ad-invariant inner product on $\mathfrak{g}$. An orbit is called semisimple if it passes through a semisimple element.

Semisimple coadjoint orbits are closed affine varieties in $\mathfrak{g}^*$. By the (polynomial) function algebra on a subvariety $M \subset \mathfrak{g}^*$ we mean the algebra $\mathcal{A}(M)$ consisting of restrictions of polynomial functions on $\mathfrak{g}^*$. In the present paper a quantization of $M$ is a flat\(^1\) $\mathbb{C}[t]$-algebra $\mathcal{A}_t(M)$ together with an isomorphism $\kappa: \mathcal{A}_t(M)/t\mathcal{A}_t(M) \to \mathcal{A}(M)$. The quantization is called $G$-equivariant if there is a $G$-action on $\mathcal{A}_t(M)$ by algebra automorphisms, an extension of the natural action on $\mathcal{A}(M)$. Such an action gives rise to a Hopf algebra action of $U(\mathfrak{g})$,

$$x \triangleright (ab) = (x \triangleright a)b + a(x \triangleright b), \quad x \in \mathfrak{g}, \quad a, b \in \mathcal{A}_t(M),$$

and vice versa, so we also call a $G$-equivariant quantization $U(\mathfrak{g})$-equivariant.

2.2. Let $\mathfrak{h}$ be a Cartan subalgebra in $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ a triangular decomposition relative to $\mathfrak{h}$. Let $\mathfrak{p}$ be a parabolic subalgebra in $\mathfrak{g}$ containing $\mathfrak{h}$ and $\mathfrak{n}^+$ and let $\mathfrak{l}$ be its Levi factor. The projection $\mathfrak{g} \to \mathfrak{h}$ along the triangular decomposition gives rise to an embedding $\mathfrak{h}^* \to \mathfrak{g}^*$. Denote by $\mathfrak{c}$ the center of $\mathfrak{l}$; then $\mathfrak{c}^*$ is identified with the annihilator of $[\mathfrak{l}, \mathfrak{l}]$ in $\mathfrak{h}^*$ via decomposition $\mathfrak{h} = [\mathfrak{l}, \mathfrak{l}] \cap \mathfrak{h} \oplus \mathfrak{c}$. Any $\lambda \in \mathfrak{c}^*$ defines a one-dimensional representation $\mathbb{C}_\lambda$ of $\mathfrak{p} \supset \mathfrak{l}$, which extends to a representation of the universal enveloping algebra $U(\mathfrak{p})$. By definition, the generalized Verma module $V_{\mathfrak{p}, \lambda}$ is the left $U(\mathfrak{g})$-module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_\lambda$. When $\mathfrak{p}$ coincides with the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$, $V_{\mathfrak{b}, \lambda}$ is the ordinary Verma module, i.e. the maximal object in the category of $U(\mathfrak{g})$-modules with the highest weight $\lambda$.

Denote by $\mathfrak{g}_t$ the Lie algebra over $\mathbb{C}[t]$ which coincides with $\mathfrak{g}[t]$ as a $\mathbb{C}[t]$-module and equipped with the Lie bracket $[x, y]_t = t[x, y]$ for all $x, y \in \mathfrak{g}$. The universal enveloping algebra $U(\mathfrak{g}_t)$ is a $G$-equivariant quantization of $\mathfrak{g}^*$. If $\mathfrak{v}$ is a subspace in $\mathfrak{g}$, we put $\mathfrak{v}_t = \mathfrak{v}[t] \subset \mathfrak{g}[t]$. Given a parabolic subalgebra $\mathfrak{p}_t \subset \mathfrak{g}_t$ and its character $\lambda$ (one dimensional representation on $\mathbb{C}[t]$) we denote by $V_{\mathfrak{p}_t, \lambda}$ the $U(\mathfrak{g}_t)$-module $U(\mathfrak{g}_t) \otimes_{U(\mathfrak{p}_t)} \mathbb{C}[t]$ and call it a generalized Verma module over $U(\mathfrak{g}_t)$.

Introduce the subset $\mathfrak{c}_{\text{reg}}^*$ of elements in $\mathfrak{c}^* \subset \mathfrak{h}^*$ whose stabilizer in $\mathfrak{g}$ is $\mathfrak{l} \supset \mathfrak{c}$. We call such elements regular. The set $\mathfrak{c}_{\text{reg}}^*$ is a dense cone in $\mathfrak{c}^*$. Let us fix an element $\lambda \in \mathfrak{c}_{\text{reg}}^*$.

**Theorem 2.1 ([DGS]).** The image $\mathcal{A}_{\mathfrak{p}_t, \lambda}$ of $U(\mathfrak{g}_t)$ in $\text{End}(V_{\mathfrak{p}_t, \lambda})$ is a $G$-equivariant quantization of the KKS bracket on the semisimple orbit $O_\lambda \subset \mathfrak{g}^*$ passing through $\lambda$.\(^1\)

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\(^1\)By flatness of a $\mathbb{C}[t]$-module we mean flatness of its $t$-adic completion over $\mathbb{C}[[t]]$. 

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4
Let \( \mathcal{J}_{p_{\nu},\lambda} \subset \mathcal{U}(\mathfrak{g}_l) \) denote the annihilator of the module \( V_{p_{\nu},\lambda} \). Theorem 2.1 says that the \( G \)-equivariant quantization of a semisimple orbit \( O_{\lambda} \) can be realized as a quotient of the algebra \( \mathcal{U}(\mathfrak{g}_l) \) by the ideal \( \mathcal{J}_{p_{\nu},\lambda} \).

3 Explicit quantization for the \( GL(n) \)-case

3.1. From now on \( \mathfrak{g} \) stands for the Lie algebra \( gl(n) \) of the complex general linear group \( G = GL(n) \). Let us specialize the definitions of the previous subsection to this case. The Cartan subalgebra \( \mathfrak{h} \) is chosen to be the subalgebra of diagonal matrices; the nilpotent subalgebras \( \mathfrak{n}^{\pm} \) consist of respectively upper- and lower-triangular matrices. Using the trace pairing, we identify the dual space \( \mathfrak{g}^* \) with \( \text{End}(\mathbb{C}^n) \) and \( \mathfrak{h}^* \) with the subspace of diagonal matrices in \( \text{End}(\mathbb{C}^n) \); the coadjoint action of the group \( G \) on \( \mathfrak{g}^* \) becomes the similarity transformation under this identification.

Let us define the set \( \{n:k\} \subset \mathbb{Z}^k \) of \( k \)-tuples \( \mathbf{m} = (m_1, \ldots, m_k) \) such that \( 0 \leq m_i \) and \( m_1 + \ldots + m_k = n \); the subset \( \{n:k\}_+ \subset \{n:k\} \) consists of \( \mathbf{m} \) with all \( m_i \) positive. Let \( \mathbb{C}^k_{\text{reg}} \) denote the subspace in \( \mathbb{C}^k \) of \( \mu = (\mu_1, \ldots, \mu_k) \) with pairwise distinct \( \mu_i \).

The Levi subalgebra \( \mathfrak{l} \subset \mathfrak{g} \) consists of block diagonal matrices, \( \mathfrak{l} = \bigoplus_{i=1}^k \mathfrak{g}_i \), where \( k \) is the number of blocks and \( \mathfrak{g}_i = gl(m_i) \). We denote by \( \mathfrak{m}(\mathfrak{l}) \) the \( k \)-tuple \( (m_1, \ldots, m_k) \in \{n:k\}_+ \). Clearly the parabolic subalgebras in \( \mathfrak{p} \) containing upper-triangular matrices are in one-to-one correspondence with elements of \( \bigcup_{k=1}^n \{n:k\}_+ \). Consider the decomposition \( \mathbb{C}^n = \mathbb{C}^{m_1} \oplus \ldots \oplus \mathbb{C}^{m_k} \) and denote by \( p_i : \mathbb{C}^n \to \mathbb{C}^{m_i} \) the corresponding projectors. The center of \( \mathfrak{p} \) is the linear space \( \mathfrak{e} = \bigoplus_{i=1}^k \mathfrak{p}_i \). Under the identification \( \mathfrak{g}^* \cong \mathfrak{g} \), the dual space \( \mathfrak{c}^* \) coincides with \( \mathfrak{e} \), and the linear isomorphism between \( \mathbb{C}^k \) and \( \mathfrak{c}^* \subset \mathfrak{h}^* \) is given by the map \( \mu \mapsto \sum_{i=1}^k \mu_i p_i \). The image of the subspace \( \mathbb{C}^k_{\text{reg}} \subset \mathbb{C}^k \) is the set of regular elements \( \mathfrak{c}^*_{\text{reg}} \subset \mathfrak{c}^* \).

3.2. The algebra \( \mathcal{U}(\mathfrak{g}_l) \) is generated by the set \( \{E^i_j\}_{i,j=1}^n \subset \mathfrak{g}_l \) of elements satisfying the relations

\[
E^i_j E^m_l - E^m_l E^i_j = t(\delta^m_j E^l_i - \delta^l_i E^m_j), \quad i, j, l, m = 1, \ldots, k. \tag{1}
\]

They can be arranged into a matrix, \( E = ||E^i_j||_{i,j=1}^n \). The matrix \( E \) can be considered as an element \( E = \sum_{i,j=1}^n E^i_j \otimes e^i_j \in \mathcal{U}(\mathfrak{g}_l) \otimes \mathbb{C} \text{End}(\mathbb{C}^n) \), where \( \{e^i_j\}_{i,j=1}^n \) is the standard base in \( \text{End}(\mathbb{C}^n) \). It is invariant with respect to the diagonal action of \( \mathcal{U}(\mathfrak{g}_l) \) on \( \mathcal{U}(\mathfrak{g}_l) \otimes \mathbb{C} \text{End}(\mathbb{C}^n) \).

We fix a multiplication in \( \text{End}(\mathbb{C}^n) \) by setting it on the base by the formula \( e^i_j e^l_m = \delta^l_j e^i_m \), where \( \delta^l_j \) is the Kronecker symbol. One can consider polynomials in \( E \) as an element of the
algebra \( \mathcal{U}(\mathfrak{g}_t) \otimes \text{End}(\mathbb{C}^n) \). Explicitly, the \( \ell \)-th power \( E^\ell \) is a matrix with the entries

\[
(E^\ell)^i_j = \sum_{\alpha_1, \ldots, \alpha_{\ell-1} = 1}^n E_j^{\alpha_1} E^{\alpha_2} \cdots E_{\alpha_{\ell-1}}^i.
\]  

(2)

We will also consider the matrix algebra \( \text{End}^o(\mathbb{C}^n) \) whose multiplication is opposite to that of \( \text{End}(\mathbb{C}^n) \). The matrix \( E \) may be thought of as an element from \( \mathcal{U}(\mathfrak{g}_t) \otimes \text{End}^o(\mathbb{C}^n) \), then its \( \ell \)-th power \( E^\circ \ell \) is explicitly

\[
(E^\circ \ell)^i_j = \sum_{\alpha_1, \ldots, \alpha_{\ell-1} = 1}^n E_j^{\alpha_1} E^{\alpha_2} \cdots E_{\alpha_{\ell-1}}^i.
\]  

(3)

In [DM2], a special series \( \{ \vartheta_\ell(\hat{m}, \mu, q^{-2}, t) \}_{\ell = 0}^\infty \) of polynomials in \( \mu \in \mathbb{C}^k \) and \( t, q^{-2} \in \mathbb{C} \) was introduced in connection with a two-parameter quantization on semisimple orbits. The notation \( \hat{m} \) stands for the vector \((\hat{m}_1, \ldots, \hat{m}_k)\), where the hat denotes the q-integers, \( \hat{m} = \frac{1-q^{-2}}{1-q^{-2}} m, m \in \mathbb{N} \). Here we will use the restriction \( \vartheta_\ell(m, \mu, t) = \vartheta_\ell(\hat{m}, q^{-2}, \mu, t)|_{q=1} \). An explicit polynomial expression for \( \vartheta_\ell(m, \mu, t) \) is presented in Appendix, formula (28).

**Definition 3.1.** Let \( m = (m_1, \ldots, m_k) \in \{ n : k \}^+ \) and \( \mu = (\mu_1, \ldots, \mu_k) \in \mathbb{C}^k \).

1. The \( \mathbb{C}[t] \)-algebra \( A_{m,\mu,t} \) is a quotient of \( \mathcal{U}(\mathfrak{g}_t) \) by the ideal specified by the relations

\[
(E - \mu_1) \cdots (E - \mu_k) = 0,
\]  

(4)

\[
\text{Tr } E^\ell = \vartheta_\ell(m, \mu, t), \quad \ell \in 1, \ldots, k - 1.
\]  

(5)

2. The \( \mathbb{C}[t] \)-algebra \( A^\circ_{m,\mu,t} \) is a quotient of \( \mathcal{U}(\mathfrak{g}_t) \) by the ideal specified by the relations

\[
(E - \mu_1)^\circ \cdots (E - \mu_k) = 0,
\]  

(6)

\[
\text{Tr } E^\circ \ell = \vartheta_\ell(m, \mu, -t), \quad \ell \in 1, \ldots, k - 1.
\]  

(7)

The algebras \( A_{m,\mu,t} \) and \( A^\circ_{m,\mu,t} \) are related by a certain transformation of parameters whose exact form will be presented in Subsection 5.2, formula (19). We will use that relation in Subsection 5.4 concerning real forms of quantum orbits.

**Theorem 3.2 ([DM2]).** Given \( \mu \in \mathbb{C}^k \) reg and \( m \in \{ n : k \}^+ \), the algebra \( A_{m,\mu,t} \) (\( A^\circ_{m,\mu,t} \)) is a \( G \)-equivariant quantization of the semisimple orbit of matrices with eigenvalues \( \mu \) of multiplicities \( m \).

This theorem was proven in [DM2] for the algebra \( A_{m,\mu,t} \). For the algebra \( A^\circ_{m,\mu,t} \) the proof is analogous. Besides, the family \( A^\circ_{m,\mu,t} \) can be obtain from \( A_{m,-\mu,t} \) via the automorphism of the \( \mathbb{C} \)-algebra \( \mathcal{U}(\mathfrak{g}_t) \) extending the map \( E_j^i \to -E_j^i, \ t \to -t \).
3.3. Let $S_k$ be the symmetric group of permutations of a $k$-element set. It acts on the family of algebras $A_{m,\mu,t}$ through the action on the $k$-tuples $m$ and $\mu$. Denote by $1$ the element $(1, \ldots, 1) \in \mathbb{C}^k$.

**Proposition 3.3.** The $\mathbb{C}[t]$-algebras $A_{m,\mu,t}$ and $A_{m',\mu',t}$ are isomorphic if and only if there is an element $\tau \in S_k$ and a complex number $b$ such that $m' = \tau(m)$ and $\mu' = \tau(\mu) + bl$. The same holds for the family $A_{m,\mu,t}$ as well.

**Proof.** The proof is straightforward in one direction. Indeed, equations (4) and (5) are symmetric with respect to permutations of the pairs $(m_i, \mu_i)$. The correspondence $E_i^j \rightarrow E_i^j - b\delta_i^j$ extends to an isomorphism $A_{m,\mu,t} \rightarrow A_{m,\mu+b,t}$, as it is seen from relations (1), (4), and (5).

Conversely, suppose $A_{m,\mu,t}$ and $A_{m',\mu',t}$ are isomorphic as $\mathbb{C}[t]$-algebras. They are quantizations of orbits characterized by eigenvalues of constituent matrices and their multiplicities. Those orbits are isomorphic as symplectic manifolds if and only if there is a permutation $\tau$ and a complex number $b$ such that $m' = \tau(m)$ and $\mu' = \tau(\mu) + bl$. 

**Remark 3.4.** Given a $\mathbb{C}[t]$-module $V$ we will often treat it as a family of $\mathbb{C}$-modules. Specialization of $V$ at a point $t = t_0$ is the $\mathbb{C}$-module $V \otimes_{\mathbb{C}[t]} \mathbb{C}$ corresponding to the $\mathbb{C}$-homomorphism $\mathbb{C}[t] \rightarrow \mathbb{C}[t]/(t-t_0) \simeq \mathbb{C}$.

3.4 (Cayley-Hamilton identity). The elements $\text{Tr } E^\ell$, $\ell = 1, \ldots, n$, generate the center $Z(g_t)$ of the algebra $\mathcal{U}(g_t)$. There is another set of generators of $Z(g_t)$, namely, the coefficients, $\{c_i\}_{i=1}^n$, of the "characteristic" polynomial equation

$$P(E) = E^n - c_1E^{n-1} + \ldots + (-1)^n c_n = 0$$

identically held in $\mathcal{U}(g_t) \otimes_{\mathbb{C}} \text{End}(\mathbb{C}^n)$. Equation (8) is a non-commutative analog of the Cayley-Hamilton identity in the classical polynomial algebra on matrices. Its existence immediately follows from representation theory arguments and the fact that $\mathcal{U}(g_t)$ is an equivariant quantization on $g^*$, see [DM2].

Let us define two characters of the center $Z(g_t)$ that will appear in what follows.

1. The assingment $\text{Tr } E^\ell \rightarrow \vartheta_\ell(m,\mu,t)$, $\ell \in \mathbb{N}$, defines a central character, [DM2], which we denote by $\chi_{m,\mu}$.

2. Given a weight $\lambda \in h^*_t$ we denote by $\chi_\lambda$ the central character whose kernel lies in the annihilator of the Verma module $V_{b_t,\lambda}$. It follows that if $\lambda$ is a character of a parabolic
subalgebra $\mathfrak{p}_t \supset \mathfrak{b}_t$, then $\ker \chi_\lambda$ lies in $\mathcal{J}_{\mathfrak{p},\lambda}$, the annihilator of the generalized Verma module $V_{\mathfrak{p},\lambda}$. One has $zv_0 = \chi_\lambda(z)v_0$, where $z \in \mathcal{Z}(\mathfrak{g}_t)$ and $v_0 \in V_{\mathfrak{p},\lambda}$ is the highest weight vector.

Let $\chi$ be a character of the center $\mathcal{Z}(\mathfrak{g}_t)$. Consider its specialization at $t \neq 0$, which is a $\mathbb{C}$-algebra homomorphism $\mathcal{Z}(\mathfrak{g}_t) \to \mathbb{C}$. We denote by $\mathfrak{R}(\chi)$ the set of roots of the polynomial

$$x^n - \chi(c_1)x^{n-1} + \ldots + (-1)^n\chi(c_n)$$

with complex coefficients $\chi(c_i)$.

**Proposition 3.5.** Given $m \in \{n:k\}_+$ and $\mu \in \mathbb{C}^k$ one has

$$\mathfrak{R}(\chi_{m,\mu}) = \left\{ \mu_1, \mu_1 - t, \ldots, \mu_1 - (m_1 - 1)t; \ldots; \mu_k, \mu_k - t, \ldots, \mu_k - (m_k - 1)t \right\}.$$  (10)

**Proof.** This follows from the construction of the functions $\vartheta_\ell(m, \mu, t)$, [DM2].

**Remark 3.6.** Relations (5) specify an ideal in $\mathcal{U}(\mathfrak{g}_t)$ which is generated by the kernel of the character $\chi_{m,\mu}$. Remark that for $k = n$ matrix polynomial (4) is obtained from (8) by the substitution $c_i \to \chi_{m,\mu}(c_i)$, as follows from (10). Thus equation (4) becomes superfluous, being a consequence of (5) and (8). This situation corresponds to an orbit of maximal rank, which is determined solely by a central character, [Kost].

4 Generalized Verma modules and quantum orbits

4.1. Theorem 2.1 describes quantization of the semisimple orbit $O_\lambda$ as a quotient of the algebra $\mathcal{U}(\mathfrak{g}_t)$ by the ideal $\mathcal{J}_{\mathfrak{p},\lambda}$. Assuming $\lambda$ to be a formal function, $\lambda = \lambda(\mu, t)$, such that $\lambda(\mu, 0) = \mu \in \mathfrak{c}^{*}_{\text{reg}}$, we obtain a family, $\mathcal{A}_{\mathfrak{p},\lambda(\mu, t)}$, of non-isomorphic quantizations of $O_\mu$. The vector $\mu$ is an element of $\mathbb{C}^k$ under the identification $\mathfrak{c}^* \sim \mathbb{C}^k$. On the other hand, the algebra $\mathcal{A}_{m,\mu,t}$, where $m = m(0)$, is a quantization of the same orbit $O_\mu$ and it is a quotient of $\mathcal{U}(\mathfrak{g}_t)$, too. The question is what dependence $\lambda(\mu, t)$ ensures an isomorphism $\mathcal{A}_{\mathfrak{p},\lambda(\mu, t)} \simeq \mathcal{A}_{m,\mu,t}$.

**Lemma 4.1 ([DM2]).** Let $\mathfrak{p}$ be a parabolic subalgebra with the center $\mathfrak{c} \simeq \mathbb{C}^k$. For any $\lambda \in \mathfrak{c}^{*}_{\text{reg}}$ there exists a polynomial in one variable,

$$p(x) = x^k - \sigma_1 x^{k-1} + \ldots + (-1)^k \sigma_k$$

with coefficients $\sigma_\ell \in \mathbb{C}[\llbracket t \rrbracket]$ such that the entries of the matrix $p(E) = \|p(E)^j\|$ lie in $\mathcal{J}_{\mathfrak{p},\lambda}$.
For $\lambda \in \mathfrak{c}_{\text{reg}}^*$, the polynomial $p$ has only simple roots. Rephrasing this lemma, the entries of the matrix $p(E) = ||p(E)|^i_j|$ annihilate the Verma module $V_{\mu,\lambda}$. Thus the coefficients $\sigma_i$ of the polynomial $p$ depend on $\lambda$ and $t$. At the same time they are symmetric polynomials of its roots $\{\mu\}$. Our goal is to find the relation between $\mu$, $\lambda$, and $t$. The present section is devoted to a proof of the following theorem.

**Theorem 4.2.** Let $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra with the Levi factor $l$ and the center $c \simeq \mathbb{C}^k$. Put $m = m(l) \in \{n : k\}^+$ and take a regular element $\lambda \in \mathfrak{c}_{\text{reg}}^*$ as a $\mathfrak{p}$-character. Then

1. the algebra $\mathcal{A}_{m,\mu,t}$ is isomorphic to $\mathcal{A}_{\mu,t}$ over $\mathbb{C}[t]$ with
   \[ \mu_i = \lambda_i - \sum_{\alpha=1}^{i-1} m_\alpha t, \quad i = 1, \ldots, k. \] (12)

2. the algebra $\mathcal{A}_{m,\nu,t}^\circ$ is isomorphic to $\mathcal{A}_{\mu,t}$ over $\mathbb{C}[t]$ with
   \[ \nu_i = \lambda_i + \sum_{\alpha=i+1}^k m_\alpha t, \quad i = 1, \ldots, k. \] (13)

The rest of the section is devoted to the proof of Theorem 4.2. We prove only statement 1; statement 2 is verified in the same manner.

4.2. Consider the coefficients $\sigma_\ell$ of $p(x)$ in (11) as the elementary symmetric polynomials in $\mu \in \mathbb{C}^k$, $\sigma_\ell(\mu) = \sum_{1 \leq i_1 < \ldots < i_\ell \leq m} \mu_{i_1} \ldots \mu_{i_\ell}$, $\ell = 1, \ldots, k$. First of all, to prove Theorem 4.2, we must show that the matrix entries $p(E)^i_j$ annihilate the generalized Verma module $V_{\mu,\lambda}$ provided condition (12) holds. Let us check the following elementary lemma.

**Lemma 4.3.** Let $W \subset \mathcal{U}(\mathfrak{g}_t)$ be a submodule with respect to the adjoint representation. The left ideal $\mathcal{J}_W = \mathcal{U}(\mathfrak{g}_t)W$ coincides with the right ideal $\mathcal{W} \mathcal{U}(\mathfrak{g}_t)$, which is therefore a two-sided ideal. Then $\mathcal{J}_W \subset \mathcal{J}_{\mu,\lambda}$ if and only if $W$ annihilates the highest weight vector of $V_{\mu,\lambda}$.

**Proof.** Let $\Delta$ and $\gamma$ denote the comultiplication and antipode of the Hopf algebra $\mathcal{U}(\mathfrak{g}_t)$. In the standard Sweedler notation with implicit summation, $\Delta(x) = x_{(1)} \otimes x_{(2)}$. Let $x \in \mathcal{U}(\mathfrak{g}_t)$ and $w \in \mathcal{W}$. Then $wx = x_{(1)} \gamma(x_{(2)}) w x_{(3)} \in \mathcal{U}(\mathfrak{g}_t)W$, which proves that $\mathcal{U}(\mathfrak{g}_t)W \subset \mathcal{W} \mathcal{U}(\mathfrak{g}_t)$. The reversed inclusion is checked similarly. This proves the first assertion of the lemma.

If $\mathcal{J}_W \subset \mathcal{J}_{\mu,\lambda}$, then $W$ annihilates the highest weight vector $v_0 \in V_{\mu,\lambda}$. Conversely, suppose $Wv_0 = 0$. An element $v \in V_{\mu,\lambda}$ can be represented as $x v_0$, where $x \in \mathcal{U}(\mathfrak{g}_t)$. Hence $Wv = (Wx)v_0 \subset \mathcal{J}_W v_0 = 0$ and therefore $\mathcal{J}_W \subset \mathcal{J}_{\mu,\lambda}$. \[\square\]
We will use Lemma 4.3 in the situation when the module $W$ is spanned by the entries of the polynomial $p(E)$, that is, $W = W_p = \text{Span}(p(E)^i_j)$.

4.3. Consider the map $\sigma : \mathbb{C}^n \to \mathbb{C}^n$, $\lambda_i \mapsto \sigma_i(\lambda)$, where $\sigma_i$ are the elementary symmetric polynomials in $\lambda \in \mathbb{C}^n$. The determinant of this map is a matrix $D(\lambda) = \|D_{ij}(\lambda)\|_{i,j=1}^n$, where $D_{ij}(\lambda) = \frac{\partial}{\partial \lambda_j}(\lambda_i)$.

Lemma 4.4. The determinant $\det D(\lambda)$ is equal to $\prod_{1 \leq i < j \leq n}(\lambda_i - \lambda_j)$.

Proof. Multiply the top row of $D(\lambda)$, which is $\|D_{1i}(\lambda)\| = (1, \ldots, 1)$, by $D(\lambda)_{1i}$ and subtract it from the $i$-th row, $i = 2, \ldots, n$. This kills the entries of the first column except for the upper one. The determinant is preserved and it equals to its minor $\det \|D_{ij}(\lambda)\|_{i,j=2}^n$. It is easy to see, that the $j$-th column of the matrix $\|D_{ij}(\lambda)\|_{i,j=2}^n$ turns zero at $\lambda_i - \lambda_j$ and, when divided by $\lambda_1 - \lambda_j$, it forms the $(j-1)$-th column of the $(n-1) \times (n-1)$ matrix $D(\lambda')$, $\lambda' = (\lambda_2, \ldots, \lambda_n)$. It remains to apply induction on $n$. \hfill $\Box$

4.4. The following lemma is a key step in the proof of Theorem 4.2.

Lemma 4.5. Let $p \in \mathfrak{g}$ be a parabolic subalgebra with the center $c \simeq \mathbb{C}^k$. There is a vector $a(p) \in \mathbb{C}^k$ such that for every $\lambda \in \mathbb{C}^k \simeq c^*$ the ideal $J_{p,\lambda}$ contains the module $W_p$ for the polynomial $p(x) = (x - \mu_1) \ldots (x - \mu_k)$ with roots $\{\mu_i\} = \{\lambda_i - a_i(p)t\}$.

Proof. First of all, let us prove this lemma assuming that $p$ is the Borel subalgebra $b = \mathfrak{b} + \mathfrak{n}^+$, i.e. for $k = n$. Denote by $p$ the polynomial (9) obtained from the characteristic polynomial $\mathcal{P}$ by substitution $c_i \to \chi_\lambda(c_i)$. Its entries annihilate the module $V_{p,\lambda}$ because $p(E)^i_j v_0 = \mathcal{P}(E)^i_j v_0 = 0$. So we have $W_p \subset J_{p,\lambda}$. We will prove the statement for $p = b$ if we show that the roots $\{\mu\}$ of the polynomial $p$ has the form stated in the lemma. It suffices to consider only $\lambda \in \mathbb{C}^n_{\text{reg}}$. The coefficients $\chi_\lambda(c_i)$ are homogeneous polynomials in $(\lambda, t)$ of degree $i$. On the other hand, they are elementary symmetric polynomials $\sigma_i(\mu)$. The map $\mathbb{C}^n \to \mathbb{C}^n$, $\mu \mapsto \sigma(\mu)$, is locally invertible at regular $\mu \in \mathbb{C}^n_{\text{reg}}$. Hence every point in $\mathbb{C}^n_{\text{reg}} \times \{0\}$ has a neighborhood $U \subset \mathbb{C}^n_{\text{reg}} \times \mathbb{C}$ and an analytic function $\psi : U \to \mathbb{C}^n$ such that $\mu = \psi(\lambda, t)$, $(\lambda, t) \in U$, are roots of the polynomial $p$. We want to show that $\psi(\lambda, t) = \lambda - at$ for some $a = a(\mathfrak{b}) \in \mathbb{C}^n$.

Observe that dilatation $E^i_j \to \frac{1}{c} E^i_j$ with $c \neq 0$ extends to a $\mathbb{C}$-algebra isomorphism $\mathcal{A}_{p,\lambda} \to \mathcal{A}_{p,ct,\lambda}$. Similarly, the dilatation $E^i_j \to \frac{1}{c} E^i_j$ extends to an isomorphism of $\mathcal{A}_{a_{\mu},t} \to \mathcal{A}_{a_{\mu},ct}$. This implies $\psi(c\lambda, ct) = c\psi(\lambda, t)$. Consider the $t$-expansion of the function $\psi$:

$$\psi(\lambda, t) = \lambda + \sum_{j>0} t^j \psi^{(j)}(\lambda), \quad (14)$$
where $\psi^{(j)} : \mathbb{C}^n \to \mathbb{C}^n$ are homogeneous functions of degree $-j + 1$. Substituting $\mu = \psi(\lambda, t)$ to $\sigma_i(\mu)$ and comparing coefficients before $t^i$, we find

$$
\sum_{l=1}^{n} D_{il}(\lambda)\psi_{l}^{(j)} + (\text{terms depending on } \psi^{(l)} \text{ with } l < j) \in \mathbb{C}^n[\lambda].
$$

Using induction on $j$ and Lemma 4.4, we see that $\psi_{l}^{(j)}$ are rational functions in $\lambda$ maybe having poles only at $\lambda_i = \lambda_l$, $i \neq l$. But $\psi^{(j)}$ are bounded at $\lambda_i = \lambda_l$ since $\mu = \psi(\lambda, t)$ are roots of the polynomial $p$ with the coefficients $\chi_{i}(c_i)$ being regular functions in $(\lambda, t)$. Therefore $\psi^{(j)} \in \mathbb{C}^n[\lambda]$ and, taking into account their homogeneity degree $-j + 1$, all they are zero except for the term $\psi^{(1)} = -a(b)$. Thus we have proven the statement for $p = b$. As a corollary, we obtained $\mathfrak{R}(\chi_{\lambda}) = \{\lambda - a(b)t\}$.

Let us consider the situation of a general parabolic subalgebra $p \supset b$. Take $\lambda \in c^*_{reg}$ and denote $\tilde{\lambda}$ its image under the canonical embedding in $c^* \subset \mathfrak{h}^*$. By Lemma 4.1, there is a polynomial $p$ of degree $k$ whose entries lie in $J_{pr,\lambda}$. Let $\mu$ be its roots. The central character $\chi_{m,\mu}$ coincides with $\chi_{\tilde{\lambda}}$, therefore $\{\mu\} \subset \{\tilde{\lambda} - a(b)t\}$ by Proposition 3.5. This proves the lemma when $\lambda \in c^*_{reg}$. But the condition $W_p v_0 = 0$ determines the coefficients of $p$ as rational functions of $(\lambda, t)$. We have already proven that they are in fact polynomials, being the elementary symmetric polynomials in the roots. Therefore the polynomial $p$ with the property $W_p \subset J_{pr,\lambda}$ does exist for all $\lambda$. Its roots $\{\mu\}$ are contained in $\{\tilde{\lambda} - a(b)t\}$, so $\mu$ is related to $\lambda$ as stated in the lemma.

**Remark 4.6.** The module $W_p$ is unique for regular $\lambda$ and sufficiently small $t$. Indeed, the condition $W_p v_0 = 0$ gives rise to a system of equations on the polynomial $p$. It goes over to the system $p(\lambda_i) = 0$, $i = 1\ldots k$, at $t = 0$. This uniquely determines $p$ up to a factor at $\lambda \in c^*_{reg}$ and $t = 0$, hence $p$ is unique for $\lambda \in c^*_{reg}$ and small $t \neq 0$.

To finish the proof of Theorem 4.2, it remains to determine the vector $a(p) \in \mathbb{C}^k$.

**4.5 (Proof of Theorem 4.2).** Let $p$ be the parabolic subalgebra with the center $c \simeq \mathbb{C}^k$ and put $m = m(l) \in \{n : k\}$. In this subsection we show that the $j$-th coordinate of the vector $a(p) \in \mathbb{C}^k$ from Lemma 4.5 is equal to $\sum_{i=1}^{j-1} m_i$. This will prove Theorem 4.2. Let us first consider the case $k = 2$. Suppose $\lambda \in c^*_{reg}$. Only two equations are independent in the system $p(E)_i^j v_0 = 0$, $i = 1, \ldots, n$, say for $i = 1$ and $i = m_1 + 1$:

$$
\begin{align*}
\lambda_1^2 - (\mu_1 + \mu_2)\lambda_1 + \mu_1\mu_2 &= 0, \\
\lambda_2^2 - (\mu_1 + \mu_2 + tm_1)\lambda_2 + \mu_1\mu_2 + tm_1\lambda_1 &= 0.
\end{align*}
$$
This system has a solution \((\lambda_1, \lambda_2) = (\mu_1, \mu_2 + tm_1)\) satisfying the condition \(\lambda = \mu\) at \(t = 0\). Thus we have proven Theorem 4.2 for \(k = 2\).

We deduce the case \(k = n\) from the studied case \(k = 2\). Recall that given an element \(\tilde{\lambda} \in \mathfrak{h}^*\) the set \(\mathcal{R}(\chi_{\tilde{\lambda}})\) is equal to \(\{\tilde{\lambda}_i - a_i(b)\}_{i=1}^n\). Putting \(\tilde{\lambda}_i = \lambda_1\) for \(i = 1, \ldots, m_1\) and \(\tilde{\lambda}_i = \lambda_2\) for \(i = m_1 + 1, \ldots, n\), we obtain a central character \(\chi_{\mathfrak{m},\mu}\) associated with the algebra \(\mathcal{A}_{\mathfrak{m},\mu,t}\), for \(\mathfrak{m} \in \{n:2\}^+_+\), and \(\mu \in \mathbb{C}^2\). We have already shown that \(\mu_1 = \lambda_1\) and \(\mu_2 = \lambda_2 - tm_1\). Thus we have, by Proposition 3.5,

\[
\mathcal{R}(\chi_{\mathfrak{m},\mu}) = \{\lambda_1 - (i-1)t\}_{i=1}^{m_1} \cup \{\lambda_2 - (m_1 + i-1)t\}_{i=1}^{m_2}.
\]

On the other hand,

\[
\mathcal{R}(\chi_{\tilde{\lambda}}) = \{\lambda_1 - a_i(b)t\}_{i=1}^{m_1} \cup \{\lambda_2 - a_{m_1+i}(b)t\}_{i=1}^{m_2}.
\]

The sets \(\mathcal{R}(\chi_{\mathfrak{m},\mu})\) and \(\mathcal{R}(\chi_{\tilde{\lambda}})\) coincide since \(\chi_{\mathfrak{m},\mu} = \chi_{\tilde{\lambda}}\). Let us put \(t = 1\) and take \(\lambda_1\) and \(\lambda_2\) to be positive and negative real numbers, respectively. We can assume them big enough in their absolute values so as to ensure coincidence of the subsets \(\{\lambda_1 - (i-1)t\}_{i=1}^{m_1} \subset \mathcal{R}(\chi_{\mathfrak{m},\mu})\) and \(\{\lambda_1 - a_i(b)t\}_{i=1}^{m_1} \subset \mathcal{R}(\chi_{\tilde{\lambda}})\). Using this argument, we can apply induction on \(m_1\) and prove the case \(k = n\) of Theorem 4.2. This gives \(a_i(b) = (i-1), i = 1, \ldots, n\).

It remains to consider the case \(2 < k < n\). Let us set \(t = 1\) and assume that the coordinates of \(\lambda \in \mathfrak{c}_{reg}^* \simeq \mathbb{C}^k\) are real and form a strictly decreasing sequence, \(\lambda_j > \lambda_{j+1}\). The vector \(\lambda \in \mathbb{C}^k\) corresponds to the vector \(\tilde{\lambda} \in \mathfrak{h}^* \simeq \mathbb{C}^n\) via the canonical embedding \(\mathbb{C}^k \simeq \mathfrak{c}^* \rightarrow \mathfrak{h}^* \simeq \mathbb{C}^n\); the coordinates of \(\tilde{\lambda}\) are \((\lambda_1, \ldots, \lambda_1; \ldots, \lambda_k, \ldots, \lambda_k)\), where each \(\lambda_i\) is taken \(m_i\) times. The set \(\mathcal{R}(\chi_{\tilde{\lambda}})\) is a union \(\bigcup_{j=1}^k \mathcal{R}_j\) of non-intersecting subsets

\[
\mathcal{R}_j = \left\{\lambda_j - \left(\sum_{l=1}^{j-1} m_l + i - 1\right)\right\}_{i=1}^{m_j}.
\]  

(15)

We have \(\text{min} \mathcal{R}_j > \text{max} \mathcal{R}_{j+1}\). On the other hand, we know from Proposition 3.5 and Lemma 4.5 that \(\mathcal{R}(\chi_{\tilde{\lambda}})\) is a union the of subsets

\[
\mathcal{R}_j' = \left\{\mu_j - (i-1)\right\}_{i=1}^{m_j} = \left\{\lambda_j - (a_j(p) + i - 1)\right\}_{i=1}^{m_j}.
\]  

(16)

We can chose such \(\lambda_j\) that \(\mathcal{R}_j'\) do not intersect and, moreover, \(\text{min} \mathcal{R}_j' > \text{max} \mathcal{R}_{j+1}'\) (for that to be true, it is enough to assume \(\lambda_j - \lambda_{j+1} > a_j(p) - a_{j+1}(p) + m_j - 1\)). Taking into account \(\# \mathcal{R}_j' = m_j = \# \mathcal{R}_j\) we conclude that \(\mathcal{R}_j' = \mathcal{R}_j\) for all \(j\). Comparing (15) and (16), we find \(a(p)\) and thus prove Theorem 4.2.
Remark 4.7. As follows from Lemma 4.5, the identity map \( g_t \to g_t \) extends to an epimorphism \( A_{m,\mu,t} \to \mathcal{U}(g_t)/J_{p_t,\lambda} \) of \( \mathbb{C} \)-algebras if the pair \( (p_t, \lambda) \) is related to the pair \( (m, \mu) \) as in (12). For regular \( \lambda \) this map is an isomorphism of \( \mathbb{C}[t] \)-algebras. It is known, \([J]\), that for almost all \( \lambda \in \mathfrak{c}^* \) the annihilator of \( V_{p,\lambda} \) is generated by a copy of adjoint representation and the kernel of a central character. Theorem 4.2 gives an explicit description of the annihilator.

4.6 (Representations of quantum orbits). This subsection is devoted to finite dimensional representations of the \( \mathbb{C} \)-algebras \( A_{m,\mu,t} \). Our principal tool is the established correspondence between the family \( A_{m,\mu,t} \) and generalized Verma modules. Note that this correspondence is not one-to-one. The algebra \( A_{m,\mu,t} \) can be realized as the algebra \( A_{p_t,\lambda} \) for some \( p_t \) and \( \lambda \) in different ways. This freedom comes out from permutations \( (m, \mu) \to (\tau(m), \tau(\mu)) \), \( \tau \in S_k \), leaving \( A_{m,\mu,t} \) invariant but changing the generalized Verma modules. The situation is exactly the same as in the case \( p = b \) when the Weyl group transitively acts on the set of the ordinary Verma modules with isomorphic annihilators.

Any representation of \( A_{m,\mu,t} \) is at the same time a representation of \( \mathcal{U}(g_t) \). Therefore to describe representations of \( A_{m,\mu,t} \) it is necessary and sufficient to determine those representation of \( \mathcal{U}(g_t) \) that are factored through the ideal specifying \( A_{m,\mu,t} \). We do it for finite dimensional representations.

Proposition 4.8. The quantum orbit \( A_{m,\mu,t} \) has a finite dimensional representation if and only if there is an element \( \tau \in S_k \) such that \( (\mu'_i - \mu'_{i+1})/t - m'_i \), where \( \mu' = \tau(\mu) \) and \( m' = \tau(m) \), are non-negative integers for \( i = 1, \ldots, k - 1 \). If exists, such a representation is unique and it is factored through a generalized Verma module.

Proof. As was already mentioned in the proof of Lemma 4.5, the correspondence \( E^i_j \to \frac{1}{c} E^i_j \), where \( c \neq 0 \), extends to an isomorphism between the algebras \( A_{m,\mu,t} \) and \( A_{m,\mu',ct} \). Therefore we can assume \( t = 1 \) in the proof.

Suppose there is such element \( \tau \in S_k \) as stated in the proposition. Take the parabolic subalgebra \( p \) with the Levi factor \( l \) such that \( m(l) = m' \). Consider the \( \mathcal{U}(g) \)-module \( V_{p,\lambda} \), where \( \lambda \) is related to \( \mu' \) by formula (12), where one should set \( t = 1 \). By assumption, the numbers \( \lambda_i = \mu'_i + \sum_{\alpha=1}^{i-1} m'_\alpha \) define a dominant integral weight of \( sl(n) \subset g \), so the module \( V_{p,\lambda} \) is projected onto a finite dimensional \( \mathcal{U}(g) \)-module \( W \) with the highest weight \( \lambda \). The homomorphism of modules induces a representation of \( A_{m,\mu,1} \simeq A_{m',\mu',1} \) on \( W \).

Conversely, suppose \( W \) is a finite dimensional module over \( A_{m,\mu,1} \). Then it is a module over \( \mathcal{U}(g) \) as well. There is a cyclic \( \mathcal{U}(g) \)-submodule in \( W \) with the highest weight \( \lambda \). We can think that this submodule coincides with \( W \). The highest weight gives rise to a central
character, $\chi_{\lambda}$. On the other hand, one has $\chi_{\lambda} = \chi_{m,\mu}$, and the set $\mathcal{R}(\chi_{\lambda}) = \{\lambda_j - (j-1)\}_{j=1}^{n}$ contains $\{\mu_i\}_{i=1}^{k}$ as a subset, by Proposition 3.5. Further, $W$ is a finite dimensional $U(\mathfrak{g})$-module with the highest weight $\lambda$, therefore $\lambda_j - \lambda_{j+1}$ are non-negative integers. The elements of $\mathcal{R}(\chi_{\lambda})$ are ordered by their real components; then the numbers $\lambda_j - (j-1)t \in \mathcal{R}(\chi_{\lambda})$, $j = 1, \ldots, n$, form a strictly decreasing sequence. The subset $\{\mu\} \subset \mathcal{R}(\chi_{\lambda})$ is ordered by inclusion, hence there is a permutation $\tau$ such that $\mu'_i > \mu'_{i+1}$, $\mu' = \tau(\mu)$. The permutation $\tau$ satisfies the condition of the theorem. Uniqueness of the module $W$ follows from the ordering on $\{\mu'\}$.

Let us show that the module $W$ is a quotient of a generalized Verma module associated with $\mathcal{A}_{m,\mu,1}$. The elements $\lambda_j - (j-1)$ belonging to the interval $[\mu'_i, \mu'_{i+1}] \subset \mathcal{R}(\chi_{\lambda})$ form an arithmetic progression with the initial term $\mu'_i$ and decrement 1, as follows from (10). Thus $\lambda_j$ is stable within this interval. Therefore $\lambda$ defines a character of the parabolic subalgebra $\mathfrak{p}$ with Levi factor $l$ such that $m(l) = m'$. Let $\rho$ denote the homomorphism $U(\mathfrak{g}) \rightarrow \text{End}(W)$. As $\dim W < \infty$, for all $x \in l$ one has $\rho(x)w_0 = \lambda(x)w_0$, where $w_0 \in W$ is the highest weight vector. Therefore the subspace $\mathbb{C}w_0 \subset W$ forms a one dimensional $U(\mathfrak{p})$-module $\mathbb{C}_\lambda$. The map $U(\mathfrak{g}) \otimes_{\mathbb{C}} \mathbb{C} \rightarrow W$, $x \otimes 1 \mapsto \rho(x)w_0$, is a homomorphism of left $U(\mathfrak{g})$-modules, which is obviously factored through the generalized Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_\lambda$. □

5 Real forms of quantum orbits

As an application of the established relation between the algebras $\mathcal{A}_{m,\mu,t}$ ($\mathcal{A}_{m,\mu,t}^c$) and the generalized Verma modules we construct real forms of quantum orbits.

5.1. An anti-linear endomorphism of a complex vector space $V$ is an additive map $f: V \rightarrow V$ satisfying $f(\beta a) = \bar{\beta} f(a)$ for any $a \in V$ and $\beta \in \mathbb{C}$; the bar stands for the complex conjugation. When $V$ is a $\mathbb{C}[t]$-module, we assume $f(t) = t$.

**Definition 5.1.** A $*$-structure on an associative algebra $A$ over $\mathbb{C}$ or $\mathbb{C}[t]$ is an anti-linear involution $*: A \rightarrow A$ that is an anti-automorphism with respect to the multiplication, $(ab)^* = b^*a^*$. Let $\mathcal{H}$ be a Hopf algebra over $\mathbb{C}$ or $\mathbb{C}[t]$ with the multiplication $m$, comultiplication $\Delta$, and antipode $\gamma$, $[\text{ChPr}]$.

**Definition 5.2.** A real form on $\mathcal{H}$ is an anti-linear involution such that

$$\theta \circ m = m \circ (\theta \otimes \theta), \quad \tau \circ \Delta \circ \theta = (\theta \otimes \theta) \circ \Delta,$$

(17)
where $\tau$ is the flip on $H \otimes H$.

When $H$ is the universal enveloping algebra $U(g)$ of a complex Lie algebra $g$, the involution restricts to $g$. The set of $\theta$-fixed points in $U(g)$ is the universal enveloping algebra $U(g_\mathbb{R})$ over $\mathbb{R}$ of the Lie algebra $g_\mathbb{R}$, a real form of $g$.

**Proposition 5.3.** Let $\theta$ be a real form of a Hopf algebra $H$ with invertible antipode $\gamma$. Then $\gamma \circ \theta \circ \gamma = \theta$.

**Proof.** Consider the map $\theta \circ \gamma \circ \theta : H \to H$. It satisfies the axioms of antipode for the opposite comultiplication and therefore equals to $\gamma^{-1}$, due to uniqueness of the antipode. \qed

This proposition implies that the composition of $\theta$ with any odd power of $\gamma$ makes $H$ an involutive algebra. We are going to define involutions on $H$-module algebras that will be compatible with real forms on $H$ in the sense of the following definition.

**Definition 5.4.** Let $\theta$ be a real form of a Hopf algebra $H$. A real form of an $H$-module algebra $A$ is a $\ast$-structure on $A$ such that

$$(x \triangleright a)^\ast = \theta(x) \triangleright a^\ast, \quad x \in H, \quad a \in A.$$

(18)

**Example 5.5.** Let $H$ be a Hopf algebra equipped with a real form $\theta$. Consider $H$ as a left module algebra over itself with respect to the adjoint action $x \triangleright y = x_{(1)}y\gamma(x_{(2)})$. Define a $\ast$-structure on $H$ by the involution $\gamma \circ \theta$. Then, $\ast$ is a real form of the self-adjoint module algebra $H$. Indeed, one has

$$(x \triangleright y)^\ast = (x_{(1)}y\gamma(x_{(2)}))^\ast = (\gamma \circ \theta \circ \gamma)(x_{(2)})y^\ast(\gamma \circ \theta)(x_{(1)}) = \theta(x_{(2)})y^\ast(\gamma \circ \theta)(x_{(1)}),$$

which is equal to $\theta(x) \triangleright y^\ast$.

5.2. Before proceeding with real forms as applied to $G$-equivariant quantization, we study a relation between the two algebras $A_{m,\mu,t}$ and $A_{m,\nu,t}^\circ$ from Definition 3.1.

**Proposition 5.6.** The identity map $g_t \to g_t$ induces an isomorphism of algebras $A_{m,\mu,t}$ and $A_{m,\nu,t}^\circ$ over $\mathbb{C}$, provided

$$\nu_i = \mu_i + (n - m_i)t, \quad i = 1, \ldots, k.$$  

(19)

**Proof.** Both algebras are quotients of $U(g_t)$ by the ideals generated by certain copies of the adjoint and trivial submodules. Denote them by $W_{\mu}$ and $W_{\nu}^\circ$ correspondingly. It is sufficient
to show that $W_\mu$ and $W^\circ_\nu$ are related by the transformation of parameters (19). With $m$ given, both modules are determined by their adjoint components, generated by the matrix coefficients of polynomials in $E$. Every monomial (2) is expressed as a linear combination of monomials (3) with coefficients being polynomial functions in $t$. A polynomial (4) can be represented as a polynomial in the sense of (3), and the coefficients of the latter will be polynomials in $\mu$ and $t$. Therefore it suffices to prove the proposition only for generic $\mu$.

Take the parabolic subalgebra $p \supset I$ such that $m(t) = m$. Assuming $\mu$ regular and $t$ small, there is a generalized Verma module such that $W_\mu \subset \mathcal{J}_{p,\lambda}$. Further, there is a unique, as emphasized in Remark 4.6, module $W^\circ_\nu$ such that $\mathcal{J}_{p,\lambda} \supset W^\circ_\nu$. Hence $W_\mu = W^\circ_\nu$, and the parameters $\mu, \lambda, \nu$ are related by transformations (12) and (13). This proves the statement for regular $\mu$ and small $t$ and therefore for all $\mu$ and $t$.

5.3. It is known, [Kn], that all real forms of the Lie algebra $gl(n, \mathbb{C})$ are isomorphic to $gl(n, \mathbb{R})$, $u^\ast(2m)$ when $n = 2m$ is even, and $u(r, s)$ with $r + s = n, r \geq s \geq 0$. When $s = 0$, $u(r, s)$ turns into the compact real form $u(n)$. The corresponding involutions are defined on generators $\{E^i_j\}_{i,j=1}^k$ as

$$\theta_{gl(n,\mathbb{R})}(E) = E, \quad \theta_{u(r,s)}(E) = -J_{(r,s)}E^i_jJ_{(r,s)}^{-1}, \quad \theta_{u^\ast(2m)}(E) = J_mE^j_iJ_m^{-1},$$

where $J_{(r,s)} = \sum_{i=1}^r e^i_i - \sum_{i=r+1}^n e^i_i$, $J_m = \sum_{i=1}^m (e^i_i + e^{i+m})$, and the prime stands for the matrix transposition.

5.4. Take $\mu \in \mathbb{C}^k, m \in \{n:k\}$ and denote by $S^{m}_{\mu}$ the stabilizer of $m$ in the symmetric group $S_k$.

**Proposition 5.7.** The map $E \to J_{(r,s)}E^i_jJ_{(r,s)}^{-1}$ extends to a $\theta_{u(r,s)}$-compatible $\ast$-real form of the algebra $A_{\mu,\bar{\mu},t}$, provided $\bar{\mu} = \tau(\mu)$ for some $\tau \in S^{m}_{\mu}$.

**Proof.** Consider $U(\mathfrak{g})$ as a module algebra over itself with respect to the adjoint representation. Define a $\theta_{u(r,s)}$-compatible $\ast$-real form on $U(\mathfrak{g})$ as in Example 5.5. It is naturally extended to a $\theta_{u(r,s)}$-compatible $\ast$-real form on $U(\mathfrak{g})[t]$ considered as a $U(\mathfrak{g})$-module algebra. This real form is restricted to the $U(\mathfrak{g})$-equivariant embedding $U(\mathfrak{g}) \to U(\mathfrak{g})[t]$. On the generators $\{E^i_j\} \subset \mathfrak{g}_t$, it is defined by the map $E \to J_{(r,s)}E^i_jJ_{(r,s)}^{-1}$. Let us prove that it induces an isomorphism $(A_{\mu,\bar{\mu},t})^* \simeq A_{\mu,\bar{\mu},t}$. Setting $J = J_{r,s}$ we have $J^2 = 1$ and $J^* = J$; hence for $m > 0$

$$(E^i_j)^* = \sum_{\alpha_1, \ldots, \alpha_{\ell-1}} E^\alpha_1 E^\alpha_2 \ldots E^\alpha_{\ell-1} \ast \sum_{\alpha_0, \ldots, \alpha_{\ell}} J^\alpha_1 E^\alpha_{\ell-1} \ldots E^\alpha_2 E^\alpha_1 J^\alpha_0.$$
In the concise matrix form this reads \((E^\ell)^* = (J E^\ell J)'\). If \(p(x)\) is a polynomial in one variable and \(\bar{p}(x)\) is obtained from \(p(x)\) by the complex conjugation of its coefficients, then \(p(E)^* \rightarrow (J \bar{p}(E)J)\). Relations (5) go over to \(\text{Tr} E^\ell = \vartheta_\ell(m, \bar{\mu}, t), \ell \in \mathbb{N}\). Therefore \((A_{m,\mu,t})^* \simeq A_m, \bar{\mu}, t = A_{m,\mu,t}**\) if \(\bar{\mu} = \tau(\mu)\) for some \(\tau \in S_k^m\). □

**Proposition 5.8.** Suppose \(-\bar{\mu} = \tau(\mu - tm) + tn\ell \in \mathbb{C}^k\) for some \(\tau \in S_k^m\). Then

1. the map \(E \rightarrow E\) extends to a \(\theta_{gl(n,\mathbb{R})}\)-compatible *-real form of \(A_{m,\mu,t}\)

2. the map \(E \rightarrow -J_m E J_m^{-1}\) extends to a \(\theta_{u*(2m)}\)-compatible *-real form of \(A_{m,\mu,t}\)

**Proof.** Let \(J\) denote either the unit matrix or \(J_m = \sum_{i=1}^{m} (e_{i+m} - e_i)\). We will consider the two cases simultaneously. Define a *-real form on the algebra \(U(g)\) in the same way as in the proof of Proposition 5.7. It is compatible with the corresponding real form of the Hopf algebra \(U(g)\). On the generators \(\{E_j\} \subset g_t\), it is defined by the map \(E \rightarrow -JEJ^{-1}\). We have

\[
(E^\ell)^i_j = \sum_{\alpha_1,\ldots,\alpha_{\ell-1}} E_j^{\alpha_1} E_{\alpha_1}^{\alpha_2} \ldots E_{\alpha_{\ell-1}}^{\alpha_\ell} \rightarrow \sum_{\alpha_0,\ldots,\alpha_\ell} (-1)^\ell J_\ell^{j} E_{\alpha_0}^{\alpha_1} E_{\alpha_1}^{\alpha_2} \ldots E_{\alpha_\ell}^{\alpha_\ell} (J^{-1})^{0}_{j0}.
\]

Thus every matrix monomial \(E^\ell\) is transformed into the matrix \((-1)^\ell J E^{\ell} J^{-1}\), therefore \((A_{m,\mu,t})^* \simeq A_m, \bar{\mu}, t^*,\) with respect to either \(sl(n,\mathbb{R})\)- or \(u*(2m)\)-involutions. Now the rest of the proof follows from (19). □

## 6 Non-commutative Connes index

6.1. The purpose of the section is to give an interpretation of certain rational functions arising in our theory of quantum orbits, [DM2]. We show that those functions give the non-commutative Connes index, [C], of basic projective modules over quantum orbits. We will consider the two-parameter \(U_q(g)\)-equivariant quantization of [DM2] (see also [DM1]) including the \(G\)-equivariant quantization as the limit case \(q \rightarrow 1\). A two-parameter quantum orbit is a quotient of the so called modified reflection equation algebra, [KSkl, IP, DM3], \(L_{q,t}\), which itself is a \(U_q(g)\)-equivariant quantization of the polynomial algebra \(g^*\). The algebra \(L_{q,t}\) is generated by the elements \(\{L^i_j\}_{i,j=1}^n\) subject to certain quadratic-linear relations turning to commutation relations (1), where \(E^i_j = \lim_{q \rightarrow 1} L^i_j\). There is a \(U_q(g)\)-equivariant generalization, \(\text{Tr}_q\), of the trace operation such that \(\text{Tr}_q L^\ell, \ell \in \mathbb{N}\), belong to the center of \(L_{q,t}\). For details the reader is referred to [DM2] and references therein.
The functions \( \vartheta_\ell (m, \mu, t) \) in the right-hand side of (5) are the specialization of certain functions \( \vartheta_\ell (\hat{m}, \mu, q^{-2}, t) \) introduced in [DM2] (see also Appendix). Here, \( \hat{m} = (\hat{m}_1, \ldots, \hat{m}_k) \) and \( \hat{m} = \frac{1-q^{-2m}}{1-q^{-2}} \) for \( m \in \mathbb{N} \). Also, we introduced in [DM2] rational functions \( C_j (\hat{m}, \mu, q^{-2}, t) \) satisfying relation (30).

**Theorem 6.1 ([DM2]).** For any \( m \in \{ n : k \} \) and \( \mu \in \mathbb{C}^k_{\text{reg}} \), the quotient \( A_{\hat{m},\mu,q,t} \) of the algebra \( L_{q,t} \) by the relations

\[
(L - \mu_1) \ldots (L - \mu_k) = 0,
\]

\[
\text{Tr}_q(L^\ell) = \vartheta_\ell (\hat{m}, q^{-2}, \mu, t), \quad \ell = 1, \ldots, k - 1,
\]

is a \( \mathcal{U}_q(g) \)-equivariant quantization on the orbit of semisimple matrices with eigenvalues \( \mu \) of multiplicities \( m \).

6.2. In the classical geometry, a vector bundle over a manifold \( M \) may be given by an idempotent \( \pi \) of the algebra \( A(M) \otimes \mathbb{C} \text{End}(V) \), where \( V \) is a finite dimensional vector space. It forms a projective \( A(M) \)-module \( \pi (A(M) \otimes \mathbb{C} V) \). If \( M \) is a \( G \)-manifold and \( V \) a \( G \)-module, a \( G \)-equivariant bundle corresponds to an invariant idempotent. For every semisimple coadjoint orbit \( O_\mu \) in \( \text{End}(\mathbb{C}^n) \) of rank \( k - 1 \) there are \( k \) projector-valued functions \( \pi_i : O_\mu \rightarrow \text{End}(\mathbb{C}^n) \). At a point \( A \in O_\mu \), they commute with \( A \) and map the linear space \( \mathbb{C}^n \) onto the \( A \)-eigenspaces. These basic vector bundles generate the Grothendieck ring of equivariant vector bundles over \( O_\mu \).

In the two-parameter quantization setting an idempotent \( \pi \) from \( A_{q,t}(M) \otimes \mathbb{C} \text{End}(V) \) defines the projective \( A_{q,t} \)-module \( \pi (A_{q,t}(M) \otimes \mathbb{C} V) \), which we consider as a quantized vector bundle over \( M \). Let us built the idempotents for the quantized basic vector bundles over the orbits.

**Proposition 6.2.** Let \( L = ||L_j||_{i,j=1}^k \) be the matrix of generators of the algebra \( A_{\hat{m},\mu,q,t} \) and suppose \( \mu \in \mathbb{C}^k_{\text{reg}} \). The elements

\[
\pi_j (L) = \prod_{i=1, i \neq j}^k \frac{L - \mu_i}{\mu_j - \mu_i} \in A_{\hat{m},\mu,q,t} \otimes \mathbb{C} \text{End}(\mathbb{C}^n), \quad j = 1, \ldots, k,
\]

are invariant projectors. One has \( \text{Tr}_q \pi_j (L) = C_j (\hat{m}, \mu, q^{-2}, t), \) \( j = 1, \ldots, k \), see (30).

**Proof.** The matrix \( L \in A_{\hat{m},\mu,q,t} \otimes \mathbb{C} \text{End}(\mathbb{C}^n) \) satisfies (20). This implies that the elements (22) are orthogonal idempotents, and \( L = \sum_{j=1}^k \mu_j \pi_j (L) \). Therefore, every positive integer
power of the matrix \( L \) has the decomposition over the basis \( \pi_j(L), \ j = 1, \ldots, k \):

\[
L^\ell = \sum_{j=1}^{k} \mu_j^\ell \pi_j(L), \quad \ell = \mathbb{N}.
\] (23)

Taking trace on both sides of this equation and using condition (21) extended for all \( \ell \in \mathbb{N} \) (see [DM2]) and representation (30), we get

\[
\sum_{j=1}^{k} \mu_j^\ell \text{Tr}_q \pi_j(L) = \sum_{j=1}^{k} \mu_j^\ell \tilde{C}_j(\hat{\mathfrak{m}}, \mu, q^{-2}, t) \text{ for all } \ell = \mathbb{N}.
\]

Since the numbers \( \{ \mu_i \} \) are pairwise distinct, this proves the statement. \( \square \)

The quantities \( \text{Tr}_q \pi_j(L) \) are central elements of \( A_{\hat{\mathfrak{m}},\mu,q,t} \). Therefore we can take the function \( \tilde{C}_j(\hat{\mathfrak{m}}, \mu, q^{-2}, t) \) as the "universal" Connes index of the family of quantum vector bundles \( \pi_j(A_{\hat{\mathfrak{m}},\mu,q,t} \otimes \mathbb{C}^n) \).

Note that the problem of \( U_q(\mathfrak{g}) \)-equivariant quantization of vector bundles on orbits as one- and two-sided projective modules over quantized function algebras was considered in [D]. An interesting problem is to construct the quantum bundles explicitly, in terms of projectors. This problem is solved for non-commutative sphere \( S^2_q \) in [GLS].

**Appendix**

In this subsection, we study the functions \( \vartheta_\ell \) entering the right-hand side of (5). They were introduced in [DM2] as specialization at \( q = 1 \) of certain functions participating in the two-parameter quantization of orbits. Assuming \( \nu, \mu \in \mathbb{C}^k \) and \( \omega \in \mathbb{C} \) consider the functions

\[
\tilde{\vartheta}_\ell(\nu, \mu, \omega) = \sum_{s=1}^{k} \omega^{s-1} \sum_{1 \leq j_1 < \cdots < j_s \leq k} \nu_{j_1} \cdots \nu_{j_s} \sum_{d \in \{\ell,s\}} \mu_{d_1}^j \cdots \mu_{d_s}^j, \quad \ell = 1, 2, \ldots
\] (24)

They satisfy the recurrent relation

\[
\tilde{\vartheta}_\ell(\nu, \mu, \omega) = \tilde{\vartheta}_\ell(\nu', \mu', \omega) + \nu_k \omega \sum_{i=1}^{m-1} \tilde{\vartheta}_{\ell-i}(\nu', \mu', \omega) \mu_k^i + \nu_k \mu_k^m,
\] (25)

where \( \mu' = (\mu_1, \ldots, \mu_{k-1}) \) and \( \nu' = (\nu_1, \ldots, \nu_{k-1}) \). Using this relation, it is easy to prove by induction on \( k \) that

\[
\tilde{\vartheta}_\ell(\nu, \mu, \omega) = \sum_{j=1}^{k} \mu_j^\ell \tilde{C}_j(\nu, \mu, \omega),
\] (26)

where

\[
\tilde{C}_j(\nu, \mu, \omega) = \nu_j + \nu_j \sum_{\ell=1}^{k-1} \omega^\ell \sum_{1 \leq i_1 < \cdots < i_\ell \leq k} \frac{\nu_{i_1} \mu_{i_1}}{\mu_j - \mu_{i_1}} \cdots \frac{\nu_{i_\ell} \mu_{i_\ell}}{\mu_j - \mu_{i_\ell}},
\] (27)
If one puts, by definition, \( \bar{\vartheta}_0(\nu, \mu, \omega) = \nu_\omega \), where \( \nu_\omega \) is determined by the equation \( (1 - \omega \nu_\omega) = \prod_{i=1}^{k}(1 - \omega \nu_i) \), then representation (26) is valid for \( \ell = 0 \) as well.

Suppose \( \nu \) has a polynomial dependence in \( \omega \) and \( \lim_{\omega \to 0} \nu(\omega) = m \). Recall that \( l \in \mathbb{C}^k \) denotes the vector \((1, \ldots, 1)\).

**Proposition 6.3.** The function \( \sum_{j=1}^{k} \mu_j C_j(\nu(\omega), \mu + \frac{t}{\omega}, \omega) \) is a polynomial in all its arguments. Its specialization \( \omega = 0 \) is a polynomial

\[
\vartheta_\ell(m, \mu, t) = \sum_{s=1}^{k} t^{s-1} \sum_{1 \leq j_1 < \ldots < j_s \leq k} m_{j_1} \ldots m_{j_s} \sum_{d \in \{t+1-s:s\}} \mu_{j_{1}}^{d_{1}} \ldots \mu_{j_{s}}^{d_{s}}.
\]

**Proof.** It seen from (27) that the functions \( \bar{C}_j(\nu(\omega), \mu + \frac{t}{\omega}, \omega) \) are regular at \( \omega = 0 \); this proves the first assertion. Consider the specialization at \( \omega = 0 \):

\[
C_j(m, \mu, t) = \bar{C}_j(\nu(\omega), \mu + \frac{t}{\omega}, \omega)|_{\omega=0}.
\]

We assume \( \mu_i \neq 0 \) for all \( i \) and put \( m/\mu = (m_1/\mu_1, \ldots, m_k/\mu_k) \). Now the statement follows from formulas (24) and (26) if one observes that \( C_j(m, \mu, t) = \bar{C}_j(m/\mu, \mu, t) \) from (27).

Specialization at \( \nu = \hat{m}, \omega = 1 - q^{-2} \) yields polynomials \( \vartheta_\ell(\hat{m}, \mu, q^{-2}, t) \) participating in the two-parameter \( \mathcal{U}_q(g) \)-equivariant quantization of orbits, [DM2]:

\[
\vartheta_\ell(m, \mu, q^{-2}, t) = \sum_{j=1}^{k} \mu_j C_j(\hat{m}, \mu, q^{-2}, t),
\]

where \( C_j(\hat{m}, \mu, q^{-2}, t) = \bar{C}_j(\hat{m}, \mu + \frac{t}{\omega}, \omega)|_{\omega=1-q^{-2}} \). The coefficients \( C_j(\hat{m}, \mu, q^{-2}, t) \) were shown above to give the non-commutative Connes index for basic homogeneous vector bundles over the two-parameter quantum orbits.

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