Experimental Measurement of the Persistence Exponent of the Planar Ising Model

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Using a twisted nematic liquid crystal system exhibiting planar Ising model dynamics, we have measured the scaling exponent ϑ which characterizes the time evolution, \( p(t) \sim t^{-\varrho} \), of the probability \( p(t) \) that the local order parameter has not changed sign up to time \( t \) after the quench. For the Ising model, \( p(t) \) is simply the fraction of spins that have not flipped up to time \( t \), or, equivalently, the probability that no interface has ever crossed a given spin. While there have been considerable theoretical and numerical efforts directed towards calculating \( \varrho \), so far there has been no experimental measurement of this exponent in spin-like systems. A kind of persistence exponent was first introduced and measured in a breath figure experiment [1], and recently for soap bubbles [2] (in this latter example \( \varrho \) takes a rather trivial value, which can be inferred on simple physical grounds [3]). In this Letter, we present the first experimental measurement of the nontrivial persistence exponent for an Ising-like system, that we find in good agreement with a recent theoretical calculation.

This power-law decay of persistence is quite ubiquitous and not just restricted to the phase ordering dynamics of the Ising or Potts models. For instance, a similar question arises in the study of Gaussian processes. Considering such a process \( m(t) \), the calculation of the probability that this Gaussian walker never crosses the origin (or changes sign) is a particularly difficult problem when this process is not Markovian [4]. For example, even for the simple diffusion equation, \( \partial_t \phi = \nabla^2 \phi \), the probability that the Gaussian variable \( \phi(x,t) \) does not change sign up to time \( t \) decays algebraically with \( t \) with a non-trivial exponent \( \varrho \) [1]. This exponent \( \varrho \) also appears in other contexts such as reaction-diffusion systems [3–5] and driven diffusive systems [6]. For a quench to the critical point of spin systems, the persistence exponent \( \varrho \) associated with the total magnetization (another non-Markovian Gaussian variable) has recently been argued to be a new non-equilibrium critical exponent [7].

For the purpose of the experimental results presented below, we will restrict ourselves to the phase ordering dynamics of the Ising model at \( T = 0 \). The Ising model at \( T = 0 \) has two ordered states: either all spins are up or they are all down. Following a quench from the high-temperature initial state where all spins are randomly up or down, the system tries to order locally. However, since the symmetry between the two ordered phases is not broken by the quench process, the two ground states compete with each other. As a result, domains of both phases form, grow with time, and the system exhibits coarsening. At late stages of coarsening, the morphology of the growing domains seems to exhibit a self-similar structure in time once all length scales are rescaled by \( L(t) \), where \( L(t) \) represents the typical linear dimension of a growing domain. The scaling hypothesis also predicts that the equal-time two-point correlation function has a simple scaling form, \( \langle \phi(0,t)\phi(\mathbf{r},t) \rangle \sim g[\mathbf{r}/L(t)] \), where the calculation of \( g(x) \) has been the subject of intense theoretical efforts [8]. Also, the length scale \( L(t) \) grows algebraically with time, \( L(t) \sim t^{1/2} \). This growth law can be simply understood by noting that the motion of the domain walls between the opposite phases is purely curvature driven [8].

The equal-time correlation function, however, only gives information about the spatial structure of the system at a given time. To obtain information about the temporal evolution, one useful measure is the unequal-time autocorrelation function, \( C(t,t') = \langle \phi(\mathbf{r},t')\phi(\mathbf{r},t) \rangle \). For \( t > t' \), this autocorrelation decays as \( C(t,t') \sim [L(t)/L(t')]^{-\lambda} \), where \( \lambda \) was argued to be a new universal non-equilibrium exponent [9].

The experimental measurement of \( \lambda \) was carried out for the first time by Mason et al. [10], using a twisted nematic liquid crystal sample bounded by two glass plates (as will be explained in detail below). The experimental value of \( \lambda \) in \( d = 2 \) was in very good agreement with their numerical simulations. The experimental measurement of \( \lambda \) in \( d = 3 \) was in good agreement with the theoretical prediction [11].

The important question, however, is whether this autocorrelation is sufficient to characterize the full temporal evolution of the system. The answer is presumably no, since the stochastic evolution process of a given spin is...
not Gaussian and not Markovian. What is the minimal set of quantities needed to specify the full distribution of this complex spatial and temporal structure? It is extremely difficult to answer this general question as the evolution of the order parameter field in general satisfies a nonlinear partial differential equation. In the absence of such information, one therefore looks for simple, easily measurable quantities that still give important information about the history of the evolution process. One such simple and natural quantity is the “persistence”, i.e., the probability \( p(t) \sim t^{-\theta} \sim [L(t)]^{-2\theta} \) that a given spin does not flip up to time \( t \).

This fraction of unflipped spins \( p(t) \), though relatively simple to measure in numerical simulations, proves extremely hard to compute theoretically. The temporal evolution of an individual spin is a non-Gaussian, non-Markovian process, and naturally, any history dependent nonlocal quantity is very hard to compute for such processes. Indeed, this persistence probability is not simply a two-point correlation function of the initial and final times, but involves all intermediate times as the process is, in general, non-Markovian. Nevertheless, for the \( d = 1 \) Ising model with Glauber dynamics at \( T = 0 \), this nontrivial exponent \( \theta \) was recently computed exactly by Derrida et al. \(^1\) who found \( \theta = 3/8 \). Unfortunately, their method cannot be extended to higher experimentally relevant dimensions. For the \( d = 2 \) spin flip dynamics of the Ising model, only numerical estimates of the exponent \( \theta \approx 0.22 \) \(^2\)\(^3\)\(^4\)\(^5\) have been available to date.

An approximate analytical method has recently been developed \(^6\) to compute \( \theta \) in any dimension for the deterministic nonlinear Landau-Ginzburg equation, which describes the \( T = 0 \) dynamics of an isotropic spin system in the continuum. It involves the following few steps. First, one assumes that the late-time dynamics of the discrete Ising spins are correctly described by the continuum noiseless model-A equation for the nonconserved order parameter field \( \phi(\mathbf{r}, t) \). In fact, for the experimental system discussed below, this continuum equation of motion for the field is certainly more appropriate \(^6\). Next, one makes a nonlinear transformation \( \phi(\mathbf{r}, t) = \sigma(m(t)) \), where \( m(\mathbf{r}, t) \) is an auxiliary field locally representing the distance to the nearest domain wall \(^2\)\(^6\) and \( \sigma \) is the equilibrium profile of a domain wall. \( \sigma \) has a sigmoid shape, which, in practice can be treated as the sign function. One then assumes that \( m(\mathbf{r}, t) \) is a true Gaussian field (an approximation becoming exact in the large \( d \) limit \(^2\)\(^6\)) which allows the calculation of the two-point correlators of both \( m \) and \( \phi \) self-consistently. This method was originally introduced by Mazenko \(^2\)\(^6\), and proves very successful in calculating the exponent \( \lambda \) or the equal-time correlator \( g(x) \). For example, in \( d = 2 \), this approximation yields \( \lambda \approx 1.289 \) to be compared with \( 1.25 \pm 0.01 \) obtained in simulations \(^4\). Once the above approximation has been made, the probability of not flipping a spin is the same as the probability that the Gaussian process \( m \), at a given point in space, does not change sign up to time \( t \), as the equilibrium interface profile is an odd function \( (\phi = \sigma(m) \sim \text{sign}(m)) \). Still, the computation of the probability of no zero crossing for the Gaussian process remains extremely hard due to its non-Markovian nature.

However, the problem of computing \( \theta \) can be mapped onto a quantum mechanics problem \(^4\) of sorts. \( \theta \) is then found to be the difference between two ground-state energies: that of a quantum-like problem with a hard wall at the origin (this constraint results from the condition that the Gaussian walker has to remain on, say, the positive axis), and the other for the same system but without the wall. In these terms, the quantum problem associated with a Markovian process is simply an harmonic oscillator with frequency \( \lambda/2 \), where \( \lambda \) is again the autocorrelation exponent. One can then develop standard variational and perturbative methods to compute the exponent \( \theta \) for a process close enough to a Markovian process. In \( d = 1 \), our approximate theory yields \( \theta \approx 0.35 \) \(^4\) as compared with the exact value \( \theta = 3/8 \) \(^1\). For \( d = 2 \), we found \( \theta \approx 0.19 \) \(^4\). This last result is also consistent with the estimate \( \theta \approx 0.186 \) obtained for \( d = 2 \) in the asymptotically exact \( d \to \infty \) limit of Mazenko theory using the independent intervals approximation for the zeros of the process \( m(t) \) \(^7\), and in recent direct simulations of the time-dependent Ginzburg-Landau (TDGL) equation (model-A) \(^8\). Note that if the process \( m(t) \) were Markovian, one could show \(^4\) that the simple equality \( \theta = \lambda/2 \approx 0.63 \) holds. The discrepancy with the actual value is an indication of strong memory effects in this system.

For comparison with theory, liquid crystals afford the possibility of designing experiments for which \( \theta \) can be easily measured. Indeed, a number of liquid crystal systems, possessing a variety of dimensionalities and types of order parameters, have been used to measure various scaling exponents characterizing the coarsening process. Such measurements have included the increase in the characteristic scaling length, \( L(t) \), in three-dimensional \(^9\)\(^10\)\(^11\) and two-dimensional \(^28\)\(^29\) systems. The twisted nematic system, exhibiting Ising-like behavior and employed here for the measurement of \( \theta \), has been previously used by Orihara and co-workers \(^28\)\(^29\) to study the behavior of the scaling length \( L(t) \) and, in addition, by Mason et al. \(^28\) to measure \( \lambda \). These studies used a liquid crystal sample placed between two glass plates whose surfaces had been treated so that, after a thermal quench from the isotropic phase to the nematic phase, the liquid crystal organized itself into domains in which the director was forced to twist clockwise or counterclockwise by \( \pi/2 \) in going from the surface of one plate to the other. The boundary between two domains of opposite twist consists of a twist disclination line. Regions of opposite twist correspond to Ising model domains in which the spins all point up or all point down. The sys-
tem relaxes viscously, driven by the line tension of the disclination lines.

The experimental apparatus used for the $\lambda$ measurements has been described in detail by us previously [21] and the data collected for this report was extracted from the video tapes obtained in these earlier Ising system studies. Briefly, the sample cell consists of a layer of liquid crystal, 20 $\mu$m thick, sandwiched between two parallel glass microscope slides. The liquid crystal used was trans-(trans)-4-methoxy-4'-n-pentyl-1,1'-bicyclohexyl (Merck, CCH-501, or equivalently ZLI-3005), with an isotropic-to-nematic phase transition of 37 $^\circ$C. The molecular orientation at each slide’s surface was forced to be in the plane of the surface and oriented unidirectionally. The surfaces of the glass plates were prepared by first dipping them into a 0.1% by weight solution of polyvinyl alcohol and then buffing the surface in one direction with a soft cloth. The two slides were mounted orthogonally to each other, thus defining a square region of liquid crystal, about 2.5 cm on a side. The orthogonal orientation of the plates with respect to one another forces the nematic liquid crystal to phase separate into two domains, as the molecular orientation undergoes either a clockwise or counterclockwise twist of $\pi/2$ radians as one goes from one plate to the other. The circular dichroic effect was used to make the domains of opposite twist visible. A square region, 1.5 mm on a side, was observed using a Nikon E Plan x4 objective and recording the images with a high-speed color recording system (NAC, HSV-400). Images were recorded every 5 ms, each frame labeled at the top with the run number and time since the recording began.

For a well percolated system undergoing self-similar coarsening, the ratio of the area occupied by domains of one particular twist to the total area as a function of time should remain constant and have the value 0.5. This ratio, averaged over the 15 runs discussed here, was 0.375 at the time of the quench, reaching within only 0.2 sec the value of 0.576 $\pm$ 0.016, at which it remained for the 200 second time interval over which data was recorded. There are several possible reasons for the deviation of this ratio from the value of 0.5. The region between domains over which the director field is distorted from the pure clockwise or counterclockwise twist is comparable to the thickness of the cell, 20 microns. This makes the optical image of the boundary between domains somewhat ill-defined. Hence, optical effects could cause us to preferentially choose one domain over the other when deciding whether a given point belongs to a region of clockwise or counterclockwise twist. This would be particularly serious at early times and probably accounts for the low value (0.375) of the ratio measured at early times. The ratio at late times is 15% higher than the expected value of 0.5 and may be due to a small bias in the domain nucleation, domain growth favoring domains of one particular twist, and, at very late times, small sample statistics. For example, if alignment of the upper and lower plate deviate from perfect orthogonality one twist direction will be favored over the other [22].

![FIG. 1. Probability $p(t)$ versus time that an individual point in the system has not switched its state at least once by time $t$. The solid line is a least-squares fit to the data and has a slope of 0.19.](image)

We measured, for each of 40 points (in an ordered, rectangular array) in each of 15 runs [23], the time $t$ after the quench when the molecular orientation first switched from one twist direction to the other. The contrast was sufficient to enable us to distinguish the orientation at a particular point immediately after the phase transition. Fig. 1 is a log-log plot showing the scaling of the probability $p(t)$ an individual point has not switched its orientation by time $t$. A least-squares fit to the data in the range 0.4 to 200 seconds gives a slope, $\theta = 0.190 \pm 0.031$, shown by the solid line. The initial approach to scaling is most likely due to the initial nonzero size of the scale length $L$ at the time of the quench and the identification bias resulting from the diffuse nature of the optical image of the domain boundaries. This initial scale length is on the order of the thickness of the cell (20 $\mu$m), resulting in a minimum domain size at the time of the quench. Also noticeable at late times is a “tail”, of reduced slope, in $p(t)$. The tail at late times could be due to several different effects: pinning of domain walls in the sample immediately outside of the region observed or nonorthogonal orientation of the glass plates (giving rise to a preferred molecular twist). The wiggles in the data are finite sample size effects reflecting the history of how macroscopic regions changed their orientation in particular runs.

In conclusion, our measured value of the persistence exponent $\theta = 0.190 \pm 0.031$ is in good agreement with the theoretical estimate $\theta \approx 0.19$ for an isotropic bidimensional spin system.

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Forty points were used in all except one run where an imperfection in one corner pinned the domain wall. In this case, the 12 points were eliminated that were in the immediate vicinity of the pinning site.