FINITE TRANSLATION SURFACES WITH MAXIMAL NUMBER OF TRANSLATIONS

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Abstract. The natural automorphism group of a translation surface is its group of translations. For finite translation surfaces of genus \( g \geq 2 \) the order of this group is naturally bounded in terms of \( g \) due to a Riemann-Hurwitz formula argument. In analogy with classical Hurwitz surfaces, we call surfaces which achieve the maximal bound Hurwitz translation surfaces. We study for which \( g \) there exist Hurwitz translation surfaces of genus \( g \).

Introduction

A finite translation surface is a closed Riemann surface together with a translation atlas \( \mu \), i.e. an atlas all of whose transition maps are translations, which is defined on \( X \) up to finitely many cone points, see Section 1 for a more detailed description and references. They very naturally show up when studying Teichmüller spaces, since every finite translation surface comes from a pair \((X, \omega)\), where \( X \) is a Riemann surface and \( \omega \) is a non zero holomorphic differential. The quadratic differential \( \omega^2 \) defines a point in the cotangent space of Teichmüller space which determines a Teichmüller disk, see e.g. [EG97]. In his seminal work in [Vee89] Veech studied translation surfaces in the context of billiards. He gave a beautiful relation between the geodesic flow on the translation surface \((X, \omega)\) and a subgroup \( \text{SL}(X, \omega) \) of \( \text{SL}_2(\mathbb{R}) \) today called Veech group that can be associated to \((X, \omega)\). A particular nice class of translation surfaces are square-tiled surfaces also called origamis which are obtained by gluing a finite number of squares along their edges via translations such that the resulting space is a connected surface. In general, translation surfaces are distinguished by the types of their singularities. The cone angle of a singularity is always a multiple of \( 2\pi \). The space of all translation surfaces which have \( n \) singularities of cone angles \((k_1 + 1) \cdot 2\pi, \ldots, (k_n + 1) \cdot 2\pi\) is called \( H(k_1, \ldots, k_n) \).

The natural automorphism group of a translation surface is the group of its translations. In this article we study the following question. Fix a natural number \( g \). What is the maximal number of translations a translation surface in genus \( g \) can have? This question can be seen as analogon for translation surfaces to a problem that Hurwitz studied in the 1890’s, namely how many automorphisms a Riemann surface of genus \( g \geq 2 \) can have. He obtained his famous upper bound of \( 84(g - 1) \) [Hur93, Abschnitt II.7.p.424]. This result from 1893 is today known as Hurwitz automorphism theorem. Since then there has been a vivid study to describe the Riemann surfaces which achieve this upper bound, see e.g. [Con90] for an overview. They are called Hurwitz surfaces and do not occur in all genera, see [Acc68] and [Mac69]. More precisely it is shown in [Lar01] that the genera in which Hurwitz
surfaces exist are as rare as perfect cubes; whereas one can deduce from [SPW06] that for the good genera there have to be a lot of Hurwitz surfaces.

Similarly as for Hurwitz surfaces there is a bound on the order of the translation groups of translation surfaces of genus $g$, namely $c(g) = 4g - 4$. It is achieved only for special $g$’s. In analogy to Hurwitz’s theory we call translation surfaces with the maximal possible number $4g - 4$ of translations Hurwitz translation surfaces. It turns out that Hurwitz translation surfaces belong to the special class of translation surfaces formed by the origamis. They are even normal origamis (see Section 1 for the definition). Furthermore they always belong to the principal stratum $H(1, \ldots , 1)$. More precisely, we obtain the characterisation of Hurwitz translation surfaces stated in Theorem 1

**Theorem 1** (proven in Section 2). Let $g \geq 2$.

i) A finite translation surface $(X, \mu)$ of genus $g$ has at most $4g - 4$ translations. It has precisely $4g - 4$ translations if and only if $(X, \mu)$ is a normal origami in the stratum $H(1, \ldots , 1)$.

ii) A finite group $G$ is the automorphism group of a Hurwitz translation surface if and only if it can be generated by two elements $a$ and $b$ such that their commutator $[a, b]$ has order 2.

We then study in which genus there exist Hurwitz translation surfaces and obtain the answer to this question in Theorem 2.

**Theorem 2** (proven in Section 3). There exists a Hurwitz translation surface of genus $g$ if and only if $g$ is odd or $g - 1$ is divisible by 3.

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1. Basics

In this section we give the basic definitions on translation surfaces and origamis which we will use. More comprehensive introductions can be found e.g. in [HS07], [HS06], [Sch05] and [Zor06]. A translations surface $(X^*, \mu)$ is a surface together with an atlas $\mu$ all of whose transition maps are translations. In this article we restrict ourselves to finite translation surfaces $(X, \mu)$, i.e. $X$ is a closed surface, $\mu$ is a translation atlas on $X^* = X \setminus \{P_1, \ldots , P_n\}$ where $P_1, \ldots , P_n$ are finitely many points of $X$, and all of the $P_i$’s are finite cone angle singularities of some cone angle $k_i \cdot 2\pi$. Such translation surfaces can always be obtained by the following handy construction: Take finitely many polygons in the Euclidean plane $\mathbb{R}^2$ such that their edges come in pairs which are parallel and have the same length. Identify for each pair these edges via a translation such that you obtain a connected oriented surface $X$. By construction $X$ carries the structure of a finite translation surface possibly having singularities in the points which come from the vertices of the polygons.
Figure 1. Translation surface obtained by gluing parallel edges of two regular pentagons

Figure 1 shows the translation surfaces built from two regular pentagons. It belongs to the famous double regular \( n \)-gon series which was already studied by Veech in [Vee89]. In the case of the double pentagon all vertices of the two polygons glue to a single point of the surface. Thus the Euler characteristic is \( \chi = 1 - 5 + 2 = -2 \) and the genus is 2. The cone angle around the one singular point obtained from the 10 vertices is \( 6\pi \), which is the sum of all interior angles of the two pentagons.

The set of all finite translation surfaces of genus \( g \) forms a space called \( \Omega M_g \), see e.g. [KZ03]. This space is stratified by the data of the angles of the singularities. More precisely the stratum \( \Omega M_g(k_1, \ldots, k_n) \) consists of the finite translation surfaces of genus \( g \) with \( n \) singularities of angle \( (k_1 + 1) \cdot 2\pi, \ldots, (k_n + 1) \cdot 2\pi \). An easy Euler characteristic calculation shows that \( k_1 + \ldots + k_n = 2g - 2 \). It follows in particular that the genus of a translation surface is at least 1.

A very special class of translation surfaces are origamis (also called \textit{square-tiled surfaces}). There are several equivalent definitions for origamis, see e.g. [Sch06, Section 1]. We will use the following ones.

**Definition 1.** An \textit{origami} \( O \) is equivalently given by one of the three following objects:

1. A translation surface \( (X, \mu) \) obtained by taking finitely many copies of the Euclidean unit squares and gluing their edges via translations, up to equivalence. Two such surfaces \( (X_1, \mu_1) \) and \( (X_2, \mu_2) \) are equivalent, if there exists a homeomorphism \( f : X_1 \to X_2 \) which is a translation with respect to the charts in \( \mu_1 \) and \( \mu_2 \).
2. A finite degree cover \( p : X \to E \) of the torus \( E \) which is ramified at most over one point, up to equivalence. Two such covers \( p_1 : X_1 \to E_1 \) and \( p_2 : X_2 \to E_2 \) are equivalent, if there exists a homeomorphism \( h : X_1 \to X_2 \) with \( p_2 \circ h = p_1 \).
3. A pair of permutations \( (\sigma_a, \sigma_b) \) in the symmetric group \( S_d \) with some \( d \in \mathbb{N} \) up to equivalence by simultaneous conjugation.

The equivalence of these definitions is e.g. described in [Sch06, Section 1]. We here just explain them for the example shown in Figure 2. The number \( d \) in (3) of Definition 1 is the number of the squares in (1) and the degree of the covering in (2).
Figure 2. A normal origami of genus 3 called the Eierlegende Wollmilchsau

Figure 2 shows an origami using the first version in Definition 1. A right edge labelled by the number $a$ is glued to the left edge of the square labelled by $a$ and similarly for the upper and the lower edges. Using the Euler characteristic formula one easily calculates that it is of genus 3. You directly see the corresponding covering $p$ from this picture: Present the torus $E$ as the surface obtained by gluing opposite edge of the Euclidean unit square. Then the cover $p$ maps each square in Figure 2 to the one square forming the torus. This gives a well-defined cover of the corresponding surface to $E$ which is ramified only over the one point obtained from the four vertices of the unit square. Also the pair of permutations in the third description of Definition 1 can be easily read off from the picture. $\sigma_a$ is the permutation which maps the number of a square to the number of its right neighbour; the permutation $\sigma_b$ maps it to the number of its upper neighbour. Thus we obtain

\[ \sigma_a = (1,2,3,4)(5,6,7,8), \quad \sigma_b = (1,8,3,6)(2,7,4,5). \]

This origami is called Eierlegende Wollmilchsau (see [HS08]) since it has several nice properties and has served as counter examples in several occasions, see e.g. [For06], [MY10] and [Möll11].

A special class of origamis are normal origamis. They are by definition origamis $p : X \to E$ for which the covering $p$ is normal, i.e. it is the quotient by a group $G$. In this case the group $G$ is finite, its order is the degree $d$ of the covering and it is the group $\text{Gal}(X/E)$ of deck transformations of the covering $p$, i.e. homeomorphisms $h$ with $p \circ h = p$. In particular the group $G$ acts transitively on the fibre of a point of the torus $E$. We may use this to label the squares by the elements in $G$: Label one square $Sq$ by id and label the square $g \cdot Sq$ by $g$. In particular, if the right neighbour of Square id is labelled by $a$ and the left neighbour by $b$, then the dual graph of the origami is the Cayley graph of $G$ with respect to the generators $a$ and $b$. Here the dual graph of the origami is the following graph. The vertices of the graph correspond to the squares of the origami and the edges of the graph are labelled by $a$ and $b$: two vertices in the graph are connected by an $a$-edge, if the corresponding squares intersect in a vertical edge, and similarly for the $b$-edges of the graph and the horizontal edges of the origami.
Example 2. Observe that the origami shown in Figure 2 is normal. More precisely the group $G$ of deck transformations is the quaternion group

$Q = \{ \pm i, \pm j, \pm k, \pm 1 \} = \langle i, j, k \rangle = \langle i, j, -1 \rangle = \langle i, -j, k \rangle = \langle j, i, -k \rangle = \langle k, -i, j \rangle = \langle -1, i, j, k \rangle$.

![Figure 3. Squares of the translation surface from Figure 2 can be labelled by the elements in the quaternion group $Q$.](image)

We finish this section by giving a formal definition of the main actor of this article.

**Definition 3.** Let $(X, \mu)$ be a finite translation surface. The group of translations $\text{Trans}(X, \mu)$ is the group of the homeomorphisms which are translations with respect to $\mu$, i.e. with respect to the charts of $\mu$ they have the form $(x, y) \mapsto (x, y) + (c_1, c_2)$ for some constant vector $(c_1, c_2)$ depending on the charts.

### 2. Hurwitz translation surfaces

In this section we show that a translation surface of genus $g$ can have at most $4g - 4$ automorphisms. We call a translation surface **Hurwitz translation surface**, if it achieves this upper bound. We prove the criterion in Theorem 1 for translation surfaces to be Hurwitz translation surface and give first examples for Hurwitz translation surfaces.

**Lemma 4.** Let $X$ be a precompact translation surface of genus $g \geq 2$. $X$ has at most $c(g) = 4g - 4$ translations. If $X$ has $4g - 4$ translations, then it lies in the stratum $H(1, \ldots, 1)$ and is an origami.

**Proof.** Let $X$ be a precompact translation surface of genus $g \geq 2$, $G = \text{Trans}(X)$ its group of translations and $d$ the order of $G$. Consider the ramified covering $p : X \rightarrow X/G$. $X/G$ is again a precompact translation surface, hence in particular its genus $g_{X/G}$ is greater or equal to 1. Let $P_1, \ldots, P_k$ on $X/G$ be the ramification points of $p$. Since $p$ is normal, all preimages of $P_i$ have the same ramification index $e_i$. Let $s_i$ be the number of preimages of $P_i$, thus we have $d = s_i e_i$ and $s_i \leq \frac{d}{e_i}$. By Riemann-Hurwitz we have for the genus $g$ and $g_{X/G}$ of $X$ and $X/G$, respectively:

\[
2g - 2 = d(2g_{X/G} - 2) + \sum_{i=1}^k s_i(e_i - 1) = d(2g_{X/G} - 2) + kd - \sum_{i=1}^k s_i e_i
\]

\[
\geq d(2g_{X/G} - 2) + kd - k^2 \frac{d}{2} = d(2g_{X/G} - 2 + k^2)\]

Observe that since $g_{X/G} \geq 1$, we have that $2g_{X/G} - 2 + \frac{k^2}{2} = 0$ if and only if $g_{X/G} = 1$ and $k = 0$. In this case we obtain from (2) that $g = 1$ which contradicts...
our assumption. Hence we have
\[ d(2g_{X/G} - 2 + \frac{k}{2}) \leq 2g - 2 \]
with \( 2g_{X/G} - 2 + \frac{k}{2} \geq \frac{1}{2} \) and thus \( d \leq 4g - 4 \). Equality holds if and only if \( 2g_{X/G} - 2 + \frac{k}{2} = \frac{1}{2} \) and thus \( s_i = \frac{d}{2} \) for all \( i \), i.e. if \( g_{X/G} = 1 \), \( k = 1 \) and \( s_1 = \frac{d}{2} \).

Thus \( p \) is a covering of the torus ramified over one point and the ramification index of the preimages are all \( e_i = 2 \). In particular \( X \) is an origami in the stratum \( H(1, \ldots, 1) \).

**Definition 5.** We say that a translation surface \( X \) is a Hurwitz translation surface (Hts), if \( X \) has \( 4g - 4 \) translations.

**Corollary 6.** It directly follows from the proof of Lemma 4 that a translation surface is a Hts if and only if it is a normal origami in the stratum \( H(1, \ldots, 1) \).

**Example 7.** The following origamis are Hurwitz translation surfaces:

1. The *Eierlegende Wollmilchsau* from Example 2 is a Hurwitz translation surface.

2. The *Escalator with 8 squares* (see Figure 4) defined by the permutations:
   \[ \sigma_a = (1, 2)(3, 4)(5, 6)(7, 8), \quad \sigma_b = (2, 3)(4, 5)(6, 7)(8, 1) \]

   The automorphism group is the dihedral group
   \[ D_4 = \langle \tau_1, \tau_2; \tau_1^2, \tau_2^2, (\tau_1 \tau_2)^4 \rangle \]
   of 8 elements. Going to the right corresponds to multiplication by \( \tau_1 \), going up corresponds to multiplication by \( \tau_2 \). The origami has four singularities of total angle \( 4\pi \) and is thus of genus 3.

   \[ \text{Figure 4. The Escalator: a normal origami of genus 3 with 8 translations} \]

3. The origami given by the following two permutations, see Figure 5:
   \[ \sigma_a = (1, 5, 7)(2, 4, 8)(11, 12, 10)(3, 6, 9), \]
   \[ \sigma_b = (1, 4)(2, 6)(3, 5)(7, 11)(8, 10)(9, 12). \]

   The origami is of genus 4. Its automorphism group is the alternating group \( A_4 \). The squares correspond to the elements in \( A_4 \) as follows:
   \[ 1 \leftrightarrow \text{id}, 2 \leftrightarrow (1, 4, 3), 3 \leftrightarrow (1, 3, 4), 4 \leftrightarrow (1, 2)(3, 4), \]
   \[ 5 \leftrightarrow (1, 2, 3), 6 \leftrightarrow (1, 2, 4), 7 \leftrightarrow (1, 3, 2), 8 \leftrightarrow (2, 4, 3) \]
   \[ 9 \leftrightarrow (1, 4)(2, 3), 10 \leftrightarrow (1, 4, 2), 11 \leftrightarrow (2, 3, 4), 12 \leftrightarrow (1, 3)(2, 4) \]
Lemma 8. Let $G$ be a finite group of order $d$ which is generated by two elements $a$ and $b$. Suppose that the commutator $[a, b]$ has order 2. $G$ acts on itself by multiplication from the right. Identify the elements of $G$ with the numbers of $\{1, \ldots, d\}$. Then each element $g$ of $G$ defines a permutation $\sigma_g$ in $S_d$. Let $O$ be the origami defined by the pair of permutations $(\sigma_a, \sigma_b)$. Then $O$ is a Hurwitz translation surface and we obtain any Hurwitz translation surface in this way.

Proof. The squares of the origami $O$ correspond to the elements of $G$. The right neighbour of the square labelled by $g$ is $g \cdot a$ and the upper neighbour is $g \cdot b$. Let $p : X \to E$ be the corresponding covering to the torus. Consider a small simple closed loop on the torus around the ramification point which goes first to the right, then up, then to the left and then down. If we lift this loop to $X$ via $p$ in a point which lies let us say in the square labelled by $g \in G$, then the lift ends in a point which lies in the square labelled by $g \cdot a \cdot b \cdot a^{-1} \cdot b^{-1}$. Hence lifting the second power of our simple closed loop closes up. This is true for any $g$, thus the ramification index of each point above the puncture of the torus is 2 and hence all singularities of $O$ are of total angle $4 \pi$. Thus $O$ is a normal origami in $H(1, \ldots, 1)$. Conversely, if we start with a normal origami in $H(1, \ldots, 1)$, its group of translations which is equal to the deck transformation group has the desired property. Hence the origami defines a Hurwitz translation surface.

Theorem 1 now directly follows from Lemma 4 and Lemma 8. Furthermore, we obtain the following non-example of Lemma 8.

Remark 9. There are no Hurwitz translation surfaces of genus 2.

Proof. Suppose there was a normal origami $X$ in $H(1, 1)$. Then $G = \text{Trans}(X)$ has order $4 \cdot 2 - 4 = 4$ and is hence abelian. But then the commutator of two elements is trivial which contradicts Lemma 8.

3. Translation Hurwitz Numbers

We have seen in Example 7 Hurwitz translation surfaces in genus 3 and genus 4 and in Remark 9 that there are none in genus 2. This naturally leads to the question in which genus there exist translation Hurwitz surfaces. This becomes via Lemma 8 a purely group theoretical question. This section is devoted to the proof of Theorem 2, which answers to this question.
We denote the cyclic group with $n$ elements by $C_n$, the commutator subgroup of a
group $G$ by $G'$, the normaliser of a subgroup $U$ of $G$ by $N_G(U)$ and the index of $U$
by $(G : U)$.

**Definition 10.** A finite group $G$ is called $tH$ (translation Hurwitz), if there exist
two elements $g, h \in G$ such that $\langle g, h \rangle = G$, and $[g, h]$ has order 2. An integer $n$ is
called $tH$, if there exists a $tH$ group of order $n$.

**Proposition 11.** An integer is $tH$ if and only if $n$ is divisible by 8 or 12.

Thus we can read off from the order of the group, that a group is not $tH$. As we will
see below it is crucial that in the non-cases, i.e. if the group is not divisible by 8
and not divisible by 12, this implies that the group is solvable. Relating arithmetic
properties of the order of a group to its structural properties is an old idea which
is carried out e.g. in [Paz59], where they deduce interesting properties of a group
as nilpotency and supersolvability just from its order.

One direction of Proposition 11 follows from the following examples.

**Proposition 12.**

1. For $a \geq 3$ we have that $2^a$ is $tH$.
2. For $b \geq 1$ we have that $4 \cdot 3^b$ is $tH$.
3. If $n$ is $tH$, and $(n, m) = 1$, then $nm$ is $tH$.

**Proof.** We provide an explicit construction for each claim.

1. Consider the group $G = C_{2^{a-1}} \rtimes_{\varphi} C_2$, where $\varphi : C_2 \to \text{Aut}(C_{2^{a-1}})$ is defined
   by $\varphi(1) : 1 \mapsto 2^{a-2} + 1$. As pair of generators we choose the standard generators
   $x = (1, 0)$ and $y = (0, 1)$. Then $[x, y] = (2^{a-2}, 0)$ has order 2.

2. The commutator of $A_4$ is $C_2 \times C_2$, thus any two elements $x, y$ of $A_4$ which do not
   commute satisfy that $[x, y]$ has order 2. Now $A_4$ is generated by $(1, 2, 3), (1, 2)(3, 4)$,
   thus $A_4$ is $tH$, and our claim holds for $b = 1$ (compare Example 7.3). For higher
   values of $b$ consider $G = A_4 \rtimes C_{3^{b-1}}$, and put $x = ((1, 2, 3), 0)$, $y = ((1, 2)(3, 4), 1)$.
   Then $[x, y] = ((1, 4)(2, 3), 0)$ is of order 2. Moreover, $\langle x, y \rangle$ contains $y^{3^{b-1}+1} = (\text{id}, 1)$
as well as $y^{3^{b-1}} = ((1, 2)(3, 4), 0)$, thus $\langle x, y \rangle = G$, and we conclude that $G$ is $tH$.

3. Let $G$ be a $tH$ group of order $n$, and let $x, y$ be suitable generators of $G$. Consider
   $G \rtimes C_m$, and put $x' = (x, 0)$, $y' = (y, 1)$. Since $(n, m) = 1$ there exist $u, v \in \mathbb{Z}$ such
   that $un + vm = 1$, thus $y^{un} = (y^u, un) = (1_G, 1)$, and we conclude that $\langle x', y' \rangle$
   contains $C_m$. Similarly $y^{vm} = (y^v, vm) = (y, 0)$, thus $\langle x', y' \rangle$ contains $G$. Hence
   $\langle x', y' \rangle = G \rtimes C_m$, and we conclude that $nm$ is $tH$. \hfill $\square$

It follows from Thompson’s classification in [Tho68] of minimal finite simple groups
that every group of order not divisible by 8 or 12 is solvable, as described in the
following: The classification in [Tho68, Section 3, Main Theorem] tells us that we
have the following list of all minimal finite simple groups: $\text{PSL}(2, 2^p)$ with $p$ a
prime, $\text{PSL}(2, 3^p)$ with $p$ an odd prim, $\text{PSL}(2, p)$ with $p$ prime, $p \geq 5$ and $p^2 + 1 \equiv 0$
mod 5, a Suzuki group $Sz(2^p)$ with $p$ an odd prime, or $\text{PSL}(3, 3)$. Thus the number of
elements of each minimal simple group is divisible by 8 or 12. Minimal simple
groups are by definition non-abelian simple groups whose proper subgroups are all
solvable. Thus any non-abelian simple group is either minimal or has a subquotient
of smaller order which is again non-abelian and simple. Iterating this will finally
lead to a minimal simple group and thus by induction also the order of each finite non-abelian simple group is divisible by 8 or 12. Finally, any finite non-solvable group $G$ has a subquotient which is non-abelian and simple, thus the order of $G$ is as well divisible by 8 or 12.

In particular we may assume for the proof of the reverse direction of Proposition 11 that $G$ is solvable. We shall repeatedly use the following, confer e.g. [Hup67, Theorem VI.1.7].

Lemma 13 (Hall). Let $G$ be a finite solvable group, $\pi$ a set of prime divisors of $|G|$. Write $|G| = \pi m$, where all prime divisors of $m$ are in $\pi$, and all prime divisors of $n$ are not in $\pi$. Then there exists a subgroup $U$ of $G$ with $|U| = n$, and all subgroups of this form are conjugate.

We call a subgroup of this form a $\pi$-Hall group. If $\pi$ consists of a single prime $p$, we write $p$ in place of $\{p\}$, and for a set $\pi$ we denote by $\pi'$ the complement of $\pi$. In particular, $2'$ denotes the set of all odd primes.

Lemma 14. Suppose that $n$ is even, but not divisible by 4. Then $G$ is not th

Proof. If $|G| \equiv 2 \pmod{4}$, then $G$ is solvable, since $|G|$ is not divisible by 4. Let $U$ be a $2'$-Hall group. Then $(G : U) = 2$, thus $U$ is normal in $G$, and $G$ projects onto $C_2$. In particular $G' \leq U$, thus $(G : G')$ is even, and $|G'|$ is odd. But then $[x, y] \in G'$ cannot have even order. \hfill $\Box$

The following statement can be concluded e.g. from [Hal64, Theorem 9.3.1]. We include the proof for the convenience of the reader. It mimics the proof of Sylow’s theorem.

Lemma 15. Let $\pi$ be a set of primes, $G$ a solvable group, $U$ a $\pi$-Hall group. Let $n$ be the number of conjugates of $U$. Then $n$ divides the $\pi'$-part of $|G|$, and there exist non-negative integers $a_p$, $p \in \pi$, such that $1 + \sum_{p \in \pi} a_p p = n$.

Proof. $G$ acts on $\Omega = \{U^g | g \in G\}$ by conjugation. This action is transitive, thus $|\Omega|$ equals the index of a point stabiliser in $G$. The stabiliser of $U$ is $N_G(U)$, hence $|\Omega| = (G : N_G(U))$. Since $U \leq N_G(U)$, we have that $|\Omega|$ divides $(G : U)$, which is the $\pi'$-part of $|G|$. The action of $U$ on $\Omega$ induces a decomposition of $\Omega$ into $U$-orbits. U itself is stable under $U$-conjugation, hence $\{U\}$ is an orbit of size 1. We claim that there exists no further orbit of size 1. Suppose that $U^9$ is stabilised by $U$. Then $U$ normalises $U^9$, hence the group $\langle U, U^9 \rangle$ generated by $U$ and $U^9$ contains $U^9$ as a normal subgroup, and $\langle U, U^9 \rangle / U^9 \cong U / (U^9 \cap U)$. Hence $|\langle U, U^9 \rangle | = \frac{|U^9|}{|U \cap U^9|}$ is a $\pi$-number, which is strictly larger than $|U|$, since $U \neq U^9$. But $|U|$ is the largest $\pi$-number dividing $G$, thus $|\langle U, U^9 \rangle |$ does not divide $|G|$, which is impossible. Hence we conclude that there exists precisely one orbit of size 1.

The size of every orbit divides $|U|$, hence the size of every orbit is either 1 or divisible by some prime divisor in $\pi$. For each orbit $o$ we pick one such prime divisor $p_o$, and we obtain an equation of the form $|\Omega| = 1 + \sum_o \text{Orbit size } p_o$. Collecting the integers $\frac{|o|}{p_o}$ we obtain the integers $a_p$ from our claim.

Corollary 16. Suppose that $|G|$ is divisible by 4, but not by 8 or 12. Then $G$ has a normal subgroup of index 4.
Proof. The divisibility condition implies that $|G|$ is solvable. Let $U$ be a $2'$-Hall group. Then $U$ has index 4. Let $n$ be the number of conjugates of $U$. Then $n$ divides 4, and we can write $n$ as
\[ n = 1 + \sum_{p | |G|, p > 2} a_p p = 1 + \sum_{p \geq 5} a_p p, \]
since by assumption $|G|$ is not divisible by 3. But every integer representable as a sum as on the right is either 0 or $\geq 5$, hence we obtain $n = 1$. But then $U$ is normal, and our claim follows. $\square$

We can now finish the proof of Proposition 11.

Proof. (of Proposition 11) The reverse direction follows from Proposition 12. Suppose now that $n$ is tH, and let $G$ be a tH-group of order $n$. Then $n$ is trivially even. The case $n \equiv 2 \pmod{4}$ is excluded by Lemma 14, thus $n \equiv 0 \pmod{4}$. If $n \equiv 0 \pmod{8}$, our claim is true, hence we may assume $n \equiv 4 \pmod{8}$. If $n$ is not divisible by 3, then from the corollary we see that $G$ contains a normal subgroup $U$ of index 4. Since every group of order 4 is abelian, we obtain $G' \leq U$. But then $(G : G')$ is divisible by 4, thus $|G'|$ is odd, and we conclude that there cannot be elements $x, y \in G$ such that $[x, y]$ has even order. Hence $|G|$ is divisible by 3, and the claim is proven. $\square$

Finally, Theorem 2 directly follows from Lemma 8 and Proposition 11.

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