The intersections of typical Besicovitch sets with lines

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Abstract

We show that a typical Besicovitch set $B$ has intersections of measure zero with every line not contained in it. Moreover, every line in $B$ intersects the union of all the other lines in $B$ in a set of measure zero.

1 Introduction

A Besicovitch set is a set $B \subseteq \mathbb{R}^n$ $(n \geq 2)$ which contains a unit line segment in every direction. Besicovitch showed that there exists a Besicovitch set of measure zero in $\mathbb{R}^2$ ([1], see also [2] Chapter 7). It is easy to see that this gives us a Besicovitch nullset in every dimension $n \geq 2$. Knowing the existence of a Besicovitch nullset it was natural to ask if it is possible to make it even smaller.

Kakeya conjecture: A Besicovitch set in $\mathbb{R}^n$ necessarily has Hausdorff dimension $n$.

This conjecture is still open except for $n = 2$ in which case it turned out to be true ([3] Davies 1971). The Kakeya conjecture is connected to several famous open questions in various fields of mathematics [4].

Tom Körner proved that if we consider a well-chosen closed subspace of $K(\mathbb{R}^2)$ in which every element contains a unit segment in every direction between $\frac{\pi}{3}$ and $\frac{2\pi}{3}$, then a typical element in this subspace is of measure zero ([5] Theorem 2.3). The union of three rotated copies of such a set is a Besicovitch set of measure zero. In this sense it is typical for a Besicovitch set to have measure zero.

There is a variation of the definition of Besicovitch set:

Definition 1.1. A Besicovitch set is a set $B \subseteq \mathbb{R}^n$ $(n \geq 2)$ which contains a line in every direction.

This gives us a variation of the Kakeya conjecture which is open as well. It is conjectured to be equivalent to the previous form. We will work with Definition 1.1 throughout this paper.

It is clear from Fubini’s theorem that if we intersect a Besicovitch nullset with lines of a fixed direction, then almost every intersection is of measure zero. We will use Baire category arguments combined with duality methods to obtain Besicovitch sets with stronger properties.

2 Preliminaries

2.1 Dual sets

We denote the orthogonal projection of the set $A \subseteq \mathbb{R}^2$ in the direction $v$ by $pr_v(A)$ (where $v$ is a nonzero vector or sometimes just its angle if it leads to no confusion). Similarly

$$P_v(A) := \left\{ \frac{x - v}{|x - v|} \in S^1 : x \in A \setminus \{v\} \right\}$$

is the radial projection of $A$ from the point $v$. We may refer to elements of $S^1$ as angles causing no confusion.
Definition 2.1. Let \( l(a, b) \) denote the line which corresponds to the equation \( y = ax + b \). We say that \( \mathcal{L} \) is the dual of \( K \subseteq \mathbb{R}^2 \) (or \( \mathcal{L} \) is coded by \( K \)) if \( \mathcal{L} = \{l(a, b) : (a, b) \in K \} \).

A well-known consequence of this definition is the following. For completeness we present the short proof.

Proposition 2.2. Let \( K \subseteq \mathbb{R}^2 \) be a set and \( \mathcal{L} \) its dual. Then the vertical sections of \( L := \bigcup \mathcal{L} \) are scaled copies of the corresponding orthogonal projections of \( K \). More precisely, \( L_x = \{(x, 1) : \text{pr}_{(-1,x)}(K)\} \).

Proof. The vertical section \( L_x \) consists of the points of the form \( ax + b \) where \( (a, b) \in K \). In other words

\[ L_x = \{ax + b : (a, b) \in K\} = \{(x, 1) \cdot (a, b) : (a, b) \in K\} = \left\{\left[(x, 1) \cdot (x, 1)^{-1}\right] \cdot (a, b) : (a, b) \in K\right\} \]

And this is exactly the orthogonal projection of \( K \) in the direction \((-1, 1)\) scaled by the constant \( |(x, 1)| \).

We need to prove a generalization of the previous observation. This generalization will play a key role in the main proof.

Proposition 2.3. Let \( \mathcal{L} \) be the dual of the set \( K \subseteq \mathbb{R}^2 \), \( L := \bigcup \mathcal{L} \), and let \( e \notin \mathcal{L} \) be a line in \( \mathbb{R}^2 \). Then the intersection \( e \cap L \) is

1. a scaled copy of an orthogonal projection of \( K \) if \( e \) is vertical,
2. otherwise it is the image of \( P_{(a_0,b_0)}(K) \setminus \{\frac{\bar{e} \cdot \vec{a}}{\lambda} \} \) by a locally Lipschitz function, where the equation of \( e \) is \( y = a_0x + b_0 \).

Proof. (1) is just the previous proposition.

(2): Note that \( \mathcal{L} \) does not contain vertical lines because it is the dual of \( K \). Then

\[ e \cap L = \{(x, y) \in \mathbb{R}^2 : \exists (a, b) \in K \ y = a_0x + b_0 = ax + b\} \]

So in the intersection \( x = \frac{b - b_0}{a_0 - a} \) holds (we have \( a \neq a_0 \) because \( e \) does not intersect lines parallel to itself). It is enough to determine the projection of \( e \cap L \) to the \( x \)-axis since \( e \cap L \) is the image of this projection by a Lipschitz function.

On the other hand, the projection of \( e \cap L \) to the \( x \)-axis is \( \left\{\frac{b - b_0}{a_0 - a} : (a, b) \in K\right\} = \left\{(-1) \cdot \frac{b - b_0}{a_0 - a} : (a, b) \in K\right\} \), which is the set of slopes of the lines connecting points of \( K \) to \((a_0, b_0)\) multiplied by \((-1)\). It is clear that this set is the image of \( P_{(a_0,b_0)}(K) \setminus \{\frac{\bar{e} \cdot \vec{a}}{\lambda} \} \) by the function \(-\tan(\varphi)\) which is locally Lipschitz.

We will need the following.

Proposition 2.4. The union of the dual of a compact set is closed.

The proof is an easy exercise, we leave it to the reader.

2.2 Special code sets

Let \( \lambda \) denote the 1-dimensional Lebesgue measure. For the main proof we need two compact sets with special properties.

The following theorem is due to Michel Talagrand [6]. For a direct proof in English, see [7] Appendix A.

Theorem 2.5. For any non-degenerate rectangle \([a, b] \times [c, d] \subseteq \mathbb{R}^2 \) there exists a compact set \( K \subseteq [a, b] \times [c, d] \) such that its projection to the \( x \)-axis is the whole \([a, b] \) interval, but in every other direction its projection is of measure zero.

Definition 2.6. A set \( A \subseteq \mathbb{R}^2 \) is invisible from a point \( a \in \mathbb{R}^2 \) if \( \lambda(P_a(A)) = 0 \).

We will use a theorem of Károly Simon and Boris Solomyak [8]:
Theorem 2.7. Let \( \Lambda \) be a self-similar set of Hausdorff dimension 1 in \( \mathbb{R}^2 \) satisfying the Open Set Condition, which is not on a line. Then, \( \Lambda \) is invisible from every \( a \in \mathbb{R}^2 \).

It is an easy exercise to check that the four corner Cantor set of contraction ratio \( \frac{1}{4} \) projects orthogonally to an interval in four different directions. It is well-known that this set satisfies the conditions of Theorem 2.7. Rotate it to have an interval as projection to the \( x \)-axis. Now by an affine transformation we can make it fit to the rectangle \([a, b] \times [c, d]\) while not losing its properties required by Theorem 2.7. By these easy observations we get the following corollary.

Corollary 2.8. For any non-degenerate rectangle \([a, b] \times [c, d] \subseteq \mathbb{R}^2\) there exists a compact set \( K \subseteq [a, b] \times [c, d] \) such that its projection to the \( x \)-axis is the whole \([a, b] \) interval, but it is invisible from every point of the plane.

### 2.3 Projections of a compact set

We will need the following two lemmas.

**Lemma 2.9.** Let \( A \) be a compact set and \( f_A : S^1 \rightarrow \mathbb{R} \), \( f_A(\varphi) = \lambda(\text{pr}_A(A)) \). Then \( f_A \) is upper semicontinuous.

Talagrand proved in [6] that \( \{f_A : A \in \mathcal{K}(\mathbb{R}^2)\} \) is the set of non-negative upper semicontinuous functions. We need only the easy direction, hence we present a proof only for that.

**Proof.** Let \( c \in \mathbb{R} \) be arbitrary. We have to verify that \( f_A^{-1}((-\infty, c)) \) is open. Let \( \varphi \) be such that \( \lambda(\text{pr}_A(A)) < c \). Since \( \text{pr}_A(A) \) is compact as well, it can be covered by finitely many open intervals \( I_j \) \( (1 \leq j \leq l) \) for which \( \lambda \left( \bigcup_{j=1}^{l} I_j \right) < c \) holds. This cover shows that \( A \) can be covered by rectangles \( R_1, \ldots, R_l \) whose projections in the direction \( \varphi \) are the intervals \( I_1, \ldots, I_l \). But for the union of finitely many rectangles it is clear that changing \( \varphi \) by a suitably small \( (\delta) \) angle we can keep the measure of its projection less than \( c \). This implies that for any \( \varphi' \in (\varphi - \delta, \varphi + \delta) \) we have

\[
\lambda(\text{pr}_{A'}(A)) \leq \lambda \left( \text{pr}_{\varphi'} \left( \bigcup_{j=1}^{l} R_j \right) \right) < c.
\]

In other words, a neighbourhood of \( \varphi \) also lies in \( f_A^{-1}((-\infty, c)) \), therefore the preimage is open. \( \square \)

**Lemma 2.10.** If \( A \subseteq \mathbb{R}^2 \) is compact, then \( F_A : \mathbb{R}^2 \setminus A \rightarrow \mathbb{R} \), \( F_A(v) = \lambda(P_v(A)) \) is upper semicontinuous.

**Proof.** Let \( c \in \mathbb{R} \). We will check that \( F_A^{-1}((-\infty, c)) \) is open. Let \( v \) be a point such that \( F_A(v) = \lambda(P_v(A)) < c \). Then by compactness we can take a finite cover of \( P_v(A) \) by open arcs \( I_1, \ldots, I_l \) such that \( \lambda \left( \bigcup_{j=1}^{l} I_j \right) < c \). This cover shows that \( A \) can be covered by \( l \) sectors \( R_1, \ldots, R_l \) of an annulus such that their radial projections from \( v \) are \( I_1, \ldots, I_l \). For the union of finitely many sectors of an annulus and a point which has a positive distance from them it is clear that moving \( v \) by a suitably small distance we can keep the measure of the radial projection of \( \bigcup_{j=1}^{l} R_j \) less than \( c \). In other words, a neighbourhood of \( v \) lies in \( F_A^{-1}((-\infty, c)) \), so it is open. \( \square \)

### 2.4 Baire category and Hausdorff distance

For the sake of clarity we assert some well-known definitions and theorems here.

**Definition 2.11.** Let \( X \) be a topological space and \( E \subseteq X \).

- \( E \) is **nowhere dense** in \( X \) if its closure has empty interior.
- \( E \) is of **first category** in \( X \) if it is the countable union of nowhere dense sets.
- \( E \) is of **second category** in \( X \) if it is not of first category.
- \( E \) is **residual** in \( X \) if its complement is of first category.

**Theorem 2.12.** *(Baire category theorem)* A complete metric space is of second category in itself.
Definition 2.13. Let $X$ be a complete metric space. The property $P(x)$ is typical in $X$ if $\{x \in X : P(x)\}$ is residual in $X$. We often formulate this in a less accurate manner: a typical $x \in X$ has the property $P(x)$.

Let $(X, d)$ be a metric space and let $K(X)$ be the set of its compact subsets. Denote the open $\delta$-neighbourhood of $A$ by $A_\delta$, and denote the closed $\delta$-neighbourhood of $A$ by $\overline{A}_\delta$.

Definition 2.14. Let $K, L \in K(X)$. The Hausdorff distance of $K$ and $L$ is
\[ d_H(K, L) := \max\{\inf\{\delta_1 \geq 0 : K \subseteq L_{\delta_1}\}, \inf\{\delta_2 \geq 0 : L \subseteq K_{\delta_2}\}\}. \]

Theorem 2.15. If $(X, d)$ is a complete metric space, then $(K(X), d_H)$ is a complete metric space as well.

3 The main theorem

We could introduce a new Besicovitch set by simply taking the dual of the compact set given by Corollary 2.8. It would have intersections of measure zero with every non-vertical line not contained in it by Proposition 2.3. However, we will go further to obtain the following stronger result:

Theorem 3.1. There exists a Besicovitch set $B = \bigcup \mathcal{L}$ (where $\mathcal{L}$ is a family of lines) in the plane such that:

1. $B$ is closed.
2. $B$ is of 2-dimensional Lebesgue measure zero.
3. For every line $e \notin \mathcal{L}$ the intersection $B \cap e$ is of 1-dimensional Lebesgue measure zero.
4. For every $e \in \mathcal{L}$ the intersection $e \cap \bigcup (\mathcal{L} \setminus \{e\})$ is of 1-dimensional Lebesgue measure zero.

Moreover, we claim that these properties are typical in the sense described below.

We work in $K([0, 1]^2)$ which is a complete metric space with the Hausdorff distance. Consider the subspace
\[ \mathcal{C} := \{K \in K([0, 1]^2) : \text{pr}_x(K) = [0, 1]\}. \]

It is easy to check that $\mathcal{C}$ is a closed subspace hence a complete metric space as well. The typicality in the main theorem means that a typical $K' \in \mathcal{C}$ codes a family of lines $\mathcal{L}'$ for which $\mathcal{L}' = \bigcup \mathcal{L}'$ is an almost Besicovitch set: the union of four rotated copies of $\mathcal{L}'$ satisfies all the properties in Theorem 3.1.

The following theorem strengthens Theorem 2.5 and it is due to Alan Chang [9]. Here we present our own proof (found independently of Chang) to provide a useful analogue for the proof of the next theorem.

Theorem 3.2. A typical element of $\mathcal{C}$ has orthogonal projections of measure zero in every non-vertical direction.

Proof. We have to prove that the set $\{K \in \mathcal{C} : \exists \varphi \in [0, \pi] \setminus \{\frac{\pi}{2}\} \quad \lambda(\text{pr}_\varphi(K)) > 0\}$ is of first category. Let $T_n = \{\varphi \in [0, \pi] : |\varphi - \frac{\pi}{2}| \geq \frac{1}{n}\}$. It suffices to show that for every $n$
\[ B_n := \left\{K \in \mathcal{C} : \exists \varphi \in T_n \quad \lambda(\text{pr}_\varphi(K)) \geq \frac{1}{n}\right\} \]
is nowhere dense in $\mathcal{C}$.

Fix a compact set $K \in \mathcal{C}$ and $\varepsilon > 0$. Denote the open ball of center $A$ and radius $\delta$ by $B_H(A, \delta)$ (with respect to the Hausdorff distance). We need to find $K' \in \mathcal{C}$ and $\varepsilon' > 0$ such that $B_H(K', \varepsilon') \subseteq B_H(K, \varepsilon)$ and $B_H(K', \varepsilon') \cap B_n = \emptyset$.

At first we construct $K'$. Take a finite $\frac{\pi}{2}$-net in $K$: $\{(x_1, y_1), \ldots, (x_N, y_N)\}$. Consider the squares of the form
\[ Q_i := \left[ x_i - \frac{\varepsilon}{3}, x_i + \frac{\varepsilon}{3} \right] \times \left[ y_i - \frac{\varepsilon}{3}, y_i + \frac{\varepsilon}{3} \right] \quad (1 \leq i \leq N). \]
Some of the squares may not lie in $[0, 1]^2$. We cut off the parts sticking out of $[0, 1]^2$ making $Q_i$ a rectangle if it is necessary. Since it was created from an $\frac{\pi}{2}$-net, $\bigcup_{i=1}^N Q_i$ covers $K$. Hence its projection to the $x$-axis is
the whole $[0,1]$. For every rectangle $Q_i$ Theorem 2.5 gives us a compact set $K_i' \subseteq Q_i$ which has orthogonal projections of measure zero in every non-vertical direction and $pr_\varphi(K_i') = pr_\varphi(Q_i)$. Now let $K' = \bigcup_{i=1}^N K_i'$.

We need to check the following:

(1) $K' \in \mathcal{C}$,

(2) $K' \in B_H(K, \varepsilon)$ and

(3) $\lambda(pr_\varphi(K')) < \frac{1}{n}$ for all $\varphi \in T_n$.

(1) This is clear since $pr_\varphi\left(\bigcup_{i=1}^N Q_i\right) = [0,1]$ and $pr_\varphi(K_i') = pr_\varphi(Q_i)$ in each $Q_i$.

(2) The following two sequences of containments prove that $d_H(K, K') < \varepsilon$.

$$K' \subseteq \bigcup_{i=1}^N Q_i \subseteq \{(x_1, y_1), \ldots, (x_N, y_N)\} \subseteq K',$$

$$K \subseteq \{(x_1, y_1), \ldots, (x_N, y_N)\} \subseteq \left(\frac{K'}{\varepsilon}\right) \subseteq K' \bigcap \frac{\varepsilon}{\varepsilon}$$

(3) $K'$ is the union of $N$ sets whose projection is of measure zero in every non-vertical direction.

Now we have to find $\varepsilon'$.

It is very easy to check that for any compact set $A$, positive real number $\delta$ and angle $\varphi$ the following holds: $pr_\varphi(A) = (pr_\varphi(A))_\delta$.

For every $\varphi$ the projection $pr_\varphi(K')$ is compact, so we have

$$\lim_{\varepsilon \to 0} \lambda\left(\frac{pr_\varphi(K')}{\varepsilon}\right) = \lambda(pr_\varphi(K')).$$

Hence there exists $\varepsilon_\varphi$ for each $\varphi \in T_n$ such that

$$\lambda\left(\frac{pr_\varphi(K')}{\varepsilon_\varphi}\right) = \lambda\left(\frac{pr_\varphi(K')}{\varepsilon_\varphi}\right) < \frac{1}{n}.$$ 

The upper semicontinuity ensured by Lemma 2.9 for $A = \frac{K'}{\varepsilon_\varphi}$ says that there exists a $\delta_\varphi$ such that for any $\varphi' \in (\varphi - \delta_\varphi, \varphi + \delta_\varphi)$ the projection is small enough: $\lambda(pr_\varphi\left(\frac{K'}{\varepsilon_\varphi}\right)) < \frac{1}{n}$. On the other hand, $T_n$ is compact, therefore it is covered by finitely many of these neighbourhoods, which gives us finitely many conditions. Hence we can choose $\varepsilon'$ so that $\lambda(pr_\varphi(K'_i)) < \frac{1}{n}$ for all $\varphi \in T_n$. Since every element of $B_H(K', \varepsilon')$ lies in $K_i'$, we proved $B_H(K', \varepsilon') \cap B_n = \emptyset$.

If it is necessary, we decrease $\varepsilon'$ further to satisfy $B_H(K', \varepsilon') \subseteq B_H(K, \varepsilon)$.

**Theorem 3.3.** A typical $K \in \mathcal{C}$ is invisible from every point of the plane.

**Proof.** The proof is very similar to the previous one. We need to prove that $\{K \in \mathcal{C} : \exists v \in \mathbb{R}^2 \quad \lambda(P_v(K)) = 0\}$ is of first category.

First observe that for any point $v \in \mathbb{R}^2$ and compact set $K \subseteq \mathbb{R}^2$

$$P_v(K) = \bigcup_{n=1}^\infty P_v\left(K \setminus B\left(v, \frac{1}{n}\right)\right),$$

which implies

$$\lambda(P_v(K)) = \lim_{n \to \infty} \lambda\left(P_v\left(K \setminus B\left(v, \frac{1}{n}\right)\right)\right).$$

Therefore, it suffices to show that

$$B_n := \left\{K \in \mathcal{C} : \exists v \in [-n, n] \times [-n, n] \quad \lambda\left(P_v\left(K \setminus B\left(v, \frac{1}{n}\right)\right)\right) \geq \frac{1}{n}\right\}$$
Fix $K \in \mathcal{C}$ and $\epsilon > 0$. Then take a finite $\frac{\epsilon}{n}$-net $\{(x_1, y_1), \ldots, (x_N, y_N)\}$ in $K$ and consider the little squares of side length $\frac{\epsilon}{n}$ around them. After chopping off the parts outside $[0,1]^2$ we get the rectangles $Q_1, \ldots, Q_N$.

Now for every $Q_i$, Corollary 2.8 gives us a compact set $K'_i \subseteq Q_i$ which is invisible from every point of the plane, and $\text{pr}_{\mathbb{R}}(K'_i) = \text{pr}_{\mathbb{R}}(Q_i)$. Let $K' = \bigcup_{i=1}^{N} K'_i$. Then $K'$ is also invisible from every point of the plane. Exactly the same argument as in the previous proof shows that $K' \in \mathcal{C}$ and $d_H(K, K') < \epsilon$ holds.

Now we have to find $\epsilon'$.

**Claim.** For every $n \in \mathbb{N}$ and $v \in [-n, n] \times [-n, n]$ there exists $\epsilon_v$ such that $\lambda\left(P_v\left(\overline{K_{\epsilon_v}'} \setminus \overline{B\left(\frac{1}{2n}\right)}\right)\right) < \frac{1}{n}$.

Fix $n$ and $v$. Restricting the radial projection to an annulus of inner radius $\frac{\epsilon_v}{4n}$ centered at $v$ it becomes a Lipschitz function with Lipschitz constant $4n$. Since $P_v\left(K' \setminus \overline{B\left(\frac{1}{4n}\right)}\right)$ is a compact set of measure zero (recall that even $K'$ is invisible from $v$), we know that

\[
\lim_{\delta \to 0} \lambda\left(\left(P_v\left(K' \setminus \overline{B\left(\frac{1}{4n}\right)}\right)\right)\right) = \lambda\left(P_v\left(K' \setminus \overline{B\left(\frac{1}{4n}\right)}\right)\right) = 0.
\]

Thus for a suitably small $\delta \leq 1$ we have $\lambda\left(P_v\left(K' \setminus \overline{B\left(\frac{1}{4n}\right)}\right)\right) < \frac{1}{n}$. Now we claim that

\[
P_v\left(K' \setminus \overline{B\left(\frac{1}{2n}\right)}\right) \subseteq P_v\left(K' \setminus \overline{B\left(\frac{1}{4n}\right)}\right)\delta.
\]

Indeed, if $x \in K' \setminus \overline{B\left(\frac{1}{4n}\right)}$, then there exists $y \in K' \setminus \overline{B\left(\frac{1}{4n}\right)}$ such that $|x - y| < \frac{\epsilon}{4n} \leq \frac{\epsilon_v}{4n}$. Therefore $|P_v(x) - P_v(y)| < \delta$ because of the Lipschitz property, and $P_v(y) \in P_v\left(K' \setminus \overline{B\left(\frac{1}{4n}\right)}\right)$, so $P_v(x) \in P_v\left(K' \setminus \overline{B\left(\frac{1}{4n}\right)}\right)\delta$. Hence $\epsilon_v = \frac{\epsilon}{4n}$ is a good choice.

If $\epsilon_v$ is suitable for $v$, then for every $v' \in B\left(\frac{1}{2n}\right)$

\[
\overline{K_{\epsilon_v}'} \setminus \overline{B\left(\frac{1}{n}\right)} \subseteq \overline{K_{\epsilon_v}'} \setminus \overline{B\left(\frac{1}{2n}\right)}
\]

therefore

\[
\lambda\left(P_{v'}\left(\overline{K_{\epsilon_v}'} \setminus \overline{B\left(\frac{1}{n}\right)}\right)\right) \leq \lambda\left(P_{v'}\left(\overline{K_{\epsilon_v}'} \setminus \overline{B\left(\frac{1}{2n}\right)}\right)\right).
\]

For $A = \overline{K_{\epsilon_v}'} \setminus \overline{B\left(\frac{1}{n}\right)}$ the function $F_A$ is upper semicontinuous on the complement of $A$ by Lemma 2.10. Hence there exists $U_v \subseteq B\left(\frac{1}{2n}\right)$ neighbourhood of $v$ such that for all $v' \in U_v$

\[
\lambda\left(P_{v'}\left(\overline{K_{\epsilon_v}'} \setminus \overline{B\left(\frac{1}{n}\right)}\right)\right) \leq \lambda\left(P_{v'}\left(K_{\epsilon_v}' \setminus \overline{B\left(\frac{1}{2n}\right)}\right)\right) = F_A(v') < \frac{1}{n}.
\]

Since $[-n, n] \times [-n, n]$ is compact, it can be covered by finitely many such neighbourhoods, therefore we may choose an $\epsilon'$ which is suitable for all $v \in [-n, n] \times [-n, n]$.

We need to prove that $B_n \cap B_H(K', \epsilon') = \emptyset$ holds. Let $L \in B_H(K', \epsilon')$ and $v \in [-n, n] \times [-n, n]$. Then $L \subseteq K_{\epsilon'}'$ hence

\[
\lambda\left(P_v\left(K_{\epsilon'}' \setminus \overline{B\left(\frac{1}{n}\right)}\right)\right) \leq \lambda\left(P_v\left(K_{\epsilon'}' \setminus \overline{B\left(\frac{1}{2n}\right)}\right)\right) < \frac{1}{n}
\]

by the choice of $\epsilon'$. Consequently, $L \notin B_n$.

Now we have two typical properties in $\mathcal{C}$ by Theorem 3.2 and Theorem 3.3, so we may merge them into one corollary.

**Corollary 3.4.** A typical element $K \in \mathcal{C}$ has orthogonal projections of measure zero in every non-vertical direction, and it is invisible from every point of the plane.

**Proof of Theorem 3.1.** Let $K'$ be a typical element in $\mathcal{C}$, $\mathcal{L}'$ be its dual and $\mathcal{L}' := \bigcup \mathcal{L}'$. Then $\mathcal{L}'$ contains a line of slope $m$ for every $m \in [0,1]$ because the slope is coded by the first coordinate and $\text{pr}_{\mathbb{R}}(K') = [0,1]$. 

(1) $L'$ is closed by Proposition 2.4.

(3) Let $e$ be any vertical line. Then its intersection with $L'$ is similar to a non-vertical orthogonal projection of $K'$ by Proposition 2.3. Therefore, it is of measure zero by Corollary 3.4. This implies (2) immediately.

Now let $e$ be any non-vertical line not in $L'$. Then its intersection with $L'$ is the image of $P_v(K') \setminus \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ by a locally Lipschitz function for some point $v \in \mathbb{R}^2 \setminus K'$ (Proposition 2.3 again). Therefore it is of measure zero by Corollary 3.4.

So $L'$ has an intersection of measure zero with every line not contained in it.

(4) Let $e \in L'$ and let $y = a_0x + b_0$ be its equation. Now $L' \setminus \{e\}$ is the dual of $K' \setminus \{(a_0, b_0)\}$, thus the intersection $e \cap \bigcup (L' \setminus \{e\})$ is the image of $P_{(a_0, b_0)}(K' \setminus \{(a_0, b_0)\}) \setminus \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ by a locally Lipschitz function (again Proposition 2.3). Therefore it is of measure zero by Corollary 3.4.

Let $B$ be the union of four rotated copies of $L'$. Finally it contains a line in every direction and we have not lost its already checked properties. The proof of the main theorem is complete. \qed

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