GEOMETRIC INEQUALITIES ON LOCALLY CONFORMALLY FLAT MANIFOLDS

PENGFEI GUAN AND GUOFANG WANG

Abstract. Through the study of some elliptic and parabolic fully nonlinear PDEs, we establish conformal versions of the quermassintegral inequality, the Sobolev inequality and the Moser-Trudinger inequality for the geometric quantities associated to the Schouten tensor on locally conformally flat manifolds.

1. Introduction

In this paper, we are interested in certain global geometric quantities associated to the Schouten tensor and their relationship to each other in conformal geometry. For an oriented compact Riemannian manifold \((M, g)\) of dimension \(n > 2\), there is a sequence of geometric functionals arising naturally in conformal geometry, which generalize the Yamabe functional, introduced by Viaclovsky in [35] as curvature integrals of the Schouten tensor. If we let \(Ric_g\) and \(R_g\) denote the Ricci tensor and the scalar curvature of \(g\) respectively, the Schouten tensor can be written as

\[
S_g = \frac{1}{n - 2} \left( Ric_g - \frac{R_g}{2(n - 1)} \cdot g \right).
\]

Let \(\sigma_k\) be the \(k\)th elementary symmetric function, and let

\[
\sigma_k(g) := \sigma_k(g^{-1} \cdot S_g)
\]

be the \(\sigma_k\)-scalar curvature of \(g\). We are interested in the following functionals defined by

\[
F_k(g) = \text{vol}(g)^{-\frac{n - 2k}{n}} \int_M \sigma_k(g) \, dg, \quad k = 0, 1, \ldots, n,
\]

where \(dg\) is the volume form of \(g\). For convenience, set \(\sigma_0(g) = 1\).

We note that \(F_0(g)\) is simply the volume of \(g\) and \(F_1(g)\) is the Yamabe functional (up to a constant multiple). Let us fix a background metric \(g_0\) and denote by \([g_0]\) the conformal class of \(g_0\). The standard Sobolev inequality states that

\[
F_0(g) \leq C_{[g_0]}(F_1^{\frac{n-1}{n}}(g)), \quad \text{for } g \in [g_0],
\]

for some constant \(C_{[g_0]}\) depending only on \([g_0]\). Equality holds if and only if the metric \(g\) is a Yamabe minimizer. For \(g \in [g_0]\), we can write \(g = e^{-2u}g_0\). Let \(S_{g_0}\) and \(S_g\) be...
the corresponding Schouten tensors of $g_0$ and $g$ respectively. There is a transformation
formula relating them:

\[ S_g = \nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}. \]

From this point of view, $F_k(g)$ in (1) resembles the $k$th quermassintegral in the theory of
convex bodies. The classical quermassintegrals suggest that there may exist similar inequalities between the geometric quantities $F_k(g)$.

The focus of this paper is the investigation of the precise relationship of the geometric
quantities defined in (1). As we will see, there exists a complete system of sharp inequalities
for $F_k$ on locally conformally flat manifolds, and the cases of equality will be characterized
by certain “extremal” metrics, just as in the case of the Yamabe problem. The main
tool used to obtain these geometric inequalities is the theory of parabolic fully nonlinear
equations, and this is one of the special features of this paper.

When $(M, g_0)$ is a locally conformally flat manifold and $k \neq n/2$, it was proved by
Viaclovsky in [35] that the critical points of $F_k$ in $[g_0]$ are the metrics $g$ satisfying the
equation

\[ \sigma_k(g) = \text{constant}. \]

For $k = n/2$, $F_{n/2}(g)$ is a constant in the conformal class (see [35]). In this case, there is
another functional which was discovered in [4]. A similar functional was also found in [8].
The functional in [4] is defined by

\[ E_{n/2}(g) = - \int_0^1 \int_M \sigma_{n/2}(g_t) u dg_t dt, \]

where $u$ is the conformal factor of $g = e^{-2u} g_0$ and $g_t = e^{-2tu} g_0$. Unlike $F_k$, $E_{n/2}$
depends on the choice of the background metric $g_0$. However, its derivative $\nabla E_{n/2}$ does not depend
on the choice of $g_0$. The critical points of $E_{n/2}$ correspond to the metrics $g$ satisfying [3]
for $k = n/2$ on locally conformally flat manifolds. These important variational properties
are unknown for general (i.e. not locally conformally flat) manifolds for $k > 2$. For this
reason, we will restrict ourselves to locally conformally flat manifolds in our study of $F_k$
in this paper. Another advantage of locally conformally flat manifolds is the availability
of the fundamental result of Schoen-Yau [30] on developing maps, which is crucial for us
in establishing the gradient estimates for the conformal flow (13) we consider below.

We will establish three types of inequalities depending on the range of $k$. More precisely,
a Sobolev type inequality [5] is established for any $k < n/2$ and a conformal quermassintegral
type inequality [5] for any $k \geq n/2$. And, for the exceptional case $k = n/2$, we establish a
Moser-Trudinger type inequality [14] for $E_{n/2}$.

Before giving precise results, let us first recall some notation and definitions. Let

\[ \Gamma_k^+ = \{ \Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n | \sigma_j(\Lambda) > 0, \forall j \leq k \}. \]

Let $\Gamma_k^{++}$ be the closure of $\Gamma_k^+$. A $C^2$ metric $g$ is said to be in $\Gamma_k^+$ if the eigenvalues of
$g^{-1} \cdot S_g(x) \in \Gamma_k^+$ for any $x \in M$ (see [14]). Such a metric is also called admissible.
Similarly, a $C^{1,1}$ metric is said to be in $\Gamma_k^+$ if the eigenvalues of $g^{-1} \cdot S_g(x) \in \Gamma_k^+$ for any $x \in M$. For convenience, we set $\sigma_0(A) = 1$ and $\sigma_0(g) = 1$. We denote

$$C_k([g_0]) = \{g \in C^{4,\alpha}[g_0] | g \in \Gamma_k^+\},$$

where $[g_0]$ is the conformal class of $g_0$. Here $0 < \alpha < 1$.

Unlike the Yamabe equation ($k = 1$ in (3)), equation (4) is fully nonlinear when $k > 1$. Moreover, from work in [5], it is necessary to assume $g \in \Gamma_k^+$ (or $\Gamma_k^- = -\Gamma_k^+$) for $k > 1$ in order to make use of elliptic theory. This is a very restrictive condition. It is proved in [14] that any locally conformally flat manifold is in $\Gamma_k^+$. For example, $S^1 \times S^{n-1}$ with the standard product metric is in $\Gamma_k^+$ for any $k < \frac{n}{2}$, and $\mathbb{H}^p \times S^{n-p}$ with the standard product metric is in $\Gamma_k^+$ for certain range of $2 \leq k$ when $p$ is sufficiently small compared to $n$. Furthermore, if $(M_1, g_1)$ and $(M_2, g_2)$ are two $n$-dimensional locally conformally flat manifolds with $g_i \in \Gamma_k^+(i = 1, 2)$ for some $1 \leq k < \frac{n}{2}$, then one can construct a locally conformally flat metric $g \in \Gamma_k^+$ on the connected sum of them (see also [13]). This type of construction yields a wide class of locally conformally flat manifolds with metrics in $\Gamma_k^+$ for each $k < \frac{n}{2}$.

We now state our main results. Since any metric $g \in \Gamma_{n/2}$ is conformally equivalent to a metric of constant sectional curvature (see [14]), we will assume, whenever we consider [4] in this paper, that $g_0$ has constant sectional curvature.

**Theorem 1.** Suppose that $(M, g_0)$ is a compact, oriented and connected locally conformally flat manifold with $g_0 \in \Gamma_{n/2}^+$ smooth and $g \in C_k$. Let $0 \leq l < k \leq n$.

**A. Sobolev type inequality:** If $0 \leq l < k < \frac{n}{2}$, then there is a positive constant $C_S = C_S([g_0], n, k, l)$ depending only on $n, k, l$ and the conformal class $[g_0]$ such that

$$\left(\mathcal{F}_k(g)\right)^{\frac{1}{n-2k}} \geq C_S \left(\mathcal{F}_l(g)^{\frac{1}{n-2l}}\right).$$

If we normalize $\int_M \sigma(g) dg = 1$, then equality holds in (5) if and only if

$$\frac{\sigma_k(g)}{\sigma_l(g)} = C_S^{n-2k}. $$

There exists a metric $g_E \in C_k$ attaining equality in (5). Furthermore,

$$C_S \leq C_{S,k,l}(\mathbb{S}^n) = \left(\begin{array}{c} n \\ k \end{array}\right)^{\frac{1}{n-2k}} \left(\begin{array}{c} n \\ l \end{array}\right)^{\frac{1}{n-2l}} \left(\frac{\omega_n^2}{2^n}\right)^{\frac{k-l}{(n-2k)(n-2l)}},$$

where $\omega_n$ is the volume of the standard sphere $\mathbb{S}^n$.

**B. Conformal quermassintegral type inequality:** If $n/2 \leq k \leq n$, $1 \leq l < k$, then

$$\left(\mathcal{F}_k(g)\right)^{\frac{1}{k}} \leq \left(\begin{array}{c} n \\ k \end{array}\right)^{\frac{1}{n}} \left(\begin{array}{c} n \\ l \end{array}\right)^{-\frac{1}{k}} \left(\mathcal{F}_l(g)^{\frac{1}{l}}\right).$$

Equality in (8) holds if and only if $(M, g)$ is a spherical space form.
Moser-Trudinger type inequality: If $k = n/2$, then

\[(9) \quad (n - 2l)\mathcal{E}_{n/2}(g) \geq C_{MT} \left\{ \log \int_M \sigma_l(g)dg - \log \int_M \sigma_l(g_0)dg_0 \right\},\]

where

\[C_{MT} = \int_M \sigma_{n/2}(g_0)dg_0 = \frac{\omega_n}{2^n} \left( \frac{n}{2} \right)^n.\]

Equality holds in (9) if and only if $(M, g)$ is a space form. The above inequality is also true for $l > k = n/2$, provided $g \in \mathcal{C}_l$.

When $l = 0$ and $k = 1$, inequality (9) is the standard Sobolev inequality (e.g., see [2]). For $l = 0$ and $1 \leq k < \frac{n}{2}$, inequality (5) provides the optimal control of the $L^n$-norm of $e^{-u}$ in terms of $\mathcal{F}_k$. For these reasons, we call (5) a Sobolev type inequality. We were informed by Professor Alice Chang that Maria Del Mar González [11] obtained inequality (7) independently for $l = 0$. We suspect that (7) should be true on general compact manifolds. Inequality (5) plays a key role in the proof of a main result in [12].

Inequality (8) is reminiscent of the classical quermassintegral inequality (e.g., see [16] for a discussion), which provides one of the motivations for this paper. In the case $n = 4, k = 2$ and $l = 1$, inequality (8) was proved earlier by Gursky in [17] for general 4-dimensional manifolds. Some cases of the inequality were also verified in [16] and [14] for locally conformally flat manifolds.

Inequality (9) is similar to the Moser-Trudinger inequality on compact Riemannian surfaces (see [32], [26], and [27]). When $l = 0$, (9) was proved by Brendle-Viaclovsky and Chang-Yang in [4] and [8] using a result in [16] on a fully nonlinear conformal flow, and was referred to by them as the Moser-Trudinger inequality. We also refer to [3] for a different form of the Moser-Trudinger inequality in higher dimensions.

The Yamabe problem is the problem of finding a metric in the conformal class $[g_0]$ which minimizes the functional $\mathcal{F}_1(g)$. The final solution of the Yamabe problem by Aubin [2] and Schoen [28] is one of the triumphs of geometric analysis. There have been several important generalizations of Yamabe type problems in other settings such as the CR Yamabe problem studied by Jerison-Lee [22]. Viaclovsky [35] considered another type of Yamabe problem: for $1 \leq k \leq n$, find a metric $g \in [g_0]$ satisfying equation (3), or equivalently the following partial differential equation for the function $u$ in local orthonormal frames with respect to $g_0$:

\[(10) \quad \sigma_k \left( u_{ij} + u_iu_j - \frac{|\nabla u|^2}{2}\delta_{ij} + S_{ij} \right) = ce^{-2ku},\]

where $S_{ij}$ are the entries of $S_{g_0}, u_i$ and $u_{ij}$ are the first and second order covariant derivatives of $u$ with respect to the local orthonormal frames. When $k > 1$, equation (10) is fully nonlinear. There has been much recent activity surrounding equation (10). Viaclovsky [35] obtained some existence results for $k = n$ under certain geometric conditions. Equation (3) was solved by Chang-Gursky-Yang [6, 7] in the case $n = 4, k = 2$. Gursky-Viaclovsky studied the equation for the negative cone case in [19], and more recently for the lower dimensional cases $n = 3, 4$ in [15]. For the locally conformally flat manifolds, equation (3)
was completely solved by Guan-Wang [16] and Li-Li [24], and for more general conformally invariant equations by Guan-Lin-Wang [12] and Li-Li [25]. We remark that for $k \geq \frac{n}{2}$, $(M, g_0)$ is conformally equivalent to a space form by [14], so any solution of $\mathcal{F}_k$ is a maximizer of $\mathcal{F}_k$. But, for $k < \frac{n}{2}$, the previous solutions to equation (3) obtained in [16, 24] are not necessarily the Yamabe type solutions. That is, they are not necessarily minimizers of the functional $\mathcal{F}_k$. Similarly, neither are the solutions of (6) on a locally conformally flat manifold obtained in [12] and [25] necessarily minimizers of a corresponding functional.

One immediate consequence of Theorem 1 is the existence of a minimizer (or maximizer) solution of the general conformal quotient equation (6). For simplicity of notation, we will denote $\frac{\sigma_k(A)}{\sigma_l(A)}$ by $\sigma_{k,l}(A)$. By the transformation formula (2), equation (6) can be expressed in the fully nonlinear form:

$$\frac{\sigma_k}{\sigma_l} \left( u_{ij} + u_iu_j - \frac{\left| \nabla u \right|^2}{2} \delta_{ij} + S_{ij} \right) = ce^{-2(k-l)u}.$$  

Corollary 1. For any $0 \leq l < k$, there is an extremal metric $g_E \in \mathcal{C}_k$ satisfying equation (11) which is a global minimizer (maximizer) of $\mathcal{F}_k$ in $\mathcal{C}_k$ with $\int_M \sigma_l(g_E)dg_E = 1$ ($E_n/2(g_E) = 1$ if $l = n/2$) when $k < n/2$ ($k > n/2$); when $k = n/2$, there is a global minimizer of $E_{n/2}$.

We will prove Theorem 1 by studying an associated flow equation. The approach of using flows to study geometric inequalities was previously considered in different contexts by various authors in [1, 9, 38, 34]. In the conformal context, a simpler flow was introduced in [16]

$$\begin{cases} \frac{dg}{dt} = - (\log \sigma_k(g) - \log r_k(g)) \cdot g, \\
g(0) = g_0, \end{cases}$$  

where $r_k(g)$ is given by

$$r_k(g) = \exp \left( \frac{1}{\text{vol}(g)} \int_M \log \sigma_k(g) \, dg \right).$$  

The global existence, regularity and convergence for the flow (12) was proved there. For the purpose of this paper, we need to deal with the following general fully nonlinear flow:

$$\begin{cases} \frac{dg}{dt} = - \left( \log \frac{\sigma_k(g)}{\sigma_l(g)} - \log r_{k,l} \right) \cdot g, \\
g(0) = g_0, \end{cases}$$  

where

$$r_{k,l} = \exp \left( \frac{\int \sigma_l(g) \log (\sigma_k(g)\sigma_l^{-1}(g)) \, dg}{\int \sigma_l(g) \, dg} \right)$$

is defined so that the flow (13) preserves $\int \sigma_l(g) \, dg$ when $l \neq n/2$ and $E_{n/2}$ when $l = n/2$. We have the following result for the flow (13).
**Theorem 2.** For any $C^{4,\alpha}$ initial metric $g_0 \in \Gamma_k^+$, the flow (13) has a global solution $g(t) \in \mathcal{C}_k$ for any $t > 0$. Moreover, there is $h \in \mathcal{C}_k$ satisfying equation (10) such that,

$$\lim_{t \to \infty} \|g(t) - h\|_{C^{4,\alpha}(M)} = 0.$$ 

Flows (12) and (13) are fully nonlinear conformal equations. The first step is to obtain a Harnack inequality (i.e., the boundedness of $|\nabla g(t)|$, see (39) below). This can be done using a fundamental result of Schoen-Yau on developing maps and the method of moving planes as in [39]. This is where the assumption of locally conformally flat is used crucially. For the flow (12), since the volume of the evolved metric is preserved, $C^0$ boundedness is a direct consequence of the Harnack inequality (see [16] and [39]). However the flow (13) may not preserve the volume. This is the major difference between the flows (12) and (13). To deal with this problem we first use the Harnack inequality (39) to bound $|\nabla^2 g(t)|$. From there we obtain a $C^0$ bound using our previous results [16] and the local estimates in [15]. The preliminaries for this are developed in Section 2.

We will also need local estimates for equation (6) in the case $0 \leq l < k < \frac{n}{2}$. These estimates are needed to prove Theorem 1.A. The existence of such local estimates is a very special feature of this type of conformal fully nonlinear equation and is crucial for the study of the elliptic equation (6).

This paper is organized as follows. In Section 2, we list some basic facts regarding $\sigma_k$ and the flow (13) and prove Theorem 1 for the case $l = 0$. In Section 3, we establish some local estimates for equation (6), which will be used in a crucial way to obtain a positive lower bound for the geometric functionals. In Section 4, we study the flow (13) and prove Theorem 2. Theorem 1 for the case $l > 0$ will be proved in Section 5. In Section 6, we discuss the best constant, which appears in part (A) of Theorem 1.

**Notation:** In the rest of the paper, $u_i, u_{ij}, \cdots$ denote the covariant derivatives of the function $u$ with respect to some local orthonormal frames of the background metric $g_0$, unless it is otherwise indicated.

**Acknowledgement:** Part of this work was carried out while the second author was visiting Mathematics Department, McMaster University, in April 2002. He would like to thank the department for its hospitality. We would also like to thank Eric Sawyer and the referees for valuable suggestions regarding the presentation of this paper.

2. **Some basic facts**

Let $\Lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$. The $k$th elementary symmetric function is defined as

$$\sigma_k(\lambda_1, \ldots, \lambda_n) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}.$$ 

A real symmetric $n \times n$ matrix $A$ is said to lie in $\Gamma_k^+$ if its eigenvalues lie in $\Gamma_k^+$. Let $A_{ij}$ be the $\{i, j\}$-entry of an $n \times n$ matrix. Then for $0 \leq k \leq n$, the $k$th *Newton transformation* associated with $A$ is defined to be

$$T_k(A) = \sigma_k(A) I - \sigma_{k-1}(A) A + \cdots + (-1)^k A^k.$$
We have

$$T_k(A)^l_j = \frac{1}{k!} \delta_i^{j_1\ldots j_k} A_{i_1j_1} \cdots A_{i_kj_k},$$

where $\delta_i^{j_1\ldots j_k}$ is the generalized Kronecker delta symbol. Here we use the summation convention. By definition,

$$\sigma_k(A) = \frac{1}{k!} \delta_i^{j_1\ldots j_k} A_{i_1j_1} \cdots A_{i_kj_k}, \quad T_k(A)^l_j = \frac{\partial \sigma_k(A)}{\partial A_{ij}}.$$ 

For $0 < l < k \leq n$, let

$$\tilde{T}_{k-1,l-1}(A) = \frac{T_{k-1}(A)}{\sigma_k(A)} - \frac{T_{l-1}(A)}{\sigma_l(A)}.$$ 

It is important to note that if $A \in \Gamma_k^+$, then $\tilde{T}_{k-1,l-1}(A)$ is positive definite. This follows the fact that the function $\frac{\sigma_k(A)}{\sigma_l(A)}$ is monotonic in $\Gamma_k^+$, that is if $B - A$ is semi-positive, then $\frac{\sigma_k(A)}{\sigma_l(A)} \leq \frac{\sigma_k(B)}{\sigma_l(B)}$. Another important fact is that $(\frac{\sigma_k(A)}{\sigma_l(A)})^{\frac{l}{k-l}}$ is concave in $\Gamma_k^+$ (e.g., see [23]). Let $\Lambda_i = (\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n)$.

**Lemma 1** (Newton-MacLaurin Inequality [20]).

$$l(n-k+1)\sigma_l(\Lambda)\sigma_{k-1}(\Lambda) \geq k(n-l+1)\sigma_k(\Lambda)\sigma_{l-1}(\Lambda).$$

**Lemma 2** (Garding’s Inequality). Let $\Lambda = (\lambda_1, \cdots, \lambda_n), \Lambda_0 = (\mu_1, \cdots, \mu_n) \in \Gamma_k^+$,

$$F(\Lambda) = \left(\frac{\sigma_k(\Lambda)}{\sigma_l(\Lambda)}\right)^{\frac{l}{k-l}}.$$ 

Then,

$$\sum_i \left\{ \frac{\sigma_{k-1}(\Lambda_i)}{\sigma_k(\Lambda)} - \frac{\sigma_{l-1}(\Lambda_i)}{\sigma_l(\Lambda)} \right\} \mu_i \geq (k-l) \frac{F(\Lambda_0)}{F(\Lambda)}.$$ 

**Proof.** The main argument of the proof follows from [5]. From the concavity of $F$ in $\Gamma_k^+$, for $\Lambda, \Lambda_0 = (\mu_1, \cdots, \mu_n) \in \Gamma_k^+$ we have

$$F(\Lambda_0) \leq F(\Lambda) + \sum_i (\mu_i - \lambda_i) \frac{\partial F(\Lambda)}{\partial \lambda_i}$$ 

$$= F(\Lambda) + \frac{1}{k-l} F(\Lambda) \sum_i \left\{ \frac{\sigma_{k-1}(\Lambda_i)}{\sigma_k(\Lambda)} - \frac{\sigma_{l-1}(\Lambda_i)}{\sigma_l(\Lambda)} \right\} (\mu_i - \lambda_i)$$ 

$$= \frac{1}{k-l} F(\Lambda) \sum_i \left\{ \frac{\sigma_{k-1}(\Lambda_i)}{\sigma_k(\Lambda)} - \frac{\sigma_{l-1}(\Lambda_i)}{\sigma_l(\Lambda)} \right\} \mu_i.$$ 

In the last equality, we have used the fact that $F$ is homogeneous of degree one. Then (15) follows. 

**Lemma 3.** A conformal class of metric $[g]$ with $[g] \cap \Gamma_k^+ \neq \emptyset$ does not have a $C^{1,1}$ metric $g_1 \in \Gamma_k^+$ with $\sigma_k(g_1) = 0$, where $\Gamma_k^+$ is the closure of $\Gamma_k^+$. 

Proof. By the assumption, there is a smooth admissible metric \( g_0 \) with \( \sigma_k(g) > 0 \). Assume by contradiction that there is a \( C^{1,1} \) metric \( g_1 \) with \( \sigma_k(g_1) = 0 \). Write \( g_1 = e^{-2u}g_0 \), so \( u \) satisfies

\[
\sigma_k \left( \nabla^2 u + du \otimes du - \frac{\nabla u^2}{2}g_0 + S_{g_0} \right) = 0. \tag{16}
\]

Let

\[
W = \left( \nabla^2 u + du \otimes du - \frac{\nabla u^2}{2}g_0 + S_{g_0} \right) \quad \text{and} \quad a_{ij}(W) = \frac{\partial \sigma_k(W)}{\partial w_{ij}}.
\]

(Here, the notation \( a_{ij}(W) = \frac{\partial \sigma_k(W)}{\partial w_{ij}} \) is usually used by analysts. More intrinsically, \( a_{ij} = T_{k-1}(W)^i_j \), where \( T_{k-1}(W) \) is the \( (k - 1) \)th Newton transformation defined above and \( W \) is viewed as the matrix \( g_0^{-1} \cdot W \).) We may assume \( u \leq 1 \) and \( u(x_0) = 1 \) for some \( x_0 \in M \), since \( u + c \) also satisfies (16) for any constant \( c \). Let \( v = e^{-u} - e^{-1} \), \( h_t = te^{-u} + (1-t)e^{-1} \), \( u_t = -\log h_t \) and \( W_t = \nabla^2 u_t + du_t \otimes du_t - \frac{\nabla u_t^2}{2}g_0 + S_{g_0} \). As in \([37]\), one can check that (a) \( W_t \in \Gamma^+_k \) and (b) \( \langle a_{ij}(W_t) \rangle \) positive definite (nonnegative definite, resp.) for all \( 0 \leq t < 1 \) \((0 \leq t \leq 1 \) resp.). We have the following

\[
0 > \sigma_k(W_1) - \sigma_k(W_0) = \sum_{ij} \left( \int_{t=0}^{1} \frac{a_{ij}(W_t)}{h_t^2} dt \right) \nabla^2 v + \sum_l b^l(t,x) \nabla^l v + dv, \tag{17}
\]

for some bounded functions \( d \) and \( b^l \), \( l = 1, \ldots, n \). Since \( v \geq 0 \) and \( v(x_0) = 0 \), this is a contradiction to the strong maximum principle. \( \blacksquare \)

For any \( 0 < k \leq n \), let

\[
\tilde{\mathcal{F}}_k(g) = \begin{cases} 
\frac{1}{n-2k} \int_M \sigma_k(g) dg, & k \neq n/2, \\
\epsilon_{n/2}(g), & k = n/2.
\end{cases} \tag{18}
\]

**Lemma 4.** Flow \((13)\) preserves \( \tilde{\mathcal{F}}_1 \), it also decreases the functional \( \tilde{\mathcal{F}}_k \). In fact, we have the following evolution identities. The evolution equations for \( \log \frac{\sigma_k}{\sigma_l} \) and \( \tilde{\mathcal{F}}_k \) are

\[
\frac{d}{dt} \log \frac{\sigma_k}{\sigma_l}(g) = \frac{1}{2} \text{tr} \{ \tilde{T}_{k-1,l-1}(S_g) \nabla^2_g \log \frac{\sigma_k}{\sigma_l}(g) \} + (k-l)(\log \frac{\sigma_k}{\sigma_l}(g) - \log r_{k,l}) \tag{19}
\]

and

\[
\frac{d}{dt} \tilde{\mathcal{F}}_k(g) = -\frac{1}{2} \int_M \left( \frac{\sigma_k}{\sigma_l}(g) - r_{k,l} \right) \left( \log \frac{\sigma_k}{\sigma_l}(g) - \log r_{k,l} \right) \sigma_l(g) dg. \tag{20}
\]

**Proof.** It is easy to verify \( \frac{d}{dt} \tilde{\mathcal{F}}_k(g) = 0 \) from the equation and the definition of \( r_{k,l} \). The first equality follows the same argument in \([16]\). We verify the second equality. When \( k \neq \frac{n}{2} \).
\[
\frac{d}{dt} \tilde{F}_k(g) = \frac{1}{2} \int_M \sigma_k(g) g^{-1} \frac{d}{dt} g \, dg \\
= -\frac{1}{2} \int_M \left( \frac{\sigma_k}{\sigma_l}(g) - r_{k,l} \right) \left( \log \frac{\sigma_k}{\sigma_l}(g) - \log r_{k,l} \right) \sigma_l(g) \, dg.
\]

The first equality was proven in [35], where the assumption of locally conformally flat is used. By [4], the above also holds for \( k = \frac{n}{2} \).

In the rest of this section we use results obtained in [16] for flow (12) to establish Theorem 1 in the case \( l = 0 \).

**Proposition 1.** Let \((M, g_0)\) be a locally conformally flat manifold with \( g_0 \in \Gamma^+_k \). We have

(a). When \( k > \frac{n}{2} \), there is a constant \( C_Q = C_Q(n, k) > 0 \) depending only on \( n \) and \( k \) such that for any metric \( g \in C_k \).

\[
\int_M \sigma_k(g) \text{vol}(g) \leq C_Q \text{vol}(g)^{\frac{n-2k}{n}}.
\]

(b). When \( k < \frac{n}{2} \), there is a constant \( C_S = C_S([g_0], n) > 0 \) such that for any metric \( g \in C_k \).

\[
\int_M \sigma_k(g) \text{vol}(g) \geq C_S \text{vol}(g)^{\frac{n-2k}{n}}.
\]

(c). If \( k = \frac{n}{2} \) and \( g_0 \) is a metric of constant sectional curvature, then for any \( g \in C_k \)

\[\mathcal{E}_{n/2}(g) \geq \frac{1}{n} C_M (\log \text{vol}(g) - \log \text{vol}(g_0)),\]

where \( C_M = \int_M \sigma_{n/2}(g_0) \, dg_0 \).

Moreover, in cases (a) and (c) the equality holds if and only if \( g \) is a metric of constant sectional curvature.

**Proof.** (c) was already proved in [4].

When \( k > \frac{n}{2} \), from [14] we know that \((M, g_0)\) is conformally equivalent to a spherical space form. In this case, it was proved in [35] that any solution of (13) for \( l = 0 \) is of constant sectional curvature. By Theorem 1 in [16] \((k > \frac{n}{2})\), for any \( g \in C_k \) there is a metric \( g_e \in C_k \) of constant sectional curvature with \( \text{vol}(g) = \text{vol}(g_e) \) and

\[
\tilde{F}_k(g) \geq \tilde{F}_k(g_e).
\]

(21)

When \( k > \frac{n}{2} \), (21) implies that

\[
\text{vol}(g)^{-\frac{n-2k}{n}} \int_M \sigma_k(g) \, dg \leq \text{vol}(g_e)^{-\frac{n-2k}{n}} \int_M \sigma_k(g_e) \, dg_e,
\]

and the equality holds if and only if \((M, g)\) is a space form. It is clear that

\[
\text{vol}(g_e)^{-\frac{n-2k}{n}} \int_M \sigma_k(g_e) \, dg_e
\]

is a constant depending only on \( n, k \). This proves (a).
It remains to prove (b). For this case, we only need to prove that
\[ \inf_{C_k \cap \{ \text{vol}(g) = 1 \}} F_k(g) =: \beta_0 > 0. \]

Assume by contradiction that \( \beta_0 = 0 \). By Theorem 1 in [16], we can find a sequence of solutions \( g_i = e^{-2u_i} g_0 \in C_k \) of (3) with \( \text{vol}(g_i) = 1 \) and \( \sigma_k(g_i) = \beta_i \) such that \( \lim_{i \to \infty} \beta_i = 0 \).

\[ \sigma_k(\nabla^2 u_i + du_i \otimes du_i - \frac{|\nabla u_i|^2}{2} g_0 + S_{g_0}) = \beta_i e^{-2u_i}. \]

Consider the scaled metric \( \tilde{g}_i = e^{-2\tilde{u}_i} g_0 \) with \( \tilde{u}_i = u_i - \frac{1}{2k} \log \beta_i \), which satisfies clearly that
\[ \sigma_k \left( \nabla^2 \tilde{u}_i + d\tilde{u}_i \otimes d\tilde{u}_i - \frac{|\nabla \tilde{u}_i|^2}{2} g_0 + S_{g_0} \right) = e^{-2k\tilde{u}_i} \]
and
\[ \text{vol}(\tilde{g}_i) = \beta_i \frac{2}{\sqrt{e}} \to 0 \quad \text{as} \quad i \to \infty. \]

By Corollary 1.2 in [15], we conclude that
\[ \tilde{u}_i \to +\infty \text{ uniformly as} \quad i \to \infty. \]

Hence \( m_i := \inf_M \tilde{u}_i \to +\infty \text{ as} \quad i \to \infty. \) Now at the minimum point \( x_i \) of \( \tilde{u}_i \), by (22),
\[ \sigma_k(S_{g_0}) \leq \sigma_k \left( \nabla^2 \tilde{u}_i + d\tilde{u}_i \otimes d\tilde{u}_i - \frac{|\nabla \tilde{u}_i|^2}{2} g_0 + S_{g_0} \right) = e^{-2km_i} \to 0. \]

This is a contradiction to the fact \( g_0 \in \Gamma^+_k \).

3. Local estimates for conformal quotient equations

In order to get a positive lower bound for the constant in (A) of Theorem 1, as in the proof of Proposition 1 in the previous section, we prove some estimates for solutions of
\[ \frac{\sigma_k}{\sigma_l}(g) = f, \]
defined locally in any open subset of \( M \) with a nonnegative \( C^2 \) function \( f \). When \( l = 0 \) local estimates were established in [15]. The same estimates hold for equation (24) when \( 0 \leq l < k \leq n \). Since the proof for the special case \( (n - k + 1)(n - l + 1) > 2(n + 1) \) is much simpler and it is suffice for the purpose of this paper, we only treat this case here. The local estimates for equation (24) in general case \( 0 \leq l < k \leq n \) will appear elsewhere. We emphasize that, in this section, we do not assume \( (M, g_0) \) is locally conformally flat.

**Theorem 3.** Let \( (M, g_0) \) be a smooth compact, \( n \)-dimensional manifold and let \( g = e^{-2u} g_0 \) be a \( C^4 \) solution of (24) in \( B_r \) with
\[ (n - k + 1)(n - l + 1) > 2(n + 1). \]
Then there exist two positive constants $c_1$ (depending only on $\|g_0\|_{C^3(B_r)}$, $n, k, l$ and $\|f\|_{C^1(B_r)}$) and $c_2$ (depending only on $\|g_0\|_{C^4(B_r)}$, $n, k, l$ and $\|f\|_{C^2(B_r)}$) such that $\forall x \in B_r^-$,

\begin{align}
\|\nabla u(x)\|^2 &\leq c_1(1 + e^{-2\inf_{B_r} u}) \\
\|\nabla^2 u(x)\| &\leq c_2(1 + e^{-2\inf_{B_r} u}).
\end{align}

In particular, if $k < \frac{n}{2}$, condition (26) holds, and hence inequalities (26) and (27) are valid in this case.

Theorem 3 has the following consequence.

**Corollary 2.** Let $k, l$ as in Theorem 3. There exists a constant $\varepsilon_0 > 0$ such that for any sequence of $C^{4,\alpha}$ solutions $u_i$ of (24) in $B_r$ with

$$
\int_{B_1} e^{-nu} \text{vol}(g_0) \leq \varepsilon_0,
$$

either

1. There is a subsequence $u_{i_1}$ uniformly converging to $+\infty$ in any compact subset in $B_r$, or
2. There is a subsequence $u_{i_1}$ converges strongly in $C^{4,\alpha}_{\text{loc}}(B_r)$. If $f$ is smooth and strictly positive in $B_r$ and $u_i$ is smooth, then $u_{i_1}$ converges strongly in $C^m_{\text{loc}}(B_r)$, $\forall m$.

The proof of Corollary 2 follows from the same lines in the proof of Corollary 1.2 in [15], the same argument can be traced back to Schoen [29], we will not repeat it here.

**Proof of Theorem 3.** The proof in [15] works for (27) as $(\frac{\sigma_k}{\sigma_l})^{\frac{1}{k-l}}$ is elliptic and concave in $\Gamma^+_k$. We only need to prove (26).

We follow the same lines of proof in [15] to prove the Theorem, together with Lemma 5 proven at the end of this section. For convenience of the reader, we will use the same notations as in [15].

Without loss of generality, we may assume $r = 1$. Let

$$
F = \left(\frac{\sigma_k}{\sigma_l}(W)\right)^{\frac{1}{k-l}} \quad \text{and} \quad F^{ij} = \frac{\partial F}{\partial w_{ij}},
$$

where $W = (\nabla^2 u + du \otimes du - \frac{\|\nabla u\|^2}{2} g_0 + S_{g_0})$. More intrinsically,

$$
F^{ij} = \frac{1}{k-l} \left(\frac{\sigma_k}{\sigma_l}(W)\right)^{\frac{1}{k-l}-1} \frac{1}{\sigma_l^2(W)} \left\{\sigma_l(W)T_{k-1}(W)^{ij} - \sigma_k(W)T_{l-1}(W)^{ij}\right\},
$$
where $T_j$ is the $j$th Newton transformation. Let $\rho$ be a test function $\rho \in C_0^\infty(B_1)$ such that
\[ \rho \geq 0, \text{ in } B_1 \quad \text{and} \quad \rho = 1, \text{ in } B_{1/2}, \]
\[ |\nabla \rho(x)| \leq 2b_0 \rho^{1/2}(x) \quad \text{and} \quad |\nabla^2 \rho| \leq b_0, \text{ in } B_1. \]

Here $b_0 > 1$ is a constant depending only on the background metric $g_0$. Set $H = \rho |\nabla u|^2$. It suffices to bound $\max_{B_1} H$. Let $x_0 \in B_1$ be a maximum point of $H$. We may assume that $W$ is diagonal at the point $x_0$ by choosing a suitable normal coordinates around $x_0$. Set $\lambda_i = w_{ii}$ and $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$. In what follows, all computations are given at the point $x_0$. We denote $\Lambda_i$ the vector with $i$th component deleted from $\Lambda$ and
\[ F^s = \frac{1}{k-l} \left( \frac{\sigma_k(\Lambda)}{\sigma_l(\Lambda)} \right)^{\frac{1}{k-l}-1} \frac{1}{\sigma_l^2(\Lambda)}. \]

At the point $x_0$, $(F^{ij})$ is diagonal and
\[ F^{ii} = F^s \{ \sigma_l(\Lambda)\sigma_{k-1}(\Lambda_i) - \sigma_k(\Lambda)\sigma_{l-1}(\Lambda_i) \}. \]

Since $W$ is diagonal at $x_0$, we have at $x_0$
\[ w_{ii} = u_{ii} + u_i^2 - \frac{1}{2} |\nabla u|^2 + S_{ii}, \quad u_{ij} = -u_iu_j - S_{ij}, \quad \forall i \neq j, \]
where $S_{ij}$ are entries of $S_{g_0}$.

We may assume that
\[ H(x_0) \geq b_0^2 A_0^2, \]
for some constant $A_0$ will be fixed later. We may also assume that
\[ |S_{g_0}|(x_0) \leq A_0^{-1} |\nabla u|^2(x_0). \]

Otherwise, we are done. Since $x_0$ is the maximum point of $H$, we can get, for any $i$,

\[ \left| \sum_{l=1}^n u_{il}u_l \right|(x_0) \leq \frac{|\nabla u|^3}{A_0}(x_0). \]

By applying maximum principle and following the same deduction in [15] (formulas (2.18) and (2.20) there), we have
\[ 0 \geq F^{ij} H_{ij} = F^{ij} \left\{ \left( -2 \frac{\rho u_{ij}}{\rho} + \rho_{ij} \right) |\nabla u|^2 + 2\rho u_{ij}u_l + 2\rho u_{il}u_{jl} \right\} \]
\[ \geq 2 \sum_{i,j,l} F^{ij} \rho u_{il}u_{jl} + \sum_{j \geq 1} F^{jj} \left\{ -10b_0^2 |\nabla u|^2 - Ce^{-\inf_{B_1} u} |\nabla u|^2 - \frac{(n + 2)^2}{A_0} \rho |\nabla u|^4 \right\} \]

where $C$ is a constant depending only on $n, k, l$, $\|f\|_{C^1(B_1)}$ and $\|g\|_{C^3(B_1)}$. Set $\tilde{u}_{ij} = u_{ij} + S_{ij}$. By [31], we have
\[ \sum_{i,j,l} F^{ij} u_{il}u_{jl} \geq \frac{1}{2} \sum_{i,l} F^{ii} \tilde{u}_{il}^2 - \frac{n}{A_0^2} |\nabla u|^4 \sum_i F^{ii}. \]
This, together with (33), implies
\[
\sum_{i,l} F^{il} \bar{u}_{il}^2 \leq \sum_{j \geq 1} F^{jj} \left\{ 10n b_0^2 |\nabla u|^2 + Ce^{-\inf \beta_i} u |\nabla u|^2 + \frac{2(n+2)^2}{A_0} \rho |\nabla u|^4 \right\}.
\]

Hence, the needed bound \( H(x_0) \leq c_1 (1 + e^{-2 \inf \beta_i} u) \) can be reduced to the verification of the following:

Claim: There is a constant \( A_0 \) (depending only on \( k \) and \( n \)) such that
\[
(34) \sum_{i,j} F^{ij} \bar{u}_{ij}^2 \geq A_0^{-\frac{2}{9}} \sum_{i} F^{ii} |\nabla u|^4.
\]

This claim is just Claim 2.5 in [15]. As in [15], the verification of the Claim is a crucial step to prove the Theorem.

Set \( \delta_0 = A_0^{-1/4} < 0.1 \). We divide the set \( I = \{1, 2, \cdots, n\} \) as in [15] into two parts:
\[
I_1 = \{ i \in I \mid u_i^2 \geq \delta_0 |\nabla u|^2 \} \quad \text{and} \quad I_2 = \{ i \in I \mid u_i^2 < \delta_0 |\nabla u|^2 \}.
\]

Clearly, \( I_1 \) is non-empty.

Case 1. There is \( j_0 \) satisfying
\[
(35) \bar{u}_{jj}^2 \leq \delta_0^2 |\nabla u|^4 \quad \text{and} \quad u_j^2 < \delta_0 |\nabla u|^2.
\]

We may assume that \( j_0 = n \). We have \(|w_{nn} + \frac{|\nabla u|^2}{2}| = |\bar{u}_{nn} + u_n^2| < 2\delta_0 |\nabla u|^2\) by (33).

From (32) and (30), we have for any \( i \in I \)
\[
|u_i (u_{ii} - (|\nabla u|^2 - u_i^2)) - \sum_l S_{il} u_l| = \left| \sum_l u_{il} u_l \right| \leq \delta_0^4 |\nabla u|^3.
\]

This, together with (31), implies
\[
(36) |u_i (u_{ii} - (|\nabla u|^2 - u_i^2))| \leq 2\delta_0^4 |\nabla u|^3,
\]

which, in turn, implies
\[
|w_{ii} - \frac{|\nabla u|^2}{2}| = |u_{ii} + u_i^2 - |\nabla u|^2 + S_{ii}| \leq 3\delta_0^2 |\nabla u|^2,
\]

for any \( i \in I_1 \). Using these inequalities, we can repeat the derivation of equation (2.38) in [15] to obtain
\[
(37) \sum_{i,l} F^{ii} \bar{u}_{il}^2 \geq \tilde{F}^1 \frac{|\nabla u|^4}{4} - \tilde{F}^1 |\nabla u|^4 + F^{nn} \frac{|\nabla u|^4}{4} + (1 - 32\delta_0^2) \frac{|\nabla u|^4}{4} \sum_i F^{ii},
\]

where \( \tilde{F}^1 = \max_{i \in I_1} F^{ii} \).

Recall that \( I_1 \) is necessarily non-empty. We may assume that \( i_0 \in I_1 \) with \( \tilde{F}^1 = F^{i_0 i_0} \). Without loss of generality, we assume that \( i_0 = 1 \). From above, we know \( w_{11} \geq w_{nn} \). By
Lemma 5 below, we have \( F_{nn} \geq \tilde{F}^{11} \) and \( \tilde{F}^{11} \leq \frac{1}{2 + c(n, k, l)} \sum_l F_l \), because \( w_{11} > 0 \). Hence, \( \delta_0^2 \) implies that
\[
\sum_{i,l} F_{ii} \tilde{u}_{il}^2 \geq \left\{ \frac{1}{4} - \frac{1}{2(2 + c(n, k, l))} - 32\delta_0^2 \right\} \sum_i F_{ii} |\nabla u|^4.
\]

The inequality (34) then follows for this case by choosing \( A_0 \geq \left( \frac{128(2 + c(n, k, l))}{c(n, k, l)} \right)^2 \).

**Case 2.** There is no \( j \in I \) satisfying (35).

We may assume that there is \( i_0 \) such that \( \tilde{u}_{i_0 i_0}^2 \leq \delta_0^2 |\nabla u|^4 \), otherwise the claim is automatically true. Assume \( i_0 = 1 \). As in Case 4 in [15], we have \( u_{11}^2 \geq (1 - 2\delta_0)|\nabla u|^2 \) and \( \tilde{u}_{jj}^2 + u_{j}^2 (|\nabla u|^2 - u_{j}^2) \geq \delta_0^2 |\nabla u|^4 \) for \( j > 1 \). From equation (2.50) in [15] and Lemma 5, we have
\[
\sum_{i,l} F_{ii} \tilde{u}_{il}^2 = \sum_i F_{ii} \left\{ \tilde{u}_{ii}^2 + u_{i}^2 (|\nabla u|^2 - u_{i}^2) \right\}
\geq \sum_{i \geq 2} F_{ii} (\tilde{u}_{ii}^2 + u_{i}^2 (|\nabla u|^2 - u_{i}^2))
\geq \delta_0^2 |\nabla u|^4 \sum_{i \geq 2} F_{ii} \geq \frac{1}{2} \delta_0^2 |\nabla u|^4 \sum_{i \geq 1} F_{ii}
\]

The Claim is verified, so the local gradient estimate (26) is proved. \( \square \)

**Lemma 5.** If \( \lambda_i \leq \lambda_j \), then \( F_{ii} \geq F_{jj} \). If \((n - k + 1)(n - l + 1) > 2(n + 1)\) and \( \lambda_1 > 0 \), then there is a positive constant \( c(n, k, l) \) such that
\[
\sum_i F_{ii} \geq (2 + c(n, k, l)) F^{11}.
\]

**Proof.** The first statement follows from the monotonicity of \( \sigma_{l-1} \) and \( \frac{\sigma_k}{\sigma_{l-1}} \), since
\[
F_{ii} = F^* \sigma_{l-1} \{ \sigma_1 (\Lambda_i) \frac{\sigma_{k-1}}{\sigma_{l-1}} (\Lambda_i) - \sigma_k (\Lambda) \}.
\]

Similarly, we have
\[
F^{11} = F^* \sigma_{l-1} \{ \sigma_1 (\Lambda_1) \frac{\sigma_{k-1}}{\sigma_{l-1}} (\Lambda_1) - \sigma_k (\Lambda) \}
\leq F^* \sigma_{l-1} \{ \sigma_1 (\Lambda) \frac{\sigma_{k-1}}{\sigma_{l-1}} (\Lambda) - \sigma_k (\Lambda) \}.
\]

From the Newton-MacLaurin inequality (14),
\[
q_0 := (n - k + 1 - \alpha_0) \sigma_1 (\Lambda) \sigma_{k-1} (\Lambda) - (n - l + 1 - \alpha_0) \sigma_k (\Lambda) \sigma_{l-1} (\Lambda) \geq 0,
\]

\[
q_0 \geq 0.
\]
where $\alpha_0 = \frac{(n-k+1)(n-l+1)}{n+1}$. Hence, we have

$$\sum_i F^{ii} = F^* \{ (n-k+1)\sigma_k(\Lambda)\sigma_{k-1}(\Lambda) - (n-l+1)\sigma_k(\Lambda)\sigma_{l-1}(\Lambda) \}$$

$$= F^* \{ q_0 + \alpha_0[s_l(\Lambda)\sigma_{k-1}(\Lambda) - \sigma_k(\Lambda)\sigma_{l-1}(\Lambda)] \}$$

$$\geq \alpha_0 F^{11}.$$ 

The Lemma follows with $c(n, k, l) = 2 - \alpha_0 > 0$. 

4. THE GLOBAL CONVERGENCE OF FLOW (13)

We treat flow (13) in this section. If $g = e^{-2u} \cdot g_0$, one may compute that

$$\sigma_k(g) = e^{2ku} \sigma_k \left( \nabla^2 u + du \otimes du - \frac{\vert \nabla u \vert^2}{2} g_0 + S_{g_0} \right).$$

Equation (13) can be written in the following form

$$2 \frac{du}{dt} = \log \frac{\sigma_k}{\sigma_l} \left( \nabla^2 u + du \otimes du - \frac{\vert \nabla u \vert^2}{2} g_0 + S_{g_0} \right) + 2(k-l)u - \log r_{k,l}(g)$$

$$u(0) = u_0.$$ 

In these equations and below, the norms and covariant derivatives are with respect to the background metric $g_0$.

Because $g_0 \in \Gamma_k^+$, the highest order term on the right hand side is uniformly elliptic. Consequently, the short time existence of flow (13) follows from the standard parabolic theory. We want to prove the long time existence and convergence. Let

$$T^* = \sup \{ T_0 > 0 \mid (13) \text{ exists in } [0, T_0] \text{ and } g(t) \in C_k \text{ for } t \in [0, T_0] \}.$$ 

For any $T < T^*$, we will establish a $C^2$ bound for the conformal factor $u$, which is independent of $T$. As in [16], we have a Harnack inequality

$$\nabla u \leq c,$$

for some positive constant $c$ independent of $T$, see a complete proof in [39]. But, as mentioned above, we can not deduce $C^0$ boundedness as in [16] because the flow (13) may not preserve the volume of the evolved metric $g(t)$. In order to to get $C^1$ bound we first use (39) to bound $\nabla^2 u$.

**Proposition 2.** There is a constant $C > 0$ independent of $T$ such that

$$\nabla^2 u \leq C.$$ 

**Proof.** Set

$$F = \log \frac{\sigma_k}{\sigma_l} \left( \nabla^2 u + du \otimes du - \frac{\vert \nabla u \vert^2}{2} g_0 + S_{g_0} \right).$$
By equation (38), 
\[ F = 2u_t - 2(k-l)u - \log r_{k,l} \]
We only need to consider the case \( k > 1 \).

From the property of \( \Gamma_2 \),
\[ |\lambda_i| \leq \max\{1, (n-2)\} \sum_{j=1}^{n} \lambda_j. \]

Therefore we only need to give a upper bound of \( \Delta u \) which dominates all other second order derivatives.

Consider \( G = \Delta u + m|\nabla u|^2 \) on \( M \times [0, T] \), where \( m \) is a large constant which will be fixed later. Note that \( \Delta u \) is analyst’s Laplacian. Without loss of generality, we may assume that the maximum of \( G \) on \( M \times [0, T] \) achieves at a point \( (x_0, t_0) \in M \times (0, T) \) and \( G(x_0, t_0) \geq 1 \). We may assume that at \( (x_0, t_0) \)

\[ 2\sigma_1(W) \geq G \geq \frac{1}{2}\sigma_1(W), \]

where \( W = \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g_0 + S_{g_0} \). Consider everything in a small neighborhood near \( x_0 \). We may consider \( W \) as a matrix with entry
\[ w_{ij} = u_{ij} + u_i u_j - \frac{1}{2}|\nabla u|^2 \delta_{ij} + S(g_0)_{ij}. \]
In the rest of the proof, \( c \) denotes a positive constant independent of \( T \), which may vary from line to line.

Since \( G \) achieves its maximum at \( (x_0, t_0) \), we have at this point

\[ G_t = \sum_l (u_{llt} + 2mu_{lt}u_t) \geq 0, \]
and
\[ G_i = \sum_l (u_{lli} + 2mu_{li}u_l) = 0, \quad \forall \ i. \]

(41) and (39) imply that at \( (x_0, t_0) \)

\[ |\sum_l u_{li}t| \leq cG. \]

By the Harnack inequality (39) and the fact \( |u_{ij}| \leq G \), we may assume that

\[ |u_{lij} - u_{ijl}| < c \quad \text{and} \quad |u_{ijkl} - u_{ijlk}| < cG, \]

where \( c \) is a constant depending only on the background metric \( g_0 \). We may assume by choosing appropriate local orthonormal frames that the matrix \( (w_{ij}) \) at \( (x_0, t_0) \) is diagonal. At the maximum point, \( G_{ij} \) is non-positive definite. Set \( F^{ij} = \frac{\partial F}{\partial w_{ij}} \). Since \( g(t) = e^{-2u(t)} g_0 \in \Gamma_k^+ \), we know that the matrix \( (F^{ij}) \) is positive. Hence in view of
and the concavity of $F$ we have
\[
0 \geq \sum_{i,j} F^{ij} G_{ij} = \sum_{i,j} F^{ij} (u_{iij} + 2mu_{iij}u_j + 2mu_{ij}u_i)
\geq \sum_{i,j,l} F^{ijl} (u_{ijl} + 2mu_{ijl}u_l - 2mu_{ilj}u_j - \frac{1}{2} |\nabla u|^2 \delta_{ij} + S(g_{ij})_{ij})
\geq -c \sum_{i} F^{ii} G + \sum_{i,j,l} F^{ijl} \{w_{ijl} - (u_{ijl} - \frac{1}{2} |\nabla u|^2 \delta_{ij} + S(g_{ij}))_{ij}\}
\geq \sum_{i} F^{ii} G + 2m \sum_{l} F_{il} u_l + \sum_{i} F^{ii} u_{i}^2 - c \sum_{i} F^{ii} G
\geq \sum_{i} F^{ii} G + 1_n G^2 \sum_{i} F^{ii} u_{i}^2 - c \sum_{i} F^{ii} G.
\]
From equation (38), \( F = 2u_t - 2(k-l)u - \log r_{k,l}(g) \). In view of (41) and (42), (45) yields
\[
0 \geq -2(k-l)G + \frac{1}{n} \sum_{i} F^{ii} G^2 + 2(m-1) \sum_{i} F^{ii} u_{i}^2 - c \sum_{i} F^{ii} G
\geq -2(k-l)G + 2(m-1) \sum_{i} F^{ii} u_{i}^2 - c \sum_{i} F^{ii} G
\geq \{-2(k-l)G + 2(m-1) \sum_{i} F^{ii} u_{i}^2\} + \frac{1}{n} \sum_{i} F^{ii} (G^2 - cG).
\]
The Proposition follows from (39), (46) and Lemma 6 below.

**Lemma 6.** There is a large constant $m > 0$ such that
\[
\frac{1}{2n} G^2 \sum_{i} F^{ii} + 2(m-1) \sum_{i} F^{ii} u_{i}^2 \geq 2(k-l)G.
\]

**Proof.** It is easy to check, from the Newton-MacLaurin inequality (14), that
\[
\sum_{i} F^{ii} u_{i}^2 = \frac{\sigma_1(W)\sigma_k(W) - (k+1)\sigma_{k+1}(W)}{\sigma_k(W)} = \frac{\sigma_1(W)\sigma_l(W) - (l+1)\sigma_{l+1}(W)}{\sigma_l(W)}
\geq (l+1)\frac{\sigma_{l+1}(W)}{\sigma_l(W)} - (k+1)\frac{\sigma_{k+1}(W)}{\sigma_k(W)} \geq c_{n,k,l} \frac{\sigma_{l+1}(W)}{\sigma_l(W)};
\]
and
\[
\sum_{i} F^{ii} = (n-k+1)\frac{\sigma_{k-1}(W)}{\sigma_k(W)} - (n-l+1)\frac{\sigma_{l-1}(W)}{\sigma_l(W)} \geq \tilde{c}_{n,k,l} \frac{\sigma_{l-1}(W)}{\sigma_l(W)};
\]
where \( c_{n,k,l} \) and \( \tilde{c}_{n,k,l} \) are two positive constant depending only on \( n, k \) and \( l \). From these two facts, we can prove the claim as follows. First, if
\[
\frac{\tilde{c}_{n,k,l}}{4n} \frac{\sigma_1(W) \sigma_{k-1}(W)}{\sigma_k(W)} \geq 4(k-l),
\]
then the claim follows from (49) and (40). Hence we may assume that
\[
\frac{\sigma_1(W) \sigma_{k-1}(W)}{\sigma_k(W)} \leq c^*_{n,k,l},
\]
for some positive constant \( c^*_{n,k,l} \) depending only on \( n, k \) and \( l \). Together with the Newton-MacLaurin inequality, it implies
\[
\sigma_{l+1}(W) \geq c^*_{n,k,l} \sigma_l(W),
\]
which, in turn, together with (48) implies
\[
\sum_i F_{ii} u^2_i \geq c^*_{n,k,l} \sigma_{l+1}(W) \geq c^*_{n,k,l} \sigma_l(W) \geq \frac{1}{2} c^*_{n,k,l} G.
\]
Hence, if we choose \( m \) large, then the lemma is true.

Now we can prove the \( C^0 \) boundedness (and hence \( C^2 \) boundedness).

**Proposition 3.** Let \( g = e^{-2u} g_0 \) be a solution of flow (13) with \( \sigma_k(g(t)) \in \Gamma^+_k \) on \( M \times [0, T^*) \). Then there is a constant \( c > 0 \) depending only on \( v_0, g_0, k \) and \( n \) (independent of \( T^* \)) such that
\[
\|u(t)\|_{C^2} \leq c, \quad \forall t \in [0, T^*).
\]

**Proof.** We only need to show the boundedness of \( |u| \). First we consider the case \( l \neq n/2 \).

By Proposition 2 and the preservation of \( \int \sigma_l(g)dg \), we have
\[
c_l = \int_M e^{(2l-n)u} \sigma_l \left( \nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0} \right) dg_0 \leq c_1 \int_M e^{(2l-n)u} dg_0.
\]
If \( l < n/2 \), then (51), together with (39), implies that \( u < c \) for some constant \( c > 0 \). On the other hand, in this case Proposition 1 gives
\[
\text{vol}(g) \leq C \left( \int_M \sigma_1(g) dg \right)^{-\frac{n}{n-2l}} = c_0 C,
\]
which, together with (39) implies \( u > c_1 \), hence \( |u| \leq C \) in this case.

If \( l > n/2 \), (51) gives a lower bound of \( u \). Suppose there is no upper bound, we have a sequence of \( u \), with \( \nabla u \) and \( \nabla^2 u \) bounded, but sup \( u \) goes to infinity (so does inf \( u \)). Set \( v = u - \inf u \), so \( v \) is bounded and so is the \( C^2 \) norm of \( v \). But, for \( \tilde{g} = e^{-2v} g_0 \), we get \( \tilde{F}_l(\tilde{g}) \) tends to 0. Take a subsequence, we get \( \sigma_l(e^{-2v} g_0) = 0 \) with \( v^* \) in \( C^{1,1} \cap \Sigma^+_k \). This is a contradiction to Lemma 3.
Then we consider the case \( l = n/2 \). In this case, \( \mathcal{E}_{n/2}(g) \) is constant. First it is easy to check that \( g_t = e^{-2u}g_0 \in \Gamma^+_n \) when \( 0 \leq t \leq 1 \) (using the fact \((1, \ldots, 1, -1) \in \Gamma^+_n \) when \( n \) even). In particular, \( \sigma_{n/2}^2(g_t) > 0 \) for \( t > 0 \). From the expression of \( \mathcal{E}_{n/2}(g) \),

\[
- \sup(u) \int_M \sigma_{n/2}^2(g) dg \leq \mathcal{E}_{n/2}(g) \leq - \inf(u) \int_M \sigma_{n/2}^2(g) dg.
\]

Since

\[
\int_M \sigma_{n/2}^2(g) dg = \int_M \sigma_{n/2}^2(g_0) dg_0,
\]

So we have

\[
- \sup(u) \int_M \sigma_{n/2}^2(g_0) dg_0 \leq \mathcal{E}_{n/2}(g) \leq - \inf(u) \int_M \sigma_{n/2}^2(g_0) dg_0.
\]

Thus, \( \inf(u) \) is bounded from above and \( \sup(u) \) is bounded from below. By (39) again, \( u \) is bounded from above and away from 0. Now we have proved the boundedness of \(|u|\) in all cases. Combining that with (39) and Proposition 2 gives a \( C^2 \) bound \( u \) independent of \( T \).

\[ \square \]

**Proposition 4.** There is a constant \( C_0 > 0 \) independent of \( T \) such that

\[
\frac{\sigma_k}{\sigma_l}(g)(t) \geq C_0, \quad \text{for} \ t \in [0, \infty).
\]

**Proof.** We follow the similar argument in [16]. Here we will make use of Lemma 2. We consider \( H = \log \frac{\sigma_k}{\sigma_l}(g) - e^{-u} \) on \( M \times [0, T] \) for any \( T < T^* \). From (13) and (19) we have

\[
\frac{dH}{dt} = \frac{1}{2} \text{tr}\{\tilde{T}_{k-1,l-1}(S_g) \nabla^2_g \log \frac{\sigma_k}{\sigma_l}(g)\} + (k - l - 1 \frac{e^{-u}}{2}) \left( \log \frac{\sigma_k}{\sigma_l}(g) - \log r_{k,l}(g) \right).
\]

Without loss of generality, we may assume that the minimum of \( H \) in \( M \times [0, T] \) achieves at \((x_0, t_0) \in M \times (0, T]\). Let \( H_j \) and \( H_{ij} \) are the first and second derivatives with respect to the background metric \( g_0 \). At this point, we have \( \frac{dH}{dt} \leq 0, 0 = H_l = \sum_{ij} F^{ij} w_{ij} + e^{-u} u_l \) for all \( l \), and \( (H_{ij}) \) is non-negative definite. Also we have \( (F^{ij}) \) is positive definite and

\[
\sum_{i,j} F^{ij} w_{ij} = \frac{1}{\sigma_k(g)} \frac{\partial \sigma_k(g)}{\partial w_{ij}} w_{ij} - \frac{1}{\sigma_l(g)} \frac{\partial \sigma_l(g)}{\partial w_{ij}} w_{ij} = k - l.
\]

Recall that in local coordinates \( \tilde{T}^i_{k-1,l-1}(S_g) = F^{ij} \) and

\[
\sum_{i,j} F^{ij} \nabla^2_g H_{ij} = \sum_{i,j} F^{ij} (H_{ij} + u_i H_j + u_j H_i - \sum_l u_l H_{i,j,l} - \delta_{ij}).
\]
It follows that at the point, for positive constants \( c \), where \( S \) are the entries of \( S(g_0) \), since \( S(g_0) \in \Gamma_k^+ \), by Lemma 2,

\[
0 \geq H_t - \frac{1}{2} \sum_{i,j} F^{ij} H_{ij}
\]

\[
= \frac{1}{2} \text{tr} \{ T_{k-1,l-1} (S_g) \nabla^2 e^{-u} \} + (k - l + \frac{1}{2} e^{-u}) \left( \log \frac{\sigma_k}{\sigma_l}(g) - \log r_{k,l}(g) \right)
\]

\[
= \frac{1}{2} \sum_{i,j} F^{ij} \{ (e^{-u})_i + u_i(e^{-u})_j + u_j(e^{-u})_i - u_l(e^{-u})_l \delta_{ij} \}
\]

\[
+ (k - l + \frac{1}{2} e^{-u}) \left( \log \frac{\sigma_k}{\sigma_l}(g) - \log r_{k,l}(g) \right)
\]

\[
= \frac{e^{-u}}{2} \sum_{i,j} F^{ij} \{ -u_{ij} - u_i u_j + |\nabla u|^2 \delta_{ij} \} + (k - l + \frac{1}{2} e^{-u}) \left( \log \frac{\sigma_k}{\sigma_l}(g) - \log r_{k,l}(g) \right)
\]

\[
= \frac{e^{-u}}{2} \sum_{i,j} F^{ij} \{ -w_{ij} + S_{ij} + \frac{1}{2} |\nabla u|^2 \delta_{ij} \} + (k - l + \frac{1}{2} e^{-u}) \left( \log \frac{\sigma_k}{\sigma_l}(g) - \log r_{k,l}(g) \right)
\]

\[
\geq \frac{e^{-u}}{2} \sum_{i,j} F^{ij} S_{ij} + (k - l + \frac{1}{2} e^{-u}) \left( \log \frac{\sigma_k}{\sigma_l}(g) - \log r_{k,l}(g) \right) - \frac{k - l}{2} e^{-u},
\]

where \( S_{ij} \) are the entries of \( S(g_0) \). Since \( S(g_0) \in \Gamma_k^+ \), by Lemma 2,

\[
(53) \quad F^{ij} S_{ij} = \left\{ \frac{1}{\sigma_k(g)} \frac{\partial \sigma_k(g)}{\partial w_{ij}} - \frac{1}{\sigma_l(g)} \frac{\partial \sigma_l(g)}{\partial w_{ij}} \right\} S_{ij} \geq (k - l) e^{2u} \left( \frac{\sigma_k(g_0)}{\sigma_l(g_0)} \right)^{\frac{1}{k-l}} \left( \frac{\sigma_k}{\sigma_l}(g) \right)^{-\frac{1}{k-l}}.
\]

By \( C^2 \) estimates, log \( r_{k,l}(g) \) is bounded from above, we have

\[
0 \geq \frac{(k - l) e^{-u}}{2} \left( \frac{\sigma_k(g_0)}{\sigma_l(g_0)} \right)^{\frac{1}{k-l}} \left( \frac{\sigma_k}{\sigma_l}(g) \right)^{-\frac{1}{k-l}}
\]

\[
+ (k - l + \frac{1}{2} e^{-u}) \left( \log \frac{\sigma_k}{\sigma_l}(g) - \log r_{k,l}(g) \right) - \frac{k - l}{2} e^{-u}
\]

\[
\geq c_1 \left( \frac{\sigma_k}{\sigma_l}(g) \right)^{-\frac{1}{k-l}} + c_2 \log \frac{\sigma_k}{\sigma_l}(g) - c_3
\]

for positive constants \( c_1, c_2 \) and \( c_3 \) independent of \( T \). It follows that there is a positive constant \( c_4 \) independent of \( T \) such that

\[
\frac{\sigma_k}{\sigma_l}(g) \geq c_4,
\]

at point \((x_0, t_0)\). Then the Proposition follows, as \(|u|\) is bounded by Proposition 3.
Proof of Theorem 2. Now we can prove Theorem 2. First, from Propositions 3 and 4 we have $T^* = \infty$. Then, by Krylov’s Theorem 23, the flow has $C^{2,\alpha}$ estimates. (20) implies that

$$\int_0^\infty \int_M (\sigma_k(g) - r_{k,l}\sigma_l(g))^2 dgdt < \infty,$$

which, in turn, implies that there is a sequence $\{t_l\}$ such that

$$\int_M (\sigma_k(g) - r_{k,l}\sigma_l(g))^2(t_l)dg \to 0$$
as $t_l \to \infty$. The above estimates imply that $g(t_l)$ converges in $C^{2,\alpha}$ to a conformal metric $h$, which is a solution of (6).

Now we want to use Simon’s argument 31 to prove that $h$ is the unique limit of flow (13) as in 16 (see also 11). Since the arguments are essentially the same, here we only give a sketch. First, with the $C^{2,\alpha}$ regularity estimates (and higher regularity estimates follows from the standard parabolic theory) established for flow (13), one can show that,

$$\lim_{t \to \infty} \|\frac{\sigma_k}{\sigma_l}(g(t)) - \beta\|_{C^{4,\alpha}(M)} = 0,$$

for some positive constant $\beta$. It is clear that $\frac{\partial}{\partial t}(h) = \beta$. By Proposition 4 and the Newton-MacLaurin inequality, there is a constant $c > 1$ such that $c^{-1} \leq \sigma_l(g(t)) \leq c$. We want to show that flow (13) is a pseudo-gradient flow, though it is not a gradient flow. The crucial step is to establish the “angle estimate” 54 for the $L^2$ gradient of some proper functionals. The estimate 54 enables one to conclude that the flow will converge to a unique stationary solution, we refer to 16 for the detailed argument.

Now we may switch the background metric to $h$ and all derivatives and norms are taken with respect to the metric $h$ (since we have all the a priori estimates for $h$).

Here we only give a proof for $l < k < n/2$. The proof for the other cases is similar after taking consideration of the corresponding functionals.

Consider a functional defined by

$$F_{k,l}(g) = \left(\int \sigma_l(g)dg\right)^{-\frac{n-2k}{n-2l}} \int_M \sigma_k(g)dg.$$

Its $L^2$-gradient is

$$\nabla F_{k,l} = -c_0((\sigma_k(g) - \tilde{r}_{k,l}(g)\sigma_l(g))e^{-nu},$$

where $c_0$ is a non-zero constant and $\tilde{r}_{k,l}(g)$ is given by

$$\tilde{r}_{k,l}(g) := \frac{\int_M \sigma_k(g)dg}{\int_M \sigma_l(g)dg},$$
which is different from \( r_{k,l} \). But it is easy to check that \( r_{k,l}(t) - \tilde{r}_{k,l}(t) \to 0 \) as \( t \to \infty \). Since \( \frac{\partial_l}{\sigma_l}(g(t)) \) is very close to a constant for large \( t \), from (20) we have

\[
\frac{d}{dt} F_{k,l}(g) \leq -c \int_M \left( \frac{\sigma_k}{\sigma_l}(g) - r_{k,l} \right) \left( \log \frac{\sigma_k}{\sigma_l}(g) - \log r_{k,l} \right) \sigma_l(g) dg
\]

\[
\leq -c \left( \int_M \left| \frac{\sigma_k}{\sigma_l}(g) - r_{k,l} \right|^2 \sigma_l(g) dg \right)^{1/2} \left( \int_M \frac{dg}{dt} \sigma_l(g) \right)^{1/2}
\]

\[
\leq -c \left( \int_M \left| \sigma_k(g) - \tilde{r}_{k,l} \right|^2 \sigma_l(g) dg \right)^{1/2} \left( \int_M \frac{dg}{dt} \sigma_l(g) \right)^{1/2}
\]

\[
\leq -c \left( \int_M \left| \sigma_k(g) - \tilde{r}_{k,l} \right| \sigma_l(g) dg \right)^{1/2} \left( \int_M \frac{dg}{dt} \sigma_l(g) \right)^{1/2}
\]

\[
\leq -c \left( \int_M \left| \nabla F_{k,l} \right|^2 dh \right)^{1/2} \left( \int_M \frac{dg}{dt} \sigma_l(g) \right)^{1/2}
\]

where \( c > 0 \) is a constant varying from line to line. The angle estimate (54) means that flow (13) is a pseudo-gradient flow. Finally from (51) we can apply the argument given in [31] to show the uniqueness of the limit as in [10].

5. Proof of Theorem 1

From the flow approach we developed, we have

**Proposition 5.** Let \((M, g_0)\) be a compact, connected and oriented locally conformally flat manifold with \(g_0 \in \Gamma^+\) and \(0 \leq l < k \leq n\). Let \(\tilde{F}_k\) defined as in (18), then there is \(g \in C_k\) satisfying equation (6) such that

\[
0 < \tilde{F}_k(g) \leq \tilde{F}_k(g_0),
\]

for any \(g \in C_k\) with \(\tilde{F}_l(g) = \tilde{F}_l(g_0)\). Moreover, if \((M, g_0)\) is conformally equivalent to a space form, then \((M, g)\) is also a space form.

**Proof.** The case \(l = 0\) has been treated in Proposition 11. We may assume \(l \geq 1\) in the rest of proof. When \((M, g_0)\) is conformally equivalent to a space form, then any solutions of (6) are metrics of constant sectional curvature (this can be proved using the method of moving plane of [10] as in [36], see Corollary 1.1 in [24] for more general statement), and hence have the same \(\tilde{F}_k\) if they have been the same \(\tilde{F}_l\). Hence the Proposition follows from Theorem 2.

Now we remain to consider the case \(k < n/2\) and \((M, g_0)\) is not conformally equivalent to a space form. We will follow the same argument in the proof of Proposition 11. Here we need the local estimates in Theorem 3 for the quotient equation (6).
First we want to show
(55) \[ \inf_{g \in \mathcal{C}_k, \mathcal{F}_1(g) = 1} \mathcal{F}_k(g) =: \beta_0 > 0. \]

Suppose \( \beta_0 = 0 \). By the result for flow (13), there is a sequence \( g_i = e^{-2u_i}g_0 \in \mathcal{C}_k \) with \( \mathcal{F}_1(g_i) = 1 \) and
\[
\frac{\sigma_k}{\sigma_l}(g_i) = \beta_i, \quad \lim_{i \to \infty} \beta_i = 0,
\]
The scaled metric \( \tilde{g}_i = e^{-2u_i}g_0 \) with \( \tilde{u}_i = u_i - \frac{1}{2(k-l)} \log \beta_i \) satisfies
(56) \[ \frac{\sigma_k}{\sigma_l}(\nabla^2 \tilde{u}_i + d\tilde{u}_i \otimes d\tilde{u}_i - \frac{|\nabla \tilde{u}_i|^2}{2}g_0 + S_{g_0}) = e^{2(k-l)\tilde{u}_i}. \]

By Proposition 1,
\[
C \text{vol}(\tilde{g}_i) \leq \mathcal{F}_1(\tilde{g}_i) = \beta_i^{n-2l} \to 0 \quad \text{as} \quad i \to \infty.
\]
Since \( \text{vol}(\tilde{g}_i) \) is tending to 0, by Corollary 2,
\[ m_i := \inf_M \tilde{u}_i \to +\infty \quad \text{as} \quad i \to \infty. \]

Now at the minimum point \( x_i \) of \( \tilde{u}_i \), by equation (56),
\[
\frac{\sigma_k}{\sigma_l}(S_{g_0}) \leq \frac{\sigma_k}{\sigma_l}(\nabla^2 \tilde{u}_i + d\tilde{u}_i \otimes d\tilde{u}_i - \frac{|\nabla \tilde{u}_i|^2}{2}g_0 + S_{g_0}) = e^{2(k-l)m_i} \to 0.
\]
This is a contradiction to the fact that \( g_0 \in \Gamma^+_k \), proving (55).

Finally we prove the existence of an extremal metric in this case. From above argument, there is a minimization sequence \( g_i \in \mathcal{C}_k \), with \( \mathcal{F}_1(g) = 1 \), and \( \frac{\sigma_k(g_i)}{\sigma_l(g_i)} = \beta_i \), with \( \beta_i \) decreasing and bound below by a positive constant. As \( (M, g_0) \) is not conformally equivalent to \( S^n \) by assumption, it follows from Theorem 1.3 in [12] (see also [25]) that the metrics converge (by taking a subsequence) to some \( g_E \) which attains the infimum \( C_S \). \[ \square \]

**Proof of (B) of Theorem 1** The cases \( l = n/2 \) and \( k = n/2 \) were considered in [16] and [14]. Hence we assume that \( k \neq n/2 \) and \( l \neq n/2 \). Let us consider
\[
\mathcal{F}_{k,l}(g) = \left( \int \sigma_l(g)dg \right)^{-\frac{n-2l}{n}} \int_M \sigma_k(g)dg.
\]
Since \( \mathcal{F}_{k,l} \) is invariant under the transformation \( g \) to \( e^{-2a}g \) for any constant \( a \), Proposition \[ \] implies that for any \( g \in \mathcal{C}_k \)
\[
\mathcal{F}_{k,l}(g) \leq \mathcal{F}_{k,l}(g_E) =: C(n, k, l).
\]
It is clear that \( C(n, k, l) \) depends only on \( n, k, l \).

Let \( c_0 = \mathcal{F}_{k,l}(g_E) \). From Proposition \[ \] we have
(57) \[ \int_M \sigma_k(g)dg \leq C(n, k, l) \left( \int_M \sigma_l(g)dg \right)^{\frac{n-2k}{n}} = C(n, k, l) \left( \int_M \sigma_l(g)dg \right)^{\gamma_k} \left( \int_M \sigma_l(g)dg \right)^{\frac{n-2}{n}} \]
where \( \gamma = \frac{n-2k}{k(n-2l)} - \frac{1}{l} \). It is clear that \( \gamma > 0 \) when \( l > n/2 \) and \( \gamma < 0 \) when \( l < n/2 \).
We first consider the case $l > n/2$. In this case, by Proposition 1, we have
\[ \int_M \sigma_l(g) dg \leq c_1 \text{vol}(g)^{\frac{n-2l}{n}}, \]
where $c_1 = \mathcal{F}_l(g_e)$. It follows that
\[ (\int_M \sigma_l(g) dg)^\gamma \leq c_0^\gamma \text{vol}(g)^{\frac{l-k}{nk}}. \]
Hence
\[ (\mathcal{F}_k(g))^{1/k} = \left(\text{vol}(g)^{-\frac{n-2k}{n}} \int_M \sigma_k(g) dg\right)^{\frac{1}{k}} \]
\[ \leq c_0^\frac{1}{k} \left(\text{vol}(g)^{-\frac{n-2l}{n}} \int_M \sigma_l(g) dg\right)^{\frac{1}{k}} \]
\[ = c_0^\frac{1}{k} (\mathcal{F}_l(g))^{1/l}. \]
The equality holds if and only if $g$ is a metric of constant sectional curvature.

Consider the case $l < n/2$. In this case, by Proposition 1, again we have
\[ \int_M \sigma_l(g) dg \geq c_1 \text{vol}(g)^{\frac{n-2l}{n}}, \]
where $c_1 = \mathcal{F}_l(g_e)$. Since $\gamma < 0$, we have (58). The same argument given in the previous case gives the same conclusion.

Finally, since $k \geq n/2$, $(M, g_0)$ is conformally equivalent to a space form (14). The existence of the extremal metric which attains the equality case follows from Proposition 5. And the constant $C(n, k, l)$ is easy to calculate.

**Proof of (C) of Theorem 1.** Let us first consider the case $l < n/2$. Let $g \in C_{n/2}$. Choose $a$ such that $\int_M \sigma_l(e^{-2a}g) d\text{vol}(e^{-2a}g) = \int_M \sigma_l(g_0) dg_0$. It is easy to see that
\[ a = \frac{1}{n-2l} \left\{ \log \int_M \sigma_l(g) dg - \log \int_M \sigma_l(g_0) dg_0 \right\}. \]
By Proposition 5, we have
\[ \mathcal{E}_{n/2}(g) = \mathcal{E}_{n/2}(e^{-2a}g) + a \int_M \sigma_{n/2}(g) dg \]
\[ \geq a \int_M \sigma_{n/2}(g_0) dg_0 \]
\[ = \frac{1}{n-2} \int_M \sigma_{n/2}(g_0) dg_0 \left\{ \log \int_M \sigma_l(g) dg - \log \int_M \sigma_l(g_0) dg_0 \right\}. \]
This proves the Theorem for the case $l < n/2$.

Now we consider the case $l > n/2$. For any $g \in C_l$ we choose
\[ a = (\int_M \sigma_{n/2}(g) dg)^{-1} \mathcal{E}_{n/2}(g) \]
such that \(E_{n/2}(e^{-2a}g) = E_{n/2}(g_0)\). Recall that \(\tilde{F}_{n/2} = F_{n/2}\). By Proposition \(\Box\) again, we have
\[
\tilde{F}_l(g) = \frac{1}{n-2l} \int_M \sigma_l(g) dg
= \frac{1}{n-2l} e^{-(2l-n)a} \int_M \sigma_l(e^{-2a}g) d\text{vol}(e^{-2a}g)
\geq \frac{1}{n-2l} e^{-(2l-n)a} \int_M \sigma_l(g_0) dg_0
= \frac{1}{n-2l} \exp \left\{ (n-2l) \left( \int_M \sigma_{n/2}(g) dg \right)^{-1} E_{n/2}(g) \right\} \int_M \sigma_l(g_0) dg_0.
\]

Since \((M, g_0)\) is conformally equivalent to a space form in this case, the existence of the extremal metric can be proved along the same line as in part (B) of the Theorem. Note that since \(n\) is even, \((M, g_0)\) is the standard sphere. The computation of \(C_{MT}\) is straightforward.

**Proof of (A) of Theorem [4]**. Inequality (5) follows from (55) in the proof of Proposition 5. The existence of the extremal metric has also proved there. As for inequality (7), since its proof is of different spirit and inequality itself is of independent interest, we will devote it in the next section (Theorem 4). The constant \(C_{S,k,l}(S^n)\) in (7) can be computed easily.

### 6. The Best Constant

In this section, we address the question of the best constant in part (A) of Theorem 1. As in the Yamabe problem (i.e., \(k = 1\) and \(l = 0\)), for \(0 \leq l < k < n/2\) we define
\[
Y_{k,l}(M, [g_0]) = \inf_{g \in C_k} (F_l(g))^{-\frac{n-2k}{n-2l}} F_k(g) = \inf_{g \in C_k} \left( \int_M \sigma_l(g) dg \right)^{-\frac{n-2k}{n-2l}} \int_M \sigma_k(g) dg.
\]

It is clear that \(Y_{k,l}(M, [g_0]) = C_{S}^{n-2k}\). In this section we prove

**Theorem 4.** For any compact, oriented locally conformally flat manifold \((M, g_0)\), we have

\[
(59) \quad Y_{k,l}(M, [g_0]) \leq Y_{k,l}(S^n, g_{S^n}),
\]

where \(g_{S^n}\) is the standard metric of the unit sphere.

When \(k = 1\) and \(l = 0\), this was proven by Aubin (e.g., see [2]) for general compact manifolds. To prove Theorem 4 we need to construct a sequence of “blow-up” functions which belong to \(C_k\). This is a delicate part of the problem.

We need two Lemmas.

**Lemma 7.** Let \(D\) be the unit disk in \(\mathbb{R}^n\) and \(ds^2\) the standard Euclidean metric. Let 
\(g_0 = e^{-2u_0} ds^2\) be a metric on \(D\) and \(g_0 \in \Gamma_k\) with \(k < n/2\). Then there is a conformal metric \(g = e^{-2u} ds^2\) on \(D\setminus\{0\}\) of positive \(\Gamma_k\)-curvature with the following properties:

1. \(\sigma_k(g) > 0\) in \(D\setminus\{0\}\).
2. \( u(x) = u_0(x) \) for \( r = |x| \in (r_0, 1] \).
3. \( u(x) = a + \log r \) for \( r = |x| \in (0, r_3) \), and some constant \( a \).

for some constants \( r_0 \) and \( r_3 \) with \( 0 < r_3 < r_0 < 1 \).

Proof. Let \( v \) be a function on \( D \) and \( \tilde{g} = e^{-2v}g_0 \). By the transformation formula of the Schouten tensor, we have

\[
S(\tilde{g})_{ij} = \nabla^2_{ij}(v + u_0) + \nabla_i(v + u_0)\nabla_j(v + u_0) - \frac{1}{2} |\nabla(v + u_0)|^2 \delta_{ij}
\]

(60)

\[
= \nabla^2_{ij}v + \nabla_i v \nabla_j v + \nabla_i v \nabla_j u_0 + \nabla_j v \nabla_i u_0 + \frac{1}{2} |\nabla v|^2 + \nabla v \nabla u_0 \delta_{ij} + S_{g_0}
\]

Here \( \nabla \) and \( \nabla^2 \) are the first and the second derivatives with respect to the standard metric \( ds^2 \). Let \( r = |x| \). We want to find a function \( v = v(r) \) with \( \tilde{g} \in \Gamma_k^+ \) and

\[
v' = \frac{\alpha(r)}{r},
\]

where \( \alpha = 1 \) near 0 and \( \alpha = 0 \) near 1. From (60) we have

\[
S(\tilde{g})_{ij} = \frac{2\alpha - \alpha^2}{2r^2}\delta_{ij} + \left( \frac{\alpha'}{r} + \frac{\alpha^2 - 2\alpha}{r^2} \right) \frac{x_i x_j}{r^2} + S(g_0)_{ij} + O(|\nabla u_0|) \frac{\alpha}{r},
\]

(61)

where \( O(|\nabla u_0|) \) is a term bounded by a constant \( C_1 \) depending only on \( \max |\nabla u_0| \). Let \( A(r) \) be an \( n \times n \) matrix with entry \( a_{ij} = S(\tilde{g})_{ij} - S(g_0)_{ij} \). Hence

\[
\sigma_k(\tilde{g}) = e^{-2k(v+u_0)} \sigma_k(A + S(g_0)).
\]

To our aim, we need to find \( \alpha \) such that \( A + S(g_0) \in \Gamma_k^+ \). Let \( \varepsilon \in (0, 1/2) \) and \( r_0 = \min\{\frac{1}{2}, C_1 \varepsilon\} \). We will choose \( \alpha \) such that

\[
(62) \quad \alpha(r) \in [0, 1] \quad \text{and} \quad \alpha(r) = 0, \quad \text{for} \quad r \in [r_0, 1].
\]

Since \( \sigma_k(\tilde{g}) = e^{2k(v+u_0)} \sigma_k(A(r) + S(g_0)) \), we want to find \( \alpha \) such that \( \sigma_k(A(r) + S(g_0)) > 0 \). It is clear to see that for \( r \in [0, r_0] \)

\[
A(r) \geq \left( \frac{2\alpha - \alpha^2 - \varepsilon \alpha}{2r^2} \delta_{ij} + \left( \frac{\alpha'}{r} + \frac{\alpha^2 - 2\alpha}{r^2} \right) \frac{x_i x_j}{r^2} \right),
\]

as a matrix. This implies that

\[
(63) \quad \sigma_k(A(r)) \geq \frac{(n-1)!}{k!(n-k)!} \left( \frac{2\alpha - \alpha^2 - \varepsilon \alpha}{2r^2} \right)^{k} \left( n - 2k + 2 \frac{r \alpha' - \varepsilon \alpha}{2\alpha - \alpha^2 - \varepsilon \alpha} \right).
\]

One can easily check that for any small \( \delta > 0 \),

\[
\alpha(r) = \frac{2(1-\varepsilon)\delta}{\delta + r^{\frac{1}{2}}}
\]

is a solution of

\[
(2 - \varepsilon)\alpha - \alpha^2 = -4(r \alpha' - \varepsilon \alpha).
\]
Now we can finish our construction of \( \alpha \). Since \( S(g_0) \in \Gamma^+_k \), by the openness of \( \Gamma^+_k \) we can choose \( r_1 \in (0, r_0) \) and an non-increasing function \( \alpha : [r_1, r_0] \subset [0, 1) \) such that \( \sigma_k(g) > 0 \) and \( \alpha(r_1) > 0 \). Now we choose a suitable \( \delta > 0 \) and \( \alpha \) in the form (64). Then find \( r_2 \in (0, r_1) \) with \( \alpha(r_2) = 1 \). It is clear that \( \sigma_k(A(r)) > 0 \) on \([r_2, r_1]\). Define \( \alpha(r) = 1 \) on \([0, r_2]\). We may smooth \( \alpha \) such that the new resulted conformal metric \( g \) satisfying all conditions in Lemma 7.

**Remark 1.** From Lemma 7, one can prove that the connected sum of two locally conformally flat manifolds \((M_1, g_1)\) and \((M_2, g_2)\) with \( g_1, g_2 \in \Gamma^+_k (k < n/2) \) admits a metric in \( \Gamma^+_k \). This is also true for general manifolds. Namely, the connected sum of two compact manifolds \((M_1, g_1)\) and \((M_2, g_2)\) with \( g_1, g_2 \in \Gamma^+_k (k < n/2) \) admits a metric in \( \Gamma^+_k \). The proof is given in [13].

**Lemma 8.** For any small constants \( \delta > 0 \) and \( \varepsilon > 0 \), there exists a function \( u : \mathbb{R}^n \setminus \{0\} \to 0 \) satisfying:

1. The metric \( g = e^{-2u} dx^2 \) has positive \( \Gamma_k \)-curvature.
2. \( u = \log(1 + |x|^2) + b_0 \) for \( |x| \geq \delta \), i.e., \( \{x \in \mathbb{R}^n \mid |x| \geq \delta\}, g \) is a part of a sphere.
3. \( u = \log|x| \) for \( |x| \leq \delta_1 \), i.e., \( \{x \in \mathbb{R}^n \mid 0 < |x| \leq \delta_1\}, g \) is a cylinder.
4. \( \operatorname{vol}(B_{\delta} \setminus B_{\delta_1}, g) \leq C \delta^{- \frac{2(n-2k)}{1-\varepsilon_0}} \).
5. \( \int_{B_{\delta} \setminus B_{\delta_1}} \sigma_k(g) d\operatorname{vol}(g) \leq C \delta^{- \frac{2(n-2k)}{1-\varepsilon_0}} \), for any \( k < n/2 \),

where \( C \) is a constant independent of \( \delta, \delta_1 = \delta \frac{3-\varepsilon_0}{1-\varepsilon_0} \) and \( b_0 \sim \frac{3-\varepsilon_0}{1-\varepsilon_0} \log \delta \).

**Proof.** Let \( \delta \in (0, 1) \) be any small constant. For any small constant \( \varepsilon_0 > 0 \), we define \( u \) by

\[
u(r) = \begin{cases} \log(1 + r^2) + b_0, & r \geq \delta \\ -\frac{2}{1-\varepsilon_0} \log \left( 1 + \frac{\delta^{3-\varepsilon_0} r^{-(1-\varepsilon_0)}}{2} \right) + \frac{3-\varepsilon_0}{1-\varepsilon_0} \log \delta & r \in (\delta_1, \delta), \\ \log r, & r \leq \delta_1, \end{cases}
\]

where \( \delta_1 = \delta \frac{3-\varepsilon_0}{1-\varepsilon_0} \) and

\[
b_0 = -\log(1 + \delta^2) - \frac{2}{1-\varepsilon_0} \log \left( 1 + \frac{\delta^2}{2} \right) + \frac{3-\varepsilon_0}{1-\varepsilon_0} \log \delta.
\]

As in the proof of Lemma 5, we write \( u'(r) = \frac{\alpha(r)}{r} \). It is easy to see that \( \alpha : \mathbb{R}_+ \to \mathbb{R}_+ \) by

\[
\alpha(r) = \begin{cases} \frac{2r^2}{1+r^2}, & r \geq \delta, \\ \frac{2\delta^{3-\varepsilon_0}}{\delta^{3-\varepsilon_0} + r^{1-\varepsilon_0}}, & r \in (\delta_1, \delta), \\ 1, & r \leq \delta_1. \end{cases}
\]
One can check all conditions in the Lemma, except the smoothness of \( u \), which is \( C^{1,1} \). We first check (1). By a direct computation, see for example (13), we have
\[
\sigma_k(e^{-2u}|dx|^2) = e^{2ku(r)} \frac{(n-1)!}{k!(n-k)!} \left( \frac{2\alpha - \alpha^2}{2r^2} \right)^k \left( n - 2k + \frac{2r\alpha'}{2\alpha - \alpha^2} \right).
\]
In the interval \((\delta_1, \delta)\), \( \alpha \in (0, 2) \) satisfies
\[
\frac{2r\alpha'}{2\alpha - \alpha^2} = -(1 - \epsilon_0).
\]
Since \( k < n/2 \), we have \( \sigma_k(e^{-2u}|dx|^2) > 0 \). One can also directly to check (4) and (5). Here we only check (5). A direct computation gives
\[
\int_{B_\delta \setminus B_{\delta_1}} \sigma_k(g)dvol(g) \leq c \int_{\delta_1}^{\delta} e^{-(n-2k)u(r)} r^{-2k} r^{n-1} dr
\]
\[
\leq c\delta^{-(n-2k)} \frac{\alpha}{1-\epsilon_0} \int_{\delta_1}^{\delta} r^{n-2k-1} dr
\]
\[
\leq c\delta^{\frac{2(n-2k)}{1-\epsilon_0}}.
\]
From our construction, we only have \( u \in C^{1,1} \). But, for \( \delta > 0 \) fixed, we can smooth \( \alpha \) so that \( u \in C^\infty \) satisfies all conditions (1)-(5).

**Proof of Theorem 4**. Let \( p \in M \) and \( U \) a neighborhood of \( p \) such that \((U, g)\) is conformally flat, namely \((U, g) = (D, e^{-2u_0}|dx|^2)\). Applying Lemma 7 we obtain a conformal metric \( u \) satisfying conditions 1)-3) in Lemma 7 with constants \( r_0, r_3 \) and \( a \). By adding a constant we may assume \( a = 0 \). Now applying Lemma 8 for any small constant \( \delta > 0 \) we have a conformal metric \( g_\delta = e^{-2u_\delta}|dx|^2 \) on \( \mathbb{R}^n \setminus \{0\} \). Consider the rescaled function
\[
\tilde{u}_\delta = u_\delta(\delta_1 x) - \log \frac{\delta_1}{r_3}.
\]
Now \( u \) and \( \tilde{u}_\delta \) are the same in \( \{0 < |x| < r_3\} \). Consider the following conformal transformation
\[
f(x) = \frac{r_3^2}{2} \frac{x}{|x|^2},
\]
which maps \( \{r_3/2 \leq |x| \leq r_3\} \) into itself and maps one of boundary components to another with opposite orientations. Now we define a new function on \( M \) by
\[
w_\delta(x) = \begin{cases} 
0, & |x| \geq r_0, \\
u - u_0, & r_3/2 \leq |x| \leq r_0, \\
\tilde{u}_\delta(f(x)) + 2\log |x| - \log \frac{r_3^2}{2} - u_0, & |x| \leq r_3/2.
\end{cases}
\]
Since $u$ and $\tilde{u}_\delta$ are the same in $\{0 < |x| < r_3\}$, it clear that $w_\delta(x)$ is smooth on $M$. Consider the conformal metric $g_\delta = e^{-2w_\delta}g$ and compute, using Lemmas 7 and 8

$$\int_M \sigma_k(g_\delta) d\text{vol}(g_\delta) = \int_{\{|x| \leq r_3/2\}} \sigma_k(g_\delta) d\sigma_\delta + O(1)$$

$$= e^{-(n-2k)\alpha_0} \int_{\mathbb{R}^n \setminus \{|x| \leq \delta\}} \sigma_k(g_{S^n}) d\text{vol}(g_{S^n}) + O(1)\delta^\frac{2(n-2k)}{1-\epsilon_0}$$

$$= \delta^{-\frac{3-\epsilon_0}{1-\epsilon_0}(n-2k)} \int_{\mathbb{R}^n \setminus \{|x| \leq \delta\}} \sigma_k(g_{S^n}) d\text{vol}(g_{S^n}) + o(\delta^{-\frac{3-\epsilon_0}{1-\epsilon_0}(n-2k)})$$

and

$$\int_M \sigma_l(g_\delta) d\text{vol}(g_\delta) = \delta^{-\frac{3-\epsilon_0}{1-\epsilon_0}(n-2l)} \text{vol}(\mathbb{R}^n \setminus \{|x| \leq \delta\}, g_{S^n}) + o(\delta^{-\frac{3-\epsilon_0}{1-\epsilon_0}(n-2l)})$$

where $g_{S^n} = \frac{1}{(1+|x|^2)} |dx|^2$ is the standard metric of the sphere and $O(1)$ is a term bounded by a constant independent of $\delta$. Now it is readily to see

$$Y_{k,l}(M) \leq \lim_{\delta \to 0} Y_{k,l}(g_\delta) \to Y_{k,l}(S^n),$$

as $\delta \to 0$. 

\section*{References}

[1] B. Andrews, \textit{Monotone quantities and unique limits for evolving convex hypersurfaces}, Internat. Math. Res. Notices \textbf{1997} (1997), 1001–1031.

[2] T. Aubin, \textit{Some nonlinear problems in Riemannian geometry}. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[3] W. Beckner, \textit{Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality}, Ann. of Math., \textbf{138} (1993), 213–242.

[4] S. Brendle and J. Viaclovsky, \textit{A variational characterization for $\sigma_{n/2}$}, preprint, 2002.

[5] L. Caffarelli, L. Nirenberg and J. Spruck, \textit{The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian}, Acta Math. \textbf{155} (1985), 261–301.

[6] A. Chang, M. Gursky and P. Yang, \textit{An equation of Monge-Ampère type in conformal geometry, and four manifolds of positive Ricci curvature}, Ann. of Math., \textbf{155} (2002), 709-787.

[7] A. Chang, Gursky and P. Yang, \textit{An a priori estimates for a fully nonlinear equation on Four-manifolds}, J. D’Analysis Math., J. Anal. Math. \textbf{87} (2002), 151–186.

[8] A. Chang and P. Yang, \textit{The inequality of Moser and Trudinger and applications to conformal geometry}, Comm. Pure & Appl. Math., \textbf{56} (2003), 1135-1150.

[9] K.-S. Chou, (K.S. Tso) \textit{On a real Monge-Ampère functional}, Invent. Math. \textbf{101} (1990), 425–448.

[10] B. Gidas, W.-M. Ni and L. Nirenberg, \textit{Symmetry and related properties via the maximum principle}, Comm. Math. Phys. \textbf{68} (1979), no. 3, 209-243.

[11] Maria Del Mar Gonzalez. Ph.D. thesis in preparation. Princeton University.

[12] P. Guan, C.S. Lin and G. Wang, \textit{Applications of method of moving planes to conformally invariant equations}, to appear in Math. Z.

[13] P. Guan, C.S. Lin and G. Wang, \textit{Schouten tensor and some topological properties}, preprint.

[14] P. Guan, J. Viaclovsky and G. Wang, \textit{Some properties of the Schouten tensor and applications to conformal geometry}, Trans A. M. S., \textbf{355} (2003), 925-933.

[15] P. Guan and G. Wang, \textit{Local estimates for a class of fully nonlinear equations arising from conformal geometry}, International Mathematics Research Notices, \textbf{2003} (2003), 1413-1432.
[16] P. Guan and G. Wang, *A fully nonlinear conformal flow on locally conformally flat manifolds*, Journal für die reine und angewandte Mathematik, 557, (2003), 219-238.

[17] M. Gursky, *The principal eigenvalue of a conformally invariant differential operator, with an application to semi-linear PDE*, Comm. Math. Phys. 207 (1999), 131–143.

[18] M. Gursky and J. Viaclovsky, *A conformal invariant related to some fully nonlinear equations*, to appear in Adv. in Math..

[19] M. J. Gursky and J. Viaclovsky, *Fully nonlinear equations on Riemannian manifolds with negative curvature*, Indiana Univ. Math. J. 52 (2003), 399–419.

[20] G. Hardy, J. Littlewood and G. Polya, *Inequalities*, Cambridge University Press, 1934.

[21] C. Hong, *A best constant and the Gaussian curvature*, Proc. Amer. Math. Soc. 97 (1986), 737–747.

[22] D. Jerison and J. Lee, *The Yamabe problem on CR manifolds*, J. Diff. Geom. 25 (1987), 167–197.

[23] N. Krylov, *Nonlinear elliptic and parabolic equations of the second order*, D. Reidel, (1987).

[24] A. Li and Y.Y. Li, *On some conformally invariant fully nonlinear equations*, to appear in Comm. Pure & Appl. Math.

[25] A. Li and Y.Y. Li, *A fully nonlinear version of the Yamabe problem and a Harnack type inequality*, math.AP/0212031.

[26] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. 20 (1970/71), 1077–1092.

[27] E. Onofri, *On the positivity of the effective action in a theory of random surfaces*, Comm. Math. Phys. 86 (1982) 321–326.

[28] R. Schoen, *Conformal deformation of a Riemannian metric to constant curvature*, J. Diff. Geome., 20 (1984), 479-495.

[29] R. Schoen, *Analytic aspects of the harmonic map problem*. Seminar on nonlinear partial differential equations (Berkeley, Calif., 1983), 321–358, Math. Sci. Res. Inst. Publ., 2, Springer, New York, 1984.

[30] R. Schoen and S. T. Yau, *Conformally flat manifolds, Kleinian groups and scalar curvature*, Invent. Math. 92 (1988), no. 1, 47–71.

[31] L. Simon, *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems*, Ann. of Math. 118 (1983), 525–571.

[32] N. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. 17 (1967), 473–483.

[33] N. Trudinger, *On the Dirichlet problem for Hessian equations*, Acta Math., 175 (1995), 151–164.

[34] N. Trudinger and X-J. Wang, *A Poincaré type inequality for Hessian integrals*. Calc. Var. Partial Differential Equations 6 (1998), no. 4, 315–328.

[35] J. Viaclovsky, *Conformal geometry, contact geometry and the calculus of variations*, Duke J. Math. 101 (2000), no. 2, 283-316.

[36] J. Viaclovsky, *Conformally invariant Monge-Ampère equations: global solutions*, Trans. AMS 352 (2000), no. 9, 4371–4379.

[37] J. Viaclovsky, *Estimates and some existence results for some fully nonlinear elliptic equations on Riemannian manifolds*, Comm. Anal. Geom. 10 (2002), 815-847.

[38] X. J. Wang, *A class of fully nonlinear elliptic equations and related functionals*, Indiana Univ. Math. J. 43 (1994), no. 1, 25–54.

[39] R. Ye, *Global existence and convergence of Yamabe flow*, J. Differential Geom. 39 (1994), no. 1, 35-50

DEPARTMENT OF MATHEMATICS, McMaster University, Hamilton, Ont. L8S 4K1, Canada.
Fax: (905)522-0935
E-mail address: guan@math.mcmaster.ca

MAX-PLANCK-INSTITUTE FOR MATHEMATICS IN THE SCIENCES, INSELSTR. 22-26, 04103 Leipzig, Germany
E-mail address: gwang@mis.mpg.de