TOPOLOGICAL MIXING OF RANDOM SUBSTITUTIONS

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ABSTRACT

We investigate topological mixing of compatible random substitutions. For primitive random substitutions on two letters whose second eigenvalue is greater than one in modulus, we identify a simple, computable criterion which is equivalent to topological mixing of the associated subshift. This generalises previous results on deterministic substitutions. In the case of recognisable, irreducible Pisot random substitutions, we show that the associated subshift is not topologically mixing. Without recognisability, we rely on more specialised methods for excluding mixing and we apply these methods to show that the random Fibonacci substitution subshift is not topologically mixing.

1. Introduction

The question of what it means for a dynamical system to be disordered is a subtle one with a rich history dating back to the birth of dynamics itself with Poincaré’s study of chaotic orbits in celestial mechanics [25]. Mathematicians measure disorder using a variety of tools including entropy, the dynamical spectrum, Lyapunov exponents, recurrence phenomena, and mixing properties. While a system deemed to be disordered is ideally identified as so by each of these measures, this is not always the case. Indeed, systems exist which are disordered according to one measure, but well-ordered according to another. Subshifts arising from random substitutions are prototypical examples.

First studied in the context of mathematical quasicrystals by Godrèche and Luck [14], the random Fibonacci substitution has the curious property of giving rise to a dynamical system which is locally disordered but globally well-ordered, characterised by the simultaneous presence of positive topological entropy and a non-trivial pure-point component in its associated diffraction spectrum. Their results immediately encourage the further study of the dynamics of the random Fibonacci substitution and related systems. In this work, we consider one such dynamical property, namely topological mixing of the associated subshift and the associated tiling space.

Random substitutions are maps which send letters from a finite alphabet to finite collections of words over the same alphabet. They have received renewed attention in recent years, following the work of Rust and Spindeler [27], in which they initiated the study of random substitutions in the context of topological dynamics and ergodic theory and they posed numerous open questions and routes
for further investigation. Since then, random substitution subshifts have been studied in terms of (among others) their frequency measures [19], diffraction spectrum [5, 15], periodicity [26], automorphism groups [13], topological and measure theoretic entropy [16, 17], and relations to shifts of finite type [18]. In their work, Rust and Spindeler determined by ad hoc methods that the subshift associated with the random period doubling substitution was not topologically mixing and they asked if a more general method could be established to determine when an arbitrary random substitution gives rise to a topologically mixing subshift. We address that question here and give an answer for large families.

For deterministic substitutions, the classification of which substitution subshifts are topologically mixing is still incomplete. Dekking and Keane studied topological mixing for deterministic substitutions in the 70s [10] and provided the first example of a minimal system which is topologically strongly mixing of order two but not order three. Importantly, they also established that the presence of mixing depends not just on the abelianisation of the substitution, but also on the order of letters. The most recent attempt at providing a full classification is the work of Kenyon, Sadun and Solomyak [20], where they were able to determine the presence of mixing for large families of substitutions (their conditions do depend only on the abelianisation). While we do not address the same question (that is, in the deterministic setting), we use their results as a guide in our investigation of topological mixing of random substitutions, and as a tool in the study itself, capitalising on the close relationship between random substitutions and their deterministic counterparts.

In Section 2, we introduce the basic definitions of primitive and compatible random substitutions, as well as topological mixing in the context of subshifts over finite alphabets. We also outline our main results and compare them with the corresponding results on deterministic substitutions established in [20]. In Section 3, we prove our main results on determining topological mixing for primitive compatible random substitutions when the second largest eigenvalue of the substitution matrix is greater than 1 in modulus. In Section 4, we show that if a primitive compatible random substitution is irreducible Pisot, then its subshift is $C$-balanced and use this to prove that irreducible Pisot random substitutions are not topologically mixing as long as a mild recognisability condition is satisfied. In Section 5, we exhibit examples for each of our main theorems to illustrate their application. Significantly, we establish that the random Fibonacci
substitution subshift is not topologically mixing; we weakened the recognisability condition enough in Section 4 for the main result to apply to this case (the random Fibonacci substitution does not satisfy full recognisability). Finally, in Section 6 we prove analogous theorems for \( \mathbb{R} \)-actions on random substitution tiling spaces.

2. Definitions and main results

2.1. Random substitution subshifts. Let \( \mathcal{A} = \{a_1, a_2, \ldots, a_d\} \) be a finite alphabet. For a word \( w = w_1 \cdots w_n \) over \( \mathcal{A} \), let \( |w| := n \) denote the length of \( w \) and let \( \mathcal{A}^n \) denote the set of all words whose length is equal to \( n \). Let \( \mathcal{A}^+ = \bigcup_{n=1}^{\infty} \mathcal{A}^n \) be the set of all non-empty words over \( \mathcal{A} \). Appending the empty word \( \epsilon \) then yields the free monoid \( \mathcal{A}^* = \mathcal{A}^+ \cup \{\epsilon\} \) of all finite words over \( \mathcal{A} \) under concatenation. A subword of a word \( w = w_1 \cdots w_n \) is a word \( w_{[i,j]} = w_i \cdots w_j \) for some \( 1 \leq i \leq j \leq n \). The set \( \mathcal{A}^\mathbb{Z} \) of all bi-infinite sequences over \( \mathcal{A} \) forms a compact metrisable space under the product topology, where \( \mathcal{A} \) is endowed with the discrete topology. The shift map \( \sigma : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z} \) given by

\[
\sigma(x)_n = x_{n+1}
\]

is then a homeomorphism. A subshift \( X \subseteq \mathcal{A}^\mathbb{Z} \) is a closed non-empty subspace of \( \mathcal{A}^\mathbb{Z} \) that is invariant under the shift action \( \sigma \). That is, \( \sigma(X) = X \). The language of a subshift \( \mathcal{L}(X) \) is the set of all subwords of elements of \( X \),

\[
\mathcal{L}(X) = \{w \in \mathcal{A}^* \mid w \text{ is a finite subword of some } x \in X\}.
\]

A deterministic substitution \( \theta \) on \( \mathcal{A} \) is a map \( \theta : \mathcal{A} \to \mathcal{A}^+ \) and uniquely extends to a map \( \theta : \mathcal{A}^+ \to \mathcal{A}^+ \) by concatenation \( \theta(w_1 \cdots w_n) := \theta(w_1) \cdots \theta(w_n) \). The action of a deterministic substitution on a bi-infinite element \( x \in \mathcal{A}^\mathbb{Z} \) is defined analogously, where the only ambiguity is where to place the origin. The usual convention is for \( \theta(x)_0 \) to be the first letter of \( \theta(x_0) \). Deterministic substitutions are well-studied [4, 12]. We are interested in a generalisation which assigns a finite set of words, rather than only a single word, to each letter of the alphabet.
Definition 1: Let $A$ be a finite alphabet, and let $\mathcal{P}(A^+)$ denote the power set of $A^+$. A random substitution is a map $\vartheta : A \to \mathcal{P}(A^+) \setminus \emptyset$ such that, for each $a \in A$, $\vartheta(a)$ is a non-empty finite set. We can extend $\vartheta$ to a function $\vartheta : A^+ \to \mathcal{P}(A^+) \setminus \emptyset$ by concatenation

$$\vartheta(v_1 \cdots v_n) = \vartheta(v_1) \cdots \vartheta(v_n) := \{w_1 \cdots w_n \mid w_i \in \vartheta(v_i), 1 \leq i \leq n\}.$$

We then extend $\vartheta$ to a function $\vartheta : \mathcal{P}(A^+) \setminus \emptyset \to \mathcal{P}(A^+) \setminus \emptyset$ by concatenation

$$\vartheta(B) := \bigcup_{w \in B} \vartheta(w).$$

Consequently, we can now take powers of $\vartheta$ by composition, so $\vartheta^k : \mathcal{P}(A^+) \setminus \emptyset \to \mathcal{P}(A^+) \setminus \emptyset$ is defined for any $k \geq 0$, where

$$\vartheta^0 := \text{Id}_{\mathcal{P}(A^+) \setminus \emptyset}$$

is the identity and

$$\vartheta^{k+1} := \vartheta \circ \vartheta^k.$$

An element of $\vartheta^k(a)$, where $a \in A$, is called a super-word of degree $k$.

Definition 2: A word $w \in A^+$ is called a realisation of $\vartheta$ on a word $v \in A^+$ if $w \in \vartheta(v)$. A word $w$ that is a realisation of $\vartheta$ on a legal word $v \in \mathcal{L}_\vartheta$ is called an inflation word. Similarly, a bi-infinite sequence $x \in A^\mathbb{Z}$ is a realisation of $\vartheta$ on a bi-infinite sequence $y \in A^\mathbb{Z}$ if $x \in \vartheta(y)$.

We say that $\vartheta$ has constant length $\ell$ if all of the super-words of degree 1 have the same length $\ell \geq 1$. A deterministic substitution $\theta$ is called a marginal of a random substitution $\vartheta$ if $\theta(a) \in \vartheta(a)$ for all $a \in A$. The random substitution $\vartheta$ is then said to be a local mixture of its marginals $\{\theta_i\}_{i \in I}$. We say that a word $w \in A^+$ is $\vartheta$-legal if there is a natural number $k$ such that $w$ is a subword of some super-word of degree $k$.

Example 3: The random Fibonacci substitution is given by

$$a \mapsto \{ab, ba\},$$

$$b \mapsto \{a\}$$

and was first introduced by Godrèche and Luck [14]. Note that $aa$ is a subword of $baa \in \vartheta(ab) \subset \vartheta^2(a)$, while $bb$ is a subword of $abba \in \vartheta(aa)$, and so is a subword of an element of $\vartheta^3(a)$. This makes both $aa$ and $bb$ $\vartheta$-legal. However, we will see that $bbb$ and $aaaaa$ are not $\vartheta$-legal.
The marginals of the random Fibonacci substitution \( \vartheta \) are given by the deterministic Fibonacci substitution \( \theta_1: a \mapsto ab, b \mapsto a \) and its reflection \( \theta_2: a \mapsto ba, b \mapsto a \). While the word \( bb \) is \( \vartheta \)-legal, it is neither \( \theta_1 \)-legal nor \( \theta_2 \)-legal, since the only way for \( bb \) to be part of an inflation word is for the first \( b \) to be part of \( \theta_1(a) = ab \) and for the second \( b \) to be part of \( \theta_2(a) = ba \).

Let \( \Phi: A^* \to \mathbb{N}_0^d: w \mapsto (|w|_{a_1}, |w|_{a_2}, \ldots, |w|_{a_d})^T \) denote the abelianisation function, where \( |w|_a \) denotes the number of occurrences of \( a \) in \( w \). That is, \( \Phi \) takes a word \( w \in A^* \) and enumerates the number of occurrences of each letter in \( w \).

**Definition 4:** Let \( \vartheta \) be a random substitution on \( A = \{a_1, a_2, \ldots, a_d\} \) and let \( \{\theta_i\}_{i \in I} \) denote the set of marginals of \( \vartheta \). We say that \( \vartheta \) is compatible if the abelianisations of its marginals coincide. That is, for each \( a \in A \), all words in \( \vartheta(a) \) have the same abelianisation. In that case, the substitution matrix \( M_{\vartheta} \) of \( \vartheta \) is given by

\[
(M_{\vartheta})_{ij} := |\vartheta(a_j)|_{a_i}
\]

for all \( 1 \leq i, j \leq d \). That is, the common abelianisation of \( \vartheta(a_i) \) determines the \( i \)th column of \( M \) (not the \( i \)th row).

A matrix \( M \) is primitive if there exists a power \( p \) such that all entries of \( M^p \) are positive. We say that the compatible random substitution \( \vartheta \) is primitive if \( M_{\vartheta} \) is primitive. In that case, each super-word of degree \( p \) contains at least one copy of each letter. If \( \vartheta \) is primitive, we let \( \lambda_1 := \lambda_{PF} \) denote the Perron–Frobenius (PF) eigenvalue of \( M_{\vartheta} \), the unique largest eigenvalue. A compatible random substitution is Pisot if \( \lambda_1 \) is a Pisot number—an algebraic integer greater than 1, all of whose algebraic conjugates lie in the open unit disk. If the characteristic polynomial of \( M_{\vartheta} \) is irreducible, then we likewise call \( \vartheta \) irreducible.

When a compatible primitive random substitution \( \vartheta \) is defined on a two-letter alphabet, we refer to the PF-eigenvalue \( \lambda_1 \) as the first eigenvalue of \( M_{\vartheta} \) and to the other eigenvalue \( \lambda_2 \) as the second eigenvalue.

**Example 5:** The random Fibonacci substitution \( \vartheta: a \mapsto \{ab, ba\}, b \mapsto \{a\} \) is compatible as

\[
\Phi(ab) = \Phi(ba) = (1, 1)^T \quad \text{and} \quad \Phi(b) = (1, 0)^T.
\]
The corresponding substitution matrix is \((\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array})\) with irreducible characteristic polynomial \(\lambda^2 - \lambda - 1\), first eigenvalue \(\lambda_1 = \frac{1+\sqrt{5}}{2}\) (the golden ratio), and second eigenvalue \(\lambda_2 = \frac{1-\sqrt{5}}{2}\). Since \(|\lambda_2| < 1\), \(\lambda_1\) is a Pisot number. Thus the random Fibonacci substitution is irreducible Pisot.

**Definition 6:** The language of \(\mathcal{L}_\vartheta\) of a random substitution \(\vartheta\) on \(\mathcal{A}\) is

\[
\mathcal{L}_\vartheta = \{ w \in \mathcal{A}^* | w \text{ is } \vartheta\text{-legal} \}.
\]

The set of length-\(n\) legal words for \(\vartheta\) is denoted \(\mathcal{L}_\vartheta^n := \mathcal{L}_\vartheta \cap \mathcal{A}^n\). The random substitution subshift of \(\vartheta\) (RS-subshift) is

\[
X_\vartheta := \{ x \in \mathcal{A}^\mathbb{Z} | w \text{ is a finite subword of } x \Rightarrow w \in \mathcal{L}_\vartheta \}.
\]

It is easy to verify that \(X_\vartheta\) is a closed, shift-invariant subspace of the full shift \(\mathcal{A}^\mathbb{Z}\), so \(X_\vartheta\) is a subshift. A primitive compatible random substitution gives rise to a non-empty RS-subshift, since each of its marginals already gives rise to a non-empty subshift.

A subspace \(Y \subset X_\vartheta\) is called **substitutive** if there exists a primitive deterministic substitution \(\theta\) such that \(Y = X_\theta\). Many results relating to the dynamics and topology of primitive RS-subshifts were presented in the works of Gohlke, Rust and Spindeler [18, 27] including the following results that will be useful later.

**Theorem 7** ([27, 18]): Let \(\vartheta\) be a primitive random substitution with a non-empty RS-subshift \(X_\vartheta\). Then:

(a) \(X_\vartheta\) contains an element with a dense shift-orbit;

(b) either \(X_\vartheta\) is substitutive or there are infinitely many distinct substitutive subspaces of \(X_\vartheta\);

(c) the union of the substitutive subspaces of \(X_\vartheta\) is dense in \(X_\vartheta\).

### 2.2. Mixing of Substitution Subshifts

In the remainder of this section, we outline the main results of our work. First, we introduce the property in which we are principally interested.

**Definition 8:** A dynamical system \((X, T)\) is said to be **topologically mixing** if for any two non-empty open subsets \(U, V \subset X\), there exists a natural number \(N\) such that for all natural numbers \(n \geq N\),

\[
T^n(U) \cap V \neq \emptyset.
\]
If $v = v_1 \cdots v_m$ is a word of length $m$ and if $n$ is an integer, the **cylinder set** $[v]_n$ is the set of $x \in X_\vartheta$ such that $x_{i+n-1} = v_i$ for $i = 1, \ldots, m$. That is, it is the set of bi-infinite words that contain the word $v$ starting at position $n$. These cylinder sets form a basis for the topology of a subshift $X$. If $\sigma : X \to X$ is the shift map, then for large $n$, the statement that

$$T^n([u]_{n_1}) \cap [v]_{n_2} \neq \emptyset$$

is equivalent to the existence of a word $uwv \in \mathcal{L}(X)$ where $uw$ has length $n + n_2 - n_1$. This allows us to recast topological mixing for subshifts in purely combinatorial terms.

**Definition 9:** A subshift $(X, \sigma)$ is said to be **topologically mixing** if for any two words $u, v \in \mathcal{L}(X)$, there exists a natural number $N$ such that for all natural numbers $n \geq N$, there exists a word $w$ of length $n$ such that $uwv \in \mathcal{L}(X)$.

Topological mixing is preserved under taking factor maps. That is, if $f : X \to Y$ is a factor map of dynamical systems and $X$ is mixing, then $Y$ is also mixing. In particular, topological mixing is preserved under topological conjugacy. It is well-known [21] that a shift of finite type $X_A$ is topologically mixing if and only if $A$ is a primitive matrix, where $A$ is the adjacency matrix of a directed graph $G$ and $X_A$ is the vertex-shift on $G$. It has been shown by Gohlke, Rust and Spindeler [18] that every topologically transitive shift of finite type can be realised up to topological conjugacy as a primitive (though often non-compatible) RS-subshift. This then gives us a class of RS-subshifts that are topologically mixing, including the full shift $A^\mathbb{Z}$ and the golden mean shift $X_A$ with $A = (\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix})$. There also exist RS-subshifts that are not topologically mixing such as the RS-subshift associated with the random period doubling substitution $\vartheta_{PD} : a \mapsto \{ab, ba\}, b \mapsto \{aa\}$ [27].

Recall that a subshift $X$ is aperiodic if $X$ contains no periodic orbits. We say that a random substitution $\vartheta$ is aperiodic if its associated RS-subshift $X_\vartheta$ is aperiodic (and similarly for deterministic substitutions). Our main results generalise those appearing in the work of Kenyon, Sadun and Solomyak [20] to the random setting.

**Theorem 10** ([20, Prop. 1.1]): Let $\theta$ be a primitive aperiodic deterministic substitution on an alphabet $A$. If the subshift $X_\theta$ is topologically mixing, then

$$\gcd\{|\theta^n(a)| : a \in A\} = 1$$

for all $n \geq 1$. 
When $|\lambda_2| > 1$ and the alphabet only consists of two letters, they were able to show that this criterion is also sufficient for topological mixing.

**Theorem 11 ([20, Thm. 1.2]):** Let $\theta$ be a primitive aperiodic deterministic substitution defined on a two-letter alphabet $\mathcal{A}$ with second eigenvalue $\lambda_2$ greater than 1 in modulus. The subshift $X_\theta$ is topologically mixing if and only if $\gcd\{|\theta^n(a)| : a \in \mathcal{A}\} = 1$ for all $n \geq 1$.

Our first goal is to extend Theorem 10 to the random case. With this in mind, we introduce an additional condition in Section 3.1 referred to as local recognisability, which means that legal words of the RS-subshift can be uniquely desubstituted given a sufficiently large legal neighbourhood of the word. This condition is an analogue of local recognisability for deterministic substitutions which is equivalent to aperiodicity in the primitive deterministic setting, thanks to a celebrated result of Mossé [24]. Unfortunately, no such result is known for random substitutions, and the situation must necessarily be more complicated as, in contrast to the deterministic case, recognisability does not follow from aperiodicity. A classic example is given by the random Fibonacci substitution whose RS-subshift is aperiodic because its PF-eigenvalue $\lambda_1 = \frac{1+\sqrt{5}}{2}$ is not an integer, but it is not recognisable [26]. It is known that the set of periodic points of a primitive RS-subshift is either empty (hence, aperiodic) or dense [27].

By assuming local recognisability, which holds automatically for aperiodic deterministic substitutions, we are able to generalise Theorem 10 to the random setting.

**Theorem 12:** Let $\vartheta$ be a primitive compatible random substitution on an alphabet $\mathcal{A}$ that is locally recognisable. If the RS-subshift $X_\vartheta$ is topologically mixing, then $\gcd\{|\vartheta^n(a)| : a \in \mathcal{A}\} = 1$ for all $n \geq 1$.

Our second main result is an extension of Theorem 11 to the random case. To prove the necessary direction, we consider the substitutive subspaces of a given RS-subshift associated with the marginals of its powers. In order to apply Theorem 11 directly, we need to show that each of these substitutive subspaces is aperiodic. We do this by providing a total classification of periodicity for primitive substitutions on two letters in Proposition 26. (To the best of our knowledge, this classification is new.)
THEOREM 13: Let \( \vartheta \) be a primitive compatible random substitution on a two-letter alphabet \( \mathcal{A} \) that is locally recognisable with second eigenvalue \( \lambda_2 \) greater than 1 in modulus. The RS-subshift \( X_\vartheta \) is topologically mixing if and only if \( \gcd\{|\vartheta^n(a)| : a \in \mathcal{A}\} = 1 \) for all \( n \geq 1 \).

We note that the local recognisability condition is only needed in one direction of the proof; our proof that the gcd condition is sufficient for topological mixing does not rely on recognisability.

For a primitive compatible random substitution \( \vartheta \) of constant length \( \ell > 1 \), by compatibility we have \( \gcd\{|\vartheta^n(a)| : a \in \mathcal{A}\} = \ell^n \) for all \( n \geq 1 \). The following then follows from Theorem 12.

COROLLARY 14: Let \( \vartheta \) be a primitive compatible constant-length random substitution on an alphabet \( \mathcal{A} \) that is locally recognisable. The RS-subshift \( X_\vartheta \) is not topologically mixing.

Note however that even if the PF-eigenvalue of a random substitution is an integer, the substitution may not be constant length (see Example 36). In the case that \(|\lambda_2| > 1\), we can extend the above corollary to the case of integral PF-eigenvalue using the following short argument.

PROPOSITION 15: Let \( \vartheta \) be a primitive compatible random substitution on a two-letter alphabet that is locally recognisable with integer first eigenvalue \( \lambda_1 \) and second eigenvalue \( \lambda_2 \) greater than 1 in modulus. Then the RS-subshift \( X_\vartheta \) is not topologically mixing.

Proof. Let \( \theta \) be a marginal of \( \vartheta \). By compatibility, \( \lambda_1 \) is also the first eigenvalue of \( \theta \). There is a classical construction, first appearing in the work of Dekking in a special case [9, Sec. V, Theorem 1] and whose general statement is folklore but sketched in several places such as [6], which produces a primitive deterministic substitution \( \theta_{cl} \) of constant length \( \ell = \lambda_1 \) such that \( X_\theta \) and \( X_{\theta_{cl}} \) are topologically conjugate. Since \( \vartheta \) is locally recognisable, the marginal \( \theta \) is also locally recognisable, and so is aperiodic by Mossé’s theorem [24]. Theorem 10 then states that \( X_{\theta_{cl}} \) is not topologically mixing, insofar as \( \theta_{cl} \) is constant-length. Thus, the subshift \( X_{\theta} \) is also not topologically mixing. By Theorem 11, there must exist some \( n \geq 1 \) such that \( \gcd\{|\theta^n(a)| : a \in \mathcal{A}\} > 1 \). But then

\[
\gcd\{|\vartheta^n(a)| : a \in \mathcal{A}\} = \gcd\{|\theta^n(a)| : a \in \mathcal{A}\} > 1,
\]

so by Theorem 13, \( X_\vartheta \) is not topologically mixing. \( \blacksquare \)
Our next main result concerns mixing properties of irreducible Pisot random substitutions. First, we show in Proposition 33 that the irreducible Pisot property implies $C$-balancedness. That is, the number of occurrences of a letter in a legal word does not differ from the statistically expected proportion by more than a uniformly bounded value.

Some well-known Pisot random substitutions such as the random Fibonacci substitution fail to be locally recognisable. However, they often satisfy a weaker condition which we call “admitting recognisable words at all levels”. For an irreducible Pisot random substitution $\vartheta$, this is enough to imply that $X_\vartheta$ is not topologically mixing via the following theorem:

**Theorem 16:** Let $\vartheta$ be primitive compatible random substitution with a $C$-balanced RS-subshift $X_\vartheta$. There exists a constant $N$ such that if $\vartheta$ admits a level-$n$ recognisable word for some $n \geq N$, then $X_\vartheta$ is not topologically mixing.

In particular, we are able to show that the random Fibonacci substitution $\vartheta$ admits recognisable words at all levels (Example 37), so $X_\vartheta$ is not topologically mixing. It should be noted that in the deterministic setting, Theorem 16 is automatic in the Pisot case thanks to the fact that, for minimal systems, topological mixing implies topological weak mixing. This is not necessarily the case for non-minimal systems such as RS-subshifts, and so we require a direct proof of non-mixing.

Finally, we turn our attention to tilings associated with our subshifts. To each letter $a_i$ we associate a labelled interval (tile) of length $\ell(a_i) = \ell_i$ and to each (possibly infinite) word we associate a concatenation of the tiles corresponding to the letters. Instead of considering a $\mathbb{Z}$-action on a space of bi-infinite words, we consider an $\mathbb{R}$-action on a space of tilings. Remarkably, the theorems for topological mixing of random substitution tilings are nearly identical to those of random substitution subshifts, and so are not repeated here, only with the condition that $\gcd\{|\vartheta^n(a)| : a \in A\} = 1$ for all $n \geq 1$ replaced by the existence of two (or more) tiles whose lengths have an irrational ratio.

### 3. Topological mixing for general primitive random substitutions

Let $\vartheta$ be a primitive random substitution on a finite alphabet $\mathcal{A}$ and let $u$ be a $\vartheta$-legal word. By the definition of $\vartheta$-legality, there is a natural number $k$ such that $u$ is a subword of a particular realisation $w_a \in \vartheta^k(a)$ for some letter $a \in \mathcal{A}$. 
For each $b \neq a$, pick such a word $w_b \in \vartheta^k(b)$ arbitrarily. Now, define the deterministic substitution $\theta_u : A \to A^+$ by $\theta_u(a) = w_a$ and $\theta_u(b) = w_b$ for each $b \in A \setminus \{a\}$. Clearly, $\theta_u$ is a primitive deterministic substitution containing $u$ as a $\theta_u$-legal word whose associated subshift $X_{\theta_u}$ is contained in $X_\vartheta$. Furthermore, the substitution matrix of $\theta_u$ is $M^k_\vartheta$, which is primitive.

This construction is taken from [18] where it was used to prove Theorem 7(c). If a substitution $\vartheta$ is a marginal of a power of $\varphi$, we call the subspace $X_\vartheta \subset X_\varphi$ a basic subspace of $\vartheta$. All basic subspaces are substitutive by construction and each $X_{\theta_u}$ is a basic subspace. The union of the basic subspaces of $\vartheta$ is then dense in $X_\varphi$ since, by the construction of the previous paragraph, every $\varphi$-legal word $u$ appears in the language of the basic subspace $X_{\theta_u}$. The basic subspaces of $X_\varphi$ will play an important role in allowing us to convert mixing properties of deterministic substitutions to random substitutions.

3.1. A sufficient condition. To prove Theorem 12, we need to introduce the concept of local recognisability. Roughly speaking, this means that if $x$ and $y$ are bi-infinite words and $x \in \varphi(y)$, then we can deduce the value of $y_0$ from the restriction of $x$ to a region of fixed size around the origin. To make this precise, we must first introduce the notions of inflation word decompositions and induced inflation word decompositions. We then prove intermediate results concerning these constructions that will be used in the proof of Theorem 12.

**Definition 17 ([13]):** Let $\vartheta$ be a random substitution and let $u \in \mathcal{L}_\vartheta$ be a legal word. Let $n \geq 1$ be a natural number. For words $u_i \in A^+$, the tuple $[u_1, \ldots, u_\ell]$ is called a $\vartheta^n$-cutting of $u$ if

$$u_1 \cdots u_\ell = u$$

and there exists a $\vartheta$-legal word $v = v_1 \cdots v_\ell$ such that:

- For $i = 2, \ldots, \ell - 1$, $u_i$ is a super-word of level $n$ built from the letter $v_i$. That is, $u_i \in \vartheta^n(v_i)$. Note that $u_i$ is a word, while $v_i$ is a single letter.
- $u_1$ is the suffix of a super-word of level $n$ built from $v_1$, and
- $u_\ell$ is the prefix of a super-word of level $n$ built from $v_\ell$.

That is, $u$ is contained in a realisation of $\vartheta^n(v)$, which is a concatenation of $n$-super-words, with each of the interior $u_i$’s being one of those $n$-super-words.
We call $v$ a root of the $\vartheta^n$-cutting and we call $([u_1, \ldots, u_\ell], v)$ the corresponding level-$n$ inflation word decomposition of $u$. If $u$ actually is a realisation of $\vartheta^n(v)$, so $u_1 \in \vartheta^n(v_1)$ and $u_\ell \in \vartheta^n(v_\ell)$, then we say that $u$ is an exact level-$n$ inflation word. Finally, we let $D_{\vartheta^n}(u)$ denote the set of all level-$n$ decompositions of the $\vartheta$-legal word $u$.

Example 18: Let $\vartheta: a \mapsto \{ab, ba\}, b \mapsto \{a\}$ be the random Fibonacci substitution. The word $aab$ has two possible $\vartheta$-cuttings $[a, ab]$ and $[a, a, b]$ with each having two distinct associated roots. The set of level-1 inflation word decompositions of $aab$ is

$$D_{\vartheta}(aab) = \{([a, ab], ba), ([a, ab], aa), ([a, a, b], bba), ([a, a, b], aba)\}.$$  

In this example, $abba$ is an exact level-1 inflation word and $abbaaaabbaaaabba$ is an exact level-1, -2, -3, and -4 inflation word. However, $bb$ is not an exact (level-1) inflation word, since any concatenation of 1-super-words containing $bb$ must also contain some $a$’s. Note that having a unique $\vartheta$-cutting does not lead to the uniqueness of roots. For example, consider $\vartheta^2: a \mapsto \{aba, baa, aab\}, b \mapsto \{ab, ba\}$. The word $bb$ has a unique $\vartheta^2$-cutting $[b, b]$, but this $\vartheta^2$-cutting can come from four distinct inflation word decompositions:

$$D_{\vartheta^2}(bb) = \{([b, b], aa), ([b, b], ab), ([b, b], ba), ([b, b], bb)\}.$$  

Similarly, having a unique root does not mean that one has a unique $\vartheta$-cutting. For example, under the random period doubling substitution

$$a \mapsto \{ab, ba\},$$

$$b \mapsto \{aa\},$$

the word $bab$ can only come from the legal word $aa$ but it has two possible $\vartheta$-cuttings; specifically,

$$D_{\vartheta}(bab) = \{([ba, b], aa), ([b, ab], aa)\}.$$  

Let $u$ be a $\vartheta$-legal word and let $u_{[i,j]}$ be a subword. An inflation word decomposition of $u$ restricts to an inflation word decomposition of $u_{[i,j]}$, which we call an induced inflation word decomposition. The idea is simple, but the precise definition is somewhat technical:
Definition 19 ([13]): Let \( \vartheta \) be a random substitution and let
\[
d = ([u_1, \ldots, u_\ell], v) \in D_\vartheta(u)
\]
be an inflation word decomposition of \( u \). For \( 1 \leq i \leq j \leq |u| - 1 \), we write \( d_{[i,j]} \) for the **induced inflation word decomposition** on the subword \( u_{[i,j]} \), defined by
\[
d_{[i,j]} = ([u_1, \ldots, u_\ell], v)_{[i,j]} = ([\hat{u}_{k(i)}, u_{k(i)}+1, \ldots, u_{k(j)}-1, \hat{u}_{k(j)}], v_{[k(i),k(j)]}),
\]
where \( 1 \leq k(i) \leq k(j) \leq \ell \) are natural numbers such that
\[
|u_1 \cdots u_{k(i)}| < i \leq |u_1 \cdots u_{k(i)}| \quad \text{and} \quad |u_1 \cdots u_{k(j)}| \leq j < |u_1 \cdots u_{k(j)+1}|
\]
\( \hat{u}_{k(i)} \) is a suffix of \( u_{k(i)} \) and \( \hat{u}_{k(j)} \) is a prefix of \( u_{k(j)} \) such that
\[
\hat{u}_{k(i)} u_{k(i)}+1 \cdots u_{k(j)}-1 \hat{u}_{k(j)} = u_{[i,j]}.
\]

Example 20: Let \( \vartheta: a \mapsto \{ab, ba\}, b \mapsto \{a\} \) be the random Fibonacci substitution. The legal word \( u = ababa \in L_\vartheta \) has exactly five level-1 inflation word decompositions given by
\[
D_\vartheta(u) = \{(a, ba, ba), (a, ba, ba), (a, ba, ba, aa), (ab, ab, a), (ab, ab, a), (ab, ab, a, aa), (ab, a, ba, aba)\}.
\]

For the subword \( u_{[2,4]} = bab \) of \( u \), the first two elements of \( D_\vartheta(u) \) yield the induced decomposition
\[
d_{[2,4]}^{(1)} = ([b], b, [a]),
\]
the next two elements yield
\[
d_{[2,4]}^{(2)} = ([b], ab, [a]),
\]
while the last element yields
\[
d_{[2,4]}^{(3)} = ([b], a, [b], aba).
\]
Thus,
\[
\# \{d_{[2,4]} \mid d \in D_\vartheta(u) \} = 3.
\]
In this example, all three possible \( \vartheta \)-cuttings of \( bab \) are induced from cuttings of \( u \).

By contrast, the word \( u' = bbaba \) has a unique inflation word decomposition \( d' = ([b], ba, ba), aaa) \) which yields a unique induced inflation word decomposition on the subword \( u'_{[2,4]} = bab \) given by \( d'_{[2,4]} = ([b], b, [a]) \). That is, when \( bab \) sits inside \( u \), the embedding tells us nothing about the \( \vartheta \)-cutting of \( bab \), but when \( bab \) sits inside \( u' \), the embedding uniquely defines the \( \vartheta \)-cutting of \( bab \).
Definition 21 ([13]): Let \( \vartheta \) be a random substitution and let \( u \in \mathcal{L}_\vartheta \) be a legal word. We say that \( u \) is **recognisable** if there exists a natural number \( N \) such that for each legal word of the form \( w = u^{(l)}uu^{(r)} \) with
\[
|u^{(l)}| = |u^{(r)}| = N,
\]
all inflation word decompositions of \( w \) induce the same inflation word decomposition of \( u \). That is, knowing the \( N \) letters to the left of \( u \) and the \( N \) letters to the right of \( u \) determines a unique induced inflation word decomposition of \( u \). We call the minimum such \( N \) the **radius of recognisability** for \( u \). If \( u \) is recognisable with respect to the \( n \)th power \( \vartheta^n \) of the random substitution, then we say that \( u \) is **level-\( n \) recognisable** with respect to \( \vartheta \).

Example 22: Let \( \vartheta : a \mapsto \{ab, ba\} \), \( b \mapsto \{a\} \) be the random Fibonacci substitution with marginals \( \theta_1 \) and \( \theta_2 \) such that \( \theta_1(a) = ab \) and \( \theta_2(a) = ba \). The word \( w = abbaaabbaaaabba \) is a level-1, -2, -3, and -4 recognisable word with radius 0 at each level, as it has the following sets of unique inflation word decompositions:
\[
\begin{align*}
D_{\vartheta}(w) &= \{([ab, ba, a, a, ab, ba, a, a, ab, ba], aabbaaabbba)\}, \\
D_{\vartheta^2}(w) &= \{([ab, ba, aab, baa, aab, ba], baaaab)\}, \\
D_{\vartheta^3}(w) &= \{([abbaa, aab, baa, aabba], abba)\}, \\
D_{\vartheta^4}(w) &= \{([abbaaab, baaaabba], aa)\}.
\end{align*}
\]

By contrast, the word \( bab \) turns out not to be recognisable at level 1 (or at any level, actually), as there exist arbitrarily long legal extensions of \( bab \) that admit both an inflation word decomposition that induces the decomposition \( ([ba, b], aa) \) of \( bab \), and another decomposition that induces \( ([b, ab], aa) \). Specifically, if \( n \) is an odd natural number, let \( u = \theta_1^n(aa) \) with the first letter removed, which is the same as \( \theta_2^n(aa) \) with the last letter removed. The middle three letters of \( u \) are \( bab \). The inflation word decomposition of \( u \) that comes from applying \( \theta_2 \) to \( \theta_2^n(aa) \) induces the decomposition \( ([ba, b], aa) \), while the decomposition of \( u \) that comes from applying \( \theta_1 \) to \( \theta_1^n(aa) \) induces \( ([b, ab], aa) \).

Although the random Fibonacci substitution has \( \vartheta \)-legal words like \( bab \) that are not recognisable, for each natural number \( n \) there also exist \( \vartheta \)-legal words that are recognisable at level \( n \). This fact, which is sufficient to establish many mixing properties of \( X_\vartheta \), will be proven in Section 5.
Definition 23: We call a random substitution $\vartheta$ **locally recognisable** if there exists a natural number $N$ such that every $\vartheta$-legal word is recognisable with radius at most $N$. The minimum such $N$ is called the **radius of recognisability** for $\vartheta$. Similarly, for any natural number $n \geq 1$, if $\vartheta^n$ is locally recognisable, we denote by $N_{\vartheta^n}$ its radius of recognisability.

Remark 24: As in the deterministic setting, there is a notion of global recognisability for random substitutions, first appearing in the work of Rust [26]. A random substitution $\vartheta$ is globally recognisable if for each bi-infinite word $x \in X_\vartheta$ there is a unique preimage $y \in X_\vartheta$ and a unique integer $n$ in a fixed bounded range such that $\sigma^n(x) \in \vartheta(y)$. Global and local recognisability are in fact equivalent for compatible random substitutions, but in practice it is usually much easier to check local recognisability than global.

By a routine inductive argument, or by invoking the equivalence with global recognisability, one can show that local recognisability for compatible random substitutions is preserved under taking powers.

**Proposition 25:** A compatible random substitution $\vartheta$ is locally recognisable if and only if $\vartheta^n$ is locally recognisable for all $n \geq 1$.  

Unfortunately, local recognisability of a random substitution is a rather strong condition; many examples, such as the random Fibonacci substitution, are not locally recognisable. However, for many proofs it is enough that a substitution admits recognisable words at all levels. We will return to this concept when we address the proof of Theorem 16.

We are now in a position to prove the first of our main results, Theorem 12. For a random substitution $\vartheta$, we write

$$|\vartheta| := \max\{|u| \mid u \in \vartheta(A)\}.$$

**Proof of Theorem 12.** We prove the contrapositive. Suppose that

$$\gcd\{|\vartheta^n(a)| : a \in A\} = p > 1$$

for some natural number $n \geq 1$. As $\vartheta$ is locally recognisable, $\vartheta^n$ is also locally recognisable by Proposition 25. So, without loss of generality, we can assume that $n = 1$. Let $N$ be the radius of recognisability for $\vartheta$, let $u$ be a $\vartheta$-legal word of length $|u| > 2N + 2|\vartheta|$, and let $u'$ be the subword of $u$ obtained by deleting the first $N$ and last $N$ letters of $u$. By local recognisability, every
inflation word decomposition of \( u \) induces the same inflation word decomposition of \( u' \). Since \(|u'| > 2|\theta|\), this unique decomposition contains a complete 1-super-word \( u'' \) in a fixed position relative to \( u' \) (and hence relative to \( u \)).

Now suppose that \( uwu \) is a \( \theta \)-legal word and consider an inflation word decomposition of \( uwu \). This induces inflation word decompositions of the first and second \( u \), giving rise to appearances of \( u'' \) spaced exactly \(|u| + |w|\) apart. However, the length of every 1-super-word is a multiple of \( p \), so \(|u| + |w|\) must be a multiple of \( p \). Since \( p > 1 \), there are arbitrarily large values of \(|w|\) that are impossible, so \( X_\theta \) is not topologically mixing.

\[ \blacksquare \]

3.2. Substitutions on two letters. In order to apply Theorem 10, it will be helpful to classify all two-letter primitive substitutions whose subshifts are periodic. A fully detailed proof appears in the thesis of the fourth author [29] and so we only present a sketch of the proof here. The constant length version of this result appears in work of Baake, Coons and Mañibo [3]. Our proof for the general case is similar in spirit to theirs. In particular, we capitalise on the absence of asymptotic pairs for periodic subshifts.

**Theorem 26:** Let \( \theta \) be a primitive deterministic substitution on \( A = \{a, b\} \); \( X_\theta \) is periodic if and only if \( \theta \) takes one of the following forms \( \{a \mapsto u^k, b \mapsto u^\ell\} \), \( \{a \mapsto (ab)^k, b \mapsto (ba)^\ell\} \), or \( \{a \mapsto (ba)^k, b \mapsto (ab)^\ell\} \) where \( u \) is a non-empty finite word and \( k, \ell \geq 0 \).

**Proof.** It is an easy exercise to show that the three above forms lead to periodic subshifts. The non-trivial step is in showing that no other form of substitution on two-letters gives rise to a periodic subshift.

Let \( \theta \) be a primitive periodic deterministic substitution. Suppose that \( \theta \) is not of the form \( a \mapsto (ab)^k, b \mapsto (ba)^\ell \) or \( a \mapsto (ba)^k, b \mapsto (ab)^\ell \). It follows then that either \( aa \) or \( bb \) is a \( \theta \)-legal word. Without loss of generality, assume that \( aa \) is legal. By periodicity and primitivity of \( \theta \), we must then have \( aa, ab, ba \in L_\theta \).

We first put the substitution into a standard form such that the leftmost letters of \( \theta(a) \) and \( \theta(b) \) are different. To do this, we take ‘conjugates’ of the substitution. That is, if \( \theta(a) = xv \) and \( \theta(b) = xw \) for \( x \in \{a, b\} \), then we replace the substitution with the conjugate substitution \( \theta': a \mapsto vx, b \mapsto wx \). It is easy to see that conjugating a primitive substitution leaves the associated subshifts fixed, so \( X_\theta = X_{\theta'} \). We may repeat the conjugation process. Either the iteration never stops, in which case \( \theta \) is of the form \( a \mapsto u^k, b \mapsto u^\ell \), or we eventually reach
a conjugate \( \theta^{(n)} : a \mapsto xv_a, b \mapsto yv_b \) with \( x \neq y \). By squaring if necessary, we can assume that \( x = a \) and \( y = b \). This new substitution \( \theta: a \mapsto au_a, b \mapsto bu_b \) for \( s, t \in \{a, b\} \) is then the standard form we may assume our substitution to be in.

Recall that \( x, y \in X \) are called an asymptotic pair if \( x \neq y \) and either

\[
d(\sigma^n(x), \sigma^n(y)) \to 0 \quad \text{or} \quad d(\sigma^{-n}(x), \sigma^{-n}(y)) \to 0 \quad \text{as} \ n \to \infty.
\]

Periodic subshifts necessarily admit no asymptotic pairs. We now consider the possible cases for the letters \( s, t \). In each case, the existence of an asymptotic pair \( x, y \in X_\theta \) will allow us to conclude that \( X_\theta \) is aperiodic and hence reach a contradiction.

For the case \( s = a, t = b \), the existence of \( a.a \) as a legal seed for producing an asymptotic pair of fixed points for the substitution is essential. This is why we had to conclude that \( aa, ab, ba \in L_\theta \) earlier. The substitution admits two (distinct) fixed points (because \( a.a \) and \( a.b \) are both legal seeds)

\[
\cdots \theta^2(u_a) \theta(u_a) au_a a \cdot a \ u_a a \ \theta(u_a a) \theta^2(u_a a) \cdots
\]

\[
\cdots \theta^2(u_a) \theta(u_a) au_a a \cdot b \ u_b b \ \theta(u_b b) \ \theta^2(u_b b) \cdots
\]

which are elements in \( X_\theta \) that are right-asymptotic to one another.

The case \( s = a, t = a \) is analogous. The case \( s = b, t = a \) reduces to the case \( s = a, t = b \) by replacing \( \theta \) with \( \theta^2 \). For the case \( s = b, t = b \), the pair of seeds \( b.a \) and \( b.b \) work (the second seed is legal because \( ab \) is a legal word and applying \( \theta \) once produces the subword \( bb \)).

**Corollary 27:** Let \( \theta \) be a primitive periodic deterministic substitution on \( A = \{a, b\} \). The second eigenvalue \( \lambda_2 \) of the substitution matrix \( M_\theta \) is either 0, 1, or \(-1\).

**Proof.** By Theorem 26, we only need to consider the three possible forms that \( \theta \) can take. If \( \theta \) is of the form \( a \mapsto u^k, b \mapsto u^\ell \), then \( M_\theta \) has rank one, with \( \lambda_2 = 0 \). For the other forms, the substitution matrix has \((1, -1)\) as a left-eigenvector with eigenvalue \( \pm 1 \).

**Lemma 28:** Let \( \vartheta \) be a primitive compatible random substitution defined on \( A = \{a, b\} \) with \( |\lambda_2| > 1 \). If \( \gcd(|\vartheta^n(a)|, |\vartheta^n(b)|) = 1 \) for all \( n \geq 1 \) then the basic subspaces of \( \vartheta \) are topologically mixing.
Proof. Let $X_\theta$ be a basic subspace of $\vartheta$. So, $\theta(a) \in \vartheta^k(a)$ and $\theta(b) \in \vartheta^k(b)$ for some natural number $k \geq 1$. The second eigenvalue of $\theta$ then has modulus $|\lambda_2^k| > 1$. In particular, this eigenvalue is not 0 or ±1, so $X_\theta$ is non-periodic. By the compatibility of $\vartheta$, we have
\[
\gcd\{|\vartheta^n(a)|, |\vartheta^n(b)|\} = \gcd\{|\vartheta^{nk}(a)|, |\vartheta^{nk}(b)|\} = 1
\]
for all $n \geq 1$. By Theorem 10, $X_\theta$ is then topologically mixing. □

We now prove Theorem 13, the generalisation of Theorem 11 to the random setting.

Proof of Theorem 13. One direction is immediate from Theorem 12. So, it remains to show that if $\gcd\{|\vartheta^n(a)|, |\vartheta^n(b)|\} = 1$ for all $n \geq 1$, then $X_\vartheta$ is topologically mixing.

Let $u$ and $v$ be arbitrary legal words. There then exist natural numbers $k_u$ and $k_v$ such that $u$ is a subword of a legal $k_u$-super-word and $v$ is a subword of a legal $k_v$-super-word. Pick $k$ bigger than the larger of $k_u$ and $k_v$. If $a_i a_j$ is a legal 2-letter word then, by primitivity, there is an element of $\vartheta^k(a_i a_j)$ that contains $u$ in the first $k$-super-word and $v$ in the second. Thus there is a $\vartheta$-legal word of the form $uwv$.

Let $\theta := \theta_{uwv}$ be a marginal of a power of $\vartheta$ such that $uwv$ is $\theta$-legal. By Lemma 28, $X_\theta$ is topologically mixing. Thus there exists a natural number $N > 0$ such that for all $n \geq N$, there is a word $w_n$ of length $n$ such that $uw_n v$ is $\theta$-legal, and hence $\vartheta$-legal. As $u$ and $v$ were chosen arbitrarily, it follows that $X_\vartheta$ is topologically mixing. □

4. Pisot random substitutions

4.1. C-BALANCEDNESS. The concept of ‘balancedness’ was first introduced by Morse and Hedlund [23] to provide one of the many classifications of Sturmian sequences [12, Ch. 6]. Here, we consider the more general notion of $C$-balanced sequences, also called sequences of bounded discrepancy. For primitive substitutions, $C$-balancedness was classified by Adamczewski [1] and for the most part, the story changes little in the study of $C$-balancedness for compatible random substitutions. Note, however, that we do not attempt a full classification of
C-balancedness in this setting. We only consider the irreducible Pisot case, which is sufficient for our needs. An approachable account of C-balancedness is presented in the work of Berthé and Bernales [7], from which we adopt the following definitions and results.

**Definition 29:** A bi-infinite sequence \( x \in \mathcal{A}^\mathbb{Z} \) is said to be **C-balanced** if there exists a constant \( C \) such that for every \( a \in \mathcal{A} \) and all pairs \( w, w' \) of subwords of \( x \) with \( |w| = |w'| \),

\[
||w|_a - |w'|_a| \leq C.
\]

A subshift \( X \subseteq \mathcal{A}^\mathbb{Z} \) is said to be **C-balanced** if there exists a constant \( C \) such that for every \( a \in \mathcal{A} \) and all pairs \( w, w' \in \mathcal{L}(X) \) with \( |w| = |w'| \),

\[
||w|_a - |w'|_a| \leq C.
\]

**Remark 30:** Note that we only consider C-balancedness with respect to letters, and not the more general notion of C-balancedness with respect to subwords, which is a uniform bound on the deviation of the number of appearances of any legal subword within pairs of words of the same length. Indeed, for our setting, this would be asking for too much from random substitutions, where such a property necessarily requires the corresponding subshift to be uniquely ergodic. This is never the case for non-degenerate RS-subshifts.

Recall that the **frequency** of a letter \( a \in \mathcal{A} \) in a bi-infinite sequence \( x \in \mathcal{A}^\mathbb{Z} \) is the limit

\[
f_a := \lim_{n \to \infty} \frac{|x_{[-n,n]}|_a}{2n+1},
\]

if it exists, and \( x \) is said to have **uniform letter-frequencies** if, for every letter \( a \in \mathcal{A} \), the convergence of \( \frac{|x_{[k \cdot (2n+1)]}|_a}{2n+1} \) towards \( f_a \) is uniform in \( k \), when \( n \) tends to infinity. The amount \( ||w|_a - f_a|w|| \) that a subword \( w \) of \( x \) deviates from the expected number of occurrences of the letter \( a \), according to the letter-frequency of \( a \) in \( x \), is called the **discrepancy** of the letter \( a \) in the word \( w \).

**Theorem 31** ([8, Prop. 2.4]): A bi-infinite sequence \( x \in \mathcal{A}^\mathbb{Z} \) is C-balanced for some \( C \) if and only if it has uniform letter frequencies and there exists another constant \( B \) such that for any finite subword \( w \) of \( x \) and any \( a \in \mathcal{A} \),

\[
||w|_a - f_a|w|| \leq B,
\]

where \( f_a \) is the frequency of \( a \).
**Theorem 32** ([7, Thm. 2.5]): Let $\theta$ be a primitive deterministic irreducible Pisot substitution. Then the subshift $X_\theta$ is $C$-balanced for some $C$.

We now extend this result to the random setting and generalise it slightly. The proof is a simple adaptation of the proof in the deterministic setting [8], generalising the usual Dumont–Thomas prefix-suffix decomposition [11] to the setting of random substitutions.

**Theorem 33:** Let $\vartheta$ be a compatible primitive random substitution such that the second-largest eigenvalue $\lambda_2$ has modulus strictly smaller than 1. Then the RS-subshift $X_\vartheta$ is $C$-balanced for some $C$.

**Proof.** Since the Perron–Frobenius eigenvalue $\lambda_1$ has multiplicity 1, and since there are no other eigenvalues of magnitude 1 or greater, all nonzero eigenvalues of $M_\vartheta$ are algebraic conjugates of $\lambda_1$ and have algebraic multiplicity 1. The characteristic polynomial of $M_\vartheta$ is a power of $\lambda$ times an irreducible Pisot polynomial and the Jordan canonical form of $M_\vartheta$ is diagonal except for blocks associated with the eigenvalue 0.

Let $|\vartheta| = k$ denote the longest inflation word for $\vartheta$. Let $m$ be the size of the largest Jordan block associated with $\lambda = 0$. A routine calculation shows that for any letters $a, b \in \mathcal{A}$, the discrepancy of $a$ in a level-$n$ inflation word in $\vartheta^n(b)$ is the sum of two pieces. The contribution from the eigenvectors with nonzero eigenvalue is bounded by $D_1|\lambda_2|^n$ for some fixed constant $D_1$. The contribution from the generalised eigenvectors with zero eigenvalue is zero if $n \geq m$, and is bounded by a constant $D_2$ if $n < m$.

Now consider an arbitrary $\vartheta$-legal word $w$ and an inflation word decomposition that includes a complete $n$-super-word with $n$ as large as possible. That is, $w$ fully contains between 1 and $2(k-1)$ complete $n$-super-words, followed by a (possibly empty) suffix $s_n$ and preceded by a (possibly empty) prefix $p_n$ such that $s_n$ and $p_n$ are both strictly contained in $n$-super-words. The suffix $s_n$ then consists of at most $k-1$ $(n-1)$-super-words followed by a further suffix $s_{n-1}$ that is strictly contained in an $(n-1)$-super-word. Continuing in this way, $s_{n-r}$ comprises at most $(k-1) (n-r-1)$-super-words and a further suffix $s_{n-r-1}$. Iterating, we get that $s_n$ is the concatenation of up to $k-1$ $(n-1)$-super-words, up to $k-1$ $(n-2)$-super-words, etc., ending with up to $k-1$ individual letters. Likewise, the prefix $p_n$ is the concatenation of up to $k-1$ letters, up to $k-1$ 1-super-words, up to $k-1$ 2-super-words, continuing through $(n-1)$-super-words.
The discrepancy of $w$ is bounded by the sum of the discrepancies of its pieces. Since there are at most $2m(k-1)$ $i$-super-words in $w$ with $i < m$, and since the generalised eigenvectors with $\lambda = 0$ contribute at most $D_2$ to each one, the total contribution from those generalised eigenvectors is at most $2m(k-1)D_2$. The contributions of the remaining eigenvectors to the discrepancy of each $i$-super-word is bounded by $D_1|\lambda_2|^i$, so the total discrepancy

$$||w||_a - f_a ||w||$$

of $a$ in $w$ is bounded above by

$$2m(k-1)D_2 + \sum_{i=0}^{n} 2(k-1)D_1|\lambda_2|^i < 2m(k-1)D_2 + \sum_{i=0}^{\infty} 2(k-1)D_1|\lambda_2|^i = 2m(k-1)D_2 + \frac{2(k-1)D_1}{1 - |\lambda_2|}.$$

By Theorem 31, this uniform bound implies that $X_\vartheta$ is $C$-balanced.

Remark 34: The most important application of Theorem 33 is to compatible primitive irreducible Pisot random substitutions such as the random Fibonacci substitution. However, it also applies to examples, such as the random Thue–Morse substitution, that have 0 as an eigenvalue.

We remark that, as a consequence of Theorem 33, bounded Rauzy fractals exist for irreducible Pisot random substitutions (and are almost surely independent of the choice of element chosen from the subshift). This potentially offers a new tool in the effort to solve the long-standing Pisot conjecture [2]. We therefore invite further exploration of these random substitution Rauzy fractals.

4.2. Non-mixing of Pisot random substitution subshifts. We are now in a position to prove Theorem 16. Since irreducible Pisot substitutions are $C$-balanced by Theorem 33, the following result also applies to those random substitutions. Recall the statement of Theorem 16.

**Theorem:** Let $\vartheta$ be a primitive compatible random substitution such that $X_\vartheta$ is $C$-balanced. There exists a constant $N$ such that if $\vartheta$ admits a level-$n$ recognisable word for some $n \geq N$, then the RS-subshift $X_\vartheta$ is not topologically mixing.
Proof. This proof has several steps:

1. Showing that the existence of a level-$n$ recognisable word implies the existence of a $\vartheta$-legal word $u$ such that, if $uwu$ is $\vartheta$-legal, then the length of $uw$ is the length of an exact level-$n$ inflation word. This step does not require $C$-balancedness.

2. Showing that for each length $L$ and natural number $n$, the set of abelianisations of words of the form $\vartheta^n(v)$, where $v$ is a $\vartheta$-legal word of length $L$, is bounded by a constant independent of $L$ and $n$. This is where $C$-balancedness comes in.

3. Showing that for sufficiently large $n$, the set of possible lengths of level-$n$ inflation words has natural density strictly less than 1. This step uses step 2. Combined with step 1, this shows that $X_\vartheta$ is not topologically mixing.

By natural density of a set $B$ of natural numbers (sometimes called the upper asymptotic density), we mean

$$\lim sup_{n \to \infty} \frac{1}{n} \# B \cap \{0, 1, \ldots, n - 1\}.$$ 

Step 1: Suppose that there exists a level-$n$ recognisable word $\hat{u}$ with recognisability radius $R$. Let $u_l$ and $u_r$ be words of length $R$ such that $u = u_l\hat{u}u_r$ is $\vartheta$-legal. By recognisability, all level-$n$ inflation word decompositions of $u$ result in the same decomposition of $\hat{u}$.

Now consider a level-$n$ decomposition of a $\vartheta$-legal word $uwu$. Since the decompositions of the two copies of $\hat{u}$ are identical, there are corresponding points in the two copies that are spaced by an exact level-$n$ inflation word. However, the spacing between corresponding points in the two copies of $\hat{u}$ is precisely the length of $uw$.

Step 2: Suppose that our alphabet $\mathcal{A}$ consists of $d$ letters. The number of a’s in a word of fixed length $L$ can take on at most $C + 1$ values, since the largest such value is at most $C$ plus the smallest. Repeating for all but one letter and noting that the sum of the entries of the abelianisation is fixed, we see that the set of abelianisations of words of length exactly $L$ has cardinality at most $(C + 1)^{d-1}$. Since the abelianisation of each element of $\vartheta^n(v)$ is just $M_\vartheta^n$ times the abelianisation of $v$, there are at most $(C + 1)^{d-1}$ abelianisations of exact level-$n$ inflation words whose roots have length exactly $L$. 

Step 3: The growth in the lengths of $n$-super-words is controlled by $\lambda_1$, the PF-eigenvalue of $M_\vartheta$. Specifically, there exist positive constants $c_1$ and $c_2$ such that every $n$-super-word has length at least $c_1 \lambda_1^n$ and at most $c_2 \lambda_1^n$. If $w$ is a $\vartheta$-legal exact level-$n$ inflation word with root $v$ and if $|w| < L$, then $|v| < L \lambda_1^{-n}/c_1$.

Since each possible length of $|v|$ gives rise to at most $(C+1)^{d-1}$ possible abelianisations of $w$, and since each abelianisation determines a unique length, the number of possible lengths of $w$ is bounded by $(C+1)^{d-1}L\lambda_1^{-n}/c_1$. As $L$ goes to $\infty$, the natural density of possible lengths of $w$ is bounded by $(C+1)^{d-1}/(c_1 \lambda_1^n)$. If we pick $N$ such that $\lambda_1^N > (C+1)^{d-1}/c_1$, then for all $n \geq N$ this density is less than 1.

5. Additional examples

Example 35: Consider the compatible random substitution

\[
\begin{align*}
  a &\mapsto \{abababa, bbaaaa\}, \\
  \vartheta: \quad b &\mapsto \{babb, bbab\}
\end{align*}
\]

whose substitution matrix is \((\begin{smallmatrix} 4 & 1 \\ 3 & 3 \end{smallmatrix})\) with eigenvalues

\[
\lambda_1 = \frac{1}{2}(7 + \sqrt{13}) \quad \text{and} \quad \lambda_2 = \frac{1}{2}(7 - \sqrt{13}) > 1.
\]

It can be verified that $\gcd\{|\vartheta^n(a_i)| : a_i \in A\} = 1$ for every $n \geq 1$. Thus, by Theorem 13, $X_\vartheta$ is topologically mixing.

Example 36: Consider the random substitution

\[
\begin{align*}
  a &\mapsto \{bb\}, \\
  \vartheta: \quad b &\mapsto \{abaaba, ababaa\}
\end{align*}
\]

Its substitution matrix is \((\begin{smallmatrix} 0 & 4 \\ 2 & 2 \end{smallmatrix})\) with eigenvalues $\lambda_1 = 4$ and $|\lambda_2| = 2 > 1$. Here

\[
\gcd\{|\vartheta(a_i)| : a_i \in A\} = \gcd\{2, 6\} = 2 > 1.
\]

Using the fact that the word $bb$ can only come from a substituted $a$, one can easily show that $\vartheta$ is locally recognisable. Thus, as a consequence of Theorem 12, the subshift $X_\vartheta$ is not topologically mixing. This can also be seen directly, since the spacing between any two appearances of $bb$ must be a 1-inflation word and so must have even length.
As previously noted, the random Fibonacci substitution is not locally recognisable. Likewise, neither are any of the random metallic means substitutions [22]. However, we do not need local recognisability to prove a lack of topological mixing. By Theorem 16, we just need to show that there exist recognisable words at a sufficiently high level. For the random Fibonacci substitution, we will do more, showing that recognisable words exist at all levels.

**Example 37:** Let $\vartheta$ be the random Fibonacci substitution given by

\[
\vartheta: \\
a \mapsto \{ab, ba\}, \\
b \mapsto \{a\}.
\]

Note that all four 2-letter words are $\vartheta$-legal. We begin with the seeds $F_0 = a|a$ and $F_1 = ab|ba$. We successively construct words $F_n = L_n|R_n \in \vartheta(F_{n-1})$ as follows, where the bar divides $F_n$ into level-$n$ super-words. Let $\phi$ be a map that:

- Replaces each $b$ with an $a$.
- Replaces each $a$ that is followed by a $b$ with $ba$.
- Replaces each $a$ that is preceded by a $b$ with $ab$.
- Replaces each $a$ that is preceded by an $a$ with $ab$ if the preceding $a$ was replaced with $ba$, and with $ba$ if the preceding $a$ was replaced with $ab$.

These rules are consistent as long as the $b$'s come in pairs (except for a possible lone $b$ at the beginning or end of a word), separated by an even number of $a$'s. This is true for $F_0$ and $F_1$, and follows inductively from the fact that $ba^{2k}b$ becomes $a(abba)^k a$.

We claim that all of the words $F_n$ are recognisable inflation words of level 1 with radius 0, with $F_{n-1}$ being the unique root of $F_n$. By induction, $F_n$ is then a recognisable inflation word of level $n$, again with radius 0, with unique root $aa$. The first few iterates and their inflation word decompositions and corresponding roots are given in the following table.

| $n$ | $F_n := L_n | R_n$ | $D_{\vartheta^n}(F_n)$ |
|-----|------------------|---------------------|
| 1   | $ab | ba$          | $\{([ab, ba], aa)\}$ |
| 2   | $baa | aab$        | $\{([baa, aab], aa)\}$ |
| 3   | $aabba | abbaa$    | $\{([aabba, abbaa], aa)\}$ |
| 4   | $abbaaaab | baaaabba$ | $\{([abbaaaab, baaaabba], aa)\}$ |

We will show by hand that $F_1$, $F_2$ and $F_3$ have unique 1-decompositions and then generalise to $F_n$. 
Since b’s come only from substituting a’s, the only way that bb can ever appear in a ϑ-legal word is as part of abba, decomposed as ([ab, ba], aa). That is, \( F_1 \) only decomposes as \( \phi(F_0) \). Note that this also shows that bbb is not ϑ-legal.

\( F_2 \) contains aaaaa. Since the middle two a’s are not adjacent to b’s, they must come from substituting b’s. Since bbb is not ϑ-legal, the outer two letters of the root must be a’s, so our root is abba = \( F_1 \). This argument also shows that aaaaa is not ϑ-legal, since the middle three a’s would have to all come from b’s and bbb is not ϑ-legal.

\( F_3 \) contains abbaabba. As previously noted, each abba can only come from substituting three a’s. Since aaaaa is not ϑ-legal, the outer letters of the root must be b’s, so the root is baaaab = \( F_2 \).

When \( n > 3 \), we simply repeat these arguments on each portion baab or baaaab of \( F_n \). Since baab is preceded and followed by b’s, it can only decompose as ([ba, ab], aa), and since baaaab contains two central a’s that can only come from b’s, this piece can only decompose as ([ba, a, a, ab], abba). The decomposition of any remaining prefixes and suffixes is determined by the fact that the root cannot contain three successive b’s or five successive a’s. Thus the only decomposition of \( F_n \) is as \( \phi(F_{n-1}) \). As ϑ is irreducible Pisot and admits recognisable words at all levels, it follows from Theorem 16 that the random Fibonacci subshift is not topologically mixing.

Incidentally, we can also build recognisable bi-infinite words as limits of the \( F_n \)’s. These words form a (deterministic) substitution subshift in their own right, a morphic copy of the Fibonacci subshift, with a replaced by \( A = baaaab \) and b replaced by \( B = baab \). This is because the map \( \phi \) sends baaaab to aabbaabbaa and sends baab to aabbbaa. After applying a conjugation that removes baa from the end of each substituted word and places it at the beginning, this is equivalent to \( A \to AB, B \to A \). The fact that such words are recognisable under \( \vartheta \), and not merely under \( \phi \), follows from the same arguments as with the finite words \( F_n \). Specifically, each abba can only come from aa and all of the remaining a’s can only come from b’s.

6. Random substitution tiling spaces

Subshifts are one way to build a dynamical system from a random substitution. An equally valid way is to build a space of tilings on which \( \mathbb{R} \) acts by translation [4, 28]. To each letter \( a_i \in \mathcal{A} \) we associate an interval of length \( \ell(a_i) \),
which we sometimes abbreviate as $\ell_i$. We call such an interval a tile. To each word $u = u_1 \cdots u_n$ we associate an interval of length $\sum_{i=1}^n \ell(u_i)$ obtained by laying tiles of type $u_1, u_2, \ldots, u_n$ end to end. We call such a concatenation of tiles a patch. A patch associated to an $n$-super-word is called an $n$-supertile. A patch is called $\vartheta$-legal if it is contained in a supertile of some order. A tiling is a bi-infinite patch. The tiling space $\Omega_{\vartheta}$ associated to the random substitution $\vartheta$ is the set of tilings with the property that every finite patch is $\vartheta$-legal.

The group $\mathbb{R}$ acts on $\Omega_{\vartheta}$ by translation. Specifically, if $T$ is a tiling and $t \in \mathbb{R}$, we define $T^t(T)$ to be the tiling obtained by moving all tiles in $T$ a distance $t$ to the left. (From the point of view of the tiles, this is equivalent to moving the origin a distance $t$.) Instead of explicitly invoking the group action $T^t$, we usually just call the result $T - t$.

Let $P$ be a finite patch, understood to lie at a specific location, and let $U$ be an open subset of $\mathbb{R}$. Then

$$Z_{P,U} = \{ S \in \Omega_{\vartheta} \mid P \subset S - y \text{ for some } y \in U \}$$

is called a cylinder set. By shifting the open set $U$, we can assume, without loss of generality, that the left endpoint of $P$ is at the origin. These cylinder sets form a basis for the topology of $\Omega_{\vartheta}$. This topology is also the metric topology coming from the big box metric, in which two tilings are considered $\epsilon$-close if, after translation by up to $\epsilon$, they agree on the interval $[-\epsilon^{-1}, \epsilon^{-1}]$.

The definition of topological mixing for $\mathbb{R}$ actions is similar to that for subshifts.

**Definition 38:** The dynamical system $(\Omega_{\vartheta}, T)$ is topologically mixing if, for any two open sets $Z_1$ and $Z_2$, there is a number $R$ such that, for all $t > R$,

$$T^t(Z_1) \cap Z_2 \neq \emptyset.$$ 

Although this definition is phrased only in terms of large positive values of $t$, topological mixing is also a property of $T^t$ for $t$ large and negative, since switching the roles of $Z_1$ and $Z_2$ is equivalent to changing the sign of $t$.

For subshifts, topological mixing is related to the number of letters in possible return words $w$ between $\vartheta$-legal words $u$ and $v$. That is, the possible lengths of words $w$ such that $uwv$ is $\vartheta$-legal. Topological mixing for tiling spaces boils down to studying the geometric lengths of the patches associated to those return words. Specifically, let $S_{u,v}$ be the set of lengths of all patches associated to words $uw$, where $uwv$ is $\vartheta$-legal.
Definition 39: A discrete subset $S \subset \mathbb{R}$ is said to be **asymptotically dense** if, for each $\epsilon > 0$, there exists a number $R_\epsilon$ such that every $t > R_\epsilon$ is within $\epsilon$ of an element of $S$.

**Proposition 40:** The tiling space $\Omega_\vartheta$ is topologically mixing if and only if, for all $\vartheta$-legal words $u$ and $v$, $S_{u,v}$ is asymptotically dense.

**Proof.** Without loss of generality, we can check topological mixing on sets of the form $Z_{P,(-\epsilon,\epsilon)}$, since the definition is invariant under translation of the sets and since intervals of size $2\epsilon$ form a basis for the topology of $\mathbb{R}$. We therefore consider the sets

$$Z_1 = Z_{P_1,(-\epsilon_1,\epsilon_1)} \quad \text{and} \quad Z_2 = Z_{P_2,(-\epsilon_2,\epsilon_2)},$$

where $P_1$ is a patch starting at the origin based on the $\vartheta$-legal word $u$ and $P_2$ is a patch starting at the origin based on the $\vartheta$-legal word $v$. The open set $Z_1 - t$ consists of all tilings that exhibit a copy of $P_1$ starting within $\epsilon_1$ of $-t$, while $Z_2$ consists of all tilings that exhibit a copy of $P_2$ starting within $\epsilon_2$ of the origin. These sets intersect if and only if there is a tiling in which the left endpoints of $P_1$ and $P_2$ are spaced between $t - (\epsilon_1 + \epsilon_2)$ and $t + (\epsilon_1 + \epsilon_2)$ apart. In other words, if and only if $t$ is within $\epsilon_1 + \epsilon_2$ of an element of $S_{u,v}$. The condition that $T^t((Z)_1) \cap (Z)_2 \neq \emptyset$ for all sufficiently large $t$ for all choices of $\epsilon_1$ and $\epsilon_2$ is equivalent to $S_{u,v}$ being asymptotically dense. 

Our results for subshifts, and those of [20], carry over to tiling spaces with one key adjustment. The condition that $\gcd\{|\theta^n(a)| : a \in A\} = 1$ for all $n \geq 1$ is replaced with the existence of letters $a$ and $b$ such that the ratio $\ell(a)/\ell(b)$ is irrational.

**Theorem 41** ([20, Prop. 1.1]): Let $\theta$ be a primitive aperiodic deterministic substitution on an alphabet $A$. If the tiling space $\Omega_\theta$ is topologically mixing, then there exist letters $a, b \in A$ such that $\ell(a)/\ell(b)$ is irrational.

**Proof.** If all the tile lengths are rational multiples of one another, then there is a basic length $\ell_0$ such that all tile lengths are multiples of $\ell_0$. But then every element of $S_{u,v}$ (where $u$ and $v$ are arbitrary $\vartheta$-legal words) is a multiple of $\ell_0$, so $S_{u,v}$ is not asymptotically dense.
Kenyon, Sadun and Solomyak [20] determined when a tiling space with a 2-letter deterministic substitution, with second eigenvalue either larger or smaller than 1, is topologically mixing. (The borderline case where $|\lambda_2| = 1$ and $\ell(a)/\ell(b)$ is irrational is subtle. There are examples where the tiling space is topologically mixing and examples where it isn’t.)

**Theorem 42** ([20, Thm. 1.2]): Let $\theta$ be a primitive aperiodic deterministic substitution defined on a two-letter alphabet $\mathcal{A} = \{a, b\}$. If $|\lambda_2| < 1$ or if the ratio $\ell(a)/\ell(b)$ of the two tile lengths is rational, then the tiling space $\Omega_\theta$ is not topologically mixing. If $|\lambda_2| > 1$ and the ratio is irrational, then the tiling space is topologically mixing.

We now generalise these results to the random setting. Many of the arguments are the same as for subshifts and as such will be abbreviated.

**Theorem 43:** Let $\Omega$ be a tiling space whose allowed sequences of tiles are given by a $C$-balanced subshift $X$. Then $\Omega$ is not topologically mixing.

**Proof.** As in the proof of Theorem 4.2, there are at most $(C + 1)^{d-1}$ abelianisations of words with a fixed number of letters, where $d$ is the size of our alphabet. If $\ell_1$ is the length of the shortest tile, then any patch of length $L$ has at most $L/\ell_1$ letters. Thus there are at most $L(C + 1)^{d-1}/\ell_1$ possible patch lengths of size $L$ or less. In particular, the number of elements of $S_{u,v}$ of size $L$ or less grows at most linearly with $L$, so $S_{u,v}$ cannot be asymptotically dense. ■

Since irreducible Pisot random substitutions are $C$-balanced, we have an immediate corollary that generalises half of Theorem 42. We highlight that this result is independent of any recognisability assumptions on the random substitution.

**Theorem 44:** Let $\vartheta$ be a primitive compatible random substitution on an alphabet $\mathcal{A} = \{a_1, \ldots, a_d\}$ with second eigenvalue $\lambda_2$ less than 1 in modulus. Then the associated RS-tiling space $\Omega_\vartheta$ is not topologically mixing.

Finally, we generalise the remainder of Theorem 42:

**Theorem 45:** Let $\vartheta$ be a primitive compatible random substitution on a two-letter alphabet with second eigenvalue $\lambda_2$ greater than 1 in modulus, such that the ratio of the two tile lengths is irrational. Then $\Omega_\vartheta$ is topologically mixing.
Proof. Let $u$ and $v$ be $\vartheta$-legal words and let $\theta$ be a marginal of $\vartheta$ such that $u$ and $v$ are $\theta$-legal. The second eigenvalue of $M_\theta$ is a positive power of $\lambda_2$ and so is larger than 1 in modulus. Applying Theorem 42 to $\Omega_\vartheta$, we see that $\Omega_\vartheta$ is topologically mixing, which implies that the set of lengths of the return patches from $u$ to $v$ in $\Omega_\vartheta$ is asymptotically dense. But that set is a subset of $S_{u,v}$, so $S_{u,v}$ is asymptotically dense. Since this is true for all pairs $(u,v)$, $\Omega_\vartheta$ is topologically mixing. ■

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