A Variational Principle for the Asymptotic Speed of Fronts of the Density Dependent Diffusion-Reaction Equation

R. D. Benguria and M. C. Depassier
Facultad de Física
P. Universidad Católica de Chile
Casilla 306, Santiago 22, Chile

Abstract

We show that the minimal speed for the existence of monotonic fronts of the equation \( u_t = (u^m)_{xx} + f(u) \) with \( f(0) = f(1) = 0, \) \( m > 1 \) and \( f > 0 \) in \( (0, 1) \), derives from a variational principle. The variational principle allows to calculate, in principle, the exact speed for general \( f \). The case \( m = 1 \) when \( f'(0) = 0 \) is included as an extension of the results.
Several problems arising in population growth \cite{1,2}, combustion theory \cite{3,4}, chemical kinetics \cite{5}, and others \cite{6}, lead to an equation of the form

\[
\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = F(\rho),
\]

where the source term $F(\rho)$ represents net growth and saturation processes. The flux $\vec{j}$ is given by Fick’s law

\[
\vec{j} = -D(\rho) \vec{\nabla} \rho,
\]

where the diffusion coefficient $D(\rho)$ may depend on the density or in simple cases be taken as a constant. In one dimension this leads to the equation

\[
\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left( D(\rho) \frac{\partial \rho}{\partial x} \right) + F(\rho).
\]  

(1a)

In what follows we shall assume that

\[
F(\rho) > 0 \quad \text{in} \quad (0,1), \quad \text{and} \quad F(0) = F(1) = 0,
\]  

(1b)

restrictions which are satisfied by several models. When the diffusion coefficient is constant and the additional requirement $F'(0) > 0$ is satisfied, the asymptotic speed of propagation of localized small perturbations to the unstable state $u = 0$ is bounded below and in some cases coincides \cite{7} with the value $c_L = 2 \sqrt{F'(0)}$ which is obtained from considerations on the linearized equation \cite{8}. However, when either $F'(0) = 0$ or $D(\rho)$ is not a constant, no hint for the speed of propagation of disturbances can be obtained from linear theory alone. A common choice for the diffusion coefficient is a power law, case with which we shall be concerned here. Therefore the equation that we study is

\[
\frac{\partial \rho}{\partial t} = (\rho^m)_{xx} + F(\rho)
\]  

(2a)

with

\[
F(0) = F(1) = 0, \quad \text{and} \quad F > 0 \quad \text{in} \quad (0,1).
\]  

(2b)

Aronson and Weinberger \cite{7,2} have shown that the asymptotic speed of propagation of disturbances from rest is the minimal speed $c^*(m)$ for which there exist monotonic travelling
fronts \( \rho(x, t) = q(x - ct) \) joining \( q = 1 \) to \( q = 0 \). The equation satisfied by the travelling fronts is

\[
(q^m)_{zz} + cq_z + F(q) = 0
\]  

with

\[
q(-\infty) = 1, \quad q > 0, \quad q' < 0 \quad \text{in} \quad (-\infty, \omega), \quad q(\omega) = 0
\]

where \( z = x - ct \). The wave of minimal speed is sharp, that is, \( \omega < \infty \) when \( m > 1 \) \cite{2}.

An explicit solution is known \cite{1, 2} for the case \( F(q) = q(1-q) \) and \( m = 2 \), the waveform is given by

\[
q(z) = \left[1 - \frac{1}{2}e^{z/2}\right]_+
\]

and it travels with speed \( c^*(2) = 1 \) (here \([x]_+ \equiv \max(x, 0)) \). Recently the derivative \( dc/dm \) at \( m = 2 \) has been calculated by two different methods. Its value is \(-7/24\) \cite{9, 10}. Other exact solutions for different choices for \( m \) and \( F \) have been given in \cite{11}.

The purpose of this work is to give a variational characterization of the minimal speed \( c^*(m) \) for Eq.(3) when \( m > 1 \), and as a byproduct for the case \( m = 1 \) when \( F'(0) = 0 \), both, cases for which no information is obtained from linear theory. The case \( m = 1 \) with \( F'(0) > 0 \) has been studied elsewhere \cite{13}. Lower bounds have been obtained on the minimal speed \( c^*(m) \) \cite{12}; the present results allow its exact calculation for arbitrary \( f \).

Since the selected speed corresponds to that of a decreasing monotonic front, we may consider the dependence of its derivative \( dq/dz \) on \( q \). Calling \( p(q) = -q^{m-1}dq/dz \), where the minus sign is included so that \( p \) is positive, we find that the monotonic fronts are solutions of

\[
p\frac{dp}{dq} - \frac{c^*}{m}p + \frac{1}{m}q^{m-1}F(q) = 0
\]  

with

\[
p(0) = p(1) = 0, \quad p > 0 \quad \text{in} \quad (0, 1).
\]

Although the wave of minimal speed is sharp and therefore \( q'(0) < 0 \), by its definition \( p(0) = 0 \) is true. We now show that the minimal speed \( c^*(m) \) follows from a variational principle whose Euler equation is Eq.(4a).
Let $g$ be a positive function such that $h = -g' > 0$. Multiplying Eq.(4a) by $g/p$ and integrating we obtain after integration by parts,

$$
\frac{c}{m} = \frac{\int_0^1 \left[ \frac{1}{m} q^{m-1} F(q) \frac{g(q)}{p(q)} + h(q)p(q) \right] dq}{\int_0^1 g(q) dq}.
$$

(5)

By Schwarz’s inequality, since, $q$, $F$, $g$ and $h$ are positive we know

$$
\frac{1}{m} q^{m-1} Fg \frac{g}{p} + hp \geq 2 \sqrt{\frac{1}{m} q^{m-1} Fgh}
$$

(6)

and therefore, replacing in Eq.(5) we have

$$
c \geq 2 \frac{\int_0^1 \sqrt{mq^{m-1} Fgh} dq}{\int_0^1 gdq}.
$$

(7)

This bound has been already given by us [12]. We now show that it is always possible to find a $g(q)$ such that the equality in Eq.(6) and therefore also in Eq.(7) holds. We do so by explicit construction of such a function $g$. The equality in Eq.(6) holds if

$$
\frac{1}{m} q^{m-1} Fg \frac{g}{p} = hp
$$

(8)

Let $v(q)$ be the positive solution of

$$
\frac{v'}{v} = \frac{c}{mp}
$$

(9a)

and choose

$$
g = \frac{1}{v'}.
$$

(9b)

We have then

$$
\frac{v''}{v} = \frac{(v')^2}{v^2} - \frac{c}{mp^2}p' = -\frac{c}{mp^2} q^{m-1} F(q)
$$

where we have used Eq.(9a) to eliminate $v'$ and Eq.(4a) to eliminate $p'$. Therefore,

$$
h = -g' = \frac{v''}{(v')^2} = \frac{1}{mp^2} q^{m-1} Fg > 0
$$

(9c)

where we have made use of Eqs.(9a) and (9b). With this expression for $h$, we can see that Eq.(8) is satisfied. In addition we must check that $g$ as we have defined it is such that its integral exists. In fact as it exists and moreover one can always normalize $g$ so that $g(0) = 1$ and $g(1) = 0$. From the definition of $g$ we obtain

$$
g(q) = \frac{mp(q)}{c} \exp \left[ -\int_{q_0}^q \frac{c}{mp} dq' \right]
$$
where $0 < q_0 < 1$. Since $p(1) = 0$ and $p$ is positive between 0 and 1 it follows that $g(1) = 0$. At zero no divergence occurs, as we now show. Call $\hat{c} = c/m$ and $f(q) = q^{m-1}F(q)/m$. Then Eq.(4a) reads

$$pp' - \hat{c}p + f = 0 \quad (10a)$$

with

$$f(0) = f(1) = 0 \quad \text{and} \quad f'(0) = 0. \quad (10b)$$

For this case Aronson and Weinberger [7] have shown that $p(q)$ approaches the fixed point $q = 0$ as $p = \hat{c}q = cq/m$. Then, near 0, $v'/v \approx 1/q$ or $v \approx q$ and from its definition $g(0) = 1$. Then the integral of $g$ exists. We have shown then

$$c^*(m) = \max 2 \int_0^1 \sqrt{mq^{m-1}Fgh} \, dq \int_0^1 g \, dq. \quad (11)$$

where the maximum is taken over all functions $g$ such that

$$g(0) = 1, \quad g(1) = 0 \quad \text{and} \quad h = -g' > 0.$$ 

It is perhaps of some interest to verify explicitly that the Euler equation for the maximizing $g$ is indeed Eq.(4a). Let us study the maximization of the functional

$$J_m(g) = 2 \int_0^1 \sqrt{mq^{m-1}Fgh} \, dq$$

where $h = -g' > 0$ subject to

$$\int_0^1 g(q) \, dq = 1.$$ 

The Euler equation for this problem is

$$\lambda + \sqrt{\frac{mq^{m-1}Fh}{g}} + \frac{d}{dq} \left( \sqrt{\frac{mq^{m-1}Fg}{h}} \right) = 0$$

where $\lambda$ is the Lagrange multiplier. Using the expression given in Eq.(9c) for $h$ we see that this is exactly Eq.(4a) with the Lagrange multiplier $\lambda = -c$.

As an application we shall consider the case $F(q) = q(1 - q)$ and $m = 2$ for which the exact solution is known. Take as the trial function $g(q) = (1 - q)^2$. Then we obtain

$$c \geq 4 \frac{\int_0^1 q(1 - q)^2 \, dq}{\int_0^1 (1 - q)^2 \, dq} = 1.$$ 

5
the exact value, which shows that this is the function $g$ for which the maximum is attained. In addition, due to the existence of the variational principle we may use the Feynman-Hellman formula to calculate the dependence of $c(m)$ on parameters of $F$. We illustrate this by applying it to the calculation of $dc/dm$ at $m = 2$. Taking the derivative of Eq.(10) with respect to $m$ we obtain

$$\frac{dc}{dm} = \frac{1}{\int_0^1 gdq} \int_0^1 \frac{ghF}{\sqrt{mFq^{m-1}gh}} [q^{m-1}(1 + m \log q)] dq.$$ 

Evaluating at $m = 2$, with $g(q) = (1 - q)^2$ we obtain

$$\frac{dc}{dm}(2) = 3 \int_0^1 q(1 - q)^2(1 + 2 \log q) dq = -\frac{7}{24},$$

the value previously obtained by other methods.

A fast estimation of the speed for other values of $m$ can be obtained with simple trial functions. In Fig. 1 we show lower bounds for $F = q(1-q)$ using as trial functions $g_1 = (1-q)^2$ and $g_2 = (1-q)$. With the first trial function we have the exact value at $m = 2$. The dotted line is the line of slope -7/24 that coincides with the tangent at $m = 2$. For larger $m$ a better estimate is obtained using $g_2$. The dashed line is the curve $\sqrt{2/m}$ which has been suggested by Newman [1] as the best fit to his numerical results. With better choice of trial functions the exact value can be approached arbitrarily close.

Finally we observe that the case $m = 1$ when $F'(0) = 0$ follows directly here. Repeating the procedure starting now from equation (10), one obtains,

$$c = \max_2 \int_0^1 \sqrt{Fgh} dq \int_0^1 gdq,$$

where the maximum is taken over all functions $g$ such that

$$g(0) = 1, \quad g(1) = 0 \quad \text{and} \quad h = -g' > 0.$$

To show this we have used $v'/v = c/p$ and $g = 1/v'$ and the asymptotic behavior described above.

1 **Acknowledgments**

We thank Prof. Dirk Meinköhn for giving us several useful references. This work was partially supported by Fondecyt project 193-0559.
References

[1] W. I. Newman, J. Theor. Biol. 85, 325, (1980)

[2] D. G. Aronson, in Dynamics and Modelling of Reacting Systems, edited by W. Stewart et al. (Academic, New York, 1980).

[3] L. E. Vulis, Thermal Regimes of Combustion Mc. Graw-Hill (1961).

[4] P. Clavin, in Annual Reviews of Fluid Mechanics, 26, 321 (1994) and references therein.

[5] S. K. Scott and K. Showalter, J. Phys. Chem. 96, 8702 (1992)

[6] W. I. Newman and C. Sagan, Icarus 46, 293, (1981)

[7] D. G. Aronson and H. F. Weinberger, Adv. Math. 30, 33 (1978).

[8] A. Kolmogorov, I. Petrovsky, and N. Piskunov, Bull. Univ. Moscow, Ser. Int. A 1, 1 (1937).

[9] D. G. Aronson and J. L. Vásquez, Phys. Rev. Lett. 72, 348 (1994).

[10] L. Y. Chen, Ph.D. Thesis, University of Illinois at Urbana-Champaign, 1994.

[11] J. J. E. Herrera, A. Minzoni, and R. Ondarza, Physica D 57, 249 (1992)

[12] R. D. Benguria and M. C. Depassier, Phys. Rev. Lett. 73, 2272 (1994).

[13] R. D. Benguria and M. C. Depassier, submitted to Commun. Math. Phys. (1994).