The approximate solutions of nonlinear Boussinesq equation

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Abstract. The homotopy analysis method (HAM) is introduced to solve the generalized Boussinesq equation. In this work, we establish the new analytical solution of the exponential function form. Applying the homotopy perturbation method to solve the variable coefficient Boussinesq equation. The results indicate that this method is efficient for the nonlinear models with variable coefficients.

1. Introduction

The nonlinear partial differential equation (NLPDE) have become a useful and exact tool to express natural phenomena of science and engineering models. Therefore, the studies of the solutions for NLPDE have great significance in searching the nonlinear natural phenomena. Considerable efforts have been made by many mathematicians and physical scientists to obtain powerful and efficient methods such as the Tanh-function method[1]; the Sine-Cosine method[2]; the Exp-function method[3]; the Adomian decomposition method[4]; the Jacobi elliptic function method[5]; the homotopy perturbation method[6]; the modified simple equation method[7]; the $(\frac{G'}{G})$ expansion method[8]; the F-expansion method[9]; the homotopy analysis method (HAM)[10] and so on. In the paper, through the homotopy analysis method and the homotopy perturbation method, solving the nonlinear Boussinesq equation

\[ u_{tt} - \alpha(t)u_{xx} - r(t)u_{xxxx} - \beta(t)(u^2)_{xx} = 0 \]

where $\alpha(t), r(t), \beta(t)$ are any function about $t$. The Boussinesq equation is a classical nonlinear equation, which describes the wave phenomenon of physics, and has been widely studied in many fields of physics. Solving the Boussinesq equation has become a hot topic in the study of nonlinear equations.

2. Basic idea of homotopy analysis method

In this section, we review the main contents of HAM briefly, we consider the nonlinear equation

\[ N[u(x,t)] = 0, \quad (2.1) \]

where $N$ is a nonlinear operator, $x$ and $t$ express independent variables, and $u(x,t)$ is an unknown function. According to the basic idea of the traditional homotopy method, we construct a
equation in the form
\[(1 - q)L[\Phi(x, t; q) - u_0(x, t)] = hqH(x, t)N[\Phi(x, t; q)], \quad (2.2)\]
where \(q \in [0, 1]\) is the embedding parameter, \(h\) is a nonzero auxiliary parameter, \(L\) is an auxiliary linear operator, \(u_0(x, t)\) is an initial guess of \(u(x, t)\); \(\Phi(x, t; q)\) is an unknown function, \(H(x, t)\) is an auxiliary function. In the above equation (2.2), we can choose \(L, H(x, t)\) and \(h\) in HAM flexibly. When the parameter \(q\) raises from 0 to 1, the solution \(\Phi(x, t; q)\) changes from \(u_0(x, t)\) to \(u(x, t)\). Expanding \(\Phi(x, t; q)\) in Taylor series with respect to \(q\), we have
\[\Phi(x, t; q) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t)q^m, \quad (2.3)\]
where
\[u_m(x, t) = \frac{1}{m!} \frac{\partial^m}{\partial q^m}\Phi(x, t; q) \bigg|_{q=0}. \quad (2.4)\]

As pointed by Liao [11], if the auxiliary objects such as \(L, H(x, t)\) and \(h\) are properly chosen, the series \(2.3\) will converge at \(q = 1\) and we has
\[u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t). \quad (2.5)\]
The above series solutions generally converge rapidly, and Liao has proved that which must be one solution of the original nonlinear Eq.(2.1). Differentiating Eq.(2.2) \(m\) times in the embedding parameter \(q\) and then setting \(q = 0\) and finally dividing by \(m!\), we have the equation in the form
\[L[u_m(x, t) - \chi_m u_{m-1}(x, t)] = hH(x, t)R_m(\vec{u}_{m-1}, x, t), \quad (2.6)\]
where
\[\vec{u}_{m-1} = \{u_0(x, t), u_1(x, t), \ldots, u_{m-1}(x, t)\}, \quad (2.7)\]
\[\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m \geq 2, \end{cases} \quad (2.8)\]
\[R_m(\vec{u}_{m-1}, x, t) = \frac{1}{m!} \frac{\partial^{m-1}}{\partial q^{m-1}}N[\Phi(x, t; q)] \bigg|_{q=0}. \quad (2.9)\]

Eq.(2.6) is a linear equation, through the iterative process, we can easily get \(u_1(x, t), u_2(x, t), \ldots, u_m(x, t)\ldots\).

### 3. Basic idea of homotopy perturbation method
Assume that a nonlinear differential equation in the form
\[A(u) - f(r) = 0, \quad r \in \Omega \quad (3.1)\]
with the boundary conditions \(B(u, \frac{\partial u}{\partial n}) = 0, \quad r \in \Gamma\), where \(A\) is a general differential operator, \(B\) is a boundary operator, \(f(r)\) is known as analytic function, and \(\Gamma\) is the boundary of the domain \(\Omega\). Dividing the operator \(A\) into two parts linear \(L\) and nonlinear \(N\). Thus Eq.(3.1) becomes
\[L(u) + N(u) - f(r) = 0. \quad (3.2)\]
On the basis of the homotopy technique\cite{6}, establishing a homotopy

\[ H(u, p) = L(u) - L(v) + p[L(v) + N(u) - f(r)], \quad p \in [0, 1], \quad r \in \Omega, \]  

(3.3)

where \( p \) is parameter, \( v \) is an initial approximation for the solution of Eq. (3.1), which is satisfied with the boundary conditions. Hence, when \( p \) increases from 0 to 1, we can obtain

\[ H(u, 0) = L(u) - L(v) \]  

(3.4)

\[ H(u, 1) = A(u) - f(r) = 0. \]  

(3.5)

Suppose that the solution of \( H(u, p) = 0 \) can be expressed as a power series with respect to \( p \):

\[ u(x, t, p) = \sum_{i=0}^{\infty} u_i(x, t)p^i = u_0 + pu_1 + p^2u_2 + \cdots. \]  

(3.6)

So when \( p = 0 \), \( u(x, t, 0) = u_0(x, t) \) is the solution of \( L(u) - L(v) = 0 \); when \( p \to 1 \), the approximate solution of \( A(u) - f(r) = 0 \) is

\[ u(x, t) = u_0 + u_1 + u_2 + \cdots. \]  

(3.7)

4. The approximate solution of the constant coefficient Boussinesq equation

Now, we study the constant coefficient nonlinear Boussinesq equation:

\[ u_{tt} - \alpha u_{xx} - ru_{xxxx} - \beta (u^2)_{xx} = 0 \]  

(4.1)

where \( \alpha, r, \beta \) are constants.

Firstly, we seek the exponential function solution of Eq. (4.1). Defining \( \xi = x + \omega t \), then Eq. (4.1) becomes in the form:

\[ \omega^2 u'' - \alpha u'' - ru''' - \beta (u^2)'' = 0 \]  

(4.2)

where the prime means the derivative in \( \xi \). Then integrating Eq. (4.2) with respect to \( \xi \), choosing constant of integration to zero, we obtain

\[ ru'' - (\omega^2 - \alpha)u + \beta u^2 = 0. \]  

(4.3)

We choose the solution in the form

\[ u(\xi) = \sum_{m=0}^{\infty} a_m e^{-m\xi} \]  

(4.4)

where \( a_m \) is a coefficient, and the linear operator

\[ L[\Phi(\xi, q)] = (\frac{\partial^2}{\partial \xi^2} - 1)\Phi(\xi, q). \]  

(4.5)

meets \( L[\alpha_1 e^{-\xi} + \beta_1 e^\xi] = 0 \), where \( \alpha_1, \beta_1 \) are constants. Based on (4.4), we choose the initial guess \( u_0(\xi) = e^{-\xi} \). From (4.3), we definite the nonlinear operator

\[ N[\Phi(\xi, q), \omega(q)] = ru'' - (\omega^2 - \alpha)\Phi + \beta \Phi^2. \]  

(4.6)
Applying the above definition, and defining $H(x, t) = 1$, we can get

$$(1-q)L[\Phi(\xi, q) - u_0(\xi)] = q\hbar N[\Phi(\xi, q), \omega(q)],$$

(4.7)

$$L[u_m(\xi) - \chi_{m+1}\bar{u}_m(\xi)] = hR_m(\bar{u}_{m-1}, \bar{\omega}_{m-1}),$$

(4.8)

where

$$\bar{u}_{m-1} = \{u_0(\xi), u_1(\xi), \cdots, u_{m-1}(\xi)\},$$

(4.9)

$$\bar{\omega}_{m-1} = \{\omega_0(\xi), \omega_1(\xi), \cdots, \omega_{m-1}(\xi)\},$$

(4.10)

and

$$R_m(\xi) = \frac{\partial^2 u_{m-1}}{\partial \xi^2} - \sum_{i=0}^{m-1} u_{m-1-i}(\sum_{j=0}^{i} \omega_{i-j} - \alpha u_{m-1} + \beta \sum_{i=0}^{m-1} u_i u_{m-1-i}).$$

(4.11)

Through the iterative process, we can get the solution of the linear differential equation (4.8) easily.

For example, when $m = 0$, Eq.(4.8) becomes

$$L[u_1(\xi)] = h[r\omega_0'' - \omega_0^2 u_0 + \alpha u_0 + \beta u_0^2].$$

(4.12)

Substituting $u_0(\xi) = e^{-\xi}$ into above equation, that is

$$L[u_1(\xi)] = h[(r - \omega_0^2 + \alpha)e^{-\xi} + \beta e^{-2\xi}].$$

(4.13)

The solution of $u_1$ involves the terms $\xi e^{-\xi}$, if $r - \omega_0^2 + \alpha \neq 0$. Thus it must be enforced to zero, i.e. $\omega_0 = \pm\sqrt{r + \alpha}$, and

$$u_1(\xi) = C_1 e^{-\xi} + \frac{\beta h}{3} e^{-2\xi},$$

(4.14)

When $m = 2$,

$$L[u_2 - u_1] = h[-\omega_0^2 u_0 + 2\omega_0 \omega_1 u_0 + r u_0'' + 2\beta u_0 u_1].$$

(4.15)

then substituting $u_0 = e^{-\xi}$, $u_1 = C_1 e^{-\xi} + \frac{\beta h}{3} e^{-2\xi}$ into Eq.(4.15), result in

$$L[u_2 - u_1] = h[-2\omega_0 \omega_1 e^{-\xi} + (r\beta h + 2\beta C_1) e^{-2\xi} + \frac{2\beta^2 h}{3} e^{-3\xi}].$$

(4.16)

Avoiding the term $\xi e^{-\xi}$, let $-2\omega_0 \omega_1 = 0$. Substituting the value of $\omega_0$ into it, we can get $\omega_1 = 0$. Then solving the linear Eq.(4.16),

$$u_2(\xi) = (C_1 + C_2) e^{-\xi} + \left(\frac{r\beta h^2 + \beta h + 2\beta C_1 h}{3}\right) e^{-2\xi} + \frac{\beta^2 h^2}{12} e^{-3\xi},$$

(4.17)

where $C_1, C_2$ are constants. At last the two-degree approximate solution of (4.2) and one-degree approximate solution of $\omega$ can be obtained as follows:

$$\omega = \omega_0 + \omega_1 = \pm\sqrt{r + \alpha},$$

(4.18)

$$u(\xi) \approx u_0 + u_1 + u_2 = (2C_1 + C_2 + 1) e^{-\xi} + \left(\frac{r\beta h^2 + 2\beta h + 2\beta C_1 h}{3}\right) e^{-2\xi} + \frac{\beta^2 h^2}{12} e^{-3\xi},$$

(4.19)

where $\xi = x + (\pm\sqrt{r + \alpha})t$.

In order to find the reasonable range of $\hbar$, we plot the $h$-curve, which is shown in Fig.1. As proved by Liao [13], the horizontal line segment is a valid region of $h$ and the solution series must converge to the exact solutions. Thus, the valid region of $h$ is $-1.4 \leq h \leq -0.6$. Fig.2 is the exponential style solution under the 10th order of approximation. Error chart is shown in Fig.3.
Figure 1. The $h$-curves of $u''_{\text{appr}}(0)$ under the 10th order of approximation when $\alpha = 1, \beta = 1, r = 1, u(0) = 1$.

Figure 2. The 10th order of $u_{\text{appr}}(\xi)$ with $h = -1, \alpha = 1, \beta = 1, r = 1, u(0) = 1$.

Figure 3. Error chart with $h = -1, \alpha = 1, \beta = 1, r = 1, u(0) = 1$.

5. The approximate solution of the variable coefficient Boussinesq equation

Next, seeking the approximate solution of the variable coefficient Boussinesq equation

$$u_{tt} - \alpha(t)u_{xx} - \beta(t)(u^2)_{xx} - r(t)u_{xxxx} = 0.$$  \hspace{1cm} (5.1)
Defining $x_1 = \sqrt{\frac{c_3}{c_1}} \int_0^t \alpha(t') \sqrt{\frac{c_3}{c_1}} \, dt'$, $t_1 = \int_0^t \frac{c_2(t')}{c_1(t')} \, dt'$, we have $c_3 = \frac{c_2(t')}{c_1(t')}$, where $c_1 \alpha(t) > 0$, $c_3 r(t) > 0$, $c_2 \neq 0$.

Eq. (5.1) becomes

$$u_{t_1 t_1} - c_1^4 u_{x_1 x_1} - c_2 (u^2)_{x_1 x_1} - c_3 u_{x_1 x_1 x_1} = 0. \quad (5.2)$$

For simplicity, we take $x = x_1$, $t = t_1$. Then (5.2) translates into

$$u_{tt} - c_1^4 u_{xx} - c_2 (u^2)_{xx} - c_3 u_{xxxx} = 0. \quad (5.3)$$

Next, we applied the homotopy perturbation method to study the approximate solution of Eq. (5.3). Firstly, we set up homotopy mapping $H(u, p) : R \times [0, 1] \to R,$

$$H(u, p) = L(u) - L(v) + p[L(v) - c_2 (u^2)_{xx} - c_1^4 u_{xx}], \quad (5.4)$$

where $R = (-\infty, +\infty)$, $v$ is the auxiliary function, and $L$ is the linear operator in the form: $L(u) = u_{tt} - c_3 u_{xxxx}$.

We can easily get the linear equation

$$v_{tt} - c_3 v_{xxxx} = 0 \quad (5.5)$$

has the following solution:

$$v = \cosh(\omega \sqrt{-\frac{1}{c_3}} \xi), c_3 > 0 \quad (5.6)$$

$$v = \cos(\omega \sqrt{-\frac{1}{c_3}} \xi) + \sin(\omega \sqrt{-\frac{1}{c_3}} \xi), c_3 < 0 \quad (5.7)$$

where $\xi = x + \omega t$, $\omega$ is any constant. One can easily observe that $H(u, 1) = 0$ and Eq. (5.3) is the same, so the solution $u(x, t)$ of Eq. (5.3) is the solution of $H(u, p) = 0$ when under the condition $p \to 1$. Let (3.6) be the solution of $H(u, p) = 0$; this series is uniformly convergent in the $p \in [0, 1]$. Thus it yields that

$$u = \sum_{i=0}^{\infty} u_i(x, t) = u_0 + u_1 + u_2 + \cdots. \quad (5.8)$$

Substituting (3.6) into the equation $H(u, p) = 0$, and matching the coefficients of the same power of $p$, we can get that

$$p^0 : L(u_0) = L(v) \quad (5.9)$$

$$p^1 : L(u_1) = -L(v) + c_2 (u_0^2)_{xx} + c_1^4 u_{0xx} = c_2 (u_0^2)_{xx} + c_1^4 u_{0xx} \quad (5.10)$$

$$p^2 : L(u_2) = 2c_2 (u_0 u_1)_{xx} + c_1^4 u_{1xx}. \quad (5.11)$$

From (5.9), we have

$$u_0(x, t) = v(x, t). \quad (5.12)$$

Defining $c_2 (u_0^2)_{xx} + c_1^4 u_{0xx} = f_1(x, t), 2c_2 (u_0 u_1)_{xx} + c_1^4 u_{1xx} = f_2(x, t)$, and using the Fourier transform method, from Eq. (5.10) (5.11), we have

$$\frac{d^2 u_{1i}}{dt^2} - \lambda^2 c_3 u_{1i} = \overline{f_1}(\lambda, t). \quad (5.13)$$

$$\frac{d^2 u_{2i}}{dt^2} - \lambda^4 c_3 u_{2i} = \overline{f_2}(\lambda, t). \quad (5.14)$$
From Eq. (5.14), as zero is the initial value, we have

\begin{equation}
\bar{u}_1(\lambda, t) = \int_0^t \frac{\tilde{f}_1}{\sqrt{-\lambda^4 c_3}} \sin \sqrt{-\lambda^4 c_3}(t - t_1) dt_1, c_3 < 0,
\end{equation}

Thus we obtain

\begin{equation}
u_1(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^t \frac{\tilde{f}_1}{\sqrt{-\lambda^4 c_3}} \sin \sqrt{-\lambda^4 c_3}(t - t_1)e^{ix\lambda x} dt_1 d\lambda, c_3 < 0.
\end{equation}

From Eq. (5.15), we assume the initial value is zero, that is

\begin{equation}
\bar{u}_{02}(\lambda, t) = \int_0^t \frac{\tilde{f}_2}{\sqrt{-\lambda^4 c_3}} \sinh \sqrt{-\lambda^4 c_3}(t - t_1) dt_1, c_3 > 0,
\end{equation}

\begin{equation}
u_2(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^t \frac{\tilde{f}_2}{\sqrt{-\lambda^4 c_3}} \sinh \sqrt{-\lambda^4 c_3}(t - t_1)e^{ix\lambda x} dt_1 d\lambda, c_3 > 0.
\end{equation}

From Eq. (5.18), as zero is the initial value, we have

\begin{equation}
u_3(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^t \frac{\tilde{f}_1 + \tilde{f}_2}{\sqrt{-\lambda^4 c_3}} \sin \sqrt{-\lambda^4 c_3}(t - t_1)e^{ix\lambda x} dt_1 d\lambda, c_3 < 0
\end{equation}

\begin{equation}u(x, t) = \cos(\omega \sqrt{-\frac{c_3}{c_3}}) + \sin(\omega \sqrt{-\frac{c_3}{c_3}}) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^t \frac{\tilde{f}_1 + \tilde{f}_2}{\sqrt{-\lambda^4 c_3}} \sin \sqrt{-\lambda^4 c_3}(t - t_1)e^{ix\lambda x} dt_1 d\lambda, c_3 < 0
\end{equation}

\begin{equation}u(x, t) = \cosh(\omega \sqrt{-\frac{c_3}{c_3}}) - \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^t \frac{\tilde{f}_1 + \tilde{f}_2}{\sqrt{-\lambda^4 c_3}} \sinh \sqrt{-\lambda^4 c_3}(t - t_1)e^{ix\lambda x} dt_1 d\lambda, c_3 > 0.
\end{equation}

6. Conclusion
This work studies the generalized Boussinesq equation by using the homotopy analysis method and the variable coefficient Boussinesq equation with the homotopy perturbation method. The solution of the generalized Boussinesq equation contains the parameter \( h \) and we can adjust the value of \( h \) to guarantee the convergence region of the solution series. With the help of some mathematical software, such as MATHEMATICA, MATLAB, we can compile program algorithm to get more accurate approximate solutions. The homotopy perturbation method deforms an nonlinear problem into a set of linear problems which are easier to solve. The two-degree approximate solution of the variable coefficient Boussinesq equation show that the homotopy perturbation method is effective to solve the variable solution equations.

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