Characterizations of the sphere by means of point-projections

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January 18, 2023

Abstract

In this work we prove the following: let $K$ be a convex body in the Euclidean space $\mathbb{R}^n$, $n \geq 3$, contained in the interior of the unit ball of $\mathbb{R}^n$, and let $p \in \mathbb{R}^n$ be a point such that, from each point of $S^{n-1}$, $K$ looks centrally symmetric and $p$ appears as the center, then $K$ is a ball.

1 Introduction

Consider a convex body $K$, i.e., a compact and convex set with non-empty interior in $\mathbb{R}^n$. As usual $\text{int } K$ and $\text{bd } K$ denote the interior and boundary of $K$. Let $\Pi \subset \mathbb{R}^n$ be a hyperplane. We denote by $S_\Pi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ the orthogonal reflection with respect to $\Pi$. We say that $K \subset \mathbb{R}^n$ is symmetric with respect to $\Pi$, or that $\Pi$ is a hyperplane of symmetry of $K$, if $S_\Pi(K) = K$. Let $B^n$ and $S^{n-1}$ denote the Euclidean unit ball and unit sphere of $\mathbb{R}^n$ with center at the origin $O$. For $u \in S^{n-1}$ and $s \in \mathbb{R}$, $s$ non-negative, we denote by $\Pi(u, s)$ the hyperplane $\{x \in \mathbb{R}^n | \langle u, x \rangle = s\}$, whose unit normal vector is $u$ and its distance to the origin is equal to $s$. Moreover, we denote by $\Pi^*(u, s)$ the open half-space $\{x \in \mathbb{R}^n | \langle u, x \rangle < s\}$. For the points $x, y \in \mathbb{R}^n$ we denote by $L(x, y)$ the line determined by $x$ and $y$, and by $[x, y]$ the segment with extreme points $x$ and $y$.

In order to establish our results we need to give the following definitions.
Definition 1. Let $K \subset \mathbb{R}^n$ be a convex body, $n \geq 3$, and let $x \in \mathbb{R}^n \setminus K$. We call the set
\[ \{ x + \lambda(y - x) | y \in K, \lambda \geq 0 \}, \]
the solid cone generated by $K$ and $x$. The boundary of this solid cone, i.e., the union of all rays starting at $x$ which does not intersect the interior of $K$, is called the cone circumscribed to $K$ with apex $x$. We denote this cone by $C_x$.

Definition 2. Let $C \subset \mathbb{R}^n$ be a convex cone with apex $x$. We say that $C$ is a symmetric cone with axis $L_x$, if there exists a line $L_x$ through $x$ such that for every $2$-dimensional plane $\Gamma$ which contains $L_x$, it holds that $L_x$ is the angle bisector of the angular region $\Gamma \cap C$. Furthermore, we say that $C$ is a right circular cone if for every hyperplane $\Pi$ orthogonal to $L_x$, with $\Pi \cap C \neq \emptyset$, the set $\Pi \cap C$ is an $(n - 1)$-dimensional ball.

If $K \subset \mathbb{R}^n$, $n \geq 3$, is an ellipsoid, then for every $x \in \mathbb{R}^n \setminus K$, the cone $C_x$ is a symmetric cone. Thus, the next conjecture (see [5]) seems very natural and should be true.

Conjecture 1. Let $K$ be a convex body contained in the interior of $B^n$, $n \geq 3$. If for every $x \in S^{n-1}$, $C_x$ is a symmetric cone, then $K$ is an ellipsoid.

Our main result is Theorem 2, which proves a special case of Conjecture 1, namely, we assume that all the axis of the cones are passing through one point. We have decided to present separately the case $n = 2$ of Theorem 2 and Theorem 1, since it could be considered as a characterisation of the circle, similar in some sense, to the main theorem in [7].

Theorem 1. Let $K$ be a convex body contained in the interior of $B^2$, and let $p$ be a point in the interior of $K$. If for every $x \in S^1$ we have that $L_x$ passes through $p$, then $K$ is a disc with center at $p$.

Theorem 2. Let $K$ be a convex body contained in the interior of $B^n$, $n \geq 3$, and let $p$ a point in the interior of $K$. If for every $x \in S^{n-1}$, $C_x$ is a symmetric cone and $L_x$ passes through $p$, then $K$ is a ball with center at $p$.

As a corollary of Theorem 2 we have the following.

Corollary 1 (Matsuura). Let $K$ be a convex body contained in the interior of $B^n$, $n \geq 3$. If for every $x \in S^{n-1}$ we have that $C_x$ is a right circular cone, then $K$ is a ball.

For the case when the apexes are in a hyperplane we have the following.

Theorem 3. Let $K \subset \mathbb{R}^n$ be a convex body, $n \geq 3$, and let $\Pi$ be a hyperplane which does not intersect the interior of $K$. If for every $x \in \Pi \setminus K$ the cone $C_x$ is a right circular cone, then $K$ is a ball.
With a restriction in the position of the hyperplane we can say a little more.

**Theorem 4.** Let \( K \subset \mathbb{R}^n \) be a convex body, \( n \geq 2 \), and let \( \Pi \) be a hyperplane tangent to \( K \). Suppose there exists a point \( p \) in the interior of \( K \) such that for every \( x \in \Pi \setminus K \) the cone \( C_x \) is symmetric and \( L_x \) passes through \( p \), then \( K \) is a ball.

We have obtained our results while exploring a family of problems concerning characterization of spheres and ellipsoids in terms of geometric properties of cones which circumscribes convex bodies. We considered the papers [1], [2], [11], [12], and particularly Conjecture 2 in [2] (which we reproduce here as Conjecture 1). Such conjecture was inspired by the following characterization of the sphere due to S. Matsuura [11] If a convex body \( K \subset \mathbb{R}^n \), \( n \geq 3 \), is contained in the interior of the region enclosed by a convex surface \( S \), and looks spherical from each point of \( S \), then \( K \) is a ball. In [5] an important evidence of the veracity of Conjecture 1 was given, namely, there was proved there that if a smooth convex body \( K \subset \mathbb{R}^n \), \( n \geq 3 \), looks centrally symmetric from every point in the exterior of \( K \) then it must be an ellipsoid.

The main result in this work is Theorem 2. Comparing with Gruber-Odor’s Theorem [5], we have reduced, substantially, the quantity of those points from where the convex body looks centrally symmetric. We assumed information of the circumscribed cones to \( K \) only from the points in a sphere. However, we add the condition that there is a point \( p \) that looks as the center of the body when it is observed from the points in such a sphere. As an immediate consequence of Theorem 2 we obtained Corollary 1 which is a special case of Matsuura’s Theorem. We only considered the case \( S = S^{n-1} \). First we proved, under Matsuura’s Theorem conditions, that all the axis of the spherical cones where the convex body is inscribed are concurrent, see Lemma 2. However, our proof of the general case, i.e., \( n > 3 \), of such restricted version of Matsuura’s Theorem can be given directly. In [11], Matsuura gives the procedure to carry out the generalization but he did not provide the complete proof.

Theorem 3 is the natural variant of Corollary 1, we replaced the sphere by a hyperplane. We decided to include Theorem 3 in this work because we did not find an explicit reference about this elementary, however, interesting result.

## 2 Proofs of Theorems 1, 2, and Corollary 1

In order to prove Theorem 1 we need to give some definitions and to prove a lemma which is interesting by itself. In what follows, the boundary of a given disc will be called a circle. Let \( \Gamma \) be a circle in the plane and let \( \Omega \) be another circle contained in the interior of \( \Gamma \). For every point \( x_0 \in \Gamma \) we define the *Poncelet-polygon* \((x_0, x_1, x_2, x_3, \ldots)\) such that the points \( x_0, x_1, x_2, x_3, \ldots \) are arranged in counter clockwise order in \( \Gamma \).
and such that the segments $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, . . . are all tangent to $\Omega$. The mapping $F$ such that $x_{i+1} = F(x_i)$ is called the Poncelet-mapping. If for a positive integer number $n$ we have that $x_n = x_0$, i.e., $x_n = F^n(x_0) = x_0$, by the well known Theorem of Poncelet (see for instance [4], or [13]) we know that for any other point $y \in \Gamma$ it holds that $y_n = F^n(y_0) = y_0$. If this is the case we say that $\Omega$ has the closure property with respect to $\Gamma$. It is also known that the map $F$ has an invariant measure and hence by Denjoy’s theorem (see for instance Theorem 12.3 in [4]) we have that $F$ is conjugate to a circle rotation. A very useful consequence of this fact is that any given circle $\Omega$, inside $\Gamma$, has either the closure property or for any point $x_0 \in \Gamma$ the set $\{x_0, x_1, x_2, x_3, \ldots\}$ is a dense set in $\Gamma$ (see [4]).

**Lemma 1.** Let $\Gamma$ be a circle with center $O$ and radius $r$ and let $p$ be a point at distance $\lambda < r$ from $O$. Then, for every two numbers $r_1 < r_2 < r - \lambda$ there exists a number $r_3$, with $r_1 < r_3 \leq r_2$, such that the circle $\Gamma_3$ with radius $r_3$ and center $p$ has not the closure property with respect to $\Gamma$.

![Figure 1: The area in Jacobi’s surface corresponding to every chord tangent to $\Gamma_2$ is constant](image)

*Proof* Consider the circle $\Gamma_2$ with center $p$ and radius $r_2$. If $\Gamma_2$ has not the closure property with respect to $\Gamma$ then we choose $r_3 = r_2$ and $\Gamma_3 = \Gamma_2$. If $\Gamma_2$ has the closure property we proceed as follows: we construct the Jacobi’s surface, i.e., the surface over the circular cylinder with base $\Gamma$ and such that for every point $x \in \Gamma$ its height is the reciprocal of the length of the tangent segment drawn from $x$ to $\Gamma_2$ (see Fig. 1). A geometric interpretation of the invariant measure assigned to the Poncelet map $F$ is
that for any chord of \( \Gamma \), to say \([a, b]\), tangent to \( \Gamma_2 \), the area of the part of the Jacobi’s surface over the arc \( \widehat{ab} \) is equal to a constant number \( S \) (see for instance [6]). From this geometric interpretation the following fact is easy to see: If \( A \) denotes the area of the whole Jacobi’s surface, then \( \Gamma_2 \) has the closure property if and only if \( \frac{S}{A} \) is a rational number. Indeed, the number \( \frac{S}{A} \) is known as the rotation number of the map \( F \).

As was proved by A. O. Lopes and M. Sebastiani in [10], the rotation number changes continuously for concentric circles which change continuously its radius. Hence, if we decrease continuously the radius \( r_2 \) and keeping the center \( p \), the ratio \( \frac{S}{A} \) changes continuously. Thus, there exist a number \( r_3 \) in the open interval \((r_1, r_2)\) such that \( \frac{S}{A} \) is irrational. It follows that the circle \( \Gamma_3 \), with center \( p \) and radius \( r_3 \) has not the closure property.

**Proof of Theorem 1.** Let \( r_1 \) be largest number such that the circle \( \Gamma_1 \) with center \( p \) and radius \( r_1 \) is contained in \( K \). Analogously, let \( r_2 \) be the smallest number such that the circle \( \Gamma_2 \) with center \( p \) and radius \( r_2 \) encloses to \( K \). If \( r_1 = r_2 \) then \( K \) is a disc. Suppose now that \( r_1 < r_2 \). Let \( x_0, x_1, x_2, x_3, \ldots \) be points in \( S^1 \) arranged in counter clockwise order and such that every segment \([x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots \) is a supporting segment of \( K \). By the condition that the angle bisectors \( L_{x_0}, L_{x_1}, L_{x_2}, L_{x_3}, \ldots \) pass through \( p \), we have that all the segments \([x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots \) are at the same distance \( r \) from the point \( p \). It follows that the circle with center \( p \) and radius \( r \) share the tangent segments \([x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots \) with \( K \). Now we apply Lemma 1 with \( \Gamma = S^1 \), hence we have that there exists a number \( r_3 \), with \( r_1 < r_3 \leq r_2 \) such that the circle \( \Gamma_3 \) has not the closure property with respect to \( S^1 \). Since the distance from \( p \) to the support lines of \( K \) takes all the values in the interval \([r_1, r_2] \), we have that there exists a support line \( \ell \) of \( K \) at distance \( r_3 \) from \( p \). Let \( x_0 \) and \( x_1 \), in counter clockwise order, be the points where \( \ell \) intersects to \( S^1 \). The Poncelet-polygon \((x_0, x_1, x_2, x_3, \ldots )\) has its sides tangent to \( \Gamma_3 \) and \( K \) simultaneously, however, the set \( \{x_0, x_1, x_2, x_3, \ldots \} \) is a dense set in \( S^1 \). It follows that every line tangent to \( \Gamma_3 \) is also a support line of \( K \); therefore, \( K \) is a disc with center \( p \).

**Proof of Theorem 2.** Let \( \Gamma \) be any 2-dimensional plane through \( p \). For every point \( x \in \Gamma \cap S^{n-1} \), the line \( L_x \) is the angle bisector between the two support lines of \( K \cap \Gamma \) through \( x \). By hypothesis, \( L_x \) passes through \( p \), hence we have the conditions of Theorem 1 and so we have that \( \Gamma \cap K \) is a disc with center at \( p \). Since this is true for every plane \( \Gamma \) through \( p \), we conclude by a theorem due to H. Busemann (see [3], pp. 91-92) that \( K \) is a ball with center at \( p \).

We first prove the following lemma and then the conclusion of the corollary follows easily.

**Lemma 2.** Under the conditions of Corollary 1, there exists a point \( p \in \text{int} \, K \) such that for every \( x \in S^{n-1} \) the axis \( L_x \) of \( C_x \) passes through \( p \).
Proof. Let $x, y \in S^{n-1}$ be any two points such that $L(x, y) \cap \text{int} K = \emptyset$. We are going to prove that $L_x \cap L_y \neq \emptyset$. Let $\Pi_1, \Pi_2$ be two support hyperplanes of $K$ containing $L(x, y)$. It is clear that $\Pi_1, \Pi_2$ are also support hyperplanes of $C_x$ and $C_y$. Let $\Pi_{1,2}$ be the hyperplane bisecting the solid dihedral angle determined by $\Pi_1, \Pi_2$ and containing $L(x, y)$. We denote by $\Sigma$ the hyperplane $\text{aff}\{L_x, \Pi_1 \cap \Pi_2\}$. Since for every hyperplane $\Gamma$, with $L_x \subset \Gamma$, the equality $S_{\Gamma}(C_x) = C_x$ holds, it follows that $S_{\Sigma}(\Pi_1)$ is a support plane of $C_x$ containing $\Pi_1 \cap \Pi_2$ and different from $\Pi_1$. Thus $S_{\Sigma}(\Pi_1) = \Pi_2$. Hence $\Pi_{1,2} = \Sigma$. Consequently, $L_x \subset \Pi_{1,2}$. In conclusion, we have that
\[
\text{aff}\{L_x, x, y\} = \bigcap \Pi_{1,2}
\]
holds, where the intersection is taken over all pairs $\Pi_1, \Pi_2$ of support hyperplanes of $K$ such that $L(x, y) \subset \Pi_1, \Pi_2$. Interchanging $x$ by $y$, due the symmetry of this argument, we conclude that
\[
\text{aff}\{L_y, x, y\} = \bigcap \Pi_{1,2}.
\]
holds, where again the intersection is taken over all pairs $\Pi_1, \Pi_2$ of support hyperplanes of $K$ such that $L(x, y) \subset \Pi_1, \Pi_2$. Therefore $\text{aff}\{L_x, x, y\} = \text{aff}\{L_y, x, y\}$. Since $L_x$ and $L_y$ can not be parallel, we get $L_x \cap L_y \neq \emptyset$.

Now, consider a convex polytope $P$ such that $K \subset P \subset \text{int} \mathbb{B}^n$. The existence of such a polytope is a well known result in Convex Geometry, see for instance Lemma 22.3 in the book by S. R. Lay [9]. Let $P_1, P_2, \ldots, P_r$, be the facets (faces of dimension $n - 1$) of $P$. Let $H_1, \ldots, H_r$, be the hyperplanes such that $P_i \subset H_i$, for every $i = 1, 2, \ldots, r$, and define $S_i$ to be the region which $H_i$ cuts off from $S^{n-1}$, for every $i = 1, 2, \ldots, r$, with $S_i$ and $K$ contained in different open half-space determined by $H_i$. By the comment made at the beginning of the proof, the family of lines $\Phi_i = \{L_x : x \in S_i\}$ has the property the every two of its members intersect, for every $i = 1, 2, \ldots, r$. Then the lines of $\Phi_i$ are concurrent in a point $p_i \in \text{int} K$. Since every two regions $S_i, S_j$, which correspond to adjacent facets, share a common region then we have that $p_i = p_j$. In this way we indeed have that there exists a point $p$ such that for every $x \in S^{n-1}$, it holds that $L_x$ passes through $p$.

Proof of Corollary [4] By Lemma [2] we have that $K$ satisfies the conditions of Theorem [2]. Therefore $K$ is a ball.

3 Proof of Theorems 3 and 4

We first note that under the conditions of Theorem [3], $K$ must be a strictly convex body, hence, for the following lemmas we consider $K$ is a strictly convex body.

Lemma 3. Let $K \subset \mathbb{R}^n$, $n \geq 2$, be a convex body and let $\Sigma$ be a hyperplane such that $\Sigma \cap \text{int} K \neq \emptyset$. If for every $(n - 2)$-dimensional affine plane $\Gamma \subset \Sigma$, with $\Gamma \cap K = \emptyset$,
the two support hyperplanes of \( K \) containing \( \Gamma \) are symmetric with respect to \( \Sigma \), then \( K \) is symmetric with respect to \( \Sigma \).

**Proof.** Consider the reflected body \( K' = S_{\Sigma}(K) \). By the hypotheses of the lemma, we have that \( K \) and \( K' \) share the same support hyperplanes except possibly for the points in \( \Sigma \cap \text{bd} \, K \). However, \( \Sigma \cap K' = \Sigma \cap K \), then we have that \( K' = K \) which means that \( K \) is symmetric with respect to \( \Sigma \). \( \square \)

**Proof of Theorem 3.** By the same argument in the beginning of the proof of Lemma 2, we have that there exists a point \( p \) such that for every \( x \in \Pi \setminus K \) the axis \( L_x \) passes through \( p \).

First we prove the theorem for dimension \( n = 3 \). We are going to prove, using Lemma 3, that each plane \( \Sigma \) passing through \( p \) is plane of symmetry for \( K \). Let \( \Sigma \) be a 2-dimensional plane, with \( p \in \Sigma \). We denote by \( L \) the intersection \( \Sigma \cap \Pi \). Let \( \Gamma \subset \Sigma \setminus K \) be a line. First, we assume that \( \Gamma \) is not parallel to \( L \). Denote by \( x \) the intersection \( \Gamma \cap L \). If \( p \in \Pi \) (see Fig. 2), then for each line \( \Pi \subset L \), \( p \in L \), and for each \( x \in L \), the axis \( L_x \) is equal to \( L \). In fact, by Lemma 2, for each \( x \in L \), the axis \( L_x \) is determined by the points \( x \) and \( p \). Since \( L = L_x \) and \( L \subset \Sigma \), we have that \( \Sigma \) is a plane of symmetry for the cone \( C_x \). In the case \( p \notin \Pi \), since \( p, x \in \Sigma \), then \( L_x \subset \Sigma \), and again \( \Sigma \) is a plane of symmetry for the cone \( C_x \). Therefore, in both cases, there exist two support planes of \( C_x \) and, consequently, of \( K \), symmetric with respect to \( \Sigma \) and containing \( \Gamma \). Analogously, the same conclusion is obtained if we assume that \( \Gamma \) is parallel to \( L \). Thus \( K \) satisfies the condition of Lemma 3. Hence \( \Sigma \) is a plane of symmetry for \( K \). By the arbitrariness of \( \Sigma \), with \( p \in \Sigma \), it follows that \( K \) must be a ball.

![Figure 2](image-url)
Now we prove the theorem for dimension $n > 3$. Let us assume that Theorem \(\ref{thm3}\) holds for dimension $n-1$. We are going to show that Theorem \(\ref{thm3}\) holds for dimension $n$. Let $\Gamma$ be an affine $(n-1)$-dimensional plane passing through $p$. Then, for all $x \in \Gamma \cap \Pi$, the axis $L_x$ is equal to $L(x, p)$, hence $L_x \subset \Gamma$. It follows that $C_x \cap \Gamma$ is a circular cone (in dimension $n-1$) that circumscribes $\Gamma \cap K$. According to the induction hypothesis $\Gamma \cap K$ is a ball. Hence all the $(n-1)$-dimensional sections of $K$ passing through $p$ are $(n-1)$-dimensional balls. Therefore, $K$ is a ball. \(\square\)

Proof of Theorem \(\ref{thm4}\). We consider first the case $n = 2$. Let $x$ be any point in $\Pi \setminus K$ and let $\ell$ be the other support line of $K$ through $x$. Let $\Omega$ be the disc with center $p$ and tangent to $\Pi$ and $\ell$. For every $x$ in $\Pi$ the angle bisector of the cone circumscribed to $\Omega$ from $x$, passes through $p$. Then, $K$ and $\Omega$ share the same support lines and so they coincide, i.e., $\Omega = K$.

Now, for dimension $n > 2$ we proceed as follows: Let $z$ be a point in $\Pi \cap K$ and let $q \in \text{bd} \, K$ be the point such that the segment $[z, q]$ contains $p$. Let $\Gamma$ be any 2-dimensional plane which contains $[z, q]$ and let $\ell \cap \Gamma$. Let $x \in \ell$ be any point. Since the axis $L_x$ passes through $p$, by the 2-dimensional case we have that $\Gamma \cap K$ is a 2-dimensional disc with center at $p$. Since $\Gamma$ is any 2-dimensional plane which contains $[z, q]$, we conclude that $K$ is a ball with center at $p$. \(\square\)

## 4 Further comments

Finally, we propose the following problem, which could be considered as the following natural step in way of the solution of Conjecture \(\ref{conj1}\).

**Conjecture 2.** Let $K$ be a convex body in the interior of $\mathbb{B}^n$, with $n \geq 3$, and let $L \subset \mathbb{R}^n$ be a line. If for every $x \in S^{n-1}$ the cone $C_x$ is a symmetric cone such that

$$L_x \cap L \neq \emptyset,$$

then $K$ is an $n$-dimensional ellipsoid and for every 3-dimensional plane $\Pi$ containing $L$, the section $\Pi \cap K$ is an ellipsoid of revolution with axis $L$.

**Acknowledgments**

We thank Sergei Tabachnikov for the helpful discussions about the proof of Theorem 1. We also thank the anonymous referee for all the comments that help to improve the paper.
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