**Plasma slab expansion into vacuum**

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**Abstract** – The problem of collisionless plasma slab expansion into vacuum is solved within a two-temperature hydrodynamic approximation in the dispersionless limit of zero Debye radius. In the framework of such an approach, the solution by the Riemann method provides a quite accurate description of the whole process of plasma dynamics. It is shown that the dispersionless approximation agrees very well with exact numerical solution of the full system of plasma hydrodynamic equations.

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**Introduction.** – Expansion of matter into vacuum is one of the canonical problems in fluid dynamics. In the framework of plasma physics, it was studied first in refs. [1,2] with the use of the collisionless kinetic equation for the slow motion of ions and under the conditions of thermal equilibrium for high-temperature electrons. This model was later modified in different directions and found applications in the explanation of experiments on the dynamics of plasma produced by interactions of very intensive laser pulses with matter (see, e.g., review article [3] and references in). As was shown in refs. [1,2] (see also [4]), self-similar expansion of the plasma which occupies initially the half-space converges very fast to the dynamics of cold ions, so in the leading approximation one can neglect dispersive effects of finite Debye radius and use the purely hydrodynamic approach for quasi-neutral plasma (see, e.g., [5]).

If a thin foil is irradiated by the laser pulse, then the plasma dynamics is not self-similar anymore, and the problem becomes much more complicated for the analytical approach, so it was studied mainly numerically (see, e.g., [6]). However, recently it has been shown that the classical Riemann method [7-9] can be successfully applied to this kind of problems. In particular, the problem of the expansion of the Bose-Einstein condensate released from a box-like trap was solved in ref. [10] and a detailed study of the Landau-Khalatnikov problem on the expansion of high-temperature hadronic matter was given in [11]. Besides that, the Riemann method was applied in ref. [12] to situations where the Riemann function is unknown. In spite of that, an approximate expression for it was derived for equations with an arbitrary type of nonlinearity. As a result, the dispersionless evolution can be studied by this method and it was shown that it yields a very accurate description of the formation of simple waves in the regions where the dispersive shock waves are not formed yet. The aim of this letter is the opposite: we consider the situation with already known Riemann function, but with the initial conditions which prevent formation of dispersive shock waves. As a result, the problem is solved for however large time of evolution including its asymptotic stage $t \to \infty$ which is important in practice. To be exact, we solve here the problem of the expansion of the plasma slab into vacuum. The dispersive effects are not essential in this case, so the hydrodynamic approximation is very accurate, as is demonstrated by the comparison of the approximate solution with the exact numerical one. Our analytical solution provides the main characteristics of the bulk of ions in the expanding plasma and simple asymptotic formulas obtained here have not been known yet.

**Formulation of the problem.** – We assume that initially plasma occupies a slab $-l \leq x \leq l$ with a uniform density $n_0$ of ions whose temperature $T_i$ is much smaller than the temperature $T_e$ of electrons ($T_e \gg T_i$), so their thermal motion can be neglected. As was indicated in the

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and are more convenient (see, e.g., [2]) the velocity distribution function degenerates exponentially fast to the δ-function corresponding to the hydrodynamic approximation. Therefore one can apply to this problem the standard system of equations (see, e.g., [5])

\[ n_T + (nv)_X = 0, \]
\[ v_T + vv_X + \frac{e}{m_i} \phi_X = 0, \]
\[ \phi_{XX} = 4\pi e^2 \left( n_0 \exp \frac{e\phi}{T_e} - n \right), \]

where \( n \) is the ion density, \( v \) is the hydrodynamic velocity of the ions, \( m_i \) is their mass, \( \phi \) is the electric potential, \( e \) is the charge of electrons, and for simplicity we assume the charge of ions being equal to \(-e\). Such a plasma is characterized by the Debye radius \( D = (T_e/4\pi e^2 n_0)^{1/2} \), ion plasma frequency \( \Omega = (4\pi e^2 n_0/m_i)^{1/2} \), and the ion-sound velocity \( c_s = (T_e/m_i)^{1/2} \). This permits one to transform the system (1) to convenient non-dimensional variables \( t = \Omega T, x = D^{-1}X, u = v/c_s, \phi = (e/T_e)\phi \) and to obtain

\[ \rho_t + (\rho u)_x = 0, \]
\[ u_t + uu_x + \frac{\rho_x}{\rho} = 0, \]
\[ \phi_{xx} = e^2 - \rho. \]

This system is still too complicated for analytical treatment and it will be solved later numerically. However, if the initial width of the slab is much greater than the Debye radius, then the dispersive effects are relatively small and we can neglect the second-order derivative in the last (Poisson) equation in system (2) and replace it by the Boltzmann equation \( \rho = e^\phi \). Then elimination of \( \phi \) yields the purely hydrodynamic system

\[ \rho_t + (\rho u)_x = 0, \]
\[ u_t + uu_x + \frac{\rho_x}{\rho} = 0, \]

which should be solved with the slab initial distribution of the density

\[ \rho(x,t=0) = \begin{cases} 1, & \text{if } |x| \leq l, \\ 0, & \text{if } |x| > l, \end{cases} \]

and \( u(x,t=0) = 0 \).

The variables \( \rho \) and \( u \) have clear physical meaning, however, in physics of nonlinear waves other variables called Riemann invariants are more convenient (see, e.g., [13]), because in unidirectional wave propagation one of them is constant. In the case of equation (3) equivalent to the equation of gas dynamics with the isothermal equation of state the Riemann invariants are calculated in a standard way and they can be written as

\[ r_{\pm} = u \pm \int_0^\rho \frac{dp}{\rho} = u \pm \ln \rho, \]

where we used the fact that in our units the sound velocity is constant: \( c_s = 1 \). The hydrodynamic equations (3) transformed to these variables take a simple symmetric form

\[ \frac{\partial r_+}{\partial t} + v_+ \frac{\partial r_+}{\partial x} = 0, \]

where the characteristic velocities are equal to

\[ v_\pm = u \pm \frac{1}{2}(r_+ + r_-) \pm 1. \]

If the Riemann invariants are found, then the physical variables are expressed in terms of them by the formulas

\[ \rho(x,t) = \exp \left( \frac{r_+(x,t) - r_-(x,t)}{2} \right), \]
\[ u(x,t) = \frac{1}{2}[r_+(x,t) + r_-(x,t)]. \]

Now we can turn to solving the formulated problem.

**Rarefaction waves.** The evolution starts with the formation of two rarefaction waves centered around the edges \( x = \pm l \) of the initial distribution. These waves propagate inward the distribution with the linear sound velocity \( c_s = 1 \) and collide at the center \( x = 0 \) at the moment \( t_c = l \). Thus, for \( t \leq l \) the wave configuration consists of two rarefaction waves with unidirectional plasma flows and therefore they belong to the class of simple waves: in the right-propagating wave \( r_+ \) is constant and in the left-propagating wave \( r_- \) is constant. The values of the constants are determined by the matching conditions at the edges propagating with sound velocities inward the region of quiescent plasma where \( u = 0 \) and \( \rho = 1 \), hence \( r_{\pm} = 0 \). Thus, one of the Riemann invariants in the simple wave regions is known and the other must obey the remaining equation (6). In our case with sharp edges of the initial distribution its solution must be self-similar and depend only on one of the variables: \((x-l)/t\) for the right-propagating wave and \((x+l)/t\) for the left-propagating wave. These solutions can be easily obtained from the resulting equations \( v_\pm = (x-l)/t \) or \( v_\pm = (x+l)/t \), respectively, and we arrive at simple formulas

\[ \rho(x,t) = \exp \left( \frac{x+l-t}{t} \right), \quad -\infty \leq x \leq -l, \]
\[ \rho(x,t) = \exp \left( -\frac{x-l+t}{t} \right), \quad l \leq x < \infty. \]

In the case of the isothermic of state the boundary with the vacuum disappears instantly and this means “infinite” velocity \( u = x/t \pm 1 \) of the plasma flow at its tails. Such a non-physical behavior is a consequence of the supposition that the thermal equilibrium of electrons is maintained permanently due to their “infinitely high” temperature. Thus, the theory breaks down for the most energetic ions and needs some modification (see, e.g., [14]). However, the number of such ions is exponentially small.
\[ \frac{\partial^2 W}{\partial r_+ \partial r_-} = 1 \left( \frac{\partial W}{\partial r_+} - \frac{\partial W}{\partial r_-} \right) = 0. \]  

At the boundary \( x_R(t) \) with the right rarefaction wave we have \( r_+ = 0, v_- = r_-/2 - 1 = (x-l)/t \), hence the left-hand side of the first equation (11) equals \(-l\), and in a similar way the left-hand side of the second equation equals \( l \).

Thus, we obtain the boundary conditions for the function \( W(r_+, r_-) \) in the hodograph plane:

\[ \frac{\partial W}{\partial r_-} = l \quad \text{at} \quad r_+ = 0, \]
\[ \frac{\partial W}{\partial r_+} = -l \quad \text{at} \quad r_- = 0. \]

**Solution by the Riemann method.** – The Euler-Poisson equation (13) with the boundary conditions (15) can be solved by the Riemann method [7–9]. Since exposition of this method applied to a similar problem has already been given in refs. [10,11], we shall not go into details here and just formulate the main principles.

Riemann showed [7] that if we wish to find the value of \( W \) at the point \( P = (r_+, r_-) \) in the hodograph plane (see fig. 2), then we should draw in this plane with coordinates \((r'_+, r'_-)\) two lines \( PA \) \((r'_+ = r_+ = \text{const})\) and \( PB \) \((r'_- = r_- = \text{const})\), which, together with \( AO \) and \( OB \) with known functions \( W(r'_+, 0) \) and \( W(0, r'_-) \) along them, form a closed contour \( C = PAOB \). Since \((r_+, r_-)\) denote the coordinates of the “observation point” \( P \), we have introduced here the notation \((r'_+, r'_-)\) for the coordinates in the hodograph plane. Following Riemann [7], we define in this plane such a vector \((V, U)\), that the integral \( \int_C (V dr'_+ + U dr'_-) = 0 \) vanishes. The components \((V, U)\) depend both on the function \( W \), which satisfies eq. (13), as well as on another function \( R = R(r'_+, r'_-, r_+, r_-) \), which must satisfy the condition of vanishing of the above integral. Riemann found that to this end one has to impose the following conditions: first, the function \( R \) must satisfy the conjugate equation which in our case has the form

\[ \frac{\partial^2 R}{\partial r'_+ \partial r'_-} + \frac{1}{4} \left( \frac{\partial R}{\partial r'_+} - \frac{\partial R}{\partial r'_-} \right) = 0. \]
second, it must satisfy the boundary conditions

\[
\frac{\partial R}{\partial r_+^\prime} - \frac{1}{4} R = 0 \quad \text{along} \quad PB, \quad (17)
\]
\[
\frac{\partial R}{\partial r_-^\prime} + \frac{1}{4} R = 0 \quad \text{along} \quad PA;
\]

third, we fix its value at the point \( P \),

\[
R(r_+, r_-; r_+, r_-) = 1. \quad (18)
\]

If such a function \( R \) is found, then the value of \( W \) at the point \( P = (r_+, r_-) \) is given by the expression

\[
W(P) = \frac{1}{2} (RW)_A + \frac{1}{2} (RW)_B + \int_A^O V dr' + \int_O^B U dr', \quad (19)
\]

where \( A = (r_+, 0), \ B = (0, r_-), \ O = (0, 0) \) and \( U, \ V \) are expressed in terms of the boundary conditions (15) by the formulas

\[
U = \frac{1}{2} \left( W \frac{\partial R}{\partial r_+^\prime} - R \frac{\partial W}{\partial r_+^\prime} \right) - \frac{1}{4} WR, \quad (20)
\]
\[
V = \frac{1}{2} \left( R \frac{\partial W}{\partial r_-^\prime} - W \frac{\partial R}{\partial r_-^\prime} \right) - \frac{1}{4} WR.
\]

Thus, to get the solution, we have to find the Riemann functions \( R \). Fortunately, for a gas with isothermal equation of state it was found by Riemann himself [7] and in our notation it can be expressed in the form

\[
R = \exp \left[ \frac{1}{4} (r_+ - \xi - r_- + \eta) \right] \times I_0 \left[ \frac{1}{2} \sqrt{(r_+ - \xi)(\eta - r_-)} \right], \quad (21)
\]

where \( I_0 \) is the Bessel function of complex argument (see, e.g., [17]). Substitution of eq. (15) into eq. (19) followed by integration by parts with account of (20) yields

\[
W(r_+, r_-) = l \exp \left( -\frac{r_+ - r_-}{4} \right) \times \left\{ \int_{r_+}^0 \left( 1 + \frac{r}{4} \right) \exp \left( \frac{r}{4} \right) I_0 \left( \frac{1}{2} \sqrt{r_- (r - r_+)} \right) dr \right. \\
\left. + \int_0^{r_-} \left( 1 - \frac{r}{4} \right) \exp \left( -\frac{r}{4} \right) I_0 \left( \frac{1}{2} \sqrt{-r_+ (r - r_-)} \right) dr \right\}. \quad (22)
\]

Once the function \( W(r_+, r_-) \) is known, the dependence of \( r_+ \) and \( r_- \) on \( x \) and \( t \) is implicitly given by the formulas (11). Then substitution of these functions into eqs. (8) yields the distributions of the physical parameters of the plasma flow. We compare in fig. 3 the analytical results (blue thin line) for the distribution of the density with the exact numerical solution (red thick line) of the system (2). As we see, the hydrodynamic approximation agrees very well with the numerical solution.

**Limiting cases.** – The solution obtained above provides the full description of plasma evolution during its expansion, but the formulas are quite complicated and therefore it is of considerable interest to obtain simpler results for some characteristic parameters of the flow. It is remarkable that the function \( t = t(r_+, r_-) \) in the hodograph representation can be obtained from the system (11) in a more direct way (see, e.g., [18]). Eliminating \( x \) from this system, we arrive immediately at the Euler-Poisson
The formula for the law of motion of the left boundary: 
\[ \frac{\partial^2 t}{\partial r_+ \partial r_-} - \frac{1}{4} \left( \frac{\partial t}{\partial r_+} - \frac{\partial t}{\partial r_-} \right) = 0, \]  
(23)
which coincides with eq. (13) for \( W \). Therefore, its solution symmetric with respect to the transformation \( r_+ \rightarrow -r_-, r_- \rightarrow -r_+ \), and satisfying the condition \( t(0, 0) = l \) can also be expressed in terms of the Bessel function:
\[ t(r_+, r_-) = l \exp \left( \frac{r_+ - r_-}{4} \right) I_0 \left( \frac{1}{2} \sqrt{-r_+ r_-} \right). \]  
(24)
At the right boundary with the rarefaction wave, the invariant \( r_+ \) vanishes and in a similar way the invariant \( r_- \) vanishes at the boundary with the left rarefaction wave (see fig. 1). Then, as follows from eq. (24), the Riemann invariants in the general solution change in the intervals
\[-4 \ln \left( \frac{l}{7} \right) \leq r_+ \leq 0, \quad 0 \leq r_- \leq 4 \ln \left( \frac{l}{7} \right), \]  
(25)
for a fixed value of \( t > l \). At the right boundary \( x_R(t) \) with \( r_+ = 0 \) the function (24) reduces to \( t(0, r_-) = l e^{-r_-^2/4} \), that is \( r_- = 4 \ln(t/l) \) here. The right rarefaction wave solution at its boundary with the general solution is given by \( x_R - (-r_-/2 - 1)t(0, r_-) = l \) and substitution of \( t(0, r_-) \) and \( r_- \) yields the law of motion of the right boundary:
\[ x_R(t) = l + 2t \ln \left( \frac{l}{7} \right) - t, \quad t \geq l. \]  
(26)
The formula for the law of motion of the left boundary \( x_L(t) \), due to the symmetry of the problem, differs from this only by the sign.

At the center \( x = 0 \) of the wave configuration we have \( u = 0 \), i.e., \( r = r_- = -r_+ \), and (24) reduces to
\[ t = t(-r, r) = le^{r^2/2} I_0 \left( \frac{r}{2} \right) = l \sqrt{\rho} I_0 \left( -\frac{1}{2} \ln \rho \right). \]  
(27)
This equation determines implicitly the dependence of the plasma density \( \rho \) on \( t \) at the distribution center. Just after the moment of collision of two rarefaction waves, when \( t - l \ll l \), we obtain
\[ \rho(0, t) \approx 1 - 2 \left( \frac{l}{l} - 1 \right) + \frac{7}{2} \left( \frac{l}{l} - 1 \right)^2, \quad \frac{t}{l} - 1 \ll 1. \]  
(28)
For asymptotically large time \( t \gg l \) the density at the center \( \rho(0, t) \) is small and we can use the asymptotic formula for the Bessel function,
\[ I_0(z) \approx \frac{e^z}{\sqrt{2\pi z}}, \quad z \gg 1, \]
which yields
\[ \rho(0, t) = e^{-T} \approx \frac{l}{l \sqrt{T}}, \quad t \gg l, \]  
(29)
with logarithmic accuracy. All these formulas are confirmed by numerical solution of the system (2) (see fig. 4).

**Conclusion.** – In this letter we have found exact analytical solution for the process of expansion of a slab of two-temperature plasma in hydrodynamic approximation. Comparison with the numerical solution of equations with account of dispersion effects, that is finite value of the Debye length, shows that this is a quite good approximation almost everywhere except for the regions of a very fast flow close to the boundaries of the plasma with a vacuum where the density is small. Thus, the obtained here solution allows one to get estimates of parameters of the plasma in the bulk of the wave configuration.

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