FROBENIUS COMPLEXES AND THE HOMOTOPY COLIMIT 
OF A DIAGRAM OF POSETS OVER A POSET

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Abstract. An affine monoid is an additive monoid which is cancellative, 
pointed and finitely generated. An affine monoid \( \Lambda \) has the partial order 
defined by \( \lambda \leq \lambda + \mu \). The Frobenius complex is the order complex 
of an open interval of \( \Lambda \) with respect to this partial order. The reduced homology of the 
Frobenius complex is related to the torsion group of the monoid algebra \( K[\Lambda] \). 

In this paper, we pay attention to homotopy types of Frobenius complexes, 
and we express the homotopy types of the Frobenius complexes of \( \Lambda \) in terms 
of those of \( \Lambda_1 \) and \( \Lambda_2 \) when \( \Lambda \) is an affine monoid obtained by gluing two 
affine monoids \( \Lambda_1 \) and \( \Lambda_2 \) with one relation. We also state an application to 
the Poincaré series of the torsion group of the monoid algebra.

1. Introduction

We consider an additive monoid which is cancellative, pointed and finitely gen-
erated, which we call an affine monoid. An affine monoid \( \Lambda \) has the partial order 
\( \leq \) defined by \( \lambda \leq \lambda + \mu \) for elements \( \lambda \) and \( \mu \) of \( \Lambda \). For a non-zero element \( \lambda \) of \( \Lambda \) 
the Frobenius complex \( F(\lambda; \Lambda) \) is the order complex \( \left\langle (0, \lambda) \right\rangle \) of the open interval 
of \( \Lambda \).

Frobenius complex is introduced by Laudal and Sletsjøe [LS], and they showed 
the isomorphism

\[
\text{Tor}_{i,\lambda}^{K[\Lambda]}(K, K) \cong H_{i-2}(F(\lambda; \Lambda); K),
\]

where \( K \) is a field. We pay attention to the Poincaré series

\[
P_\Lambda(t, z) = P_K^{K[\Lambda]}(t, z) = \sum_{i \in \mathbb{N}} \sum_{\lambda \in \Lambda} \dim K \cdot \text{Tor}_{i,\lambda}^{K[\Lambda]}(K, K) \cdot t^i z^\lambda
\]

of this torsion group.

Clark and Ehrenborg [CE] took notice of homotopy types of Frobenius com-
plexes, and determined the homotopy types of the Frobenius complexes of \( \Lambda \) by 
using discrete Morse theory when \( \Lambda \) is the submonoid \( \langle a, b \rangle \) of \( \mathbb{N} \) generated by two 
relatively prime integers \( a \) and \( b \) [CE Theorem 4.1], and when \( \Lambda \) is the submonoid 
\( \langle a + nd | n \in \mathbb{N} \rangle \) of \( \mathbb{N} \) generated by the arithmetic sequence for two relatively prime 
integers \( a \) and \( d \) [CE Theorem 5.1].

Tounai [Tou] expressed the homotopy types of the Frobenius complexes of \( \Lambda[r/r] \) 
in terms of the Frobenius complexes of \( \Lambda \), where \( \Lambda[r/r] \) denotes the affine monoid 
obtained from \( \Lambda \) by adjoining the formal \( r \)-th part of a reducible element \( \rho \) of \( \Lambda \).

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algebra; Poincaré series.
This result is an extension of [CE Theorem 4.1], and the proof is based on basic tools of homotopy theory for posets and CW complexes.

The main theorem of this paper is an extension of [Tou, Theorem 3.1], and express the homotopy types of the Frobenius complexes of Λ in terms of the Frobenius complexes of Λ₁ and Λ₂ when Λ is the affine monoid obtained from the direct sum Λ₁ ⊕ Λ₂ of two affine monoids by identifying a reducible element ρ₁ of Λ₁ with a reducible element ρ₂ of Λ₂. As a corollary, the Poincaré series of Λ is expressed as

\[ P_\Lambda(t, z) = P_{\Lambda_1}(t, z) \cdot P_{\Lambda_2}(t, z) - t \cdot z^{\rho} \]

where ρ denotes the equivalence class of ρ₁ and ρ₂ in Λ. The case z = 1 of this formula is a special case of [Avr, Proposition 3.3.5.(2)], and Avramov [Avr] implies that the graded version of this proposition will be also proven in an algebraic way.

To prove the main theorem we use the notion of homotopy colimit, which is established by Bousfield and Kan [BK]. We refer to [BWW], [Tho], [WZZ] and [ZZ] about homotopy colimits.

2. Preliminaries

2.1. Posets. A partially ordered set (poset for short) is a set P together with a partial order ≤. The order complex \( \|P\| \) of a poset P is the simplicial complex whose vertices are elements of P and whose simplices are non-empty finite chains of P. For posets P and Q a map \( f: P \to Q \) is a poset map if \( p \leq p' \) implies \( f(p) \leq f(p') \) for any \( p, p' \in P \). The simplicial map induced by a poset map f is denoted by \( \|f\| \).

Proposition 2.1 (Quillen [Qui, 1.3]). Let \( f, g: P \to Q \) be poset maps. If \( f \leq g \) holds, that is, \( f(p) \leq g(p) \) holds for each \( p \in P \), then the induced maps \( \|f\| \) and \( \|g\| \) are homotopic.

Lemma 2.2. If a poset P has a maximum element, then the order complex \( \|P\| \) is contractible.

Proof. The constant map c to the maximum element satisfies \( id_P \leq c \), and thus the identity map on \( \|P\| \) is homotopic to a constant map. □

Lemma 2.3. Let P be a poset and S a subset of P. If \( S^{\leq p} = \{ s \in S \mid s \leq p \} \) has a maximum element for each \( p \in P \), then \( \|S\| \) is a deformation retract of \( \|P\| \).

Proof. Let \( i: S \to P \) be the inclusion map. Define \( r: P \to S \) by

\[ r(p) = \max S^{\leq p} \quad (p \in P). \]

Then r is a poset map, and satisfies \( r \circ i = id_S \) and \( i \circ r \leq id_P \), which proves the lemma. □

Theorem 2.4 (Quillen [Qui, 1.6]). Let \( f: P \to Q \) be a poset map. If \( f^{-1}(Q_{\geq q}) \) is contractible for each \( q \in Q \), then the induced map \( \|f\|: \|P\| \to \|Q\| \) is a homotopy equivalence.

The following is a slightly modified version of [BWW Theorem 2.5], and is proved in the same way.
**Theorem 2.5** (Björner-Wachs-Welker [BWW Theorem 2.5]). Let \( f: P \rightarrow Q \) be a poset map, and assume that \( Q \) has a minimum element \( m \). If the inclusion \( \| f^{-1}(Q_{\leq q}) \| \rightarrow \| f^{-1}(Q_{\geq q}) \| \) is homotopic to the constant map to a point \( c_q \) of \( \| f^{-1}(Q_{\geq q}) \| \) for each non-minimum element \( q \) of \( Q \), then
\[
\| P \| \simeq \bigvee_{q \in Q} \| Q_{>q} \| * \| f^{-1}(Q_{\geq q}) \|,
\]
where the wedge is formed by identifying \( q \in \| Q_{>m} \| * \| f^{-1}(Q_{\leq m}) \| \) with \( c_q \in \| Q_{>q} \| * \| f^{-1}(Q_{\leq q}) \| \) for each non-minimum element \( q \) of \( Q \).

**Lemma 2.6** (Walker [Wal Theorem 5.1.(d)]). Let \( P \) and \( Q \) be posets, \( p,p' \in P \) and \( q,q' \in Q \). Assume that \( p < p' \) and \( q < q' \). Then the order complex \( \| (p,q),(p',q') \|_P \times Q \| \) of the open interval of the product poset is homeomorphic to the suspension of the join \( \| (p,p')_P \| * \| (q,q')_Q \| \) of the order complexes of the open intervals.

### 2.2. Homotopy colimits over a finite poset.

A diagram over a poset \( P \) (\( P \)-diagram for short) in a category \( C \) is a functor \( D: P \rightarrow C \). For \( p,q \in P \) with \( p \leq q \) the induced morphism \( D(p) \rightarrow D(q) \) of \( C \) is denoted by \( D^p_q \).

Let \( D \) be a diagram of topological spaces over a finite poset \( P \), that is, \( D \) is a \( P \)-diagram in the category of topological spaces. The **homotopy colimit** of \( D \) is defined by
\[
hocolim D = \coprod_{p \in P} \| P_{\geq p} \| \times D(p)
\]
where \( \sim \) is the equivalence relation generated by
\[
\| P_{\geq p} \| \times D(p) \ni (a,x) \sim (a,D^p_q(x)) \in \| P_{\geq q} \| \times D(q)
\]
for \( p,q \in P \) with \( p \leq q \), \( a \in \| P_{\geq q} \| \), and \( x \in D(p) \).

Note that a natural homeomorphism \( \alpha: D \rightarrow E \) between diagrams induces a homeomorphism \( \hocolim D \simeq \hocolim E \). Moreover, it is known that a natural homotopy equivalence induces a homotopy equivalence \( \hocolim D \simeq \hocolim E \) [WZZ 3.7].

For a \( P \)-diagram \( X \) of posets by composing the order complex functor we obtain a \( P \)-diagram of topological spaces, denoted by \( \| X \| \).

**Theorem 2.7** (Thomason [Tho Theorem 1.2]). Let \( X \) be a diagram of posets over a finite poset \( P \). Then the homotopy colimit of \( \| X \| \) is homotopy equivalent to the order complex of the poset \( P \bowtie X \) defined by
\[
P \bowtie X = \coprod_{p \in P} X(p)
\]
\[
= \{ (p,x) \mid p \in P, x \in X(p) \}
\]
and
\[
(p,x) \leq (q,y) \iff p \leq q \text{ and } X^p_q(x) \leq y.
\]

#### 2.3. Frobenius complexes.

In this paper \( \mathbb{N} \) denotes the additive monoid \( \mathbb{Z}_{\geq 0} \) of non-negative integers. An **affine monoid** is an additive monoid \( \Lambda \) which satisfies the three following condition.

1. \( \Lambda \) is cancellative, that is, \( \lambda + \mu = \lambda' + \mu \) implies \( \lambda = \lambda' \) for any \( \lambda, \lambda', \mu \in \Lambda \).
2. \( \Lambda \) is pointed, that is, \( \lambda + \mu = 0 \) implies \( \lambda = \mu = 0 \) for any \( \lambda, \mu \in \Lambda \).
(3) \( \Lambda \) is finitely generated, that is, there exist finite elements \( \alpha_1, \ldots, \alpha_d \) of \( \Lambda \) such that for any element \( \lambda \) of \( \Lambda \) can be written as \( \lambda = m_1\alpha_1 + \cdots + m_d\alpha_d \) for some \( m_1, \ldots, m_d \in \mathbb{N} \).

For example, a finitely generated submonoid of \( \mathbb{N}^d \) is an affine monoid.

An affine monoid \( \Lambda \) has the Frobenius order \( \leq \) defined by

\[
\lambda \leq \nu \iff \text{there exists } \mu \in \Lambda \text{ satisfying } \lambda + \mu = \nu.
\]

For a non-zero element \( \lambda \) of \( \Lambda \) we define the Frobenius complex \( \mathcal{F}(\lambda; \Lambda) \) by

\[
\mathcal{F}(\lambda; \Lambda) = \left\| (0, \lambda) \right\|,
\]

where \( (0, \lambda) \) denotes the open interval of \( \Lambda \) with respect to the Frobenius order.

Note that \( (0, \lambda) \) is finite \cite{tou2}. We also define

\[
\mathcal{F}(0; \Lambda) = S^{-2},
\]

where \( S^{-2} \) is just a formal symbol, not a topological space.

**Proposition 2.8.** The Frobenius complexes of \( \mathbb{N} \) satisfy

\[
\mathcal{F}(0; \mathbb{N}) = S^{-2},
\]

\[
\mathcal{F}(1; \mathbb{N}) = S^{-1},
\]

\[
\mathcal{F}(n; \mathbb{N}) = \text{pt} \quad (n \geq 2).
\]

**Proof.** The case \( n = 0 \) follows by definition, and the case \( n = 1 \) from \( (0, 1) = \emptyset \).

If \( n \geq 2 \), then we have

\[
\mathcal{F}(n; \mathbb{N}) = \left\| (0, n) \right\| = \left\| (1, n) \right\|, \quad \text{pt}.
\]

\( \square \)

**Lemma 2.9.** Let \( \Lambda_1 \) and \( \Lambda_2 \) be affine monoids. Then

\[
\mathcal{F}(\lambda_1 + \lambda_2; \Lambda_1 \oplus \Lambda_2) \approx \mathcal{F}(\lambda_1; \Lambda_1) \oplus \mathcal{F}(\lambda_2; \Lambda_2)
\]

for \( \lambda_1 \in \Lambda_1 \) and \( \lambda_2 \in \Lambda_2 \), where \( X \oplus Y \) denotes the suspension of the join \( X \ast Y \) for topological spaces \( X \) and \( Y \), and let \( S^{-2} \oplus X = X \oplus S^{-2} \).

**Proof.** It follows from Lemma 2.6. \( \square \)

**Lemma 2.10.** Let \( \Lambda \) be an affine monoid and \( \rho \) a non-zero element of \( \Lambda \). Then for any \( \lambda \in \Lambda \) there uniquely exist \( \ell \in \mathbb{N} \) and \( \hat{\lambda} \in \Lambda \) which satisfy \( \ell \rho + \hat{\lambda} = \lambda \) and \( \hat{\lambda} \not\geq \rho \).

**Proof.** Let \( \lambda \in \Lambda \). By \cite{tou2} Lemma 2.8, there exists a maximum \( \ell \) satisfying \( \ell \rho \leq \lambda \). Then we can take \( \hat{\lambda} \in \Lambda \) satisfying \( \ell \rho + \hat{\lambda} = \lambda \). The maximality of \( \ell \) implies \( \hat{\lambda} \not\geq \rho \). We next show the uniqueness. Let \( \ell \rho + \hat{\lambda} = \ell' \rho + \hat{\lambda}' \) and \( \hat{\lambda}, \hat{\lambda}' \not\geq \rho \). We can assume that \( \ell \leq \ell' \). Cancellativity implies \( \hat{\lambda} = (\ell' - \ell) \rho + \hat{\lambda}' \). Since \( \hat{\lambda} \not\geq \rho \), we have \( \ell = \ell' \), and thus \( \hat{\lambda} = \hat{\lambda}' \). \( \square \)

The composition poset \( C(\lambda; \Lambda) \) is the poset of non-trivial ordered partitions of \( \lambda \) in \( \Lambda \), that is,

\[
C(\lambda; \Lambda) = \left\{ [\xi^{(1)}|\cdots|\xi^{(k)}] \mid k \geq 2, \xi^{(i)} \in \Lambda \setminus \{0\}, \sum_{i=1}^{k} \xi^{(i)} = \lambda \right\}
\]

and the order is generated by

\[
[\xi^{(1)}|\cdots|\xi^{(i)} + \xi^{(i+1)}|\cdots|\xi^{(k)}] \leq [\xi^{(1)}|\cdots|\xi^{(i)}|\xi^{(i+1)}|\cdots|\xi^{(k)}]
\]

for \( [\xi^{(1)}|\cdots|\xi^{(k)}] \in C(\lambda; \Lambda) \) with \( k \geq 3 \) and \( i = 1, \ldots, k - 1 \).
Proposition 2.11. For an affine monoid \( \Lambda \) and a non-zero element \( \lambda \) of \( \Lambda \) the Frobenius complex \( \mathcal{F}(\lambda; \Lambda) \) is homeomorphic to the order complex of the composition poset \( C(\lambda; \Lambda) \).

Proof. Define the map \( \Phi \) from \( C(\lambda; \Lambda) \) to the face poset of \( \| (0, \lambda)_{\Lambda} \| \) by
\[
\Phi\left(\{\xi(1) \cdots \xi(k)\}\right) = \{\xi(1) < \xi(1) + \xi(2) < \cdots < \xi(1) + \cdots + \xi(k-1)\}.
\]
Then the inverse of \( \Phi \) is given by
\[
\Phi^{-1}\left(\{\mu_0 < \cdots < \mu_k\}\right) = [\mu_0 | \mu_1 - \mu_0 | \cdots | \mu_k - \mu_{k-1}| \lambda - \mu_k].
\]
We can check that both \( \Phi \) and \( \Phi^{-1} \) are order-preserving. Thus \( \| C(\lambda; \Lambda) \| \) is isomorphic to the barycentric subdivision of \( \| (0, \lambda)_{\Lambda} \| \). \( \square \)

A monoid homomorphism \( \varphi: \Lambda \to \Lambda' \) between two affine monoids is proper if \( \varphi(\lambda) = 0 \) implies \( \lambda = 0 \) for any \( \lambda \in \Lambda \). Note that if \( \varphi \) is proper, then \( \varphi \) is strictly order-preserving, but not necessarily injective. A proper homomorphism \( \varphi: \Lambda \to \Lambda' \) induces poset maps
\[
\varphi: (0, \lambda)_{\Lambda} \to (0, \varphi(\lambda))_{\Lambda'},
\]
\[
\varphi_*: C(\lambda; \Lambda) \to C(\varphi(\lambda); \Lambda')
\]
for a non-zero element \( \lambda \) of \( \Lambda \) in obvious ways.

Lemma 2.12. Let \( \varphi: \Lambda \to \Lambda' \) be a proper homomorphism between affine monoids, \( \lambda \) a non-zero element of \( \Lambda \), and \( \lambda' = \varphi(\lambda) \in \Lambda' \). For \( \xi \in C(\lambda; \Lambda) \) and \( \eta \in C(\lambda'; \Lambda') \) which satisfy \( \varphi_*(\xi) \geq \eta \) there uniquely exists \( \xi' \in C(\lambda; \Lambda) \) satisfying \( \xi \geq \xi' \) and \( \varphi_*(\xi') = \eta \).

Proof. We use the poset isomorphism \( \Phi \) defined in the proof of Proposition 2.11.
Let \( \{\mu_1 < \cdots < \mu_k\} \) and \( \{\mu'_1 < \cdots < \mu'_k\} \) be non-empty chains of \( (0, \lambda)_{\Lambda} \) and \( (0, \lambda')_{\Lambda'} \), respectively. Assume that \( \varphi(\{\mu_1, \ldots, \mu_k\}) \) contains \( \{\mu'_1, \ldots, \mu'_k\} \). Since \( \varphi \) is strictly order-preserving, there uniquely exists a subchain of \( \{\mu_1, \ldots, \mu_k\} \) whose image by \( \varphi \) coincides with \( \{\mu'_1, \ldots, \mu'_k\} \). \( \square \)

An additive equivalence relation on an affine monoid \( \Lambda \) is an equivalence relation which preserves the addition, that is, \( x \sim x' \) and \( y \sim y' \) imply \( x + y \sim x' + y' \) for any \( x, x', y, y' \in \Lambda \). Clearly, there exist a smallest additive equivalence relation which contains a given relation.

An element \( \rho \) of an affine monoid \( \Lambda \) is reducible if there exist non-zero elements \( \sigma \) and \( \tau \) of \( \Lambda \) satisfying \( \sigma + \tau = \rho \).

2.4. Poincaré series. Let \( \Lambda \) be an affine monoid, and fix a field \( K \).

Theorem 2.13 (Laudal-Sletsjøe [LS]). There is an isomorphism
\[
\text{Tor}_{i,\Lambda}^{K[\Lambda]}(K, K) \cong \tilde{H}_{i-2}(\mathcal{F}(\lambda; \Lambda); K)
\]
for each \( i \in \mathbb{N} \) and \( \lambda \in \Lambda \).

The Poincaré series of \( \Lambda \) is defined by
\[
P_\Lambda(t, z) = \sum_{i \in \mathbb{N}} \sum_{\lambda \in \Lambda} \beta_i(\lambda; \Lambda) t^i z^\lambda,
\]
where
\[
\beta_i(\lambda; \Lambda) = \dim_K \text{Tor}_{i,\Lambda}^{K[\Lambda]}(K, K) \quad (i \in \mathbb{N}, \ \lambda \in \Lambda).
\]
By the previous theorem, we have

\[(2.1) \quad \beta_i(\lambda; \Lambda) = \tilde{\beta}_{i-2}(\mathcal{F}(\lambda; \Lambda)),\]

where \(\tilde{\beta}_i(X)\) denotes the \(i\)-th reduced Betti number of a topological space \(X\), that is,

\[\tilde{\beta}_i(X) = \dim_K \tilde{H}_i(X; K).\]

We also define

\[\tilde{\beta}_i(S^{-2}) = \begin{cases} 1 & \text{if } i = -2, \\ 0 & \text{otherwise}. \end{cases}\]

\[Lemma 2.14. \text{ Let } X \text{ and } Y \text{ be topological spaces. Then } \tilde{\beta}_{i-2}(X \otimes Y) = \sum_{j+k=i} \tilde{\beta}_{j-2}(X) \cdot \tilde{\beta}_{k-2}(Y).\]

**Proof.** By [Mil, Lemma 2.1], we have

\[\tilde{H}_{i-2}(X \otimes Y; K) \cong \tilde{H}_{i-3}(X \ast Y; K) \cong \bigoplus_{j+k=i-4} \tilde{H}_j(X; K) \otimes \tilde{H}_k(Y; K) \cong \bigoplus_{j+k=i} \tilde{H}_{j-2}(X; K) \otimes \tilde{H}_{k-2}(Y; K).\]

\[Proposition 2.15. \text{ Let } \Lambda_1 \text{ and } \Lambda_2 \text{ be affine monoids. Then } \]

\[P_{\Lambda_1 \oplus \Lambda_2}(t, z) = P_{\Lambda_1}(t, z) \cdot P_{\Lambda_2}(t, z).\]

**Proof.** By the equation (2.1), Lemma 2.9 and the previous lemma, we have

\[\beta_i(\lambda_1 + \lambda_2; \Lambda_1 \oplus \Lambda_2) = \tilde{\beta}_{i-2}(\mathcal{F}(\lambda_1 + \lambda_2; \Lambda_1 \oplus \Lambda_2)) = \tilde{\beta}_{i-2}(\mathcal{F}(\lambda_1; \Lambda_1) \oplus \mathcal{F}(\lambda_2; \Lambda_2)) = \sum_{j+k=i} \tilde{\beta}_{j-2}(\mathcal{F}(\lambda_1; \Lambda_1)) \cdot \tilde{\beta}_{k-2}(\mathcal{F}(\lambda_2; \Lambda_2)) = \sum_{j+k=i} \beta_j(\lambda_1; \Lambda_1) \cdot \beta_k(\lambda_2; \Lambda_2).\]

Thus

\[P_{\Lambda_1 \oplus \Lambda_2}(t, z) = \sum_{i \in \mathbb{N}} \sum_{\lambda_1 \in \Lambda_1} \sum_{\lambda_2 \in \Lambda_2} \beta_i(\lambda_1 + \lambda_2; \Lambda_1 \oplus \Lambda_2) t^j z^L \lambda_1 + \lambda_2 = \sum_{i \in \mathbb{N}} \sum_{\lambda_1 \in \Lambda_1} \sum_{\lambda_2 \in \Lambda_2} \sum_{j+k=i} \beta_j(\lambda_1; \Lambda_1) \cdot \beta_k(\lambda_2; \Lambda_2) t^j z^L \lambda_1 + \lambda_2 = \sum_{j \in \mathbb{N}} \sum_{\lambda_1 \in \Lambda_1} \beta_j(\lambda_1; \Lambda_1) t^j z^L \lambda_1 \cdot \sum_{k \in \mathbb{N}} \beta_k(\lambda_2; \Lambda_2) t^k z^L \lambda_2 = P_{\Lambda_1}(t, z) \cdot P_{\Lambda_2}(t, z).\]
3. The main theorem

Let $\Lambda_1$ and $\Lambda_2$ be affine monoids, and let $\rho_1$ and $\rho_2$ be reducible elements of $\Lambda_1$ and $\Lambda_2$, respectively. Let $\Lambda$ be the quotient of the direct sum $\Lambda_1 \oplus \Lambda_2$ modulo the smallest additive equivalence relation $\sim$ satisfying $\rho_1 \sim \rho_2$. We denote the equivalence class of $\rho_1$ and $\rho_2$ simply by $\rho$, and define

$$\hat{\Lambda}_1 = \{ \lambda_1 \in \Lambda_1 \mid \lambda_1 \not\sim \rho_1 \}$$

$$\hat{\Lambda}_2 = \{ \lambda_2 \in \Lambda_2 \mid \lambda_2 \not\sim \rho_2 \}.$$

**Proposition 3.1.** The following hold.

1. The quotient $\Lambda$ has the additive monoid structure inherited from $\Lambda_1 \oplus \Lambda_2$.
2. With the above monoid structure $\Lambda$ is an affine monoid.
3. For any element $\lambda$ of $\Lambda$ there uniquely exist $n \in \mathbb{N}$, $\hat{\lambda}_1 \in \hat{\Lambda}_1$ and $\hat{\lambda}_2 \in \hat{\Lambda}_2$ which satisfy $n\rho + \hat{\lambda}_1 + \hat{\lambda}_2 = \lambda$.

**Proof.** The proof is straightforward. \hfill \Box

**Theorem 3.2.** The Frobenius complexes of $\Lambda$ satisfy

$$\mathcal{F}(\lambda; \Lambda) \simeq \bigvee_{\ell + \lambda_1 + \lambda_2 = \lambda} S^{2\ell-2} \otimes \mathcal{F}(\lambda_1; \Lambda_1) \otimes \mathcal{F}(\lambda_2; \Lambda_2)$$

for $\lambda \in \Lambda$, where $\ell$ runs in $\mathbb{N}$, $\lambda_1$ in $\Lambda_1$ and $\lambda_2$ in $\Lambda_2$.

**Proof.** Take $n \in \mathbb{N}$, $\hat{\lambda}_1 \in \hat{\Lambda}_1$ and $\hat{\lambda}_2 \in \hat{\Lambda}_2$ which satisfy $\lambda = n\rho + \hat{\lambda}_1 + \hat{\lambda}_2 \in \Lambda$. Let $P$ be the poset of non-empty subset of $[n] = \{0, \ldots, n\}$ ordered by reverse inclusion, that is,

$$p \leq q \iff p \supseteq q \quad (p, q \in P).$$

Define

$$\hat{\Lambda} = \Lambda_1 \oplus \mathbb{N}\alpha_1 \oplus \cdots \oplus \mathbb{N}\alpha_n \oplus \Lambda_2.$$

For $p \in P$ let $\Lambda_p$ be the quotient monoid of $\hat{\Lambda}$ modulo the additive equivalence relation $\sim_p$ generated by

$$\alpha_i \sim_p \alpha_{i+1} \quad (i \in [n] \setminus p),$$

where $\alpha_0$ and $\alpha_{n+1}$ denote $\rho_1$ and $\rho_2$, respectively, and let $\hat{\phi}_p: \hat{\Lambda} \to \Lambda_p$ be the canonical surjection. Note that for $p = \{p_0 < \cdots < p_s\} \in P$ the composition

$$(\hat{\lambda} = \hat{\lambda}_1 + \alpha_1 + \cdots + \alpha_n + \hat{\lambda}_2 \in \hat{\Lambda})$$

$$\Lambda_1 \oplus \mathbb{N}\alpha_{p_1} \oplus \cdots \oplus \mathbb{N}\alpha_{p_s} \oplus \Lambda_2 \leftrightarrow \hat{\Lambda} \xrightarrow{\hat{\phi}_p} \Lambda_p$$

is an isomorphism. For $p, q \in P$ with $p \leq q$ let $\varphi_q^p: \Lambda_p \to \Lambda_q$ be the homomorphism satisfying $\varphi_q^p \circ \hat{\phi}_p = \hat{\phi}_q$. Note that $\varphi_q^p$ is proper. Define

$$\hat{\lambda} = \hat{\lambda}_1 + \alpha_1 + \cdots + \alpha_n + \hat{\lambda}_2 \in \hat{\Lambda},$$

and for $p \in P$ let $\lambda_p = \hat{\phi}_p(\hat{\lambda})$. Then $\varphi_q^p$ sends $\lambda_p$ to $\lambda_q$ for $p, q \in P$ with $p \leq q$.

Let $X$ and $Y$ be the $P$-diagrams of posets defined as follows. For $p \in P$ let

$$X(p) = (0, \lambda_p)_\Lambda$$

$$Y(p) = C(\lambda_p; \Lambda_p)$$

and for $p, q \in P$ with $p \leq q$ let $X_q^p: X(p) \to X(q)$ and $Y_q^p: Y(p) \to Y(q)$ be the poset maps induced by $\varphi_q^p$. By Proposition 2.11 two $P$-diagrams $\|X\|$ and $\|Y\|$ are naturally homeomorphic. By Theorem 2.17 we have

$$\|P \ltimes X\| \simeq \hocolim \|X\| \approx \hocolim \|Y\| \simeq \|P \ltimes Y\|. $$
The rest of proof is divided into two parts: In the first part, we show that $\|P \times Y\|$ is homotopy equivalent to $\mathcal{F}(\lambda; \Lambda)$. In the second part, we show that $\|P \times X\|$ is homotopy equivalent to the right-hand side of the theorem.

Let $\tilde{\pi}: \hat{\Lambda} \to \Lambda$ be the homomorphism defined by

$$
\tilde{\pi}(\lambda_1) = \lambda_1 \quad (\lambda_1 \in \Lambda_1)
$$

$$
\tilde{\pi}(\alpha_i) = \rho \quad (i = 1, \ldots, n)
$$

$$
\tilde{\pi}(\lambda_2) = \lambda_2 \quad (\lambda_2 \in \Lambda_2).
$$

For $p \in P$ let $\pi^p: \Lambda_p \to \Lambda$ be the homomorphism satisfying $\pi^p \circ \hat{\varphi}_p = \tilde{\pi}$. Then $\tilde{\pi}$ is proper and sends $\hat{\lambda}$ to $\lambda$, and thus $\pi^p$ is proper and sends $\Lambda_p$ to $\lambda$. Let $f: P \times Y \to C(\lambda; \Lambda)$ be the poset map defined by

$$
f((p, \xi)) = \pi^p_*(\xi) \quad ((p, \xi) \in P \times Y).
$$

We now show that $f$ induces a homotopy equivalence by using Theorem 2.14. Let $\zeta = [\zeta^{(1)}] \cdots [\zeta^{(k)}] \in C(\lambda; \Lambda)$. It suffices to show that $\|f^{-1}(\geq \zeta)\|$ is contractible, where $f^{-1}(\geq \zeta)$ is an abbreviation for $f^{-1}(C(\lambda; \Lambda) \geq \zeta)$. Let $(p, \xi) \in f^{-1}(\geq \zeta)$, that is, $\pi^p_*(\xi) \geq \zeta$. By Lemma 2.12 there uniquely exists $\xi' \in C(\lambda_1: \Lambda_p)$ which satisfies $\xi' \leq \xi$ and $\pi^p_*(\xi') = \zeta$. Then $(p, \xi')$ is a maximum element of $f^{-1}(\xi) \leq (p, \xi)$, since for $(q, \eta) \in f^{-1}(\xi) \leq (p, \xi)$ the uniqueness of $\xi'$ implies $\varphi_q^p(\eta) = \xi'$ and thus $(q, \eta) \leq (p, \xi')$. By Lemma 2.13 $\|f^{-1}(\xi)\|$ is a deformation retract of $\|f^{-1}(\geq \zeta)\|$.

Using the bijection of Proposition 3.1 (3), define the map $\text{rem}_1: \Lambda \to \hat{\Lambda}_1$ as the composition

$$
\Lambda \cong \mathbb{N}_0 \times \hat{\Lambda}_1 \times \hat{\Lambda}_2 \xrightarrow{\text{proj}} \hat{\Lambda}_1.
$$

Using the isomorphism 3.1 and the bijection of Lemma 2.10 also define the map $\text{rem}^p: \Lambda_p \to \hat{\Lambda}_1$ as the composition

$$
\Lambda_p \cong \Lambda_1 \oplus \mathbb{N}_0 \alpha_{p_1} \oplus \cdots \oplus \mathbb{N}_0 \alpha_{p_s} \oplus \Lambda_2 \xrightarrow{\text{proj}} \Lambda_1 \cong \mathbb{N}_0 \times \hat{\Lambda}_1 \xrightarrow{\text{proj}} \hat{\Lambda}_1
$$

for $p = \{p_0 < \cdots < p_s\} \in P$. Then we can check that

$$
\text{rem}_1 \circ \pi^p = \text{rem}^p.
$$

Let

$$
\ell = \max\left\{ \ell \in \mathbb{N} \mid \ell \rho_1 \leq \sum_{i=1}^k \text{rem}_1(\zeta^{(i)}) \right\},
$$

and let

$$
S = \{(p, \xi) \in f^{-1}(\zeta) \mid \ell \in p \}.
$$

We now show that $\|S\|$ is a deformation retract of $\|f^{-1}(\xi)\|$ by using Lemma 2.13. Let $(p, \xi) \in f^{-1}(\zeta)$, where $p = \{p_0 < \cdots < p_s\}$ and $\xi = [\xi^{(1)}] \cdots [\xi^{(k)}]$. Then we have

$$
\lambda_p = \sum_{i=1}^k \xi^{(i)} \geq \sum_{i=1}^k \text{rem}_1(\xi^{(i)}) = \sum_{i=1}^k \text{rem}_1(\zeta^{(i)}) \geq \ell \rho_1.
$$

Thus at least $\ell$ of $\alpha_1, \ldots, \alpha_n$ are identified with $\rho_1$, which implies $\ell \leq p_0$. If $\ell = p_0$ holds, then $(p, \xi)$ itself is a maximum element of $S \leq (p, \xi)$. Assume that $\ell < p_0$, and let $\bar{p} = \{\ell\} \cup p$. Using the isomorphism 3.1 and Lemma 2.10 we can see that there uniquely exist $\bar{\xi}^{(i)} \in \Lambda_{\bar{p}}$ which satisfies $\varphi_{\bar{p}}^p(\bar{\xi}^{(i)}) = \xi^{(i)}$ and $\bar{\xi}^{(i)} \not\geq \rho_1$. 

for each $i$. Then $\bar{\zeta} = [\bar{\xi}^{(1)}|\cdots|\bar{\xi}^{(k)}]$ satisfies $\bar{\zeta} \in C(\lambda_\ell; \Lambda_\ell)$, $\varphi^{\varphi}_{p,\ast}(\bar{\zeta}) = \xi$, and thus $(\bar{p}, \bar{\zeta}) \in S^{\leq (p, \xi)}$. Moreover, the set 
\[ \{ \xi' \in C(\lambda_\ell; \Lambda_\ell) \mid \varphi^{\varphi}_{p,\ast}(\xi') = \xi \} \]
consists of the unique element $\bar{\xi}$. For $(q, \eta) \in S^{\leq (p, \xi)}$ we have $q \leq \bar{p}$ and $\varphi^{\varphi}_{p,\ast}(\eta) = \bar{\xi}$ which imply $(q, \eta) \leq (\bar{p}, \bar{\zeta})$. Hence $(\bar{p}, \bar{\zeta})$ is a maximum element of $S^{\leq (p, \xi)}$. Thus $\|S\|$ is a deformation retract of $\|f^{-1}(\bar{\zeta})\|$.

Let us construct a maximum element of $S$. Using the isomorphism (3.1) and Lemma [2.10], we can see that there uniquely exists $\bar{\zeta}^{(i)} \in \Lambda_\ell$ which satisfies $\pi^{\varphi}_{\ell}(\bar{\zeta}^{(i)}) = \bar{\xi}^{(i)}$ and $\bar{\zeta}^{(i)} \not\simeq p_1$ for each $i$. Then $\bar{\zeta} = [\bar{\xi}^{(1)}|\cdots|\bar{\xi}^{(k)}]$ satisfies $\bar{\zeta} \in C(\lambda_\ell; \Lambda_\ell)$, $\pi^{\varphi}_{\ell}(\bar{\zeta}) = \bar{\xi}$, and thus $\{(\ell), \bar{\zeta}\} \in S$. Moreover, the set 
\[ \{ \xi' \in C(\lambda_\ell; \Lambda_\ell) \mid \pi^{\varphi}_{\ell}(\xi') = \bar{\xi} \} \]
consists of the unique element $\bar{\zeta}$. For $(q, \eta) \in S$ we have $q \leq \ell$ and $\varphi^{\varphi}_{\ell}(\eta) = \bar{\zeta}$, which imply $(q, \eta) \leq (\ell, \bar{\zeta})$. Hence $\{(\ell), \bar{\zeta}\}$ is a maximum element of $S$, and thus $\|S\|$ is contractible. From the above we have 
\[ \|f^{-1}(\bar{\zeta})\| \simeq \|f^{-1}(\zeta)\| \simeq \|S\| \simeq pt. \]

Thus we conclude that 
\[ \|P \times Y\| \simeq \|C(\lambda; \Lambda)\| \simeq F(\lambda; \Lambda). \]

Next, we consider $P \times X$. Define $g : P \times X \to P$ by 
\[ g(p, x) = p \quad ((p, x) \in P \times X). \]

Let us check that $g$ satisfies the assumption of Theorem [2.5]. Clearly, $P$ has the minimum element $[n] = \{0, \ldots, n\}$. In the case $n = 0$ the assumption is trivially satisfied. Assume that $n \geq 1$. Let $p = \{p_0 < \cdots < p_s\} \in P$ with $p > [n]$. Note that 
\[ g^{-1}(P^{<p}) = P^{<p} \times X \]
\[ = \{(q, \mu) \mid q \in P^{<p}, \mu \in X(q)\} \]
\[ g^{-1}(P^{\leq p}) = P^{\leq p} \times X \]
\[ = \{(q, \mu) \mid q \in P^{\leq p}, \mu \in X(q)\}. \]

Moreover, the poset map $P^{<p} \times X \to X(p)$ which sends $(q, \mu)$ to $X^q_\mu(\mu)$ induces a homotopy equivalence, since the canonical injection $X(p) \hookrightarrow P^{<p} \times X$ induces a homotopy inverse. Thus it suffices to show that the composition 
\[ \|P^{<p} \times X\| \to \|P^{<p} \times X\| \to \|X(p)\| \]
is homotopic to a constant map. This map is induced by the poset map 
\[ \varphi : P^{<p} \times X \to X(p) \]
which sends $(q, \mu)$ to $\varphi^{q}_{p}(\mu)$. By definition, we have 
\[ \|X(p)\| = F(\lambda_\ell; \Lambda_\ell) \]
using the isomorphism (3.3) 
\[ \approx F(p_0\alpha_1 + \lambda_1 + (p_1 - p_0)\alpha_{p_1} + \cdots + (p_s - p_{s-1})\alpha_{p_s} + (n - p_s)\rho_2 + \lambda_2; \]
\[ \Lambda_1 \oplus \Lambda_{p_1} \oplus \cdots \oplus \Lambda_{p_s} \oplus \Lambda_2) \]
using Lemma 2.9
\[ \approx \mathcal{F}(p_0 \rho_1 + \hat{\lambda}_1; \Lambda_1) \otimes \mathcal{F}(p_1 - p_0; \mathbb{N}) \otimes \cdots \]
\[ \cdots \otimes \mathcal{F}(p_s - p_{s-1}; \mathbb{N}) \otimes \mathcal{F}((n - p_s) \rho_2 + \hat{\lambda}_2; \Lambda_2). \]

By Proposition 2.8 if \( p = \{p_0, \ldots, p_s\} \) does not have the form \( \{\ell_1, \ell_1 + 1, \ldots, \ell_1 + \ell\} \), then \( \|X(p)\| \) is contractible, and thus \( \|\varphi\| \) is homotopic to a constant map. Assume that \( p = \{\ell_1, \ell_1 + 1, \ldots, \ell_1 + \ell\} \), and let \( \ell_2 = n - (\ell_1 + \ell) \). Then we have
\[ \|X(p)\| \approx \mathcal{F}(\ell_1 \rho_1 + \hat{\lambda}_1; \Lambda_1) \otimes S^{-1} \otimes \cdots \otimes S^{-1} \otimes \mathcal{F}(\ell_2 \rho_2 + \hat{\lambda}_2; \Lambda_2) \]
\[ \approx S^{\ell-2} \otimes \mathcal{F}(\ell_1 \rho_1 + \hat{\lambda}_1; \Lambda_1) \otimes \mathcal{F}(\ell_2 \rho_2 + \hat{\lambda}_2; \Lambda_2). \]

Take non-zero elements \( \sigma_1 \) and \( \tau_1 \) of \( \Lambda_1 \) satisfying \( \sigma_1 + \tau_1 = \rho_1 \). Similarly, take non-zero elements \( \sigma_2 \) and \( \tau_2 \) of \( \Lambda_2 \) satisfying \( \sigma_2 + \tau_2 = \rho_2 \). For \( q \in P^{<p} \) let \( \psi_p^q: \Lambda_q \rightarrow \Lambda_p \) be the homomorphism defined by
\[ \psi_p^q(\mu_1) = \mu_1 \quad (\mu_1 \in \Lambda_1) \]
\[ \psi_p^q(\alpha_i) = \begin{cases} \sigma_1 & \text{if } \min q < i \leq \ell_1 \\ \sigma_2 & \text{if } \ell_1 + \ell < i \leq \max q \\ \alpha_i & \text{otherwise} \end{cases} \]
\[ \psi_p^q(\mu_2) = \mu_2 \quad (\mu_2 \in \Lambda_2). \]

Then \( \psi_p^q \) is well-defined, proper and satisfies \( \psi_p^q \leq \varphi_p^q \). Moreover, \( \psi_p^r \leq \psi_p^q \circ \varphi_q^r \) holds for \( q, r \in P \) with \( r \leq q < p \). Define the map \( \psi: P^{<p} \times X \rightarrow X(p) \) by
\[ \varphi((q, \mu)) = \psi_p^q(\mu) \quad ((q, \mu) \in P^{<p} \times X). \]

Then \( \psi \) is a poset map, and satisfies \( \psi \leq \varphi \) and thus \( \|\psi\| \approx \|\varphi\| \). Therefore it suffices to show that the image of \( \|\psi\| \) is contractible in \( \|X(p)\| \). For \( q \in P^{<p} \) either \( \ell_1 + \ell < \max q \) holds. By the definition of \( \psi_p^q \), \( \min q < \ell_1 \) implies \( \psi_p^q(\lambda_q) \leq \lambda_p - \tau_1 \), and \( \ell_1 + \ell < \max q \) implies \( \psi_p^q(\lambda_q) \leq \lambda_p - \tau_2 \). If \( \ell_1 = 0 \), then \( \ell_1 + \ell < \max q \) holds for each \( q \in P^{<p} \), and thus the image of \( \|\psi\| \) is contained in \( \|0, \lambda_p - \tau_2\| \Lambda_p \). Similarly, in the case \( \ell_2 = 0 \) the image of \( \|\psi\| \) is contained in \( \|0, \lambda_p - \tau_1\| \Lambda_p \). Assume that both \( \ell_1 \) and \( \ell_2 \) are positive. Then we have
\[ \text{image}\|\psi\| = \|\text{image }\psi\| \]
\[ = \bigcup_{q \in P^{<p}} \psi_p^q((0, \lambda_q) \Lambda_q) \]
\[ \subseteq \|0, \lambda_p - \tau_1\| \Lambda_p \cup \|0, \lambda_p - \tau_2\| \Lambda_p \].

Moreover,
\[ \|0, \lambda_p - \tau_1\| \Lambda_p \cap \|0, \lambda_p - \tau_2\| \Lambda_p = \|0, \lambda_p - \tau_1 - \tau_2\| \Lambda_p \].

Since each of \( \|0, \lambda_p - \tau_1\| \Lambda_p \), \( \|0, \lambda_p - \tau_2\| \Lambda_p \) and their intersection is contractible, so is their union. Hence in any case the image of \( \|\psi\| \) is contained in a contractible set.

By Theorem 2.5 we have
\[ \|P \times X\| \simeq \bigvee_{p \in P} \|P_{>p}\| * \|g^{-1}(P^{<p})\| \]
Corollary 3.3. The Poincaré series of

\[ P_{\Lambda}(t, z) = \frac{P_{\Lambda_{1}}(t, z) \cdot P_{\Lambda_{2}}(t, z)}{1 - t^{2}z^{\rho}}. \]

Proof. By the previous theorem, we have

\[ \beta_i(\lambda; \Lambda) = \tilde{\beta}_{i-2}(\mathcal{F}(\lambda; \Lambda)) \]

\[ = \tilde{\beta}_{i-2} \left( \bigvee_{\ell_{p}+\lambda_{1}+\lambda_{2} = \lambda} S^{2\ell - 2} \otimes \mathcal{F}(\lambda_{1}; \Lambda_{1}) \otimes \mathcal{F}(\lambda_{2}; \Lambda_{2}) \right) \]

\[ = \sum_{\ell_{p}+\lambda_{1}+\lambda_{2} = \lambda} \tilde{\beta}_{i-2} \left( S^{2\ell - 2} \otimes \mathcal{F}(\lambda_{1}; \Lambda_{1}) \otimes \mathcal{F}(\lambda_{2}; \Lambda_{2}) \right) \]

\[ = \sum_{\ell_{p}+\lambda_{1}+\lambda_{2} = \lambda} \sum_{m+j+k=i} \tilde{\beta}_{m-2} \left( S^{2\ell - 2} \cdot \tilde{\beta}_{j-2} \left( \mathcal{F}(\lambda_{1}; \Lambda_{1}) \right) \right) \cdot \tilde{\beta}_{k-2} \left( \mathcal{F}(\lambda_{2}; \Lambda_{2}) \right). \]

Thus

\[ P_{\Lambda}(t, z) = \sum_{\ell \in \mathbb{N}} \sum_{\lambda \in \Lambda} \tilde{\beta}_{i} \left( \mathcal{F}(\lambda_{1}; \Lambda_{1}) \right) \cdot \tilde{\beta}_{j} \left( \mathcal{F}(\lambda_{2}; \Lambda_{2}) \right) t^{\ell}z^{\lambda}. \]

\[ = \sum_{\ell \in \mathbb{N}} \sum_{\lambda \in \Lambda} \tilde{\beta}_{i} \left( \mathcal{F}(\lambda_{1}; \Lambda_{1}) \right) \cdot \tilde{\beta}_{j} \left( \mathcal{F}(\lambda_{2}; \Lambda_{2}) \right) t^{\ell}z^{\lambda}. \]

\[ = \sum_{\ell \in \mathbb{N}} t^{2\ell}z^{\rho} \cdot \sum_{\lambda \in \Lambda} \sum_{\lambda_{1}+\lambda_{2} = \lambda} \tilde{\beta}_{j} \left( \mathcal{F}(\lambda_{1}; \Lambda_{1}) \right) \cdot \tilde{\beta}_{k} \left( \mathcal{F}(\lambda_{2}; \Lambda_{2}) \right) t^{2\ell+j}z^{\lambda_{1}+\lambda_{2}}. \]

\[ = \frac{P_{\Lambda_{1}}(t, z) \cdot P_{\Lambda_{2}}(t, z)}{1 - t^{2}z^{\rho}}. \]

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