Scaling Relations and Topological Quadruple Points in Light-Matter Interactions with Anisotropy and Nonlinear Stark Coupling

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Universality is a common quality in different physical parameters that is rooted in the deep nature of physical systems. Scaling relation is a typical universality for critical phenomena around a quantum phase transition, while topological classification provides another type of universality essentially different from the critical universality. Both classes of universalities can be present in a single-qubit system with light-matter interactions, as exhibiting generally in the fundamental quantum Rabi model with anisotropy not only for linear coupling but also for nonlinear Stark coupling (NSC). In low frequencies different levels of scaling relations are extracted, holding for anisotropic or/and NSCs, locally or globally. At finite frequencies such a critical universality breaks down and diversity is dominant. However, common topological feature of the ground state can be extracted from the node number, which yields a topological class of universality amidst the critical diversity. Both conventional and unconventional topological transitions emerge, with their meeting, which never occurs in linear interaction, enabled by the nonlinear coupling to form topological quadruple points which are found to be spin-invariant points. Sensitivity analysis indicates that the NSC can be another applicable approach to manipulate topological transitions in addition to coupling anisotropy.

1. Introduction

In the frontiers of modern quantum physics and quantum technologies, the past decade has seen the great wave evoked by the extraordinary experimental progresses[1–2] and tremendous theoretical efforts[3–5] on the studies of light-matter interactions. The remarkable realization of the ultra-strong[15,16] and even deep-or adaptations are made. Extraordinary experimental progresses[1,2] and tremendous technologies, the past decade has seen the great wave evoked by the quantum Rabi model (QRM)[5], which is a most fundamental model of light-interactions, has triggered an intensive dialogue between mathematics and physics[18] and leads to a boom of theoretical developments[4,5,19–63]. Without mentioning the ubiquitous role of light-matter interaction and its broad relevance to quantum optics, quantum information and quantum computation[1,4,64–67] quantum metrology[36–38,68] condensed matter[27,28] and relativistic systems[69] the explosively-growing investigations have yielded abundant findings in the QRM and its extensions, such as hidden symmetry[60–61] various patterns of symmetry breaking[26,27,29] few-body quantum phase transitions[5,22,29–70] multicriticalities and multiple points[26–28] universality classification[24,25,27,52] spectral collapse[33–35,45,48] photon blockade effect[39,40] spectral conical intersections[42] classical-quantum correspondence[21] single-qubit conventional and unconventional topological phase transitions[27,28] and so forth.

An intriguing phenomenon most relevant in coupling enhancement may be quantum phase transition (QPT)[5,22–27,70] Generally speaking, QPTs are transitions of ground states (GSs) induced by a variation of some non-thermal parameter[71]. In contrast to thermal fluctuations in classical phase transitions, QPTs are regarded to be driven by quantum fluctuations and traditionally lie in the thermodynamic limit in condensed matter. Interestingly, the QRM as a few-body system also exhibits a QPT[22,23,25] in the low-frequency limit, that is, $\omega/\Omega \to 0$ where $\omega$ is the bosonic frequency and $\Omega$ is the atomic level splitting or tunneling strength. It was also suggested that whether the transition should be termed quantum or not is a matter of taste by taking the negligible quantum fluctuations in the photon vacuum state into account[22]. Nevertheless, when critical universality is a character often born with QPTs as in the condensed matter, it has been shown that the anisotropic QRM manifests a universal scaling relation in the critical exponent that can be really bridged to the thermodynamic limit[24].

Opposite to universality is diversity which represents the quality to be diverse or different. With the opposite qualities universality and diversity are apparently antagonists. Unexpectedly, the universality scenarios in the anisotropic QRM demonstrate that they can turn to support each other. Indeed, the aforementioned
critical universality of scaling relation needs the condition of low frequency limit, while at finite frequencies the universal scaling relation breaks down and the system properties are dominated by diversity. However, amidst the diversity a new universality classification can be found from the topological structure of the GS wave function.\[^{[27]}\] In fact, such universality-diversity-universality scenarios involve two different kinds of universalities: one is critical universality, while the other is topological universality. Note that such scenarios occur in the anisotropic QRM which is linear in the light-matter interaction, one may wonder whether the universalities are simply a special case or hold more generally, for example, in a nonlinear coupling.

To get more robust universalities we consider the QRM with both anisotropy and the nonlinear Stark coupling in the present work. We consider both the low-frequency limit and the finite-frequency case. In the low-frequency limit we analytically obtain the phase boundaries of QPTs and extract different levels of scaling relations which are valid respectively in various anisotropic couplings or for both anisotropic and nonlinear Stark couplings, locally around transitions or globally for all critical regimes. At finite frequencies, indeed the critical universality collapses and diversity dominates, while topological phase transitions (TPTs) emerge. Both conventional and unconventional TPTs respectively with and without gap closing are present. Their different sensitivities in response to the nonlinear Stark coupling enable the forming of topological quadruple points, while it never occurs in linear interaction. A further analysis by composite phase diagrams with hexapole points reveals that the topological quadruple points are actually spin-invariant points.

The paper is organized as follows. Section 2 introduces the anisotropic QRM with nonlinear Stark coupling and addresses the symmetry in quadrature representation. In Section 3 methods are introduced to obtain analytic boundaries of QPTs in the low-frequency limit. Different levels of scaling relations are extracted. Section 4 shows the breakdown of the critical universality and arising of diversity at finite frequencies. Section 5 presents topological classifications at finite frequencies, with findings of topological quadruple points, composite hexapole points, and invariant points. Section 6 is devoted to mechanism clarifications. Conclusions and discussions are finally given in Section 7.

### 2. Model and Symmetry

The standard QRM has a linear and isotropic interaction, while in experimental setups extended versions of QRM are often applied. Indeed, coupling anisotropy plays an important role in ultrastrong couplings\[^{[6,31]}\] and is highly tunable.\[^{[32]}\] On the other hand, a so-called Stark nonlinear coupling can be added and realized with adjustable amplitude and sign.\[^{[30,70-73]}\] In fact, a faithful scheme has been proposed to realize a generalized QRM including simultaneously both the anisotropic linear coupling and the nonlinear Stark coupling (see Hamiltonian (13) in ref. [73]) with possible access to any regime of strong, ultrastrong, or deep-strong couplings. Note that in such an engineered system the nonlinear Stark coupling is also tunable on an equal footing with the linear coupling. Motivated by these practical progresses, we consider the QRM with both the anisotropy and the nonlinear Stark coupling as described by the following Hamiltonian

\[
H = \omega a^\dagger a + \frac{\Omega}{2} \sigma_x + \chi \omega \sigma_z + H_x
\]

(1)

\[
H_x = g\left(\tilde{\sigma}_y a^\dagger a + \lambda (\tilde{\sigma}_x a^\dagger + \tilde{\sigma}_x a)\right)
\]

(2)

Here \(\omega\) is the frequency of a bosonic mode created (annihilated) by \(a^\dagger\) (\(a\)), while \(\Omega\) is atomic level splitting in cavity systems or tunneling strength in superconducting circuit systems with the qubit (spin) represented by the Pauli matrix \(\sigma_z\). The linear coupling strength is controlled by \(g\). The anisotropy \(\lambda\) tunes the ratio of the rotating-wave terms and the counter-rotating terms, with \(\lambda = 1\) and \(\lambda = 0\) retrieving the QRM\[^{[17]}\] and the Jaynes-Cummings model (JCM)\[^{[26]}\] respectively. Note we have adopted the spin notation as in ref. [51], in which \(\sigma_z = \pm\) labels two flux states in flux-qubit circuit systems.\[^{[6,31]}\] In such a spin notation, the spin raising and lowering operators on \(\sigma_z\) basis are expressed by \(\tilde{\sigma}_y = (\sigma_x \pm i\sigma_y)/\sqrt{2}\), while one can recover the conventional form by a spin rotation \(\{\sigma_z,\sigma_x,\sigma_y\} \rightarrow \{\sigma_z, -\sigma_y, \sigma_x\}\) around the axis \(\vec{x} \pm \vec{z}\). The \(\chi\) term denotes the nonlinear Stark coupling with a limitation \(|\chi| \leq 1\) beyond which the system energy would be negatively unbound thus unphysical.

It should be noted that neither the anisotropy nor the nonlinear Stark coupling breaks the parity so that the model preserves the parity symmetry, with \(H\) commuting with the parity operator \(\tilde{P} = \sigma_z (-1)^{\vec{p}}\). The parity symmetry is relevant for symmetry-protected TPTs\[^{[27,28]}\] as in condensed matter\[^{[78-81]}\] while there is a hidden symmetry breaking of spin inversion or space inversion for the symmetry-breaking QPT in the GS.\[^{[27]}\]

Changing to the quadrature representation by \(a^\dagger = (\vec{x} - i\vec{y})/\sqrt{2}\), \(a = (\vec{x} + i\vec{y})/\sqrt{2}\) with momentum \(\vec{p} = -i\frac{\partial}{\partial x}\) will facilitate our analysis in the effective position space

\[
H = \frac{\omega}{2} \vec{p}^2 + v_\omega(x) + H_x \sigma^+ + H_x \sigma^-
\]

(3)

\[
H_x = \frac{(\Omega - \chi \omega)}{2} \frac{g_0}{\sqrt{2}} \vec{x} + \frac{\chi \omega}{2} \left(\vec{x}^2 + \vec{p}^2\right)
\]

(4)

It should be noted here that, differently from \(\tilde{\sigma}_z\), the spin raising and lowering in \(\sigma_x = \sigma^+ + \sigma^-\), \(\sigma_y = -i(\sigma_z - \sigma_x)\) are now on \(\sigma_z = \pm\) basis. We have defined \(g_0 = \sqrt{2}g_{\sigma^z}/\omega\) for \(g_0 = \frac{1}{\sqrt{2}}\) and \(g_0 = \frac{(\delta g_0)}{2\omega}\) thus \(g_0^2\) is effectively the amplitude of spin-dependent displacement in harmonic potentials \(v_\omega(x) = \omega(x + g_0^2\sigma_z^2)/2 + \epsilon_0^2\) where \(\epsilon_0 = -\frac{1}{\sqrt{2}}\) in \(\vec{p}^2\). In such a representation the \(\Omega\) term effectively plays the role of spin flipping in the spin \(\sigma_x\) space and the role of tunneling in the effective position space.\[^{[24,51]}\]

The \(g_0\) term takes the form \(\sqrt{2}g_0\sigma_z\) as the Rashba spin-orbit coupling in nanowires\[^{[80-85]}\] or the equal-weight mixture\[^{[86,87]}\] of the linear Dresselhaus \((\vec{p},\sigma_z + \vec{p},\sigma_z)\) and Rashba \((\vec{p},\sigma_z - \vec{p},\sigma_z)\) spin-orbit couplings in condensed matter\[^{[88,89]}\] and cold atomic gases.\[^{[86,87,90]}\]

The Hamiltonian can be rewritten in \(x-p\) dual forms

\[
H_x = \frac{\omega}{2} \left[\left(-i \frac{\partial}{\partial x} + g_0^2\sigma_z\right)^2 + (\vec{x} + g_0^2\sigma_z)^2\right]
\]

\[
+ \left(\frac{\Omega - \chi \omega}{2}\right) \left(\sigma_x^2 + \sigma_y^2\right) + \epsilon_0^2
\]

(5)
where \( \epsilon_x = -\alpha_0(1 + g_0^2 + g_0^2)/2 \) and \( z = i \frac{\dot{z}}{2} \). From \( H_x \) and \( H_y \) one sees that \( \lambda > 0 \) and \( \lambda < 0 \) regimes are symmetric under the spin rotation and transform to momentum space \( \{ \sigma_x, \sigma_y, \sigma_z \} \rightarrow \{ \sigma_y, -\sigma_z, \sigma_x \} \), \( x \rightarrow -p \), and \( \lambda \rightarrow -\lambda \).\(^{[27]} \) Phase transitions in the GS would involve a linear coupling of order as the critical point in the absence of the nonlinear Stark coupling \( g^2_n = \frac{2}{\lambda_n} g^{[24,27]}_x \) with \( g_x = \sqrt{\omega_p/2} \), in the low frequency limit \( \omega/\Omega \rightarrow 0 \) or that of a TPT, \( g^{[24]}_x = \frac{1}{\lambda_n} g^{[27]}_x \) at finite frequencies. At low frequencies, the contribution of \( \nu_y(x) \) and \( \xi^2 \) terms are of leading order\(^{[26]} \) \( \Omega \) while the \( g_y \) term and \( \xi^2 \) term are of subdominant orders \( (\omega \Omega)^{1/2} \) and \( \omega^3 \). Thus, the \( \lambda > 0 \) regime is \( x \)-type in the sense \( (\xi^2) \) is more dominant than \( (\xi^2) \), with the main characters more conveniently described by \( H_x \). At finite frequencies, the \( g_y \) term has some self-cancellation effect due to oscillation as seen later on, while larger amplitudes of \( g_y \) than \( g_0 \) still favor an \( x \)-type state in \( \lambda > 0 \) regime. Hereafter, unless specially mentioned, we shall focus on \( \lambda > 0 \) regime while one has similar results with a \( p \)-type state by \( H_y \) in the momentum space for \( \lambda < 0 \) regime.

### 3. QPTs and Scaling Relations in Low-Frequency Limit

We shall first study the low-frequency limit to extract GS phase diagrams and critical scaling relations. We figure out the full phase diagrams numerically by the exact diagonalization\(^{[26]} \) while to obtain analytic phase boundaries and find different scaling relations we need some analytic methods. For the latter purpose, we fall back on a semiclassical variational method for \( \lambda \neq 0 \) and the exact solution at \( \lambda = 0 \), as described in this section. We will get different levels of scaling relations and eliminate a singular behavior at \( \lambda = 0 \). The obtained analytic phase boundaries will also provide a reference to fix the invariant points at finite frequencies in next section.

#### 3.1. Explicit Solution and Energy at \( \lambda = 0 \)

##### 3.1.1. General Solution at Any Frequencies

The explicit exact solution is available for the JCM at \( \lambda = 0 \) in linear coupling\(^{[20,31]} \) while here we shall address in the presence of the nonlinear Stark coupling. Setting \( \lambda = 0 \) drops the counter-rotating terms in the Hamiltonian (1) so that the eigenstates only involve at most two bases in the following form

\[
\psi^{(s)}_n = \left( C^{(s)}_{n\sigma} |n - 1, \uparrow \rangle \sigma_x + C^{(s)}_{n\uparrow} |n, \uparrow \rangle \sigma_x \right) / \sqrt{N},
\]

\[
\psi_0 = |0, \uparrow \rangle \sigma_x
\]

where \( n = 1, 2, \ldots \) and \( \uparrow, \downarrow \) are spin states of \( \sigma_x \) as labeled by the subscript of the basis. The coefficients on the basis are explicitly given by

\[
C^{(s)}_{n\sigma} = e^{-\frac{\epsilon_x}{2}} \sqrt{\frac{\epsilon_x + n \sigma^2}{2}},
\]

\[
C^{(s)}_{n\uparrow} = g \sqrt{n}
\]

where \( \epsilon_x = (n - 1/2)\omega \) and \( e_\uparrow = \frac{1}{2}(\Omega/\omega) + (n - 1/2)\chi \omega \) and \( \sqrt{N} = C^{(s)}_{n\uparrow} + C^{(s)}_{n\downarrow} \) is the normalization factor. Corresponding to the above states one can get the eigenenergies

\[
E^{(s)}_{S\rightarrow X, \uparrow} = e_\uparrow \pm \sqrt{e_\uparrow^2 + n \sigma^2}
\]

\[
E^{(s)}_{S\rightarrow X, \downarrow} = -\frac{\Omega}{2}
\]

respectively. Note the energy in branch \( E^{(s)}_{S\rightarrow X, \uparrow} \) is higher than \( E^{(s)}_{S\rightarrow X, \downarrow} \) thus the GS lies in the competition among the corresponding states in branch \( \psi^{(s)}_n \) as well as \( \psi_0 \), as shown in Figure 1a where the orange thin lines are \( E^{(s)}_{S\rightarrow X, \uparrow} \) (plotted every five levels), with \( E^{(s)}_{S\rightarrow X, \downarrow} \) being the horizontal one, while the final GS is indicated by the blue line.

#### 3.1.2. Photon Number in Low-Frequency Limit

In the low-frequency limit \( \omega/\Omega \rightarrow 0 \), the spacing of the quantum number \( n \) becomes small relatively to the characteristic number \( n_0 = \chi^2/2 = \Omega/(4\alpha) \), where \( \chi = \sqrt{2g_0/\omega} = \sqrt{\Omega/(2\omega)} \) which is the order of photon number induced by a coherent state in the displaced potential. In such a situation we can approximately regard the energy \( E^{(s)}_{S\rightarrow X, \uparrow} \) as a continuous function of \( n \). As illustrated in Figure 1b, the minimization of \( E^{(s)}_{S\rightarrow X, \uparrow} \) with respect to \( n \) gives an optimal quantum number

\[
n_{\text{min}} = \frac{1 + \chi}{2\chi} - \frac{(g_x^2 + 4\chi)^2}{8\chi^2\omega} + \frac{g_y^2}{8\chi^2} \sqrt{\frac{g_y^2 + 8\chi^2}{1 - \chi^2}}
\]

with \( \chi = \chi(1 - 1 + \chi)^{n_0}/\Omega^2 \). The integer number \( n_0 \) nearest to \( n_{\text{min}} \) should be the discrete quantum number of the GS which can obtained by level crossing \( E^{(s)}_{S\rightarrow X, \uparrow} = E^{(s)}_{S\rightarrow X, \downarrow} \) (see expression of \( n_0 \) in Appendix A).

In the low-frequency limit, both \( n_{\text{min}} \) and \( n_j \) approach to

\[
n_{\text{min}} = \frac{n_j}{n_j} - \frac{g_x^2 + 2\chi}{2\chi^2} + \frac{g_y^2}{2\chi^2} \sqrt{\frac{g_y^2 + 8\chi^2}{1 - \chi^2}}
\]

Figure 1c compares \( n_{\text{min}} \) (orange), \( n_j \) (blue) with the expectation of photon number \( \langle \hat{n} \rangle = (\langle n - 1 \rangle C^{(s)}_{n\sigma} + nC^{(s)}_{n\uparrow})/N \) at a frequency \( \omega = 0.01 \Omega \), they all coincide. The result (Equation (14)) will be later on used for discussion on the singular point in scaling relation.

#### 3.2. Semiclassical Variational Method for \( \lambda > 0 \)

Now let us discuss the QPTs in \( \lambda > 0 \) regime, while one gets the same results for \( \lambda < 0 \) regime in the momentum space. Note the
We see that at infinity in the low-frequency limit with nonlinear Stark coupling. Here $\omega = 0.01 \Omega$, $\chi = 0.4$ for (a,b) and $\lambda = 0.5$ for (d–f). a) Analytic energy spectrum versus $g$ for excited states $E^{n=1}$ (orange thin lines) and ground state (GS) $E_{GS}$ (blue thick line). b) $E^{n=1}$ as a continuous (blue solid) or discrete (green dots) function of $n$ at $g = 3g_c$, with optimized number $n_{\text{min}}$ marked by the vertical dashed line. c) Discrete quantum number $n_j$ in $E^{n=1}$ (blue), optimized number $n_{\text{min}}$ (orange) and the expectation of photon number $\langle \hat{n} \rangle$ (black). d) Spin configuration before transition at $\chi = 0.54$, the arrows mark the positions in the effective potential for spin-up (blue, $v_+$) spin-down (orange, $v_-$) components. e) Spin configuration after transition in $\nu_x$ at $\chi = 0.74$ with variational displacements normalized by $\zeta$ from the potential-bottom positions $\pm g'_z$ (dashed). f) Variational energy $\varepsilon$ versus the displacement $x$ at $g = 0.8g_c$, and $\lambda = 0.5$, for $\chi = 0.54$ (blue, upper), 0.64 (dashed, middle), 0.74 (orange, lower) and the critical point is $\chi_c = 0.64$.

Critical coupling is of order $g_c$ which yields a potential energy $\nu_{\nu_x}$ of order $\Omega$. In the low frequency limit, the kinetic energy is of order $\omega$, thus being relatively negligible. The Rashba spin-orbit coupling term has a strength $g_y$, which is of order $\omega^{1/2}$ at critical couplings, also being negligible. Thus, in the leading order, we can reduce the model to a semiclassical Hamiltonian for the GS which has a zero momentum as the GS of a classical particle while the quantum part is kept in spin space:\[26]

$$H_x \rightarrow H_{SC}^x = \frac{\omega}{2} \left( x + g'_x \sigma_z \right)^2 + \varepsilon_{SC}^x + \left( \frac{\Omega}{2} + \frac{\chi \omega}{2} x^2 \right) \sigma_z$$  \hspace{1cm} (15)

where $\varepsilon_{SC}^x = -\frac{1}{2} g'_x \omega$. For the semiclassical approximation we have dropped the zero-point energy $\omega \nu_x$ in $\nu = \frac{\hbar}{2}(x^2 + p^2) - \frac{\omega}{2}$ and the Stark term. From an alternative angle $\frac{\nu}{\nu_x}$ also can be dropped due to negligible order in the low-frequency limit. One gets similar reduced Hamiltonian $H_{SC}^x$ of $H_x$ for $\lambda < 0$ regime with the position $x$ replaced by the momentum $p$ and $g'_x$ changed to be $g'_z$.

We see that at infinity $x \rightarrow \infty$ the energy would be dominated by

$$H_{SC} \rightarrow \frac{\omega}{2} \left( 1 + \chi \sigma_z \right) x^2 + \varepsilon_{SC}^x$$  \hspace{1cm} (16)

which is negatively unbound for $|x| > 1$, thus this regime is unstable and unphysical as mentioned in Section 2. Hereafter we shall focus on the physical regime $|x| \leq 1$.

To obtain the explicit energy one can rewrite $H_{SC}$ in a matrix form

$$H_{SC} = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{11} \end{pmatrix}$$

where

$$h_{11} = \frac{\omega}{2} \left( x + g'_z \right)^2 + \varepsilon_{SC}^z$$

$$h_{12} = \frac{\omega}{2} \left( x - g'_z \right)^2 + \varepsilon_{SC}^{z'}$$

$$h_{11} = \frac{\Omega}{2} + \frac{\chi \omega}{2} x^2$$

Diagonalization of $H_{SC}$ gives two energies with the lower one being

$$\varepsilon = \frac{\omega}{2} \left( x^2 + g'_z^2 \right) - \frac{\omega}{2} \sqrt{4g''_z x^2 + \left( x^2 + \left( \frac{\Omega}{\omega} \right)^2 \right)^2 + \varepsilon_{SC}^{z''}}$$  \hspace{1cm} (18)

which is still variational as the spatial position $x$ has not yet been optimized. Minimization with respect to $x$

$$\frac{\partial \varepsilon}{\partial x} = 0$$  \hspace{1cm} (20)

leads to two solutions for the most favorable position

$$x_{m}^n = 0$$  \hspace{1cm} (21)

$$|x_m^n| = \sqrt{\frac{2g''_z + \chi \frac{\Omega}{\omega}}{\chi} \left( 1 - x^2 \right)} - \left( \frac{2g''_z + \chi \frac{\Omega}{\omega}}{\chi} \right) x$$  \hspace{1cm} (22)
Actually at a small coupling strength $|\lambda| \ll 1$, the GS phase diagrams in the low-frequency limit. a,c,e) Density plots in the $g$-$\lambda$ plane and the $g$-$\chi$ plane for $(\hat{x}^2) - (\hat{p}^2)$ and b,d,f) the spin expectation $\langle \sigma_z \rangle$ with a fixed $\chi = 0.4$ (a,b), $\chi = -0.5$ (c,d), and $\lambda = 0.5, 0.75, 1.0$ (e,f). Here we define $\Omega = 0.01 \Omega$. To visualize the numerical boundaries better, $(\hat{x}^2) - (\hat{p}^2)$ is scaled by $x^2_{\chi \lambda}$, where $x^2_{\chi \lambda} = g^c_\chi (\mu_\chi)$ for $\lambda > 0 (\lambda < 0)$, and timed with sign$(1 - (\hat{x}^2) - (\hat{p}^2))/x^2_{\chi \lambda}$, while in (d) the amplitude is multiplied by $|\langle \sigma_z \rangle|^{1/2}$. Apart from the JC line (horizontal long-dashed), the dotted, dashed, and dot-dashed curves are analytic boundaries $g^{\lambda,\chi}_c$ (Equation (25)), $g^{\lambda}_c$ (Equation (35)), and $g^{\lambda,\chi}_g$ (Equations (32) and (33)), respectively.

After substituting $|x_m|$ into Equation (19) we arrive at the explicit final energies

$$E^{\lambda}_c = -\frac{\Omega}{2}$$

$$E^{\lambda}_c = E^{\lambda}_c - \frac{g^c_\chi (2 - \chi^2) \omega + \chi \Omega}{2 \sqrt{\chi^2}}$$

$$g^{\lambda}_c = \frac{g^c_\chi (1 - \chi^2) \omega + \chi \Omega}{2 \sqrt{\chi^2}}$$

$$g^{\lambda}_c = \frac{g^c_\chi (1 - \chi^2) \omega + \chi \Omega}{2 \sqrt{\chi^2}}$$

which actually are the energies before and after the phase transition, respectively, as discussed in the following.

3.3. Phase Diagrams and Critical Scalings in Low-Frequency Limit

3.3.1. Phase Transition and Critical Boundary

Actually at a small coupling strength $|x_m|$ is imaginary so the only physical solution is $x^0_m$. Indeed, in such a situation $\varepsilon$ has a single minimum which is located at the origin, as illustrated by the blue (upper) line in Figure 1f. The variational energy remains in the mono-minimum profile till a second-order phase transition is triggered at a critical point

$$g^{\lambda,\chi}_c = \frac{2 \sqrt{(1 - \chi) \varepsilon}}{|1 + |\lambda||} g^c_\chi = \varepsilon (1 - \chi) g^c_\chi$$

$$|\lambda^{\chi,\chi}_c| = \frac{2 \sqrt{(1 - \chi) \varepsilon}}{g^c_\chi}$$

$$|\chi^{\lambda,\chi}_c| = \frac{1 - (1 + |\lambda|)^2 g^2_\chi}{4 g^c_\chi}$$

after which a double-minimum structure shows up. The black dotted line in Figure 1f shows the case right at the critical point, with a flat bottom at the origin. After the critical point, as demonstrated by the orange (lower) line, two degenerate minima appear at $\pm |x^0_m|$ while $x^0_m$ becomes an unstable local maximum at the origin. Note the expressions for the critical point are general for both $\chi$ and $\lambda$, setting $\varepsilon = 0$ recovers the boundary for the anisotropic QRM [24,27]

$$g^{\lambda}_c = \frac{2}{1 + |\lambda|} g^c_\chi$$

in the absence of the Stark coupling. Setting $\lambda = 0$, the general critical boundary $g^{\chi,\chi}_c$ reduces to the Stark-JC critical point

$$g^{\chi,\chi,\chi,\chi}_c = 2 g^c_\chi \sqrt{(1 - \chi)}$$

which is also exactly obtained by the level crossing

$$E^{\chi,\chi,\chi,\chi}_c = E^{\chi,\chi,\chi,\chi}_c$$

In Figure 2 we show the phase diagrams of the expectation $\langle \hat{x}^2 \rangle - (\hat{p}^2)$ (a,c,e) and the spin expectation $\langle \sigma_z \rangle$ (b,d,f) at a low frequency $\omega = 0.01 \Omega$. The dotted curves denote the analytic critical boundary (Equations (25)–(27)) which agrees well with the second-order-like transition in numerics. Figure 2a,e indicates the case at $\lambda > 0$ regime, with a first-order boundary (dotted) $(\hat{p}^2)/(\hat{x}^2)$ is vanishing relatively in $\lambda > 0$ regime. The ratio of $(\hat{x}^2)$ and $(\hat{p}^2)$ is reversed in $\lambda < 0$ regime, with a first-order boundary (long-dashed) at $\lambda = 0$. This scenario forms a tricritical point which is moving toward smaller-coupling direction for a positive $\chi$ and toward larger-coupling direction for a negative $\chi$ as compared with the case at $\chi = 0$ marked by the dot in Figure 2a,e.

Here we see that the phase transitions are induced by variations of the non-thermal parameters $g$, $\lambda$, or $\chi$ in the ground state as conventional QPTs at zero temperature. Usually QPTs occur in thermodynamic systems, while phase transitions in few-body
systems as here are also viewed as QPTs. Conventionally QPTs are regarded to be driven by quantum fluctuations. Although few-body systems of light-matter interaction have quantum fluctuations of single physical quantities as the position and the momentum in the amplitude-squeezing/phase-squeezing transitions, the quantum fluctuations concerning QPTs refer to the fluctuation of different parts of Hamiltonian. It was suggested that whether the transition should be termed quantum or not is a matter of taste by finding negligible quantum fluctuations in the photon vacuum before the transition. Nevertheless, the system exhibits scaling relations around the transitions as will be discussed later on, which resembles the conventional QPTs. Such an issue about the concept of QPT with the controversial views in light-matter systems deserves further special discussions, which we would address in some other works. Here we simply adopt the term of QPTs.

A practical implication of our phase diagrams may lie in the few-body critical quantum metrology. Few-body critical quantum sensors benefit from high controllability and do not suffer from the difficulty to reach equilibrium in thermodynamic systems. It has been shown that the QPT in the QRM can provide a high-precision resource for critical quantum metrology. Note that the QRM has only one critical point at $g = g_s$ which would much limit the application regime. Here our phase diagrams indicate that the both highly tunable anisotropy and Stark coupling yield a continuous critical boundary, ranging from $g = 0$ to $g = 2\sqrt{2}g_s$ as indicated by Equation (25), which can provide a global critical resource for the few-body quantum metrology.

### 3.3.2. Adiabatic Boundary

In the absence of the nonlinear Stark coupling, $|\chi_{m}\rangle$ never goes beyond bottom of the bare potential $v_{s}$, that is, the displacement renormalization ratio $\xi = x_{m}/x_{s}$ as indicated in Figure 1e, where $x_{m} = g_{s}$ for $\lambda > 0$ and $x_{m} = g_{s}$ for $\lambda < 0$, is always smaller than 1. Now in the presence of the Stark coupling, it may be more favorable to go farther away from the origin, and finally beyond the potential bottom at a boundary

$$|\chi_{c}^{+}| = -1 + 2\sqrt{\frac{1 - \chi^{2}}{1 + \chi^{2}}} \frac{g_{s}}{g}$$

$$|\chi_{c}^{-}| = -1 + 2\sqrt{\frac{1 - \chi^{2}}{1 + \chi^{2}}} \frac{g_{s}}{g}$$

$$g_{c}^{+} = \frac{2g_{s}}{1 + |\lambda|} \sqrt{\frac{1 - \chi^{2}}{1 + \chi^{2}}}$$

$$g_{c}^{-} = \frac{2g_{s}}{1 + |\lambda|} \sqrt{\frac{1 - \chi^{2}}{1 + \chi^{2}}}$$

for $\chi > 0$ and $\chi < 0$, respectively. We show this boundary by the dot-dashed lines in Figure 2a,c,e, as compared with the numeric boundary of $\text{sign}[1 - \langle \hat{x}^{2} \rangle - \langle \hat{p}^{2} \rangle] / \langle x_{m}^{2} \rangle$ (note here $\langle \hat{p}^{2} \rangle$ is negligible in $\lambda > 0$ regime as later on proven in Section 3.5). Along this boundary the effective particle keeps staying at the potential bottom position, which is adiabatic in the sense that the particle is always following the potential.

### 3.3.3. Vanishing-$\langle \sigma_{s} \rangle$ Boundary and Coincidence with Adiabatic Boundary

As mentioned in Section 2, $\langle \sigma_{s} \rangle$ reflects flipping in the spin space and tunneling in the spatial space. Figure 2b,d shows $\langle \sigma_{s} \rangle$ which, unlike $\langle \hat{x}^{2} \rangle - \langle \hat{p}^{2} \rangle$, is symmetric with respect to $\lambda$. From the phase diagrams of $\langle \sigma_{s} \rangle$ we find another phase boundary in $\lambda < 0$ regime, as plotted by the dashed lines in Figure 2d,f, which separates the positive and negative regimes of $\langle \sigma_{s} \rangle$ at

$$|\lambda_{c}^{\sigma_{s}}| = -1 + 2\sqrt{\frac{2}{|\lambda|}} \frac{g_{s}}{g}$$

$$g_{c}^{\sigma_{s}} = \frac{2g_{s}}{|\lambda| + 1} \sqrt{2} |\lambda|$$

as extracted by $\langle \sigma_{s} \rangle = 0$ (see analytic expression of $\langle \sigma_{s} \rangle$ in Equation (39)).

Comparing Figure 2c,f, one may notice the vanishing-$\langle \sigma_{s} \rangle$ boundary coincides with the second adiabatic boundary, $g_{c}^{\chi} = g_{c}^{\chi_{c}}$, as also confirmed by Equations (35) and (33). In fact, under a negative $\chi$ the Stark-coupling energy is counteracting with the tunneling energy, as one can see from the $\Omega$ term and the $x^1$ term in Equation (15). The vanishing-$\langle \sigma_{s} \rangle$ boundary marked by different parameters in Equations (34)-(36) is the point where the Stark coupling energy $E_{21} = \omega(\sigma_{s})$ and the tunneling energy $E_{T} = x\omega(\sigma_{s})$ are canceling. Indeed at this point, not only the expectation of spin flipping $\sigma_{s}$ itself is vanishing, but also the effective coefficients of $\sigma_{s}$ cancel:

$$\langle \sigma_{s} \rangle = 0, \quad \Omega + \frac{2}{\chi^{\omega}} x^{2} = 0$$

at $x = x_{m}^{\lambda}$ and $g = g_{c}^{\chi_{c}}$. Consequently the Stark coupling and tunneling term does not come to effect here and only the bare potential $v_{s}$ play the role.

Besides realizing that the sign reversal of $\langle \sigma_{s} \rangle$ only occurs in the negative-$\chi$ regime, as indicated by the square root $\sqrt{-2/\chi}$ in (Equation (34)), we also see that the two boundaries $g_{c}^{\chi_{c}}$ and $g_{c}^{\chi_{c}}$ for $\chi = -1$, as demonstrated by the Figure 2f.

### 3.4. Scaling Relations at Fixed Stark Couplings

There exist some scaling relations for the critical behavior. The scaling relation forms a universality of critical properties in the linear anisotropic QRM, while some more general universality can be found in the presence of the nonlinear Stark coupling. We first consider the case at fixed nonlinear Stark couplings. In
the low-frequency limit, after the transition at $g_c^{(λ)}$, we see the effective spatial position in $n$-order

$$\left(\hat{g}_n^{λ}\right)^{2/n^2} x_n^{λ} = \left(-\frac{\tilde{g}_n^{λ} + \chi}{X'} \pm \sqrt{\frac{\tilde{g}_n^{λ} + 2X'}{1 - \chi'}}\right)^{1/2}$$  \tag{38}$$

for the positive-$λ$ region and similarly for the negative-$λ$ region, while it is vanishing before the transition. Here we have defined $x_n^{λ} = \sqrt{2g_c^{λ}/ω}$ and $\tilde{g}_n^{λ} = g_c^{λ} = \sqrt{1 - \tilde{g}_n^{λ'}}$, where $\tilde{g}_n^{λ'} = g_c^{λ'}/\sqrt{g_c^{λ'}/g_c^{λ}}$. So, the scaled expression in (Equation (38)) is a function of $\tilde{g}_n^{λ}$ or $\tilde{g}_n^{λ'}$, thus being universal for all values of $λ$.

**Figure 3a** illustrates the numerical expectation $\langle \hat{g}_n^{λ} \rangle$ for different anisotropies at a given Stark coupling $λ = 0.4$ for a low frequency $ω = 0.01$. One sees that with the scaling $g_c^{λ}/\sqrt{2g_c^{λ}/ω}$ for the coupling strength and $(\hat{g}_n^{λ})^2/x_n^{λ}$ for the quadratic position all data of different anisotropies collapse into a single line, except the JC case $λ = 0$. The analytic scaling relation (Equation (38)) is plotted by the black dashed line which coincides with the numerics.

We also find the critical scaling relation for the spin expectation

$$\langle σ_n^{λ}⟩ = \frac{1}{\chi'} \left(-1 + \tilde{g}_n^{λ} \sqrt{\frac{1 - \chi'}{2X' + \tilde{g}_n^{λ'}}}\right)$$  \tag{39}$$

which is also independent of $λ$ and agrees well with the numerical scaling data, as shown in Figure 3b where all anisotropy cases collapse into a single line, including $λ = 0$. We see that the scaling of $\langle σ_n^{λ}⟩$ with respect to the anisotropy is more universal than $(\hat{g}_n^{λ})^2$, $\langle \hat{p}_n^{λ}⟩$ in the sense there is no discontinuity at the point $λ = 0$ in $\langle σ_n^{λ}⟩$. The difference comes from the fact that $(\hat{g}_n^{λ})^2$ and $\langle \hat{p}_n^{λ}⟩$ suffer from a spontaneous symmetry breaking of the duality exchange\cite{27} while $\langle σ_n^{λ}⟩$ remains unaffected in the duality exchange.

3.5. Symmetry Breaking and Singularity in the $⟨\hat{g}^2⟩$-Scaling Relation Around $λ = 0$

As seen from $H_{sc}$ and $H_p$ in Section 2 the $λ = 0$ case has an $x$--$p$ duality symmetry which is however broken once a non-zero value of $λ$ is introduced. It is this symmetry breaking that leads to the singular behavior of $λ = 0$ in the scaling relation of $(\hat{g}_n^{λ})^2$ aforementioned in Section 3.4. Here we shall clarify the mechanism more explicitly.

For the $λ = 0$ case, from the exact wavefunction (Equation (7)) we see the expectations are indeed equal in accordance with the $x$--$p$ duality symmetry, $\langle \hat{g}_n^{λ} \rangle = \langle \hat{p}_n^{λ} \rangle = \frac{(n - \frac{1}{2})C_{n+\frac{1}{2}} + (n + \frac{1}{2})C_{n-\frac{1}{2}}}{N}$ with $n$ substituted by $n_{min}$ in Equation (13), which approach to

$$\langle \hat{g}_n^{λ} \rangle_{low} \approx \langle \hat{p}_n^{λ} \rangle_{low} = \frac{g_c^{λ}}{\sqrt{2g_c^{λ}/ω + \tilde{g}_n^{λ'}}} + \sqrt{\frac{1 - \chi'}{2X'}} \sqrt{\frac{1 + \chi'}{2X'}}$$  \tag{40}$$

in the low-frequency limit. Figure 3a shows the agreements of analytic result (blue dotted line) of Equation (40) and the numerics (red dots).

Once away from the $λ = 0$ line, the symmetry breaking leads to an imbalance of $(\hat{g}_n^{λ})^2$ and $\langle \hat{p}_n^{λ}⟩$, as we can see from a comparison of the energy $H_{sc}^{λ}$ and $H_p^{λ}$ in Equation (15). In fact, the minimized energy of $H_{sc}^{λ}$ and $H_p^{λ}$ as in Equation (24) can be unified to be a same function of $g_c^{λ'}$, $E_{sc}^{λ}(g_c^{λ'})$ with $g_c^{λ'} = g_c^{λ}$ for $H_{sc}^{λ}$ and $g_c^{λ'} = g_c^{λ}$ for $H_p^{λ}$, which is a decreasing function (see Appendix B). Note in the positive-$λ$ regime $g_c^{λ} = \frac{(n+\frac{1}{2})}{\frac{1}{2}g}$ has a larger value than $g_c^{λ} = \frac{(n-\frac{1}{2})}{\frac{1}{2}g}$, consequently $H_{sc}^{λ}$ provides a lower energy than $H_p^{λ}$ due to the decreasing function $E_{sc}^{λ}(g_c^{λ'})$. As a result, the GS from $H_{sc}^{λ}$ has a vanishing $\langle \hat{p}_n^{λ} \rangle$ but a finite $(\hat{g}_n^{λ})^2$ in Equation (38) that is twice of $\langle \hat{g}_n^{λ} \rangle_{low}$ in Equation (40), as in the contrast displayed in Figure 3a. Reversely in the negative-$λ$ regime, $(\hat{g}_n^{λ})^2$ is vanishing and $\langle \hat{p}_n^{λ}⟩$ is finite as $g_c^{λ}$ is larger than $g_c^{λ}$. From the above discussion we see that the singular behavior in the $(\hat{g}_n^{λ})^2$-scaling relation is a consequence of the difference from
half weights of $\langle \chi^2 \rangle$ and $\langle \hat{p}^2 \rangle$ for $\lambda = 0$, full weight of $\langle \hat{p}^2 \rangle$ for $\lambda > 0$, and full weight of $\langle \hat{p}^2 \rangle$ for $\lambda < 0$, in the $x - \rho$ duality symmetry breaking. We can get rid of the singular behavior and get a more unified scaling relation

$$\frac{\langle \hat{p}^2 \rangle}{x^2} = 2 \left( \frac{g_2 + X}{x^2} + \frac{g_3}{x^2} \right)$$

which holds for any anisotropy including $\lambda = 0$, as shown in Figure 4a.

3.6. Local Scaling Relations for Various Stark Couplings around Transition

In the last two sections we have extended the scaling relation with respect to anisotropy from the absence to the presence of non-linear Stark coupling. A more general scaling relation would be universal not only for all anisotropies but also for various Stark couplings. The $\lambda$-universal scaling relations, Equations (38) and (39), are however not unified for different values of $\chi$, as shown in Figure 3c,d. A general scaling relation universal for both $\lambda$ and $\chi$ holding globally for any strength of coupling is not readily available. Nevertheless, since critical exponent depends on the behavior in the vicinity of the transition, we can extract some scaling relations around the transition by expansion

$$\frac{(1 - \lambda) \langle \hat{p}^2 \rangle}{2x^2} = 2 \frac{d \hat{p}^2_{x,\lambda,\omega}}{\hat{p}^2_{x,\lambda,\omega}} - d \hat{p}^2_{x,\lambda,\omega} + O(d \hat{p}^2_{x,\lambda,\omega})$$

$$\langle \sigma \rangle = -1 + 2 \frac{d \hat{p}^2_{x,\lambda,\omega}}{\hat{p}^2_{x,\lambda,\omega}} - 3 \frac{d \hat{p}^2_{x,\lambda,\omega}}{\hat{p}^2_{x,\lambda,\omega}} + O(d \hat{p}^2_{x,\lambda,\omega})$$

where $d \hat{p}^2_{x,\lambda,\omega} = 1 - 2 \frac{X}{1 + X} - 1$, which are pure functions of $d \hat{p}^2_{x,\lambda,\omega}$ independent of $\lambda$, $\chi$, $\omega$ in the first two orders. We present a comparison of the analytic scaling in Equations (42) and (43) with the numerical data in Figure 3e,f. The comparison shows that the scaling relations basically hold for different values of $\chi$ indeed, except near the unphysical limit $\chi = 1$ due to the singular third-order term which takes the form of $2d \hat{p}^2_{x,\lambda,\omega}(2 - 3 \chi)/(1 - \chi)$.

3.7. Global Scaling Relation for Various Stark Couplings After Transition

Still, a more robust scaling relation can be obtained for $\langle \sigma \rangle$, universal for $\lambda$, $\chi$ and low frequencies without limitation of $\chi$ or the critical regime around the transition. In fact we find that the following scaling relation after the transition

$$\frac{\langle \sigma \rangle - \langle \sigma \rangle_0}{X^{-2} - 1} = \frac{1}{2X\hat{p}^2_{x,\lambda} + 1} = \frac{1}{2(\chi^{-1} - 1)\hat{p}^2_{x,\lambda} + 1}$$

where $\langle \sigma \rangle_0 = -1/\chi$ is a shift. Figure 3g shows the scaling relation in Equation (44) as a function of $\chi^{-1}/\hat{p}^2_{x,\lambda}$ or $\chi^{-1} - 1/\hat{p}^2_{x,\lambda}$. Here in the figure the horizontal symbols are the numeric data before the phase transition, while in the critical regime after the phase transition all data in different values of $\chi$ collapse into a same line which coincides with the analytic scaling (Equation (44)). Note here the values of $\chi$ are positive, while negative $\chi$ also has a similar scaling behavior but in a different branch. Nevertheless both positive and negative $\chi$ can be finally unified in a scaling as a function of $\chi^{-1}/\hat{p}^2_{x,\lambda}$ or $(\chi^{-1} - 1)\hat{p}^2_{x,\lambda}$, as in Figure 3h. Note that the variation in $\lambda$ has been scaled without any limitation as shown both numerically and analytically in Section 3.4, thus the scaling (Equation (44)) is universally valid for both $\lambda$ and $\chi$.

Usually the order parameter of a phase, such as $\langle \hat{p}^2 \rangle$ in Equation (42), starts with a zero value at a continuous phase transition, while $\langle \sigma \rangle$ is different as it already has a finite value $\langle \sigma \rangle_0 = -1$ before the transition. Thus one needs to subtract or shift this finite value to get a local critical exponent around the transition as in Equation (43). However, a naive direct shift does not reach a more demanding global scaling. Instead, we see here the adapted shift $\langle \sigma \rangle_0$ enables us to see the global scaling. This also indicates that transitions under different system parameters may enter a universal scaling relation at different points as in Figure 3g,h, not necessarily limited to a same entering point as in Figure 3c,d. Note that critical universality around a transition conventionally only requires the same critical exponent while the proportionality constant is not concerned.[92] For scaling relations one has to go a step further to also unify the proportionality constants by a same form of analytic coefficient (which is not necessarily always achievable, for example, $\langle \hat{p}^2 \rangle$ has a singular proportionality constant at $\lambda = 0$ as discussed in Section 3.5).
while a global scaling relation would require even more stringent conditions. Our finding of the global scaling (Equation (44)) by the adapted shift provides an unconventional way of thought to seek novel classes of scaling relations.

These scaling relations addressed in previous sections indicate that the properties in the critical regime obey a universal law, despite that the parameters are different in the anisotropy, the Stark coupling ratio, and the frequencies (in low frequencies).

4. Breaking Down of Critical Universality and Diversity-Dominating Situation at Finite Frequencies

As addressed in Sections 3.4–3.6, all levels of the scaling relations are valid under the condition of low-frequency limit. At finite frequencies the critical universality will break down and different scenarios arise. We also apply the exact diagonalization\[^{26}\] at finite frequencies as the numerics in low-frequency case. Indeed, as illustrated in Figure 4b at a finite frequency \( \omega = 0.5 \, \Omega \) for a fixed Stark coupling \( X = 0.2 \), the expectations \( \langle \hat{\chi}^2 \rangle + \langle \hat{p}^2 \rangle \) in different ratios of anisotropy are not collapsing into a single line any longer. We see that \( \langle \hat{\chi}^2 \rangle + \langle \hat{p}^2 \rangle \) in different \( \lambda \) not only increases in various gradients but also fragment into disconnected sections. As the section breakings actually emerging are a series of first-order phase transitions. Note here that fixing the Stark coupling as in Figure 3a,b is the lowest level of scaling. Now even the lowest level of scaling relation has broken down, not to mention the collapse of the higher levels of scaling relations under both various anisotropies and different nonlinear Stark couplings in Figure 3c–h. Thus we see the critical universality collapses and the system properties are dominant by diversity which, opposite to universality, is actually diversity that makes a rich and colorful world. How course, unlike universality which attracts more interest, diversity may be not a fruitful physical concept if mentioned alone despite it is actually diversity that makes a rich and colorful world. However, the diversity here describes a situation that the critical universality collapses and as the antagonist to universality the dominating diversity seems to mean that there would be no universality any more. It is just such a spurious feeling of no universality that makes it a surprise that new universality can be found amidst the diversity as we shall discuss in the coming section. In fact, the new universality is not the Landau class but topological class which is essentially different. Contrary to the critical universality which needs to neglect the \( g_\chi \hat{p}^2 \) terms, the topological class of universality relies on their contributions, especially the \( g_\chi \hat{p}^2 \) term is similar to the Rashba spin-orbit term which often plays a key role for topological transitions in condensed matter. In other words, although the \( g_\chi \hat{p}^2 \) terms are detrimental to the critical universality, they are however favorable for both critical diversity and topological universality. This is also the reason why diversity and universality, conceptually the antagonists, can counter-intuitively coexist and even support each other here.

5. Topological Classification at Finite Frequencies

Although the critical universality in low frequencies breaks down at finite frequencies and the properties are diversified, among the diversity we can still extract some common feature but from the topological structure of the GS wave function.\[^{27,28}\] Indeed, within each emerging phase at finite frequencies the GS wave function has a same node number, that is, the number of zeros \( n_x \). Wave functions with different node numbers are topologically different in the sense that, by fixing a node number \( n_x \), one cannot go to another \( n_x \) state by continuous shape deformation of the wave function, just as one cannot change a torus into a sphere by a continuous deformation which is a well-known illustration for topological difference. When a common example of a topological invariant is the number of holes in an object, here the topological invariant is the number of nodes in a GS wave function. Nodes of polynomial functions are also related to topological Galois theory in connecting algebra to topology.\[^{93}\] Moreover, differently from 1D spinless systems confined in a potential, which generally have no transition of nodes in the ground state due to the constant absence of nodes under the strict constraint of the no-node theorem,\[^{94}\] here the systems in light-matter interactions involve the spin nontrivially. It is recently found that the alternate nodes in two spin components drive spin windings and the node number of the ground state is equal to the effective spin winding number, which endows the node number a more explicit topological character in a physical way.\[^{95}\] Although the conventional QRM is also restricted by the no-node theorem, the coupling anisotropy can result in nodes.\[^{28}\] When spin-orbit coupling is often fundamentally responsible for TPTs in condensed matter,\[^{82–85,96,97}\] here as mentioned above Hamiltonian \( 4 \) the anisotropic \( g_\chi \) term also plays a role of linear Rashba/Dresselhaus spin-orbit coupling and the Stark term is a kind of nonlinear spin-orbit coupling, which lays the physical basis for the analogous\[^{27}\] of TPTs in our systems. The node number is the same universally for all system parameters within a phase. This leads us to a topological classification which is not only valid for the linear anisotropic QRM\[^{27,28}\] but also in the presence of the nonlinear Stark coupling as shown in the following. On the other hand, nonlinear coupling will lead us to new phenomena unexpected in linear coupling, such as topological quadruple points, composite sextuple points, and \((\sigma_z)\)-invariant points.

5.1. Conventional Topological Transitions with Gap Closing

Figure 5a–c shows the first excitation gap \( \Delta \) in the \( g_\chi-\lambda \) plane for \( \omega = 0.5 \, \Omega \) at a fixed Stark coupling ratio \( \chi = 0.1, 0.4, -0.3 \). We see
that some series of boundaries emerge where the gap is actually closing and re-opening. Figure 5d–f shows the phase diagrams of parity correspondingly, with the negative and positive parities represented by the colors in blue and red, respectively. Comparing Figure 5d–f with Figure 5a–c we see that the parity is reversed at the gap closing boundaries. Note the parity has only two values which are not enough to distinguish the series phases that emerge with the series of transitions. Something beyond the parity is needed to understand the nature of the transitions, which turns out to be topological structure of the wave function. Indeed, the node number $n_z$ of the GS wave function changes across each boundary of the gap closing and parity reversal, as shown by Figure 5g–i where the numbers mark $n_z$ of different phases. These transitions are analogs of the conventional TPTs that occurs at gap closing without symmetry breaking.\(^{[27]}\)

### 5.2. Unconventional Topological Transitions without Gap Closing

Besides the conventional TPTs with gap closing, unconventional TPTs may also occur without gap closing. Figure 5i shows the phase diagram of node number at a negative Stark coupling ratio $\chi = -0.3$. We see that, besides the transitions at the gap closing and parity reversal corresponding to Figure 5c,f, there are two boundaries that have no match of either gap closing or parity change. Figure 6 shows the phase diagrams in the $g - \chi$ plane at fixed anisotropy strengths $\lambda = 0.15 (a,c,e)$ and $\lambda = 0.5 (b,d,f)$. We see that besides the conventional transitions with gap closing and parity reversal, a transition boundary of node number without gap closing is also showing up in the negative-$\chi$ regime.

These additional transitions are analogs of the unconventional TPTs without gap closing in condensed matter which may occur in some particular situations, such as in the presence of a strong electron–electron interaction in the quantum spin Hall effect\(^{[98]}\) or in the presence of disorder with Berry curvature separation in the quantum anomalous Hall effect.\(^{[99]}\) It should be worthwhile to mention that, compared with the difficulty in controlling electron–electron interaction or spin-orbit coupling in these topological materials in condensed matter, the systems of light-matter interactions considered the present work have the advantage of high tunability of system parameters which provides great flexibility to manipulate phase transitions including the unconventional TPTs.

### 5.3. Topological Quadruple Points

In the absence of the nonlinear Stark coupling the unconventional TPTs occur in $\lambda > 1$ regime.\(^{[28]}\) Here we see that the nonlinear Stark coupling is bringing the unconventional TPTs from the regime beyond the QRM line ($\lambda = 1$) to the intermediate regime $0 < \lambda < 1$ between the QRM and the JCM which are the most fundamental models in light-matter interactions. The entering of the unconventional TPTs into the intermediate
regime has two consequences: On the one hand, the TPTs can occur in the isotropic QRM with weak couplings, as indicated by Figure 5i; On the other hand, some topological quadruple points are formed, as one finds in Figures 5i and 6e,f.

Note that the node number characterizes the topological structure of the wave function within a same spin component, while the parity reflects the relative structure between the two spin components. In the phase labels of Figures 5i and 6e,f we have combined the node number and the parity to distinguish the phases better from each other. We see that four topological phases meet at a topological quadruple point, for example, in Figure 5i, marked by dots around which two phases have \( n_Z = 0 \) and the other two have \( n_Z = 1 \) while the parity is different for the phases with a same node number. Such topological quadruple points mean the boundary crossing of the conventional TPTs with gap closing and unconventional TPTs without gap closing, which never happens for the linear interaction in the absence of the nonlinear Stark coupling. [28]

5.4. Composite Phase Diagrams: Multicriticality, Composite Quadruple Points, Composite Sextuple Points

We have seen in Figure 5 that \( P \) and \( n_Z \), are symmetric with respect to the sign reversal of \( \lambda \) as it is also true for \( \langle \sigma_x \rangle, \zeta \) discussed in Section 3.3, while \( \langle \hat{x}^2 \rangle - \langle \hat{y}^2 \rangle \) is antisymmetric in Figure 2a,c. Here at a finite frequency \( \zeta \) is extracted by the ratio of the main-peak position of \( \psi \) (x) and the potential-bottom position \( g'_s \) for \( \lambda > 0 \) \( g'_s \) for \( \lambda < 0 \). Combining these quantities for an overview will expose some underlying features.

5.4.1. Hexacritical Point

In Figure 7d we present a density plot for the composite quantity \( (n_Z + 1)^{1/4} P(\langle \hat{x}^2 \rangle - \langle \hat{y}^2 \rangle) \text{sign}(\langle \sigma_z \rangle) \) in the \( g: \lambda \) plane at \( \omega = 0.5 \Omega \) and \( \chi = -0.3 \). Along the symmetric \( \lambda = 0 \) line, one sees first a hexacritical point around \( g = 2.3g \), which is the crossing point of the first-order boundary (white dashed line, see Equations (45) and (48)) and the second-order boundary \( g^{s, \sigma} \) (black dotted curves, Equation (25)) as more reflected by the amplitude of \( \langle \sigma_x \rangle \) in Figure 7a.

5.4.2. Composite Quadruple/Sextuple Points

Along the \( \lambda = 0 \) line following the aforementioned hexacritical point are a composite quadruple point around \( g = 3.5g \), and a composite sextuple point around \( g = 4.5g \), while increasing \( g \) one would see more composite quadruple points beyond the plotting range. The composite sextuple point is actually a quadruple point (not topological quadruple point) in \( n_Z P(\langle \hat{x}^2 \rangle - \langle \hat{p}^2 \rangle) \) but the sign-reversal boundary of \( \langle \sigma_x \rangle \) renders it to be a sextuple-like point. The \( \langle \sigma_x \rangle \)-sign-reversal boundary as shown in Figure 7b also leads to another composite quadruple point away from the \( \lambda = 0 \) line around \( \{g/g_s, \lambda\} = [3.8, 0.2] \) in Figure 7c.

5.4.3. Meeting of Second-Order Transition and Unconventional TPT

The composite multiple points addressed above are located at the conventional TPT boundaries which are in principle of first order with gap closing. Another two composite quadruple points we did not stress are the crossing points of the boundary \( g^{s, \sigma} \) (dotted curves) and the unconventional TPT boundary, around \( g/g_s, \lambda = [1.4, \pm 0.6] \) as marked by the empty squares in Figure 7d. The critical transition at \( g^{s, \sigma} \) is second-order, which is softened at finite frequencies but still has a remnant of superradiant transition in photon number. [28] The unconventional TPT would be infinite-order.

5.5. Composite Phase Diagrams: Topological Quadruple Point being \( \langle \sigma_x \rangle \)-Invariant Point

Apart from the aforementioned composite sextuple point formed from the non-topological quadruple point and the \( \langle \sigma_x \rangle \)-sign-reversal boundary, more special is another composite sextuple point around \( \{g/g_s, \lambda\} = [3.36, 0.538] \) (with its dual point at
\[ \begin{align*}
\{3.36, -0.538\} & \text{ in } \lambda < 0 \text{ regime}, \text{ as marked by the dots in Figure 7d. This second composite sextuple point previously was the first topological quadruple point of Figure 5i addressed in Section 5.3 and now we see it happens that the conventional TPT boundary, the unconventional TPT boundary and the } (\sigma_x)-\text{sign-reversal boundary are all crossing at the topological quadruple point to form a sextuple-like point.}

\text{The } (\sigma_x)-\text{sign-reversal boundary is also vanishing-}(\sigma_x)-\text{boundary indicated by the bright line Figure 7a. Note here the frequency is finite, while the vanishing-}(\sigma_x)-\text{boundary in the low-frequency limit, } g^{s}_{c} \text{ in Equation (34), is plotted as the dot-dashed line Figure 7b. Particularly, the topological quadruple remains invariant when the other vanishing-}(\sigma_x)-\text{points are moving away from dot-dashed line in the variation of frequency. Thus we find this topological quadruple point is a } (\sigma_x)-\text{invariant point. Moreover, it is also an adiabatic-invariant as similarly displayed in Figure 7c where the dot-dashed line is adiabatic boundary } g^{s}_{c} \text{ in the low-frequency limit, Equation (33), while the color change around the dot-dashed line indicates the } \zeta = 1 \text{ boundary at the finite frequency.}

\text{In Figure 7e one can also see the sextuple-like point in a composite phase diagram of } (n_x + 1)P[\text{sign}((\sigma_x)) + \frac{1}{2}]) \text{ in the } g - \lambda \text{ plane at a fixed } \omega = 0.5 \text{ and } \lambda = 0.5. \text{ As shown by Figure 7f, the sextuple degeneracy will be lifted if it is located close to the Stark-JC critical boundary } g^{JC-\text{Stark}} \text{ (Equation (29) as plotted by dashed lines in Figure 7e,f). Here, unlike the leading two-peak structure both before and after transition for points away from } g^{JC-\text{Stark}}, \text{ the GS wave function is however of one-peak structure before the transition and two-peak structure after } \zeta^{(27)} \text{ which leads to different vanishing-}(\sigma_x)-\text{points thus the dislocation of the } (\sigma_x)-\text{boundaries. In contrast, the topological quadruple point is more robust and still survives there despite of the breakdown of the sextuple degeneracy.}

5.6. Analytic Expressions of the First Topological Boundary and Topological Quadruple Point

\text{By adding the Stark term to the treatment on the } \sqrt{T} \text{ transition in the polaron picture,}^{(27)} \text{ we can get an analytic boundary for the first conventional TPT in the leading order (see Appendix C)}

\[ g^{s}_{T1} = \frac{2\sqrt{2}}{\sqrt{(1 + \lambda)(2 + \chi) - \lambda(2 - \chi)}} g^{s}_{c}, \] \( (45) \)

\[ \lambda^{T1} = \frac{2\sqrt{1 - 2(2 - \chi)g^{s}_{c} / g^{2} + \chi}}{2 - \chi} \] \( (46) \)

\[ \chi^{T1} = \frac{2(4 - (1 - \chi^{2})g^{3}_{c} / g^{2})}{(1 + |\lambda|)g^{3}_{c} / g^{2}} \] \( (47) \)

\text{which provides an analytic confirmation with a direct insight about the node variation at the TPT. Indeed, as seen analytically in Appendix C, similar to the anisotropic QRM}^{(27)} \text{ the first conventional TPT in the presence of both the anisotropy and the Stark coupling also occurs with parity reversal and level crossing, when the wave function is braided and a node is created around the origin.}
From exact solution \(^3,100\) we can also get an accurate analytic boundary
\[
g_{T^L,E} = \frac{2\sqrt{1 - \lambda^2}}{\sqrt{(1 + \lambda^2) - \lambda^2 (1 - \lambda^2)}} g_c \tag{48}
\]
\[
\lambda_{T^L,E} = \sqrt{(1 + \lambda^2) - \frac{4}{g_c^2}} \tag{49}
\]
\[
\chi_{T^L,E} = \frac{1 + \lambda^2}{8} g_c^2 + \sqrt{\frac{1 + \lambda^2}{8} g_c^2 - \frac{g_c^2}{2 g_p^2}} \tag{50}
\]

Besides recovering \(g_{T^L,0} = \frac{2}{\sqrt{1 - \lambda^2}} g_c \) at \( \lambda = 0 \), both \( g_{T^L} \) and \( g_{T^L,E} \) agree with the numeric results at a finite \( \lambda \), as indicated by the dashed lines in Figure 7a–d, except for some discrepancy around \( \lambda = 0 \) for \( g_{T^L} \) at a large \( \lambda \).

Combining Equations (48)–(50) and Equations (34)–(36), we find the analytic locations of the topological quadruple points
\[
g_{TQ} = \frac{\sqrt{2(1 - \lambda^2)}}{\sqrt{\lambda^2}} g_c, \quad \lambda_{TQ} = \pm \frac{1 + \lambda^2}{1 - \lambda^2} \tag{51}
\]
\[
g_{\chi_{TQ}} = \frac{2\sqrt{2}}{\sqrt{1 - \lambda^2}} g_c, \quad x_{\chi_{TQ}} = \frac{1 - |\lambda|}{1 + |\lambda|} \tag{52}
\]

under a given Stark coupling and under a fixed anisotropy ratio respectively, which are plotted as dots and coincide with numerics in Figures 5 and 7.

### 5.7. Topological Quadruple Points in Large \( \lambda \)

So far we have focused on \(|\lambda| \leq 1\) regime, while topological quadruple points can also emerge in large-\( \lambda \) regime. Figure 8a,b displays the phase diagrams of \( n_2 \), together with \( \chi \) represented by overlies and underlines, respectively under a given Stark coupling ratio \( \chi = 0.2 \) (a) and at a fixed anisotropy strength \( \lambda = 2.0 \) (b). For a confirmation and a more direct view, the parity is also explicitly plotted in Figure 8c at \( \lambda = 2.0 \). The conventional TPTs occur between phases \( n_2 \) and \( \overline{n_2} \), while the unconventional ones lie on the boundaries between phases \( \overline{n_2} \) and \( n_2 \pm 1 \) or between \( n_2 \) and \( \overline{n_2} \pm 1 \). As one sees from Figure 8a the conventional TPT boundaries remain almost unmovig in adding the Stark coupling as compared with the \( \chi = 0 \) boundary (dotted line). This scenario is confirmed by Figure 8b where the conventional TPTs are not much affected in the vicinity of \( \chi = 0 \), unless a large amplitude of \( \chi \) is involved. In a strong contrast, the unconventional TPTs (0/1 boundary) are very sensitive to the variation of \( \chi \), as one compares with the dashed line which represents the unconventional TPT boundary at \( \chi = 0 \). In the absence of the Stark coupling, the conventional and unconventional TPTs do not cross each other.\(^{28}\) Now in adding the Stark coupling, the slow motion of the conventional TPTs and the fast moving of the unconventional TPTs result in the boundary crossing and thus bring about the topological quadruple points.

### 6. Mechanisms

To get an understanding for some key features of the different TPTs and the topological quadruple points, in Figure 9 we show the profiles of \( \psi_+ (x), \psi_- (x) \) and \( \partial_x \psi_+, \hat{p}^2 \psi_-, \hat{x}^2 \psi_- \) in space for GSs. They contribute to the tunneling and different interacting parts in the GS energy
\[
E_{\psi} = \frac{\Omega}{2} \int \psi_+ (x) \psi_- (x) dx \tag{53}
\]
\[
E_{\partial} = \sqrt{2} (-g_p) \int \partial_x \psi_+ (x) \psi_- (x) dx \tag{54}
\]
\[
E_{\hat{p}^2} = \frac{\chi \Omega}{2} \int \psi_+ (x) \hat{p}^2 \psi_- (x) dx \tag{55}
\]
\[
E_{\hat{x}^2} = \frac{\chi \Omega}{2} \int \psi_+ (x) \hat{x}^2 \psi_- (x) dx \tag{56}
\]

which involve subtle competitions.

### 6.1. TPTs and Quadruple Points in \( \lambda > 1 \) Regime

#### 6.1.1. Node from Infinity Side and Unconventional TPT

Figure 9a–c illustrates some typical points of different phases 0, 1, 2 in Figure 8 in \( \lambda > 1 \) regime with \( \chi = 0.1 \). The state change from panel (b) to panel (c) of Figure 9 illustrates a conventional TPT, while the state variation from panel (a) to panel (b) represents...
the unconventional TPT. The former has a node added around the origin $x = 0$, the latter introduces a node from the infinity side of the wave packet. From panel (b) and (c) of Figure 9 the node braids the wave function so that the parity is reversed. States in opposite parities belong to different energy levels, the parity changeover in the GS can occur only via level crossing. Consequently such a transition is always accompanied with a gap closing which is the condition of the conventional TPT.\[^{[27,78,96,97,101]}\]

Differently in the variation between panels (a) and (b) of Figure 9, the node from the infinity side does not affect the parity and the energy can evolve smoothly. Indeed the energy variation induced by the node introduction is approaching to zero as the density is exponentially decaying with distance in the infinity direction. As a result, such a node status change does not connect to other parity states which are staying away with an energy gap. Thus, such a node changeover takes place in a gapped situation, which accounts for the unconventional TPT.\[^{[28,98,99]}\] Although the above simple analysis provides a direct understanding for the conventional and unconventional TPTs, when the node from the infinity side is favorable to induce the unconventional TPT depends on specific parameters and we need to look into the detailed energy competitions among different interacting parts as in the following.

Around the isotropic line as in Figure 9a with $\lambda = 1.1$ and $g = 2.6g_s$, the amplitude of $g_s(1 - \lambda)g/2$ is small so that $E_z$ plays a more dominant role which favors a nodeless state with $n_z = 0$ which has opposite signs of $\psi_+ (x)$ and $\psi_- (x)$ in all positions. In a larger $\lambda$ as in Figure 9b with $\lambda = 2.0$ and $g = 2.6g_s$, note $(-g_s)$ is positive here which also favors opposite signs of $\partial_x \psi_+$ and $\psi_-(x)$. The larger contribution of $E_z$ leads to the negative-peak replacement of $\psi_-$ by $\partial_x \psi_-$ in alignment (as indicated by vertical dashed line) with the positive peak of $\psi_+(x)$ on the left side to get a lower energy. A node introduction from the infin-

**Figure 9.** Mechanism analysis: GS wave function and energy competitions. $\psi_+$ (blue solid), $\psi_-$ (blue dotted), $\partial_x \psi_+$ (orange), $\hat{P}^2 \psi_-$ (green), $\hat{X}^2 \psi_-$ (red) in $x$ space at a) $\chi = 0.1$, $\lambda = 1.1$, $g = 2.6g_s$ ($P = -1$, $n_z = 0$), b) $\chi = 0.1$, $\lambda = 2.0$, $g = 2.6g_s$ ($P = -1$ and $n_z = 1$), c) $\chi = 0.1$, $\lambda = 2.0$, $g = 3.3g_s$ ($P = 1$ and $n_z = 2$), d) $\chi = -0.3$, $\lambda = 0.538$, $g = 3.355g_s$ ($P = -1$ and $n_z = 0$), e) $\chi = -0.3$, $\lambda = 0.8$, $g = 3.0g_s$ ($P = -1$ and $n_z = 1$), and f) $\chi = -0.3$, $\lambda = 0.4$, $g = 3.6g_s$ ($P = 1$ and $n_z = 1$). The vertical dashed lines mark the peak positions of $\psi_+$. In all panels $\omega = 0.5\Omega$ and except (d) the amplitude $A_m$ in each line is amplified by $A_m^2$ for better visibility.

6.1.2. Unconventional TPT in $\lambda > 1$ Regime Sensitive to $\chi$

In the nonlinear Stark parts, $\hat{P}^2 \psi_-$ is oscillating to cancel itself to a large extent, thus the main contribution lies in $\hat{X}^2 \psi_-$. The Stark coupling to $\partial_x \psi_-$ is also a reason why it is x-type in $\lambda > 0$ regime. Note, with a node from the infinity side, the tails of $\hat{X}^2 \psi_-$ has a same sign as $\psi_+(x)$ as in Figure 9b which is unfavorable for $E_z$ with a positive $\chi$. In this sense, $\hat{X}^2 \psi_-$ is counteracting with $\partial_x \psi_-$ in such a node introduction. Consequently, one needs a larger $\lambda$ to strengthen the $(-g_s)$ term $E_z$ to trigger the unconventional TPT. This accounts for the far boundary moving of the unconventional TPT from around $\lambda = 0$ (dashed line in Figure 8a) in the absence of the Stark coupling to a larger $\lambda (\mathcal{O})/1$ boundary around $\lambda = 1.7$ in Figure 8a) in the presence of a positive Stark coupling $\chi$.

6.1.3. Conventional TPT in $\lambda > 1$ Regime Unaffected by $\chi$

On the other hand, for the conventional TPT, Figure 9c shows the state at $\lambda = 2.0$ and a larger linear coupling $g = 3.5g_s$ after the $1/2$ transition from state in Figure 9b. Such a conventional TPT introduces a node around the origin $x = 0$, thus accompanied with gap closing and parity reversal. In such a situation the leading variation lies around the origin while the farther parts remain little affected. Note that $\hat{X}^2 \psi_-$ and $\hat{P}^2 \psi_-$ have similar decreasing amplitudes but opposite signs around the origin both...
before and after the transition, which leads to a cancellation effect. As a result, $\hat{x}^2\psi_+$ and $\hat{p}^2\psi_-$ together do not play much role in this conventional TPT, unless one increases $\chi$ much to multiply their difference. This explains the little moving of the conventional TPT boundaries in the variations of Stark coupling as in Figure 8 (dotted line and $\sqrt{2}$ boundary).

6.1.4. Topological Quadruple Points

Since the conventional TPTs keeps almost unmoved while the unconventional TPT is sensitive to the introduction of the Stark coupling, their boundary meeting naturally occurs. The final boundary crossing gives rise to the topological quadruple points.

6.2. TPTs and Quadruple Points in $\chi < 1$ Regime

6.2.1. Unconventional TPT in $\chi < 1$ Regime with Negative $\chi$

Now we look at the $\chi < 1$ regime with a negative $\chi$, as in Figure 5i. The nodeless state ($n_x = 0$) in a small $g$ is similar to Figure 9a with peak alignment of $\psi_+(x)$ and $\psi_-(x)$ due to the dominant $E_{\text{lin}}$. Figure 9e shows a nodal state in Figure 5i after the 0/1 unconventional TPT. Here ($-g_0$) is negative, different signs of $\partial_x\psi_-(x)$ and $\psi_+(x)$ are unfavorable for lowering the energy of $E_{\text{lin}}$. A node introduced from the infinity side as in Figure 9e would not only bring wave-packet tails with same signs of $\psi_-(x)$ and $\psi_+(x)$ to increase $E_{\text{lin}}$ but also lead to larger tails of $\partial_x\psi_-(x)$ and $\psi_+(x)$ with different signs on the right than the tails with same signs on the left to raise $E_{\text{lin}}$, so there is no unconventional TPT in the absence of Stark coupling. However, in the presence of a negative $\chi$, on both sides $\hat{x}^2\psi_-$ has tails with same signs as $\psi_+$, as in Figure 9e, which reduces the energy from $E_{\text{lin}}$ and makes the unconventional TPT possible. Therefore, the unconventional TPT boundary moves from $\chi > 1$ regime to $\chi < 1$ regime as in Figure 5i.

6.2.2. Conventional TPT in $\chi < 1$ Regime Depending on $\chi$

In contrast to the $\chi$-insensitiveness in $\chi > 1$ regime the conventional TPT in $\chi < 1$ regime depends much on $\chi$ as shown in Figures 5 and 6. Figure 9f shows a state in the $\bar{T}$ phase of Figure 5i. Comparing with Figure 9c one sees there is no aforementioned cancellation effect of $\hat{x}^2\psi_-$ and $\hat{p}^2\psi_-$ around the origin. This is because the distance of left and right wavepackets depends on $g_0 = (1 + \lambda)\delta g/2$ which is much smaller in $\chi < 1$ regime so that there is more overlap between left and right wavepackets. Since the conventional TPT comes from the node number variation around the origin, the transition boundary is then much influenced by the Stark coupling with the enlarged difference of $\hat{x}^2\psi_-$ and $\hat{p}^2\psi_-$.

6.2.3. Invariant Point

Actually Figure 9e,f takes the points along the boundary where $\langle \sigma_x \rangle$ vanishes and changes the sign in Figure 7a,b. In these cases the wavefunction is finite in amplitude on both sides, while the vanishing of $\langle \sigma_x \rangle$ comes from the cancellation between same-sign and opposite-sign parts of $\psi_+(x)$ and $\psi_-(x)$ with a certain position of the node. Such a cancellation depends on the frequency $\omega$ since the size of wavepackets will vary with the frequency[26] to affect the cancellation situation. In contrast, the status of the topological quadruple point is distinctive, as one side of wavepacket is completely flat as demonstrated by Figure 9d. The vanishing of $\langle \sigma_x \rangle$ at the topological quadruple point results from the vanishing local product of $\psi_+(x)$ and $\psi_-(x)$ rather than the cancellation. In such a situation, other terms $\partial_x\psi_-, \hat{p}^2\psi_-, \hat{x}^2\psi_-$ do not come to effect either. Thus, the GS effectively behaves like a non-interacting particle in displaced harmonic potential $\langle \psi_0(x) \rangle$ in Equation (3)), with the particle location adiabatically being the potential bottom position, which is the reason why here also $\zeta = 1$. Note such a status effectively being the GS of a displaced harmonic potential remains the same for different frequencies, this topological quadruple point appears as an invariant point in the sense the vanishing value of $\langle \sigma_x \rangle$ and adiabatic value $\zeta = 1$ remain unchanged when the frequency is varying.

7. Conclusions and Discussions

We have investigated the critical universality and topological universality in light-matter interactions via a thorough study on the first excitation gap and the GS of the QRM generally in the presence of interaction anisotropy and nonlinear Stark coupling.

In the low-frequency limit, we have obtained both numerically and analytically all phase boundaries of the QPTs in the GS as well as the adiabatic boundaries and the vanishing $\langle \sigma_x \rangle$ boundaries. We have extracted various scaling relations in which physical properties collapse into the same line, respectively for different anisotropy ratios under finite Stark coupling and variations of both anisotropy and Stark coupling, locally around the QPTs or globally for all coupling regions after the transitions. These scaling relations form different levels of critical universalities. It may be worthy to mention that usually critical universality concerns a same critical exponent around the transition while same coefficients are not required.[52] Here, the scaling relations with same-line collapsing and more global range provide a stricter universality in some sense. Furthermore, our finding of the global scaling by the adapted shift provides an unconventional way of thought to seek novel classes of scaling relations.

At finite frequencies, the critical universality breaks down and the diversity comes to dominate. Amidst the diversity we have extracted the topological classifications which form a new universality essentially different from the critical universality. The critical universality involves the second-order transitions, while the topological universality here classifies the phases in the emerging first-order transitions for the conventional TPTs with gap closing or the infinite-order transitions for the unconventional TPTs without gap closing. Moreover, the universality-diversity-universality process demonstrates that although universality and diversity are antagonists by nature, counter-intuitively they can acquire coexistence and mutual support. We stress that both the critical universality and the topological classification hold not only for the linear interaction but also in the presence of nonlinear Stark coupling, thus yielding a more robust scenario of universalities.
While the conventional TPTs and the unconventional TPTs never meet in linear QRM,[28] the presence of the nonlinear coupling enables boundary crossings of the conventional and unconventional TPTs, which brings about the appearance of topological quadruple points. The composite phase diagrams in combination with the vanishing-$(\sigma_y)$ and adiabatic boundaries further display the multicityclicity, composite quadruple points and composite hexape points. In particular, we reveal that the topological quadruple points in the intermediate anisotropy regime are in fact $(\sigma_y)$-invariant points and adiabatically-invariant points in varying the frequency. This indicates that the locations of such topological quadruple points can be detected by invariant spin-flipping or tunneling points when one tunes the frequency.

Our phase diagrams and sensitivity analysis with respect to the nonlinear Stark coupling demonstrate that in addition to the anisotropy the nonlinear coupling provides another approach to manipulate both the critical QPTs and the TPTs. Especially, the unconventional TPTs are quite sensitive in response to the nonlinear coupling. Note the few-body QPTs have a great potential for applications in critical quantum metrology.[92] Indeed, the regime around the QPT of the linear QRM provides a critical resource for high-precision measurements. While the linear QRM suffers from the limitation of one point of QPT, our phase diagrams demonstrate that the nonlinear QRM with the Stark coupling possesses a continuous and wide distribution range of QPTs which ensures a global high-precision resource for critical quantum metrology.[92] while the scaling relations indicate that similar orders of high precision would be available under a scaling factor. On the other hand, it is recently found that the node number in the topological classification as discussed in the present work has a correspondence to spin windings and the spin texture changes sign at nodes,[95] which makes the topological information detectable. This provides possibility for designing topological quantum devices or sensors, while both conventional and unconventional TPTs might have potential applications since both gap-closing[37] and gapped[68] situations are applicable. In particular, the regime around the topological quadruple points is topologically sensitive to parameter variations of the linear coupling, the anisotropy and the Stark coupling, which might also provide a special sensitivity resource for designing topological quantum devices. The frequency-invariant topological quadruple point has some additional advantage in need of tuning the frequency from low frequency limit to more practical finite frequencies while keeping the same transitions and properties around. Such possible topological quantum devices would benefit much from the high tunability of few-body systems, as the parameters of anisotropy,[6,33,72] and Stark coupling[30,73-75] here, and be free of the difficulty of reaching equilibrium for large systems from condensed matter as sensors.

Experimentally in superconducting circuit systems[77,102,101] with ultra-strong/deep-strong couplings[1,6-15,104,105] the effective position $x$ and momentum $p$ are realistically the flux and charge of Josephson junctions and the spin can be also implemented by flux qubit, the nodal status might be detected by interference devices and magnetometer.[103] Such systems have the advantage that different interactions can be well designed.[32,71,102] In practice, the interaction anisotropy is highly tunable[6,31,72] and on an equal footing[25] the nonlinear Stark coupling can also be realized with adjustable amplitude and sign,[71-75] which could provide feasible platforms for possible tests or potential applications of our results.

**Appendix A: Expression of Ground-State Quantum Number of JCM by Level Crossing**

After the first transition the integer quantum number number $n_j$ for the GS of the JCM can be obtained by level crossing $E^{(n_j)}_{JC-Stark} = E^{(n_j-1)}_{JC-Stark}$ at $\bar{g}_s$:

$$
\bar{g}_s = \sqrt{-x + C_1 \alpha + \sqrt{1 - 2(1+x)\alpha + C_1 \alpha^2}} \tag{A1}
$$

$$
n_j = \frac{1}{2} \frac{g_s^2 + 4\alpha}{8x^2\alpha} + \sqrt{\frac{2}{C_1\alpha}} \sqrt{\frac{2}{1-x^2}} \tag{A2}
$$

where $d = 16\alpha^2(1-x^2)^2/(g^2_s\Omega^2)$, $C_1 = (1+x)[1+2(1-x)n]/2$, $C_2 = (1+x)[9 - 7x + 4(1-x)(n-1)(n+2)]$.

**Appendix B: Decreasing Function of Unified Energy $E_{SC}^{\lambda}(g_{zy}')$ of $H_{SC}^f$ and $H_{SC}^{p}$**

To facilitate the energy comparison for $H_{SC}^f$ (Equation (15)) in position space and $H_{SC}^{p}$ in momentum space, we unify their minimized energies in Equation (24) as a same function of $g_{zy}^\lambda$.

$$
E_{SC}^{\lambda}(g_{zy}') = -\frac{1}{2} \frac{g_{zy}^2 (2 - x^2 \omega + \Omega)}{2\lambda^2} + \frac{g_{zy}^2 (1-x^2)\omega}{\sqrt{g_{zy}^2 + \frac{\Omega^2}{\lambda^2}} - \frac{\sqrt{1-x^2}}{\lambda}} \tag{B1}
$$

with $g_{zy}' = g_{zy}'(H_{SC}^f)$ and $g_{zy}^\lambda = g_{zy}'(H_{SC}^{p}) E_{SC}^{\lambda}$ as a function of $g_{zy}'$ has a maximum point $g_{zy,max}$ at

$$
g_{zy,max} = g_{zy,SC} = \sqrt{(1-x)\Omega/(2\omega)} \tag{B2}
$$

which happens to be the critical point $g_{zy,SC}$ for the transition as decided by $E_{SC}^{\lambda} = E_{SC}^{\lambda (1)}$. After the transition $E_{SC}^{\lambda}$ becomes a decreasing function of $g_{zy}'$, which can be seen from the derivative $dE_{SC}^{\lambda}/dg_{zy}' = d_\lambda - d_\alpha$, where $d_\lambda - d_\alpha = -C_g(g_{zy}'^2 - g_{zy,max}'^2)(g_{zy}'^2 + (1 + x)\Omega^2/(2\omega))$ and $C_g = 4\alpha^2/\Lambda^2$ (with $g_{zy,max}'$), being negative after $g_{zy,max}'$. Thus, a larger value of $g_{zy,max}'$ provided by $g_{zy}'$ and $g_{zy}'$ will be more favorable for the candidate of the GS, which yields a lower energy for $H_{SC}^f$ in $\lambda > 0$ regime but for $H_{SC}^{p}$ in $\lambda < 0$ regime as discussed in Section 3.5.

**Appendix C: Derivation in Polaron Picture for $g_{T1}^{(X)}$ at the First Conventional TPT**

In this Appendix we provide a brief derivation for $g_{T1}^{(X)}$ in Equations (45)-(47) in the polaron picture.[23,26-28,44] We can decompose the wave function into $n_{\Psi}$ number of polarons $\Psi_{\nu}(x) = \sum_{\nu=0}^{n_{\Psi}} \Psi_{\nu}(x)$, with weight $w_{\nu}$ for $\nu(x) = e^{\nu(x)\epsilon_0 \frac{\epsilon(x)C_0}{2}} (g_{\nu}(x)/\lambda)^4$, where $\zeta_0$ and $\zeta_\nu$ denote variational displacement renormalization and frequency renormalization in the Gaussian-like wave packets.[23,28,44] The first conventional TPT only involves two polarons denoted by $a, \beta$ with wave function $\Psi_a(x) = a(x) \Psi_{\nu}(x)$ and $\Psi_\beta(x) = \beta(x) \Psi_{\nu}(x)$, while the parity is reversed from $P = -1$ to $P = 1$ which braids the wave function and creates a node around the origin.[27,28] Note
here the displacement of polaron $\alpha$ is in the same direction as the bare potential $\nu_\beta(x)$, while that of polaron $\beta$ is in the opposite direction due to the tunneling effect;\textsuperscript{[23]} that is, $\zeta_\alpha > 0$ while $\zeta_\beta < 0$. The different energy parts can be calculated by

$$E_K = -\frac{e_0}{2N_p} \langle \varphi_\alpha | \hat{x}^2 | \varphi_\alpha \rangle - 2P\alpha \beta \langle \varphi_\alpha | \hat{d}_\varphi | \varphi_\beta \rangle + \beta^2 \langle \varphi_\beta | \hat{d}_\varphi^2 | \varphi_\beta \rangle \quad (C1)$$

$$E_\varphi = \frac{1}{N_p} \left[ \langle \varphi_\alpha | \hat{v}_\alpha \varphi_\alpha \rangle - 2P\alpha \beta \langle \varphi_\alpha | \hat{d}_\varphi \varphi_\beta \rangle + \beta^2 \langle \varphi_\beta | \hat{d}_\varphi \varphi_\beta \rangle \right] \quad (C2)$$

$$E_{\Omega} = \frac{\Omega}{2N_p} \hat{P} \left[ \langle \varphi_\alpha | \hat{\nabla}_\varphi \varphi_\alpha \rangle - 2P\alpha \beta \langle \varphi_\alpha | \hat{d}_\varphi \varphi_\beta \rangle + \beta^2 \langle \varphi_\beta | \hat{d}_\varphi \varphi_\beta \rangle \right] \quad (C3)$$

$$E_{\Omega} = \sqrt{2} \langle \varphi_\alpha | \hat{v}_\alpha \varphi_\alpha \rangle - 2P\alpha \beta \langle \varphi_\alpha | \hat{d}_\varphi \varphi_\beta \rangle + \beta^2 \langle \varphi_\beta | \hat{d}_\varphi \varphi_\beta \rangle \quad (C4)$$

$$E_{\Omega} = \frac{\Omega}{2N_p} \hat{P} \left[ \langle \varphi_\alpha | \hat{\nabla}_\varphi \varphi_\alpha \rangle - 2P\alpha \beta \langle \varphi_\alpha | \hat{d}_\varphi \varphi_\beta \rangle + \beta^2 \langle \varphi_\beta | \hat{d}_\varphi \varphi_\beta \rangle \right] \quad (C5)$$

$$E_{\Omega} = -\frac{\Omega}{2N_p} \hat{P} \left[ \langle \varphi_\alpha | \hat{\nabla}_\varphi \varphi_\alpha \rangle - 2P\alpha \beta \langle \varphi_\alpha | \hat{d}_\varphi \varphi_\beta \rangle + \beta^2 \langle \varphi_\beta | \hat{d}_\varphi \varphi_\beta \rangle \right] \quad (C6)$$

where $\varphi_\varphi(x) \equiv \varphi_\alpha(-x)$ and $N_p = a^2 + 2 \beta^2 - 2P\alpha \beta \langle \varphi_\alpha | \varphi_\beta \rangle$ is normalization factor. Since $\beta$ is small relative to $a$ due to higher potential at a large $g$, the leading order of the energy competition around the transition lies in the $a^2$ terms. Also, in addition to the small weight $\beta$, the wave-packet overlap $\langle \varphi_\alpha | \varphi_\beta \rangle$ in $N_p$ is negligible at large $g$ so that the leading order of $N_p$ is unaffected by the transition. We also see that the leading terms of $E_K$ and $E_{\Omega}$ do not change after the transition as $P$ only affects the second terms. Thus, the transition occurs at level crossing of the leading total interaction energy, including $E_{\Omega}$, $E_\varphi$ and the full Stark coupling energy $E_{\Omega} + E_\varphi$. $E_{\Omega}$ should be vanished which is as remains even with its sign ($P = \pm 1$ as the coefficient) reversed. So at the transition we have

$$\frac{\Omega - \Omega}{2} \langle \varphi_\alpha | \hat{v}_\alpha \varphi_\alpha \rangle - \sqrt{2P\alpha \beta \langle \varphi_\alpha | \hat{d}_\varphi \varphi_\beta \rangle + \beta^2 \langle \varphi_\beta | \hat{d}_\varphi \varphi_\beta \rangle} \right) \langle \varphi_\alpha | \hat{\nabla}_\varphi \varphi_\alpha \rangle - \frac{\Omega}{2} \langle \varphi_\alpha | \hat{\nabla}_\varphi \varphi_\alpha \rangle = 0 \quad (C7)$$

A large $g$ at the first TPT leads to large displacements so that the overlap between polarons $\alpha, \beta$ becomes small and the displacement basically follows the potential, thus we can approximately set $\zeta_\alpha \approx 1$ and $\zeta_\beta \approx -1$. On the other hand, the first TPT is accompanied with an amplitude-squeezing/phase-squeezing transition\textsuperscript{[24]} so we can set $\zeta_\alpha \approx \zeta_\beta \approx 1$ at the transition. Thus we obtain explicitly

$$\langle \varphi_\alpha | \hat{v}_\alpha \varphi_\alpha \rangle = e^{z_\alpha \hat{a}_\alpha^\dagger}, \quad \langle \varphi_\alpha | \hat{d}_\varphi \varphi_\beta \rangle = e^{z_\beta \hat{a}_\beta^\dagger} g_\beta^\dagger$$

$$\langle \varphi_\alpha | \hat{\nabla}_\varphi \varphi_\alpha \rangle = \langle \varphi_\alpha | \hat{\nabla}_\varphi \varphi_\alpha \rangle, \quad \langle \varphi_\alpha | \hat{\nabla}_\varphi \varphi_\alpha \rangle \approx \frac{1}{2} z_\beta \hat{a}_\beta^\dagger (2g_\beta^2 - 1) \quad (C9)$$

Substitution into Equation (C7) leads to the transition boundary $\delta_{\alpha 1}^\dagger$, $\delta_{\beta 1}^\dagger$, $x_{T1}$, $x_{T2}$ Equations(45)–(47) in the main text. The above derivation assumes a large $g$ which is valid for negative $\chi$ as illustrated by Figure 7a, while in positive-$\chi$ regime it becomes invalid in approaching to $\chi = +1$ as smaller $g$ is involved.

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Conflict Of Interest

The author declares no conflict of interest.
