Improved estimates for extinction probabilities and times to extinction for populations of tsetse (Glossina spp)

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S1 Text

Proof of equation (7)

When \( k = 0 \), we obtain

\[
p_0 = \epsilon \lambda^\nu (1 - \lambda^\tau) \sum_{n=0}^{\infty} \binom{n}{0} (\lambda^\tau \beta)^n (\frac{1}{\beta} - \varphi P)^n
\]

\[
= \epsilon \lambda^\nu (1 - \lambda^\tau) \sum_{n=0}^{\infty} [\lambda^\tau (1 - \beta \varphi P)]^n
\]

\[
= \epsilon \lambda^\nu (1 - \lambda^\tau) \left[ \frac{1}{(1 - \lambda^\tau (1 - \beta \varphi P))} \right]
\]

\[
= \frac{\epsilon \lambda^\nu (1 - \lambda^\tau)}{(1 - \lambda^\tau (1 - \beta \varphi P))}.
\]

When \( k = 1 \), we obtain

\[
p_1 = \epsilon \lambda^\nu (1 - \lambda^\tau) \varphi P \sum_{n=1}^{\infty} \binom{n}{1} (\lambda^\tau \beta)^n (\frac{1}{\beta} - \varphi P)^{n-1}
\]

\[
= \epsilon \lambda^\nu (1 - \lambda^\tau) \varphi P \sum_{n=1}^{\infty} n(\lambda^\tau \beta)^n (\frac{1}{\beta} - \varphi P)^{n-1}.
\]

If we let \( a = \lambda^\tau \beta, b = (\frac{1}{\beta} - \varphi P) \) and \( F = \sum_{n=1}^{\infty} na^n b^{n-1} \), this implies that

\[
F = ab^0 + 2a^2 b + 3a^3 b^2 + \ldots
\]

\[
abF = a^2 b + 2a^3 b^2 + \ldots
\]

\[
(1 - ab)F = ab^0 + a^2 b + a^3 b^2 + \ldots
\]

\[
ab(1 - ab)F = a^2 b + a^3 b^2 + a^4 b^3 + \ldots = \frac{a^2 b}{(1 - ab)}
\]

\[
F = \frac{a}{(1 - ab)^2} = \frac{\lambda^\tau \beta}{(1 - \lambda^\tau (1 - \beta \varphi P))^2}.
\]
Thus, the final solution for $p_1$ becomes
\[
p_1 = \epsilon \lambda^\nu (1 - \lambda^\tau) \varphi^F P = \epsilon \lambda^\nu (1 - \lambda^\tau) \varphi^P \left( \frac{\lambda^\tau \beta}{(1 - \lambda^\tau(1 - \beta \varphi^P))^2} \right) = \frac{\epsilon \lambda^{\nu+\tau}(1 - \lambda^\tau) \beta \varphi^P}{(1 - \lambda^\tau(1 - \beta \varphi^P))^2}.
\]

When $k = 2$, we obtain
\[
p_2 = \epsilon \lambda^\nu (1 - \lambda^\tau) \varphi^{2P} \sum_{n=2}^{\infty} \left( \frac{n}{2} \right) (\lambda^\tau \beta)^n \left( \frac{1}{\beta} - \varphi^P \right)^{n-2}
= \epsilon \lambda^\nu (1 - \lambda^\tau) \varphi^{2P} \sum_{n=2}^{\infty} \left[ \frac{n(n-1)}{2} (\lambda^\tau \beta)^n \left( \frac{1}{\beta} - \varphi^P \right)^{n-2} \right].
\]

Also letting $a = \lambda^\tau \beta$, $b = \left( \frac{1}{\beta} - \varphi^P \right)$ and $G = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} a^n b^{n-2}$, we have
\[
G = a^2 b^0 + 3a^3 b + 6a^4 b^2 + 10a^5 b^3 + \ldots,
abG = a^3 b + 3a^4 b^2 + 6a^5 b^3 + \ldots,
(1 - ab)G = a^2 b^0 + 2a^3 b + 3a^4 b^2 + 4a^5 b^3 + \ldots
ab(1 - ab)G = a^3 b + 2a^4 b^2 + 3a^5 b^3 + 4a^6 b^4 + \ldots
ab(1 - ab)^2 G = (1 - ab)^2 G = a^2 b^0 + a^3 b + a^4 b^2 + a^5 b^3 + \ldots
ab(1 - ab)^2 G = a^3 b + a^4 b^2 + a^5 b^3 + a^6 b^4 + \ldots = \frac{a^3 b}{(1 - ab)}.
\]

Thus, the final solution for $p_2$ becomes
\[
p_2 = \epsilon \lambda^\nu (1 - \lambda^\tau) \varphi^{2P} G = \epsilon \lambda^\nu (1 - \lambda^\tau) \varphi^{2P} \left( \frac{(\lambda^\tau \beta)^2}{(1 - \lambda^\tau(1 - \beta \varphi^P))^3} \right) = \frac{\epsilon \lambda^{\nu+2\tau}(1 - \lambda^\tau) \beta^2 \varphi^{2P}}{(1 - \lambda^\tau(1 - \beta \varphi^P))^3}.
\]

Thus, in general
\[
p_k = \frac{\epsilon \lambda^{\nu+k\tau}(1 - \lambda^\tau) \beta^k \varphi^{kP}}{(1 - \beta \lambda^\tau(\frac{1}{\beta} - \varphi^P))^{k+1}}, \text{ for } k > 0.
\]
Proof of equations (8) and (10)

\[
P_{(k>0)} = \sum_{k=1}^{\infty} \frac{\epsilon \lambda^\nu k}{(1 - \beta \lambda^\nu (\frac{1}{\beta} - \varphi^P)) k+1} (1 - \lambda^\nu (1 - \beta \varphi^P))
\]

\[
= \frac{\epsilon \lambda^\nu (1 - \lambda^\nu)}{1 - \lambda^\nu (1 - \beta \varphi^P)} \sum_{k=1}^{\infty} \left[ \frac{\lambda^\nu \beta \varphi^P}{1 - \lambda^\nu (1 - \beta \varphi^P)} \right]^k
\]

\[
= \frac{\epsilon \lambda^\nu (1 - \lambda^\nu)}{1 - \lambda^\nu (1 - \beta \varphi^P)} \left[ \frac{\lambda^\nu \beta \varphi^P}{1 - \lambda^\nu (1 - \beta \varphi^P)} \right]
\]

\[
= \frac{\epsilon \lambda^\nu (1 - \lambda^\nu)}{1 - \lambda^\nu (1 - \beta \varphi^P)} \left[ \frac{\lambda^\nu \beta \varphi^P}{1 - \lambda^\nu (1 - \beta \varphi^P)} \right]
\]

\[
= \frac{\epsilon \lambda^\nu (1 - \lambda^\nu)}{1 - \lambda^\nu (1 - \beta \varphi^P)}
\]

Thus, the probability that a female tsetse fly does not produce any surviving female offspring before she dies is given by:

\[
p_0 = 1 - p_{(k>0)} = 1 - \frac{\epsilon \lambda^{\nu+\tau} \beta \varphi^P}{1 - \lambda^\nu (1 - \beta \varphi^P)}.
\]

Extinction probability, \(\phi(\theta)\) is:

\[
\phi(\theta) = \sum_{k=0}^{\infty} p_k \theta^k = p_0 + \sum_{k=1}^{\infty} p_k \theta^k.
\]

\[
\phi(\theta) = 1 - \frac{\epsilon \lambda^{\nu+\tau} \beta \varphi^P}{1 - \beta \lambda^\nu (1 + (1 - \varphi^P))} + \frac{\epsilon \lambda^\nu (1 - \lambda^\nu)}{1 - \lambda^\nu (1 - \varphi^P)} \sum_{k=1}^{\infty} \left[ \frac{\lambda^\nu \beta \varphi^P \theta}{1 - \lambda^\nu (1 - \varphi^P)} \right]^k
\]

\[
= 1 - \frac{\epsilon \lambda^{\nu+\tau} \beta \varphi^P}{1 - \lambda^\nu (1 - \beta \varphi^P)} + \frac{\epsilon \lambda^\nu (1 - \lambda^\nu)}{1 - \lambda^\nu (1 - \beta \varphi^P)} \left[ \frac{\lambda^\nu \beta \varphi^P \theta}{1 - \lambda^\nu (1 - \beta \varphi^P)} \right]
\]

\[
= 1 - \frac{\epsilon \lambda^{\nu+\tau} \beta \varphi^P}{1 - \lambda^\nu + \beta \lambda^\nu \varphi^P} + \frac{\epsilon \lambda^\nu (1 - \lambda^\nu)}{1 - \lambda^\nu + \beta \lambda^\nu \varphi^P} \left[ \frac{\lambda^\nu \beta \varphi^P \theta}{1 - \lambda^\nu + \beta \lambda^\nu \varphi^P - \beta \lambda^\nu \varphi^P \theta} \right].
\]
Setting $A = 1 - \lambda^r$, $B = \beta \lambda^r \varphi^P$ and $C = 1 - \epsilon \lambda^r$, we obtain

$$
\phi(\theta) = 1 - \frac{\epsilon \lambda^r B}{A + B} + \frac{\epsilon \lambda^r A}{A + B} \left[ \frac{B \theta}{A + B - B \theta} \right]
$$

$$
= \frac{(A + B)(A + B - B \theta) - \epsilon \lambda^r B (A + B - B \theta) + \epsilon A B \lambda^r \theta}{(A + B)(A + B - B \theta)}
$$

$$
= \frac{(A + B)(A + B - B \theta) - \epsilon \lambda^r B (A + B) + \epsilon \lambda^r B^2 \theta + \epsilon \lambda^r A B \theta}{(A + B)(A + B - B \theta)}
$$

$$
= \frac{(A + B)(A + B - B \theta) - (A + B)(\epsilon \lambda^r B) + (A + B) \epsilon \lambda^r B \theta}{(A + B)(A + B - B \theta)}
$$

$$
= \frac{A + B(1 - \theta - \epsilon \lambda^r + \epsilon \lambda^r \theta)}{A + B(1 - \theta)}
$$

$$
= \frac{A + B(1 - \theta)(1 - \epsilon \lambda^r)}{A + B(1 - \theta)}
$$

$$
= \frac{A + BC(1 - \theta)}{A + B(1 - \theta)}.
$$

Proof of equations (12) and (13)

$$
M_1 = \sum_{k=0}^{\infty} k p_k
$$

$$
= \sum_{k=0}^{\infty} k \epsilon \lambda^r + \kappa \tau (1 - \lambda^r) \beta^k \varphi^k P
$$

$$
= \sum_{k=0}^{\infty} k \epsilon \lambda^r (1 - \lambda^r) \frac{1}{1 - \beta \lambda^r (\frac{1}{\beta} - \varphi^P)}
$$

$$
= \frac{\epsilon \lambda^r (1 - \lambda^r)}{1 - \beta \lambda^r (\frac{1}{\beta} - \varphi^P)} \sum_{k=0}^{\infty} k \left[ \frac{\lambda^r \beta \varphi^p}{1 - \beta \lambda^r (\frac{1}{\beta} - \varphi^P)} \right]^k.
$$

Using the sum of power series, that is $\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}$ to simplify the terms not involving the summation sign, we obtain

$$
M_1 = \frac{\epsilon \lambda^r (1 - \lambda^r)}{1 - \lambda^r (1 - \beta \varphi^P)} \left[ \frac{\lambda^r \beta \varphi^p}{1 - \lambda^r (1 - \beta \varphi^P)} \right] \left[ \frac{1}{\lambda^r (1 - \beta \varphi^P)^2} \right]
$$

$$
= \frac{\epsilon \lambda^r (1 - \lambda^r)}{1 - \lambda^r (1 - \beta \varphi^P)} \left[ \frac{\lambda^r \beta \varphi^p}{\lambda^r (1 - \beta \varphi^P)^2} \right]
$$

$$
= \frac{\epsilon \lambda^r (1 - \lambda^r)}{1 - \lambda^r (1 - \beta \varphi^P)} \left[ \frac{\lambda^r \beta \varphi^p (1 - \lambda^r (1 - \beta \varphi^P))}{\lambda^r (1 - \beta \varphi^P)^2} \right]
$$

$$
= \frac{\epsilon \lambda^r (1 - \lambda^r) \beta \varphi^p}{(1 - \lambda^r)^2}
$$

$$
= \frac{\epsilon \lambda^r (1 - \lambda^r) \beta \varphi^p}{(1 - \lambda^r)^2}.
$$
\[ M_2 = \sum_{k=0}^{\infty} k^2 p_k \]
\[ = \sum_{k=0}^{\infty} k^2 \epsilon \lambda^\nu (1 - \lambda^\tau) \beta^k \varphi^k p^k \]
\[ = \frac{\epsilon \lambda^\nu (1 - \lambda^\tau)}{1 - \lambda^\tau (1 - \beta \varphi^p)} \sum_{k=0}^{\infty} k^2 \left[ \frac{\lambda^\tau \beta \varphi^p}{1 - \lambda^\tau (1 - \beta \varphi^p)} \right]^k. \]

Using the sum of power series, that is \[ \sum_{n=0}^{\infty} n^2 x^n = \frac{x + x^2}{(1-x)^3} \] to simplify the terms not involving the summation sign, we have

\[ M_2 = \frac{\epsilon \lambda^\nu (1 - \lambda^\tau)}{1 - \lambda^\tau (1 - \beta \varphi^p)} \left[ \frac{\lambda^\tau \beta \varphi^p (1 - \lambda^\tau (1 - \beta \varphi^p)) + (\lambda^\tau \beta \varphi^p)^2}{(1 - \lambda^\tau (1 - \beta \varphi^p))^2} \right] \]
[\[=\frac{\epsilon \lambda^\nu (1 - \lambda^\tau)}{1 - \lambda^\tau (1 - \beta \varphi^p)} \left[ \frac{\lambda^\tau \beta \varphi^p (1 - \lambda^\tau (1 - \beta \varphi^p))}{(1 - \lambda^\tau (1 - \beta \varphi^p))^3} \right] \]
[\[=\frac{\epsilon \lambda^\nu (1 - \lambda^\tau)}{1 - \lambda^\tau (1 - \beta \varphi^p)} \left[ \frac{\lambda^\tau \beta \varphi^p (1 - \lambda^\tau (1 - 2\beta \lambda \varphi^p))}{(1 - \lambda^\tau)^3} \right] \]
[\[=\epsilon \lambda^\nu + \epsilon \lambda^\nu (1 - \lambda^\tau (1 - 2\beta \lambda \varphi^p)) \right]. \]