THE FRAMEWORK, CAUSAL AND CO-COMPACT STRUCTURE OF SPACE-TIME

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Abstract. We introduce a canonical, compact topology, which we call weakly causal, naturally generated by the causal site of J. D. Christensen and L. Crane, a pointless algebraic structure motivated by certain problems of quantum gravity. We show that for every four-dimensional globally hyperbolic Lorentzian manifold there exists an associated causal site, whose weakly causal topology is co-compact with respect to the manifold topology and vice versa. Thus, the causal site has the full information about the topology of space-time, represented by the Lorentzian manifold. In addition, we show that there exist also non-Lorentzian causal sites (whose causal relation is not a continuous poset) and so the weakly causal topology and its de Groot dual extends the usual manifold topology of space-time beyond topologies generated by the traditional, smooth model. As a source of inspiration in topologizing the studied causal structures, we use some methods and constructions of general topology and formal concept analysis.

1. Introduction

The belief that the causal structure of space-time is one of its most fundamental underlying structures is almost as old as the idea of the relativistic space-time itself, but how is it related to the topology of space-time? By tradition, there is no doubt regarding the topology of space-time at least locally. It is usually considered to be locally homeomorphic with the Cartesian power of the real line, equipped with the Euclidean topology. More recently, however, there appeared ideas of discrete and pointless models of space-time in which the causal structure is introduced axiomatically and so independently on the locally Euclidean models. Is, in these cases, the axiomatic causal structure rich enough to carry also the full topological information? In addition, after all, how the topology that we perceive around us and which is essentially and implicitly at the background of many physical phenomena, may arise? The perceived topology of the universe belongs to our reality similarly as light, matter or gravitation, but the process of its generation, certainly a fascinating, uninvestigated phenomenon, unfortunately has been ignored for a long time.

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The previous observation about $\mathbb{R}$ can be easily extended to any locally compact Hausdorff topological space $(X, \tau)$, covering the most of the usual relativistic models of space-time. If the original topology $\tau$ is compact Hausdorff, then both topologies coincide, so we can restrict our further consideration to the non-compact case only. Then its counterpart -- the dual topology $\tau^G$, is compact, $T_1$, superconnected, and what is also important, it is strictly weaker than $\tau$ but it coincides with $\tau$ on every compact subset -- physically speaking, at finite distances. Explaining the more complicated objects and phenomena by the simpler ones, is a widely accepted and fundamental principle of science. In this sense, the co-compact topology $\tau^G$ may be regarded as more fundamental for the natural world than the Euclidean topology of Minkowski space or than the usual topology $\tau$ of Lorentzian manifolds, since although it is simpler, it still contains the full information about the usual topology in its compact subsets.

Since for a locally compact Hausdorff space we have $\tau = \tau^GG$, the preference of $\tau^G$ over $\tau$ seems to be rather a matter of the topologist’s taste, or a modification of the well-known dilemma “which came first -- chicken or the egg”. However, there is an indication that the compact $T_1$ topologies could be more fundamental for explanation of the physical phenomenon of “generating the topology” than the locally compact Hausdorff topologies. The reason lies in the properties of the structure called a causal site, introduced by J. D. Christensen and L. Crane in [2]. Causal site is a very general, pointless algebraic structure, capturing the behavior of the light cones in relativity, motivated by certain problems and developments in quantum gravity. It is a natural generalization of R. Sorkin’s causal sets, based on a modification of the Grothendieck topology [1]. In the next part of the paper we will find out that every causal site generates, in a very natural way, a compact $T_1$ topology (and, in a contrast, not a non-compact locally compact Hausdorff topology).

However, we will show even more than that -- the causal site can be selected in such a way that the generated topology will be the co-compact topology of the topology of any four-dimensional globally hyperbolic Lorentzian manifold. Considering this construction, it should be noted that usually it is not a problem to choose certain regions of space-time as the topological subbase for closed or open sets and reconstruct from them the de Groot dual of the original topology or even the original topology itself, as well as the underlying set of points. The difficult part consists in selecting appropriately these regions (with the corresponding binary relations) and proving, that they satisfy all the axioms of a compatible causal site.

Nevertheless, the main purpose and sense of the algebraic structure of causal sites do not consist in another, perhaps modified formulation of general relativity. The structure was designed in order to formalize some alternative, very general (and perhaps hypothetical) models of space-time, very different from the models that one can meet in classical general relativity.
Our construction is then a contribution in a topologization of these models of the generalized space-time, lying beyond of the traditional, smooth model.

As a tool for topologization of various models of reality we will introduce a general construction, suitable for equipping a set of objects with a topology-like structure, using the inner, natural and intuitive relationships between them.

2. Topological Prerequisites

Throughout this paper, we mostly use the usual terminology of general topology, for which the reader is referred to [4] or [8], with one exception – in a consensus with a modern approach to general topology, we no longer assume the Hausdorff separation axiom as a part of the definition of compactness. This is especially affected by some recent motivations from computer science, but also the contents of the paper [14] confirms that such a modification of the definition of compactness is a relevant idea. Thus, we say that a topological space is compact, if every open cover of the space has a finite subcover, or equivalently, if every centered system of closed sets or a closed filter base has a non-empty intersection. Note that by the well-known Alexander’s subbase lemma, the general closed sets may be replaced by more special elements of any closed subbase for the topology.

We have already mentioned the co-compact or the de Groot dual topology, which was first systematically studied probably at the end of the 60’s by J. de Groot and his coworkers, J. M. Aarts, H. Herrlich, G. E. Strecker and E. Wattel. The initial paper is [11]. About 20 years later, motivated by research in domain theory, the co-compact topology again came to the center of interest of some topologists and theoretical computer scientists. During discussions in the community the original definition due to de Groot was slightly changed to its current form, inserting a word “saturated” to the original definition (a set is saturated, if it is an intersection of open sets, so in a $T_1$ space, all sets are saturated). Let $(X, \tau)$ be a topological space. The topology, generated by the family of all compact saturated sets, used as the base for the closed sets, we denote by $\tau^G$ and call it co-compact or de Groot dual with respect to the original topology $\tau$. In [19] J. Lawson and M. Mislove stated question, whether the sequence, containing the iterated duals of the original topology, is infinite or the process of taking duals terminates after finitely many steps with topologies that are dual to each other. In 2001 the first author solved the question and proved that only four topologies may arise (see [16]).

The following theorem summarizes the previously mentioned facts important for understanding the main results, contained in Section 5. The theorem itself is not new, under slightly different terminology the reader can essentially find it in [11]. A more general result, equivalently characterizing the topologies satisfying $\tau = \tau^{GG}$, the reader may find in the second author’s
Theorem 2.1. Let \((X, \tau)\) be a non-compact, locally compact Hausdorff topological space. Then

(i) \(\tau^G \subseteq \tau\),
(ii) \(\tau = \tau^{GG}\),
(iii) \((X, \tau^G)\) is compact and superconnected,
(iv) the topologies induced from \(\tau\) and \(\tau^G\) coincide on every compact subset of \((X, \tau)\).

Proof. The topology \(\tau^G\) has a closed base which consists of compact sets. Since in a Hausdorff space all compact sets are closed, we have (i).

Let \(C \subseteq X\) be a closed set with respect to \(\tau\), to show that \(C\) is compact with respect to \(\tau^G\), let us take a non-empty family \(\Phi\) of compact subsets of \((X, \tau)\), such that the family \(\{C\} \cup \Phi\) has f.i.p. Take some \(K \in \Phi\). Then the family \(\{C \cap K\} \cup \{C \cap F\} \mid F \in \Phi\) also has f.i.p. in a compact set \(K\), so it has a non-empty intersection. Hence, also the intersection of \(\{C\} \cup \Phi\) is non-empty, which means that \(C\) is compact with respect to \(\tau^G\). Consequently, \(C\) is closed in \((X, \tau^{GG})\), which means that \(\tau \subseteq \tau^{GG}\). The topology \(\tau^{GG}\) has a closed base consisting of sets which are compact in \((X, \tau^G)\). Take such a set, say \(H \subseteq X\). Let \(x \in X \setminus H\). Since \((X, \tau)\) is locally compact and Hausdorff, for every \(y \in H\) there exist \(U_y, V_y \in \tau\) such that \(x \in U_y, y \in V_y\) and \(U \cap V = \emptyset\), with \(\text{cl} U_y\) compact. Denote \(W_y = X \setminus \text{cl} U_y\). We have \(y \in V_y \subseteq W_y\), so the sets \(W_y, y \in H\) cover \(H\). The complement of \(W_y\) is compact with respect to \(\tau\), so \(W_y \in \tau^G\). The family \(\{W_y \mid y \in H\}\) is an open cover of the compact set \(H\) in \((X, \tau^G)\), so it has a finite subcover, say \(\{W_{y_1}, W_{y_2}, \ldots, W_{y_k}\}\). Denote \(U = \bigcap_{i=1}^k U_{x_i}\). Then \(U \cap H = \emptyset\), \(x \in U \subseteq X \setminus H\), which means that \(X \setminus H \in \tau\) and \(H\) is closed in \((X, \tau)\). Hence, \(\tau^{GG} \subseteq \tau\), an together with the previously proved converse inclusion, it gives (ii).

Let us show (iii). Take any collection \(\Psi\) of compact subsets of \((X, \tau)\) having f.i.p. They are both compact and closed in \((X, \tau)\), so \(\bigcap \Psi \neq \emptyset\). Then \((X, \tau^G)\) is compact. Let \(U, V \in \tau^G\) and suppose that \(U \cap V = \emptyset\). The complements of \(U, V\) are compact in \((X, \tau)\) as intersections of compact closed sets in a Hausdorff space. Then \((X, \tau)\) is compact as a union of two compact sets, which is not possible. Hence, it holds (iii).

Finally, take a compact subset \(K\) and a closed subset \(C\) of \((X, \tau)\). Then \(K \cap C\) is compact in \((X, \tau)\) and hence closed in \((X, \tau^G)\). Thus, the topology on \(K\) induced from \(\tau^G\) is finer than the topology induced from \(\tau\). Together with (i), we get (iv).
3. CAUSAL SITES AND THEIR PROPERTIES

In this section we will study the axiomatized causality relationships. Philosophically, the section is inspired by the work of R. Sorkin and his notion of causal set, [27]. However, we will use a slightly different notion, due to J. D. Christensen and L. Crane, motivated by certain problems in quantum gravity.

Recall that a causal site \((\mathcal{S}, \sqsubseteq, \prec)\) defined by J. D. Christensen and L. Crane in [2] is a set \(\mathcal{S}\) of regions equipped with two binary relations \(\sqsubseteq, \prec\), where \((\mathcal{S}, \sqsubseteq)\) is a partial order having the binary suprema \(\sqcup\) and the least element \(\bot \in \mathcal{S}\), and \((\mathcal{S} \setminus \{\bot\}, \prec)\) is a strict partial order (i.e., anti-reflexive and transitive), linked together by the following axioms, which are satisfied for all regions \(a, b, c \in \mathcal{S}\):

(i) \(b \sqsubseteq a\) and \(a \prec c\) implies \(b \prec c\),
(ii) \(b \sqsubseteq a\) and \(c \prec a\) implies \(c \prec b\),
(iii) \(a \prec c\) and \(b \prec c\) implies \(a \sqcup b \prec c\).

(iv) There exists \(b_a \in \mathcal{S}\), called cutting of \(a\) by \(b\), such that
(1) \(b_a \prec a\) and \(b_a \sqsubseteq b\);
(2) if \(c \in \mathcal{S}\), \(c \prec a\) and \(c \sqsubseteq b_a\) then \(c \sqsubseteq b_a\).

Lemma 3.1. Let \((\mathcal{S}, \sqsubseteq, \prec)\) be a causal site with \(L \subseteq \mathcal{S} \setminus \{\bot\}\) linearly ordered by the relation \(\sqsubseteq\), \(|L| \geq 2\). Then \(L\) is an anti-chain with respect to \(\prec\).

Proof. Let \(a, b \in L \setminus \{\bot\}, a \neq b\). Without loss of generality, we may choose the denotation of the elements \(a, b\) such that \(b \sqsubseteq a\). Suppose for a moment, that \(a \prec b\). Then by the axiom (i) it should be \(b \prec b\). Similarly, if \(b \prec a\), by the axiom (ii) we get again that \(b \prec b\). But this is impossible, since \(\prec\) is anti-reflexive on \(\mathcal{S} \setminus \{\bot\}\) by definition. Hence, the subset \(L \subseteq \mathcal{S} \setminus \{\bot\}\) is an anti-chain with respect to \(\prec\). □

Lemma 3.2. Let \((\mathcal{S}, \sqsubseteq, \prec)\) be a causal site. Then \(\bot\) is the least element of \(\mathcal{S}\) with respect to \(\prec\), and \(\bot \prec \bot\).

Proof. Let \(a \in \mathcal{S}\) be an arbitrary element. By the axiom (iv), there exists \(a_a \in \mathcal{S}\) such that \(a_a \prec a\) and \(a_a \sqsubseteq a\). By Lemma 3.1, this is not possible if \(a_a \in \mathcal{S} \setminus \{\bot\}\) since then \(L = \{a_a, a\}\) would be a two-element linearly ordered subset of \(\mathcal{S} \setminus \{\bot\}\). Hence \(a_a = \bot\) and so \(\bot \prec \bot\). □

Corollary 3.1. Let \((\mathcal{S}, \sqsubseteq)\) be a linearly ordered set with the least element \(\bot\). Then there exists a unique binary relation \(\prec\) on \(\mathcal{S}\), such that \((\mathcal{S}, \sqsubseteq, \prec)\) is a causal site. In this relation, \(\mathcal{S} \setminus \{\bot\}\) is an anti-chain and \(\bot\) is the least element of \((\mathcal{S}, \prec)\).

It is not difficult to show, that for every inclusion relation \(\sqsubseteq\), the corresponding causal relation \(\prec\) always exists. One can simply define \(\prec\) on \(\mathcal{S} \setminus \{\bot\}\) as anti-chain and take \(\bot\) as the least element of \((\mathcal{S}, \prec)\) similarly as in the previous example. It follows from Lemma 3.2 that such a causal
Corollary 3.2. Let \((\mathcal{S}, <)\) be a strictly linearly ordered set, \(|\mathcal{S}| \geq 3\). Then there is no binary relation \(\sqsubseteq\) such that \((\mathcal{S}, \sqsubseteq, <)\) is a causal site.

Proof. Suppose conversely, that \((\mathcal{S}, \sqsubseteq, <)\) is a causal site with \(<\) linear, \(|\mathcal{S}| \geq 3\). There exist \(b, c \in \mathcal{S} \setminus \{\bot\}\), \(b \neq c\). Since \((\mathcal{S}, \sqsubseteq)\) has binary suprema by the definition of a causal site, there exist \(a = b \sqcup c \in \mathcal{S} \setminus \{\bot\}\). Clearly, \(b, c \sqsubseteq a\). Since \(b \neq c\), at least one of the elements \(b, c\) differs from \(a\). Let \(L\) be the two-element set containing \(a\) and that element from \(\{b, c\}\), different from \(a\). Then \(L\) is a chain with respect to \(\sqsubseteq\) as well as \(<\), which is not possible by Lemma 3.1. □

Theorem 3.1. Let \((\mathcal{S}, \sqsubseteq, <)\) be a causal site and let \(N(x) = \{y | y \in \mathcal{S}, y \notin x \prec y\}\). Then for every distinct \(x_1, x_2, \ldots, x_k \in \mathcal{S} \setminus \{\bot\}\), where \(k \geq 2\),

\[
\bigcap_{i=1}^{k} N(x_i) \neq \emptyset.
\]

Proof. Let \(x_1, x_2, \ldots, x_k \in \mathcal{S}\) and denote \(s = x_1 \sqcup x_2 \sqcup \cdots \sqcup x_k\). It is clear that \(x_i \subseteq s\) for every \(i = 1, 2, \ldots, k\). Suppose, for some \(i = 1, 2, \ldots, k\), that it holds \(x_i < s\) or \(s < x_i\). Because of anti-reflexivity of the relation \(<\) on \(\mathcal{S} \setminus \{\bot\}\), the equality \(x_i = s\) is possible only for \(s = \bot\), which immediately gives that \(x_1 = x_2 = \cdots = x_k = \bot\). However, this contradicts to the assumptions of the theorem. Hence, \(x_i \neq s\). Then \(L = \{x_i, s\} \subseteq \mathcal{S} \setminus \{\bot\}\), \(|L| = 2\) is linearly ordered by \(\sqsubseteq\), so by Lemma 3.1 \(L\) is an anti-chain with respect to \(<\). But this is a contradiction to our assumption that \(x_i < s\) or \(s < x_i\). Hence, \(s \in \bigcap_{i=1}^{k} N(x_i) \neq \emptyset\). □

The previous theorem gives a number of examples illustrating, how a causal site does not look like.

Corollary 3.3. Let \(M \subseteq \mathcal{S} \setminus \{\bot\}\) be a finite set such that for each \(x \in \mathcal{S}\), there exists some \(m \in M\) with \(x < m\) or \(m < x\). Then there is no relation \(\sqsubseteq\) such that \((\mathcal{S}, \sqsubseteq, <)\) is a causal site.

Because of anti-reflexivity of \(<\), there may exist finite causal sites in spite of the corollary (Corollary 3.1 yields such a construction). However, the limitation is very strong, as we will see later.

If not specified otherwise, let \(\preceq\) be the denotation of the reflexive closure of the binary relation \(<\). Recall that an atom in a poset with the least element \(\bot\) is a minimal element among all elements different from \(\bot\) (that is, atoms are the immediate successors of the least element). Then, for instance, \((\mathcal{S}, \preceq)\) cannot be a finite lattice for \(|\mathcal{S}| > 2\) or even any other poset with the top element, which is not an atom. It also cannot be an infinite ‘ladder’ whose Hasse diagram is on the Figure 1, since one can let,
for instance, $M = \{2, 3\}$. There are also many other possibilities, which we leave to the reader as an easy exercise.

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
5 & 4 & 3 & 2 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\bot & \bot & \bot & \bot
\end{array}
\]

Figure 1.

The interesting and very complex problem of characterization of those posets $(\mathcal{S}, \preceq)$ for which the causal site $(\mathcal{S}, \sqsubseteq, \prec)$ exists for some appropriate order relation $\sqsubseteq$ (and, perhaps, have also some physical counterpart in the reality) exceeds the aim of this paper. However, as we will see later, for every four-dimensional globally hyperbolic Lorentzian manifold there exists an appropriate causal site generated by its causal structure. So it is natural to ask whether there exist also some ‘non-Lorentzian’ causal sites – those, which cannot be properly associated with a Lorentzian manifold or its subset. In the following proposition we will describe a construction which yields such examples. For any set $S$, by $S^F$ we denote the family of all finite subsets of $S$.

**Proposition 3.1.** Let $(P, \leq)$ be any poset. For every $K, L \in P^F$ we put

\[K \prec L \text{ if and only if } k < l \text{ for every } (k, l) \in K \times L.\]

Then $(P^F, \subseteq, \prec)$ is a causal site.

**Proof.** It is obvious that the conditions (i) – (iii) from the definition of the causal site are satisfied. We will verify (iv). Let $A, B \in P^F$. We put

\[B_A = \{b | b \in B, b < a \text{ for every } a \in A\}.\]

It is clear that $B_A \prec A$, $B_A \subseteq B$. Take any $C \in P^F$, such that $C \prec A$, $C \subseteq B$. Let $c \in C$. Then $c \in B$, $c < a$ for all $a \in A$, so $C \subseteq B_A$. \(\square\)

Varying the poset $(P, \leq)$, one can generate causal sites which cannot be Lorentzian. The easiest way it is to take $P$ with cardinality greater than it can have any connected Lorentzian manifold. We will construct another, a more suitable example. At first, let us recall some notions. Let $(P, \leq)$ be a poset. We say that $x$ is *way below* $y$, and write $x \ll y$ for $x, y \in P$, if for every directed set $D \subseteq P$ with a supremum $\sup D \in P$, the relation $y \leq \sup D$
implies that there exists some $d \in D$ with $x \leq d$. We say that the poset $(P, \leq)$ is continuous, if for every $x \in P$, the set $\downarrow \{x\} = \{t \in P, t \ll x\}$ is directed and $x = \text{sup} \downarrow \{x\}$. For more detail, the reader is referred to [12]. A continuous poset $(P, \leq)$ is called bicontinuous, if it is continuous also with respect to the inverse order relation $\geq$, and the corresponding way-below relations are inverse to each other. For a more precise definition, the reader is referred to [20].

**Corollary 3.4.** Let $(P, \leq)$ be a poset with the Hasse diagram as on the Figure 2, in which two infinitely countable chains $\{1, 3, \ldots\}$ and $\{2, 4, \ldots\}$ have their common supremum $\omega$ and the infimum $\bot$. Then $(P^F, \subseteq, \prec)$ is a causal site in which $(P^F, \preceq)$ is not a continuous poset.

![Figure 2](image)

**Proof.** Let $(P, \leq)$ be a poset defined by the Hasse diagram on the Figure 2. Proposition 3.1 ensures that $(P^F, \subseteq, \prec)$ is a causal site. It remains to show that $(P^F, \preceq)$ is not a continuous poset. Indeed, $\{\omega\}$ is an upper bound of the chains $L_1 = \{\{1\}, \{3\}, \ldots\}$ and $L_2 = \{\{2\}, \{4\}, \ldots\}$. Let $D \in P^F$ be an an upper bound for $L_i$ for $i = 1$ or $i = 2$ respectively. Then for every $k = 1, 2, \ldots$ and $x \in D$ it follows $2k - 1 < x$ or $2k < x$, respectively. In the both cases, this is possible only for $x = \omega$, so $\text{sup} L_1 = \{\omega\} = \text{sup} L_2$.

Now, let $K \triangleleft \{\omega\}$ for some $K \in P^F$. By the definition of the way-below relation, there exist $m, n \in \mathbb{N}$ such that $K \preceq \{2m - 1\}$, $K \preceq \{2n\}$. Then $K$ cannot contain any element of $\mathbb{N}$, since otherwise it would be odd and even at the same time. Hence, $K = \{\bot\}$. Then $\{\omega\}$ is clearly not the supremum of $\downarrow \{\omega\}$, so the poset $(P^F, \preceq)$, as well as $(P, \leq)$, is not continuous. □

Corollary 3.4 shows, that there exist causal sites which are not covered by the theory of P. Panangaden and K. Martin, who studied causality via
domain theory (see [20] and [21]). In their approach, the causal relation is a bicontinuous poset and the corresponding counterparts of regions are the intervals with respect to the way-below relation. Since P. Panangaden and K. Martin proved that every globally hyperbolic Lorentzian manifold can be represented in this way, Corollary 3.4 also illustrates that there exist also ‘non-Lorentzian’ causal sites. On the other hand, there is no evidence that the ‘regions’ constructed from the way-below intervals satisfy the axioms of a causal site.

The previous corollaries illustrate that in a causal site the ‘inclusion’ relation \( \sqsubseteq \) and the ‘causal’ relation \( \prec \) highly depend one on each other. Once a causal site is given, its corresponding topology is fully determined by the ‘inclusion’ relation \( \sqsubseteq \), but as it is shown by Corollary 3.2 for a given causal relation \( \prec \), the relation \( \sqsubseteq \) cannot be arbitrary. In Corollary 3.2, the relation \( \prec \) is too strong to allow the existence of an appropriate inclusion relation \( \sqsubseteq \).

These facts yield a reply for a certain criticism, complaining that the topology derived from \( \sqsubseteq \) completely ignores causality and so it is unphysical. This is certainly not true, since for a given, existing causal site \((S, \sqsubseteq, \prec)\), the relations \( \sqsubseteq \) and \( \prec \) are not independent, and a properly chosen causal site has a real, physical sense. On the contrary, our approach leaves open the possibility that also the other, uninvestigated topologies, compatible with the causal relation, could be potentially useful.

**Example 3.1.** Consider a set \( Q \) of particles. An interaction between two particles \( p, q \in Q \) may be interpreted as an element \((p, q)\) of the Cartesian product \( Q \times Q \). But since it is natural to admit that various particles may interact several times, we rather represent their interaction as the triple \((p, q, n)\) \( \in Q \times Q \times \mathbb{N} \).

Suppose that for every particle \( p \in Q \) there exists a poset \((Q(p), \leq_p)\) such that \( Q(p) \subseteq Q \times Q \times \mathbb{N} \) and \( \pi_1(Q(p)) = p \), where \( \pi_1 : Q \times Q \times \mathbb{N} \to Q \) is the projection to the first component of the product \( Q \times Q \times \mathbb{N} \).

The poset \((Q(p), \leq_p)\) represents the chronological order of the interactions of the particle \( p \) with other particles from \( Q \). Note that \((Q(p), \leq_p)\) need not linear if one admits possible existence of alternative histories of particle interactions. Let \((P, \leq)\) be the disjoint sum of the posets \((Q(p), \leq_p), p \in Q\). By Proposition 3.1 \((P^F, \sqsubseteq, \prec)\) forms a causal site, which, in some cases, need not be Lorentzian. \(\square\)

4. **How to Topologize Everything**

As it has been recently noted in [14], nature or physical universe, whatever it is, has probably no existing, real points like in the classical Euclidean geometry (or, at least, we cannot be completely sure of that). Points, as
a useful mathematical abstraction, are infinitesimally small and thus cannot be measured or detected by any physical way. However, what we can be sure that really exists, there are various locations, containing concrete physical objects. In this paper we will call these locations places. Various places can overlap, they can be merged, embedded or glued together, so the theoretically understood virtual “observer” can visit multiple places simultaneously. For instance, the Galaxy, the Solar system, the Earth, (the territory of) Europe, Brno (a beautiful city in the Czech Republic, the place of the first author’s home), Mogilev (a charming and scenic town in Belarus, the second author’s home), the room in which the reader is present just now, is a simple and natural example of places conceived in our sense. Certainly, in this sense, one can be present at many of these places at the same time, and, also certainly, there exist pairs of places (e.g. Brno and Mogilev), where the simultaneous presence of any physical objects is not possible. Thus, the presence of various physical objects connects these primarily free objects – our places – to the certain structure, which we call a framework.

Note that it does not matter that the places are, at the first sight, determined rather vaguely or with some uncertainty. They are conceived as elements of some algebraic structure, without any additional geometrical or metric structure and as we will see later, the “uncertainty” could be partially eliminated by the relationships between them. Let’s now give the precise definition.

**Definition 4.1.** Let $P$ be a set, $\pi \subseteq 2^P$. We say that $(P, \pi)$ is a framework. The elements of $P$ we call places, the set $\pi$ we call framology.

Although every topological space is a framework by the definition, the elementary interpretation of a framework is very different from the usual interpretation of a topological space. The elements of the framology are not primarily considered as neighborhoods of places, although it seems to be also very natural. If $P$ contains all the places that are or can be observed, the framology $\pi$ contains the list of observations of the fact that the virtual “observer” or some physical object that “really exists” (whatever it means), can be present at some places simultaneously. The structure which $(P, \pi)$ represents arises from these observations.

Let us introduce some other useful notions.

**Definition 4.2.** Let $(P, \pi)$ and $(S, \sigma)$ be frameworks. A mapping $f : P \to S$ satisfying $f(\pi) \subseteq \sigma$ we call a framework morphism.

**Definition 4.3.** Let $(P, \pi)$ be a framework, $\sim$ an equivalence relation on $P$. Let $P_{\sim}$ be the set of all equivalence classes and $g : P \to P_{\sim}$ the corresponding quotient map. Then $(P_{\sim}, g(\pi))$ is called the quotient framework of $(P, \pi)$ (with respect to the equivalence $\sim$).

**Definition 4.4.** A framework $(P, \pi)$ is $T_0$ if for every $x, y \in P$, $x \neq y$, there exists $U \in \pi$ such that $x \in U$, $y \notin U$ or $x \notin U$, $y \in U$. 
Definition 4.5. Let \((P, \pi)\) be a framework. Denote \(P^d = \pi\) and \(\pi^d = \{\pi(x) \mid x \in P\}\), where \(\pi(x) = \{U \mid U \in \pi, x \in U\}\). Then \((P^d, \pi^d)\) is the dual framework of \((P, \pi)\). The places of the dual framework \((P^d, \pi^d)\) we call abstract points or simply points of the original framework \((P, \pi)\).

The framework duality is a simple but handy tool for switching between the classical point-set representation (like in topological spaces) and the point-less representation, introduced above.

Some Examples. There are a number of natural examples of mathematical structures satisfying the definition of a framework, including non-oriented graphs, topological spaces (with open maps as morphisms), measurable spaces or texture spaces of M. Diker [5]. Among physically motivated examples, we may mention the Feynman diagrams with particles in the role of places, and interactions as the associated abstract points. Very likely, certain aspects of the string theory, related to general topology, can also be formulated in terms of the framework theory.

It should be noted that the notion of a framework is a special case of the notion of the formal context, due to B. Ganter and R. Wille [9], sometimes also referred as the Chu space [3]. Recall that a formal context is a triple \((G, M, I)\), where \(G\) is a set of objects, \(M\) is a set of attributes and \(I \subseteq G \times M\) is a binary relation. Thus, the framework \((P, \pi)\) may be represented as a formal context \((P, \pi, \in)\), where objects are the places and their attributes are the abstract points.

Even though the theory and methods of formal concept analysis may be a useful tool also for our purposes, we prefer the topology-related terminology that we introduced in this section because it seems to be more close to the way, how mathematicians and physicists usually understand to the notion of space-time. Nevertheless, we completely share the opinion of R. Wille expressed in [28], that the abstract mathematical disciplines, like the lattice theory and other, should be returned more close to their potential users. We hope that this section will contribute to the same purpose regarding general topology.

It also seems that frameworks are closely related to the notion of partial metric due to S. Matthews [22], but these relationships will be studied in a separate paper.

Proposition 4.1. Let \((P, \pi)\) be a framework. Then \((P^d, \pi^d)\) is \(T_0\).

Proof. Denote \(S = \pi, \sigma = \{\pi(x) \mid x \in P\}\), so \((S, \sigma)\) is the dual framework of \((P, \pi)\). Let \(u, v \in S, u \neq v\). Since \(u, v \in 2^P\) are different sets, either there exists \(x \in u\) such that \(x \notin v\), or there exists \(x \in v\) such that \(x \notin u\). Then \(u \in \pi(x)\) and \(v \notin \pi(x)\), or \(v \in \pi(x)\) and \(u \notin \pi(x)\). In both cases there exists \(\pi(x) \in \sigma\), containing one element of \(\{u, v\}\) and not containing the other. \(\square\)
Theorem 4.1. Let \((P, \pi)\) be a framework. Then \((P^{dd}, \pi^{dd})\) is isomorphic to the quotient of \((P, \pi)\). Moreover, if \((P, \pi)\) is \(T_0\), then \((P^{dd}, \pi^{dd})\) and \((P, \pi)\) are isomorphic.

Proof. We denote \(R = P^d = \pi, \rho = \pi^d = \{\pi(x) \mid x \in P\}, S = R^d = \rho, \sigma = \rho^d = \{\rho(x) \mid x \in R\}\). Then \((S, \sigma)\) is the double dual of \((P, \pi)\). It remains to show, that \((S, \sigma)\) is isomorphic to some quotient of \((P, \pi)\).

For every \(x \in P\), we put \(f(x) = \pi(x)\). Then \(f : P \to S\) is a surjective mapping. It is easy to show, that \(f\) is a morphism. Indeed, if \(U \in \pi\), then \(f(U) = \{\pi(x) \mid x \in U\} = \{\pi(x) \mid x \in P, U \in \pi(x)\} = \{V \mid V \in \rho, U \in V\} = \rho(U) \in \sigma\). Therefore, \(f(\pi) \subseteq \sigma\), which means that \(f\) is an epimorphism of the framework \((P, \pi)\) onto \((S, \sigma)\).

Now, we define \(x \sim y\) for every \(x, y \in P\) if and only if \(f(x) = f(y)\). Then \(\sim\) is an equivalence relation on \(P\). For every equivalence class \([x] \in P_\sim\) we put \(h([x]) = f(x)\). The mapping \(h : P_\sim \to S\) is correctly defined, moreover, it is a bijection. The verification that \(h\) is a framework isomorphism is standard, but, because of completeness, it has its natural place here. The quotient framology on \(P_\sim\) is \(g(\pi)\), where \(g : P \to P_\sim\) is the quotient map. The quotient map \(g\) satisfies the condition \(h \circ g = f\). Let \(W \in g(\pi)\). There exists \(U \in \pi\) such that \(W = g(U)\). Then \(h(W) = h(g(U)) = f(U) \in \sigma\). Hence \(h(g(\pi)) \subseteq \sigma\), which means that \(h : P_\sim \to S\) is a framework morphism. Conversely, let \(W \in \sigma = \{\rho(U) \mid U \in \pi\}\). We will show that \(h^{-1}(W) \in g(\pi)\). By the previous paragraph, \(\rho(U) = f(U)\) for every \(U \in \pi\), so there exists \(U \in \pi\), such that \(W = f(U) = h(g(U))\). Since \(h\) is a bijection, it follows that \(h^{-1}(W) = g(U) \in g(\pi)\). Hence, also \(h^{-1} : S \to P_\sim\) is a framework morphism, so the frameworks \((P_\sim, g(\pi))\) and \((S, \sigma)\) are isomorphic.

Now let us consider the special case when \((P, \pi)\) is \(T_0\). Suppose that \(f(x) = f(y)\) for some \(x, y \in P\). Then \(\pi(x) = \pi(y)\), which is possible only when \(x = y\). Then the relation \(\sim\) is the diagonal relation, and the quotient mapping \(g\) is an isomorphism. \(\square\)

Corollary 4.1. Every framework arise as dual if and only if it is \(T_0\).

Corollary 4.2. For every framework \((P, \pi)\), it holds \((P^{dd}, \pi^{dd}) \cong (P^{ddd}, \pi^{ddd})\).

Note, that the framework structure is suitable for addressing the compatibility problem of various scales in physics and their different models. Since the points of the Universe probably do not exist in reality (although they are a useful mathematical abstraction), the abstract points of a framework only express certain relationships between places, which – in a contrast to points – can be really observed and which exclusively exist in the physical reality. Then various framologies and various topological models may peacefully coexist with help of the framework duality on a given set \(P\) of places.

Definition 4.6. Let \((P, \pi)\) be a framework, \((X, \tau)\) be a topological space with the family \(C\) of closed sets. We say that \((X, \tau)\) is an open (closed,
respectively) topological model for \((\mathcal{P}, \pi)\), if there exist a framework \((\mathcal{S}, \sigma)\) isomorphic to \((\mathcal{P}, \pi)\) and set \(X' \subseteq X\) such that \(\mathcal{S} \subseteq \tau\) \((\mathcal{S} \subseteq \mathcal{C}, \text{ respectively})\) and \(\sigma = \{\{A| A \in \mathcal{S}, x \in A\}| x \in X'\}\).

**Example 4.1.** Let \(p_i, i \in \mathbb{Z}\) be pairwise distinct elements. We put \(\mathcal{P} = \{p_i| i \in \mathbb{Z}\}, \pi = \{\{p_i, p_{i+1}\}| i \in \mathbb{Z}\}\). Then the framework \((\mathcal{P}, \pi)\) has many open as well as closed topological models, including the real line \(\mathbb{R}\) equipped with the Euclidean topology and the Khalimsky line, that is, the set \(\mathbb{Z}\) with the topology generated by its open base \(\tau_K = \{\ldots, \{1\}, \{1, 2, 3\}, \{3\}, \{3, 4, 5\}, \{5\}, \ldots \}\). For an easy proof, one can the places \(p_i\) identify with non-empty, open (or closed, respectively) overlapping sets in a topological space, such that \(p_i\) has a non-empty intersection only with \(p_{i-1}, p_i\) and \(p_{i+1}\). In addition, in case of Khalimsky topology, one can simulate various scales by taking \(p_i\) with more or less elements, although the original framework \((\mathcal{P}, \pi)\) still remains the same.

5. **Topologies from Causal Sites**

In this section we show that the notion of a framework, introduced and studied in the previous section, has some real utility and sense. In a contrast to simple examples mentioned above, from a properly defined framework we will be able to construct a topological structure with a real, physical meaning.

Consider a causal site \((\mathcal{P}, \sqsubseteq, \prec)\) and let us define appropriate framework structure on \(\mathcal{P}\). We say that a subset \(\mathcal{F} \subseteq \mathcal{P}\) set is centered, if for every \(x_1, x_2, \ldots, x_k \in \mathcal{F}\) there exists \(y \in \mathcal{P}\), \(y \neq \bot\) satisfying \(y \sqsubseteq x_i\) for every \(i = 1, 2, \ldots, k\). If \(\mathcal{L} \subseteq 2^\mathcal{P}\) is a chain of centered subsets of \(\mathcal{P}\) linearly ordered by the set inclusion \(\subseteq\), then \(\bigcup \mathcal{L}\) is also a centered set. Then every centered \(\mathcal{F} \subseteq \mathcal{P}\) is contained in some maximal centered \(\mathcal{M} \subseteq \mathcal{P}\). Let \(\pi\) be the family of all maximal centered subsets of \(\mathcal{P}\). Now, consider the framework \((\mathcal{P}, \pi)\) and its dual \((\mathcal{P}^d, \pi^d)\). Let \((X, \tau)\) be the topological space with \(X = P^d = \pi\) and the topology \(\tau\) generated by its closed subbase (that is, a subbase for the closed sets) \(\pi^d\).

**Theorem 5.1.** The topological space \((X, \tau)\), corresponding to the framework \((\mathcal{P}^d, \pi^d)\) and the causal site \((\mathcal{P}, \sqsubseteq, \prec)\), is compact \(T_1\).

**Proof.** By the well-known Alexander’s subbase lemma, for proving the compactness of \((X, \tau)\) it is sufficient to show, that any subfamily of \(\pi^d\) having the f.i.p., has nonempty intersection. The subbase for the closed sets of \((X, \tau)\) has the form \(\pi^d = \{\pi(x)| x \in \mathcal{P}\}\), so any subfamily of \(\pi^d\) can be indexed by a subset of \(\mathcal{P}\). Let \(\mathcal{F} \subseteq \mathcal{P}\) and suppose that for every \(x_1, x_2, \ldots, x_k \in \mathcal{F}\) we have 

\[
\pi(x_1) \cap \pi(x_2) \cap \cdots \cap \pi(x_k) \neq \emptyset.
\]
Then there exists $U \in \pi$ such that $U \in \pi(x_1) \cap \pi(x_2) \cap \cdots \cap \pi(x_k)$, so $x_i \in U$ for every $i = 1, 2, \ldots, k$. Since $U$ is a (maximal) centered family, there exists $\perp \neq y \in P$ such that $y \subseteq x_i$ for every $i = 1, 2, \ldots, k$. Thus, $F$ is a centered family, contained in some maximal centered family $M \subseteq P$. Then we have $M \in \pi$, so

$$M \in \bigcap_{x \in M} \pi(x) \subseteq \bigcap_{x \in F} \pi(x) \neq \emptyset.$$ 

Hence, $(X, \tau)$ is compact.

Let $U, V \in X = \pi$, $U \neq V$. Since both are maximal centered subfamilies of $P$, none of them can contain the other one. So, there exist $x, y \in P$ such that $x \in U \setminus V$ and $y \in V \setminus U$. Then $U \in \pi(x)$, $V \notin \pi(x)$, $V \in \pi(y)$, $U \notin \pi(y)$. Thus, $X \setminus \pi(x)$, $X \setminus \pi(y)$ are open sets in $(X, \tau)$ containing just one of the points $U, V$. So the topological space $(X, \tau)$ satisfies the $T_1$ axiom. \hfill \Box

Note that the topology, generated by a causal site by the way described in this section, we call the weakly causal topology.

6. Causal Sites from Lorentzian Manifolds

In this section we will construct a causal site from an arbitrary, four-dimensional Lorentzian manifold which does not admit of a closed non-spacelike curve. We will also show, that the weak topology, generated from this causal site, is the de Groot dual of the manifold topology.

For most notions and terminology used in this section we refer the reader to the book [13] by S. W. Hawking and G. F. R. Ellis. Throughout this section, if not specified otherwise, by $M$ we will denote a connected, four-dimensional, Hausdorff, $C^\infty$-differentiable manifold, equipped with a Lorentz metric $g : M \times M \to \mathbb{R}$, a metric tensor of signature $+2$ on $M$. In addition, we will suppose that $M$ satisfies the causality condition, that is, there are no closed non-spacelike curves in $M$, or in other words, that $M$ is globally hyperbolic.

The following two lemmas we will use as our starting point. They summarize some results which one can find in [13] (especially in Section 4.5).

**Lemma 6.1.** Let $U \subseteq M$ be a convex normal coordinate neighborhood. The following statements hold:

(i) Every $p \in U$ can be reached from some $q \in U$ by a timelike, future-oriented curve and from some $r \in U$ by a timelike, past-oriented curve.

(ii) If $p \in U$ can be reached from $q \in U$ by a non-spacelike curve but not by a timelike curve, then $p$ lies on a null geodesic from $q$.

For the proof, the reader is referred to [13].
Lemma 6.2. Let $p, q \in M$, $p \neq q$. Then the following statements are fulfilled:

(i) If $p$ and $q$ are joined by a non-spacelike curve $\gamma$, which is not a null geodesic, they can also be joined by a timelike curve, say $\zeta$.

(ii) Moreover, if $\gamma$ lies totally in a normal convex coordinate neighborhood $U$, then $\zeta$ also can be chosen in such a way that it totally lies in $U$.

Note that for the proof, the reader is referred to [13], where only (i) is explicitly stated. The proof technique used, however, allows showing also the part (ii).

Lemma 6.3. Let $V \subseteq M$ be convex. Then also $\text{cl} V$ is convex.

Proof. Suppose for a moment that there exists a geodesic $\zeta : [0, 1] \rightarrow M$ connecting the points $p_0, q_0 \in \text{cl} V$ which does not lie totally in $\text{cl} V$. The geodesics in $M$ depend differentiably and hence continuously on their initial and terminal points. Therefore, there exists a continuous function $f : [0, 1] \times U \times W$, where $U$, $W$ are certain open neighborhoods of the points $p_0$, $q_0$, such that $f(t, p_0, q_0) = \zeta(t)$, and $f(t, p, q)$ is the geodesic connecting the general points $p \in U$, $q \in W$, defined on the interval $[0, 1]$. It means that $f(0, p, q) = p$ and $f(1, p, q) = q$.

By our initial assumption, there exists $t_0 \in [0, 1]$ such that $f(t_0, p_0, q_0) = \zeta(t_0) \in M \setminus \text{cl} V$. From continuity of $f$ it follows that there exist open $U_0 \subseteq U$, $W_0 \subseteq W$ such that $p_0 \in U_0$, $q_0 \in W_0$ and for every $(p, q) \in U_0 \times W_0$ it holds $f(t_0, p, q) \in M \setminus \text{cl} V$. However, since $p_0, q_0 \in \text{cl} V$, there exist $p_1 \in U_0 \cap V \neq \emptyset$, $q_1 \in W_0 \cap V \neq \emptyset$. There is only one geodesic connecting the points $p_1, q_1 = f(t, p_1, q_1)$, and because of convexity of $V$, whole $f(t, p_1, q_1)$ is contained in $V$. This contradicts to our previous conclusion $f(t_0, p, q) \in M \setminus \text{cl} V$, following from the continuity of $f$. \hfill \Box

Lemma 6.4. Let $p \in M$ and let $\varphi : U \rightarrow \mathbb{R}^4$ be a normal coordinate system at $p$ with the coordinate functions $x^i$, $\varphi(p) = 0$. Then there exists arbitrarily small $\varepsilon > 0$ such that the open set $V_\varepsilon = \{ q \mid q \in U, \sum_{i=1}^4 (x^i(q))^2 < \varepsilon \}$ has the following property: If $\zeta : [0, 1] \rightarrow M$ is a geodesic emanating from $r \in V_\varepsilon$ with $\zeta(t_0) \in \text{fr} V_\varepsilon$, then for every open neighborhood $W$ of $\zeta(t_0)$ there exists some $t \in [0, t_0]$ such that $\zeta(t) \in V_\varepsilon \cap W$.

Proof. Let us choose $\varepsilon > 0$ sufficiently small such that $V_\varepsilon$ is convex and the symmetric $(0, 2)$-tensor having the components $\delta_{ij} - \Gamma^k \delta_i x^k$ (where $\Gamma^k_{ij}$ are the Christoffel symbols) is positively definite on $\text{cl} V_\varepsilon$. In fact, the second assumption is standardly used for the proof of convexity of $V_\varepsilon$ (for example, in [23]). However, we will slightly modify the standard technique for our purposes.

Suppose conversely that there exists an open neighborhood $W$ of $\zeta(t_0)$ such that $\zeta([0, t_0]) \cap V_\varepsilon \cap W = \emptyset$. Without loss of generality we may assume
that \( W \) is convex. Since \( \text{cl} V \) is convex by Lemma 6.3, \( \zeta([0, t_0]) \subseteq \text{cl} V \), which means that \( \zeta([0, t_0]) \cap W \subseteq \text{fr} V \). By continuity of \( \zeta \), \( \zeta^{-1}(W) \) is an open neighborhood of \( t_0 \) in the topological space \([0, 1]\), and so there exists some \( w \in \zeta^{-1}(W) \cap \{0, t_0\} \). Then \( \zeta(w) \in \zeta([0, t_0]) \cap W \) and since \( W \) is convex, it follows \( \zeta([w, t_0]) \subseteq \zeta([0, t_0]) \cap W \subseteq \text{fr} V \). Consider the function

\[
(6.1) \quad f(t) = \sum_{i=1}^{4} (x^i \circ \zeta)^2(t).
\]

For every \( t \in [w, t_0] \) it holds \( f(t) = \varepsilon^2 \). On the other hand,

\[
(6.2) \quad \frac{d^2 f}{dt^2} = 2 \sum_{i=1}^{4} \left( \left( \frac{d(x^i \circ \zeta)}{dt} \right)^2 + (x^i \circ \zeta) \cdot \frac{d^2(x^i \circ \zeta)}{dt^2} \right),
\]

from which using the geodesic equation for \( \zeta \)

\[
(6.3) \quad \frac{d^2(x^i \circ \zeta)}{dt^2} + \Gamma_{jk}^i \frac{d(x^j \circ \zeta)}{dt} \cdot \frac{d(x^k \circ \zeta)}{dt} = 0
\]

it is easy to get

\[
(6.4) \quad \frac{d^2 f}{dt^2} = 2 \left( \delta_{ij} - \Gamma_{ij}^k x^k \right) \frac{d(x^i \circ \zeta)}{dt} \cdot \frac{d(x^j \circ \zeta)}{dt} > 0,
\]

since the tensor \( \delta_{ij} - \Gamma_{ij}^k x^k \) is positively definite on \( \text{cl} V \). But this is a contradiction with the constant value \( \varepsilon^2 \) of the function \( f \) on the interval \([w, t_0]\).

\[\square\]

**Lemma 6.5.** For every \( p \in M \) there exists an open local base \( \sigma_p \subseteq \tau \) at \( p \), such that every \( V \in \sigma_p \) has the following properties:

(i) \( V \) is convex.

(ii) \( K = \text{cl} V \) is compact.

(iii) There exists a normal convex coordinate neighborhood \( U \) such that \( K \subseteq U \).

(iv) For every \( r \in \text{int} K \), \( K \cap J^-(r) \) is regular closed.

**Proof.** Let us choose a normal coordinate system \( \varphi : U \to \mathbb{R}^4 \) at \( p \). The set \( \varphi(U) \) is open in the Euclidean topology of \( \mathbb{R}^4 \), so \( \varphi(p) \) has an open neighborhood, say \( O \), such that \( \text{cl} O \subseteq \varphi(U) \) and \( \text{cl} O \) is compact. Hence, \( S = \varphi^{-1}(O) \) is open in \( M \), \( H = \varphi^{-1}(\text{cl} O) \) is compact and hence closed in the Hausdorff topological space \( M \), and \( S \subseteq H \). We put \( \sigma_p = \{ V_\varepsilon | \varepsilon > 0, V_\varepsilon \subseteq S \} \), where \( V_\varepsilon \) is the convex open neighborhood of \( p \) whose existence is ensured by Lemma 6.3. Since \( V_\varepsilon \subseteq S \subseteq H \subseteq U \), \( \text{cl} V_\varepsilon \subseteq H \) is compact and contained in \( U \). It is also clear that \( \sigma_p \) is a local base at \( p \), so the conditions (i), (ii) and (iii) are satisfied.

To show (iv), let us take some fixed \( V \in \sigma_p \) and denote \( K = \text{cl} V \). We will show that \( K \cap J^-(r) = \text{cl}(\text{int}(K \cap J^-(r))) \). Consider the set \( U \cap J^-(r) \). It is shown in [13] that the boundary of \( J^-(r) \) in \( U \) is formed by null geodesics, so \( U \cap J^-(r) \) is a closed topological subspace of \( U \). Hence, there exists a
closed set \( H \subseteq M \) such that \( U \cap J^-(r) = U \cap H \). Then, \( K \cap J^-(r) = K \cap U \cap J^-(r) = K \cap U \cap H = K \cap H \), which is a closed set in \( M \). Therefore, 
\[
\text{cl}(\text{int}(K \cap J^-(r))) \subseteq K \cap J^-(r).
\]

Conversely, take any \( x \in K \cap J^-(r) \) and select an open neighborhood, say \( W \), of \( x \). We will show that \( W \) meets \( \text{int}(K \cap J^-(r)) \). In any case, \( x \) can be an inner or a boundary point of \( J^-(r) \). If \( x \in \text{int} J^-(r) \), then \( W \cap \text{int} J^-(r) \) is also an open neighborhood of \( x \) which must meet \( V \) since \( x \in K = \text{cl} V \).

Then \( \emptyset \neq W \cap \text{int} J^-(r) \cap V \subseteq W \cap \text{int} J^-(r) \cap \text{int} K = W \cap (\text{int}(K \cap J^-(r))) \), so \( W \) meets \( \text{int}(K \cap J^-(r)) \). Now, suppose the other possibility, that \( x \in \text{fr} J^-(r) \). Then \( x \) lies on a null geodesic, say \( \zeta \), emanating from \( r \). From Lemma 6.4, it follows that there exists \( x' \in W \cap V \subseteq W \cap \text{int} K \), also lying on \( \zeta \). However, \( W \cap \text{int} K \) is an open neighborhood of \( x' \), which must meet \( \text{int} J^-(r) \) since \( x \) is a boundary point of \( J^-(r) \). Then, \( \emptyset \neq W \cap \text{int} J^-(r) \cap V \subseteq W \cap \text{int} (K \cap J^-(r)) \), so again \( W \) meets \( \text{int}(K \cap J^-(r)) \).

Hence \( x \in \text{cl}(\text{int}(K \cap J^-(r))) \). It follows \( K \cap J^-(r) \subseteq \text{cl}(\text{int}(K \cap J^-(r))) \), which completes the proof.

Let \( x, y \in M \). We put \( x \leq y \) if \( x = y \) or there exists a non-spacelike future-oriented curve \( \psi : [0,1] \to M \) with \( \psi(0) = x, \psi(1) = y \). It is clear that \( \leq \) is a reflexive and transitive relation, so it is a preorder on \( M \). The relation need not necessarily be antisymmetric.

**Definition 6.1.** For every \( p \in M \) we define \( J^-(p) = \{ x | x \in M, x \leq p \} \), \( J^+(p) = \{ x | x \in M, p \leq x \} \). For \( F, G \subseteq M \) we put
\[
F \cap G = \bigcap_{p \in F} J^+(p) \cap \bigcap_{q \in G} J^-(q).
\]

We say that the set \( F \cap G \) is a multi-diamond if the following conditions are satisfied:

(i) \( F, G \) are non-empty and finite.

(ii) \( F \cap G \) is compact.

(iii) There exists a normal convex coordinate neighborhood \( U_{F,G} \subseteq M \) and an open set \( V_{F,G} \) with \( K_{F,G} = \text{cl} V_{F,G} \subseteq U_{F,G} \) compact such that \( F \cup G \subseteq K_{F,G} \), \( F \cap G \subseteq \text{int} K_{F,G} \).

(iv) For every \( r \in \text{int} K_{F,G} \), the set \( K_{F,G} \cap J^-(r) \) is regular closed.

Let \( \mathcal{D} \) be a family of all such multi-diamonds \( F \cap G \) (with non-empty interior) and let \( \mathcal{P} = \mathcal{D}^c \) be the family of all finite unions of elements of \( \mathcal{D} \) (including the union of the empty collection, so we admit \( \emptyset \in \mathcal{P} \)) The family \( \mathcal{D} \) is closed under finite intersections and similarly, \( \mathcal{P} \) is closed under finite unions by its definition. Let \( A, B \in \mathcal{P} \). Then there exist multi-diamonds
$C_1, C_2, \ldots, C_n \in \mathcal{D}$ and $D_1, D_2, \ldots, D_m \in \mathcal{D}$ such that $A = \bigcup_{i=1}^{n} C_i$ and $B = \bigcup_{j=1}^{m} D_j$. Then

$$A \cap B = (\bigcup_{i=1}^{n} C_i) \cap (\bigcup_{j=1}^{m} D_j) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (C_i \cap D_j),$$

and since $C_i \cap D_j \in \mathcal{D}$, we also have $A \cap B \in \mathcal{P}$. Hence, $\mathcal{P}$ is also closed under finite intersections.

**Lemma 6.6.** Let $W \subseteq M$ be an open set. Then for every $p \in W$ there exists $A \in \mathcal{P}$ with $p \in \text{int} A$ and $A \subseteq W$.

**Proof.** Take an open set $W \subseteq M$ and $p \in W$. By Lemma 6.5, there exists a convex open neighborhood $V \subseteq W$ of $p$ such that $K = \text{cl} V$ is compact, contained in some normal convex coordinate neighborhood $U$ and such that $K \cap J^-(r)$ is a regular closed set for every $r \in \text{int} K$. It is also clear that there exist $u, v \in V$, $u \leq v$, with $p \in \text{int}(\{u\} \diamond \{v\}) \subseteq \{u\} \diamond \{v\} \subseteq V \subseteq W$. Furthermore, the set $A = \{u\} \diamond \{v\}$ is compact as a closed subspace of the compact set $K$. By Definition 6.1, $A$ is a multi-diamond, and so $A \in \mathcal{P}$. $\square$

**Corollary 6.1.** The family $\{\text{int} A | A \in \mathcal{P}\}$ is a base of the topology of $M$.

**Lemma 6.7.** The family $\mathcal{P}$ is a closed base for the co-compact topology on $M$.

**Proof.** The co-compact topology $\tau_M^G$ on $M$ is generated by its open base, which is formed by the complements of sets, compact in the manifold topology $\tau_M$. Let $K \subseteq M$ be compact. Denote $U = M \setminus K$. Take a point $x \in U$. Since $M$ is a $T_1$ space, $M \setminus \{x\}$ is an open set with respect to the manifold topology. By Lemma 6.6, for every $y \in K$, there exists $A_y \in \mathcal{P}$ such that $y \in \text{int} A_y \subseteq M \setminus \{x\}$, which also means that $x \notin A_y$.

Since $K$ is compact, there exist $y_1, y_2, \ldots, y_k \in K$ such that

$$K \subseteq \bigcup_{i=1}^{k} \text{int} A_y.$$

Then

$$x \in \bigcap_{i=1}^{k} (M \setminus A_{y_i}) = M \setminus \bigcup_{i=1}^{k} A_{y_i} \subseteq M \setminus K = U,$$

and the closed set $\bigcup_{i=1}^{k} A_{y_i}$ is an element of $\mathcal{P}$. Hence, every set $U$, which is open with respect to $\tau_M^G$, is a union of complements of elements of $\mathcal{P}$, which are closed in the same topology. Then $\mathcal{P}$ forms a closed base for $\tau_M^G$. $\square$

**Lemma 6.8.** Let $A = F \diamond G$ be a multi-diamond, $p \in M$, $A_p = (F \setminus \{p\}) \diamond G$. Let $K = K_{F,G}$ be the compact set introduced in Definition 6.1. Then for each $x \in O_p$ where

$$O_p = \{x | x \in K, A_p \cap J^+(x) = A\}$$

...
there exists a maximal element \( m \in O_p \) such that \( x \leq m \).

Proof. At first, let us show that the set \( O_p \) is nonempty. If \( p \in F \), then \( A_p \cap J^+(p) = A \) and so \( p \in O_p \). On the other hand, if \( p \notin F \), it follows \( A_p = A \), so the condition \( A_p \cap J^+(x) = A \) is equivalent to \( A \subseteq J^+(x) \). But then \( \emptyset \neq F \subseteq O_p \).

Now, let \( L \subseteq O_p \subseteq K \) be a non-empty linearly ordered chain with respect to \( \leq \). We will show that \( L \) has an upper bound in \( O_p \). Consider the net \( id_L(L, \leq) \). Since \( K \) is compact \( id_L(L, \leq) \) has a cluster point, say \( p_L \in K \). Suppose that there is some \( l \in L \) such that \( p_L \notin J^+(l) \). Since the set \( K \cap J^+(l) \) is closed, there exists an open neighborhood \( U \) of \( p_L \) with \( U \cap K \cap J^+(l) = \emptyset \). By the definition of the cluster point, there exists \( m \in L, l \leq m \), such that \( m \in U \). Then \( m \in U \cap K \cap J^+(m) \), but this is not possible since \( J^+(m) \subseteq J^+(l) \). Hence, \( p_L \in \bigcap_{l \in L} J^+(l) \), which means that \( p_L \) is an upper bound of \( L \) in \( K \).

It remains to show that \( p_L \in O_p \), which is equivalent to verify that \( A_p \cap J^+(p_L) = A \). Let \( l \in L \subseteq O_p \subseteq K \). Then \( A_p \cap J^+(l) = A \) and since \( l \leq p_L \), we have \( J^+(p_L) \subseteq J^+(l) \). We will show that \( A \subseteq J^+(p_L) \). Suppose conversely, that there exists some \( r \in A \setminus J^+(p_L) \). Since \( K \cap J^+(p_L) \) is closed (and \( M \) certainly is a metrizable topological space) there exists \( \varepsilon > 0 \) such that \( B_\varepsilon(r) \cap K \cap J^+(p_L) = \emptyset \). Since by Definition \( 6.1 \), \( A \subseteq \text{int} K \), without loss of generality we may select \( \varepsilon > 0 \) in such a way that \( B_\varepsilon(r) \subseteq K \).

Since \( K \) is contained in a normal convex coordinate neighborhood by Definition \( 6.1 \), for every \( x \in K \) there exist a unique geodesic \( \zeta_x \) emanating from \( x \) and terminating at \( r \) and a unique geodesic, say \( \eta_x \), connecting \( x \) with \( p_L \). Moreover, if \( x \in K \cap \text{int} J^-(r) \), the geodesic \( \zeta_x \) must be timelike and future-oriented.

Denote by \( f(x) \) the terminal point of the geodesic \( \zeta'_x \) emanating from \( p_L \) of the same length as \( \zeta_x \), with the corresponding tangent vector at \( p_L \) paralelly transported along \( \eta_x \) from \( x \). Then \( f(x) \) depends differentiably and hence continuously on \( x \), moreover, on the compact set \( K \) the mapping \( f(x) \) is uniformly continuous by the Heine-Cantor theorem. Hence, there exists some \( \delta > 0 \) such that if for some \( p \in K \), \( x \in B_\delta(p) \), then \( f(x) \in B_\delta(f(p)) \). Since \( p_L \) is a cluster point of the net \( id_L(L, \leq) \), there exists \( z \in B_\delta(p_L) \cap L \). Since \( z \in L \), it follows \( r \in A \subseteq J^+(z) \), which means that \( z \in J^-(r) \). Hence \( B_\delta(p_L) \cap K \cap J^{-}(r) \neq \emptyset \) and since \( K \cap J^{-}(r) \) is a regular closed set, there exists some \( x \in B_\delta(p_L) \cap \text{int}(K \cap J^{-}(r)) = B_\delta(p_L) \cap \text{int} K \cap \text{int} J^{-}(r) \). Now, \( x \) is in the domain of \( f \) and by its uniform continuity, \( f(x) \in B_\varepsilon(f(p_L)) \). From the construction of the function \( f \) and the uniqueness of the geodesics \( \zeta_x \) and \( \zeta'_x \) it follows \( f(p_L) = r \). Since \( f(x) \) is the terminal point of the non-spacelike future-oriented curve \( \zeta'_x \), emanating from \( p_L \), it follows that \( f(x) \in B_\varepsilon(r) \cap J^+(p_L) \subseteq K \), which is a contradiction. Hence, \( A \subseteq J^+(p_L) \). Then

\[
A = A \cap J^+(p_L) \subseteq A_p \cap J^+(l) = A.
\]
and so \( p_L \in O_p \) is the upper bound of the chain \( L \). Let \( M_p \) be the set of all maximal elements of \( O_p \) (with respect to \( \leq \)). By Zorn’s Lemma, for every \( x \in O_p \) there exists \( m \in M_A \) such that \( x \leq m \).

Since \( p_L \in O_p \), every element of \( F \) can be replaced by an element, which is maximal among those, yielding the same set \( F \cap G \).

**Lemma 6.9.** Let \( U \subseteq M \) be a normal convex neighborhood, \( p, q \in U \) points that are connected by a null geodesic. Then the following conditions hold:

(i) \( U \cap \text{int} J^−(p) \cap J^+(q) = \varnothing \),

(ii) \( U \cap J^−(p) \cap \text{int} J^+(q) = \varnothing \),

(iii) The set \( U \cap \text{fr} J^−(p) \cap \text{fr} J^+(q) \) is equal to the closed geodesical segment between the points \( p, q \).

*Proof.* (i) Suppose conversely that there is some \( z \in U \cap \text{int} J^−(p) \cap J^+(q) \). Then there exists a future-oriented, non-spacelike path connecting \( q \) with \( z \) and a future-oriented, timelike path connecting \( z \) with \( p \), so there exists a non-spacelike path, say \( \zeta \), from \( q \) to \( p \), which lies totally in \( U \). Since \( \zeta \) is not a null geodesic, \( p \notin \text{fr} J^+(q) \), so \( p \in \text{int} J^+(q) \). Hence, there exists a timelike geodesics joining the points \( p, q \). Then \( p, q \) are connected by two different geodesics, which is not possible in the normal convex neighborhood \( U \).

(ii) is analogous to (i).

(iii) Since \( U \) is geodesically convex, it is clear that the geodesical segment \( \zeta \), connecting \( p, q \) is totally contained in \( U \) and every its point is a boundary point of \( J^−(p), J^+(q) \). Conversely, let \( x \in U \cap \text{fr} J^−(p) \cap \text{fr} J^+(q) \). Then, there is a future-oriented null geodesic emanating from \( q \) to \( x \) and a future-oriented null geodesic connecting \( x \) with \( p \). Suppose that \( x \) does not lie on \( \zeta \). Then there exists a future-oriented, non-spacelike path, say \( \gamma \), from \( q \) to \( p \), totally lying in \( U \), which is not a null geodesic. By Lemma 6.2, \( \gamma \) can be varied to a timelike geodesic, connecting the points \( p, q \) in \( U \). Then \( p, q \) are connected by two different geodesics, which certainly is not possible.

*□*

**Lemma 6.10.** Let \( A = F \cap G \in D \) be a nonempty multi-diamond, \( K = K_{F,G} \) be the compact set introduced in Definition 6.7. Then there exists a finite set \( M_A \) containing all maximal elements of \( O_A = \{ x \mid x \in K, A \subseteq J^+(x) \} \).

*Proof.* Suppose \( A \) having the form \( A = F \cap G = \bigcap_{p \in F} J^+(p) \cap \bigcap_{q \in G} J^−(q) \), where \( F, G \) are proper finite subsets of \( M \). By Lemma 6.8 without loss of generality we may assume, that the set \( F \) contains only maximal elements among those, giving the same value for \( A = F \cap G \).

Let \( m \) be a maximal element of \( O_A \). The cone \( J^+(m) \) contains \( A \), which is a closed compact set under the assumptions stated in the lemma. Then also \( \text{fr} A \subseteq J^+(m) \). Suppose for a moment, that \( \text{fr} A \subseteq A \subseteq \text{int} J^+(m) \). Then for every \( x \in A \) there exists a timelike future-oriented curve \( \zeta_x : [0, 1] \to M \) with \( \zeta_x(0) = m, \zeta_x(1) = x \). Denote \( V_x = \text{int} J^+(\zeta_x(\frac{1}{2})) \). Then \( x \in V_x \),
so \( \Omega = \{ V_x \mid x \in A \} \) is an open cover of the compact set \( A \). Hence, \( \Omega \) has a finite subcover, say \( \{ V_{x_1}, V_{x_2}, \ldots, V_{x_k} \} \). The set \( \bigcap_{i=1}^{k} \text{int} J^{-}(\zeta_{x_i}(\frac{1}{2})) \) is an open neighborhood of \( m \), so by Lemma 6.1 there exists an open set \( U \subseteq M \) such that \( m \in U \subseteq \bigcap_{i=1}^{k} \text{int} J^{-}(\zeta_{x_i}(\frac{1}{2})) \) and some \( r \in U \) which can be reached from \( m \) by a timelike, future-oriented curve. Let \( x \in A \). There exists some \( j \in \{ 1, 2, \ldots, k \} \) such that \( x \in V_{x_j} = J^{+}(\zeta_{x_j}) \) and since also \( r \in J^{-}(\zeta_{x_j}) \), there exists a timelike future-oriented curve which joins \( r \) with \( x \). It means that \( A \subseteq J^{+}(r) \), which contradicts to the maximality of \( m \). Hence, \( \text{fr} J^{+}(m) \) meets \( \text{fr} A \). On the other hand, \( \text{fr} J^{+}(m) \) cannot meet the interior of \( A \) since the opposite leads to a nonempty intersection of \( \text{int} A \) with the complement of \( J^{+}(m) \), which then contradicts to \( A \subseteq J^{+}(m) \). The boundary of \( A \) can be decomposed into the union \( \text{fr} A = C_1 \cup C_2 \cup \cdots \cup C_n \), where each \( C_i \) is a closed compact (and smooth) piece of the boundary \( E_i \) of \( J^{+}(q) \) or \( J^{-}(q) \) for some \( q \in F \cup G \). Consider the set \( C_i \cap \text{fr} J^{+}(m) \) for some \( i = 1, 2, \ldots, n \), for which \( C_i \cap \text{fr} J^{+}(m) \) is nonempty. It can contain only finitely many points or, otherwise, it constitutes a submanifold of \( M \) of the dimension equal to 1, 2 or 3. We will distinguish several possibilities, and in each of them we will show that \( m \in F \cup G \).

(i) At first, consider the most simple case, that \( \dim(C_i \cap \text{fr} J^{+}(m)) = 3 \). Since both \( \text{fr} J^{+}(m) \) and \( C_i \) are submanifolds of \( M \) of dimension 3, on some non-empty open subset of \( M \) their corresponding equations must coincide, so the cones \( J^{+}(m) \), \( J^{+}(q) \) are the same. But then \( m = q \in F \cup G \).

Now, suppose that \( \dim(C_i \cap \text{fr} J^{+}(m)) < 3 \). Then the submanifolds \( \text{fr} J^{+}(m) \) and \( C_i \) can either touch or cut one another.

(ii) The first case means that \( \dim(C_i \cap \text{fr} J^{+}(m)) = 1 \) and the corresponding three-dimensional tangent vector spaces \( T_p C_i, T_p \text{fr} J^{+}(m) \) coincide at any point \( p \in C_i \cap \text{fr} J^{+}(m) \). Take some fixed \( p \in C_i \cap \text{fr} J^{+}(m) \), \( p \notin F \cup G \cup \{ m \} \), and some coordinate system \( \varphi = (x^1, x^2, x^3, x^4) \), where we also denote \( x^4 = t \), defined on some open coordinate neighborhood \( U_p \) of \( p \), such that \( \varphi(p) = (x^1_p, x^2_p, x^3_p, t_p) \).

Recall that by \( J^{+}(q) \), where \( q \in F \cup G \) is some proper element, we previously denoted the cone having the boundary \( E_i = \text{fr} J^{+}(q) \) containing \( C_i \). Let \( \sigma : M \times M \rightarrow \mathbb{R} \) be the Synge’s world function (see [24]), defined by

\[
(6.5) \quad \sigma(x, x') = \frac{1}{2}(\lambda_1 - \lambda_0) \int_{\lambda_0}^{\lambda_1} \left( g_{ij} \circ \beta \right)(\lambda) \frac{\partial(x^i \circ \beta)}{\partial \lambda} \frac{\partial(x^j \circ \beta)}{\partial \lambda} \, d\lambda,
\]

where the integral is evaluated on the geodesic \( \beta \) with the affine parameter \( \lambda \), linking \( x, x' \) such that \( \beta(\lambda_0) = x' \) and \( \beta(\lambda_1) = x \). It is not difficult to show that Synge’s world function vanishes on a geodesic if and only if it is a null geodesic, [24]. Then the equations of \( C_i, \text{fr} J^{+}(m) \) on \( U_p \) are

\[
(6.6) \quad \sigma(x, q) = 0, \quad \sigma(x, m) = 0.
\]
The fact that $T_p C_i = T_p \text{fr} J^+(m)$ means that the normal vectors to $T_p C_i$, $T_p \text{fr} J^+(m)$ coincide, so

$$\frac{\partial}{\partial x^i} \sigma(\varphi^{-1}(x^1(p), \ldots, x^4(p)), q) = \frac{\partial}{\partial x^i} \sigma(\varphi^{-1}(x^1(p), \ldots, x^4(p)), m).$$

(6.7)

However, the quantities $\frac{\partial}{\partial x^i} \sigma \left( \varphi^{-1}(x^1(p), \ldots, x^4(p)), q \right)$, apart from a linear factor, express also the covariant components of the tangent vector of the geodesics emanating from $\zeta \in \{q, m\}$ to $p$. Hence, (6.7) means that $p$, $q$, $m$ lie on the same geodesic, say $\zeta$. From $p \in C_i$ it follows that $p$ is connected with $q$ by a null geodesic, so by the standard theory of differential equations it coincides with $\zeta$, which hence is null. Therefore, $q \leq m$ or $m \leq q$.

(ii$_1$) Suppose that $E_i = \text{fr} J^-(q)$. Since $\emptyset \neq A \subseteq J^-(q) \cap J^+(m)$, it follows $m \leq q$. By the construction of $A$, there exists a normal convex neighborhood $U$, containing $q$, $m$ and $A$. Suppose that there exist some $x \in A \cap \text{int} J^-(q)$. Then $U \cap \text{int} J^-(q)$ is an open neighborhood of $x$. By (i) of Lemma 6.9 it is not possible that $x \in J^+(m)$, which is a contradiction with $A \subseteq J^+(m)$. Hence, $A \cap \text{int} J^-(q) = \emptyset$, which means that $A \subseteq \text{fr} J^-(q)$. By a similar way, from (ii) of Lemma 6.9 it follows, that $A \subseteq \text{fr} J^+(m)$, so $A \subseteq \text{fr} J^-(q) \cap \text{fr} J^+(m)$. By (iii) of Lemma 6.9 $A$ is a subset of the geodesical segment, connecting the points $q$, $m$. Since $A$ is closed, from maximality of $m$ it follows $m \in A$, which implies that $m$ is an upper bound of $F$. Then $J^+(m) \subseteq \bigcap_{b \in F} J^+(a)$, so

$$J^+(m) \cap \bigcap_{b \in G} J^-(b) \subseteq A = \bigcap_{a \in F} J^+(a) \cap \bigcap_{b \in G} J^-(b) \subseteq J^+(m) \cap \bigcap_{b \in G} J^-(b),$$

where the last inclusion follows from the fact that $A \subseteq J^+(m)$, and clearly $A \subseteq \bigcap_{b \in G} J^-(b)$. Hence, $A = J^+(m) \cap \bigcap_{b \in G} J^-(b)$. By the assumption stated in the first paragraph of this proof, $F = \{m\}$.

(ii$_2$) Now, suppose that $C_j \subseteq E_j = \text{fr} J^+(q)$. Consequently, it is not possible that $q \leq m$ and $q \neq m$ since $q$ would not be a maximal element among those giving the same value for the set $A = F \cap G$. Indeed, then $A \subseteq J^+(m) \subseteq J^+(q)$, which implies that $A = J^+(m) \cap \bigcap_{b \in G} J^-(b) \subseteq J^+(q) \cap A_q = A$, where we denoted $A_q = (F \setminus \{q\}) \cap G$ as in Lemma 6.8. Then $q$ could be replaced by the greater element $m$, which is a contradiction. Hence, it holds $m \leq q$. Then $A \subseteq J^+(q) \subseteq J^+(m)$ and from the assumption of maximality of $m$ it follows that $m = q \in F$.

(iii) The second case means that the submanifolds $\text{fr} J^+(m)$ and $C_i$ cut each other and $\text{dim}(C_i \cap \text{fr} J^+(m)) = 2$. Let $p \in C_i \cap \text{fr} J^+(m)$. The set $C_i \cap \text{fr} J^+(m)$ is infinite, so $p$ can be chosen in such a way that $p \notin F \cup G \cup \{m\}$. We again select a coordinate system $\varphi = (x^1, x^2, x^3, x^4)$, where we also denote $x^4 = t$, defined on some open coordinate neighborhood $U_p$ of $p$, such that $\varphi(p) = (x^1_p, x^2_p, x^3_p, t_p)$. There exists a smooth function of two variables $\chi(u, v)$ parameterizing $C_i \cap \text{fr} J^+(m)$ on some neighborhood of $p$ and restricting the domain of $\varphi$ if necessary we may assume that this
neighborhood is $U_p$. Further, without loss of generality we may assume that $\chi(u, v)$ is defined on an open set of the form $I \times I$, where $I$ is an open interval containing $0$, and $\chi(0, 0) = p$.

Since $\text{fr} J^+(m)$ does not meet $\text{int} A$, the non-empty intersection $C_i \cap \text{fr} J^+(m)$ must be separated from the open set $U_p \cap \text{int} A \subseteq \text{int} A$ by another boundary piece $C_j$ on some open neighborhood $V_p$ of $p$. Restricting the domains of $\varphi$ and $\chi$ if necessary we may ensure that for simplicity $V_p = U_p$ and $\chi(I \times I) \subseteq C_i \cap C_j \cap \text{fr} J^+(m)$. Recall that in our previous notation, $C_i, C_j$ are pieces of the boundaries $E_i, E_j$ of some cones $J^{\pm}(q), J^{\pm}(r)$, respectively, where $q, r \in F \cup G$. Hence, the points $m, q, r$ lie on null geodesics emanating from the same point $\chi(u, v)$.

We will analyze the situation in the tangent vector space $T_p M$ of $M$. Let $\sigma : M \times M \to \mathbb{R}$ be the Synge’s world function. Then

\begin{equation}
\sigma(\chi(u, v), z) = 0
\end{equation}

for every fixed $z \in \{m, q, r\}$ and varying $(u, v) \in I \times I$. After differentiating $\sigma$ with respect to $u$ and $v$, using the chain rule we get

\begin{equation}
\frac{\partial (\sigma \circ \varphi^{-1})}{\partial x^i} \cdot \frac{\partial (x^i \circ \chi)}{\partial u} = 0
\end{equation}

and

\begin{equation}
\frac{\partial (\sigma \circ \varphi^{-1})}{\partial x^i} \cdot \frac{\partial (x^i \circ \chi)}{\partial v} = 0,
\end{equation}

where all derivatives are calculated at $p$ for each $z \in \{m, q, r\}$. The quantities \(\frac{\partial (\sigma \circ \varphi^{-1})}{\partial x^i}\), apart from a linear factor, are the components of the rescaled tangent vector of the null geodesics emanating from $p$ to $z$ with respect to the covariant basis in $T_p M$. On the other hand, \(\frac{\partial (x^i \circ \chi)}{\partial u}\) and \(\frac{\partial (x^i \circ \chi)}{\partial v}\) are linearly independent generators of the two-dimensional tangent vector space $T_p(C_i \cap C_j \cap \text{fr} J^+(m))$.

As it follows from $6.9$ and $6.10$ considered only numerically, both sets of arithmetic vectors \(\left\{\frac{\partial (\sigma \circ \varphi^{-1})}{\partial x^i}\right\}, \left\{\frac{\partial (x^i \circ \chi)}{\partial u}, \frac{\partial (x^i \circ \chi)}{\partial v}\right\}\) are mutually orthogonal with respect to the standard dot product. Since $g_{ik}$ works as the transition matrix between covariant and usual, contravariant components of a vector in $T_p M$, the tangent vectors, say $m^i, q^i, r^i$, corresponding to the geodesics emanating from $p$ to $m, q, r$, respectively, are linearly dependent. On the other hand, linearly dependent $m^i, q^i$ would imply the points $p, q, m$ lying on the same geodesic, whose segment would be contained in $\chi(I \times I) \subseteq C_i \cap \text{fr} J^+(m)$. Then $m^i, q^i$ would be linear combinations of $\frac{\partial (x^i \circ \chi)}{\partial u}$ and $\frac{\partial (x^i \circ \chi)}{\partial v}$, which then would imply a contradiction with $6.9$ and $6.10$. Beside this, it is not possible that $m = q$ since it would lead to $\dim(C_i \cap \text{fr} J^+(m)) = 3$.

Hence, $m^i, q^i$ are linearly independent and so there exist $a, b \in \mathbb{R}$, $(a, b) \neq (0, 0)$, such that $r^i = am^i + bq^i$. Further, from the fact that $m^i, q^i, r^i$ are
(non-zero) null vectors, it follows
\begin{equation}
0 = g_{ik} r^i r^k = g_{ik} (am^i + bq^i) (am^k + bq^k) = \\
g_{ik} (a^2 m^i m^k + abm^i q^k + abm^k q^i + b^2 q^i q^k) = \\
a^2 g_{ik} m^i m^k + 2abg_{ik} m^i q^k + b^2 g_{ik} q^i q^k = 2abg_{ik} m^i q^k,
\end{equation}
which implies
\begin{equation}
abg_{ik} m^i q^k = 0
\end{equation}

Now, let us choose the coordinate system \( \varphi = (x^1, x^2, x^3, x^4) \) in such a way that the non-diagonal elements of the metric \( g_{ik} \) vanish and \( g_{11} = g_{22} = g_{33} = 1, g_{44} = -1 \) at \( p \). Then, for \( a \neq 0 \neq b \), the equation (6.12) will have the form
\begin{equation}
3 \sum_{i=1}^{3} m^i q^i = m^4 q^4
\end{equation}
and since \( m^i, q^i \) are null vectors, also it holds
\begin{equation}
\sum_{i=1}^{3} (m^i)^2 = (m^4)^2, \quad \sum_{i=1}^{3} (q^i)^2 = (q^4)^2.
\end{equation}

Then
\begin{equation}
\left( \sum_{i=1}^{3} m^i q^i \right)^2 = \left[ \sum_{i=1}^{3} (m^i)^2 \right] \cdot \left[ \sum_{i=1}^{3} (q^i)^2 \right],
\end{equation}
but this is possible only if there exists some \( c \in \mathbb{R}, c \neq 0 \), such that \( q^i = cm^i \) for \( i = 1, 2, 3 \) (as it follows from the properties of the well-known Cauchy-Bunyakovsky-Schwarz inequality). Further, (6.13) and (6.14) imply
\begin{equation}
(q^4)^2 = \sum_{i=1}^{3} (q^i)^2 = c m^i q^i = c \sum_{i=1}^{3} m^i q^i = cm^4 q^4
\end{equation}
and since \( q^4 \neq 0 \), it holds \( q^4 = cm^4 \). But it means that the vectors \( m^i, q^i \) are collinear, which we already excluded. Hence, it holds \( a = 0, b \neq 0 \) or \( a \neq 0, b = 0 \).

The first case means that \( r^i = bq^i \), so the points \( q, r \) lie on the same null geodesic as \( p \), but \( q = r \) is not possible, since by our previous considerations \( C_i, C_j \) belong to the boundaries of different cones \( J^\pm(q), J^\pm(r) \). Take any element \( p' \in C_i \cap C_j \). Suppose, for certainty, that \( r \leq q \), so the null geodesic segment emanating from \( r \) and terminating at \( q \) is future-oriented. Then there exists a non-timelike path from \( r \) to \( p' \), say \( \eta \), consisting of the null geodesic segment from \( r \) to \( q \) and the null geodesic segment emanating from \( q \) and terminating at \( p' \). Suppose that \( \eta \) is not a null geodesic path. Then, by Lemma 6.2 there exist a timelike curve \( \zeta \) connecting \( r \) with \( p' \). Since \( p' \in C_j \),
the point \( p' \) also lies on a null geodesic, say \( \gamma \), emanating from \( r \). But this is not possible on a normal convex neighborhood \( U = U_{FG} \), containing \( F \cup G \) and \( F \cap G \) by Definition 6.1. Indeed, if so, then the timelike curve \( \zeta \) could be replaced by the longest timelike curve, connecting \( p' \) with \( r \), which would be also a geodesic. Then there would exist two different geodesics between \( p' \) and \( r \), however, by the standard theory of differential equations, it is not possible on \( U \). Hence, \( \eta = \gamma \) is the only geodesic path connecting \( p' \) and \( r \).

But, that path contains totally the set \( C \) contradicts to the assumption that \( \dim(r, C) = 0 \), which means that the points \( p \) and \( m \) as \( a \) is a geodesic. Then there would exist two different geodesics between \( p' \) and \( r \), however, by the standard theory of differential equations, it is not possible on \( U \). Hence, \( \eta = \gamma \) is the only geodesic path connecting \( p' \) and \( r \).

But, that path contains totally the set \( C_i \cap C_j \), so \( \dim(C_i \cap C_j) = 1 \), which contradicts to the assumption that \( \dim(C_i \cap \text{fr} J^+(m)) = \dim \chi(I \times I) = 2 \).

The case of \( q \leq r \) could be solved analogically, only by exchanging the roles of \( q, r \). Finally, it remains only the possibility that \( a \neq 0 \) and \( b = 0 \), which means that the points \( m, r \) are on the same null geodesic as \( p \), which implies that \( r \leq m \) or \( m \leq r \).

(iii) Using Lemma 6.3, we can similarly as in (ii) show that \( E_j = \text{fr} J^-(r) \), \( r \in G \). Then \( A \) is contained in the geodesical segment between the points \( m, r \), but this contradicts to the main assumption of (iii), that \( \dim(C_i \cap \text{fr} J^+(m)) = 2 \).

Hence, \( C_j \subseteq E_j = \text{fr} J^+(r) \). It is not possible that \( r \leq m \) and \( r \neq m \) since \( r \) would not be a maximal element among those giving the same value for the set \( A = F \cup G \). Indeed, then \( A \subseteq J^+(m) \subseteq J^+(r) \), which implies that \( A = J^+(m) \cap A \subseteq J^+(m) \cap A_r \subseteq J^+(r) \cap A_r = A_r \), where we denoted \( A_r = (F \setminus \{r\}) \cap G \) (as in Lemma 6.8). Then \( r \) could be replaced by the greater element \( m \), which is a contradiction. Hence, it holds \( m \leq r \).

Then \( A \subseteq J^+(r) \subseteq J^+(m) \) and from the assumption of maximality of \( m \) it follows that \( m = r \in F \).

Now, let \( A, B \in \mathcal{P} \). We put \( A \prec B \) if \( A \neq B \) and for every \( a \in A, b \in B \), \( a \leq b \).

**Theorem 6.1.** \( (\mathcal{P}, \subseteq, \prec) \) is a causal site.

**Proof.** First of all, we need to show that \( \prec \) is a transitive binary relation on the set \( \mathcal{P} \setminus \{\emptyset\} \) (the anti-reflexivity of \( \prec \) follows directly from the definition). Suppose that \( A \prec B \) and \( B \prec C \), where \( A, B, C \) are non-empty elements of \( \mathcal{P} \). Let \( a \in A, c \in C \). Since \( B \neq \emptyset \), there is some \( b \in B \). By definition, \( a \leq b \leq c \), and since \( \leq \) is a transitive relation, we have \( a \leq c \). Suppose that \( A = C \). Then \( A \prec B \) and \( B \prec A \). It means that \( x \leq y \) and \( y \leq x \) for all \( x \in A \) and \( y \in B \). Since \( A \neq B \), it is possible to choose some concrete \( x = y \). Then it follows that there exist two future-oriented non-spacelike curves, where the first one emanates from \( x \) and terminates at \( y \), and the second curve vice-versa. Then \( M \) admits of a closed non-spacelike curve, which is not possible. It means that \( A \neq C \) and so the relation \( \prec \) is transitive.

The conditions (i)-(iii) of the definition of the causal site are only immediate consequences of the properties of the set inclusion and the fact that \( M \)
does not admit of the closed non-spacelike curves. We leave their verification to the reader. It remains to check the last axiom (iv).

At first, suppose that \( A \) consists of only a single multi-diamond. Denote

\[
O_A = \{ x \mid x \in D, A \subseteq J^+(x) \}.
\]

Let \( M_A \) be the set of all maximal elements of \( O_A \) (with respect to the order \( \leq \)). Choosing \( p \notin F \) in Lemma 6.8 we can conclude that for every \( x \in O_A \) there exists \( m \in M_A \) such that \( x \leq m \). We put

\[
A_\perp = \bigcup_{m \in M_A} J^-(m),
\]

and for \( B \in \mathcal{P}, B \neq A \) we denote

\[
B_A = B \cap A_\perp.
\]

By Lemma 6.10, \( M_A \) is finite, so \( B_A \in \mathcal{P} \). Let \( b \in B_A, a \in A \). By the definition of \( B_A \), there exists some \( m \in M_A \) with \( b \in J^-(m), \) so \( b \leq m \). We also have \( a \in A \subseteq J^+(m) \), so \( m \leq a \). Then \( b \leq a \), which implies \( B_A \subseteq A \). Suppose that \( C \prec A, C \subseteq B \) for some \( C \in \mathcal{P} \). Let \( c \in C \). If \( a \in A \), then \( c \leq a \), which gives \( a \in J^+(c) \). Therefore, \( A \subseteq J^+(c) \). Then \( c \in O_A, \) so there exists \( m \in M_A \), such that \( c \leq m \). Then \( c \in J^-(m) \subseteq A_\perp \). Hence, \( C \subseteq A_\perp \), which together with \( C \subseteq B \) gives the requested inclusion \( C \subseteq B_A \). Then \( B_A \) is the cutting of \( B \) by \( A \).

Now, consider the general case that \( A = \bigcup_{i=1}^n A_i \), where \( A_i \in \mathcal{D} \) are non-empty multi-diamonds for \( i \in \{1, 2, \ldots, n\} \). For \( B \in \mathcal{P} \) we put

\[
B_A = \bigcap_{i=1}^n B_{A_i}.
\]

Since \( B_{A_i} \subseteq B \), it is clear that \( B_A \subseteq B \). Also we have \( B_{A_i} \prec A_i \) for every \( i \in \{1, 2, \ldots, n\} \). Take \( b \in B_A \) and \( a \in A \). It follows that \( b \leq a \), since there exists \( i \in \{1, 2, \ldots, n\} \) such that \( a \in A_i \), and \( b \in B_A \subseteq B_{A_i} \), \( B_{A_i} \prec A_i \). Suppose that \( B_A = A \). Then, for \( i \in \{1, 2, \ldots, n\}, A_i \subseteq A = B_A \subseteq B_{A_i} \), which implies \( A_i \subseteq B_{A_i} \prec A_i \). By the already verified condition (i), it follows \( A_i \prec A_i \). Since the multi-diamonds \( A_i \) are non-empty, this is not possible.

Now, consider \( C \in \mathcal{P} \) such that \( C \prec A, C \subseteq B \). Then, certainly, \( C \prec A_i \) for \( i \in \{1, 2, \ldots, n\} \) by (ii). But then \( C \subseteq B_{A_i} \) for \( i \in \{1, 2, \ldots, n\} \) and so \( C \subseteq \bigcap_{i=1}^n B_{A_i} = B_A \). Hence, \( B_A \) is the cutting of \( B \) by \( A \).

Let \( \pi \) be the family of all maximal centered subsets of \( \mathcal{P} \).

**Theorem 6.2.** The topological space \((X, \tau)\) corresponding to the framework \((\mathcal{P}^d, \pi^d)\) is homeomorphic to \( M \) equipped with the co-compact topology.
Proof. As we already defined before, \( X = \mathcal{P}^d = \pi \). We will show that any point \( p \in M \) defines a maximal centered subset of \( \mathcal{P} \), say \( f(p) = \{C \mid C \in \mathcal{P}, p \in C \} \). The family \( f(p) \) obviously is centered, since \( \mathcal{P} \) is closed under finite intersections and \( f(p) \) contains those elements of \( \mathcal{P} \), which contain \( p \). Let \( Q \) be another centered family such that \( f(p) \subseteq Q \subseteq \mathcal{P} \). Suppose that there is some \( F \in Q \), such that \( p \notin F \). The set \( M \setminus F \) is an open neighborhood of \( p \) with respect to the manifold topology \( \tau_M \). By Lemma 6.6 there exists \( A \in \mathcal{P} \) with \( x \in \text{int} A \subseteq A \subseteq M \setminus F \). Then \( A \cap F = \emptyset \), which contradicts to the assumption that \( Q \) is centered. Thus, all elements of \( Q \) contain \( p \), which means that \( Q = f(p) \). Now it is clear that \( f(p) \) is a maximal centered subfamily of \( \mathcal{P} \).

Conversely, let \( Q \in \pi \). By Lemma 6.7 the elements of \( Q \) are closed with respect to the co-compact topology on \( M \). Since \( M \) is compact with respect to its co-compact topology, \( \bigcap_{F \in Q} F \neq \emptyset \). Suppose that there exist \( \{x, y\} \subseteq \bigcap_{F \in Q} F \), where \( x \neq y \). Since \( M \) is a \( T_1 \) space, \( M \setminus \{y\} \) is an open neighborhood of \( x \). By Lemma 6.6 there is some \( B \in \mathcal{P} \) with \( x \in \text{int} B \subseteq B \subseteq M \setminus \{y\} \). The collection \( Q \cup \{B\} \subseteq \mathcal{P} \) is centered and since \( y \notin B \), it is an extension of \( Q \). But this contradicts to the assumption that \( Q \) is maximal. Thus, the intersection \( \bigcap_{F \in Q} F \) contains only one element, say \( g(Q) \). Consequently we have \( g(f(p)) = p \) and \( f(g(Q)) = Q \). Hence, the mappings \( f : M \to X \) and \( g : X \to M \) are bijections, inverse to each other.

Furthermore, for \( A \in \mathcal{P} \) we have \( g^{-1}(A) = \{Q \mid Q \in X, g(Q) \in A\} = \{Q \mid Q \in \pi, Q \in f(A)\} = \{Q \mid Q \in \pi, A \subseteq Q\} = \pi(A) \), which is a subbasic closed set in \((X, \tau)\). Then \( g : X \to M \) is continuous.

Now, take a set \( \pi(B) \), where \( B \in \mathcal{P} \), from the closed base \( \pi^d \) of \( \tau \). Then \( f^{-1}(\pi(B)) = \{p \mid p \in M, f(p) \in \pi(B)\} = \{p \mid p \in M, B \in f(p)\} \). For every \( p \in f^{-1}(\pi(B)) \), \( f(p) \) is a maximal centered subfamily of \( \mathcal{P} \), containing the set \( B \). As we have shown above, its intersection contains the only element \( g(f(p)) = p \). So \( f^{-1}(\pi(B)) = \{p \mid p \in M, p \in B\} = B \). Since \( B \) is a compact set with respect to the manifold topology \( \tau_M \) on \( M \), it is closed with respect to the co-compact topology on \( M \) and so the map \( f : M \to X \) is continuous. Hence, the spaces \((X, \tau)\) and \( M \), equipped with the co-compact topology, are homeomorphic.

7. Finite approximations

It is well-known that Albert Einstein asked the question if the Universe is finite in the sense of volume, mass and energy (see, for example, [7]). Whatever it is the correct answer, all our experience with the Universe is finite, since in a finite time one can do only a finite number of observations, measurements or experiments. Hence, everything what we know about the Universe, is a result of extrapolation of that our finite experience. Thus, it may have some sense to study which framework or topological structures may arise from a process of forming and completing of a family of finite
frameworks, representing our growing, but still finite experience with the Universe.

**Definition 7.1.** Let \((X, \alpha)\) be a framework, \(Y \subseteq X\). Denote \(\beta = \{U \cap Y \mid U \in \alpha\}\). Then \((Y, \beta)\) is called the induced subframework of \((X, \alpha)\).

We put \(\pi_K = \{U \cap K \mid U \in \pi\}\) for every finite \(K \subseteq \mathcal{P}\). Obviously, if \(K, L\) are finite subsets of \(\mathcal{P}\) and \(K \subseteq L\), \((K, \pi_K)\) is an induced subframework of \((L, \pi_L)\) and both are induced subframeworks of the original framework \((\mathcal{P}, \pi)\). The collection of finite frameworks \((\mathcal{P}_K, \pi_K)\) is directed by the set inclusion. Let \(\sigma = \{W \mid W \subseteq \mathcal{P}, W \cap K \in \pi_K\) for every finite \(K \subseteq \mathcal{P}\}\).

It could seem that it would be a good idea to approximate \((\mathcal{P}, \pi)\) by \((\mathcal{P}, \sigma)\). However, as we will show later, \((\mathcal{P}, \sigma)\) may contain too many abstract points (that is, elements of \(\sigma\)) in comparison to \((\mathcal{P}, \pi)\). Let \(\lambda \subseteq \sigma\) be a chain linearly ordered by the set inclusion. We put \(L = \bigcup \lambda\). Clearly, \(L \subseteq \mathcal{P}\). If \(K \subseteq \mathcal{P}\) is finite, then also the set \(L \cap K = \bigcup_{W \in \lambda}(W \cap K)\) is finite. Denote \(L \cap K = \{x_1, x_2, \ldots, x_k\}\). Then for every \(i \in \{1, 2, \ldots, k\}\), there is some \(W_i \in \lambda\) with \(x_i \in W_i\). But \(\lambda\) is a chain, so there is the greatest element, say \(W_m \in \sigma\), among all \(W_1, W_2, \ldots, W_k\) with respect to \(\subseteq\). Then \(L \cap K = W_m \cap K \in \pi_K\). Thus, \(L \in \sigma\), so \(L\) is the upper bound of the chain \(\lambda\). By Zorn’s Lemma, every element \(W \in \sigma\) is contained in some maximal element \(M \in \sigma\). Let \(\mu \subseteq \sigma\) be the set of all maximal elements of \(\sigma\). The framework \((\mathcal{P}, \mu)\) could be another candidate for an approximation of \((\mathcal{P}, \pi)\).

**Example 7.1.** Let \(\mathcal{P} = \mathbb{N}\) and let \(\pi\) be the set of all finite subsets of \(\mathcal{P}\). Then, respecting the previous denotations, \(\sigma = 2^\mathcal{P}\), and \(\mu = \{\mathcal{P}\}\).

**Proof.** It is obvious, that \(\sigma \subseteq 2^\mathcal{P}\). Let \(W \in 2^\mathcal{P}\). For every finite \(K \subseteq \mathcal{P}\), \(W \cap K \in \pi_K = 2^K\), so \(W \in \sigma\). However, the set \(\sigma = 2^\mathcal{P}\) has only one maximal element with respect to the set inclusion, \(\mathcal{P}\). □

The following theorem now describes the approximation properties of our construction under very general topological conditions.

**Theorem 7.1.** Let \((X, \tau)\) be a topological \(T_1\) space, \(\mathcal{C}\) the family of all closed sets. Let \((\mathcal{P}, \pi)\) be the dual framework of \((X, \mathcal{C})\). Then the dual of \((\mathcal{P}, \mu)\) generates the Wallman compactification of \((X, \tau)\). More precisely, \(\mu^d\) is a closed subbase of \(\omega_X\).
Proof. We have \( \mathcal{P} = \mathcal{C} \) and \( \pi = \{ \mathcal{C}(x) \mid x \in X \} \), where \( \mathcal{C}(x) = \{ C \mid C \in \mathcal{C}, x \in C \} \). We will show that every element of \( \sigma \) is a family of closed sets of the topological space \((X, \tau)\), having f.i.p. Let \( W \in \sigma \) and let \( K \subseteq W \) be finite. Then \( K = W \cap K \in \pi_K \), so there exists \( y \in X \) such that \( K = \mathcal{C}(y) \cap K \). Then \( K \subseteq \mathcal{C}(y) \), which gives \( y \in \bigcap K \neq \emptyset \). Hence, \( W \) has f.i.p and its closedness follows from the fact that \( W \subseteq \mathcal{P} = \mathcal{C} \).

Let us show that \( \pi \subseteq \mu \). Suppose that for some \( W \in \sigma \) we have \( \mathcal{C}(x) \subseteq W \). Since \((X, \tau)\) is a \( T_1 \) space, \( \{ x \} \in \mathcal{C}(x) \subseteq W \). But \( W \) has f.i.p, so every its element must contain \( x \). Then \( W \in \mathcal{C}(x) \), so \( W = \mathcal{C}(x) \). Therefore, \( \mathcal{C}(x) \) is a maximal element of \( \sigma \), that is, \( \pi \subseteq \mu \).

Let \( \eta \) be the family of all maximal collections of closed sets having f.i.p. Note that, in other words, \( \eta \) is the family of all ultra-closed filters on \((X, \tau)\). We will show that \( \eta = \mu \). As the first step, we will prove that \( \eta \subseteq \mu \).

Let \( U \in \eta \). Take any finite \( K \subseteq \mathcal{P} = \mathcal{C} \) and denote \( L = U \cap K \). The set \( L \) contains only finitely many elements of \( U \). The family \( U \) has f.i.p., so \( \bigcap L \neq \emptyset \). Denote \( D = \bigcup (K \setminus L) \). The set \( D \) is closed (and could be possibly empty, if \( K = \emptyset \)). Suppose that \( \bigcap L = \bigcap (L \cup \{ D \}) \). Consider the family \( U \cup \{ D \} \). If \( F \subseteq U \cup \{ D \} \) is finite, then \( F \setminus \{ D \} \) and also \( F \setminus \{ D \} \cup L \) are finite subsets of \( U \), so \( \emptyset \neq \bigcap (F \setminus \{ D \} \cup L) = \bigcap (F \setminus \{ D \} \cup L) = \bigcap (F \setminus \{ D \} \cup L) \setminus F \).

Then \( U \cup \{ D \} \) has f.i.p. In particular, \( D \neq \emptyset \), which implies that \( K \neq L \). Then \( K \setminus L \neq \emptyset \), and \( U \cap (K \setminus L) = \emptyset \). It follows from the maximality of \( U \) that \( D = \bigcup (K \setminus L) \notin U \). Then \( U \cup \{ D \} \) is a strictly greater family than \( U \). This contradicts to the maximality of \( U \). Therefore, there is some \( z \in X \) such that \( z \in \bigcap L, z \notin D \). Then \( L \subseteq \mathcal{C}(z) \), but \( (K \setminus L) \cap \mathcal{C}(z) = \emptyset \). That means \( U \cap K = L = L \cap \mathcal{C}(z) = K \cap \mathcal{C}(z) \subseteq \pi_K \). Then \( U \in \sigma \). By definition of the set \( \mu \), there exists \( W \in \mu \) such that \( U \subseteq W \). But as we have shown above, \( W \) is a family of closed sets having f.i.p. By maximality of \( U \), we have \( U = W \), so \( U \in \mu \). Therefore, \( \eta \subseteq \mu \).

Conversely, let \( U \in \mu \). Because also \( U \in \sigma \), the family \( \sigma \) consists of closed sets and has f.i.p. Then there exists some \( W \in \eta \) with \( U \subseteq W \). By the previous paragraph, we have \( W \in \mu \subseteq \sigma \). But \( U \) is a maximal element of \( \sigma \), so \( U = W \) and \( U \in \eta \). Together we finally have \( \mu = \eta \).

Now, consider the framework \((\mathcal{P}^d, \mu^d)\). It holds \( \mathcal{P}^d = \mu, \mu^d = \{ \mu(C) \mid C \in \mathcal{C} \} \), where \( \mu(C) = \{ U \mid U \in \mu, C \in U \} \). Consider the topological space \((Y, \vartheta)\), where \( Y = \mathcal{P}^d \) and its topology is generated by its closed subbase \( \mu^d \). Consider \( \Psi \subseteq \mathcal{C} \), such that for every \( C_1, C_2, \ldots, C_k \in \Psi \) it holds

\[
\mu(C_1) \cap \mu(C_2) \cap \cdots \cap \mu(C_k) \neq \emptyset.
\]

There exist \( U \in \mu \) (depending on the selection of \( C_1, C_2, \ldots, C_k \)), such that \( C_1, C_2, \ldots, C_k \subseteq U \), so \( \emptyset \neq \bigcap_{i=1}^k C_i \in U \). Then \( \Psi \) has f.i.p., so there exists a maximal family \( W \subseteq \mathcal{C} = \mathcal{P} \), having f.i.p. and containing \( \Psi \). By the previous paragraph, \( W \in \mu \). Now, if \( C \in \Psi \), then also \( C \in W \), which gives
\[ W \in \mu(C) \] and so 
\[ W \in \bigcap_{C \in \Psi} \mu(C) \neq \emptyset. \]

Therefore, \((Y, \vartheta)\) is compact.

Finally, consider the mapping \(f : X \to Y\), where \(f(x) = C(x)\). Clearly, \(f\) is an injection. Indeed, for \(x \neq y\) we have \(\{x\} \in C(x)\) but \(\{x\} \notin C(y)\), so \(f(x) \neq f(y)\). Let \(C \in \mathcal{C}\). Then 
\[ f^{-1}(\mu(C)) = \{x \in X \mid x \in X, f(x) \in \mu(C)\} = \{x \in X \mid \exists C \in \mathcal{C}(x) \} = \{x \in X \mid x \in C\} = C, \]
so \(f\) is continuous. Further, for any \(D \in \mathcal{C}\), 
\[ f(D) = \{C(x) \mid x \in D\} = \{C(x) \mid x \in X, C(x) \in \mu(D)\} = f(X) \cap \mu(D), \]
so \(f\) is also a closed mapping.

Then \(f\) is a homeomorphous embedding of \((X, \tau)\) to the compact space \((Y, \vartheta)\). Moreover, \(f(X) = \pi\), so the elements of \(X\) and the families \(C(x)\), which constitute the principal ultra-closed filters generated by the elements of \(X\), may be identified. Consider the set \(\mu \setminus \pi\). Then every its element \(W \in \mu \setminus \pi\) is vanishing (that is, non-convergent) – otherwise, because of maximality, \(W = C(z)\), where \(z\) is the unique element from \(\bigcap W\), which would imply \(W \in \pi\). But then \(Y\) is the underlying set of the Wallman compactification of \((X, \tau)\) and 
\[ \mu^d = \{\mu(C) \mid C \in \mathcal{C}\} \]
is its closed base. \(\square\)

Among others the previous theorem means that for a compact \(T_1\) topological space, its approximation by a suitable family of finite frameworks may achieve arbitrary precision.

8. Some Final Remarks in Historical Context

The progress in mathematical and theoretical physics witnesses that various applications of topology in physics may be far-reaching and illuminating. It could be very difficult to track down the origins of such applications, but one of the first attempts may be associated with the year 1914, when A. A. Robb came with his axiomatic system for Minkowski space \(M\), analogous to the well-known axioms of Euclidean plane geometry. In [25] he essentially proved that the geometrical and topological structure of \(M\) can be reconstructed from the underlying set and a certain order relation among its points. As it is noted in [6], some prominent mathematicians and physicists criticized the use of locally Euclidean topology in mathematical models of the space-time. Perhaps as a reflection of these discussions, approximately at the same time when de Groot wrote his papers on co-compactness duality, there appeared two interesting papers [29] and [30], in which E. C. Zeeman studied an alternative topology for Minkowski space. The Zeeman topology, also referred as the fine topology, is the finest topology on Minkowski space, which induces the 3-dimensional Euclidean topology on every space-axis and the 1-dimensional Euclidean topology on the time-axis. Among other interesting properties, it induces the discrete topology on every light ray. A. Kartsaklis in [15] studied connections between topology and causality.
He attempted to axiomatize causality relationships on a point set, equipped with three binary relations, satisfying certain axioms, by a structure called a causal space. He also introduced so called chronological topology, the coarsest topology, in which every non-empty intersection of the chronological future and the chronological past of two distinct points of a causal space is open.

In the camp of quantum gravity, there appeared similar efforts and attempts to get some gain from studying the underlying structure of space-time – topological, geometrical or discrete – however, significantly later. The possible motivation is explained, for instance, in [26] by C. Rovelli. It is pointed out that the loop quantum gravity leads to a view of the space-time geometry extremely different from the smooth model at the shortest scale level. Also the topology of space-time at Planck scales could be very different from that we meet in our everyday experience and which has been originally extrapolated from the fundamental concepts of the continuous and differentiable mathematics. Thus, the usual properties and attributes of the space-time, like its Hausdorffness or metrizability may not be satisfied (for a groundbreaking paper, see [14]). The most important source of inspiration for our paper was the work [2] of J. D. Christensen and L. Crane. Motivated by certain requirements of their research in quantum gravity, these authors developed a novel axiomatic system for the generalized space-time, called causal site, qualitatively different from the previous, similar attempts. The notion itself is a successful synthesis of two other notions, a Grothendieck site (which is a small category equipped with the Groethendieck topology) [1] and a causal set of R. Sorkin [27]. One of the most important merits of the new axiomatic system it is that the causal site is a pointless structure, not unlike to certain well-known concepts of pointless topology and locale theory. We should also mention the work of K. Martin and P. Panangaden who studied causality with help of the theoretical background of domain theory (see, for example, [20] and [21] and our note in Section 5 after Corollary 3.4).

We close the paper by a summary of the main ideas and results:

(i) Every causal site generates, by a canonical way, an associated compact $T_1$ topology (and, naturally, also its de Groot dual). We call this topology weakly causal topology.

(ii) The axioms of a causal site connect both of the relations $\subseteq, \prec$ together to one organic unit, so they are no longer independent. However, the strengths of their connection may vary. Some candidates for a ‘causal’ relation $\prec$ admits of more different appropriate ‘inclusion’ relations $\subseteq$, and for some of them any corresponding ‘inclusion relation may not exist.

(iii) For every globally hyperbolic Lorentzian manifold there exists a causal site, whose weakly causal topology is the de Groot dual of the manifold topology and vice versa, that is, the manifold topology
is also the de Groot dual of the weakly causal topology. Thus, the weakly causal topology and so also the causal site itself has the full information about the usual, manifold topology.

(iv) The weakly causal topology is weaker than the manifold topology, but in the sense of mathematical analysis, it is capable of doing the same job as the manifold topology. Both of the topologies coincide on compact subsets, and so on finite distances, so there is no physical way how to distinguish between them. Both of these topologies are possible extrapolations of the human’s finite experience with the universe, but because it seems that nature follows a certain ‘principle of minimality’ in the natural laws, the weakly causal topology may be considered more natural or fundamental.

As it had been remarked by de Groot in [11], from the philosophical point of view, the co-compact topology is naturally related to the notion of potential infinity – in a contrast to the notion of actual infinity, which is mostly used in the traditional mathematical approach. To illustrate the difference, consider a countably infinite sequence $x_1, x_2, \ldots$ of points lying on a straight line in space or space-time, with the constant distance between $x_i$ and its successor $x_{i+1}$. In the usual, Euclidean topology, the sequence is divergent and it approaches to an improper point at infinity. To make it convergent, one need to embed the space into its compactification (for instance, the Alexandroff one-point compactification is a suitable one). The points completed by the compactification then appear at the infinite distance from any other point of the space. On the other hand, the co-compact topology, which locally coincides with the usual topology, is already compact and superconnected, so the sequence $x_1, x_2, \ldots$ lies eventually in each neighborhood of every point.

(v) In this sense, the universe is ‘co-compact’: the usual, manifold topology is co-compact with respect to the weakly causal topology and vice versa.

(vi) There exist causal sites, which cannot be generated by any Lorentzian manifold, and whose causal relation or its reflexive closure is not a continuous poset. These causal sites still generate their weakly causal topology and its de Groot dual, which is a corresponding counterpart of the manifold topology of the relativistic space-time. In such a case, however, the weakly causal topology need not satisfy the identity $\tau = \tau^{GG}$ and so up to four topologies may arise as the de Groot dual iterations of the weakly causal topology.

(vii) Although it is (so far) unknown if non-Lorentzian causal sites have any physical sense, they could be potentially used for investigation of various physical relationships on the particle level, beyond the smooth space-time models, as in quantum gravity, in very short distances (below the Planck length) or in cases in which the metric
properties of the space-time collapse to one-point singularity (and its topological properties still may have some continuation).

(viii) In Section 4 we developed a general method suitable for topologizing various situations, including physics and computer science. The method is also useful for working with various scales using different and 'incompatible' topological models ('smooth' vs. 'discrete') if considered in the traditional point-set approach.

(ix) In Section 7 we proved that a compact $T_1$ topological space can be expressed by a limit of finite framework approximations. Since the weakly causal topology, naturally generated by every causal site, is compact $T_1$, this topology can be reached by the process of finite approximations as a limit. According to Section 6 the co-compact topology on Lorentzian globally hyperbolic manifolds, for example, can be reached in this way.

(x) It could be proven that open diamonds also form a causal site, which generates the manifold topology of a globally hyperbolic Lorentzian manifold directly, without the necessity of using the de Groot dual. However, this approach would give much weaker results. The topology arising from regions used as a subbase for open sets does not have, in general, any interesting or significant properties. In particular, it need not be compact $T_1$ even for a special causal site, generated by a Lorentzian, globally hyperbolic manifold. As a consequence, the topology need not be approximable by the family of finite frameworks, described in Section 7. This also indicates, that the perceived topology, generated from our extrapolated finite experience with causal relationships in the Universe, is rather the co-compact topology, than its de Groot dual, the usual, locally Euclidean manifold topology.

References

1. Artin M., Grothendieck Topology, Notes on a Seminar, Harward University Press (1962), pp. 1-95.
2. Christensen J. D., Crane L., Causal Sites as Quantum Geometry, J. Math. Phys. 46 (2005), 122502-122523.
3. Chen X., Li Q., Formal Topology, Chu Space and Approximable Concept, CLA 2005 (Bělohlávek R., Snášel V., eds.), 158-165.
4. Császár Á., General Topology, Akademiaia Kiadó, Budapest 1978, pp. 1-487.
5. Diker M., One-point compactification of ditopological texture spaces, Fuzzy Sets and Systems 147 (2004), 233-248.
6. Domiaty R. Z., Remarks to the problem of defining a topology by its homeomorphism group, Proceedings of the fourth Prague topological symposium, 1976, Part B.: Contributed papers, Praha (1977), 99-110.
7. Einstein A., Relativity. The Special and General Theory, Pi Press, New York (2005), pp. 1-259.
8. Engelking R., General topology, Heldermann Verlag, Berlin (1989), pp. 1-532.
9. Ganter B., Wille, R., \textit{Formal Concept Analysis}, Springer-Verlag, Berlin (1999), pp.1-285.
10. Gratzer G., \textit{General Lattice Theory}, Birkhauser Verlag, Berlin (2003), pp. 1-663.
11. de Groot J., \textit{An Isomorphism Principle in General Topology}, Bull. Amer. Math. Soc. 73 (1967), 465-467.
12. Gierz G., Hofmann K. H., Keimel K., Lawson J. D., Mislove M. W., Scott D. S., \textit{Continuous Lattices and Domains}, Cambridge University Press (2003), pp. 1-591.
13. Hawking S. W., Ellis G.F.R., \textit{The Large Scale Structure of Space-time}, Cambridge University Press, Cambridge (1973), pp. 1-391.
14. Heller M., Pysiak L., Sasin W., \textit{Geometry of non-Hausdorff spaces and its significance for physics}, J. Math. Phys. 52, 043506 (2011), 1-7.
15. Kartsaklis A., \textit{Topological considerations in the foundations of quantum and time theories}, Proceedings of the fourth Prague topological symposium, 1976, Part B.: Contributed papers, Praha (1977), 207-213.
16. Kovár M. M., \textit{At most 4 topologies can arise from iterating the de Groot dual}, Topology and Appl. 130 (1) (2003), 175-182.
17. Kovár M. M., \textit{On iterated de Groot dualizations of topological spaces}, Topology and Appl. 146-147 (1)(2005), 83-89.
18. Krein M. G., Rutman M. A. \textit{Linear operators leaving invariant a cone in a Banach space}, UMN 3 no. 1(23) (1948), 3-95 (Russian).
19. Lawson J.D., Mislove M., \textit{Problems in domain theory and topology}, Open problems in topology (van Mill J., Reed G. M. eds.), North-Holland, Amsterdam (1990), pp. 349-372.
20. Martin K., Panangaden P., \textit{A Domain of Spacetime Intervals in General Relativity}, Commun. in Math. Phys. 267 (3)(2006), 563-586.
21. Martin K., Panangaden P., \textit{Spacetime geometry from causal structure and measurement}, Proceedings of Symposia in Applied Mathematics, 1-20.
22. Matthews S. G., \textit{Partial Metric Topology}, Papers on General Topology and Applications, Eight Summer Conference at Queens College (S. Andima et.al. eds.), Ann. N. Y. Acad. Sci. 728 (1994), 183-197.
23. O’Neill, \textit{Semi-Riemannian Geometry}, Academic Press, New York (1983), pp. 1-482.
24. Poisson E., \textit{The Motion of Point Particles in Curved Spacetime}, Living Rev. Relativity 14,7 (2011), [http://www.livingreviews.org/lrr-2011-7] pp. 1-190.
25. Robb A. A., \textit{Geometry of Time and Space}, Cambridge University Press, 1936, pp. 1-408. (the second edition of \textit{A theory of Time and Space}, Cambridge University Press, 1914)
26. Rovelli C., \textit{Quantum Gravity}, Cambridge University Press, 2005, pp. 1-458.
27. Sorkin R. D., \textit{Causal Sets: Discrete Gravity}, [gr-qc/0309009]
28. Wille, R., \textit{Restructuring Lattice Theory: An Approach Based on Hierarchies of Concepts}, I. Rival (ed.), Ordered Sets (1982), 445-470.
29. Zeeman E. C., \textit{Causality implies the Lorentz group}, J. Math. Phys. 5 (1964), no.4, 490-493.
30. Zeeman E. C., \textit{The topology of Minkowski space}, Topology 6 (1967), 161-170.