Enumeration of One-Nodal Rational Curves
in Projective Spaces

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Abstract
We give a formula computing the number of one-nodal rational curves that pass through an appropriate collection of constraints in a complex projective space. We combine the methods and results from three different papers.

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1 Introduction

Enumerative algebraic geometry is a field of mathematics that dates back to the nineteenth century. However, many of its most fundamental problems remained unsolved until the early 1990s. For example, let $d$ be a positive integer and $\mu = (\mu_1, \ldots, \mu_N)$ an $N$-tuple of linear subspaces of $\mathbb{P}^n$ of codimension at least two such that

$$\text{codim}_C\mu \equiv \sum_{l=1}^{l=N} \text{codim}_C\mu_l - N = d(n+1) + n - 3.$$ 

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If the constraints $\mu$ are in general position, denote by $n_d(\mu)$ the number of rational degree-$d$ curves that pass through $\mu_1, \ldots, \mu_N$. This number is finite and depends only on the homology classes of the constraints. If $d=1$, it can be computed using Schubert calculus; see [GH]. All but very-low-degree numbers $n_d(\mu)$ remained unknown until [KM] and [RT] derived a recursive formula for these numbers. In this paper, we prove

**Theorem 1.1** Suppose $n \geq 3$, $d \geq 1$, and $\mu = (\mu_1, \ldots, \mu_N)$ is an $N$-tuple of proper subvarieties of $\mathbb{P}^n$ in general position such that

$$\text{codim}_{\mathbb{C}} \mu \equiv \sum_{i=1}^{l=N} \text{codim}_{\mathbb{C}} \mu_i = N = d(n+1) - 1.$$  \hspace{1cm} (1.1)

Then the number of degree-$d$ rational curves that have a simple node and pass through the constraints $\mu$ is given by

$$n^{(1)}_d(\mu) = 1/2 (RT_{1,d}(\mu_1; \mu_2, \ldots, \mu_N) - CR_1(\mu)),$$

where

$$CR_1(\mu) = \sum_{k=1}^{2k \leq n+1} (-1)^{k-1}(k-1)! \sum_{l=0}^{n+1-2k} \binom{n+1}{l} \langle a^t \eta_{n+1-2k-l}, [\bar{V}_k(\mu)] \rangle.$$  

The symplectic invariant $RT_{1,d}(\cdot; \cdot)$ and the top intersections $\langle a^t \eta_{n+1-2k-l}, [\bar{V}_k(\mu)] \rangle$ are computable via algorithms described elsewhere.

| $n$ | $3$ | $4$ | $5$ | $5$ | $6$ |
|----|----|----|----|----|----|
| $d$ | $4$ | $4$ | $4$ | $6$ | $6$ |
| $\mu$ | (5,5) | (5,1,4) | (5,1,0,4) | (2,1,1,7) | (2,1,1,1,6) |
| $n^{(1)}_d(\mu)$ | 1,800 | 1,800 | 1,800 | 20,340 | 20,340 |

For the purposes of this table, we assume that the constraints $\mu_1, \ldots, \mu_N$ are linear subspaces of $\mathbb{P}^n$ of codimension at least two. We describe such a tuple $\mu$ of constraints by listing the number of linear subspaces of codimension $2, \ldots, n$ among $\mu_1, \ldots, \mu_N$.

In the statement of Theorem 1.1, $RT_{1,d}(\cdot; \cdot)$ denotes the genus-one degree-$d$ symplectic invariant of $\mathbb{P}^n$ defined in [RT]. This invariant can be expressed in terms of the numbers $n_d(\cdot)$; see [RT]. In particular, it is computable. Brief remarks concerning the meaning of $RT_{1,d}(\cdot; \cdot)$ can be found at the beginning of Section 3.

The compact oriented topological manifold $\bar{V}_k(\mu)$ consists of unordered $k$-tuples of stable rational maps of total degree $d$. Each map comes with a special marked point $\infty_i$. All these marked points are mapped to the same point in $\mathbb{P}^n$. In particular, there is a well-defined evaluation map

$$\text{ev}: \bar{V}_k(\mu) \rightarrow \mathbb{P}^n,$$

which sends each tuple of stable maps to the value at one of the special marked points. We also require that the union of the images of the maps in each tuple intersect each of the constraints $\mu_1, \ldots, \mu_N$. In fact, the elements in the tuple carry a total of $N$ marked points, $y_1, \ldots, y_N$, in
addition to the $k$ special marked points. These marked points are mapped to the constraints $\mu_1, \ldots, \mu_N$, respectively. Roughly speaking, each element of $\tilde{V}_k(\mu)$ corresponds to a degree-$d$ rational curve in $\mathbb{P}^n$, which has at least $k$ irreducible components, and $k$ of the components meet at the same point in $\mathbb{P}^n$. The precise definition of the spaces $\tilde{V}_k(\mu)$ can be found in Subsection 2.2.

The cohomology classes $a$ and $\eta_l$ are tautological classes in $\tilde{V}_k(\mu)$. In fact,

$$a = ev^*c_1(O(1_{\mathbb{P}^n})).$$

Let $\tilde{V}'_k(\mu)$ be the oriented topological manifold defined as $\tilde{V}_k(\mu)$, except without specifying the marked points $y_1, \ldots, y_N$ mapped to the constraints $\mu_1, \ldots, \mu_N$. Then, there is well-defined forgetful map,

$$\pi: \tilde{V}_k(\mu) \rightarrow \tilde{V}'_k(\mu),$$

which drops the marked points $y_1, \ldots, y_N$ and contracts the unstable components. The cohomology class $\eta_l \in H^2(\tilde{V}_k(\mu))$ is the sum of all degree-$l$ monomials in the elements of the set

$$\{\pi^*\psi_{0_1}, \ldots, \pi^*\psi_{0_k}\} \subset H^2(\tilde{V}_k(\mu)).$$

As common in algebraic geometry, $\psi_{0_i}$ denotes the first chern class of the universal cotangent line bundle for the marked point $0_i \in \tilde{V}'_k(\mu)$. In Subsection 2.2, we give a definition of $\eta_l$ that does not involve the projection map $\pi$. An algorithm for computing the intersection numbers involved in the statement of Theorem 1.1 is given in Subsection 5.7 of [Z2]. It is closely related to the algorithm of [P2] for computing intersections of tautological classes in moduli spaces of stable rational maps into $\mathbb{P}^n$.

If $n=2$, we denote by $n_d^{(1)}(\mu)$ the number of rational degree-$d$ curves passing through the constraints counted with a choice of the node on each curve. The formula of Theorem 1.1 gives

$$n_d^{(1)}(\mu) = \left(\frac{d-1}{2}\right)n_d(\mu).$$

(1.2)

This identity is clear, since the arithmetic genus of every degree-$d$ curve in $\mathbb{P}^2$ is $(d-1)_2$. Equation (1.2) is used in [P1] to count genus-one plane curves with complex structure fixed. More precisely, if $\mu$ is a tuple of constraints in $\mathbb{P}^n$ satisfying condition (1.1), let $n_{1,d}(\mu)$ denote the number of genus-one degree-$d$ curves that pass through the constraints $\mu$ and have a fixed generic complex structure on the normalization, i.e. its $j$-invariant is different from 0 and 1728. The key step in [P1] is to show that

$$n_{1,d}(\mu) = n_d^{(1)}(\mu),$$

(1.3)

if $\mu$ is a tuple of $3d-1$ points in $\mathbb{P}^2$. One of the main ingredients in proving Theorem 1.1 is Proposition 4.1, which states that (1.3) is valid for any tuple $\mu$ that satisfies condition (1.1). Note that the numbers listed in the above table are consistent with (1.3) and facts of classical algebraic geometry. In particular, the image of every degree-4 map from a genus-one curve to $\mathbb{P}^5$ lies in a $\mathbb{P}^3$ and the image of every degree-6 map lies in a $\mathbb{P}^5$; see [ACGH, p116]. Thus, the first three numbers in the table should be the same, and the last two numbers should be the same. The proof of Proposition 4.1 extends the degeneration argument of [P1] and builds up on modifications
described in [1]. We work with the moduli space $\overline{\mathcal{M}}_{1,N}(\mathbb{P}^n, d)$ of stable degree-$d$ maps from genus-one $N$-pointed curves into $\mathbb{P}^n$ and study what happens in the limit to the maps that pass through the constraints $\mu$ as the $j$-invariant of the domain tends infinity, i.e. the domain degenerates to a rational curve with two points identified.

Proposition 1.1 is not useful for determining the numbers $n_{1,d}(\mu)$ in $\mathbb{P}^n$ if $n \geq 3$, since the right-hand side of (1.3) is unknown. Computation of $n_{1,d}(\mu)$ for all projective spaces is the subject of [1], where an entirely different approach is taken. The main step in computing these numbers is taken. The topological tools of Theorem 1.1 follows immediately from Propositions 3.1 and 4.1. Their proofs are mutually independent. Section 4 uses some of the notation defined in Subsection 2.2. The topological tools of Subsection 2.1, the descriptive notation of Subsection 2.2, and the structure theorem of Subsection 2.3 are integral to the computations of Section 3.

We obtain the expression of Theorem 1.1 for the number $CR_1(\mu)$ in Section 3; see Proposition 2.2. The remaining step is to express this number of zeros topologically. In general, if the linear part of an affine map $\psi$ does not vanish, it is easy to determine the signed cardinality of $\psi^{-1}(0)$; see Lemma 2.3. The approach of [1] is to replace the linear part $\alpha$ of the affine part under consideration by a nonvanishing linear map over a space obtained from $\mathcal{V}_1'(\mu)$ by sequence of blowups and then to express the resulting intersection number in terms of intersection numbers on the spaces $\mathcal{V}_k'(\mu)$. The main problem with this approach is that the new linear map is not described in [1] and it is not clear how to construct it in general. In addition, the normal bundles of certain spaces needed for the second part of this approach are given incorrectly; see Lemma 2.8 or equation (2.27) in [1] for example. Both of these statements can be corrected without affecting the computability of the intersection numbers, but presumably with a change in the final result.

If $n=2$, no blowup is needed. If $n=3, 4$, the zero set of $\alpha$ is a complex manifold and the “derivative” of $\alpha$ in the normal direction along $\alpha^{-1}(0)$ is nondegenerate. In such cases, only one blowup is needed and a linear map with the required properties can be constructed fairly easily. Furthermore, Lemma 2.8 of [1] requires no correction in the $n=2, 3, 4$ cases, while equation (2.27) is never used. If $n=2, 3$, $CR_1(\mu)$ and $n_{1,d}(\mu)$ are then expressed in terms of the numbers $n_{d'}(\mu')$, with $d' \leq d$ and $\mu'$ related to $\mu$. Several numbers $n_{1,d}(\mu)$ for $\mathbb{P}^4$ are given in [1] as well. However, no topological formula, like that of Theorem 1.1, is given for $CR_1(\mu)$ or $n_{1,d}(\mu)$ for $\mathbb{P}^n$ with $n \geq 4$ and no number $n_{1,d}(\mu)$ is given for $\mathbb{P}^n$ with $n \geq 5$.

Theorem 1.1 follows immediately from Propositions 1.1 and 4.1. Their proofs are mutually independent. Section 3 uses some of the notation defined in Subsection 2.2. The topological tools of Subsection 2.1, the descriptive notation of Subsection 2.2, and the structure theorem of Subsection 2.3 are integral to the computations of Section 3.
In brief, we enumerate one-nodal rational curves from genus-one fixed-complex-structure invariants. Can a similar approach be used with higher-genera enumerative invariants? Let $\mu$ be an $N$-tuple of proper subvarieties of $\mathbb{P}^n$ in general position such that

$$\text{codim}_C \mu = d(n + 1) - n.$$ 

Denote by $n_{2,d}(\mu)$ the number of genus-two degree-$d$ curves that pass through the constraints $\mu$ and have a fixed generic complex structure on the normalization. Let $n_d^{(3)}(\mu)$, $\tau_d(\mu)$, and $T_d(\mu)$ denote the number of rational two-component curves connected at three nodes, of rational curves with a triple point, and of rational curves with a tacnode, respectively. If $n = 2$, we take $n_d^{(3)}(\mu)$ to be the number of two-component rational curves with a choice of three nodes common to both components. In all cases, the curves have degree-$d$ and pass through the constraints $\mu$. Completing the degeneration argument of [KQR], it is shown in [Z1] that

$$n_{2,d}(\mu) = 6(n_d^{(3)}(\mu) + \tau_d(\mu) + T_d(\mu)), \tag{1.4}$$

if $\mu$ is a tuple of $3d - 2$ points in $\mathbb{P}^2$. The arguments of [KQR] and [Z1] should extend to show that equation (1.4) is valid for arbitrary constraints $\mu$ in all projective spaces. On the other hand, $n_{2,d}(\mu)$ for $\mathbb{P}^3$ is computed in [Z2] and the method extends at least to $\mathbb{P}^4$. Thus, in those two cases, we should be able to express the sum of the numbers $n_d^{(3)}(\mu)$, $\tau_d(\mu)$, and $T_d(\mu)$ in terms of intersection numbers of the spaces $\bar{V}_k(\mu)$. The relation (1.4) is obtained by considering a degeneration to a specific singular genus-two curve. Perhaps, different relations can be obtained by considering degeneration to other singular genus-two curves. With enough different relations, we would be able to compute the numbers $n_d^{(3)}(\mu)$, $\tau_d(\mu)$, and $T_d(\mu)$ at least for $\mathbb{P}^3$ and $\mathbb{P}^4$.

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2 Background

2.1 Topology

We begin by describing the topological tools used in the next section. In particular, we review the notion of contribution to the euler class of a vector bundle from a (not necessarily closed) subset of the zero set of a section. We also recall how one can enumerate the zeros of an affine map between vector bundles. These concepts are closely intertwined. Details can be found in Section 3 of [Z2].

Throughout this paper, all vector bundles are assumed to be complex and normed. If $F \to \mathcal{M}$ is a smooth vector bundle, closed subset $Y$ of $F$ is small if it contains no fiber of $F$ and is preserved under scalar multiplication. If $Z$ is a compact oriented zero-dimensional manifold, we denote the signed cardinality of $Z$ by $\pm |Z|$. If $k$ is an integer, we write $\lfloor k \rfloor$ for the set of positive integers not exceeding $k$.

Definition 2.1 Suppose $F, \mathcal{O} \to \mathcal{M}$ are smooth vector bundles.

1. If $F = \bigoplus_{i=1}^k F_i$ and $\underline{d} = (d_1, \ldots, d_k)$ is a $k$-tuple of positive integers, bundle map $\alpha : F \to \mathcal{O}$ is a
polynomial of degree \( d \) if for each \( i \in [k] \) there exists

\[
p_i \in \Gamma(\mathcal{M}; F_i^{\otimes d_i} \otimes \mathcal{O}) \quad \text{for} \ i \in [k] \quad \text{s.t.} \quad \alpha(v) = \sum_{i=1}^{i=k} p_i(v)^{d_i} \quad \forall v = (v_i)_{i \in [k]} \in \bigoplus_{i=1}^{i=k} F_i.
\]

(2) If \( \alpha: F \to \mathcal{O} \) is a polynomial, the rank of \( \alpha \) is the number

\[
\text{rk} \alpha = \max \{ \text{rk}_b \alpha: b \in \mathcal{M} \}, \quad \text{where} \quad \text{rk}_b \alpha = \dim \{ \text{Im} \alpha_b \}.
\]

Polynomial \( \alpha: F \to \mathcal{O} \) is of constant rank if \( \text{rk}_b \alpha = \text{rk} \alpha \) for all \( b \in \mathcal{M} \); \( \alpha \) is nondegenerate if \( \text{rk}_b \alpha = \text{rk} F \) for all \( b \in \mathcal{M} \).

(3) If \( \Omega \) is an open subset of \( F \) and \( \phi: \Omega \to \mathcal{O} \) is a smooth bundle map, bundle map \( \alpha: F \to \mathcal{O} \) is a dominant term of \( \phi \) if there exists \( \varepsilon \in C^0(F; \mathbb{R}) \) such that

\[
|\phi(v) - \alpha(v)| \leq \varepsilon(v)|\alpha(v)| \quad \forall v \in \Omega \quad \text{and} \quad \lim_{v \to 0} \varepsilon(v) = 0.
\]

Dominant term \( \alpha: F \to \mathcal{O} \) of \( \phi \) is the resolvent of \( \phi \) if \( \alpha \) is a polynomial of constant rank.

(4) \( \phi: \Omega \to \mathcal{O} \) is hollow if there exist dominant term \( \alpha \) of \( \phi \) and splittings \( F = F^- \oplus F^+ \) and \( \mathcal{O} = \mathcal{O}^- \oplus \mathcal{O}^+ \) such that \( \alpha(F^+) \subset \mathcal{O}^+ \), \( \alpha^- \equiv \pi^- \circ (\alpha|F^-) \) is a constant-rank polynomial, where \( \pi^-: \mathcal{O}^+ \to \mathcal{O}^+ \) is the projection map, and \( (\text{rk} \alpha^- + \frac{1}{2} \dim \mathcal{M}) < \text{rk } \mathcal{O}^- \).

The base spaces we work with in the next two sections are closely related to spaces of rational maps into \( \mathbb{P}^n \) of total degree \( d \) that pass through the \( N \) constraints \( \mu_1, \ldots, \mu_N \). From the algebraic geometry point of view, spaces of rational maps are algebraic stacks, but with a fairly obscure local structure. We view these spaces as mostly smooth, or \( ms \)-, manifolds: compact oriented topological manifolds stratified by smooth manifolds, such that the boundary strata have (real) codimension at least two. Subsection 2.3 gives explicit descriptions of neighborhoods of boundary strata and of the behavior of certain bundle sections near such strata. We call the main stratum \( \mathcal{M} \) of \( ms \)-manifold \( \mathcal{M} \) the smooth base of \( \mathcal{M} \). Definition 3.7 in [Z2] also introduces the natural notions of \( ms \)-maps between \( ms \)-manifolds, \( ms \)-bundles over \( ms \)-manifolds, and \( ms \)-sections of \( ms \)-bundles.

**Definition 2.2** Let \( \tilde{M} = \mathcal{M}_n \sqcup \bigsqcup_{i=0}^{n-2} \mathcal{M}_i = \mathcal{M} \sqcup \bigsqcup_{i=0}^{n-2} \mathcal{M}_i \) be an \( ms \)-manifold of dimension \( n \).

(1) If \( Z \subseteq \mathcal{M}_i \) is a smooth oriented submanifold, a **normal-bundle model for** \( Z \) **is a tuple** \( (F, Y, \vartheta) \), **where**

(1a) \( F \to Z \) is a smooth vector bundle and \( Y \) is a small subset of \( F \);

(1b) for some \( \delta \in C^\infty(\mathcal{Z}; \mathbb{R}^+) \), \( \vartheta: F_\delta - (Y - Z) \to \mathcal{M} \) is a continuous map such that

(1b-i) \( \vartheta|_{Y - Z} \to \mathcal{M} \) is a homeomorphism onto an open neighborhood of \( Z \) in \( \mathcal{M} \);

(1b-ii) \( \vartheta|_Z \) is the identity map, and \( \vartheta: F_\delta - Y - Z \to \mathcal{M} \) is an orientation-preserving diffeomorphism on an open subset of \( \mathcal{M} \).

(2) A **closure** of normal-bundle model \( (F, Y, \vartheta) \) for \( Z \) is a tuple \( (\bar{Z}, \bar{F}, \bar{\pi}) \), **where**

(2a) \( \bar{Z} \) is an \( ms \)-manifold with smooth base \( Z \);

(2b) \( \bar{\pi}: \bar{Z} \to \bar{M} \) is an \( ms \)-map such that \( \bar{\pi}|_Z \) is the identity;

(2c) \( \bar{F} \to \bar{Z} \) is an \( ms \)-bundle such that \( \bar{F}|_Z = F \).

We use a normal-bundle model for \( Z \) to describe the behavior of bundle sections over \( \mathcal{M} \) near \( Z \). In particular, if \( \alpha: E \to \mathcal{O} \) is an \( ms \)-polynomial, we call \( Z \) an \( \alpha \)-**regular** subset of \( \mathcal{M} \) if for some normal-bundle model \( (F, Y, \vartheta) \) for \( Z \), \( \vartheta^* \alpha \) can be approximated, by a constant-rank polynomial
Let \( F \oplus E \rightarrow \mathcal{O} \); see Definition 3.9 in [Z2]. Polynomial \( \alpha : E \rightarrow \mathcal{O} \) is regular if \( \mathcal{M} \) can be decomposed into finitely many \( \alpha \)-regular subsets. If \( \text{rk } E + \frac{1}{2} \dim \mathcal{M} = \text{rk } \mathcal{O} \), for a generic \( \nu \in \Gamma(\mathcal{M}; \mathcal{O}) \), the zero set of the polynomial map

\[
\psi_{\alpha, \nu} : E \rightarrow \mathcal{O}, \quad \psi_{\alpha, \nu}(v) = \nu + \alpha(v),
\]

is a zero-dimensional oriented submanifold of \( E|\mathcal{M} \). By Lemma 3.10 in [Z2], if \( \alpha \) is a regular polynomial, \( \psi_{\alpha, \nu}^{-1}(0) \) is a finite set for a generic choice of \( \nu \), and \( N(\alpha) \equiv \sum |\psi_{\alpha, \nu}^{-1}(0)| \) is independent of such a choice of \( \nu \).

As described below, counting the zeros of \( \psi_{\alpha, \nu} \) involves determining the contribution \( C_Z(s) \) to the Euler class of a bundle \( V \) from a subset \( Z \) of the zero set of a section \( s \) of \( V \). In the cases we encounter in Section 3, \( Z \) decomposes into disjoint, and usually non-compact, complex manifolds \( Z_i \) near which the behavior of \( s \) can be understood. Then \( C_Z(s) = \sum C_{Z_i}(s) \), where \( C_{Z_i}(s) \) is the \( s \)-contribution of \( Z_i \) to \( e(V) \). This is the signed number of elements of \( \{ s + \nu \}^{-1}(0) \) that lie very close to \( Z_i \), where \( \nu \in \Gamma(\mathcal{M}; V) \) is a small generic perturbation of \( s \). The manifolds \( Z_i \) we encounter fall in one of the two categories described below.

**Definition 2.3** Suppose \( \mathcal{M} \) is an ms-manifold of dimension 2n, \( V \rightarrow \mathcal{M} \) is an ms-bundle of rank \( n \), \( s \in \Gamma(\mathcal{M}; V) \), and \( Z \subset s^{-1}(0) \).

1. \( Z \) is \( s \)-hollow if there exist a normal-bundle model \((F, Y, \theta)\) for \( Z \) and a bundle isomorphism \( \vartheta_V : \vartheta^*V \rightarrow \pi^*_V \mathcal{O} \), covering the identity on \( F_\delta - (Y - Z) \), such that
   - \( \vartheta_V| F_\delta - (Y - Z) \) is smooth and \( \vartheta_V| Z \) is the identity;
   - \( \text{(1b) the map } \phi \equiv \vartheta_V \circ \vartheta^* : F_\delta - (Y - Z) \rightarrow V \) is hollow.
2. \( Z \) is \( s \)-regular if there exist a normal-bundle model \((F, Y, \theta)\) for \( Z \) with closure \((\bar{Z}, \bar{F}, \pi)\), regular polynomial \( \alpha : \bar{F} \rightarrow \pi^*V \), and a bundle isomorphism \( \vartheta_V : \vartheta^*V \rightarrow \pi^*_V \mathcal{O} \) covering the identity on \( F_\delta - (Y - Z) \), such that
   - \( \text{(2a) } \vartheta_V| F_\delta - (Y - Z) \) is smooth and \( \vartheta_V| Z \) is the identity;
   - \( \text{(2b) } \alpha| Z \) is nondegenerate and \( \vartheta_V| Z \) is the resolvent for \( \phi \equiv \vartheta_V \circ \vartheta^* : F_\delta - (Y - Z) \rightarrow V \), and \( Y \) is preserved under scalar multiplication in each of the components of \( F \) for the splitting corresponding to \( \alpha \) as in (1) of Definition 2.4.

**Proposition 2.4** Let \( V \rightarrow \mathcal{M} \) be an ms-bundle of rank \( n \) over an ms-manifold of dimension 2n. Suppose \( \mathcal{U} \) is an open subset of \( \mathcal{M} \) and \( s \in \Gamma(\mathcal{M}; V) \) is such that \( s|\mathcal{U} \) is transversal to the zero set.

1. If \( s^{-1}(0) \cap \mathcal{U} \) is a finite set, \( \sum |s^{-1}(0) \cap \mathcal{U}| = \langle e(V), [\mathcal{M}] \rangle - C_{\mathcal{M} - \mathcal{U}}(s) \).
2. If \( \mathcal{M} - \mathcal{U} = \bigcup_{i=1}^{i=k} Z_i \), where each \( Z_i \) is \( s \)-regular or \( s \)-hollow, then \( s^{-1}(0) \cap \mathcal{U} \) is finite, and

\[
\sum |s^{-1}(0) \cap \mathcal{U}| = \langle e(V), [\mathcal{M}] \rangle - C_{\mathcal{M} - \mathcal{U}}(s) = \langle e(V), [\mathcal{M}] \rangle - \sum_{i=1}^{i=k} C_{Z_i}(s).
\]

If \( Z_i \) is \( s \)-hollow, \( C_{Z_i}(s) = 0 \). If \( Z_i \) is \( s \)-regular and \( \alpha_i : \bar{F}_i \rightarrow V \) is the corresponding polynomial,

\[
C_{Z_i}(s) = \langle e(\pi^*V/\alpha_i(\bar{F}_i)), [\bar{Z}_i] \rangle.
\]
This is Corollary 3.13 in [2]. Proposition 2.4 reduces the problem of computing \( C_{Z_i}(s) \) for an \( s \)-regular manifold \( Z_i \) to counting the zeros of a polynomial map between two vector bundles. The general setting for the latter problem is the following. Suppose \( E, O \to \mathcal{M} \) are ms-bundles such that \( \text{rk } E + \frac{1}{2} \dim \mathcal{M} = \text{rk } O \), and \( \alpha: E \to O \) is a regular polynomial. Let \( \tilde{\nu} \in \Gamma(\mathcal{M}; O) \) be such that the map

\[
\psi_{\alpha, \tilde{\nu}} \equiv \tilde{\nu} + \alpha: E \to O
\]

is transversal to the zero set in \( O \) on \( E|\mathcal{M} \), and all its zeros are contained in \( E|\mathcal{M} \). Then \( N(\alpha) \equiv \pm |\psi_{\alpha, \tilde{\nu}}^{-1}(0)| \) depends only on \( \alpha \). If the rank of \( E \) is zero, then clearly

\[
N(\alpha) = \pm |\psi_{\alpha, \tilde{\nu}}^{-1}(0)| = \langle e(O), [\mathcal{M}] \rangle.
\]

If the rank of \( E \) is positive and \( \tilde{\nu} \) is generic, it does not vanish and thus determines a trivial line subbundle \( \mathbb{C}\tilde{\nu} \) of \( O \). Let \( O^\perp = O/\mathbb{C}\tilde{\nu} \) and denote by \( \alpha^\perp \) the composition of \( \alpha \) with the quotient projection map. If \( E \) is a line bundle and \( \alpha \) is linear,

\[
N(\alpha) = \pm |\psi_{\alpha, \tilde{\nu}}^{-1}(0)| = \langle e(E^\ast \otimes O^\perp), [\mathcal{M}] \rangle - C_{\alpha^\perp(0)}(\alpha^\perp).
\]

By Proposition 2.4, computation of \( C_{\alpha^\perp(0)}(\alpha^\perp) \) again involves counting the zeros of polynomial maps, but with the rank of the new target bundle, i.e. \( E^\ast \otimes O^\perp \), one less than the rank of the original one, i.e. \( O \). Subsection 3.3 in [2] reduces the problem of determining \( N(\alpha) \) in all other cases to the case \( E \) is a line bundle and \( \alpha \) is linear. Thus, at least in reasonably good cases, \( N(\alpha) \) can be determined after a finite number of steps.

The next lemma summarizes the results of Subsection 3.3 in [2] in the case the original map \( \alpha: E \to O \) is linear. This case suffices for our purposes. We denote by

\[
\alpha' \in \Gamma(\mathbb{P}E; \text{Hom}(\gamma_E, \pi_E^* O))
\]

the section induced by \( \alpha \). Let \( \lambda_E = c_1(\gamma_E) \).

\[\text{Lemma 2.5} \quad \text{Suppose } \mathcal{M} \text{ is an ms-manifold and } E, O \to \mathcal{M} \text{ are ms-bundles such that } \]

\[
\text{rk } E + \frac{1}{2} \dim \mathcal{M} = \text{rk } O.
\]

If \( \alpha \in \Gamma(\mathcal{M}; E^\ast \otimes O) \) and \( \tilde{\nu} \in \Gamma(\mathcal{M}; O) \) are such that \( \alpha \) is regular, \( \tilde{\nu} \) has no zeros, the map

\[
\psi_{\alpha, \tilde{\nu}} \equiv \tilde{\nu} + \alpha: E \to O
\]

is transversal to the zero set on \( E|\mathcal{M} \), and all its zeros are contained in \( E|\mathcal{M} \), then \( \psi_{\alpha, \tilde{\nu}}^{-1}(0) \) is a finite set, \( |\psi_{\alpha, \tilde{\nu}}^{-1}(0)| \) depends only on \( \alpha \), and

\[
N(\alpha) \equiv \pm |\psi_{\alpha, \tilde{\nu}}^{-1}(0)| = \langle c(O)c(E)^{-1}, [\mathcal{M}] \rangle - C_{\alpha^\perp(0)}(\alpha^\perp).
\]

Furthermore, if \( n = \text{rk } E \),

\[
\lambda_E^n + \sum_{k=1}^{k=n} c_k(E)\lambda_E^{n-k} = 0 \in H^{2n}(\mathbb{P}E) \text{ and } \langle \mu \lambda_E^{n-1}, \mathbb{P}E \rangle = \langle \mu, [\mathcal{M}] \rangle \quad \forall \mu \in H^{2n-2}(\mathcal{M}). \tag{2.1}
\]
2.2 Notation

In this subsection, we describe the most important notation used in this paper. Some of the notation is only sketched; see Section 2 in [Z3] for more details.

If $I_1$ and $I_2$ are two sets, denote the disjoint union of $I_1$ and $I_2$ by $I_1 + I_2$. We set

$$\infty = (0, 0, -1) \in S^2 \subset \mathbb{R}^3 \quad \text{and} \quad e_\infty = (1, 0, 0) \in T_\infty S^2.$$  

Let $q_N: \mathbb{C} \to S^2 \subset \mathbb{R}^3$ be the stereographic projection mapping the origin in $\mathbb{C}$ to the north pole. We identify $\mathbb{C}$ with $S^2 - \{\infty\}$ via the map $q_N$.

**Definition 2.6** A finite partially ordered set $I$ is a linearly ordered set if for all $i_1, i_2, h \in I$ such that $i_1, i_2 < h$, either $i_1 \leq i_2$ or $i_2 \leq i_1$.

A linearly ordered set $I$ is a rooted tree if it has a unique minimal element, i.e. there exists $\hat{0} \in I$ such that $\hat{0} \leq i$ for all $i \in I$.

If $I$ is a linearly ordered set, let $\hat{I}$ be the subset of the non-minimal elements of $I$. For every $h \in \hat{I}$, denote by $\iota_h \in I$ the largest element of $I$ which is smaller than $h$. Suppose $I = \bigcup_{k \in K} I_k$ is the splitting of $I$ into rooted trees such that $k$ is the minimal element of $I_k$. If $\hat{I} \not \subseteq I$, we define the linearly ordered set $I + \hat{I}$ to be the set $I + \hat{I}$ with all partial-order relations of $I$ along with the relations $k < \hat{I}$ and $\hat{I} < h$ if $h \in I_k$.

If $S$ is a (possibly singular) complex curve and $M$ is a finite set, a $\mathbb{P}^n$-valued bubble map with $M$-marked points is a tuple

$$b = (S, M, I; x, (j, y), u),$$

where $I$ is a linearly ordered set, and

$$x: \hat{I} \to S \cup S^2, \quad j: M \to I, \quad y: M \to S \cup S^2, \quad \text{and} \quad u: I \to C_\infty(S; \mathbb{P}^n) \cup C_\infty(S^2; \mathbb{P}^n)$$

are maps such that

$$x_h \in \begin{cases} S^2 - \{\infty\}, & \text{if } \iota_h \in \hat{I}; \\ S, & \text{if } \iota_h \not \in \hat{I}; \end{cases} \quad y_l \in \begin{cases} S^2 - \{\infty\}, & \text{if } j_l \in \hat{I}; \\ S, & \text{if } j_l \not \in \hat{I}; \end{cases} \quad u_i \in \begin{cases} C_\infty(S^2; \mathbb{P}^n), & \text{if } i \in \hat{I}; \\ C_\infty(S; \mathbb{P}^n), & \text{if } i \not \in \hat{I}; \end{cases}$$

and $u_h(\infty) = u_{\iota_h}(x_h)$ for all $h \in \hat{I}$. We associate such a tuple with Riemann surface

$$\Sigma_b = \left( \bigsqcup_{i \in I} \Sigma_{b,i} \right) \bigg/ \sim, \quad \text{where} \quad \Sigma_{b,i} = \begin{cases} \{i\} \times S^2, & \text{if } i \in \hat{I}; \\ \{i\} \times S, & \text{if } i \not \in \hat{I}, \end{cases}$$

with marked points $(j_l, y_l) \in \Sigma_{b, j_l}$, and continuous map $u_b: \Sigma_b \to \mathbb{P}^n$, given by $u_b|\Sigma_{b,i} = u_i$ for all $i \in I$. We require that all the singular points of $\Sigma_b$ and all the marked points be distinct. Furthermore, if $S = S^2$, all these points are to be different from the special marked point $(\hat{0}, \infty) \in \Sigma_{b, \hat{0}}$. In addition, if $\Sigma_{b,i} = S^2$ and $u_i|S^2 = 0 \in H_2(\mathbb{P}^n; \mathbb{Z})$, then $\Sigma_{b,i}$ must contain at least two singular and/or marked points of $\Sigma_b$ other than $(i, \infty)$. If $S \neq S^2$, but $S$ is unstable, $u_i$ must satisfy a similar stability condition whenever $\Sigma_{b,i} = S$. In particular, if $S$ is a torus or a circle of spheres and the restriction of $u_i$ to a component $S_h$ of $S$ is homologically zero, $S_h$ contains at least one
marked point of $\Sigma_b$. Two bubble maps $b$ and $b'$ are equivalent if there exists a homeomorphism $\phi: \Sigma_b \to \Sigma_{b'}$ such that $u_b = u_{b'} \circ \phi, \phi(j_l, y_l) = (j'_l, y'_l)$ for all $l \in M$ and $\phi|_{\Sigma_{b,i}}$ is holomorphic for all $i \in I$.

The general structure of bubble maps is described by tuples $\mathcal{T} = (S, M, I; j, d)$, with $d_i \in \mathbb{Z}$ specifying the degree of the map $u_b$ on $\Sigma_{b,i}$. We call such tuples bubble types. Bubble type $\mathcal{T}$ is simple if $I$ is a rooted tree; $\mathcal{T}$ is basic if $\hat{I} = \emptyset$ and $d_i \neq 0$ for all $i \in I$; $\mathcal{T}$ is semiprimitive if $\chi_h \neq 1$, $d_i = 0$, and $d_h \neq 0$, for all $h \in \hat{I}$. The above equivalence relation on the set of bubble maps induces an equivalence relation on the set of bubble types. For each $h, i \in I$, let

$$D_i \mathcal{T} = \{ h \in \hat{I}: i < h \}, \quad \bar{D}_i \mathcal{T} = D_i \mathcal{T} \cup \{ i \}, \quad H_i \mathcal{T} = \{ h \in \hat{I}: \chi_h = i \}, \quad M_i \mathcal{T} = \{ l \in M: j_l = i \},$$

$$\chi \mathcal{T} h = \begin{cases} 0, & \text{if } \forall i \in I \text{ s.t. } h \in D_i \mathcal{T}, d_i = 0; \\
1, & \text{if } d_h \neq 0, \text{ but } \forall i \in I \text{ s.t. } h \in D_i \mathcal{T}, d_i = 0; \\
2, & \text{otherwise} \end{cases} \quad \chi(\mathcal{T}) = \{ h \in I: \chi \mathcal{T} h = 1 \}.$$

Denote by $\mathcal{H}_\mathcal{T}$ the space of all holomorphic bubble maps with structure $\mathcal{T}$.

The automorphism group of every bubble type $\mathcal{T}$ we encounter in the next two sections is trivial. Thus, every bubble type discussed below is presumed to be automorphism-free.

If $S$ is a circle of spheres, we denote by $\mathcal{M}_\mathcal{T}$ the set of equivalence classes of bubble maps in $\mathcal{H}_\mathcal{T}$. For each bubble type $\mathcal{T} = (S^2, M, I; j, d)$, let

$$\mathcal{U}_\mathcal{T} = \{ [b]: b = (S^2, M, I; x, (j, y), u) \in \mathcal{H}_\mathcal{T}, u_{i_1}(\infty) = u_{i_2}(\infty) \forall i_1, i_2 \in I - \hat{I} \}.$$

Then there exists $\mathcal{B}_\mathcal{T} \subset \mathcal{H}_\mathcal{T}$ such that $\mathcal{U}_\mathcal{T}$ is the quotient of a subset $\mathcal{B}_\mathcal{T}$ of $\mathcal{H}_\mathcal{T}$ by a $\tilde{G}_\mathcal{T} \equiv (S^1)^I$-action. Denote by $\mathcal{U}_\mathcal{T}^{(0)}$ the quotient of $\mathcal{B}_\mathcal{T}$ by $G_\mathcal{T} \equiv (S^1)^I \subset \tilde{G}_\mathcal{T}$. Then $\mathcal{U}_\mathcal{T}$ is the quotient of $\mathcal{U}_\mathcal{T}^{(0)}$ by the residual $G^{(0)}_\mathcal{T} \equiv (S^1)^{I - \hat{I}} \subset \tilde{G}_\mathcal{T}$ action. Corresponding to these quotients, we obtain line orbi-bundles $\{ L_i \mathcal{T} \to \mathcal{U}_\mathcal{T}: i \in I \}$. Let

$$\mathcal{F}_\mathcal{T} = \bigoplus_{h \in \hat{I}} \mathcal{F}_h \mathcal{T} \to \mathcal{U}_\mathcal{T}, \quad \text{where } \mathcal{F}_h \mathcal{T} = L_h \mathcal{T} \otimes L^*_h \mathcal{T}.$$ 

Denote by $\mathcal{F}_\mathcal{T}^{(0)}$ the open subset of $\mathcal{F}_\mathcal{T}$ consisting of vectors with all components nonzero.

Gromov topology on the space of equivalence classes of bubble maps induces a partial ordering on the set of bubble types and their equivalence classes such that the spaces

$$\bar{\mathcal{U}}^{(0)}_\mathcal{T} = \bigcup_{\mathcal{T} \leq \mathcal{T}'} \mathcal{U}^{(0)}_{\mathcal{T}'} \quad \text{and} \quad \bar{\mathcal{U}}_{\mathcal{T}} = \bigcup_{\mathcal{T} \leq \mathcal{T}'} \mathcal{U}_{\mathcal{T}'}$$

are compact and Hausdorff. The $G^{*}_{\mathcal{T}}$-action on $\mathcal{U}^{(0)}_{\mathcal{T}}$ extends to an action on $\bar{\mathcal{U}}^{(0)}_{\mathcal{T}}$, and thus the line orbi-bundles $L_i \mathcal{T} \to \mathcal{U}_\mathcal{T}$ with $i \in I - \hat{I}$ extend over $\bar{\mathcal{U}}_{\mathcal{T}}$. These bundles can be identified with the universal tangent line bundles for appropriate sections of the universal bundle over $\mathcal{U}_\mathcal{T}$. The evaluation maps

$$\text{ev}_l: \mathcal{H}_\mathcal{T} \to \mathbb{P}^n, \quad \text{ev}_l((S, M, I; x, (j, y), u)) = u_{j_l}(y_l),$$
descend to all the quotients and induce continuous maps on \( \bar{U}_T \) and \( \bar{U}_T^{(0)} \). If \( \mu = \mu_M \) is an \( M \)-tuple of subvarieties of \( \mathbb{P}^n \), let
\[
\mathcal{M}_T (\mu) = \{ b \in \mathcal{M}_T : \text{ev}_l (b) \in \mu_l \ \forall l \in M \}
\]
and define spaces \( \mathcal{U}_T (\mu) \), \( \bar{U}_T (\mu) \), etc. in a similar way. If \( S = S^2 \), we define another evaluation map,
\[
ev : \mathcal{B}_T \longrightarrow \mathbb{P}^n \ \text{by} \ev ((S^2, M, I; x, (j, y), u)) = u_0 (\hat{0}),
\]
where \( \hat{0} \) is any minimal element of \( I \). This map descends to \( \bar{U}_T^{(0)} \) and \( \bar{U}_T \). If \( \mu = \mu_{\tilde{M}} \) is an \( \tilde{M} \)-tuple of constraints, let
\[
\bar{U}_T (\mu) = \{ b \in \bar{U}_T : \text{ev}_l (b) \in \mu_l \ \forall l \in M \cap \tilde{M}, \ \text{ev}(b) \in \mu_i \ \forall l \in M - \tilde{M} \}
\]
and define \( \bar{U}_T^{(0)} (\mu) \), etc. similarly.

Suppose \( T = (S^2, M, I; j, d) \) is a bubble type, \( k \in I - \hat{I} \), and \( M_0 \) is nonempty subset of \( M_k T \). Let \( T / M_0 = (S^2, I, M - M_0; j | (M - M_0), d) \). Define \( T(M_0) \equiv (S^2, M, I + k \ \hat{1}; j', d') \) by
\[
j'_l = \begin{cases} 
k, & \text{if } l \in M_0; \\
\hat{1}, & \text{if } l \in M_k T - M_0; \\
j_l, & \text{otherwise};
\end{cases}
\quad d'_i = \begin{cases} 
0, & \text{if } i = k; \\
d_k, & \text{if } i = \hat{1}; \\
d_i, & \text{otherwise}.
\end{cases}
\]
The tuples \( T / M_0 \) and \( T(M_0) \) are bubble types as long as \( d_k \neq 0 \) or \( M_0 \neq M_0 T \). Then,
\[
\bar{U}_{T(M_0)} (\mu) = \tilde{\mathcal{M}}_{0, \{1\} + M_0} \times \bar{U}_{T/M_0} (\mu), \tag{2.2}
\]
where \( \tilde{\mathcal{M}}_{0, \{1\} + M_0} \) denotes the Deligne-Mumford moduli space of rational curves with \( \{\hat{0}, \hat{1}\} + M_0 \)-marked points. If \( T \) is a basic bubble type, let
\[
c_1(L_k^* T) \equiv c_1(L_k^* T) - \sum_{\emptyset \neq M_0 \subset M_k T} PD_{\bar{U}_T (\mu)} [\bar{U}_{T(M_0)}(\mu)] \in H^2 (\bar{U}_T (\mu)). \tag{2.3}
\]
This cohomology class is well-defined; see Subsection 5.2 in [Z2].

We are now ready to explain the claim of Theorem [□]. Let \( n, d, N \) and \( \mu \) be as in the statement of the theorem. If \( k \geq 1 \) and \( m \geq 1 \), denote by \( \bar{V}_{k, m} (\mu) \) the disjoint union of the spaces \( \bar{U}_T (\mu) \) taken over equivalence classes of basic bubble types \( T = (S^2, [N] - M_0, I; j, d) \) with \( |M_0| = m \), \( |I| = k \), and \( \sum d_k = d \). Let \( \bar{V}_k (\mu) = \bar{V}_{k, 0} (\mu) \). We define the spaces \( \mathcal{V}_{k, m} (\mu) \) similarly. Let
\[
\{ c_1(L_i^*: i \in [k]) \}, \{ c_1(L_i^*: i \in [k]) \} \subset H^2 (\bar{V}_{k, m} (\mu); \mathbb{Z})
\]
be given by
\[
\{ c_1(L_i^*: i \in [k]) : i \in I \} = \{ c_1(L_i^*: i \in I) \}; \quad \{ c_1(L_i^*: i \in [k]) : i \in I \} = \{ c_1(L_i^*: i \in I) \},
\]
and
whenever \( T \) is as above. We denote by \( \eta_i, \tilde{\eta}_i \in H^2(\hat{V}_{k,m}(\mu); \mathbb{Z}) \) the sum of all degree-\( l \) monomials in \( \{c_1(L_i^*): i \in [k]\} \) and in \( \{c_1(L^*_i): i \in [k]\} \), respectively. For example,
\[
\eta_2 = c_1^2(L^*_1) + c_1^2(L^*_2) + c_1(L^*_1)c_1(L^*_2) \in H^4(\hat{V}_{k,m}(\mu); \mathbb{Z}).
\]
Finally, let \( a = ev^*c_1(\gamma^*_{pn}) \in H^2(\hat{V}_{k,m}(\mu); \mathbb{Z}) \), where \( \gamma^*_{pn} \rightarrow \mathbb{P}^n \) denotes the tautological line bundle.

We next describe a generalization of the splitting (2.2) which is used in computations in Section 3. If \( T = (S^2, I, [N] - M_0; \bar{v}, \bar{l}) \) is a bubble type, let
\[
\mathcal{T} = (S^2, I, [N] - \bar{M}_0; j([N] - \bar{M}_0), \bar{d} [\bar{I}]), \quad \text{where} \quad \bar{I} = I - \{i \in I - \bar{I}: d_i = 0\}, \quad \bar{M}_0 = M_0 \cup \bigcup_{i \in I - \bar{I}} M_i T.
\]
Note that if \( T \) is semiprimitive, \( \mathcal{T} \) is basic. Furthermore,
\[
\begin{align*}
\mathcal{U}_T(\mu) &= \prod_{i \in I - \bar{I}} \mathcal{M}_{0, H, T + M_i T} \times \mathcal{U}_T(\mu), \quad (2.4) \\
\bar{\mathcal{U}}_T(\mu) &= \prod_{i \in I - \bar{I}} \bar{\mathcal{M}}_{0, H, T + M_i T} \times \bar{\mathcal{U}}_T(\mu), \quad (2.5)
\end{align*}
\]
where \( \mathcal{M}_{0, H, T + M_i T} \) denotes the main stratum of \( \bar{\mathcal{M}}_{0, H, T + M_i T} \). If \( i \in I - \bar{I} \), by definition, the bundle \( L_i T \rightarrow \bar{\mathcal{U}}_T(\mu) \) is the pullback by the projection map of the bundle
\[
L_0 T_i^{(0)} \rightarrow \bar{\mathcal{M}}_{0, H, T + M_i T} = \bar{\mathcal{U}}_T^{(0)}, \quad \text{where} \quad T_i^{(0)} = (S^2, H_i T + M_i T, \{0\}; \bar{0}, 0).
\]
We call the latter bundle the tautological line bundle over \( \bar{\mathcal{M}}_{0, H, T + M_i T} \). This is the universal tangent line at the marked point \( \bar{0} \in \bar{\mathcal{M}}_{0, H, T + M_i T} \).

Finally, if \( X \) is any space, \( F \rightarrow X \) is a normed vector bundle, and \( \delta: X \rightarrow \mathbb{R} \) is any function, let
\[
F_{\delta} = \{(b, v) \in F: |v|_b < \delta(b)\}.
\]
Similarly, if \( \Omega \) is a subset of \( F \), let \( \Omega_{\delta} = F_{\delta} \cap \Omega \). If \( v = (b, \bar{v}) \in F \), denote by \( b_v \) the image of \( \bar{v} \) under the bundle projection map, i.e. \( b \) in this case.

2.3 A Structural Description

We now describe the structure of the spaces \( \hat{V}_{k,m}(\mu) \) and the behavior of certain bundle sections over \( \hat{V}_{k,m}(\mu) \) near the boundary strata.

If \( b = (S^2, M, I; x, (j, y), u) \in \mathcal{B}_T \) and \( k \in I \), let
\[
\mathcal{D}_{T,k} b = d u_k |_{e_{\infty}} e_{\infty}.
\]
If \( T^* \) is a basic bubble type, the maps \( \mathcal{D}_{T,k} \) with \( T < T^* \) and \( k \in I - \bar{I} \) induce a continuous section of \( ev^* T \mathbb{P}^n \) over \( \bar{\mathcal{U}}_T^{(0)} \) and a continuous section of the bundle \( L_k^* T^* \otimes ev^* T \mathbb{P}^n \) over \( \mathcal{U}_T^* \), described by
\[
\mathcal{D}_{T^*, k}[b, c_k] = c_k \mathcal{D}_{T,k} b, \quad \text{if} \quad b \in \mathcal{U}_T^{(0)}, \quad c_k \in \mathbb{C}.
\]
Proposition 2.7 Suppose \( p > 2, \ n \geq 2, \ d \geq 1, \ N \geq 1, \ \mu = (\mu_1, \ldots, \mu_N) \) is an \( N \)-tuple of proper subvarieties of \( \mathbb{P}^n \) in general position, such that

\[
\text{codim}_{\mathcal{C}} \mu \equiv \sum_{l=1}^{l=N} \text{codim}_{\mathcal{C}} \mu_l - N = d(n+1) - 1,
\]

and \( M_0 \) is a subset of \([N]\). If \( T^* = (S^2, [N]-M_0, I^*; j^*, d^*) \) is a basic bubble type such that \( \sum d_i^* = d \), the space \( \mathcal{U}_T^*(\mu) \) is an ms-manifold of (real) dimension \( 2(n+1-2|I^*| - |M_0|) \) and \( L_k \mathcal{T}^* \) for \( k \in I^* \) and \( ev^* \mathcal{T} \mathbb{P}^n \) are ms-bundles over \( \mathcal{U}_T^*(\mu) \). If \( T = (S^2, [N]-M_0, I; j, d) < T^* \), there exist \( \delta, C \in C^\infty(\mathcal{U}_T(\mu); \mathbb{R}^+) \) and a homeomorphism

\[
\gamma^\mu_T: \mathcal{F}T_\delta \longrightarrow \mathcal{U}_T^*(\mu),
\]

onto an open neighborhood of \( \mathcal{U}_T(\mu) \) in \( \mathcal{U}_T^*(\mu) \) such that \( \gamma^\mu_T|_{\mathcal{U}_T(\mu)} \) is the identity and \( \gamma^\mu_T|_{\mathcal{F}^0T_\delta} \) is an orientation-preserving diffeomorphism onto an open subset of \( \mathcal{U}_T(\mu) \). Furthermore, with appropriate identifications,

\[
\left| D_{T^*,k} \gamma^\mu_T(v) - \alpha_{T,k}(\rho_T(v)) \right| \leq C(b_T)[v] \frac{1}{|T^*|} \rho_T(v) \quad \forall v \in \mathcal{F}T_\delta,
\]

where

\[
\rho_T(v) = ((\tilde{v}_h)_{h \in \chi(T)}) \in \tilde{\mathcal{F}}T \equiv \bigoplus_{h \in \chi(T)} L_h T \otimes L_{i_h} T; \quad \tilde{v}_h = \bigotimes_{i \in I, h \in D_i T} v_i; \quad \tilde{i}_h \in I - I, \ h \in D_i T
\]

\[
\alpha_{T,k}((\tilde{v}_h)_{h \in \chi(T)}) = \sum_{h \in I_k \cap \chi(T)} D_{T,k} \tilde{v}_h,
\]

and \( I_k \subset I \) is the rooted tree containing \( k \).

This is a special case of Theorem 2.8 in [22]; see also the remark following the theorem. The dimension of \( \mathcal{U}_T^*(\mu) \) is obtained as follows:

\[
\frac{1}{2} \dim \mathcal{U}_T^*(\mu) = \dim_{\mathcal{C}} \mathcal{U}_T^*(\mu) = \sum_{i \in I^*} (d_i(n+1) + n - 2) - (|I^*| - 1)n - (\text{codim}_{\mathcal{C}}\mu + |M_0|)
\]

\[
= n + 1 - 2|I^*| - |M_0|.
\]

The analytic estimate on \( D_{T^*,k} \) is crucial for implementation of the topological tools of Subsection 2.4 in Subsection 3.1. If \( T \) is semiprimitive, the bundle \( \mathcal{F}T = \tilde{\mathcal{F}}T \) and the section \( \alpha_T = \alpha_T \circ \rho_T \) extend over \( \mathcal{U}_T(\mu) \) via the decomposition (2.3). In terms of the notions of Subsection 2.1, \( (\mathcal{F}T, \mathcal{F}T - \mathcal{F}^0T, \gamma_T^\mu) \) is a normal-bundle model for \( \mathcal{U}_T^*(\mu) \subset \mathcal{U}_T^*(\mu) \). This normal-bundle model admits a closure if \( T \) is semiprimitive. Note that \( \mathcal{F}T \) is not usually the normal bundle of \( \mathcal{U}_T^*(\mu) \) in \( \mathcal{U}_T^*(\mu) \) if both spaces are viewed as algebraic stacks; see [23]. Proposition 2.7 implies only that the restrictions to \( \mathcal{U}_T(\mu) \) of \( \mathcal{F}T \) and of the normal bundle of \( \mathcal{U}_T^*(\mu) \) in \( \mathcal{U}_T^*(\mu) \) are isomorphic as topological vector bundles.

For any \( k, m \in \mathbb{Z} \), we define bundle \( E_{k,m} \rightarrow \tilde{V}_{k,m}(\mu) \) and homomorphism \( \alpha_{k,m}: E_{k,m} \rightarrow ev^* \mathcal{T} \mathbb{P}^n \) over \( \tilde{V}_{k,m}(\mu) \) by

\[
E_{k,m}|_{\mathcal{U}_T^*(\mu)} = \bigoplus_{i \in I^*} L_i T^*, \quad \alpha_{k,m}((v_i)_{i \in I^*}) = \sum_{i \in I^*} D_{T^*,i} v_i,
\]

whenever \( T^* = (S^2, [N]-M_0, I^*; j^*, d^*) \) is a basic bubble type such that \( \sum d_i^* = d \), \(|I^*|=k\), and \(|M_0|=m\). The following lemma will be used in Section 3.
Lemma 2.8 Suppose \( n \geq 2, d \geq 1, N \geq 1, \) and \( \mu = (\mu_1, \ldots, \mu_N) \) is an \( N \)-tuple of proper subvarieties of \( \mathbb{P}^n \) in general position such that \( \text{codim}_\mathbb{C} \mu = d(n + 1) - 1 \). If \( T = (S^2, [N] - M_0, I; j, d) \) is a bubble type such that \( \mathcal{U}_T(\mu) \subset \bar{V}_{k,m}(\mu) \), the restriction of \( \alpha_{k,m} \) to the subbundle

\[
E^\perp_T = \bigoplus_{i \in \chi(T) - I} L_i T \subset E_{k,m}
\]

is nondegenerate over \( \mathcal{U}_T(\mu) \).

Proof: The linear map \( \alpha_{k,m} \) has full rank on \( E^\perp_T \) over \( \mathcal{U}_T(\mu) \) if and only if the section

\[
\{ \alpha_{k,m}|E^\perp_T \}' \in \Gamma(\mathbb{P}E^\perp_T|\mathcal{U}_T(\mu); \gamma^*_{E^\perp_T} \otimes \text{ev}^* T \mathbb{P}^n)
\]

has no zeros. Note that

\[
\dim \mathbb{C} \mathbb{P}E^\perp_T|\mathcal{U}_T(\mu) \leq \dim \mathbb{C} V_k(\mu) + (k - 1) = n - k < n.
\]

Thus, it is enough to show that \( \{ \alpha_{k,m}|E^\perp_T \}' \) is transversal to the zero set in \( \mathbb{P}E^\perp_T|\mathcal{U}_T(\mu) \) if the constraints \( \mu \) are in general position. This last fact is immediate from Lemma 2.9.

Lemma 2.9 If \( u: S^2 \rightarrow \mathbb{P}^n \) is a holomorphic map of positive degree and \( e_\infty \in T_\infty S^2 \) is a nonzero vector, the linear maps

\[
H^0_\partial(S^2; u^* T \mathbb{P}^n) \rightarrow T_{u(\infty)} \mathbb{P}^n, \quad \xi \mapsto \xi(\infty),
\]

\[
\{ \xi \in H^0_\partial(S^2; u^* T \mathbb{P}^n); \xi(\infty) = 0 \} \rightarrow T_{u(\infty)} \mathbb{P}^n, \quad \xi \mapsto \nabla e_\infty \xi,
\]

are onto.

This lemma is well-known; see Corollary 6.3 in [Z2] for example.

3 Computations

3.1 Topology

In this section, we prove

Proposition 3.1 Suppose \( n \geq 2, d \geq 1, \) and \( \mu = (\mu_1, \ldots, \mu_N) \) is an \( N \)-tuple of proper subvarieties of \( \mathbb{P}^n \) in general position such that

\[
\text{codim}_\mathbb{C} \mu \equiv \sum_{l=1}^{l=N} \text{codim}_\mathbb{C} \mu_l - N = d(n + 1) - 1.
\]

Then the number of degree-\( d \) genus-one curves that have a fixed generic complex structure on the normalization and pass through the constraints \( \mu \) is given by

\[
n_{1,d}(\mu) = \frac{1}{2} (RT_{1,d}(\mu_1; \mu_2, \ldots, \mu_N) - CR_1(\mu), \quad \text{where}
\]

\[
CR_1(\mu) = \sum_{k=1}^{2k \leq n+1} (-1)^{k-1}(k-1)! \sum_{l=0}^{n+1-2k} \binom{n+1}{l} \langle d^l \eta_{n+1-2k-l}, [\bar{V}_k(\mu)] \rangle.
\]
We use the topological tools of Subsection 2.1 and the analytic estimate of Proposition 2.7 to
deduce Proposition 3.1 from Proposition 3.2. The main step is Lemma 3.3; the rest of this section
is essentially combinatorics.

Proposition 3.2 Suppose \( n \geq 2, \ d \geq 1, \) and \( \mu = (\mu_1, \ldots, \mu_N) \) is an \( N \)-tuple of proper subvarieties
of \( \mathbb{P}^n \) in general position such that \( \text{codim}_{\mathbb{C}} \mu = d(n+1) - 1 \). Then the number of degree-\( d \) genus-
one curves that have a fixed generic complex structure on the normalization and pass through the
constraints \( \mu \) is given by

\[
n_{1,d}(\mu) = \frac{1}{2} \left( RT_{1,d}(\mu_1; \mu_2, \ldots, \mu_N) - CR_1(\mu) \right), \quad \text{where} \quad CR_1(\mu) = N(\alpha_{1,0}),
\]
i.e. \( CR_1(\mu) \) is the number of zeros of the affine map

\[
\psi_{\alpha_{1,0}, \nu} : E_{1,0} = L_1 \to ev^*\mathbb{P}^n, \quad \psi_{\alpha_{1,0}, \nu}(v) = \nu + \alpha_{1,0}(v),
\]
over \( \bar{\nu}_1(\mu) \) for a generic section \( \bar{\nu} \in \Gamma(\bar{\nu}_1(\mu); ev^*\mathbb{P}^n) \).

Proposition 3.2 is basically the main result of the analytic part of [I]. The exact statement is not
made in [I], but it can be deduced from the arguments in [I] by comparing with the methods of [Z2].

The general meaning of Proposition 3.2 is the following. The number \( RT_{1,d}(\mu_1; \mu_2, \ldots, \mu_N) \) can
be viewed as the “euler class” of a bundle \( \Gamma^{0,1} \) over a closure \( C^\infty \) of the space \( C^\infty \) of smooth maps
from a fixed elliptic curve that pass through the constraints \( \mu_1, \ldots, \mu_N \); see [I]. Then,

\[
2n_{1,d}(\mu) = |\bar{\sigma}^{-1}(0) \cap C^\infty| = RT_{1,d}(\mu_1; \mu_2, \ldots, \mu_N) - \sum C_{\mathcal{M}_T(\mu)}(\bar{\sigma}), \quad (3.1)
\]
where \( \{ \mathcal{M}_T(\mu) \} \) are complex finite-dimensional, usually non-compact, manifolds that stratify
\( \bar{\sigma}^{-1}(0) \cap (C^\infty - C^\infty) \). Equation (3.1) is an infinite-dimensional analogue of (2) of Proposition 2.4.

In the finite-dimensional case, computation of a contribution to the euler class from an \( s \)-regular stratum \( Z \) of the zero set of section \( s \) reduces to counting the zeros of a polynomial map
between finite-rank vector bundles over \( \bar{Z} \), unless \( Z \) is \( s \)-hollow. The goal in the infinite-dimensional
\underline{case} under consideration is a reduction to the same problem and involves an adoption of the
obstruction-bundle idea of [I]. It turns out that \( C_{\mathcal{M}_T(\mu)}(\bar{\sigma}) = 0 \) for all but one stratum \( \mathcal{M}_T(\mu) \) of
\( \bar{\sigma}^{-1}(0) \cap (C^\infty - C^\infty) \). The number \( CR_1(\mu) \) described by Proposition 3.2 is the contribution
\( C_{\mathcal{M}_T(\mu)}(\bar{\sigma}) \) from the only stratum \( \mathcal{M}_T(\mu) \) of \( \bar{\sigma}^{-1}(0) \cap (C^\infty - C^\infty) \) that does contribute to the “euler
class” \( RT_{1,d}(\mu_1; \mu_2, \ldots, \mu_N) \) of \( \Gamma^{0,1} \).

As Subsection 2.1 suggests, computation of \( N(\alpha_{1,0}) \) may require going through a possibly large
tree of steps. We construct this tree as follows. Each node is a tuple \( \sigma = (r; k, m; \phi) \), where \( r \geq 0 \) is
the distance to the root \( \sigma_0 = (0; 1, 0; \cdot) \), \( k \geq 1 \), and \( m \geq 0 \). The tree satisfies the following properties.
If \( r > 0 \) and \( \sigma^* = (r-1; k^*, m^*; \phi^*) \) is the node from which \( \sigma \) is directly descendent, we require
that \( k^* \leq k, m^* \leq m \), and at least one of the inequalities is strict. Furthermore, \( \phi \) specifies a
splitting of the set \( [k] \) into \( k^* \)-disjoint subsets and an assignment of \( m - m^* \) of the elements of the set \( [m] = \{(1, 1), \ldots, (1, m)\} \) to these subsets. This description inductively constructs an infinite
tree. However, we will need to consider only the nodes \( \sigma = (r; k, m; \phi) \) with \( 2k + m \leq n + 1 \). We will
write \( \sigma \vdash \sigma' \) to indicate that \( \sigma \) is directly descendent from \( \sigma' \).
For each node in the above tree, we now define a linear map between vector bundles over an ms-manifold. If \( \sigma = (r; k, m; \phi) \), let \( \{ \sigma_s = (s; k_s, m_s; \phi_s) : 0 \leq s \leq r \} \) be the sequence of nodes such that \( \sigma_r = \sigma \) and \( \sigma_s \mid \sigma_{s-1} \) for all \( s > 0 \). Put

\[
\tilde{V}_\sigma = \tilde{V}_{k,m}(\mu), \quad E_\sigma = E_{k,m} \rightarrow \tilde{V}_\sigma, \quad \alpha_\sigma = \alpha_{k,m}, \quad \chi_\sigma = Y_\sigma \times \tilde{V}_\sigma, \quad X_{\sigma,s} = Y_{\sigma,s} \times \tilde{V}_\sigma,
\]

where \( Y_\sigma = Y_{\sigma,r}, \quad Y_{\sigma,0} = \{ pt \}, \quad Y_{\sigma,s} = \mathbb{P}F_s \times Y_{\sigma,s-1} \) if \( s > 0 \),

\[
\mathcal{M}_\sigma = \bigcap_{i \in \text{Im } \phi} \mathcal{M}_{0,i+\phi-1(i)}, \quad F_\sigma = \bigoplus_{i \in \text{Im } \phi} \gamma_{\sigma;i} \rightarrow \mathcal{M}_\sigma.
\]

For the purposes of the last line above, we view \( \phi \) as a map from \([k] - [k^*]\) and a subset of \([m]\) to \([k^*]\) in the notation of the previous paragraph. Then, \( \gamma_{\sigma;i} \rightarrow \mathcal{M}_{0,i+\phi-1(i)} \) is the tautological line bundle; see Subsection 2.2. Denote by \( \gamma_{F_{\sigma,0}} \) the (trivial) line bundle over \( Y_{\sigma,0} \). Let

\[
\mathcal{O}_\sigma = \mathcal{O}_{\sigma,r}, \quad \mathcal{O}_{\sigma,0} = \text{ev}^*\text{TP}^n, \quad \mathcal{O}_{\sigma,s} = \mathcal{O}_{\sigma,s-1} / \text{Im } \tilde{\nu}_{\sigma,s-1} \quad \text{if } s > 0,
\]

where \( \tilde{\nu}_{\sigma,s} \in \Gamma(X_{\sigma,s}; \text{Hom}(\gamma_{F_{\sigma,s}}, \mathcal{O}_{\sigma,s})) \) is a generic section. Since \( k_s - 1 \leq k_s, m_s - 1 \leq m_s \), and one of the inequalities is strict,

\[
\frac{1}{2} \dim X_{\sigma,s} \leq \frac{1}{2} \dim X_\sigma = (n+1-2k-m) + \sum_{s=1}^{s=r} (|\text{Im } \phi_s| - 1) = n - k - r < \text{rk } \mathcal{O}_{\sigma,0} - r.
\]

Thus, we see inductively that each bundle \( \mathcal{O}_{\sigma,s} \) is well-defined and a generic section \( \tilde{\nu}_{\sigma,s} \) of \( \text{Hom}(\gamma_{F_{\sigma,s}}, \mathcal{O}_{\sigma,s}) \) does not vanish. Let \( \pi_\sigma : \text{ev}^*\text{TP}^n \rightarrow \mathcal{O}_\sigma \) be the projection map. We define

\[
\tilde{\alpha}_\sigma \in \Gamma(X_\sigma; \text{Hom}(\gamma_{F_{\sigma}} \otimes E_\sigma; \gamma_{F_{\sigma}} \otimes \mathcal{O}_\sigma)), \quad \text{by } \{ \tilde{\alpha}_\sigma(\tau \otimes v) \} (w) = \tau(w) \cdot \pi_\sigma \alpha_\sigma(v) \in \mathcal{O}_\sigma.
\]

Note that \( \tilde{\alpha}_{\sigma_0} = \alpha_{1,0} \).

**Lemma 3.3** For every node \( \sigma^* \),

\[
N(\tilde{\alpha}_{\sigma^*}) = \{ c(\gamma_{F_{\sigma^*}} \otimes \mathcal{O}_{\sigma^*}) c(\gamma_{F_{\sigma^*}} \otimes E_{\sigma^*})^{-1}, [X_{\sigma^*}] \} = \sum_{\sigma \mid \sigma^*} N(\tilde{\alpha}_\sigma).
\]

**Remark:** For a dense open subset of tuples \( \{ \tilde{\nu}_{\sigma,s} \} \), the corresponding linear map \( \alpha_\sigma \) constructed above is regular and \( N(\alpha_\sigma) \) is independent of the choice of \( \{ \tilde{\nu}_{\sigma,s} \} \). What we need is that for every bubble type \( \mathcal{T} \) such that \( U_\mathcal{T}(\mu) \subset V_{\nu_{\sigma,m_r}}(\mu) \) the intersection of the image of the linear map

\[
\alpha_\mathcal{T} \in \Gamma(Y_\sigma \times U_\mathcal{T}(\mu); \text{Hom}(\bigoplus_{i \in \chi(\mathcal{T})} L_i \mathcal{T}, \text{ev}^*\text{TP}^n)), \quad \alpha_\mathcal{T}(v) = \sum_{i \in \chi(\mathcal{T})} D_{\mathcal{T},i} v_i,
\]

with the subbundle

\[
\text{Im } \tilde{\nu}_{\sigma,0} \oplus \ldots \oplus \text{Im } \tilde{\nu}_{\sigma,r-1} \subset \mathcal{O}_{\sigma,0} = \text{ev}^*\text{TP}^n
\]

is \( \{0\} \). The fact that this condition is satisfied for a dense open subset of tuples \( \{ \tilde{\nu}_{\sigma,s} \} \) follows by a dimension count as above, along with an argument similar to the proof of Lemma 3.10 in [Z2].

**Proof of Lemma 3.3** (1) By Lemma 2.3

\[
N(\tilde{\alpha}_{\sigma^*}) = \{ c(\gamma_{F_{\sigma^*}} \otimes \mathcal{O}_{\sigma^*}) c(\gamma_{F_{\sigma^*}} \otimes E_{\sigma^*})^{-1}, [X_{\sigma^*}] \} - C_{\tilde{\alpha}_{\sigma^*}}(0)(\tilde{\alpha}_{\sigma^*}^{\perp}).
\]
Let $\sigma^* = (r^*, k^*, m^*, \phi^*)$. By Lemma 2.8, $\tilde{\alpha}_{\sigma^*}^{-1}(0)$ is the union of the sets

$$Z^J_T \equiv \mathcal{Y}_{\sigma^*} \times \left( \mathbb{P}ET^J - \bigcup_{J' \subseteq J} \mathbb{P}ET^{J'} \right), \quad \text{where} \quad E^J_T = \bigoplus_{i \in J} L_i T \rightarrow \mathcal{U}_T(\mu),$$

taken over non-basic bubble types $T = (S^2, [N] - M_0, I; j, d)$, with $|I - \hat{I}| = k^*$, $|M_0| = m^*$, and $\sum d_i = d$, and nonempty subsets $J$ of $I - \hat{I} - \chi(T)$.

(2) The map $\gamma^J_T$ of Proposition 2.7 induces an orientation-preserving homeomorphism $\gamma^J_{Z_T}$ between open neighborhoods of $Z^J_T$ in

$$N Z^J_T \equiv FT \oplus \gamma^*_E T \otimes (ET^J - \hat{I} - \chi(T)) \oplus \mathbb{P}ET^\perp \rightarrow Z^J_T$$

and in $\mathcal{Y}_{\sigma^*} \times \mathbb{P}E_{\sigma^*}$. Furthermore, for some $\delta, C \in C^{\infty}(Z^J_T; \mathbb{R}^+)$, with appropriate identifications,

$$|\tilde{\alpha}_{\sigma^*}^l(\gamma^J_{Z_T}(b; v, u)) - \alpha^J_{\gamma^J_T}(\rho^J_{\gamma^J_T}(b; v, u))| \leq C(b) |v|^{\frac{1}{2}} |\rho^J_{\gamma^J_T}(b; v, u)| \quad \forall (v, u) \in N Z^J_T, \quad (3.3)$$

where $\rho^J_{\gamma^J_T} : N Z^J_T \rightarrow \tilde{N} Z^J_T \equiv \bigoplus_{h \in \chi(T)} \tilde{N}_h Z^J_T$, $\tilde{N}_h Z^J_T = \begin{cases} L^*_h T \otimes L_h T, & \text{if } h \in \hat{I}, \tilde{i}_h \in J; \\ \gamma^*_E T \otimes L_h T, & \text{otherwise}; \end{cases}$

$$\rho^J_{\gamma^J_T}(v, u) = \begin{cases} \rho_{\gamma^J_T}(v), & \text{if } h \in \hat{I}, \tilde{i}_h \in J; \\ u_{\tilde{i}_h} \otimes \rho_{\gamma^J_T}(v), & \text{if } h \in \hat{I}, \tilde{i}_h \notin J; \\ u_{\tilde{i}_h} & \text{if } h \in \chi(T) - \hat{I}; \end{cases}$$

$$\alpha^J_{Z_T} \in \Gamma \left( Z^J_T; \text{Hom}(\tilde{N} Z^J_T, \text{Hom}(\gamma^*_E T \otimes \gamma^*_E T, \gamma^*_E T \otimes \mathcal{O}_{\sigma^*})) \right)$$

$$\{ \{ \alpha^J_{Z_T}(\rho^J_{\gamma^J_T}(v, u)) \} \{ \tau \otimes \tilde{v} \} \} \{ w \} = \tau(w) \cdot \pi_{\sigma^*} \left( \sum_{i \in J} \{ \alpha^J_{T,i}(\rho_T v) \} \{ \tilde{u}_i \} \right)$$

$$+ \sum_{i \in \chi(T) - \hat{I}} \mathcal{D}_{T,i}(u_{\tilde{i}} \tilde{v}) + \sum_{i \in \hat{I} - \chi(T) - J} \{ \alpha^J_{T,i}(\rho_T v) \} \{ u_i \tilde{v} \} \in \mathcal{O}_{\sigma^*}.$$

Above $\rho_{\gamma^J_T}$ denotes the $h$th component of $\rho_T$, i.e. $\tilde{v}_h$ in the notation of Proposition 2.7. Note that the section $\alpha^J_{Z_T}$ is well-defined. By Lemma 2.8 and the splitting 2.4, possibly applied several times, $\pi_{\rho_{\gamma^J_T}} \circ \alpha^J_{Z_T}$ has full rank on every fiber of $\tilde{N} Z^J_T$, provided the sections $\{ \tilde{v}_\sigma^s : 0 \leq s \leq r^* \}$ are generic. Then by (3.3),

$$|\pi^1_{\rho_{\gamma^J_T}} \circ \tilde{\alpha}_{\sigma^*}^l(\gamma^J_{Z_T}(b; v, u)) - \pi^1_{\rho_{\gamma^J_T}} \circ \alpha^J_{Z_T}(\rho^J_{\gamma^J_T}(b; v, u))| \leq C(b) |v|^{\frac{1}{2}} |\rho^J_{\gamma^J_T}(b; v, u)| \quad (3.4)$$

for all $(v, u) \in N Z^J_T$. Thus, $\pi^1_{\rho_{\gamma^J_T}} \circ \alpha^J_{Z_T}$ is the resolvent for $\gamma^J_{Z_T}(\pi^1_{\rho_{\gamma^J_T}} \circ \tilde{\alpha}_{\sigma^*}^l)$; see Definition 2.1. If $T$ is not semiprimitive or $J \neq I - \hat{I} - \chi(T)$, the rank of $\tilde{N} Z^J_T$ is less than the rank of $N Z^J_T$. It follows that $Z^J_T$ is $\pi^1_{\rho_{\gamma^J_T}} \circ \tilde{\alpha}_{\sigma^*}^l$-hollow and

$$C^J_{Z_T}(\tilde{\alpha}_{\sigma^*}^l) = 0 \quad \text{if } T \text{ is not semiprimitive or } J \neq I - \hat{I} - \chi(T) \quad (3.5)$$

by Proposition 2.4.

(3) On the other hand, by the analytic estimate (3.4), Proposition 2.4 and the splitting 2.3,

$$C^J_{Z_T}(\tilde{\alpha}_{\sigma^*}^l) = N(\alpha_{\sigma^*, T}) \quad \text{if } T \text{ is semiprimitive and } J = I - \hat{I} - \chi(T), \quad (3.6)$$

where $\alpha_{\sigma^*, T} \in \Gamma(\mathcal{Y}_{\sigma^*, T} \times \mathcal{U}_T(\mu); \text{Hom}(N Z_{\sigma^*, T}, \gamma^*_E T \otimes \mathcal{O}_{\sigma^*, T}))$, $\mathcal{Y}_{\sigma^*, T} = \mathcal{Y}_{\sigma^*} \times \mathbb{P}ET$. 

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\[
ET = \bigoplus_{i \in I} \gamma_{\mathcal{T};i} \longrightarrow \mathcal{M}_{\sigma^*,\mathcal{T}} = \prod_{i \in I} \mathcal{M}_{0,H_i,\mathcal{T}+M_i}\mathcal{T}, \quad N\mathcal{Z}_{\sigma^*,\mathcal{T}} = \bigoplus_{i \in I} \gamma_{\mathcal{T};i}^* \otimes L_{h_i,\mathcal{T}} \oplus \bigoplus_{h \in I} \gamma_{\mathcal{T};i}^* \otimes L_{h_\mathcal{T}},
\]

\[
O_{\sigma^*,\mathcal{T}} = O_{\sigma^*}/\mathrm{Im}\, \tilde{\nu}_{\sigma^*} \approx \gamma_{F_{\sigma^*}}^* \otimes \left( (\gamma_{F_{\sigma^*}}^* \otimes O_{\sigma^*}) / \mathbb{C} \nu_{\sigma^*} \right),
\]

\[
\{ \alpha_{\sigma^*,\mathcal{T}}(u \otimes v) \} (\tilde{v}) = \pi_{\sigma^*,\mathcal{T}} \left( \sum_{i \in I} \omega_\tilde{i}(v_i) (D_{\mathcal{T},i}v_i) + \sum_{i \in \chi(\mathcal{T}) - \tilde{I}} u_i(\tilde{v}) (D_{\mathcal{T},i}v_i) \right) \in O_{\sigma^*,\mathcal{T}},
\]

\[
\gamma_{\mathcal{T};i} \longrightarrow \mathcal{M}_{0,H_i,\mathcal{T}+M_i}\mathcal{T} \text{ is the tautological line bundle, and } \pi_{\sigma^*,\mathcal{T}}: ev^*T\mathbb{P}^n \longrightarrow O_{\sigma^*,\mathcal{T}} \text{ is the quotient projection map. We next observe that}
\]

\[
N(\alpha_{\sigma^*,\mathcal{T}}) = N(\tilde{\alpha}_{\sigma^*,\mathcal{T}}), \quad \text{where}
\]

\[
k = |\chi(\mathcal{T})| = |\tilde{I}|, \quad m = m^* + \sum_{i \in I - \chi(\mathcal{T})} |M_i|, \quad \text{and } \{ \tilde{\alpha}_{\sigma^*,\mathcal{T}}(\tau \otimes v) \}(w) = \tau(w) \cdot \pi_{\sigma^*,\mathcal{T}}(\omega_{\tilde{\sigma},k,m}(v)).
\]

The reason for the equality (3.7) is the following. For a generic \( \tilde{v} \in \Gamma(\mathcal{Y}_{\sigma^*,\mathcal{T}} \times \mathcal{U}_{\mathcal{T}}(\mu); \gamma_{E_{\mathcal{T}}}^* \otimes O_{\sigma^*,\mathcal{T}}) \), the affine maps

\[
\psi_{\alpha_{\sigma^*,\mathcal{T}},\tilde{v}} \equiv \tilde{v} + \alpha_{\sigma^*,\mathcal{T}}: N\mathcal{Z}_{\sigma^*,\mathcal{T}} \longrightarrow \gamma_{E_{\mathcal{T}}}^* \otimes O_{\sigma^*,\mathcal{T}} \quad \text{and} \quad \psi_{\tilde{\alpha}_{\sigma^*,\mathcal{T}},\tilde{v}} \equiv \tilde{v} + \tilde{\alpha}_{\sigma^*,\mathcal{T}}: \gamma_{E_{\mathcal{T}}}^* \otimes E_{k,m} \longrightarrow \gamma_{E_{\mathcal{T}}}^* \otimes O_{\sigma^*,\mathcal{T}}
\]

have no zeros over the complement of \( \mathcal{Z}_{\mathcal{T}}^c \), since it is a finite union of smooth manifolds of dimension less than that of \( \mathcal{Z}_{\mathcal{T}}^c \). There is a canonical identification of the line bundle \( \gamma_{E_{\mathcal{T}}}^* \) with each line bundle \( \gamma_{\mathcal{T};i} \) over \( \mathcal{Z}_{\mathcal{T}}^c \). This identification induces a bijection between the zeros of the two affine maps that lie over \( \mathcal{Z}_{\mathcal{T}}^c \). The identity (3.7) follows from this argument along with

\[
N(\alpha_{\sigma^*,\mathcal{T}}) = \pm |\psi_{\alpha_{\sigma^*,\mathcal{T}},\tilde{v}}^{-1}(0)| \quad \text{and} \quad N(\tilde{\alpha}_{\sigma^*,\mathcal{T}}) = \pm |\psi_{\tilde{\alpha}_{\sigma^*,\mathcal{T}},\tilde{v}}^{-1}(0)|.
\]

(4) From equations (3.2), (3.3), (3.6), and (3.7), we conclude that

\[
N(\tilde{\alpha}_{\sigma^*}) = \langle c(\gamma_{F_{\sigma^*}}^* \otimes O_{\sigma^*}) c(\gamma_{F_{\sigma^*}}^* \otimes E_{\sigma^*})^{-1}, [X_{\sigma^*}] \rangle - \sum_{(k,m) > (k^*,m^*)} \sum_{i \in I - \chi(\mathcal{T})} |M_i| |M_i| = m - m^* \sum_{\sigma^*, \sigma^*} N(\tilde{\alpha}_{\sigma})
\]

\[
= \langle c(\gamma_{F_{\sigma^*}}^* \otimes O_{\sigma^*}) c(\gamma_{F_{\sigma^*}}^* \otimes E_{\sigma^*})^{-1}, [X_{\sigma^*}] \rangle - \sum_{\sigma^*, \sigma^*} N(\tilde{\alpha}_{\sigma}).
\]

The inner sum on the first line above is taken over all equivalence classes of semiprimitive bubble types \( \mathcal{T} = (S^2, N - M_0, I; j, d) \) such that \(|I - \tilde{I}| = k^*, |M_0| = m^*, \text{ and } \sum d_i = d\). Condition \((k,m) > (k^*,m^*)\) means that \( k \geq k, m \geq m^* \) and one of the inequalities is strict.

**Lemma 3.4** For every node \( \sigma = (r; k, m; \phi) \) and positive integer \( s \leq r - 1 \),

\[
\langle c(O_{\sigma,s+1}) c(E_{\sigma})^{-1}, [X_{\sigma,s}] \rangle = \langle c(O_{\sigma,s}) c(E_{\sigma})^{-1}, [X_{\sigma,s-1}] \rangle,
\]

where \( \{ \sigma_s \} \) is the sequence corresponding to \( \sigma \) defined in the paragraph preceding Lemma 2.3.

**Proof:** Since \( O_{\sigma,s+1} \approx O_{\sigma,s} / \gamma_{F_{\sigma,s}} \),

\[
\{ c(O_{\sigma,s+1}) c(E_{\sigma})^{-1} \}_{\dim X_{\sigma,s}} = \sum_{l=0}^{\dim X_{\sigma,s}} \sum_{l_1 + l_2 = l} (\sum_{l_1} X_{F_{\sigma,s}} l_2 (O_{\sigma,s}) \{ c(E_{\sigma})^{-1} \}_{\dim X_{\sigma,s-2l}}\ (3.8)
\]
By construction, $\lambda_{F_\sigma} \in H^*(\mathbb{P}F_\sigma)$, while $c(O_{\sigma,s}), c(E_\sigma) \in H^*(\mathcal{X}_{\sigma,s-1})$. Thus, (3.8) gives

$$
\{c(O_{\sigma,s+1})c(E_\sigma)^{-1}\}_{\dim \mathcal{X}_{\sigma,s}} = \lambda_{F_\sigma}^{n_{\sigma}} \sum_{l=0}^{\dim \mathcal{X}_{\sigma,s}} c_{l-n_{\sigma}}(O_{\sigma,s}) \{c(E_\sigma)^{-1}\}_{\dim \mathcal{X}_{\sigma,s-2l}}
$$

(3.9)

where $n_{\sigma} = \dim \mathbb{P}F_{\sigma}$. By (2.4),

$$
\langle \lambda_{F_\sigma}^{n_{\sigma}}, [\mathbb{P}F_\sigma] \rangle = \langle c(F_\sigma)^{-1}, [\mathcal{M}_\sigma] \rangle = \prod_{i \in \Im \phi_s} \langle c(\gamma_{\sigma_s};i)^{-1}, [\mathcal{M}_0;i+\phi^{-1}_{s}(i)] \rangle = 1.
$$

(3.10)

The last identity is a consequence of (1) of Lemma 3.11. The claim follows from (3.8)-(3.10).

**Corollary 3.5** For every node $\sigma = (r,k,m; \phi)$,

$$
\langle c(\gamma_{F_\sigma}^* \otimes O_{\sigma})c(\gamma_{F_\sigma}^* \otimes E_\sigma)^{-1}, [\mathcal{X}_{\sigma}] \rangle = \langle c(e^{*}T\mathbb{P}^n)c(E_{k,m})^{-1}, [\hat{V}_{k,m}(\mu)] \rangle.
$$

**Proof:** Since $rk O_{\sigma} = rk E_\sigma + \frac{1}{2} \dim \mathcal{X}_{\sigma}$, we can identify $E_\sigma$ with a subbundle of $O_{\sigma}$. Then,

$$
c(\gamma_{F_\sigma}^* \otimes O_{\sigma})c(\gamma_{F_\sigma}^* \otimes E_\sigma)^{-1} = c(\gamma_{F_\sigma}^* \otimes O_{\sigma}/\gamma_{F_\sigma}^* \otimes E_\sigma) = c(\gamma_{F_\sigma}^* \otimes (O_{\sigma}/E_\sigma)) \Rightarrow
$$

$$
\{c(\gamma_{F_\sigma}^* \otimes O_{\sigma})c(\gamma_{F_\sigma}^* \otimes E_\sigma)^{-1}\}_{\dim \mathcal{X}_{\sigma}} = \sum_{l=0}^{\dim \mathcal{X}_{\sigma}} \lambda_{F_\sigma}^{l} \{c(O_{\sigma})c(E_\sigma)^{-1}\}_{\dim \mathcal{X}_{\sigma-2l}}.
$$

(3.11)

Similarly to the proof of Lemma 3.4, (3.11) gives

$$
\langle c(\gamma_{F_\sigma}^* \otimes O_{\sigma})c(\gamma_{F_\sigma}^* \otimes E_\sigma)^{-1}, [\mathcal{X}_{\sigma}] \rangle = \langle c(O_{\sigma}), c(E_\sigma)^{-1}, [\mathcal{X}_{\sigma,r-1}] \rangle
$$

(3.12)

$$
= \langle c(O_{\sigma,r}), c(E_\sigma)^{-1}, [\mathcal{X}_{\sigma,r-1}] \rangle.
$$

Applying Lemma 3.4 to the last expression in (3.12) and using $O_{\sigma,1} \approx (e^{*}T\mathbb{P}^n)/\mathbb{C}$, we obtain

$$
\langle c(\gamma_{F_\sigma}^* \otimes O_{\sigma})c(\gamma_{F_\sigma}^* \otimes E_\sigma)^{-1}, [\mathcal{X}_{\sigma}] \rangle = \langle c(O_{\sigma,1}), c(E_\sigma)^{-1}, [\mathcal{X}_{\sigma,0}] \rangle = \langle c(e^{*}T\mathbb{P}^n)c(E_{k,m})^{-1}, [\hat{V}_{k,m}(\mu)] \rangle.
$$

We now combine Lemma 3.3 and Corollary 3.5 to obtain a topological formula for the number $N(\alpha_{1,0})$. For any integers $k$ and $k^*$, let $\theta_k^{k^*}$ denote the number of ways of splitting a set of $k^*$-elements into $k$ nonempty subsets. For every pair $(k^*, m^*) \geq (1,0)$ of integers, we define $\Theta(k^*, m^*)$ inductively by

$$
\Theta(1,0) = 1, \quad \Theta(k^*, m^*) = - \sum_{(1,0) \leq (k,m) < (k^*,m^*)} \binom{m^*}{m}^{k^* - m} \theta_k^{k^*} \Theta(k,m) \text{ if } (k^*,m^*) > (1,0).
$$

(3.13)

**Corollary 3.6** With notation as above,

$$
N(\alpha_{1,0}) = \sum_{(1,0) \leq (k,m)} \Theta(k,m) \sum_{l=0}^{n+1-(2k+m)} \binom{n+1}{l} \langle d^{n+1-(2k+m)-l}, [\hat{V}_{k,m}(\mu)] \rangle.
$$
Proof: By Lemma 3.3 and Corollary 3.5,
\[ N(\alpha_{1,0}) = N(\tilde{\alpha}_{1,0}) = \sum_{(1,0) \leq (k,m)} \Theta(k,m) \langle c(ev^*T_P^n)c(E_{k,m})^{-1}, [\tilde{V}_{k,m}(\mu)] \rangle. \] (3.14)

Since \( E_{k,m} = \bigoplus L_i \),
\[ c(E_{k,m})^{-1} = \prod_{i=1}^{\infty} (1 + c_1(L_i))^{-1} = \prod_{i=1}^{\infty} \sum_{l=0}^{\infty} c_1(L_i^*) = \sum_{l=0}^{\infty} \tilde{\eta}_l. \] (3.15)

The last equality above is immediate from the definition of \( \tilde{\eta}_l \); see Subsection 2.2. The claim follows from (3.14) and (3.15), along with \( c(ev^*T_P^n) = (1+a)^n+1 \).

3.2 Combinatorics

In this subsection, we show that the topological expression for \( N(\alpha_{1,0}) \) given in Corollary 3.6 is the same as the topological expression for \( CR_1(\mu) \) given in Proposition 3.1. This fact is immediate from Corollary 3.10. We start by proving an explicit formula for the numbers \( \Theta(k,m) \).

Lemma 3.7 If \( (k,m) \geq (1,0) \), \( \Theta(k,m) = (-1)^{k+m-1}k^m(k-1)! \).

(1) We first start verify this formula in the case \( k = 1 \). By (3.13),
\[ \Theta(1,0) = 1, \quad \Theta(1,m^*) = -\sum_{m=0}^{m^*-1} \binom{m^*}{m} \Theta(1,m) \quad \text{if} \quad m^* > (1,0). \] (3.16)

We need to show that \( \Theta(1,m) = (-1)^m \). If \( m = 0 \), this is the case. Suppose \( m^* \geq 1 \) and \( \Theta(1,m) = (-1)^m \) for all \( m < m^* \). Then, by (3.16),
\[ \Theta(1,m^*) = -\sum_{m=0}^{m^*-1} \binom{m^*}{m} \Theta(1,m) = -\sum_{m=0}^{m^*} \binom{m^*}{m} (-1)^m + (-1)^{m^*} = -(1-1)^{m^*} = (-1)^{m^*}, \]
as needed.

(2) We now verify the formula in the general case. It is easy to see from the definition of \( \theta_{k^*}^k \) in the previous subsection that
\[ \theta_{k^*}^k = 1 \quad \text{if} \quad k \geq 1 \quad \text{and} \quad \theta_{k^*}^k = k\theta_{k^*}^{k-1} + \theta_{k^*}^{k-1} \quad \text{if} \quad k \geq 2. \] (3.17)

Suppose \( k^* \geq 2 \) and the claimed formula holds for all \( (k,m) \) with \( (1,0) \leq (k,m) < (k^*,m^*) \). Then by (3.13),
\[ \Theta(k^*,m^*) = -\sum_{(1,0) \leq (k,m) < (k^*,m^*)} \binom{m^*}{m} k^{m^*-m} \theta_{k^*}^k \Theta(k,m) \]
\[ = k^{m^*} \sum_{(1,0) \leq (k,m) < (k^*,m^*)} (-1)^{k+m} \binom{m^*}{m} \theta_{k^*}^k (k-1)! \] (3.18)
Using (3.17), we obtain
\[ \sum_{(1,0) \leq (k,m) < (k^*, m^*)} (-1)^{k+m} \binom{m^*}{m} \theta_{k^*}^k (k-1)! = \sum_{(1,0) \leq (k,m) < (k^*, m^*)} (-1)^{k+m} \binom{m^*}{m} (k \theta_{k^*-1}^k + \theta_{k^*-1}^{k-1})(k-1)! \]
\[ = \sum_{m=0}^{m^*-1} (-1)^m \binom{m^*}{m} \sum_{k=1}^{k^*} (-1)^k (\theta_{k^*-1}^k! + \theta_{k^*-1}^{k-1}(k-1)!) + (-1)^{m^*} \sum_{k=1}^{k^*-1} (-1)^k (\theta_{k^*-1}^k! + \theta_{k^*-1}^{k-1}(k-1)). \]

Note that
\[ \sum_{k=1}^{k^*} (-1)^k (\theta_{k^*-1}^k! + \theta_{k^*-1}^{k-1}(k-1)!) = \sum_{k=1}^{k^*} (-1)^k \theta_{k^*-1}^k! - \sum_{k=0}^{k^*-1} (-1)^k \theta_{k^*-1}^{k-1}! = 0; \quad (3.20) \]
\[ \sum_{k=1}^{k^*-1} (-1)^k (\theta_{k^*-1}^k! + \theta_{k^*-1}^{k-1}(k-1)!) = \sum_{k=1}^{k^*-1} (-1)^k \theta_{k^*-1}^k! - \sum_{k=0}^{k^*-2} (-1)^k \theta_{k^*-1}^{k-1}! = (-1)^{k^*-1}(k^*-1)!, \]

since \( c_{k^*-1} = 0, c_{k^*-1}' = 1, \) and \( c_0 = 0 \) if \( k^* > 1. \) Combining equations (3.18) - (3.20), we verify the claimed identity for \( (k, m) = (k^*, m^*). \)

We next need to relate the intersection numbers \( \alpha^i \bar{\eta}^i \) and \( \alpha^i \eta^i \). We break the computation into several steps.

**Lemma 3.8** Suppose \( \mathcal{T} = (S^2, M, I; j, d) \) is a basic bubble type, \( i \in I, \) and \( M_i \subset M_\mathcal{T}. \) Then, under the splitting (2.2), with \( \hat{T} = \mathcal{T}/M_i, \)
\[ c_1(L^*_i \mathcal{T}) | \hat{U}_{\mathcal{T}(M_i)}(\mu) = \begin{cases} \gamma_1^* \times 1, & \text{if } i' = i; \\ 1 \times c_1(L^*_i \mathcal{T}), & \text{if } i' \neq i; \end{cases} \]
\[ c_1(L^*_i \mathcal{T}) | \hat{U}_{\mathcal{T}(M_i)}(\mu) = 1 \times c_1(L^*_i \hat{T}). \]

Proof: The first identity and the case \( i' \neq i \) of the second identity are immediate from the definitions.

In the remaining case, by (2.3), we have
\[ c_1(L^*_i \mathcal{T}) | \hat{U}_{\mathcal{T}(M_i)}(\mu) = c_1(L^*_i \mathcal{T}) | \hat{U}_{\mathcal{T}(M_i)}(\mu) - \sum_{\theta_M \subset M_\mathcal{T}} \text{PD}_{\hat{U}_M(\mu)} \hat{U}_{\mathcal{T}(M'_i)}(\mu) | \hat{U}_{\mathcal{T}(M_i)}(\mu). \]
\[ (3.21) \]

By definition of the spaces,
\[ \text{PD}_{\hat{U}_M(\mu)} \hat{U}_{\mathcal{T}(M'_i)}(\mu) | \hat{U}_{\mathcal{T}(M_i)}(\mu) = \begin{cases} 0, & \text{if } M'_i \nsubseteq M_i \text{ and } M_i \nsubseteq M'_i; \\ 1 \times \text{PD}_{\hat{U}_M(\mu)} \hat{U}_{\mathcal{T}(M'_i-M_i)}(\mu), & \text{if } M'_i \nsubseteq M_i; \\ \text{PD}_{\hat{U}_M(\mu)} \hat{U}_{\mathcal{T}(M'_i-M_i)} \times 1, & \text{if } M_i \nsubseteq M'_i. \end{cases} \]
\[ (3.22) \]

where \( M_0 = (S^2, \hat{I} + M_i, \{i\}; i, 0), \) i.e. \( \hat{U}_{M_0} = \hat{M}_{0,1+M_i}. \) Plugging (3.22), (2) of Lemma 3.11, and the case \( i' = i \) of the first statement of this lemma into (3.21), we obtain the remaining claim.

**Corollary 3.9** For all \( k \geq 1, m \geq 0, \) and \( l \geq 0, \)
\[ \langle \alpha^i \bar{\eta}^i_{n+1-(2k+m)-l}, [\hat{V}_{k,m}(\mu)] \rangle = \sum_{m^* \geq m} \binom{m^*}{m} k^{m^*-m} \langle \alpha^i \eta^i_{n+1-(2k+m*)-l}, [\hat{V}_{k,m^*}(\mu)] \rangle. \]
Proof: Let \( \mathcal{T} = (S^2, [N] - M_0, I; j, d) \) be a basic bubble type such that \(|I| = k\), \(|M_0| = m\), and \(\sum d_i = d\). By Lemma 3.8 and (1) of Lemma 3.11,

\[
\langle a^l \eta_{n+1-(2k+m)-l}, [\hat{U}_T(\mu)] \rangle = \sum_{M_0 \subset M_0' \subset [N]} \langle a^l \eta_{n+1-(2k+|M_0'|-l)}, [\hat{U}_{T/M_0'}(\mu)] \rangle,
\]

where \( T/M_0' = (S^2, [N] - M_0', I; j, d) \). The claim is obtained by summing (3.23) over all equivalence classes of bubble types \( \mathcal{T} \) of the above form.

Corollary 3.10 For all \( k \geq 1 \) and \( l \geq 0 \),

\[
\sum_{m \geq 0} \Theta(k, m) \langle a^l \eta_{n+1-(2k+m)-l}, [\hat{V}_k(m(\mu))] \rangle = (-1)^{k-1} (k-1)! \langle a^l \eta_{n+1-2k-l}, [\hat{V}_k(1)] \rangle.
\]

Proof: By Lemma 3.7 and Corollary 3.9,

\[
\sum_{m \geq 0} \Theta(k, m) \langle a^l \eta_{n+1-(2k+m)-l}, [\hat{V}_k(m(\mu))] \rangle
\]

\[
= (-1)^{k-1}(k-1)! \sum_{m \geq 0} \sum_{m' \geq m} (-1)^m \binom{m^*}{m} k^m \langle a^l \eta_{n+1-(2k+m^*-l)}, [\hat{V}_k(m^*(\mu))] \rangle
\]

\[
= (-1)^{k-1}(k-1)! \sum_{m' \geq 0} k^{m^*} \left( \sum_{m \leq m^*} (-1)^m \binom{m^*}{m} \right) \langle a^l \eta_{n+1-(2k+m^*-l)}, [\hat{V}_k(m^*(\mu))] \rangle
\]

\[
= (-1)^{k-1}(k-1)! \langle a^l \eta_{n+1-2k-l}, [\hat{V}_{k,0}(\mu)] \rangle,
\]

since

\[
\sum_{m \leq m^*} (-1)^m \binom{m^*}{m} k^m = (1 - 1)^{m^*} = 0 \quad \text{if} \quad m^* \neq 0.
\]

Lemma 3.11 (1) If \( J \) is a finite set of cardinality at least two, \( \langle c_1^{[J] - 2}(\gamma^*_j), [\mathcal{M}_{0,J}] \rangle = 1 \), where \( \gamma_j \rightarrow \mathcal{M}_{0,J} \) is the tautological line bundle.

(2) If \( \mathcal{T} = (S^2, M, I; j, d) \) is a basic bubble type, \( i \in I \), and \( M_i \) is nonempty subset of \( M_i \mathcal{T} \), under the splitting (2.3),

\[
PD_{\bar{U}_T(\mu)} \bar{U}_T(M_i)(\mu) \bar{U}_T(M_i)(\mu) = -1 \times c_1(L^* \mathcal{T}) + c_1(\gamma^*_{T,i}) \times 1 - \sum_{\emptyset \neq M'_i \subsetneq M_i} PD_{\bar{U}_T} \bar{U}_{T_0}(M_i - M'_i) \times 1,
\]

where \( T_0 = (S^2, \hat{1} + M_i, \{i\}; i, 0) \) and \( \mathcal{T} = T / M_i \).

Proof: (1) Both statements are straightforward consequences of well-known facts in algebraic geometry; see [2]. In our notation, \( \mathcal{M}_{0,J} \) is the Deligne-Mumford moduli space of rational curves with points marked by the set \( \{0\} + J \) and \( c_1(\gamma^*_j) = \psi_0 \). Thus, if \( j_1, j_2 \in J \) and \( j_1 \neq j_2 \),

\[
c_1(\gamma^*_j) = \psi_0^* = \sum_{\emptyset \neq J' \subset J \setminus \{j_1, j_2\}} PD_{\bar{U}_T} \bar{U}_{T_0}(J'),
\]

(3.24)
where $T_0 = (S^2, J, \{i\}; i, 0)$. Since $c_1(\gamma_{r})|U_{T_0(p')} = c_1(\gamma_{r+1}')$ under the decomposition (2.2), the first claim of the lemma follows from (3.24).

(2) Equation (3.24) implies that for any $\hat{1} \in J$,

$$c_1(\gamma_{r}) + \psi_1 = \sum_{\emptyset \neq J', J - \{1\}} \text{PD}_{U_{T_0}} U_{T_0}(J').$$

If $T, i,$ and $M_i$ are as in (2) of the lemma, under the splitting (2.2),

$$\text{PD}_{U_{T}(\mu)} U_{T(M_i)}(\mu) = -\psi_1 \times 1 - 1 \times \psi_0.$$ (3.26)

The second claim of the lemma follows from (3.25), applied with $J = \{\hat{1}\} + M_i$, and (3.26), since $1 \times \psi_0 = 1 \times c_1(L_i T)$.

4 Comparison of $n^{(1)}_d(\mu)$ and $n_{1,d}(\mu)$

4.1 Summary

In this section, we prove

**Proposition 4.1** Suppose $n \geq 2$, $d \geq 1$, and $\mu = (\mu_1, \ldots, \mu_N)$ is an $N$-tuple of proper linear subspaces of $\mathbb{P}^n$ in general position such that $\text{codim}_{\mathbb{P}^n} \mu = d(n+1) - 1$. Then

$$n^{(1)}_d(\mu) = n_{1,d}(\mu).$$

Denote by $\overline{M}_{1,1}$ the Deligne-Mumford moduli space of stable genus-one curves with one marked point and by $M_{1,1}$ the main stratum of $\overline{M}_{1,1}$, i.e. the complement of the point $\infty$ in $\overline{M}_{1,1}$. The elements of $M_{1,1}$ parameterize (equivalence classes of) smooth genus-one curves with one marked point. The point $\infty \in \overline{M}_{1,1}$ corresponds to a sphere with one marked point and with two other points identified.

Denote by $\overline{M} = \overline{M}_{1,N}(\mathbb{P}^n, d)$ the moduli space of stable degree-$d$ maps from $N$-pointed genus-one curves to $\mathbb{P}^n$. Let

$$\overline{M}(\mu) = \{b \in \overline{M} : \text{ev}_l(b) \in \mu \forall l \in [N]\}.$$

We denote by $\pi: \overline{M} \to \overline{M}_{1,1}$ the forgetful functor sending each stable map $b = [S, [N], I; x, (j, y), u]$ to the one-marked curve $[S, y_1]$ and contracting all unstable components of $(S, y_1)$. The resulting complex curve is either a torus or a sphere with two points identified. For any $\sigma \in \overline{M}_{1,1}$, let

$$\overline{M}_\sigma = \pi^{-1}(\sigma), \quad \overline{M}_\sigma(\mu) = \overline{M}_\sigma \cap \overline{M}(\mu).$$

If the $j$-invariant $\sigma$ is different from infinity, i.e. the stable curve $C_\sigma$ corresponding to $\sigma$ is smooth, the cardinality of $\overline{M}_\sigma(\mu)$ is $|\text{Aut}(C_\sigma)|$ times the number of genus-one degree-$d$ curves with $j$-invariant $\sigma$ that pass through the constraints $\mu$, i.e.

$$|\overline{M}_\sigma(\mu)| = 2n_{1,d}(\mu).$$ (4.1)
For every Lemma 4.3 a purely analytic proof can be found in [RT]. See Corollary 6.5 in [Z2] for example. The lemma follows from (4.2) by standard arguments.

\[ b = (S, [N], \{0\}; , (0, y), u) \in \mathcal{M}_\infty (\mu) \]

such that \( \Sigma_b \) is a sphere with two points identified and for every \( \sigma \in \mathcal{M}_{1,1} \) sufficiently close to \( \infty \), there exists a unique stable map \( b(\sigma) \in \mathcal{M}_\sigma (\mu) \) close to \( b \) in \( \mathcal{M} \), see Lemma 4.3. Since the number of stable maps

\[ b = (S, [N], \{0\}; , (0, y), u) \in \mathcal{M}_\infty (\mu) \]

such that \( \Sigma_b \) is a sphere with two points identified is \( 2n_d^{(1)} (\mu) \), Proposition 4.1 follows from the two lemmas, the corollary, and equation (4.1).

### 4.2 Dimension Counts

In this subsection, we show that if

\[ [b] = [S, [N], I; x, (j, y), u] \in \mathcal{M}_\infty (\mu) \]

and \( u_0 = u_b | S \) is not constant, then \( \Sigma_b = S \) is a sphere with two points identified; see Lemma 4.3. This lemma is proved by dimension counting. We then observe that for each such stable map \( b \) and every \( \sigma \in \mathcal{M}_{1,1} \) sufficiently close to \( \infty \), there exists a unique stable map \( b(\sigma) \in \mathcal{M}_\sigma (\mu) \) close to \( b \) in \( \mathcal{M} \); see Lemma 4.3.

**Lemma 4.2** If \([b] = [S, [N], I; x, (j, y), u] \in \mathcal{M}_\infty (\mu) \) and \( u_0 = u_b | S \) is not constant, then \( \Sigma_b = S \) is a sphere with two points identified.

**Proof:** Suppose \( T = (S, [N], I; j, d) \) is a simple bubble type such that \( S \) is a circle of \( k \) spheres, \( d_0 \neq 0 \), and \( \sum d_i = d \). Let \( U_T, \mathfrak{d}_0 \), denote the subspace of \( U_T \) such that the nonconstant restrictions of \( u_b \) to the components of \( S \) have degrees \( d_{0,1}, \ldots, d_{0,k'} \) for all \( b \in U_T. \mathfrak{d}_0 \). We must have \( \sum d_{0,l} = d_0 \).

Then, the dimension of \( \dim U_T, \mathfrak{d}_0 (\mu) \) is given by

\[
\left( \sum_{l=1}^{k'} (d_{0,l}(n+1)+n-1) - nk' + \sum_{i \in I} (d_i(n+1)+n-2-(n-1)) + (N-(k-k')) \right) - (\text{codim} \mu + N)
\]

\[ = 1 - |k| - |I|. \]

Thus, \( U_T (\mu) = \emptyset \) unless \( k = 1 \) and \( I = \emptyset \), i.e., \( \Sigma_b = S \) is a sphere with two points identified.

**Lemma 4.3** For every \([b] = [S, [N], \{0\}; , (0, y), u] \in \mathcal{M}_\infty (\mu) \) such that \( S \) is a sphere with two points identified, there exists neighborhood \( U_b \) of \( \infty \) in \( \mathcal{M}_{1,1} \) and \( W_b \) of \( b \) in \( \mathcal{M}_{1,1} (\mathbb{P}^n, d) \) such that

\[ \mathcal{N}_\infty (\mu) \cap W_b = 1 \quad \forall \sigma \in U_b - \{\infty\}. \]

**Proof:** Since \( d \geq 1 \),

\[ H^1(S; u_b^* \mathcal{O}(1_{\mathbb{P}^n})) = (n+1)H^1(S; u_b^* \mathcal{O}(1_{\mathbb{P}^n})) = 0, \quad (4.2) \]

see Corollary 6.5 in [22] for example. The lemma follows from (4.2) by standard arguments. A purely analytic proof can be found in [11].
4.3 A Property of Limits in $\overline{\mathcal{M}}_{1,N}(\mathbb{P}^n, d)$

Suppose $\{\sigma_k\} \subset \mathcal{M}_{1,1}$ converges to $\infty \in \overline{\mathcal{M}}_{1,1}$ and $b_k \in \overline{\mathcal{M}}_{\sigma_k}$ converges to

$$[b] = [S, [N], I; x, (j, y), u] \in \overline{\mathcal{M}}_{\infty}$$

such that $u_b | S$ is constant. In this subsection, we describe a condition such a limit $b$ must satisfy; see Lemma 4.4. This lemma is the key part of Section 3. Its proof extends the argument of $[P]$ for the $n=2$ case and makes use of the explicit notation described in Subsection 2.2. We conclude by observing that no element of $\overline{\mathcal{M}}_{\infty}(\mu)$ can satisfy this condition if the constraints $\mu$ are in general position.

**Lemma 4.4** Suppose

$$[b] = [S, [N], I; x, (j, y), u] \in \bigcup_{\sigma \in \mathcal{M}_{1,1}} \overline{\mathcal{M}}_{\sigma} \cap \mathcal{M}_T,$$

where $T = (S, [N], I; j, d)$ is a simple bubble type such that $S$ is a circle of spheres and $d_0 = 0$. Then the dimension of the linear span of the set $\{du_h |_{\infty, \infty} : h \in \chi(T)\}$ is less than $|\chi(T)|$.

**Proof:** (1) By the algebraic geometry definition of stable-map convergence, there exist

(i) a one-parameter family of curves $\tilde{\kappa} : \tilde{\mathcal{F}} \to \Delta$ such that $\Delta$ is a neighborhood of 0 in $\mathbb{C}$, $\tilde{\mathcal{F}}$ is a smooth space, $\tilde{\kappa}^{-1}(0) = \Sigma_{\delta}$, and $\Sigma = \tilde{\kappa}^{-1}(t)$ is a smooth genus-one curve for all $t \in \Delta^* \equiv \Delta - \{0\}$; (ii) a holomorphic map $\tilde{u} : \tilde{\mathcal{F}} \to \mathbb{P}^n$ such that $\tilde{u}|_{\tilde{\kappa}^{-1}(0)} = u_b$.

This family $\tilde{\kappa} : \tilde{\mathcal{F}} \to \Delta$ can be obtained from another family of curves $\kappa_0 : \mathcal{F}_0 \to \Delta$ that satisfies (i), except $\kappa_0^{-1}(0) = S$, by a sequence of blowups at smooth points of the central fiber as we now describe. Choose an ordering $\prec$ of the set $I$ consistent with its partial ordering. If $h \in I$, let

$$I^h = \{i \in I : i \prec h\}, \quad i(h) = \max I^h \text{ if } h \in \hat{I}, \quad I^{(h)} = I^h \cup \{h\}, \quad M(h) = \{t \in [N] : j_t \leq h\},$$

$$b(h) = (S^2, M(h), I^{(h)}; x|\tilde{\kappa}^{(h)}, (j, y)|M(h), u|I^{(h)}).$$

Suppose $h \in \hat{I}$ and we have constructed a one-parameter family of curves $\kappa_{i(h)} : \mathcal{F}_{i(h)} \to \Delta$ that satisfies (i), except $\kappa_{i(h)}^{-1}(0) = \Sigma_{\delta(i(h))}$. Let $\mathcal{F}_h$ be the blowup of $\mathcal{F}_{i(h)}$ at the smooth point of $(t_h, x_h)$ of $\Sigma_{\delta(i(h))}$ and let $\kappa_h : \mathcal{F}_h \to \Delta$ be the induced projection map. Choose coordinates $(t, w_h)$ near $(t_h, x_h) \in \mathcal{F}_{i(h)}$ such that $d\kappa_{i(h)} \frac{\partial}{\partial w_h} = 0$, i.e. $w_h$ is a coordinate in $\kappa_{i(h)}^{-1}(t)$ for $t \in \Delta$ sufficiently small. We define coordinates $(t, z_h)$ on a neighborhood in $\mathcal{F}_h$ of the complement of the node of the new exceptional divisor by

$$(t, z_h) \to (t, w_h = t z_h, [1, z_h]).$$

For a good choice of the family $\kappa_0 : \mathcal{F}_0 \to \Delta$, $\tilde{\mathcal{F}} = \mathcal{F}_h^*$ and $\tilde{\pi} = \pi_h^*$, where $h^*$ is the largest element of $I$ with respect to the ordering $\prec$.

(2) Let $\psi \in H^0(S; \omega_S)$ be a nonzero differential, i.e. $\psi$ is a holomorphic $(1, 0)$-form on the components of $S$, which has simple poles at the singular points of $S$ with residues that add up to zero at each node. Then, for each $h \in H_0\mathcal{T}$, there exists $a_h \in \mathbb{C}^*$ such that

$$\psi|_{(0, w_h)} = a_h (1 + o(1)) dw_h.$$

Thus, we can extend $\psi$ to a family of elements $\psi \in H^0(\Sigma_t; \omega_{\Sigma_t})$ such that

$$\psi|_{(t, w_h)} = a_h (1 + o(1)) dw_h, \quad \text{with } a_h \in \mathbb{C}^*. \quad (4.3)$$
If $h \in \hat{I}$, let $|h| = \{i \in I : i < h\}$. Denote by $\hat{h}$ the element of $H_0 T$ such that $h \in \hat{D}_h T$. By (1.3), we have

$$\psi|_{(t,z_h)} = \hat{t}^{|h|} a_{\hat{h}} (1 + o(1)) dz_h,$$  \hspace{2cm} (4.4)

with $a_{\hat{h}} \in \mathbb{C}^*$.  

(3) Let $H_1$ and $H_2$ be any two hyperplanes in $\mathbb{P}^n$ that intersect the image of $u_b$ transversally and miss the image of the nodes of $\Sigma_b$. Then for all $t$ sufficiently small and $i = 1, 2$,  

$$u_t^{-1}(H_i) = \{z^{(i)}_{j,h}(t), \ldots, z^{(i)}_{d',h_d}\} \subset \Sigma_t, \quad \text{where} \quad h_j \in \hat{I}, \quad z^{(i)}_{j,h}(t) = z^{(i)}_{j,h}(0) + o(1_t),$$  \hspace{2cm} (4.5)

$$z^{(i)}_{j,h}(0) \in \Sigma_{h,h}, \quad \text{and} \quad u_t = \hat{u}|\Sigma_t.$$  

Since $\sum z^{(1)}_{h}(t)$ and $\sum z^{(2)}_{j,h}(t)$ are linearly equivalent divisors in $\Sigma_t$,  

$$\sum_{j=1}^{j=d} \int_{t} z^{(2)}_{j,h}(t) \psi = 0 \quad \forall t \in \Delta^*,$$  \hspace{2cm} (4.6)

where each line integral is taken inside of an appropriate coordinate chart $(t, z_h)$. Plugging (4.4) and (4.5) into (4.6) gives  

$$\sum_{j=1}^{j=d} \int_{t} a_{\hat{h}} (z^{(2)}_{j,h}(t) - z^{(1)}_{j,h}(t) + o(1_t)) = 0 \quad \forall t \in \Delta^*.$$  \hspace{2cm} (4.7)

Let $k = \min \{|h| : h \in \chi(T)\}$; then $k = \min \{|h| : j \in [d]\}$. Thus, dividing equation (4.7) by $t^k$ and then taking the limit as $t \to 0$, we conclude that  

$$\sum_{|h_j|=k} a_{\hat{h}} z^{(1)}_{j,h}(0) = \sum_{|h_j|=k} a_{\hat{h}} z^{(2)}_{j,h}(0).$$  \hspace{2cm} (4.8)

(4) Equality (4.8) holds for a dense subset of pairs $(H_1, H_2)$. The consequences of this fact can be interpreted as follows. For each $h \in \hat{I}$, let $[u_h, v_h]$ be homogeneous coordinates on $\Sigma_{h,h}$ such that $z_h = v_h/u_h$. Each map $u_h$ corresponds to an $(n + 1)$-tuple of homogeneous polynomials  

$$p_{h,i} = \sum_{l=0}^{l=d_h} p_{h,i,l} t^l v^{d-l}, \quad i = 0, \ldots, n, \quad p_{h,i,l} \in \mathbb{C}.$$  

Equality (4.8) implies that there exists $K \in \mathbb{C}$ such that  

$$\sum_{|h|=k,d_h \neq 0} a_{\hat{h}} \sum_{l=0}^{l=n} c_l p_{h,i,l} = K \quad \forall [c_0, \ldots, c_n] \in \mathbb{P}^n.$$  \hspace{2cm} (4.9)

On the other hand, $u_{h_1}(\infty) = u_{h_2}(\infty)$ for all $h_1, h_2 \in \chi(T)$. Thus, for all $h_1, h_2 \in \chi(T)$, there exists $K_{h_1,h_2} \in \mathbb{C}^* - \{0\}$ such that  

$$\left(p_{h_1,0:d_{h_1}}, \ldots, p_{h_1,n:d_{h_1}}\right) = K_{h_1,h_2} \left(p_{h_2,0:d_{h_2}}, \ldots, p_{h_2,n:d_{h_2}}\right).$$

It follows that (4.9) is equivalent to  

$$\sum_{|h|=k,d_h \neq 0} a_{\hat{h}} \sum_{l=0}^{l=n} c_l p_{h,i,l} = K \quad \forall c_i \in \mathbb{C} \implies$$  \hspace{2cm} (4.10)

$$\sum_{|h|=k,d_h \neq 0} a_{\hat{h}} p_{h,i,l} = K p_{h,i,l}, \quad i = 0, \ldots, n.$$
where $h_1$ is a fixed element of the set $\{ h \in \hat{I} : |h| = k, d_h \neq 0 \}$ and $\tilde{a}_h \in \mathbb{C}^*$. It is straightforward to deduce from (4.10) that

$$\sum_{|h| = k, d_h \neq 0} \tilde{a}_h du_h |_{\infty e_{\infty}} = 0.$$ 

The lemma is now proved, since $\{ h \in \hat{I} : |h| = k, d_h \neq 0 \} \subset \chi(T)$.

**Corollary 4.5** Suppose

$$[b] = [S, [N], \hat{I}; x, (j, y), u] \in \bigcup_{\sigma \in \mathcal{M}_{1,1}} \mathcal{M}_\sigma \cap \mathcal{M}_\infty(\mu).$$

Then $u_b|S$ is not constant.

**Proof:** Suppose $u_b|S$ is constant. Let

$$\hat{I} = \{ i \in I : \chi_{\hat{I}} \neq 0 \} \subset \hat{I}, \quad M_0 = \bigcup_{i \in I - \hat{I}} M_i, \quad \hat{x} = x|\hat{I}, \quad (\hat{j}, \hat{y}) = (j, y)|([N] - M_0), \quad \hat{d} = d|\hat{I}, \quad \hat{u} = u|\hat{I};$$

$$\tilde{T} = (S^2, [N] - M_0, \hat{I}; \hat{j}, \hat{d}), \quad \tilde{b} = (S^2, [N] - M_0, \hat{I}; \hat{x}, (\hat{j}, \hat{y}), \hat{u}).$$

Then, $\tilde{T}$ is a bubble type such that $\sum \hat{d}_i = d$ and $\hat{d}_i > 0$ for all $i \in I - \hat{I}$. The latter property implies that $\chi(\tilde{T}) = \hat{I} - \hat{I}$. Furthermore, $\hat{b} \in \mathcal{U}_{\tilde{T}}(\mu)$. By Lemma 4.4, the linear map

$$\alpha|_{\chi(\tilde{T}), [M_0]} : \bigoplus_{i \in \chi(\tilde{T})} L_i \tilde{T} \longrightarrow \text{ev}^* T\mathbb{P}^n, \quad \alpha|_{\chi(\tilde{T}), [M_0]}(v) = \sum_{i \in \chi(\tilde{T})} \mathcal{D}_{\tilde{T}, i} v_i,$$

does not have full rank at $\tilde{b}$. However, this is impossible by Lemma 2.8.

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