Characterizing the combinatorics of distributed EPR pairs for multi-partite entanglement

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Abstract

We develop protocols for preparing a GHZ state and, in general, a pure multi-partite maximally entangled state in a distributed network with apriori quantum entanglement between agents using classical communication and local operations. We investigate and characterize the minimal combinatorics of the sharing of EPR pairs required amongst agents in a network for the creation of multi-partite entanglement. We also characterize the minimal combinatorics of agents in the creation of pure maximal multi-partite entanglement amongst the set $N$ of $n$ agents in a network using apriori multi-partite entanglement states amongst subsets of $N$. We propose protocols for establishing multi-partite entanglement in the above cases.

1 Introduction

Quantum entanglement is one of the most remarkable aspects of quantum physics. Two particles in an entangled state behave in some respects as a single entity even if the particles are physically separated by a great distance. The entangled particles exhibit what physicists call non-local effects. Such non-local effects were alluded to in the famous 1935 paper by Einstein, Podolsky, and Rosen [1] and were later referred to as spooky actions at a distance by Einstein [2]. In 1964, Bell [3] formalized the notion of two-particle non-locality in terms of correlations amongst probabilities in a scenario where measurements are performed on each
particle. He showed that the results of the measurements that occur quantum physically can be correlated in a way that cannot occur classically unless the type of measurement selected to be performed on one particle affects the result of the measurement performed on the other particle. The two particles thus correlated maximally are called EPR pairs or Bell pairs. Non-local effects, however, without being supplemented by additional quantum or classical communication, do not convey any signal and therefore the question of faster than light communication does not arise.

Entanglement is a key resource for quantum information processing and spatially separated entangled pairs of particles have been used for numerous purposes like teleportation [4], superdense coding [5] and cryptography based on Bell’s Theorem [6], to name a few. An EPR channel (a bipartite maximally entangled distributed pair of entangled particles) can be used in conjunction with a classical communication channel to send an unknown quantum state of a particle to a distant particle. The original unknown quantum state is destroyed in the process and reproduced at the other end. The process does not copy the original state; it transports a state and thus does not violate the quantum no-cloning theorem [7]. This process is called teleportation and was proposed by Bennett et al. in their seminal work [4]. In teleportation, a quantum communication channel is simulated using a classical channel and an EPR channel. A classical channel can also be simulated using a quantum channel and an EPR channel using superdense coding as proposed by Bennett and Wiesner [5]. Two qubits (classical bits) are compressed in to a qubit and sent through an EPR channel to the distant party, who then recovers the qubits by local operations.

The use of EPR pairs for cryptography was first proposed by Ekert in 1991 [6]. He proposed a protocol based on generalized Bell’s theorem for quantum key distribution between two parties. The two parties share an EPR pair in advance. They do a computational basis measurement on their respective qubits and the measurement result is then used as the one bit shared key. While the measurement result is maximally uncertain, the correlation between their results is deterministic. Based on similar principles, a multiparty quantum key distribution protocol using EPR pairs in a distributed network and its proof of unconditional security has been proposed by Singh and Srikanth [8]. Apart from these applications, entanglement has been used in several other applications such as cheating bit commitment [9], broadcasting of entanglement [10] and testing Bell’s inequalities [3, 11, 12].

Just as two distant particles could be entangled forming an EPR pair, it is also possible to entangle three or more separated particles. One example (called GHZ state) is due to Greenberger, Horne and Zeilinger [13]; here three particles are entangled. A well-known manifestation of multipartite entanglement is in testing nonlocality from different directions [13, 14, 15, 16]. Recently, it has also been used for many multi-party computation and communication tasks [17, 18, 19, 20, 21] and multi-party cryptography [22, 23, 24, 25, 26] and [27]. Buhrman, Cleve and van Dam [17], make use of three-party entanglement and demonstrate the existence of a function whose computation requires strictly lesser classical
communication complexity compared to the scenario where no entanglement is used. Brassard et al. in [19], show that prior multipartite entanglement can be used by $n$ agents to solve a multi-party distributed problem, whereas no classical deterministic protocol succeeds in solving the same problem with probability away from half by a fraction that is larger than an inverse exponential in the number of agents. Buhrman et al. [18] solves an $n$-party problem which shows a separation of $n$ versus $\Theta(n \log n)$ bits between quantum and classical communication complexity. For one round, three-party problem this article also proved a difference of $(n + 1)$ versus $((3/2)n + 1)$ bits between communication with and without initial entanglement. In [21], Pal et al. present a fair and unbiased leader election protocol using maximal multi-partite entanglement.

Quantum teleportation strikingly underlines the peculiar features of the quantum world. All the properties of the quantum state being transferred are retained during this process. So it is natural to ask whether one qubit of an entangled state can be teleported while retaining its entanglement with the other qubit. The answer is, not surprisingly, in the affirmative. This is called entanglement swapping. Yurke and Stoler [28] and Zukowski et al [29] have shown that through entanglement swapping one can entangle particles that do not even share any common past. This idea has been generalized to the tripartite case by Zukowski et al. in [30] and later to the multipartite case by Zeilinger et al. [31] and Bose et al. [24]. Zeilinger et al. presented a general scheme and realizable procedures for generating GHZ states out of two pairs of entangled particles from independent emissions. They also proposed a scheme for observing four-particle GHZ state and their scheme can directly be generalized to more particles. To create a maximally entangled state of $(n + m - 1)$ particles from two groups, one of $n$ maximally entangled particles and the other of $m$ maximally entangled particles, it is enough to perform a controlled operation between a particle from the first group and a particle from the second group and then a measurement of the target particle. We observe that if particles are distributed in a network then a single cbit of communication is required for broadcasting the measurement result to construct the desired $(n + m - 1)$ maximally entangled state using local operations. In [24], Bose et al. have generalized the entangled swapping scheme of Zukowski et al. in a different way. In their scheme, the basic ingredient is a projection onto a maximally entangled state of $N$ particles. Each of the $N$ users needs to share a Bell pair with a central exchange. The central exchange then projects the $N$ qubits with it on to an $N$ particle maximally entangled basis, thus leaving the $N$ users in an $N$ partite maximally entangled state. However, we note that in order to get a desired state, the measurement result obtained by the central exchange must be broadcast so that the end users can appropriately apply requisite local operations. This involves $N$ cbits of communication.

In this paper, we consider the problem of creating pure maximally entangled multi-partite states out of Bell pairs distributed in a communication network from a physical as well as a combinatorial perspective. We investigate and characterize the minimal combinatorics of the distribution of Bell pairs and show how this combinatorics gives rise to resource minimiza-
tion in long-distance quantum communication. We present protocols for creating maximal multi-partite entanglement. The first protocol (see Theorem 2.1) enables us to prepare a GHZ state using two Bell pairs shared amongst the three agents with the help of two cbits of communication and local operations with the additional feature that this protocol involves all the three agents dynamically. Such a protocol with local dynamic involvement in creating entanglement may find applications in cryptographic tasks. The second protocol (see Theorem 3.3) entails the use of $O(n)$ cbits of communication and local operations to prepare a pure $n$-partite maximally entangled state in a distributed network of Bell pairs; the requirement here is that the pairs of nodes sharing EPR pairs must form a connected graph. We show that a spanning tree structure (see Theorem 3.2) is the minimal combinatorial requirement for creating multi-partite entanglement. We also characterize the minimal combinatorics of agents in the creation of pure maximal multi-partite entanglement amongst the set $N$ of $n$ agents in a network using apriori multi-partite entanglement states amongst subsets of $N$. This is done by generalizing the EPR graph representation to an entangled hypergraph and the requirement here is that the entangled hypergraph representing the entanglement structure must be connected.

This paper is organized as follows. In Section 2, we present our protocol I to prepare a GHZ state from two Bell pairs involving all the three agents dynamically and compare our protocol in the light of existing schemes. Section 3 is devoted to characterizing the spanning tree combinatorics of Bell pairs for preparing a pure $n$-partite maximally entangled state. We develop our protocol II for this purpose. In Section 4 we generalize the results of Section 3 to the setting where subsets of agents in the network share apriori pure multi-partite maximally entangled states. Finally in Section 5 we compare our scheme of Section 3 with the multipartite entanglement swapping scheme of Bose et al. [24], observe the similarity between Helly-type theorems and the combinatorics developed in Sections 3 and 4 and conclude with a few remarks on open research directions.

2 Preparing a GHZ state from two EPR pairs shared amongst three agents

In this section we consider the preparation of a GHZ state from two EPR pairs shared amongst three agents in a distributed network. We establish the following theorem.

**Theorem 2.1** If any two pairs of the three agents $A$ (Alice), $B$ (Bob) and $C$ (Charlie) share EPR pairs (say the state $\left(\left|00\right> + \left|11\right>\right)/\sqrt{2}$) then we can prepare a GHZ state $\left(\left|000\right> + \left|111\right>\right)/\sqrt{2}$ amongst them with two bits of classical communication, while involving all the three agents dynamically.

Proof: The proof follows from the Protocol I.

*The Protocol I:* Without loss of generality let us assume that the sharing arrangement is as in Figure 2.1. $A$ shares an EPR pair with $B$ and another EPR pair with $C$ but $B$ and $C$
A shares an EPR pair with each of B and C. This means that we have the states $(|0_{a1}0_b⟩ + |1_{a1}1_b⟩)/\sqrt{2}$ and $(|0_{a2}0_c⟩ + |1_{a2}1_c⟩)/\sqrt{2}$ where subscripts $a1$ and $a2$ denote the first and second qubits with A and subscripts $b$ and $c$ denote qubits with B and C, respectively.

Our aim is to prepare $(|0_{a1}0_b0_c⟩ + |1_{a1}1_b1_c⟩)/\sqrt{2}$ or $(|0_{a2}0_b0_c⟩ + |1_{a2}1_b1_c⟩)/\sqrt{2}$. We need three steps to do so.

*Step 1:* A prepares a third qubit in the state $|0⟩$. We denote this state as $|0_{a3}⟩$ where the subscript $a3$ indicates that this is the third qubit of A.

*Step 2:* A prepares the state $(|0_{a1}0_b0_{a3}⟩ + |1_{a1}1_b1_{a3}⟩)/\sqrt{2}$ using the circuit in Figure 2.2.

*Step 3:* A sends her third qubit to C with the help of the EPR channel $(|0_{a2}0_c⟩ + |1_{a2}1_c⟩)/\sqrt{2}$. A straight forward way to do this is through teleportation. This method however, does not involve both of B and C dynamically. By a party being dynamic we mean that the party is involved in applying the local operations for the completion of the transfer of the state of
third qubit to create the desired GHZ state.

We use our new and novel teleportation circuit as shown in Figure 2.3 where both B and C are dynamic. The circuit works as follows. A has all her three qubits with her and can do any operation she wants to be performed on them. Initially the five qubits are jointly in the state $|\phi_1\rangle$. A first applies a controlled NOT gate on her second qubit controlling it from her third qubit changing $|\phi_1\rangle$ to $|\phi_2\rangle$. Then she measures her second qubit yielding measurement result $M_2$ and bringing the joint state to $|\phi_3\rangle$. She then applies a Hadamard gate on her third qubit and the joint state becomes $|\phi_4\rangle$. A measurement on the third qubit is then done by her yielding the result $M_1$ and bringing the joint state to $|\phi_5\rangle$. She then applies a NOT (Pauli’s X operator) on her first qubit, if $M_2$ is 1. Now she sends the measurement results $M_2$ to B and $M_1$ to C. B applies an X gate on his qubit if he gets 1 and C applies a Z gate (Pauli’s Z operator) if he gets 1. The order in which B and C apply their operations does not matter. The final state is $|\phi_7\rangle$. The circuit indeed produces the GHZ state between $A$, $B$ and $C$ as can be seen from the detailed mathematical explanation of the circuit given below. It can be noted that this protocol requires two cbits of communication.

The above circuit can be explained as follows:

$$|\phi_1\rangle = (|0_{a1}0_b0_{a3}\rangle + |1_{a1}1_b1_{a3}\rangle)(|0_{a2}0_c\rangle + |1_{a2}1_c\rangle)/2;$$

$$|\phi_2\rangle = [(|0_{a1}0_b0_{a3}\rangle(|0_{a2}0_c\rangle + |1_{a2}1_c\rangle) + |1_{a1}1_b1_{a3}\rangle(|1_{a2}0_c\rangle + |0_{a2}1_c\rangle))/2.$$ 

**Case 1: $M_2 = 0$**

$$|\phi_3\rangle = (|0_{a1}0_b0_{a3}0_{a2}0_c\rangle + |1_{a1}1_b1_{a3}0_{a2}1_c\rangle)/\sqrt{2};$$

$$= (|0_{a1}0_b0_{a3}0_{a2}0_c\rangle + |1_{a1}1_b1_{a3}0_{a2}1_c\rangle)/\sqrt{2},$$

$$|\phi_4\rangle = (|0_{a1}0_b0_{a3}0_{a2}0_c\rangle + |0_{a1}0_b1_{a3}0_{a2}c\rangle + |1_{a1}1_b0_{a3}1_{a2}c\rangle - |1_{a1}1_b1_{a3}1_{a2}c\rangle)/2);$$

$$= ([|0_{a1}0_b0_{a3}0_{a2}0_c\rangle + |1_{a1}1_b1_{a3}1_{a2}c\rangle])0_{a3}\rangle + (|0_{a1}0_b0_{a3}0_{a2}0_c\rangle - |1_{a1}1_b1_{a3}1_{a2}c\rangle)/2)/2.$$

When $M_1 = 0$,

$$|\phi_5\rangle = (|0_{a1}0_b0_{a3}0_{a2}0_c\rangle + |1_{a1}1_b1_{a3}0_{a2}0_c\rangle)/\sqrt{2},$$

$$|\phi_6\rangle = (|0_{a1}0_b0_{a3}0_{a2}0_c\rangle + |1_{a1}1_b1_{a3}0_{a2}0_c\rangle)/\sqrt{2},$$

$$|\phi_7\rangle = (|0_{a1}0_b0_{a3}0_{a2}0_c\rangle + |1_{a1}1_b1_{a3}0_{a2}0_c\rangle)/\sqrt{2}.$$ 

When $M_1 = 1$,

$$|\phi_5\rangle = (|0_{a1}0_b0_{a3}0_{a2}0_c\rangle - |1_{a1}1_b1_{a3}0_{a2}0_c\rangle)/\sqrt{2},$$

$$|\phi_6\rangle = (|0_{a1}0_b0_{a3}0_{a2}0_c\rangle - |1_{a1}1_b1_{a3}0_{a2}0_c\rangle)/\sqrt{2};$$
Figure 2.3: Circuit for creating a GHZ state from two EPR pairs with dynamic involvement of both $B$ and $C$. 
When $M_1 = 0$,

\[|\phi_5\rangle = (|0_a0_b1_c\rangle + |1_a1_b0_c\rangle)|0_a3\rangle|1_a2\rangle/\sqrt{2},\]

\[|\phi_6\rangle = (|1_a1_b1_c\rangle + |0_a10_b0_c\rangle)|0_a3\rangle|1_a2\rangle/\sqrt{2},\]

\[|\phi_7\rangle = (|1_a1_b1_c\rangle + |0_a10_b0_c\rangle)|0_a3\rangle|1_a2\rangle/\sqrt{2}.\]

When $M_1 = 1$,

\[|\phi_5\rangle = (|0_a10_b1_c\rangle - |1_a1_b0_c\rangle)|1_a3\rangle|1_a2\rangle/\sqrt{2},\]

\[|\phi_6\rangle = (|1_a1_b1_c\rangle - |0_a10_b0_c\rangle)|1_a3\rangle|1_a2\rangle/\sqrt{2},\]

\[|\phi_7\rangle = ( - |1_a1_b1_c\rangle - |0_a10_b0_c\rangle)|1_a3\rangle|1_a2\rangle/\sqrt{2} = (|1_a1_b1_c\rangle + |0_a10_b0_c\rangle)|1_a3\rangle|1_a2\rangle/\sqrt{2}.\]

The roles of $B$ and $C$ are symmetrical. Nevertheless, there is a condition on what operations they should perform when they get a single cbit from $A$. $B$ performs an $X$ and $C$ performs a $Z$ operation, as required. We set a cyclic ordering $A \rightarrow B \rightarrow C \rightarrow A$. Let $A$ be the one sharing EPR pairs with the other two; $A$ is the first one in the ordering. The second one is $B$, and he must perform an $X$ operation when he gets a single cbit from $A$. The third one is $C$, and he must perform a $Z$ operation on his qubit when he gets a single cbit from $A$. If $B$ is the one sharing EPR pairs with the other two then $C$ applies an $X$ on his qubit after getting a cbit from $B$ and, $A$ applies $Z$ on her qubit after getting a cbit from $B$ and so on.

As we mentioned in the introduction, methods for creating a GHZ state from Bell pairs have also been discussed by Zukowski et al. [30] and Zeilinger et al. [31]. First one uses three Bell pairs for this purpose, therefore our protocol seems better than theirs in the sense that it uses only two Bell pairs. The later, however, uses only two Bell pairs and only one cbit of
communication and seems to be better than our method at first sight. However, the most interesting fact and the motivation for developing our protocol is the dynamic involvement of both $B$ and $C$ which was lacking in the above methods. It might me highly desired in many multi-party interactive quantum protocols and multi-party cryptography (viz. secret sharing) that both $B$ and $C$ take part actively, say for fairness. By fairness we mean that every party has an equal chance for participating and effecting the protocol in probabilistic sense. It should be interesting to implement this in practical situation.

3 Preparing a pure $n$-partite maximally entangled state from EPR pairs shared amongst $n$ agents

Definition 1 (EPR graph): Suppose there are $n$ agents. We denote them as $A_1, A_2, ..., A_n$. Construct an undirected graph $G = (V, E)$ as follows:

$$V = \{A_i : i = 1, 2, 3, ..., n\},$$
$$E = \{\{A_i, A_j\} : A_i \text{ and } A_j \text{ share an EPR pair}, 1 \leq i, j \leq n; i \neq j\}.$$

We call the graph $G = (V, E)$, thus formed, the EPR graph of the $n$ agents.

Our definition should not be confused with the entangled graph proposed by Plesch and Buzek [32, 33]. In entangled graph edges represent any kind of entanglement and not necessarily maximal entanglement and therefore there is no one to one correspondence between graphs and states. EPR graph is unique up to different EPR pairs. Moreover, we are not concerned with classical correlations which are also represented by different kind of edges in entangled graphs. In an EPR graph, two vertices are connected by an edge if and only if they share an EPR pair.

Definition 2 (spanning EPR tree): We call an EPR Graph $G = (V, E)$ as spanning EPR tree when the undirected graph $G = (V, E)$ is a spanning tree [34].

We are now ready to develop protocol II to create the $n$-partite maximally entangled state $(|000...0\rangle + |111...1\rangle)/\sqrt{2}$ along a spanning EPR tree. The protocol uses only $O(n)$ qubits of communication and local operations. We do not use any qubit communication after the distribution of EPR pairs to form a spanning EPR tree.

Protocol II: Let $G = (V, E)$ be the spanning EPR tree. Since $G$ is a spanning EPR tree, it must have a vertex say $T = A_t$ which has degree one (the number of edges incident on a vertex is called its degree). Note that vertices of $G$ are denoted $A_i$, where $1 \leq i \leq n$. Let $S = A_s$ be the unique vertex connected to $T$ by an edge in $G$ and $L$ be the set of all vertices of $G$ having degree one. The vertex $T$ and its only neighbor $S$ are vertices we start with. We eventually prepare the $n$-partite maximally entangled state using the three steps summarized below. In the first step, one qubit (say $1$) is broadcasted by $S$ to signal other $(n - 1)$ parties that the protocol for the preparation of the $n$-partite entangled state is about to commence. The second step creates the GHZ state between $S = A_s, T = A_t$ and another neighbour $R = A_r$ of $S$. The third step is the main inductive step where multi-partite entanglement...
Figure 3.4: Entangling a new qubit in agent $i$. 
states are created in a systematic manner over the spanning EPR tree. At the end of step 3, when the \( n \)-partite entangled state is ready, one cbit of broadcating from all the \((k - 1)\) elements of \( L \setminus \{ T \} \) (terminal or degree one vertices of \( G \)) is expected. The \((k - 1)\)th such cbit indicates that the protocol is over and that maximally entangled state is ready. The details are stated below.

**Step 1:** \( S \) broadcast one classical bit to signal the other \((n - 1)\) agents that the preparation of an \( n \)-partite entangled state is going to be started and they must not use their EPR pairs for a qubit teleportation amongst themselves. In other words, they must save their EPR pairs in order to use them for the preparation of the \( n \)-partite entangled state.

**Step 2:** Clearly \( S \) must be connected to a vertex \( R \) (say \( A_r \)) other than \( T \) by an edge in \( G \), otherwise \( G \) will not be a spanning EPR tree. A GHZ state among \( S, T \) and \( A_r \) is created. The GHZ state can be prepared either by using usual teleportation circuit, the symmetric circuit of protocol I or by the Zeilinger et al. scheme. Thus using the EPR pairs \((|0_{s,t}0_{t,s}\rangle + |1_{s,t}1_{t,s}\rangle)/\sqrt{2}\) and \((|0_{s,r}0_{r,s}\rangle + |1_{s,r}1_{r,s}\rangle)/\sqrt{2}\), we prepare the GHZ state \((|0_{s,t}0_{t,s}0_{r,s}\rangle + |1_{s,t}1_{t,s}1_{r,s}\rangle)/\sqrt{2}\). Here the double subscript \( i, j \) denotes that in preparing the given state, EPR pairs among the agents \( A_i \) and \( A_j \) have been used. Here, \( A_s = S, A_r = R \) and \( A_t = T \).

**Step 3:** Suppose we are currently at vertex \( A_i \) and we have already prepared the \( m \)-partite maximally entangled state, say

\[
(|0_{i1,j1}0_{i2,j2}0_{i3,j3}...0_{im,jm}\rangle + |1_{i1,j1}1_{i2,j2}1_{i3,j3}...1_{im,jm}\rangle)/\sqrt{2},
\]

where \( i1 = s, j1 = t, i2 = t, j2 = s, i3 = r, j3 = s \) and \( i = ir \) for some \( 1 \leq r \leq m \).

The vertex \( A_i \) starts as follows. As soon as he gets two cbits from one of his neighbors, he completes the operations required for the success of teleportation and starts processing as follows. If \( A_i \in L \) then \( A_i \) broadcast a single cbit. Otherwise, (when \( A_i \notin L \)) let \( A_{k1}, A_{k2}, ..., A_{kp} \) be the vertices connected to \( A_i \) by an edge in \( G \) such that \( k1, k2, ..., kp \) are not in the already entangled set with vertex indices \( \{i1, i2, i3, ..., im\} \). \( A_i \) takes an extra qubit and prepares this qubit in the state \(|0_1\rangle \) denoted by \(|0_i\rangle \). He then prepares the state

\[
(|0_{i1,j1}0_{i2,j2}0_{i3,j3}...0_{im,jm}0_i\rangle + |1_{i1,j1}1_{i2,j2}1_{i3,j3}...1_{im,jm}1_i\rangle)/\sqrt{2}
\]

using the circuit in Figure 3.4. Finally, he teleports his extra qubit to \( A_{k1} \) using the EPR pair \((|0_{l,k1}0_{k1,i}\rangle + |1_{l,k1}1_{k1,i}\rangle)/\sqrt{2}\), thus enabling the preparation of the \((m + 1)\)-partite maximally entangled state:

\[
(|0_{i1,j1}0_{i2,j2}0_{i3,j3}...0_{im,jm}0_{k1,i}\rangle + |1_{i1,j1}1_{i2,j2}1_{i3,j3}...1_{im,jm}1_{k1,i}\rangle)/\sqrt{2}.
\]

\( A_i \) repeats this until no other vertex, which is connected to it by an edge in \( G \), is left.

Step 3 is repeated until one cbit each from the elements of \( L \) (except for \( T \)) is broadcasted, indicating that all vertices in \( L \) as well as in \( V \setminus L \) have got entangled.

Note that more than one vertex might be processing Step 3 at the same time. This however does not matter since local operations do not change the reduced density matrix of
other qubits. Moreover, while processing the Step 3 together, such vertices will no longer be directly connected by an edge in $G$.

Now we determine the communication complexity of protocol II, the number of cbits used in creating the $n$-partite maximally entangled state. Step 1 involves one cbit broadcast by $S$ to signal the initiation of the protocol. To create the GHZ state in Step 2, at most 2 cbits is required. In Step 3, teleportation is used to create an $(m + 1)$-partite maximally entangled state from that of $m$-partite. Such $(n - 3)$ teleportation steps are used in this Step entailing $2(n - 3)$ cbits of communication. Finally, $(k - 1)$ cbits are broadcast by terminal vertices (except $T$). Thus the total cbits used in protocol II is $1 + 2 + 2(n - 3) + k - 1 = 2n + k - 4 \leq 2n + n - 1 - 4 = 3n - 5 = O(n)$.

Protocol II leads to the following interesting theorem.

**Theorem 3.1** If the combinatorial arrangement of distributed EPR pairs amongst $n$ agents forms a spanning EPR tree, then the $n$-partite maximally entangled state $(|000\ldots0\rangle + |111\ldots1\rangle)/\sqrt{2}$ can be prepared amongst them with $O(n)$ bits of classical communication.

Theorem 3.1 thus gives a sufficient condition for preparing a maximally entangled $n$-partite state in a distributed network of EPR pairs. In order to prove this sufficiency, we have also developed two more protocols which require $O(n)$ cbits of communication. The first two steps of these protocols are essentially the same as that of Protocol II. The first protocol involves all the agents already entangled in each iteration in Step 3, where, a circuit very similar to the symmetric teleportation circuit (Figure 2.3) of Protocol I is used. The classical communication cost is $(2n - 4)$ bits. The second protocol uses a generalization of the method of Zeilinger et al. in each iteration of Step 3 and requires $(2n - 3)$ cbits of communication. In this paper we have presented only Protocol II instead of these two protocols because of simplicity and the direct use of teleportation.

The question of interest now is that of determining the minimal structure or combinatorics of the distribution of EPR pairs necessary for creating the $n$-partite maximally entangled state. In other words, we wish to characterize necessary properties to be satisfied by the EPR graph for this purpose. We argue below that the EPR graph, indeed, must contain a spanning EPR tree, and must therefore be connected. We assume for the sake of contradiction that the EPR graph $G$ is not connected. Then, it must have at least two components, say $C_1$ and $C_2$. No member of $C_1$ is connected to any member of $C_2$ by an edge in $G$. This means that no member of $C_1$ is sharing an EPR pair with any member of $C_2$. Suppose a protocol $P$ can create a pure $n$-partite maximally entangled state starting from the disconnected EPR graph $G$ of $n$ agents. If we are able to create an $n$-partite maximally entangled state using protocol $P$ with this structure using only classical communication and local operations, it is easy to see that we will also be able to create an EPR pair between two parties that were not earlier sharing any EPR pair, using just local operations and classical communication. This can be done as follows. Let $A$ be the first party that possesses all the qubits of his group (say $C_1$) and $B$ be the second party that possesses all the qubits of his
group (say $C_2$). Now the protocol $P$ is run on this structure to create the $n$-partite maximally entangled state. Then, $A$ ($B$) disentangles all of his qubits except one by reversing the circuit in Figure 3.4; this leaves $A$ and $B$ sharing an EPR pair. This means that two parties which were never sharing an EPR pair are able to share it just by local operations and classical communication (LOCC). This is forbidden by fundamental laws in quantum information theory (LOCC cannot increase the expected entanglement [35]), hence $G$ must be connected. Note that no qubit communication is permitted after the formation of EPR graph $G$. We present this necessary condition in the following theorem.

**Theorem 3.2** A necessary condition that the $n$-partite maximally entangled state $(|000...0⟩ + |111...1⟩)/\sqrt{2}$ be prepared in a distributed network permitting only EPR pairs for pairwise entanglement between agents is that the EPR graph of the $n$ agents must be connected.

It can be noted that after the preparation of the state $(|000...0⟩ + |111...1⟩)/\sqrt{2}$, any other pure $n$-partite maximally entangled state can also be prepared by just using local operations. We also know that any connected undirected graph contains a spanning tree [34]. Thus a connected EPR graph will contain a spanning EPR tree. With this observations, we combine the above two theorems in the following theorem.

**Theorem 3.3** Amongst $n$ agents in a communication network permitting only pairwise entanglement in the form of EPR pairs, a pure $n$-partite maximally entangled state can be prepared if and only if the EPR graph of the $n$ agents is connected.

### 4 Entangling a set of agents from entangled states of subsets: Combinatorics of general entanglement structure

In the previous section we have presented the necessary and sufficient condition for preparing a pure multi-partite maximally entangled state in a distributed network of EPR pairs (see Theorem 3.3). However, agents may not be connected by EPR pairs in a general network. We assume that subsets of agents may be sharing pure maximally entangled states. So, some triples of agents may be GHZ entangled, some pairs of agents may share EPR pairs and some subsets of agents may share even higher dimensional entangled states.

Now we develop the combinatorics of multi-partite entanglement within subsets of agents required to prepare multi-partite entanglement between all the agents. When we were dealing only with EPR pairs in the case of EPR graphs or spanning EPR trees, we used the simple graph representation. Now subsets of the set of all agents may be in multi-partite entangled states and therefore we use a natural representation for such entanglement structures with hypergraphs as follows.
Let $S$ be the set of $n$ agents in a communication network. Let $E \subset S$, $|E| = k$. Suppose $E$ is such that the $k$ agents in $E$ are in a $k$-partite pure maximally entangled state. Let $E_1, E_2, ..., E_m$ be such subsets of $S$, each having a pure maximally entangled shared state amongst its agents. Note that the sizes of these subsets may be different. Consider the hypergraph $H = (S, F)$ \[36\] such that $F = \{E_1, E_2, ..., E_m\}$. We call such a hypergraph $H$, an entangled hypergraph of the $n$ agents. In standard hypergraph notation the elements of $F$ are called hyperedges of $H$. Now we present the necessary and sufficient condition for preparing a $n$-partite pure maximally entangled state in such networks, given entanglements as per the entangled hypergraph. We need the definition of a hyperpath in a hypergraph: a sequence of $j$ hyperedges $E_1, E_2, ..., E_j$ in a hypergraph is called a hyperpath from a vertex $a$ to a vertex $b$ if (i) $E_i$ and $E_{i+1}$ have a common vertex (agent) for all $1 \leq i \leq j - 1$ (ii) $a$ and $b$ are agents in $S$ (iii) $a \in E_1$ and (iv) $b \in E_j$. If there is a hyperpath between every pair of vertices of $S$ in a hypergraph $H$ then we say that $H$ is connected.

**Theorem 4.1** Given $n$ agents in a communication network and an entangled hypergraph, a pure $n$-partite maximally entangled state can be prepared amongst the $n$ agents if and only if the entangled hypergraph is connected.

The proof of this theorem is based on the following Protocol III.

The Protocol III: We assume without loss of generality that $n > |E_1| \geq |E_2| ... \geq |E_m|$. We maintain the set $F$ and $R = S \setminus F$ where $F$ contains the agents already entangled in the $|F|$-partite pure maximally entangled state. Initially, $F = E_1$ and $R = \{E_2, E_3, ..., E_m\}$. We repeat the following steps until $F = S$. Choose $E_i \in R$ with minimum $i$ such that $F$ and $E_i$ have at least one common agent and $E_i$ is not in $F$; let the smallest index common element between $E_i$ and $F$ be the agent $A_j$. (Since the entangled hypergraph is connected, there is always such a hyperedge $E_i$.) We can now use the method of Zeilinger et al. to create an $(N + M - 1)$-partite maximum entangled state from two groups, one containing $N = |F|$ agents and the other containing $M = |E_i|$ agents. The measurement is processed by $A_j$. So, an $(|F| + |E_i| - 1)$-partite entanglement state is prepared from amongst the members of $F$ and $E_i$. If $F$ and $E_i$ share only one common agent then we are done. Otherwise, each member common to $F$ and $E_i$ other than $A_j$ will have two qubits each from the $|F| + |E_i| - 1$ entangled qubits. These qubits must be disentangled using a circuit same as the reverse of circuit in Figure 3.4. Now the members of $F$ and $E_i$ remain entangled in $(|F| + |E_i| - |F \cap E_i|)$-partite state, each holding exactly one qubit. Finally, we set $F = F \cup E_i$ and $R = R \setminus E_i$.

The proof of necessity is similar to that of the proof of necessity in Theorem 3.2. For the sake of contradiction assume that the entangled hypergraph $H$ is not connected. Then there is no hyperpath between two agents (say $a$ and $b$), implying the existence of at least two components $C_1$ and $C_2$ in $H$, with no member of $C_1$ sharing a hyperedge of entanglement with any member of $C_2$. Suppose a protocol $P$ can create a pure $n$-partite maximally entangled state starting from the disconnected entangled hypergraph $H$ of $n$ agents. If we are able to create an $n$-partite maximally entangled state using protocol $P$ with this structure using only classical communication and local operations, it is easy to see that we will also be able to create an EPR pair between two parties that were not earlier sharing any EPR pair, using
just local operations and classical communication. This is forbidden by fundamental laws in quantum information theory (LOCC cannot increase the expected entanglement [35]). Hence $H$ must be connected. This completes the proof of Theorem 4.1.

5 Concluding remarks

We compare our method (Protocol II) of generating multipartite maximally entangled states with that of Bose et al. [24]. The scheme of Bose et al. works as follows. Each agent needs to share a Bell pair with a central exchange in the communication network of $n$ agents. The central exchange then projects the $n$-qubits with him, on to the $n$-partite maximally entangled basis. This leaves the $n$ agents in a $n$-partite maximally entangled state. Thus, the two basic requirements of their scheme are a central exchange and a projective measurement on a multi-partite maximally entangled basis. The central exchange essentially represents a star topology in a communication network and allows certain degree of freedom to entangle particles belonging to any set of users only if the necessity arises. However, a real time communication network may not always be a star network, in which case, we may need to have several such central exchanges. Of course, one will also be interested in setting up such a network with minimum resources, especially in the case of a long distance communication network. Issues involved in the design of such central exchanges such as minimizing required resources, are of vital interest while dealing with real communication networks. Such networks may be called Quantum Local Area Network (Q-LAN or Non-LAN, a Non-Local LAN) or Quantum Wide Area Network (Q-WAN or Non-WAN). Our scheme presented in Section 3 addresses these issues. We have shown in Theorem 3.3 that the spanning EPR tree is the minimal combinatorial requirement for this purpose. The star topology is a special case of the spanning EPR tree where the central exchange is one of the agents. It is therefore clear that the star network requirements of the scheme of Bose et al. provides a sufficient condition where as the requirement in our spanning EPR tree scheme is the most general and minimal possible structure.

Our scheme also helps in minimizing resources. Our spanning tree topology has been used by Singh and Srikanth [8] for this purpose. They assign weights to the edges of the EPR graph based on the resources (such as quantum repeaters, etc.) needed to build that particular edge. Then, a minimum spanning EPR tree represents the optimized requirement. They also use this topology for multi-party quantum cryptography to minimize the size of the sector that can be potentially controlled by an eavesdropper. Thus our topology seems to be a potential candidate for building a long distance quantum communication network (such as in a Non-LAN or Non-WAN).

The second basic ingredient of the scheme of Bose et al. is the projection on a multi-partite maximally entangled basis. As they point out, the circuit for such a measurement is an inverse of the circuit that generates a maximally entangled state from a disentangled input in the computational basis. In a communication network involving a large number of
agents, this entails a lot of work to be done on part of the central exchange while the agents are idle. In our scheme, work is distributed amongst the agents. Moreover, the $n$-qubit joint measurement on the entangled basis in the scheme of Bose et al. seems to be well high-impossible from a practical standpoint given the current technology, whereas all the practical requirements of our scheme (Protocol II) can be met using current technology (using telecom cables to distribute entanglement etc).

The projection used by the central exchange in the scheme of Bose et al. may lead to any of the $2^n$ possible $n$-partite maximally entangled states. For practical purposes, one might be more interested in a particular state. To get the desired state, the measurement result must be broadcast by the central exchange. The $2^n$ possible states can be represented by a $n$ bit number and thus the communication complexity involved in their scheme is $n$ cbits, essentially the same as that of ours asymptotically. Therefore, our scheme is comparable to their scheme also in terms of communication complexity. It can also be noted at this point that, in our topology, even the method of Zeilinger et al. for creating $(m+1)$-partite maximally entangled state from a $m$-partite maximally entangled state becomes applicable. The use even reduces the communication complexity by some cbits but still requires $2n-3$ cbits which is $O(n)$. As it can be observed, all these schemes require $O(n)$ cbits of communication. Whether there is an $\Omega(n)$ lower bound on the cbit communication complexity for preparing $n$-partite a pure maximally entangled state given a spanning EPR tree remains open for further research.

The results in Theorem 3.3 and Theorem 4.1 are similar to the classical theorem by Helly [37] in convex geometry. Helly’s theorem states that a collection of closed convex sets in the plane must have a non-empty intersection if each triplet of the convex sets from the collection has a non-empty intersection. In one dimension, Helly’s theorem ensures a non-empty intersection of a collection of intervals if each pair of intervals has a non-empty intersection. In our case (Theorems 3.3 and 4.1), there is similar combinatorial nature; if $n$ agents are such that each pair has a shared EPR pair, then (with linear classical communication cost) a pure $n$-partite state with maximum entanglement can be created entangling all the $n$ agents. As stated in Theorem 3.1, the case is stronger because just $(n-1)$ EPR pairs suffice. Due to this similarity in combinatorial nature, we call our results in Theorem 3.3 and Theorem 4.1 quantum Helly-type theorems.

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