SOLVING 1ODES WITH FUNCTIONS

L.G.S. Duarte\textsuperscript{a}, L.A.C.P. da Mota\textsuperscript{a,*}, A.B.M.M. Queiroz\textsuperscript{a,b}

\textsuperscript{a}Universidade do Estado do Rio de Janeiro, Instituto de Física, Depto. de Física
Teórica, 20559-900 Rio de Janeiro – RJ, Brazil
\textsuperscript{b}Inmetro - Instituto Nacional de Metrologia, Qualidade e Tecnologia.

Abstract

Here we present a new approach to deal with first order ordinary differential equations (1ODEs), presenting functions. This method is an alternative to the one we have presented in [1]. In [2], we have establish the theoretical background to deal, in the extended Prelle-Singer approach context, with systems of 1ODEs. In this present paper, we will apply these results in order to produce a method that is more efficient in a great number of cases. Directly, the solving of 1ODEs is applicable to any problem presenting parameters to which the rate of change is related to the parameter itself. Apart from that, the solving of 1ODEs can be a part of larger mathematical processes vital to dealing with many problems.

Keywords: 1ODEs, Symbolic Computation, Elementary functions

1. Introduction

Apart from their mathematical interest, dealing with first order differential equation (1ODEs) has its more direct application. Any physical (or, for that matter, any scientific question) that can be formulate as a relationship between the rate of change of some quantity of interest and the quantity itself can be formulated as solving an 1ODE (at some stage). There are many well known phenomena which fall into this category: Concentration

\textsuperscript{*}Corresponding author

L.G.S. Duarte and L.A.C.P. da Mota wish to thank Fundação de Amparo à Pesquisa do Estado do Rio de Janeiro (FAPERJ) for a Research Grant.

\textit{Email addresses:} lgsduarte@gmail.com.br (L.G.S. Duarte), lacpdamota@uerj.br or lacpdamota@gmail.com (L.A.C.P. da Mota), andream.melo@gmail.com (A.B.M.M. Queiroz)

Preprint submitted to Elsevier

October 3, 2017
and dilution problems are quickly remembered examples. Population models are another such example. More specifically, of non-linear 1ODEs and, as an final example, Astrophysics \[3\], etc. There are many others and there can be many more lurking around the corner of the many scientific endeavours being pursuit. In this sense, any novel mathematical approach, technique to deal with such differential equations are welcome and could prove vital to solve a fundamental question. More so in our case here where we allow for the case where elementary functions can be presented on the 1ODE.

In \[2\], we have introduced a theoretical approach for finding the (possible) elementary invariants for a 3D systems of 1ODE\[4\]. The whole method and algorithms were a follow up of many results we have produced on the line of Darbouxian approaches for solving and/or reducing ODEs \[1, 9, 10, 11, 12, 13, 14\]. There has been already 6 years or so since we have completed the work presented on \[2\] and we have produced some more work on those lines since then, but nothing directly related to systems.

In \[19\], we have picked it up again and we have developed an new approach to deal with 3D-systems of 1ODEs, this time basing our method on our (so-called) \(S\)-function (introduced in \[13\] and used by us and other people \[21, 21, 22, 23, 24, 25, 26, 27\]), that improves greatly the efficiency of the method introduced in \[2\], for many cases.

These results concerning 3D-systems will set the stage for the method we are presenting here. The motivation for the present work is twofold: First, we will place the dealing with 1ODEs with function in the same (theoretical) footing, in regard to the Prelle-Singer approach, as the methods we have for rational 1ODEs \[28\], rational 2ODEs \[13, 15, 16, 17, 18\] and (since \[2\]) for 3D-systems with Elementary Invariants.

We will start by reviewing the main results introduced in \[2\]. Then, in sections \(2\), \(3\), \(4\) and \(5\), we will present the whole method and algorithm, explaining how we are going to use the ideas pertaining solving 3D systems to solve 1ODEs, pointing out the limitations of our approach. We proceed then to show some examples of the usage of the method. After that, in section \(6\) and \(7\), we point out the efficiency of this new method in comparison to some powerful solvers. Finally, we conclude and point out directions of how to further our work.

Let us start by briefly reviewing both methods to deal with 3D-systems

\[\text{\footnotesize \[1\] For other interesting approaches please see \[4, 5, 6, 7, 8\]}\]
that we have just mentioned.

1.1. 3D systems of 1ODEs with elementary invariants

In this section, we will introduce some basic concepts involving 3D polynomial systems of 1ODEs and how we have managed to produce some results that were the analogue of previous results for 2D systems. These results allowed for the production of a semi-algorithm to deal with a class of 3D polynomial systems of 1ODEs presenting, at least, one elementary first integral\(^2\).

It is important that we do that summary in order to set the stage for the new theoretical results\(^2\) and better understand the advances we had to make to achieve it.

Consider the generic 3D system:

\[
\begin{align*}
\frac{dx}{dt} & = \dot{x} = f(x, y, z), \\
\frac{dy}{dt} & = \dot{y} = g(x, y, z), \\
\frac{dz}{dt} & = \dot{z} = h(x, y, z). 
\end{align*}
\]

(1)

where \(f, g\) and \(h\) are all polynomials in \((x, y, z)\).

A function \(I(x, y, z)\) is a differential invariant of the system (1) if \(I(x, y, z)\) is constant over all solution curves of (1), i.e., \(\frac{dI}{dt} = 0\).

Thus, over the solutions, one has:

\[
\frac{dI}{dt} = \partial_x I \dot{x} + \partial_y I \dot{y} + \partial_z I \dot{z} = f \partial_x I + g \partial_y I + h \partial_z I = 0.
\]

(2)

Defining the Darboux operator (see, equation (3)) associated with (1), we can write the condition for a function \(I(x, y, z)\) to be a first integral of the 3D system (1) as \(D[I] = 0\), finally leading to equation (4) below:

\[
D \equiv f \partial_x + g \partial_y + h \partial_z ,
\]

(3)

\[
dt = \frac{dx}{f} = \frac{dy}{g} = \frac{dz}{h}
\]

(4)

\(^2\)Results to be introduced in sub-section 1.3.
where \( f, g \) and \( h \) are the polynomials introduced in equation (1). So, by defining the 1-forms \( \alpha \) and \( \beta \) as \( \alpha \equiv g \, dx - f \, dy \) and \( \beta \equiv h \, dx - f \, dz \), one can see that they are null over the solutions of the system. So, one has:

\[
\alpha \equiv g \, dx - f \, dy = 0 \quad \text{and} \quad \beta \equiv h \, dx - f \, dz = 0.
\]

leading to the following conclusion (please see [2], for details):

\[
dI = r \alpha + s \beta
\]

where \( r \) and \( s \) are functions of \((x, y, z)\).

In a way, it could be said that the task at hand (to solve the 3D system) is “finding” \( r \) and \( s \) thus allowing us to have \( I \) via quadratures: From (6) we have

\[
D[I] = dI = r \, (g \, dx - f \, dy) + s \, (h \, dx - f \, dz)
\]

implying that

\[
I_x = -r \, g - s \, h \\
I_y = r \, f \\
I_z = s \, f
\]

So,

\[
I(x, y, z) = \int (-r \, g - s \, h) \, dx + \int \left( r \, f - \frac{\partial}{\partial y} \int (-r \, g - s \, h) \, dx \right) dy + \\
\int \left\{ s \, f - \frac{\partial}{\partial z} \left[ \int (-r \, g - s \, h) \, dx + \int \left( r \, f - \frac{\partial}{\partial y} \int (-r \, g - s \, h) \, dx \right) dy \right] \right\} dz.
\]

Of course, stating that if determine \( r \) and \( s \) we will be able to use (9) to finde the desired Differential invariant seems an empty statement without the means to do so. In [2] we have produced this method. Without demonstrating it again here, these steps to finding \( r \) and \( s \) are:

**Definition:** Let \( I \) be a first order invariant for the system (1). The function defined by \( S := I_y/I_y' = r/s \) is called the \( S \) function associated with the system via the first integral \( I \).

**Theorem 1:** Consider a 3D polynomial system of first order ODEs (1) that presents an elementary first integral \( I \). If \( s/r \) (\( r \) and \( s \) defined by equation
is a rational function of \((x,y,z)\) (i.e., \(s/r = P/Q\) where \(P\) and \(Q\) are polynomials that do not have any common factors), then \(f \frac{D[r/Q]}{r/Q}\) is a polynomial.

The results by Prelle and Singer in \[28\] also apply:

If a 3D polynomial system of first order ODEs \([\text{7}]\) presents an elementary first integral then it possesses one of the form

\[
I = W_0 + \sum_{i>0} c_i \ln(W_i),
\]

(10)

where the \(W_i\) are algebraic functions of \((x,y,z)\). That led to the following theorem:

**Theorem 2:** Consider a 3D system of autonomous polynomial first order ODEs \([\text{7}]\), that presents an elementary first integral. Then, from the result by Prelle and Singer it presents one of the form \(I = W_0 + \sum_{i>0} c_i \ln(W_i)\), where the \(W_i\) are algebraic functions of \((x,y,z)\). If \(r/s\) (\(r\) and \(s\) are defined above) is a rational function of \((x,y,z)\) (i.e., \(s/r = P/Q\) where \(P\) and \(Q\) are polynomials that do not have any common factors), then \(r/Q\) can be written as

\[
\frac{r}{Q} = \prod_i p_i^{n_i}
\]

(11)

where \(p_i\) are irreducible polynomials in \((x,y,z)\) and \(n_i\) are non-zero rational numbers.

From the theorem just above, one can produce the following corollary:

**Corollary 1:** The polynomials \(p_i\) (see theorem 2 above) are Darboux polynomials of the operator \(D \equiv f \partial_x + g \partial_y + h \partial_z\) (i.e., \(D[p_i]^{1/p_i}\) is a polynomial) or they are factors of the polynomial \(f\).

1.2. An Algorithm

First of all, we would like to comment about the title of this subsection. Why “an algorithm” and not “the algorithm”? In \[2\], we have pursued one of the possibilities presented on corollary 1 (above), namely that \(p_i|D[p_i]\). The other possibility, that \(p_i\) could as well be factors of the polynomial
was not considered. This extra “road” will also be overlooked here in this paper. In the future, we will pursue this way and also the possibilities that we could equally well have used that \((g \, dx = f \, dy, g \, dz = h \, dy)\) or \((h \, dx = f \, dz, h \, dy = g \, dz)\). instead of the one we have used: \(g \, dx = f \, dy\) and \(h \, dx = f \, dz\).

Recapitulating (briefly) this possible algorithm, we will display the following equations that are the two pillars where the whole method stands:

\[
f \, P \, \frac{D[T]}{T} = f \, P \sum n_i q_i = -f \, D[P] - P \left( f \, f_x + g \, f_y + f \, h_z \right) - Q \left( f \, g_z - g \, f_z \right), \tag{12}
\]

and

\[
f \, Q \, \frac{D[T]}{T} = f \, Q \sum n_i q_i = -f \, D[Q] - Q \left( f \, f_x + f \, g_y + h \, f_z \right) - P \left( f \, h_y - h \, f_y \right). \tag{13}
\]

Basically, these equations came from the use of the compatibility conditions \((I_{xy} = I_{yx}, I_{xz} = I_{zx}, I_{yz} = I_{zy})\) and \((8)\).

In words, our proposed algorithm (see [2]) can be explained as follows: The first and most costly step is the determination of the Darboux Polynomials \(p_i\) and of the co-factors \(q_i\) to a certain degree \(deg\). By inspection on (12) and (13), if that step is taken, it remains for us to determine \(n_i, P\) and \(Q\), where \(n_i\) are constants and \(P\) and \(Q\) are polynomials. We note that if we construct two polynomials \(P\) and \(Q\) (in \((x, y, z)\)) of generic degrees \(deg_P\) and \(deg_Q\) and solve (12,13) for these coefficients and for \(n_i\), we would have determined \(n_i, P\) and \(Q\) and, consequently, \(T = \prod_i p_i^{n_i}\). Using that \(r/Q = T\), we determine \(r\). Since, by doing that, \(r\) and \(s \ (s = r/Q)\) would have been found, we can use (9) to obtain the first integral \(I(x, y, z)\) by quadratures.

That is the jest of the method, now we will conclude by presenting the algorithm in the form of steps:

- **Steps of the Algorithm**
  1. Set \(deg = 0\).
  2. Set \(degQ = 0\) and \(degP = 0\).
  3. Increase \(deg\): \(deg = deg + 1\).
  4. Construct generic polynomials \(p\) and \(q\) (in \((x, y, z)\)) of degrees \(deg\) and \(MAX(deg f, deg g, deg z) - 1\).
5. Construct the operator $D$ and determine the Darboux polynomials and the associated co-factors, up to degree $\text{deg}$, for the operator $D$.

6. Increase $\text{deg}_Q$ and $\text{deg}_P$: $\text{deg}_Q = \text{deg}_Q + 1$ and $\text{deg}_P = \text{deg}_P + 1$.

7. Construct generic polynomials $Q$ and $P$ (in $(x, y, z)$) of degrees $\text{deg}_Q$ and $\text{deg}_P$.

8. Try to solve equations (12, 13) for the coefficients defining $Q$ and $P$ and for $n_i$.

9. If we are successful, go to step 10. In the opposite case, if $\text{deg}_Q < \text{deg}$, we return to step 6. If $\text{deg}_Q = \text{deg}$, return to step 3.

10. Check if $s$ and $r$ satisfy the compatibility equations ($I_{xy} = I_{yx}$, $I_{yz} = I_{zy}$ and $I_{xz} = I_{zx}$) using equation (8). In the affirmative case go to step 11. In the negative case, return to step 6.

11. Calculate the associated first integral using (9).

1.3. Second Approach

In the previous subsection, we have re-presented the method we have introduced in [2]. Here, we are going to mention a second possibility we have recently developed [19]. It is not similar to the one just presented. It is based on our $S$-function [13]. It is not a follow-up of the cases we have just mentioned that we did not pursue when introducing the method above (see section (1.2)), despite being based on some of the theoretical developments in in [2] and that is why both approaches are being here presented as subsections as the summary of theoretical aspects.

This new approach proved to be more efficient than the previous one in a greater many number of cases. This is due to the fact that first approach relies on the determination of the pertinent Darboux polynomials and that can be computationally costly for these cases, rendering the whole approach unpractical. This new approach has other interesting features as well, but this is another story.

Basically, our new approach is based on the following equations:

The first equation is obtained considering that $I_o$ and $I_s$ are (respectively) the differential invariants for a 2ODE and a 3D:

$$D_o[I_o] \equiv N \partial_x[I_o] + z N \partial_y[I_o] + M \partial_z[I_o] = 0$$

$$D_s[I_s] \equiv f \partial_x[I_s] + g \partial_y[I_s] + h \partial_z[I_s] = 0$$

(14)
where

\[ D_o \equiv N \partial_x + z N \partial_y + M \partial_z \]
\[ D_s \equiv f \partial_x + g \partial_y + h \partial_z \]  \hspace{1cm} (15)

where \( M \) and \( N \) are the Numerator and denominator for the said 2ODE.

Formally considering these invariants the same: we would (after some algebra) get:

\[ I_o = I_s = I \Rightarrow \]
\[ \Rightarrow N \partial_y[I] + \left( \frac{Nh - fM}{g - zf} \right) \partial_z[I] = 0 \]  \hspace{1cm} (16)

finally,

\[ S = -\frac{\partial_y[I]}{\partial_z[I]} \Rightarrow \text{using this into equation (16)} \]
\[ \Rightarrow (-N S (g - zf) + (Nh - fM)) \partial_z[I] = 0 \]  \hspace{1cm} (17)

Where it is implied that \( g - zf \neq 0 \) and \( f \neq 0 \). Equation (17) above is one of the equations we have mentioned that will be the basis of this new approach.

The second equation needed to fully determine the \( S \)-function and get on with this approach is as follows (fully demonstrated on [20]:

\[ D[S] = (S)^2 \left( \frac{f g_z - g f_z}{f} \right) + (S) \frac{g f_y + f h_z - f g_y - h f_z}{f} - \frac{f h_y - h f_y}{f} \]  \hspace{1cm} (18)

Once we have determined the \( S \)-function, we can, using its definition, finding the solution to the equation

\[ \frac{dz}{dy} = -S(x, y, z) \]  \hspace{1cm} (19)

It is easy to demonstrate that the the differential operator associated with this differential equation (19), namely \((D_o[I] = (\partial_y - S \partial_z)[I])\), applied to the invariant to the 1ODE system under consideration \( I(x, y, z) = C \), would yield zero. It is important to stress that this does not mean that by solving the above equation (19) we would have found the Invariant we seek. Please
remember that we have considered \( x \) as a parameter in that equation (19), so it is not certain that things will work out properly. In order to complete the task, we have to do the following:

The relation of the general solution of the associated 1ODE (19), \( G(x, y, z) = K \) to the first integral \( I(x, y, z) \) is given by \( I(x, y, z) = F(x, G) \), such that \( F \) satisfies:

\[
D_x[I] = \frac{\partial F}{\partial x} + \left( \frac{\partial G}{\partial x} + z \frac{\partial G}{\partial y} + \frac{M}{N} \frac{\partial G}{\partial z} \right) \frac{\partial F}{\partial G} = 0. 
\]

(20)

The above results and idea were demonstrated on [20] but here we are applying them in the context of the 3D systems. In a very short summary, we use equations (17 and 18) to determine the \( S \)-function and equations (19 and 20) to find the differential Invariant.

Both methods to find the differential Invariant summarized above (1.2 and 1.3) are tailored to find the differential Invariants to 3D-systems of 1ODEs. They both have their strength and weaknesses. The later has proved more efficient in the sense that there are more cases where it fares better than the “old” one compared to the opposite, but both of them have their place in the arsenal to deal with 3D-systems.

The important aspect we intend to stress now is that the results presented on [2] put the dealing with 3D-systems of 1ODEs in the same footing as the dealings with 1ODEs and 2ODEs in the context of the Prelle-Singer approach. In [2], we have also introduced the method summarized in (1.2). Whether one uses this method or the one summarized in (1.3) and introduced in [19] or, for that matter, any method to solve the 3D-system, the important thing is that the dealing with 1ODEs with functions, as we expect to make it clear below, will be in the same footing as well. Of course, we will dwell on the practical side of actually solving, finding the Invariant for that type of systems as well. That will bring us back to the methods described on (1.2) and (1.3).

2. New approach to deal with 1ODEs presenting functions

The title of this section suggests that there is, at least, one other method to deal with 1ODEs presenting functions. Of course, there are many. What it is meant by the title is another method to deal with 1ODEs presenting functions within the Prelle-Singer approach (Darbouxian). The previous one implied is presented on [1]. In these papers, we have produced an heuristic (effective) approach. It is pointed out that the theoretical foundations we
have used at the time were not solid. The work was recognized by the practical applicability, what is worth by itself, Here we will establish a method that has a sound theoretical basis besides practicability.

2.1. The idea behind the method

Consider the following 1ODE:

\[
\frac{dy}{dx} = \frac{M(x, y, f(x))}{N(x, y, f(x))},
\]

where \(M\) and \(N\) are polynomials in \((x, y, f(x))\).

In principle, outside the extension we have produced in [], there is no place in the Prelle- Singer approach for treating these (non-rational) equations. Let us remind ourselves of the following:

In general, any 1ODE of the form of equation (21) is equivalent to the following 2D system:

\[
\begin{align*}
\frac{dy}{dt} &= M(x, y, f(x)) \\
\frac{dx}{dt} &= N(x, y, f(x))
\end{align*}
\]

(22)

We then go a step further. Let us take this 2D system and go one more dimension: Let us simplify notation first. From now on, we will use \(u = f(x)\). Now, besides the system (22) above, let us consider the following 3D system:

\[
\begin{align*}
\frac{dy}{dt} &= M(x, y, u) \\
\frac{dx}{dt} &= N(x, y, u) \\
\frac{du}{dt} &= \frac{du}{dx} \frac{dx}{dt} = N(x, y, u) \frac{du}{dx}
\end{align*}
\]

(23)

Now, we claim that, apart from the system (22), the system (23) is also directly related to the 1ODE (21). Note that, in the third equation, we explicitly use that \(\frac{dx}{dt} = N(x, y, u)\) and also \(\frac{du}{dx}\). Furthermore, he system (23) is a 3D system of polynomial\(^3\) 1ODEs, differently from system (22), which is a 2D system. We are going to exploit this in what follows.

\(^3\)More about the validity of that later in the paper
One final observation before we continue, of course, in the above, we have mention only the possible function of $x$. But, of course, we could have a function of both $x$ and $y$. So the more general case of the system \((23)\) would be:

\[
\begin{align*}
\frac{dy}{dt} &= M(x, y, u) \\
\frac{dx}{dt} &= N(x, y, u) \\
\frac{du}{dt} &= N(x, y, u) \frac{du}{dx} + M(x, y, u) \frac{du}{dy}
\end{align*}
\]

In most of what follows, we are going to consider the case of $f(x)$ for the sake of simplicity of the formulas to be displayed and to really focus on other concepts. But, surely, generalizations such as the one that led to the system \((24)\) (instead of the system \((23)\)) are always a possibility.

3. Example of the application of the ideia above: Exponentials

One obvious example to start with is the example of the exponential function, for its simplicity. The most simple case within this contet is the function $u = e^x$.

For these case, system \((23)\) reads:

\[
\begin{align*}
\frac{dy}{dt} &= M(x, y, u) \\
\frac{dx}{dt} &= N(x, y, u) \\
\frac{du}{dt} &= N(x, y, u) u
\end{align*}
\]

The case spans a whole family of cases: Let us consider the following. In principle, the above case cover “only” the cases where the function $e^x$ appears in the system, but if any function of the general expression $e^{p(x)}$, where $p(x) =$ any polynomail in $x$ appears (just this exponential), will be easily included as a possibility within our present method, let us see:

\[
\begin{align*}
\frac{du}{dt} &= \frac{dx}{dt} \frac{du}{dx} = N(x, y, u) \frac{du}{dx} = N(x, y, u) \frac{dp(x)}{dx} u \Rightarrow 
\end{align*}
\]
and the system would become

\[
\frac{dy}{dt} = M(x, y, u) \\
\frac{dx}{dt} = N(x, y, u) \\
\frac{du}{dt} = N(x, y, u)(a_n(n)x^{n-1} + a_{n-1}(n-1)x^{n-2} + \ldots)u
\]  

Another possibility that springs to mind is that of a system with multiple “exponential functions” of the form \( f_i = e^{a_i x + b_i} \), where \( a_i \) is an integer and \( b_i \) any constant. How come? Are we not dealing with cases of a single function that we have called \( u \)? Not exactly. Let us see:

We actually have that:

\[
f_i = e^{a_i x + b_i} = (\prod_{i=1..a_i} e^x).e^{b_i} = (\prod_{i=1..a_i} e^x).C_i = (u^{a_i}).C_i
\]  

and the whole problem can be dealt with using the single function \( u = f(x) = e^x \) yet again, no matter how many \( f_i \) we encounter on the ODE. The system will be written using \( u \) to many different powers and that is it.

The same form (basically) of the system (25), being actually a more general form of it.

We could now present examples of the cases just enumerated but, before that and before analyzing the effectiveness of the method, we would like to point out that, surely, one of the shortcomings of the approach here proposed is the fact that we can deal with any \( f(x) = e^{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0} \) present on the system. But we can not deal with system presenting two (or more)
such functions (e.g., \( f(x) = e^{x^2} \) and \( g(x) = e^{x^3-1} \)) if the exponents of such exponential functions are not linear (please see case just introduced above (30)).

But let us first concern ourselves with another shortcoming: what about \( f(x) = e^{r(x)} \), where \( r(x) \) is a rational function of \( x \)? Can we deal with that case within our framework? The answer to that question will prove very important for the scope of the method.

3.1. Dealing with denominators

What about the case where \( u = f(x) = e^{r(x)} \), where \( r(x) = \frac{p_1(x)}{p_2(x)} \) is a rational function of \( x \)? and, therefore, \( p_1(x) \) and \( p_2(x) \) are polynomial functions that have no common factors (let us consider that)? For that general expression for the function on the 1ODE, the correspondent 3D system of differential first order equations would look like:

\[
\begin{align*}
\frac{dy}{dt} &= M(x, y, u) \\
\frac{dx}{dt} &= N(x, y, u) \\
\frac{du}{dt} &= N(x, y, u) \left( \frac{p_1 \frac{dp_2}{dx} - p_2 \frac{dp_1}{dx}}{p_2^2} \right) u
\end{align*}
\]  

(31)

where \( p_1(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 \) and \( p_2(x) = b_m x^m + b_{m-1} x^{m-1} + \ldots + b_0 \).

Clearly, the system (31) is not covered by the methods and results presented so far, where only polynomial systems are dealt with. Please keep in mind that we are looking for a 3D system that is correspondent to a particular 1ODE. So, take heart, we can solve this situation by doing the following transformation:

\[
\begin{align*}
M(x, y, u) &= (b_m x^m + b_{m-1} x^{m-1} + \ldots + b_0)^2 M(x, y, u) \\
N(x, y, u) &= (b_m x^m + b_{m-1} x^{m-1} + \ldots + b_0)^2 N(x, y, u)
\end{align*}
\]  

(32)

So, the system

\[
\begin{align*}
\frac{dy}{dt} &= M^*(x, y, u) \\
\frac{dx}{dt} &= N^*(x, y, u)
\end{align*}
\]  

(33)
would still be correspondent to the 1ODE under study since \( \frac{M}{N} = \frac{M^*}{N^*} \). The missing equation to complete the correspondency is the one defining \( u \), the third equation in all our systems so far. That would be (using the above system (33)):

\[
\frac{du}{dt} = \frac{dx}{dt} \frac{du}{dx} = N^*(x, y, u) \frac{du}{dx} = N(x, y, u)(p_1 \frac{dp_2}{dx} - p_2 \frac{dp_1}{dx})u
\]

(34)

and the whole system reads:

\[
\frac{dy}{dt} = M^*(x, y, u) \\
\frac{dx}{dt} = N^*(x, y, u) \\
\frac{du}{dt} = \frac{dx}{dt} \frac{du}{dx} = N^*(x, y, u) \frac{du}{dx} = N(x, y, u)(p_1 \frac{dp_2}{dx} - p_2 \frac{dp_1}{dx})u
\]

(35)

and it is a polynomial 3D differential system.

This maneuver will prove to greatly improve the applicability of the approach hereby presented. Let us deal with that in the next sections.

4. The Logarithm

In the present section, we are going to extend the above ideas to the case where \( u = f(x) = \ln(R(x)) \) where \( R(x) \) is a rational function of \( x \). The technique, introduced in the section (3.1) justa above, allows us to deal with the \( R(x) \) function and still be dealing with a 3D polynomial system. Let us see: For the 1ODE

\[
\frac{dy}{dx} = \frac{M(x, y, u)}{N(x, y, u)}
\]

(36)

where \( M \) and \( N \) are polynomials in \((x, y, f(x))\) and \( u = \ln(R(x)) \) where \( R(x) = \frac{p_4(x)}{p_3(x)} \) is a rational function of \( x \) and, of course, \( p_3 \) and \( p_4 \) are co-prime polynomials.

In that case, the 3D system associated with the 1ODE (36) is:
\[
\frac{dy}{dt} = M(x, y, u) \\
\frac{dx}{dt} = N(x, y, u) \\
\frac{du}{dt} = \frac{dx
du}{dt
dx} = N(x, y, u)\frac{du}{dx} = N(x, y, u)\left(\frac{\frac{d}{dx}p_3(x)p_4(x) - p_3(x)\frac{d}{dx}p_4(x)}{p_4(x)p_3(x)}\right)
\]
and again we are facing the “denominator” problem. This time it is resolved by doing:

\[
M(x, y, u)^* = p_4(x)p_3(x) M(x, y, u) \\
N(x, y, u)^* = p_4(x)p_3(x) N(x, y, u)
\]

in turn, this would imply the third equation to became:

\[
\frac{du}{dt} = \frac{dx
du}{dt
dx} = N^*(x, y, u)\frac{du}{dx} = N(x, y, u)(\left(\frac{d}{dx}p_3(x)p_4(x) - p_3(x)\frac{d}{dx}p_4(x)\right))
\]

and the whole system reads:

\[
\frac{dy}{dt} = M^*(x, y, u) \\
\frac{dx}{dt} = N^*(x, y, u) \\
\frac{du}{dt} = N(x, y, u)\left(\frac{d}{dx}p_3(x)p_4(x) - p_3(x)\frac{d}{dx}p_4(x)\right)
\]

and it is a polynomial 3D differential system.

5. Trigonometry

Perhaps the most important consequence (in the sense of the scope implied) of the idea introduced in subsection [3.1], namely that we can re-define \(M\) and \(N\) and still be able to represent the IODE as a 3D differential system,
\[ f(x) \text{ as a function of } e^{ix} \text{ as a function of } u \]

| Function | Expression 1 | Expression 2 |
|----------|--------------|--------------|
| \( \sin(x) \) | \(-1/2 i (e^{ix} - e^{-ix})\) | \(-1/2 i (u^2 - 1)\) |
| \( \cos(x) \) | \(1/2 e^{ix} + 1/2 e^{-ix}\) | \(1/2 \frac{u}{u^2 + 1}\) |
| \( \sec(x) \) | \(2 (e^{ix} + e^{-ix})^{-1}\) | \(2 \frac{u}{u^2 + 1}\) |
| \( \tan(x) \) | \(-i \frac{(e^{ix} - e^{-ix})}{e^{ix} + e^{-ix}}\) | \(-i \frac{u^2 - 1}{u^2 + 1}\) |

Table 1: Examples of Trigonometric functions written as a function of imaginary exponentials

is the application of it to the trigonometric function. The reason for that is that we can represent any trigonometric function as written in terms of:

\[
\begin{align*}
  u & = e^{ix} \\
  1/u & = e^{-ix}
\end{align*}
\]

i.e., we can use a single function \( f(x) = u \) to represent all trigonometric functions.

Let us remind ourselves how to do that (please, see table (1)):

One can easily see from table (1) that denominators will appear on this scenario (trigonometry). But, using the idea of redefining \( M \) and \( N \), as we have been presenting, we can accommodate these cases as well. Let us digress about that a little bit. All the trigonometric functions present, as they are differentiated, derivatives that are written (again) in terms of the same set of trigonometric functions, that can be written as functions of \( u \) (as we have tried to exemplify on the table (1)). So, in principle, we can use the same method and redefine the numerator and the denominator of the 1ODE under study in order to generate a 3D equivalent differential system.

5.1. Method 1

We are not going to go into too much detail here, since there are many possibilities that are covered by the trigonometric functions and the idea of expressing them as exponentials. But, basically, what will happen is the following:

Consider the following 3D system and that it is associated to an 1ODE in the manner we have been doing so far:
\[ \frac{dy}{dt} = M(x, y, f(x)) \]
\[ \frac{dx}{dt} = N(x, y, f(x)) \]
\[ \frac{du}{dt} = \frac{df}{dx} \frac{dx}{dt} = N(x, y, u) \frac{du}{dx} \]  \hspace{1cm} (42)

Furthermore, let us consider that the function \( f(x) \) is a trigonometric. For example, one function such as those displayed on table (1). So, by considering that \( u = e^{ix} \), one can write that \( f(x) = g(u) \) and that \( g(u) \) can be written as \( g(u) = \frac{p_5(u)}{p_6(u)} \), where the \( p_i \)s are polynomials, as can be seen from table (1).

So, in principle, depending of the general expression of \( M(x, y, f(x)) \) and \( N(x, y, f(x)) \) and of \( f(x) = g(u) \), the system (42) can be written as:

\[ \frac{dy}{dt} = M(x, y, u) \frac{du}{dx} \]
\[ \frac{dx}{dt} = N(x, y, u) \frac{du}{dx} \]
\[ \frac{du}{dt} = \frac{df}{du} \frac{dx}{du} \frac{du}{dx} = (iu) \frac{df}{du} N(x, y, u) \]  \hspace{1cm} (43)

Since \( \frac{df}{du} \) can be written as \( \frac{df}{du} = \frac{p_9(u)}{p_{10}(u)} \), where the \( P_i \)s are polynomials, one can get:

\[ \frac{dy}{dt} = M^*(x, y, u) \]
\[ \frac{dx}{dt} = N^*(x, y, u) \]
\[ \frac{du}{dt} = \frac{df}{du} \frac{dx}{du} \frac{du}{dx} = (iu) \frac{df}{du} N^*(x, y, u) \]  \hspace{1cm} (44)

where

\[ M^*(x, y, u) = M(x, y, u) p_7(u) p_8(u) p_{10}(u) \]
\[ N^*(x, y, u) = N(x, y, u) p_7(u) p_8(u) p_{10}(u) \]  \hspace{1cm} (45)
and so the system finally can be written as:

\[
\begin{align*}
\frac{dy}{dt} &= M^*(x, y, u) \\
\frac{dx}{dt} &= N^*(x, y, u) \\
\frac{du}{dt} &= \frac{df}{du} \frac{du}{dx} \frac{dx}{dt} = (iu)p_7(u)p_8(u)p_9(u)p_{10}(u)N(x, y, u)
\end{align*}
\]

that is a 3D polynomial system.

5.2. Method 2

One can notice that the above reasoning proves the concept that the trigonometric functions, present as they may be in the 1ODE, can be regarded as a 3D polynomial system. Again, in (46), we have concentrated on the case where the trigonometric function is only a function of \(x\), in order to clarify the main aspect we are trying to convey. Of course, one can have \(u(y)\) or \(u(x, y)\) as another possibilities. But, as we are going to try and explain below, the above approach is not the only way to proceed in a “practical” manner. There is at least one other way to proceed by really using, from the start, that all the trigonometric functions can be expressed as a function of \(u(x) = e^{ix}\) (or similar \(u's\) such as \(u(y) = e^{iy}\)) that is to, before embarking in redefining the \(M's\) and \(N's\), we substitute the trigonometric functions for the associated expressions as a function of \(u\) (please see table (1)). In that case, the redefining of the denominator and the numerator of the 1ODE can be “less costly”, for some cases, in the sense that this procedure might produce polynomial terms (both in the numerator and denominator) of lesser degree in \((x, y, u)\) than if we had proceeded as previously (equation (46) and before) implied. For instance, the third equation in (46) would read:

\[
\frac{du}{dt} = \frac{du}{dx} \frac{dx}{dt} + \frac{du}{dy} \frac{dy}{dt} = (iu)(M''(x, y, u) + N''),
\]

where \(N''\) and \(M''\) are the denominator and numerator already written as a function of \(u\), instead of what is present in equation (46). The term \(\frac{df}{du}\) would not be present (there would not be any function of \(u\) left). All would be already expressed only using the single function \(u(x)\) \((u(y), etc\). Furthermore, the Method 1 (see subsection 5.1 above) is designed to work for a single trigonometric function whereas the method 2 can work for any number of trigonometric functions. The interesting aspect of these ideas is that, as we shall demonstrate below (in sections (6) and (7)), in the examples, this concept works! The issuing 3D system represents the original 1ODE in either way we chose to proceed.
5.3. Hyperbolic

The reader can easily infer that the same set of ideas and procedures (used above for the trigonometric functions) can be applied to the hyperbolic functions. This time, every one of the functions can be written in terms of a single function \( u = e^x \). Please see table (2).

6. Examples of the Usage of the Method

In this section, we will present simple examples of our method at work, solving rational 1ODEs presenting functions. In order for the reader to better understand the methods and ideas introduced above.

6.1. First example

Let us start with an example that deals with the \( \ln \):

\[
\frac{d}{dx} y (x) = \frac{x-y(x)-\ln(x-5)}{-x-4}
\]

Using the ideas introduced in section (4), particularly, equation (37), we get the following 3D system:

\[
\begin{align*}
\frac{dy}{dt} &= M(x, y, u) = -x + y(x) + \ln(x - 5) \\
\frac{dx}{dt} &= N(x, y, u) = x + 4 \\
\frac{du}{dt} &= \frac{dx \, du}{dt \, dx} = N(x, y, u) \frac{du}{dx} = \\
&= N(x, y, u) \left( \frac{\frac{d}{dx} p_3(x)}{p_4(x) p_3(x)} p_4(x) - p_3(x) \frac{d}{dx} p_4(x) \right) \\
&= \frac{(x + 4)}{(x - 5)} 
\end{align*}
\]
So, as can be seen from the third equation, some redefining is needed. So, the 3D system we will actually use would read:

\[
\begin{aligned}
\frac{dy}{dt} &= (-x + y + u)(x - 5) \\
\frac{dx}{dt} &= (x + 4)(x - 5) \\
\frac{du}{dt} &= (x + 4)
\end{aligned}
\]

where \( u \) stands for \( \ln(x - 5) \). That's the 3D polynomial system we aimed for.

Applying the techniques we have developed to solve such systems (actually to find its differential invariant), one gets such invariant to be:

\[
\frac{-1}{9} \ln(x - 5) \left( x - 10 \ln(x + 4) x - 5 \ln(x - 5) - 40 \ln(x + 4) - 9 y - 36 \right)
\]

\[x + 4\]

and, solving for \( y(x) \), one gets:

\[
y(x) = \left( -\frac{10}{9} \ln(x + 4) - 4 (x + 4)^{-1} + \frac{1}{9} \ln(x - 5) (x - 5) + C \right) (x + 4)
\]

which is the solution to the 1ODE under study.

6.2. Second example

Now, trying one equation with an exponential function:

\[
\frac{dy}{dx} y(x) = \frac{x - y(x) - e^{x^2+x}}{x}
\]

Using the ideas introduced in section (3), we get the following 3D system:

\[
\begin{aligned}
\frac{dy}{dt} &= M(x, y, u) = x - y - e^{x^2+x} = x - y - u \\
\frac{dx}{dt} &= N(x, y, u) = x \\
\frac{du}{dt} &= (2x + 1) e^{x^2+x} x = (2x + 1) u x
\end{aligned}
\]

(49)
where \( u \) stands for \( e^{x^2+x} \). One can see that, for the present case, no redefining is needed.

Applying the techniques we have developed to solve such systems (actually to find its differential invariant), one gets such invariant to be:

\[
\frac{i \sqrt{\pi} e^{-1/4} \text{erf} (ix + 1/2 i) e^{x^2+x} + x^2 e^{x(x+1)} - 2 x y e^{x(x+1)}}{e^{x^2+x}}
\]

and, solving for \( y(x) \), one gets:

\[
y(x) = \frac{-1/2 \left( -i \sqrt{\pi} e^{-1/4+x^2+x} \text{erf} (ix + 1/2 i) - x^2 e^{x(x+1)} + C e^{x(x+1)} \right) e^{-x(x+1)}}{x}
\]

which is the solution to the 1ODE under study.

6.3. Third example

Let us now deal with a simple trigonometric case:

(Example for the usage of Method 2 - section 5.2 above)

\[
\bullet \; \frac{d}{dx} y(x) = x - \sin(x) - \cos(x) - y(x)
\]

Using the ideas introduced in section 5 and the description of both methods we suggest for the trigonometric case (please see subsection 5.1 and 5.2 we will actually use the following 1ODE instead:

\[
\begin{align*}
\frac{d}{dx} y(x) &= x + \frac{1/2 i \left( (u(x))^2 - 1 \right)}{u(x)} - 1/2 \frac{(u(x))^2 + 1}{u(x)} - y(x) \\
\frac{d}{dx} y(x) &= 1/2 \frac{i (u(x))^2 - (u(x))^2 - 2 y(x) u(x) + 2 x u(x) - 1 - i}{u(x)}
\end{align*}
\]

So, the 3D system we will actually use would read:

\[
\begin{align*}
\frac{dy}{dt} &= i u^2 - 2 y u - u^2 + 2 x u - 1 - i \\
\frac{dx}{dt} &= 2 u \\
\frac{du}{dt} &= (2 i) u^2
\end{align*}
\]
where $u$ stands for $e^{ix}$.

Applying the techniques we have developed to solve such systems (actually to find its differential invariant), one gets such invariant to be:

$$(e^{(1-i)x}). \left( i (u(x))^2 + iu(x) + 2 xu(x) - 2 yu(x) - i - u(x) \right)^{-1}$$

and, solving for $y(x)$, one gets:

$$y(x) = -1/4 \frac{-2 i e^{(1-i)x} (u(x))^2 - 4 e^{(1-i)x} u(x) x + 2 i e^{(1-i)x} + 4 e^{(1-i)x} u(x) + C}{e^{(1-i)x} u(x)}$$

substituting $u(x) = e^{ix}$, one finally produces the solution to the 1ODE under study:

$$y(x) = -1/4 \frac{-2 i (e^{ix})^2 - 4 x e^{ix} + 2 i + 4 e^{ix} + C e^{(-1+i)x}}{e^{ix}}$$

The reader might want to have this answer as function of the “normal” trigonometric equations. In order to do that, it is only necessary to substitute the transformations in table (1) and get the (equivalent) solution to the 1ODE:

$$y(x) = -1/4 \left( 4 e^x \sin(x) - 4 e^x x + C + 4 e^x \right) e^{-x}$$

7. Theoretical versus Practical results

In the previous section, we have present examples of the method in order to exemplify the results previously displayed, making them a little clearer, we hope. Here, we are going to point out that the method introduced can be of a “practical” value as well. Besides placing the solving 1ODEs with functions (in the manner and with the limitations we have emphasized) in the same theoretical basis as the rational 1ODEs in the context of the Prelle-Singer approach.
What do we mean by “practical” value? The addition of any mathematical method to the arsenal available to the researcher or user alike is always welcome. Furthermore, if said method can solve “hard” 1ODEs, i.e., 1ODEs that many methods find it difficult.

Below, we are going to present a few cases where the powerful methods that are implemented on the \texttt{dsolve} in built Maple command fails to solve the 1ODE. The first one will be presented in somewhat more detail, showing more of the intermediary steps in order for the reader to have a “feeling” of the kind of differential equation that appears on the method, for a case where some other powerful methods fail.

### 7.1. First 1ODE

The first example in this section is one presenting an exponential function in the 1ODE. It is not very elaborate 1ODE and also not very long. But it eludes the \texttt{dsolve} command. Below, the 1ODE:

\[
\frac{d}{dx}y(x) = - \left( \frac{x^2 e^{\frac{(y(x))^2}{x}} - x^2 y(x) + (y(x))^3}{x (-2 (y(x))^2 + x)} \right)
\]

That will produce the associated 3D system:

\[
\begin{align*}
\frac{dy}{dt} &= M(x, y, u) = -x^2 y + x^2 z + y^3 \\
\frac{dx}{dt} &= N(x, y, u) = -x(-2y^2 + x) \\
\frac{du}{dt} &= N(x, y, u) \frac{du}{dx} + M(x, y, u) \frac{du}{dy} = -z(2xy - 2xz - y)y
\end{align*}
\]

The correspondent \(S\)-function is:

\[
S = -\frac{z}{y}
\]

That produces the associated differential equation (please, see equation (53))

\[
\frac{d}{dy}z(y) = \frac{z(y)}{y}
\]

Solving this, one finds the \(G(x, y, z)\) function for this case:

\[
G(x, y, z) = \frac{z}{y}
\]
Following to solving equation (20), one gets:

\[ Inv = \frac{(y - z) e^{-x}}{z} \]  
(57)

Finally, leading to the following differential invariant for the system (and 1ODE):

\[ \left( y(x) - e^{\frac{y(x)}{2}} \right) e^{-x} \left( e^{\frac{y(x)}{2}} \right)^{-1} = C1 \]

and corresponding solution for the 1ODE:

\[ y(x) = (e^{x} \cdot C1 + 1) \frac{1}{\sqrt{-2 \left( e^{x} \cdot C1 + 1 \right)^{2} x^{-1} \left( \text{LambertW} \left( -2 \left( e^{x} \cdot C1 + 1 \right)^{2} x^{-1} \right) \right)^{-1}}} \]

7.2. Second 1ODE

Let us consider the following 1ODE now:

\[ \frac{dy}{dx}(x) = - \frac{(\ln(xy(x) - 1)^2 xy(x) - \ln(xy(x) - 1) x (y(x))^2 - (\ln(xy(x) - 1))^2 + y(x) \ln(xy(x) - 1) - (y(x))^2}{\ln(xy(x) - 1) xy(x) - xy(x) - xy(x) - \ln(xy(x) - 1)} \]

That will produce the associated 3D system:

\[ \frac{dy}{dt} = M(x, y, u) = xy^2 z - xyz^2 + y^2 - yz + z^2 \]
\[ \frac{dx}{dt} = N(x, y, u) = xyz - xy - z \]
\[ \frac{du}{dt} = N(x, y, u) \frac{du}{dx} + M(x, y, u) \frac{du}{dy} = (xy - xz + y)z \]  
(58)

Leading to the following differential invariant for the system (and 1ODE):

\[ \frac{(y(x) - \ln(xy(x) - 1)) e^{-x}}{\ln(xy(x) - 1)} = C1 \]
and corresponding solution for the 1ODE:

\[ y(x) = e^x \left( C1 + (e^x)^{-1} \right) \left( -LambertW \left( -e^{\frac{x}{e^{-x}C1+1}} (x\cdot C1 \cdot e^x + x)^{-1} \right) + \frac{1}{x (e^x C1 + 1)} \right) \]

The fact is that the method succeed in solving the 1ODE, via solving the associated 3D system. But, trying to understand a possible reason for that and the fact that dsolve failed, we might argue that the strength of our method, in cases like the present one, lies on the fact that the associated 3D system is “simple”, none of the polynomials \( f, g \) and \( h \) were not of a extremely high degree, despite the fact that the original 1ODE was very long and the function present was not simple. These facts might have contributed to disturb the methods implemented on dsolve.

7.3. Third 1ODE

Let us conclude with a trigonometric example:

\[ \frac{d}{dx} y(x) = \left( \tan \left( \frac{x-y(x)}{xy(x)} \right) \right)^2 - x \tan \left( \frac{x-y(x)}{xy(x)} \right) + xy(x) + 1 \left( y(x) \right)^2 \]

That will produce the associated 3D system:

(\text{example for the usage of Method 1 - section 5.1 above})

\[ \begin{align*}
\frac{dy}{dt} &= M(x, y, u) = -x^2 y + x^2 z + y^3 \\
\frac{dx}{dt} &= N(x, y, u) = -x(-2y^2 + x) \\
\frac{du}{dt} &= N(x, y, u) \frac{du}{dx} + M(x, y, u) \frac{du}{dy} = -z(2xy - 2xz - y) \end{align*} \]

Leading to the following differential invariant for the system (and 1ODE):

\[ - \left( y(x) - \tan \left( \frac{x - y(x)}{xy(x)} \right) \right) \cdot x^{-1} = C1 \]

and corresponding solution for the 1ODE:
Again, the fact is that our method succeed in finding an invariant for the system and 1ODE. In this present case, we have used what we have called method 2 for dealing with trigonometric functions in section (5), subsection (5.2). There was only one trigonometric function, albeit being an elaborated one. So, this approach (method 2) tends to work better. Also again, the fact that our method converts the problem of solving 1ODEs with complicated functions into solving 3D polynomial systems, trading complexity for extra dimensions, proved a practical way to proceed.

8. Conclusion

In this paper, we have introduced an approach to deal with 1ODEs with elementary functions. We have been working on the solving or reducing odes for a while now and our main initial motivation to doing what we had stared to do here was to put the dealing with such 1ODEs in the same theoretical footing as we have for rational 1ODEs \[28\], 2ODEs \[13\] and 3D systems \[2\].

When we do what we suggest in section \(2\), we believe that we have done so, since in \([\ref{10}]\) we have established the theoretical basis for the case of 3D polynomial systems and, in section \(2\) and in the following sections, we have argued for the correspondence of the said 1ODEs with 3d systems, in many cases.

We have tried to make it plain that the proposed approach has its limitations. For instance, in the range of functions (in the 1ODE) that can be dealt with in the approach hereby introduced.

One can deal with exponential functions, any number of them, as long as the exponents are linear in one of the variables (i.e., \(f_i = e^{a_ix+b_i} = (\prod_{i=1..a} e^{x}).e^{b_i} = (\prod_{i=1..a} e^{x}).C_i = (u^{a_i}).C_i\). We are left with dealing with the single function \(u = e^x\), no matter how many such function \(f_i\) we encounter on the 3D system. Actually, that is the whole point: Being able to deal with one function only (that is, broadly speaking, the role the third equation on the 3D system play, define the “one function”). Therefore, we can deal with a slightly broader case than the one we have just mentioned. If one has a single monomial, we can deal with it, we can convert them all into one single functions. Let us see: Consider \(f_i = e^{a_ix^k+y^m+b_i} = (\prod_{i=1..a} e^{x^k}).e^{b_i} = \)
\[(\prod_{i=1}^{a} e^{x^i y^i}).C_i = (u^{a_i}).C_i,\] where (this time) \(u = e^{x^i y^i}\). Again, we are left with a single function \(u\). We can work with exponentials of any rational function with the nice idea we have introduced that enables the method hereby presented to deal with “undesired” denominators (see section (3.1)). We have discussed the topic somewhat else in subsection (3) but, in order not to forget anything, perhaps it is better to point out the following: we can deal with any case where we can reduce it to one single function! It is better perhaps exemplify what we can not do. We, for instance can not deal with a 3D system present, say, two exponentials such as, namely, \(e^{x^2 - 1}\) and \(e^{y(x)}\), etc.

Regarding the logarithmic functions, we have concentrated the discussion in section (4) around the question of being a logarithmic function of any rational function of \(x, y, z\), that was the focus there. So, if the only function present on 3D system is \(d\), such that \(u = \ln(R(x))\) where \(R(x) = \frac{p_3(x)}{p_4(x)}\) is a rational function of \(x\) and, of course, \(p_3\) and \(p_4\) are co-prime polynomials, we are in business. But, remind ourselves of the real request, namely that we have only one function present on the 1ODE (in order to be able to convert it into the correspondent 3D system), we can use the properties of the logarithmic functions to broaden the scope of the method a little bit. Consider that:

\[
\ln(a) + \ln(b) = \ln(ab) \quad \text{and that} \quad k\ln(a) = \ln(a^k) \quad (60)
\]

So, if we have a an 1ODE with many logarithmic functions, as long as they are logarithmic functions of powers of the same monomial, we are in business. See the example below:

\[
\frac{dy(x)}{dx} = \frac{x^2 - y(x) + \ln(x)}{3 - \ln(x^3) - x\ln(x^2)} = \frac{x^2 - y(x) + \ln(x)}{3 - 3\ln(x) - 2x\ln(x)} \quad (61)
\]

So, even thought (at first) it seemed that we would not be able to deal with the 1ODE, since it presented three different logarithmic functions, we see that, after a little manipulation, we could re-write the 1ODE with just one function. That is a simple example, but we think that the reader got the jest of the idea we are trying to convey.

Again, perhaps is more useful to remind the reader of what are the actual limitations, instead of enumerating all the possible cases where we can use the present method. We can not deal with different (more than one) logarithmic functions present on the 1ODE. So, we can not deal with cases where, for
example, the following two functions appear on the same 1OD: \( \ln(x^2) \) and \( \ln(x - 1) \), etc...

Finally, in what concerns both trigonometric and hyperbolic functions, as we have left it plain in subsection (5.1 and 5.2), we can deal with any number of those functions if they all have the same arguments, for instance, \( \tan(R(x)) \), \( \sin(R(x)) \), etc...where \( R(x) \) is a rational function of \( (x, y) \). Of course, some combinations of different arguments can be acceptable. Please consider that, for both trigonometric and hyperbolic cases, it boils down to writing the functions in terms of exponential functions, \( e^{tx} \) for the trigonometric case and \( e^{r} \) for the hyperbolic functions. So, as we have explained just above, some combinations can be re-arranged to be put in terms of a single exponential/ But, again, let us not dwell in trying to predict all of those exceptional cases and focus on the cases where we certainly can not work within our new method here presented. We can not treat cases where, for example, \( \sin(x) \) and \( \cos(x^2 - x) \) appear simultaneously on the 1ODE.

We hope that the above considerations has succeeded in conveying the fact that, despite its limitations, the results hereby introduced are of interest both theoretical and “practical”. Our initial motivation was fulfilled for a great many cases. Many 1ODEs can now be treated within the theoretical Prelle-Singer framework and, some of those, are solved by our efforts here and elude other (powerful) solvers.

For future endeavours, we intend to extend our results for ND-systems, where \( N \) is bigger than 3. By doing so, apart from dealing with systems more completely, we will be able to extend the possibilities for dealing with 1ODEs as well.

References

[1] J. Avellar, L.G.S. Duarte, A.C.P. da Mota, PSsolver: A Maple implementation to solve first order ordinary differential equations with Liouvillian solutions, Computer Physics Communications Volume 183, Issue 10, October 2012, Page 2313

[2] L.G.S. Duarte and A.C.P. da Mota, 3D polynomial dynamical systems with elementary first integrals, Journal of Physics A: Mathematical and Theoretical, Volume 43, Number 6, (2010).

[3] J.-M. Huré, F. Hersant A new equation for the mid-plane potential of
power law disks Volume 467 / No 3 (June I 2007) Astronomy & Astrophysica (A&A), 467 3 (2007) 907-910

[4] J. Llibre and C. Valls Polynomial, rational and analytic first integrals for a family of 3-dimensional Lotka-Volterra systems Z. Angew. Math. Phys. 62 (2011), 761–777.

[5] J. Llibre and X. Zhang Darboux theory of integrability in Cn taking into account the multiplicity J. of Differential Equations 246 (2009), 541–551.

[6] Jaume Llibre, Regilene Oliveira & Claudia Valls On the Darboux integrability of a three-dimensional forced–damped differential system Pages 473-494 Published online: 11 Sep 2017 Journal of Nonlinear Mathematical Physics, Volume 24, 2017 - Issue 4

[7] A. Ghose Choudhury, Partha Guha, Barun Khanra On the Jacobi Last Multiplier, integrating factors and the Lagrangian formulation of differential equations of the Painlevé–Gambier classification J. Math. Anal. Appl. 360 (2009) 651–664

[8] M C Nucci Jacobi Last Multiplier and Lie Symmetries: A Novel Application of an Old Relationship Journal of Nonlinear Mathematical Physics, Volume 12, 2005 - Issue 2. Pages 284-304.

[9] L.G.S. Duarte, S.E.S.Duarte and L.A.C.P. da Mota, A method to tackle first order ordinary differential equations with Liouvillian functions in the solution, in J. Phys. A: Math. Gen., 35 3899-3910 (2002).

[10] L.G.S. Duarte, S.E.S.Duarte and L.A.C.P. da Mota, Analyzing the Structure of the Integrating Factors for First Order Ordinary Differential Equations with Liouvillian Functions in the Solution, J. Phys. A: Math. Gen., 35 1001-1006 (2002).

[11] L.G.S. Duarte, S.E.S.Duarte, L.A.C.P. da Mota and J.F.E. Skea, Extension of the Prelle-Singer Method and a MAPLE implementation, Computer Physics Communications, Holanda, v. 144, n. 1, p. 46-62 (2002).

[12] J. Avellar, L.G.S. Duarte, S.E.S. Duarte, L.A.C.P. da Mota, Integrating First-Order Differential Equations with Liouvillian Solutions via Quadratures: a Semi-Algorithmic Method, Journal of Computational and Applied Mathematics 182, 327-332, (2005).
[13] L.G.S. Duarte, S.E.S. Duarte, L.A.C.P. da Mota and J.F.E. Skea, Solving second order ordinary differential equations by extending the Prelle-Singer method, J. Phys. A: Math. Gen., 34 3015-3024 (2001).

[14] J. Avellar, L.G.S. Duarte, S.E.S. Duarte and L.A.C.P. da Mota, A semi-algorithm to find elementary first order invariants of rational second order ordinary differential equations, Appl. Math. Comp., 184 2-11 (2007).

[15] L. Cairó and J. Llibre, Darboux Integrability for 3D Lotka-Volterra systems, J. Phys. A: Math. Gen., 33 2395-2406 (2000).

[16] C. Christopher, Invariant algebraic curves and conditions for a center, Proc. R. Soc. Edin. A, 124 1209 (1994).

[17] C. Christopher, Liouvillian first integrals of second order polynomial differential equations. Electron. J. Differential Equations, No. 49, 7 pp. (electronic) (1999).

[18] C. Christopher and J. Llibre, Integrability via invariant algebraic curves for Planar polynomial differential systems, Ann. Differential Equations, 16, no. 1, 5-19 (2000).

[19] L.G.S. Duarte, J.P.C. Eiras, L.A.C.P. da Mota The search for Invariants for 3D Systems of 1ODEs - a new Method and Integrability Analysis https://arxiv.org/abs/1708.08893

[20] J. Avellar, M.S. Cardoso, L.G.S. Duarte, L.A.C.P. da Mota Dealing with Rational Second Order Ordinary Differential Equations where both Darboux and Lie Find It Difficult: The S-function Method, http://arxiv.org/abs/1707.09007.

[21] V.K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, On the complete integrability and linearization of certain second order nonlinear ordinary differential equations. Proceedings of the Royal Society London Series A, 461, Number 2060, 2005.

[22] M. Lakshmanan and S. Rajasekar, Nonlinear Dynamics: Integrability, Chaos and Patterns. New York: Springer-Verlag (2003).

[23] Ajey K. Tiwari, S. N. Pandey, V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan Acta Mechanica, July 2016, Volume 227, Issue 7, pp 2039–2051.
[24] R. Mohanasubha, V. K. Chandrasekar, M. Senthilvelan, M. Lakshmanan Interplay of symmetries and other integrability quantifiers in finite-dimensional integrable nonlinear dynamical systems Published 22 June 2016.DOI: 10.1098/rspa.2015.0847

[25] P. R. Gordoa, A. Pickering and M. Senthilvelan, The Prelle-Singer method and Painleve hierarchies, J. Math. Phys., 55, 053510 (2014)

[26] C Muriel and J L Romero First integrals, integrating factors and λ-symmetries of second-order differential equations Journal of Physics A: Mathematical and Theoretical, Volume 42, Number 36

[27] C Muriel and J L Romero Contribution to the Special Issue “Symmetries of Differential Equations: Frames, Invariants and Applications” Nonlocal Symmetries, Telescopic Vector Fields and λ-Symmetries of Ordinary Differential Equations SIGMA 8 (2012), 106

[28] M. Prelle and M. Singer, Elementary first integral of differential equations. Trans. Amer. Math. Soc., 279 215 (1983).