Scalar-field cosmological and collapse models with general self-interaction potentials

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Abstract.

We present the results of the investigation of a wide class of self-interacting, self-gravitating homogeneous scalar fields models, characterized by quite general conditions on the scalar field potential, and including both asymptotically polynomial and exponential behaviors. We show that the generic evolution is always divergent in a finite time, and this result is used to construct cosmological models as well as radiating collapsing star models of the Vaidya type – for the latter it turns out that black holes are generically formed.

1. Introduction

It is well known that scalar fields as sources of self-gravitating models play an important role as models of the early universe, but it has been one of the first test bed model for gravitational collapse also. However, there are crucial differences in the way the two situations has been treated. First, consider the cosmological models. Here, the scalar field is always assumed as self-interacting; in other words, the cosmological expansion is ”driven” by a non-vanishing scalar field potential, and the ”free” case (i.e. the case of the quadratic field Lagrangian) corresponds, as is well known, just to the case of a dynamical cosmological constant. Thus, in cosmology, the presence of a non vanishing field potential is a key point, and indeed many efforts have been made to study the possible dynamical behaviors of the models in dependence of the choice of the potential, and even to try to put in evidence possible large-scale observable effects (see e.g. [19, 20] and references therein). The situation is even more intriguing when issues from string theories come into play; indeed, here one is lead to test simple models which are asymptotically anti-De Sitter, and the potential for the scalar field might have, as a consequence, ”non-standard” behaviors (we shall come back on this interesting issue later on). All in all, self-interacting fields with a non trivial potential are a key ingredient in cosmological models and therefore, of course, in the nature of the cosmological singularities.

Consider, now, the ”astrophysical counterpart” of the cosmological solutions, that is, gravitational collapse. Here, as is well known, there still exist, unsolved, the issue of the cosmic censorship, that is whether the singularities which form at the end state of the collapse are always covered by an event horizon, or not [6, 12]. The scalar field model of course proposes itself a good test-bed also in this different scenario. However, in this context the scalar fields sources have usually been considered as free, i.e. minimally coupled to gravity via equivalence principle only. For this very special case the existence of naked singularities has been shown,
but it has been also shown that such singularities are in a sense non generic with respect to the choice of the initial data [3, 4]. Only in recent years, a few results have been added to this scenario with the study of gravitational collapse of homogeneous self-interacting scalar fields. In these papers formation of naked singularities has been found [7, 10], but with the choice of very special potentials.

All in all, in both applications (cosmology, and gravitational collapse) it would be especially welcome a treatment of self-interacting scalar field dynamics able to treat the models in a unified manner, and to predict the qualitative behavior in dependence of the choice of the potential, viewed as an element of a space of admissible "equations of state" for the matter source. It is the aim of the present paper to provide this unified framework, at least in the special case of homogeneous scalar fields. Indeed, we study here the dynamical behavior of such fields for a very general class of potentials, which satisfies simple physical requirements - the potentials are bounded from below and the weak energy condition is satisfied - as well as a few more technical assumptions to be discussed below.

In the case of cosmological models, our results widely extend the recent relevant results obtained by Rendall ([17, 18]) and by Miritzis ([13]). In the case of collapsing models, we rely on slightly more general assumptions on the potential than the ones used by Foster [5], that however, using a dynamical system approach, actually imposes further conditions on some coordinate transformation of the unknown variables, that we do not need here. Of course, whether this result can actually be shown to hold also in the much more difficult case of both inhomogeneous and self-interacting scalar fields remains an open problem.

2. Expanding and collapsing models

We consider homogeneous, spatially flat spacetimes

\[
g = -dt \otimes dt + a^2(t) \left[ dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3 \right],
\]

where gravity is coupled to a scalar field \( \phi \), self-interacting with a scalar potential \( V(\phi) \). We introduce the following conditions on \( V \):

(i) \( V(\phi) \) is a \( C^2 \) function bounded from below;
(ii) The critical points of \( V \) are isolated. They are either minimum points or non degenerate maximum points.

Denoting the energy density of the scalar field by

\[
\epsilon = \frac{1}{2} \dot{\phi}^2 + V(\phi),
\]

we will consider the following system in the unknown functions \( a(t), \phi(t) \):

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{\epsilon}{3},
\]

\[
\ddot{\phi} + V'(\phi) = -3\frac{\dot{a}}{a} \dot{\phi},
\]

Introducing the sign function

\[
\chi(t) := \text{sgn}(\dot{a}(t)),
\]

describing an expanding (respectively, collapsing or re-collapsing) model at time \( t \), our final system is composed by

\[
\frac{\dot{a}}{a} = \chi \sqrt{\frac{\epsilon}{3}},
\]

\[
\ddot{\phi} + V'(\phi) = -\chi \sqrt{3\epsilon} \dot{\phi}.
\]
Let us observe that, using (2) and (4b), the following relation can be seen to hold:

\[ \dot{\epsilon} = -\chi \sqrt{3} \sqrt{\epsilon} \dot{\phi}^2, \quad (5) \]

that is \( \epsilon(t) \) is monotone in each interval in which \( a \) is.

We shall divide our discussion of the properties of the solutions into the two following sections, the first dealing with the expanding case and the other with the collapsing case. Of course, the two cases are not disconnected, because in the expanding case, where \( \epsilon(t) \) is decreasing, there might be the possibility of reaching a vanishing \( \epsilon \) in a finite time \( t_0 \). If this happens, the model will be ruled from \( t_0 \) onward by the equations for the collapsing situation, where \( \epsilon(t) \) is increasing, and it will actually be proved to be divergent at some finite time for almost every choice of the initial data, thus yielding a singularity. Moreover this fact, and of course the particular form of system (4a)–(4b) allows us to perform the same analysis to study also backwards qualitative behavior of the solution, concluding that the expanding model originates from a big–bang singularity.

We conclude the section stating a couple of results – see [8] for the proof – that will be used throughout all the paper. The first essentially states that the velocity of all (finite energy) solutions which extend infinitely in the future asymptotically vanishes: if \( \phi(t) \) a solution of (4b) that can be extended for all \( t > 0 \), then

\[ \epsilon(t), V'(\phi(t)) \text{ bounded } \Rightarrow \lim_{t \to +\infty} \dot{\phi}(t) = 0, \quad (6) \]

and, in addition,

\[ V''(\phi(t)) \text{ bounded } \Rightarrow \lim_{t \to +\infty} V'(\phi(t)) = 0. \quad (7) \]

The second result shows local existence/uniqueness of solutions with initial zero-energy: let \( \phi_0, v_0 \) such that \( v_0^2 + 2V(\phi_0) = 0 \). Then, there exists \( t_* > 0 \) such that the Cauchy problem

\[
\begin{cases}
\ddot{\phi}(t) = -V'(\phi(t)) + \sqrt{\frac{3}{2}(\dot{\phi}(t)^2 + 2V(\phi(t)))} \dot{\phi}(t), \\
\phi(0) = \phi_0, \\
\dot{\phi}_0 = v_0,
\end{cases}
\quad (8)
\]

has a unique solution \( \phi(t) \) defined in \([0, t_*]\) with the property

\[ \epsilon(t) = \frac{3}{4} \left( \int_0^t \dot{\phi}(s)^2 \, ds \right)^2, \quad \forall t \in [0, t_*]. \quad (9) \]

Moreover if \( (\phi_{0,m}, v_{0,m}) \to (\phi_0, v_0), (v_{0,m})^2 + 2V(\phi_{0,m}) = 0 \) and \( \phi_m \) is the solution of (8) with initial data \( (\phi_{0,m}, v_{0,m}) \) satisfying condition (9), then \( \phi_m \to \phi \) with respect to the \( C^2 \)-norm in the interval \([0, t_*]\).

3. Qualitative analysis of the expanding models

In this section we analyze the global behavior of the solutions of (4a), (4b) in the expanding \( (\chi = 1) \) case. Throughout the present section we shall assume the following further condition on \( V \):

\[ \lim_{|\phi| \to \infty} V(\phi) = +\infty. \quad (10) \]
We begin our study considering a solution $\phi$ of (4b) with $\chi = 1$ such that $\epsilon(0) = \frac{1}{2}\dot{\phi}(0)^2 + V(\phi(0)) > 0$. Due to (5), there is $T > 0$ such that $\sqrt{3} \int_0^T \dot{\phi}(s)^2 \, ds = 2\sqrt{\epsilon(0)}$ or

$$\int_0^t \dot{\phi}(s)^2 \, ds < 2\sqrt{\epsilon(0)}^3, \quad \forall t > 0.$$  

In the latter case, $\epsilon(t) \in [0, \epsilon_0]$, $\forall t > 0$. Since $V$ is bounded from below we obtain that the solution is defined for all $t > 0$. Then, by (10) we have that $\phi(t)$ is bounded and (6) implies that

$$\lim_{t \to +\infty} \dot{\phi}(t) = 0, \quad \lim_{t \to +\infty} V'(\phi(t)) = 0.$$  

By the hypotheses made on $V(\phi)$ we immediately see that $\phi$ must be convergent to a critical point of $V$ and we conclude that either there $\exists \tilde{T} = T(\phi(0), \phi(0)) > 0$ such that $\epsilon(T) = 0$, or $\phi(t)$ is defined in $\mathbb{R}^+$ and there exists $\phi_*$ critical point of $V$ such that:

$$\lim_{t \to +\infty} \phi(t) = \phi_*, \quad \lim_{t \to +\infty} \dot{\phi}(t) = 0.$$  

We now proceed to study the two possible cases described above starting from the behavior of the solutions for which $\epsilon$ vanishes in a finite time $T > 0$. Observe first that this situation may happen only if $V(\phi(T)) \leq 0$. However, we can show that if $\inf V < 0$ this situation is generic. Indeed, reversing time direction in (8), the $C^2$–norm convergence result of (unique) solutions of (8) with initial zero–energy stated before allows for genericity of expanding solutions such that the energy $\epsilon(t)$ vanishes at some finite time $T$, in the sense that solutions of this kind exist, and that the set of initial data leading to solutions such that $V(\phi(T)) < 0$ is open. Of course, for $t > T$ the system will be ruled by the equations for the collapsing situation discussed in Section 4 below.

Now, here remains to see what happens if $\phi(t)$ converges to a critical point for $V(\phi)$. In the following, we shall study first the behavior of the solution near a local minimum point of $V$.

Let $\phi_*$ be a local minimum point of $V$ (not necessarily non degenerate). Let us consider a sublevel of the potential $\{V(\phi) \leq V\}$, with $V(\phi_*) < V$, such that the connected component containing $\phi_*$ does not have critical points other that $\phi_*$. It is easy to show that a solution $\phi(t)$ with $\phi(0)$ in this connected component and $\epsilon(0) \leq V$ is defined on all $\mathbb{R}^+$ and

$$\lim_{t \to +\infty} \phi(t) = \phi_*, \quad \lim_{t \to +\infty} \dot{\phi}(t) = 0.$$  

To prove the above fact, it is crucial to observe that $\epsilon(t)$ is is decreasing for expanding solutions, to show that the solution satisfies hypotheses of (6).

Whenever $V(\phi_*) = 0$ further information about the asymptotic behavior of the solution near the minimum point $\phi_*$ can be obtained. In [18] it is indeed proved the oscillation of $\phi$ around the minimum point using asymptotic analysis. If $\phi_*$ is a non degenerate minimum point we can prove the oscillatory character also using an alternative argument as follows. Suppose, by contradiction, that $\phi$ does not oscillate around $\phi_*$, this means that there exists $t_1 > 0$ such that

$$\phi(t) \neq \phi_*, \forall t \geq t_1.$$  

To fix our ideas suppose $\phi(t) > \phi_* \forall t \geq t_1$. Then, if $t \geq t_1$ and $\dot{\phi}(t) = 0$ it is $\ddot{\phi}(t) = V'(\phi(t)) < 0$ (since $V' > 0$ in a right neighborhood of $\phi_*$). This implies that there exists $t_2 > 0$ such that

$$\dot{\phi}(t) < 0, \forall t \geq t_2.$$  

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Moreover there are not sequences $t_k \to +\infty$ where $\ddot{\phi}(t_k) = 0$. Otherwise we should have
$$
\ddot{\phi}(t_k) = -(V''(\phi(t_k)) + \sqrt{3}\dot{\phi}(t_k)) = -(V''(\phi(t_k)) - 12(\phi(t_k))^2)\dot{\phi}(t_k)),
$$
so $\ddot{\phi}(t_k) < 0$ for any $k$ sufficiently large, getting a contradiction. Therefore there exists $t_3 > 0$ such that
$$
\ddot{\phi}(t) > 0, \forall t \geq t_3.
$$
Then, by (4b),
$$
-V'(\phi) - \sqrt{3}\dot{\phi} > 0, \forall t \geq t_3,
$$
that implies (recalling that $\dot{\phi} < 0$)
$$
3\dot{\phi}^4 + 6V(\phi)^2 - (V')^2 \geq 0, \forall t \geq t_3,
$$
and
$$
\dot{\phi}^2 \geq \sqrt{V^2 + (V')^2 / 3} - V, \forall t \geq t_3.
$$
Now $V(\phi) = a(\phi - \phi_*)^2 + o(\phi - \phi_*)^2$ with $a > 0$, so there exists $b > 0$ and $t_4 > 0$ such that
$$
|\dot{\phi}| \geq b\sqrt{\phi - \phi_*}, \forall t \geq t_4,
$$
(recall that $\phi(t) > \phi_*$). Finally $\forall t \geq t_4$
$$
\dot{\phi} \leq -b\sqrt{\phi - \phi_*},
$$
and integrating we obtain,
$$
\sqrt{\phi(t) - \phi_*} - \sqrt{\phi(t_4) - \phi_*} \leq -\frac{b}{2}t, \forall t \geq t_4,
$$
in contradiction with $\phi(t) \geq \phi_*$ for any $t \geq t_4$.

Now let us consider the case when $\phi_*$ is a local non degenerate maximum point of $V$, with nonnegative critical value. Therefore, it follows immediately by classical theory on stability of nonlinear dynamical system in a neighborhood of an equilibrium point (see e.g. [16]) that there are only two solutions $\phi_1$ and $\phi_2$ of (4b) such that
$$
\lim_{t \to +\infty} \phi_i(t) = \phi_*, \lim_{t \to +\infty} \dot{\phi_i}(t) = 0, i = 1, 2. \quad (12)
$$
The proof relies on the study of the first order system associated to (4b)
$$
\begin{cases}
\dot{x} = -V'(y) - \sqrt{3E(x,y)}x, \\
\dot{y} = x,
\end{cases} \quad (13)
$$
where $E(x,y) = \frac{1}{2}x^2 + V(y)$. Note that the function $\sqrt{3E(x,y)x}$ is not of class $C^1$ in a neighborhood of $(0, y_*)$ whenever $V(y_*) = 0$. However the quantity $\sqrt{3E(x,y)x}$ is an infinitesimal of order greater than 1 in $(0, y_*)$, therefore the classical theory can be adapted to this case, studying the linear system
$$
\begin{cases}
\dot{\alpha} = -V''(y_*)\beta, \\
\dot{\beta} = \alpha,
\end{cases}
$$
obtaining the same conclusion of the case $V(y_*) > 0$.

The growth condition (10) on the potential has been introduced to avoid cases when $V(\phi)$ admits a flat plateau at infinity; on such situations the function $\phi(t)$ may possibly also diverge as $t \to \infty$, with $\epsilon(t)$ approximating the plateau of $V(\phi)$. In this case, the critical point at infinity is non hyperbolic and arguments leading for instance to (12) do not apply anymore.

With an argument that uses of the Center Manifold Theorem, a result of local asymptotical instability for critical point at infinity is found in [14, 15], where the potential is supposed to be $V_\infty(1 - e^{-\sqrt{3/2}\phi})^2$. It can be seen that the Center Manifold Theorem applies to extend results of [14, 15] to a more general situation where, as $\phi \to +\infty$, $V(\phi)$ has a finite limit, $V'(\phi) \to 0^+$, and

$$\lim_{\phi \to +\infty} \frac{V''(\phi)}{V'(\phi)}$$

exists finite (and negative).

We can summarize the results of this section as follows: let be $\phi$ a solution of (4b) where $\chi = 1$, with initial data such that $\epsilon(0) > 0$. Then, one of the two mutually exclusive situations occur:

either

$\exists T = T(\phi(0), \dot{\phi}(0)) > 0$ such that $\epsilon(T) = \frac{1}{2}(\dot{\phi}(T))^2 + V(\phi(T)) = 0,$

and the set of initial data such that (14) is satisfied with $V(\phi(T)) < 0$ is open and not empty, while there are only a finite number of solutions satisfying (14) with $V(\phi(T)) = 0$,

or

$\exists \phi_*$ critical point of $V$ with $V(\phi_*) \geq 0$ such that $\lim_{t \to \infty} \phi(t) = \phi_*$, and $\lim_{t \to \infty} \dot{\phi}(t) = 0,$

and the measure of the initial data set such that $\phi(t) \to \phi_*$ maximum point is zero, while the set of initial data such that $\phi(t) \to \phi_*$ minimum point is open and not empty.

Moreover, situation (14) occurs only if $\inf V < 0$.

4. Qualitative analysis of the collapsing models

The aim of the present section is to study the qualitative behavior of the solution of (4a)–(4b) in the collapsing case ($\chi = -1$), for which we will require that $V(\phi)$ satisfies some further conditions. We assume that $V$ is a potential satisfying assumptions stated at the beginning of Section 2 such that $V'(\phi)$ is eventually positive (respectively, negative) as $\phi \to +\infty$ (resp., $\phi \to -\infty$). Moreover, defining the function

$$u(\phi) := \frac{V'(\phi)}{\sqrt{6V(\phi)}},$$

we assume that $V$ satisfies the growth conditions

$$\limsup_{\phi \to \pm \infty} |u(\phi)| < 1,$$

$$\exists \lim_{\phi \to \pm \infty} u'(\phi) = 0.$$
Under the above assumptions made on \( V(\phi) \), except at most for a measure zero set of initial data satisfying weak energy condition, there exists \( t_s \in \mathbb{R} \) such that the scalar field solution becomes singular at \( t = t_s \), that is \( \lim_{t \to t_s^-} \epsilon(t) = +\infty \), and \( \lim_{t \to t_s^-} a(t) = 0 \). Moreover \( \lim_{t \to t_s^-} \phi(t) = +\infty \).

The proof of this result, that can be found in its full details in [8], consists of several steps. Here we shall give a sketch.

- Since we consider only data which satisfy the weak energy condition and thus, it must be \( \epsilon_0 = \epsilon(0) = \frac{1}{2} \phi^2(0) + V(\phi(0)) \geq 0 \).
- Using existence/uniqueness results of solutions of (8) with initial zero–energy, \( \epsilon_0 > 0 \) up to shifting initial time of observation. Note that \( \epsilon(t) \) is increasing, and therefore, defined
  \[
  t_s = \sup I \in \mathbb{R}^+ \cup \{+\infty\},
  \]
  where \( I \) is the maximal interval of definition of the solution, the limit \( \lim_{t \to t_s^-} \epsilon(t) \) exists.
- Up to shifting initial time of observation again, and up to a zero–measured set of initial data, all critical points of \( V(\phi) \) can be thought as contained in the sublevel \( V(\phi) < \epsilon_0 \). Indeed, if that is not the case, and \( \phi(t) \) is a solution such that \( \epsilon_0 \leq \epsilon(t) \leq V(\phi_s) \) where \( \phi_s \) is a critical point for \( V \), then (6) applies, to find \( \lim_{t \to +\infty} \dot{\phi}(t) = 0 \), \( \lim_{t \to +\infty} V'(\phi(t)) = 0 \). By the assumptions made on the potential, \( \phi(t) \) converges to a critical point of \( V \), that we will call \( \phi_s \) again for sake of simplicity. Whenever \( \phi_s \) is a (non degenerate) maximum point we can study the linearization of the first order system equivalent to (4b) in a neighborhood of the equilibrium point \((0, \phi_s)\) obtaining a result totally analogous to (12).
- Moreover using again the existence/uniqueness result of solutions of (8) with initial zero–energy (in a “time-reversed” version) we see if \( \phi_s \) is a minimum point and \( \phi \) starts with initial data close to \((0, \phi_s)\), then \( \phi \) moves far away from \( \phi_s \). To sum up, if the evolution is constrained in a potential well below some critical point, we obtain either a non generic situation, or a contradiction.
- \( \phi(t) \) is unbounded. This can be seen by a contradiction argument – if \( \phi \) was bounded, it can be proven that \( \phi \) would also be, and it would be \( t_s = +\infty \). That would give, in view of (6), that \( \lim_{t \to +\infty} \dot{\phi}(t) = 0 \), which would imply \( \lim_{t \to +\infty} V(\phi(t)) = \lim_{t \to +\infty} \epsilon(t) > \epsilon_0 \), namely \( \phi(t) \) moves in a region where \( V \) is invertible, and so it converges to some \( \phi_s \). Then by (4b), \( \phi(t) \to -V'(\phi_s) \neq 0 \), that is a contradiction.
- \( \lim_{t \to t_s^-} |\phi(t)| = +\infty \). This can be proved with a contradiction argument, where is crucial to observe that the function
  \[
  \rho(t) := \frac{2V(\phi(t))}{\phi(t)^2},
  \]
  satisfies the equation
  \[
  \dot{\rho} = \sqrt{6} \phi \rho \sqrt{1 + \rho} \left( u(\phi) \sqrt{1 + \rho} - \text{sgn}(\dot{\phi}) \right).
  \]
  \( \lim_{t \to t_s^-} \rho(t) = 0 \). Indeed, supposing \( \phi \) positively diverging, (20) allows to see \( \rho \) as a function of \( \phi \), and to study late time behavior of \( \rho \) it suffices to study the solutions of the ODE
  \[
  \frac{d \rho}{d \phi} = \rho \sqrt{6(1 + \rho)} \left( u(\phi) \sqrt{1 + \rho} - 1 \right),
  \]
  which are indefinitely extendable to the right, and except for a zero–measured set of initial data, \( \rho(t) \to 0 \).
- \( t_s < +\infty \), and \( \lim_{t \to t_s^-} \epsilon(t) = +\infty \), \( \lim_{t \to t_s^-} a(t) = 0 \). That the solutions cannot be indefinitely extended follows from the relation \( \dot{\epsilon}(t) = 2\sqrt{3} \epsilon^{3/2} \frac{1}{1 + \rho(t)} \), and the other results are consequence of (4a), (5).
Figure 1. Behavior of the function $\phi(t)$, $\epsilon(t)$, and $a(t)$ with potential given by $V(\phi) = 2x^4 - x^3 - 6x^2 + 10$. The initial conditions are $\phi_0 = 0$, $\dot{\phi}_0 = 1$, $a_0 = 1$, $\chi_0 = 1$.

5. Examples

Let us now review some examples, where the potential satisfies the conditions assumed at the beginning of Section 4. We stress that this class contains the potentials which are usually considered in the physical applications, and it obviously suffices for $V$ to diverge at infinity to be also in the class studied in Section 3.

To begin, let us consider potentials with polynomial leading term at infinity (i.e. $\lambda^2 \phi^{2n}$). For instance, for a quartic potential

$$V(\phi) = -\frac{1}{2}m^2 \phi^2 + \lambda^2 \phi^4,$$

(22)

with $\lambda, m \neq 0$, the function $u(\phi)$ goes as $\sqrt{3/2} \phi^{-1}$ for $\phi \to +\infty$, and all conditions listed above hold.

Figure 1 shows the behavior of the scalar, under the action of a potential from this class, that is expanding at initial time. Backwards time analysis amounts to study a collapsing situation, as we have already pointed out, and a singularity forms in the past in a finite amount of comoving time. On the other side, since the absolute minimum is at a strictly positive critical value, we expect an indefinite expansion forward in time, as it actually happens. The scalar field approaches one of the local minimum of $V(\phi)$, say $\phi_*$, and the energy $\epsilon$ converges to $V(\phi_*)$.

When a local minimum $\phi_*$ such that $V(\phi_*) = 0$ exists, the oscillating behavior of an expanding solution discussed in Section 3 generically takes place; in this case, the energy $\epsilon(t)$ converges to zero. A situation of this kind is represented in Figure 2. Of course, time backwards analysis in this case is completely similar to the previous case, and a big bang singularity exists again.

Finally, when the potential also attains negative values, as explained Section 3, recollapse generically happens, in the sense that this situation is stable under perturbation of the initial data. Figure 3 shows the behavior in this case: the energy of an expanding solution vanishes at some $T > 0$, and for $t > T$ the solution recollapses, to form a singularity in the future – a singularity in the past is also formed, of course, as in the other cases.

In the case of exponential potentials, the results hold for asymptotic behaviors with leading
Figure 2. The oscillating behavior occurs when minima at zero critical value exist. In this case, $V(\phi) = (x^2 - 1)^2$, and the initial conditions are $\phi_0 = 0$, $\dot{\phi}_0 = 1$, $a_0 = 1$, $\chi_0 = 1$.

Figure 3. When the potential allows for negative value, expanding solutions may recollapse. In this case, $V(\phi) = 2x^4 - x^3 - 6x^2 + 4$, and the initial conditions are $\phi_0 = 0$, $\dot{\phi}_0 = 1$, $a_0 = 1$, $\chi_0 = 1$.

term at infinity of the form $V_0 e^{\sqrt{\lambda} |\phi|}$ with $\lambda < 1$. For instance for

$$V(\phi) = V_0 \cosh(\sqrt{6\lambda} \phi),$$

(23)

where $V_0, \lambda > 0$, the quantity $u(\phi)$ goes like $\lambda$ and so (17) is verified if $\lambda < 1$.

6. Gravitational collapse models

Homogeneous scalar fields may be used as sources for collapsing models, provided that we can find a suitable matching with an exterior spacetime. The natural choice for the exterior is the
decreasing. It follows that

\[ \kappa > 0 \]

In the analysis of collapsing case, the function \( \rho \) has been observed to vanish, in the late time behavior, for almost every choice of the initial data. Now we want to reconsider the non–generic situation, where the limit \( \lim_{t \rightarrow \frac{t_0}{2}} u(\phi(t)) \sqrt{1 + \rho(\phi(t))} \) equals 1. Actually, if the potential \( V(\phi) \) satisfies the condition

\[ \lim_{\phi \rightarrow \pm \infty} |u(\phi)| > 0, \]

the same argument can be adapted also to this (non generic) case, to prove the formation of the singularity in a finite amount of time. Indeed, in this case \( \rho \) is bounded above, so that one can find a constant \( \kappa > 0 \) such that \( \dot{\rho} > \kappa \dot{\phi}^{3/2} \) in a left neighborhood of \( \sup \hat{\mathcal{I}} \).

Moreover, recalling equation (25), if

\[ \lim_{\phi \rightarrow \pm \infty} |u(\phi)| > \frac{\sqrt{3}}{\dot{\phi}}, \]

the argument seen above also applies here, and the collapse ends into a black hole. In particular, observe that in this case \( 2 - \rho \) is bounded away from zero, and positive. On the other side, if

\[ \lim_{\phi \rightarrow \pm \infty} |u(\phi)| < \frac{\sqrt{3}}{\dot{\phi}}, \]

then \( \dot{a} \) is eventually positive, and this implies that \( \dot{a} \) is bounded, so that the formation of the apparent horizon is forbidden. As a consequence, the collapse, in this non–generic situation, ends into a naked singularity.
6.2. Examples
The potential in exponential form of Example (23) is such that \( u(\phi) \) behaves like a positive constant \( \lambda \), and then it always gives rise to a black hole. The – non generic – data producing the solution such that \( \lim_{\phi \to -\infty} \rho(\phi) = \frac{1}{\lambda^2} - 1 \) are discussed in [7, 10]; a choice of \( \lambda < \frac{\sqrt{2}}{3} \) makes the limit \( \rho_\infty \) greater than 2, and the apparent horizon cannot form, resulting in a naked singularity. In [11] it is argued that cosmic censorship is restored for these models when loop quantum gravity modifications are taken into account – however, by the aforesaid, it should be stressed that even at a classical level these naked singularities are the outcome of a special choice of initial data.

Let \( V \) a potential in polynomial form, as the ones treated in Example (22). We already know that almost every choice of initial data forms a singularity, which happens to be a black hole. Note that, in the general case of a polynomial potential with leading term \( \lambda^2 \phi^{2n} \), with \( n > 2 \), it can be shown that every initial data gives rise to a singularity, although the function \( u(\phi) \) diverges and therefore the above arguments cannot be applied. Indeed, if \( \sqrt{\rho + 1} \) goes like \( u(\phi)^{-1} \) – that in this case behaves like \( \frac{n}{\sqrt{\phi}} \), then using (19) it can be seen that \( \phi \approx \phi^{n-1} \), and then if \( n > 2 \) the solution must diverge in a finite amount of time. The non genetically data yielding this particular situation, moreover, forbid apparent horizon formation, and so the resulting singularity is naked.

6.3. Collapse in higher order gravity theories
Scalar fields arise also when higher order gravity theories, described by Lagrangians of the form \( L = f(R)\sqrt{-g} \), are considered. Indeed [1, 2], under the conformal transformation \( g_{\mu \nu} \to f'(R)g_{\mu \nu} \), fields equations reduce to Einstein field equations with a scalar field (interacting with a potential) as a matter source. The function \( f \) determines the form of the potential. In paper [9] this situation is analyzed where, before performing the conformal transformation, a perfect fluid matter contribution \( p = (\gamma - 1)\rho \) \((0 \leq \gamma < 2) \) is added to the Lagrangian. Interestingly enough, it is proved that this is a fact generically takes place when \( \gamma > 2/3 \) – yet producing a lower order curvature divergence with respect to the energy of the scalar field.

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