A canonical structure on the tangent bundle of a pseudo- or para-Kähler manifold

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Abstract It is a classical fact that the cotangent bundle \( T^*M \) of a differentiable manifold \( M \) enjoys a canonical symplectic form \( \Omega^* \). If \((M, J, g, \omega)\) is a pseudo-Kähler or para-Kähler \( 2n \)-dimensional manifold, we prove that the tangent bundle \( TM \) also enjoys a natural pseudo-Kähler or para-Kähler structure \((\tilde{J}, \tilde{g}, \Omega)\), where \( \Omega \) is the pull-back by \( g \) of \( \Omega^* \) and \( \tilde{g} \) is a pseudo-Riemannian metric with neutral signature \((2n, 2n)\). We investigate the curvature properties of the pair \((\tilde{J}, \tilde{g})\) and prove that: \( \tilde{g} \) is scalar-flat, is not Einstein unless \( g \) is flat, has nonpositive (resp. nonnegative) Ricci curvature if and only if \( g \) has nonpositive (resp. nonnegative) Ricci curvature as well, and is locally conformally flat if and only if \( n = 1 \) and \( g \) has constant curvature, or \( n > 2 \) and \( g \) is flat. We also check that (i) the holomorphic sectional curvature of \((\tilde{J}, \tilde{g})\) is not constant unless \( g \) is flat, and (ii) in \( n = 1 \) case, that \( \tilde{g} \) is never anti-self-dual, unless conformally flat.

Keywords Tangent bundle · Pseudo-Kähler geometry · Para-Kähler geometry · Self-duality · Anti-self-duality

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1 Introduction

It is a classical fact that given any differentiable manifold $\mathcal{M}$, its cotangent bundle $T^*\mathcal{M}$ enjoys a canonical symplectic structure $\Omega^*$. Moreover, given a linear connection $\nabla$ on a manifold $\mathcal{M}$, (e.g. the Levi-Civita connection of a Riemannian metric), the bundle $TT\mathcal{M}$ splits into a direct sum of two subbundles $H\mathcal{M}$ and $V\mathcal{M}$, both isomorphic to $T\mathcal{M}$. This allows to define an almost complex structure $J$ by setting $J(X_h, X_v) := (-X_v, X_h)$, where, for $X \in TT\mathcal{M} = H\mathcal{M} \oplus V\mathcal{M}$, we write $X \simeq (X_h, X_v) \in T\mathcal{M} \times T\mathcal{M}$. Analogously, one may introduce a natural almost para-complex (or bi-Lagrangian) structure, setting $J'(X_h, X_v) := (X_v, X_h)$.

It is also well known that the tangent bundle of a Riemannian manifold $(\mathcal{M}, g)$ can be given a natural Riemannian structure, called Sasaki metric. A simple way to understand this construction, which extends verbatim to the case of a pseudo-Riemannian metric $g$ with signature $(p, m - p)$, is as follows: using the splitting $TT\mathcal{M} = H\mathcal{M} \oplus V\mathcal{M}$, we set:

$$G((X_h, X_v), (Y_h, Y_v)) := g(X_h, Y_h) + g(X_v, Y_v).$$

This metric has signature $(2p, 2(m - p))$ and is well behaved with respect to $J$ in two ways: (i) $G$ is compatible with $J$, i.e. $G(\cdot, \cdot) = G(J(\cdot), J(\cdot))$, and (ii) the symplectic form $\Omega := G(J(\cdot), \cdot)$ is nothing but the pull-back of $\Omega^*$ by the musical isomorphism between $T\mathcal{M} \cong_g T^*\mathcal{M}$. In other words, the triple $(J, G, \Omega)$ defines an “almost pseudo-Kähler” structure\(^1\) on $T\mathcal{M}$.

Unfortunately, this construction suffers two flaws: $J$ is not integrable unless $\nabla$ is flat and the metric $G$ is somewhat “rigid”: for example, if $G$ has constant scalar curvature, then $g$ is flat (see [14]). We refer to [4,15] and the survey [8] for more detail on the Sasaki metric.

Another construction can be made in the case where $\mathcal{M}$ is complex (resp. para-complex): in this case both $T\mathcal{M}$ and $T^*\mathcal{M}$ enjoy a canonical complex (resp. para-complex) structure which are defined as follows: given a family of holomorphic (resp. para-holomorphic\(^2\)) local charts $\varphi : \mathcal{M} \to \mathcal{U} \subset \mathbb{C}^{2n}$ on $\mathcal{M}$, we define holomorphic (resp. para-holomorphic) local charts $\tilde{\varphi} : T\mathcal{M} \to \mathcal{U} \times \mathbb{C}^{2n}$ by $\tilde{\varphi}(x, V) = (\varphi(x), d\varphi_x(V))$, $\forall (x, V) \in T\mathcal{M}$ for the tangent bundle, and $\tilde{\varphi} : T^*\mathcal{M} \to \mathcal{U} \times \mathbb{C}^{2n}$ by $\tilde{\varphi}(x, \xi) = (\varphi(x), ((d\varphi_x)^{-1})(\xi))$, $\forall (x, \xi) \in T^*\mathcal{M}$ for the cotangent bundle. In the first section, we shall see that if $\mathcal{M}$ is merely almost complex (resp. almost para-complex), then a more subtle argument allows to define again a canonical almost complex structure (resp. almost para-complex structure) on $T\mathcal{M}$. On the other hand, we shall prove in the second section that if $\mathcal{M}$ is pseudo- or para-Kähler, the corre-

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\(^1\) We might also define an “almost para-Kähler” structure on $T\mathcal{M}$ by introducing the para-Sasaki metric

$$G'((X_h, X_v), (Y_h, Y_v)) := g(X_h, Y_h) - g(X_v, Y_v).$$

This metric has neutral signature $(m, m)$ ($m$ being the dimension of $\mathcal{M}$), is compatible with $J'$ and verifies $\Omega := -G'(J', \cdot, \cdot)$.

\(^2\) The terminology split-holomorphic is sometimes used.
Corresponding structure on $T\mathcal{M}$ can also be constructed using the splitting $H\mathcal{M} \oplus V\mathcal{M}$ induced by the Levi-Civita connection of the Kählerian metric.

Combining the canonical symplectic structure $\Omega^*$ of $T^*\mathcal{M}$ with the canonical complex (resp. para-complex) structure $\tilde{J}^*$ just defined, it is natural to introduce a 2-tensor $\tilde{g}^*$ by the formula

$$\tilde{g}^* := \Omega^*(., \tilde{J}^*).$$

However, it turns out that $\Omega^*$ is not compatible with $\tilde{J}^*$, since it turns out that $\Omega^*(\tilde{J}^*, \tilde{J}^*) = -\varepsilon \Omega^*$ instead of the required formula $\Omega^*(\tilde{J}^*, \tilde{J}^*) = \varepsilon \Omega^*$ (here and in the following, in order to deal simultaneously with the complex and para-complex cases, we define $\varepsilon$ to be such that $(\tilde{J}^*)^2 = -\varepsilon \text{Id}$, i.e. $\varepsilon = 1$ in the complex case and $\varepsilon = -1$ in the para-complex case). It follows that the tensor $\tilde{g}^*$ is not symmetric and therefore we failed in constructing a canonical pseudo-Riemannian structure on $T^*\mathcal{M}$.

On the other hand, the same idea works well if one considers, instead of the cotangent bundle, the tangent bundle of a pseudo- or para-Kähler manifold $(\mathcal{M}, J, g)$, thus obtaining a canonical pseudo- or para-Kähler structure. The purpose of this note is to investigate in detail this construction and to study its curvature properties. The results are summarized in the following:

**Main Theorem** Let $(\mathcal{M}, J, g, \omega)$ be a pseudo- or para-Kähler manifold. Then $T\mathcal{M}$ enjoys a natural pseudo- or para-Kähler structure $(\tilde{J}, \tilde{g}, \Omega)$ with the following properties:

- $\tilde{J}$ is the canonical complex or para-complex structure of $T\mathcal{M}$ induced from that of $\mathcal{M}$;
- $\Omega$ is the pull-back of $\Omega^*$ by the metric isomorphism $T\mathcal{M} \cong_g T^*\mathcal{M}$;
- The pseudo-Riemannian metric $\tilde{g}$ can be recovered from $\tilde{J}$ and $\Omega$ by the equation $\tilde{g}(., .) := \Omega(., \tilde{J})$;
- According to the splitting $TT\mathcal{M} = H\mathcal{M} \oplus V\mathcal{M}$ induced by the Levi-Civita connection of $g$, the triple $(\tilde{J}, \tilde{g}, \Omega)$ takes the following expression:

$$\tilde{J}(X_h, X_v) := (JX_h, JX_v)$$
$$\tilde{g}((X_h, X_v), (Y_h, Y_v)) := g(X_v, JY_h) - g(X_h, JY_v)$$
$$\Omega((X_h, X_v), (Y_h, Y_v)) := g(X_v, Y_h) - g(X_h, Y_v);$$

- The pseudo-Riemannian metric $\tilde{g}$ has the following properties:
  (i) $\tilde{g}$ has neutral signature neutral $(2n, 2n)$ and is scalar flat;
  (ii) $(T\mathcal{M}, \tilde{g})$ is Einstein if and only if $(\mathcal{M}, g)$ is flat, and therefore $(T\mathcal{M}, \tilde{g})$ is flat as well;
  (iii) the Ricci curvature $\tilde{\text{Ric}}$ of $\tilde{g}$ has the same sign as the Ricci curvature $\text{Ric}$ of $g$;
  (iv) $(T\mathcal{M}, \tilde{g})$ is locally conformally flat if and only if $n = 1$ and $g$ has constant curvature, or $n > 2$ and $g$ is flat; if $n = 1$, $\tilde{g}$ is always self-dual, so anti-self-duality is equivalent to conformal flatness;
  (v) the pair $(\tilde{J}, \tilde{g})$ has constant holomorphic curvature if and only if $g$ is flat.
Remark 1 We use in (iv) the general property that four-dimensional neutral pseudo-Kähler or para-Kähler manifolds are self-dual if and only if their scalar curvature vanishes. This is analogous to the case of Kähler four-dimensional manifolds, except that self-duality is exchanged with anti-self-duality. A proof of this statement is given in Theorem 7.2 in the appendix.

This result is a generalization of previous work on the tangent bundle of a Riemannian surface (see [2, 9, 10]). The authors wish to thank Brendan Guilfoyle for his valuable suggestions and comments.

2 Almost complex and para-complex structures on the tangent bundle

Given a manifold $M$ endowed with an almost complex or almost para-complex structure $J$, it is only natural to ask whether its tangent or cotangent bundle inherit such a structure. The answer is positive:

Proposition 1 Let $(M, J)$ be an almost complex (resp. para-complex) manifold. Then its tangent bundle admits a canonical almost complex (resp. para-complex) structure $\tilde{J}$. Furthermore, if $J$ is complex (resp. para-complex), so is $\tilde{J}$.

Remark 2 Such a result has been proven already by Lempert and Szöke [13] for the tangent bundle in the almost complex case. Their construction uses the jets over $M$ and is quite a bit more technical than our proof. However it gives an interesting interpretation of the meaning of $\tilde{J}$. We shall see below in Proposition 2 a different and simpler way of defining and understanding $\tilde{J}$, provided $M$ is a pseudo- or para-Kähler manifold.

Proof We prove the result using coordinate charts, which amounts to showing that $\tilde{J}$ can be defined independently of any change of variable. Let $y = \varphi(x)$ be a local change of coordinates on $\mathbb{R}^n$ and write $\xi$ and $\eta$ respectively for the tangent coordinates induced by the charts (i.e. $\sum_i \xi^i \partial/\partial x^i = \sum_i \eta^i \partial/\partial y^i$). The change of tangent coordinates at $x$ is $\xi \mapsto \eta = d\varphi(x)\xi$, in other words $\varphi$ induces a chart $\Phi$ on $\mathbb{R}^{2n}$, $\Phi : (x, \xi) \mapsto (\varphi(x), d\varphi(x)\xi)$. The tangent coordinates at $(x, \xi)$ (resp. $(y, \eta)$) are denoted by $(X, \Xi)$ (resp. $(Y, H)$) and the change of (doubly) tangent coordinates is

$$d\Phi(x, \xi) : (X, \Xi) \mapsto (Y, H) = (d\varphi(x)X, d^2\varphi(x)(X, \xi) + d\varphi(x)\Xi).$$

Assume moreover that we have a $(1, 1)$ tensor, which reads in the $x$ coordinate as the matrix $J(x)$ and in the $y$ coordinate as the matrix $J'(\varphi(x)) = d\varphi(x) \circ J(x) \circ (d\varphi(x))^{-1}$. Equivalently for any $X$ and $Y = d\varphi(x)X$, we have $J'(\varphi(x))d\varphi(x)X = d\varphi(x)J(x)X$. Differentiating this equality along $\xi$ yields

$$(D_{d\varphi(x)\xi}J')(\varphi(x))d\varphi(x)X + J'(\varphi(x))d^2\varphi(x)(X, \xi)$$

$$= d\varphi(x)(D_\xi J)(x)X + d^2\varphi(x)(J(x)X, \xi),$$

where $(D_\xi J)(x)$ denotes in this proof the directional derivative of the matrix $J$ at $x$ in the direction $\xi$ (not a covariant derivative).
We now define the $(1, 1)$ tensor $\tilde{J}$ in the $(x, \xi)$ coordinate by

$$\tilde{J}(x, \xi) : (X, \Xi) \mapsto (J(x)X, J(x)\Xi + D_\xi J(x)X).$$

Let us prove that this definition is coordinate-independent (for greater readability we will often write $J$, $J'$ for $J(x)$, $J'(y)$). Using (1) and the symmetry of the second order differential $d^2\varphi(x)$,

$$d\Phi(x, \xi)(J(X, \Xi)) = d\Phi(x, \xi)(JX, J\Xi + D_\xi J(x)X)$$
$$= (d\varphi(x)JX, d^2\varphi(x)(JX, \xi) + d\varphi(x)(J\Xi + D_\xi J(x)X))$$
$$= (J'Y, J'd\varphi(x)\Xi$$
$$+ (D_{d\varphi(x)\xi}J')(\varphi(x))d\varphi(x)(X, \xi))$$
$$= (J'Y, J'(d\varphi(x)\Xi + d^2\varphi(x)(X, \xi))$$
$$+ (D_{d\varphi(x)\xi}J')(\varphi(x))d\varphi(x)X)$$
$$= (J'Y, J'H + D_\eta J'(y)Y) = \tilde{J}'(y, \eta)(Y, H),$$

where $\tilde{J}'$ denotes the map corresponding to $\tilde{J}$ in the $(y, \eta)$ coordinates. Consequently the tensor on $\mathcal{M}$ extends naturally to $T\mathcal{M}$.

We have so far defined a $(1, 1)$ tensor on $T\mathcal{M}$ without extra assumptions. Suppose now that $J$ is an almost complex (resp. para-complex) structure, so that $J^2 = -\varepsilon \text{Id}$. Differentiating this property yields $J D_\xi J + (D_\xi J) J = 0$. Then

$$\tilde{J}^2(X, \Xi) = (J^2X, J(J\Xi + D_\xi JX) + D_\xi J(JX))$$
$$= (-\varepsilon X, -\varepsilon \Xi + J(dJ\xi)X + (dJ\xi)(JX)) = -\varepsilon(X, \Xi)$$

so that $\tilde{J}$ is also an almost complex (resp. para-complex) structure.

Finally if $J$ is a complex (resp. para-complex) structure then we can use complex (resp. para-complex) coordinate charts, which amounts to saying that $J$ is a constant matrix. Then $\tilde{J}$ defined in the associated charts on $T\mathcal{M}$ takes a simpler expression, and is also constant:

$$\tilde{J}(x, \xi) : (X, \Xi) \mapsto (JX, J\Xi)$$

and that characterizes a complex (resp. para-complex) structure. □

Remark 3 Finding a similar almost-complex structure on $T^*\mathcal{M}$ is much more difficult, and may not be true in all generality. The Reader will note that, whenever $\mathcal{M}$ is endowed with a pseudo-Riemannian metric, we have a musical correspondence between $T\mathcal{M}$ and $T^*\mathcal{M}$, and $\tilde{J}$ induces a corresponding structure $\tilde{J}$ on $T^*\mathcal{M}$. However different metrics will yield different structures on $T^*\mathcal{M}$. There is one unambiguous case, which will be the setting in the remainder of this article, namely when $J$ is integrable.
3 The Kähler structure

Let $\mathcal{M}$ be a differentiable manifold. We denote by $\pi$ and $\pi^*$ the canonical projections $T\mathcal{M} \to \mathcal{M}$ and $T\mathcal{M}^* \to \mathcal{M}$. The subbundle $\ker(d\pi) := V\mathcal{M}$ of $TT\mathcal{M}$ (it is thus a bundle over $T\mathcal{M}$) will be called the vertical bundle.

Assume now that $\mathcal{M}$ is equipped with a linear connection $\nabla$. The corresponding horizontal bundle is defined as follows: let $\bar{X}$ be a tangent vector to $T\mathcal{M}$ at some point $(x_0, V_0)$. This implies that there exists a curve $\gamma(s) = (x(s), V(s))$ such that $(x(0), V(0)) = (x_0, V_0)$ and $\gamma'(0) = \bar{X}$. If $X \notin V\mathcal{M}$ (which implies $x'(0) \neq 0$), we define the connection map (see [2,6]) $K : TT\mathcal{M} \to T\mathcal{M}$ by $K\bar{X} = \nabla_{x'(0)}V(0)$, where $V$ denotes the Levi-Civita connection of the metric $g$. If $X$ is vertical, we may assume that the curve $\gamma$ stays in a fiber so that $V(s)$ is a curve in a vector space. We then define $K\bar{X}$ to be simply $V'(0)$. The horizontal bundle is then $\text{Ker}(K)$ and we have a direct sum

$$TT\mathcal{M} = H\mathcal{M} \oplus V\mathcal{M} \simeq T\mathcal{M} \oplus T\mathcal{M}$$

$$\bar{X} \simeq (\Pi\bar{X}, K\bar{X}). \quad (2)$$

Here and in the following, $\Pi$ is a shorthand notation for $d\pi$.

**Lemma 1** [6] Given a vector field $X$ on $(\mathcal{M}, \nabla)$ there exists exactly one vector field $X^h$ and one vector field $X^v$ on $T\mathcal{M}$ such that $(\Pi X^h, KX^h) = (X, 0)$ and $(\Pi X^v, KX^v) = (0, X)$. Moreover, given two vector fields $X$ and $Y$ on $(\mathcal{M}, \nabla)$, we have, at the point $(x, V)$:

$$[X^v, Y^v] = 0,$$

$$[X^h, Y^v] = (\nabla_X Y)^v \simeq (0, \nabla_X Y),$$

$$[X^h, Y^h] \simeq ([X, Y], -R(X, Y)V),$$

where $R$ denotes the curvature of $\nabla$ and we use the direct sum notation (2).

The Reader should not confuse the horizontal lift $X^h$, which is a vector field on $T\mathcal{M}$ constructed from a vector field $X \in \mathfrak{X}(\mathcal{M})$, with the notation $\bar{X}_h = \Pi\bar{X}$ denoting the horizontal part of $\bar{X} \in \mathfrak{X}(T\mathcal{M})$. Similarly, the vertical lift $X^v$ is not the vertical projection $\bar{X}_v = K\bar{X}$.

We say that a vector field $\bar{X}$ on $T\mathcal{M}$ is projectable if it is constant on the fibres, i.e. $(\Pi\bar{X}, K\bar{X})(x, V) = (\Pi\bar{X}, K\bar{X})(x, V')$. According to the lemma above, it is equivalent to the fact that there exists two vector fields $X_1$ and $X_2$ on $\mathcal{M}$ such that $\bar{X} = (X_1)^h + (X_2)^v$.

Assume now that $\mathcal{M}$ is equipped with a pseudo-Riemannian metric $g$, i.e. a non-degenerate bilinear form. By the non-degeneracy assumption, we can identify $T^*\mathcal{M}$ with $T\mathcal{M}$ by the following (musical) isomorphism:

$$\iota : T\mathcal{M} \to T^*\mathcal{M}$$

$$(x, V) \mapsto (x, \xi),$$
where $\xi$ is defined by

$$\xi(W) = g(V, W), \quad \forall \ W \in T_x \mathcal{M}.$$ 

The Liouville form $\alpha \in \Omega^1(T^* \mathcal{M})$ is the 1-form defined by $\alpha(x, \xi)(\bar{X}) = \xi(d\pi^*(\bar{X}))$, where $\bar{X}$ is a tangent vector at the point $(x, \xi)$ of $T^* \mathcal{M}$. The canonical symplectic form on $T^* \mathcal{M}$ is defined to be $\Omega^* := -d\alpha$. There is an elegant, explicit formula for the symplectic form $\Omega := \iota^*(\Omega^*)$ in terms of the metric $g$ and the splitting induced by the Levi-Civita connection $\nabla$ (see [1, 12]):

**Lemma 2** Let $\bar{X}$ and $\bar{Y}$ be two tangent vectors to $T\mathcal{M}$; we have

$$\Omega(\bar{X}, \bar{Y}) = g(K\bar{X}, \Pi\bar{Y}) - g(\Pi\bar{X}, KY).$$

**Proposition 2** Let $(\mathcal{M}, J, g)$ be a pseudo- or para-Kähler manifold. The canonical structure $\tilde{J}$ satisfies

$$\tilde{J}\bar{X} \simeq \tilde{J}(\Pi\bar{X}, K\bar{X}) = (J\Pi\bar{X}, JK\bar{X}).$$

**Corollary 1** Let $(\mathcal{M}, J, g)$ be a pseudo- or para-Kähler manifold. The 2-tensor $\tilde{g}(., .) := \Omega(., \tilde{J}.)$ satisfies

$$\tilde{g}(\bar{X}, \bar{Y}) = g(K\bar{X}, J\Pi\bar{Y}) - g(\Pi\bar{X}, JK\bar{Y}).$$

Moreover, $\tilde{g}$ is symmetric and therefore defines a pseudo-Riemannian metric on $T\mathcal{M}$.

**Proof of Proposition 2** Let us write the splitting of $TT\mathcal{M}$ in a local coordinate $x$ as in the proof of Proposition 1 (3). The Levi-Civita connection is expressed through its connection form $\mu$: $\nabla_X Y = dY(X) + \mu(X)Y$. Consequently, if $(X, \Xi) \in T(x, \xi)T\mathcal{M}$, $\Pi(X, \Xi) = X$ and $K(X, \Xi) = \Xi + \mu(X)\xi$. Thus

$$\Pi(\tilde{J}(X, \Xi)) = JX : \text{and} \ K(\tilde{J}(X, \Xi)) = J(x)\Xi + (dJ(x)\xi)X + \mu(J(x)X)\xi.$$ 

Because $J$ is integrable, we may choose $x$ to be a complex coordinate, so that $J$ is a constant endomorphism, and $dJ(x)\xi$ vanishes. Because $\mathcal{M}$ is Kähler, we know that $\mu(X)$ commutes with $J$. However, $\nabla$ being without torsion, $\mu(X)Y = \mu(Y)X$, so

$$K(\tilde{J}(X, \Xi)) = J\Xi + J\mu(X)\xi = JK(X, \Xi).$$

□

**Corollary 2** The symplectic form $\Omega$ is compatible with the complex or para-complex structure $J$.

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3 The Reader should be aware of the conflicting notation: the splitting of $TT\mathcal{M} \simeq \mathbb{R}^{4n}$ as $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$ induced by the coordinate charts (e.g. $\bar{X} \simeq ((x, \xi), (X, \Xi))$) differs a priori from the connection-induced splitting $\bar{X} \simeq (\Pi\bar{X}, K\bar{X})$.
Proof Using Lemma 2, we compute

\[
\Omega(\tilde{J}\tilde{X}, \tilde{J}\tilde{Y}) = g(K\tilde{J}\tilde{X}, \Pi\tilde{J}\tilde{Y}) - g(\Pi\tilde{J}\tilde{X}, K\tilde{J}\tilde{Y}) \\
= g(JK\tilde{X}, J\Pi\tilde{Y}) - g(J\Pi\tilde{X}, JK\tilde{Y}) \\
= \varepsilon g(K\tilde{X}, \Pi\tilde{Y}) - \varepsilon g(\Pi\tilde{X}, K\tilde{Y}) \\
= \varepsilon \Omega(\tilde{X}, \tilde{Y}).
\]

\[\Box\]

4 The Levi-Civita connection of \(\tilde{g}\)

The following lemma describes the Levi-Civita connection \(\tilde{\nabla}\) of \(\tilde{g}\) in terms of the direct decomposition of \(TT\mathcal{M}\), the Levi-Civita connection \(\nabla\) of \(g\) and its curvature tensor \(R\).

Lemma 3 Let \(\tilde{X}\) and \(\tilde{Y}\) be two vector fields on \(T\mathcal{M}\) and assume that \(\tilde{Y}\) is projectable, then at the point \((x, V)\) we have

\[
(\tilde{\nabla}_{\tilde{X}}\tilde{Y})|_{V} = (\nabla_{\Pi\tilde{X}}\Pi\tilde{Y}, \nabla_{\Pi\tilde{X}}K\tilde{Y} - T_1(\Pi\tilde{X}, \Pi\tilde{Y}, V)),
\]

where

\[
T_1(X, Y, V) = \frac{1}{2}(R(X, Y)V - \varepsilon R(V, JX)JY - \varepsilon R(V, JY)JX)
\]

Moreover, if \(\mathcal{M}\) is a pseudo-Riemannian surface with Gaussian curvature \(c\), we have

\[
T_1(X, Y, V) = \begin{cases} 
-2cg(V, X)Y & \text{in the Kähler case,} \\
+2cg(V, Y)X & \text{in the para-Kähler case.}
\end{cases}
\]

Proof We use Lemma 1 together with the Koszul formula:

\[
2\tilde{g}(\tilde{\nabla}_{\tilde{X}}\tilde{Y}, \tilde{Z}) = \tilde{X}\tilde{g}(\tilde{Y}, \tilde{Z}) + \tilde{Y}\tilde{g}(\tilde{X}, \tilde{Z}) - \tilde{Z}\tilde{g}(\tilde{X}, \tilde{Y}) + \tilde{g}([\tilde{X}, \tilde{Y}], \tilde{Z}) \\
- \tilde{g}([\tilde{X}, \tilde{Z}], \tilde{Y}) - \tilde{g}([\tilde{Y}, \tilde{Z}], \tilde{X}),
\]

where \(X, Y\) and \(Z\) are three vector fields on \(T\mathcal{M}\). From the fact that \([X^v, Y^v]\) and \(\tilde{g}(X^v, Y^v)\) vanish we have:

\[
2\tilde{g}(\tilde{\nabla}_{X^v}Y^v, Z^v) = X^v\tilde{g}(Y^v, Z^v) + Y^v\tilde{g}(X^v, Z^v) - Z^v\tilde{g}(X^v, Y^v) \\
+ \tilde{g}([X^v, Y^v], Z^v) - \tilde{g}([X^v, Z^v], Y^v) - \tilde{g}([Y^v, Z^v], X^v) \\
= 0.
\]
Moreover, taking into account that \( \tilde{g}(Y^v, Z^h) \) and similar quantities are constant on the fibres, we obtain

\[
2\tilde{g}(\tilde{\nabla}X^v Y^v, Z^h) = X^v \tilde{g}(Y^v, Z^h) + Y^v \tilde{g}(X^v, Z^h) - Z^h \tilde{g}(X^v, Y^v) \\
+ \tilde{g}([X^v, Y^v], Z^h) - \tilde{g}(X^v, Z^h], Y^v) - \tilde{g}([Y^v, Z^h], X^v)
\]

\[
= -\tilde{g}(-(\nabla Z)X^v, Y^v) - \tilde{g}(-(\nabla Z)Y^v, X^v)
\]

\[
= 0.
\]

From these last two equations we deduce that \( \tilde{\nabla}X^v Y^v \) vanishes. Analogous computations show that \( \tilde{\nabla}X^v Y^h \) vanishes as well. From Lemma 1 and the formula \([\bar{X}, \bar{Y}] = \tilde{\nabla}X \bar{Y} - \tilde{\nabla}Y \bar{X}, \) we deduce that

\[
\tilde{\nabla}X^v Y^v \simeq (0, \nabla X Y).
\]  (3)

Finally, introducing

\[
T_1(X, Y, V) := \frac{1}{2}(R(X, Y)V - \varepsilon R(V, JY)JX - \varepsilon R(V, JX)JY),
\]

we compute that

\[
2\tilde{g}(\tilde{\nabla}X^h Y^v, Z^h) = -g(R(X, Y)V, JZ) + g(R(X, Z)V, JY) + g(R(Y, Z)V, JX)
\]

\[
= -g(R(X, Y)V, JZ) + g(R(V, JY)X, Z) + g(R(V, JX)Y, Z)
\]

\[
= -g(R(X, Y)V, JZ) + \varepsilon g(R(V, JY)JX, JZ) + \varepsilon g(R(V, JX)JY, JZ)
\]

\[
= -g(2T_1(X, Y, V), JZ)
\]

and

\[
\tilde{g}(\tilde{\nabla}X^h Y^v, Z^v) = -g(\nabla X Y, JZ),
\]

from which we deduce that

\[
\tilde{\nabla}X^h Y^v(V) = (\nabla X Y, -T_1(X, Y, V)).
\]  (4)

From (3) and (4) we deduce the required formula for \( \tilde{\nabla}_X \bar{Y} \).

If \( n = 1 \), we have \( R(X, Y)Z = c(g(Y, Z)X - g(X, Z)Y) \), hence the tensor \( T_1 \) becomes:

\[
2T_1(X, Y, V) = R(X, Y)V + \varepsilon JR(V, JX)Y + \varepsilon JR(V, JY)X
\]

\[
= c(g(Y, V)X - g(X, V)Y
\]

\[
- \varepsilon J(g(JX, Y)X - g(JY, X)V - g(Y, V)JY)
\]

\[
= c(g(Y, V)X - g(X, V)Y
\]

\[
- \varepsilon J(g(JX, Y)JX + g(Y, V)X + g(JY, X)JV + g(Y, V)Y)
\]

\[
= c((1 - \varepsilon)g(V, Y)X - (1 + \varepsilon)g(V, X)Y).
\]
Remark 4 It should be noted that covariant derivatives with respect to a projectable vertical field $X^v$ always vanish.

**Proposition 3** The structure $\tilde{J}$ is parallel with respect to $\tilde{\nabla}$.

**Proof** It can be seen as a trivial consequence of the fact that $\tilde{J}$ is complex (resp. para-complex) and $\Omega$ is closed, but can also be checked directly, using the equivariance properties of $J$ w.r.t. the connection $\nabla$ and the curvature tensor $R$. Using the definition of $\tilde{J}$ and Lemma 3, $\tilde{\nabla}_X \tilde{J} Y$ is obvious provided $T_1(X, JY, V) = JT_1(X, Y, V)$. That is indeed the case since

$$2(T_1(X, JY, V) - JT_1(X, Y, V)) = R(X, JY)V + R(V, JX)Y + R(V, Y)JX - R(X, Y)JV - R(V, JX)Y - R(V, JY)X = R(X, JY)V + R(JY, V)X + J(R(V, Y)X + R(Y, X)V) = R(V, X)JY + JR(X, Y)V = 0,$$

where we have used Bianchi’s identity. 

5 Curvature properties of $(\tilde{J}, \tilde{g})$

5.1 The Riemannian curvature tensor of $\tilde{g}$

**Proposition 4** The curvature tensor $\tilde{R}m := -\tilde{g}(\tilde{R}, \cdot, \cdot)$ of $\tilde{g}$ at $(x, V)$ is given by the formula

$$\tilde{R}m(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = g(T_2(\Pi\bar{X}, \Pi\bar{Y}, \Pi\bar{Z}, V), J\Pi\bar{W}) - Rm(\Pi\bar{X}, \Pi\bar{Y}, \Pi\bar{Z}, JK\bar{W}) - Rm(\Pi\bar{X}, \Pi\bar{Y}, J\bar{K}\bar{Z}, \Pi\bar{W}) - Rm(\Pi\bar{X}, JK\bar{Y}, \Pi\bar{Z}, \Pi\bar{W}) + Rm(JK\bar{X}, \Pi\bar{Y}, \Pi\bar{Z}, \Pi\bar{W}),$$

where

$$T_2(X, Y, Z, V) := (\nabla_X T_1)(Y, Z, V) - (\nabla_Y T_1)(X, Z, V).$$

Moreover, $(T\mathcal{M}, \tilde{g})$ is scalar flat and the Ricci tensor of $\tilde{g}$ is

$$\tilde{R}ic(\bar{X}, \bar{Y}) = 2Ric(\Pi\bar{X}, \Pi\bar{Y}).$$

**Corollary 3** $(T\mathcal{M}, \tilde{g})$ is Einstein if and only if $(\mathcal{M}, g)$ is flat. Moreover $(T\mathcal{M}, \tilde{g})$ has nonnegative (resp. nonpositive) Ricci curvature if and only if $(\mathcal{M}, g)$ has nonnegative (resp. nonpositive) Ricci curvature as well.

**Proof of Proposition 4** We will compute the curvature tensor for projectable vector fields, and need only do so for the following six cases, due to the symmetries of
Remark 4 simplifies computations greatly, since most vertical derivatives vanish, except when the derived vector field is not projectable. In particular \( \widetilde{R}(X^v, Y^v) \) vanishes as endomorphism, hence:

\[
\begin{aligned}
\widetilde{Rm}(X^v, Y^v, Z^v, W^v) &= 0 \\
\widetilde{Rm}(X^v, Y^v, Z^v, W^h) &= 0 \\
\widetilde{Rm}(X^v, Y^v, Z^h, W^v) &= 0
\end{aligned}
\]

To obtain the last three combinations, let us first derive \( \widetilde{R}(X^h, Y^h)Z^h \). This is more delicate since we have to covariantly differentiate non-projectable quantities. Indeed

\[
\begin{aligned}
\widetilde{R}(X^h, Y^h)Z^h &= \tilde{\nabla}_h \tilde{\nabla}_{Y^h} Z^h - \tilde{\nabla}_{Y^h} \tilde{\nabla}_h Z^h - \tilde{\nabla}_{[X^h, Y^h]} Z^h \\
&= \tilde{\nabla}_h (\nabla_Y Z, -T_1(Y, Z, V)) - \tilde{\nabla}_{Y^h} (\nabla_X Z, -T_1(X, Z, V)) \\
&\quad - \tilde{\nabla}_{[X,Y]} Z^h \\
&= (\nabla_X \nabla_Y Z, -T_1(X, \nabla_Y Z, V)) - D_{X^h}(0, T_1(Y, Z, V)) \\
&\quad - (\nabla_Y \nabla_X Z, -T_1(Y, \nabla_X Z, V)) + D_{Y^h}(0, T_1(X, Z, V)) \\
&\quad - (\nabla_{[X,Y]} Z, -T_1([X, Y], Z, V)) \\
&= (R(X, Y)Z, 0) \\
&\quad - (0, T_1(X, \nabla_Y Z, V) - T_1(Y, \nabla_X Z, V) - T_1([X, Y], Z, V)) \\
&\quad - \tilde{\nabla}_{X^h}(0, T_1(Y, Z, V)) + \tilde{\nabla}_{Y^h}(0, T_1(X, Z, V))
\end{aligned}
\]

Recalling the lemma\(^4\) in [11], there exists a vector field \( U \) on \( M \) such that \( U(x) = V \) and \( (\nabla_X U)(x) = 0 \). Then the vertical lift of \( T_1(Y, Z, U) \) is seen to agree to first order with

\[
(x, V) \mapsto (0, T_1(X(x), Z(x), V))
\]

thus allowing us to use the formula in Lemma 3:

\[
\begin{aligned}
\tilde{\nabla}_{X^h}(0, T_1(Y, Z, \cdot)) &= \tilde{\nabla}_{X^h}(T_1(Y, Z, U)^v) \\
&= (0, \nabla_X(T_1(Y, Z, U))) \\
&= (0, (\nabla_X T_1)(Y, Z, U) + T_1(\nabla_X Y, Z, U) \\
&\quad + T_1(Y, \nabla_X Z, U) + T_1(Y, Z, \nabla_X U))
\end{aligned}
\]

which, evaluated at \( (x, V) \), yields

\[
\begin{aligned}
\tilde{\nabla}_{X^h}(0, T_1(Y, Z, \cdot))|_{(x, V)} &= (0, (\nabla_X T_1)(Y, Z, V) + T_1(\nabla_X Y, Z, V) + T_1(Y, \nabla_X Z, V)).
\end{aligned}
\]

\(^4\) Note that computations in [11] are done for the Sasaki metric, hence direct results do not apply.
Summing up,

\[
\tilde{R}(X^h, Y^h)Z^h|_{(x, V)} = (R(X, Y)Z, -T_1(X, \nabla_Y Z, V) + T_1([X, Y], Z, V) - (\nabla_X T_1)(Y, Z, V) - T_1(\nabla_Z Y, V, Z) - T_1(\nabla_X Z, V) + T_1(\nabla_Y X, Z, V) + T_1(\nabla_Y Z, V))
\]

\[
= (R(X, Y)Z, -(\nabla_X T_1)(Y, Z, V) + (\nabla_Y T_1)(X, Z, V))
\]

\[
= (R(X, Y)Z, -T_2(X, Y, Z, V)).
\]

From that we deduce directly

\[
\tilde{Rm}(X^h, Y^h, Z^h, W^v) = -Rm(X, Y, Z, JW)
\]

\[
\tilde{Rm}(X^h, Y^h, Z^h, W^h)|_{(x, V)} = g(T_2(X, Y, Z, V), JW).
\]

On the other hand, using repeatedly Remark 4,

\[
\tilde{Rm}(X^h, Y^v, Z^h, W^v) = \tilde{g}(\tilde{\nabla}_X h \tilde{\nabla}_Y v W^v - \tilde{\nabla}_Y v \tilde{\nabla}_X h W^v - \tilde{\nabla}_{[X, Y]} v W^v, Z^h)
\]

\[
= \tilde{g}(-\tilde{\nabla}_Y v(0, \nabla_X W, Z^h) = \tilde{g}(0, Z^h) = 0.
\]

The claimed formula is easily deduced using the symmetries of the curvature tensor.

In order to calculate the Ricci curvature of \( \tilde{g} \), we consider a Hermitian pseudo-orthonormal basis \((e_1, \ldots, e_{2n})\) of \( T_x M \), i.e. \( g(e_a, e_b) = \epsilon_{ab} \delta_{ab} \), where \( \epsilon_a = \pm 1 \), and \( e_{n+a} = J e_a \). In particular, \( \epsilon_{n+a} = \epsilon \epsilon_a \). This gives a (non-orthonormal) basis of \( T_{(x, V)} T M \):

\[
\tilde{e}_a := (e_a)_h \quad \tilde{e}_{2n+a} := (e_a)^v.
\]

A calculation using Corollary 1 shows that the expression of \( \tilde{g} \) in this basis is:

\[
[\tilde{g}_{\mu v}]_{1 \leq \mu, v \leq 4n} := \begin{pmatrix}
0 & 0 & 0 & \Delta \\
0 & 0 & -\Delta & 0 \\
0 & -\Delta & 0 & 0 \\
\Delta & 0 & 0 & 0
\end{pmatrix},
\]

where \( \Delta = \epsilon \text{diag}(\epsilon_1, \ldots, \epsilon_n) = \text{diag}(\epsilon_{n+1}, \ldots, \epsilon_{2n}) \). It follows that \( \tilde{\text{Ric}}(X^v, Y^v) \) and \( \tilde{\text{Ric}}(X^h, Y^v) \) vanish.
Moreover, noting that $\tilde{\bar{g}}^{\mu\nu} = \tilde{g}_{\mu\nu}$,

$$\tilde{\text{Ric}}(X^h, Y^h) = \sum_{\mu, \nu=1}^{4n} \tilde{g}^{\mu\nu} \tilde{\text{Rm}}(X^h, \tilde{e}_\mu, Y^h, \tilde{e}_\nu)$$

$$= \sum_{a=1}^{n} \varepsilon \varepsilon_a \left( \tilde{\text{Rm}}(X^h, (e_a)^h, Y^h, (J e_a)^h) - \tilde{\text{Rm}}(X^h, (J e_a)^h, Y^h, (e_a)^h) \right)$$

$$- \tilde{\text{Rm}}(X^h, (e_a)^v, Y^h, (J e_a)^h) + \tilde{\text{Rm}}(X^h, (J e_a)^v, Y^h, (e_a)^h))$$

$$= \sum_{a=1}^{n} \varepsilon \varepsilon_a \left( -\text{Rm}(X, e_a, Y, J^2 e_a) + \text{Rm}(X, J e_a, Y, J e_a) \right)$$

$$+ \text{Rm}(Y, J e_a, X, J e_a) - \text{Rm}(Y, e_a, X, J^2 e_a))$$

$$= 2 \sum_{a=1}^{n} (\varepsilon_a \text{Rm}(X, e_a, Y, e_a) + \varepsilon_{a+n} \text{Rm}(X, e_{a+n}, Y, e_{a+n}))$$

$$= 2 \sum_{k=1}^{2n} \varepsilon_k \text{Rm}(X, e_k, Y, e_k) = 2 \text{Ric}(X, Y).$$

We see easily that $\tilde{\text{Ric}}$ vanishes whenever one of the vectors is along the vertical fiber, thus the expected formula.

Finally the scalar curvature

$$\overline{\text{Scal}} = \sum_{\mu, \nu=1}^{4} \tilde{g}^{\mu\nu} \overline{\text{Ric}}(\tilde{e}_\mu, \tilde{e}_\nu) = 0,$$

since $\tilde{\bar{g}}^{\mu\nu}$ vanishes as soon as both $\tilde{e}_\mu, \tilde{e}_\nu$ are both horizontal. □

5.2 The Weyl curvature tensor of $\tilde{g}$

**Proposition 5** The Weyl tensor $\tilde{\tilde{W}}$ at $(x, V)$ is given by

$$\tilde{\tilde{W}}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) = \overline{\text{Rm}}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W})$$

$$- \frac{1}{2n-1} (\text{Ric}(\Pi \tilde{X}, \Pi \tilde{Z}) \tilde{g}(\tilde{Y}, \tilde{W}) + \text{Ric}(\Pi \tilde{Y}, \Pi \tilde{W}) \tilde{g}(\tilde{Y}, \tilde{W})$$

$$- \text{Ric}(\Pi \tilde{X}, \Pi \tilde{W}) \tilde{g}(\tilde{Y}, \tilde{Z}) - \text{Ric}(\Pi \tilde{Y}, \Pi \tilde{Z}) \tilde{g}(\tilde{X}, \tilde{W})).$$

In particular, if $n = 1$,

$$\tilde{\tilde{W}}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) = g(T_2(\Pi \tilde{X}, \Pi \tilde{Y}, \Pi \tilde{Z}, V), J \Pi \tilde{W})$$

**Corollary 4** $(T \tilde{M}, \tilde{g})$ is locally conformally flat if and only if $n = 1$ and $g$ has constant curvature, or $n \geq 2$ and $g$ is flat.
Remark 5 This result has been proved in the case \( n = 1 \) and \( \varepsilon = 1 \) in [9].

Proof of Proposition 5 Since the scalar curvature vanishes, we have

\[
\bar{\nabla} = \nabla - \frac{1}{4n-2} \nabla \otimes \bar{g},
\]

where \( \otimes \) denotes the Kulkarni–Nomizu product. Recall that \( \bar{\nabla}(X, Y) = 0 \) if one of the two vectors \( X \) and \( Y \) is vertical. Consequently

\[
\bar{\nabla} \otimes \bar{g}(X, Y, Z, W) = 2(\nabla(\Pi X, \Pi Z) \bar{g}(Y, W) + \nabla(\Pi Y, \Pi Z) \bar{g}(X, W)) - \nabla(\Pi X, \Pi Z) \bar{g}(Y, Z) - \nabla(\Pi Y, \Pi Z) \bar{g}(X, W)).
\]

The expression of the Weyl tensor follows easily.

In the case \( n = 1 \) of a surface with Gaussian curvature \( c \), we have \( \nabla(X, Y) = cg(X, Y) \) and \( \nabla(X, Y, Z, W) = c(g(X, Z)g(Y, W) - g(X, W)g(Y, Z)) \). Hence, using Proposition 4, the expression of Weyl tensor simplifies and we get the claimed formula.

\( \Box \)

Proof of Corollary 4 We first deal with the case \( n = 1 \). Lemma 3 implies that \( T_1(X, Y, Z) = -2cg(Z, X)X \) when \( \varepsilon = 1 \) (resp. \( 2cg(Z, X)X \) when \( \varepsilon = -1 \)). Therefore, if \( \varepsilon = 1 \),

\[
T_2(X, Y, Z, W) = \nabla_X T_1(Y, Z, W) - \nabla_Y T_1(X, Z, W)
\]

\[
= -2(X.c)g(W, Y)Z + 2(Y.c)g(W, X)Z
\]

\[
= 2g((Y.c)X - (X.c)Y, W)Z,
\]

which vanishes if and only if \( (X.c)Y = (Y.c)X \) for all vectors \( X, Y \), i.e. the curvature \( c \) is constant. Analogously, if \( \varepsilon = -1 \),

\[
T_2(X, Y, Z, W) = \nabla_X T_1(Y, Z, W) - \nabla_Y T_1(X, Z, W)
\]

\[
= 2(X.c)g(W, Z)Y - 2(Y.c)g(W, Z)X
\]

\[
= 2((X.c)Y - (Y.c)X)g(W, Z),
\]

which again vanishes if and only if the curvature \( c \) is constant.

Assume now that \( (T, M, \bar{g}) \) is conformally flat with \( n \geq 2 \). Thus in particular

\[
\bar{\nabla}(X^h, Y^h, Z^h, W^v)
\]

\[
= -\nabla(X, Y, Z, JW)
\]

\[
- \frac{1}{2n-1}(-\nabla(X, Z)g(Y, JW) + \nabla(Y, Z)g(X, JW))
\]

vanishes, so

\[
\nabla(X, Y, Z, JW) = \frac{1}{2n-1}(-\nabla(X, Z)g(Y, JW) + \nabla(Y, Z)g(X, JW)).
\]
(Observe that this equation always holds if $\mathcal{M}$ is a surface.) Let us apply the symmetry property of the curvature tensor to this equation with $Z = X$ and $JW = Y$, assuming furthermore that $X$ and $Y$ are two non-null vectors:

$$0 = (2n - 1)(Rm(X, Y, X, Y) - Rm(Y, X, Y, X))$$

$$= \text{Ric}(X, X)g(Y, Y) - \text{Ric}(Y, X)g(X, Y)$$

$$- \text{Ric}(Y, Y)g(X, X) + \text{Ric}(X, Y)g(Y, X)$$

$$= \text{Ric}(X, X)g(Y, Y) - \text{Ric}(Y, Y)g(X, X).$$

Hence

$$\frac{\text{Ric}(X, X)}{g(X, X)} = \frac{\text{Ric}(Y, Y)}{g(Y, Y)}.$$

The set of non null vectors being dense in $T\mathcal{M}$, it follows by continuity that $g$ is Einstein. We deduce that

$$\text{Rm}(X, Y, X, Y) = \frac{1}{2n - 1}(\text{Ric}(X, X)g(Y, Y) - \text{Ric}(Y, X)g(X, Y))$$

$$= c(g(X, X)g(Y, Y) - g(X, Y)g(Y, X)),$$

so $g$ has constant curvature. But since $\mathcal{M}$ is Kähler and has dimension $2n \geq 4$, it must be flat. $\Box$

Finally, we recall the general result linking the Weyl tensor to the scalar curvature in dimension four: for a neutral pseudo-Kähler or para-Kähler metric, self-duality is equivalent to scalar flatness (see Theorem 7.2 in annex). We can therefore conclude

**Corollary 5** In dimension four ($n = 1$), the metric $\tilde{g}$ is anti-self-dual if and only the curvature $c$ of $g$ is constant.

**Proof** Thanks to Proposition 4, we know that $\tilde{g}$ is scalar flat, hence self-dual ($W^-$ vanishes identically). In order for $\tilde{g}$ to be also anti-self-dual, the Weyl tensor has to vanish completely, which amounts, following Corollary 4, to having constant (sectional) curvature $c$ on $\mathcal{M}$. $\Box$

5.3 The holomorphic sectional curvature of $(\tilde{J}, \tilde{g})$

**Proposition 6** $(\tilde{J}, \tilde{g})$ has constant holomorphic sectional curvature if and only if $g$ is flat.

**Proof** Define the holomorphic sectional curvature tensor of $\tilde{g}$ by $\widetilde{\text{Hol}}(\tilde{X}) := \widetilde{\text{Rm}}(\tilde{X}, \tilde{JX}, \tilde{X}, \tilde{JX})$. Writing any doubly tangent vector $\tilde{X}$ as the sum of a horizontal and a vertical factor, we will compute $\widetilde{\text{Hol}}(X^h + Y^v)$. We deduce from Proposition 4 that $\text{Rm}$ vanishes whenever two or more entries are vertical. Hence, using the
antisymmetric properties of the Riemann tensor w.r.t. the complex or para-complex structure,

\[ \overline{\text{Hol}}(X^h + Y^v) = \overline{\text{Rm}}(X^h, JX^h, X^h, JX^h) \\
+ \overline{\text{Rm}}(X^h, JX^h, X^h, JY^v) + \overline{\text{Rm}}(X^h, JX^h, Y^v, JX^h) \\
+ \overline{\text{Rm}}(X^h, JY^v, JX^h, X^h, JX^h) + \overline{\text{Rm}}(Y^v, JX^h, X^h, JX^h) \\
= \overline{\text{Rm}}(X^h, JX^h, X^h, JX^h) + 4\overline{\text{Rm}}(X^h, JX^h, X^h, JY^v) \\
= g(T_2(X, JX, X, V) - 4\varepsilon R(X, Y)X, JX). \]

In particular,

\[ \overline{\text{Hol}}(X^v) = 0 \]
\[ \overline{\text{Hol}}(X^h + X^v) = g(T_2(X, JX, X, V), JX) \]
\[ \overline{\text{Hol}}(X^h + (JX)^v) = g(T_2(X, JX, X, V), JX) + 4\varepsilon \text{Hol}(X). \]

It follows from the first equation that if \( \overline{\text{Hol}} \) is constant, it must be zero. Hence, from the second and third equation we deduce that \( \text{Hol} \) must vanish, i.e. \( g \) is flat. \( \square \)

### 6 Examples

The simplest examples where we may apply the construction above is where \((\mathcal{M}, J, g, \omega)\) is the plane \( \mathbb{R}^2 \) equipped with the flat metric \( g := dq_1^2 + \varepsilon dq_2^2 \) and the complex or para-complex structure \( J \) defined by \( J(\partial_{q_1}, \partial_{q_2}) = (-\varepsilon \partial_{q_2}, \partial_{q_1}) \). In other words, \( \mathbb{R}^2 \) is identified with the complex plane \( \mathbb{C} \) or the para-complex plane \( \mathbb{D} \).

We recall that \( \mathbb{D} \), called the algebra of double numbers, is the two-dimensional real vector space \( \mathbb{R}^2 \) endowed with the commutative algebra structure whose product rule is given by

\[ (u, v)(u', v') = (uu' + vv', uv' + u'v). \]

The number \((0, 1)\), whose square is \((1, 0)\) and not \((-1, 0)\), will be denoted by \( \tau \).

We claim that in the complex case \( \varepsilon = 1 \), the structure \((\tilde{J}, \tilde{g}, \Omega)\) just constructed on \( T\mathbb{C} \) is equivalent to that of the standard complex pseudo-Euclidean plane \((\mathbb{C}^2, \tilde{J}, \langle ., . \rangle_2, \omega_1)\), where \( \tilde{J} \) is the canonical complex structure, \((z_1 = x_1 + iy_1, z_2 = x_2 + iy_2)\) are the canonical coordinates and

\[ \langle ., . \rangle_2 := -dx_1^2 - dx_2^2 + dx_2^2 + dy_2^2 \]
\[ \omega_1 := -dx_1 \wedge dy_1 + dx_2 \wedge dy_2. \]

To see this, it is sufficient to consider the following complex change of coordinates

\[ \begin{cases} 
  z_1 := \frac{\sqrt{3}}{2}((p_1 + ip_2) + i(q_1 + iq_2)) \\
  z_2 := \frac{\sqrt{3}}{2}((p_1 + ip_2) - i(q_1 + iq_2)),
\end{cases} \]
which preserves the symplectic form, since we have

$$\omega_1 := -dx_1 \land dy_1 + dx_2 \land dy_2 = dq_1 \land dp_1 + dq_2 \land dp_2 = \Omega,$$

where $\Omega$ is the canonical symplectic form of $T^*\mathbb{C} \simeq_\mathbb{C} T\mathbb{C}$. The metric of a pseudo-Kähler structure being determined by the complex structure and the symplectic form through the formula $\tilde{g} = \Omega(., \tilde{J}.,)$, we have the required identification.

Analogously, in the para-complex case $\varepsilon = -1$, the structure $(\tilde{J}, \tilde{g}, \Omega)$ constructed on $T\mathbb{D}$ is equivalent to that of the standard para-complex plane $(\mathbb{D}^2, \tilde{J}, \langle ., . \rangle_\mathbb{D}, \omega_\mathbb{D})$, where $\tilde{J}$ is the canonical para-complex structure, $(w_1 = u_1 + \tau u_1, w_2 = u_2 + \tau y_2)$ are the canonical coordinates and

$$\langle ., . \rangle_\mathbb{D} := du_1^2 - dv_1^2 + du_2^2 - dv_2^2,$$

$$\omega_\mathbb{D} := du_1 \land dv_1 + du_2 \land dv_2.$$

Here we have to be careful with the identification of $T^*\mathbb{D}$ with $T\mathbb{D}$: since the metric $g$ is $dq_1^2 - dq_2^2$, we have $q_1 := dp_1 \simeq_g \partial_{p_1}$ and $q_2 := dp_2 \simeq_g -\partial_{q_2}$. Hence $\Omega^* = dq_1 \land dp_1 + dq_2 \land dp_2$ and $\Omega = dq_1 \land dp_1 - dq_2 \land dp_2$. Introducing the change of para-complex coordinates

$$\begin{align*}
    w_1 &:= \sqrt{2}(p_1 + \tau p_2) - \tau(q_1 + \tau q_2) \\
    w_2 &:= \sqrt{2}(\tau p_1 + p_2 + (q_1 + \tau q_2)),
\end{align*}$$

we check that

$$\omega_\mathbb{D} = du_1 \land dv_1 + du_2 \land dv_2 = dq_1 \land dp_1 - dq_2 \land dp_2 = \Omega,$$

hence we obtain the identification between $(T\mathbb{D}, \tilde{J}, \tilde{g}, \Omega)$ and $(\mathbb{D}^2, \tilde{J}, \langle ., . \rangle_\mathbb{D}, \omega_\mathbb{D})$. Of course the metrics considered in these two examples are flat.

The next simplest examples of pseudo-Riemannian surfaces are the two-dimensional space forms, namely the sphere $S^2$, the hyperbolic plane $H^2 := \{x_1^2 + x_2^2 - x_3^2 = -1\}$ and the de Sitter surface $dS^2 := \{x_1^2 + x_2^2 - x_3^2 = 1\}$. Their tangent bundles enjoy a interesting geometric interpretation (see [9]): the tangent bundle $TS^2$ is canonically identified with the set of oriented lines of Euclidean three-space:

$$L(\mathbb{R}^3) \ni \{V + tx | t \in \mathbb{R}\} \simeq (x, V - \langle V, x \rangle_0 x) \in TS^2.$$

Analogously, the tangent bundle $T H^2$ is canonically identified with the set of oriented negative (timelike) lines of three-space endowed with the metric $\langle ., . \rangle_1 := dx_1^2 + dx_2^2 - dx_3^2$:

$$\mathbb{L}^3_{1,-} \ni \{V + tx | t \in \mathbb{R}\} \simeq (x, V - \langle V, x \rangle_1 x) \in T H^2,$$
Finally, the tangent bundle $TdS^2$ is canonically identified with the set of oriented positive (spacelike) lines of three-space endowed with the metric $\langle ., . \rangle_1$:

$$L^3_{1,+} \ni \{V + tx | t \in \mathbb{R}\} \cong (x, V - \langle V, x \rangle_1 x) \in TdS^2.$$ 

Observe that the metric constructed on $T\mathbb{S}^2$ (resp. $T\mathbb{H}^2$) has non-negative (resp. non-positive) Ricci curvature.

Appendix: the Weyl tensor in the pseudo-Kähler or para-Kähler cases

The Riemann curvature tensor $Rm$ of a pseudo-Riemannian manifold $\mathcal{N}$ may be seen as a symmetric form $R$ on bivectors of $\Lambda^2 T\mathcal{N}$ (see [3] for references). Splitting $R$ along the eigenspaces $\Lambda^+ \oplus \Lambda^-$ of the Hodge operator $\ast$ on $\Lambda^2 T\mathcal{N}$, yields the following block decomposition

$$R = \begin{pmatrix} W^+ + \frac{\text{Scal}}{12} I & Z^* \\ Z & W^- + \frac{\text{Scal}}{12} I \end{pmatrix}$$

where $Z^*$ denotes the adjoint w.r.t. the induced metric on $\Lambda^2 T\mathcal{N}$, so that $W = W^+ \oplus W^-$, the Weyl tensor seen as a 2-form on $\Lambda^2 T\mathcal{N}$, is the traceless, Hodge-commuting part of the Riemann curvature operator $R$. Hence the following formula

$$W = Rm - \frac{1}{2} \text{Ric} \otimes g + \frac{\text{Scal}}{12} g \otimes g.$$ 

If, additionally, $\mathcal{N}$ is a four dimensional Kähler manifold, then

**Theorem 7.1** $W^+$ can be written as a multiple of the scalar curvature by a parallel non-trivial 2-form on $\Lambda^2 T\mathcal{N}$.

See Prop. 2 in [5] for a proof and the explicit formula for the tensor involved. We do not need it explicitly since we are only interested in the following

**Corollary 6** $(\mathcal{N}, g, J)$ is anti-self-dual ($W^+ = 0$) if and only if the scalar curvature vanishes.

The result extends to the two cases considered in this article: (1) neutral pseudo-Kähler manifolds and (2) para-Kähler manifolds, with a slight twist: $W^+$ is replaced by $W^-$. Precisely:

**Theorem 7.2** Let $(\mathcal{N}, g, J)$ be a four dimensional manifold endowed with a pseudo-Kähler neutral metric (respectively a para-Kähler metric, necessarily neutral). Then the Weyl tensor $W$ commutes with the Hodge operator and $\mathcal{N}$ is self-dual ($W^- = 0$) if and only if the scalar curvature vanishes.

The result for neutral pseudo-Kähler manifolds is probably known and relates to representation theory (see [3] for introduction and references), but since we could
not find an explicit proof in the literature\textsuperscript{5}, we will give a simple one below. To our knowledge, the proof for the para-Kähler case is new (albeit similar).

A.1 The pseudo-Kähler case

We will write explicitly the Weyl tensor in a given positively oriented orthonormal frame, denoted by \((e_1, e_1', e_2, e_2')\), where \(e_1' = Je_1, e_2' = Je_2\), \(g(e_1) = g(e_1') = -1\) and \(g(e_2) = g(e_2') = +1\). (For brevity, \(g(X)\) denotes the norm \(g(X, X)\).) The pseudo-metric \(g\) extends to bivectors, has signature \((2, 4)\), and will be again denoted by \(g\):

\[
g(e_a \wedge e_b) = g(e_a)g(e_b) - g(e_a, e_b)^2 = g(e_a)g(e_b), \quad \text{so that} \quad \mathcal{B} = (e_1 \wedge e_1', e_1 \wedge e_2, e_1' \wedge e_2', e_1' \wedge e_2, e_2' \wedge e_2') \text{ is an orthonormal frame of } \Lambda^2, \quad \text{with } g(e_a \wedge e_b) = -1, \quad \text{except for } g(e_1 \wedge e_1') = g(e_2 \wedge e_2') = +1. \quad \text{(Note that the other convention, taking } -g \text{ does not change the induced metric on } \Lambda^2.)
\]

Since the volume \(e_1 \wedge e_1' \wedge e_2 \wedge e_2'\) is positively oriented, we construct an orthonormal eigenbasis for the Hodge star on \(\Lambda^2 T\mathcal{N}^2\):

\[
\begin{align*}
E_1^\pm &= \frac{\sqrt{2}}{2} (e_1 \wedge e_1' \pm e_2 \wedge e_2') \\
E_2^\pm &= \frac{\sqrt{2}}{2} (e_1 \wedge e_2 \pm e_1' \wedge e_2') \\
E_3^\pm &= \frac{\sqrt{2}}{2} (e_1 \wedge e_2' \mp e_1' \wedge e_2)
\end{align*}
\]

so that \(\Lambda^\pm\) is generated by \(E_1^\pm, E_2^\pm, E_3^\pm\).

The Kähler condition implies

\[
\text{Rm}(\text{J}X, \text{J}Y, Z, T) = \text{Rm}(X, Y, Z, T) = \text{Rm}(X, Y, JZ, JT),
\]

because \(J\) is isometric and parallel. The matrix of the symmetric 2-form \(R\) in the orthonormal frame \(\mathcal{B}\) is

| \(e_{11}'\) | \(e_{12}\) | \(e_{12}'\) | \(e_{12}''\) | \(e_{12}''\) | \(e_{12}''\) |
|---|---|---|---|---|---|
| \(e_{11}'\) | \(R_{11}'1\) | \(R_{11}'1\) | \(R_{11}'1\) | \(R_{11}'1\) | \(R_{11}'1\) |
| \(e_{12}\) | \(R_{1212}\) | \(R_{1212}\) | \(R_{1212}\) | \(R_{1212}\) | \(R_{1212}\) |
| \(e_{12}'\) | \(R_{1212}\) | \(R_{1212}\) | \(R_{1212}\) | \(R_{1212}\) | \(R_{1212}\) |
| \(e_{12}''\) | \(R_{1212}\) | \(R_{1212}\) | \(R_{1212}\) | \(R_{1212}\) | \(R_{1212}\) |
| \(e_{12}''\) | \(R_{1212}\) | \(R_{1212}\) | \(R_{1212}\) | \(R_{1212}\) | \(R_{1212}\) |

where \(e_{ab}\) stands for \(e_a \wedge e_b\), for greater legibility. We have written the matrix as a table for clarity and to make symmetries more obvious, and because \(R\) is symmetric

\textsuperscript{5} On the contrary, some authors seem to imply that scalar flatness is equivalent to anti-self-duality, see \cite{7}). However this contradiction could possibly come from a different choice of orientation, which would exchange self-dual with anti-self-dual.
we need only write half the matrix. We have used the internal symmetries of $R$, to choose among equivalent coefficients the ones lowest in the lexicographic order of the indices.

The Weyl tensor satisfies some of the $J$-symmetries of $R$: indeed

$$\text{Ric}(JX, JY) = \sum_{i=1}^{4} g(e_i) \text{Rm}(JX, e_i, JY, e_i) = \sum_{i=1}^{4} g(e_i) \text{Rm}(X, Je_i, Y, Je_i)$$

$$= \sum_{i=1}^{4} g(Je_i) \text{Rm}(X, Je_i, Y, Je_i) = \text{Ric}(X, Y)$$

because $(Je_i)$ is again an orthonormal basis. In particular, this invariance implies $r_{11'} = \text{Ric}(e_1, e_1') = r_{1'} = -r_{11}$, so $r_{11'}$ vanish (and so does $r_{22'}$). For the Kulkarni–Nomizu product,

$$\text{Ric} \otimes g(JX, Y, Z, T) = \text{Ric}(JX, Z)g(Y, T) + \text{Ric}(Y, T)g(JX, Z) - \text{Ric}(JX, T)g(Y, Z) - \text{Ric}(Y, Z)g(JX, T)$$

$$= -\text{Ric}(X, JZ)g(JY, JT) - \text{Ric}(JY, JT)g(X, JZ) + \text{Ric}(X, JT)g(JY, JZ) + \text{Ric}(JY, JZ)g(X, JT)$$

$$= -\text{Ric} \otimes g(X, JY, JZ, JT)$$

so

$$\text{Ric} \otimes g(JX, JY, Z, T) = -\text{Ric} \otimes g(X, J^2Y, JZ, JT) = \text{Ric} \otimes g(X, Y, JZ, JT).$$

Hence the following symmetries (fewer than for $R_m$) in the coefficients of $\text{Ric} \otimes g$, $g \otimes g$ and $R_m$, and therefore $W$:

| $e_{11'}$ | $e_1 \wedge e_2$ | $e_{12'}$ | $e_1' \wedge e_2$ | $e_{12}'$ | $e_{22}'$ |
|---|---|---|---|---|---|
| $W_{11'11'}$ | $W_{11'12}$ | $W_{11'12'}$ | $W_{11'12} = -W_{11'12'}$ | $W_{11'21'2'} = W_{11'12}$ | $W_{11'22'}$ |
| $e_{12}$ | $W_{1212}$ | $W_{1212'}$ | $W_{1212} = -W_{1212'}$ | $W_{121'2'} = W_{1212}$ | $W_{1222'}$ |
| $e_{12}'$ | $W_{12'12}$ | $W_{12'12'}$ | $W_{12'12} = -W_{12'12'}$ | $W_{12'1'2'} = -W_{1212}$ | $W_{12'22'}$ |
| $e_{1'2}$ | $W_{1'21'2'} = W_{12'12'}$ | $W_{1'21'2'} = -W_{1212'}$ | $W_{1'21'2'} = -W_{1212}$ | $W_{1'2'22'} = W_{1212}$ | $W_{1'2'2'2'} = W_{1222'}$ |
| $e_{1'2'}$ | $W_{1'2'12'} = W_{1212}$ | $W_{1'2'12'} = W_{1212}$ | $W_{1'2'12'} = W_{1212}$ | $W_{1'2'22'} = W_{1222'}$ | $W_{1'2'2'2'} = W_{1222'}$ |
| $e_{22'}$ | | | | | |

Expanding on the above eigenbasis of $\Lambda^+ \oplus \Lambda^-$ (which differs from the one in the positive definite case) yields the following Weyl tensor coefficients, which we have simplified using the symmetries above (up to a factor $1/2$ due to normalization):
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\[ E^+ + E^+ + E^+ + E^- + E^- + E^- \]

\[ (W_{11'11'} + W_{22'22'}) + 2(W_{11'12} + W_{1222'}) + 2(W_{11'12'} + W_{12'22'}) = 2(W_{11'12} + W_{1222'}) + 2(W_{11'12'} + W_{12'22'}) \]

Further simplifications come from computing \( W \), and using

\[ Scal = -r_{11} - r_{1'1'} + r_{22} + r_{2'2'} = 2(r_{22} - r_{11}) \]

\[ = 2(-(-R_{11'11'} + R_{1212} + R_{12'12'}) + (-R_{1212} - R_{1'21'2} + R_{22'22'})) \]

\[ = 2(R_{11'11'} - 2(R_{1212} + R_{12'12'}) + R_{22'22'}) \]

First prove that the Hodge star commutes with \( W \) by considering \( W(\Lambda^+, \Lambda^-) \):

\[ W_{11'11'} = R_{11'11'} + \frac{1}{2}(r_{11} + r_{1'1'}) + \frac{Scal}{6} = R_{11'11'} + r_{11} + \frac{Scal}{6} \]

\[ = R_{1212} + R_{12'12'} + \frac{Scal}{6} \]

\[ W_{22'22'} = R_{22'22'} - \frac{1}{2}(r_{22} + r_{2'2'}) + \frac{Scal}{6} = R_{22'22'} - r_{22} + \frac{Scal}{6} \]

\[ = R_{1212} + R_{12'12'} + \frac{Scal}{6} \]

so that \( W_{11'11'} - W_{22'22'} = 0 \). Similarly

\[ W_{11'12} = R_{11'12} + \frac{r_{1'2}}{2}, \quad W_{1222'} = R_{1222'} + \frac{r_{12'}}{2} = R_{1222'} - \frac{r_{1'2}}{2} \]
so

\[
W_{11'12} - W_{1222'} = R_{11'12} - R_{1222'} + r_{1'2} = 0
\]
\[
W_{11'12'} = R_{11'12'} + \frac{r_{1'2}}{2} = R_{11'12'} + \frac{r_{12}}{2}, \quad W_{12'22'} = R_{12'22'} - \frac{r_{12}}{2},
\]
\[
W_{11'12'} - W_{12'22'} = R_{11'12'} - R_{12'22'} + r_{12} = 0.
\]

That proves that \( W \) is block-diagonal.

The \( W^- \) term satisfies

\[
W_{11'11'} + W_{22'22'} - 2W_{11'22'} = R_{11'11'} + r_{11} + R_{22'22'} - r_{22} + \frac{\text{Scal}}{3} - 2R_{11'22'}
\]
\[
= R_{11'11'} + R_{22'22'} - 2R_{11'22'} - \frac{\text{Scal}}{6}
\]
\[
= R_{11'11'} + R_{22'22'} - 2(R_{1212} + R_{12'12'})
\]
\[
= \frac{\text{Scal}}{2} - \frac{\text{Scal}}{6} = \frac{\text{Scal}}{3}
\]

using the first Bianchi identity (and the invariance of \( R_m \)):

\[
R_{11'22'} = -R_{12'12'} - R_{21'12'} = R_{12'12'} + R_{1212}.
\]
\[
W_{1212} - W_{121'2} = R_{1212} + \frac{r_{22} - r_{11}}{2} - \frac{\text{Scal}}{6} - R_{121'2} = \frac{\text{Scal}}{4} - \frac{\text{Scal}}{6}
\]
\[
W_{12'12'} + W_{12'12} = R_{12'12'} + \frac{\text{Scal}}{4} - \frac{\text{Scal}}{6} + R_{12'12} = \frac{\text{Scal}}{12}
\]
\[
W_{1212'} + W_{121'2} = R_{1212'} + \frac{r_{22'}}{2} + R_{121'2} - \frac{r_{11'}}{2} = \frac{1}{2}(r_{22'} - r_{11'}) = 0.
\]

Finally,

\[
W^- = \text{Scal} \begin{pmatrix} 1/3 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 \end{pmatrix} = \frac{\text{Scal}}{6} \text{Id} + \frac{\text{Scal}}{6} E_1^- \otimes E_1^-
\]

(and indeed this matrix is traceless w.r.t. the pseudo-metric \( g \)). One should note that the above expression differs from the Riemannian case, where

\[
W^+ = \text{Scal} \begin{pmatrix} 1/3 & -1/6 & -1/6 \\ -1/6 & 1/6 & -1/6 \end{pmatrix} = -\frac{\text{Scal}}{6} \text{Id} + \frac{\text{Scal}}{3} E_1^+ \otimes E_1^+.
\]
We let the Reader check that in the neutral case, the $W^+$ part is not a multiple of the scalar curvature, which completes the proof of Theorem 7.2.

A.2 The para-Kähler case

The computations are almost identical, but the results differ from the pseudo-Kähler setup, because the para-complex structure $J$ is now an anti-isometry: $R(JX, JY)Z = -R(X, Y)Z$. We pick an orthonormal basis $(e_1, e_1', e_2, e_2')$ with $e_1' = Je_1$, $e_2' = Je_2$, and $g(e_1) = g(e_2) = +1$, $g(e_1') = g(e_2') = -1$. The frame $B = (e_1 \wedge e_1', e_1 \wedge e_2, e_1 \wedge e_2', e_1' \wedge e_2, e_1' \wedge e_2', e_2 \wedge e_2')$ of $\Lambda^2 T\Lambda$ is also orthonormal w.r.t. the induced metric on $\Lambda^2$, again denoted by $g$, which has signature $(2, 4)$: $g(e_a \wedge e_b) = g(e_a)g(e_b) = -1$, except for $g(e_1 \wedge e_2) = g(e_1' \wedge e_2') = +1$.

An orthonormal eigenbasis for the Hodge operator is the following:

$$
\begin{align*}
E_1^\pm &= \frac{\sqrt{3}}{2} (e_1 \wedge e_1' \mp e_2 \wedge e_2') \\
E_2^\pm &= \frac{\sqrt{3}}{2} (e_1 \wedge e_2 \mp e_1' \wedge e_2') \\
E_3^\pm &= \frac{\sqrt{3}}{2} (e_1 \wedge e_2' \mp e_1' \wedge e_2)
\end{align*}
$$

where the $E_a^+$ (resp. $E_a^-$) span $\Lambda^+$ (resp. $\Lambda^-$). (Note the sign differences w.r.t. the pseudo-Kähler case.)

Since $J$ is anti-isometric and parallel,

$$Rm(JX, JY, Z, T) = -Rm(X, Y, Z, T) = Rm(X, Y, JZ, JT).$$

Hence the following symmetries of the Riemannian curvature operator $R$, expressed in the frame $B$ (for symmetry reasons and greater legibility, lower left coefficients are not written in this and the subsequent matrices):

|     | $e_11'$ | $e_{12}$ | $e_{12}'$ | $e_{11'} \wedge e_2$ | $e_{11'}2'$ | $e_{22}'$ |
|-----|---------|----------|-----------|---------------------|-------------|-----------|
| $e_{11'}$ | $R_{11'11'}$ | $R_{11'12}$ | $R_{11'12}'$ | $R_{11'12}$ | $R_{11'12}'$ | $R_{11'12}$ |
| $e_{12}$ | $R_{1212}$ | $R_{1212}'$ | $R_{1212}$ | $R_{1212}'$ | $R_{1212}$ | $R_{1212}'$ |
| $e_{12}'$ | $R_{12'12}$ | $R_{12'12}'$ | $R_{12'12}$ | $R_{12'12}'$ | $R_{12'12}$ | $R_{12'12}'$ |
| $e_{11'}$ | $R_{11'12}$ | $R_{11'12}'$ | $R_{11'12}$ | $R_{11'12}'$ | $R_{11'12}$ | $R_{11'12}'$ |
| $e_{12}$ | $R_{1212}$ | $R_{1212}'$ | $R_{1212}$ | $R_{1212}'$ | $R_{1212}$ | $R_{1212}'$ |
| $e_{12}'$ | $R_{12'12}$ | $R_{12'12}'$ | $R_{12'12}$ | $R_{12'12}'$ | $R_{12'12}$ | $R_{12'12}'$ |

(Note again the similarity with the pseudo-Kähler case: only a few signs change.)
The Weyl tensor satisfies some of the J-symmetries of $R_m$ since

$$\text{Ric}(JX, JY) = \sum_{i=1}^{4} g(e_i) R_m(JX, e_i, JY, e_i) = \sum_{i=1}^{4} g(e_i) R_m(X, Je_i, Y, Je_i)$$

$$= - \sum_{i=1}^{4} g(Je_i) R_m(X, Je_i, Y, Je_i) = -\text{Ric}(X, Y)$$

since $(Je_i)$ is also an orthonormal basis. In particular this invariance implies $r_{11'} = r_{11'',}$ so $r_{11'}$ vanishes (and so does $r_{22''}$). Finally,

$$\frac{\text{Scal}}{2} = r_{11} + r_{22} = -R_{11'11''} + 2(R_{1212} - R_{12'12''}) - R_{22'22''}.$$ 

The Kulkarni–Nomizu product $\text{Ric} \otimes g$ satisfies

$$\text{Ric} \otimes g(JX, Y, Z, T) = \text{Ric}(JX, Z)g(Y, T) + \text{Ric}(Y, T)g(JX, Z)$$

$$- \text{Ric}(JX, T)g(Y, Z) - \text{Ric}(Y, Z)g(JX, T)$$

$$= \text{Ric}(X, JZ)g(JY, JT) + \text{Ric}(JY, JT)g(X, JZ)$$

$$- \text{Ric}(X, JT)g(JY, JZ) - \text{Ric}(JY, JZ)g(X, JT)$$

$$= \text{Ric} \otimes g(X, JY, JZ, JT)$$

so

$$\text{Ric} \otimes g(JX, JY, Z, T) = \text{Ric} \otimes g(X, J^2 Y, JZ, JT) = \text{Ric} \otimes g(X, Y, JZ, JT)$$

and the same property holds for $g \otimes g$. Hence the following symmetries (fewer than for $R_m$) in the coefficients of $\text{Ric} \otimes g$, $g \otimes g$ and $R_m$, and therefore $W$:

| $e_{11'}$ | $e_{12}$ | $e_{12''}$ | $e_{1'2}$ | $e_{1'2''}$ | $e_{22''}$ |
|---------|---------|-----------|---------|---------|---------|
| $W_{11'11''}$ | $W_{11'12}$ | $W_{11'12''}$ | $W_{11'1'2}$ | $W_{11'1'2''}$ | $W_{11'22''}$ |
| $W_{12}$ | $W_{1212}$ | $W_{1212''}$ | $W_{121'2}$ | $W_{121'2''}$ | $W_{1222''}$ |
| $W_{12'2'}$ | $W_{12'2''}$ | $W_{12'2''}$ | $W_{12'22''}$ | $W_{12'22''}$ | $W_{12'22''}$ |
| $W_{1'2'}$ | $W_{1'2'2'}$ | $W_{1'2'2''}$ | $W_{1'2'2''}$ | $W_{1'2'2''}$ | $W_{1'2'2''}$ |
| $W_{22''}$ | $W_{22''}$ | $W_{22''}$ | $W_{22''}$ | $W_{22''}$ | $W_{22''}$ |

Let us now express $W$ in the Hodge basis defined earlier, using the above symmetries (up to a factor $1/2$ due to normalization).
Canonical structure on the tangent bundle

| $E_i^+$ | $E_i^-$ | $E_i^+$ |
|---------|---------|---------|
| $E_1^+$ | $W_{11'}11' - W_{22'22'}$ | $2(W_{11'}12 - W_{122'}22')$ |
| $E_2^+$ | $2(W_{11'}12 + W_{122'}22')$ | $2(W_{1212} - W_{121'2'})$ |
| $E_3^+$ | $2(W_{11'}12' + W_{122'22'})$ | $2(W_{122'12'} - W_{122'1'})$ |

| $E_i^-$ | $E_i^-$ | $E_i^-$ |
|---------|---------|---------|
| $E_1^-$ | $W_{11'}11' - W_{22'22'}$ | $0$ |
| $E_2^-$ | $2(W_{11'}12 + W_{122'}22')$ | $0$ |
| $E_3^-$ | $2(W_{11'}12' + W_{122'22'})$ | $0$ |

Only three terms in the off-block-diagonal part are not obviously zero.

\[
W_{11'}11' = R_{11'}11' - \frac{1}{2}(-r_{11} + r_{11'}) - \frac{\text{Scal}}{6} = R_{11'}11' + r_{11} - \frac{\text{Scal}}{6}
\]

\[
W_{22'22'} = R_{22'22'} - \frac{1}{2}(-r_{22} + r_{22'}) - \frac{\text{Scal}}{6} = R_{22'22'} + r_{22} - \frac{\text{Scal}}{6}
\]

but \( r_{11} = -R_{11'}11' + R_{1212} - R_{12'12'} \) and \( r_{22} = R_{2212} - R_{22'12'} - R_{222'22'} = R_{1212} - R_{12'12'} - R_{222'22'} \) so that

\[
W_{11'}11' - W_{22'22'} = R_{11'}11' - R_{22'22'} + r_{11} - r_{22} = 0.
\]

Similarly

\[
W_{11'}12' + W_{122'22'} = R_{11'}12' - \frac{r_{11'}}{2} + R_{122'2'} + \frac{r_{12'}}{2} = R_{11'}12' + R_{122'2'} - r_{11'} = 0
\]

\[
W_{11'}12' + W_{122'22'} = R_{11'}12' - \frac{r_{12'}}{2} + R_{122'2'} + \frac{r_{12'}}{2} = R_{11'}12' + R_{12'22'} + r_{12} = 0
\]

which proves that \( W \) is block-diagonal, i.e. commutes with the Hodge operator.
Let us now look more closely at the $W^-$ term

$$
\begin{pmatrix}
W_{11/11'} + W_{22/22'} + 2W_{11/22'} \\
2(W_{1212} + W_{121'2'}) \\
2(W_{121'2'} + W_{1212'})
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
2(W_{1212} + W_{121'2'})
\end{pmatrix}
$$

$$
W_{11/11'} + W_{22/22'} + 2W_{11/22'}
= R_{11/11'} + r_{11} - \frac{\text{Scal}}{6} + R_{22/22'} + r_{22} - \frac{\text{Scal}}{6} + 2R_{11/22'}
= R_{11/11'} + R_{22/22'} + 2R_{11/22'} + \frac{\text{Scal}}{2} - \frac{\text{Scal}}{3}
= R_{11/11'} + R_{22/22'} + 2(-R_{1212} + R_{12/12'}) + \frac{\text{Scal}}{6} = -\frac{\text{Scal}}{3}
$$

where we have used the first Bianchi identity (and the invariance of $Rm$)

$$
R_{11/22'} = -R_{1/212'} - R_{211'2'} = R_{12/12'} - R_{1212'}
$$

$$
W_{1212} + W_{121'2'} = R_{1212} - \frac{r_{22} + r_{11}}{2} + \frac{\text{Scal}}{6} + R_{121'2'}
= R_{1212} - \frac{\text{Scal}}{4} + \frac{\text{Scal}}{6} + R_{121'2'} = -\frac{\text{Scal}}{12}
$$

$$
W_{12'/12'} + W_{12/1'2} = R_{12/12'} + \frac{\text{Scal}}{4} - \frac{\text{Scal}}{6} + R_{12'/1'2} = \frac{\text{Scal}}{12}
$$

$$
W_{12'/12'} + W_{121'2} = R_{1212'} - \frac{r_{22'}}{2} + R_{121'2} - \frac{r_{11'}}{2} = 0.
$$

Finally,

$$
W^- = \text{Scal} \begin{pmatrix}
-1/3 \\
-1/6 \\
1/6
\end{pmatrix}
$$

vanishes if and only if $\text{Scal} = 0$. (The Reader will check that this matrix is indeed traceless w.r.t. the pseudo-metric $g$.)

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