EQUIRESIDUAL ALGEBRAIC GEOMETRY OVER AN ARBITRARY COMMUTATIVE FIELD

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Abstract. We introduce the first bases of algebraic geometry over any commutative field $k$ inside the affine spaces $k^n$ themselves, rather than in an algebraically closed extension of $k$ or an equivalent setting. This concrete approach relies on the transposition in non-algebraically closed fields of McKenna’s idea of (Galois-theoretic) normic forms, which are homogeneous polynomials with no non-trivial zeros, and builds upon an “equiresidual” generalisation of Hilbert’s Nullstellensatz and an associated radical in finitely generated $k$-algebras. It is natural to work out the usual algebraic constructions surrounding affine algebraic geometry inside $k^n$ by using a new type of algebras over $k$ which correspond to “canonical” localisations of $k$-algebras, associated to the set of polynomials over $k$ with no inner zero. The theory leads to a fruitful characterisation of the sections of the sheaf of regular functions over an affine algebraic set, in that it permits us to dualise the (equiresidual) affine algebraic varieties over $k$ using an analogue of reduced algebras of finite type and a maximal spectrum functor.

1. Introduction and Preliminaries

Is it possible to develop a relevant algebraic geometry over any commutative field, i.e. without the hypothesis that the field is algebraically closed? The usual answer is: yes, embed your favorite field $k$ into an algebraically closed field $K$ (sometimes with infinite transcendence degree, as in Weil’s approach, see Chapter 10 of [11] for instance), and do the algebraic geometry in $K$ with parameters in $k$. Or, to be more fashionable, work in a suitable category of schemes over $k$, considered itself as a one-element scheme (see Chapter II of [12] for the principle). A third and subtle possibility is to consider algebraic spaces over $k$, built from maximal spectra of finitely generated $k$-algebras (see Chapter 11 in [17]). All these solutions have one thing in common: one comes down to classical algebraic geometry over algebraically closed fields over $k$, virtually considering rational points of “geometric objects” in all finite or finitely generated extensions of $k$, and a form of another of Hilbert’s Nullstellensatz is implied. Another solution is, in certain very specific cases, to develop whole analogues of complex algebraic geometry, by using some specific features of the field or a related family of fields often identified by a set of (first order) axioms. This is the case of the formidable example of real algebraic geometry ([10]), where one abstracts the essential properties of $\mathbb{R}$ which make it possible to develop a peculiar approach to algebraic geometry in it, with its unique and additional features, and then develop the theory in the category of real closed fields and related algebraic structures. This kind of situation is often strongly connected to first order logic and model-theoretic considerations. In particular, one knows in some core examples how to interpret the model-theoretic notion of quantifier elimination as some analogue of Chevalley’s theorem on constructible sets (see Proposition 5.2.2 in [10] for real algebraic geometry, and [8] for a $p$-adic analogue). As we were considering the basics of a wide generalisation of this second approach (expanding the ideas underlying our preceding [5] and [6]), which will hopefully appear in its time, it struck us that our first question is a very legitimate one, and should be given a definite and simple answer, but in the same spirit as basic linear algebra and affine geometry are done over any field, or as basic algebraic geometry is done over any

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algebraically closed field. We thus wish to develop some relevant algebraic geometry over any commutative field \( k \) in an intrinsic manner, and in particular without working explicitly or implicitly in algebraically closed fields containing \( k \) or in a related axiomatisable family of fields. Interesting connexions between algebraic geometry and positive logic revealed how to do this by purely algebraic means, i.e. without model-theoretic methods. At least, it is possible for a start to generalise the theory to as far as algebraic varieties, as to encompass for instance all quasi-projective varieties, which we believe is a very good start. In this present work we want to expound the foundations of this approach, algebraic and affine, saving the theory of algebraic varieties for a further publication.

In section \( 2 \), we explain why a certain equiresidual Nullstellensatz (Theorem \( 2.4 \)) holds in every commutative field. This rests on an analogue of a model-theoretic lemma of McKenna about the existence in all non-algebraically closed fields of homogeneous polynomials having only the trivial zero. Characterising the maximal ideals of finitely generated algebras over a field \( k \) which have points rational in \( k \), which we call special, we define an analogue of the classical radical of an ideal - at least for finitely generated \( k \)-algebras - the equiresidual radical. We also define the key algebraic construction which we will use, the canonical localisation of an algebra over the base field, which applied to localisation at one element leads to an essential characterisation of the equiresidual radical (Theorem \( 2.18 \)).

In section \( 3 \) we first develop the abstract counterpart of canonical localisation, the notion of a \(*\)-algebra over a field, which is the “right”category of algebras in which it is suitable to work out this inner algebraic geometry in general, in connection with special \( *\)-algebras - a counterpart of reduced algebras as they appear in the classical affine algebro-geometric context; it is the occasion to introduce and characterise the special ideals, which are equal to their equiresidual radical. Secondly, we establish the usual “dictionary”between specific ideals and algebraic sets in affine spaces. Thirdly, we carefully study the algebras of sections of the sheaf of regular functions over an affine algebraic subvariety. Here lies our core result, Theorem \( 3.17 \): the affine sheaves of regular functions are sheaves of special \(*\)-algebras, and their algebras of sections are essentially the canonical localisations of the usual coordinate algebras. In section \( 4 \) we first introduce a natural category of locally ringed spaces over a base field which locally look like affine algebraic subvarieties, thus containing a subcategory of equiresidual affine algebraic varieties, the abstract counterparts of affine subvarieties; we also give a corresponding abstract characterisation of the algebras of global sections of the structure sheaves of these, the affine \(*\)-algebras, i.e. the special \(*\)-algebras of finite type as such. Secondly, we show that a natural maximal spectrum functor turns these algebras into affine algebraic equivarieties. Finally, building upon section \( 3 \) we prove that the global sections functor and the maximal spectrum functor are indeed a duality between both categories (Theorem \( 4.15 \)).

Preliminaries and conventions. All rings and fields considered are implicitly unitary and commutative and we use some standard notation, terminology and folklore from commutative algebra and algebraic geometry, which we briefly review and complete. If \( k \) is a field and \( I \) is an ideal of a polynomial algebra \( k[ X_1 , \ldots , X_n ] \), the corresponding (affine) algebraic set of \( k^n \) is noted \( \mathcal{Z}(I) = \{ P \in k^n : \forall f \in I, f(P) = 0 \} \). These algebraic sets of \( k^n \) are the closed sets of a Noetherian topology called the Zariski topology; recall that in general any nonempty open subset of an irreducible closed set is dense and irreducible (\( 12 \), Example 1.1.3). If \( V \subseteq k^n \) is an algebraic set, the coordinate ring (or algebra) of \( V \) is the \( k \)-algebra \( k[V] := k[X_1 , \ldots , X_n ]/ \mathcal{I}(V) \), where \( \mathcal{I}(V) = \{ f \in k[ X ] : \forall P \in V, f(P) = 0 \} \); any element \( f \in k[V] \) defines a function \( V \to k, P \mapsto f(P) \), for any \( F \in k[ X ] \) such that \( F + \mathcal{I}(V) = f \). By definition of the induced Zariski topology on \( V \), any basic open subset will be denoted by \( D_V(f) = \{ P \in V : f(P) \neq 0 \} = V - \mathcal{Z}_V(f) \) for a certain \( f \in k[V] \). If \( U \subseteq V \) is an open subset (for the induced Zariski topology on \( V \)), a function \( f : U \to k \)
is called regular at \( P \in V \), if there exists an open neighbourhood \( U_P \subseteq U \) of \( P \) in \( U \) and elements \( g, h \in k[V] \) such that for all \( Q \in U_P \), \( h(Q) \neq 0 \) and \( f(Q) = g(Q)/h(Q) \); \( f \) is called regular (over \( U \)) if \( f \) is regular at every \( P \in U \) (notice that \( f \) is then continuous).

The set of regular functions over \( U \) is written \( \mathcal{O}_U \), it is a \( k \)-algebra of finite type, and \( \mathcal{O}_V \) is a sheaf of \( k \)-algebras, called the sheaf of regular functions on \( V \). An affine algebraic subvariety of \( k^n \) if a pair \((V, \mathcal{O}_V)\), where \( V \subseteq k^n \) is an algebraic set and \( \mathcal{O}_V \) is its sheaf of regular functions. An element of the stalk \( \mathcal{O}_{V, P} \) of \( \mathcal{O}_V \) at \( P \in V \) will be noted \([g, U] \), where \( P \in U \subseteq V \) and \( g \in \mathcal{O}_V(U) \).

The following proposition - which we will refer to as the “small lemma” - should be folkloric but we have never read it elsewhere (in usual textbooks on algebraic geometry, it is proved on algebraically closed fields as a consequence of Hilbert’s Nullstellensatz, see [12], Theorem 1.3.2 for instance!):

**Proposition 1.1 (“Small lemma”).** For any affine algebraic subvariety \( V \subseteq k^n \), for any \( P \in V \), we have \( \mathcal{O}_{V, P} \cong k[V]_{m_P} \) for \( m_P = \{ f \in k[V] : f(P) = 0 \} \). In particular, the structural morphism \( k \rightarrow \mathcal{O}_{V, P} \) is an isomorphism.

**Proof.** Write \( A = k[V] \) and \( m = m_P \). Let \([f] \in \mathcal{O}_P = \mathcal{O}_{V, P} \): there is a neighbourhood \( U \) of \( P \) in \( V \) and \( a/g \in A \) with \( g \neq 0 \) and \( f(Q) = a(Q)/g(Q) \) for every \( Q \in U \), whence \( U \subseteq D_{V}(g) \) and we may assume that \( U = D_{V}(g) \) with the same data. As \( g \in A - m \), define \( \varphi([f]) := a/g \in A_m \) : if \([f] = [b/h] \in \mathcal{O}_P \), there exists a basic open neighbourhood of \( U' = D_{V}(l) \subseteq U \) of \( P \) in \( V \) on which \( a/g = b/h \); we have \( D_{V}(l) \subseteq D_{V}(g) \cap D_{V}(h) = D_{V}(gh) \) and \( l \in A - m \), the regular map defined by \((ah - bg)/gh \) on \( D_{V}(l) = D_{V}(ghl) \) is zero, hence \( ahl - bgl \) is also zero on \( D_{V}(l) \), whereas for \( Q \in V - D_{V}(l) \), we have \( gh/l(Q) \neq 0 \), and therefore \((ahl - bgl)(ghl) \) is zero on \( V \), and hence in \( A \). As \( gh \not\in m \), we get \( ahl - bgl = 0 \) in \( A_m \), whereby \( a/g = b/h \) in \( A_m \) and \( \varphi \) is well defined, and obviously a \( k \)-morphism. Finally, if \( a/g \in A_m \), the regular map defined by \( a/g \) on \( D_{V}(g) \) has \( \varphi([a/g, D_{V}(g)]) = a/g \), and if \( \varphi([f]) = 0 \) with \( f \) defined as before on \( D_{V}(g) \) by \( a/g \) say, as \( a/g = 0 \) in \( A_m \) there is \( h \in A - m \) with \( ha = 0 \) in \( A \), whence \([a/g] = [ah/ghl_{D_{V}(gh)}] = 0 \), and \( \varphi \) is an isomorphism. The isomorphism \( \mathcal{O}_{V, P} \cong k[V]_{m_P} \) induces a residual isomorphism \( \mathcal{O}_{V, P} \cong k[V]_{m_P} / \mathfrak{m}_P k[V]_{m_P} \) over \( k \), and by definition of \( m_P \) this last is \( k \)-isomorphic to \( k \). \( \square \)

If \( W \subseteq k^m \) is another algebraic set, a regular morphism from \( V \) to \( W \) is a map \( f : V \rightarrow W \) such that there exist \( f_1, \ldots, f_m \in k[V] \) with \( f(P) = (f_1(P), \ldots, f_m(P)) \) for all \( P \in V \). Any regular morphism \( f = (f_1, \ldots, f_m) : V \rightarrow W \) induces in turn a \( k \)-algebra morphism \( k[f] : k[W] \rightarrow k[V] \) in the usual way : to every \( g \in G + \mathcal{J}(W) \in k[W] \) we associate \( G(F_1, \ldots, F_m) + \mathcal{J}(V) \), if \( f_i = F_i + \mathcal{J}(V) \) for \( i = 1, \ldots, m \), and this defines a full and faithful functor \( k[-] \) from the dual category of affine algebraic sets and regular morphisms into the category of reduced \( k \)-algebras of finite type. If \( f : V \rightarrow W \) is a regular morphism of affine algebraic subvarieties, for every open subset \( U \subseteq W \) and for every \( s \in \mathcal{O}_W(U) \), we have \( f \circ s \in \mathcal{O}_V(f^{-1}U) \), the map \( f^\#: s \in \mathcal{O}_W(U) \mapsto f \circ s \in \mathcal{O}_V(f^{-1}U) \) is a morphism of \( k \)-algebras and the \( f^\# \)'s define a sheaf morphism \( f^\# : \mathcal{O}_W \rightarrow f_* \mathcal{O}_V \). For every \( P \in V \) we have a residual \( k \)-morphism \( f^\#_P : \mathcal{O}_{W, f(P)} \rightarrow \mathcal{O}_{V, P} \) induced by the universal property of stalks considered as inductive limits, which is local by the small lemma \([11]\) so \( (f, f^\#) : (W, \mathcal{O}_W) \rightarrow (V, \mathcal{O}_V) \) is a morphism of locally ringed spaces in \( k \)-algebras, and we have a functor \( f \mapsto (f, f^\#) \) from the dual category of affine algebraic subvarieties to locally ringed spaces in \( k \)-algebras. If \( A \) is any ring, we note \( Spm(A) \) the maximal spectrum of \( A \), i.e. the set of all maximal ideals of \( A \), implicitly topologised as usual by taking as basic open sets the subsets of the form \( D(f) = \{ m \in Spm(A) : f \not\in m \} \); this is the Zariski topology on \( A \). We refer the reader to Chapter II of \([12]\), for instance, about generalities on sheaves and locally ringed spaces. Not being an original algebraic geometry, we also apologise to the educated reader for any clumsiness in notation, conception, or reference, and for any presumption of demonstrating anything which is already well known to the specialist.
2. The Equiresidual Nullstellensatz and its Associated Radical

Normic forms and the ˇAquinullstellensatz. In a context loaded with first order logic, McKenna ingeniously introduces the notion of a normic form in a first order theory of fields ([16], Lemma 4), which permits him to deal with the characterisation of analogues of the radical of an ideal. We adapt his definition as the following

Definition 2.1. If $k$ is a field, a normic form over $k$ is a homogeneous polynomial $P(X_1, \ldots, X_n)$ with coefficients in $k$, such that the only $\pi \in k^n$ for which $P(\pi) = 0$ is $\overline{0}$.

Remark 2.2. This is a priori only an analogue of McKenna’s notion, but both have a common generalisation thanks to basic positive logic (see [4]).

In general, the only constant normic form over a field $k$ is $0$, and the normic forms in one variable are the nonzero monomials. Normic forms are useful - at least in non-algebraically closed fields - in order to reduce the description of algebraic sets to sets of zeros of a unique polynomial, so in this respect they become interesting with at least two variables. Notice that if $k$ is algebraically closed and $P \in k[X_1, \ldots, X_n]$ with $n \geq 2$, $P$ always has a nontrivial zero, so $k$ does not have such forms! The miracle is that however, by elementary Galois theory they always exist over any other field:

Proposition 2.3 (Normic forms over non-algebraically closed fields). If $k$ is a field, not algebraically closed, then there exist normic forms of an arbitrary number of variables over $k$.

Proof. We adapt the proof of McKenna ([16], Lemma 4) to the present context. Composing polynomials and substituting zeros for certain variables, it suffices to show that there exists a normic form in two variables over $k$. As $k$ is not algebraically closed, there exists a proper algebraic extension $k \to k(\alpha)$ of $k$ and we distinguish two cases. First, if $k$ is separably closed, we have $\text{char}(k) = p > 0$ and by Proposition V.6.1 in [14], there exists $m \in \mathbb{N}$ such that $\alpha^p^m \in k$; we choose $m$ minimal with this property, we have $m > 0$ and $\alpha^{p^m - 1} \not\in k$, and we let $N(X,Y) := X^{p^m} - \alpha^{p^m} Y^{p^m} \in k[X,Y]$. If $p = 2$, we have $N(X,Y) = (X^{2^{m-1}} - \alpha^{2^{m-1}} Y^{2^{m-1}})(X^{2^{m-1}} + \alpha^{2^{m-1}} Y^{2^{m-1}})$ and if $a,b \in k$ are such that $N(a,b) = 0$ with $b \neq 0$, distinguishing cases we have $\alpha^{2^{m-1}} \in k$, which is impossible, so $b = 0$, and also $a = 0$. If $p \neq 2$, we have $N(X,Y) = (X - \alpha Y)^{p^m}$, and if $a,b \in k$ and $N(a,b) = 0$ with $b \neq 0$, we have $a/b = \alpha \in k$, which contradicts the choice of $\alpha$, so $b = 0$ and also $a = 0$, and therefore $N(X,Y)$ is a normic form over $k$. Secondly, if $k$ is not separably closed, we may assume that $\alpha$ is separable algebraic over $k$ and any splitting field $k \to K$ for $\alpha$ is a finite separable algebraic extension by Theorem V.4.4 of [14], so a Galois extension, generated by a single element $\beta$ by Abel’s Theorem ([14], Theorem V.4.6) : we have $K = k[\beta] = k(\beta)$ and the polynomial $N(X,Y) := \prod_{\sigma \in \text{Gal}(K/k)} (X - \beta^\sigma Y)$ is a member of $k[X,Y]$ by the fundamental theorem of Galois theory ([14], Theorem VI.1.1). Let again $a,b \in k$ with $N(a,b) = 0$ : if $b \neq 0$, as $\prod_{\sigma \in \text{Gal}(K/k)} (a - \beta^\sigma b) = 0$ there exists $\tau \in \text{Gal}(K/k)$ such that $\beta^\tau = a/b \in k$, which is impossible (all the conjugates of $\beta$ generate $K$ over $k$). We conclude that $b = 0$, so $a = 0$ also, therefore $N(X,Y)$ is a normic form over $k$.

Combining this phenomenon with the exclusion, in finitely generated algebras over a field $k$, of ideals which contain certain functions with no zero rational over $k$, we may generalise Hilbert’s Nullstellensatz as the following

Theorem 2.4 (“ˇAquinullstellensatz”). Let $k$ be any field, $A$ a finitely generated $k$-algebra, and $S$ the set of all $f \in A$ such that $\varphi(f) \neq 0$ for all $k$-morphisms $\varphi : A \to k$. Every ideal $I$ of $A$ disjoint from $S$ and maximal as such is a maximal ideal such that $A/I \cong k$ (and reciprocally).
Proof. If $k$ is algebraically closed, then $S = k$ and the result is a consequence of Hilbert's Nullstellsatz, so we now suppose that $k$ is not algebraically closed. If $P \in k[X] = k[X_1, \ldots, X_n]$ has a zero $[\overline{f}] = \overline{f} + I$ in $A/I$ for $\overline{f} \in A^n$, we have $P(\overline{f}) \in I$, and as $I \cap S = \emptyset$, there exists a $k$-morphism $\varphi : A \to k$ such that $P(\varphi(\overline{f})) = \varphi(P(\overline{f})) = 0$, and $P$ already has a zero in $k$. In particular, if $I = (P_1, \ldots, P_m)$ and $N(X_1, \ldots, X_n)$ is a normic form for $k$ by Proposition 2.4 as $N(P_1, \ldots, P_m)$ has a zero in $A/I$, it has a zero in $k$ by what precedes, and as $N$ is a normic form, $I$ itself has a zero in $k$, corresponding by evaluation to a $k$-morphism $e : A/I \to k$. Now the composite $k$-morphism $\varphi : A \to A/I \to k$ has $I \subseteq \ker(\varphi)$, and if $P \in \ker(\varphi)$, by definition we have $P \notin S$, so $\ker(\varphi) \cap S = \emptyset$ : by maximality of $I$ with this last property, we have $I = \ker(\varphi)$, so $e : A/I \to k$ is an isomorphism, and $I$ is maximal. □

Remark 2.5. i) Finiteness is needed in both cases, in the first for the application of Hilbert's Nullstellsatz, in the second for the application of a normic form to a finitely generated ideal.
ii) For any $k$-algebra $A$ and ideal $I$ of $A$, if $\varphi : A/I \cong k$ is an isomorphism, $\varphi$ is necessarily the inverse of the structural morphism $k \to A/I$, so the ideals of the statement are exactly those for which $k \cong A/I$.

Corollary 2.6. If $k$ is any field, $V \subseteq k^n$ is an affine algebraic subvariety, and $S = \{g \in k[V] : \forall P \in V, g(P) \neq 0\}$, an ideal $I$ of $k[V]$ has a zero in $V$ if and only if $I \cap S = \emptyset$.

Proof. If $I$ has a zero in $V$, certainly we have $I \cap S = \emptyset$. Conversely, if $I \cap S = \emptyset$, by Noetherianity of $k[V]$ there exists an ideal $m$ of $k[V]$, containing $I$, disjoint from $S$, and maximal with this property : by Theorem 2.4 the structural morphism $k \to k[V]/m$ is an isomorphism, which means that $I$ has a zero in $V$. □

As a first significant geometric consequence of the Aquirullstellsatz, we may characterise the global sections of the sheaf of regular functions on an irreducible affine algebraic subvariety, a result which we will generalise in section 6.

Proposition 2.7. If $V$ is irreducible and $k\{V\} := k[V]_S$, where $S = \{g \in k[V] : \forall P \in V, g(P) \neq 0\}$, then $\Gamma(V, \mathcal{O}_V) \cong k\{V\}$.

Proof. Let $f \in \Gamma(V, \mathcal{O}_V)$ and for every $P \in V$, $U_P \subseteq V$ an open neighbourhood of $P$ in $V$ such that $f|_{U_P} \equiv uP/vP$ (i.e. such that $vP(Q) \neq 0$ and $f(Q) = uP(Q)/vP(Q)$ for all $Q \in U_P$), with $u_P, v_P \in k[V]$. As $v_P \neq 0$ for all $P$, define a map $\varphi : \Gamma(V, \mathcal{O}_V) \hookrightarrow k(V)$ by $f \mapsto uP/vP$ for any $P$; if $P, Q \in V$, as $V$ is irreducible $U_P$ is dense in $V$, so $O := U_P \cap U_Q \neq \emptyset$ and for every $R \in O$ we have $f(R) = uP/vP(R) = uQ(R)/vQ(R)$ so $u_P v_Q|_O = u_Q v_P|_O$ and by density of $O \neq \emptyset$ in $V$, as the diagonal $\Delta_V$ is closed in $V \times V$, we have $u_P v_Q = u_Q v_P$ in $k[V]$, and therefore $u_P/v_P = u_Q/v_Q$ and $\varphi$ is well defined, and obviously a $k$-morphism. If $\varphi(f) = u_P/v_P = 0$, we have $u_P = 0$ in $k(V)$, so $f|_{U_P} \equiv 0$ and as $f$ is continuous and $U_P$ is dense, as $\Delta_V$ is closed again we have $f \equiv 0$, and $\varphi$ is injective : denote by $A$ its isomorphic image in $k(V)$ and note that by definition, we have $k\{V\} \subseteq A$. Now let $I$ be the ideal of $k[V]$ generated by the $v_P$’s, $P \in V$ : if $I \cap S = \emptyset$, by the Aquirullstellsatz 2.4 $I$ has a rational point $P \in \cap S$, $r \in \mathbb{N}$, $P_1, \ldots, P_r \in V$ and $\alpha_1, \ldots, \alpha_r \in k[V]$ with $v = \sum_{i=1}^r \alpha_i u_{P_i}$, from which we get, in $A$, $\varphi(f)v = \sum_i \alpha_i \varphi(f)(u_{P_i}) = \sum_i \alpha_i u_{P_i}$, and therefore $\varphi(f) = (1/v)\sum_i \alpha_i u_{P_i} \in k\{V\}$, and we conclude that $k\{V\} = A$, i.e. $\Gamma(V, \mathcal{O}_V) \cong k\{V\}$. □

Equiwalradicals and canonical localisation. If $k[X] = k[X_1, \ldots, X_n]$ is a polynomial algebra and $I$ is an ideal of $k[X]$, the elements of $\mathcal{E}(I)$ are in bijection with the $k$-morphisms $\varphi : k[X] \to k$ such that $I \subseteq \ker(\varphi)$; we let $e_P : k[X] \to k$ be the evaluation morphism at $P \in k^n$. In other words, if $S = \{f \in k[X] \mid \forall \varphi : k[X] \to k, \varphi(f) \neq 0\}$, for
every point \( P \in \mathcal{Z}(I) \) we have \( \text{Ker}(e_P) \cap S = \emptyset \), and conversely every maximal ideal disjoint from \( S \) and containing \( I \) has the form \( e_P \) for \( P \in \mathcal{Z}(I) \) by Theorem 2.3. It follows that \( \mathcal{J}(\mathcal{Z}(I)) \), which is the kernel of the product \( k\)-morphism \( e_I : k[\mathcal{X}] \to k[\mathcal{X}] \) of the morphisms \( e_P \)'s for \( P \in \mathcal{Z}(I) \), is the intersection of all maximal ideals of \( k[\mathcal{X}] \) containing \( I \) and disjoint from \( S \). Abstracting this notion we adopt the following

**Definition 2.8.** If \( A \) is a \( k \)-algebra, say that a maximal ideal \( m \) of \( A \) is special if the structural morphism \( k \to A/m \) is an isomorphism. If \( I \) is any ideal of \( A \), the equiradical of \( I \), or equiradical of \( I \), noted \( \sqrt[\mathcal{J}]{I} \), is the intersection of all special maximal ideals of \( A \) containing \( I \).

**Remark 2.9.** i) If \( A \) is of finite type and \( S = \{ f \in A | \forall \varphi : A \to k, \varphi(f) \neq 0 \} \) as before, then by the Äquinullstellensatz (2.A) a maximal ideal \( m \) of \( A \) is special if and only if \( m \cap S = \emptyset \).

ii) If \( A \) is the coordinate algebra \( k[V] \) of some affine algebraic subvariety \( V \subseteq k^n \), then the equiradical of an ideal \( I \) of \( A \) is nothing else than \( \mathcal{J}(\mathcal{Z}_V(I)) = \{ f \in A : \forall P \in \mathcal{Z}_V(I), f(P) = 0 \} \). All this could seem trivial, were it not for the existence of normic forms which make it possible to “encode”this information in the multiplicative set \( S \) in case \( k \) is not algebraically closed.

iii) Another solution is to save the expression “special maximal ideal” for a maximal ideal \( m \) such that \( A/m \) preserves the algebraic signature (Definition 2.10), as in [4]. Both notions coincide for finitely generated \( k \)-algebras, so we keep it this way in order to connect with the general concept of a special algebra (Definition 3.3).

If \( A \) is a \( k \)-algebra, the set \( S \) as defined above is multiplicative; in case \( A \) is of finite type, by what precedes we may identify the special maximal ideals of \( A \) by localisation. This leads to a transposition of the usual algebraic constructions surrounding classical algebraic geometry into these kind of localised algebras, which we begin to study here using a more convenient description of \( A \) of \( k \)-algebras, say that a maximal ideal \( m \) of \( A \) is special if and only if \( m \cap S = \emptyset \).

**Definition 2.10.** i) The algebraic signature of \( k \) is the set \( \mathcal{D} \) of all polynomials in finitely many variables \( k \) which have no zero rational in \( k \).

ii) If \( A \) is a \( k \)-algebra, we note \( M_A \) the multiplicative subset of all \( D(\overline{\alpha}) \) for \( D \in \mathcal{D} \) and \( \overline{\alpha} \in A \), and we call \( A_M := A_{M_A} \) the canonical localisation of \( A \).

**Remark 2.11.** i) The algebraic signature is an analogue of McKenna’s “determining sets” ([16], Theorem 2). As with normic forms, both notions have a common natural generalisation using positive logic (see [11] again).

ii) If \( k \) is algebraically closed, then \( \mathcal{D} = k^* \), so for every \( k \)-algebra \( A \), we have \( M_A \cong k^* \) and \( A \cong A_M \).

**Lemma 2.12.** If \( A \) is a finitely generated \( k \)-algebra and \( J \) is an ideal of \( A \), then \( J \cap S = \emptyset \iff J \cap M_A = \emptyset \).

**Proof.** It suffices to prove it for \( A = k[\mathcal{X}] = k[X_1, \ldots, X_n]/I \). If \( f \in J \cap S \), write \( f = f + I \) with \( F \in k[\mathcal{X}] \), and let \( P_i : i = 1, \ldots, m \) be finitely many generators of \( I \). Suppose \( k \) is algebraically closed, by definition of \( S \) the ideal \( (F, I) \) of \( k[\mathcal{X}] \) has no zero in \( k \); by Hilbert’s Nullstellensatz we have \( 1 \in \sqrt{(F, I)} \), in other words there are polynomials \( G, H_i \in k[\mathcal{X}] \) such that \( 1 = GF + \sum H_i P_i \), whence \( 1 = gf \) in \( A \), for \( g = G + I \); it follows that \( 1 \in J \), so \( J = A \) and \( J \cap M_A \neq \emptyset \). Suppose \( k \) is not algebraically closed, and \( N(Y, Z_1, \ldots, Z_m) \) is an appropriate normic form over \( k \) by Proposition 2.3: by definition of \( S \), \( F \) and the \( P_i \)'s have no common zero in \( k \), so the polynomial \( D = N(F, P_i : i) \) has no zero in \( k \) and is therefore a member of \( \mathcal{D} \). It follows that \( g := N(f, \overline{\alpha}) = N(F, P_i : i) + I = D(\overline{\alpha} + I) \)
is both a member of $J$ (as a $k$-linear combination of powers of $f$) and a member of $M_A$. Conversely, suppose $f \in J \cap M_A$, then $f = D(\varphi)$ for some $D \in \mathcal{P}$ and $\varphi = \varphi f + 1$; if \( \varphi : A \to k \) is a k-morphism, we have $\varphi(f) = \varphi(D(\varphi)) = D(\varphi(\mathfrak{g})) \neq 0$ by definition of $\mathcal{P}$, so $f \in J \cap S$, which is not empty, and the lemma is proved.

\begin{proof}

\end{proof}

**Remark 2.13.** Keeping in mind the first point of Remark 2.9, we now see that the special maximal ideals of a finitely generated $k$-algebra $A$ are the maximal ideals which are disjoint from $M_A$. Beware that this is not true in general $k$-algebras (see Example 3.2).

**Proposition 2.14.** For every finitely generated $k$-algebra $A$, we have $A_M \cong A_S$.

\begin{proof}

As $M_A \subseteq S$, it suffices to show by the universal properties of $A_M$ and $A_S$ that every member of $S$ becomes invertible in $A_M$. We have $A_M = \bigcap \{ m^e : m \in \text{Sp}(A_M) \}$, and $m \in \text{Sp}(A_M) \iff m = nA_M$ for $n$ disjoint from $M_A$ and maximal as such $\iff m = nA_M$ for $n$ disjoint from $S$ and maximal as such (by Lemma 2.12) $\iff m = nA_M$ for $n$ maximal and special by Theorem 2.4. Now let $s \in S$; $s$ belongs to no special maximal ideal of $A$, so by what precedes $s$ is invertible in $A_M$ and the proposition is proved.

\end{proof}

**Remark 2.15.** A direct proof in the non-algebraically closed case along [2.12] is interesting: if the members of $M_A$ are invertible, an element of $S$ has the form $F + I$ with $(F, I)$ having no zero in $k$, so $N(f, \mathcal{F})$ is in $M_A$, so is invertible; now $N(f, \mathcal{F})$ is precisely the monomial where only the variable corresponding to $f$ occurs, with a power $\geq 1$, so that inverting $N(f, \mathcal{F})$ entails inverting $f$.

The following lemma is a generalisation of the existence of "rational points" (i.e. morphisms to the base field) for any non-trivial finitely generated algebra over an algebraically closed field.

**Lemma 2.16.** If $A$ is a finitely generated $k$-algebra, then $A_M \neq 0$ if and only if there exists a $k$-morphism $A_M \to k$.

\begin{proof}

It suffices to prove the direct sense. If $A_M \neq 0$, there exists by Noetherianity a maximal ideal $m$ of $A_M$, and we let $p := m \cap A$, an ideal of $A$ disjoint from $M_A$, and maximal as such. By Lemma 2.12 $p$ is disjoint from $S$, and maximal as such, so by the Äquivalenzsatz $2.3$ $p$ is maximal and $A/p \cong k$. As $A_M/m \cong (A/p)_M \cong k$ as $k$-algebras, $m$ is the kernel of a morphism $A_M \to k$ and the proof is complete.

\end{proof}

In order to characterise the equiradical in finitely generated $k$-algebras $A$, we are going to use localisation at one element. If $a \in A$, we let $\Sigma_a$ be the multiplicative subset generated by all elements of the form $a^m D^\#(b, a^n)$, for $D(\mathcal{F}) \in \mathcal{P}$ of degree $d$ say, $D^\#(\mathcal{X}, \mathcal{Y}) = Y^d D(\mathcal{X}/\mathcal{Y})$ the homogenisation of $D$, $m, n \in \mathbb{N}$ and $b$ an appropriate tuple from $A$. We note $A_{(a)}$, the localisation $\Sigma_a^{-1}A$.

**Lemma 2.17.** If $A$ is a finitely generated $k$-algebra and $a \in A$, the map $c/a^m D^\#(b, a^n) \in A_{(a)} \mapsto (c/a^m)/a^m D(\mathcal{F}/a^n) \in (A_{(a)})_M$ is a $k$-isomorphism $A_{(a)} \cong (A_{(a)})_M$.

\begin{proof}

Let $l_a : A \to A_{(a)}$ be the localisation at $a$, $l_M : A_{(a)} \to (A_{(a)})_M$ the canonical localisation and $f_a : A \to A_{(a)}$ the localisation at $\Sigma_a$. As $a \in (A_{(a)})^\times$, there exists a unique morphism $\varphi_a : A \to (A_{(a)})_M$ such that $\varphi_a \circ l_a = f_a$. For $D(\mathcal{F}) \in \mathcal{P}$ and $\mathcal{F}/a^n$ a corresponding tuple in $A_{(a)}$, we have $D(\mathcal{F}/a^n) = (1/a^{md}) D^\#(\mathcal{F}/a^n)$, an element of $A_{(a)}$ which becomes invertible in $A_{(a)}$. By the universal property of $l_M$ (as a morphism of $A$-algebras), there exists a unique $A$-morphism $\varphi : (A_{(a)})_M \to A_{(a)}$ such that $\varphi \circ l_M = \varphi_a$, and by the universal property of $l_a$ this is the unique such that $\varphi \circ l_M \circ l_a = f_a$. The other way round, any non-negative power $a^m$ of $a$ is invertible in $(A_{(a)})_M$ and in $A_{(a)}$ we have $D^\#(\mathcal{F}, a^n) = a^m D(\mathcal{F}/a^n)$, which also becomes invertible in $(A_{(a)})_M$. By the universal property of $f_a$, there exists a unique $\psi : A_{(a)} \to (A_{(a)})_M$ such that $\psi \circ f_a = l_M \circ l_a$. We have $\psi \varphi f_a = \psi f_a = l_M l_a$. 

\end{proof}
and by the universal properties of localisation this entails \( \psi \varphi = 1 \); likewise, we have \( \varphi \psi f_a = f_a \) and for the same reason we have \( \varphi \psi = 1 \), so that \( \varphi \) and \( \psi \) are reciprocal isomorphisms. Now by definition, we have \( \varphi((c/a^n)/D(b/a^n)) = ca^{ad}/a^{md}D\ISCO{b}{a^n} \) and \( \psi(c/a^n)D\ISCO{b}{a^n} = (c/a^n)/a^{nd}D(b/a^n) \). The maps are represented on the following diagram:

\[
\begin{array}{cccc}
A & \xrightarrow{l_a} & A_a & \xrightarrow{l_M} \ (A_a)_M \\
\downarrow f_a & & \downarrow \varphi_a & \uparrow \psi \\
A_{(a)} & = & A_{(a)} & = A_{(a)}.
\end{array}
\]

\[\square\]

**Theorem 2.18.** For a finitely generated \( k \)-algebra \( A \) and an ideal \( I \) of \( A \), we have \( \sqrt{I} = \{a \in A : I \cap \Sigma_a \neq \emptyset\} \).

**Proof.** Suppose \( a \notin \sqrt{I} \) : by definition there exists a special maximal ideal \( \mathfrak{m} \) of \( A \) containing \( I \) and such that \( a \notin \mathfrak{m} \). For all \( m,n \in \mathbb{N} \), we have \( a^m,a^n \notin \mathfrak{m} \) and as \( A/\mathfrak{m} \cong k \), for all \( D \in \mathcal{D} \) with degree \( d \) and appropriate \( \overline{b} \in A \) we have \( a^mD\ISCO{b}{a^n} \notin \mathfrak{m} \) (otherwise \( D(\overline{b}/[\overline{a^n}]) = (1/\overline{a^n})D\ISCO{b}{[\overline{a^n}]} = 0 \) in \( A/\mathfrak{m} \), contradicting the choice of \( D \) and \( \mathfrak{m} \)), so \( I \cap \Sigma_\mathfrak{m} = 0 \) by primality of \( \mathfrak{m} \). Conversely, if \( I \cap \Sigma_\mathfrak{m} = 0 \), then \( A_{(a)/\Sigma_a^{-1}I} = 0 \) and as \( (A/I)_{(a+1)} \cong \Sigma_a^{-1}(A/I) \cong A_{(a)}/\Sigma_a^{-1}I \), there exists a morphism \( A_{(a)}/\Sigma_a^{-1}I \rightarrow k \). Indeed, we have \( (A/I)_{(a+1)} \cong ((A/I)_{a+1})_M \) by Lemma 2.17 and as \( (A/I)_{a+1} \) is finitely generated over \( k \), we may apply Lemma 2.16. Let then \( \mathfrak{m} \) be the kernel of the composite morphism \( A \rightarrow A/I \rightarrow (A/I)_{(a+1)} \rightarrow k \) : we have \( a \notin \mathfrak{m} \), and as \( \mathfrak{m} \) is special we get \( a \notin \sqrt{I} \). \[\square\]

This theorem is the key ingredient to the characterisation of the algebras of sections of regular functions over an open subset of an affine algebraic subvariety, which is the core result of the next section and of the article.

### 3. Affine algebraic subvarieties

**\( \ast \)-Algebras and special algebras.** The following definition captures the intrinsic algebraic properties of canonical localisations.

**Definition 3.1.** Say that a \( k \)-algebra \( A \) is a \( \ast \)-algebra (over \( k \)) if every element of \( M_A \) is invertible in \( A \).

**Example 3.2.** For every irreducible affine algebraic subvariety \( V \subseteq k^n \), \( k(V) \) is a \( \ast \)-algebra : if \( D \in \mathcal{D} \) and \( f/g \in k(V) \), we have \( D(f/g) = D\ISCO{f}{g}.1 = (1/g^d)D\ISCO{f}{g} \) and as \( g \neq 0 \), we have \( D\ISCO{f}{g} \neq 0 \) by Lemma 3.4 so \( D(f/g) \in k(V)^\times \).

**Lemma 3.3.** If \( A \) is a \( k \)-algebra and \( l_M : A \rightarrow A_M \) its canonical localisation, then \( A \) is a \( \ast \)-algebra if and only if \( l_M \) is an isomorphism. In particular, \( A_M \) is a \( \ast \)-algebra for every \( k \)-algebra \( A \).

**Proof.** If \( A \) is a \( \ast \)-algebra, then for every \( k \)-morphism \( f : A \rightarrow B \) with \( f(M_A) \subseteq B^\times \), there exists a unique \( g : A \rightarrow B \) such that \( g \circ 1_A = f \), so \( 1_A \) has the universal property of \( l_M \), which is therefore an isomorphism. Conversely, if we assume that \( l_M : A \rightarrow A_M \) is an isomorphism, it suffices to show that \( A_M \) in general is a \( \ast \)-algebra. Let thus \( D(\overline{a}) \in \mathcal{D} \) and \( \overline{a}/m \in A_M \) an appropriate tuple : \( m \) has the form \( D_1(\overline{a})_1 \) for \( D_1 \in \mathcal{D} \) and \( D(\overline{a}/m) = D\ISCO{1}{m}.1 = (1/m^d)D\ISCO{1}{m} \). Let \( D_2(\overline{a},\overline{b}) = D\ISCO{1}{1} \) : if \( \overline{b},\overline{b}_1 \in k \), we have \( D_1(\overline{b}_1) \neq 0 \), hence \( D\ISCO{1}{b}D_1(\overline{b}_1) \neq 0 \) (otherwise \( D(\overline{b}/D_1(\overline{b}_1)) = (1/D_1(\overline{b}_1)^d)D\ISCO{1}{b}D_1(\overline{b}_1) = 0 \)), so \( D_2 \in \mathcal{D} \), and therefore \( m^dD(\overline{a}/m) = D_2(\overline{a},\overline{b})D_1(\overline{a}) \in A_M^\times \), whence \( D(\overline{a}/m) \in A_M \), so \( A_M \) is a \( \ast \)-algebra. \[\square\]
Remark 3.4. i) The key ingredient of the proof is borrowed from [9], Theorem 2.1.
ii) By the properties of localisation, to every morphism of k-algebras \( \varphi : A \to B \), we may associate a morphism of \( * \)-algebras \( \varphi_M : A_M \to B_M \) in an obvious way. Canonical localisation is thus a functor from the k-algebras to \( * \)-algebras, left adjoint to the forgetful functor. This last category has many interesting properties, being in particular locally finitely presentable (see [1] for instance). We will not go into the category-theoretic detail here, but we will use a notion of \( * \)-algebra of finite type (as such) in section [4].

The following very simple definition, inspired by the first order theory of quasivarieties (the curious reader might want to have a glance at sections 9.1 and 9.2 of [13]), generalises reduced algebras over (algebraically closed) fields.

Definition 3.5. Say that a k-algebra \( A \) is special if \( A \) embeds as a k-algebra into a power of \( k \). Say that an ideal \( I \) of a k-algebra \( A \) is special if \( I = \sqrt{I} \).

It is obvious that a maximal ideal \( m \) of \( A \) is special in the sense of the present definition if and only if it is in the sense of Definition 2.3.

Lemma 3.6. If \( A \) is a k-algebra and \( I \) an ideal of \( A \), then \( A/I \) is special if and only if \( I \) is special.

Proof. Suppose \( A/I \) is special : there exists a set \( S \) and an embedding \( \varphi : A/I \to k^S \) of k-algebras. If \( a \in A - I \), we have \( \varphi(a+I) \neq 0 \), so there exists \( s \in S \) such that \( p_s \circ \varphi(a) \neq 0 \), where \( p_s : k^S \to k \) is the \( s \)-th projection. It follows that \( a \notin m := \text{Ker}(p_s \circ \varphi \circ \pi_1) \), for \( \pi_1 : A \to A/I \) the canonical projection; as \( m \) is special, we have \( a \notin \sqrt{I} \), so \( I = \sqrt{I} \). Conversely, if \( I = \sqrt{I} \), then by definition of \( \sqrt{I} \) the quotient \( A/I = A/\sqrt{I} \) embeds into \( k^S \), where \( S \) is the set of special maximal ideals containing \( I \), so \( A/I \) is special.

Lemma 3.7. An ideal \( I \) of a k-algebra \( A \) is special (i.e. \( I = \sqrt{I} \)) if and only if for all \( D(\pi) \in \mathcal{D}, \pi, b \in A \) and \( n, m \in \mathbb{N} \) such that \( b^m D^\#(\pi, b^n) \in I \), we have \( b \in I \). In particular, an algebra \( A \) is special if and only if for all \( D(\pi) \in \mathcal{D} \) and \( \pi, b \in A \) such that \( b^m D^\#(\pi, b^n) = 0 \), we have \( b = 0 \).

Proof. Suppose \( I \) is special : by Lemma 3.3, \( A/I \) is special, so there exists an embedding \( \varphi : A/I \to k^S \) for a set \( S \). Let \( D(\pi) \in \mathcal{D} \), \( \pi, b \in A \) and \( m, n \in \mathbb{N} \) be such that \( b^m D^\#(\pi, b^n) \in I \) : write \( \pi_I = \pi + I \), \( b_I = b + I \), for each \( s \in S \) we have \( \varphi b_I(s)^m D^\#(\varphi \pi_I(s), \varphi b_I(s)^n) = 0 \) in \( k \). If \( \varphi b_I(s) \neq 0 \), then \( D(\varphi \pi_I(s)/\varphi b_I(s)^n) = D(\varphi \pi_I(s), \varphi b_I(s)^n) = 0 \), which contradicts the definition of \( \mathcal{D} \), so \( \varphi b_I(s) = 0 \) for all \( s \), whence \( \varphi(b_I) = 0 \) and therefore \( b_I = 0 \), i.e. \( b \in I \). Conversely, suppose the property holds, and let \( b \notin I \). If \( D(\pi) \in \mathcal{D} \), \( \pi \in A \) and \( n \in \mathbb{N} \), suppose that \( D(\pi/[b^n]) = 0 \) in \( B := (A/I)_b \) : we get \( D^\#/([\pi], [b^n]) = 0 \), thus there exists \( m \in \mathbb{N} \) such that \( b^m D^\#(\pi, b^n) = 0 \) in \( A/I \), whence \( b \in I \), which contradicts our assumption. We get \( D([\pi]/[b^n]) = 0 \), so \( 0 \notin M_B \), whence \( M_B = 0 \) and by Lemma 2.16 there exists a \( k \)-morphism \( B_M \to k \) : the kernel \( m \) of the composite morphism \( A \to (A/I)_b \to B_M \to k \) is special and contains \( I \) but not \( b \), so \( b \notin \sqrt{I} \). We conclude that \( I = \sqrt{I} \), i.e. that \( I \) is special.

Remark 3.8. Special ideals may as well be characterised by the homogeneous signature of \( k \), which is the set \( \mathcal{H} \) of all homogeneous polynomials \( P \) over \( k \) with no non-trivial zero rational in \( k \) : an ideal \( I \) is special if and only if for every \( P(\pi, y) \in \mathcal{H} \) and \( \pi, b \in A \) such that \( P(\pi, b) \in I \), we have \( b \in I \). We do not need this here but we will expand on the subject in [1].

The total ring of fractions of a coordinate ring generalises the function field of an irreducible affine subvariety; we need to check that the construction preserves the fact of being a special algebra.
Proposition 3.9. If $A$ is a special $k$-algebra, then the total ring $ΦA$ of fractions of $A$ is a special $*$-algebra, as well as the canonical localisation $A_M$.

Proof. Let $D(π) ∈ D$ and $π/s, b/s ∈ ΦA$ with the same denominator : if $D^#(π/s, b/s) = 0$, we have $D^#(π, b) = s^dD^#(π/s, b/s) = 0$ in $A$ (for $d = deg(D)$), whence $b = 0$ by Lemma 3.7, so $b/s = 0$, and by the same lemma $ΦA$ is special. Furthermore, let $D(π) ∈ D$ and $π/s$ an appropriate tuple from $ΦA$ : if $b ∈ A$ and $D^#(π, s)b = 0$ in $A$, either $D^#$ is constant (and nonzero) and thus $b = 0$, or $deg(D^#) = d > 0$ and $0 = D^#(π, s)b^d = D^#(bπ, bs)$ and by Lemma 3.7, again, we get $bs = 0$, whence $b = 0$ because $s$ is simplifiable: it follows that $D^#(π, s)$ is simplifiable, so $D(π/s) = (1/s^d)D^#(π, s) ∈ (ΦA)^*$, which is therefore a $*$-algebra. Now let $m ∈ MA$ and $f ∈ A$ such that $fm = 0$; as $A$ is special, by definition there exists an embedding $ϕ : A → kS$ for a set $S$, and for each $s ∈ S$ we have $ϕ(f)(s).ϕ(m)(s) = 0$ in $k$; write $m = D(π)$ : as $ϕ(m)(s) = D(ϕ(π)(s))$, we get $ϕ(f)(s) = 0$ by definition of $D$, so $ϕ(f) = 0$ and $f = 0$, i.e. $m$ is simplifiable. It follows that $A_M$ embeds into $ΦA$, and is special as a subalgebra of a special algebra. □

Points and subvarieties in affine space. In classical algebraic geometry (i.e. over an algebraically closed field $k$), we have a well known correspondence between algebraic subsets of $k^n$ and radical ideals of $k[X_1, ..., X_n]$ ([12], Corollary I.1.4). This is true in general if we replace radical ideals by special ideals. We begin with the case of points (recall that if $(a_1, ..., a_n) ∈ k^n$ and $k[x_1, ..., x_n] = k[X_1, ..., X_n]/(X_1 - a_1, ..., X_n - a_n)$, in $k[π]$ we have $x_i = a_i$ for each $i$, so the structural morphism $k → k[π]$ is an isomorphism and $(X_1 - a_1, ..., X_n - a_n)$ is a maximal ideal).

Lemma 3.10. For every $n ∈ N$, the map $P ∈ k^n → Ker(e_P)$ is a bijection between the points of the affine $n$-space and the special maximal ideals of $k[X_1, ..., X_n]$, which are therefore of the form $(X_1 - a_1, ..., X_n - a_n)$ for $a_1, ..., a_n ∈ k$ (and the reciprocal bijection is given by $m → π(m)$).

Proof. Write $A = k[X_1, ..., X_n]$. If $P = (a_1, ..., a_n)$, we have $(X_1 - a_1, ..., X_n - a_n) = Ker(e_P)$ by what precedes; if $Q = (b_1, ..., b_n)$ and $Q ≠ P$, it follows that $Ker(e_P) ≠ Ker(e_Q)$, and the map is injective. As for surjectivity, if $m$ is a special maximal ideal of $A$, we have $m = √m$ by definition, so by Theorem 2.3 we have $J(√m) = √m = m ≠ A$, whence $√m(π) ≠ 0$ and there exists a zero $P$ of $m$ in $k^n$, so that $Ker(e_P) ⊆ m$: by maximality of $Ker(e_P)$, we have $Ker(e_P) = m$ and the map is surjective. □

Points are particular cases of irreducible subvarieties, which have in general the same useful characterisation as in algebraically closed fields. Recall that if $V$ is an affine subvariety and $f ∈ k[V]$, we let $2_V(f) = \{P ∈ V : f(P) = 0\}$.

Lemma 3.11. If $V ⊆ k^n$ is an affine subvariety, then $V$ is irreducible if and only if $Γ(V, O_V)$ is an integral domain, if and only if $k[V]$ is an integral domain.

Proof. Suppose $V$ is irreducible and $f, g ∈ J(V) := Γ(V, O_V)$ are such that $fg = 0$ : for all $P ∈ V$, we have $f(P)g(P) = 0$, so $V = 2_V(f) ∪ 2_V(g)$; as $V$ is irreducible and $2_V(f), 2_V(g)$ are closed, we have $V = 2_V(f)$ or $V = 2_V(g)$, i.e. $f = 0$ or $g = 0$, and $J(V)$ is an integral domain, as well as $k[V]$, which embeds into $J(V)$. Next, suppose $k[V]$ is an integral domain, and let $V = V_1 ∪ V_2$, with $V_1 = 2_V(I_1)$ and $V_2 = 2_V(I_2)$ for $I_1, I_2$ ideals of $k[V]$, and distinguish two cases : if $I_1 = (0)$, then $V = V_1$, whereas if $I_1 ≠ (0)$, there exists $f ∈ I_1$, $f ≠ 0$; for every $P ∈ V$ and $g ∈ I_2$ we now have $f(g(P)) = 0$ (because $V = V_1 ∪ V_2$), so $f = 0$ and as $k[V]$ is integral, we have $g = 0$, and therefore $I_2 = (0)$ and $V = V_2$. We conclude that $V$ is irreducible. □

Let $I$ be an ideal of $k[X_1, ..., X_n]$ : we have $J(√I) = √I$ by the Aequillnullstellensatz [2.3], so $2_I(I) = 2_I(√I)$, and thus every algebraic set is the zero set of a special ideal. The correspondence is thus given as the following
Proposition 3.12. The map \( I \mapsto \mathcal{Z}(I) \) induces an order-reversing bijection between special ideals of \( k[X] \) and algebraic sets of \( k^n \), which restricts to a bijection between prime and special ideals and irreducible algebraic sets, which restricts to a bijection between maximal and special ideals and points of \( k^n \).

Proof. Suppose \( I, J \) are special and \( \mathcal{Z}(I) = \mathcal{Z}(J) \) : we have \( \mathcal{I}(\mathcal{Z}(I)) = \mathcal{I}(\mathcal{Z}(J)) \), so \( I = \sqrt{J} = \sqrt{J} = J \) by what precedes, so the map is injective on special ideals. If \( V = \mathcal{Z}(I) \subseteq k^n \) is an algebraic subset, we have seen that \( V = \mathcal{Z}(\sqrt{I}) \) so the map is surjective on special ideals, it is a bijection. By Lemma 3.11 a special ideal \( I \) is prime if and only if \( \mathcal{Z}(I) \) is irreducible, which establishes the second part of the statement. Finally, a special ideal \( I \) is maximal if and only if it is a special maximal ideal, if and only if \( \mathcal{Z}(m) \) is a point by Lemma 3.10.

Remark 3.13. The picture may be completed as usual by a description of the topological closure \( \mathfrak{S} = \mathcal{Z}(\mathcal{I}(S)) \) of any subset \( S \subseteq k^n \), and by the relativisation of the correspondence to any affine subvariety.

Sheaves of regular functions. If \( V \subseteq k^n \) is an affine algebraic subvariety and \( h, h' \in k[V] \), we have

\[ (\ast) \ D_V(h) \subseteq D_V(h') \iff \mathcal{Z}_V(h) \supseteq \mathcal{Z}_V(h') \iff \sqrt{h} \subseteq \sqrt{(h')} \iff h \in \sqrt{(h')} \iff \exists \alpha \in \Sigma_h \cap (h') \]

(by definition of \( \sqrt{\cdot} \) and Theorem 2.11). Now let \( g, h \in k[V] \) and \( \alpha \in \Sigma_h \) : if \( P \in D_V(h) \) we have \( h(P) \neq 0 \) and one easily checks that \( \alpha(P) \neq 0 \), so \( h(P)\alpha(P) \neq 0 \), and \( gh/\alpha \) defines a regular function on \( D_V(h) \), i.e. an element of \( \mathcal{O}_V(D_V(h)) \).

Lemma 3.14. The map \( P \in D_V(h) \mapsto g(P)/\alpha(P) \) is zero on \( D_V(h) \) if and only if \( gh = 0 \) in \( k[V] \).

Proof. If \( gh/\alpha \equiv 0 \) on \( D_V(h) \), then as \( h = 0 \) on \( Z_V(h) = V - D_V(h) \), we have \( gh \equiv 0 \) on \( V \), i.e. \( gh = 0 \) in \( k[V] \). Conversely, if \( gh = 0 \), then for every \( P \in V \) we have \( g(P)/h(P) = 0 \), so \( g(P) = 0 \) if \( P \in D_V(h) \), as well as \( g(P)/h(P)\alpha(P) \).

Although the next lemma should be considered as folklore, we include it for the sake of completeness.

Lemma 3.15. For any affine algebraic subvariety \( V \subseteq k^n \), any basic open subset of \( V \) is isomorphic, as a locally ringed space in \( k \)-algebras, to an affine algebraic subvariety of \( k^{n+1} \).

Proof. Write \( V = \mathcal{Z}(I) \) for \( I \) an ideal of \( k[X] = k[X_1, \ldots, X_n] \). A basic open subset of \( V \) has the form \( D_V(h) = \{ P \in V : h(P) \neq 0 \} \) with \( h \in k[V] \), and the restriction of \( \mathcal{O}_V \) to \( D_V(h) \) is \( \mathcal{O}_V \) itself; if \( H \in k[\overline{X}] \) with \( h = H + \mathcal{I}(V) \), for \( W = \mathcal{Z}(I, Y H - 1) \subseteq k^{n+1} \) and the sheaf \( \mathcal{O}_W \) of regular functions on \( W \), one easily checks that the projection map \( \varphi : W \to D_V(h) \) is a homeomorphism. Now if \( U \subseteq D_V(h) \) is open and \( f \in \mathcal{O}_V(U) \), we let \( g : (\overline{\tau}, b) \in \overline{\varphi}^{-1}U \mapsto f(\overline{\tau}) \in k ; \) as \( f \) is regular, for each \( \overline{\tau} \in U \) there is an open \( \overline{U}_{\overline{\tau}} \subseteq U \) and \( L, M \in k[\overline{X}] \) such that \( \overline{\tau} \in \overline{U}_{\overline{\tau}} \) and for all \( (\tau, d) \in \overline{\varphi}^{-1}U_{\overline{\tau}} \), \( g(\tau, d) = L(\tau)/M(\tau) \), which shows that \( g \in \mathcal{O}_{\overline{U}}(\varphi^{-1}U) \), and if we let \( \varphi_{\#}(f) := g \) we have defined a morphism of \( k \)-algebras and a natural transformation \( \varphi_{\#} : \mathcal{O}_V|_{D_V(h)} \to \varphi_*\mathcal{O}_W \). The other way round, if \( g \in \varphi_*\mathcal{O}_W(U) = \mathcal{O}_W(\varphi^{-1}U) \), we let \( f : \overline{\tau} \in \overline{U} \mapsto g(\overline{\tau}, 1/H(\overline{\tau})) \) ; for each \( (\overline{\tau}, b) \in \overline{\varphi}^{-1}U \) there is an open \( U_{\overline{\tau}, b} \subseteq \overline{\varphi}^{-1}U \) and \( L, M \in k[\overline{X}, Y] \) such that \( (\overline{\tau}, b) \in U_{\overline{\tau}, b} \) and for all \( \overline{\tau} \in \overline{U}_{\overline{\tau}, b} \) we have \( f(\overline{\tau}) = L(\overline{\tau}, 1/H(\overline{\tau}))/M(\overline{\tau}, 1/H(\overline{\tau})) = (H(\overline{\tau})^\deg L - \deg M) L(\overline{\tau} H(\overline{\tau}), 1, H(\overline{\tau}))/M^\#(\overline{\tau} H(\overline{\tau}), 1, H(\overline{\tau})) \), so \( f \) is regular and if we let \( \psi_{\#}(g) := f \) we have defined a morphism of \( k \)-algebras and a natural transformation \( \psi_{\#} : \varphi_*\mathcal{O}_W \to \mathcal{O}_V|_{D_V(h)} \), and one checks that \( \varphi_{\#} \) are \( \psi_{\#} \) mutually inverse isomorphisms.
Lemma 3.16. Every affine algebraic subvariety is compact for the Zariski topology.

Proof. Let $V \subseteq \mathbb{k}^n$ be such a subvariety, and suppose that $V = \bigcup_i D_V(f_i)$, a cover by basic open subsets. We have $\emptyset = \bigcap_i \mathcal{V}(f_i) = \mathcal{V}(\bigcap_i k[V] f_i)$, so $k[V] = \mathcal{A}(\emptyset) = \mathcal{A}(\mathcal{V}(\bigcap_i k[V] f_i))$ by Theorem 2.4 and thus by Theorem 2.18 there exists $m \in \Sigma_1 = M_k[V]$, a finite subset $I_0$ of $I$ and $(a_i : i \in I_0)$ in $k[V]$ such that $m = \sum_{i \in I_0} a_i f_i$, and therefore $\emptyset = \mathcal{V}(m) = \mathcal{V}(\bigcap_{i \in I_0} k[V] f_i)$, whence $V = \bigcap_{i \in I_0} D_V(f_i)$. □

So far we have defined a natural map $k[V]_{h > 0} \to \Gamma(D_V(h), \mathcal{O}_V)$, $g/\alpha \mapsto [P \mapsto g(P)/\alpha(P)]$, which is an injective k-morphism: if the member on the right is zero, then the regular map defined by $gh/\alpha$ is zero on $D_V(h)$, so $gh^2 = 0$ in $k[V]$ by Lemma 3.14 and $g/\alpha = gh^2/h^2\alpha = 0$ in $k[V]_{h > 0}$. The following theorem is our core result, bringing together the preceding algebraic theory and the affine geometric theory, and is inspired by [17], Proposition 3.6(a).

Theorem 3.17. The morphism $k[V]_{h > 0} \to \Gamma(D_V(h), \mathcal{O}_V)$ is an isomorphism. In particular, the sheaf $\mathcal{O}_V$ is a sheaf of special *-algebras over $V$.

Proof. As for the first assertion, it only remains to prove that the morphism is surjective. Let $f \in \Gamma(D_V(h), \mathcal{O}_V)$: there exists an open cover $D_V(h) = \bigcup_i U_i$, as well as $g_i, h_i \in k[V]$ for each $i$, such that for every $i$ and $P \in U_i$, $h_i(P) \neq 0$ and $f(P) = g_i(P)/h_i(P)$. Replacing the $U_i$’s by basic open subsets, we may suppose that $U_i = D_V(a_i)$ for all $i$, with $a_i \in k[V]$: we have $D_V(a_i) \subseteq D_V(h_i)$ and by (1), for every $i$ there exists $\alpha_i \in \Sigma_{a_i}$ and $g'_i \in k[V]$ with $\alpha_i = g'_i h_i$; on $D_V(a_i)$, $f$ is represented by $g_i g'_i/h_i g'_i = g_i g'_i a_i/h_i a_i$: replacing $g_i$ by $g_i g'_i a_i$, and $h_i$ by $a_i \alpha_i$, as $D_V(a_i) = D_V(a_i \alpha_i)$ we may suppose that $U_i = D_V(h_i)$ for all $i$: we have $D_V(h) = \bigcup_i D_V(h_i)$, and $f$ is represented on $D_V(h_i)$ by $g_i/h_i$. By Lemmas 3.15 and 3.16 $D_V(h)$ is compact so we may suppose that this cover is finite, and as the functions represented by $g_i/h_i$ and $g_j/h_j$ on $D_V(h_i) \cap D_V(h_j) = D_V(h_i h_j)$ are equal, we have $(g_i h_j - g_j h_i)/h_i h_j \equiv 0$ on $D_V(h_i h_j)$, whence by Lemma 3.14 $h_i h_j (g_i h_j - g_j h_i) = 0$ in $k[V]$, i.e. $h_i h_j g_i = h_i h_j g_j$. Now we have $D_V(h) = \bigcup_{i=1}^m D_V(h_i) = \bigcup_{i=1}^m D_V(h_i^2)$, so $\mathcal{V}(h_i) = \mathcal{V}(h_i^2, h_i)$, whence $h \in \mathcal{A}(\mathcal{V}(h_i^2, h_i)) = \sqrt{(h_i^2, h_i)}$ by the Čech-nullstellensatz [2.4] again, and by Theorem 2.18 there exists $\alpha \in \Sigma_h$ and $a_i \in k[V]$ such that $\Sigma_h a_i h_i^\alpha = 0$ : we want to show that $f$ is represented on $D_V(h_i)$ by $\Sigma_i a_i g_i h_i^\alpha$: Let $P \in D_V(h_i)$: for each $j$ such that $P \in D_V(h_j)$, we have $h_j^2 \Sigma_i a_i g_i h_i = \Sigma_i a_i g_i h_i^2 = h_j h_i \alpha$ by what precedes. Now $f$ is represented on $D_V(h_j)$ by $g_j/h_j$, so $f h_j \equiv g_j$ on $D_V(h_j)$, on which therefore we have $f h_j h_j^\alpha \equiv g_j h_j^\alpha \equiv h_j^2 \Sigma_i a_i g_i h_i$ as maps. As $h_j^2(P) \neq 0$ for each $P \in D_V(h_i)$, on $D_V(h_i)$ we have $f \alpha \equiv \Sigma_i a_i g_i h_i$ as maps, so that $f$ is represented on $D_V(h_i)$ by $\Sigma_i a_i g_i h_i/\alpha$, and as this is true for every $j$, this is true on $D_V(h)$, so finally the morphism $k[V]_{h > 0} \to \Gamma(D_V(h), \mathcal{O}_V)$ is surjective, it is an isomorphism. As for the second assertion, for every open subset $U \subseteq V$, we have $U = \bigcup \{D_V(f) : D_V(f) \subseteq U, f \in k[V]\}$, and therefore $\mathcal{O}_V(U) = \lim_{D(f) \subseteq U} \mathcal{O}_V(D(f))$. Now by what precedes each $\mathcal{O}_V(D(f))$ is a *-algebra by Lemmas 2.17 and 3.3 and a special algebra as well, because $D_V(f)$ is isomorphic to an affine algebraic subvariety by Lemma 3.15 as *-algebras and special algebras are clearly closed under projective limits, $\mathcal{O}_V(U)$ is a special *-algebra. □

Corollary 3.18. For every affine algebraic subvariety $V \subseteq \mathbb{k}^n$, the k-algebra $\Gamma(V, \mathcal{O}_V)$ of everywhere regular functions on $V$ is isomorphic to $k[V]_M$, by the natural k-morphism $k[V]_M \to \Gamma(V, \mathcal{O}_V)$.

Proof. By Theorem 3.17 we have $\Gamma(V, \mathcal{O}_V) = \Gamma(D_V(1), \mathcal{O}_V)) \cong k[V]_{(1)} = k[V]_M$. □

Remark 3.19. i) This is the generalisation over an arbitrary field of the characterisation of the algebra of global sections of the sheaf of regular functions on an affine algebraic subvariety (if $k$ is algebraically closed, this algebra is essentially $k[V]$).
ii) Of course, in general $k[V]_M$ is bigger than $k[V]$. For instance, if $k$ is a subfield of $\mathbb{R}$, the rational function $x \in k \mapsto 1/x^2 + 1$ is regular over $k$, but the rational fraction $1/X^2 + 1$ is not in $k[X]$, the coordinate algebra of $k$, so $(1/X^2 + 1) \in k[X]_M - k[X]$.

4. Equiresidual Affine Algebraic Varieties over a Field

**Equiresidual varieties and affine \(^*\)-algebras.** Let $k$ be any field. The co-restriction to special $k$-algebras of the coordinate algebra functor $k[-]$ is in fact a duality:

**Proposition 4.1.** The functor $k[-]$ is an duality between the categories of affine algebraic subvarieties of $k$ and finitely generated special $k$-algebras.

**Proof.** We focus on essential surjectivity, so let $A$ be a special $k$-algebra of finite type: $A$ is isomorphic to an algebra of the form $k[\mathcal{X}]/I$, where $\mathcal{X} = (x_1, \ldots, x_n)$. Consider the affine algebraic subvariety $V := \mathcal{V}(I) \subseteq k^n$: we have $k[V] := k[\mathcal{X}]/\mathcal{I}(V) = k[\mathcal{X}]/\sqrt{I}$ (by Theorem 2.4) $= k[\mathcal{X}]/\mathcal{I}$ (by Lemma 3.6) because $\mathcal{A}$ is special; in short, $A$ is isomorphic to $k[V]$, and $k[-]$ is an equivalence. \(\square\)

As in the classical context, we may want to work with a category of locally ringed spaces in $k$-algebras which are locally isomorphic to affine algebraic subvarieties (among which we will find the "equiresidual" version of algebraic equivarieties, see [3]). This category will comprise the spaces which are essentially affine subvarieties, like basic open subsets of these. We start from a broad definition which we will use in subsequent work.

**Definition 4.2.** i) An equiresidual variety over $k$, or equiviariety over $k$ for short, is a locally ringed space in $k$-algebras $(V, O_V)$, such that for every $P \in V$, there exists an open neighbourhood $U$ of $P$ in $V$ for which $(U, O_V|U)$ is isomorphic to an affine algebraic subvariety.

ii) An (equiresidual) affine algebraic variety over $k$ or affine algebraic equiviariety for short, is an equiviariety over $k$ which is isomorphic to an affine algebraic subvariety of $k$.

We note $EVar_k^n$ the category of equiresidual affine algebraic varieties over $k$, with arrows the morphisms of locally ringed spaces in $k$-algebras. As we have seen in the introduction, any regular morphism of affine subvarieties "is" naturally a morphism of equiviarieties.

**Remark 4.3.** i) If $V$ is an equiviariety over $k$, for $f \in \Gamma(V, O_V)$ write $[f]_P \in O_V|_P$ its germ at $P$. If $U$ is an open neighbourhood of $P$ such that $(U, O_V|_U)$ is isomorphic to an affine algebraic subvariety, we have $O_V|_U \equiv O_V|_P$, by the small lemma 1.1 the structural morphism $k \to O_V|_P$ (the residual field of $O_V|_P$) is an isomorphism.

ii) In this situation, if $U \subseteq V$ is an open subset and $v \in O_V(U)$ and for each $P \in U$, $[v]_P \neq 0$ in $O_V|_P$, we have $v \in O_V(U)^\times$. Indeed, it suffices to prove this for $V$ an affine subvariety, in which case this is obvious.

Let $(\varphi, \varphi^\#) : V \to W$ be a morphism of affine algebraic equiviarieties over $k$, with $\varphi^\# : O_W \to O_V$. We have the $k$-algebra morphism $\varphi^\# : O_W(W) \to \varphi_* O_V(W) = O_V(\varphi^{-1}(W)) = O_V(V)$, and we let $J(\varphi) = \varphi^\#$. This obviously defines a functor from the dual of $EVar_k^n$ to the category of $k$-algebras. By Proposition 4.1 the category $EVar_k^n$ and the category of special $k$-algebras of finite type are dual, but if $(V, O_V)$ is an affine algebraic subvariety of $k^n$, the algebra of sections $J(V) = \Gamma(V, O_V)$ is, as we have seen in Corollary 3.13, naturally isomorphic to the $*-$algebra $k[V]_M$, which is not in general isomorphic to $k[V]$. The duality of Proposition 4.1 cannot therefore be extended to $EVar_k^n$ by taking the only natural functor we have in mind, which is $J$, the global section functors, as it is the case in the classical setting. This suggests searching for another duality, by introducing the following types of $k$-algebras:

**Definition 4.4.** Say that a $*-$algebra $A$ over $k$ is:

i) affine, if it is isomorphic to the algebra of global sections of regular functions on an
affine algebraic subvariety of \( k \)

ii) of finite \(*\)-type if there exists a surjective \( k \)-morphism of the form \( k[X_1, \ldots, X_n] \to A \)
(or equivalently if \( A \) is isomorphic to the canonical localisation of a \( k \)-algebra of finite type). By Corollary 3.13, an affine \(*\)-algebra is of finite \(*\)-type.

**Remark 4.5.** We have an analogue of the strong form of Hilbert’s Nullstellensatz : if \( k \to K \) is a field extension with \( K \) also a \(*\)-algebra of finite \(*\)-type, then the extension is an isomorphism. Indeed, we have a \( k \)-isomorphism \( \text{Spm} A \to \text{Spec} K \) with \( m \) a special maximal ideal of \( k[X] \) by 2.3 and 2.12 whence the sequence of \( k \)-isomorphisms \( k \to \text{Spm} A \to \text{Spec} K \) is isomorphic to \( \text{Spm} A \to \text{Spec} K \).

Now the functor \( J \) has values in the category \( \text{Aff}_k \) of affine \(*\)-algebras. The obvious properties of such algebras almost readily suggest their following characterisation, which will be used to build the duality :

**Proposition 4.6.** If \( A \) is a \( k \)-algebra, the following are equivalent :

i) \( A \) is an affine \(*\)-algebra over \( k \)

ii) \( A \) is a special \(*\)-algebra of finite \(*\)-type over \( k \).

**Proof.** (i)\( \Rightarrow \)(ii) Let \( V \subseteq k^n \) be an affine algebraic subvariety of \( k^n \) say, such that \( A \cong \Gamma(V, \mathcal{O}_V) \) : we have \( \Gamma(V, \mathcal{O}_V) \cong k[V]_M \) by Corollary 3.13 and as \( k[V] \) is special, so is \( k[V]_M \) by Proposition 3.9. Now the surjective morphism \( k[X] \to k[V] \) gives by canonical localisation a surjective morphism \( k[X]_M \to k[V]_M \), whence (ii).

(ii)\( \Rightarrow \)(i) Let \( \varphi : k[X_1, \ldots, X_n] \to A \) be a surjective \( k \)-morphism, and \( B = \varphi(k[X_1, \ldots, X_n]) \) : we have \( A \cong B_M \) and as \( A \) is special, so is \( B \) as a subalgebra; as a special \( k \)-algebra of finite type, \( B \) is isomorphic to a coordinate algebra of an affine algebraic subvariety by Proposition 3.11 for instance the subvariety \( V = \mathcal{Z}(I) \), for \( I = \text{Ker}(\varphi_{k[X]}(\mathcal{O}) \). In particular, we have \( A \cong B_M \cong \Gamma(V, \mathcal{O}_V) \) by Corollary 3.13, therefore \( A \) is an affine \(*\)-algebra over \( k \).

\( \square \)

**Maximal spectra of affine \(*\)-algebras.** The functor \( J \) is a duality if and only if it has a right adjoint \( K : \text{Aff}_k \to (\text{EVar}_k)^\circ \) such that \( I_d \cong J \circ K \) and \( I_d \cong K \circ J \), in which case we have an adjoint equivalence (1.5, Theorem IV.4.1); this adjoint is provided by the maximal spectrum of affine \(*\)-algebras. In general, let \( A \) be any \( k \)-algebra : following the same line as for algebraic spaces (17, Chapter 11), we may define a natural sheaf \( \mathcal{O}_X \) on \( X = \text{Spm}(A) \) as follows : for every open \( U \subseteq X \) for the Zariski topology, we let \( \mathcal{O}_X(U) \) be the set of all maps \( s : U \to \coprod_{m \in U} A/m \), such that for all \( m \in U \), \( s(m) \in A/m \) and there exists an open neighbourhood \( U_m \subseteq U \) of \( m \) and \( u_m, v_m \in A \) for which \( s|_{U_m} = [u_m]/[v_m] \) (by which we mean that for every \( n \in U_m \) we have \( v_m \notin n \) and \( g(n) = [u_m]/[v_m] \) in \( A/n \)). Now let \( (V, \mathcal{O}_V) \) be a (general) equivalence over \( k \) and \( A = \Gamma(V, \mathcal{O}_V) \) (a \(*\)-algebra by Theorem 3.17), and let \( P \in V : \) as the structural morphism \( k \to \mathcal{O}_{V,P} \) is an isomorphism (see Remark 1.3), the natural morphism \( \Gamma(V, \mathcal{O}_V) \to \mathcal{O}_{V,P} \to \mathcal{O}_{V,P} \) is surjective, with kernel \( m_P = \{ g \in A : [g]P = 0 \} \) by definition and we have a \( k \)-isomorphism \( i_P : A/m_P \cong \mathcal{O}_{V,P} \),

\[ [g] \mapsto [g]P. \]

It is easy to see that the map \( \varphi_V : P \in V \mapsto m_P \in X : = \text{Spm}(A) \) is continuous, and we may define a sheaf morphism \( (\varphi_V)^* : \mathcal{O}_X \to (\mathcal{O}_V)^* \) as follows. If \( U \subseteq X \) is open and \( s \in \mathcal{O}_X(U) \), for each \( m \in U \) let \( u_m, v_m \in A = \Gamma(V, \mathcal{O}_V) \) as before; if \( P \in V \), write \( O_P = \varphi_V^{-1}U_{m_P} \) : for each \( Q \in O_P \), we have \( m_Q \subseteq U_{m_P} \) and \( [b_{m,P}] \neq 0 \) in \( A/m_Q \), so \( i_Q([b_{m,P}]) = [b_{m,P}]Q \neq 0 \) in \( \mathcal{O}_{V,Q} \), whence \( b_{m,P}|O_Q \in \mathcal{O}_V(O_P)^* \) by Remark 4.3 and we define an element of \( \mathcal{O}_V(O_P) \) by \( t_P := a_{m,P}|O_P/b_{m,P}|O_P \).

**Lemma 4.7.** If \( s, t \in \mathcal{O}_V(U) \) are such that for each \( P \in U, [s]P = [t]P \) in \( \mathcal{O}_{V,P} \), then \( s = t \).
Proof. By definition of an equiarrity and the local character of equality for sections of a sheaf, it suffices to prove this for \( V \) an affine algebraic subvariety of \( k^n \), in which case this is obviously true because \( s \) and \( t \) have functions with values in \( k \).

Let \( P, Q \in V \) : if \( R \in O_P \cap O_Q \), we have \( m_R \in U_{m_R} \cap U_{m_Q} \), so that \( s(m_R) = [u_{m_P}]/[v_{m_Q}] = [u_{m_Q}]/[v_{m_R}] \) in \( A/m_R \), and thus \( u_{m_R}v_{m_Q} - u_{m_Q}v_{m_R} \in m_R \), which means that \( [u_{m_P}]/[v_{m_Q}] = [u_{m_Q}]/[v_{m_R}] \) in \( O_{V,R} \), and as this is true for each \( R \in O_P \cap O_Q \), by Lemma 4.7 we have \( t_P|_{O_P \cap O_Q} = t_Q|_{O_P \cap O_Q} \), so the \( t_P \)'s define a unique section \( t \in O_V(U) \) and we let \( (\varphi_V)^\#(s) := t \). We have defined a map \( (\varphi_V)^\# : \mathcal{O}_X(U) \to \mathcal{O}_V(\varphi^{-1}_V U) = (\varphi_V)^\#(\mathcal{O}_V(U)) \), and one checks that it is a morphism of \( k \)-algebras and defines a natural transformation \( (\varphi_V)^\# : \mathcal{O}_X \to (\varphi_V)^\#, \mathcal{O}_V \), so we have a morphism of locally ringed spaces of \( k \)-algebras and defines a natural transformation \( (\varphi_V, (\varphi_V)^\#) : (V, \mathcal{O}_V) \to (X, \mathcal{O}_X) \). Indeed, with the same notations the induced morphism on the stalk at \( P \) is \( (\varphi_V)^\#_P : \mathcal{O}_{X,m_P} \to \mathcal{O}_{V, \mathcal{O}_P}, [s, U] \to [u_{m_P}]/[v_{m_P}] \), and we may assume that \( U = U_P \); as \( k \to \mathcal{O}_{X,m_P} \) is an isomorphism, \( (\varphi_V)^\#_P \) is local. We now turn to maximal spectra of \(*\)-algebras of finite \(*\)-type.

**Proposition 4.8.** For every finitely generated \( k \)-algebra \( A = k[X_1, \ldots, X_n]/I \), the map \( P \in \mathcal{P}(I) \mapsto m_P = \{f/g \in A_M : f(P) = 0\} \in \mathcal{Spm}(A_M) \) is a homeomorphism.

Proof. Write \( V = \mathcal{P}(I), P = (a_1, \ldots, a_n) \in V(I) \) and \( e_P : A \to k \) be the evaluation at \( P \), \( f \mapsto f(P) \), factoring out the evaluation \( e_P : k[X_1, \ldots, X_n] \to k \). By Lemma 4.10 the map \( \varphi : P \in V \mapsto \text{Ker}(e_P) \) is obviously bijective. As the canonical localisation \( t_M : A \to A_M \) exchanges the special maximal ideals of \( A \) and the maximal ideals of \( A_M \) by Lemma 2.12 the map \( \varphi_M : V \to \mathcal{Spm}(A_M), P \mapsto \text{Ker}(e_P)_M = \{f/g \in A_M : f(P) = 0\} \), is also a bijection. Let \( D(f/m) = \{m_M \in \mathcal{Spm}(A_M) : f/m \notin m_M\} \) be a basic open subset of \( \mathcal{Spm}(A_M) \) : we have \( \varphi_M^{-1}(D(f/m)) = \{P \in V : f \notin \text{Ker}(e_P)\} = \{P \in V : f(P) \neq 0\} = D_V(f) \leq V \), an open subset, so \( \varphi_M \) is continuous; and if \( D_V(f) = \{P \in V : f(P) \neq 0\} \) is a basic open subset of \( V \), we have \( \varphi_M(D(f)) = \{m_M \in \mathcal{Spm}(A_M) : f/M \notin m_M\} = \{m_M \in \mathcal{Spm}(A_M) : f \notin m \} = D(f/1) = \{ \text{basic open subset of } \leq \text{Spm}(A_M) \}, \) so \( \varphi_M^{-1} \) is continuous, and \( \varphi_M \) is a homeomorphism. \( \square \)

As Proposition 4.8 suggests, the maximal spectrum turns affine \(*\)-algebras into affine algebraic equivalences:

**Proposition 4.9.** If \( A \) is an affine \(*\)-algebra and \( X = \mathcal{Spm}(A) \), then for the sheaf \( \mathcal{O}_X \) of regular functions on \( X \) as defined above, \( (X, \mathcal{O}_X) \) is an equiarrdual affine algebraic variety.

Proof. By definition, if \( A \) is an affine \(*\)-algebra it is isomorphic to \( \Gamma(V, \mathcal{O}_V) \) for \( V \subseteq k^n \) an affine algebraic subvariety, so we may assume that \( A = \Gamma(V, \mathcal{O}_V) \). By Corollary 3.18 and Proposition 4.8, the above map \( \varphi_V : P \in V \mapsto m_P = \{g \in \Gamma(V, \mathcal{O}_V) : g(P) = 0\} \in \mathcal{Spm}(A) \) is a homeomorphism, and we want to show that \( \varphi_V^{-1} \) is an isomorphism. If \( U \subseteq X \) is open, \( s \in \mathcal{O}_X(U) \) is represented on the open neighbourhood \( U_{m_P} \subseteq U \) of \( m_P \) for each \( P \in \varphi_V^{-1}U \) and \( (\varphi_V^{-1}U)(s) = 0 \), for every \( P \in U \) we have \( u_{m_P}(P)/v_{m_P}(P) = 0 \), so \( u_{m_P} = 0 \) and \( v_{m_P} \in m_P \), and therefore \( s = 0 \) by its local characterisation on each \( U_{m_P} \), so \( \varphi_V^{-1}U \) is injective. As for surjectivity, let \( g \in \mathcal{O}_V(\varphi^{-1}U) \) : for each \( P \in \varphi^{-1}U \) there exists an open \( O_P \subseteq \varphi^{-1}(U) \) and \( u_{m_P}, v_{m_P} \in k[V] \), such that \( P \in O_P \) and \( g|_{O_P} \equiv u_{m_P}/v_{m_P} \) in the (of regular functions). Applying \( \varphi \), \( U_{m_P} := \varphi(O_P) \subseteq U \) is an open neighbourhood of \( m_P \), \( u_{m_P} \) and \( v_{m_P} \) define regular functions on \( V \) (we keep the same notation), and \( u_{m_P}(Q) \neq 0 \) for all \( Q \in O_P \), i.e. \( v_{m_P} \notin m_Q \). Now let \( s(m \in U) := [u_{m_P}]/[v_{m_P}] \in A/m \), for the unique \( P \in V \) such that \( m = m_P \); as before, if \( Q \in O_P \), as \( g|_{O_Q} \equiv u_{m_P}/v_{m_P} \) and \( g|_{O_Q} \equiv u_{m_Q}/v_{m_Q} \) we have \( u_{m_P}(Q)/v_{m_P}(Q) = u_{m_Q}(Q)/v_{m_Q}(Q) \in k \). This means that \( u_{m_P}, v_{m_Q} - u_{m_Q}v_{m_P} \in m_Q \), so \( u_{m_P}/v_{m_Q} = u_{m_Q}/v_{m_P} \in A/m \). Therefore \( s \) has a
constant description on $U_{mp}$ for every $P \in U$, i.e. $s \in \mathcal{O}_X(U)$; now obviously we have $(\varphi_V^\#)_{U}(s) = g$, so $(\varphi_V^\#)_{U}$ is surjective, $\varphi_V^\#$ is an isomorphism, and $(X, \mathcal{O}_X)$ is an affine algebraic equivarities.

As for functoriality, if $f : A \to B$ is a $k$-morphism of $\ast$-algebras of finite $\ast$-type, we have a natural continuous map $f^* : Spm(B) \to Spm(A)$, which sends a maximal ideal $m$ of $B$ to $m := f^{-1}(m)$. Consider the induced $\omega$-residual isomorphism, we may now define a morphism of sheaves $(f^*)^\# : \mathcal{O}_X \to \varphi_* \mathcal{O}_Y$ as follows: if $U \subseteq X$ is open, we let $(f^*)^\#(g \in \mathcal{O}_X(U)) := [n \in (f^*)^{-1}(U) \to (f/n)(g((f^*)^\#(n))] \in \mathcal{O}_Y((f^*)^{-1}U) = (f^*)_* \mathcal{O}_Y(U)$. One checks that this is well defined and that for $n \in (f^*)^{-1}U$, $(f^*)^\#(g)$ is represented on $(f^*)^{-1}U_{f^n(a)}$ by $(f(u_{f^n(a)})/f(v_{f^n(a)}))$ if $g$ is represented on $U_{f^n(a)}$ by $[u_{f^n(a)}]/[v_{f^n(a)}]$. In particular, by Proposition 4.9 we have a functor $K : f \mapsto (f^*, (f^*)^\#)$ from the category $\ast Aff_k$ of affine $\ast$-algebras over $k$, into the dual category $(EVar_k^\circ)^\circ$ of affine algebraic equivarities over $k$.

### The affine adjoint duality

The last matter of business is to show that $J$ and $K$ are mutual quasi-inverses, i.e. to define natural isomorphisms $Id \cong J \circ K$ and $Id \cong K \circ J$. The proof of the following proposition is tedious but straightforward.

**Proposition 4.10.** The morphisms $(\varphi_V, (\varphi_V)^\#) : (V, \mathcal{O}_V) \to (KJ(V), \mathcal{O}_{KJ(V)})$, for $V \in EVar_k^\circ$, define a natural transformation $\varphi$ from $Id$ to $KJ$.

Now if $V$ is any affine algebraic equivarities, there exists an isomorphism between $V$ and an affine algebraic subvariety $W$ of a $k^n$ say; we have seen in the proof of Proposition 4.9 that $\varphi_W$ is a homeomorphism, so $\varphi_V$ is also a homeomorphism by Proposition 4.10. We are now able to prove the

**Proposition 4.11.** The pair $(\varphi_V, (\varphi_V)^\#) : (V, \mathcal{O}_V) \to (X, \mathcal{O}_X)$ is an isomorphism, and thus it defines a natural isomorphism $\varphi : Id \cong K \circ J$ of endofunctors of $EVar_k^\circ$.

**Proof.** Suppose that $U \subseteq X$ is open, $s \in \mathcal{O}_X(U)$, and $t = (\varphi_V^\#)_{U}(s) = 0$. If $m \in U$, there exists a unique $P \in \varphi_V^{-1}U$ such that $m = mp = \varphi(P)$, and by definition we have $t_{|U_P} = (up|_{\varphi_V^{-1}U_P})/(up|_{\varphi_V^{-1}U_P})$ for a representation $up|_{U_P}$ of $s$ on $U_P$. As $t = 0$ by hypothesis, we have $t_{|\varphi_V^{-1}U_P} = 0$, so $up|_{\varphi_V^{-1}U_P} = 0$, and in the residual field $\overline{\mathcal{O}_V,U_P}$ we get $|up|_{\varphi_V^{-1}U_P} = 0$, whence $|up| = i_P^{-1}([up|_{\varphi_V^{-1}U_P}] = 0$ in $J(V)/mP$ and finally $s(p) = |up|/(|up|) = 0$. It follows that $s = 0$, so $((\varphi_V^\#)_{U}$ is injective and $(\varphi_V)^\#$ is a monomorphism. As for surjectivity, we first suppose that $V$ is an affine algebraic subvariety of $k^n$ say and let $t \in \mathcal{O}_V(\varphi_V^{-1}U)$ ; we have $t = (\varphi_V^{-1}U) \to k$ and for each $P \in O := \varphi_V^{-1}U$ there exists $OP \subseteq O$ and $ap, bp \in k[V]$ such that $P \in OP$ and $t_{|OP} = ap|_{OP}/bp|_{OP}$, where $ap \in J(V)$ is the global section defined by $ap$. Let $s : U \to \bigcup_{m \in \mathcal{O}_V} \mathcal{O}_V(m)/m$, $m = mp \mapsto [ap]/[bp]$; if $n \in U_P = \varphi_V(OP)$, we have $n = mQ$ for $Q \in OP$, so $t(Q) = ap(Q)/bp(Q)$ but also $t(Q) = aQ(Q)/bQ(Q)$, whence $apQ - aQ - bp \in Ker(eq : k[V] \to k)$, and therefore $\tilde{a}pQ - \tilde{a}Qbp \in mQ \subseteq J(V)$, so $s(n) = s(mQ) = [\tilde{a}Q]/[bQ] = [\tilde{a}p]/[bp]$, and finally $s \in \mathcal{O}_X(U)$, as it has a locally constant description. Let now $P \in \mathcal{O}_U$ : if we note $u := (\varphi_V^\#)_{U}(s)$, by definition we have $u_{|\mathcal{O}_P} = \tilde{a}P|_{\mathcal{O}_P}/\tilde{b}P|_{\mathcal{O}_P}$, which is exactly $t_{|OP}$; by characterisation of a sheaf, we have $t = u$, so $(\varphi_V^\#)_{U}$ is surjective, and hence $(\varphi_V, (\varphi_V)^\#)$ is an isomorphism. In the general case, if $V$ is an affine algebraic equivarities there exists an affine subvariety $W$ and an isomorphism $\psi = (\varphi, \psi^\#) : W \to V$, and applying what precedes and the functoriality of Proposition
the diagram following:

\[
\begin{array}{ccc}
V & \xrightarrow{\varphi_V} & KJ(V) \\
\psi & \uparrow & \quad \KJ(\psi) \\
W & \xrightarrow{\varphi_W} & KJ(W)
\end{array}
\]

commutes and \( \varphi_V = KJ(\psi) \circ \varphi_W \circ \psi^{-1} \) is an isomorphism, so \( \varphi : Id \cong K \circ J \) and the proof is complete. \( \square \)

The second transformation is easier to describe. Let \( A \) be an affine \( * \)-algebra and \( f_A : A \to JK(A) \) be defined by \( f_A(a \in A) := [m \in KA = Spm(A) \to [a] \in A/m] \); it is obvious that \( f_A \) is a morphism of \( k \)-algebras.

**Proposition 4.12.** The \( k \)-algebra morphisms \( f_A : A \to JK(A) = \Gamma(Spm(A), \mathcal{O}_{Spm(A)}) \), for \( A \in \ast \text{Aff}^a \), define a natural transformation \( f : Id \to J \circ K \).

**Proof.** Let \( \varphi : A \to B \) be a morphism of affine \( * \)-algebras over \( k \) and \( (X, \mathcal{O}_X) := K(A) \), \( (Y, \mathcal{O}_Y) := K(B) \) : the morphism \( K\varphi : Y \to X \) is defined by \( K\varphi : n \in \Gamma(V, \mathcal{O}_Y) \mapsto \varphi^{-1}n \) and the sheaf morphism \( (K\varphi)^\sharp : \mathcal{O}_X \to (K\varphi)_\sharp \mathcal{O}_Y \), for every open \( U \subseteq X \), by \( (K\varphi)^\sharp (s \in \mathcal{O}_X(U)) : n \in (K\varphi)^{-1}U \mapsto (\varphi/n) \circ s \varphi^{-1}n \), for \( \mathcal{O}_n : A/\varphi^{-1}n \to B/n \) the quotient morphism; the morphism \( JK(\varphi) : JK(A) = \Gamma(X, \mathcal{O}_X) \to JK(B) = \Gamma(Y, \mathcal{O}_Y) \) is then simply \( (K\varphi)^\sharp Y : s \mapsto [n \in Y \mapsto \varphi/n \circ (\varphi)^{-1}n] \). It follows that for each \( a \in A \), we have \( JK\varphi \circ f_A(a) = JK\varphi([m \in X \to [a] \in A/m]) = [n \in Y \to \varphi/n([a] \in A/\varphi^{-1}n)] \). By definition of \( \varphi/n \), it is the section \( n \in Y \mapsto [\varphi(a)] \in B/n \), which is exactly \( f_B \circ \varphi(a) \), so \( f \) is a natural transformation.

**Lemma 4.13.** If \( A \) is an affine \( * \)-algebra, then the Jacobson radical of \( A \) is \( (0) \).

**Proof.** It suffices to prove it for \( A = k[V]_M \cong \Gamma(V, \mathcal{O}_V) \), for \( V \subseteq k^n \) an affine algebraic subvariety. Suppose that \( f/g \) is in the Jacobson radical of \( k[V]_M \) : in particular, for every \( P \in V \) we have \( f/g \in m_P k[V]_M \), where \( m_P \) is the maximal ideal of \( P \in k[V]_M \). In other words, for every \( P \in V \) we have \( f(P) = 0 \), and thus \( f = 0 \) in \( k[V]_M \). In particular, \( f/g = 0 \) and the Jacobson radical of \( k[V]_M \) is \( (0) \). \( \square \)

**Proposition 4.14.** For every affine \( * \)-algebra \( A \), the \( k \)-algebra morphism \( f_A : A \to JK(A) \) is an isomorphism. In particular, the maps \( f_A \) define a natural isomorphism \( Id \cong J \circ K \) of endofunctors of \( \ast \text{Aff}^a_k \).

**Proof.** First, suppose that \( A = \Gamma(V, \mathcal{O}_V) \) for an affine algebraic equivaridity \( (V, \mathcal{O}_V) \), \( X = Spm(A) \) and \( f_A : A \to JK(A) = \Gamma(X, \mathcal{O}_X) \). Let \( a \in A \) be such that \( f_A(a) = 0 \) : for each \( P \in V \), we have \( [a] = 0 \) in \( A/m_P \), so \( a \in m_P \); it follows that \( a \) is in the Jacobson radical of \( A \), which is zero by Lemma 4.13 so \( a = 0 \), and \( f_A \) is injective. As for surjectivity, let \( \varphi_V : V \to X \) be the homeomorphism \( P \mapsto m_P \), \( s \in \Gamma(X, \mathcal{O}_X) \) and \( t = (\varphi_V)^\sharp X (s) \) : by definition, for every \( P \in V \) there exists an open \( O_P \subseteq V \) and \( a_P, b_P \in A \) such that \( t|_{O_P} = a_P|_{O_P}/b_P|_{O_P} \), and \( s|_{\mathcal{O}_P} = [a_P]/[b_P] \). We have \( f_A(t) : m \in X \to [t] \in A/m_P \), so let \( m \in X : \varphi_V \) is a homeomorphism, write \( c = m_P \) for a unique \( P \in V \), and via the isomorphism \( i_P : J(V)/m_P \cong \mathcal{O}_{V,P} \) we may write \( i_P([t]) = [t]_P = [t|_{O_P}]_P = [a_P|_{O_P}/b_P|_{O_P}]_P = [a_P]/[b_P] = i_P(s(m_P)) \), whence \( [t] = s(m_P) \) in \( A/m_P = A/m \), and as this is true for every \( m \in X \), finally we have \( f_A(t) = s \), and \( f_A \) is surjective : it is an isomorphism. In the general case, any affine \( * \)-algebra is by definition isomorphic to an algebra of the form \( \Gamma(V, \mathcal{O}_V) \), so as in the end of the proof of Proposition 4.11 \( f_A \) is an isomorphism as well. \( \square \)

Assembling Propositions 4.11 and 4.12 we get the duality theorem:

**Theorem 4.15.** The global sections functor \( J : (\text{EV} \text{ar}^a_k)^0 \to \ast \text{Aff}^a_k \) is a duality between the categories \( \text{EV} \text{ar}^a_k \) and \( \ast \text{Aff}^a_k \), with adjoint the maximal spectrum functor \( K \).
Conclusions

So far we have built a robust theory of equiresidual affine algebraic varieties over any field and a promising extension of the usual commutative algebra surrounding affine algebraic geometry. We have laid the groundwork for a theory of algebraic equivarieties, which we will develop in a forthcoming publication ([3]), and in which we will expound the important particular case of quasi-projective equivarieties. From this point on, several directions may be pursued. First it is desirable and possible to pursue this subject further and investigate some usual constructions and theorems from classical algebraic geometry in the present setting. For a start, we hope to tackle shortly the study of simple points and tangent spaces, of étale morphisms of algebraic equivarieties, and of differential forms, thanks to the formalism of canonical localisations and $*$-algebras. Further explorations may concern normal varieties and other subjects pertaining to the classical theory, which will have to be reinterpreted in the equiresidual approach. Another obvious series of questions lies in the potential application of the present general theory to the study of the “inner” algebraic geometry of any particular field, using the tools and concepts presented here. A good start would be to sketch some general features of algebraic geometry over the field $\mathbb{Q}$ of rational numbers without working in its algebraic closure. In this perspective, normic forms have played a fundamental role in the present work, but were only used as a ”tool” for the Akinullstellensatz and the Equiradical, whereas in general homogeneous polynomials with only the trivial zero may serve to characterise the special ideals (Remark 3.8). We plan to explore deeper this topic, hopefully connecting through Galois theory the present approach to algebraic geometry in an algebraic closure or a separable closure of the ground field. The example of $\mathbb{Q}$ would again be a good landmark. From another point of view, Proposition 4.8 shows that the maximal spectrum is well behaved with respect to all $*$-algebras of finite $*$-type, and not only with respect to affine, i.e. special, ones, which are reduced. Along this line of thought, after some background on algebraic equivarieties we will naturally be led to an equiresidual version of the algebraic spaces, which would permit the use of infinitesimals in a mild formalism avoiding for the moment the need of an analogue or generalisation of scheme theory.

It is also possible to give back to first order logic what we borrowed and expressed here in the form of pure commutative algebra. We will consider this in [4], which deals with “positive algebraic geometry”, an interplay between the present (equiv)algebraic geometry, positive logic and quasivarieties, laying the foundation for algebraic geometry in fields considered in the light of model theory. With some background on étale morphisms of affine equivarieties, we will hopefully build on this foundation in order to study a ubiquitous type of theories of fields which appear in connexion to number theory (real closed fields, $p$-adically closed fields, complete theories of pseudo-algebraically closed fields,...), and which have been recognised in [16] and systematised in [7] thanks to the work of Robinson ([18]), joining forces with the tradition of coherent logic and connecting with topos theory. The archetypical example of pseudo-algebraically closed fields will fall into this field of investigation and we hope that the present work and some elements of this program will be of some use to the theory of “Field Arithmetics”, where one’s particular interest lies in algebraic geometry over many fields which are not algebraically closed.

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