Borel summation of the semi-classical expansion of the partition function associated to the Schrödinger equation\textsuperscript{*†}

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Abstract

Let $H$ be a perturbation of the semi-classical harmonic oscillator on $\mathbb{R}^\nu$. We prove that the partition function associated to the Hamiltonian $H$ is the Borel sum of its $h$-expansion.

1 Introduction

-1 Let $\nu \geq 1$. Let $\omega_1, \ldots, \omega_\nu > 0$ and $h > 0$. Let

$$
H := -h^2 \partial_\nu^2 + \frac{1}{4}(\omega x)^2 - c(x)
$$

$$
:= -h^2(\partial_{x_1}^2 + \cdots + \partial_{x_\nu}^2) + \frac{1}{4}(\omega_1^2 x_1^2 + \cdots + \omega_\nu^2 x_\nu^2) - c(x_1, \ldots, x_\nu).
$$

(1.1)

Under suitable assumptions on the potential $c$, the operator $H$ is self-adjoint on $L^2(\mathbb{R}^\nu)$ with discrete spectrum and the following partition function

$$
\Theta_H(t, h) := \text{Tr}(e^{-tH}) = \sum_{n=1}^{+\infty} e^{-t \lambda_n(h)/h}
$$

is well defined on $]0, +\infty[^2$. Here

$$
\lambda_1(h) \leq \lambda_2(h) \leq \cdots \leq \lambda_n(h) \leq \cdots
$$

denote the eigenvalues of the operator $H$. Let $\Theta_H^{\text{conj}}(t, h)$ be defined by the identity

$$
\Theta_H(t, h) = e^{te^{c(0)}/h} \times \Theta_H^{\text{conj}}(t, h).
$$

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Then, under suitable assumptions on the potential $c$, for $t \in [0, +\infty[$, the following expansion holds

$$
\Theta_H^{\text{conj}}(t, h) = a_0(t) + a_1(t)h + \cdots + a_{r-1}(t)h^{r-1} + h^r O_{h \rightarrow 0+}(1). \quad (1.2)
$$

In this paper we give conditions on the potential $c$ so that this expansion is Borel summable (with respect to $h$) and its Borel sum is equal to $\Theta_H^{\text{conj}}(t, h)$ (see Theorem 2.3 for a more precise statement). A Borel summation statement concerning the heat kernel is also proved (see Proposition 2.2).

We also give and use a Borel summation statement for multidimensional Gaussian integrals (Proposition 6.3). This statement can be viewed as a consequence of Proposition 6.6 which deal with so-called hypergeometric vection transforms. These transforms map $\hat{\mathcal{N}}_{R, K}$ (see Definition 6.2), the space of summable symbols, into itself. Hypergeometric vection transforms are related to vection transforms which satisfy a similar property. Vection transforms play a role in celeration theory [E4]. See Section 6 and Appendix B for more details.

Quantum mechanics gives many examples of divergent expansions [Si]. Let us focus on semi-classical expansions related to the Schrödinger equation. In this case, an important motivation is to describe quantum quantities with the help of classical quantities [B-B]. For instance the coefficients

$$
c(0), a_0(t), \ldots, a_r(t), \ldots
$$

are classical quantities whereas $\Theta_H(t, h)$ is a quantum quantity (see also introduction in [Ha4]). How to recover a quantum quantity with the help of the coefficients of its $h$-expansion? Notice that asymptotic points of view do not provide an answer to this question. A Borel summation viewpoint is used by Voros [V1] and Delabaere, Dillinger, Pham [D-D-P] for the study of the one dimensional Schrödinger operator $-h^2 \frac{d^2}{dx^2} + V(x)$, where the potential $V$ is polynomial. In particular, a study of tunnelling is proposed in [D-D-P]. This involves a non-elementary Borel summation process.

A semi-classical interpretation can be proposed for the small time expansion of the heat kernel or the partition function (see for instance [Ha4]). Actually there are a lot of similarities between Borel summability of $h$-expansions and small time (high temperature) expansions [Ha4].

We also find technical similarities between the study of the groundstate energy of the massless spin-boson model [A] and our work. In this paper, the description of the groundstate energy is viewed with the help of the heat kernel, the interaction between the bosonic quantum field and the spin model is viewed as a perturbation (without renormalization), and above all, the tree graph equality plays an important role (we give more details about this equality and its use in the sequel).

The operator $H$ describes the behaviour of a quantum particle moving in a classical field. The interaction between the particle and the field is understood via the deformation formula. This formula, which can be derived from the Dyson expansion [On], is a multiple scattering expansion (see [Fe, Figure 1]).
The convergence of the $h$-expansion of the heat kernel is studied with the use of this formula and with the help of the tree graph equality, an identity used in statistical mechanics and quantum field theory [Br], dealing with an infinite number of particles. The Hamiltonian $H$ does not involve, namely, second quantization but our method, through the deformation formula, involves virtual particles which copy the studied particle at different times (see [Ha4, Appendix, Formula 5.1]).

We find similarities between the use of the tree graph equality and the arborification-coarborification process, an important tool, for instance, when linearization of vector fields or diffeomorphisms are considered [E2, E-V]. In both cases, in a general setting, by an algebraic way, one can rearrange terms in order to restore convergence.

-3- Let us comment the assumptions on the function $c$. The function $c$ is viewed as the Fourier transform of some Borel measure $\mu$:

$$c(x) = \int_{\mathbb{R}^N} \exp(i x \cdot \xi) d\mu(\xi).$$

- Proving a Borel summation statement for the heat kernel

$$\langle x| e^{-\frac{t}{h} H} |y \rangle$$

(see Proposition 2.2) requires the assumption (2.2). This assumption holds for instance if $\nu = 1$, $c(x) = k \cos x$ where $k$ is an arbitrary real number and $T$ is small enough. Here, $\mu := \frac{k}{2}(\delta_{\xi=1} + \delta_{\xi=-1})$. This allows one to build a potential

$$V := \frac{1}{4}(\omega x)^2 - c(x)$$

with an arbitrary number of wells. Of course, the Borel summation statement for the corresponding heat kernel holds for small values of $t$.

Notice that giving a Borel summation statement for the the Green function or resolvent (see [B-B]) with the same assumptions on the potential seems out of reach with our method. Since, under suitable assumptions,

$$\langle x|(H + \lambda)^{-1}|y \rangle = \int_0^\infty e^{-\lambda t} \langle x|e^{-tH}|y \rangle dt$$

$$= \int_0^\infty e^{-\lambda \frac{t}{h}} \langle x|e^{-\frac{t}{h} H}|y \rangle \frac{dt}{h},$$

the knowledge of the Green function through the heat kernel involves large values of $t$; this is consistent with the above remarks.

- Proving a Borel summation statement for the partition function $\Theta_{H}^{\text{conj}}$ (see Theorem 2.3) requires also the assumption (2.2) and the supplementary assumptions (2.6), (2.7), (2.8). These assumptions hold for instance if $\nu = 1$,

$$c(x) = k e^{-x^2/2}, \varepsilon = 1/4,$$

(1.3)
\[ k \in \mathbb{R} \text{ and } |k| \text{ is small enough. Here } d\mu(\xi) = k(2\pi)^{-1/2}e^{-\xi^2/2}d\xi. \] Notice that the potential \( V \) given by the above choice for \( c \) (\( k \neq 0 \)) has no real critical points except 0 if \( |k| \) is small enough: we do not consider tunnelling.

We do not give a Borel summation statement concerning the eigenvalues associated to the operator \( H \): all complex critical times of the potential \( V \) certainly must be taken into account for such a result. Let us comment this. Under suitable assumptions on \( V \), the first eigenvalue (ground state energy) satisfies

\[ \lambda_1(h) = \lim_{t \to +\infty} \frac{1}{t} \log \Theta_H(t, h), \]

thus involving the partition function for large values of \( t \). Let us consider again the example given by (1.3). Then the potential \( V \) has at least one complex critical point different from 0. But the larger the parameter \( t \) is, the smaller the parameter \( k \) must be chosen such that the non-zero critical points are far from the real space and do not interfere with the Borel summation process. This qualitatively explains the utility of (2.6).

4- Let us outline the proof of our result. We use the deformation formula (see [Ha4, Ha6]) which gives a representation of the conjugate heat kernel \( p^{\text{conj}} \) defined by

\[ \langle x | e^{-tH} | y \rangle = p^{\text{harm}}(t, x, y, h) \times p^{\text{conj}}(t, x, y, h); \]

here \( p^{\text{harm}} \) denotes the heat kernel of the operator \(-h^2\partial_x^2 + \frac{1}{4}(\omega_x)^2\). By the deformation formula, \( p^{\text{conj}} \) is viewed as a divergent expansion with respect to \( \frac{1}{h} \) and \( h \). We then consider the formal logarithm of this expansion. This yields

\[ p(t, x, y, h) = h^{-\nu/2} \prod_{\nu=1}^{\nu} \left( \frac{\omega_u}{4\pi \text{ sh}(\omega_u, t)} \right)^{1/2} \times e^{-\phi(t, x, y)/h + w(t, x, y, h)} \]  

(1.4)

where the function \( \phi \) does not depend of \( h \) and the function \( w \) is a divergent expansion with respect to \( h \). These functions are defined through iterated integrals, inherited from the deformation formula, and a sum indexed by trees, inherited from the tree graph equality. We prove that the \( h \)-expansion of the function \( e^w \) is Borel summable. The partition function satisfies

\[ \Theta_H(t, h) = \int_{\mathbb{R}^\nu} p(t, x, x, h)dx. \]

and by (1.4)

\[ p(t, x, x, h) = h^{-\nu/2} \prod_{\nu=1}^{\nu} \left( \frac{\omega_u}{4\pi \text{ sh}(\omega_u, t)} \right)^{1/2} \times e^{-\Phi_t(x)/h + w_t(x, h)} \]

where \( \Phi_t(x) := \phi(t, x, x) \) and \( w_t(x, h) := w(t, x, x, h) \). We then establish the following results.

1. The function \( \Phi_t \) is analytic on a particular neighbourhood of \( \mathbb{R}^\nu \) in \( \mathbb{C}^\nu \).
2. A Morse lemma holds for $\Phi_t$ on this neighborhood.

3. The $h$-expansion of the function $e^{\omega t}$ is Borel summable, uniformly in $(x, y)$. Finally a Borel summation statement for multidimensional Gaussian integrals (Proposition 6.3) is proved. Then the proof of the Borel summability of the expansion with respect to $h$ of the partition function can be achieved.

Let us add the following remarks.

The function $\phi_t$ is a solution of a first order non-linear partial differential equation, the Hamilton Jacobi (eikonal) equation. The explicit form of the solution allows to avoid the method of characteristics, a standard way to solve this equation. In a heuristic setting and if $\omega = 0$, a similar formulation of the solution (without taking into account the tree-graph equality, which restores convergence) can be found in [Fu-Os-Wi]; however, we use the formalism developed in [Ha3].

The Borel summability of the $h$-expansion of the function $e^{\omega}$ comes from the Borel summability of the expansion of $w$. For this purpose, we proceed as in [Ha4]: the deformation formula gives an explicit Borel transform of the function $w$ based on Bessel functions. Here the tree graph equality plays a central role (see the assertion 3 in Remark 3.4).

A global version of the Morse lemma allows to express the function $\Phi_t$ in a suitable way for the Borel summation statement dealing with Gaussian integrals.

We express the function $c$ as the Fourier transform of a measure $\mu$: it is a convenient way to use the deformation formula [It, Ha4]. Our statement involves two norms on $\mu$, denoted by $M_{\mu, \varepsilon}$ and $M_{\mu}'$ (see Definition 2.1). We make the following assumptions.

- The norm $M_{\mu, \varepsilon}$ is small. This implies the Borel summability of the function $e^{\omega}$. This assumption seems natural when a perturbation viewpoint and Borel summation are considered: it is also used in the proof of Borel summability of the small time expansion of the heat kernel (see [Ha4]).

- The function $c$ is $\mathbb{R}$-valued on $\mathbb{R}^\nu$, its first order derivatives vanish at the origin and the norm $M_{\mu}'$ is small (Theorem 2.3). This implies the analyticity of $\Phi_t$ and the Morse lemma on a neighbourhood of $\mathbb{R}^\nu$ in $\mathbb{C}^\nu$. For this last step, we use the following: $\Phi_t$ is a global perturbation of the function

\[
x \mapsto \frac{1}{2} \sum_{\nu=1}^{\nu} \frac{\omega_{\nu}}{\text{sh}(\omega_{\nu} t)} (\text{ch}(\omega_{\nu} t) - 1) x_{\nu}^2,
\]

which is the trace on the diagonal of $\mathbb{R}^{2\nu}$ of the phase (up to a factor $1/h$) of $p_{\text{harm}}$ (see (1.1)). This explains why we assume that the measure $\mu$ admits a differentiable density with respect to Lebesgue measure (see the definition of $M_{\mu}'$).

Here the problem is non-linear and therefore assuming that the data are small is not surprising. As we saw above, the assumptions on the perturbation $c$ are highly restrictive. However this perturbation kills the symmetries due to the choice of the harmonic oscillator.
2 Notation and main results

For $z = |ze^{i\theta} \in \mathbb{C}, \theta \in \pi - \pi,\pi]$, we denote $z^{1/2} := |z|^{1/2}e^{i\theta/2}$. For $z \in \mathbb{C}$, we denote $sh z := \frac{1}{2}(e^z - e^{-z})$, $ch z := \frac{1}{2}(e^z + e^{-z})$, $th z := sh z / ch z$ (if $ch z \neq 0$).

For $z, z' \in \mathbb{C}^\nu$, let $z \cdot z' := z_1z'_1 + \cdots + z_\nu z'_\nu$, $z^2 := z \cdot z$, $\bar{z} := (\bar{z}_1, \ldots, \bar{z}_\nu)$, $\lambda z := (\lambda z_1, \ldots, \lambda z_\nu)$, $|z| := (z \cdot z)^{1/2}$ (if $z \in \mathbb{R}^\nu$, $|z| = \sqrt{z^2}$). We extend the first two notations to operators such as $\partial_z = (\partial_{z_1}, \ldots, \partial_{z_\nu})$. Let $a \in \mathbb{C}^\nu$.

We denote by $a$ (bold character) the linear transformation defined on $\mathbb{C}^\nu$ by

$$z \mapsto az = (a_1z_1, \ldots, a_\nu z_\nu)$$

where $a_1, \ldots, a_\nu$ (respectively $z_1, \ldots, z_\nu$) denote the coordinates of the vector $a$ (respectively $z$) in the standard basis of $\mathbb{C}^\nu$. For $R > 0$, let

$$D_R := \{z \in \mathbb{C}^\nu | |z| < R\}.$$ 

If $U$ and $V$ are two subsets of $\mathbb{C}^\nu$, let $U + V := \{u + v | u \in U, v \in V\}$. We denote $D_{U,R} := U + D_R$. For instance

$$D_{B_{\mathbb{R}^\nu},R} := \{a + ib | a \in \mathbb{R}^\nu, b \in \mathbb{R}^\nu, |b| < R\}.$$ 

We also denote by $(e_1, \ldots, e_\nu)$ the standard basis of $\mathbb{C}^\nu$ and by $(e_1^*, \ldots, e_\nu^*)$ its dual basis. For $A = (a_{j,k})_{1 \leq j,k \leq \nu}$ a $\nu \times \nu$ matrix with complex entries, we denote $|A| := \sup_{|z|=1} |Az|$ and $|A|_{\infty} := \max_{1 \leq j,k \leq \nu} |a_{j,k}|$. Then $|A| \leq |\nu| |A|_{\infty}$.

We set $A^*$ for the transpose of the matrix $A$ and $I$ denotes the identity matrix.

Let $\Omega$ be an open domain in $\mathbb{R}^m \times \mathbb{C}^m$. Let $F$ be a finite dimensional space. We denote by $A_{F}(\Omega)$ (respectively $A(\Omega)$) the space of $F$-valued (respectively $\mathbb{C}$-valued) analytic functions on $\Omega$. If $I$ is an interval of $\mathbb{R}$, $C^k(I, A_{F}(\Omega))$ denotes the space of functions $f$ defined on $I$ with values in $A_{F}(\Omega)$ such that $f, \ldots, f^{(k)}$ exist and are continuous. We always consider these spaces with their standard Frelchet structure (the semi-norms are indexed by compact sets and eventually differentiation order).

Let $\mathcal{B}$ be the collection of all Borel sets on $\mathbb{R}^m$. A $\mathbb{C}$-valued measure $\mu$ on $\mathbb{R}^m$ is a $\mathcal{B}$-valued function on $\mathcal{B}$ satisfying the classical countable additivity property [Ru]. We denote by $|\mu|$ the positive measure defined by

$$|\mu|(E) = \sup \sum_{j=1}^{\infty} |\mu(E_j)|(E \in \mathcal{B}),$$

the supremum being taken over all partition $\{E_j\}$ of $E$. In particular $|\mu|(\mathbb{R}^m) < \infty$. If there exists some measurable function $\rho_\mu$ such that $d\mu(\xi) = \rho_\mu(\xi)d\xi$, then $d|\mu|(\xi) = |\rho_\mu(\xi)|d\xi$.

We refer to [Ha4] for a rigorous definition of Borel and Laplace transform. Roughly speaking, assuming that $f$ (respectively $\hat{f}$) is a function of a complex variable $h$ (respectively $\zeta$), $f$ is the Laplace transform of $\hat{f}$ if

$$f(h) = \int_0^{+\infty} \hat{f}(\zeta)e^{-\frac{\zeta h}{h}}d\zeta,$$

(2.1)
whereas the (formal) Borel transform of the formal power series $\hat{f} = \sum_{r \geq 0} a_r h^r$ is defined by

$$\hat{f}(\zeta) = \sum_{r=0}^{\infty} \frac{a_r}{r!} \zeta^r.$$  

With suitable assumptions, these two transforms are inverse of each other. We say that a formal power series $\tilde{f} = \sum_{r \geq 0} a_r h^r$ is Borel summable if its Borel transform $\hat{f}$ has a non-vanishing radius of convergence near 0, has an analytic continuation (still denoted by $\hat{f}$) on a domain $D_{\mathbb{R}^+, \kappa}$ with $\kappa > 0$ and is exponentially dominated on this domain. The Borel sum of $\tilde{f}$ is by definition the Laplace transform of this analytic continuation (see [Ha4] for rigorous definitions). In the whole paper, sums indexed by an empty set are, by convention, equal to zero.

**Definition 2.1** Let $\varepsilon > 0$ and let $\mu$ be a $\mathbb{C}$-valued measure on $\mathbb{R}^\nu$. Let us denote

$$M_\mu := \int_{\mathbb{R}^\nu} e^{5|\xi|d_{\nu}||\mu||}(\xi),$$

$$M_{\mu, \varepsilon} := \int_{\mathbb{R}^\nu} e^{\varepsilon x^2 + 5|\xi|d_{\nu}||\mu||}(\xi).$$

If there exists a differentiable function on $\mathbb{R}^\nu$ such that

$$d\mu(\xi) = \rho_{\mu}(\xi)d\xi,$$

let us denote

$$M'_{\mu} := \int_{\mathbb{R}^\nu} \max(|\rho_{\mu}(\xi)|, |\partial_{\xi_1}\rho_{\mu}(\xi)|, \ldots, |\partial_{\xi_\nu}\rho_{\mu}(\xi)|) e^{5|\xi|d_{\nu}||}\xi.$$

If $M_{\mu} < \infty$, we shall associate to $\mu$ the operator $H$ defined by (1.1) where

$$c(x) = \int_{\mathbb{R}^\nu} \exp(ix \cdot \xi)d\mu(\xi).$$

Notice that the function $c$ is $\mathbb{C}$-valued analytic and bounded on $\mathbb{R}^\nu$.

In the following proposition, we give a Borel summation statement concerning the heat kernel associated to the operator $H$.

Let $\kappa > 0$. Let

$$S_\kappa := \{z \in \mathbb{C}||\mathcal{I}mz^{1/2}|^2 < \kappa\} = \{z \in \mathbb{C}|\Re e z > \frac{1}{4\kappa}\mathcal{I}m^2 z - \kappa\}.$$

$S_\kappa$ is the interior of a parabola (see [Ha4, fig 2.2.]) and

$$D_{\mathbb{R}^+, \kappa} \subset S_\kappa.$$

We also denote

$$\mathbb{C}^+ := \{z \in \mathbb{C}|\Re e z > 0\}.$$
Proposition 2.2 Let $T, \varepsilon, \omega_1, \ldots, \omega_\nu > 0$. Let $\mu$ be a $\mathbb{C}$-valued measure on $\mathbb{R}^\nu$ such that
\[ 4T^2 e^T M_{\mu, \varepsilon} < 1. \] (2.2)
Then there exist $\kappa, K_1 > 0$, $\phi \in C^0([0, T], \mathcal{A}(\mathbb{R}^{2\nu}))$ and $\hat{W} \in C^0([0, T], \mathcal{A}(\mathbb{R}^{2\nu} \times S_\kappa))$ such that, for every $(t, x, y) \in [0, T] \times \mathbb{R}^{2\nu}$,
- for every $\sigma \in S_\kappa$,
\[ |\hat{W}(t, x, y, \sigma)| \leq K_1 e^{K|\sigma|^{1/2}}, \] (2.3)
- for every $h \in \mathbb{C}^+$
\[ \langle x|e^{-\frac{i}{h} H}|y \rangle = h^{-\nu/2} e^{-\frac{i}{h} \phi(t, x, y)} \int_0^{+\infty} \hat{W}(t, x, y, \sigma) e^{-\frac{i}{h} \sigma} d\sigma. \] (2.4)

Let us assume that the function $c$ takes its values in $\mathbb{R}$. Then the operator $H$ defined by (1.1) is self-adjoint on $L^2(\mathbb{R}^\nu)$, its spectrum is discrete and its partition function $\Theta_H$ is well defined on $[0, +\infty]^2$.

For $\omega_1, \ldots, \omega_\nu > 0$, we denote
\[ \omega_\flat := \min(\omega_1, \ldots, \omega_\nu), \quad \omega_\sharp := \max(\omega_1, \ldots, \omega_\nu). \]
The next theorem is the main goal of this paper.

Theorem 2.3 Let $T, \varepsilon, \omega_1, \ldots, \omega_\nu > 0$. Let $\mu$ be a $\mathbb{C}$-valued measure on $\mathbb{R}^\nu$ such that
\[ 4T^2 e^T M_{\mu, \varepsilon} < 1, \] (2.5)
\[ \frac{\omega_\sharp (1 + \omega_\flat T) \sinh(\frac{\omega_\flat T}{\omega_\sharp}) M'_\mu}{\omega_\flat^2 (1 - 4T^2 M_\mu)} < \alpha_\nu, \] (2.6)
where the constant $\alpha_\nu$ only depends on the dimension $\nu$. Moreover, let us assume that
\[ \tilde{\rho}_\mu(\xi) = \rho_\mu(-\xi), \] (2.7)
\[ \int_{\mathbb{R}^\nu} \xi_1 d\mu(\xi) = \cdots = \int_{\mathbb{R}^\nu} \xi_\nu d\mu(\xi) = 0, \] (2.8)
Let $T_0 \in [0, T]$. Then there exist $\kappa, K, K_1 > 0$ and $\theta \in C^0([T_0, T], \mathcal{A}(S_\kappa))$ such that, for every $t \in [T_0, T]$,
- for every $\sigma \in S_\kappa$,
\[ |\hat{\theta}(t, \sigma)| \leq K_1 e^{K|\sigma|^{1/2}}, \] (2.9)
- for every $h \in \mathbb{C}^+$,
\[ \Theta_H^{\text{conj}}(t, h) = \int_0^{+\infty} \hat{\theta}(t, \sigma) e^{-\frac{i}{h} \sigma} d\sigma. \] (2.10)

\[ \text{See Proposition 5.2.} \]
Therefore the expansion (1.2) is Borel summable with respect to $h$ and its Borel sum is equal to $\Theta_H^{\text{conj}}(t, h)$.

Remark 2.4

- Let $V$ be the potential defined by
  
  \[ V(x) := \frac{(\omega x)^2}{4} - c(x). \]

(2.4) means that $V|_{\mathbb{R}^n}$ takes real values and (2.8) means that

\[ \partial_{x_1} V(0) = \cdots = \partial_{x_n} V(0) = 0. \]

- If $c$ is the null function then
  
  \[ \Theta_H^{\text{conj}}(t, h) = \Theta_H(t, h) = \prod_{\nu=1}^{\nu} \frac{1}{e^{\omega \nu t/2} - e^{-\omega \nu t/2}}. \]

- Both of the proofs of Proposition 2.2 and Theorem 2.3 use the tree graph equality. However, proving Proposition 2.2 is easier than proving Theorem 2.3. For the proof of the theorem, the Gaussian Borel summation statement (see section 6) is necessary and a Morse lemma is needed (see Proposition 5.2).

3 Mould formalism and combinatorics related to graphs

Let $\Omega$ be an arbitrary set. Let us denote by $\text{seq}(\Omega)$ (respectively $\mathcal{P}_0(\Omega)$) the set of finite (eventually empty) ordered sequences of elements of $\Omega$ (respectively finite subsets of $\Omega$). Let $\mathcal{A}$ be a commutative $\mathbb{C}$-algebra. Let $\mathcal{M}^\text{cl}(\Omega) = \mathcal{A}^{\text{seq}(\Omega)}$ (respectively $\mathcal{M}^\text{ab}(\Omega) = \mathcal{A}^{\mathcal{P}_0(\Omega)}$). Equipped with the following sum and product

\[ C = A + B \iff C^{\lambda_1, \ldots, \lambda_r} = A^{\lambda_1, \ldots, \lambda_r} + B^{\lambda_1, \ldots, \lambda_r}, \]

\[ C = A \times B \iff C^{\lambda_1, \ldots, \lambda_r} = \sum_{i=0}^r A^{\lambda_1, \ldots, \lambda_i} B^{\lambda_{i+1}, \ldots, \lambda_r}, \]

respectively

\[ C = A + B \iff C^I = A^I + B^I, \]

\[ C = A \times \text{sym} \ B \iff C^I = \sum_{J \cup K = I \land J \cap K = \emptyset} A^J B^K, \]

$\mathcal{M}^\text{cl}(\Omega)$ and $\mathcal{M}^\text{ab}(\Omega)$ are algebras. For instance

\[ C^{(1,2)} = A^{(1,2)} B^\emptyset + A^{(1)} B^{(2)} + A^{(2)} B^{(1)} + A^\emptyset B^{(1,2)}. \]
The algebra $\mathcal{M}_{ab}(\Omega)$ is commutative. Elements of $\mathcal{M}_{cl}(\Omega)$ or $\mathcal{M}_{ab}(\Omega)$ are called moulds. We denote by 1 the identity element of $\mathcal{M}_{ab}(\Omega)$. Then $1^I = 1$ and $1^I = 0$ if $|I| \geq 1$. Let $A \in \mathcal{M}_{ab}(\Omega)$ be such that $A^\emptyset = 0$ and let $\varphi \in \mathbb{C}[[H]]$. Then the mould $\varphi(A)$ is well defined (only finite sums occur in its definition). The following elementary fact will be useful. Let $A \in \mathcal{M}_{ab}(\Omega)$ be such that $A^\emptyset = 0$ and let $\varphi \in \mathcal{BV}[H]$. Then the mould $\varphi(A)$ is well defined (only finite sums occur in its definition).

**Remark 3.1** A lot of important symmetries occur in $\mathcal{M}_{cl}(\Omega)$. In particular, a mould $A$ is said to be symmetrical if for every sequence $\varsigma^1$ and $\varsigma^2$

$$A^{\varsigma^1}A^{\varsigma^2} = \sum_{\varsigma \in \text{sh}(\varsigma^1, \varsigma^2)} A^\varsigma$$

where $\text{sh}(\varsigma^1, \varsigma^2)$ denotes the set of all sequences $\varsigma$ obtaining from $\varsigma^1$ and $\varsigma^2$ under shuffling. The following lemma (Lemma 3.2) is related to this fundamental notion (see also [Ha3, Prop.3.1]). See [E1, E2, E-V] for general aspects of the mould formalism.

For $m = 1, 2, \ldots$, let

$$T_m := \{(s_1, \ldots, s_m) \in [0, 1]|0 < s_1 < \cdots < s_m < 1\} \times \mathbb{R}^{\nu m}$$

and let us denote $T_{\infty} := \{\emptyset\} \cup T_1 \cup T_2 \cup \cdots$. Let $\mathcal{M}_{pre}$ be the algebra of $\mathcal{A}$-valued functions $f$ defined on $T_{\infty}$, such that $f|_{T_m}$ is measurable for every $m \geq 1$, equipped with the trivial sum and the following commutative product

$$h = f \times_{\text{pre}} g \iff h(s_1, \ldots, s_m; \xi_1, \ldots, \xi_m) = \sum_{J \cup K = \{1, \ldots, m\}} f(s_{j_1}, \ldots, s_{j_p}; \xi_{j_1}, \ldots, \xi_{j_p}) g(s_{k_1}, \ldots, s_{k_q}; \xi_{k_1}, \ldots, \xi_{k_q}).$$

Here $j_1 < \cdots < j_p$ (respectively $k_1 < \cdots < k_q$) denote the elements of $J$ (respectively $K$). Let $\lambda$ be a $\mathcal{C}$-valued Borel measure defined on $[0, 1] \times \mathbb{R}^{\nu}$ and let us denote by $\lambda^\otimes m$ (respectively $|\lambda|^\otimes m$) the Borel measure defined on $T_m$ by

$$d\lambda^\otimes m(s, \xi) = d\lambda(s_1, \xi_1) \cdots d\lambda(s_m, \xi_m),$$

$$d|\lambda|^\otimes m(s, \xi) = d|\lambda|(s_1, \xi_1) \cdots d|\lambda|(s_m, \xi_m)$$

(respectively).

Let $\mathcal{M}_{1}^{\text{pre}}$ be the subalgebra of $\mathcal{M}_{\text{pre}}$ of all functions $f$ such that for every $m \geq 1$ $f|_{T_m}$ is integrable on $T_m$ with respect to $|\lambda|^\otimes m$. 

10
Lemma 3.2  The mapping $\Phi$

\[
\mathcal{M}_1^{pre} \rightarrow \mathcal{A}[[H]]
\]

\[
f \mapsto \sum_{m \geq 0} H^m \int_{T^m} f(s_1, \ldots, s_m; \xi_1, \ldots, \xi_m) d\lambda^\otimes m(s, \xi)
\]

is an algebra morphism.

Proof  Let $m \geq 1$. One gets

\[
\sum_{p+q=m} \int_{T^p} f(s_1, \ldots, s_p; \xi_1, \ldots, \xi_p) d\lambda^\otimes p(s, \xi) \times \int_{T^q} g(s_1, \ldots, s_q; \xi_1, \ldots, \xi_q) d\lambda^\otimes q(s, \xi)
\]

\[
= \int_{T^m} (f \times_{pre} g)(s_1, \ldots, s_m; \xi_1, \ldots, \xi_m) d\lambda^\otimes m(s, \xi).
\]

It is a consequence of the shuffling property concerning the following characteristic functions:

\[
1_{0<v_1<\ldots<v_p<1} \times 1_{0<v_1<\ldots<v_q<1} = \sum_{(w_1, \ldots, w_{p+q}) \in \text{sh}(u_1, \ldots, u_p), (v_1, \ldots, v_q)} 1_{0<w_1<\ldots<w_{p+q}<1}.
\]

Then the mapping defined in Lemma 3.2 is a multiplicative morphism. □

3.1 Some identities

Let $I$ be a subset of $\mathbb{N}$ such that $2 \leq |I| < \infty$. We denote by $\mathcal{G}_I$ the set of (unordered) connected graphs on $I$. A connected graph with no cycles is called a (unordered) tree and we denote by $\mathcal{T}_I$ the set of trees on $I$. For instance

\[
\mathcal{T}_{\{1,2,4\}} = \left\{ \left\{ [1, 2], [1, 4] \right\}, \left\{ [1, 2], [2, 4] \right\}, \left\{ [1, 4], [2, 4] \right\} \right\}.
\]

If $|I| = n$ then $|\mathcal{T}_I| = n^{n-2}$. Let $g \in \mathcal{G}_I$. An element $\ell$ of $g$ is called a edge and $\ell = [j, k]$ where $j, k \in I, j \neq k$ and we always assume that $j < k$ by convention.

Proposition 3.3  (tree graph equality) For $1 \leq j < k < \infty$, let $\tilde{z}_{j,k} \in \mathbb{C}$. Let $A$ and $B$ be the moulds defined by

\[
A^\emptyset = 1, \ A^{\{j\}} = 1, \ B^\emptyset = 0, \ B^{\{j\}} = 1
\]

and for $I \subset \mathbb{N}^*$, $2 \leq |I| < \infty$,

\[
A^I = \exp \left( \sum_{j, k \in I, j < k} \tilde{z}_{j,k} \right),
\]

\[
(B)^I = \sum_{g \in \mathcal{T}_I} \left( \prod_{[j, k] \in g} \tilde{z}_{j,k} \right) \int_{\theta \in [0, 1]^g} e^{\sum_{j, k \in I, j < k} \theta_{j,k} \tilde{z}_{j,k} \theta} d|I|-1 \theta
\]
(the tree $g$ contains $|I|-1$ elements). Here

$$d^{|I|-1} := \prod_{[j,k] \in g} d\theta_{[j,k]}$$

and

$$\theta_{j,k,g} = \min_{[p,q]} \theta_{[p,q]}$$

where $[p,q]$ runs over all edges belonging to the unique path joining $j$ and $k$ in the tree $g$. Then

$$A = e^B.$$  \hfill (3.3)

Remark 3.4

1. Here is an example illustrating the definition of $\theta_{j,k,g}$. Let

$$g = \{ [1,2], [2,3], [2,8], [5,8], [5,6], [4,6] \}.$$  

Then $\theta_{2,6,g} = \min(\theta_{2,8}, \theta_{5,8}, \theta_{5,6})$.

2. The exponential in (3.3) is defined by

$$e^A := \sum_{m \geq 0} \frac{1}{m!} A \times_{\text{sym}} \cdots \times_{\text{sym}} A \mbox{ m times}.$$  

3. The mould $B$ has a simpler expression

$$B^I = \sum_{g \in \mathcal{T}_I} \prod_{[j,k] \in g} (e^{\tilde{z}_{j,k}} - 1).$$  \hfill (3.4)

But $|\mathcal{G}_I| \approx 2^{|I|(|I|-1)/2}$ whereas $|\mathcal{T}_I| = |I|^{3} - 2 = e^{(|I|-2)\ln(|I|)}$. Therefore (3.4) is more efficient than (3.3) to prove the convergence of series containing terms like $B^I$. However, (3.3) can be useful for explicit computations [Fu-Os-Wi, Ha3].

Remark 3.5 Equality (3.3) is proved in [B-K] or [Br] (set $u_{k,l}(s) = -\tilde{z}_{k,l}1_{[0,1]}(s)$ and $t = 1$ in [B-K, Th.3.1]). Let us give the idea of this proof translated in the combinatorial language of moulds (see also Section 7 in [Br] where the definition of the product $\times_{\text{sym}}$ is given). Let us consider the moulds defined by

$$A^g_t = 1, A^j_t = 1, B^0_t = 0, B^j_t = 1,$$

$$A^I_t := \exp \left( t \sum_{j,k \in I} \tilde{z}_{j,k} \right),$$

$$(B^I_t) := \sum_{g \in \mathcal{T}_I} \left( \int_{0}^{t} \right)^{|I|-1} \left( \prod_{[j,k] \in g} \tilde{z}_{j,k} e^{\sum_{j,k \in I,j<k} (t-t_{j,k}) \tilde{z}_{j,k}} dt_{[j,k]} \right),$$
Here \([p, q]\) runs over all edges belonging to the unique path joining \(j\) and \(k\) in the tree \(g\). Let us notice that \(e^{B_t}|_{t=0} = A_t|_{t=0}\) and that the equality (3.3) is equivalent to \(e^{B_t}|_{t=1} = A_t|_{t=1}\). The mould \(A_t\) satisfies the differential equation

\[
\partial_t A_t = \left( \sum_{j < k} \tilde{z}_{j,k} \Delta_j \Delta_k \right) A_t
\]

where \(\Delta_j\) is the elementary mould’s derivation defined by \((\Delta_j A)^I = 1_{j \in I} A^I\). Therefore, since \(\Delta_j\) is a derivation and \(\times_{\text{sym}}\) is commutative, (3.3) is satisfied if

\[
\partial_t B_t = \left( \sum_{j < k} \tilde{z}_{j,k} \Delta_j \Delta_k \right) B_t + \left( \sum_{j < k} \tilde{z}_{j,k} \Delta_j B_t \times_{\text{sym}} \Delta_k B_t \right).
\]

Differentiating the integrand in (3.5) with respect to \(t\) explains the first term of the right hand side of (3.6). The core of the proof is to check that the non-linear term in (3.6) comes from the differentiation of the upper limit of the integrals in (3.5).

Let \(h \in \mathbb{C}^*\) and for \(j, k = 1, 2, \ldots, j \leq k\), let \(z_{j,k} \in \mathbb{C}\). Let \(C_h \in M^\text{ab}(\mathbb{N}^*)\) be defined by

\[
C_h^\varnothing = 1, C_h^I = \frac{1}{h^{|I|}} \times \int e^{h \vartheta z_{j,k} \Delta_k} d\vartheta,
\]

Later we shall consider the behaviour of quantities involving the mould \(C_h\) when \(h\) goes to 0. The next proposition simplifies their study. Let \(E, R_h \in M^\text{ab}(\mathbb{N}^*)\) be defined by

\[
E^\varnothing = 0, E^\{j\} = 1, E^I = 2^{|I|-1} \sum_{g \in \mathcal{T}_I} \prod_{[j, k] \in g} z_{j,k},
\]

\[
R_h^\varnothing = 0, R_h^\{j\} = z_{j,j} \int_0^1 e^{h \vartheta z_{j,j}} d\vartheta,
\]

\[
R_h^I = 2^{|I|-1} \sum_{g \in \mathcal{T}_I} \left( \prod_{[j, k] \in g} z_{j,k} \right) \times \int_{[0,1]^n \times [0,1]} \left( \sum_{j \in I} z_{j,j} + 2 \sum_{j, k \in I, j < k} \theta_{j,k,g} z_{j,k} \right) d|I|-1 d\vartheta.
\]

Remark 3.6 The mould \(E\) does not depend on \(h\) and the mould \(R_h\) (unlike the mould \(C_h\)) is not singular when \(h\) goes to 0.

Proposition 3.7

\[
C_h = \exp \left( \frac{1}{h} E + R_h \right).
\]
Proof One gets
\[(\log C_h)^\emptyset = 0, \quad (\log C_h)^\{j\} = h^{-1} e^{hz_{j,j}}.\]
Then the identity \((\log C_h)^I = \left(\frac{1}{h} E + R_h\right)^I\) for \(|I| \leq 1\) is straightforward. Let \(D_h\) be the mould defined by
\[D_h^\emptyset = 0, \quad D_h^\{j\} = 1, \quad D_h^I = e^{2h \sum_{j,k \in I, j < k} z_{j,k}}.\]
Let \(I \subset \mathbb{N}^*\) be such that \(2 \leq |I| < \infty\). Then
\[(\log C_h)^I = h^{1-|I|} \prod_{j \in I} e^{hz_{j,j}} \times (\log(1 + D_h))^I\]
The last equality is obtained by choosing \(\tilde{z}_{j,k} = 2hz_{j,k}\) in Proposition 3.3. Then choosing \(\varphi = h\left(z_{j,j} + 2 \sum_{j,k \in I, j < k} \theta_{j,k,g} z_{j,k}\right)\) in the identity
\[e^\varphi = 1 + \varphi \int_0^1 e^{\varphi \theta} d\theta\]
yields the decomposition \((\log C_h)^I = \left(\frac{1}{h} E + R_h\right)^I\). This proves Proposition 3.7. □

Corollary 3.8 Let \(\alpha\) be a \(\mathbb{C}\)-valued function on \([0, 1]^2 \times \mathbb{R}^{2\nu}\). Let \(f, g \in \mathcal{M}^{\text{pre}}\) be defined by
\[f(s_1, \ldots, s_m, \xi_1, \ldots, \xi_m) = \left(\frac{1}{h} E + R_h\right)^{1,\ldots,m} \bigg|_{z_{j,k} = \alpha(s_j, s_k, \xi_j, \xi_k)}\]
and
\[g(s_1, \ldots, s_m, \xi_1, \ldots, \xi_m) := C_h^{1,\ldots,m} \bigg|_{z_{j,k} = \alpha(s_j, s_k, \xi_j, \xi_k)}.
\]
Then
\[g = \exp_{\text{pre}}(f)\]
(\(\exp_{\text{pre}}\) is defined according to the product \(\times_{\text{pre}}\)).

4 Deformation formulas
Let \((x, y) \in \mathbb{C}^{2\nu}, t \in \mathbb{C}^*, \ h \in \mathbb{C}^*\) and \(\omega_1, \ldots, \omega_\nu > 0\). Let \(p^{\text{harm}}\) be defined by
\[p^{\text{harm}} = (4\pi h)^{-\nu/2} \prod_{\omega = 1}^{\nu} \left(\frac{\omega_i}{\text{sh}(\omega_i t)}\right)^{1/2} \times\]
\begin{equation}
\exp\left(-\frac{1}{4\hbar} \sum_{\nu=1}^{\nu} \frac{\omega_{\nu}}{\mathrm{sh}(\omega_{\nu} t)} \left(\chi h(\omega_{\nu} t)(x_{\nu}^2 + y_{\nu}^2) - 2x_{\nu} y_{\nu}\right)\right). \tag{4.1}
\end{equation}

Then, by a variant of Mehler's formula,

\[ h\partial_t p_{\text{harm}} = \left(h^2 \partial_x^2 - \frac{\omega x}{4}\right) p_{\text{harm}}, \quad p_{\text{harm}}|_{t=0^+} = \delta_{x=y}. \]

Let \( \xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^\nu_m. \) Let

\[ |\xi| := |\xi_1| + \cdots + |\xi_m|, \]

and for \( s \in [0, 1], \nu \in \{1, \ldots, \nu\}, \)

\[ q_{t, \nu}(s) := \frac{1}{\mathrm{sh}(\omega_{\nu} t)} \left(\mathrm{sh}(\omega_{\nu} t s) x_{\nu} + \mathrm{sh}(\omega_{\nu} t(1-s)) y_{\nu}\right), \]

\[ q_{t}(s) := (q_{t,1}(s), \ldots, q_{t,\nu}(s)), \]

and for \( s = (s_1, \ldots, s_m) \in [0, 1]^m \)

\[ q^m_{t}(s) \cdot \xi := q_{t}(s_1) \cdot \xi_1 + \cdots + q_{t}(s_m) \cdot \xi_m. \]

Let \( \Omega_t, \xi \otimes_m \xi \) be defined by

\[ \Omega_t, \xi \otimes_m \xi := \sum_{j, k=1}^{m} \sum_{\nu=1}^{\nu} \frac{\mathrm{sh}(\omega_{\nu} t s_j \wedge k)}{\omega_{\nu} \mathrm{sh}(\omega_{\nu} t)} \xi_{j,\nu} \xi_{k,\nu}. \]

We also need the following generalization. Let \( m \geq 2. \) Let \( g \in \mathcal{T}_m \) and \( \theta \in [0, 1]^g. \) Let \( \theta_{j,k,g} \) be defined by \( \theta_{j,k,g}. \) Let \( \Omega_{t}^{g,\theta}, \xi \otimes_m \xi \) be defined by

\[ \Omega_{t}^{g,\theta}, \xi \otimes_m \xi := \sum_{j=1}^{m} \sum_{\nu=1}^{\nu} \frac{\mathrm{sh}(\omega_{\nu} t s_j) \mathrm{sh}(\omega_{\nu} t(1-s_j))}{\omega_{t} \mathrm{sh}(\omega_{t})} \xi_{j,\nu} \xi_{j,\nu}. \]

\[ 2 \sum_{1 \leq j < k \leq m} \theta_{j,k,g} \sum_{\nu=1}^{\nu} \frac{\mathrm{sh}(\omega_{\nu} t s_j) \mathrm{sh}(\omega_{\nu} t(1-s_k))}{\omega_{t} \mathrm{sh}(\omega_{t})} \xi_{j,\nu} \xi_{k,\nu}. \]

**Lemma 4.1** Let \( \omega_1, \ldots, \omega_{\nu}, t \geq 0. \) Let \( g \) and \( \theta \) be as above. Let \( (s_1, \ldots, s_m) \in [0, 1]^m \) such that \( 0 < s_1 < \cdots < s_m < 1 \) and let \( \xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^\nu_m. \) Then

\[ 0 \leq \Omega_t, \xi \otimes_m \xi \leq \frac{m t}{4} (\xi_1^2 + \cdots + \xi_m^2), \tag{4.2} \]

\[ 0 \leq \Omega_{t}^{g,\theta}, \xi \otimes_m \xi \leq \frac{m t}{4} (\xi_1^2 + \cdots + \xi_m^2). \tag{4.3} \]
Proof Let \( \dot{\omega} \geq 0 \). We claim that the \( m \times m \) matrix \( M \) defined by

\[
M_{j,k} = \frac{\text{sh}(\dot{\omega}ts_j \wedge k) \text{sh}(\dot{\omega}(1-s_j \lor k))}{\dot{\omega}t \text{sh}(\dot{\omega}t)}
\]

is symmetric non-negative. Let \( \zeta_1, \ldots, \zeta_m \in \mathbb{R} \) and let \( u : [0,1] \rightarrow \mathbb{R} \) defined by

\[
u(s) := \sum_{j=1}^{m} \frac{\text{sh}(\dot{\omega}ts_j \wedge s) \text{sh}(\dot{\omega}(1-s_j \lor s))}{\dot{\omega}t \text{sh}(\dot{\omega}t)} \zeta_j.
\]

Let \( \mu \) be the \( \mathbb{R} \)-valued Borel measure on \([0,1]\)

\[
\mu := \sum_{j=1}^{m} \zeta_j \delta_{s_j}.
\]

Then \( u \) is continuous, piecewise differentiable on \([0,1]\) and (see also [Ha4])

\[
\begin{cases}
-\frac{d^2 u}{ds^2} + (\dot{\omega}t)^2 u = \mu \\
u(0) = u(1) = 0
\end{cases}
\]

Then

\[
\sum_{j,k=1}^{m} M_{j,k} \zeta_j \zeta_k = \int_{0}^{1} u d\mu = \int_{0}^{1} \left\{ \left( \frac{du}{ds} \right)^2 + (\dot{\omega}t)^2 u^2 \right\},
\]

which proves the claim. Now, by choosing \( \dot{\omega} = \omega_1, \ldots, \dot{\omega} = \omega_n \),

\[
\Omega_{t,\xi} \otimes \mu, \xi \geq 0.
\]

Let us prove the non-negativity of \( \Omega^{0,\theta}_{t,\xi} \otimes \mu, \xi \). By the same argument, without loss of generality, we may choose \( \nu = 1 \). We shall use Lemma 8.1 (see Appendix). Let \((u_{j,k})_{j,k \in \{1, \ldots, m\}}\) be the real symmetric matrix defined by \( u_{j,j} = 1 \) and, for \( j \neq k \), \( u_{j,k} := \theta_{j \wedge k, j \lor k, q} \). For \( j, k = 1, \ldots, m \), let us denote by \( (j, k)_g \) the unique path in the tree \( g \) joining \( j \) and \( k \). Let \( q = 1, \ldots, m \). Let us denote by \( q_0 \) the unique element belonging to \( (j, k)_q \) such that \( (j, k)_q \cap (q_0, q)_g = \{ q_0 \} (q_0 = q \) if \( q \in (j, k)_q \). The path \( (j, k)_g \) is covered by the union of \( (j, q)_g \) and \( (k, q)_g \). Therefore, by (3.2), (8.1) holds and, by Lemma 8.1

\[
\Omega_{t,\xi}^{q,\theta} \otimes \mu, \xi \geq 0.
\]

Let us prove the upper bound in (4.2) and (4.3). Since \( s_j \wedge k \leq s_j \lor k \) and

\[
\frac{\text{sh}(x \wedge A) \text{sh}(x(1-B))}{x \text{sh}(x)} \leq \frac{1}{4} \text{ for arbitrary } x \in \mathbb{R} \text{ and } 0 \leq A < B \leq 1,
\]

one gets

\[
M_{j,k} \leq \frac{1}{4}.
\]

(4.4)
Then (4.2) holds. Since the parameters $\theta_{j,k,g}$ are bounded by 1, (4.3) also holds.

**Proposition 4.2** Let $\omega_1, \ldots, \omega_\nu > 0$. Let $h \in \mathbb{C}$ such that $\text{Re} h > 0$. Let $\mu$ be a $\mathbb{C}$-valued measure on $\mathbb{R}^\nu$. Let us assume that for every $R > 0$
\[
\int_{\mathbb{R}^\nu} \exp(R|\xi|) d|\mu|(\xi) < \infty. \tag{4.5}
\]
Let
\[
c(x) = \int_{\mathbb{R}^\nu} \exp(ix \cdot \xi) d\mu(\xi).
\]
Let $v$ be defined by
\[
v = 1 + \sum_{m \geq 1} v_m,
\]
\[
v_m(t, x, y, h) = \left(\frac{t}{h}\right)^m \int_{0 < s_1 < \cdots < s_m < 1} \int_{\mathbb{R}^{\nu m}} e^{-h\Omega_t \xi \otimes_m \xi} e^{iq^m(s) \cdot \xi} d\nu^m(a) \mu^\otimes(\xi) d^m s.
\tag{4.6}
\]
In (4.6), $d^m s$ denotes $ds_1 \cdots ds_m$ and $d\nu^m \mu^\otimes(\xi)$ denotes $d\mu(\xi_m) \cdots d\mu(\xi_1)$. Then, denoting $v_0 = 1$, for every $m \geq 1$,
\[
\begin{cases}
h(\partial_t - \frac{2}{p^{\text{harm}} m} \partial_x p^{\text{harm}} \cdot \partial_x) v_m = h^2 \partial_x^2 v_m + c(x) v_{m-1} \\
v_m|_{t=0^+} = 0
\end{cases} \tag{4.7}
\]
Moreover $v \in C^1([0, +\infty], \mathcal{A}(\mathbb{C}^{2\nu} \times \mathbb{C}^+))$ and the function $u := p^{\text{harm}} v$ is the solution of
\[
\begin{cases}
h\partial_t u = h^2 \partial_x^2 u - \frac{(\omega x)^2}{4} u + c(x) u \\
u|_{t=0^+} = \delta_{x=y}
\end{cases} \tag{4.8}
\]
**Proof** For small values of $t$, this proposition can be viewed as a consequence of Theorem 2.1 in [Ha6]. The proof is very similar to the one of [Ha4, Proposition 4.7] and is left to the reader. The convergence of the integrals defining the function $v$ for arbitrary values of $t$ uses the non-negativity of $\Omega_t \xi \otimes_n \xi$ (Lemma 4.1) and the estimate (4.11). \qed

**Remark 4.3** Considering simultaneously complex values of $h$ ($\text{Re} h > 0$) and complex values of $t$ (in a small conical neighbourhood of $\mathbb{R}^+$) certainly needs an extra assumption (as in [Ha7, Prop. 4.5, case 2]). Therefore we only consider smoothness (and not analyticity) of the function $v$ with respect to $t$.

**Remark 4.4** Let us assume that the function $c$ is $\mathbb{R}$-valued. Let $t, h \in [0, +\infty]$ and $x, y \in \mathbb{R}^\nu$. Then $q^m(t) \in \mathbb{R}^{\nu m}$ and $\Omega_t \xi \otimes_m \xi \geq 0$. Therefore
\[
|v_m(t, x, y, h)| \leq \frac{1}{m!} \left(\frac{t}{h}\right)^m \left(\int_{\mathbb{R}^\nu} d|\mu|(\xi) \right)^m
\]
and \( v(t, \cdot, h) \in L^\infty(\mathbb{R}^{2\nu}) \). Therefore, since \( p^{\text{harm}}(t, \cdot, h) \in L^2(\mathbb{R}^{2\nu}) \), \( u(t, \cdot, h) \) belongs to \( L^2(\mathbb{R}^{2\nu}) \) and the operator \( e^{-\frac{1}{2}H} \) is Hilbert-Schmidt. Then the operator \( e^{-\frac{1}{2}H} = e^{-\frac{1}{2}H} \times e^{-\frac{1}{2}H} \) is trace class and

\[
\Theta_H(t, h) = \int_{\mathbb{R}^e} u(t, x, x, h) dx.
\]

In formula (4.10), the function \( v \) looks very singular with respect to \( h \). Considering \( v \) as the exponential of a new function will allow us to deal with this singularity (see Remark 3.6). We need some definitions.

**Definition 4.5** Let \( \omega \in \mathbb{R}^\nu \) and let \( \mu \) be as in Proposition 4.2. For every \( (t, x, y) \in \mathbb{R} \times \mathbb{C}^{2\nu} \), let us denote

\[
Q_1(t, x, y) = \int_{0<s_1<1} \int_{\mathbb{R}^e} e^{iq(t \cdot s_1)} \xi d\mu(\xi) ds_1
\]

and, for \( m = 2, 3, \ldots \)

\[
Q_m(t, x, y) = (-2)^{m-1} \sum_{g \in \mathcal{R}_m} \int_{0<s_1<\ldots<s_m<1} \int_{\mathbb{R}^{\nu m}} \mathcal{Y}_t^{g, \xi} e^{iq(t \cdot s_1)} \xi \, d\mu(\xi) \, ds
\]

where

\[
\mathcal{Y}_t^{g, \xi} := \prod_{[j, k] \in g} \left( \sum_{i=1}^{\nu} \frac{\text{sh}(\omega_i t s_j) \text{sh}(\omega_i t (1 - s_k))}{\omega_i t \text{sh}(\omega_i t)} \xi_{j, \omega_i, k, \omega_i} \right).
\]

For every \( t \in \mathbb{R} \), \( s \in [0, 1] \) and \( v \in \{1, \ldots, \nu\} \), let

\[
\varpi_{t, v}(s) := \frac{1}{\text{sh}(\omega_v t)} \left( \text{sh}(\omega_v t s) + \text{sh}(\omega_v t(1 - s)) \right),
\]

\[
\varpi_t(s) := (\varpi_{t, 1}(s), \ldots, \varpi_{t, \nu}(s))
\]

Then

\[
\frac{1}{\text{ch}(\frac{\varpi_t}{2})} \leq \varpi_{t, v}(s) \leq 1 \quad (4.9)
\]

and, for every \( R > 0 \),

\[
\forall x \in D_{R^e, R}, |\text{Im}(q^m_{t_i}|_{y=x}(s) \cdot \xi)| \leq |\text{Im}x| \times |\xi|_1 \leq R|\xi|_1. \quad (4.10)
\]

\[
\forall (x, y) \in D^2_{R^e, R}, |\text{Im}(q^m_{t_i}(s) \cdot \xi)| \leq (|\text{Im}x| + |\text{Im}y|) \times |\xi|_1 \leq 2R|\xi|_1. \quad (4.11)
\]

Therefore the functions \( Q_m \) are well defined on \( \mathbb{R} \times \mathbb{C}^{2\nu} \). In the case \( t \omega = 0 \), these quantities are studied in [Ha1, Ha3]. We need upper bounds for some quantities which depend on the functions \( Q_m \), for large values of \( m \). We often use the following elementary inequalities. Let \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m \) and let \( a_1, \ldots, a_m > 0 \). Let us denote \( |\alpha| := \alpha_1 + \cdots + \alpha_m \). Then

\[
\frac{a_{\alpha_1}^{\alpha_1}}{\alpha_1!} \cdots \frac{a_{\alpha_m}^{\alpha_m}}{\alpha_m!} \leq e^{a_1 + \cdots + a_m} \quad (4.12)
\]
and, for every $\lambda = 1, 2, \ldots$,
\[
\frac{a_1^{\alpha_1} \cdots a_m^{\alpha_m}}{\alpha_1! \cdots \alpha_m!} (a_1^\lambda + \cdots + a_m^\lambda) \leq (|\alpha| + m)(|\alpha| + \lambda)^{\lambda - 1} e^{a_1 + \cdots + a_m}.
\]

Lemma 4.6 Let $\omega_1, \ldots, \omega_\nu \geq 0$ and let $\mu$ be as in Proposition 4.2. Then, for every $m \geq 1$, $Q_m \in C^1(\mathbb{R}, A(C^{2\nu}))$. Moreover, for every $R > 0$ and every $(t, x, y) \in \mathbb{R} \times D^{2, R}_{\nu, R}$,
\[
|Q_m(t, x, x)| \leq \left(\frac{4}{\mathcal{R}^\nu} e^{(1+R)|\xi|d^\nu |\mu|(\xi)}\right)^m,
\]
\[
|Q_m(t, x, y)| \leq \left(\frac{4}{\mathcal{R}^\nu} e^{(1+2R)|\xi|d^\nu |\mu|(\xi)}\right)^m,
\]
\[
|\partial_t(t^{2m-1}Q_m(t, x, y))| \leq |t|^{2m-2} \times \left(1 + \omega_1|t| \max(|x|, |y|)\right) \left(\frac{4}{\mathcal{R}^\nu} e^{(1+2R)|\xi|d^\nu |\mu|(\xi)}\right)^m.
\]

Proof
-1- The proof of (4.14), (4.15) and (4.16) (using (4.21) and (4.22)) are straightforward if $m = 1$ and we now assume $m \geq 2$.

-2- Let us check (4.15) and (4.16). By (4.11)
\[
|Q_m(t, x, y)| \leq \frac{2m-1}{4^{m-1}m!} \sum_{g \in T_m} \int_{\mathbb{R}^m} |\xi_{j_1}| |\xi_{k_1}| \cdots |\xi_{j_{m-1}}| |\xi_{k_{m-1}}| e^{2R|\xi|d^\nu |\mu|^\otimes(\xi)}
\]
where $[j_1, k_1], \ldots, [j_{m-1}, k_{m-1}]$ denote the $m - 1$ edges of the graph $g$. For $g \in T_m$, let us denote by $d_1, \ldots, d_m$ the degrees of the vertices $1, \ldots, m$. Then $d_1 + \cdots + d_m = 2(m - 1)$ and, if $d_1, \ldots, d_m$ satisfy the previous equality, the number of trees with vertices $1, \ldots, m$ such that the $d^\nu(1) = d_1, \ldots, d^\nu(m) = d_m$ is equal to $\frac{(m-2)!}{(d_1-1)! \cdots (d_m-1)!} [\text{Co}$. Therefore
\[
|Q_m(t, x, y)| \leq \frac{2^{1-m}}{m(m-1)} \sum_{d_1, \ldots, d_m=2m-2} \int_{\mathbb{R}^m} |\xi_{j_1}|^{d_1} \cdots |\xi_{j_{m-1}}|^{d_{m-1}} |\xi_{k_{m-1}}|^{d_m} e^{2R|\xi|d^\nu |\mu|^\otimes(\xi)}
\]
\[
\leq \frac{2}{m(m-1)} \sum_{d_1, \ldots, d_m=2m-2} \int_{\mathbb{R}^m} |\xi_{j_1}|^{d_1} \cdots |\xi_{j_{m-1}}|^{d_{m-1}} |\xi_{k_{m-1}}|^{d_m} e^{2R|\xi|d^\nu |\mu|^\otimes(\xi)},
\]

19
since, under the assumption \(d_1 + \cdots + d_m = 2m - 2\),
\[
d_1 \times \cdots \times d_m \leq \left(\frac{d_1 + \cdots + d_m}{m}\right)^m \leq 2^m. \tag{4.17}
\]

Then, by (4.12) and since
\[
\left|\{(d_1, \ldots, d_m) \in (\mathbb{N}^*)^m | d_1 + \cdots + d_m = 2m - 2\}\right| = \left(\frac{2m - 3}{m - 1}\right) \leq 2^{2m-3},
\]
one gets
\[
|Q_m| \leq \frac{2^{2m-2}}{m(m-1)} \int_{\mathbb{R}^m} e^{(1+2R)|\xi|} d^m |\mu|^\otimes(\xi) \leq \frac{1}{4m(m-1)} (4M)^m \tag{4.19}
\]
where
\[
M := \int_{\mathbb{R}^\nu} e^{(1+2R)|\xi|} d^\nu |\mu|(|\xi|). \tag{4.20}
\]
This proves (4.15). By (4.10), one gets (4.14).

-3- For proving (4.16), we need the following inequalities. Let \(s \in [0,1]\) and let \(A, B \in [0,1]\) such that \(A < B\). Then, for every \(\omega \geq 0\) and every \(t \in \mathbb{R}\),
\[
\left|\frac{d}{dt} \left(\frac{\text{sh}(\omega ts)}{\text{sh}(\omega t)}\right)\right| \leq \frac{1}{2} \omega, \tag{4.21}
\]
\[
\left|\frac{d}{dt} \left(\frac{\text{sh}(\omega tA) \text{sh}(\omega t(1-B))}{\omega \text{sh}(\omega t)}\right)\right| \leq \frac{1}{2}. \tag{4.22}
\]

One has
\[
\partial_t(t^{m-1}Q_m) = mt^{m-1} \times t^{m-1}Q_m + R_m \tag{4.23}
\]
where
\[
R_m := (-2)^{m-1} t^m \sum_{g \in T_m} \int_{0 < s_1 < \ldots < s_m < 1} \int_{\mathbb{R}^m} \left(\partial_t(t^{m-1}Y_t^{g, \xi}) + it^{m-1}Y_t^{g, \xi} \partial_t(q_t^{m}(s) \cdot \xi))e^{iq_t^{m}(s) \cdot \xi} d^m |\mu|^\otimes(\xi) d^m s.
\]
By (4.4) and (4.22)
\[
|\partial_t(t^{m-1}Y_t^{g, \xi})| \leq (m - 1) \times \left(\frac{|t|}{4}\right)^{m-2} \times \frac{1}{2} \times |\xi_{j_1}||\xi_{k_1}| \cdots |\xi_{j_{m-1}}||\xi_{k_{m-1}}|.
\]
By (4.3) and (4.21)
\[
|\partial_t(t^{m-1}Y_t^{g, \xi} \partial_t(q_t^{m}(s) \cdot \xi))| \leq \left(\frac{1}{4}\right)^{m-1} |\xi_{j_1}||\xi_{k_1}| \cdots |\xi_{j_{m-1}}||\xi_{k_{m-1}}| \times \omega_t \tilde{R} |\xi|.
\]
where \(\tilde{R} = \frac{1}{2}(|x| + |y|)\). Then
\[
|\partial_t(t^{m-1}Y_t^{g, \xi}) + it^{m-1}Y_t^{g, \xi} \partial_t(q_t^{m}(s) \cdot \xi)| \leq
\]

where
\(\begin{align*}
\left( \frac{|t|}{4} \right)^{m-2} \left( m - \frac{1}{2} + \frac{\omega}{|t|} R \right) |\xi_{j_k}| |\xi_{k_1}| \cdots |\xi_{j_{m-1}}| |\xi_{k_{m-1}}|
\end{align*}\)
and, by (4.17)
\[|R_m| \leq \frac{2^{2m-1} |t|^m}{m(m-1)} \left( \frac{|t|}{4} \right)^{m-2} \sum_{d_1 + \cdots + d_m = 2m-2} \int_{\mathbb{R}^m} \frac{|\xi_1|}{d_1!} \cdots \frac{|\xi_m|}{d_m!} m_{d_1} \cdots m_{d_m} \times \left( \frac{m - 1}{2} + \frac{\omega}{|t|} R \right) \left| 2^R \xi_1 \cdot d^{\nu m} |\mu|^{\otimes}(\xi) \right|
\leq \frac{2^{2m-1} |t|^m}{m(m-1)} \left( \frac{|t|}{4} \right)^{m-2} \left( \frac{2m - 3}{m - 1} \right) (A1 + A2)
\]
where \(A1 := \frac{m - 1}{2} \times M^m\) by (4.12) and \(A2 := \frac{\omega}{|t|R} \times (3m - 2) \times M^m\) by using (4.13). Then
\[|R_m| \leq \frac{(4M)^m |t|^{2m-2}}{m} \left( \frac{1}{2} + \frac{\omega}{|t|} R \right)
\]
and by (4.23) and (4.19)
\[|\partial_t \left( t^{2m-1} Q_m \right) | \leq (4M)^m |t|^{2m-2} (1 + \omega |t|R).
\]
This proves (4.10).

\[\Box\]

**Definition 4.7** Let \(\omega_1, \ldots, \omega_r \geq 0\) and let \(\mu\) be as in Proposition 4.2. For every \((t, x, y, h) \in \mathbb{R}^+ \times \mathbb{C}^{2r} \times \mathbb{C}^+\), let us denote
\[w_1(t, x, y, h) := -t \int_{0 < s_1 < 1} \int_{\mathbb{R}^r} \int_0^1 \Omega_i \cdot \xi_1 \cdot \xi_1 e^{-h \xi_1 \cdot \xi_1} d\theta d\mu(\xi_1) ds
\]
and for \(m \geq 2\)
\[w_m(t, x, y, h) := 2^{m-1} (-1)^m t^{2m-1} \sum_{\gamma \in T_m} \int_{0 < s_1 < \cdots < s_m < 1} \int_{\mathbb{R}^r} \int_{[0,1]^m \times [0,1]} \Omega_i^\gamma \cdot \xi_1 e^{-h \xi_1 \cdot \xi_1} \cdots e^{-h \xi_1 \cdot \xi_1} d\theta d\mu(\xi_1) d\theta d\mu(\xi_1) d\theta d\mu(\xi_1)
\]
By Lemma 4.11 and (4.11) the functions \(w_m\) are well defined on \([0, +\infty[ \times \mathbb{C}^{2r} \times \mathbb{C}^+\).

**Lemma 4.8** Let \(\omega_1, \ldots, \omega_r \geq 0\) and let \(\mu\) be as in Proposition 4.2. Then, for every \(m \geq 1\),
\[w_m \in \mathcal{C}^1 ([0, +\infty[, \mathcal{A}(\mathbb{C}^{2r} \times \mathbb{C}^+)).
\]
Moreover, for every \((t, x, y, h) \in [0, +\infty[ \times D_{\mathbb{R}^r, \mathbb{R}^r} \times \mathbb{C}^+\),
\[|w_m(t, x, y, h)| \leq m \left( 4 t^2 \int_{\mathbb{R}^r} e^{(1 + 2R)|\xi|} d^r |\mu|() \right)^m. \quad (4.24)
\]

21
Proof  The proof is similar to the one of Lemma 4.6 and we only focus on the differences between the two proofs. By (4.2), (4.24) holds for $m = 1$. Let $m \geq 2$. By Lemma 4.1

$$|w_m| \leq \frac{2m-3}{M} \left( \frac{2m-3}{m-1} \right) \sum_{g \in T} \int_{R^m}$$

By (4.13)

$$\left| \frac{\xi_1}{d_1} \cdots \frac{\xi_m}{d_m} \left( \frac{2m}{d_1 + \cdots + d_m = 2m-2} \right) \int_{R^m} \right| \leq (3m-2)(2m)e^{\xi_1} \leq 6m^2e^{\xi_1}. \quad (4.25)$$

Then

$$|w_m| \leq \frac{3m^2}{m} \left( \frac{2m-3}{m-1} \right) \sum_{d_1 + \cdots + d_m = 2m-2} \int_{R^m}$$

where $M$ is given by (4.20). This proves (4.24). By dominated convergence theorem

$$w_m \in C^1([0, +\infty[, A(\mathbb{C}^{2\nu} \times \mathbb{C}^+)).$$

□

Lemma 4.9  Let $\omega_1, \ldots, \omega_\nu \geq 0$ and let $\mu$ be as in Proposition 4.2. For $m \geq 1$, let us define

$$v_m^\varphi(t, x, y, h) := \frac{t^{2m-1}}{h} Q_m(t, x, y) + w_m(t, x, y, h).$$

Then

$$\left( \partial_t - \frac{2}{p_{\text{harm}}} \partial_x p_{\text{harm}} \cdot \partial_x \right) v_m^\varphi = \partial_x v_m^\varphi + \sum_{p + q = m} \partial_x v_p^\varphi \cdot \partial_x v_q^\varphi + c(x) \delta_{m=1} \quad (4.26)$$

and $v_m^\varphi|_{t=0} = 0$.

Proof  Let $E$ and $R_h$ be the moulds defined in subsection 3.1. Let $f \in M^{\text{pre}}$ be defined by

$$f^\varphi = 0,$$

$$f(s_1, \ldots, s_m, \xi_1, \ldots, \xi_m) = \left( \frac{1}{h} E + R_h \right)^{\{1, \ldots, m\}}$$

22
where, for \(1 \leq j \leq k \leq m\),
\[
z_{j,k} := - \sum_{\nu=1}^{\nu} \frac{\text{sh}(\omega_{\nu}, t s_j) \text{sh}(\omega_{\nu}, t(1 - s_k))}{\omega_{\nu} \text{sh}(\omega_{\nu}, t)} \xi_{j,\nu} \xi_{k,\nu}.
\]

Let \(\lambda\) be the Borel measure defined on \([0, 1] \times \mathbb{R}^\nu\) by
\[
d\lambda(s, \xi) = te^{iq(t)s} a^{\nu} \mu(\xi) ds.
\]

Then
\[
\frac{i^{2m-1}}{h} Q_m + w_m = \int_{0 < s_1 < \ldots < s_m < 1} \int_{\mathbb{R}^m} f(s_1, \ldots, s_m; \xi_1, \ldots, \xi_m) d^{\nu m} \lambda \otimes (s, \xi).
\]
By Corollary 3.8
\[
\exp_{\text{pre}}(f)(s_1, \ldots, s_m; \xi_1, \ldots, \xi_m) = \frac{1}{h^m} \exp(-h \Omega_t, \xi \otimes_m \xi).
\]

Let \(\tilde{\nu}\) be the formal series with respect to \(H\) defined by
\[
\tilde{\nu} := \exp \left( \sum_{m=1}^{+\infty} v_m H^m \right).
\]

Then
\[
\tilde{\nu} = \exp(\Phi(f)) = \Phi(\exp_{\text{pre}} f) = \sum_{m=0}^{+\infty} v_m H^m.
\]
The second equality holds since \(\Phi\) is a morphism (Lemma 3.2) and the third one uses the definition (4.6) of \(v_m\). Then (4.7) implies (4.26).

**Proposition 4.10** Let \(\omega_1, \ldots, \omega_\nu, T > 0\). Let \(\mu\) be as in Proposition 4.2 and let us assume that
\[
4T^2 M_\mu < 1.
\]

Let
\[
\varphi(t, x, y) := \sum_{m=1}^{+\infty} i^{2m-2} Q_m(t, x, y),
\]
\[
w(t, x, y, h) := \sum_{m=1}^{+\infty} w_m(t, x, y, h),
\]
and
\[
\phi := \frac{1}{4} \left( \sum_{\nu=1}^{\nu} \frac{\omega_{\nu}}{\text{sh}(\omega_{\nu}, t)} (\text{ch}(\omega_{\nu}, t)(x_{\nu}^2 + y_{\nu}^2) - 2x_{\nu} y_{\nu}) \right) - t \varphi.
\]

Then
\[
\varphi \in C^1([-T, T], \mathcal{A}(D^2_{\mathbb{R}^\nu, 2})) \quad \text{and} \quad w \in C^1([0, T], \mathcal{A}(D^2_{\mathbb{R}^\nu, 2} \times \mathbb{C}^+)).
\]
Moreover
\[ u(t, x, y, h) := (4\pi h)^{-\nu/2} \prod_{v=1}^{\nu} \frac{\omega_v^{1/2}}{\text{sh}^{1/2}(\omega_v t)} \times e^{-\phi/h + w} \] (4.27)
satisfies (4.8).

\textbf{Proof}  
By dominated convergence theorem, by choosing \( R = 2 \) in Lemma 4.6 and Lemma 4.8,
\[ \varphi \in C^1([-T,T|, A(D_{R,2}^{\nu,2})) \text{ and } w \in C^0([0,T], A(D_{R,2}^{\nu,2} \times \mathbb{C}^+)). \]
The quantity \( |\partial_t w_m| \) is bounded by
\[ P(m, |h|, |x|, |y|)t^{2m-1} \left( 4 \int_{R^\nu} e^{\frac{5}{4} |\xi|} |d\nu| (\xi) \right)^m, \]
where \( P \) is a polynomial with respect to its arguments (this step is left to the reader). This implies the \( C^1 \)-regularity of \( w \).

By Lemma 4.9, \( v^\diamond \) satisfies (4.26). Therefore
\[ v^\diamond := \frac{t}{R} \varphi + w = \sum_{m \geq 1} v^\diamond_m \]
satisfies
\[ \left( \partial_t - \frac{2}{p_{\text{harm}}} \partial_x p_{\text{harm}} \cdot \partial_x \right) v^\diamond = \partial_x^2 v^\diamond + \partial_x v^\diamond \cdot \partial_x v^\diamond + c(x). \]
Then
\[ \left( \partial_t - \frac{2}{p_{\text{harm}}} \partial_x p_{\text{harm}} \cdot \partial_x \right) e^{v^\diamond} = \partial_x^2 e^{v^\diamond} + c(x) e^{v^\diamond} \]
and the function \( u \) satisfies (4.8). \( \Box \)

\textbf{Remark 4.11} Let \( y \in D_{R,2}^{\nu,2} \). Then the function \((t,x) \mapsto \phi(t, x, y)\) satisfies
\[ \left\{ \begin{array}{l}
\partial_t \phi + (\partial_x \phi)^2 = \frac{(\omega_x)^2}{4} - c(x) \\
\left( \phi - \frac{(x-y)^2}{4t} \right) \bigg|_{t=0} = 0
\end{array} \right. \] (4.28)
for \((t, x) \in [-T, T[-\{0\}] \times D_{R,2}^{\nu,2}.

5 Borel summation preliminary statements

The following lemma will be useful for the proof of a Morse Lemma concerning the function \( \phi|_{y=x} \) (see Proposition 4.10).
Lemma 5.1 Let $\mu$ be as in Definition 2.1 and Proposition 4.2. Let us assume that $M'_\mu < \infty$ and $4T^2M_\mu < 1$. Then, for every $(t, x) \in ]0, T[ \times D_{\mathbb{R}^N_4}$,

$$|\varphi(t, x, x)| \leq \frac{4M_\mu}{1 - 4T^2M_\mu},$$

(5.1)

$$\max_{1 \leq \delta \leq \nu} |x_\delta \varphi(t, x, x)| \leq \exp\left(\frac{\omega_2 T}{2}\right) \times \frac{M'_\mu}{1 - 4T^2M_\mu}$$

(5.2)

Proof Let $(t, x) \in ]0, T[ \times D_{\mathbb{R}^N_4}$.

-1- Let us check (5.1). Since

$$\varphi(t, x, x) = \sum_{m=1}^{+\infty} t^{2m-2}Q_m(t, x, x)$$

and by choosing $R = 4$ in (4.14),

$$|\varphi(t, x, x)| \leq \frac{4M_\mu}{1 - 4T^2M_\mu}.$$ 

This proves (5.1).

-2- Let us check (5.2). By Definition 4.5

$$x_\delta Q_1(t, x, x) = \int_{0<s_1<1} \int_{\mathbb{R}^N} x_\delta e^{ix_1}(\varpi(s_1)\xi_1) F(s_1, \xi_1) d\xi_1 ds_1$$

where $F(s_1, \xi_1) = \rho\mu(\xi_1)$. If $m \geq 2$,

$$x_\delta Q_m(t, x, x) = (-2)^{m-1} \sum_{g \in T_m} \int_{0<s_1<\ldots<s_m<1} \int_{\mathbb{R}^N} x_\delta F(s, \xi) e^{ix_1}(\varpi(s_1)\xi_1 + \ldots + \varpi(s_m)\xi_m) d^{Nm} \xi d^m s$$

where

$$F(s, \xi) := T_1^{q_{\xi}} \rho\mu(\xi_1) \cdots \rho\mu(\xi_m).$$

Let

$$D := \frac{1}{m} \sum_{p=1}^{m} \frac{1}{\varpi_{t, \delta}(s_p)} \partial_{\xi_{p, \delta}}.$$

By integration by parts,

$$x_\delta \int_{\mathbb{R}^m} F \times e^{ix_{1, \delta}}(\varpi_{t, \delta}(s_1)\xi_{1, \delta} + \ldots + \varpi_{t, \delta}(s_m)\xi_{m, \delta}) d\xi_{1, \delta} \cdots d\xi_{m, \delta} =$$

$$\int_{\mathbb{R}^m} DF \times e^{ix_{1, \delta}}(\varpi_{t, \delta}(s_1)\xi_{1, \delta} + \ldots + \varpi_{t, \delta}(s_m)\xi_{m, \delta}) d\xi_{1, \delta} \cdots d\xi_{m, \delta}.$$
Therefore
\[ x_\delta Q_1(t, x, x) = i \int_{0 < s_1 < 1} \int_{\mathbb{R}^\nu} e^{i\omega}(\omega(t(s_1)) \xi_1) \left( \partial_{\xi_1, \delta} \rho_\mu(\xi_1) \right) d\xi_1 \frac{ds_1}{\omega(t(s_1))} \]

and, if \( m \geq 2 \), denoting by \( F_g \) the quantity \( F \) in order to emphasize its dependence on the tree \( g \),
\[ x_\delta Q_m(t, x, x) = i(-2)^{m-1} \sum_{g \in T_m} \int_{0 < s_1 < \ldots < s_m < 1} \int_{\mathbb{R}^\nu^m} DF_g \times e^{i\omega}(\omega(t(s_1)) \xi_1 + \ldots + \omega(t(s_m)) \xi_m) d\nu^m \xi d^m s. \]

By (4.9) and (4.10),
\[ |x_\delta Q_1(t, x, x)| \leq \text{ch} \left( \frac{\omega_{\mu t}}{2} \right) M'_\mu. \]

Let us assume that \( m \geq 2 \). Using (4.1) and (4.9), one can show
\[ |\Upsilon_{t, \eta}^\xi| \leq \frac{1}{4^{m-1} m} |\xi_{1m}| |\xi_{1}||\xi_{j_{m-1}}||\xi_{j_{m-1}}|; \]
\[ |D\Upsilon_{t, \eta}^\xi| \leq \frac{\text{ch} \left( \frac{\omega_{\mu t}}{2} \right)}{4^{m-1} m} \left( \partial_{u_1} + \ldots + \partial_{u_m} \right) \left( u_{j_1}, u_{j_1}, \ldots, u_{j_{m-1}}, u_{j_{m-1}} \right) |u_{j_1} = |\xi_{1m}| \ldots, |u_{j_{m-1}} = |\xi_{j_{m-1}}| \right| \]
where \((j_1, k_1), \ldots, (j_{m-1}, k_{m-1})\) denote the \( m-1 \) edges of the graph \( g \) \((j_p < k_p)\).

Let \( d_1, \ldots, d_m \) be the degrees of the vertices \( 1, \ldots, m \) of the graph \( g \). Then
\[ \frac{|DF_g|}{d_1! \cdots d_m!} \leq \frac{1}{4^{m-1}} (A1 + A2) \]
where
\[ A1 := \frac{\text{ch} \left( \frac{\omega_{\mu t}}{2} \right)}{m} \frac{|\xi_1|^{d_1}}{d_1!} \cdots \frac{|\xi_m|^{d_m}}{d_m!} \times \]
\[ \left( |\partial_{\xi_1, \delta} \rho_\mu(\xi_1) \rho_\mu(\xi_2) | \cdots | \rho_\mu(\xi_m) | + \cdots + | \rho_\mu(\xi_1) | \cdots | \rho_\mu(\xi_m) | \right) \left( \partial_{\xi_1, \delta} \rho_\mu(\xi_1) \right) \cdots \left( \partial_{\xi_1, \delta} \rho_\mu(\xi_m) \right), \]
\[ A2 := \frac{\text{ch} \left( \frac{\omega_{\mu t}}{2} \right)}{m} \left( \partial_{u_1} + \ldots + \partial_{u_m} \right) \left( \frac{u_1^{d_1}}{d_1!} \cdots \frac{u_m^{d_m}}{d_m!} \right) \left| u_{j_1} = |\xi_{1m}| \ldots, u_{j_{m-1}} = |\xi_{j_{m-1}}| \right| \times \]
\[ |\rho_\mu(\xi_1) | \cdots | \rho_\mu(\xi_m) |. \]

Using (4.12), one gets
\[ A1 \leq \frac{1}{m} \text{ch} \left( \frac{\omega_{\mu t}}{2} \right) e^{\|\xi_1\|} \times \]
\[ \left( |\partial_{\xi_1, \delta} \rho_\mu(\xi_1) | \cdots | \rho_\mu(\xi_2) | + \cdots + | \rho_\mu(\xi_1) | \cdots | \rho_\mu(\xi_m) | \right) \left( \partial_{\xi_1, \delta} \rho_\mu(\xi_1) \right) \cdots \left( \partial_{\xi_1, \delta} \rho_\mu(\xi_m) \right), \]
\[ A2 \leq \text{ch} \left( \frac{\omega_{\mu t}}{2} \right) e^{\|\xi_1\|} |\rho_\mu(\xi_1) | \cdots | \rho_\mu(\xi_m) | \]

since
\[ \left( \partial_{u_1} + \ldots + \partial_{u_m} \right) \left( \frac{u_k^{d_k}}{d_k!} \right) = \frac{u_k^{d_k-1}}{(d_k-1)!}. \]
Then
\[ \frac{|DF_\rho|}{d_1! \cdots d_m!} \leq \frac{2}{4m-1} \text{ch}(\frac{\omega t}{2}) e^{4|\xi|} \times \]
\[ \frac{1}{m!} \left( |\rho_\mu(\xi_1)| \cdot |\rho_\mu(\xi_2)| \cdots |\rho_\mu(\xi_m)| + \cdots + |\rho_\mu(\xi_1)| \cdots |\rho_\mu(\xi_{m-1})| |\rho_\mu(\xi_m)| \right) \]  
(5.3)

where
\[ |\rho_\mu(\eta)| := \max(|\rho_\mu(\eta)|, |\partial_{\eta_1} \rho_\mu(\eta)|, \ldots, |\partial_{\eta_\nu} \rho_\mu(\eta)|) \]
for \( \eta \in \mathbb{R}^\nu \). Then, by (4.10) and (5.3),
\[ |x_\delta Q_m(t,x,x)| \leq \frac{2^{m-1}}{m!} \sum_{g \in T_m} \int_{0 < s_1 < \ldots < s_m < 1} \int_{\mathbb{R}^m} |DF_\rho| \times e^{4|\xi|} d^m \xi m^s \]
\[ \leq \frac{2^{m-1}}{m!} \times \left( \sum_{g \in T_m} d_1! \cdots d_m! \right) \times \frac{2}{4m-1} \text{ch}(\frac{\omega t}{2}) M'_\mu M_m^{m-1}. \]

But, by (4.17) and (4.18)
\[ \sum_{g \in T_m} d_1! \cdots d_m! = \sum_{d_1 + \cdots + d_m = 2m-2} \frac{(m-2)!}{(d_1-1)! \cdots (d_m-1)!} \times d_1! \cdots d_m! \]
\[ \leq (m-2)! \times 2^m \times 2^{2m-3}. \]

Then
\[ |x_\delta Q_m(t,x,x)| \leq 2^{4m-1} \frac{4m-1}{m(m-1)} \text{ch}(\frac{\omega t}{2}) M'_\mu M_m^{m-1}. \]

Then, for every \( m \geq 1 \),
\[ |x_\delta Q_m(t,x,x)| \leq \text{ch}(\frac{\omega t}{2}) M'_\mu (4M_\mu)^{m-1}. \]

This proves (5.2). \( \square \)

The following Morse lemma gives a convenient expression for \( \phi \big|_{y=x} \) (see Proposition 4.10 for the definition of the function \( \phi \)). We denote by \( D_\omega \) the following \( \nu \times \nu \) diagonal matrix
\[ D_\omega := \frac{1}{\sqrt{2}} \begin{pmatrix} \omega_1^{1/2} \text{th}^{1/2}(\frac{\omega_1 t}{2}) & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \omega_\nu^{1/2} \text{th}^{1/2}(\frac{\omega_\nu t}{2}) \end{pmatrix}. \]
(5.4)

Moreover, if \( W \) is a \( \mathbb{C}^\nu \)-valued function analytic function defined on some open set of \( \mathbb{C}^\nu \), we denote by \( \partial_x W(x) \) the \( \nu \times \nu \) matrix \( (\partial_{x_1} W(x), \ldots, \partial_{x_\nu} W(x)) \).
Proposition 5.2 Let $\alpha_\nu > 0$ such that $\alpha = \alpha_\nu$ satisfies (5.18) and (5.20) below (the number $\alpha_\nu$ only depends on $\nu$). Let $\mu$ be as in Definition 2.1 and Proposition 4.2. Let us assume that $M'_\mu < \infty$, that (2.7), (2.8) are satisfied and

$$\int_{\mathbb{R}^\nu} d\mu(\xi) = 0.$$  

Let $\omega_1, \ldots, \omega_\nu, T > 0$ such that

$$4T^2 M_\mu < 1,$$  

$$\text{ch} \left( \frac{\omega \cdot T}{2} \right) \frac{\omega^2 (1 + \omega \cdot T)}{\omega^3 (1 - 4T^2 M_\mu)} < \alpha.$$  

Then there exists $\Lambda \in C^0([0, T] \times D_{\mathbb{R}^\nu, 1})$ such that

1. $\Lambda|_{[0, T] \times D_{\mathbb{R}^\nu}}$ is $\mathbb{R}^\nu$-valued and

$$\sup_{(t,x) \in [0, T] \times D_{\mathbb{R}^\nu, 1}} |\partial_x \Lambda(t, x)| \leq \frac{1}{2},$$  

2. for every $(t, x) \in [0, T] \times D_{\mathbb{R}^\nu, 1}$, $\phi|_{y=x} = \theta^2(t, x)$ where

$$\theta(t, x) := D_\omega (x + \Lambda(t, x)).$$

Proof We assume that (5.5) holds and that, for some arbitrary $\alpha$,

$$\text{ch} \left( \frac{\omega \cdot T}{2} \right) \frac{\omega^2 (1 + \omega \cdot T)}{\omega^3 (1 - 4T^2 M_\mu)} < \alpha.$$  

Let $(t, x) \in [0, T] \times D_{\mathbb{R}^\nu, 1}$.

**1**- First, we look for a condition on $\alpha$ providing the existence of the function $\Lambda$ such that (5.7) holds. Let us denote $\Phi(t, x) := \phi|_{y=x}$ (the function $\Phi$ is well defined by Proposition 4.10). We claim that

$$\Phi(t, 0) = \partial_{x_1} \Phi(t, 0) = \cdots = \partial_{x_\nu} \Phi(t, 0) = 0.$$  

This fact can be proved either by checking that, for $m \geq 1$,

$$Q_m(t, 0, 0) = \partial_{x_1} Q_m(t, 0, 0) = \cdots = \partial_{x_\nu} Q_m(t, 0, 0) = 0$$

(notice that each tree of $\mathcal{T}_m$ has at least two vertices of degree 1 if $m \geq 2$) or by using

$$c(0) = \partial_{x_1} c(0) = \cdots = \partial_{x_\nu} c(0) = 0$$

and that $\phi$ satisfies (4.28).
\[
\frac{\dot{\omega}}{\sin(\omega t)} (\cosh(\omega t) - 1) = \dot{\omega} \text{th}\left(\frac{\omega t}{2}\right)
\]
for \(\omega \in \mathbb{R}\) and by the Taylor formula,
\[
\Phi(t, x) = x \cdot (B(t, x) \cdot x)
\]
where
\[
B(t, x) := \int_0^1 (1 - u) (\partial_z \otimes \partial_z) \Phi(t, z) |_{\omega = u x} du
\]
\[
= D_\omega^2 - t \int_0^1 (1 - u) (\partial_z \otimes \partial_z) \varphi(t, z, z) |_{\omega = u x} du
\]
\[
= D_\omega (1 + R(t, x)) D_\omega
\]
and
\[
R(t, x) := -t D_\omega^{-1} \left( \int_0^1 (1 - u) (\partial_z \otimes \partial_z) \varphi(t, z, z) |_{\omega = u x} du \right) D_\omega^{-1}.
\] (5.10)

Let \(A > 0\), let \(\beta = 1, \ldots, \nu\) and let \(\psi\) be an analytic function on \(D_{\mathbb{R}^v, A+1}\). By the Cauchy formula
\[
\sup_{z \in D_{\mathbb{R}^v, A}} |\partial_z^\beta \psi(z)| \leq \sup_{z \in D_{\mathbb{R}^v, A+1}} |\psi(z)|.
\] (5.11)

Let \(z \in D_{\mathbb{R}^v, 1}\). Let \(\beta, \gamma = 1, \ldots, \nu\). Then, by (5.1) and (5.11),
\[
|\partial_z^\beta \partial_z^\gamma \varphi(t, z, z)| \leq \frac{4M_\mu}{1 - 4T^2 M_\mu}.
\]

Then
\[
|\partial_z \otimes \partial_z \varphi(t, z, z) |_{\omega = u x} |_{\infty} \leq \frac{4M_\mu}{1 - 4T^2 M_\mu}.
\]

Since the matrix \(D_\omega^{-1}\) is diagonal,
\[
|R(t, x)|_{\infty} \leq t |D_\omega^{-1}|_{\infty}^2 \times \frac{2M_\mu}{1 - 4T^2 M_\mu}
\]
\[
\leq \frac{t}{\omega_\flat \text{th}\left(\frac{\omega_\flat t}{2}\right)} \times \frac{4M_\mu}{1 - 4T^2 M_\mu}.
\]

For every \(\theta \in [0, +\infty]\),
\[
\frac{\theta}{\text{th} \theta} \leq 1 + 2\theta.
\] (5.12)

Then
\[
|R(t, x)|_{\infty} \leq \frac{8M_\mu(1 + \omega_\flat T)}{\omega_\flat^2 (1 - 4T^2 M_\mu)}
\] (5.13)

and by (5.8)
\[
|R(t, x)|_{\infty} < 8\alpha.
\] (5.14)
For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, let us denote by $\mathcal{M}^\mathbb{K}_{\text{sym}}$ the space of $\nu \times \nu$ symmetric matrices with entries in $\mathbb{K}$. Since the map $\mathcal{M}^\mathbb{R}_{\text{sym}} \to \mathcal{M}^\mathbb{C}_{\text{sym}}$, $C \mapsto C^2$ is a real analytic local diffeomorphism near $C = 1$, one can find $U \subset \mathcal{M}^\mathbb{R}_{\text{sym}}$ a neighbourhood of $1$ and a real analytic local diffeomorphism

$$\Xi : U \to \mathcal{M}^\mathbb{R}_{\text{sym}}$$

such that, denoting $S^{1/2} := \Xi(S)$, $\mathbb{1}^{1/2} = \mathbb{1}$ and

$$S^{1/2} S^{1/2} = S$$

(5.15)

for $S \in U$. Let $B_{\infty}^\mathbb{C}(1, \rho) \subset \mathcal{M}^\mathbb{C}_{\text{sym}}$ be the open ball of center $\mathbb{1}$ and radius $\rho$ (with respect to the norm $| \cdot |_\infty$). By analytic continuation, there exists $\rho_\mathbb{K} > 0$ such that $\Xi$ is analytic and (5.15) is satisfied on $B_{\infty}^\mathbb{C}(1, \rho_\mathbb{K})$. Let $\partial \Xi : \mathcal{E}$ be the tangent map associated to $\Xi$ at the point $C$. Since $\partial \Xi : \mathcal{E} = \frac{1}{2} H$, we can choose $\rho_\mathbb{K}$ small enough such that for every $S \in B_{\infty}^\mathbb{C}(1, \rho_\mathbb{K})$ and every $H \in \mathcal{M}^\mathbb{C}_{\text{sym}}$,

$$|S^{1/2} - \mathbb{1}| \leq \nu |S - \mathbb{1}|_\infty,$$

(5.16)

$$|\partial \Xi \cdot H| \leq \nu |H|_\infty.$$

(5.17)

Let us assume

$$8\alpha < \rho_\mathbb{K}.$$

(5.18)

By (5.10) and Proposition 4.10, $R \in C^0([0, T[, \mathcal{A}_{\mathcal{M}^\mathbb{C}_{\text{sym}}}(D_{R(\ast, \cdot)})]$ and by (5.14)

$$|R(t, x)|_\infty < \rho_\mathbb{K}.$$

Then

$$B(t, x) = D_\nu \left( \mathbb{1} + R(t, x) \right)^{1/2} \left( \mathbb{1} + R(t, x) \right)^{1/2} D_\nu.$$

Let

$$\theta(t, x) := \left( \mathbb{1} + R(t, x) \right)^{1/2} D_\nu x.$$

(5.19)

Then

$$\phi |_{y = x} = x \cdot (B(t, x)x) = \theta^2(t, x).$$

Let

$$\Lambda(t, x) := D_\nu^{-1} \left( \left( \mathbb{1} + R(t, x) \right)^{1/2} - \mathbb{1} \right) D_\nu x.$$

Then (5.17) holds. Notice also that $\Lambda \in C^0([0, T[, \mathcal{A}_{\mathcal{E}}(D_{R(\ast, \cdot)})]$. 

-2- We want to prove that (5.6) holds. We shall need an additional condition on $\alpha$. One has

$$\partial_{x, \gamma} \Lambda(t, x) = D_\nu^{-1} \left( \left( \mathbb{1} + R(t, x) \right)^{1/2} - \mathbb{1} \right) D_\nu e_\gamma + D_\nu^{-1} \left( \partial_{x, \gamma} \left( \mathbb{1} + R(t, x) \right)^{1/2} \right) D_\nu x.$$

By (5.16) and (5.13)

$$\left| D_\nu^{-1} \left( \left( \mathbb{1} + R(t, x) \right)^{1/2} - \mathbb{1} \right) D_\nu e_\gamma \right| \leq \nu \left| D_\nu^{-1} \right| \left| R(t, x) \right|_\infty \left| D_\nu \right| |e_\gamma| \leq \frac{8 M_\mu (1 + \omega_T T)}{\omega_\mathbb{K} (1 - 4 T^2 M_\mu)} \times \frac{\omega_\mathbb{K}}{\omega_\mathbb{K}}$$

30
since
\[ |D^{-1}_\omega|D_\omega| = \frac{\omega^{1/2} \text{th}^{1/2} \left( \frac{\omega T}{2} \right)}{\omega^{1/2} \text{th}^{1/2} \left( \frac{\omega T}{2} \right)} \leq \frac{\omega}{\omega_b}. \]

By (5.17)
\[ |D^{-1}_\omega \partial_{x_\gamma} (1 + R(t,x))^{1/2} D_\omega x| \leq \nu \frac{\omega}{\omega_b} |\partial_{x_\gamma} R(t,x)|_{\infty} |x| \leq \nu^{3/2} \frac{\omega}{\omega_b} \max_{1 \leq \delta \leq \nu} |x_\delta \partial_{x_\gamma} R(t,x)|_{\infty}. \]

By (5.10)
\[ x_\delta \partial_{x_\gamma} R(t,x) := -2tD^{-1}_\omega \left( \int_0^1 (1-u)z_\delta \partial_{z_\gamma} (\partial_z \otimes \partial_z) \varphi(t,z,z)|_{z=ux} du \right) D^{-1}_\omega \]
Let \( z \in D_{B_\nu \cdot 1}. \) Since
\[ z_\delta \partial_{x_\gamma} \partial_{z_\beta} \partial_{z_\alpha} \psi = \partial_{x_\gamma} \partial_{z_\beta} \partial_{z_\alpha} z_\delta \psi - (\delta_\beta = \delta \partial_{z_\beta} \partial_{z_\alpha} + \delta_\alpha = \delta \partial_{z_\alpha} \partial_{z_\beta} + \delta_\gamma = \delta \partial_{z_\gamma} \partial_{z_\beta} \partial_{z_\alpha}) \psi \]
and by (5.11), (5.12), (5.13), one gets
\[ |z_\delta \partial_{x_\gamma} \partial_{z_\beta} \partial_{z_\alpha} \varphi(t,z,z)| \leq \frac{M'_{\mu}}{1 - 4T^2 M_{\mu}} \text{ch} \left( \frac{\omega T}{2} \right) + 3 \times \frac{4M_{\mu}}{1 - 4T^2 M_{\mu}} \]
Then
\[ |x_\delta \partial_{x_\gamma} R(t,x)|_{\infty} \leq \frac{26M'_{\mu}(1 + \omega T)}{\omega_b^2(1 - 4T^2 M_{\mu})} \text{ch} \left( \frac{\omega T}{2} \right). \]
Then
\[ |\partial_{x_\gamma} \Lambda(t,x)| \leq 34\nu^{3/2} \text{ch} \left( \frac{\omega T}{2} \right) \frac{\omega}{\omega_b^2} (1 + \omega T)M'_{\mu} \]
and
\[ |\partial_{x_\gamma} \Lambda(t,x)| \leq 34\nu^{5/2} \text{ch} \left( \frac{\omega T}{2} \right) \frac{\omega}{\omega_b^2} (1 + \omega T)M'_{\mu}. \]
Let us assume that
\[ 34\nu^{5/2} \alpha < \frac{1}{2}. \]
Then, by (5.8), \( |\partial_{x_\gamma} \Lambda(t,x)| < \frac{1}{2}. \) This proves (5.6). \( \square \)

The following proposition gives a Borel summability property concerning the function \( w \) (see Proposition 4.10 for the definition of this function).
Proposition 5.3 Let \(\omega_1, \ldots, \omega_\nu, T, R, \varepsilon > 0\). Let \(\mu\) be a \(\mathbb{C}\)-valued Borel measure defined on \(\mathbb{R}^\nu\) such that

\[4T^2e^TM_{\mu, \varepsilon} < 1.\]  \hfill (5.21)

Then there exist \(\kappa, K, K_1 > 0\) and a function \(\hat{W} \in C^0([0, T], \mathcal{A}(D_{R^2, 1/2}^2 \times S_\kappa))\) satisfying, for every \((t, x, y) \in [0, T] \times D_{R^2, 1/2}^2\),

\[\forall \sigma \in S_\kappa, |\hat{W}(t, x, y, \sigma)| \leq K_1e^{K|\sigma|^{1/2}}\]  \hfill (5.22)

and

\[\forall h \in \mathbb{C}^+, \exp(w(t, x, y, h)) = \int_0^{+\infty} e^{-\sigma/h} \hat{W}(t, x, y, \sigma) \frac{d\sigma}{h}.\]  \hfill (5.23)

Proof

-1a- We first prove that there exist \(\kappa, K_2 > 0\) and a function \(\hat{w} \in C^0([0, T], \mathcal{A}(D_{R^2, 1/2}^2 \times S_{2\kappa}))\) such that, for \((t, x, y, \sigma) \in [0, T] \times D_{R^2, 1/2}^2 \times S_{2\kappa}\),

\[|\hat{w}(t, x, y, \sigma)| \leq K_2\]  \hfill (5.24)

and, for \(h \in \mathbb{C}^+\),

\[w(t, x, y, h) = \int_0^{+\infty} e^{-\sigma/h} \hat{w}(t, x, y, \sigma) \frac{d\sigma}{h}.\]  \hfill (5.25)

We proceed as in [Ha4]: for \(B \geq 0\), the Borel transform of the function \(h \to e^{-Bh}\) is \(\sigma \to J(B\sigma)\) where

\[J(z) := \sum_{n \geq 0} (-1)^n \frac{z^n}{(n!)^2} = \int_0^\pi \cos(2z^{1/2}\sin(\varphi)) \frac{d\varphi}{\pi}.\]

Therefore, let

\[\hat{w}_1(t, x, y, \sigma) := -t \int_0^{s_1} \int_{\mathbb{R}^\nu} \Omega_t, \xi \otimes_1 \xi J((\partial_\Omega_t, \xi \otimes_1 \xi)\sigma) e^{iq_1^{(s_1)} \xi_1} d\mu(\xi_1) ds_1\]

and for \(m \geq 2\)

\[\hat{F}_m(t, x, y, \sigma) := 2^{m-1}t^{2m-1}Y_t^{q, \xi}\Omega_t^{q, \theta} \xi \otimes_\nu_\mu \xi J((\partial_\Omega_t^{q, \theta}, \xi \otimes_\nu_\mu \xi)\sigma) e^{iq^{(s)}_1 \xi_1} d\mu(\xi_1) ds_1\]

\[\hat{w}_m(t, x, y, \sigma) := (-1)^m \sum_{g \in \Gamma_m} \int_0^{s_1} \int_{\mathbb{R}^\nu} \int_{[0, 1]^{q \times [0, 1]}} \int_{[0, 1]^{\nu \times [0, 1]}} \hat{F}_m(t, x, y, \sigma) d\nu d\mu(\xi) d\mu(\xi) ds_1 \]

32
(we shall check later that these integrals are convergent). Let us choose
\[ \kappa = \frac{\varepsilon}{2}. \]  
(5.26)

-1b- We claim that, for \( m \geq 1 \), \( \hat{w}_m \in C^0([0,T],\mathcal{A}(D^2_{R^*\cdot 1/2} \times S_{2\kappa})) \),
\[ |\hat{w}_m(t,x,y,\sigma)| \leq m(4T^2e^{T\mu\varepsilon})^m, \]  
and for \( h \in \mathbb{C}^+ \)
\[ w_m(t,x,y,h) = \int_{0}^{+\infty} e^{-\sigma/h} \hat{w}_m(t,x,y,\sigma) \frac{d\sigma}{h}. \]  
(5.28)

We shall use the following estimate (see [Ha4]). For every \( B \geq 0 \)
\[ |J(B\sigma)| \leq \exp(2B^{1/2}|\text{Im}\sigma^{1/2}|). \]  
(5.29)

In the sequel, we assume that \( x,y \in D_{R^*\cdot 1/2}, t \in [0,T] \) and \( \sigma \in S_{2\kappa} \).

-1c- Let \( m = 1 \). Since \( \sigma \in S_{2\kappa} \) and by (5.26),
\[ |\hat{w}_1(t,x,y,\sigma)| \leq m(4T^2e^{T\mu\varepsilon})^m, \]  
(5.27)
and for \( h \in BV^+ \)
\[ \hat{w}_1(t,x,y,h) = \int_{0}^{+\infty} e^{-\sigma/h} \hat{w}_1(t,x,y,\sigma) \frac{d\sigma}{h}. \]  
(5.28)

We shall use the following estimate (see [Ha4]). For every \( B \geq 0 \)
\[ |J(B\sigma)| \leq \exp(2B^{1/2}|\text{Im}\sigma^{1/2}|). \]  
(5.29)

In the sequel, we assume that \( x,y \in D_{R^*\cdot 1/2}, t \in [0,T] \) and \( \sigma \in S_{2\kappa} \).

-1d- Let \( m \geq 2 \). By (1.3), (5.29) and (5.30)
\[ |\hat{F}_m(t,x,y,\sigma)| \leq e^{mt\varepsilon}(\xi_1 + \cdots + \xi_m)^{1/2}. \]  
(5.28)

Therefore, by (4.11), (4.3) and (4.3),
\[ |\hat{F}_m(t,x,y,\sigma)| \leq \frac{2^{m-1}t^{2m-1}}{4m-1} \prod_{j,k \in g} |\xi_j||\xi_k| \times \frac{mt}{4}(\xi_1 + \cdots + \xi_m)^{1/2} \times e^{mt\varepsilon}(\xi_1 + \cdots + \xi_m)^{1/2} \times e^{\varepsilon(\xi_1 + \cdots + \xi_m)}. \]
Then, by the dominated convergence theorem, our regularity claim holds. Moreover, by mimicking the proof of Lemma 4.6,

$$\left| \hat{w}_m(t, x, y, \sigma) \right| \leq \frac{2^{2m-1}}{m(m-1)} \sum_{d_1, \ldots, d_m=2m-2} \left| \frac{\xi_1}{d_1!} \cdots \frac{\xi_m}{d_m!} \right| \times \frac{mt}{4} (\xi_1^2 + \cdots + \xi_m^2) e^{mt} e^\varepsilon (\xi_1^2 + \cdots + \xi_m^2) e^{\left| \xi_1 \right| d_1 \theta d d^{m-1} \left| \theta d \right|} \left| \mu \right| \otimes (\xi),$$

which yields, by (4.13) and (4.18),

$$\left| \hat{w}_m(t, x, y, \sigma) \right| \leq \frac{2^{2m-3}}{2(m-1)} (3m - 2)(2m) t^{2m} e^{mt} M_{\mu, \varepsilon}^m.$$

This proves (5.27). Since the Laplace transform of the function $\sigma \mapsto J(B\sigma)$ is the function $h \mapsto e^{-Bh}$, one gets (5.28).

-1e- Let us assume that (5.21) holds. By (5.27), the function $\hat{w}$ defined by

$$\hat{w}(t, x, y, \sigma) = \sum_{m \geq 1} \hat{w}_m(t, x, y, \sigma)$$

belongs to the space $C^0\left(\mathbb{R}^+, A(D_{R^+}^{2,2/1} \times S_2)\right)$ and

$$\left| \hat{w}(t, x, y, \sigma) \right| \leq \frac{4T^2 e^{T M_{\mu, \varepsilon}^m}}{(1 - 4T^2 e^{T M_{\mu, \varepsilon}^m})^2}.$$

Therefore (5.24) holds and, by (5.28), (5.25) is also satisfied.

-2- Since there exists $\rho > 0$ such that $S_\kappa + D_{\rho} \subset S_2$, and by the Cauchy formula, the function $\partial_{\sigma} \hat{w}$ is bounded on $]0, T[ \times D_{R^+}^{2,2/1} \times S_2$. Then, by a parameter-dependent version of Proposition 8.2 (see Appendix), there exists a function $\hat{W} \in C^0\left(\mathbb{R}^+, A(D_{R^+}^{2,2/1} \times S_2)\right)$ satisfying (5.22) and (5.23). □

6 A Gaussian Borel summation statement

6.1 A multidimensional statement

Definition 6.1 Let $R, K > 0$. Let $N_{R,K}^{\nu}$ be the set of analytic functions on $D_{R^+}^{2,2/1} \times D_{R^+}^{2,2/1} \times D_{R^+}^{2,2/1}$ satisfying

$$a \in N_{R,K}^{\nu} \iff \forall (R', K') \in \mathbb{R} \times [K, +\infty], \exists C > 0, \forall (x, \sigma) \in D_{R^+}^{2,2/1} \times D_{R^+}^{2,2/1},$$

$$\left| a(x, \sigma) \right| \leq C e^{K'(|x|^2 + |\sigma|)}.$$  (6.1)
Definition 6.2 Let $R, K > 0$. Let $\hat{N}_{R,K}$ be the set of analytic functions on $D_{R^+, R}$ satisfying

$$\hat{c} \in \hat{N}_{R,K} \iff \forall (R', K') \in [0, R] \times ]K, +\infty[, \exists C > 0, \forall \tau \in D_{R^+, R'},$$

$$|\hat{c}(\tau)| \leq C e^{K'|\tau|}. \quad (6.2)$$

Notice that the space $\hat{N}_{R,K}$ is invariant under the transformations $\hat{c} \mapsto \partial_\tau \hat{c}, \tau \hat{c}$.

Proposition 6.3

1. Let $R, K > 0$. For every $a \in N_{R,K}^\nu$, there exists $\hat{b} \in \hat{N}_{R,K}$ such that, for $h \in \mathbb{C}, \Re(e^{\frac{1}{h}}) > K$,

$$\int_{R^v} \int_0^{+\infty} e^{-(x^2 + \sigma)/h} a(x, \sigma) \frac{dx d\sigma}{h^{1+v/2}} = \int_0^{+\infty} \hat{b}(\tau) e^{-\tau/h} d\tau. \quad (6.3)$$

2. Let us assume that there exist $\kappa', C', K' > 0$ such that the function $a$ is analytic on $D_{R^v, \sqrt{\kappa}} \times S_{\kappa'}$ and

$$\forall (x, \sigma) \in D_{R^v, \sqrt{\kappa}} \times S_{\kappa'}, |a(x, \sigma)| \leq C' e^{K'|\sigma|^{1/2}}. \quad (6.4)$$

Then the function $\hat{b}$ is analytic on $S_{\kappa''}$, for every $K'' > K'$ and $\kappa'' < \kappa'$, there exists $C'' > 0$ such that

$$\forall \tau \in S_{\kappa''}, |\hat{b}(\tau)| \leq C'' e^{K''|\tau|^{1/2}} \quad (6.5)$$

and (6.3) holds for $h \in \mathbb{C}^+$.

6.2 Hypergeometric vection transforms

In this subsection, we introduce transforms exhibiting the analytic content of Proposition 6.3 and useful for its proof.

Definition 6.4 Let $R, K > 0$. Let $\mathcal{H}_{1/2 \rightarrow 1}$ and $\mathcal{H}_{1 \rightarrow 1/2}$ be the operators defined on $\hat{N}_{R,K}$ by

$$\mathcal{H}_{1/2 \rightarrow 1}(\hat{c})(\tau) := \frac{2}{\pi} \int_0^{\pi/2} \hat{c}(\tau \sin^2 \theta) d\theta,$$

$$\mathcal{H}_{1 \rightarrow 1/2}(\hat{c})(\tau) := \int_0^{\pi/2} \hat{c}(\tau \sin^2 \theta) \sin \theta d\theta + 2\tau \int_0^{\pi/2} \hat{c}'(\tau \sin^2 \theta) \sin^3 \theta d\theta.$$

35
Proposition 6.5 Let $R, K > 0$. For every $\hat{a} \in \mathcal{N}_{R,K}$ (respectively $\hat{b} \in \mathcal{N}_{R,K}$) there exists a unique $b \in \mathcal{N}_{R,K}$ (respectively $\hat{a} \in \mathcal{N}_{R,K}$) such that, for $h \in \mathbb{C}$, $\Re(e^{h}) > K$,

$$2 \int_{0}^{+\infty} \hat{a}(\tau^2) e^{-\tau^2/h} \frac{d\tau}{(\pi h)^{1/2}} = \int_{0}^{+\infty} b(\tau) e^{-\tau/h} d\tau.$$ 

Moreover $\hat{b}(\tau) = \mathcal{H}_{1/2} \hat{a}(\tau)$ and $\hat{a}(\tau) = \mathcal{H}_{1/2}(\hat{b})(\tau)$.

**Proof** Since

$$2 \int_{0}^{+\infty} \hat{a}(\tau^2) e^{-\tau^2/h} \frac{d\tau}{(\pi h)^{1/2}} = \frac{1}{\Gamma(1/2)} \int_{0}^{+\infty} \hat{a}(\zeta) e^{-\zeta/h} \frac{d\zeta}{h^{1/2} \zeta^{1/2}}.$$ 

Proposition 6.5 is a consequence of the following proposition with the choice $\gamma = \frac{1}{2}$.

Proposition 6.6 Let $\gamma \in ]0,1[$. Let $R, K > 0$. For every $\hat{a} \in \mathcal{N}_{R,K}$ (respectively $\hat{b} \in \mathcal{N}_{R,K}$) there exists a unique $b \in \mathcal{N}_{R,K}$ (respectively $\hat{a} \in \mathcal{N}_{R,K}$) such that, for $h \in \mathbb{C}$, $\Re(e^{h}) > K$,

$$\frac{1}{\Gamma(\gamma)} \int_{0}^{+\infty} \hat{a}(\tau) e^{-\tau/h} \left(\frac{\tau}{h}\right)^{\gamma-1} \frac{d\tau}{h} = \int_{0}^{+\infty} b(\tau) e^{-\tau/h} d\tau.$$ 

(6.6)

Moreover $\hat{b}(\tau) = \mathcal{H}_{\gamma-1}(\hat{a})(\tau)$ and $\hat{a}(\tau) = \mathcal{H}_{1-\gamma}(\hat{b})(\tau)$ where

$$\mathcal{H}_{\gamma-1}(\hat{a})(\tau) := \frac{\sin(\pi \gamma)}{\pi} \int_{0}^{1} \hat{a}(\tau u) u^{\gamma-1}(1-u)^{-\gamma} du,$$

$$\mathcal{H}_{1-\gamma}(\hat{b})(\tau) := \gamma \int_{0}^{1} \hat{b}(\tau u)(1-u)^{\gamma-1} du + \tau \int_{0}^{1} \hat{b}'(\tau u)(1-u)^{\gamma-1} du$$

(hypergeometric vector transforms).

**Proof** By the definition of the operators $\mathcal{H}_{\gamma-1}$ and $\mathcal{H}_{1-\gamma}$, the space $\mathcal{N}_{R,K}$ is invariant under these operators. See also Section 9.2 (Appendix). For two continuous functions $\hat{f}_{1}, \hat{f}_{2}$ on $[0, +\infty[$ such that $\tau^\rho \hat{f}_{k}(\tau) \rightarrow 0$ for some $\rho \in ]0,1[$, let $*$ be the convolution product defined by

$$\hat{f}_{1} \ast \hat{f}_{2}(\tau) := \int_{0}^{\tau} \hat{f}_{1}(\tau_{1}) \hat{f}_{2}(\tau - \tau_{1}) d\tau_{1}.$$ 

(6.7)

Let $\hat{a} \in \mathcal{N}_{R,K}$. One has

$$\int_{0}^{+\infty} \hat{a}(\tau) e^{-\tau/h} \left(\frac{\tau}{h}\right)^{\gamma-1} d\tau = h^{1-\gamma} \times \int_{0}^{+\infty} \hat{a}(\tau) e^{-\tau/h} \tau^{\gamma-1} d\tau = \frac{1}{\Gamma(1-\gamma)} \int_{0}^{+\infty} \tau^{-\gamma} e^{-\tau/h} d\tau \times \int_{0}^{+\infty} \tau^{\gamma-1} \hat{a}(\tau) e^{-\tau/h} d\tau = \frac{1}{\Gamma(1-\gamma)} \int_{0}^{+\infty} \hat{A}(\tau) e^{-\tau/h} d\tau.$$ 

36
where the function \( \hat{A} \) denotes the convolution product of the functions \( \tau \mapsto \hat{a}(\tau), \tau^{-\gamma} \). But

\[
\hat{A}(\tau) = \int_0^\tau \tau_1^{\gamma-1} \hat{a}(\tau_1)(\tau - \tau_1)^{-\gamma} d\tau_1 = \int_0^1 \hat{a}(\tau u) u^{\gamma-1}(1 - u)^{-\gamma} du.
\]

This proves the existence of a function \( \hat{b} \in \hat{N}_{R,K} \) satisfying (6.6) and justifies the definition of \( \mathcal{H} \mathcal{V}_{\gamma \rightarrow 1} \).

Let \( \hat{b} \in \hat{N}_{R,K} \). A function \( \hat{a} \) satisfies (6.6) if and only if

\[
\int_0^{+\infty} \hat{a}(\tau) e^{-\tau/h} \tau^{\gamma-1} d\tau = h^{-1} \times \hat{I}(\gamma) h^\gamma \times \int_0^{+\infty} \hat{b}(\tau) e^{-\tau/h} d\tau.
\]

(6.8)

Since

\[
\hat{I}(\gamma) h^\gamma = \int_0^{+\infty} \tau^{\gamma-1} e^{-\tau/h} d\tau,
\]

the right hand side of (6.8) is equal to

\[
\frac{1}{h} \int_0^{+\infty} \hat{B}(\tau) e^{-\tau/h} d\tau = \int_0^{+\infty} \partial_\tau (\hat{B}(\tau)) e^{-\tau/h} d\tau
\]

where the function \( \hat{B} \) denotes the convolution product of the functions \( \tau \mapsto \hat{b}(\tau), \tau^{\gamma-1} \). But

\[
\tau^{1-\gamma} \partial_\tau (\hat{B}(\tau)) = \tau^{1-\gamma} \partial_\tau \left( \int_0^\tau \hat{b}(\tau_1)(\tau - \tau_1)^{\gamma-1} d\tau_1 \right)
\]

\[
= \tau^{1-\gamma} \partial_\tau \left( \tau^\gamma \int_0^1 \hat{b}(\tau u)(1 - u)^{\gamma-1} du \right)
\]

\[
= \gamma \int_0^1 \hat{b}(\tau u)(1 - u)^{\gamma-1} du + \tau \int_0^1 \hat{b}'(\tau u)(1 - u)^{\gamma-1} du.
\]

This proves the existence of a function \( \hat{a} \in \hat{N}_{R,K} \) satisfying (6.6) and justifies the definition of \( \mathcal{H} \mathcal{V}_{\gamma \rightarrow 1} \).

**Remark 6.7** Proposition 6.6 is also related to fractional calculus. Let \( \alpha > 0 \). Let \( f \) be a function continuous on \([0, \infty[\) and integrable on \([0, 1]\). For every \( x \in [0, \infty[ \), let

\[
\mathcal{I}^\alpha_{0,x} f := \frac{1}{\Gamma(\alpha)} \int_0^x (x - x_1)^{\alpha-1} f(x_1) dx_1
\]

(see also [M-R]). Then, if \( \gamma \in [0, 1[ \) and \( \hat{c} \in \hat{N}_{R,K} \),

\[
\mathcal{H} \mathcal{V}_{\gamma \rightarrow 1} (\hat{c})(\tau) = \frac{1}{\Gamma(\gamma)} \times \mathcal{I}^{\gamma-1}_{0,\tau} (\tau^{\gamma-1} \hat{c}),
\]

37
\[ \mathcal{H}_{1\to\gamma}(\hat{c})(\tau) = \Gamma(\gamma) \tau^{1-\gamma} \partial_\tau(I_{0,\gamma}^\nu \hat{c}). \]

The fact that the transforms \( \mathcal{H}_{1\to\gamma} \) and \( \mathcal{H}_{\gamma\to1} \) are inverse each to other is straightforward with the above formulas. One can also prove Proposition 6.6 by using a fractional integration by parts.

### 6.3 Proof of Proposition 6.3

Let \( R, K > 0 \) and \( a \in \mathcal{N}^\nu_{R,K} \). Let

\[
I(h) := \int_{\mathbb{R}^\nu} \int_0^{+\infty} e^{-(x^2 + \sigma)/h} a(x, \sigma) \frac{d^\nu x d\sigma}{h^{1+\nu/2}}.
\]

By the definition of \( \mathcal{N}^\nu_{R,K} \), the above integral is well defined if \( \text{Re}(h/\pi) > K \).

Moreover

\[
I(h) = \int_{\mathbb{R}^\nu} \int_0^{+\infty} e^{-(x^2 + \sigma)/h} A(x', \sigma) \frac{d^\nu-1 x' d\sigma}{h^{1+(\nu-1)/2}}
\]

where

\[
A(x', \sigma) := \int_R e^{-x'^2/h} a(x_1, x', \sigma) \frac{dx_1}{h^{1/2}}
\]

\[
= \frac{\pi^{1/2}}{2} \times 2 \int_0^{+\infty} e^{-x'^2/h} \left( a(x_1, x', \sigma) + a(-x_1, x', \sigma) \right) \frac{dx_1}{h^{1/2}}.
\]

By Proposition 6.5

\[
A(x', \sigma) = \int_0^{+\infty} e^{-x^2/h} B(\tau_1, x', \sigma) \frac{d\tau_1}{h}
\]

where

\[
B(\tau_1, x', \sigma) := \frac{1}{\pi^{1/2}} \int_0^{\pi/2} \left( a(\tau_1^{1/2} \sin \theta, x', \sigma) + a(-\tau_1^{1/2} \sin \theta, x', \sigma) \right) d\theta.
\]

Then, by iterating this argument,

\[
I(h) = \int_{[0, +\infty[^{\nu+1}} e^{-(\tau_1 + \cdots + \tau_\nu + \sigma)/h} d(\tau_1, \ldots, \tau_\nu, \sigma) \frac{d\tau_1 \cdots d\tau_\nu d\sigma}{h^{\nu+1}}
\]

where

\[
d(\tau_1, \ldots, \tau_\nu, \sigma) := \frac{1}{\pi^{\nu/2}} \int_{[0, \pi/2]^{\nu}} \sum_{\varepsilon_1 = \pm, \ldots, \varepsilon_\nu = \pm} a(\varepsilon_1 \tau_1^{1/2} \sin \theta_1, \ldots, \varepsilon_\nu \tau_\nu^{1/2} \sin \theta_\nu, \sigma) d\theta_1 \cdots d\theta_\nu.
\]

Then

\[
I(h) = \int_{0}^{+\infty} \tilde{b}(\tau) e^{-\tau/h} \frac{d\tau}{h}
\]

38
where

\[
\hat{b}(\tau) := \partial_\nu \left( \tau^{1/2} \int_{\epsilon_1 = \ldots = \epsilon_{\nu} = \pm 1} \frac{1}{\pi^{\nu/2}} \sum a(\epsilon_1 u_1^{1/2} \sin \theta_1 \tau^{1/2}, \ldots, \epsilon_{\nu} u_\nu^{1/2} \sin \theta_\nu \tau^{1/2}, u_{\nu+1} \tau) d\theta_1 \cdots d\theta_\nu du_1 \cdots du_\nu) \right).
\]

Let us remind that \( z \in D_{R^+, R} \Rightarrow |Im z|^{1/2} < R^{1/2} \Rightarrow \pm z^{1/2} \in D_{R, \sqrt{R}} \). Then, since \( u_1 + \cdots + u_\nu \leq 1 \),
\[
\tau \in D_{R^+, R} \Rightarrow (\epsilon_1 u_1^{1/2} \sin \theta_1 \tau^{1/2}, \ldots, \epsilon_{\nu} u_\nu^{1/2} \sin \theta_\nu \tau^{1/2}, u_{\nu+1} \tau) \in D_{R^+, \sqrt{R} \times D_{R^+, R}}.
\]

Then the analyticity of the function \( a \) on \( D_{R^+, \sqrt{R} \times D_{R^+, R}} \) implies the analyticity of the function \( \hat{b} \) on \( D_{R^+, R} \). The function \( a \) satisfies (6.1), thus the function \( \hat{b} \) satisfies (6.2) (we use in particular that the space \( \mathcal{N}_{R, K} \) is invariant under \( \hat{u} \mapsto \partial_\nu \hat{u}, \tau \hat{u} \)). Let us assume that the function \( a \) satisfies (6.3). Then, since
\[
\tau \in S_{\nu} \Rightarrow (\epsilon_1 u_1^{1/2} \sin \theta_1 \tau^{1/2}, \ldots, \epsilon_{\nu} u_\nu^{1/2} \sin \theta_\nu \tau^{1/2}, u_{\nu+1} \tau) \in D_{R^+, \sqrt{K} \times S_{\nu}},
\]
the function \( \hat{b} \) is analytic on \( S_{\nu} \). Moreover \( \hat{b} \) satisfies (6.5). This proves Proposition (6.3).

7 Proofs of Proposition 2.2 and Theorem 2.3

7.1 Proof of Proposition 2.2

By (2.2), one can use Proposition 4.10. Then, by (4.27),
\[
\langle y | e^{-\frac{1}{2} H} | x \rangle = h^{-\nu/2} e^{-\Phi(t, x, y) / h} \times \left( \prod_{\nu=1}^{\nu} \frac{\omega_{\nu}^{1/2}}{\sqrt{4\pi \sinh^{1/2}(\omega_{\nu} t)}} \right) e^{w(t, x, y, h)}.
\]

By (2.2), one can use Proposition 6.3. This proves Proposition 2.2.

7.2 Proof of Theorem 2.3

Without loss of generality, one can assume \( c(0) = 0 \). Then \( \Theta_H^{\text{conj}} = \Theta_H \). Let \( u \) be the solution of (4.8). For \( t, h > 0 \), one has (see Remark 4.4)
\[
\Theta_H(t, h) = \int_{\mathbb{R}^n} u(t, x, x, h) dx.
\]

By Proposition 4.10 there exist \( \phi \in C^0([0, T], \mathcal{A}(D^2_{R^+, 1})) \) and
\[
w \in C^0([0, T], \mathcal{A}(D^2_{R^+, 1} \times \mathbb{C}^+))
\]
such that
\[
u(t, x, x, h) := (4\pi h)^{-\nu/2} \prod_{\nu=1}^{\nu} \left( \frac{\omega_{\nu}}{\sinh(\omega_{\nu} t)} \right)^{1/2} \times \exp \left( -\frac{1}{h} \phi |_{y=x} \times \exp (w |_{y=x}) \right).
\]

39
By Proposition 5.2 there exists \( \Lambda \in C^0([0,T], \mathcal{A}(D_{R^\nu,1})) \), such that \( \phi|_{y=x} = \theta^2(t,x) \) where \( \theta(t,x) := D_\omega \varphi(t,x) \), \( \varphi(t,x) := x + \Lambda(t,x) \) (see 5.4 for the definition of the matrix \( D_\omega \)). By Proposition 5.3 there exist \( \kappa > 0 \) and a function \( \hat{W} \in C^0([0,T], \mathcal{A}(D_{R^\nu,1/2} \times S_\kappa)) \) (choose \( \hat{W}(t,x,\sigma) := W(t,x,\sigma) \)) such that
\[
\exp(w|_{y=x}) = \int_0^{+\infty} e^{-\sigma/h} \hat{W}(t,x,\sigma) \frac{d\sigma}{h}
\]
Therefore
\[
\Theta_H(t,h) = \prod_{\nu=1}^\nu \left( \frac{\omega_\nu}{4\pi \text{sh}(\omega_\nu t)} \right)^{1/2} \times \int_{R^\nu} \int_0^{+\infty} e^{-\left(\theta^2(t,x) + \sigma\right)/h} \hat{W}(t,x,\sigma) \frac{d\nu y d\sigma}{h^{1+\nu/2}}.
\]
Since the assertion I of Proposition 5.2 is satisfied, one can use a parameter-dependent version of Proposition 8.3 (see Appendix). Therefore \( \varphi(t,\cdot)|_{R^\nu} \) and \( \theta(t,\cdot)|_{R^\nu} \) are global diffeomorphisms from \( R^n \) onto \( R^n \). Then
\[
\Theta_H(t,h) = \int_{R^\nu} \int_0^{+\infty} e^{-\left(y^2 + \sigma\right)/h} a(t,y,\sigma) \frac{d\nu y d\sigma}{h^{1+\nu/2}}.
\]
where
\[
a(t,y,\sigma) := \epsilon \prod_{\nu=1}^\nu \left( \frac{\omega_\nu}{4\pi \text{sh}(\omega_\nu t)} \right)^{1/2} \times \frac{\hat{W} \left( t, (\theta(t,\cdot)|_{R^\nu})^{-1}(y), \sigma \right)}{\text{det} \left( (\partial_\nu \theta(t,\cdot)) \circ (\theta(t,\cdot)|_{R^\nu})^{-1}(y) \right)}
\]
and \( \epsilon \) is the sign of the above determinant. By Proposition 8.3 \( (\varphi(t,\cdot)|_{R^\nu})^{-1} \) maps \( D_{R^\nu,1/2} \) into \( D_{R^\nu,1} \). By 5.6
\[
\sup_{(t,x) \in [0,T] \times D_{R^\nu,1}} \left| \text{det} \left( \partial_\nu \varphi(t,x) \right) \right|^{-1} < \infty.
\]
Moreover 5.2 holds. Let \( T_0 \in [0,T] \). Since \( \theta(t,x) = D_\omega \varphi(t,x) \), there exists \( \kappa', C', K' > 0 \) such that \( a \in C^0([T_0,T], \mathcal{A}(D_{R^\nu,\sqrt{\nu}} \times S_{\kappa'})) \) and, for every \( t \in [T_0,T] \),
\[
\forall (y,\sigma) \in D_{R^\nu,\sqrt{\nu}} \times S_{\kappa'}, \left| a(t,y,\sigma) \right| \leq C' e^{K'|\sigma|^{1/2}}
\]
(notice that, since the matrix \( D_\omega \) vanishes for \( t = 0 \), \( \kappa' \) goes to 0 and \( C' \) goes to \( \infty \) when \( T_0 \) goes to \( 0 \)). By a parameter-dependent version of Proposition 6.3 since the function \( a(t,\cdot) \) satisfies 6.3, one gets 2.17 and 2.10. This proves Theorem 2.3

8 Appendix A

Here is a statement about a non-negativity property which is useful when the tree graph equality is considered. Such a result is well known (see [A-R, Th.IV-5]).
Lemma 8.1 Let $I$ be a non-empty finite set. Let $M := (M_{j,k})_{j,k \in I}$ be a real symmetric non-negative matrix. Let $(u_{j,k})_{j,k \in I}$ be a symmetric matrix with coefficients in $[0, +\infty[$ such that, for every $j, k \in I$, $j \neq k$,

\[
\begin{cases}
\min_{l \in I} u_{l,l} \geq u_{j,k} \\
\forall q \in I - \{j, k\}, u_{j,k} \geq \min(u_{j,q}, u_{k,q})
\end{cases} \tag{8.1}
\]

Let $M^n$ be the matrix defined by

\[M^n_{j,k} = u_{j,k}M_{j,k}.\]

Then the matrix $M^n$ is symmetric non-negative.

**Proof** We prove the lemma by induction on $|I|$. The statement is straightforward if $|I| \leq 2$. Let us assume the lemma is proved for $|I| < n$ and let $I$ be a subset such that $|I| = n$. Let $u_{\text{min}} = \min_{j,k \in I} u_{j,k}$. Let $\mathcal{I}$ be the set containing every pair $(I_1, I_2)$ such that $\emptyset \neq I_1 \subset I$, $\emptyset \neq I_2 \subset I$, $I_1 \cap I_2 = \emptyset$ and

\[
\forall i_1 \in I_1, \forall i_2 \in I_2, u_{i_1, i_2} = u_{\text{min}}. \tag{8.2}
\]

Let $(I_1, I_2) \in \mathcal{I}$ be such that $|I_1| + |I_2|$ is maximal. Let us assume that $I_1 \cup I_2 \neq I$. Let $j \in I - I_1 \cup I_2$. Then there exist $i_1 \in I_1$ and $i_2 \in I_2$ such that $u_{i_1,j} > u_{\text{min}}$ and $u_{i_2,j} > u_{\text{min}}$. Then, by (8.1)

\[u_{i_1,i_2} \geq \min(u_{i_1,j}, u_{i_2,j}) > u_{\text{min}}\]

which contradicts (8.2). Thus $I_1 \cup I_2 = I$. Let $v_{j,k} := u_{j,k} - u_{\text{min}}$. Then

\[M^n_{j,k} = u_{\text{min}}M_{j,k} + 1_{j,k \in I_1} v_{j,k} M_{j,k} + 1_{j,k \in I_2} v_{j,k} M_{j,k}. \]

(8.1) always holds if the coefficients $u_{j,k}$ are replaced by the coefficients $v_{j,k}$ and the set $I$ is replaced by some subset. Then the matrices $(v_{j,k} M_{j,k})_{j,k \in I_1}$ and $(v_{j,k} M_{j,k})_{j,k \in I_2}$ are symmetric non-negative. Then, by the above decomposition, the matrix $M^n$ is symmetric non-negative. \qed

For the reader’s convenience we now verify the Borel summability of the exponential of a Borel summable expansion in the setting of the paper. The fact that Borel summability is preserved by composition with analytic functions is well known in similar settings.

**Proposition 8.2** Let $U \subset \mathbb{C}$ be an open convex neighbourhood of $\mathbb{R}^+$. Let $K_2 > 0$. Then there exist $K_1, K > 0$ satisfying the following property. For every analytic function $\hat{a}$ on $U$ such that

\[\max(\{ |\hat{a}(0)|, \sup_{\sigma \in U} |\hat{a}'(\sigma)| \}) \leq K_2 \]

there exists an analytic function $\hat{b}$ on $U$ satisfying

\[\sum_{j,k \in I} M_{j,k} x_j x_k \geq 0 \text{ for every } (x_j)_{j \in I} \in \mathbb{R}^I.\]

\[\text{Non-negativity means } \sum_{j,k \in I} M_{j,k} x_j x_k \geq 0 \text{ for every } (x_j)_{j \in I} \in \mathbb{R}^I.\]
1. for every \( \sigma \in U \)
\[ |\hat{b}(\sigma)| \leq K_{1}e^{K|\sigma|^{1/2}}, \]

2. for every \( h \in \mathbb{C}^{+} \)
\[ \exp\left(\int_{0}^{+\infty} e^{-\sigma/h}\hat{a}(\sigma)\frac{d\sigma}{h}\right) = \int_{0}^{+\infty} e^{-\sigma/h}\hat{b}(\sigma)\frac{d\sigma}{h}. \]

**Proof** Let \( h \in \mathbb{C}^{+} \). By integration by parts,
\[ \int_{0}^{+\infty} e^{-\sigma/h}\hat{a}(\sigma)\frac{d\sigma}{h} = \hat{a}(0) + \int_{0}^{+\infty} e^{-\sigma/h}\hat{a}'(\sigma)d\sigma. \]
Then
\[ \exp\left(\int_{0}^{+\infty} e^{-\sigma/h}\hat{a}(\sigma)\frac{d\sigma}{h}\right) = e^{\hat{a}(0)}\left(1 + \sum_{n \geq 1} \frac{1}{n!}A_n(h)\right) \tag{8.3} \]
where
\[ A_n(h) := \left(\int_{0}^{+\infty} e^{-\sigma/h}\hat{a}'(\sigma)d\sigma\right)^n \]
\[ = \int_{0}^{+\infty} e^{-\sigma/h}(\hat{a}')^{*,n}(\sigma)d\sigma \]
where
\[ (\hat{a}')^{*,n}(\sigma) := (\hat{a}' \ast \cdots \ast \hat{a}')^{n}(\sigma) \quad \text{(n times)} \]
\[ = \int_{\sigma_1 + \cdots + \sigma_n = \sigma} \hat{a}'(\sigma_1) \cdots \hat{a}'(\sigma_n)d\sigma_1 \cdots d\sigma_n \]
(see [6.7] for the definition of \( \ast\)). Let
\[ \hat{A}_n(\sigma) := \int_{0}^{\sigma} (\hat{a}')^{*,n}(\sigma')d\sigma' \]
\[ = \sigma^n \int_{u_1 + \cdots + u_n \leq 1} \hat{a}'(u_1\sigma) \cdots \hat{a}'(u_n\sigma)du_1 \cdots du_n. \]
Then, for \( \sigma \in U \),
\[ |\hat{A}_n(\sigma)| \leq \frac{K_n}{n!}|\sigma|^n \]
and, by integration by parts,
\[ A_n(h) = \int_{0}^{+\infty} e^{-\sigma/h}\hat{A}_n(\sigma)\frac{d\sigma}{h} \tag{8.4} \]
Let \( \hat{b} \) be the analytic function on \( U \) defined by
\[ \hat{b}(\sigma) = e^{\hat{a}(0)}\left(1 + \sum_{n \geq 1} \frac{1}{n!}\hat{A}_n(\sigma)\right). \]
Then
\[
|\hat{b}(\sigma)| \leq e^{\hat{a}(0)} \sum_{n \geq 0} \frac{1}{n!} K_2^n |\sigma|^n
\]
\[
\leq e^{K_2} \sum_{n \geq 0} \frac{1}{(2n)!} \left( 2K_2^{1/2} |\sigma|^{1/2} \right)^{2n}
\]
\[
\leq e^{K_2} \exp \left( 2K_2^{1/2} |\sigma|^{1/2} \right).
\]

Hence the function \( \hat{b} \) satisfies the assertion. By (8.3) and (8.4),
\[
\exp \left( \int_0^{+\infty} e^{-\sigma/h} \hat{a}(\sigma) \frac{d\sigma}{h} \right) = e^{\hat{a}(0)} \int_0^{+\infty} e^{-\sigma/h} \left( 1 + \sum_{n \geq 1} \frac{1}{n!} \hat{A}_n(\sigma) \right) \frac{d\sigma}{h}
\]
\[
= \int_0^{+\infty} e^{-\sigma/h} \hat{b}(\sigma) \frac{d\sigma}{h},
\]
and the assertion 2 is also satisfied. □

We also use in the paper the following result.

**Proposition 8.3** Let \( r > 0 \). Let \( \Lambda : D_{\mathbb{R}^\nu, r} \rightarrow \mathbb{C}^\nu \) be an analytic function such that

1. the function \( \Lambda_{\mathbb{R}^\nu} \) is \( \mathbb{R}^\nu \)-valued,
2. the matrix \( \partial_x \Lambda(x) := (\partial_{x_1} \Lambda(x), \ldots, \partial_{x_\nu} \Lambda(x)) \) satisfies
\[
M := \sup_{x \in D_{\mathbb{R}^\nu, r}} |\partial_x \Lambda(x)| < 1. \tag{8.5}
\]

Let \( \varphi \in A_{\mathbb{C}^\nu}(D_{\mathbb{R}^\nu, r}) \) be defined by \( \varphi(x) = x + \Lambda(x) \). Then \( \varphi|_{\mathbb{R}^\nu} \) is a global diffeomorphism from \( \mathbb{R}^\nu \) onto \( \mathbb{R}^\nu \). Moreover \( (\varphi|_{\mathbb{R}^\nu})^{-1} \) admits an analytic continuation\(^{3}\) on \( D_{\mathbb{R}^\nu, r(1-M)} \) and
\[
(\varphi|_{\mathbb{R}^\nu})^{-1}(D_{\mathbb{R}^\nu, r(1-M)}) \subset D_{\mathbb{R}^\nu, r}.
\]

**Proof** By (8.5), \( \varphi \) is a local diffeomorphism. Let us prove that \( \varphi \) is injective. Let \( x_1, x_2 \in D_{\mathbb{R}^\nu, r} \). Then
\[
\varphi(x_2) - \varphi(x_1) = \left( \text{Id} + A(x_1, x_2) \right)(x_2 - x_1) \tag{8.6}
\]
where
\[
A(x_1, x_2) := \int_0^1 (\partial_z \Lambda)(z)|_{z=x_1+u(x_2-x_1)} du.
\]
But the matrix \( \text{Id} + A(x_1, x_2) \) is invertible since
\[
|A(x_1, x_2)| \leq \int_0^1 |\partial_z \Lambda(z)||_{z=x_1+u(x_2-x_1)} du \leq M < 1.
\]
\(^{3}\)we choose the same notation for this continuation.

43
Then, by (8.6), \( \varphi \) is injective.

Let \( x_0 \in \mathbb{R}^\nu \) and let \( y_0 = \varphi(x_0) \). Let \( R > 0 \) and let \( (x, y) \in B(x_0, r) \times B(y_0, R) \). One has

\[
y = \varphi(x) \iff y-y_0 = \varphi(x)-\varphi(x_0) \iff (\text{Id}+A(x_0, x))^{-1}(y-y_0) = (x-x_0). \quad (8.7)
\]

Let us assume that \( 1 - M \times R < r \).

Let \( \rho \in ]\frac{1}{1-M}, r[ \). Then \( x \mapsto x_0 + (\text{Id}+A(x_0, x))^{-1}(y-y_0) \) maps \( B(x_0, \rho) \) into \( B(x_0, \rho) \) and has a fixed point by the Brouwer theorem since \( B(x_0, \rho) \) is compact and convex. Therefore, for every \( y \in B(y_0, R) \), there exists \( x \in B(x_0, \rho) \) satisfying (8.7). Then

\[
\forall x_0 \in \mathbb{R}^\nu, B(\varphi(x_0), r(1-M)) \subset \varphi(B(x_0, r)).
\]

The above inclusion holds if \( B(z, \varepsilon) \) denotes the real or complex ball of center \( z \in \mathbb{R}^\nu \) and radius \( \varepsilon > 0 \). In particular, \( \varphi|_{\mathbb{R}^\nu} \) is an open and closed map. Therefore \( \varphi|_{\mathbb{R}^\nu}(\mathbb{R}^\nu) = \mathbb{R}^\nu \). Since \( \varphi|_{\mathbb{R}^\nu} \) is injective, \( \varphi|_{\mathbb{R}^\nu} \) is a global diffeomorphism from \( \mathbb{R}^\nu \) onto \( \mathbb{R}^\nu \). Moreover

\[
\bigcup_{x_0 \in \mathbb{R}^\nu} B(\varphi(x_0), r(1-M)) \subset \bigcup_{x_0 \in \mathbb{R}^\nu} \varphi(B(x_0, r)).
\]

Then

\[
D_{\mathbb{R}^\nu, r(1-M)} \subset \varphi(D_{\mathbb{R}^\nu, r}).
\]

Therefore \( (\varphi|_{\mathbb{R}^\nu})^{-1} \) admits an analytic continuation on \( D_{\mathbb{R}^\nu, r(1-M)} \) and

\[
(\varphi|_{\mathbb{R}^\nu})^{-1}(D_{\mathbb{R}^\nu, r(1-M)}) \subset D_{\mathbb{R}^\nu, r}.
\]

\[\square\]

9 Appendix B

In this section we propose an interpretation of the tools used in Section 6 with respect to standard Borel summation concepts.

9.1 Gaussian integrals

For \( \theta \in ]0, \pi[ \) and \( R > 0 \), let

\[
C_{<, \theta} := \{ re^{i\varphi} \in \mathbb{C} | r > 0, \varphi \in ]-\theta, \theta[ \} \\
C_{R, <, \theta} := \{ re^{i\varphi} \in \mathbb{C} | r \in ]0, R[ , \varphi \in ]-\theta, \theta[ \},
\]
Definition 9.1 Let $\alpha \in ]0, 2[$. We say that

- $f$ is analytic on $\mathcal{C}_{\rho, \alpha(\frac{\pi}{2} + \varepsilon)}$,
- there exist $a_0, a_1, \ldots \in \mathbb{C}$, $R_0, R_1, \ldots$ analytic functions on $\mathcal{C}_{\rho, \alpha(\frac{\pi}{2} + \varepsilon)}$ such that, for every $r \geq 0$, for every $x \in \mathcal{C}_{\rho, \alpha(\frac{\pi}{2} + \varepsilon)}$,
  \[ f(x) = a_0 + \cdots + a_{r-1}x^{r-1} + R_r(x), \]  
  \[ |R_r(x)| \leq K \frac{\Gamma(1+\alpha r)}{\kappa^{\alpha r}}|x|^r. \]

- $\hat{f}$ is analytic on $D_{\kappa^\alpha} \cup \mathcal{C}_{\rho, \alpha\varepsilon}$,
- for every $\xi \in D_{\kappa^\alpha} \cup \mathcal{C}_{\rho, \alpha\varepsilon}$
  \[ |\hat{f}(\xi)| \leq Ke^{\frac{1}{\alpha} |\xi|^\alpha}. \]

One has (Watson’s lemma)

Theorem 9.2 Let $\alpha \in ]0, 2[$.

- If $f$ verifies $\mathcal{P}_{\text{wat}, \alpha}$, then
  \[ \hat{f}(\xi) := \sum_{r=0}^{\infty} \frac{a_r \xi^r}{\Gamma(1+\alpha r)} \]  
  admits an analytic continuation which verifies $\hat{\mathcal{P}}_{\text{wat}, \alpha}$.

- If $\hat{f}$ verifies $\hat{\mathcal{P}}_{\text{wat}, \alpha}$, then
  \[ f(x) := \int_{0}^{+\infty} \hat{f}(\xi)e^{-\frac{1}{\alpha} \xi^\alpha}d\xi \left(\frac{\xi^{1/\alpha}}{x^{1/\alpha}}\right) = \int_{0}^{+\infty} \hat{f}(x\xi^\alpha)e^{-\xi}d\xi \]  
  verifies $\mathcal{P}_{\text{wat}, \alpha}$.

- $\hat{f}$ given by \(9.4\) is called the $\alpha$-Borel transform of $f$. $f$ given by \(9.5\) is called the $\alpha$-Laplace transform of $\hat{f}$. These two transforms are inverse each to other.
We say that a function \( f \) satisfying \( \mathcal{P}_{\text{wat}, \alpha} \) is \( \frac{1}{\alpha} \mathbb{R}^+ \)-summable. The notion of \( \frac{1}{\alpha} \)-summability is related to the notion of critical time and celeration’s theory [Bals, E3, Ma-Ra]. Notice that two different indices \( \alpha, \alpha' \) may yield two different notions of Borel-summability [Lo].

Let \( \varepsilon, \kappa, \rho, K > 0 \). Let \( f \) be an analytic function on

\[
\mathcal{U} := C_{\rho, \prec, \frac{1}{\alpha} (\varepsilon \pi + \varepsilon)} \cup e^{i\pi} C_{\rho, \prec, \frac{1}{\alpha} (\varepsilon \pi + \varepsilon)}
\]
satisfying (9.1) and (9.2) for \( \alpha = \frac{1}{2} \) and \( x \in \mathcal{U} \). For \( s \in C_{\rho, \prec, \frac{1}{\alpha} + \varepsilon} \), let us define

\[
f_{\text{ev}}(s) = \frac{1}{2} (f(s^{1/2}) + f(-s^{1/2})),
\]

\[
f_{\text{odd}}(s) = \frac{s^{1/2}}{2} (f(s^{1/2}) - f(-s^{1/2})).
\]

Then the functions \( f_{\text{ev}} \) and \( f_{\text{odd}} \) are \( \frac{1}{\alpha} \mathbb{R}^+ \)-summable and one can easily describe the \( \frac{1}{\alpha} \mathbb{R}^+ \)-summable function \( f \) with the help of the \( 1-\mathbb{R}^+ \)-summable functions \( f_{\text{ev}} \) and \( f_{\text{odd}} \) since

\[
f(x) = f_{\text{ev}}(x^2) + \frac{1}{x} f_{\text{odd}}(x^2).
\]

Of course, this process does not hold for an arbitrary \( \frac{1}{\alpha} \mathbb{R}^+ \)-summable function. Let us now consider what happens in the Borel plane. By Theorem 9.2 there exists a function \( \hat{f} \) satisfying \( \mathcal{P}_{\text{wat}, 1/2} \) such that

\[
f(x) = \int_0^{+\infty} \hat{f}(\xi) e^{-\frac{x^2}{4\xi}} d\xi = \int_0^{+\infty} \hat{f}(x\xi) e^{-\xi^2} d\xi (\xi^2).
\]

By the properties of the function \( f \), there exist \( \varepsilon, \kappa, \rho, K > 0 \) such that the function \( \hat{f} \) is analytic on \( D_{\kappa, \prec, \frac{1}{\alpha} \varepsilon} \cup e^{i\pi} C_{\kappa, \prec, \frac{1}{\alpha} \varepsilon} \) and satisfies (9.3) on \( D_{\kappa, \prec, \frac{1}{\alpha} \varepsilon} \cup e^{i\pi} C_{\kappa, \prec, \frac{1}{\alpha} \varepsilon} \). Let \( \hat{f}_{\text{ev}} \) and \( \hat{f}_{\text{odd}} \) be the analytic functions defined on \( D_{\kappa} \cup C_{\prec, \varepsilon} \) by

\[
\hat{f}_{\text{ev}}(\zeta) := \frac{1}{2} (\hat{f}(\zeta^{1/2}) + \hat{f}(-\zeta^{1/2})),
\]

\[
\hat{f}_{\text{odd}}(\zeta) := \zeta^{1/2} (\hat{f}(\zeta^{1/2}) - \hat{f}(-\zeta^{1/2})).
\]

Then

\[
\hat{f}(\xi) = \hat{f}_{\text{ev}}(\xi^2) + \frac{1}{2\xi} \hat{f}_{\text{odd}}(\xi^2)
\]

and

\[
f_{\text{ev}}(s) = \int_0^{+\infty} \hat{f}_{\text{ev}}(\xi) e^{-\frac{s^2}{4\xi}} d\xi,
\]

\[
f_{\text{odd}}(s) = \int_0^{+\infty} \hat{f}_{\text{odd}}(\xi^2) e^{-\frac{s^2}{4\xi}} d\xi (\xi^2). \quad (9.6)
\]

In the last expression, we recognize the integral occurring in Proposition 6.5. Then Proposition 6.5 allows one to express the integral in (9.6) as a standard Laplace transform: the function \( f_{\text{odd}} \) is \( 1-\mathbb{R}^+ \)-Borel summable.
9.2 A remark about hypergeometric vection transforms

In this section, we do not give rigorous statements for the sake of conciseness. Here we consider the following choice for the Laplace and the Borel transform. If \( \hat{f} \) is a function of a complex variable \( \zeta \), we denote

\[
L \hat{f}(x) = \int_0^{+\infty} \hat{f}(\zeta)e^{-\zeta x}d\zeta.
\]

Let \( \sigma \in [0, +\infty[. \) Then the Borel transform \( B \), which is the inverse transform of \( L \), satisfies

\[
B(x^\sigma)(\zeta) = \frac{\zeta^{\sigma-1}}{\Gamma(\sigma)}.
\]

With this choice, the pointwise product is mapped on the natural convolution product defined by (6.7). Let \( \gamma \in [0, 1[. \) Let

\[
\hat{b}(\zeta) := \sum_{n \geq 0} \hat{b}_n \zeta^n.
\]

Then

\[
b(x) := L(\hat{b})(\zeta) = \sum_{n \geq 0} \Gamma(n + 1)\hat{b}_n x^{n+1}
\]

and

\[
b(x) = x^{1-\gamma} \sum_{n \geq 0} \Gamma(n + 1)\hat{b}_n x^{n+\gamma}.
\]  \( (9.7) \)

Then

\[
\hat{b}(\zeta) = B(x^{1-\gamma})(\zeta) * \left( \sum_{n \geq 0} \frac{\Gamma(n + 1)\hat{b}_n}{\Gamma(n + \gamma)} \zeta^{n+\gamma-1} \right).
\]

This induces a natural factorization:

\[
\hat{b}(\zeta) = B(x^{1-\gamma})(\zeta) * \left( B(x^\gamma)(\zeta) \times \hat{a}(\zeta) \right)
\]

where

\[
\hat{a}(\zeta) := \sum_{n \geq 0} \frac{\Gamma(\gamma)\Gamma(n + 1)}{\Gamma(n + \gamma)} \hat{b}_n \zeta^n.
\]

Notice that

\[
\mathcal{H}V_{\gamma \rightarrow 1}(\hat{a})(\zeta) = B(x^{1-\gamma})(\zeta) * \left( B(x^\gamma)(\zeta) \times \hat{a}(\zeta) \right).
\]

Then the content of Proposition 6.6 is related to the factorization (9.7). Notice also that, for every \( n \in \mathbb{N}, \)

\[
\mathcal{H}V_{\gamma \rightarrow 1}(\zeta^n) = \frac{\Gamma(n + \gamma)}{\Gamma(\gamma)\Gamma(n + 1)} \zeta^n
\]

= \[
\frac{(\gamma)_n}{n!} \zeta^n
\]
where

\[(\gamma)_n := \gamma(\gamma + 1) \cdots (\gamma + n - 1).\]

Let \(a, b, c \in [0, 1].\) Then hypergeometric vection transforms give, for instance, the following decomposition of the standard hypergeometric function \(\, _2F_1\)

\[
\, _2F_1(a, b; c; z) := \sum_{n \geq 0} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}
= \mathcal{HV}_{a \to 1} \circ \mathcal{HV}_{b \to 1} \circ \mathcal{HV}_{1 \to c} \left( \frac{1}{1-z} \right).
\]

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