GLOBAL BEHAVIOR OF BIFURCATION CURVES FOR THE
NONLINEAR EIGENVALUE PROBLEMS WITH PERIODIC
NONLINEAR TERMS

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ABSTRACT. We consider the bifurcation problem
\[-u''(t) = \lambda (u(t) + g(u(t)), \ t \in I := (-1, 1), \ u(\pm 1) = 0,\]
where \(g(u) \in C^1(\mathbb{R})\) is a periodic function with period \(2\pi\) and \(\lambda > 0\) is a bifurcation parameter. It is known that, under the appropriate conditions on \(g\), \(\lambda\) is parameterized by the maximum norm \(\alpha = \|u_\lambda\|_\infty\) of the solution \(u_\lambda\) associated with \(\lambda\) and is written as \(\lambda = \lambda(\alpha)\). If \(g(u)\) is periodic, then it is natural to expect that \(\lambda(\alpha)\) is also oscillatory for \(\alpha \gg 1\). We give a simple condition of \(g(u)\), by which we can easily check whether \(\lambda(\alpha)\) is oscillatory and intersects the line \(\lambda = \pi^2/4\) infinitely many times for \(\alpha \gg 1\) or not.

1. Introduction. This paper is concerned with the following nonlinear eigenvalue problems
\[-u''(t) = \lambda (u(t) + g(u(t)), \ t \in I := (-1, 1), \ u(t) > 0, \ t \in I, \ u(-1) = u(1) = 0,\]
where \(\lambda > 0\) is a parameter. We assume that \(g(u)\) satisfies the following conditions.

(A.1) \(g(u) \in C^1(\mathbb{R})\) and \(u + g(u) > 0\) for \(u > 0\).
(A.2) \(g(u + 2\pi) = g(u)\) for \(u \in \mathbb{R}\).

It is known from [15] that, under the condition (A.1), for any given \(\alpha > 0\), there exists a unique solution pair \((\lambda, u_\alpha)\) of (1.1)–(1.3) with \(\alpha = \|u_\alpha\|_\infty\) and \(\lambda\) is parameterized by \(\alpha\) as \(\lambda = \lambda(\alpha)\). Furthermore, \(\lambda(\alpha)\) is continuous in \(\alpha > 0\). We write \(\lambda = \lambda(g, \alpha)\), since \(\lambda\) also depends on \(g\).

The study of the structures of the bifurcation curves is one of the main topics in bifurcation analysis, and there are quite many works concerning the properties of bifurcation diagrams. We refer to [1-4, 6, 7, 10, 14, 16, 17] and the references therein. In particular, the qualitative properties of the oscillatory bifurcation diagrams have been studied intensively. We refer to [8, 11, 12, 13, 18, 19] and the references therein.

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therein. In this paper, we focus on the study whether \( \lambda(g, \alpha) \) inherits the oscillatory properties of \( g(u) \) or not if \( g(u) \) is a periodic function.

To clarify our intention, we consider the typical example \( g_0(u) = (1/2) \sin u \), which satisfies (A.1)–(A.2). Recently, the following asymptotic formula for \( \lambda(g_0, \alpha) \) as \( \alpha \to \infty \) has been obtained in [18].

**Theorem 1.0 ([18, Theorem 1.2]).** Let \( g_0 = (1/2) \sin u \). Then as \( \alpha \to \infty \),

\[
\lambda(g_0, \alpha) = \frac{\pi^2}{4} - \frac{\pi}{2\alpha} \sqrt{\frac{\pi}{2\alpha}} \sin \left( \alpha - \frac{1}{4} \pi \right) + O(\alpha^{-2}).
\]  

We see from Theorem 1.0 that \( \lambda(g_0, \alpha) \) satisfies the following oscillatory property (OP).

\( \text{(OP)} \) \( \lambda(g, \alpha) \to \pi^2/4 \) as \( \alpha \to \infty \), and it intersects the line \( \lambda = \pi^2/4 \) infinitely many times for \( \alpha \gg 1 \).

![Fig. 1: \( \lambda(g, \alpha) \) with (OP)](image1)

The purpose of this paper is to establish a simple condition, from which we understand immediately whether \( \lambda(g, \alpha) \) satisfies (OP) or not. For instance, we consider the following typical example. Let \( \delta, \epsilon > 0 \) be small fixed constants. We consider \( \psi \in C^1(\mathbb{R}) \) satisfying \( \psi(t) > 0 \) for \( t \in I_\delta := (\pi/2 - \delta, \pi/2 + \delta) \) and \( \psi(t) = 0 \) for \( [-\pi, \pi] \setminus I_\delta \), and \( g_\epsilon(u) := \sin u + \epsilon \psi(u) \) for \( u \in [-\pi, \pi] \), \( g_\epsilon(u + 2\pi) = g_\epsilon(u) \) for \( u \in \mathbb{R} \). Clearly, \( g_\epsilon(u) \) satisfies (A.1)–(A.2). However, it seems difficult to distinguish whether \( g(u) \) satisfies (OP) or not.

![Fig. 2: graph of \( \sin x + \epsilon \psi(x) \)](image2)

Since \( g(u) \) is bounded in \( \mathbb{R} \) by (A.2), it is clear that \( \lambda(g, \alpha) \to \pi^2/4 \) as \( \alpha \to \infty \). Therefore, the essential point is to find the condition whether \( \lambda(g, \alpha) \) intersects the line \( \lambda = \pi^2/4 \) infinitely many times for \( \alpha \gg 1 \).

Now we state our main results.
Theorem 1.1. Assume that $g(u)$ satisfies (A.1)–(A.2). Then as $\alpha \to \infty$,
\[
\lambda(g, \alpha) = \frac{\pi^2}{4} - \frac{\pi a_0}{2\alpha} - \frac{1}{2\alpha} \sqrt{\frac{\pi}{2\alpha}} \sum_{n=1}^\infty \frac{c_n}{n^{3/2}} + O(\alpha^{-2}),
\]
where
\[
a_0 := \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) d\theta,
\]
\[
c_n := \int_{-\pi}^{\pi} g'(\theta) \cos \left( n(\theta - \alpha) + \frac{3}{4}\pi \right) d\theta, \quad (n \in \mathbb{N}).
\]

As a corollary of Theorem 1.1, we obtain a meaningful result for the asymptotic property of $\lambda(g, \alpha)$.

Corollary 1.2. Assume that $g(u)$ satisfies (A.1)–(A.2). If $a_0 \neq 0$, then $\lambda(g, \alpha)$ does not satisfy (OP).

We apply Corollary 1.2 to $\lambda(g, \alpha)$. In this case, we have
\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} g_0(\theta) d\theta = \frac{1}{\pi} \int_{-\pi}^{\pi} (\sin \theta + c\psi(\theta)) d\theta = \frac{c}{\pi} \int_{-\pi}^{\pi} \psi(\theta) d\theta > 0.
\]

Therefore, by (1.5), $\lambda(g, \alpha)$ does not satisfy (OP).

Theorem 1.1 is also useful to determine $g(u)$ satisfies (OP). For instance, let $g_1(u) = (\sin u + \sin 2u)/4$. We show that $\lambda(g_1, u)$ satisfies (OP) in Example 2.2 in Section 2.

The method to study the local behavior of $\lambda(\alpha)$ has been already established in [17] and [18], since the time-map method and Taylor expansion work very well in this case. To understand the total structure of $\lambda(g, \alpha)$, we show the following asymptotic formulas for completeness.

Theorem 1.3. Assume (A.1)–(A.2). Furthermore, assume that $g \in C^2$ near $u = 0$.

(i) Assume that $g(0) \neq 0$. Then as $\alpha \to 0$,
\[
\lambda(g, \alpha) = \frac{2\alpha}{g(0)} \left\{ 1 + A_1 \alpha + A_2 \alpha^2 + o(\alpha^2) \right\},
\]
where
\[
A_1 = -\frac{5}{6g(0)}(1 + g'(0)), \quad A_2 = \frac{32}{45} \left( 1 + g'(0) \right)^2 - \frac{11}{30} \frac{g''(0)}{g(0)}.
\]

(ii) Assume that $g(0) = 0$ and $g'(0) > -1$. Then as $\alpha \to 0$,
\[
\lambda(g, \alpha) = \frac{1}{1 + g'(0)} \left( \frac{\pi^2}{4} - \frac{\pi g''(0)}{3(1 + g'(0))} \alpha + o(\alpha) \right).
\]

The proof of Theorem 1.1 is given by the combination of time-map method, Fourier expansion and the asymptotic formulas for some special functions.

2. Proof of Theorem 1.1. In this section, let $\alpha \gg 1$. For simplicity, we write $\lambda = \lambda(g, \alpha)$. Furthermore, we denote by $C$ the various positive constants independent of $\alpha$. We put
\[
G(u) := \int_0^u g(s) ds.
\]
It is known that if \((u, \lambda) \in C^2(\tilde{I}) \times \mathbb{R}_+\) satisfies (1.1)–(1.3), then
\[
\begin{align*}
u(t) &= u(-t), \quad 0 \leq t \leq 1, \quad (2.2) \\
u(0) &= \max_{-1 \leq t \leq 1} u(t) = \alpha, \quad (2.3) \\
u'(t) &> 0, \quad -1 < t < 0. \quad (2.4)
\end{align*}
\]
We construct the well known time-map (2.7) below (cf. [18]). By (1.1), we have
\[
\{u''(t) + \lambda (u(t) + g(u(t)))\} u'(t) = 0.
\]
By this and putting \(t = 0\), we obtain
\[
\frac{1}{2} u'(t)^2 + \lambda \left(\frac{1}{2} u(t)^2 + G(u(t))\right) = \text{constant} = \lambda \left(\frac{1}{2} \alpha^2 + G(\alpha)\right).
\]
This along with (2.4) implies that for \(-1 \leq t \leq 0,
\[
u'(t) = \sqrt{\lambda} \sqrt{\alpha^2 - u(t)^2 + 2(G(\alpha) - G(u(t)))}.
\]
It follows from (A.2) that \(|g(u)| \leq C\) for \(u \in \mathbb{R}\). Then for \(0 \leq s \leq 1\), we have
\[
\frac{|G(\alpha) - G(\alpha s)|}{\alpha^2(1 - s^2)} = \int_0^s g(t) dt \leq \frac{C \alpha(1 - s)}{\alpha^2(1 - s^2)} \leq C \alpha^{-1}.
\]
By (2.5), (2.6), putting \(s := u(t)/\alpha\) and Taylor expansion, we obtain
\[
\sqrt{\lambda} = \int_{-1}^1 \sqrt{\alpha^2 - u(t)^2 + 2(G(\alpha) - G(u(t)))} dt
= \int_0^1 \frac{1}{\sqrt{1 - s^2 + 2(G(\alpha) - G(\alpha s))/\alpha^2}} ds
= \int_0^1 \frac{1}{\sqrt{1 - s^2}} \sqrt{1 + 2(G(\alpha) - G(\alpha s))/(\alpha^2(1 - s^2))} ds
= \int_0^1 \frac{1}{\sqrt{1 - s^2}} \left\{1 - \frac{G(\alpha) - G(\alpha s)}{\alpha^2(1 - s^2)} + O(\alpha^{-2})\right\} ds
:= \frac{\pi}{2} - \frac{1}{\alpha^2} K(\alpha) + O(\alpha^{-2}),
\]
where
\[
K(\alpha) := \int_0^1 \frac{G(\alpha) - G(\alpha s)}{(1 - s^2)^{3/2}} ds.
\]
We calculate \(K(\alpha)\) by using the asymptotic formulas for some special functions.

Let \(J_\nu(z), Y_\nu(z), J_\nu'(z), E_\nu(z)\) and \(\Gamma(z)\) be the Bessel functions, Neumann functions, Anger functions, Weber functions and Gamma functions. For \(z \gg 1\), we have
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(cf. [9, p. 929, p. 958])

\[ J_1(z) = \sqrt{2} z \left\{ 1 + R_1 \cos \left( z - \frac{3}{4} \pi \right) - \left[ \frac{1}{2} \Gamma \left( \frac{3}{2} \right) + R_2 \right] \sin \left( z - \frac{3}{4} \pi \right) \right\}, \quad (2.9) \]

\[ J_{-1}(z) = \sqrt{2} z \left\{ 1 + R_1 \cos \left( z + \frac{1}{4} \pi \right) - \left[ \frac{1}{2} \Gamma \left( \frac{3}{2} \right) + R_2 \right] \sin \left( z + \frac{1}{4} \pi \right) \right\}, \quad (2.10) \]

\[ Y_1(z) = \sqrt{2} z \left\{ 1 + R_1 \sin \left( z - \frac{3}{4} \pi \right) + \left[ \frac{1}{2} \Gamma \left( \frac{3}{2} \right) + R_2 \right] \cos \left( z - \frac{3}{4} \pi \right) \right\}, \quad (2.11) \]

\[ Y_{-1}(z) = \sqrt{2} z \left\{ 1 + R_1 \sin \left( z + \frac{1}{4} \pi \right) + \left[ \frac{1}{2} \Gamma \left( \frac{3}{2} \right) + R_2 \right] \cos \left( z + \frac{1}{4} \pi \right) \right\}, \quad (2.12) \]

where

\[ R_1 < \left| \frac{\Gamma \left( \frac{3}{2} \right)}{8 \Gamma \left( \frac{1}{2} \right) z^2} \right|, \quad |R_2| < \left| \frac{\Gamma \left( \frac{3}{2} \right)}{48 \Gamma \left( -\frac{1}{2} \right) z^3} \right|, \quad (2.13) \]

\[ J_{\pm 1}(z) = J_{\pm 1}(z), \quad (2.14) \]

\[ E_{\pm 1}(z) = -Y_{\pm 1}(z) \mp \frac{2}{\pi z^2} + O(z^{-4}). \quad (2.15) \]

Furthermore, it is known that under the conditions (A.1)–(A.2),

\[ g(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad (2.16) \]

holds for \( x \in \mathbb{R} \) and the right hand side of (2.16) converges to \( g(x) \) uniformly on \( \mathbb{R} \), since (A.1) implies that \( g' \in L^2(-\pi, \pi) \). Here,

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \cos n\theta d\theta = -\frac{1}{n\pi} \int_{-\pi}^{\pi} g'(\theta) \sin n\theta d\theta \quad (2.17) \]

\[ := -\frac{1}{n} \tilde{a}_n \quad (n \in \mathbb{N}_0), \]

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \sin n\theta d\theta = \frac{1}{n\pi} \int_{-\pi}^{\pi} g'(\theta) \cos n\theta d\theta \quad (2.18) \]

\[ := \frac{1}{n} \tilde{b}_n \quad (n \in \mathbb{N}). \]

We obtain (2.17) and (2.18) by using integration by parts, since \( g(-\pi) = g(\pi) \) by (A.2).

Lemma 2.1. As \( \alpha \to \infty \),

\[ K(\alpha) = \frac{1}{2} a_0 \alpha + \frac{1}{\pi} \sqrt{\frac{\pi \alpha}{2}} \sum_{n=1}^{\infty} \frac{c_n}{n^{1/2}} + O(\alpha^{-1/2}). \quad (2.19) \]
Proof. We put $s = \sin \theta$ in (2.8). Then by integration by parts, we obtain

$$K(\alpha) = \int_0^{\pi/2} \frac{1}{\cos^2 \theta} (G(\alpha) - G(\alpha \sin \theta)) d\theta$$

(2.20)

$$= \int_0^{\pi/2} (\tan \theta)'(G(\alpha) - G(\alpha \sin \theta)) d\theta$$

$$= [\tan \theta(G(\alpha) - G(\alpha \sin \theta))]_0^{\pi/2} + \alpha \int_0^{\pi/2} g(\alpha \sin \theta) \sin \theta d\theta.$$  

By l'Hôpital's rule, we obtain

$$\lim_{\theta \to \pi/2} \frac{G(\alpha) - G(\alpha \sin \theta)}{\cos \theta} = \lim_{\theta \to \pi/2} \frac{\alpha g(\alpha \sin \theta) \cos \theta}{\sin \theta} = 0. \quad (2.21)$$

For $n = \mathbb{N}$, we put

$$U_n := \int_0^{\pi/2} \cos(n\alpha \sin \theta) \sin \theta d\theta, \quad V_n := \int_0^{\pi/2} \sin(n\alpha \sin \theta) \sin \theta d\theta. \quad (2.22)$$

By this, (2.16)–(2.18) and (2.20), we obtain

$$K(\alpha) = \alpha \int_0^{\pi/2} g(\alpha \sin \theta) \sin \theta d\theta$$

(2.23)

$$= \alpha \int_0^{\pi/2} \left\{ \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\alpha \sin \theta) + \sum_{n=1}^{\infty} b_n \sin(n\alpha \sin \theta) \right\} \sin \theta d\theta$$

$$= \alpha \left\{ \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \int_0^{\pi/2} \cos(n\alpha \sin \theta) \sin \theta d\theta \right. \right.$$  

$$+ \sum_{n=1}^{\infty} b_n \int_0^{\pi/2} \sin(n\alpha \sin \theta) \sin \theta d\theta \right\}$$

$$= \alpha \left\{ \frac{1}{2}a_0 - \sum_{n=1}^{\infty} \frac{1}{n} \tilde{a}_n U_n + \sum_{n=1}^{\infty} \frac{1}{n} \tilde{b}_n V_n \right\}.$$  

Put $\theta = \pi/2 - \phi$ in (2.22). Then by (2.9)–(2.12), (2.14), (2.15) and [9, p.425], we obtain

$$U_n = \int_0^{\pi/2} \cos(n\alpha \cos \phi) \cos \phi d\phi$$

(2.24)

$$= \frac{\pi}{4} (\mathbf{E}_1(n\alpha) - \mathbf{E}_{-1}(n\alpha))$$

$$= \frac{\pi}{4} (-Y_1(n\alpha) + Y_{-1}(n\alpha) + O((n\alpha)^{-2}))$$

$$= \frac{\pi}{4} \left( -\sqrt{\frac{2}{\pi n\alpha}} \sin \left( n\alpha - \frac{3}{4}\pi \right) + \sqrt{\frac{2}{\pi n\alpha}} \sin \left( n\alpha + \frac{1}{4}\pi \right) \right) + O((n\alpha)^{-3/2})$$

$$= -\sqrt{\frac{\pi}{2n\alpha}} \sin \left( n\alpha - \frac{3}{4}\pi \right) + O((n\alpha)^{-3/2}),$$
\[ V_n = \int_0^{\pi/2} \sin(n\alpha \cos \phi) \cos \phi d\phi \]  
\begin{align*}
V_n &= \frac{\pi}{4} \{ J_1(n\alpha) - J_{-1}(n\alpha) \} \\
&= \frac{\pi}{4} \{ J_1(n\alpha) - J_{-1}(n\alpha) \} \\
&= \pi \left\{ \sqrt{\frac{2}{n\pi \alpha}} \cos \left( n\alpha - \frac{3}{4}\pi \right) - \sqrt{\frac{2}{n\pi \alpha}} \cos \left( n\alpha + \frac{1}{4}\pi \right) \right\} + O((n\alpha)^{-3/2}) \\
&= \sqrt{\frac{2}{2n\alpha}} \cos \left( n\alpha - \frac{3}{4}\pi \right) + O((n\alpha)^{-3/2}).
\end{align*}

By (2.17), (2.18) and (2.23)–(2.25), we obtain
\[ K(\alpha) = \alpha \left\{ \frac{1}{2} a_0 + \frac{\pi}{2\alpha} \sum_{n=1}^{\infty} \left( \tilde{a}_n \sin \left( n\alpha - \frac{3}{4}\pi \right) + \tilde{b}_n \cos \left( n\alpha - \frac{3}{4}\pi \right) \right) \frac{1}{n^{3/2}} \right\} \\
+ O \left( \alpha^{-1/2} \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \right) \\
= \alpha \left\{ \frac{1}{2} a_0 + \frac{1}{\pi} \sqrt{\frac{2}{2\alpha}} \sum_{n=1}^{\infty} \left( \tilde{c}_n \right) \frac{1}{n^{3/2}} \right\} + O(\alpha^{-1/2}).
\]

Thus the proof is complete. \( \square \)

By (2.7) and Lemma 2.1, we obtain Theorem 1.1. \( \square \)

**Example 2.2.** Let \( g_1(u) = (\sin u + \sin(2u))/4 \). Then \( g_1(u) \) satisfies \( \text{(OP)} \). Indeed, in this case, it is clear that \( a_n = 0 \ (n \in \mathbb{N}_0) \) and \( b_n = 0 \ (n \geq 3) \). Therefore, we see from (2.18) that
\[ \begin{align*}
\hat{b}_1 &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{4} \cos \theta + 2 \cos 2\theta \cos \theta d\theta = \frac{1}{4}, \\
\hat{b}_2 &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{4} \cos \theta + 2 \cos 2\theta \cos 2\theta d\theta = \frac{1}{2}.
\end{align*} \]

By this, (2.23), (2.25) and [9, pp. 30], we obtain
\[ K(\alpha) = \alpha \left\{ \tilde{b}_1 V_1 + \frac{1}{2} \tilde{b}_2 V_2 \right\} = \frac{1}{4} \alpha (V_1 + V_2) \]  
\begin{align*}
K(\alpha) &= \frac{\alpha}{4} \left\{ \sqrt{\frac{\pi}{2\alpha}} \cos \left( \alpha - \frac{3}{4}\pi \right) + \sqrt{\frac{\pi}{4\alpha}} \cos \left( 2\alpha - \frac{3}{4}\pi \right) \right\} + O(\alpha^{-3/2}) \\
&= \frac{\sqrt{\alpha \pi}}{8} \left\{ \sqrt{2} \cos \left( \alpha - \frac{3}{4}\pi \right) + \cos \left( 2\alpha - \frac{3}{4}\pi \right) + O(\alpha^{-1}) \right\}.
\end{align*}

By this and (2.7), we obtain
\[ \lambda(\alpha) = \frac{\pi^2}{4} - \frac{\pi^{3/2}}{8\alpha^{3/2}} \left\{ \sqrt{2} \cos \left( \alpha - \frac{3}{4}\pi \right) + \cos \left( 2\alpha - \frac{3}{4}\pi \right) \right\} + O(\alpha^{-2}). \]

For instance, if we put \( \alpha = n\pi + (3\pi)/4 \ (n \in \mathbb{N}, \ n \gg 1) \), then we can easily check that \( \lambda(\alpha) \) satisfies \( \text{(OP)} \).
3. **Proof of Theorem 1.3.** The local behavior of $\lambda(\alpha)$ is easy to calculate, since Taylor expansion and the time-map method work very well. We only prove Theorem 1.3 (i) for completeness.

**Proof of Theorem 1.3 (i).** Since $g(0) \neq 0$, by (A.1), we see that $g(0) > 0$. We put

$$M_\alpha(u) := 2g(0)(\alpha - u) + (1 + g'(0))\alpha^2 - u^2 + \frac{1}{3}(1 + o(1))g''(0)(\alpha^3 - u^3).$$

By (1.1), (2.5) and Taylor expansion, for $0 < u \ll 1$ and $-1 \leq t \leq 0$, we have

$$u'_\alpha(t) = \sqrt{\lambda(\alpha)}M_\alpha(u_\alpha(t)).$$

By this and putting $s = u_\alpha(t)/\alpha$, we obtain

$$\sqrt{\lambda(\alpha)} = \int_{-1}^{0} \frac{u'_\alpha(t)}{\sqrt{M_\alpha(u_\alpha(t))}} dt = \int_{0}^{1} \sqrt{2g(0)(1 - s) + (1 + g'(0))\alpha(1 - s^2)} + (1 + o(1))g''(0)\alpha^2(1 - s^3)/3 ds$$

$$= \frac{\alpha g(0)}{2g(0)} \int_{0}^{1} \frac{1}{\sqrt{1 - s}} \sqrt{1 + \frac{1 + g'(0)}{2g(0)}\alpha(1 + s) + \frac{1 + o(1)}{6g(0)}g''(0)\alpha^2(1 + s + s^2)} ds$$

$$= \frac{\alpha g(0)}{2g(0)} \int_{0}^{1} \frac{1}{\sqrt{1 - s}} \left[ 1 - \frac{1 + g'(0)}{4g(0)}\alpha(1 + s) - \frac{g''(0)}{12g(0)}\alpha^2(1 + s + s^2) \right]$$

$$+ \frac{3}{32g(0)} \left(1 + g'(0)\right)^2 \alpha^2(1 + s)^2 + o(\alpha^2) \right] ds$$

This implies (1.9). Thus the proof is complete. \qed

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