Asymptotic Interactions of Critically Coupled Vortices

N. S. Manton*
Department of Applied Mathematics and Theoretical Physics
University of Cambridge
Wilberforce Road, Cambridge CB3 0WA, England
and
J. M. Speight†
Department of Pure Mathematics
University of Leeds
Leeds LS2 9JT, England

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*e-mail address: N.S.Manton@damtp.cam.ac.uk
†e-mail address: J.M.Speight@leeds.ac.uk
Abstract

At critical coupling, the interactions of Ginzburg-Landau vortices are determined by the metric on the moduli space of static solutions. Here, a formula for the asymptotic metric for two well separated vortices is obtained, which depends on a modified Bessel function. A straightforward extension gives the metric for \( N \) vortices. The asymptotic metric is also shown to follow from a physical model, where each vortex is treated as a point-like particle carrying a scalar charge and a magnetic dipole moment of the same magnitude. The geodesic motion of two well separated vortices is investigated, and the asymptotic dependence of the scattering angle on the impact parameter is determined. Formulae for the asymptotic Ricci and scalar curvatures of the \( N \)-vortex moduli space are also obtained.
1 Introduction

Critically coupled Ginzburg-Landau vortices and BPS monopoles are two of the most studied examples of topological solitons in field theory [8]. Vortices are particle-like solutions of the abelian Higgs theory in two dimensions and BPS monopoles are solutions of a Yang-Mills-Higgs theory in three dimensions.

At critical coupling (separating the Type I and Type II regimes of superconductivity), vortices exert no static forces on each other, and there are static multi-vortex solutions. These satisfy the planar Bogomolny equations [3]

\[ D_1\phi + iD_2\phi = 0 \]  
\[ B + \frac{1}{2}(|\phi|^2 - 1) = 0, \]

and the boundary condition \(|\phi| \to 1\) as \(|x| \to \infty\). Here, \(D_i\phi = \partial_i\phi + iA_i\phi\) is the covariant derivative of the complex scalar field \(\phi\), and \(B = \partial_1A_2 - \partial_2A_1\) is the magnetic field in the plane. Taubes showed that an \(N\)-vortex solution is uniquely determined by specifying \(N\) points where \(\phi\) is zero [19, 8]. The moduli space of \(N\)-vortex solutions, \(\mathcal{M}_N\), is therefore the configuration space of \(N\) unordered points in the plane, which is a smooth \(2N\)-dimensional manifold. Using the first Bogomolny equation, the gauge potential can be eliminated, and the second equation can then be written in terms of the gauge invariant field \(h = \log |\phi|^2\) as

\[ \nabla^2 h - \frac{1}{4}\pi \sum_{r=1}^{N} \delta(x - y_r). \]

The points \(\{y_r : 1 \leq r \leq N\}\) are the locations of the vortices, where \(\phi\) vanishes and \(h\) has a logarithmic singularity. The boundary condition is \(h \to 0\) as \(|x| \to \infty\). Taubes showed that \(h\) approaches 0 exponentially fast [8].

Static BPS monopoles are solutions in \(\mathbb{R}^3\) of the Bogomolny equations

\[ B_i = D_i\Phi, \]

where \(D_i\Phi\) is the covariant derivative of the adjoint Higgs field and \(B_i\) is the Yang-Mills magnetic field. For fields of finite energy there is a well-defined monopole number \(N\), and the moduli space of \(N\)-monopole solutions for gauge group \(SU(2)\) is a \(4N\)-dimensional smooth manifold. It is not so simple as in the vortex case to say precisely what the moduli signify without introducing some additional structures (e.g. Donaldson’s rational map), but for well separated monopoles there are four moduli associated with each of them. Three specify the location of the monopole, and the fourth is an internal phase angle.

We are interested not just in static solutions but also in time-dependent ones. We suppose the complete Lagrangian, both for vortices and for monopoles, is the Lorentz invariant extension of the static energy function, with a kinetic term quadratic in the time derivatives of the fields. In the vortex case, the Lagrangian density is that of the abelian Higgs theory at critical coupling

\[ \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\phi D^\mu\phi - \frac{1}{8}(|\phi|^2 - 1)^2, \]
where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu \phi = \partial_\mu \phi + i A_\mu \phi$. The vortices can move at arbitrary speeds less than the speed of light. For monopoles the situation is similar, but the theory is non-abelian. High speed collisions of either vortices or monopoles are complicated, involving substantial energy transfer to radiation modes of the fields, and amenable only to numerical simulation. However, collisions at slow speeds can be treated adiabatically, using the geodesic approximation [2]. The idea here is that the moduli space of static solutions acquires a natural metric by restricting the kinetic terms of the field theory Lagrangian to motion tangent to the moduli space, and it can be shown that the geodesic trajectories on moduli space accurately model the field theory dynamics of the solitons. An argument for this was given in [9]. It was justified rigorously for vortex motion [17] and for monopole motion [18] by Stuart. It is fairly clear now that the geodesic approximation is the formal non-relativistic limit of the field theory dynamics of the solitons, where radiation is neglected.

Having recognized the importance of the metric on moduli space, it becomes desirable to calculate it. This is not so easy. For monopoles there is an explicit understanding of the metric only for one monopole, where the metric is flat, and for two monopoles, where it was calculated by Atiyah and Hitchin [1]. For vortices there is a general formula due to Samols [14], which we shall use below. Again, for one vortex the metric is flat, but even for two vortices the metric is not known explicitly.

However, it is possible to calculate the explicit asymptotic metric on the moduli space for $N$ well separated monopoles. There are now two approaches to this calculation. The first is physically motivated, and not quite rigorous [10, 7]. The monopoles are treated as point-like objects, carrying a magnetic charge and a scalar charge of equal magnitude. These charges are regarded as sources for auxiliary linear fields through which the monopoles interact. (Monopoles, in reality, are smooth but nonlinear, with a core radius of order 1, but provided their separation is much greater than 1, a linearization of the fields appears justified.) For monopoles at rest, the magnetic forces exactly cancel the scalar forces, so there is no net force. For monopoles in relative motion, the magnetic and scalar forces are not identical, because of their different Lorentz transformation properties, and there are net forces which cause the monopoles to scatter non-trivially. The effects can be encapsulated in an $N$-particle Lagrangian with a purely kinetic term, quadratic in velocities. The coefficient matrix of this quadratic form defines the metric on the moduli space for well separated monopoles. An alternative approach is due to Bielawski, who calculated rigorously the asymptotic form of the Nahm data associated with well separated monopoles, and from this calculated the asymptotic metric on moduli space [2]. These approaches give the same result, and they are consistent with the Atiyah-Hitchin metric for two monopoles, whose asymptotic form was derived in [6]. The asymptotic $N$-monopole metric, like the true metric on the $N$-monopole moduli space, is hyperkähler, but unlike the true metric it has singularities when the monopoles come close together.

These results on monopoles motivated the present work. Here we give an explicit expression for the metric on the $N$-vortex moduli space, for $N$ well separated vortices. Furthermore we calculate it in two ways. Our first approach is to take Samols’ general formula and evaluate the quantities occurring there by a method of matched asymptotic expansions. Essentially, we solve eq. (1.3) for two well separated vortices, calculating the effect of one vortex on the other at linear order, and from the solution determine the asymptotic 2-vortex metric. It is
straightforward to generalize the 2-vortex metric to the $N$-vortex metric. We need to assume that for well separated vortices the field $h$ far from the vortex cores obeys the linearization of (1.3), namely the Helmholtz equation

$$\nabla^2 h - h = 0,$$  \hspace{1cm} (1.6)

and that the relevant solution is a linear superposition of the solutions due to the $N$ vortices separately. Corrections due to the nonlinear terms neglected in eq. (1.6) are of higher exponential order in the separations. However, a careful treatment of this point is lacking, and would require a considerable refinement of Taubes’ estimate of the exponential decay of solutions.

Our second approach is the more physical. It is a variant of the calculation involving point-like monopoles, and regards well separated vortices as point-like sources interacting via auxiliary linear fields. A study of the static forces between vortices that are close to critical coupling shows that well separated vortices can be regarded as particles each carrying a scalar charge and a magnetic dipole moment (thought of as perpendicular to the plane which the vortex inhabits) [16]. For critically coupled vortices, the magnitudes of the scalar charge and the dipole are the same, and the static forces due to them cancel. For vortices in motion, the scalar and magnetic forces do not exactly cancel, but result in velocity-dependent forces. Again there is an effective Lagrangian for two well separated vortices which is purely kinetic, and from this the metric can be read off. The extension to $N$ vortices is as before.

The asymptotic metric for vortices, which involves the Bessel function $K_0$, has some similarities to the true metric. It has the same isometries, and like the true metric, it is Kähler. As for monopoles, the asymptotic metric becomes singular as the vortices approach one another closely, since it is not positive definite if the minimum vortex separation is below a certain critical value ($2.21$ to two decimal places in the case $N = 2$). Of course, the asymptotic metric is not valid in this region.

It remains open to rigorously prove that our formula gives the asymptotic metric on the $N$-vortex moduli space, but the known results for monopoles make this conjecture plausible. Using our formula we can calculate the scattering of two vortices that do not approach close to each other. The leading exponentially small expression for the scattering angle can be obtained exactly.

This paper is organized as follows. In Section 2 we obtain the asymptotic 2-vortex metric, and its generalization to $N$ vortices, along with the asymptotic Ricci and scalar curvatures. In Section 3 we rederive the metric using the model of vortices as point-like particles. In Section 4 we discuss the scattering of two vortices using the asymptotic metric.

2 Well Separated Vortices – Field and Metric

The key to the metric on $M_N$, the $N$-vortex moduli space, is the equation (1.3), whose solutions determine the static $N$-vortex fields. It is sometimes convenient to use a complex coordinate $z = x + iy$ for a general point in the plane, and to denote the vortex locations
correspondingly by \( \{ Z_r : 1 \leq r \leq N \} \). Eq. (1.3) becomes

\[
\nabla^2 h - e^h + 1 = 4\pi \sum_{r=1}^{N} \delta(z - Z_r) \tag{2.1}
\]

where \( \nabla^2 = 4\frac{\partial^2}{\partial z \partial \bar{z}} \). Around \( Z_r \), the function \( h(z, \bar{z}) \) has the local expansion

\[
h = \log |z - Z_r|^2 + a_r + \frac{1}{2} \bar{b}_r(z - Z_r) + \frac{1}{2} b_r(\bar{z} - \bar{Z}_r) \nonumber + \bar{c}_r(z - Z_r)^2 - \frac{1}{4} (z - Z_r)(\bar{z} - \bar{Z}_r) + c_r(\bar{z} - \bar{Z}_r)^2 + \ldots , \tag{2.2}
\]

where \( a_r \) is real, and \( b_r, c_r \) complex. Taubes proved that this series, with the logarithmic term removed, is a convergent Taylor expansion. The logarithmic term and the coefficient \( \frac{1}{4} \) are determined by the equation locally, but the remaining coefficients are not. They depend on the positions of the other vortices, but not in an explicitly known way. Most important for us is the coefficient \( b_r \).

Samols’ formula for the metric on \( \mathcal{M}_N \) is

\[
g = \pi \sum_{r,s=1}^{N} \left( \delta_{rs} + \frac{2}{\partial Z_r} \frac{\partial b_s}{\partial \bar{Z}_s} \right) \, dZ_r d\bar{Z}_s . \tag{2.3}
\]

The functions \( b_r \) obey the symmetry relation

\[
\frac{\partial b_s}{\partial Z_r} = \frac{\partial \bar{b}_r}{\partial \bar{Z}_s} \tag{2.4}
\]

and from this it follows that the metric is not only real, but also Kähler. Invariance of the metric under a translation of all the vortices implies that \( \sum b_r = 0 \), and rotational invariance implies that \( \sum Z_r b_r \) is real.

For well separated vortices, we assume that \( h \) is exponentially small except in a core region with radius of order 1 around each vortex, and there \( h \) has an approximate, local circular symmetry. It follows that if the minimum separation of any pair is \( L \gg 1 \), then the \( \delta_{rs} \) term dominates the metric, and the correction is of order \( e^{-L} \). The metric is therefore approximately flat.

Let us now concentrate on two vortices, and denote their positions by

\[
Z_1 = Z + \sigma e^{i\theta}, \quad Z_2 = Z - \sigma e^{i\theta} . \tag{2.5}
\]

It follows from the symmetry of the 2-vortex field around the centre of mass \( Z \), or from the properties of the functions \( b_r \) mentioned above, that in this case \( b_1 = b(\sigma) e^{i\theta} \) and \( b_2 = -b_1 \), where \( b(\sigma) \) is a real function. Samols’ formula implies that the moduli space metric is

\[
g = 2\pi \, dZ d\bar{Z} + \eta(\sigma) (d\sigma^2 + \sigma^2 \, d\theta^2) \tag{2.6}
\]

where

\[
\eta(\sigma) = 2\pi \left( 1 + \frac{1}{\sigma} \frac{d}{d\sigma} \left( \sigma b(\sigma) \right) \right) . \tag{2.7}
\]
The relative motion of two vortices takes place on the reduced moduli space where \( Z \) is fixed. This is a surface of revolution. The range of the coordinates is \( 0 \leq \sigma < \infty \) and \(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\), with \( \theta = -\frac{\pi}{2} \) and \( \theta = \frac{\pi}{2} \) identified. The range of \( \theta \) is \( \pi \) and not \( 2\pi \) because the vortices are identical. Therefore, the surface is asymptotically conical, rather than planar.

So far, our exposition has been a summary of known results, but now we show how to calculate the leading asymptotic correction to the conical metric. We return to the equation (1.3), and consider first the circularly symmetric solution \( h_0 \) for a single vortex at the origin.

In terms of polar coordinates \((\rho, \varphi)\), the equation satisfied by \( h_0 \), for \( \rho > 0 \), is

\[
\frac{d^2 h_0}{d\rho^2} + \frac{1}{\rho} \frac{dh_0}{d\rho} - e^{h_0} + 1 = 0.
\]

(2.8)

The boundary conditions are \( h_0 \sim 2 \log \rho \) for small \( \rho \), and \( h_0 \to 0 \) as \( \rho \to \infty \). The Taylor expansion of \( h_0 - 2 \log \rho \) about \( \rho = 0 \) involves only even powers of \( \rho \). For large \( \rho \), eq. \( 2.8 \) has the linearized form

\[
\frac{d^2 h_0}{d\rho^2} + \frac{1}{\rho} \frac{dh_0}{d\rho} - h_0 = 0,
\]

(2.9)

the modified Bessel equation of zeroth order, so

\[
h_0(\rho) \sim \frac{q}{\pi} K_0(\rho),
\]

(2.10)

where \( q \) is a constant. The corrections to this asymptotic approximation are expected to be suppressed by order \( e^{-\rho} \). By numerical integration of the nonlinear equation \( 2.8 \), it has been determined that \( q = -10.6 \). Recently, Tong has given an argument involving dualities in string theory which strongly suggests that \( q = -2\pi \delta^2 \), in agreement with the numerical result \( 20 \). There is as yet no direct proof of this using \( 2.8 \).

Next, let us consider the perturbation of the solution \( h_0 \) due to other, distant vortices, still assuming that one vortex, which we label as vortex 1, is precisely at the origin. Let us write \( h = h_0 + h_1 \), where \( h_1 \) is small in the neighbourhood of vortex 1. The linearization of eq. \( 1.3 \) implies that

\[
(\nabla^2 - e^{h_0}) h_1 = 0.
\]

(2.11)

The operator acting on \( h_1 \) has no singularity at the origin, so \( h_1 \) is smooth there, and the logarithmic singularity of \( h \) is carried entirely by \( h_0 \). Since \( h_0 \) is circularly symmetric, we can separate variables and write

\[
h(\rho, \varphi) = h_0(\rho) + h_1(\rho, \varphi)
\]

\[
:= h_0(\rho) + \frac{1}{2} f_0(\rho) + \sum_{n=1}^{\infty} \left( f_n(\rho) \cos n\varphi + g_n(\rho) \sin n\varphi \right),
\]

(2.12)

where \( f_n \) obeys the equation

\[
\frac{d^2 f_n}{d\rho^2} + \frac{1}{\rho} \frac{df_n}{d\rho} - \left( e^{h_0} + \frac{n^2}{\rho^2} \right) f_n = 0,
\]

(2.13)

and \( g_n \) obeys the same equation. \( f_n \) is nonsingular at \( \rho = 0 \) and has a series expansion \( f_n = \alpha_n \rho^n + \ldots \). Similarly, \( g_n = \beta_n \rho^n + \ldots \). The expansion \( 2.12 \) is consistent with the general
expansion of \( h \) around vortex 1, that is, \( (2.2) \) with \( r = 1 \) and \( Z_1 = 0 \). By identifying the terms linear in \( \rho \), we find that \( b_1 \), the coefficient we are interested in, is given by \( b_1 = \alpha_1 + i\beta_1 \).

From now on, therefore, we just consider equation (2.13) for \( f_1 \), that is,

\[
\frac{d^2 f_1}{d\rho^2} + \frac{1}{\rho} \frac{df_1}{d\rho} - \left( e^{h_0} + \frac{1}{\rho^2} \right) f_1 = 0.
\] (2.14)

For large \( \rho \), this simplifies to

\[
\frac{d^2 f_1}{d\rho^2} + \frac{1}{\rho} \frac{df_1}{d\rho} - \left( 1 + \frac{1}{\rho^2} \right) f_1 = 0,
\] (2.15)

which is valid for \( 1 \ll \rho \ll L \), where \( L \) is the distance from the origin to the next-nearest vortex. Note that the difference between the coefficients in eqs. (2.14) and (2.15) is \( e^{h_0} - 1 \), which is smooth, finite, and exponentially localized. We therefore suppose that the asymptotic form of the solutions of (2.14) are exact solutions of (2.15). This is supported by the results used in various examples of scattering theory, and which follow from Levinson’s theorem [4]; however a result of the precise type we require, involving perturbations of Bessel’s equation, appears to us not to have been established. Eq. (2.15) is the modified Bessel equation of first order, whose general solution is a linear combination of the functions \( K_1(\rho) \) and \( I_1(\rho) \).

Let us now assume that there is just one other vortex, vortex 2, whose location (in Cartesian coordinates) is \((-2\sigma, 0)\), with \( \sigma \gg 1 \). In this case, \( h \) is reflection symmetric under \( \varphi \mapsto -\varphi \), so in the Fourier series for \( h_1 \) all the functions \( g_n \) vanish. In particular, \( b_1 = \alpha_1 \), and is real.

In the region far from both vortex centres, the equation (1.3) linearizes to

\[
\nabla^2 h - h = 0
\] (2.16)

and is solved by the linear superposition of the fields due to each vortex separately

\[
h(\rho, \varphi) = \frac{q}{\pi} K_0(\rho) + \frac{q}{\pi} K_0 \left( \sqrt{4\sigma^2 + 4\sigma \rho \cos \varphi + \rho^2} \right).
\] (2.17)

The argument of the second \( K_0 \) function is the distance to vortex 2 from the point with polar coordinates \((\rho, \varphi)\). By separation of variables, the general solution of the Helmholtz equation (2.16), regular at \( \rho = 0 \) and with the reflection symmetry \( \varphi \mapsto -\varphi \), is a linear combination of the functions \( I_n(\rho) \cos n\varphi \). The function \( K_0 \left( \sqrt{4\sigma^2 + 4\sigma \rho \cos \varphi + \rho^2} \right) \) is such a solution (whereas \( K_0(\rho) \) is not, being singular at \( \rho = 0 \)), so

\[
K_0 \left( \sqrt{4\sigma^2 + 4\sigma \rho \cos \varphi + \rho^2} \right) = k_0 I_0(\rho) + 2 \sum_{n=1}^{\infty} k_n I_n(\rho) \cos(n\varphi)
\] (2.18)

for some real constants \( k_n \).

Note that an important special solution of (2.16) is \( e^x = e^{\rho \cos \varphi} \), and its expansion

\[
e^{\rho \cos \varphi} = I_0(\rho) + 2 \sum_{n=1}^{\infty} I_n(\rho) \cos(n\varphi)
\] (2.19)
defines the functions $I_n(\rho)$. Combining the series for the exponential function with trigonometric identities, one can compute the leading terms in the series expansions for $I_n$. These are also given in standard references, e.g. [5]. It is sufficient for us to record that

$$I_0(\rho) = 1 + \ldots, \quad I_1(\rho) = \frac{1}{2} \rho + \ldots \quad (2.20)$$

We can now return to (2.18) and determine $k_1$, the coefficient we need. The Taylor expansion (in $\rho$) of the two sides gives

$$K_0(2\sigma) - K_1(2\sigma)\rho \cos \varphi + \ldots = k_0 + k_1 \rho \cos \varphi + \ldots \quad (2.21)$$

where we have used the identity $K_1 = -K_0'$, and the results above for $I_0$ and $I_1$. So $k_0 = K_0(2\sigma)$ and

$$k_1 = -K_1(2\sigma). \quad (2.22)$$

With this result, we can now match the Fourier expansion of (2.17), the linearized field $h$ due to the two vortices, valid outside their cores, with the Fourier expansion of $h = h_0 + h_1$ near vortex 1. In the range $1 < \rho < 2\sigma$, we find

$$\frac{q}{\pi} \left( K_0(\rho) + K_0(2\sigma)I_0(\rho) \right) - \frac{2q}{\pi} K_1(2\sigma)I_1(\rho) \cos \varphi + \ldots$$

$$= \frac{q}{\pi} K_0(\rho) + \frac{1}{2} f_0(\rho) + f_1(\rho) \cos \varphi + \ldots \quad (2.23)$$

Therefore, $f_1(\rho)$ has the asymptotic form

$$f_1(\rho) = -\frac{2q}{\pi} K_1(2\sigma)I_1(\rho) \quad (2.24)$$

and there is no $K_1(\rho)$ piece. The further terms on both sides of (2.23) involve $\cos n\varphi$ with $n > 1$, and could be determined from higher order terms in the Taylor expansion (2.21), but we do not need these.

The last step is to extrapolate the function $f_1$ into the core region of vortex 1. It is rather remarkable that this can be done, because the equation satisfied by $f_1$, namely (2.14), is not a standard equation, and the coefficient $e^{h_0}$ is not known explicitly. However, one solution of (2.14) is known. It is

$$\tilde{f}_1 = \frac{d}{d\rho} h_0 \quad (2.25)$$

This can be verified by differentiating (2.8), the nonlinear equation for $h_0$. The interpretation of this solution is that it corresponds to the translational zero mode of vortex 1. If the centre of that vortex is infinitesimally translated by $\epsilon$ in the $x$-direction, then the field of vortex 1 becomes $h = h_0(\rho - \epsilon \cos \varphi)$ to first order in $\epsilon$. So $h = h_0 + h_1$ where $h_1 = -\epsilon \tilde{f}_1 \cos \varphi$ and $\tilde{f}_1 = \frac{d}{d\rho}$. Since $h_0 = \frac{\sigma}{\pi} K_0(\rho)$ asymptotically, it follows that $\tilde{f}_1 = -\frac{\sigma}{\pi} K_1(\rho)$ asymptotically. Similarly, since $h_0 \sim 2 \log \rho$ for small $\rho$, it follows that $\tilde{f}_1 \sim 2/\rho$ for small $\rho$. By contrast, the solution $f_1$ of (2.14) that really interests us has the asymptotic behaviour $-\frac{2q}{\pi} K_1(2\sigma)I_1(\rho)$ for large $\rho$, and the finite linear behaviour $b_1 \rho$ for small $\rho$, where $b_1$ is to be found.
Now, equation (2.24) has a Wronskian identity
\[ \rho \left( f_1 \frac{df_1}{d\rho} - \tilde{f}_1 \frac{d\tilde{f}_1}{d\rho} \right) = \text{constant}, \quad (2.26) \]
relating the two solutions \( f_1 \) and \( \tilde{f}_1 \). Using the asymptotic forms of \( f_1 \) and \( \tilde{f}_1 \), and the Wronskian identity for the modified Bessel functions
\[ \rho \left( K_1 \frac{dI_1}{d\rho} - I_1 \frac{dK_1}{d\rho} \right) = 1, \quad (2.27) \]
we deduce that the constant in (2.26) is \( \frac{q^2}{2\pi^2} K_1(2\sigma) \). Evaluating (2.26) near \( \rho = 0 \) we deduce, finally, that
\[ b_1 = \frac{q^2}{2\pi^2} K_1(2\sigma). \quad (2.28) \]

We can now use this result to calculate the 2-vortex metric. In the above calculation, vortex 1 was at the origin and vortex 2 at \((-2\sigma,0)\). From (2.25) we see that \( Z = -\sigma \) and \( \theta = 0 \), so
\[ b(\sigma) = \frac{q^2}{2\pi^2} K_1(2\sigma). \quad (2.29) \]
Therefore the prefactor \( \eta \) in the metric (2.6) is
\[ \eta(\sigma) = 2\pi \left( 1 - \frac{q^2}{\pi^2} K_0(2\sigma) \right), \quad (2.30) \]
where we have used (2.7) and the identity \( K_1'(s) + K_1(s)/s = -K_0(s) \). The complete asymptotic 2-vortex metric is
\[ g = 2\pi dZd\bar{Z} + 2\pi \left( 1 - \frac{q^2}{\pi^2} K_0(2\sigma) \right) (d\sigma^2 + \sigma^2 d\theta^2). \quad (2.31) \]
We shall investigate the geodesics of this metric in section 4, and hence determine how vortices scatter.

To extend (2.31) to the asymptotic \( N \)-vortex metric is not hard. Let us use the complex coordinates of the vortices \( Z_r \) and introduce the notation \( Z_{rs} := Z_r - Z_s \). The flat part of the metric (2.3) can be reexpressed as
\[ \pi \sum_{r=1}^{N} dZ_r d\bar{Z}_r = N \pi dZ d\bar{Z} + \frac{\pi}{2N} \sum_{r \neq s} dZ_{rs} d\bar{Z}_{rs}, \quad (2.32) \]
where \( Z \) is the centre of mass coordinate
\[ Z = \frac{1}{N} (Z_1 + Z_2 + \ldots + Z_N). \quad (2.33) \]
Note that the differentials \( dZ_{rs} \) are not all linearly independent.
To find the remaining part of the metric, we need to find $b_s$ and its derivatives. The solution of the Helmholtz equation \ref{eq:2.16} becomes a linear superposition of the fields due to the $N$ vortices. The asymptotic matching of $h$ in the neighbourhood of the $s$-th vortex can be carried out as before. This leads to the following expression for $b_s$ that is a linear superposition of the effects of the other $N - 1$ vortices,

$$b_s = \frac{q^2}{2\pi^2} \sum_{r \neq s} K_1(|Z_{sr}|) \frac{Z_{sr}}{|Z_{sr}|}.$$  \hfill (2.34)

Each term is the obvious generalization of \ref{eq:2.28}, combined with the orientational phase factor $Z_{sr}/|Z_{sr}|$ which reduces to $e^{i\theta}$ for two vortices.

Because of translational invariance,

$$\sum_{r=1}^{N} \frac{\partial b_s}{\partial Z_r} = 0 ,$$  \hfill (2.35)

so

$$\frac{\partial b_s}{\partial Z_s} = -\sum_{r \neq s} \frac{\partial b_s}{\partial Z_r}$$  \hfill (2.36)

(no summation over $s$). For $r \neq s$, we find, differentiating \ref{eq:2.34} with respect to $Z_r$ and keeping $\bar{Z}_r$ fixed, that

$$\frac{\partial b_s}{\partial Z_r} = \frac{q^2}{4\pi^2} K_0(|Z_{sr}|) .$$  \hfill (2.37)

Eq. \ref{eq:2.37} combined with \ref{eq:2.36} gives

$$\sum_{r,s=1}^{N} \frac{\partial b_s}{\partial Z_r} dZ_r d\bar{Z}_s = \frac{q^2}{4\pi^2} \sum_{r \neq s} K_0(|Z_{sr}|) (dZ_r - dZ_s) d\bar{Z}_s .$$  \hfill (2.38)

Since $K_0(|Z_{sr}|) = K_0(|Z_{rs}|)$, we symmetrize over the contributions of these two terms, obtaining

$$\sum_{r,s=1}^{N} \frac{\partial b_s}{\partial Z_r} dZ_r d\bar{Z}_s = -\frac{q^2}{8\pi^2} \sum_{r \neq s} K_0(|Z_{rs}|) dZ_{rs} d\bar{Z}_{rs} .$$  \hfill (2.39)

Putting these ingredients together, we obtain our final expression for the asymptotic $N$-vortex metric

$$g = N \pi dZ d\bar{Z} + \pi \sum_{r \neq s} \left( \frac{1}{2N} - \frac{q^2}{4\pi^2} K_0(|Z_{rs}|) \right) dZ_{rs} d\bar{Z}_{rs} .$$  \hfill (2.40)

For two vortices, located at the points \ref{eq:2.5}, this reduces to \ref{eq:2.31}. Since the coefficients in the asymptotic metric depend only on the magnitudes of the vortex separations, it is clear that it is translationally and rotationally symmetric. The structure of the metric as a small perturbation of the flat Euclidean metric becomes clearer if we eliminate the centre of mass coordinate $Z$ using \ref{eq:2.33}:

$$g = \pi \sum_{r} dZ_r d\bar{Z}_r - \frac{q^2}{4\pi} \sum_{r \neq s} K_0(|Z_{rs}|) dZ_{rs} d\bar{Z}_{rs} .$$  \hfill (2.41)
One way to see that this metric is Kähler is to note that eq. (2.4) is satisfied, since (2.36) and (2.37) imply that \( \frac{\partial b}{\partial Z} \) is real and symmetric. More explicitly, the asymptotic Kähler form is

\[
\omega = \frac{iN\pi}{2} dZ \wedge d\bar{Z} + \frac{i\pi}{2} \sum_{r \neq s} \left( \frac{1}{2N} - \frac{q^2}{4\pi^2} K_0(|Z_{rs}|) \right) dZ_{rs} \wedge d\bar{Z}_{rs} .
\]

(2.42)

Since the 1-forms \( dZ \), \( dZ_{rs} \) are closed, one finds that

\[
d\omega = -\frac{iq^2}{16\pi} \sum_{r \neq s} K_0(|Z_{rs}|) \left( \bar{Z}_{rs} dZ_{rs} \wedge d\bar{Z}_{rs} + Z_{rs} d\bar{Z}_{rs} \wedge dZ_{rs} \right) = 0 ,
\]

(2.43)

so \( \omega \) is closed. The Kähler potential is

\[
\pi \sum_{r=1}^{\infty} Z_r \bar{Z}_r - \frac{q^2}{\pi} \sum_{r \neq s} K_0(|Z_{rs}|) .
\]

(2.44)

The Kähler form is of direct interest in certain non-relativistic models of vortex dynamics [13]. Such models have first order dynamics in time, and it is conjectured that slow vortex dynamics is well approximated by a Hamiltonian flow on the \( N \)-vortex moduli space, where the symplectic structure is precisely this Kähler form. Clearly the closure of \( \omega \) is crucial for this to make sense.

The curvature properties of soliton moduli spaces are of some interest. For example, the scalar curvature of \( \mathcal{M}_N \) is relevant to quantum \( N \)-soliton dynamics [12], while in the case of monopoles, Ricci flatness of \( \mathcal{M}_N \) was the key property exploited in Atiyah and Hitchin’s construction of the metric for \( N = 2 \). In order to compute the asymptotic Ricci tensor for the \( N \)-vortex metric (2.41), it is convenient to write \( g \) as

\[
g = \sum_{r,s} g_{rs} dZ_r d\bar{Z}_s = \sum_{r,s} \pi (\delta_{rs} + h_{rs}) dZ_r d\bar{Z}_s ,
\]

(2.45)

and work up to linear order in the perturbation \( h \). It is a standard result in Kähler geometry [21] that the Ricci tensor associated with \( g \) is

\[
R = -\sum_{r,s} \frac{\partial^2 \log G}{\partial Z_r \partial Z_s} dZ_r d\bar{Z}_s
\]

(2.46)

where \( G \) is the determinant of the hermitian coefficient matrix \( g_{rs} \). In this case,

\[
G = \det (\mathbb{1} + h) = \pi^N (1 + \text{tr} h + \cdots)
\]

\[
\Rightarrow \log G = N \log \pi + \sum_r h_{rr} + \cdots
\]

\[
= N \log \pi - \frac{q^2}{2\pi^2} \sum_{r \neq s} K_0(|Z_r - Z_s|)
\]

(2.47)

from (2.41). Equations (2.46) and (2.47) together with Bessel’s equation imply that

\[
R = \frac{q^2}{8\pi^2} \sum_{r \neq s} K_0(|Z_r - Z_s|) dZ_{rs} d\bar{Z}_{rs}.
\]

(2.48)

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One sees that the $N$-vortex Ricci tensor is asymptotically positive semi-definite, its two-dimensional null space being tangent to the translation orbits in $\mathcal{M}_N$ (that is, a vector is null if and only if it generates a rigid translation of the $N$-vortex system). Tracing $R$, one obtains the scalar curvature,

$$\text{Scal} = \sum_{r,s} g^{rs} R_{rs} = \frac{q^2}{4\pi^3} \sum_{r \neq s} K_0(|Z_r - Z_s|) + \cdots,$$

(2.49)

whence one sees that $\mathcal{M}_N$ is asymptotically scalar positive. It is an interesting open question whether the true metric on $\mathcal{M}_N$ has similar curvature positivity properties. The numerical results of Samols for $N=2$ suggest that it may [13].

### 3 The Point Source Formalism

In this section we rederive the asymptotic 2-vortex metric from a more physical viewpoint. The idea is that, viewed from afar, a static vortex looks like a solution of a linear field theory with a point source at the vortex centre. We will see that the appropriate point source is a composite scalar monopole and magnetic dipole in a Klein-Gordon/Proca theory. If physics is to be model independent, the forces between vortices should approach those between the corresponding point particles in the linear theory as their separation grows. This idea, which originated in the context of monopole dynamics [10], has already been successfully used to obtain an asymptotic formula for static intervortex forces away from critical coupling [16]. The present application is somewhat more subtle since we are required to analyze the interaction between point sources moving along arbitrary trajectories. We handle the problem perturbatively: using a mixture of Lorentz invariance and conservation properties we obtain expressions for a moving point source and the Klein-Gordon/Proca field it induces correct up to acceleration terms. From these we construct the interaction Lagrangian for one moving point source interacting with the field induced by another. This Lagrangian is purely kinetic, i.e. quadratic in velocities, and hence may naturally be reinterpreted as the energy associated with geodesic flow on the asymptotic 2-vortex moduli space. The extension to $N$-vortex dynamics is entirely trivial.

In this section $x^\mu = (x^0, x^1, x^2)$ denotes a space-time point. $x^0 = t$ is the time and $\mathbf{x} = (x^1, x^2)$ denotes a spatial point. To linearize the abelian Higgs theory (1.5), we choose the gauge so that the scalar field $\phi$ is real. Since vortices have nontrivial winding at infinity, this requires a gauge transformation which is singular at the vortex centre. This need not concern us since we seek only to replicate the local, far field behaviour of the vortex in the linear theory. In this gauge, the vacuum is $\phi = 1$, so we define $\phi = 1 + \psi$ and linearize in $\psi$. The resulting Lagrangian density is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \psi \partial^\mu \psi - \frac{1}{2} \psi^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} A_\mu A^\mu + \kappa \psi - j_\mu A_\mu$$

(3.1)

where $\kappa$ is the scalar charge density and $j$ the electromagnetic current density. These will be chosen to replicate the vortex asymptotics. The corresponding field equations are

$$\Box + 1) \psi = \kappa$$

(3.2)

$$(\Box + 1) A_\mu = j_\mu$$

(3.3)
where $\Box = \partial_\mu \partial^\mu = \partial_t^2 - \nabla^2$, and we assume that $j$ is a conserved current, $\partial_\mu j^\mu = 0$.

In this gauge, the scalar field of a single vortex located at the origin is $\phi = \exp(\frac{1}{2}h_0)$ where $h_0$ satisfies \ref{eq:2.10}, so

$$\phi = 1 + \frac{1}{2}h_0 + \ldots \sim 1 + \frac{q}{2\pi}K_0(|x|) \quad (3.4)$$

for large $|x|$ by eq. \ref{eq:2.10}. Hence we seek a point source $\kappa$ so that the solution of \ref{eq:3.2} is $\psi = \frac{q}{2\pi}K_0(|x|)$. Since the static Klein-Gordon equation (Helmholtz equation) in two dimensions has Green’s function $K_0$,

$$(-\nabla^2 + 1)K_0(|x|) = 2\pi\delta(x), \quad (3.5)$$

one sees that

$$\kappa = q\delta(x). \quad (3.6)$$

For a static vortex, the time component of the gauge potential $A_0$ vanishes. The asymptotic behaviour of its spatial components $A_i$ is determined by the first Bogomolny equation \ref{eq:1.1} which, on linearization, implies

$$\partial_1 \psi - A_2 + i(\partial_2 \psi + A_1) = 0. \quad (3.7)$$

Hence

$$A = (A^1, A^2) = (\partial_2, -\partial_1)\psi = -\frac{q}{2\pi}k \times \nabla K_0(|x|) \quad (3.8)$$

where we have introduced $k$, the unit vector in a fictitious $x^3$-direction orthogonal to the physical plane. It follows that the point source which reproduces the asymptotic vortex gauge field in eq. \ref{eq:3.3} is

$$(j^0, j) = (0, -qk \times \nabla \delta(x)). \quad (3.9)$$

The physical interpretation of \ref{eq:3.6} and \ref{eq:3.9} is that the point particle corresponding to a single vortex at rest is a composite consisting of a scalar monopole of charge $q$ and a magnetic dipole of moment $q$. We shall refer to this composite as a (static) point vortex.

The interaction between two arbitrary (possibly time-dependent) composite sources $(\kappa_1, j_1)$ and $(\kappa_2, j_2)$ in this linear theory is described by the Lagrangian

$$L_{\text{int}} = \int d^2x (\kappa_1 \psi_2 - j_{(1)}^{\mu} A_{(2)\mu}) \quad (3.10)$$

where $(\psi, A)$ are the fields induced by $(\kappa, j)$ according to the wave equations \ref{eq:3.2} and \ref{eq:3.3}. This is obtained by extracting the cross terms in $\int d^2x \mathcal{L}$ where $(\kappa, j) = (\kappa_1, j_1) + (\kappa_2, j_2)$ and $(\psi, A) = (\psi_1, A_1) + (\psi_2, A_2)$ by linearity. Although \ref{eq:3.10} looks asymmetric under interchange of sources, it is not, as may be shown using \ref{eq:3.2} and \ref{eq:3.3} and integration by parts. If the sources are chosen to be static point vortices, that is, translated versions of \ref{eq:3.6} and \ref{eq:3.9}, one finds that $L_{\text{int}} = 0$, so static point vortices exert no net force on one another at critical coupling, in agreement with the nonlinear theory.

We seek to compute $L_{\text{int}}$ in the case where the two sources represent point vortices moving along arbitrary trajectories in $\mathbb{R}^2$. To do so, we must construct a time-dependent point source representing a vortex moving along some curve $y(t)$, say. The construction is guided by two principles: first, in the case of motion at constant velocity, the source should reduce to \ref{eq:3.6},
in the vortex’s rest frame; second, for any trajectory the vector source $j$, which represents
the vortex’s electromagnetic current density, must remain a conserved current. The result will
be correct up to quadratic order in velocity and linear order in acceleration.

It is straightforward to calculate Lorentz boosted versions of the sources (3.3), (3.4). Let $\xi^\mu$ denote the rest frame coordinates and assume that at time $t = 0$, the point vortex lies at
the origin, $x = 0$, and is moving with velocity $u$. Then, by decomposing $x$ and $\xi$ into their
components parallel and perpendicular to $u$, one finds that at this time, $\xi^\parallel = \gamma(u)x^\parallel$ (where
$\gamma(u) = (1 - u^2)^{-\frac{1}{2}}$ is the usual contraction factor) while $\xi^\perp = x^\perp$. Hence
\[
\xi = \gamma(u)\frac{x \cdot u}{u} u + \left(x - \frac{x \cdot u}{u} u\right) = x + \frac{1}{2}(x \cdot u)u + \ldots
\]  
(3.11)

where the ellipsis denotes discarded terms of order $u^4$ or greater. We shall not persist in so
denoting these terms. Rather, we shall include an ellipsis only where further negligible terms
have been dropped to obtain the given expression. Since $\kappa$ is a Lorentz scalar, the boosted
scalar monopole is $\kappa(x) = \kappa_{\text{static}}(\xi) = q\delta(\xi)$. To interpret this delta function as a distribution
on the $x$-plane, note that for any test function $f$,
\[
\int d^2x f(x) \delta(\xi) = \int d^2 \xi \left| \frac{\partial x}{\partial \xi} \right| f(x(\xi)) \delta(\xi) = (1 - \frac{1}{2}u^2) f(0) + \ldots.
\]  
(3.12)

Therefore $\delta(\xi) = (1 - \frac{1}{2}u^2)\delta(x)$ and so, at $t = 0$,
\[
\kappa(x) = q(1 - \frac{1}{2}u^2)\delta(x).
\]  
(3.13)

The boosted dipole is more subtle, since $j$ itself transforms as a Lorentz vector. The rest
frame source is
\[
j_{\text{static}} = (j^0_{(0)}, j^\parallel_{(0)}) = (0, -qk \times \nabla_{\xi}\delta(\xi)).
\]  
(3.14)

To obtain the laboratory frame source, one must perform a Lorentz boost on this with velocity
$-u$. Explicitly,
\[
j^0(x) = u\gamma(u)j^\parallel_{(0)}(\xi(x)) = u \cdot j_{(0)}(\xi(x)) + \ldots
\]  
(3.15)
\[
j^\parallel(x) = \gamma(u)j^\parallel_{(0)}(\xi(x)) = (1 + \frac{1}{2}u^2)j^\parallel_{(0)}(\xi(x)) + \ldots
\]  
(3.16)
\[
j^\perp(x) = j^\perp_{(0)}(\xi(x)).
\]  
(3.17)

We may combine (3.15) and (3.16) into a single equation for $j$ by using the same polarization
trick as in (3.11), yielding
\[
j(x) = j_{(0)}(\xi(x)) + \frac{1}{2}(j_{(0)}(\xi(x)) \cdot u) u.
\]  
(3.18)

Now
\[
\nabla_{\xi}\delta(\xi) = \left(\frac{\partial x}{\partial \xi}\right)_\xi (1 - \frac{1}{2}u^2)\delta(x) = \left[(1 - \frac{1}{2}u^2)\nabla - \frac{1}{2}u(u \cdot \nabla)\right] \delta(x) + \ldots
\]  
(3.19)
Hence
\[ j^0(x) = q(k \times u) \cdot \nabla \delta(x) + \ldots \] (3.20)
\[ j(x) = -q(1 - \frac{1}{2}u^2)k \times \nabla \delta(x) + \frac{1}{2} q[u \cdot (k \times u)] \cdot \nabla \delta(x) \]
\[ = -qk \times \nabla \delta(x) + q(k \times u) \cdot \nabla \delta(x). \] (3.21)

By replacing \( x \) by \( x - y(t) \) and \( u \) by \( \dot{y}(t) \) in (3.13), (3.20) and (3.21) we obtain expressions for the instantaneously Lorentz boosted point vortex travelling along an arbitrary trajectory. In particular,
\[ j_{\text{boost}} = q((k \times \dot{y}) \cdot \nabla, -k \times \nabla + (k \times \dot{y}) \dot{y} \cdot \nabla) \delta(x - y). \] (3.22)

Note that
\[ \partial_\mu j^\mu_{\text{boost}} = q(k \times \ddot{y}) \cdot \nabla \delta(x - y) + \ldots \neq 0 \] (3.23)
so \( j_{\text{boost}} \) is not the moving point source we seek. Rather, we must add a correction \( j_{\text{acc}} \) to \( j_{\text{boost}} \) of order \( \dot{y} \) to enforce current conservation. Such a term will vanish in the case of motion at constant velocity and hence does not conflict with the required Lorentz properties of \( j \). We make the simplest choice, namely
\[ j_{\text{acc}} = (0, -qk \times \ddot{y} \delta(x - y)). \] (3.24)

Though we have chosen \( j^0_{\text{acc}} = 0 \), any function of order \( |\ddot{y}| \) would do since \( \partial_0 j^0_{\text{acc}} \) is automatically of negligible order. It turns out that this ambiguity has no bearing on our calculation because \( j^0_{\text{acc}} \) makes no contribution to \( L_{\text{int}} \) at order \( (\text{velocity})^2 \).

To summarize, the point vortex moving along a trajectory \( y(t) \) is represented by a composite point source
\[ \kappa(t, x) = q(1 - \frac{1}{2}|\dot{y}|^2) \delta(x - y) \] (3.25)
\[ j(t, x) = q((k \times \dot{y}) \cdot \nabla, -k \times \nabla + (k \times \dot{y}) \dot{y} \cdot \nabla - k \times \ddot{y}) \delta(x - y). \] (3.26)

The second task in the calculation of \( L_{\text{int}} \) is to construct the fields \((\psi_{(2)}, A_{(2)})\) induced by the second moving vortex. Were the field equations (3.2) and (3.3) massless, we could simply use retarded potentials, making a suitable expansion in time derivatives. Since they are not massless, we need a substitute for this procedure. We handle the problem by introducing formal temporal Fourier transforms of the fields and sources, as follows. Let \( \psi(t, x) \) be the field induced by the time varying source \( \kappa(t, x) \) according to (3.2), and define Fourier transforms \( \tilde{\psi} \) and \( \tilde{\kappa} \) with variable \( \omega \) dual to \( t \),
\[ \psi(t, x) := \int_{-\infty}^{\infty} d\omega e^{i\omega t} \tilde{\psi}(\omega, t), \quad \kappa(t, x) := \int_{-\infty}^{\infty} d\omega e^{i\omega t} \tilde{\kappa}(\omega, t). \] (3.27)
Then (3.24) implies
\[ [-\nabla^2 + (1 - \omega^2)] \tilde{\psi} = \tilde{\kappa}, \] (3.28)
so for each value of $\omega$, $\tilde{\psi}(\omega, \cdot)$ satisfies the static inhomogeneous Klein-Gordon equation with mass $\sqrt{1 - \omega^2}$ and source $\tilde{\kappa}(\omega, \cdot)$. Comparing with (3.30) one sees that (3.28) is solved, at least formally, by convolution of $\tilde{\kappa}(\omega, \cdot)$ with the (suitably scaled) Green’s function $K_0$,

$$\tilde{\psi}(\omega, x) = \frac{1}{2\pi} \int d^2x' K_0(\sqrt{1 - \omega^2}|x - x'|) \tilde{\kappa}(\omega, x'). \tag{3.29}$$

Expanding the Green’s function in $\omega$ and truncating at order $\omega^2$ yields

$$\tilde{\psi}(\omega, x) = \frac{1}{2\pi} \int d^2x' \left[ K_0(|x - x'|) - \frac{\omega^2}{2}|x - x'|K'_0(|x - x'|) \right] \tilde{\kappa}(\omega, x'). \tag{3.30}$$

From this we obtain $\psi$ by (3.27),

$$\psi(t, x) = \frac{1}{2\pi} \int d^2x' \left[ K_0(|x - x'|)\kappa(t, x') + \Upsilon(|x - x'|)\partial_t^2\kappa(t, x') \right] \tag{3.31}$$

where we have defined

$$\Upsilon(s) := \frac{1}{2}sK'_0(s). \tag{3.32}$$

Note that truncating the expansion in $\omega$ is, in effect, the same as neglecting higher time derivatives of $\kappa$, and hence eventually of $\Upsilon(t)$ in our application. No claim of rigour is attached to the above Fourier transform manoeuvre. One should regard it as a convenient algebraic shorthand for generating a perturbative solution of (3.2). Direct substitution of (3.31) into (3.2) confirms that this really is a solution up to higher time derivative terms ($\partial^3_t\kappa$ etc.). Since equation (3.3) is formally identical to (3.2), the vector field induced by a time dependent source $j$ is

$$A^\mu(t, x) = \frac{1}{2\pi} \int d^2x' d^2x'' \left[ K_0(|x - x'|)j^\mu(t, x') + \Upsilon(|x - x'|)\partial_t^2 j^\mu(t, x') \right] + \ldots. \tag{3.33}$$

Having obtained $(\psi(2), A(2))$ induced by the time varying source $(\kappa(2), j(2))$ we may compute the Lagrangian governing its interaction with another source $(\kappa(1), j(1))$ by substitution of (3.31) and (3.33) into (3.10). The result is

$$L_{\text{int}} = \frac{1}{2\pi} \int d^2x \int d^2x' \left\{ K_0(|x - x'|) \left[ \kappa(1)(t, x)\kappa(2)(t, x') - j^\mu(1)(t, x)j(2)_\mu(t, x') \right] \right. \left. - \Upsilon(|x - x'|) \left[ \partial_t\kappa(1)(t, x)\partial_t\kappa(2)(t, x') - \partial_t j^\mu(1)(t, x)\partial_t j(2)_\mu(t, x') \right] \right\} \tag{3.34}$$

where a total time derivative has been discarded. It remains to substitute the point vortex sources (3.25), (3.26) for vortices moving along trajectories $y(t)$ and $z(t)$ into (3.31) and evaluate the integrals. Explicitly,

$$L_{\text{int}} = \frac{1}{2\pi} \int d^2x \int d^2x' [\kappa(1)\kappa(2) + j(1) \cdot j(2) - j^0(1)j^0(2)]K_0(|x - x'|)$$

$$- \frac{1}{2\pi} \int d^2x \int d^2x' [\partial_t\kappa(1)\partial_t\kappa(2) + \partial_j(1) \cdot \partial_j(2) - \partial_j^0(1)\partial_j^0(2)]\Upsilon(|x - x'|) \tag{3.35}$$
where

\[
\kappa_{(1)K(2)} = \frac{q^2}{2} \left[ 1 - \frac{1}{2} (|\dot{y}|^2 + |\dot{z}|^2) \right] \delta(x - y) \delta(x' - z)
\]

\[
\dot{j}_{(1)} \cdot \dot{j}_{(2)} = \frac{q^2}{2} \left( \nabla_y \cdot \nabla_z - (\dot{z} \cdot \nabla_y)(\dot{z} \cdot \nabla_z) - (\dot{y} \cdot \nabla_y)(\dot{y} \cdot \nabla_z) - \dot{z} \cdot \nabla_y - \dot{y} \cdot \nabla_z \right) \delta(x - y) \delta(x' - z)
\]

\[
\partial_t \kappa_{(1)} \partial_t \kappa_{(2)} = \frac{q^2}{2} \left( \dot{y} \cdot \nabla_y (\dot{z} \cdot \nabla_z) - \dot{z} \cdot \nabla_y (\dot{y} \cdot \nabla_z) \right) \delta(x - y) \delta(x' - z)
\]

\[
\partial_t \dot{j}_{(1)} \cdot \partial_t \dot{j}_{(2)} = \frac{q^2}{2} \left( \dot{y} \cdot \nabla_y (\dot{z} \cdot \nabla_z) \right) \delta(x - y) \delta(x' - z)
\]

\[
\partial_t \dot{j}_{(1)} \partial_t \dot{j}_{(2)} = 0.
\]

We have discarded terms of order \(|\dot{y}|^3\), \(|\dot{y}||\dot{z}|\) etc., and systematically used

\[
\nabla \delta(x - y) = -\nabla_y \delta(x - y), \quad \nabla' \delta(x' - y) = -\nabla_z \delta(x' - z).
\]

All the integrals are thus rendered trivial, so

\[
L_{\text{int}} = \frac{q^2}{2\pi} \left\{ 1 - \frac{1}{2} (|\dot{y}|^2 + |\dot{z}|^2) + \nabla_y \cdot \nabla_z - (\dot{z} \cdot \nabla_y)(\dot{z} \cdot \nabla_z) - (\dot{y} \cdot \nabla_y)(\dot{y} \cdot \nabla_z) - \dot{z} \cdot \nabla_y - \dot{y} \cdot \nabla_z - (\kappa \cdot \nabla_y)(\kappa \cdot \nabla_z) \right\} K_0(|y - z|)
\]

\[
= \frac{q^2}{2\pi} \left[ -\frac{1}{2} |\dot{y}|^2 + |\dot{z}|^2 + (\dot{y} \cdot \nabla_z) + (\dot{z} \cdot \nabla_y) - \dot{y} \cdot \nabla_z \right] \Lambda(|y - z|) + L_{\text{acc}}.
\]

In (3.41), \(\alpha, \beta\) are any vectors with \(\beta\) independent of \(y\), and \(\Lambda\) denotes the function

\[
\Lambda(s) := K_0(s) - 2 \frac{K_0'(s)}{s}.
\]

Also, we have used the decomposition of \(\alpha, \beta\) relative to the orthonormal frame

\[
n^\parallel = \frac{y - z}{|y - z|}, \quad n^\perp = k \times n^\parallel.
\]

Equations (3.39) and (3.40) give useful cancellations in lines 1 and 3 of (3.38). All but two of the remaining differential operators (namely, the acceleration terms) are of the form (3.41) for suitable choice of \(\alpha\) and \(\beta\). Hence, we may repeatedly apply (3.41), yielding
where $L_{\text{acc}}$ represents the remaining acceleration terms. In fact

$$L_{\text{acc}} = -\frac{q^2}{2\pi} (\ddot{z} \cdot \nabla y + \dot{y} \cdot \nabla z) K_0(|y - z|) = \frac{q^2}{2\pi} (\dot{y} - \dot{z}) \cdot (y - z) K_0'(|y - z|)$$

$$= -\frac{q^2}{2\pi} \left[ |\dot{y} - \dot{z}|^2 \frac{K_0'(y - z)}{|y - z|} + (\ddot{y} - \ddot{z}) \cdot (y - z) K_0(|y - z|) \right] + \text{total time derivative.} \quad (3.45)$$

Substituting (3.45) and (3.42) into (3.44) finally yields

$$L_{\text{int}} = -\frac{q^2}{4\pi} |\dot{y} - \dot{z}|^2 K_0(|y - z|). \quad (3.46)$$

Our point particle model of 2-vortex dynamics is completed by adding to $L_{\text{int}}$ the usual nonrelativistic free Lagrangian for two particles of mass $\pi$ (the vortex rest energy), so

$$L = \frac{\pi}{2} (|\dot{y}|^2 + |\dot{z}|^2) - \frac{q^2}{4\pi} |\dot{y} - \dot{z}|^2 K_0(|y - z|). \quad (3.47)$$

We may define centre of mass and relative coordinates

$$\mathbf{R} = \frac{1}{2} (y + z), \quad \mathbf{r} = \frac{1}{2} (y - z) \quad (3.48)$$

so that

$$L = \pi |\dot{\mathbf{R}}|^2 + \pi \left( 1 - \frac{q^2}{\pi^2} K_0(2|\mathbf{r}|) \right) |\dot{\mathbf{r}}|^2. \quad (3.49)$$

Motion under this Lagrangian coincides with geodesic flow on the 2-vortex moduli space with respect to the asymptotic metric (2.31) of section 2: we simply identify $Z = R^1 + iR^2$ and $\sigma e^{i\theta} = r^1 + ir^2$.

Extension of this treatment to the case of $N$ well separated vortices is trivial due to the underlying linearity of our point-particle model. To the standard Lagrangian for $N$ free particles of mass $\pi$, moving along trajectories $y_r(t)$, $r = 1, 2, \ldots, N$ say, one simply adds one copy of $L_{\text{int}}$, as in eq. (3.46), for each unordered pair of distinct particles, or equivalently, one copy of $\frac{1}{2}L_{\text{int}}$ for each ordered distinct pair. The result is

$$L = \frac{\pi}{2} \sum_r |\dot{y}_r|^2 - \frac{q^2}{8\pi} \sum_{r \neq s} K_0(|y_r - y_s|)|\dot{y}_r - \dot{y}_s|^2. \quad (3.50)$$

The asymptotic formula for the $N$-vortex metric readily follows. Defining complex coordinates $Z_r = y_r^1 + iy_r^2$, their differences $Z_{rs} = Z_r - Z_s$, and holomorphic 1-forms $dZ_{rs} = dZ_r - dZ_s$, one sees that

$$g = \pi \sum_r dZ_r d\bar{Z}_r - \frac{q^2}{4\pi} \sum_{r \neq s} K_0(|Z_{rs}|) dZ_{rs} d\bar{Z}_{rs}, \quad (3.51)$$

which coincides with (2.31).
4 Two-Vortex Scattering

The relative motion of two vortices, in the geodesic approximation, is determined by the purely kinetic Lagrangian

\[ L = \frac{1}{2} \eta(\sigma)(\dot{\sigma}^2 + \sigma^2 \dot{\theta}^2) \]  

(4.1)

where the ranges of \( \sigma \) and \( \theta \) are as in section 2. Samols calculated the function \( b(\sigma) \) and hence \( \eta(\sigma) \) numerically, and using this found the geodesic motion of two vortices [14]. The geodesic motion has two constants of integration: the energy \( E \), which is \( L \) itself, and the angular momentum \( \ell \), which equals \( \eta(\sigma)\sigma^2 \dot{\theta} \). Using these, one can find \( d\theta/d\sigma \), and the scattering angle can be determined by integration. It depends on the impact parameter \( a \), given by the motion of the vortices as they approach from infinity. There the energy is \( \pi v^2 \) and the angular momentum is \( 2\pi av \).

The result for the scattering angle as a function of impact parameter agrees well with numerical simulations of 2-vortex scattering using the complete field equations [11, 15]. The motion is repulsive, and the scattering angle increases monotonically as the impact parameter decreases, from zero when the impact parameter is infinite, up to \( \frac{\pi}{2} \) in a head-on collision. The field dynamics begins to significantly differ from the geodesic motion only if the vortex speeds exceed about half the speed of light.

Now that we have obtained the asymptotic form of the 2-vortex metric, with \( \eta(\sigma) \) given by eq. (2.30), it is possible to estimate the asymptotic scattering. Unfortunately, the exact relation between scattering angle and impact parameter for this metric is given by a rather intractable integral. Rather than evaluate this numerically, we adopt a different strategy.

Since our asymptotic metric is only valid at large vortex separations, it can only be used with confidence to evaluate the scattering angle at large impact parameter. Here the vortex trajectories are almost straight, and the scattering angle small. We can find the approximate scattering angle by a perturbative calculation. It is convenient to use Cartesian coordinates \( x = \sigma \cos \theta \) and \( y = \sigma \sin \theta \). The Lagrangian becomes

\[ L = \frac{1}{2} \eta(\sigma)(\dot{x}^2 + \dot{y}^2) \]  

(4.2)

where \( \sigma = \sqrt{x^2 + y^2} \). The equations of motion are

\[ \eta(\sigma)\ddot{x} + \frac{\eta'(\sigma)}{2\sigma}(x\dot{x}^2 + 2y\dot{x}\dot{y} - x\dot{y}^2) = 0 \]  

(4.3)

\[ \eta(\sigma)\ddot{y} + \frac{\eta'(\sigma)}{2\sigma}(y\dot{y}^2 + 2x\dot{x}\dot{y} - y\dot{x}^2) = 0 \]  

(4.4)

We may assume that the motion is approximately along the line \( x = a \), with \( y \) increasing from \(-\infty \) to \( \infty \) at approximately constant speed \( v \). (This is in fact the trajectory of one of the vortices; the other moves in the opposite direction along the line \( x = -a \).) The initial value of \( \dot{x} \) is taken to be zero, and by calculating the small change in \( \dot{x} \) we can calculate the scattering angle. We work to leading order in the small quantity \( \exp(-2a) \). \( \eta' \) is of first order in \( \exp(-2a) \) along the trajectory, so at this order \( \ddot{x} \) is given by the term proportional to \( \dot{y}^2 \) in eq. (4.3), and the coefficient \( \eta \) multiplying \( \ddot{x} \) can be approximated by \( 2\pi \). \( \eta'\dot{x} \) is negligible.
\( \ddot{y} \) is of order \( \exp(-2a) \) too, but the consequent change of speed in the \( y \)-direction along the trajectory can be neglected. Thus, it is a sufficiently good approximation to take the solution of eq. (4.4) to be \( y = vt \), and to simplify eq. (4.3) to

\[
\ddot{x} = \frac{av^2 \eta'(\sigma)}{4\pi \sigma}.
\] (4.5)

The total change in \( \dot{x} \) is therefore

\[
\Delta \dot{x} = \frac{av^2}{4\pi} \int_{-\infty}^{\infty} \frac{\eta'(\sigma)}{\sigma} dt,
\] (4.6)

and the scattering angle, assuming it is small, is

\[
\Theta = \frac{1}{v} \Delta \dot{x}.
\] (4.7)

Expressing \( \Delta \dot{x} \) as an integral over \( y \), using \( y = vt \) and \( \sigma = \sqrt{a^2 + y^2} \), and also \( \eta'(\sigma) = \frac{4q^2}{\pi}K_1(2\sigma) \), we find

\[
\Theta = \frac{q^2a}{\pi^2} \int_{-\infty}^{\infty} K_1(2\sqrt{a^2 + y^2}) \frac{dy}{a^2 + y^2}.
\] (4.8)

\[
= -\frac{q^2}{2\pi^2} \frac{d}{da} \int_{-\infty}^{\infty} K_0(2\sqrt{a^2 + y^2}) dy.
\] (4.9)

Remarkably, this gives the simple result

\[
\Theta = \frac{q^2}{2\pi} \exp(-2a).
\] (4.10)

The integral can be understood as follows. The planar Helmholtz equation with source at the origin

\[
(-\nabla^2 + 4)\chi = 2\pi \delta(x)
\] (4.11)

has the exponentially decaying solution \( \chi = K_0(2\sqrt{x^2 + y^2}) \). Integrating the equation with respect to \( y \) we find that \( \tilde{\chi} = \int_{-\infty}^{\infty} K_0(2\sqrt{x^2 + y^2}) dy \) satisfies

\[
\left(-\frac{d^2}{dx^2} + 4\right) \tilde{\chi} = 2\pi \delta(x)
\] (4.12)

and hence \( \int_{-\infty}^{\infty} K_0(2\sqrt{x^2 + y^2}) dy = \frac{2}{\pi} \exp(-2|x|) \).

If we substitute the numerical value of \( q \), we can compare the dependence of scattering angle on impact parameter with Samols' result. The agreement is good for \( a \geq 2 \). This is shown in Fig. [1]

We expect corrections to this calculation. These are partly due to the neglected terms in (4.3) and (4.4), and partly due to corrections to our asymptotic metric.

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Figure 1: Scattering angle $\Theta$ against impact parameter $a$ for 2-vortex scattering in the geodesic approximation. Dashed curve: Samols’ numerical implementation; solid curve: our perturbative approximation.

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