Localization and topological transitions in non-Hermitian SSH models

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(Dated: January 12, 2023)

We study the localization and topological transitions of non-Hermitian SSH models, where the non-Hermiticities are introduced by the complex quasi-periodic hopping and the nonreciprocal hopping. The winding numbers of energy and the Lyapunov exponents in analytic form are obtained to characterize the transition points. Under the open boundary condition, two delocalization transitions are found due to the competition between the boundary localization and the Anderson localization. Although the second transition can be characterized by the winding numbers, all the two delocalization transitions don’t accompany the topological transition. The second delocalization transition is dominated by the quasi-periodic hopping due to the lack of boundary locations which is similar to the periodic boundary case. Furthermore, the on-site non-Hermiticity and non-Hermitian skin effect are detrimental to the boundary localization and topological transitions. However, the non-Hermitian skin effect enhances the Anderson localizations. The above analyses are verified by calculating the energy gap and the inverse of the participation ratio numerically.

I. INTRODUCTION

As the topological edge states are protected by the symmetry, the topological phases are expected to be immune to the perturbations of disorders [1–3]. Nevertheless, the strong disorders induce the destructive interference of scattered waves and induce a transition from the topological nontrivial phase to the Anderson localized phase [4, 5]. The competition of the boundary localization and bulk localization leads to the delocalization. The disorders connect the two unrelated quantum states which gives rise to the topological Anderson insulators [6–8]. Recently, the ability to engineer non-Hermitian Hamiltonian and the related observations of unconventional topological edge modes attracts a great interest to extend topological band theory to non-Hermitian systems [9]. The interplays between the topology and non-Hermiticity result in a plethora of exotic phenomena that have no Hermitian counterparts, e.g., the Weyl exceptional ring [10], and the point gap [11]. The non-Hermitian topological Anderson insulator becomes an interesting topic naturally.

The Anderson localizations are generally studied with the Aubry-André-Harper (AAH) model which is the periodical lattices modulated by quasi-periodical potential. Quasicrystals constitute an intermediate phase between fully periodic lattices and fully disordered media, showing a long-range order without periodicity [12, 13]. This solves the problems of reliably controlling disorders in solid-state systems where the Anderson localization could only be studied indirectly or through numerical simulations. As the AAH model can be mapped to the lattice version of the two-dimensional integer Hall effect problem, it gives the opportunity to study the Anderson localization and topological properties simultaneously [14, 15].

The exact localization-delocalization transition can be obtained by the duality transformation between real and momentum spaces [16, 20].

When the non-Hermiticities are further introduced to the AAH model by the complex on-site quasiperiodic potentials, it was found that the self-dual symmetry still determines the transitions from topological nontrivial phase to localized phase [21–23]. The quantum phase transitions still have the topological nature characterized by the winding numbers. However, when the non-Hermiticity is from nonreciprocal hopping, the induced non-Hermitian skin effect leads to the asymmetry localized states [24, 31, 32]. The Anderson transition point must be rescaled by the skin effect.

Up to now, the studies of the non-Hermitian topological Anderson transitions mainly focus on the AAH models [22, 28, 29, 32, 33]. The Su-Schrieffer-Heeger (SSH) model for polyacetylene is the simplest one-dimensional topological insulator with the chirality symmetry [34]. As many non-Hermitian models can be mapped to various non-Hermitian versions of the extend SSH models [35], they have become the typical model to study the non-Hermitian effects. In particular, the nonreciprocal hopping induces the non-Hermitian skin effect and leads to the general bulk-boundary correspondence [36, 37]. The non-Hermitian skin effect localizes all states at boundary whether zero-energy edge states or bulk states. However, the applied disorders on the model result in the Anderson localization on the bulk. The localizations of the model are expected to be the competition between the boundary localization and the bulk localization. The influence of non-Hermitian skin effect on disordered SSH model is explored and the result shows that the non-Hermiticity can enhance the topological phase against disorders by increasing bulk gaps [31]. The onset of a mobility edge and the topological phase transition in the disordered SSH chain connected to two external baths was investigated the Lindblad equation method [52]. The localization/delocalization of the disordered chain can be

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recovered the scaling properties of the nonequilibrium stationary current. However, their phase boundaries in analytical forms are still lacking.

In this paper, we study the non-Hermitian generalized SSH model, where the non-Hermiticities are introduced by both the complex quasiperiodic modulation hopping and the nonreciprocal hopping. The main results of this article include the following: (i) We find the equivalence and the universality of the models connected by the similarity transformation. (ii) The winding numbers and the Lyapunov exponents in analytic form are obtained to characterize the topological transition and localization transition. (iii) We clarify the origin of the two delocalization transitions. (iv) The influence of non-Hermiticities to the localizations and topological phase transitions is given.

The rest of paper is organized as follows. In Sec. II we present the Hamiltonian of generalized non-Hermitian SSH model and its various equivalent models. We explain the equivalence of the hopping terms and elucidate why the complex quasiperiodic modulation hopping can be applied in any of the terms. We also clarify how the nonreciprocal hopping can be transformed from one form to another by the similarity transformations. The winding numbers and the Lyapunov exponents are derived in Sec. III to characterize the topological transition and the Anderson transition. In Sec. IV the phase diagram of the model is obtained by calculating the inverse of participation ratio. A brief summary is presented in Sec. V

II. MODEL

We consider the non-Hermitian SSH model [pictorially shown in Fig. I (a)] \[^{34,52}\]. The Hamiltonian is described by

\[
H = \Psi^\dagger \hat{h} \Psi = \sum_n [t_1 a_n^\dagger b_n + t_1 b_n^\dagger a_n] + (t_2 - \gamma_2) a_{n+1}^\dagger b_n + (t_2 + \gamma_2) b_n^\dagger a_{n+1} + (t_3 - \gamma_3) a_n^\dagger b_{n+1} + (t_3 + \gamma_3) b_{n+1}^\dagger a_n,
\]

where \(\Psi^\dagger = (a_1^\dagger, b_1^\dagger, \ldots, a_L^\dagger, b_L^\dagger)\). \(a_n^\dagger\) and \(b_n^\dagger\) \((a_n\) and \(b_n)\) are the creation (annihilation) operators of a spinless fermion at lattice site \(n\). \(t_1, t_2\) and \(t_3\) are the hopping amplitudes. The parameters \(\gamma_2\) and \(\gamma_3\) induce the asymmetric coupling. The model in Eq. (I) has chiral symmetry \(S\hat{h}S^{-1} = -\hat{h}\), with operator \(S = (1, 1, \cdots, -1, -1, \cdots)\). The chiral symmetry ensures the eigenvalues \((E, -E)\) pairs [54].

The hopping amplitude \(t_1\) is set to be site dependent, i.e.

\[
t_1 = t_{1,n} = V_n = 2V\cos(2\pi \beta n + i\alpha),
\]

with the strength \(V\). \(\beta\) is an irrational number which is used to characterize the quasiperiodicity. It usually takes

\[
\phi = \frac{\gamma}{\alpha},
\]

the inverse of golden ratio \(\beta = (\sqrt{5} - 1)/2\), \(\alpha\) characterizes the non-Hermiticity of the quasiperiodic potential. The quasiperiodic term \(t_{1,n}\) acts as the disorders and induces localizations of the states.

It should emphasize that the setting of \(t_1 = t_{1,n}\) is not special since the \(t_1, t_2\) and \(t_3\) are equivalent and the quasiperiodicity can be set to any of the terms. The reason is the model in Fig. II (a) is equivalent to the two-leg ladder model [pictorially shown in Fig. II (b)]. The model remains unchanged when exchanging \(t_2\) and \(t_3\) terms. Exchanging \(t_1\) and \(t_3\) terms, the AB lattice of the model becomes merely the BA lattice which corresponds to relabeling \(b_n \rightarrow b_{n+1}\).

The introduction of parameters \(\gamma_2\) and \(\gamma_3\) do not lose the universality since the asymmetry of hopping \(t_2 \pm \gamma_2 = t_{3\pm e^{\pm \phi}}\) and \(t_3 \pm \gamma_3 = t_{3\pm e^{\pm \phi}}\) can be shifted from one term to another with the similarity transformation under the open boundary condition (OBC) \[^{52}\]. After the similarity transformations, we obtain the equivalent models. For example, under the OBC, the Hamiltonian of this model in Eq. (I) can be transformed into

\[
\hat{H} = \Psi^\dagger \hat{h} \Psi = \sum_n [t_1 \hat{a}_n^\dagger \hat{b}_n + t_1 \hat{b}_n^\dagger \hat{a}_n] + (t_2 - \gamma_2) \hat{a}_{n+1}^\dagger \hat{b}_n + (t_2 + \gamma_2) \hat{b}_n^\dagger \hat{a}_{n+1} + (t_3 - \gamma_3) \hat{a}_n^\dagger \hat{b}_{n+1} + (t_3 + \gamma_3) \hat{b}_{n+1}^\dagger \hat{a}_n,
\]

by the similarity transformation \(\hat{h} = S_1^{-1}hS_1\) and \(\hat{\Psi} = S_1^{-1}\Psi\) with

\[
S_1 = \text{diag}\{r_1, r_1^2, r_1^2, \cdots, r_1^{L/2}, r_1^{L/2}\}.
\]

When \(r_1 = e^{i\phi}\), the non-Hermiticity in \(t_3\) term is killed and the non-Hermiticity is transformed into \(t_2\) term completely. However, when \(r_1 = e^{-i\phi}\), the non-Hermiticity in \(t_2\) term is eliminated and the non-Hermiticity remains.

FIG. 1. (a) Non-Hermitian SSH model and (b) its equivalent two-leg ladder model.
in \( t_3 \) term. In such case \( [r_1 = e^{-\phi_2}] \), further doing the transformation \( h = S_2^{-1} \hat{h} S_2 \) and \( \tilde{\Psi} = S_2^{-1} \tilde{\Psi} \) with

\[
S_2 = \text{diag}\{1, r_2, r_2^2, r_2^3, \cdots, r_2^{L/2-1}, r_2^{L/2-1}, r_2^{L/2}, \cdots\},
\]

and \( r_2 = e^{-(\phi_2+\phi_3)/2} \), the model in Eq. (3) becomes

\[
\hat{H} = \tilde{\Psi}^\dagger \hat{h} \tilde{\Psi} = \sum_{n} [t_1 r_2^{-1} a_n^\dagger b_n + t_1 r_2 b_n^\dagger a_n \\
+ t_2 a_{n+1}^\dagger b_n + t_2 b_n^\dagger a_{n+1} + t_3 a_{n+1}^\dagger b_{n+1} + t_3 b_{n+1}^\dagger a_{n+1}].
\]

The non-Hermiticity in \( t_3 \) term is transformed into \( t_1 \) term.

It is easy to define the unity operator \( U \) which transforms the operator in Eq. (11) to

\[
\Phi^\dagger = \Psi^\dagger U^\dagger = \left( a_1^\dagger, b_1^\dagger, \cdots, a_L^\dagger, b_L^\dagger \right) U^\dagger \\
= \left( a_1^\dagger, a_2^\dagger, \cdots, a_L^\dagger, b_1^\dagger, b_2^\dagger, \cdots, b_L^\dagger \right) \\
= \left( u^\dagger, v^\dagger \right).
\]

With the unity operator \( U \), the Hamiltonian \( h \) in Eq. (11) under the PBC can be transformed into two block off-diagonal form

\[
\tilde{h} = U^\dagger h U = -i \tilde{h}_+ \sigma_+ + i \tilde{h}_- \sigma_- 
\]

here

\[
\tilde{h}_+ = \begin{pmatrix}
t_1 & t_2 - \gamma_3 & t_2 - \gamma_2 \\
t_2 - \gamma_2 & t_1 & \ddots \\
t_3 - \gamma_3 & t_2 - \gamma_3 & t_3 - \gamma_3 \\
\end{pmatrix},
\]

\[
\tilde{h}_- = \begin{pmatrix}
t_1 & t_2 + \gamma_2 & t_2 + \gamma_3 \\
t_2 + \gamma_2 & t_1 & \ddots \\
t_3 + \gamma_3 & t_2 + \gamma_3 & t_3 + \gamma_3 \\
\end{pmatrix},
\]

and \( \sigma_\pm = (\sigma_x \pm i \sigma_y) / 2 \). Under the OBC, there aren’t non-diagonal elements in the matrix \( \tilde{h}_+ \) and \( \tilde{h}_- \).

With the block off-diagonal form, the eigenvalue problem turns into

\[
\tilde{h}_+ \tilde{h}_- u_n = E_n^a u_n, \\
\tilde{h}_- \tilde{h}_+ v_n = E_n^b v_n.
\]

\( u_n \) and \( v_n \) can be treated as the single-particle states, and have the same properties.

Replacing \( t_2 \to -t - \Delta, t_3 \to -t + \Delta, \gamma_2 = \gamma_3 = \gamma \) and \( t_4 \to V_j \), the non-Hermitian SSH Hamiltonian in Eq. (17) is mapped to non-Hermitian AAH models with p-wave pairing [31]

\[
H = \sum_j \left[ -(t + \gamma) c_j^\dagger c_{j+1} - (t - \gamma) c_{j+1}^\dagger c_j \\
+ \Delta c_j c_{j+1} + \Delta c_{j+1}^\dagger c_j \right] + \sum_j V_j c_j^\dagger c_j \\
= \tilde{\Psi}_L^\dagger \tilde{h} R,
\]

in the Majorana fermion representation. \( c_j^\dagger \) is the creation operator of a spinless fermion at lattice site \( j \) and \( t \) is the hopping amplitude and set as the unit energy (\( t = 1 \)). \( \Delta \) is the p-wave pairing amplitude. The Majorana fermion operator

\[
\tilde{\Psi}_L^\dagger = (\nu_1^A, \nu_2^A \cdots \nu_L^A, \nu_1^B, \nu_2^B \cdots \nu_L^B),
\]

where \( \nu_j^A \equiv c_j + c_j^\dagger \) and \( \nu_j^B \equiv i(c_j - c_j^\dagger) \) are operators of two Majorana fermions belonging to one physical site. They satisfy relations \( \nu_j^A^\dagger = \nu_j^A \) and \( \{\nu_j^A, \nu_j^B\} = 2\delta_{jk}\delta_{\kappa\lambda} \) \( (\kappa, \lambda = A, B) \) [53,54]. In the case of \( \gamma = 0 \), localization and topological phase transitions of model [39] have been studied in Ref. [31].

III. THE TOPOLOGICAL TRANSITION AND THE ANDERSON TRANSITION

A. Winding number

Considering the periodicity of AB lattice and the modulated quasiperiodic potential, the topological transitions can in principle be discussed in GBZ with the technique proposed in Ref. [36–50]. The topological transition can also be studied with its topological equivalent model in conventional BZ introduced in Ref. [22]. However, when the quasiperiodic potential in Eq. (12) is used to depict the disorder, a large order of Fibonacci number must be used to ensure the long term disorder which results in the complexity of the discussion. Alternatively, a phase \( \delta \) is usually added in the complex quasiperiodic potential

\[
V_j = 2V\cos(2\pi\beta j + \alpha + \delta /L),
\]

to introduce an additional dimension. The winding number of energy is often used to characterize the topological nontrivial phase and defined as

\[
\nu = \lim_{L \to \infty} \oint \frac{dk}{2\pi i} \delta_\phi \left[ \ln \det (H) \right],
\]

here the Hamiltonian \( H \) is defined under the PBC [11]. \( \nu \) can be used to depict how the complex spectral trajectory \( E \) encircles a base energy 0 in the complex energy plane, when \( \delta \) changes from 0 to \( 2\pi \).

With the block off-diagonal Hamiltonian in Eq. (7), the winding number is given by [55,56]

\[
\nu = \lim_{L \to \infty} \oint \frac{dk}{2\pi i} \delta_\phi \left[ \ln \det \left( \tilde{h}_{k,+} \tilde{h}_{k,-} \right) \right],
\]

which is the summation of winding numbers of winding vectors \( \text{det}(\tilde{h}_{k,+}) \) and \( \text{det}(\tilde{h}_{k,-}) \) in the complex energy plane. In the large limit \( L \to \infty \), \( \text{det}(\tilde{h}_{k,+}) \) and \( \text{det}(\tilde{h}_{k,-}) \) have the analytical forms which are given by

\[
\text{det}(\tilde{h}_{\pm}) = -L^{L/2} (-1)^{L/4+1} V^L e^{L\alpha - i\delta} + P_{\pm},
\]
The transfer matrix of the whole system is given by $T$ to topological transitions. 

$$P_{\pm} = \max(\ell_2 \ell_3 e^{\pm(\phi_2 + \phi_3)}, V) \left[ \begin{array}{c} V_1 \sqrt{\xi} \\ 1 \end{array} \right],$$

The detailed calculations can consult the Refs. [29, 31, 52, 53]. $P_{\pm}$ can be neglected under the PBC. The winding number is given by $\nu = \nu_+ + \nu_-,

$$\nu_{\pm} = \Theta(V e^\alpha - \tau_{\pm}),$$

here $\Theta(x)$ is the step function. The topological transition points are given by $V_{T,\pm} = \tau_{\pm} e^{-\alpha}$ which indicate that the onsite non-Hermiticity $\alpha$ is always detrimental to topological transitions.

### B. Lyapunov exponent

The transition from extended to localized phases can be clarified by the Lyapunov exponent $\eta$ which is the inverse of localization length. In the localized phase, the wave function is exponentially decayed in the localization center $n_0$, i.e. $\exp(-\eta |n-n_0|)$.

Under the weak disorder perturbations, the energy-band structure remains unchanged. For the topological nontrivial system, the nontrivial edge states are almost localized at several sites. In the strong disorder case, the energy-band structure will have a breakdown. The localization is determined by Anderson localization. The definition of Lyapunov exponent of the nontrivial state is based on the transfer matrix approach [60]. If the system under OBC hosts a topological nontrivial state, the block off-diagonal Hamiltonian in Eq. (7) has the zero-energy mode which should satisfy $\hbar_\nu v_\nu = 0$. In the transfer matrix form, the equation can be rewritten as

$$\left( \begin{array}{c} v_{j+1} \\ v_j \end{array} \right) = T_j \left( \begin{array}{c} v_j \\ v_{j-1} \end{array} \right),$$

with

$$T_j = \left( \begin{array}{cc} V_j & -t_3 e^{-\gamma_3} \\ t_3 - \gamma_3 & 0 \end{array} \right).$$

The transfer matrix of the whole system is given by $T = \Pi_{j=1}^L T_j$ with the two eigenvalues $\lambda_1$ and $\lambda_2$. Supposing $|\lambda_1| < |\lambda_2|$, the Lyapunov exponent of the edge states is defined as

$$\eta = - \lim_{L \to \infty} \frac{\ln |\lambda_2|}{L}. \quad (13)$$

In order to determine $\eta$, we perform a similarity transformation $ST_jS^{-1} = \xi^{-1/2}T_j$ with $S = \text{diag}(\xi^{1/4}, \xi^{-1/4})$ and

$$\xi = \frac{t_3 - \gamma_3}{t_2 - \gamma_2}.$$

As a result, the localization transition point is $V = e^{-\alpha} \tau_-$ which equals to the topological transition point $V_{L,\nu}$. The numerical analysis in Sec. [16] will verify $V_{L,\nu}$ is the first delocalization transition point of the edge state which indicates the topological transition accompany the breakdown of zero-energy edge states.

Under the PBC, the calculation of Lyapunov exponent can consult the Ref. [31]. We can obtain two values $\eta_+ = \ln \frac{V e^\alpha}{\xi^{1/2}}$ which are the same as the winding numbers. Since the log function needs $V e^\alpha > \tau_\pm$, we can get two localization transition points $V_{L,\pm} = e^{-\alpha} \tau_\pm$ which also indicate that the onsite non-Hermiticity $\alpha$ is detrimental to localization transitions.
IV. PHASE DIAGRAM

We diagonalize the eigenfunction in Eq. (8) numerically to obtain the OBC spectra shown in Fig. 2 (a1) and PBC spectra in Fig. (b1) respectively. In the numerical analysis, \( t_2 = 1 \), \( t_3 = 0.8 \), \( \alpha = 0.4 \) and \( \gamma = 1/3 \). Comparing the two energy spectra, the zero-energy states are found in the OBC and PBC chains. To estimate the topological phase transition, we plot semilog of the lowest \( |E_n^2| \) of the two spectra in Fig. 2 (a2) and (b2) under the two kinds of boundary conditions. It finds that the open/close of the gap in Fig. 2 (b2) is accompanied by the emerging of zero-energy in Fig. 2 (a2). Hence, the dip of the energy in Fig. 2 (b1) can be taken as an indicator to characterize the topological phase transition of the model.

The first transition point in Fig. 2 (a1) is consistent with the prediction by the formula \( V_{T,-} = \tau_+ e^{-\alpha} = (t_2 - \gamma) e^{-\alpha} = 0.4469 \). Although the second transition point \( V_{T,+} = \tau_+ e^{-\alpha} = (t_2 + \gamma) e^{-\alpha} = 0.8938 \) don’t accompany the gap closing of the OBC spectra in Fig. 2 (a1), the small dip appears in PBC energy spectra in Fig. 2 (b1). In fact, with increasing the disorder intensity \( V \) across \( V_{T,-} \), the nontrivial edge state is destroyed. Further increasing the disorder intensity \( V \) across \( V_{T,+} \), the system has been in the topological trivial phase. As a result, the topological transition point is \( V_{T,-} \). We will see that the dip near \( V_{T,+} \) is related to the Anderson transition.

We calculate the lattice length \( L \) dependence of two transition points in Fig. 2 (c) under the two kinds of boundary conditions. It finds that the topological transition approaches \( V_{T,\pm} = \tau_\pm e^{-\alpha} \) under the OBC when \( L \) is larger than 900. Under the PBC however, the chain length \( L > 500 \) is enough to ensure the calculation accuracy. So \( L \) is taken to be 1200 in all the numerical calculations. To further verify the topological transition of the model, we plot semilog of the lowest \( |E_n^2| \) of the PBC spectra in Fig. 2 (d1) with different parameters. The amplitude of the quasiperiodic potential \( V \) is scaled by \( V_{T,-} = (d_2) \) and \( V_{T,+} \) (d3) respectively. The two dips collapse to the points which indicate the effectiveness of \( V_{T,\pm} = \tau_\pm e^{-\alpha} \).

In order to characterize the localization of an normalized state \( u_n(x_m) \), the inverse of the participation ratio (IPR) is defined as

\[
\text{IPR}_n = \sum_{m=1}^{L} |u_n(x_m)|^4.
\]

For an extended state, \( \text{IPR}_n \) is of the order \( 1/L \), whereas it approaches to 1 for a localized state. When \( u_n \) is used to calculate \( \text{IPR}_n \), the conclusions remain unchanged. In order to characterize the localization of the whole system, the mean inverse of the participation ratio \( \overline{\text{IPR}} = \sum_n \text{IPR}_n / L \) is defined.

To study the localization of nontrivial topological
phase, we calculate \( \text{IPR}_1 \) corresponding to the lowest \( |E_n^2| \) shown in Fig. 3(a). As a demonstration of the topological phase, we also plot zero-energy number versus \( V \) in (b). In the low \( V \) region near \( V_{L,1} = e^{-\alpha} (t_2 - \gamma_2) = 0.4469 \), a dip \( V \simeq 0.45 \) dives in Fig. 3(a). The appearance of the dip (delocalization) accompanying the topological transition is due to the localization competition between nontrivial edge states and the disorders. The competition leads to the most extended case of the particle distributions (the smallest \( \text{IPR}_1 \)) in the transition point. As expected, the zero-energy edge states [Fig. 3(c1) and (c2)] indicate the system in the topological nontrivial phase when \( V < V_{T,+} \). Further increasing \( V > e^{-\alpha} (t_2 - \gamma_2) \), the system has been in the Anderson localization phase. We get the other transition point \( V_{L,-} = e^{-\alpha} (t_2 + \gamma_2) \). In the strong disorder region \( V \simeq V_{L,-} = 0.8938 \), the other dip appears at \( V = 0.89 \) due to the the Anderson localization only.

We calculate the averaged inverse participation ratios (IPR) over the right eigenstates \( u_n \) in Eq. (3) under two kinds of boundary condition in Fig. 3(b). In the low \( V \) region near \( V_{L,-} = e^{-\alpha} \epsilon_1 = e^{-\alpha} \sqrt{(t_2 - \gamma)} (t_3 - \gamma) = 0.3739 \), a deep \( V \simeq 0.38 \) dives in Fig. 3(b) under the OBC which is different from that in Fig. 3(a). The appearance of the dip indicates the transition of the whole system doesn’t accompany the topological transition where the contribution of edge zero-energy states is negligible. The delocalization transition is due to the localization competition between boundary localization from the non-Hermitian skin effect and the Anderson localization from the disorders. Further increasing \( V > e^{-\alpha} \epsilon_+ \), the system has been in the Anderson localization phase. We get the other transition point \( V_{L,+} = e^{-\gamma} \tau_+ = 0.89 \). Across the dip, IPR increases remarkably and the system in the Anderson localization phase shown in Fig. 3(e1) and (e2). In the intermediate region, \( V_{L,-} < V < V_{T,+} \), the spatial distribution of wave functions is shown in Fig. 3(d1) and (d2).

This transition also occurs under the PBC where IPR increases rapidly across the transition point [blue-circle in Fig. 3(b)]. In such case, the non-Hermitian skin effect and zero-energy edge states disappear. Therefore, there isn’t the boundary-localization nature. The Anderson localization is completely due to the disorders which lead to the destructive interference of scattered waves. The consistence of the second transition points in Fig. 3 (a) and (b) indicates the delocalizations have the same origin which is unrelated to the topological transition although it can be characterized by the winding number. According to the formula \( V_{L,+} = (t_2 + \gamma) e^{-\alpha} \), a small \( V \) is needed to obtain the same Anderson transition point as the Hermitian case \( \gamma = 0 \). We conclude that the non-Hermitian skin effect enhances the Anderson localization.

FIG. 3. Localization transitions. (a) IPR of the lowest \( |E_n^2| \) state \( u_n \) versus \( V \) and number of zero-energy state versus \( V \). (b) IPR of the state \( u \) versus \( V \). The second row and the third row are the wave-function \( u_n \) and \( v_n \) of the lowest \( |E_n^2| \) for the three different regions \( V \) under the OBC.
V. SUMMARY

In summary, we have studied the localization and topological phase transitions of non-Hermitian SSH models, where the non-Hermiticities are introduced by the complex quasiperiodic hopping and the nonreciprocal hopping. Under the OBC, increasing the intensity of the complex quasiperiodic hopping destroys the boundary localizations, and then drives the system into the Anderson localization. The competitions lead to the delocalization transitions. Due to non-Hermitian skin effect, the localization transitions are not necessarily accompanied by the topological transitions. By analyzing the winding number of energy and the Lyapunov exponent, we find the non-Hermitian skin effect is detrimental to the topological phase transitions, and enhances the Anderson localization. However, the on-site non-Hermiticity is always detrimental to the Anderson localization and topological phase transitions. Since the models have many topological equivalent models, the results we studied are useful to further study the topological Anderson insulator experimentally.

ACKNOWLEDGMENTS

This work was supported by Hebei Provincial Natural Science Foundation of China (Grant No. A2012203174, No. A2015203387), Science and Technology Project of Hebei Education Department, China (Grant No. ZD202020) and National Natural Science Foundation of China (Grant No. 10974169, No. 11304270).

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