On extracting physical content from asymptotically flat spacetime metrics

C Kozameh\(^1\), E T Newman\(^2\) and G Silva-Ortigoza\(^3\)

\(^1\) FaMaF, University of Cordoba, Cordoba, Argentina
\(^2\) Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, PA 15260, USA
\(^3\) Facultad de Ciencias Físico Matemáticas, de la Universidad Autónoma de Puebla, Apartado Postal 1152, 72001 Puebla, Pue., México

Received 14 February 2008, in final form 13 April 2008
Published 19 June 2008
Online at stacks.iop.org/CQG/25/145001

Abstract
A major issue in general relativity, from its earliest days to the present, is how to extract physical information from any solution or class of solutions to the Einstein equations. Though certain information can be obtained for arbitrary solutions, e.g., via geodesic deviation, in general, because of the coordinate freedom, it is often hard or impossible to do. Most of the time information is found from special conditions, e.g. degenerate principle null vectors, weak fields close to Minkowski space (using coordinates close to Minkowski coordinates), or from solutions that have symmetries or approximate symmetries. In the present work, we will be concerned with asymptotically flat spacetimes where the approximate symmetry is the Bondi–Metzner–Sachs group. For these spaces the Bondi 4-momentum vector and its evolution, found from the Weyl tensor at infinity, describes the total energy–momentum of the interior source and the energy–momentum radiated. By generalizing the structures (shear-free null geodesic congruences) associated with the algebraically special metrics to asymptotically shear-free null geodesic congruences, which are available in all asymptotically flat spacetimes, we give kinematic meaning to the Bondi 4-momentum. In other words, we describe the Bondi vector and its evolution in terms of a center of mass position vector, its velocity and a spin vector, all having clear geometric meaning. Among other items, from dynamic arguments, we define a unique (at our level of approximation) total angular momentum and extract its evolution equation in the form of a conservation law with an angular momentum flux.

PACS number: 03.50.De
1. Introduction

From the very earliest days of general relativity (GR), the issue of extracting the physical meaning or content in the solutions of the Einstein equations has been a difficult problem [1] and usually has been solved only in special cases or special situations, e.g. in the presence of symmetries or approximate (asymptotic) symmetries. Probably the best-studied case is that of asymptotically flat spacetimes where an asymptotic symmetry exists, namely the Bondi–Metzner–Sachs (BMS) group, and where, at null infinity, the total energy–momentum of the interior source was identified as well as gravitational radiation, i.e. energy and momentum loss. These identifications, in terms of the asymptotic components of the Weyl tensor, were made via group theoretical arguments combined with physical insight into the dynamics of the radiation process. However the problem of gravitational multipole moments and in particular the mass dipole and the angular momentum has proved to be difficult. At the present time, using symmetry arguments, there are several different proposed definitions [2–5] of these quantities with little apparent use of dynamic considerations. (To our knowledge only in linear theory or in stationary cases has the issue of quadrupole moments been addressed.) In the present work, using largely a dynamic argument and verifying it by a symmetry consideration, we return to the issue of recognizing the mass-dipole moment and angular momentum hidden in the asymptotic Weyl tensor. From this dynamic argument the mass-dipole moment and angular momentum, using the Bondi energy–momentum 4-vector, are found. Both kinematic expressions and dynamic equations (equations of motion) are determined.

In flat-space Maxwell theory, the related issues are much simpler though some questions do remain. The charge, obtained by a 2D surface integral of the asymptotic Maxwell field, is a constant and lies in the scalar representation of the Lorentz group; the dipole moments (electric and magnetic) also obtained by 2D surface integrals of the asymptotic Maxwell field (though the results depend on a time direction and the 2-surface chosen for the integral) also lie in a finite dimensional representation of the Lorentz group. For a static or stationary system, it is easy to define the center of charge (so that the electric dipole moment associated with it vanishes) and a constant magnetic dipole moment. Recently it was shown [6] that for general asymptotically flat Maxwell fields (with non-vanishing total charge) one could find a unique complex worldline in complex Minkowski space, referred to as the complex center of charge, from which the dynamic electric and magnetic moments could be obtained.

It is this construction of the complex center of charge for the Maxwell field that we generalize to GR. This leads, in a very unorthodox manner, to a complex center of mass. The real part is identified with the usual center of mass while the imaginary part is the specific spin-angular momentum, i.e. the spin per unit mass. The argument leading to this complex worldline is based largely on dynamics and analogies with electrodynamics and special solutions of GR, e.g., the algebraically special metrics and in particular the charged spinning metric [7]. It is applied to both the asymptotic vacuum Einstein equations and the Einstein–Maxwell equations. For ease of presentation, most of the analysis will be for the vacuum case. The results for the Einstein–Maxwell case will be presented towards the end of section 3 without detailed derivation.

In order to get a better perspective on this work and on our point of view, several comments might be of some use.

Both Maxwell theory and GR are considered to be fundamental physical theories (in addition to quantum theory which so far appears incompatible with GR) and thus in principle this construction is applicable to the gravitational field in the neighborhood of any massive body, from elementary particles to galaxies. Though GR is almost always applied to astrophysical situations, there is in principle no reason that it could not be applied at some level
to laboratory masses or even to elementary particles. If the individual masses are sufficiently far apart, we could even consider their gravitational fields at (relatively) large distances as being asymptotically flat and use the general theory of asymptotically flat spacetimes for their analysis. The theory of asymptotically flat spacetimes appears to be the best tool to define isolated bodies in GR and, as in Newton’s theory, the notion of isolated bodies is an approximation of reality that is useful in general, and specifically for defining center of mass, intrinsic angular momentum, etc.

It is this point of view we adopt in this work. Though there is no rigorous way to make physical identification with the GR variables, we will consider three very similar alternative identifications—some more intuitive than others—and see what are the theoretical consequences—and then argue for the last one. The idea is then to look at the dynamic predictions of each of the alternatives. At the linear level they all agree: it is in the consideration of the nonlinear terms that we see the differences and see how one of the alternatives, probably the least intuitive one, comes closest to what we would physically expect.

The starting point for the discussion is the asymptotic Bianchi identities. We have assumed that the Einstein or the Einstein/Maxwell equations have been integrated in the asymptotic region using the standard Bondi–Sachs–Newman–Penrose asymptotic [8–11] (peeling theorem) behavior. When the relevant integrations have been completed we are left with several evolution equations, namely the asymptotic Bianchi identities (the Bondi supplementary conditions) and asymptotic Maxwell equations [12], that become our object for analysis and interpretation.

In section 2, we review the relevant ideas about asymptotically flat spacetimes (Einstein or Einstein/Maxwell) at null infinity [12]. More specifically, we discuss the structure of null infinity (and its complexification), i.e. Penrose’s $\mathcal{I}^+$. We then review a variety of subjects: e.g. what ‘lives’ on $\mathcal{I}^+$, the asymptotic Weyl tensor with the Bianchi identities and the Bondi shear, other related structures as cuts of $\mathcal{I}^+$, past light cones and angle fields. The spherical harmonic decomposition of the various fields is then described. In this context, we discuss the Bondi 4-momentum and the energy/momentum loss theorem.

In section 3, we examine three alternative suggestions for extracting physical information from the Weyl tensor components in the asymptotic Bianchi identities. A central structure in this extraction process comes from the dynamics associated with the Bondi 4-momentum. More specifically, we want first to identify the mass dipole (or center of mass) and angular momentum. Then, second, we want to establish dynamical laws for these quantities, i.e. we want equations of motion for the center of mass and evolution equations—with fluxes—for the angular momentum. They come directly from the energy–momentum loss equations and a certain reality condition. These dynamical laws are given at future null infinity and therefore are intended to apply to the entire isolated system. Interactions between different bodies cannot be considered in this framework.

The first and most ‘obvious’ or intuitive model for the selection for the mass dipole and angular momentum turns out to lack certain essential dynamic features and has little or no geometric meaning. We consider the first (and second) approaches as ‘toy’ models, introduced in order to motivate and clarify the third and primary model. The second and the third approaches which are based on specific choices of complex slicing of future null infinity, referred to as (complex) cut functions, lead to very similar dynamics. The nod goes to the third choice, it being more ‘natural’ in its similarity or analogy to the algebraically special metrics and to its unity and geometric attractiveness. They are both based on a geometric construction of complex cut functions. One of them, the second, is more appropriate to flat space; the other (which does reduce to the second in flat space) is far better suited to the general asymptotically flat case. In the latter case we give an extended description of the predictions
and dynamic results including the results for the Einstein–Maxwell equations. These results include kinematic expressions for the Bondi mass and momentum, a center of mass and its equations of motion and a definition of angular momentum that includes both spin and orbital terms with a conservation law with an angular momentum flux.

In section 4 we analyze the invariance, under the BMS group, of the third approach to this method of assigning physical meaning to the variables. We briefly discuss the representation theory of the Lorentz group due to Gelfand et al [13, 14], and in particular apply their description of the invariant finite dimensional subspaces of the infinite dimensional representations to our physical assignments. As a simple example, we apply the representation theory to the Bondi energy–momentum 4-vector and show that it is indeed a Lorentzian 4-vector. The details are given in appendix A.2.

In section 5, we summarize and discuss our results, including the results for the Einstein–Maxwell case. We also speculate on possible consequences of these results. One of our more interesting results—worth speculating on—is that a simple geometric condition on the Einstein–Maxwell fields leads to the Dirac value of the gyromagnetic ratio.

Our method for the study of physical identifications is based on the use of spherical harmonic expansions so that the coefficients in these expansions become identified with physical objects. It is clear from the nonlinearity of the theory that it is impossible to work with the exact infinite expansions and, thus, we must work with a truncated series. Specifically, we expand everything up to and including the $l = 2$ harmonics and then include only second-order products. (In one place, where the result is physically so attractive, we have included a third-order term.) Almost all of the nonlinear terms in our results arise from the (frequent) use of Clebsch–Gordan expansions of spherical harmonic products.

2. Preliminaries

2.1. Null infinity

After integrating [8–12] the Einstein (or Einstein–Maxwell) equations along null surfaces in the spin-coefficient formalism for a large affine parameter, $r$, we are left in the limit of large $r$ with several evolution equations [12] (for the asymptotic Weyl tensor components) that are referred to as the ‘asymptotic Bianchi identities’. (These equations are analogous to the remaining Maxwell equations [12] after all the radial integrations have been performed.) This limit of large $r$ has been formalized by Penrose into the idea of the future null boundary of spacetime, a null 3-surface, and referred to as $\mathcal{I}^+$. Our remaining equations are thus a set of differential equations on $\mathcal{I}^+$. This null boundary has the topology of $S^2 \times \mathbb{R}$ and an assignment of (Bondi) coordinates $(u, \zeta, \bar{\zeta})$, with $u$ on the $\mathbb{R}$ part and $(\zeta, \bar{\zeta})$ as complex stereographic coordinates on $S^2$. The freedom in the choice of these Bondi coordinates, known as the BMS group, is geometrically the asymptotic symmetry group. The BMS group is composed of two parts, the supertranslations

$$\tilde{u} = u + \alpha(\zeta, \bar{\zeta}), \quad \tilde{\zeta} = (\zeta, \bar{\zeta})$$

with $\alpha(\zeta, \bar{\zeta})$ an arbitrary smooth function on the sphere and the Lorentz transformations [8, 9, 14–16] given by

$$\tilde{u} = Ku$$

$$K = \frac{1 + \zeta \bar{\zeta}}{(a\zeta + b)(\bar{a}\zeta + \bar{b}) + (c\zeta + d)(\bar{c}\zeta + \bar{d})}$$

$$\tilde{\zeta} = \frac{a\zeta + b}{c\zeta + d}; \quad ad - bc = 1.$$
Though we are dealing with real spacetimes and \( \mathcal{I}^+ \) is a real 3-surface, since we assume that all the relevant functions are analytic, it is useful to allow \( u \) to take on complex values close to the real and to allow \( \bar{\zeta} \) to deviate slightly from the complex conjugate of \( \zeta \).

In addition to the Bondi coordinates, we also have a null tetrad \((l^a, n^a, m^a, \overline{m}^a)\) that is associated with the Bondi coordinates (and does change with a BMS coordinate transformation) where \( n \) is tangent to the null generators of \( \mathcal{I}^+ \), \( m^a \) and \( \overline{m}^a \) are tangent to the \( u = \text{constant} \) slices of \( \mathcal{I}^+ \) while \( l \) is the null vector normal to the slices pointing to the spacetime along the null surface \( u = \text{constant} \).

### 2.2. Further structures

There are several important structures associated with \( \mathcal{I}^+ \) that we now describe.

- In addition to the Bondi ‘slicing’ of \( \mathcal{I}^+ \) given by the ‘cuts’, \( u = \text{constant} \), one can take an arbitrary real one-parameter, \( s \), family of slices given by real cut functions,
  \[
  u = G(s, \zeta, \bar{\zeta}),
  \]
  or their generalization to analytic complex cut functions
  \[
  u = X(\tau, \zeta, \bar{\zeta})
  \]
  with inverse functions
  \[
  \tau = T(u, \zeta, \bar{\zeta})
  \]
  where the complex \( \tau \) must be able to be chosen so that \( u \) is real. The freedom in the choice of the parameter \( \tau: \hat{\tau} = F(\tau) \) with \( F \) analytic will be used later to normalize a physical variable.

- At every point of \( \mathcal{I}^+ \) there is the past light cone of rays going back into the interior. The directions are labeled by the complex stereographic angle \((L, \overline{L})\) with the zero value taken along \( l \) and the infinity along \( n \). Any complex angle field \( L(\zeta, \bar{\zeta}) \), i.e., a stereographic angle given for each point on \( \mathcal{I}^+ \), can be expressed \([17]\) in terms of a complex cut function, restricted to real \( u \)'s by
  \[
  L(u, \zeta, \bar{\zeta}) = \delta(\tau)X(\tau, \zeta, \bar{\zeta})
  \]
  \[
  \tau = T(u, \zeta, \bar{\zeta}).
  \]

The subscript \((\tau)\) means the application of \( \delta \) holding \( \tau \) constant.

- We will have considerable use for the local Lorentz transformation (null rotations) at each point of \( \mathcal{I}^+ \) parametrized by the arbitrary angle field
  \[
  L = L(u, \zeta, \bar{\zeta})
  \]
  that preserves the vector \( n \), namely
  \[
  l^* = l + L\overline{m} + \overline{L}m + L\overline{L}n
  \]
  \[
  m^* = m + Ln
  \]
  \[
  n^* = n.
  \]

For each point on \( \mathcal{I}^+ \), the null vector \( l^* \) determines a null geodesic extending backwards into the spacetime so that the field of \( l^* \)'s determines a null geodesic congruence of the spacetime.
2.3. What lives on $I^+$?

The following functions are defined [12] on $I^+$:

- $\sigma = \sigma(u, \zeta, \zeta)$, the asymptotic shear of the null geodesic congruence with the Bondi tangent vector $l$.
- $\psi_1(u, \zeta, \zeta)$ and $\psi_2(u, \zeta, \zeta)$ are the leading terms of two tetrad components of the Weyl tensor, $\psi_1 = \psi_1^0/r^4 + \cdots$ and $\psi_2 = \psi_2^0/r^3 + \cdots$.

With

\[
\psi_1 = -C_{abcd}n^an^b\psi_{abcd}^0 \quad \psi_2 = -C_{abcd}n^an^b\psi_{abcd}^0.
\]

Under the tetrad transformation, equation (7), the Weyl components $\psi_1^0$ and $\psi_2^0$ transform as

\[
\psi_1^0 = \psi_1^0 - 3L\psi_2^0 + 3L^2d(\sigma) + L^3\sigma
\]

\[
\psi_2^0 = \psi_2^0 - 2Ld(\sigma) - L^2\sigma.
\]

$\psi_1^0$ and $\psi_2^0$ satisfy the asymptotic Bianchi identities

\[
(\psi_1^0) = -\delta\psi_2^0 + 2\sigma \delta(\sigma)
\]

\[
(\psi_2^0) = -\delta^2(\sigma) - \sigma(\sigma)
\]

\[
\psi_2^0 - \psi_2^0 = \delta^2(\sigma) - \delta^2(\sigma) + (\sigma)(\sigma) - (\sigma)(\sigma).
\]

The last two equations can be rewritten in terms of the 'mass aspect', $\Psi$, as

\[
\Psi = \sigma \bar{\sigma},
\]

\[
\Psi = -\bar{\Psi} = \psi_0^0 + \delta^2(\sigma) + (\sigma)(\sigma).
\]

When a Maxwell field is present, these equations become modified and the Maxwell equations must be included [12]:

\[
(\psi_1^0) = -\delta\psi_2^0 + 2\sigma \delta(\sigma) + 2k\phi_0^0(\phi_2^0)
\]

\[
(\psi_2^0) = -\delta^2(\sigma) - \sigma(\sigma) + k\phi_0^0(\phi_2^0)
\]

\[
\bar{\Psi} = \Psi = \psi_0^0 + \delta^2(\sigma) + (\sigma)(\sigma),
\]

\[
k = 2Gc^{-4}
\]

\[
(\phi_0^0) + \delta\phi_0^0 - \sigma\phi_2^0 = 0,
\]

\[
(\phi_1^0) + \delta\phi_2^0 = 0.
\]

2.4. Asymptotically shear-free null geodesic congruences

- If the angle field $L(u, \zeta, \zeta)$ satisfies the differential equation [17, 18]

\[
\delta L + LL = \sigma(u, \zeta, \bar{\zeta}),
\]

the null geodesic congruence determined by the null vector field given by equation (7) is asymptotically shear free. It has been shown earlier [17] that solutions to equation (21) that are regular on $I^+$, i.e. have no infinities, are given by the construction as follows.
\[ L(u, \zeta, \bar{\zeta}) \text{ is given parametrically by} \]
\[ L(u, \zeta, \bar{\zeta}) = \delta_{\ell \gamma} X(\tau, \zeta, \bar{\zeta}) \]  
\[ u = X(\tau, \zeta, \bar{\zeta}), \]  
where \( X(\tau, \zeta, \bar{\zeta}) \) is found by first solving the ‘good cut’ equation \[ N^2 Z = \sigma(Z, \zeta, \bar{\zeta}), \]  
whose solutions are known to depend on four arbitrary complex parameters, \( \varepsilon^a \), i.e. \( Z = Z(\varepsilon^a, \zeta, \bar{\zeta}) \). By choosing an arbitrary worldline in the parameter space (H-space \[19\]), i.e. \( \varepsilon^a = \xi^a(\tau) \), \( X(\tau, \zeta, \bar{\zeta}) \) is determined by \[ u = X(\tau, \zeta, \bar{\zeta}) = Z(\xi^a(\tau), \zeta, \bar{\zeta}). \]  
We thus have that every regular solution to the asymptotically shear-free condition, equation \(21\), is determined by an arbitrary complex worldline in a four-complex dimensional parameter space. The freedom, mentioned earlier, in the choice of \( \tau \) is used later to give a normalization to \( v^a(\tau) \equiv \partial_\tau \xi^a \).

We could reverse the procedure just described and assume that \( u = X(\tau, \zeta, \bar{\zeta}) \) was given rather than \( \sigma(u, \zeta, \bar{\zeta}) \). Then \( \sigma(u, \zeta, \bar{\zeta}) \) and \( L(u, \zeta, \bar{\zeta}) \) could be determined parametrically by \[ u = X(\tau, \zeta, \bar{\zeta}) \]
\[ L(u, \zeta, \bar{\zeta}) = \delta_{\ell \gamma} X(\tau, \zeta, \bar{\zeta}) \]
\[ \sigma(u, \zeta, \bar{\zeta}) = \frac{N}{2} \delta_{\ell \gamma} X(\tau, \zeta, \bar{\zeta}) \]  
with \( \xi^a = (\xi^0(\tau), \xi^1(\tau)) \). The complex parameter \( \tau \) is given by \( \tau = T(u, \zeta, \bar{\zeta}) \), equation \(6\). Note the important point that when \( \tau \) is replaced by \( T \), the spherical harmonic decomposition of those variables becomes non-trivial since it involves products of different spherical harmonics.

In the special case of flat space (with \( \sigma(u, \zeta, \bar{\zeta}) = 0 \)), the asymptotically shear-free congruences become shear-free congruences and the regular solutions to equation \(21\), given parametrically, become
\[ u = X(\tau, \zeta, \bar{\zeta}) = \xi^a(\tau) \hat{l}_a(\zeta, \bar{\zeta}) = \frac{1}{\sqrt{2}} \xi^0(\tau) - \frac{1}{2} \xi^1(\tau) Y^0_1(\zeta, \bar{\zeta}) + \cdots \]
\[ L(u, \zeta, \bar{\zeta}) = \xi^1(\tau) Y^0_1(\zeta, \bar{\zeta}) - 6 \xi^1(\tau) Y^0_{21}(\zeta, \bar{\zeta}) + \cdots \]
\[ \sigma(u, \zeta, \bar{\zeta}) = 24 \xi^{11}(\tau) Y^2_{21} + \cdots \]  
with \( \hat{l}_a(\zeta, \bar{\zeta}) \) as a flat-space null vector on the ‘light cone’ composed of the \( l = 0, 1 \) harmonics. In other words, the regular flat-space shear-free null geodesic congruences are determined by an analytic complex curve in complex Minkowski space. One can reinterpret the Minkowski space curve as a curve in the space of asymptotic complex Poincaré translations. The advantage of this latter interpretation is that it applies just as well to the asymptotically flat spacetimes.
2.5. **Spherical harmonic decomposition**

All the functions on $\mathcal{M}$ that we are dealing with have a spin weight $(s)$ and most have a definite conformal weight $(w)$. They thus can be expanded in the spin-weighted tensor harmonics [20]. Denoting the spin and conformal weights by $(s, w)$ as a subscript, e.g. $W_{(s,w)}$, we have

\[
X \equiv X_{(0,1)} = \frac{1}{\sqrt{2}} \xi^0(\tau) - \frac{1}{4} \xi^1(\tau) Y^0_{li} (\xi, \zeta) + \xi^i(\tau) Y^0_{ij} (\xi, \zeta) + \cdots
\]

\[
\sigma \equiv \sigma_{(2, -2)} = 24 \xi^i(\tau) Y_{2j}^2 + \cdots
\]

\[
L \equiv L_{(1, -1)} = \xi^i(\tau) Y_{1j}^i (\xi, \zeta) - 6 \xi^i(\tau) Y_{2j}^i (\xi, \zeta) + \cdots
\]

\[
\psi_0^0 \equiv \psi_{0(0, -3)} = \psi_{0}^0 Y_{li}^0 + \psi_{2j}^0 Y_{2j}^0 + \cdots
\]

\[
\psi_1^0 \equiv \psi_{1(1, -3)} = \psi_{1i}^0 Y_{li}^1 + \psi_{1ij}^0 Y_{2ij}^1 + \cdots
\]

\[
\psi \equiv \Psi_{(0, -3)} = \psi_{0}^0 + \psi_{i}^i Y_{li}^0 + \psi_{ij}^i Y_{2ij}^0 + \cdots
\]

\[
\phi_0^0 = \phi_{0i}^0 Y_{li}^1 + \phi_{0ij}^0 Y_{2ij}^1 + \cdots
\]

\[
\phi_1^0 = Q + \phi_{0i}^0 Y_{li}^0 + \phi_{1ij}^0 Y_{2ij}^0 + \cdots
\]

\[
\phi_2^0 = \phi_{2i}^0 Y_{li}^{-1} + \phi_{2ij}^0 Y_{2ij}^{-1} + \cdots
\]

The basic idea is to try to give physical meaning or significance to the harmonic coefficients. An important well-known example of this is Bondi’s identification of the $l = (0, 1)$ parts of the mass aspect with the total energy–momentum 4-vector of the interior sources, $(M^c, P^i)$, which is explicitly given by

\[
\Psi = -\frac{2\sqrt{2}G}{c^3} - \frac{6G}{c^3} P^i Y_{li}^0 + \cdots
\]

From the $l = 0$ harmonic coefficient in equation (13), one obtains the Bondi mass loss equation which allows us to identify the coefficient $\xi^i(\tau)$ as proportional to the second derivative of the mass quadrupole.

A few other identifications coming from Maxwell theory are that $Q$ is the Coulomb charge, while $\phi_{0i}^0$ is proportional to the complex electromagnetic dipole moment (electric + $i$ magnetic), $\phi_{2i}^0$ is proportional to the second time derivative of $\phi_{0i}^0$ while $\phi_{2ij}^0$ is proportional to the third derivative of the quadrupole moment.

*Very roughly speaking*, we make the approximate identification of $\psi_{li}^0$ (which is complex) with the complex gravitational dipole moment, i.e. with the mass dipole plus ‘$i$’ times the angular momentum.

3. **Identifications**

Our starting point for the physical identifications is equation (10)

\[
\left( \psi_{li}^0 \right) = -\partial \psi_{2i}^0 + 2\sigma \partial(\sigma)
\]

and the spherical harmonic decomposition. We noted that the $l = 1$ part of $\psi_{2i}^0$ was proportional to the Bondi 3-momentum, $\hat{P}$. The linearized version of equation (46) leads to the fact that the $l = 1$ part of $\left( \psi_{li}^0 \right)$ is also proportional to $\hat{P}$. This in turn suggests that $\psi_{li}^0$ itself should be, at least in the linearized version, proportional to the mass-dipole, $M\hat{X}$, where $\hat{X}$ should
be identified with the position vector of the center of mass so that \( M \vec{X} = \vec{P} \). This is our fundamental observation. It will be analyzed and generalized.

We begin by postulating three different models or methods for the identification. Though we have one method (the third) that we consider to be fundamental and correct, nevertheless we felt that at least a few others should be explored to see if they would be reasonable choices. Our criteria for selecting a model are that (i) it should predict already known laws or reasonable new laws, (ii) it should have a clear geometric foundation and have a logical consistency and (iii) it should agree with special cases, mainly the algebraically special metrics or analogies with the flat-space Maxwell theory.

The first (naive) approach is to simply assume that the Bondi 3-momentum is given by \( \vec{P} = M \vec{X} = M \vec{V} \). The second approach is based on the flat-space transformation (translation) properties of dipoles, i.e. transforming to the center of mass or charge, while the third approach generalizes this to asymptotically flat spacetimes. The first two do not satisfy our three criteria while the third one does. We nevertheless felt that it was worthwhile to see how the increasing sophistication of the methods led to improved results.

**Remark 1.** We have (for notational simplicity) totally abused standard notation. We allow the indices \((i, j, k, l, \ldots)\), which are Euclidean, to be raised and lowered with impunity and allow repeated indices, e.g. \( v_i \xi^i \), to indicate summation.

**Remark 2.** Though \( u \) is the conventional Bondi time, it is more appropriate to use \( w = \sqrt{2}u \), it being the retarded time. Derivatives with respect to \( u \) are denoted by dot (\( \cdot \)), while \( w \) derivatives are given by a prime (\( ' \)), i.e. \( w(\cdot) = \sqrt{2}(\cdot) \).

**Claim.** A very important computational issue is to find the inversion of equation (36), i.e. to find \( \tau = T(u, \xi, \zeta) \). The reason for its importance is that we must be able to explicitly eliminate \( \tau \) and replace it by the Bondi \( u \) (or \( w \)) in the parametric expressions for \( L(u, \xi, \zeta) \), i.e. in the expression for the complex worldline \( \xi^a(\tau) \). The approximate inversion (linear) is given [21] by

\[
\tau = T(u, \xi, \zeta) = w + \sqrt{2} \xi^i(w) Y^0_{ij}(\xi, \zeta) - \sqrt{2} \xi_{ij}(w) Y^0_{ij}(\xi, \zeta) + \cdots .
\]  

(47)

### 3.1. The first identification method

For our first (toy) model, we **assume** that the Bondi momentum (at least at linear order) has the standard kinematic form, namely

\[ \vec{P} = M \vec{V}, \]

(48)

and try to find the consequences. From our perspective this already is unsatisfactory, since we would like this kinematic expression to follow directly from the Einstein equations rather than to assume it. In fact, in the second and third models this relationship is a derived result. Nevertheless, from a heuristic point of view, we believe it worthwhile to show the line of reasoning—in this simpler case—before it is used later for the preferred models. In addition, roughly speaking, from this first model we can see what consequences can be anticipated.

Thus from the above argument, we take \( \psi^0_{ij} \) to have the form

\[ \psi^0_{ij} = \alpha M \lambda_{ij}, \]

(49)

where \( \alpha \) is a constant to be determined so that \( P = MV \), \( M \) is the Bondi mass and \( \lambda_i \) is a *complex position* 3-vector in some unknown space (hopefully) to be determined. The real part of \( \lambda_i \) is tentatively associated with the ‘center of mass position’ and its derivative with the ‘center of mass’ velocity, i.e. \( \dot{\lambda_i} = v_i \Rightarrow \dot{V} \).
Using the harmonic expansions, equations (39), (40) and Clebsch–Gordan products (see appendix A.3) equation (46) becomes
\[ \psi_{2k}^0 = \frac{\sqrt{2}}{2c} \psi_{1k}^0 - \frac{36 \times 64}{5c} \xi_{ij}^{m'} \xi_{ij}^m \epsilon_{ijkl}, \] (50)
while the expansion of the mass aspect, equation (41), is
\[ \psi = \Upsilon + \frac{16 \times 36 \sqrt{2}}{5} \xi_{ij}^i \xi_{ij}^j + \left( \frac{\psi_{0}^2}{5} + \frac{i(32)(36)}{5} \xi_{ijkl}^{ij} \epsilon_{ijkl} \right) Y_{1k}^0 + \cdots. \] (51)

Using equations (51)–(49) we identify, from equation (45), the mass and 3-momentum
\[ M = -\frac{c^2}{2\sqrt{2}G} \Upsilon = -\frac{288 c}{5} \xi_{ij}^{ij} \xi_{ij}^{ij}, \] (52)
\[ p^k = M \lambda_k + i \frac{192 c}{5} \xi_{ij}^{m'} \xi_{ij}^m \epsilon_{ijkl}, \] (53)
\[ \alpha = -\frac{12G}{\sqrt{2}c^2}. \] (54)
Writing the variables as
\[ \Upsilon = \Upsilon_R + i \Upsilon_I \quad \lambda_k = \lambda_{Rk} + i \lambda_{Ik} \quad \xi_{lm} = \xi_{Rlm} + i \xi_{Im} \quad \xi_{lm}^{mr} = \xi_{Rlm}^{mr} + i \xi_{Im}^{mr}, \] the reality conditions, equation (14), with
\[ M_0 = -\frac{c^2}{2\sqrt{2}G} \Upsilon_R \] become
\[ M = M_0 - \frac{36 \cdot 4}{5} \frac{c}{G} \left( \xi_{Rij}^{ij} \xi_{Rij}^{ij} + \xi_{Rij}^{ij} \xi_{Rij}^{ij} \right), \] (52)
\[ \Upsilon_I = \frac{(24)^2 \sqrt{2}}{5c} \left( \xi_{Rij}^{ij} v_{ij}^{ij} - \xi_{Rij}^{ij} v_{ij}^{ij} \right), \] (53)
\[ p^k = M \lambda_{Rk} + \frac{(24)^2 c^2}{5G} \left( \xi_{Rij}^{ij} v_{ij}^{ij} - \xi_{Rij}^{ij} v_{ij}^{ij} \right) \epsilon_{ijkl}, \] (54)
\[ M \lambda_{Ik} = -\frac{192}{5G} c^3 \left( v_{ij}^{ij} \xi_{Rij}^{ij} + v_{ij}^{ij} \xi_{Rij}^{ij} \right) \epsilon_{ijkl}. \] (55)
Finally, equation (11),
\[ \left( \psi_{2}^{0} \right)^* = -\frac{\alpha}{\sigma} \left( \psi_{2}^{0} \right)^*, \] after using the Clebsch–Gordan expansion for \( \sigma \left( \overline{\sigma} \right)^* \), yields, for the \( l = 0, 1 \) terms, the mass and momentum loss expressions,
\[ M' = -\frac{288c}{5G} (v_{ij}^{ij} v_{ij}^{ij} + v_{ij}^{ij} v_{ij}^{ij}) \] (56)
\[ p^{ij} = F^k = \frac{192c^2}{5G} (v_{ij}^{ij} v_{ij}^{ij} - v_{ij}^{ij} v_{ij}^{ij}) \epsilon_{ijkl}. \] (57)
There are now two identifications that can be made immediately from these expressions. From the quadrupole energy loss relations [25]
\[ M' = -\frac{G}{5c} (Q_{\text{Mass}}^{ij} Q_{\text{Mass}}^{ij} + Q_{\text{Spin}}^{ij} Q_{\text{Spin}}^{ij}), \] (58)
we can relate \(\xi_{ij}\) to the mass and spin quadrupole moments by
\[
\xi_{ij} = (\xi_{ij} + i\xi_{ij}) = \frac{G}{12\sqrt{2}c^4} (Q_{\text{Mass}}^{ij} + iQ_{\text{Spin}}^{ij}).
\]
(59)

This identification applies to all three models.

Furthermore, we can substitute the expression for \(-\vec{P}\), i.e. equation (54), into the momentum loss equation, equation (57), obtaining equations of motion for the ‘position’ vector \(\lambda_i\).

\[
M\lambda''_R - \frac{(24)^2c^2}{5G} (\xi_{il}R v_{ij} - \xi_{ij}R v_{il})\epsilon_{ijk} = \frac{192c^2}{5G} (v_{ij}R v_{ij} - v_{ij}R v_{ij})\epsilon_{ijk}.
\]
(60)

This can be interpreted as Newton’s second law, with the second term on the left coming from the gravitational radiation reaction and the right side from the ‘rocket’ or recoil force due to momentum loss.

For the last identification in this model we consider equation (55). From the analogy with the rotating mass solution (with and without charge [7]) and results, which will be described later, we identify
\[
S_k = Mc\lambda_{ik}
\]
as the spin angular momentum so that equation (55) becomes the angular momentum loss equation,
\[
S'_k = -\frac{(24)^2c^3}{5G} (\xi_{ij}R v_{ij} + \xi_{ij}I v_{ij})\epsilon_{ijk}.
\]

Summarizing the results for this physical model, we have assumed that the (complex) \(l = 1\) part of \(\psi^{(1)}_{\lambda k}\) was proportional to the mass dipole moment and the spin angular momentum.

The Bianchi identity equation (46) allowed us to find a kinematic expression for the Bondi linear momentum in terms of the mass dipole moment (or center of mass) while the Bianchi identity, equation (11), yielded the equations of motion. The \(l = 1\) reality condition became the angular momentum loss equation.

Though this model does lead to physical identifications that are quite reasonable, it nevertheless contains several very unsatisfactory features. First of all, there is no hint of a geometric meaning for the ‘position’ vector \(\lambda_k\). What space is it in? It was pulled out of thin air, just so that the momentum could be written as \(\vec{P} = MV\). The constant \(\alpha\) was arbitrary until determined by the kinematic meaning of \(\vec{P}\). A second deficiency is that the angular momentum is given only by the spin without any suggestion of an orbital part. Lastly, this identification is at total odds with the identifications obtained for the algebraically special metrics [21, 26–28] or with the analogous flat space Maxwell identification with the center of charge [6].

We will see that the third model does correct these deficiencies.

3.2. The second identification method

Our second and third identification models are based on complex curves; in the second model these curves are associated with flat-space shear-free null geodesic congruences while in the third model they are associated with the asymptotically shear-free null geodesic congruences of the relevant spacetime. These curves, as we mentioned earlier, are geometrically well defined as complex worldlines in the space of complex Poincaré translations at \(\mathcal{I}^+\). The ideas involved are not obvious and do lie outside of the usual default understandings and thus it takes a bit of ‘getting used to’. Basically, the idea is to generalize the standard action of moving
the origin of coordinates to a new point (the center of mass) so that the mass dipole vanishes for that origin. The generalization consists of making these transformations (translations) complex and moving to the complex center of mass, so that not only does the mass dipole vanish there, but also the angular momentum. This type of procedure, which worked perfectly, in linearized gravity and Maxwell theory [22, 23] must now be applied to asymptotically flat spacetimes.

We begin with the observation that most attempts [2] at defining the angular momentum have the imaginary part of the \( l = 1 \) harmonic of \( \psi_0^1 \) as part of the definition.

**Remark 3.** Most of the angular momentum definitions based on symmetry arguments start with the \( l = 1 \) part of \( \psi_0^1 \) and augment it with terms quadratic in the shear and its derivatives. Later in this section, we will return to the issue of these quadratic terms. See appendix A.4.

The central part of our argument is to use the \( l = 1 \) part of the null rotated Weyl component, \( \psi_0^1 \), from equation (8), i.e.

\[
\psi_0^1 = \psi_0^1 - 3L\psi_0^2 + 3L^2\psi_0^3 - L^3\psi_0^4,
\]

as the transformed (translated) complex dipole moment and then set the \( l = 1 \) part of \( \psi_0^1 \) to zero (partially) determine \( L \). (In flat space, this type of operation really moves the origin of Bondi light cones to a new complex origin in complex Minkowski space so that the center of charge and the magnetic dipole both vanish.) In other words, we will set \( \psi_0^1 |_{l=1} = 0 \) so that the \( l = 1 \) part of \( \psi_0^1 \) is given by

\[
\psi_0^1 |_{l=1} = (3L\psi_2^0 - 3L^2\psi_3^0 + L^3\psi_4^0) |_{l=1} \equiv 3(L\psi_2^0) |_{l=1}.
\]

It is this relationship that had, in the first model, its counterpart in the arbitrary assumption of equation (49). Now it has a justification and a geometric meaning. Also, it does not contain an arbitrary factor \( \alpha \).

Our second and third models depend on two different, but related, choices of \( L(u, \zeta, \bar{\zeta}) \). For the third model, we choose \( L(u, \zeta, \bar{\zeta}) \) to determine an arbitrary asymptotically shear-free null geodesic congruence, via equations (22) and (25), while for the second model we choose \( L(u, \zeta, \bar{\zeta}) \) associated with flat-space shear-free null geodesic congruences, equations (33) and (32). \( L(u, \zeta, \bar{\zeta}) \) in either case is determined by an arbitrary complex worldline \( \xi^a(\tau) \), which contains information about the center of mass worldline and the spin. Though the third model is more consistent and has a more logical basis, for two reasons we first discuss model 2. It is easier to work with and the final results are very similar to those of the third model.

Using the flat-space \( L(u, \zeta, \bar{\zeta}) \), working with terms up to the second order, with the frequent use of Clebsch–Gordan expansions, we find, from equation (62), that

\[
\psi_{i1}^0 = 3\sqrt{2}\left[\xi^i + i\epsilon_{ijkl}\xi^k\xi^l\right] + i\frac{3\sqrt{2}}{2}\epsilon_{ijl}\psi_{jl}^0\xi^i = -\frac{18}{5}\psi_{2j}^0\xi^j.
\]

There are two things to immediately note: (1) since \( \Upsilon \) is proportional to the mass, the first term has exactly the same form as in the first model with \( \xi^i \) now replacing \( \lambda^i \) as the ‘complex center of mass’ and (2) \( \psi_{i1}^0 \) is fully determined without the constant \( \alpha \). In addition, new terms have appeared, some of which will have immediate physical meaning. From the Bianchi identity, equation (10), using equation (63) and the reality conditions, we have for the \( l = 0 \) component of \( \psi_{2}^0 \)

\[
M = M_0 - \frac{288c}{5G}\left(\frac{e^{j\ell}e^{j\ell'}}{s^R_s^R e^{j\ell'} + s^R_s^\ell e^{j\ell'}}\right)
\]

\[
M_0 = -\frac{c^2}{2\sqrt{2}G}\Upsilon_R(w)
\]
\[ \Upsilon_I = \frac{24(24)\sqrt{2}}{5c} \left( \xi_R^{ij} \xi_R^{ji} - \xi_I^{ij} \xi_R^{ji} \right), \]

which are identical to those of the first model. However for the \( l = 1 \) components, new terms appear.

For the momentum \( \vec{P} \), we have

\[ P^k = M_0v^k_R + \frac{M_0}{c} \left( v^j_R v^l_I - \xi_I^{jli} \right) \epsilon_{i+j+l}, \quad (64) \]

\[ R^k = -\frac{3(24)c^2}{2G} \left( \xi_R^{ik} \xi_R^{ji} + \xi_I^{jli} \xi_R^{ji} \right) - \frac{24c^2}{5G} \left( \xi_I^{jli} \xi_R^{ji} - \xi_R^{ik} \xi_I^{jli} \right) \epsilon_{i+j+l}. \quad (65) \]

The \( l = 1 \) part of the reality condition yields an expression that consists of two types of terms, total time derivatives and the others. We define it as the conservation of angular momentum equation:

\[ J^k = -\frac{(24)c^3}{5G} \left( \xi_R^{ik} \xi_R^{ji} + \xi_I^{jli} \xi_R^{ji} \right) \epsilon_{i+j+l}, \quad (66) \]

\[ J^k = M_0 \xi_I^k + M_0 \left( \xi_R^{ik} v^j_I - \xi_I^{jli} \right) \epsilon_{i+j+l} = -\frac{3(24)c^3}{5\sqrt{2}G} \left( \xi_I^{jli} \xi_R^{ji} - \xi_R^{ik} \xi_I^{jli} \right). \quad (67) \]

The angular momentum flux is the same as in the first model. However, the angular momentum itself has much more physical content. The first term

\[ S^k = M_0 \xi_I^k \quad (68) \]

is again identified as the intrinsic spin, while the second term is precisely the orbital angular momentum, \( \mathbf{r} \times \mathbf{p} \), while the term \( -M_0 \xi_R^{ik} v^j_I \epsilon_{i+j+l} \) is the contribution from spin-precession. The last term is new, involving dipole and quadrupole coupling. This could be considered as a prediction of the theory though it is probably untestable. With these identifications, going back to \( P^k \) in equation (64), we observe that the second term

\[ M_0 \xi_R^{ik} v^j_I \epsilon_{i+j+l} = c^{-2} S^I v^j_R \epsilon_{i+j+l} = c^{-2} (S^I v)_k \]

is the Mathisson–Papapetrou contribution to the linear momentum.

**Remark 4.** Many of the kinematic expressions for the physical quantities have been given up to the second order. When the dynamic equations are used, it often turns out that these quantities are really of a higher order and should be neglected for second-order considerations. We felt that for purposes of understanding their kinematic role, their inclusion here was important.

The equations of motion for the position vector, \( \xi_R^k \), are obtained by substituting \( \vec{P} \) from equation (64) into the momentum loss equation

\[ p^{kl} = F^k = \frac{192c^2}{5G} \left( v^j_I v^l_I - v^l_I v^j_I \right) \epsilon_{i+j+l}, \]

yielding

\[ M_0 \epsilon^{kl} - \frac{M_0}{c} \left( \xi_R^{ik} v^j_I + \xi_I^{jli} \right) \epsilon_{i+j+l} + R^k = \frac{384c^2}{5G} v^j_I v^l_I \epsilon_{i+j+l}. \]

This again is Newton’s second law but with spin-coupling forces, radiation reaction and momentum recoil.
Though this second model is far superior to the first model, it still has certain deficiencies. It does not yield the correct dynamics for the algebraically special metrics. In other words, the dynamical equations would differ from those of the algebraically special metrics. Also, there is a lack of logical consistency in this treatment. We have used $L(u, \zeta, \bar{\zeta})$ for the flat-space shear-free null geodesic congruence. We should have used $L(u, \zeta, \bar{\zeta})$ associated with the asymptotically shear-free congruence, i.e. $L(u, \zeta, \bar{\zeta})$ from equation (22). We would then have the null rotated Weyl component, $\psi_0^{ij}$, based on an asymptotically shear-free null congruence so that the quadratic terms in the shear, which would normally appear in the definition of angular momentum, would be now absent (see remark 3), and our procedure of setting $\psi_0^{ij} = 0$ would be consistent with all definitions of the center of mass and angular momentum at the complex ‘origin’.

3.3. The third identification method or physical model

The third method is, as we mentioned earlier, basically the same as the second method. So rather than repeat with the vacuum ‘asymptotically shear-free’, $L(u, \zeta, \bar{\zeta})$, we will give the results for the asymptotically flat Einstein–Maxwell equations. Since the method and arguments are so similar to those of method 2 and the detailed calculations are so long, we will simply summarize the results.

The physical identifications are substantially changed by the presence of the Maxwell field. For the vacuum case, our results apply to the general asymptotically flat situation. For asymptotically flat GR, with the Maxwell field, the situation is different. It turns out that the Weyl tensor condition (62) has a counterpart for the Maxwell tensor. In other words, the $l = 1$ part of a null rotated $\phi_0^0$, i.e. $\phi_0^0$ when set to zero, determines a different complex worldline (the complex center of charge) for which the electric and magnetic dipole moments vanish. We consider only the special case where the gravitational and electromagnetic worldlines coincide. In some sense, this coincidence of worldlines implies that the source has a restricted structure and is relatively ‘simple’. We take this as the meaning of a gravitational elementary particle.

We start with a very general cut function and its inverse that came from a given Bondi shear, i.e. from equations (25) and (22):

$$u \equiv \frac{w}{\sqrt{2}} = X = \frac{1}{\sqrt{2}}\xi^0(\tau) - \frac{1}{2}\xi^i(\tau)Y^0_{ij}(\zeta, \bar{\zeta}) + \xi^{ij}(\tau)Y^0_{2ij}(\zeta, \bar{\zeta}) + \cdots ,$$  

$$\tau = T(u, \zeta, \bar{\zeta}) = w + \sqrt{2}\xi^i(w)Y^0_{ij}(\zeta, \bar{\zeta}) - \sqrt{2}\xi^{ij}(w)Y^0_{2ij}(\zeta, \bar{\zeta}) + \cdots$$  

$$L = \xi^i(\tau)Y^0_{ij}(\zeta, \bar{\zeta}) - 6\xi^{ij}(\tau)Y^0_{2ij}(\zeta, \bar{\zeta})$$  

$$\xi^a = (\xi^0, \xi^i(\tau) = (\tau, \xi^i(\tau) = \xi^a_R(w) + iv^a_I(w))$$  

$$\xi'^a = v^a(w) = v^a_R(w) + iv^a_I(w).$$

After eliminating $\tau$, equation (62) yields the expression for $\psi_{ij}^0$:

$$\psi_{ij}^0 = 3 \Gamma \left[ \xi^i(w) + i\frac{3}{2}v^k\xi^j + N^i \right] + \frac{3\sqrt{2}}{2}\epsilon_{lij}\psi_{0,j}^0 \xi^l - \frac{18}{5}\psi_{2,j}^0 \xi^j$$

$$= \frac{6\sqrt{2}}{5}v^k\xi^j - \frac{18\sqrt{2}}{5}v^k\xi^j + \frac{144}{5}\epsilon_{mnj}v^j\xi^m.$$

14
\[ \phi_0^i = \sqrt{2} Q \left[ v^i(w) + \frac{1}{2} \epsilon_{ijkl} v^j x^k + N^i \right] + i \sqrt{2} Q \epsilon_{kij} v^j x^k - \frac{2}{5} (\phi_{0k}^i \xi^k) \]
\[ - \frac{72}{5} Q v^i v^j + \frac{24 \sqrt{3}}{15} \epsilon_{ijkl} \phi_{0lj}^i \xi^k - \frac{24 \sqrt{3}}{3} \epsilon_{ijkl} (\phi_{0ik}^j \xi^m) \]
\[ \phi_0^i = -2 Q \left[ v^i + \frac{1}{2} \epsilon_{ijkl} v^j x^k y^l + N^i \right] - i 2 Q \epsilon_{kij} (v^j x^k) y^l + \frac{2 \sqrt{2}}{5} (\phi_{0k}^i \xi^k) \]
\[ + \frac{72 \sqrt{3}}{5} Q (v^i v^j)' - \frac{48}{15} i \epsilon_{ijkl} (\phi_{0lj}^i \xi^k) + i \frac{48}{5} \epsilon_{ijkl} (\phi_{0ik}^j \xi^m) \].

From the Bianchi identity, equation (15), we obtain
\[ P^k = M v^k + M_0 \frac{v^k}{c} (v_E^k v_E^l - \xi_E^k v_R^l - \xi_R^k v_E^l) \epsilon_{ijk} - \frac{2}{3c^2} v^k R^k + \frac{2}{3c^2} \left[ \xi_E^k v_E^l - \xi_R^k v_E^l + \xi_R^k v_R^l \right] \epsilon_{ijk} + \Pi^k \]
\[ \Pi^k = - \frac{M}{c^2} \left( \frac{6 \sqrt{3}}{5} \left[ 8 (\xi_E^k v_E^l - \xi_R^k v_R^l) + 3 (v_E^k v_R^l - v_R^k v_E^l) \right] + \frac{144}{5} (v^k v^l - v^l v^k) \epsilon_{ijk} \right) \]
\[ + \frac{Q^2}{3c^4} \left( \frac{12}{5} (v_R^k v_E^l - v_R^l v_E^k) \epsilon_{ijk} + \frac{3}{5} \left( v_R^k v_E^l - v_R^l v_E^k \right) \epsilon_{ijk} \right) \]
\[ - \frac{12 \sqrt{3}}{5} (\xi_E^k v_E^l - \xi_R^k v_R^l) \epsilon_{ijk} + \frac{24}{5 \sqrt{2}} \left( D_E^i \xi_E^l + D_M^i \xi_R^l \right) \epsilon_{ijk} \]
\[ - \frac{1}{5} \left( v^k v_E^l - v^l v_E^k \right) + \frac{1}{5} \left( v^k v_R^l - v^l v_R^k \right) \epsilon_{ijk} \]
\[ + \frac{4}{45 \sqrt{2}} \left( D_E^i v_E^l - D_E^l v_E^i \right) \epsilon_{ijk} = \frac{36 \sqrt{3}}{5G} (\xi_E^k v_E^l + \xi_R^k v_R^l) \epsilon_{ijk} \]
\[ + \frac{2 (42) c^2}{5G} (v^k v_E^l + \xi_R^k v_R^l) \epsilon_{ijk} \].

\( D_E^i \) and \( D_M^i \) are respectively the electric quadrupole and magnetic quadrupole moments found from the \( l = 2 \) radiation term in the solution of the Maxwell equations. All nonlinear terms involving the quadrupole terms are gathered into \( \Pi^k \).

The vanishing of the imaginary part of the reality condition yields the relations
\[ J^k = \frac{2Q^2}{3c^4} (v_E^k v_E^l + v_R^k v_R^l) \epsilon_{ijk} + \frac{1}{90c^3} (D_E^i v_E^l + D_M^i v_M^l) \epsilon_{ijk} \]
\[ - \frac{(24)^2 c^3}{5G} (\xi_E^k v_E^l + \xi_R^k v_R^l) \epsilon_{ijk}, \]
\[ \text{where } J^k, \text{ identified (from the dynamics rather than through the conventional symmetry argument) as the total angular momentum, is given by} \]
\[ J^k = Mc v_R^k + M (\xi_E^k v_E^l - \xi_R^k v_R^l) \epsilon_{ijk} + \frac{2Q^2}{3c^4} v^l - \frac{2Q^2}{3c^4} (\xi_R^k v_R^l + 2 \xi_R^l v_R^k) \epsilon_{ijk} + K^k \]
\[ K^k = -M \left( \frac{6\sqrt{2}}{5} \right) \left[ 8(\xi_R^k v_j^i + \xi_j^k v_R^i) + 3(v_R^k \xi_j^i + v_j^i \xi_R^k) \right] - \frac{144}{5} \left( v_R^{iij} - v_j^{iij} \right) \varepsilon_{ijk} \]
\[ - \frac{Q^2}{3c^3} \left( -18(6)\frac{\sqrt{2}}{5} (v_R^{iik} - v_j^{iik}) + \frac{96\sqrt{2}}{5} (v_R^{ij} v_j^i + v_i^i v_R^j) \right) \]
\[ - \frac{12\sqrt{2}}{5} (\xi_R^i v_j^i + \xi_j^i v_R^i) + \frac{(36)\sqrt{2}}{5} (v_R^{iij} + v_j^{iij} \xi_R^i) \]
\[ - \frac{288}{5} (v_R^{iij} v_j^i - v_j^{iij} \xi_R^i) + \frac{Q}{15c^3} (\xi_i^j D_M^{jim} + \xi_i^j D_E^{jim} + 2v_j^i D_E^{jim} + 3v_R^k D_M^{jim} - 3v_j^i D_M^{jim} \varepsilon_{ijk}) \]
\[ + 4\sqrt{2}(2\xi_R^i D_E^{jim} - 2\xi_i^j D_M^{jim} + 3v_R^k D_M^{jim} - 3v_j^i D_M^{jim}) \varepsilon_{ijk} \]
\[ - \frac{36\sqrt{2}c^3}{5G} (\xi_i^j - \xi_j^i \xi_R^i). \]  

The justification for calling this the angular momentum is the same as in the previous model, coming from the sum of terms interpretable as the spin, orbital and precessional moments.

In the absence of a Maxwell field, \( J^i \) is simply proportional to the imaginary part of \( \psi_{1i}^0 \).
\[ J^i = -\frac{\sqrt{2}c^3}{12G} \varepsilon_{1i} \psi_{1i}^0. \]  

**Remark 5.** In the published literature [4], there are ambiguities in the definition of angular momentum. For us, because of our approximations, these ambiguities disappear, i.e. the ambiguous terms are of a higher order. See appendix A.4.

Our last results arise from the mass/energy loss equation and the momentum loss equation, the latter being the dynamic equations for \( \xi_R^i \). By substituting \( P^k \) of equation (77) into the Bianchi identity, equation (16), we find
\[ M' = -\frac{G}{5c^3} (Q_{\text{Mass}}^{jim} Q_{\text{Mass}}^{jim} + Q_{\text{Spin}}^{jim} Q_{\text{Spin}}^{jim}) - \frac{2Q^2}{3c^3} (v_R^k v_j^i + v_j^i v_R^k) \]
\[ - \frac{1}{180c^3} (D_M^{jim} D_M^{jim} + D_E^{jim} D_E^{jim}) \]  

\[ P_k^i = F_k = \frac{2G}{15c^6} \left( Q_{\text{Spin}}^{jim} Q_{\text{Mass}}^{jim} - Q_{\text{Mass}}^{jim} Q_{\text{Spin}}^{jim} \right) \varepsilon_{ilk} - \frac{Q^2}{3c^4} (v_R^k v_j^i - v_j^i v_R^k) \varepsilon_{ilk} \]
\[ + \frac{Q}{15c^3} (v_R^{iij} D_E^{jim} + v_j^{iij} D_M^{jim}) + \frac{1}{540c^6} (D_E^{jim} D_M^{jim} - D_M^{jim} D_E^{jim}) \varepsilon_{ilk} \]  

where we have now replaced the \( \xi_i^j \) by the more physical variables, the quadrupoles, \( (Q_{\text{Mass}}, Q_{\text{Spin}}) \) via
\[ \xi_i^j = (\xi_R^i + i\xi_E^i) = \frac{G}{12\sqrt{2}c^4} (Q_{\text{Mass}}^{jim} + iQ_{\text{Spin}}^{jim}). \]

Writing out the equations of motion in detail is long and not completely enlightening. There are many nonlinear terms whose meanings, other than that they are interpretable as a gravitational radiation reaction, are not clear. Instead we will write out a truncated version of equation (83), hiding these terms in a single symbol, \( R^k \), so that we have
\[ M v_R^k + v_k^i M' - \frac{2Q^2}{3c^3} v_R^{iij} + R^k = F^k. \]  

There are several things to note here. First of all, strictly speaking, we should ignore the term \( v_R^k M' \) since it is of third order because \( M' \) already is of second order from equation (56).
We keep it, understanding its suspect nature, because it is potentially so important. For the moment ignore $R^k$ and $F^k$ and consider only the second term in the mass loss, i.e. the electric dipole energy loss, so that equation (85) becomes

$$M v_R^k \frac{2 Q^2}{3 c^3} v^l R_{kl} - 2 Q^2 \frac{3 c}{2} v_R^{k''} = 0.$$  

(86)

These [24, 25] are the classical equations of motion for a charged particle that contains the well-known electromagnetic radiation reaction force and exhibit the unstable run-away behavior. (The cubic term, though not sufficiently large to stabilize the equation, has the correct sign for stabilization.) This classical equation has been obtained without the usual model building or mass renormalization. Returning to equation (85), we see that in addition to $R^k$ and $F^k$, there are the extra terms in $M'$ coming from gravitational and electromagnetic quadrupole radiation. It is very hard to see the consequences of these terms though one can see that they are on the side of stabilization. It is then easy to conjecture that coupling electrodynamics with general relativity stabilizes the equations of motion.

There are other comments concerning our results that should be made.

• The mass/energy loss equation contains both the gravitational radiation quadrupole expression (this has been adjusted by definition) and the classical electromagnetic dipole and quadrupole radiation expressions that come straight from the construction. This allows us to identify

$$\mu^k = Q \xi^k$$  

(87)

as the magnetic dipole moment, so that with the spin definition, $S^k = M c \xi^k$, we obtain the Dirac value of the gyromagnetic ratio, i.e.

$$g = 2.$$  

(88)

• The angular momentum expression, equation (80), contains the spin, the orbital angular momentum and a precession contribution. There is now a prediction that there is a charge-spin contribution to the total $J$, i.e. $\frac{2 Q^2}{4 M c} S^k$, as well as higher order corrections.

• Probably the strongest argument for the validity of this approach to extracting physical information from the asymptotic field is the observation that in the angular momentum loss equation, equation (79), the dipole contribution to the angular momentum flux coming just from the electromagnetic field, i.e.

$$\frac{2 Q^2}{3 c^3} (v^l R_{kl} + v^l R_{kl}) \epsilon_{ijk},$$  

(89)

exactly coincides with the classical electrodynamic angular momentum flux [25]. This result was obtained with no a priori expectations.

4. BMS invariance

Our results concerning the identifications of physical quantities from the asymptotic fields were all obtained in an arbitrary but specific Bondi coordinate/tetrad system. The question is: what are the relations between the same quantities but calculated in a different Bondi coordinate/tetrad system? In other words, we want to know the transformation properties of our physical variables under the action of the BMS group [8, 9, 14–16]. As we pointed out in section 2, the BMS group is composed of two parts, the supertranslations and the Lorentz transformations, given respectively by

$$\tilde{u} = u + \alpha (\zeta, \bar{\zeta})$$

$$\tilde{\zeta}, \tilde{\bar{\zeta}} = (\zeta, \bar{\zeta}).$$  

(90)
with \( \alpha(\zeta, \bar{\zeta}) \) as an arbitrary smooth function on the sphere now considered to be small, i.e. as a first-order quantity, and
\[
\hat{\mathbf{u}} = K \mathbf{u}
\]
\[
K = \frac{1 + \zeta \bar{\zeta}}{(a\zeta + b)(\bar{a}\zeta + \bar{b}) + (c\zeta + d)(\bar{c}\zeta + \bar{d})}
\]
\[
\hat{\zeta} = \frac{a\zeta + b}{c\zeta + d}; \quad ad - bc = 1.
\]
with \((a, b, c, d)\) as the complex parameters of \(SL(2,C)\). The invariance under supertranslations is actually subtle and requires a bit of thought while the invariance under the Lorentz group, though straightforward, requires some more technical background. Since all our calculations were done under the assumption of second-order perturbations of a Reissner–Nordstrom background, we must keep the BMS transformations small, i.e. close to the identity.

4.1. The supertranslations

We first treat the supertranslation invariance.

Starting on \( \mathcal{I}^+ \) with some arbitrary but given Bondi coordinates \((u, \zeta, \bar{\zeta})\) and tetrad \((l, n, m, \bar{m})\), we saw that there was a null rotation, equation (7),
\[
l^* = l + L\bar{m} + Lm + L\bar{L}n
\]
\[
m^* = m + Ln
\]
\[
n^* = n
\]
that was determined by the choice of the angle field \( L(u, \zeta, \bar{\zeta}) \). \( L(u, \zeta, \bar{\zeta}) \) was then determined first by the asymptotic shear-free condition, leaving the freedom in the \( l = (0, 1) \) harmonic coefficients, and then by the requirement that the \( l = 1 \) harmonic component of the ‘rotated’ Weyl component, \( \psi_0^{0*} \), i.e. \( \psi_1^{0*}\big|_{l=1} = 0 \), should vanish. The important point to note is that the Weyl tensor component \( \psi_1^{0*} \) is determined by the new tetrad, \((l^*, n^*, m^*, \bar{m}^*)\). It is a geometric structure given independent of the choice of coordinates, depending on the fixed known (*)-tetrad. If we have a second Bondi system with Bondi tetrad \((\hat{l}, \hat{n}, \hat{m}, \bar{\hat{m}})\), it will have been obtained by a different null rotation via some other angle field, \( \hat{L} \), from the \((l, n, m, \bar{m})\) tetrad. This simply means that there is a different null rotation, now going from the \( (\hat{\cdot}) \)-tetrad to the previously determined \((\cdot)\)-tetrad. \( \psi_1^{0*}\big|_{l=1} \) thus remains zero when this harmonic is extracted from \( \psi_1^{0*} \) when ‘\( u \)’ is held constant. There however is a serious issue that must be raised. When we extracted the \( l = 1 \) part of \( \psi_1^{0*} \), it was done at a constant value of the Bondi ‘\( u \)’. But now, after the supertranslation, the harmonic decomposition should be done at constant ‘\( \hat{u} \)’. However, it turns out that, because of the first-order BMS supertranslation, \( \hat{u} = u + \alpha(\zeta, \bar{\zeta}) \); the \( \psi_1^{0*}\big|_{l=1} \) part of \( \psi_1^{0*} \), now obtained by holding ‘\( \hat{u} \)’ constant, remains zero up to the second order. So up to our accuracy, our results are supertranslation invariant. The more difficult issue of how to deal with finite supertranslations will be discussed in the conclusion.

4.2. Lorentz transformations

To deal with the Lorentz subgroup of the BMS group requires a review of the representation theory.

The theory of the representations of the Lorentz group was beautifully described by Gelfand et al [13] using homogeneous functions of two complex variables as the representation space. We will summarize these ideas using an equivalent method [14], namely by using spin-weighted functions of the sphere as the representation spaces. Representations are labeled,
in the notation of Gelfand, Graev and Vilenkin, by two numbers \((n_1, n_2)\) or by \((s, w)\), with \((n_1, n_2) = (w - s + 1, w + s + 1)\). 's' is referred to as the spin weight and 'w' as the conformal weight [29] (sometimes called 'boost weight'). The representations are referred to as \(D_{(n_1, n_2)}\). The special case of irreducible unitary representations, which occur when \((n_1, n_2)\) are not integers, are not of interest to us and will not be discussed. We will consider only the case when \((n_1, n_2)\) are integers so that \((s, w)\) can then take on integer or half-integer values. If \(n_1\) and \(n_2\) are both positive integers or both negative integers, we have, respectively, the positive or negative integer representations. The representation space, for each \((s, w)\), is given in [14].

Rather than giving the full description of these invariant subspaces, which is available elsewhere, we will confine ourselves to few cases which are relevant to us.

1. For \(s = 0\) and \(w = 1\), the harmonic, \(l = (0, 1)\), form the invariant subspace. Applying this to the cut function, \(X(\tau, \xi, \bar{\xi})\), we obtain from

\[
X = \frac{1}{\sqrt{2}} \xi^0(\tau) - \frac{1}{2} \xi^i(\tau) Y^0_{1i}(\xi, \bar{\xi}) + \xi^{ij}(\tau) Y^0_{2ij}(\xi, \bar{\xi}) + \cdots
\]

invariant subspace

\[
= \frac{1}{\sqrt{2}} \xi^0(\tau) - \frac{1}{2} \xi^i(\tau) Y^0_{1i}(\xi, \bar{\xi})
\]

(96) (97)

\(\xi^a(\tau) = (\xi^0(\tau), \xi^i(\tau)) = \) Lorentz vector.

(98)

This allows us to single out, in a Lorentz-invariant manner, the four \(l = (0, 1)\) harmonic coefficients of the cut function \(X(\tau, \xi, \bar{\xi})\) as a complex position vector.

2. The mass aspect

\[
|\Psi| = \Psi_{(0,-3)} \equiv |\Psi^0 + \Psi^i Y^0_{1i} + \Psi^{ij} Y^0_{2ij} + \cdots|
\]

(99)
is an $s = 0$ and $w = -3$, $((n_1, n_2) = (-2, -2))$, quantity. The factor space is isomorphic to the finite dimensional positive integer space $((n_1, n_2) = (2, 2))$ and hence the harmonics of $l = (0, 1)$ lie in the invariant subspace. From the isomorphism (which does change the numerical coefficients), we can construct functions of the form equation (97), which, in turn, lead to

$$P^a = (Mc, P^i) = \text{Lorentz vector}.$$  

This gives the justification for calling the $l = (0, 1)$ harmonics of the mass aspect a Lorentzian 4-vector, $P^a$. Technically, the Bondi 4-momentum is a co-vector but we have allowed ourselves a slight notational irregularity.

(3) The Weyl tensor component, $\psi^0_{i1}$, has $s = 1$ and $w = -3$, $((n_1, n_2) = (-3, -1))$. The associated finite dimensional factor space is isomorphic to the finite part of the $s = -1, w = 1, ((n_1, n_2) = (3, 1))$ representation. We have that

$$\psi^0_{i1} \equiv \psi^0_{i1(1,-3)} = \psi^0_{i1} y^1_{i1} + \psi^0_{ij} y^1_{2ij} + \cdots$$  \hspace{1cm} (100)$$

leads to the invariant subspace

$$\text{Invariant subspace} \equiv \psi^0_{i1} y^{-1}_{i1}.$$  

The question of what finite tensor transformation does this correspond to is slightly more complicated than that of the previous examples of Lorentzian vectors. In fact, it corresponds to the Lorentz transformations applied to (complex) self-dual antisymmetric two-index tensors. As an example from Maxwell theory, from a given $E$ and $B$, the Maxwell tensor, $F^{ab}$, and then its self-dual version

$$W^{ab+} = F^{ab} + i F^{*ab}$$

can be constructed. A Lorentz transformation applied to $W^{ab+}$ is equivalent [25] (see appendix A.2) to the same transformation applied to

$$\psi^0_{i1} = (E + i B)i.$$  \hspace{1cm} (101)$$

These observations allow us to assign an invariant physical meaning to our identifications of the position vector, $\xi^a$, the Bondi momentum, $P^a$, and the angular momentum, $J^i$.

5. Discussion

Starting from a very unorthodox point of view, we have tried to describe in the context of GR (either in the vacuum case or for the Einstein–Maxwell equations), the equations of motion of an isolated charged, massive body (our gravitational elementary particle) that possesses both an intrinsic spin and quadrupole moments and can radiate both gravitational and electromagnetic radiations. The point of view arises by considering a generalization of the algebraically special metrics where one can identify physical quantities and determine their evolution from the asymptotic field equations.

For the algebraically special metrics, the shear-free null geodesic congruence that is associated with these metrics automatically assigned kinematic variables (e.g., a position vector and an intrinsic spin) to the Bondi energy–momentum 4-vector. The Bondi evolution equations then became the equations of motion. Our generalization consisted in observing that the existence of a shear-free null geodesic congruence could be generalized to the existence of an asymptotically shear-free null geodesic congruence. Applying the same physical identifications as with the shear-free congruences leads in exactly the same manner to the more general equations of motion.
To test out if this result was accidental and, perhaps, other physical assignments would lead to similar equations of motion, we tried two alternate strategies. They both were found to be lacking. They were not as natural or as physically meaningful as the third identification method.

Since we are far from any of the standard or default approaches to the description of motion in GR, it would be appropriate to summarize our results.

Looking at the asymptotic Einstein–Maxwell equations, the asymptotic shear-free conditions lead, in general, to two different complex worldlines in the space of complex Poincaré translations acting on \( \mathcal{I}^+ \), one from the Weyl tensor and the other from the Maxwell tensor. We have considered only the special case where the two worldlines coincide, defining this case as ‘elementary particles in GR’.

Some results among others are as follows.

- The mass has a kinematic correction term dependent on the variable quadrupole moment. This could perhaps be considered as a prediction.
- The Bondi linear 3-momentum is expressed in kinematic variables, e.g. \( M v^k \), \( \frac{2Q}{\nu^2} v^\nu \) and the Mathisson–Papapetrou spin coupling, among others.
- The imaginary part of the complex position vector is identified with the specific intrinsic spin angular momentum. From the solutions to Maxwell’s equations, the magnetic moment is seen to be the charge, \( q \), times the imaginary part of the position vector. This agrees with the algebraically special charged spinning metric and leads us to the Dirac value of the gyromagnetic ratio, i.e. \( g = 2 \). Though earlier we have defined a ‘gravitational elementary particle’ from this result, it should be noted that in the elementary particle community [30], it has been speculated that all charged elementary particles with spin have this property. One then has the question: what, if any, is the relationship between the two types of particle?
- One of the strongest arguments for our interpretations comes from the \( l = 1 \) part of the reality conditions, which is interpreted as the dynamics (the conservation law) for the total angular momentum. There is a total time derivative term of a quantity, \( J^k \), that we define as the total angular momentum. It contains the spin, the orbital angular momentum and a precession term. The angular momentum flux contains three terms which come, respectively, from the gravitational quadrupole radiation, the electromagnetic quadrupole radiation and a term arising from the electromagnetic (electric and magnetic) dipole radiation. This latter term is identical to that calculated purely from electromagnetic theory [25].
- From the Bondi mass loss equation, we can identify from the flux terms the gravitational quadrupole but also see that our identification of the electromagnetic dipole moments agrees with the predicted dipole energy loss.
- From the Bondi momentum loss, we obtain the equations of motion. In a sense, we ‘derive’ Newton’s second law, \( F = M v \), where the force is a combination of the electromagnetic radiation reaction [24, 25], gravitational radiation reaction and a ‘rocket’ recoil force from the electromagnetic and gravitational momentum loss.
- Finally, we showed that each of the quantities that were identified as physical variables transformed appropriately under the Lorentz group, i.e. as Lorentzian tensorial objects.
- Our results follow from the existence of a well-defined geometric structure, namely the UCF, a unique one-complex parameter family of slices of null infinity. This suggests that the higher order coefficients in the harmonic expansion of the UCF, e.g. \( \xi^{ij}, \xi^{ijk}, \ldots \), should be identified with the time derivatives of the higher multipole moments.
There are other unfamiliar terms that could be thought of as predictions of this theoretical construct. How to possibly measure them is not at all clear.

- One interesting physical prediction concerns the contribution that the charge makes to the total angular momentum. Looking at the equation defining $J^k$, equation (80), we see the (linear) contribution from

$$\frac{2Q^2}{3c^2} v^k = \frac{2Q}{3c^2} D^k_M,$$

which is a coupling between the charge and the changing magnetic dipole moment.

The final item to be discussed concerns the invariance under the supertranslation subgroup of the BMS group.

An obvious question concerning the material described here concerns the issue of the extraction of the $l = (0, 1)$ harmonics from different Weyl tensor components or from the universal cut function, $X$. We have consistently performed the extraction of the harmonic components on the cuts, $u = \text{const.}$, or on neighboring cuts, $\hat{u} = \text{const.} + \Delta$, that are close to the first set of cuts. To second order, the results are unchanged. If we do go to arbitrary cuts, there is no reason for the extraction to lead to the same results. Our results thus appear to depend on the choice of cuts. In fact there is a canonical choice of cuts, i.e. a special one-parameter family of cuts, labeled by ‘$s$’, on which the extraction should always be performed and for which there is no ambiguity. In the text, the $u = \text{const.}$ cuts were sufficiently close to the canonical choice, so that, to second order, they were the same. In fact, in principle, we should have been doing all our calculations on the $s = \text{const.}$ cuts. It was, however, easier doing it with $u = \text{const.}$

The question then is: what are these canonical cuts? Returning to the complex universal cut function

$$u = X(\tau, \zeta, \bar{\zeta})$$

we earlier saw that $\tau$ had to be chosen so that $u$ had real values. If we write

$$\tau = s + i\Lambda(s, \zeta, \bar{\zeta}), \quad (102)$$

then one can show (see appendix A.1) that $\Lambda(s, \zeta, \bar{\zeta})$ can be chosen so that

$$u = X(s + i\Lambda(s, \zeta, \bar{\zeta}), \zeta, \bar{\zeta}) = \hat{X}(s, \zeta, \bar{\zeta}) \quad (103)$$

is a real function of the real variable ‘$s$’. This is the construction of our canonical slicing.

**Acknowledgments**

GSO acknowledges the financial support from CONACYT and Sistema Nacional de Investigadores (SNI-México). CK thanks CONICET and SECYTUNC for support.

**Appendix A**

**A.1. The canonical slicing**

To construct the canonical slicing, we begin with the complex UCF

$$u = X(\tau, \zeta, \bar{\zeta})$$

$$= \frac{i}{\sqrt{2}} \xi^0(\tau) - \frac{i}{2} \xi^i(\tau) Y^0_{li}(\zeta, \bar{\zeta}) + \xi^{ij}(\tau) Y^0_{2ij}(\zeta, \bar{\zeta}) + \cdots \quad (A.1)$$

and write

$$\tau = s + i\lambda \quad (A.2)$$
with $s$ and $\lambda$ as real. The cut function can then be rewritten as

$$u = X(\tau, \zeta, \bar{\zeta}) = X(s + i\lambda, \zeta, \bar{\zeta}) = \chi_R(s + i\lambda, \zeta, \bar{\zeta}) + i\chi_I(s + i\lambda, \zeta, \bar{\zeta}),$$

(A.3)

with real $\chi_R(s, \lambda, \zeta, \bar{\zeta})$ and $\chi_I(s, \lambda, \zeta, \bar{\zeta})$. $\chi_R(s, \lambda, \zeta, \bar{\zeta})$ and $\chi_I(s, \lambda, \zeta, \bar{\zeta})$ are easily calculated from $X(\tau, \zeta, \bar{\zeta})$ by

$$\chi_R(s, \lambda, \zeta, \bar{\zeta}) = \frac{1}{2}[X(s + i\lambda, \zeta, \bar{\zeta}) + X(s + i\lambda, \zeta, \bar{\zeta})],$$

$$\chi_I(s, \lambda, \zeta, \bar{\zeta}) = \frac{1}{2}[X(s + i\lambda, \zeta, \bar{\zeta}) - X(s + i\lambda, \zeta, \bar{\zeta})].$$

(A.4)

By setting

$$\chi_I(s, \lambda, \zeta, \bar{\zeta}) = 0$$

(A.5)

and solving for

$$\lambda = \Lambda(s, \zeta, \bar{\zeta}),$$

(A.6)

we obtain the real slicing

$$u = \chi_R(s, \Lambda(s, \zeta, \bar{\zeta}), \zeta, \bar{\zeta}).$$

(A.7)

Remark 6. We remark without proof [6] that using the gauge freedom described early for the choice of the parameter $\tau$, we can normalize the real velocity vector, $v^a_R(s) = \xi^a_R(s = \tau)$, to 1, i.e. $\eta_{ab}v^a_R(s)v^b_R(s) = 1$.

A.2. Lorentzian tensors

In section 4 we pointed out that a certain spin and conformal weighted function on the sphere carried finite dimensional representations of the Lorentz group, i.e. they carried information about Lorentzian tensor objects. As examples of this, we will work out two specific cases.

Starting with the Lorentz transformation

$$\hat{\zeta} = \frac{a\zeta + b}{c\zeta + d}, \quad ad - bc = 1$$

(A.8)

$$e^{i\lambda} = \frac{c\zeta + d}{\bar{c}\zeta + \bar{d}}$$

(A.9)

$$K = \frac{1 + \bar{\zeta}\zeta}{[(a\zeta + b)(\bar{a}\zeta + \bar{b}) + (c\zeta + d)(\bar{c}\zeta + \bar{d})]}$$

(A.10)

$$\tilde{\eta}_{(r,w)}(\hat{\zeta}, \bar{\zeta}) = e^{i\lambda}K^w\eta_{(r,w)}(\zeta, \bar{\zeta}),$$

(A.11)

we choose the special transformation

$$\zeta = a^2\zeta$$

$$e^{i\lambda} = \frac{d}{\bar{d}} = \frac{\bar{a}}{a}$$

(A.12)

$$K = \frac{a\bar{a}(1 + \bar{\zeta}\zeta)}{[(a\bar{a})^2\zeta\bar{\zeta} + 1]}.$$

Example 1. $s = 0$ and $w = 1$

Applying this special transformation to the invariant subspace of an $s = 0$ and $w = 1$ quantity, e.g. to $u = X(\tau, \zeta, \bar{\zeta})$, we have
\[ \hat{\eta}_{(0,1)}(\xi, \bar{\xi}) = K \eta_{(0,1)}(\xi, \bar{\xi}) \]  \hspace{1cm} (A.13)

\[ \hat{\xi}^a l_a(\xi, \bar{\xi}) = K \xi^a l_a(\xi, \bar{\xi}) \]  \hspace{1cm} (A.14)

\[ l_a(\xi, \bar{\xi}) = \frac{\sqrt{\xi}}{2} \left( 1, \frac{\xi + \bar{\xi}}{1 + \xi \bar{\xi}}, -1, \frac{\xi - \bar{\xi}}{1 + \xi \bar{\xi}}, -1 + \xi \bar{\xi} \right). \]  \hspace{1cm} (A.15)

After using equation (A.12) in equation (A.14) and comparing the coefficients of 

\( (1, \xi, \bar{\xi}, \xi \bar{\xi}), \) \hspace{1cm}

we have the Lorentz transformation 

\[ \hat{\xi}^0 = \frac{1}{2} (a \bar{\alpha} + a^{-1} \alpha^{-1}) \xi^0 + \frac{1}{2} (a^{-1} \bar{\alpha}^{-1} - a \bar{\alpha}) \xi^3 \]  \hspace{1cm} (A.16)

\[ \hat{\xi}^3 = \left( \frac{1}{2} a \bar{\alpha} + \frac{1}{2} a^{-1} \alpha^{-1} \right) \xi^3 + \left( \frac{1}{2} a^{-1} \bar{\alpha}^{-1} - \frac{1}{2} a \bar{\alpha} \right) \xi^0 \]  \hspace{1cm} (A.17)

\[ \hat{\xi}^1 - i \hat{\xi}^2 = \frac{\bar{\alpha}}{a} (\xi^1 - i \xi^2) \]  \hspace{1cm} (A.18)

\[ \hat{\xi}^1 + i \hat{\xi}^2 = \frac{a}{\bar{\alpha}} (\xi^1 + i \xi^2). \]  \hspace{1cm} (A.19)

Since \( \frac{\bar{\alpha}}{a} \) can be written as \( e^{i \phi} \), we have a spatial rotation in the \( (1, 2) \) plane. Then by identifying 

\[ \left( 1 - \frac{\bar{\alpha} \alpha}{\bar{\alpha} \alpha} \right)^{-\frac{1}{2}} = \frac{1}{2} (a \bar{\alpha} + a^{-1} \alpha^{-1}), \]  \hspace{1cm} (A.20)

we have the Lorentz transformation 

\[ \hat{\xi}^0 = \frac{\xi^0}{(1 - \frac{\bar{\alpha} \alpha}{\bar{\alpha} \alpha})^{\frac{1}{2}}} + \frac{\bar{\alpha} \xi^3}{(1 - \frac{\bar{\alpha} \alpha}{\bar{\alpha} \alpha})^{\frac{1}{2}}}, \]  \hspace{1cm} (A.21)

\[ \hat{\xi}^3 = \frac{\xi^3}{(1 - \frac{\bar{\alpha} \alpha}{\bar{\alpha} \alpha})^{\frac{1}{2}}} + \frac{\bar{\alpha} \xi^0}{(1 - \frac{\bar{\alpha} \alpha}{\bar{\alpha} \alpha})^{\frac{1}{2}}}. \]  \hspace{1cm} (A.22)

We see that the special fractional linear transformation \( \hat{\xi} = a^2 \xi \) corresponds to the standard Lorentz transformation with a spatial rotation.

**Example 2** \( s = -1 \) and \( w = 1 \) coming from the \( s = 1 \) and \( w = -3 \) isomorphism.

Applying \( \hat{\xi} = a^2 \xi \) to the \( s = -1 \) and \( w = 1 \) case, e.g. to the invariant factor space of 

\( V_1 = V_1(1, -1), V_1 = V_1, V_1 = V_1, \ldots \), we have 

\[ \hat{\eta}_{(-1,1)} = \hat{\psi}_0 l_0(\xi, \bar{\xi}) \]  \hspace{1cm} (A.23)

\[ m_0(\xi, \bar{\xi}) = \sqrt{2} \left( 0, 1 - \xi^2, -i(1 + \xi^2), 2 \xi \right). \]

Comparing the coefficients of \( (1, \xi, \xi^2), \) we find that 

\[ \hat{\psi}_1 = \frac{1}{2} (a^2 + a^{-2}) \psi_1 + \frac{1}{2} (a^2 - a^{-2}) \psi_1, \]  \hspace{1cm} (A.24)

\[ = \frac{\psi_1}{(1 - \frac{\bar{\alpha} \alpha}{\bar{\alpha} \alpha})^{\frac{1}{2}}} - i \frac{\bar{\alpha} \psi_1}{(1 - \frac{\bar{\alpha} \alpha}{\bar{\alpha} \alpha})^{\frac{1}{2}}}. \]  \hspace{1cm} (A.25)
\[
\psi_{1.2}^0 = \frac{1}{2} [a^{-2} + a^2] \psi_{1.2}^0 + \frac{1}{2} i [a^2 - a^{-2}] \psi_{1.1}^0
\]
\[
= \frac{\psi_{1.2}^0}{(1 - \frac{a^2}{a^{-2}})^2} + i \frac{\psi_{1.1}^0}{(1 - \frac{a^2}{a^{-2}})^2},
\]
(A.26)

If we had identified \(\psi_{l_i}^0\) with a Maxwell field via
\[
\psi_{l_i}^0 = (\psi_{1.1}^0, \psi_{1.2}^0, \psi_{1.3}^0) = (E + i B),
\]
then equation (A.24) would be equivalent to a Lorentz transformation of the Maxwell tensor \(F^{ab}\). The six real components of \(\psi_{l_i}^0\) thus correspond to a skew-symmetric Lorentzian tensor [25].

### A.3. Products of spin-s harmonics

For completeness, we give several of the relevant Clebsch–Gordan products that were used. We have left out terms with \(l\)-values greater than 2:

\[
Y_{2k}^2 Y_{2j}^{-2} = \frac{\delta_{jk} \delta_{jl}}{5} + \frac{i \sqrt{2}}{5} \delta_{jl} \epsilon_{ijk} Y_{1i}^0 - \frac{1}{7} \delta_{lj} Y_{2k}^0,
\]
\[
Y_{2k}^0 Y_{2j}^0 = \frac{24}{5} \delta_{kj} \frac{1}{7} \delta_{lj} Y_{2k}^0,
\]
\[
Y_{2k}^{-1} Y_{2j}^0 = -\frac{i 12 \sqrt{2}}{5} \delta_{lk} \frac{12}{7} \delta_{lj} Y_{2k}^0,
\]
\[
Y_{2k}^1 Y_{2j}^0 = \frac{i 12 \sqrt{2}}{5} \delta_{lk} \frac{12}{7} \delta_{lj} Y_{2k}^0,
\]
\[
Y_{1j}^1 Y_{1j}^{-1} = \frac{1}{3} \delta_{ij} - \frac{i \sqrt{2}}{4} \epsilon_{ijk} Y_{1k}^0 - \frac{1}{12} Y_{2j}^2,
\]
\[
Y_{2j}^2 Y_{1k}^{-1} = \frac{3}{5} \delta_{jk} Y_{1i}^0 - \frac{i \sqrt{2}}{6} \epsilon_{ijkl} Y_{2j}^0,
\]
\[
\bullet Y_{2j}^2 Y_{1k}^0 = \frac{i 2 \sqrt{2}}{3} \epsilon_{ijkl} Y_{2j}^0,
\]
\[
Y_{2j}^2 Y_{2m}^0 = \frac{-24}{7} \delta_{ij} Y_{2mj}^0,
\]
\[
\bullet Y_{2j}^2 Y_{2m}^{-1} = \frac{2 \sqrt{2}}{5} i \delta_{im} \epsilon_{ijf} Y_{1f}^0 + \frac{6}{7} \delta_{lj} Y_{2m}^1,
\]
(A.28)
\[
Y_{2j}^{-1} Y_{1k}^1 = \frac{3}{5} Y_{1i}^0 \delta_{jk} + \frac{i \sqrt{2}}{6} \epsilon_{jkl} Y_{2l}^0,
\]
\[
\bullet Y_{2j}^{-1} Y_{1k}^0 = \frac{3}{5} Y_{1i}^0 \delta_{jk} - \frac{i \sqrt{2}}{6} \epsilon_{jkl} Y_{2l}^0,
\]
\[
Y_{1k}^1 Y_{1i}^0 = \frac{1}{\sqrt{2}} \epsilon_{jkl} Y_{1l}^0 + \frac{1}{2} Y_{2k}^1,
\]
\[
\bullet Y_{1k}^{-1} Y_{1i}^0 = -\frac{1}{\sqrt{2}} \epsilon_{jkl} Y_{1l}^{-1} + \frac{1}{2} Y_{2k}^{-1},
\]
\[
Y_{1k}^0 Y_{2j}^0 = -\frac{6}{5} Y_{1i}^0 \delta_{jk} + i \sqrt{2} \epsilon_{klj} Y_{1j}^0,
\]
\[
\bullet Y_{1k}^{-1} Y_{2j}^1 = -\frac{6}{5} Y_{1i}^{-1} \delta_{jk} - i \sqrt{2} \epsilon_{klj} Y_{2j}^{-1},
\]
\[
Y_{2l}^0 Y_{1i}^0 = \frac{6}{5} Y_{1i}^0 \delta_{lm} - \frac{i}{3} \sqrt{2} \epsilon_{ijm} Y_{2j}^0,
\]
\[
\bullet Y_{2l}^{-1} Y_{1i}^0 = \frac{6}{5} Y_{1i}^{-1} \delta_{lk} + \frac{i}{3} \sqrt{2} \epsilon_{ikl} Y_{2j}^{-1},
\]
\[
Y_{1i}^0 Y_{2j}^0 = \frac{2}{5} \delta_{ij} + \frac{1}{3} Y_{2j}^0,
\]
\[
\bullet Y_{1i}^{-1} Y_{2j}^1 = \frac{12}{5} \delta_{ij} Y_{1k}^0.
\]

25
A.4. Angular momentum ambiguities

As we mentioned earlier in the text there have been ambiguities, described in the literature [4], in the definition of the asymptotic angular momentum, $J^k$. In our notation (omitting the Maxwell field), the ambiguities are in the arbitrary choice of the constant $p$ in the expression

$$J^k = -\sqrt{\frac{2c^3}{G}} \psi_{ik}^0 + p \frac{c^3}{G} \text{Im} \left( \sigma \partial \sigma + \frac{1}{2} \partial (\sigma \sigma) \right)_k. \quad (A.29)$$

The default choices appear to be either 2, 1 or 0. The present work does not influence or help resolve the ambiguity since to second order, the expression

$$\text{Im} \left( \sigma \partial \sigma + \frac{1}{2} \partial (\sigma \sigma) \right)_k = i \frac{3}{10} (24)^2 \sqrt{2} \left( \xi^{ij} \xi^{ij} + \bar{\xi}^{ij} \bar{\xi}^{ij} \right) \epsilon_{ijk} \quad (A.30)$$

vanishes.

One might have thought that our flux law (still omitting the Maxwell field)

$$J^k = -\frac{(24)^2 c^3}{5G} \left( \xi_R^{ij} \xi_R^{ij} + \bar{\xi}_R^{ij} \bar{\xi}_R^{ij} \right) \epsilon_{ijk}, \quad (A.31)$$

would have an ambiguity in the flux, i.e. a total derivative, arising from the use of the chain rule, e.g.

$$\epsilon_{ijk} \xi_R^{ij} \bar{\xi}_R^{ij} = \epsilon_{ijk} \left( \xi_R^{ij} \xi_R^{ij} \right) - \epsilon_{ijk} \xi_R^{ij} \bar{\xi}_R^{ij} = -\epsilon_{ijk} \xi_R^{ij} \bar{\xi}_R^{ij}.$$

This apparent ambiguity disappears since these total derivatives are equivalent to the expression in equation (A.30) and again vanish identically.

It is possible that if our calculations were repeated, but done to third order, the ambiguities could be resolved.

References

[1] Weyl H 1922 Space-Time-Matter (London: Methuen)
[2] Szabados L B 2005 Quasi-local energy–momentum and angular momentum in GR: a review article Living Rev. Rel. 7
[3] Heller A D 2007 Gen. Rel. Grav. 39 2125–47
[4] Dray T and Streubel M 1984 Class. Quantum Grav. 1 15–26
[5] Ashtekar A and Winicour J 1982 J. Math. Phys. 23 2410–7
[6] Kozameh C, Newman E T and Silva-Ortigoza G 2007 Class. Quantum Grav. 24 5479–93
[7] Newman E T, Couch E, Chinnapared K, Exton A, Prakash A and Torrence R 1965 J. Math. Phys. 6 918
[8] Bondi H, van der Burg M G J and Metzner A W K 1962 Proc. R. Soc. A 269 21
[9] Sachs R 1962 Proc. R. Soc. 270 103
[10] Newman E T and Penrose R 1962 J. Math. Phys. 3 566–768
[11] Newman E T and Unti T 1962 J. Math. Phys. 3 891
[12] Newman E T and Tod K P 1980 General Relativity and Gravitation vol 2 ed A Held (New York: Plenum)
[13] Gelfand I M, Graev M I and Vilenkin N Ya 1966 Generalized functions. Integral Geometry and Problems of Representation Theory vol 5 (New York: Academic)
[14] Held A, Newman E T and Posadas R 1970 J. Math. Phys. 11 3145
[15] Sachs R 1962 Proc. R. Soc. 270 103
[16] Penrose R 1963 Phys. Rev. Lett. 10 66
[17] Kozameh C and Newman E T 2005 Class. Quantum Grav. 22 4659–65
[18] Aronsen D and Newman E T 1972 J. Math. Phys. 13 1847–51
[19] Hansen R, Newman E T, Penrose R and Tod K P 1978 Proc. R. Soc. A 363 445
[20] Newman E T and Silva-Ortigoza G 2006 Class. Quantum Grav. 23 497–509
[21] Kozameh C, Newman E T and Silva-Ortigoza G 2007 Class. Quantum Grav. 24 1955–79
[22] Newman E T 2002 Phys. Rev. D 65 104005
[23] Newman E T 2004 Class. Quantum Grav. 21 3197–221
[24] Thirring W 1958 Principles of Quantum Electrodynamics (New York: Academic) p 25
[25] Landau L D and Lifschitz E M 1962 Classical Theory of Fields (Reading, MA: Addison-Wesley)
[26] Robinson I and Trautman A 1962 Proc. R. Soc. A 289 463
[27] Talbot C J 1969 Commun. Math. Phys. 13 45
[28] Kozameh C, Newman E T and Silva Ortigoza G 2006 Class. Quantum Grav. 23 6599–620
[29] Exton A, Newman E T and Penrose R 1969 J. Math. Phys. 10 1566
[30] Rivas M, Aguirregabiria J M and Hernandez A 1999 Phys. Lett. A 257 21–5