ON MULTIDIMENSIONAL MANDELBROT’S CASCADES

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Dedicated to Jean-Pierre Kahane, with admiration

Abstract. Let \( Z \) be a random variable with values in a proper closed convex cone \( C \subset \mathbb{R}^d \), \( A \) a random endomorphism of \( C \) and \( N \) a random integer. Given \( N \) independent copies \((A_i, Z_i)\) of \((A, Z)\) we define a new random variable \( \hat{Z} = \sum_{i=1}^N A_i Z_i \). Let \( T \) be the corresponding transformation on the set of probability measures on \( C \). We study existence and properties of fixed points of \( T \). Previous one dimensional results on existence of fixed points of \( T \) as well as on homogeneity of their tails are extended to higher dimensions.

1. Introduction

We consider the vector space \( V = \mathbb{R}^d \) endowed with a scalar product \( \langle x, y \rangle \) and the corresponding norm \( x \rightarrow |x| \). We fix a proper closed convex subcone \( C \subset V \) with nonzero interior \( C^0 \neq \emptyset \) and we write \( C_+ = C \setminus \{0\} \). Let \( S \) be the subsemigroup of \( \text{End} V \) which consists of \( a \in \text{End} V \) such that \( a(C) \subset C \) and \( S_+ \) the open subsemigroup of \( S \) defined by \( S_+ = \{ a \in S; \text{Ker} a \cap C = \{0\} \} \). Each element \( a \) of \( S_+ \) has a positive dominant eigenvalue \( r(a) \) and a corresponding dominant eigenvector \( v_a \in C_+ \).

We denote by \( M^1(E) \) the space of probability measures on a measurable space \( E \). Let \( \mu \) be a probability measure on \( S_+ \) i.e. \( \mu \in M^1(S_+) \). Let \( A \) be a random endomorphism distributed according to \( \mu \), \( N \) a random integer, \( Z \) a \( C_+ \)-valued random vector such that \( A, N \) and \( Z \) are independent. We consider \( N \) independent copies \((A_i, Z_i) (1 \leq i \leq N)\) of \((A, Z)\) and the new random variable \( \hat{Z} \) defined by

\[
\hat{Z} = \sum_{i=1}^N A_i Z_i.
\]

The corresponding transformation \( \rho \rightarrow T\rho \) is defined on \( M^1(C_+) \), where \( \rho \) is the law of \( Z \) and \( T\rho \) the law of \( \hat{Z} \). We observe that \( T \) commutes with the action of \( \mathbb{R}^*_+ \) on measures defined by extension of the dilations \( x \rightarrow tx \) (\( t \in \mathbb{R}^*_+ \)).

We are interested in nontrivial fixed points of \( T (\rho \neq \delta_0) \) and their tails. For \( d = 1 \), fixed points of analogous transformations have been considered by Durrett and Liggett [DL], Holley [Ho], Spitzer [S] and others in connection with the study of invariant measures for infinite systems of particles in interaction. Equation (1.1) is a limiting case of the more complex equations in [Ho, S]; it corresponds to the case where the particles are randomly moving on a tree instead of the \( n \)-dimensional lattice, and the random interaction is given by \( A_i \). The matricial equation \( (d > 1) \) can be also interpreted in this context with ‘coloured’ particles numbered from 1 to \( d \). Also, if \( d = 1, N \) is constant and
NEA = 1, the fixed point equation with finite mean plays an important role in the context of construction of a large class of self-similar random measures [GMW, L1]. Independently of [Ho, S], and in the context of random fractals various questions on equation (1.1) were considered (d = 1) by Benoit Mandelbrot [M] and some of them were solved by Kahane, Peyrière [KP]. The question on the homogeneity of the tails of the fixed points was studied by Guivarch [G] (see also [L2, RTV]). Equation (1.1) appeared also in the context of branching random walks; see Biggins [Bi] for closely related work.

Here we consider the situation d > 1, and we prove the analogues of the results of [KP, G]. If the cone C is C = \mathbb{R}^d_+, then S is the semigroup of nonnegative matrices. Another special case, we have in mind, is the cone C of real positive definite matrices. Then S contains the set of mappings \( M \rightarrow aMa^t \), where \( a \) is an orthogonal transformation or an upper triangular matrix. More generally C can be a symmetric cone (e.g. the light cone, see [FK]) or a homogeneous cone [V].

Our framework is as follows. We assume, as in [KP], that \( A \) and \( N \) have finite expectations: \( \mathbb{E}[|A|] + \mathbb{E}[N] < \infty \) and we denote \( c = \mathbb{E}[N] \in \mathbb{R}_+ \), \( m = \mathbb{E}[A] \in S \setminus \{0\} \). If \( \rho \in M^1(C_+) \) has a finite mean \( \rho_1 \) the same is true for \( T\rho \) and the mean \( (T\rho)_1 \) of \( T\rho \) is \( (T\rho)_1 = cm\rho_1 \). Hence \( T \) preserves \( M^1(C) \), the convex subset of \( M^1(C) \) of elements with nonzero finite mean.

In the present paper we studied, provided finite expectation of \( A \), the convergence of iterations of (1.1), the fixed points of \( T \) in \( M^1(C) \) and their asymptotics at infinity of \( C \). If \( Z \) and \( \hat{Z} \) have the same law \( \rho \) with finite expectation, by abuse of language, \( Z \) will be called a fixed point of (1.1).

Clearly, if \( Z \) is a fixed point of (1.1), then the law \( \rho \) of \( Z \) satisfies \( T\rho = \rho \). Also, the condition \( T\rho = \rho \) implies \( \rho_1 = (T\rho)_1 = cm\rho_1 \), hence the spectral radius \( r(m) \) of \( m = \mathbb{E}A \) is \( \frac{1}{c} \). We assume below this condition to be satisfied.

Conversely, one can construct a fixed point of (1.1) as follows. Let \( \mathcal{T} \) be a tree with root \( o \), a typical vertex (resp. edge) will be denoted by \( \gamma \) (resp. \( \alpha \)). We identify \( \gamma \) with the path from \( o \) to \( \gamma \), we denote by \( |\gamma| \) the length of this path and by \( N_\gamma \) the number of children of \( \gamma \). If an element \( A_\alpha \) of \( S_+ \) is associated to each edge \( \alpha \) of \( \mathcal{T} \), we obtain a labelled tree \( \mathcal{\hat{T}} \).

For every vertex \( \gamma \) of \( \mathcal{\hat{T}} \), we form the product of matrices \( P_\gamma = A_{\alpha_1}A_{\alpha_2}\ldots A_{\alpha_n} \), where \( \alpha_1\alpha_2\ldots\alpha_n \) is the path from \( o \) to \( \gamma \). We assume that \( A_\alpha \) (resp. \( N_\alpha \)) are independent copies of \( A \) (resp. \( N \)) and that all \( A_\alpha \), \( N_\gamma \) are independent.

Let \( \mathcal{F}_n \) be the \( \sigma \)-field generated by all \( N_\gamma \), \( A_\alpha \) where \( |\gamma| \leq n \) and \( \alpha \) is the edge connecting \( \gamma \) to \( \gamma' \) being a child of \( \gamma \) with \( |\gamma'| \leq n \).

If \( v \in C_+ \) is a dominant eigenvector of \( \mathbb{E}[A] = m \), we consider the sequence of random variables \( Y_n = \sum_{|\gamma| = n} P_\gamma v \) called Mandelbrot’s cascade, and the shifted version \( Y_n^i \) of \( Y_n \), where \( i \in \mathbb{N} \setminus \{0\} \) is a child of \( o \). Then \( Y_{n+1} = \sum_{i=1}^{N_0} A_i Y_n^i \). Also, since \( N \) and \( A \) are independent,

\[
\mathbb{E}[Y_{n+1}|\mathcal{F}_n] = c \sum_{|\gamma| = n} P_\gamma \mathbb{E}[A]v = cr(m)Y_n = Y_n.
\]

Then \( Y_n \) is a \( C_+ \)-valued martingale, hence \( Y_n \) converges a.e. to \( Y \in C \). By what has been said above, in the limit we have

\[
Y = \sum_{i=1}^{N_0} A_i Y^i
\]

with \( Y^i = \lim_{n \to \infty} Y_n^i \). Also \( \mathbb{E}[Y] \in C \) and \( Y_i \)'s are independent with the same law as \( Y \). So, if \( Y \neq 0 \), we have a solution of (1.1) and we are led to discuss below the nondegeneracy of \( Y \) (Theorem 2.2) and its tail (Theorem 2.3).

Before describing the general case, we recall the previous results concerning the one-dimensional situation. If \( d = 1 \) and \( C = \mathbb{R}^d_+ \), nondegeneracy of \( Y \) was discussed by Kahane, Peyrière [KP]. Behavior of \( Y \) at infinity depends on the properties of the function \( v(\theta) = \mathbb{E}[\sum_{i=1}^N A_i^\theta] \) (here \( A_i \) and \( N \) can be dependent), see [DL]. Then if \( v(1) = 1, v'(1) < 0 \) and \( v(\chi) = 1 \) for some \( \chi > 1 \),...
Guivarc’h [G] and Liu [L2] proved that if \( \text{supp} \mu \) is non arithmetic, then the tail of \( Y \) is homogeneous at infinity i.e. \( \lim_{t \to \infty} t^n \mathbb{P}[Y > t] \) exists and it is positive. Recently, also asymptotic properties of solutions of (1.1) in the boundary case, when \( \nu'(1) = 0 \) were described (see Biggins, Kyprianou [BK] and Buraczewski [Bu]).

2. Notations and statements of results

We denote \( C_+ = C \setminus \{0\} \) with \( d > 1 \), \( S_+ \) as defined above, \( C_1 = \{x \in C; |x| = 1\} \). Let \( C^0 \) be the interior of \( C \), \( S^0 = \{a \in S; aC_+ \subset C^0\} \). We observe that \( S \) is a proper closed convex subcone of \( \text{End} V \) and, as it is easily shown, \( S^0 \) is the interior of \( S \). For \( a \in S \) we write \( \iota(a) = \inf_{x \in C_1} |ax| \) and, by definition of \( S_+ \), we have that \( \iota(a) > 0 \) for \( a \in S_+ \). Let \( C^* = \{x \in V; \langle x, y \rangle \geq 0 \) for any \( y \in C\} \). Observe that \( C^* \) is a proper closed subcone of \( V \), so we will adopt for \( C^* \) similar notations as for \( C \). We write \( \ast^\nu \) for the dual map of \( a \) defined by \( \langle \ast^\nu x, y \rangle = \langle x, ay \rangle \) (\( x, y \in V \)), hence \( \ast^\nu C^* \subset C^* \). We will consider subsemigroups \( \Gamma \) of \( S \) that satisfy condition \( (C) \) (compare [H]), i.e.:

a) no subspace \( W \subset V \) with \( W \cap C \neq \{0\} \) satisfies \( GW \subset W \)

b) \( \Gamma \cap S^0 \neq \emptyset \).

This condition is an analogue in the cone context of the so called i-p condition in [GL1]. If it holds for \( \Gamma \) and \( C \), it is also valid for \( \Gamma^*, C^* \). If \( a \in S \), we denote by \( r(a) \) its spectral radius, i.e. \( r(a) = \lim_{n \to \infty} |a^n|^\frac{1}{n} \). Let \( \mu \in M^1(S) \) be the law of the random endomorphism \( A \). We denote by \( \text{supp} \mu \) the smallest closed subsemigroup of \( S \) which contains \( \text{supp} \mu \). We will always assume that \( \text{supp} \mu \) satisfies condition \( (C) \) and \( m = \mathbb{E}[A] = \int a \mu(da) \) exists. We observe that if \( s \geq 0 \), the number \( \kappa(s) = \lim_{n \to \infty} \mathbb{E}[|A_n \ldots A_1|^s]^\frac{1}{n} \) is well defined on the interval \( I_\mu = \{s \geq 0; \mathbb{E}[|A|^s] < \infty\} \). Indeed, since \( u_n(s) = \mathbb{E}[|A_n \ldots A_1|^s] \) is submultiplicative, \( u_n^n(s) \) converges to \( \inf_{k \geq 1} \mathbb{E}[|A_k \ldots A_1|^s]^\frac{1}{n} \). Also, the number \( \kappa^\nu(s) = \lim_{n \to \infty} \mathbb{E}[|A_n^\nu \ldots A_1^\nu|^s]^\frac{1}{n} \) is well defined under the same conditions and \( \kappa^\nu(s) = \kappa(s) \), since \( |A|^s = |A| \). In particular, \( \kappa(1) < \infty \), i.e. \( 1 \in I_\mu \) and \( \log \kappa(s) \) is convex on \( I_\mu \).

As will be seen below, condition \( (C) \) implies that \( m = \mathbb{E}A \) has a unique dominant eigenvector \( v \in C^0_1 \) with \( mv = r(m)v \). The same is true for \( m^\ast \) and we denote by \( v^\ast \) the unique eigenvector of \( m^\ast \) in \( C^0_1 \). It will be shown that \( \kappa(1) = r(m) = r(m^\ast) \). By abuse of notation we denote also by \( v^\ast \) the function \( v^\ast(x) = \langle v^\ast, x \rangle \) on \( C \) and we observe that the convolution operator \( P \) on \( C \) defined by

\[
P\phi(x) = \int_S \phi(ax)\mu(da)
\]

admits an eigenfunction \( v^\ast \) with the eigenvalue \( r(m) \). Then we consider the Markov operator \( Q \) on \( C \) defined by

\[
Q\phi = \frac{1}{r(m)v^\ast}P(\phi v^\ast)
\]

and we will prove below that \( Q \) has a unique invariant measure of the form \( \pi \otimes I \) where \( \pi \in M^1(C_1) \) and \( I(dt) = \frac{dt}{\tau} \). We will be led to consider also the analogous Markov operator on \( C^* \) defined by

\[
P_x \Phi(x) = \int_S \Phi(ax)\mu(da).
\]

Then we obtain the following generalization of the result of Kahane and Peyrière [KP]:

**Theorem 2.2.** Assume that the semigroup \( \text{supp} \mu \) satisfies condition \( C \) above, the dominant eigenvalue \( r(m) \) of \( \mathbb{E}[A] \) satisfies \( \mathbb{E}[N]r(m) = 1 \), \( N \geq 2 \) a.s. and \( \mathbb{E}[N^2] < \infty \). Then the following are equivalent:

1. \( \mathbb{E}[Y] = v \).
2. \( \mathbb{E}[Y] \neq 0 \).
3. There exists a fixed point \( Z \) of (1.1) with \( \mathbb{E}[Z] < \infty \) and \( \mathbb{E}Z \neq 0 \).
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4.

The derivative $\kappa'(1^-) < 0$.

The derivative $\kappa'(1^-)$ is given by the following formula

$$
\kappa'(1^-) = \frac{1}{r(n)} \int \frac{\langle v^*, ax \rangle}{\langle v^*, x \rangle} \log \langle v^*, ax \rangle \mu(da) \pi(dx).
$$

Moreover if $s > 1$, $E[|Z|^s] < \infty$ if and only if $s$ satisfies $\kappa(s)E[N] < 1$.

If $C = \mathbb{R}_+^d$, then repeating the argument given in [G] one can prove that (1.1) has a unique fixed point $Z$, such that $|EZ| = 1$.

The next theorem contains an asymptotics of the fixed point $\rho$ on $C_+$ obtained in the theorem above being a generalization of one dimensional results of Guivarc’h [G] and Liu [L2].

**Theorem 2.3.** Suppose hypothesis of Theorem 2.2 are satisfied. Assume additionally that

- $N = c \geq 2$ is constant;
- $\chi > 1$ exists with $\kappa(\chi) = 1/N$;
- $E[|A|^N \log |A|]$, $E[|A|^N \log \iota(A)]$ are both finite;
- the law of $Z$ gives zero mass to every affine subspace intersecting the cone $C_+$, i.e. for any $r > 0$ and any $u \in C_+^*$, $P([Z, u] = r) = 0$.

Then for every $u \in C^*$

$$
\lim_{t \to \infty} t^\chi P([Z, u] > t) = D(u) > 0,
$$

where $D(u)$ is a $\chi$-homogeneous $P_\ast$-harmonic positive function, uniquely defined up to a positive coefficient.

**Remark.** The last hypothesis is satisfied e.g. if the support of the measure $\mu$ consists of invertible matrices (see Lemma (5.17)).

Notice, that in view of the result of Basrak, Davis and Mikosch [BDM], if $\chi$ is noninteger, the latter result implies that $\rho$ is regularly varying, i.e. the family of measures $t^\chi \delta_t \cdot \rho$ converges weakly on $\mathbb{R}_+^d \setminus \{0\}$ to a $P$-harmonic $\chi$-homogeneous Radon measure.

In the whole paper we will denote by $C$ the cone and sometimes, by abuse of notation also continuous functions. We will use symbol $D$ to denote auxiliary constants, which will appear in the sequel.

3. Transfer Operators on Projective Spaces

In this section we describe some ergodic properties of a class of transfer operators on $C_+$. We start with the convolution operator $P$ on $C_+$ defined in (2.1) If $\phi(x) = \psi(\pi)|x|^s$ with $\psi \in C(C_1)$ and $\pi = \frac{x}{|x|} \in C_1$ we get an operator $P^s$ on $C(C_1)$ defined by

$$
P^s \psi (x) = \int_S |ax|^s \psi(a \cdot x) \mu(da),
$$

where $a \cdot x = \frac{ax}{|ax|} \in C_1$. We will be led also to consider the operators $P^s_\ast$ on $C(C_+^* \ast)$ defined by

$$
P^s_\ast \phi (x) = \int_S |a^* x|^s \phi(a^* \cdot x) \mu(da), \quad \psi \in C(C_+^* \ast).
$$

If $s \geq 0$, these operators have nice contraction properties on suitable functional spaces. These properties are studied in [GL1, GL2] in the slightly different context of invertible maps and projective spaces, under the so called condition i-p for irreducibility and proximality. Here, since $C$ is a proper convex cone, $C_1$ can be considered as a convex part of the projective space $\mathbb{P}(V)$. However the
elements of $S_+$ are not invertible in general i.e. $S_+ \not\subset \text{GL}(V)$. Hence we are led to define $\Gamma$-invariance of a subspace $W$ as $\Gamma W \subset W$ instead of $\Gamma W = W$ as in [GL1]. However, with this extended concept, the ergodic type results of [GR] (see also [GU]) remain valid here. In condition C for a semigroup $\Gamma \subset S$, we don’t require irreducibility, hence $\Gamma$ can leave invariant subspace $W \subset V$ with $W \cap C = \{0\}$, but this fact plays no role in our analysis.

Let us give more details. We define the Birkhoff distance $\beta$ on $C_1$ as follows: if $x, y \in C_1$, $x \neq y$ and $u, v$ are the intersections of the line $xy$ with the boundary of $C$, we write $\beta(x, y) = \log(uvxy)$, where $(uvxy)$ is the cross ratio of $u, v, x, y$. If $a \in S$, we have $a(C) \subset C$, and since $C$ is convex, the Brouwer theorem implies the existence of $v_a \in C_1$, $\lambda_a > 0$ with $a v_a = \lambda_a v_a$. If $a \in S^0$, $a \cdot C_1$ is a compact subset of $C_1^0$, hence the Birkhoff distance $\beta$ on $C_1$ is strictly contract ed by a coefficient $\rho < 1$ under the action of $a$; hence $(v_a, \lambda_a)$ as above are unique, $v_a \in C^0$ and we write $a_+ = v_a$. If $a \in S_+, x, y \in C_1$ we have $\beta(a \cdot x, a \cdot y) \leq \beta(x, y)$

We denote $\Lambda(\Gamma) = \text{cl}\{a^+ : a \in \Gamma \cap S^0\}$. Then the irreducibility condition in $C$ implies that $\Lambda(\Gamma)$ generates $V$ as a subspace. Also $C$ implies a strong irreducibility condition, i.e. if $W$ is a supp-$\mu$-invariant finite union of subspaces with $W \cap C \neq \{0\}$, then $W = V$. In fact let $x \in W \cap C$ and $a \in \text{supp}\mu \cap S^0$. Then we have $\lim_{n \to \infty} a^n \cdot x = a^\ast \in \Lambda(\text{supp}\mu)$. Hence $W \cap C = \Lambda(\text{supp}\mu)$. Since $\Lambda(\text{supp}\mu)$ generates $V$ we conclude $W = V$. This property is essential in dealing with measures on $C_1$. We will say that $\nu \in M^1(C_1)$ is proper if $\nu(W) = 0$ for any subspace $W \subset V$.

Then by translation of the arguments of [Fu, GR, GU] we get

**Proposition 3.1.** Assume $\Gamma$ is a subsemigroup of $S$ which satisfies condition $C$. Then $\Lambda(\Gamma)$ is the unique $\Gamma$-minimal subset of $C_1$. If $\mu \in M^1(S)$ is such that $\Gamma = \text{supp}\mu$ satisfies $C$, then there exists a unique $\mu$-stationary measure $\nu$ on $C_1$. The measure $\nu$ is proper and supp$\nu = \Lambda(\text{supp}\mu)$.

The following Proposition plays an essential role below in the study of the iterates of $P^\ast$. The proof is given in [GR] (see also [GU]) in the context of condition i-p (Corollary 4.8). It remains valid here once the irreducibility concept is properly adjusted as above.

**Proposition 3.2.** Assume $d > 1$ and $\Gamma$ is a subsemigroup of $S$, which satisfies condition $C$. Then $\{\lambda_a : a \in \Gamma \cap S^0\}$ generates a dense subgroup of $\mathbb{R}_+^d$.

Also we have

**Theorem 3.3.** Assume $\mu \in M^1(S_+)$ is such that $\text{supp}\mu$ satisfies condition $C$, and let $s \in I_\mu$. Then the equation $P^\ast \phi = \kappa(s)\phi$ with $\phi \in C(C_1)$ has a unique normalized solution $\phi = e^s (|e^s|_\infty = 1)$. The function $e^s$ is strictly positive and $\pi$-Hölder with $\pi = \text{inf}\{1, s\}$. There exists a unique $\nu^s \in M^1(C_1)$ with $P^\ast \nu^s = \kappa(s)\nu^s$ and we have supp$\nu^s = \Lambda(\text{supp}\mu)$. If $\nu^s \in M^1(C_1^0)$ satisfies $P^\ast \nu^s = \kappa(s)\nu^s$, then $e^s(x)$ is proportional to $\int (x,y)^t \nu^s(dy)$. In the same way the equation $P^\ast \Phi = \kappa(s)\Phi$ has a unique normalized solution $\Phi = e^\ast (|e^\ast|_\infty = 1)$ and $e^\ast(x)$ is proportional to $\int (x,y)^t \nu^s(dy)$.

The map $s \to \nu^s$ (resp. $s \to e^s$) is continuous in the weak topology (resp. uniform topology) and the function $s \to \kappa(s)$ is strictly convex on $I_\mu$. We have the uniform convergence on $C(C_1)$:

$$\lim_{n \to \infty} \frac{(P^s)^n \phi}{\kappa^n(s)} = \frac{\nu^s(\phi)}{\nu^s(e^s)} e^s.$$  

**Proof.** The proof is the same as the proof of Theorem 2.6 in [GL2] and is inspired by [K1]. We sketch the main arguments as follows.

Find $\sigma \in M^1(C_1)$ with $P^\ast \sigma = \kappa \sigma$, using the Schauder - Tychonov theorem for the compact convex set $M^1(C_1)$, show $\sigma$ is proper and deduce that $\kappa = \kappa(s)$. Take $\nu^\ast = \sigma$ and $\nu^\ast \in M^1(C_1)$ with $P^\ast \nu^\ast = \kappa(s)\nu^\ast$. Define $\tilde{e}^\ast(x) = \int (x,y)^t \nu^\ast(dy)$ and $e^\ast = \frac{\tilde{e}^\ast}{|\tilde{e}^\ast|_\infty}$. Then on $C_1$ one has $P^\ast e^\ast = \kappa(s)e^\ast$.

Replace $P^\ast$ by the Markov operator $Q^s$ on $C_1$ defined by $Q^s \phi = \frac{1}{\kappa(s)e^s} P^s(\phi e^s)$. Show, using $\Gamma$-minimality of $\Lambda(\text{supp}\mu)$ and Proposition 3.1 that any $\phi \in C(C_1)$ which satisfies $Q^s \phi = z\phi$
\(|z| = 1\) is constant and \(z = 1\). This gives us uniqueness of \(e^s\). Show, by direct calculation, the equicontinuity of \(((Q^n)\phi; n \in \mathbb{N})\) for any \(\phi \in C(C_1)\) and deduce the uniqueness of the stationary measure \(\pi^s\) of \(Q^s\). The uniqueness of \(\nu^s\) follows, hence the fact that \(\text{supp} \nu^s = \Lambda(\text{supp} \mu)\) follows by another use of Schauder - Tykhonov theorem in \(M^1(\Lambda(\text{supp} \mu))\). The uniform convergence of the iterates of \(Q^s\) follows from a classical theorem of M. Rosenblatt [R] applied to \(Q^s\), since \(z = 1\) is the unique eigenvalue of \(Q^s\) with modulus 1 as follows from Proposition 3.2 as indicated above. \(\square\)

The following lemma seems to be just a technical one, will play a substantial role in the sequel

**Lemma 3.4.** The function \(\tau(x) = \inf_{x \in \mathbb{N}} \frac{|ax|}{|x|}\) is strictly positive on \(\mathbb{N}\). On every compact subset \(K \subset \mathbb{N}\), \(\inf_{y \in K} \tau(y) > 0\).

**Proof.** We observe that the subset of End \(V, S_+^1 = \{a \in S_+; |a| = 1\}\) is relatively compact, hence its closure \(\overline{S}_+^1\) is compact. If \(a \in S_+^1\), then \(\text{Ker} a \cap \mathbb{N} = \emptyset\), hence \(\text{Ker} a \cap \mathbb{N} = \emptyset\) if \(a \in \overline{S}_+^1\). It follows that, if \(x \in \mathbb{N}\), \(|ax| > 0\) for any \(a \in \overline{S}_+^1\). Since \(a \rightarrow |ax|\) is continuous on \(\overline{S}_+^1\), and for any \(a \in S_+\), \(|a|\in \overline{S}_+^1\), it follows \(\tau(x) = \inf_{x \in \mathbb{N}} \frac{|ax|}{|x|} > 0\) is attained. The same argument is valid for the function \((a, x) \rightarrow |ax|\) on \(\overline{S}_+^1 \times K\), hence \(\inf_{x \in K} \tau(x) > 0\). \(\square\)

For \(s = 1\) explicit formulae for \(\kappa(1)\) and \(e^1\) can be obtained.

**Lemma 3.5.** Let \(m = \int a\mu(da)\). Then for some \(n \geq 1\), \(m^n \in \mathbb{N}\) and if \(\nu^s \in C_1^s\) is the dominant eigenvector of \(m^s\) we have: \(\kappa(1) = r(m), e^1(x) = \frac{(\nu^s, x)}{|\nu^s|_\infty}\).

**Proof.** Since \([\text{supp} \mu] \cap \mathbb{N} \neq \emptyset\), there exists \(n \in \mathbb{N}\) such that \(\mu^n\) is a barycenter with nonzero coefficients of \(\mu_0 \in M^1(\mathbb{N})\) and \(\mu_1 \in M^1(\mathbb{N})\):

\[
\mu^n = u\mu_0 + (1 - u)\mu_1, \quad u \in (0, 1).
\]

Then

\[
m^n = \int a\mu^n(da) = u \int a\mu_0(da) + (1 - u) \int a\mu_1.
\]

If \(x \in C\):

\[
m^n x = u \int ax\mu_0(da) + (1 - u) \int ax\mu_1(da).
\]

Clearly \(ax \in \mathbb{N}\) if \(a \in \mathbb{N}\), hence by convexity of \(\mathbb{N}\) and \(C\): \(\int ax\mu_0(da) \in \mathbb{N}, \int ax\mu_1(da) \in \mathbb{N}\). Since \(u > 0\), we get \(m^n x \in \mathbb{N}\). The existence and uniqueness of the dominant eigenvector \(\nu \in C_1^s\) for \(m^n\) follows. Then, since \(m\) and \(m^n\) commute, \(\nu\) is also the unique dominant eigenvector for \(m\):

\[
\int \nu(x)\mu(da) = r(m)\nu.
\]

Similar results are valid for \((m^n)^n\) and \(m^n\): \(m^n\) has a unique dominant eigenvector \(\nu^n\) which belongs to the interior of \(C_1^n\) and \(m^n\nu^n = r(m)\nu^n\). Since \(\nu^n\) is in the interior of \(C_1^n\), Lemma 3.4 gives a constant \(D\) such that for any \(a \in S_+: |a| = |a^n| \leq D|a^*\nu^n|\). The same argument as in the proof of the Lemma gives that, if \(x \in C^n_1\), there exists \(D' > 0\) such that for any \(v' \in C_1^s\): \(|v'| \leq D'(v', x)\). It follows \(|a| \leq DD'(a^\nu, x)\). Hence for any \(n \geq 1\),

\[
|E[A_n \ldots A_1]| \leq DD'(E[A^n]\nu^s, x) = DD'r^n(m)\langle v^s, x\rangle.
\]

In the limit \(\kappa(1) \leq r(m)\). On the other hand, for any \(n \geq 1\), \(|E[A^n]| \leq E|A_n \ldots A_1|\), hence in the limit \(r(m) \leq \kappa(1)\), and finally \(r(m) = \kappa(1)\).

Considering the continuous function \(v^s\) on \(C\) defined by \(v^s(x) = \langle v^s, x\rangle\), we have

\[
Pv^s(x) = \int \langle v^s, ax\rangle\mu(da) = \langle m^n\nu^s, x\rangle = r(m)\langle v^s, x\rangle,
\]
hence $Pv^* = r(m)v^*$. The uniqueness in Theorem 3.3 gives $e^s(x) = \frac{(\alpha^s(x))}{|\alpha^s(x)|}$.

In order to go further in the study of $\kappa(s)$ and $e^s$, we introduce new probability spaces. If $s > 0$, $x \in C_1$, $a \in S$ we write

$$q_n^s(x,a) = \frac{|ax|^s e^s(a \cdot x)}{\kappa^n(s)} e^s(x).$$

In particular if $n = 1$, $q_1^s(x,a) = q^s(x,a)$. If $\omega \in \Omega = S^\mathbb{N}$, we write $S_n(\omega) = a_n \ldots a_1$, $q_n^s(x,\omega) = q_n^s(x,S_n(\omega))$ and we observe that

$$\int q_n^s(x,\omega)\mu^\otimes n(d\omega) = 1.$$  

We denote by $\theta$ the shift on $\Omega$. Also the system of probability measures $q_n^s(x,\cdot)\mu^\otimes n$ is a projective system, hence we can define by Kolmogorov extension theorem its projective limit $Q^s_\omega$ on $\Omega$. We write $Q^s = \int Q^s_\omega\pi^s(dx)$ and we observe that $Q^s$ is shift invariant and ergodic since $\pi^s$ is the unique $Q^s$-stationary measure measure on $C_1$. We denote by $E^s_\omega$ the corresponding expectation symbol.

Then we have the following analogue of Theorem 3.11 in [GL2]:

**Theorem 3.7.** Assume that $\mu$ satisfies condition $C$, $s \in I_\mu$ and $|a|^s \log |a|$, $|a|^s \log \nu(a)$ are $\mu$-integrable. Then, for any $x \in C_1$, $Q^s_\omega$ a.e. and $Q^s_\omega$ a.e.

$$\alpha(s) = \lim_{n \to \infty} \frac{1}{n} \log |S_n(\omega)x| = \lim_{n \to \infty} \frac{1}{n} \log |S_n(\omega)|,$$

where

$$\alpha(s) = \int \log |ax|q^s(x,a)\pi^s(dx)\mu(da).$$

Furthermore the derivative of $\kappa$ exists and is continuous on $I_\mu$. The left derivative of log $\kappa(s)$ is finite and given by $\alpha(s) = \frac{\kappa'(s^-)}{\kappa(s)}$. In particular, if $\chi > 1$ and $\kappa(\chi) = \kappa(1)$, then $\kappa'(\chi^-) \in (0, \infty)$.

Here, in view of the cone situation, more elementary arguments than in [GL2] can be used, hence we adapt the proof of [GL2] as follows.

**Lemma 3.9.** There exists $d_1 > 0$, such that for any $x \in C_1$: $Q^s_\omega \leq d_1Q^s$.

**Proof.** From continuity and positivity of $e^s$ on $C_1$, there exists $d_2 > 0$ such that for any $x, y \in C_1$: $e^s(y) \leq d_2e^s(x)$. Hence, for any $a \in S^+$:

$$q_n^s(x,a) = \frac{1}{\kappa^n(s)} e^s(a \cdot x) |ax|^s \leq \frac{D}{\kappa^n(s)} |a|^s.$$

Since $\pi^s$ is proper, we have, with some $d_3 > 0$ and $d_1 > 0$

$$|a|^s \leq d_3 \int |ay|^s \pi^s(dy),$$

$$q_n^s(x,a) \leq \frac{d_2d_3}{\kappa^n(s)} \int |ay|^s \pi^s(dy) \leq d_1 \int q_n^s(y,a)\pi^s(dy).$$

Let $\psi$ be a nonnegative function on $\Omega$ depending of $n$ first coordinates. Then

$$Q^s_\omega(\psi) = \int \psi(\omega)q_n^s(x,S_n(\omega))\mu^\otimes n(d\omega)$$

$$\leq d_1 \int \int \psi(\omega)q_n^s(y,S_n(\omega))\pi^s(dy)\mu^\otimes n(d\omega)$$

$$= d_1 Q^s(\psi).$$
In view of the arbitrariness of \( n \) and \( \psi \) the conclusion follows.

\[ \square \]

**Lemma 3.10.** For any \( x \in C_1 \) we have \( Q^* \text{ a.e. } \inf_{n \in \mathbb{N}} \frac{|S_n(x)|}{|S_n(x)|} > 0. \)

Proof. We follow arguments of [K1, H]. Condition \( C \) implies existence of \( k \in \mathbb{N}^* \), and \( a_1, \ldots, a_k \in \text{supp} \mu \) such that their product \( a_k \cdots a_1 \) belongs to \( S^0 \). Since \( q_k(x, \omega) > 0 \) we have \( \mu^{\otimes k} \text{ - a.e., } Q_k^* (S_k(\omega) \in S^0) > 0. \) Then Lemma 3.9 implies \( Q^* (S_k(\omega) \in S^0) > 0. \) Let \( \Omega' = \{ \omega \in \Omega; S_k(\omega) \in S^0 \} \), \( T'(\omega) = \inf \{n \geq 1; \theta^n \omega \in \Omega' \} \), \( T(\omega) = \inf \{m \geq 0; S_m(\omega) \in S^0 \} \). Since \( S_{n+1}(\omega) = S_k(\theta^n \omega) S_n(\omega), \) we have \( T(\omega) \leq k + T'(\omega). \) Since \( Q^* (\Omega') > 0 \) and \( \theta \) is ergodic with respect to the invariant measure \( Q^* \), Birkhoff’s theorem implies \( T'(\omega) < \infty \text{ a.e.}, \) hence \( T(\omega) < \infty \text{ Q}^* \text{ - a.e.} \) If \( n \geq T \), we can write

\[ S_n(\omega) x = A_n \cdots A_{T-1} A_T \cdots A_1 x \]

hence, since \( A_T \cdots A_1 x \in C^0 \) and using Lemma 3.4,

\[ |S_n(\omega) x| \geq |A_n \cdots A_{T+1}| c(A_T \cdots A_1 x) \geq |S_n(\omega)| \frac{\tau(A_T \cdots A_1 x)}{|A_T \cdots A_1|}, \]

i.e.

\[ \frac{|S_n(\omega) x|}{|S_n(\omega)|} \geq \frac{\tau(A_T \cdots A_1 x)}{|A_T \cdots A_1|} > 0, \quad Q^* \text{ a.e.} \]

On the other hand

\[ \inf_{\theta^k \omega \in \Omega} \frac{|S_n(\omega) x|}{|S_n(\omega)|} > 0. \]

It follows \( Q^* \text{ a.e.}: \)

\[ \inf_{n \in \mathbb{N}} \frac{|S_n(\omega) x|}{|S_n(\omega)|} > 0. \]

\[ \square \]

**Proof of Theorem 3.7.** We consider the space \( \mathcal{O} = C_1 \times \Omega \) and the extended shift \( \hat{\theta} \):

\[ \hat{\theta}(x, \omega) = (a_1 \cdot x, \theta \omega). \]

Then the probability measure

\[ \hat{Q}^* = \int (\delta_x \otimes \delta \omega) Q_n^* (d\omega) \pi^* (dx) \]

is \( \hat{\theta} \text{ - invariant and ergodic since } \mathcal{O} \text{ can be identified with the space of paths of the Markov chain defined by } Q^* \text{ and using Theorem 3.3, } \pi^* \text{ is its unique stationary measure. We observe that } f(x, \omega) = \log |a_1(\omega) x| \text{ satisfies } \log \nu(\Delta) \leq f(x, \omega) \leq \log |a|, \text{ hence the } \mu \text{ - integrability of } f(x, \omega). \)

Then Birkhoff’s theorem gives \( Q^* \text{ - a.e.:} \)

\[ \lim_{n \to \infty} \frac{1}{n} \log |S_n(x)| = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{\infty} f \circ \hat{\theta}^k(x, \omega) = \hat{Q}^* (f) = \alpha(s). \]

On the other hand the subadditive ergodic theorem can be applied to the ergodic system \((\Omega, \theta, Q^*)\) and the sequence \( \log |S_n(\omega)| \):

\[ \lim_{n \to \infty} \frac{1}{n} \log |S_n(\omega)| = \alpha_s, \quad Q^* \text{ a.e.} \]

where the convergence is also valid in \( L^1(Q^*). \)

For arbitrary \( x \in C_1 \) and using Lemma 3.10

\[ \lim_{n \to \infty} \frac{1}{n} \log |S_n(x)| = \lim_{n \to \infty} \frac{1}{n} \log |S_n(\omega)| + \lim_{n \to \infty} \frac{1}{n} \log \frac{|S_n(\omega)|}{|S_n(\omega)|} = \alpha_s, \quad Q^* \text{ a.e.} \]
Lemma 3.9 gives $Q^n_x \leq d_1 Q^n_s$, hence the above convergence is also valid $Q^n_x$ a.e., hence $\hat Q^n$ a.e. Then the above convergences imply $\alpha_s = \alpha(s)$, i.e. the first part of Theorem 3.7.

In order to prove the second part we show

$$\log \kappa(s) = \int_0^s \alpha(t) dt.$$ 

We consider

$$v_n(s) = \frac{1}{n} \log \left( \int |ax| \mu^n(da) \right).$$

We observe that

$$\frac{1}{\kappa^n(s)} \int |ax| \mu^n(da) = e^s(x) (Q^n s(1/e^s)(x)$$

is bounded from below and from above. Hence $\log \kappa(s) = \lim_{n \to \infty} v_n(s)$. Furthermore, Theorem 3.3 gives the uniform convergence of $(Q^n s) \phi$ to $\pi^n(\phi)$, if $\phi \in C(C_1)$. Hence

$$\lim_{n \to \infty} \frac{1}{\kappa^n(s)} \int |ax| \mu^n(da) = e^s(x) \pi^n(1/e^s).$$

We write

$$v'_n(s) = \frac{1}{n \kappa^n(s)} \int |ax| \log |ax| \mu^n(da),$$

and we observe that

$$\frac{1}{\kappa^n(s)} \int |ax| \log |ax| \mu^n(da) = e^s(x) E^n_s \left[ \frac{1}{e^s(S^n \cdot x) \log |S^n x|} \right].$$

From the $L^1(Q^n_x)$ convergence above we have

$$\lim_{n \to \infty} \int \left| \frac{1}{n} \log |S^n (\omega) x| - \alpha(s) \right| Q^n_x (d\omega) = 0.$$ 

Applying again Theorem 3.3 to $\phi \in C(C_1)$:

$$\lim_{n \to \infty} \frac{1}{n} E^n_s \left[ \phi(S^n \cdot x) \log |S^n x| \right] = \alpha(s) \pi^n(\phi),$$

hence taking $\phi = \frac{1}{e^s}$:

$$\lim_{n \to \infty} \frac{1}{n \kappa^n(s)} \int |ax| \log |ax| \mu^n(da) = e^s(x) \alpha(s) \pi^n(1/e^s).$$

Therefore, from the expression of $v'_n(s)$

$$\lim_{n \to \infty} v'_n(s) = \alpha(s).$$

Clearly $v_n(t)$ is convex and has a continuous derivative on $[0, s]$, hence $v'_n(0) \leq v'_n(t) \leq v'_n(s)$ and $v_n(s) = \int_0^s v'_n(t) dt$. By dominated convergence we conclude

$$\log \kappa(s) = \lim_{n \to \infty} v_n(s) = \int_0^s \alpha(t) dt.$$ 

The expression of $\alpha(s)$, the continuity in $s$ of $q^s(x, a)$, $\pi^s$ and the inequality $\log \bar{\alpha} \leq \log |ax| \leq \log |a|$ allow to conclude that the left derivative of $log \kappa(s)$ is equal to $\alpha(s)$. Since $\alpha(s)$ is continuous on $I_{\mu}$, we get also that $\kappa(s)$ has a continuous derivative on $I_{\mu}$, and $\kappa'(s) = \alpha(s)$, if $s \in [0, s_{\infty})$. The convexity of $\log \kappa(s)$ gives for $\kappa(s) = 1, \kappa'(\alpha^-) > 0$. \[\square\]
4. MANDELBROT’S CASCADES - PROOF OF THEOREM 2.2

Proof of Theorem 2.2. Equivalence of conditions (1), (2) and (3) follows easily from the construction of Mandelbrot’s cascade. For the remaining part of the proof we will apply arguments of Kahane and Peyrière [KP] to our settings. First we will prove that (4) implies (1). Assume $\kappa'(1^-) < 0$. We will use the following inequality

$$
\left( \sum_{i=1}^k y_j \right)^h \geq \sum_{i=1}^k y_j^h - 2(1-h) \sum_{i<j}(y_iy_j)^{\frac{h}{2}}
$$

which is valid for $y_i > 0$ and $h \in (1-\varepsilon, 1]$ for some small $\varepsilon$ independent on $k$ (see [KP], Lemma C).

Then, for $x_i \in C$, $i = 1, \ldots, k$ we obtain

$$
\tilde{e}^h\left( \sum_{i=1}^k x_i \right) = \int_{C_1} \left\langle \sum_{i=1}^k x_i, u \right\rangle^h \nu_u^h(du)
$$

$$
\geq \sum_{i=1}^k \tilde{e}^h(x_i) - 2(1-h) \sum_{i<j} \int_{C_1^*} \left\langle x_i, u \right\rangle^h \left\langle x_j, u \right\rangle^{\frac{h}{2}} \nu_u^h(du)
$$

$$
\geq \sum_{i=1}^k \tilde{e}^h(x_i) - 2(1-h) \sum_{i<j} \left[ \int_{C_1^*} \left\langle x_i, u \right\rangle^h \nu_u^h(du) \right] \left[ \int_{C_1^*} \left\langle x_j, u \right\rangle^h \nu_u^h(du) \right]^{\frac{h}{2}}
$$

$$
= \sum_{i=1}^k \tilde{e}^h(x_i) - 2(1-h) \sum_{i<j} \tilde{e}^h(x_i)^{\frac{h}{2}} \tilde{e}^h(x_j)^{\frac{h}{2}}.
$$

Hence, for $Y_n$ and $Y_{n-1}^i$ defined in the Introduction, we have

$$
\tilde{e}^h(Y_n) = \tilde{e}^h\left( \sum_{i=1}^N A_i Y_{n-1}^i \right) \geq \sum_{i=1}^N \tilde{e}^h(A_i Y_{n-1}^i) - 2(1-h) \sum_{i<j} \tilde{e}^h(A_i Y_{n-1}^i)^{\frac{h}{2}} \tilde{e}^h(A_j Y_{n-1}^j)^{\frac{h}{2}}.
$$

Taking expected value of both sides, we obtain

$$
\mathbb{E}[\tilde{e}^h(Y_n)] = \mathbb{E}\left[ \mathbb{E}[\tilde{e}^h(Y_n) \mid N] \right]
$$

$$
\geq \mathbb{E}\left[ N\kappa(h)\mathbb{E}[\tilde{e}^h(Y_{n-1})] - N(N-1)(1-h)\left( \mathbb{E}[\tilde{e}^h(A_1 Y_{n-1}^1)^{\frac{h}{2}}] \right)^2 \mid N \right]
$$

$$
= \mathbb{E}[N\kappa(h)\mathbb{E}[\tilde{e}^h(Y_{n-1})] - (1-h)\mathbb{E}[N(N-1)]\left( \mathbb{E}[\tilde{e}^h(A_1 Y_{n-1}^1)^{\frac{h}{2}}] \right)^2
$$

Notice that for every $y \in C_1^*$, $\langle Y_n, y \rangle$ is a martingale, so $\langle Y_n, y \rangle^h$ is a supermartingale hence $\mathbb{E}[\langle Y_n, y \rangle^h] \leq \mathbb{E}[\langle Y_{n-1}, y \rangle^h]$. Integrating both sides with respect to $\nu_u^h(dy)$ we obtain $\mathbb{E}[\tilde{e}^h(Y_n)] \leq \mathbb{E}[\tilde{e}^h(Y_{n-1})]$. Therefore

$$
\left( \mathbb{E}[\tilde{e}^h(A_1 Y_{n-1}^1)^{\frac{h}{2}}] \right)^2 \geq \frac{(\mathbb{E}[N\kappa(h) - 1])\mathbb{E}[\tilde{e}^h(Y_{n-1})]}{\mathbb{E}[N(N-1)(1-h)]}
$$
and going with \( h \) to the left limit at 1, we obtain

\[
\left( \mathbb{E} \left[ \left( \int_{C_1} (A_0 Y_{n-1}, u) \nu^*_n(du) \right)^{\frac{1}{2}} \right] \right)^2 = \left( \mathbb{E} \left[ e^{1/2}(A_0 Y_{n-1}) \right] \right)^2 \\
\geq - \frac{\kappa'(1\cdot)}{\mathbb{E}[N(N-1)]} \mathbb{E}[e^{1/2}(Y_{n-1})] \\
= - \frac{\kappa'(1\cdot)}{\mathbb{E}[N(N-1)]} \int_{C_1} (Y_{n-1}, u) \nu^*_n(du) \\
= - \frac{\kappa'(1\cdot)}{\mathbb{E}[N(N-1)]} \int_{C_1} (v, u) \nu^*_n(du) = |v|^2 \cdot D > 0.
\]

Define now \( W_n = \int_{C_1} (A_0 Y_n, u) \nu^*_n(du) \). Then \( W_n \) is a positive martingale and it converges pointwise to \( W = \int_{C_1} (A_0 Y, u) \nu^*_1(du) \). Hence the sequence \( (W_n)^{\frac{1}{2}} \) converges pointwise to \( (W)^{\frac{1}{2}} \) and we will prove that the convergence holds also in the norm. For this purpose it is enough to observe that the family of random variables \( \{ (W_n)^{\frac{1}{2}} \} \) is uniformly integrable. Indeed since for any positive \( x \)

\[
x \mathbb{P}[W_n > x] \leq \mathbb{E}[W_n] = \frac{1}{\mathbb{E}[N]} \int_{C_1} (v, u) \nu^*_n(du) = D_2
\]

and

\[
\mathbb{E}[W_n] = P^1 e^1(v) = \frac{e^1(v)}{\mathbb{E}[N]} = D_3
\]

we have

\[
\lim_{x \to \infty} \sup_n \mathbb{E}\left[ (W_n)^{\frac{1}{2}} 1_{W_n > x} \right] \leq \lim_{x \to \infty} \sup_n \left( \mathbb{E}[W_n] \right)^{\frac{1}{2}} \mathbb{P}[W_n > x]^{\frac{1}{2}} \leq \lim_{x \to \infty} \frac{\sqrt{D_3}}{\sqrt{x}} \cdot D_2 = 0.
\]

Therefore, since \( W_n^{\frac{1}{2}} \) is a supermartingale bounded in \( L^2 \),

\[
\mathbb{E}\left[ (W)^{\frac{1}{2}} \right] = \lim_{n \to \infty} \mathbb{E}\left[ (W_n)^{\frac{1}{2}} \right] \geq \sqrt{D}
\]

and so, the random variable \( Y \) cannot be degenerate.

Finally we prove (3) implies (4). We proceed as in [KP]. Suppose now \( Z \) is a fixed point of (1.1).

We will use the following inequalities

\[
(x + y)^h \leq x^h + hy^h, \quad \text{for} \quad x \geq y > 0, 0 < h < 1
\]

and

\[
\mathbb{E}[X^h 1_{X \geq x}] \geq \varepsilon \mathbb{E}[X^h],
\]

for real valued independent random variables \( X, X' \), \( 0 < h < 1 \) and some \( \varepsilon > 0 \) (see Lemmas A and B in [KP])

Let \( \{ Z_i \}_{i \in \mathbb{N}} \) be a sequence of independent copies of \( Z \). We have

\[
\tilde{e}^h(A_1 Z_1 + A_2 Z_2) \leq \int_{C_1} 1_{\{ (A_1 Z_1, u) \leq (A_2 Z_2, u) \}} (h(A_1 Z_1, u)^h + (A_2 Z_2, u)^h) \nu^*_n(du) \\
+ \int_{C_1} 1_{\{ (A_1 Z_1, u) > (A_2 Z_2, u) \}} ((A_1 Z_1, u)^h + h(A_2 Z_2, u)^h) \nu^*_n(du).
\]
For \( h \leq 1 \) the function \( \tilde{e}^h \) is subadditive hence
\[
\mathbb{E}[\tilde{e}^h(Z)] = \mathbb{E} \left[ \mathbb{E} \left[ \tilde{e}^h \left( \sum_{i=1}^{N} A_i Z_i \right) \right] \right] \\
\leq \mathbb{E} \left[ \sum_{i=3}^{N} \mathbb{E} \left[ \tilde{e}^h(A_i Z_i) + \mathbb{E} \left[ \tilde{e}^h(A_1 Z_1 + A_2 Z_2) \right] \right] \right] \\
\leq \mathbb{E}\[N\] \mathbb{E}[\tilde{e}^h(A_1 Z_1)] - 2(1-h)\mathbb{E} \left[ \int_{C_1} 1_{\{(A_1 Z_1, u) \leq (A_2 Z_2, v)\}} (A_1 Z_1, u)^h \nu^h(du) \right] \\
\leq \mathbb{E}\[N\] \kappa(h) \mathbb{E}[\tilde{e}^h(Z)] - 2(1-h) \varepsilon \kappa(h) \mathbb{E}[\tilde{e}^h(Z)].
\]
Hence
\[
2\varepsilon \kappa(h) \mathbb{E}[\tilde{e}^h(Z)] \leq \frac{\mathbb{E}[N]\kappa(h) - 1}{1-h} \cdot \mathbb{E}[\tilde{e}^h(Z)]
\]
and passing with \( h \) to 1 from below
\[
\frac{2\varepsilon \mathbb{E}[\tilde{e}^1(Z)]}{\mathbb{E}[N]} \leq -\mathbb{E}[N] \kappa'(1^-) \mathbb{E}[\tilde{e}^1(Z)].
\]
Then, since \( \mathbb{E}[\tilde{e}^1(Z)] \) is nonzero
\[
\kappa'(1^-) \leq -\frac{2\varepsilon}{\mathbb{E}[N]^2}.
\]
\[\square\]

**Lemma 4.1.** \( \mathbb{E}|Z|^h < \infty \) if and only if \( \kappa(h) < \frac{1}{\mathbb{E}[N]} \).

**Proof.** As above be denote by \( \{Z_i\}_{i \in \mathbb{N}} \) a sequence of independent copies of \( Z \). If \( \mathbb{E}|Z|^h < \infty \) then also \( \mathbb{E}[\tilde{e}^h(Z)] < \infty \), thus
\[
\mathbb{E}[\tilde{e}^h(Z)] = \mathbb{E} \left[ \int_{C_1} \left\langle \sum_{i=1}^{N} A_i Z_i, u \right\rangle^h \nu^h(du) \right] \\
\geq \mathbb{E} \left[ \sum_{i=1}^{N} \mathbb{E}[\tilde{e}^h(A_i Z_i)] \right] = \mathbb{E}[N] \kappa(h) \mathbb{E}[\tilde{e}^h(Z)]
\]
and since \( \mathbb{E}[\tilde{e}^h(Z)] \neq 0 \), we deduce \( \kappa(h) < \frac{1}{\mathbb{E}[N]} \).

We omit the converse implication since the argument is exactly the same as in [KP], p. 137. \[\square\]

5. The Renewal Equation and Proof of Theorem 2.3

5.1. **Kesten’s renewal theorem.** The main tool we will use to prove Theorem 2.3 is Kesten’s renewal theorem [K2] and here we are going to check that its assumptions are satisfied in our settings. For reader’s convenience we state here all the details in the case the state space \( S \) in [K2] is the compact subset \( C_1 \) of \( S^{d-1} \) endowed with the metric \( d(x, y) = |x - y| \). First we introduce some definitions.

Fix \( x \in C_1 \) and define \( X_0(\omega) = x \), \( X_n(\omega) = a_n(\omega) \cdot X_{n-1}(\omega) = S_n(\omega) \cdot x \) and \( V_n(\omega) = \log |S_n(\omega)x| = \sum_{i=1}^{n} U_i(\omega) \), for \( U_i(\omega) = \log |a_n(\omega)X_{n-1}(\omega)| \). Let \( F(dt|x, y) \) be the conditional law of \( U_1 \), given \( X_0 = x \), \( X_1 = y \), i.e.
\[
Q_x \left[ X_1 \in A, U_1 \in B \right] = \int_A \int_B F(dt|x, y)Q(x, dy).
\]
Next we define a family of subsets of $C_1$

$$D_k = \left\{ x \in C_1 : \mathbb{Q}_x \left[ V_m \geq \frac{m}{k}, \forall m \geq k \right] \geq \frac{1}{2} \right\}.$$  

Of course $D_k$ is an increasing family. We put $D_0 = \emptyset$. We say that a function $g : C_1 \times \mathbb{R} \to \mathbb{R}$ is directly Riemann integrable (dRi) if it is $\mathcal{B}(C_1) \times \mathcal{B}(\mathbb{R})$ measurable and satisfies

$$\sum_{k=0}^{\infty} \sum_{t=0}^{\infty} (k+1) \sup \left\{ |g(x,t)| : x \in D_{k+1} \setminus D_k, l \leq t \leq l+1 \right\} < \infty$$  

and if for every fixed $x \in C_1$ and the function $t \mapsto g(x,t)$ is Riemann integrable on $[-L,L]$, for $0 < L < \infty$.

**Theorem 5.2 ([K2]).** Assume the following conditions are satisfied:

- **Condition I.1** There exists $\pi \in M^1(C_1)$ such that $\pi Q = \pi$ and for every open set $O$ with $\pi(O) > 0$, $\mathbb{Q}_x [X_n \in O$ for some $n] = 1$ for every $x \in C_1$.
- **Condition I.2**

  $$\int |t| F(dt|x,y) Q(x,dy) \pi(dx) < \infty$$

  and for all $x \in C_1$, $\mathbb{Q}_x$ a.e.:

  $$\lim_{n \to \infty} \frac{V_n}{n} = \alpha = \int t F(dt|x,y) Q(x,dy) \pi(dx) > 0.$$

- **Condition I.3** There exists a sequence $\{\zeta_i\} \subset \mathbb{R}$ such that the group generated by $\zeta_i$ is dense in $\mathbb{R}$ and such that for each $\zeta_i$ and $\lambda > 0$ there exists $y = y(i,\lambda) \in C_1$ with the following property: for each $\varepsilon > 0$ there exists $A \in \mathcal{B}(C_1)$ with $\pi(A) > 0$ and $m_1, m_2 \in \mathbb{N}, \tau \in \mathbb{R}$ such that for any $x \in A$:

  $$\mathbb{Q}_x \left\{ d(X_{m_1}, y) < \varepsilon, |V_{m_1} - \tau| \leq \lambda \right\} > 0$$

  $$\mathbb{Q}_x \left\{ d(X_{m_2}, y) < \varepsilon, |V_{m_2} - \tau - \zeta_i| \leq \lambda \right\} > 0$$

- **Condition I.4** For each fixed $x \in C_1$, $\varepsilon > 0$ there exists $r_0 = r_0(x,\varepsilon)$ such that all real valued $f \in \mathcal{B}(C_1 \times \mathbb{R})$ and for all $y$ with $d(x,y) < r_0$ one has:

  $$\mathbb{E}_x f(X_0, V_0, X_1, V_1, \ldots) \leq \mathbb{E}_x f^\varepsilon(X_0, V_0, X_1, V_1, \ldots) + \varepsilon |f|_{\infty},$$

  $$\mathbb{E}_x f(X_0, V_0, X_1, V_1, \ldots) \leq \mathbb{E}_y f^\varepsilon(X_0, V_0, X_1, V_1, \ldots) + \varepsilon |f|_{\infty},$$

  where

  $$f^\varepsilon(x_0, v_0, x_1, v_1, \ldots) = \sup \{ f^\varepsilon(y_0, u_0, y_1, u_1, \ldots) : |x_i - y_i| + |v_i - u_i| < \varepsilon \text{ if } i \in \mathbb{N} \}$$

If a function $g : C_1 \times \mathbb{R} \to \mathbb{R}$ is jointly continuous and directly Riemann integrable, then for every $x \in C_1$

$$\lim_{t \to \infty} \mathbb{E}_x \left[ \sum_{n=0}^{\infty} g(X_n, t - V_n) \right] = \frac{1}{\alpha} \int_{C_1} \pi_s(dy) \int_{\mathbb{R}} g(y, s) ds.$$  

From now we will study the measures $\mathbb{Q}_x^\chi$ for $\chi > 1$ defined in the statement of Theorem 2.3, i.e. such that $\kappa(\chi) = \frac{1}{\chi}$. They will play the role of $\mathbb{Q}_x$ in the Kesten renewal theorem. Let $\alpha = \alpha(\chi)$ (see (3.8) for the definition of $\alpha(\chi)$). The existence of $\chi$ is assumed in Theorem 2.3.

**Proposition 5.5.** Under hypotheses of Theorem (2.3), conditions I.1 - I.4 are satisfied by the measures $\mathbb{Q}_x^\chi$. 
Moreover the statement above remains valid if we replace a small and
\[ \lambda \]
fix where \( \Gamma = \text{supp} \)

The second part follows immediately from Theorem 3.7.

By definition, \( x \)

Therefore if \( A \) take \( g \) for appropriately chosen function \( 5.2. \) Proof of Theorem 2.3.

\[ (K1), p. 217-218). \]

Lemma 5.6. \[ \text{If } h \in C_b(C_1 \times \mathbb{R}) \text{ satisfies} \]

\[ \sum_{l=-\infty}^{\infty} \sup \left\{ |h(x,t)| : x \in C_1, t \in [l,l+1] \right\} < \infty, \]

then \( h \) is directly Riemann integrable (with respect to the measure \( Q^x \)).
We will show that the number \( k \) that implies that the set \( \{w_i\} \) is indeed finite and (5.1) implies (5.1).

Take \( \delta = \frac{1}{2r+1} \), where \( d_1 \) is as in Lemma 3.9. Define a random variable \( Z(x) = \inf_{n \in \mathbb{N}} \frac{|S_n(x)|}{|S_n|} \). In view of Lemma 3.10, \( Z(x) \) is strictly positive, \( \mathbb{Q} \) a.e. Let \( \{w_i\}_{i \in \mathbb{N}} \) be a dense countable subset of \( C_1 \). Then, for every \( w_i \) there exists \( \varepsilon(w_i) > 0 \) such that
\[
\mathbb{Q}^\chi[Z(w_i) \leq \varepsilon(w_i)] 
\leq \frac{\delta}{2r+1}.
\]
By compactness of \( C_1 \) there exists a finite subset \( \{x_1, \ldots, x_K\} \) of the sequence \( \{w_i\} \) such that the balls \( B(x_i, \varepsilon(x_i)/2) \) cover \( C_1 \) and moreover
\[
\mathbb{Q}^\chi[Z(x_i) \leq \varepsilon(x_i), \text{ for some } i \leq K] < \frac{\delta}{2}.
\]
Since, by Theorem 3.7, \( \lim_{n \to \infty} \frac{1}{n} \log |S_n(\omega)x_i| = \alpha > 0 \), \( \mathbb{Q}^\chi \) a.s. for every \( i \), there exists \( k \) such that the set
\[
\Omega_1 = \left\{ \omega : \frac{\log |S_n(\omega)x_i|}{n} > \frac{2}{k} \text{ and } Z(x_i) > \varepsilon(x_i) \text{ for } n \geq k, 1 \leq i \leq K \right\}
\]
satisfies
\[
\mathbb{Q}^\chi(\Omega_1) > 1 - \delta.
\]
We will show that the number \( k \) is exactly the index we are looking for.

Now take arbitrary \( y \in C_1 \) and \( x_i \) such that \( y \in B(x_i, \frac{\varepsilon(x_i)}{2}) \). Notice that
\[
\left| \frac{|S_n(\omega)y|}{|S_n(\omega)x_i|} - 1 \right| \leq \frac{\varepsilon(x_i)}{2} \frac{|S_n(\omega)|}{|S_n(\omega)x_i|} \leq \frac{1}{2} \text{ for } \omega \in \Omega_1, n \geq k.
\]
Notice \( |\log x| < 2|x-1| \) for \( x \in (\frac{1}{2}, \frac{3}{2}) \), hence
\[
\log \frac{|S_n(\omega)y|}{|S_n(\omega)x_i|} \geq -1 \text{ for } \omega \in \Omega_1, n \geq k,
\]
that implies
\[
\frac{|S_n(\omega)y|}{n} \geq \frac{|S_n(\omega)x_i|}{n} - \frac{1}{n} > \frac{1}{k} \text{ for } \omega \in \Omega_1, n \geq k.
\]
Therefore \( \mathbb{Q}^\chi(\Omega_2) > 1 - \delta \) for
\[
\Omega_2 = \left\{ \omega : \frac{\log |S_n(\omega)y|}{n} > \frac{1}{k} \text{ for } n > k \right\}
\]
and finally, by Lemma 3.9
\[
\mathbb{Q}^\chi_\delta(\Omega_2) = 1 - \mathbb{Q}^\chi_\delta(\Omega_2^c) \geq 1 - d_1 \mathbb{Q}^\chi(\Omega_2^c) \geq 1 - d_1 \delta = \frac{1}{2},
\]
thus \( y \in D_k \).

To prove Theorem 2.3 we need to consider the action of matrices on the dual cone \( C^* \). Therefore we are led to define on \( \Omega \) measures \( \mathbb{Q}^{\chi,\beta}_\delta \), using the function \( \phi^{\chi}_\delta \) instead of the function \( \phi^\chi \), and the corresponding expectation symbol \( E^{\beta,\chi}_\delta \). Since the condition \( \mathcal{C} \) is valid for \( \sup\mu^\ast \) and \( \mathcal{C} \) all the results proved up to now concerning measures \( \mathbb{Q}^\chi \) hold also for \( \mathbb{Q}^{\chi,\beta}_\delta \).

Fix \( u \in C^*_\chi \). Let \( X \) be a solution of (1.1). We denote by \( X_1, \ldots, X_N \) independent copies of \( X \) and by \( A_1, \ldots, A_N \) independent random variables distributed according to \( \mu \). Recall that we assume \( N \) to be a constant. We define two probability measures on \( \mathbb{R}^+ \), \( \nu_n \) - the law of \( \langle X, u \rangle \) and \( \beta_n \) - the law of \( \sum_{i=2}^N \langle A_i, X_i, u \rangle \). For \( h(t) = t \), let \( \eta_n = h\nu_n \) be a positive measure on \( \mathbb{R}^+ \). Denote \( \phi(u) = \eta_n(\mathbb{R}^+) = \nu_n(h) \) for \( u \in C^*_\chi \). By Theorem 2.2 the function \( \phi \) is well defined. One can prove
that $\phi$ is an eigenfunction of $P^1$, thus $\phi = D e^*_x$ for some positive constant $D$, but this fact will not play any role in our considerations.

Define $f(u, r) = \eta_u(r, \infty)$. Our aim is to prove that the limit $\lim_{r \to \infty} r^{\chi-1} f(u, r)$ exists. Indeed, we will prove a stronger result. Namely, let

$$G(u, t) = \frac{e^{t(\chi-1)}}{e^*_x(u)} f(u, e^t),$$

be a function on $C^*_T \times \mathbb{R}$. Then our aim is to show that for every $u \in C^*_T$ the limit

$$\lim_{t \to \infty} G(u, t)$$

exists, is strictly positive and does not depend on $u$.

First we prove that $G$ is a potential of some function $g$.

**Lemma 5.8.** We have

$$G(u, t) = \sum_{n=0}^{\infty} E^{\chi, \ast}_u \{ g(X_n, t - V_n) \},$$

for

$$g(u, t) = \frac{N}{e^*_x(u)} \int_{\mathbb{R}_+^T} e^{t(\chi-1)} \mathbb{E} \left[ 1_{(e^t - u, e^t)} (\langle AX, u \rangle) (AX, u) \right] \beta_u(dy).$$

In particular the function $g$ is positive.

**Proof.** On the set of measurable function on $C^*_T \times \mathbb{R}$ we define the Markov operator $\Theta$ by

$$\Theta h(u, t) = E^{\chi, \ast}_u \{ h(X_1, t - V_1) \}$$

and let

$$g(u, t) = G(u, t) - \Theta G(u, t).$$

We will prove below that $g$ satisfies (5.10). Notice first that

$$\Theta G(u, t) = E^{\chi, \ast}_u \left[ G(A \cdot u, t - \log |Au|) \right]$$

$$= E \left[ \frac{1}{e^*_x(A^* \cdot u)} \frac{e^{t(\chi-1)} f(A^* \cdot u, e^t)}{e^*_x(A^* \cdot u)} \frac{1}{e^*_x(A^* \cdot u)} \frac{e^*_x(A^* \cdot u)}{e^*_x(u)} \right]$$

$$= \frac{N e^{t(\chi-1)}}{e^*_x(u)} E \left[ 1_{((e^t - u, e^t), \infty]} (\langle X, A^* \cdot u \rangle) (X, A^* \cdot u) \right]$$

$$= \frac{N e^{t(\chi-1)}}{e^*_x(u)} E \left[ 1_{(e^t, \infty]} (\langle AX, u \rangle) (AX, u) \right].$$

Therefore

$$G(u, t) = \frac{1}{e^*_x(u)} e^{t(\chi-1)} f(u, e^t)$$

$$= \frac{1}{e^*_x(u)} e^{t(\chi-1)} E \left[ 1_{(e^t, \infty]} (\langle X, u \rangle) (X, u) \right]$$

$$= \frac{N}{e^*_x(u)} \int_{\mathbb{R}_+^T} e^{t(\chi-1)} E \left[ 1_{(e^t, \infty]} (\sum_{i=1}^{N} A_i X_i, u) \right] \beta_u(dy)$$

$$= \Theta G(u, t) + g(u, t).$$
Combining (5.11) with (5.12) and (5.13) we obtain (5.10). Next iterating the equation (5.11) we obtain

\[ G(u, t) = \Theta^n G(u, t) + g(u, t) + \Theta g(u, t) + \cdots + \Theta^{n-1}(g)(u, t). \]

Therefore to prove (5.9) it is enough to show that \( \Theta^n G \) converges to 0 as \( n \) goes to \( \infty \). Notice

\[
\Theta^n G(u, t) = \mathbb{E}^*_u \left[ G(X_n, t - V_n) \right] = \mathbb{E}^*_u \left[ G(X_n, t - V_n) \frac{e^z(S_n \cdot u)}{\kappa_n(\chi)} \right]
\]

Combining (5.11) with (5.12) and (5.13) we obtain (5.10). Next iterating the equation (5.11) we obtain

\[
(5.14)
\]

\[
\Theta^n G(u, t) = \mathbb{E}^*_u \left[ G(X_n, t - V_n) \frac{e^z(S_n \cdot u)}{\kappa_n(\chi)} \right]
\]

\[
= \frac{N^n}{e^z(u)} \mathbb{E}^*_u \left[ e^{t(x-1)} \left( (X, X_n) \right) e^z(S_n \cdot u) \right]
\]

\[
= \frac{N^n e^{t(x-1)}}{e^z(u)} \mathbb{E}^*_u \left[ 1_{(e^t, \infty)}(S_n X, u) \right]
\]

Let us estimate the expected value. Choose a positive \( p \) satisfying \( \max \{ 1, \chi - 1/2 \} < p < \chi \). Then for \( \varepsilon < \frac{1}{N} \), by independence of \( S_n \) and \( X \) we have

\[
\mathbb{E}_u \left[ 1_{(e^t, \infty)}(S_n X, u) \right] \leq \sum_{k=0}^{\infty} \mathbb{P} \left[ e^{(k+1)\varepsilon} \leq (S_n X, u) \leq e^{2k+1} \right] e^{(k+1)\varepsilon}
\]

\[
\leq \sum_{k=0}^{\infty} \mathbb{P} \left[ (S_n X, u) \geq e^{2k} \right] e^{2k+1}
\]

\[
\leq \sum_{k=0}^{\infty} e^{2k+1} \mathbb{E} \left[ |S_n|^p \right] \mathbb{E} \left[ |X|^p \right]
\]

\[
\leq D e^{(1-p)t} \left( \frac{1}{N} - \varepsilon \right)^n \mathbb{E} \left[ |X|^p \right],
\]

(we use here the definition of \( \kappa \) and Lemma 4.1). Therefore for fixed \( t \) and \( u \)

\[
\lim_{n \to \infty} \Theta^n G(u, t) \leq \lim_{n \to \infty} DN^n \left( \frac{1}{N} - \varepsilon \right)^n e^{(1-p)t} \mathbb{E} \left[ |X|^p \right]
\]

and letting \( n \) go to infinity in (5.14) we prove the Lemma. \( \square \)

Now we would like to apply the renewal theorem (Theorem 5.2), however we do not know whether the function \( g \) is jointly continuous and directly Riemann integrable. To overcome this difficulty we proceed as Goldie [Go], i.e. given a function \( h \) on \( C^*_t \times \mathbb{R} \) we define the smoothing operator

\[
\tilde{h}(u, t) = \int_{-\infty}^{t} e^{s-t} h(u, s) ds.
\]

From now, instead of \( g \) and \( G \) we will consider their smoothed versions \( \tilde{g} \) and \( \tilde{G} \). It follows immediately from (5.9) that \( G \) is the potential of \( \tilde{g} \):

\[
\tilde{G}(u, t) = \sum_{n=0}^{\infty} \mathbb{E}^*_u \tilde{g}(X_n, t - V_n).
\]

But now it turns out that \( \tilde{g} \) is both jointly continuous and directly Riemann integrable. Therefore in view of Kesten’s renewal theorem we are able to describe behavior of \( \tilde{G} \) at infinity.

Continuity of \( \tilde{g} \) is a consequence of the last hypothesis of Theorem 2.3:

**Lemma 5.16.** Under hypotheses of Theorem 2.3 the function \( \tilde{g} \) is jointly continuous.
Proof. Since for every positive \( r \) and \( u \in C^*_b \), \( \mathbb{P}[(X,u) = r] = 0 \), it follows that the function \( u \mapsto \mathbb{E}[1_{(r,\infty)}((X,u))(X,u)] \) is continuous. Then continuity of \( \tilde{g} \) follows immediately from the definition of \( G \) and (5.11) but with \( g,G \) replaced by \( \tilde{g},\tilde{G} \). \( \square \)

Now we prove that assumptions of Theorem 2.3 can be weaken.

**Lemma 5.17.** If the support of the measure \( \mu \) consists of invertible matrices and condition \( C \) is satisfied, then for all positive \( r \) and any \( u \in C^*_b \), \( \mathbb{P}[(X,u) = r] = 0 \).

**Proof.** Let \( \nu \) be a fixed point of \( T \). We consider an affine recursion associated with \( (T,\nu) \) and we use the same argument as in [BDG], Lemma 2.5, with a few complements. Let \( Z, Z_i \) (\( 1 \leq i \leq N \)) be i.i.d random variables with law \( \nu \), \( B = \sum_{i=2}^{N} A_i Z_i \). Let us write the fixed point equation (1.1) as \( Z = d A Z_1 + B \). Denote by \( \eta \) the law of \( B \) and let us consider the probability measure \( p = \mu \otimes \eta \) on the affine group \( H = GL(V) \times V \).

We first show that the action of the support of \( p \) on \( V \) has no fixed points. Otherwise, for some \( x \in V \) and \( p \) a.e., \( (a,b) \in H \) we have \( x = ax + b \). Hence \( \mu \) a.e., for some fixed \( y, ax = y \) and \( b = x - y \). In other words \( \eta \) is the unit Dirac mass at \( x = y \). This gives that the law of \( A_1 Z_1 \) is the Dirac unit mass at \( \frac{a-b}{a} \), hence \( \nu = \delta_x \) with \( \text{supp} \mu z = \frac{a-b}{a} \). By definition of \( \nu \) we have \( \mu \otimes \eta \) a.e.: \( z = (a_1 + \cdots + a_N)z \), in particular for each \( a \in \text{supp} \mu \), \( N a z = z \). Then, if \( z \neq 0 \), the line \( W \) generated by \( z \) is a nontrivial \( \text{supp} \mu - \text{invariant subspace with} \ W \cap C \neq \emptyset \), which contradicts condition \( C \), since \( d > 1 \). Since \( \text{supp} \nu \subset C_+ \), \( \nu = \delta_0 \) is excluded.

Let \( W \) be the set of affine subspaces with positive \( \mu \)-mass and minimal dimension. If \( W,W' \in W \) and \( W \neq W' \), then \( \dim(W \cap W') < \dim W \), hence \( \nu(W \cap W') = 0 \). Since \( \sum_{W \in W} \nu(W) \leq 1 \), it follows that \( \text{sup} \\{ \nu(W): W \in W \} = \nu(W_0) \) for some \( W_0 \in W \) and the set \( W_0 \) of such \( W_0 \)'s is finite. From the fixed point equation, we have if \( W_0 \in W_0, \nu(W_0) = \int (h \nu)(W_0) dp(h) \). Since \( h \) is invertible we have \( \dim h^{-1}(W_0) = \dim W_0 \), hence \( \nu(h^{-1}W_0) \leq \nu(W_0) \). Then the equation above gives \( h^{-1}W_0 \cap W_0 \) hence the finite set \( W_0 \) is invariant under the action of the subgroup of \( H \) generated by \( \text{supp} \mu \).

Let us show that \( \bigcap_{W_0 \in W_0} W_0 \neq \emptyset \). Let \( h \in H \) with \( h a = ax + b, a \in \text{supp} \mu, r(a) < 1, b \in \text{supp} \eta \). Then for some \( k \in \mathbb{N}, |a^k| < 1 \) and any \( W_0 \in W_0 \): \( h^k W_0 = W_0 \). Hence \( h^k \) has a unique fixed point \( h_+ \) which is attracting for any \( y \in V \), \( \lim_{n \to \infty} h^{kn} y = h_+ \in \text{supp} \nu \). On the other hand if \( y \in W_0 \in W_0 \), \( h^{kn} y \in W_0 \). In the limit we get \( h_+ = \bigcap_{W_0 \in W_0} W_0 \neq \emptyset \). Then the affine subspace \( D = \bigcap_{W_0 \in W_0} W_0 \) is invariant under the action of \( \text{supp} \mu \) and \( D \cap C_+ \neq \emptyset \). If \( \dim D > 0 \), the direction \( D \) is invariant under the action of \( \text{supp} \mu \) and \( D \cap C_+ \neq \emptyset \). If \( D = \emptyset \), the fixed point equation (1.1) is an analog of Lemma 9.1 in [Go], which cannot be used in our settings, since our definition of direct integrability involves uniform estimates with respect to \( u \).

The next lemma says that because \( g \) is integrable, its smoothed version \( \tilde{g} \) is directly Riemann integrable. This is an analog of Lemma 9.1 in [Go], which cannot be used in our settings, since our definition of direct integrability involves uniform estimates with respect to \( u \).

**Lemma 5.18.** The function \( \tilde{g} \) is directly Riemann integrable.

**Proof.** We will use here the following formula for \( \tilde{g} \), which is an immediate consequence of (5.10) and the definition of the smoothing operator,

\[
\tilde{g}(u,t) = \frac{N}{e^t e^2(u)} \int_0^{e^t} \int_{\mathbb{R}^+} r^{\chi^{-1}} \mathbb{E}\left[ 1_{(r-y,r)}((AX,u))(AX,u) \right] \beta_u(dy)dr.
\]
Recall that $\tilde{g}$ is positive. In view of Lemma 5.6 it is enough to prove
$$
\tilde{g}(u, t) \leq De^{-\varepsilon|t|}.
$$

For negative $t$ we just write
$$
\tilde{g}(u, t) \leq De^{-\varepsilon|t|}.
$$

For positive $t$ we fix $p$ very close to $\chi$ ($1 < p < \chi$) and $\eta$ satisfying
$$
1 - \frac{1}{\chi} < \eta < \min\left\{\frac{1}{\chi + 1 - p}, 1 + p - \chi\right\}
$$
(in particular $\eta < 1$). We first estimate
$$
\tilde{g}(u, t) \leq D\left(g_1(u, t) + g_2(u, t)\right),
$$
for
$$
g_1(u, t) = \frac{1}{e^t} \int_0^{e^t} r^{\chi - 1} dr \mathbb{E}\left[\langle AX, u \rangle\right] \beta_u(dy)dr,
$$
$$
g_2(u, t) = \frac{1}{e^t} \int_0^{e^t} \int_0^{\eta t} r^{\chi - 1} \mathbb{E}\left[\langle AX, u \rangle\right] \beta_u(dy)dr.
$$

Then
$$
g_1(u, t) \leq \frac{1}{e^t} \int_0^{e^t} r^{\chi - 1} dr \beta_u\left\{y : y > \frac{\eta t}{2}\right\} \mathbb{E}\left[\langle AX, u \rangle\right]
\leq De^{(\chi - 1)t} e^{-\eta t} \int_{\mathbb{R}^+} y^{\eta} \beta_u(dy),
$$
hence if we choose $q \in \left(\frac{\chi - 1}{\eta}, \chi\right)$ then the expression above can be estimated by $De^{-\varepsilon_1 t}$ for some $\varepsilon_1 > 0$. Indeed the last integral is finite since by Lemma 4.1 we have
$$
\int_{\mathbb{R}^+} y^{\eta} \beta_u(dy) \leq \mathbb{E}\left[\left(\sum_{i=2}^N A_i X_i, u\right)^q\right] \leq \mathbb{E}\left[\langle X, u \rangle^q\right] < \infty.
$$

To estimate $g_2$ we fix $y < \frac{\eta t}{2}$. We will first prove

$$
(5.19) \quad \frac{1}{e^t} \int_0^{e^t} r^{\chi - 1} \mathbb{E}\left[\langle AX, u \rangle\right] \beta_u(dy)dr \leq De^{-\varepsilon_2 t} + De^{-t} y^\chi.
$$

We will use the inequality
$$
\mathbb{E}\left[\langle AX, u \rangle\right] \leq Dr^{1-p},
$$
which was proved in the previous lemma (compare (5.15)).
By (5.15) we have
\[
\frac{1}{c^t} \int_0^{e^t} r^{1\chi-1} E \left[ 1_{(r-y,r)}(yX, u) \right] dr \\
= \frac{1}{c^t} \int_y^{e^t} r^{1\chi-1} E \left[ 1_{(r-y,\infty)}(yX, u) \right] dr + \frac{1}{c^t} \int_0^y r^{1\chi-1} E \left[ 1_{(0,\infty)}(yX, u) \right] dr \\
- \frac{1}{c^t} \int_0^{e^t-y} r^{1\chi-1} E \left[ 1_{(r,\infty)}(yX, u) \right] dr - \frac{1}{c^t} \int_{e^t-y}^{e^t} r^{1\chi-1} E \left[ 1_{(r,\infty)}(yX, u) \right] dr \\
\leq \frac{1}{c^t} \int_y^{e^t} \left( r^{1\chi-1} - (r-y)^{1\chi-1} \right) E \left[ 1_{(r-y,\infty)}(yX, u) \right] dr + \frac{1}{c^t} \int_0^y r^{1\chi-1} dr \ E[(AX, u)] \\
\leq \frac{D}{c^t} \int_y^{e^t} r^{1\chi-1} \left( 1 - (1 - \frac{y}{r})^{1\chi-1} \right) (r-y)^{1\chi-1} dr + De^{-\varepsilon t} y^{1\chi-1}.
\]

To estimate the first integral we divide it into two parts: the integral over the interval \((y, 2y)\) and the second one over \((2y, e^t)\). We study each of them separately. To estimate the first integral we write:
\[
\frac{1}{c^t} \int_y^{2y} r^{1\chi-1} \left( 1 - (1 - \frac{y}{r})^{1\chi-1} \right) (r-y)^{1\chi-1} dr \\
\leq \frac{2^{1\chi-1} y^{1\chi-1}}{c^t} \int_y^{2y} (r-y)^{1\chi-1} dr \\
\leq De^{-\varepsilon t} y^{1\chi-1} \leq De^{(\chi+1\chi-1)t} \leq De^{-\varepsilon t}.
\]

To handle with the second one we will use the following inequality, valid for \(0 \leq a \leq \frac{1}{2}\), being a consequence of the mean value theorem:
\[
1 - (1 - a)^{1\chi-1} \leq Da,
\]
for some constant \(D\) depending only on \(\chi\). We have:
\[
\frac{1}{c^t} \int_{2y}^{e^t} r^{1\chi-1} \left( 1 - (1 - \frac{y}{r})^{1\chi-1} \right) (r-y)^{1\chi-1} dr \\
\leq \frac{Dy}{c^t} \int_{2y}^{e^t} r^{1\chi-1} \frac{y}{r} (r-y)^{1\chi-1} dr \\
\leq \frac{Dy}{c^t} \int_{2y}^{e^t} r^{1\chi-1} (r-y)^{1\chi-1} dt \leq Dy e^{(\chi-1)\chi-1} t \\
\leq De^{(\chi+\chi-1)t} \leq De^{-\varepsilon t}.
\]

Thus, we obtain (5.19). Finally to estimate \(g_2\) we choose \(\varepsilon < \min\{\chi - 1, 1\}\) and write
\[
g_2(u, t) \leq \int_0^{e^t} \left( De^{-\varepsilon t} + De^{-\varepsilon t} y^{1\chi-1} \right) \beta_u(dy) \\
\leq De^{-\varepsilon t} + De^{(\varepsilon + 1\chi - 1)t} \int_{\mathbb{R}^+} y^\chi y^{\varepsilon} \beta_u(dy) \leq De^{-\varepsilon t}.
\]

Finally we are able to conclude the main result.

**Proof of Theorem 2.3.** By Kesten’s renewal theorem [K2]
\[
\lim_{t \to \infty} \bar{G}(u, t) = \frac{1}{\alpha(\chi)} \int_{C^\chi} \int_{\mathbb{R}} g(y, s) ds \pi^\chi y^\chi(dy) = D_+ > 0.
\]
Hence
\[ \lim_{t \to \infty} \frac{1}{e^t} \int_0^{e^t} r^{\chi-1} f(u, r) dr = e^\chi(u) D_+. \]

'Unsmoothing' the function \( \tilde{G} \) (Goldie [Go], Lemma 9.3) we obtain
\[ \lim_{t \to \infty} t^{\chi-1} \eta_u(t, \infty) = \lim_{t \to \infty} t^{\chi-1} f(u, t) = e^\chi(u) D_+. \]

Finally by Theorem 2, Chapter VIII of Feller [Fe] we deduce
\[ \lim_{t \to \infty} t^{\chi} P[(X, u) > t] = \lim_{t \to \infty} t^{\chi} \nu_u(t, \infty) = e^\chi(u) \frac{(\chi-1)D_+}{\chi}. \]

The result follows since \( e^\chi \) is \( P_* \)-harmonic and \( \chi \)-homogeneous. \( \square \)

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