The Directed Power Graph of a group is a graph whose vertex set is the elements of the group, with an edge from x to y if y is a power of x. The Power Graph of a group can be obtained from the directed power graph by disorienting its edges. This article discusses properties of cliques, cycles, paths, and coloring in power graphs of finite groups. A construction of the longest directed path in power graphs of cyclic groups is given, along with some results on distance in power graphs. We discuss the cyclic subgroup graph of a group and show that it shares a remarkable number of properties with the power graph, including independence number, completeness, number of holes etc., with a few exceptions like planarity and Hamiltonian.

1. Introduction

The power graph of finite groups has been studied in [4], [5], [6], [8]. In this work, we denote the group of integers mod n under addition by $\mathbb{Z}_n$, or $\mathbb{Z}/n\mathbb{Z}$. These notations are used interchangeably in this article. It can be shown that any cyclic group of order n is isomorphic to $\mathbb{Z}_n$.

A graph is a pair $\Gamma = (V(\Gamma), E(\Gamma))$ where $V(\Gamma)$ is a non-empty set, called the vertex set whose elements are called vertices, and $E(\Gamma)$ is a (possibly empty) set consisting of sets of pairs of elements of $V(\Gamma)$ called the edge set, whose elements are called edges. If $e \in E(\Gamma)$ is an edge and $v \in e$, then e is said to be incident to v. The degree of a vertex v is the number of edges incident to v. If $\{v_1, v_2\} \in E(\Gamma)$, then $v_1$ and $v_2$ are said to be adjacent. Where there is no ambiguity, $V(\Gamma)$ is sometimes denoted as just V, and similarly $E(\Gamma)$ as E. A directed graph or digraph is a pair $\Gamma = (V(\Gamma), E(\Gamma))$ where $V(\Gamma)$ is a non-empty vertex set, and $E$ is a (possible empty) set consisting of ordered pairs of elements in $V(\Gamma)$ called the edge set.

Let $G$ be a group. The power graph of $G$, denoted by $\Gamma(G)$ is defined as the graph with vertex set consisting of the elements of $G$, and edge set $E = \{\{x, y\} \mid x \neq y \text{ and } \langle x \rangle \leq \langle y \rangle \text{ or } \langle y \rangle \leq \langle x \rangle\}$. The directed
power graph of $G$, denoted $g(G)$ is defined as the graph with vertex set consisting of the elements of $G$, and edge set $E = \{(x, y) \mid x \neq y$ and $\langle y \rangle \leq \langle x \rangle\}$. 

2. Some Properties of Power Graphs of Groups

A graph $\Gamma$ is called connected if for any two vertices $u$ and $v$ in $\Gamma$, there is a path joining $u$ and $v$. In a connected graph $\Gamma$, the distance between $u$ and $v$, denoted $d_\Gamma(u, v)$, can be defined as the length of the shortest path joining $u$ and $v$. The eccentricity of a vertex $u$ is the maximum distance between $u$ and any other vertex in $\Gamma$. The radius of $\Gamma$ is the minimum eccentricity of a vertex in $\Gamma$, and the diameter of $\Gamma$ is the maximum eccentricity of a vertex in $\Gamma$. A central vertex in $\Gamma$ is a vertex with eccentricity equal to the radius of $\Gamma$. The center of $\Gamma$ is the set of central vertices in $\Gamma$.

**Proposition 2.1.** It is well known that the Power graphs of finite groups are connected.

**Proof.** Let $G$ be a group with power graph $g(G)$ and identity $e$. Let $g$ be an arbitrary element of $G$. Since $G$ is a group, $\langle e \rangle \leq \langle g \rangle$, so in the power graph of a group there is an edge between the identity and every other group element. \[\Box\]

Note that the distance between the identity and any other vertex is 1, in the power graph of a group, giving the following corollary.

**Corollary 2.2.** The radius of the power graph of a group is 1, and the center of the power graph of a group of order $n$ is the set of vertices with degree $n - 1$.

The center of a group $G$ is the set of elements of $G$ which commute with all other elements of $G$. The following proposition gives a relation between the center of a power graph of a group and the center of the group.

**Proposition 2.3.** Let $G$ be a group of order $n$ with power graph $g(G)$. The vertices in the center of the power graph $g(G)$ are in the center of the group $G$.

**Proof.** Let $g \in G$ and suppose $\text{deg}_{g(G)}(g) = n - 1$. Then for any other $h \in G$, either $\langle h \rangle \leq \langle g \rangle$, or $\langle g \rangle \leq \langle h \rangle$, either way $g$ and $h$ commute. Hence, $g$ is in the center of $G$. \[\Box\]

Since power graphs of groups are connected and the identity is adjacent to every other element, there is a path of length at most 2 between any two vertices of a power graph of a group, so the diameter of the
power graph of a group is 2, unless the power graph is complete, in which case it is 1.

**Proposition 2.4.** Let $G$ be a non trivial Abelian group of order $n$. The center of $\mathfrak{g}(G)$ has cardinality

1. $\phi(n) + 1$, if $G$ is cyclic and $n \neq p^k$ for any prime $p$ and positive integer $k$.
2. $n$, if $G$ is cyclic and $n = p^k$ for some prime $p$ and positive integer $k$.
3. 1, if $G$ is non-cyclic.

**Proof.**

(1): First let $G$ be a cyclic group whose order is $n$, and suppose $n$ is not a power of a prime. Note, all the $\phi(n)$ many generators of $G$ and the identity are in the center of $\mathfrak{g}(G)$, so the center has size at least $\phi(n) + 1$. Since $n$ is not a power of a prime, $n$ has at least two distinct prime factors. Consider any $b \in G$, where $b$ is neither a generator of $G$, nor the identity of $G$. So, if order of $b$ is $m$, then $m < n$. Hence, by converse of Lagrange’s Theorem over finite Abelian group, there exists $c \in G$, where order of $c$ is $r$ and $r$ does not divide $m$, $m$ does not divide $r$. So, there is no edge between $b$ and $c$. Since $b$ is an arbitrary non-generator of $G$, which is not the identity of $G$ either, the cardinality result follows.

(2): Now suppose $G$ is cyclic and it’s order is $n = p^k$ for some prime $p$ and positive integer $k$. Then, the graph being complete, by [5], the center has cardinality $n$.

(3): Let, $G$ be not cyclic. Then, the result follows by the proposition 3.7 in the next section.

3. Some results relating Composition series in power graphs

The Jordan-Hölder theorem asserts that any two composition series of a given group have the same length, and isomorphic factor groups up to permutation. The length of a composition series of a group $G$ is called its composition length and denoted as $\ell(G)$. The composition length $\ell(G)$ is also the maximum length a normal series of $G$ can have. Then a group $G$ can be partitioned in the following way, let $X_i$ be the set of all elements of $G$ which generate a cyclic subgroup with composition length $i$. Then $G = \bigcup_i X_i$, and $X_i \cap X_j = \emptyset$ for any $i \neq j$. 
Lemma 3.1. The composition length of an Abelian group is the sum of the exponents of the order of the group, when written as a product of prime numbers.

Proof. An Abelian group is simple only if its order is prime. Then if $|G| = p_1^{k_1}p_2^{k_2} \cdots p_n^{k_n}$, then each $G_i/G_{i+1}$ must have prime order, that is $|G_i| = p_i|G_{i+1}|$. Then each $G_{i-1}$ has an order whose exponents sum to one less than those in the order of $G_i$, so there must be $k_1 + k_2 + \cdots + k_n$ inclusions in a maximal length normal series of an Abelian group. □

Proposition 3.2. If $X_i$ is non-empty, then $X_{i-1}$ is non-empty.

Proof. Let $x \in X_i$ and $\langle x \rangle \supset \langle x_{i-1} \rangle \supset \cdots \supset \langle x_2 \rangle \supset \{e\}$ be a composition series. Then there is a normal series of length $i-1$ from $\langle x_{i-1} \rangle$. Suppose there were a normal series of length $i$ or greater from $\langle x_{i-1} \rangle$, then $\langle x \rangle \supset \langle x_{i-1} \rangle \supset \cdots \supset \{e\}$ is a normal series from $\langle x \rangle$ of length greater than $i$, contradicting the assumption that $x \in X_i$. □

Corollary 3.3. Removing subgroups from the beginning of a composition series of length greater than zero from a cyclic subgroup leaves a composition series.

Proposition 3.4. If some vertex in $X_i$ is adjacent to all other vertices in $X_i$, then $X_i$ is a clique in $\mathfrak{g}(G)$.

Proof. If $x_1, x_2 \in X_i$ and they are adjacent in $\mathfrak{g}(G)$, then either $\langle x_1 \rangle \leq \langle x_2 \rangle$ or $\langle x_2 \rangle \leq \langle x_1 \rangle$. Without loss of generality suppose $\langle x_2 \rangle$ is a proper subgroup of $\langle x_1 \rangle$. Then there exists a composition series $\langle x_2 \rangle \supset \langle x_3 \rangle \supset \cdots \supset \{e\}$ of length $i$ and $\langle x_1 \rangle \supset \langle x_2 \rangle \supset \cdots \supset \{e\}$ is a normal series of length $i+1$, contradicting the assumption that $x_1 \in X_i$, so it must be the case that $\langle x_2 \rangle = \langle x_1 \rangle$. Then, if any element of $X_i$ is adjacent to all other elements of $X_i$ in $\mathfrak{g}(G)$, then all elements of $X_i$ generate the same subgroup and form a clique in $\mathfrak{g}(G)$. □

Proposition 3.5. If $X_i$ is a clique in $\mathfrak{g}(G)$, then $X_{i+1}$ is a clique in $\mathfrak{g}(G)$.

Proof. Suppose $X_i$ is a clique in $\mathfrak{g}(G)$ and $X_{i+1}$ is not a clique in $\mathfrak{g}(G)$, and let $x \in X_i$. Then there exist two distinct elements in $X_{i+1}$, say $y$ and $z$, with orders $p|x|$ and $q|x|$ respectively where $p$ and $q$ are prime and $p \neq q$. Write $|x| = p_1^{k_1}p_2^{k_2} \cdots p_n^{k_n}$, with $\sum_{t=1}^{n} k_t = i$. Then $\langle y \rangle$ has a subgroup of order $p_1^{k_1-1}p_2^{k_2} \cdots p_n^{k_n}$, and $\langle z \rangle$ has a subgroup of order $q_1^{k_1-1}p_2^{k_2} \cdots p_n^{k_n}$. Both of these subgroups are in $X_i$, but since $p \neq q$ at least one of the subgroups is not equal to $\langle x \rangle$, contradicting the assumption that $X_i$ is a clique in $\mathfrak{g}(G)$. □
Proposition 3.6. If $X_i$ is a clique in $g(G)$ and $x \in X_i$ and $\langle x \rangle \neq p^k$ for some prime $p$ and positive integer $k$, then $X_{i+1}$ is empty.

Proof. Suppose $x \in X_i$, $\langle x \rangle \neq p^k$, so $|x|$ has at least two distinct prime factors, say $p$ and $q$. Suppose there exists a $y \in X_{i+1}$, then $pq | |y|$, but then $|y| = pqm$ for some $m$ (possible equal to $p$ or $q$), and in that case $\langle y \rangle$ has subgroups of orders $pm$ and $qm$, both in $X_i$, contradicting the assumption that $X_i$ is a clique in $g(G)$. □

Proposition 3.7. If $G$ is a finite Abelian group and not cyclic, then $\max_i X_i$ is not a clique in $g(G)$.

Proof. Since $G$ is a finite non-cyclic Abelian group, $G \cong \mathbb{Z}_{p_1}^{k_1} \times \mathbb{Z}_{p_2}^{k_2} \times \cdots \times \mathbb{Z}_{p_t}^{k_t}$, where not all of the $p_i$'s are distinct. Then, $G \cong \mathbb{Z}_m \times \mathbb{Z}_{q_1}^{l_1} \times \cdots \times \mathbb{Z}_{q_r}^{l_r}$, where $p_i^{k_i} | m$, $\cdots$, $p_t^{k_t} | m$, and $m = q_1^{l_1}q_2^{l_2} \cdots q_r^{l_r}$ where $q_1, q_2, \cdots, q_r$ are relatively prime. Notice that $g = (1, 0, \cdots, 0)$ and $g' = (1, 1, 0, \cdots, 0)$ both have the maximum possible order in $G$, that is $m$. Also, notice that $\langle g \rangle \neq \langle g' \rangle$, so $g$ and $g'$ are both in $\max_i X_i$ but not adjacent to each other, so $\max_i X_i$ is not a clique in $g(G)$. □

Since $\max_i X_i$ is not a clique in $g(G)$, then no $X_i$ is a clique in $g(G)$. Since $G = \bigcup_i X_i$, no non-identity vertex in $g(G)$ has degree $n - 1$.

4. Perfect Graphs

A clique in a graph $\Gamma$ is a subset $C$ of the vertices of $\Gamma$, such that any two vertices in $C$ are adjacent. The size of the largest clique in $\Gamma$ is called the clique number of $\Gamma$, and is denoted $\omega(\Gamma)$.

A vertex coloring of a graph is an assignment of labels (called colors) to the vertices of a graph $\Gamma$ such that no two adjacent vertices share the same color. The smallest number of colors which can be used in a coloring of $\Gamma$ is called the chromatic number of $\Gamma$, and is denoted $\chi(\Gamma)$.

A graph $\Gamma$ is called perfect if for every induced subgraph $\Gamma_i$ of $\Gamma$, $\omega(\Gamma_i) = \chi(\Gamma_i)$. It was conjectured by Berge in 1961, and was proved by Chudnovsky et. al. in [2] that graphs are perfect if and only if they contain no odd holes or odd anti-holes of odd length greater than 3. A hole in a graph is a cycle such that no two vertices in the cycle are joined by an edge which does not itself belong to the cycle. An anti-hole is the edge complement of a hole.

Theorem 4.1 (Strong Perfect Graph Theorem). A graph is perfect if and only if it contains no odd holes or odd anti-holes of length greater than 3.

Let $G$ be a group with power graph $g(G)$. Then $g(G)$ can contain no holes of odd length. To prove this a few short lemmas are used.
Lemma 4.2 (Path). Let $G$ be a group with directed power graph $\vec{g}(G)$. If a path exists between two vertices then they are adjacent.

Proof. Denote the start of the path as vertex $a$ and label the vertices along the path as $a_1, a_2, \ldots, a_m$. Then $a_1 = a^n, a_2 = a_1^{n_1}, a_3 = a_2^{n_2} \ldots a_m = a_{m-1}^{n_{m-1}}$. Then $a_m = a^{n_1 n_2 \cdots n_{m-1}}$, so there is an edge from $a$ to $a_m$. $\square$

Lemma 4.3 (Strong Path). Let $G$ be a group with directed power graph $\vec{g}(G)$, and let $\vec{g}(G)$ contain a directed path of length $n$, then the vertices making up the directed path form a clique of size $n$ in $g(G)$.

Proof. The proof follows from the previous (Path) Lemma. $\square$

The strong path lemma shows that whenever there is a path of length $n$ in $\vec{g}(G)$ there is a clique of size $n$ consisting of the same vertices in $g(G)$. Here it will be shown that the converse is true as well, that is, whenever there is a clique of size $n$ in $g(G)$, then those vertices are traversable by a path in $\vec{g}(G)$. Path-clique equivalence in power graphs of finite groups is a consequence of Rédei’s theorem [3].

As per our need, later in our work, we reproduced the proof of the following well known theorem here one more time.

Theorem 4.4 (Rédei’s Theorem). Every orientation of a complete graph contains a directed Hamiltonian path.

Theorem 4.5 (Path-clique Equivalence). Let $G$ be a group with directed power graph $\vec{g}(G)$ and undirected power graph $g(G)$. Then whenever there is a path in $\vec{g}(G)$, its constituent vertices form a clique in $g(G)$, and whenever there is a clique in $g(G)$, its vertices are traversable by a path in $\vec{g}(G)$.

Proof. The proof of the first direction is given above, here we show that a clique in $g(G)$ is traversable by a path in $\vec{g}(G)$. Let $g(G)$ contain a clique of size $\alpha$. The proof is by induction on $\alpha$. If $\alpha = 1$, then the clique is traversable by the path consisting of only the single vertex in the clique.

Suppose the result holds for $0 < \alpha \leq k$, and let there exist in $g(G)$ a clique of size $k + 1$. By the induction hypothesis, a clique of size $k$ is traversable by a directed path, so at most one vertex is excluded from the longest path through the clique. Proceed along that directed path through these vertices and label them in the order they are encountered, $v_1, v_2, v_3, \ldots, v_k$. If for any $v_i$ in the clique both $(v_i, v_{k+1})$ and $(v_{k+1}, v_i)$ are edges in $\vec{g}(G)$, then the sequence can be modified from $(\cdots v_1, v_{i+1} \cdots)$ to $(\cdots v_1, v_{k+1}, v_{i+1} \cdots)$ adding $v_{k+1}$ to the path. Suppose no two-sided edges exist in the clique. If $(v_{k+1}, v_1)$ is an edge in
\[ \bar{g}(G), \text{ then } v_{k+1} \text{ can be added to the beginning of the existing path. If } (v_k, v_{k+1}) \text{ is an edge in } \bar{g}(G), \text{ then } v_{k+1} \text{ can be added to the end of the existing path. Suppose neither of these are edges in } \bar{g}(G). \text{ If } (v_k+1, v_2) \text{ is an edge is } \bar{g}(G), \text{ then } v_{k+1} \text{ can be inserted in the path between } v_1 \text{ and } v_2. \text{ Then, if } (v_{k+1}, v_2) \text{ is not an edge in } \bar{g}(G), (v_2, v_{k+1}) \text{ must be an edge in } \bar{g}(G). \text{ Proceeding in this way we get the desired directed path in } \bar{g}(G). \]

\[ \square \]

**Figure 1.** An illustration of Theorem 4.5. Only relevant edges from the new vertex are shown in each case.

(A) An edge in both directions between \( v_{k+1} \) and any other vertex implies \( v_{k+1} \) can be added to the path

(B) If there is an edge from \( v_k \) to \( v_{k+1} \), then \( v_{k+1} \) can be added to the end of the path. Here \( k = 5 \).

(C) If there is an edge from \( v_{k+1} \) to \( v_1 \), then \( v_{k+1} \) can be added to beginning of the the path. Here \( k = 5 \).

(D) If none of the other cases hold, then there must exist a pair of vertices in between which \( v_{k+1} \) can be placed.

It is well known that Power graphs of groups are perfect, which has been proved by using Theorem 2.

**5. The Cyclic Subgroup Graph**

Let \( G \) be a group and define the relation \( \sim \) on \( G \) by \( x \sim y \) if \( \langle x \rangle = \langle y \rangle \). Define the graph \( \tilde{C}(G) \) by \( V(\tilde{C}(G)) = G/\sim \) and \( (A, B) \in E(\tilde{C}(G)) \) if there exists elements \( b \in B \) and \( a \in A \) such that \( \langle b \rangle \leq \langle a \rangle \) and \( \langle b \rangle \neq \langle a \rangle \).
Also define a weight function $w : V(\overline{C}(G)) \rightarrow \mathbb{N}$ by $w(A) = |A|$. Then $\overline{C}(G)$ is a directed acyclic graph with a similar structure to the directed power graph $\overline{g}(G)$.

**Proposition 5.1.** The weight of the path with the largest weight in $\overline{C}(G)$ is the length of the longest path in $\overline{g}(G)$.

**Proof.** The vertices in $\overline{C}(G)$ with only out-edges represent generators of the maximal cyclic subgroups of $G$. As in the proof of Theorem 3 above any longest path in $G$ must start with these vertices. If this maximal cyclic subgroup has order $n$ then a vertex adjacent to it represents generators of a cyclic subgroup of order $\frac{n}{d}$ where $d$ is a divisor of $n$, and if the subgroup is maximal $d$ will be a prime divisor of $n$. Then the sum of the weights of the vertices in a path from a maximal cyclic subgroup of $G$ of order $n$ through all of its maximal subgroups of maximum order to the trivial subgroup will be given by $\Psi(n)$, the length of the longest path in $\overline{g}(G)$.

When $G$ is cyclic there is no ambiguity in naming vertices in $\overline{C}(G)$ by their corresponding isomorphic group $\mathbb{Z}_n$, for example

**Figure 3.** $\overline{C}(\mathbb{Z}_{18})$ with vertex weights shown in parentheses.

The non-oriented graph $C(G)$ also contains a lot of the information in the power graph in a smaller form. For example $d_{\overline{g}(G)}(u, v) = d_{C(G)}([u]_\sim, [v]_\sim)$ as long as $[u]_\sim \neq [v]_\sim$ and the independence number $\alpha(\overline{g}(G)) = \alpha(C(G))$.

**Proposition 5.2.** The cyclic subgroup graph is isomorphic to an induced subgraph of the power graph.
Proof. Define \( f : V(C(G)) \to (V \overline{g}(G)) \) by \( f([x]_\sim) = x' \), for a fixed choice of the corresponding equivalence class representative \( x' \). The result follows by the definition of cyclic subgroup graph. \( \square \)

**Proposition 5.3.** \( d_{\overline{g}(G)}(u, v) = d_{C(G)}([u]_\sim, [v]_\sim) \) as long as \( [u]_\sim \neq [v]_\sim \).

**Proof.** Let \( G \) be a group and let \( u, v \in G \) with \( d_{\overline{g}(G)}(u, v) = k \). Let \( P = \{u, u_1, u_2, \ldots, u_k, v\} \) denote the shortest path from \( u \) to \( v \) in \( \overline{g}(G) \). It must be the case that for each \( u_i, u_j \in P, [u_i]_\sim \neq [u_j]_\sim \) otherwise deleting the \( u_i \) or \( u_j \) we get a shorter path in \( g(G) \) which is a contradiction. Then \( \{[u]_\sim, [u_1]_\sim, \ldots, [u_k]_\sim\} \) is a path from \( [u]_\sim \) to \( [v]_\sim \) in \( C(G) \). Suppose there is a shorter path from \( [u]_\sim \) to \( [v]_\sim \) in \( C(G) \), and denote that path \( P_1 = \{[u]_\sim, [y_1]_\sim, \ldots, [y_j]_\sim, [v]_\sim\} \) where \( j < k \). Then \( \{u, y_1, \ldots, y_j, v\} \) is a path in \( g(G) \) which is shorter than \( P \), a contradiction. Then \( \{[u]_\sim, [u_1]_\sim, \ldots, [u_k]_\sim\} \) is the shortest path from \( [u]_\sim \) to \( [v]_\sim \) in \( C(G) \) meaning \( d_{C(G)}([u]_\sim, [v]_\sim) = k = d_{\overline{g}(G)}(u, v) \). \( \square \)

**Proposition 5.4.** Let \( G \) be a group. Elements \( \{g_1, g_2, \ldots, g_k\} \) form an independent set in \( \overline{g}(G) \) if and only if \( \{[g_1]_\sim, [g_2]_\sim, \ldots, [g_k]_\sim\} \) is an independent set in \( C(G) \).

**Proof.** Suppose \( I = \{g_1, g_2, \ldots, g_k\} \) forms an independent set in \( \overline{g}(G) \). Then for each \( g_i, g_j \in I, [g_i]_\sim \not\sim [g_j]_\sim \) and \( [g_i]_\sim \not\sim [g_j]_\sim \). So, \( \{[g_i]_\sim, [g_j]_\sim\} \not\in E(C(G)) \), giving an independent set of size \( k \) in \( C(G) \).

Now suppose \( \{[g_1]_\sim, [g_2]_\sim, \ldots, [g_k]_\sim\} \) is an independent set in \( C(G) \). Then \( \{g_1, g_2, \ldots, g_k\} \) is an independent set in \( \overline{g}(G) \). \( \square \)

**Corollary 5.5.** \( \alpha(\overline{g}(G)) = \alpha(C(G)) \).

**Proposition 5.6.** \( \overline{g}(G) \) is complete if and only if \( C(G) \) is complete.

**Proof.** Suppose \( \overline{g}(G) \) is complete and let \( [u]_\sim \) and \( [v]_\sim \) be arbitrary vertices in \( C(G) \). Since \( u \in [u]_\sim \) and \( v \in [v]_\sim \) and \( \{u, v\} \in E(\overline{g}(G)) \), \( \langle u \rangle \leq \langle v \rangle \) or \( \langle v \rangle \leq \langle u \rangle \), so \( \{[u]_\sim, [v]_\sim\} \in E(C(G)) \). Since \( [u]_\sim \) and \( [v]_\sim \) were arbitrary, \( C(G) \) must be complete.

Now suppose \( \overline{g}(G) \) is not complete, then there exist vertices \( u \) and \( v \) such that \( \{u, v\} \not\in E(\overline{g}(G)) \). Then \( \{u, v\} \) is an independent set in \( \overline{g}(G) \) and consequently \( \{[u]_\sim, [v]_\sim\} \) is an independent set in \( C(G) \). Then there exist elements \( [u]_\sim \) and \( [v]_\sim \) in \( C(G) \) such that \( \{[u]_\sim, [v]_\sim\} \not\in E(C(G)) \) so \( C(G) \) is not complete. \( \square \)

**Proposition 5.7.** Let \( u_1, u_2, \ldots, u_{2k} \) be vertices in \( \overline{g}(G) \), then \( u_1, \ldots, u_{2k} \) is a hole in \( \overline{g}(G) \) if and only if \( [u_1]_\sim, [u_2]_\sim, \ldots, [u_{2k}]_\sim \) is a hole in \( C(G) \).

**Proof.** Let \( P = \{u_1, u_2, \ldots, u_{2k}\} \) be a hole in \( \overline{g}(G) \), then each \( u_i \in P \) has exactly two neighbors in \( P \), call them \( u_{i+1} \) and \( u_{i-1} \). Then either
Proposition 5.9. \( g \) is perfect graphs.

Proposition 5.10. \( g \) is claw free if and only if \( C(G) \) is so.

Proposition 5.11. A vertex \( x \in g(Z_n) \) is simplicial if and only if \( [x]_{\sim} \in C(Z_n) \) is simplicial.

Proof. The result follows as induced subgraph of a complete graph is complete.

Proposition 5.12. If \( C(G) \) is Hamiltonian, then \( g(G) \) is also Hamiltonian.

Proof. Suppose \( C(G) \) is Hamiltonian, and let \( \{[u_1]_{\sim}, [u_2]_{\sim}, \ldots, [u_2]_{\sim}, [u_1]_{\sim} \} \) be a Hamiltonian cycle in \( C(G) \). \([u_1]_{\sim}\) is a clique in \( g(G) \) since for \( g_1, g_2 \in [u_i]_{\sim} \), \( \langle g_1 \rangle = \langle g_2 \rangle \), so \( \langle g_1 \rangle \leq \langle g_2 \rangle \), so \( \{g_1, g_2\} \in E(g(G)) \), so denote \( [u_i]_{\sim}\) by \( \{u_{i_1}, u_{i_2}, \ldots, u_{i_{k_i}}\} \), then \( \{u_{11}, u_{21}, u_{31}, \ldots, u_{k_{i_1}1}, u_{12}, u_{22}, \ldots, u_{k_{i_2}2}, \ldots, u_{1n}, u_{2n}, \ldots, u_{kn_n} u_{11}\} \) is a Hamiltonian cycle in \( g(G) \).

Singh and Devi showed in [1] that the cyclic subgroup graph of cyclic groups of non-prime order is Hamiltonian in. Power graphs of groups of prime order are complete, and therefore Hamiltonian for all orders except for 2, so we note the following corollary by using Rédei’s theorem, [3].

Corollary 5.13. Let \( G \cong \mathbb{Z}_n \) be a cyclic group with \( n \neq 2 \), then \( g(G) \) is Hamiltonian.

Since the cyclic subgroup subgraph is isomorphic to an induced subgraph of the power graph, if \( g(G) \) is planner, then so is \( C(G) \). For example, consider \( G = \mathbb{Z}_0 \). Here \( g(G) \) being a complete graph with 9 vertices it is not planner. \( C(G) \) being a triangle is planner.
6. Chordless Cycles

It has now been shown that power graphs contain no holes of odd-length. Here it will be shown that for arbitrary even integer \( n \), there exists a finite group whose power graph contains a hole of length \( n \).

**Proposition 6.1.** Let \( n \) be an even integer, then for even \( n > 4 \), the power graph of a cyclic group will contain a hole of length \( n \), if the order of the group has \( \frac{n}{2} \) distinct prime factors. The power graph of the group will contain a hole of length 4, if the order of the group has at least two prime factors of multiplicity two or more.

**Proof.** First consider the case that \( n = 4 \). Then in the group \( \mathbb{Z}_{p^2q^2} \) there is a subgraph consisting of the vertices \( p, pq, q, p^2 \). This subgraph will be a hole of length 4.

Now consider the case that \( n \geq 6 \), and take primes \( p_1, p_2, \ldots, p_n \). Then the group contains a hole of length \( n \), namely subgraph consisting of vertices

\[
p_1 - p_1p_2 - p_2 - p_2p_3 - p_3 - \cdots - p_2^2 \cdots p_n^2 - p_1 - p_1
\]

Then this subgraph is a hole of length \( n \) in \( g(\mathbb{Z}_{p_1p_2\ldots p_n}) \).

**Note:** Following the proof of the theorem, it is possible to create many holes of even length permuting the positions of the primes and allowing various exponent of them.

**Proposition 6.2.** If the power graph of a finite cyclic group \( G \) contains a hole of length \( n \), then \( |G| \) has at least \( \frac{n}{2} \) distinct prime factors.

**Proof.** Suppose \( g(\mathbb{Z}_m) \) contains a hole of length \( n \). This cycle in the corresponding directed power graph consists of \( \frac{n}{2} \) vertices with only out-edges and \( \frac{n}{2} \) vertices with only in-edges. Each vertex with out-edges is non-adjacent to each other vertex with out-edges, so certainly if \( x, y \) are group elements represented by vertices in the hole with out-edges, then \( |x| \) does not divide \( |y| \), and \( |y| \) does not divide \( |x| \). Then the order of each of these \( \frac{n}{2} \) vertices with out-edges has a prime factor which is not shared by the other \( \frac{n}{2} \) vertices with out-edges. By Lagrange’s theorem, the order of each element must divide the order of the group, so the order of the group must contain at least \( \frac{n}{2} \) prime factors.

It has been shown that for an arbitrary even integer \( n \), a finite group can be found whose power graph contains a hole of length \( n \). The proof relied on the fact that there were \( \frac{n}{2} \) primes dividing the order of the group so that the multiples of the primes were a subset of their multiples in \( \mathbb{Z} \). Then \( g(\mathbb{Z}) \) contains holes of any even length. In fact
since there are infinitely many prime numbers, there will be an infinite number of holes of any even length in $g(\mathbb{Z})$.

A necessary and sufficient condition for the existence of a hole of length $n$ in finite cyclic groups has been given above. This is not a necessary condition for the existence of holes of length $n$ in general Abelian groups. Consider the Abelian group $\mathbb{Z}_{12} \times \mathbb{Z}_{12}$. This group has order $144 = 2^4 \cdot 3^2$. Consider the subgraph of $\mathbb{Z}_{12} \times \mathbb{Z}_{12}$ consisting of the elements $(1, 0), (2, 0), (1, 6), (3, 6), (1, 2), (2, 4), (1, 8), (3, 0)$. These elements (in order around the cycle) form a hole of length 8.

**Proposition 6.3.** Let $G$ be an Abelian group whose order is $p^k$ for some prime $p$ and positive integer $k$. Then $g(G)$ can contain no hole.

**Proof.** The result follows as the power graph is complete in this case. □

**Theorem 6.4.** An element whose order is a power of a prime cannot be a vertex with out-edges in a hole of even length.

**Proof.** Suppose, that is not the case. Then, $g(G)$ contains a hole with an element $x$ of order $p^n$ as a vertex with out-edges in the hole. Let $y$ and $z$ be the elements adjacent to $x$. Then $y$ and $z$ are of orders $p^l$ and $p^m$ respectively, where $l, m < n$. Without loss of generality, suppose $l < m$, then $\langle y \rangle \subset \langle z \rangle$ as each are proper subgroups of $\langle x \rangle$ which is a primary cyclic group. So there is a chord from $y$ to $z$ giving a contradiction. □

**Proposition 6.5.** Power graphs of groups can contain no anti-holes of length greater than 4.

**Proof.** Let $G$ be a group and suppose that $g(G)$ contains an anti-hole of length $n$ greater than 4. Claim: All vertices in the anti hole have in-degree zero or out-degree zero in the corresponding directed power graph. Proof of the claim: First note that, if we consider $n = 4$, then the claim follows clearly. Arbitrarily choose a vertex $d$ in the anti hole. There must be a vertex $s_1$ in the antihole adjacent to $d$. Without loss of generality, let the source of the corresponding edge be $s_1$ and the destination vertex $d$. Now choose an edge between a vertex $s_2$, which is non-adjacent to vertex $s_1$, and adjacent to $d$. Such an edge must exist as we assume $n > 4$, since no two vertices can share the same pair of non-adjacent vertices. The direction of this edge must be from $s_2$ to $d$ since if it were from $d$ to $s_2$ then a directed path would exist between $s_1$ and $s_2$, which by the path lemma would make them adjacent. Repeat the process for a vertex non-adjacent to one of either $s_1$ or $s_2$ and adjacent to $d$ again. The stopping point of this process is when all
n – 3 vertices adjacent to vertex d have been selected. Following this procedure we see that d has out-degree zero. Hence the claim follows. Now power graph of a group being perfect, here n > 4 means n ≥ 6, as it can’t have any anti-hole of odd length. So, for any arbitrary vertex d in the anti-hole, degree of d = n – 3 ≥ 3. So, it is possible to choose two vertices v1 and v2 in the neighborhood of d that are adjacent to each other. Without loss of generality, if we assume d has in degree zero, then as there is at least one directed edge between v1 and v2, we get a contradiction to the above claim. Hence the result follows. □

7. Completeness

Here an alternative proof of the well known result regarding completeness of power graphs of cyclic groups of prime-power order in terms of the strong path lemma is presented.

Proposition 7.1. The power graph of a cyclic group of order \( p^n \) where p is prime and n is a non-negative integer is complete.

Proof. The proof is by induction on n. By the Strong path lemma it suffices to show that there exists a directed path through all vertices in the power graph. When n = 0 the graph consists of one vertex and the result follows.

Suppose a directed path exists through all vertices in the power graphs of cyclic groups of order \( p^k \) for 0 ≤ k < n. Let \( G \cong \mathbb{Z}_{p^k} \). Let x, y denote two arbitrary generators of \( G \), then in \( g(G) \) there is an edge from x to y and there is an edge from y to x, as both elements are members of G and therefore generated by each other. Then, there is a directed path between all generators of G. This path can be extended towards a generator of a subgroup of order \( p^{k-1} \), which contains a directed path through all of it’s vertices by the induction hypothesis. As every subgroup of G divides the order of G by Lagrange’s theorem, every subgroup of G is properly contained inside the subgroup of order \( p^{k-1} \), so there exists a directed path through the entire vertex set of \( g(G) \).

By the strong path lemma, there is a clique of size \( p^n \) in the cyclic group of order \( p^n \), so the graph is complete. □

8. Chromatic Number of Power Graphs of Cyclic Groups

Proposition 8.1. Let G be the cyclic group of order n and let \( g(G) \) be its power graph. Let H denote the set of non-generators of G. Let \( \chi(\Gamma) \) denote the chromatic number of a graph \( \Gamma \). Then \( \chi(g(G)) = \phi(n) + \chi(H) \) where \( H \) is the subgraph of G consisting of vertices representing
non-generators and the edges between them and $\phi$ is Euler’s totient function.

**Proof.** Since elements in $G \setminus H$ generate $G$, for any $g \in G \setminus H$ and $x \in G$, $x \in \langle g \rangle$, so vertex $g$ is adjacent to all elements in $g(G)$. Then no color used in a coloring of the portion of $g(G)$ consisting of elements of $G \setminus H$ can be used in a coloring of $H$. Additionally that portion of the graph is internally complete as well so the subgraph of $g(G)$ consisting of elements of $G \setminus H$ is isomorphic to $K_{\phi(n)}$ and has chromatic number $\phi(n)$. Since every color used in the coloring of $H$ is distinct from every color in the subgraph of $g(G)$ consisting of elements of $G \setminus H$, and $H$ has chromatic number $\chi(H)$, $\chi(g(G))$ is at most $\phi(n) + \chi(H)$. Also since the colors used in the colorings of the two subgraphs of $g(G)$ are distinct, $\chi(g(G))$ cannot be less than $\phi(n) + \chi(H)$, so $\chi(g(G)) = \phi(n) + \chi(H)$. □

**Corollary 8.2.** Let $G \cong \mathbb{Z}_n$ then $\phi(n) < \chi(g(G)) \leq n$.

The result above gives a lower bound for the chromatic number of a power graph of a cyclic group by identifying a clique of a known size in every cyclic group. By the strong path lemma, the existence of a directed path of length $k$ implies that the vertices along the path also make up a clique of length $k$. If $m$ is the length of the longest directed path in a group $G \cong \mathbb{Z}_n$, then the chromatic number of the cyclic group of order $n$ is at least as large as the length of that path.

**Proposition 8.3.** Let $G$ be a group and let $m$ be the length of a directed path in $\mathcal{g}(G)$. Then $m \leq \chi(g(G)) \leq n$.

In a cyclic group $G$ of order $n$, a path $m$ of length longer than $\phi(n)$ can be constructed as follows. Follow the path of length $\phi(n)$ through the generators of $G$. After visiting the last generator, follow the path to a generator of a proper subgroup of $G$, which must exist as a path exists from a generator of $G$ to every element of $G$. As any two generators of any group have edges from each vertex to the other, each of the generators of this subgroup can be added to the path. This process can than be continued for a subgroup of this subgroup and so on until the the identity element is added to the path. In fact, the longest path through any cyclic subgroup will be of this form, a descending chain of generators of proper subgroups.

**Theorem 8.4.** Let $\mathbb{Z}/n\mathbb{Z}$ be the cyclic group of order $n$, and let $S_k$ be the set of generators of $\mathbb{Z}/k\mathbb{Z}$. The longest path through $\mathbb{Z}/n\mathbb{Z}$ will be of the form

$$s_{n1}, s_{n2}, \ldots s_{n\phi(n)}, d \cdot s_{n\phi(n)}^1, d \cdot s_{n\phi(n)}^2, d \cdot s_{n\phi(n)}^3, \ldots (dd_1\ldots d_j) \cdot s_{\frac{n}{dd_1d_2\ldots d_j}}^{\phi(d_1d_2\ldots d_j)}$$
where \( d, d_1, d_2 \ldots d_j \) are prime divisors of \( n \) with \( d \leq d_1 \leq d_2 \leq \ldots \leq d_j \).

and \( s_{ix} \in S_i \).

**Proof.** Suppose a path longer than the one given exists. Such a path necessarily begins with the set of elements of order \( n \), that is the elements of \( S_n \), since if it did not these elements could simply be added to the beginning of the path to make a new longer path. After every element of order \( n \) is added to the path an element \( g_1 \) from a proper subgroup of \( \mathbb{Z}/n\mathbb{Z} \) can be added to the path, but this element will have order \( \frac{n}{d} \) where \( d \) is a divisor of \( n \). In fact this element will generate a subgroup \( \langle g_1 \rangle \) of order \( \frac{n}{d} \) which is isomorphic to \( \mathbb{Z}/\frac{n}{d}\mathbb{Z} \) with the mapping given by \( z \in \mathbb{Z}/\frac{n}{d}\mathbb{Z} \rightarrow d \cdot g \in \langle g_1 \rangle \). Then for each of the \( \varphi(\frac{n}{d}) \) generators \( z \in z \in \mathbb{Z}/\frac{n}{d}\mathbb{Z} \), the corresponding element \( d \cdot z \in \langle g_1 \rangle \) must be added to the path, since if it were not added to the path a new path could be constructed with these elements following (or preceding) \( g_1 \) which is longer. The same process can be repeated from \( \langle g_1 \rangle \), add an element from a subgroup of \( \langle g_1 \rangle \) and all of the other generators of the same subgroup. In this way generators of a descending chain of subgroups are added to the path, terminating with the generator of the trivial group, the identity element.

Since the longest path must be in the form of generators of a chain of subgroups, it remains to be shown that by always choosing the largest possible proper subgroup whose generators to add to the path, the path size is maximized. That is, by traversing the subgroups of \( \mathbb{Z}/n\mathbb{Z} \) in the order

\[
\mathbb{Z}/n\mathbb{Z} \rightarrow d\mathbb{Z}/n\mathbb{Z} \rightarrow d \cdot d_1\mathbb{Z}/n\mathbb{Z} \rightarrow \ldots \rightarrow (d \cdot d_1 \cdot \ldots \cdot d_j)\mathbb{Z}/n\mathbb{Z}
\]

where \( d \leq d_1 \leq \ldots \leq d_j \), the number of elements added to the path is as large as possible. To see this, some properties of Euler’s totient function are examined. First observe that increasing any single prime factor in a number will increase the totient of that number, that is if \( n = p_1 p_2 \ldots p_k \) where \( p_1 \leq p_2 \leq \ldots \leq p_k \) and \( m = p_2 p_2 \ldots p_{2k} \) where \( p_{1x} = p_{2x} \) for all \( x \) except one, and at that one index \( p_{1x} > p_{1x} \), then \( \phi(m) \geq \phi(n) \). It is known that the totient function is multiplicative over relatively prime arguments, that is \( \phi(ab) = \phi(a)\phi(b) \) if \( \gcd(a, b) = 1 \), so we can write \( \phi(n) \) and \( \phi(m) \) as \( \Pi_{x \in X} \phi(p_{1x}^k) \) and \( \Pi_{y \in Y} \phi(p_{2y}^k) \) respectively where \( X \) is the set of prime factors of \( n \) and \( Y \) is the set of prime factors of \( m \). Then both \( \phi(n) \) and \( \phi(m) \) can be divided by the prime factors and multiplicities for which they agree, leaving only the prime factors which differ between \( n \) and \( m \). Then observe that \( \phi(p^k) < \phi(q^k) \) if \( p < q \) are distinct primes. Also \( \phi(p^k) \leq \phi(p^{k-1})q \) if \( p < q \) since \( p^k - p^{k-1} \leq (p^{k-1} - p^{k-2})(q - 1) \). Then it is clear that if \( n \) is the order of a cyclic group the subgroups must be traversed in
the order above to make the number of elements in the path as large as possible.

Here the largest path through $\bar{g}(\mathbb{Z}/n\mathbb{Z})$ has been constructed, which will have a length equal to the size of the largest clique in $g(\mathbb{Z}/n\mathbb{Z})$, by the path-clique equivalence theorem. Since power graphs are perfect, the size of the largest clique in a power graph is also equal to its chromatic number. Then the following result is true for power graphs of cyclic groups.

**Corollary 8.5.** Let $G \cong \mathbb{Z}/n\mathbb{Z}$ be a group with power graph $g(G)$. Also let $d_1 \leq d_2 \leq \ldots \leq d_k$ be (not necessarily distinct) prime divisors of $n$ then $\chi(g(G)) = \phi(n) + \phi\left(\frac{n}{d_1}\right) + \phi\left(\frac{n}{d_1d_2}\right) + \ldots + \phi\left(\frac{n}{d_1d_2\ldots d_k}\right)$.

Here the proof of Theorem 4.5 did not depend on the fact that $G$ was a cyclic group in any way, except that the path constructed through the elements of $G$ could always be started with an element of order $n$, where $n$ is the order of $G$. In a general group, there are no elements with order equal to the order of the group, so there are many possibilities for where to begin the longest path. Let $g_1, g_2, \ldots g_n$ be elements of a group $G$, and define $\Psi(n) = \phi(n) + \phi\left(\frac{n}{d_1}\right) + \phi\left(\frac{n}{d_1d_2}\right) + \ldots + \phi\left(\frac{n}{d_1d_2\ldots d_k}\right)$, where $d_1, d_2, \ldots d_k$ are prime divisors of $n$, then $\chi(G) = \max_i \{\Psi(g_i)\}$.

### 9. Chordallity of Power Graph of Cyclic Groups

**Proposition 9.1.** Consider $n \neq p^m$ for some prime $p$ and a positive integer $m$. If a vertex $k \in g(\mathbb{Z}_n)$ is simplicial then $\gcd(k, n) \neq 1$.

**Proof.** Let $k \in g(\mathbb{Z}_n)$ be simplicial with $\gcd(k, n) = 1$. Then $k$ generates $\mathbb{Z}_n$. So, $k$ being adjacent to every $x \in g(\mathbb{Z}_n)$, $g(\mathbb{Z}_n)$ turns to be complete, which is a contradiction as $n \neq p^m$. □

Converse of the above result is not true in general. For example: Consider $\mathbb{Z}_{12}, \gcd(6, 12) \neq 1$. Though, 6 is not a simplicial vertex in $g(\mathbb{Z}_{12})$ as because 2, 3 are adjacent to 6, but they are not adjacent to each other.

**Proposition 9.2.** $g(\mathbb{Z}_n)$ is chordal if and only if $n = p^m$ for some prime $p$ and positive integer $m$ or $n = p^mq$, for two distinct primes $p, q$ and positive integer $m$.

**Proof.** If $n = p^m$, then $g(\mathbb{Z}_n)$ being complete is chordal. If $n = p^mq$, then subgroup diagram of the group $\mathbb{Z}_n$ is chordal. And hence, $C(\mathbb{Z}_n)$ is also so, as the subgroup diagram is a subgraph of $C(\mathbb{Z}_n)$. Thus, $g(\mathbb{Z}_n)$ is chordal. Conversely, if $n \neq p^m, p^mq$, then $n$ has at least two distinct prime factors $p, q$ with powers $m, n$ where both $m, n$ are positive.
integers bigger than or equal to 2. In that case, \( p - pq - q - p^2q - p \) is a chordless cycle in \( g(Z_n) \) giving the graph as non-chordal.

**Proposition 9.3.** A vertex in \( C(Z_n) \) other than \( [n]_\sim \) and \( [1]_\sim \) is simplicial iff it has only one parent and one child.

**Proof.** Let \( x \) be a vertex in \( C(Z_n) \) other than \( [n]_\sim \) and \( [1]_\sim \). If it has more than one parent, then any two parents are not adjacent. Similarly, if it has more than one child then any two children are not adjacent to each other.

Conversely let \( x \) be a simplicial vertex. Then, it can neither have more one parent, nor more than one child.

Note that, even if \( g(Z_n) \) is not chordal for \( n = p^m q^r \) where \( m, r \geq 2 \) and \( p, q \) are distinct primes, they have simplicial vertices namely \( p^m, q^r \), as because they are simplicial in the corresponding \( C(Z_n) \).

Thus, we can now state the following result:

**Theorem 9.4.** If \( n = \prod_{i=1}^k p_i^{\alpha_i} \), then whenever \( k \geq 3 \), \( g(Z_n) \) does not have any simplicial vertex.

**Proof.** \( n \) and 1 are not simplicial. Other wise, \( g(Z_n) \) will be complete, giving more than one prime factor of \( n \) which contradicts the hypothesis of the theorem. On the other hand, if we consider any vertex in \( g(Z_n) \) namely \( m = \prod_{i=1}^k p_i^{\beta_i} \), then the equivalence classes of \( p_j p_i, p_i p_k \) are two parents of that of \( p_j \) in \( C(Z_n) \). So, the vertex is not simplicial in \( C(Z_n) \) and hence is not simplicial in \( g(Z_n) \).

\( \square \)

10. Power graph of \( U_n \) and \( Q_n \)

For any positive integer \( n \), let \( U_n \) be the multiplicative group of integers modulo \( n \) and let \( Q_n \) be its subgroup of quadratic residue modulo \( n \). Then, we have the following results.

**Proposition 10.1.** \( g(Q_n) \) is not planner whenever

i. \( n = p \) or \( 2p \), where \( p \) is a prime bigger than 37.

ii. \( n = p^m \) or \( 2p^m \), where \( p \geq 7, m \geq 2 \) or else, \( p = 3 \) or 5 and \( m \geq 2 \).

**Proof.**

i. For \( n = p \) or \( 2p \), the cardinality of \( Q_n \) is \( \mu = \frac{\phi(n)}{2} = \frac{p-1}{2} \) by Lemma 7.3 of [7]. In that case \( Q_n \) is cyclic as \( U_n \) is so. Hence by lemma 4.7 in [5], \( g(Q_n) \) is not planner if \( \phi\left(\frac{p-1}{2}\right) > 7 \), that is if \( \frac{p-1}{2} > 18 \), that is if \( p > 37 \).


Proposition 10.2. \( g(\mathbb{Q}_n) \) is planner if \( n \) divides 240.

Proof. If \( n \) divides 240, then \( g(\mathbb{U}_n) \) is planner by theorem 4.10 in [5]. Hence by proposition 4.5 in [5], \( g(\mathbb{Q}_n) \) is also planner. \( \square \)

Proposition 10.3. Let \( n = p^m \) or \( n = 2p^m \), where \( p \) is a Fermat prime. Then, \( g(\mathbb{U}_n) \) is chordal iff \( m \leq 2 \) or \( p = F_0 = 3 \).

Proof. Let \( n = p^2 \) or \( n = 2p^2 \) where \( p \) is a Fermat prime. Then, by Corollary 6.14 in [7], \( \mathbb{U}_n \) is isomorphic to \( \mathbb{Z}_p \times \mathbb{Z}_{p-1} = \mathbb{Z}_p \times \mathbb{Z}_{2^m} = \mathbb{Z}_{2^m} \), for some positive integer \( m \) as \( p \) is a Fermat Prime. Hence the graph is chordal by proposition 9.2. In a similar way, for \( n = p \) or \( n = 2p \) where \( p \) is a Fermat prime, \( \mathbb{U}_n \) is isomorphic to \( \mathbb{Z}_{p-1} = \mathbb{Z}_{2^m} \) for some positive integer \( m \) and hence is chordal by proposition 9.2. Now let \( p = F_0 = 3 \). Then, if \( n = p^m \) or \( n = 2p^m \), \( \mathbb{U}_n \) is isomorphic to \( \mathbb{Z}_{p^m-1} \times \mathbb{Z}_{2^i} \), \( i \geq 2 \). Thus the power graph is not chordal by proposition 9.2. \( \square \)

Corollary 10.4. For a positive integer \( n \), if \( p^m \) or \( 2p^m \) divides \( n \), where \( p \) is a Fermat prime bigger than 3 and \( m \geq 3 \), then \( g(\mathbb{U}_n) \) is not chordal.

Proposition 10.5. Let, \( n \) be an odd integer which is not square free and no Fermat prime be a factor of \( n \). Then, \( g(\mathbb{U}_n) \) is not Chordal.

Proof. As \( n \) is not square free, there is an odd prime \( p \), which is not a Fermat prime and \( p^m \) divides \( n \) for some \( m \geq 2 \). In that case, as in Corollary 6.14 of [7], \( \mathbb{Z}_{p^m-1}(p-1) \) appears inside the direct product decomposition of \( \mathbb{U}_n \), where \( p - 1 \) is not a power of 2 since \( p \) is not a Fermat prime. So, \( p - 1 \) being it has an odd prime factor \( q \) where \( p \neq q \), thus \( p^{m-1}(p - 1) \) contains at least 3 distinct odd primes and hence the power graph is not chordal. \( \square \)

Proposition 10.6. Let \( n \) be an even integer. Then, if

i. \( 2^f \parallel n \) (that means \( f \) is the largest integer so that \( 2^f \) divides \( n \)), where \( f \geq 4 \) and if there are at least two distinct primes \( p \) and \( q \) and positive integers \( \alpha, \beta \), where neither of \( p, q \) are Fermat’s prime, then \( g(\mathbb{U}_n) \) is not chordal.
ii. If $2^f \parallel n, f \geq 1$ and $p^\alpha, q^\beta \parallel n$, where $\alpha, \beta \geq 2$ and neither of $p, q$ are Fermat’s prime, then $g(U_n)$ is not chordal.

Proof. The proof follows by Corollary 6.14 in [7] and by our proposition 9.2.

We have analogous result regarding chordality for $g(Q_n)$.

**Proposition 10.7.** Let $n = p^m$ or $n = 2p^m$, where $p$ is a Fermat prime. Then, $g(Q_n)$ is chordal iff $m \leq 2$ or $p = F_0 = 3$ or $p = F_1 = 5$.

Proof. The proof follows by using Lemma 7.1 and Corollary 6.14 in [7] as in that case, $Q_n = \mathbb{Z}_{\phi(p^m)} = \mathbb{Z}_{p^{m-1}(p-1)}$. Hence, by similar method as in the proof of proposition 10.5, the result follows.

**Corollary 10.8.** For a positive integer $n$, if $p^m$ or $2p^m$ divides $n$, where $p$ is a Fermat prime bigger than 3, 5 and $m \geq 3$, then $g(Q_n)$ is not chordal.

**Proposition 10.9.** Let, $n$ be an odd integer which is not square free and no Fermat prime be a factor of $n$. Then, $g(Q_n)$ is not Chordal.

Proof. The proof follows as in proposition 10.5 and Lemma 7.1 in [7].

**Proposition 10.10.** Let $n$ be an even integer. Then, if

i. $2^f \parallel n$ where $f \geq 5$ and if there are at least two distinct primes $p$ and $q$ and positive integers $\alpha, \beta$, where neither of $p, q$ are Fermat’s prime, then $g(Q_n)$ is not chordal.

ii. If $2^f \parallel n, f \geq 2$ and $p^\alpha, q^\beta \parallel n$, where $\alpha, \beta \geq 2$ and neither of $p, q$ are Fermat’s prime, then $g(Q_n)$ is not chordal.

Proof. The proof follows as in proposition 10.6.

Finally, we conclude with a question: Precisely, for what positive integers $n, g(U_n)$ and $g(Q_n)$ are chordal?

**Acknowledgment:** Authors acknowledge Dr. M. K. Sen for his helpful suggestions and advice towards this work.

**References**

[1] J. John Arul Singh and S. Devi, *Cyclic Subgroup Graph of a Finite Group*, International Journal of Pure and Applied Mathematics, 111, 403-408, 2016.

[2] Maria Chudnovsky, Neil Robertson, Paul D. Seymour, Robin Thomas. *The Strong Perfect Graph Theorem*, Annals of Mathematics 164 No. 1, 51-229, 2006.

[3] J.A. Bondy and U.S.R. Murty, *Graph Theory* Springer, 48, 2008.

[4] P. J. Cameron, S. Ghosh, *The power graph of a finite group*, Discrete Mathematics, Volume 311, Issue 13, 6 July 2011, Pages 1220-1222
[5] I. Chakrabarty, S. Ghosh, M.K.Sen, *Undirected power graphs of semigroups*, Semigroup Forum, 78(2009), 410–426

[6] A. V. Kelarev, S. J. Quinn, *A combinatorial property and power graphs of groups*, D. Dornberger, G. Eigenthaler, M. Goldstern, H.K. Kaiser, W. More, W.B. Mueller (Eds.), Contrib. General Algebra 12, Springer-Verlag (2000), pp. 229-235 58. Arbeitstagung Allgemeine Algebra (Vienna University of Technology, June 3–6, 1999)

[7] G. A. Jones and J. M. Jones, *Elementary Number Theory*, ISBN 978-3-540-76197-6, ISBN 978-1-4471-0613-5, DOI 10.1007/978-1-4471-0613-5

[8] P. J. Camerona, *The power graph of a finite group, II*, Journal of Group Theory, Volume 13: Issue 6

Department of Mathematics and Statistics, University of Toledo, Main Campus, Toledo, OH 43606-3390

Email address: Amrita.Acharyya@utoledo.edu

Department of Mathematics and Statistics, University of Toledo, Main Campus, Toledo, OH 43606-3390

Email address: Allen.Williams2@rockets.utoledo.edu