THE DIHEDRAL RIGIDITY CONJECTURE FOR $n$-CUBES

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Abstract. We prove the following comparison theorem for metrics with nonnegative scalar curvature, also known as the dihedral rigidity conjecture by Gromov: for $n \leq 7$, if an $n$-dimensional cube has nonnegative scalar curvature and weakly mean convex faces, then its dihedral angle cannot be everywhere non-obtuse, unless it is isometric to an Euclidean rectangular solid. The proof relies on the idea of slicing a polyhedron with free boundary minimal surfaces, an extension of Schoen-Yau’s proof of the positive mass theorem.

1. Introduction

In a paper [Gro14] from 2014, Gromov proposed the first steps towards understanding Riemannian manifolds with scalar curvature bounded below. He suggested that a polyhedron comparison theorem should play a role analogous to that of the Alexandrov’s triangle comparisons for spaces with sectional curvature lower bounds [ABN86]. Precisely, let $M^n$ be a convex polyhedron in Euclidean space, and $g$ a metric on $M$. Denote the Euclidean metric $g_0$. Gromov made the following conjecture (see section 2.2 of [Gro14], and section 7, Question $F_1$ of [Gro18a]):

Conjecture 1.1 (The dihedral rigidity conjecture). Suppose $(M,g)$ has nonnegative scalar curvature and weakly mean convex faces, and along the intersection of any two adjacent faces, the dihedral angle of $(M,g)$ is not larger than the (constant) dihedral angle of $(M,g_0)$. Then $(M,g)$ is isometric to a flat Euclidean polyhedron.

When $n = 2$, Conjecture 1.1 follows directly from the Gauss-Bonnet formula. In fact, given a Riemann surface $(M^2,g)$, the Gauss curvature of $K_g > 0$ everywhere if and only if there exists no geodesic triangle with total inner angle smaller than $2\pi$. This fact is generalized by Alexandrov [ABN86] in the study of sectional curvature lower bounds in all dimensions.

As a first step towards Conjecture 1.1, Gromov studied the case for cubes, and obtained the following theorem:

Theorem 1.2 ([Gro14]). Let $M = [0, 1]^n$ be a cube, and $g$ be a Riemannian metric on $M$. Then $(M,g)$ cannot simultaneously satisfy:

1. The scalar curvature of $g$ is positive;
2. Each face of $M$ is strictly mean convex with respect to the outward normal vector field;
(3) Everywhere the dihedral angle between two faces of $M$ is acute.

The crucial observation is that conditions (2) and (3) may be interpreted as $C^0$ properties of the metric $g$. Thus, Gromov proposed a possible definition of $R \geq 0$ for $C^0$ metrics:

$$R(g) \geq 0 \iff \text{there exists no cube } M$$

with mean convex faces and everywhere acute dihedral angle.

Gromov’s proof (or at least a sketch of proof) of Theorem 1.2 is based on an elegant doubling argument and the fact that the $n$-dimensional torus admits no metric with positive scalar curvature. However, carrying out the argument in detail seems challenging, and requires a deep understanding of positive scalar curvature metrics with stratified singularities. We refer the readers to [LM18] for a detailed discussion, and [Kaz19] for some recent progress. Also, the doubling construction relies on the fact that the cube is the fundamental domain of $\mathbb{Z}^n$ action on $\mathbb{R}^n$, hence not applicable to general polytopes. Most importantly, this argument cannot handle the rigidity statement in Conjecture 1.1. In fact, even in the Euclidean space, it is unknown whether one can perturb a flat convex polyhedron, such that the faces are still minimal surfaces, while the dihedral angles between them remain the same. See section 1.5 of [Gro14].

In dimension 3, Conjecture 1.1 is verified by the author [Li17] for a large collection of polytopes, including the cubes and all 3-simplices. The idea is to relate Conjecture 1.1 with a natural geometric variational problem of capillary type. The primary scope of this paper is to extend this idea and prove Conjecture 1.1 for cubes when $n \leq 7$. In fact, we will prove that cubes are dihedrally rigid among a much larger class of manifolds, which are defined as overcubic manifolds.

**Definition 1.3.** Let $Q^n$ be a smooth manifold. A manifold $M^n$ with non-empty boundary is called overcubic, if $M^n$ is the domain enclosed by $2n$ smooth hypersurfaces $F_1, \cdots, F_{2n}$ of $Q$ intersecting transversely, and there exists a map $\phi : M \to [0,1]^n$ of degree one, such that for each $k < n$, any $k$-face of $M$ maps to a $k$-faces of $[0,1]^n$.

Throughout the paper, we denote the (closed) faces of $M$ by $F_1, \cdots, F_{2n}$.

We now state the main theorem of this paper:

**Theorem 1.4.** Assume $(M^n, g)$ is an overcubic Riemannian manifold with a $C^{2,\alpha}$ metric $g$, $n \leq 7$, such that

1. The scalar curvature of $g$ is nonnegative;
2. Each face of $M$ is weakly mean convex;
3. The dihedral angles between adjacent faces of $M$ is everywhere equal to $\pi/2$.

The same idea seems also relevant to Conjecture 1.1 for $n$-dimensional simplices. However, the major technical difficulty is due to the lack of a satisfactory regularity theory of capillary surfaces. See Section 6 for more discussions.
Then \((M,g)\) is isometric to an Euclidean rectangular solid.

**Remark 1.5.** In the statement of Theorem 1.4, we put the stronger assumption that the dihedral angle of \(M^n\) is everywhere equal to \(\pi/2\). However, one may reduce the general case where the dihedral angles are merely non-obtuse to the case treated by Theorem 1.4 via a bending construction by Gromov (section 11 of [Gro18b]). We state the relevant results in Appendix A.

As an important feature of the theorem, we do not make any a priori topological assumptions other than that \(M\) is overcubic. The fact that \(M\) is contractible is a consequence, rather than a presumption, of the geometric conditions (1)-(3). This extra generality plays an essential role in the proof, as we implement area minimizing currents in the proof, whose topological type is a priori not well controlled.

Now let us discuss the structure of the proof of Theorem 1.4. Let \(M^n\) be an overcubic manifold with conditions (1)-(3). Take two opposite faces \(F_1, F'_1\). Namely, take the pre-image of \(\{0\} \times \mathbb{R}^{n-1}\) and \(\{1\} \times \mathbb{R}^{n-1}\) of the map \(\phi\). Consider the variational problem

\[
I = \inf \{\mathcal{H}^{n-1}(\partial D \cap M) : D \text{ is an open set of finite perimeter containing } F_1, D \cap F'_1 = \emptyset\}. \tag{1.1}
\]

By the Federer-Fleming compactness theorem, \(I\) is always achieved by the mass of an integral current \(\Sigma^{n-1} = \partial D \cap M\). However, (1.1) is a variational problem with weakly mean convex barriers \(F_1, F'_1\). Thus \(\Sigma\) may not be a priori stationary with respect to all deformations. To overcome this, we establish a strong maximum principle for varifolds with free boundary:

**Theorem 1.6.** Given an over cubic Riemannian polyhedron \((M^n, g)\) and two opposite faces \(F_1, F'_1\) as above. Let \(V\) be an \((n-1)\)-dimensional varifold in \(M\) with free boundary on \(\partial M \setminus (F_1 \cup F'_1)\). Then either \(\text{spt } V\) is disjoint from \(F_1\) and \(F'_1\), or \(\text{spt } V\) entirely coincides with \(F_1\) (or \(F'_1\)). In the latter case, \(F_1\) (or \(F'_1\)) is a minimal surface.

We remark that the interior maximum principles of minimal surfaces has been extensively studied in various settings, see, for example, [Sim87, SW89, Whi11, Wic14]. Recently, Li-Zhou established a weak maximum principle on the boundary of a stationary varifold, see [LZ17]. Theorem 1.6 is a further extension of these results.

When \(n \leq 7\), by the regularity theory in [Ede16] (which we summarize in Theorem 2.3) and the elliptic regularity theory developed in Appendix B, we are able to conclude that \(\Sigma\) is a \(C^{2,\alpha}\) hypersurface. Since each \(k\)-face of \(\Sigma\) is homologous to the corresponding \(k\)-face of \(F_1\), \(\Sigma\) is an overcubic manifold of dimension \((n-1)\). Inspired by the celebrated positive mass theorem in higher dimensions (Schoen-Yau [SY79]), we then argue that \(\Sigma\) can be conformally deformed so that the geometric conditions (1)-(3) are also preserved. Inductively, we are able to construct a free boundary minimal
slicing \[ \Sigma^2 \subset \cdots \subset \Sigma^{n-1} \subset M \]
of \( M \), such that each \( \Sigma^j \) itself is an overcubic manifold satisfying the conditions (1)-(3). In dimension 2, the Gauss-Bonnet formula guarantees that \( \Sigma^2 \) is flat. Using induction, we obtained a minimal surface proof of Theorem 1.2.

To perform the rigidity analysis, we extend an elegant idea of Carlotto-Chodosh-Eichmair \[\text{CCE16}\] in the study of effective positive mass theorems, which was inspired on earlier works on deformations of Riemannian metrics with other curvature conditions (see \[\text{Ehr76, AR89, Liu13}\]). Precisely, through an argument involving a family of well-chosen local conformal deformations, we prove that \( M \) contains a dense collection of free boundary area minimizing surfaces, whose boundary is also dense on \( \partial M \). This is enough to guarantee that \( M \) is isometric to an Euclidean rectangular solid. We also mention here related rigidity phenomena involving scalar curvature \[\text{CG00, BBN10, CEM18}\] and the survey article by Brendle \[\text{Bre12}\].

A central difficulty throughout this paper is due to the singular nature of a polyhedron. However, when all the dihedral angles are \( \pi/2 \), we gain several advantages in the analysis. For instance, we prove the following curvature estimate necessary for our rigidity analysis, which is of interest on its own:

**Theorem 1.7.** Let \((M^n, g)\) be an overcubic manifold such that the dihedral angles of \( M \) is everywhere \( \pi/2 \), \( 3 \leq n \leq 7 \). Suppose \( \Sigma^{n-1} \) is an embedded stable free boundary minimal surface in \( M^n \) with area bounded by \( \Lambda \). Then the curvature estimate

\[ \sup_\Sigma |A_\Sigma| < C \]

holds, where \( C \) depends on \((M, g)\) and \( \Lambda \).

Theorem 1.7 is an extension of the classical curvature estimates of Schoen-Simon-Yau \[\text{SSY75}\] and Schoen-Simon \[\text{SS81}\] in the interior, and the recent work by Guang-Li-Zhou \[\text{GLZ18}\] for free boundary surfaces with smooth boundary.

Based on the recent progress of the positive mass theorem in all dimensions \[\text{SY17}\], we speculate that the singular analysis in \[\text{SY17}\] may be helpful to generalize Theorem 1.4 to all dimensions. Also, the same minimal slicing idea suggests a possible approach to Conjecture 1.1 for other polytopes. However, the main difficulty is the lack of regularity of capillary surfaces (see \[\text{DPM15}\] for some recent development). We briefly discuss these questions in section 6.

This paper is organized as follows. In section 2, we review some preliminary geometric properties of Riemannian polyhedra, as well as free boundary minimal surfaces. In section 3, we prove the strong maximum principle (Theorem 1.6) for free boundary varifolds. In section 4, we prove the curvature estimate (Theorem 1.7) for embedded free boundary stable minimal hypersurfaces. We then prove Theorem 1.4 in section 5. Finally, we remark
some possible further developments regarding Conjecture [LZ17] in section 6. We include Gromov’s bending construction in Appendix A, and develop an elliptic regularity theory in cubicle domains in Appendix B.

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2. Preliminaries of Riemannian polyhedron and free boundary minimal surfaces

In this section we review some preliminary facts on the Riemannian geometry in a polyhedron, and on free boundary minimal surface in domains with piecewise smooth boundary. We also include a regularity theorem for free boundary minimal hypersurfaces in polyhedron domains.

2.1. Local geometry of a Riemannian polyhedron. Let $M^n$ be a Riemannian polyhedron, and $\{F_j\}_{j=1}^N$ be its faces. We assume all adjacent faces meet everywhere at $\pi/2$. Let $p \in \partial M$ be a point that lies on the intersection of $k$ faces of $M$. That is, suppose $F_1, \ldots, F_k$ be $k$ faces meeting transversely, and $p \in \cap_{i=1}^k F_i$. Thus, a local neighborhood of $p$ in $M$ is diffeomorphic to the cubicle region $\{(x_1, \ldots, x_n) : x_1^2 + \cdots + x_n^2 < r^2, x_j \geq 0, j = 1, \ldots, k\}$, where under this diffeomorphism, the face $F_j$ got mapped onto $\{x_j = 0\}$. We first establish a local foliation result that is essentially taken from Lemma 2.1 in [LZ17].

**Lemma 2.1.** There exists a constant $\delta > 0$, a neighborhood $U$ of $p$ in $M$, and a foliation $F_1^s$ with $s \in [0, \delta)$, of $U$, such that $F_1^0 = F_1 \cap U$, and each $F_1^s$ meets $F_j$ orthogonally, $j = 1, \ldots, k$.

**Proof.** For $j = 2, \ldots, k$, let $x_j$ be the signed distance function in $M$ to the face $F_j$, such that $x_j > 0$ in the interior of $M$. Let $(x_1, x_{k+1}, \ldots, x_n)$ be a local Fermi coordinate system of the smooth submanifold $\cap_{j=2}^k F_j$, such that $x_1$ is the distance to the submanifold $\cap_{j=1}^k F_j$. This gives a local coordinate system of $U$ with the property that $\partial_j$ is normal to $F_j$. We can express $F_1$ as a graph $x_1 = f(x_2, \ldots, x_n)$, where $f$ is a function defined in the Euclidean cubicle domain $\{(x_1, \ldots, x_n) : x_1^2 + \cdots + x_n^2 < r^2, x_j \geq 0, j = 1, \ldots, k\}$, with the property that $f = 0$ along $\partial F_1$, and $\frac{\partial f}{\partial x_j} = 0$ along $F_1 \cap F_j$, $j \geq 2$. The translated graphs $x_1 = f(x_2, \ldots, x_n) + s, s \in [0, \delta)$ then gives a local foliation $F_1^s$ which meets $F_2, \ldots, F_k$ orthogonally along their intersections. \[\square\]
As a direct corollary, for each $j = 2, \cdots, k$, the same neighborhood $U$ can also be foliated by leaves $\{F^s_j\}$, $s \in [0, \delta)$, where each $F^s_j$ is everywhere orthogonal to $F_1$ along their intersection.

**Lemma 2.2.** There exists a constant $\delta > 0$ and a neighborhood $U$ of $p$ in $M$, such that for each $j = 2, \cdots, k$, there exists a foliation $\{F^s_j\}_{s \in [0, \delta)}$ of $U$, with the following properties:

1. $F^0_j = F_j \cap U$, and for every $s, t \in [0, \delta)$, $F^s_j$ and $F^t_1$ intersect orthogonally.
2. For any $i, j \in \{2, \cdots, k\}$, $s \in [0, \delta)$, $F^s_j$ and $F^s_i$ intersect orthogonally along $F^s_1$.

**Proof.** Perform induction on the dimension $n$. When $n = 2$, the result follows directly from Lemma 2.1. Suppose the assertion holds for dimension $n - 1$. Take the foliation $\{F^t_1\}_{t \in [0, \delta)}$ constructed in Lemma 2.1. For a point $q \in U$, let $F^t_1$ be the unique leaf that contains $q$, and $\nu(q)$ be the unit normal vector field of $F^t_1$, properly chosen such that $\nu(q)$ is continuous, and is equal to the inward unit normal vector field on $F^0_1 = F_1$. By construction, $\nu(q)$ is tangential to $\partial M$ whenever $q \in \partial M$. The vector field $\nu(q)$ is integrable in $U$, and its integral curves (parametrized by $\{\Gamma_q \mid q \in F^t_1 \cap U\}$) give an one-dimensional foliation of $U$.

Observe that $F^t_1$ itself is a Riemannian cubicle manifold of dimension $(n - 1)$, with faces $F_j \cap F^t_1$, $j = 2, \cdots, k$, meeting orthogonally along their intersections, and $q$ lies on the intersection of all the faces. By inductive hypothesis, for each $j$, there is a foliation $(F^s_j \cap F^t_1)$, $s \in [0, \delta)$ of a neighborhood $U \cap F^t_1$, such that when $i \neq j$, $(F^s_j \cap F^t_1)$ meets $F^s_i \cap F^s_1$ orthogonally. Now consider the submanifold of $U$ given by

$$F^s_j = \bigcup_{q \in (F^s_j \cap F^t_1)^s} \Gamma_q.$$

By possibly taking the intersection of a smaller neighborhood, $\{F^s_j\}$ is a foliation of $U$. Conclusion (1) follows from the fact that each curve $\Gamma_q$ is orthogonal to $F^s_1$. Conclusion (2) follows from the induction hypothesis. \(\square\)

### 2.2. Preliminaries of free boundary minimal surfaces

We briefly describe the geometry of free boundary minimal surfaces. Let $(M, g)$ be an $n$-dimensional compact Riemannian manifold with Lipschitz boundary. Assume also that $\partial M$ is piecewise $C^{2,\alpha}$ smooth. Suppose $(\Sigma, \partial \Sigma) \subset (M, \partial M)$ is an embedded hypersurface. We say $(\Sigma, \partial \Sigma)$ is an embedded free boundary minimal hypersurface, if the interior of $\Sigma$ is minimal (having zero mean curvature), and $\Sigma$ meets $\partial M$ orthogonally.

Free boundary minimal hypersurfaces are the critical points of $(n - 1)$-dimensional volume functional of $(M, g)$ among class of all embedded hypersurfaces. Given a vector field $Y$ in $M$ which is tangential to $\partial M$, let $\psi_t$ be the one-parameter family of diffeomorphisms generated by $Y$. Then for $\varepsilon$
small enough, $\psi_t(\Sigma)$ is a smoothly embedded hypersurface in $M$. The first variational formula implies that
\[
\frac{d}{dt} \bigg|_{t=0} \mathcal{H}^{n-1}(\psi_t(\Sigma)) = -\int_{\Sigma} X \cdot \vec{H} dV + \int_{\partial \Sigma} X \cdot \eta dS,
\]
where $\vec{H}$ is the mean curvature vector field of $\Sigma$, $\eta$ is the outward conormal vector field of $\partial \Sigma \subset \Sigma$, $dV$ and $dS$ are the induced volume forms on $\Sigma$ and $\partial \Sigma$, respectively. It is then clear that $\Sigma$ is a critical point of the $(n-1)$-dimensional volume functional, if and only if $\mathcal{H}^{n-1}(\psi_t(\Sigma))$ vanishes for all admissible vector fields $Y$, which is equivalent to the fact that $\vec{H} \equiv 0$ and that $\Sigma$ meets $\partial M$ orthogonally.

The second variational formula for a free boundary minimal hypersurface is given as
\[
\frac{d^2}{dt^2} \bigg|_{t=0} \mathcal{H}^{n-1}(\psi_t(\Sigma)) = Q(f, f)
\]
\[
= \int_{\Sigma} |\nabla \Sigma f|^2 - (|A_{\Sigma}|^2 + \text{Ric}_M(\nu, \nu)) f^2 dV - \int_{\partial \Sigma} \Pi_{\partial M}(\nu, \nu) f^2 dA.
\]
Here $\nu$ is the unit normal vector field of $\Sigma$ in $M$, $f = Y \cdot \nu$ is the normal component of the variation, $A_{\Sigma}$ is the second fundamental form of $\Sigma$ in $M$, $\Pi$ is the second fundamental form of $\partial M$ in $M$, $\text{Ric}_M$ is the Ricci curvature of $M$. We have adopted the sign convention that $A_{\Sigma} > 0$ for convex hypersurfaces in $\mathbb{R}^n$. In particular, the unit 2-sphere in $\mathbb{R}^3$ has constant mean curvature $2$.

Call a two-sided free boundary minimal hypersurface $\Sigma$ stable, if its second variation is always nonnegative. By (2.2), $\Sigma$ is stable, if and only if $Q(f, f) \geq 0$ for any smooth function $f$.

2.3. Regularity of free boundary area minimizing currents. In [EL], the authors established a regularity theory for free boundary varifolds in locally convex domains. We briefly describe the results relevant to this paper. Given an integer $k \geq 2$, let $\Omega_0 \subset \mathbb{R}^k$ be a polyhedral cone enclosed by $k$ hypersurfaces of $\mathbb{R}^k$ through $0^k$, such that the dihedral angle between each pair of faces is not larger than $\pi/2$. Suppose $l \geq 1$, and $\Omega = \Omega_0 \times \mathbb{R}^l$. Denote $n = k + l$. Let $\Phi : B_1(0^n) \rightarrow \mathbb{R}^n$ be a $C^{2,\alpha}$ mapping with
\[
\Phi(0) = 0, \quad D\Phi(0)|_0 = Id, \quad |\Phi - Id|_{C^{2,\alpha}(B_1)} \leq \Gamma \leq 1.
\]
Let $\Omega_\Phi = \Phi(\Omega)$.

Assume $\Sigma$ be an area minimizing currents in $\Phi(\Omega)$, such that $0 \in \text{spt} \Sigma$, and the associated measure $\mu_{\Sigma}$ satisfies $\mu_{\Sigma}(\partial \Omega) = 0$. Assume also that $\Sigma$ has free boundary, in the sense that $\Sigma(X) = 0$ for all vectors $X \in C^1_0(B_1, \mathbb{R}^n)$ which are tangential to $\Omega_\Phi$. We then have the following theorem:

**Theorem 2.3.** There is an $\alpha_0(\Omega_0)$, $\delta(\Omega_0, n)$, and some linear subspace $\mathbb{R}^{l-1} \subset \mathbb{R}^l$, such that for any $\alpha \in (0, \alpha_0)$, if $\Gamma \leq \delta^2$, then there exists a...
function $u : (\Omega_0 \times \mathbb{R}^{l-1}) \cap B_{1/2}(0) \to \mathbb{R}$, so that
$$\Sigma \cap B_{1/2} = \text{[graph } u\text{]}.$$
we constructed before, we prove a similar boundary maximum principle in a Riemannian polyhedron.

**Proposition 3.1.** Let $M^n$ be an overcubic Riemannian manifold, such that each pair of intersecting faces meet at constant angle $\pi/2$. Suppose $p \in \cap_{j=1}^k F_j$ is a corner point, and $F_1$ is strictly mean convex at $p$. Then $p$ is not contained in the support of any $(n-1)$ dimensional varifold $V$ which is supported in $M$ and stationary with free boundary and barrier $F_1$.

**Proof.** The proof here closely resembles those in [Whi10] and [LZ17]. Suppose there is a $(n-1)$ dimensional free boundary stationary varifold $V$ with barrier $F_1$ whose support contains $p$, we will construct a vector field $X$, $X$ is supported in a neighborhood $U$ containing $p$, tangential to $\partial M$ away from $F_1$, pointing inward $M$ on $F_1$, and $\delta V(X) < 0$.

We first place $M$ in a larger Riemannian manifold near $p$. Precisely, for a coordinate chart $U$ centered at $p$ and a diffeomorphism $\phi : \{(x_1, \cdots , x_n) : x_1^2 + \cdots + x_n^2 < r^2, x_j \geq 0, j = 1, \cdots , k\} \to U$, extend the metric tensor smoothly into $B_r(0)$, such that the hypersurfaces $\{x_j = 0\}_{j=1}^k$ are still mutually orthogonal. Denote $M^*$ the region $\phi(\{(x_1, \cdots , x_n) : x_j \geq 0, j = 2, \cdots , k\})$. With slight abuse of notation, we denote the neighborhood $\phi(B_r(0))$ in the larger Riemannian manifold also by $U$.

For $\varepsilon > 0$ small enough, define a hypersurface $\Gamma \subset \partial M^* \cap U$ by letting

$$\Gamma = \{x \in \partial M : \text{dist}_\partial M(x, F_1 \cap \partial M^*) = \varepsilon \text{dist}_\partial M(x, p)\}.$$  

As in [LZ17], using the coordinate system in Lemma 2.1, one can easily find a face $\overline{F}_1 \subset U$ with $\partial \overline{F}_1 = \Gamma$. Notice that $\overline{F}_1$ and $F_1$ touches in second order at $p$. By construction, the region separated by $\overline{F}_1$ in $M^*$ is an overcubic Riemannian manifold in $U$, where all pair of faces meet orthogonally. Furthermore, since we assume that $F_1$ is strictly mean convex at $p$, by taking $\varepsilon > 0$ small enough, $\overline{F}_1$ is also strictly mean convex at $p$.

Since $\overline{F}_1$ and $F_j$, $j \geq 2$, intersects pairwise orthogonally, by Lemma 2.2 there exists a local foliation $\overline{F}_1^s$, $F_j^s$, $s \in [0, \delta)$ such that each $F_j^s$ meets each $F_j^t$ orthogonally. Let $s$ be the function in $U$ such that $s(q)$ is the unique $s$ where $q \in \overline{F}_1^s$. Let $\nu$ be the inward pointing unit normal vector field of $\overline{F}_1^s$, defined in $U$. By construction in Lemma 2.2 $\nabla s = \psi \nu$ for some function $\psi$, and $\psi = 1$ along $F_j$, $j = 2, \cdots , k$.

Define a vector field $X$ on $M^*$ by letting $X(q) = \phi(s(q))\nu(q)$, where $\phi(s)$ is a cutoff function defined by

$$\phi(s) = \begin{cases} e^{1/(s-\varepsilon)}, & 0 \leq s \leq \varepsilon, \\ 0, & s \geq \varepsilon. \end{cases}$$

In $U$, $X$ is a vector field tangential to $\partial M$ along $F_j$, and is inward pointing on $F_1$, provided that $\varepsilon$ is sufficiently small.

Let $\{e_i\}_{i=1}^n$ be a local frame of unit vectors such that $e_1 = \nu$, $e_j$ be the inward pointing unit normal vector field of $F_j^s$, $j = 2, \cdots , k$. We
can take \( \{e_1, e_{k+1}, \ldots , e_n\} \) to be an orthonormal frame, also orthogonal to \( \{e_2, \ldots , e_k\} \). Also at \( p \), \( \{e_i\}_{i=1}^n \) is orthonormal. Define a bilinear form \( Q \) on \( T_q M^* \) by letting \( Q(u,v) = \langle \nabla_u X, v \rangle (q) \). We calculate the components of \( Q \). Since each leaf \( F_1^r \) is orthogonal to each \( F_j^r \), \( j \geq 2 \), we have that \( e_j(s) = 0 \). We calculate the components of \( Q \) as follows.

First, \( Q(e_1, e_1) = Q(\nu, \nu) = \langle \nabla_\nu (\phi \nu), \nu \rangle = \nu (\phi) = \phi' \psi \).

For \( i, j \geq 2 \), \( Q(e_i, e_j) = \langle \nabla_{ei_i} (\phi \nu), e_j \rangle = -\phi A^{F_i^s}(e_i, e_j) \). Here \( A^{F_i} \) is the second fundamental form of the hypersurface \( F_1 \), taken with respect to the unit normal \( \nu \).

For \( i \geq 2 \), \( Q(e_i, e_1) = Q(e_i, \nu) = \langle \nabla_{ei_i} (\phi \nu), \nu \rangle = \phi \langle \nabla_{ei_i} \nu, \nu \rangle = 0 \).

For \( i \geq 2 \), \( Q(e_i, e_i) = Q(\nu, e_i) = \langle \nabla_\nu (\phi \nu), e_i \rangle = \phi \langle \nabla_\nu \nu, e_i \rangle \). If we further have that \( i \leq k \), then this expression is also equal to \( \phi A^{F_i^s}(\nu, \nu) \).

Since \( F_1 \) is a small \( C^1 \) perturbation of \( F_1 \), by taking \( \varepsilon \in (0, \varepsilon_0) \) and \( \delta \in (0, \delta_0) \) both sufficiently small, we have the following uniform estimates:

\[
|A^{F_i^s}|, |\langle \nabla_\nu \nu, e_i \rangle|, |A^{F_i^s}| < K,
\]

for some constant \( K = K(\varepsilon_0, \delta_0) \). Also, by construction of \( \phi \), it is straightforward to check that \( \phi' \leq -\frac{1}{2} \phi \). Also, since \( \psi = 1 \) at \( p \), by taking \( \delta \) sufficiently small, we have that \( \psi \geq \frac{1}{2} \) in \( U \).

To finish the proof, we verify that for any \((n-1)\) dimensional subspace \( P \subset T_q M^* \), \( \text{tr}_P Q < 0 \). Let \( c_0 > 0 \) be a lower bound of the mean curvature of \( F_1^s \) for \( s \in [0, \delta] \). If \( P = T_q F_1^s \), then \( \text{tr}_P Q \leq -c_0 < 0 \), since \( F_1^s \) is strictly mean convex. If \( P \notin T_q F_1^s \), since \( P \) and \( T_q F_1^s \) are two \((n-1)\) dimensional subspaces of \( \mathbb{R}^n \), we can find an orthonormal basis \( v_1, \ldots , v_{n-1} \) of \( P \), such that \( v_1, \ldots , v_{n-2} \in T_q F_1^s \), \( v_{n-1} \notin T_q F_1^s \), and we have an orthogonal decomposition \( v_{n-1} = \cos \theta v_0 + \sin \theta \nu \), here \( v_0 \in T_q F_1^s \) and \( v_0 \perp \nu \), \( j = 1, \ldots , n-2 \). In particular, \( v_0, \ldots , v_{n-2} \) is an orthonormal basis of \( T_q F_1^s \), and \( v_0 \perp \nu \). We express \( v_0 \) as a linear combination of \( \{e_2, \ldots , e_n\} \) (which is an orthogonal basis of \( T_q F_1^s \)):

\[
v_0 = \sum_{j=2}^n a_j e_j.
\]

Notice that \( \{e_j\}_{j=2}^n \) is not orthonormal in \( U \), but \( \langle e_i, e_j \rangle \to \delta_{ij} \), if we take the neighborhood \( U \) small enough. Therefore, we conclude that \( |a_j| < 2 \), for each \( j = 2, \ldots , n \).

By a direct calculation as in \([LZ17]\) (Lemma 3.2, Page 9), we have:
THE DIHEDRAL RIGIDITY CONJECTURE FOR $n$-CUBES

$$\text{tr}_P Q = \sum_{i=1}^{n-1} Q(v_i, v_i)$$
$$= \sum_{i=1}^{n-2} Q(v_i, v_i) + \cos^2 \theta Q(v_0, v_0) + \sin \theta \cos \theta(Q(\nu, v_0) + Q(v_0, \nu))$$
$$+ \sin^2 \theta Q(\nu, \nu)$$
$$= \sum_{i=0}^{n-1} Q(v_i, v_i) + \sin^2 \theta(Q(\nu, \nu) - Q(v_0, v_0))$$
$$+ \sin \theta \cos \theta(Q(\nu, v_0) + Q(v_0, \nu))$$
$$= -\phi H^{F_1} + \sin^2 \theta \left(\phi' \psi + \phi A^{F_1}(v_0, v_0)\right) + \sin \theta \cos \theta Q(\nu, v_0)$$
$$\leq -c_0 \phi + \sin^2 \theta \phi \left(-\frac{1}{2\varepsilon^2} + K\right) + \sin \theta \cos \theta \sum_{j=2}^{n} a_j Q(\nu, e_j)$$
$$\leq -c_0 \phi + \phi \left(-\frac{\sin^2 \theta}{2\varepsilon^2} + K\sin^2 \theta + 2(n-1)K|\sin \theta \cos \theta|\right).$$

(3.1)

Lastly, it is straightforward to check (see Lemma 3.3 in [LZ17]) that when $\varepsilon_0 > 0$ is sufficiently small (depending on $K, n$) and $\varepsilon \in [0, \varepsilon_0)$,

$$\max_{\theta \in [0, \pi]} \left[-\frac{\sin^2 \theta}{2\varepsilon^2} + K\sin^2 \theta + 2(n-1)K|\sin \theta \cos \theta|\right] < \frac{c_0}{2}.$$

This gives that $\text{tr}_P Q < 0$ for all $(n-1)$ dimensional subspace $P$. In particular, $\delta V(X) < 0$ whenever $\text{spt} V \cap U \neq \emptyset$. This finishes the proof.

3.2. A strong maximum principle. We proceed to prove a strong maximum principle for free boundary varifolds. Precisely, suppose that $V$ is a stationary varifold with free boundary in $M$ and with barrier $F_1$, where $F_1$ meets the other faces of $M$ orthogonally, and $F_1$ is weakly mean convex. As discussed above, if $V$ is a $C^2$ hypersurface, $V$ and $F_1$ cannot have any common point, unless they coincide entirely. Here we prove this fact for all varifolds $V$. The argument here is a generalization of a beautiful idea of Solomon-White [SW89].

**Proposition 3.2.** With the same assumptions in Proposition 3.1 except that now $F_1$ is only assumed to be weakly mean convex. Then the support of any $(n-1)$ dimensional free boundary stationary varifold $V$ cannot contain any point $p$ on $F_1$, unless $\text{spt} V = F_1$.

**Proof.** It suffices to prove that if $\text{spt} V$ contains a point $p \in F_1$, it contains a neighborhood of $p$ in $F_1$. By [SW89], we only need to consider the case
when \( p \in \partial F_1 \). The main idea is that, if a sufficiently small neighborhood of \( p \) in \( F_1 \) is not entirely contained in \( \text{spt} \, V \), then we may deform the barrier \( F_1 \) to \( \tilde{F}_1 \), so that it still meets other faces orthogonally, and the \( V \) still lies on one side of \( \tilde{F}_1 \), but \( \tilde{F}_1 \) is strictly mean convex with respect to outward unit normal, violating Proposition 3.1. The deformed surface \( \tilde{F}_1 \) will be given as the graph of a function.

We first extend \( M \) beyond its boundary, near \( p \), as in the proof of Proposition 3.1. Let \( U \) be a small neighborhood of \( p \) in \( M \). Take the foliation \( F^s_1 \) constructed in Lemma 2.1 and let \( \nu \) be the inward unit normal vector field of \( F^s_1 \). Let \( \phi_t \), \( t \in [0, \delta] \), be the 1-parameter family of diffeomorphisms generated by \( \nu \). Then \( \nu \) is tangential to \( \partial M \) on \( \partial M \setminus F_1 \). Let \( \text{Fix } \alpha \in (0, 1) \). For a function \( u \in C^{2,\alpha}(F_1) \), denote \( F_u \) be the surface given by

\[
F_u = \{ (\phi_u(q)) : q \in F_1 \}.
\]

We denote the mean curvature of \( F_u \) (with respect to outward unit normal vector field) and the angle between \( F_u \) and \( \partial M \) by \( H_u \) and \( \theta_u \), respectively. By a direct calculation (see, for instance, the appendix of Amb15), the linearized operators of \( H_u \) and \( \theta_u \) are given by:

\[
\begin{align*}
\partial_t|_{t=0} H_{(tu)} &= \Delta u + (\text{Ric}_M(\nu, \nu) + |A|^2)u, \\
\partial_t|_{t=0} \cos \theta_{(tu)} &= -\frac{\partial u}{\partial \eta} + \langle \nu, \nabla X \rangle u.
\end{align*}
\]

(3.2)

Here \( X \) denotes the outward unit normal vector field of \( \partial M \). Thus, we are going to construct a strictly mean convex barrier with free boundary.

Choose \( U = B_p(p) \) small enough so that the operator \( \Delta + (\text{Ric}_M(\nu, \nu) + |A|^2) \) is invertible. We may also assume that with the same choice \( p \), \( \text{spt} \, V \) does contain all the points in \( F_1 \cap \partial B_p(p) \). We may further assume, without loss of generality, that \( \rho = 1 \) (otherwise rescale around the point \( p \)). Let \( g \) be a smooth nonnegative function on \( \partial B_1(p) \cap F_1 \) which is supported in

\[
\partial B_1(p) \cap F_1 \setminus (\text{spt} \, V \cup F_2 \cdot \cdot \cdot \cup F_n),
\]

and is not identically zero. For sufficiently small \( \varepsilon > 0 \), \( s, t \in [-\delta, \delta] \), there exists a function \( u_{s,t,\varepsilon} : F_1 \cap U \to \mathbb{R} \) that solves

\[
\begin{cases}
(\Delta + (\text{Ric}_M(\nu, \nu) + |A|^2))u_{s,t,\varepsilon} = s & \text{in } F_1 \cap B_1(p), \\
\frac{\partial u_{s,t,\varepsilon}}{\partial \eta} = \langle \nu, \partial_{\nu} X \rangle u_{s,t,\varepsilon} & \text{on } \partial F_1 \cap B_1(p), \\
u_{s,t,\varepsilon} = \varepsilon g + t & \text{on } F_1 \cap \partial B_1(p).
\end{cases}
\]

(3.3)

By the regularity theory in Appendix B, \( u_{s,t,\varepsilon} \in C^{2,\alpha}(F_1 \cap B_1) \). Thus there exists a surface \( \tilde{F}_{(u_{s,t,\varepsilon})} \), constructed by the implicit function theorem, and is a small perturbation of \( F_{(u_{s,t,\varepsilon})} \), such that \( \tilde{F}_{(u_{s,t,\varepsilon})} \) has free boundary on \( \partial M \), and when \( s > 0 \), the mean curvature of \( \tilde{F}_{(u_{s,t,\varepsilon})} \) is positive. Since \( \text{spt} \, V \) lies strictly above \( F_1 \), we may take \( \varepsilon > 0 \) sufficiently small, such that \( \text{spt} \, V \) also lies strictly above \( \tilde{F}_{u_{s,t,\varepsilon}} \). Fix this value of \( \varepsilon \) and let \( u_{s,t} = u_{s,t,\varepsilon} \). Since \( u_{0,0}(q) \geq 0 \) (but not identically 0) on \( \partial F_1 \cap \partial B_1(p) \), by the Hopf maximum
principle, \( u_{0,0} > 0 \) in \( F_1 \cap B_1(0) \). Now fix an \( s > 0 \) such that \( u_{s,0} > 0 \) in \( B_1(0) \).

The function \( u_{s,0} \) then defines a free boundary surface \( \tilde{F}_{(u_{s,0})} \) with positive mean curvature that lies entirely above \( F_1 \), but on the boundary \( F_1 \cap \partial B_1(p) \), \( \text{spt} \ V \) lies above \( \tilde{F}_{(u_{s,0})} \). Now let \( t_0 \) be the smallest value of \( t \) for which \( \tilde{F}_{(u_{s,t})} \) intersects \( \text{spt} \ V \). Since \( p \in \text{spt} \ V \) and \( \tilde{F}_{u_{s,0}} \) lies above \( F_1 \), \( t_0 < 0 \). Denote \( \tilde{F}_1 = \tilde{F}_{u_{s,t_0}} \). We observe:

- \( \text{spt} \ V \) intersects \( \tilde{F}_1 \) in \( B_1(p) \);
- over \( \partial B_1 \), \( \text{spt} \ V \) lies above \( \tilde{F}_1 \);
- \( \tilde{F}_1 \) has strictly positive mean curvature.

These contradicts the weak maximum principle obtained by Proposition 3.1.

\[ \Box \]

4. CURVATURE ESTIMATE FOR STABLE FREE BOUNDARY HYPERSURFACES IN OVERCUBIC MANIFOLDS

**Proposition 4.1.** Suppose \( 0 \leq k < n \leq 7 \), \( \Omega = [0, \infty)^k \times \mathbb{R}^{n-k} \) be the intersection of \( k \) half spaces in \( \mathbb{R}^n \). Assume \( \Sigma \subset \Omega \) is a properly embedded stable minimal hypersurface with free boundary and Euclidean volume growth, that is,

\[ \mathcal{H}^{n-1}(\Sigma \cap B_r(0)) < C_0 r^{n-1}, \]

holds for all \( r > 0 \), with some \( C_0 > 0 \). Then \( \Sigma = P \cap \Omega \) for some hyperplane \( P \subset \mathbb{R}^n \).

**Proof.** Denote \( \bar{\Sigma} \) the hypersurface obtained by reflecting \( \Sigma \) across the coordinate planes \( x_j = 0 \), \( j = 1, \ldots, k \). Then \( \bar{\Sigma} \) is a minimal hypersurface without boundary in \( \mathbb{R}^n \). By the regularity theory in \([GJ86]\), \( \bar{\Sigma} \) is smooth up to the smooth part of \( \partial \Omega \). Therefore \( \bar{\Sigma} \) is smooth away from the its intersection with the codimension two singular strata \( E = \{x_i = x_j = 0, 1 \leq i < j \leq k\} \), and is stable in \( \mathbb{R}^n \setminus E \) with respect to deformations that are even with respect to each \( x_j, j = 1, \ldots, k \). Since the set \( \mathbb{R}^n \setminus E \) is connected, by Theorem 1 of \([FCS80]\), stability of \( \bar{\Sigma} \) is equivalent to that

\[ \lambda^*_1(D) > 0, \]

for any compact domain \( D \subset \mathbb{R}^n \setminus E \) which is symmetric about each \( x_j = 0 \), \( j = 1, \ldots, k \). Here \( \lambda^*_1(D) \) is the first Dirichlet eigenvalue of the Jacobi operator, taken within functions which are even about \( x_j, j = 1, \ldots, k \):

\[ \lambda^*_1(D) = \inf \left\{ \frac{\int_{\Sigma \cap D} (|\nabla f|^2 - |A|^2 f^2)}{\int_{\Sigma \cap D} f^2} : f \in C^\infty_c(D), f(x_j) = f(-x_j), j = 1, \ldots, k \right\}. \]  

(4.1)
Since the domain $\Sigma \cap D$ is symmetric about each $x_j = 0$, and that the first Dirichlet eigenfunction is positive in the interior of $\Sigma \cap D$, a standard maximum principle implies that the first Dirichlet eigenfunction $\varphi$ is even with respect to each $x_j$, $j = 1, \cdots, k$. In particular, this implies that $\lambda_1^*(D) = \lambda_1(D)$, where $\lambda_1(D)$ is the first Dirichlet eigenvalue of the Jacobi operator among all $H^1_0(\Sigma \cap D)$. By [FCS80], this implies that $\Sigma$ is stable in $\mathbb{R}^n \setminus E$. Since $\Sigma \cap E$ is of codimension two in $\Sigma$, we can use [SS81] and conclude that $\tilde{\Sigma}$ is smooth everywhere, and stability further implies that it is a hyperplane.

We proceed to prove the curvature estimate. In [GLZ18], Guang-Li-Zhou proved a monotonicity formula for free boundary minimal surfaces up to the smooth part of the boundary. Specifically, let $E$ be the union of edges as before, then for any properly embedded stable free boundary minimal hypersurface $\Sigma$ with an area upper bound $\Lambda$,

$$\sup_{x \in \Sigma \setminus E} |A_{\Sigma}|(x) \cdot \text{dist}_g(x, E) \leq C(M, g, \Lambda).$$

We observe that Proposition 4.1 combined with a standard blow up argument, give us a stronger curvature estimate for properly embedded free boundary minimal hypersurfaces in overcubic Riemannian manifolds. We start by reviewing the following monotonicity formula. For a proof, see (section 3 of [GLZ18] and [EL]).

**Lemma 4.2.** Let $M^n$ be an overcubic Riemannian manifold, $V^k$ a free boundary stationary varifold in $M$. For any point $q \in \partial M$, there exists a radius $r_0 = r_0(M, q)$, and a constant $\Lambda > 0$, such that

$$\exp(\Lambda r) \frac{\|V\|(B_q(r))}{r^k}$$

is an increasing function for $r \in (0, r_0)$.

We now prove the following local curvature estimate. Theorem 1.7 follows from a straightforward covering argument.

**Proposition 4.3.** Let $3 \leq n \leq 7$. Suppose $(M^n, g)$ is an overcubic manifold whose dihedral angle is everywhere $\pi/2$, and $g$ is a $C^{2,\alpha}$ metric. Suppose $(\Sigma, \partial \Sigma) \subset (M, \partial M)$ is an embedded free boundary stable minimal hypersurface, $C^{2,\alpha}$ to its corners, satisfying the volume bound $\mathcal{H}^{n-1}(\Sigma \cap B_p(R)) \leq \Lambda$. Then

$$\sup_{x \in \Sigma \cap B_p(\frac{R}{2})} |A_{\Sigma}|(x) \leq C(\Lambda, M),$$

where $C > 0$ is a constant depending on $\Lambda$ and $M$. Moreover, for a compact family of choices of $C^{2,\alpha}$ metrics $g$ and $\Lambda \in \mathbb{R}_+$, the constant $C$ can be chosen uniformly.
By possibly taking a further subsequence (which we still denote by 
Therefore, there exists a sequence $x_i \in \Sigma_i \cap B_p(R)$ such that $|A_{\Sigma_i}|(x_i) \to \infty$. By possibly taking a further subsequence (which we still denote by $\{x_i\}$), assume that $x_i \to x \in B_p(\frac{2R}{3})$. By the interior curvature estimate \cite{SS81} of Schoen-Simon, $x \in \partial M$. The surfaces $\Sigma_i$ must also have a non-empty boundary component on $\partial M \cap B_p(R)$.

We next use a standard point-picking argument and blow up the surface $\Sigma_i$ at some appropriate points. Define

$$r_i = (\sup_{x \in \Sigma_i \cap B_p(R)} |A_{\Sigma_i}|(x))^{-1/2}.$$

We have $r_i \to 0$ and $r_i |A_{\Sigma_i}|(x_i) \to \infty$ as $i \to \infty$. Choose $y_i \in \Sigma_i \cap B_{x_i}(r_i)$ so that it achieves the maximum of

$$\sup_{y \in \Sigma_i \cap B_{x_i}(r_i)} |A_{\Sigma_i}|(y) \cdot \text{dist}_M(y, \partial B_{x_i}(r_i)).$$

By definition, this maximum is achieved in the interior of $\Sigma_i \cap B(x_i, r_i)$. Let $r_i' = r_i - \text{dist}_M(y_i, x_i)$. Define $\lambda_i := |A_{\Sigma_i}|(y_i)$. Then

$$\lambda_i r_i' = |A_{\Sigma_i}|(y_i) \cdot \text{dist}_M(y_i, \partial B_{y_i}(r_i'))$$

$$\geq |A_{\Sigma_i}|(x_i) \cdot \text{dist}_M(x_i, \partial B_{x_i}(r_i))$$

$$= r_i |A_{\Sigma_i}|(x_i) \to \infty.$$

Hence $\lambda_i \to \infty$ as $r_i' \to 0$. Define the blow up $\eta_i$ on $M$ by $\eta_i(z) := \lambda_i(z - y_i)$. Denote $(M_i, \partial M_i) = (\eta_i(M), \partial M)$. From the choice of $\lambda_i$ and $y_i$, we obtain a sequence of embedded stable free boundary minimal hypersurfaces

$$(\Sigma'_i, \partial \Sigma'_i) = (\eta_i(\Sigma_i), \eta_i(\partial \Sigma_i)) \subset (M'_i, \partial M'_i).$$

Moreover, $|A_{\Sigma_i'}|(0) = 1$ for every $i$, and $\sup_{x \in \Sigma_i'} |A_{\Sigma_i'}| < 2$, provided that $i$ is sufficiently large.

The manifolds $M'_i = \eta_i(M)$, equipped with the induced metric, converge, in the sense of $C^{2,\alpha}$ Cheeger-Gromov to a region $\Omega$, where up to a rigid motion of $\mathbb{R}^n$, $\Omega = [0, \infty)^k \times \mathbb{R}^{n-k}$ for some integer $k \in [0, n - 1]$. Note that each part $\{x_j = 0\} \cap \Omega$ is the blow up limit of a certain face of $M$. Denote these faces $F_j$, $j = 1, \cdots, k$. Then for each $j$, we have two types of convergence as $i \to \infty$:

- Type I: $\lim\inf_{i \to \infty} \lambda_i \text{dist}_M(y_i, F_j) = \infty$;
- Type II: $\lim\inf_{i \to \infty} \lambda_i \text{dist}_M(y_i, F_j) < \infty$.
For Type I convergence, the rescaled boundary component $\eta_i(F_j')$ escape to infinity as $i \to \infty$, and there disappear in the limit. For Type II convergence, after passing to a subsequence, $\partial M' \cap \eta_i(F_j')$ convergence locally uniformly smoothly to some hyperplane in $\mathbb{R}^n$.

When $i$ is sufficiently large, $\sup_{x \in \Sigma_i'}|A_{\Sigma_i'}| < 2$. Hence $\Sigma_i'$ can be represented locally as a graph of a function $u_i$ over its tangent plane. In the local Fermi coordinates, the functions $u_i$ are $C^2$ functions that satisfy an elliptic PDE in the form (see equation (7) of [Sim87])

$$\begin{cases} \text{div}((a_i + Id) \nabla u_i) + b_i \cdot \nabla u_i + c_i u_i = 0 & \text{in } \Omega', \\ \frac{\partial u_i}{\partial \nu} = 0 & \text{on } \partial \Omega'. \end{cases} \quad (4.2)$$

Here the coefficients $a_i, b_i, c_i$ are $C^\alpha$ functions with uniform $C^\alpha$ norms, and $\Omega = [0, \infty)^{k'} \times \mathbb{R}^{n-k'-1}$ is some cubical domain.

Now $|A_{\Sigma_i'}| < 2$ implies a uniform $C^2$ bound on the graphing functions $u_i$, which further implies that $u_i$ satisfies a locally uniform $C^{2,\alpha}$ upper bound in $\Omega'$ (see Appendix B). We conclude that $\Sigma_i'$ converges to some limit surface $\Sigma_\infty$ locally $C^{2,\alpha}$ smoothly. The monotonicity formula of $\Sigma_i$ guarantees that $\Sigma_\infty$ has Euclidean volume growth. Therefore $\Sigma_\infty \subset \Omega'$ is a stable free boundary minimal surface with Euclidean volume growth, and $|A_{\Sigma_\infty}|(0) = 1$. This is a contradiction, since Proposition 4.1 implies that such $\Sigma_\infty = \Omega \cap P$ for some hyperplane $P$. □

5. RIGIDITY AND SPLITTING OF MINIMAL SLICING

In this section we are going to prove Theorem 1.4 through induction. We first verify the argument when $n = 2$. In this case, Theorem 1.4 is direct corollary of the Gauss-Bonnet theorem: if $(M^2, g)$ is overcubic of dimension two, then

$$\int_M K_g dA + \int_{\partial M} k_g ds + \sum_{j=1}^{4} (\pi - \alpha_j) = 2\pi \chi(M) \leq 2\pi.$$ 

Here $\alpha_j$, $j = 1, \cdots, 4$ denote the dihedral angles. Assuming that $K_g \geq 0$, $k_g \geq 0$ and $\alpha_j \leq \pi/2$, the only possibility is then $K_g = 0$, $k_g = 0$ and each $\alpha_j = \pi/2$. Therefore $M$ is isometric to a flat Euclidean square.

Suppose, as induction hypothesis, that all $C^{2,\alpha}$ overcubic manifolds $(M, g)$ of dimension $(n-1)$ are isometric to an Euclidean rectangular solid, if they satisfy:

- The scalar curvature of $g$ is nonnegative;
- The faces of $M$ are weakly mean convex with respect to outward normal vector fields;
- The dihedral angles along adjacent faces of $M$ is everywhere at most $\pi/2$.

We are going to prove that the same statement holds for overcubic manifolds of dimension $n$. Take a stable free boundary minimal hypersurface
(\Sigma, \partial \Sigma) \subset (M, \partial M)$, where $\Sigma = \partial D$ is produced by minimizing the functional in (1.1). By section \ref{section:regularity}, $\Sigma$ is an embedded, $C^{2,\alpha}$ regular minimal hypersurface of $M$. Since $\Sigma = \partial D \cap M$ for some open set $D$ containing a face $F_1$ of $M$, $\Sigma$ is overcubic of dimension $(n-1)$. The stability inequality implies that

$$Q(f, f) = \int_{\Sigma} |\nabla f|^2 - (|A_{\Sigma}|^2 + \text{Ric}(\nu, \nu)) f^2 dV - \int_{\partial \Sigma} \Pi_{\partial M}(\nu, \nu) f^2 dS \geq 0,$$

(5.1)

for any smooth function $f$ on $\Sigma$.

By the Gauss equation, we have

$$\text{Ric}(\nu, \nu) + |A_{\Sigma}|^2 = \frac{1}{2}(R_M - R_{\Sigma} + |A_{\Sigma}|^2),$$

(5.2)

where $R_M, R_{\Sigma}$ are the scalar curvature of $M, \Sigma$, respectively. The second variation form therefore becomes

$$Q(f, f) = \int_{\Sigma} |\nabla f|^2 - \frac{1}{2}(R_M - R_{\Sigma} + |A_{\Sigma}|^2) f^2 dV - \int_{\partial \Sigma} \Pi_{\partial M}(\nu, \nu) f^2 dS.$$  

(5.3)

Since $\Sigma$ is stable, the principal eigenvalue of the variational problem (see section 2 of \cite{MNS17}, or \cite{Sch06}) associated to the second variation form satisfies:

$$\lambda_1 = \inf_{\varphi \in L^2(\Sigma)} \frac{Q(\varphi, \varphi)}{\int_{\Sigma} \varphi^2 dV} \geq 0.$$  

(5.4)

The associated eigenfunction $\varphi \in W^{1,2}(\Sigma)$ satisfies, in the weak sense, the elliptic equation

$$\begin{cases}
\Delta_{\Sigma} \varphi + \frac{1}{2}(R_M - R_{\Sigma} + |A_{\Sigma}|^2) \varphi = -\lambda_1 \varphi & \text{in } \Sigma, \\
\frac{\partial \varphi}{\partial \nu} = \Pi(\nu, \nu) \varphi & \text{on } \partial \Sigma.
\end{cases}$$

(5.5)

It is easy to verify that equation (5.5) satisfies the following compatibility condition as needed in Proposition \ref{prop:compatibility}. Notice that this condition is necessary for the solution $\varphi$ to (5.5) to have Hölder continuous second derivatives in $\Sigma$.

**Lemma 5.1.** Suppose $F_i, F_j$ are two adjacent faces of $M$ meeting orthogonally, where $\Sigma \cap F_i, \Sigma \cap F_j$ are non-empty. Let $e_i, e_j$ be the outward normal vector field of $F_i, F_j$ in $M$. Then

$$\nabla_{e_j}(\Pi_{F_i}(\nu, \nu)) = \nabla_{e_i}(\Pi_{F_j}(\nu, \nu)).$$

holds along the intersection $F_i \cap F_j \cap \Sigma$.

**Proof.** Fix a point $p \in F_i \cap F_j \cap \Sigma$. Extend $e_i(p), e_j(p), \nu(p)$ to be a normal coordinate frame in a neighborhood of $p$, such that at $p$, all covariant derivatives are zero. We apply the Codazzi equation on the hypersurface $F_i \subset M$, and obtain

$$\nabla_{e_j}(\Pi_{F_i}(\nu, \nu)) = \nabla_{e_i}(\Pi_{F_j}(\nu, \nu)).$$
Since the local coordinates is normal,
$$\nabla e_j(\Pi F_i(\nu, \nu)) = \nabla \Pi F_i(e_j, \nu, \nu) = \nabla \Pi F_i(\nu, e_j, \nu) = \nabla \nu \langle \nabla e_i e_j, \nu \rangle$$
holds at \( p \). Similarly, \( \nabla e_i(\Pi F_j(\nu, \nu)) = \nabla \nu \langle \nabla e_j e_i, \nu \rangle \) at \( p \). Notice that \( \nabla e_i e_j = \nabla e_j e_i \), hence the conclusion of lemma holds.

Given Lemma 5.1, the solution \( \varphi \) to (5.5) is in \( C^{2,\alpha}(\Sigma) \), by virtue of the Proposition B.3 in Appendix B.

Let \( g_1 \) be the induced metric on \( \Sigma \). Using the induction hypothesis, we prove the following property of \( \Sigma \).

**Proposition 5.2.** The hypersurface \( \Sigma \), equipped with the induced metric \( g_1 \), is isometric to an Euclidean rectangular solid. Moreover, \( \Sigma \subseteq M \) is infinitesimally rigid: we have that
$$\text{Ric}_M(\nu, \nu) = 0, \quad |A\Sigma| = 0 \quad \text{on} \quad \Sigma,$$
and \( \Pi_{\partial M}(\nu, \nu) = 0 \) on \( \partial \Sigma \).

**Proof.** Let \( \varphi \) be a positive function on \( \Sigma \), under the conformal change of metric
$$g_2 = \varphi^{\frac{2}{n-2}} g_1,$$
the scalar curvature changes by
$$R(g_2) = \varphi^{-\frac{n}{n-2}} \left( -2\Delta \varphi + R(g_1) \varphi + \frac{n-1}{n-2} \frac{|
abla \varphi|^2}{\varphi} \right); \quad (5.6)$$

Where \( \Delta \) and \( \nabla \) are taken with respect to the metric \( g_1 \). The mean curvature of \( \partial \Sigma \subseteq \Sigma \) with respect to the outward unit conormal vector field changes via
$$H_{\partial \Sigma}(g_2) = \varphi^{-\frac{n}{n-2}} \left( H_{\partial \Sigma}(g_1) + \frac{1}{\varphi} \frac{\partial \varphi}{\partial \nu} \right). \quad (5.7)$$

Now we choose \( \varphi > 0 \) to be the solution to (5.5). Then the scalar curvature satisfies
$$R(g_2) = \varphi^{-\frac{n}{n-2}} \left( (R_M + |A\Sigma|^2 + \lambda_1) \varphi + \frac{n-1}{n-2} \frac{|
abla \varphi|^2}{\varphi} \right) \geq 0. \quad (5.8)$$

The boundary mean curvature becomes
$$H_{\partial \Sigma}(g_2) = \varphi^{-\frac{n}{n-2}} (H_{\partial \Sigma}(g_1) + \Pi(\nu, \nu)).$$

Let \( \{e_j\}_{j=1}^{n-2} \) be an orthonormal frame in an open set of \( \partial \Sigma \). Since \( \Sigma \) meets \( \partial M \) orthogonally, the conormal vector \( \eta \) of \( \partial \Sigma \) in \( \Sigma \) is the same as the conormal vector of \( \partial M \) in \( M \). Therefore \( e_j, j = 1, \ldots, n-2, \nu \) and \( \eta \) forms an orthogonal basis in an open neighborhood of \( \partial M \). Therefore, we verify that
$$H_{\partial \Sigma}(g_1) + \Pi(\nu, \nu) = \sum_{j=1}^{n-2} \langle \nabla e_j, \eta \rangle + \langle \nabla \nu, \eta \rangle = \overline{H}_{\partial M}. \quad (5.9)$$
Here $\Pi_{\partial M}$ represents the mean curvature of $\partial M$ in $M$, with respect to the outward unit normal vector field $\nu$. We therefore conclude that

$$H_{\partial \Sigma}(g_2) = \varphi^{-\frac{1}{n-2}}\Pi_{\partial M} \geq 0. \tag{5.10}$$

Observe that the dihedral angles of $\Sigma$ are equal to the dihedral angles of $M$, since $\Sigma$ meets $\partial M$ orthogonally. Moreover, the conformal deformation does not change dihedral angles. We therefore conclude, by (5.8) and (5.10), that $(\Sigma, g_2)$ is an overcubic manifold of dimension $(n - 1)$ with nonnegative scalar curvature, weakly mean convex faces, and everywhere non-obtuse dihedral angles. By induction, $(\Sigma, g_2)$ is isometric to an Euclidean rectangular solid. Tracking equalities in (5.8) and (5.10), we conclude that, with respect to metric $g_1$:

$$R_M = 0, \quad A_\Sigma = 0, \quad \lambda_1 = 0, \quad \nabla \varphi = 0 \quad \text{on } \Sigma,$$

and $\Pi_{\partial M} = 0$ on $\partial \Sigma$.

In particular, $\varphi$ is a constant function. This implies that $(\Sigma, g_1)$ is also isometric to a flat Euclidean rectangular solid. The fact that $(\Sigma, g_1)$ is infinitesimally rigid then follows. □

We then proceed to the proof of Theorem 1.4. The basic idea is to conformally deform the metric locally near the infinitesimally rigid hypersurface $\Sigma$, and solve the variational problem (1.1) in a new metric. As a result, we are able to conclude that each side of $\Sigma$ contains a dense collection of flat Euclidean rectangular solids. This important technique first appeared in [CCE16], and were also used in [CEM18].

**Proof of Theorem 1.4.** Let $(M^n, g)$ be an overcubic manifold as in the assumption of Theorem 1.4. $\Sigma^{n-1} \subset M$ be a free boundary area minimizing hypersurface. Suppose $M_+ \subset M$ is a region separated by $\Sigma$. By the strong maximum principle (Proposition 3.2), either $\Sigma$ is disjoint with $F_1$ and $F'_1$, or $\Sigma$ completely coincides with $F_1$ or $F'_1$. In the latter case, $\Sigma = F_1$ (or $F'_1$ itself is an area minimizing hypersurface). Since $g$ is a $C^{2,\alpha}$ metric in $\overline{M}$, by the regularity result Proposition 3.4, $\Sigma$ is a $C^{2,\alpha}$ hypersurface of $M$.

Fix $r_0 > 0$ small to be determined later. Fix a point $p \in M_+$, such that $\text{dist}_g(p, \Sigma) \in (1.5r_0, 2.5r_0)$ and $\text{dist}_g(p, \partial M) > 4r_0$. Denote by $r(x)$ the distance function to $p$ with respect to the metric $g$. We take $r_0$ small enough, such that the function $\frac{\partial r}{\partial g} > 0$ in $\Sigma \cap B_{3r_0}(p)$.

**Claim:** There exists an $\varepsilon > 0$ and a family of Riemannian metrics $\{g(t)\}_{t \in (0, \varepsilon)}$ on $M$ in the conformal class of $g$, with the following properties:

1. $g(t) \to g$ smoothly as $t \to 0$;
2. $g(t) = g$ on $M \setminus B_{3r_0}(p)$;
3. $g(t) \leq g$ as metrics on $M$, with strict inequality on $B_{3r_0}(p) \setminus B_{r_0}(p)$.
4. $R(g(t)) > 0$ on $B_{3r_0}(p) \setminus B_{r_0}(p)$. 

(5) Surface $\Sigma$ is weakly mean convex and strictly mean convex at one interior point with respect to the metric $g(t)$, and the unit normal vector field pointing outward from $M_+$. The construction of these conformal metrics follows from a direct generalization of the Appendix J of [CCE16], which we briefly describe below for the sake of completeness.

Let $f \in C^\infty(\mathbb{R})$ be a non-positive function supported in $[0, 3]$, such that in $(1, 3)$,

$$f(s) = -\exp\left(\frac{16n + 8}{s}\right).$$

This is to make sure that $f$ satisfies the inequalities $f'(s) > 0$, $(4n - 1)f'(s) + sf''(s) < 0$ for $s \in (1, 3)$.

Let $r_0 < \text{inj}(M, g)$ be small enough to guarantee that in any geodesic ball $B_{3r_0}(p)$, the inequality $\Delta_g(\text{dist}_g^2(\cdot, q)) < 8n$ is satisfied. On $M$, define $v(x) = r_0^2 f(r(x)/r_0)$. In $B_{3r_0}(p) \setminus B_{r_0}(p)$, we then check that

$$\Delta_g v = r_0 f'\left(\frac{r}{r_0}\right) \Delta_g r + f''\left(\frac{r}{r_0}\right)$$

$$< (4n - 1)\frac{r_0}{r} f'\left(\frac{r}{r_0}\right) + f''\left(\frac{r}{r_0}\right) < 0. \quad (5.11)$$

The metric $g(t)$ is then defined as $g(t) = (1 + tv)^\frac{n-2}{n} g$. We verify that conditions (1)-(5) are satisfied by $\{g(t)\}$. Conditions (1)-(3) are straightforward. To see (4), we note that

$$R(g(t)) = (1 + tv)^{-\frac{n+2}{n-2}} \left(\frac{4(n - 1)t}{n - 2} \Delta_g v + R(g)(1 + tv)\right) > 0.$$ 

To see (5), we notice that along $\Sigma$,

$$H_{\Sigma, g(t)} = (1 + tv)^{-\frac{n}{n-2}} \left( H_{\Sigma, g} + \frac{2(n - 1)t}{n - 2} \frac{\partial v}{\partial v}\right) \geq 0.$$ 

The claim is proved.

To proceed, observe that $(M_+, g(t))$ is an overcubic manifold that satisfies all the conditions satisfied by $(M, g)$. Denote $\Sigma$ by the face $F_1$, and its opposite face by $F'_1$. We consider the variational problem

$$I = \inf \left\{ \mathcal{H}^{n-1}(\partial D \cap M_+)_{g(t)} : D \text{ is an open set of finite perimeter containing } F_1, D \cap F'_1 = \emptyset. \right\}$$

By the strong maximum principle (Proposition 3.2) and the regularity theory, $I$ is achieved by a $C^{2,\alpha}$ area minimizing hypersurface $\Sigma_t$. Notice that $\Sigma_t$ is disjoint from $\Sigma$ by the convexity condition (5) of the metric $g(t)$. Also $\Sigma_t$ must intersect $B_{3r_0}(p)$, otherwise we would have:

$$|\Sigma|_g \leq |\Sigma_t|_g = |\Sigma_t|_{g_t} \leq |\Sigma|_{g_t} < |\Sigma|_g,$$
contradiction (where the last inequality comes from condition (4)). Also, Proposition \ref{prop:5.2} implies that $R(g(t)) \equiv 0$ on $\Sigma_t$. Since $R(g(t)) > 0$ in $B_{3\rho_0}(p) \setminus B_{\rho_0}(p)$, we conclude that $\Sigma_t \cap B_{\rho_0}(p) \neq \emptyset$. Consider the family of free boundary area minimizing surfaces $\{\Sigma_t\}_{t \in (0, \varepsilon)}$. By the curvature estimate (ADD REF), $\Sigma_t$ subsequentially converges in $C^{2,\alpha}$ to a free boundary area minimizing surface $\Sigma' \subset M_+$. Moreover, $\Sigma' \cap B_{\rho_0}(p) \neq \emptyset$. In particular, $\Sigma'$ and $\Sigma$ are disjoint.

This argument, carried out on each of the free boundary minimizing hypersurface, with varying choices of the small radius $r_0$ and point $p$, implies that the whole region $M_+$ (and $M \setminus M_+$ likewise) contains a collection of free boundary area minimizing hypersurfaces $\{\Sigma^\rho\}_{\rho \in A}$, where $\bigcup_{\rho \in A} \Sigma^\rho$ is dense in $M_+$. By Proposition \ref{prop:5.2}, each $\Sigma^\rho$ is isometric to an Euclidean rectangular solid, and is also infinitesimally flat in $M$. We also observe that $\partial \Sigma^\rho$ is also dense in the boundary. Precisely, for any point $q \in \partial M^+$ and any $\varepsilon > 0$, there is a free boundary area minimizing surface $\Sigma^\rho$ that intersects $B_\varepsilon(q) \cap M_+$, $\rho \in A$. By the curvature estimate (Theorem \ref{thm:1.7}), $\Sigma^\rho$ must also intersects $\partial M^+$ in some $B_{C\varepsilon}(q)$, for some uniform constant $C$ (independent of the choice of $q$ and $\varepsilon$).

Pick a hypersurface $\Sigma^\rho$. For notational simplicity, assume without loss of generality that $\rho = 0$ and $\Sigma^0 = \Sigma$. Extend the unit normal vector field $\nu$ of $\Sigma$ to a smooth vector field $X$, such that $X$ is tangential on $\partial M$. Let $\phi_t$ be the one-parameter family of diffeomorphisms generated by $\phi_t$. For $\rho$ sufficiently small, $\Sigma^\rho$ can be represented as a graph

$$\{\phi_{u^\rho}(x) : x \in \Sigma\}.$$ 

Fix a point $x_0 \in \Sigma$. Arguing as (2) in \cite{Liu13}, the functions $\{u^\rho(x)/u(x_0)\}$ converges in $C^{2,\alpha}(\Sigma)$, as $\rho \to 0$, to a function $u$ that satisfies

$$(\nabla^2 u)(X,Y) + Rm_M(\nu,X,Y,\nu) = 0,$$

for all tangential vectors $X,Y$. Taking trace, $\Delta_{\Sigma} u = 0$. By (3.3), $\frac{\partial u}{\partial \eta} = 0$ on $\partial \Sigma$. We therefore conclude that $u = 0$ on $\Sigma$, and hence $Rm_M(\nu,X,Y,\nu) = 0$. Using the Gauss-Codazzi equations on $\Sigma$, and the fact that $|A_{\Sigma}| = 0$, we then obtain that $Rm_M = 0$ on $\Sigma$. By density of $\{\Sigma^\rho\}$, we conclude that $M_+$ is flat. By Proposition \ref{prop:5.2}, $\Pi = 0$ on each face of $\partial \Sigma^\rho$. Since $\Sigma^\rho$ is dense in $\partial M_+$, we conclude that each face of $\partial M_+$ is totally geodesic. This implies that $M_+$ is isometric to an Euclidean rectangular solid. Combined with a similar argument for $M \setminus M_+$, we conclude that $(M,g)$ is isometric to an Euclidean rectangular solid.

\hfill \Box

6. Remarks and open questions

We finish the paper with a few remarks on possible future developments of Conjecture \ref{conj:1.1}. 
6.1. Dihedral rigidity for \( n \)-simplices. Let \( P \subset \mathbb{R}^n \) be a flat simplex, equipped with the Euclidean metric \( g_0 \). We call a manifold with non-empty boundary over-simplicial, if there exists a degree one map \( \phi : M \rightarrow P \), such that when restricted to each \( k \)-face of \( M \), \( \phi \) is a degree one map onto a \( k \)-face of \( P \). Let \( g \) be a Riemannian metric on \( M \). Conjecture 1.1 for general \( n \)-simplices can be stated as following:

**Conjecture 6.1.** Suppose \( (M^n, g) \) is an over-simplicial Riemannian polyhedron with faces \( \{F_j\}_{j=1}^{n+1} \), and \( P \) be a flat \( n \)-simplex. Suppose that:

- The scalar curvature of \( g \) is non-negative;
- Each face of \( M \) is weakly mean convex with respect to the outward unit normal vector field;
- The dihedral angle of \( M \) along any edge \( E = F_i \cap F_j \) is not larger than the (constant) dihedral angle of \( P \) along \( \phi(E) \).

Then up to scaling, \( (M, g) \) is isometric to \( (P, g_0) \).

In dimension \( n = 3 \), the author confirmed Conjecture 6.1 in [Li17]. We now discuss a possible approach for Conjecture 6.1 in higher dimensions. Take a vertex \( p \in M \), and the vertex \( \phi(p) \in P \). For \( j = 1, \cdots, n+1 \), denote the faces of \( P \) by \( \tilde{F}_j = \phi(F_j) \). Let \( F_1 \) be the unique face of \( M \) that does not contain \( p \), so that \( \tilde{F}_1 \) is the opposite face of \( \phi(p) \) in \( P \). We denote the dihedral angle between \( \tilde{F}_j \) and \( \tilde{F}_1 \) by \( \gamma_j, j = 2, \cdots, n+1 \).

Inspired by [Li17], we consider a capillary energy functional

\[
F(D) = \mathcal{H}^{n-1}(\partial D \cap \tilde{M}) - \sum_{j=2}^{n+1} \cos \gamma_j \mathcal{H}^{n-1}(\partial D \cap F_j).
\]

Here \( D \subset M \) is an open set of finite perimeter that contains \( p \), and is disjoint from \( F_1 \). We then consider the variational problem

\[
I = \inf \{ F(D) : D \subset M \text{ is an open set of finite perimeter, } p \in D, D \cap F_1 = \emptyset \}. \tag{6.1}
\]

We speculate that a similar argument as in section 2 gives a maximum principle for the function \( F \), and the minimizer \( D \) is either disjoint from \( F_1 \), or \( D = M \). Now if \( I \) is achieved by an open set \( D \) with smooth boundary \( \Sigma^{n-1} = D \cap \tilde{M} \), then a argument using conformal deformation, similar to section 5, should imply that there exists a conformal factor \( u \), such that \( (\Sigma, u^{-2} g) \) is an over-simplicial manifold with the assumptions of Conjecture 6.1 satisfied. We thus obtain a capillary minimal slicing

\[
\Sigma^2 \subset \cdots \subset \Sigma^{n-1} \subset M
\]

where each \( \Sigma^j \) is an over-simplicial polyhedron of dimension \( j \), satisfying conditions (1)-(3) in Conjecture 6.1. The same rigidity argument in section 5 then implies that \( (M, g) \) is flat.

The major difficulty in this argument the lack of a satisfying regularity theory for the minimizer \( \Sigma \). To the author’s best knowledge, the sharpest
partial regularity theory of capillary hypersurfaces is due to De Philippis-Maggi in [DPM15], where they proved that on the smooth part of $\partial M$, the singularities of $\partial \Sigma$ is of Hausdorff dimension at most $(n - 4)$. Conjecturally, the singularities should be of Hausdorff dimension at most $(n - 8)$. Of course, it would be more challenging to study the regularity of $\Sigma$ at a corner point.

6.2. Dimensions $n \geq 8$. The recent solution to the high dimensional positive mass theorem by Schoen-Yau [SY17] extends the minimal slicing technique to all dimensions. It is therefore natural to speculate that a singular analysis may also work for free boundary minimal slicing. Of course, treating singularities along the boundary and the corner of the free boundary hypersurface requires new ideas. Moreover, the argument relies on the fact that $R > 0$ in the manifold. Therefore, carrying out the rigidity analysis seems challenging.

Appendix A. Bending construction of hypersurfaces

In this section we review a bending construction due to Gromov. For more details, we refer the readers to page 701, section 11.3 of [Gro18b]. Our objective is the following proposition, which reduces the Conjecture 1.1 to the statement of Theorem 1.4, by fixing the dihedral angles between faces of a polyhedron.

Proposition A.1. Assume $(M^n, g)$ is a Riemannian polyhedron with a $C^\infty$ metric $g$, such that

1. Each face of $M$ is weakly mean convex;
2. There exists a smooth function $\alpha$ defined on the $(n - 2)$-skeleton $E$ of $M$, which is constant along each edge $F_i \cap F_j$, such that for any point $q \in E$, the dihedral angle $q$ is less than or equal to $\alpha(q)$.

Then there exists a Riemannian polyhedron $M' \subset M$, such that with the induced metric, $M'$ satisfies that

1. Each face of $M'$ is weakly mean convex;
2. The dihedral angle at every $q \in E$ is equal to $\alpha(q)$.

Gromov’s idea of proof of Proposition A.1 is by induction, which we very briefly describe as follows. Take one face $F_0 \subset M$. We fix a vector field $X$ in $M$, such that $X$ is transversal to $\Sigma_0$ and points inward $M$, but tangential to all the other faces of $M$. Let $\phi = \phi(x,t)$ be the flow generated by the vector field $X$. Let $f$ be a smooth function defined on $\Sigma_0$. Define

$$F_f = \{\phi(p, f(p) : x \in F_0)\}.$$  

By the tubular neighborhood theorem, $\Sigma_f$ is isotopic to $\Sigma_0$ when $\|f\|_{C^0}$ is small enough. Hence by replacing the face $F_0$ by $F_f$ and keeping all the other faces the same, we may obtain a new polyhedron $M' \subset M$. It remains to check the conditions on mean curvature and on dihedral angle. Let $H_f$ and $\theta_f$ be the mean curvature (with respect to outward unit normal) and
the dihedral angle of the surface $F_j$. The linearized operator of $H_f$ and $\theta_f$ is then given by equation (3.3):

$$
\partial_t|_{t=0}H_{(t)} = \Delta f + (\text{Ric}_M(\nu, \nu) + |A|^2)f,$$

$$
\partial_t|_{t=0}\cos \theta_{(t)} = -\frac{\partial f}{\partial \eta} + \langle \nu, \nabla_\nu X \rangle f.
$$

(A.1)

Denote $\Lambda > 0$ be an upper bound of $\text{Ric}_M(\nu, \nu) + |A|^2$ and $\langle \nu, \nabla_\nu X \rangle$. Then one may choose a function $f$ such that $|f|_{C^0}$ is sufficiently small, but $\Delta f \geq 2\Lambda |f|$, $\frac{\partial f}{\partial \eta} \leq -2\Lambda |f|$. (See, for instance, the function $f$ in the proof of Theorem 1.4.) By the implicit function theorem, we then may increase the mean curvature and the dihedral angle the same time. Notice that during this process, we do not change the dihedral angle between other faces.

Once we have fixed the dihedral angle between $F_0$ and other faces, we may repeat this process and inductively increase the dihedral angle between all pairs of faces.

A.1. Application to dihedral rigidity in dimension three. As an application of Proposition A.1, we may generalize the main results of [Li17] to all polyhedra of cone or prism types. In Theorem 1.4 and 1.5 of [Li17], for the purpose of regularity of the capillary functional minimizer, an extra condition on the dihedral angles was needed. Let us briefly recall these results. Given an Euclidean flat cone or prism $P$ and a Riemannian polyhedron $(M^3, g)$ diffeomorphic to $P$, let $F_j, j = 1, \ldots, k$ be the side faces of $M$, $F'_j$ be the side faces of $P$. Let $B'$ be the base face of $P$. Denote the (constant) dihedral angle between $F'_j$ and $B'$ by $\gamma_j$. Then in Theorem 1.4 and 1.5 of [Li17], it was assumed that

$$
|\pi - (\gamma_j + \gamma_{j+1})| < \angle(F_j, F_{j+1}).
$$

(A.2)

Theorem A.2 (Theorem 1.4 of [Li17]). Assume $P$ is an Euclidean cone or prism, $(M^3, g)$ is a Riemannian polyhedron diffeomorphic to $P$, such that (A.2) holds. Then $(M, g)$ cannot simultaneously satisfy that $R(g) \geq 0$ in the interior of $M$, each face of $M$ is mean convex, and that the dihedral angles of $M$ is everywhere less than the corresponding dihedral angle of $P$.

Notice that Proposition A.1 enables us to increase the dihedral of a Riemannian polyhedron, and at the same time do not decrease face mean curvature and the interior scalar curvature. Therefore the lower bound in condition (A.2) may, without loss of generality, be dropped.

In Theorem 1.5 of [Li17], another condition, namely $\gamma_j \in (0, \pi/2]$ for $j = 1, \ldots, k$, is assumed, in order to get the full dihedral rigidity statement. In the case that $P$ is a simplex, this condition is automatically true: consider all the four distances from a vertex of the simplex $P$ to its opposite face. Suppose $v$ is the vertex with the least distance among the four. Regarding $P$ a cone with vertex $v$, then its side faces (faces that contain $v$) meet the base face (the face that does not contain $v$) at dihedral angles in $(0, \pi/2)$.
In particular, we conclude that the dihedral rigidity conjecture holds for all three dimensional simplices, without any extra conditions:

**Theorem A.3.** Let $P$ be a simplex in $\mathbb{R}^3$, $g_0$ is the Euclidean metric on $P$. Suppose $g$ a Riemannian metric on $P$, such that $R(g) \geq 0$ in the interior of $M$, each face of $M$ is weakly mean convex, and the dihedral angles of $(P,g)$ is everywhere not larger than the corresponding dihedral angle of $(P,g_0)$. Then up to a scaling, $(P,g)$ is isometric to $(P,g_0)$.

**Appendix B. Elliptic regularity in cubicle domains**

We prove some regularity results for second order elliptic equations in cubicle domains. Let $(M^n,g)$ be an n-cubic Riemannian manifold with faces $F_i$, $i = 1, \cdots, 2n$. We assume that each face $F_j$ is a $C^{2,\alpha}$ hypersurface, and each pair of adjacent faces $F_i, F_j$ intersects transversely at constant angle $\pi/2$. The Riemannian metric $g$ is also assumed to be $C^{2,\alpha}$ in $\overline{M}$. Consider an elliptic equation

\[
\begin{cases}
Lu = 0 & \text{in } M, \\
\frac{\partial u}{\partial \nu} = qu & \text{on } \partial M.
\end{cases}
\]

Here, we assume that $L$ is a linear elliptic operator, such that the second order part is Laplacian. (In particular, we will apply this appendix to study regularity of minimal surfaces in $M$, and regularity of solutions to $(5.5)$.) We also assume that $q$ is a function, $C^{1,\alpha}$ when restricted on each face of $M$, and satisfies the compatibility condition

\[
\frac{\partial}{\partial \nu_j}(q|_{F_i}) = \frac{\partial}{\partial \nu_i}(q|_{F_j})
\]

along the intersection $F_i \cap F_j$, for each pair of adjacent faces.

Since $(M,g)$ is a Lipschitz domain, by classical theory in elliptic PDE, we know that any weak solution to $(B.1)$ is in $C^{6,\alpha}(\overline{M})$, and is locally $C^{2,\alpha}$ in the interior of $M$ and on the smooth part of $\partial M$ (namely, on the $(n-1)$ dimensional strata of $\partial M$). Hence it suffices to consider $(B.1)$ on the corners of $M$ (namely, on $d$ dimensional strata of $\partial M$ with $0 \leq d \leq n-2$). Since the regularity theory is local in nature, we can fix a point $p$ at the corner. We first construct a good coordinate system in which the metric is particularly adapted to studying regularity.

Precisely, suppose a neighborhood of $p$ in $M$ is diffeomorphic to the cubicle domain

\[
\Omega = \{(x_1, \cdots, x_n) : x_j \geq 0, j = 1, \cdots, k\},
\]

for some integer $k \in [2, n]$. (In the case where $k = 1$, $p$ is on the smooth part of the boundary, and the solution is automatically $C^{2,\alpha}$.) Then $p$ lies on the intersection of $k$ faces of $M$, say $F_1, \cdots, F_k$. The as a first step, we review the following important local foliation structure near $p$, which is a straightforward consequence of Lemma 2.1.
Proposition B.1. There exists a constant $\delta > 0$, a neighborhood $U$ of $p$ in $M$ and foliations $\{F_1^i\}, \ldots, \{F_k^i\}$, with each $s_j \in [0, \delta)$, of $U$, such that $F_j = F_j^0$, and $F_i^s$ intersects $F_j^0$ orthogonally for every $i$ and $j$.

Using this local foliation, we define a coordinate system in $U$ as follows. Denote $N = \bigcap_{j=1}^k F_j$ be the intersection of the faces near $p$. Take a geodesic normal coordinate of $N$ in a neighborhood of $p$, and denote it by $(x_{k+1}, \cdots, x_n)$. For $j = 1, \cdots, k$, define the coordinate of a nearby point $q$ by $x_j(q) = s_j$, if $q \in F_j^{s_j}$. Thus, the coordinate system $(x_1, \cdots, x_n)$ provides a diffeomorphism of $U$ and the cubicle domain $\Omega \cap B_\rho(0) \subset \mathbb{R}^n$. After a possible scaling, we assume that $\rho = 1$. By the local foliation structure, at each point $q \in U$, $j = 1, \cdots, k$, the vector $\partial_j$ is normal to $F_j^{s_j}$. Thus $g_{ij} = 0$ on $F_i$, whenever $1 \leq i \leq k$ and $i \neq j$. In this coordinate system, we consider elliptic equations of the form

$$\begin{cases}
  a_{ij}u_{ij} + b_iu_i + cu = f & \text{in } \Omega \cap B_1, \\
  \frac{\partial u}{\partial c} = 0 & \text{on } \partial \Omega \cap B_1, \\
  u = g & \text{on } \Omega \cap \partial B_1.
\end{cases}$$

where the coefficients satisfies

$$A^{-1}|\xi|^2 \leq a_{ij}\xi^i\xi^j \leq A|\xi|^2$$

$$a_{ij}, b_i, c \in C^{0,\alpha}(\Omega \cap B_1) \quad \text{with} \quad |a_{ij}|_{0,\alpha}, |b_i|_{0,\alpha}, |c|_{0,\alpha} \leq \Lambda. \quad (B.3)$$

$$a_{ij} = 0 \quad \text{on } \{x_i = 0\}, \text{whenever } j \neq i.$$

We note that the equations necessary for our geometric argument satisfies this property:

Remark B.2. In local coordinates, the Laplacian operator takes the form

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \partial_i \left( \sqrt{\det g} g^{ij} \partial_j u \right) = g^{ij}u_{ij} + \text{lower order terms}.$$

Since $g_{ij} = 0$ on $F_i$ for $1 \leq i \leq k$, $1 \leq j \leq n$ and $i \neq j$, $(B.3)$ is satisfied.

Remark B.3. We will see that the minimal surface equation also satisfies the assumptions $(B.3)$. By Theorem 2.5, a free boundary area minimizing hypersurface in $M$ is locally a $C^{1,\alpha}$ graph over its tangent plane. The minimal surface equation takes the form

$$F(g_{ij}, u, u_i)u_{ij} + \text{lower order terms} = 0$$

with Neumann boundary conditions. Here $F$ is a smooth function depending analytically on the metric $g_{ij}$, the graphical function $u$ and its first derivatives. In particular, if $u_i = 0$ along $x_i = 0$, and $j \neq i$, $F_{ij} = 0$. As a result, any $C^{1,\alpha}$ free boundary minimal hypersurface in a cubicle domain is actually $C^{2,\alpha}$.

Proposition B.4. Suppose $u \in C^{1,\alpha}(\Omega \cap B_1) \cap W^{2,2}(\Omega \cap B_1)$ is a weak solution to $(B.2)$, where the coefficient satisfies $(B.3)$, and $f \in C^{0,\alpha}(\Omega \cap B_1)$, $g \in C^0(\Omega)$. Then $u \in C^{2,\alpha}(\Omega \cap \overline{B_1/2})$. 

Proof. By the standard regularity theory, one has $C^{2,\alpha}$ regularity for $u$ in the interior of $\Omega \cap B_1$ and along the smooth part of $\partial \Omega \cap B_1$. We first rewrite the equation as

$$a_{ij}u_{ij} = f - bu_i - cu,$$

with the same boundary conditions.

Consider the even extension $\bar{u}$ of $u$, defined in $B_1$, by letting

$$\bar{u}(x_1, \ldots, x_n) = u(|x_1|, \ldots, |x_k|, x_{k+1}, \ldots, x_n).$$

Since $u \in C^{1,\alpha}(\Omega \cap B_1)$ and $\frac{\partial u}{\partial \nu} = 0$ on $\partial \Omega \cap B_1$, $\bar{u} \in C^{1,\alpha}(B_1)$. We prove that $\bar{u}$ is the weak solution to an elliptic equation with $C^{0,\alpha}$ coefficients in $B_1$. Fix a point $x = (x_1, \ldots, x_n)$. For each pair of $(i, j)$, denote an integer $\tau_{ij} \in \{0, 1, 2\}$ as follows: if $i = j$, then $\tau_{ij} = 0$; otherwise, $\tau_{ij}$ is the number of the elements $k \in \{i, j\}$ where $x_k < 0$. Define

$$\bar{g}_{ij}(x_1, \ldots, x_n) = (-1)^{\tau_{ij}}(x_1, \ldots, x_n)g_{ij}(|x_1|, \ldots, |x_k|, x_{k+1}, \ldots, x_n).$$

Notice that $\bar{g}_{ij}$ is a $C^{0,\alpha}$ function in $B_1$. In fact, across each face $x_k = 0$, either $\bar{g}_{ij}$ is an even extension of $g_{ij}$, or $\bar{g}_{ij}$ is an odd extension with $g_{ij} = 0$ along $x_k$.

Similarly, for each $i$, define $\tau_i \in \{0, 1\}$ as follows: if $x_i \geq 0$ then $\tau_i = 0$; otherwise let $\tau_i = 1$. Then define

$$\bar{b}_i(x_1, \ldots, x_n) = (-1)^{\tau_i}(x_1, \ldots, x_n)b(|x_1|, \ldots, |x_k|, x_{k+1}, \ldots, x_n).$$

We observe that the function $\bar{b}_i \bar{u}_i$ is a $C^{0,\alpha}$ function in $B_1$. Indeed, along each $x_k = 0$, either $i \neq k$ and $\bar{b}_i \bar{u}_i$ is an even extension, or $i = k$ and $u_i = 0$ along $x_i = 0$.

Finally we just define $\bar{c}, \bar{f}, \bar{g}$ as the even extension of $c, f, g$. Then $\bar{u}$ is a weak solution to

$$\bar{g}^{ij}\bar{u}_{ij} = \bar{f} - \bar{b}_i \bar{u}_i - \bar{c}\bar{u} \text{ in } B_1, \quad \bar{u} = \bar{g} \text{ on } \partial B_1.$$

Where the coefficients are in $C^{0,\alpha}(\overline{B_1})$. Therefore $\bar{u} \in C^{2,\alpha}(B_1) \cap C^0(\overline{B_1})$ by standard elliptic regularity theory.

We are now ready to conclude the regularity needed for the paper.

**Corollary B.5.** Assume $3 \leq n \leq 7$. Let $(M^n, g)$ be an over cubic Riemannian manifold with $g$ a $C^{2,\alpha}$ metric, and the dihedral angle is everywhere $\pi/2$. Let $\Sigma^{n-1}$ be a properly embedded volume minimizing hypersurface of $M$ with free boundary. Then $\Sigma$ is a $C^{2,\alpha}$ graph over its tangent plane everywhere.

**Corollary B.6.** Let $M$ be as above. Suppose $u \in W^{1,2}(M)$ is a weak solution to

$$\begin{cases}
\Delta_g u + cu = f & \text{in } M, \\
\frac{\partial u}{\partial \nu} = q_j u & \text{on } F_j.
\end{cases}$$

Here $c, f \in C^{0,\alpha}(\overline{M})$, and $q_j \in C^{1,\alpha}(\overline{F_j})$ satisfy the compatibility condition

$$\frac{\partial}{\partial \nu_j} q_i = \frac{\partial}{\partial \nu_i} q_j$$

in $\Omega$. Therefore $u \in C^{2,\alpha}(B_1) \cap C^0(\overline{B_1})$ by standard elliptic regularity theory.
along the intersection $F_i \cap F_j$. Then $u \in C^{2,\alpha}(\overline{M})$.

Proof. By standard elliptic regularity in Lipschitz domains, $u$ is $C^{0,\alpha}$ smooth. Since the functions $q_j$ satisfies the compatibility condition, there exists a $C^{2,\alpha}$ function in the interior of $M$, and on the smooth part of $\partial M$. Also, since $M$ is locally convex, it follows that $u \in C^{1,\alpha}(\overline{M})$.

Since $\{q_j\}$ satisfies the compatibility condition, there exists a function $v_0 \in C^{2,\alpha}(\overline{M})$ such that $\frac{\partial v_0}{\partial \nu} = q_j$ on $F_j$. Let $v = e^{v_0}$. Then it is straightforward to check that the function $w = \frac{u}{v} \in W^{2,2}(M)$ is a weak solution to an equation in the form

\[
\begin{cases}
\Delta_g w + \vec{b} \cdot w + cw = f & \text{in } M, \\
\frac{\partial w}{\partial \nu} = 0 & \text{on } \partial M.
\end{cases}
\]

Since $w \in C^{1,\alpha}(\overline{M})$, by possibly subtracting $lw$ on both sides, where $l$ is a large constant, we may also without loss of generality assume that the constant term $c$ is non-positive. Near each point $p$ at the corner of $M$, consider the equation

\[
\begin{cases}
\Delta_g w_0 + \vec{b} \cdot w_0 + cw_0 = f & \text{in } M \cap B_\rho(p), \\
\frac{\partial w_0}{\partial \nu} = 0 & \text{on } \partial M \cap B_\rho(p), \\
w_0 = w & \text{on } M \cap \partial B_\rho(p).
\end{cases}
\]

By Proposition B.3, the solution $w$ is in $C^{2,\alpha}(M \cap B_{\rho/2}(p))$. On the other hand, the maximum principle implies that $w = w_0$ in $M \cap B_\rho(p)$. We then conclude that $w \in C^{2,\alpha}(\overline{M})$, and hence $u \in C^{2,\alpha}(\overline{M})$. \qed

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