Almost representations of Lie algebras and quantization

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Abstract

We introduce almost representations of compact Lie algebras, and establish an Ulam-stability type phenomenon: every almost representation is close to a genuine irreducible representation. As an application, we prove that all geometric quantizations of the two-dimensional sphere are conjugate in the semi-classical limit up to a small error.

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1 Introduction and main results

The goal of this article is twofold. Our first objective is to establish an Ulam-stability type phenomenon for representations of Lie algebras: under certain assumptions, every skew-Hermitian almost representation of a compact Lie

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algebra is close to a genuine irreducible representation. While a similar problem has been studied for representations of groups \([10, 16, 8]\), to the best of our knowledge such a question has not been addressed in the framework of Lie algebras. In fact, we present two versions of this stability, associated to two different notions of an almost representation.

In Section 2, we analyze the case of the Lie algebra \(\mathfrak{su}(2)\). An \(n\)-dimensional unitary irreducible representation of this algebra is defined by a triple of skew-Hermitian \(n \times n\) matrices \(X_i, i \in \mathbb{Z}/3\mathbb{Z}\), satisfying the commutation relation \([X_i, X_{i+1}] = X_{i+2}\), and such that the Casimir element \(-\sum_i X_i^2\) equals \(\frac{n^2-1}{4}\). We prove that any almost representation, i.e., a triple of matrices for which the above characterization holds approximately, is close to a genuine irreducible representation.

In Section 3, we work with another definition of almost representation, valid for general compact Lie algebras involving the Casimir element in the adjoint representation. As explained in Remark 3.3, although the latter version is more general, the former version is sharper, and is necessary for applications to geometric quantization of the two-sphere.

Geometric quantization is a mathematical recipe behind the quantum-classical correspondence, a fundamental physical principle stating that quantum mechanics contains classical mechanics in the limiting regime when the Planck constant \(\hbar\) tends to zero. In Section 4, we will use the Ulam-stability result of Section 2 to show that all geometric quantizations of the sphere, satisfying the axioms of Definition 1.4, are conjugate to each other up to an error of order \(O(\hbar)\).

Let us pass to precise formulations. For a finite-dimensional Hilbert space \(H\), write \(\| \cdot \|_{\text{op}}\) for the operator norm on the space \(\mathfrak{su}(H)\) of skew-Hermitian operators acting on \(H\). Recall that the Lie algebra \(\mathfrak{su}(2)\) has real dimension 3, and admits a basis \(L_1, L_2, L_3 \in \mathfrak{su}(2)\) satisfying the commutation relations

\[
[L_j, L_{j+1}] = L_{j+2} \quad \text{for all} \quad j \in \mathbb{Z}/3\mathbb{Z}.
\]

(1.1)

An irreducible representation is a linear map \(\rho : \mathfrak{su}(2) \to \mathfrak{su}(H)\) preserving the commutation relations and such that the triple of skew-Hermitian operators \(X_j := \rho(L_j), j \in \mathbb{Z}/3\mathbb{Z}\), do not preserve any proper subspace of \(H\). As well known in such a case, writing \(n := \dim H\) for the complex dimension of \(H\), we have

\[
X_1^2 + X_2^2 + X_3^2 = -\frac{n^2 - 1}{4} \mathbb{1}.
\]

(1.2)
Our first main result is as follows.

**Theorem 1.1.** For every $c \in \mathbb{R}$ and $r > 0$, there exist $k_0 \in \mathbb{N}$ and $C > 0$ such that the following holds. Let $H$ be a finite-dimensional Hilbert space, and assume that there exist $k \in \mathbb{N}$ with $k \geq k_0$ and a triple of operators $x_i \in \mathfrak{su}(H), i \in \mathbb{Z}/3\mathbb{Z}$, such that

\[
\text{(R1)} \quad \left\| x_1^2 + x_2^2 + x_3^2 + \left( \frac{k^2}{4} + \frac{k}{2} \right) I \right\|_{op} \leq r; \\
\text{(R2)} \quad \left\| [x_j, x_{j+1}] - x_{j+2} \right\|_{op} \leq r/k \quad \text{for} \quad j \in \mathbb{Z}/3\mathbb{Z}.
\]

Then

(I) $c \in \mathbb{Z}$;

(II) $k/2 - C \leq \|x_j\|_{op} \leq k/2 + C$ for all $j \in \mathbb{Z}/3\mathbb{Z}$.

If in addition

\[
\dim H < 2(k + c), \tag{1.3}
\]

then

(III) $\dim H = k + c$;

(IV) there exists an irreducible representation $\rho : \mathfrak{su}(2) \to \mathfrak{su}(H)$ such that

\[
\|x_j - \rho(L_j)\|_{op} \leq C. \tag{1.4}
\]

The proof of this theorem is given in Section 2. Note that assumption $(1.3)$ on the dimension of $H$ is optimal in order to guarantee the irreducibility of $\rho$. Indeed, the direct sum of two $k$-dimensional irreducible representations is a $2k$-dimensional genuine representation with $c = 0$. Note also that for genuine irreducible representations of $\mathfrak{su}(2)$ assumption (R1) holds with $r = 1/4$ by $(1.2)$, while (R2) is valid for any $r$.

**Conjecture 1.2.** The analogue of Theorem 1.1 holds for all real compact Lie algebras, with the operator in the left hand side of (R1) replaced by the Casimir element.

**Remark 1.3.** Our proof of Theorem 1.1 uses the explicit description of representations of $\mathfrak{su}(2)$ for all $k \in \mathbb{N}$, and new ideas are needed in order to find a uniform proof for all compact Lie algebras in this setting.
In Section 3, we consider another notion of an irreducible almost representation $t : \mathfrak{g} \to \mathfrak{su}(H)$ of a compact Lie algebra $\mathfrak{g}$. Take any orthonormal basis $e_1, \ldots, e_n$ in $\mathfrak{g}$ with respect to the Killing form. We define, in the context of almost representations, a counterpart of the Casimir element in the adjoint representation, called almost-Casimir, by

$$
\Gamma : \mathfrak{su}(H) \longrightarrow \mathfrak{su}(H)
$$

$$
\sigma \mapsto -\sum_{i=1}^{n} [[\sigma, t(e_i)], t(e_i)] .
$$

(1.5)

We define almost representations as linear maps $t : \mathfrak{g} \to \mathfrak{su}(H)$ which satisfy approximate commutation relations, and such that $\Gamma$ is invertible. Theorem 3.2 below provides an upper bound for the distance between such a $t$ and a genuine irreducible representation of $\mathfrak{g}$ in terms of the operator norm of the inverse of $\Gamma$. Here we adapt a Newton-type method as in [16].

In Section 4, we apply Theorem 1.1 to an analysis of geometric quantization. In the case when the classical phase space is represented by a closed (i.e., compact without boundary) symplectic manifold $(M, \omega)$, geometric quantization is a linear correspondence $f \mapsto T_\hbar(f)$ between classical observables, i.e., real functions $f \in C^\infty(M)$ on the phase space $M$, and quantum observables, i.e., Hermitian operators $T_\hbar(f) \in \mathcal{L}(H_\hbar)$ on a complex Hilbert space $H_\hbar$. This correspondence is assumed to respect, in the leading order as the Planck constant $\hbar$ tends to 0, a number of basic operations. In this paper, we consider the sphere $S^2$ as a symplectic manifold endowed with its standard area form of the total area $2\pi$, and write $\{\cdot, \cdot\}$ for the associated Poisson bracket.

**Definition 1.4.** A geometric quantization of the sphere associated to a sequence $H_k$, $k \in \mathbb{N}$ of finite-dimensional complex Hilbert spaces, is a collection of $\mathbb{R}$-linear maps $T_k : C^\infty(S^2) \to \mathcal{L}(H_k)$ with $T_k(1) = 1$ for all $k \in \mathbb{N}$, satisfying the following axioms as $k \to +\infty$,

- (P1) $\|T_k(f)\|_{op} = \|f\|_\infty + O(1/k)$;
- (P2) $[T_k(f), T_k(g)] = \frac{i}{\hbar} T_k(\{f, g\}) + O(1/k^2)$;
- (P3) $T_k(f)T_k(g) = T_k(fg + \frac{1}{\hbar} C_1(f, g) + \frac{1}{\hbar^2} C_2(f, g)) + O(1/k^3)$.

\footnote{Our convention for the Poisson bracket is $\{f, g\} := -\omega(sgrad f, sgrad g)$ for all $f, g \in C^\infty(M)$, where $sgrad f$ is the Hamiltonian vector field of $f$ defined by $t_{sgrad f/\omega} + df = 0$.}
In axiom (P3), the map $T_k$ is extended to a map $C^\infty(S^2, \mathbb{C}) \to \text{End}(H_k)$ by $\mathbb{C}$-linearity, and functionals $C_1, C_2 : C^\infty(S^2) \times C^\infty(S^2) \to C^\infty(S^2, \mathbb{C})$ are bi-differential operators. The remainders are understood in the sense of the operator norm, uniformly in the $C^N$-norms of $f, g \in C^\infty(S^2)$ for some $N \in \mathbb{N}$.

In Definition 1.4, the integer $k \in \mathbb{N}$ represents a quantum number, and should be thought as inversely proportional to the Planck constant. Then the limit $k \to +\infty$ describes the so-called semi-classical limit, when the scale gets so large that we recover the laws of classical mechanics from those of quantum mechanics. In particular, the axiom (P2) is the celebrated Dirac condition, relating the Poisson bracket on classical observables to the commutator bracket on quantum observables.

Example 1.5. The existence of geometric quantizations of the sphere was established by Bordemann, Meinrenken and Schlichenmaier [3], using the theory of Boutet de Monvel and Guillemin [4]. Their construction is called Berezin-Toeplitz quantization, and goes as follows. Consider $S^2 = \mathbb{C}P^1$ as a complex manifold, and write $L$ for the dual of the tautological line bundle. For any $k \in \mathbb{N}$, write $L^k$ for the $k$-th tensor power of $L$, and define the Hilbert space $H_k$ as the space of all global holomorphic sections of $L^k$, which can be identified with the space of all homogeneous polynomials on $\mathbb{C}P^1$ of degree $k$. The space $H_k$ lies in the Hilbert space of all $L^2$-sections of $L^k$ equipped with the canonical Hermitian product, and satisfies $\dim H_k = k + 1$. With this language, the Toeplitz operators $T_k(f) \in \mathcal{L}(H_k)$ act by composition of the multiplication by $f \in C^\infty(M)$ and the orthogonal projection to $H_k$. Note that by a shift $k \to k + m - 1$ of the parameter $k$, this construction produces a discrete family of geometric quantizations of the sphere depending on $m \in \mathbb{N}$ and satisfying $\dim H_k = k + m$.

While the construction given above is rather straightforward, verification of the axioms of Definition 1.4 is highly non-trivial. For comprehensive introductions to the Berezin-Toeplitz quantization, see for instance [18, 19, 23]. As explained in Theorem 4.1, a slight modification of this construction leads to the Kostant-Souriau quantization of the sphere, which induces a representation of $\mathfrak{su}(2)$ when restricted to the coordinate functions of $S^2 \subset \mathbb{R}^3$.

Definition 1.4 of a geometric quantization readily extends to arbitrary closed symplectic manifolds $(M, \omega)$, and the construction of Berezin-Toeplitz
quantization described above can be extended to general closed Kähler manifolds admitting a prequantum line bundle $L$. An important ingredient in this construction is a choice of a complex structure $J$ on $M$ making $\omega$ a Kähler form. The Berezin-Toeplitz quantizations associated to two distinct complex structures are essentially different, so that even in the case of the sphere, this construction produces a variety of examples. As shown by Ma and Marinescu [20], Xu [27] and Charles [6], for such quantizations, the bi-differential operator $C_1(f, g)$ is proportional to the Hermitian product of the Hamiltonian vector fields of $f$ and $g$, while the coefficient $C_2$ involves the Ricci curvature.

In Section 4, we use Theorem 1.1 to establish the following classification result on geometric quantizations of the sphere.

**Definition 1.6.** Two geometric quantizations $T_k$ and $Q_k$ with families of Hilbert spaces $\{H_k\}$ and $\{H'_k\}$, $k \in \mathbb{N}$, respectively, are called *semi-classically equivalent*, if for $k$ big enough $\dim H_k = \dim H'_k$, and there exists a sequence of unitary operators $U_k : H_k \to H'_k$ such that for any $f \in C^\infty(S^2)$,

$$\|U_k^{-1}Q_k(f)U_k - T_k(f)\|_{op} = O(1/k),$$

as $k \to +\infty$.

**Theorem 1.7.** Let $T_k : C^\infty(S^2) \to \mathcal{L}(H_k)$, $k \in \mathbb{N}$, be a geometric quantization of the sphere, and assume that

$$\limsup_{k \to +\infty} \dim H_k/k < 2.$$ (1.7)

Then there exists an integer $m \in \mathbb{Z}$ such that for all $k \in \mathbb{N}$ big enough, we have

$$\dim H_k = k + m.$$ (1.8)

Furthermore, any other geometric quantization $Q_k : C^\infty(S^2) \to \mathcal{L}(H'_k)$, $k \in \mathbb{N}$, with $\dim H'_k = k + m$ for all $k$ big enough, is semi-classically equivalent to $T_k$.

Note that for any $m \in \mathbb{N}$, a geometric quantization of the sphere satisfying (1.8) can be realized through the construction of Example 1.5. We thus get the following corollary.
**Corollary 1.8.** Under the dimension assumption \((1.7)\), every geometric quantization of the sphere is semi-classically equivalent to a Berezin-Toeplitz quantization of Example 1.5.

For Berezin-Toeplitz quantizations associated with different complex structures on the sphere, Corollary 1.8 follows from the work of Charles [5]. His result actually holds for Berezin-Toeplitz quantizations of general closed Kähler manifolds, when one varies the complex structure. This leads to the following question.

**Question 1.9.** Is it true that two geometric quantizations of a closed symplectic manifold with sequences of Hilbert spaces of the same dimension are semi-classically equivalent?

Even for the sphere \(S^2\), it is not clear to what extent the dimension assumption \((1.7)\) can be relaxed. An affirmative answer to Conjecture 1.2 should yield an affirmative answer to Question 1.9 in the case of coadjoint orbits of general compact Lie groups, at least with the appropriate assumption on the dimension.

**Remark 1.10.** An additional *trace axiom* for geometric quantizations, which is satisfied for Berezin-Toeplitz quantizations, and which we discuss in the second part of Section 4, yields dimension inequality \((1.7)\) of Theorem 1.7. For geometric quantizations of closed \(2d\)-dimensional symplectic manifolds \((M, \omega)\), this trace axiom implies that as \(k \to +\infty\),

\[
\dim H_k = \left( \frac{k}{2\pi} \right)^d \text{Vol}(M, \omega) + O(k^{d-1}) .
\]

This reflects the physical principle that \(\dim H_k\) approximately equals the maximal number of pair-wise disjoint quantum cells, i.e. cubes of volume \((2\pi \hbar)^d\), inside the classical phase space \((M, \omega)\). For the two-dimensional sphere of the total area \(2\pi\), formula \((1.9)\) reads \(\dim H_k = k + O(1)\). Therefore, inequality \((1.7)\) is satisfied.

A different, albeit related, mathematical model of quantization is *deformation quantization*, which is an \(\hbar\)-linear associative algebra on the space \(C^\infty(M)[[\hbar]]\) such that

\[
f \ast g = fg + \hbar C_1(f, g) + \hbar^2 C_2(f, g) + \cdots ,
\]

\[(1.10)\]
where $C_1(f, g) - C_1(g, f) = \{f, g\}, f, g \in C^\infty(M)$, and $\{\cdot, \cdot\}$ stands for the Poisson bracket [2]. Here the Planck constant $\hbar$ plays the role of a formal deformation parameter, and the operation (1.10) is called a star-product. In Section 4 we also consider the extension (4.32) of axiom (P3) to an asymptotic expansion up to $O(1/k^m)$ for any $m \in \mathbb{N}$, and such an expansion defines a star product via the formal relation $T_\hbar(f)T_\hbar(g) = T_\hbar(f \ast g)$ with $\hbar = 1/k$.

In particular, the Berezin-Toeplitz quantizations described above satisfy this extension of axiom (P3), and thus induce a star-product over $M$ [3, 24, 11].

While deformation quantizations of closed symplectic manifolds are completely classified up to star equivalence given by formula (4.36) below, the classification of geometric quantizations up to conjugation and an error of order $O(\hbar^m)$ with given $m \in \mathbb{N}$ is not yet completely understood, and Theorem 1.1 solves this problem in the case of the sphere for $m = 1$.

In Theorem 4.3 we express the asymptotics of the trace (4.24) in terms of the coefficient $C_2$ of axiom (P3). In Corollary 4.4 we show that for geometric quantizations inducing a deformation quantization (1.11), Theorem 4.3 implies the equality of the usual trace with the canonical trace of the induced star product up to $O(1/k)$, defined by formula (4.37) below.

## 2 Almost representations of $\mathfrak{su}(2)$

Let $H$ be a Hilbert space of complex dimension $\dim H = n \in \mathbb{N}$. A triple of skew-Hermitian operators $X_1, X_2, X_3 \in \mathfrak{su}(H)$ is said to generate an irreducible representation of $\mathfrak{su}(2)$ if they satisfy the commutation relations (1.1) and do not preserve any proper subspace of $H$. From the basic representation theory of $\mathfrak{su}(2)$, this is equivalent with the fact that

$$X_1^2 + X_2^2 + X_3^2 = -\left(\frac{n^2 - 1}{4}\right) \mathbb{1} \quad \text{and}$$

$$[X_j, X_{j+1}] = X_{j+2} \quad \text{for all } j \in \mathbb{Z}/3\mathbb{Z}. \quad (2.1)$$

Furthermore, there exists an orthonormal basis $\{e_j\}_{j=1}^n$ of $H$ in which we have

$$X_3 = \text{diag}\left(i \frac{n-1}{2}, i \left(\frac{n-1}{2} - 1\right), \cdots, -i \frac{n-1}{2}\right), \quad (2.2)$$

Setting the ladder operators to be

$$Y_\pm := \pm iX_1 + X_2 \in \text{End}(H), \quad (2.3)$$
they are described by their coefficients in the orthonormal basis above for all $1 \leq j, m \leq n$ by

$$
\langle Y_{\pm m}, e_j \rangle = 0 \text{ if } j \neq m \pm 1 \text{ and } \\
\langle Y_{\pm m}, e_{m \pm 1} \rangle = \sqrt{\frac{n^2}{4} - \frac{1}{4} - \left(\frac{n - 1}{2} - m\right)^2 \mp \left(\frac{n - 1}{2} - m\right)}.
$$

Conversely, if we have operators $X_3, Y_+, Y_- \in \text{End}(H)$ satisfying (2.2) and (2.4) in an orthonormal basis, then setting $X_1 := i(Y_- - Y_+)/2$ and $X_2 := (Y_+ + Y_-)/2$, we get three operators $X_1, X_2, X_3 \in \mathfrak{su}(2)$ generating an irreducible representation of $\mathfrak{su}(2)$ on $H$.

The rest of the section is dedicated to the proof of Theorem 1.1. Let us first establish the following Lemma on the existence of quasimodes.

**Lemma 2.1.** Let $A \in \text{End}(H)$ be Hermitian, and assume that $v, w \in H$ and $\alpha \in \mathbb{C}$ satisfy

$$
Av = \alpha v + w.
$$

Then there exists $\lambda \in \text{Spec}(A)$ satisfying

$$
|\lambda - \alpha| \leq \frac{\|w\|}{\|v\|}.
$$

Furthermore, for any $\delta > 0$, let $V_\delta \subset H$ be the direct sum to the eigenspaces of eigenvalues $\eta \in \text{Spec}(A)$ satisfying $|\eta - \alpha| < \delta$. Then there exists $e \in V_\delta$ with $\|e\| = 1$ such that

$$
\left\| v - \|v\|e \right\| \leq 2\frac{\|w\|}{\delta}.
$$

**Proof.** As $A \in \text{End}(H)$ is Hermitian, there is an orthonormal basis $\{e_j\}_{j=1}^n$ of $H$ such that $A$ can be diagonalized, with real eigenvalues $\{\lambda_j\}_{j=1}^n$. Consider $v, w \in H$ and $\alpha \in \mathbb{C}$ satisfying formula (2.5). Then we have

$$
\|w\| = \|(A - \alpha)v\| \geq \min_{1 \leq j \leq n} |\lambda_j - \alpha| \|v\|,
$$

which implies that there exists $1 \leq m \leq n$ such that $\lambda_m$ satisfies formula (2.6).
Fix now $\delta > 0$. Then formula (2.5) implies
\[
\|w\|^2 = \sum_{1 \leq j \leq n} |\lambda_j - \alpha|^2 |\langle v, e_j \rangle|^2 \geq \sum_{|\lambda_j - \alpha| \geq \delta} |\lambda_j - \alpha|^2 |\langle v, e_j \rangle|^2 \\
\geq \delta^2 \|v - \sum_{|\lambda_m - \alpha| < \delta} \langle v, e_m \rangle e_m\|^2,
\]
(2.9)
Write $\tilde{e} := \sum_{|\lambda_m - \alpha| < \delta} \langle v, e_m \rangle e_m \in V_\delta$. Then this implies in particular that
\[
\|w\| \geq \delta \|v - \tilde{e}\| \geq \delta \|\|v\| - \|\tilde{e}\|\|
\]
(2.10)
Taking $e := \tilde{e}/\|\tilde{e}\|$, we then get
\[
\|v - \|v\| e\| \leq \|v - \tilde{e}\| + \|\|v\| - \|\tilde{e}\|\|
\]
\[
\leq 2 \|w\|/\delta.
\]
(2.11)
This proves the result.

Before starting with the proof, let us compare some basic consequences of the axioms (R1) and (R2) of Theorem 1.1 with the basic theory of representations of $\mathfrak{su}(2)$ described at the beginning of the Section. For any $k \in \mathbb{N}$, introduce the ladder operators
\[
y_\pm := \pm ix_1 + x_2 \in \text{End}(H_k),
\]
(2.12)
which satisfy $y_\pm^* = -y_\mp$. Then axiom (R2) translates to
\[
\|\pm iy_\pm - [x_3, y_\pm]\|_{op} = \mathcal{O}(1/k).
\]
(2.13)
On the other hand, one has
\[
y_+y_- = x_1^2 + x_2^2 + i[x_1, x_2] , \\
y_-y_+ = x_1^2 + x_2^2 - i[x_1, x_2],
\]
(2.14)
so that axioms (R1) and (R2) imply
\[
\|y_+y_- + \frac{(k + c)^2}{4} \mathbb{1} + x_3^2 - ix_3\|_{op} = \mathcal{O}(1),
\]
\[
\|y_-y_+ + \frac{(k + c)^2}{4} \mathbb{1} + x_3^2 + ix_3\|_{op} = \mathcal{O}(1).
\]
(2.15)
Proof of Theorem 1.1. In what follows we consider the integer \( k \) from the formulation of the theorem as a large parameter, while the constants \( c \) and \( \epsilon \) are fixed. We denote the Hilbert space by \( H_k \) in order to emphasize the dependence on \( k \). All the estimates in the proof are performed with respect to the Hilbert norm \( \| \cdot \| \) on \( H_k \) as \( k \to +\infty \).

Let \( \lambda_k \in \mathbb{R} \) be the highest eigenvalue of the Hermitian endomorphism \(-ix_3 \in \text{End}(H_k)\), and let \( e_k \in H_k \) with \( \|e_k\| = 1 \) be such that
\[ x_3 e_k = i \lambda_k e_k . \] (2.16)

Using formula (2.13), we get the estimate
\[ x_3 (y+e_k) = i (\lambda_k + 1)y_e + \mathcal{O}(1/k) . \] (2.17)

Applying Lemma 2.1 to \( A = -ix_3, v = y+e_k, w = \mathcal{O}(1/k) \), and using the fact that \( \lambda_k \in \mathbb{R} \) is the highest eigenvalue of \(-ix_3\), we get the estimate \( \|y+e_k\| = \mathcal{O}(1/k) \). Using now formula (2.15) and Cauchy-Schwartz inequality, this implies
\[ \mathcal{O}(1/k^2) = \|y+e_k\|^2 = -\langle y-y+e_k, e_k \rangle = \frac{(k+c)^2}{4} - \lambda_k^2 - \lambda_k + \mathcal{O}(1) , \] (2.18)

which readily leads to the estimate
\[ \lambda_k = \frac{k + c - 1}{2} + \mathcal{O}(1/k) . \] (2.19)

Let us now estimate the other eigenvalues of \(-ix_3\) by descending induction using the lowering operators. For the first step of the induction, first note that via formula (2.13) and Cauchy-Schwartz inequality, the estimate (2.19) implies
\[ \|y-e_k\|^2 = -\langle y-y-e_k, e_k \rangle = \frac{(k+c)^2}{4} - \lambda_k^2 + \lambda_k + \mathcal{O}(1) = k + \mathcal{O}(1) , \] (2.20)

so that in particular \( y-e_k \in H_k \) does not vanish for \( k \in \mathbb{N} \) big enough. On the other hand, formula (2.13) implies
\[ x_3(y-e_k) = i (\lambda_k - 1)y_e + \mathcal{O}(1/k) , \] (2.21)

so that applying Lemma 2.1 to \( A = -ix_3, v = y_e, w = \mathcal{O}(1/k) \), we get an eigenvalue \( \lambda_{k-1} \in \mathbb{R} \) of \(-ix_3\) such that
\[ \lambda_{k-1} = \frac{k + c - 1}{2} - 1 + \mathcal{O}(1/k^{3/2}) . \] (2.22)
For the induction step, we will need the basic fact that for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $k \in \mathbb{N}$ and all $\lambda \in \mathbb{R}$

$$\frac{k + c - 1}{2} + \epsilon < \lambda < \frac{k - 1}{2} \quad \text{implies} \quad \frac{(k + c)^2}{4} - \lambda^2 + \lambda > \delta k. \quad (2.23)$$

Let us now assume by induction that for some $m \in \mathbb{N}$ with $0 < m < k + c - 1$, there is $e_{k-m} \in H_k$ with $\|e_{k-m}\| = 1$ such that $x_3 e_{k-m} = i \lambda_{k-m} e_{k-m}$ and

$$\lambda_{k-m} = \frac{k + c - 1}{2} - m + m \mathcal{O}(1/k^{3/2}). \quad (2.24)$$

Then using formula (2.15) as in (2.20), the inequality (2.23) for $\lambda = \lambda_{k-m}$ and $\epsilon > 0$ small enough, together with the fact that the last term of (2.24) satisfies $|m \mathcal{O}(1/k^{3/2})| < \epsilon/2$ for $k \in \mathbb{N}$ big enough, implies

$$\|y - e_{k-m}\|^2 = \frac{(k + c)^2}{4} - \lambda_{k-m}^2 + \lambda_{k-m} + \mathcal{O}(1) \geq \delta k + \mathcal{O}(1). \quad (2.25)$$

Then again by formula (2.13) we have

$$x_3(y - e_{k-m}) = i(\lambda_{k-m} - 1)y - e_{k-m} + \mathcal{O}(1/k), \quad (2.26)$$

so that applying Lemma 2.1 again, we get an eigenvalue $\lambda_{k-m+1}$ of $-ix_3$ satisfying

$$\lambda_{k-m-1} = \lambda_{k-m} - 1 + \mathcal{O}(1/k^{3/2}), \quad (2.27)$$

which implies (2.24) with $m$ replaced by $m + 1$. Using also formula (2.19) for the highest eigenvalue, We thus get by induction starting with (2.22) that for all $m \in \mathbb{N}$ such that $0 \leq m < k + c$, we have the estimate

$$\lambda_{k-m} = \frac{k + c - 1}{2} - m + \mathcal{O}(1/\sqrt{k}). \quad (2.28)$$

Let us now write $\lambda_- \in \mathbb{R}$ for the lowest eigenvalue of $-ix_3$. The argument leading to the estimate (2.19) using $y_-$ instead of $y_+$ leads to the estimate

$$\lambda_- = -\frac{k + c - 1}{2} + \mathcal{O}(1/k). \quad (2.29)$$

On the other hand, as $\lambda_-$ is the lowest eigenvalue, we must have $\lambda_- \leq \lambda_{k+m}$ for all $0 \leq m < k + c$. By comparing (2.29) with (2.28) for $m = \lfloor k + c \rfloor$ we
get that \( c - |c| = O(1/\sqrt{k}) \), and hence \( c \) is an integer. This proves statement (I) of Theorem 1.1.

Observe that we also established that \( \|x_3\|_{op} = k/2 + O(1) \). Replacing \( x_3 \) by \( x_1 \) and \( x_2 \) and applying the same reasoning, we get statement (II).

We can then assume \( c \in \mathbb{Z} \), and through the shift \( k \mapsto k + c \), we will assume without loss of generality that \( c = 0 \). Using the estimate (2.28), we get a set of eigenvalues of \(-ix_3\) parametrized by \( m \in \mathbb{N} \) with \( 0 \leq m \leq k - 1 \), which are pairwise distinct for \( k \in \mathbb{N} \) big enough. This implies that \( \dim H_k \geq k \).

Let us now fix \( k \in \mathbb{N} \) big enough and assume that \( \dim H_k \geq k + 1 \). Let \( E \subset H_k \) be the direct sum of 1-dimensional eigenspaces associated with each of the eigenvalues (2.28), so that \( \dim E = k \) for \( k \in \mathbb{N} \) big enough, and \( E \) is a proper subspace of \( H_k \). In particular, there exists an eigenvalue \( \tilde{\lambda} \) of \(-ix_3\) admitting an eigenvector \( e_{\tilde{\lambda}} \notin E \). First note that the eigenvalue \( \tilde{\lambda} \) has to lie between the highest eigenvalue (2.19) and the lowest eigenvalues (2.29). Furthermore, if we write \( \tilde{c} := \frac{\lambda_k^+ - \tilde{\lambda}}{2} \),

we can repeat the induction process above, but starting with \( \tilde{\lambda} \) instead of \( \lambda_k \), to produce eigenvalues \( \tilde{\lambda}_m \) for all \( m \in \mathbb{N} \) with \( 0 \leq m < k - \tilde{c} \), satisfying the estimate

\[
\tilde{\lambda}_m = \frac{k - \tilde{c} - 1}{2} - m + O(1/\sqrt{k}) .
\] (2.30)

Once again, as we must have \( \tilde{\lambda}_m \geq \lambda_- \) for all \( 0 \leq m < k - \tilde{c} \), this implies that there exists an integer \( m_0 \in \mathbb{N} \) such that \( \tilde{r} = m_0 + O(1/\sqrt{k}) \), and if \( k \in \mathbb{N} \) is big enough, we have

\[
\left| \tilde{\lambda} - \lambda_{k-m_0} \right| < \frac{1}{4} .
\] (2.31)

For any \( m \in \mathbb{N} \), write

\[
V_m := \bigoplus_{|\lambda - \lambda_{k-m}| < \frac{1}{4}} E_{\lambda} ,
\] (2.32)

where \( E_{\lambda} := \{ v \in H_k \mid -ix_3v = \lambda v \} \) for all \( \lambda \in \mathbb{R} \). Now by assumption, there exists an eigenvector \( e_{\tilde{\lambda}} \in H_k \) associated with \( \tilde{\lambda} \) which does not belong to a 1-dimensional eigenspace associated with \( \lambda_{k-m_0} \). Thus using the inequality (2.31), we have

\[
\dim V_{m_0} \geq 2 .
\] (2.33)
Note that for $k \in \mathbb{N}$ big enough, we have either $m_0 > 0$ or $m_0 < k - 1$ (or both). Without loss of generality, let us assume that $m_0 < k - 1$. Let $f_1, f_2 \in V_{m_0}$ be eigenvectors such that $\|f_1\| = \|f_2\| = 1$ and $\langle f_1, f_2 \rangle = 0$. Then applying formula (2.13) and Lemma 2.1 with $A = -ix_3, v = y - f_j$ for $j = 1, 2$ and $w = O(1/k)$, we know that there exists eigenvectors $\tilde{f}_1, \tilde{f}_2 \in V_{m_0+1}$ with $\|\tilde{f}_1\| = \|\tilde{f}_2\| = 1$ such that for $j = 1, 2$, we have

$$\|y - f_j - y - f_j\| \tilde{f}_j = O(1/k). \tag{2.34}$$

On the other hand, using formula (2.15) the fact that $f_1 \in V_{m_0}$ is an eigenvalue of $-ix_3$, we have

$$\langle y - f_1, y - f_2 \rangle = -\langle y_+ y - f_1, f_2 \rangle = O(1). \tag{2.35}$$

Furthermore, formula (2.25) shows that for $j = 1, 2$, we have

$$\|y - f_j\|^2 \geq \frac{k}{2} + O(1). \tag{2.36}$$

This shows

$$\langle \tilde{f}_1, \tilde{f}_2 \rangle = \frac{1}{\|y - f_1\|} \frac{1}{\|y - f_2\|} \langle y - f_1, y - f_2 \rangle + O(1/k^2) = O(1/k^2), \tag{2.37}$$

so that $\tilde{f}_1, \tilde{f}_2 \in V_{m_0+1}$ are linearly independant for $k \in \mathbb{N}$ big enough. This shows $\dim V_{m_0+1} \geq 2$.

Now if we have $m_0 + 1 < k - 1$, one can repeat the same process replacing $V_{m_0}$ by $V_{m_0+1}$, and we get that $\dim V_{m_0+2} \geq 2$. This shows by induction that for all $m \in \mathbb{N}$ with $m_0 \leq m \leq k - 1$, we have $\dim V_m \geq 2$. On the other hand, if $m_0 > 0$, we can repeat the same process on $V_{m_0}$ using $y_-$ instead of $y_+$ to get $\dim V_{m_0-1} \geq 2$. Thus again by induction, we finally get that for all $m \in \mathbb{N}$ with $0 \leq m \leq k - 1$, we have $\dim V_m \geq 2$. Now by definition (2.32), the subspaces $V_m$ are pairwise orthogonal for each $m \in \mathbb{N}$, and we have

$$\dim \bigoplus_{0 \leq m \leq k-1} V_m \geq 2k. \tag{2.38}$$

This contradicts the assumption $\dim H_k \geq 2k$ for all $k \in \mathbb{N}$, and proves the statement (III) of Theorem 1.1.

Assuming without loss of generality that $c = 0$, the argument above shows in particular that all eigenvalues of $-ix_3$ are simple and given by (2.28) for
all \( m \in \mathbb{N} \) with \( 0 \leq m \leq k - 1 \). Then for any normalized eigenvector \( e_m \in H_k \) of \(-ix_3\) associated with \( \lambda_m \), we can apply the second statement of Lemma 2.1 to \( A = -ix_3 \), \( v = y_\cdot e_m \) and \( w = \mathcal{O}(1/k) \), so that via formulas (2.25), (2.26) and Cauchy-Schwartz inequality, we get a normalized eigenvector \( e_m - 1 \in H_k \) of \(-ix_3\) associated with \( \lambda_m - 1 \) satisfying

\[
\langle y_\cdot e_m, e_m - 1 \rangle = \langle \|y_\cdot e_m\| e_m - 1, e_m - 1 \rangle + \mathcal{O}(1/k) = \|y_\cdot e_m\| + \mathcal{O}(1/k). \tag{2.39}
\]

Starting with any eigenvector \( e_k \) of \(-ix_3\) associated with \( \lambda_k \), we thus construct an orthonormal eigenbasis \( \{e_j\}_{j=1}^k \) for \( x_3 \) associated to the sequence of eigenvalues \( \{\lambda_j\}_{j=1}^k \) and satisfying formula (2.39) for all \( 1 \leq m \leq k \). Let us now note that for any \( \lambda \in \mathbb{R} \), using in particular formula (2.23), we get

\[
-k - 1/2 + \epsilon < \lambda < k - 1/2 \quad \text{implies} \quad \left| \frac{d}{d\lambda} \left( \sqrt{\frac{k^2}{4} - \frac{1}{4} - \lambda^2 + \lambda} \right) \right| = \mathcal{O}(\sqrt{k}), \tag{2.40}
\]

Now that by the first line of equation (2.25) and formula (2.28), we get for all \( 1 \leq m \leq k \) that

\[
\|y_\cdot e_m\| = \sqrt{\frac{k^2}{4} - \frac{1}{4} - \left( \frac{k - 1}{2} - m \right)^2 + \left( \frac{k - 1}{2} - m \right) + \mathcal{O}(1)}. \tag{2.41}
\]

On the other hand, for all \( j \neq m + 1 \), using formula (2.13) and Cauchy-Schwartz inequality, we get

\[
i\langle y_\cdot e_m, e_j \rangle = \langle [x_3, y_\cdot] e_m, e_j \rangle + \mathcal{O}(1/k) = i(\lambda_j - \lambda_m) \langle y_\cdot e_m, e_j \rangle + \mathcal{O}(1/k). \tag{2.42}
\]

Now formula (2.19) implies that \( |\lambda_m - \lambda_j - 1| \geq 1/2 \) as soon as \( j \neq m + 1 \) and \( k \in \mathbb{N} \) big enough, so that we get

\[
\langle y_\cdot e_m, e_j \rangle = \mathcal{O}(1/k) \quad \text{for} \quad j \neq m + 1. \tag{2.43}
\]

Replacing \( y_\cdot \) by \( y_+ \) in the reasoning above via formula (2.13) and (2.15), we get for all \( 1 \leq j, m \leq k \) in the same way

\[
\langle y_+ e_m, e_j \rangle = \mathcal{O}(1/k) \quad \text{for} \quad m \neq j + 1, \tag{2.44}
\]

\[
\langle y_\cdot e_m, e_{m+1} \rangle = \sqrt{\frac{k^2}{4} - \frac{1}{4} - \left( \frac{k - 1}{2} - m \right)^2 - \left( \frac{k - 1}{2} - m \right) + \mathcal{O}(1)}.
\]
In the orthonormal basis \( \{e_j\}_{j=1}^k \) of \( H_k \) constructed above and following (2.2) and (2.1), let us now set

\[
X_3 := \text{diag} \left( i \frac{k-1}{2}, i \left( \frac{k-1}{2} - 1 \right), \ldots, -i \frac{k-1}{2} \right),
\]

\[
Y_{\pm} e_m := \sqrt{\frac{k^2}{4} - \frac{1}{4} - \left( \frac{k-1}{2} - m \right)^2} \mp \left( \frac{k-1}{2} - m \right) e_{m\pm 1},
\]

for all \( 1 \leq m \leq k \). By the basic representation theory of \( \mathfrak{su}(2) \) described at the beginning of the section and the definition (2.12) of \( y_{\pm} \), to show Theorem 1.1, it suffices to show that \( \|x_3 - X_3\|_{op} = \mathcal{O}(1) \) and \( \|y_{\pm} - Y_{\pm}\|_{op} = \mathcal{O}(1) \). Now formula (2.24) implies immediately that

\[
\|x_3 - X_3\|_{op} = \mathcal{O}(1/\sqrt{k}).
\]

On the other hand, for all \( 1 \leq j, m \leq k \), formulas (2.39), (2.41), (2.43) and (2.45) yield

\[
\langle (y_{\pm} - Y_{\pm})e_j, e_m \rangle = \mathcal{O}(1/k) \quad \text{for} \quad m \neq j \pm 1,
\]

\[
\langle (y_{\pm} - Y_{\pm})e_m, e_{m\pm 1} \rangle = \mathcal{O}(1).
\]

Decompose the matrix into \( y_{\pm} - Y_{\pm} = A + B \), where all coefficients of \( A \) vanish except \( A_{m, m\pm 1} = \mathcal{O}(1) \) for all \( 1 \leq m \leq k \) and where \( B_{jm} = \mathcal{O}(1/k) \) for all \( 1 \leq j, m \leq k \). Then we readily get \( \|A\|_{op} = \mathcal{O}(1) \), while by Cauchy-Schwartz we compute

\[
\|B\|_{op}^2 = \max_{\|v\|=1} \sum_{j=1}^k \sum_{m=1}^k |B_{jm} \langle e_m, v \rangle|^2
\]

\[
\leq k \max_{1 \leq j \leq k} \sum_{m=1}^k |B_{jm}|^2
\]

\[
\leq k^2 \mathcal{O}(1/k^2) = \mathcal{O}(1).
\]

By the triangle inequality this gives

\[
\|y_{\pm} - Y_{\pm}\|_{op} \leq \|A\|_{op} + \|B\|_{op} = \mathcal{O}(1).
\]

Thus we get the statement (IV) of Theorem 1.1. This concludes the proof.
3 Almost representations of compact Lie algebras

In this section, we propose an alternative notion of irreducibility of almost representations in the context of general compact Lie algebras, and present another version of the Ulam-type statement: irreducible almost-representations can be approximated by a genuine representation.

Let \((g, \{\cdot, \cdot\})\) be a real compact \(n\)-dimensional Lie algebra. This means that it is semi-simple and that its Killing form \(\langle \cdot, \cdot \rangle\) is negative definite. Consider an orthonormal basis \(\{e_j\}_{j=1}^{n}\) of \(g\) such that for all \(1 \leq j, k \leq n\), we have
\[
\langle e_j, e_k \rangle = -\delta_{jk}.
\] (3.1)

Let \(H\) be a complex Hilbert space of finite dimension. Recall that \(\| \cdot \|_\text{op}\) denotes the operator norm on the space \(\mathfrak{su}(H)\) of skew-Hermitian operators. For an operator \(A : \mathfrak{su}(H) \to \mathfrak{su}(H)\) we write \(|||A|||\) for its operator norm with respect to the operator norm on \(\mathfrak{su}(H)\).

**Definition 3.1.** A linear map \(t : g \to \mathfrak{su}(H)\) is called a \((\mu, K, \epsilon)\)-almost representation of \((g, \{\cdot, \cdot\})\) if the following assumptions hold:

- For all \(1 \leq j, k \leq n\), the defect
  \[
  \alpha_{jk} := t(\{e_j, e_k\}) - [t(e_j), t(e_k)]
  \] (3.2)
  satisfies \(\epsilon := \max_{j, k} \|\alpha_{jk}\|_\text{op} ;\)
- \(K := \max_j \|t(e_j)\|_\text{op} ;\)
- The almost-Casimir operator \(\Gamma\) defined by \([165]\) is invertible with \(\mu := |||\Gamma^{-1}|||\).

**Theorem 3.2.** Let \((g, \{\cdot, \cdot\})\) be a real semi-simple compact finite dimensional Lie algebra. Then for any \(c > 0\), there exists a constant \(\gamma > 0\) with the following property. Given any \((\mu, K, \epsilon)\)-almost representation \(t : g \to \mathfrak{su}(H)\) with \(\epsilon \leq \gamma \min(\mu^{-2}K^{-2}, \mu^{-1}, 1)\), there exists a representation \(\rho : g \to \mathfrak{su}(H)\) such that for all \(1 \leq j \leq n\),
\[
\|t(e_j) - \rho(e_j)\|_\text{op} \leq c \mu K \epsilon .
\] (3.3)
Remark 3.3. Although more general, this result has a number of drawbacks as compared to Theorem [1.1] in the case \( g = \mathfrak{su}(2) \). First, it is unclear to us how to estimate \( \mu \) in the case of geometric quantizations of the sphere. Second, even if we have an ansatz \( \mu \sim 1 \) and \( \|x_j\| \sim k \sim \dim H \), as it should be for an irreducible \( k \)-dimensional representation, the existence of a nearby genuine representation is guaranteed only when the defect \( \epsilon \lesssim k^{-2} \), as opposed to a less restrictive assumption \( \epsilon \lesssim k^{-1} \) provided by Theorem [1.1].

Discussion on almost irreducibility: For representations, the invertibility of the adjoint Casimir \( \Gamma \) is equivalent to irreducibility. In fact, note that the definition of almost-Casimir given in (1.3) extends to any collection \( X = \{x_1, \ldots, x_n\} \) of operators in \( \mathfrak{su}(H) \) by the formula

\[
\Gamma \sigma := - \sum_{i=1}^{n} [[\sigma, x_i], x_i].
\]

Furthermore,

\[
\text{tr}(\Gamma(\sigma) \sigma) = \sum_{i=1}^{n} \text{tr} ( [\sigma, x_i]^2) ,
\]

and hence \( \Gamma \sigma = 0 \) if and only if \( \sigma \) commutes with all the operators from \( X \). In particular, \( \Gamma \) is invertible if and only if the operators from \( X \) possess a common proper invariant subspace. With this in mind, we are going to compare \( \mu(X) := |||\Gamma^{-1}| | \) with another quantity of geometric flavor which can be interpreted as a magnitude of irreducibility. Put

\[
d(X) := \min_{\Pi} \max_{j} \|(1 - \Pi) x_j \Pi\|_{op},
\]

where \( \Pi \) runs over all orthogonal projectors to proper subspaces \( V \subset H \), and \( j \in \{1, 2, 3\} \). Intuitively speaking, smallness of \( d \) yields that the corresponding subspace \( V \) is almost invariant.

To this end, denote by \( \mathcal{X} \) the space of all collections \( X \) whose almost-Casimir is invertible. We say that two positive functions on \( \mathcal{X} \) are equivalent if their ratio is bounded away from 0 and \(+\infty\) by two constants which depend on \( \dim H \).

Proposition 3.4. The functions \( \mu^{-1/2} \) and \( d \) are equivalent.
Sketch of the proof: Denote by $\|A\|_2 := \sqrt{\text{tr}(A^*A)}$ the Hilbert-Schmidt norm of an operator, and by $\lambda_1(X)$ the first eigenvalue of $-\Gamma$. The standard inequalities between the Hilbert-Schmidt norm and the operator norm imply that $\mu^{-1}$ is equivalent to $\lambda_1$. The claim follows from the inequalities
\[ d(X)^2 \leq C_1(k)\lambda_1(X) , \]
and
\[ \lambda_1(X) \leq C_2(k)d(X)^2 . \]
In order to prove inequality (3.5), take an eigenvector $A$ of $\Gamma$ with $\|A\|_2 = 1$ corresponding to the first eigenvalue. Since $\text{tr}A = 0$, the spectrum of $A$ can be written as the union of two clusters lying at distance at least $\sim k^{-2}$ apart. Let $\Pi$ be the spectral projection corresponding to one of them. Since by (3.4) $A$ almost commutes with $x_j$ up to $\epsilon$, one readily deduces from Lemma 2.1 on quasimodes that the image of $\Pi$ is almost invariant under $x_j$. This yields (3.5).

Inequality (3.6) follows from the identity
\[ -(\Gamma(\Pi), \Pi) = 2 \sum_{i=1}^{n} \| [x_i, \Pi] \|_2^2 , \]
which holds true for every orthogonal projector $\Pi$.

The details of the argument are left to the reader. \hfill \Box

It would be interesting to find sharp bounds on the ratio of $\mu^{-1/2}$ and $d$ in terms of $\dim H$. At the moment, we cannot compute them even for genuine irreducible representations.

Proof of Theorem 3.2. To simplify the notations, we will often write $x_j := t(e_j)$ for all $1 \leq j \leq n$. All the estimates in the proof are with respect to the operator norm of $\mathfrak{su}(H)$ and only depend on $(\mathfrak{g}, \{\cdot,\cdot\})$.

For a linear map $a : \mathfrak{g} \to \mathfrak{su}(H)$, define an approximate Elienberg-Chevalley coboundary $d_t a : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{su}(H)$ by
\[ d_t a(g, h) := [t(g), a(h)] - t(h), a(g)] - a(\{g, h\}) . \]

The proof follows the Newton-type iterative process due to Kazhdan [16] adapted to the context of Lie algebras. At the first step we try to find a linear map $a : \mathfrak{g} \to \mathfrak{su}(H)$ so that
\[ \overline{t}(g) := t(g) + a(g) \]
is a genuine representation. This yields equation
\[ \alpha(g, h) - dt a(g, h) - [a(g), a(h)] = 0 . \] (3.9)

Ignore the third, quadratic in \( a \) term, and solve the linearized equation \( dt a = \alpha \). As we will see, the almost representation \( \tilde{t} := t + a \) is closer to a genuine representation. Repeating the process, we get in the limit the desired genuine representation approximating the original almost representation \( t \).

To make this precise, we have to solve the linearized homological equation \( dt a = \alpha \). This is done by using an effective approximate version of Whitehead’s Lemma (see p.88–89 of [14]).

Consider the anti-symmetric 2-form \( \alpha : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{su}(H) \) defined for any \( g, h \in \mathfrak{g} \) by
\[ \alpha(g, h) := t(\{g, h\}) - [t(g), t(h)] \] (3.10)
and the 1-form \( a : \mathfrak{g} \to \mathfrak{su}(H) \) defined for any \( g \in \mathfrak{g} \) by
\[ a(g) := -\sum_{i=1}^{n} \Gamma^{-1}[\alpha(g, e_i), x_i] . \] (3.11)

**Lemma 3.5.** For all \( j, k = 1, \ldots n \)
\[ \alpha(e_j, e_k) = dt a(e_j, e_k) + O(\mu^2 K^2 \epsilon^2) . \] (3.12)

The lemma is proved at the end of this section.

Let us now consider the linear map \( \overline{t} : \mathfrak{g} \to \mathfrak{su}(H) \) defined for all \( g \in \mathfrak{g} \) by
\[ \overline{t}(g) := t(g) + a(g) , \] (3.13)
and set \( \overline{x}_j := \overline{t}(e_j) \) for all \( 1 \leq j \leq n \). Then for all \( 1 \leq j \leq n \), by formula (3.11) for \( a(e_j) \) we have
\[ \overline{x}_j \leq K(1 + O(\mu \epsilon)) . \] (3.14)

On the other hand, considering for all \( 1 \leq j, k \leq n \) the defect
\[ \overline{\alpha}_{jk} := \overline{t}(\{e_j, e_k\}) - [\overline{t}(e_j), \overline{t}(e_k)] , \] (3.15)
we see from (3.9) that
\[ \overline{\alpha}_{jk} = \alpha_{jk} - dt a(e_j, e_k) - [a(e_j), a(e_k)] = O(\mu^2 K^2 \epsilon^2) . \] (3.16)
Finally, consider the almost-Casimir operator $\Gamma : \mathfrak{su}(H) \to \mathfrak{su}(H)$ defined as in (1.5) with $x_k$ replaced by $\tilde{t}(e_k)$ for all $1 \leq k \leq n$. Then we get
\[
\Gamma = \Gamma + \epsilon(1 + \mu \epsilon)O(\mu K^2) = \Gamma \left(1 + \epsilon(1 + \mu \epsilon)O(\mu^2 K^2)\right).
\]
This implies that for any $\delta > 0$, there exists a constant $\gamma > 0$ such that if $\epsilon(1 + \mu \epsilon) \leq \gamma/\mu^2 K^2$, then $\Gamma$ is invertible and for all $\sigma \in \mathfrak{su}(H)$, its inverse satisfies
\[
\|\Gamma^{-1}(\sigma)\|_{op} \leq (1 + \delta)\mu \|\sigma\|_{op}.
\]
This, together with the estimates (3.14) and (3.16), shows that for any $\delta > 0$, there exists $\gamma > 0$ such that if $\epsilon \leq \gamma \min(\mu - 1, 1)$, the linear map $\tilde{t} : g \to \mathfrak{su}(H)$ is an $(\mu, K, \epsilon)$-almost representation with $\mu \leq \mu(1 + \delta)$, $K \leq K(1 + \delta)$ and $\epsilon \leq \epsilon \delta$.

Taking $\delta > 0$ such that $\delta < (1 + \delta)^{-4}$, we get that $\epsilon \leq \gamma \min(\mu^{-2} K^{-2}, \mu^{-1}, 1)$, and we can reiterate the construction above with the $(\mu, K, \epsilon)$-almost representation $\tilde{t} : g \to \mathfrak{su}(H)$ instead of $t : g \to \mathfrak{su}(H)$. At the $N$-th iteration, we get a $(\mu_N, K_N, \epsilon_N)$-almost representation $t_N : g \to \mathfrak{su}(H)$ with
\[
\mu_N \leq \mu(1 + \delta)^N, \quad K_N \leq K(1 + \delta)^N \quad \text{and} \quad \epsilon_N \leq \epsilon \delta^N.
\]
Writing $a_N : g \to \mathfrak{su}(H)$ for the 1-form defined as in (3.14) for $t_N : g \to \mathfrak{su}(H)$, for all $1 \leq j \leq n$ we get
\[
t_N(e_j) = t_{N-1}(e_j) + a_N(e_j) = t(e_j) + \sum_{k=1}^{N} a_k(e_j)
\]
\[
= t(e_j) + \sum_{k=1}^{N} ((1 + \delta)^2 \delta)^k O(\mu K \epsilon),
\]
and the sum of the last line converges as $N \to +\infty$ for $\delta > 0$ small enough. As $\epsilon_N \to 0$, the limit map $\rho : g \to \mathfrak{su}(H)$ is a genuine representation, satisfying the inequality (3.3) by (3.17).

**Proof of Lemma 3.5.** First note that by definition, for any $1 \leq i, j, k \leq n$, we have
\[
0 = [\alpha_{jk}, x_i] + \alpha([e_j, e_k], e_i) + [\alpha_{ki}, x_j]
+ \alpha([e_k, e_i], e_j) + [\alpha_{ij}, x_k] + \alpha([e_i, e_j], e_k).
\]
Taking the bracket of this identity with \( x_i \) and following the computations of [14, p.90], this implies that for all \( 1 \leq j, k \leq n \), we have

\[
\Gamma \alpha_{jk} = \sum_{i=1}^{n} \left( [\alpha(\{e_j, e_k\}, e_i), x_i] + 
\left[
[\alpha_{ki}, x_i], x_j
- [\alpha_{ji}, x_i], x_k
\right]\right)
- A_{jk},
\]  

(3.18)

with

\[
A_{jk} := - \sum_{i=1}^{n} \left( [\alpha_{ki}, [x_j, x_i]] - [\alpha(e_k, \{e_i, e_j\}), x_i]
- [\alpha_{ji}, [x_k, x_i]] + [\alpha(e_j, \{e_i, e_k\}), x_i] \right).
\]  

(3.19)

Applying \( \Gamma^{-1} : \mathfrak{su}(H) \rightarrow \mathfrak{su}(H) \) on both sides of the equality (3.18) and recalling the definition (3.11) of \( a : \mathfrak{g} \rightarrow \mathfrak{su}(H) \), we get

\[
\alpha_{jk} = [x_j, a(e_k)] - [x_k, a(e_j)] - a(\{e_i, e_j\}) - B_{jk} - \Gamma^{-1} A_{jk},
\]  

(3.20)

with

\[
B_{jk} := - \sum_{i=1}^{n} \left( \Gamma^{-1}[[\alpha_{ki}, x_i], x_j] - \Gamma^{-1}[\alpha_{ki}, x_i], x_j\right)
- \Gamma^{-1}[[\alpha_{ji}, x_i], x_k] + \Gamma^{-1}[\alpha_{ji}, x_i], x_k\right).
\]  

(3.21)

Let us now estimate the terms (3.19) and (3.21). First note that as the Killing form \( \langle \cdot, \cdot \rangle \) is Ad-invariant and by the explicit formula (3.1), we have

\[
- \sum_{i=1}^{n} [\alpha(e_k, \{e_i, e_j\}), x_i] = \sum_{i=1}^{n} \left( \sum_{l=1}^{n} \langle \{e_i, e_j\}, e_l \rangle [\alpha_{kl}, x_i] \right)
= \sum_{l=1}^{n} [\alpha_{kl}, \sum_{i=1}^{n} \langle \{e_j, e_l\}, e_i \rangle x_i] = - \sum_{l=1}^{n} [\alpha_{kl}, t(\{e_j, e_l\})]
\]  

(3.22)

\[
= - \sum_{l=1}^{n} [\alpha_{kl}, [x_j, x_l]] + \sum_{l=1}^{n} [\alpha_{kl}, \alpha_{jl}] = - \sum_{l=1}^{n} [\alpha_{kl}, [x_j, x_l]] + O(\epsilon^2).
\]

Comparing with formula (3.19) for \( A_{jk} \), this implies that

\[
\Gamma^{-1} A_{jk} = O(\mu \epsilon^2).
\]  

(3.23)
On the other hand, following [114, p. 78] for any \( g \in \mathfrak{g} \) and \( 1 \leq j \leq n \), using the Killing form in the same way than in (3.22) we get

\[
\Gamma[g, x_j] = -\sum_{i=1}^{n} [[[g, x_j], x_i], x_i] - \sum_{i=1}^{n} [[[g, x_j], x_i], x_i] - \sum_{i=1}^{n} [[[g, x_i], x_j], x_i]
\]

\[
= [\Gamma, g, x_j] - C_j(g) - \sum_{i=1}^{n} [[[g, t({e_j, e_i})], x_i], x_i] - \sum_{i=1}^{n} [[[g, x_i], t({e_j, e_i})], x_i] = [\Gamma, g, x_j] - C_j(g),
\]

with

\[
C_j(g) := -\sum_{i=1}^{n} [[[g, x_j], x_i], x_i] - \sum_{i=1}^{n} [[[g, x_i], x_j], x_i].
\]

In particular, for any \( 1 \leq i, j, k, l \leq n \), we have

\[
\Gamma^{-1}[[\alpha_{ij}, x_k], x_l] - \Gamma^{-1}[[\alpha_{ij}, x_k], x_l] = \Gamma^{-1}([[\alpha_{ij}, x_k], x_l] - \Gamma^{-1}[[\alpha_{ij}, x_k], x_l]) = \Gamma^{-1}C_l(\Gamma^{-1}[[\alpha_{ij}, x_k]]) = O(\mu^2 K^2 \epsilon^2).
\]

Comparing with formula (3.21) for \( B_{jk} \), we thus get

\[
B_{jk} = O(\mu^2 K^2 \epsilon^2). \tag{3.24}
\]

Then via the estimates (3.23) and (3.24), the identity (3.20) becomes

\[
\alpha_{jk} = [x_j, a(e_k)] - [x_k, a(e_j)] - a({e_k, e_j}) + O(\mu^2 K^2 \epsilon^2).
\]

This completes the proof of the lemma. \( \square \)

4 Equivalence of quantizations of the sphere

The basic strategy of the proof of Theorem [114] is to use our Theorem [114] on almost representations of \( \mathfrak{su}(2) \) to show that any geometric quantization of the sphere in the sense of Definition [114] is semi-classically equivalent to a geometric quantization generating irreducible representations of \( \mathfrak{su}(2) \) when restricted to the Cartesian coordinate functions \( u_1, u_2, u_3 \in C^\infty(S^2) \) of \( S^2 \subset \mathbb{R}^3 \). Recall in fact that these functions satisfy the commutation relation

\[
\{u_j, u_{j+1}\} = -2u_{j+2}, \tag{4.1}
\]

for all \( j \in \mathbb{Z}/3\mathbb{Z} \). The following theorem shows that such a geometric quantization indeed exists, as expected from any reasonable notion of quantization.
Theorem 4.1. [3] There exists a geometric quantization of the sphere \( S_k : C^\infty(S^2) \to \mathcal{L}(H_k) \), \( k \in \mathbb{N} \), with \( \dim H_k = k + 1 \) for all \( k \in \mathbb{N} \), such that the operators \( X_1, X_2, X_2 \in \mathfrak{su}(H_k) \), defined for all \( j \in \mathbb{Z}/3\mathbb{Z} \) by

\[
X_j := \frac{ik}{2} S_k(u_j), \quad (4.2)
\]
generate an irreducible representation of \( \mathfrak{su}(2) \) on \( H_k \) for all \( k \in \mathbb{N} \).

Proof. It is a consequence of a result of Tuynman [25, Th.2.1] that the Berezin-Toeplitz quantization \( T_k : C^\infty(S^2) \to \mathcal{L}(H_k) \), \( k \in \mathbb{N} \), constructed in [3] (cf. Example 1.5 above) induces the Kostant-Souriau quantization \( S_k : C^\infty(S^2) \to \mathcal{L}(H_k) \), \( k \in \mathbb{N} \), through the following formula for all \( f \in C^\infty(S^2) \),

\[
S_k(f) := T_k \left(f + \frac{1}{4k} \Delta f\right), \quad (4.3)
\]
where \( \Delta \) is the Laplace-Beltrami operator of \( S^2 \) associated with the Kähler metric. It is straightforward to check that it again satisfies all the axioms of Definition 1.4, and it is a basic property of Kostant-Souriau quantization that in the case of the canonical complex structure of \( (S^2, \omega) \), its restriction on coordinate functions generate an irreducible representation of \( \mathfrak{su}(2) \) on \( H_k \) for all \( k \in \mathbb{N} \) (for a comprehensive account of Kostant-Souriau geometric quantization, see [26, Chap. 9]).

Let us recall some preliminaries needed for the proof of Theorem 1.7. A bi-differential operator \( C : C^\infty(M) \times C^\infty(M) \to C^\infty(M) \) is called a Hochschild cocycle if for all \( f_1, f_2, f_3 \in C^\infty(M) \), we have

\[
f_1 C(f_2, f_3) - C(f_1 f_2, f_3) + C(f_1, f_2 f_3) - C(f_1 f_2) f_3 = 0. \quad (4.4)
\]

We will write

\[
C_-(f, g) := \frac{C(f, g) - C(g, f)}{2} \quad \text{and} \quad C_+(f, g) := \frac{C(f, g) + C(g, f)}{2},
\]

for the anti-symmetric and symmetric part of \( C \).

Assume now that \( T_k : C^\infty(S^2) \to \mathcal{L}(H_k) \), \( k \in \mathbb{N} \), satisfy the axioms of Definition 1.4. The associativity of composition of operators implies that the bi-differential \( C_1 \) appearing in axiom (P3) is Hochschild cocycle. Furthermore, the axiom (P2) is equivalent with the fact that

\[
C_1^-(f, g) = \frac{i}{2} \{f, g\}, \quad (4.5)
\]
for all \( f, g \in C^\infty(S^2) \). Then formula (4.4) for \( C_1^+ \) is a consequence of the Leibniz rule for the Poisson bracket, and this shows that \( C_1^+ \) is a symmetric Hochschild cocycle. Then by [12, Prop. 2.14], it is a Hochschild coboundary, meaning that there exists a differential operator \( D \) vanishing on constants such that for \( f, g \in C^\infty(S^2) \), we have

\[
C_1^+(f, g) = D(f)g + fD(g) - D(fg) .
\] (4.6)

Furthermore, the axiom (P1) implies that the operator \( T_k(f) \in \text{End}(H_k) \) is Hermitian for all \( k \in \mathbb{N} \) big enough if and only if \( f \in C^\infty(M, \mathbb{C}) \) is real valued. As the square of a Hermitian operator is Hermitian, the axiom (P3) then shows that \( C_1^+ \) is a real-valued bi-differential operator, so that \( D \) has real coefficients.

Let us now assume that \( C_1^+ \equiv 0 \). Then using the associativity of composition of operators, one readily checks that \( C_2^- \) also satisfies formula (4.4) in that case, so that it is an anti-symmetric Hochschild cocycle. Then by [12, Prop. 2.14], there exists a 2-form \( \alpha \in \Omega^2(S^2, \mathbb{C}) \) so that for all \( f, g \in C^\infty(M) \), we have

\[
C_2^-(f, g) = \frac{i}{2} \alpha(\text{sgrad} f, \text{sgrad} g) .
\] (4.7)

Furthermore, by axiom (P1) as above and the fact that the commutator of Hermitian operators is Hermitian, the axiom (P3) implies that the bi-differential operator \( iC_2^- \) is real valued, so that \( \alpha \) is a real 2-form.

The proofs of Theorem 1.7 and 4.3 are based on a natural operation on quantizations, which we call a change of variable. Specifically, given a geometric quantization \( T_k : C^\infty(S^2) \to \mathcal{L}(H_k) \), \( k \in \mathbb{N} \), and a differential operator \( D : C^\infty(S^2) \to C^\infty(S^2) \), set

\[
T_k^D(f) := T_k\left(f + \frac{1}{k} D f\right) ,
\] (4.8)

for all \( f \in C^\infty(S^2) \) and all \( k \in \mathbb{N} \). Then one readily checks that the maps \( T_k^D : C^\infty(S^2) \to \mathcal{L}(H_k) \), \( k \in \mathbb{N} \), satisfy the axioms of Definition 1.4 and that for any \( f \in C^\infty(S^2) \), we have the estimate

\[
\|T_k(f) - T_k^D(f)\|_{op} = O(1/k) ,
\] (4.9)

as \( k \to +\infty \). We will write \( C_{1,D} \) and \( C_{2,D} \) for the associated bi-differential operators of axiom (P3).
We will use the operation of change of variables to reduce the proof of Theorems 1.7 to a class of remarkable quantizations, described by the following result.

**Lemma 4.2.** For any geometric quantization \( T_k : C^\infty(S^2) \to \mathcal{L}(H_k), k \in \mathbb{N}, \) there exists a differential operator \( D : C^\infty(S^2) \to C^\infty(S^2) \) vanishing on constants such that the bi-differential operators of axiom (P3) associated with the induced quantization \( T^D_k : C^\infty(S^2) \to \mathcal{L}(H_k), k \in \mathbb{N}, \) satisfy

\[
C^+_{1,D}(f, g) = 0 \quad \text{and} \quad C^-_{2,D}(f, g) = -c \{f, g\},
\]

for all \( f, g \in C^\infty(S^2), \) where \( c \in \mathbb{R} \) is constant.

**Proof.** One readily computes that a change of variable \( (4.8) \) associated to a differential operator \( D : C^\infty(S^2) \to C^\infty(S^2) \) acts on the bi-differential operators \( C^+_{1,D} \) and \( C^-_{2,D} \) via the following formula, for all \( f, g \in C^\infty(S^2), \)

\[
C^+_{1,D}(f, g) = C^+_{1}(f, g) + D(f)g + fD(g) - D(fg),
\]

\[
C^-_{2,D}(f, g) = C^-_{2}(f, g) + \frac{i}{2} \left( \{D(f), g\} + \{f, D(g)\} - D(\{f, g\}) \right).
\]

In particular, formula \( (4.6) \) shows that there is an operator \( D \) satisfying \( C^+_{1,D} \equiv 0 \), determined up to the addition of a derivation \( \delta : C^\infty(S^2) \to C^\infty(S^2). \) Let now \( D : C^\infty(S^2) \to C^\infty(S^2) \) be such that that \( C^+_{1,D} \equiv 0, \) and let \( \alpha_D \in \Omega^2(S^2, \mathbb{R}) \) be the two form of formula \( (4.7) \) associated with \( C^-_{2,D}. \) Then if we set

\[
c := \frac{1}{2\pi} \int_{S^2} \alpha_D,
\]

we know that there exists a 1-form \( \theta \in \Omega^1(S^2, \mathbb{R}) \) such that

\[
\alpha_D = c \omega + d\theta.
\]

On the other hand, for all \( f, g \in C^\infty(M), \) we have by definition

\[
d\theta(\text{sgrad} f, \text{sgrad} g) = \{\theta(\text{sgrad} f), g\} + \{f, \theta(\text{sgrad} g)\} - \theta(\text{sgrad}\{f, g\}).
\]

Then if we consider the derivation \( \delta : C^\infty(S^2) \to C^\infty(S^2) \) defined for all \( f \in C^\infty(S^2) \) by \( \delta f := -\theta(\text{sgrad} f), \) formulas \( (4.11) \) and \( (4.13) \) imply

\[
C^-_{2,D+\delta}(f, g) = c \omega(\text{sgrad} f, \text{sgrad} g) = -c \{f, g\},
\]

and \( C^+_{1,D+\delta} = C^+_{1,D} \equiv 0. \) This shows the result. \( \square \)
Proof of Theorem 1.7. Without loss of generality, $H_k = H'_k$ for all $k$ big enough. Using the estimate (4.9) and Lemma 4.2, we see that it suffices to establish Theorem 1.7 for geometric quantizations for which there is a constant $c \in \mathbb{R}$ such that $C^+_1 \equiv 0$ and $C^-_2 = -c \{\cdot, \cdot\}$. Given such a quantization, one readily checks from Definition 1.4 and the commutation relations (4.1) that the assumptions of Theorem 1.1 are satisfied for the constant $c \in \mathbb{R}$ and the operators $x_1, x_2, x_3 \in \mathfrak{su}(H_k)$ defined for all $k \in \mathbb{N}$ and $j \in \mathbb{Z}/3\mathbb{Z}$ by

$$x_j := \frac{ik}{2} \frac{k}{k-c} T_k(u_j), \quad (4.16)$$

where $u_1, u_2, u_3 \in C^\infty(S^2)$ are the Cartesian coordinates of $S^2 \subset \mathbb{R}^3$. As the assumption $\limsup_{k \to +\infty} \dim H_k/k < 2$ implies in particular that $\dim H_k < 2(k + c)$ for all $k \in \mathbb{N}$, it follows that $c \in \mathbb{Z}$ and that $\dim H_k = k + c$ for all $k \in \mathbb{N}$ big enough, which proves the first statement (1.8).

Furthermore, Theorem 1.1 implies that there exist operators $X_1, X_2, X_3 \in \mathfrak{su}(H_k)$ generating an irreducible representation of $\mathfrak{su}(2)$ such that for all $1 \leq j \leq 3$,

$$\|x_j - X_j\|_{op} = O(1). \quad (4.17)$$

Recall that any two irreducible representations of $\mathfrak{su}(2)$ with same dimension are isomorphic, and consider the quantization $S_k : C^\infty(S^2) \to \mathcal{L}(H_k), k \in \mathbb{N}$ of Theorem 4.1. Then there exist unitary operators $U_k : H_k \to H_k$ for all $k \in \mathbb{N}$ such that the quantization $Q_k := U_k^{-1} S_{k+c-1} U_k : C^\infty(S^2) \to \mathcal{L}(H_k)$, $k \in \mathbb{N}$, satisfies

$$X_j = i \frac{(k + c - 1)}{2} Q_k(u_j),$$

for all $j \in \mathbb{Z}/3\mathbb{Z}$ and $k \in \mathbb{N}$. Observe that

$$\|Q_k(u_j)\|_{op} = 1 \ \forall j \in \mathbb{Z}/3\mathbb{Z}, \quad (4.18)$$

by the definition of $X_j$ and (2.2).

In order to establish (1.6), it suffices to show that for all $f \in C^\infty(S^2)$,

$$\|T_k(f) - Q_k(f)\|_{op} = O(1/k). \quad (4.19)$$

First, the estimate (4.19) holds for $f = u_j$ for all $j \in \mathbb{Z}/3\mathbb{Z}$ by construction. Indeed, by (4.17),

$$\left\| \frac{k^2}{2(k-c)} T_k(u_j) - \frac{k + c - 1}{2} Q_k(u_j) \right\|_{op} = O(1).$$
Dividing by $k^2/(2(k - c))$ and using that $\|Q_k(u_j)\|_{op} = 1$ by (4.18), we get the claim.

Next, assume by induction that there exist constants $C > 0$ and $M \in \mathbb{N}$, depending only on the quantizations, such that for any polynomial $P_n \in C^\infty(S^2)$ of degree $n \in \mathbb{N}$ in the coordinates functions $u_j \in C^\infty(S^2)$ for every $1 \leq j \leq 3$, we have

$$\|T_k(P_n) - Q_k(P_n)\|_{op} \leq \frac{nC}{k} \|P_n\|_{C^M}.$$  (4.20)

By axioms (P1) and (P3), for all $1 \leq j \leq 3$ we have

$$\|T_k(u_j P_n) - T_k(u_j) T_k(P_n)\|_{op} \leq \frac{C_0}{k} \|u_j\|_{C^N} \|P_n\|_{C^N},$$  (4.21)

and the same holds for $Q_k$, for constants $C_0 > 0$ and $N \in \mathbb{N}$ depending only on the quantizations. Combining (4.18) together with the induction hypothesis (4.20), and the sub-multiplicativity of the operator norm, we get that

$$\|T_k(u_j P_n) - Q_k(u_j P_n)\|_{op}$$

$$\leq \|T_k(u_j) T_k(P_n) - Q_k(u_j) Q_k(P_n)\|_{op} + \frac{C_0}{k} \|u_j\|_{C^N} \|P_n\|_{C^N}$$

$$\leq \|(T_k(u_j) - Q_k(u_j)) Q_k(P_n)\|_{op}$$

$$+ \|Q_k(u_j)(T_k(P_n) - Q_k(P_n))\|_{op} + \frac{C_0}{k} \|u_j\|_{C^N} \|P_n\|_{C^N}$$

$$\leq \frac{C_1C_0}{k} \|P_n\|_{C^M} + \frac{nC}{k} \|P_n\|_{C^M} + \frac{C_0}{k} \|u_j\|_{C^N} \|P_n\|_{C^N},$$  (4.22)

where the constant $C_1 > 0$ is the biggest constant appearing in the estimate (4.17), for all $1 \leq j \leq 3$. Using the fact $\|u_j\|_{C^N} \|P_n\|_{C^N} \leq \|u_j P_n\|_{C^{2N}}$, we can choose $C = C_0(1 + C_1)$ and $M = 2N$ in (4.21) to see that it holds with $n$ replaced by $n + 1$, thus for all $n \in \mathbb{N}$ by induction.

On the other hand, for any $n \in \mathbb{N}$, the $n$-th eigenfunction of the Laplace-Beltrami operator $\Delta$ acting on $C^\infty(S^2, \mathbb{R})$ associated with the round metric is the restriction on the sphere of a Legendre polynomial of degree $\leq n$. Take now any $f \in C^\infty(S^2)$, and take its spectral decomposition $f = \sum_{n \in \mathbb{N}} \phi_n$ into eigenfunctions of $\Delta$. Since $f$ is smooth, the norms $\|\phi_n\|_{C^N}$ decay faster than

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any power of \( n \), so that there exists \( C' > 0 \) such that
\[
\|T_k(f) - Q_k(f)\|_{op} \leq \frac{C}{k} \sum_{n \in \mathbb{N}} n \|\phi_n\|_{C^N} \leq \frac{C'}{k} .
\] (4.23)

This shows formula (4.19), hence formula (1.6).

Note that this proof shows in particular that the constant \( c \in \mathbb{R} \) appearing in Lemma 4.2 is an integer, uniquely determined by the condition \( \dim H_k = k + c \), for all \( k \in \mathbb{N} \) big enough. This fact can be refined for geometric quantizations \( T_k : C^\infty(S^2) \to \mathcal{L}(H_k) \), \( k \in \mathbb{N} \), satisfying the following additional axiom: there exists a function \( R \in C^\infty(S^2) \) such that for all \( f \in C^\infty(S^2) \), we have
\[
\text{tr} T_h(f) = \frac{k}{2\pi} \int_{S^2} f R_k \omega ,
\] (4.24)
where \( R_k = 1 + \frac{1}{k} R + \mathcal{O}(1/k^2) \), as \( k \to +\infty \). We then have the following result, relating this trace with the coefficient \( C^- \).

**Theorem 4.3.** Let \( T_k : C^\infty(S^2) \to \mathcal{L}(H_k) \), \( k \in \mathbb{N} \), be a geometric quantization with \( C^+_1 \equiv 0 \) and satisfying the trace axiom (4.24). Then we have
\[
C^-(f, g) = -\frac{i}{2} R \{ f, g \} .
\] (4.25)

**Proof.** Let \( T_k : C^\infty(S^2) \to \mathcal{L}(H_k) \), \( k \in \mathbb{N} \), be a geometric quantization satisfying the trace axiom (4.24) and with \( C^+_1 \equiv 0 \), and recall the form \( \alpha \in \Omega^2(S^2, \mathbb{R}) \) of formula (4.7). Let \( c \in \mathbb{R} \) and \( \theta \in \Omega^1(S^2, \mathbb{R}) \) be such that \( \alpha = c \omega + d\theta \), as in formula (4.13), and write
\[
d\theta =: R_\theta \omega ,
\] (4.26)
with \( R_\theta \in C^\infty(S^2) \). Considering the change of variable (4.8) induced by the derivation \( \delta : C^\infty(S^2) \to C^\infty(S^2) \) defined by \( \delta f := -\theta(\text{grad } f) \), we compute
\[
\int_{S^2} \delta f \omega = -\int_{S^2} f \ d\theta = \int_{S^2} R_\theta f \omega .
\] (4.27)

Then one readily computes that the induced quantization \( T_k^\delta : C^\infty(S^2) \to \mathcal{L}(H_k) \), \( k \in \mathbb{N} \), also satisfies the trace axiom (4.24), where that function...
$R \in C^\infty(S^2)$ is replaced by the function $R_\delta := R - R_\theta$. On the other hand, we know from the proof of Lemma 4.2 that $C^+_{1,\delta} = C^+_1 \equiv 0$ and $C^-_{2,\delta} = -c \{\cdot, \cdot\}$, and from the proof of Theorem 1.7 above that $c \in \mathbb{R}$ is an integer satisfying $\dim H_k = k + c$ for all $k \in \mathbb{N}$ big enough. Applying formula (4.24) to $f = 1$ and using that $T^\delta_k(1) = 1$, we get

$$\frac{1}{2\pi} \int_{S^2} R_\delta \omega = c.$$  \hspace{1cm} (4.28)

On the other hand, using the axioms (P2) and (P3), we get for any $f, g \in C^\infty(S^2)$ that as $k \to +\infty$,

$$i \left(1 - \frac{c}{k}\right) \text{tr} T^\delta_k(\{f, g\}) = k \text{tr} \left([T^\delta_k(f), T^\delta_k(g)] + O(1/k^3)\right)$$

$$= O(1/k).$$ \hspace{1cm} (4.29)

Now as every function with zero mean can be written as a sum of Poisson brackets (see e.g. [1, Theorem 1.4.3]), we get that

$$\int_{S^2} f \omega = 0 \implies \text{tr} T^\delta_k(f) = O(1/k) \text{ as } k \to +\infty.$$ \hspace{1cm} (4.30)

Using formula (4.24) again, we see that this is possible if and only $R_\delta$ is constant, equal to $c \in \mathbb{Z}$ by formula (4.28). We thus have $R = c - R_\theta$, by formulas (4.14) and (4.26), we get

$$C^-_2(f, g) = -c \{f, g\} + R_\theta \{f, g\} = -R \{f, g\}.$$ \hspace{1cm} (4.31)

This gives the result. \hfill \square

This result is of specific interest in the theory of deformation quantization of the Poisson algebra $(C^\infty(S^2), \{\cdot, \cdot\})$. To see this, consider the following extension of axiom (P3), for all $m \in \mathbb{N}$ and $f, g \in C^\infty(S^2)$,

$$T_k(f)T_k(g) = T_k \left(fg + \sum_{j=1}^{m-1} \frac{1}{k^j} C_j(f, g)\right) + O(1/k^m),$$ \hspace{1cm} (4.32)

as $k \to +\infty$, for a collection of bi-differential operators $C_j$ for all $j \in \mathbb{N}$. Together with the other axioms of Definition 1.4, this induces a differential star product $*$ on the ring of formal power series $C^\infty(S^2, \mathbb{C})[[\hbar]]$, with formal
parameter $\hbar$. Specifically, such a quantization defines an associative $\mathbb{C}[\hbar]$-linear product $\ast$ on $C^\infty(S^2, \mathbb{C})[[\hbar]]$, which is unital for $1 \in C^\infty(S^2)$ and satisfies $f \ast g - g \ast f = i\hbar \{f, g\} + O(\hbar^2)$, by the following formula for all $f, g \in C^\infty(S^2)$,

$$f \ast g = fg + \sum_{j=1}^{\infty} \hbar^j C_j(f, g).$$ (4.33)

Setting $\hbar = 1/k$, we see that (4.32) reads as a star product axiom

$$T_k(f)T_k(g) = T_k(f \ast g),$$ (4.34)

where this equality is understood as an asymptotic expansion with respect to the operator norm.

Working with formal power series in $\hbar$, one can extend the notion (4.8) of a change of variable over any subset $U \subset S^2$ as a map $A : C^\infty(U, \mathbb{C})[[\hbar]] \to C^\infty(U, \mathbb{C})[[\hbar]]$ satisfying $A(1) = 1$ and

$$A(f) := f + \sum_{j=1}^{+\infty} \hbar^j D_j f,$$ (4.35)

for all compactly supported $f \in C^\infty(U)$, where $D_j$ are differential operators for all $j \in \mathbb{N}$. This acts on a star product $\ast$ via the formula

$$f \ast_A g := A^{-1}(A(f) \ast A(g)),$$ (4.36)

where $\ast_A$ is defined on compactly supported functions $f, g \in C^\infty(U, \mathbb{C})$. In the theory of deformation quantization, this is also called a star-equivalence. For change of variables of the form $A(f) = f + \hbar D f$ for any $f \in C^\infty(S^2)$, one readily checks that $\ast_A$ is the star product (4.34) associated to the geometric quantization $T_k^D : k \in \mathbb{N}$, of (4.8).

Following [21 §1, p.229] (see also [17 §2, p.220]), one can define the canonical trace of a differential star product $\ast$ as the map $\text{tr}_h : C^\infty(S^2)[[\hbar]] \to \mathbb{C}[[\hbar]]$ such that for any $f \in C^\infty(S^2)$ supported over a contractible Darboux chart $U \subset S^2$, we have

$$\text{tr}_h(f) = \frac{1}{2\pi\hbar} \int_X A_U(f) \omega,$$ (4.37)

where $A_U : C^\infty(U)[[\hbar]] \to C^\infty(U)[[\hbar]]$ is a change of variable making $\ast$ equal to the usual Moyal-Weyl star product over $\mathbb{R}^{2n}$ in these Darboux charts. We
will not need the full definition of the Moyal-Weyl star product, but only that it satisfies $C_1^+ = C_2^- = 0$. The following result is then a consequence of Theorem 4.3.

**Corollary 4.4.** Let $T_k : C^\infty(S^2) \to \mathcal{L}(H_k)$, $k \in \mathbb{N}$, be a geometric quantization satisfying the trace axiom (4.24) and the star product axiom (4.34). Then for all $f \in C^\infty(S^2)$, we have the following asymptotic expansion as $k \to +\infty$,

$$\text{tr} T_k(f) = \text{tr}_{1/k}(f) + O(1/k).$$

**Proof.** Take $f \in C^\infty(S^2)$ to be compactly supported in a Darboux chart $U \subset S^2$, and let $A_U : C^\infty(U)[[\hbar]] \to C^\infty(U)[[\hbar]]$ be a local change of variable making the induced star product (4.33) equal to the Moyal-Weyl star product. Let us write

$$A_U(f) = f + \hbar D_U f + O(\hbar^2),$$

(4.38)

and write $\tilde{C}_1$ and $\tilde{C}_2$ for the bi-differential operators of (4.33) associated with the star product $\ast_{A_U}$ over $U$. Note that terms of order $\hbar^2$ and more do not affect $\tilde{C}_1^+$ and $\tilde{C}_2^-$, and by formula (4.11), the condition $\tilde{C}_1^+ \equiv 0$ determines $D_U : C^\infty(U) \to C^\infty(U)$ up to a derivation. In particular, by the trace axiom (4.24), one sees that both the usual trace and the canonical trace change the same way under a change of variable of the form (4.8). By Lemma (4.2), it suffices to show the result for quantizations which already satisfy $C_1^+ \equiv 0$.

Let then $T_k : C^\infty(S^2) \to \mathcal{L}(H_k)$, $k \in \mathbb{N}$, be a geometric quantization with $C_1^+ \equiv 0$ and satisfying the trace axiom (4.24), so that we are under the hypotheses of Theorem 4.3. Then by formula (4.11), the condition $\tilde{C}_1^+ \equiv 0$ implies that $D_U : C^\infty(U) \to C^\infty(U)$ has to be a derivation in that case. Furthermore, formulas (4.11) and (4.14) show that in order to also have $\tilde{C}_2^- \equiv 0$, this derivation has to be of the form $D_U f := -\theta(\text{grad} f)$ for all compactly supported $f \in C^\infty(U)$, where $\theta \in \Omega^1(S^2, \mathbb{R})$ satisfies

$$C_2^-(f, g) = \frac{i}{2} d\theta(\text{grad} f, \text{grad} g),$$

(4.39)

for all compactly supported $f, g \in C^\infty(U)$. Note that this is compatible with formula (4.7), as all 2-forms over a contractible open set $U \subset S^2$ are exact. Then by definition (4.37) of the canonical trace, for all $f \in C^\infty(S^2)$ with
compact support in $U \subset X$, we then have

$$
\text{tr}_h(f) = \frac{1}{2\pi h} \int_X (f - h \theta(\text{grad} f)) \omega + O(h^2)
$$
\begin{equation}
= \frac{1}{2\pi h} \int_X (f + hf_R U) \omega + O(h^2),
\end{equation}

where $R_U \in \mathcal{C}^\infty(U)$ is defined by the formula

$$
d\theta =: R_U \omega|_U,
$$
\begin{equation}
(4.41)
\end{equation}

so that in particular, by formula (4.39), for all compactly supported $f, g \in \mathcal{C}^\infty(U)$, we have

$$
C_2^{-}(f, g) = -\frac{i}{2} R_U \{f, g\}.
$$
\begin{equation}
(4.42)
\end{equation}

By Theorem 4.3, the trace $\text{tr} T_k(f)$ is given by the last term in formula (4.40), and hence coincides with the canonical trace $\text{tr}_h(f)$ up to $O(1/k)$. This completes the proof of the corollary.

Corollary 4.4 naturally leads to the following Conjecture.

**Conjecture 4.5.** Let $T_k : \mathcal{C}^\infty(S^2) \to \mathcal{L}(H_k)$, $k \in \mathbb{N}$, be a geometric quantization of the sphere satisfying trace axiom (4.24) and the star product axiom (4.34). Then for all $f \in \mathcal{C}^\infty(S^2)$ and $m \in \mathbb{N}$, we have the following asymptotic expansion as $k \to +\infty$,

$$
\text{tr} T_k(f) = \text{tr}_{1/k}(f) + O(1/k^m).
$$

The trace axiom (4.24) is a basic property of Berezin-Toeplitz quantizations of closed Kähler manifolds, and the fact that these quantizations satisfy the expansion (4.32) has been shown by Schlichenmaier in [24]. Then Conjecture 4.5 for Berezin-Toeplitz quantizations of closed Kähler manifolds has been established by Hawkins in [13, Cor. 10.5].

**Remark 4.6.** As explained for instance in [12, §6], there exists a notion of characteristic class for differential star-products $*$ over symplectic manifolds, which has been introduced by Deligne in [7] as an element $c(*)$ of the affine space $h^{-1} [\omega] + H^2(M, \mathbb{R})[[h]]$. By the work of Fedosov [9] and Nest and Tsygan [21, 22], this class is known to classify star-products up to $star$-equivalence (4.36). Then we have the relation

$$
c(*) = h^{-1} [\omega] + c [\omega] + \mathcal{O}(\hbar),
$$
\begin{equation}
(4.43)
\end{equation}

33
where \( c \in \mathbb{R} \) is the constant produced from \( T_k : C^\infty(S^2) \to \mathcal{L}(H_k), \) \( k \in \mathbb{N} \) by Lemma 4.2. Then for geometric quantizations satisfying star product axiom (4.34), the proof of Theorem 1.7 computes this constant to be an integer via the formula \( \dim H_k = k + c \) for all \( k \in \mathbb{N} \). The Deligne-Fedosov class of Berezin-Toeplitz quantizations of closed Kähler manifolds has been computed by Hawkins in [13, Th. 10.6] and Karabegov and Schlichenmaier in [15].

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References

[1] Banyaga, A., The Structure of Classical Diffeomorphism Groups, Mathematics and its Applications, 400. Kluwer Academic Publishers Group, Dordrecht, 1997.

[2] Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., and Sternheimer, D., Quantum mechanics as a deformation of classical mechanics, Lett. Math. Phys. 1 (1977), no. 6, 521–530.

[3] Bordemann, M., Meinrenken, E., and Schlichenmaier, M., Toeplitz quantization of Kähler manifolds and \( gl(N), N \to \infty \) limits, Comm. Math. Phys. 165 (1994), 281–296.

[4] Boutet de Monvel, L., and Guillemin, V., The spectral theory of Toeplitz operators, Annals of Mathematics Studies, vol. 99, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1981.

[5] Charles, L., Semi-classical properties of geometric quantization with metaplectic correction, Comm. Math. Phys. 270 (2007), no. 2, 445–480.

[6] Charles, L., Subprincipal symbol for Toeplitz operators. Lett. Math. Phys. 106 (2016), no. 12, 1673-1694.

[7] Deligne, P., Déformations de l’Algèbre des Fonctions d’une Variété Symplectique: Comparaison entre Fedosov et De Wilde, Lecomte, Sel. Math. New Ser., 1 (1995), no. 4, 667–697.
[8] De Chiffre, M., Glebsky, L., Lubotzky, A., Thom, A., Stability, cohomology vanishing, and nonapproximable groups, Forum of Mathematics, Sigma 8 (2020), e18.

[9] Fedosov, B. V., A simple geometrical construction of deformation quantization, J. Differential Geom. 40 (1994), no. 2, 213–238.

[10] Grove, K., Karcher, H. and Ruh, E.A., Jacobi fields and Finsler metrics on compact Lie groups with an application to differentiable pinching problems, Math. Ann., 211 (1974), 7–21.

[11] Guillemin, V., Star products on compact pre-quantizable symplectic manifolds, Lett. Math. Phys. 35 (1995), 85–89.

[12] Gutt, S., and Rawnsley, J., Equivalence of star products on a symplectic manifold: An introduction to Deligne’s Čech cohomology classes, J. Geom. Phys. 29 (1999), 347–392.

[13] Hawkins, E., Geometric quantization of vector bundles and the correspondence with deformation quantization, Comm. Math. Phys. 215 (2000), no. 2, 409–432.

[14] Jacobson, N., Lie Algebras, Republication of the 1962 original. Dover Publications, Inc., New York, 1979.

[15] Karabegov, A. V., and Schlichenmaier, M., Identification of Berezin-Toeplitz deformation quantization, J. Reine Angew. Math. 540 (2001), 49–76.

[16] Kazhdan, D., On $\epsilon$-representations, Israel J. Math. 43 (1982), 315 –323.

[17] Karabegov, A. V., On the canonical normalization of a trace density of deformation quantization, Lett. Math. Phys. 45 (1998), 217–228.

[18] Le Floch, Y., A Brief Introduction to Berezin-Toeplitz Operators on Compact Kähler Manifolds, Springer, 2018.

[19] Ma, X., and Marinescu, G., Toeplitz operators on symplectic manifolds, J. Geom. Anal. 18 (2008), 565–611.

[20] Ma, X., and Marinescu, G., Berezin-Toeplitz quantization on Kähler manifolds, J. Reine Angew. Math. 662 (2012), 1–56.
[21] Nest, R., and Tsygan, B., *Algebraic index theorem*, Comm. Math. Phys. 172 (1995), no. 2, 223-262.

[22] Nest, R., and Tsygan, B., *Algebraic index theorem in families*, Adv. Math. 113 (1995), no. 2, 151–205.

[23] Schlichenmaier, M., *Berezin-Toeplitz quantization for compact Kähler manifolds. A review of results*, Adv. Math. Phys., 2010, 927280.

[24] Schlichenmaier, M., *Deformation quantization of compact Kähler manifolds by Berezin-Toeplitz quantization*, Conférence Moshé Flato 1999, Vol. II (Dijon), Math. Phys. Stud., vol. 22, Kluwer Acad. Publ., Dordrecht, 2000, pp. 289–306.

[25] Tuynman, G. M., *Quantization: towards a comparison between methods*, J. Math. Phys. 28 (1987), no. 12, 2829–2840.

[26] Woodhouse, N. M. J., *Geometric quantization*, second ed., Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1992, Oxford Science Publications.

[27] Xu, H., *An explicit formula for the Berezin star product*, Lett. Math. Phys. 101 (2012), 239 – 264.

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