A Functional Central Limit Theorem for the Becker–Döring Model

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Abstract We investigate the fluctuations of the stochastic Becker–Döring model of polymerization when the initial size of the system converges to infinity. A functional central limit problem is proved for the vector of the number of polymers of a given size. It is shown that the stochastic process associated to fluctuations is converging to the strong solution of an infinite dimensional stochastic differential equation (SDE) in a Hilbert space. We also prove that, at equilibrium, the solution of this SDE is a Gaussian process. The proofs are based on a specific representation of the evolution equations, the introduction of a convenient Hilbert space and several technical estimates to control the fluctuations, especially of the first coordinate which interacts with all components of the infinite dimensional vector representing the state of the process.

Keywords Becker–Döring equations · Functional central limit theorem · SDEs in Hilbert space

1 Introduction

Polymerization is a key phenomenon in several important biological processes. Macromolecules proteins, also called monomers, may be assembled randomly into several aggregated states called polymers or clusters. These clusters can themselves be fragmented into monomers and polymers at some random instants. The fluctuations of the number of polymerized monomers analyzed in this paper is an important characteristic of polymerization processes in general.
1.1 The Becker–Döring Model

We investigate the fluctuations of a stochastic version of the Becker–Döring model which is a classical mathematical model to study polymerization. The Becker–Döring model describes the time evolution of the distribution of cluster sizes in a system where only additions (coagulation mechanism) or removals (fragmentation) of one monomer from a cluster are possible. A cluster of size 1 is a monomer and clusters of size greater than 2 are polymers. Under Becker–Döring model, coagulation and fragmentation are simple synthesis and decomposition reactions: a polymer of size $k$ may react with a monomer to form a polymer of size $k+1$ at kinetic rate $a_k$; a polymer of size $k+1$ may break down into a polymer of size $k$ and a monomer at kinetic rate $b_{k+1}$, i.e.,

$$\frac{1}{(k+1)} \xrightarrow{a_k} (k+1).$$

The ODEs associated to the deterministic version of the Becker–Döring model have been widely studied in physics since 1935, see Becker and Döring [3]. This is an infinite system of ordinary differential equations of $c(t) = (c_k(t), k \in \mathbb{N}^+)$, given by

$$\begin{cases}
    \frac{dc_1}{dt}(t) = -2J_1(c(t)) - \sum_{k \geq 2} J_k(c(t)), \\
    \frac{dc_k}{dt}(t) = J_{k-1}(c(t)) - J_k(c(t)), \quad k \geq 1,
\end{cases}$$

(BD)

with $J_k(c) = a_k c_1 c_k - b_{k+1} c_{k+1}$ if $c = (c_k) \in \mathbb{R}_{+}^{\mathbb{N}^+}$. For $k \geq 1$, $c_k(t)$ represents the concentration of clusters of size $k$ at time $t$. The conditions on existence/uniqueness of solutions for the Becker–Döring equations (BD) have been extensively investigated. See Ball et al. [2], Niethammer [22] and Penrose [24,25].

The evolution equations satisfied by $(c(t))$ can be rewritten under a more compact form as,

$$dc(t) = \tau \circ s(c(t)) \, dt,$$

where $s$ is a mapping from $\mathbb{R}_{+}^{\mathbb{N}^+} \to \mathbb{R}_{+}^{\mathbb{N}^+}$: for any $k \in \mathbb{N}^+$ and $c \in \mathbb{R}_{+}^{\mathbb{N}^+}$

$$s_{2k-1}(c) = a_k c_1 c_k \quad \text{and} \quad s_{2k}(c) = b_{k+1} c_{k+1};$$

and $\tau$ is a linear mapping from $\mathbb{R}_{+}^{\mathbb{N}^+} \to \mathbb{R}_{+}^{\mathbb{N}^+}$: for any $z \in \mathbb{R}_{+}^{\mathbb{N}^+}$ and $k \geq 2$,

$$\begin{cases}
    \tau_1(z) = - \sum_{i \geq 1} (1 + 1_{i=1}) z_{2i-1} + \sum_{i \geq 2} (1 + 1_{i=2}) z_{2i-2}, \\
    \tau_k(z) = z_{2k-3} - z_{2k-2} - z_{2k-1} + z_{2k}.
\end{cases}$$

As it will be seen this representation will turn out to be very useful to derive the main results concerning fluctuations.

1.2 Becker–Döring ODEs and Polymerization Processes

This set of ODEs is used to describe the evolution of the concentration $c_i(t)$, $i \geq 1$, of polymers of size $i$. The classical framework assumes an initial state with only polymers of size 1, monomers. In a biological context, experiments show that the concentration of polymers of size greater than 2 stays at 0 until some instant, defined as the lag time, when the polymerized mass grows very quickly to reach its stationary value. With convenient
parameters estimations, these ODEs can be used to describe first order characteristics such as mean concentration of polymers of a given size. The use of systems of ODEs to describe the evolution of polymerization processes started with Oosawa’s pioneering work in 1962, see Oosawa and Asakura [23] for example. Morris et al. [21] presents a quite detailed review of the classical sets of ODEs used for polymerization processes. As it can be seen and also expected, the basic dynamics of the Becker–Döring model of adding/removing a monomer to/from a polymer occupy a central role in most of these mathematical models. See also Prigent et al. [26] and Hingant and Yvinec [15] for recent developments in this domain.

Outside the rapid growth of polymerized mass at the lag time, the other important aspect observed in the experiments is the high variability of the instant when it occurs, the lag time, from an experiment to another. This is believed to explain, partially, the variability of the starting point of diseases associated to these phenomena in neural cells, like Alzheimer’s disease for example. See Xue et al. [32]. Hence if the deterministic Becker–Döring ODEs describes the first order of polymerization through a limiting curve, the fluctuations around these solutions will give a characterization of the variability of the processes itself. Up to now the mathematical studies of these fluctuations are quite scarce, the stochastic models analyzed include generally only a finite number of possible sizes for the polymers. See Szavits et al. [30], Xue et al. [32] and Eugène et al. [12] for example. To study these important aspects, we have to introduce a stochastic version of the Becker–Döring model.

1.3 The Stochastic Becker–Döring Model

The polymerization process is described as a Markov process \((X^N(t)) := (X^N_k(t), 1 \leq k \leq N)\), where \(X^N_k(t)\) is the total number of clusters of size \(k\) at time \(t\), it takes values in the state space,

\[
S^N := \left\{ x \in \mathbb{N}^N \mid \sum_{k=1}^N k x_k = N \right\},
\]

of configurations of polymers with mass \(N\).

For any \(x \in S^N\) and \(1 \leq k < N\), the associated jump matrix \(Q^N = (q^N(\cdot, \cdot))\) is given by

\[
\begin{align*}
q^N(x, x-e_1-e_k+e_{k+1}) &= a_k x_1 (x_k - 1_{[k=1]}) / N, \\
q^N(x, x+e_1+e_k-e_{k+1}) &= b_{k+1} x_{k+1},
\end{align*}
\]

where \((e_k, k \in \mathbb{N}^+)\) is the standard orthonormal basis of \(\mathbb{N}^{N^+}\). In other words, a monomer is added to a given polymer of size \(k\) at rate \(a_k x_1 / N\) and a monomer is detached from a polymer of size \(k\) at rate \(b_k\). Note that if \(N\) is interpreted as a “volume”, the quantity \(X^N_1(t) / N\) can be seen as the concentration of monomers.

There are few studies of this important stochastic process. The large scale behavior of the stochastic Becker–Döring model, when \(N\) gets large, is an interesting and challenging problem. Given the transition rates, one can expect that the deterministic Becker–Döring equations (BD) give the limiting equations for the concentration of the different species of polymers, i.e. for the convergence in distribution

\[
\lim_{N \to +\infty} \left( X^N_k(t) / N, k \geq 1 \right) = (c_k(t), k \geq 1).
\]

Such a first order analysis is achieved in Jeon [17]. This result is in fact proved for a more general model, the Smoluchowski coagulation–fragmentation model. The Becker–Döring
model is a special case of Smoluchowski model, see Aldous [1] for a survey on the coalescence models. The stochastic approximation of the pure Smoluchowski coalescence equation through Marcus–Lushnikov process is investigated in Norris [16]. There are also results on the relation of stochastic Becker–Döring model and the deterministic Lifshitz–Slyozov equation, see Deschamps et al. [9] for example.

1.4 The Main Contributions

In this paper, we investigate the fluctuations of the Becker–Döring model, i.e. the limiting behavior of the $\mathbb{R}^N$-valued process

$$W^N(t) := \left( \frac{1}{\sqrt{N}} \left( X^N(t) - Nc(t) \right) \right)$$

is analyzed. We prove that, under appropriate conditions, the fluctuation process $(W^N(t))$ converges for the Skorohod topology to a $L^2(w)$-valued process $(W(t))$ which is the strong solution of the SDE

$$dW(t) = \tau \left( \nabla s(c(t)) \cdot W(t) \right) dt + \tau \left( \text{Diag} \left( \sqrt{s(c(t))} \right) \cdot d\beta(t) \right),$$

where

1. $L^2(w)$ is a Hilbert subspace of $\mathbb{R}^{N^+}$, see Definition 2 of Sect. 3;
2. The operators $s(\cdot)$ and $\tau(\cdot)$ are defined respectively by Relations (2) and (3);
3. $\nabla s(c)$ is the Jacobian matrix $\nabla s(c) = \left( \frac{\partial s_i}{\partial c_j}(c) \right), c \in \mathbb{R}^{N^+}$;
4. $\beta(t) = (\beta_k(t), k \in \mathbb{N}^+)$ is a sequence of independent standard Brownian motions in $\mathbb{R}$;
5. $\text{Diag}(v)$ is a diagonal matrix whose diagonal entries are the coordinates of the vector $v \in \mathbb{R}^{N^+}$.

See Theorem 3 in Sect. 3 for a precise formulation of this result. Note that the drift part of (5) is the gradient of the Becker–Döring equation (1).

Under appropriate assumptions, see Ball et al. [2], the Becker–Döring equations (1) have a unique fixed point $\tilde{c}$. If $c(0) = \tilde{c}$, the asymptotic fluctuations around this stationary state are then given by $(\tilde{W}(t))$. It is shown in Proposition 2 that this process, the solution of SDE (5), can be represented as

$$\tilde{W}(t) = T(t)\tilde{W}(0) + \int_0^t T(t-s)\tau \left( dB(s) \right),$$

where $T$ is the semi-group associated with linear operator $\tau \circ \nabla s(c)$ and $(B(t))$ is a $\tilde{Q}$-Wiener process in $L_2(w)$ where $\tilde{Q} = \text{Diag} (w_n s_n(c), n \geq 1)$. In particular if the initial state $\tilde{W}(0)$ is deterministic, then $(\tilde{W}(t))$ is a Gaussian process. For the definitions and properties of $\tilde{Q}$-Wiener process and Gaussian processes in Hilbert spaces, see Sections 3.6 and 4.1 in Da Prato and Zabczyk [8] for example.

1.5 Literature

For the fluctuation problems in the models with finite chemical reactions, results are well known, see Kurtz [18,19] for example. However, in the Becker–Döring model, there are...
countable many species and reactions, where the results for the finite reactions are not directly applicable. See the Open Problem 9 in Aldous [1] for the description of fluctuation problem in the general Smoluchowski coagulation model. For this reason, the studies of fluctuations of the Becker–Döring models are quite scarce. Ranjbar and Rezakhanlou [27] investigated the fluctuations at equilibrium of a family of coagulation–fragmentation models with a spatial component. They proved that, at equilibrium, the limiting fluctuations can be described by an Ornstein–Uhlenbeck process in some abstract space. Their results rely on balance equations which hold because of the stationary framework. Durrett et al. [10] gave the stationary distributions for all reversible coagulation–fragmentation processes. Then they provided the limits of the mean values, variances and covariances of the stationary densities of particles of any given sizes when the total mass tends to infinity.

Concerning fluctuations of infinite dimensional Markov processes, several examples have received some attention, in statistical physics mainly. In the classical Vlasov model, the first result on the central limit theorem seems to be given by Braun and Hepp [5] in 1977. They investigated the fluctuations around the trajectory of a test particle. The central limit theorem for the general McKean–Vlasov model when the initial measures of the system are products of i.i.d. measures is proved by Sznitman [31] in 1984. For the Ginzburg–Landau model on \( \mathbb{Z} \), where the evolution of the state at a site \( i \) depends only on the state of its nearest neighbors \( i \pm 1 \). The independent (Brownian) stochastic fluctuations at each site do not depend of the state of the process. The hydrodynamic limit, i.e. a first order limit, is given by Guo et al. [14]. The fluctuations around this hydrodynamic limit live in an infinite dimensional space. Zhu [34] proved that, at equilibrium, the limiting fluctuations converge to a stationary Ornstein–Uhlenbeck process taking values in the dual space of a nuclear space. In the non-equilibrium case, the fluctuations have been investigated by Chang and Yau [7], the limiting fluctuations can be described as an Ornstein–Uhlenbeck in a negative Sobolev space. For more results on related fluctuation problems in statistical mechanics, see Spohn [28,29]. Fluctuations of an infinite dimensional Markov process associated to load balancing mechanisms in large stochastic networks have been investigated in Graham [13], Budhiraja and Friedlander [6].

One of the difficulties of our model is the fact that the first coordinate \( (X_1^N(t)) \) of our Markov process, the number of monomers, interact with all non-null coordinates of the state process. Recall that monomers can react with all other kinds of polymers. This feature has implications on the choice of the Hilbert space \( L^2(\omega) \) chosen to formulate the SDE (10) and, additionally, several estimates have to be derived to control the stochastic fluctuations of the first coordinate. This situation is different from the examples of the Ginzburg–Landau model in [7,28,34], since each site only have interactions with a finite number of sites, or in the stochastic network example [6] where the interaction range is also finite. It should be also noted that our evolution equations are driven by a set of independent Poisson processes whose intensity is state dependent which is not the case in the Ginzburg–Landau models for which the diffusion coefficients are constant. For this reason Lipschitz properties have to be established for an appropriate norm for several functionals, it complicates the already quite technical framework of these problems.

### 1.6 Outline of the Paper

Section 2 introduces the notations, assumptions and the stochastic model as well as the evolution equations. Section 3 investigates the problem of existence and uniqueness of the solution of the SDE (5). The proof of the fluctuations at equilibrium is also given. Section 4 gives the proof of the main result, Theorem 3, the convergence of the fluctuation processes.
2 The Stochastic Model

In this section we introduce the notations and assumptions used throughout this paper. The stochastic differential equations describing the evolution of the model are introduced.

For any $h \in \mathbb{R}^+$, $(N^i_h)_{i=1}^{\infty}$ denotes a sequence of independent Poisson processes with intensity $h$. The stochastic Becker–Döring equation with aggregation rates $(a_k, k \in \mathbb{N}^+)$ and fragmentation rates $(b_k, k \in \mathbb{N}^+)$ can be expressed as the solution of the SDEs

$$dX^N_1(t) = -\sum_{k \geq 1} (1 + \mathbb{I}_{[k=1]}) \sum_{i=1} \mathcal{N}_{ak/N}^i(dt) + \sum_{k \geq 2} (1 + \mathbb{I}_{[k=2]}) \sum_{i=1} \mathcal{N}_{bk}^i(dt),$$

$$dX^N_k(t) = \sum_{i=1} \mathcal{N}_{ak-1/N}^i(dt) - \sum_{i=1} \mathcal{N}_{bk}^i(dt),$$

and for all $k > N$, $X^N_k(t) \equiv 0$, with $f(t-)$ being the limit on the left of the function $f$ at $t>0$.

In order to separate the drift part and the martingale part, it is convenient to introduce the corresponding martingales $(D^N(t))$. For $k \geq 1$, let

$$D^N_{2k-1}(t) = \int_0^t \left( X^N_k(u-) X^N_{k+1}(u-) - \mathbb{I}_{[k=1]} \right) \mathcal{N}_{ak/N}^i(dt) - \frac{1}{N} a_k X^N_1(u) (X^N_k(u) - \mathbb{I}_{[k=1]}) du,$$

$$D^N_{2k}(t) = \int_0^t \left( \sum_{i=1} \mathcal{N}_{bk+1}^i(dt) - b_{k+1} X^N_{k+1}(u) du \right).$$

Clearly, for every fixed $i \in \mathbb{N}^+$, $(D^N_i(t))$ is local martingale in $\mathbb{R}$ with previsible increasing process

$$(D^N_i(t)) = \left( N \int_0^t s_i \left( \frac{X^N(u)}{N} \right) du - \mathbb{I}_{[i=1]} \int_0^t a_1 \frac{X^N_1(u)}{N} du \right),$$

where the operator $s = (s_i, i \geq 1)$ is defined by Relation (2), additionally the cross-variation processes are null, i.e., for any $i \neq j$,

$$\langle D^N_i, D^N_j \rangle(t) = 0.$$

This is in fact one of the motivations to introduce the variables $(D^N_k(t))$ and the functionals $(\tau(\cdot))$ and $(s(\cdot))$.

A simple calculation shows that, for any $N$, the process $(X^N(t))$ satisfies the relation

$$X^N(t) = X^N(0) + N \int_0^t \tau \left( s \left( \frac{X^N(u)}{N} \right) \right) du$$

$$+ \tau(D^N(t)) + 2a_1 \int_0^t \left( \frac{X^N(u)}{N} \right) e_1 du,$$

where $\tau$ is the stopping time.
where $\tau$ is defined by Relation (3) and $e_1 = (1, 0, 0, \ldots)$. Therefore the fluctuation process $(W^N(t))$, see Relation (4), satisfies

$$W^N(t) = W^N(0) + \frac{1}{\sqrt{N}} \tau \left( D^N(t) \right) + \frac{2a_1}{\sqrt{N}} \int_0^t \left( \frac{X_1^N(u)}{N} \right) e_1 \, du$$

$$+ \frac{1}{2} \int_0^t \tau \left( \nabla s \left( \frac{X^N(u)}{N} \right) \cdot W^N(u) + \nabla s(c(u)) \cdot W^N(u) \right) \, du.$$  

(8)

If we let $N$ go to infinity, then $(X^N(t)/N)$ converges to $c(t)$ and provides that, for all $i \geq 1$,

$$\lim_{N \to +\infty} \left( \frac{D^N_i}{\sqrt{N}}(t) \right) = \int_0^t s_i(c(u)) \, du.$$  

We can expect that the process $(D^N(t)/\sqrt{N})$ converges to a stochastic integral

$$\int_0^t \text{Diag} \left( \sqrt{s(c(u))} \right) \cdot d\beta(u),$$

with $(\beta(t))$ defined in Relation (5) in the introduction. See page 339 in Ethier and Kurtz’s [11] for example. Then, formally, the limiting process for $(W^N(t))$, provided that it exists, would be the solution of the SDE (5).

In order to give the well-posedness of the infinite dimensional process (5), we introduce the following notations.

**Definition 1** Let $\mathcal{X}_+$ be the phase space of the Becker–Döring model with initial density less than 1,

$$\mathcal{X}_+ := \left\{ c = (c_k, k \in \mathbb{N}^+) \mid \forall k, \ c_k \geq 0; \ \sum_{k \geq 1} kc_k \leq 1 \right\}.$$  

For $c(0) \in \mathcal{X}_+$, the solution of the Becker–Döring equation (BD) is a continuous function taking values in $\mathcal{X}_+$.

**Definition 2** One assumes that $w = (w_n)$ is a fixed non-decreasing sequence of positive real numbers such that

- $\|1/w\|_l := \sum_{n \geq 1} 1/w_n < \infty$;
- $\lim_{n \to \infty} w_n^{1/n} \leq 1$;
- There exists a constant $\gamma_0$ such that for all $n \in \mathbb{N}^+$, $w_{2n} \leq \gamma_0 w_n$.

One denotes by $L_2(w)$ the associated $L_2$-space,

$$L_2(w) := \left\{ z \in \mathbb{R}^{\mathbb{N}^+} \mid \sum_{n=1}^{\infty} w_n z_n^2 < \infty \right\},$$

its inner product is defined by, for $z, z' \in L_2(w)$

$$\langle z, z' \rangle_{L_2(w)} = \sum_{n=1}^{\infty} w_n z_n z'_n,$$

and its associated norm is $\|z\|_{L_2(w)} = \sqrt{\langle z, z \rangle_{L_2(w)}}$.

An orthonormal basis $(h_n)_{n=1}^{\infty}$ of $L_2(w)$ is defined as, for $n \geq 1$,

$$h_n = (0, \ldots, 1/\sqrt{w_n}, 0, \ldots).$$  

(9)
As it will be seen in Sect. 3, $L_2(w)$ is a convenient space to ensure a boundedness property of the linear mapping $\tau \circ \nabla s(\cdot)$, which is essential for the study of fluctuation process (5).

We now turn to the conditions on the rates and the initial state of the Becker–Döring equations.

**Assumptions**

(a) The kinetic rates $(a_k), (b_k)$ are positive and bounded, i.e.,

$$\Lambda := \max_{k \geq 1} \{a_k, b_k\} < \infty.$$

(b) There exists a positive increasing sequence $r \in \mathbb{R}^+_{\infty}$, such that

- $r_k \geq w_k$, for all $k \geq 1$;
- There exists a fixed constant $\gamma_r$, such that $r_{2k} \leq \gamma_r r_k$, for all $k \geq 1$;
- $(w_k / r_k)$ is a decreasing sequence converging to 0,

where $(w_n)$ is the sequence introduced in Definition 2.

The initial state of the process $(X_k^N(0))$ is assumed to satisfy

$$\sup_N \mathbb{E} \left( \sum_{k \geq 1} r_k X_k^N(0) \right) < \infty.$$

(c) The initial state of the first order process, $c(0) = (c_k(0)) \in \mathcal{X}_+$, is such that

$$\sum_{k \geq 1} w_k c_k(0) < \infty \quad \text{and} \quad \lim_{N \to \infty} \mathbb{E} \left( \sum_{k \geq 1} \frac{X_k^N(0)}{N} - c_k(0) \right) = 0$$

hold.

(d) The initial state of the centered process, the random variables $W^N(0), N \geq 1$, defined by Relation (4) satisfies the relation

$$C_0 := \sup_N \mathbb{E} \left( \|W^N(0)\|_{L_2(w)}^2 \right) < +\infty,$$

and the sequence $(W^N(0))$ converges in probability to an $L_2(w)$-valued random variable $W(0)$.

For a detailed discussion of Assumption (a), see “Remark on the Assumptions on the Coefficients $(a_i)$ and $(b_i)$” section. Assumption (b) gives conditions on the moments of the initial state of the process. For example, by taking $r_k = k^\beta$ with $\beta >$, one can study the central limit problem in the Hilbert space $L_2(w)$ with weights $w_k = k^\alpha$ for some $\alpha \in (1, \beta)$. Note that this condition is more, but not much more, demanding than the conservation of mass relation

$$\sum_{k \geq 1} k X_k^N(t)/N \equiv 1,$$

which is always satisfied. This assumption gives therefore a quite large class of initial distributions for which a central limit theorem holds.

**Some Notations**

For any $T > 0$, let $D_T := D([0, T], L_2(w))$ be the space of càdlàg functions on $[0, T]$ taking values in $L_2(w)$. Since $L_2(w)$ is a separable and complete space, there exists a metric on $D_T$, such that $D_T$ is a separable and complete space. See the Chapter 3 in Billingsley [4] for details.
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Let $\mathcal{L}(L_2(w))$ be the set of linear operators on $L_2(w)$ and the $\| \cdot \|_{\mathcal{L}(L_2(w))}$ be the associated norm for linear operators, i.e. for any $f \in \mathcal{L}(L_2(w))$,

$$
\| f \|_{\mathcal{L}(L_2(w))} = \sup_{z \in L_2(w), \| z \|_{L_2(w)} \leq 1} \| f(z) \|_{L_2(w)}.
$$

For any $f, g \in \mathcal{L}(L_2(w))$, $f \circ g$ denotes the composition. We call $A \in \mathcal{L}(L_2(w))$ to be in a trace class if there exist a constant $C$, such that for all $z \in L_2(w)$,

$$
\| Az \|_{L_2(w)} \leq C \| z \|_{L_2(w)}
$$

and it has a finite trace, i.e.,

$$
\text{Tr} A := \sum_{n \geq 1} \langle Ah_n, h_n \rangle_{L_2(w)} < \infty.
$$

3 SDE in Hilbert Space

In this section, we are going study the existence and uniqueness of process

$$
W(t) = W(0) + \int_0^t \tau \left( \nabla s(c(u)) \cdot W(u) \right) du + \int_0^t \tau \left( \text{Diag} \left( \sqrt{s(c(u))} \right) \cdot d\beta(u) \right)
$$

when Assumptions (a) and (c) hold.

Clearly, the process $(W(t))$ has the form of a stochastic differential equation in the infinite dimensional space $\mathbb{R}^{N^+}$. It has a stochastic integral

$$
\mathcal{M}(t) := \int_0^t \tau \left( \text{Diag} \left( \sqrt{s(c(u))} \right) \cdot d\beta(u) \right)
$$

with respect to $\beta(t) = \sum_{n \geq 1} \beta_n(t)e_n$, which is a cylindrical Wiener process (c.f. Yor [33]) in the Hilbert space $L^2(\mathbb{R}^{N^+})$. Here $(c(t))$ is the unique solution of Becker–Döring equation (1) with initial state $c(0)$.

However, the mapping $\tau$ is unbounded in $L^2(\mathbb{R}^{N^+})$. Therefore, we consider the SDE (10) in the Hilbert space $L_2(w)$ (Definition 2). In the following, we are going to show that

1. For any $c \in \mathcal{X}_+$, the linear mapping $\tau \circ \nabla s(c) : L_2(w) \rightarrow L_2(w)$ is bounded (and therefore Lipschitz);
2. The stochastic process $(\mathcal{M}(t))$ is well-defined in $L_2(w)$ and is a martingale.

Finally, by using the results in Section 7.1 of Da Prato and Zabczyk [8], we give the proof of the existence and uniqueness of the solutions of the Eq. (10).

**Lemma 1** $\tau$ is a continuous linear mapping from $L_2(w)$ to $L_2(w)$: there exists a finite constant $\gamma_{\tau}(w)$, such that for any $z \in L_2(w)$,

$$
\| \tau(z) \|_{L_2(w)} \leq \gamma_{\tau}(w) \| z \|_{L_2(w)}.
$$

**Proof** One only needs to verify that $\tau$ is bounded. For any $z \in L_2(w)$,

$$
\| \tau(z) \|_{L_2(w)}^2 = R_1 + R_2,
$$

$\square$ Springer
where
\[
R_1 = w_1 \left( - \sum_{k \geq 1} (1 + \mathbb{1}_{k=1}) z_{2k-1} + \sum_{k \geq 2} (1 + \mathbb{1}_{k=2}) z_{2k-2} \right)^2,
\]
\[
R_2 = \sum_{k \geq 2} w_k \left( z_{2k-3} - z_{2k-2} - z_{2k-1} + z_{2k} \right)^2.
\]
By using Cauchy–Schwarz inequality, we have
\[
R_1 \leq 4 w_1 \sum_{k \geq 1} \frac{1}{w_k} \|z\|^2_{L_2(w)} = 4 w_1 \left\| \frac{1}{w} - \|z\|^2_{L_2(w)},
\right.
\]
and by using the increasing property of weights \(w\)
\[
R_2 \leq 4 \sum_{k \geq 2} w_k \left( z_{2k-3}^2 + z_{2k-2}^2 + z_{2k-1}^2 + z_{2k}^2 \right)
\]
\[
\leq 4 \sum_{k \geq 2} \left( w_{2k-3} z_{2k-3}^2 + w_{2k-2} z_{2k-2}^2 + w_{2k-1} z_{2k-1}^2 + w_{2k} z_{2k}^2 \right) + w_2 z_1^2
\]
\[
\leq (8 + w_2/w_1) \|z\|^2_{L_2(w)}.
\]
In conclusion, one has \(\|\tau(z)\|_{L_2(w)} \leq \gamma_t(w) \|z\|_{L_2(w)}\), for
\[
\gamma_t(w) = \left( 4 w_1 \left\| \frac{1}{w} \right\|_{L_1} + 8 + \frac{w_2}{w_1} \right)^{1/2}.
\]

Lemma 2
Under Assumption (a), for any \(c \in X_+\) that satisfies \(\sum_{k \geq 1} w_k c_k < \infty\), there exists a finite constant \(\gamma(c, w)\), such that for any \(x \in L_2(w)\),
\[
\|\nabla s(c) \cdot x\|_{L_2(w)} \leq \gamma(c, w) \|x\|_{L_2(w)}.
\]
Moreover, under Assumption (a) and (c), if \((c(t))\) is the solution of the Becker–Döring equation (1) with initial state \(c(0)\), then for any finite time \(T\), one has
\[
\sup_{u \leq T} \gamma(c(u), w) < + \infty.
\]
Proof
\[
\|\nabla s(c) \cdot x\|^2_{L_2(w)} = \sum_{k \geq 1} w_{2k-1} a_k^2 (x_1 c_k + c_1 x_k)^2 + \sum_{k \geq 2} w_{2k-2} b_k^2 x_k^2
\]
\[
\leq 2 \gamma_0 \Lambda^2 \left( \sum_{k \geq 1} w_k c_k^2 \right) x_1^2 + 2 \gamma_0 \Lambda^2 \sum_{k \geq 1} w_k x_k^2
\]
\[
+ \gamma_0 \Lambda^2 \sum_{k \geq 2} w_k x_k^2 \leq \gamma(c, w) \|x\|^2_{L_2(w)},
\]
where for any \(c \in X_+\)
\[
\gamma(c, w) = \gamma_0 \Lambda^2 \left( 3 + 2 \sum_{k \geq 1} w_k c_k \right).
\]
Theorem 2.2 in Ball et al. [3] gives that \( \sup_{u \leq T} \sum_{k} c_k(u) < \infty \) under Assumption (a) and (c).

\[ \Box \]

**Remark on the Assumptions on the Coefficients \((a_i)\) and \((b_i)\)**

We recall the main results for the existence and uniqueness of a solution of the deterministic Becker–Döring ODEs.

1. From Theorem 2.2 in Ball et al. [2], existence and uniqueness hold under the condition

\[
a_i \left( g_{i+1} - g_i \right) = O(g_i) \quad \text{and} \quad \sum_{i=1}^{+\infty} g_i c_i(0) < +\infty,
\]

for a positive increasing sequence \( g \) satisfying \( \min_i (g_{i+1} - g_i) \geq \delta > 0 \). For example, when \( g_i = i^2 \), i.e., \( a_i = O(i) \) and the second moment of the initial state is bounded, the Becker–Döring equation is well-posed.

2. Conditions from Theorem 2.1 in Laurençot and Mischler [20], there exists some constant \( K > 0 \) such that

\[
a_i - a_{i-1} < K \quad \text{and} \quad b_i - b_{i-1} < K,
\]

for any \( i \in \mathbb{N} \).

It should be noted that Laurençot and Mischler [20] have a stronger conditions on the coefficients but, contrary to Ball et al. [2], not any on the initial state.

The known conditions necessary to get the convergence of the first order of the stochastic Becker–Döring model to the solution of the deterministic Becker–Döring ODEs are more demanding, they are given by Theorem 2 in Jeon [17], they are

\[
\lim_{i \to \infty} \frac{a_i}{i} = 0 \quad \text{and} \quad \lim_{i \to \infty} b_i = 0.
\]

Note that a law of large numbers in this paper requires growth rates \((a_i)\) to be sublinear and break rates \((b_i)\) to be vanishing.

For second order convergence, additional conditions seem however to be necessary to establish the well-posedness of the limiting fluctuation process defined by SDE (5). First of all, for any \( c \in \mathcal{X}_+ \), the linear operator \( \tau \circ \nabla s(c) \) does not seem to have monotonicity or symmetry properties that could give an alternative construction of the solution of SDE (5). Hence, boundedness properties of the linear operator \( \tau \circ \nabla s(c) \) have to be used to get existence and uniqueness results of the solution of SDE (5).

For any \( k \geq 2 \), it is easy to check that the \( k \)th coordinate of the vector \( \tau \circ \nabla s(c) \cdot h_k \) is

\[
- (b_k + a_k c_1) / \sqrt{w_k},
\]

in particular the relation

\[
\| \tau \circ \nabla s(c) \|_{L_2(w)} \geq \| \tau \circ \nabla s(c) \cdot h_k \|_{L_2(w)} \geq (b_k + a_k c_1)^2
\]

for all \( k \), holds. Hence the operator \( \tau \circ \nabla s(c) \) is unbounded in any weighted \( L_2 \) space if a boundedness property for the sequences \((a_i)\) and \((b_i)\) does not hold. By using similar arguments, we observe that the operator \( \tau \circ \nabla s(c) \) is unbounded in the state spaces \( l_1 \) and \( l_{\infty} \) as well if this property does not hold.

**Proposition 1** If Assumption (a) and (c) hold, then the process \((M(t))\) is a well-defined continuous, square-integrable, martingale.
Proof By definition of the stochastic integral with respect to a cylindrical Wiener process (Chapter 4 in Da Prato and Zabczyk [8]), it is sufficient to verify that,
\[
\int_0^T \sum_{n \geq 1} \| \tau \circ \text{Diag} \left( \sqrt{s(c(u))} \right) \cdot e_n \|_{L^2(w)}^2 \, du < \infty.
\]
By using Lemma 1,
\[
\| \tau \circ \text{Diag} \left( \sqrt{s(c(u))} \right) \cdot e_n \|_{L^2(w)}^2 \leq \gamma \tau(w)^2 \text{Diag} \left( \sqrt{s(c(u))} \right) \cdot e_n \|_{L^2(w)}^2 = \gamma \tau(w)^2 w_n s_n(c(u)).
\]
Therefore,
\[
\int_0^T \sum_{n \geq 1} \| \tau \circ \text{Diag} \left( \sqrt{s(c(u))} \right) \cdot e_n \|_{L^2(w)}^2 \, du \\
\leq \gamma \tau(w)^2 \int_0^T \sum_{k \geq 1} (w_{2k-1} a_k c_1(u) c_k(u) + w_{2k} b_{k+1} c_{k+1}(u)) \, du \\
\leq \gamma_0 \gamma \tau(w)^2 \Lambda T \sup_{s \leq T} \sum_{k \geq 1} w_k c_k(s) < \infty.
\]
The last inequality is valid under Assumption (a) and (c). See Ball et al. [3] for details. \( \square \)

**Theorem 1** For any measurable \( L^2(w) \)-valued random variable \( W(0) \), the SDE (10) has a unique strong solution in \( L^2(w) \). If, in addition, for \( p \geq 1 \), \( \mathbb{E} \| W(0) \|_{L^2(w)}^{2p} < \infty \), then
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \| W(s) \|_{L^2(w)}^{2p} \right) < \infty.
\]

**Proof** In Proposition 1, we proved that the martingale part \( \mathcal{M}(t) \) of the SDE (10) is well-defined and continuous. By using Lemmas 1 and 2, the drift part is Lipschitz and linear on any \( x \in C([0, T], L^2(w)) \):
\[
\| \rho(\nabla s(c(u)) \cdot x(u)) \|_{L^2(w)} \leq \gamma \tau(w) \sup_{0 \leq u \leq T} \rho(c(u), w) \sup_{0 \leq u \leq T} \| x(u) \|_{L^2(w)}.
\]
Therefore, by using the results in Section 7.1 of Da Prato and Zabczyk [8], we can obtain the strongly existence, uniqueness, continuous and bounded results of SDE (10). \( \square \)

In Ball et al. [2] it is shown that, under the assumptions
\[
z_s^{-1} := \limsup_{i \to \infty} R_i^{1/i} < \infty \quad \text{and} \quad \sup_{0 \leq z < z_s} \sum_{i \geq 1} i R_i z^i > 1, \tag{11}
\]
where, for \( k \geq 1 \), \( R_k = \prod_{i=2}^{k} (a_{i-1}/b_i) \), then the equation
\[
\sum_{k \geq 1} k R_k z^k = 1
\]
has a unique solution \( z = \tilde{c}_1 \). Moreover if \( \tilde{c}_k = R_k (\tilde{c}_i)^i \), then \( (\tilde{c}_i)_{i \geq 1} \) is the unique fixed point of Becker–Döring equations (BD).
Proposition 2 (Fluctuations at equilibrium) Under condition (11), at equilibrium, the strong solution of SDE (10) can be represented as
\[
\tilde{W}(t) = T(t)\tilde{W}(0) + \int_0^t T(t-s)\tau\left( dB(s) \right),
\]
where \((T(t))\) is the semi-group associated with linear operator \(\tau \circ \nabla s(\tilde{c})\) and \((B(t))\) is a \(\tilde{Q}\)-Wiener process in \(L_2(w)\) where
\[
\tilde{Q} = \text{Diag} (w_n s_n(\tilde{c}), n \geq 1).
\]
Moreover, the stochastic convolution part
\[
\tilde{W}_{sc}(t) := \int_0^t T(t-s)\tau dB(s)
\]
is Gaussian, continuous in mean square. Its previsible increasing process is given by
\[
\left( \langle \tilde{W}_{sc}(t) \rangle \right) = \left( \int_0^t T(r)\tau \tilde{Q} (T(r)\tau)^* dw \right).
\]

Proof At equilibrium, the SDE (10) becomes
\[
\tilde{W}(t) = \tilde{W}(0) + \int_0^t \tau (\nabla s(\tilde{c}) \cdot \tilde{W}(u)) du + \int_0^t \tau (\text{Diag}(\sqrt{s(\tilde{c})}) \cdot d\beta(u)).
\]
(13)
It is a linear equation with additive noise. The noise part can be expressed by \(\tau(B(t))\) where
\[
B(t) := \int_0^t \text{Diag}(\sqrt{s(\tilde{c})}) \cdot d\beta(u) = \sum_{n \geq 1} \sqrt{w_n s_n(\tilde{c})} \beta_n(t) h_n.
\]
Recall that \((h_n)\) defined by (9) is an orthonormal basis in \(L_2(w)\). By definition of \(z_s\) and the assumption \(\lim_{n \to \infty} w_n^{1/n} = 1\), if \(0 \leq z < z_s\), then
\[
\sum_{n \geq 1} w_n R_n z_n < \infty
\]
holds. Let
\[
\tilde{Q} = \text{Diag} (w_n s_n(\tilde{c}), n \geq 1),
\]
then it is trace class, i.e.,
\[
\text{Tr} \tilde{Q} = \sum_{n \geq 1} w_n s_n(\tilde{c}) \leq 2\gamma_0 \Lambda \sum_{n \geq 1} w_n \tilde{c}_n < \infty.
\]
Therefore, \(B(t)\) is a \(\tilde{Q}\)-Wiener process in \(L_2(w)\) (c.f. Section 4.1 in [8]).

Again, by using Lemmas 1 and 2, we can see that the linear operator \(\tau \circ \nabla s(\tilde{c})\) is bounded, and therefore, the associated semi-group
\[
T(t) := e^{\tau \circ \nabla s(\tilde{c}) t}
\]
is uniformly continuous and satisfies that for any \(z \in L_2(w)\),
\[
\| T(t)z \|_{L_2(w)} \leq e^{\gamma(\tau)(w)(\gamma(\tilde{c},w)t)} \| z \|_{L_2(w)}.
\]
Therefore, on any finite time interval \([0, T]\),
\[
\int_0^T \text{Tr} \left( T(r) \tau \tilde{Q} \tau^* T^*(r) \right) dr = \int_0^T \| T(r) \tau \tilde{Q} \tau^* h_n \|^2_{L_2(w)} dr \\
\leq \gamma_T (w)^2 T e^{2 \gamma_T (w) \gamma(c, w)^T} \text{Tr} \tilde{Q} < \infty.
\]

By applying the Theorem 5.4 in Da Prato and Zabczyk [8], the process (12) is the unique weak solution of SDE (13). By Theorem 5.2 in [8], the stochastic convolution part \((\tilde{W}_{sc}(t))\) is Gaussian, continuous in mean square, has a predictable version and for all \(t \in [0, T]\),
\[
\langle \tilde{W}_{sc}(t) \rangle = \int_0^t T(r) \tau \tilde{Q} \tau^* T^* \langle r \rangle dr.
\]

To show that the process defined by Relation (12) is also a strong solution, we use Theorem 5.29 in [8]. It is sufficient to check
\[
\sum_{n \geq 1} \| \tau \nabla_s (\tilde{c}) \tau \tilde{Q} \tau^* h_n \|^2_{L_2(w)} < \infty.
\]

Then, stochastic integral \(\tau \circ \nabla_s (\tilde{c}) \cdot \tilde{W}_{sc}(t)\) is a well defined continuous square-integrable martingale on \([0, T]\). The process \(\tau \circ \nabla_s (\tilde{c}) \cdot \tilde{W}(t)\) has square integrable trajectories. Therefore, \((\tilde{W}(t))\) is a strong solution. The proposition is proved. \(\Box\)

### 4 Convergence of the Fluctuation Processes

Recall that the fluctuation processes \((W^N(t))\) satisfies relation (8),
\[
W^N(t) = W^N(0) + \frac{1}{\sqrt{N}} \tau \left( D^N(t) \right) + \frac{2 a_1}{\sqrt{N}} \int_0^t \left( \frac{X^N_1(u)}{N} \right) e_1 du \\
+ \frac{1}{2} \int_0^t \tau \left( \nabla_s \left( \frac{X^N(u)}{N} \right) \cdot W^N(u) + \nabla_s (c(u)) \cdot W^N(u) \right) du.
\]

The goal of this section is to prove that, when \(N\) is going to infinity, the process \((W^N(t))\) is converging in distribution to \((W(t))\), the solution of the SDE (10). We will first prove some technical lemmas and the convergence of scaled process \((X^N(t)/N)\). Then, we will prove the tightness and convergence of the local martingales \((D^N(t))\) and, with the help of these results, we will get the tightness of \((W^N(t))\) and identify the limit.

**Lemma 3** Under Assumptions (a) and (b) then, for any \(T > 0\), one has
\[
\zeta_T := \sup_N \sup_{t \leq T} \mathbb{E} \left( \sum_{k \geq 1} \frac{r_k X^N_k(t)}{N} \right) < + \infty, \tag{14}
\]
\[
\kappa_T := \sup_N \mathbb{E} \left( \sup_{t \leq T} \sum_{k \geq 1} \frac{|M^N_k(t)|}{\sqrt{N}} \right) < + \infty, \tag{15}
\]
where \(M^N(t) = \tau (D^N(t))\).
Proof By using the SDE (7), we get that, for any $N$,

\[
\mathbb{E} \left( \sum_{k \geq 1} \frac{r_k X_k^N(t)}{N} \right) = \mathbb{E} \left( \sum_{k \geq 1} \frac{r_k X_k^N(0)}{N} \right) + \mathbb{E} \int_0^t r_2 \left( a_1 \frac{X_1^N(u)(X_1^N(u)-1)}{N^2} - b_2 \frac{X_2^N(u)}{N} \right) du \\
+ r_1 \mathbb{E} \int_0^t \left( - \sum_{i \geq 1} (1 + \mathbb{1}_{i=1}) a_i X_i^N(u)(X_i^N(u)-1) \right) \mathbb{1}_{i=1} du \\
+ \mathbb{E} \int_0^t \left( \frac{X_i^N(u)}{N} \sum_{i \geq 2} (r_{i+1}-r_i) a_i X_i^N(u) + \sum_{i \geq 3} (r_{i-1}-r_i) b_i X_i^N(u) \right) du \\
\leq \mathbb{E} \left( \sum_{k \geq 1} \frac{r_k X_k^N(0)}{N} \right) + (r_2 + r_1) \Lambda T + \gamma_r \Lambda \mathbb{E} \int_0^t \sum_{i \geq 1} r_i X_i^N(u) du,
\]

where $\Lambda$ is defined in Assumption (a). We apply Gronwall’s inequality, then, for any $t \leq T$,

\[
\mathbb{E} \left( \sum_{k \geq 1} \frac{r_k X_k^N(t)}{N} \right) \leq e^{r_2 \Lambda T} \left( \mathbb{E} \left( \sum_{k \geq 1} \frac{r_k X_k^N(0)}{N} \right) + (r_2 + r_1) \Lambda T \right).
\]

Therefore, there exists a constant $\xi_T$, such that the Relation (14) holds.

For the other inequality, we first see

\[
\mathbb{E} \left( \sup_{t \leq T} \sum_{k \geq 1} \frac{|\mathcal{M}_k^N(t)|}{\sqrt{N}} \right) \\
\leq \sum_{k \geq 1} \frac{1}{w_k} + \mathbb{E} \left( \sup_{t \leq T} \sum_{k \geq 1} \frac{|\mathcal{M}_k^N(t)|}{\sqrt{N}} \right) \mathbb{1} \left\{ \frac{|\mathcal{M}_k^N(t)|}{\sqrt{N}} \geq \frac{1}{w_k} \right\} \\
\leq 2 + \mathbb{E} \left( \sup_{t \leq T} \sum_{k \geq 1} \frac{w_k |\mathcal{M}_k^N(t)|}{\sqrt{N}} \right)^2 \leq 2 + \mathbb{E} \left( \sum_{k \geq 1} \frac{w_k (|\mathcal{M}_k^N(T)|)^2}{N} \right).
\]

By using the expression of the increasing process (6) and the definition of $\tau$, we get

\[
\sum_{k \geq 2} \frac{w_k}{N} \left( \mathcal{M}_k^N(T) \right) \leq \Lambda \int_0^T \sum_{k \geq 1} (w_{k+1} \mathbb{1}_{(k \geq 2)} + w_k) \frac{X_k^N(u)}{N} du \\
+ \Lambda \int_0^T \sum_{k \geq 2} (w_k + w_{k-1} \mathbb{1}_{(k \geq 3)}) \frac{X_k^N(u)}{N} du.
\]

It implies

\[
\sup_N \mathbb{E} \left( \sum_{k \geq 2} \frac{w_k}{N} \left( \mathcal{M}_k^N(T) \right) \right) \leq w_1 \Lambda T + (3 + \gamma_0) \Lambda T \sup_N \mathbb{E} \sum_{k \geq 2} \frac{w_k}{N} X_k^N(t)
\leq w_1 \Lambda T + (3 + \gamma_0) \Lambda T \xi_T.
\]

For $k = 1$,

\[
\left( \frac{\mathcal{M}_1^N(T)}{N} \right) \leq 2 \int_0^T \sum_{i \geq 1} \left( a_i \frac{X_1^N(u)X_i^N(u)}{N^2} + b_{i+1} \frac{X_{i+1}^N(u)}{N} \right) du \leq 4 \Lambda T.
\]
Therefore, there exists a positive constant $\kappa_T$, such that

$$
\sup_N \mathbb{E} \left( \sup_{t \leq T} \sum_{k \geq 1} \frac{|M^N_k(t)|}{\sqrt{N}} \right) \leq \kappa_T
$$

holds.

Now we prove the law of large numbers under Assumptions (a), (b) and (c) in the $L^1$-norm. This result will be used in the proof of Theorem 3. We should note that we could not directly apply the Theorem 2 of Jeon [17] since it requires the conditions $\lim_{i \to \infty} a_i / i = 0$ and $\lim_{i \to \infty} b_i = 0$.

**Theorem 2** (Law of large numbers) Under Assumptions (a), (b) and (c), for any $T > 0$, one has

$$
\lim_{N \to \infty} \mathbb{E} \sup_{t \leq T} \sum_{k \geq 1} \left| \frac{X^N_k(t)}{N} - c_k(t) \right| = 0,
$$

where $(c(t))$ is the unique solution of Becker–Döring equation (BD) with initial state $c(0) \in \mathcal{X}_+$.

**Proof** From SDE (7), we have that

$$
\frac{X^N(t)}{N} - c(t) = \frac{X^N(0)}{N} - c(0) + \int_0^t \tau \left( s \left( \frac{X^N(u)}{N} \right) - s(c(u)) \right) du + \frac{2a_1}{N} \int_0^t \left( \frac{X^N_1(u)}{N} \right) e_1 du + \frac{M^N(t)}{N}.
$$

With a simple calculation, we get the relation

$$
\sum_{k \geq 1} \left| \frac{X^N_k(t)}{N} - c_k(t) \right| \leq \sum_{k \geq 1} \left| \frac{X^N_k(0)}{N} - c_k(0) \right| + \frac{2\Lambda t}{N} + \int_0^t 8\Lambda \sum_{k \geq 1} \left| \frac{X^N_k(u)}{N} - c_k(u) \right| du + \sum_{k \geq 1} \left| \frac{M^N_k(t)}{N} \right|,
$$

and, by Gronwall’s inequality and relation (15), we have

$$
\lim_{N \to \infty} \mathbb{E} \sup_{t \leq T} \sum_{k \geq 1} \left| \frac{X^N_k(t)}{N} - c_k(t) \right| = 0.
$$

**Proposition 3** Under Assumptions (a) and (b), the sequence of local martingales

$$
\left( \frac{1}{\sqrt{N}} D^N(t) \right) = \left( \frac{1}{\sqrt{N}} D^N_i(t) \right)_{i \geq 1}
$$

is tight for the convergence in distribution in $\mathcal{D}_T$. 

\( \square \) Springer
Proof. Recall that for all $i > N$, $(D_i^N(t)) \equiv 0$ and $(X_i^N(t)) \equiv 0$. By using Jensen’s and Doob’s inequalities, for any fixed $T$ and $N$, we get

$$
\mathbb{E} \left( \sup_{t \leq T} \frac{1}{\sqrt{N}} D^N(t) \right) \leq \mathbb{E} \left( \sup_{t \leq T} \frac{1}{\sqrt{N}} D^N(t)^2 \right)^{\frac{1}{2}}
$$

$$
= \sqrt{\mathbb{E} \left( \sup_{t \leq T} \sum_{i=1}^{N} w_i \frac{1}{N} D_i^N(t)^2 \right)} \leq \sqrt{\sum_{i=1}^{N} w_i \mathbb{E} \left( (D_i^N(T))^2 \right)}.
$$

From the expression of the increasing processes (6), we have

$$
\frac{1}{N} \sum_{k \geq 1} w_{2k-1} \mathbb{E} \left( (D_{2k-1}^N(T))^2 \right) \leq \gamma_0 A \int_0^T \mathbb{E} \left( \sum_{k \geq 1} w_k X_k(u) \right) \, du,
$$

and

$$
\frac{1}{N} \sum_{k \geq 2} w_{2k-2} \mathbb{E} \left( (D_{2k-2}^N(T))^2 \right) \leq \gamma_0 A \int_0^T \mathbb{E} \left( \sum_{k \geq 2} w_k X_k(u) \right) \, du.
$$

Therefore, by using inequality (14), one gets

$$
\sup_N \mathbb{E} \sup_{t \leq T} \frac{1}{\sqrt{N}} D^N(t) \leq 2\gamma_0 A T \sup_N \mathbb{E} \left( \sum_{k \geq 1} w_k X_k(u) \right) < \infty,
$$

it gives that

$$
\lim_{a \to \infty} \limsup_{N \to \infty} \mathbb{P} \left( \sup_{t \leq T} \frac{1}{\sqrt{N}} D^N(t) \geq a \right) \leq \lim_{a \to \infty} \sup_N \mathbb{E} \left( \sup_{t \leq T} \frac{1}{\sqrt{N}} D^N(t) \right) = 0.
$$

Since the Skorohod distance is weaker than the uniform distance in $\mathcal{D}_T$, we estimate the modulus of continuity with uniform distance to prove the tightness in the Skorohod space. For any $\varepsilon > 0$, $\delta > 0$ and any $L \in \mathbb{N}$, we have

$$
\mathbb{P} \left( \sup_{t, s \leq T, |t-s| < \delta} \left\| \frac{1}{\sqrt{N}} D^N(t) - \frac{1}{\sqrt{N}} D^N(s) \right\|_{L_2(w)} \geq \varepsilon \right) \leq \mathbb{P} \left( \sup_{i \leq L} \sum_{i > L} r_i \frac{1}{N} D_i^N(t)^2 \geq \varepsilon^2 \frac{r_L}{4w_L} \right) + \mathbb{P} \left( \sup_{t, s \leq T, |t-s| < \delta} \sum_{i=1}^{L} w_i \left( \frac{1}{\sqrt{N}} D_i^N(t) - \frac{1}{\sqrt{N}} D_i^N(s) \right)^2 \geq \varepsilon^2 \right).
$$

By using the inequality (14),

$$
\mathbb{P} \left( \sup_{i \leq L} \sum_{i > L} r_i \frac{1}{N} D_i^N(t)^2 \geq \varepsilon^2 \frac{r_L}{4w_L} \right) \leq \frac{4w_L}{\varepsilon^2 r_L} \mathbb{E} \left( \sum_{i > L} r_i \frac{1}{N} (D_i^N(T))^2 \right) \leq \frac{8\gamma r w_L A}{\varepsilon^2 r_L} \sup_{t \leq T} \mathbb{E} \left( \sum_{i \geq 1} r_i X_i^N(t) \right) \leq \frac{8\gamma r w_L A}{\varepsilon^2 r_L} T \xi_T.
$$
By using the Assumption (b), \( \lim_{k \to \infty} w_k / r_k = 0 \), for any constant \( \eta > 0 \), there exist a constant \( L \), such that
\[
\mathbb{P} \left( \sup_{t \leq T, i \geq L} \sum_{i=1}^{L} r_i \frac{1}{N} D_i^N(t)^2 \geq \frac{\varepsilon^2 r_L}{4 w_L} \right) \leq \frac{\eta}{2}.
\]
The processes \( (D_i^N(t)/\sqrt{N}) \) \( i = 1, \ldots, L \), live in finite dimensional space and each of them has an increasing process that is uniformly continuous almost surely. Therefore, for \( N \) large enough, we have
\[
\mathbb{P} \left( \sup_{t,s \leq T, |t-s| < \delta} \sum_{i=1}^{L} w_i \left( \frac{1}{\sqrt{N}} D_i^N(t) - \frac{1}{\sqrt{N}} D_i^N(s) \right)^2 \geq \frac{\varepsilon^2}{2} \right) \leq \frac{\eta}{2},
\]
consequently, by using Theorem 13.2 in Billingsley [4], the sequence of processes \( (D_i^N(t)/\sqrt{N}) \) is tight in \( D_T \).

**Proposition 4** For the convergence in distribution of random process, uniformly on compact sets,
\[
\lim_{N \to +\infty} \left( \frac{1}{\sqrt{N}} D_i^N(t) \right) = (D(t)) := \left( \int_0^t \text{Diag} \left( \sqrt{s(c(u))} \right) \cdot d\beta(u) \right).
\]

**Proof** From the previous proposition, we have that, for \( T > 0 \), the sequence \( (D_i^N(t)/\sqrt{N}) \) is tight in \( D([0, T], L_2(w)) \). Let \( D'(t) \) be a possible limit. For any \( d \in \mathbb{N}^* \), let \( \mathcal{P}_d \) be the projection from \( \mathbb{R}^{\mathbb{N}^*} \) to \( \mathbb{R}^d \), i.e., for any \( z \in \mathbb{R}^{\mathbb{N}^*} \), \( \mathcal{P}_d(z) = (z_1, \ldots, z_d) \). It is easy to check that \( (\mathcal{P}_d(D'(t))) \) is a limit of a subsequence of \( (\mathcal{P}_d(D_i^N(t)/\sqrt{N})) \) for the weak convergence in probability in the \( L_2 \)-norm. By using Theorem 1.4 page 339 of Ethier and Kurtz [11], we know that for any \( d \in \mathbb{N}^* \), the equality \( (\mathcal{P}_d(D'(t))) = (\mathcal{P}_d(D(t))) \) holds in distribution. Hence, by Kolmogorov’s theorem, we have the equality in distribution \( (D'(t)) = (D(t)) \). \( \square \)

We can now state our main result.

**Theorem 3** (Functional Central Limit Theorem) Under Assumptions (a)–(d), then the fluctuation process \( (W_i^N(t)) \) defined by Relation (4) converges in distribution to the \( L_2(w) \)-valued process \( (W(t)) \), the unique strong solution of the SDE (10).

**Proof** From relation (8), process \( (W_i^N(t)) \) satisfies
\[
W_i^N(t) = W_i^N(0) + \frac{1}{\sqrt{N}} \tau \left( D_i^N(t) \right) + \int_0^t \tau \circ \nabla s \left( c(u) \right) \cdot W_i^N(u) \, du
+ \frac{2a_1}{\sqrt{N}} \int_0^t \frac{X_i^N(u)}{N} \, e_1 \, du + \int_0^t \tau \left( \Delta_i^N(u) \right) \, du
\]
where
\[
\Delta_i^N(u) = \frac{1}{2} \left( \nabla s \left( \frac{X_i^N(u)}{N} \right) + \nabla s \left( c(u) \right) \right) \cdot W_i^N(u) - \nabla s \left( c(u) \right) \cdot W_i^N(u).
\]
By direct calculations, we have that for \( k \geq 1 \)
Thanks to Gronwall’s lemma, for any $t$, we have
\[ \left\| \Delta^N_{2k-1}(u) \right\|_{L^2(w)} = \left( 1 + \mathbb{E}_{k=1} \right) a_k \left( \frac{X^N_1(u)}{N} - c_1(u) \right) W^N_k(u), \]
and then
\[ \left\| \Delta^N(u) \right\|_{L^2(w)} \leq 2\sqrt[10]{\gamma} A \left\| \frac{X^N_1(u)}{N} - c_1(u) \right\| \left\| W^N(u) \right\|_{L^2(w)} \leq 2\sqrt[10]{\gamma} A \left\| W^N(u) \right\|_{L^2(w)}. \]

Let $\Gamma = \gamma_T(w) (\sup_{u \leq T} \gamma'(c(u, w)) + 2\sqrt[10]{\gamma} A)$, by using Lemmas 1 and 2, for all $t \leq T$, one has
\[ \left\| W^N(t) \right\|_{L^2(w)} \leq \left\| W^N(0) \right\|_{L^2(w)} + \gamma_T(w) \frac{1}{\sqrt{N}} \left\| D^N(t) \right\|_{L^2(w)} + \frac{2\sqrt[10]{\gamma} A t}{\sqrt{N}} + \int_0^t \Gamma \left\| W^N(u) \right\|_{L^2(w)} \, du. \]

Thanks to Gronwall’s lemma, for any $t \leq T$, we have
\[ \sup_N \mathbb{E} \sup_{t \leq T} \left\| W^N(t) \right\|_{L^2(w)}^2 \leq 3e^{2\Gamma T} \left( \sup_N \mathbb{E} \left\| W^N(0) \right\|_{L^2(w)}^2 + \gamma_T(w)^2 \sup_N \mathbb{E} \sup_{t \leq T} \frac{1}{N} \left\| D^N(t) \right\|_{L^2(w)}^2 + \frac{4\sqrt[10]{\gamma} A^2 T^2}{N} \right). \]

In the proof of Proposition 3, we have shown that
\[ \sup_N \mathbb{E} \sup_{t \leq T} \frac{1}{N} \left\| D^N(t) \right\|_{L^2(w)}^2 < \infty, \]
therefore, if
\[ \sup_N \mathbb{E} \left\| W^N(0) \right\|_{L^2(w)}^2 \leq C_0, \]
then there exists a finite constant $C_T$ such that
\[ \sup_N \mathbb{E} \sup_{t \leq T} \left\| W^N(t) \right\|_{L^2(w)}^2 \leq C_T. \]

For the tightness of $(W^N(t))$, for any $\varepsilon > 0$, $\eta > 0$, by using Lemmas 1 and 2, we get that
\[ \mathbb{P} \left( \sup_{t, s \leq T, |t - s| < \delta} \left\| W^N(t) - W^N(s) \right\|_{L^2(w)} \geq \varepsilon \right) \leq \mathbb{P} \left( \Gamma \delta \sup_{u \leq T} \left\| W^N(u) \right\|_{L^2(w)} \geq \frac{\varepsilon}{2} \right) \]
\[ + \mathbb{P} \left( \sup_{t, s \leq T, |t - s| < \delta} \left\| \frac{D^N(t)}{\sqrt{N}} - \frac{D^N(s)}{\sqrt{N}} \right\|_{L^2(w)} \geq \frac{\varepsilon}{4\gamma_T(w)} \right) + \mathbb{P} \left\{ \frac{2\sqrt[10]{\gamma} A \lambda \delta}{\sqrt{N}} > \frac{\varepsilon}{4} \right\} \]
holds. Choose $\delta > 0$ such that
\[ \delta < \frac{\sqrt{\eta \varepsilon}}{2CT}, \]
then, for $\delta \in (0, \delta_1)$ and $N \geq 1$, \[
\mathbb{P}\left( \Gamma \delta \sup_{u \leq T} \|W^N(u)\|_{L^2(w)} \geq \frac{\varepsilon}{2} \right) \leq \left( \frac{2\Gamma \delta}{\varepsilon} \right)^2 \mathbb{E} \sup_{u \leq T} \|W^N(u)\|_{L^2(w)}^2 < \frac{\eta}{2}.
\]

According to the proof of Proposition 3, there exist $\delta_2 > 0$ and $N_0$, such that, for $\delta \in (0, \delta_2)$ and $N > N_0$, the relation \[
\mathbb{P}\left( \sup_{t, s \leq T, |t - s| < \delta} \left\| \frac{D^N(t)}{\sqrt{N}} - \frac{D^N(s)}{\sqrt{N}} \right\|_{L^2(w)} \geq \frac{\varepsilon}{4\gamma(w)} \right) \leq \frac{\eta}{2}
\]
holds. In conclusion, for any $\delta < \delta_1 \wedge \delta_2 \wedge (\varepsilon/(8 \Lambda \sqrt{w}))$ and $N > N_0$, \[
\mathbb{P}\left( \sup_{t, s \leq T, |t - s| < \delta} \|W^N(t) - W^N(s)\|_{L^2(w)} \geq \varepsilon \right) \leq \eta.
\]

It is then easy to check that \[
\lim_{a \to \infty} \limsup_{N \to \infty} \mathbb{P}\left( \sup_{t \leq T} \|W^N(t)\|_{L^2(w)} \geq a \right) = 0.
\]

Therefore, the process $(W^N(t))$ is tight in $D([0, T], L^2(w))$. To identify the limit, note that \[
\mathbb{E} \sup_{t \leq T} \left\| \int_0^T \tau \left( \Delta^N(u) \right) \, du \right\|_{L^2(w)} \leq \gamma_T(w) \mathbb{E} \int_0^T \|\Delta^N(u)\|_{L^2(w)} \, du
\]
\[
\leq 2\sqrt{T\gamma_0\gamma_T(w)} T \Lambda \mathbb{E} \left( \sup_{u \leq T} \left( \frac{X^N(u)}{N} - c_1(u) \right) \right) \left( \mathbb{E} \sup_{u \leq T} \|W^N(u)\|_{L^2(w)}^2 \right)^{1/2}
\]
\[
\leq 2\sqrt{T\gamma_0\gamma_T(w)} T \Lambda \left( 2\mathbb{E} \sup_{u \leq T} \left( \frac{X^N(u)}{N} - c_1(u) \right)^2 \right)^{1/2}
\]
\[
\leq 2\sqrt{T\gamma_0\gamma_T(w)} T \Lambda \left( 2\mathbb{E} \sup_{u \leq T} \left( \frac{X^N(u)}{N} - c_1(u) \right)^2 \right)^{1/2}.
\]

By using Theorem 2, this term is vanishing as $N$ goes to infinity. From Proposition 4, one conclude that any limit of $(W^N(t))$ satisfies SDE (10). The theorem is proved. \qed

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