AN IMPROVED SUBEXPOENTIAL BOUND FOR ON-LINE CHAIN PARTITIONING

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ABSTRACT. Bosek and Krawczyk exhibited an on-line algorithm for partitioning an on-line poset of width $w$ into $w^{14\lg w}$ chains. They also observed that the problem of on-line chain partitioning of general posets of width $w$ could be reduced to First-Fit chain partitioning of $(2w^5 + 1)$-ladder-free posets of width $w$, where an $m$-ladder is the transitive closure of the union of two incomparable chains $x_1 \leq \ldots \leq x_m, y_1 \leq \ldots \leq y_m$ and the set of comparabilities $\{x_1 \leq y_1, \ldots, x_m \leq y_m\}$. Here, we improve the subexponential upper bound to $w^{6.5w + O(1)}$ with a simplified proof, exploiting the First-Fit algorithm on ladder-free posets.

1. INTRODUCTION

An on-line poset $P^\prec$ is a triple $(V, \leq_P, \prec)$, where $P = (V, \leq_P)$ is a poset and $\prec$ is a linear order of $V$, called the presentation order of $P$. An on-line chain partitioning algorithm is a deterministic algorithm $A$ that assigns the vertices $v_1 \prec \cdots \prec v_n$ of the on-line poset $P^\prec$ to disjoint chains $C_1, \ldots, C_n$ so that for each $i$, the chain $C_i$ to which the $i$th vertex $v_i$ is assigned, is determined solely by the subposet $P[v_1, \ldots, v_i]$ induced by the first $i$ vertices $v_1 \prec \cdots \prec v_i$. This formalizes the scenario in which the algorithm $A$ receives the vertices of $P$ one at a time, and at the time a vertex is received, irrevocably assigns it to one of the chains. Let $\chi_A(P^\prec)$ denote the number of (nonempty) chains that $A$ uses to partition $P^\prec$, and $\chi_A(P) = \max_{\prec}(\chi(P^\prec))$ over all presentation orders $\prec$ for $P$. For a class of posets $\mathcal{P}$, let $\text{val}_A(\mathcal{P}) = \max_{P \in \mathcal{P}}(\chi_A(P))$ and $\text{val}(\mathcal{P}) = \min_{A}(\text{val}_A(\mathcal{P}))$ over all on-line chain partitioning algorithms $A$. This paper is motivated by the problem of bounding $\text{val}(\mathcal{P}_w)$, where $\mathcal{P}_w$ is the class of finite posets of width $w$ (allowing countably infinite posets with $w$ finite in $\mathcal{P}_w$ would not effect known results).

By Dilworth’s Theorem [8], every poset with finite width $w$ can be partitioned into $w$ chains, and this is best possible. However this bound cannot be achieved on-line. In 1981, Kierstead [15] proved that $4w - 3 \leq \text{val}(\mathcal{P}_w) \leq 2^{w-1}$, and asked whether $\text{val}(\mathcal{P}_w)$ is polynomial in $w$. It was also noted that the arguments could be modified to provide a super linear lower bound. Shortly after, Szemerédi proved a quadratic lower bound (see [16]). Up to a few years ago, there were only small improvements. In 1997 Felsner [12] proved that $\text{val}(\mathcal{P}_2) \leq 5$, and in 2008 Bosek [1] proved that $\text{val}(\mathcal{P}_3) \leq 16$. Bosek et al. [2] improved the lower bound to $(2 - o(1))(w+1)$. In 2010 Bosek and Krawczyk made a major advance by proving a subexponential bound.

Theorem 1 (BB & TK [3]). $\text{val}(\mathcal{P}_w) \leq w^{14\lg w}$. 

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Date: October 14, 2014.
Key words and phrases. partially ordered set, poset, first-fit, on-line chain partition, ladder, regular poset.
†Research of these authors is supported by Polish National Science Center (NCN) grant 2011/03/B/ST6/01367.
‡Research of this author is supported in part by NSF grant DMS-0901520.
Building on techniques and ideas from the authors in [3] and [22], we will improve this bound:

**Theorem 2.** \( \text{val}(P_w) \leq w^{6.5 \log w + O(1)} \).

The First-Fit algorithm is employed in Theorems 1 and 2. Perhaps the simplest of all on-line chain partitioning algorithms, First-Fit assigns each new vertex \( v_i \) to the chain \( C_j \), with the least index \( j \in \mathbb{Z}^+ \) such that for all \( h < i \) if \( v_h \in C_j \) then \( v_h \) is comparable to \( v_i \). It was observed in [15] that \( \text{val}_{\text{FF}}(P_w) = \infty \) (see [16] for details) for any \( w > 1 \). The poset used to show this fact contains substructures that will be important to this paper, so we present it here.

**Lemma 3.** For every positive integer \( n \) there exists an on-line poset \( R_n \) with width 2 such that \( \chi_{\text{FF}}(R_n) = n \).

**Proof.** We define the on-line poset \( R_n = (X, \leq, \prec) \) as follows. The poset \( R_n \) consists of \( n \) chains \( X^1, \ldots, X^n \) with

\[
X^k = x_k^k \leq_R x_{k-1}^k \leq_R \cdots \leq_R x_2^k \leq_R x_1^k
\]

and the additional comparabilities and incomparabilities given by:

\[
x_i^k \geq_R X^1 \cup X^2 \cup \cdots \cup X^{k-2} \cup \{x_{k-1}^{k-1}, x_{k-2}^{k-1}, \ldots, x_1^{k-1}\}
\]

\[
x_i^k \parallel_R \{x_i^{k-1}, x_{i-2}^{k-1}, \ldots, x_1^{k-1}\}.
\]

Note that the superscript of a vertex indicates to which chain \( X^k \) it belongs and the subscript is its index within that chain. The example of \( R_5 \) is illustrated in Figure 1. The presentation order \( \prec \) is given by \( X^1 \prec \cdots \prec X^n \), where the order \( \prec \) on the vertices of \( X^k \) is the same as \( \leq_R \) on \( X^k \).

Observe that \( X^{k-2} \leq_R X^k \). Hence, the width of \( R_n \) is 2. By induction on \( k \) one can show that each vertex \( x_i^k \) is assigned to chain \( C_i \).

\[ \square \]

**Figure 1.** Hasse diagrams of \( R_5 \) and \( L_m \).
Despite Lemma 3 the analysis of the performance of First-Fit on restricted classes of posets has proved very useful and interesting. For posets $P$ and $Q$, we say $P$ is $Q$-free if $P$ does not contain $Q$ as an induced subposet. Let $\text{Forb}(Q)$ denote the family of $Q$-free posets, and $\text{Forb}_w(Q)$ denote the family of $Q$-free posets of width at most $w$. Slightly abusing notation, we write $\text{val}_{\text{FF}}(Q, w)$ for $\text{val}_{\text{FF}}(\text{Forb}_w(Q))$.

Let $s$ denote the total order (chain) on $s$ vertices, and $s + t$ denote the width 2 poset consisting of disjoint copies of $s$ and $t$ with no additional comparabilities or vertices. It is well known [13] that the class of interval graphs is equal to $\text{Forb}(2 + 2)$. First-Fit chain partitioning of interval orderings has applications to polynomial time approximation algorithms [17, 18] and Max-Coloring [24].

The first linear upper bound $\text{val}_{\text{FF}}(2 + 2, w)$ was proved by Kierstead in 1988. This was improved later to $\text{val}_{\text{FF}}(2 + 2, w) \leq 26w$ in [20]. In 2004 Pemmaraju, Raman, and K. Varadarajan [24] introduced a beautiful new technique to show $\text{val}_{\text{FF}}(2 + 2, w) \leq 10w$, and this was quickly improved to $\text{val}_{\text{FF}}(2 + 2, w) \leq 8w$ [7, 23]. In 2010 Kierstead, D. Smith, and Trotter [21, 25] proved $5(1 - o(1))w \leq \text{val}_{\text{FF}}(2 + 2, w)$. In 2010 Bosek, Krawczyk, and Szczypka [6] proved that $\text{val}_{\text{FF}}(t + t, w) \leq 3tw^2$. This result plays an important role in the proof of Theorem 1.

In fact, there is a general result. In 2010 Bosek, Krawczyk, and Matecki proved the following:

**Theorem 4 (BB, TK & GM [5]).** For every width 2 poset $Q$ there exists a function $f_Q$ such that $\text{val}_{\text{FF}}(Q, w) \leq f_Q(w)$.

Lemma 3 shows that the theorem cannot be extended to posets $Q$ with width greater than 2.

In [3], Bosek and Krawczyk drew attention to ladders. For a positive integer $m$, we say poset $L$ is an $m$-ladder (or $L = L_m$) if its vertices consist of two disjoint chains $x_1 <_L x_2 <_L \cdots <_L x_m$ and $y_1 <_L y_2 <_L \cdots <_L y_m$ with $x_i <_L y_i$ for all $i \in [m]$ and $y_i \not<_L x_j$ if $i < j \leq m$. We provide a Hasse diagram of $L = L_m$ in Figure 1. Notice that for two consecutive chains $X^i$ and $X^{i+1}$ of $R_n$, the set $X^i \cup (X^{i+1} - x_i^{i+1})$ induces the ladder $L_i$ in $R_n$. The vertices $x_1, x_2, \ldots, x_m$ are the lower leg and the vertices $y_1, y_2, \ldots, y_m$ are the upper leg of $L_m$. The vertices $x_i$, $y_i$ together form the $i$-th rung of $L_m$. When considering ladders, we will restrict our attention to ladders with more than one rung. The family $\text{Forb}(L_1)$ is the family of antichains, which are rather uninteresting when considering on-line chain partitioning. The following observation is the cause for investigation into the performance of First-Fit on ladder-free posets.

**Observation 5 (BB & TK [3]).** If $\text{val}_{\text{FF}}(L_m, w)$ is bounded from above by a function in $m$ and $w$, then $\text{val}(P_w)$ is bounded from above by a function in $w$.

Kierstead and Smith [22] found the preliminary results for bounding $\text{val}_{\text{FF}}(L_m, w)$, by restriction of $m$ and then by restriction of $w$, showing $\text{val}_{\text{FF}}(L_2, w) = w^2$ and $m - 1 \leq \text{val}_{\text{FF}}(L_m, 2) \leq 2m$. We will present proof of the bounds on $\text{val}_{\text{FF}}(L_m, w)$ without restrictions on $m$ or $w$, which will allow us to exploit Observation 5.

**Lemma 6.** For $m, w \in \mathbb{Z}^+$, we have $\text{val}_{\text{FF}}(L_m, w) \leq w^{2.5 \lg(2w) + 2\lg m}$.

To provide a loose outline for the proof of Theorem 2 we will define and examine regular posets. We will then show that we can reduce the general on-line chain partitioning problem to on-line chain partitioning of regular posets. Showing that any width $w$ regular poset belongs to $\text{Forb}(L_{2w^2+1})$ and the proof of Lemma 6 will complete the theorem. Afterward, we will look into the limitations of our methods by showing some regular posets have $m$-ladders where $m$ is quadratic in the width of the poset and show the following superpolynomial bound on $\text{val}_{\text{FF}}(L_m, w)$.

**Lemma 7.** For $m, w \in \mathbb{Z}^+$ with $m > 1$, we have $w^{\lg(m-1)/(m - 1)} \leq \text{val}_{\text{FF}}(L_m, w)$. 


2. Notation and Grundy Colorings

Let \( P = (V, \leq_P) \) be a poset with \( u, v \in P \). We will rarely make mention of \( V \) and use \( u \in P \) to mean \( u \in V \) and \( P - u \) to mean \( V - u, \leq_P \mid_{V - u} \). The upset of \( u \) in \( P \) is \( U_P(u) = \{ v : u <_P v \} \), the downset of \( u \) in \( P \) is \( D_P(u) = \{ v : v <_P u \} \), and the incomparability set of \( u \) in \( P \) is \( I_P(u) = \{ v \mid_P u \} \). The closed upset and closed downset of \( u \) in \( P \) are, respectively, \( U_P[u] = U_P(u) + u \) and \( D_P[u] = D_P(u) + u \). We also define \( [u, v]_P = U_P[u] \cap D_P[v] \). For \( U \subseteq V \), we similarly define \( D_P(U) = \bigcup_{u \in U} D_P(u) \) and \( U_P(U) = \bigcup_{u \in U} U_P(u) \), as well as \( D_P(U) = D_P(U) \cup U \) and \( U_P(U) = U_P(U) \cup U \). If \( U' \subseteq V \), we take \( [U, U']_P = U_P[U] \cap D_P[U'] \). The subposet of \( P \) induced by \( U \) is the poset \( (U, \leq_P \mid_{V - U}) \). We also denote this by \( P[U] \). If \( U_P(u) = \emptyset \), then \( u \) is maximal. If \( D_P(u) = \emptyset \), then \( u \) is minimal. If \( D_P[u] = P \), then \( u \) is maximum, greatest, or largest. If \( U_P[u] = P \), then \( u \) is minimum, least, or smallest. Let \( \text{Max}_P(U) \) be the set of maximal vertices in \( P[U] \) and \( \text{Min}_P(U) \) be the set of minimal vertices in \( P[U] \). In an abuse of notation, we use \( \text{Max}_P(P) \) and \( \text{Min}_P(P) \) to represent \( \text{Max}_P(V) \) and \( \text{Min}_P(V) \), respectively.

A width(\( P \))-chain partition of \( P \) is a Dilworth partition of \( P \). If there is a Dilworth partition of \( P \) so that vertices \( u \) and \( v \) are in the same chain, then we say \( uv \) is a \( P \)-Dilworth edge or simply a Dilworth edge if \( P \) is clear from context.

Since a chain in a poset \( P \) corresponds to an independent set in its cocomparability graph, we will use the terms chain partition and coloring interchangeably (as well as chain and color, on-line chain partitioning algorithm and on-line coloring algorithm, etc.).

With respect to First-Fit coloring, describing the presentation order of an on-line poset is cumbersome. To avoid confusion, we introduce the idea of the Grundy coloring.

**Definition 8.** Let \( P \) be a poset and \( n \) a positive integer. The function \( g : P \rightarrow [n] \) is an \( n \)-Grundy coloring of \( P \) if the following three conditions hold.

1. (G1) For each \( i \in [n] \), the set \( \{ u \in P : g(u) = i \} \) is a chain in \( P \).
2. (G2) For each \( i \in [n] \), there is some \( u \in P \) so that \( g(u) = i \) (i.e.: \( g \) is surjective).
3. (G3) If \( v \in P \) with \( g(v) = j \), then for all \( i \in [j - 1] \) there is some \( u \in I_P(v) \) such that \( g(u) = i \).

Often, we will call the elements of \([n]\) colors. If \( u \in P \) and \( g(u) = i \), we will say \( u \) is colored with \( i \). Let the color class \( i \) be the chain \( P_i(g) = \{ u \in P : g(u) = i \} \). If we are only concerned with one coloring function, we will shorten this to \( P_i \). If \( Q \) is a subposet of \( P \) and \( Q \cap P_i \neq \emptyset \), then color \( i \) appears on \( Q \).

Let \( u, v \in P \). If \( u \parallel_P v \) and \( g(u) < g(v) \), we will say \( u \) is a \( g(u) \)-witness for \( v \) under \( g \). If we are only concerned with one coloring function, this will be shortened to \( g(u) \)-witness. If we are not concerned with a specific color or the color is specified in another way, we will simply say \( u \) is a witness for \( v \).

If \( Q \) is a subposet of \( P \) and \( g \) is an \( n \)-Grundy coloring of \( P \), we will abuse notation and use \( g \) for the function \( g|_Q : Q \rightarrow [n] \) (i.e.: the function \( g \) with domain restricted to \( Q \)). Note that \( g \) might not be an \( n \)-Grundy coloring of \( Q \).

Our interest in Grundy coloring is justified by the next lemma.

**Lemma 9.** If \( P \) is a poset, then \( P \) has an \( n \)-Grundy coloring if and only if \( P \) has a presentation \( \prec \) so that \( \chi_{\text{FF}}(P \prec) = n \). Consequently, \( \chi_{\text{FF}}(P) \) is equal to the largest \( n \) so that \( P \) has an \( n \)-Grundy coloring.

**Proof.** Let \( P \) be a poset and \( g \) be an \( n \)-Grundy coloring of \( P \). We build presentation \( \prec \) based on \( g \). Let \( P_1, P_2, \ldots, P_n \) be the color classes of \( g \). Set \( P_1 \prec P_2 \prec \cdots \prec P_n \). For each \( i \in [n] \), the order \( \prec \) on the vertices of \( P_i \) may be chosen arbitrarily. If \( g(v) = j \), First-Fit will assign \( v \) to \( C_j \) because, for each \( i < j \), there is some vertex \( u \in C_i \) so that \( u \parallel_P v \) with \( u \prec v \). Hence, \( \chi_{\text{FF}}(P \prec) = n \).

Suppose we have presentation \( \prec \) so that \( \chi_{\text{FF}}(P \prec) = n \). For each vertex \( u \), let \( g(u) \) be the index of the chain to which \( u \) is assigned by First-Fit. We will show the conditions of Definition 8 are
satisfied. The result of First-Fit is a chain partition, so \((G1)\) holds. Each chain used by First-Fit is nonempty, so \((G2)\) is satisfied. Suppose \(g(v) = j\). Then \(v\) was assigned to chain \(C_j\). By the definition of First-Fit, for each \(i < j\), there is some \(u \in C_i\) so that \(u \parallel_p v\) with \(u < v\) (i.e.: \(u\) is an \(i\)-witness), so \((G3)\) holds as well. \(\square\)

3. Regular Posets and the Reduction of the General Problem

In this section we explore regular posets, a family of on-line posets first introduced and studied by Bosek and Krawczyk \[3, 4\] (the roots of regular posets are in the local game used by Bosek in \[1\] to show \(\text{val}(P_3) \leq 16\)). Regular posets can be seen as a restricted, more structured on-line posets, that already have proved to have properties useful in designing efficient on-line algorithms for their colorings. The essence is that, despite these advantages, the regular posets are still general in the sense that any efficient on-line coloring algorithm for regular posets translates into an efficient on-line coloring algorithm.

To define regular posets we need some preliminaries. Let \(\mathcal{M}\) be the set of maximum antichains in a poset \(P\). In the set \(\mathcal{M}\) we introduce \(\sqsubseteq_p\) relation

\[A \sqsubseteq_p B \text{ if } A \subseteq D_p[B] \text{ (or equivalently } B \subseteq U_p[A]).\]

If \(A \sqsubseteq_p B\) and \(A \neq B\), we write \(A \sqcap_p B\). In \[9\] Dilworth showed that \((\mathcal{M}, \sqsubseteq_p)\) is a lattice with the meet and the join defined as

\[(1) \quad A \sqcap B = \text{Min}_P\{A \cup B\},\]
\[(2) \quad A \sqcup B = \text{Max}_P\{A \cup B\}.\]

For a given chain \(\{A_1, A_2, \ldots, A_n\}\) in \((\mathcal{M}, \sqsubseteq_p)\) we define \(p(i)\) so as \(A_{p(i)}\) is the \(\sqsubseteq_p\)-greatest element in \(\{A_1, \ldots, A_{i-1}\}\) such that \(A_{p(i)} \sqsubseteq_p A_i\) and similarly, we define \(s(i)\) so as \(A_{s(i)}\) is the \(\sqsubseteq_p\)-least element in \(\{A_1, \ldots, A_{i-1}\}\) such that \(A_i \sqsubseteq_p A_{s(i)}\). Note that \(s(i)\) (\(p(i)\)) is not defined when \(A_i\) is maximum (resp. minimum) in \((\{A_1, \ldots, A_i\}, \sqsubseteq_p)\).

A poset \(P = (V, \leq_p)\) is bipartite if the set \(V\) can be partitioned into two disjoint antichains \(A, B\) such that \(A \sqsubseteq_p B\) – such the poset is denoted by \((A, B, \leq_p)\). A bipartite poset \(P = (A, B, \leq_p)\) is a core if \(|A| = |B|\) and for any comparable pair \(x \leq_p y\) with \(x \in A\) and \(y \in B\), \(xy\) is a Dilworth edge (see Figure \[2\]. Informally, we think of a core as a bipartite poset whose Hasse diagram is a balanced bipartite graph in which each edge is included in some perfect matching.

Now we are ready to define regular posets. Since the formal definition is quite involved, we comment on it just after it is stated.

**Definition 10.** A tuple \((P, A_1, \ldots, A_n)\) is a regular poset of width \(w\) if \(P = (V, \leq)\) is a poset of width \(w\), \(A_1, \ldots, A_n\) is the sequence of maximum antichains in \(P\), called the presentation order of \(P\), such that \(P\) and \(A_1, \ldots, A_n\) satisfy the conditions \((R1)-(R5)\).

\[(R1)\] \(V = A_1 \cup A_2 \cup \cdots \cup A_n\).
\[(R2)\] \(A_i \cap A_j = \emptyset\) for each \(i \neq j\).
\[(R3)\] The set \(\{A_1, A_2, \ldots, A_n\}\) is linearly ordered under \(\sqsubseteq_p\).
\[(R4)\] The bipartite poset \(P[A_i \cup A_{s(i)}] \text{ (resp. } P[A_{p(i)} \cup A_i]\) is a core, provided \(A_{s(i)}\) (resp. \(A_{p(i)}\)) exists.
\[(R5)\] Suppose \(x \leq_p y\) with \(x \in A_i\) and \(y \in A_j\). If \(i > j\) then there is some \(z \in A_{s(i)}\) such that \(x \leq_p z \leq_p y\); otherwise there is some \(z \in A_{p(j)}\) such that \(x \leq_p z \leq_p y\).

A regular poset \((P, A_1, \ldots, A_n)\) can be viewed as a restricted on-line poset that realizes a slightly different scenario of presenting vertices. The sequence \(A_1, \ldots, A_n\) plays the similar role as the presentation order in on-line posets and describes the scenario in which in the \(i\)-th round all the points from the antichain \(A_i\) are presented to an on-line coloring algorithm at once, and the on-line
algorithm must assign colors to the points from $A_i$ basing solely on $P[A_1 \cup \ldots \cup A_i]$. The conditions \((R1)-(R2)\) guarantee that such the interpretation is feasible.

The conditions \((R1)-(R3)\) provide that the antichains $A_1, \ldots, A_n$ induce a layer structure in the poset $P$, with the order of layers (antichains) determined by $\sqsubseteq_P$-relation (the order of layers is usually different than the order in which the layers are presented) – see Figure 3.

The conditions \((R4)-(R5)\) impose the restrictions on the relation $\leq_P$ between points that appear in successively presented layers. The antichains $A_{s(i)}$ and $A_{p(i)}$ are the layers that are, respectively, just above and just below the layer $A_i$ at the moment $A_i$ is presented. Note that $A_{s(i)}$ (resp. $A_{p(i)}$) is not defined when $A_i$ is maximum (resp. minimum) at the moment it is presented. In particular, the condition \((R4)\) ensures that at every stage in the presentation of $P$, the bipartite subposets induced by every two $\sqsubseteq_P$-consecutive layers are cores. Finally, the last condition \((R5)\) asserts that at the time $A_i$ is presented, every $<_P$-comparability between a point in $A_i$ and a point from outside $A_i$ must be implied by transitivity on some point from $A_{s(i)}$ or from $A_{p(i)}$.

We color regular posets using First-Fit algorithm that receives the points from every presented antichain in arbitrarily order; hence regular poset can be treated as on-line poset in this case. In fact, the analysis of First-Fit exploits some structural properties of regular posets, which allow us to assume any presentation order of all points.

Denote the class of all regular posets of width $w$ by $R_w$. Lemma 11 was first proved in [4], however we include its slightly simplified proof for the sake of the completeness of the paper.

**Lemma 11.** $\text{val}(P_w) \leq \text{val}(R_1) + \ldots + \text{val}(R_w)$.

**Proof.** We will proceed by induction of $w$. For $w = 1$ we are done, since the family of on-line posets of width 1 coincides with the family of regular posets of width 1. The inductive hypothesis says that there is an on-line algorithm that uses at most $\text{val}(R_1) + \ldots + \text{val}(R_{w-1})$ colors on every on-line poset of width at most $w - 1$. To complete the proof we show an on-line algorithm that uses at most $\text{val}(R_1) + \ldots + \text{val}(R_w)$ colors on every on-line poset of width $w$.

Let $P^\prec = (V, \leq, \prec)$ be an on-line poset of width $w$. Roughly speaking, our algorithm partitions the consecutive points of $P^\prec$ into two sets: $X$ and $V \setminus X$. The points inserted to the set $X$ induce in $P$ the subposet of width at most $w - 1$, and hence they are colored with at most $\text{val}(R_1) + \ldots + \text{val}(R_{w-1})$ colors by the algorithm asserted by the inductive hypothesis. On the other hand, every

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**Figure 2.** Posets $P$ and $Q$ are cores of width 5. $R$ is not a core since $xy$ is not a Dilworth edge.
point not included to \( \bar{X} \) has the property that it is contained in some maximum antichain (of size \( w \)) at the moment it is presented. The algorithm will use the lattice structure of the maximum antichains of \( P \) to convert the maximum antichains containing the points from \( V \setminus \bar{X} \) into a regular poset of width \( w \). Then, colors of the points from \( V \setminus \bar{X} \) will be derived from the on-line coloring of the regular poset, for which the algorithm needs at most \( \text{val}(\mathcal{R}_w) \) additional colors. Concluding, the algorithm uses at most \( \text{val}(\mathcal{R}_1) + \ldots + \text{val}(\mathcal{R}_w) \) colors in total.

During the execution the algorithm maintains an auxiliary structure \((\bar{X}, \mathcal{A}, i)\), initially set to \((\emptyset, \emptyset, 0)\), based on which the colors are assigned to the successive points in \((V, \prec)\). Suppose we are at the moment when a point \( x \) of \((V, \prec)\) is being presented; Algorithm 1 shows how the structure \((\bar{X}, \mathcal{A}, i)\) is modified just after receiving \( x \).

**Algorithm 1:** Updating the structure \((\bar{X}, \mathcal{A}, i)\) after receiving \( x \).

1: if width\((P[\bar{X} + x])\) < \( w \) then
   2: \( \bar{X} = \bar{X} + x \)
   3: else
   4: \( \mathcal{A}_x \) = an antichain of size \( w \) in \( P[\bar{X} + x] \) containing \( x \)
   5: if \( x \in U_P(A) \) for some \( A \in \mathcal{A} \) then
   6: \( \mathcal{A}_d = \max_{\subseteq P}\{A \in \mathcal{A} : x \in U_P(A)\} \)
   7: else
   8: \( \mathcal{A}_d = \mathcal{A}_x \)
   9: end if
10: if \( x \in D_P(A) \) for some \( A \in \mathcal{A} \) then
11: \( \mathcal{A}_u = \min_{\subseteq P}\{A \in \mathcal{A} : x \in D_P(A)\} \)
12: else
13: \( \mathcal{A}_u = \mathcal{A}_x \)
14: end if
15: \( i = i + 1 \)
16: \( x_i = x \)
17: \( \mathcal{A}_i = \mathcal{A}_d \lor (\mathcal{A}_u \land \mathcal{A}_x) \)
18: \( \mathcal{A} = \mathcal{A} \cup \{\mathcal{A}_i\} \)
19: end if
From Algorithm \(1\) it is clear that
\[
\text{width}(P[\bar{X}]) < w
\]
at each stage in the presentation of \(P\). It follows that the on-line poset \((\bar{X}, \leq_p |_{\bar{X}}, \prec_{|\bar{X}})\) can be colored with at most \(\text{val}(R_1) + \ldots + \text{val}(R_{w-1})\) colors by the on-line algorithm asserted by the inductive hypothesis.

To color the points from outside \(\bar{X}\) we need some more work. In particular, the algorithm will use the set \(A\) to define a regular poset of width \(w\), which further will be used to color the points from outside \(\bar{X}\). Assume that the set \(V \setminus \bar{X}\) has exactly \(n\) elements, i.e., \(x_1, \ldots, x_n\) are the successive points of \((V, \prec)\) which did not pass the conditional test in Line \(1\). First, we prove some simple claims.

**Claim 11.A.** For every \(i \in [n]\), the set \(A_i\) is the maximum antichain in \(P\) such that \(x_i \in A_i\). Moreover, the antichains \(A_i\) for \(i \in [n]\) are pairwise different.

**Proof.** The fact that \(A_i\) is the maximum antichain follows directly from the fact that the set of maximum antichains in \(P\), equipped with \(\lor\) and \(\land\) operations, is a lattice. To show that \(x_i \in A_i\), note that \(x \in U_P(A_d)\) if \(x\) passes the conditional test in Line \(5\) and \(x \in D_P(A_u)\) if \(x\) passes the conditional test in Line \(10\) – see Figure 4.

The second part of the claim follows by the fact that for every \(i, j \in [n]\) such that \(j < i\), \(x_i \in A_i\) and \(x_i\) is not an element of \(A_j\).

By a simple induction argument on the size of \(A\) one can verify the following

**Claim 11.B.** The antichains in \(A\) are linearly ordered under \(\sqsubseteq_P\) relation.

![Figure 4. Constructing \(A_i\) based on \(A_x, A_d,\) and \(A_u\).](image)

For every \(i \in [n]\), just after the execution of Algorithm \(1\) for \(x_i\), the algorithm sets
\[
B_i = \{(u, A_i) : u \in A_i\}, \quad U_i = \bigcup_{j=1}^{i} B_j
\]
and extends (initially empty) the relation \(\leq_U\) on the set \(U_i\) so as for every \((u, A_i) \in B_i\) and \((v, A_j) \in B_j\) for \(j < i\) we have
\[
(u, A_i) \leq_U (v, A_j) \iff u \leq_P v \text{ and } A_i \sqsubseteq A_j,
\]
\[
(u, A_i) \geq_U (v, A_j) \iff u \geq_P v \text{ and } A_i \sqsupseteq A_j.
\]
We leave the reader to verify that: \(U = (U_n, \leq_U)\) is a partial order of width \(w\), \(B_i\) is a maximum antichain in \(U\) for \(i \in [n]\), and the sequence \((U, B_1, \ldots, B_n)\) satisfies the conditions \([R1]-[R3]\) from the definition of regular posets.

In order to ensure the properties \([R4]-[R5]\), the algorithm will resign from some comparabilities \(\leq_U\) between the points in \(B_i\) and from \(U_{i-1}\), thus obtaining the new comparability relation \(\leq_R\). Since \((U, B_1, \ldots, B_n)\) satisfies \([R1]-[R3]\) we will use \(B_{s(i)}\) and \(B_{p(i)}\) with the meaning as defined in
the definition of regular posets. For every \( u \in B_i \) the algorithm updates the relation \( \leq_R \) between \( u \) and the points in \( U_i \) so as:

(i) For every \( v \in B_i, u \leq_R v \) iff \( u = v \).

(ii) For every \( u \in B_{si}(i) \), \( u \leq_R v \) iff \( uv \) is a Dilworth edge in \((B_i, B_{si}(i), \leq_U)\).

(iii) For every \( v \in B_p(i), v \leq_R u \) iff \( vuv \) is a Dilworth edge in \((B_p(i), B_i, \leq_U)\).

(iv) There are no other \( \leq_R \)-comparabilities between \( u \) and the points in \( U_{i-1} \) than those implied by transitivity of \( \leq_R \) on the points from \( B_{si}(i) \) or from \( B_p(i) \).

See Figure 5 for an illustration how the relation \( \leq_R \) is derived from \( \leq_U \). Note that \( \leq_R \) between the points in \( B_i \) and in \( U_i \) can be computed basing solely on the relation \( \leq_U \) restricted to \( U_i \). Note also that the bipartite poset \((A_{pi}(i), A_i, \leq_R) \mid [A_i, A_{si}(i), \leq_R] \) if exists has width \( w \), since \((A_{pi}(i), A_i, \leq_U)\) has a Dilworth partition of size \( w \), and the edges of every Dilworth partition of \((A_{pi}(i), A_i, \leq_U)\) are in \( \leq_R \).

![Figure 5. Hasse diagrams of \( U[B_p(i) \cup B_i \cup B_{si}(i)] \) and \( R[B_p(i) \cup B_i \cup B_{si}(i)] \).](image)

Now, we prove two claims that will eventually imply \((U_n, \leq_R), B_1, \ldots, B_n\) is a regular poset of width \( w \). The first claim follows directly from the definition of \( \leq_R \).

**Claim 11.C.** The following statements hold:

1. \( \leq_R \subseteq \leq_U \).
2. For every \( i \in [n] \), the bipartite poset \((B_i, B_{si}(i), \leq_R)\) is a core of width \( w \) provided \( B_{si}(i) \) exists.
3. For every \( i \in [n] \), the bipartite poset \((B_p(i), B_i, \leq_R)\) is a core of width \( w \) provided \( B_p(i) \) exists.

**Claim 11.D.** The relation \( \leq_R \) establishes a partial order in the set \( U_n \).

**Proof.** Clearly, by the definition of \( \leq_R \) it follows that \( \leq_R \) is reflexive and antisymmetric. We need to check that \( \leq_R \) is transitive.

We will prove by induction on \( i \) that the relation \( \leq_R \) restricted to \( U_i \) is transitive, \( i \in [n] \). The claim holds for \( i = 1 \). Suppose the claim holds for all \( j < i \). We show that \( \leq_R \) restricted to \( U_i \) is indeed transitive. Suppose we have \( y, u, z \in U_i \) with \( y \leq_R u \) and \( u \leq_R z \). By the definition of \( \leq_R \) we may focus only on the case when \( u \in B_i \) and \( y, z \in U_{i-1} \). By \([ii][iv]\) there are vertices \( y' \in B_{pi}(i) \) and \( z' \in B_{si}(i) \) so that \( y \leq_R y' \leq_R u \) and \( u \leq_R z' \leq_R z \). Since \( y' \) is a Dilworth edge in \((B_{pi}(i), B_i, \leq_U)\), it follows that there is a Dilworth chain partition of \((B_{pi}(i), B_i, \leq_U)\) in which \( y'u \) are in the same chain. For the same reason, there is a Dilworth chain partition of \((B_i, B_{si}(i), \leq_U)\) in which \( uz' \) are in the same chain. Joining two Dilworth chain partitions of \((B_{pi}(i), B_i, \leq_U)\) and \((B_i, B_{si}(i), \leq_U)\) results in a Dilworth partition of \((B_{pi}(i) \cup B_i \cup B_{si}(i))\) in which \( \{y', u, z'\} \) is a Dilworth chain. So there is a Dilworth partition of \((B_{si}(i), B_{pi}(i), \leq_U)\) in which \( y'z' \) is a Dilworth edge. It follows that \( y' \leq_R z' \) by \([ii][iii]\) and inductive hypothesis, since \( p(i), s(i) < i \) and \( B_{pi}(i) \) and both \( B_{si}(i) \) were \( \leq_U \)-consecutive before \( B_i \) has appeared. Finally, \( y \leq_R z \) follows by inductive hypothesis, since \( y, y', z, z' \in U_{i-1} \).
By Claims 11.C and 11.D we deduce that \((U_n, \leq_R)\) is a partial order of width \(w\). Indeed, by composing Dilworth partitions of successive adjacent cores in \((U_n, \leq_R)\) we obtain a partition of \((U_n, \leq_R)\) into \(w\) chains.

We leave the reader to check that \(((U_n, \leq_R), B_1, \ldots, B_n)\) satisfies (R1)-(R3). The remaining conditions (R4)-(R5) follow directly from the definition of \(\leq_R\).

Finally, we use an on-line algorithm that colors the regular poset \(((U_n, \leq_R), B_1, \ldots, B_n)\) with at most \(\text{val}(R_w)\) colors. Just after the point \(x_i\) has been presented, we set the color of \(x_i\) as the same as the color of \((x_i, A_i)\) from \(B_i\) in the regular poset \(((U_n, \leq_R), B_1, \ldots, B_n)\). In this way the colors of \(\{x_1, \ldots, x_n\}\) induce a chain partition in \(P[\{x_1, \ldots, x_n\}]\) since \((y, A) \leq_R (y', A')\) implies \(y \preceq_P y'\) (by Claim 11.C and the definition of \(\leq_U\)).

Clearly, the presented algorithm is on-line, as the color of every successive point \(x\) from \((V, \prec)\) is determined just after receiving \(x\), solely by the subposet \(P[\{z \in V : z \preceq x\}]\).

\[\square\]

4. First-Fit Coloring of Regular Posets

We split this section into two parts. In the first part we take a look into ladders that appear in regular posets. Precisely, we show that any regular poset of width \(w\) is a member of \(\text{Forb}(L_{2w^2+1})\). In the second part of the section we analyse the First-Fit performance on ladder-free posets. In particular, we enclose a shortened and corrected proof of Lemma 6. The original proof can be found in [22] but with a small mistake in calculations. The section ends with the proof of Theorem 2 which is the essence of our considerations.

For the first part of this section assume that \((P, A_1, \ldots, A_n)\) is a regular poset of width \(w\), \(P = (V, \preceq)\). For any \(x \in P\), let \(A(x)\) denote the unique antichain in \(\{A_1, \ldots, A_n\}\) containing \(x\).

The first proposition collects some additional properties of \((P, A_1, \ldots, A_n)\).

**Proposition 12.** For any \(r, s \in [n]\) so that \(A_r \sqcap_P A_s\) the following holds

(R6) \(P[A_r \cup A_s]\) is a core,

(R7) Let \(t\) be an integer such that \(A_r \sqcap_P A_t \sqcap_P A_s\) and \(A_t\) is the earliest presented antichain among these in \([A_t, A_s]_P\) (in \([A_r, A_t]_P\)). Then, for any \(x \preceq_P y\) with \(x \in A_r\) and \(y \in A_s\) there is \(z \in A_t\) such that \(x \preceq_P z \preceq_P y\).

**Proof.** For a pair of two antichains \(A_i \sqcap_P A_j\) we define \(\text{len}(i, j)\) as the number of antichains \(A_t\) such that \(A_i \sqcap_P A_t \sqcap_P A_j\) and \(t < \max(i, j)\). Note, that \(\text{len}(i, j) = 0\) means that \(s(i) = j\) if \(i > j\) and \(p(j) = i\) if \(i < j\). We prove the claim by induction on \(\text{len}(r, s)\).

For the proof of (R6) observe that the base step for \(\text{len}(r, s) = 0\) holds by (R4). Let \(l = \text{len}(r, s) > 0\) and assume (R6) holds for all pairs \((r', s')\) with \(\text{len}(r', s') < l\). Suppose that \(r > s\); the other case can be proved similarly. It follows that \(s(r)\) is defined, \(s(r) \neq r\) and \(s(r) \neq s\). To prove \(P[A_r \cup A_s]\) is a core take any \(x \in A_r\) and \(y \in A_s\) with \(x \preceq_P y\). By (R5) there is some \(z \in A_{s(r)}\) such that \(x \preceq_P z \preceq_P y\). By the inductive hypothesis, both \(P[A_r \cup A_{s(r)}]\) and \(P[A_{s(r)} \cup A_s]\) are cores, since \(\text{len}(r, s(r)), \text{len}(s(r), s) < l\). Let \(C\) be a Dilworth partition of \(P[A_r \cup A_{s(r)}]\) with \(x\) and \(z\) in the same chain and \(D\) be a Dilworth partition of \(P[A_{s(r)} \cup A_s]\) with \(y\) in the same chain. It is easy to see that \(\{(C \cup D) - (C \cap D) : C \in C, D \in D, C \cap D \neq \emptyset\}\) is a Dilworth partition of \(P[A_r \cup A_s]\) with \(x\) and \(y\) in the same chain. It follows that \(P[A_r \cup A_s]\) is a core, which completes the proof of (R6).

Now we turn our attention to (R7). First, we note that (R7) is trivial for \(t = r\) or \(t = s\). Therefore, the base of the induction for \(\text{len}(r, s) = 0\) is satisfied. Again, let \(\text{len}(r, s) > 0\) and assume (R7) holds for all smaller cases. Suppose \(A_t\) is the earliest presented antichain among these in \([A_t, A_s]_P\). The other case can be handled analogously.

Fix \(x \in A_r\) and \(y \in A_s\) such that \(x \preceq_P y\). We consider two cases: \(r > s\) and \(r < s\). Suppose first that \(r > s\), which means, in particular, that \(s(r)\) exists. Let \(r' = s(r)\). Since \(A_r \sqcap_P A_t \sqcap_P A_s\)
by \( r > s \geq t \) and since \( A_r, A_{s(r)} \) are \( \sqsubseteq_P \) consecutive at the moment \( A_r \) is presented, we have that \( A_r \sqsubseteq_P A_t \). By \([R5]\) there is some \( x' \in A_r \) such that \( x \leq_P x' \leq_P y \). By inductive hypothesis for \( A_r \sqsubseteq_P A_s \), there is \( z \in A_t \) such that \( x' \leq_P z \leq_P y \). Thus \( z \leq_P z \leq_P y \), which proves \([R7]\) for that case. Suppose now that \( r < s \). Note that \( s' = p(s) \) is defined and \( A_t \sqsubseteq_P A_{s'} \) since \( t < s \) and \( A_{s'}, A_s \) are \( \sqsubseteq_P \) consecutive at the moment \( A_s \) is presented. By \([R5]\) there is some \( y' \in A_{s'} \) such that \( x \leq_P y' \leq_P y \). Observe that \( A_t \) is still the earliest presented antichain among these in \([A_t, A_{s'})P\). Therefore, by inductive hypothesis for \( A_r \sqsubseteq_P A_{s'} \), there is \( z \in A_t \) such that \( x \leq_P z \leq_P y' \). It follows that \( x \leq_P z \leq_P y \), which completes the proof of \([R7]\). □

Now we show that ladders appearing in the regular poset \((P, A_1, \ldots, A_n)\) cannot have to many rungs. We split this task into two parts: first we prove an upper bound of the size of a so-called canonical ladder in \((P, A_1, \ldots, A_n)\), then we use it to provide an upper bound for the size of any ladder in \((P, A_1, \ldots, A_n)\).

A set \( L \subseteq V \) induces a canonical \( m \)-ladder in \((P, A_1, \ldots, A_n)\) if \( P[L] \) is an \( m \)-ladder such that \( A(y_i) \sqsubseteq_P A(x_{i+1}) \) for each \( i \in [m - 1] \), where \( x_iy_i \) is the \( i \)-th rung of \( P[L] \) for \( i \in [m] \).

**Proposition 13.** If \( L \subseteq V \) induces a canonical \( m \)-ladder in \( P \), then \( m \leq w \).

**Proof.** Let \( x_1, x_2, \ldots, x_m \) and \( y_1, y_2, \ldots, y_m \) be the lower and the upper leg of \( L \), respectively (in particular, \( y_i \parallel_P x_i \) for \( i \in [2, m] \)). We show, by induction on \( i \), that the existence of \( L \) in \( P \) yields

\[
|U_P[y_i] \cap A(y_i)| \geq i \quad \text{for} \quad i \in [m].
\]

Clearly, from \([4]\) it follows that \( m \leq w \).

For \( i = 1 \), we have \( U_P[y_1] \cap A(y_1) = \{y_1\} \) and \([4]\) holds. Now, fix \( 1 < i \leq m \) and assume that \([4]\) holds for \( i - 1 \). As \( L \) is a canonical ladder, we have \( A(y_{i-1}) \subseteq_P A(x_i) \). By \([R6] \) \( P[A(y_{i-1}) \cup A(x_i)] \) is a core, and hence there is \( z \in A(y_{i-1}) \) such that \( z \leq_P x_i \). Note that we must have \( z \parallel_P y_i \). Indeed, \( y_i \leq_P z \) implies \( y_i \leq_P z \leq_P x_i \), contradicting \( y_i \parallel_P x_i \), and \( z \leq_P y_i \) implies \( z < P \leq_P y_{i-1} \), contradicting \( z \in A(y_{i-1}) \). Let \( S = U_P[y_i] \cap A(y_{i-1}) \). By the inductive hypothesis, we have \( |S| \geq i - 1 \). By \([R6] \) \( P[A(y_{i-1}) \cup A(y_i)] \) is a core with a Dilworth edge \( zy_i \), where the latter fact follows by \( z \leq_P x_i \leq_P y_i \). Let \( C \) be a Dilworth partition of \( P[A(y_{i-1}) \cup A(y_i)] \) with \( z \) and \( y_i \) in the same chain. Each vertex of \( S \) is matched in \( C \) to a distinct vertex of \( A(y_i) \), different than \( y_i \) (see Figure 6) as \( z \notin S \). Consequently, \( |U_P[y_i] \cap A(y_i)| \geq |S| + 1 \geq i - 1 + 1 = i \). This proves \([4]\).

![Figure 6](image)

**Figure 6.** The intersections of \( U_P[y_1] \) with \( A(y_{i-1}) \) and \( A(y_i) \).

□

In the next section, in Lemma 18 we will show that not all ladders in regular posets are canonical. However, the existence of ladders of large size in \((P, A_1, \ldots, A_n)\) implies the existence of canonical ladders of large size in \((P, A_1, \ldots, A_n)\), which cannot be too large by the previous proposition. To show this we need one proposition.
Proposition 14. Assume $L \subseteq V$ induces a $(2w+1)$-ladder in $P$ with the lower leg $x_1, x_2, \ldots, x_{2w+1}$ and the upper leg $y_1, y_2, \ldots, y_{2w+1}$. Then $A(y_1) \subseteq_P A(x_{2w+1})$.

Proof. Assume to the contrary that $A(x_{2w+1}) \nsubseteq_P A(y_1)$. It follows that $A(x_1) \subseteq_P A(x_2) \subseteq_P \ldots \subseteq_P A(x_{2w+1}) \subseteq_P A(y_1) \subseteq_P A(x_{2w+1})$.

Let $t \in [n]$ be the least integer so that $A(x_{w+1}) \subseteq_P A_t \subseteq_P A(y_{w+1})$, i.e., $A_t$ is the earliest presented antichain among these in $[A(x_{w+1}), A(y_{w+1})]_P$. By the linear order under $\subseteq_P$, we must have $A(x_{2w+1}) \nsubseteq_P A_t$ or $A_t \subseteq_P A(y_1)$.

Assume $A_t \subseteq_P A(y_1)$, see Figure 7. Consider the comparabilities $x_i \leq_P y_i$ for $i \in [w+1]$. Observe that $A_t$ is still the earliest presented antichain among these in $[A_t, A(y_1)]_P$. By the choice of $t$, we can apply (R7) to obtain $z_i \in A_t$ such that $x_i \leq_P z_i \leq_P y_i$. Take $i, j \in [w+1]$ such that $i < j$. If $z_i = z_j$, then we get $x_j \leq_P y_i$ by transitivity on $z_i$. However, it is not possible as $L$ is a ladder. It follows that all $z_i$ for $i \in [w+1]$ are distinct, but this can not be the case as $|A_t| = w$.

The case $A(x_{2w+1}) \nsubseteq A(t)$ can be handled similarly. 

We end the first part of this section with the following lemma.

Lemma 15. If $(P, A_1, \ldots, A_n)$ is a regular poset of width $w$, then $P \in \text{Forb}(L_{2w^2+1})$.

Proof. Assume $L_{2w^2+1}$ is a ladder in $P$ with the lower leg $x_1, x_2, \ldots, x_{2w^2+1}$, and the upper leg $y_1, y_2, \ldots, y_{2w^2+1}$. By Proposition 13 for any $i, j \in [2w^2+1]$ with $j - i \geq 2w$ we must have $A(y_i) \subseteq_P A(x_j)$. Thus, the subposet induced by the vertices

$$\bigcup_{0 \leq i \leq w} \{x_{2wi+1}, y_{2wi+1}\}$$

is a canonical ladder with $w + 1$ rungs, which contradicts Proposition 13. 

\[\square\]
Now that we have an upper bound on the number of rungs in a ladder in a regular poset, we will examine the performance of First-Fit in posets in the family $\text{Forb}(L_m)$.

We start with some preliminaries. Assume that $P = (V, \leq_P)$ is a poset of width $w$ such that $P \in \text{Forb}(L_m)$. Let $g : P \rightarrow [n]$ be an $n$-Grundy coloring of $P = (V, \leq_P)$, let $P_i = \{ x \in V : g(x) = i \}$ for $i \in [n]$. Given $P$ and $g$, we draw the Hasse diagram of $P$ in such a way that all the points from $P_i$ are on a vertical line and the line containing the points from $P_j$ is to the left of the line containing the points from $P_i$ for $i < j$. Having this in mind, we say a sequence $x_1, \ldots, x_k$ of elements from $V$ is an ascending chain under $P$ and $g$ (see Figure 8) if

$$x_1 <_P x_2 <_P \ldots <_P x_k \text{ and } g(x_1) < g(x_2) < \ldots < g(x_k).$$

Similarly, a sequence $x_1, \ldots, x_k$ of elements from $V$ is a descending chain under $P$ and $g$ if

$$x_1 >_P x_2 >_P \ldots >_P x_k \text{ and } g(x_1) > g(x_2) > \ldots > g(x_k).$$

The following assertion follows from the famous Erdős-Szekeres Theorem.

**Proposition 16.** Let $s, t \in \mathbb{N}$ and let $C$ be a chain in $P$. If the length of every ascending sequence of elements from $C$ is at most $s$, and the length of every descending sequence of elements from $C$ is at most $t$, then $|C| \leq s \cdot t$.

The next proposition exploits the fact that the poset $P$ is in $\text{Forb}(L_m)$ (see Figure 8).

**Proposition 17.** Let $x_1, \ldots, x_k$ be an ascending (a descending) chain in $P$ and $y_1, \ldots, y_{k-1}$ be a sequence of elements from $V$ such that $y_i$ is a $g(x_i)$-witness of $x_{i+1}$ for $i \in [k-1]$. Then the following holds:

(C1) $x_i <_P y_i$ (resp. $x_i >_P y_i$) for $i \in [k-1]$.

(C2) $y_i \not<_P x_j$ (resp. $y_i \not>_P x_j$) for all $i, j$ such that $1 \leq i < j \leq k$.

(C3) If $y_i \|$ $x_j$ for all $i, j$ such that $1 \leq i < j \leq k$, then $k \leq m(w-1)$.

**Proof.** We consider only the case when $x_1, \ldots, x_k$ is the ascending chain; the remaining one can be proved analogically.

To show (C1) observe that $x_i <_P x_{i+1} \parallel_P y_i$. But, since $x_i$ and $y_i$ are comparable under $\leq_P$, the only possibility is $x_i <_P y_i$ – see Figure 8. To prove (C2) note that $y_i >_P x_j$ implies $y_i >_P x_j >_P x_{i+1}$, which contradicts $y_i$ is a witness for $x_{i+1}$.

To show (C3) assume to the contrary that $k > m(w-1)$. It implies $k \geq (m-1)(w-1) + 2$ as $m \geq 2$. Note that the subposet of $P$ induced by the elements $y_1, \ldots, y_{k-1}$ has width at most $w - 1$ as $y_i \parallel x_k$ for $i \in [k-1]$ by the assumption of (C3). Since $P\{y_1, \ldots, y_{k-1}\}$ has at least $(m-1)(w-1) + 1$ elements, by Dilworth Theorem it follows there is a subsequence $y_{i_1}, \ldots, y_{i_m}$ of $y_1, \ldots, y_k$ that induces a chain in $P$. We show that $y_{i_1} \leq_P \ldots \leq_P y_{i_m}$, which by (C1) and the assumption of (C3) implies $\{x_{i_1}, y_{i_1}, \ldots, x_{i_m}, y_{i_m}\}$ induces an $m$ ladder in $P$ with $y_{i_j} x_{i_j}$ being its $j$-th rung. However, this is a contradiction, as $P \in \text{Forb}(L_m)$. To show $y_{i_j} <_P y_{i_t}$ for $i_j < i_t$, note that $y_{i_j} >_P y_{i_t}$ yields $y_{i_j} >_P x_{i_j}$ by $y_{i_j} >_P x_{i_t}$, which contradicts (C2).

Lemma 6 was first shown in [22], however we incorporate it for the sake of the completeness of the paper.

**Lemma 6.** For $m, w \in \mathbb{Z}^+$, we have $\text{val}_{FF}(L_m, w) \leq w^{2.5 \lg(2w)+2 \lg m}$.

**Proof.** We argue by induction on $w = \text{width}(P)$. The base step $w = 1$ is trivial. Fix $w$ and assume the lemma holds for all smaller values of $w$.

Let $P = (V, \leq_P)$ be a poset of width $w$ such that $P \in \text{Forb}(L_m)$ and $g : V \rightarrow [n]$ be an $n$-Grundy coloring of $P$ with $n = \text{val}_{FF}(L_m, w)$. To complete the proof we must show that

$$n \leq w^{2.5 \lg(2w)+2 \lg m}.$$
Let $A$ be the set of maximum antichains in $P$. Select $A \in A$ so that

$$\min_{a \in A} g(a) = \max \min_{b \in B} g(b).$$

In other words, $A$ is a maximum antichain so that the smallest color of a vertex in $A$ is as large as possible. Set

$$N = \min_{a \in A} g(a).$$

By our choice of $A$, the subposet of $P$ induced by the vertices in $P_{N+1} \cup P_{N+2} \cup \cdots \cup P_n$ has width at most $w - 1$. By the inductive hypothesis, at most $\text{val}_F(L_m, w - 1)$ colors appear on this subposet. Hence,

$$n \leq N + \text{val}_F(L_m, w - 1).$$

Consider any $i \in [N - 1]$ and let $x$ be the greatest vertex of the chain $P_i$. Since $A$ is a maximum antichain, $x$ is comparable to some $a \in A$. If $x <_P a$, then $P_i <_P a$. Consequently, $a$ has no $i$-witness under $g$, which cannot be the case as $g$ is a Grundy coloring of $P$. It follows that $x$ is greater than some vertex of $A$. Similarly, the least vertex of $P_i$ is less than some vertex of $A$. We conclude

$$U_P(A) \cap P_i \neq \emptyset \text{ and } D_P(A) \cap P_i \neq \emptyset.$$ 

For $i \in [N - 1]$, define $q_i^\downarrow$ to be the greatest $q \in P_i \cap D_P(A)$ and $q_i^\uparrow$ to be the smallest $q \in P_i \cap U_P(A)$. They exist by (5), and satisfy

$$q_i^\downarrow <_P q_i^\uparrow \text{ and } [q_i^\downarrow, q_i^\uparrow]_P \cap P_i = \{q_i^\downarrow, q_i^\uparrow\},$$

i.e., $q_i^\downarrow$ and $q_i^\uparrow$ are consecutive in $P_i$. Indeed, note that $q_i^\downarrow \neq q_i^\uparrow$ and $q_i^\downarrow <_P q_i^\uparrow$ would yield a comparability between two different points from $A$. Moreover, $[q_i^\downarrow, q_i^\uparrow]_P \cap P_i \neq \{q_i^\downarrow, q_i^\uparrow\}$ would contradict $A$ is a maximum antichain. Note also that

$$A \subseteq I(q_i^\downarrow) \cup I(q_i^\uparrow),$$

which is equivalent that any $a \in A$ is incomparable with either $q_i^\downarrow$ or $q_i^\uparrow$. Indeed, $q_i^\downarrow \leq_P a \leq_P q_i^\uparrow$ would imply that $a$ is comparable with all elements in $P_i$ by (7). Consequently, $a$ has no $i$-witness under $g$, which is impossible as $g$ is a Grundy coloring of $P$.

We say a vertex

$$x \in P \text{ has property } (*) \text{ if } |I(x) \cap A| \geq w/2.$$
By \([5]\) and by the pigeonhole principle, at least one of \(q_i^\downarrow\) or \(q_i^\uparrow\) has property \((*)\); we select one of them, denote it by \(q_i\), and call it a near witness for color \(i\). If \(q_i \in D_P(A)\) then let \(r_i\) be the smallest vertex of the chain \(P_i\) with property \((*)\); otherwise (when \(q_i \in U_P(A)\)) let \(r_i\) be the greatest vertex of \(P_i\) with property \((*)\). We call \(r_i\) a far witness for color \(i\), and the pair \((q_i, r_i)\) a corresponding pair for color \(i\). Note that it is possible that \(q_i = r_i\).

Let \(R = \{r_1, r_2, \ldots, r_{N-1}\}\). The next claim gives an upper bound on the number of elements in the set \(R\).

**Claim 17A.** \(|R| \leq w^2m^2(w - 1)^2 \text{val}_{FF}(L_m, \lfloor w/2 \rfloor)\).

*Proof.* Since \(A\) is a maximum antichain in \(P\), disjoint with \(R\), the sets \(R \cap U_P(A)\) and \(R \cap D_P(A)\) partition the set \(R\). By Dilworth’s Theorem and by the pigeonhole principle, there is a chain \(S\) in \(R \cap U_P(A)\) such that

\[
|R \cap U_P(A)| \leq |S| \cdot w. 
\]

We bound the size of \(S\) by showing \(S\) can not contain large monotonic sequences and then by applying the Erdős-Szekeres theorem. Precisely, we will show that

- \((T1)\) the size of any ascending sequence in \(S\) is at most \(m(w - 1)\),
- \((T2)\) the size of any descending sequence in \(S\) is at most \(\frac{w}{2}m(w - 1)\text{val}_{FF}(L_m, \lfloor w/2 \rfloor)\).

Applying \((T1)\) and \((T2)\) to Proposition \([16]\) we get

\[
|S| \leq \frac{w}{2}m^2(w - 1)^2 \text{val}_{FF}(L_m, \lfloor w/2 \rfloor),
\]

which plugged into \([9]\) yields

\[
|R \cap U_P(A)| \leq \frac{w^2}{2}m^2(w - 1)^2 \text{val}_{FF}(L_m, \lfloor w/2 \rfloor).
\]

Using the same bound on \(|R \cap D_P(A)|\) we get our claim. What remains is to show \((T1)\) and \((T2)\).

To show \((T1)\) let \(x_1, \ldots, x_k\) be any ascending sequence in \(S\). Let \(y_i\) be a \(g(x_i)\)-witness for the vertex \(x_{i+1}\) for \(i \in \{k - 1\}\). Note that now we are in a position to apply Proposition \([17]\) for the sequences \(x_1, \ldots, x_k\) and \(y_1, \ldots, y_{k-1}\). In particular, we will show that \(y_i \parallel_P x_j\) for all \(i < j\), which will yield \(k \leq m(w - 1)\) by \((C3)\). Since \(y_i >_P x_j\) is not possible by \((C2)\) we assume \(y_i <_P x_j\). Then \(D_P(y_i) \cap A \subseteq D_P(x_j) \cap A\). Since \(x_i\) is a far witness for color \(g(x_i)\), and \(y_i\) is greater than \(x_i\) in the chain \(P_{g(x_i)}\) by \((C1)\), it follows that \(y_i\) does not satisfy \((*)\) property. It means that \(|D_P(y_i) \cap A| > \frac{w}{2}\), and consequently \(|D_P(x_j) \cap A| > \frac{w}{2}\). So \(x_j\) does not satisfy \((*)\) property, which is a contradiction as \(x_j\) is a far witness for color \(g(x_j)\) satisfies \((*)\) property.

To show \((T2)\) we need some more work. Let \(z_1, \ldots, z_k\) be a descending sequence in \(S\). Let \(P'\) be the subposet of \(P\) induced by \(D_P[z_1] \cap U_P[A]\). We show that

\[
|P'| = \text{the width of } P' \text{ is at most } w/2.
\]

Let \(B\) be a maximum antichain in \(P'\) and let \(A' = D_P[B] \cap A\). Because \(D_P[B] \subseteq D_P[z_1]\) and \(z_1\) has property \((*)\), we must have \(|A'| \leq w/2\). If \(|A'| < |B|\), then the antichain \((A \setminus A') \cup B\) has more than \(w\) vertices, which is impossible as \(\text{width}(P') = w\). So we have \(\text{width}(P') = |B| \leq |A'| \leq w/2\).

For each \(i \in \{k\}\), let \((w_i, z_i)\) be the corresponding pair for color \(g(z_i)\). Note that \(w_i <_P z_i \leq_P z_1\) for \(i \in \{k\}\). By Dilworth’s Theorem, there is a chain \(T\) in \(\{w_1, \ldots, w_k\}\) such that

\[
k \leq \frac{w}{2} \cdot |T|.
\]

We show that

- \((T3)\) the size of any descending sequence in \(T\) is at most \(m(w - 1)\),
- \((T4)\) the size of any ascending sequence in \(T\) is at most \(\text{val}_{FF}(L_m, \lfloor w/2 \rfloor)\).
Let $g$ be a $g(s_i)$-witness of $s_{i+1}$ for $i \in [l-1]$. Now we are at a position to apply Proposition $17$ for the sequences $s_1, \ldots, s_l$ and $t_1, \ldots, t_{l-1}$. We show that $t_i \parallel_P s_j$ for all $i, j$ such that $1 \leq i < j \leq l$, which by $(C3)$ gives $l \leq m(w-1)$. Again, $t_i \parallel_P s_j$ is not possible by $(C2)$. Assume that $s_j \parallel_P t_i$. Since $s_i$ is a near witness for color $g(s_i)$, and $t_i \parallel_P s_i$, we have $t_i \in D_P(A)$. By $s_j \parallel_P t_i$, we get $s_j \in D_P(A)$, which is a contradiction with $s_j \in U_P(A) \cap D_P(z_1)$.

To show $(T3)$ let $u_1, \ldots, u_l$ be an ascending sequence in $T$, and let $(u_i, v_i)$ be a corresponding pair for color $g(u_i)$ for $i \in [l]$. Since $u_i$ is the near witness and $v_i$ is the far witness for color $g(u_i)$ and $u_i, v_i \in U_P(A)$, we have $u_i \leq_P v_i$. Consequently,

\begin{equation}
\begin{aligned}
    u_1 \parallel_P u_2 \parallel_P \cdots \parallel_P u_l \leq_P v_l \parallel_P v_{l-1} \parallel_P \cdots \parallel_P v_1.
\end{aligned}
\end{equation}

Let $U_i = [u_i, v_i] \cap P_{g(u_i)}$ and $U = \bigcup_{i=1}^l U_i$. Let $Q$ be the subposet of $P'$ induced by the points from the set $U$ and let $g' : U \rightarrow [l]$ be a function defined:

$$g'(x) = i \text{ iff } x \in U_i$$

Now it is an easy exercise to check that $g'$ is an $l$-Grundy coloring of $Q$. Since $Q$ is $L_m$-free as a subposet of $P$, we get $l \leq \text{val}_P(L_m, \lfloor \frac{w}{2} \rfloor)$.

Using Claim $17.A$ and the equation $5$ with $n = \text{val}_P(L_m, w)$, we have

$$\text{val}_P(L_m, w) \leq m^2 w^4 \text{ val}_P \left( L_m, \left\lfloor \frac{w}{2} \right\rfloor \right) + \text{val}_P(L_m, w-1).$$

Applying this recursion repeatedly to the second term, with $\text{val}_P(L_m, 1) = 1$, we obtain

$$\text{val}_P(L_m, w) \leq 1 + \sum_{2 \leq k \leq w} m^2 k^4 \text{ val}_P \left( L_m, \left\lfloor \frac{k}{2} \right\rfloor \right) \leq w m^2 w^4 \text{ val}_P \left( L_m, \left\lfloor \frac{w}{2} \right\rfloor \right).$$

From this it is quite easy to see (use induction to verify) that

$$\text{val}_P(L_m, w) \leq m^{2 \lg w} w^{2.5 \lg(2w) + 2 \lg m}.$$ 

Hence $\text{val}_P(L_m, w) \leq w^{2.5 \lg(2w) + 2 \lg m}$. \hfill $\square$

The proof of our main theorem (we restate the statement below) is now a matter of collecting our results.

**Theorem 2.** $\text{val}(P_w) \leq w^{6.5 \lg w + O(1)}$.

**Proof of Theorem 2.** Every regular poset of width $w$ is in $\text{Forb}(L_{2w^2+1})$ by Lemma $15$ and thus by the previous lemma is colored with at most $w^{2.5 \lg(2w) + 2 \lg(2w^2+1)} = w^{6.5 \lg w + O(1)}$ colors by First-Fit. The algorithm asserted by Lemma $11$, together with First-Fit coloring of regular posets, colors every on-line poset of width $w$ with at most $w \cdot w^{6.5 \lg w + O(1)} = w^{6.5 \lg w + O(1)}$.
colors.

5. Limitations of Our Methods

Loosely speaking, two major parts of the proof of our main theorem rely on limiting the number of rungs in a ladder within a regular poset and the performance of First-Fit on the family Forb($L_m$). Here, we will show that our general upper bound for the on-line coloring problem cannot be greatly improved with our current methods.

In the first part of the section we show that the assertion of Lemma 15 cannot be improved. Although $L_{2w^2+1}$ is not a subposet of any width $w$ regular poset, we show that there are regular posets of width $w$ that contain ladders whose number of rungs is quadratic in $w$.

**Lemma 18.** For each integer $w \geq 2$, there is a regular poset $(P, A)$ so that width$(P) = w$ and $P$ contains $L_{w\lfloor (w+2)/2 \rfloor}$ as an induced subposet.

**Proof.** To simplify the description of $P$ we assume that the elements of every antichain $A$ in $A$ is enumerated with successive numbers from $[w]$. It allows us to refer uniquely to elements of $A$ using the term the $i$-th element of $A$ for $i \in [w]$. Additionally, in our figures we draw the $i$-th element of $A$ to the left of the $j$-th element of $A$ for $i < j$.

Consider two antichains $A = \{u_1, \ldots, u_w\}$ and $B = \{v_1, \ldots, v_w\}$, where $u_i$ and $v_i$ are the $i$-th elements of $A$ and $B$, respectively. We say $(A, B, \leq)$ is a core of:

- **type $I$** if for all $i, j \in [w]$
  
  \[ u_i \leq v_j \text{ iff } i = j, \]

- **type $S_k$** for $k \in [w]$ if for all $i, j \in [w]$
  
  \[ u_i \leq v_j \text{ iff } (i = 1 \text{ and } j \in [k]) \text{ or } (i \in [2, k] \text{ and } j \in [i - 1, i]), \]

- **type $T_k$** for $k \in [w]$ if for all $i, j \in [w]$
  
  \[ u_i \leq v_j \text{ iff } (i = w \text{ and } j \in [n - k + 1]) \text{ or } (i \in [n - k] \text{ and } j \in [i - 1, i]). \]

It is straightforward to verify that bipartite posets of types $I$, $S_k$ and $T_k$ are cores. See Figure 9 for examples.

**Figure 9.** Hasse diagrams of $I$, $S_6$, and $T_3$ for $w = 6$. 17
Now we construct an auxiliary regular poset \((Q, A_1, \ldots, A_{2w+1})\), based on which the regular poset \((P, A)\) is built. At every step in the presentation of \((Q, A_1, \ldots, A_{2w+1})\), every two \(\sqsubseteq_Q\)-consecutive antichains induce a core, one of type: \(I, S_k\) or \(T_k\) for \(k \in [w]\).

Recall that in regular posets the antichains \(A_{s(i)}\) and \(A_{p(i)}\), if exist, denote the antichains that are respectively just above and just below \(A_i\) at the moment \(A_i\) is presented. To define the relation \(\leq_Q\) in \(Q\) we need only to determine the relation \(\leq_Q\) between \(A_i\) and \(A_{s(i)}\) and between \(A_{p(i)}\) and \(A_i\) at the moment \(A_i\) is presented; the other comparabilities will follow by transitivity – see \([R5]\).

Below are the rules how \(\leq_Q\) is determined for the successively presented antichains \(A_1, \ldots, A_{2w+1}\).

1. \(A_2\) is set so that
   - \(s(2) = 1\) and \((A_2, A_1, \leq_Q)\) is of type \(S_w\),
   - \(p(2)\) is not defined.

2. For \(i \in [3, w + 1]\) the antichain \(A_i\) is set so that
   - \(s(i) = 1\) and \((A_i, A_{i-1}, \leq_Q)\) is of type \(S_{w-i+2}\),
   - \(p(i) = i - 1\) and \((A_{i-1}, A_i, \leq_Q)\) is of type \(I\).

3. \(A_{w+2}\) is set so that
   - \(s(w + 2)\) is not defined,
   - \(p(w + 2) = 1\) and \((A_1, A_{w+2}, \leq_Q)\) is of type \(T_w\).

4. For \(i \in [w+3, 2w + 1]\) the antichain \(A_i\) is set so that
   - \(s(i) = i - 1\) and \((A_i, A_{i-1}, \leq_Q)\) is of type \(I\),
   - \(p(i) = 1\) and \((A_1, A_i, \leq_Q)\) is of type \(T_{2w-i+2}\).

The above rules imply the following relations between the antichains \(A_1, \ldots, A_{2w+1}\) in the poset \(Q\):

\[A_2 \sqsubseteq_Q A_3 \sqsubseteq_Q \ldots \sqsubseteq_Q A_{w+1} \sqsubseteq_Q A_1 \sqsubseteq_Q A_{w+2} \sqsubseteq_Q A_{2w} \sqsubseteq_Q A_{2w+1} \sqsubseteq_Q A_{w+2} \sqsubseteq_Q \ldots \sqsubseteq_Q A_1\]

Although it is tedious to verify that \((Q, A_1, \ldots, A_{2w+1})\) is indeed a width \(w\) regular poset, it is straightforward and we leave it to the reader.

Let \(V = \bigcup_{i=1}^{2w+1} A_i\), \(\sqsubseteq = A_2\), \(\top = A_{w+2}\). For every \(i \in [w]\) we denote by:

- \(x_i\) – the first point in \(A_{i+1}\),
- \(y_i\) – the \(w\)-th point in \(A_{2w+2-i}\),
- \(b_i\) – the \(i\)-th point in \(\sqsubseteq\),
- \(t_i\) – the \(i\)-th point in \(\top\),

and finally we let \(X = \{x_1, \ldots, x_w\}\) and \(Y = \{y_1, \ldots, y_w\}\). By inspection we may easily check the following properties of \(Q\).

\((P1)\) \(x_1 <_Q \ldots <_Q x_w\) and \(y_1 <_Q \ldots <_Q y_w\).

Moreover, for any \(i, j \in [w]\):

\((P2)\) If \(i \leq j\) then \(x_i <_Q y_j\), otherwise \(x_i \parallel_Q y_j\).

\((P3)\) If \(j < i \leq j + 2\) or \(i = 1\) or \(j = w\) then \(b_i \leq_Q t_j\), otherwise \(b_i \parallel_Q t_j\).

\((P4)\) If \(j = w\) then \(y_i <_Q t_j\), otherwise \(y_i \parallel_Q t_j\).

\((P5)\) If \(i = 1\) then \(b_i <_Q x_j\), otherwise \(b_i \parallel_Q x_j\).

Now, we are ready to describe the regular poset \((P, A)\). The poset \(P\) will consists of \(h = \lfloor(w + 2)/2\rfloor\) copies of \(Q\). We will use the same variable names to denote elements (sets) in the copies of \(Q\) in \(P\) as those introduced for \(Q\); however, we add the superscript \(i\) to specify that a variable refers to an element (a set) from the \(i\)-th copy of \(Q\). Formally, the poset \(P = (V, \leq_P)\) is defined such that \(V = \bigcup_{i=1}^{h} V^i\) and \(\leq_P\) is the transitive closure of

\[(\leq_{Q1} \cup \ldots \cup \leq_{Qn}) \cup \{(t^i_j, b^i_{j+1}) : i \in [w], j \in [h-1]\}.

Finally, the presentation order \(A\) of \(P\) is set so as:
(i) the antichains from the $i$-th copy of $Q$ are presented before the antichains from the $j$-th copy of $Q$ for $i < j$,
(ii) and the order of the antichains within every copy of $Q$ remains the same.

Again, checking that $(P, A)$ is a regular poset of width $w$ is straightforward; an example of $(P, A)$ is shown in Figure 10.

**Figure 10.** The construction of $P$ for $w = 5$.

To finish the proof of the lemma we show that

$$(13) \quad \text{the set } \bigcup_{j=1}^{h} (X^j \cup Y^j) \text{ induces an } (w \cdot h)\text{-ladder in } P,$$

with $x^j_i y^j_i$ being its $((j - 1)h + i)$-th rung. Clearly, we have

$$(14) \quad X^1 \prec_P \ldots \prec_P X^h \text{ and } Y^1 \prec_P \ldots \prec_P Y^h$$

by the definition of $\le_P$. Finally, we will show that for all $i, j \in [h]$:

$$(15) \quad X^i \prec_P Y^j \text{ if } i < j \text{ and } X^i \parallel_Q Y^j \text{ if } i > j.$$

Note that the case when $i = j$ is handled by [P2]. Clearly, by [P1], (14), (15), and [P2] we deduce (13). What remains is to prove (15). Assume that $i < j$. Clearly, by (P2) it follows that $X^i$ is less than the greatest element in $Y^i$. Consequently, $X^i \prec_P Y^j$ by (14). Assume $i > j$. We
consider only the case $i = h$ and $j = 1$; the remaining ones are even easier to prove. First note that every comparability between a point in $Y_1$ and a point in $X_h$ needs to be implied by transitivity on some point from $\bot^h$. Note that $D_P(X_h) \cap \bot^h$ contains only the first element of $\bot^h$ by \[(P5)\] By \[(P3)\] and \[(P4)\] note that the set $U_P(Y_1) \cap A_i^j$ contains exactly $2i - 3$ last elements in $\bot^i$ for $i \in [2, h]$. Plugging $h = \lfloor (w + 2)/2 \rfloor$ to the last observation we get $U_P(Y_1) \cap \bot^h$ contains not more than $2\lfloor (w + 2)/2 \rfloor - 3 < w - 1$ last elements from $\bot^h$. In particular, $U_P(Y_1) \cap \bot^h$ does not contain the first element of $\bot^h$. It follows that $X^h \parallel P \ Y^1$.

\[\square\]

In the last part of this section we give the lower bound from Lemma \[\square\]. For the upcoming construction we remind the definition of the lexicographical product of two posets. For posets $P$ and $Q$, the lexicographical product $P \cdot Q$ is the poset with vertices $\{(p, q) : p \in P, q \in Q\}$ and order $\leq_{P \cdot Q}$, where

\[(p_1, q_1) \leq_{P \cdot Q} (p_2, q_2) \text{ if either } p_1 <_P p_2 \text{ or } (p_1 = p_2 \text{ and } q_1 \leq_Q q_2).\]

Informally, we may think of $P \cdot Q$ as the poset $P$ where each vertex has been “inflated” to a copy of $Q$. It is well known that

\begin{equation}
\text{width}(P \cdot Q) = \text{width}(P) \cdot \text{width}(Q).
\end{equation}

The following two simple properties (we left the proof for the reader) are the key in the proof of Lemma \[\square\]. For $p, r \in P$ and $u, v, s \in Q$ we have:

\begin{align*}
(17) & \quad \text{If } ((p, u) \leq_{P \cdot Q} (r, s)) \text{ or } (p, u) \geq_{P \cdot Q} (r, s)) \text{ and } (r, s) \parallel_{P \cdot Q} (p, v), \text{ then } p = r. \\
(18) & \quad \text{If } (p, u) \leq_{P \cdot Q} (r, s) \leq_{P \cdot Q} (p, v), \text{ then } p = r.
\end{align*}

Now, we restate the statement and provide the proof of

**Lemma 7.** For $m, w \in \mathbb{Z}^+$ with $m > 1$, we have $w^{1g(m-1)/(m-1)} \leq \text{val}_{FF}(L_m, w)$.

**Proof.** Fix $m \in \mathbb{Z}^+$ with $m > 1$. Let $R$ be the poset $R_{m-1}$ as defined in the proof of Lemma \[\square\]. For technical reasons we would like $R$ to have the least and the greatest element. Vertex $x_{i}^m$ is already the greatest in $R$, but there is no least element in $R$. Therefore we extend $R$ to $P$ by adding a new element $\hat{0}$ which is below entire $R$. The greatest element in $P$ is still $x_{1}^{m-1}$, which we denote by $\hat{1}$.

It is a simple exercise to see that $P$ also satisfies the statement of Lemma \[\square\] i.e., $\text{width}(P) = 2$ and $\chi_{FF}(P) \geq \chi_{FF}(R) \geq m - 1$. As $R$ is an induced subposet of $P$ we have $I_P(\hat{0}) = \emptyset$ and $|I_P(x_i^k)| = k < m - 1$ for $1 \leq i \leq k < m - 1$ and $|I_P(x_i^{m-1})| = i - 1 < m - 1$ for $i \in [m - 1]$. Observe that in a ladder $L_m$, the lowest vertex of the upper leg is always incomparable to $m - 1$ vertices. Hence, there is no vertex in $P$ that can serve as the lowest vertex of the upper leg of an $m$-ladder and thus

\begin{equation}
P \in \text{Forb}(L_m).
\end{equation}

We are prepared to build a poset $Q_k \in \text{Forb}(L_m)$ with $n$-Grundy coloring so that $\text{width}(Q_k) = 2^k$ and $n \geq (m - 1)^k$. Poset $Q_k$ is defined by the following rules:

- (Q1) $Q_0$ is a single vertex $z$.
- (Q2) $Q_{k+1} = P \cdot Q_k$.

Note that $Q_1$ and $P$ are isomorphic and so we will treat $Q_1$ as $P$. The next two properties are the consequence of the definition of $Q_k$, equation (16) and the fact that $P$ has the least and the greatest element with $\text{width}(P) = 2$. For each $k \in \mathbb{N}$

- (Q3) $Q_k$ has a minimum vertex and a maximum vertex,
- (Q4) $\text{width}(Q_k) = 2^k$.
Claim 18.A. For each $k \in \mathbb{N}$, $Q_k \in \text{Forb}(L_m)$.

Proof. We will use induction on $k$. For our bases, we see $k = 0$ is trivial and $k = 1$ is established by (19). Take $k > 1$ and suppose the inductive hypothesis holds for all smaller cases. Assume $L$ is an $m$-ladder in $Q_k$ with the lower leg $(a_1, u_1) \triangleleft Q_k (a_2, u_2) \triangleleft \ldots \triangleleft Q_k (a_m, u_m)$ and the upper leg $(b_1, v_1) \triangleleft Q_k (b_2, v_2) \triangleleft Q_k \ldots \triangleleft Q_k (b_m, v_m)$. If all vertices of $L$ are pairwise different in the first coordinate, then these vertices would induce an $m$-ladder in $P$, which violates (19). Hence, at least two vertices of $L$ share a first coordinate, say $p \in P$. Let $Q' = \{(p, q) : q \in Q_{k-1}\}$ and note that $Q'$ and $Q_{k-1}$ are isomorphic. Let $0$ and $1$ to be the minimum and the maximum, respectively, vertices of $Q'$ (which exist by (Q3)).

Assume for a while, $Q$ contains two vertices of the lower leg of $L$, i.e., there are $i < j \in [m]$ so that $(a_i, u_i), (a_j, u_j) \in Q'$ with $a_i = a_j = p$. From the definition of a ladder, we know $(a_i, u_i) \leq Q_k (b_i, v_i) \parallel Q_k (a_j, u_j)$. By (17) we have $b_i = p$ and thus $Q'$ contains $(b_i, v_i)$, a vertex of the upper leg of $L$. For similar reasons, if $Q'$ contains two vertices of the upper leg of $L$, then it has to have one of the lower leg of $L$. Therefore, there are $(a_i, u_i), (b_j, v_j) \in Q'$, vertices of the lower and the upper leg of $L$, respectively. We see $(a_i, u_i) \leq Q_k (a_m, u_m) \parallel Q_k (b_j, v_j)$ (if $j < m$) or $(a_i, u_i) \leq Q_k (a_j, u_j) \parallel Q_k (b_j, v_j)$ (if $j = m$). In the former case we use (17) and in the latter case (18) to show $(a_m, u_m) \in Q'$. Similarly, $(a_i, u_i) \parallel Q_k (b_i, v_1) \leq Q_k (b_j, v_j)$ (if $i > 1$) or $(a_i, u_i) \leq Q_k (b_i, v_1) \leq Q_k (b_j, v_j)$ (if $i = 1$). Again, using (17) or (18), we have $(b_1, v_1) \in Q'$.

For any vertex $(r, s)$ in $L$ so that $(r, s) \notin \{(a_1, u_1), (b_m, v_m)\}$, we have either $(b_1, v_1) \parallel Q_k (a_m, u_m)$ or $(b_1, v_1) \parallel Q_k (a_m, u_m)$. By (17) we deduce $(r, s) \in Q'$. Finally, the vertices

$$\{0, (a_2, u_2), (a_3, u_3), \ldots, (a_m, u_m), (b_1, v_1), (b_2, v_2), \ldots, (b_{m-1}, v_{m-1}), 1\} \subseteq Q'$$

induce an $m$-ladder in $Q'$, which contradicts the inductive hypothesis, proving the claim.

Claim 18.B. $\chi_{FF}(Q_{k+1}) \geq (m - 1)\chi_{FF}(Q_k)$.

Proof. We already know $P$ contains two vertices of $(m - 1)$-Grundy coloring, say $\text{f}$. Let $g$ be a $n$-Grundy coloring of $Q_k$. Define $h : Q_{k+1} \rightarrow [(m - 1)n]$ by $h((p, q)) = (\text{f}(p) - 1)n + g(q)$. We will show $h$ is an $((m - 1)n)$-Grundy coloring of $Q_{k+1}$. For that we need to prove (G1)-(G3) of Definition 8.

It is easy to check that a function $(f, g) \rightarrow (f - 1)n + g$ is a bijection between $[m - 1] \times [n]$ and $[(m - 1)n]$. Since $\text{f}$ and $g$ are surjective, then also $h$ must be surjective. Thus, $h$ satisfies (G2). To show (G1) suppose $h((p, q)) = h((r, s))$. This implies that $\text{f}(p) = \text{f}(r)$ and $g(q) = g(s)$. By (G1) of $\text{f}$ and $g$, two pairs of vertices $p, r$ and $q, s$ are comparable respectively in $P$ and in $Q_k$. Therefore, by the definition of the lexicographical product, vertices $(p, q)$ and $(r, s)$ are comparable in $Q_{k+1}$ and condition (G1) holds for $h$.

![Figure 11. Simplified Hasse diagram of $Q_{k+1}$ with $m = 4$.](image)
Consider \((r, s) \in Q_{k+1}\) so that \(h((r, s)) = j > 1\) and take any \(i < j\). We will show \((r, s)\) has an \(i\)-witness in \(Q_{k+1}\) which will prove [G3]. There are unique integers \(c \in [m - 1]\) and \(d \in [n]\) so that \(j = (c - 1)n + d\) and \(f(r) = c, g(s) = d\). Similarly, we can find \(a \in [m - 1]\) and \(b \in [k]\) so that \(i = (a - 1)n + b\). As \(i < j\), we must have \(a \leq c\).

Suppose \(a = c\), then \(b < d\). As \(g\) satisfies [G3], there is some \(q \in Q_k\) so that \(g(q) = b\) and \(q \parallel Q_k s\). By the definition of lexicographical product, \((r, q) \parallel Q_{k+1} (r, s)\). Observe \(h((r, q)) = i\) and then \((r, q)\) is the desired witness.

The case \(a < c\) is similar. This time we use [G3] of \(f\) to get \(p \in P\) so that \(f(p) = a\) and \(p \parallel P r\). Take any \(q \in Q_k\) so that \(g(q) = b\) (\(q\) exists by [G2] of \(g\)). Again, by the definition of lexicographical product, \((p, q) \parallel Q_{k+1} (r, s)\). Finally, as \(h((p, q)) = i\), we deduce \((p, q)\) is the desired witness in this case.

Claim [18,3] with \(\chi_{\text{FF}}(Q_0) = 1\) implies \(\chi_{\text{FF}}(Q_k) \geq (m - 1)^k\). Note that \(\text{width}(Q_{\lfloor \lg w \rfloor})\) could be less then \(w\). But we can always add some isolated vertices to \(Q_{\lfloor \lg w \rfloor}\) to get width \(w\) poset \(Q'\) so that \(\chi_{\text{FF}}(Q') \geq \chi_{\text{FF}}(Q_{\lfloor \lg w \rfloor})\). This finally shows

\[
\text{val}_{\text{FF}}(L_m, w) \geq \chi_{\text{FF}}(Q_{\lfloor \lg w \rfloor}) \geq (m - 1)^{\lfloor \lg w \rfloor} \geq \frac{w^{\lfloor \lg (m - 1) \rfloor}}{m - 1}.
\]

\[
\square
\]

Lemmas [18] and [7] show that the upper bound of \(\text{val}(P_w)\) cannot be pushed below \(w^{\lfloor \lg w \rfloor}\) using our current methods.

6. Concluding Remarks

Although we have improved the upper bound for \(\text{val}(P_w)\), our current methods cannot bring it down to a polynomial bound without some major changes. Perhaps improvements in the understanding of regular posets could lead us to a subfamily or more interesting forbidden substructures. We could also examine on-line coloring algorithms other than First-Fit to reduce the number of colors used on the family \(\text{Forb}(L_m)\).

We may look beyond the scope of \(\text{val}(P_w)\). So far, the reduction to regular posets has only been studied on general posets. We might ask what the results of the algorithm from Lemma [11] are when we start with a poset from \(\text{Forb}(L)\) (for some poset \(L\)). It is interesting to ask what analogues of Lemma [11] could be built. For instance, could an analogue for cocomparability graphs be created? Already, Kierstead, Penrice, and Trotter [19] have shown that a cocomparability graph can be colored on-line using a bounded number of colors. However, this bound is so large that it was not computed. Perhaps methods similar to the reduction to regular posets could be created.

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