Etudes for the inverse spectral problem

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Funding information
NSF, Grant/Award Numbers: DMS-1500821, DMS-1954085; Ministry of Science and Higher Education of the Russian Federation, Grant/Award Number: No. 075-15-2021-602

Abstract
In this note, we study inverse spectral problems for canonical Hamiltonian systems, which encompass a broad class of second-order differential equations on a half-line. Our goal is to extend the classical results developed in the work of Marchenko, Gelfand–Levitan, and Krein to broader classes of canonical systems and to illustrate the solution algorithms and formulae with a variety of examples. One of the main ingredients of our approach is the use of truncated Toeplitz operators, which complement the standard toolbox of the Krein–de Branges theory of canonical systems.

MSC 2020
30D05, 34L05, 42A05 (primary)

1 | INTRODUCTION

The main object of this note is a canonical Hamiltonian system of differential equations on a half-line. Such systems were studied around the middle of the last century by M. G. Krein, who was first to find multiple connections between spectral problems for such systems and structural problems in certain spaces of entire functions. The rich complex analytic content of Krein’s theory was further developed in the work of L. de Branges. At present, the Krein–de Branges (KdB) theory became a standard tool in the area of spectral problems. The theory experiences a new peak of popularity in the last 20 years due to its connections to other areas such as the theory of orthogonal polynomials, number theory, random matrices and the nonlinear Fourier transform \cite{1, 2, 6, 8, 14, 16, 17, 29, 34}. In addition to the original book by de Branges \cite{7} and a chapter in the book by Dym and McKean \cite{10}, more recent monographs by Remling \cite{31} and Romanov \cite{32} contain the basics of KdB-theory and further references.
A regular half-line canonical (Hamiltonian) system (CS) is the equation
\[ \Omega \ddot{X} = z H X \quad \text{on } [0, \infty). \] (1.1)

Here the Hamiltonian \( H = H(t) \) is a given \( 2 \times 2 \) matrix-function satisfying
\[ H \in L^1_{\text{loc}}[0, \infty), \quad H \neq 0 \quad \text{a.e.,} \quad H \geq 0 \quad \text{a.e.} \]

The first relation means that the entries of \( H \) are integrable on each finite interval. Systems satisfying this condition are called regular. The matrix \( \Omega \) in (1.1) is the symplectic matrix
\[ \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \]
and \( z \in \mathbb{C} \) is the “spectral parameter”. The unknown function \( X = X(t, z) \) is a two-dimensional vector-function on \([0, \infty)\).

Several important classes of second-order differential equations, including Schrödinger and Sturm-Liouville equations, Dirac and Zaharov-Shabat systems, string equations and orthogonal polynomials can be rewritten as CS, see, for instance, [9, 30, 31] or [32], which makes a study of such systems even more general.

Spectral problems for differential operators ask to find spectral data for a given operator (direct problem) or recover an operator from its spectral data (inverse problem). Such problems stem from theoretical physics and lead to deep questions in analysis and adjacent areas of mathematics. Among the classical results in the area of inverse problems are existence and uniqueness theorems for Schrödinger equations by Borg and Marchenko [4, 5, 23, 24] and further extensions to broader classes of Dirac systems by Gelfand, Levitan, Gazymov, Krein, and their collaborators [11, 19].

Our goal in this paper is to extend classical theory to broader classes of CS and to provide explicit algorithms for the inverse spectral problem in those classes. In our approach, spectral measures become symbols of truncated Toeplitz operators, bringing inverse spectral problems into the general framework of the Toeplitz approach to problems of the uncertainty principle in harmonic analysis recently developed in [20–22, 27].

Truncated Toeplitz operators, defined in Subsection 4.1, are bounded and invertible in Paley–Wiener (PW) spaces of entire functions if and only if their symbols are sampling measures in PW-spaces. We show that the class of CS whose spectral measures are PW-sampling is broader than those classes considered in classical theories and use the inverse Toeplitz operators to solve the inverse spectral problem.

One of the key parts of the Gelfand–Levitan theory is the statement that the Fourier transform \( \hat{\mu} \) of a spectral measure \( \mu \) of a regular Dirac system has the form \( \delta_0 + \phi \), where \( \delta_0 \) is the Dirac point-mass at 0 and \( \phi \) is a locally summable function on \( \mathbb{R} \). Recalling that the Fourier transform of the Lebesgue measure \( m \) on \( \mathbb{R} \) is \( \delta_0 \) (after a proper normalization), the theory considers systems that are close, in some sense, to the free system whose spectral measure is \( m \). The Gelfand–Levitan approach to the inverse spectral problem then uses the invertibility of the operator
\[ T : L^2([0, L]) \to L^2([0, L]), \quad Tf = f + \phi \ast f, \]
to solve the problem. The invertibility of \( T \) is established via the Hilbert–Fredholm Lemma and uses the property that \( T \) is of the form \( I + K \), where \( K \) is compact. The compactness of the
convolution operator $Kf = \phi \ast f$ follows from the local summability of $\phi$, which is an important condition in the classical theory.

Our approach replaces the Fredholm invertibility of $T$ with invertibility of the truncated Toeplitz operator $L_\mu$, see Subsection 4.1. We show that spectral measures satisfying the condition

$$\hat{\mu} = \delta_0 + \phi$$

(1.2)

form a subclass of sampling measures in PW-spaces, see Lemma 3.1. (We will call (1.2) Gelfand–Levitan condition since it appears in their work [11, 19].) The algorithm then proceeds with the use of the inverse operator $L^{-1}_\mu$ that exists when $\mu$ is PW-sampling.

This approach connects inverse spectral problems with the study of sampling measures in PW-spaces, see [25] for results and further references. While describing sampling measures in a fixed space PW_\alpha is a deep and difficult problem, measures that are sampling for all PW-spaces admit an elementary description, see Theorem 3.1.

Our algorithm takes an especially compact form in the case when the spectral measure is periodic, see Subsections 6.1–6.5. In this case, the inverse spectral problem connects with the theory of orthogonal polynomials on the unit circle, see Theorem 6.3.

Given a CS with a PW-sampling spectral measure corresponding to a fixed initial condition at the origin, it is natural to ask if the spectral measures corresponding to other initial conditions are also PW-sampling. One of our examples (see Subsection 5.3) shows that generally this is not true. Theorem 5.3 gives necessary and sufficient conditions for other spectral measures (measures from the same Aleksandrov–Clark [AC] family) to be PW-sampling.

Our approach allows us to extend the classical results developed in the work of Marchenko, Gelfand–Levitan, and Krein to broader classes of canonical systems and to illustrate the solution algorithms and formulae with a variety of examples in the last section of the paper.

The paper is organized as follows.

- In Subsection 2.1, we review the basics of KdB-theory.
- Subsection 2.2 discusses the definition and main properties of spectral measures of CS.
- In Subsection 2.3, we define representing measures for de Branges (dB) spaces generalizing Parseval’s identity.
- In Subsection 2.4, we discuss the Weyl transform that generalizes the Fourier transform in the settings of KdB-theory.
- Subsection 2.5 contains the definition and basic formulae related to AC measures.
- In Subsection 2.6, we discuss PW-spaces as a particular case of dB-spaces corresponding to a free system.
- In Subsection 3.1, we introduce PW-sampling measures. Theorem 3.1 gives an elementary description of PW-class.
- Subsection 3.2 provides basic examples of PW-sampling measures.
- In Subsection 3.3, we discuss the class of measures appearing the classical Gelfand–Levitan theory. Lemma 3.1 shows that such measures form a subclass of PW-class.
- Subsection 3.4 introduces the main class of CS considered in this paper, the systems whose spectral measures are PW-sampling. We formulate Theorems 3.2 and 3.4 showing that PW-sampling measures correspond to Hamiltonians with a.e. nonvanishing determinants.
- In Subsection 3.5, we show how symmetries of the Hamiltonian relate to symmetries of the spectral measure.
• The special case of Krein’s string equation, which corresponds to CS with diagonal Hamiltonians, is discussed in Subsection 3.6.
• In Subsection 4.1, we define truncated Toeplitz operators, one of the main tools in our algorithms.
• Using Toeplitz operators we are able to supply the proofs of Lemma 3.1 and Theorem 3.4 in Subsection 4.2.
• Subsection 4.3 shows how to obtain reproducing kernels of dB spaces from sinc functions using inverse Toeplitz operators.
• In Subsection 4.4, we provide an algorithm for the recovery of the upper left element of a Hamiltonian $H$ from the spectral measure.
• To recover off-diagonal terms of $H$ we define conjugate reprokernels in Subsection 4.5.
• Subsection 4.6 contains Theorem 4.5, which gives the formula for the generalized Hilbert transform, the unitary operator sending a chain of dB-spaces to the dual chain corresponding to a different boundary condition.
• In Subsection 4.7, we provide an algorithm for the recovery of the off-diagonal terms of $H$.
• In Subsection 4.8, we give formulae for the Fourier transform of the reproducing kernel at 0, which is used in our algorithms.
• Section 5 discusses algorithms of recovery of the lower right element of $H$.
• Subsection 5.2 discusses the relation between AC-dual measures and a change of initial condition in a given system.
• In Subsection 5.3, we give an example of a PW-sampling measure whose AC-dual is not PW-sampling. Theorem 5.3 gives a necessary and sufficient condition for a PW-sampling measure to have only PW-sampling measures as duals.
• Subsection 6.1 starts the discussion of the periodic case.
• Subsection 6.2 introduces a matrix of trigonometric moments for a periodic measure.
• Subsection 6.3 contains a simplified algorithm for the recovery of the upper left element of $H$ in the periodic case.
• Subsection 6.4 gives a similar algorithm for the recovery of off-diagonal terms.
• In Subsection 6.5, we discuss connections of the periodic case with the theory of orthogonal polynomials.
• In Subsections 7.1–7.10, we provide a variety of examples illustrating our methods and formulae.

2  CANONICAL SYSTEMS AND SPECTRAL MEASURES

2.1  Transfer matrices, Hermite–Biehler functions, and de Branges spaces

Instead of a two-dimensional vector function $X$ one may look for a $2 \times 2$ matrix-valued solution $M = M(t, z)$ solving (1.1). Such a matrix valued function satisfying the initial condition $M(0, z) = I$ is called the transfer matrix or the matrizenant of the system. The columns of the transfer matrix $M$ are the solutions for the system (1.1) satisfying the initial conditions $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (Neumann) and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (Dirichlet) at 0. As a general rule, we denote

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$  (2.1)
An entire function $F(z)$ belongs to the Hermite–Biehler (HB) class if

$$|F(z)| > |F(\bar{z})| \text{ for all } z \in \mathbb{C}_+. $$

We say that an entire function is real if it is real on $\mathbb{R}$.

For each fixed $t$, the entries of the transfer matrix $M$ of the system (1.1), $A(z) = A(t, z) \equiv A_t(z)$, $B(z), C(z)$ and $D(z)$ are real entire functions. The functions

$$E := A - iC, \quad \tilde{E} := B - iD$$

belong to the HB class, see, for instance, [31].

For an entire function $G$ we denote by $G^\#$ its Schwartz reflection with respect to $\mathbb{R}$, $G^\#(z) = \tilde{G}(\bar{z})$. We denote by $H^2$ the standard Hardy space in the upper half-plane.

For every HB function $F$ one can consider the de Branges (dB) space $\mathcal{B}(F)$, a Hilbert space of entire functions defined as

$$\mathcal{B}(F) = \left\{ G \mid G \text{ is entire}, \frac{G}{F}, \frac{G^\#}{F} \in H^2 \right\}.$$  

The Hilbert space structure in $\mathcal{B}(F)$ is inherited from $H^2$:

$$\langle G, H \rangle_{\mathcal{B}(F)} = \left\langle \frac{G}{F}, \frac{H}{F} \right\rangle_{H^2} = \int_{-\infty}^{\infty} G(t)\tilde{H}(t)\frac{dt}{|F(t)|^2}. $$

It follows that

$$\mathcal{B}(F) \subset L^2(1/|F|^2, \mathbb{R}).$$

One of the important properties of dB-spaces is that they admit an equivalent axiomatic definition.

**Theorem 2.1** [7]. Suppose that $H$ is a Hilbert space of entire functions that satisfies

(A1) $F \in H, F(\lambda) = 0 \Rightarrow F(z)^{z-\bar{\lambda}} \in H$ with the same norm,

(A2) $\forall \lambda \notin \mathbb{R}$, the point evaluation $F \mapsto F(\lambda)$ is a bounded linear functional on $H$,

(A3) $F \mapsto F^\#$ is an isometry in $H$.

Then $H = \mathcal{B}(E)$ for some $E \in HB$.

It is not difficult to show that in every dB-space $\mathcal{B}(E)$, (A2) actually holds for all $\lambda \in \mathbb{C}$. Thus, as follows from the Riesz representation theorem, for each $\lambda \in \mathbb{C}$ there exists $K(\lambda, \cdot) \in B(H)$ such that for any $F \in \mathcal{B}(E)$,

$$F(\lambda) = \langle F, K(\lambda, \cdot) \rangle_{\mathcal{B}(E)}. $$

The function $K(\lambda, z)$ is called the reproducing kernel (reprokernel) for the point $\lambda$. In the case of the dB-space $\mathcal{B}(E)$, $K(\lambda, z)$ has the formula

$$K(\lambda, z) = \frac{1}{2\pi i} \frac{E(z)E^\#(\bar{\lambda}) - E^\#(z)E(\bar{\lambda})}{\bar{\lambda} - z} = \frac{1}{\pi} \frac{A(z)C(\bar{\lambda}) - C(z)A(\bar{\lambda})}{\bar{\lambda} - z}, $$

where $A = (E + E^\#)/2$ and $C = (E^\# - E)/2i$ are real entire functions such that $E = A - iC$. 

The functions $E$, $\tilde{E}$ corresponding to a canonical system (1.1) give rise to the family of dB-spaces
\[ B_t = B(E(t, \cdot)), \quad \tilde{B}_t = B(\tilde{E}(t, \cdot)). \]

A value $t$ is $\mathcal{H}$-regular if it does not belong to an open interval on which $\mathcal{H}$ is a constant matrix of rank one. The spaces $B_t, \tilde{B}_t$ form chains, that is, $B_s \subset B_t$ for $s < t$ and the inclusion is isometric for regular $t$ and $s$.

We denote by $K_t(z, w)$ and $\tilde{K}_t(z, w)$ their reprokernels; sometimes, when $t$ is fixed, we write $K_w(z)$ and $\tilde{K}_w(z)$.

We denote by $\Pi$ the Poisson measure on $\mathbb{R}$, $d\Pi(x) = \frac{dx}{1+x^2}$. We will denote by $m$ the Lebesgue measure on $\mathbb{R}$.

An entire function $F$ belongs to the Cartwright class $C_a$ if it has exponential type at most $a$ and $\log |F| \in L^1(\Pi)$.

A dB-space $B(E)$ is called regular if for any $F \in B(E)$ and any $w \in \mathbb{C}$, $(F(z) - F(w))/(z - w) \in B(E)$. Any regular space $B(E)$ is a subspace of $C_a$ for some $a > 0$, see [10, 31].

Regular systems (1.1) give rise to chains of regular dB-spaces $B_t, \tilde{B}_t$.

**Remark 2.1.** A more general class of $2 \times 2$ systems are self-adjoint systems (SAS) defined as
\[ \Omega \dot{X} = z \mathcal{H}X + QX \quad \text{on} \quad [0, \infty). \]

Here $\Omega$ and $\mathcal{H}$ are as before and $Q$ is another given $2 \times 2$ locally summable symmetric real matrix function. An SAS is a general form of a second-order symmetric real system. It is well-known that every regular system of this form is equivalent to a regular CS in the sense that they have the same dB spaces $B_t$ and therefore same spectral measures, as defined in the next section. An SAS can be reduced to a regular CS via a substitution, see [31, 32].

One of the important classes of SAS is the class of Dirac systems for which $\mathcal{H} = I$. Such systems play an important role in applications and constitute one of the main objects in the study of spectral problems, see, for instance, [18]. The class of canonical systems considered in this note includes the systems corresponding to regular Dirac systems.

### 2.2 Spectral measures

There are several ways to introduce spectral measures of canonical systems. We will make a simplifying assumption that the system has no “jump intervals,” that is, intervals on which the Hamiltonian is rank one and constant. In this case, all $t \in [0, \infty)$ are $\mathcal{H}$-regular and all inclusions $B_s \subset B_t, \tilde{B}_s \subset \tilde{B}_t$ are isometric. We can make this assumption because we will be mostly concerned with the case $\det \mathcal{H} \neq 0$ a.e.

We informally identify absolutely continuous measures with functions writing $\mu = f$ for the measure satisfying $d\mu(x) = f(x)dx$. In particular, instead of writing $\mu = cm$ for constant multiples of the Lebesgue measure we write $\mu = c$.

A measure $\mu$ on $\mathbb{R}$ is called Poisson-finite ($\Pi$-finite) if
\[ \int \frac{|\mu|(x)}{1 + x^2} < \infty. \]
A measure on $\mathring{R} = \mathbb{R} \cup \{\infty\}$ is $\Pi$-finite if it is a sum of a $\Pi$-finite measure on $\mathbb{R}$ and a finite point mass at infinity.

By definition, a positive measure $\mu$ on $\mathbb{R}$ is a spectral measure of the CS (1.1) with the initial condition $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at $t = 0$ if

$$ \forall t, \quad B_t \overset{\text{iso}}{\subset} L^2(\mu). $$

(The definition is slightly more complicated in presence of jump intervals for the Hamiltonian, which we do not allow in this paper.) It is well-known that spectral measures of regular CS are $\Pi$-finite, see, for instance, [31]. In a similar way, using $\tilde{B}_t$, one can define a spectral measure $\tilde{\mu}$ for the initial condition $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Conversely, one of the main results of the KdB theory says that every positive $\Pi$-finite measure is a spectral measure of a regular CS. In general, the corresponding system may not satisfy $\det H \neq 0$ a.e., the restriction we are assuming in this article. Also, HB functions corresponding to the systems considered in this paper have no zeros on the real line. We will assume this restriction in our general discussions of dB-spaces. (If $E$ vanishes at some point of $\mathbb{R}$, then all functions in $B(E)$ must vanish at the same point, as follows from the definition. PW-type spaces discussed in this note clearly do not have such a property.)

Every regular canonical system has a spectral measure; in fact for $\mu$ we can take any limit point of the family of measures $|E_t|^{-2}$ as $t \to \infty$, [7]. The spectral measure may or may not be unique. It is unique if and only if

$$ \int \text{trace } \mathcal{H}(t)dt = \infty. $$ (2.2)

The case when the spectral measure is unique is called the limit point case and the case when it is not, the limit circle case.

Finally, let us mention that spectral measures are invariant with respect to “time” parameterizations, that is, a change of variable $t$ in the initial system (1.1) via an increasing homeomorphism $t \mapsto s(t)$ does not change the spectral measure.

### 2.3 De Branges measures

If $E$ is an HB-function nonvanishing on $\mathbb{R}$, then one can define a continuous branch of the argument of $E$, $\phi_E(x) = -\arg E(x)$. The function $\phi_E$ is called the phase function for the space $B(E)$. It is not difficult to show that $\phi_E$ is a growing real analytic function, $\phi' > 0$ on $\mathbb{R}$.

We call a sequence of real points discrete if it has no finite accumulation points. A measure on $\mathring{R} = \mathbb{R} \cap \{\infty\}$ is discrete if its support on $\mathbb{R}$ is a discrete sequence.

For a dB-space $B(E)$, there exists a standard one-parameter family of discrete measures on $\mathbb{R}$ such that $B(E) \overset{\text{iso}}{=} L^2(\mu)$. Let $\alpha \in \mathbb{C}$, $|\alpha| = 1$ and denote by $t_n$ the sequence of points on $\mathbb{R}$ such that

$$ \frac{E^\#(t_n)}{E(t_n)} = \alpha. $$
In terms of the phase function, \( \phi(t_n) = \arg \alpha \mod \pi \). Consider the measure

\[
\mu_\alpha = \sum_n \frac{\pi}{\phi'(t_n)|E(t_n)|^2}\delta_{t_n}.
\]

(2.3)

Then \( B(E) \overset{\text{iso}}{=} L^2(\mu_\alpha) \), for all values of \( \alpha \) except possibly one, see, for instance, [7]. It is well-known that in the case considered in this paper, when \( E = E_t \) is an HB function corresponding to a regular det-normalized CS, such an exceptional value of \( \alpha \) does not exist.

If \( E_t \) is the family of HB-functions corresponding to a system (1.1) with a fixed initial condition at 0, and \( E = E_T \) for some fixed \( T > 0 \), then the family of measures \( \mu_\alpha \) constructed as above is the family of spectral measures of the system restricted to the finite interval \((0, T)\) with a corresponding self-adjoint boundary condition at 0 and various self-adjoint boundary conditions at \( T \).

The Cauchy integrals of the measures \( \mu_\alpha \) can be expressed through the elements of the transfer matrix \( M \). The following formula will be useful to us in Subsection 5.2.

**Lemma 2.1.** Let \( M \) be a transfer matrix (2.1) of a CS (1.1). Let \( E = A - iC \) be the HB function corresponding to the first column of \( M \) and let the representing measures \( \mu_\alpha \) for \( B(E) \) be defined as in (2.3). Suppose that \( B(E) \overset{\text{iso}}{=} L^2(\mu_{-1}) \). Then

\[
\mathcal{K}_{\mu_{-1}}(z) := \frac{1}{\pi} \int \left[ \frac{1}{\zeta - z} - \frac{\zeta}{1 + \zeta^2} \right] d\mu_{-1}(\zeta) = -\frac{B}{A} + \text{const}.
\]

Note that, in particular, the measure \( \mu_{-1} \) is supported at the zeros of \( A(z) \), which can also be seen straight from its definition.

**Proof.** The inequality \( K_z(z) = ||K_z||^2 > 0 \) for the reproducing kernel \( K_z \) of the space \( \mathcal{B}(A + iB) \) implies that \( -\frac{B}{A} \) has positive imaginary part in \( C_+ \). By the Herglotz representation theorem,

\[
-\frac{B}{A} = \mathcal{K}\nu + \text{const}
\]

for some positive measure \( \nu \) on \( \mathbb{R} \). As \( -B/A \) is analytic outside of the zeros of \( A \), \( \nu \) is a discrete measure whose support on \( \mathbb{R} \) is at \( \{A = 0\} = \{t_n\} \), the same as \( \mu_{-1} \). As \( B(E) \overset{\text{iso}}{=} L^2(\mu_{-1}) \), the reproducing kernels \( K_{t_n} \) form an orthogonal basis in \( \mathcal{B}(E) \), which implies that \( \nu \) cannot have a pointmass at infinity, see, for instance, the proof of Theorem 22 in [7]. Examining the behavior of \(-B/A\) near \( t_n \) we find that \( \nu \) has pointmasses of the size \(-\pi B(t_n)/A'(t_n)\) at \( t_n \). As

\[
\phi = -\arg E = -\arctan\left(-\frac{C}{A}\right) \mod \pi,
\]

we have

\[
\phi' = \frac{1}{1 + (C/A)^2} \frac{C'A - A'C}{A^2} = \frac{1}{|E|^2} (C'A - A'C).
\]
As \( A(t_n) = 0 \),
\[
\phi'(t_n) = -\frac{A'C}{|E|^2}(t_n)
\]
and
\[
\frac{\pi}{\phi'(t_n)|E(t_n)|^2} = -\frac{\pi}{A'(t_n)C(t_n)}.
\]
As \( \det M \equiv 1 \), \( C(t_n) = 1/B(t_n) \) and
\[
\nu(t_n) = -\pi \frac{B(t_n)}{A'(t_n)} = \pi \frac{\pi}{A'(t_n)C(t_n)} = \frac{\pi}{\phi'(t_n)|E(t_n)|^2} = \mu_{-1}(t_n).
\]

\[\square\]

2.4 | The Weyl transform

On every finite interval \([0,t]\) we define the Hilbert space \( L^2(\mathcal{H}, [0,t]) \) associated with the Hamiltonian \( \mathcal{H} \geq 0 \). The space consists of of 2-dim vector functions on \([0,t]\) with the inner product

\[
\langle f, g \rangle_{L^2_{\mathcal{H}, [0,t]}} = \int_0^t \langle Hf, g \rangle \, dt.
\]

To give an alternative definition of a spectral measure for a CS we can consider the Weyl transform

\[
\mathcal{W} : \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \mapsto F(z) = \frac{1}{\sqrt{\pi}} \int_0^\infty \left\langle H(t) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} A_t(\bar{z}) \\ C_t(\bar{z}) \end{pmatrix} \right\rangle \, dt,
\]

first defined on test functions, and call \( \mu \) a spectral measure if \( \mathcal{W} \) extends to an isometry \( L^2(\mathcal{H}, [0,t]) \to L^2(\mu) \) for every \( t \geq 0 \). Note that then
\[
B_t = \mathcal{W} L^2(\mathcal{H}, [0,t]) \tag{2.4}
\]
and the reprokernels \( K_\lambda(z) \) of \( B_t \) are obtained as Weyl transforms of the solutions,
\[
K_\lambda = \mathcal{W}X_\lambda. \tag{2.5}
\]

The spectral measure \( \tilde{\mu} \) can be similarly defined using the functions \( B \) and \( D \) in place of \( A \) and \( C \) in the above formula. Spectral measures corresponding to other self-adjoint boundary conditions can also be defined using proper modifications of either one of our two definitions.

2.5 | Aleksandrov–Clark and dual measures

Let \( \varphi \) be a function from the unit ball of \( H^\infty(C_+) \) (such functions are often called Schur functions). Associated with every such \( \varphi \) is the family of \( \Pi \)-finite positive measures \( \{\sigma_\alpha\}_{\alpha \in \mathbb{T}} \) on \( \hat{\mathbb{R}} \) defined via
the formula
\[ \Re \alpha + \varphi(z) \alpha - \varphi(z) = py + \frac{1}{\pi} \int \frac{yd\sigma^p_\alpha(t)}{(x-t)^2 + y^2}, \quad z = x + iy, \tag{2.6} \]

where the number \( \pi p \) can be interpreted as the point mass of \( \sigma^p_\alpha \) at \( \infty \). Such measures \( \sigma^p_\alpha \) are called AC measures for \( \varphi \). Families of AC-measures appear in complex function theory and applications to perturbation theory and spectral problems, see, for instance, [28].

At most one of the measures \( \{\sigma^p_\alpha\}_{\alpha \in \mathbb{T}} \) has a point mass at infinity: \( \sigma_\alpha(\infty) \neq 0 \) if and only if \( \varphi(\infty) = \alpha \), in the sense of nontangential limit, and \( \varphi - \alpha \in L^2(\mathbb{R}) \).

It follows from the definition that all of the measures \( \sigma^p_\alpha \) are singular if and only if one of them is singular if and only if \( \varphi \) is inner. All measures are discrete if and only if one of them is discrete if and only if \( \varphi \) is a meromorphic inner function (MIF), an inner function that can be extended meromorphically to the whole plane.

If \( \varphi = \theta \) is inner, then
\[ \sigma_\alpha \cdot 1_{\mathbb{R}} = 2\pi \sum_{\theta(\xi) = \alpha} \frac{\delta_\xi}{|\theta'(\xi)|}, \tag{2.7} \]
as can be deduced from the definition.

Every HB-function \( E(z) \) gives rise to a MIF, \( \theta_E = E^# / E \). In this case, the last formula is closely related to (2.3) and the measures involved in the two formulae satisfy
\[ \sigma_\alpha = |E|^2 \mu_\alpha. \]

We will call two \( \Pi \)-finite positive measures \( \mu \) and \( \tilde{\mu} \) on \( \mathbb{R} \) (AC-) dual if there exists a Schur function \( \varphi \) such that \( \mu = \sigma^p_{\varphi^{-1}} \) and \( \tilde{\mu} = \sigma^p_1 \). We discuss dual measures in Subsection 5.2 in more detail.

Connection between duality of measures, as defined above, to spectral problems is provided by the property that if \( \mu \) is a spectral measure of a CS with the Neumann initial condition at 0, then the spectral measure \( \tilde{\mu} \) of the same system with the Dirichlet initial condition is dual to \( \mu \). Similarly, measures satisfying (2.6) with other \( \alpha \) correspond to other initial conditions for the same system. Note also that any two measures \( \sigma_\alpha \) and \( \sigma_\beta, \alpha \neq \beta \), from the same AC-family are dual up to constant multiples, that is, there exist \( C_1, C_2 > 0 \) such that \( C_1 \sigma_\alpha \) and \( C_2 \sigma_\beta \) are dual.

### 2.6 Paley–Wiener spaces

Let us discuss the “free” case \( \mathcal{H} \equiv I \).

This case illustrates the connection between the Weyl transform and the Fourier transform. We will use the following definition for the Fourier transform in \( L^2(\mathbb{R}) \):
\[ (Ff)(\xi) \equiv \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int e^{-i\xi x} f(x)dx, \]
first defined on test functions and then extended to a unitary operator \( \mathcal{F}L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) via Parseval’s theorem. The PW space \( \text{PW}_t \) of entire functions is defined as the image

\[
\text{PW}_t = \mathcal{F}L^2[-t, t].
\]

By the PW theorem, \( \text{PW}_t \) can be equivalently defined as the space of entire functions of exponential type at most \( t \) that belong to \( L^2(\mathbb{R}) \). The Hilbert structure in \( \text{PW}_t \) is inherited from \( L^2(\mathbb{R}) \).

Verifying the axioms in Theorem 2.1, one can show that any \( \text{PW}_a \) is a dB-space. Alternatively, one can arrive at the same conclusion using the first definition of a dB-space with \( E(z) = e^{-iaz} \in \text{HB} \).

For the transfer matrix of the free system \( \mathcal{H} = I \) we have

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix}
\cos tz & -\sin tz \\
\sin tz & \cos tz
\end{pmatrix}.
\]

Using the standard notation \( S(z) = e^{iz} \),

\[
E_t(z) = S^{-t}(z) \equiv e^{-itz},
\]

and the spectral measure of \( \mathcal{H} \equiv I \) is \( \mu = m \), the Lebesgue measure on \( \mathbb{R} \). The spaces \( \mathcal{B}_t \) are PW-spaces,

\[
\text{PW}_t = \mathcal{B}(S^{-t})
\]

The reproducing kernels of \( \text{PW}_t \) are (constant multiples of) sinc functions

\[
K_t(z, w) = \frac{1}{\pi} \text{sinc}_t(z - \bar{w}) = \frac{1}{\pi} \frac{\sin t(z - \bar{w})}{z - \bar{w}}.
\]

We will denote the kernel at \( w = 0 \) by

\[
k^*_t(z) := \frac{1}{\pi} \frac{\sin tz}{z}.
\]

The Weyl transform in the free case \( \mathcal{H} \equiv I \) becomes the classical Fourier transform:

\[
\mathcal{W} : \begin{pmatrix}
f_1 \\
f_2
\end{pmatrix} \mapsto F(z) = \frac{1}{\sqrt{\pi}} \int_0^\infty \left\langle \mathcal{H}(t) \begin{pmatrix}
f_1 \\
f_2
\end{pmatrix}, \begin{pmatrix}
A_t(\bar{z}) \\
C_t(\bar{z})
\end{pmatrix} \right\rangle dt
\]

\[
= \frac{1}{\sqrt{\pi}} \int_0^\infty \left\langle \begin{pmatrix}
f_1 \\
f_2
\end{pmatrix}, \begin{pmatrix}
\cos tz \\
\sin tz
\end{pmatrix} \right\rangle dt = \frac{1}{\sqrt{\pi}} \int_0^\infty (f_1 \cos tz + f_2 \sin tz) dt
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-itz} dt = \hat{f}(z),
\]
where the function \( f \) is defined on \( \mathbb{R} \) as

\[
  f(t) = \begin{cases} 
    \frac{1}{\sqrt{2}}(f_1(t) + if_2(t)) & \text{for } t > 0 \\
    \frac{1}{\sqrt{2}}(f_1(t) - if_2(t)) & \text{for } t < 0
  \end{cases}
\]

It follows that the Lebesgue measure \( m \) on \( \mathbb{R} \) is the spectral measure for the free system. The measure is unique because \( \text{trace } \mathcal{H} \equiv 2 \) and (2.2) holds.

3 | PW-SAMPLING MEASURES AND HAMILTONIANS OF PW-TYPE

3.1 | PW-sampling measures

By definition, a positive measure \( \mu \) on \( \mathbb{R} \) is PW-sampling (\( \mu \in \text{PW} \)) if it is sampling for all PW spaces \( \text{PW}_t \):

\[
  \forall t \; \exists C > 0, \; \forall f \in \text{PW}_t, \; C^{-1} \| f \| \leq \| f \|_{L^2(\mu)} \leq C \| f \|.
\]

Note that any PW-sampling measure must satisfy

\[
  \sup_{x \in \mathbb{R}} \mu((x, x + 1)) < \infty, \tag{3.1}
\]

because otherwise

\[
  \sup_{s \in \mathbb{R}} \| f(\cdot - s) \|_{L^2(\mu)} = \infty
\]

for every nonzero \( f \in \text{PW}_a \).

Indeed, first notice that for any such \( f \) there exists an interval \((x, x + \delta), \, \delta > 0\), on which \( |f| > \varepsilon > 0 \). If (3.1) is not satisfied then there exists a sequence of intervals \( I_n = (x_n, x_n + \delta) \) such that \( \mu(I_n) \to \infty \). But then

\[
  \| f(\cdot - (x_n + \delta/2)) \|_{L^2(\mu)}^2 \geq \varepsilon^2 \mu(I_n) \to \infty,
\]

which shows that the norms are not equivalent because

\[
  \| f(\cdot - (x_n + \delta/2)) \|_{\text{PW}_a} = \text{const.}
\]

It follows from (3.1) that a PW-sampling measure is II-finite. If \( \mu = h \cdot m \) with \( h^{\pm 1} \in L^\infty \), then \( \mu \) is obviously a PW-sampling measure. More examples will be given at the end of this section.

Although the description of sampling measures for a fixed \( \text{PW}_a \)-space is a difficult problem (see, for instance, [25]), PW-sampling measures, as defined above, admit a complete metric characterization.

Given \( \mu \) and \( \delta > 0 \) we say that an interval \( l \subset \mathbb{R} \) is \( \delta \)-massive with respect to \( \mu \) if

\[
  \mu(l) \geq \delta \text{ and } |l| \geq \delta.
\]

The \( \delta \)-capacity of an interval \( I \subset \mathbb{R} \) with respect to \( \mu \), denoted by \( C_\delta(I) \), is the maximal number of disjoint \( \delta \)-massive intervals intersecting \( I \).
Theorem 3.1. $\mu$ is PW-sampling if and only if

(i) for any $x \in \mathbb{R}$, $\mu(x, x+1) \leq \text{const}$;
(ii) for any $t > 0$ there exist $L$ and $\delta$ such that for all $I$, $|I| \geq L$,

$$C_d(I) \geq t|I|.$$ 

To prove Theorem 3.1, we need to recall the following well-known results.

A sequence $\Lambda = \{\lambda_n\} \subset \mathbb{C}$ is called separated if $|\lambda_m - \lambda_n| > c > 0$ for a fixed $c$ and all $m \neq n$. A separated sequence is sampling for $PW_a$ if

$$||f(\lambda_n)||_{L^2} \asymp ||f||_{PW_a}$$

for $f \in PW_a$. Sampling measures on $\mathbb{R}$ and real sampling sequences are closely related.

Proposition 3.1 [25]. A positive measure $\mu$ on $\mathbb{R}$ is sampling for $PW_a$ if and only if

1. $\mu((x, x + 1)) < C < \infty$ for some $C$ and all $x \in \mathbb{R}$ and
2. there exists a sampling sequence $\{\lambda_n\}$ for $PW_a$ and $\delta > 0$ such that the points $\lambda_n$ have disjoint neighborhoods $U(\lambda_n)$ satisfying $\mu(U_n) > \delta$.

For $\Lambda \subset \mathbb{R}$ define the lower Beurling uniform density of $\Lambda$ as

$$D_-(\Lambda) = \lim_{L \to \infty} \min_{x \in \mathbb{R}} \frac{\#(\Lambda \cap (x, x + L))}{L}.$$ 

Proposition 3.2. A real separated sequence $\Lambda$ is sampling for $PW_a$ if $\pi D_-(\Lambda) > a$ and not sampling if $\pi D_-(\Lambda) < a$.

This statement is essentially contained in the work of Beurling [3]; see also [13] for a different version.

Proof of Theorem 3.1. Let $\mu$ satisfy the conditions of the theorem and let $t$ be a positive number. Then for $\delta = \delta(t)$ from (ii) one can choose a sequence of disjoint $\delta$-massive intervals whose centers $\lambda_n$ form a sequence $\Lambda$ satisfying $D_-(\Lambda) > t/2$. Hence, it is a sampling sequence for $PW_{t/2}$ by Proposition 3.2. Together with (i), Proposition 3.1 implies that $\mu$ is a sampling measure for $PW_{t/2}$.

In the opposite direction, if (i) does not hold, then $\mu$ is not a sampling measure for any $PW_a$ as was discussed above. Suppose that (i) holds but $\delta$ from (ii) does not exist for some $t > 0$. If $\mu$ is sampling for $PW_{t+\varepsilon}$, then by Proposition 3.1 one can choose $U(\lambda_n)$ such that $\Lambda = \{\lambda_n\}$ is sampling for $PW_{t+\varepsilon}$. Then $D(\Lambda) > t$ by Proposition 3.2. As the sequence $\Lambda$ is separated, one can choose $U(\lambda_n)$ to be $\delta$-massive and obtain a contradiction. Therefore, $\mu$ is not sampling for $PW_{t+\varepsilon}$. □

For a PW-sampling measure $\mu$, we denote by $PW_a(\mu)$ the Hilbert spaces of entire functions that have the same elements as the spaces $PW_a$ but the $L^2(\mu)$ scalar product. From the axiomatic characterization of dB-spaces, it follows that $PW_a(\mu)$ are dB-spaces. By construction, all spaces $PW_a(\mu)$ are embedded isometrically in $L^2(\mu)$.
3.2  Basic examples of PW-sampling measures

(1) As was mentioned before, any measure $\mu$ of the form

$$d\mu = w(x)dx, \ w(x) \approx 1$$

is PW-sampling. The upper bound $w < C$ can be replaced with (3.1). This can be seen directly from the estimate

$$||f'||_\infty \leq \alpha \sqrt{2\alpha ||f||_{PW-a}},$$

which holds for any $f \in PW_a$.

(2) Any measure that decays to 0 at infinity, that is, any measure satisfying

$$\mu((x, x + 1)) \to 0 \text{ as } x \to \infty,$$

is not PW-sampling. To see this, note that shifts of the sinc function have the same PW-norm but

$$\left\| \frac{\sin(z - N)}{z - N} \right\|_{L^2(\mu)}^2 \lesssim \sum_{n \in \mathbb{Z}} \frac{\mu((n, n + 1))}{(N - n)^2 + 1} \to 0 \text{ as } N \to \infty.$$

In particular, any finite measure is not PW-sampling.

(3) Let $\mu = m + \sum_{n=0}^{\infty} \alpha_n \delta_{x_n}$, where $0 < \alpha_n < C$ and $\{x_n\} \subset \mathbb{R}$ is a separated sequence. Then $\mu$ is PW-sampling.

Also, if $\mu \in PW$ and $\nu \geq 0$ decays at infinity, in particular if $\nu$ has compact support, then $\mu + \nu \in PW$. This and further examples of that kind are obtained from the following argument.

If $\mu$ satisfies (ii) from Theorem 3.1 and $\nu$ is any nonnegative measure, then $\mu + \nu$ satisfies (ii). If $\mu$ and $\nu$ both satisfy (i), then so does $\mu + \nu$. Hence, if $\mu \in PW$ and $\nu \geq 0$ satisfies (i), then $\mu + \nu \in PW$.

In particular, addition of finite positive measures with compact support or locally finite positive measures decaying at infinity does not change the PW-sampling property.

(4) It follows from Theorem 3.1 that a positive periodic measure is PW-sampling if and only if it has locally infinite support. In particular, the measures $\mu(x) = 1 + a \cos x, \ a \in [-1, 1]$, considered in Section 7, are PW-sampling, but a measure that has only finitely many point masses on each period is not.

(5) Any discrete measure $\mu$ satisfying

$$D_-(\text{supp } \mu) < \infty$$

is not PW-sampling, as follows from Propositions 3.1 and 3.2. In particular, any measure supported on a separated sequence is not PW-sampling.

(6) PW-sampling measures of the form $\mu = 1_S \cdot m, \ S \subset \mathbb{R}$ are described by the Paneah–Logvinenko–Sereda theorem [19, 26]. A set $S \subset \mathbb{R}$ is relatively dense if there exists $r, \delta > 0$ such that

$$|(x - r, x + r) \cap S| > \delta$$
for all $x$. An equivalent reformulation of the theorem in one dimension (the case studied by Paneah) is that $\mu$ is PW-sampling if and only if $S$ is relatively dense. This statement also follows from Theorem 3.1.

3.3 Relations to Gelfand–Levitan theory

We will say that a positive $\Pi$-finite nondiscrete measure $\mu$ is a GL-measure ($\mu \in GL$) if it satisfies the Gelfand–Levitan-type condition

$$\mathcal{F}\mu = C\delta_{0} + f, \quad (3.2)$$

for some function $f \in L_{1 loc}(\mathbb{R})$ and $C > 0$. Here the Fourier transform $\mathcal{F}\mu$ is understood in the sense of distributions. The condition means that $\mu$ is, in some sense, close to a constant multiple of the Lebesgue measure, $\frac{C}{\sqrt{2\pi}m}$, whose Fourier transform is $C\delta_{0}$. As was mentioned in the introduction, measures of this class are considered in the classical Gelfand–Levitan theory of inverse spectral problems. The relation between the class of PW-sampling measures and the class GL defined above is established by the following statement.

**Lemma 3.1.**

$$GL \subset PW.$$ 

The inverse inclusion clearly does not hold. The measure $1 + \cos x$ considered in Section 7 is obviously PW-sampling but not GL because its Fourier transform consists of three pointmasses. Furthermore, any periodic measure with locally infinite support is a PW-sampling measure, but not a GL-measure, unless it is a constant multiple of $m$. For a nonperiodic example, If $\nu$ is any even finite measure with a nontrivial singular part, then $f = \hat{\nu}$ is a real bounded function and $\mu = (C + f)m \in PW$ for large enough $C$. But $\mu \notin GL$ because $\mathcal{F}(\mu) = D\delta_{0} - \nu$ and $\nu \notin L_{1 loc}$.

The proof of Lemma 3.1 is postponed until Subsection 4.1.

**Remark 3.1.** The measures considered in Gelfand–Levitan theory (see, for instance, [18, chapter XII]) are the spectral measures of Dirac systems with regular potentials on the half-line. Such measures have nontrivial absolutely continuous parts, satisfy (3.2) and some additional conditions.

The class GL as defined above is the class of measures for which some of the methods of the classical theory may still work, although it is substantially broader than the set of measures actually considered in [18] or [30].

As will be seen in the proof of Lemma 3.1, the restriction that a GL-measure should not be discrete can be further weakened to the condition that it should not be supported on a zero set of a PW-function (a sequence of finite Beurling–Malliavin exterior density, see, for instance, [15]).

3.4 Canonical systems of PW-type

We say that two Hilbert spaces $H_{1}$ and $H_{2}$ are equal as sets, and write $H_{1} = H_{2}$, if the spaces have the same elements, but not necessarily the same scalar product. We say that a dB-space $B$ has
exponential type $T$ if $T$ is the smallest constant with the property that all functions in $B$ have exponential type at most $T$.

A formula by Krein, see, for instance, [31, 32], allows one to calculate the type $T$ of a dB-space $B_t$ corresponding to a Hamiltonian $H(t)$:

$$T = \int_0^t \sqrt{\det H(t)}dt.$$  

(3.3)

Two SAS are equivalent if the corresponding chains of dB-spaces $B^1_t$ and $B^2_t$ satisfy

$$B^1_t = B^2_{s(t)}$$

(norms and all) for some increasing homeomorphism $s : \mathbb{R}_+ \to \mathbb{R}_+$. The systems are strongly equivalent if $s(t) = t$. Basic results of KdB theory imply that two CS are equivalent if and only if they have the same spectral measures.

The condition $\mathcal{H} \neq 0$ a.e., imposed on every CS (1.1), together with (2.4), implies that for any $t_2 > t_1 \geq 0$, $B_{t_1} \subset B(t_2)$.

Let $\mathcal{H}$ be a Hamiltonian of a canonical system (1.1). We say that $\mathcal{H}$ is of PW-type ($\mathcal{H} \in \text{PW}$) if for any $t > 0$ there exists $s = s(t) > 0$ such that $s(t) \to \infty$ as $t \to \infty$ and

$$B_t(H) \doteq \text{PW}_s.$$  

(3.4)

We call the corresponding system a PW-type system.

**Theorem 3.2.** If $\mathcal{H}$ is of PW-type, then

$$\det \mathcal{H} \neq 0 \text{ a.e., } \int_0^\infty \sqrt{\det \mathcal{H}(t)}dt = \infty.$$  

(3.5)

Also, we have

$$B_t \doteq \text{PW}_{s(t)}, \quad s(t) \doteq \int_0^t \sqrt{\det(\mathcal{H})}.$$  

Proof. We can assume that $\mathcal{H}$ is trace-normalized. Let $\mu$ be a spectral measure corresponding to $\mathcal{H}$. Then $\mu \in \text{PW}$ and $\det \mathcal{H} \neq 0$ a.e., see the proof of Theorem 3.4 in Subsection 4.2. The equation in (3.5) follows from $s(t) \to \infty$ and Krein’s type formula, together with the remaining equations in the statement.

The change of time $t \mapsto s = s(t)$ in the above theorem allows us to transform any PW-type system, or more generally any canonical system satisfying (3.5), into a canonical system with

$$\det \mathcal{H} = 1 \text{ a.e.}$$

We will call such systems det-normalized. A regular (locally summable) Hamiltonian will remain regular under det-normalization.
Theorem 3.3. Consider a regular CS with the Hamiltonian $\mathcal{H}$, $\det \mathcal{H} \neq 0$ a.e. Then there exists an equivalent regular system with the Hamiltonian $\tilde{\mathcal{H}}$ such that $\det \tilde{\mathcal{H}} = 1$ a.e.

Proof. The function

$$s(t) = \int_0^t \sqrt{\det \mathcal{H}}$$

is a $W^{1,1}_{\text{loc}}$-diffeomorphism because $\sqrt{\det \mathcal{H}}$ is in $L^1_{\text{loc}}$ and $\det \mathcal{H} > 0$ a.e. Consider the inverse diffeomorphism $t(s)$ and set

$$\tilde{\mathcal{H}}(s) = (\mathcal{H} \circ t)(s)t'(s).$$

Then $\tilde{\mathcal{H}} \geq 0$, is locally integrable, and $\det \tilde{\mathcal{H}} = 1$ a.e. Clearly, the systems corresponding to $\tilde{\mathcal{H}}$ and $\mathcal{H}$ are equivalent. $\square$

As was mentioned before, a change of time variable, in particular det-normalization, does not affect the spectral measure of the system.

The converse of the last theorem is not true: our next example shows that there are det-normalized canonical systems that are not of PW-type, see also [2].

Example 3.1. Consider the Hamiltonian

$$\mathcal{H} = \begin{pmatrix} t^m & 0 \\ 0 & t^{-m} \end{pmatrix}, \quad m \in (-1, 1),$$

and denote

$$m = 2\nu - 1.$$ Let $J_{\nu}$ denote the Bessel function of the first kind and let

$$J_{\nu}(\lambda) = \lambda^{\nu} F_{\nu}(\lambda).$$

Then $F_{\nu}$ is an entire function. For the solution $(A/C)$ of the CS with the Hamiltonian $\mathcal{H}$ we have:

Lemma 3.2.

$$A = F_{\nu-1}(zt), \quad C = t^{2\nu} z F_{\nu}(zt).$$

The lemma can be established by direct calculations.

We will use the following asymptotic of the Bessel functions on the positive axis:

$$J_{\nu}(\lambda) = \frac{\cos \left[ \lambda - (2\nu + 1) \frac{\pi}{4} \right]}{\sqrt{\lambda}} + O\left( \lambda^{-3/2} \right), \quad \lambda \to \infty.$$
The amplitudes of $|A|^2$ and $|C|^2$ are $t^m x^{-m}$ and $t^{-m} x^{-m}$, respectively. Hence, for each fixed $t$, 
\[
\frac{1}{|E(x)|^2} \approx x^m, \quad x \to \infty.
\]

It follows that the measure $|E_t(x)|^{-2} dx$ is not sampling for PW$_t$ for any $t$, except in the case $m = 0$. Therefore, the corresponding spaces $B_t$ are not equal to PW$_t$ as sets and the system is not of PW-type.

**Remark 3.2.**

(a) All regular Dirac systems are of PW-type as follows from Lemma 3.1 and the results of [19], see also [2].

(b) Characterization of PW-Hamiltonians is a difficult problem even in the diagonal case. So far we know that
\[
\det H \neq 0 \not\Rightarrow \text{PW}
\]

but
\[
\det H \neq 0, \quad H \in W^{1,1}_{\text{loc}} \Rightarrow \text{PW},
\]

as follows from the property that regular Dirac systems correspond to CS with $W^{1,1}_{\text{loc}}$-Hamiltonians.

(c) We will also see that
\[
\begin{pmatrix}
  h_{11} & h_{12} \\
  h_{12} & h_{22}
\end{pmatrix} \in \text{PW} \not\Rightarrow \begin{pmatrix}
  h_{22} & -h_{12} \\
  -h_{12} & h_{11}
\end{pmatrix} \in \text{PW}.
\]

Example 5.1. The second Hamiltonian corresponds to the switch between the Dirichlet and Neumann initial conditions, see Subsection 3.5.

The next statement relates PW-type systems and PW-sampling measures.

**Proposition 3.3.** Suppose $\det(H) = 1$ a.e., and let $\mu$ be the (unique) spectral measure of the corresponding CS. Then
\[
\mu \in (\text{PW}) \iff H \in (\text{PW}).
\]

Moreover, if either holds, then
\[
\forall t, \quad B_t(H) = \text{PW}_t(\mu).
\]

**Proof.** A $\Pi$-finite measure $\mu$ admits a unique, up to parameterization, chain of regular de Branges spaces isometrically embedded in $L^2(\mu)$, as follows from the results of [7]. As both $B_t(H)$ and PW$_s(\mu)$ form such chains, we must have
\[
B_t(H) = \text{PW}_s(\mu) \equiv \text{PW}_s.
\]
By Krein’s formula, the space $B_t$ consists of functions of exponential type at most

$$
\int_0^t \sqrt{\det H(x)} dx,
$$

which is equal to $t$ due to the det-normalization. As $PW_s(\mu)$ consists of functions of type $s$, we must have $B_t(H) = PW_t(\mu)$.

By one of the main theorems of KdB theory, every positive $\Pi$-finite measure on $\mathbb{R}$ is a spectral measure for a canonical system with a locally summable (trace-normalized) Hamiltonian [7]. We obtain the following corollary.

**Theorem 3.4.** Let $\mu$ be a PW-sampling measure. Then there exists a det-normalized $H$ such that $\mu$ is its spectral measure. Two det-normalized Hamiltonians have the same spectral measure if and only if there is $k \in \mathbb{R}$ such that

$$
\hat{H} = H_k : = T_k^* H T_k, \quad T_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.
$$

We postpone the proof until Subsection 4.2.

**Remark 3.3.** In the above formula,

$$
H_k = \begin{pmatrix} h_{11} & h_{12} + kh_{11} \\ h_{12} + kh_{11} & h_{22} + 2kh_{12} + k^2h_{11} \end{pmatrix}
$$

and

$$
M_k := T_k^{-1} M T_k = \begin{pmatrix} A - kC & B + kA - kD - k^2C \\ C & kC + D \end{pmatrix}
$$

is the matrierzant for $H_k$.

### 3.5 Symmetries

For a Hamiltonian

$$
\mathcal{H} = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix},
$$

we denote

$$
\tilde{\mathcal{H}} = \begin{pmatrix} h_{11} & -h_{12} \\ -h_{12} & h_{22} \end{pmatrix} \quad \text{and} \quad \overline{\mathcal{H}} = \begin{pmatrix} h_{22} & -h_{12} \\ -h_{12} & h_{11} \end{pmatrix}.
$$

For a measure $\mu$, $\tilde{\mu}$ is the measure satisfying $\tilde{\mu}(e) = \mu(-e)$, $e \subseteq \mathbb{R}$, and $\overline{\mu}$ is the AC-dual measure as defined in Subsection 2.5.

We denote by $\mu_{\mathcal{H}}$ the spectral measure corresponding to the system with the Hamiltonian $\mathcal{H}$. 
Lemma 3.3.

(i) The involution $H \mapsto \bar{H}$ descends to the involution $\mu \mapsto \bar{\mu}$ for the corresponding spectral measures: $\mu = \mu_H$, $\bar{\mu} = \bar{\mu_H}$.

(ii) The involution $H \mapsto \bar{H}$ descends to the involution $\mu \mapsto \bar{\mu}$. In other words,

$$\mu_H = \bar{\mu_H}.$$

Proof.

(i) Let $\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $\tau^2 = \text{id}$ and $\Omega \tau = -\tau \Omega$. Note that

$$\tau H \tau = \bar{H}.$$ 

Computation shows that $N(z) = M(\tau)(-z)$, where $M(\tau) := \tau M \tau$, is the matrizant of $\bar{H}$: from $\tau \Omega M \tau = z \tau H M \tau$ we get

$$-\Omega M(\tau) = z \tau H \tau ^{\tau} M \tau = z \bar{H} M(\tau).$$

Furthermore,

$$N(x) = \begin{pmatrix} A(-z) & -B(-z) \\ -C(-z) & D(-z) \end{pmatrix}$$

and

$$|E_N(x)|^2 = |A(-x) + iC(-x)|^2 = |E(-x)|^2.$$ 

It follows that $\bar{\mu}$ is the spectral measure of $\bar{H}$.

(ii) We use the same argument but with $\tau = \Omega$. We have

$$\Omega (\Omega M \Omega) = z \Omega H M \Omega = z (-\Omega H \Omega)(\Omega M \Omega).$$

It follows that

$$\Omega M \Omega = \begin{pmatrix} -D & C \\ B & -A \end{pmatrix}$$

is the matrizant of $\bar{H} = -\Omega H \Omega$. □

3.6 | Even measures and diagonal Hamiltonians

We call a measure $\mu$ on $\mathbb{R}$ even if $\mu = \bar{\mu}$, that is, $\mu(x) = \mu(-x)$.

Theorem 3.5. If the Hamiltonian $H$ is diagonal, then the spectral measure $\mu_H$ is even. Conversely, if $\mu$ is an even positive $\Pi$-finite measure, then there exists a diagonal Hamiltonian $H$ such that $\mu = \mu_H$. 
Proof. The first part follows from the last lemma. Alternatively, we can argue as follows. Let \( H = \text{diag}(h_1, h_2) \), then the system is
\[
\dot{C} = z h_1 A, \quad \dot{A} = -z h_2 C.
\]
Denote
\[
\tilde{A}(z, t) = A(-z, t), \quad \tilde{C}(z, t) = -C(-z, t).
\]
We have \( \tilde{A} = A \) and \( \tilde{C} = C \) because the functions satisfy the same system and the same ICs. We derive the reality condition
\[
E(-z) = E^\#(z),
\]
and therefore the symmetry of \( \mu \).

As follows from Theorem 3.4 and Lemma 3.3, if the measure is symmetric, then there is a number \( k \) such that
\[
\bar{H} - H = \begin{pmatrix} 0 & kh_{11} \\ kh_{11} & * \end{pmatrix},
\]
which implies
\[
-2h_{12} = kh_{11},
\]
so the ratio of the functions \( h_{12} \) and \( h_{11} \) is constant. Applying Theorem 3.4 again, we see that the Hamiltonian can be made diagonal. \( \square \)

Remark 3.4. CS with diagonal Hamiltonians are equivalent to so-called Krein’s strings, see the book by Dym and McKean [10]. In particular, the converse statement in Theorem 3.4 also follows from Krein’s theorem which says that if \( \mu \) is a \( \Pi \)-finite even measure, then it is a spectral measure of a string.

Combining the last statement with Theorem 3.4, in the PW-situation we arrive to the following conclusion.

**Theorem 3.6.** There is a 1-to-1 correspondence between even PW-sampling measures and diagonal det-normalized Hamiltonians of PW-type.

**4 \ | \ SOLUTION FORMULAE FOR THE INVERSE PROBLEM**

**4.1 \ | \ Truncated Toeplitz operators**

Let \( t > 0 \) and let \( \mu \) be a positive measure such that \( \text{PW}_t \subset L^2(\mu) \). Then we can define a bounded operator
\[
L \equiv L_{\mu,t} : \text{PW}_t \to \text{PW}_t
\]
by the (symbolic) formula
\[ f \mapsto P_t(f \mu), \]
where \( P_t \) is the “projection” to \( \mathcal{P}W_t \). The formula makes perfect sense if \( \mu \in L^\infty \) but we need to interpret it in terms of quadratic forms in the general case:
\[ \forall f, g \in \mathcal{P}W_t, \quad (Lf, g) = \int f \overline{g} d\mu. \quad (4.1) \]
Clearly, \( L \) is a well-defined bounded nonnegative operator.

The following statement can be deduced directly from the definition.

**Lemma 4.1.** \( L_{\mu,t} \) is an invertible (positive) operator if and only if \( \mu \) is sampling for \( \mathcal{P}W_t \).

In regard to the family of \( \mathcal{P}W \)-sampling measures, we obtain the following

**Corollary 4.1.** If \( \mu \in \mathcal{P}W \), then the operator \( L_\mu \) defined by (4.1) is bounded and invertible in every \( \mathcal{P}W_t, t > 0 \).

Consider now the dB spaces
\[ B_t = \mathcal{P}W_t(\mu) \]
and the operators
\[ j = j_t : \mathcal{P}W_t \to B_t, \quad f \mapsto f. \]

**Theorem 4.1.** If \( \mu \in \mathcal{P}W \), then for any \( t > 0 \),
\[ j^*_t j_t = L_\mu. \]

**Proof.** If \( f, g \in \mathcal{P}W_t \), then
\[ (j^*_t j f, g)_{\mathcal{P}W} = (jf, j g)_B = (jf, j g)_\mu = \int f \overline{g} d\mu. \]
\[ \square \]

### 4.2 Postponed proofs

Using the operators \( L_\mu \), we can now supply the proofs of the lemma on GL-measures and the theorem on Hamiltonians with \( \mathcal{P}W \) spectral measures.

**Proof of Lemma 3.1.** Let \( \mu \in GL, \hat{\mu} = \delta_0 + \phi, \phi \in L^1_{loc}. \)

Let us first show that \( L_\mu \) has trivial kernel in every \( \mathcal{P}W_a \). If \( f \in \mathcal{P}W_a, f \neq 0 \) is in the kernel, then the measure \( f \mu \) annihilates \( \mathcal{P}W_a \). In particular,
\[ \int f^* f d\mu = \int |f|^2 d\mu = 0, \]
which implies that \( \mu \) is a discrete measure supported on the zero set of \( f \). As GL-measures are nondiscrete, we obtain that \( L_\mu \) is injective on every \( \mathcal{P}W_a \).
Let $\mu_a$ be the measure such that $\hat{\mu}_a = 1_{[-2a,2a]} \hat{\mu}$. Then $\mu_a = (1 + \psi)m$ where $\psi = F^{-1}(\phi_a)$, $\phi_a = 1_{[-2a,2a]} \phi$, is a bounded function. Note that the restriction of $L_\mu$ onto $\operatorname{PW}_a$ is equal to $L_{\mu_a}$. Therefore, $L_{\mu_a}$ has trivial kernel in $\operatorname{PW}_a$.

Note that

$$F(L_{\mu_a}f) = 1_{[-a,a]}(\hat{f} + \hat{f} * \phi_a) = \hat{f} + 1_{[-a,a]} \hat{f} * \phi_a.$$ 

Hence, the operator

$$L_{\mu_a} : L^2([-a,a]) \to L^2([-a,a]), \quad L_{\mu_a} \hat{f} = F(L_{\mu_a}f),$$

is of the Fredholm form

$$L_{\mu_a} = I + K, \quad Kf = P_a(f * \phi_a),$$

where $P_a$ is the orthogonal projection $L^2([-3a,3a]) \to L^2([-a,a])$. As $\phi_a$ is summable, and $P_a$ is bounded, $K$ is compact. As the kernel of $L_{\mu_a}$ is trivial, so is the kernel of $L_{\mu_a}$. By the Fredholm alternative, $L_{\mu_a}$ is invertible.

As $\phi_a$ is summable, $L_{\mu_a}$ is bounded.

Hence, $L_\mu$ is bounded and invertible in any $\operatorname{PW}_a$. By Lemma 4.1, $\mu \in \operatorname{PW}$.

Let us now prove Theorem 3.4.

We denote by $\operatorname{PW}_{[a,b]}(\mu)$ the orthogonal difference $\operatorname{PW}_b(\mu) \ominus \operatorname{PW}_a(\mu)$. We will need the following.

**Lemma 4.2.** Let $\mu \in \operatorname{PW}$. Then for any $0 \leq a < b \leq c$ there exists a constant $d = d(c,\mu) > 0$ such that for any $f \in \operatorname{PW}_{[a,b]}(\mu)$

$$\int_a^b |\hat{f}(x)|^2 dx > d ||f||_2^2.$$

**Proof.** Let $f \in \operatorname{PW}_{[a,b]}(\mu)$. Note that then $f \in \operatorname{PW}_b$. Denote by $f_a$ the orthogonal projection of $f$ onto $\operatorname{PW}_a$ in $\operatorname{PW}_b$ and let $f_b = f - f_a$. Then the integral in the statement is $||f_b||_2^2$.

Note that $f \in \operatorname{PW}_{[a,b]}(\mu)$ is equivalent to $L_{\mu,b}f \perp \operatorname{PW}_a$, which implies that $\operatorname{supp} F(L_{\mu,b}f) \cap (-a,a) = \emptyset$. As $L_{\mu,a}$ is invertible,

$$||L_{\mu,b}f_b||_2^2 = \int_{-b}^b |F(L_{\mu,b}f_b)(x)|^2 dx \geq \int_{-a}^a |F(L_{\mu,b}f_b)(x)|^2 dx = \int_{-a}^a |F(L_{\mu,b}(f - f_a))(x)|^2 dx = \int_{-a}^a |F(L_{\mu,b}f_a)(x)|^2 dx = ||L_{\mu,a}f_a||_2^2 > d_1 ||f_a||_2^2,$$

for some $d_1 = d_1(c)$. As $L_{\mu,b}$ is bounded, it follows that

$$||f_b||_2^2 > d_2 ||f_a||_2^2,$$

for some $d_2 = d_2(c)$. \qed
Proof of Theorem 3.4. As PW-sampling measures are $\Pi$-finite, $\mu$ is the spectral measure of some regular trace-normalized canonical system with Hamiltonian $\mathcal{H}$. We need to show that the set $\{t \mid \det \mathcal{H}(t) = 0\}$ has Lebesgue measure zero. If that is the case, we can renormalize the system with the change of variable

$$s(t) = \int_0^t \sqrt{\det \mathcal{H}}.$$ 

To show that $|\{t \mid \det \mathcal{H}(t) = 0\}| = 0$, recall that any $\Pi$-finite measure admits a unique chain of regular de Branges spaces. As both $\text{PW}_s(\mu)$ and the spaces $\mathcal{B}_t$ corresponding to $\mathcal{H}$ constitute such chains, $\text{PW}_s = \mathcal{B}_t$ for some $s = s(t)$. According to Krein’s formula for the exponential type (3.3), $s(t)$ is defined as above.

Assume now that $|\{t \mid \det \mathcal{H}(t) = 0\} \cap (0, T)| > 0$ for some $T > 0$. Let $f \in L^2(\mathcal{H})$ be a nonzero function such that

$$\text{supp } f \subset \{t \mid \det \mathcal{H}(t) = 0\} \cap (0, T).$$

It follows from the type formula that for any $\varepsilon > 0$ one can choose disjoint intervals

$$[a_k, b_k] \subset (0, T) \setminus \text{supp } f, \quad 0 \leq a_1 < b_1 < a_2 < \ldots < b_n \leq T$$

such that

$$\sum_{k=1}^n s(b_k) - s(a_k) = s(T) - \varepsilon.$$ 

Put $F = \mathcal{W}f$. Then $F \in \text{PW}_{s(T)}$ and $\text{supp } \hat{F} \subset [0, s(T)]$. According to the last lemma,

$$\int_0^{s(a_1)} |\hat{F}(x)|^2 dx + \int_{s(a_2)}^{s(b_1)} |\hat{F}(x)|^2 dx + \cdots + \int_{s(b_n)}^{s(T)} |\hat{F}(x)|^2 dx > d||F||_2^2.$$ 

Recall that the Weyl transform $\mathcal{W}$ is unitary and therefore $||F||_2^2 > 0$. As

$$\varepsilon = |[0, s(T)] \setminus \cup [s(a_k), s(b_k)]|$$

can be chosen arbitrarily small, we obtain a contradiction.

For the second part of the statement, notice that if $\mathcal{H}$ and $\tilde{\mathcal{H}}$ have the same PW-sampling spectral measure, then the det-normalization for both Hamiltonians implies $\mathcal{B}_t = \tilde{\mathcal{B}}_t = \text{PW}_s(\mu)$ for all $t$. Then for the HB functions $E_t = A_t - iC_t$ and $\tilde{A}_t - i\tilde{C}_t$ we have

$$\begin{pmatrix} \tilde{A}_t \\ \tilde{C}_t \end{pmatrix} = R^{-1} \begin{pmatrix} A_t \\ C_t \end{pmatrix}$$

for some constant matrix $R \in \text{SL}(2, \mathbb{R})$, see, for instance, [31], Theorem 2.2.

Note that if $\begin{pmatrix} A_t \\ C_t \end{pmatrix}$ is a solution to a CS with a Hamiltonian $\mathcal{H}$ then $R^{-1} \begin{pmatrix} A_t \\ C_t \end{pmatrix}$ is a solution to a system with the Hamiltonian $R^* \mathcal{H}R$. Here we use the property that any $\text{SL}(2, \mathbb{R})$ matrix is symplectic, that is, $R^* \Omega R = \Omega$. 
From the uniqueness of systems on \((0, t)\) with the same HB-function corresponding to \(t\) (see, for instance, Theorem 13 in [32]), it follows that all \(\begin{pmatrix} \tilde{A}_s \\ \tilde{C}_s \end{pmatrix}\), \(s < t\) satisfy (4.2) with \(t = s\) and the same constant matrix \(R\). Choosing larger \(t\) we conclude that (4.2) must hold with the same \(R\) for all \(t\). Finally, the property that
\[
\begin{pmatrix} A_t \\ C_t \end{pmatrix}(0) = \begin{pmatrix} \tilde{A}_t \\ \tilde{C}_t \end{pmatrix}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]
together with \(R \in \text{SL}(2, \mathbb{R})\) imply that \(R\) is upper triangular with 1 on the diagonal.

4.3 Reprokernels

Let \(\mu\) be a sampling measure for \(PW_t\) for some fixed \(t > 0\). Consider the de Branges space \(B = PW_t(\mu), B \doteq PW_t\). Then the truncated Toeplitz operator \(L = L_{\mu,t}, L : PW_t \to PW_t\) defined as in the last section is bounded and invertible. For \(w \in \mathbb{C}\) we denote by \(K_w\) and \(\hat{K}_w\) the reprokernels in \(B\) and \(PW_t\), respectively.

**Theorem 4.2.** For all \(w \in \mathbb{C}\),
\[
j[L^{-1} \hat{K}_w] = K_w.
\]

**Proof.** For any \(f\) and \(g\) in \(PW_t\), we have
\[
(jf, jg)_{B} = (f, Lg)_{PW}.
\]
For \(g = j^{-1}K_w\), we obtain
\[
(jf, K_w)_{B} = (f, Lj^{-1}K_w)_{PW}.
\]
The left-hand side is equal to
\[
f(w) = (f, K_w)_{PW},
\]
and as \(f\) is arbitrary, we get
\[
Lj^{-1}K_w = K_w.
\]

In the formulae for the inverse spectral problem the last statement will only be used in the case \(w = 0\). We will use the notations \(k_t, \hat{k}_t\) for the reprokernels \(K_0, \hat{K}_0\) in \(B_t\) and \(PW_t\), respectively. As
\[
\hat{k}_t(z) = \frac{1}{\pi} \frac{\sin tz}{z},
\]
Theorem 4.2 implies
\[
k_t = j_t \left[ L_{\mu,t}^{-1} \hat{k}_t \right]. \quad (4.3)
\]
4.4 Recovering the term $h_{11}(t)$

In this section, we obtain a formula that recovers the element $h_{11}$ of the Hamiltonian

$$\mathcal{H} = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix}$$

in CS (1.1) from the spectral measure.

Note that for det-normalized CS with even spectral measures this amounts to a full solution for the inverse spectral problem as $h_{12} \equiv 0$ by Theorem 3.5 and $h_{11} h_{22} = 1$. As was mentioned before, systems with even measures represent Krein’s string operators.

As before, we assume that $\mu$ is a PW-sampling measure. Recall that by Theorem 3.4, $h_{11}$ is determined by $\mu$ in a unique way (unlike $h_{22}$ or $h_{12}$).

**Theorem 4.3.** Let $\mu \in \text{PW}$ be the spectral measure of a system (1.1) with the Hamiltonian $\mathcal{H}$. Then $t \mapsto k_t(0)$ is an absolutely continuous function and

$$h_{11}(t) = h_\mu(t) := \pi \frac{d}{dt} k_t(0),$$

where the kernel $k_t$ is given by (4.3).

**Proof.** From (2.5), we obtain the “Christoffel-Darboux” formula

$$\int_0^t \langle \mathcal{H}(s)X_z(s),X_w(s) \rangle \, ds = \pi K_w(z,t).$$

It follows that

$$\langle \mathcal{H}(t)X_z(t),X_w(t) \rangle = \pi K_w(z,t).$$

If $z = w = 0$, then $X_0(t) = (1,0)^T$, and

$$\langle \mathcal{H}(t)X_0(t),X_0(t) \rangle = h_{11}(t).$$

□

**Remark 4.1.** The nesting property $B_{t_1} \subset B_{t_2}$ for $t_1 < t_2$, together with

$$k_t(0) = \sup \{ f(0) | f \in B_t, ||f||_{B_t} = 1 \},$$

implies that the function $t \mapsto k_t(0)$ is increasing, so the derivative in the definition of $h_\mu$ exists a.e. The same can be seen from our proof. The det-normalization implies $h_\mu > 0$ a.e.

**Corollary 4.2.** If $\mu \in \text{PW}$ is even, then

$$\mathcal{H} = \begin{pmatrix} h_\mu & 0 \\ 0 & h_\mu^{-1} \end{pmatrix},$$

where $h_\mu$ is from the last theorem, is the unique det-normalized diagonal Hamiltonian such that $\mu$ is the spectral measure of (1.1).
4.5 Conjugate kernels

Let $\mathcal{H}$ be a det-normalized PW Hamiltonian and let $B_t$ and $\tilde{B}_t$ be the corresponding chains of dB-spaces generated by HB functions $E_t = A_t - iC_t$ and $\tilde{E}_t = B_t - iD_t$. We drop the subindex $t$ in our formulae when $t$ is fixed.

As before, we denote by $K_t(z, w)$ and $\tilde{K}_t(z, w)$ the corresponding reprokernels, sometimes using abbreviated notations $K_w(z)$, $\tilde{K}_w(z)$ when $t$ is fixed. Due to a special role played by reprokernels at 0, we also use the notation

$$k_t(z) := K_t(z, 0) = \frac{1}{\pi} \frac{C_t(z)}{z}.$$  

We now introduce “conjugate” kernels $\tilde{L}_w(z) \equiv L_t(z, w) \in \tilde{B}_t$ defined as

$$\tilde{L}_w(z) = \frac{1}{\pi} \left[ D(z)A(\tilde{w}) - B(z)C(\tilde{w}) \right] - 1.$$ 

In particular,

$$\tilde{L}_t(z) := L_t(z, 0) = \frac{1}{\pi} \frac{D(z) - 1}{z}.$$ 

As before, we denote by $X_z(t)$ and $Y_z(t)$ the solutions to CS (1.1) with Neumann and Dirichlet initial conditions correspondingly,

$$X_z(t) := \begin{pmatrix} A_t(z) \\ C_t(z) \end{pmatrix}, Y_z(t) := \begin{pmatrix} B_t(z) \\ D_t(z) \end{pmatrix}.$$ 

The following relation presents a special case of Lagrange identities.

**Lemma 4.3.**

$$\pi \tilde{L}_t(z, w) = (Y_z, X_w)_{L^2(\mathcal{H}, [0, t])}$$

To prove the lemma, we first establish the following relation for the transfer matrix.

**Lemma 4.4.**

$$\frac{\partial}{\partial t} [M_t^*(w)\Omega M_t(z)] = (z - \tilde{w})M_t^*(w)\Omega M_t(z).$$

**Proof.**

$$\frac{\partial}{\partial t} [M_t^*(w)\Omega M_t(z)] = +M_t^*(w)\Omega \frac{\partial}{\partial t} M_t(z) + \frac{\partial}{\partial t} [M_t^*(w)\Omega] M_t(z)$$

$$= zM_t^*(w)\mathcal{H} M_t(z) - \tilde{w}M_t^*(w)\mathcal{H}^* M_t(z) = (z - \tilde{w})M_t^*(w)\Omega M_t(z).$$

**Proof of Lemma 4.3.**

$$(Y_z, X_w)_{L^2(\mathcal{H}, [0, t])} = \int_0^t \langle HY_z(s), X_w(s) \rangle ds$$

$$= \int_0^t \left\langle \mathcal{H} M_z(s) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, M_s(w) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle ds$$
\[ \int_0^t \left\langle M_s^*(w) \Omega M_s(z) \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle ds \]

\[ \int_0^t \left\langle \frac{\partial}{\partial s} M_s^*(w) \Omega M_s(z) \frac{z - \overline{w}}{z - \overline{w}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle ds \]

\[ \frac{\partial}{\partial s} [M_t^*(w)] \Omega M_t(z) - 1 \]

\[ \frac{[D(z)A(w) - B(z)C(w)] - 1}{z - \overline{w}}. \]

**Corollary 4.3.**

\[ h_{12}(t) = \pi \frac{d}{dt} \tilde{l}(0). \]

**Proof.** If \( z = w = 0 \), then

\[ (Y_z, X_w)_{L^2(\Omega, [0, t])} = \int_0^t \left\langle \mathcal{H} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = \int_0^t h_{12}. \]

Our next goal is to express the conjugate kernel \( \tilde{l} \) in terms of the spectral measure that will allow us to recover the element \( h_{12} \) of the Hamiltonian via the last corollary, see Theorem 4.6.

**Theorem 4.4.** For each \( t \), there exists a linear unitary operator

\[ T_t : B_t \to \tilde{B}_t \]

such that

\[ \forall w \in \mathbb{C}, \quad K_w \mapsto L_w. \]

If \( t_1 < t_2 \), then \( T_{t_1} \) coincides with the restriction of \( T_{t_2} \) to \( B_{t_1} \).

**Proof.** The operator \( T_t \) can be defined via the formula

\[ \tilde{W}_t = T_t \circ W_t, \]

where \( W_t : L^2(\mathcal{H}, [0, t]) \to B_t \) and \( \tilde{W}_t : L^2(\mathcal{H}, [0, t]) \to \tilde{B}_t \) are the restrictions of the corresponding Weyl transforms.

**Remark 4.2.** We emphasize the operators \( T_t \) (and the spaces \( \tilde{B}_t \)) are not determined by the spectral measure; they also depend on the choice of \( \mathcal{H} \). If we replace \( \mathcal{H} \) by \( \tilde{\mathcal{H}} = \mathcal{H}_k \) (see Remark 3.3), then we get

\[ \tilde{T}_t = T_t + kI. \]

### 4.6 Hilbert transform

For a \( \Pi \)-finite measure \( \mu \) and \( f \in L^2(|\mu|) \), we will use the notation

\[ K(f \mu)(z) = \frac{1}{\pi} \int \frac{f(s) \, d\mu(s)}{s - z}, \]
\[
\mathcal{K}_\mu(z) = \frac{1}{\pi} \int \left[ \frac{1}{s-z} - \frac{s}{1+s^2} \right] d\mu(s),
\]
where \( z \in \mathbb{C} \setminus \mathbb{R} \). If \( f \in L^2(\mu) \) is an entire function, then we define
\[
H^\mu f = K(f \mu) - f \mathcal{K}_\mu.
\]
It is clear that \( H^\mu f \) extends to an entire function:
\[
(H^\mu f)(z) = \frac{1}{\pi} \int \left[ \frac{f(s) - f(z)}{s-z} + \frac{sf(z)}{1+s^2} \right] d\mu(s).
\]

**Example 4.1.** If \( \mu = m \), then \( H^\mu f \) is the analytic continuation of the Hilbert transform
\[
Hf(z) = \frac{1}{\pi} \text{v.p.} \int_{\mathbb{R}} \frac{f(s) ds}{s-z}
\]
and
\[
H^\mu : \text{PW}_t \rightarrow \text{PW}_t
\]
is a unitary operator.

**Remark 4.3.** If \( \mu \) is a periodic locally finite measure, then
\[
\int \frac{s}{1+s^2} d\mu(s) = c
\]
exists and \( H^\mu \) takes on the form
\[
H^\mu f = Kf \mu - fK\mu + cf = \frac{1}{\pi} \int \frac{f(s) - f(z)}{s-z} d\mu(s) + cf.
\]

**Theorem 4.5.** Let \( \mathcal{H} \) be a det-normalized PW Hamiltonian, and let \( \mu \) be the spectral measure. Let \( B_1, \tilde{B}_1 \) be the corresponding dB-spaces and let \( T_t \) be the unitary operators from Theorem 4.4. Then there exists a real constant \( c = c(\mathcal{H}) \) such that
\[
\forall t, \quad T_t = (H^\mu - cI) \bigg|_{B_1}.
\]

**Remark 4.4.** Note that \( c(\mathcal{H}_k) = c(\mathcal{H}) - k \), where \( \mathcal{H}_k \) is like in Remark 3.3.

**Proof.**

(1) Let us fix \( t > 0 \). Let \( \nu = \mu_{-1} \) be the representing measure for the space \( B_{t_0} \) for some \( t_0 > t \) and \( \alpha = -1 \), as defined in Subsection 2.3. Let us first show that
\[
T_t = (H^\nu - cI) \bigg|_{B_1}.
\]
Indeed, for $F \in B_i$,

\[ T_i F(z) = \langle T_i F, K_z \rangle_{B_i} = \langle T_i F, T_i L_z \rangle_{B_i} = \langle F, L_z \rangle_{B_i} \]

\[ = \int F(s) \frac{[A(s)D(z) - C(s)B(z)] - 1}{s - z} d\nu(s) \]

\[ = -K(F\nu)(z) - B(z) \int \frac{F(s)C(s)}{s - z} d\nu(s), \]

because by construction $\nu = \mu_{-1}$ is supported at the zeros of $A$. On the other hand,

\[ F(z) = \langle F, K_z \rangle_{B_i} = -\int F(s) \frac{A(s)C(z) - C(s)A(z)}{s - z} d\nu(s) \]

\[ = A(z) \int \frac{F(s)C(s)}{s - z} d\nu. \]

Therefore,

\[ T_i F = -K(F\nu) - B \frac{A}{F}. \]

It is left to recall that by Lemma 2.1

\[ \mathcal{K}\nu = \mathcal{K}\mu_{-1} = -\frac{B}{A} + \text{const}. \]

(2) We will now show that $H^\nu = H^\mu$ on $B_i$.

Fix any function $G \in B_i$ such that $G(i), G(-i) \neq 0$. Consider the expression

\[ \int \left[ \frac{F(s) - F(z)}{s - z} - \frac{1}{2} \left( \frac{F(z)}{G(i)} \frac{G(s) - G(i)}{s - i} + \frac{F(z)}{G(-i)} \frac{G(s) - G(-i)}{s + i} \right) \right] d\nu(s). \quad (4.4) \]

Let us denote the function under the integral by $S(z, s)$. Note that as $B_i$ is regular, for each fixed $z$ the functions $S(z, w)$ and $wS(z, w)$ belong to $B_i$ as functions of $w$. Hence, the integral must remain the same if $\nu$ is replaced with any measure $\eta$ such that $B_i \subset L^2(\eta)$. Indeed, let $Q \in B_i$ be any function such that $Q(0) = 1$. Once again, as $B_i$ is regular, $R(z) = (Q(z) - 1)/z \in B_i$. Therefore,

\[ \int S(z, s)d\eta(s) = \int S(z, s)(\tilde{Q}(s) - s\tilde{R}(s))d\eta(s) \]

\[ = \langle S(z, s), Q(s) \rangle_\eta - \langle sS(z, s), R(s) \rangle_\eta \]

\[ = \langle S(z, s), Q(s) \rangle_\nu - \langle sS(z, s), R(s) \rangle_\nu. \]

As $\mu$ is the spectral measure, it can replace $\nu$ in $(4.4)$. It is left to notice that when $\nu = \mu$ the expression in $(4.4)$ is equal to

\[ K F \mu(z) - F(z) \mathcal{K}\mu(z) + F(z) \int \left[ \frac{1}{G(i)} \frac{G(s)}{s - i} + \frac{1}{G(-i)} \frac{G(s)}{s + i} \right] d\mu. \]

Recall that $G$ was a fixed function, independent of $F$. It remains to put

\[ c_\mu = d_\mu - d_\nu, \]
where $d_\mu$ is equal to the last integral and $d_\nu$ is equal to the last integral with $\mu$ replaced by $\nu$. □

4.7 | Recovering the term $h_{12}(t)$

**Theorem 4.6.** Let $\mu \in \text{PW}$, and let the functions $k_t \in \text{PW}_t(\mu)$ be obtained from $\mu$ by via the formula (4.3). Define $\tilde{l}_t = H^\mu k_t$. Then $\mu$ is the spectral measure of the Hamiltonian

$$
\mathcal{H} = \begin{pmatrix}
h^\mu & g^\mu \\
g^\mu & \frac{g^\mu}{1 + g^2_\mu}
\end{pmatrix},
$$

where

$$
g^\mu(t) := \pi \frac{d}{dt} \tilde{l}_t(0).
$$

**Proof.** Let $\tilde{\mathcal{H}}$ be the Hamiltonian provided by Theorem 3.4. Uniqueness of the regular chain of dB-spaces in $L^2(\mu)$ implies that $\tilde{\mathcal{H}} = \text{PW}_t(\mu)$. Let $\tilde{T}_t$ be the operators from Theorem 4.4. Denote

$$
\tilde{l}_t = \tilde{T}_t k_t.
$$

By Theorem 4.5, there is a real number $c$ such that $\tilde{T}_t = H^\mu - cI$ on $\tilde{\mathcal{H}}$, so

$$
\tilde{l}_t = \tilde{l}_t - ck_t.
$$

Differentiating, we get

$$
\tilde{h}_{12} = g^\mu - ch^\mu.
$$

It follows that

$$
\mathcal{H} = T_c^* \tilde{\mathcal{H}} T_c.
$$

□

4.8 | Equations for the Fourier transform of $k_t$

How to compute the functions $h^\mu$ and $g^\mu$ from Theorem 4.5? Sometimes it is helpful to work with the functions

$$
\psi_t := \hat{k}_t,
$$

so that

$$
\frac{1}{\sqrt{2\pi}} \int \psi_t = k_t(0).
$$

If $f \in \text{PW}_t$, then

$$
f(0) = \frac{1}{\sqrt{2\pi}} \int \hat{f}(\xi) d\xi
$$

while also

$$
f(0) = (f, \hat{k}_t)_{\text{PW}_t} = (\hat{f}, \mathcal{F}(k_t))_{L^2(-t,t)}.
$$
This leads us to the well-known formula for the Fourier transform of the sinc function:

$$\mathcal{F} \hat{k}_t = \mathcal{F} \left( \frac{\sin tz}{\pi z} \right) = \frac{1}{\sqrt{2\pi}} 1_{(-t, t)}.$$ 

The Fourier transform $\psi_t$ of the reprokernel $k_t$ can be found using the following statement.

**Theorem 4.7.** $\psi = \psi_t$ satisfies

$$\psi \ast \hat{\mu} = 1 \quad \text{on} \quad (-t, t)$$

and

$$\psi = 0 \quad \text{on} \quad \mathbb{R} \setminus [-t, t].$$

**Proof.** We denote by $P_t$ the orthogonal projection $P_t : L^2(\mathbb{R}) \to \text{PW}_t$. Somewhat formally we interpret the equation $L_{\mu} k_t = \hat{k}_t$ as

$$P_t(k_t \mu) = \hat{k}_t,$$

and taking Fourier transforms, we obtain

$$1_{(-t,t)} \mathcal{F}(k_t \mu) = \frac{1}{\sqrt{2\pi}} 1_{(-t,t)}.$$

Note that

$$\mathcal{F}(k_t \mu) = \frac{1}{\sqrt{2\pi}} \hat{\mu} \ast \psi_t.$$ 

**Remark 4.5.** For $\mu \in \text{PW}$ and $f \in \text{PW}_a$ the map

$$g \mapsto \int g \hat{f} \, d\mu$$

defines a bounded linear functional on any $\text{PW}_t$, $t > 0$. It follows that the restriction of $\hat{f} \ast \hat{\mu}$ to any interval $(-t, t)$ belongs to $L^2([-t, t])$. In particular, so does $\psi_t \ast \hat{\mu}$ because $k_t \in \text{PW}_t(\mu) \equiv \text{PW}_t$.

## 5 DIFFERENT INITIAL CONDITIONS

### 5.1 The term $h_{22}(t)$

Let $H$ be a Hamiltonian such that $\det(H) = 1$ a.e. and let $\tilde{\mu}$ be the spectral measure of $H$ for the initial condition $(0, 1)^T$. Assume that $\tilde{\mu} \in (\text{PW})$. Then we can recover the term $h_{22}$ from $\tilde{\mu}$ exactly as we did it for $h_{11}$ and the spectral measure $\mu$:

$$h_{22}(t) = \pi \frac{d}{dt} \tilde{k}_i(0), \quad \tilde{k}_i := j_1 \left[ L_{\tilde{\mu},1}^{-1} \tilde{k}_i \right]$$

or simply,

$$h_{22} = H^{\tilde{\mu}}.$$
Theorem 5.1. Let $\mu$ and $\bar{\mu}$ be the spectral measures of the det-normalized Hamiltonian $H$ corresponding to the Neumann and Dirichlet boundary conditions at $0$ correspondingly. Assume that both $\mu$ and $\bar{\mu}$ are PW-sampling measures. Then there exists a real number $c$ such that

$$
H = \begin{pmatrix}
\mu & \mu - ch \\
ch & \bar{\mu}
\end{pmatrix},
$$

Corollary 5.1. If $H$ is diagonal and both $\mu$ and $\bar{\mu}$ are PW-sampling measures, then

$$
h\mu h\bar{\mu} = 1 \quad \text{a.e. on } \mathbb{R}_+.
$$

5.2 AC duality

In view of the last theorem, we would like to describe all possible measures $\bar{\mu}$ for a given $\mu$.

As was defined in Subsection 2.5, $\bar{\mu}$ is AC-dual to $\mu$ if there exists a function $\phi \in H^\infty(C_+)$, $\|\phi\|_\infty \leq 1$, such that

$$
\mu = \sigma^{\phi}_{-1}, \quad \bar{\mu} = \sigma^{\phi}_{1},
$$

that is,

$$
P_{\mu} = \Re \frac{1 - \phi}{1 + \phi}, \quad P_{\bar{\mu}} = \Re \frac{1 + \phi}{1 - \phi}. \quad (5.1)
$$

For simplicity, we will assume $\mu(\infty) = \bar{\mu}(\infty) = 0$; as we mentioned earlier our spectral measures will satisfy this condition. Given $\mu$ there is a 1-parameter family of dual measures $\bar{\mu} = \sigma^{\phi_b}_{1}$ where

$$
\phi_b = \frac{i - K\mu - b}{i + K\mu + b}. \quad (b \in \mathbb{R}). \quad (5.2)
$$

We will sometimes use the notation $\bar{\mu}_b$ if $\mu = \sigma^{\phi_b}_{-1}$, $\bar{\mu}_b = \sigma^{\phi_b}_{1}$ for $\phi_b$ as in (5.2). In particular,

$$
P_{\bar{\mu}_b} = \Re \frac{i}{K\mu + b}. \quad (5.3)
$$

Theorem 5.2. Let $\mu \in \text{PW}$ be given. Then $\bar{\mu}$ is an AC-dual measure of $\mu$ if and only if there exists a Hamiltonian $H$ such that $\mu$ and $\bar{\mu}$ are the spectral measures of $H$ with the Neumann and Dirichlet initial conditions, respectively.

Proof.

(i) Let $\mu$ and $\bar{\mu}$ be the spectral measures of some $H$, and let $\left( \begin{array}{cc}
A & B \\
C & D
\end{array} \right)$ be the transfer matrix of $H$. For each $t > 0$ we define the measure $\mu_t \equiv \mu_{E_t}$ as the representing measure for the space $B(A_t - iC_t)$ supported on $\{A_t = 0\}$ and $\bar{\mu}_t \equiv \mu_{E_t}$ as the representing measure for the space $B(B_t - iD_t)$ supported on $\{B_t = 0\}$. If we denote

$$
\phi_t = \frac{A_t - iB_t}{A_t + iB_t},
$$

then

$$
\mu_t = \sigma^{\phi_t}_{-1}, \quad \bar{\mu}_t = \sigma^{\phi_t}_{1}.
$$
ETUDES FOR THE INVERSE SPECTRAL PROBLEM

(note that $\mathcal{K}_t = -B/A$ like in Lemma 2.1). As $\mu_t$ is a representing measure for $B_t = B(E_t)$, we have

$$B_s \subset L^2(\mu_s) \text{ for all } s \leq t.$$  

As $\mu$ is the unique measure for which the above inclusion holds for all $s > 0$ and as $\cup_t B_t$ is dense in $L^2(\mu)$, we obtain that $\mu_t \rightarrow \mu$, $*$-weakly as $t \rightarrow \infty$. Similarly, $\tilde{\mu}_t \rightarrow \tilde{\mu}$. As

$$\phi_t = \frac{1 + i\mathcal{K}_t}{1 - i\mathcal{K}_t},$$

the functions $\phi_t$ tend to $\phi$,

$$\phi = \frac{1 + i\mathcal{K}_t}{1 - i\mathcal{K}_t}$$

pointwise in $\mathbb{C}_+$. Then $\mu$, $\tilde{\mu}$ and $\phi$ satisfy (5.1) and we see that $\mu$ and $\tilde{\mu}$ are dual.

(ii) Consider $\hat{H} = H_k$ defined as in Remark 3.3. Then $\hat{A} = A$ and $\hat{B} = B + kA$, so

$$\hat{\phi}_t = \frac{A_t - ikA_t - iB_t}{A_t + ikA_t + iB_t}.$$  

Recalling that, according to Lemma 2.1, $\mathcal{K}\mu = -B/A + \text{const}$ and using (5.2), we obtain all AC-dual measures as $k$ ranges over $\mathbb{R}$. □

5.3 PW-sampling in AC-families

The statement of Theorem 5.1 includes an assumption that both measures $\mu$ and $\tilde{\mu}$ are PW-sampling. This brings up a natural question of whether this assumption is redundant, i.e., if $\mu \in \text{PW}$ implies $\tilde{\mu} \in \text{PW}$.

Our next goal is to study this question in more detail. First we give an example showing that the answer, in general, is negative.

We denote by sign $x$ the sign-function defined as $\pm 1$ on $\mathbb{R}_\pm$ and as 0 at 0.

Example 5.1. Consider $\mu(x) = 2 + \text{sign } x$. Clearly, $\mu \in \text{PW}$. Let $\phi \in H^\infty$, $||\phi||_{\infty} \leq 1$ be the function such that $\phi(i) \in \mathbb{R}$ and (2.6) is satisfied for $\mu$ and $\tilde{\mu}$ with $\alpha = \mp 1$, respectively. Then

$$\mathcal{K}\mu = i\frac{1 - \phi}{1 + \phi} = -\frac{1}{\mathcal{K}\tilde{\mu}}.$$  

As $\mathcal{K}\mu(z) \rightarrow \infty$ as $z \rightarrow \infty$ the boundary values of $\mathcal{K}\tilde{\mu}(z)$ on $\mathbb{R}$ tend to 0 at $\pm \infty$. It follows that

$$\tilde{\mu}((x, x + 1)) \leq \int_x^{x+1} |\mathcal{K}\tilde{\mu}(t)| dt \rightarrow 0 \text{ as } x \rightarrow \pm \infty.$$  

As was discussed in Subsection 3.1, this implies $\tilde{\mu} \notin \text{PW}$.

Remark 5.1. This example has the following interpretation: even though $\tilde{\mu}$ is not a PW-measure, the corresponding inverse spectral problem can be solved by switching to its dual $\mu \in \text{PW}$. 
If, on the other hand, all (or two) of the measures in an AC-family are PW-sampling then a combination of Theorems 4.6 and 5.1 provides a continuum of often rather nontrivial identities between solutions to the inverse spectral problems corresponding to different measures from the family.

We now formulate a necessary and sufficient condition for all measures dual to a given $\mu \in \text{PW}$ to be PW-sampling.

**Theorem 5.3.** Let $\mu \in \text{PW}$. All measures dual to $\mu$ are PW-sampling if and only if there exists $d \neq 0$ such that $\mathcal{K}\mu(z)$ is bounded on the line $\{\Re z = d\}$.

Note that $\mathcal{K}\mu$ is bounded on some line $\{\Re z = d\}$, $d \neq 0$, if and only if $\mathcal{K}\mu$ is bounded on every such line.

First we establish the following

**Lemma 5.1.** Let $\nu$ be a positive measure on $\mathbb{R}$,

$$\frac{1}{\pi} \int \frac{d\nu(x)}{1 + x^2} \leq C,$$

and let $I_1, I_2, I_3$ be three adjacent intervals, enumerated left to right, $|I_j| = L > 0$. Let $I = I_1 \cup I_2 \cup I_3$ be an open interval. Suppose that

1. $|\mathcal{K}\nu| < C$ on $\mathbb{R} + i$;
2. $\nu(I_1), \nu(I_3) > \varepsilon$.

Let $\tilde{\nu} = \tilde{\nu}_b, b \in \mathbb{R}$ be a dual measure. Then $\tilde{\nu}(I) > d$, where $d > 0$ depends only on $C, L, \varepsilon$ and $b$, but not on the position of the intervals on $\mathbb{R}$.

**Proof.** First, let $I$ be an interval centered at zero (i.e., we do not aim to prove that $d$ does not depend on the position of $I$ at this stage). Denote by $d_0$ the infimum of $\tilde{\nu}(I)$ taken over all $\nu$ satisfying the conditions. Suppose that $d$ is zero and that $\nu^n$ is a sequence of measures such that $\tilde{\nu}^n_b(I) \to 0$. Choosing a subsequence of $\nu^n$ if necessary, we assume that $\nu^n$ converges $*$-weakly in the space of $\Pi$-finite measures on $\mathbb{R}$ to a $\Pi$-finite measure $\eta$. Note that $\eta$ must still satisfy (2), and therefore is nontrivial. The relation between Cauchy integrals for dual measures implies $\tilde{\nu}^n_b \to \tilde{\eta}_b$. As $I$ is open, $\tilde{\eta}(I) = \lim \tilde{\nu}^n(I) = 0$. As $\tilde{\eta}$ is a positive measure vanishing on $I$, $\mathcal{K}\tilde{\eta}$ increases monotonically on $I$. It follows that $\eta$ may have only one point-mass on $I$, which contradicts (2). Therefore, $d > 0$.

Now let $I$ be an arbitrary interval and let us show that because of 1), $d$ does not depend on its position on $\mathbb{R}$.

Suppose that there exists a sequence $I^n$ of open intervals and a sequence $\mu^n$ of measures such that each of them satisfies the conditions of the lemma in place of $\mu$ and $I$, but $\tilde{\nu}^n_b(I^n) \to 0$. Suppose that $I^n$ is centered at $x^n$. Consider the sequence of measures $\nu^n = \mu^n(x - x^n)$. As

$$\mathcal{K}\nu^n(z) = \mathcal{K}\mu^n(z - x^n) + b^n,$$

where

$$b^n = \int \frac{t}{1 + t^2} d\nu(t) - \int \frac{t - x^n}{1 + (t - x^n)^2} d\nu(t) = 3\mathcal{K}\nu(i) - 3\mathcal{K}\nu(i + x^n),$$

we obtain $\tilde{\nu}^n_{b-b^n} = \tilde{\nu}_b(z - x^n)$. Note that because of 1), $|b^n| < 2C$. 


Choosing a subsequence if necessary, we can assume that \( b^n \to b_0 \). It follows from (5.2) and (5.1) that \( P \nu^n_{b-b_0} - P \nu^n_{b-b_n} \to 0 \) pointwise in \( C_+ \). Hence, \( \nu^n_{b-b_0} - \nu^n_{b-b_n} \to 0 \), *-weakly and \( \nu^n_{b-b_0}(I) - \nu^n_{b-b_n}(I) \to 0 \). By our choice of the measures \( \nu^n = \mu^n(x-x^n) \), \( \nu^n_{b-b_n}(I) \to 0 \), which implies \( \nu^n_{b-b_0}(I) \to 0 \). As \( \nu^n \) satisfy the conditions of the lemma on the interval \( I \) centered at 0, this contradicts the first part of the proof.

**Proof of Theorem 5.3.** Let \( \mu \in \text{PW} \) satisfy the conditions of the theorem. Then \( \bar{\mu}((x,x+1)) \) is uniformly bounded. Indeed, if it is not, then \( P\bar{\mu} \) is unbounded on \( \mathbb{R} + id \). Let \( x_n \in \mathbb{R} \) be a sequence of points such that \( P\bar{\mu}(x_n + id) \to \infty \). It follows that \( \phi(x_n + id) \to 1 \) and \( P\mu(x_n + id) \to 0 \). Then one can choose intervals \( J_n \) centered at \( x_n \) such that \( |J_n| \to \infty \) and \( \mu(J_n) \to 0 \). Existence of such intervals contradicts \( \mu \in \text{PW} \) because it implies

\[
\|f(x-x_n)\|_{L^2(\mu)} \to 0
\]

for any \( f \in \text{PW}_a \).

Let \( I_1, I_2, I_3 \) be consequent \( \delta \)-massive intervals for \( \mu \), from the statement of Theorem 3.1. Then by the last lemma \( I = I_1 \cup I_2 \cup I_3 \) is a \( \delta' \)-massive interval for \( \bar{\mu} \) for some \( \delta' > 0 \). As \( \mu \) is sampling for every \( \text{PW}_{3a} \), \( \bar{\mu} \) is sampling for every \( \text{PW}_a \).

In the opposite direction, suppose that \( K\mu \) is unbounded on some \( \mathbb{R} + i\varepsilon \). Similar to above, \( P\bar{\mu} \) is then not bounded from below by a positive constant, which implies the existence of intervals \( J_n \) such that \( |J_n| \to \infty \) but \( \bar{\mu}(J_n) \to 0 \). Hence, \( \bar{\mu} \notin \text{PW} \).



### 6 PERIODIC SPECTRAL MEASURES

#### 6.1 Measures from \( M^+(\mathbb{T}) \)

Let \( M^+(\mathbb{T}) \) be the set of all finite positive measures on the unit circle \( \mathbb{T} \sim [0,2\pi] \) with infinite support. We identify measures on the circle with \( 2\pi \)-periodic measures on \( \mathbb{R} \), \( M^+(\mathbb{T}) \subset M^+(\mathbb{R}) \), and define \( M_{\text{even}}^+(\mathbb{T}) \subset M^+(\mathbb{R}) \) as the subset of even periodic measures.

Notice that all measures in \( M^+(\mathbb{T}) \) are \( \Pi \)-finite. Furthermore, as was mentioned before, it follows from Theorem 3.1 that \( M^+(\mathbb{T}) \subset \text{PW} \). Let us list two more important properties of \( M^+(\mathbb{T}) \) pertaining to our problems.

**Proposition 6.1.** If \( \mu \in M^+(\mathbb{T}) \), then all dual measures are in \( M^+(\mathbb{T}) \)

**Proof.** Periodicity of \( \bar{\mu}_b \) follows from (5.2) and (5.1). A well-known property of dual measures is that supports of \( \mu \) and \( \bar{\mu} \) alternate, that is, between any two points of one support there is a point of another. It follows that, as the support \( \mu \) is infinite, so is the support of \( \bar{\mu} \).

**Proposition 6.2.** If \( \mu \in M_{\text{even}}^+(\mathbb{T}) \), then there is a unique \( \tilde{\mu} \in M_{\text{even}}^+(\mathbb{T}) \) such that \( \mu \) and \( \tilde{\mu} \) are dual: \( \tilde{\mu} = \bar{\mu}_b \), \( b = -\mathcal{R} \mathcal{K} \mu(0) \).

**Proof.** From (5.1), we see that in order for \( \bar{\mu}_b \) to be even, \( \mathcal{R} \frac{1+\phi_b}{1-\phi_b} \) must be even. From (5.3), we obtain that

\[
P\bar{\mu}_b = \mathcal{R} \frac{i}{\mathcal{K} \mu + b} = \frac{P\mu}{|\mathcal{K} \mu + b|^2}.
\]
In the last expression, $P\mu$ is even because $\mu$ is even. In order for the whole fraction to be even, $|K\mu + b|$ must be even. If $b = -\Im K\mu(0)$, then

$$\Im (K\mu + b) = \Im K\mu = P\mu$$

is even, $\Re (K\mu + b)$ is odd and $|K\mu + b|$ is even. For any other value of $b$, $\Re (K\mu(0) + b) \neq 0$, $\Re (K\mu + b)$ is not odd and $[\Re (K\mu + b)]^2$ is not even. Hence, $|K\mu + b|$ is not even. \[\square\]

Note that for the unique even dual measure $\tilde{\mu}$ we have

$$K\tilde{\mu} = -\frac{1}{K\mu}, \quad P\tilde{\mu} = -\Re \frac{1}{K\mu}.$$  

(6.1)

### 6.2 | Moments

If $\mu \in M^+(\mathbb{T})$, we can consider the Fourier series

$$\mu \sim \sum_{-\infty}^{\infty} \gamma_k e^{ikx},$$

where

$$\gamma_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikx} \, d\mu(x) = \alpha_k + i\beta_k$$

are the “trigonometric moments” of $\mu$. Clearly,

$$\gamma_{-k} = \overline{\gamma_k},$$

so

$$\mu \sim \gamma_0 + 2 \sum_{1}^{\infty} (\alpha_k \cos kx - \beta_k \sin kx),$$

and $\mu$ is even if and only if all moments are real, that is, all $\beta_k = 0$.

The moments can be arranged into the Toeplitz matrix

$$\Gamma(\mu) = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots \\ \gamma_{-1} & \gamma_0 & \gamma_1 & \cdots \\ \gamma_{-2} & \gamma_{-1} & \gamma_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

We will denote by $\Gamma_n(\mu)$ the $n \times n$ matrix in the upper left corner of $\Gamma$. It is well-known that a sequence $\{\gamma_k\}$ is the sequence of moments of some $\mu \in M^+(\mathbb{T})$ if and only if

$$\forall n, \quad \det \Gamma_n > 0.$$  

Finally let us recall the relation between the moments of a periodic measure and its Fourier transform: if $\mu \sim \sum \gamma_k e^{ikx}$, then

$$\hat{\mu} = \sqrt{2\pi} \sum_{-\infty}^{\infty} \gamma_k \delta_k.$$
6.3 Computation of $h^\mu(t)$

Recall that if $\mu \in \text{PW}$, then

$$h_\mu(t) := \pi \frac{d}{dt} \left[ L_{\mu, t}^{-1} \circ k_t(0) \right].$$

If $A = (a_{jk})$ is a matrix we use the notation $\Sigma[A]$ for the sum of the elements of $A$:

$$\Sigma[A] = \sum_{j,k} a_{jk}.$$

**Theorem 6.1.** If $\mu \in M^+(\mathbb{T})$, then the function $h^\mu(t)$ is locally constant on $\mathbb{R}_+ \setminus \frac{1}{2} \mathbb{N}$, that is,

$$h_\mu(t) = h_0, h_1, h_2, \ldots \text{ on } (0, \frac{1}{2}), \left( \frac{1}{2}, 1 \right), \left( 1, \frac{3}{2} \right), \ldots,$$

and

$$h_n = \Sigma [\Gamma_{n+1}(\mu)^{-1}] - \Sigma [\Gamma_n(\mu)^{-1}], \quad h_0 = \Sigma [\Gamma_1^{-1}] = \gamma_0^{-1}.$$

**Proof.**

As

$$\hat{\mu} = \sqrt{2\pi} \sum \gamma_k \delta_k,$$

the Fourier transform $\phi_t$ of $k_t$ satisfies the relations in Theorem 4.7. Let

$$f = f_t = \sqrt{2\pi} \psi_t.$$

Then $f$ satisfies

$$\sum \gamma_k f(\xi - k) = 1 \text{ on } (-t, t) \text{ and } f = 0 \text{ on } \mathbb{R} \setminus [-t, t]. \quad (6.2)$$

Recall

$$h^\mu(t) = \pi \frac{d}{dt} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi_t = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} f_t.$$

Let us now find $f_t$ from (6.2).

Case $0 < t < \frac{1}{2}$. If $\xi \not\in (-t, t)$, then $f(\xi) = 0$. If $\xi \in (-t, t)$, then $\xi \pm 1, \xi \pm 2, \ldots$ are not in $(-t, t)$, so

$$\sum \gamma_k f(\xi - k) = \gamma_0 f(\xi) \text{ must be } = 1.$$

Thus,

$$f_t = \gamma_0^{-1} 1_{(-t, t)}, \quad \int f_t = 2\gamma_0^{-1}, \quad h^\mu(t) = \gamma_0^{-1}.$$

Case $\frac{1}{2} < t < 1$. Consider the partition of the interval $(-t, t)$ by the points from $\pm t + \mathbb{Z}$:

$$-t < t - 1 < -t + 1 < t.$$

We claim that $f = f_t$ is constant on each interval of the partition. For instance, let $\xi \in (-t, t - 1)$. Then $f(\xi + k) = 0$ unless $k = 0$ or $k = 1$, so equating the value of $f * \hat{\mu}$ to 1 on the first and third
intervals of the partition we get
\[ \gamma_0 f(\xi) + \gamma_{-1} f(\xi + 1) = 1 \]
and
\[ \gamma_0 f(\xi + 1) + \gamma_{1} f(\xi) = 1. \]
The linear system for \( f(\xi) \) and \( f(\xi + 1) \) have the matrix \( \Gamma_2 \), which implies
\[ \begin{pmatrix} f(\xi) \\ f(\xi + 1) \end{pmatrix} = \Gamma_2^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \]
In particular, it follows that the values \( f(\xi) \) and \( f(\xi + 1) \) are constant and
\[ f(\xi) + f(\xi + 1) = \Sigma[\Gamma^{-1}_2]. \]
(Here we use the formula \( \Sigma[M] = M \vec{1} \cdot \vec{1}, \) where \( \vec{1} = (1, 1, ..., 1)^T \).)
On the other hand, if \( \xi \) is in the middle interval \((t - 1, -t + 1)\), then
\[ f(\xi) = \gamma_0^{-1} = \Sigma[\Gamma_1^{-1}] \]
by the same argument as in the previous case. Note that the middle interval is shrinking with velocity 2, while the two other intervals are expanding with velocity 2. Integrating \( f \) from \(-t\) to \( t\) and then differentiating the result with respect to \( t \) we get
\[ h^\mu(t) = \Sigma[\Gamma_2^{-1}] - \Sigma[\Gamma_1^{-1}]. \]
Let us now consider the general case \( \frac{n}{2} < t < \frac{n+1}{2} \).
There is a simple way to visualize the partition of the interval \((-t, t)\), see the picture below. In the \((\xi, t)\)-plane consider the interval (blue) on the line \( t = \text{const} \) lying in the sector \( t > |\xi| \). The points of the partition are given by intersections of the interval with the lines \( t = \pm \xi + n, \) \( n \in \mathbb{Z} \), which form a square lattice in the sector.
If we enumerate the $2n + 1$ intervals of the partition from left to right, then odd numbered intervals

$$I_1 = (-t, t-n), I_3 = (-t + 1, t - (n - 1)), \ldots, I_{2n+1} = (-t + n, t)$$

are expanding while the even numbered intervals

$$I_2 = (t - n, -t + 1), I_4 = (t - (n - 1), -t + 2), \ldots, I_{2n} = (t - 1, -t + n)$$

are shrinking. If $\xi \in I_1$ then $\xi + k \in I_{2k+1}$, $k = 1, 2, \ldots, n$. Equating the values of $f \ast \mu$ on each odd interval to 1, we obtain the system of $n + 1$ linear equations

$$
\begin{pmatrix}
  f(\xi) \\
  f(\xi + 1) \\
  \vdots \\
  f(\xi + n)
\end{pmatrix}
= 
\Gamma^{-1}_{n+1}
\begin{pmatrix}
  1 \\
  1 \\
  \vdots \\
  1
\end{pmatrix},
$$

from which we deduce

$$f(\xi) + \cdots + f(\xi + n) = \Sigma[\Gamma^{-1}_{n+1}]$$

(6.3)

for the values on the expanding intervals. Similarly for $\xi \in I_2$,

$$
\begin{pmatrix}
  f(\xi) \\
  f(\xi + 1) \\
  \vdots \\
  f(\xi + (n - 1))
\end{pmatrix}
= 
\Gamma^{-1}_{n}
\begin{pmatrix}
  1 \\
  1 \\
  \vdots \\
  1
\end{pmatrix},
$$

which yields

$$f(\xi) + \cdots + f(\xi + (n - 1)) = \Sigma[\Gamma^{-1}_{n}]$$

(6.4)

for the values on the shrinking intervals. Integrating $f$ from $-t$ to $t$ and differentiating the result with respect to $t$, we obtain

$$h^{\mu}(t) = \Sigma[\Gamma^{-1}_{n+1}] - \Sigma[\Gamma^{-1}_{n}]$$

as claimed. \qed

### 6.4 Computation of $g^\mu(t)$

Let $\mu \in M^+(\mathbb{T})$ and let

$$
\Delta(\mu) = 
\begin{pmatrix}
  0 & \gamma_1 & \gamma_2 & \cdots \\
  -\gamma_1 & 0 & \gamma_1 & \cdots \\
  -\gamma_2 & -\gamma_1 & 0 & \cdots \\
  \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
$$
where, as before, $\gamma_k$ are trigonometric moments of $\mu$. We denote by $\Delta_n$ the $n \times n$ matrix in the upper left corner of $\Delta$.

**Theorem 6.2.** If $\mu \in M^+(\mathbb{T})$, then the function $g^\mu(t)$ is locally constant on $\mathbb{R}_+ \setminus \frac{1}{2} \mathbb{N}$, that is,

$$g^\mu(t) = g_0, \ g_1, \ g_2, \ldots \quad \text{on} \quad \left(0, \frac{1}{2}\right), \ \left(\frac{1}{2}, 1\right), \ \left(1, \frac{3}{2}\right), \ldots$$

and

$$g_n = \sum \Delta_{n+1} \Gamma^{-1}_{n+1} - \sum \Delta_n \Gamma^{-1}_n.$$

**Proof.** To use Theorem 4.6, we need to calculate $\tilde{l}_i(0)$. Once again we will switch to the Fourier transform and use the identity $\tilde{l}_i(0) = \int_{-t}^{t} F(\tilde{l}_i).$ Recall that $\tilde{l}_i = H^\mu k_i$. For $\mu \in M^+(\mathbb{T}),$

$$H^\mu k_i = Kk_i \mu - k_i K\mu + ck_i,$$

see Remark 4.3. The last term can be disregarded, see Theorem 3.4 and Remark 3.3.

The first term $Kk_i \mu$ is equal to the Hilbert transform $Hf_\mu$ on $\mathbb{R}$. Recall that $F(k_i \mu) = 1$ on $(-t, t)$ and that $F(Hg)(s) = \text{sign } s \cdot \hat{g}$. Hence,

$$\int_{-t}^{t} F(Kk_i \mu)(s)ds = \int_{-t}^{t} \text{sign } s \cdot F(k_i \mu)ds = \int_{-t}^{t} \text{sign } s \ ds = 0.$$

Therefore,

$$\tilde{l}_i(0) = \int_{-t}^{t} F(k_i K\mu)(s)ds = \int_{-t}^{t} f_i \ast (\text{sign } s \cdot \hat{\mu}(s))ds,$$

where $f_i = \hat{k}_i$, like in the last proof.

Suppose that $t \in (\frac{n}{2}, \frac{n+1}{2})$. Let $I_k$, $k = 1, 2, \ldots, 2n + 1$ be the intervals of the partition of $(-t, t)$ like in the last proof. As was established there, $f_i$ is constant on the intervals $I_k$ and its values on the odd and even intervals are given by (6.3) and (6.4), respectively. It follows that the values of $q = f_i \ast (\text{sign } s \cdot \hat{\mu}(s))$ on odd $I_k$ are given by

\[
\begin{pmatrix}
q(\xi) \\
q(\xi + 1) \\
\vdots \\
q(\xi + n)
\end{pmatrix} = \Delta_{n+1} \Gamma^{-1}_{n+1}
\begin{pmatrix}
1 \\
1 \\
\vdots \\
1
\end{pmatrix},
\]

for $\xi \in I_1$ and for even $I_k$ by

\[
\begin{pmatrix}
q(\xi) \\
q(\xi + 1) \\
\vdots \\
q(\xi + (n - 1))
\end{pmatrix} = \Delta_{n} \Gamma^{-1}_{n}
\begin{pmatrix}
1 \\
1 \\
\vdots \\
1
\end{pmatrix},
\]

for $\xi \in I_2$. Finishing like in the last proof, we obtain the formula in the statement. □
6.5  Canonical systems and orthogonal polynomials on the unit circle

In this section, we study the connection between the inverse spectral problems for CS with periodic spectral measures and orthogonal polynomials on the unit circle. We show that the formula for $h_n$ from Theorem 6.1 can be interpreted as point evaluations of such polynomials.

For $\mu \in M_+(\mathbb{T})$ we denote by ONP($\mu$) the family of polynomials of $z$ on the unit circle orthonormal in $L^2(\mu)$. By $\phi_n \in \text{ONP}(\mu)$ we denote the polynomial of degree $n$.

**Theorem 6.3.** Let $\mu \in M^+(\mathbb{T})$ and let $\{\phi_n\}$ be ONP($\mu$). We have

$$h_n^\mu = |\phi_n(1)|^2.$$  

In the proof, we will use the following notation. We denote

$$\mathcal{P}_n(\mu) = \text{span}\{1, \ldots, z^n\} \subset L^2(\mu).$$

Then

$$K_n^w \equiv K_n(z, w) = \sum_{j=0}^n \phi_j(z)\overline{\phi_k(w)}$$

is the reprokernel of $\mathcal{P}_n(\mu)$.

In this section, $m = m_\mathbb{T}$ stands for the normalized Lebesgue measure on $\mathbb{T}$, $m(\mathbb{T}) = 1$. For ONP($m$) we have $\phi_n(z) = z^n$ and

$$\hat{K}(z, w) = \sum_{j=0}^n z^j \overline{w}^j.$$  

If $w = 1$, then we will write $k_n(z)$ for $K_n(z, 1)$, in particular

$$\hat{k}_n(z) = \sum_{j=0}^n z^j.$$  

(6.5)

We consider truncated Toeplitz operators on $\mathcal{P}_n = \mathcal{P}_n(m)$:

$$T_\mu = T_n^{\mu} : \mathcal{P}_n \rightarrow \mathcal{P}_n$$

defined by

$$(T_\mu p, q) = \int p\overline{q} \, d\mu \quad \forall \, p, q \in \mathcal{P}_n.$$  

For absolutely continuous measures $\mu$ with $L^2(m)$-densities $T_\mu : p \mapsto P_n(p\mu)$ where $P_n$ is the orthogonal projection onto $\mathcal{P}_n$ in $L^2(m)$. Note that the matrix of $T_n$ in the basis $\{z^j\}$ is

$$T_{jk} = (Tz^k, z^j) = \int z^k \overline{z}^j \, d\mu = \gamma_{j-k},$$

which is the matrix $\Gamma_{n+1}^T(\mu)$, the transpose of $\Gamma_{n+1}(\mu)$ from Subsection 6.2.
Lemma 6.1. If $j_n : P_n \to P_n(\mu)$ denote the “identity” embedding operators, then

\[ j_n^* j_n = T_\mu \text{ on } P_n. \]

Proof.

\[ (j_n^* j_n p, q) = (j_n p, j_n q)_\mu = \int p \bar{q} \, d\mu = (T_\mu p, q). \]

Lemma 6.2.

\[ \forall w \in \mathbb{C}, \quad j_n \left[ T_\mu^{-1} K_w \right] = K_w \]

In particular,

\[ j_n \left[ T_\mu^{-1} k_n \right] = k_n. \]

Proof. Recall that

\[ \forall p, q \in P_n, \quad (j p, j q)_\mu = (T_\mu p, q). \]

Let $p = j^{-1} K_w$. Then we have

\[ (K_w, q) = \bar{q}(w) = (K_w, j q)_\mu = (T j^{-1} K_w, q). \]

Proof of Theorem 6.3. We will first show that

\[ \Sigma(\Gamma_{n+1}^{-1}(\mu)) = k_n(1). \]

Indeed, for an $n \times n$ matrix $A$ and the standard basis $e_0, \ldots, e_n$ in $\mathbb{R}^n$,

\[ \Sigma(A) = \sum_j \sum_k (A e_j, e_k) = (A \mathbf{1}, \mathbf{1}), \text{ where } \mathbf{1} = \sum_{j=0}^n e_j. \]

Applying this to the matrix $\Gamma_{n+1}^T(\mu)$ of $T_\mu$ with respect to the basis $1, \ldots, z^n$ and taking into account that $\mathbf{1} = k_n$, see (6.5), we get

\[ \Sigma(\Gamma_{n+1}^{-1}(\mu)) = \Sigma \left( \Gamma^{-1}_{n+1}(\mu) \right) = \left( T_\mu^{-1} k_n, k_n \right) = k_n(1). \]

To finish the proof of the theorem, we observe that

\[ k_n(1) = K_n(1, 1) = \sum_{j=0}^n |\phi_j(1)|^2, \]

and

\[ h_n = \Sigma \left( \Gamma_{n+1}^{-1} \right) - \Sigma \left( \Gamma_n^{-1} \right) = k_n(1) - k_{n-1}(1) = |\phi_n(1)|^2. \]

Theorem 6.3 allows us to describe the change of $h_n$ corresponding to a shift of a periodic spectral measure. For $\eta \in \mathbb{T}$ we define $\mu^{(\eta)}(\xi) = \mu(\eta \xi)$. 


Corollary 6.1.

\[ h^{(\eta)}_n = |\phi_n(\eta)|^2. \]

Proof. ONP of \( \mu^{(\eta)} \) are the polynomials \( \phi_n(\eta) \).

\[ \Box \]

7 | EXAMPLES OF SPECTRAL PROBLEMS

The goal of this section is to illustrate our formulae with explicit examples and calculations.

7.1 | Constant Hamiltonians

Let us first consider spectral problems, inverse and direct, for systems with constant Hamiltonians.

Lemma 7.1. Let the constants \( h_1 > 0, h_2 > 0, g \in \mathbb{R} \) satisfy \( h_1 h_2 = 1 + g^2 \), and

\[ H = \begin{pmatrix} h_1 & g \\ g & h_2 \end{pmatrix} \in SL(2, \mathbb{R}). \]

Then

\[ M(t, z) = \begin{pmatrix} C - gS & -h_2S \\ h_1S & C + gS \end{pmatrix}, \]

where \( C := \cos zt \) and \( S := \sin zt \).

Proof. Just verify \( M(0) = I \) and \( \Omega \dot{M} = zHM \) using \( \det(H) = 1 \). To guess the solution, use undetermined coefficients.

\[ \Box \]

Corollary 7.1. \( \mu = h_1^{-1} \).

Proof. Notice that the solution can be rewritten as

\[ \begin{pmatrix} u_z(t) \\ v_z(t) \end{pmatrix} \begin{pmatrix} C - gS \\ h_1S \end{pmatrix} = \begin{pmatrix} 1 - g \\ 0 \end{pmatrix} \begin{pmatrix} C \\ S \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{h_1}} & -\frac{g}{\sqrt{h_2}} \\ 0 & \frac{1}{\sqrt{h_1}} \end{pmatrix} \begin{pmatrix} \sqrt{h_1}C \\ \sqrt{h_1}S \end{pmatrix} \]

As the determinant of the last 2 \( \times \) 2 matrix is 1, by Theorem 2.2 from [31] \( F_t(z) = \sqrt{h_1}C - i\sqrt{h_1}S \) generates the same chain of dB spaces as \( E_t(z) = u_z(t) - iv_z(t) \). But

\[ B(F_t) = B\left( \sqrt{h_1}e^{-izt} \right), \]

and \( h_1^{-1} = 1/|F_t|^2 \) is the spectral measure.

\[ \Box \]

Alternatively, to prove the last corollary we can apply the general algorithm described in Subsection 6.3 to the even measure \( \mu = a, a = h_1^{-1} \). We have \( \Gamma(\mu) = aI \), so \( \Gamma_n^{-1} = 1/a_nI_n \) and \( S(\Gamma_n^{-1}) = \frac{n}{a} \).

Then

\[ h_n = \frac{n+1}{a} - \frac{n}{a} = \frac{1}{a}. \]
To practice calculating $h_2$ using the dual measure, notice that for $\mu = a, i\mathcal{K}\mu + ib = a + ib$ and $\mathcal{P}\bar{\mu}; = \bar{a} := \frac{a}{a^2 + b^2}$. We have

$$h = h^\mu = \frac{1}{a}, \quad g^\mu = 0, \quad \bar{h} = h^\bar{\mu} = \frac{1}{\bar{a}}, \quad g^\bar{\mu} = 0.$$ 

Hamiltonians corresponding to the pair of measures $\mu, \bar{\mu}$ are given by the constant matrix

$$H = \begin{pmatrix} h & kh \\ kh & \bar{h} \end{pmatrix}, \quad k \in \mathbb{R}.$$ 

### 7.2 Spectral measure $\mu = 1 + \cos x$

Let us solve the inverse spectral problem for this particular measure. Recall that by Theorem 6.1, for a $2\pi$-periodic even measure $\mu = 1 + \cos x$ the Hamiltonian is diagonal and constant on

$$\left(0, \frac{1}{2}\right), \left(\frac{1}{2}, 1\right), \left(1, \frac{3}{2}\right), \ldots$$

To find the values of the Hamiltonian on these intervals using Theorem 6.1, notice that

$$\Gamma(\mu) = \begin{pmatrix} 1 & 1 & \cdots & 0 \\
\frac{1}{2} & 1 & \cdots & \frac{1}{2}
\vdots & \vdots & \ddots & \vdots \\
0 & \frac{1}{2} & \cdots & 1
\end{pmatrix},$$

$$\Gamma^{-1}_1 = (1), \quad \Gamma^{-1}_2 = \frac{4}{3} \begin{pmatrix} 1 & -\frac{1}{2} \\
-\frac{1}{2} & 1
\end{pmatrix}, \ldots$$

and

$$\Sigma_1 = 1, \Sigma_2 = \frac{4}{3}, \ldots$$

for $\Sigma_n = \Sigma_1 [\Gamma^{-1}_n]$. We conclude that the values of $h_0, h_1, h_2, \ldots$ are

$$1, \frac{1}{3}, \frac{2}{3}, \frac{2}{5}, \frac{3}{5}, \frac{3}{7}, \frac{4}{7}, \frac{4}{9}, \ldots$$

$$\ldots, h_{2n} = \frac{n + 1}{2n + 1}, h_{2n+1} = \frac{n + 1}{2n + 3}, \ldots$$

As the Hamiltonian $H$ is diagonal and $\det H = 1$, the values for the second diagonal term can be calculated as $1/h_n$.

From (6.1), we can find the evendual measure $\bar{\mu}$:

$$\mathcal{P}\mu = \mathfrak{R}(1 + e^{iz}), \quad \mathcal{P}\bar{\mu} = \mathfrak{R} \frac{1}{1 + e^{iz}} = \frac{1}{2} + \frac{1}{2} \mathfrak{R} \frac{1 - S}{1 + S},$$
The spectral problem.

\[ \bar{\mu} = \frac{1}{2} + \pi \sum_{-\infty}^{\infty} \delta_{\pi+2\pi k}, \quad \Gamma(\bar{\mu}) = \begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \cdots \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \frac{1}{2} & \cdots \\ \frac{1}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]

Computation gives \( \bar{h}_0 = 1, \bar{h}_1 = 3, \bar{h}_2 = \frac{3}{2}, \bar{h}_3 = \frac{5}{2}, \) and so on, as expected from the above calculation for \( h_k \).

Note that one can find \( h_n \) for \( \mu = 1 + \cos x \) and many other periodic measures using tables of orthogonal polynomials on the unit circle available in the literature, together with Theorem 6.3. To find \( \bar{h}_n \), one can apply Theorem 6.3 to the dual measure, see Subsection 7.4. Further examples based on that approach will be given below, see Subsections 7.5 and 7.6.

### 7.3 Four steps by hand

Suppose \( \mu \) is even (then the moments \( \gamma_k \) are real). As before, denote \( \Sigma_n = \Sigma[\Gamma^{-1}_n] \).

**Lemma 7.2.**

\[ \Sigma_1 = \frac{1}{\gamma_0}, \quad \Sigma_2 = \frac{2}{\gamma_0 + \gamma_1}, \quad \Sigma_3 = \frac{3\gamma_0 - 4\gamma_1 + \gamma_2}{(\gamma_0 + \gamma_2)\gamma_0 - 2\gamma_1^2}, \]

\[ \Sigma_4 = \frac{2(2\gamma_0 - \gamma_1 - 2\gamma_2 + \gamma_3)}{(\gamma_0 + \gamma_3)(\gamma_0 + \gamma_1) - (\gamma_1 + \gamma_2)^2}. \]

**Proof.** The first formula is obvious. The second follows from

\[ \Gamma^{-1}_2 = \frac{1}{\gamma_0^2 - \gamma_1^2} \begin{pmatrix} \gamma_0 & -\gamma_1 \\ -\gamma_1 & \gamma_0 \end{pmatrix} \]

Third formula: let \( \tilde{x} \) be defined by

\[ \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 \\ \gamma_1 & \gamma_0 & \gamma_1 \\ \gamma_2 & \gamma_1 & \gamma_0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \]

By symmetry we have \( x_1 = x_3 \), and

\[ \Sigma_3 = (\Gamma^{-1}_3, \tilde{1}) = x_1 + x_2 + x_3 = 2x_1 + x_2. \]

The variables \( 2x_1 \) and \( x_2 \) satisfy the system

\[ \frac{\gamma_0 + \gamma_2}{2}(2x_1) + \gamma_1 x_2 = 1, \quad \gamma_1(2x_1) + \gamma_0 x_2 = 1 \]
(the first two equations in the $3 \times 3$ system above), and we have

$$\Sigma_3 = 2x_1 + x_2 = \Sigma(M^{-1}),$$

where $M$ is the $2 \times 2$ matrix of the last system. The derivation of the forth formula is as simple – we use the symmetry $x_1 = x_4$ and $x_2 = x_3$, so again we only need to invert a $2 \times 2$ matrix. \hfill \square

**Example 7.1.** If $\gamma_0 = 1$, $\gamma_1 = \frac{1}{2}$, $\gamma_2 = 0$, then we have

$$\Sigma_1 = 1, \quad \Sigma_2 = \frac{4}{3}, \quad \Sigma_3 = 2, \quad \Sigma_4 = \frac{12}{5},$$

and if $\gamma_0 = 1$, $\gamma_1 = -\frac{1}{2}$, $\gamma_2 = \frac{1}{2}$, then

$$\Sigma_1 = 1, \quad \Sigma_2 = 4, \quad \Sigma_3 = \frac{11}{2}.$$

### 7.4 Finding dual measures

Recall that if $\phi$ is a Schur function, then we define Clark’s measures $\sigma_{\alpha}$ on $\hat{\mathbb{T}}$ for $\alpha \in \mathbb{T}$ via the equation

$$P\sigma_{\alpha} = \Re \frac{\alpha + \phi}{\alpha - \phi},$$

see Subsection 2.5.

Let us find all dual measures for $\mu = 1 + \cos x$ using the formulae from Subsection 5.2. It is clear the corresponding Schur function $\phi$ is a periodic nonconstant function and therefore $\phi - \alpha \notin L^2(\mathbb{R})$ for any $\alpha$. Hence, the dual measures have no masses at $\infty$.

For $S(z) := e^{iz}$ we have

$$P\mu = \mathfrak{F} \mathcal{K} \mu = \Re(1 + S) \text{ and } \mathcal{K} \mu = i(1 + S).$$

Hence, by (5.3),

$$P\bar{\mu}_b = \Re \frac{1}{1 + S - ib}. \quad (7.1)$$

Case $b = 0$ (the only choice such that $\bar{\mu}$ is even):

$$P\bar{\mu} = \frac{1}{2} + \frac{1}{2} \Re \frac{1 - S}{1 + S} = P\left[ \frac{1}{2} + \frac{1}{2} \sigma_{-1} \right],$$

and by (2.7),

$$\bar{\mu} = \frac{1}{2} + \pi \sum_{n \in \mathbb{Z}} \delta_{(2n+1)\pi}.$$

Let us also find the moments of $\bar{\mu}$:

$$\gamma_k = \frac{1}{4\pi} \int_0^{2\pi} e^{-ikx} \, dx + \frac{1}{2} \sum_{n = -\infty}^{\infty} \int_0^{2\pi} e^{-ikx} \delta_{(2n+1)\pi}(x) = I + II.$$
In $II$, only the term with $n = 0$ has an atom in $(0, 2\pi)$, so

$$II = \frac{1}{2} e^{-ik\pi} = \frac{1}{2}(-1)^k.$$ 

Of course, $I = \frac{1}{2}$ if $k = 0$ and $I = 0$ otherwise.

Case $b \neq 0$. From (7.1) one can conclude that $P\tilde{\mu}_b$ is bounded near $\mathbb{R}$ and therefore $\tilde{\mu}$ is absolutely continuous. Hence, on $\mathbb{R}$,

$$\tilde{\mu} = \Re \frac{1}{1 + S - ib} = \Re \frac{1}{1 + \cos x + i(sin x - b)} = \frac{1 + \cos x}{2 + 2\cos x - 2b \sin x + b^2}.$$ 

(Note that formally setting $b = 0$ in the last formula we get the wrong answer because $\tilde{\mu}_0$ is not absolutely continuous) To find the moments, we may use geometric progression,

$$\tilde{\mu} = \Re \left[ \frac{1}{1 - ib} \frac{1}{1 + \frac{e^{ix}}{1 - ib}} \right] = \Re \left[ \frac{1}{1 - ib} - \frac{e^{ix}}{(1 - ib)^2} + \frac{e^{2ix}}{(1 - ib)^3} - \ldots \right],$$ 

and we get

$$\gamma_0 = \frac{1}{1 + b^2}, \quad \gamma_1 = -\frac{1}{2 (1 - ib)^2}, \quad \gamma_2 = \frac{1}{2 (1 - ib)^3}, \quad \ldots$$ 

(and $\gamma_{-k} = \overline{\gamma_k}$). This follows from the representations like

$$\Re \frac{e^{ix}}{(1 - ib)^2} = \frac{1}{2} \frac{e^{ix}}{(1 - ib)^2} + \frac{1}{2} \frac{e^{-ix}}{(1 + ib)^2}.$$ 

Note that the formal limit $b = 0$ gives the right moments.

Using the moments and Theorem 6.1 one can recover the Hamiltonians (in the case $b = 0$ it was done in Subsection 7.2).

Generalizing the example from Subsection 7.2, let us consider $\mu = 1 + a \cos x$ where $|a| < 1$. Then $P\mu = \Re(1 + aS)$, and

$$P\tilde{\mu} = \Re \frac{1}{1 + aS}$$

for a symmetric case ($b = 0$). Thus,

$$\tilde{\mu} = \frac{1 + a \cos x}{1 + 2a \cos x + a^2},$$

and

$$\Gamma(\tilde{\mu}) = \left\{ 1, -\frac{a}{2}, \frac{a^2}{2}, \frac{a^3}{2}, \ldots \right\}.$$ 

(Here we use the notation $M = \{a_1, a_2, \ldots\}$ for a symmetric Toeplitz matrix $M$ with the first row $a_1, a_2, \ldots$.)

This follows from the representation $\tilde{\mu} = \Re[1 - aS + a^2S^2 - \ldots]$. Note that we get the right moments in the limit $a = 1$. Of course,

$$\Gamma(\mu) = \left\{ 1, \frac{a}{2}, 0, 0, \ldots \right\}.$$
7.5  The Poisson measure

Fix $a \in \mathbb{D}$ and consider the $\mu \in M^+(\mathbb{T})$, $\mu = P_a$, where $P_a$ is the Poisson kernel on $\mathbb{T}$,

$$P_a(\xi) = \frac{1 - |a|^2}{|\xi - a|^2}.$$

Recall that a polynomial is called monic if its main coefficient is equal to 1.

**Lemma 7.3 (Szego, Bernstein).** The monic orthogonal polynomials on $\mathbb{T}$ with respect to $P_a$ are

$$\Phi_0(z) = 1, \quad \Phi_n(z) = z^n - az^{n-1}, \quad (n \geq 1),$$

are orthogonal with respect to $\mu$. Their norms are

$$\|\Phi_0\|_\mu = 1, \quad \|\Phi_n\|_\mu^2 = 1 - |a|^2.$$

**Proof.** If $n > k \geq 1$, then

$$(\Phi_n, \Phi_k)_\mu = \int_{\mathbb{T}} (z^n - az^{n-1})(z^{-k} - \bar{a}z^{1-k})P_a(z) = a^{n-k} + |a|^2a^{n-k} - aa^{n-k-1} - \bar{a}a^{n-k+1} = 0.$$

Also,

$$(\Phi_n, \Phi_0)_\mu = \int_{\mathbb{T}} (z^n - az^{n-1})P_a = a^n - aa^{n-1} = 0.$$

Finally,

$$\|\Phi_n\|^2 = \int_{\mathbb{T}} (1 + |a|^2 - az - \bar{a}z)P_a = 1 + |a|^2 - |a|^2 - |a|^2 = 0.$$

For ONP, we have

$$\phi_0(z) = 1, \quad \phi_n(z) = \frac{z^{n-1}(z - a)}{\sqrt{1 - |a|^2}} \quad (n \geq 1).$$

**Corollary 7.2.**

$$h_0 = 1, \quad h_n = \frac{|1 - a|^2}{1 - |a|^2} \quad (n \geq 1).$$

**Remarks.**

(a) *The periodic measure $\mu^{\mathbb{R}}(x) = \mu^{\mathbb{T}}(e^{ix})$ is even if $\mu^{\mathbb{T}}(z) = \mu^{\mathbb{T}}(z)$. This is the case when $a$ is real; then*

$$h_n = \frac{1 - a}{1 + a} \quad (n \geq 1).$$
(b) Shift of the spectral measure, $\mu(\xi) \mapsto \mu_{\eta}(\xi) = \mu(\eta \xi)$, $\eta \in \mathbb{T}$, results in the Hamiltonian with

$$h_n^{(\eta)} = \frac{|\eta - a|^2}{1 - |a|^2} \quad (n \geq 1).$$

If $a$ is real, then

$$h_n^{(-1)} = \frac{1 + a}{1 - a} = h_n^{-1},$$

which suggests that the measures $P_a$ and $P_{-a}$ are dual. This relation can also be established directly for all $a \in \mathbb{T}$:

**Lemma 7.4.** For any $a \in \mathbb{D}$, the measures $P_{a \mathbb{R}}$ and $P_{-a \mathbb{R}}$ are dual.

**Proof.** The harmonic function

$$\mathfrak{R} \frac{1 + \bar{a} S}{1 - \bar{a} S} = \frac{1 - |a|^2 |S|^2}{|1 - \bar{a} S|^2}$$

is positive in the upper half-plane and its boundary values are $P_{a \mathbb{R}}(x)$. One of the dual measures can then be found from

$$\mathfrak{R} \frac{1 - \bar{a} S}{1 + \bar{a} S} = P_{-a} \quad \text{(on } \mathbb{R}).$$

□

**Corollary 7.3.** We have

$$h_n \tilde{h}_n = \frac{|1 - a^2|^2}{(1 - |a|^2)^2} \quad (n \geq 1)$$

and therefore

$$g_n^2 = h_n \tilde{h}_n - 1 = \frac{4(\Im a)^2}{(1 - |a|^2)^2}.$$

To illustrate our formulae further let us use the above calculations and obtain the measure $P_a$ in the case of real $a$ solving the direct spectral problem. In this case, the Hamiltonian is diagonal and, as was shown above, equal to $I$ on $(0, \frac{1}{2})$ and to

$$\begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix}$$

on the rest of $\mathbb{R}_+$, where

$$h = \frac{1 - a}{1 + a}.$$

Using the formulae for constant Hamiltonians from Subsection 7.1, we obtain that the matrizant is equal to

$$\begin{pmatrix} \cos(z/2) & -\sin(z/2) \\ \sin(z/2) & \cos(z/2) \end{pmatrix}$$
on $(0, \frac{1}{2})$ and to
\[
\begin{pmatrix}
\cos zt & -h^{-1} \sin zt \\
h \sin zt & \cos zt
\end{pmatrix}
\begin{pmatrix}
\cos(z/2) & -\sin(z/2) \\
\sin(z/2) & \cos(z/2)
\end{pmatrix}
\]
at “time” $\frac{1}{2} + t$.

The corresponding (normalized) HB function is
\[
E = A - iC,
\]
where
\[
A = \cos zt \cos(z/2) - h^{-1} \sin zt \sin(z/2), \quad C = h \sin zt \cos(z/2) - \cos zt \sin(z/2).
\]
The function $E$ has the same dB space as the function
\[
E^* = \sqrt{h} A - i \frac{1}{\sqrt{h}} C.
\]
A small miracle happens in the following computation:
\[
|E^*|^2 = h |A|^2 + \frac{1}{h} |C|^2 = h \cos^2 \frac{z}{2} + \frac{1}{h} \sin^2 \frac{z}{2},
\]
with the final expression showing no dependence on $t$. It follows that the spectral measure is
\[
\mu(x) = \frac{1}{|E^*(x)|^2} = \frac{1}{h \cos^2 \frac{x}{2} + \frac{1}{h} \sin^2 \frac{x}{2}}.
\]
This is equal to
\[
P_a(x) = \frac{1 - a^2}{(\cos x - a)^2 + \sin^2 x}.
\]

## 7.6 Delta measure plus a constant

Let now $\gamma \in (0, 1)$ be a parameter, and consider $\mu \in M^+(\mathbb{T})$ defined as
\[
\mu^\top = (1 - \gamma) + 2 \pi \gamma \delta_1.
\]
As before, we identify $\mu$ with a periodic measure on $\mathbb{R}$,
\[
\mu = (1 - \gamma) + 2 \pi \gamma \sum_n \delta_{2 \pi n}.
\]

**Lemma 7.5** [33]. *The monic orthogonal polynomials on $\mathbb{T}$ with respect to $\mu$ are*
\[
\Phi_0(z) = 1, \quad \Phi_n(z) = z^n - \alpha_{n-1} (1 + \cdots + z^{n-1}), \quad (n \geq 1),
\]
*where*
\[
\alpha_n = \frac{\gamma}{1 + n \gamma}.
\]
The norms are
\[ \|\Phi_n\|^2 = 1 - n\alpha_{n-1}^2, \quad (n \geq 0). \]

As the measure is even, the corresponding Hamiltonian is diagonal with the diagonal entries on \((n, n + \frac{1}{2})\) given by Theorem 6.3:

**Corollary 7.4.**
\[
\begin{align*}
    h_0 &= 1, \\
    h_n &= \frac{\|\Phi_n(1)\|^2}{\|\Phi_n\|^2} = \frac{(1 - n\alpha_{n-1})^2}{1 - n\alpha_{n-1}^2}, \\
    \tilde{h}_0 &= 1, \\
    \tilde{h}_n &= \frac{1}{h_n} = \frac{1 - n\alpha_{n-1}^2}{(1 - n\alpha_{n-1})^2}.
\end{align*}
\]

### 7.7 Spectral measure \( \mu = \alpha + \beta \pi \delta_0, \alpha > 0, \beta \geq 0 \)

Let us now consider this example of a nonperiodic even spectral measure. Using the formulae
\[
\hat{\delta} = \frac{1}{\sqrt{2\pi}} \cdot 1, \quad \hat{1} = \sqrt{2\pi} \cdot \delta,
\]
we obtain
\[
\hat{\mu} = \alpha \sqrt{2\pi} \cdot \delta + \frac{\beta \pi}{\sqrt{2\pi}} \cdot 1.
\]

Next, we use Theorem 4.7 to find the Fourier transform \( \psi_t \) of the reproducing kernel. As \( \psi \ast 1 = \int \psi \), Theorem 4.7 gives
\[
\forall x \in (-t, t), \quad \alpha \sqrt{2\pi} \psi_t(x) + \frac{\beta \pi}{\sqrt{2\pi}} \int_{-t}^{t} \psi_t = 1.
\]

Thus,
\[
\psi_t = c(t)1_{(-t, t)}, \quad \int_{-t}^{t} \psi_t = 2tc(t),
\]
and
\[
\alpha \sqrt{2\pi} c(t) + \frac{\beta \pi}{\sqrt{2\pi}} 2tc(t) = 1,
\]
so
\[
c(t) = \frac{1}{\alpha \sqrt{2\pi} + \frac{\beta \pi}{\sqrt{2\pi}} 2t} = \frac{\sqrt{2\pi}}{2\pi \alpha + 2\pi t \beta}
\]
and
\[
k_t(0) = \frac{1}{\sqrt{2\pi}} \int \psi_t = \frac{2tc(t)}{\sqrt{2\pi}} = \frac{1}{\pi} \frac{t}{\alpha + t\beta}.
\]

It follows that
\[
h^\mu(t) = \pi \frac{d}{dt} k_t(0) = \frac{d}{dt} \frac{t}{\alpha + t\beta} = \frac{\alpha}{(\alpha + t\beta)^2}.
\]
and that
\[
H(t) = \begin{pmatrix}
\frac{\alpha}{(\alpha+t\beta)^2} & 0 \\
0 & \frac{(\alpha+t\beta)^2}{\alpha}
\end{pmatrix}
\]
is a unique diagonal Hamiltonian with spectral measure \(\mu\).

Remarks 7.1.

\begin{itemize}
\item Note that in the limiting case \(\alpha \to 0\) (and \(\beta = 1\)):
\[\mu(x) \to \pi \delta(x), \quad h^\mu(t) \to \delta(t).\]

Indeed,
\[
\int_0^\infty \frac{\alpha}{(\alpha + t)^2} \, dt = -\frac{\alpha}{\alpha + t} \bigg|_{t=0}^\infty = 1.
\]
This shows how using our methods developed for PW-measures one can solve the inverse problem for a non-PW measure \(\mu = \pi \delta\).

\item The dual spectral measure of \(H\) is
\[\tilde{\mu}(x) = \frac{\alpha x^2}{\alpha^2 x^2 + \beta^2}.
\]

\textbf{Proof.}
\[
P\mu(z) = \alpha + \beta \frac{y}{x^2 + y^2} = \Re \left[ \alpha + \frac{i\beta}{z} \right]
\]
We have
\[
\frac{1}{\alpha + \frac{i\beta}{z}} = \frac{z}{\alpha z + i\beta},
\]
on \(\mathbb{R}\) the real part is \(\frac{\alpha x^2}{\alpha^2 x^2 + \beta^2}\). \(\square\)

\item It is a good exercise to show directly that if \(\tilde{\mu} = x^2/(1 + x^2)\), then \(h^{\tilde{\mu}} = (1 + t)^2\). Note that
\[
P \left(\frac{1}{1 + x^2}\right)(\xi) = \sqrt{\frac{\pi}{2}} e^{-|\xi|},
\]
and we need to solve equations like
\[
\psi + ce^{-|\xi|}*\psi = 1
\]
(this convolution is called the Beurling transform of \(\psi\) in [12]). Hint:
\[
\int \psi_t(y) e^{-|x-y|} \, dy = \int_{-t}^x + \int_{t}^x.
\]
7.8 Adding an eigenvalue at the origin

Here is a generalization of the last example studied with different methods by Winkler in [35].

**Theorem 7.1.** Let $\mu$ be a PW-sampling measure. For $r \geq 0$, let $\mu_r = \mu + r \pi \delta$. Then

$$h^{\mu_r}(t) = \frac{h^{\mu}(t)}{1 + r \int_0^t h^{\mu}(s) \, ds}.$$  

**Example 7.2.** $\mu = \alpha m$, $h^{\mu} = \alpha^{-1}$, and

$$h^{\mu_r}(t) = \frac{\alpha^{-1}}{(1 + \alpha^{-1}rt)^2} = \frac{\alpha}{(\alpha + rt)^2}.$$  

**Proof.** Fix $r > 0$. Let $\psi_t$ and $f_t$ be solutions of the equations from Theorem 4.7,

$$\psi_t \ast \hat{\mu}_r = 1, \quad f_t \ast \hat{\mu}_r = 1 \quad \text{on} \quad (-t, t),$$

so that

$$h^{\mu} = \sqrt{\frac{\pi}{2}} \int \psi_t$$

and

$$h^{\mu_r} = \sqrt{\frac{\pi}{2}} \int f_t.$$  

(7.2)

From $f \ast \hat{\mu}_r = 1$, we find

$$f_t \ast \hat{\mu} + r \sqrt{\frac{\pi}{2}} \int f_t = 1,$$

or

$$f_t \ast \hat{\mu} = 1 - c(t), \quad c(t) = r \sqrt{\frac{\pi}{2}} \int f_t.$$  

It follows that

$$f_t = (1 - c(t))\psi_t,$$

where $c(t)$ satisfies the equation

$$c(t) = r \sqrt{\frac{\pi}{2}} (1 - c(t)) \int \psi_t.$$  

Hence, $c(t) = a/(1 + a)$ where $a(t) = r \sqrt{\pi/2} \int \psi_t$ and

$$\frac{d}{dt} a(t) = rh^{\mu}(t), \quad a(0) = 0.$$  

From (7.2), we have

$$h^{\mu_r}(t) = \frac{d}{dt} \frac{c(t)}{r},$$

which yields the statement. \qed
Remark 7.1. Note that $\mu_{x+y}$ can be obtained as $\mu_{x+y} = \mu_x + y\delta_0$. For $h_x = h_\mu$ the last theorem yields the equation

$$h_{x+y}(t) = \frac{h_x(t)}{(1 + y \int_0^t h_x)^2}$$

with the initial condition $h_0 = h^\mu$.

7.9 Spectral measure $\mu = \alpha + \beta \pi \delta_\lambda$

Here $\alpha > 0$, $\beta > 0$, and $\lambda \in \mathbb{R}$ are parameters. Unlike Subsection 7.7, if $\lambda \neq 0$, then the measure is not even.

(i) We want to find

$$h_\mu = \sqrt{\frac{\pi}{2}} \frac{d}{dt} \int \psi_t,$$

where

$$\hat{\mu} * \psi_t = 1 \quad \text{on} \ (-t, t).$$

We will use the notation $e_\lambda(x) = e^{i\lambda x}$. We have

$$\hat{\mu} = \alpha \sqrt{2\pi} \delta_0 + \frac{\pi \beta}{\sqrt{2\pi}} e^{-\lambda}.$$

Note that

$$(e_{-\lambda} * \psi)(x) = e_{-\lambda}(x) \int e_\lambda \psi,$$

so

$$\psi_t = \left[ \frac{1}{\alpha \sqrt{2\pi}} - \frac{\beta}{2\alpha} c(t) e_{-\lambda} \right] \cdot 1_{(-t,t)},$$

where

$$c(t) = \int e_\lambda \psi_t.$$

Also,

$$\int_{-t}^t e_\lambda = \frac{2 \sin \lambda t}{\lambda}.$$ 

From the last three equations, we get

$$c(t) = \sqrt{\frac{2}{\pi}} \frac{1}{\alpha + \beta t} \frac{\sin \lambda t}{\lambda},$$

and

$$\sqrt{\frac{\pi}{2}} \int \psi_t = \frac{t}{\alpha} - \frac{\beta}{\alpha \alpha + \beta t} \left( \frac{\sin \lambda t}{\lambda} \right)^2.$$
Hence, 

\[ h_\mu(t) = \frac{d}{dt} \left[ \frac{t}{\alpha} - \frac{\beta}{\alpha + \beta t} \left( \frac{\sin \lambda t}{\lambda} \right)^2 \right]. \]

For example, if \( \alpha = \beta = 1 \), then

\[ h_\mu(t) = \sin^2 \lambda t + \left[ \cos \lambda t - \frac{\sin \lambda t}{\lambda(1 + t)} \right]^2. \]

(ii) We now turn to the computation of \( g_\mu \) for

\[ \mu = \alpha + \pi \beta \delta \lambda. \]

To compute \( g_\mu \), we need

\[
\begin{align*}
\ell_t(0) &= \frac{1}{2\pi i} \int_{-t}^{t} \psi_t(\epsilon) (\sigma e_{-\lambda}) d\tau = \frac{\beta}{2i \sqrt{2\pi}} \int_{-t}^{t} \psi_t(\epsilon) (\sigma e_{-\lambda}) d\tau - \frac{\beta^2 c(t)}{4\alpha i \sqrt{2\pi}} \int_{-t}^{t} (1_{[-t,t]} e_{-\lambda}) (\sigma e_{-\lambda}) d\tau.
\end{align*}
\]

The following formula can be established via direct calculations.

**Lemma 7.6.**

\[
\int_{-t}^{t} (1_{[-t,t]} e_{\lambda}) (\sigma e_{-\mu}) = \frac{4}{i(\lambda - \mu)} \left[ \text{sinc}_t(\lambda) - \cos(\lambda - \mu) t \text{sinc}_t(\mu) \right].
\]

Using the lemma to continue the calculation of \( \ell_t(0) \),

\[
\begin{align*}
\ell_t(0) &= \frac{-\beta}{\alpha \lambda \pi} \left[ t - \cos \lambda t \text{sinc}_t(\lambda) \right] - \frac{\beta^2 c(t)}{\alpha \lambda \sqrt{2\pi}} \left[ t \cos \lambda t - \text{sinc}_t(\lambda) \right] \\
&= \frac{-\beta}{\alpha \lambda \pi} \left[ t - \cos \lambda t \text{sinc}_t(\lambda) \right] - \sqrt{\frac{2}{\pi}} \frac{\sin(\lambda t)}{\lambda + \beta t} \frac{\beta^2}{\alpha \lambda \sqrt{2\pi}} \left[ t \cos \lambda t - \text{sinc}_t(\lambda) \right] \\
&= \frac{-\beta}{\alpha \pi \lambda} \left[ t - \frac{\cos \lambda t \sin \lambda t}{\lambda} + \frac{\beta t}{\alpha + \beta t} \frac{\cos \lambda t \sin \lambda t}{\lambda} - \frac{\beta}{\alpha + \beta t} \frac{\sin^2 \lambda t}{\lambda^2} \right] \\
&= \frac{-\beta}{\alpha \pi \lambda} \left[ t - \frac{\alpha}{\alpha + \beta t} \cos \lambda t \sin \lambda t - \frac{\beta}{\alpha + \beta t} \frac{\sin^2 \lambda t}{\lambda^2} \right].
\end{align*}
\]

Now,

\[
\begin{align*}
g_\mu(t) &= \frac{d}{dt} \ell_t(0) \\
&= \frac{-\beta}{\alpha \pi \lambda} \left[ 1 - \frac{\alpha}{\alpha + \beta t} \cos 2\lambda t + \frac{\alpha \beta}{(\alpha + \beta t)^2} \frac{\cos \lambda t \sin \lambda t}{\lambda} - \frac{\beta}{\alpha + \beta t} \frac{\sin 2\lambda t}{\lambda} + \frac{\beta^2}{(\alpha + \beta t)^2} \frac{\sin^2 \lambda t}{\lambda^2} \right] \\
&= \frac{-\beta}{\alpha \pi \lambda} \left[ 1 - \frac{\alpha}{\alpha + \beta t} \cos 2\lambda t - \frac{(\alpha \beta + 2\beta^2) \sin 2\lambda t}{(\alpha + \beta t)^2} + \frac{\beta^2}{(\alpha + \beta t)^2} \frac{\sin^2 \lambda t}{\lambda^2} \right].
\end{align*}
\]
Let us simplify the above formula in the particular case $\alpha = \beta = \lambda = 1$:

$$g^{\mu}(t) = -\frac{1}{\pi} \left[ 1 - \frac{\cos 2t}{1 + t} - \frac{(1 + 2t) \sin 2t}{(1 + t)^2} - \frac{\sin^2 t}{(1 + t)^2} \right]$$

$$= \frac{(1 + t)2 \cos 2t + (1 + 2t) \sin 2t - (1 - \cos 2t)}{2\pi(1 + t)^2} - \frac{1}{\pi}$$

$$= \frac{2t(\cos 2t + \sin 2t) + 3 \cos 2t + \sin 2t - 1}{2\pi(1 + t)^2} - \frac{1}{\pi}$$

$$= \frac{(2t + 1)(\cos 2t + \sin 2t) + 2 \cos 2t - 1}{2\pi(1 + t)^2} - \frac{1}{\pi}.$$  

### 7.10 Adding multiple solitons

Let us solve the inverse problem for

$$\mu = \alpha + \sum_n \pi \beta_n \delta_{\lambda_n}.$$  

For its Fourier transform, we have

$$\hat{\mu} = \sqrt{2\pi \alpha} \delta_0 + \sum_n \frac{\pi \beta_n}{\sqrt{2\pi}} e^{-\lambda_n},$$  

where $e_s(t) = e^{ist}$.

As before, let $\psi = \psi_1$ be the Fourier transform of the reproducing kernel $k_t(\cdot) = K_t(0, \cdot) \in B_t$. Theorem 4.7 then gives

$$\hat{\mu} * \psi = 1 \text{ on } (-t, t),$$

which can be rewritten as

$$\sqrt{2\pi \alpha} \psi(x) + \sum_n c_n(t) \frac{\pi \beta_n}{\sqrt{2\pi}} e^{-\lambda_n}(x) = 1,$$  

(7.3)

where

$$c_n(t) = \int_{\mathbb{R}} e_{\lambda_n} \psi = \sqrt{2\pi} \hat{\psi}(-\lambda_n).$$

Solving (7.3) for $\psi$, we get

$$\psi(x) = \frac{1}{\sqrt{2\pi \alpha}} - \sum_n \frac{\beta_n}{2\alpha} c_n(t)e_{-\lambda_n}(x).$$  

(7.4)

Let us recall the notation for the sinc function:

$$\text{sinc}_t(x) = \frac{\sin(tx)}{x}.$$  

The reprokernel of $PW_t$ is $\hat{k}_t(w, z) = \frac{1}{\pi} \text{sinc}_t(z - \bar{w})$.  

\[\text{MAKAROV and POLTORATSKI}\]
**Lemma 7.7.**

\[
\int_{-t}^{t} e_{\lambda} = \frac{2 \sin \lambda t}{\lambda} = 2 \text{sinc}(\lambda)
\]

\[
\int_{-t}^{t} e_{\lambda} e^{-\mu} = \frac{2 \sin(\lambda - \mu)t}{\lambda - \mu} = 2 \text{sinc}(\lambda - \mu).
\]

Using (7.4),

\[
c_n(t) = \int_{-t}^{t} e_{\lambda_n} \psi_t = \int e_{\lambda_n} \left[ \frac{1}{\alpha \sqrt{2\pi}} - \sum_k \frac{\beta_k}{2\alpha} c_k(t)e^{-\lambda_k} \right] =
\]

\[
\frac{2}{\alpha \sqrt{2\pi}} \text{sinc}(\lambda_n) - t \frac{\beta_n}{\alpha} c_n(t) - \sum_{k \neq n} \frac{\beta_k}{\alpha} c_k(t) \text{sinc}(\lambda_n - \lambda_k).
\]

The last equation can be rewritten as

\[
(\alpha + \beta_n t)c_n + \sum_{k \neq n} \text{sinc}(\lambda_n - \lambda_k)\beta_k c_k = \frac{2}{\sqrt{2\pi}} \text{sinc}(\lambda_n).
\]  (7.5)

Let us introduce the following two matrices,

\[
B = \begin{bmatrix}
\beta_1 & 0 & \ldots & 0 & 0 & 0 \\
0 & \beta_2 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & \beta_{n-2} & 0 & 0 \\
0 & 0 & \ldots & 0 & \beta_{n-1} & 0 \\
0 & 0 & \ldots & 0 & 0 & \beta_n
\end{bmatrix},
\]

and

\[
S_t = (\text{sinc}(\lambda_j - \lambda_k))_{1 \leq j, k \leq n},
\]

and two vectors

\[
L_t = \sqrt{\frac{2}{\pi}} \begin{bmatrix}
\text{sinc}(\lambda_1) \\
\text{sinc}(\lambda_2) \\
\vdots \\
\text{sinc}(\lambda_n)
\end{bmatrix}, \text{ and } C(t) = \begin{bmatrix}
c_1(t) \\
c_2(t) \\
\vdots \\
c_n(t)
\end{bmatrix}.
\]

Our calculations above lead to the following

**Theorem 7.2.**

\[
(\alpha + S_t B)^{-1} L_t = C(t).
\]
It follows that \( h_{11} = h^\mu \) can now be calculated via the following formula, which concludes the solution in the case when \( \mu \) is even.

**Corollary 7.5.**

\[
h^\mu(t) = \frac{1}{\pi \alpha} - \frac{1}{\sqrt{2\pi \alpha}} \frac{d}{dt} \mathcal{B}(\alpha + S_i B)^{-1} L_i, L_i > .
\]

In the general case, the off-diagonal term of \(\mathcal{H}, g^\mu\), can be calculated using the formulae of Subsection 4.7. These calculations will be presented elsewhere.

### 7.11 Dual solitons

Let us, once more, consider a measure \( \nu = \alpha + \beta \pi \delta_0 \). The dual measure \( \mu = \tilde{\nu} \) is

\[
\mu = \frac{\alpha x^2}{\alpha^2 x^2 + \beta^2} = \frac{1}{\alpha} \left( 1 - \frac{\beta^2}{\alpha^2 x^2 + \beta^2} \right) = \frac{1}{\alpha} \left( 1 - \frac{1}{1 + \beta^2 x^2} \right).
\]

Let us carry on with the case

\[
\alpha = \beta = 1, \mu = 1 - \frac{1}{1 + x^2}.
\]

As

\[
\frac{1}{1 + x^2} = \sqrt{\frac{2\pi}{2}} e^{-|y|} \text{ and } \hat{1} = \sqrt{2\pi} \delta_0,
\]

the convolution equation from Theorem 4.7 becomes

\[
\psi - \frac{1}{2} (e^{-x} F(x) + e^x G(x)) = \frac{1}{\sqrt{2\pi}}, \quad (7.6)
\]

where

\[
F(x) = \int_{-x}^{x} \psi(y) e^y dy \text{ and } G(x) = \int_{x}^{t} \psi(y) e^{-y} dy. \quad (7.7)
\]

Note that as \( \psi \) is even, \( G(x) = F(-x) \). Next notice that

\[
F'(x) = \psi e^x \text{ and } G'(x) = -\psi e^{-x}. \quad (7.8)
\]

In particular,

\[
F(-t) = 0, \psi(0) = F'(0) \text{ and from (7.6) } F'(0) - F(0) = \frac{1}{\sqrt{2\pi}}. \quad (7.9)
\]
Multiplying (7.6) by $e^{-x}$, we get
\[ F' e^{-2x} - \frac{1}{2} (e^{-2x} F + G) = \frac{1}{\sqrt{2\pi}} e^{-x}. \]

Differentiating the last equation, we get
\[ F'' e^{-2x} - 2F' e^{-2x} - \frac{1}{2} (-2e^{-2x} F + e^{-2x} F' + G') = -\frac{1}{\sqrt{2\pi}} e^{-x}. \]

From (7.8),
\[ F'' e^{-2x} - 2F' e^{-2x} - \frac{1}{2} (-2e^{-2x} F + e^{-2x} F' - e^{-2x} F') = -\frac{1}{\sqrt{2\pi}} e^{-x}, \]
and
\[ F''' - 2F'' + F = -\frac{1}{\sqrt{2\pi}} e^x. \]

A particular solution is $-\frac{1}{2\sqrt{2\pi}} x^2 e^x$ and the homogeneous solution is $Ce^x + Dx e^x$. Hence,
\[ F = Ce^x + Dx e^x - \frac{1}{2\sqrt{2\pi}} x^2 e^x. \]

From (7.9),
\[ e^t F(-t) = C - Dt - \frac{t^2}{2\sqrt{2\pi}} = 0, \quad \text{and} \quad F'(0) - F(0) = D = \frac{1}{\sqrt{2\pi}}, \]
which gives
\[ C = \frac{2t + t^2}{2\sqrt{2\pi}} \]
and
\[ F = \frac{1}{2\sqrt{2\pi}} ((2t + t^2)e^x + 2xe^x - x^2 e^x). \]

Next, using (7.8),
\[ F' = \frac{1}{2\sqrt{2\pi}} ((2t + t^2)e^x + 2e^x + 2xe^x - x^2 e^x - 2xe^x) \]
and
\[ \psi = \frac{1}{2\sqrt{2\pi}} ((2t + t^2) + 2 + 2x - x^2 - 2x) = \frac{1}{2\sqrt{2\pi}} ((2t + t^2) + 2 - x^2). \]

Then
\[ \int_{-t}^{t} \psi_t(x) dx = \frac{1}{2\sqrt{2\pi}} \left( 4t^2 + 2t^3 + 4t - \frac{2}{3} t^3 \right) = \frac{1}{2\sqrt{2\pi}} \left( \frac{4}{3} t^3 + 4t^2 + 4t \right). \]
Finally,

\[ h^t = \frac{d}{dt} \int_{-t}^{t} \psi_t(x) dx = \frac{1}{2\sqrt{2\pi}} (4t^2 + 8t + 4) = \sqrt{\frac{2}{\pi}} (t + 1)^2. \]

ACKNOWLEDGEMENTS

The first author is supported by NSF Grant DMS-1500821. The second author is supported by NSF Grant DMS-1954085. The work on Section 7 in the fall of 2021 was supported by the Ministry of Science and Higher Education of the Russian Federation, agreement No. 075-15-2021-602.

JOURNAL INFORMATION

The Journal of the London Mathematical Society is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

REFERENCES

1. R. Bessonov, Sampling measures, Muckenhoupt Hamiltonians, and triangular factorization, Int. Math. Res. Not. (2018), no. 12, 3744–3768.
2. R. Bessonov and R. Romanov, An inverse problem for weighted Paley-Wiener spaces, Inverse Probl. 32 (2016), no. 11, 115007, 15 pp.
3. A. Beurling, Collected works, vol. 2, Harmonic analysis, Birkhauser, Boston, 1989, pp. 341–365.
4. G. Borg, Eine Umkehrung der Sturm-Liouvilleischen Eigenwertaufgabe, Acta Math. 79 (1946), 1–96.
5. G. Borg, Uniqueness theorems in the spectral theory of \( y'' + (\lambda - q(x))y = 0 \), Proc. 11th Scandinavian Congress of Mathematicians, Johan Grundt Tanums Forlag, Oslo, 1952, pp. 276–287.
6. J.-F. Burnol, Two complete and minimal systems associated with the zeros of the Riemann zeta function, J. Théor. Nombres Bordeaux 16 (2004), 6594.
7. L. De Branges, Hilbert spaces of entire functions, Prentice-Hall, Englewood Cliffs, NJ, 1968.
8. S. Denisov, Continuous analogs of polynomials orthogonal on the unit circle and Krein systems, International Mathematics Research Surveys 2006 (2006), 54517, 148 pp.
9. H. Dym, An introduction to de Branges spaces of entire functions with applications to differential equations of the Sturm-Liouville type, Adv. Math. 5 (1971), 395–471.
10. H. Dym and H. P. McKean, Gaussian processes, function theory and the inverse spectral problem, Academic Press, New York, 1976
11. I. M. Gelfand and B. M. Levitan, On the determination of a differential equation from its spectral function (Russian), Izvestiya Akad. Nauk SSSR, Ser. Mat. 15 (1951), 309–360; English translation in Amer. Math. Soc. Translation, Ser. 2 1 (1955), 253–304.
12. V. Havin and B. Jöricke, The uncertainty principle in harmonic analysis, Springer, Berlin, 1994.
13. S. Jaffard, A density criterion for frames of complex exponentials, Mich. Math. J. 38, no. 91, 339–348.
14. M. G. Krein, On a difficult problem of operator theory and its relation to classical analysis, talk at the Jubilee scientific session of the Moscow Mathematical Society, Moscow, 1964.
15. P. Koosis, The logarithmic integral, vols. I & II, Cambridge University Press, Cambridge, 1988.
16. J. Lagarias, Zero spacing distributions for differenced \( L \)-functions, Acta Arith. 120 (2005), no. 2, 159–184.
17. J. Lagarias, The schrödinger operator with morse potential on the right halfline, Commun. Number Theory Phys. 3 (2009), no. 2, 323–361.
18. B. M. Levitan and I. S. Sargsjan, Sturm–Liouville and Dirac operators, Kluwer, Dordrecht, 1991.
19. V. N. Logvinenko and Y. F. Sereda, Equivalent norms in spaces of entire functions of exponential type, Teor. funk-tsii, funkts. analiz i ich prilozhenia 20 (1971), pp. 62–78 (Russian).
20. N. Makarov and A. Poltoratski, Meromorphic inner functions, Toeplitz kernels, and the uncertainty principle, Perspectives in analysis, Springer, Berlin, 2005, pp. 185–252.
21. N. Makarov and A. Poltoratski, *Beurling-Malliavin theory for Toeplitz kernels*, Invent. Math. **180** (2010), no. 3, 443–480.
22. N. Makarov and A. Poltoratski, *Two spectra theorem with uncertainty*, J. Spectr. Theory **9**, no. 4, 1249–1285.
23. V. Marchenko, *Some questions in the theory of one-dimensional linear differential operators of the second order, I*, Trudy Mosk. Mat. Obsch. **1** (1952), 327–420.
24. V. Marchenko, *Sturm-Liouville operators and applications*, Birkhauser, Basel, 1986.
25. J. Ortega-Cerdá and K. Seip, *Fourier frames*, Ann. Math. **155** (2002), 789–806.
26. B. P. Paneah, *On some theorems of Paly-Wiener type*, Dokl. Akad. Nauk **138** (1961), pp. 47–50 (Russian).
27. A. Poltoratski, *Toeplitz approach to problems of the uncertainty principle*, CBMS series, Amer. Math. Soc., Providence, RI, 2015.
28. A. Poltoratski and D. Sarason, *Aleksandrov–Clark measures*, Recent advances in operator-related function theory, Contemp. Math., vol. 393, Amer. Math. Soc., Providence, RI, 2006, pp. 1–14.
29. A. Poltoratski, *Pointwise convergence of the non-linear Fourier transform*, arXiv:2103.13349, 2021.
30. C. Remling, *Schrödinger operators and de Branges spaces*, J. Funct. Anal. **196** (2002), 323–394.
31. C. Remling, *Spectral theory of canonical systems*, De Gruyter, Berlin, 2018.
32. R. Romanov, *Canonical systems and de Branges spaces*, arXiv:1408.6022, 2014.
33. B. Simon, *Orthogonal polynomials on the unit circle, parts I and II*, vol. 54, American Mathematical Society Colloquium Publications, Providence, RI, 2004.
34. B. Valkó and B. Virág, *The many faces of the stochastic zeta function*, Geom. Funct. Anal. **32** (2022), no. 5, 1160–1231.
35. H. Winkler, *On transformation of canonical systems*, I. Gohberg and H. Langer (eds.), Operator theory and boundary eigenvalue problems, vol. 80, Operator Theory: Advances and Applications, Birkhäuser, Basel, 1995, pp. 276–288.