CRITICAL BEHAVIOUR AND SCALING FUNCTIONS FOR THE THREE-DIMENSIONAL $O(6)$ SPIN MODEL WITH EXTERNAL FIELD

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We numerically investigate the three-dimensional $O(6)$ model on $12^3$ to $120^3$ lattices. From Binder’s cumulant at vanishing magnetic field we obtain the critical coupling $J_c = 1.42865(5)$ and verify this value with the $\chi^2$-method. The universal value of Binder’s cumulant at this point is $g_r(J_c) = -1.94456(10)$. At the critical coupling we find the critical exponents $\nu = 0.818(5)$, $\beta = 0.425(2)$ and $\gamma = 1.604(6)$ from a finite size scaling analysis. We also determine the finite-size-scaling function on the critical line and the equation of state. Our $O(6)$-result for the equation of state is compared to the Ising, $O(2)$ and $O(4)$ results.

1. Introduction

Our goal is the determination of universal quantities of the three-dimensional $O(6)$-invariant nonlinear $\sigma$-model, especially critical exponents and scaling functions. These quantities shall eventually be compared to those of 2-flavour staggered QCD with adjoint fermions. The relevant symmetry of this model is $SU(4)$, so it should be in the same universality class as the $O(6)$ spin model. The Hamiltonian of our model is defined as

$$\beta \mathcal{H} = -J \sum_{<x,y>} \vec{\phi}_x \cdot \vec{\phi}_y - \vec{H} \cdot \sum_x \vec{\phi}_x.$$  (1)

Here $x$ and $y$ are the nearest-neighbour sites on a three-dimensional hypercubic lattice, $\vec{\phi}_x$ is a 6-component unit vector at site $x$ and $\vec{H}$ is the external magnetic field. The coupling constant $J$ is considered as inverse temperature, therefore $J = 1/T$. At $H = |\vec{H}| \neq 0$ the order parameter, the

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magnetisation $M$, is defined by

$$M = \langle \phi^\parallel \rangle,$$

(2)

where $\phi^\parallel$ is the longitudinal (parallel to $\vec{H}$) component of $\vec{\phi}$. At $H = 0$ this quantity vanishes for all couplings on the lattice, but we can use

$$M = \langle |\vec{\phi}| \rangle$$

(3)
as an approximate order parameter. We also measure the susceptibility $\chi$ and Binder’s cumulant $g_r$, defined by

$$\chi = V(\langle \phi^2 \rangle - M^2),$$

(4)

$$g_r = \frac{\langle (\phi^2)^2 \rangle}{\langle \phi^2 \rangle^2} - 3.$$  

(5)

We have simulated the model with a Wolff-Cluster-Algorithm, using a ghostspin to emulate the external magnetic field. We used hypercubic lattices with periodic boundaries and linear extensions $L$ between 12 and 120. Near the critical coupling $J_c$ we have made up to 200000 measurements per data point, elsewhere about 20000 measurements. Between the measurements we have used up to 400 cluster updates in the broken phase and up to 1500 updates in the symmetric phase.

2. Determination of the critical coupling

At $H = 0$ the Binder cumulant can be described by the finite-size-scaling function

$$g_r = Q_g(tL^{1/\nu}; L^{-\omega}),$$

(6)

where $t = \frac{T - T_c}{T_c}$ is the reduced temperature and $\omega$ is the leading irrelevant exponent. Therefore $g_r$ should be independent of $L$ at the critical point $t = 0$ apart from corrections due to irrelevant scaling fields. The left plot in Fig. 1 shows our reweighted data, with the dotted lines representing the jackknife error corridors. Although the intersection points coincide within the errors, there are some small corrections. As the value of $\omega$ is unknown, we used Binder’s approximation

$$\frac{1}{J_{ip}} = \frac{1}{J_c} + \frac{c_2}{\log b}$$

(7)
to extrapolate the intersection points $J_{ip}(L, b)$ of two lattices with sizes $L$ and $L' = bL$. Fitting the results for the lattice sizes $L = 12, 16, 20, 24, 30, 36$
to a constant value we find
\[ \frac{1}{J_c} = 0.699960(14) \Rightarrow J_c = 1.42865(5). \]  
(8)

We have checked this result with the \( \chi^2 \)-method described in Ref. 3. From this value we derive the universal quantity \( g_r(J_c) = -1.94456(10) \).

The result for \( J_c \) is comparable with the result \( J_c = 1.42895(6) \) of Butera and Comi\(^4\) using a high temperature expansion, but their errors seem to be underestimated.

3. The critical exponents \( \beta, \gamma \) and \( \nu \)

For the determination of the critical exponents we use the following scaling relations at \( t = H = 0 \):

\[ M = L^{-\beta/\nu} (a_0 + a_3 L^{-\omega}) \]
\[ \chi = L^{\gamma/\nu} (b_0 + b_3 L^{-\omega}) \]
\[ \frac{\partial g_r}{\partial J} = L^{1/\nu} (d_0 + d_3 L^{-\omega}) \]

Fitting these relations to our data at lattices in the range \( L = 12 - 72 \) we get \( \beta/\nu = 0.519(2) \) and \( \gamma/\nu = 1.961(3) \) from \( M \) and \( \chi \). The error of these quantities includes a variation of \( \omega \) between 0.5 and 1.0. From the derivative of \( g_r \) we get \( 1/\nu = 1.223(5) \). Here we neglect the correction term, because it is zero within the errors. From these results we can calculate the critical exponents \( \beta = 0.425(2), \gamma = 1.604(6) \) and \( \nu = 0.818(5) \). They fulfill the hyperscaling relations and are in complete agreement with the results of Butera and Comi\(^4\).
4. The equation of state

In the vicinity of $T_c$ the critical behaviour can be described by the universal
equation of state. It can be written in the form

$$M = h^{1/\delta} f_G(z'), \quad z' = t'/h^{1/\delta}$$

(12)

$t'$ and $h$ are the normalized reduced temperature $t' = (T - T_c)/T_0$ and
magnetic field $h = H/H_0$ with the normalization conditions $f_G(0) = 1$ and
$f_G(z') = (-z')^\beta$ as $z' \to -\infty$. To get the normalization constants $H_0$ and
$T_0$ we have to determine the critical amplitudes of the magnetisation on the
critical isotherm and on the coexistence line. The infinite volume behaviour
of $M$ is given by

$$M(T, H) = d_c H^{1/\delta} (1 + d_1 H^{\omega_{nc}})$$

(13)

including the leading correction term. From hyperscaling relations we get
for the exponents $\delta = 4.780(22)$ and $\nu_c = 0.4031(24)$. With this ansatz,
using only the data of the largest lattice at each value of $H$, we get for the
critical amplitude $d_c = 0.642(1)$, which translates to $H_0 = d_c^{-\delta} = 8.3(1)$.

In the broken phase the magnetisation is described by the ansatz

$$M(T < T_c, H) = M(T, 0) + c_1 (T - T_c)^{1/2} + c_2 (T - T_c) H.$$ 

(14)

The $H^{1/2}$ term is due to the Goldstone effect. As we have already seen
with the $O(2)^5$ and $O(4)^6$ spin model, this ansatz fits very well and allows
the extrapolation of data at nonzero $H$ to $H = 0$. Fitting the results for
$M(T, 0)$ for several couplings ($J = 1.45, 1.47, 1.5, 1.55$ and $1.6$) to the ansatz

$$M(T \leq T_c, 0) = B (T_c - T)^{1/2} [1 + b_1 (T_c - T)^{\omega_{nc}} + b_2 (T_c - T)] 

(15)$$

we get $B = 1.22(1)$, which leads to $T_0 = B^{-1/\beta} = 0.63(1)$.

The left plot in Fig. 2 shows the scaling function $f_G(z')$. In the broken phase ($z' < 0$) the solid lines are reweighted data from different $J$-values.

There are visible corrections, so we use the ansatz

$$Mh^{-1/\delta} = f_G(z') + k^{\omega_{nc}} f^{(1)}_G(z') + k^{2\omega_{nc}} f^{(2)}_G(z')$$

(16)

to obtain the universal value of $f_G(z')$ (dashed line). In the symmetric
phase the corrections are negligible. In the right plot of Fig. 2 we show a
comparison of the $O(6)$ scaling functions to those of some other $O(N)$ mod-
els, which have been determined in Refs. 7 (Ising), 5 ($O(2)$) and 6 ($O(4)$).

As one can see, the functions get steeper with increasing $N$. These func-
tions can eventually be used to determine the universality class of staggered
2-flavour QCD both with fundamental or with adjoint fermions. But until
now QCD data has too low statistics to distinguish between these scaling functions, and also the used lattice sizes are too small in QCD to be sure that one has reached the infinite volume limit.

5. Finite-size-scaling function at $J_c$

Because of the small lattice sizes used in QCD, finite-size-scaling functions are a better tool for the comparison of spin models to QCD. At $T_c$ and for small $H$ the different lattices should scale like

$$M(T_c, H, L) = L^{-\beta/\nu} Q_M(HL^{\beta/\nu})$$

(17)

where $Q_M$ is a universal scaling function. Our results are shown in the right plot in Fig. 1. The solid line shows the asymptotic behaviour in the limit $L \to \infty$

$$Q_M(z) = Q_\infty(z) = d_c z^{1/\delta},$$

(18)

which is observable for $z = H L^{\beta/\nu} \gtrsim 40$.

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