COMPUTING CHARACTERISTIC NUMBERS USING FIXED POINTS

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This article has its genesis in a course that Raoul Bott gave at Harvard in the fall of 1996 shortly before his official retirement. The topic of the course was equivariant cohomology, which is simply the cohomology of a group action. The course was to culminate in the equivariant localization formula, discovered by Berline and Vergne, and independently by Atiyah and Bott, around 1982. When a manifold has a torus action, the equivariant localization formula, while formulated in equivariant cohomology, is a powerful tool for doing calculations in the ordinary cohomology of the manifold. It extends and simplifies Bott’s work several decades earlier on the relationship between characteristic numbers and the zeros of a vector field (5).

In one of the lectures during the course, Raoul Bott considered an action of a circle on a projective space and computed its equivariant cohomology from the fixed points using the Borel localization theorem. After class, he asked me if I could do the same for a homogeneous space $G/H$ of a compact, connected Lie group $G$ by a closed subgroup $H$ of maximal rank, under the natural action of a maximal torus. Unbeknownst to me at the time, and perhaps to him also, this problem had been solved earlier and is in retrospect not so difficult (see [1] and [9], which contain much more than this). As was his wont, Raoul liked to understand everything ab initio and in his own way.

Using Bott’s method, I worked out the equivariant cohomology ring of $G/H$ from the fixed points of the torus action. I saw then that some of the lemmas I developed for this calculation may be used, in conjunction with the equivariant localization formula, to calculate the ordinary (as well as the equivariant) characteristic numbers of $G/H$.

The idea of relating integration of ordinary differential forms to integration of equivariant differential forms is folklore. It is implicit in Atiyah–Bott ([2, Section 8]) and explicitly stated for the Chow ring in Edidin–Graham ([11, Proposition 5, p. 627]). When the fixed points are isolated and the manifold is compact, by converting ordinary integration to equivariant integration, one can hope to obtain the original integral as a finite sum over the fixed points of the action. While the idea is simple, its execution in specific examples is not necessarily so. The explicit formulas for the characteristic numbers of $G/H$ obtained here are involved but beautiful, and serve as an affirmation of the power and versatility of the equivariant localization formula. Throughout this

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project, I met with Raoul many times over a period of several years. This article is a testimony to his generosity with his time, ideas, and friendship, and so I think it is particularly appropriate as a contribution to a volume on his mathematical legacy. To make the article more self-contained and in an effort to emulate Raoul Bott’s style, about half of the article is exposition of known results, for example, the computation from scratch of the ordinary cohomology rings of $G/T$ and $G/H$. Although the results are known, I have also included the computation of the equivariant cohomology rings of $G/T$ and $G/H$, partly because Bott’s method of using the Borel localization theorem may be new—at least I have not seen it in the literature—and may be applicable to other situations. I think of this article as an application of the equivariant localization formula to one nontrivial example. It is nonetheless a key example, since every orbit of every action is a homogeneous space. In the hope that the article could be understood by a graduate student with a modicum of knowledge of equivariant cohomology, perhaps someone who has read Bott’s short introduction to equivariant cohomology [7], I have allowed myself the liberty of being more expository than in a typical research article.

Throughout this paper, $H^*(\ )$ stands for singular cohomology with rational coefficients. The main technical results are the restriction formulas (Proposition 10 and Theorem 15) and the Euler class formulas (Proposition 13 and Theorem 19), which allow us to apply the Borel localization theorem and the equivariant localization formula to compact homogeneous spaces.

Using techniques of symplectic quotients, Shaun Martin ([14, Proposition 7.2]) has obtained a formula for the characteristic numbers of a Grassmannian similar to our Proposition 23. Some of the ideas of this paper, in particular that of looking at the fixed points of the action of $T$ on $G/T$, have parallels in the Atiyah–Bott proof of the Weyl character formula ([6]).

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1. Line Bundles on $G/T$ and on $BT$

Let $G$ be a compact, connected Lie group and $T$ a maximal torus in $G$. The cohomology classes on the homogeneous space $G/T$ and on the classifying space $BT$ all come from the first Chern class of complex line bundles on these spaces. Both $G/T$ and $BT$ are base spaces of principal $T$-bundles: $G/T$ is the base space of the principal $T$-bundle $G \to G/T$ and $BT$ is the base space of a universal principal $T$-bundle $ET \to BT$. For this reason, we will first construct complex line bundles on the base space of an arbitrary principal $T$-bundle.

A character of the torus $T$ is a multiplicative Lie group homomorphism of $T$ into $\mathbb{C}^*$. Let $\hat{T}$ be the group of characters of $T$, with the multiplication of characters written additively: for $\alpha, \beta \in \hat{T}$ and $t \in T,$

$$t^{\alpha+\beta} := t^\alpha t^\beta = \alpha(t)\beta(t).$$

Denote by $\mathbb{C}_\gamma$ the complex vector space $\mathbb{C}$ with an action of $T$ given by the character $\gamma: T \to \mathbb{C}^*$. A character $\alpha: T \to \mathbb{C}^*$ has the form $\alpha(t_1, \ldots, t_\ell) = t_1^{n_1} \cdots t_\ell^{n_\ell},$ where $t_i \in S^1$ and $n_i \in \mathbb{Z}$ ([10, Proposition 8.1, p. 107]). So the character group $\hat{T}$ is isomorphic to $\mathbb{Z}^\ell$. 

Suppose a torus $T$ of dimension $\ell$ acts freely on the right on a topological space $X$ so that $X \to X/T$ is a principal $T$-bundle. By the mixing construction, a character $\gamma$ on $T$ associates to the principal bundle $X \to X/T$ a complex line bundle $L(X/T, \gamma)$ over $X/T$:

$$L_{\gamma} := L(X/T, \gamma) := X \times_T \mathbb{C}_\gamma := (X \times \mathbb{C}_\gamma)/T,$$

where $T$ acts on $X \times \mathbb{C}_\gamma$ by

$$(x, v)t = (xt, \gamma(t^{-1})v).$$

The equivalence class of $(x, v)$ is denoted $[x, v]$. It is easy to check that as complex line bundles over $X/T$,

$$L_{\alpha + \beta} \cong L_\alpha \otimes L_\beta.$$

Hence, $L_{-\alpha} \cong L_\alpha^\vee$, the dual bundle of $L$.

The first Chern class of an associated complex line bundle defines a homomorphism of abelian groups

$$c: \hat{T} \to H^2(X/T), \quad \gamma \mapsto c_1(L(X/T, \gamma)).$$

Let $\text{Sym}(\hat{T})$ be the symmetric algebra with rational coefficients generated by the additive group $\hat{T}$. The group homomorphism (1) extends to a ring homomorphism

$$c: \text{Sym}(\hat{T}) \to H^*(X/T),$$

called the characteristic map of $X/T$, sometimes denoted $c_{X/T}$.

We apply the construction of the associated bundle of a character $\gamma \in \hat{T}$ to two situations:

(i) The classifying space $BT$. Applied to the universal bundle $ET \to BT$, this construction yields line bundles

$$S_{\gamma} := L(ET/T, \gamma) = L(BT, \gamma)$$

over $BT$ and cohomology classes $c(\gamma) = c_1(S_{\gamma})$ in $H^2(BT)$. The characteristic map

$$c: \text{Sym}(\hat{T}) \to H^*(BT)$$

is a ring isomorphism, since both $\text{Sym}(\hat{T})$ and $H^*(BT)$ are polynomial rings in $\ell$ generators and the generators correspond. If $\chi_1, \ldots, \chi_\ell$ is a basis for the character group $\hat{T}$ and $u_i = c_1(S_{\chi_i})$, then $\text{Sym}(\hat{T})$ is the polynomial ring $\mathbb{Q}[\chi_1, \ldots, \chi_\ell]$, and

$$H^*(BT) = H^*(BS^1 \times \cdots \times BS^1) \cong H^*(BS^1) \otimes_\mathbb{Q} \cdots \otimes_\mathbb{Q} H^*(BS^1) \cong \mathbb{Q}[u_1, \ldots, u_\ell],$$

because $BS^1 \cong \mathbb{C}P^\infty$.

(ii) The homogeneous space $G/T$. Applied to the principal $T$-bundle $G \to G/T$, this construction yields line bundles

$$L_{\gamma} := L(G/T, \gamma)$$

over $G/T$.

For each character $\gamma$ on the torus $T$, the relationship of the line bundle $S_{\gamma}$ over $BT$ and the line bundle $L_\alpha$ over $G/T$ is as follows. The universal $G$-bundle $EG \to BG$ factors through $EG \to EG/T \to BG$. The total space $EG$ is a contractible space on which $G$ acts freely. A fortiori, $EG$ is also a contractible space on which the subgroup
$T$ acts freely. Hence, $EG = ET$. It follows that $(EG)/T = (ET)/T = BT$, so there is a commutative diagram

$$
\begin{array}{ccc}
G & \rightarrow & ET \\
\downarrow & & \downarrow \\
G/T & \rightarrow & BT \\
\downarrow & & \downarrow \\
pt & \rightarrow & BG,
\end{array}
$$

representing $G/T$ as a fiber of the fiber bundle $BT \rightarrow BG$ and the principal $T$-bundle $G \rightarrow G/T$ as the restriction of the principal $T$-bundle $ET \rightarrow BT$ from $BT$ to $G/T$. Hence, the associated bundle $L_\gamma = G \times_\gamma \mathbb{C}$ is the restriction of the associated bundle $S_\gamma = ET \times_\gamma \mathbb{C}$ from $BT$ to $G/T$.

2. The actions of the Weyl group

The Weyl group of a maximal torus $T$ in the compact, connected Lie group $G$ is $W = N_G(T)/T$, where $N_G(T)$ is the normalizer of $T$ in $G$. The Weyl group is a finite reflection group.

We use $w$ to denote both an element of $W$ and a lift of the element to the normalizer $N_G(T)$. The Weyl group $W$ acts on the character group $\hat{T}$ of $T$ by

$$(w \cdot \gamma)(t) = \gamma(w^{-1}tw).$$

This action induces an action on $\text{Sym}(\hat{T})$ as ring isomorphisms. Let $R$ be the polynomial ring

$$R = \text{Sym}(\hat{T}) = \mathbb{Q}[\chi_1, \ldots, \chi_\ell]$$

and $R^W$ the subring of $W$-invariant polynomials.

If the Lie group $G$ acts on the right on a space $X$, then the Weyl group $W$ acts on the right on $X/T$ by

$$r_w(xT) = (xT)w = xwT.$$ 

This action of $W$ on $X/T$ induces by the pullback an action of $W$ on the line bundles over $X/T$ and also on the cohomology ring $H^*(X/T)$: if $L$ is a line bundle on $X/T$ and $a \in H^*(X/T)$, then

$$w \cdot L = r_w^*L, \quad w \cdot a = r_w^*a,$$

where we use the same notation $r_w^*$ for the pullback of a line bundle and for the pullback of a cohomology class.

**Proposition 1.** The action of the Weyl group $W$ on the associated line bundles over $X/T$ is compatible with its action on the characters of $T$; more precisely, for $w \in W$ and $\gamma \in \hat{T}$,

$$r_w^*(X/T, \gamma) \simeq L(X/T, w \cdot \gamma).$$

**Indication of proof.** Define $\varphi: r_w^*(X/T, \gamma) \rightarrow L(X/T, w \cdot \gamma)$ by

$$\varphi(xT, [xw, v]) = [x, v].$$

If we use a different representative $xt$ to represent $xT$, then

$$(xT, [xw, v]) = (xtT, [xtw, v']).$$
and it is easily verified that \( v' = \gamma(w^{-1}t^{-1}w)v \). Hence,
\[
\varphi(xT, [xtw, v']) = [xt, v'] = [xt, (w \cdot \gamma)(t^{-1})v] = [x, v].
\]
This shows that \( \varphi \) is well defined. It has the obvious inverse map
\[
\psi([x, v]) = (xT, [xw, v]).
\]
□

**Corollary 2.** The characteristic map \( c : \text{Sym}(\hat{T}) \to H^*(X/T) \) is \( W \)-equivariant.

**Proof.** Let \( \gamma \) be a character on the torus \( T \). Writing \( L_\gamma \) instead of \( L(X/T, \gamma) \), it follows from the proposition that
\[
r_w^*c(\gamma) = r_w^*c_1(L_\gamma) = c_1(r_w^*L_\gamma) = c_1(L_{w^{-1}}) = c(w \cdot \gamma). \]
\]
□

**Corollary 3.** For \( \gamma \) a character on \( T \) and \( w \) an element of the Weyl group \( W \),
\[
w \cdot c_1(S_\gamma) = c_1(S_{w^{-1}}).
\]

**Proof.** This is a special case of the preceding corollary with \( X = ET \). □

### 3. Fiber Bundles with Fiber \( G/T \)

Let \( G \) be a compact, connected Lie group, \( T \) a maximal torus, and \( W = N_G(T)/T \) the Weyl group of \( T \) in \( G \). Suppose \( G \) acts freely on the right on a topological space \( X \) so that \( X \to X/G \) is a principal \( G \)-bundle. Then the natural projection \( X/T \to X/G \) is a fiber bundle with fiber \( G/T \). We will have frequent occasion to call on the following topological lemma.

**Lemma 4.** The rational cohomology of \( X/G \) is the subspace of \( W \)-invariants of the rational cohomology of \( X/T \):
\[
H^*(X/G) \simeq H^*(X/T)^W.
\]

The proof is based on the following two facts from [13]:

**Fact 1.** The compact homogeneous space \( G/T \) has a cellular decomposition into even-dimensional cells indexed by the Weyl group.

This is a consequence of the well-known Bruhat decomposition (see [13, p. 35]) using the fact that a compact homogeneous space \( G/T \) has a complex description \( G_C/B = G/T \), where \( G_C \) is the complexification of \( G \) and \( B \) is a Borel subgroup containing \( T \). It implies that \( H^*(G/T) \) vanishes in odd degrees and that the Euler characteristic of \( G/T \) is \( \chi(G/T) = |W| \).

**Fact 2.** If \( N = N_G(T) \) is the normalizer of \( T \) in \( G \), then \( G/N \) is acyclic.

**Proof.** The projection \( G/T \to G/N \) is a regular covering map with group \( W = N/T \). Hence, \( H^*(G/N) = H^*(G/T)^W \). It follows that like \( G/T \), \( G/N \) also has nonzero cohomology classes only in even degrees. Since
\[
\chi(G/N) = \frac{1}{|W|} \chi(G/T) = \frac{1}{|W|} |W| = 1,
\]
\( H^0(G/N) = \mathbb{Q} \) and \( H^k(G/N) = 0 \) for \( k > 0 \). □

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1Fact 1 may also be obtained from Morse theory; there is a proof in [4] for \( G \) compact, connected, and simply connected.
Proof of Lemma 4. Factor \( X/T \to X/G \) into \( X/T \to X/N \to X/G \). Because \( G/N \) is acyclic, the constant function 1 defines a global cohomology class on \( X/N \) that restricts to a generator of cohomology on each fiber \( G/N \) of \( X/N \to X/G \). By the Leray–Hirsch theorem, \( H^*(X/N) \cong H^*(X/G) \).

Since \( X/T \to X/N \) is a regular covering with group \( W = N/T \), \( H^*(X/N) \cong H^*(X/T)^W \). Thus, \( H^*(X/G) \cong H^*(X/T)^W \). \( \square \)

4. Cohomology Rings of \( G/T \) and \( G/H \)

Let \( EG \to BG \) and \( ET \to BT \) be universal bundles for the compact, connected Lie group \( G \) and its maximal torus \( T \). Since \( ET = EG, BG = (EG)/G \) and \( BT = (EG)/T \). So the natural projection \( BT \to BG \) is a fiber bundle with fiber \( G/T \).

Choose a basis \( \chi_1, \ldots, \chi_\ell \) for the character group \( \hat{T} \), let \( S_{\chi_i} \) be the associated complex line bundles over \( BT \), and set \( u_i = c_1(S_{\chi_i}) \). As noted earlier, if \( R = \text{Sym}(T) \), then \( H^*(BT) = \mathbb{Q}[u_1, \ldots, u_\ell] \cong \mathbb{Q}[\chi_1, \ldots, \chi_\ell] \cong R \).

By Lemma 4 the cohomology of \( BG \) is the subring of \( W \)-invariants in \( H^*(BT) \):

\[
H^*(BG) = H^*(BT)^W = R^W.
\]

Theorem 5. Let \( R^W_+ \) be the submodule of \( R^W \) generated by all homogeneous elements of positive degree, and \( (R^W_+)^r \) the ideal in \( R \) generated by \( R^W_+ \). Then \( H^*(G/T) \cong R/(R^W_+) \).

Proof. Consider the spectral sequence of the fiber bundle \( BT \to BG \) with fiber \( G/T \). Since both the base \( BG \) and the fiber \( BT \) have only even-dimensional cohomology, all the differentials \( d_r \) vanish for \( r \geq 2 \) and the spectral sequence degenerates at the \( E_2 \) term (\([8, \text{Theorem 14.14 and Theorem 14.18}]\)). Therefore,

\[
H^*(BT) = E_\infty \cong E_2 \cong H^*(BG) \otimes_\mathbb{Q} H^*(G/T).
\]

In the picture of \( E_2 \), \( H^*(G/T) \) is the zeroth column and \( H^*(BG) \) is the bottom row.

\[
H^*(G/T) \cong \frac{H^*(BT)}{\bigoplus_{p>0} H^p(BG)} = \frac{R}{(R^W_+)}.
\]

From the picture of \( E_2 \), one sees that the kernel of the restriction to the fiber: \( H^*(BT) \to H^*(G/T) \) is the shaded area, which is the ideal generated by \( \bigoplus_{p>0} H^p(BG) = R^W_+ \). Hence,

\[
H^*(G/T) \cong \frac{H^*(BT)}{\bigoplus_{p>0} H^p(BG)} = \frac{R}{(R^W_+)}.
\]
This isomorphism is a priori a module isomorphism, but because the restriction map
\( H^*(BT) \to H^*(G/T) \) is a ring homomorphism, the module isomorphism \([2]\) is in fact a ring isomorphism. \( \square \)

In terms of the basis \( \chi_1, \ldots, \chi_\ell \) for \( \mathbb{T} \), let \( L_{\chi_i} \) be the line bundle over \( G/T \) associated to the character \( \chi_i \) and let \( y_i = c_1(L_{\chi_i}) \in H^2(G/T) \). If \( i : G/T \to BT \) is the inclusion map as a fiber of \( BT \to BG \), then as noted in Section 1, \( L_{\chi_i} = i^*S_{\chi_i} \) and

\[
y_i = c_1(L_{\chi_i}) = c_1(i^*S_{\chi_i}) = i^*c_1(S_{\chi_i}) = i^*u_i.
\]

Let \( H \) be a closed, connected subgroup of maximal rank in the compact, connected Lie group \( G \), and \( T \subset H \) a maximal torus. Denote by \( W_G \) and \( W_H \) the Weyl groups of \( T \) in \( G \) and in \( H \) respectively.

**Theorem 6.** If \( R = H^*(BT) \), \( R_{W_H} \) the subring of \( W_H \)-invariant elements, and \( (R^W_G) \) the ideal generated by \( W_G \)-invariant elements of positive degree in \( R_{W_H} \), then

\[
H^*(G/H) \cong \frac{R_{W_H}}{(R^W_G)}.
\]

**Proof.** The natural projection \( G/T \to G/H \) is a fiber bundle with fiber \( H/T \). By Lemma 3 and Theorem 5

\[
H^*(G/H) \cong H^*(G/T)^{W_H} \cong \left( \frac{R}{R_{W_G}} \right)^{W_H} = \frac{R_{W_H}}{(R^W_G)}.
\] \( \square \)

5. **Equivariant cohomology and equivariant characteristic classes**

Suppose a topological group \( G \) acts on the left on a topological space \( M \), and \( EG \to BG \) is the universal principal \( G \)-bundle. Since \( G \) acts freely on \( EG \), the diagonal action of \( G \) on \( EG \times M \),

\[
(e, x)g = (eg, g^{-1}x),
\]

is also free. The space \( M_G := EG \times_G M := (EG \times M)/G \) is called the *homotopy quotient* of \( M \) by \( G \), and its cohomology \( H^*(M_G) \) the *equivariant cohomology* of \( M \) under the \( G \)-action, denoted \( H^*_G(M) \). For the basics of equivariant cohomology, we refer to \([7]\) or \([12]\).

A \( G \)-equivariant vector bundle \( E \to M \) induces a vector bundle \( EG \to M_G \). An *equivariant characteristic class* \( c^G(E) \) of \( E \to M \) is defined to be the corresponding ordinary characteristic class \( c(EG) \) of \( EG \to M_G \).

By the definition of homotopy quotient, \( M_G \to BG \) is a fiber bundle with fiber \( M \). Let \( i : M \to M_G \) be the inclusion map as a fiber. We say that a cohomology class \( \tilde{\alpha} \in H^*_G(M) \) is an *equivariant extension* of \( \alpha \in H^*(M) \) if \( i^*\tilde{\alpha} = \alpha \). For example, if \( E \to M \) is a \( G \)-equivariant vector bundle, then any characteristic class \( c(E) \) has an equivariant extension \( c^G(E) \), since

\[
i^*c^G(E) = i^*c(EG) = c(i^*EG) = c(E).
\]

(3)

Now suppose \( G \) is a compact, connected Lie group with maximal torus \( T \). We associate to a character \( \gamma : T \to \mathbb{C}^* \) the complex line bundle \( L_\gamma \) on \( G/T \):

\[
L_\gamma = G \times_\gamma \mathbb{C}.
\]
There is an action of the torus $T$ on $L_\gamma$:

$$t \cdot [g,v] = [tg, v], \quad \text{for } [g,v] \in G \times \gamma \mathbb{C},$$

where $[g,v]$ is the equivalence class of $(g,v) \in G \times \mathbb{C}$. This action is compatible with the action of $T$ on $G/T$ in the sense that the diagram

$$\begin{array}{c}
L_\gamma \\
\downarrow \\
G/T
\end{array} \quad \begin{array}{c}
t \\
\downarrow \\
t
\end{array} \quad \begin{array}{c}
L_\gamma \\
\downarrow \\
G/T
\end{array}$$

commutes for all $t \in T$. Therefore, $L_\gamma \to G/T$ is a $T$-equivariant line bundle and induces a line bundle $(L_\gamma)_T$ over the homotopy quotient $(G/T)_T$. Fix a basis $\chi_1, \ldots, \chi_\ell$ for the characters of $T$ and let $y_i = c_1((L_{\chi_i})_T) \in H^2_\gamma(G/T)$. These are equivariant extensions of the cohomology classes $y_i := c_1(L_{\chi_i}) \in H^2(G/T)$.

The $T$-equivariant cohomology of a point is

$$H^*_T(pt) = H^*(BT) = \mathbb{Q}[u_1, \ldots, u_\ell], \quad u_i = c_1(S_{\chi_i}). \quad (4)$$

For any $T$-space $M$, let $\pi: G/T \to pt$ be the constant map. The pullback map $\pi^*: H^*_T(pt) \to H^*_T(M)$ makes $H^*_T(M)$ into an algebra over the polynomial ring $\mathbb{Q}[u_1, \ldots, u_\ell]$.

6. **Additive structure of $H^*_T(G/T)$**

Consider the action of $T$ on $G/T$ by left multiplication. Write $W = W_G = W_G(T)$ for the Weyl group of $T$ in $G$. By Theorem 4, the rational cohomology ring of $G/T$ is

$$H^*(G/T) = \mathbb{Q}[y_1, \ldots, y_\ell]/(R^W_+) = R/(R^W_+),$$

where $R = \mathbb{Q}[y_1, \ldots, y_\ell] \simeq H^*(BT)$ and $(R^W_+)$ is the ideal generated by the $W$-invariant polynomials of positive degree in $R$. In particular, the cohomology $H^*(G/T)$ has only even-dimensional cohomology classes. For dimensional reasons, viz., $H^{odd}(G/T) = H^{odd}(BT) = 0$, all the differentials $d_2, d_3, \ldots$ in the spectral sequence of the fiber bundle

$$G/T \to (G/T)_T \to BT \quad (5)$$

vanish, and additively

$$H^*((G/T)_T) = E_2\text{-term of the spectral sequence}$$

$$= H^*(BT) \otimes_\mathbb{Q} H^*(G/T) \simeq R \otimes_\mathbb{Q} H^*(G/T) \simeq R \otimes_\mathbb{Q} (R/(R^W_+))$$

$$= \mathbb{Q}[u_1, \ldots, u_\ell] \otimes_\mathbb{Q} (\mathbb{Q}[y_1, \ldots, y_\ell]/(R^W_+)).$$

This shows that $H^*_T(G/T)$ is a free $R$-module of rank equal to $\dim H^*(G/T)$.

Moreover, because the differentials $d_r, r \geq 2$, in the spectral sequence are all vanish, the classes $y_1, \ldots, y_\ell$ extend to global classes on $(G/T)_T$. Indeed, since $y_i = c_1(L_{\chi_i})$ is the first Chern class of a $T$-equivariant line bundle on $G/T$, by 1 its global extension is $\tilde{y}_i = c_1((L_{\chi_i})_T)$ in $H^*_T(G/T)$. Thus, $H^*_T(G/T)$ is generated as a $\mathbb{Q}$-algebra by $u_1, \ldots, u_\ell, \tilde{y}_1, \ldots, \tilde{y}_\ell$, and it remains only to determine the relations they satisfy.
7. Restriction to a fixed point in $G/T$

Although it is possible to give a shorter derivation of the equivariant cohomology ring $H^*_T(G/T)$ (see [9]), we will determine the ring structure of $H^*_T(G/T)$ from the fixed points of the action of $T$ on $G/T$ using the following localization theorem of Borel. This approach leads to a restriction formula (Proposition 10) that will be useful in our subsequent computation of characteristic numbers.

Theorem (Borel localization theorem, [13, Proposition 2, p. 39]) Suppose a torus $T$ acts on a manifold $M$ with fixed point set $F$. Let $i_F: F \hookrightarrow M$ be the inclusion map. Then both the kernel and cokernel of the restriction homomorphism $i^*_F: H^*_T(M) \to H^*_T(F)$ are torsion $H^*(BT)$-modules.

To apply this theorem to the action of $T$ on $G/T$ by left multiplication, it is necessary to know the fixed point set $F$ of the action as well as the restriction homomorphism $i^*_F: H^*_T(G/T) \to H^*_T(F)$.

Proposition 7. For a compact Lie group $G$ with maximal torus $T$, let $N_G(T)$ be the normalizer of $T$ in $G$ and let $T$ act on $G/T$ by left multiplication. Then the fixed point set $F$ of the action is $W_G = N_G(T)/T$, the Weyl group of $T$ in $G$.

Proof. The coset $xT$ is a fixed point if

\[ txT = xT \text{ for all } t \in T \]
\[ x^{-1}txT = T \text{ for all } t \in T \]
\[ x^{-1}tx \in T \]
\[ x \in N_G(T). \]

At a fixed point $w$ in $G/T$, the group $T$ acts on the fiber $(L_\gamma)_w$ of the line bundle $L_\gamma$. Therefore, the fiber $(L_\gamma)_w$ is a complex representation of $T$.

Lemma 8. At the fixed point $w = xT \in W_G$, the torus $T$ acts on the fiber of the line bundle $L_\gamma$ as the representation $w \cdot \gamma$, i.e., $(L_\gamma)_w = \mathbb{C}_{w \cdot \gamma}$.

Proof. The fiber of $L_\gamma$ at a fixed point $xT$ consists of elements of the form $[x,v] \in G \times \mathbb{C}$. If $t \in T$, then

\[ t \cdot [x,v] = [tx,v] = [x(x^{-1}tx),v] = [x, \gamma(x^{-1}tx)v] = [x, (w \cdot \gamma)(t)v]. \]

Hence, under the identification $[x,v] \leftrightarrow v$, the torus $T$ acts on the fiber $(L_\gamma)_w$ as the representation $w \cdot \gamma$.

Lemma 9. Let $w$ be a point in the Weyl group $W_G \subset G/T$ and $i_w: \{w\} \hookrightarrow G/T$ the inclusion map. For a character $\gamma$ of $T$, the restriction of the line bundle $(L_\gamma)_T$ from $(G/T)_T$ to $\{w\}_T \cong BT$ is given by

\[ (i_w)^*_T(L_\gamma)_T \cong S_{w \cdot \gamma}, \]

where $S_{w \cdot \gamma}$ is the complex line bundle on $BT$ associated to the character $w \cdot \gamma$. 

PROOF. By Lemma 8 the restriction of the line bundle \( L_\gamma \) to the fixed point \( w \) gives rise to a commutative diagram of \( T \)-equivariant maps

\[
\begin{array}{ccc}
\mathbb{C}_{w,\gamma} & \hookrightarrow & L_\gamma \\
\downarrow & & \downarrow \\
\{w\} & \hookrightarrow & G/T.
\end{array}
\]

Taking the homotopy quotients results in the diagram

\[
\begin{array}{ccc}
(C_{w,\gamma})_T & \hookrightarrow & (L_\gamma)_T \\
\downarrow & & \downarrow \\
BT & \hookrightarrow & (G/T)_T.
\end{array}
\]

But \((C_{w,\gamma})_T = ET \times T \mathbb{C}_{w,\gamma}\) is precisely the line bundle \( S_{w,\gamma} \) over \( BT \) associated to the character \( w \cdot \gamma \) of \( T \).

To avoid a plethora of subscripts, we write \((i_w)_T^*\) also as \( i_w^* \). To describe the restriction \( i_w^*: H^*(G/T) \to H^*(F) \), we need to describe \( i^*_w(u_i) \) and \( i^*_w(\tilde{y}_i) \) for each \( w \in W \).

**Proposition 10 (Restriction formula for \( G/T \)).** At a fixed point \( w \in W \), let

\[
i_w^*: H^*_T(G/T) \to H^*_T\{w\}
\]

be the restriction map in equivariant cohomology. Then

(i) \( i_w^*u_i = u_i \).

(ii) For any character \( \gamma \in \hat{T} \), \( i_w^*c_1^T(L_\gamma) = w \cdot c_1(S_\gamma) \). In particular, if \( \tilde{y}_i = c_1^T(L_{\chi_i}) \) and \( u_i = c_1(S_{\chi_i}) \), then \( i_w^*\tilde{y}_i = w \cdot u_i \).

**Proof.** (i) Let \( \pi: G/T \to \{w\} \) be the constant map. Since \( \pi \circ i_w = 1_w \), the identity map on \( \{w\} \), \( i_w^*\pi^* = 1_{H^*_T(G/T)} \). The elements \( u_i \) in \( H^*_T(G/T) \) are really \( \pi^*u_i \), so

\[
i_w^*u_i = i_w^*\pi^*u_i = u_i.
\]

(ii) By the naturality of \( c_1 \) and by Lemma 9

\[
i_w^*c_1((L_\gamma)_T) = c_1(i_w^*(L_\gamma)_T) = c_1(S_{w,\gamma})
= w \cdot c_1(S_\gamma) \quad \text{(by Corollary 3)}.
\]

Now take \( \gamma \) to be \( \chi_i \).

\[
\square
\]

8. **The equivariant cohomology rings of \( G/T \) and \( G/H \)**

Suppose \( G \) is a compact, connected Lie group with maximal torus \( T \). Choose a basis \( \chi_1, \ldots, \chi_l \) of the character group \( \hat{T} \). Let \( L_{\chi_i} \) be the associated line bundles over \( G/T \), and \( y_i = c_1(L_{\chi_i}) \in H^2(G/T) \) and \( \tilde{y}_i = c_1^T(L_{\chi_i}) \in H^2_T(G/T) \) be their ordinary and equivariant first Chern classes. Similarly, let \( S_{\chi_i} \) be the associated line bundles over \( BT \), and \( u_i = c_1(S_{\chi_i}) \in H^2(BT) \) their first Chern classes. Recall \( R = H^*(BT) = \mathbb{Q}[u_1, \ldots, u_l] \).
Theorem 11. (i) The equivariant cohomology ring of $G/T$ under the action of $T$ on $G/T$ by left multiplication is

$$H_T^*(G/T) \cong \frac{\mathbb{Q}[u_1, \ldots, u_\ell, \tilde{y}_1, \ldots, \tilde{y}_\ell]}{\mathfrak{J}},$$

where $\mathfrak{J}$ is the ideal in $\mathbb{Q}[u_1, \ldots, u_\ell, \tilde{y}_1, \ldots, \tilde{y}_\ell]$ generated by $b(\tilde{y}) - b(u)$ for all polynomials $b \in R_T^W$, the $W_G$-invariant polynomials of positive degree.

(ii) If $H$ is a closed subgroup of $G$ containing the maximal torus $T$, and $T$ acts on $G/H$ by left multiplication, then

$$H_T^*(G/H) \cong \frac{\mathbb{Q}[u_1, \ldots, u_\ell] \otimes \mathbb{Q} \langle \mathbb{Q}[\tilde{y}_1, \ldots, \tilde{y}_\ell]^W \rangle}{\mathfrak{J}},$$

where $\mathfrak{J}$ is the ideal in $\mathbb{Q}[u_1, \ldots, u_\ell] \otimes \mathbb{Q} \langle \mathbb{Q}[\tilde{y}_1, \ldots, \tilde{y}_\ell]^W \rangle$ generated by $b(\tilde{y}) - b(u)$ for all polynomials $b \in R_T^W$.

Proof. By Proposition 7 the fixed point set $F$ of the action of $T$ on $G/T$ is the Weyl group $W = W_G$. For each $w \in W$, by the restriction formula (Proposition 10), $i_w^* \tilde{y}_i = w \cdot u_i$. Hence, if $b(u) = b(u_1, \ldots, u_\ell)$ is a $W$-invariant polynomial with coefficients in $\mathbb{Q}$, then

$$i_w^* b(\tilde{y}_1, \ldots, \tilde{y}_\ell) = b(w \cdot u_1, \ldots, w \cdot u_\ell) = b(u_1, \ldots, u_\ell).$$

With $\pi : G/T \to \{ w \}$ being the constant map and $i_w : \{ w \} \to G/T$ the inclusion map, $\pi \circ i_w = 1$. Thus,

$$b(u) = i_w^* \pi^* b(u).$$

It follows that

$$i_w^* b(\tilde{y}) = b(u) = i_w^* \pi^* b(u),$$

or

$$i_w^* (b(\tilde{y}) - \pi^* b(u)) = 0.$$

As is customary, in $H_T^*(G/T)$, we identify $\pi^* b(u)$ with $b(u)$, so we can write

$$i_w^* (b(\tilde{y}) - b(u)) = 0.$$

Since this is true at any fixed point $w \in F$,

$$i_F^* (b(\tilde{y}) - b(u)) = 0,$$

where $i_F : F \hookrightarrow G/T$ is the inclusion map.

Let $R = H^*(BT)$. By the Borel localization theorem, the kernel of the restriction map

$$i_F^* : H_T^*(G/T) \to H_T^*(F)$$

is a torsion $R$-module. Since $H_T^*(G/T)$ is a free $R$-module, the kernel must be 0. Therefore, $i_F^*$ is injective. So we obtain the relations

$$b(\tilde{y}) - b(u) = 0$$

in $H_T^*(G/T)$ for all $W$-invariant polynomials $b(u_1, \ldots, u_\ell)$.

Let $\mathfrak{J}$ be the ideal $(b(\tilde{y}) - b(u))$ in $\mathbb{Q}[u_1, \ldots, u_\ell, \tilde{y}_1, \ldots, \tilde{y}_\ell]$ generated by $b(\tilde{y}) - b(u)$ for all polynomials $b \in R_T^W$. Then there is a surjective ring homomorphism

$$\phi : \frac{\mathbb{Q}[u_1, \ldots, u_\ell, \tilde{y}_1, \ldots, \tilde{y}_\ell]}{\mathfrak{J}} \cong \frac{R \otimes \mathbb{Q} R}{\mathfrak{J}} \to H_T^*(G/T).$$

(7)
From the spectral sequence of the fibering \((5)\), we know that \(H^*_T(G/T)\) is a free \(R\)-module of rank equal to \(\dim H^*(G/T)\). To prove that \(\phi\) is an isomorphism, it suffices to show that \((R \otimes_R R)/\mathfrak{J}\) is a free \(R\)-module of the same rank, for in that case the kernel of \(\phi\), being of rank 0, is a torsion submodule of a free module and is therefore the zero module. Note that

\[
\frac{R \otimes_R R}{\mathfrak{J}} \simeq R \otimes_{R^W} R,
\]

since the tensor product over \(R^W\) is obtained from the tensor product over \(Q\) by dividing out by the ideal generated by elements of the form \(b \otimes 1 - 1 \otimes b\) for \(b \in R^W\). Moreover,

\[
R \otimes_{R^W} R \simeq (R \otimes_Q Q) \otimes_{R^W} R \simeq \left( R \otimes_Q \left( \frac{R^W}{(R^W)^+} \right) \right) \otimes_{R^W} R
\]

\[
\simeq R \otimes_Q \left( \frac{R^W}{(R^W)^+} \right) \otimes_{R^W} R \simeq R \otimes_Q \left( \frac{R}{(R^W)^+} \right) \simeq R \otimes_Q H^*(G/T) \quad \text{(Theorem 5)},
\]

So \((R \otimes_R R)/\mathfrak{J}\) is a free \(R\)-module of rank equal to \(\dim H^*(G/T)\). This proves that \(\phi\) is an isomorphism.

(ii) Since the fiber bundle \(G/H \to G/T\) with fiber \(H/T\) is \(T\)-equivariant, it induces a map in homotopy quotients \((G/H)_T \to (G/T)_T\), which is also a fiber bundle with fiber \(H/T\). By Lemma \(4\),

\[
H^*_T(G/H) = H^*_T(G/T)^{WH} = \frac{Q[u_1, \ldots, u_\ell] \otimes_Q (Q[\tilde{y}_1, \ldots, \tilde{y}_\ell])}{\mathfrak{J}},
\]

where as before, \(\mathfrak{J}\) is the ideal generated by \(b(\tilde{y}) - b(u)\) for all \(b \in R^W\). \(\square\)

9. The tangent bundle of \(G/H\)

Let \(G\) be a Lie group with Lie algebra \(\mathfrak{g}\), and \(H\) a closed subgroup with Lie algebra \(\mathfrak{h}\).

**Proposition 12** (See also [16, p. 130]). *The tangent bundle of \(G/H\) is diffeomorphic to the vector bundle \(G \times_H (\mathfrak{g}/\mathfrak{h})\) associated to the principal \(H\)-bundle \(G \to G/H\) via the adjoint representation of \(H\) on \(\mathfrak{g}/\mathfrak{h}\).*

**Proof.** For \(g \in G\), let \(\ell_g : G/H \to G/H\) denote left multiplication by \(g\). Then \(\ell_g\) is a diffeomorphism and its differential

\[
(\ell_g)_* : T_{eH}(G/H) \to T_{gH}(G/H)
\]

is an isomorphism.

Now define

\[
\varphi : G \times_H (\mathfrak{g}/\mathfrak{h}) \to T(G/H)
\]

by

\[
\varphi([g, v]) = (\ell_g)_*(v).
\]
To show that \( \varphi \) is well-defined, pick another representative of \([g, v]\), say \([gh, \text{Ad}(h^{-1})v]\) for some \( h \in H \). Then

\[
(\ell_{gh})_*(\text{Ad}(h^{-1})v) = (\ell_{gh})_*(\ell_{h^{-1}})_*(r_h)_*v = (\ell_g)_*(\ell_{h^{-1}})_*(r_h)_*v = (\ell_g)_*v,
\]

since right multiplication \( r_h \) on \( G/H \) is the identity map.

Since \( \dim G/H = \dim \mathfrak{g}/\mathfrak{h} \), \( \varphi \) is a surjective bundle map of two vector bundles of the same rank, and so it is an isomorphism. \( \square \)

10. **Equivariant Euler class of the normal bundle at a fixed point of \( G/T \)**

In this section \( G \) is a compact, connected Lie group, \( T \) a maximal torus in \( G \), \( \mathfrak{g} \) and \( \mathfrak{t} \) their respective Lie algebras, and \( W = N_G(T)/T \) the Weyl group of \( T \) in \( G \). The adjoint representation of \( T \) on \( \mathfrak{g} \) decomposes \( \mathfrak{g} \) into a direct sum

\[
\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta^+} \mathbb{C} \alpha,
\]

where \( \Delta^+ \) is a choice of positive roots.

Let \( w \in W \) be a fixed point of the left multiplication action of \( T \) on \( G/T \), and \( \nu_w \) the normal bundle of \( \{w\} \) in \( G/T \). The normal bundle at a point is simply the tangent space. Let \( \ell(w) \) be the length of \( w \) and \((-1)^w := (-1)^{\ell(w)}\) the sign of \( w \).

**Proposition 13** (Euler class formula for \( G/T \)). The equivariant Euler class of the normal bundle \( \nu_w \) at a fixed point \( w \in W \) of the left action of \( T \) on \( G/T \) is

\[
e^T(\nu_w) = w \cdot \left( \prod_{\alpha \in \Delta^+} c_1(S_\alpha) \right) = (-1)^w \prod_{\alpha \in \Delta^+} c_1(S_\alpha) \in H^*(BT).
\]

**Proof.** By Proposition 12 the tangent bundle of \( G/T \) is the homogeneous vector bundle associated to the adjoint representation of \( T \) on \( \mathfrak{g}/\mathfrak{t} = \bigoplus_{\alpha \in \Delta^+} \mathbb{C} \alpha \). In the notation of Section 11 with \( L_\alpha \) being the complex line bundle over \( G/T \) associated to the character \( \alpha \) of \( T \),

\[
T(G/T) \cong G \times_T (\mathfrak{g}/\mathfrak{t}) \cong G \times_T \left( \bigoplus_{\alpha \in \Delta^+} \mathbb{C} \alpha \right) \cong \bigoplus_{\alpha \in \Delta^+} L_\alpha.
\]

(9)

By (9) and Lemma 8 at a fixed point \( w \) the normal bundle \( \nu_w \) is

\[
\nu_w = T_w(G/T) = \bigoplus_{\alpha \in \Delta^+} (L_\alpha)_w \cong \bigoplus_{\alpha \in \Delta^+} \mathbb{C}_{w,\alpha}.
\]
It follows that the equivariant Euler class of $\nu_w$ is

$$e^T(\nu_w) = e^T \left( \bigoplus_{\alpha \in \Delta^+} C_{w \cdot \alpha} \right) = e^T \left( \bigoplus_{\alpha \in \Delta^+} (C_{w \cdot \alpha})_T \right)$$

$$= e \left( \bigoplus_{\alpha \in \Delta^+} S_{w \cdot \alpha} \right) \quad \text{(by the definition of } S_{\alpha})$$

$$= \prod_{\alpha \in \Delta^+} c_1(S_{w \cdot \alpha}) = \prod_{\alpha \in \Delta^+} w \cdot c_1(S_{\alpha}) \quad \text{(Corollary 3)}$$

$$= w \cdot \left( \prod_{\alpha \in \Delta^+} c_1(S_{\alpha}) \right) \quad \text{($w$ is a ring homomorphism)}$$

Each simple reflection $s$ in the Weyl group $W = N_G(T)/T$ carries exactly one positive root to a negative root. Note that $c_1(S_{-\alpha}) = c_1(S_{\alpha}^\vee) = -c_1(S_{\alpha})$. Hence,

$$s \cdot \left( \prod_{\alpha \in \Delta^+} c_1(S_{\alpha}) \right) = \prod_{\alpha \in \Delta^+} c_1(S_{s \cdot \alpha}) = - \prod_{\alpha \in \Delta^+} c_1(S_{\alpha}).$$

Since $w$ is the product of $\ell(w)$ reflections,

$$w \cdot \left( \prod_{\alpha \in \Delta^+} c_1(S_{\alpha}) \right) = (-1)^{\ell(w)} \prod_{\alpha \in \Delta^+} c_1(S_{\alpha}). \quad \square$$

11. Fixed points of $T$ acting on $G/H$

The quotient space $W_G/W_H$ can be viewed as a subset of $G/H$ via

$$\frac{W_G}{W_H} \cong \frac{N_G(T)}{N_H(T)} = \frac{N_G(T)}{N_G(T) \cap H} \to \frac{G}{H}.$$ 

**Proposition 14.** Under the action of $T$ on $G/H$ by left multiplication, the fixed point set $F$ is $W_G/W_H$.

**Proof.** The coset $xH$ is a fixed point

- iff $txH = xH$ for all $t \in T$
- iff $x^{-1}txH = H$ for all $t \in T$
- iff $x^{-1}tx \in H$ for all $t \in T$
- iff $x^{-1}Tx \subset H$.

Since any two maximal tori in $H$ are conjugate by an element of $H$, there is an element $h \in H$ such that

$$h^{-1}x^{-1}Txh = T.$$ 

Therefore, $xH \in N_G(T)$, and $xH = xhH$. Thus any fixed point can be represented as $yH$ for some $y \in N_G(T)$. 
Conversely, if \( y \in N_G(T) \), then \( yH \) is a fixed point of the action of \( T \) on \( G/H \), since \( TyH = yTH = yH \). It follows that there is a surjective map

\[
N_G(T) \twoheadrightarrow F \subset G/H,
\]

with fiber \( N_G(T) \cap H \) and hence a bijection

\[
\frac{N_G(T)}{N_G(T) \cap H} \simeq F.
\]

\[ \square \]

12. Restriction to a fixed point of \( G/H \)

**Theorem 15** (Restriction formula for \( G/H \)). With \( H^*_T(G/H) \) as in Theorem 11(ii) and \( w \) a fixed point in \( W_G/W_H \), the restriction map \( i^*_w \) in equivariant cohomology

\[
i^*_w: H^*_T(G/H) \to H^*_T(\{w\}) = \mathbb{Q}[u_1, \ldots, u_\ell]
\]

is given by

\[
i^*_w u_i = u_i, \quad i^*_w f(\tilde{y}) = w \cdot f(u)
\]

for any \( W_H \)-invariant polynomial \( f(\tilde{y}) \in \mathbb{Q}[\tilde{y}_1, \ldots, \tilde{y}_\ell]^{W_H} \).

**Proof.** Write \( w = xH \in N_G(T)/(N_G(T) \cap H) \subset G/H \), with \( x \in N_G(T) \). The commutative diagram of \( T \)-equivariant maps

\[
\begin{array}{ccc}
\{xT\} & \xrightarrow{i_{xT}} & G/T \\
\downarrow{\sigma_x} & & \downarrow{\sigma} \\
\{xH\} & \xrightarrow{i_{xH}} & G/H
\end{array}
\]

induces a commutative diagram in equivariant cohomology

\[
\begin{array}{ccc}
H^*(BT) & \xleftarrow{i^*_{xT}} & H^*_T(G/T) \\
\downarrow{\sigma^*_x} & & \downarrow{\sigma^*} \\
H^*(BT) & \xleftarrow{i^*_{xH}} & H^*_T(G/H)
\end{array}
\]

Since \( \sigma^*: H^*_T(G/H) \to H^*_T(G/T) \) is an injection and

\[
\sigma^*_{xT}: H^*_T(\{xH\}) = H^*(BT) \to H^*(BT) = H^*_T(\{xT\})
\]

is the identity map, the restriction \( i^*_{xH} \) is given by the same formula as the restriction \( i^*_{xT} \). The theorem then follows from the restriction formula for \( G/T \) (Proposition 10).

\[ \square \]

13. Pullback of an associated bundle

Assume that \( \rho: H \to GL(V) \) is a representation of the group \( H \) on a vector space \( V \) over any field. By restriction, one obtains a representation of the maximal torus \( T \).
Proposition 16. Let $\sigma: G/T \to G/H$ be the projection map. Under $\sigma$ the associated bundle $G \times_H V$ pulls back to $G \times_T V$:

$$\sigma^*(G \times_H V) \simeq G \times_T V.$$ 

Proof. Let $[g,v]_T$ and $[g,v]_H$ denote the equivalence classes of $(g,v)$ in $G \times_T V$ and $G \times_H V$ respectively. Define $\varphi: G \times_T V \to \sigma^*(G \times_H V)$ by

$$[g,v]_T \mapsto (gT, [g,v]_H).$$

If $(gT, [g',v']_H)$ is any element of $\sigma^*(G \times_H V)$, then $gH = g'H$. Hence, $g' = gh$ for some $h \in H$ and

$$(gT, [g',v']_H) = (gT, [gh,v']_H) = (gT, [g,hv']_H) = \varphi([g,hv']_T),$$

which shows that $\varphi$ is surjective. A surjective bundle map between two vector bundles of the same rank is an isomorphism. □

14. Pulling back the tangent bundle of $G/H$ to $G/T$

Let $\mathfrak{g}, \mathfrak{h},$ and $\mathfrak{t}$ be the Lie algebras of $G, H,$ and $T$ respectively. Under the adjoint representation of $H$ the Lie algebra $\mathfrak{g}$ decomposes into a direct sum of $H$-modules

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}.$$ 

By Proposition 12 the tangent bundle of $G/H$ is the associated bundle

$$T(G/H) \simeq G \times_H \mathfrak{m}.$$ (10)

Restricting the adjoint representation to the maximal torus $T$, the Lie algebras of $H$ and $G$ decompose further into a sum of $T$-modules

$$\mathfrak{h} = \mathfrak{t} \oplus \left( \bigoplus_{\alpha \in \Delta^+(H)} \mathbb{C}_\alpha \right),$$

and

$$\mathfrak{g} = \mathfrak{t} \oplus \left( \bigoplus_{\alpha \in \Delta^+(H)} \mathbb{C}_\alpha \right) \oplus \left( \bigoplus_{\alpha \in \Delta^+ \setminus \Delta^+(H)} \mathbb{C}_\alpha \right),$$

where $\Delta^+(H)$ denotes a choice of positive roots for $H$, $\Delta^+$ a choice of positive roots for $G$ containing $\Delta^+(H)$, and $\Delta^+ \setminus \Delta^+(H)$ the complement of $\Delta^+(H)$ in $\Delta^+$. Hence, as a $T$-module,

$$\mathfrak{m} = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta^+(H)} \mathbb{C}_\alpha.$$ (11)

Proposition 17. Under the natural projection $\sigma: G/T \to G/H$, the tangent bundle $T(G/H)$ pulls back to a sum of associated line bundles:

$$\sigma^* T(G/H) \simeq \bigoplus_{\alpha \in \Delta^+ \setminus \Delta^+(H)} L_\alpha.$$
Proof.  
\[ \sigma^*T(G/H) \simeq \sigma^*(G \times_H \mathfrak{m}) \simeq G \times_T \mathfrak{m} \quad \text{(by (10) and Proposition 16)} \]
\[ = G \times_T \left( \bigoplus_{\alpha \in \Delta^+ \setminus \Delta^+(H)} \mathbb{C}_\alpha \right) \quad \text{(by (11))} \]
\[ = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta^+(H)} L_\alpha \quad \text{(definition of } L_\alpha) \quad \Box \]

15. The normal bundle at a fixed point of \( G/H \)

At a fixed point \( w = xH \) of \( G/H \), with \( x \in N_G(T) \), the action of \( T \) on \( G/H \) induces an action of \( T \) on the tangent space \( T_w(G/H) \).

Proposition 18. At a fixed point \( w = xH \) of \( G/H \), the tangent space \( T_w(G/H) \) decomposes as a representation of \( T \) into
\[ T_w(G/H) \simeq \bigoplus_{\alpha \in \Delta^+ \setminus \Delta^+(H)} \mathbb{C}_{w\cdot \alpha}. \]

Proof. Let \( \sigma: G/T \to G/H \) be the projection map. Then
\[ T_{xH}(G/H) = (T(G/H))_{xH} \quad \text{(fiber of tangent bundle at } xH) \]
\[ = (\sigma^*T(G/H))_{xT} \]
\[ = \left( \bigoplus_{\alpha \in \Delta^+ \setminus \Delta^+(H)} (L_\alpha)_{xT} \right) \quad \text{(Proposition 17)} \]
\[ \simeq \bigoplus_{\alpha \in \Delta^+ \setminus \Delta^+(H)} \mathbb{C}_{w\cdot \alpha} \quad \text{(Lemma 8)} \quad \Box \]

Theorem 19 (Euler class formula for \( G/H \)). At a fixed point \( w \in W_G/W_H \) of the left action of \( T \) on \( G/H \), the equivariant Euler class of the normal bundle \( \nu_w \) is given by
\[ e^T(\nu_w) = w \cdot \left( \prod_{\alpha \in \Delta^+ \setminus \Delta^+(H)} c_1(S_\alpha) \right). \]

Proof. The normal bundle \( \nu_w \) at the point \( w \) is the tangent space \( T_w(G/H) \). By the multiplicativity of the Euler class and the fact that the Euler class of a complex line bundle is its first Chern class, the equivariant Euler class of \( \nu_w \) is
\[ e^T(\nu_w) = e^T(T_w(G/H)) = e^T \left( \bigoplus_{\alpha \in \Delta^+ \setminus \Delta^+(H)} \mathbb{C}_{w\cdot \alpha} \right) \quad \text{(Proposition 18)} \]
\[ = \prod_{\alpha \in \Delta^+ \setminus \Delta^+(H)} c^T_1(\mathbb{C}_{w\cdot \alpha}) = \prod_{\alpha \in \Delta^+ \setminus \Delta^+(H)} c_1((\mathbb{C}_{w\cdot \alpha})_T) \]
\[ = \prod_{\alpha \in \Delta^+ \setminus \Delta^+(H)} c_1(S_{w\cdot \alpha}) = w \cdot \left( \prod_{\alpha \in \Delta^+ \setminus \Delta^+(H)} c_1(S_\alpha) \right). \quad \Box \]
16. Equivariant characteristic numbers of $G/H$ and $G/T$

Suppose a torus $T$ acts on a compact oriented manifold $M$. Let $\pi: M \to \text{pt}$ be the constant map and $\pi_*: H^*_T(M; \mathbb{R}) \to H^*_T(\text{pt}; \mathbb{R}) = H^*(BT; \mathbb{R})$ the push-forward or integration map in equivariant cohomology. For any class $\tilde{\eta} \in H^*_T(M; \mathbb{R})$, the equivariant localization formula computes $\pi_*\tilde{\eta}$ in terms of an integral over the fixed point set $F$ ([2], [3]). In case the fixed points are isolated, the equivariant localization formula states that

$$\pi_*\tilde{\eta} = \sum_{p \in F} \frac{i_p^*\tilde{\eta}}{e^T(\nu_p)}. \quad (12)$$

On the right-hand side the calculation is performed in the fraction field of $H^*(BT; \mathbb{R})$ and so is a priori a rational function of $u_1, \ldots, u_\ell$, but it is part of the theorem that the sum will be a polynomial in $u_1, \ldots, u_\ell$ since the left-hand side $\pi_*\tilde{\eta} \in H^*(BT; \mathbb{R})$ is a polynomial in $u_1, \ldots, u_\ell$.

Although the equivariant localization formula is stated for real cohomology, by viewing rational cohomology as a subset of real cohomology, we can apply the equivariant localization formula to rational cohomology classes. Now suppose $G$ is a compact, connected Lie group, $T$ a maximal torus in $G$, and $H$ a closed subgroup of $G$ containing $T$. For $\tilde{\eta} \in H^*_T(G/H; \mathbb{Q})$, $\pi_*\tilde{\eta}$ is a priori a real class in $H^*(BT; \mathbb{R})$. However, the explicit formula (13) below shows that it is in fact a rational class in $H^*(BT; \mathbb{Q})$.

For $M = G/H$ with the standard torus action, by Theorem (11ii), an element of $H^*_T(G/H; \mathbb{Q})$ is of the form

$$\tilde{\eta} = \sum (\pi^*a_i(u))f_i(\tilde{y}),$$

where $a_i(u) \in \mathbb{Q}[u_1, \ldots, u_\ell]$ and $f_i(\tilde{y}) \in \mathbb{Q}[\tilde{y}_1, \ldots, \tilde{y}_\ell]^{WH}$. By the projection formula,

$$\pi_*\tilde{\eta} = \sum a_i(u)\pi_*f_i(\tilde{y}).$$

Thus, to calculate $\pi_*: H^*_T(G/H; \mathbb{Q}) \subset H^*_T(G/H; \mathbb{R}) \to H^*(BT; \mathbb{R})$, it suffices to calculate $\pi_*f(\tilde{y})$ for $f(\tilde{y}) = f(\tilde{y}_1, \ldots, \tilde{y}_\ell)$ a $WH$-invariant polynomial with coefficients in $\mathbb{Q}$. Since $\tilde{y}_1, \ldots, \tilde{y}_\ell$ are all equivariant characteristic classes, $\pi_*f(\tilde{y})$ is called an equivariant characteristic number of $G/H$. With the aid of the restriction formula (Theorem 15) and the Euler class formula (Theorem 19), the equivariant localization formula (12) gives for any $f(\tilde{y}) \in \mathbb{Q}[\tilde{y}_1, \ldots, \tilde{y}_\ell]^{WH}$,

$$\pi_*f(\tilde{y}) = \sum_{w \in W_G/W_H} w \cdot \frac{f(u)}{\prod_{a \in \Delta^+ \setminus \Delta^+(H)} c_1(S_a)} \cdot f(\tilde{y}). \quad (13)$$

In this formula, $f(u) = f(u_1, \ldots, u_\ell)$ is obtained from $f(\tilde{y})$ by replacing $\tilde{y}_i$ by $u_i$. Since the left-hand side $\pi_*f(\tilde{y}) \in H^*(BT; \mathbb{R})$ is a polynomial in $u_1, \ldots, u_\ell$ with real coefficients, so is the right-hand side. But the right-hand side clearly has rational coefficients. Hence, $\pi_*f(\tilde{y}) \in H^*(BT; \mathbb{Q})$.

For $G/T$,

$$H^*_T(G/T) = \frac{\mathbb{Q}[u_1, \ldots, u_\ell, \tilde{y}_1, \ldots, \tilde{y}_\ell]}{(b(\tilde{y}) - b(u) \mid b \in R^+_+ W_G)},$$

and the fixed point set is $W_G$. For $f(\tilde{y}) \in \mathbb{Q}[\tilde{y}_1, \ldots, \tilde{y}_\ell]$, by the restriction formula (Proposition 11) and the Euler class formula (Proposition 13) for $G/T$,

$$\pi_*f(\tilde{y}) = \sum_{w \in W_G} w \cdot \frac{f(u)}{\prod_{a \in \Delta^+} c_1(S_a)} = \frac{\sum_{w \in W_G} (-1)^w w \cdot f(u)}{\prod_{a \in \Delta^+} c_1(S_a)}. \quad (14)$$
17. Ordinary integration and equivariant integration

We state a general principle (Proposition 20), well known to the experts, relating ordinary integration and equivariant integration.

Proposition 20. Let \( M \) be a compact oriented manifold of dimension \( n \) on which a compact, connected Lie group \( G \) acts and let \( \pi_* : H^*_G(M; \mathbb{R}) \to H^*(BG; \mathbb{R}) \) be equivariant integration. Suppose a cohomology class \( \eta \in H^n(M; \mathbb{R}) \) has an equivariant extension \( \tilde{\eta} \in H^n_G(M; \mathbb{R}) \). Then

\[
\int_M \eta = \pi_\ast \tilde{\eta}.
\]

Remark 21. For a torus action the right-hand side \( \pi_\ast \tilde{\eta} \) of (15) can be computed using the equivariant localization formula in terms of the fixed point set \( F \) of \( T \) on \( M \). In case the fixed points are isolated, this gives

\[
\int_M \eta = \sum_{p \in F} \frac{i_p^\ast \tilde{\eta}}{e^T(\nu_p)}.
\]

Proof of Proposition 20. The commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{i} & MG \\
\tau \downarrow & & \downarrow \pi \\
pt & \xrightarrow{j} & BG
\end{array}
\]

induces by the push-pull formula ([12, p. 158]) a commutative diagram in cohomology

\[
\begin{array}{ccc}
H^n(M; \mathbb{R}) & \xrightarrow{i^\ast} & H^n_G(M; \mathbb{R}) \\
\tau_* & & \pi_* \\
H^0(pt; \mathbb{R}) & \xrightarrow{j^\ast} & H^0(BG; \mathbb{R}).
\end{array}
\]

In degree 0, the restriction \( j^\ast : H^0(BG; \mathbb{R}) \to H^0(pt; \mathbb{R}) = \mathbb{R} \) is an isomorphism. Hence, if \( \eta \) has degree \( n \) in \( H^n(M; \mathbb{R}) \), then by the commutative diagram (16) and the push-pull formula

\[
\int_M \eta = \tau_* \eta = \tau_* i^\ast \tilde{\eta} = j^\ast \pi_* \tilde{\eta} = \pi_* \tilde{\eta}.
\]

Since all ordinary characteristic classes of \( G \)-vector bundles have equivariant extensions, all ordinary characteristic numbers of \( G \)-vector bundles can be computed from the equivariant localization formula.

Fix a basis \( \chi_1, \ldots, \chi_\ell \) of the character group \( \hat{T} \) of the maximal torus \( T \) in the compact, connected Lie group \( G \). Let \( H \) be a closed subgroup containing \( T \) in \( G \). On \( G/T \), we have associated bundles \( L_{\chi_i} := G \times_T C_{\chi_i} \). Let \( y_i = c_1(L_{\chi_i}) \in H^2(G/T; \mathbb{Q}) \). Let \( R \) be the polynomial ring \( \mathbb{Q}[y_1, \ldots, y_\ell] \). By Theorem [6]

\[
H^*(G/H; \mathbb{Q}) = \frac{R^{W_H}}{(R_+^{W_G})} = \frac{Q[y_1, \ldots, y_\ell]^{W_H}}{(R_+^{W_G})}.
\]
Theorem 22. Let $f(y) \in \mathbb{Q}[y_1, \ldots, y_r]^W_H$ be a $W_H$-invariant polynomial of degree $\dim G/H$, where each $y_i$ has degree 2. Then the characteristic number $\int_{G/H} f(y)$ of $G/H$ is given by

$$\int_{G/H} f(y) = \pi_* f(\tilde{y}) = \sum_{w \in W_G/W_H} w \cdot \left( \frac{f(u)}{\prod_{\alpha \in \Delta^+ \setminus \Delta^+(H)} c_1(S_\alpha)} \right).$$

Proof. Since $\tilde{y}_i = c_1^T(L_{\chi_i})$ is an equivariant extension of $y_i$, the cohomology class $f(\tilde{y}) \in H^*_T(G/H; \mathbb{Q})$ is an equivariant extension of $f(y)$. Combining Proposition 20 and (13), the formula for the ordinary characteristic numbers of $G/H$ follows. As noted earlier, (13) shows that if $f(\tilde{y})$ is a rational class, then so is $\pi_* f(\tilde{y})$. \hfill $\Box$

18. Example: the complex Grassmannian

In this example, we work out the $T$-equivariant cohomology ring as well as the characteristic numbers of the complex Grassmannian $G(k, n)$ of $k$-planes in $\mathbb{C}^n$. As a homogeneous space, $G(k, n)$ can be represented as $G/H$, where $G$ is the unitary group $U(n)$ and $H$ is the closed subgroup $U(k) \times U(n-k)$.

A maximal torus contained in $H$ is

$$T = U(1) \times \cdots \times U(1) = \left\{ t = \begin{bmatrix} t_1 & \cdots & t_n \\ \end{bmatrix} \mid t_i \in U(1) \right\}.$$

A basis for the characters of $T$ is $\chi_1, \ldots, \chi_n$, with $\chi_i(t) = t_i$. The characters $\chi_i$ define line bundles $S_{\chi_i}$ over the classifying space $BT$. We let $u_i = c_1(S_{\chi_i}) \in H^2(BT)$. A choice of positive roots for $G$ and for $H$ is

- $\Delta^+ = \{ \chi_i - \chi_j \mid 1 \leq i < j \leq n \}$,
- $\Delta^+(H) = \{ \chi_i - \chi_j \mid 1 \leq i < j \leq k \} \cup \{ \chi_i - \chi_j \mid k + 1 \leq i < j \leq n \}$.

Therefore,

$$\Delta^+ \setminus \Delta^+(H) = \{ \chi_i - \chi_j \mid 1 \leq i \leq k, k + 1 \leq j \leq n \}.$$

If $\alpha = \chi_i - \chi_j$, then

$$c_1(S_\alpha) = c_1(S_{\chi_i} \otimes S_{\chi_j}^*) = c_1(S_{\chi_i}) - c_1(S_{\chi_j}) = u_i - u_j. \quad (17)$$

The Weyl groups of $T$ in $G$ and $H$ are

$$W_G = S_n,$$  
$$W_H = S_k \times S_{n-k}.$$

(Notation: $S_\alpha, S_\chi$ are line bundles over $BT$ associated to the characters $\alpha$ and $\chi$, but $S_k, S_n$ are symmetric groups.) A permutation in the symmetric group $S_n$ is a bijection $w: \{1, \ldots, n\} \to \{1, \ldots, n\}$,

$$w(1) = i_1, \ldots, w(k) = i_k, w(k+1) = j_1, \ldots, w(n) = j_{n-k}, \quad (18)$$

where $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_{n-k})$ are two complementary multi-indices, i.e., $I \cup J = \{1, \ldots, n\}$. The equivalence class of $w$ in $S_n/(S_k \times S_{n-k})$ has a unique representative with $I = (i_1 < \cdots < i_k)$ and $J = (j_1 < \cdots < j_{n-k})$ both strictly increasing.
By Theorem (11(ii)), the rational equivariant cohomology ring of the Grassmannian $G(k,n)$ under the torus action is

$$H^*_T(G(k,n)) = \frac{\mathbb{Q}[u_1,\ldots,u_n] \otimes \mathbb{Q} \left[\tilde{y}_1,\ldots,\tilde{y}_k,\tilde{y}_{k+1},\ldots,\tilde{y}_n\right]^{S_k \times S_{n-k}}}{I},$$  

where $I$ is the ideal in $\mathbb{Q}[u_1,\ldots,u_n] \otimes \mathbb{Q} \left[\tilde{y}_1,\ldots,\tilde{y}_k,\tilde{y}_{k+1},\ldots,\tilde{y}_n\right]^{S_k \times S_{n-k}}$ generated by $b(\tilde{y}) - b(u)$ for all symmetric polynomials $b(\tilde{y}) \in \mathbb{Q}[\tilde{y}_1,\ldots,\tilde{y}_n]$. Let $\sigma_r$ be the $r$th elementary symmetric polynomial. Since every symmetric polynomial is a polynomial in the elementary symmetric polynomials, $I$ is also the ideal generated by $\sigma_r(\tilde{y}) - \sigma_r(u)$ for $r = 1,\ldots,n$. Thus, we may write

$$H^*_T(G(k,n)) = \frac{\mathbb{Q}[u_1,\ldots,u_n] \otimes \mathbb{Q} \left[\tilde{y}_1,\ldots,\tilde{y}_k,\tilde{y}_{k+1},\ldots,\tilde{y}_n\right]^{S_k \times S_{n-k}}}{(1 + \tilde{y}_1)^{r_1}(1 + u_1)\cdots(1 + \tilde{y}_n)^{r_n}(1 + u_n)}$$  

In this formula, the notation $(p(u,\tilde{y}))$ means the ideal generated by the homogeneous terms $p(u,\tilde{y})$.

If $S$ and $Q$ are the universal sub- and quotient bundles over $G(k,n)$, then setting

$$\tilde{s}_r = c^*_r(S) = \sigma_r(\tilde{y}_1,\ldots,\tilde{y}_k), \quad \tilde{q}_r = c^*_r(Q) = \sigma_r(\tilde{y}_{k+1},\ldots,\tilde{y}_n),$$

we have

$$1 + \tilde{s}_1 + \cdots + \tilde{s}_k = \prod_{i=1}^{n-k}(1 + \tilde{y}_i), \quad \text{and} \quad 1 + \tilde{q}_1 + \cdots + \tilde{q}_{n-k} = \prod_{i=k+1}^{n}(1 + \tilde{y}_i).$$

Thus (20) can be rewritten in the form

$$H^*_T(G(k,n)) = \frac{\mathbb{Q}[u_1,\ldots,u_n,\tilde{s}_1,\ldots,\tilde{s}_k,\tilde{q}_1,\ldots,\tilde{q}_{n-k}]}{(1 + \tilde{s}_1 + \cdots + \tilde{s}_k)(1 + \tilde{q}_1 + \cdots + \tilde{q}_{n-k}) - \prod_{i=1}^{n-k}(1 + u_i)}.$$  

Using the relation

$$1 + \sum_{i=1}^{n-k}\tilde{q}_i = \frac{\prod_{i=1}^{n-k}(1 + u_i)}{1 + \tilde{s}_i},$$

one can eliminate all the $\tilde{q}_i$ from $H^*_T(G(k,n))$: in other words, $H^*_T(G(k,n))$ is generated as an algebra over $\mathbb{Q}[u_1,\ldots,u_n]$ by $\tilde{s}_1,\ldots,\tilde{s}_k$ with relations given by terms of degree $\geq 2(n-k)$ in (21). In computing degrees, keep in mind that $\deg u_i = 2$ and $\deg \tilde{s}_i = \deg \tilde{q}_i = 2k$.

Similarly, in this notation, the rational cohomology ring of the Grassmannian $G(k,n)$ is

$$H^*(G(k,n)) = \frac{\mathbb{Q}[y_1,\ldots,y_k,y_{k+1},\ldots,y_n]^{S_k \times S_{n-k}}}{(1 + y_1)^{r_1}(1 + y_k)^{r_k}} = \frac{\mathbb{Q}[s_1,\ldots,s_k,q_1,\ldots,q_{n-k}]}{(1 + s_1 + \cdots + s_k)(1 + q_1 + \cdots + q_{n-k})},$$

where $s_r = c_r(S) = \sigma_r(y_1,\ldots,y_k)$ and $q_r = c_r(Q) = \sigma_r(y_{k+1},\ldots,y_n)$.

**Proposition 23.** The characteristic numbers of $G(k,n)$ are

$$\int_{G(k,n)} c_1(S)^{m_1} \cdots c_k(S)^{m_k} = \sum_I \prod_{i=1}^{m_1} \sigma_r(u_{i_1},\ldots,u_{i_k})^{m_r} \prod_{j \in J} (u_{i_j} - u_j),$$

where $\sum m_r = k(n-k)$, $I$ runs over all multi-indices $1 \leq i_1 < \cdots < i_k \leq n$ and $J$ is its complementary multi-index.
Proof. In Theorem \([22]\) take \(f(y)\) to be \(\prod_{r=1}^{k} c_r(S)^{m_r} = \prod_{r=1}^{k} \sigma_r(y_1, \ldots, y_k)^{m_r}\) and 
\[ w = (i_1, \ldots, i_k, j_1, \ldots, j_{n-k}) \]
as in \([18]\). Because \(w \cdot u_1 = u_{i_1}, \ldots, w \cdot u_k = u_{i_k}\),
\[ w \cdot f(u) = \prod_{r=1}^{k} \sigma_r(u_{i_1}, \ldots, u_{i_k})^{m_r}. \]
By \([17]\),
\[ \prod_{\alpha \in \Delta^+ \backslash \Delta^+(H)} c_1(S_{\alpha}) = \prod_{i=1}^{k} \prod_{j=k+1}^{n} (u_i - u_j). \]
By \([18]\),
\[ w \cdot \left( \prod_{\alpha \in \Delta^+ \backslash \Delta^+(H)} c_1(S_{\alpha}) \right) = w \cdot \left( \prod_{i=1}^{k} \prod_{j=k+1}^{n} (u_i - u_j) \right) = \prod_{i \in I} \prod_{j \in J} (u_i - u_j). \]

One of the surprising features of the localization formula is that although the right-hand side of \([22]\) is apparently a sum of rational functions of \(u_1, \ldots, u_n\), the sum is in fact an integer.

**Example.** As an example, we compute the characteristic numbers of \(CP^2 = G(1, 3)\). The rational cohomology of \(CP^2\) is \(H^*(CP^2) = \mathbb{Q}[x]/(x^3)\), generated by \(x = c_1(S^2) = -c_1(S)\). By Proposition \([23]\),
\[ \int_{CP^2} x^2 = \int_{G(1,3)} c_1(S)^2 = \sum_{i=1}^{3} \frac{u_i^2}{\prod_{j \neq i} (u_i - u_j)} \]
\[ = \frac{u_1^2}{(u_1 - u_2)(u_1 - u_3)} + \frac{u_2^2}{(u_2 - u_1)(u_2 - u_3)} + \frac{u_3^2}{(u_3 - u_1)(u_3 - u_2)}, \]
which simplifies to 1, as expected.

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