ON THE THETA DIVISOR OF SU(2,1).

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Introduction

Among the objects which are interestingly related to a complex projective curve $C$, one certainly finds the varieties

$$SU_C(r, d).$$

These are the moduli spaces for semistable vector bundles on $C$ having rank $r$ and fixed determinant $L \in \text{Pic}^d(C)$. In this paper we assume $C$ is smooth, connected and of genus $g \geq 2$, then $SU_C(r, d)$ is projective and its Picard group is isomorphic to $\mathbb{Z}$ (cfr. [DN]). Therefore we can consider the ample generator $H_{r,d}$ of $\text{Pic} SU_C(r,d)$ and the corresponding rational map

$$h_{r,d}: SU_C(r, d) \to \mathbb{P} H^0(H_{r,d})^*.$$

In general, as soon as $r \geq 3$, not very much is known on the behaviour of this map. For instance, one does’nt know under which conditions $h_{r,d}$ is a morphism, nor what is the degree of $h_{r,d}$ onto its image, (see, for a survey on this matter, [B3]). The situation becomes better in the case $r = 2$. In our paper we contribute to this case by showing the following:

**Theorem 1** $h_{2,1}$ is an embedding.

Of course this implies the same property for $h_{2,2d+1}$. The proof relies on a method we will explain in a moment and on a second result we prove, which is of some independent interest:

**Theorem 2** Let $F$ be a stable rank two vector bundle of degree $2g + 1$. If $e \in \text{Pic}^0(C)$ is general then $F(e)$ is globally generated and $h^0(F) = 3$.

To put in perspective theorem 1, let us recall briefly some known facts about the maps $h_{2,d}$.

At first $h_{2,d}$ is an embedding if $d$ is even and $C$ has general moduli. This result is due to Laszlo, ([L1]). If $g$ is $\leq 3$ and $d$ is even, $h_{2,d}$ and its image have been completely described by Narasimhan and Ramanan for every curve $C$, ([NR1], [NR2]).

If $d$ is odd and $C$ is hyperelliptic, an explicit description of $h_{2,d}(SU_C(2, d))$ has been given by Desale and Ramanan, ([D-R]). From this description it follows that $h_{2,d}$ is an embedding in the hyperelliptic case, therefore the same property holds for a general $C$.

Notice also that, according to Beauville, $h_{2,d}$ has degree one unless $C$ is hyperelliptic of genus $\geq 3$ and $d$ is even, (cfr. [B1] and J[B2] remarqueJ(3.14)).

On the other hand, knowing for which curves $h_{2,d}$ fails to be an embedding is somehow a delicate question. If $d$ is even, a partial answer can be summarized in the following way (cfr. [B1], [BV]):

1. Assume $C$ is hyperelliptic of genus $g \geq 3$, then $h_{2,d}: SU(2, d) \to h_{2,d}(S(2, d))$ is a finite double covering.

2. Assume $C$ is not hyperelliptic of genus $g \geq 3$, then $h_{2,d}$ is injective and its restriction to the stable locus $SU_C^s(2, d)$ of $SU_C(2, d)$ is an embedding.
Since $SU_C(2, d) - SU^*_C(2, d) = SingSU_C(2, d)$, what it is still missing is the description of the tangent map $d(h_{r,d})$ at points of $SingSU_C(2, 0)$.

If $d$ is odd the situation is less complicate and our previous theorem 1 says that $h_{2,d}$ is always an embedding. Let us sketch the method we will use for proving this theorem and the way the paper is organized.

Of course, if $d$ is odd, we can identify $SU_C(2, d)$ to the moduli space $X$ of stable rank two vector bundles with determinant $\omega_C(p)$, where $p$ is a given point of $C$. Then we consider in $X$ the family of open sets

$$X_l, \quad l \in Pic^1(C),$$

parametrizing bundles $E$ such that $F = E(l)$ is globally generated and has general cohomology. One can show that for such an $F$ the natural determinant map

$$w : \wedge^2 H^0(F) \to H^0(detF)$$

is injective. Since $h^0(F) = 3$, the space

$$W = Im(w)$$

is an element of the Grassmannian $G_l$ of 3-dimensional subspaces of $H^0(\omega_C(2l + p))$, $(\omega_C(2l + p) \cong detF)$. This construction defines a rational map

$$g_l : X \to G_l$$

sending the moduli point of $E$ to $W$. Since $F$ is globally generated, the linear system $|W|$ is base-point-free. Let

$$f_W : C \to \mathbb{P}^2$$

be the morphism defined by $|W|$, it is a standard fact that

$$f_W^*T_{\mathbb{P}^2}(-1) \cong F.$$

Therefore we have reconstructed $F$ starting from $W$: this implies that $g_l$ is birational and moreover that $g_l$ is biregular at the moduli point of $E$. Assume the two following facts are true:

(a) $g_l$ is defined by a linear system which is contained in $|\mathcal{L}_X|$, $\mathcal{L}_X$ being the ample generator of $PicX$.

(b) $X = \bigcup X_l, \quad l \in Pic^1(C)$.

Then we can conclude that the map associated to $\mathcal{L}_X$ is an embedding.

In section 2 the map $g_l$ and its regular locus are studied in detail, in particular we show (a). In section 3 we show theorem 2, that is (b). Let $\Theta$ be a symmetric theta-divisor in the Jacobian $J$ of $C$, the proof of theorem 2 involves the pencil

$$P_E \subset |2\Theta|$$

which is naturally associated to a stable vector bundle $E$ of determinant $\omega_C(p)$, (see section 1 and [B2]). Actually theorem 2 turns out to be equivalent to the following statement:

*for each $E$, $P_E$ has no fixed component.*
Therefore we analyze carefully the base locus $B_E$ of $P_E$ and finally show that $\dim B_E = g - 2$. The proof of this fact is technical. The essential reason for it seems to be the following property:

let $Z \subset \text{Pic}^d(C)$ be a family of subbundles of $E$ having positive degree $d$ then $\dim Z \leq g - d - 1$.

This property is related to Martens theorem, (cfr. [ACGH]), and to a result of Lange-Narasimhan saying that $E$ has at most finitely many linear subbundles of maximal degree, ([LN] Prop. 4.1).

In section 4 we study some more geometry of the base locus $B_E$, the main remark being that $B_E$ is reducible and splits in two components interchanged by $-1$ multiplication. Let $W_1 \subset J$ be the image of $C$ under the Abel map. Each component of $B_E$ parametrizes the family of translates of $W_1$ which are contained in an element of the pencil $P_E$. In view of this, we conclude the paper with the following question:

*describe the locus of pencils $P \subset |2\Theta|$ with reducible base locus.*

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1 Preliminary results and notations.

In this section we fix our notations. Moreover we recollect some results which will be frequently used. We will always denote by

\[(1.1)\]
\[X\]

the moduli space of stable rank two vector bundles with determinant \(\omega_C(p)\), where \(p\) is a given point of \(C\). \(X\) is a smooth, irreducible projective variety of dimension \(3g - 3\) and \(PicX\) is isomorphic to \(\mathbb{Z}\). By definition, the generalized theta divisor of \(X\) is the ample generator

\[(1.2)\]
\[L_X\]

of \(PicX\). The map associated to \(L_X\) will be

\[(1.3)\]
\[\phi : X \to P_X,\]

where

\[P_X = PH^0(L_X)^*\]

By Verlinde formula \(dim P_X = 2^{g-1}(2^g - 1) - 1\). We reserve the notation

\[\xi\]

for points of \(X\). With some abuse, the same notation will be used for the vector bundle corresponding to \(\xi\). On the other hand, we will deal frequently with the moduli space

\[Y\]

of semistable rank two vector bundles of determinant \(\omega_C\). A point on \(Y\), or a bundle parametrized by this point, will be

\[\eta\]

We recall that \(Y\) is irreducible of dimension \(3g - 3\) and that \(SingY\) is the locus of non stable points. \(SingY\) is the image of the morphism \(f : Pic^{g-1}(C) \to Y\) which sends \(L \in Pic^{g-1}(C)\) to \(\eta = L \oplus M\), \(M = \omega_C \otimes L^*\). It turns out that \(SingY\) is the Kummer variety of the Jacobian of \(C\). As in the case of \(X\), \(PicY\) is isomorphic to \(\mathbb{Z}\) and its ample generator

\[L_Y\]

is called the generalized theta divisor of \(Y\). \(L_Y\) defines a map

\[\psi : Y \to P_Y,\]

where \(P_Y = PH^0(L_Y)^*\) and \(dim P_Y = 2^{2g} - 1\).

Let

\[(1.4)\]
\[\Theta \subset J = Pic^0(C)\]
be a symmetric theta divisor, there is a fundamental relation between the linear system \(|2\Theta|\) and the maps \(\phi\) and \(\psi\). This relation has been explained by Beauville in [B1] and [B2]. We need to summarize the most important aspects of it.

At first, there exists an identification

\[(1.5) \quad P_Y = |2\Theta|\]

which relies on Wirtinger duality between \(H^0(O_J(2\Theta))\) and \(H^0(O_J(2\Theta))^*\). Let \(\eta \in Y\), \(\eta\) defines in \(J\) a divisor

\[(1.6) \quad D = \{ e \in J/h^0(\eta(e)) \geq 1 \},\]

with a natural structure of determinant scheme (cfr. [B1]). It turns out that

\[D \in |2\Theta|\]

and that

\[(1.8) \quad \psi(\eta) = D.\]

Let

\[\xi \in X,\]

the next construction associates to \(\xi\) a line in \(P_Y\), that is a pencil of \(2\Theta\)-divisors.

**CONSTRUCTION 1.9** Let \(C_p\) be the structure sheaf of the point \(p\), \(\xi\) defines the family of exact sequences

\[
0 \longrightarrow \eta_\lambda \longrightarrow \xi \overset{\lambda}{\longrightarrow} C_p \longrightarrow 0,
\]

where \(\lambda \in \mathbb{P}^1 = \mathbb{P}Hom(\xi, C_p) = \mathbb{P}\xi^*_p\).

Since \(\xi\) is stable, it follows that \(\eta_\lambda\) is a semistable rank two vector bundle of determinant \(\omega_C\). Therefore \(\eta_\lambda\) defines the divisor

\[\psi(\eta_\lambda) = D_\lambda \in |2\Theta|\].

We can consider the map

\[\psi_\xi : \mathbb{P}^1 \to |2\Theta|\]

sending \(\lambda\) to \(D_\lambda\). One can show that \(\psi_\xi\) is a linear embedding, ([B2]). In particular the line \(\psi_\xi(\mathbb{P}^1)\) is a pencil of \(2\Theta\)-divisor.

**DEFINITION 1.10** The previous line is the pencil associated to \(\xi\). It will be denoted by

\[P_\xi = \{ D_\lambda/\lambda \in \mathbb{P}^1 \}.\]

\(P_\xi\) is a fundamental object for the study of the map \(\phi\). Note that \(P_\xi\) is a point of the Grassmannian \(Grass(2, H^0(O_J(2\Theta)))\), hence of its Plucker space

\[(1.11) \quad \mathbb{P} = \mathbb{P} \wedge^2 H^0(O_J(2\Theta)).\]
**DEFINITION 1.12** \( \phi_p : X \to \mathbf{P} \) is the morphism which is so defined:

\[
\phi_p(\xi) = P_\xi.
\]

\( \phi_p \) has been studied by Beauville in [B2]. One has

\[
\phi^*_p \mathcal{O}_{\mathbf{P}}(1) = \mathcal{L}_Y,
\]

therefore \( \phi^*_p \) induces an homomorphism between global sections

\[
F_p : H^0(\mathcal{O}_{\mathbf{P}}(1)) \to H^0(\mathcal{L}_Y).
\]

Since the two spaces have the same dimension, it follows \( \phi = \phi_p \) as soon as \( F_p \) is an isomorphism. It is not always true that \( \phi_p \) is an embedding nor that \( F_p \) an isomorphism, (cfr. [B2]). A clear example is given by the case when \( C \) is hyperelliptic and \( p \) is a Weierstrass point. Then \( \phi_p \) has degree two onto its image while \( \phi \) is an embedding, ([B2], 3.14).

Let \( \eta \) be a point of \( Y \) and let \( D = \psi(\eta) \), we want to make some remarks on the singular locus of \( D \). If \( e \in \text{Pic}^0(C) \) we denote as

\[
b_e : C \times C \to \text{Pic}^0(C)
\]

the difference map \((x, y) \mapsto e + x - y\). Assume

\[
h^0(\eta(e)) \geq 2,
\]

then certainly \( e \in D \). As a corollary of Laszlo’s theorem on the singularities of the generalized theta divisor ([L2]), it follows that \( e \in \text{Sing}D \). We want to show some slight variation of this corollary.

**PROPOSITION 1.13** Assume \( b_e(C \times C) \) is not contained in \( D \). Then:

1. \( h^0(\eta(e)) = 2 \) and the determinant map \( v : \wedge^2 H^0(\eta(e)) \to H^0(\omega_C(2e)) \) is injective.
2. Let \( d = \text{div}(s) = \Sigma y_i \), where \( s \) is a generator of \( \text{Im}(v) \). In \( CJ \times C \) consider the divisor

\[
\Gamma = \Sigma C \times \{y_i\};
\]

Then \( \Gamma \) is a component of \( b^*_e D \).

**Proof.** (1) Assume \( h^0(\eta(e)) \geq 3 \), then \( h^0(\eta(e + x - y)) \geq 1 \) and \( e + x - y \in D \) for every \((x, y) \in C \times C\): a contradiction. Assume \( h^0(\eta(e)) = 2 \) and \( v \) not injective, then \( \eta(e) \) contains a line bundle \( L \) with \( h^0(L) = 2 \). Therefore \( h^0(\eta(e - x)) \geq 1 \) for each \( x \in C \) and hence \( b_e(CJ \times C) \subset D \): a contradiction again.

To show (2) it suffices to fix a general \( x \in C \) and compute that \( b^*_e D \) restricted to \( \{x\} \times C \) contains the restriction of \( \Gamma \). For such an \( x \) we have \( h^0(\eta(e + x)) = 2 \). Otherwise it would follow \( h^0(\eta(x)) \geq 3 \) generically and hence \( b_e(CJ \times C) \subset D \). Let \( s_1, s_2 \) be a basis of \( H^0(\eta(e + x)) \), it is a standard fact that \( \text{div}(s_1 \wedge s_2) \) is the divisor \( b^*_e D \) restricted to \( \{x\} \times C \). In other words \((x - C) \cdot D = \text{div}(s_1 \wedge s_2)\), where \( x - C \) is the image of \( C \) by the Abel map \( y \mapsto x - y \). From the exact sequence

\[
0 \to \eta(e) \to \eta(e + x) \to \eta(e + x)_x \to 0
\]
we have $\text{div}(s_1 \wedge s_2) = d + 2x$ and the result follows.

Note that $e$ is a singular point of the curve $b_e(\Gamma)$. Let

$$< d > \subset \mathbf{P}^{g-1} = \mathbf{P} H^0(\omega_C)^*$$

be the linear span of $\phi_* d$, where $\phi$ is the canonical map for $C$. The projectivized tangent space to $b_e(\Gamma)$ at $e$ contains $< d >$, (cfr. [ACGH]). Since $\mathcal{O}_C(d) = \omega_C(2e)$ it follows $< d > = \mathbf{P}^{g-1}$ provided $\mathcal{O}_C(2e) \neq \mathcal{O}_C$. In this case $e$ is a singular point of every hypersurface $M \subset \mathbf{P} H^0(C)$ such that $b_e^* M$ contains $\Gamma$. We have shown the following

**COROLLARY 1.14** Assume $\eta(e)$ satisfies condition (1) of the previous proposition. Let $M$ be a component of $D$ and let $\mathcal{O}_C(2e) \neq \mathcal{O}_C$. If $b_e^* M$ contains $\Gamma$, then $e \in \text{Sing}_M$.

We end this section by stating some basic properties which are related to elementary transformations.

Let $\mathcal{C}_p$ be the structure sheaf of a point $p \in C$, in the following we consider any exact sequence

(1.15) $$0 \longrightarrow \eta \longrightarrow \xi \longrightarrow \mathcal{C}_p \longrightarrow 0,$$

where $\eta$ and $\xi$ are rank two vector bundles. We assume that the morphism of sheaves $\lambda$ is defined up to a non zero constant factor. Fixing $\xi$, the family of all exact sequences (1.15) is parametrized by

$$\mathbf{P}^1 = \mathbf{P} \text{Hom}(\xi, \mathcal{C}_p).$$

Let $q \subset H^0(\xi)$ be a 1-dimensional vector space and let $q(p)$ be its image in the fibre $\xi_p$. Assume $q(p)$ is not zero, then $\xi_p/q(p) \cong \mathcal{C}_p$ and we can associate to $q$ the surjective morphism

$$\pi_q = f \cdot e : \xi \rightarrow \mathcal{C}_p,$$

where $f : \xi_p \rightarrow \xi_p/q(p)$ is the quotient map and $e : \xi \rightarrow \xi_p$ is the natural evaluation. $\pi_q$ is defined up to a non zero constant factor and its construction is linear. That is

$$\pi_q = \pi(q),$$

where

(1.16) $$\pi : \mathbf{P} H^0(\xi) \rightarrow \mathbf{P} \text{Hom}(\xi, \mathcal{C}_p)$$

is a linear map of center $\mathbf{P} H^0(\xi(-p))$. The Kernel of the map

(1.17) $$h^0(\pi_q) : H^0(\xi) \rightarrow H^0(\mathcal{C}_p)$$

is spanned by $q$ and $H^0(\xi(-p))$. 

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The sequence (1.15) induces the exact commutative diagram

\[
\begin{array}{c}
0 \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
H^0(\eta_\lambda) & \longrightarrow & H^0(\xi) & \longrightarrow & C \\
(\eta\lambda)_p & \longrightarrow & \xi_p & \longrightarrow & C & \longrightarrow & 0
\end{array}
\]

(1.18)

where the vertical arrows are the evaluation maps.

**PROPOSITION 1.19**

(1) If $\xi$ is stable, then $\eta_\lambda$ is semistable for each $\lambda \in P^1$.

(2) If $e_p : H^0(\xi) \rightarrow \xi_p$ is surjective, then $h^0(\eta_\lambda) = h^0(\xi) - 1$, for each $\lambda \in P^1$.

(3) Let $L \in \text{Pic}C$, then the following conditions are equivalent:

(i) $h^0(\eta_\lambda \otimes L) \geq 1$ for each $\lambda \in P^1$.

(ii) $h^0(\xi \otimes L(-p)) = 1$ or $h^0(\xi \otimes L) \geq 2$.

**Proof**

(1) Obvious. (2) In the previous diagram (1.18) the latter vertical arrow is an isomorphism. Therefore $\rho_\lambda$ is surjective if and only if $e_p$ is surjective, this implies the result. (3) Tensor the sequence (1.15) by $L$ and consider the corresponding diagram (1.18). Then $h^0(\eta_\lambda \otimes L) = \text{dim Ker}(\rho_\lambda \otimes L)$. Note that this dimension is non zero for each $\lambda$ if and only if the following two conditions hold: (1) $h^0(\xi \otimes L) \geq 1$, (2) either $e_pJ \otimes L$ is zero or

\[
\text{Im}(e_p \otimes L) \cap \text{Ker}(\rho_\lambda,p) \neq (0) \ \forall \lambda \in P^1.
\]

This happens if and only if $h^0(\xi \otimes L) \geq 1$ and $e_p \otimes L$ is zero or surjective. Then the result follows.

**PROPOSITION 1.20**

Assume $h^0(\xi(-y)) = 1$ for some $y \in C$. Then there exists $\lambda \in P^1 = \text{PHom}(\xi, C_p)$ such that the sequence

\[
0 \longrightarrow \eta_\lambda \longrightarrow \xi \longrightarrow \lambda \longrightarrow C_p \longrightarrow 0
\]

is exact and $h^0(\eta_\lambda(-y)) \neq 0$. $\lambda$ is unique if $h^0(\xi(-y - p)) = 0$, otherwise one has $h^0(\eta_\lambda(-y)) \neq 0$ for each $\lambda$.

**Proof**

Tensor the exact sequence (1.15) by $O_C(-y)$ and consider the corresponding diagram (1.18). If $e_p(-y) : H^0(\xi(-y)) \rightarrow \xi_p(-y)$ is zero, it follows that $h^0(\eta_\lambda(-y)) \neq 0$ for each $\lambda$. Moreover $e_p(-y)$ is zero if and only if $h^0(\xi(-p - y)) \neq 0$. If $e_p(-y)$ is not zero, then its image $q(p)$ is one dimensional. As above let $\pi_q : \xi(-y) \rightarrow C_p$ be the surjective morphism associated to $q(p)$. Putting $\lambda = \pi_q(y)$ one obtains the unique $\lambda$ such that $h^0(\eta_\lambda(-y)) \neq 0$.

Let

\[
V \subset H^0(\xi)
\]

be a 3-dimensional vector space and let

\[
w : \wedge^2 V \rightarrow H^0(\text{det}\xi)
\]
be the determinant map, in the following we assume that $w$ is injective.
Consider the 3-dimensional vector space

$$W = \text{Im}(w) \subset H^0(\det \xi)$$

and the rational map

$$(1.21) \quad f_W : C \to \mathbf{P}^2 = PW^*$$

defined by $W$. Then we have

**PROPOSITION 1.22** Assume that the evaluation map $V \otimes \mathcal{O}_C \to \xi$ is surjective, then

$$f_W^* T_{\mathbf{P}^2}(-1) = \xi.$$ 

**Proof** Well known.

**FURTHER CONVENTIONS AND NOTATIONS**
- Let $l \in \text{Pic}(C)$ and let $d \in \text{Div}(C)$. When no confusion arises we will use freely $l(d)$ or $l + d$ to denote $l \otimes \mathcal{O}_C(d)$.
- $C(m)$ is the $m$-symmetric product of $C$. Let $l \in \text{Pic}^n(C)$, then $l - C(m)$ denotes the image of $C(m)$ in $\text{Pic}^{n-m}(C)$ by the Abel map $d \to l - d$.
- Let $A$ be an abelian variety and let $Z \subset A$. $Z_e$ is the translate of $Z$ by $e \in A$.
- $-Z = \{-z, z \in Z\}$.
- $V^*$ is the dual of the vector bundle (vector space) $V$. $\mathbf{P}V$ denotes the projectivization of $V$. $\text{Proj}(V) = \mathbf{P}V^*$. 


2 Very ampleness of $L_X$.

We want to study the morphism

\[(2.1) \quad \phi : X \to P_X,\]

defined by the ample generator of $PicX$. For doing this conveniently, we begin by constructing a family of auxiliary rational maps

\[(2.2) \quad g_l : X \to G_l, \quad l \in Pic^1(C).\]

**DEFINITION 2.3** Let $(\xi, l) \in X \times Pic^1(C)$. We say that $(\xi, l)$ satisfies condition (+) if the following three properties hold:

1. $h^0(\xi(l)) = 3$,
2. $\xi(l)$ is globally generated,
3. the determinant map $w_l : \wedge^2 H^0(\xi(l)) \to H^0(det \xi(l))$ is injective.

By definition

\[(2.4) \quad X_l = \{ \xi \in X/ (\xi, l) \text{ satisfies (+)} \} .\]

Actually, as we are going to show, condition (+3) follows from (+1) and (+2).

**PROPOSITION 2.5** Assume a pair $(\xi, l)$ satisfies conditions (+1) and (+2), then the determinant map

\[w_l : \wedge^2 H^0(\xi(l)) \to H^0(det \xi(l))\]

is injective.

**Proof** Since $h^0(\xi(l)) = 3$, every vector $v \in \wedge^2 H^0(\xi(l))$ is indecomposable. Let $v = s_1 \wedge s_2$, $v \neq 0$. If $w_l(v) = 0$, there exists an exact sequence

\[(2.6) \quad 0 \to A \to \xi(l) \to B \to 0\]

such that $A$ is a line bundle and the image of $H^0(A)$ in $H^0(\xi(l))$ contains $s_1, s_2$. Passing to the long exact sequence, it follows that the image of $H^0(\xi(l))$ in $H^0(B)$ is generated by one element $b$. Let $x$ be a point such that $b(x) = 0$. Then the image of $H^0(\xi(l))$ in $\xi(l)_x$ is contained in $A_x$ and hence $\xi(l)$ is not globally generated. This contradiction shows that $w_l$ is injective.

The next proposition will be useful in various situations.

**PROPOSITION 2.6** For each $l \in Pic^1(C)$ there exists a projective curve

\[\Gamma_l \subset X\]

such that:

1. the set $(X - X_l) \cap \Gamma_l$ is finite,
2. each point of $\Gamma_l$ satisfies conditions (+1) and (+3),

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(3) $h^0(\xi(l-p)) = 1$ for each $\xi \in \Gamma_l$.

Proof. Observe that the set

$$U_l = \{ M \in \text{Pic}^{g-1}(C) | h^0(M(l-p)) = 0, h^0(M(l+p-x)) = 1, \forall x \in C \}$$

is open and non empty for each $l \in \text{Pic}^1(C)$. Let $\eta_o = M \oplus N$, where $M$ and $N = \omega_C \otimes M^*$ are choosen in $U$. Then $\eta_o(l+p)$ is globally generated, $h^0(\eta_o(l+p)) = 4$ and $h^0(\eta_o(l-p)) = 0$. By semicontinuity we can replace the semistable $\eta_o$ by a stable $\eta$ satisfying the same properties.

Applying construction (1.9) to $\eta(p+l)$, we obtain a family of exact sequences

$$0 \longrightarrow \xi_\lambda(l) \longrightarrow \eta(p+l) \longrightarrow \lambda \longrightarrow C_p \longrightarrow 0,$$

where $\lambda \in \mathbb{P}^1$. Since each $\xi_\lambda$ is stable, this defines a morphism

$$\gamma : \mathbb{P}^1 \to X$$

such that $\gamma(\lambda) = \xi_\lambda$. We denote by $\Gamma_l$ the image of $\gamma$.

By proposition (1.19)(2) we have $h^0(\xi_\lambda(l)) = 3$, i.e. every point of $\Gamma_l$ satisfies property (+1). In principle it is possible that $\xi_\lambda$ is not globally generated for every $\lambda$. We show that this is not the case if $\eta$ is sufficiently general. Let $D \in |2\Theta|$ be the divisor defined by $\eta$. If $\eta$ is general $D$ is reduced and the curve $l-C$ is transversal to $D$. Assuming this it follows that $D \cap (l-C)$ is a set of $2g$ distinct points

$$\{ l-y_1, \ldots, l-y_{2g} \} \quad (y_i \in C)$$

satisfying the condition $h^0(\eta(l-y_i)) = 1$. Moreover one has $h^0(\eta(l-x)) = 0$ for $x \neq y_1, \ldots, y_{2g}$. Under these assumptions we check the values of $\lambda$ for which $\xi_\lambda(l)$ is not globally generated.

If $\xi_\lambda(l)$ is not globally generated at $x$, it follows $h^0(\xi_\lambda(l-x)) = h^0(\eta(l+p-x)) = 2$. But then $\eta(l+p-x)$ is not globally generated at $p$ by proposition 1.19(2). Hence $h^0(\xi_\lambda(l-x)) = 1$ and $x$ belongs to $\{ y_1, \ldots, y_{2g} \}$.

Since we are assuming $h^0(\eta(l-p)) = 0$, it follows $h^0(\eta(l-p-x)) = 0$ for each $x \in C$. Then the evaluation $H^0(\eta(l-x)) \to \eta_p(l-x)$ has one-dimensional image $q(p)$ and $q(p)$ defines a surjective morphism $\pi_q : \eta(l-x) \to C_p$, as in (1.15). Twisting by $O_C(p+x)$ we finally obtain

$$\pi_q(x) : \eta(l+p) \to C_p.$$
satisfies the following condition: $\text{Ker}(w)$ does not contain indecomposable vectors. If the genus of $C$ is $\geq 3$ this follows from [BV] theorem 3.3(1), where it is shown that $\text{Ker}(w) = (0)$. In genus two a proof can be given by a simple dimension count, (cfr. [Br]). The inclusion of the subbundle $\xi_\lambda(l)$ in $\eta(l+p)$ induces the standard commutative diagram

\[
\begin{array}{ccc}
\wedge^2 H^0(\xi_\lambda) & \rightarrow & \wedge^2 H^0(\eta(l+p)) \\
\downarrow w_\lambda & & \downarrow w \\
H^0(\omega_C(2l+p)) & \rightarrow & H^0(\omega_C(2l+2p))
\end{array}
\]

where the top arrow is injective and preserves indecomposable vectors. The vertical arrows are just the determinant maps. Since $w$ is injective, $w_\lambda$ must be injective for each $\lambda$. Therefore each point of $\Gamma_l$ satisfies property (+3) and the proof of (2) is completed.

To show (3) observe that $h^0(\eta(l)) = 2$, because we are assuming $h^0(\eta(l-p)) = 0$. Then $\eta(l)$ is globally generated at the point $p$. From proposition (1.19)(2), it follows $h^0(\xi_\lambda(l-p)) = 1$ for each $\lambda$, this implies (3).

From (1) of the previous proposition we obtain immediately

**COROLLARY 2.7** For each $l \in \text{Pic}^1(C)$, $X_l$ is a non empty open set.

**REMARK 2.8** For any $\xi \in X$ one can also consider the set

$$U_\xi = \{ l \in \text{Pic}^1(C) / (\xi, l) \text{ satisfies (+)} \}.$$

It turns out that $U_\xi$ is not empty, the more difficult proof of this fact will be the main result of section 3.

Let $l \in \text{Pic}^1(C)$, we consider the line bundle

\[(2.9)\quad M_l = \omega_C(2l+p)\]

and the Grassmann variety

\[(2.10)\quad G_l = \text{Grass}(3, H^0(M_l))\]

of 3-dimensional subspaces. The Poincaré bundle

$$E \rightarrow X_l \times C,$$

defines on $X_l$ the sheaf

$$\mathcal{H} = \pi_{1*}(E \otimes \pi_2^* \mathcal{O}_C(l)),$$

where $\pi_i, i = 1, 2$ denotes the projection onto the $i$-th factor. Since $h^0(\xi(l)) = 3$ for each $\xi \in X_l$, $\mathcal{H}$ is a rank three vector bundle. The fibre of $\mathcal{H}$ at $\xi$ is

$$\mathcal{H}_\xi = H^0(\xi(l)).$$

A standard construction yields a map of vector bundles

$$w : \wedge^2 \mathcal{H} \rightarrow H^0(M_l) \otimes \mathcal{O}_C$$
such that the fibrewise map
\[ w_\xi : \wedge^2 H^0(\xi(l)) \to H^0(M_l) \]
is exactly the determinant map. Note that, by our assumptions on \( X_l \), \( w \) is an injective map of vector bundles. In particular \( w \) defines a rational map

\[ (2.11) \quad g_l : X \to G_l \]
such that
\[ g_l(\xi) = \text{Im}(w_\xi), \]
for each \( \xi \in X_l \). This is the family of rational maps we want to consider.

**PROPOSITION 2.12**

(1) \( g_l : X \to G_l \) is birational. Moreover
\[ g_l/X_l : X_l \to g_l(X_l) \]
is a biregular morphism.

(2) \( g_l \) is defined at each point \( \xi \) such that \( (\xi, l) \) satisfies conditions \((+1)\) and \((+3)\).

**Proof**

(1) Note that \( \dim X = \dim G_l = 3g - 3 \) and that both \( X \) and \( G_l \) are smooth varieties. Since \( X_l \) is open it suffices to show that the morphism \( g_l/X_l \) is injective. Then, by Zariski main theorem, \( g_l/X_l : X_l \to g_l(X_l) \) is biregular. If \( \xi \in X_l \), the image of the determinant map \( w_\xi \) is a 3-dimensional vector space \( W \subset H^0(M_l) \) and \( \xi(l) \) is globally generated. The latter condition implies that the linear system of divisors defined by \( W \) is base-point-free. Consider the morphism
\[ f : C \to \mathbb{P}^2 \]
associated to \( W \). By proposition (1.22) we have
\[ f^* T_{\mathbb{P}^2}(-1) \cong \xi. \]
Now assume \( g_l(\psi) = g_l(\xi) \) for some \( \psi \in X_l \). Then \( \text{Im}(w_\psi) = \text{Im}(w_\xi) = W \). Hence \( \psi \cong f^* T_{\mathbb{P}^2}(-1) \cong \xi \) and \( g_l/X_l \) is injective.

(2) Immediate consequence of the definitions.

As a next step we will construct a commutative diagram

\[ (2.13) \quad X \xrightarrow{\phi_p} \mathbb{P} \]
\[ g_l \downarrow \quad \downarrow p_l \]
\[ G_l \xrightarrow{u_l} \mathbb{P}_l \]
such that:
- \( p_l : \mathbb{P} \to \mathbb{P}_l \) is a linear map between projective spaces,
- \( u_l : G_l \to \mathbb{P}_l \) is induced by a linear map between the Pluecker space of \( G_l \) and \( \mathbb{P}_l \).

We recall that \( \phi_p \) is the morphism defined in 1.14 of section 1, (cfr. [B2]).
At first we define $p_l$: in $J$ one has the curve
\begin{equation}
C_l = \{l - x, \quad x \in C\},
\end{equation}
we consider the restriction map
\begin{equation}
\rho_l : H^0(\mathcal{O}_J(2\Theta)) \to H^0(\mathcal{O}_{C_l}(2\Theta))
\end{equation}
and its wedge product
\begin{equation}
\wedge^2(\rho_l) : \wedge^2H^0(\mathcal{O}_J(2\Theta)) \to \wedge^2H^0(\mathcal{O}_{C_l}(2\Theta)).
\end{equation}
We know that the target space of $\phi_p$ is
\begin{equation}
P = P \wedge^2H^0(\mathcal{O}_J(2\Theta)),
\end{equation}
let
\begin{equation}
P_l = P \wedge^2H^0(\mathcal{O}_{C_l}(2\Theta)),
\end{equation}
\textbf{DEFINITION 2.17} $p_l : P \to P_l$ is the projectivization of $\wedge^2(\rho_l)$.
To define $u_l$ we observe that
\begin{equation}
\mathcal{O}_{C_l}(2\Theta) \cong M_l(-p),
\end{equation}
leaving the proof as an exercise. This isomorphism yields an obvious identity
\begin{equation}
P_l = P \wedge^2H^0(M_l(-p)).
\end{equation}
On the other hand there exists a standard exact sequence
\begin{equation}
0 \longrightarrow \wedge^3H^0(M_l(-p)) \longrightarrow \wedge^3H^0(M_l) \longrightarrow \wedge^2H^0(M_l(-p)) \longrightarrow 0
\end{equation}
such that, for each indecomposable vector $s_1 \wedge s_2 \wedge s_3$, one has
\begin{equation}
f(s_1 \wedge s_2 \wedge s_3) = \Sigma s_i(p)(s_j \wedge s_k), \quad i = 1, 2, 3.
\end{equation}
Note that $f$ is uniquely defined up to a non zero constant factor. Let set
\begin{equation}
\Pi_l = P(\wedge^3H^0(M_l)),
\end{equation}
$\Pi_l$ is the Plucker space of $G_l$. We denote the projectivization of $f$ as
\begin{equation}
\overline{\pi}_l : \Pi_l \to P_l.
\end{equation}
Notice that $f$ is preserving indecomposable vectors $v = s_1 \wedge s_2 \wedge s_3$. Indeed $v = c(t_1 \wedge t_2 \wedge t_3)$, where at least two of the $t_i$'s vanish at $p$, hence $f(v)$ is indecomposable. Therefore the image of a point

$$W \in G_l$$

by $\varpi_l$ is a point of the Grassmannian $GrassH^0(2, M_l)$. Such a point has an obvious geometrical interpretation, namely

$$(2.22) \quad \varpi_l(W) = W(-p)$$

where $W(-p)$ is the subspace of sections of $W$ vanishing at $p$. Let

$$(2.23) \quad \Lambda_l = P \wedge^3 H^0(M_l - p))$$

be the center of the projection $\varpi_l$, it is clear that

$$(2.24) \quad \Lambda_l \cap G_l = \{ W \in G_l / p \text{ is a base point of } |W| \}.$$  

**Definition 2.25** $u_l : G_l \to P_l$ is the restriction of $\varpi_l$ to $G_l$.

The next theorems 2.26 and 3.1 imply the main theorem of this section.

**Theorem 2.26** The diagram

$$\begin{array}{ccc}
X & \xrightarrow{\phi_p} & P \\
g_l \downarrow & & \downarrow p_l \\
G_l & \xrightarrow{u_l} & P_l
\end{array}$$

is commutative.

Let us point out the geometric meaning of this theorem: consider the pencil of $2\Theta$-divisors $P_\xi$ which is associated to $\xi \in X$. For $\xi$ general the restriction of $P_\xi$ to $C_l$ is a line

$P$

in $|M_l(-p)|$, that is a point of the Plucker embedding of the Grassmannian

$Grass(2, H^0(M_l(-p)))$ in $P_l$. Note that

$$P = p_l(\phi_p(\xi)).$$

On the other hand we consider the determinant map $w_l : \wedge^2 H^0(\xi(l)) \to H^0(M_l)$ and its image $W$. From $W$ we obtain the two-dimensional vector space $W(-p) \subset H^0(M_l(-p))$. From (2.22) we know that

$$PW(-p) = u_l(g_l(\xi)).$$

The previous theorem says

$$(2.27) \quad P = PW(-p).$$
Proof of theorem 2.26 Let us show the equivalent condition (2.27). As usual consider the family of exact sequences

\[
0 \to \eta_\lambda(l) \to \xi(l) \to \omega_{C^p(l)} \to 0
\]
defined in (1.9). We can assume that \( \xi \in X_l \). Then, by proposition (1.19)(2), it follows that \( h^0(\eta_\lambda(l)) = 2 \) for each \( \lambda \in \mathbb{P}^1 \). Moreover the determinant map

\[
v_\lambda : \wedge^2 H^0(\eta_\lambda) \to H^0(\omega_{C^p(2l)})
\]
is injective because the same holds for \( w_l : \wedge^2 H^0(\xi(l)) \to H^0(\omega_{C^p(2l+p)}) \).

Let \{s_\lambda, t_\lambda\} be a basis of \( H^0(\eta_\lambda) \) and let \{s'_\lambda, t'_\lambda\} be its image in \( H^0(\xi(l)) \). Then

\[
d_\lambda + p = d'_\lambda,
\]
where \( d_\lambda \) is the divisor of \( v_\lambda(s_\lambda \wedge t_\lambda) \) and \( d'_\lambda \) is the divisor of \( w_l(s'_\lambda \wedge t'_\lambda) \). It is well known that

\[
D_\lambda \cdot C_l = d_\lambda,
\]
where \( D_\lambda \) is the \( 2\Theta \)-divisor associated to \( \eta_\lambda \). This implies (2.27).

**Theorem 3.1** Let \( \xi \) be any point of \( X \). Then, for \( l \) general in \( \text{Pic}^1(C) \), \( \xi(l) \) is globally generated and \( h^0(\xi(l)) = 3 \).

From this theorem and proposition (2.5) it follows that

\[
X = \bigcup X_l, \quad l \in \text{Pic}^1(C).
\]

The proof of theorem 3.1 will be the content of section 3.

Let us remind that \( G_l \subset \Pi_l \) via Plücker embedding. Therefore the target space of \( g_l \) is \( \Pi_l \). We are now in position to show that the map

\[
g_l : X \to \Pi_l
\]
is defined by a linear system contained in \( |L_X| \), where \( L_X \) is the generalized theta divisor.

**Proposition 2.28** There exists a linear map of projective spaces

\[
\pi_l : P_X \to \Pi_l
\]
such that

\[
g_l = \pi_l \cdot \phi_p.
\]

Proof The rational map \( p_l \cdot \phi_p : X \to P_l \) is defined by \( h_0, \ldots, h_r \) independent sections of \( H^0(L_X), \quad (r = \text{dim}P_l) \). Let \( \pi_l : \Pi_l \to P_l \) be the linear projection defined above in (2.21). Since the diagram in theorem (2.26) is commutative, we have \( p_l \cdot \phi_p = g_l \cdot \pi_l \). Therefore the map \( g_l : X \to G_l \subset \Pi_l \) is defined by the independent sections

\[
ch_0, \ldots, ch_r, h_{r+1}, \ldots h_s \in H^0(L_X^\otimes m),
\]
with \( c \in H^0(L_X^{\otimes (m-1)}) \). The linear system \( |H| \) spanned by \( ch_0, \ldots, ch_r, h_{r+1}, \ldots, h_s \) has no fixed components. Indeed, we know from proposition (2.7)(2) that \( g_l \) is regular at every point \( \xi \) satisfying conditions (+1) and (+3) of definition (2.3). Moreover, as in proposition (2.6), we can construct a projective curve \( \Gamma \) which is entirely contained in the set of points satisfying (+1) and (+3). Hence \( |H| \) has no base point on \( \Gamma \). Since \( \text{Pic} X \cong \mathbb{Z} \), it follows that \( |H| \) has no fixed components.

Now we want to show that \( c \) is constant. Let \((x_0: \cdots: x_s)\) be projective coordinates on \( \Pi_l \) such that \( x_0 = ch_0, \ldots, x_r = ch_r, x_{r+1} = h_{r+1}, \ldots, x_s = h_r \) are the equations of \( g_l \). Then the center for the projection \( \pi_l \) is \( \Lambda_l = \{x_0 = \ldots x_r = 0\} \). It is clear that \( g_l(\text{div}(c)) \subset \Lambda_l \cap G_l \).

On the other hand we have already remarked in (2.24) that \( W \in \Lambda_l \cap G_l \) if and only if \( p \) is a base point for the linear system \( |W| \). Therefore every point \( \xi \in \text{div}(c) \) which is not in the base locus of \( |H| \) satisfies

\[
h^0(\xi(l-p)) \geq 2.
\]

By semicontinuity, the latter condition holds at each point of \( \text{div}(c) \). By proposition (2.6)(3), there exists a projective curve \( \Gamma \) satisfying the condition: \( h^0(\xi(l-p)) = 1 \) for each \( \xi \in \Gamma \). But then \( \text{div}(c) \cap \Gamma = \emptyset \) and \( c \) is a non zero constant.

Finally we can show

**THEOREM 2.29** Let \( C \) be any curve of genus \( g \geq 2 \). Then the generalized theta divisor of \( SU(2,1) \) is very ample.

**Proof** We must show that \( \phi \) is an embedding. Assume

\[
\phi(\xi) = \phi(\xi'),
\]

for two points \( \xi, \xi' \in X \). By theorem (3.1) and proposition (2.5) both \( \xi \) and \( \xi' \) are in the same open set \( X_l \), provided \( l \) is sufficiently general. Since, by proposition (2.12), \( g_l/X_l \) is injective it follows \( \xi \cong \xi' \). Hence \( \phi \) is injective.

Assume

\[
(d\phi)_\xi(v) = 0
\]

for a point \( \xi \in X \) and a tangent vector \( v \in T_{X,\xi} \). Note that the linear projection \( \pi_l \) is defined at \( \phi(\xi), \) provided \( l \) is general. Indeed \( \pi_l \) is defined at \( \phi(\xi) \) if no element of the pencil \( P_\xi \) contains \( C_l \). The latter condition is true for a general \( l \). This is an immediate consequence of the following property: the family of curves \( C - l \) which are contained in an element \( D \in P_\xi \) has dimension \( \leq g - 2 \), (cfr. [BV] 5.10). Therefore we have

\[
(d\pi_l)_{\phi(\xi)} \cdot (d\phi)_\xi = (dg_l)_\xi.
\]

By theorem (3.1) \( \xi \in X_l \) for \( l \) general. Then, by proposition (2.12)(1), \( (dg_l)_\xi \) is injective. Hence \( v = 0 \) and \( (d\phi)_\xi \) is injective.
The main technical result.

The purpose of this section is to show the following theorem:

\textbf{THEOREM 3.1} Let \( \xi \) be a stable rank two vector bundle on \( C \) and let \( \det \xi \in Pic^{2g-1}(C) \). Then, for a general \( l \in Pic^1(C) \), the following conditions are satisfied:
- \( h^0(\xi(l)) = 3 \),
- \( \xi(l) \) is globally generated.

It suffices to show the theorem when \( \det \xi = \omega_C(p) \), \( (p \in C) \), therefore we will assume as usual \( \xi \in X \).

The proof of the theorem is given at the end of the section. Preliminarily, we introduce some definitions and lemmas.

\textbf{DEFINITION 3.2} Let \( \xi \in X \), we fix the following notations:
- \( E_\xi = \{ e \in Pic^0(C)/h^0(\xi(e)) \geq 2 \} \).
- \( V_\xi = \{ e \in Pic^0(C)/h^0(\xi(e - y)) \geq 1, \forall y \in C \} \).
- \( H_\xi = \{ l \in Pic^1(C)/l - C \subset E_\xi \} \).

By definition \( E_\xi \) is the exceptional locus of \( \xi \).

Let \( e \in E_\xi \). Tensoring by \( O_C(e) \) the usual sequence

\begin{align*}
0 \longrightarrow \eta_\lambda \longrightarrow \xi \longrightarrow \lambda \longrightarrow C_p \longrightarrow 0 \quad (\lambda \in \mathbb{P}^1)
\end{align*}

and passing to the long exact sequence, we obtain

\[ 0 \rightarrow H^0(\eta_\lambda(e)) \rightarrow H^0(\xi(e)) \rightarrow C \rightarrow .... \]

Then it follows \( h^0(\eta_\lambda(e)) \geq 1 \) and hence \( e \in D_\lambda \), where \( D_\lambda \) is the divisor of \( \eta_\lambda \). We have shown the following

\textbf{LEMMA 3.4}

(i) \( E_\xi \) is in the base locus of the pencil \( P_\xi \).
(ii) \( \dim E_\xi \leq g - 1 \).

Our main problem will be to exclude \( \dim E_\xi = g - 1 \).

\textbf{REMARK} Let \( \mathcal{P} \rightarrow Pic^0(C) \times C \) be a Poincaré bundle and let \( \mathcal{E} = \mathcal{P} \otimes \pi_2^* \xi \), where \( \pi_i \) denotes the projection onto the \( i \)-th factor. Of course we can consider the standard exact sequence

\[ O \rightarrow \mathcal{E}(-D) \rightarrow \mathcal{E} \rightarrow \mathcal{E} \otimes O_D \rightarrow 0, \]

where \( D = Pic^0(C) \times \{ p \} \). Applying the functor \( \pi_{1*} \) to this sequence, it follows that \( E_\xi \) is the Support of \( R^1 \pi_{1*} \mathcal{E} \), in particular \( E_\xi \) has dimension \( \geq g - 2 \). We will study \( E_\xi \) in the next section.
**Lemma 3.5** \( \dim H_\xi \leq g - 2 \).

*Proof* Let \( l \in H_\xi \) and let \( y \in C \). Tensoring (3.3) by \( \mathcal{O}_C(l - y) \) and passing to the long exact sequence, we obtain

\[
0 \to H^0(\eta_\lambda(l - y)) \to H^0(\xi(l - y)) \to C \to \ldots
\]

Then \( h^0(\eta_\lambda(l - y)) \geq 1 \) and the curve \( l - C \) is contained in the divisor \( D_\lambda \) of \( \eta_\lambda \). This shows that \( H_\xi \subset H_{\eta_\lambda} = \{ l \in \text{Pic}^1(C)/l - C \subset D_\lambda \} \). On the other hand it is known that \( \dim H_{\eta_\lambda} \leq g - 2 \), ([BV] 5.10). This implies the result.

**Corollary 3.6** \( h^0(\xi(l)) = 3 \) for a general \( l \in \text{Pic}^1(C) \).

*Proof* Let \( H^1_\xi = \{ l \in \text{Pic}^1(C)/h^0(\xi(l)) \geq 4 \} \). For each \( l \in H^1_\xi \) the curve \( l - C \) is contained in \( E_\xi \). Hence \( H^1_\xi \) is contained in \( H_\xi \) and, by the previous lemma, \( \dim H^1_\xi \leq g - 2 \).

**Lemma 3.7**

(i) \( V_\xi \subset E_\xi \).

(ii) \( \dim V_\xi \leq g - 2 \).

*Proof* (i) By definition a point \( e \) in \( V_\xi \) satisfies the condition \( h^0(\xi(e - y)) \geq 1 \), \( \forall y \in C \). It is obvious that this implies \( h^0(\xi(e)) \geq 2 \).

(ii) Let \( V \subset V_\xi \) be an irreducible component of dimension \( \geq g - 1 \), consider the difference map

\[
\delta : V \times C \to \text{Pic}^{-1}(C), \quad (\delta(e, y) = e - y).
\]

Since \( \dim V \geq g - 1 \) it follows that \( \delta \) is surjective. The surjectivity of \( \delta \) implies

\[
h^0(\xi(-l)) \geq 1 \quad \forall l \in \text{Pic}^1(C).
\]

From Riemann-Roch, Serre duality and the isomorphism \( \omega_C \otimes \xi^*(l) \cong \xi(l - p) \) it follows that the previous condition is equivalent to

\[
h^0(\xi(l - p)) \geq 2 \quad \forall l \in \text{Pic}^1(C).
\]

But then \( E_\xi = \text{Pic}^1(C) \), against lemma (3.4).

In the next proposition we bound the dimension of a family of positive linear subbundles of \( \xi \).

**Proposition 3.10** Let \( Z \subset \text{Pic}^d(C) \) be a closed subset and let \( d \geq 1 \). Assume

\[
(3.11) \quad h^0(\xi \otimes L^*) \geq 1, \forall L \in Z.
\]

Then \( d \leq g - 1 \) and moreover \( \dim Z \leq g - 1 - d \).

*Proof* We will say that any closed set satisfying (3.11) is a family of subbundles of \( \xi \). By stability \( d \leq g - 1 \), let us show that \( \dim Z \leq g - d - 1 \).
There is no restriction to assume $Z$ irreducible. To obtain a contradiction we assume

$$\dim Z \geq g - d,$$

then we consider the closed set

$$\tilde{Z} = \{(L, x) \in Z \times C/h^0(\xi \otimes L^*(-x)) \geq 1\}$$

and its first projection

$$Z_n \subset Z.$$

**CLAIM** There exists a family $T \subset \text{Pic}^k(C)$ of subbundles of $\xi$ such that

$$\dim T \geq g - k \quad \text{and} \quad T - T_n \neq \emptyset.$$  

Therefore, up to replacing $Z$ by $T$, we can assume

$$Z - Z_n \neq \emptyset.$$  

To show the claim assume $Z = Z^n$ and consider the sum map

$$\phi : \tilde{Z} \to \text{Pic}^{d+1}(C), \quad (\phi(L, x) = L(x)).$$

$\phi$ defines a family

$$Z_1 = \phi(\tilde{Z})$$

of subbundles of $\xi$ of degree $d + 1$. Since $Z = Z^n$, it follows $\dim \tilde{Z} \geq \dim Z$. On the other hand each fibre of $\phi$ is at most one dimensional, hence

$$\dim Z_1 \geq \dim \tilde{Z} - 1 \geq \dim Z - 1 \geq g - (d + 1).$$

If $Z_1 - Z_1^n$ is non empty we replace $Z$ by the family $T = Z_1$. If $Z_1 - Z_1^n$ is empty we go on with the same construction. After a finite number $s$ of steps, we obtain the required family $T$ in $\text{Pic}^{d+s}(C)$. This shows the claim.

Let $e \in \text{Pic}^0(C)$ and let $Z_e$ be the translate of $Z$ by $e$. Then $Z_e$ is a family of subbundles of $\xi(e)$. By lemma 3.4 $h^0(\xi(e)) = 1$ if $e$ is general. Therefore, up to a translation, we can assume

$$h^0(\xi) = 1.$$  

Let

$$W_d$$

be the image of the Abel map $a : C(d) \to \text{Pic}^d(C)$. Up to translating $Z$ by $e$, we can also assume that $W_d$ is transversal to $Z$ and $Z_n$. As a consequence of the previous assumptions it follows that

$$W_d \cap (Z - Z_n)$$

is non empty. We want to remark that $W_d \cap (Z - Z_n)$ contains two distinct points. This is obvious if $\dim Z > g - d$. Assume $\dim Z = g - d$. Then, by transversality, we have
\[ W_d \cap Z^n = \emptyset. \]  Moreover the cardinality of \( W_d \cap Z \) equals the intersection index \( (W_d, Z) \).

It is well known that the latter cannot be one in the Jacobian of \( C \).

Now we can complete the proof: let \( A_1, A_2 \) be distinct points of \( W_d \cap (Z - Z^n) \), then \( A_i = O_C(a_i) \), where \( a_i \) is effective. We have \( h^0(\xi(-a_i)) = h^0(\xi) = 1, i = 1, 2 \). Hence a non zero section \( s \in H^0(\xi) \) vanishes on \( \text{Supp}a_1 \cup \text{Supp}a_2 \). But then \( h^0(\xi(-a_1 - x)) = 1 \) for some \( x \in \text{Supp}a_2 \) and \( A_1 \in Z^n \): a contradiction.

**REMARK** The proposition implies the following property: Let \( E \) be a stable rank two vector bundle and let \( s(E) = \deg E - 2d \), where \( d \) is the maximal degree for a linear line subbundle of \( E \). If \( s(E) = 1 \) the family \( Z \) of all subbundles of degree \( d \) is finite. Indeed, tensoring \( E \) and its maximal subbundles by a suitable \( L \in \text{Pic}(C) \), we can assume \( \text{det} E = \omega_C(p) \) and \( d = g - 1 \). The property was already proved by Lange-Narasimhan in [LN] 4.2, where they also compute the number of maximal subbundles.

Now we fix an irreducible component

\[ B \]

of the exceptional locus \( E_\xi \) and we assume that

\[ \dim B = g - 1. \]  

\[ (3.12) \]

**LEMMA 3.13** If \( B \) exists a general point \( e \in B \) satisfies the following conditions (*) and (**)。

(*) \( h^0(\xi(e)) = 2 \) and the determinant map

\[ v : \wedge^2 H^0(\xi(e)) \to H^0(\text{det}\xi(e)) \]

is injective,

(**) let \( d_e \in | \text{det}\xi(e) | \) be the divisor defined by \( \text{Im}(v) \). For each \( y \in \text{Supp}d_e \) the curve

\[ C_y = \{ e + x - y/x \in C \} \]

is not contained in \( B \).

**Proof** Let \( e \in \text{Pic}^0(C) \), it is easy to check that \( e \in V_\xi \) if and only if one of the following conditions is satisfied:

1. \( h^0(\xi(e)) \geq 3 \),
2. \( h^0(\xi(e)) = 2 \) and the determinant map \( v : J \wedge^2 H^0(\xi(e)) \to H^0(\text{det}\xi(e)) \) is zero.

Assume \( e \) is a general point of \( B \). Then \( e \) is not in \( V_\xi \) because \( \dim V_\xi \leq g - 2 \). Hence \( h^0(\xi(e)) = 2 \) and the map

\[ v : \wedge^2 H^0(\xi(e)) \to H^0(\text{det}\xi(e)) \]

is injective. This shows that \( e \) satisfies (*).

To show that \( e \) satisfies (**) consider the two natural projections

\[ (3.14) \]

\[ C \xleftarrow{\alpha} C \times C \xrightarrow{\beta} C \]
and the exact sequence
\[(3.15) \quad 0 \to \alpha^*\xi(e - y) \to \alpha^*\xi(e - y) \otimes \mathcal{O}_{C \times C}(\Delta) \to \alpha^*\xi(e - y) \otimes \mathcal{O}_\Delta(\Delta) \to 0,\]
where \(\Delta \subset C \times C\) is the diagonal and \(y \in Suppd_e\).

Applying the functor \(\beta_*\) to the previous sequence, one obtains:
\[(3.16) \quad 0 \to H^0(\xi(e - y)) \otimes \mathcal{O}_C \to R^0 \to \omega_C^* \otimes \xi(e - y) \to H^1(\xi(e - y)) \otimes \mathcal{O}_C \to R^1 \to 0,\]
where
\[R^i = R^i\beta_*(\alpha^*\xi(e - y) \otimes \mathcal{O}_{C \times C}(\Delta)).\]

Since \(y \in Suppd_e\), we have \(h^0(\xi(e - y)) \geq 1\) and hence \(h^1(\xi(e - y)) \geq 2\). Moreover, for a general \(x \in C\), it holds
\[R^0_x = H^0(\xi(e + x - y)) \quad , \quad R^1_x = H^1(\xi(e + x - y)).\]

To obtain a contradiction let's assume
\[C_y \subset B\]
for a general point \(e \in B\). Then the rank of \(R^0\) is 2 and the rank of \(R^1\) is 1. This implies that, in the previous sequence 3.16, the image of the coboundary map
\[c : \omega_C^* \otimes \xi(e - y) \to H^1(\xi(e - y)) \otimes \mathcal{O}_C\]
has rank (at most) one. On the other hand, by Serre duality, the dual map
\[c^* : H^0(\xi(e + y - p)) \otimes \mathcal{O}_C \to \xi(e + y - p).\]
is just the evaluation map. Since the sheaf \(Im(c)\) has rank one \(c^*\) is not generically surjective. Therefore the image of \(c^*\) is a line bundle
\[(3.17) \quad N \subset \xi(e + y - p)\]
such that \(h^0(N) \geq 2\). In particular it follows that
\[e + y - p \in V_\xi.\]

We want to show that this yields a contradiction. Consider the closed sets
\[\tilde{B} = \{(e, y) \in B \times C / y \in Suppd_e\}\]
and
\[T = \{e + y - p \in Pic^0(C) / (e, y) \in \tilde{B}\}.\]

From the previous remarks it is clear that \(T \subset V_\xi\). On the other hand
\[T = \phi(\tilde{B}),\]
where \( \phi : \tilde{B} \to \text{Pic}^0(C) \) is the morphism sending \((e, y)\) to \(e + y - p\). The fibre of \( \phi \) at \( f = \phi(e, y) \) is \( \phi^{-1}(f) = \{(e + y' - y, y') : y' J \in C\} \). Then, since \( \dim \tilde{B} \geq g - 1 \), it follows

\[
\dim T = g - 2.
\]

Let us see that this is impossible. Consider the closed set

\[
W = \{(N, f)/f \in T, NJ \subset \xi(f), N \in W^1_d\},
\]

where \( W^1_d \subset \text{Pic}^d(C), 2 \leq d \leq g - 1 \), is the Brill-Noether locus. From the previous remarks it follows that the projection of \( W \) onto \( T \) is surjective. Hence

\[
\dim W \geq g - 2.
\]

The difference map

\[
\delta : W \to \text{Pic}^d(C)
\]

yields a family

\[
Z = \delta(W)
\]

of linear subbundles of \( \xi \) of positive degree. For the fibre of \( \delta \) at a point \( L = \delta(N, f) \) we have

\[
\delta^{-1}(L) \subseteq \{(N', N - N' + f) : N' \in W^1_d, \} \cong W^1_d.
\]

By Martens theorem \( \dim W^1_d \leq d - 2 \), \( \dim W^1_d \leq d - 3 \) if \( C \) is not hyperelliptic). This implies \( \dim Z \geq g - d \), against proposition 3.10.

**Lemma 3.18** If \( B \) exists a general point \( e \in B \) satisfies the following condition (***):

(1) There exist two points \( y_1, y_2 \in C \) such that

\[
h^0(\xi(e - y_1)) = h^0(\xi(e - y_2)) = 1, \quad h^0(\xi(e - y_1 - y_2)) = 0.
\]

(2) If \( h^0(\xi(e - p)) = 0 \), there exist two distinct points \( \lambda_1, \lambda_2 \in \text{P} \text{Hom}(\xi, C_p) \) such that

\[
h^0(\eta_{\lambda_i}(e - y_i)) = 1 \quad i = 1, 2
\]

where \( \eta_{\lambda_i}(e) \) is defined by \( \lambda_i \) via the exact sequence

\[
0 \longrightarrow \eta_{\lambda_i}(e) \longrightarrow \xi(e) \longrightarrow \frac{\lambda_i}{\lambda_i} \longrightarrow C_p(e) \longrightarrow 0.
\]

(3) If \( h^0(\xi(e - p)) \geq 1 \), there exists \( \lambda \in \text{P} \text{Hom}(\xi, C_p) \) such that \( h^0(\eta_{\lambda}(e)) = 2 \), where \( \eta_{\lambda}(e) \) is defined by \( \lambda \) via the exact sequence

\[
0 \longrightarrow \eta_{\lambda}(e) \longrightarrow \xi(e) \longrightarrow \frac{\lambda}{\lambda} \longrightarrow C_p(e) \longrightarrow 0.
\]
In particular the determinant map $v_\lambda : \wedge^2 H^0(\eta_\lambda(e)) \to H^0(\omega_C(2e))$ is injective. Proof
By lemma 3.13 conditions (*) and (**) are satisfied on a dense open set

$$U \subset B.$$ 

For $e \in U$ the determinant map $v_e : \wedge^2 H^0(\xi(e)) \to h^0(det\xi(e))$ is injective and $h^0(\xi(e)) = 2$. Therefore $e$ defines the effective divisor

$$d_e = div(s_e) \in |\omega_C(2e + p)|,$$

where $s_e$ is a generator of $Im(v_e)$.

**CLAIM 3.19** Assume $e$ is general in $U$. Then $\text{Supp}(d_e)$ contains at least $g - 1$ distinct points $y_1, \ldots, y_{g - 1}$, these points are different from $p$.

Proof of the claim. We consider the morphism $b : U \to C(2g - 1)$ sending $e$ to $d_e$ and the Abel map $a : C(2g - 1) \to \text{Pic}^{2g - 1}(C)$. Note that

$$a \cdot b : U \to \text{Pic}^{2g - 1}(C)$$

is simply the restriction to $U$ of the morphism

$$h : \text{Pic}^0(C) \to \text{Pic}^{2g - 1}(C),$$

where $h(e) = det\xi(e) = \omega_C(2e + p)$. $h$ is finite of degree $2^{2g}$, therefore

$$dim b(U) = dim U = g - 1.$$

Let

$$d_e = k_1y_1 + \cdots + k_r y_r,$$

where $y_1, \ldots, y_r$ are distinct points and $k_i \geq 1$. Then, for $e$ general in $U$, $r \geq g - 1$; otherwise we would have $dim b(U) < g - 1$. For the same reason $r \geq g$ if $\text{Supp}\, d_e$ constantly contains a point $p$.

Let $y_1 \in \text{Supp}\, d_e$, if $h^0(\xi(e - y_1)) \geq 2$ the curve

$$C_y = \{e + x - y_1, x \in C\}$$

is contained in $E_\xi$. Since $e$ is general we can assume that $B$ is the only component of $E_\xi$ through $e$. Hence $C_y \subset B$, which is impossible because $e$ satisfies condition (**) of lemma (3.13). This contradiction implies that $h^0(\xi(e - y_1)) = 1$, $(\forall y_1 \in \text{Supp}\, d_e)$. Now assume $d_e = k_1y_1 + \cdots + k_r y_r$ with $r > g - 1$, then fix a non zero section $s_1$ vanishing at $y_1$. $s_1$ defines a saturated subbundle $\mathcal{O}_C(d_1)$ of $\xi(e)$, where $d_1$ is an effective divisor such that $y_1 \leq d_1 \leq d_e$. By stability $deg d_1 \leq g - 1$. Hence the set $Z = \text{Supp}\, d_e - \text{Supp}\, d_1$ is not empty and we can choose a non zero section $s_2$ vanishing at a point $y_2 \in Z$. By construction $\{s_1, s_2\}$ is a basis of $H^0(\xi(e))$, moreover $s_1(y_2), s_2(y_1)$ are not zero. Therefore $h^0(\xi(e - y_1 - y_2)) = 0$.

Finally we assume $r = g - 1$. Then, for general $e$, $\text{Supp}\, d_e$ consists of $g - 1$ distinct points. Fixing $s_1$ and $d_1$ as above, we have that either $d_1 = x_1 + \ldots + x_{g - 1}$ or $\text{Supp}\, d_e - \text{Supp}\, d_1 \neq \emptyset$. In the latter case we repeat the previous argument. In the former case we can construct as in section 1 an exact sequence

$$0 \to \eta(e - d_1) \to \xi(e - d_1) \to \mathcal{C}_p(e - d_1) \to 0$$

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such that $h^0(\eta(e - d_i)) \geq 1$. Then $\eta$ is semistable not stable and its $2\Theta$-divisor is $D = \Theta_\alpha + \Theta_{-\alpha}$, for some $\alpha \in \text{Pic}^0(C)$. Since $D \in P_\xi$ and no component of $D$ can move in a pencil, it follows that the base locus of $P_\xi$ has codimension two. Hence $B$ cannot exist and this case is impossible. This shows part (1) of condition (**). To show part (2) we take the two previous sections $s_1, s_2 \in H^0(\xi(e))$. Since $s_i$ vanishes at $y_i$, there exists an exact sequence

$$0 \longrightarrow \eta_{\lambda_i}(e) \longrightarrow \xi(e) \longrightarrow \mathbf{C}_p(e) \longrightarrow 0,$$

such that $h^0(\eta_{\lambda_i}(e - y_i)) \geq 1, (i = 1, 2)$. This sequence is constructed from the evaluation of $s_i$ at $p$ as in section 1, (1.20). Since $h^0(\xi(e - p)) = 0$, we have $\lambda_1 \neq \lambda_2$ and $h^0(\eta_{\lambda_i}(e - y_i)) = 1$. This follows from (1.16) and proposition 1.19(2).

To show part (3) observe that the evaluation map $e_p : H^0(\xi(e)) \rightarrow \xi(e)_p$ is one dimensional because $h^0(\xi(e - p)) = 1$. Then the image of $e_p$ defines the required $\lambda$ as in (1.18). In particular, it is obvious that the determinant $v_{\lambda}$ is injective, because the same holds for $v : \wedge^2 H^0(\xi(e)) \rightarrow H^0(\omega_C(2e + p))$.

**THEOREM 3.20** $\dim E_\xi \leq g - 2$ for each $\xi \in X$.

*Proof* Let $B \subset E_\xi$ be an irreducible component of dimension $g - 1$ and let

$$P_\xi = \{D_\lambda, \lambda \in \mathbf{P}^1\}$$

be the pencil associated to $\xi$. Since $E_\xi$ is in the base locus of $P_\xi$, we have

$$D_\lambda = M_\lambda + F \quad \forall \lambda \in \mathbf{P}^1,$$

where $\{M_\lambda, \lambda \in \mathbf{P}^1\}$ is a pencil with no fixed components and $F$ is an effective divisor containing $B$.

Let $e$ be a general point of $B$, we can assume that $e$ satisfies the conditions (*), (**), (***) which have been defined in lemmas 3.13 and 3.18. Moreover we can also assume that $e$ satisfies the following conditions:
- $e$ is not in the base locus of the pencil $P_M = \{M_\lambda, \lambda \in \mathbf{P}^1J\}$,
- $e$ is smooth for the unique element of $P_M$ passing through $e$,
- $B$ is the unique irreducible component of $F$ passing through $e$.

Assume $h^0(\xi(e - p)) = 0$. Since (***) holds, there are two distinct points $\lambda_1, \lambda_2 \in \text{P}Hom(\xi, \mathbf{C}_p)$ such that

$$h^0(\eta_{\lambda_i}(e - y_i)) = 1 \quad i = 1, 2$$

where $\eta_{\lambda_i}(e)$ is defined by $\lambda_i$ via the exact sequence

$$0 \longrightarrow \eta_{\lambda_i}(e) \longrightarrow \xi(e) \longrightarrow \mathbf{C}_p(e) \longrightarrow 0.$$

Now observe that the condition $h^0(\eta_{\lambda_i}(e - y_i)) = 1$ implies

$$C_{y_i} = e + C - y_i \subset D_{\lambda_i}, \quad (i = 1, 2).$$
Note that the curve $C_{y_i}$ is not contained in $F$. Indeed $B$ does not contain $C_{y_i}$ because $e$ satisfies (**). On the other hand $e$ is a point of $C_{y_i}$ and every irreducible component of $F$ which is different from $B$ does not contain $e$. Therefore $C_{y_i}$ is not contained in $F$ and hence

$$C_{y_i} \subset M_{\lambda_i}, \quad (i = 1, 2).$$

This implies that $e$ is in the base locus of the pencil $\{M_{\lambda}, \lambda \in \mathbb{P}^1\}$: a contradiction.

Finally, assume $h^0(\xi(e - p)) \geq 1$, then we know from condition (***) (3) that there exists an exact sequence

$$0 \longrightarrow \eta_\lambda(e) \longrightarrow \xi(e) \overset{\lambda}{\longrightarrow} C_p(e) \longrightarrow 0.$$

such that $h^0(\eta_\lambda(e)) = 2$ and the determinant map $v_\lambda$ is injective. Obviously we can choose $\mathcal{O}_C(2e) \neq \mathcal{O}_C$, so that $\eta_\lambda(e)$ satisfies all the assumptions of corollary 1.14. Let $M = M_\lambda$ be the corresponding element of the pencil $P_M$. With the same notations of Proposition (1.13), we consider the curve

$$b_e^*(M + F) \subset C \times C.$$

Recall that $b_e^*(M + F)$ contains the divisor $\Gamma = \Sigma C \times \{y_i\}$; where, in our case, $\Sigma y_i = d_e - p$. Since $F$ does not contain $C_{y_i}$, $b_e^*F$ does not contain $C \times \{y_i\}$. Hence $\Gamma$ is contained in $b_e^*M$. Then, by the corollary (1.14), $e$ is a singular point of $M$: a contradiction.

Finally we can give a

**PROOF OF THEOREM 3.1**

Proof Let $\sigma : \text{Pic}^0(C) \times C \rightarrow \text{Pic}^1(C)$ be the sum map. It is clear that

$$\sigma(E_\xi \times C) = Y,$$

where

$$Y = \{l \in \text{Pic}^1(C)/h^0(\xi(l - x)) \geq 2, \text{ for some } x \in C\}.$$

On the other hand $l \in Y$ if and only if $\xi(l)$ is not globally generated or $h^0(\xi(l)) \geq 4$. By the previous theorem 3.20 we have $\dim Y \leq \dim (E_\xi \times C) \leq g - 1$. Hence the complement of $Y$ is not empty and theorem 3.1 follows.
4 Pencils of $2\Theta$-divisors.

In this section we describe some geometry of the base locus

$$B_{\xi}$$

of a pencil $P_{\xi}$. In particular we will see that $B_{\xi}$ is reducible if $\xi$ is general, a component of it being the exceptional locus

$$E_{\xi} = \{ e \in \text{Pic}^0(C)/h^0(\xi(e)) \geq 2 \}.$$ 

$B_{\xi}$ is considered with its natural structure of scheme, it is useful to recall that $B_{\xi} = -B_{\xi}$.

**Lemma 4.3** For each $\xi$ one has

$$B_{\xi} = E_{\xi} \cup -E_{\xi}.$$ 

In particular $E_{\xi}$ is a component of $B_{\xi}$, moreover $E_{\xi} \neq -E_{\xi}$ if $\xi$ is general.

*Proof* It is clear that $E_{\xi} \cup -E_{\xi} \subset B_{\xi}$. Let $e \in B_{\xi}$ be in the complement of $E_{\xi}$, then $h^0(\xi(e)) = 1$. Since $e$ is in the base locus of $P_{\xi}$, we have $h^0(\eta(\lambda(e))) = 1$ for each $\lambda \in \mathbb{P}^1$. Hence, from proposition (1.19), it follows $h^0(\xi(e-p)) = 1$ and finally $h^1(\xi(e-p)) = 2$. But then $e \in -E_{\xi}$ because $h^0(\xi(-e)) = h^1(\xi(e-p)) = 2$. Hence $B_{\xi} = E_{\xi} \cup -E_{\xi}$.

To complete the proof, let us produce one $\xi$ such that $E_{\xi} \neq -E_{\xi}$. At first there is no problem to construct a stable, globally generated rank two vector bundle $\eta(l)$ having very ample determinant $\omega_C(2l) \in \text{Pic}^2(C)$ such that $h^0(\eta(l)) = 3$. For this just choose $\eta(l) = f_{W}^{*}T\mathbb{P}^1(-1)$, where $f_W : C \to \mathbb{P}^2$ is the morphism associated to a general 3-dimensional subspace $W \subset H^0(\omega_C(2l))$. We can also assume that $f_W(C)$ has no cusp so that $h^0(\eta(l-2p)) = 0$. Let $e = l-p$, by (1.19) $\eta(l)$ induces a family of exact sequences

$$0 \longrightarrow \xi_{\lambda}(e) \longrightarrow \eta(l) \longrightarrow \mathbb{C}_p \longrightarrow 0,$$

where $\xi_{\lambda}$ is stable and $h^0(\xi_{\lambda}(e)) = 2$, $\forall \lambda \in \mathbb{P}^1$. Passing to the corresponding long exact sequences we have $h^1(\xi_{\lambda}(e)) = 1$ iff $h^0(\lambda) : H^0(\eta(l)) \to \mathbb{C}_p$ is surjective. On the other hand we have $h^0(\xi_{\lambda}(-e)) = h^1(\xi_{\lambda}(e))$. Note that $h^0(\eta(e)) = 1$ and that $h^0(\eta(e-p)) = 0$. Therefore, applying proposition (1.20), there exists a unique $\lambda_{o}$ such that $h^0(\lambda_{o})$ is not surjective. Let $\lambda \neq \lambda_{o}$, then $h^0(\xi_{\lambda}(-e)) = 1$ and hence $E_{\xi_{\lambda}} \neq -E_{\xi_{\lambda}}$.

**Remark.** It is possible that $E_{\xi} = -E_{\xi}$ in some special situations. For instance if $C$ is hyperelliptic and $\xi \cong i^{*}\xi$, $(i$ hyperelliptic involution), it follows $-E_{\xi} = i^{*}E_{\xi} = E_{\xi}$.

As a component of $B_{\xi}$, $E_{\xi}$ has a natural structure of $(g-2)$-dimensional scheme:

**Proposition 4.4** Assume $\xi$ is general. Then $E_{\xi}$ is reduced and the intersection $E_{\xi} \cap -E_{\xi}$ is proper. Hence $B_{\xi}$ is reduced.

*Proof* Let $Z$ be the Zariski closure of $B_{\xi} - E_{\xi}$. By the previous lemma we have shown $-Z \subseteq E_{\xi}$. Hence the intersection $Z \cap -Z$ is proper. We show that $-Z$ is reduced.
and and equal to $E_\xi$. Let us consider the surface $S_l = \{l + p - x - y, \quad x + y \in C(2)\}$. In the next lemma (4.9) it is shown that the set 

$$S_l \cap Z$$

consists of exactly $2g(g - 1)$ distinct points, provided $\xi$ and $l$ are general. On the other hand, computing intersection numbers, we obtain 

$$(S_l, Z) + (S_l, -Z) = 2(S_l, Z) \leq (S_l, \omega_C) = 4g(g-1),$$

that is $(S_l, Z) \leq 2g(g - 1)$. Hence $(S_l, Z) = 2g(g-1)$ and $Z$ is reduced.

To show $-Z = E_\xi$ observe that $-Z \subseteq E_\xi$, and moreover that $(S_l, -Z) = (S_l, E_\xi)$. Then $-Z = E_\xi$, because $S_l$ is a positive cycle.

**COROLLARY 4.5** $E_\xi$ has cohomology class $[2\Theta^2]$.

We want to explain the geometrical meaning of the number

$$(4.6) \quad 2g(g - 1) = (E_\xi, S_l) = 2[\Theta^g \over (g - 2)!]$$

appearing in the proof of the proposition. With the usual notations we fix a general $W \in G_l$, where $G_l$ is the Grassmannian $\text{Grass}(3, H^0(M_l))$ and $M_l = \omega_C(2l + p)$. Let

$$(4.8) \quad f_W : C \to \mathbb{P}^2 = \mathbb{P}W^*$$

be the map defined by $W$. Since $M_l$ is very ample, we can assume that:
- $|W|$ is base-point-free,
- $f_W : C \to f_W(C)$ is birational and the singular points of $f_W(C)$ are ordinary nodes.

This follows from generic projection lemma. Of course we can also assume

$W = g_l(\xi)$

for some $\xi \in X_l$, (same notations of section 2). Computing the number $\delta$ of nodes of $f_W(C)$, we obtain $\delta = 2g(g - 1)$.

**LEMMA 4.9** Let $\xi$ be as above and let $Z$ be the Zariski closure of $B_\xi - E_\xi$. There exists a natural bijection

$$(4.10) \quad b^- : \text{Sing}f_W(C) \to S_l \cap Z,$$

where $S_l$ is the surface $S_l = \{l + p - x - y, \quad x + y \in C(2)\} \subset \text{Pic}^0(C)$.

*Proof* Let $\text{Sing}f_W(C) = \{o_1, \ldots, o_\delta\}$, $(\delta = 2g(g - 1))$. For each $o_i$ we consider the two branches

$$(x_i, y_i) = f_W^{-1}(o_i),$$

and the point $e_i = l + p - x_i - y_i \in S_l$. Let us show that

$$(4.10) \quad S_l \cap Z = \{e_1, \ldots, e_\delta\}.$$
For $l$ general we have $S_l \cap Z \cap E_\xi = \emptyset$. Therefore we can assume $h^0(\xi(e)) = 1$, for each $e \in Z \cap S_l$. Moreover we know that a point $e$ satisfying $h^0(\xi(e)) = 1$ belongs to $B_\xi$ iff $h^0(\xi(e-p)) = 1$. Therefore, for a point $e = l + p - x - y \in S_l$, it follows

$$e \in Z \iff h^0(\xi(l - x - y)) = 1.$$  

It is easy to see that the latter condition is satisfied if and only if $W(x) = W(y)$. That is if $o = W(x) = W(y)$ is a node of $f_W(C)$. Then

$$Z \cap S_l = \{e_1 \ldots e_\delta\}.$$  

This defines a bijective map $b^- : \text{Sing}f_W(C) \to (E_\xi) \cap S_l$ sending $o$ to $l + p - f^*_W o$.

In the proof of proposition (4.4) we have shown that $Z = -E_\xi$ if $\xi$ is general. Therefore propositions (4.4) and (4.9) imply the following

**PROPOSITION 4.11** Assume the pair $(\xi, l) \in X \times C$ is general. Let $W = g_l(\xi)$ and let $S_l = l + p - C(2)$. Then there is a bijection

$$b^- : \text{Sing}f_W(C) \to (E_\xi) \cap S_l$$  

sending $o \in \text{Sing}f_W(C)$ to $b^-(o) = l + p - f^*_W o$.

We want to complete the picture by a characterization of

$$E_\xi \cap S_l.$$  

We will only sketch this, leaving the proofs as an exercise.

For a general pair $(\xi, l)$ one has $h^0(\xi(l)) = 5$. We consider the ruled surface

$$R = \text{Proj}(\xi)$$  

and the map

$$g : R \to \mathbb{P}^4 = \mathbb{P}H^0(\xi(l + p))^*,$$  

which is induced by the evaluation $H^0(\xi(l + p)) \otimes \mathcal{O}_C \to \xi(l + p)$. We assume that

$$g : R \to g(R)$$  

is a birational morphism and that $\text{Sing}f_W(R)$ consists of finitely many apparent double points, (that is points $o$ such that $g^*o$ is reduced of length two). One can show that a general pair $(\xi, l)$ satisfies this assumption. Again, the formula for the number $\delta$ of apparent double points says $\delta = 2g(g - 1)$. Let $\pi : R \to C$ be the natural projection, for each $o \in \text{Sing}(g(R))$ we have a point

$$e = l + p - x - y, \text{ where } \pi_* g^* o = x + y \in C(2).$$  

It is easy to check that $h^0(\xi(e)) \geq 2$, therefore $e \in E_\xi \cap S_l$.

**PROPOSITION 4.12** Assume the pair $(\xi, l) \in X \times C$ is general. Let
Let $g : R \to \mathbf{P}^4$ be as above and let $S_l = l + p - C(2)$. Then there exists a natural bijection $b^+ : \text{Sing}_W(C) \to E_\xi \cap S_l$

sending $o \in \text{Sing}_W(C)$ to $b^+(o) = l + p - \pi_* g \ast o$.

Finally we describe how $E_\xi$ parametrizes the curves $l - C$ which are contained in a divisor $D_\lambda \in P_\xi$. To a point $e \in E_\xi$ we associate the curve $C_e = \{e + p - x, \; x \in C\}$, passing through $e$. $C_e$ is contained in at least one $D_\lambda$ of the pencil $P_\xi$. Indeed, since $h^0(\xi(-e - p)) \geq 1$, there exists at least one exact sequence

$$
0 \longrightarrow J_{\eta_\lambda} \longrightarrow \xi \longrightarrow \mathbf{C}_p \longrightarrow 0
$$

such that $h^0(\eta_\lambda(-e - p)) \geq 1$. For such a $\lambda$ we have $C_e \subseteq D_\lambda$. Let $F_\xi = \{e \in \text{Pic}^0(C)/e + p - C \subset D_\lambda, \text{for some } \lambda\}$,

$F_\xi$ is a closed set. From the previous remarks it follows

$$
E_\xi \subseteq F_\xi.
$$

Assume the intersection scheme

$$
\Delta_e = C_e \cdot B_\xi
$$

is smooth and finite. Then, since $P_\xi$ is a pencil of $2\Theta$-divisors and $(\Theta, C_e) = g$, $C_e \cap B_\xi$ consists of exactly $2g$ points. The distribution of these points is the following: $2g - 1$ points on $-E_\xi$, the point $e$ on $E_\xi$, (see the next proposition).

Let $e \in E_\xi$, assume that the determinant map $v : \wedge^2 H^0(\xi(e)) \to H^0(\omega_C(2e))$ is injective and that $h^0(\xi(e)) = 2$. As in lemma (3.13), the image of $v$ defines the divisor

$$
d_e \in |\omega_C(2e)|.
$$

One can easily check that, as a divisor on $C$,

$$
\Delta_e = d_e + p.
$$

**PROPOSITION 4.13** $E_\xi = F_\xi$ if $C$ is not hyperelliptic.

**Proof** Assume $e \in F_\xi$. It suffices to show that, for some $\lambda \in \mathbf{P}^1$, one has $h^0(\eta_\lambda(-e - p)) \geq 1$. This implies $h^0(\xi(-e - p)) \geq 1$ and hence $e \in E_\xi$. Let $Y$ be an irreducible component of $F_\xi$, consider the closed set

$$
\tilde{Y} = \{(e, \lambda) \in Y \times \mathbf{P}^1/h^0(\eta_\lambda(-e - p)) \geq 1\}
$$

and the complement $U$ of its projection in $Y$. We must show that $U$ is empty. Assume $e \in U$. Then, for some $\lambda = o$, we have:
Note that (ii) is equivalent to \( h_0(\eta_0(e + p)) = 2 \). Conditions (i) and (ii) imply that \( \eta_0(e + p) \) contains a subbundle \( L \in W^1_d \), where \( W^1_d \) is the Brill-Noether locus. Clearly, the same holds for \( \xi(e + p) \). Consider the set of pairs

\[
T = \{(e, L) \in Y \times W^1_d / h_0(\xi(e + p) \otimes L^*) \geq 1\},
\]

for the dimension of \( T \) we have \( \dim T \geq \dim U = \dim Y \). The image of \( T \) under the map \( (e, L) \rightarrow L(-e - p) \) is a family \( Z \) of line subbundles of \( \xi \) having degree \( d - 1 \). Since the fibre of this map is isomorphic to \( W^1_d \), it follows

\[
\dim Z \geq \dim Y - \dim W^1_d.
\]

On the other hand, by proposition (3.10), we have

\[
\dim Z \leq g - (d - 1) - 1 = g - d.
\]

Observe that \( F_\xi = \cup H_\lambda \), where \( H_\lambda = \{ e \in \text{Pic}^0(C) / e + p - C \subset D_\lambda \} \). It is known that \( \dim H_\lambda \geq g - 3 \), moreover the dimension of \( \{ e \in E_\xi / e + p - C \subset E_\xi \} \) is \( \leq g - 3 \), (cfr. [BV] proposition (5.10) and lemma (5.14)). Therefore it follows

\[
\dim Y \geq g - 2.
\]

Assume \( C \) is not hyperelliptic. Then, by Martens theorem, \( \dim W^1_d \leq d - 3 \). This implies \( g - d \geq \dim Z \geq \dim Y - \dim W^1_d \geq g - d + 1 \), which is a contradiction. Therefore \( U \) is empty and \( F_\xi = E_\xi \).

**Remark** \( E_\xi \) defines a divisor \( D_\xi \) which is naturally associated to \( \xi \). Let \( C = \{(e, f) \in E_\xi \times \text{Pic}^0(C) / f \in C \} \), then

\[
D_\xi = \pi_\ast C
\]

where \( \pi : E_\xi \times \text{Pic}^0(C) \rightarrow E_\xi \) is the first projection.

Let \( G = \text{Grass}(2, H^0(\mathcal{O}_J(2\Theta))) \) and let \( X_C \) be the moduli space of stable rank two vector bundles of determinant \( \omega_C(p) \), where \( p \in C \). We can consider the morphism \( f : X_C \rightarrow G \) such that \( f(\xi) = \phi_p(\xi) \) if \( \det \xi = \omega_C(p) \). It is natural to end this section with the following

**Problem** Is it true that \( f(X_C) \) is an irreducible component of the variety of pencils \( P \in G \) having reducible base locus?

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