EXPONENTIAL DECAY FOR THE LINEAR ZAKHAROV-KUZNETSOV EQUATION WITHOUT CRITICAL DOMAINS RESTRICTIONS

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Abstract. Initial-boundary value problems for the linear Zakharov-Kuznetsov equation posed on bounded rectangles are considered. Spectral properties of a stationary operator are studied in order to show that the evolution problem posed on a bounded rectangle has no critical restrictions on its size. Exponential decay of regular solutions is established.

1. Introduction

We are concerned with initial-boundary value problems (IBVPs) posed on bounded rectangles for the Zakharov-Kuznetsov (ZK) equation

\[ u_t + u_x + uu_x + u_{xxx} + u_{xyy} = 0 \]  

(1.1)

which is a two-dimensional analog of the well-known Korteweg-de Vries (KdV) equation

\[ u_t + uu_x + u_{xxx} = 0 \]  

(1.2)

with clear plasma physics applications [25].

Equations (1.1) and (1.2) are typical examples of so-called dispersive equations which jointly with the Schrödinger and other equations attract considerable attention of both pure and applied mathematicians in the past decades. The KdV equation is probably more studied in this context. The theory of the initial-value problem for (1.2) posed on the whole real line is considerably advanced today [2, 10, 11] and the references therein.

Recently, due to physics and numerics needs, publications on initial-boundary value problems for dispersive equations have been appeared
In particular, it has been discovered that the KdV equation posed on a bounded interval possesses an implicit internal dissipation. This allowed to prove the exponential decay rate of small solutions for \( (1.2) \) posed on bounded intervals without adding any artificial damping term [19]. Similar results were proved for a wide class of dispersive equations of any odd order with one space variable [8].

However, \( (1.2) \) is a satisfactory approximation for real waves phenomena while the equation is posed on the whole line \( (x \in \mathbb{R}) \); if cutting-off domains are taken into account, \( (1.2) \) is no longer expected to mirror an accurate rendition of reality. More correct equation in this case (see, for instance, [3]) should be written as

\[
\dot{u} + u_x + uu_x + u_{xxx} = 0. 
\tag{1.3}
\]

Indeed, if \( x \in \mathbb{R}, \ t > 0 \), the linear traveling term \( u_x \) in \( (1.3) \) can be easily scaled out by a simple change of variables; but it can not be safely ignored for problems posed on both finite and semi-infinite intervals without changes in the original domain.

Once bounded domains are considered as a spatial region of waves propagation, their sizes appear to be restricted by certain critical conditions. An important result regarding these conditions is the explicit description of a spectrum-related countable critical set [20]

\[
\mathcal{N} = \left\{ \frac{2\pi}{\sqrt{3}} \sqrt{k^2 + kl + l^2} ; \quad k, l \in \mathbb{N} \right\}.
\tag{1.4}
\]

While studying the controllability and stabilizability of solutions for \( (1.3) \), the set \( \mathcal{N} \) provides qualitative difficulties when the length of a spatial interval coincides with some of its elements [20]. More recent results on control and stabilizability for the KdV equation can be found in [21, 22].

Quite recently, the interest on dispersive equations became to be extended to the multi-dimensional models such as Kadomtsev-Petviashvili (KP) and ZK equations. As far as the ZK equation is concerned, the results on both IVP and IBVP can be found in [6, 7, 9, 15, 16, 17, 24]. Our work has been inspired by [23] where \( (1.1) \) has been considered on a strip bounded in the \( x \) variable. Studying this paper, we have found that the term \( u_{xyy} \) in \( (1.1) \) delivers additional dissipation which may ensure decay of solutions. For instance, the term \( u_{xyy} \) provides exponential decay of small solutions in a channel-type domain; namely, in a half-strip unbounded in \( x \) direction [14]. However, there are restrictions on a width of a channel.

It has been showed in [5] that the above restrictions are stipulated by the spectral properties of the corresponding stationary operator. More
precisely, considering linearized (1.1) posed on a rectangle
\[ \mathcal{D} = (0, L) \times (0, B) \subset \mathbb{R}^2 \]
with the simplest Dirichlet-type boundary data, one can see that stabilizability of solutions fails if \( L > 0 \) and \( B > 0 \) solve
\[
\left( \frac{2\pi}{L\sqrt{3}} \sqrt{k^2 + kl + l^2} \right)^2 + \left( \frac{\pi n}{B} \right)^2 = 1, \tag{1.5}
\]
i.e., if \( \mathcal{D} \) is of a critical size, likewise in the case of KdV posed on an interval. In other words, (1.5) is a 2D generalization of (1.4).

The following question arises naturally:

- Are there some physically reasonable mechanisms which help to avoid the critical restrictions for the ZK equation?

In the present paper we show that there are specific physically reasonable boundary conditions such that corresponding IBVP with no size restrictions on a domain possess solutions that decay exponentially; at least in a linear framework. We exploit effectively the dissipative role of the term \( u_{xyy} \) which apparently is due to the elliptic properties of a stationary operator considering as applied to \( u_x \).

The main goal of our paper is to establish the existence and uniqueness of global-in-time regular solutions of linearized (1.1) posed on bounded rectangles with a special type boundary condition on the part \( \{y = B\} \) of a spatial domain, and the exponential decay rate of these solutions independent of critical size limitations.

The paper has the following structure. Section 1 is Introduction. Section 2 contains formulation of the problem and auxiliaries. In Section 3, we prove the existence theorem and preliminary estimates. In Section 4, we provide forthcoming estimates to establish our principal stabilization result.

2. Problem and preliminaries

Let \( L, B, T \) be finite positive numbers. Define
\[ \mathcal{D} = \{(x, y) \in \mathbb{R}^2 : x \in (0, L), \ y \in (0, B)\}, \quad \mathcal{D}_T = \mathcal{D} \times (0, T). \]

We consider in \( \mathcal{D}_T \) the following IBVP:
\[
L^D u \equiv u_t + u_x + u_{xxx} + u_{xyy} = 0, \quad \text{in} \ \mathcal{D}_T; \tag{2.1}
\]
\[
u(x, 0, t) = 0, \ u(x, B, t) = u_{xy}(x, B, t), \quad x \in (0, L), \ t > 0; \tag{2.2}
\]
\[
u(0, y, t) = u(L, y, t) = u_x(L, y, t) = 0, \quad y \in (0, B), \ t > 0; \tag{2.3}
\]
\[
u(x, y, 0) = u_0(x, y), \quad (x, y) \in \mathcal{D}, \tag{2.4}
\]
where \( u_0 : \mathcal{D} \to \mathbb{R} \) is a given function.
Hereafter subscripts $u_x$, $u_{xy}$, etc. denote the partial derivatives, as well as $\partial_x$ or $\partial^2_{xy}$ when it is convenient. Operators $\nabla$ and $\Delta$ are the gradient and Laplacian acting over $\mathcal{D}$. By $(\cdot, \cdot)$ and $\| \cdot \|$ we denote the inner product and the norm in $L^2(\mathcal{D})$, and $\| \cdot \|_{H^k}$ stands for the norm in the $L^2$-based Sobolev spaces.

The following result will be useful:

**Lemma 2.1.** For arbitrary $L > 0$, $B > 0$ let $u : [0, L] \times [0, B] \to \mathbb{C}$ be a regular solution to the eigenvalue problem

$$u_x + u_{xxx} + u_{xyy} = \lambda u, \quad (x, y) \in \mathcal{D}, \quad \lambda \in \mathbb{C}; \quad (2.5)$$

$$u(x, 0) = 0, \quad u(x, B) = u_{xy}(x, B), \quad x \in (0, L); \quad (2.6)$$

$$u(0, y) = u_x(0, y) = u(L, y) = u_x(L, y) = 0, \quad y \in (0, B). \quad (2.7)$$

Then $u \equiv 0$.

**Proof.** Performing $y \mapsto B - y$ and continuing $u$ by zero to all $x \in \mathbb{R}$, the result for $\lambda = 0$ follows by Holmgren’s uniqueness theorem \[1\]. If $\lambda \neq 0$, the function

$$\hat{u}(\xi, y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \pi(x, y)e^{-i\xi x} \, dx,$$

where

$$\pi : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}, \quad \pi(x, y) = 1_{[0,L]}u(x, y)$$

solves

$$\hat{u}_{yy} + \left(1 - \xi^2 - \frac{\lambda}{i\xi}\right)\hat{u} = 0, \quad \hat{u}|_{y=0} = \hat{u}_y|_{y=0} = 0.$$ 

Therefore, $\hat{u} \equiv 0$ which gives $u \equiv 0$, as well. \[\square\]

### 3. Existence Theorem

In this section we state the existence results for problems in $\mathcal{D}_T$.

**Theorem 3.1.** Let $u_0 \in H^3(\mathcal{D})$ be a given function such that

$$u_0|_{y=B} = u_{0xy}|_{y=B}, \quad u_0|_{y=0} = u_0|_{x=0,L} = u_0|x=L = 0.$$ 

Then for all finite positive $L, B, T$ there exists a unique regular solution to $(2.1)$-$(2.4)$ such that

$$u \in L^\infty(0, T; H^3(\mathcal{D}));$$

$$u_t \in L^\infty(0, T; L^2(\mathcal{D}))$$
and for all \( t \in (0, T) \) it holds
\[
\|u\|^2(t) + \int_0^t \left[ \|u_x\|^2(s) + \|u_y\|^2(s) \right] ds \\
+ \int_0^t \left\{ \int_0^B u_x^2(0, y, s) \, dy + 2 \int_0^L (1 + x)u^2(x, B, s) \, dx \right\} ds \\
\leq (1 + L + T)\|u_0\|^2. \tag{3.1}
\]

Proof. To prove this theorem we use the classical semigroup approach and appropriate boundary estimates. First we write (2.1)-(2.4) as an abstract evolution equation
\[
\frac{d}{dt} u = Au \tag{3.2}
\]
subject to the initial condition
\[
u(0) = u_0. \tag{3.3}
\]

Consider the space
\[D(A) = \left\{ v \in H^3(D) : v(x, 0) = 0, v(x, B) = v_{xy}(x, B), \\
v(0, y) = v_x(0, y) = v(L, y) = v_x(L, y) = 0 \right\} \]
and the closed linear operator \( A : D(A) \to L^2(D) \) defined by
\[Av = -v_x - \Delta v_x. \]

Let \( v \in D(A) \). Then
\[(Av, v) = -\frac{1}{2} \int_0^B v_x^2(0, y) \, dy - \int_0^L v^2(x, B) \, dx \leq 0.\]

On the other hand, for the adjoint operator \( A^* \) defined as
\[A^*w = \Delta w_x + w_x \]
with the domain
\[D(A^*) = \left\{ w \in H^3(D) : w(x, 0) = 0, w(x, B) = -w_{xy}(x, B), \\
w(0, y) = w_x(0, y) = w(L, y) = w_x(L, y) = 0 \right\} \]
it holds
\[(w, A^*w) = -\frac{1}{2} \int_0^B w_x^2(L, y) \, dy - \int_0^L w^2(x, B) \, dx \leq 0 \]
which means that both \( A \) and \( A^* \) are dissipative. By semigroup theory (see, for instance, [18]), the operator \( A \) generates a strongly continuous
semigroup of contractions \( \{ S(t) \}_{t \geq 0} \) on \( L^2(\mathcal{D}) \). Then for all \( u_0 \in D(A) \) there exists a unique solution \( u(t) = S(t)u_0 \) for (3.2), (3.3) satisfying

\[
u \in C([0, T]; D(A)) \cap C^1((0, T); L^2(\mathcal{D}))\]

and

\[
\|u\|_{C([0, T]; D(A))} \leq \|u_0\|_{D(A)}.
\]

Due to the structure of (2.1), one concludes that the second inclusion above is in fact closed, i.e. \( u \in C^1([0, T]; L^2(\mathcal{D})) \). To see that \( u(x, y, t) \) satisfies (3.1), we need the following estimates.

3.1. **Estimate I.** Multiply (2.1) by \( u \) and integrate over \( \mathcal{D} \) to obtain

\[
(L^D u, u) \equiv \frac{1}{2} \frac{d}{dt}\|u\|^2(t) + \frac{1}{2} \int_0^B u_x^2(0, y, t) \, dy + \int_0^L u^2(x, B, t) \, dx = 0, \quad t \in (0, T).
\]

Integrating in \( t \in (0, T) \) gives

\[
\|u\|(t) \leq \|u_0\|, \quad t \in [0, T]
\]  

(3.4)

and

\[
\Phi(T, L, B) \equiv \frac{1}{2} \int_0^T \int_0^B u_x^2(0, y, t) \, dy \, dt + \int_0^T \int_0^L u^2(x, B, t) \, dx \, dt \leq \frac{1}{2}\|u_0\|^2.
\]  

(3.5)

3.2. **Estimate II.** Multiplying (2.1) by \((1 + x)u\), we get

\[
(L^D u, (1 + x)u) \equiv \frac{1}{2} \frac{d}{dt} \int_0^L \int_0^B (1 + x)u^2 \, dx \, dy + \int_0^L \int_0^B \left[ \frac{3}{2} u_x^2 + \frac{1}{2} u_y^2 - \frac{1}{2} u^2 \right] \, dx \, dy + \frac{1}{2} \int_0^B u_x^2(0, y, t) \, dy + \int_0^L (1 + x)u^2(x, B, t) \, dx = 0.
\]  

(3.6)
Integrating in \( t \in (0, T) \), (3.6) becomes
\[
\|\sqrt{1 + x u}\|_2(t) + \int_0^t \left[ 3\|u_x\|^2(s) + \|u_y\|^2(s) \right] ds \\
+ \int_0^t \left\{ \int_0^B u_x^2(0, y, s) dy + 2 \int_0^L (1 + x)u^2(x, B, s) dx \right\} ds \\
= \|\sqrt{1 + x u_0}\|_2^2 + \int_0^t \|u\|^2(s) ds \\
(3.7)
\]
which yields (3.1).

\( \square \)

4. Exponential decay

To prove the exponential decay of the \( L^2 \) norm of solutions, we need the observability inequality and the uniform estimate for solution in \( L^2(0, T; L^2) \).

4.1. Estimate III. We multiply (2.1) by \((T - t)u\) and integrate over \( \mathcal{D}_T \) to obtain
\[
\int_0^T (L^T u, (T - t)u) dt \equiv \frac{1}{2} \int_0^T \|u\|^2(t) dt - \frac{T}{2} \|u_0\|^2 \\
+ \frac{1}{2} \int_0^T (T - t) \int_0^B u_x^2(0, y, t) dy dt \\
+ \int_0^T (T - t) \int_0^L u^2(x, B, t) dx dt = 0.
\]
Hence,
\[
\|u_0\|^2 \leq \frac{1}{T} \int_0^T \|u\|^2(t) dt + 2\Phi(T, L, B) \\
(4.1)
\]
with \( \Phi(T, L, B) \) defined in (3.5).

4.2. Estimate IV. In conditions of Theorem 3.1 it holds
\[
\frac{1}{T} \int_0^T \|u\|^2(t) dt \leq M\Phi(T, L, B), \\
(4.2)
\]
where a constant \( M \) does not depend on \( T > 0 \).

Proof. Indeed, if (4.2) is false, then there exists a sequence \( u_n(x, y, t) \) of solutions to (2.1)-(2.4) such that
\[
\int_0^T \|u_n\|^2(t) \geq n\Phi_n(T, L, B) \\
(4.3)
\]
with
\[ \Phi_n(T, L, B) \equiv \int_0^T \left\{ \frac{1}{2} \int_0^B u_{xx}^2(0, y, t) \, dy + \int_0^L u_n^2(x, B, t) \, dx \right\} \, dt. \]

Since \( \int_0^T \|u_n\|^2 \, dt \leq T\|u_0\|^2 \), then \( \Phi_n(T, L, B) \to 0 \) as \( n \to \infty \). By the
properties of solutions there exists a subsequence \( \Phi_{n_k} \to \Phi \). Therefore, \( \Phi(T, L, B) \equiv 0 \) which implies
\[ u_x(0, y, t) = u(x, B, t) = 0. \]

Due to Lemma 2.1 this means \( u \equiv 0 \) in \( \mathcal{D}_T \). This contradicts to \( u_0 \)
being arbitrary, and (4.2) is thereby true. \( \square \)

We now prove the main result of this work.

**Theorem 4.1.** Let all the conditions of Theorem 3.1 hold. Then
\[ \|u\|(t) \leq K\|u_0\| \exp\{-\gamma t\} \quad \text{for all } t > 0. \]

**Proof.** Combing (4.1) and (4.2) one has
\[ \|u_0\|^2 \leq C\Phi \text{ with } C = M + 2. \]

Therefore,
\[ (1+C)\|u\|^2(t) = (1+C) \left[ \|u_0\|^2 - 2\Phi \right] \leq C\|u_0\|^2 - (2+C)\Phi \leq C\|u_0\|^2. \]

Thus
\[ \|u\|(t) \leq K\|u_0\| \exp\{-\gamma t\} \]

with
\[ K = \frac{1+C}{C} \text{ and } \gamma = -\ln \frac{C}{1+C}. \]

\( \square \)

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