Solutions of the wave equation for the spinless particle in the static de Sitter metric are examined. Two pairs of linearly independent ones are singled out: moving and standing waves; those being connected through a Kummer transformation. A commonly accepted algorithm for calculating a reflection coefficient $R_{\epsilon j}$ on the background of the de Sitter space-time model is analyzed. It is shown that the algorithm employed here determines $R_{\epsilon j}$ in a way coupled with an accompanying limitation on quantum number $\epsilon \rho/\hbar c \gg j$, where $\rho$ is a curvature radius. After having considered that limitation, the $R_{\epsilon j}$ identically vanish. It is argued that the calculation of the reflection coefficient $R_{\epsilon j}$ is not required at all because any barrier in an effective potential curve does not exist here; the latter is valid irrespective of the particle’s spin and mass. The result is the same for electromagnetic and Dirac fields.

This paper is an updated and more comprehensive version of the old paper

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1 Introduction

Examining fundamental particle fields on the background of expanding universe, in particular de Sitter and anti de Sitter models, has a long history; special value of these geometries consists in their simplicity and high symmetry groups underlying them which makes us to believe in existence of exact analytical treatment for some fundamental problems of classical and quantum field theory in these curved spaces. In particular, there exist special representations for fundamental wave equations, Dirac’s and Maxwell’s, which are explicitly invariant under symmetry groups $SO(4,1)$ and $SO(3,2)$ for these models. In the most of the literature, when dealing with a spin 1 field in de Sitter models they use group theory approach. Many of the most important references are given below (it is not exhaustive bibliography, which should be enormous): Dirac [1, 2], Schrödinger [3, 4], Lubanski–Rosenfeld [5], Goto [6], Ikeda [7], Nachtmann [8], Chernikov–Tagirov [9], Gehenniau–Schombld [10], Borner–Durr [11], Tugov [12], Fushchych–Krivsky [13], Chevalier [15], Castagnino [16, 19], Vidal [17], Adler [18], Schnirman–Oliveira [20], Tagirov [21], Rior- dan [22], Pestov–Chernikov–Shavoxina [23], Candelas–Raine [24], Schombld–Spindel [25, 26], Dowker–Crichley [27], Avis-Isham–Storey [28], Brugarino [29], Fang-Frnsdal [30], Angelopoulos et al. [31], Burges [32], Deser–Nepomechie [33], Dullemond–Beveren [34], Gazeau [35], Allen [36], Fefferman–Graham [37], Flato–Frnsdal–Gazeau [38], Allen–Jacobsen [39], Allen–Folacci [40], Sanchez [41], Pathinayake–Vilenkin–Allen [42], Gazeau–Hans [43], Bros–Gazeau–Moschella [44], Takook [45], Pol’shin [46, 47, 48], Gazeau–Takook [49], Takook [50], Deser–Waldron [51, 52], Spradlin–Strominger–Volovich [53], Cai–Myung–Zhang [54], Garidi–Huguet–Renaud [55], Pol’shin [56], Behroozi et al [57], Huguet–Queva–Renaud [58], Garidi et al [59], Huguet–Queva–Renaud [60], Delghani et al [61], Moradi–Rohani–Takook [62], Faci et al [63].

In a number of works, they have examined exact solutions of the wave equations on the background of the de Sitter space and in particular the problem of passing quantum-mechanical particles of various sorts (with different spin values and masses) through de Sitter horizon: Lohiya–Panchapakesan [69, 70], Khanal–Panchapakesan [71, 72], Khanal [73, 74], Hawking–Page Otschik [75], Motolla [76], Bogus–Otschik–Red’kov [77], Mishima–Nakayama [78, 79], Polarski [79], Suzuki–Takasugi [80], Suzuki–Takasugi–Umetsu [81, 82, 83].

In that studying, among other things, they had used a special definition for the so-called reflection and transmission coefficients; then they had calculated corresponding quantities which in turn were used in considering the Hawking’s formula [64, 65, 66, 67] as applied to the de Sitter space-time model

\[
\frac{\Gamma_{\varepsilon j}}{\exp\left(\frac{2\pi E\rho}{\hbar c}\right) - 1} = \Gamma_{\varepsilon j} + R_{\varepsilon j} = 1,
\]

where $R_{\varepsilon j}$ and $\Gamma_{\varepsilon j}$ are reflection and transmission coefficients, respectively. Furthermore, on these lines, some rather complicated analytical expressions for those coefficients depending on the particle’s spin, mass, and quantum numbers ($\varepsilon, j$) have been found. In contrast to most of considerations, the work by Otschik [75] has predicted (the case $S = 1/2, m \neq 0$ was meant) that $R_{\varepsilon j} = 0, \Gamma_{\varepsilon j} = 1$. It should be noted that the outcome of the work [75], $R_{\varepsilon j} = 0$, has appeared accidental one in the sense that we have obtained the zero-value for a quantity that needs to be calculated.

To date, any detailed analysis of possible explanation for that discrepancy has not been given yet. The present paper aims to investigate just those aspects of the problem. In the author’s opinion, the problem to be solved is to show up the main points in conventional algorithm for finding those $R_{\varepsilon j}$ and $\Gamma_{\varepsilon j}$, as applied to the de Sitter space-time, which provides subsequently the source of trouble. Such revealing study seems important because this general algorithm for calculating coefficients $R$ and $\Gamma$ was adopted (rather formally and without due caution) from
other and more complicated (and really barrier-contained) situations where any exact solutions are not known): Staroninskiy – Churilov [84, 85], Teukolsky – Press [86, 87, 88], Bardeen – Press [89], Bardeen – Carter – Hawking [90], Unruh [91], Fabbri [92], Wald [93], Boulware [94], —age [95] + [96] + [97], Chandrasekhar – Detweiler [98], Matzner – Michael [99], Guven [100], Bekenstein – Meisels [101], Martellini – Treves [102], Jyer – Kumar [103], Hawking – Page [104], Chandrasekhar [105].

Let us outline the content. In Sec. 2, we briefly reexamine early known solutions of scalar field in the de Sitter static coordinates’ background. In so doing, certain their properties, the most material to the following, are itemized. In particular, four different solutions are singled out for special treatment (see Sec. 3)

\[ \Psi_{\text{stand.}}^{\text{reg.}}(x) = e^{-i \epsilon t} f(r) Y_{jm}(\theta, \phi), \]
\[ \Psi_{\text{stand.}}^{\text{sing.}}(x) = e^{-i \epsilon t} g(r) Y_{jm}(\theta, \phi), \]
\[ \Psi_{\text{run.}}^{\text{out.}}(x) = e^{-i \epsilon t} U_{\text{run.}}^{\text{out.}}(r) Y_{jm}(\theta, \phi), \]
\[ \Psi_{\text{run.}}^{\text{in.}}(x) = e^{-i \epsilon t} U_{\text{run.}}^{\text{in.}}(r) Y_{jm}(\theta, \phi); \]

the notation used will be spelled out below. Here, \( f \) and \( g \) are real-valued and linearly independent solutions of the matter radial equation (see (2.3)); respectively regular and singular ones at \( r = 0 \). The functions \( U_{\text{run.}}^{\text{out.}} \) and \( U_{\text{run.}}^{\text{in.}} \) again are linearly independent ones, complex-valued, and conjugated to each other. As a matter of fact, these pairs \( f(r), g(r) \) and \( U_{\text{run.}}^{\text{out.}}(r), U_{\text{run.}}^{\text{in.}}(r) \) can be related to each other by means of a linear transformation. So, there are some grounds for thinking that the above solutions \( \Psi_{\text{stand.}}^{\text{reg.}}(x) \) and \( \Psi_{\text{stand.}}^{\text{sing.}}(x) \) represent standing waves, as well as \( \Psi_{\text{run.}}^{\text{out.}}(x) \) and \( \Psi_{\text{run.}}^{\text{in.}}(x) \) describe running waves. Besides, as noted in Sec. 4, asymptotic behavior of all these solutions agrees with the used terminology. In addition, we will demonstrate that as the de Sitter curvature radius tends to infinity (\( \rho \rightarrow \infty \)), these de Sitter space-time model’s solutions are reduced to the well known standing and running waves in the flat space-time background.

In addition, as shown in Sec. 5, a radial component of the scalar particle’s conserved current \( J_r(x) \) vanishes for the solutions \( \Psi_{\text{stand.}}^{\text{reg.}}(x) \) and \( \Psi_{\text{stand.}}^{\text{sing.}}(x) \), whereas to the \( \Psi_{\text{run.}}^{\text{out.}}(x) \) and \( \Psi_{\text{run.}}^{\text{in.}}(x) \) there correspond non-zero current waves; moreover, the relation \( (J_r)^{\text{in.}} = - (J_r)^{\text{out.}} \) holds.

Finally, it should be added that a form itself of the effective potential curve, with no barrier, for the scalar particle’s radial equation (this matter is treated in more detail in Sec. 6) might be able to refute absolutely the formulation itself about the need to study the process of passing quantum-mechanical particles through de Sitter horizon.

So, the situation seems like as if the whole of the flat space-time model’s case is carried, only with slight alterations, over de Sitter’s model. Correspondingly, quite justified suspicions that some of the early established results on the process of particles’ penetration through the de Sitter horizon are partly wrong might arise. However, it seems difficult to discern at once which part of the early performed study gives rise to subsequently misleading conclusions. In the same time. the mere claim that those results cannot be correct could be hardly enough.

In sec. 7 we discuss how approximations in the form of wave functions can influence physical results. In sec. 8–10 we extend results to the case of electromagnetic field in de Sitter space. Finally, in sec. 11–12 we briefly consider the problem for spin 1/2 field in de Sitter space.

The main goal of the present paper is to clarify the whole situation with the particle penetration through de Sitter horizon. The paper is partly a pedagogical review, so we intend to describe in detail the mathematics used to treat particles in presence of curved space-time background. Our treatment contains much technical details that may be of small interest for skillful readers but helpful for beginners.
2 Solutions of a radial equation

Conformally invariant wave equation for a massive scalar field $\Phi(x)$ in de Sitter space has a form ($M = mcρ/\hbar$, $\rho$ is the curvature radius):

$$\left( \frac{1}{\sqrt{-g}} \partial_\alpha \sqrt{-g} g^{\alpha\beta} \partial_\beta + 2 + M^2 \right) \Psi(x) = 0.$$  (2)

We shall analyze solutions of that equation in the static coordinates

$$dS^2 = \Phi dt^2 - \frac{dr^2}{\Phi} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) , \quad 0 \leq r \leq 1 , \quad \Phi = 1 - r^2 .$$  (3)

Taking $\Psi(x)$ in the form of a spherical wave

$$\Psi(x) = e^{-i\tau} f(r) Y_{jm}(\theta, \phi) , \quad \tau = E\rho/\hbar$$

for $f(r)$ we get

$$\frac{d^2 f}{dr^2} + \left( \frac{2}{r} + \frac{\Phi'}{\Phi} \right) \frac{df}{dr} + \left( \frac{\epsilon^2}{\Phi^2} - \frac{M^2 + 2}{\Phi} - \frac{j(j+1)}{fr^2} \right) f = 0 .$$  (4)

All solutions of that equation (4) can be expressed in terms of hypergeometric functions. To this end, after introducing a new variable

$$r^2 = z ,$$

the equation reads as

$$\left[ 4(1-z) \frac{d^2}{dz^2} + (6-10z) \frac{d}{dz} + \frac{\epsilon^2}{1-z} - M^2 - 2 - \frac{j(j+1)}{z} \right] f = 0 .$$  (5)

and further taking the substitution

$$f(z) = z^\kappa (1-z)^\sigma F(z) ,$$

where parameters ($\kappa$, $\sigma$) are to be

$$\kappa = j/2 , \quad -(j+1)/2 , \quad \sigma = \pm i\epsilon/2 ,$$

for $S(z)$ we get

$$z(1-z)F'' + [c - (a + b + 1)z] F' - ab F = 0$$

where ($a$, $b$, $c$) satisfy

$$c = 2\kappa + 3/2 , \quad a + b = 2\kappa + 2\sigma + 3/2 ,$$

$$ab = \kappa^2 + 2\kappa\sigma + \sigma^2 + \frac{3}{2}\kappa + \frac{3}{2}\sigma + \frac{M^2 + 4}{2} .$$

Taking $\kappa = j/2$ and $\sigma = -i\epsilon/2$, for ($a$, $b$, $c$) we get a regular (at $r = 0$) solution

$$\kappa = j/2 , \quad \sigma = -i\epsilon/2 , \quad c = j + 3/2 , \quad f(z) = z^{j/2} (1-z)^{-i\epsilon/2} F(a, b, c; z) ,$$

$$a = \frac{3/2 + j + i\sqrt{M^2 - 1/4 - i\epsilon}}{2} , \quad b = \frac{3/2 + j - i\sqrt{M^2 - 1/4 - i\epsilon}}{2} .$$  (6)

In turn, choosing $\kappa = -(j+1)/2$ and $\sigma = -i\epsilon/2$, we will obtain a singular (at $r = 0$) solution

$$\kappa = -(j+1)/2 , \quad \sigma = -i\epsilon/2 , \quad c = -j + 1/2 ,$$

$$\alpha = \frac{1/2 - j + i\sqrt{M^2 - 1/4 - i\epsilon}}{2} ,$$

$$\beta = \frac{1/2 - j - i\sqrt{M^2 - 1/4 - i\epsilon}}{2} ,$$

$$g(z) = z^{-(j+1)/2} (1-z)^{-i\epsilon/2} F(\alpha, \beta, \gamma; z) .$$  (7)
3 Standing and running waves

It is easily verified that the regular solution above, first, represents a real-valued function; second, it can be resolved into two complex-valued (and conjugate to one another) solutions of the same equation \((4)\). Applying here the terminology used in the ordinary case of spherical waves in the flat space-time model, one might say that the regular standing wave is certain superposition of two running waves; one of the latter travels to the de Sitter horizon (out-wave), another runs backwards it (in-wave). The same is valid for the singular standing wave: the \(g(r)\) is a real-valued alike, and it can be resolved into a linear combination of the same complex-valued solutions (certainly with other coefficients).

Now, let us examine this matter in full detail. First we shall consider the regular solution. To realize a required decomposition, it is sufficient to use one of the so-called Kummer’s relationships

\[
U_1 = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} U_2 + \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} U_6 ,
\]

where \(U_1, U_2, U_6\) represents three different solution of the same hypergeometric equation; those three are determined by

\[
U_1 = F(a, b, c; z) , \ U_2 = F(a, b, a + b - c + 1; 1 - z) , \ U_6 = F(c - a, c - b, c - a - b + 1; 1 - z) .
\]

Applying that relation \((8)\) to the solution \(f(z)\) as \(U_1\), we shall get

\[
f(z) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} U_{\text{run.}}^{\text{out.}}(z) + \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} U_{\text{run.}}^{\text{in.}}(z) ,
\]

where

\[
U_{\text{run.}}^{\text{out.}}(z) = z^{j/2} (1 - z)^{-i\epsilon/2} F(a, b, a + b - c + 1; 1 - z) , \ U_{\text{run.}}^{\text{in.}}(z) = z^{j/2} (1 - z)^{+i\epsilon/2} F(c - a, c - b, c - a - b + 1; 1 - z) .
\]

Noting that

\[
a^* = (a - a) , \ b^* = (c - b) , \ (a + b - c)^* = -(a + b - c)
\]

one can conclude that those two functions \(U_{\text{run.}}^{\text{out.}}(z)\) and \(U_{\text{run.}}^{\text{in.}}(z)\) are referred to each other by the operation of complex conjugation

\[
(U_{\text{run.}}^{\text{out.}}(z))^* = U_{\text{run.}}^{\text{in.}}(z) .
\]

Besides, the function \(f(z)\) being a sum of two complex-conjugate expressions (see \((10)\) and \((12)\)) is a real-valued one. Hence, one can get

\[
f(z) = 2 \Re \left[ \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} U_{\text{out.}}(z) \right] = 2 \Re \left[ \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} U_{\text{in.}}(z) \right] .
\]

To obtain an analogous expansion for the singular function \(g(z)\) \((2.6)\), one ought to apply the same Kummer’s formula \((8)\), but now having replaced \((a, b, c)\) by \((\alpha, \beta, \gamma)\):

\[
U_1 = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} U_2 + \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)} U_6 ,
\]

(13)
that is appropriate for asymptotic analysis:

\[ U_1 = F(\alpha, \beta, \gamma; z), \quad U_2 = F(\alpha, \beta, \alpha + \beta - \gamma + 1; 1 - z), \]
\[ U_6 = F(\gamma - \alpha, \gamma - \beta, \gamma - \alpha - \beta + 1; 1 - z). \]  

Thus we arrive at

\[ g(z) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} U_{\text{run}}^{\text{out}}(z) + \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)} U_{\text{run}}^{\text{in}}(z), \]  

where

\[ U_{\text{run}}^{\text{out}}(z) = z^{j/2} (1 - z)^{-\epsilon/2} F(a + 1 - \gamma, \beta + 1 - \gamma, \alpha + \beta + 1 - \gamma; 1 - z), \]
\[ U_{\text{run}}^{\text{in}}(z) = z^{j/2} (1 - z)^{+\epsilon/2} F(1 - \alpha, 1 - \beta, \gamma + 1 - \alpha - \beta; 1 - z) \]  

and the following relations

\[ \alpha + 1 - \gamma = a, \quad \beta + 1 - \gamma = b, \quad \alpha + \beta - \gamma + 1 = a + b + 1 - c, \]
\[ (1 - \alpha) = (c - a), \quad (1 - \beta) = (c - b) \]

are taken into account. Again, the function \( g(z) \) is real-valued, and one can get

\[ g(z) = 2\text{Re} \left[ \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} U_{\text{run}}^{\text{out}}(z) \right] = 2\text{Re} \left[ \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)} U_{\text{run}}^{\text{in}}(z) \right]. \]  

4 Asymptotic behavior and flat space-time limit

Now let us consider how the above four solutions behave at two natural asymptotics: at the origin \( r \sim 0 \) and at the horizon’s region \( r \sim 1 \). First, let us single \( U_{\text{run}}^{\text{out}} \) out for that treatment:

\[ U_{\text{run}}^{\text{out}}(z) = z^{j/2} (1 - z)^{-\epsilon/2} F(a, b, a + b - c + 1; 1 - z). \]  

Again, applying the formula (18) to \( U_{\text{run}}^{\text{out}}(z) \) as \( U_1 \), we arrive at the following expansion

\[ U_{\text{run}}^{\text{out}}(z) = \frac{\Gamma(a + b + 1 - c)\Gamma(1 - c)}{\Gamma(b - c + 1)\Gamma(a - c + 1)} z^{j/2} (1 - z)^{-\epsilon/2} F(a, b, c; z) \]
\[ + \frac{\Gamma(a + b + 1 - c)\Gamma(c - 1)}{\Gamma(a)\Gamma(b)} z^{-(j+1)/2} (1 - z)^{-\epsilon/2} F(b - c + 1, a - c + 1, -c + 2; z). \]  

that is appropriate for asymptotic analysis:

\[ U_{\text{run}}^{\text{out}}(r \sim 0) \sim \frac{1}{r^{j+1}}, \quad U_{\text{run}}^{\text{out}}(r \sim 1) \sim (1 - r^2)^{-\epsilon/2}. \]  

Introducing a new radial variable \( r^* \):

\[ r^* = \frac{\rho}{2} \ln \frac{1 + r}{1 - r}, \quad r = \frac{\exp(2r^*/\rho) - 1}{\exp(2r^*/\rho) + 1}, \quad r \in [0, \infty) \]

one can easily reexpress the second asymptotic relation in (20) in the form (\( \epsilon = E\rho/\hbar c \))

\[ U_{\text{run}}^{\text{out}}(r^* \sim \infty) \sim \left(2^{-iE\rho/\hbar c}\right) \exp(\pm iEr^*/\hbar c). \]  

6
Analogously, one can find asymptotics for the in-wave:

\[ U_{\text{run.}}^{\text{in.}}(r \sim 0) \sim \frac{1}{r^{j+1}}, \quad U_{\text{run.}}^{\text{in.}}(r \sim 1) \sim (1 - r^2)^{+i/2} \] (22)

or

\[ U_{\text{run.}}^{\text{in.}}(r^* \sim \infty) \sim \left(2^{+iE\rho/\hbar c}\right) \exp((-iEr^*/\hbar c)) . \] (23)

Proceeding with that analysis for the standing waves, we can easily get

\[ f(r \sim 0) \sim r^j, \quad g(r \sim 0) \sim \frac{1}{r^{j+1}}, \] (24)

\[ f(r \sim 1) \sim \left[ \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} (1-r^2)^{-i/2} + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-r^2)^{+i/2} \right] \]

\[ = 2 \text{Re} \left[ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} 2^{+iE\rho/\hbar c} \exp(-iEr^*/\hbar c) \right], \quad (25) \]

\[ g(r \sim 1) \sim \left[ \frac{\Gamma(\gamma)\Gamma(\gamma-a-\beta)}{\Gamma(\gamma-a)\Gamma(\gamma-\beta)} (1-r^2)^{-i/2} + \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} (1-r^2)^{+i/2} \right] \]

\[ = 2 \text{Re} \left[ \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} 2^{+iE\rho/\hbar c} \exp(-iEr^*/\hbar c) \right]; \quad (26) \]

Now, we take up examination of a flat space-time limit as applied to the de Sitter’s solutions. For definiteness we single out the above out-wave. It will convenient to start with its representation according to (19). At this, one essential step should have been done in advance; namely, one ought to have rewritten the above-used expressions for parameters \((a, b, c)\) with a curvature radius \(\rho\) apparently separated and set aside (a subsequent limiting procedure will be realized just through that parameter: \(\rho \rightarrow \infty\) )

\[ a = \frac{p + 1 - i\epsilon \rho + i\sqrt{\rho^2M^2 - 1/4}}{2}, \]

\[ b = \frac{p + 1 - i\epsilon \rho - i\sqrt{\rho^2M^2 - 1/4}}{2}, \]

\[ (a - c + 1) = \frac{-p + 1 - i\epsilon \rho + i\sqrt{\rho^2M^2 - 1/4}}{2}, \]

\[ (b - c + 1) = \frac{-p + 1 - i\epsilon \rho - i\sqrt{\rho^2M^2 - 1/4}}{2}, \]

\[ p = j + 1/2; \quad (a + b + c) = (1 - i\epsilon \rho), \quad (27) \]

and \(\lim_{\rho \to \infty}(\rho^2z) = R^2, \) where \(R\) is the usual radial coordinate of the flat space-time.

Using a conventional series representation for a hypergeometric function

\[ F(a, b; c; z) = \left[ 1 + \frac{ab}{c} \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^2}{2!} + \ldots \right], \quad (|z| < 1) \] (28)

and further taking into account two limiting relations

\[ \lim_{\rho \to \infty} \left[ \frac{(a+n)(b+n)}{\rho^2} \frac{z^2\rho^2}{2!} \right] = -(\epsilon^2 - M^2) \frac{r^2}{4}, \]

\[ \lim_{\rho \to \infty} \left[ \frac{(a-c+1)(b-c+1)}{\rho^2} \frac{z^2\rho^2}{2!} \right] = -(\epsilon^2 - M^2) \frac{r^2}{4}. \]
one comes to \( (\epsilon^2 - M^2) \equiv k^2 \):

\[
\lim_{p \to \infty} F(a, b, c; z) = \Gamma(1 + p) \sum_{n=0}^{\infty} \frac{(-k^2 r^2/4)^n}{n! (1 + n + p)}; \quad (29)
\]

\[
\lim_{p \to \infty} F(b - c + 1, a - c + 1, -c + 2; z) = \Gamma(1 - p) \sum_{n=0}^{\infty} \frac{(-k^2 r^2/4)^n}{n! (1 + n - p)}. \quad (30)
\]

Now, with the use of a known expansion for the Bessel functions

\[
J_p(x) = \left(\frac{x}{2}\right)^p \sum_{n=0}^{\infty} \frac{(ix/2)^{2n}}{n! (1 + n + p)}
\]

from (29)–(30) one gets

\[
\lim_{p \to \infty} \left[ \rho^j r^j (1 - r^2)^{-i/2} F(a, b, c; z) \right] = \left(\frac{2}{k}\right)^p \Gamma(1 + p) \frac{J_p(kr)}{\sqrt{r}},
\]

\[
\lim_{p \to \infty} \left[ \rho^{-j-1} r^{-j-1} (1 - r^2)^{-i/2} \left(\frac{2}{k}\right)^{-p} \Gamma(1 - p) \frac{J_{-p}(kr)}{\sqrt{r}} \right]. \quad (31)
\]

Further, taking in mind (27) and (31)–(31), for \( U_{\text{out.}}^\text{AF} \), at the limit \( \rho \to \infty \), we obtain (A is yet a non-fixed constant)

\[
\lim_{\rho \to \infty} A U_{\text{out.}}^\text{AF}(z) = \lim_{\rho \to \infty} A \times
\]

\[
\frac{\Gamma(-i\epsilon + 1) \Gamma(-p) \Gamma(1 + p) (2/k)^p J_p(kr) \rho^{-p+1/2}}{\left[ \frac{1}{\Gamma[1/2(\sqrt{\rho^2 M^2 - 1/4} - i\epsilon + p + 1)]} \Gamma[1/2(\sqrt{\rho^2 M^2 - 1/4} + i\epsilon + p + 1)] \right]} \frac{1}{\sqrt{r}} + \frac{\Gamma(-i\epsilon + 1) \Gamma(p) \Gamma(1 - p) (2/k)^{-p} J_{-p}(kr) \rho^{p+1/2}}{\left[ \frac{1}{\Gamma[1/2(\sqrt{\rho^2 M^2 - 1/4} - i\epsilon - p - 1)]} \Gamma[1/2(\sqrt{\rho^2 M^2 - 1/4} + i\epsilon - p - 1)] \right]} \frac{1}{\sqrt{r}}.
\]

Now, using the known relation for \( \Gamma \)-function

\[
\Gamma(p) \Gamma(1 - p) = \frac{\pi}{\sin(\pi p)},
\]

and introducing shortening notation

\[
-\frac{i\epsilon + \sqrt{\rho^2 M^2 - 1/4}}{2} \sim i \frac{M - \epsilon}{2} = \rho z_1, \quad \frac{-i\epsilon - \sqrt{\rho^2 M^2 - 1/4}}{2} \sim i \frac{M + \epsilon}{2} = \rho z_2
\]

we change (32) into

\[
\lim_{\rho \to \infty} A U_{\text{out.}}^\text{AF}(z) = \lim_{\rho \to \infty} A \times \frac{\pi \Gamma(-i\epsilon + 1)}{\sin(\pi p) \sqrt{r}} \left[ -\frac{(2/k)^p J_p(kr) \rho^{-p+1/2}}{\Gamma(\rho z_1 - \frac{1}{2}) \Gamma(\rho z_2 - \frac{1}{2})} + (p \to -p) \right]. \quad (33)
\]

the designation \( (p \to -p) \) means that here else one term is to be placed, which is derived from the previous one with a change \( p \to -p \). The constant \( A \) will be chosen to guarantee the existence of a finite limit in (4.9); so it is

\[
\lim_{p \to \infty} A = \lim_{p \to \infty} \frac{\Gamma(a - \frac{c-1}{2}) \Gamma(b - \frac{c-1}{2})}{\Gamma(a + b - c + 1)} = \lim_{p \to \infty} \frac{\Gamma(\rho z_1 + 1/4) \Gamma(\rho z_2 + 1/4)}{\Gamma(-i\epsilon + 1)}. \quad (34)
\]
One may carry out further calculations solely with the first term in (33) (it will be designated as $F(p)$; the second term’s contribution will be determined by the first through a change $p \to -p$)

$$F(p) = \lim_{p \to \infty} \left[ \frac{-\pi}{\sin(\pi p)} \frac{1}{\sqrt{r}} \frac{(2/k)^p J_p(kr)}{2^{p-1/2}} \frac{\Gamma(\rho z_1 + 1/4) \Gamma(\rho z_2 + 1/4)}{\Gamma(\rho z_1 - 1/2) \Gamma(\rho z_2 - 1/2)} \right].$$

(35)

So, let us dwell on the term $F(p)$. Applying else one time the above formula referring $\Gamma$-s with arguments $p$ and $(1 - p)$, one can produce

$$F(p) = \lim_{p \to \infty} \left[ \frac{-\pi}{\sin(\pi p)} \frac{1}{\sqrt{r}} \frac{(2/k)^p J_p(kr)}{2^{p-1/2}} \frac{\Gamma(\rho z_1 + 1/4) \sin(\rho z_2 + 1/2)}{\Gamma(\rho z_1 + 1/2) \sin(\rho z_2 + 1/4)} \frac{\Gamma(-\rho z_2 + 1/2)}{\Gamma(-\rho z_2 + 3/4)} \right]$$

(36)

which after having used the known asymptotic relationship for $\Gamma$-functions

$$x \to \infty, \ |\arg x| < \pi \to \frac{\Gamma(x + \alpha)}{\Gamma(x + \beta)} \sim x^{\alpha - \beta}$$

and taken into account the equality ($(\epsilon + M) > 0$)

$$\frac{\sin(\rho z_2 + 1/2)}{\sin(\rho z + 1/4)} = \frac{\exp[\pi \rho(M+\epsilon)/2] + i\pi 1/2]}{\exp[\pi \rho(M+\epsilon)/2] + i\pi/4]} - \frac{\exp[\pi \rho(M+\epsilon)/2] - i\pi 1/2]}{\exp[\pi \rho(M+\epsilon)/2] - i\pi/4]} = \lim_{\rho \to \infty} \exp[i\pi(p/2 + 1/2)]$$

provides us with the expansion

$$F(p) = \lim_{p \to \infty} \left[ \frac{-\pi}{\sin(\pi p)} \frac{1}{\sqrt{r}} \frac{(2/k)^p J_p(kr)}{2^{p-1/2}} \frac{\exp[(\rho z_1)^{p/2 - 1/4} (\rho z_2)^{p/2 - 1/4}] \exp[i\pi(p/2 + 1/2)]}{\pi} J_p(kr) \right]$$

from where it follows

$$F(p) = \sqrt{\frac{2}{kr}} \exp[i\pi(p/2 + 1/2)] \frac{\pi}{\sin(\pi p)} J_p(kr).$$

(37)

Remembering the second term’s contribution $(p \to -p)$, the final limiting expression for $A U_{out}(z)$ will be

$$\lim_{\rho \to \infty} A U_{out}(z) = 1/\epsilon^{j+1} \sqrt{\frac{2}{kr}} H_{j+1/2}^{(1)}(kr)$$

(38)

where $H_{j+1/2}^{(1)}(kr)$ denotes the Hankel function

$$H_{j+1/2}^{(1)}(x) = \frac{ip}{\sin(\pi p)} [ e^{ip\pi} J_p(x) - J_{-p}(x) ] .$$

(39)

The right-hand part of the relation (38) provides quite standard representation for an expanding spherical wave (going to the spatial infinity) in flat space-time model. Thus, the notation in the Sitter space model

$$\Phi^{\text{sing. stand.}}, \Phi^{\text{reg. stand.}}, \Phi^{\text{out. run.}}, \Phi^{\text{in. run.}}$$

is correct and consistent with that used in the flat Minkowski space.

In connection with that limiting passage from the curved to flat space-time, one point of noticeable importance for the following should be noted. Because of the curved metric (33) practically coincides with the flat (Minkowski’s) one as soon as one gets far enough from the horizon
(0 ≤ r << ρ), one might expect little difference between "curved" and "flat" solutions of the respective (scalar particle's) equations: \( f_{\text{curve}}(r) \approx f_{\text{flat}}(r) \). Being applied to a solution of the kind of extending ones, that claim can be formulated as follows: the approximation

\[
A U_{\text{run}}(z) \sim \frac{1}{\vartheta + r} \sqrt{\frac{2}{kr}} H_{j+1/2}^{(1)}(kr)
\]

is quite valid. However, as will be seen, such a suggestion is not completely right: some limitations accompanying (40) will be readily discerned.

To clear up this matter, we shall compare carefully two respective equations:

- **in the de Sitter model**, \( \Phi = e^{-i\epsilon t} f_\epsilon(r) Y_{jm}(\theta, \phi) \),

\[
\left[ \frac{d^2}{dr^2} + \frac{2(1-2r^2)}{r(1-r^2)} \frac{d}{dr} + \frac{\epsilon^2}{(1-r^2)^2} - \frac{M^2 + 2}{1-r^2} - \frac{j(j+1)}{r^2} \right] f_\epsilon = 0 ;
\]

- **in the Minkowski model**, \( \Phi^0 = e^{-i\epsilon t} f_{\epsilon 0}^0(r) Y_{jm}(\theta, \phi) \),

\[
\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + \epsilon^2 - M^2 - \frac{j(j+1)}{r^2} \right] f_{\epsilon 0}^0 = 0 .
\]

At the region \( r << \rho \), the equation (41) may be rewritten as follows

\[
\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \left( 1 - \frac{r^2}{\rho^2} + \ldots \right) + \epsilon^2 \left( 1 + \frac{2r^2}{\rho^2} + \ldots \right) - \left( M^2 + \frac{2}{\rho^2} \right) \left( 1 + \frac{r^2}{\rho^2} + \ldots \right) - \frac{j(j+1)}{r^2} \frac{r^2}{\rho^2} \right] f_\epsilon = 0 .
\]

On comparing (42) with (41) one may conclude that those practically coincide if and only if two inequalities

\[
r << \rho \text{ and } \left( \epsilon^2 - M^2 - \frac{j(j+1)}{r^2} \right) \gg \left( \frac{j(j+1)}{\rho^2} + \frac{2}{\rho^2} \right);
\]

hold; from the latter it follows

\[
\epsilon^2 - M^2 \gg \frac{j(j+1) + 2}{\rho^2} .
\]

That is, the difference between the particle's energy \( \epsilon \) and its mass \( m \) should be great enough as compared with a scale given by \( (j/\rho) \), where \( \rho \) denotes the curvature radius, and \( j \) is an orbital momentum number. At \( j = 0 \), the relation \( (\epsilon^2 - M^2) \gg 2/\rho^2 \) ought to be held. The curvature parameter \( \rho \) is quite a big but finite one, whereas no limitations on \( j \) exist.

5 Standing, running waves and conserved current \( J_\alpha(x) \)

As an additional argument to justify the above-used terminology let us consider properties of the scalar particle's conserved current

\[
J_\alpha(x) = i(\Phi^* \nabla_\alpha \Phi - \Phi^* \nabla_\alpha \Phi) .
\]
Using a general spherical wave’s representation $\Phi(x) = e^{-i\epsilon t} f(r) Y_{jm}(\theta, \phi)$, for current’s components $J_\alpha(x)$ one gets

$$J_t = 2\epsilon |f|^2 |\Theta_{jm}|^2, \quad J_\phi = -2m |f|^2 |\Theta_{jm}|^2,$$

$$J_r = i(f^* \frac{d}{dr} f - f \frac{d}{dr} f^*) |\Theta_{jm}|^2, \quad J_\theta = 0$$

whence it follows that standing and running solutions (in the above terminology) are characterized, respectively, by

$$(J_r)_{\text{reg.stand.}} = 0, \quad (J_r)_{\text{sing.stand.}} = 0, \quad (J_r)_{\text{out.run.}} = -(J_r)_{\text{in.run.}}.$$

Those properties of a radial current component are in agreement with our notation which prescribes consider $\Phi_{\text{run.(in.)}}$-solutions as relating to radial in- and out-streams of particles, whereas $\Phi_{\text{reg.(sing.)}}$ represent radially immobile particles’ states. It seems likely that analogous relationships conforming to the anticipated behavior of such waves will be established if we turn to the energy-momentum tensor referring to those solutions. Thus, the terminology exploited above seems quite reasonable and appropriate.

6 On transmittivity and reflectivity of the de Sitter horizon

From the preceding material it might be concluded that the phenomenon itself of any reflection of scalar particles by the Sitter event horizon should not be observed whatsoever. Added credence to that opinion may be given by qualitative but quite heuristic consideration of the problem in terms of an one-dimensional Schrödinger’s like equation with an effective potential $U(r)$. Indeed, the matter radial equation (4) can be taken into the form (the variable $r^*$ is used again)

$$\left[ \frac{d^2}{dr^{*2}} + (\epsilon^2 - U(r^*)) \right] G(r^*) = 0; \quad (47)$$

where $G(r^*)$ is given by

$$f(r) = \mu(r^*) G(r^*) , \quad \left( \frac{d}{dr^*} \ln \mu = \frac{(1-r)(1+2r)}{\rho} \right),$$

and an effective potential $U(r^*)$ is determined according to

$$U(r^*) = \frac{1 - r^2}{\rho^2} \left[ 4(1-r) + \frac{r}{1+r} + m^2 \rho^2 + \frac{j(j+1)}{r^2} \right]. \quad (48)$$

As readily verified, that potential (48) creates an effective repulsive force $-dU/dr^*$ at every spatial point ($r \leq 1; r^* \in [0, +\infty]$) of the de Sitter space-time. On other words, the center $r = 0$ effectively repulses the scalar particle everywhere and no barrier appears between the center and event horizon region $r \sim \rho$:

$$F_{r^*} \equiv -\frac{dU}{dr^*} = \frac{1 - r^2}{\rho^2} + \left[ 2r \left( \frac{j(j+1)}{r^2} + m^2 \rho^2 \right) + \frac{r}{1+r} + 4(1-r) \right] + (1-r^2) \left( \frac{2j(j+1)}{r^3} + 4 - \frac{1}{(1+r)^2} \right) > 0.$$
In a region near to the horizon \(r^* \sim +\infty\), the potential \(U(r^*)\) tends to zero, and correspondingly solutions (running waves are meant here for definiteness) look like exponents

\[
G(r^*) \sim \exp(\pm i\epsilon r^*);
\]

the latter boundary behavior may be naturally interpreted as associated with waves passing to (+) and from (−) the de Sitter horizon \((r \sim 1\) or \(r^* \sim \infty\)).

After having examined this Schrödinger’s-like equation, it has become absolutely evident that the quantity such as a reflection coefficient \(R_{cj}\) cannot be correctly determined. However, it would not be sufficiently enough if only this single statement had been formulated. In so doing, undoubtedly, a bit of vagueness would still remain. Because of this, now let us reexamine closely (if not narrowly) an algorithm for calculating such a quantity \(R_{cj}\) on the background of the de Sitter space-time model. For definiteness, let us consider the case of massless particles.

A wave going to the horizon is described by the following radial function (see (19) at \(m^2 = 0\) and \(r = R/\rho\))

\[
U_{\text{ran.}}^{\text{out.}}(R) = \left[ \frac{\Gamma(a + b + 1 - c) \Gamma(1 - c)}{\Gamma(b - c + 1) \Gamma(a - c + 1)} \frac{1}{r^{j+1}} (1 - r^2)^{-i\epsilon \rho/2} F(a, b, c; r^2) \right. \\
\left. \frac{\Gamma(a + b + 1 - c) \Gamma(c - 1)}{\Gamma(a) \Gamma(b)} \frac{1}{r^{j+1}} (1 - r^2)^{-i\epsilon \rho/2} F(b - c + 1, a - c + 1, -c + 2; r^2) \right] 
\]

where

\[
a = \frac{j - i\epsilon \rho}{2}, \quad b = \frac{j + 1 - i\epsilon \rho}{2}, \quad c = j + 3/2 = p + 1.
\]

At great distance away from the horizon \((R << \rho)\), for the \(U_{\text{ran.}}^{\text{out.}}(R)\) one has an approximated representation

\[
U_{\text{ran.}}^{\text{out.}}(R) \sim \left[ \frac{\Gamma(a + b + 1 - c) \Gamma(1 - c)}{\Gamma(b - c + 1) \Gamma(a - c + 1)} \frac{1}{\rho^j} \left(\frac{2}{\epsilon}\right)^p \Gamma(1 + p) \frac{J_p(eR)}{\sqrt{R}} \right. \\
\left. \frac{\Gamma(a + b + 1 - c) \Gamma(c - 1)}{\Gamma(a) \Gamma(b)} \frac{1}{\rho^{j+1}} \left(\frac{2}{\epsilon}\right)^{-p} \Gamma(1 - p) \frac{J_{-p}(eR)}{\sqrt{R}} \right],
\]

but one ought to remember about accompanying limitation (44) on quantum numbers (see above the limiting passage from the curved to flat space-time model):

\[
R << \rho, \quad j << \epsilon \rho.
\]

This means that to be correct one will must take the limits

\[
\lim_{\epsilon \rho >> j} = \left[ \frac{\Gamma(a + b + 1 - c) \Gamma(1 - c)}{\Gamma(b - c + 1) \Gamma(a - c + 1)} \right], \quad \lim_{\epsilon \rho >> j} = \left[ \frac{\Gamma(a + b + 1 - c) \Gamma(c - 1)}{\Gamma(a) \Gamma(b)} \right]
\]

if one is going to exploit that expansion (50) subsequently in a noticeably significant physical context. For the moment, setting away the need to realize additionally the above limiting passages, let us proceed with calculations providing “a required” analytical expression for \(R_{cj}\), following the standard pattern. So, taking into consideration an asymptotic formula for Bessel functions

\[
x >> \nu^2 : \quad J(x) \sim \frac{\Gamma(2\nu + 1)}{\Gamma(\nu + 1) \Gamma(\nu + 1/2)} \frac{1}{\sqrt{x}} \times \left[ \exp \left( +i(x - \frac{\pi}{2}(\nu + \frac{1}{2}) \right) \right] + \exp \left( -i(x - \frac{\pi}{2}(\nu + \frac{1}{2})) \right)
\]

(53)
we get

\[ j < J^2 << \epsilon R << \epsilon \rho : \]

\[
U_{\text{out}}(R) \sim \left[ \frac{e^{+i\epsilon R}}{\epsilon R} \left( A \exp(-i\frac{\pi}{2}(p + \frac{1}{2})) + B \exp(-i\frac{\pi}{2}(-p + \frac{1}{2})) \right) + \frac{e^{-i\epsilon R}}{\epsilon R} \left( A \exp(+i\frac{\pi}{2}(p + \frac{1}{2})) + B \exp(+i\frac{\pi}{2}(-p + \frac{1}{2})) \right) \right],
\]

(54)

where \( A \) and \( B \) are

\[
A = \frac{\Gamma(a + b + 1 - c) \Gamma(1 - c)}{\Gamma(b - c + 1) \Gamma(a + c + 1)} 2^{-j-1} \frac{\Gamma(2p + 1)}{\epsilon\rho^j \Gamma(p + 1/2)},
\]

(55)

\[
B = \frac{\Gamma(a + b + 1 - c) \Gamma(c - 1)}{\Gamma(a) \Gamma(b)} (\epsilon\rho)^j \Gamma(p + 1/2).
\]

The reflection coefficient \( R_{\epsilon j} \) by definition, is a square modulus of the ratio of an amplitude at \( e^{-i\epsilon R}/\epsilon R \) to an amplitude at \( e^{+i\epsilon R}/\epsilon R \); thus we shall find

\[
R_{\epsilon j} = \left| i \frac{B}{A} \frac{e^{+i\epsilon p}}{e^{-i\epsilon p} + 1} \right|^2.
\]

(56)

It remains to show that, after having into account the limitation \( \epsilon\rho >> j \), this expression for \( R_{\epsilon j} \) will identically vanish. For \( A/B \) one has

\[
\frac{A}{B} = \frac{1}{(2\epsilon\rho)^{2j+1}} \frac{\Gamma(a) \Gamma(b)}{(a - c + 1) \Gamma(b - c + 1)} \frac{\Gamma(1 - c)}{\Gamma(c - 1)} \frac{\Gamma(-p + 1/2)}{\Gamma(p + 1/2)} \frac{\Gamma(2p + 1)}{\Gamma(-2p + 1)}. \]

(57)

After using the formulas

\[
\Gamma(p) \Gamma(1 - p) = \frac{\pi}{\sin(\pi p)}, \quad \frac{\Gamma(2x)}{\Gamma(x)} = \Gamma(x + 1/2) \frac{2^{2x-1}}{\Gamma(1/2)},
\]

(58)

we find

\[
\frac{\Gamma(1 - c)}{\Gamma(c - 1)} \frac{\Gamma(-p + 1/2)}{\Gamma(p + 1/2)} \frac{\Gamma(2p + 1)}{\Gamma(-2p + 1)} = -2^{4j+2}.
\]

(59)

Further, applying the asymptotic relation

\[
\{ x \to \infty , \ | \arg x | < \pi \} \Rightarrow \frac{\Gamma(x + \alpha)}{\Gamma(x + \beta)} \sim x^{\alpha - \beta}
\]

we get

\[
\lim_{\epsilon\rho >> j} \frac{\Gamma(a) \Gamma(b)}{(a - c + 1) \Gamma(b - c + 1)} = \left( \frac{\epsilon\rho}{2} \right)^{2j+1} (-i)^{2p}.
\]

(60)

Substituting (59) and (60) into (57), for \( A/B \) we finally arrive at a simple result

\[
\lim_{\epsilon\rho >> j} \frac{A}{B} = -(-i)^{2p}
\]

and therefore the expected identical zero for \( R_{\epsilon j} \) arises, \( R_{\epsilon j} \equiv 0 \).

In other words, analytical expressions for \( R_{\epsilon j} \) existing in the literature are due to not taking the all required limits in the final expressions. It should be stressed that, as can be seen from the present study, certain analogous limiting procedures as well are to be performed in other and more complicated physical cases. Correspondingly, the possible underestimating of all set of limiting conditions in every separate situation , coupled with calculation of final analytical expressions of coefficients \( R \) and \( \Gamma \), might be source for partly incorrect conclusions.
7 Approximation influencing physical results

To prevent anybody from possible errors let us discuss the formula \(50\) giving decomposition for a wave going to de Sitter horizon at far distant domain and consider possible discrepancies arising from the use of incorrect approximate expressions for wave functions. Indeed, instead of correct decomposition \(50\) let us write its (by hand) modification through introducing at \(J_p(\epsilon R)\) and \(J_{-p}(\epsilon R)\) some coefficients \((1 + \Delta_p)\) and \((1 + \Delta_{-p})\) (to correct formula there correspond \(\Delta_p = 0\) and \(\Delta_{-p} = 0\)):

\[
U_{\text{out}}(R) \sim \left[ (1 + \Delta_p) \frac{\Gamma(a + b + 1 - c)}{\Gamma(b - c + 1)\Gamma(a - c + 1)} \frac{1}{\rho^p} \left(\frac{2}{\epsilon}\right)^p \Gamma(1 + p) \frac{J_p(\epsilon R)}{\sqrt{R}} \right] \\
+ (1 + \Delta_{-p}) \frac{\Gamma(a + b + 1 - c)}{\Gamma(a)\Gamma(b)} \rho^{p+1} \left(\frac{2}{\epsilon}\right)^{-p} \Gamma(1 - p) \frac{J_{-p}(\epsilon R)}{\sqrt{R}}.
\]

(62)

Correspondingly, instead of \(A\) and \(B\) according to \(55\) we have

\[
bA \rightarrow A' = (1 + \Delta_p) A, \quad B \rightarrow B' = (1 + \Delta_{-p}) B,
\]

and further

\[
\lim_{\epsilon^\rho \gg j} \frac{A}{B} \rightarrow \lim_{\epsilon^\rho \gg j} \frac{A'}{B'} = \lim_{\epsilon^\rho \gg j} \frac{1 + \Delta_p}{1 + \Delta_{-p}} \left[ -(-i)^{2p} \right].
\]

(64)

Therefore, now we arrive at a new expression for reflection coefficient (compare with \(56\))

\[
R'_{\epsilon j} = \frac{(A'/B')((+i)^{2p} + 1)}{(A'/B')((-i)^{2p} + 1)^2} = \frac{\Delta_p - \Delta_{-p}}{2 + \Delta_p + \Delta_{-p}}.
\]

(65)

In other words, just an error in approximate form for a wave function leads us to a quite ”physical result” Such incorrectness may be presented if any uses approximate solutions for wave equations

\[
R \sim \lim \left| \frac{1 + \Delta_p}{1 + \Delta_{-p}} \frac{A}{B} (+i)^{2p} + 1 / \frac{1 + \Delta_p}{1 + \Delta_{-p}} \frac{A}{B} (-i)^{2p} + 1 \right|^2.
\]

(66)

8 On Maxwell equations in general covariant tetrad Majorana – Oppenheimer form on the background of de Sitter space

Below we show that the main results can be extended to the case of electromagnetic field. At this we will use an old and almost unusable in the literature approach by Riemann – Silberstein – Majorana – Oppenheimer in general covariant tetrad form. Maxwell equations in Riemann space can be presented as one matrix equation \(106\) \(107\) \(108\)

\[
\alpha^c \left( e^\rho_c \partial_\rho + \frac{1}{2} j^{ab}_{\gamma abc} \right) \Psi = J(x),
\]

\[
\alpha^0 = -iI, \quad \Psi = \begin{bmatrix} 0 \\ E + icB \end{bmatrix}, \quad J = \frac{1}{\epsilon_0} \begin{bmatrix} \rho \\ i j \end{bmatrix},
\]

\[
\alpha^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \alpha^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \alpha^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},
\]

\]

(67)
or
\[-i (\epsilon^{(0)}_a \partial_a + \frac{1}{2} j^{ab}_a \gamma_{ab0}) \Psi + \alpha^k (\epsilon^{(k)}_a \partial_a + \frac{1}{2} j^{ab}_a \gamma_{abk}) \Psi = J(x). \] (68)

Allowing for identities
\[
\frac{1}{2} j^{ab}_a \gamma_{ab0} = [s_1 (\gamma_{230} + i \gamma_{010}) + s_2 (\gamma_{310} + i \gamma_{020}) + s_3 (\gamma_{120} + i \gamma_{030})],
\]
\[
\frac{1}{2} j^{ab}_a \gamma_{abk} = [s_1 (\gamma_{23k} + i \gamma_{01k}) + s_2 (\gamma_{31k} + i \gamma_{02k}) + s_3 (\gamma_{12k} + i \gamma_{03k})],
\]
where
\[
s_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \tau_1 & 0 \\ \tau_1 & 0 & 0 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad s_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \]
\[
\tau_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \tau_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \tau_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\] (69)
and using the notation
\[
e^{(0)}_a \partial_a = \partial_0 (\cdot), \quad e^{(k)}_a \partial_a = \partial_k (\cdot), \quad a = 0, 1, 2, 3,
\]
\[
(\gamma_{01a}, \gamma_{02a}, \gamma_{03a}) = v_a, \quad (\gamma_{23a}, \gamma_{31a}, \gamma_{12a}) = p_a,
\] (70)
eq. (68) in absence of sources reduces to
\[-i [\partial_0 + s(p_0 + iv_0)] \Psi + \alpha^k [\partial_k + s(p_k + iv_k)] \Psi = 0,
\] (71)
Let us consider this equation in the de Sitter static metric and tetrad
\[
dS^2 = \Phi \, dt^2 - \frac{dr^2}{\Phi} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad \Phi = 1 - r^2,
\]
\[
e^{(0)}_a = (\frac{1}{\sqrt{\Phi}} 0, 0, 0), \quad e^{(3)}_a = (0, \sqrt{\Phi}, 0, 0),
\]
\[
e^{(1)}_a = (0, 0, \frac{1}{r}, 0), \quad e^{(2)}_a = (1, 0, 0, \frac{1}{r \sin \theta}),
\]
\[
\gamma_{030} = \frac{\Phi'}{2\sqrt{\Phi}}, \quad \gamma_{311} = \frac{\sqrt{\Phi}}{r}, \quad \gamma_{322} = \frac{\sqrt{\Phi}}{r}, \quad \gamma_{122} = \frac{\cos \theta}{r \sin \theta},
\] (72)
we get to explicit form of a matrix equation
\[
[-i \frac{\partial_1}{\sqrt{\Phi}} + \sqrt{\Phi} (\alpha^2 \partial_\theta + \frac{\alpha^1 s_2 - \alpha^2 s_1}{r} + \frac{\Phi'}{2\Phi} s_3) + \frac{1}{r} \Sigma_{\theta, \phi}] \begin{bmatrix} 0 \\ \psi \end{bmatrix} = 0,
\]
\[
\Sigma_{\theta, \phi} = \frac{\alpha^1}{r} \partial_\theta + \alpha^2 \frac{\partial_\phi + s_3 \cos \theta}{\sin \theta}.
\] (73)
It is convenient to have the spin matrix $s_3$ as diagonal, which is reached by simple linear transformation to the known cyclic basis
\[
\Psi' = U_4 \Psi, \quad U_4 = \begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix},
\]
\[
U = \begin{bmatrix} -1/\sqrt{2} & i/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & i/\sqrt{2} & 0 \end{bmatrix}, \quad U^{-1} = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix},
\] (74)
15
so that

\[
\begin{align*}
\tau_1' &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & -i \\ 0 & -i & 0 \end{pmatrix}, \\
\tau_2' &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\tau_3' &= -i \begin{pmatrix} +1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\
\alpha_1' &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -i \\ 0 & -i & 0 \end{pmatrix}, \\
\alpha_2' &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & -i \\ -i & 0 & 1 \end{pmatrix}, \\
\alpha_3' &= -i \begin{pmatrix} 0 & 0 & 1 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{align*}
\]

Eq. (73) becomes

\[
\begin{align}
&\frac{-i\partial_t}{\sqrt{\Phi}} + \sqrt{\Phi}(\alpha_1' \partial_r + \frac{\alpha_1' s_{r2} - \alpha_2' s_3'}{r} + \frac{\Phi'}{2\Phi} s_3') + \frac{1}{r} \Sigma_{\theta,\phi}' \bigg| 0_{\psi'} \bigg) = 0, \\
&\Sigma_{\theta,\phi}' = \frac{\alpha_1'}{r} \partial_\theta + \alpha_2' \frac{\partial_\phi + s_3' \cos \theta}{\sin \theta}.
\end{align}
\]

9 Separating the variables and Wigner functions

Spherical waves with \((j, m)\) quantum numbers should be constructed as follows

\[
\psi = e^{-i\omega t} \begin{vmatrix} 0 \\ f_1(r)D_{-1} \\ f_2(r)D_0 \\ f_3(r)D_{+1} \end{vmatrix}
\]

where the shorted notation for Wigner \(D\)-functions [112, 113] is used: \(D_{\sigma} = D_{j,m,\sigma}(\phi, \theta, 0)\), \(\sigma = -1, 0, +1\); \(j, m\) determine total angular moment. We adhere notation developed in [116, 117], [119]; before similar techniques was applied by Dray [116, 117], Krolikowski and Turski [118], Turski [119] ; many years ago such a tetrad basis was used by Schrödinger [109] and Pauli [110] when looking at the problem of single-valuedness of wave functions in quantum theory. In the literature equivalent techniques of spin-weighted harmonics Goldberg et al [114] (se also in [115]) preferably is used though equivalence of both approach is known [116, 117].

With the use the recursive relations [113] (below \(\nu = \sqrt{j(j+1)}\), \(a = \sqrt{(j-1)(j+2)}\))

\[
\begin{align*}
\partial_\theta D_{-1} &= \frac{1}{2}(aD_{-2} - \nu D_0), \\
\frac{m - \cos \theta}{\sin \theta} D_{-1} &= \frac{1}{2}(aD_{-2} + \nu D_0), \\
\partial_\theta D_0 &= \frac{1}{2}(\nu D_{-1} - \nu D_{+1}), \\
\frac{m}{\sin \theta} D_0 &= \frac{1}{2}(\nu D_{-1} + \nu D_{+1}), \\
\partial_\theta D_{+1} &= \frac{1}{2}(\nu D_0 - aD_{+2}), \\
\frac{m + \cos \theta}{\sin \theta} D_{+1} &= \frac{1}{2}(\nu D_0 + aD_{+2}),
\end{align*}
\]

we get (the factor \(e^{-i\omega t}\) is omitted)

\[
\Sigma_{\theta,\phi}' \Psi' = \frac{\nu}{\sqrt{2}} \begin{vmatrix} (f_1 + f_3)D_0 \\ -i f_2 D_{-1} \\ i (f_1 - f_3)D_0 \\ +i f_2 D_{+1} \end{vmatrix}
\]

\[16\]
Turning to Maxwell equation (75), we arrive at the radial system

\[
\begin{align*}
\text{(1)} \quad & \sqrt{\Phi} \left( \frac{d}{dr} + \frac{2}{r} \right) f_2 + \frac{1}{r} \frac{\nu}{\sqrt{2}} (f_1 + f_3) = 0, \\
\text{(2)} \quad & -\frac{\omega}{\sqrt{\Phi}} - i \sqrt{\Phi} \frac{d}{dr} - i \frac{\Phi'}{2\sqrt{\Phi}} f_1 - i \frac{\nu}{r} \sqrt{2} f_2 = 0, \\
\text{(3)} \quad & -\frac{\omega}{\sqrt{\Phi}} f_2 + i \frac{\nu}{r} \sqrt{2} (f_1 - f_3) = 0, \\
\text{(4)} \quad & -\frac{\omega}{\sqrt{\Phi}} + i \sqrt{\Phi} \frac{d}{dr} + i \frac{\Phi'}{2\sqrt{\Phi}} f_3 + i \frac{\nu}{r} \sqrt{2} f_2 = 0. 
\end{align*}
\] (79)

Combining equations (2) and (4), instead of (79) we get

\[
\begin{align*}
\text{(2) + (4)} \quad & -\frac{\omega}{\sqrt{\Phi}} (f_1 + f_3) - i(\sqrt{\Phi} \frac{d}{dr} + \frac{\Phi'}{2\sqrt{\Phi}}) (f_1 - f_3) = 0, \\
\text{(2) - (4)} \quad & -\frac{\omega}{\sqrt{\Phi}} (f_1 - f_3) - i(\sqrt{\Phi} \frac{d}{dr} + \frac{\Phi'}{2\sqrt{\Phi}}) (f_1 + f_3) - \frac{2i}{r} \frac{\nu}{\sqrt{2}} f_2 = 0, \\
\text{(3)} \quad & -\frac{\omega}{\sqrt{\Phi}} f_2 + i \frac{\nu}{r} \sqrt{2} (f_1 - f_3) = 0, \\
\text{(1)} \quad & \sqrt{\Phi} \left( \frac{d}{dr} + \frac{2}{r} \right) f_2 + \frac{1}{r} \frac{\nu}{\sqrt{2}} (f_1 + f_3) = 0.
\end{align*}
\] (79)

It is easily verified that equation (1) is an identity when allowing for remaining ones. So independent equations are

\[
\begin{align*}
\text{(1)} \quad & -\frac{\omega}{\sqrt{\Phi}} f_2 + i \frac{\nu}{r} \sqrt{2} (f_1 - f_3) = 0, \\
\text{(2) - (4)} \quad & -\frac{\omega}{\sqrt{\Phi}} (f_1 - f_3) - i(\sqrt{\Phi} \frac{d}{dr} + \frac{\Phi'}{2\sqrt{\Phi}}) (f_1 + f_3) - \frac{2i}{r} \frac{\nu}{\sqrt{2}} f_2 = 0, \\
\text{(3) + (4)} \quad & -\frac{\omega}{\sqrt{\Phi}} (f_1 + f_3) - i(\sqrt{\Phi} \frac{d}{dr} + \frac{\Phi'}{2\sqrt{\Phi}}) (f_1 - f_3) = 0, \\
\text{(4) + (3)} \quad & -\frac{\omega}{\sqrt{\Phi}} (f_1 - f_3) - i(\sqrt{\Phi} \frac{d}{dr} + \frac{\Phi'}{2\sqrt{\Phi}}) (f_1 + f_3) - \frac{2i}{r} \frac{\nu}{\sqrt{2}} f_2 = 0. 
\end{align*}
\] (80)

Let us introduce new functions:

\[
f = \frac{f_1 + f_3}{\sqrt{2}}, \quad g = \frac{f_1 - f_3}{\sqrt{2}},
\]

then eqs. (80) read

\[
\begin{align*}
f_2 &= i\frac{\nu}{\omega} \sqrt{\Phi} g, \quad -\frac{\omega}{\Phi} f - i \left( \frac{d}{dr} + \frac{1}{r} + \frac{\Phi'}{2\Phi} \right) g = 0, \\
-\frac{\omega^2}{\Phi} g - i\omega \left( \frac{d}{dr} + \frac{1}{r} + \frac{\Phi'}{2\Phi} \right) f + \frac{\nu^2}{r^2} g &= 0. 
\end{align*}
\] (81)

The system (81) is simplified by substitutions

\[
g(r) = \frac{1}{r\sqrt{\Phi}} G(r), \quad f(r) = \frac{1}{r\sqrt{\Phi}} F(r),
\]

\[17\]
and it gives

\[ f_2 = \frac{iv}{\omega} \frac{1}{r^2} G(r), \quad i\omega F = \Phi \frac{d}{dr} G, \]

\[ i\omega \frac{d}{dr} F + \frac{\omega^2}{\Phi} G - \frac{\nu^2}{r^2} G = 0, \quad (82) \]

So we have arrived at a single differential equation for \( G(r) \):

\[ \frac{d^2 G}{dr^2} + \frac{\Phi'}{\Phi} \frac{dG}{dr} + \left( \frac{\omega^2}{(1-r^2)^2} - \frac{j(j+1)}{r^2(1-r^2)} \right) G = 0. \quad (83) \]

or taking \( \Phi = 1 - r^2 \)

\[ \frac{d^2 G}{dr^2} - \frac{2r}{1-r^2} \frac{dG}{dr} + \left( \frac{\omega^2}{(1-r^2)^2} - \frac{j(j+1)}{r^2(1-r^2)} \right) G = 0. \quad (84) \]

This equation coincides with that following from (4), if one translates eq. (4) at \( M = 0 \) to new function \( f(r) = r^{-1} \phi(r) \):

\[ \frac{d^2 \phi}{dr^2} + \frac{2r}{1-r^2} \frac{d\phi}{dr} + \left( \frac{\omega^2}{(1-r^2)^2} - \frac{j(j+1)}{r^2(1-r^2)} \right) \phi = 0. \quad (85) \]

With the variable \( z = r^2 \), eq. (85) gives

\[ 4z(1-z) \frac{d^2 F}{dz^2} + 2(1-3z) \frac{dF}{dz} + \left( \frac{\omega^2}{1-z} - \frac{j(j+1)}{z} \right) F = 0 \quad (86) \]

and after substitution \( G = z^a(1-z)^b F(z) \) we arrive at

\[ 4z(1-z) \frac{d^2 F}{dz^2} + 4 \left[ 2a + \frac{1}{2} - (2a + 2b + \frac{3}{2}) \right] \frac{dF}{dz} + \\
+ \left[ \frac{4a^2 - 2a - j(j+1)}{z} + \frac{4b^2 + \omega^2}{1-z} - 4(a+b)(a+b+\frac{1}{2}) \right] F = 0. \quad (87) \]

Requiring

\[ a = \pm \frac{j+1}{2}, \quad b = \pm \frac{\omega}{2}, \quad \omega > 0; \]

we get

\[ z(1-z) \frac{d^2 F}{dz^2} + \left[ 2a + \frac{1}{2} - (2a + 2b + \frac{3}{2}) \right] \frac{dF}{dz} - (a+b)(a+b+\frac{1}{2}) F = 0, \quad (88) \]

parameters of hypergeometric function are given by

\[ \alpha = a + b, \quad \beta = a + b + \frac{1}{2}, \quad \gamma = 2a + \frac{1}{2}. \quad (89) \]

Evidently, these solutions coincide with those described in Sec. 2.
10 Spin 1/2 particle

Dirac equation (the notation according [108] is used)

\[ i\gamma^c (\epsilon^{ac}_0 \partial_a + \frac{1}{2} \sigma^{ab} \gamma_{abc}) - M ] \Psi = 0 \]  

(90)

in static coordinates and tetrad of the the Sitter space takes the form

\[ \{ i \frac{\gamma^0}{\sqrt{\Phi}} \partial_t + i \sqrt{\Phi} \gamma^3 ( \partial_r + \frac{1}{r} \Phi' + \frac{1}{4r} \Sigma^\kappa_{\theta,\phi} - M ) \} \Psi(x) = 0 . \]  

(91)

Below the spinor basis will be used

\[ \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \gamma^j = \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix}, i\sigma^{12} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}. \]

Allowing for \( \gamma^1 \sigma^{31} + \gamma^2 j^{32} = \gamma^3 \), \( \gamma^0 \sigma^{03} = \gamma^3 / 2 \), eq. (91) reads

\[ \{ i \frac{\gamma^0}{\sqrt{\Phi}} \partial_t + i \sqrt{\Phi} \gamma^3 ( \partial_r + \frac{1}{r} \Phi' + \frac{1}{4r} \Sigma^\kappa_{\theta,\phi} - M ) \} \Psi(x) = 0 . \]  

(92)

One can simplify the task with the help of substitution \( \Psi(x) = r^{-1} \Phi^{-1/4} F(x) \):

\[ ( i \frac{\gamma^0}{\sqrt{\Phi}} \partial_t + i \sqrt{\Phi} \gamma^3 \partial_r + \frac{1}{r} \Sigma^\kappa_{\theta,\phi} - M ) F(x) = 0 . \]  

(93)

Spherical waves are constructed trough substitution \( D^j_{m,\sigma}(\phi, \theta, 0) \equiv D_\sigma \):

\[ \Psi_{ejm}(x) = \frac{e^{-i\epsilon t}}{r} \begin{vmatrix} f_1(r) D_{-1/2}^j \\ f_2(r) D_{1/2}^j \\ f_3(r) D_{-1/2}^j \\ f_4(r) D_{1/2}^j \end{vmatrix} . \]  

(94)

With the use of recursive relations [113]

\[ \partial_\theta D_{-1/2}^j = a D_{-1/2}^j - b D_{1/2}^j, \quad -m - \frac{1}{2} \cos \theta \frac{D_{1/2}^j}{\sin \theta} = -a D_{-1/2}^j - b D_{3/2}^j, \]
\[ \partial_\theta D_{-3/2}^j = b D_{-3/2}^j - a D_{1/2}^j, \quad -m + \frac{1}{2} \cos \theta \frac{D_{1/2}^j}{\sin \theta} = -b D_{-3/2}^j - a D_{3/2}^j, \]

where \( a = (j + 1)/2, \quad b = (1/2) \sqrt{(j - 1/2)(j + 3/2)} \), we get

\[ \Sigma_{\theta,\phi} \Psi_{ejm}(x) = i \nu \frac{e^{-i\epsilon t}}{r} \begin{vmatrix} -f_3(r) D_{-1/2}^j \\ + f_4(r) D_{1/2}^j \\ + f_2(r) D_{-1/2}^j \\ - f_1(r) D_{1/2}^j \end{vmatrix}, \quad \nu = (j + 1/2) . \]  

(95)

19
And further we arrive at radial system
\[
\frac{\epsilon}{\sqrt{\Phi}} f_3 - i\sqrt{\Phi} \frac{d}{dr} f_3 - \frac{\nu}{r} f_4 - M f_1 = 0 , \quad \frac{\epsilon}{\sqrt{\Phi}} f_4 + i\sqrt{\Phi} \frac{d}{dr} f_4 + \frac{\nu}{r} f_3 - M f_2 = 0 , \quad \frac{\epsilon}{\sqrt{\Phi}} f_1 + i\sqrt{\Phi} \frac{d}{dr} f_1 - \frac{\nu}{r} f_2 - M f_3 = 0 , \quad \frac{\epsilon}{\sqrt{\Phi}} f_2 - i\sqrt{\Phi} \frac{d}{dr} f_2 - \frac{\nu}{r} f_1 - M f_4 = 0 .
\] (96)

To simplify the system let us diagonalize \( P \)-operator. In Cartesian basis it is \( \hat{\Pi}_C = i\gamma^0 \otimes \hat{P} \), after translating to spherical tetrad it looks
\[
\hat{\Pi}_{sph.} = \begin{vmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{vmatrix} \otimes \hat{P} .
\] (97)

From the equation \( \hat{\Pi}_{sph.} \Psi_{jm} = \Pi \Psi_{jm} \) it follows \( \Pi = \delta (1)^{j+1}, \delta = \pm 1 \)

\[
f_4 = \delta f_1 , \quad f_3 = \delta f_2 , \quad \Psi(x)_{\epsilon j m \delta} = \frac{e^{-i\epsilon t}}{r} \begin{vmatrix}
f_1(r) & D_{-1/2} \\
f_2(r) & D_{+1/2} \\
\delta f_2(r) & D_{-1/2} \\
\delta f_1(r) & D_{+1/2}
\end{vmatrix} .
\] (98)

Allowing for (98), we simplify the system (96):

\[
(\sqrt{\Phi} \frac{d}{dr} + \frac{\nu}{r}) f + (\frac{\epsilon}{\sqrt{\Phi}} + \delta M) g = 0 , \quad (\sqrt{\Phi} \frac{d}{dr} - \frac{\nu}{r}) g - (\frac{\epsilon}{\sqrt{\Phi}} - \delta M) f = 0 ;
\] (99)

where instead of \( f_1 \) and \( f_2 \) new functions are used
\[
f = \frac{f_1 + f_2}{\sqrt{2}} , \quad g = \frac{f_1 - f_2}{i\sqrt{2}} .
\]

For definiteness, let us consider eqs. (99) at \( \delta = +1 \) (formally to the second case \( \delta = -1 \) there corresponds the change \( M \Longrightarrow -M \)):

\[
(\sqrt{\Phi} \frac{d}{dr} + \frac{\nu}{r}) f + (\frac{\epsilon}{\sqrt{\Phi}} + M) g = 0 , \quad (\sqrt{\Phi} \frac{d}{dr} - \frac{\nu}{r}) g - (\frac{\epsilon}{\sqrt{\Phi}} - M) f = 0 .
\] (100)

Here there arise additional singularities at the points \( \epsilon + \sqrt{\Phi} M = 0, \epsilon - \sqrt{\Phi} M = 0 \). Correspondingly, equation for \( f(r) \) has the form

\[
\frac{d^2}{dr^2} f - \left( \frac{2r}{1 - r^2} - \frac{Mr}{\sqrt{1 - r^2} (\epsilon + M \sqrt{1 - r^2})} \right) \frac{d}{dr} f + \left( \frac{\epsilon^2}{(1 - r^2)^2} - \frac{M^2}{1 - r^2} - \frac{\nu(\nu + 1)}{r^2(1 - r^2)} - \frac{\nu}{(1 - r^2) \sqrt{1 - r^2}} + \frac{M\nu}{\sqrt{1 - r^2} (\epsilon + M \sqrt{1 - r^2})} \right) f = 0 .
\] (101)
However, there exists possibility to move these singularities away through special transformation upon the functions \( f(r), g(r) \) (see [75]). To this end, as a first step, let us introduce a new variable \( r = \sin \rho \), eqs. (100) look simpler

\[
\begin{align*}
\left( \frac{d}{d\rho} + \frac{\nu}{\sin \rho} \right) f + \left( \frac{\epsilon}{\cos \rho} + M \right) g &= 0, \\
\left( \frac{d}{d\rho} - \frac{\nu}{\sin \rho} \right) g - \left( \frac{\epsilon}{\cos \rho} - M \right) f &= 0.
\end{align*}
\]

(102)

Summing and subtracting two last equations, we get

\[
\begin{align*}
\frac{d}{d\rho}(f + g) + \frac{\nu}{\sin \rho}(f - g) - \frac{\epsilon}{\cos \rho}(f - g) + M(f + g) &= 0, \\
\frac{d}{d\rho}(f - g) + \frac{\nu}{\sin \rho}(f + g) + \frac{\epsilon}{\cos \rho}(f + g) - M(f - g) &= 0.
\end{align*}
\]

(103)

Introducing two new functions

\[
f + g = e^{-i\rho/2}(F + G), \quad f - g = e^{+i\rho/2}(F - G),
\]

(104)

one translates \(103\) into

\[
\begin{align*}
\frac{d}{d\rho}e^{-i\rho/2}(F + G) + \frac{\nu}{\sin \rho}e^{+i\rho/2}(F - G) - \frac{\epsilon}{\cos \rho}e^{+i\rho/2}(F - G) + M e^{-i\rho/2}(F + G) &= 0, \\
\frac{d}{d\rho}e^{+i\rho/2}(F - G) + \frac{\nu}{\sin \rho}e^{-i\rho/2}(F + G) + \frac{\epsilon}{\cos \rho}e^{-i\rho/2}(F + G) - M e^{+i\rho/2}(F - G) &= 0,
\end{align*}
\]

or

\[
\begin{align*}
\frac{d}{d\rho}(F + G) - \frac{i}{2}(F + G) + \frac{\nu}{\sin \rho}(\cos \rho + i \sin \rho)(F - G) - \frac{\epsilon}{\cos \rho}(F - G) + M(F + G) &= 0, \\
\frac{d}{d\rho}(F - G) + \frac{i}{2}(F - G) + \frac{\nu}{\sin \rho}(\cos \rho - i \sin \rho)(F + G) + \frac{\epsilon}{\cos \rho}(F + G) - M(F - G) &= 0.
\end{align*}
\]

(105)

Now summing and subtracting two last equations, we get

\[
\begin{align*}
\left( \frac{d}{d\rho} + \nu \frac{\cos \rho}{\sin \rho} - i \epsilon \frac{\sin \rho}{\cos \rho} \right) F + \left( \epsilon + M - i \nu - \frac{i}{2} \right) G &= 0, \\
\left( \frac{d}{d\rho} - \nu \frac{\cos \rho}{\sin \rho} + i \epsilon \frac{\sin \rho}{\cos \rho} \right) G + \left( -\epsilon + M + i \nu - \frac{i}{2} \right) F &= 0.
\end{align*}
\]

(105)

This system can be simplified by substitutions

\[
F(\rho) = \sin^{-\nu} \rho \cos^{-i\epsilon} \rho \varphi(\rho), \quad G(\rho) = \sin^{+\nu} \rho \cos^{-i\epsilon} \rho \Gamma(\rho),
\]

\[
\sin^{-2\nu} \rho \cos^{-2i\epsilon} \rho \frac{d}{d\rho} \varphi + (\epsilon + M - i \nu - \frac{i}{2}) \Gamma = 0, \\
\sin^{+2\nu} \rho \cos^{+2i\epsilon} \rho \frac{d}{d\rho} \Gamma + (-\epsilon + M + i \nu - \frac{i}{2}) \varphi = 0.
\]

(106)
Let us specify a 2-order differential equations for $\varphi$

$$\frac{d^2}{d\rho^2} + (-2\nu \frac{\cos \rho}{\sin \rho} + 2i \epsilon \frac{\sin \rho}{\cos \rho}) \frac{d}{d\rho} + [\epsilon (\epsilon - i\nu)^2 - (M - i/2)^2] \varphi = 0.$$ 

Turning back to the variable $r = \sin \rho$

$$(1 - r^2)\frac{d^2\varphi}{dr^2} + \left[-\frac{2\nu}{r} + (2\nu - 1 + 2i\epsilon) r\right] \frac{d\varphi}{dr} + \left[\epsilon (\epsilon - i\nu)^2 - (M - i/2)^2\right] \varphi = 0,$$

and translating the equation to the variable $z = r^2$ one obtains

$$z(1 - z)\frac{d^2\varphi}{dz^2} + \left[\frac{1}{2} - \nu\right] - (1 - \nu - i\epsilon) z \frac{d\varphi}{dz} - \frac{1}{4}\left[(M - i/2)^2 - (\epsilon - i\nu)^2\right] \varphi = 0. \quad (107)$$

The later is of hypergeometric type with parameters defined by

$$\gamma = \frac{1}{2} - \nu, \quad \alpha + \beta = -\nu - i\epsilon, \quad \alpha\beta = \frac{1}{4}\left[(M - i/2)^2 - (\epsilon - i\nu)^2\right];$$

from whence it follows

$$\alpha = \frac{-\nu - i\epsilon + (iM + 1/2)}{2}, \quad \beta = \frac{-\nu - i\epsilon - (iM + 1/2)}{2}.$$ 

Thus, solution are are given by (remembering, $\nu = j + 1/2$)

$$\alpha = \frac{-j - i(\epsilon + M)}{2}, \quad \beta = \frac{-j - 1 - i(\epsilon - M)}{2},$$

$$\gamma = -j = -1/2, -3/2, ... \quad \varphi(r) = F(\alpha, \beta, \gamma, z = r^2). \quad (108)$$

The main conclusions about penetration of the spin 1/2 particle through de Sitter horizon remain the same: they coincides with results for $S = 0, 1$ fields.

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