Discrete mathematics as a resource for developing scientific activity in the classroom

Ximena Colipan1 · Alvaro Liendo2

Accepted: 6 May 2022 / Published online: 6 June 2022
© FIZ Karlsruhe 2022

Abstract
In this paper we present a theoretical discussion on how problems issued from discrete mathematics can develop the attitudes, skills, and knowledge required for scientific mathematical activity in the classroom. We do so via the research situation for the classroom model by performing mathematical and didactical analyses of a problem issued from discrete geometry, and we present the results of a preliminary experiment conducted with Chilean students between the ages of 11 and 13. Our chosen problem asked students to tile a fixed square with a finite number of smaller squares. From the theoretical analysis and the preliminary experimentation, we conclude that this problem, and problems issued from discrete mathematics in general, can induce genuine mathematical activity in lower-secondary school students. In particular, we conclude that this problem is effective in developing the knowledge, skills and attitudes advocated in the Chilean mathematics curriculum.

Keywords Research situations for the classroom · Discrete mathematics for the classroom · Development of skills and attitudes · Mathematical problem solving

1 Introduction
According to professional mathematicians, mathematical activity essentially consists of solving problems for which we do not know a priori a method with which to do so. This creative process begins by selecting a problem that captivates the mathematician’s interest, followed by concentrating efforts on finding and testing strategies to solve it (Nimier, 1989).

Mathematicians follows many steps to achieve this end. On the one hand, they solve particular cases, similar but simpler problems, change the outlook and create models. They formulate, prove and disprove conjectures and even change the problem itself to obtain another problem that they can solve. On the other hand, there is a social component to mathematical activity where the mathematician looks for previous research regarding similar problems, discussing and eventually collaborating with colleagues who share interest in the problem. All of these actions can be applied in any order. They are repeated as many times as is necessary to satisfy the curiosity of the mathematician, by rendering one of the following outcomes: an answer to the problem, a counterexample to the problem, or the realization that the problem is out of reach at this time (Schoenfeld, 2020).

During the last decades, the school curricula in mathematics worldwide have focused on the acquisition of knowledge and the development of skills and attitudes that allow students to put the actions described above into practice. However, finding problems that engage students in these actions so as to achieve these desired skills and attitudes is rather involved. A common mistake made during the process of finding such problems in school settings is taking an exercise familiar to the students and setting it up differently or disguising it in an artificial context (Grenier & Payan, 2002).

In this paper we show that problems issued from discrete mathematics are appropriate for developing the skills and attitudes required by the school curriculum, even in countries where discrete mathematics is absent as a formal area of study.

Indeed, discrete mathematics is an area in which many notions and concepts are easily accessible to students, ranging from elementary school to university level (Hart & Martin, 2016; Hart & Sandefur, 2018). Moreover, it has interactions with many other areas, which allow the teacher
to introduce notions of discrete mathematics while developing these areas. For instance, it has strong interactions with geometry, algebra, logic, set theory, number theory, combinatorics, among others.

In view of these facts, it seems pertinent to propose problems from discrete mathematics to school-aged students so that, without any prior knowledge of the subject, they can engage in genuine mathematical activity, thus developing the skills and attitudes required in the curriculum in Chile.

2 Discrete mathematics

Discrete mathematics deals with objects that can be defined using a finite set of objects and relations. In recent decades there has been special interest in introducing discrete mathematics into school curricula. In fact, several studies presented in the volume by Rosenstein et al. (1997) advised that discrete mathematics should be taught in K-12 classrooms. In particular, it has been shown that discrete mathematics provides an opportunity to focus on how mathematics is taught, as well as giving teachers new ways of looking at mathematics and new ways of making it accessible to their students. Moreover, Grenier & Payan (1998) showed how discrete structures are often easier for students to grasp than are continuous structures.

A special issue of ZDM in 2004 was devoted to discrete mathematics and proof in the high school and undergraduate curriculum. From this issue we highlight two works, as follows: by DeBellis & Rosenstein (2004), proposing that discrete mathematics should be viewed not only as a collection of new and interesting mathematical topics, but, more importantly, as a vehicle for providing teachers with a new way to think about traditional topics and new strategies for engaging their students in the study of mathematics; and by Goldin (2004), who stated that discrete mathematics serves the twofold purpose of providing students who struggle in mathematics with a clean start and an easily accessible new problem, while simultaneously providing a challenge for more advanced students.

Discrete mathematics not being included in the curriculum, as in the case of Chile where this study was conducted, is not an obstacle since it is not so much the content of discrete mathematics that makes it a vehicle for improving mathematics instruction, but the opportunity that the content offers to engage people in mathematical activity (Devaney, 2018; Lockwood & Reed, 2018). Nevertheless, there is interest in adding discrete mathematics to the curriculum for the following reasons: it is easy to work on proofs in discrete mathematics (Heinze et al., 2004; Rosenstein et al., 1997), since they are well adapted to the study of algorithms and recursivity (Coliman, 2018; Sandefur et al., 2018); and they naturally lead to the production of mathematical models (Coenen et al., 2018).

Attitudinal dynamics have also been studied from the perspective of discrete mathematics in the classroom. In particular, Goldin (2004, 2018) concluded that discrete mathematics is a powerful tool for improving problem-solving attitudes because it develops the affective components of decision-making processes, such as persistence with a problem, self-confidence, willingness to take mathematical risks, frustration, etc.

3 Mathematical practices in the classroom

In recent decades, in several studies on the teaching and learning of mathematics, researchers have investigated teaching methods centered on the student.

For instance, problem-based learning was introduced in Schoenfeld’s (1985) influential work studying the methods that mathematicians use to solve problems. In particular, Schoenfeld identified a certain uniformity in the way mathematicians solve problems, which can be decomposed into processes, strategies and understandings.

Another such method, called inquiry-based learning, is defined loosely as a way of teaching in which students are invited to work in ways similar to those in which mathematicians and scientists work (Artigue & Blomhøj, 2013). One of the main features of this method is ‘authenticity’, meaning that learning happens while solving problems from real life by applying the scientific method.

One last method called discovery learning (Swan, 2006), occurs whenever the learner is not provided with the target information or conceptual understanding and must find it independently and with only the provided materials (Alfieri et al., 2011). Such an approach is not restricted to mathematics but open to all the sciences (Minner et al., 2010).

In all these studies, the authors concluded that classes should be planned so that students engage in a genuine mathematical practice mimicking the activities of mathematicians engaged in research (Maaß & Artigue, 2013). Furthermore, in school settings such student-centered teaching techniques highly resemble the characteristics of mathematical problem solving. In these cases, the learner is an active agent exploring aspects of the subject by using them to address situations as a scientist or mathematician might (Schoenfeld, 2020; Schoenfeld & Kilpatrick, 2008).

There is much research concerning problem solving in the literature, but our focus is the global process of engaging students in genuine mathematical activity where, in particular, all the didactical variables are left to be chosen by the one who solves the problem instead of the one who poses it. Our focus in centered on the resolution
process instead of the solution. Hence, we do not expect the students to find a final answer to the initial problem. Such research has a precedent in the work of the team ‘Math-à-Modeler’ in France. This team has succeeded in implementing these kinds of activities at different school levels. Among the productions of this group we highlight a few results. Grenier et al. (2017) showed that problems surrounding tiling activities with material polyominoes allow for the resolution of particular cases, the emergence of conjectures and even the construction of sketches of proof. Ouvrier-Buffet (2006) applied situations from discrete mathematics in working with students who had no previous knowledge of the subject, in order to induce the production of pertinent definitions during research. Giroud (2011) explored the role played by experimentation in the mathematical activity in the classroom, with the aim of identifying the epistemological and didactical conditions that favor the practice of such experimental approach.

3.1 The skills and knowledge required in problem solving

According to Giroud (2011) and Dias (2014), the skills at work in problem solving can be grouped together in what we call the experimental approach in mathematics. This approach can be described as comprising the following skills.

- Questioning. The initial question is not the only question in scientific activity in mathematics. New questions arise as we try to solve a problem, and such questions lead to new developments. Most commonly in scientific activity in mathematics, the final production concerns a derived question instead of the initial question.
- Observation and experimentation. While researching mathematics we often begin with an observation of a phenomenon that catches our attention. Later, we actively experiment to obtain new information from the question at hand. Experimentation is also commonly used to test the validity of a hypothesis or conjecture.
- Hypothesis and conjecture. A hypothesis is a proposition that we enunciate without regard to its validity, while a conjecture is a proposition that we claim to be true, but for which we do not yet possess a proof.
- Proving. The final step in the experimental approach is unique to mathematics. After confirming the validity of a conjecture via experimentation, we need to produce a proof so that, regardless of the variables being chosen in an experimentation yet to be conducted, the outcome will nevertheless be that our conjecture holds. After this step, a conjecture is now called a theorem.

We understand knowledge to be the mathematical objects, concepts, the inner relations of these objects, and the concepts that are required to understand and solve a problem. In some cases, these objects and relations must be created or discovered in the process of solving the problem.

3.2 The attitudes of problem solving

Moyer et al. (2018) stated that the teaching and learning of mathematics is deeply tied to affective factors such as beliefs, emotions, values and attitudes. There is no generally accepted definition of attitudes, but most definitions include one or more of three components cognitive, affective, and behavioral. Goldin (2009) stated that the affective system is central to mathematical processing, distinguishing between local and global affect. Global affective traits include attitudes toward mathematics in general. On the other hand, local affective aspects include anxiety, satisfaction, enjoyment, and surprise, while coping with a particular mathematical task (Cai & Leikin, 2020). Hannula (2015) defined a system with mathematics-related emotions experienced when actually doing mathematics, including self-esteem and confidence. Collaborative learning has also been studied in the context of affect. Saadati and Reyes (2019) showed that collaborative learning affects attitudes positively while learning mathematics.

The Chilean curriculum promotes a set of attitudes oriented to the affective and social development of the students, including the following: approaching the search for solutions in a flexible and creative way; showing curiosity and interest in solving mathematical challenges; showing effort, perseverance and rigor in the resolution of problems; working responsibly and proactively in teams; and exercising critical thinking. These are the attitudes we consider in this study.

We summarize the practice of mathematical activity in the following diagram (Fig. 1), which shows the interactions between skills, attitudes and knowledge in problem resolution.

4 Mathematical activity in the classroom: the SiRC model

Starting with a well-chosen problem, it is possible to induce genuine mathematical activity in the classroom (Colipan, 2018), that is, to facilitate the scientific practice of mathematics equivalent to that of a professional mathematician. We hypothesize that this allows for the harmonious interaction between the knowledge, skills and attitudes presented above.

In our study we applied the model ‘Research situations for the classroom’ usually called SiRC as an acronym for its
French name ‘Situations de recherche pour la classe’. The main goal of the SiRCs model is to guide students of all levels, from elementary school to undergraduate, in real mathematical practices, giving them the opportunity to develop research in an autonomous way (Grenier & Payan, 2002). This goal can be accomplished by building on elementary mathematical knowledge. The main differences from other methods such as ‘mathematical problem solving’ (Schoenfeld, 1985), ‘situation-problème’ (Douady, 1986), ‘problème ouvert’ (Arsac et al., 1988) and ‘situations adidactiques’ (Brousseau, 1998) are as follows: in the SiRC model there is not necessarily a final answer to the initial problem; the mathematical concepts at play are not predefined and are not restricted a priori; moreover, they are at the service of the problem solver in searching for a solution, since there is no particular knowledge or notion to be taught or exercised during the SiRC, proofs are the only validation method, and the students are left in charge of them.

The SiRC model was first presented by Grenier & Payan (2002) as a research situation obeying the following constraints:

- The research situation must be included in a research problem, and even if the problem has been completely solved, it must be sufficiently related to unsolved problems. Indeed, the hypothesis is made that this close proximity to questions that are unsolved, not only for the students but also for the teacher and the presenter of the situation, is fundamental to the way the students will face the situation.
- The initial question is easy to access. For the question to be easily accessed by the students, the problem must lie outside of over-formalized branches of mathematics. It must be the situation itself that brings the students into the mathematical aspects of the problem.
- Initial problem-solving strategies exist without the need for mathematical knowledge beyond the students’ reach. Such strategies are not required to bring about a complete resolution of the problem, but some particular cases should be easily handled with such strategies. The knowledge needed to approach the situation must be kept elementary and reduced as much as possible, even if more developed techniques may be required to reach a full solution.
- Several strategies for going forward with the research are available, and multiple, possibly conflicting, developments are possible. This multiplicity is required from both the point of view of the activity (new constructions, proofs, computations) and the point of view of the mathematical notions required.
- A solution to a particular case immediately brings about a new question. The problem may be extended without limit by the variables left open for the students. A counterexample does not finish the problem. Instead, it simply changes the question.

The expected learning outcomes in a SiRC are those corresponding to the experimental approach that we group together as a trio, namely, problem, conjecture, and proof. In this way, the transition between each of these groups sets in motion the constituent skills of the scientific activity in mathematics, such as the following:

(a) **Problem-conjecture**

Rephrasing the problem in their own words, asking questions, representing the situation graphically or otherwise, spotting similarities with other known problems, creating extensions and restrictions of the problem, dividing the problem into sub-cases, studying particular cases, finding patterns, formulating hypotheses.

(b) **Conjecture-proof**

Developing argumentation, studying the dependence relations between the objects in the problem, creating pertinent notations, arguing by exhaustion of cases, finding counterexamples, reasoning by induction, reasoning by deduction, arguing by contradiction, contrasting the proofs with the conjectures, creating extensions and restrictions of the problem.

(c) **Proof-problem**

Contrasting a proof with the original problem, arguing by exhaustion of cases, verifying that all cases have been considered, shaping an idea of solution, communicating a result or problem, applying a result to a different problem, communicating convincingly, creating extensions and restrictions of the problem, stating new problems that appear during proofs.
The transitions described above intervene in the research process of a SiRC, and they are some of the tools used in the analysis of the work produced by the students who partook in our preliminary experimentation.

Previous research conducted on the SiRC model showed that situations within the frame of discrete mathematics provide the opportunity to induce real mathematical practices in the students, allowing them to develop research in an autonomous way (Grenier et al., 2017). For instance, Colipan (2016, 2018) turned problems from combinatorial game theory into the SiRC model with the aim of developing problem resolution skills, introducing recursive thinking, and inducing the production of proofs by exhaustion of cases. Grenier et al. (2017) developed SiRCs concerning tilings using polyominoes in order to introduce the notion of existence theorems and develop algorithmic and recursive thinking. Finally, Gernier & Tanguay () created SiRCs using regular polyhedra in space to induce the production of definitions and proofs.

Taking all this into account, we selected a problem from discrete mathematics to be put in the framework of the SiRC model.

4.1 The problem

The guiding problem we selected originated in discrete geometry, an area of discrete mathematics having overlap with Euclidean geometry, which is present in the Chilean curriculum where this experiment was conducted. The problem is as follows:

Given a fixed polytope $P$, find all the positive integers $n$ such that $P$ admits a partition in exactly $n$ polytopes similar to $P$.

Recall that a polytope is a bounded region of a Euclidean space given by the intersection of finitely many half spaces. The problem of studying the partitions of a polytope is ubiquitous in mathematics and was even present on the famous list of problems presented by David Hilbert at the ICM in 1900 (Hilbert, 1901). The problem we analyzed is the restricted form of the above in which the polytopes are squares. Hence, the problem takes the following form:

Find all the positive integers $n$ such that a square admits a tiling with exactly $n$ squares.

The problem is well adapted to be turned into a SiRC. Indeed, it has already been used as such before by Grenier et al. (2017). For instance, an initial question to initiate independent research by the students may be asking if such tiling is possible with 6, 7 or 8 squares. Moreover, other problems may be derived from the original problem, for instance, requiring that all the squares in the tiling must be the same size. The problem also allows for extensions by changing squares to another polygon, or even increasing the dimension by changing squares to cubes.

5 Methodology

To analyze the devolution and validation of the scientific activity induced by the problem, we used a methodology with qualitative focus based in didactic engineering (Artigue, 1988). The registry is based on case studies with internal validation and founded on the confrontation of the analyses a priori and a posteriori.

The descriptive and predictive analysis was completed using a mathematical and didactical analysis of the problem based on the transitions in the process of resolution of a SiRC described in Sect. 4. We later performed a preliminary experiment whose objective was to determine the attitudes, skills and knowledge induced during practice. In the a posteriori analysis, we considered the resolution strategies implemented by the students, the results found (conjectures, counterexamples, methods and proofs) and the attitudes induced during the resolution process, contrasting this with the a priori analysis.

5.1 Participants

The preliminary experiment was conducted in Chile, where the school curriculum in mathematics since 2012, following the worldwide trend, is mainly focused on the development of skills and attitudes. The Chilean curriculum includes a topic on tiling for students between 11 and 13 years old. Hence, we conducted the study with students of these ages.

The experiment was conducted with 20 students attending a voluntary extracurricular activity event intended for students who are motivated to work on challenging problems in their school. No extra credit was awarded for this activity. The total duration of the experiment was four hours divided in four 1-h sessions, taking place between August and September 2020. The participants were randomly divided into six groups with three students, and one group with two students. Our experimental work began after the inception of the Covid-19 pandemic, so we were forced to conduct the experiment online using the platform Zoom.

5.2 Data collection

In coherence with our objectives, we applied ethnographic methods including interviews, and data analysis based on the students’ productions. We developed 4 observations of 60 min that were conducted by three ‘observers’ taking field notes. The platform Zoom allowed us to join all groups virtually in
order to register their interactions and to verify that the task was fully understood by the students. As a criterion for methodological rigor, observers were instructed to provide only neutral answers without offering any hint towards the resolution of the problem.

We conducted the analysis of the data using an inductive process involving the productions of the students, which included screen captures during the experiment, pictures sent via WhatsApp, audio recorded during the sessions, and text generated in a PowerPoint presentation during the last session.

One of the observers was the regular teacher for this group of students. The teacher had experience with problems presented with the SiRC model from a previous activity. The other two observers were the authors of this paper.

### 5.3 Protocols and scripts for data collection

The problem was introduced to the students through the discussion of two sub-problems that worked as examples for the main problem. This introduction was presented both orally and using an electronic whiteboard. The sub-problems presented to the students were the sub-cases with $n = 4$, and $n = 6$, and were phrased as follows:

- Is it possible to tile a square with four squares?
- Is it possible to tile a square with six squares?

The solution for the case where $n = 4$ was quickly evident to every group. Nevertheless, the representations of the problem for $n = 6$ took longer to appear. One of the first questions the students asked the observers is whether the squares of the tiling must be of equal size. Before the experiment, the observers were instructed to answer this question by saying that this is a hypothesis to be considered, so both cases can be studied. Once the solutions for $n = 6$ were presented by the students, the general problem was introduced to them in the following form: for which values of $n$ (with $n$ a positive integer) is it possible to tile a square with $n$ squares.

### 5.4 Data analysis

For the analysis of the data, we employed theoretical categories (Ballesteros Velázquez & Encina, 2018), specifically, elements coming from the SiRC model with emphasis in discrete mathematics. In our analysis, we identified the singularities in each learning trajectory, and, at the same time, we extracted the common patterns that were of a nature that was divergent from the explanation given by the students.

### 6 Analysis a priori

The solution of the problem is that a tiling with exactly $n$ squares can be found for all positive integers except for $n = 2$, $n = 3$ and $n = 5$. We present here a proof of the existence of such tilings. The cases where the tiling does not exist can be proven with simple geometric arguments. We omit this proof due to lack of space.

#### 6.1 Study of the cases where the tiling exists

We construct the required tiling in the cases where $n$ is different from 2, 3 and 5. We begin the analysis with three remarks:

1. We first remark that for all positive integers $n$ that are perfect squares $n = k^2$ with $k \geq 1$ there is an easy solution with tiles of equal size. The following figure shows the case $n = 16$ (Fig. 2).

2. Moreover, for any positive even integer $n = 2k$ with $k \geq 2$ there is a solution obtained by replacing $(k - 1)^2$ squares in the solution with $k^2$ squares above by a single square. The following figure shows the case $n = 8$ (Fig. 3).

This indeed gives a solution for all $n = 2k$ with $k \geq 2$ since
\[ k^2 - (k - 1)^2 + 1 = k^2 - k^2 + 2k - 1 + 1 = 2k \]

3. Finally, whenever we have a solution with \( n \) squares, we can obtain a solution with \( n + 3 \) squares by replacing one of the squares with four squares having half the side of the original square. The following figure shows the case \( n = 4 \) (Fig. 4).

These three remarks allow us to obtain solutions for all integers with the exception of \( n = 2, n = 3 \) and \( n = 5 \). Indeed, according to the second remark, we have a solution for all \( n = 2k \) with \( k \geq 2 \). By means of remark 3, we have a solution for all \( n = 2k + 3 \) with \( k \geq 2 \). This gives solutions for all positive integers except for \( n = 2, n = 3 \) and \( n = 5 \).

### 6.2 Mathematical notions and knowledge at play

While regarding this problem as a SiRC, several mathematical notions come into play. It must be stressed that these notions are not considered learning goals during a SiRC. The mathematical notions that we identify in this problem are as follows: geometric properties, such as area, characterization of rectangles and squares, tilings, etc.; and arithmetic operation in the integers, such as addition, multiplication, congruence modulo 3 and parity.

These notions will or will not appear during the experiment depending on the knowledge available to the students and the resolution processes that they attempt. Moreover, depending on the resolution strategy taken by the students, the problem allows for the developing of different types of reasoning, such as inductive, algorithmic and geometric reasoning, as shown in the types of resolution of the problem given above.

### 6.3 Skills of scientific activity in mathematics

The analysis of the skills that the students manifested was developed in three stages according to the transitions between problem, conjecture and proof introduced in Sect. 4.

![Fig. 4 Tiling a square with \( n + 3 \) squares of unequal sizes](image)

### 6.3.1 Problem-conjecture

In the first approach, the students perform a random exploration of the problem so as to solve it for values of \( n \) chosen without any particular aim. In this stage, which is called random experimentation, the students identify certain features of the problem. For instance, they may realize that it is possible to produce a tiling with \( n = 4 \) using only squares of the same size, or that for \( n = 6 \) there is a tiling, but it seems impossible to find a tiling with squares of the same size. The students may also realize at this stage that it seems impossible to find a solution for \( n \) equal to 2, 3, and 5.

In a second stage, called inductive experimentation, the students choose particular values of \( n \) with the aim of finding conjectures. For instance, they may choose to solve for \( n = 9 \) and \( n = 25 \) and conjecture that a tiling with all the squares of the same size is possible if \( n \) is a perfect square. They may also realize that they can further subdivide one square in a solution for \( n = 4 \) to reach a solution for \( n = 7 \).

Finally, solving the problem for several particular cases can lead to conjectures for a general case, for instance solving for all \( n \) greater that 6 and smaller than 12 may be enough to solve all the affirmative cases.

### 6.3.2 Conjecture-proof

For small values of \( n \), a conjecture can be proved with the help of a drawing. for general values of \( n \), a formal written proof like the one given above may be out of reach for the students. Nevertheless, the students may explain how a solution is constructed. This would push them into the social component of mathematical activity and engage them in algorithmic and inductive reasoning. Finally, even if the problem is discrete in nature, it requires certain notions of geometric reasoning in order to prove that tilings do not exist when \( n \) is 2, 3 or 5.

### 6.3.3 Proof-problem

While solving randomly for different values of \( n \), each solution naturally leads to the problem of solving for a new value of \( n \). Moreover, whenever an algorithm to construct tilings is found, this naturally leads to the question of which values of \( n \) can be achieved using this algorithm. Finally, even after solving the proposed problem completely, new generalizations of the problem arise, such as exchanging the square for a different polygon or increasing the dimension to consider a cube or another polytope.
6.4 The importance of a problem issued from discrete mathematics

The discrete nature of the problem provides fertile ground for the spontaneous appearance of inductive and algorithmic reasoning. Indeed, there is a quite natural operation of replacing a square in a tiling by four squares with half the side of the original one to obtain a new tiling with 3 more tiles. This operation is intrinsically algorithmic, and the reasoning used with this algorithm is inductive. This approach combined with informal induction and particular solutions for \( n = 1, 6 \) and 8 provides a proof of the existence of the tilings for all \( n \) different from 2, 3 and 5. We remark that a similar approach can be applied to the generalized problem where we replace the square by a triangle. In this case a triangle can also be divided into four similar triangles by means of all three midesgments.

Another notion that arises naturally from the discrete nature of the problem is the exhaustion of cases since it is possible to solve the problem for small values of \( n \) without much complication.

On the other hand, Goldin (2004, 2018) gave a list of interesting features shared by problems from discrete mathematics other than their discrete nature, and our problem satisfies all of them. Indeed, the problem can be introduced directly from a natural context, such as tiling the floor of a room, without the need for any mathematical notions. The problem admits some easy particular cases such as tilings with 1, 4 or 9 squares. As our mathematical analysis shows, interesting mathematical notions arise when we attempt to solve the problem. Finally, the problem can be explored and solved even with only the basic notions of mathematics that are learned during the early stages of education.

6.5 The problem and the Chilean Curriculum

The Chilean mathematics curriculum focuses on knowledge, skills, and attitudes. The knowledge corresponding to discrete mathematics is mostly absent from the Chilean curriculum with the sole exceptions of certain notions that intersect with probability theory such as permutations and combinations in 8th grade, and algorithms in the last years of secondary education (MINEDUC, 2015).

Nevertheless, this absence can be used as an advantage in the development of a SIRC since it allows one to avoid considering the mathematical notions required by discrete mathematics as learning goals (DeBellis & Rosenstein, 2004). The knowledge at play in the particular problem described above is accessible to students in Chile starting from 6th grade.

The skills to be developed in the Chilean school curriculum in mathematics are problem-solving, transitioning between different levels of representation, mathematical modeling, and argumentation and communication. All these skills are supported by the problem that we have chosen.

The school curriculum in Chile promotes six attitudes that must be developed transversally with the outlined knowledge and skills (MINEDUC, 2015). Our problem addresses all of them as we explain now.

1. Approaching the quest for solutions in a flexible and creative way.
2. Showing interest and curiosity towards solving problems, with confidence in their own capabilities even when a solution cannot be found quickly.
3. Showing perseverance and effort in the quest for solutions.
4. Working in teams proactively and responsibly.
5. Evidencing critical thinking when assessing the evidence found in the resolution of a problem.
6. Using communication technologies responsibly and effectively and giving credit for other people’s work when appropriate.

Our hypothesis is that the first, second, third and fifth attitudes in this list appear naturally when solving our problem due to its discrete nature. The fourth attitude in this list is induced by having the students work in teams when attempting to find a solution. Finally, the sixth attitude in this list arises naturally given the online context in which our experiment was conducted, due to the sanitary emergency caused by the Covid-19 pandemic.

7 Experimental results

7.1 Skills developed in the triple problem-conjecture-proof

7.1.1 Problem-conjecture

All of the groups could state conjectures, but not all of them were correct. The research strategies applied by the students were not the same in all groups. Some of them remained focused on the study of particular cases while others progressed to generalizations.

We now present an overview of the questioning, the research strategies and the conjectures that emerged in the groups during the sessions.

(a) Study of tilings with tiles of the same size.

Groups B, C and G studied which tilings can be obtained with tiles of the same size. We describe now the strategy followed by group B. During the first session, group B studied the number of squares that appear
in increasingly large squares as in the following picture (Fig. 5).

This strategy arose naturally with the aid of the pattern in the paper they had available by increasing in one unit the size of the square and then multiplying the length and width to determine the total number of tiles.

In the second session, the group was asked to explain what they had found. Below is an English translation of the partial script.

O: Can you explain what you have done?
B1: We made squares and then we marked the squares that were inside.
B2: We did that all the way up to 13!
O: And what did you find?
B2: We were increasing the length and width and the number of squares inside also increased.
B3: That is \(x\times x\), \(x^2\), that is how we found the \(x^2\).
O: And what does that mean? Sorry, I didn’t understand.
B2: What B3 is saying is that we realized that multiplying width and length of the square, the result gives the total number of squares of size 1.
O: ahhh, ok! and so B3, what numbers did you find?
B3: This gives us 1, 4 9, and so on until 169 that is 13 times 13.
B1: The pattern is \(x^2\), when the side is \(x\)… the power of 2.
O: Great, could you please write what you have just told me in a sentence? Without drawing the examples so that all the groups can understand the result you found.

B2: Only with words? Uffff, that is hard, it is always hard to do that.
O: You can do it! You just explained it to me!
B1: mmmm, yes. Ok.

The conjecture that group B produced is the following: When dividing a square with side \(x\) into squares of unit 1 and increasing the length and width of the square \(x\) in one unit, we can find all of \(x^2\):

\[
\begin{align*}
\text{Resultado n° 1:} & \\
\text{Al Dividir un cuadrado de lado } x \text{ en cuadrados de unidad 1 e ir aumentando en una unidad el ancho y el alto del cuadrado } x, & \text{podemos encontrar todos los } x^2
\end{align*}
\]

(b) Study of tilings with squares of different sizes.

This kind of tiling was studied by all the groups in different levels of depth. All the groups were able to state conjectures regarding their findings. We concentrate our presentation of results in the findings of group D.

During the sessions, group D found tilings for all natural numbers smaller or equal to 10 by trial and error. In particular, they found tilings for \(n = 6\) and \(n = 7\) which allowed them to state two conjectures that we show here. Below is an English translation of selected sections of the script.

First session:
O: Very good! You already have 6 and 8! How did you find them?
D1: I had for 6 and for 8 we need two more. I enlarged the square I had for 6 by one above and one to the side and the big square remains, but it is larger (he shows his notebook) … It is the same as for 6 but bigger. [See Fig. 6 containing their drawings for \( n = 6 \) and \( n = 8 \) for better understanding of the students’ argument].
O: Could you find which numbers you obtain with this kind of tilings?
D3: the next one is 11… ah, no! it is 10… wait… let’s see…
O: You tell me if you find anything!
Second session:
D1: (D1 sends pictures via WhatsApp since he has bad internet signal. D1 can hear us but we don’t hear what D1 says).
O: Can you tell me what have you found?
D2: We did it before! I almost spent all the pages in my notebook (laughs).
D3: They are the even numbers!
O: It seems so!
D2: Yes, they are!
O: But 2 is also an even number and it is not there.
D3: 2 can’t be done since you find rectangles.
O: So, what numbers can you form with this kind of tilings?
D3: ah… all even without 2.

O: Could you please write a sentence describing the result you found?
D2: Like a formula?
O: As you wish, but we need that all the groups understand what you found by reading this sentence.
The conjecture that group D stated is the following:
Adding a square in the upper row and a another in the right column the number of squares that appear in each figure, are an even number. (Agregando un cuadrado en la fila superior y un en la columna derecha la cantidad de cuadrado que aparecen en cada figura, son un número par.)

While D2 and D3 were writing the sentence demanded by the Observer, D1 is looking for a tiling for the odd numbers. The following partial script happened in the minute 38 of the second session.
D2: D1 called me. He found for 7 (D2 shows a tiling for 7 in the screen).
O: Oh, great. What numbers can you find if you keep tiling this way?
D3: Making crosses? Ah, again like a pattern!
O: Yes.
D3: Like this? D3 shows the notebook… mmm… we find 10!
D1 writes via WhatsApp: I already did it. I send pictures.
O: Look, D1 found that it gives 7, 10, 13, 16.
D1 writes via WhatsApp: it gives odd and even numbers.
O: Ok, I let you research to see what you find with this configuration.

Before the third session, the group spontaneously sent the statement of their conjecture and images of the results that they found in their second line of research.

Third session:
O: You worked a lot. What do you have to tell me?
D2: Well, we did the squares in the squares and we found 4, 7, 10, 13, 16, 19, 22, 25… we did it until 37.
O: Great! And is there any relation among those numbers?
D1: I found the pattern 3 k-2.
O: So, you found a pattern. Very well… (D2 interrupts with excitement).
D2: Because you were going to ask us for the pattern!
O: (Laughs) Yes, that is true. And can you tell me how did you find it? D2, can you explain?
D2: I did the x + 3 first, but it only gives the 4, then gives 5 and that is not right… and D1 found it and it worked.
D1: I thought and said that there is a 3 k not a x + 3 alone since it doesn’t work… But there if it is 1 it gives 3 and that one has no tiling.
O: What is 1?
D1: the k in 3 k and then I saw that with 2 it gives 6 and if I remove 2 it gives 4. That is how I tried 3 k-2 and it worked for all the numbers. Phew, I thought a lot!
D2: And I verified and it is right.
O: Great! It looks as if you found another result.

D3: We are too smart! (laughs).
O: Ok, write what you found so you can show it to the other groups.
The conjecture that group D stated is the following (Fig. 7):

A square with side $n$ by $n$ can be tilled with $3x - 2$ squares.” (Un cuadrado de lado $n$ por $n$ se puede embaldosar con $3x - 2$ cuadrados.)

7.1.2 Conjecture-proof

For small values of $n$, the method of proof used by the students was to draw the tilings making sure that the geometric properties of the representation in the drawings were correct, for instance, by marking the size of the different squares in

![Fig. 7 Further investigation by group D](image1)

![Fig. 8 Erroneous solutions](image2)
the drawing. The observers would often inquire about the correctness of the geometric properties of all the drawings (regardless if they were correct or not). For instance, in the cases $n = 6$ in the following figures (Fig. 8).

These solutions were quickly invalidated by the members of the group after an intervention by an observer asking how they can be sure that all the supposed squares involved are indeed squares. The method of proof by exhaustion of cases was also applied by several groups; most notably Group F provided solutions for all values of $n \leq 30$ except for $n = 2, 3$ and 5 (Fig. 9).

The students that participated in this study had not learned proof by induction yet in their curriculum. Hence, the production of formal proofs by induction, of their conjectures, was out of their reach. Instead, they provided intuitive proofs by inductive reasoning in the form of a generic example, that is, an example that is representative enough of the general case so as to allow anyone that understands such an example to give a solution in any other case (Colipan, 2018). For instance, the following is the idea of proof given by Group D for their conjecture stated in the third session (Fig. 10).

The students’ validation of these conjectures by recursive reasoning seems to validate the hypothesis of Stylianides et al. (2016) that problems without a clear statement of what needs to be proven may lead the students to produce proofs by induction, by intuitively applying a case-by-case exploration to find the induction step of the proof.

Proof-problem.

At the beginning of the experiment, the students were asked to find tilings for $n = 4$ and $n = 6$. This naturally led the groups to divide the problem in two: the values of $n$ for which tilings exists with tiles of equal size and with tiles of different sizes. Furthermore, as shown in the partial scripts reproduced above, some groups worked to describe algebraically the different values of $n$ that could be obtained with three different tiling algorithms. In this last case, this work was induced by the observers.

On the other hand, once an algorithm was obtained for all $n > 2$ even, the problem of tilings with odd numbers arose naturally in the groups. Nevertheless, the conjecture that gives tilings for all integers $n = 3k + 1$ with $k \geq 0$ did not lead the students to investigate whether similar algorithms lead to solutions with $n = 3k + 2$ and $n = 3k + 3$.

In the final presentation of the findings where all participants were together in the same virtual room, they were asked by the observers about further lines of research in the form of the question: If you want to continue research on the same lines as this problem, what would you research? The answer given spontaneously by a student and expanded upon by the whole group was to change the squares into other regular polygons such as tiling a triangle with triangles and tiling a hexagon with hexagons.

The problem was not completely solved by the groups. Moreover, when the groups found an algorithm to produce solutions for an infinite family of numbers, they often would study this algorithm for the remainder of the duration of the experiment without returning to the original problem. Indeed, once an algorithm was found, most groups would consume all the remaining time of the experiment looking for a formula for the cases covered by the algorithm. We hypothesize that finding a full solution for a particular case was a strong motivation for the students.

### 7.2 Mathematical notions and attitudes that emerged in the experiment

One of the mathematical notions that emerged during the experiment was the notion of parity, and more generally, the notion of congruence modulo an integer, namely 2 and
3 in our case. Also, the notion of a perfect square in the integers appeared when dealing with tilings with squares of the same size. Another notion that emerged naturally was that of restriction when students found an algorithm that could provide a tiling with any even positive integer different from 2.

The notion of generalization from particular cases to the general case arose in the work of the students by the construction of algorithms that iterate a certain procedure to produce new tilings for other integers. Of course, such constructions also evidence the appearance of the notions of a recurrence relation since a new solution is constructed from a previous one.

The construction of new solutions led to algebraic formulas for the number of tiles in a configuration. For instance, Group D arrived at the formula $3x - 2$ as discussed above. This suggests that problems such as ours that engage recursive thinking can induce the development of algebraic thinking. This result is in agreement with the findings of previous research. For instance, our students show evidence of the triple passage from geometrical to numerical to algebraic notions studied by Rivera and Rossi Becker (2008), who found the same behavior for a simpler geometrical problem, which incidentally resulted in the same algebraic formula as in our case. Furthermore, producing algebraic results for a seemingly non-algebraic problem has strong implications for the mathematical empowerment of the students as we can see from the last partial script from Group D. This result agrees with the final conclusion in the work by Amit and Neria (2008). Finally, Carraher et al. (2008) provided results of a study of the process that occurred naturally with our students, going from the geometric situation to patterns to algebra, and Yeap and Kaur (2008) studied the intermediate step that arises when going from simple particular cases to a global result.

On the geometrical side, the properties of a square and its difference from other rectangles were evidenced by the students providing tilings with non-square rectangles that were invalidated by their teammates. This was also evidenced in the attempts at proofs that tilings with 2 and 3 squares do not exist.

Regarding the attitudes, the students were divided into groups of at most three students, in line with the fourth attitude promoted by the Chilean curriculum. This choice of the group size was made considering previous work that indicated that this social organization encourages the students to verbalize their ideas (Pruner & Liljedahl, 2021; Saadati & Reyes, 2019). Such verbalization has a positive influence in the representation of the problem, the exchange of ideas, the debate of solutions, the construction of methods, and the statement of conjectures; all this mimicks the process in the mathematical activity in which researchers submit their findings to their colleagues in their research team. This is a motivating factor for research, thus avoiding lack of motivation.

We could verify that the choice of the size of the groups was appropriate even if the experiment was conducted by videoconference.

Concerning the attitudes in the Chilean curriculum, they emerged spontaneously, as expected, given the discrete nature of the problem, and the way in which it was presented by the SiRC.

The first attitude, flexibility and creativity in the quest for solutions, the second one, showing interest and curiosity, and the third one, perseverance and effort, appeared naturally. In particular, the motivation to solve the problem arose quickly in the students without the need for intervention by the observers. Moreover, the students showed flexibility, creativity and curiosity by dividing the problem into sub-problems, by formulating conjectures, and by finding algorithms.

The fourth attitude, working in teams, also appeared naturally due to the setup of the experiment. The fifth attitude, critical thinking, was also evidenced in teamwork while students were validating and invalidating their own conjectures.

Finally, concerning the sixth attitude in the Chilean curriculum, it was originally not considered in the design of the experiment, but after the inception of the COVID-19 pandemic, this experiment, like the entire Chilean educational system, moved into an online setting. Hence, the attitude of using communication technologies responsibly and effectively ended up being fundamental to our work.

## 8 Conclusions

Our main conclusion in this mostly theoretical research is that genuine mathematical activity is possible by applying the SiRC model using problems from discrete mathematics. What is more, the attitudes and skills included in the Chilean mathematical curriculum appear spontaneously in the students while participating in a SiRC. The use of problems from discrete mathematics in the SiRC model makes the mathematical thinking appear naturally, bringing into play not only skills, but also the creativity of the students, as they seek and provide diverse solutions to the problem being considered.

The problem we considered from discrete mathematics in the SiRC model highlights even further the key idea that the most important part of the mathematical process is likely the quest for solutions rather than the answer itself. As often happens in the work of a professional mathematician, the satisfaction with the scientific activity comes from finding a solution, any solution, not necessarily one for the original problem. Our problem in the SiRC model reproduces this feature in the life of a mathematician, where, starting with a problem, the research process itself brings the seeker to a
different problem, possibly in another area of mathematics that may lie outside their domain of comfort.

In this paper we presented the SiRC model, and we used it to analyze the problem of tiling squares with smaller squares. We presented a detailed mathematical and didactically analysis of this problem. We conclude from this theoretical analysis that the problem is easy to access with only basic mathematical knowledge, even if more advanced knowledge may also be applied. Moreover, several strategies exist to help students advance in the resolution of the problem.

Despite the theoretical nature of the paper, we presented the results of a preliminary experiment with a reduced group of Chilean students. A few groups in the experiment, after finding a particular algorithm to produce tiling of a certain shape, remained in this line of research studying the tilings that can be obtained by applying this algorithm. This feature commonly appears in the practice of professional mathematics, where, starting from a broad problem, certain subcases captivate one’s attention, becoming the sole focus of the research, thus causing one to forget the initial problem completely.

Another finding we extract from this study is that the problem is indeed attractive to students, and this causes them to show evidence of the attitudes and skills without further need for external intervention.

This research is part of an ongoing 3-year project. We intend to conduct further experimentation with similar problems now that the students in Chile have returned to the classrooms.

Acknowledgements We would like to thank the three anonymous referees and the editors for their extremely valuable suggestions for improving this manuscript. This work was partially funded by the National Agency for Research and Development (ANID) via FOND-ECYT Iniciación Grant number 11190254 where the first named author is the principal investigator.

References

Alfieri, L., Brooks, P., Aldrich, N., & Tenenbaum, H. (2011). Does discovery-based instruction enhance learning? Journal of Educational Psychology, 103, 1–18.

Amit, M., & Neria, D. (2008). “Rising to the challenge”: Using generalization in pattern problems to unearth the algebraic skills of talented pre-algebra students. ZDM – the International Journal of Mathematics Education, 40(1), 111–129.

Arsac, G., Germain, G., & Mante, M. (1988). Problème ouvert et situation-problème. IREM de Lyon.

Artigue, M. (1988). Ingénierie didactique. Recherches En Didactique Des Mathématiques, 9(3), 281–308.

Artigue, M., & Blomhøj, M. (2013). Conceptualizing inquiry-based education in mathematics. ZDM – the International Journal of Mathematics Education, 45(6), 797–810.

Ballesteros Velázquez, B., & Encina, J. M. (2018). Reflexión crítica de la investigación cualitativa desde la perspectiva de los estudiantes. Campo Abierto Revista De Educación, 37(1), 51–64.

Brousseau, G. (1998). Théorie des situations didactiques. La pensée sauvage éditions.

Cai, J., & Leikin, R. (2020). Affect in mathematical problem posing: Conceptualization, advances, and future directions for research. Educational Studies in Mathematics, 105, 287–301.

Carradori, D. W., Martinez, M. V., & Schliemann, A. D. (2008). Early algebra and mathematical generalization. ZDM – the International Journal of Mathematics Education, 40(1), 3–22.

Cartier, L. (2008). A propos du théorème d’Euler et des parcours élévi- riens dans les graphes. Petit x, 76, 27–53.

Coenen, T., Hof, E., & Verhoeef, N. (2018). Combinatorial reasoning to solve problems. In E. Hart & J. Sandefur (Eds.), Teaching and learning discrete mathematics worldwide: Curriculum and research (pp. 69–79). Springer.

Colipan, X. (2016). Desarrollo de la actividad científica en clases a través del estudio de juegos combinatorios, el ejemplo del juego del chocolate. Boletín De Educação Matemática, 30(55), 691–712.

Colipan, X. (2018). Mathematical research in the classroom via combinatorial games. In E. Hart & J. Sandefur (Eds.), Teaching and learning discrete mathematics worldwide: Curriculum and research. ICME-13 Monographs (pp. 215–228). Springer.

DeBellis, V., & Rosenstein, J. (2004). Discrete mathematics in primary and secondary schools in the United States. ZDM Mathematics Education, 36(2), 46–55.

Devaney, R. (2018). Discrete dynamical systems: A pathway for students to become enchanted with mathematics. Teaching and learning discrete mathematics worldwide: Curriculum and research (pp. 137–144). Springer.

Dias, T. (2014). Des mathématiques expérimentales pour révéler le potentiel de tous les élèves. La Nouvelle Revue De L’adaptation Et De La Scolarisation, 65, 151–161.

Douady, R. (1986). Jeux de cadres et dialecte outil-objet. Recherches En Didactique Des Mathématiques, 7(2), 5–31.

Giroud, N. (2011). Etude de la démarche expérimentale dans les situ- ations de recherche pour la classe. [Thèse de doctorat en Mathématiques, Université de Grenoble | (NNT : 2011GRENM056).

Goldin, G. (2004). Problem solving heuristics, affect, and discrete mathematics. ZDM Mathematics Education, 36(2), 56–60.

Goldin, G. A. (2009). The affective domain and students’ mathe- matical inventiveness. Creativity in mathematics and the education of gifted students (pp. 181–194). Brill Sense.

Goldin, G. (2018). Discrete mathematics and the affective dimension of mathematical learning and engagement. In E. Hart & J. Sandefur (Eds.), Teaching and learning discrete mathematics worldwide: Curriculum and research. ICME-13 Monographs (pp. 53–65). Springer.

Greiner, D., & Payan, C. (1998). Spécificités de la preuve et de la modélisation en Mathématiques Discrètes. Recherches En Didac- tique Des Mathématiques, 18(2), 59–100.

Greiner, D., & Payan, C. (2002). Situations de recherche en classe: essai de caractérisation et proposition de modélisation. Paris. Actes du séminaire national de didactique de mathématiques (pp. 189–205). IREM de Paris 7 et ARDM.

Greiner, D., Barche, R., Barbe, H., Belfara, E., Bicaïs, Y., Charlot, G., Decauwert, M., Deraux, M., Gezer, T., Meilhan, J., & Mouton, F. (2017). Situations de recherche pour la classe, pour le collège et le lycée... et au-delà. Expérimenter, conjecturer et raisonner en Mathématiques. IREM de Grenoble.

Greiner, D., & Tanguay, D. (2008). L’angle dièdre, notion incontournable dans les constructions pratiques et théoriques des polyèdres réguliers. Petit x, 78, 26–52.

Greiner, D., & Tanguay, D. (2010). Experimentation and proof in a solid geometry teaching situation. For the Learning of Mathematics, 30(3), 36–42.
Hannula, M. S. (2015). Emotions in problem solving. Selected regular lectures from the 12th international congress on mathematical education (pp. 269–288). Springer.

Hart, E., & Martin, W. (2016). Discrete mathematics is essential mathematics in a 21st century school curriculum. In 13th International congress on mathematical education. Springer.

Hart, E., & Sandefur, J. (Eds.). (2018). Teaching and learning discrete mathematics worldwide: Curriculum and research. ICME-13 Monographs. Springer.

Heinze, A., Anderson, I., & Reiss, K. (2004). Discrete mathematics and proof in the high school. ZDM Mathematics Education, 36(2), 44–45.

Hilbert, D. (1901). Mathematical problems. Bulletin of the American Mathematical Society, 8, 437–479.

Kortenkamp, U. (2004). Experimental mathematics and proofs in the classroom. ZDM Mathematics Education, 36(2), 61–66.

Lockwood, E., & Reed, Z. (2018). Reinforcing mathematical concepts and developing mathematical practices through combinatorial activity. Teaching and learning discrete mathematics worldwide: Curriculum and research (pp. 93–110). Springer.

Maaß, K., & Artigue, M. (2013). Implementation of inquiry-based learning in day-to-day teaching: A synthesis. ZDM – the International Journal on Mathematics Education, 45(6), 779–795.

MINEDUC. (2015). Bases Curriculares Matemática; Resource document. Ministerio de Educación República de Chile. Retrieved from: https://www.curriculumnacional.cl/614/articulos-37136_bases.pdf. Accessed 10 Aug 2020

Moyer, J. C., Robison, V., & Cai, J. (2018). Attitudes of high-school students taught using traditional and reform mathematics curricula in middle school: A retrospective analysis. Educational Studies in Mathematics, 98(2), 115–134.

Nimier, J. (1989). Entretiens avec des mathématiciens, Lyon: IREM de Lyon.

Ouvrier-Buffet, C. (2006). Exploring mathematical definition construction processes. Educational Studies in Mathematics, 63, 259–282.

Ouvrier-Buffet, C. (2015). Quelles sont les conceptions d’élèves, d’enseignants, de mathématiciens contemporains sur la définition ? Qu’en est-il de l’activité de définition ? Vers un modèle de l’activité de définition en mathématiques. Repères IREM, No, 100, 5–24.

Pruner, M., & Liljedahl, P. (2021). Collaborative problem solving in a choice-affluent environment. ZDM Mathematics Education, 53(4), 753–770.

Rivera, F. D., & Rossi Becker, J. (2008). Middle school children’s cognitive perceptions of constructive and deconstructive generalizations involving linear figural patterns. ZDM – the International Journal on Mathematics Education, 40(1), 65–82.

Rosenstein, J., Franzblau, D., & Roberts, F. (Eds.). (1997). Discrete mathematics in the schools. DIMACS series in discrete mathematics and theoretical computer science (Vol. 36). American Mathematical Society and NCTM.

Saadati, F., & Reyes, C. (2019). Collaborative learning to improve problem-solving skills: A relation affecting the attitude toward mathematics. In P. Felmer, P. Liljedahl, & B. Knichu (Eds.), Problem solving in mathematics instruction and teacher professional development: Research in Mathematics Education. Springer.

Sandefur, J., Somers, K., & Dance, R. (2018). How recursion supports algebraic understanding. Teaching and learning discrete mathematics worldwide: Curriculum and research (pp. 145–162). Springer.

Schoenfeld, A. (1985). Mathematical problem solving. Academic Press.

Schoenfeld, A. (2020). Mathematical practices, in theory and practice. ZDM Mathematics Education, 52(6), 1163–1175.

Schoenfeld, A., & Kilpatrick, J. (2008). Toward a theory of proficiency in teaching mathematics. International handbook of mathematics teacher education (Vol. 2, pp. 321–354). Brill Sense.

Stylianides, G. J., Sandefur, J., & Watson, A. (2016). Conditions for proving by mathematical induction to be explanatory. The Journal of Mathematical Behavior, 43, 20–34.

Swan, M. (2006). Collaborative learning in mathematics: A challenge to our beliefs and practices. National Research and Development Centre for Adult Literacy and Numeracy National Institute of Adult Continuing Education.

Yeap, B. H., & Kaur, B. (2008). Elementary school students engaging in making generalisation: A glimpse from a Singapore classroom. ZDM – the International Journal on Mathematics Education, 40(1), 55–64.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.