On the density at integer points of a system comprising an inhomogeneous quadratic form and a linear form

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Abstract
We prove an analogue of the Oppenheim conjecture for a system comprising an inhomogeneous quadratic form and a linear form in 3 variables using dynamics on the space of affine lattices.

1 Introduction

In this paper, we study the values taken at integer points for a pair consisting of an inhomogeneous quadratic form and a linear form in 3 variables. Let $Q$ be a nondegenerate indefinite quadratic form on $\mathbb{R}^n$. We say that $Q$ is irrational if $Q$ is not proportional to a quadratic form with integer coefficients. It is a famous theorem of Margulis [20] resolving an old conjecture of Oppenheim that for an irrational, indefinite, nondegenerate quadratic form $Q$ in $n \geq 3$ variables, $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$. We refer to [3] for a nice introduction to the problem and Margulis’ proof which involves dynamics on homogeneous spaces, and to [22] for a survey. Subsequently, there have been rapid developments in this subject, quantitative versions were proved in [8–10] and recently effective versions have been established in [4,11,12,19]. Inhomogeneous quadratic forms have been studied in [13,14,23,24].
Inhomogeneous quadratic forms

Let $Q'$ be an inhomogeneous quadratic form on $\mathbb{R}^n$, i.e. $Q'$ is a degree two polynomial in $n$ variables. Then $Q'$ can be written as

$$Q'(x) = Q(x) + L(x) + c \quad \forall \ x \in \mathbb{R}^n$$

where $Q$ is a homogeneous quadratic form on $\mathbb{R}^n$, $L$ is a linear form on $\mathbb{R}^n$ and $c \in \mathbb{R}$. We say that $Q'$ is indefinite and nondegenerate if $Q$ is indefinite and nondegenerate respectively. The form $Q'$ is said to be irrational if either $Q$ is irrational as a homogeneous quadratic form or $L$ is irrational, i.e. not a scalar multiple of a form with integer coefficients. A particular kind of inhomogeneous quadratic form is defined as follows. Let $Q$ be a nondegenerate homogeneous quadratic form on $\mathbb{R}^n$ and $\xi \in \mathbb{R}^n$. Define the inhomogeneous quadratic form $Q_\xi$ by

$$Q_\xi(x) = Q(x + \xi) \quad \text{for} \quad x \in \mathbb{R}^n.$$  \hspace{1cm} (1.1)

It is easy to see that $Q_\xi$ is irrational iff either $Q$ is irrational as a homogeneous quadratic form or $\xi$ is an irrational vector, i.e. not a scalar multiple of a vector with integer coordinates.

In [23], Margulis and Mohammadi proved quantitative forms of Oppenheim’s conjecture for inhomogeneous forms. Their work contains the qualitative density as a special case. In particular, Theorem 1.4 in [23] implies that for an indefinite, irrational, nondegenerate inhomogeneous quadratic form $Q_\xi$ in $n \geq 3$ variables, $Q_\xi(\mathbb{Z}^n)$ is dense in $\mathbb{R}$.

Systems of forms

The problem of density at integer values for systems of forms dates back to Dani and Margulis [7]. They proved that for a 3 variable quadratic form $Q$ and a linear form $L$,

$$\{(Q(x), L(x)) : x \in \mathbb{Z}^3\}$$

is dense in $\mathbb{R}^2$ if no nonzero linear combination of $Q$ and $L^2$ is rational, and the plane $\{L = 0\}$ is tangent to the surface $\{Q = 0\}$. In [5], Dani proved that if the surface $\{Q = 0\}$ and the plane $\{L = 0\}$ intersect transversally, the density can fail for a set pairs of full Hausdorff dimension. The work of Dani and Margulis was generalised by Gorodnik [15] who studied pairs comprising a quadratic and linear form in dimensions greater than 3. Subsequently, he studied systems of quadratic forms in [16]. Further progress on systems comprising a quadratic and linear form was made in [6] by Dani. In a related direction, Lazar [17] studied the density of a pair comprising a quadratic and linear form at $S$-integer points, see also the recent paper [18]. Sargent [27], studied the density of linear forms at integer points on a quadratic surface.

Results

It is a natural question to investigate the density at integer values of systems consisting of inhomogeneous forms. We take the first step in this paper by investigating a pair consisting an inhomogeneous form and a linear form. Our main theorem is:

**Theorem 1.1** Let $Q_\xi$ be an inhomogeneous, nondegenerate and indefinite quadratic form in 3 variables and let $L$ be a linear form on $\mathbb{R}^3$. Suppose that:

1. the plane $\{x \in \mathbb{R}^3 \mid L(x) = 0\}$ is tangential to the cone $\{x \in \mathbb{R}^3 \mid Q(x) = 0\}$ and
2. any non-zero linear combination of $Q_\xi$ and $L^2$ is an irrational quadratic form.

Then $\{(Q_\xi(x), L(x)) \mid x \in \mathbb{Z}^3\}$ is a dense subset of $\mathbb{R}^2$.

Remark:

1. Our proof uses the strategy of Margulis, currently the only available strategy for density problems involving forms in low variables, and involves dynamics of group actions on the space of affine lattices in $\mathbb{R}^3$. Condition (1) in Theorem 1.1 implies that the joint stabilizer of the inhomogeneous form and the linear form is a unipotent group and so the corresponding action is subject to Ratner’s theorems. As in the case of Dani’s result [5] referred to above, if the plane $\{x \in \mathbb{R}^3 \mid L(x) = 0\}$ intersects the cone $\{x \in \mathbb{R}^3 \mid Q(x) = 0\}$ transversally, we expect that the density will fail for a full Hausdorff dimension set of pairs.

2. Condition (2) is natural to assume for density.

3. The original Oppenheim conjecture can be easily reduced to 3 variables but this is not the case for systems of forms. It would be interesting to study analogues of Theorem 1.1 in higher dimensions. It should be possible to approach this problem via an application of Ratner’s theorem. However, the analysis of intermediate subgroups which is already quite complicated in 3 variables, will probably be much more difficult.

Along the way, we need several lemmata which can also be used to study the Oppenheim conjecture for a single inhomogeneous quadratic form, and so we take the opportunity to present a self contained proof of the following theorem.

**Theorem 1.2** Let $Q_\xi$ be an indefinite, irrational and non-degenerate quadratic form in $n$ variables, $n \geq 3$. Then $Q_\xi(\mathbb{Z}^n)$ is dense in $\mathbb{R}$.

As noted above, Theorem 1.2 is already implied by the work of Margulis and Mohammadi [23], so we make no claims to originality as regards Theorem 1.2.

**2 Notation**

This paper is heavy on notation, so we are devoting this section to defining the various groups that will play a role in subsequent chapters. We have a natural action of $\text{SL}(3, \mathbb{R}) \ltimes \mathbb{R}^3$ on $\mathbb{R}^3$ given by

$$(g, v).x = gx + v$$

where $(g, v) \in \text{SL}(3, \mathbb{R}) \ltimes \mathbb{R}^3$ and $x \in \mathbb{R}^3$.

**Definition 2.1** Given inhomogeneous quadratic forms $Q_\xi$ and $Q'_\xi$, on $\mathbb{R}^3$, say $Q_\xi$ is equivalent to $Q'_\xi$, denoted by $Q_\xi \sim Q'_\xi$, iff there exists $(g, v) \in \text{SL}(3, \mathbb{R}) \ltimes \mathbb{R}^3$ and $\lambda \in \mathbb{R}\{0\}$ such that $\lambda Q_\xi((g, v).x) = Q'_\xi(x) \forall x \in \mathbb{R}^3$.

Given an inhomogeneous, indefinite and nondegenerate quadratic form $Q_\xi$, it is easy to see that $Q_\xi \sim Q_0$, where $Q_0(x) = x_1^2 + x_2^2 - x_3^2$. Indeed, since $Q$ is an indefinite, nondegenerate and homogeneous quadratic form in 3 variables, its signature is either $(2, 1)$ or $(1, 2)$ and hence there exists $\lambda \in \mathbb{R}\{0\}$ and $g \in \text{SL}(3, \mathbb{R})$ such that $\lambda Q(gx) = Q_0(x)$. Let $v = -\xi$. Then, $\lambda Q_\xi((g, v).x) = \lambda Q(gx) = Q_0(x)$ which gives $Q_\xi \sim Q_0$. 

(Springer)
Definition 2.2 Let \( Q_\xi, Q'_\xi \) be inhomogeneous quadratic forms and \( L, L' \) be linear forms on \( \mathbb{R}^3 \). We say that the pairs \((Q_\xi, L)\) and \((Q'_\xi, L')\) are equivalent iff there exists \( \lambda, \mu \in \mathbb{R} \setminus \{0\} \) and \((g, v) \in \text{SL}(3, \mathbb{R}) \times \mathbb{R}^3\) such that \( \lambda Q_\xi((g, v).x) = Q'_\xi(x) \) and \( \mu L((g, v).x) = L'(x) \).

Definition 2.3 For an inhomogeneous quadratic form \( Q_\xi \) and a linear form \( L \) on \( \mathbb{R}^3 \), define

\[
\text{SO}(Q_\xi) := \{(g, v) \in \text{SL}(3, \mathbb{R}) \times \mathbb{R}^3 \mid Q_\xi((g, v).x) = Q_\xi(x) \ \forall \ x \in \mathbb{R}^3 \},
\]

\[
\text{SO}(L) := \{(g, v) \in \text{SL}(3, \mathbb{R}) \times \mathbb{R}^3 \mid L((g, v).x) = L(x) \ \forall \ x \in \mathbb{R}^3 \},
\]

and

\[
\text{SO}(Q_\xi, L) = \text{SO}(Q_\xi) \cap \text{SO}(L).
\]

For a subgroup \( H \) of \( G \), \( H^0 \) denotes the identity component of \( H \) and \( N(H) \) denotes the normalizer of \( H \) in \( G \). We set \( G = \text{SL}(3, \mathbb{R}) \times \mathbb{R}^3, \Gamma = \text{SL}(3, \mathbb{Z}) \times \mathbb{Z}^3 \) and \( H = \text{SO}(2, 1)^0 \times \{0\} \). Note that \( \Gamma \) is a nonuniform lattice in \( G \).

Let

\[
V_1 = \left\{ \begin{pmatrix} 1 & t^2 \mu^2 & \mu \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}, \quad V = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} : a, b \in \mathbb{R} \right\},
\]

\[
W = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}, \quad \text{and} \quad D = \left\{ \begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{-1} \end{pmatrix} : t \in \mathbb{R} \setminus \{0\} \right\}.
\]

For \( t \in \mathbb{R} \), let

\[
v(t) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad N = \left\{ \begin{pmatrix} a & b & c \\ 0 & a^{-2} & d \\ 0 & 0 & a \end{pmatrix} : a \in \mathbb{R} \setminus \{0\}, b, c, d \in \mathbb{R} \right\},
\]

\[
N_1 = \left\{ \begin{pmatrix} a^2 & b & c \\ 0 & a^{-5} & d \\ 0 & 0 & a^3 \end{pmatrix} : a \in \mathbb{R} \setminus \{0\}, b, c, d \in \mathbb{R} \right\},
\]

\[
N_2 = \left\{ \begin{pmatrix} a^3 & b & c \\ 0 & a^{-5} & d \\ 0 & 0 & a^2 \end{pmatrix} : a \in \mathbb{R} \setminus \{0\}, b, c, d \in \mathbb{R} \right\}.
\]

Let

\[
Q_1 = \left\{ g = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} : A \text{ is a } 3 \times 2 \text{ matrix such that } g \in \text{SL}(3, \mathbb{R}) \right\}.
\]
On the density at integer points of a system comprising...
\[ \mathcal{Q}_1 = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & a & d \\ 0 & e & -a \end{pmatrix} : a, b, c, d, e \in \mathbb{R} \right\} , \quad \mathcal{Q}_2 = \left\{ \begin{pmatrix} a & b & c \\ e & -a & d \\ 0 & 0 & 0 \end{pmatrix} : a, b, c, d, e \in \mathbb{R} \right\} . \]

For \( t \in \mathbb{R} \), let
\[ \mathcal{R}_t = \left\{ \begin{pmatrix} a & b & 0 \\ c & 0 & b - 2tc \\ 0 & c & -a \end{pmatrix} : a, b, c, \in \mathbb{R} \right\} , \]
and for \( \beta \in \mathbb{R} \setminus \{0\} \), set
\[ \mathcal{P}_\beta = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} , \begin{pmatrix} c \\ d \\ b\beta \end{pmatrix} : a, b, c, \in \mathbb{R} \right\} . \]

For \( \alpha \in \mathbb{R} \), set
\[ \mathcal{I}_t = \left\{ \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 0 \alpha \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\} . \]

Finally, we set
\[ \mathcal{S}_\alpha = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 0 \alpha \\ 0 \end{pmatrix} : a, b, c, \in \mathbb{R} \right\} , \]
and
\[ \mathcal{B}_\alpha = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} , \begin{pmatrix} d \\ a\alpha \\ 0 \end{pmatrix} : a, b, c, \in \mathbb{R} \right\} . \]

For a subgroup \( C \) of \( \text{SL}(3, \mathbb{R}) \), denote by \( C \ltimes \mathbb{R} \) and \( C \ltimes \mathbb{R}^2 \) the subgroups of \( G \) consisting of elements
\[ \left\{ \begin{pmatrix} h & a \\ 0 & \end{pmatrix} : a \in \mathbb{R}, h \in C \right\} , \text{ and } \left\{ \begin{pmatrix} h & a \\ 0 & \end{pmatrix} : a, b \in \mathbb{R}, h \in C \right\} , \]
respectively.

## 3 Preparatory Lemmata

In this section, we prove some lemmata required for proving the theorems.
Lemma 3.1 With notation as in (2.3), we have that
\[
\text{SO}(Q_\xi, L) = \{ (g, g\xi - \xi) \mid g \in \text{SO}(Q, L) \}.
\]

Proof It is easy to see that for \( g \in \text{SO}(Q, L) \), \((g, g\xi - \xi) \in \text{SO}(Q_\xi, L) \). Conversely, suppose \((g, v) \in \text{SO}(Q_\xi, L) \). Then for \( x \in \mathbb{R}^3 \), \( Q_\xi ((g, v)x) = Q_\xi(x) \) and \( L((g, v)x) = L(x) \) which implies that \( Q(gx + v + \xi) = Q(x + \xi) \) and \( L(gx + v) = L(x) \). This gives that
\[
Q(gy + v + \xi - g\xi) = Q(y) \forall y \in \mathbb{R}^3.
\]
Let \( \xi' = v + \xi - g\xi \). Then for every \( y \in \mathbb{R}^3 \), \( Q(gy + \xi') = Q(y) \) which implies that
\[
Q(gy) + Q(\xi') + 2(\xi')^t A\xi' = Q(y),
\]
where \( A \) denotes the symmetric matrix corresponding to the quadratic form \( Q \). This gives that for every \( y \in \mathbb{R}^3 \), \( Q(gy) = Q(y) \) and \((\xi')^t A\xi' = 0\) which further shows that \( g \in \text{SO}(Q) \) and \( \xi' = v + \xi - g\xi = 0 \). Therefore \( v = g\xi - \xi \). Substituting for \( v \) in \( L(gx + v) = L(x) \) gives \( L(gy) = L(y) \) \( \forall y \in \mathbb{R}^3 \). Hence \( g \in \text{SO}(Q, L) \) and \( v = g\xi - \xi \) thus proving the lemma. \( \Box \)

Remark 3.2 Taking \( L = 0 \) in the above lemma gives \( \text{SO}(Q_\xi) = \{ (g, g\xi - \xi) \mid g \in \text{SO}(Q) \} \).

Lemma 3.3 With notation as above,
\[
\text{SO}(Q_\xi)^o = (g, -\xi)H(g, -\xi)^{-1}
\]
where \( g \in \text{SL}(3, \mathbb{R}) \) is such that \( \lambda Q(gx) = Q_0(x) \) for some \( \lambda \in \mathbb{R} \setminus \{0\} \).

Proof Since \( \lambda Q(gx) = Q_0(x) \), we have that \( \text{SO}(Q) = g \text{SO}(2, 1)g^{-1} \). Let \( h \in \text{SO}(2, 1) \). It is straight forward to compute that \( (g, -\xi)(h, 0)(g, -\xi)^{-1} = (ghg^{-1}, ghg^{-1}\xi - \xi) \). Then,
\[
Q_\xi ((ghg^{-1}, ghg^{-1}\xi - \xi)x) = Q(ghg^{-1}(x + \xi))
\]
\[
= Q(x + \xi) \quad \because ghg^{-1} \in \text{SO}(Q)
\]
\[
= Q_\xi(x).
\]
Therefore \((g, -\xi) \text{SO}(2, 1) \times \{0\}(g, -\xi)^{-1} \subseteq \text{SO}(Q_\xi) \).

Now, let \((g', v) \in \text{SO}(Q_\xi) \). By Remark 3.2, we get that \( g' \in \text{SO}(Q) \) and \( v = g'\xi - \xi \). Since \( \text{SO}(Q) = g \text{SO}(2, 1)g^{-1} \), there exists \( h \in \text{SO}(2, 1) \) such that \( g' = ghg^{-1} \). Therefore
\[
(g', v) = (ghg^{-1}, ghg^{-1}\xi - \xi) = (g, -\xi)(h, 0)(g, -\xi)^{-1}.
\]

Hence,
\[
\text{SO}(Q_\xi) \subseteq (g, -\xi) \text{SO}(2, 1) \times \{0\}(g, -\xi)^{-1}.
\]

Taking the identity components, we get that \( \text{SO}(Q_\xi)^o = (g, -\xi)H(g, -\xi)^{-1} \). \( \Box \)

For a subset \( S \) of \( G \), we denote by \( \overline{S} \) its Zariski closure.

Lemma 3.4 Let \( S \) be a subset of \( \text{SL}(3, \mathbb{Z}) \times \mathbb{R}^3 \). Then \( \overline{S} \) is defined over \( \mathbb{Q} \).

Proof Suppose \( \overline{S} \) is the set of zeroes of \( \mathcal{S} \) for some \( \mathcal{S} \subseteq \mathbb{P}^n \), where \( \mathbb{P}^n \) denotes the set of polynomials of degree \( \leq n \). Then the subspace \( \{ f \in \mathbb{P}^n \mid f(S) = 0 \} \) is defined by linear equations with rational coefficients, since \( S \subseteq \text{SL}(3, \mathbb{Z}) \times \mathbb{R}^3 \). As \( \mathcal{S} \subseteq \{ f \in \mathbb{P}^n \mid f(S) = 0 \} \), we get that \( \overline{S} \) is defined over \( \mathbb{Q} \). \( \Box \)
Lemma 3.5 Let $Q$ be an indefinite and nondegenerate quadratic form. If $\text{SO}(Q_\xi)^o$ is defined over $\mathbb{Q}$, then $Q_\xi$ is not an irrational quadratic form.

Proof Firstly, we will show that if $\text{SO}(Q_\xi)^o = \text{SO}(Q_\xi')^o$, then $\xi = \xi'$ and there exists $c \in \mathbb{R}$ such that $\sigma = c\sigma'$ where $\sigma$ and $\sigma'$ are the symmetric matrices corresponding to $Q$ and $Q'$ respectively. Let $(h, v) \in \text{SO}(Q_\xi)^o$. Then by Remark 3.2, $h \in \text{SO}(Q)$ and $v = h\xi - \xi$. Since $\text{SO}(Q_\xi)^o = \text{SO}(Q_\xi')^o$, $(h, v)$ also lies in $\text{SO}(Q_\xi')^o$ which implies that $h \in \text{SO}(Q')$ and $v = h\xi' - \xi'$. Consider,

$$(h, v)(\sigma, -\xi)(\sigma', -\xi')^{-1}(h, v)^{-1} = (h, v)(\sigma, -\xi)(h', 0)(h', 0)^{-1}(\sigma', -\xi')^{-1}(h, v)^{-1} = (h\sigma h', v - h\xi')(h\sigma' h', v - h\xi')^{-1} = (\sigma, \xi)(\sigma', -\xi')^{-1}.$$ 

This implies that $(\sigma, -\xi)(\sigma', -\xi')^{-1}$ lie in the centralizer of $\text{SO}(Q_\xi)^o$.

We now

Claim: The centralizer of $\text{SO}(Q_\xi)^o$ in $\text{GL}(3, \mathbb{R}) \ltimes \mathbb{R}^3$ is $\{(cI, (c - 1)\xi) \mid c \in \mathbb{R}\setminus\{0\}\}$ where $I$ denotes the identity matrix.

Proof of the claim. Let $(A, v) \in \text{GL}(3, \mathbb{R}) \ltimes \mathbb{R}^3$ be such that $(A, v)$ commutes with every element of $H$. Then $A$ lies in the centralizer of $\text{SO}(2, 1)^o$ and $hv = v$ for every $h \in \text{SO}(2, 1)^o$. From (Lemma 2.2 (ii), chapter 6, [1]), it follows that $A = cI$ for some $c \in \mathbb{R}$ and $v = 0$. Therefore, the centralizer of $H$ is $\{(cI, 0) \mid c \in \mathbb{R}\setminus\{0\}\}$. Since $Q$ is indefinite and nondegenerate, there exists $\lambda \in \mathbb{R}\setminus\{0\}$ and $g \in \text{SL}(3, \mathbb{R})$ such that $\lambda Q(gx) = Q_0(x)$. Hence by Lemma 3.3, $\text{SO}(Q_\xi)^o = (g, -\xi)\text{H}(g, -\xi)^{-1}$. Therefore, the centralizer of $\text{SO}(Q_\xi)^o$ is

$$\{(g, -\xi)(cI, 0)(g, -\xi)^{-1} \mid c \in \mathbb{R}\setminus\{0\}\} = \{(cI, (c - 1)\xi) \mid c \in \mathbb{R}\setminus\{0\}\}$$

thereby proving the claim.

Therefore, there exists $c \in \mathbb{R}\setminus\{0\}$ such that $(\sigma, -\xi)(\sigma', -\xi')^{-1} = (cI, (c - 1)\xi)$ which gives that $\sigma = c\sigma'$. Since $\text{SO}(Q_\xi)^o = \text{SO}(Q_\xi')^o$, the claim implies that $\xi = \xi'$.

Now, let $\phi \in \text{Aut}(\mathbb{C}/\mathbb{Q})$. By $\phi(Q)$ we mean the quadratic form obtained by applying $\phi$ to the coefficients of $Q$ and $\phi(\xi)$ is the vector obtained by applying $\phi$ to each coordinate of $\xi$. Then $\text{SO}(\phi(Q_\phi(\xi))^o = \phi(\text{SO}(Q_\xi)^o) = \text{SO}(Q_\xi')^o$ (Since $\text{SO}(Q_\xi)^o$ is defined over $\mathbb{Q}$). Therefore, there exists $\alpha_\phi \in \mathbb{R}\setminus\{0\}$ such that $\phi(\sigma) = \alpha_\phi \sigma$ and $\phi(\xi) = \xi$ where $\sigma$ is the matrix corresponding to the quadratic form $Q$. By taking a scalar multiple, we can assume that one of the matrix entries of $\sigma$ is rational. Then, as $\phi$ fixes that coefficient, we get that $\alpha_\phi = 1$. Hence $\phi(\sigma) = \sigma$ and $\phi(\xi) = \xi$ for every $\phi \in \text{Aut}(\mathbb{C}/\mathbb{Q})$. Since the fixed point set of $\text{Aut}(\mathbb{C}/\mathbb{Q})$ is $\mathbb{Q}$, we get that $Q$ is a scalar multiple of a rational form and $\xi$ is a rational vector thus proving that $Q_\xi$ is not an irrational quadratic form. \hfill $\Box$

The following Lemma is well known, see for instance (exercise 17, 1.2, [29]).

Lemma 3.6 $\mathfrak{so}(2, 1)$ is a maximal Lie subalgebra of $\mathfrak{sl}(3, \mathbb{R})$.

Lemma 3.7 The only closed connected subgroups of $G$ containing $H$ are $H$, $\text{SO}(2, 1)^o \ltimes \mathbb{R}^3$, $\text{SL}(3, \mathbb{R}) \ltimes \{0\}$ and $G$.

Proof Denote by $\mathfrak{g}$ and $\mathfrak{h}$, the Lie algebras of $G$ and $H$ respectively. We will show that the only Lie subalgebras of $\mathfrak{g}$ containing $\mathfrak{h}$ are $\mathfrak{so}(2, 1) \ltimes \mathbb{R}^3$, $\mathfrak{sl}(3, \mathbb{R}) \ltimes \{0\}$ and $\mathfrak{g}$. The Lemma will follow from the correspondence between Lie groups and Lie algebras. Let $\mathfrak{f}$ be a Lie subalgebra of $\mathfrak{g}$ such that $\mathfrak{h} \subseteq \mathfrak{f} \subseteq \mathfrak{g}$. Let $P$ be the projection map from $\mathfrak{g}$ to $\mathfrak{sl}(3, \mathbb{R})$. Then, $P(\mathfrak{f})$ is a Lie subalgebra of $\mathfrak{sl}(3, \mathbb{R})$ containing $\mathfrak{so}(2, 1)$. Since $\mathfrak{so}(2, 1)$ is a maximal
Lie subalgebra of $\mathfrak{sl}(3, \mathbb{R})$ (by Lemma 3.6), $P(\mathfrak{f})$ is either equal to $\mathfrak{sl}(3, \mathbb{R})$ or $\mathfrak{so}(2, 1)$. We examine these cases separately.

Case 1: $P(\mathfrak{f}) = \mathfrak{so}(2, 1)$.
Since $\mathfrak{so}(2, 1) \nsubseteq \{0\} \subseteq \mathfrak{f}$, there exists an element $(g, v) \in \mathfrak{f}$ such that $(g, v) \notin \mathfrak{h}$. The assumption $P(\mathfrak{f}) = \mathfrak{so}(2, 1)$ implies that $g \in \mathfrak{so}(2, 1)$. Since $(g, v) \notin \mathfrak{h}$, we have that $v \neq 0$. As $(g, 0) \in \mathfrak{f}$, we get $(g, v) - (g, 0) = (0, v) \in \mathfrak{f}$. Therefore, for all $g \in \mathfrak{so}(2, 1)$,
\[
[(g, 0), (0, v)] = (0, gv) \in \mathfrak{f}
\]
Since $\mathfrak{so}(2, 1)$ acts irreducibly on $\mathbb{R}^3$, we get that $(0, w) \in \mathfrak{f}, \forall \ w \in \mathbb{R}^3$. Hence, $\forall \ g \in \mathfrak{so}(2, 1)$ and $\forall \ w \in \mathbb{R}^3$, we have that
\[
(g, 0) + (0, w) = (g, w) \in \mathfrak{f}
\]
Therefore, $\mathfrak{so}(2, 1) \nsubseteq \mathbb{R}^3 \subseteq \mathfrak{f}$ and since $P(\mathfrak{f}) = \mathfrak{so}(2, 1)$, we get that $\mathfrak{f} = \mathfrak{so}(2, 1) \nsubseteq \mathbb{R}^3$.

Case 2: $P(\mathfrak{f}) = \mathfrak{sl}(3, \mathbb{R})$.
Assume that for some $g \in \mathfrak{sl}(3, \mathbb{R}) \backslash \mathfrak{so}(2, 1)$, we have that $(g, 0) \notin \mathfrak{f}$. Since the Lie subalgebra generated by $\mathfrak{so}(2, 1)$ and $g$ is $\mathfrak{sl}(3, \mathbb{R})$ (as $\mathfrak{so}(2, 1)$ is a maximal subalgebra of $\mathfrak{sl}(3, \mathbb{R})$), we get $(g, 0) \in \mathfrak{f} \forall \ g \in \mathfrak{sl}(3, \mathbb{R})$. Therefore, $\mathfrak{sl}(3, \mathbb{R}) \nsubseteq \{0\} \subseteq \mathfrak{f}$. If $\mathfrak{sl}(3, \mathbb{R}) \nsubseteq \{0\} \subseteq \mathfrak{f}$ then
\[
(0, v) \in \mathfrak{f} \text{ for some non-zero } v \text{ which implies } (0, w) \in \mathfrak{f} \forall \ w \in \mathbb{R}^3
\]
as in Case 1 and hence $\mathfrak{f} = \mathfrak{sl}(3, \mathbb{R}) \nsubseteq \mathbb{R}^3$ which is a contradiction. Therefore, $\mathfrak{f} = \mathfrak{sl}(3, \mathbb{R}) \nsubseteq \{0\}$.

Now, suppose for every $g \in \mathfrak{sl}(3, \mathbb{R}) \backslash \mathfrak{so}(2, 1)$, we have that $(g, 0) \notin \mathfrak{f}$. Then for every $g \in \mathfrak{sl}(3, \mathbb{R}) \backslash \mathfrak{so}(2, 1)$, there exists a non-zero element $v_g \in \mathbb{R}^3$, such that $(g, v_g) \in \mathfrak{f}$ since $P(\mathfrak{f}) = \mathfrak{sl}(3, \mathbb{R})$. Let
\[
\begin{align*}
g_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, & g_2 &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{pmatrix}, & g &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{pmatrix} \\
h &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & k &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} 
\end{align*}
\]
Since $g_1, g_2, g \in \mathfrak{sl}(3, \mathbb{R}) \backslash \mathfrak{so}(2, 1)$, there exist non zero elements $v_{g_1}, v_{g_2}, v_g$ in $\mathbb{R}^3$ such that
\[
(g_1, v_{g_1}), (g_2, v_{g_2}), (g, v_g) \in \mathfrak{f}
\]
Since $h, k \in \mathfrak{so}(2, 1)$ we have that $(h, 0), (k, 0) \in \mathfrak{f}$. Therefore,
\[
[(h, 0), (g_1, v_{g_1})], [(h, 0), (g_2, v_{g_2})], [(k, 0), (g, v_g)] \in \mathfrak{f}
\]
which implies that
\[
(\{h, g_1\}, hv_{g_1}), (\{h, g_2\}, hv_{g_2}), (\{k, g\}, kv_g) \in \mathfrak{f}.
\]
It is straightforward to check that $[h, g_1], [h, g_2], [k, g] \in \mathfrak{so}(2, 1)$. Therefore
\[
(0, hv_{g_1}), (0, hv_{g_2}), (0, kv_g) \in \mathfrak{f}.
\]
Now let
\[ v_{g_1} = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \quad v_{g_2} = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}, \quad v_g = \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix}, \]
then
\[ h v_{g_1} = \begin{pmatrix} b_1 \\ -a_1 \\ 0 \end{pmatrix}, \quad h v_{g_2} = \begin{pmatrix} b_2 \\ -a_2 \\ 0 \end{pmatrix}, \quad k v_g = \begin{pmatrix} c_3 \\ -c_3 \\ a_3 - b_3 \end{pmatrix}. \]
If one among \( a_1, b_1, a_2, b_2 \) is non-zero, then either \( h v_{g_1} \) or \( h v_{g_2} \) is non-zero and hence \( (0, v) \in \mathcal{f} \) for some non zero element \( v \). As in case 1, this implies that \( \mathfrak{s}o(2, 1) \ltimes \mathbb{R}^3 \subset \mathcal{f} \). Similarly, if either \( c_3 \) is non-zero or \( a_3 \neq b_3 \), then \( k v_g \) is non-zero and hence \( (0, v) \in \mathcal{f} \) for some non-zero \( v \) which again implies \( \mathfrak{s}o(2, 1) \ltimes \mathbb{R}^3 \subset \mathcal{f} \). Since \( P(\mathcal{f}) = \mathfrak{s}l(3, \mathbb{R}) \), we get \( \mathfrak{f} = \mathfrak{s}l(3, \mathbb{R}) \ltimes \mathbb{R}^3 \) which is a contradiction. Now, suppose \( a_1 = b_1 = a_2 = b_2 = c_3 = 0 \) and \( a_3 = b_3 \) then \( v_{g_1} = \begin{pmatrix} 0 \\ 0 \\ c_1 \end{pmatrix}, \quad v_{g_2} = \begin{pmatrix} 0 \\ 0 \\ c_2 \end{pmatrix}, \quad \text{and} \quad v_g = \begin{pmatrix} a_3 \\ a_3 \\ 0 \end{pmatrix} \). It is easy to compute that
\[ \left[ (g, v_g), (g_1, v_{g_1}) \right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} c_1 - a_3 \\ -a_3 \\ -2c_1 \end{pmatrix} \in \mathcal{f}, \]
and
\[ \left[ (g_2, v_{g_2}), (g_1, v_{g_1}) \right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} c_1 \\ 0 \\ 2(c_2 - c_1) \end{pmatrix} \in \mathcal{f}. \]
Hence their difference, which is \( (0, v) \) for some non zero \( v \), lies in \( \mathcal{f} \). This implies that \( \mathfrak{s}o(2, 1) \ltimes \mathbb{R}^3 \subset \mathcal{f} \) which again gives \( \mathfrak{f} = \mathfrak{s}l(3, \mathbb{R}) \ltimes \mathbb{R}^3 \) since \( P(\mathfrak{f}) = \mathfrak{s}l(3, \mathbb{R}) \), a contradiction. \( \square \)

4 Proof of Theorem 1.2

In this section, we prove Theorem 1.2, i.e. the Oppenheim conjecture for inhomogeneous forms.

Reduction to the case of \( n = 3 \)

Using induction on \( n \), it follows from the following Lemma that it is enough to prove the theorem for the case of \( n = 3 \).

**Lemma 4.1** Let \( Q_\xi \) be an indefinite, irrational and nondegenerate quadratic form in \( n \) variables, \( n \geq 3 \). Then there exists a rational hyperplane \( L \) such that the restriction of \( Q_\xi \) to \( L \) is indefinite, irrational and nondegenerate.

**Proof** Since \( Q_\xi \) is irrational, either \( Q \) is an irrational quadratic form or \( \xi \) is an irrational vector. Firstly, assume that \( Q \) is irrational. Then from (Lemma 2.1, chapter 6, [1]), it follows
that there exists a rational hyperplane \( L \) such that restriction of \( Q \) to \( L \) is indefinite, irrational and nondegenerate. This implies that restriction of \( Q_{\xi} \) to \( L \) is indefinite, irrational and nondegenerate.

Now, assume that \( Q \) is not irrational. Then \( \xi \) has to be an irrational vector. Since \( Q \) is indefinite and nondegenerate, we can find a rational hyperplane \( L \) such that restriction of \( Q \) to \( L \) is indefinite and nondegenerate (This is a part of the proof of (Lemma 2.1, chapter 6, [1])). Then the restriction of \( Q_{\xi} \) to \( L \) is irrational (Since \( \xi \) is irrational), indefinite and nondegenerate. \( \square \)

Using the above stated lemmas, we now prove Theorem 1.2 when \( n = 3 \).

**Proof** Let \( g \in \text{SL}(3, \mathbb{R}) \) and \( \lambda \in \mathbb{R} \setminus \{0\} \) be such that \( \lambda Q(gx) = Q_0(x) \). Since \( H = \text{SO}(2, 1)^{\circ} \rtimes \{0\} \) is generated by unipotent elements (Lemma 2.2, chapter 6 of [1]) and \( \Gamma \) is a lattice in \( G \), we may apply Ratner’s orbit closure theorem (See Theorem (1.1.14) and Remark(1.1.19) of [29]) which tells us that there is a closed connected subgroup \( F \) of \( G \) such that

1. \( H \subset F \),
2. the image \( [F.(g, -\xi)^{-1}] \) of \( F.(g, -\xi)^{-1} \) in \( G/\Gamma \) is closed and has finite \( F \)-invariant measure,
3. the closure of \( [H.(g, -\xi)^{-1}] \) is equal to \( [F.(g, -\xi)^{-1}] \).

By Lemma 3.7, \( F \) is either \( H, \text{SO}(2, 1)^{\circ} \rtimes \mathbb{R}^3, \text{SL}(3, \mathbb{R}) \rtimes \{0\} \) or \( G \).

**Case 1:** Suppose \( F \) is either \( \text{SO}(2, 1)^{\circ} \rtimes \mathbb{R}^3 \) or \( \text{SL}(3, \mathbb{R}) \rtimes \{0\} \) or \( G \). Then observe that \( F.(g, -\xi)^{-1}\mathbb{Z}^3 = \mathbb{R}^3 \). Hence,

\[
\lambda Q_{\xi}((g, -\xi)^{-1}\mathbb{Z}^3) = \frac{\lambda}{\lambda} Q_{\xi}( (g, -\xi)^{-1} \mathbb{Z}^3 ) = \frac{\lambda}{\lambda} Q_0((g, -\xi)^{-1} \mathbb{Z}^3) = \frac{\lambda}{\lambda} (\text{SL}(3, \mathbb{Z}) \times \mathbb{Z}^3, \mathbb{Z}^3 = \mathbb{Z}^3 ]
\]

\[
= Q_0(H(g, -\xi)^{-1}\mathbb{Z}^3) = \text{by (3), } H(g, -\xi)^{-1}\mathbb{R}^3 = H(g, -\xi)^{-1}\mathbb{Z}^3
\]

\[
= Q_0(F(g, -\xi)^{-1}\mathbb{Z}^3) = \mathbb{R}^3
\]

Therefore, \( Q_{\xi}(\mathbb{Z}^3) \) is dense in \( \mathbb{R} \).

**Case 2:** Suppose \( F = H \).

We will show that \( Q_{\xi} \) cannot be an irrational quadratic form. By (2), \( \text{ SO}(2, 1)^{\circ} \) is closed in \( G/\Gamma \) and has finite \( H \)-invariant measure. This implies that \( \text{SO}(2, 1)^{\circ} \rtimes \text{SL}(3, \mathbb{R}) \rtimes \{0\} \) is a lattice in \( (g, -\xi)^{-1}\text{SO}(Q_{\xi}) \) (By Lemma 3.3). Denote by \( \Gamma(g, \xi) = (g, -\xi)^{-1}\text{SO}(Q_{\xi}) \) (By Lemma 3.3). By the Borel density theorem, all unipotent elements of \( \text{SO}(Q_{\xi})^{\circ} \) lie in the Zariski closure of \( \text{SL}(3, \mathbb{R}) \rtimes \{0\} \). Since \( \text{SO}(Q_{\xi})^{\circ} \) is generated by its unipotent elements (Since it is a conjugate of \( H \) and \( H \) is generated by unipotent elements), we get that \( \text{SO}(Q_{\xi})^{\circ} = \Gamma_{(g, \xi)}, \) where \( \Gamma_{(g, \xi)} \) denotes the Zariski closure of \( \Gamma(g, \xi) \). Since \( \Gamma(g, \xi) \subseteq \Gamma, \) \( \Gamma(g, \xi) \) is defined over \( \mathbb{Q} \) (By Lemma 3.4) and hence \( \text{SO}(Q_{\xi})^{\circ} \) is defined over \( \mathbb{Q} \). This implies that \( Q_{\xi} \) is not an irrational quadratic form (By Lemma 3.5). \( \square \)
5 Proof of Theorem 1.1

In this section, we denote by $Q_0$ the quadratic form defined by $Q_0(x) = 2x_1x_3 - x_2^2$.

**Lemma 5.1** Let $Q_\xi$ be an inhomogeneous, non-degenerate and indefinite quadratic form and $L$ be a linear form on $\mathbb{R}^3$. Suppose that the plane $\{x \in \mathbb{R}^3 \mid L(x) = 0\}$ is tangential to the cone $\{x \in \mathbb{R}^3 \mid Q(x) = 0\}$. Then there exists $\alpha \in \mathbb{R}$ such that $(Q_\xi, L) \sim ((Q_0)_{(0,0,\alpha)}, L_0)$ where $Q_0(x) = 2x_1x_3 - x_2^2$, $L_0(x) = x_3$ and $\{(Q_\xi(x), L(x)) \mid x \in \mathbb{R}^3\} = \mathbb{R}^2$.

**Proof** Since the plane $\{x \in \mathbb{R}^3 \mid L(x) = 0\}$ is tangential to the cone $\{x \in \mathbb{R}^3 \mid Q(x) = 0\}$, there exists $\lambda, \mu \in \mathbb{R}\setminus\{0\}$ and $g \in \text{SL}(3, \mathbb{R})$ such that $\forall \, x \in \mathbb{R}^3$, $\lambda Q(gx) = Q_0(x)$ and $\mu L(gx) = L_0(x)$ where $Q_0(x) = 2x_1x_3 - x_2^2$ and $L_0(x) = x_3$. Let $\alpha = \mu L(\xi)$ and $v = g \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix} - \xi$. Then it can be easily seen that $\lambda Q_\xi((g, v)\cdot x) = (Q_0)_{(0,0,\alpha)}(x)$ and $\mu L((g, v)\cdot x) = L_0(x)$ and hence $(Q_\xi, L)$ is equivalent to $((Q_0)_{(0,0,\alpha)}, L_0)$. Therefore, $\{(Q_\xi(x), L(x)) \mid x \in \mathbb{R}^3\} = \{(Q_0)_{(0,0,\alpha)}(x), L(x)) \mid x \in \mathbb{R}^3\} = \mathbb{R}^2$.

**Lemma 5.2** With notation as in section 2,
1. the only closed connected unimodular subgroups of $G$ containing $H_0$ are:

| $V_1 \times \{0\}$ | $V_1 \times \mathbb{R}$ | $V_1 \times \mathbb{R}^2$ | $V_1 \times \mathbb{R}^3$ |
|-------------------|---------------------|---------------------|---------------------|
| $V \times \{0\}$  | $V \times \mathbb{R}$ | $V \times \mathbb{R}^2$ | $V \times \mathbb{R}^3$ |
| $W \times \{0\}$  | $W \times \mathbb{R}$ | $W \times \mathbb{R}^2$ | $W \times \mathbb{R}^3$ |
| $v(t) SO(Q_0)^{\circ} v(t)^{-1} \times \{0\}$ | $v(t) SO(Q_0)^{\circ} v(t)^{-1} \times \mathbb{R}^3$ | $Q_1 \times \{0\}$ | $Q_1 \times \mathbb{R}$ |
| $Q_1 \times \mathbb{R}^3$ | $Q_2 \times \{0\}$ | $Q_2 \times \mathbb{R}^2$ | $Q_2 \times \mathbb{R}^3$ |
| $N^0 \times \{0\}$ | $N^0 \times \mathbb{R}^3$ | $N^0 \times \mathbb{R}^2$ | $N^0 \times \mathbb{R}^2$ |
| SL(3, $\mathbb{R}$) $\times \{0\}$ | SL(3, $\mathbb{R}$) $\times \mathbb{R}^3$ | $P_\beta$ for $\beta \in \mathbb{R} \setminus \{0\}$ |

2. for $\alpha \in \mathbb{R}\setminus\{0\}$, the only closed connected subgroups of $G$ containing $H_\alpha$ are:

| $V_1 \times \mathbb{R}^2$ | $V_1 \times \mathbb{R}^3$ | $V \times \mathbb{R}^2$ | $V \times \mathbb{R}^3$ |
|----------------------|----------------------|----------------------|----------------------|
| $W \times \mathbb{R}^2$ | $W \times \mathbb{R}^3$ | $v(t) SO(Q_0)^{\circ} v(t)^{-1} \times \mathbb{R}^3$ | $Q_1 \times \mathbb{R}^3$ |
| $Q_2 \times \mathbb{R}^2$ | $Q_2 \times \mathbb{R}^3$ | $N^0 \times \mathbb{R}^3$ | $N^0 \times \mathbb{R}^2$ |
| SL(3, $\mathbb{R}$) $\times \mathbb{R}^3$ | $A_\alpha$ | $B_\alpha$ | $P_\beta$ for $\beta \in \mathbb{R} \setminus \{0\}$ |
Proof The lemma follows from the classification of Lie subalgebras of $sl(3, \mathbb{R}) \times \mathbb{R}^3$. There has of course been extensive work on this subject, we use the paper [28] of Winternitz which is well suited for our purpose. Namely, by using the subalgebra classification algorithm (2.4, [28]) and from Table 1 of [28], one can compute that the only unimodular subalgebras of $sl(3, \mathbb{R}) \times \mathbb{R}^3$ containing $H_\alpha$ are: $\mathfrak{V}_1 \times \{0\}$, $\mathfrak{V}_1 \times \mathbb{R}$, $\mathfrak{V}_1 \times \mathbb{R}^2$, $\mathfrak{V}_1 \times \mathbb{R}^3$, $\mathfrak{V} \times \{0\}$, $\mathfrak{V} \times \mathbb{R}$, $\mathfrak{V} \times \mathbb{R}^2$, $\mathfrak{V} \times \mathbb{R}^3$, $\mathfrak{W} \times \{0\}$, $\mathfrak{W} \times \mathbb{R}$, $\mathfrak{W} \times \mathbb{R}^2$, $\mathfrak{W} \times \mathbb{R}^3$, $\mathfrak{R}_1 \times \{0\}$, $\mathfrak{R}_1 \times \mathbb{R}$, $\mathfrak{R}_1 \times \mathbb{R}^2$, $\mathfrak{R}_1 \times \mathbb{R}^3$, $\mathfrak{V}_2 \times \{0\}$, $\mathfrak{V}_2 \times \mathbb{R}$, $\mathfrak{V}_2 \times \mathbb{R}^2$, $\mathfrak{V}_2 \times \mathbb{R}^3$, $\mathfrak{N} \times \{0\}$, $\mathfrak{N} \times \mathbb{R}$, $\mathfrak{N}_2 \times \mathbb{R}^2$, $sl(3, \mathbb{R}) \times \{0\}$, $sl(3, \mathbb{R}) \times \mathbb{R}^3$, $\mathfrak{P}$ for $\beta \in \mathbb{R} \setminus \{0\}$. By taking the Lie subgroups corresponding to these Lie subalgebras, part 1 of the Lemma follows.

Similarly for part(2), one can show that the only unimodular subalgebras of $sl(3, \mathbb{R}) \times \mathbb{R}^3$ containing $H_\alpha$ for $\alpha \neq 0$ which is the Lie algebra of $H_\alpha$ are $\mathfrak{V}_1 \times \mathbb{R}^2$, $\mathfrak{V}_1 \times \mathbb{R}^3$, $\mathfrak{V} \times \mathbb{R}^2$, $\mathfrak{V} \times \mathbb{R}^3$, $\mathfrak{W} \times \mathbb{R}^2$, $\mathfrak{W} \times \mathbb{R}^3$, $\mathfrak{R}_1 \times \mathbb{R}^3$, $\mathfrak{Q}_1 \times \mathbb{R}^3$, $\mathfrak{Q}_2 \times \mathbb{R}^3$, $\mathfrak{N} \times \mathbb{R}^3$, $\mathfrak{N}_2 \times \mathbb{R}^2$, $sl(3, \mathbb{R}) \times \mathbb{R}^3$, $\mathfrak{P}$ for $\beta \in \mathbb{R} \setminus \{0\}$. By the correspondence between Lie subgroups and Lie algebras, the conclusion of the Lemma holds.

We are now ready for the proof of Theorem 1.1.

Proof By lemma 5.1, there exists $\lambda, \mu \in \mathbb{R} \setminus \{0\}$ and $(g, v) \in SL(3, \mathbb{R}) \times \mathbb{R}^3$ such that $\lambda Q_\xi((g, v).x) = (Q_0)_{(0,0,\alpha)}(x)$ and $\mu L((g, v).x) = L_0(x)$. Then by Lemma 3.1, it is straight forward to check that

$$SO((Q_0)_{(0,0,\alpha)}, L_0) = \left(\begin{array}{ccc} 1 & \frac{\lambda}{2} & 0 \\ 0 & 1 & \frac{\lambda}{2} \\ 0 & 0 & 1 \end{array}\right) = H_\alpha$$

and hence $SO(Q_\xi, L) = (g, v)H_\alpha(g, v)^{-1}$.

Since $H_\alpha$ is a unipotent subgroup of $G$, by Ratner’s orbit closure theorem (Theorem (1.1.14) and Remark(1.1.19) of [29]) there is a closed connected subgroup $F_\alpha$ of $G$ such that

1. $H_\alpha \subset F_\alpha$
2. the image $[F_\alpha . (g, v)^{-1}]$ of $F_\alpha . (g, v)^{-1}$ in $G / \Gamma$ is closed and has finite $F_\alpha$- invariant measure.
3. the closure of $[H_\alpha . (g, v)^{-1}]$ is equal to $[F_\alpha . (g, v)^{-1}]$ in $G / \Gamma$.

Let $x = (g, v)^{-1} \Gamma \in G / \Gamma$. By (2), $F_\alpha x$ is closed and has finite $F_\alpha$-invariant measure which implies that $F_\alpha$ contains a lattice and hence it is unimodular. Define $f : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$f(x) = (Q_\xi(x), L(x)).$$

Then

$$f(Z^3) = f((g, v)H_\alpha(g, v)^{-1}\Gamma Z^3) \supseteq f((g, v)H_\alpha(g, v)^{-1}\Gamma Z^3) = f((g, v)F_\alpha(g, v)^{-1}Z^3).$$

Case 1: Suppose $L(\xi) = 0$. Then $\alpha = 0$ and by Lemma 5.2, $F_0$ has to be one of the subgroups $V_1 \times \{0\}$, $V_1 \times \mathbb{R}$, $V_1 \times \mathbb{R}^2$, $V_1 \times \mathbb{R}^3$, $V \times \{0\}$, $V \times \mathbb{R}$, $V \times \mathbb{R}^2$, $V \times \mathbb{R}^3$, $W \times \{0\}$, $W \times \mathbb{R}$, $W \times \mathbb{R}^2$, $W \times \mathbb{R}^3$, $Q_1 \times \{0\}$, $Q_1 \times \mathbb{R}$, $Q_1 \times \mathbb{R}^3$, $Q_2 \times \{0\}$, $Q_2 \times \mathbb{R}^2$, $Q_2 \times \mathbb{R}^3$, $N^0 \times \{0\}$, $N^0 \times \mathbb{R}$, $N^0 \times \mathbb{R}^2$, $SL(3, \mathbb{R}) \times \{0\}$, $SL(3, \mathbb{R}) \times \mathbb{R}^3$, $P_\beta$ for $\beta \in \mathbb{R} \setminus \{0\}$. 

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If $F_0$ is one of the subgroups $V_1 \ltimes \mathbb{R}^3$, $V \ltimes \mathbb{R}^3$, $W \ltimes \mathbb{R}^3$, $Q_1 \ltimes \{0\}$, $Q_1 \ltimes \mathbb{R}$, $Q_1 \ltimes \mathbb{R}^3$, $v(t) \text{SO}(Q_0) v(t)^{-1} \ltimes \mathbb{R}^3$, $N^0 \ltimes \mathbb{R}^3$, $\text{SL}(3, \mathbb{R}) \ltimes \{0\}$, $\text{SL}(3, \mathbb{R}) \ltimes \mathbb{R}^3$, $P_\beta$ then it can be easily verified that $(g, v) F_0 (g, v)^{-1} \mathbb{Z}^3 = \mathbb{R}^3$. Therefore, $f(\mathbb{R}^3) \subseteq f(\mathbb{Z}^3)$. Since $f(\mathbb{R}^3) = \mathbb{R}^2$ by Lemma 5.1, the conclusion of the Theorem holds.

Let $P : \text{SL}(3, \mathbb{R}) \ltimes \mathbb{R}^3 \to \text{SL}(3, \mathbb{R})$ denote the natural projection. Since $F_{0x}$ is closed and has finite $F_0$-invariant measure, $(g, v) F_0 (g, v)^{-1} \cap \Gamma$ is a lattice in $(g, v) F_0 (g, v)^{-1}$. Assume that $F_0$ is generated by unipotent elements. Then by Borel density theorem (Proposition 4.7.1, [29]) $(g, v) F_0 (g, v)^{-1} \cap \Gamma$ is Zariski dense in $(g, v) F_0 (g, v)^{-1}$. By Lemma 3.4, the Zariski closure of $(g, v) F_0 (g, v)^{-1} \cap \Gamma$ is defined over $\mathbb{Q}$ and hence $(g, v) F_0 (g, v)^{-1}$ is defined over $\mathbb{Q}$. Therefore $P((g, v) F_0 (g, v)^{-1}) = g P(F_0) g^{-1}$ is also defined over $\mathbb{Q}$. Hence its normalizer $N(g P(F_0) g^{-1}) = g N(P(F_0)) g^{-1}$ is defined over $\mathbb{Q}$.

Suppose $F_0 = v(t) \text{SO}(Q_0)^{\circ} v(t)^{-1} \ltimes \{0\}$. Since $F_0$ is a conjugate of $H$ and $H$ is generated by unipotent elements, by the above argument we get that $(g, v) F_0 (g, v)^{-1}$ is defined over $\mathbb{Q}$. It can be checked that

$$\text{SO}(Q_\xi)^{\circ} = \text{SO}(Q_{\xi} - 2tL^2)^{\circ} = (g, v) F_0 (g, v)^{-1}$$

where $Q' = Q - 2tL^2$. Hence by Lemma 3.5, $Q'_\xi$ is not an irrational quadratic form which implies that $Q_{\xi} - 2tL^2$ is not an irrational quadratic form which is a contradiction.

Suppose $F_0$ is such that $P(F_0)$ is either $W$ or $Q_2$. Since $N(P(F_0))$ is a parabolic subgroup defined over $\mathbb{Q}$ and $g N(P(F_0)) g^{-1}$ is also a parabolic subgroup defined over $\mathbb{Q}$ by (Theorem 20.9, [2]), there exists $\theta \in \text{SL}(3, \mathbb{Q})$ such that

$$\theta g N(P(F_0)) g^{-1} \theta^{-1} = N(P(F_0)).$$

Therefore $\theta g$ normalises $N(P(F_0))$ which implies that $\theta g \in N(P(F_0))$ since the normalizer of a parabolic subgroup is the subgroup itself (Theorem 11.16, [2]). Let $\theta g = h$ where $h \in N(P(F_0))$. Then

$$L^2(x) = L_0^2((g, v)^{-1} x)$$

$$= L_0^2(g^{-1} x - g^{-1} v)$$

$$= L_0^2(h^{-1} (\beta x - g^{-1} v)$$

$$= (\beta(q_1 x_1 + q_2 x_2 + q_3 x_3) + c)^2,$$

for some $\beta, c \in \mathbb{R}$ and $q_1, q_2, q_3 \in \mathbb{Q}$. Hence $L^2$ is not an irrational quadratic form.

Now, let $F_0$ be such that $P(F_0) = V_1$. Since $N(V_1) \subseteq \mathbb{D}^* V$, $g \mathbb{D}^* V g^{-1}$ is defined over $\mathbb{Q}$ and hence its unipotent radical $g \mathbb{V} g^{-1}$ is also defined over $\mathbb{Q}$ (0.23, [21]). Again, since $N(V) = 1 \mathbb{D}^* W$, we get that $g \mathbb{D}^* W g^{-1}$ is defined over $\mathbb{Q}$ and hence its unipotent radical $g \mathbb{V} g^{-1}$ is defined over $\mathbb{Q}$. Similarly, when $P(F_0) = V$, it follows that $g \mathbb{V} g^{-1}$ is defined over $\mathbb{Q}$. By the argument as before, this gives that $L^2$ is not an irrational quadratic form.

If $F_0 = N^0 \ltimes \{0\}$, then since $F_0 x$ is closed, $F_0 \cap (g, v)^{-1} \Gamma (g, v)$ is a lattice in $F_0$. Since $W \ltimes \{0\}$ is the unipotent radical of $F_0$ and $F_0$ is solvable, by (Corollary 8.25, [25]) we get that $W \ltimes \{0\} \cap (g, v)^{-1} \Gamma (g, v)$ is a lattice in $W \ltimes \{0\}$. This implies that $(W \ltimes \{0\}) x$ is closed. Similarly if $F_0 = N_1^0 \ltimes \mathbb{R}$ we get that $(W \ltimes \mathbb{R}) x$ is closed and if $F_0 = N_2^0 \ltimes \mathbb{R}^2$ then $(W \ltimes \mathbb{R}^2) x$ is closed. In each of these cases using the same argument as when $P(F_0) = W$, one can show that $L^2$ is not an irrational quadratic form.
Case 2: Suppose $L(\xi) \neq 0$. Then $\alpha \neq 0$ and by Lemma 5.2, $F_\alpha$ is one of the subgroups $V_1 \times \mathbb{R}^2, V_1 \times \mathbb{R}^3, V \times \mathbb{R}^2, V \times \mathbb{R}^3, W \times \mathbb{R}^2, W \times \mathbb{R}^3, v(t)\text{SO}(\mathbb{R})^0v(t)^{-1} \times \mathbb{R}^3, Q_1 \times \mathbb{R}^3, Q_2 \times \mathbb{R}^2, Q_2 \times \mathbb{R}^3, N^0 \times \mathbb{R}^3, N^0 \times \mathbb{R}^3, N^1 \times \mathbb{R}^2, N^1 \times \mathbb{R}^3, SL(3, \mathbb{R}) \times \mathbb{R}^3, A_\alpha, B_\alpha, P_\beta$ for $\beta \in \mathbb{R}\{0\}$. If $F_\alpha$ is one of the subgroups $V_1 \times \mathbb{R}^3, V \times \mathbb{R}^3, W \times \mathbb{R}^3, v(t)\text{SO}(\mathbb{R})^0v(t)^{-1} \times \mathbb{R}^3, Q_1 \times \mathbb{R}^3, Q_2 \times \mathbb{R}^3, N^0 \times \mathbb{R}^3, N^0 \times \mathbb{R}^3, N^1 \times \mathbb{R}^2, N^1 \times \mathbb{R}^3, SL(3, \mathbb{R}) \times \mathbb{R}^3, A_\alpha, B_\alpha, P_\beta$ for $\beta \in \mathbb{R}\{0\}$ then $(g, v)F_\alpha(g, v)^{-1}Z^3 = \mathbb{R}^3$. Hence $f(\mathbb{Z}^3) \supseteq f(\mathbb{R}^3)$ which implies $f(\mathbb{Z}^3) = \mathbb{R}^2$, since by Lemma 5.1, $f(\mathbb{R}^3) = \mathbb{R}^2$.

If $F_\alpha$ is such that $P(F_\alpha) = V_1, V, W$ or $Q_2$, then by the same argument as in case 1, we get that $L^2$ is not an irrational quadratic form.

If $F_\alpha = N^0_2 \times \mathbb{R}^3$, then again by the same argument as in case 1, we get that $L^2$ is not an irrational quadratic form.

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