CLASSIFICATION OF SOLVABLE LEIBNIZ ALGEBRAS WITH NATURALLY GRADED FILIFORM NILRADICAL

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Abstract. In this paper we show that the method for describing solvable Lie algebras with given nilradical by means of non-nilpotent outer derivations of the nilradical is also applicable to the case of Leibniz algebras. Using this method we extend the classification of solvable Lie algebras with naturally graded filiform Lie algebra to the case of Leibniz algebras. Namely, the classification of solvable Leibniz algebras whose nilradical is a naturally graded filiform Leibniz algebra is obtained.

1. Introduction

Leibniz algebras were introduced at the beginning of the 90s of the past century by J.-L. Loday in [8]. They are a “non-commutative” generalization of Lie algebras. Leibniz algebras inherit an important property of Lie algebras which is that the right multiplication operator on an element of a Leibniz algebra is a derivation. Active investigations on Leibniz algebra theory show that many results of the theory of Lie algebras can be extended to Leibniz algebras. Of course, distinctive properties of non-Lie Leibniz algebras have also been studied [5, 6].

In fact, for a Leibniz algebra we have the corresponding Lie algebra, which is the quotient algebra by the two-sided ideal I generated by the square elements of a Leibniz algebra. Notice that this ideal is the minimal one such that the quotient algebra is a Lie algebra and in the case of non-Lie Leibniz algebras it is always non trivial (moreover, it is abelian).

From the theory of Lie algebras it is well known that the study of finite dimensional Lie algebras was reduced to the nilpotent ones [7, 9]. In Leibniz algebras case we have an analogue of Levi’s theorem [6]. Namely, the decomposition of a Leibniz algebra into a semidirect sum of its solvable radical and a semisimple Lie algebra is obtained. The semisimple part can be described from simple Lie ideals and therefore, the main problem is to study the solvable radical, i.e. in a similar way as in the case of Lie algebras, the description of Leibniz algebras is reduced to the description of the solvable ones. The analysis of works devoted to the study of solvable Lie algebras (for example [11, 13, 14], where solvable Lie algebras with various types of nilradical were studied, such as naturally graded filiform and quasi-filiform algebras, abelian, triangular, etc.) shows that we can also apply similar methods to solvable Leibniz algebras with a given nilradical. Some results of Lie algebras theory generalized to Leibniz algebras [4] allow to apply the technique of description of solvable extensions of nilpotent Lie algebras to the case of Leibniz algebras.

The aim of the present paper is to classify solvable Leibniz algebras with naturally graded filiform nilradical. Thanks to the works [5] and [14], we already have the classification of naturally graded filiform Leibniz algebras.

In order to achieve our goal we organize the paper as follows: in Section 2 we give some necessary notions and preliminary results about Leibniz algebras and solvable Lie algebras with naturally graded filiform radical. Section 3 is devoted to the classification of solvable Leibniz algebras whose nilradical is a naturally graded filiform Lie algebra and in Section 4 we describe, up to isomorphisms, solvable Leibniz algebras whose nilradical is a naturally graded filiform non-Lie Leibniz algebra.

Throughout the paper vector spaces and algebras are finite-dimensional over the field of the complex numbers. Moreover, in the table of multiplication of an algebra the omitted products are assumed to be zero and, if it is not noticed, we shall consider non-nilpotent solvable algebras.

2. Preliminaries

In this section we give necessary definitions and preliminary results.
Definition 2.1. An algebra $(L, [-,-])$ over a field $F$ is called a Leibniz algebra if for any $x, y, z \in L$ the so-called Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

holds.

From the Leibniz identity we conclude that the elements $[y, x]$, $[x, y]$, and for any $x, y \in L$, lie in $\text{Ann}_x(L) = \{ x \in L \mid [y, x] = 0, \text{ for all } y \in L \}$, the right annihilator of the Leibniz algebra $L$. Moreover, we also get that $\text{Ann}_y(L)$ is a two-sided ideal of $L$.

The two-sided ideal $\text{Center}(L) = \{ x \in L \mid [x, y] = 0 \text{ for all } y \in L \}$ is said to be the center of $L$.

Definition 2.2. A linear map $d: L \rightarrow L$ of a Leibniz algebra $(L, [-,-])$ is said to be a derivation if for all $x, y \in L$, the following condition holds:

$$d([x, y]) = [d(x), y] + [x, d(y)].$$

For a given element $x$ of a Leibniz algebra $L$, the right multiplication operators $R_x: L \rightarrow L, R_x(y) = [y, x], y \in L$, are derivations. This kind of derivations are said to be inner derivations. Any Leibniz algebra $L$ has associated the algebra of right multiplications $R(L) = \{ R_x \mid x \in L \}$. $R(L)$ is endowed with a structure of Lie algebra by means of the bracket $[R_x, R_y] = R_x R_y - R_y R_x = R_{[x,y]}$. Moreover, there is an antisymmetric isomorphism between $R(L)$ and the quotient algebra $L/\text{Ann}_x(L)$.

Definition 2.3. For a given Leibniz algebra $(L, [-,-])$ the sequences of two-sided ideals defined recursively as follows:

$$L^1 = L, \; L^{k+1} = [L^k, L], \; k \geq 1, \; L^{[1]} = L, \; L^{[n+1]} = [L^n, L], \; s \geq 1.$$

are said to be the lower central and the derived series of $L$, respectively.

Definition 2.4. A Leibniz algebra $L$ is said to be nilpotent (respectively, solvable), if there exists $n \in \mathbb{N}$ (respectively, $m \in \mathbb{N}$) such that $L^n = 0$ (respectively, $L^m = 0$). The minimal number $n$ (respectively, $m$) with such property is said to be the index of nilpotency (respectively, of solvability) of the algebra $L$.

Evidently, the index of nilpotency of an $n$-dimensional algebra is not greater than $n + 1$.

Definition 2.5. An $n$-dimensional Leibniz algebra $L$ is said to be null-filiform if $\dim L^i = n+1-i$, $1 \leq i \leq n+1$.

Evidently, null-filiform Leibniz algebras have maximal index of nilpotency.

Theorem 2.6 ([5]). An arbitrary $n$-dimensional null-filiform Leibniz algebra is isomorphic to the algebra

$$NF_n: \; \{ e_i, e_1 \} = e_{i+1}, \; 1 \leq i \leq n-1,$$

where $\{ e_1, e_2, \ldots, e_n \}$ is a basis of the algebra $NF_n$.

Actually, a nilpotent Leibniz algebra is null-filiform if and only if it is one-generated algebra. Notice that this notion has no sense in Lie algebras case, because they are at least two-generated.

Definition 2.7. An $n$-dimensional Leibniz algebra $L$ is said to be filiform if $\dim L^i = n-i$, for $2 \leq i \leq n$.

Now let us define a naturally graduation for a filiform Leibniz algebra.

Definition 2.8. Given a filiform Leibniz algebra $L$, put $L_i = L^i/L^{i+1}, 1 \leq i \leq n-1, \text{ and } \text{gr}(L) = L_1 \oplus L_2 \oplus \cdots \oplus L_{n-1}$. Then $[L_i, L_j] \subseteq L_{i+j}$ and we obtain the graded algebra $\text{gr}(L)$. If $\text{gr}(L)$ and $L$ are isomorphic, then we say that an algebra $L$ is naturally graded.

Thanks to [14] it is well known that there are two types of naturally graded filiform Lie algebras. In fact, the second type will appear only in the case when the dimension of the algebra is even.

Theorem 2.9 ([14]). Any complex naturally graded filiform Lie algebra is isomorphic to one of the following non isomorphic algebras:

$$n_{n,1}: \; [e_i, e_1] = -[e_1, e_i] = e_{i+1}, \; 2 \leq i \leq n-1.$$

$$Q_{2n}: \; \begin{cases} 
[e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leq i \leq 2n-2, \\
[e_i, e_{2n+1-i}] = -[e_{2n+1-i}, e_i] = (-1)^i e_{2n}, & 2 \leq i \leq n.
\end{cases}$$
In the following theorem we recall the classification of the naturally graded filiform non-Lie Leibniz algebras given in [5].

**Theorem 2.10** ([5]). Any complex $n$-dimensional naturally graded filiform non-Lie Leibniz algebra is isomorphic to one of the following non isomorphic algebras:

$$F^1_n = \begin{cases} [e_i, e_1] = e_3, \\ [e_i, e_1] = e_{i+1}, \end{cases} \quad 2 \leq i \leq n-1, \quad F^2_n = \begin{cases} [e_i, e_1] = e_3, \\ [e_i, e_1] = e_{i+1}, \end{cases} \quad 3 \leq i \leq n-1.$$

**Definition 2.11.** The maximal nilpotent ideal of a Leibniz algebra is said to be the nilradical of the algebra.

Notice that the nilradical is not the radical in the sense of Kurosh, because the quotient Leibniz algebra by its nilradical may contain a nilpotent ideal (see [7]).

All solvable Lie algebras whose nilradical is the naturally graded filiform Lie algebra $n_{n,1}$ are classified in [12]. Further solvable Lie algebras whose nilradical is the naturally graded filiform Lie algebra $Q_{2n}$ are classified in [2].

Using the above classifications, we shall give the classification of solvable non-Lie Leibniz algebras whose nilradical is a naturally graded filiform Lie algebra.

It is proved that the dimension of a solvable Lie algebra whose nilradical is isomorphic to an $n$-dimensional naturally graded filiform Lie algebra is not greater than $n+2$. Below, we present their classification.

In order to agree with the tables of multiplications of algebras in Theorems 2.9 and 2.10 we make the following change of basis in the classification of [12]:

$$e'_i = e_{n+1-i}, \quad 1 \leq i \leq n, \quad x = -f.$$ We also use different notation to denote the algebras that appear in [12]. That way the results would be:

**Theorem 2.12** ([12]). There are three types of solvable Lie algebras of dimension $n+1$ with nilradical isomorphic to $n_{n,1}$, for any $n \geq 4$. The isomorphism classes in the basis $\{e_1, \ldots, e_n, x\}$ are represented by the following algebras:

$$S_{n+1}(\alpha, \beta) : \begin{cases} [e_i, e_1] = -[e_i, e_i] = e_{i+1}, \\ [e_i, x] = -[x, e_i] = ((i-2)\alpha + \beta) e_i, \\ [e_1, x] = -[x, e_1] = \alpha e_1. \end{cases}$$

The mutually non-isomorphic algebras of this type are $S_{n+1,1}(\beta) = S_{n+1}(1, \beta)$ and $S_{n+1,2} = S_{n+1}(0, 1)$.

$$S_{n+1,3} : \begin{cases} [e_i, e_1] = -[e_i, e_i] = e_{i+1}, \\ [e_i, x] = -[x, e_i] = (i-1) e_i, \\ [e_1, x] = -[x, e_1] = e_1 + e_2. \end{cases}$$

$$S_{n+1,4}(a_3, a_4, \ldots, a_{n-1}) : \begin{cases} [e_i, e_1] = -[e_i, e_i] = e_{i+1}, \\ [e_i, x] = -[x, e_i] = e_i + \sum_{l=i+2}^{n} a_{l+1-i} e_l, \\ 2 \leq i \leq n, \end{cases}$$

where the first non-vanishing parameter $\{a_3, \ldots, a_{n-1}\}$ can be assumed to be equal to 1.

**Theorem 2.13** ([12]). There exists only one class of solvable Lie algebras of dimension $n+2$ with nilradical $n_{n,1}$. It is represented by a basis $\{e_1, e_2, \ldots, e_n, x, y\}$ and the Lie brackets are

$$S_{n+2} : \begin{cases} [e_i, e_1] = -[e_i, e_i] = e_{i+1}, \\ [e_i, x] = -[x, e_i] = (i-2) e_i, \\ [e_1, x] = -[x, e_1] = e_1, \\ [e_i, y] = -[y, e_i] = e_i, \\ 2 \leq i \leq n. \end{cases}$$

Now we recall the classification given in [2] after the following change of basis:

$$e'_1 = -e_1, \quad x' = -Y_1, \quad y' = -Y_2.$$
Therefore, we have that the dimension of $R$ is $n + 1$ with nilradical isomorphic to $Q_{2n}$ is isomorphic to one of the following algebras:

$Q_{2n+1,1}(\alpha):$

\[
\begin{aligned}
&[e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leq i \leq 2n-2, \\
&[e_i, e_{2n+1-i}] = -[e_{2n+1-i}, e_i] = (-1)^i e_{2n}, & 2 \leq i \leq n, \\
&e_1, x = -[x, e_1] = e_1, \\
&e_i, x = -[x, e_i] = (i - 2 + \alpha) e_i, & 2 \leq i \leq 2n-1, \\
&e_{2n}, x = -[x, e_{2n}] = (2n - 3 - 2\alpha) e_{2n}.
\end{aligned}
\]

$Q_{2n+1,2}:$

\[
\begin{aligned}
&[e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leq i \leq 2n-2, \\
&[e_i, e_{2n+1-i}] = -[e_{2n+1-i}, e_i] = (-1)^i e_{2n}, & 2 \leq i \leq n, \\
&e_1, x = -[x, e_1] = e_1 + \varepsilon e_{2n}, & \varepsilon = 0, 1, \\
&e_i, x = -[x, e_i] = (i - n) e_i, & 2 \leq i \leq 2n-1, \\
&e_{2n}, x = -[x, e_{2n}] = e_{2n}.
\end{aligned}
\]

$Q_{2n+1,3}(\alpha):$

\[
\begin{aligned}
&[e_{i}, e_{1}] = -[e_{1}, e_{i}] = e_{i+1}, & 2 \leq i \leq 2n-2, \\
&[e_{i}, e_{2n+1-i}] = -[e_{2n+1-i}, e_{i}] = (-1)^{i} e_{2n}, & 2 \leq i \leq n, \\
[e_{2i}, x] = -[x, e_{2i}] = e_{2i} + \sum_{k=2}^{2n-3-i} \alpha_{2k+1} e_{2k+1+i}, & 0 \leq i \leq 2n-6, \\
[e_{2n-i}, x] = -[x, e_{2n-i}] = e_{2n-i}, & i = 1, 2, 3, \\
[e_{2n}, x] = -[x, e_{2n}] = 2 e_{2n}.
\end{aligned}
\]

**Proposition 2.14** ([2]). Any solvable Lie algebra of dimension $2n+1$ with nilradical isomorphic to $Q_{2n}$ is isomorphic to one of the following algebras:

\[
\begin{aligned}
&[e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leq i \leq 2n-2, \\
&[e_i, e_{2n+1-i}] = -[e_{2n+1-i}, e_i] = (-1)^i e_{2n}, & 2 \leq i \leq n, \\
&e_1, x = -[x, e_1] = e_1, \\
&e_i, x = -[x, e_i] = (i - 2 + \alpha) e_i, & 2 \leq i \leq 2n-1, \\
&e_{2n}, x = -[x, e_{2n}] = (2n - 3 - 2\alpha) e_{2n}.
\end{aligned}
\]

Let $R$ be a solvable Leibniz algebra with nilradical $N$. We denote by $Q$ the complementary vector space of the nilradical to the algebra $R$. Let us consider the restrictions to $N$ of the right multiplication operator on an element $x \in Q$ (denoted by $R_{x|N}$). If the operator $R_{x|N}$ is nilpotent, then we assert that the subspace $(x + N)$ is a nilpotent ideal of the algebra $R$. Indeed, since for a solvable Leibniz algebra $R$ we get the inclusion $R^2 \subseteq N$ [4], and hence the subspace $(x + N)$ is an ideal. The nilpotency of this ideal follows from the Engel’s theorem for Leibniz algebras [4]. Therefore, we have a nilpotent ideal which strictly contains the nilradical, which is in contradiction with the maximality of $N$. Thus, we obtain that for any $x \in Q$, the operator $R_{x|N}$ is a non-nilpotent derivation of $N$.

Let $\{x_1, \ldots, x_m\}$ be a basis of $Q$, then for any scalars $\{\alpha_1, \ldots, \alpha_m\} \in \mathbb{C} \setminus \{0\}$, the matrix $\alpha_1 R_{x_1|N} + \ldots + \alpha_m R_{x_m|N}$ is not nilpotent, which means that the elements $\{x_1, \ldots, x_m\}$ are nil-independent [10]. Therefore, we have that the dimension of $Q$ is bounded by the maximal number of nil-independent derivations of the nilradical $N$. Moreover, similarly to the case of Lie algebras, for a solvable Leibniz algebra $R$ the inequality $\dim N \geq \frac{\dim R - 1}{2}$ holds.

### 3. Solvable Leibniz algebras whose nilradical is a Lie algebra

It is not difficult to see that if $R$ is a solvable non-Lie Leibniz algebra with nilradical isomorphic to the algebras $n_{n,1}$ or $Q_{2n}$, then the dimension of $R$ is also not greater than $n + 2$ and $2n + 2$, respectively.

Let $n_{n,1}$ or $Q_{2n}$ be the nilradical of a solvable Leibniz algebra $R$. Since the ideal $I = \langle \{[x, x] \mid x \in R\} \rangle$ is contained in $\text{Ann}_r(R)$, then $I$ is abelian, hence it is contained in the nilradical. Taking into account the multiplication in $n_{n,1}$ (respectively $Q_{2n}$) we conclude that $I = \langle \{e_n\} \rangle$. 


Having in mind that an \((n + 1)\)-dimensional algebra \(R\) is solvable, then the quotient algebra \(R/I\) is also a solvable Lie algebra with nilradical \(n_{n,1}\) (whose lists of tables of multiplication are given in Theorems 2.12 and 2.13).

**Case** \(n_{n,1}\). Let us assume that \(R\) has dimension \(n + 1\), then the table of multiplication in \(R\) will be equal to the table of multiplication of \(S_{n+1,i}\), \((i = 1, 2, 3, 4)\), except the following products:

\[
\begin{align*}
[e_1, x] &= \alpha_1 e_1 + \gamma_4 e_n, & [e_2, x] &= \beta_1 e_2 + \gamma_5 e_n, \\
[x, e_1] &= -\alpha_1 e_1 + \gamma_1 e_n, & [x, e_2] &= -\beta_1 e_2 + \gamma_2 e_n, & [x, x] &= \gamma_3 e_n,
\end{align*}
\]

where \(\{\gamma_1 + \gamma_4, \gamma_2 + \gamma_5, \gamma_3\} \neq \{0, 0, 0\}\).

Note that taking the change of basis

\[
e_1' = \alpha e_1 + \gamma_4 e_n, \quad e_2' = \beta_1 e_2 + \gamma_5 e_n
\]

we can assume that \(\gamma_4 = \gamma_5 = 0\), i.e., \([e_1, x] = \alpha e_1\) and \([e_2, x] = \beta_1 e_2\).

It is not difficult to see that, for the omitted products, the antisymmetric identity holds, i.e.

\[
\begin{align*}
[e_i, e_1] &= -[e_1, e_i] = e_{i+1}, & 2 \leq i \leq n - 1, \\
[e_i, x] &= -[x, e_i] = 0, & 3 \leq i \leq n.
\end{align*}
\]

We have \([e_n, x] = 0\) because \(0 = [x, e_n] = -[e_n, x]\).

Consider

\[
0 = [x, e_n] = [x, [e_{n-1}, e_1]] = [[x, e_{n-1}], e_1] = [(x, e_1), e_{n-1}] = -(n - 2 + \beta) e_n.
\]

In the list of Theorem 2.12 only the algebra \(S_{n+1,1}(\beta)\) is representative of the class for which the equality \([e_n, x] = 0\) holds. This class is defined by \(\beta = 2 - n\).

Therefore, in the case of \(\dim R = n + 1\) whose nilradical is \(n_{n,1}\), we have the following family:

\[
R_{n+1,1}(\gamma_1, \gamma_2, \gamma_3) : \begin{cases}
[e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leq i \leq n - 1, \\
[e_1, x] = e_1, \\
[x, e_1] = -e_1 + \gamma_1 e_n, \\
[e_2, x] = (2 - n) e_2, \\
[x, e_2] = (n - 2) e_2 + \gamma_2 e_n, \\
[e_i, x] = -[x, e_i] = (i - n) e_i, & 3 \leq i \leq n - 1, \\
[x, x] = \gamma_3 e_n,
\end{cases}
\]

where \(\{\gamma_1, \gamma_2, \gamma_3\} \neq \{0, 0, 0\}\).

Applying a similar argument and the table of multiplication of the algebra in Theorem 2.13 we conclude that solvable non-Lie Leibniz algebras of dimension \(n + 2\) with nilradical \(n_{n,1}\) do not exist.

**Theorem 3.1.** Any \((n + 1)\)-dimensional solvable Leibniz algebra with nilradical \(n_{n,1}\) is isomorphic to one of the following pairwise non isomorphic algebras:

\[
R_{n+1,1}(0, 0, 1), \quad R_{n+1,1}(0, 1, 0), \quad R_{n+1,1}(1, 1, 0), \quad R_{n+1,1}(1, 0, 0).
\]

**Proof.** We consider the general change of basis in the family \(R_{n+1,1}(\gamma_1, \gamma_2, \gamma_3)\):

\[
e_i' = \sum_{i=1}^{n} A_i e_i, \quad e_2' = \sum_{i=1}^{n} B_i e_i, \quad x' = D x + \sum_{i=1}^{n} C_i e_i,
\]

where \((A_1 B_2 - B_1 A_2) D \neq 0\).

Using \([e_i', e_{i+1}] = e'_{i+1}, 2 \leq i \leq n - 1\), the table of multiplication of \(R_{n+1,1}(\gamma_1, \gamma_2, \gamma_3)\) and an induction, we obtain

\[
e_i' = A_1^{i-3} \sum_{j=i}^{n} (A_1 B_{j+2-i} - B_1 A_{j+2-i}) e_j, \quad 3 \leq i \leq n.
\]

From the equalities

\[
0 = [e_3', e_2'] = B_1 \sum_{j=4}^{n} (A_1 B_{j-2} - B_1 A_{j-2}) e_j
\]

we have \(B_1 = 0\).

Consider the multiplications
\[ [e'_1, x'] = A_1 D e_1 - D \sum_{i=2}^{n-1} A_i (n - i) e_i + \sum_{i=3}^{n} (A_{i-1} C_1 - A_1 C_{i-1}) e_i \]
\[ = A_1 D e_1 - A_2 D (n - 2) e_2 + \sum_{i=3}^{n-1} (A_{i-1} C_1 - A_1 C_{i-1} - (n - i) A_i D) e_i \]
\[ + (A_{n-1} C_1 - A_1 C_{n-1}) e_n. \]

On the other hand
\[ [e'_1, x'] = e'_1 = \sum_{i=1}^{n} A_i e_i. \]

Comparing the coefficients of the basic elements we derive:
\[ D = 1, \quad A_2 = 0, \quad A_{i+1} = \frac{A_1 C_i - A_i C_{i-1}}{i - n - 1}, \quad 2 \leq i \leq n - 2, \quad A_n = A_1 C_{n-1} - A_{n-1} C_n. \]

From the equalities
\[ -(n - 2) \sum_{i=2}^{n} B_i e_i = -(n - 2) e'_2 = [e'_1, x'] = \left[ \sum_{i=2}^{n} B_i e_i, x + \sum_{i=1}^{n} C_i e_i \right] \]
\[ = - \sum_{i=2}^{n-1} B_i (n - i) e_i + C_1 \sum_{i=3}^{n} B_{i-1} e_i \]
\[ = - B_2 (n - 2) e_2 + \sum_{i=3}^{n-1} (B_{i-1} C_1 - B_i (n - i)) e_i + B_{n-1} C_1 e_n, \]

we deduce the following restrictions:
\[ B_i = (-1)^i \frac{B_2 C_{i-1}^{n-2}}{(i - 2)!}, \quad 3 \leq i \leq n. \]

In an analogous way, comparing coefficients at the basic element \( e_n \) in the equalities, we obtain:
\[ \gamma_3 A_1^{n-2} B_2 e_n = \gamma_3' e'_n = [x', x'] = (\gamma_3 + C_1 \gamma_1 + C_2 \gamma_2) e_n \]
and so
\[ \gamma_3' = \frac{\gamma_3 + C_1 \gamma_1 + C_2 \gamma_2}{A_1^{n-2} B_2}. \]

With a similar argument,
\[ -e'_1 + A_1^{n-2} B_2 \gamma'_1 e_n = -e'_1 + \gamma'_1 e'_n = [x', e'_1] = -e'_1 + A_1 \gamma_1 e_n. \]

and
\[ -(n - 2) e'_2 + A_1^{n-2} B_2 \gamma'_2 e_n = (n - 2) e'_2 + \gamma'_2 e'_2 = [x', e'_2] = (n - 2) e'_2 + B_2 \gamma_2 e_n \]
we obtain
\[ \gamma'_1 = \frac{\gamma_1}{A_1^{n-3} B_2} \quad \text{and} \quad \gamma'_2 = \frac{\gamma_2}{A_1^{n-2}}. \]

Now we shall consider the possible cases of the parameters \( \{ \gamma_1, \gamma_2, \gamma_3 \} \).

**Case 1.** Let \( \gamma_1 = 0 \). Then \( \gamma'_1 = 0 \).
If \( \gamma_2 = 0 \), then \( \gamma'_2 = 0 \) and \( \gamma'_3 = \frac{-C_1}{A_1^{n-2} B_2} \neq 0 \). Putting \( B_2 = \frac{-C_1}{A_1^{n-2} B_2} \), then we have that \( \gamma'_3 = 1 \), so the algebra is \( R_{n+1,1}(0,0,1) \).
If \( \gamma_2 \neq 0 \), then putting \( A_1 = -\sqrt[n]{\gamma_2} \) and \( C_2 = -\frac{2\gamma_2}{\gamma_2} \), we get \( \gamma'_2 = 1 \) and \( \gamma'_3 = 0 \), i.e. we obtain the algebra \( R_{n+1,1}(0,1,0) \).

**Case 2.** Let \( \gamma_1 \neq 0 \). Then putting \( B_2 = \frac{-C_1}{A_1^{n-3}} \) and \( C_1 = -\frac{2\gamma_2 + C_2 \gamma_3}{\gamma_1} \), we have:
\[ \gamma'_1 = 1, \quad \gamma'_2 = \frac{\gamma_2}{A_1^{n-2}}, \quad \gamma'_3 = 0. \]

If \( \gamma_2 \neq 0 \), then putting \( A_1 = -\sqrt[n]{\gamma_2} \) we have that \( \gamma'_2 = 1 \), so we obtain the algebra \( R_{n+1,1}(1,1,0) \).
If \( \gamma_2 = 0 \), then we get the algebra \( R_{n+1,1}(1,0,0) \).
Theorem 3.2. Any derivation of the algebra \( R_{2n+1} \) isomorphic to one of the following pairwise non isomorphic algebras:

\[
R_{2n+1,1} : \begin{cases}
[e_i, e_j] = -[e_1, e_i] = e_{i+1}, & 2 \leq i \leq 2n-2, \\
[e_i, e_{2n+1-i}] = -[e_{2n+1-i}, e_i] = (-1)^i e_{2n}, & 2 \leq i \leq n, \\
[e_1, x] = e_1, & \\
x, e_1 = -e_1 + \gamma_1 e_n, & \\
[e_2, x] = \frac{2n-3}{2} e_2, & \\
[x, e_2] = -\frac{2n-3}{2} e_2 + \gamma_2 e_n, & 3 \leq i \leq 2n-1, \\
[e_i, x] = -[x, e_i] = \frac{2n+2i-7}{2} e_i, & 3 \leq i \leq 2n-1, \\
[x, x] = \gamma_3 e_n,
\end{cases}
\]

where \( (\gamma_1, \gamma_2, \gamma_3) \neq (0, 0, 0) \).

\[
\text{Theorem 3.2. Any } (2n+1)\text{-dimensional solvable Leibniz algebra with nilradical } Q_{2n} \text{ is isomorphic to one of the following pairwise non isomorphic algebras:}
\]
\[
R_{2n+1,1}(0, 0, 1), \quad R_{2n+1,1}(0, 1, 0), \quad R_{2n+1,1}(1, 1, 0), \quad R_{2n+1,1}(1, 0, 0).
\]

\[
\text{Proof. The proof is carried out by applying similar arguments as in the proof of Theorem 3.1.}\]

4. SOLVABLE LEIBNIZ ALGEBRAS WHOSE NILRADICAL IS A NON-LIE LEIBNIZ ALGEBRA

In the following proposition we describe derivations of the algebra \( F_n^1 \).

Proposition 4.1. Any derivation of the algebra \( F_n^1 \) has the following matrix form:

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_{n-1} & \alpha_n \\
0 & \alpha_1 + \alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_{n-1} & \beta \\
0 & 0 & 2\alpha_1 + \alpha_2 & \alpha_3 & \ldots & \alpha_{n-2} & \alpha_{n-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & (n-1)\alpha_1 + \alpha_2
\end{pmatrix}
\]

\[
\text{Proof. Let } d \text{ be a derivation of the algebra. We set }
\]
\[
d(e_1) = \sum_{i=1}^{n} \alpha_i e_i, \quad d(e_2) = \sum_{i=1}^{n} \beta_i e_i.
\]

From the equality
\[
0 = d([e_1, e_2]) = [d(e_1), e_2] + [e_1, d(e_2)] = \beta_1 e_3
\]

we get \( \beta_1 = 0 \).

Further we have
\[
d(e_3) = d([e_1, e_1]) = [d(e_1), e_1] + [e_1, d(e_1)] = (2\alpha_1 + \alpha_2) e_3 + \sum_{i=3}^{n-1} \alpha_i e_{i+1}.
\]

On the other hand
\[
d(e_3) = d([e_2, e_1]) = [d(e_2), e_1] + [e_2, d(e_1)] = (\alpha_1 + \beta_2) e_3 + \sum_{i=3}^{n-1} \beta_i e_{i+1}.
\]

Therefore, \( \beta_2 = \alpha_1 + \alpha_2, \beta_i = \alpha_i, \quad 3 \leq i \leq n-1. \)

With similar arguments applied on the products \( [e_i, e_1] = e_{i+1} \) and with an induction on \( i \), it is easy to check that the following identities hold for \( 3 \leq i \leq n \):

\[
d(e_i) = ((i-1)\alpha_1 + \alpha_2) e_i + \sum_{j=i+1}^{n} \alpha_{j-i+2} e_j, \quad 3 \leq i \leq n.
\]
From Proposition 4.1 we conclude that the number of nil-independent outer derivations of the algebra $F_n^1$ is equal to two. Therefore, by arguments after Proposition 2.15 we have that any solvable Leibniz algebra whose nilradical is $F_n^1$ has dimension either $n + 1$ or $n + 2$.

4.1. Solvable Leibniz algebras with nilradical $F_n^1$.

Below we present the description of such Leibniz algebras when dimension is equal to $n + 1$.

**Theorem 4.2.** An arbitrary $(n + 1)$-dimensional solvable Leibniz algebra with nilradical $F_n^1$ is isomorphic to one of the following pairwise non-isomorphic algebras:

\[
R_1(\alpha) = \begin{cases} 
[e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, \quad 2 \leq i \leq n - 1, \\
[e_1, x] = -e_1, & [e_2, x] = -e_2 + \alpha e_n, \quad \alpha \in \{0, 1\} \\
x, e_1 = e_1, & [e_i, x] = -(i - 1)e_i, \quad 3 \leq i \leq n.
\end{cases}
\]

\[
R_2(\alpha_4, \ldots, \alpha_{n-1}, \alpha) = \begin{cases} 
[e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, \quad 2 \leq i \leq n - 1, \\
[e_1, x] = e_2 + \sum_{i=4}^{n-1} \alpha_i e_i + \alpha e_n, & [e_2, x] = e_2 + \sum_{i=4}^{n-1} \alpha_i e_i, \\
x, e_1 = e_1, & [e_i, x] = e_i + \sum_{j=i+2}^{n} \alpha_{j-i+2} e_j, \quad 3 \leq i \leq n.
\end{cases}
\]

\[
R_3(\alpha_4, \ldots, \alpha_{n-1}) = \begin{cases} 
[e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, \quad 2 \leq i \leq n - 1, \\
[e_1, x] = e_2 + \sum_{i=4}^{n-1} \alpha_i e_i, & [e_2, x] = e_2 + \sum_{i=4}^{n-1} \alpha_i e_i + e_n, \\
x, e_1 = e_1, & [e_i, x] = e_i + \sum_{j=i+2}^{n} \alpha_{j-i+2} e_j, \quad 3 \leq i \leq n.
\end{cases}
\]

Moreover, the first non-vanishing parameter $\{\alpha_4, \ldots, \alpha_{n-1}\}$ in the algebras $R_2(\alpha_4, \ldots, \alpha_{n-1}, \alpha)$ and $R_3(\alpha_4, \ldots, \alpha_{n-1})$ can be scaled to 1.

**Proof.** From Theorem 2.10 and arguments after Proposition 2.15 we know that there exists a basis $\{e_1, e_2, \ldots, e_n, x\}$ such that the multiplication table of the algebra $F_n^1$ is completed with the products coming from $R_{x, e_i} = (e_i, 1 \leq i \leq n$, i.e.

\[
[e_1, x] = \sum_{i=1}^{n} \alpha_i e_i, \quad [e_2, x] = (\alpha_1 + \alpha_2) e_2 + \sum_{i=3}^{n-1} \alpha_i e_i + \beta e_n, \\
[e_i, x] = ((i - 1)\alpha_1 + \alpha_2) e_i + \sum_{j=i+1}^{n} \alpha_{j-i+2} e_j, \quad 3 \leq i \leq n.
\]

Finally, we consider the remainder products as follows:

\[
[x, e_1] = \sum_{i=1}^{n} \beta_i e_i, \quad [x, e_2] = \sum_{i=1}^{n} \gamma_i e_i, \quad [x, x] = \sum_{i=1}^{n} \delta_i e_i.
\]

From the chain of equalities

\[
0 = [x, e_3] = [x, [e_2, e_1]] = [[x, e_2], e_1] - [[x, e_1], e_2] = [[x, e_2], e_1] = (\gamma_1 + \gamma_2) e_3 + \sum_{i=1}^{n} \gamma_{i-1} e_i
\]

we conclude that $\gamma_2 = -\gamma_1$, $\gamma_i = 0$, $3 \leq i \leq n - 1$.

Since $\gamma_1 e_3 = [e_1, [x, e_2]] = [[e_1, x], e_2] - [[e_1, e_2], x] = 0$, then $\gamma_1 = 0$.

The identity

\[
[e_1, [x, e_1]] = [[e_1, x], e_1] - [[e_1, e_1], x]
\]

implies $\beta_1 = -\alpha_1$.

Applying the Leibniz identity to the elements of the form $\{x, x, e_2\}$ and $\{x, e_2, x\}$, we conclude that:

\[
\begin{array}{l}
((n - 1)\alpha_1 + \alpha_2) \gamma_n = 0, \\
(n - 2)\alpha_1 \gamma_n = 0.
\end{array}
\]
Note that $\gamma_n = 0$ (otherwise $\alpha_1 = \alpha_2 = 0$ and then we get a contradiction with the non-nilpotency of the derivation $D$ (see Proposition 4.1)).

Now we are going to discuss the possible cases of the parameters $\alpha_1$ and $\alpha_2$.

**Case 1.** Let $\alpha_1 \neq 0$. Then taking the following change of basis:

$$x' = -\frac{1}{\alpha_1} x, \quad e'_1 = e_1 - \frac{1}{\alpha_1} \sum_{i=2}^{n} \beta_i e_i, \quad e'_i = -\frac{1}{\alpha_1}((-\alpha_1 + \beta_2) e_i + \sum_{j=i+1}^{n} \beta_{j-i+2} e_j), \quad 2 \leq i \leq n,$$

we obtain

$$[e_1, e_1] = e_3, \quad [e_1, x] = \sum_{i=1}^{n} \mu_i e_i, \quad [e_i, e_1] = e_{i+1}, \quad 2 \leq i \leq n - 1,$$

$$[x, e_1] = e_1, \quad [e_2, x] = \sum_{i=1}^{n} \eta_i e_i, \quad [x, e_2] = 0, \quad [x, x] = \sum_{i=1}^{n} \theta_i e_i.$$

From the equalities

$$0 = [[e_1, e_2], x] = [e_1, [e_2, x]] + [e_1, [e_1, x]] = [e_1, \sum_{i=1}^{n} \eta_i e_i] = \eta_1 e_3 \text{ we have } \eta_1 = 0.$$

Consider

$$[e_3, x] = [[e_1, e_1], x] = [e_1, [e_1, x]] + [e_1, [e_1, x]] = \mu_1 e_3 + \mu_1 e_3 + \sum_{i=3}^{n-1} \mu_i e_{i+1} = (2\mu_1 + \mu_2) e_3 + \sum_{i=3}^{n-1} \mu_i e_{i+1}.$$

On the other hand

$$[e_3, x] = [[e_2, e_1], x] = [e_2, [e_1, x]] + [e_2, [e_1, x]] = \mu_1 e_3 + \eta_2 e_3 + \sum_{i=3}^{n-1} \eta_i e_{i+1} = (\mu_1 + \eta_2) e_3 + \sum_{i=3}^{n-1} \eta_i e_{i+1}.$$

The comparison of both linear combinations implies that:

$$\eta_2 = \mu_1 + \mu_2, \quad \eta_i = \mu_i, \quad 3 \leq i \leq n - 1,$$

that it is to say:

$$[e_2, x] = (\mu_1 + \mu_2) e_2 + \sum_{i=3}^{n-1} \mu_i e_i + \eta_n e_n \quad \text{and} \quad [e_3, x] = (2\mu_1 + \mu_2) e_3 + \sum_{i=3}^{n-1} \mu_i e_{i+1}.$$

Now we shall prove the following equalities by an induction on $i$:

$$[e_i, x] = ((i - 1)\mu_1 + \mu_2) e_i + \sum_{j=i+1}^{n} \mu_j e_{j+1}, \quad 3 \leq i \leq n. \quad (1)$$

Obviously, the equality holds for $i = 3$. Let us assume that the equality holds for $3 < i < n$, and we prove it for $i + 1$:

$$[e_{i+1}, x] = [[e_i, e_1], x] = [e_i, [e_1, x]] + [e_i, [e_1, x]] = \mu_1 e_{i+1} + (i - 1)\mu_1 e_3 + \sum_{j=i+1}^{n} \mu_j e_{j+1} = (i\mu_1 + \mu_2) e_{i+1} + \sum_{j=i+2}^{n} \mu_j e_j;$$

so the induction proves the equalities (1) for any $i, 3 \leq i \leq n$.

Applying the Leibniz identity to the elements $\{e_1, x, e_1\}, \{e_1, x, x\}, \{x, e_1, x\}$, we deduce that:

$$\mu_1 = -1, \quad \mu_2 = \theta_1 = 0, \quad \theta_i = \mu_{i+1}, \quad 2 \leq i \leq n - 1.$$
Below we summarize the table of multiplication of the algebra

\[
\begin{align*}
[e_1, e_1] &= e_3 & [e_i, e_1] &= e_{i+1}, & 2 \leq i \leq n-1, \\
[e_1, x] &= -e_1 + \sum_{i=3}^{n} \mu_i e_i, & [e_2, x] &= -e_2 + \sum_{i=3}^{n-1} \mu_i e_i + \eta_n e_n, \\
[x, e_1] &= e_1, & [e_i, x] &= -(i-1) e_i + \sum_{j=1+1}^{n} \mu_{j-i+2} e_j, & 3 \leq i \leq n, \\
[x, x] &= \sum_{i=2}^{n-1} \mu_{i+1} e_i + \theta_n e_n.
\end{align*}
\]

Let us take the change of basis in the following form:

\[
e'_1 = e_1 + \sum_{i=3}^{n} A_i e_i, \quad e'_2 = e_2 + \sum_{i=3}^{n} A_i e_i, \quad e'_i = e_i + \sum_{j=i+1}^{n} A_{j-i+2} e_j, \quad 3 \leq i \leq n, \quad x' = \sum_{i=2}^{n-1} A_{i+1} e_i + B e_n + x.
\]

where

\[
A_3 = \mu_3, \quad A_i = \frac{1}{i-2} (\mu_i + \sum_{j=3}^{i-1} A_j \mu_{i-j+2}), \quad 4 \leq i \leq n \quad \text{and} \quad B = \frac{1}{n-1} (\theta_n + \sum_{j=3}^{n} A_j \mu_{n-j+3}).
\]

Then

\[
[x', e'_1] = \left[ \sum_{i=2}^{n-1} A_{i+1} e_i + B e_n + x, e_1 \right] = e_1 + \sum_{i=3}^{n} A_i e_i = e'_1,
\]

\[
[e'_1, x'] = [e_1, x] + \sum_{i=3}^{n} A_i [e_i, x] = -e_1 + \sum_{i=3}^{n} \mu_i e_i + \sum_{i=3}^{n} A_i \left( -(i-1) e_i + \sum_{j=i+1}^{n} \mu_{j-i+2} e_j \right)
\]

\[
= -e_1 - \sum_{i=3}^{n} A_i e_i + \sum_{i=3}^{n} \mu_i e_i - \sum_{i=3}^{n} A_i (i-2) e_i + \sum_{i=3}^{n} A_i \left( \sum_{j=i+1}^{n} \mu_{j-i+2} e_j \right)
\]

\[
= -e_1 - \sum_{i=3}^{n} A_i e_i + \sum_{i=3}^{n} \mu_i e_i - \sum_{i=3}^{n} A_i (i-2) e_i + \sum_{i=4}^{n} \left( \sum_{j=3}^{i-1} A_j \mu_{j-i+2} \right) e_i
\]

\[
= -e_1 - \sum_{i=3}^{n} A_i e_i + (\mu_3 - A_3) e_3 + \sum_{i=4}^{n-1} \left( -A_i (i-2) + \mu_i + \sum_{j=3}^{i-1} A_j \mu_{j-i+2} \right) e_i
\]

\[
= -e_1 - \sum_{i=3}^{n} A_i e_i = -e'_1
\]

\[
[e'_2, x'] = [e_2, x] + \sum_{i=3}^{n} A_i [e_i, x] = -e_2 + \sum_{i=3}^{n-1} \mu_i e_i + \eta_n e_n + \sum_{i=3}^{n} A_i \left( -(i-1) e_i + \sum_{j=i+1}^{n} \mu_{j-i+2} e_j \right)
\]

\[
= -e_2 - \sum_{i=3}^{n} A_i e_i + \sum_{i=3}^{n-1} \mu_i e_i + \eta_n e_n - \sum_{i=3}^{n} A_i (i-2) e_i + \sum_{i=3}^{n} A_i \left( \sum_{j=i+1}^{n} \mu_{j-i+2} e_j \right)
\]

\[
= -e_2 - \sum_{i=3}^{n} A_i e_i + \sum_{i=3}^{n-1} \mu_i e_i + \eta_n e_n - \sum_{i=3}^{n} A_i (i-2) e_i + \sum_{i=4}^{n} \left( \sum_{j=3}^{i-1} A_j \mu_{j-i+2} \right) e_i
\]

\[
= -e_2 - \sum_{i=3}^{n} A_i e_i + (\mu_3 - A_3) e_3 + \sum_{i=4}^{n-1} \left( -A_i (i-2) + \mu_i + \sum_{j=3}^{i-1} A_j \mu_{j-i+2} \right) e_i
\]

\[
+ \left( \eta_n - (n-2) A_n + \sum_{i=3}^{n-1} A_i \mu_{n-i+2} \right) e_n = -e'_2 + \eta e'_n,
\]
\[ [x', x'] = \sum_{i=2}^{n-1} A_{i+1}[e_i, x] + B[e_n, x] + [x, x] \]

\[ = \sum_{i=2}^{n-1} A_{i+1} \left( -(i - 1)e_i + \sum_{j=i+1}^{n} \mu_{j-i+2}e_j \right) - B(n-1)e_n + \sum_{i=2}^{n-1} \mu_{i+1}e_i + \theta_ne_n \]

\[ = -\sum_{i=2}^{n-1} A_{i+1}(i - 1)e_i + \sum_{i=2}^{n-1} \mu_{i+1}e_i - B(n-1)e_n + \theta_ne_n + \sum_{i=2}^{n-1} A_{i+1} \left( \sum_{j=i+1}^{n} \mu_{j-i+2}e_j \right) \]

\[ = -\sum_{i=2}^{n-1} A_{i+1}(i - 1)e_i + \sum_{i=2}^{n-1} \mu_{i+1}e_i - B(n-1)e_n + \theta_ne_n + \sum_{i=3}^{n} \left( \sum_{j=3}^{i} A_{j}\mu_{i-j+3} \right) e_i \]

\[ = (\mu_3 - A_3)e_2 + \sum_{i=3}^{n-1} \left( -A_{i+1}(i - 1) + \mu_{i+1} + \sum_{j=3}^{i} A_{j}\mu_{i-j+3} \right) e_i \]

\[ + \sum_{j=3}^{n} \left( -B(n-1) + \theta_n + \sum_{j=3}^{n} A_{j}\mu_{n-j+3} \right) e_n = 0 . \]

With a similar induction as the given for equations (11), it is easy to check that the following equalities hold:

\[ [e_i, x] = -(i - 1)e_i, \quad 3 \leq i \leq n. \]

Thus, we obtain the following table of multiplication:

\[
\begin{align*}
[e_1, e_1] &= e_3, \quad [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n - 1, \\
[e_1, x] &= -e_1, \quad [e_2, x] = -e_2 + \eta e_n, \\
[x, e_1] &= e_1, \quad [e_i, x] = -(i - 1)e_i, & 3 \leq i \leq n.
\end{align*}
\]

Now we take the general change of basis in the following form:

\[
\begin{align*}
e_i' &= \sum_{i=1}^{n} A_i e_i, \\
e_i' &= A_i^{-1}2 \left( (A_1 + A_2) e_i + \sum_{j=i+1}^{n} A_j e_j \right), & 2 \leq i \leq n, \\
x' &= \sum_{i=1}^{n} B_i e_i + B_{n+1} x, \quad \text{where } A_1(A_1 + A_2)B_{n+1} \neq 0.
\end{align*}
\]

Then from the equalities

\[ \sum_{i=1}^{n} A_i e_i = e_1' = [x', e_1'] = A_1B_{n+1}e_1 + A_1(B_1 + B_2)e_3 + A_1 \left( \sum_{j=4}^{n} B_{j-1}e_j \right) \]

we obtain:

\[ B_{n+1} = 1, \quad A_2 = 0, \quad B_1 + B_2 = A_3/A_1, \quad B_i = A_{i+1}/A_1, \quad 3 \leq i \leq n - 1. \]

Similarly, from

\[ -A_1 e_1 - \sum_{i=3}^{n} A_i e_i = -e_1' = [e_1', x'] = -A_1 e_1 + (A_1B_1 - 2A_3)e_3 + \sum_{i=4}^{n} (B_1A_{i-1} - (i - 1)A_i)e_i \]

we obtain:

\[ B_1 = A_3/A_1, \quad A_i = \frac{A_3A_{i-1}}{(i - 2)A_1}, \quad 4 \leq i \leq n, \]

consequently

\[ B_2 = 0, \quad A_i = \frac{A_3^{i-2}}{(i - 2)!A_1^{i-3}}, \quad 4 \leq i \leq n. \]

Consider the product \([e_i', x']\), namely:
we derive
we can assume that
we can assume that
Therefore, the parameter \( \eta' \) satisfies the relation \( \eta' = \frac{1}{A_1^{n-2}} \eta \).
If \( \eta = 0 \), then \( \eta' = 0 \). If \( \eta \neq 0 \), then choosing \( A_1 \) such that \( A_1^{n-2} = \eta \), we conclude \( \eta' = 1 \). Thus the algebras \( R_1(\alpha) \), for \( \alpha \in \{0, 1\} \), are obtained.

Case 2. Let \( \alpha_1 = 0, \alpha_2 \neq 0 \). Then making the following change of basis

we can assume that \( [x, e_1] = \beta_2 e_2 \).
From the identity
we derive
consequently, \( \beta_2 = 0, \delta_i = 0, 2 \leq i \leq n - 1 \).
Making the change of basis
we can assume that \( [x, x] = 0 \).
Summarizing, we obtain the following table of multiplication of the algebra in this case:

Now we shall study the behaviour of the parameters in this family of algebras under the general change of basis in the form (2).
Then the equalities

imply \( B_1 = -B_2, B_i = 0, 3 \leq i \leq n - 1 \).
Now we shall express the product \([e'_1, x']\) as a linear combination of the basis \(\{e_1, e_2, \ldots, e_n, x\}\), namely:

\[
[e'_1, x'] = \left[\sum_{i=1}^{n} A_i e_i, B_1 e_1 + B_{n+1} x\right]
= B_1 \left((A_1 + A_2)e_3 + \sum_{i=4}^{n} A_{i-1} e_i\right)
+ B_{n+1} \left(A_1 \sum_{i=2}^{n} \alpha_i e_i + A_2 \left(\sum_{i=2}^{n-1} \alpha_i e_i + \beta e_n\right) + \sum_{i=3}^{n} A_i \sum_{j=1}^{n} \alpha_{j-i+2} e_j\right)
= B_1 (A_1 + A_2)e_3 + \sum_{i=4}^{n} B_1 A_{i-1} e_i
+ B_{n+1} (A_1 \sum_{i=2}^{n} \alpha_i e_i + B_{n+1} A_2 \sum_{i=2}^{n-1} \alpha_i e_i + B_{n+1} \sum_{i=3}^{n} \sum_{j=3}^{i} A_j \alpha_{i-j+2} e_i
= B_{n+1} (A_1 + A_2) \alpha_2 e_2 + \left((A_1 + A_2) (B_1 + B_{n+1} \alpha_3) + B_{n+1} A_3 \alpha_2\right) e_3
+ \sum_{i=4}^{n} \left(B_1 A_{i-1} + B_{n+1} (A_1 + A_2) \alpha_i + \sum_{j=3}^{i} B_{n+1} A_j \alpha_{i-j+2}\right) e_i
+ \left(B_1 A_{n-1} + B_{n+1} (A_1 \alpha_n + A_2 \beta + \sum_{i=3}^{n} A_i \alpha_{n-i+2})\right) e_n.
\]

On the other hand

\[
[e'_1, x'] = \sum_{i=2}^{n} \alpha'_i e'_i = \sum_{i=2}^{n} \alpha'_i A_{i-2} (A_1 + A_2) e_i + \sum_{j=i+1}^{n} A_{j-i+2} e_j
= \sum_{i=2}^{n} \alpha'_i A_{i-2} (A_1 + A_2) e_i + \sum_{i=3}^{n} \sum_{j=3}^{i} A_{i-j} \alpha_{i-j+2} e_i
= \alpha'_2 (A_1 + A_2) e_2 + (A_1 (A_1 + A_2) \alpha'_3 + A_2 \alpha'_3) e_3 + \sum_{i=4}^{n} \left(\alpha'_i A_{i-2} (A_1 + A_2) + \sum_{j=3}^{i} A_{i-j} \alpha_{i-j+2}\right) e_i.
\]

Comparing coefficients at the basic elements in both combinations, we obtain the following relations:

\[
\alpha'_2 (A_1 + A_2) = B_{n+1} (A_1 + A_2) \alpha_2,
A_1 (A_1 + A_2) \alpha'_3 + A_3 \alpha'_2 = (A_1 + A_2) (B_1 + B_{n+1} \alpha_3) + B_{n+1} A_3 \alpha_2,
\alpha'_i A_{i-2} (A_1 + A_2) + \sum_{j=3}^{i} A_{i-j} \alpha_{i-j+2} = B_1 A_{i-1} + B_{n+1} (A_1 + A_2) \alpha_i + \sum_{j=3}^{n} B_{n+1} A_j \alpha_{i-j+2},
4 \leq i \leq n - 1,
A_{n-2} \alpha'_n (A_1 + A_2) + \sum_{j=3}^{n} A_{n-j} \alpha_{n-j+2} = B_1 A_{n-1} + B_{n+1} (A_1 \alpha_n + A_2 \beta + \sum_{i=3}^{n} A_i \alpha_{n-i+2}).
\]

The simplification of these relations implies the following identities:

\[
\alpha'_2 = B_{n+1} \alpha_2, \quad \alpha'_3 = \frac{B_1 + \alpha_3 B_{n+1}}{A_1}, \quad \alpha'_i = \frac{B_{n+1} \alpha_i}{A_{i-2}}, \quad 4 \leq i \leq n - 1, \quad \alpha'_n = \frac{(\alpha_n A_1 + \beta A_2) B_{n+1}}{A_{n-2} (A_1 + A_2)}.
\]

Analogously, considering the product \([e'_2, x']\), we get the relation:

\[
\beta' = \frac{\beta B_{n+1}}{A_{n-2}},
\]

and

\[
[x', x'] = -\frac{\beta B_1 - \alpha_n B_1 - \alpha_2 B_n) B_{n+1}}{A_{n-2} (A_1 + A_2)} e_n.
\]
Since \( [x', x'] = 0 \), then \( B_n = \frac{\beta B_1 - \alpha_n B_1}{\alpha_2} \).

Setting \( B_{n+1} = 1/\alpha_2 \) and \( B_1 = \frac{\alpha_2}{\alpha_3/\alpha_2} \), then we derive that \( \alpha_2' = 1 \), \( \alpha_3' = 0 \).

If \( \beta = 0 \) and \( \alpha_n = 0 \), then \( \beta' = 0 \) and we obtain the algebra \( R_2(\alpha_4, \ldots, \alpha_{n-1}, 0) \).

If \( \beta = 0 \) and \( \alpha_n \neq 0 \), then putting \( A_2 = \frac{\alpha_n - \alpha_2 A_1^{n-2}}{\alpha_2 A_1^{n-3}} \), we have \( \alpha_n' = 1 \) and so we obtain the algebra \( R_2(\alpha_4, \ldots, \alpha_{n-1}, 1) \).

If \( \beta \neq 0 \), then choosing

\[
A_1 = \frac{n - \beta}{\alpha_2}, \quad A_2 = -\frac{A_1 \alpha_n}{\beta},
\]

we obtain \( \beta' = 1 \), \( \alpha_n' = 0 \) and the algebra \( R_3(\alpha_4, \ldots, \alpha_{n-1}) \).

Now we shall consider the case when the dimension of a solvable Leibniz algebra with nilradical \( F_n \) is equal to \( n + 2 \).

**Theorem 4.3.** It does not exist any \((n + 2)\)-dimensional solvable Leibniz algebra with nilradical \( F_n \).

**Proof.** From the conditions of the theorem, we have the existence of a basis \( \{e_1, e_2, \ldots, e_n, y\} \) such that the table of multiplication of \( F_n \) remains. The outer non-nilpotent derivations of \( F_n \), denoted by \( \mathcal{R}_{x, F_n} \) and \( \mathcal{R}_{y, F_n} \), are of the form given in Proposition 4.1 with the set of entries \( \{\alpha_i, \gamma\} \) and \( \{\beta_i, \delta\} \), respectively, where \( \{e_i, x\} = \mathcal{R}_{x, F_n}(e_i) \) and \( \{e_i, y\} = \mathcal{R}_{y, F_n}(e_i) \).

Taking the following change of basis:

\[
x' = \frac{\beta_2}{\alpha_1 \beta_2 - \alpha_2 \beta_1} x - \frac{\alpha_2}{\alpha_1 \beta_2 - \alpha_2 \beta_1} y, \quad y' = -\frac{\beta_1}{\alpha_1 \beta_2 - \alpha_2 \beta_1} x + \frac{\alpha_1}{\alpha_1 \beta_2 - \alpha_2 \beta_1} y,
\]

we may assume that \( \alpha_1 = \beta_2 = 1 \) and \( \alpha_2 = \beta_1 = 0 \).

Therefore we have the products

\[
[e_1, x] = e_1 + \sum_{i=3}^{n} \alpha_i e_i, \quad [e_2, x] = e_2 + \sum_{i=3}^{n-1} \alpha_i e_i + \gamma e_n, \quad [e_i, x] = (i - 1) e_i + \sum_{j=i+1}^{n} \alpha_j e_j, \quad 3 \leq i \leq n,
\]

\[
[e_1, y] = e_2 + \sum_{i=3}^{n} \beta_i e_i, \quad [e_2, y] = e_2 + \sum_{i=3}^{n-1} \beta_i e_i + \delta e_n, \quad [e_i, y] = e_i + \sum_{j=i+1}^{n} \beta_j e_j, \quad 3 \leq i \leq n.
\]

Applying similar arguments as in Case 1 of Theorem 12 and taking into account that the products \( \{e_1, y\}, \{e_2, y\}, \{e_i, y\} \) will not be changed under the bases transformations which were used there, we obtain the products:

\[
[e_1, x] = e_1, \quad [e_2, x] = e_2 + \gamma e_n, \quad [e_i, x] = (i - 1) e_i, \quad 3 \leq i \leq n, \quad [x, e_1] = -e_1.
\]

Let us introduce the notations:

\[
[y, e_1] = \sum_{i=1}^{n} \eta_i e_i, \quad [y, e_2] = \sum_{i=1}^{n} \theta_i e_i, \quad [y, y] = \sum_{i=1}^{n} \tau_i e_i, \quad [x, y] = \sum_{i=1}^{n} \sigma_i e_i, \quad [y, x] = \sum_{i=1}^{n} \rho_i e_i.
\]

From the Leibniz identity

\[
[e_1, [y, e_1]] = [[[e_1, y], e_1], e_1] - [[[e_1, e_1], y], e_1]
\]

we get \( \eta_1 = 0 \).

Note that we can assume \( [y, e_1] = \eta_2 e_2 \) (by changing \( y' = y - \sum_{i=2}^{n-1} \eta_1 e_1 \)).

Due to

\[
[y, [e_1, e_2]] = [[[y, e_1], e_2], e_1] - [[[y, e_2], e_1], e_1]
\]

we obtain \( \theta_2 = -\theta_1, \quad \theta_i = 0, \quad 3 \leq i \leq n - 1 \).

Since \( [e_1, [y, e_2]] = [[[e_1, y], e_2], e_2] - [[[e_1, e_2], y], e_2] \), then we have \( \theta_1 = \theta_2 = 0 \). Moreover, the Leibniz identity \( [y_1, [y, e_2]] = [[[y, y], e_2], e_2] - [[[y, e_2], y], e_2] \) implies that \( \theta_n = 0 \), i.e., \( [y, e_2] = 0 \).

From the following chain of equalities

\[
0 = \eta_2 [y, e_2] = [y, \eta_2 e_2] = ([y, y], e_1] - [[[y, e_1], y]
\]

\[
= (\tau_1 + \tau_2) e_3 + \sum_{i=4}^{n} \tau_{i-1} e_i - \eta_2 [e_2, y] = (\tau_1 + \tau_2) e_3 + \sum_{i=4}^{n} \tau_{i-1} e_i - \eta_2 (e_2 + \sum_{i=3}^{n-1} \beta_i e_i + \delta e_n)
\]

we have that the Leibniz algebra \( F_n \) is a 3-dimensional solvable Leibniz algebra with nilradical \( F_n \).
we derive that
\[ y_2 = 0, \quad \tau_2 = -\tau_1, \quad \tau_1 = 0, \quad 3 \leq i \leq n - 1. \]

Therefore, we have \([y, e_1] = 0\) and \([y, y] = \tau_1 e_1 - \tau_1 e_2 + \tau_n e_n\).

Consider the Leibniz identity
\[ [x, [y, e_1]] = [[x, y], e_1] - [[x, e_1], y] \]
then we get
\[ -e_2 - \sum_{i=3}^{n} \beta_i e_i = (\sigma_1 + \sigma_2)e_3 + \sum_{i=3}^{n-1} \sigma_i e_{i+1}. \]

Thus, we have a contradiction with the assumption of the existence of an algebra under the conditions of the theorem. \(\square\)

4.2. Solvable Leibniz algebras with nilradical \(F_n^2\).

In this section we describe solvable Leibniz algebras with nilradical \(F_n^2\), i.e. solvable Leibniz algebras \(R\) which decompose in the form \(R = F_n^2 \oplus Q\).

Proposition 4.4. An arbitrary derivation of the algebra \(F_n^2\) has the following matrix form:

\[
D = \begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_{n-1} & \alpha_n \\
0 & \beta & 0 & 0 & \cdots & 0 & \gamma \\
0 & 0 & 2\alpha_1 & \alpha_3 & \cdots & \alpha_{n-2} & \alpha_{n-1} \\
0 & 0 & 0 & 3\alpha_1 & \cdots & \alpha_{n-3} & \alpha_{n-2} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & (n-1)\alpha_1
\end{pmatrix}
\]

Proof. The proof follows by straightforward calculations in a similar way as the proof of Proposition 4.1. \(\square\)

Remark 4.5. It is an easy task to check that the number of nil-independent derivations of the algebra \(F_n^2\) is equal to 2.

Corollary 4.6. The dimension of a solvable Leibniz algebra with nilradical \(F_n^2\) is either \(n + 1\) or \(n + 2\).

Theorem 4.7. An \((n + 1)\)-dimensional solvable Leibniz algebra with nilradical \(F_n^2\) is isomorphic to one of the following pairwise non-isomorphic algebras:

\[
R_1(\alpha) : \begin{cases}
[e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, \quad 3 \leq i \leq n - 1, \\
[e_1, x] = -e_1, & [e_i, x] = -(i-1)e_i, \quad 3 \leq i \leq n, \\
[x, e_1] = e_1, & [x, x] = \alpha e_2, \quad \alpha \in \{0, 1\}.
\end{cases}
\]

\[
R_2(\alpha) : \begin{cases}
[e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, \quad 3 \leq i \leq n - 1, \\
[e_1, x] = -e_1, & [e_i, x] = -(i-1)e_i, \quad 3 \leq i \leq n, \\
[x, e_1] = e_1, & [e_2, x] = \alpha e_2, \quad \alpha \neq 0.
\end{cases}
\]

\[
R_3 : \begin{cases}
[e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, \quad 3 \leq i \leq n - 1, \\
[e_1, x] = -e_1, & [e_i, x] = -(i-1)e_i, \quad 3 \leq i \leq n, \\
x, e_1] = e_1, & [e_2, x] = (1-n)e_2 + e_n.
\end{cases}
\]

\[
R_4(\alpha) : \begin{cases}
[e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, \quad 3 \leq i \leq n - 1, \\
[e_1, x] = -e_1, & [e_i, x] = -(i-1)e_i, \quad 3 \leq i \leq n, \\
x, e_1] = e_1, & [e_2, x] = -\alpha e_2, \quad \alpha \neq 1,
\end{cases}
\]

\[
R_5(\alpha) : \begin{cases}
[e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, \quad 3 \leq i \leq n - 1, \\
[e_1, x] = -e_1 - \alpha e_2, \quad \alpha \in \{0, 1\}, & [e_i, x] = -(i-1)e_i, \quad 3 \leq i \leq n, \\
x, e_1] = e_1 + \alpha e_2, & [e_2, x] = -e_2.
\end{cases}
\]
implies that \( x_0 \) is 16 J. M. CASAS, M. LADRA, B. A. OMIROV AND I. A. KARIMJANOV

\[
R_6(\alpha_3, \alpha_4, \ldots, \alpha_n, \lambda, \delta) : \begin{cases}
[e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, & 3 \leq i \leq n - 1, \\
[e_1, x] = \sum_{i=3}^{n} \alpha_i e_i, & [e_i, x] = \sum_{j=i+1}^{n} \alpha_{j-i+2} e_j, & 3 \leq i \leq n - 1, \\
x, e_2 = \delta e_2, & \\
[x, e_2] = \lambda e_n, & [x, x] = e_2, \\
\delta \in \{0, -1\}.
\end{cases}
\]

In the algebra \( R_6(\alpha_3, \alpha_4, \ldots, \alpha_n, \lambda, \delta) \) the first non vanishing parameter \( \{\alpha_3, \alpha_4, \ldots, \alpha_n, \lambda, \delta\} \) can be scaled to 1.

**Proof.** Let \( R \) be a solvable Leibniz algebra satisfying the conditions of the theorem, then there exists a basis \( \{e_1, e_2, \ldots, e_n, x\} \), such that \( \{e_1, e_2, \ldots, e_n\} \) is the standard basis of \( F^2_n \), and for non nilpotent outer derivations of the algebra \( F^2_n \) we have that \( [e_i, x] = R_{e_i,F_n}(e_i), 1 \leq i \leq n, \)

Due to Proposition 1.3 we can assume that

\[
[e_1, x] = \sum_{i=1}^{n} \alpha_i e_i, \quad [e_2, x] = \beta_2 e_2 + \beta_n e_n, \quad [e_i, x] = (i-1)\alpha_1 e_i + \sum_{j=i+1}^{n} \alpha_{j-i+2} e_j, \quad 3 \leq i \leq n.
\]

Let us introduce the following notations:

\[
x, e_1 = \sum_{i=1}^{n} \gamma_i e_i, \quad [x, e_2] = \sum_{i=1}^{n} \delta_i e_i, \quad [x, x] = \sum_{i=1}^{n} \lambda_i e_i.
\]

Considering the Leibniz identity for the elements \( \{e_1, x, x\} \), \( \{e_1, x, e_1\} \), \( \{x, e_2, e_1\} \) we obtain \( \lambda_1 = 0, \gamma_1 = -\alpha_1 \) and \( [x, e_2] = \delta_2 e_2 + \delta_n e_n \). By setting \( e_2' = \delta_2 e_2 + \delta_n e_n \) we can assume that \( [x, e_2] = \delta e_2 \).

Now we distinguish the following possible cases:

**Case 1.** Let \( \alpha_1 \neq 0 \). Then the following change of basis

\[
x' = \frac{1}{\gamma_1} x, \quad e_1' = e_1 + \frac{1}{\gamma_1} \sum_{j=3}^{n} \gamma_j e_j, \quad e_2' = e_2, \quad e_i' = e_i + \frac{1}{\gamma_1} \sum_{j=i+1}^{n} \gamma_{j-i+2} e_j, \quad 3 \leq i \leq n,
\]

implies that \( [x', e_1'] = e_1' + \gamma e_2' \) (where \( \gamma = \frac{\alpha_1}{\gamma_1} \)) and the rest of products remains unchanging.

From the equalities:

\[
e_1 + \gamma(1 + \delta) e_2 = [x, [x, e_1]] = [[x, x], e_1] - [[x, e_1], x] = \sum_{i=4}^{n} \lambda_{i-1} e_i - \sum_{i=1}^{n} \alpha_i e_i - \gamma \beta_2 e_2 - \gamma \beta_n e_n
\]

we deduce that

\[
\alpha_1 = -1, \quad \alpha_3 = 0, \quad \alpha_2 = -\gamma(1 + \delta + \beta_2), \quad \lambda_i = \alpha_{i+1}, \quad 3 \leq i \leq n - 2, \quad \text{and} \quad \lambda_{n-1} = \alpha_n + \gamma \beta_n.
\]

In addition, if we take the following change of basis:

\[
e_1' = e_1 + \sum_{i=4}^{n} A_i e_i, \quad e_2' = e_2, \quad e_i' = e_i + \sum_{j=i+2}^{n} A_{j-i+2} e_j, \quad 3 \leq i \leq n, \quad x' = \sum_{i=4}^{n} A_i e_i + B e_n + x,
\]

where \( A_j = \frac{1}{2} \alpha_j, \quad j = 4, 5, \quad A_i = \frac{1}{i-2}(\alpha_i + \sum_{j=4}^{i-2} A_j \alpha_{i-j+2}), \quad 6 \leq i \leq n \) and \( B = \frac{1}{n-1}(\lambda_n + \sum_{j=4}^{n-1} A_j \alpha_{n-j+3}) \), then we have

\[
[e_1', x'] = -e_1' + \alpha_2 e_2', [e_2', x'] = \beta_2 e_2' + \beta_n e_n', [x', x'] = \lambda_2 e_2' + \gamma \beta_n e_{n-1}, [e_i', x'] = -(i-1) e_i', \quad 3 \leq i \leq n.
\]

Finally, we obtain the following table of multiplication of the algebra \( R \):

\[
\begin{cases}
[e_1, x] = -e_1 - \gamma(1 + \delta + \beta_2) e_2, & [e_2, x] = \beta_2 e_2 + \beta_n e_n, \\
x, e_1 = e_1 + \gamma e_2, & [e_i, x] = -(i-1) e_i, \quad 3 \leq i \leq n, \\
x, e_2 = \delta e_2, & [x, x] = \lambda_2 e_2 + \gamma \beta_n e_{n-1}.
\end{cases}
\]

Considering the Leibniz identity for the elements \( \{x, x, e_2\} \), \( \{x, x, x\} \), \( \{x, e_1, x\} \), we obtain:

\[
\delta \beta_n = \delta (\delta + \beta_2) = \delta \lambda_2 = \gamma \delta (\delta + \beta_2) = 0.
\]

Notice that if \( e_2 \in \text{Ann}_r(R) \), then \( \dim \text{Ann}_r(R) = n - 1 \) and if \( e_2 \notin \text{Ann}_r(R) \), then \( \dim \text{Ann}_r(R) = n - 2 \).

Now we analyze the following possible subcases:
Case 1.1. Let $e_2 \in \text{Ann}_r(R)$. Then $\delta = 0$ and making the change $e'_1 = e_1 + \gamma e_2$ we can assume that $[x, e_1] = e_1$.

In this case, we must consider two new subcases:

Case 1.1.1. Let $e_2 \in \text{Center}(R)$. Then $\dim \text{Center}(R) = 1$ and $\beta_2 = \beta_n = 0$. Then we have two options: if $\lambda_2 = 0$, then we get the split algebra $R_1(0)$; if $\lambda_2 \neq 0$, then we obtain the algebra $R_1(1)$ by scaling the basis.

Case 1.1.2. Let $e_2 \not\in \text{Center}(R)$. Then $\dim \text{Center}(R) = 0$ and $(\beta_2, \beta_n) \neq (0, 0)$.

Let us take the following general change of basis:

$$
eq \sum_{i=1}^{n} A_i e_i, \quad e'_2 = \sum_{i=1}^{n} B_i e_i, \quad e'_1 = A_1^{-2} \left( \sum_{j=i+1}^{n} A_j e_i + \sum_{j=i+1}^{n} A_j e_j \right), \quad 3 \leq i \leq n, \quad x' = \sum_{i=1}^{n} C_i e_i + C_{n+1} x,$$

where $(A_1 B_2 - A_2 B_1) C_{n+1} \neq 0$.

From $0 = [e'_2, e'_1] = [e'_2, e'_2]$, we obtain that $B_1 = 0$, $B_i = 0$, $3 \leq i \leq n - 1$, i.e. $e'_2 = B_2 e_2 + B_n e_n$ and $A_1 B_2 \neq 0$.

The equalities

$$e'_1 = [x', e'_1] = A_1 C_1 e_1 + \sum_{i=4}^{n} A_1 C_{i-1} e_i + A_1 C_{n+1} e_1,$$

imply that

$$C_{n+1} = 1, \quad A_2 = 0, \quad A_3 = A_1 C_1, \quad A_i = A_1 C_{i-1}, \quad 4 \leq i \leq n.$$

Similarly, from

$$B_2 \beta'_2 e_2 + (B_n \beta'_2 + \beta'_n A_n^{-1}) e_n = \beta'_2 e'_2 + \beta'_n e'_n = [e'_2, x'] = B_2 \beta_2 e_2 + (B_2 \beta_n - (n - 1) B_n) e_n,$$

and

$$A_2 B_2 e_2 + \lambda_2 B_n e_2 = \lambda_2 e'_2 = [x', x'] = (\lambda_2 + C_2 \beta_2) e_2 + (C_1^2 - 2 C_3) e_3 + \sum_{i=4}^{n-1} (C_{i+1} - (i - 1) C_i) e_i + (C_{n+1} - (n - 1) C_n + C_2 \beta_n) e_n$$

we obtain $C_i = \frac{1}{(i - 1)!} C_i^{-1}$, $3 \leq i \leq n - 1$ and

$$\beta'_2 = \beta_2, \quad \beta'_n = \frac{B_2 \beta_n - B_n (\beta_2 + n - 1)}{A_1}, \quad \lambda_2 = \frac{C_i}{B_2}, \quad \lambda_2 B_n = C_1 C_{n+1} - (n - 1) C_n + C_2 \beta_n.$$

Now we must distinguish two subcases:

Case 1.1.2.1. Let $\beta_2 = 1 - n$. We put $C_2 = \frac{\lambda_2}{1 - n}, \quad C_3 = \frac{C_1 C_{n+1} + C_2 \beta_n}{n - 1}$, then we get $\lambda_2 = 0$ and $\beta'_n = \frac{B_2 \beta_n}{A_1}$.

If $\beta_2 = 0$, then we get the algebra $R_2(\alpha)$ for $\alpha = 1 - n$.

If $\beta_2 \neq 0$, then making $A_1 = \frac{1}{\sqrt{\beta_2 n} B_2}$, we obtain $\beta'_n = 1$ and the algebra $R_3$.

Case 1.1.2.2. Let $\beta_2 \neq 1 - n$. Taking the change $B_n = \frac{B_2 \beta_n}{\beta_2 + n - 1}$, we obtain $\beta_n = 0$. Since $\beta_2 \neq 0$, we set $C_2 = \frac{-\lambda_2}{\beta_2}, \quad C_n = \frac{C_1 C_{n+1} + C_2 \beta_n}{n - 1}$, and we get $\lambda_2 = 0$, i.e. the algebra $R_2(\alpha)$ is obtained, for $\alpha \notin \{1 - n, 0\}$.

Case 1.2. Let $e_2 \not\in \text{Ann}_r(R)$. Then $\delta \neq 0$ and $\beta_2 = -\delta, \beta_n = \lambda_2 = 0$.

Let us consider the general change of basis in the following form:

$$e'_1 = \sum_{i=1}^{n} A_i e_i, \quad e'_2 = \sum_{i=1}^{n} B_i e_i, \quad e'_1 = A_1^{-2} (A_1 e_i + \sum_{j=i+1}^{n} A_j e_j), \quad 3 \leq i \leq n, \quad x' = \sum_{i=1}^{n} C_i e_i + C_{n+1} x,$$

where $(A_1 B_2 - A_2 B_1) C_{n+1} \neq 0$.

Then from $0 = [e'_2, e'_1] = [e'_2, e'_2]$, we derive that $B_1 = 0$, $B_i = 0$, $3 \leq i \leq n - 1$, i.e. $e'_2 = B_2 e_2 + B_n e_n$ and $A_1 B_2 \neq 0$.

Similarly, from the equations:

$$e'_1 + \gamma' e'_2 = [x', e'_1] = A_1 C_{n+1} e_1 + C_{n+1} (A_1 \gamma + A_2 \delta) e_2 + A_1 C_1 e_3 + \sum_{i=4}^{n} A_1 C_i e_i$$

and

$$\delta' (B_2 e_2 + B_n e_n) = \delta' e'_2 = [x', e'_2] = B_2 \delta e_2$$

we obtain $C_i = \frac{1}{(i - 1)!} C_i^{-1}$, $3 \leq i \leq n - 1$ and

$$\beta'_2 = \beta_2, \quad \beta'_n = \frac{B_2 \beta_n - B_n (\beta_2 + n - 1)}{A_1}, \quad \lambda_2 = \frac{C_i}{B_2}, \quad \lambda_2 B_n = C_1 C_{n+1} - (n - 1) C_n + C_2 \beta_n.$$
we obtain
\[ C_{n+1} = 1, \quad A_3 = A_1 C_1, \quad A_i = A_1 C_{i-1}, \quad 4 \leq i \leq n - 1, \]
\[ \gamma' = \frac{A_1 \gamma + A_2 (\delta - 1)}{B_2}, \quad A_1 C_{n-1} = A_n + \gamma' B_n, \quad \delta' = \delta, \quad \delta ' B_n = 0. \]

Now we distinguish the following two subcases:

**Case 1.2.1.** Let \( \delta \neq 1 \). Then by the substitution \( A_2 = -\frac{A_1 \gamma}{\delta - 1} \), \( A_n = A_1 C_{n-1} \) into the above conditions, we get \( \gamma' = 0 \) and the algebra \( R_4(\alpha) \).

**Case 1.2.2.** Let \( \delta = 1 \). Then \( B_n = 0 \). In the case of \( \gamma = 0 \), we get \( \gamma' = 0 \). In the case \( \gamma \neq 0 \), by putting \( B_2 = A_1 \gamma \) and \( A_n = A_1 C_{n-1} - B_n \), we get \( \gamma' = 1 \). Thus, the algebras \( R_5(\alpha), \alpha \in \{0,1\} \), are obtained.

**Case 2.** Let \( \alpha_1 = 0 \). Then \( \beta_2 \neq 0 \) and by replacing \( x \) by \( x' = \frac{1}{\beta_2} x \), we can assume \([e_2, x'] = e_2 + \beta_n e_n]\.

Under these conditions, the table of multiplication of the solvable algebra \( R \) has the form:

\[
\begin{align*}
[e_1, x] &= \sum_{i=2}^{n} \alpha_i e_i, \quad [e_2, x] = e_2 + \beta_n e_n, \\
[x, e_1] &= \sum_{i=2}^{n} \gamma_i e_i, \quad [e_i, x] = \sum_{j=i+1}^{n} \alpha_{j-i+2} e_j, \quad 3 \leq i \leq n - 1, \\
[x, e_2] &= \delta e_2, \quad [x, x] = \sum_{i=2}^{n} \lambda_i e_i.
\end{align*}
\]

Making the transformation \( x' = x - \gamma_3 e_1 - \sum_{i=3}^{n-1} \gamma_i e_i \), we can assume that \([x, e_1] = \gamma e_2\).

Similarly as above, we obtain the conditions:

\[ \gamma (\delta + 1) = \alpha_2 \delta - \gamma = \beta_n \delta = \delta (\delta + 1) = \lambda_2 \delta = 0. \]

Now we distinguish the following subcases depending on the possible values of the parameter \( \delta \):

**Case 2.1.** Let \( \delta \neq 0 \). Then \( \dim \mathrm{Ann}_r (R) = n - 2 \) and \( \beta_n = \lambda_2 = 0 \), \( \delta = -1 \), \( \alpha_2 = -\gamma \). By means of the change of the basis element \( e_1' = e_1 + \gamma e_2 \), we can suppose that \([x', e_1] = 0\).

**Case 2.2.** Let \( \delta = 0 \). Then \( \dim \mathrm{Ann}_r (R) = n - 1 \) and \( \gamma = 0 \). Taking the general change of bases as in the above considered cases, we derive the conditions for the parameters under the following basis transformation:

\[ \alpha_i' = \frac{\alpha_i}{A_1^{i-2}}, \quad 3 \leq i \leq n, \quad \lambda_n' = \frac{\lambda_n}{A_1^{n-1}}. \]

Consequently, we deduce the algebra \( R_6(\alpha_3, \alpha_4, \ldots, \alpha_n, \lambda, -1) \).

**Case 2.2.** Let \( \delta = 0 \). Then \( \dim \mathrm{Ann}_r (R) = n - 1 \) and \( \gamma = 0 \). Taking the change of basis \( e_2' = e_2 + \beta_n e_n \), we can assume that \([e_2, x] = e_2\) and by the change \( x' = x - \lambda_2 e_2 \), we can also suppose that \([x, x] = \lambda_n e_n\). Therefore, we have the products

\[
[e_1, x] = \sum_{i=2}^{n} \alpha_i e_i, \quad [e_2, x] = e_2, \quad [e_i, x] = \sum_{j=i+1}^{n} \alpha_{j-i+2} e_j, \quad 3 \leq i \leq n - 1, \quad [x, x] = \lambda_n e_n.
\]

Applying similar arguments to general transformation of bases, we have

\[ \alpha_i' = 0, \quad \alpha_i' = \frac{\alpha_i}{A_1^{i-2}}, \quad 3 \leq i \leq n, \quad \lambda_n' = \frac{\lambda_n}{A_1^{n-1}}. \]

Thus, we obtain the algebra \( R_6(\alpha_3, \alpha_4, \ldots, \alpha_n, \lambda, 0) \).

**Theorem 4.8.** An arbitrary \((n+2)\)-dimensional solvable Leibniz algebra with nilradical \( F_{n}^{2} \) is isomorphic to one of the following non isomorphic algebras:

\[
L_1 : \begin{cases}
[e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, & 3 \leq i \leq n - 1, \\
[e_1, x] = e_1, & [x, e_1] = -e_1, \\
[e_2, y] = -[y, e_2] = e_2, & [e_i, x] = (i-1)e_i, & 3 \leq i \leq n,
\end{cases}
\]
Let 

\[
L_2 : \begin{cases}
[e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, \quad 3 \leq i \leq n - 1, \\
[e_1, x] = e_1, & [x, e_1] = -e_1, \\
[e_2, y] = e_2, & [e_i, x] = (i - 1)e_i, \quad 3 \leq i \leq n.
\end{cases}
\]

Proof. Let

\[
R_{x, y|F_n^2} = \begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_{n-1} & \alpha_n \\
0 & \beta & 0 & 0 & \ldots & 0 & \gamma \\
0 & 0 & 2\alpha_1 & \alpha_3 & \ldots & \alpha_{n-2} & \alpha_{n-1} \\
0 & 0 & 0 & 3\alpha_1 & \ldots & \alpha_{n-3} & \alpha_{n-2} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & (n-1)\alpha_1
\end{pmatrix}
\]

and

\[
R_{y, x|F_n^2} = \begin{pmatrix}
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \ldots & \lambda_{n-1} & \lambda_n \\
0 & \mu & 0 & 0 & \ldots & 0 & \nu \\
0 & 0 & 2\lambda_1 & \lambda_3 & \ldots & \lambda_{n-2} & \lambda_{n-1} \\
0 & 0 & 0 & 3\lambda_1 & \ldots & \lambda_{n-3} & \lambda_{n-2} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & (n-1)\lambda_1
\end{pmatrix}
\]

be two nil independent outer derivation of the algebra $F_n^2$.

Taking the change of the basic elements $x, y$ similar to 45, we can assume that $\alpha_1 = \mu = 1$, $\lambda_1 = \beta = 0$.

Thus, we have the products:

\[
[e_1, x] = e_1 + \sum_{i=1}^{n} \alpha_i e_i, \quad [e_2, x] = \gamma e_n, \quad [e_i, x] = (i - 1)e_i + \sum_{j=i+1}^{n} \alpha_{j-i+1} e_j, \quad 3 \leq i \leq n,
\]

\[
[e_1, y] = \sum_{i=1}^{n} \lambda_i e_i, \quad [e_2, y] = e_2 + \nu e_n, \quad [e_i, y] = \sum_{j=i+1}^{n} \lambda_{j-i+1} e_j, \quad 3 \leq i \leq n.
\]

Applying similar reasonings and changes of bases which we have used in Theorem 4.7 we obtain isomorphism classes of algebras whose representative elements are the $L_1$ and $L_2$.

\[\square\]

**Remark 4.9.** In fact, the algebra $L_1$ is a direct sum of the ideals $(NF_{n-1} + \langle x \rangle)$ and $\langle e_2, y \rangle$, where the sum $NF_{n-1} + \langle x \rangle$ is a solvable Leibniz algebra with nilradical $NF_{n-1}$ and $\langle e_2, y \rangle$ is a two-dimensional solvable Lie algebra. The algebra $L_2$ is a direct sum of the ideals $(NF_{n-1} + \langle x \rangle)$ and $\langle e_2, y \rangle$, where $\langle e_2, y \rangle$ is a two-dimensional solvable non-Lie Leibniz algebra. Thus, from Theorem 4.8, we conclude that any $(n + 2)$-dimensional solvable Leibniz algebra with nilradical $F_n^2$ is split.

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