ON GENERALIZED SELF-DUALITY EQUATIONS TOWARDS
SUPERSYMMETRIC QUANTUM FIELD THEORIES OF FORMS

Laurent Baulieu\textsuperscript{†‡} and Céline Laroche\textsuperscript{†}
(baulieu, laroche@lpthe.jussieu.fr)

\textsuperscript{†}LPTHE, Paris, France.
Universités Pierre et Marie Curie (Paris 6) et Denis Diderot (Paris 7)
and
\textsuperscript{‡}CERN, Genève, Switzerland.

ABSTRACT

We classify possible “self-duality” equations for $p$-form gauge fields in space-time dimension up to $D = 16$, generalizing the pioneering work of Corrigan \textit{et al.} (1982) on Yang-Mills fields ($p = 1$) in $4 < D \leq 8$. We impose two crucial requirements. First, there should exist a $2(p + 1)$-form $T$ invariant under a sub-group $H$ of $SO_D$. Second, the representation for the $SO_D$ curvature of the gauge field must decompose under $H$ in a relevant way. When these criteria are fulfilled, the “self-duality” equations can be candidates as gauge functions for $SO_D$-covariant and $H$-invariant topological quantum field theories. Intriguing possibilities occur for $D \geq 10$ for various $p$-form gauge fields.

\textsuperscript{†}URA CNRS 280, Boîte 126, Tour 16, 1\textsuperscript{er} étage, 4 place Jussieu, F-75252 Paris Cedex 05, France.
\textsuperscript{‡}TH-Division, CERN, 1211 Genève 23, Switzerland.
1 Introduction

A large class of topological quantum field theories (denoted as $TQFT$s) for a $p$-form gauge field $A_{(p)}$ with $(p + 1)$-form curvature $F_{(p+1)}$ can be understood as theories which explore the moduli space of (anti-)self-duality equations

$$\pm F_{\mu_1\cdots\mu_{p+1}} = \frac{1}{(D/2)!} \varepsilon_{\mu_1\cdots\mu_{p+1} \mu_{p+2} \cdots \mu_D} F_{\mu_{p+2} \cdots \mu_D}. \quad (1.1)$$

Assuming that the gauge symmetry of the topological theory is arbitrary, local redefinitions of the classical fields, the self-duality equations (1.1) are relevant as topological gauge-fixing functions. With the method of $BRST$ quantization, one determines the $BRST$-invariant Lagrangian to be used in the path integral formulation of the $TQFT$ as well as the observables [1], [2]. Moreover, the topological $BRST$ symmetry can often be (un)twisted to recover an ordinary space-time supersymmetry [3], [4].

The self-duality equations (1.1) rely on the existence of the totally antisymmetric tensor $\varepsilon_{\mu_1\cdots\mu_D}$. Thus, one expects that Yang-Mills-like $TQFT$’s can only be constructed for $p$-form gauge fields satisfying the condition $2(p + 1) = D$. However, relaxing some constraints, one foresees the possibility for new classes of $TQFT$’s involving $p$-forms in various dimensions $D$. This seems an interesting perspective to us: it could provide effective field-theories on the world-volume of branes (see for instance [4], [5], [6], [7]); or a 12-dimensional $TQFT$ for a 3-form gauge field, which could be candidate for the elusive “$F$-theory”. Furthermore, new instanton solutions could be of interest, as for $p = 1$, $D = 8$ [8].

Such theories reduce the $D$-dimensional Euclidean invariance down to invariance under a sub-group $H \subset SO_D$ (with Lie algebra $h$), while preserving the $SO_D$ covariance. From a geometrical point of view, in curved space, $H$ would be the holonomy group of the $D$-dimensional manifold over which the supersymmetric $TQFT$ is constructed. This generalization is done by changing $\varepsilon_{\mu_1\cdots\mu_D}$ in (1.1) into a more general $2(p + 1)$-tensor $T$. For $D > 2(p + 1)$, $T$ cannot be invariant (i.e. covariantly constant in curved space) under $SO_D$, but rather under one of its sub-groups $H \subset SO_D$. The determination of such invariant tensors $T$ was first worked out by Corrigan et al. in [9], for Yang-Mills fields in dimensions up to eight.
In the same spirit of [4], our purpose here is thus to exhibit equations of the type

$$\lambda F_{\mu_1 \cdots \mu_{p+1}} = T^{\nu_1 \cdots \nu_{p+1}} F_{\nu_1 \cdots \nu_{p+1}} (1.2)$$

where $\lambda$ is to be understood as one of $T$’s eigenvalues. Keeping in mind that we wish (1.2) to be relevant as a gauge-fixing function, we expect that the field equations remain a consequence of the Bianchi identity together with (1.2). This implies that we ought to restrict ourselves to totally antisymmetric tensors $T$. Therefore, equation (1.2) can be called a “generalized self-duality equation” by analogy with the generic case (1.1) where $2(p+1) = D$. Furthermore, since it only involves one gauge field, we call (1.2) of “pure” type, as opposed to self-duality equations that would mix the curvatures for various gauge fields.

If a $D$-dimensional TQFT can be constructed from (1.2), its Lagrangian will only be $SO_D$-covariant and $H$-invariant. Its supersymmetry will be a topological BRST-symmetry, where all the fermions are ghosts. However, it may happen that the full $SO_D$-invariance be recovered from a twist of the various fields involved in the theory, and the ordinary supersymmetry from untwisting the topological BRST symmetry (a further dimensional reduction can be necessary). An example of this scenario is the one in [4], where a covariant eight-dimensional TQFT is built for a Yang-Mills field, with the $SO_8$ invariance broken down to a $G_2 \subset SO_7 \subset SO_8$ invariance: the action explicitly depends on an $SO_8$ 4-form $T$, invariant under $G_2$ only. Nevertheless, untwisting the corresponding Lagrangian enables to absorb this dependence into field redefinitions, giving rise to the ordinary Lorentz invariant supersymmetric action in eight dimensions. The latter is the one that is obtained by dimensional reduction of the usual $N = 1$, $D = 10$ super-Yang-Mills theory [4], [3].

Determining $H$-invariant tensors $T$ can be done using the branching rules for irreducible representations of semi-simple Lie algebras (we used the tables in the book from W.G.McKay, J.Patera [11]: they provide the information as to semi-simple algebras, while abelian factors can be found in [11]). Furthermore, we are interested in a situation where a TQFT can be constructed, with the self duality equation (1.2) as gauge function. In this perspective, (1.2) must ideally give a number of independent equations on the curvature, matching the number $N_{eq}$ of gauge invariant components of $A_{(p)}$. As we shall see, the realization of the requirements towards a TQFT is non trivial, as soon as it comes to cases other than the generic $2(p+1) = D$ for even $D$. It even came as a surprise to us that we found possibilities for other $(p, D)$.

Our paper has two aspects. On one hand, we give the methodology to probe the possible
relevance of such theories, enlightening the selection of $h$ and $T$ for given $(p, D)$'s. On another
hand, we implement the method and explicitly determine the eligible sub-algebras $h$ for each
possible value of $p$ and $D$, using the tables in [11] and unpublished results concerning higher
dimensional cases.\footnote{We gratefully thank J. Patera for private communication of those.}

There are limitations in our analysis. First, the existing tables give the decompositions of the $SO_D$-representations into maximal semi-simple sub-algebras. We have thus restricted our
analysis to maximal sub-algebras $h$ (except for $su_{D/2} \subset so_D$ cases that we obtain explicitly by complexification). This could lead us to miss possible solutions. For instance, the interesting
octonionic solution of non-maximal $g_2 \subset so_7 \subset so_8$ for $(p = 1, D = 8)$ can be extracted from the
solutions of the non-maximal (and thus not recorded in [11]) sub-algebra $su_4 \subset so_8$. However,$g_2$ is maximal within $so_7$, and the octonionic solution can nevertheless be found from [10],
through a slightly different and more refined analysis (see subsection 2.1).

Second, we shall not consider in this paper the possible mixing of curvatures in the self-
duality equation (1.2). Mixed self-duality equations involving several gauge fields occur, for
instance, from dimensional reduction of a “pure” case like (1.2). An example was observed in [4];
the “pure” $(p = 1, D = 8)$ equation gives, when reduced to 4 dimensions, a non abelian version
of the Seiberg-Witten equations, involving a spinor and a Yang-Mills field. We emphasize that
other mixed self-duality equations may exist (that do not come from dimensional reduction of
a “pure” case), but we shall not consider them here.

Despite these points, and despite the fact that we only exhibit the possible $h$’s without any
certainty that the corresponding $TQFT$ can be given a sense, we nevertheless believe that the results presented here are already quite interesting by themselves. In particular, they point
out the diversity of the possibilities, depending on the various values of $D$. They also indicate
the specificity of $D = 8$ which is the only case where an exceptional group, $G_2$, arises in our
analysis (the only orthogonal groups that have $G_2$ as a maximal sub-group are $SO_7$ and $SO_{14}$,
but in the later case, the requirements we impose cannot be realized).

The organization is as follows: in section 2 are given the arguments for the selection of the
invariance algebra $h$ and for the conditions on $T$ together with its $\lambda$’s. Section 3 contains our
results: we display the possible invariance sub-algebra $h$ for each pair $(p, D)$ and comment on
the peculiarities of each case we have explored.
2 Analysis of the Generalized Self-Duality Equation

Our aim for this section is to probe the various criteria to be fulfilled for the self-duality equation (1.2) to count for $N_{eq}$ gauge fixing conditions, and to be appropriate for building TQFT’s of various $(p, D)$. Let us first consider equation (1.2) from a general point of view, keeping in mind that we seek a systematic check of the following requirements:

i) the $SO_D$ form $T$ should be an $H \subset SO_D$-invariant;

ii) the $SO_D$ irreducible representation for $F$ should contain an $N_{eq}$-dimensional representation in its decomposition under $H$;

iii) the eigenvalues $\lambda$ should have a suitable degeneracy, allowing (1.2) to count for $N_{eq}$ independent equations on $F$;

iv) the properties of $T$ should enable to put the action under a Yang-Mills-like form.

We shall detail the points i), ii) in the following subsection. Criteria iii) will be the object of subsection 2.2, and the remaining criteria iv) of subsection 2.3.

2.1 On the selection of the invariance sub-algebra $h$

Before describing the method, let us set up our notations: we consider a $p$-form gauge field in $D$ space-time dimensions. Its curvature is a $(p_F = p + 1)$-form, and has $d_F = \binom{D}{p_F}$ independent components. Consequently, the invariant form $T$, relevant to write (1.2), is a $p_T$-form with $p_T = 2p_F = 2(p + 1)$. Since it is chosen totally antisymmetric, $T$ has $d_T = \binom{D}{p_T}$ independent components. Thus, for a given $p$, we seek a $2(p+1)$-form that is invariant under a sub-group $H$ of $SO_D$, while $T$ (resp. $F$) lies in a representation of $SO_D$ of dimension $d_T$ (resp. $d_F$).

The first check i) is to examine whether an $SO_D$ $p_T$-form can be left invariant by the transformations of some sub-group $H \subset SO_D$. It consists in inspecting, in [10], the decomposition of the $so_D$-representation $d_T$ upon irreducible representations of all the possible sub-algebras $h$. We then select the $h$’s that leave the $p_T$-form invariant, i.e. the sub-algebra under which the decomposition of $d_T$ produces a singlet 1. Notice that we restrict ourselves to maximal sub-algebras, except for the particular case of a sub-algebra $h$ leaving $d_T$ irreducible. In this case, we pick up the sub-algebra $\tilde{h} \subset h$ that leaves $T$ invariant. This is the situation of the $(p = 1, D = 8)$ case, where the 4-form $T$ can be decomposed into self and anti-self-dual parts, irreducible under $so_7$. More generally, this occurs for $D = 4(p + 1)$. 

Once such a sub-algebra \( h \) (or \( \tilde{h} \)) has been selected, we impose that it satisfies criteria \( ii \):
to end up with a proper \( TQFT \) for the \( p \)-form gauge field, the self-duality equations (1.2) must
count for as many equations as there are of gauge-invariant components of \( A_{(p)} \) (the longitudinal
gauge degrees of freedom can be fixed with the conventional \( BRST \) formalism for forms [2]). A
\( p \)-form gauge field has a total of \( \binom{D}{p} \) components, but is defined up to an anticommuting gauge
symmetry parameter \( \xi_{(p-1)} \). The latter has a total of \( \binom{D}{p-1} \) components, but is also defined up
to a commuting gauge symmetry parameter \( \xi_{(p-2)} \) with \( \binom{D}{p-2} \) components, and so on. Thus,
the number \( N_{eq} \) of gauge-invariant components of \( A_{(p)} \) is the number of components of a \( p \)-form
in \((D - 1)\) dimensions, that is

\[
N_{eq} = \left( \frac{D}{p-1} \right) - \left( \frac{D}{p-2} \right) + \cdots \pm \left( \frac{D}{0} \right) = \left( \frac{D-1}{p} \right).
\] (2.1)

Therefore, criteria \( ii \) consists in checking whether the \( so_D \)-representation \( d_F \) contains, in its
\( h \)-decomposition \( d_F = \bigoplus d_{F^i} \ (i = 1 \cdots n) \), a representation \( N_{eq} = \bigoplus d_{F^j} \) with total dimen-
sion \( N_{eq} \). The right number of constraints on the gauge field \( A_{(p)} \) can then be obtained by
imposing (1.2) if the latter allows to project \( F \) on the space corresponding to \( (d_F - N_{eq}) \). In
this final sorting of the sub-algebras, we do not impose that \( N_{eq} \), nor \( (d_F - N_{eq}) \), be irreducible
representations of \( h \). Allowing these representations to be a sum of irreducible representations
does not lead to inconsistencies: (1.2) is then expressed as independant sets of equations that
involve independant combinations of \( F \)’s components.

\( T \) can be understood as a linear operator, mapping the space of \((p + 1)\)-forms into itself;
and since it has been chosen a singlet of \( h \), Schur’s lemma implies that \( T \) acts block diagonally
in each of the \( d_{F^i} \)’s (\( T \) being an invariant of \( H \), its action on an irreducible representation of
\( H \) remains in the same representation). Each \( d_{F^i} \) is associated with an eigenvalue \( \lambda^i \), and one
can always choose an appropriate basis to diagonalize \( T \) in each \( d_{F^i} \) so that it reads:

\[
T_{diag} = \begin{pmatrix}
[\lambda^1 I_{(1)}] \\
[\lambda^2 I_{(2)}] \\
\vdots \\
[\lambda^i I_{(i)}] \\
\vdots
\end{pmatrix}
\] (2.2)

where \( I_{(i)} \) denotes the relevant \( (d_{F^i}) \)-dimensional identity operator involved in the associated
sub-space.\footnote{This actually holds as long as each \( d_{F^i} \) appears only once in the decomposition \( d_F = \bigoplus d_{F^i} \). If a given}
Taking into account the possible degeneracy of the eigenvalues, the \((\lambda_1, \cdots, \lambda_i, \cdots, \lambda_n)\) take \(m\) distinct values \((\mu_1, \cdots, \mu^a, \cdots, \mu^m)_{(m \leq n)}\). One can then re-write \(T\) in terms of the \(\mu\)'s, and define the \(h\)-invariant operator \(P_{\mu^a}\), key to the gauge fixing, as \(P_{\mu^a} = T - \mu^a I(a)\). According to the degeneracy of the \(\lambda\)'s, each \(\mu^a\) is now associated to one or a sum of representations (with the same \(\lambda^j\): \(d_a = \bigoplus d_F^{\lambda^j}\), of total dimension \(d_a\). Imposing \(P_{\mu^a} F = 0\) projects \(F\) on the sub-space corresponding to \(d_a\). It amounts to cancelling the projection of \(F\) onto all the other sub-spaces corresponding to \(\mu^b \neq \mu^a\). This gives a number of independent equations on \(F\) equal to the total dimension \(\sum d_b = (d_F - d_a)\) of the \((\bigoplus d_b)\) representation. Hence, \(P_{\mu^a} F = 0\) may be suitable as a gauge fixing function in the sense described above, provided that it counts for \(N_{eq}\) equations, i.e. provided that \((d_F - d_a) = N_{eq}\).

Before taking this any further (we shall come back on this point in subsection 2.3), let us first probe the conditions that one has on the \(\lambda^i\)'s, and in particular the issue of criteria \(iii)\).

### 2.2 On the degeneracy of the eigenvalues \(\lambda\)

Suppose that one has found a sub-algebra \(h\) and a form \(T\) satisfying criteria \(i)\), \(ii)\); and consider only one of the sub-spaces upon which \(F\) decomposes. To compute the corresponding eigenvalue \(\lambda^i\), one can apply \(T\) on \((1.2)\) once more to get:

\[
(\lambda^i)^2 F_{\mu_1 \cdots \mu_{p+1}} = T_{\mu_1 \cdots \mu_{p+1}} \mu_1 \cdots \mu_{p+1} T_{\nu_1 \cdots \nu_{p+1}} \nu_1 \cdots \nu_{p+1} T_{\sigma_1 \cdots \sigma_{p+1}} F_{\sigma_1 \cdots \sigma_{p+1}}.
\]

One obtains an equation for \(\lambda^i\) if one knows how \((T^2)_{\mu_1 \cdots \mu_{p+1}}^{\sigma_1 \cdots \sigma_{p+1}}\) expands upon \(h\)-invariant tensors, and how these act on \(F\). Unlike \(T\), \(T^2\) lies in a reducible representation \(d_{T^2}\) of \(so_D\): it transforms like the product of two \(p_F\)-forms, \(d_{T^2} \equiv d_F \otimes d_F\), that one first expands as a sum of irreducible representations of \(so_D\). Second, \(T^2\) is a tensor of \(so_D\), necessarily \(h\)-invariant because \(T\) is, so it can only expand as a linear combination of \(h\)-invariant tensors. One is sure to find, in \(T^2\), at least the trace part \(\delta_{[\mu_1}^{\nu_1} \cdots \delta_{\mu_{p+1}]^{\sigma_{p+1}}\sigma_1 \cdots \sigma_{p+1}}\), as well as a \(T_{\mu_1 \cdots \mu_{p+1}}^{\sigma_1 \cdots \sigma_{p+1}}\) term, but there might be others. Generically, let us write the decomposition of \(d_{T^2}\) upon \(h\)-invariant irreducible representations of \(so_D\) as

\[
d_{T^2} \equiv d_F \otimes d_F = 1 \oplus d_T \oplus \left( \bigoplus_m d_{T_m} \right).
\]

representation has multiplicity \(m\) higher than one, then \(T\)'s action on these \(m\) identical representations is one single block (instead of \(m\)), that is not necessarily diagonal.

7
Whenever $T$ and the trace term are the only tensors to appear in (2.4), one has

\[(T^2)_{\mu_1 \cdots \mu_{p+1}}^{\nu_1 \cdots \nu_{p+1}} = A\delta_{[\mu_1} \delta_{\cdots \mu_{p+1}]}^{[\nu_1 \cdots \nu_{p+1}]} + BT_{\mu_1 \cdots \mu_{p+1}}^{\nu_1 \cdots \nu_{p+1}} \]

(2.5)

and the $\lambda^i$’s are solutions of

\[(\lambda^i)^2 = A(p + 1)! + B\lambda^i, \tag{2.6}\]

where the trace part appears with a non-vanishing coefficient $A$. The statement that $A \neq 0$ ensures that $\lambda^i \neq 0$. Equation (2.6) tells us a lot more: as long as the hypothesis of the unicity of $T$ in (2.4) holds, the $\lambda^i$’s can only take two possible values, whatever number of $h$-irreducible representations is involved in the decomposition of $F$. One can thus expect some quite restrictive degeneracy of the $\lambda$’s.

The only possibility for the $\lambda$’s to be allowed more than two values is when other invariant tensors than $T$ appear in (2.5). To each of those invariant tensors $T_m$ is associated a set of eigenvalues. Reproducing the same string of arguments we have used for $T^2$, one gets the decomposition of each $(T_m)^2$ upon the set of invariant tensors. These lead to a set of equations for the eigenvalues of $T$ together with the eigenvalues of the $T_m$’s.

In general, we expect a high level of degeneracy for the $\lambda$’s. Unfortunately, to be sure that (1.2) eventually counts for $N_{eq}$ constraints on $F$, one would need to know, case by case, the detailed spectrum of the $\lambda^i$’s. Since the explicit construction of the $T$ tensor is not our goal here, we shall not take this any further, but rather turn to requirement iv).

2.3 On the relevance of the self-duality equation towards a TQFT

As to the relevance of the self-duality equation, there are two things to be kept in mind: in all the cases we expose in the coming section, the check ii) ensures that one can always write $N_{eq}$ $H$-invariant equations. However, it is not at all systematic that these equations can be imposed with the sole constraint (1.2). Even if it is the case, it is not guaranted that the form $T$ is suitable to recover a Yang-Mills like action, plus topological terms, when the gauge function is squared. Notice that one can always find an $h$-diagonal matrix with the spectrum

\[3\text{Indeed, if one contracts the remaining indices to get } (T^2)_{\mu_1 \cdots \mu_{p+1}}^{\nu_1 \cdots \nu_{p+1}}, \text{ the only term that survives is the trace term. Furthermore, and since we are considering Euclidean spaces, its coefficient } A \text{ cannot be zero unless } T \text{ itself is zero.} \]
of eigenvalues suitable to project the curvature on the relevant sub-spaces. However, the later
does not necessarily correspond to an antisymmetric tensor of $SO_D$ and thus is not relevant in
view of $iv)$.

Criteria $iv)$ involves non trivial properties of $T$. The most favorable case is when $T$ is
the only tensor to appear in the right hand side of (2.4), and when $P_\mu F = 0$ projects on
the relevant sub-space. The $N_{eq}$ equations coming from (1.2) can then be written as duality
relations between the “electric” and “magnetic” parts of the curvature:

$$F_{Di_1\cdots i_p} = c_{i_1\cdots i_p j_1\cdots j_{p+1}} F_{j_1\cdots j_{p+1}},$$

$$c_{i_1\cdots i_p j_1\cdots j_{p+1}} \equiv T_{Di_1\cdots i_p j_1\cdots j_{p+1}},$$

with $(i_1, i_2, \cdots, j_1, \cdots)$ being $(D-1)$-dimensional indices. Therefore, when the gauge function
is squared, using (2.5), one gets the appropriate terms $F \wedge \ast F$ and $(\ast T) \wedge F \wedge F$ in the action
for the $TQFT$:

$$\left| \lambda F_{\mu_1\cdots \mu_{p+1}} + T_{\mu_1\cdots \mu_{p+1}}^{\nu_1\cdots \nu_{p+1}} F_{\nu_1\cdots \nu_{p+1}} \right|^2$$

$$= \alpha \left| F_{\mu_1\cdots \mu_{p+1}} \right|^2 + \beta (T_{\mu_1\cdots \mu_{p+1}}^{\nu_1\cdots \nu_{p+1}} F_{\mu_1\cdots \mu_{p+1}} F_{\nu_1\cdots \nu_{p+1}})$$

$$\mapsto \alpha (F \wedge \ast F) + \beta (\ast T \wedge F \wedge F)$$

with the $|(D-2(p+1)) \equiv (D-p_T)|$-form $(\ast T)$ defined as $\epsilon_{\mu_1\cdots \mu_{p+1}} T^{\mu_1\cdots \mu_{p+1}} = (\ast T)_{\mu_{p+1}+1\cdots \mu_D}$. The way supersymmetric terms are added to (2.9), giving a $BRST$-invariant action, is a straightforward generalization of the work detailed in [4].

For less ideal cases, the degeneracy of the $\lambda$’s can have drastic consequences on the relevance
of (1.2) as a gauge function. Very often, the self-duality equations amount to a number of
independent conditions exceeding $N_{eq}$. These are the most common cases, for which the $N_{eq}$-dimensional sub-space shares its eigenvalue $\lambda$ with other eigenspaces. Another situation is when
the $N_{eq}$-dimensional sub-space contains different eigenspaces which are associated with different
eigenvalues $(\lambda^1 \cdots \lambda^k)$. Here, one can tentatively use as gauge function

$$\left( \prod_{i=1}^k P_{\lambda^i} \right) F \equiv \left( \prod_{i=1}^k (T_{\mu_1\cdots \mu_{p+1}}^{\nu_1\cdots \nu_{p+1}} - \lambda^i \delta_{[\mu_1}^{[\nu_1} \cdots \delta_{\mu_{p+1}]^{\nu_{p+1}]} ) \right) F_{\nu_1\cdots \nu_{p+1}}.$$
have an unusual Lagrangian, that possibly depends on these other invariant tensors. Notice that the additional $T_m$-terms are not topological, since the $T_m$’s do not correspond to totally antisymmetric tensors of SO$_D$; these terms can nevertheless be of interest.

For a given sub-group $H$ satisfying our criteria, we often found only one singlet in the decomposition of $dT$. To the later corresponds a given spectrum of the eigenvalues $\lambda^i$ (i.e. a given set of $\mu^a$), often such that the number of equations obtained on the components of $F$ is greater than $N_{eq}$. An example is the case of $h \equiv so_{D-1}$, with $2(p + 1) = (D - 1)$, in odd space-time dimension $D$. Here, (1.2) amounts to canceling $F_{i_1 \ldots i_p}$ ($N_{eq}$ equations) and the $\frac{1}{2}(D-1)$ anti-self-dual components of $F_{i_1 \ldots i_p+1}$ ($i_1, i_2, \ldots$ are $(D - 1)$ dimensional indices).

Another example is the $h \equiv so_{d} \times so_{D-d} \subset so_D$ case. Here, the only possible $T$ again has eigenvalues $\pm 1$ and leads to more than $N_{eq}$ constraints on the curvature.

However, there are cases where $dT$ gives several singlets in its decomposition under $h$, giving more flexibility as to the spectrum of the eigenvalues. The general solution for $T$ is then expressed as a linear combination of the corresponding tensors: $T = \sum_k \alpha_k T^{(k)}$, with $\lambda = \sum_k \alpha_k \lambda^{(k)}$. Thus, one has the opportunity of adjusting the coefficients $\alpha_k$, such that the eventual spectrum allows to project $F$ precisely on the relevant sub-spaces. The situation of having several $T$’s occurs in few and privileged cases, but there are some examples.

Notice that relaxing the condition that $h$ be maximal would increase the probability to have several independent invariant tensors $T$. Indeed, let $h'$ be a sub-algebra of $h$, itself being maximally embedded in $so_D$. In the chain of decompositions of $dT$, singlet pieces can come from the various “upper level” components of $T$: under $h \subset so_D$, $dT$ decomposes into $1 + dT'$, and under $h' \subset h$, $dT'$ can give rise to additional singlet pieces. We shall point out such a privileged situations in the next section, for instance in the eight-dimensional case.

Following the steps described above, we have obtained non trivial solutions, that we are now going to expose.
3 Results and Comments

We present our various results under the form of a table exploring the possibilities for $p_T = 2(p + 1) \leq D$ up to $D = 16$. This table gathers the invariance sub-algebras $h$ which fulfill our conditions $i), ii)$. Each possibly relevant case will be separately detailed afterwards (though, again, $T$ is not explicitly constructed here, so that we are not in a position to check $iv)$. 

| $D$  | $p = 1$ | $p = 2$ | $p = 3$ | $p = 4$ | $p = 5$ | $p = 6$ | $p = 7$ |
|------|---------|---------|---------|---------|---------|---------|---------|
| 4    | $su_2 \times su_2$ |         |         |         |         |         |         |
| 5    | $su_2 \times su_2$ |         |         |         |         |         |         |
| 6    |         |         | $su_4$ |         |         |         |         |
| 7    | $su_2 \times su_2 \times su_2$ |         | $su_4$ |         |         |         |         |
| 8    | $g_2 \subset so_7^*$ | $so_8$ |         |         |         |         |         |
| 9    |         |         |         | $su_2 \times su_4$ | $so_8$ |         |         |
| 10   | $su_2 \times su_2 \times su_4$ | $su_5$ | $so_{10}$ |         |         |         |         |
| 11   | $su_2$ |         | $su_2$ | $so_{10}$ |         |         |         |
| 12   | $su_4 \times sp_4 \subset so_{11}^*$ | $su_6$ | $so_{12}$ |         |         |         |         |
|      |         |         | $su_2 \times sp_6$ |         |         |         |         |
| 13   |         |         | $su_2$ | $so_{12}$ |         |         |         |
| 14   |         |         |         |         | $su_7$ | $so_{14}$ |         |
| 15   | $su_2$ |         | $su_2$ | $su_4$ | $su_2$ | $so_{14}$ |         |
|      |         |         | $su_4$ |         | $su_4$ |         |         |
|      |         |         | $su_2 \times sp_4$ |         |         |         |         |
| 16   |         | $sp_4 \times sp_4$ | $so_{16}$ |         | $su_8$ | $sp_4 \times sp_4$ | $so_{16}$ |
|      |         | $su_2 \subset so_{15}^*$ |         |         | $sp_4 \times sp_4$ | $su_2 \times sp_8$ |         |
|      |         | $su_4 \subset so_{15}^*$ |         |         |         |         |         |
|      |         | $su_2 \times sp_4 \subset so_{15}^*$ |         |         |         |         |         |
|      |         | $so_7 \times so_8 \subset so_{15}^*$ |         |         |         |         |         |

The index ($^*$) indicates the particular cases $4(p + 1) = D$, where $T$ can be decomposed into self- and anti-self dual parts, so that $df_T$ is not irreducible under $so_D$, but rather under $so_{(D-1)}$.

Let us remind the isomorphisms between simple Lie algebras: $su_2 \sim so_3 \sim sp_2$, $sp_4 \sim so_5$, $su_4 \sim so_6$. 

11
In the above table, the generic self-duality (1.1) systematically appears on the diagonal for even \( D = 2(p + 1) \). Similarly, one has the odd dimensional \((D - 1) = 2(p + 1)\) briefly discussed in sub-section 2.3.

In 5, 7 and 9 dimensions, the possibilities are of the aforementioned type \( h \equiv so_{D-1} \) or \( so_d \times so_{D-d} \), for which \( \lambda = \pm 1 \) and \( T \) does not project \( F \) properly: one gets more than \( N_{eq} \) constraints on the gauge field. Details for the solutions \( p = 1 \) of \( D = 5, 7 \) can be found in [9]. The other cases with \( h \equiv so_d \times so_{D-d} \) or \( so_{D-1} \) occurring in higher dimensions are analogous. Therefore, we only comment on the cases \((D = 8, p = 1)\) and \( D \geq 10 \) where more peculiar sub-algebras \( h \) occur. Furthermore, we display all the \( h \)-decompositions of \( d_F \) representations in the tables of the appendix.

### 3.1 8-dimensional cases

Apart from the standard solution (1.1) of a 3-form, one can have \( su_4 \) or \( g_2 \) invariance for the 4-form \( T \) of the Yang-Mills case \((p = 1)\). These are the cases of interest in [1], [4], [5]. For \( p = 1, D = 8 \), \( T \) is a \( D/2 \)-form and can thus be naturally decomposed into self and anti-self-dual parts. While \( d_F = 28 \), one seeks \( N_{eq} = 7 \) equations, where the 4-form \( T \), being itself a solution for (1.1), transforms as a \( 70 \sim 35^+ + 35^- \) of \( so_8 \).

We first consider the \( g_2 \subset so_7 \) theory. Under \( so_7 \), \( 35^+ \) and \( 35^- \) remain irreducible, whereas \( d_F \) obviously decomposes into

\[
28 = 7 + 21. \tag{3.1}
\]

Checking the decomposition of \( so_7 \)'s \( 35 \) under its sub-algebras selects \( g_2 \), under which

\[
35 = 1 + 7 + 27,
\]
\[
21 = 7 + 14,
\]
\[
7 = 7: \text{irreducible}. \tag{3.2}
\]

The spectrum for \( T \)'s eigenvalues allows to impose exactly 7 equations (i.e. to project \( F \) onto a \( 7 + 14 \)). Indeed, one can check that \( T \) is the only invariant tensor upon which \( T^2 \) can decompose, so that it admits two eigenvalues (see sub-section 2.2), namely \( \lambda_1 = -1, \lambda_2 = 3 \) (see [4]). Moreover, \( T_{\text{ijk}} \) \((i,j,k\) are 7-dimensional indices\) is antisymmetric and thus traceless.
with respect to the $SO_7$ metric. Therefore, the only possibility for the degeneracy of the $\lambda$'s is the appropriate one: $21\lambda_1 + 7\lambda_2 = 0$, and imposing (1.2) with $\lambda = \lambda_1$ leads to the 7 equations on $F$. This case was first studied by Corrigan et al. [9] and the corresponding $TQFT$ was constructed in [4]. The set of 7 self-duality equations obtained are

\begin{align*}
F_{8i} &= c_{ijk} F_{jk}, \\
c_{ijk} &= T_{8ijk}
\end{align*}

(3.3)

where the $c_{ijk}$ ($i, j, k = 1 \cdots 7$) are the structure constants for octonions. More explicitly, in terms of 8-dimensional components of the curvature, they read

\begin{align*}
F_{12} + F_{34} + F_{56} + F_{78} &= 0, \\
F_{13} + F_{42} + F_{57} + F_{86} &= 0, \\
F_{14} + F_{23} + F_{76} + F_{85} &= 0, \\
F_{15} + F_{62} + F_{73} + F_{48} &= 0, \\
F_{16} + F_{25} + F_{38} + F_{47} &= 0, \\
F_{17} + F_{82} + F_{35} + F_{48} &= 0, \\
F_{18} + F_{27} + F_{63} + F_{54} &= 0.
\end{align*}

(3.4)

Let us now turn to the case $\mathfrak{h} \equiv \mathfrak{su}_4$ (it is not maximal and thus not recorded in the tables from [10]). It enables to complexify the 8-dimensional indices (with $z_a \equiv x_a + ix_{a+4}$, $a = 1 \cdots 4$):

\begin{align*}
F_{\mu\nu} &\rightarrow (F_{z_au}, \overline{F_{z_au}}, F_{z_au}), \\
28 &= \begin{pmatrix} 6 & 6 & 15 & 1 \end{pmatrix}.
\end{align*}

(3.5)

(3.6)

To end up with 7 equations, the projection of $F$ on a $(1 + 6)$ has to be cancelled. This is possible because there are three independant $\mathfrak{su}_4$ invariant 4-forms (see appendix B in [9]), and one considers $T$ a linear combinaison of them. The coefficients can be adjusted at will to end up with the self-duality equations:

\begin{align*}
\text{(1 real eqs)} &\quad \sum_{a=1}^4 F_{z_au} = 0, \\
\text{(6 real eqs)} &\quad F_{za} z_b + \frac{1}{2} \epsilon_{abcd} F_{z_c} z_d = 0.
\end{align*}

(3.7)

(3.8)
Moreover, one can observe that equation (3.7) is the imaginary part of the complex equation

\[ \sum_{a=1}^{4} \left( \partial_{z^a} A_{\overline{z}^a} + \frac{1}{2} [A_{z^a}, A_{\overline{z}^a}] \right) = 0. \] (3.9)

The real part of (3.9) gives the Landau gauge condition \( \partial_\mu A_\mu = 0 \). It is remarkable that the former equations (3.7)–(3.8), when re-expressed in terms of \( SO_8 \) components \( F_{\mu\nu} \), give the set of equations (3.4) obtained through \( g_2 \).

Notice that if one considers the maximal \( su_4 \times u_1 \) rather than \( su_4 \), only one \( T \) is possible, and one cannot adjust its eigenvalues. Equation (1.2) then leads to the sets of 13, 16 or 27 equations of [9] and criteria ii) cannot be satisfied. This illustrates the situation explained in subsection 2.3, where more flexibility comes from relaxing the hypothesis that \( h \) be maximal.

### 3.2 10-dimensional cases

Here, apart from the case of a 2-form (of the aforementioned type \( so_d \times so_{D-d} \equiv so_4 \times so_6 \)), we have as a non-standard solution a 3-form gauge field with \( T \) invariant under \( su_5 \) (here again, the maximal sub-group is rather \( su_5 \times u_1 \subset so_{10} \)). The 4-form curvature is a \( 210 \), the 8-form \( T \) is a \( 45 \) and the self-duality equation should count for 84 conditions (\( so_{10} \) has two \( 210 \), only one of which is the 4-form we are interested in). This case can be investigated through complexification of the 10-dimensional indices (with \( z_a \equiv x_a + i x_{a+5} \)):

\[ F_{\mu\nu\rho\sigma} \rightarrow (F_{z_a z_b z_c z_d}, F_{z_a z_b z_c z_e}, F_{z_a z_b z_c z_f}, F_{z_a z_b z_c z_g}, F_{z_a z_b z_c z_h}), \] (3.10)

\[ 210^A = (5^+ + 5^-) + (40^+ + 40^-) + (10^+ + 10^-) + 75 + 24 + 1. \] (3.11)

An 84-dimensional sum of representations can be exhibited in various ways.

### 3.3 11-dimensional cases

Two interesting possibilities arise, for a Yang-Mills field as well as a 3-form, both involving the so-called principal \( su_2 \) of \( so_{11} \) as the invariance sub-algebra (there are many \( su_2 \) sub-algebras in \( so_D \), but the only one that is maximal is the principal \( su_2 \) for odd \( D \)). Indeed, our first two requirements are fulfilled: the 1-form’s curvature is a \( 55 \) and the 3-form’s curvature a \( 330 \), while

\[ 55 = 10 + 15 + 11 + (7 + 3)^p \quad \text{for } p = 1, \] (3.12)
\[330 = 23 + 21 + 19 + 3 \times 17 + 3 \times 13 + 2 \times 11 + 3 \times 9 + 7 + 1\]
\[+ (29 + 25 + 21 + 2 \times 15 + 3 \times 5)^*\quad \text{for } p = 3. \quad (3.13)\]

For the 1-form and 3-form, one needs respectively \( N_{\text{eq}} = 10 \) and 120 equations. There are several possibilities for the corresponding sum of representations, one of which we point out by \((\cdots)^*\) in \((3.12)-(3.13)\).

Notice that one can expect more than two eigenvalues for \( T \) in both cases (other \( su_2 \)-invariant tensors appear in the decomposition of \( T^2 \)). However, as to the \( p = 3 \) case, in view of the number of representations involved in the decomposition for \( d_F \), we do not know whether the spectrum of the eigenvalues can allow to project exactly on the complementary representation of \( N_{\text{eq}} \) in \((3.13)\).

### 3.4 12-dimensional cases

In addition to the generic case \([1.1]\) of a 5-form, \( D = 12 \) allows \( p = 2 \) and 3. For \( p = 2 \), the \((D = 8, p = 1)\) pattern is reproduced: since it is a \( D/2 \)-form, \( T \) can be decomposed into self and anti-self-dual parts \((462^+ + 462^-)\), which remain irreducible under \( so_{11} \). Moreover, the 3-form curvature lies in the \( 220 \) of \( so_{12} \), which, under \( so_{11} \), automatically gives rise to the crucial \( N_{\text{eq}} = 55 \) dimensional representation:

\[220 = 55 + 165. \quad (3.14)\]

Inspecting \( d_T \)'s decompositions under \( so_{11} \)'s maximal sub-algebras selects \( su_4 \times sp_4 \sim so_6 \times so_5 \).

The corresponding representations for the curvature \( F \) behave under \( su_4 \times sp_4 \) as:

\[55 = (15, 1) + (6, 5) + (1, 10) \quad (3.15)\]
\[165 = (10^+, 1) + (10^-, 1) + (15, 5) + (6, 10) + (1, 10). \quad (3.16)\]

Unlike the \((p = 1, D = 8)\) case, \( T \) is not the only invariant tensors to appear in \( T^2 \) so there can be more than two values for \( \lambda \). We do not know whether their spectrum enables to get the wanted 55 equations from \((1.2)\).

The remaining case \( p = 3 \) admits two possible invariance sub-algebras, namely \( su_6 \) (non-maximal) and \( su_2 \times sp_6 \). The decomposition for \( d_F = d_T = 495 \) under \( su_6 \) is obtained by complexification of the indices and an \( N_{\text{eq}} = 165 \)-dimensional representation can be extracted. As for the case of \( p = 2 \), other invariant tensors appear in the right hand side of \((2.4)\).
3.5 13- and 14- dimensional cases

As for the 11-dimensional 1- and 3-forms, for \( p = 3 \) in 13 dimensions, it is the principal \( su_2 \) that arises. The representation \( 1287 \) for the 8-form \( T \) has two singlet components under \( su_2 \). This suggests that one has some freedom to arrange the eigenvalues, but we do not know whether \( F \)’s projection onto the \( N_{eq} = 220 \)-dimensional reducible representation can be cancelled.

The \((D = 14, \ p = 5)\) \( su_7 \) possibility is another case of complexification of the indices and is easily worked out: \( T \) has only one singlet component and an \( N_{eq} = 1287 \)-dimensional representation can be found in the decomposition for \( d_F = 3003 \).

3.6 15-dimensional cases

\( D = 15 \) offers quite a few possibilities, for all possible form degree except the 2-form gauge field. The 6-form is of the aforementioned type \( h \equiv so(D-1) \).

For the case of a Yang-Mills field, \( F \) lies in the representation \( 105 \) of \( so_{15} \) and \( T \) in the \( 1365 \). The later gives two singlets in its decomposition under the maximal \( su_2 \subset so_{15} \).

In the case of a 3-form gauge field, three maximal sub-algebras are suitable, namely, \( su_2, su_4 \) and \( su_2 \times sp_4 \). Here, \( d_T = 6435 \) gives four singlets under \( su_2 \) against only one under the last two. The decompositions of \( d_F = 1365 \) can be found in the appendix, where the \( N_{eq} = 364 \)-dimensional representation is singled out. It involves an increasing number of low-dimensional representations of \( h \), thus the fulfilling of \( iv \) seems more and more difficult to us. However, for the case of \( su_2 \), \( T \) being a linear combination of four invariant tensors, there is more freedom for a relevant choice of the eigenvalues.

3.7 16-dimensional cases

In 16 dimensions, for \( p = 3 \), one has the case of \( 4(p + 1) = D \). It gives several possible sub-algebras \( h \subset so_{15} \subset so_{16} \). Unlike the eight and twelve dimensional analogues, here \( T \) also contains a singlet in its decomposition under \( sp_4 \times sp_4 \), maximal within \( so_{16} \). Finally, one has a 5-form with either \( su_8 \) (complexification of the indices), \( sp_4 \times sp_4 \) or \( su_2 \times sp_8 \). For all these cases, \( T \) has only one singlet component.
4 Conclusion

Recent developments have indicated that $TQFT$’s in various dimensions, i.e. theories with twisted supersymmetries, may play an important role. The case of the eight dimensional Yang-Mills $TQFT$ has indicated the relevance of a theory defined on a manifold with special holonomy group $H \subset SO_D$. The key to such $TQFT$’s is the existence of $H$-invariant “self-duality equations” (1.2) for the curvature of the gauge field: via BRST quantization, they can serve as gauge fixing functions, provided they satisfy certain restrictions.

In this paper, we have pointed out restrictive conditions to be fulfilled by (1.2) to possibly admit instanton-like solutions and determine $D$-dimensional $TQFT$’s. We have not restricted ourselves to Yang-Mills fields, but have considered $p$-form gauge fields, in view of their importance in the definition of field theories for branes. Following the first steps exhibited in our analysis, we have established a table, listing for each $p$ and $D \leq 16$, the possible subgroups $H \subset SO_D$ for which an $SO_D$-covariant and $H$-invariant self-duality equation can exist. We did not go any further in building the $TQFT$’s, but we believe that for some high dimensional cases, interesting and possibly unfamiliar theories could arise.

Acknowledgments:

We are grateful to R. Stora for interesting discussions related to this work. We also thank O. Babelon, E. Ragoucy, P. Roche, M. Talon for fruitful comments, and J. Patera for communication of unpublished results.
### Appendix

Here, we give the $h$-decomposition of $d_F$ for all the relevant cases for $D \geq 10$; one possibility for the $N_{eq}$-dimensional reducible representation is singled out with $(\cdots)^*$.

| $D = 10$ | $p = 2$, $d_F = 120$, $d_T = 210$, $N_{eq} = 36$ |
| --- | --- |
| $su_2 \times su_2 \times su_4$ | $120 = (1,3,6)+ (3,1,6)+ (2,2,15)+ (1,1,10^+)+ (1,1,10^-)+ (2,2,1)$ |
| $su_5$ | $210 = (10^+ +10^- +10^- +24^+)^* + 75 + 40^+ + 5^+ + 5^- + 1$ |
| $D = 11$ | $p = 1$, $d_F = 55$, $d_T = 330$, $N_{eq} = 10$ |
| $su_2$ | $55 = (7+3)^* + 10 + 15 + 11$ |
| $su_4 \times sp_4$ | $330 = (29+25+21+2 \times 15+3 \times 5)^* + 23+21 + 19+3 \times 17+3 \times 13 + 2 \times 11+ 3 \times 9+7+1$ |
| $D = 12$ | $p = 2$, $d_F = 220$, $d_T = 462 + 462$, $N_{eq} = 55$ |
| $so_{15}$ | $220 = 55^* + 165$ |
| $su_4 \times sp_4$ | $55 = (15,1)+ (6,5)+ (1,10)$, $165 = (10^+ +10^- +15^+ +6 \times 10)+ (1,10)$ |
| $su_6$ | $495 = (105^+ + 2 \times 15^+ + 2 \times 15^-)^* + 180 + 150^+ + 35 + 1$ |
| $su_2 \times sp_6$ | $495 = (1,90)+ (5,14^+)+ (5,1)^* + (3,70)+ (3,21)+ (3,14^+)+ (1,14^+)+ (1,1)$ |
| $D = 13$ | $p = 3$, $d_F = 715$, $d_T = 1287$, $N_{eq} = 220$ |
| $su_2$ | $715 = (37+33+31+2 \times 29+27+25+9)^* + 2 \times 25+ 2 \times 21 + 3 \times 19+4 \times 17$ $+ 3 \times 15+ 5 \times 13+ 2 \times 11+ 3 \times 9+ 2 \times 7+ 3 \times 5+ 2 \times 1$ |
| $D = 14$ | $p = 5$, $d_F = 3003$, $d_T = 91$, $N_{eq} = 1287$ |
| $su_7$ | $3003 = (490^+)+ 392+ 224^+ + 112^+ + 48+ 21^+)^* + 784+ 490^+ + 224^+ + 112^- + (35^+ + 35^-)+ 21^- + 7^+ + 2^+ + 1$ |
| $D = 15$ | $p = 1$, $d_F = 105$, $d_T = 1365$, $N_{eq} = 14$ |
| $su_2$ | $105 = (11+2)^* + 27+ 23+ 19+ 15+ 7$ |
| $su_2$ | $1365 = (45+41+29+ 2 \times 37+ 35+ 3 \times 33+ 31)^* + 31+ 4 \times 29+ 3 \times 27+ 5 \times 25$ $+ 4 \times 23+ 6 \times 21 + 4 \times 19+ 6 \times 17+ 4 \times 15+ 6 \times 13+ 3 \times 11+ 5 \times 9+ 2 \times 7+ 4 \times 5+ 2 \times 1$ |
| $su_4$ | $1365 = (105+ 175+ 84)^* + (256^+ + 256^-)^* + 175 + 84 + (35^+ + 35^-)+ (45^+ + 45^-)+ 2 \times 20 + 2 \times 15$ |
| $su_2 \times sp_4$ | $1365 = ((3,35)^+)+ (3,35)+ (5,14)+ (1,14)^* + (3,81)+ (7,35)+ (5,35)+ (1,35)^* + (7,10)+ (2,5)+ (3,14)+ (5,10)+ 2 \times (3,10)+ (5,5)+ (1,5)+ (6,1)+ (1,1)$ |
\[
p = 4, \, d_F = 3003, \, d_P = 3003, \, N_{eq} = 1001
\]

\[
\text{su}_4
\]
\[
3003 = (300 + 280^+ + 105 + 175 + 45^+ + 45^- + 20 + 2 \times 15 + 1)^* + 280^\mp + 2 \times 256^+ + 2 \times 256^- + 2 \times 175 + 2 \times 84 + 2 \times 45^+ + 2 \times 45^- 
\]

\[
\text{su}_2
\]
\[
5005 = (55 + 51 + 49 + 2 \times 47 + 2 \times 45 + 4 \times 43 + 3 \times 41 + 6 \times 39 + 5 \times 37 + 8 \times 35 + 8 \times 33 + 12 \times 31 + 23 + 7 + 3)^* + 37 + 10 \times 29 + 14 \times 27 + 13 + 25 + 15 + 23 + 14 + 21 + 18 + 19 + 13 + 17 + 12 + 13 + 13 + 14 + 11 + 9 \times 9 + 11 + 7 \times 4 + 5 \times 3 
\]

\[
\text{su}_4
\]
\[
5005 = (2 \times 300 + 256^+ + 256^- + 5 \times 175 + 15)^* + 729 + 2 \times (280^+ + 280^-) + (256^+ + 256^-) + (35^+ + 35^-) + 3 \times 84 + 3 \times (45^+ + 45^-) + 20 + 2 \times 15 
\]

\[
D = 16
\]
\[
p = 3, \, d_F = 1820, \, d_P = 6435 + 6435, \, N_{eq} = 455
\]

\[
\text{sp}_4 \times \text{sp}_4
\]
\[
1820 = [(35, 10) + (10, 10) + (5, 1)]^* + (14, 14) + (10, 35) + (35, 5) + (5, 35) + (14, 14) + (10, 10) + (5, 10) + (10, 5) + (14, 14) + (5, 10) + (14, 14) + (5, 10) + (14, 14) + (5, 10) + (14, 14) + (1, 1) 
\]

\[
\text{so}_5
\]
\[
1820 = 1365 + 455 
\]

\[
\text{su}_2
\]
\[
455 = 37 + 33 + 31 + 29 + 27 + 2 \times 25 + 23 + 21 + 2 \times 19 + 2 \times 17 + 2 \times 15 + 3 \times 13 + 11 + 2 \times 9 + 7 + 5 + 1 
\]
\[
1365 = 45 + 41 + 39 + 2 \times 37 + 35 + 33 + 3 \times 31 + 4 \times 29 + 3 \times 27 + 5 \times 25 + 4 \times 23 + 6 \times 21 + 4 \times 19 + 6 \times 17 + 4 \times 15 + 6 \times 13 + 3 \times 11 + 5 \times 9 + 2 \times 7 + 4 \times 5 + 2 \times 1 
\]

\[
\text{su}_4
\]
\[
455 = 175 + 84 + 45^+ + 45^- + 20 + 15 + 1 
\]
\[
1365 = 256^+ + 256^- + 105 + 2 \times 175 + 2 \times 84 + 35^+ + 35^- + 45^+ + 45^- + 2 \times 20 + 2 \times 15 
\]

\[
\text{su}_2 \times \text{sp}_4
\]
\[
455 = (5, 35) + (1, 20) + (3, 35) + (7, 10) + (3, 10) + (5, 5) + (3, 5) + (1, 1) 
\]
\[
1365 = (3, 81) + (5, 35) + (5, 35) + (7, 35) + (1, 35) + (3, 35) + (5, 14) + (7, 10) + (9, 5) + (3, 14) + (5, 10) + (1, 14) + 2 \times (3, 10) + (5, 5) + (1, 5) + (5, 1) + (1, 1) 
\]

\[
\text{so}_7 \times \text{so}_8
\]
\[
455 = (7, 28) + (21, 8)^\vee + (1, 56)^C + (35, 1) 
\]
\[
1365 = (21, 28) + (7, 56)^C + (35, 8)^C + (1, 35)^\vee + (1, 35)^C + (35, 1) 
\]

\[
\text{su}_8
\]
\[
p = 5, \, d_F = 8008, \, d_P = 1820, \, N_{eq} = 3003 
\]

\[
8008 = (1255 + 420^\mp + 70 + 20 + 28 \pm)^* + (15, 12)^\pm + 1512^- 
\]
\[
+ 720 + (378^\mp + 378^-) + 420^\mp + 28^\mp + (28^\mp + 28^-) + 1 
\]

\[
\text{sp}_4 \times \text{sp}_4
\]
\[
8008 = [(35, 35) + (81, 5) + (10, 35) + (5, 81) + (14, 14) + (81, 1) + (1, 81) + (14, 10) + (10, 10) + 2 \times (10, 1)]^* + (35^\mp + 30^\mp + 10 + 20 + 35^\mp + 2 \times (35, 5) + 2 \times (35, 5) + 2 \times (5, 35) + 2 \times (5, 35) + 2 \times (10, 10) + 2 \times (10, 10) + 2 \times (10, 10) + 2 \times (10, 10) + 2 \times (10, 10) + 2 \times (10, 10) + 2 \times (10, 10) + 2 \times (10, 10) 
\]

\[
\text{su}_2 \times \text{sp}_8
\]
\[
8008 = [(3, 315) + (3, 315) + (3, 42) + (7, 27) + (3, 27) + (3, 1)]^* + (1, 825) + (3, 729(2)) 
\]
\[
+ (3, 303) + (3, 315) + (3, 360) + (3, 42) + (3, 27) + (1, 36) + (3, 27) + (7, 1) 
\]

19
References

[1] E. Witten, *Topological Quantum Field Theory*, Comm. Math. Phys. **117**(1988)353.

[2] L. Baulieu, I. M. Singer, *Topological Yang-Mills Symmetry*, Nucl. Phys. Proc. Suppl. **15B**(1988)12; L. Baulieu, *On the Symmetries of Topological Quantum Field Theories*, Int. Jour. Mod. Phys. **A10**(1995)4483, hep-th/9504015.

[3] E. Witten, *Supersymmetric Yang-Mills theory on a four-manifold*, in Jour. Math. Phys. **35**(1994)5101, hep-th/9403195.

[4] L. Baulieu, H. Kanno and I. M. Singer, *Cohomological Yang-Mills Theory in Eight Dimensions*, Talk given at APCTP Winter School on Dualities in String Theory, (Sokcho, Korea), February 24-28, 1997, hep-th/9705127; *Special Quantum Field Theories in Eight and Other Dimensions*, hep-th/9704167, to appear in Comm. Math. Phys.

[5] B. S. Acharya, M. O’Loughlin, *Self-Duality in D ≤ 8 dimensional Euclidean Gravity*, hep-th/9612182; B. S. Acharya, J. M. Figueroa-O’Farrill, M. O’Loughlin, B. Spence, hep-th/9707118; B. S. Acharya, M. O’Loughlin, B. Spence, hep-th/9705138.

[6] C. Hull, *Higher Dimensional Yang-Mills Theories and Topological Terms*, hep-th/9710165.

[7] M. J. Duff, J. M. Evans, R. R. Khuri, J. X. Lu, R. Minasian, *The Octonionic Membrane*, Phys. Lett. **B412**(1997)281, hep-th/9706124.

[8] D. B. Fairlie, J. Nuys, Jour. Phys. **A17**(1984)2867; S. Fubini, H. Nicolai, Phys. Lett. **155B**(1985)369; T. A. Ivanova, A. D. Popov, Lett. Math. Phys. **24**(1992)85; Theor. Math. Phys. 94(1993)225; T. A. Ivanova, Phys. Lett. **B315**(1993)277; M. Günayden, H. Nicolai, Phys. Lett. **B351**(1995)507;

[9] E. Corrigan, C. Devchand, D. B. Fairlie, J. Nuyts, *First Order Equations for Gauge Fields in Spaces of Dimension Greater Than Four*, in Nucl. Phys. **B214**(1983)452.

[10] W. G. Mckay, J. Patera, *Tables of Dimensions, Indices, and Branching Rules for Representations of Simple Lie Algebras*, Lecture Notes in Pure and Applied Mathematics, Volume 69, Ed. Marcel Dekker Inc., New York and Basel; J. Patera, *private communication*.

[11] R. Slansky, *Group Theory for Unified Model Building*, Phys. Rev. **79**(1981)1.