We study the effects of weak columnar and point disorder on the vortex-lattice phase transitions in high temperature superconductors. The combined effect of thermal fluctuations and of quenched disorder is investigated using a simplified cage model. For columnar disorder the problem maps into a quantum particle in a harmonic + random potential. We use the variational approximation to show that columnar and point disorder have opposite effect on the position of the melting line as observed experimentally. Replica symmetry breaking plays a role at the transition into a vortex glass at low temperatures.

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There is a lot of interest in the physics of high temperature superconductors due to their potential technological applications. In particular these materials are of type II and allow for partial magnetic flux penetration. Pinning of the magnetic flux lines (FL) by many types of disorder is essential to eliminate dissipative losses associated with flux motion. In clean materials below the superconducting temperature there exist a 'solid ' phase where the vortex lines form a triangular Abrikosov lattice \([1]\). This solid can melt due to thermal fluctuations and the effect of impurities. In particular known observed transitions are into a flux liquid at higher temperatures via a melting line (ML) \([2]\), and into a vortex glass at low temperature \([3]\, [4]\, [5]\) in the presence of disorder- the so called entanglement line (EL) \([6]\). Point disorder has been induced by electron irradiation (1 GeV Xe or 0.9 GeV Pb). It turns out that the flux melting transition persists in the presence of either type of disorder, but its position shifts depending on the disorder type and strength.

A significant difference has been observed between the effects of columnar and point disorder on the location of the ML. Weak columnar defects stabilize the solid phase with respect to the vortex liquid phase and shift the transition to higher fields, whereas point-like disorder destabilizes the vortex lattice and shifts the melting transition to lower fields. In this paper we attempt to provide an explanation to this observation. The case of point defects has been addressed in a recent paper by Ertas and Vinokur \([8]\). Here we use a different approach to analyze the cage-model Hamiltonian. We use the replica method together with the variational approximation. In the case of columnar defects our approach relies on our recent analysis of a quantum particle in a random potential \([9]\). We compare the effect of the two types of disorder with each other and with results of recent experiments.

Assume that the average magnetic field is aligned along the \(z\)-axis. Following EN we describe the Hamiltonian of a single FL whose position is given by a two-component vector \(\mathbf{r}(z)\) (overhangs are neglected) by:

\[
H = \int_0^L dz \left\{ \frac{\varepsilon}{2} \left( \frac{d\mathbf{r}}{dz} \right)^2 + V(z, \mathbf{r}) + \frac{\mu}{2} \right\}.
\]  

(1)

Here \(\varepsilon = \varepsilon_0 / \gamma^2\) is the line tension of the FL, \(\gamma^2 = m_\perp / m_\parallel\) is the mass anisotropy, \(\varepsilon_0 = (\Phi_0 / 4 \pi \lambda)^2\) (\(\lambda\) being the penetration length), and \(\mu \approx \varepsilon_0 / a_0^2\) is the effective spring constant (setting the cage size) due to interactions with neighboring FLs, which are at a typical distance of \(a_0 = \sqrt{\Phi_0 / B}\) apart.

For the case of columnar (or correlated) disorder, \(V(z, \mathbf{r}) = V(\mathbf{r})\) is independent of \(z\), and

\[
\langle V(\mathbf{r})V(\mathbf{r}') \rangle \equiv -2f((\mathbf{r} - \mathbf{r}')^2 / 2) = g \varepsilon_0^2 \xi^2 \delta_\xi^2(\mathbf{r} - \mathbf{r}').
\]  

(2)

where

\[
\delta_\xi^2(\mathbf{r} - \mathbf{r}') \approx 1 / (2\pi \xi^2) \exp(-(\mathbf{r} - \mathbf{r}')^2 / 2\xi^2),
\]  

(3)

and \(\xi\) is the vortex core diameter. The dimensionless parameter \(g\) is a measure of the strength of the disorder.

On the other hand for point–disorder, \(V\) depends on \(z\) and

\[
\langle V(z, \mathbf{r})V(z', \mathbf{r}') \rangle = \tilde{\Delta} \varepsilon_0^2 \xi^3 \delta_\xi^2(\mathbf{r} - \mathbf{r}')\delta(z - z').
\]  

(4)

The quantity that measures the transverse excursion of the FL is

\[
\Delta(\mathbf{r}, \mathbf{r}') = \int d\mathbf{r}'' \delta_\xi(\mathbf{r} - \mathbf{r}'').
\]

(5)

The minus sign is a convention to define the effective force felt by the vortex. 

\[
\langle V(\mathbf{r})\rangle = \int d\mathbf{r}' \int d\mathbf{r}'' \delta_\xi(\mathbf{r} - \mathbf{r}') \delta_\xi(\mathbf{r}' - \mathbf{r}'') = \varepsilon_0^2 \xi^2 \delta_\xi(\mathbf{r}) \delta_\xi(\mathbf{r} - \mathbf{r}')
\]

(6)

\[
\frac{d^2 H}{d\mathbf{r}} = -2f((\mathbf{r} - \mathbf{r}')^2 / 2).
\]  

(7)

We will use the following notation and conventions: \(\delta_\xi(\mathbf{r} - \mathbf{r}')\) is a delta function of the vortex position and \(\xi\) is the vortex core diameter. 

\[
\Delta(\mathbf{r}, \mathbf{r}') \equiv \int d\mathbf{r}'' \delta_\xi(\mathbf{r} - \mathbf{r}'').
\]

(8)

The minus sign is a convention to define the effective force felt by the vortex.
\[ u_0^2(\ell) \equiv \langle |r(z) - r(z + \ell)|^2 \rangle / 2, \quad (5) \]

Let us now review the connection between a quantum particle in a random potential and the behavior of a FL in a superconductor. The partition function of the former is just like the partition sum of the FL, provided one make the identification

\[ h \rightarrow T, \quad \beta h \rightarrow L, \quad (6) \]

Where \( T \) is the temperature of the superconductor and \( L \) is the system size in the \( z \)-direction. \( \beta \) is the inverse temperature of the quantum particle. We are interested in large fixed \( L \) as \( T \) is varied, which corresponds to high \( \beta \) for the quantum particle when \( h \) (or alternatively the mass of the particle) is varied. Variable \( z \) is the so called Trotter time. This is the picture we will be using for the case of columnar disorder.

For the case of point-disorder the picture we use is that of a directed polymer in the presence of a random potential plus an harmonic potential as used by EN.

The main effect of the harmonic (or cage) potential is to cap the transverse excursions of the FL beyond a confinement length \( \ell^* \approx a_0/\gamma \). The mean square displacement of the flux line is given by

\[ u^2(T) \approx u_0^2(\ell^*), \quad (7) \]

The location of the melting line is determined by the Lindemann criterion

\[ u^2(T_m(B)) = c_L^2 a_0^2, \quad (8) \]

where \( c_L \approx 0.15 - 0.2 \) is the phenomenological Lindemann constant. This means that when the transverse excursion of a section of length \( \approx \ell^* \) becomes comparable to a finite fraction of the interline separation \( a_0 \), the melting of the flux solid occurs.

We consider first the case of columnar disorder. In the absence of disorder it is easily obtained from standard quantum mechanics and the correspondence [2], that when \( L \rightarrow \infty \),

\[ u^2(T) = \frac{T}{\sqrt{\varepsilon_\mu}} \left(1 - \exp(-\ell^* \sqrt{\mu/\varepsilon})\right) = \frac{T}{\sqrt{\varepsilon_\mu}} (1 - e^{-1}), \quad (9) \]

from which we find that

\[ B_m(T) \approx \frac{\Phi_0 \varepsilon_\varepsilon \gamma^2}{\gamma^2 \ell^2 T^2}. \quad (10) \]

When we turn on disorder we have to solve the problem of a quantum particle in a random quenched potential. This problem has been recently solved using the replica method and the variational approximation [3]. Let us review briefly the results of this approach. In this approximation we chose the best quadratic Hamiltonian parametrized by the matrix \( s_{ab}(z - z') \):

\[ h_n = \frac{1}{2} \int_0^L dz \sum_a \frac{[\tilde{r}_a^2 + \mu r_a^2]}{2} \]
\[ - \frac{1}{2T} \int_0^L dz \int_0^L dz' \sum_{a,b} s_{ab}(z - z') r_a(z) \cdot r_b(z'). \quad (11) \]

Here the replica index \( a = 1 \ldots n \), and \( n \rightarrow 0 \) at the end of the calculation. This Hamiltonian is determined by stationarity of the variational free energy which is given by

\[ \langle F \rangle_R/T = \langle H_n - h_n \rangle_{h_n} - \ln \int [dr] \exp(-h_n/T), \quad (12) \]

where \( H_n \) is the exact \( n \)-body replicated Hamiltonian. The off-diagonal elements of \( s_{ab} \) can consistently be taken to be independent of \( z \), whereas the diagonal elements are \( z \)-dependent. It is more convenient to work in frequency space, where \( \omega \) is the frequency conjugate to \( z \).

\[ \omega_j = (2\pi/L)j, \text{with } j = 0, \pm 1, \pm 2, \ldots \]

Assuming replica symmetry, which is valid only for part of the temperature range, we can denote the off-diagonal elements of \( \tilde{s}_{ab}(\omega) = (1/T) \int_0^L dz e^{i \omega z} s_{ab}(z) \), by \( \tilde{s}(\omega) = \tilde{s}_\omega.0 \). Denoting the diagonal elements by \( \tilde{s}_d(\omega) \), the variational equations become:

\[ \tilde{s} = 2L \frac{L}{T} \tilde{f} \left( \frac{2T}{\mu L} + \frac{2T}{L} \sum_{\omega \neq 0} \frac{1}{\varepsilon \omega^2 + \mu - \tilde{s}_d(\omega')} \right) \quad (13) \]

\[ \tilde{s}_d(\omega) = \tilde{s} - 2L \frac{L}{T} \int_0^L d\zeta \left(1 - e^{i \omega \zeta}\right) \times \]

\[ \tilde{f} \left( \frac{2T}{L} \sum_{\omega' \neq 0} \frac{1 - e^{-i \omega' \zeta}}{\varepsilon \omega'^2 + \mu - \tilde{s}_d(\omega')} \right). \quad (14) \]

where \( \tilde{f}'(y) \) denotes the derivative of the "dressed" function \( f(y) \) which is obtained in the variational scheme from the random potential's correlation function \( f(y) \) (see eq. \( 12 \)), and in 2+1 dimensions is given by:

\[ \tilde{f}(y) = -g \frac{\xi^2 \varepsilon^2}{4\pi} \frac{1}{\xi^2 + y} \quad (15) \]

The full equations, taking into account the possibility of replica-symmetry breaking are given in ref. [3]. In terms of the variational parameters the function \( u_0^2(\ell^*) \) is given by

\[ u_0^2(\ell^*) = \frac{2T}{L} \sum_{\omega' \neq 0} \frac{1 - \cos(\omega' \ell^*)}{\varepsilon \omega'^2 + \mu - \tilde{s}_d(\omega')} \quad (16) \]

This quantity has not been calculated in ref. [3]. There we calculated \( \langle r^2(0) \rangle \) which does not measure correlations along the \( z \)-direction.
In the limit \( L \to \infty \) we were able to solve the equations analytically to leading order in \( g \). In that limit eq. (14) becomes (for \( \omega \neq 0 \)):

\[
\hat{s}_d(\omega) = \frac{4}{\mu} \tilde{f}''(b_0) - \frac{2}{T} \int_0^\infty d\omega \frac{1 - \cos(\omega \xi)}{2\pi \xi \omega^2 + \mu - \hat{s}_d(\omega)}
\]

with

\[
C_0(s) = 2T \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1 - \cos(\omega \xi)}{2\pi \xi \omega^2 + \mu - \hat{s}_d(\omega)}
\]

(17)

and \( b_0 \) given by a similar expression with the cosine term missing in the numerator of eq. (18).

Defining

\[
\tau = T / \sqrt{\mu}, \quad \alpha = \tau / (\xi^2 + \tau),
\]

\[
f_1(\alpha) = 1/(1 - \alpha) - (1/\alpha) \log(1 - \alpha),
\]

\[
f_2(\alpha) = \frac{1}{\alpha} \sum_{k=1}^{\infty} (k + 1)\alpha^k / k^3
\]

(19) \quad (20) \quad (21)

\[
a^2 = f_1(\alpha)/f_2(\alpha), \quad A = -\tilde{f}''(\tau) f_1^2(\alpha)/f_2(\alpha)/\mu,
\]

\[
s_\infty = \tilde{f}''(\tau) (4 + f_1(\alpha))/\mu,
\]

(22) \quad (23)

a good representation of \( \tilde{s}_d(\omega) \), (\( \omega \neq 0 \)) with the correct behavior at low and high frequencies is

\[
\tilde{s}_d(\omega) = s_\infty + A\mu/\xi (\xi^2 + a^2)\mu.
\]

(24)

(notice that this function is negative for all \( \omega \)). Substituting in eq. (18) and expanding the denominator to leading order in the strength of the disorder, we get:

\[
u_0^2(\ell) = \rho(\sqrt{\mu} / \mu) = \tau(1 - A/(a^2 - 1)^2 / \mu)
\]

\[
\times (1 - e^{-\ell/\ell'}) + \tau A/(a^2 - 1)^2 / \mu
\]

\[
\times (1 - e^{-2\ell/\ell'}) + \tau(2\mu) / (s_\infty + A/(a^2 - 1))
\]

\[
\times (1 - e^{-\ell/\ell'} - (\ell/\ell') e^{-\ell/\ell'})
\]

(25)

In order to plot the results we measure all distances in units of \( \xi \), we measure the temperature in units of \( \epsilon_0 \xi \), and the magnetic field in units of \( \Phi_0 / \xi^2 \). We observe that the spring constant \( \mu \) is given in the rescaled units by \( B \) and \( a_0 = 1/\sqrt{\rho} \). We further use \( \gamma = 1 \) for the plots.

Fig. 1 shows a plot of \( \sqrt{\rho(\ell^*)/a_0} \) vs. \( T \) for zero disorder (curve a) as well as for \( g/2\pi = 0.02 \) (curve b). We have chosen \( B = 1/900 \). We see that the disorder tends to align the flux lines along the columnar defects, hence decreasing \( u^2(T) \). Technically this happens since \( \tilde{s}_d(\omega) \) is negative. The horizontal line represents a possible Lindemann constant of 0.15.

In Fig. 2 we show the modified melting line \( B_m(T) \) in the presence of columnar disorder. This is obtained from eq. (8) with \( c_L = 0.15 \). We see that it shifts towards higher magnetic fields.

For \( T < T_c \approx (\epsilon_0 \xi / \gamma) [g^2 \epsilon_0/(16\pi^2 \mu \xi^2)]^{1/6} \), there is a solution with RSB but we will not pursue it further in this paper. This temperature is at the bottom of the range plotted in the figures for columnar disorder. We will pursue the RSB solution only for the case of point disorder, see below. The expression (23) becomes negative for very low temperature. This is an artifact of the truncation of the expansion in the strength of the disorder.

For the case of point defects the problem is equivalent to a directed polymer in a combination of a random potential and a fixed harmonic potential. This problem has been investigated by MP [10], who were mainly concerned with the limit of \( \mu \to 0 \). In this case the variational quadratic Hamiltonian is parametrized by:

\[
h_n = \frac{1}{2} \int_0^L dz \sum_{a} (\partial^2_{x^2} + \mu x^2) + \frac{1}{2} \int_0^L dz \sum_{a,b} s_{ab} r_a(z) \cdot r_b(z),
\]

(26)

with the elements of \( s_{ab} \) all constants as opposed to the case of columnar disorder.

The replica symmetric solution to the variational equations is simply given by:

\[
s = s_d = 2\xi / T \tilde{f}'(\tau)
\]

(27)

\[
u_0^2(\ell) = 2T \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1 - \cos(\omega \xi)}{\omega^2 + \mu} \left( 1 + \frac{s_d}{\epsilon \omega^2 + \mu} \right)
\]

(28)

and hence

\[
u_0^2(\ell) = \tau (1 - e^{-\ell/\ell'}) + \tau s_d / (2\mu)
\]

\[
\times (1 - e^{-\ell/\ell'} - (\ell/\ell') e^{-\ell/\ell'}).\]

(29)

In eq. (27) \( \tilde{f} \) is the same function as defined in eq. (13) with \( g \) replaced by \( \Delta \). As opposed the case of columnar disorder, in this case \( s_d \) is positive and independent of \( \omega \), and hence the mean square displacement \( u_0^2(\ell^*) \) is bigger than its value for zero disorder. Fig. 1 curve c shows a plot of \( u_0^2(\ell^*)/a_0 \) vs. \( T \) for \( A/2\pi = 0.8 \). Again \( B = 1/900 \). For \( T < T_{cp} \approx (\epsilon_0 \xi / \gamma) (A/2\pi)^{1/3} \) it is necessary to break replica symmetry as shown by MP [10]. This means that the off-diagonal elements of the variational matrix \( s_{ab} \) are not all equal to each other. MP worked out the solution in the limit of \( \mu \to 0 \), but it is not difficult to extend it to any value of \( \mu \). We have worked out the first stage RSB solution which is all is required for a random potential with short ranged correlations. The analytical expression is not shown here for lack of space. The solution is represented by curve d in Fig. 1 which consists of upward triangles.

The modified melting line in the presence of disorder is indicated by the curve c in Fig. 2 for \( T > T_{cp} \). For \( T < T_{cp} \) the so called entanglement line is represented by curve d of filled squares. The value of the magnetic
field $B_m(T_{cp}) \approx (\Phi_0/\xi^2)(\gamma\Delta/2\pi)^{-2/3}c_4^L$ gives a reasonable agreement with the experiments.

The analytical expressions given in eqs. (25), (29), though quite simple, seem to capture the essential feature required to reproduce the position of the melting line. The qualitative agreement with experimental results is remarkable, especially the opposite effects of columnar and point disorder on the position of the melting line. The 'as grown' experimental results are corresponding to very small amount of point disorder, and thus close to the line of no disorder in the figures. At low temperature, the entanglement transition is associated in our formalism with RSB, and is a sort of a spin-glass transition in the sense that many minima of the random potential and hence free energy, compete with each other. In this paper we worked out the one-step RSB for the case of point disorder. The experiments show that in the case of columnar disorder the transition into the vortex glass seems to be absent. This has to be further clarified theoretically. We have shown that the cage model together with the variational approximation reproduce the main feature of the experiments. Effects of many body interaction between vortex lines which are not taken into account by the effective cage model seem to be of secondary importance. Inclusion of such effects within the variational formalism remains a task for the future.

For point disorder, in the limit of infinite cage ($\mu \to 0$), the variational approximation gives a wandering exponent of 1/2 for a random potential with short ranged correlations [10], whereas simulations give a value of 5/8 [11]. This discrepancy does not seem of importance with respect to the conclusions obtained in this paper. Another point to notice is that columnar disorder is much more effective in shifting the position of the melting line as compared for point disorder in the range of parameters considered here. We have used a much weaker value of correlated disorder to achieve a similar or even larger shift of the melting line than for the case of point disorder. The fact that the random potential does not vary along the z-axis enhances its effect on the vortex lines.

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Figure Captions: Fig1: Transverse fluctuations in the cage model for (a) no disorder (b) columnar disorder (c) point disorder (d) RSB for point disorder. Fig. 2: Melting line for (a) no disorder (b) columnar disorder (c) point disorder (d) entanglement line for point disorder.
