Algebraically closed real geodesics on n-dimensional ellipsoids are dense in the parameter space and related to hyperelliptic tangential coverings

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Abstract

The closedness condition for real geodesics on n–dimensional ellipsoids is in general transcendental in the parameters (semi-axes of the ellipsoid and constants of motion). We show that it is algebraic in the parameters if and only if both the real and the imaginary geodesics are closed and we characterize such double–periodicity condition via real hyperelliptic tangential coverings. We prove the density of algebraically closed geodesics on n–dimensional ellipsoids with respect to the natural topology in the 2n–dimensional real parameter space. In particular, the approximating sequence of algebraic closed geodesics on the approximated ellipsoids may be chosen so to share the same values of the length and of the real period vector as the limiting closed geodesic on the limiting ellipsoid.

Finally, for real doubly–periodic geodesics on triaxial ellipsoids, we show how to evaluate algebraically the period mapping and we present some explicit examples of families of algebraically closed geodesics.

1 Introduction

Integrability of the geodesic motion on a triaxial ellipsoid Q was proven in 1838 by Jacobi [19] who reduced the system to hyperelliptic quadratures; moreover Weierstrass [40] integrated the system in terms of theta–functions on a genus 2 hyperelliptic curve. The geodesic flow has many interesting geometric properties: in particular, each geodesic on Q oscillates between the two lines of intersection of Q with a confocal hyperboloid Q_c (caustic) and by a theorem by Chasles [8] all the tangent lines to the geodesics are also tangent to Q_c. The generic geodesic is quasi–periodic and, in case a geodesic on Q is closed, then all the geodesics on Q tangent to the same confocal hyperboloid are also closed and have the same length.

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The theorem of Chasles generalizes to $n$–dimensional quadrics $Q$ and the set of common tangent lines to $n$ confocal quadrics plays an important role in the study of the geodesics on any of such quadrics and in the reformulation of integrability of the system in the modern language of algebraically integrable systems (see Moser [27, 28], Knörrer [21, 22] and Audin [3]). In particular, in [22] Knörrer settled the so–called Moser–Trubowitz isomorphism between the geodesics on quadrics and the stationary solutions to the Korteweg de Vries equation (KdV).

One of the consequences of Chasles theorem is that, when a geodesic on $Q$ is closed, all the geodesics sharing the same values of the constants of motion are closed and of the same length (see for instance [20]). The condition for a geodesic on an $n$–dimensional quadric $Q$ to be closed is then expressed as a certain linear combination of integrals of holomorphic differentials on a hyperelliptic curve. Such condition is transcendental in the parameters of the problem (semiaxes of the quadric $Q$ and parameters of the caustics) and, by the Moser–Trubowitz isomorphism, it is equivalent to impose that the stationary solutions of the KdV are real periodic in $x$.

Characterization of the set for which the closedness property of real geodesics on $n$–dimensional ellipsoids is algebraic in the parameters A natural question is then: is it possible to settle extra conditions so that the closedness property (2.7) of the geodesic be algebraic in the parameters (semiaxes of the quadric $Q$ and the constants of motions)? In [13, 1], we found a set of sufficient conditions in the complex setting: we introduced and characterized a family of algebraic closed geodesics associated to hyperelliptic tangential covers.

The results in the above papers indicate that the closedness property is algebraic (in the parameters) if the periodicity condition is essentially one–dimensional in the complex setting. In the algebraic-geometric setting, this in turn means that the closedness condition is algebraic (in the parameters) if it is equivalent to the inversion of an elliptic integral.

In the present paper, we complete the characterization of algebraically closed geodesics: we restrict ourselves to the real setting and we settle the necessary and sufficient conditions so that the closedness property be algebraic in the real parameters (semiaxes of the ellipsoid and constants of motion).

In particular, we prove that the closedness condition is algebraic in the parameters if and only if both the real and the imaginary geodesics on the $n$–dimensional ellipsoid are closed. The double periodicity condition we introduce here for the real geodesics on ellipsoids is modelled after a similar condition by Mc Kean and van Moerbeke [26] for the real Hill problem.

Then we explicitly show that the double–periodicity condition is equivalent to the existence of a real hyperelliptic tangential cover [38, 35, 36], thus completing the study started in [13, 1].

The conclusion is then the following: the closedness property is algebraic in the parameters of the problem (square semiaxes of the ellipsoid and constants of the motion) if and only if the double–periodicity condition holds and, in such a case,
the closedness property is equivalently expressed by an elliptic integral associated
to the elliptic curve in the hyperelliptic tangential covering.

We remark that the appearance of hyperelliptic tangential covers is natural,
since their role in the topological classification of elliptic KdV solitons in the complex
moduli space of hyperelliptic curves is well known after Treibich-Verdier[35]-[39] and
the Moser-Trubowitz isomorphism ensures a relation with the geodesic problem.

Since the classification of closed geodesics on real quadrics (and of real KdV
elliptic solitons) are of a certain interdisciplinary interest and the double-periodicity
property of geodesics on ellipsoids is not invariant under general birational trans-
formations, we explicitly describe such coverings for the geodesic problem. In par-
ticular, we investigate the real structure of the elliptic curve of the covering and we
show that the associated lattice is rectangular (i.e. all of the ramification points of
the elliptic curve are real).

We remark also that it is appropriate to call doubly-periodic the geodesics as-
sociated to hyperelliptic tangential covers, since the coordinates and momenta are
doubly–periodic in the length parameter \( s \), that is they are expressed in terms of
eLLiptic functions of \( s \); moreover it is also appropriate to call algebraic the doubl-
periodic geodesics, since the closedness property is algebraic in the parameters (semi-
axes of the ellipsoid and constants of motion).

**The density property** The second set of questions we characterize in the present
paper concerns the density characterization of algebraically closed geodesics. We
show that it is possible to approximate a given real closed geodesics on a given
eLLipsoid with a sequence of real algebraically closed geodesics on perturbed ellip-
soids with perturbed constants of motion. Moreover, such approximate algebraically
closed geodesics may be chosen so to share the same length and/or the same value
of the period vector as the limiting geodesic.

Our estimates are optimal in the sense that we are able to count the number
of parameters which may be kept fixed in the approximation process of real closed
geodesics on a given \( n \)-dimensional ellipsoid via a sequence of doubly–periodic real
closed geodesics on perturbed ellipsoids.

For instance, in the simplest case (geodesics on triaxial ellipsoids), Theorem 4.5
implies that we may keep fixed one parameter: indeed there are four parameters
(the three semi-axes and the caustic parameter), two conditions (length and period
mapping of the real closed geodesic to be approximated algebraically) and one ex-
tra condition (the approximating geodesics satisfy the double–periodicity condition,
i.e. the period mapping of the associated imaginary geodesic has to be rational).
Similarly Theorem 4.6 implies that we may keep fixed two parameters if we allow
the length of approximating algebraic geodesic to vary a little.

The density characterization follows from a theorem by McKean and van Mo-
erbeke [26] which allows the construction of a locally invertible analytic map from
the set of the parameters of the problem (the semi-axes and the caustic parameters)
to the quasi-periods associated to the geodesics on \( n \)-dimensional ellipsoids.
The case of triaxial ellipsoids: the period mapping and the examples

We then specialize to the case $n = 2$ (triaxial ellipsoids), where a more detailed characterization of doubly periodic geodesics is possible since the associated two dimensional (complex) torus is isogenous to the product of two elliptic curves. The first elliptic curve is associated to the hyperelliptic tangential covering, while the properties of the second covering have been discussed by Colombo et al. [9].

In particular, we show that the period mapping of a doubly periodic real geodesic is algebraic in the parameters of the problem and may be computed using the topological character of the second covering.

We also work out the reality condition for geodesics on triaxial ellipsoids associated to degree $d = 3, 4$ hyperelliptic tangential covering and we compute the period mapping using the topological character of the associated second covering.

Finally, we prove the existence of real doubly-periodic geodesics associated to the one-parameter family of degree two coverings with the extra automorphism group $D_8$ [17, 2]. In this case the two elliptic curves of the covering are isomorphic, i.e. they have the same $j$–invariant, and the geodesics are doubly-periodic for a dense set in the parameter space. In view of the above discussion, for such values of the parameter, the given hyperelliptic curve also admits another cover which is hyperelliptic tangential of degree $d > 2$ (see Figure 4 for an explicit example).

The plan of the paper

The plan of the paper is the following: in the next section we summarize some well known facts about geodesics on $n$–dimensional real quadrics. In section 3, we introduce doubly–periodic closed geodesics, hyperelliptic tangential covers and we characterize the algebraic condition of closedness; in section 4 we present the density results; in sections 5 and 6 we specialize to algebraic closed geodesics on triaxial ellipsoids, we characterize algebraically the period mapping and we present the examples.

Since the classification of closed geodesics on real quadrics and of real KdV elliptic solitons are of a certain interdisciplinary interest, we have tried at best to report our results in a way comprehensible also to not experts in the theory of Riemann surfaces.

2 Closed geodesics on ellipsoids

The Jacobi problem of the geodesic motion on an $n$-dimensional ellipsoid

$$Q : \left\{ \frac{X_1^2}{a_1} + \cdots + \frac{X_{n+1}^2}{a_{n+1}} = 1 \right\}$$

is well known to be integrable and to be linearized on a covering of the Jacobian of a genus $n$ hyperelliptic curve (see [27]). Namely, let $l$ be the natural parameter of the geodesic and $\lambda_1, \ldots, \lambda_n$ be the ellipsoidal coordinates on $Q$ defined by the formulas

$$X_i = \sqrt{\frac{(a_i - \lambda_1) \cdots (a_i - \lambda_n)}{\prod_{j \neq i} (a_i - a_j)}}, \quad i = 1, \ldots, n + 1. \quad (2.1)$$
Then, denoting $V_i = \dot{X}_i = dX_i/dl$, $i = 1, \ldots, n+1$ and $\dot{\lambda}_k = d\lambda_k/dl$, $k = 1, \ldots, n$, the corresponding velocities, the total energy $\frac{1}{2}(V_1^2 + \cdots + V_{n+1}^2)$ takes the Stäckel form

$$H = -\frac{1}{8} \sum_{k=1}^{n} \frac{\nu_k^2}{\prod_{i=1}^{n+1} (\lambda_k - a_i)}.$$

According to the Stäckel theorem, the system is Liouville integrable. Upon fixing the constants of motion $H = h_1, c_1, \ldots, c_{n-1}$ and after the re-parametrization

$$dl = \lambda_1 \cdots \lambda_n \frac{ds}{\sqrt{8h_1}},$$

the evolution of the $\lambda_k$ is described by quadratures which involve $n$ independent holomorphic differentials on a genus $n$ hyperelliptic curve whose affine part takes the form

$$\Gamma : \{ \nu^2 = -\lambda \prod_{i=1}^{n+1} (\lambda - a_i) \prod_{k=1}^{n-1} (\lambda - c_k) \} = \{ \nu^2 = -\prod_{i=0}^{2n} (\lambda - b_i) \},$$

where we set the following notation throughout the paper

$$\{ 0, a_1 < \cdots < a_{n+1}, c_1 < \cdots < c_{n-1} \} = \{ b_0 = 0 < b_1 < \cdots < b_{2n} \}.$$

**Remark 2.1** Following [21, 3], the reality condition for geodesics on ellipsoids is equivalent to either $c_i = b_{2i}$ or $c_i = b_{2i+1}$, $i = 1, \ldots, n-1$.

A first consequence is that, given the ellipsoid $Q$ the values of the real constants of motion $c_i$s can’t take arbitrary values. As an example, in the simplest case $n = 2$ (triaxial ellipsoid), given the square semiaxes $0 < a_1 < a_2 < a_3$ the real constant of motion $c$ satisfies either $a_1 < c < a_2$ or $a_2 < c < a_3$.

On the other side, it also implies that given a $(2n)$-tuple $0 < b_1 < \cdots < b_{2n}$ (i.e. given the hyperelliptic curve $\Gamma$), there are a finite number of mechanical configurations associated to it. For instance, again in the simplest case $n = 2$, to the 4-tuple $b_1 < b_2 < b_3 < b_4$ there are associated either the geodesics with constant of motion $c = b_2$ on the ellipsoid with square semiaxes $b_1 < b_3 < b_4$ or the geodesics with constant of motion $c = b_3$ on the ellipsoid with square semiaxes $b_1 < b_2 < b_4$.

**Remark 2.2** Throughout the paper, for any given curve $\Gamma$ with all real branch points as in (2.3), we use the following basis of holomorphic differentials

$$\omega_j = \frac{\lambda^j d\lambda}{w}, \quad j = 1, \ldots, n,$$

and the homological basis $\alpha_i, \beta_i$, $i = 1, \ldots, n$ (see Figure 1), so that the periods $\int_{\alpha_i} \omega_j \in \mathbb{R}$, $i, j = 1, \ldots, n$.

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1 for the necessary definitions and classical properties of hyperelliptic curves we refer to [12, 15]
Then, the quadrature gives rise to the Abel–Jacobi map of the \( n \)-th symmetric product \( \Gamma^{(n)} \) to the Jacobian variety of \( \Gamma \),

\[
\int_{P_0}^{P_n} \omega_j + \cdots + \int_{P_0}^{P_n} \omega_j = \begin{cases} 
  s + \text{const.}, & \text{for } j = 1, \\
  \text{const.}, & \text{for } j = 2, \ldots, n,
\end{cases}
\]

where and \( P_0 \) is a fixed basepoint and \( P_k = (\lambda_k, w_k) \in \Gamma, k = 1, \ldots, n \). Then, the geodesic motion in the new parametrization is linearized on the Jacobian variety of \( \Gamma \). Its complete theta-functional solution was presented in [40] for the case \( n = 2 \), and in [21] for arbitrary dimensions, whereas a topological classification of real geodesics on quadrics was made in [3]. In particular, the constants of motion \( c_1, \ldots, c_{n-1} \) have the following geometrical meaning (see [8, 27]): the corresponding geodesics are tangent to the quadrics \( Q_{c_1}, \ldots, Q_{c_{n-1}} \) of the confocal family

\[
Q_c = \left\{ \frac{X_1^2}{a_1 - c} + \cdots + \frac{X_{n+1}^2}{a_{n+1} - c} = 1 \right\}.
\]

**Closed geodesics and real Hill curves** Let \( \alpha_i, \beta_i, i = 1, \ldots, n \) be the conventional homological basis depicted in Figure 1. Since we are interested in the reality problem, it is not restrictive to take \( b_{2i-1} < \lambda_i < b_{2i}, i = 1, \ldots, n \), in the quadratures (2.6). Then the real geodesic associated to (2.6) is closed if and only if there exist non trivial \( m_i \in \mathbb{Z}, i = 1, \ldots, n \) and a real non vanishing \( T > 0 \) such that

\[
\sum_{i=1}^{n} m_i \oint_{\alpha_i} \omega_1 = T, \quad \sum_{i=1}^{n} m_i \oint_{\alpha_i} \omega_j = 0, \quad j = 2, \ldots, n, \quad (2.7)
\]

where the basis of differentials has been introduced in (2.5). From (2.7), it is self-evident that, if a geodesic on \( Q \) is closed, then all the geodesics sharing the same constants of motion \( c_1, \ldots, c_{n-1} \) are closed. In the following, we call Hill a hyperelliptic curve as in (2.3) for which (2.7) holds.

As it is well known, Hill curves originally arose from the study of isospectral classes connected with the periodic Korteweg–de Vries equation (see [11, 18, 24, 25, 29, 23, 26, 6]). Let \( \mathcal{H}_n^R \) be the real component of the moduli space of the non singular
genus \( n \) hyperelliptic curves with maximal number \( n + 1 \) of connected components, so that all the branch points are real and distinct, \( b_0 = 0 < b_1 < \cdots < b_{2n} \), then (2.7) is equivalent to require that \( \Gamma \) is a real Hill curve up to the Moser–Trubowitz isomorphism, which is associated to the birational transformation \( z = 1/\lambda \) (which exchanges the branch points at 0 and \( \infty \)).

In particular, in [26], it is proven that real Hill curves are dense in the moduli space of curves \( \mathcal{H}_n^R \). A similar statement holds true also for real closed geodesics; however, since the set of equations (2.7) are transcendental in the branch points of \( \Gamma \), they are of little use for the search of parameters corresponding to closed geodesics. For real geodesics on ellipsoids, the above discussion may be summarized in the following classical result:

**Proposition 2.3** For any fixed choice of the square semiaxes \( a_1 < \cdots < a_{n+1} \) and for any \( n \)-tuple \( \zeta_i, \ i = 1, \ldots, n \) such that \( a_1 < \zeta_1 < a_2 < \cdots < a_n < \zeta_n < a_{n+1} \), there is a dense set \( I \subset [\zeta_1, \zeta_2] \times \cdots \times [\zeta_{n-1}, \zeta_n] \) (in the natural topology of \( \mathbb{R}^{n-1} \)), such that \( \forall (c_1, \ldots, c_{n-1}) \in I \), there exist nontrivial integers \( m_i, \ i = 1, \ldots, n \) and a real \( T > 0 \) such that (2.7) holds.

The above statement takes into account of the reality condition settled in Remark 2.1 and exhausts all possibilities accordingly for real closed geodesics on \( n \)-dimensional ellipsoids \( Q \).

### 3 Doubly–periodic closed geodesics, hyperelliptic tangential covers and algebraic condition of periodicity in the real parameter space

In the following we consider real geodesics in the regular case when all square semiaxes and constants of motion take distinct values. The periodicity condition (2.7) is transcendental in the parameters of the problem (the square semiaxes \( a_1, \ldots, a_n \) of the ellipsoid and constants of motion \( c_1, \ldots, c_{n-1} \)). So a natural question is: is it possible to settle extra conditions so that the periodicity condition (2.7) becomes algebraic in the parameters?

In [13] we introduced and characterized a family of algebraic closed geodesics associated to hyperelliptic tangential covers in the complex setting. Fedorov [13] proved that such geodesics are a connected component of the intersection of the quadric \( Q \) with an algebraic surface in \( \mathbb{R}^n \). For triaxial ellipsoids this surface is an elliptic or rational curve and the explicit description of the algebraic surface in terms of elliptic \( \mathcal{P} \)-Weierstrass functions in special cases of such coverings was given in [13]. In [1], we computed the explicit expression of the coordinates \( X_i(s) \) in terms of one-dimensional theta-functions and applied such results also to describe periodic orbits of an integrable billiard.

In this section, we complete the characterization of algebraically closed geodesics, we restrict ourselves to the real setting and we settle the necessary and sufficient conditions so that the closedness property be algebraic in the real parameters (semiaxes of the ellipsoid and constants of motion).
The conclusion is the following one: the periodicity condition (2.7) is algebraic in the parameters of the problem if and only if it is equivalent to the inversion of a single integral; by Jacobi inversion problem the latter integral has to be elliptic. The form of the periodicity condition implies that the elliptic curve is the one associated to the hyperelliptic curve via the hyperelliptic tangential covering. Finally, under our hypotheses, $\mathcal{E}$ has a real structure and we prove that the associated lattice is rectangular (i.e. all of the finite branch points of $\mathcal{E}$ are real).

Indeed, we introduce and characterize a double periodicity condition for the geodesics on ellipsoids which is modelled after a similar condition for the real Hill problem by McKean and van Moerbeke [26]. Then we explicitly show that this condition is equivalent to the existence of a hyperelliptic tangential cover (explicitly described in Definition 3.5).

The theorems 3.6 [13] and 3.7 imply that the periodicity condition (2.7) is algebraic in the parameters of the problem if and only if the real closed geodesics are doubly-periodic.

We remark that it is appropriate to call such geodesics doubly-periodic, since the coordinates and momenta, $X_i(s), V_i(s), i = 1, \ldots, n + 1$, are doubly-periodic in $s$, that is they are expressed in terms of elliptic functions of $s$; moreover it is also appropriate to call algebraic the doubly-periodic geodesics, since the closedness property is algebraic in the parameters (semiaxes of the ellipsoid and constants of motion).

The appearance of hyperelliptic tangential covers is natural, since their role in the topological classification of elliptic KdV solitons in the complex moduli space of hyperelliptic curves is well known after Treibich-Verdier [35]-[39] and the Moser-Trubowitz isomorphism ensures a relation with the geodesic problem.

Since the double-periodicity property of geodesics on ellipsoids is not invariant under general birational transformations (see Lemma 3.3), we explicitly describe such coverings for the geodesic problem and we characterize their real structure.

The plan of the section is the following: we first introduce the double-periodicity condition and characterize it via a dual curve; then we explicitly construct the hyperelliptic tangential cover associated to the double-periodicity condition and give the necessary and sufficient conditions so that the closedness condition is algebraic in the parameters.

**Definition 3.1 (The double periodicity condition)** A hyperelliptic curve $\Gamma$ with real branch points as in (2.7) is associated to doubly-periodic closed geodesics if and only if the real periodicity condition holds, that is there exists a non trivial real cycle $\alpha = \sum_{i=1}^{n} m_i \alpha_i$, such that
\[
\oint_{\alpha} \omega_1 = 2T, \quad \oint_{\alpha} \omega_j = 0, \quad j = 2, \ldots, n, \tag{3.1}
\]
and there exists a non trivial imaginary cycle $\beta = \sum_{i=1}^{n} m_i' \beta_i$, such that
\[
\oint_{\beta} \omega_1 = 2\sqrt{-1}T', \quad \oint_{\beta} \omega_j = 0, \quad j = 2, \ldots, n, \tag{3.2}
\]
for some non-zero real $T, T'$. 

The conditions (3.1) and (3.2) mean that both the real and the imaginary geodesics on the ellipsoid $Q$ are closed.

**The dual curve** To any given real curve $\Gamma$ like in (2.3) McKean and van Moerbeke [26] associate a dual real curve $\Gamma'$ with reflected branch points so that the real KdV elliptic soliton is doubly-periodic in $x$ if and only if both $\Gamma$ and $\Gamma'$ are real Hill curves (for the Hill operator). From the algebraic-geometric point of view, the birational transformation which sends branch points of $\Gamma$ into those of $\Gamma'$ is uniquely defined by the requirement that it exchanges real and imaginary periods and it transforms holomorphic differentials vanishing at the infinity ramification point of $\Gamma$ to holomorphic differentials vanishing at the infinity ramification point of $\Gamma'$. From the analytical point of view, the real solutions to the Hill problem associated to the dual curve $\Gamma'$ correspond precisely to the imaginary solutions for the corresponding problem on $\Gamma$.

We remark the nontriviality of this construction: birationally equivalent curves are identified in the moduli space of hyperelliptic curves; however, the topological characterization of the real solutions to the Hill problem for KdV is not invariant under general birational transformations. The same remark holds in the case of closed geodesics.

Below we apply the same idea to the case of the geodesic problem on ellipsoids: the analogous construction maps the imaginary geodesics on the ellipsoid $Q$ with constants of motion $c_1, \ldots, c_{n-1}$, to the real geodesics on a dual ellipsoid $Q'$ with constants of motion $c'_1, \ldots, c'_{n-1}$. So the real geodesics on $Q$ are closed and doubly periodic, if and only if both the real geodesics on $Q$ and $Q'$ are closed for the given constants of motion.

To identify the dual curve $\Gamma'$ (i.e. the dual ellipsoid $Q'$ and the dual constants of motion $c'_1, \ldots, c'_{n-1}$), we recall that the Moser-Trubowitz isomorphism exchanges the infinity ramification point of the hyperelliptic curve associated to the classical Hill problem, with the $(0, 0)$ finite ramification point of the hyperelliptic curve associated to the geodesic problem.

Moreover, either the real (3.1) or the imaginary (3.2) periodicity conditions for the geodesic problem are equivalent to require that, given a hyperelliptic curve as in (2.3), there exists a nontrivial cycle $\gamma$ such that $\int_\gamma \omega = 0$, for all holomorphic differentials $\omega$ vanishing at the branch point $(0,0)$.

**Lemma 3.2** Let $\Gamma : \mu^2 = -\prod_{k=0}^{2n} (\lambda - b_k)$ be as in (2.3), let $P_0 = (0,0)$ and let $\omega_k = \lambda^{k-1}/\mu$, $k = 1, \ldots, n$, be the basis of holomorphic differentials introduced in (2.5). Then the holomorphic differential $\omega$ vanishes at the branch point $P_0$ if and only if it is a linear combination of the holomorphic differentials $\omega_2, \ldots, \omega_n$.

**Sketch of the proof:** Let $\tau$ be the local coordinate in a neighborhood of $P_0 = (0,0)$ such that $\tau(P_0) = 0$, then $\omega_k \approx A\tau^{2k-2}d\tau$, $k = 1, \ldots, n$, where $A = \ldots$. 

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Finally, to construct the dual curve $\Gamma'$ for the geodesic problem, we must identify the birational transformations which preserve the form of (2.3) and transform holomorphic differentials vanishing at the ramification point $(0,0) \in \Gamma$ to holomorphic differentials vanishing at the ramification point $(0,0) \in \Gamma'$.

**Lemma 3.3** The class of birational transformations between $\Gamma : \{\mu^2 = -\prod_{k=0}^{2n} (\lambda - b_k)\}$ and $\Gamma' : \{\nu^2 = -\prod_{k=0}^{2n} (\rho - b'_k)\}$ which transform holomorphic differentials vanishing at the ramification point $(0,0) \in \Gamma$ to holomorphic differentials vanishing at the ramification point $(0,0) \in \Gamma'$ has the following two generators: $\rho = \kappa \lambda$ and $\rho = b_1 \lambda / (\lambda - b_1)$.

The first transformation is a homogeneous rescaling of all of the parameters of the problem (square semiaxes and constants of the motion) and it preserves the real periodicity condition. The latter transformation is the analog of the one introduced by McKean and van Moerbeke [26] for the periodic KdV problem and it exchanges the real and the imaginary cycles.

Finally, the statement below gives a simple characterization of doubly–periodic geodesics and is the analog of a theorem in [26] for the Hill problem.

**Theorem 3.4** Let $\Gamma = \{\mu^2 = -\lambda \prod_{k=1}^{2n} (\lambda - b_k) \equiv -\lambda \prod_{i=1}^{n+1} (\lambda - a_i) \prod_{k=1}^{n-1} (\lambda - c_k)\}$ be a real hyperelliptic curve as in (2.3) and $\Gamma' : \{\nu^2 = -\prod_{i=0}^{2n} (\rho - b'_i) \equiv -\lambda \prod_{i=1}^{n+1} (\lambda - a'_i) \prod_{k=1}^{n-1} (\lambda - c'_k)\}$ be the real hyperelliptic curve whose branch points $b'_i$ are related to the $b_j$s by the birational transformation $\rho = \frac{b_1 \lambda}{\lambda - b_1}$.

Let $Q = \{X_1^2/a_1 + \cdots + X_{n+1}^2/a_{n+1} = 1\}$ and $Q' = \{X_1^2/a'_1 + \cdots + X_{n+1}^2/a'_{n+1} = 1\}$.

Then the real geodesics associated on $Q$ with constants of motion $c_1, \ldots, c_{n-1}$ are doubly–periodic if and only if the real geodesics respectively associated to $Q$ (with constants of motion $c_1, \ldots, c_{n-1}$) and to $Q'$ (with constants of motion $c'_1, \ldots, c'_{n-1}$) are closed.

In view of Remark [2.1], to any given $(2n)$-tuple $b'_1, \ldots, b'_{2n}$, there are associated a finite number of dual ellipsoids and dual constants of motion. Clearly the Theorem [3.4] implies that the real geodesics on $Q'$ be closed, for any admissible dual ellipsoid $Q'$ and constants of motion $c'_1, \ldots, c'_{n-1}$ associated to $b'_1, \ldots, b'_{2n}$.

In the last section, we apply Theorem [3.4] both to compute the period mapping associated to families of coverings and to compute the parameters of Example [6.3].
Hyperelliptic tangential covers and the algebraic periodicity condition for closed geodesics

In this paragraph, we prove that the double-periodicity condition is necessary and sufficient for the algebraic characterization of the closedness property of real geodesics on $n$-dimensional ellipsoids. The statement follows from the fact that the double periodicity condition settled by equations (3.1) and (3.2) is equivalent to the existence of a real rectangular hyperelliptic tangential cover defined in Definition 3.5.

Hyperelliptic tangential covers \cite{35}-\cite{38} have originally appeared in connection with the topological classification of the $x$ doubly-periodic solutions of the Korteweg-de Vries (KdV) equation $u_t = 6uu_x - u_{xxx}$. Due to the Moser–Trubowitz isomorphism, we get a natural relation between the classification of real doubly-periodic geodesics and the relevant class of periodic potentials associated to the Hill operator $-\partial_x^2 + u(x,t)$, depending on the parameter $t$ (due to the impossibility of citing all relevant contributions in this field we limit to cite \cite{11, 10, 18, 24, 25, 4}).

We recall that a solution to the KdV equation of the form $u(x,t) = 2\sum_{j=1}^{N} P(x - q_j(t)) + c$ is called a KdV-elliptic soliton. $u(x,t)$ is a KdV-elliptic soliton if and only if $\sum_{1\leq j \leq N, j \neq k} P'(q_j(t) - q_k(t)) = 0$, $k = 1, \ldots, N$ \cite{4}. Any KdV-elliptic soliton is uniquely associated with a marked hyperelliptic curve $(X, P)$ of positive genus $g$ equipped with a projection $\pi : X \mapsto \mathcal{E}$ the so called hyperelliptic tangential cover such that $P$ is a smooth Weierstrass point of $X$ and the canonical images of $(X, P)$ and $(\mathcal{E}, Q)$ in the Jacobian of $X$ are tangent at the origin \cite{36}.

The problem of classifying all hyperelliptic tangential covers in the complex moduli space of genus $g$ hyperelliptic curves and to characterize the associated KdV-elliptic solitons has been successfully considered in a series of papers by Treibich and Verdier \cite{35}-\cite{39}. We refer to \cite{38} for an account of the vast literature on the subject.

In particular, a different approach to the classification problem of KdV-elliptic solitons has been developed by Krichever \cite{23} based on the theory of one point Baker–Akhiezer functions, while Gesztesy and Weikard \cite{14} give an analytic characterization of elliptic finite-gap potentials. Finally, explicit examples of families of such coverings have been worked out by many authors (see in particular \cite{32, 35, 33}).

In \cite{1, 13}, hyperelliptic tangential covers were first considered in connection to doubly-periodic closed geodesics on $n$-dimensional (complex) quadrics and explicit examples were worked out. In particular, a theorem by Fedorov \cite{13} implies if the curve $(\Gamma, P_0)$ is a (complex) hyperelliptic tangential cover, then the geodesics on the associated quadric are (complex) doubly periodic.

Here we restrict ourselves to real hyperelliptic curves $\Gamma$ with all finite branch points real. For such curves we call the hyperelliptic tangential covering real (resp. real rectangular, real rhombic), if the elliptic curve $\mathcal{E}$ has a real structure (resp. with rectangular, rhombic period lattice).

The theorem by Fedorov may be easily rephrased so to hold in the case of real tangential coverings. Moreover, here we prove the reverse statement: if the double-periodicity condition (3.1) and (3.2) hold, then the associated algebraic curve is a
real hyperelliptic tangential cover. Finally, in the latter case we show that it is always possible to associate to the hyperelliptic curve for which the double periodicity condition holds, a real rectangular hyperelliptic tangential covering.

The conclusion is then that the double–periodicity condition is necessary and sufficient for the algebraic characterization of the closedness property of real geodesics on $n$–dimensional ellipsoids.

**Definition 3.5 Real rectangular hyperelliptic tangential coverings** Let $\Gamma : \{ \mu^2 = -\prod_{k=0}^{2n} (\lambda - b_k) \}$ be as in (2.3), let $P_0 = (0,0)$ and let $\omega_k = \lambda^{k-1}/\mu d\lambda$, $k = 1, \ldots , n$, be the basis of holomorphic differentials introduced in (2.5). Let $A = 2 \left( \sqrt{-\prod_{j=1}^{2n} b_j} \right)^{-1}$ be as in the proof of Lemma 3.2.

The curve $\Gamma$ admits a canonical embedding into its Jacobian variety $\text{Jac}(\Gamma)$ by the map $P \mapsto A(P) = \int_{P_0}^P (\omega_1, \ldots , \omega_n)^T$, so that $P_0$ is mapped into the origin of the Jacobian and $U = \frac{d}{d\tau} A(P) \big|_{P = P_0} = (A, 0, \ldots , 0)$, is the tangent vector of $\Gamma \subset \text{Jac}(\Gamma)$ at the origin.

Assume that $\Gamma$ is an $N$–fold covering of an elliptic curve $E$, which we represent in the canonical Weierstrass form

$$E = \{(P'(u))^2 = 4P^3(u) - g_2P(u) - g_3 \equiv 4(P(u) - e_1)(P(u) - e_2)(P(u) - e_3)\}.$$ 

Assume that under the covering map $\pi : \Gamma \mapsto E$, $P_0$ is mapped to $Q_0$ the infinite point of $E$ and choose $u$ as local coordinate.

The covering from the marked curve $(\Gamma, P_0)$ to $(E, Q_0)$ is hyperelliptically tangential if $E$ admits the following canonical embedding to $\text{Jac}(\Gamma)$, $u \mapsto uU$, so that the embedding of $\Gamma$ and $E$ are tangent at the origin.

We call $(\Gamma, P_0)$ a real hyperelliptic tangential covering if the above holds and the elliptic curve $E$ has a real structure (i.e. the period lattice associated to $E$ is either rectangular or rhombic).

We call the real hyperelliptic tangential covering $(\Gamma, P_0)$ rectangular if moreover all the finite branch points of $E$ are real (so the lattice associated to $E$ is rectangular). Otherwise, we call the real hyperelliptic tangential covering rhombic.

For the geodesic problem, the existence of a real hyperelliptic tangential covering implies the double-periodicity condition by the following theorem.

**Theorem 3.6** If $(\Gamma, P_0)$ is a real hyperelliptic tangential cover, then the associated geodesics are closed and doubly–periodic.

The above theorem was originally proven by Fedorov [13] in the complex setting; indeed if $(\Gamma, P_0)$ is a hyperelliptic tangential cover, then the complex geodesics on the quadric $Q$ satisfy a double–periodicity condition. His argument may be easily modified so to hold in the real setting. We remark that we get the double-periodicity condition (3.1)–(3.2) either if the real hyperelliptic tangential covering is rectangular or rhombic.
The above theorem settles a sufficient condition for the algebraicity of the closedness property of real geodesics on ellipsoids. Next theorem implies that such condition is also necessary; so that we get the complete characterization of algebraically closed geodesics via the double–periodicity condition.

We now prove the converse to Theorem 3.6.

**Theorem 3.7** Let \( \Gamma = \{ \mu^2 = -\lambda \prod_{i=1}^{n+1} (\lambda - a_i) \prod_{k=1}^{n-1} (\lambda - c_k) \equiv -\lambda \prod_{k=1}^{2n} (\lambda - b_k) \} \) be the hyperelliptic curve associated to the geodesics on the ellipsoid \( Q = \{ X_1^2/a_1 + \cdots + X_{n+1}^2/a_{n+1} = 1 \} \) with constants of motion \( c_1, \ldots, c_{n-1} \). Let \( \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \) be the conventional canonical homological basis depicted in Figure 1 and let \( \omega_j, j = 1, \ldots, n \) be the basis of holomorphic differentials introduced in (2.5).

If the doubly–periodicity conditions (3.1) and (3.2) hold, then \( (\Gamma, P_0) \) is a real rectangular hyperelliptic tangential cover.

**Proof of Theorem 3.7** The doubly–periodicity conditions (3.1) and (3.2) hold if and only if there exist two cycles \( \alpha = \sum_{i=1}^{n} m_i \alpha_i \) and \( \beta = \sum_{i=1}^{n} m'_i \beta_i \), such that

\[
\oint_\alpha \omega_j = \begin{cases} T, & j = 1, \\ 0, & j = 2, \ldots, n, \end{cases} \quad \oint_\beta \omega_j = \begin{cases} \sqrt{-1}T', & j = 1, \\ 0, & j = 2, \ldots, n. \end{cases} \tag{3.3}
\]

The above equations imply that \( \omega_2, \ldots, \omega_n \) are the \( (n-1) \) independent holomorphic differentials vanishing at \( P_0 = (0,0) \) and possess a maximal system of \( (2n-2) \) independent periods. Then by Poincaré reducibility theorem [31], there exist an elliptic curve \( \mathcal{E} \) and a \( (n-1) \)-dimensional Abelian subvariety \( \mathcal{A}_{n-1} \) such that \( \text{Jac}(\Gamma) \) is isogenous to the direct product \( \mathcal{E} \times \mathcal{A}_{n-1} \). Since \( P_0 = (0,0) \) is among the Weierstrass points of \( \Gamma \), the covering \( \pi : \Gamma \rightarrow \mathcal{E} \) is tangent at the Weierstrass point \( P_0 \) [39] [36].

Since all of the Weierstrass points of the curve \( \Gamma \) are real (see 2.3) and since the double periodicity condition (3.3) ensures the rational dependence between the real periods (associated to the \( \alpha \) cycle) and the rational dependence between the imaginary periods (associated to \( \beta \)), we easily conclude that the hyperelliptic tangential covering has a real structure.

We now explicitly construct such covering in order to investigate the real structure associated to \( \mathcal{E} \). The tangency condition and the (3.3) ensure the existence of two real numbers \( A, B \), of a holomorphic differential \( \Omega_1 = \omega_1 + \sum_{j=2}^{n} c_j \omega_j \), and of constants \( k_1, \ldots, k_n, h_1, \ldots, h_n \in \mathbb{Z} \), such that

\[
\oint_\alpha \Omega_1 = 2k_j A, \quad \oint_\beta \Omega_1 = 2h_j \sqrt{-1}B, \quad j = 1, \ldots, n.
\]

Since \( \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \) form a homological basis, any other period of \( \Omega_1 \) is an integer combination of \( 2A \) and \( 2\sqrt{-1}B \). In particular,

\[
T = \oint_\alpha \Omega_1 = A \sum_{j=1}^{n} k_j m_j, \quad \sqrt{-1}T' = \oint_\beta \Omega_1 = \sqrt{-1}B \sum_{j=1}^{n} h_j m'_j.
\]
We now investigate the real structure of the covering. Let us fix \( P_0 = (0,0) \in \Gamma \) as basepoint, let \( z = \int_{P_0}^P \Omega_1, P \in \Gamma \). Then \( z \in \mathcal{T} = \mathbb{C}/\Lambda \), the one-dimensional torus with period lattice \( \Lambda \) generated by \( 2A, 2\sqrt{-1}B \).

Finally let \( \mathcal{P}(z) \equiv \mathcal{P}(z|A, \sqrt{-1}B) \) be the Weierstrass \( \mathcal{P} \)-function with half-periods \( A, \sqrt{-1}B \) and \( \mathcal{E} : \{(\mathcal{P}'(z))^2 = 4 \prod_{k=1}^{3} (\mathcal{P}(z) - e_k)\} \) the elliptic curve in Weierstrass normal form with finite branch points \( e_1 = \mathcal{P}(A), e_2 = \mathcal{P}(A + \sqrt{-1}B) \) and \( e_3 = \mathcal{P}(\sqrt{-1}B) \).

Then, the covering \( \pi : \Gamma \mapsto \mathcal{E} \) is real rectangular and tangential at \( P_0 = (0,0) \) by construction. \( \square \)

We remark that there is a certain freedom in the construction of the curve \( \mathcal{E} \) and of the covering, due to the isogeneity between \( \text{Jac}(\Gamma) \) and \( \mathcal{E} \times A_{n-1} \). For instance, if we introduce the complex conjugate numbers \( C_{\pm} = A \pm \sqrt{-1}B \), we may associate to \( (\Gamma, P_0) \) a real rhombic hyperelliptic tangential covering.

Theorem 3.7 means that the double-periodicity condition is algebraic in the parameters of the problem (the square semiaxes \( a_1, \ldots, a_{n+1} \) and the constants of motion \( c_1, \ldots, c_{n-1} \)), since it may be equivalently expressed in terms of elliptic integrals associated to the covering.

Theorem 3.6 means that for the special class of geodesics on ellipsoids associated to a real hyperelliptic tangential covering \( (\Gamma, P_0) \), the periodicity condition \( (3.1) \)

\[
\sum_{i=1}^{n} m_i \int_{\alpha_i} \omega_1 = 2T, \quad \sum_{i=1}^{n} m_i \int_{\alpha_i} \omega_j = 0, \quad j = 2, \ldots, n,
\]

is algebraic in the parameters of the problem, since the covering \( \pi \) imposes algebraic relations among the branch points of \( \mathcal{E} \) and the ramifications points of \( \Gamma \) (square semiaxes \( a_1, \ldots, a_{n+1} \) and constants of motion \( c_1, \ldots, c_{n-1} \)), and the real (resp. imaginary) periodicity condition is expressible as a real (resp. imaginary) elliptic integral on \( \mathcal{E} \).

We thus get the following

**Corollary 3.8** Let \( \Gamma = \{ \mu^2 = -\lambda \prod_{i=1}^{n+1} (\lambda - a_i) \prod_{k=1}^{n-1} (\lambda - c_k) \equiv -\lambda \prod_{k=1}^{2n} (\lambda - b_k) \} \) be the hyperelliptic curve associated to the geodesics on the ellipsoid \( Q = \{ X_1^2/a_1 + \cdots + X_{n+1}^2/a_{n+1} = 1 \} \) with constants of motion \( c_1, \ldots, c_{n-1} \). Let \( \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \) be the conventional canonical homological basis depicted in Figure 1 and let \( \omega_j = \lambda^{j-1}d\lambda/\mu, j = 1, \ldots, n \) be the basis of holomorphic differentials introduced in \( (2.7) \). Let \( P_0 = (0,0) \in \Gamma \).

Then the closedness property \( (2.7) \)

\[
\sum_{i=1}^{n} m_i \int_{\alpha_i} \omega_1 = 2T, \quad \sum_{i=1}^{n} m_i \int_{\alpha_i} \omega_j = 0, \quad j = 2, \ldots, n,
\]

is algebraic in the parameters of the problem \( a_1, \ldots, a_{n+1}, c_1, \ldots, c_{n-1} \) (square semiaxes and constants of motion), if and only if there exists a non trivial imaginary
cycle $\beta = \sum_{i=1}^{n} m'_i \beta_i$, such that

$$\oint_{\beta} \omega_1 = 2\sqrt{-1}T', \quad \oint_{\beta} \omega_j = 0, \quad j = 2, \ldots, n.$$ 

In the latter case, $(\Gamma, P_0)$ is a real rectangular hyperelliptic tangential cover.

The Corollary is perfectly consistent with the Treibich–Verdier characterization of elliptic solitons of the Korteweg–de Vries equations (we refer in particular to [39] for a discussion of the dimension of the real moduli space associated to either the periodic or double–periodic stationary solution to the KdV equation).

On the other side the Corollary implies that the periodicity condition for the geodesic problem will stay transcendental for any over type of covering: for instance the periodicity condition will stay transcendental, if $\Gamma$ as in (2.3) is a hyperelliptic tangential cover with marked point $P_j = (b_j, 0)$, for some $j = 1, \ldots, 2n$ or if $\Gamma$ is a degree $d = 2$ covering (the degree of a hyperelliptic tangential cover is at least 3).

In particular, in the last section we prove the existence of doubly–periodic closed geodesics related to degree 2 coverings with extra automorphisms and we give an explicit example (see Figure 4): in view of Corollary 3.8 in such case the curve admits also a hyperelliptic tangential cover, and then an infinite number of coverings by a classical theorem by Picard [30].

4 Density of doubly–periodic closed geodesics

In this section, we prove that the algebraic condition of real closed geodesics settled in the previous section, is fulfilled on a dense set of parameters (the square semi-axes $a_1, \ldots, a_{n+1}$ and the constants of motion $c_1, \ldots, c_{n-1}$) with respect to the natural topology over the reals. So it is possible to characterize algebraically dense sets of real closed geodesics on ellipsoids and to approximate real closed geodesics on given ellipsoid by sequences of algebraically closed (i.e. doubly-periodic) geodesics on perturbed ellipsoids with perturbed constants of motion.

We remark that, such approximate algebraically closed geodesics may be chosen so to share the same length and/or the same value of the period vector as the limiting geodesic.

Our estimates are optimal in the sense that we are able to count the number of parameters which may be kept fixed in this approximation scheme. For instance, in the simplest case (geodesics on triaxial ellipsoids), Theorem 4.5 implies that we may keep fixed one parameter: indeed we have four parameters (the three semi-axes and the caustic parameter), two conditions originating from the limiting closed geodesics (length $T$ and period mapping $m_1/m_2$) and one extra condition (the approximating geodesics have rational value of the imaginary period mapping $m'_1/m'_2$ which approximates the irrational quasi–period of the limiting imaginary geodesic). Similarly Theorem 4.6 implies that we may keep fixed two parameters (since we also perturb the length of the approximating algebraic geodesics).
The proofs of the density results rely on a theorem by McKean and van Moerbeke for the Hill problem \[26\]. Using their idea, we define a quasi-period vector \((x, y) \equiv (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n}\) associated to any real and imaginary geodesics. Using the Riemann bilinear relations, such quasi-period vector may be explicitly computed using the periods of two meromorphic differentials. The theorem by \[26\] (originally stated for the Hill problem), ensures that the map from the parameter space \((a_1, \ldots, a_{n+1}, c_1, \ldots, c_{n-1})\) to the quasi periods \((x, y)\) is analytic and locally invertible.

**Density of algebraically closed geodesics** For an easier comparison with the density characterization of KdV-elliptic solitons, we also report the following characterization of hyperelliptic tangential covers in the complex moduli space of hyperelliptic curves due to Colombo et al. \[9\]. Their theorem implies immediately that real closed geodesics may be approximated by complex doubly-periodic geodesics.

**Theorem 4.1** \[9\] Hyperelliptic tangential covers of genus \(n\) are dense in the complex moduli space \(H_n\) of the hyperelliptic curves of genus \(n\).

To prove the density statement (Theorem 4.3) for real doubly periodic geodesics on \(n\)-dimensional ellipsoids with respect to the real parameter space, we apply the ideas used by McKean and VanMoerbeke in \[26\] for the Hill problem. We report their theorem below in a version suitable for the geodesics problem and then show that any real closed geodesics on a given ellipsoid may be approximated by real doubly-periodic geodesics on perturbed ellipsoids.

**Theorem 4.2** \[26\] Let \(\Gamma = \{\mu^2 = -\lambda \prod_{i=1}^{n+1} (\lambda - a_i) \prod_{k=1}^{n-1} (\lambda - c_k) \equiv -\lambda \prod_{k=1}^{2n} (\lambda - b_k)\}\) be as in (2.3). Let \((x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n}\) be defined by

\[
\begin{align*}
\sum_{i=1}^{n} x_i \oint_{\alpha_i} \omega_j &= \begin{cases} 1, & \text{for } i = 1, \\ 0, & \text{for } i = 2, \ldots, n, \end{cases} \\
\sum_{i=1}^{n} y_i \oint_{\beta_i} \omega_j &= \begin{cases} \sqrt{-1}, & \text{for } i = 1, \\ 0, & \text{for } i = 2, \ldots, n. \end{cases}
\end{align*}
\tag{4.1}
\]

Then, (4.1) define a real analytic locally invertible map from open sets in the parameter space \((b_1, \ldots, b_{2n})\) to open sets in the quasi-period space \((x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n)\). In particular, a small perturbation of the real branch points of \(\Gamma\) will make the point \((x, y)\) rational.

If we compare (4.1) with the double-periodicity condition settled in (3.1) and (3.2),

\[
\begin{align*}
\sum_{i=1}^{n} m_i \oint_{\alpha_i} \omega_j &= \begin{cases} T, & \text{for } i = 1, \\ 0, & \text{for } i = 2, \ldots, n, \end{cases} \\
\sum_{i=1}^{n} m'_i \oint_{\beta_i} \omega_j &= \begin{cases} \sqrt{-1}T', & \text{for } i = 1, \\ 0, & \text{for } i = 2, \ldots, n. \end{cases}
\end{align*}
\]

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we easily conclude that if the point \((x, y)\) is rational, then the double periodicity condition is satisfied. Then the following density property of algebraically closed geodesics holds.

**Theorem 4.3** Given a real closed geodesic on the ellipsoid \(Q = \{X_1^2/a_1 + \cdots + X_{n+1}^2/a_{n+1} = 1\}\) with caustic parameters \(c_j, j = 1, \ldots, n - 1\), for any \(\epsilon > 0\) sufficiently small, there exist \(a_1, \ldots, a_{n+1}, c_1^\epsilon, \ldots, c_{n-1}^\epsilon \in \mathbb{R}\) such that

\[
\sum_{j=1}^{n-1} (c_j - c_j^\epsilon)^2 + \sum_{i=1}^{n+1} (a_i - a_i^\epsilon)^2 < \epsilon
\]

and the geodesics on \(Q' = \{X_1^2/a_1^\epsilon + \cdots + X_{n+1}^2/a_{n+1}^\epsilon = 1\}\) with caustic parameters \(c_j^\epsilon, j = 1, \ldots, n - 1\), are real doubly periodic.

**Proof:** Let \(\Gamma\) be the real Hill curve associated to the closed geodesics on the ellipsoid \(Q\) with caustic parameters \(c_1, \ldots, c_{n-1}\) so that the set of equations (3.1) hold. Let \(\epsilon_0 = \frac{1}{2} \min \{b_j - b_{j-1}, j = 1, \ldots, 2n\}\) where \(\{b_1 < \cdots < b_{2n}\} = \{a_1 < \cdots < a_{n+1}, c_1 < \cdots, c_{n-1}\}\).

\(\Gamma',\) the dual curve to \(\Gamma\) introduced in Theorem 4.2, is associated to a dual ellipsoid \(Q'\) which possesses real quasi–periodic closed geodesics, so that the vector \(y = (y_1, y_2) \in \mathbb{R}^2\).

Similarly to [26], we introduce the differential of the second kind \(\Omega_2^{(0)}\) with vanishing \(\beta_j\) periods, with a double pole at \(P_0 = (0, 0)\) and the following normalization. Let \(\tau\) be the local coordinate in a neighborhood of \(P_0 = (0, 0)\) such that \(\tau(P_0) = 0\), then \(\Omega_2^{(0)} \approx (2\pi A)^{-1} \tau^{-2} d\tau\), \(k = 1, \ldots, n\), where \(A = 2 \left(\sqrt{-\prod_{j=1}^{2n} b_j}\right)^{-1}\) is the constant defined in Definition 3.5. Let

\[
y_j = \oint_{\alpha_j} \Omega_2^{(0)}, \quad j = 1, \ldots, n,
\]

be the \(\alpha\)-period vector of \(\Omega_2^{(0)}\). Then applying Riemann bi–linear identities to \(\omega_l, l = 2, \ldots, n\) and to \(\tilde{u} = \oint_{P_0} \Omega_2^{(0)}\), we immediately conclude that

\[
\sum_{j=1}^{n} y_j \oint_{\beta_j} \omega_1 = \sqrt{-1}, \quad \sum_{j=1}^{n} y_j \oint_{\beta_j} \omega_l = 0, \quad l = 2, \ldots, n.
\]

Finally, applying Theorem 4.2 we may perturb the curve \(\Gamma\) so that on \(\Gamma'\) (with \(\epsilon < \epsilon_0\))

\[
\sum_{j=1}^{n} x_j^{(e)} \oint_{\alpha_j^{(e)}} \omega_1^{(e)} = 1, \quad \sum_{j=1}^{n} y_j^{(e)} \oint_{\beta_j^{(e)}} \omega_1^{(e)} = \sqrt{-1},
\]

\[
\sum_{j=1}^{n} x_j^{(e)} \oint_{\alpha_j^{(e)}} \omega_l^{(e)} = 0, \quad \sum_{j=1}^{n} y_j^{(e)} \oint_{\beta_j^{(e)}} \omega_l^{(e)} = 0, \quad l = 2, \ldots, n,
\]

for rational vector \((x^{(e)}, y^{(e)})\). According to Theorem 3.7 \((\Gamma^{(e)}, P_0)\) is a hyperelliptic tangential cover. \(\square\)
Remark 4.4 It is easy to verify that the vectors \((x, y)\) in (4.1) correspond to a hyperelliptic tangential cover if and only if \(x = (x_1, \ldots, x_n)\) has rationally dependent components and the same holds for \(y = (y_1, \ldots, y_n)\) (that is the requirement that \((x, y)\) be rational may be weakened, without loosing the algebraicity of the closedness condition of the associated geodesics).

In view of the above remark, it is possible to optimize the density characterization of doubly periodic closed geodesics. Indeed it is possible to modify the proof of the above theorem so that the ellipsoids \(Q, Q'\) share the same value of the greatest square semiaxis \(a_{n+1} = a'_{n+1}\), and the perturbed real doubly-periodic closed geodesics on \(Q'\) have the same length and the same period vector as the initial real closed geodesics on \(Q\), i.e. \((x_1, \ldots, x_n) = (x'_1, \ldots, x'_n)\).

Theorem 4.5 Let \(\Gamma = \{\mu^2 = -\lambda(\lambda - c) \prod_{i=1}^{n+1} (\lambda - a_i) \prod_{j=1}^{n-1} (\lambda - c_j)\}\), be a real Hill curve so that the real geodesics on the ellipsoid \(Q = \{X_1^2/a_1 + \cdots + X_n^2/a_n + X_{n+1}^2/a_{n+1} = 1\}\) with caustic parameters \(c_1, \ldots, c_{n-1}\) are closed and have length \(T\).

Then, there exists a sequence \(\{a^{(k)}_1, \ldots, a^{(k)}_n, c^{(k)}_1, \ldots, c^{(k)}_{n-1}\} \in \mathbb{R}^{2n-1}\) such that

\[
\lim_{k \to +\infty} c^{(k)}_j = c_j, \quad (j = 1, \ldots, n-1), \quad \lim_{k \to +\infty} a^{(k)}_i = a_i, \quad (i = 1, \ldots, n),
\]

and the geodesics on \(Q^{(k)} = \{X_1^2/a^{(k)}_1 + \cdots + X_n^2/a^{(k)}_n + X_{n+1}^2/a_{n+1} = 1\}\) with caustic parameters \(c^{(k)} = (c^{(k)}_1, \ldots, c^{(k)}_{n-1})\) are doubly–periodic, with same length \(T\) and with the same value of the period vector as the closed geodesics on \((Q, c_1, \ldots, c_{n-1})\).

Proof: The proof follows from a straightforward adaptation of the argument in Theorem 4.2 since the jacobian determinant of the real analytic map there defined is not vanishing, also its restriction to a generic \(2n - 1\)-dimensional subvariety will not vanish locally. To fix ideas, we choose the subvariety \(b_{2n} = a_{n+1} = \text{const.}\).

Let \(\Gamma\) be real Hill, let \(\omega_1, \ldots, \omega_n\) be the holomorphic basis of differentials defined in (2.5) and \(\alpha_i, \beta_i, \ i = 1, \ldots, n\) the homological basis as in Remark 2.2.

Let \(\Omega^{(0)}\) be the normalized meromorphic differential of the second kind with double pole at \(P_0 = (0, 0)\), vanishing \(\beta\) periods, as in the proof of Theorem 4.3 and let \((y_1, \ldots, y_n)\) be its \(\alpha\) period vector.

Let \(\epsilon_0 = \frac{1}{2} \min\{b_j - b_{j-1}, \ j = 1, \ldots, 2n\}\), where, as usual \(\{b_1 < \cdots < b_{2n}\} = \{a_1, \ldots, a_{n+1}, c_1, \ldots, c_{n-1}\}\).

Then the geodesics on \(Q\) with caustic parameters \(c_1, \ldots, c_{n-1}\) are real closed and satisfy the periodicity condition

\[
f_1(b_1, \ldots, b_{2n}) \equiv \sum_{i=1}^{n} m_i \int_{\alpha_i} \omega_1 = 0, \quad f_j(b_1, \ldots, b_{2n}) \equiv \sum_{i=1}^{n} m_i \int_{\alpha_i} \omega_j = 0, \quad j = 2, \ldots, n.
\]
Let \( b_{2n}, m_1, \ldots, m_n, T \) be fixed. As a consequence of Theorem 4.2, the \( n \) equations 
\[ f_j = 0, \quad j = 1, \ldots, n \]
are locally analytically invertible near the point \((b_1, \ldots, b_{2n-1})\) and there exist \( n \) analytic functions \( \hat{b}_r = \hat{b}_r(b_1, \ldots, b_{n-1}) \), \( r = n, \ldots, 2n-1 \), on the 
\((n-1)\)-dimensional ball \( B_0 \) centered at \((b_1, \ldots, b_{n-1})\) and of radius \( \epsilon < \epsilon_0 \).

On the initial curve \( \Gamma \),
\[
g_j \equiv y_j / y_n = \int_{b_{2j-1}}^{b_{2j}} \Omega_2^{(0)} / \int_{b_{2n-1}}^{b_{2n}} \Omega_2^{(0)}, \quad j = 1, \ldots, n-1
\]
take some real value \( \tau_j, \quad j = 1, \ldots, n-1 \) and are real analytic in \( \hat{b}_1, \ldots, \hat{b}_{n-1} \) on the 
ball \( B_0 \), again by Theorem 4.2.

Then, there exists a sequence \( (b_1^{(k)}, \ldots, b_{n-1}^{(k)}) \in B_0 \) converging to \((b_1, \ldots b_{n-1})\) such that
\[
\lim_{k \to +\infty} b_r^{(k)} = \lim_{k \to +\infty} \hat{b}_r(b_1^{(k)}, \ldots, b_{n-1}^{(k)}) = b_r, \quad r = n, \ldots, 2n-1;
\]
\[
g_j(b_1^{(k)}, \ldots, b_{n-1}^{(k)}) \in Q, \quad j = 1, \ldots, n-1; \tag{4.3}
\]
\[
\lim_{k \to +\infty} g_j(b_1^{(k)}, \ldots, b_{n-1}^{(k)}) = \tau_j, \quad j = 1, \ldots, n-1.
\]

Finally, for any \( k \), by construction, the corresponding hyperelliptic curve \( \Gamma^{(k)} = \{ \mu^2 = -\lambda \prod_{j=1}^{2n} (\lambda - b_j^{(k)}) \} \) is a hyperelliptic tangential cover with marked point
\( P_0 = (0,0) \) and the associated geodesics have the same length and the same period vector as the initial one associated to \( \Gamma \). Indeed equations (4.2) ensure that on \( \Gamma^{(k)} \) the period vector and the length of the real geodesics be preserved; by (4.3), the imaginary period vector \( (y_1^{(k)}, \ldots, y_n^{(k)}) \) has rationally dependent components for all \( k \) which, approximate the rationally independent components of the imaginary quasi-period of the limiting imaginary geodesics, so that, by construction, the limiting real closed geodesics are those associated to \( \Gamma \).

Finally, if we just require to preserve the period vector of the geodesics and allow that the length of the approximating geodesics vary, i.e. if we just require
\[
(x_2/x_1, \ldots, x_n/x_1) = (x_2^e/x_1^e, \ldots, x_n^e/x_1^e),
\]
we may keep fixed two square semiaxes, for instance the smallest and the greatest one, \( a_1 = a_1^e \) and \( a_{n+1} = a_{n+1}^e \) and we get
the following statement.

**Theorem 4.6** Let \( \Gamma = \{ \mu^2 = -\lambda(\lambda - c) \prod_{i=1}^{n+1} (\lambda - a_i) \prod_{j=1}^{n-1} (\lambda - c_j) \} \), be a real Hill curve
so that the real geodesics on the ellipsoid \( Q = \{ X_1^2/a_1 + \cdots + X_{n+1}^2/a_{n+1} = 1 \} \) with
caucus parameters \( c_1, \ldots, c_{n-1} \) are closed and have length \( T \).

Then, there exists a sequence \( \{ a_2^{(k)}, \ldots, a_n^{(k)}, c_1^{(k)}, \ldots, c_{n-1}^{(k)} \} \in \mathbb{R}^{2n-2} \) such that
\[
\lim_{k \to +\infty} c_j^{(k)} = c_j, \quad (j = 1, \ldots, n-1), \quad \lim_{k \to +\infty} a_i^{(k)} = a_i, \quad (i = 2, \ldots, n),
\]
and the geodesics on $Q^{(k)} = \{ X_1^2/a_1 + X_2^2/a_2^{(k)} + \cdots + X_n^2/a_n^{(k)} + X_{n+1}^2/a_{n+1}^{(k)} = 1 \}$ with caustic parameters $c^{(k)} = (c_1^{(k)}, \ldots, c_{n-1}^{(k)})$ are doubly-periodic, with same value of the period vector as the closed geodesics on $(Q, c_1, \ldots, c_n)$.

The proof is a straightforward modification of the one for Theorem 4.5 and we omit it.

Remark In [1], we used the algebraic characterization of closed geodesics associated to hyperelliptic tangential covers to construct periodic billiard trajectories of an integrable billiard on a quadric $Q$ with elastic impacts on a confocal quadric $Q_d$. The results we have presented in this section may be applied to this billiard model and imply the algebraic characterization of a dense set of its periodic orbits.

5 The algebraic computation of the period mapping in the case $n = 2$

In the special case of triaxial ellipsoids ($n = 2$), a stronger characterization of doubly-periodic closed geodesics holds. In particular, we show below that the period mapping of a doubly periodic closed geodesic, which measures the ratio between oscillation and winding for a geodesics, is algebraic in the parameters of the problem and that it may be explicitly computed using the second covering associated to the hyperelliptic curve. Indeed, the 2-dimensional $\text{Jac}(\Gamma)$ is isogenous to the product of two elliptic curves $E_1 \times E_2$.

The second covering plays a relevant role also in the case of elliptic solitons. Airault et al. [4] discovered a remarkable link between the pole dynamics of the KdV elliptic solutions with the initial data in the form of the Lamé potential and the dynamics of Calogero–Moser particle system [7]. In the genus 2 case, the topological characterization of the covering ramified at $P_0$ reduces the problem of describing the pole dynamics to the search of solutions of certain algebraic equations related to the covering and to the inversion of elliptic integrals [5, 33].

Below we first recall the definition of the period mapping and some classical results. Then we show how to compute the period mapping explicitly using the topological character of the associated second covering. Unfortunately there do no exist general theorems which characterize topologically such families of coverings. As an application, we compute the value of the period mapping for some special classes of coverings in the next section.

Closed geodesics on triaxial ellipsoids and the period mapping In the case $n = 2$ (geodesics on triaxial ellipsoids), Proposition 2.3 implies that for any fixed choice of the semiaxes $0 < a_1 < a_2 < a_3$ there is a dense set $I \subseteq [a_1, a_3] \setminus \{a_2\}$ such that for all $c \in I$ the hyperelliptic curve $\Gamma : \{ \mu^2 = -\lambda(\lambda - c) \prod_{i=1}^{3} (\lambda - a_i) \}$ is Hill.
The application
\[
c \mapsto \varphi(c) = \begin{cases} 
2 \oint_{\alpha_2} \omega_2 : 2 \oint_{\alpha_1} \omega_2, & a_1 < c < a_2 < a_3, \\
2 \oint_{\alpha_1} \omega_2 : 2 \oint_{\alpha_2} \omega_2, & a_1 < a_2 < c < a_3,
\end{cases}
\]
measures the ratio between oscillation and winding for a geodesic with parameter \(c\) and it is called the period mapping (see [20]). Comparing the above definition with (3.1) and Proposition 2.3, it is evident that the geodesic with parameter \(c\) is closed if and only if \(\varphi(c)\) is rational. A closed geodesic is called simple if it has no self-intersections. To be simple closed, only a single winding is allowed; hence \(\varphi(c)\) must be an integer greater than one. The following theorems explain under which condition there do exist topologically non–trivial simple closed geodesics.

**Theorem 5.1** [20] Let \(a_1 < a_2 < a_3\) be fixed and \(c \in [a_1, a_3] \setminus \{a_2\}\). Then \(\varphi(c)\) is a monotone decreasing function of \(c\). If \(c \in [a_1, a_2]\), then \(\varphi(c) > 1\) and \(\lim_{c \to a_2} \varphi(c) = 1\).

Moreover, let \(t = a_1/a_3\) be fixed and \(\sigma = a_2/a_3 \in ]t, 1[\). Then, \(\varphi(a_1)\) is a monotone increasing function of \(\sigma\) with upper limit \(\sqrt{a_3/a_1}\) and lower limit 1.

**Theorem 5.2** [20] On an ellipsoid \(\{ \sum_{i=1}^{3} X_i^2/a_i = 1 \}\), there exist non standard simple closed geodesics (i.e. simple closed geodesics different from the three principal ellipses), if and only if \(\varphi(a_1) > 2\).

More precisely, for each integer value \(\varphi(c) \in ]1, \varphi(a_1)[\), the projection of the flow lines yields closed geodesics which wind once around the \(X_1\)–axis while performing \(\varphi(c)\) many oscillations. Their length is greater than the length of the middle ellipse in the \((X_1, X_3)\)–plane.

**The second covering** In the case \(n = 2\), \(\text{Jac}(\Gamma)\) is isogenous to the product of two elliptic curves \(E_1 \times E_2\) and the second covering is ramified at \(P_0 = (0, 0)\) according to the following proposition by Colombo et al.

**Proposition 5.3** [9] Let \(\Gamma\) be a genus 2 curve which covers an elliptic curve \(\pi_1 : \Gamma \mapsto \mathcal{E}_1\) and let \(\pi_2 : \Gamma \mapsto \mathcal{E}_2\) be another covering so that \(\text{Jac}(\Gamma) \approx E_1 \times E_2\). Then \(\pi_i\) is tangential exactly at the points where \(\pi_j\) is ramified \(i \neq j\).

We briefly turn back to the double-periodicity condition in the special case of geodesics on triaxial ellipsoids so to construct directly the second covering associated to the double–periodicity condition.

**Proposition 5.4** Let \(\Gamma = \{ \mu^2 = -\lambda(\lambda - c) \prod_{i=1}^{3} (\lambda - a_i) \equiv -\lambda \prod_{k=1}^{4} (\lambda - b_k) \}\) be the genus 2 hyperelliptic curve associated to the geodesics on the triaxial ellipsoid \(Q = 21\).
\( \{ X_1^2/a_1 + X_2^2/a_2 + X_3^2/a_3 = 1 \} \) with caustic parameter \( c \). Let \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) be the conventional canonical homological basis depicted in Figure 1 and let \( \omega_1 = d\lambda/\mu \), \( \omega_2 = \lambda d\lambda/\mu \), be the basis of holomorphic differentials introduced in (2.5).

Suppose that on \( \Gamma \) as above, the double periodicity condition (3.1) and (3.2) holds

\[
\int_{\alpha_1} \omega_1 + m_2 \int_{\alpha_2} \omega_1 = 2T, \quad \int_{\beta_1} \omega_1 + m_2 \int_{\beta_2} \omega_1 = 2\sqrt{-1}T',
\]

for some non-zero real \( T, T' \). Then, there exists a covering

\[
\pi_2 : \Gamma \mapsto \mathcal{E}_2,
\]

ramified of order 3 at \( P_0 = (0, 0) \) and such that

\[
\pi_2^*(\Omega_2) = \kappa \omega_2,
\]

where \( \Omega_2 \) is the normalized holomorphic differential on \( \mathcal{E}_2 \) and \( \kappa \) is a numerical constant.

Proof: The double–periodicity conditions (3.1) and (3.2) imply the existence of two cycles \( \alpha = m_1 \alpha_1 + m_2 \alpha_2 \) and \( \beta = m'_1 \beta_1 + m'_2 \beta_2 \), such that

\[
\int_{\alpha} \omega_2 = 0, \quad \int_{\beta} \omega_2 = 0. \tag{5.2}
\]

In (5.2) it is not restrictive to suppose that \((m_1, m_2)\) (respectively \((m'_1, m'_2)\)), be relative prime integer numbers.

Inspection of (5.2) implies that all of the periods of \( \omega_2 \) are integer multiples of two periods \( S, \sqrt{-1}S' \) of \( \omega_2 \) and this is sufficient to prove the existence of a covering \( \pi_2 : \Gamma \mapsto \mathcal{E}_2 \).

Indeed, let \((m_1, m_2)\) (respectively \((m'_1, m'_2)\)), be relative prime integer numbers and let \( 2S = \int_{\alpha_1} \omega_2/|m_2| \) (resp. \( 2\sqrt{-1}S' = \int_{\beta_1} \omega_2/|m'_2| \)). By Bezout identity, there exist integers \( p_j, p'_j \), \( j = 1, 2 \) such that \( p_1 m_1 - p_2 m_2 = 1 \) (resp. \( p'_1 m'_1 - p'_2 m'_2 = 1 \) so that \( 2S, 2\sqrt{-1}S' \) are indeed periods of \( \omega_2 \) and any other period \( \int_{\alpha} \omega_2 \) is an integer multiple of \( 2S, 2\sqrt{-1}S' \).

Let now fix \( P_0 = (0, 0) \in \Gamma \) as basepoint, let \( z = \int_{P_0} \omega_2, P \in \Gamma \). Then, by Poincaré reducibility theorem, \( z \in \mathcal{T} = \mathbb{C}/\Lambda \), the one–dimensional torus with period lattice \( \Lambda \) generated by \( 2S, 2\sqrt{-1}S' \).

Finally let \( \mathcal{P}(z) \equiv \mathcal{P}(z; S, \sqrt{-1}S') \) be the Weierstrass \( \mathcal{P} \)-function with half-periods \( S, \sqrt{-1}S \) and \( \mathcal{E}_2 : \{ (\mathcal{P}'(z))^2 = 4 \prod_{k=1}^{3} (\mathcal{P}(z) - E_k) \} \) the elliptic curve in Weierstrass normal form with finite branch points \( E_1 = \mathcal{P}(S), E_2 = \mathcal{P}(S + \sqrt{-1}S') \) and \( E_3 = \mathcal{P}(\sqrt{-1}S') \).
Then, the covering $\pi_2 : \Gamma \mapsto E_2$ has degree $d$ and, introducing local coordinates at $P_0 \in \Gamma$, it is straightforward to verify that it is ramified of order three at $P_0 = (0, 0)$ (the latter remark implies $d \geq 3$).

Of course, by Theorem 3.7 we already know that there exists a covering $\pi_1 : \Gamma \mapsto E_1$ which is hyperelliptic tangential at $P_0$. The second covering constructed above is ramified exactly at $P_0$ in agreement with Proposition 5.3.

The second covering and the period mapping  We now show that the topological type of the second covering (which is ramified at $P_0$ of order 3) is naturally linked to the topological classification of the associated real closed geodesics (period mapping).

**Definition 5.5 (topological characteristic of the second covering)** The topological characteristic of a covering is a sequence of four integer numbers $(\nu_0, \nu_1, \nu_2, \nu_3)$ which count the number of Weierstrass points of $\Gamma$ in the preimage of the four branch point of $E_2$, with the exception of $P_0 = (0, 0) \in \Gamma$, the Weierstrass point at which the second covering is ramified, and with the usual convention that $\nu_0$ is associated to the branch point of $E_2$ at infinity.

For a given $\Gamma = \{\mu^2 = -\lambda \prod_{j=1}^4 (\lambda - b_j)\}$, the computation of the period mapping amounts to identify the two integer numbers $m_1, m_2$ such that

$$m_1 \int_{\alpha_1} \omega_2 + m_2 \int_{\alpha_2} \omega_2 = 0.$$ 

Let $\pi_2 : \Gamma \mapsto E_2$ be the second covering, where $E_2 = \{W^2 = 4Z^3 - G_2 Z - G_3 \equiv 4 \prod_{i=1}^3 (Z - E_i)\}$ is represented in the canonical Weierstrass form. In our setting the curves $\Gamma$ and $E$ are real with maximal number of real connected components, so that it makes sense to call $\alpha$ the real cycle associated to $E$.

From the proof of Proposition 5.4, we know that the pull-back of the holomorphic differential on $E_2$ is $dZ/W = \omega_2$. So we may conclude that

$$\int_{\alpha_i} \omega_2 = \kappa_i \int_{\alpha} dZ/W, \quad i = 1, 2,$$  

where the integer numbers $\kappa_1, \kappa_2$ satisfy $m_1 \kappa_1 + m_2 \kappa_2 = 0$.

Finally, $\kappa_1, \kappa_2$ are uniquely associated to the topological characteristic of the covering $\pi_2$. To compute them it is sufficient to compute the preimages of the four branch points $E_0, E_1, E_2, E_3$ of $E_2$. Since in our setting the covering is real, $\pi_2^{-1}(E_i)$ are either the branch points of $\Gamma$ or real points on the curve $\Gamma$ or come in complex conjugate pairs.

Then it is self–evident that, whenever we know the topological characteristic of second covering, we may compute $\kappa_1$ and $\kappa_2$. Unfortunately, we do not possess such complete piece of information in the general case. Anyway, for any degree $d$, there
exist a finite number of families of hyperelliptic tangential coverings so that only a finite number of topological characteristic are possible and, consequently, only a finite number of values of the period mapping may be realized. In the next section we discuss the case in which the degree of the covering is either 3 or 4. When the degree of the covering is 5, there exist two families of hyperelliptic tangential coverings (see [33]) and there exist real doubly–periodic geodesics associated to such coverings either simple or with 1,2,3 or 4 self–intersections. Since the complexity of the computations increases with the degree of the covering, we shall report the degree $d = 5$ case in detail in a subsequent publication.

6 Examples and applications

Explicit examples of hyperelliptic tangential covers when the genus of the hyperelliptic curve is $n \leq 8$ have been worked out (see for instance [38] and references therein).

In this section, we impose the reality conditions for algebraic closed geodesics on triaxial ellipsoids for the families of degree $d = 3, 4$ hyperelliptic tangential covers and we determine the possible values of the period mapping using the topological character of the second covering. For a comparison with the case of elliptic KdV solitons, we refer to Smirnov [32, 33] or to Belokolos and Enol’ski [5].

Finally in the last subsection, we prove the existence of doubly–periodic closed geodesics related to degree 2 coverings with extra automorphisms and we give an explicit example (see Figure 4): in view of Corollary 3.8 in such case the curve admits also a hyperelliptic tangential cover, and then an infinite number of coverings by a classical theorem by Picard [30]. The same family of coverings has also been considered by Taimanov [34] in relation to elliptic KdV solitons.

Remark 6.1 In all examples, we adopt the following convention: $0 < a_1 < a_2 < a_3$ are the semiaxes of the triaxial ellipsoid $Q = \{X_1^2/a_1 + X_2^2/a_2 + X_3^2/a_3 = 1\}$ and $c$ is the parameter of the confocal quadric to which the geodesic is tangent, so that the finite branch points of the associated hyperelliptic curve $\Gamma$ are \{b_0 = 0 < b_1 < b_2 < b_3 < b_4 = \{0, c, a_i, i = 1, \ldots, 3\}.

For an easier comparison of our results with $d$-elliptic KdV solitons, we first impose that the hyperelliptic tangential $d : 1$ covering $(\mathcal{G}, P_\infty) \mapsto (\mathcal{E}, Q)$ be associated to real KdV-solitons, where $\mathcal{G} : \{w^2 = -\prod_{k=1}^{5}(z - z_k)\}$ and $P_\infty$ is the branch point of $\mathcal{G}$ at infinity. Then, by Moser–Trubowitz isomorphism, the curves $\mathcal{G}$ and $\Gamma$ are birationally equivalent and the following relation among the finite branch points $z_k$s of $\mathcal{G}$ and the finite branch points $b_j$ of $\Gamma$ holds:

$$\{z_1, z_2, z_3, z_4, z_5\} = \{\beta, \beta + \frac{1}{b_j}, j = 1, \ldots, 4\}, \quad \text{where} \quad \beta = \min\{z_k, k = 1, \ldots, 5\}. \quad (6.1)$$
6.1 Hyperelliptic tangential covers of degree 3.

Description of the hyperelliptic tangential covering and reality problem for doubly–periodic closed geodesics

The tangential 3:1 covering \( G \rightarrow E \) is associated to 3-elliptic KdV solutions and dates back to the works of Hermite and Halphen (16). For the closed geodesics problem, we require that the genus 2 curve \( \Gamma \) be birationally equivalent to \( G = \{ w^2 = -\frac{1}{4}(4z^3 - 9g_2z - 27g_3)(z^2 - 3g_2) \} \) which covers the elliptic curve \( E_1 = \{ W^2 = 4Z^3 - g_2Z - g_3 \} \), where the covering is given by the relations

\[
Z = \frac{1}{9} z^3 - \frac{27g_3}{z^2 - 3g_2}, \quad W = \frac{2}{27} \frac{w(z^3 - 9g_2z + 54g_3)}{(z^2 - 3g_2)^2}.
\]

The holomorphic differential on \( E_1 \) is the pull–back of the holomorphic differential \( \frac{dZ}{W} = -\frac{3}{2} z \frac{dz}{w} \) on \( G \).

The \( z_j \)s are related to the branch points of \( E_1 \), \( e_j \), \( j = 1, \ldots, 3 \), by

\[
\beta \equiv z_1 = -\sqrt{3g_2} < z_2 = 3e_1 < z_3 = 3e_2 < z_4 = 3e_3 < z_5 = \sqrt{3g_2},
\]

where \( e_1 < e_2 < e_3 \).

**Proposition 6.2** Let \( 0 < a_1 < a_2 < a_3 \) and \( c \in [a_1, a_3 \setminus \{a_2\}] \) be given. Then the geodesic flow on the ellipsoid \( Q \) tangent to the confocal quadric \( Q_c \) is doubly periodic and related, up to birational transformation, to the 3:1 covering \( \mathcal{G} \rightarrow \mathcal{E}_1 \) if and only if

\[
\frac{1}{c^2} + \frac{1}{a_2^2} + \frac{1}{a_3^2} - 2\left( \frac{1}{ca_2} + \frac{1}{ca_3} + \frac{1}{a_2a_3} \right) = 0, \quad a_1 = \frac{3ca_2a_3}{2(a_2a_3 + c(a_2 + a_3))}. \quad (6.2)
\]

If (6.2) holds, then the branch points on \( \mathcal{E}_1 \) are \( \{e_1, e_2, e_3\} = \left\{ \frac{2a_1 - c}{6a_1c}, \frac{2a_1 - a_i}{6a_1a_i}, \quad i = 2, 3 \right\} \),

\[
\beta = -\frac{1}{2a_1} \text{ and } g_2 = -\frac{1}{12a_1^2}.
\]

(6.2) may be inverted and we get \( a_1, a_3 \) parametrically in function of \( a_2, c \) or \( a_2, c \) in function of \( a_1, a_3 \).

**Corollary 6.3** Let \( a_2, c > 0 \) be given and \( a_2 \neq c \). Then (6.2) is equivalent to

\[
a_3 = \left( \frac{1}{\sqrt{a_2}} - \frac{1}{\sqrt{c}} \right)^{-2}, \quad a_1 = \frac{3a_2c}{4(a_2 + c - \sqrt{ca_2})}.
\]

Let \( 0 < a_1 < a_3 \) be given. Then (6.2) is equivalent to

\[
\frac{1}{a_2}, \quad \frac{1}{c} = \pm \frac{1}{2\sqrt{a_3}} + \sqrt{\frac{4}{3a_1} - \frac{3}{4a_3}}.
\]
The second covering and the period mapping  Let $\Gamma = \{ \mu^2 = -\lambda \prod_{k=1}^{4} (\lambda-b_k) \}$, then the second 3:1 covering $\pi_2 : \Gamma \to \mathcal{E}_2$ has topological characteristic $(0,3,1,1)$ (see [5, 33]).

In this case the explicit expression of the covering $\Pi_2 : \mathcal{G} \mapsto \mathcal{E}_2$ is known [33] and it is given by the maps

$$Z = -\frac{1}{4}(4z^3 - 9g_2z - 9g_3), \quad W = -\frac{1}{2}w \left( 4z^2 - 3g_2 \right)$$

and the moduli of $\mathcal{E}_2$ are $G_2 = \frac{27}{4}(g_3^2 + 9g_3^3)$, $G_3 = -\frac{243}{8}g_3(3g_3^2 - g_2^3)$. The finite branch points of $\mathcal{E}_2$ are $E_1 = -9/2g_2$, $E_2 = 9/4g_3 + 3/4g_2\sqrt{3g_2}$ and $E_3 = -E_1 - E_2$ and satisfy $E_2 < E_1 < E_3$.

Using the birational transformation $z = 1/\lambda - \sqrt{3g_2}$, we find the explicit expression of the covering $\pi_2 : \Gamma \mapsto \mathcal{E}_2$. It is ramified of order 3 at $b_0 = 0$ (and mapped to infinity by $\pi_2$), that is $\pi_2^{-1}(E_\infty) = \{ P_0, P_0, P_0 \}$ and

$$\pi_2^{-1}(E_1) = \{ b_2, b_3, b_4 \}, \quad \pi_2^{-1}(E_2) = \{ b_1, P_\pm \}, \quad \pi_2^{-1}(E_3) = \{ b_\infty, Q_\pm \},$$

where $b_\infty$ denotes the infinite ramification point of $\Gamma$, $b_j$s are the finite ramification points of $\Gamma$ (with a slight abuse of notation, we use the same symbol for the point on the curve and its $\lambda$ coordinate), $P_\pm$ and $Q_\pm$ are the real points on $\Gamma$ such that $\lambda(P_\pm) = 2/\sqrt{3g_2}$ and $\lambda(Q_\pm) = 2/\sqrt{27g_2}$. Finally, it is easy to check that

$$\lambda(P_\pm) \in ]b_3, b_4[, \quad \lambda(Q_\pm) \in ]b_2, b_3[, \quad \lambda(P_\pm) \in ]b_3, b_4[, \quad \lambda(Q_\pm) \in ]b_2, b_3[,$$

so that

$$\oint_{\alpha_1} \omega_2 = \oint_{b_1} \omega_2 = \frac{2}{4} \oint_{E_1} dZ/W = \frac{2}{4} \oint_{E_2} dZ/W = \frac{1}{2},$$

and finally, comparing the definition of period mapping (5.1) with (5.3), we conclude that the period mapping is either $2 : 1$ or $1 : 2$.

Also the dual curve $\Gamma'$ defined in Theorem 3.4 is a hyperelliptic tangential cover of degree $d = 3$ and the branch points of $\Gamma'$ still satisfy Proposition 6.2 so that the algebraic real closed geodesics associated to the dual curve have period mapping $2 : 1$ or $1 : 2$.

We have thus proven the following

**Lemma 6.4** The closed geodesics associated to a real curve $\Gamma$ which is a 3:1 hyperelliptical tangential cover, either have period mapping $2 : 1$ or $1 : 2$. 

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6.2 Hyperelliptic tangential covers of degree 4.

Description of the hyperelliptic tangential covering and reality problem for doubly–periodic closed geodesics In this case, we require $\Gamma$ to be birationally equivalent to $G = \left\{ w^2 = -\prod_{i=1}^{5}(z - z_i) \right\}$, where

$$
\begin{align*}
  z_1 &= 6e_j, \\
  z_{2,3} &= -e_k - 2e_j \pm 2\sqrt{(7e_j + 2e_k)(e_j - e_k)}, \\
  z_{4,5} &= -e_l - 2e_j \pm 2\sqrt{(7e_j + 2e_l)(e_j - e_l)}.
\end{align*}
$$

(6.3)

$G$ covers the elliptic curve $E_1$ with moduli $g_2, g_3$, $E_1 = \{ W^2 = 4Z^3 - g_2Z - g_3 = 4\prod_{s=1}^{3}(Z - e_s) \} \subset (Z, W)$, and the covering is given by the relations

$$
Z = e_j + \frac{(z^2 - 3ze_j - 7e_j^2 - 27e_1^2e_3^2)^2}{4(z - 6e_j)(2z - 15e_j)^2}, \quad \frac{dZ}{W} = \frac{(2z - 3e_j) dz}{w}.
$$

The explicit expression of this covering has been found by Belokolos and Enolski [5] (see also [33]). The reality condition for $z_i, i = 1, \ldots, 5$ is $H_j^2 = \prod_{k\neq j}^{3}(e_j - e_k) > 0$, that is $e_j$ is either $e_1$ or $e_3$ in (6.3). If $e_j = e_1$ in (6.3), then $\beta = 6e_1$; if $e_j = e_3$ in (6.3), then $\beta = -2e_2 - 2\sqrt{(7e_2 + 2e_3)(e_3 - e_2)}$. In both cases, we give necessary and sufficient conditions using the following notation

$$
f_1 = 1/b_1, \quad f_2 = 1/b_2, \quad f_3 = 1/b_3, \quad f_4 = 1/b_4.
$$

![Figure 2.](image_url)

**Proposition 6.5** Let $0 < f_4 < f_3 < f_2 < f_1$ as defined above. Then the geodesic flow on the ellipsoid $Q$ tangent to the confocal quadric $Q_c$ is doubly periodic and
related, up to birational transformation, to the 4:1 covering $G \to E_1$ introduced above, if and only if either condition (A) or (B) below is fulfilled:

\[
(A) \begin{cases} 
81(f_2^2 + f_3^2) + 62f_1^2 f_3^2 + 36(f_1 + f_4)^2 - 108(f_2 + f_3)(f_1 + f_4) = 0, \\
81(f_1^2 + f_4^2) + 62f_1^2 f_4^2 + 36(f_2 + f_3)^2 - 108(f_2 + f_3)(f_1 + f_4) = 0.
\end{cases}
\]

\[
(B) \begin{cases} 
100f_4 f_3 + 8f_1 (f_4 + f_3 + 8f_1 - 9f_2) - (9f_4 + 9f_3 - 6f_2)^2 = 0, \\
36(f_4 + 6f_3 - 9f_2)^2 + 8f_1 (9f_4 + 9f_3 - 8f_1 + f_2) = 0.
\end{cases}
\]

If (A) holds, then $\beta = 6e_1$ and the branch points of $E_1$ are

\[e_1 = \frac{1}{30} (f_1 + f_2 + f_3 + f_4), \quad e_2 = \frac{4}{15} (f_1 + f_4) - \frac{7}{30} (f_2 + f_3), \quad e_3 = -e_2 - e_1.\]

If (B) holds, then the branch points of $E_1$ are

\[e_3 = \frac{2}{15} f_1 - \frac{1}{30} (f_2 + f_3 + f_4), \quad e_2 = \frac{f_1}{15} + \frac{4f_2}{15} - \frac{7}{30} (f_3 + f_4), \quad e_1 = -e_2 - e_3.\]

and $\beta = -\frac{1}{5} (f_1 + f_2 + f_3 + f_4)$.

Conditions (A) and (B) may be expressed in the following equivalent way

**Proposition 6.6** (A') Let $\sigma_{\pm} > 0$ and such that $2/3\sigma_+ < \sigma_- < \sigma_+$. Let

\[\gamma_+ = \frac{9}{25} \left( \frac{3}{2}\sigma_+ - \sigma_- \right)^2, \quad \gamma_- = \frac{9}{25} \left( \frac{3}{2}\sigma_- - \sigma_+ \right)^2.\]
Then \(0 < f_4 < f_3 < f_2 < f_1\) satisfy (A) if and only if \(f_2, f_3\) (resp. \(f_1, f_4\)) are the roots of
\[
x^2 - \sigma_+ x + \gamma_+ = 0, \quad \text{resp. } x^2 - \sigma_- x + \gamma_- = 0.
\]
In this case, the moduli of the elliptic curve are
\[
g_2 = \frac{19}{75}(\sigma_+ + \sigma_-)^2 - \sigma_+ \sigma_-,
\quad g_3 = \frac{28}{3375}(\sigma_+ + \sigma_-)^3 - \frac{\sigma_+ \sigma_-}{30}(\sigma_+ + \sigma_-).
\]

(B') Let \(-\sigma_+ < \sigma_- < 0\) and define
\[
\gamma_+ = \frac{1}{5}\sqrt{16\sigma_+^2 - 18\sigma_- \sigma_+ - 9\sigma_-^2},
\quad \gamma_- = \frac{1}{5}\sqrt{16\sigma_+^2 + 2\sigma_+ \sigma_- - 14\sigma_-^2}.
\]

Then \(0 < f_4 < f_3 < f_2 < f_1\) satisfy (B) if and only if
\[
f_1 = \sigma_+ + \gamma_+,
\quad f_2 = 2\gamma_+,
\quad f_3 = \frac{\sigma_-}{2} + \gamma_+ + \gamma_-,
\quad f_4 = \frac{\sigma_-}{2} + \gamma_+ - \gamma_-.
\]
If the above equation is satisfied, the moduli of the elliptic curve are
\[
g_2 = \frac{19}{75}\sigma_-^2 - \frac{2}{75}\sigma_- \sigma_+ + \frac{4}{75}\sigma_+^2,
\quad g_3 = \frac{28}{3375}\sigma_+^3 + \frac{8}{3375}\sigma_-^3 - \frac{37}{1125}\sigma_- \sigma_+ - \frac{2}{1125}\sigma_- \sigma_+^2.
\]

**Proposition 6.7** Let \(\Gamma = \{\mu^2 = -\lambda \prod_{k=1}^{4}(\lambda - b_k)\}\) be a real 4:1 hyperelliptic tangential cover verifying Proposition 6.5 (A). Then the branch points of the dual curve \(\Gamma'\) introduced in Theorem 3.4 satisfy Proposition 6.5 (B) and vice versa.

**The second covering and the period mapping** Let \(\Gamma = \{\mu^2 = -\lambda \prod_{k=1}^{4}(\lambda - b_k)\}\), then the second 4:1 covering \(\pi_2 : \Gamma \to \mathcal{E}_2\) has topological characteristic \((1, 2, 2, 0)\) and its explicit expression is given in \([31, 33]\).

Let \(\mathcal{E}_2 = \{\mathcal{W}^2 = 4 \prod_{i=1}^{3}(\mathcal{Z} - E_i)\}\), \(E_1 < E_2 < E_3\), then proceeding as for the case of the degree 3:1 covering, after some ugly and straightforward computations, we arrive to the following conclusion. If Proposition 6.5 (A) holds, \(b_0 = 0\) is an order 3 ramification point mapped to infinity by \(\tau_2, b_1, b_4 \in \pi_2^{-1}(E_1), b_2, b_3 \in \pi_2^{-1}(E_2)\) and the infinity point of \(\Gamma\) maps to the infinity of \(\mathcal{E}_2\). Finally, computing the solutions to the equation \((\lambda, \mu) = \pi_2^{-1}(E_j), j = 1, 2\) we find real points with \(\lambda\) coordinate in \(]b_3, b_4[\), and we conclude that
\[
\oint_{\alpha_2} \omega_2 = \frac{1}{2} \int_{b_1}^{b_2} \omega_2 = \frac{1}{2} \int_{E_1}^{E_2} \mathcal{Z}/\mathcal{W} = \oint_{\alpha} \mathcal{Z}/\mathcal{W} = 1/3.
\]

That is, the period mapping is either 3 : 1 or 1 : 3.
If Proposition 6.5 (B) holds, using Proposition 6.7 and proceeding as above, we get
\[ \oint_{\alpha_1} \omega_2 = 2 \int_{b_1}^{b_2} \omega_2 = 2 \int_{E_1}^{E_2} \frac{dZ}{W} = \frac{1}{2} \] and we conclude that the period mapping is either \(2 : 1\) or \(1 : 2\). We have thus proven

**Proposition 6.8** The closed geodesics associated to a curve \(\Gamma\) which is a 4:1 hyperelliptical tangential cover, have period mapping \(3 : 1\) or \(1 : 3\) in case Proposition 6.5 (A) holds, and have period mapping \(2 : 1\) or \(1 : 2\) in case Proposition 6.5 (B) holds.

**Figures 2 and 3:** In figure 2 we present closed geodesics with period mapping \(1 : 3\) \((a_1 < a_2 < c < a_3)\) and \(3 : 1\) \((a_1 < c < a_2 < a_3)\) associated to the hyperelliptic curve
\[ \Gamma = \{ \mu^2 = -\lambda(\lambda - 1.453)(\lambda - 1.483)(\lambda - 4.434)(\lambda - 84.967) \}, \]
which is a 4:1 hyperelliptic tangential cover corresponding to \(\sigma_+ = 2.7, \sigma_- = 2.1\) so that Proposition 6.5 (A) is satisfied.

In figure 3 we present closed geodesics with period mapping \(1 : 2\) \((a_1 < a_2 < c < a_3)\) and \(2 : 1\) \((a_1 < c < a_2 < a_3)\) associated to the hyperelliptic curve
\[ \Gamma = \{ \mu^2 = -\lambda(\lambda - 0.0996)(\lambda - 0.1012)(\lambda - 0.150)(\lambda - 4.5510) \}, \]
which is a 4:1 hyperelliptic tangential cover corresponding to \(\sigma_+ = -3, \sigma_- = 5.1\) so that Proposition 6.5 (B) is satisfied.

**6.3 Doubly–periodic closed geodesics related to degree 2 coverings with extra automorphisms**

In this section we prove the existence of a family of doubly–periodic closed geodesics on triaxial ellipsoids parametrized by \(\tau^2 \in \mathbb{Q}\) related to the family of genus two hyperelliptic curves \(\Gamma\) which covers 2:1 two isomorphic elliptic curves \(E_{1,2}\) (this family of coverings has also been considered in relation to doubly–periodic KdV solutions by I. Taimanov [34]).

The parameter \(\tau\) is the moduli of the elliptic curve \(E_1\). Since it is not possible to determine algebraically the branch points of an elliptic curve in function of the moduli or vice versa, the condition on \(\tau\) is transcendental. However Theorem 3.7 implies that for such values of the parameter \(\tau^2\), \(\Gamma\) is also a hyperelliptic tangential cover of another curve \(E_3\), so that in principle it should be possible to express such condition also algebraically. Indeed we have been able to work out an explicit example (Figure 4) associated to the real intersection of this one parameter family of degree 2 coverings with extra automorphisms with the two-parameter family of degree 3 hyperelliptic tangential covers characterized in subsection 4.1.
Description of the covering  The hyperelliptic curve

$$G_\alpha = \{ \ w^2 = z(z^2 - \alpha^2)(z^2 - 1/\alpha^2) \ \} ,$$

covers 2:1 the elliptic curve

$$E_1 = \{ \ W^2 = Z(Z-1)(\kappa_\alpha^2 Z - 1) \ \} , \quad \kappa_\alpha^2 = \frac{(\alpha + 1)^2}{2(\alpha^2 + 1)} ,$$

and the covering $\Pi_1 : G_\alpha \mapsto E_1$ is given by

$$Z = -\frac{2(1 + \alpha^2)z}{\alpha(z - \alpha)(z - 1/\alpha)}, \quad W = \sqrt{-\frac{2(1 + \alpha^2)}{\alpha} \frac{(z + 1)w}{(z - \alpha)^2(z - 1/\alpha)^2}} ,$$

and, moreover, $\frac{dZ}{W} = -\sqrt{\frac{2(1+\alpha^2)}{\alpha} \frac{(z-1)dz}{w}}$

There exists a second 2:1 cover $\Pi_2 : G_\alpha \mapsto E_2$, with

$$E_2 = \{ \ \bar{W}^2 = \bar{A}_\alpha \bar{Z}(\bar{Z} - 1)(\kappa_\alpha^2 \bar{Z} - 1) \ \} , \quad \bar{A}_\alpha = -\frac{2(\alpha + 1)^2(\alpha^2 + 1)}{(\alpha - 1)^4} ,$$

$$Z = \frac{(z - \alpha)(z - 1/\alpha)}{(z - 1)^2} , \quad \bar{W} = \sqrt{-\frac{2\alpha^2 + 1}{\alpha} \frac{y}{(x - 1)^3}}$$

and $\frac{d\bar{Z}}{\bar{W}} = \frac{(\alpha-1)^2}{\sqrt{2\alpha(\alpha^2 + 1)}} \frac{(z+1)dz}{w}$. Clearly $E_1$ and $E_2$ are isomorphic since they have the same $j$-invariant (see for instance [2]).

Now, let $\alpha > 1$. Under the birational transformation,

$$\lambda = 1/(z + \alpha), \quad \mu = \frac{y}{\sqrt{1 - \alpha^2(z + \alpha)^2}} ,$$

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\[ \mathcal{G}_\alpha \text{ is equivalent to} \]
\[ \Gamma_\alpha = \{ \mu^2 = -\lambda(\lambda - \frac{1}{2\alpha})(\lambda - \frac{1}{\alpha})(\lambda - \frac{\alpha}{\alpha^2 - 1})(\lambda - \frac{\alpha}{\alpha^2 + 1}) \}. \] (6.4)

Since \( 0 < \frac{1}{2\alpha} < \frac{\alpha}{\alpha^2 + 1} < \frac{1}{\alpha} < \frac{\alpha}{\alpha^2 - 1} \), \( \Gamma_\alpha \) may be interpreted as the hyperelliptic curve associated either to the geodesics on the ellipsoid \( Q_0 \) of semiaxes \( \frac{1}{2\alpha}, \frac{\alpha}{\alpha^2 + 1}, \frac{\alpha}{\alpha^2 - 1} \) and tangent to the confocal quadric \( Q_c, c = \frac{1}{\alpha} \), or to the geodesics on the ellipsoid \( Q_0 \) of semiaxes \( \frac{1}{2\alpha}, \frac{1}{\alpha}, \frac{\alpha}{\alpha^2 - 1} \) and tangent to the confocal quadric \( Q_c, c = \frac{\alpha}{\alpha^2 + 1} \).

The family of hyperelliptic curves \( \Gamma_\alpha \) is rather exceptional. Indeed, the birational transformation \( \rho = a_1 \lambda/(\lambda - a_1) \) introduced in Lemma 3.3 just permutes the branch points so that \( \Gamma \) coincides with its dual curve \( \Gamma' \). Using Theorem 3.4 we immediately get

**Proposition 6.9** The real geodesics associated to \( \Gamma_\alpha \) are closed if and only if they are doubly–periodic. In the latter case \( \Gamma_\alpha \) coincides with its dual.

**A transcendental condition for doubly–periodic closed geodesics** We now discuss the existence of such doubly–periodic closed geodesics for \( \Gamma_\alpha \). Using the above formulas it is easy to check that

\[ \frac{dZ}{\sqrt{Z(Z-1)(Z-\kappa_{\alpha}^2)}} = 2\rho_{\alpha} \frac{((\alpha+1)\lambda - 1)d\lambda}{\mu}, \]

\[ \frac{d\tilde{Z}}{\sqrt{\tilde{Z}(\tilde{Z}-1)(\tilde{Z}-\kappa_{\alpha}^2)}} = 2i\rho_{\alpha} \frac{((\alpha-1)\lambda - 1)d\lambda}{\mu}, \]

where \( \rho_{\alpha} = \sqrt{\frac{(1+\alpha^2)}{4\alpha(\alpha-1)(\alpha^2+1)}} \). Now let \( P_1 = (\lambda_1, \mu_1), P_2 = (\lambda_2, \mu_2) \in \Gamma_\alpha \) and set

\[ U_i = 2 \int_{P_0}^{P_i} \frac{dZ}{\sqrt{Z(Z-1)(Z-\kappa_{\alpha}^2)}}, \quad \tilde{U}_i = 2 \int_{P_0}^{P_i} \frac{d\tilde{Z}}{\sqrt{\tilde{Z}(\tilde{Z}-1)(\tilde{Z}-\kappa_{\alpha}^2)}}, \quad (i = 1, 2). \]

Then, the quadrature of the geodesics flow

\[ \sum_{i=1}^{2} \int_{P_0}^{P_i} \frac{d\lambda}{\mu} = s + \text{const.}, \quad \sum_{i=1}^{2} \int_{P_0}^{P_i} \frac{\lambda d\lambda}{\mu} = \text{const.}, \]

is equivalent to

\[ U_1 + U_2 = -\rho_{\alpha}s + c, \quad \tilde{U}_1 + \tilde{U}_2 = -\sqrt{-1}(\rho_{\alpha}s + \tilde{c}), \]

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with \( c, \tilde{c} \) constants. Using the addition theorem for elliptic integrals, the above equations may be inverted and we get

\[
P(U_1|\tau_\alpha) + P(U_2|\tau_\alpha) = P(-\rho s + c|\tau_\alpha),
\]

where \( 0 < \tau_\alpha < 1 \) is the moduli of \( E_1 \). Using the identity \( P(\sqrt{-1}U|\tau_\alpha) = P(U|\tau_\alpha) \), (6.5) is equivalent to

\[
P(U_1|\tau_\alpha) + P(U_2|\tau_\alpha) = P(-\rho s + c|\tau_\alpha),
\]

(6.6)

Then, the geodesic is doubly periodic if and only if \( \tau_\alpha^2 \in \mathbb{Q} \) and, in such a case, the parameter \( s \) may be eliminated from (6.6) using the addition theorem for elliptic functions. In view of theorem 3.7, then \( \Gamma_\alpha \) also possesses a hyperelliptic tangential cover of convenient degree \( d \) (actually it possesses an infinite number of coverings following [30]). We have thus proven the following

**Theorem 6.10** Let \( \Gamma_\alpha \) be the one parameter family of hyperelliptic curves described above and let \( 0 < \tau_\alpha < 1 \) be the moduli of \( E_1 \). Then, the geodesics associated to \( \Gamma_\alpha \) are doubly–periodic if and only if \( \tau_\alpha^2 \in \mathbb{Q} \). In the latter case, there exist an integer \( d \geq 3 \) and an elliptic curve \( E^{(d)} \) such that \( (\Gamma_\alpha, P_0) \) is a \( d : 1 \) hyperelliptic tangential cover over \( E^{(d)} \).

From Theorem 6.10 and the argument used to prove Proposition 6.9, we immediately conclude the following.

**Corollary 6.11** Suppose that the geodesics associated to \( \Gamma_\alpha \) are doubly–periodic. Then the period mapping of the real and imaginary closed geodesics are either equal or reciprocal to each other.

The above Corollary settles quite restrictive conditions on the possible hyperelliptic tangential coverings associated to \( \Gamma_\alpha \). For instance there cannot exist \( d = 4 \) hyperelliptic tangential coverings associated to \( \Gamma_\alpha \), since the curve and its dual would possess different values of the period mapping for that degree of the covering (compare Propositions 6.7 and 6.8 for the \( d = 4 \) hyperelliptic coverings with Proposition 6.9). Below we construct explicitly a \( d = 3 \) hyperelliptic tangential covering in the family \( \Gamma_\alpha \).

**Figure 4.** We show an example of doubly–periodic closed geodesic associated to a covering satisfying Theorem 6.10. This example possesses rather exceptional and intriguing properties. Let \( \alpha = \sqrt{2/\sqrt{3}} \) and \( \Gamma_\alpha \) as in (6.4), then \( \Gamma_\alpha \) covers 2:1 the elliptic curve

\[
E_1^{(2)} = \{ \ W^2 = Z(Z - 1)(\kappa_\alpha^2 Z - 1), \ \} ,
\]

where \( \kappa_\alpha^2 = 1/2 + 2\sqrt{2\sqrt{3} - \sqrt{6}\sqrt{3}}. \)
Moreover, the branch points of $\Gamma_\alpha$

$$a_1 = 1/(2\alpha), \quad a_2 = \alpha/(\alpha^2 + 1), \quad a_3 = \alpha/(\alpha^2 - 1), \quad c = 1/\alpha,$$

also satisfy (6.2) in Proposition 6.2. That is $\Gamma_\alpha$ is a degree 3 hyperelliptic tangential cover over the elliptic curve

$$\mathcal{E}_1^{(3)} = \{w^2 = 4z^3 - 2/9\sqrt{3}z\}.$$  

We remark that the $j$-invariant of $\mathcal{E}_1^{(3)}$ takes the exceptional value 1728, that is the elliptic curve $\mathcal{E}_1^{(3)}$ possesses non trivial automorphisms of order two (see [2] and references therein).

Finally for the second 3:1 cover we find $G_2 = g_2$, $G_3 = g_3 = 0$, that is $\mathcal{E}_2^{(3)} = \mathcal{E}_1^{(3)}$.

As expected the closed geodesics have period mapping 1:2 (one self–intersection), if we exchange $c$ and $a_2$ we get period mapping 2:1 and simple closed geodesics.

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