From Coalescing Random Walks on a Torus to Kingman’s Coalescent

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Abstract
Let $T^d_N$, $d \geq 2$, be the discrete $d$-dimensional torus with $N^d$ points. Place a particle at each site of $T^d_N$ and let them evolve as independent, nearest-neighbor, symmetric, continuous-time random walks. Each time two particles meet, they coalesce into one. Denote by $C_N$ the first time the set of particles is reduced to a singleton. Cox (Ann Probab 17:1333–1366, 1989) proved the existence of a time-scale $\theta_N$ for which $C_N/\theta_N$ converges to the sum of independent exponential random variables. Denote by $Z^N_t$ the total number of particles at time $t$. We prove that the sequence of Markov chains $(Z^N_{t/\theta_N})_{t \geq 0}$ converges to the total number of partitions in Kingman’s coalescent.

Keywords Interacting particle systems · Martingale problem · Markov chain model reduction · Kingman’s coalescent

Mathematics Subject Classification 82C22 · 60K35 · 60F99

1 Introduction

Fix $d \geq 2$, and denote by $T^d_N = \{0, \ldots, N-1\}^d$ the discrete, $d$-dimensional torus with $N^d$ points. Consider independent, nearest-neighbor, symmetric, continuous-time coalescing
random walks evolving on $\mathbb{T}_N^d$. This dynamics can be informally described as follows. Place a particle at each point of $\mathbb{T}_N^d$. Each particle evolves, independently from the others, as a continuous-time random walk which jumps from $x$ to $x \pm e_j$ with probability $1/2^d$, where the summation is taken modulo $N$ and $\{e_1, \ldots, e_d\}$ stands for the canonical basis of $\mathbb{R}^d$. Whenever a particle jumps to a site occupied by another particle, the two particles coalesce into one.

Let $C_N$ be the first time the set of particles is reduced to a singleton, and let $s_N = N^d$ in dimension $d \geq 3$, $s_N = N^2 \log N$ in dimension 2. Cox [6] proved that $C_N/s_N$ converges in distribution to a random variable $\tau$ which can be expressed as

$$\tau = \sum_{k \geq 2} T_k, \quad (1.1)$$

where $(T_k)_{k \geq 2}$ is a sequence of independent, exponential random variables whose expectations are given by

$$E[T_n] = \frac{2}{n(n-1)}, \quad \text{for } n \geq 2.$$

This result directs us to Kingman’s coalescent [10], a dynamic which describes a continuous-time Markov process on the equivalence relations of

$\mathbb{N} = \{1, 2, \ldots\}$. Here we focus our interest in the process $(N_t)_{t \geq 0}$ which records the number of equivalence classes in Kingman’s coalescent. The process $N_t$ is a pure death process on $\mathbb{N} \cup \{\infty\}$, starting at $\infty$, finite at any positive time, and jumping from $k$ to $k-1$ at rate $k(k-1)/2$. A path of $(N_t)_{t \geq 0}$ can be sampled as follows. Recall the definition of the random variables $(T_n)_{n \geq 2}$, and set $T_1 = \infty$. Note that with probability one $\sum_{n=2}^{\infty} T_n < \infty$ and so

$$\left[ \sum_{n=k+1}^{\infty} T_n, \sum_{n=k}^{\infty} T_n \right], \quad k \in \mathbb{N},$$

turns to be a partition of $(0, \infty)$. Set $N_0 = \infty$ and, for every $t > 0$ and $k \geq 1$, define

$$N_t = k \iff \sum_{n=k+1}^{\infty} T_n \leq t < \sum_{n=k}^{\infty} T_n. \quad (1.2)$$

Notice that this process is not continuous at $t = 0$ unless every neighborhood of $\infty \in \mathbb{N} \cup \{\infty\}$ has finite complement.

We shall use an alternative description of this process, more suitable to our purposes. Consider the bijection

$$\{1, 2, \ldots, \infty\} \rightarrow S := \{1, 1/2, 1/3, \ldots, 0\}$$

$$x \mapsto 1/x,$$

taking $\infty$ to 0, and endow $S$ with the standard differential structure inherited by the real line. The first result of this article characterizes the law of

$$\mathcal{X}_t = 1/N_t, \quad t \geq 0, \quad \text{(where } 1/\infty = 0) \quad (1.3)$$
as the unique solution of a martingale problem.

The second main result of the article asserts that in an appropriate time-scale the process which records the [inverse of the] total number of particles at a given time converges in the
Skorohod topology to $X_t$. This result sharpens and extends previous ones. In dimension 2, Zähle et al. [18] proved the convergence of the one-dimensional distribution of $N_t$ provided the particles are initially spread out. Also in dimension 2 and under similar assumptions, Heuer and Sturm [8] proved that the partition structure of a coalescent process converges to the partition structure of the Kingman’s coalescent. In particular, this shows the weak convergence of $X_t$ in an appropriate time scale, provided the particles are initially spread out. In dimension $d \geq 3$, Limic and Sturm [14] proved the convergence of $N_t$, excluding a neighborhood of $t = 0$. By adopting the view point of a martingale problem, we are able to handle the convergence in the Skorohod topology for all times and avoid the assumption that particles are initially spread out.

Since Cox’ article [6], the asymptotic behavior of the coalescence time $C_N$ has been the subject of several papers. Consider a connected graph $G_N$ with $N$ vertices. If $G_N$ is the complete graph, the distribution of $C_N$ can be computed exactly and the process which records the total number of particles is Markovian. This example is called the mean-field model, and one expects that, under some mixing conditions on the random walk on the graph $G_N$, the asymptotic behavior of the coalescence time $C_N$ resembles the one of the mean field model.

Denote by $h_N$ the expected hitting time of a vertex for a random walker starting from the stationary distribution of the random walk, and by $t_N$ the expected meeting time of two independent random walks over $G_N$, both starting from the stationarity state. Aldous and Fill [1, Chap. 14] conjectured in Open Problem 12 that under some mixing conditions $E[C_N]$ is of the same order as $h_N$, as in the mean-field case.

Durrett [7] proved mean field behavior in a small world random graph and Cooper, Frieze and Radzik [5, Theorem 8] in random $d$-regular graphs. Oliveira [15,16] showed that under some reasonable mixing conditions $C_N/t_N$ converges to $\tau$, the random time introduced in (1.1), in transitive, reversible, irreducible Markov chains.

Our motivation to consider this problem comes from the theory of metastable Markov chains. We proposed in [2,3] a general method, based on the characterization of Markov processes as solutions of martingale problems, to show that projections of Markov chains on smaller state spaces are asymptotically Markovian. Coalescing random walks fit perfectly in this framework, as it is expected that the total number of particles evolves asymptotically as Kingman’s coalescent.

This article leaves some open questions. It would certainly be interesting to extend the results presented here to the random graphs covered by Oliveira [16] or to non-reversible dynamics, but also to consider the dynamics which keeps track of the total number of particles which coalesced with each particle present at a given time. This later dynamics is related to a Wright-Fisher diffusion, already examined by Cox [6] and Chen et al. [4].

2 Notation and Results

Denote by $p$ the probability measure on $\mathbb{Z}^d$ given by

\[ p(x) = \frac{1}{2d} \text{ if } x \in \{ \pm e_1, \ldots, \pm e_d \}, \quad \text{and } p(x) = 0 \text{ otherwise} . \quad (2.1) \]

Let $E_N$ be the family of nonempty subsets of $\mathbb{T}_N^d$. The coalescing random walks introduced in the previous section is the $E_N$-valued, continuous-time Markov chain, represented by $\{A_N(t) : t \geq 0\}$, whose generator $L_N$ is given by

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\[(L_N f)(A) = \sum_{x \in A} \sum_{y \not\in A} p(y - x) (f(A_x, y) - f(A)) + \sum_{x \in A} \sum_{y \in A} p(y - x) (f(A_x) - f(A)) , \tag{2.2}\]

where \(A_{x,y}\) (resp. \(A_x\)) is the set obtained from \(A\) by replacing the point \(x\) by \(y\) (resp. removing the element \(x\)):

\[
A_{x,y} = [A \setminus \{x\}] \cup \{y\}, \quad A_x = A \setminus \{x\}.
\]

### 2.1 Kingman’s Coalescent

Recall from the previous section the definition of Kingman’s coalescent \((N_t)_{t \geq 0}\) and the definition of the set \(S\).

Let \(D(\mathbb{R}_+, S)\) be the space of \(S\)-valued, right-continuous trajectories with left-limits, endowed with the Skorohod topology. The respective coordinate maps are denoted by

\[
X_t : D(\mathbb{R}_+, S) \rightarrow S, \quad t \geq 0.
\]

Consider the canonical filtration

\[
\mathcal{G}_t := \sigma(X_s : 0 \leq s \leq t), \quad t \geq 0.
\]

It is known that \(\mathcal{G}_\infty := \sigma(X_t : t \geq 0)\) coincides with the corresponding Borel \(\sigma\)-field on \(D(\mathbb{R}_+, S)\). Let \(C^1(S)\) be the set of functions \(f : S \rightarrow \mathbb{R}\) of class \(C^1\), that is \(f \in C^1(S)\) is the restriction to \(S\) of a continuously differentiable function defined on a neighborhood of \(S\).

For each \(f \in C^1(S)\) define \(\mathcal{L} f : S \rightarrow \mathbb{R}\) as

\[
(\mathcal{L} f)(y) := \begin{cases} 
\frac{n}{2} \left( f \left( \frac{1}{n} - 1 \right) - f \left( \frac{1}{n} \right) \right), & \text{if } y = \frac{1}{n} \text{ and } n \geq 2, \\
0, & \text{if } y = 1, \\
(1/2) f'(0), & \text{if } y = 0.
\end{cases} \tag{2.3}
\]

Note that

\[
\lim_{k \to +\infty} (\mathcal{L} f)(y_k) = (\mathcal{L} f)(0), \quad \text{for } y_k \in S \text{ with } \lim_{k \to +\infty} y_k = 0; \tag{2.4}
\]

this makes \(\mathcal{L} f\) a continuous function. Also observe that the equality \((\mathcal{L} f)(1) = 0\) says that once the process hits 1, it stays there forever.

The following proposition guarantees existence and uniqueness for the \((C^1(S), \mathcal{L})\)-martingale problem and that \((X_t)_{t \geq 0}\), defined in (1.3), provides the unique solution starting at \(0 \in S\).

**Proposition 2.1** For each \(x \in S\), there exists a unique solution for the \((C^1(S), \mathcal{L})\)-martingale problem starting at \(x\). That is, there exists a unique probability measure \(P_x\) on the measurable space \((D(\mathbb{R}_+, S), \mathcal{G}_\infty)\) such that \(P_x[X_0 = x] = 1\) and, for every \(f \in C^1(S)\),

\[
f(X_t) - \int_0^t (\mathcal{L} f)(X_s) \, ds, \quad t \geq 0, \tag{2.5}
\]

is a \(P_x\)-martingale with respect to \((\mathcal{G}_t)_{t \geq 0}\). Moreover, \(P_0\) coincides with the law of \((X_t)_{t \geq 0}\).
2.2 Main Result

Recall that $E_N$ stands for the set of nonempty subsets of $\mathbb{T}^d_N$. Consider the partition of $E_N$ according to the number of elements:

$$E_N = \bigcup_{n \in \mathbb{N}} E^n_N, \quad \text{where} \quad E^n_N := \{ A \subset \mathbb{T}^d_N : |A| = n \}, \quad n \in \mathbb{N}, \quad (2.6)$$

and $|A|$ stands for the number of elements of $A$. Let $\Psi_N : E_N \to S$ be the projection corresponding to partition (2.6)

$$\Psi_N(A) = 1/|A|, \quad A \in E_N.$$

For each $A \in E_N$, let $P^N_A$ denote a probability measure under which the process $(A_N(t))_{t \geq 0}$ corresponds to a coalescing random walk on $\mathbb{T}^d_N$ starting at $A$, i.e. a Markov chain with state space $E_N$ and generator $L_N$ (defined in (2.2)) such that $P^N_A[A_N(0) = A] = 1$. When $A = \mathbb{T}^d_N$, we denote $P^N_A$ simply by $P^N$. Expectation with respect to $P^N_A$, $P^N$ is represented by $E^N_A$, $E^N$, respectively.

Consider two independent random walks $(x^N_t)_{t \geq 0}$ and $(y^N_t)_{t \geq 0}$ on $\mathbb{T}^d_N$, both with jump probability given by $p(\cdot)$, starting at the stationary (thus the uniform) distribution. Let $\theta_N$ be the expected meeting time:

$$\theta_N := E[ \min\{ t \geq 0 : x^N_t = y^N_t \} ].$$

Since $x^N_t - y^N_t$ evolves as a random walk speeded-up by 2, $\theta_N$ represents the expectation of the hitting time of the origin for a simple symmetric random walk speeded-up by 2 which starts from the stationary state. By [2, Proposition 6.10], we may express this expectation in terms of capacities. Sharp bounds for the capacity then provide an asymptotic formula for $\theta_N$.

Consider a continuous-time, random walk $(x_t)_{t \geq 0}$ on $\mathbb{Z}^d$ with jump probabilities given by (2.1) and which starts from the origin. Assume that $d \geq 3$, and denote by $\tau_1$ the time of the first jump, $\tau_1 = \inf\{ t \geq 0 : x_t \neq 0 \}$, and by $H^+$ the return time to the origin: $H^+ = \inf\{ t \geq \tau_1 : x_t = 0 \}$. Let $v_d$ be the escape probability: $v_d = P[H^+ = \infty]$. By the argument presented in the previous paragraph, by [9, Corollary 6.8] in dimension $d \geq 3$, and by [9, Corollary 6.12] in dimension 2,

$$\lim_{N \to \infty} \frac{\theta_N}{N^d} = \frac{1}{2v_d} \quad \text{in dimension } d \geq 3,$$

$$\lim_{N \to \infty} \frac{\theta_N}{N^2 \log N} = \frac{1}{\pi} \quad \text{in dimension } d = 2. \quad (2.7)$$

The factor 2 in the denominator appears because the process has been speeded-up by 2. In particular, in $d = 2$, $1/\pi$ should be understood as $(1/2)(2/\pi)$.

Consider the rescaled reduced process

$$X_N(t) = \Psi_N(A_N(\theta_N t)) \quad \text{for } t \geq 0. \quad (2.8)$$

Notice that $X_N(t)$ is not a Markov chain, but only a hidden Markov chain. Denote by $\mathcal{P}^N$ the probability law on $(D(\mathbb{R}_+, S), \mathcal{G}_\infty)$ induced by the reduced process $(X_N(t))_{t \geq 0}$ under $P^N$ (i.e. starting from all vertices in $\mathbb{T}^d_N$ occupied). The main result of this article reads as follows

**Theorem 2.2** For every $d \geq 2$, the sequence of measures $\mathcal{P}^N$ converges to $\mathcal{P}_0$. 

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It follows from Theorem 2.2 that, under $\mathbb{P}^N$,

$$(\mathbb{X}_N(t))_{t \geq 0} \xrightarrow{\text{Law}} (\mathbb{X}_t)_{t \geq 0}, \quad \text{for } d \geq 2.$$  

The scaling limit for the coalescing times obtained in [6] immediately follows from these results.

**Remark 2.3** The proofs apply, with minor modifications, to the case in which the jump probability $p(\cdot)$ is symmetric and has finite range. It also applies if the initial condition $\mathbb{T}_N^d$ is replaced by a finite set $A = \{x_1, \ldots, x_n\}$ whose points are scattered: $\|x_i - x_j\| \geq a_N$ for $1 \leq i \neq j \leq n$, where $a_N$ is the sequence introduced in (3.3). This problem has been addressed in [14].

### 2.3 Sketch of the Proof

The proof is divided in two steps. We first show that the sequence $(\mathbb{P}^N)$ is tight, and then we guarantee uniqueness of limit points by proving that every limit point solves the $(C^1(S), \mathcal{L})$-martingale problem.

For the later step, consider a smooth function $f : \mathbb{R} \to \mathbb{R}$, and denote by $M_N(t)$ the martingale given by

$$f(\mathbb{X}_N(t)) - f(\mathbb{X}_N(0)) - \int_0^t \theta_N L_N(f \circ \Psi_N)(A_N(s\theta_N)) \, ds.$$  

Since

$$L_N(f \circ \Psi_N)(A) = R(A) \left\{ f\left(\frac{w}{1-w}\right) - f(w) \right\},$$  

where $w = \Psi_N(A)$, and $R(A)$ is the jump rate given by

$$R(A) = \sum_{x \in A} \sum_{y \in A \setminus \{x\}} p(y - x), \quad (2.9)$$

the martingale $M_N(t)$ can be written as

$$f(\mathbb{X}_N(t)) - f(\mathbb{X}_N(0)) - \theta_N \int_0^t R(A_N(s\theta_N)) \left\{ f\left(\frac{\mathbb{X}_N(s)}{1-\mathbb{X}_N(s)}\right) - f(\mathbb{X}_N(s)) \right\} ds.$$  

If the martingale $M_N(t)$ were expressed in terms of the process $\mathbb{X}_N$, that is if $\theta_N R(A_N(s\theta_N)) = r(\mathbb{X}_N(s))$, we could pass to the limit and argue that

$$f(\mathbb{X}(t)) - f(\mathbb{X}(0)) - \int_0^t r(\mathbb{X}(s)) \left\{ f\left(\frac{\mathbb{X}(s)}{1-\mathbb{X}(s)}\right) - f(\mathbb{X}(s)) \right\} ds.$$  

(2.10)

is a martingale for every limit point $\mathbb{P}^*$ of the sequence $\mathbb{P}^N$. This result together with the uniqueness of solutions of the martingale problem (2.10) on $(D(\mathbb{R}^+, S), G_\infty)$ would yield the uniqueness of limit points.

The previous argument evidences that the main point of the proof consists in “closing” the martingale $M_N(t)$ in terms of the reduced process $\mathbb{X}_N(s)$, that is, that the major difficulty lies in the proof of the existence of a function $r : S \to \mathbb{R}$ such that

$$\int_0^t \left\{ \theta_N R(A_N(s\theta_N)) - r(\mathbb{X}_N(s)) \right\} g(\mathbb{X}_N(s)) ds \to 0.$$  

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for all smooth functions \( g : \mathbb{R} \to \mathbb{R} \). This is the so-called “replacement lemma” or the “local ergodic theorem”. One has to replace a function \( \theta_N R(A) \) which does not vanish only in a tiny portion of the state space (in the present context for subsets of \( (T^d_N)^n \) which contain at least two neighboring points) and which is very large (here of order \( \theta_N \)) when it does not vanish, by a function of order 1 in the entire space.

The statement of the local ergodic theorem requires some notation. Denote by \( D(\mathbb{R}+,E_N) \) the right-continuous trajectories \( \omega : \mathbb{R}+ \to E_N \) which have left-limits. Let

\[
\rho(\frac{1}{n}) := \lambda(n) := \binom{n}{2}, \quad n \geq 2.
\]

**Proposition 2.4** Let \( F : \mathbb{N} \to \mathbb{R} \) be a function which eventually vanishes: there exists \( k_0 \geq 1 \) such that \( F(k) = 0 \) for all \( k \geq k_0 \). Let \( t_0 > 0 \) and let \( (B^N_n : D(\mathbb{R}+,E_N) \to \mathbb{R}; \quad N \geq 1) \) be a sequence of uniformly bounded functions, with each \( B^N \) measurable with respect to \( \sigma(A_N(s\theta_N) : 0 \leq s \leq t_0) \). Then, for every \( t > t_0 \),

\[
\lim_{N \to \infty} E_N \left[ B^N \int_{t_0}^t \{ \theta_N R(A_N(s\theta_N)) - n_{s\theta_N} \} F(|A_N(s\theta_N)|) ds \right] = 0,
\]

where \( n_s = \lambda(|A_N(s)|) \).

### 2.4 Mixing Time

The mixing conditions sufficient for some of our results are stated in terms of the mixing time defined in this subsection.

Let \( E \) be a countable state space. We denote by \( \| \mu - \nu \|_{TV} \) the total variation distance between two probability measures, \( \mu, \nu \) on \( E \):

\[
\| \mu - \nu \|_{TV} := \frac{1}{2} \sum_{a \in E} | \mu(a) - \nu(a) |.
\]

Let \( Y \) be a continuous time Markov chain over \( E \) and denote by \( p^Y_t(\cdot, \cdot) \) the transition probabilities of \( Y \) at time \( t \). Suppose that \( Y \) has a unique invariant probability measure \( \pi_Y \). The **mixing time** of \( Y \) is defined by

\[
t^Y_{\text{mix}}(\varepsilon) := \inf \left\{ t > 0 : \sup_{a \in E} \| p^Y_t(a, \cdot) - \pi_Y \|_{TV} \leq \varepsilon \right\},
\]

and

\[
t^Y_{\text{mix}} := t^Y_{\text{mix}}(1/4).
\]

It is well known that when \( Y \) is irreducible and \( E \) is finite there exists a unique invariant probability measure, and the mixing time is finite. Moreover, for all \( \varepsilon \in (0, 1/2) \) we have

\[
t^Y_{\text{mix}}(\varepsilon) \leq C t^Y_{\text{mix}} \ln \left( \frac{1}{\varepsilon} \right),
\]

for a universal constant \( C > 0 \). This is proven in [13, Sect. 4.5] for discrete time, but the same works in our context.

We denote by \( t^N_{\text{mix}} \) the mixing time of the continuous time, simple, symmetric random walk on \( T^d_N \). From (2.12) it follows that taking \( (s_N)_N \) such that \( t^N_{\text{mix}} \ll s_N \),

\[
\lim_{N \to \infty} \max_{x \in T^d_N} \| p^N_{s_N}(x, \cdot) - \pi_N \|_{TV} = 0,
\]
where \( p_t^N(\cdot, \cdot) \) denotes the transition probabilities of the random walk on \( \mathbb{T}_N^d \), and \( \pi_N \) is its invariant probability measure.

Finally, we know (cf. [13, Sects. 5.3 and 7.4]) that there exist constants \( 0 < c(d) < C(d) < \infty \) such that

\[
c(d)N^2 \leq t_{\text{mix}}^N \leq C(d)N^2,
\]

therefore (2.13) works for any \((s_N)^2\) such that \( N^2 \ll s_N \).

The article is organized as follows. In Sect. 3, we present the results on coalescing random walks needed in the proof of Proposition 2.4, which is presented in the following section. In Sect. 5, we prove Theorem 2.2 and, in Sect. 6, Proposition 2.1.

### 3 Coalescing Random Walks on \( \mathbb{T}_N^d \)

We present in this section some results on coalescing random walks obtained by Cox [6]: Propositions 3.1, 3.5 and 3.6. We start with some notation.

Throughout this section, \( P^N_x \) represents the distribution of a \( \mathbb{T}_N^d \)-valued random walk, speeded-up by 2, whose jump probability is \( p(\cdot) \), introduced in (2.1), and initial position is \( x \). Denote by \( p_t(x,y) = P^N_x[x(t) = y] \) the transition probabilities of this process and by \( \pi_N \) its stationary state, which is the uniform measure on \( \mathbb{T}_N^d \).

The first result, Proposition 4 in [6], provides a bound on the expectation of the number of particles still present at time \( t \). Let

\[
g_N(t) = \begin{cases} N^2 t^{-1} \log(1 + t) & d = 2, \\ N^d/t & d \geq 3. \end{cases}
\]

**Proposition 3.1** There exists a finite constant \( c_d \) such that

\[
E^N[|A_N(t)|] \leq c_d \max\{1, g_N(t)\}
\]

for all \( t > 0, N \geq 1 \).

Recall from (2.6) that we denote by \( E^N_n \) the subsets of \( \mathbb{T}_N^d \) with \( n \) elements. Denote by \( \tau_j, j \geq 1 \), the time when the process \( A_N(t) \) is reduced to a set of \( j \) elements:

\[
\tau_j = \inf \{ t \geq 0 : |A_N(t)| = j \} = \inf \{ t \geq 0 : A_N(t) \in E^N_j \}.
\]

**Lemma 3.2** There exists a finite constant \( C_0 \) independent of \( N \) such that for all \( j \geq 2 \),

\[
\max_{A \in E^N_j} \frac{1}{\theta_N^j} E^N_A[\tau_{j-1}] \leq C_0.
\]

**Proof** Fix two points \( x, y \) in \( A \) and denote by \( \tau_{x,y} \) the first time these particles meet: \( \tau_{x,y} = \inf\{ t > 0 : x(t) = y(t) \} \). Since \( \tau_{j-1} \leq \tau_{x,y} \), and since the difference \( x(t) - y(t) \) evolves as a random walk speeded-up by 2, the expectation appearing in the statement of the lemma is bounded by

\[
\max_{x \in E^N_n} \frac{1}{\theta_N^j} E^N_x[H_0],
\]

where \( H_0 \) represents the hitting time of the origin. By [13, Proposition 10.13], this quantity is bounded by a finite constant independent of \( N \). \( \square \)
It follows from the previous result that for every \( j \geq 2 \),
\[
\lim_{M \to \infty} \limsup_{N \to \infty} \max_{A \in \mathcal{E}_N^1} P_N^A[\tau_{j-1} \geq M \theta_N] = 0 .
\] (3.2)

Hereafter, the symbol \( \alpha_N \ll \beta_N \), for two non-decreasing sequences \( \alpha_N, \beta_N \), means that \( \alpha_N/\beta_N \to 0 \). Denote by \( a_N \) an increasing sequence such that \( 1 \ll a_N \ll N \). In dimension 2, assume further that \( N/\sqrt{\log N} \ll a_N \). Denote by \( \mathcal{G}_N(n, a_N) \) the scattered subsets of \( E_N \).

These are the sets \( A = \{y_1, \ldots, y_n\} \) in \( \mathcal{E}_N^1 \) such that
\[
\min_{i \neq j} |y_i - y_j| \geq a_N .
\] (3.3)

**Lemma 3.3** For every \( n \geq 2 \), \( t > 0 \),
\[
\lim_{N \to \infty} \max_{A \in \mathcal{E}_N^1} P_N^A\left[A_N(t \theta_N) \notin \mathcal{E}_N^1 \cup \bigcup_{k=2}^n \mathcal{G}_N(k, a_N)\right] = 0 .
\]

**Proof** Since \( n \) is finite and since the difference of two random walks evolves as a random walk speeded-up by 2, this assertion follows from the claim that for every \( t > 0 \)
\[
\lim_{N \to \infty} \max_{\theta \in \mathcal{T}_N^d} P_N^\theta\left[H_0 < t \theta_N, |x(t \theta_N)| \leq a_N\right] = 0 .
\]

By the Markov property, the previous probability is bounded by
\[
E_N^x\left[P_N^{x(t \theta_N/2)}\left[|x(t \theta_N/2)| \leq a_N\right]\right].
\]

Recall from the beginning of this section that \( \pi_N \) represents the stationary state of the random walk on \( \mathcal{T}_N^d \). The previous expectation is less than or equal to
\[
P_N^{\pi_N}\left[|x(t \theta_N/2)| \leq a_N\right] + 2 \|\pi_N(\cdot) - p_{t \theta_N/2}(x, \cdot)\|_{TV} ,
\]
where \( p_t(x, y) \) represents the transition probabilities of a random walk evolving on \( \mathcal{T}_N^d \) speeded-up by 2. The first term is bounded by \( C_0(a_N/N)^d \to 0 \), while the second one vanishes because of (2.13), since \( \theta_N \gg t_{\text{mix}}^N \). \( \square \)

**Corollary 3.4** For every \( t > 0 \),
\[
\lim_{N \to \infty} P_N\left[A_N(t \theta_N) \notin \mathcal{E}_N^1 \cup \bigcup_{k=2}^N \mathcal{G}_N(k, a_N)\right] = 0 .
\]

**Proof** Fix \( t > 0 \), and let \( \mathcal{H}_s = \{A_N(s \theta_N) \notin \mathcal{E}_N^1 \cup \bigcup_{k=2}^N \mathcal{G}_N(k, a_N)\} \), \( s > 0 \). Clearly, for every \( M > 0 \),
\[
P_N[\mathcal{H}_t] \leq P_N[|A_N(t \theta_N/2)| \leq M, \mathcal{H}_t] + P_N[|A_N(t \theta_N/2)| > M] .
\]

By Proposition 3.1, the second term is bounded by \( C(d, t)/M \), where \( C(d, t) \) is a constant depending only on \( d \) and \( t \). This happens because \( g_N(t \theta_N/2) \) is bounded uniformly in \( N \); remember (2.7) and the definition of \( g_N \). Hence, by the Markov property,
\[
P_N[\mathcal{H}_t] \leq \max_{2 \leq k \leq M} \max_{A \in \mathcal{E}_N^1} P_N^A[\mathcal{H}_t/2] + \frac{C(d, t)}{M} .
\]  

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By Lemma 3.3, the first term on the right-hand side vanishes as $N \to \infty$ for every $M \geq 2$. This proves the corollary.

**Proposition 3.5** For every $2 \leq j < k$,

$$\lim_{N \to \infty} \max_{A \in \mathcal{G}_N(k, a_N)} P^N_A \left[ A_N(\tau_j) \notin \mathcal{G}_N(j, a_N) \right] = 0.$$  

**Proof** Fix $2 \leq j < k$. By (3.2), it is enough to prove that for all $M > 0$,

$$\lim_{N \to \infty} \max_{A \in \mathcal{G}_N(j, a_N)} P^N_A \left[ A_N(\tau_j) \notin \mathcal{G}_N(j, a_N), \tau_j \leq M \theta_N \right] = 0.$$  

This is exactly assertions (3.7) and (3.8) in [6]. \hfill \Box

Denote by $\pi_n^N$, $n \geq 2$, the uniform measure on $\mathcal{E}_N^n$. Recall the definition of $\lambda(\cdot)$ given in (2.11). The next proposition is a weak version of [6, Theorem 5].

**Proposition 3.6** For all $j \geq 2$,

$$\lim_{N \to \infty} P^N_{\pi_j^N} \left[ \tau_{j-1} \geq t \theta_N \right] = e^{-\lambda(j) t}.$$  

It follows from the previous result that for every $n \geq 1$,

$$\lim_{\delta \to 0} \limsup_{N \to \infty} P^N_{\pi_n^N} \left[ \tau_n \leq \delta \theta_N \right] = 0. \quad (3.4)$$  

Indeed, fix $n \geq 1$ and consider a set $A \in \mathcal{E}_N^{n+1}$. Since $A \subset \mathcal{T}_N^d$, $P^N[A_n \leq \delta \theta_N] \leq P^N_A[A_n \leq \delta \theta_N]$. Averaging over $A$ with respect to $\pi_n^{n+1}$ we obtain that $P^N[A_n \leq \delta \theta_N] \leq P^N_{\pi_n^{n+1}}[A_n \leq \delta \theta_N]$. By Proposition 3.6, the previous quantity vanishes as $N \to \infty$ and then $\delta \to 0$.

Denote by $\gamma_N$ a sequence much larger than the mixing time and much smaller than the hitting time:

$$t_N^{\max} \ll \gamma_N \ll \theta_N. \quad (3.5)$$  

The existence of this sequence is a consequence of (2.14) and (2.7). Let $(\ell_N : n \geq 1)$ be a sequence such that $1 \ll \ell_N \ll N$. In dimension 2, we assume that $N^\alpha \ll \ell_N \ll N$ for all $0 < \alpha < 1$, so that

$$\lim_{N \to \infty} \frac{\ell_N}{N} = 0, \quad \lim_{N \to \infty} \frac{\log \ell_N}{\log N} = 1. \quad (3.6)$$  

Note that in dimension 2 the conditions imposed on $\ell_N$ are weaker than the ones assumed on $a_N$ in [6, Theorem 4].

**Lemma 3.7** For every $n \geq 2$,

$$\lim_{N \to \infty} \max_{A \in \mathcal{G}_N(\ell_N)} P^N_A \left[ \tau_{n-1} \leq \gamma_N \right] = 0.$$  

**Proof** The probability is bounded by

$$\binom{n}{2} \max_{\|x\| \leq \ell_N} P^N_x \left[ H_0 \leq \gamma_N \right],$$  

where, recall, $H_0$ stands for the hitting time of the origin. Since $\gamma_N \ll \theta_N$, by equation (6.18) in [9], this expression vanishes in the limit. \hfill \Box
In the next lemma we compare the dynamics $A_N(t)$ with the one of independent random walks. Fix $n \geq 2$, and denote by $(x_n^a(t))_{t \geq 0}$, the evolution of $n$ independent random walks on $\mathbb{T}_N$ with jump probabilities $p(\cdot)$ given by (2.1). The stationary state of this dynamics, denoted by $\pi^a_N$, is the product measure on $[\mathbb{T}_N^d]^n$ in which each component is the measure $\pi_N$.

Denote by $p_{t}^N(x, y)$ the transition probabilities of $x_n^a(t)$, and by $t_{\text{mix}}^{N,n}$ the corresponding mixing time. Since the dynamics amounts to the evolution of a random walk on $\mathbb{T}_N^d$, thanks to (2.14), there exist constants $0 < c(d, n) < C(d, n) < \infty$ such that $c(d, n) N^2 \leq t_{\text{mix}}^{N,n} \leq C(d, n)N^2$.

Denote by $x_j(t) \in \mathbb{T}_N^d$ the $j$-th coordinate of $x_n^a(t)$, $1 \leq j \leq n$. Up to time $\tau_{n-1}$ the process $A_N(t)$ evolves as $\{x_n^a(t)\} := \{x_1(t), \ldots, x_n(t)\}$. More precisely, fix $A = \{a_1, \ldots, a_n\} \in \mathcal{E}_N^n$, and let

$$\mathcal{E}_N^a := \bigcup_{k=1}^n e_k^N.$$ 

There exists a probability measure on $D(\mathbb{R}^+, \mathcal{E}_N^a \times (\mathbb{T}_N^d)^n)$, denoted by $\hat{\mathbb{P}}_A$, which fulfills the following conditions. The distribution of the first, resp. second, coordinate corresponds to the transition probabilities $\pi^a_N$ and $\pi^a_N$, respectively. Fix $n \geq 2$. Let $F_N : \mathcal{E}_N^a \to \mathbb{R}$ be a sequence of uniformly bounded functions, $\|F\| := \sup_{N \geq 1} \max_{A \in \mathcal{E}_N^n} |F_N(A)| < \infty$, and let $(\beta_N)_{N \geq 1}$ be a non-negative sequence. Then, for every $A = \{a_1, \ldots, a_n\} \in \mathcal{E}_N^n$,

$$\mathbb{E}_A^N \left[ F_N(A_N(\beta_N)) \mathbb{1}_{\{\tau_{n-1} > \beta_N\}} \right] = -\mathbb{E}_A^N \left[ F_N(\{x_n^a(\beta_N)\}) \mathbb{1}_{\{\tau_{n-1} \leq \beta_N, \{x_n^a(\beta_N)\} \in \mathcal{E}_N^n\}} \right] + R_N,$$

where

$$|R_N| \leq \|F\| \left\{ 2 \|p_{t}^N(a, \cdot) - \pi^a_N(\cdot)\|_{TV} + c_N \right\}$$

and $\lim_{N \to \infty} c_N = 0$.

**Proof** Fix $A = \{a_1, \ldots, a_n\} \in \mathcal{E}_N^n$. We may rewrite the expectation appearing in the statement of the lemma as

$$\hat{\mathbb{E}}_A^N \left[ F_N(A_N(\beta_N)) \mathbb{1}_{\{\tau_{n-1} > \beta_N\}} \right].$$

Since $A_N(t) = \{x_n^a(t)\}$ in the time interval $[0, \tau_{n-1}]$, we may replace in the previous equation $A_N(\beta_N)$ by $\{x_n^a(\beta_N)\}$ and then add the indicator function of the set $\{x_n^a(\beta_N)\} \in \mathcal{E}_N^n$. After these replacements, the previous expression becomes

$$\hat{\mathbb{E}}_A^N \left[ F_N(\{x_n^a(\beta_N)\}) \mathbb{1}_{\{\{x_n^a(\beta_N)\} \in \mathcal{E}_N^n\}} \right] - \hat{\mathbb{E}}_A^N \left[ F_N(\{x_n^a(\beta_N)\}) \mathbb{1}_{\{\tau_{n-1} \leq \beta_N, \{x_n^a(\beta_N)\} \in \mathcal{E}_N^n\}} \right].$$

We estimate the first term. Recall that we denote by $p_{t}^N(x, y)$ the transition probabilities of $x_n^a(t)$. With this notation, we may write this term as

$$\sum_{x \in [\mathbb{T}_N^d]^n} F_N(\{x\}) \mathbb{1}_{\{x \in \mathcal{E}_N^n\}} \pi^a_N(x) + R_N^{(1)}.$$
where
\[ |R_N^{(1)}| \leq 2 \|F\| \|p^{(n)}_{\beta_N}(a, \cdot) - \pi_N^{\otimes n}(\cdot)\|_{TV}.\]
and \(a = (a_1, \ldots, a_n)\).

To bound the first term of the penultimate formula, recall that we denote by \(\pi_N^n\) the uniform measure on \(\mathcal{E}_N^n\). Let
\[R_{N,n}^{(2)} := \sum_{A \in \mathcal{E}_N^n} \left| \pi_N^n(A) - \sum_{x \in [\mathcal{T}_N^n]} 1\{x = A\} \pi_N^{\otimes n}(x) \right|.\]
(3.7)

An elementary computation shows that \(\lim_{N \to \infty} R_{N,n}^{(2)} = 0\) for every \(n \geq 2\). The assertion of the lemma follows from the previous estimates.

The next lemma is a consequence of [6, Theorem 5] in dimension \(d \geq 3\). In dimension 2 it is a slight generalization since our assumptions on \(\ell_N\) are weaker.

**Lemma 3.9** Let \(\ell_N\) be a sequence satisfying the conditions introduced above (3.6). Then, for all \(t > 0\),
\[\lim_{N \to \infty} \max_{A \in \mathcal{G}(n, \ell_N)} \left| P^N_A\left[ \tau_{n-1} \geq t\theta_N \right] - e^{-\lambda(n)t} \right| = 0.\]

**Proof** We present the proof in dimension \(d = 2\). The one in higher dimension is analogous. Fix a set \(A = \{a_1, \ldots, a_n\} \in \mathcal{G}(n, \ell_N)\) and a sequence \(1 \ll t_N \ll \log N\). Recall from the previous lemma the definition of the measure \(\hat{P}_N^A\). Since the first coordinate evolves as \(A_N(t)\),
\[P^N_A\left[ \tau_{n-1} \geq t\theta_N \right] = \hat{P}_N^A\left[ \tau_{n-1} \geq t\theta_N \right].\]
By the Markov property,
\[\hat{P}_N^A\left[ \tau_{n-1} \geq t\theta_N \right] = \hat{E}_N^A\left[ \hat{P}_{A_N(\gamma_N)}^N\left[ \tau_{n-1} \geq t\theta_N - \gamma_N \right] 1\{\tau_{n-1} > \gamma_N\} \right],\]
where \(\gamma_N = t_N N^2\).

We apply Lemma 3.8 with \(\beta_N = \gamma_N\) to estimate the right-hand side. Let \(F_N : \mathcal{E}_N^{\leq n} \to \mathbb{R}\) be the function defined by
\[F_N(A) = P^N_A\left[ \tau_{n-1} \geq t\theta_N - \gamma_N \right], \quad A \in \mathcal{E}_N^n,\]
and \(F_N(A) = 0\) for \(A \notin \mathcal{E}_N^n\). By Lemma 3.8, the right hand side of the penultimate formula is equal to
\[P^N_{\pi_N^n}\left[ \tau_{n-1} \geq t\theta_N - \gamma_N \right] + R_N,\]
where
\[|R_N| \leq \hat{P}_A^N\left[ \tau_{n-1} \leq \gamma_N \right] + 2 \|p^{(n)}_{\gamma_N}(a, \cdot) - \pi_N^{\otimes n}(\cdot)\|_{TV} + C_N,\]
with \(\lim_{N \to \infty} C_N = 0\).

Each term of the previous expression is negligible. In the first one, we may replace \(\hat{P}_A^N\) by \(P_A^N\), and apply Lemma 3.7 to conclude that this expression vanishes as \(N \to \infty\). The second one also vanishes in the limit because \(\gamma_N \gg t_{\text{mix}}^N\) and \(t_{\text{mix}}^{N,n}\) is of the same order as \(t_{\text{mix}}^N\). To complete the proof of the lemma, as \(\gamma_N \ll \theta_N\), it remains to apply Proposition 3.6.

Recall the properties of the sequence \(a_N\) introduced in (3.3). By the previous result, for all \(k > j \geq 2\),
\[\lim_{N \to \infty} \max_{A \in \mathcal{G}(k,a_N)} \left| P_A^N\left[ \tau_{j-1} - \tau_j \geq t\theta_N \right] - e^{-\lambda(j)t} \right| = 0.\]
(3.8)
Indeed, by Proposition 3.5, we may intersect the event appearing inside the probability with the set \(A_N(\tau_j) \in \mathcal{G}_N(j, a_N)\). Then, applying the strong Markov property at time \(\tau_j\) we reduce assertion (3.8) to Lemma 3.9.

The next result together with the previous lemma entails the convergence of \(\mathbb{E}_{A_N}^N[\tau_{n-1}/\theta_N]\) to \(\lambda(n)^{-1}\) for any sequence \(A_N \in \mathcal{G}(n, \ell_N)\).

**Lemma 3.10** For every \(n \geq 2\), \(m \geq 1\), there exists a finite constant \(C(n, m)\) such that for all \(N \geq 1\),

\[
\max_{A \in \mathcal{E}_N^n} \mathbb{E}_{A}^N[(\tau_{n-1}/\theta_N)^m] \leq C(n, m).
\]

**Proof** By the Markov property, for all \(k \geq 1\),

\[
\max_{A \in \mathcal{E}_N^n} \mathbb{P}_{A}^N[\tau_{n-1}/\theta_N \geq k] \leq \left( \max_{A \in \mathcal{E}_N^n} \mathbb{P}_{A}^N[\tau_{n-1}/\theta_N \geq 1] \right)^k.
\]

We claim that

\[
\max_{A \in \mathcal{E}_N^n} \mathbb{P}_{A}^N[\tau_{n-1} \geq \theta_N] \leq \mathbb{P}_{A}^N[\tau_{n-1} \geq \theta_N/2] + \delta_N. \tag{3.9}
\]

where \(\delta_N \to 0\). Indeed, fix \(A = \{a_1, \ldots, a_n\} \in \mathcal{E}_N^n\), and apply the Markov property to obtain that

\[
\mathbb{P}_{A}^N[\tau_{n-1} \geq \theta_N] = \mathbb{E}_{A}^N\left[ \mathbb{P}_{A}^N[\tau_{n-1} \geq \theta_N/2] 1_{\{\tau_{n-1} \geq \theta_N/2\}} \right].
\]

Let \(F_N : \mathcal{E}_N^n \to \mathbb{R}\) be the function defined by

\[
F_N(A) = \mathbb{P}_{A}^N[\tau_{n-1} \geq \theta_N/2], \quad A \in \mathcal{E}_N^n,
\]

and \(F_N(A) = 0\) for \(A \notin \mathcal{E}_N^n\). Since \(F_N\) is non-negative, by Lemma 3.8, the right-hand side of the penultimate formula is bounded above by

\[
\mathbb{P}_{\pi_N}^N[\tau_{n-1} \geq \theta_N/2] + 2 \left\| \mu_{\pi_N}(a, \cdot) - \pi_N(\cdot) \right\|_{TV} + c_N,
\]

where \(a = (a_1, \ldots, a_n)\). Assertion (3.9) follows from the facts that \(\theta_N \gg t_{\text{mix}}^N\) and that \(t_{\text{mix}}^N\) is of the same order as \(t_{\text{mix}}^N\).

By Proposition 3.6, under the measure \(\mathbb{P}_{\pi_N}^N\), \(\tau_{n-1}/\theta_N\) converges weakly to an exponential random variable of parameter \(\lambda(n)\). Thus, the right-hand side of (3.9) converges to \(e^{-\lambda(n)/2} < 1\). Therefore, there exists \(\delta < 1\) such that for all \(N \geq 1\),

\[
\max_{A \in \mathcal{E}_N^n} \mathbb{P}_{A}^N[\tau_{n-1}/\theta_N \geq k] \leq \delta^k.
\]

This proves the lemma. \(\square\)

**Corollary 3.11** For every \(n \geq 2\),

\[
\lim_{N \to \infty} \max_{A \in \mathcal{G}_N(n, \ell_N)} \left| \frac{1}{\theta_N} \mathbb{E}_{A}^N[\tau_{n-1}] - \frac{1}{\lambda(n)} \right| = 0.
\]

**Proof** Fix a sequence \(A_N \in \mathcal{G}_N(n, \ell_N)\), \(N \geq 1\). The convergence in law of the sequence \(\tau_{n-1}/\theta_N\) under the measure \(\mathbb{P}_{A_N}^N\) to an exponential random variable of parameter \(\lambda(n)\) follows from Lemma 3.9. By the previous lemma the sequence \(\tau_{n-1}/\theta_N\) is uniformly integrable. \(\square\)

Recall that we denote by \((e_1, \ldots, e_d)\) the canonical basis of \(\mathbb{R}^d\).
Lemma 3.12 Assume that $d \geq 3$ and $n \geq 2$. Let $(B_N)_{N}$ be a sequence of sets whose elements
$B_N$ are in $\mathcal{S}_N(n - 1, \ell_N)$. From each $B_N$ we obtain $A_N \in \mathcal{E}_N^R$ by adding a particle on an
empty neighbouring site to an existing particle in $B_N$. i.e.

$$A_N = B_N \cup \{x_N + u_N\} \in \mathcal{E}_N^R,$$

where $u_N \in \{e_j, -e_j : j = 1, \ldots, d\}$ and $x_N \in B_N$. For all $t > 0$,

$$\lim_{N \to \infty} P_{A_N}^{N}[\tau_{n-1} \geq t\theta_N] = v_d e^{-\lambda(n)t}. \quad (3.10)$$

Proof Denote by $x(t), y(t)$ the position at time $t$ of the particle initially at $x_N, x_N + u_N$,
respectively. Let $D_r, r \geq 0$, be the first time the distance between these particles attains $r$: $D_r = \inf\{t > 0 : \|x(t) - y(t)\| = r\}$, and let $H = D_0 \wedge D_{\ell_N}$. As $\ell_N \ll N$ and since
a symmetric random walk reaches a distance $R_N$ by a time of order $R_N^2$, an elementary
computation shows that

$$\lim_{N \to \infty} P_{A_N}^{N}[H > N^2] = 0.$$ 

We may therefore insert the set $\{H \leq N^2\}$ in the probability appearing in equation (3.10).
Let $N$ be large enough so that $t\theta_N > N^2$. For such $N$, $\{H \leq N^2\} \cap \{D_0 < D_{\ell_N}\} \cap \{\tau_{n-1} \geq t\theta_N\} \subset \{\tau_{n-1} \leq N^2\} \cap \{\tau_{n-1} \geq t\theta_N\} = \emptyset$. Hence, for $N$ large enough,

$$P_{A_N}^{N}[\tau_{n-1} \geq t\theta_N] = P_{A_N}^{N}[H \leq N^2, D_0 > D_{\ell_N}, \tau_{n-1} \geq t\theta_N] + o_N(1),$$

where $o_N(1) \to 0$ as $N \to \infty$.

By the Markov property, the probability on the right hand side is equal to

$$E_{A_N}^{N} \left[ 1 \{H \leq N^2, D_0 > D_{\ell_N}, \tau_{n-1} \geq N^2\} P_{A_N(N^2)}^{N}[\tau_{n-1} \geq t\theta_N - N^2] \right].$$

On the event $\{\tau_{n-1} \geq N^2\}$, we may replace the distribution of $A_N(N^2)$ by the one of the
position at time $N^2$ of $n$ independent random walks starting from $A_N$. After this replacement,
we may insert in the expectation the indicator of the set $\{A_N(N^2) \in \mathcal{S}_N(n, \ell_N)\}$ because the
probability of the complement vanishes as $N \to \infty$ [indeed, whatever the initial position of a random walk, the probability that it is at distance $\ell_N$ from the origin at time $N^2$ vanishes].
After this insertion, we write the previous expectation as

$$e^{-\lambda(n)t}P_{A_N}^{N}[H \leq N^2, D_0 > D_{\ell_N}, A_N(N^2) \in \mathcal{S}_N(n, \ell_N), \tau_{n-1} \geq N^2] + R_N,$$

where the absolutely value of $R_N$ is bounded by

$$\max_{A \in \mathcal{S}_N(n, \ell_N)} \left| P_{A}^{N}[\tau_{n-1} \geq t\theta_N - N^2] - e^{-\lambda(n)t} \right|.$$ 

By Lemma 3.9, this expression vanishes as $N \to \infty$. Hence, up to this point we proved that the probability appearing in (3.10) is equal to

$$e^{-\lambda(n)t}P_{A_N}^{N}[H \leq N^2, D_0 > D_{\ell_N}, A_N(N^2) \in \mathcal{S}_N(n, \ell_N), \tau_{n-1} \geq N^2] + o_N(1).$$

On the set $\{H \leq N^2, D_0 > D_{\ell_N}, \tau_{n-1} \leq N^2\}$ two particles which were at distance
at least $\ell_N$ met in a time interval of length bounded by $N^2$. Indeed, the time $\tau_{n-1}$ may correspond to the coalescence of two particles on the set $B_N \setminus \{x_N\}$ or one particle in the set $B_N \setminus \{x_N\}$ and one in the set $\{x_N, x_N + u_N\}$. In both cases, these particles were initially at distance at least $\ell_N$ from each other. The time $\tau_{n-1}$ may also correspond to the coalescence
of the particles initially at \(x_N, x_N + u_N\). In this case, at time \(H \leq N^2 \land D_0\) these particles were at distance \(\ell_N\).

By Lemma 3.7 with \(n = 2\), the probability that two particles which are at distance \(\ell_N\) meet before time \(N^2\) vanishes as \(N \to \infty\). We may therefore remove from the previous probability the event \(\{\tau_{n-1} \geq N^2\}\). We may also remove, as explained above in the proof, the events \(\{H \leq N^2\}\) and \((A_N(N^2) \in \mathcal{G}_N(n, \ell_N)\), so that

\[
P^N_{A_N}[\tau_{n-1} \geq t\theta_N] = e^{-\lambda(n)t} P^N_{A_N}[D_0 > D_{\ell_N}] + o_N(1).
\]

As \(N \to \infty\), this latter probability converges to the escape probability, denoted by \(v_d\), which proves the lemma. \(\square\)

The next result follows from the previous lemma and from the uniform integrability provided by Lemma 3.10.

**Corollary 3.13** Assume that \(d \geq 3\) and \(n \geq 2\). Let \((B_N)_N\) be a sequence of sets whose elements \(B_N\) are in \(\mathcal{G}_N(n-1, \ell_N)\). From each \(B_N\) we obtain \(A_N \in \mathcal{E}_N^n\) by adding a particle on an empty neighbouring site to an existing particle in \(B_N\). i.e.

\[A_N = B_N \cup \{x_N + u_N\} \in \mathcal{E}_N^n,\]

where \(u_N \in \{e_j, -e_j : j = 1, \ldots, d\}\) and \(x_N \in B_N\). Then,

\[
\lim_{N \to \infty} \frac{1}{\theta_N} \mathbb{E}^N_{A_N}[\tau_{n-1}] = \frac{v_d}{\lambda(n)}.
\]

By (2.7), the previous limit can be written as

\[
\lim_{N \to \infty} \frac{1}{N^d} \mathbb{E}^N_{A_N}[\tau_{n-1}] = \frac{1}{2\lambda(n)}.
\]  \(\text{(3.11)}\)

We turn to the 2-dimensional case.

**Lemma 3.14** Assume that \(d = 2\) and \(n \geq 2\). Let \((B_N)_N\) be a sequence of sets whose elements \(B_N\) are in \(\mathcal{G}_N(n-1, \ell_N)\). From each \(B_N\) we obtain \(A_N \in \mathcal{E}_N^n\) by adding a particle on an empty neighbouring site to an existing particle in \(B_N\). i.e.

\[A_N = B_N \cup \{z_N + u_N\} \in \mathcal{E}_N^n,\]

where \(u_N \in \{e_j, -e_j : j = 1, \ldots, d\}\) and \(z_N \in B_N\). Then,

\[
\lim_{N \to \infty} \frac{1}{N^2} \mathbb{E}^N_{A_N}[\tau_{n-1}] = \frac{1}{2\lambda(n)}.
\]

**Proof** Fix a sequence of sets \(A_N\) satisfying the hypotheses of the lemma. Enumerate the points of \(A_N = \{x_1, \ldots, x_n\}\) in such a way that \(x_1 = z_N, x_2 = z_N + u_N\). Denote by \(x_i(t)\) the position at time \(t\) of the random walks initially at \(x_i\).

Let \((\ell_N : N \geq 1), (m_N : N \geq 1)\) be the sequences \(\ell_N = N/(\log N)^{4}, m_N = N/\log N\). Notice that both sequences fulfill the conditions above (3.6). Let \(T_{1,2}\) be the first time the difference \(x_1(t) - x_2(t)\) reaches the distance \(\ell_N\), \(T_{1,2} = \inf\{t > 0 : \|x_1(t) - x_2(t)\| \geq \ell_N\}\), and denote by \(T_i, 1 \leq i \leq n,\) the first time the particle \(x_i\) reaches a distance \(m_N\) from its original position: \(T_i = \inf\{t > 0 : \|x_i(t) - x_i(0)\| \geq m_N\}\). The proof of the lemma relies on the estimates (3.12), (3.13) and (3.14).

Since the difference \(x_1(t) - x_2(t)\) evolves as a random walk speeded-up by 2,

\[
\mathbb{E}^N_{A_N}[T_{1,2}] = \mathbb{E}^N_{e_1}[\bar{D}_{\ell_N}],
\]

\(\square\) Springer
where $\hat{D}_{\ell_N}$ is the first time the particle reaches a distance $\ell_N$ from the origin, and $P_{e_1}^N$ represents the distribution of a symmetric, nearest-neighbor random walk speeded-up by 2, starting from $e_1$. We couple a speeded-up random walk like this with a random walk in $\mathbb{Z}$ in the following way: we think of $\mathbb{Z}$ as a vertical and ordered collection of points. Each time the speeded-up random walker moves in a vertical direction, our walker in $\mathbb{Z}$ repeats the same motion, and we do nothing in $\mathbb{Z}$ when our speeded-up random walker in $\mathbb{Z}^2$ moves in an horizontal direction. Let $H_{t_N}$ be the first time the random walker in $\mathbb{Z}$ is at distance at least $\ell_N$ from the origin. Then $E_{e_1}^N[\hat{D}_{\ell_N}] \leq \bar{E}_0[H_{t_N}]$, where $\bar{P}_0$ represents the distribution of the coupled random walk in $\mathbb{Z}$. This implies that $E_{e_1}^N[\hat{D}_{\ell_N}] \leq C_0 \ell_N^2$ for some constant $C_0$ independent of $N$. Hence,

$$
\lim_{N \to \infty} \frac{1}{N^2} E_{A_N}^N[T_{1,2}] = 0 .
$$

(3.12)

For every $1 \leq i \leq n$, and every sequence $(S_N)_{N \geq 1}$ of non-negative numbers,

$$
P_{A_N}^N[T_i \leq S_N] = \bar{P}_0^N[\hat{D}_{m_N} \leq S_N] = \bar{P}_0^N\left[ \sup_{t \leq S_N} \|x(t)\| \geq m_N \right],
$$

where $\bar{P}_0^N$ stands for the distribution of a nearest-neighbor, symmetric, random walk starting from the origin. The difference with respect to $P_0^N$ is that the random walk is not speeded-up by 2 under $\bar{P}_0^N$. An elementary random walk estimation yields that the right hand side multiplied by $\log N$ vanishes as $N \to \infty$ if we choose $S_N = N^2/(\log N)^4$ [indeed, replace first $\sup_{t \leq S_N} \|x(t)\|$ by $\|x(S_N)\|$, then apply Markov inequality for the square, and estimate $E_{\bar{P}_0}^N[\|x(S_N)\|^2]$ by $C_0 S_N$. This argument yields that the probability is bounded by $C_0 S_N/m_N^2 \leq C_0/(\log N)^2$. Hence, with this definition for $S_N$, for all $1 \leq i \leq n$,

$$
\lim_{N \to \infty} (\log N) P_{A_N}^N[T_i \leq S_N] = 0 .
$$

In contrast,

$$
P_{A_N}^N[T_{1,2} \geq S_N] = P_{e_1}^N[\hat{D}_{\ell_N} \geq S_N] = P_{e_1}^N\left[ \sup_{t \leq S_N} \|x(t)\| \leq \ell_N \right].
$$

Another elementary random walk estimation yields that the right hand side multiplied by $\log N$ vanishes for the same choice of the sequence $S_N$. Hence,

$$
\lim_{N \to \infty} (\log N) P_{A_N}^N[T_{1,2} \geq S_N] = 0 .
$$

It follows from the last two estimates that

$$
\lim_{N \to \infty} (\log N) P_{A_N}^N[T_{1,2} \geq \min_i T_i] = 0 .
$$

(3.13)

Denote by $\tau_{i,j}$, $1 \leq i \neq j \leq n$, the first time the particles $x_i$, $x_j$ meet, $\tau_{i,j} = \inf\{t > 0 : x_i(t) = x_j(t)\}$. The arguments used to derive (3.13) show that for all pairs $\{i, j\} \neq \{1, 2\}$,

$$
\lim_{N \to \infty} (\log N) P_{A_N}^N[T_{1,2} \geq \tau_{i,j}] = 0 .
$$

(3.14)

We are now in a position to prove the lemma. By the strong Markov property,

$$
E_{A_N}^N[\tau_{n-1}] = E_{A_N}^N\left[ T_{1,2} + \tau_{n-1} \circ \varnothing_{T_{1,2}} \right] 1\{T_{1,2} < \tau_{n-1}\} + E_{A_N}^N[\tau_{n-1} 1\{\tau_{n-1} < T_{1,2}\}]
$$

$$
= E_{A_N}^N\left[ E_{A_N}^N[\tau_{n-1} | T_{1,2}] \right] 1\{T_{1,2} < \tau_{n-1}\} + E_{A_N}^N[\tau_{n-1} 1\{\tau_{n-1} < T_{1,2}\}] .
$$

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The second term is bounded by $E_{A_N}^N[T_{1,2}]$. By (3.12), this expectation divided by $N^2$ vanishes as $N \to \infty$. On the other hand,

$$
\frac{1}{N^2} E_{A_N}^N \left[ E_{A_N(T_{1,2})}^N [\tau_{n-1}] \mathbb{1}\{T_{1,2} \geq \min_i T_i, T_{1,2} < \tau_{n-1}\} \right] \\
\leq \sup_{A \in \mathcal{E}_N^N} \frac{1}{\log N} \frac{1}{N^2} E_{A}^N [T_{n-1}] (\log N) P_{A_N}^N [T_{1,2} \geq \min_i T_i].
$$

This expression vanishes as $N \to \infty$ because, by Lemma 3.10, the first term is uniformly bounded and, by (3.13), the second term tends to 0.

Up to this point, we proved that

$$
\lim_{N \to \infty} \frac{1}{N^2} E_{A_N}^N [\tau_{n-1}] = \lim_{N \to \infty} \frac{1}{N^2} E_{A_N}^N \left[ E_{A_N(T_{1,2})}^N [\tau_{n-1}] \mathbb{1}\{T_{1,2} \leq \min_i \tau_{n-1}, T_i\} \right].
$$

On the set $\{T_{1,2} < \min_i T_i\}$, $A_N(T_{1,2})$ belongs to $\mathcal{G}_N(n, \ell N)$. Hence, by Corollary 3.11 and by (2.7),

$$
\frac{1}{N^2} E_{A_N(T_{1,2})}^N [\tau_{n-1}] = (\log N) \frac{\pi^{-1}}{\lambda(n)} \left[ 1 + o_N(1) \right],
$$

so that

$$
\lim_{N \to \infty} \frac{1}{N^2} E_{A_N}^N [\tau_{n-1}] = \frac{\pi^{-1}}{\lambda(n)} \lim_{N \to \infty} (\log N) P_{A_N}^N [T_{1,2} < \min_i \tau_{n-1}, T_i].
$$

By (3.13), in the previous expression we may remove the indicator of the set $\{T_{1,2} < \min_i T_i\}$. By (3.14), we may also exclude the sets $\{\tau_i \leq T_{1,2}\}$ for $i, j \neq 1, 2$. Hence, the previous expression is equal to

$$
\frac{\pi^{-1}}{\lambda(n)} \lim_{N \to \infty} (\log N) P_{A_N}^N [T_{1,2} < 1] = \frac{\pi^{-1}}{\lambda(n)} \lim_{N \to \infty} (\log N) P_{e_1}^N [D_{\ell N} < H_0],
$$

where $H_0$ represents the hitting time of the origin. By [9, Lemma 6.10], the previous expression is equal to $1/(2\lambda(n))$, which completes the proof of the lemma.

Recall the definition of the jump rate $R$ introduced in (2.9).

**Lemma 3.15** For every $n \geq 2$,

$$
\lim_{N \to \infty} \sum_{A \in \mathcal{E}_N^N} \pi_A^n (A) E_{A}^N [\tau_{n-1}] R(A) = 1.
$$

**Proof** Since $R(A) = 0$ unless $A$ contains two nearest-neighbor points, for all sets $A$ such that $R(A) > 0$, $E_A^N [\tau_{n-1}] \leq E_{e_1} [H_0]$, where $H_0$ represents the hitting time of the origin. By [13, Proposition 10.13], this latter expectation is bounded by $C_0 N^d$. Observe also that $\pi_A^n (A) = 1/\binom{N^d}{n}$ for all the sets in the previous sum.

First consider sets $A = \{x + u_N\} \cup B$, where $u_N \in \{e_j, -e_j : j = 1, \ldots, d\}$, $B \in \mathcal{G}_N(n-1, \ell N)$, and $x \in B$. The number of such sets $A$ is $(n-1)d \binom{N^d}{n-1} [1 + o_N(1)]$. For them $R(A) = 1/d$, and, by (3.11) and Lemma 3.14,

$$
E_A^N [\tau_{n-1}] = \frac{N^d}{2\lambda(n)} [1 + o_N(1)].
$$
Hence, the sum alluded to above, restricted to this kind of sets, is equal to
\[
[1 + o_N(1)] (n - 1) d \frac{N^d}{n^d - 1} \frac{N^d}{2 \lambda(n)} \frac{1}{d} = [1 + o_N(1)] \frac{n(n - 1)}{2 \lambda(n)}.
\]

Now consider the other sets, where there are more than one pair of nearest-neighbor points. Note that \( R(A) \) is uniformly bounded and \( E_N^A [\tau_{n-1}] \leq C_0 N^d \). The number of sets we are considering is of order \( N^d(n - 2) \). From these we deduce that the sum in Lemma 3.15, restricted to these sets, goes to zero as \( N \) grows.

Therefore, the sum in Lemma 3.15 equals the last displayed expression. The result follows from the definition of \( \lambda(n) \) given in (2.11).

\[\square\]

### 4 Local Ergodicity

We prove in this section Proposition 2.4. The proof is divided in a sequence of lemmata.

**Lemma 4.1** For every \( n \geq 2 \), there exists a finite constant \( C(n) \) such that
\[
\max_{A \in E^N_n} E^N_A \left[ \int_0^{\tau_{n-1}} R(A_N(s)) \, ds \right] \leq C(n).
\]

**Proof** Recall the definition of \( R \) in (2.9). Since \( R(B) = 0 \) if \( |B| = 1 \),
\[
\int_0^\infty R(A_N(s)) \, ds = \int_0^{\tau_1} R(A_N(s)) \, ds.
\]
It is therefore enough to prove that for each \( n \geq 2 \), there exists a finite constant \( C(n) \) such that
\[
\max_{A \in E^N_n} E^N_A \left[ \int_0^{\tau_{n-1}} R(A_N(s)) \, ds \right] \leq C(n).
\]

Fix \( n \geq 2 \) and a set \( A = \{x_1, \ldots, x_n\} \) in \( \mathbb{E}_N^n \). Denote by \( x_i(s) \) the position at time \( s \) of the particle \( x_i \) and by \( \tau_{i,j} \) the collision time of particles \( i \) and \( j \): \( \tau_{i,j} = \inf\{t > 0 : x_i(t) = x_j(t)\} \).

As
\[
\int_0^{\tau_{n-1}} R(A_N(s)) \, ds \leq \sum_{i \neq j} \int_0^{\tau_{i,j}} 1\{|x_i(s) - x_j(s)| = 1\} \, ds,
\]
it is enough to estimate
\[
E^N_{\{x_i, x_j\}} \left[ \int_0^{\tau_{i,j}} 1\{|x_i(s) - x_j(s)| = 1\} \, ds \right].
\]

As the difference evolves as a random walk speeded-up by 2, it is enough to bound, for \( x \in \mathbb{T}_N^d \),
\[
E^N_x \left[ \int_0^{H_0} \mathbb{1}\{x(s) = e_1\} \, ds \right],
\]
where \( H_0 \) stands for the hitting time of the origin. This integral represents the time spent at \( e_1 \) before hitting the origin. In particular, it is bounded by a geometric sum of independent exponential random variables, which completes the proof of the lemma. \[\square\]
Remark 4.2  It follows from last lemma and the strong Markov property at time $\tau_n$ that there exists a finite constant $C(n)$ such that

$$\max_{A \in E_n} E^N_A \left[ \int_0^\infty R(A_N(s)) 1\{|A_N(s)| \leq n\} \, ds \right] \leq C(n).$$

Recall the definition of the sequence $a_n$ introduced in (3.3), and that $\pi^n_N$ represents the uniform measure in $E^n_N$.

Lemma 4.3  For every $n \geq 2$,

$$\lim_{N \to \infty} \max_{A \in \mathcal{G}_N(n,a_N)} E^N_A \left[ \int_0^{\tau_{n-1}} R(A_N(s)) \, ds \right] = 0.$$

Proof  The goal is to replace the initial condition $A$ by the pseudo-invariant measure $\pi^n_N$ and then to apply Corollary 4.6. To carry out this strategy, we remove from the time integral an interval large enough for the process to relax and small enough not to interfere with the overall value of the time integral.

Fix a set $A$ in $\mathcal{G}_N(n,a_N)$, enumerate its elements, $A = \{x_1, \ldots, x_n\}$, and denote by $x_i(t)$ the position at time $t$ of the particle initially at $x_i$. Let $D_1$ be the first time two particles are at distance 1 from each other: $D_1 = \inf \{t \geq 0 : \|x_i(t) - x_j(t)\| = 1 \text{ for some } i \neq j\}$. Note that $R(A_N(s)) = 0$ for $s < D_1$ and that $D_1 \leq \tau_{n-1}$.

Let $\gamma_N$ be the sequence introduced in (3.5). We claim that

$$\lim_{N \to \infty} \max_{A \in \mathcal{G}_N(n,a_N)} E^N_A \left[ 1\{\tau_{n-1} \leq \gamma_N\} \int_0^{\tau_{n-1}} R(A_N(s)) \, ds \right] = 0.$$

Indeed, as $R(A_N(s)) = 0$ for $s < D_1$ and $D_1 \leq \tau_{n-1}$, we may replace the lower limit in the integral by $D_1$ and include in the indicator the condition $D_1 \leq \gamma_N$ to bound the previous expectation by

$$E^N_A \left[ 1\{D_1 \leq \gamma_N\} \int_0^{\tau_{n-1}} R(A_N(s)) \, ds \right].$$

By the strong Markov property, this expression is bounded by

$$E^N_A \left[ D_1 \leq \gamma_N \right] \max_{B \in E_n^N} E^N_B \left[ \int_0^{\tau_{n-1}} R(A_N(s)) \, ds \right].$$

By Lemma 4.1 the above expectation is bounded, and by equation (6.18) in [9] the probability vanishes as $N \to \infty$ uniformly in $A \in \mathcal{G}_N(n,a_N)$. Note that in dimension $d \geq 3$, by equation (6.6) in [9], the result (6.18) holds for any sequence $l_N$ such that $1 \ll l_N \ll N$. This proves the claim.

Denote by $\partial_s : D(\mathbb{R}_+, E^*_N) \to D(\mathbb{R}_+, E^*_N)$, $s \geq 0$, the time translation operators such that $(\partial_s \omega(t))(s) = \omega(t+s)$ for all $t \geq 0$. It follows from the previous assertion that we may introduce the indicator of the set $\{\gamma_N < \tau_{n-1}\}$ in the expectation appearing in the statement of the lemma. After the inclusion in the expectation of the indicator of the set $\{\gamma_N < \tau_{n-1}\}$, in the upper limit of the integral rewrite $\tau_{n-1}$ as $\gamma_N + \tau_{n-1} - \gamma_N$ and apply the Markov property to get that the expectation is equal to

$$E^N_A \left[ 1\{\gamma_N < \tau_{n-1}\} \int_0^{\gamma_N} R(A_N(s)) \, ds \right] + E^N_A \left[ 1\{\gamma_N < \tau_{n-1}\} E^N_{A(\gamma_N)} \left[ \int_0^{\tau_{n-1} - \gamma_N} R(A_N(s)) \, ds \right] \right].$$
We claim that the first term vanishes as $N \to \infty$, uniformly in $A \in \mathcal{G}_N(n, a_N)$. Recall the definition of the hitting time $D_1$. If $\gamma_N \leq D_1$, the expression inside the expectation vanishes since $R(A_N(s)) = 0$ for $s \leq D_1$. We may therefore assume that $D_1 \leq \gamma_N$. We may also replace the lower limit of the integral by $D_1$ and the upper limit by $\tau_{n-1}$ to find out that the first term in (4.2) is bounded by (4.1). Since the expectation in (4.1) vanishes as $N \to \infty$, uniformly in $A \in \mathcal{G}_N(n, a_N)$, the claim is proved.

It remains to examine the second expectation in (4.2). To apply Lemma 3.8, let $F_N : \mathcal{E}_N^n \to \mathbb{R}$ be the function given by

$$F_N(B) = \mathbb{E}_B^N \left[ \int_0^{\tau_{n-1}} R(A_N(s)) \, ds \right], \quad B \in \mathcal{E}_N^n, \quad N \geq 1,$$

(4.3)

$F_N(B) = 0$ for $B \not\in \mathcal{E}_N^n$. By Lemma 4.1, $F_N$ is uniformly bounded, $\|F_N\| \leq C(n)$ for all $N \geq 1$, and therefore fulfills the condition of Lemma 3.8. Hence, by this result, the second term in (4.2) can be written as

$$\mathbb{E}_N^N \left[ \int_0^{\tau_{n-1}} R(A_N(s)) \, ds \right] + R_N,$$

where the absolute value of the remainder $R_N$ is bounded by

$$C(n) \left\{ \text{p}_N^N(\tau_{n-1} \leq \gamma_N) + 2 \|p_{\gamma_N}^{(n)}(a, \cdot) - \pi_N^{\otimes a} (\cdot)\|_{TV} + c_N \right\}.$$

In this formula, $a = (a_1, \ldots, a_n), a_j$ are the elements of $A$ and $c_N$ a constant which vanishes as $N \to \infty$. Since $\gamma_N \gg N_{\text{max}}$, the second term inside braces vanishes as $N \to \infty$, uniformly in $A \in \mathcal{E}_N^n$. By Lemma 3.7, the first term inside braces vanishes as $N \to \infty$, uniformly in $A \in \mathcal{G}_N(n, a_N)$. To complete the proof of the lemma, it remains to apply Corollary 4.6. \qed

**Lemma 4.4** Let $F : \mathbb{N} \to \mathbb{R}$ be a function which eventually vanishes: there exists $k_0 \geq 0$ such that $F(k) = 0$ for all $k > k_0$. For all $t > 0$, $n > 1,$

$$\lim_{N \to \infty} \max_{A \in \mathcal{G}_N(n, a_N)} \left| \mathbb{E}_A^N \left[ \int_0^{|\theta_N|} \left\{ R(A_N(s)) - \theta_N^{-1} n_s \right\} F(|A_N(s)|) \, ds \right] \right| = 0.$$

**Proof** Fix $n \geq 2$ and $A$ in $\mathcal{G}_N(n, a_N)$. Since $R(A'), \lambda(|A'|)$ vanish for $|A'| = 1$, if $k_0 \leq 1$ there is nothing to prove. Assume, therefore, that $k_0 \geq 2$. Since $F(k) = 0$ for $k > k_0$, we may start the integral from $\tau_{n_0}$, where $n_0 = n \wedge k_0$. If $t \theta_N \leq \tau_{n_0}$, the integral vanishes. We may therefore insert inside the expectation the indicator function of the set $\{t \theta_N \geq \tau_{n_0}\}$, which can be written as the disjoint union of the sets $\{\tau_j < t \theta_N \wedge \tau_1 \leq \tau_{j-1}\}, 2 \leq j \leq n_0$. Hence, the time-integral appearing in the statement of the lemma can be written as

$$\sum_{j=2}^{n_0} \left[ \int_{\tau_{n_0}}^{\tau_{j-1}} \right] \hat{R}(A_N(s)) F(|A_N(s)|) \, ds$$

(4.4)

$$- \sum_{j=2}^{n_0} \left[ \int_{t \theta_N}^{\tau_{j-1}} \right] \hat{R}(A_N(s)) F(|A_N(s)|) \, ds,$$

where $\hat{R}(A) = R(A) - \theta_N^{-1} \lambda(|A|)$.

We consider each term separately. Write the integral appearing in the first line as a sum of integrals on the intervals $[\tau_i, \tau_{i-1}]$ and sum by parts to obtain that the first expression is equal to
\[
\sum_{i=2}^{n_0} 1\{\tau_i < t\theta_N \land \tau_1 \leq \tau_1\} F(i) \int_{\tau_i}^{\tau_{i-1}} \hat{R}(A_N(s)) \, ds ,
\]

where we used the fact that \(F\) is constant in the time interval \([\tau_i, \tau_{i-1})\). Remove from the indicator the condition \(\{t\theta_N \land \tau_1 \leq \tau_1\}\), which is always satisfied, and replace \(\{\tau_i < t\theta_N \land \tau_1\}\) by \(\{\tau_i < t\theta_N\}\). Fix \(2 \leq i \leq n\), disregard the constant \(F(i)\), and consider the expectation with respect to \(P_A^N\):

\[
E_A^N \left[ 1\{\tau_i < t\theta_N\} \int_{\tau_i}^{\tau_{i-1}} \hat{R}(A_N(s)) \, ds \right].
\]

(4.5)

We claim that

\[
\lim_{N \to \infty} E_A^N \left[ 1\{A_N(\tau_i) \notin \mathcal{G}_N(i, a_N)\} \int_{\tau_i}^{\tau_{i-1}} \{R(A_N(s)) - \theta_N^{-1}n_s\} \, ds \right] = 0 .
\]

Indeed, by the strong Markov property, the absolute value of the previous expectation is less than or equal to

\[
P_A^N \left[ A_N(\tau_i) \notin \mathcal{G}_N(i, a_N) \right] \max_{B \in \mathcal{E}_N^i} \left\{ E_B^N \left[ \int_{0}^{\tau_{i-1}} R(A_N(s)) \, ds \right] + \lambda(i) \theta_N^{-1} E_B^N \left[ \tau_{i-1} \right] \right\} .
\]

By Lemma 3.10 and 4.1, the maximum is bounded. On the other hand, since \(A\) belongs to \(\mathcal{G}_N(n, a_N)\), by Proposition 3.5, the probability vanishes as \(N \to \infty\), which proves the claim.

We may therefore insert in (4.5) the indicator of the set \(\{A_N(\tau_i) \in \mathcal{G}_N(i, a_N)\}\). By the strong Markov property, this expectation is equal to

\[
E_A^N \left[ 1\{\tau_i < t\theta_N, A_N(\tau_i) \in \mathcal{G}_N(i, a_N)\} E_{A_N(\tau_i)}^N \left[ \int_{0}^{\tau_{i-1}} \hat{R}(A_N(s)) \, ds \right] \right] .
\]

By Lemma 4.3 and 3.15,

\[
\lim_{N \to \infty} E_N^B \left[ \int_{0}^{\tau_{i-1}} R(A_N(s)) \, ds \right] = 1
\]

uniformly for \(B \in \mathcal{G}_N(i, a_N)\). By Corollary 3.11, as \(N \to \infty\), \(\lambda(i) E^N_B[\tau_{i-1}/\theta_N]\) converges to 1 uniformly for \(B \in \mathcal{G}_N(i, a_N)\).

It remains to examine the second expression in (4.4). The argument is similar to the one presented above. Fix \(2 \leq j \leq n_0\) and take the expectation with respect to \(P_A^N\) for \(A \in \mathcal{G}_N(n, a_N)\). Since \(\tau_1 \geq \tau_j\), we may remove \(\tau_1\) from the indicator. For \(j = 2\) the set becomes \(\{\tau_2 < t\theta_N\}\), while for \(2 < j \leq n_0\) it is given by \(\{\tau_j < t\theta_N \leq \tau_{j-1}\}\). In the first case, to unify the notation, we insert the condition \(t\theta_N \leq \tau_1\). This is possible because the integral vanishes if this bound is not fulfilled.

We claim that

\[
\lim_{N \to \infty} E_A^N \left[ 1\{\mathcal{G}_N\} \int_{t\theta_N}^{\tau_{j-1}} \{R(A_N(s)) - \theta_N^{-1}n_s\} \, ds \right] = 0 ,
\]

where \(\mathcal{G}_N\) is the set \(\{\tau_j < t\theta_N \leq \tau_{j-1}, A_N(t\theta_N) \notin \mathcal{G}_N(j, a_N)\}\). The proof of this claim is identical to the one produced below (4.5). Observe that on the set \(\{\tau_{j-1} \geq t\theta_N\}\) we may write \(\tau_{j-1}\) as \(t\theta_N + \tau_{j-1} \circ \theta_{t\theta_N}\). Apply the Markov property at time \(t\theta_N\), estimate the conditional expectation by the supremum over all sets in \(\mathcal{E}_N^j\), and apply Lemma 3.10 and 4.1, and Lemma 3.3 (instead of Proposition 3.5).
After inserting in the expectation the indicator of the set \( \{ A_N(t_0 \theta_N) \in \mathcal{G}_N(j, a_N) \} \), applying the Markov property at time \( t_0 \theta_N \), the expectation becomes

\[
E^N_A \left[ 1[\mathcal{M}_N] E^N_{A_N(t_0 \theta_N)} \left[ \int_0^{T_{j-1}} \bar{R}(A_N(s)) \, ds \right] \right],
\]

where \( \mathcal{M}_N = \{ \tau_j < t_0 \theta_N \leq \tau_{j-1}, A_N(t_0 \theta_N) \in \mathcal{G}_N(j, a_N) \} \). By the first part of the proof, this expression vanishes as \( N \to \infty \).

\textbf{Proof of Proposition 2.4} Fix \( \varepsilon > 0 \). In view of Proposition 3.1, choose \( M \in \mathbb{N} \) such that \( \mathbb{P}^N[|A_N(t_0 \theta_N)| > M] \leq \varepsilon \). Let \( W_N(A) = \theta_N R(A) - \lambda(|A|) F(|A|) \). There exists a finite constant \( C(F, B, t) \) such that

\[
\left| E^N \left[ B^N 1[|A_N(t_0 \theta_N)| > M] \int_{t_0}^t W_N(A_N(s \theta_N)) \, ds \right] \right| \leq C(F, B, t) \varepsilon . \tag{4.6}
\]

To prove this assertion, apply the Markov property to write the expectation appearing in the left-hand side as

\[
E^N \left[ B^N 1[|A_N(t_0 \theta_N)| > M] E^N_{A(t_0 \theta_N)} \left[ \int_0^{t-t_0} W_N(A_N(s \theta_N)) \, ds \right] \right].
\]

We claim that the absolute value of the expectation with respect to \( \mathbb{P}^N_{A(t_0 \theta_N)} \) is bounded by a constant depending on \( F \) and \( t \). On the one hand, the function \( \lambda(|A|) F(|A|) \) is bounded because \( F(k) = 0 \) for all \( k \) large enough. On the other hand, since \( F \) vanishes outside a finite subset of \( \mathbb{N} \), by Remark 4.2, the expectation of the time integral of \( \theta_N R(A_N(s \theta_N)) F(A_N(s \theta_N)) \) is bounded. This proves the claim.

It follows from this claim that the absolute value of the expectation appearing in the last displayed equation is bounded by

\[
C(F, B, t) \mathbb{P}^N[|A_N(t_0 \theta_N)| > M],
\]

Assertion (4.6) follows from the choice of \( M \).

A similar argument, using Corollary 3.4 instead of Proposition 3.1, proves that for all \( N \) sufficiently large

\[
\left| E^N \left[ B^N 1 \left\{ A_N(t_0 \theta_N) \notin \mathcal{E}_N^1 \cup \bigcup_{k=2}^{N^d} \mathcal{G}_N(k, a_N) \right\} \times \int_{t_0}^t W_N(A_N(s \theta_N)) \, ds \right] \right| \leq C(F, B, t) \varepsilon .
\]

It follows from the previous two estimates that we may restrict our attention to the expectation

\[
E^N \left[ B^N 1[\mathcal{M}_N(M, t_0)] \int_{t_0}^t W_N(A_N(s \theta_N)) \, ds \right],
\]

where \( \mathcal{M}_N(M, t_0) = \{ |A_N(t_0 \theta_N)| \leq M \searrow A_N(t_0 \theta_N) \in \mathcal{E}^1_N \cup \bigcup_{k=2}^M \mathcal{G}_N(k, a_N) \} \). Applying the Markov property at time \( t_0 \theta_N \) yields that the absolute value of the previous expectation is bounded by

\[
C(B) \max_{A \in \mathcal{E}^1_N \cup \bigcup_{k=2}^M \mathcal{G}_N(k, a_N)} \left| E^N_A \left[ \int_0^{t-t_0} W_N(A_N(s \theta_N)) \, ds \right] \right|,
\]
where the constant $C(B)$ is an upper bound for $|B^N| : N \in \mathbb{N})$. This expression vanishes as $N \to \infty$ by Lemma 4.4, which completes the proof of the proposition. \hfill \square

4.1 Equilibrium Expectation of Hitting Times

We conclude this section with a result on the equilibrium expectation of hitting times. Let $X_t$ be a reversible, irreducible, continuous-time Markov chain on a finite set $E$. Denote by $\pi$ the unique stationary state and by $H_B, B \subset E$, the hitting time of the set $B$: $H_B = \inf\{t \geq 0 : X_t \in B\}$. Denote by $\mathbb{P}_x$ the distribution of the Markov chain $X_t$ starting from $x$. Expectation with respect to $\mathbb{P}_x$ is represented by $\mathbb{E}_x$. As usual, for a probability measure $\mu$ on $E$, $\mathbb{P}_\mu = \sum_{x \in E} \mu(x) \mathbb{P}_x$.

**Lemma 4.5** For all subsets $B$ of $E$, and all functions $f : E \to \mathbb{R}$,

$$\mathbb{E}_\pi \left[ \int_0^{H_B} f(X_s) \, ds \right] = \sum_{x \in E} \pi(x) \, f(x) \, \mathbb{E}_x[H_B].$$

**Proof** Denote by $(Y_k)_{k \geq 0}$ the skeleton of the chain $X_t$. This is the discrete-time Markov chain which keeps track of the sequence of elements of $E$ visited by the process. Denote by $\lambda(x), x \in E$, the holding rate at $x$. Representing the process $X_t$ in terms of the chain $Y_k$ and independent, mean-one, exponential random variables (cf. Sect. 6 of [2]), the first expectation appearing in the statement of the lemma can be written as

$$\mathbb{E}_\pi \left[ \sum_{k=0}^{h_B-1} \frac{f(Y_k)}{\lambda(Y_k)} \right] = \sum_{k \geq 0} \sum_{x \notin B} \sum_{y \notin B} \pi(x) \frac{f(y)}{\lambda(y)} \mathbb{P}_x[Y_k = y, h_B > k],$$

where $h_B$ stands for the hitting time of the set $B$ by the Markov chain $Y_k: h_B = \min\{j \geq 0 : Y_j \in B\}$. By reversibility, since

$$\pi(x)\lambda(x)\mathbb{P}_x[Y_1 = y] = \pi(y)\lambda(y)\mathbb{P}_y[Y_1 = x], \quad x, y \in E,$$

the previous expression is equal to

$$\sum_{k \geq 0} \sum_{x \notin B} \sum_{y \notin B} \pi(y) \frac{\lambda(x)}{\lambda(x)} \mathbb{P}_y[Y_k = x, h_B > k] = \sum_{y \in E} \pi(y) \, f(y) \, \mathbb{E}_y \left[ \sum_{k=0}^{h_B-1} \frac{1}{\lambda(Y_k)} \right].$$

The last expectation is equal to $\mathbb{E}_y[H_B]$, which completes the proof of the lemma. \hfill \square

**Corollary 4.6** For every $n \geq 2$,

$$\lim_{N \to \infty} \left| \mathbb{E}_\pi^N \left[ \int_0^{\tau_{n-1}} R(A_N(s)) \, ds \right] - \sum_{B \in \mathcal{E}_N^m} \pi_n^N(B) \, \mathbb{E}_B^N[\tau_{n-1}] \, R(B) \right| = 0.$$

**Proof** Let $F : \mathcal{E}_N^{\leq n} \to \mathbb{R}$ be the function given by (4.3), and recall that it is uniformly bounded. The expectation appearing in the statement of the lemma is equal to $E_{\pi_N^N}[F]$. By (3.7) and since $F$ vanishes on $\mathcal{E}_N^m$, $m < n$, and is uniformly bounded, this expectation is equal to $E_{\pi_N^N}[F] + c_N$, where $\lim_N c_N = 0$.

By definition of $F$,

$$E_{\pi_N^N}[F] = \sum_{x \in [\mathcal{E}_N^m]^n} \pi_N^N(x) \, 1\{x \in \mathcal{E}_N^m\} \, \mathbb{E}_x^N \left[ \int_0^{\tau_{n-1}} R(A_N(s)) \, ds \right].$$

\(\square\) Springer
Up to time $\tau_{n-1}$ the evolution of $A_N(s)$ corresponds to the evolution of $n$ independent particles. We may thus replace $A_N(s)$ by $\{x^n_N(s)\}$ inside the expectation, where $\tau_{n-1}$ represents in this context the first time two particles meet. The previous sum is thus equal to

$$\sum_{\mathbf{x} \in [\mathbb{T}^d_N]^n} \pi^{\otimes n}_N(\mathbf{x}) \mathbf{1}\{\mathbf{x} \in \mathcal{E}_N^n\} \tilde{E}_x^N \left[ \int_0^{\tau_{n-1}} R((x^n_N(s))) \, ds \right],$$

where $\tilde{P}_x^N$ represents the distribution of $x^n_N$ starting from $x$.

Since $\tau_{n-1} = 0$ if the process $x^n_N(s)$ starts from a configuration $\mathbf{x}$ such that $\{\mathbf{x}\} \notin \mathcal{E}_N^n$, we may remove the indicator in the previous sum. As the process is reversible and $\pi^{\otimes n}_N$ is its unique stationary state, by Lemma 4.5, the sum is equal to

$$\sum_{\mathbf{x} \in [\mathbb{T}^d_N]^n} \pi^{\otimes n}_N(\mathbf{x}) \tilde{E}_x^N[\tau_{n-1}] R((\mathbf{x})].$$

As $\tau_{n-1} = 0$ if the process $x^n_N(s)$ starts from a configuration $\mathbf{x}$ such that $\{\mathbf{x}\} \notin \mathcal{E}_N^n$, we may restrict the sum to configurations $\mathbf{x}$ such that $\{\mathbf{x}\} \in \mathcal{E}_N^n$. For such a configuration, $\tilde{E}_x^N[\tau_{n-1}] = E_N^{(\mathbf{x})}[\tau_{n-1}]$. Hence, the last sum is equal to

$$\sum_{\mathbf{x} \in [\mathbb{T}^d_N]^n} \pi^{\otimes n}_N(\mathbf{x}) E_N^{(\mathbf{x})} \left[ \tau_{n-1} \right] R((\mathbf{x})) = \sum_{\mathbf{x} \in \mathcal{E}_N^n} E_N^{(\mathbf{x})}[\tau_{n-1}] R(A) \sum_{\mathbf{x} = A} \pi^{\otimes n}_N(\mathbf{x}),$$

where the last sum is performed over all configuration $\mathbf{x} \in [\mathbb{T}^d_N]^n$ such that $\{\mathbf{x}\} = A$. Comparing $\sum_{\mathbf{x} = A} \pi^{\otimes n}_N(\mathbf{x})$ with $\pi^n_N(A)$ yields that the previous sum is equal to

$$(1 + O(N^{-d})) \sum_{A \in \mathcal{E}_N^n} E_N^{(A)}[\tau_{n-1}] R(A) \pi^n_N(A),$$

where $O(N^{-d})$ is a sequence of numbers whose absolute value is bounded by $C_0 N^{-d}$ for some finite constant $C_0$. By Lemma 3.15, the sum converges to 1. In particular, the term $O(N^{-d})$ times the sum is negligible. This completes the proof of the corollary. \qed

### 5 Proof of Theorem 2.2

The proof of Theorem 2.2 is divided in two steps. We show in Lemma 5.3 that the sequence $(\mathcal{P}^N)_N$ is tight, and in Lemma 5.1 that all limit points solve the $(C^1(S), \mathcal{L})$-martingale problem introduced in Proposition 2.1. We prove Proposition 2.1 and, therefore, complete the proof of Theorem 2.2, in Section 6.

Denote by $\mathbb{P}_A^N$, $A \in \mathbb{E}_N$, the probability measure on $D(\mathbb{R}_+, E_N)$ induced by the Markov chain $A_N(t)$ speeded-up by $\theta_N$ starting from $A$. When $A = \mathbb{T}_N^d$, we denote $\mathbb{P}_A^N$ simply by $\mathbb{P}^N$. Expectation with respect to $\mathbb{P}_A^N$, $\mathbb{P}^N$ are represented by $\mathbb{E}_A^N$ and $\mathbb{E}^N$, respectively. Note that

$$\mathcal{P}^N = \mathbb{P}^N \circ \hat{\Psi}^{-1}_N,$$

where $\hat{\Psi}_N : D(\mathbb{R}_+, E_N) \to D(\mathbb{R}_+, S)$ is given by $[\hat{\Psi}_N(\omega)](t) = \Psi_N(\omega(t))$.

In the next lemmata, expectation with respect to $\mathcal{P}^N$, $\mathcal{P}$ are represented by $E_{\mathcal{P}^N}$, $E_{\mathcal{P}}$, respectively.
Lemma 5.1 Let $\mathcal{P}$ be a limit point of the sequence $(\mathcal{P}^N)_{N}$, and let $f : \mathbb{R} \to \mathbb{R}$ be a function in $C^{1}$ which is constant in a neighborhood of the origin: there exists $\delta > 0$ such that $f(x) = f(0)$ for $x \leq \delta$. Then, under $\mathcal{P}$, the process defined by (2.5) is a martingale.

Proof. Assume without loss of generality that $(\mathcal{P}^N)_{N}$ converges to $\mathcal{P}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a function in $C^{1}$ which is constant in a neighborhood of the origin. Denote by $M_{N}(t)$ the $\mathcal{P}^{N}$-martingale given by

$$f(\mathcal{X}_{N}(t)) - f(\mathcal{X}_{N}(0)) - \int_{0}^{t} \theta_{N} L_{N}(f \circ \Psi_{N}) A_{N}(s \theta_{N}) \, ds,$$

where $\mathcal{X}_{N}(t) = \Psi_{N}(A_{N}(t \theta_{N}))$. Since

$$L_{N}(f \circ \Psi_{N})(A) = R(A) \left\{ \frac{w}{1-w} - f(w) \right\},$$

where $w = \Psi_{N}(A)$, and $R(A)$ is the jump rate introduced in (2.9), the martingale $M_{N}(t)$ can be written as

$$f(\mathcal{X}_{N}(t)) - f(\mathcal{X}_{N}(0)) - \theta_{N} \int_{0}^{t} R(A_{N}(s \theta_{N})) \left\{ f\left(\frac{\mathcal{X}_{N}(s)}{1-\mathcal{X}_{N}(s)}\right) - f(\mathcal{X}_{N}(s)) \right\} \, ds.$$

Fix $0 \leq t_{0}$, $k \geq 1$, $0 \leq s_{1} < \cdots < s_{k} \leq t_{0}$, a bounded function $G : \mathbb{R}^{k} \to \mathbb{R}$, and let $B^{N} = G(\mathcal{X}_{N}(s_{1}), \ldots, \mathcal{X}_{N}(s_{k}))$. Since $M_{N}$ is a martingale, for every $t_{0} \leq t$,

$$\mathbb{E}^{N} \left[ B^{N} \{ M_{N}(t) - M_{N}(t_{0}) \} \right] = 0.$$

By Proposition 2.4, in the integral part of the martingale we may replace the rate $\theta_{N} R(A_{N}(s \theta_{N}))$ by $\lambda(|A_{N}(s \theta_{N})|) = r(\mathcal{X}_{N}(s))$ to obtain that

$$\lim_{N \to \infty} \mathbb{E}^{N} \left[ B^{N} \{ \hat{M}_{N}(t) - \hat{M}_{N}(t_{0}) \} \right] = 0 , \quad (5.2)$$

where

$$\hat{M}_{N}(t) = f(\mathcal{X}_{N}(t)) - f(\mathcal{X}_{N}(0)) - \int_{0}^{t} r(\mathcal{X}_{N}(s)) \left\{ f\left(\frac{\mathcal{X}_{N}(s)}{1-\mathcal{X}_{N}(s)}\right) - f(\mathcal{X}_{N}(s)) \right\} \, ds.$$

Notice that the process $\hat{M}_{N}(t)$ is expressed as a function of $\mathcal{X}_{N}$. Therefore, in view of (5.1), we may replace in (5.2) the probability $\mathbb{P}^{N}$ by $\mathcal{P}^{N}$ and write

$$\lim_{N \to \infty} E_{\mathcal{P}^{N}} \left[ B^{N} \{ \hat{M}_{N}(t) - \hat{M}_{N}(t_{0}) \} \right] = 0 ,$$

Since, by assumption, $(\mathcal{P}^{N})_{N}$ converges to $\mathcal{P}$,

$$E_{\mathcal{P}} \left[ B^{N} \{ \hat{M}_{N}(t) - \hat{M}_{N}(t_{0}) \} \right] = 0 .$$

This shows that (2.5) is a martingale under $\mathcal{P}$ and completes the proof of the lemma.

We turn to the tightness of $(\mathcal{P}^{N})_{N}$. Remember that for $w \in D(\mathbb{R}^{n} , S)$, the modified modulus of continuity is defined as

$$\tilde{\omega}(w, t, \delta) := \inf_{\Delta} \max_{k} \sup_{t_{k} \leq s < t_{k+1}} \| w(s) - w(r) \| , \quad t > 0 , \quad \delta > 0 ,$$

where the infimum extends over all partitions $\Delta = \{ 0 = t_{0} < t_{1} < \cdots < t_{\ell} < t \}$ such that $t_{k+1} - t_{k} \geq \delta$ for $k = 1, \ldots, \ell - 1$. It is well known (see for instance [11, Theorem 4.8.1]) that the tightness follows from

\( \square \) Springer
(1) for any $t \in \mathbb{R}_+$, the sequence $\left(\mathbb{X}_N(t)\right)_N$ is tight in $S$; and

(2) for all $\varepsilon > 0$, $t > 0$,

$$\lim_{\delta \to 0} \sup_N \mathbb{P}^N[\tilde{\omega}(\mathbb{X}_N, t, \delta) > \varepsilon] = 0.$$  \hspace{1cm} (5.3)

Since $\mathbb{X}_N(t) \in S$ for all $t \in \mathbb{R}_+$ and $S$ is compact, condition (1) holds immediately thanks to Prohorov’s criterion. Denote by $\sigma_j$, $j \geq 1$, the hitting time of $1/j$: $\sigma_j = \inf\{t \geq 0 : \mathbb{X}(t) = 1/j\}$.

**Lemma 5.2** Condition (2) follows from

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \mathbb{P}^N[\sigma_{j-1} - \sigma_j \leq \delta] = 0, \quad \forall j \geq 2.$$ \hspace{1cm} (5.4)

**Proof** Assume that (5.4) holds, fix $\varepsilon > 0$, $t > 0$, $\eta > 0$ and choose $n \in \mathbb{N}$ such that $1/n \leq \varepsilon$. By Proposition 3.1 and by the Markov inequality

$$\mathbb{P}^N[\mathbb{X}_N(t) \leq 1/n] = \mathbb{P}^N[A_{N(t\theta_N)} \geq n] \leq \frac{\mathbb{E}^N[A_{N(t\theta_N)}]}{n} \leq \frac{C(t, d)}{n},$$

where $C(t, d)$ is a positive constant depending only on $t$ and $d$. Then, increasing $n$ if necessary, we can assume that

$$\mathbb{P}^N[\sigma_n < t] > 1 - \eta/3.$$  

Our assumption implies that there are $\delta_0 > 0$ and $M \in \mathbb{N}$ such that

$$\mathbb{P}^N[\sigma_{j-1} - \sigma_j \geq \delta_0, \text{ for all } j \in \{2, 3, \ldots, n\}] > 1 - \eta/3, \quad \forall N > M.$$  

Let $m := \min\{j \geq 1 : \sigma_j < t\}$. On the set $\{\sigma_n < t\}$, define the random partition $\Delta := \{0 = t_0 < t_1 = \sigma_n < \cdots < t_{\ell} = \sigma_m < t\}$. Since $\mathbb{X}_N(r)$ is constant in the intervals $[\sigma_j, \sigma_{j-1})$, using this partition we deduce that

$$\tilde{\omega}(\mathbb{X}_N, t, \delta) \leq 1/n \leq \varepsilon, \quad \forall \delta < \delta_0, \quad N > M,$$

on the event

$$\{\sigma_n < t\} \cap \{\sigma_{j-1} - \sigma_j \geq \delta_0, \text{ for all } j \in \{2, 3, \ldots, n\}\},$$

that has probability at least $1 - 2\eta/3$. Hence

$$\sup_{N > M} \mathbb{P}^N[\tilde{\omega}(\mathbb{X}_N, t, \delta) > \varepsilon] < 2\eta/3, \quad \forall \delta < \delta_0.$$  

On the other hand, it is clear that there is $\delta_1 > 0$ such that

$$\mathbb{P}^N[\tilde{\omega}(\mathbb{X}_N, t, \delta) > \varepsilon] < \eta/3, \quad N \leq M, \quad \forall \delta < \delta_1.$$  

Therefore

$$\sup_N \mathbb{P}^N[\tilde{\omega}(\mathbb{X}_N, t, \delta) > \varepsilon] < \eta, \quad \forall \delta < \min\{\delta_0, \delta_1\},$$

which completes the proof, since $\eta > 0$ was arbitrary. \hfill \Box

We complete the proof of the tightness in the next lemma.

**Lemma 5.3** The sequence of measures $\left(\mathbb{P}^N\right)_N$ is tight.
Proof By Lemma 5.2 it is enough to show (5.4). In terms of the measure $P_N$, the probability appearing in (5.4) can be rewritten as

$$P_N[\tau_{j-1} - \tau_j \leq \delta \theta_N] .$$

Fix $\epsilon > 0$ and $M > j$. In view of (3.4), choose $\alpha > 0$ small enough for $P_N[\tau_M \leq 3\alpha \theta_N] \leq \epsilon$ for all $N$ sufficiently large. By Proposition 3.1, choose $K \geq M$ such that $P_N[|A_N(\alpha \theta_N)| \geq K] \leq \epsilon$ for all $N$ sufficiently large. Hence, the probability appearing in (5.4) is less than or equal to

$$P_N[|A_N(\alpha \theta_N)| \leq K, \tau_M \geq 3\alpha \theta_N, \tau_{j-1} - \tau_j \leq \delta \theta_N] + 2\epsilon .$$

By Lemma 3.3, this expression is less than or equal to

$$P_N[|A_N(\alpha \theta_N)| \leq K, A_N(2\alpha \theta_N) \in \mathcal{G}_N, \tau_M \geq 3\alpha \theta_N, \tau_{j-1} - \tau_j \leq \delta \theta_N] + 3\epsilon .$$

By the Markov property, this sum is bounded by

$$\max_{M \leq N \leq K} \max_{A \in \mathcal{G}_N(n,\alpha \theta_N)} P_N[\tau_{j-1} - \tau_j \leq \delta \theta_N] + 3\epsilon .$$

By Propositions 3.5, 3.6 and the strong Markov property at time $\tau_j$, the first term of the previous expression vanishes as $N \uparrow \infty$ and $\delta \to 0$. \qed

6 Uniqueness

In order to state the uniqueness result as it has been used in Sect. 5 we need to introduce the subset $D_0 \subseteq C^1(S)$ of functions $f : S \to \mathbb{R}$ which are constant on a neighborhood of zero: $f \in D_0$ if and only if for some $k(f) \in \mathbb{N}$ we have

$$f(0) = f(1/k) , \quad \forall k > k(f) .$$

We shall say that a probability measure $P$ on the measurable space $(\mathcal{D}_0, L)$ (resp. $(C^1(S), L)$)-martingale problem if

$$M^f_t := f(X_t) - \int_0^t (L_f)(X_s) ds , \quad t \geq 0$$

is a $P$-martingale for every $f \in \mathcal{D}_0$ (resp. $f \in C^1(S)$). In addition, we say that $P$ is starting at $x \in S$ whenever $P\{X_0 = x\} = 1$.

6.1 Uniqueness on $S \setminus \{0\}$

For each $k \in \mathbb{N}$, let $P_{1/k}$ be the law on $(\mathcal{D}(\mathbb{R}_+, S), \mathcal{G}_\infty)$ of a Markov process on $S$ starting at $1/k$ and with transition rates

$$q\left(\frac{1}{n}, \frac{1}{n-1}\right) = \binom{n}{2}, \quad \text{for } 2 \leq n \leq k$$

and zero elsewhere. By Dinkyn’s martingales, the process

$$f(X_t) - \int_0^t (L^k f)(X_s) ds , \quad t \geq 0$$

is a $P$-martingale for every $f \in \mathcal{D}_0$ (resp. $f \in C^1(S)$). In addition, we say that $P$ is starting at $x \in S$ whenever $P\{X_0 = x\} = 1$. \qed
is a $\mathcal{P}_{1/k}$-martingale, for all $f : S \to \mathbb{R}$, where

$$\mathcal{L}^k f(x) := \begin{cases} \binom{n}{2} \left( f \left( \frac{1}{n-1} \right) - f \left( \frac{1}{n} \right) \right), & \text{if } x = \frac{1}{n} \in \left[ \frac{1}{k}, \frac{1}{2} \right], \\ 0, & \text{otherwise}. \end{cases}$$

In particular, $\mathcal{P}_{1/k}$ is a solution of the $(\mathcal{D}_0, \mathcal{L}^k)$-martingale problem. Moreover, uniqueness for this problem can be obtained by standard methods so that

**Remark 6.1** For each $k \in \mathbb{N}$, $\mathcal{P}_{1/k}$ is the unique solution of the $(\mathcal{D}_0, \mathcal{L}^k)$-martingale problem starting at $1/k$.

Since $\mathcal{P}_{1/k}\{X_t \geq 1/k, \ \forall \ t \geq 0\} = 1$ and

$$\mathcal{L}^k f(x) = \mathcal{L} f(x), \text{ for all } x \geq 1/k$$

we may then replace $\mathcal{L}^k$ by $\mathcal{L}$ in (6.2). Therefore,

**Remark 6.2** For each $x \in S \setminus \{0\}$, $\mathcal{P}_x$ is a solution of the $(C^1(S), \mathcal{L})$, and so, also the $(\mathcal{D}_0, \mathcal{L})$-martingale problem.

We now prove that, for all $x \in S \setminus \{0\}$, $\mathcal{P}_x$ is actually the unique solution for both martingale problems when starting at $x$. Of course, it is enough to prove this assertion for $(\mathcal{D}_0, \mathcal{L})$. In virtue of Remark 6.1, it suffices to prove that under any such solution $X_t \geq 1/k, \ \forall \ t \geq 0$ almost surely.

**Lemma 6.3** For each $x \in S \setminus \{0\}$, $\mathcal{P}_x$ is the unique solution of the $(\mathcal{D}_0, \mathcal{L})$-martingale problem starting at $x \in S$.

**Proof** Fix some $x = 1/k$ and let $\mathcal{P}$ be a probability satisfying the assumption. Consider the $(\mathcal{G}_t)$-stopping time

$$\tau := \min\{t \geq 0 : X_t < 1/k\}.$$

Since

$$\int_0^{\tau \land \tau} \mathcal{L} f(X_s) ds = \int_0^t \mathcal{L}^k f(X_{s\land \tau}) ds, \ \forall t \geq 0,$$

then

$$f(X_{t \land \tau}) - \int_0^t \mathcal{L}^k f(X_{s\land \tau}) ds, \ \forall t \geq 0$$

is a $\mathcal{P}$-martingale, for any $f \in \mathcal{D}_0$. Equivalently, if $X^\tau : D(\mathbb{R}_+, S) \to D(\mathbb{R}_+, S)$ denotes the measurable map defined by

$$X_t \circ X^\tau = X_{t \land \tau}, \ \forall t \geq 0,$$

then the law of $X^\tau$ under $\mathcal{P}$, denoted by $\mathcal{P} \circ (X^\tau)^{-1}$, turns out to be a solution of the $(\mathcal{D}_0, \mathcal{L})$-martingale problem. By Remark 6.1 we conclude that

$$\mathcal{P} \circ (X^\tau)^{-1} = \mathcal{P}_{1/k},$$

which in turn implies that

$$\mathcal{P}(X_{t \land \tau} \geq 1/k, \ \forall t \geq 0) \equiv \mathcal{P}_{1/k}(X_t \geq 1/k, \ \forall t \geq 0).$$

Since the right hand side above equals one, then $\mathcal{P}(\tau = \infty) = 1$ and so

$$\mathcal{P} \circ (X^\tau)^{-1} = \mathcal{P}.$$ (6.5)

The desired result follows from (6.4) and (6.5).
6.2 A Strong Markov Property

As our next step, we prove Lemma 6.4 below which relates any solution of the \((D_0, \mathcal{L})\)-martingale problem with laws \(\{P_x\}_{x \in S \setminus \{0\}}\) we just introduced.

Let \(\vartheta : \mathbb{R}_+ \times D(\mathbb{R}_+, S) \to D(\mathbb{R}_+, S)\) be the measurable map defined by
\[
X_t \circ \vartheta(s, \cdot) = X_s + t(\cdot), \quad \text{for all } t, s \geq 0.
\]

In addition, given any \((\mathcal{G}_t)\)-stopping time \(\tau\) we define \(\vartheta_\tau : D(\mathbb{R}_+, S) \to D(\mathbb{R}_+, S)\) as
\[
\vartheta_\tau(\omega) := \begin{cases} \vartheta(\tau(\omega), \omega), & \text{if } \tau(\omega) < \infty, \\ \omega, & \text{otherwise}. \end{cases}
\]

Consider the system of neighborhoods of 0 \(\in S\)
\[A_k := \{x \in S : x < 1/k\}, \quad k \in \mathbb{N},\]
and their corresponding exit times
\[\sigma_k := \inf\{t \geq 0 : X_t \in S \setminus A_k\}, \quad k \in \mathbb{N}.\]

Since \(A_k\) and \(S \setminus A_k\) are closed subsets then every \(\sigma_k\) is a stopping time and
\[X_{\sigma_k} \geq 1/k \quad \text{on } \{\sigma_k < \infty\}. \quad (6.6)\]

**Lemma 6.4** Let \(P\) be any solution of the \((D_0, \mathcal{L})\)-martingale problem and let \(k \in \mathbb{N}\). For any \(C \in \mathcal{G}_\infty\), we have
\[
P\{\vartheta_{\sigma_k} \in C, \sigma_k < \infty\} = \int_{\{\sigma_k < \infty\}} P X_{\sigma_k}(\omega)(C) \ P(d\omega). \quad (6.7)
\]
(Recall observation (6.6).)

**Proof** Fix \(k \in \mathbb{N}\) and let \(\{Q_\omega : \omega \in D(\mathbb{R}_+, S)\}\) be a conditional probability distribution of \(P\) given \(\mathcal{G}_{\sigma_k}\) such that for all \(\omega \in D(\mathbb{R}_+, S)\) we have
\[Q_\omega(A) = \delta_\omega(A), \quad \forall A \in \mathcal{G}_{\sigma_k}. \quad (6.8)\]

The existence of such \(\{Q_\omega\}\) is established in [17, Theorem 1.3.4] for a space of continuous paths but the same proof applies for \(D(\mathbb{R}_+, S)\). Taking conditional expectation with respect to \(\mathcal{G}_{\sigma_k}\) in the left hand side below we have
\[
P\{\vartheta_{\sigma_k} \in C, \sigma_k < \infty\} = \int_{\{\sigma_k < \infty\}} Q_\omega(\vartheta_{\sigma_k} \in C) \ P(d\omega). \]

Applying (6.8) we get \(Q_\omega(\sigma_k = \vartheta_k(\omega)) = 1\) for all \(\omega\) and so the right hand side above equals
\[
\int_{\{\sigma_k < \infty\}} Q_\omega(\vartheta_{\sigma_k}(\omega) \in C) \ P(d\omega). \quad (6.9)
\]

Now, we relate \(\{Q_\omega\}\) to \(\{P_x\}_{x \in S \setminus \{0\}}\). For each \(f \in D_0\), we know that the process \((M^f_t)\) defined in (6.1) is a \(P\)-martingale. Then, in virtue of [17, Theorem 1.2.10], for each \(f \in D_0\) there exists some \(A_f \in \mathcal{G}_{\sigma_k}\) with \(P[A_f] = 1\) such that, for all \(\omega \in A_f \cap \{\sigma_k < \infty\},\)
\[M^f_t\) is a \(Q_\omega\)-martingale after time \(\sigma_k(\omega), \quad (6.10)\]
i.e. \( \Omega_\omega[M^f_{t_2} | G_{t_1}] \equiv M^f_{t_1}, \) whenever \( \sigma_k(\omega) \leq t_1 < t_2, \) where \( \Omega_\omega[\cdot \mid \cdot] \) stands for conditional expectation with respect to \( \Omega_\omega. \) It follows from (6.10) that,

\[
(M^f_t) \text{ is a } \Omega_\omega \circ (\vartheta_{\sigma_k(\omega)})^{-1}-\text{martingale}.
\]  

(6.11)

Let us consider the countable subset of \( \tilde{D}_0 \)

\[
\tilde{D}_0 := \{ f \in D_0 : f(x) \text{ is a rational number for all } x \in S \}
\]

and denote \( A := \bigcap_{f \in \tilde{D}_0} A_f. \) Then, (6.11) implies that, for all \( \omega \in A \cap \{ \sigma_k < \infty \}, \)

\( \Omega_\omega \circ (\vartheta_{\sigma_k(\omega)})^{-1} \) is a solution of the \( (\tilde{D}_0, L) \)-martingale problem.

But, given any \( f \in D_0, \exists (f_n) \in \tilde{D}_0 \) such that \( f_n \to f \) and \( Lf_n \to Lf, \) both pointwise, and such that

\[
\sup_{n \geq 1} \max_{x \in S} (|f_n(x)| + |Lf_n(x)|) < \infty.
\]

By using this approximation it is easy to conclude that, for all \( \omega \in A \cap \{ \sigma_k < \infty \}, \)

\( \Omega_\omega \circ (\vartheta_{\sigma_k(\omega)})^{-1} \) is a solution of the \( (D_0, L) \)-martingale problem. (6.12)

On the other hand, for all \( \omega \in \{ \sigma_k < \infty \}, \)

\[
\Omega_\omega \circ (\vartheta_{\sigma_k(\omega)})^{-1}\{X_0 = X_{\sigma_k(\omega)}\} = \Omega_\omega\{X_{\sigma_k(\omega)} = X_{\sigma_k(\omega)}\} = 1
\]

(we applied (6.8) in the last equality.) Namely, for all \( \omega \in \{ \sigma_k < \infty \}, \)

\[
\Omega_\omega \circ (\vartheta_{\sigma_k(\omega)})^{-1} \text{ is starting at } X_{\sigma_k(\omega)} \in S \setminus \{0\} ,
\]

(6.13)

where we used observation (6.6) for the last assertion. We may now conclude from (6.12), (6.13) and the uniqueness result established in Lemma 6.3 that

\[
\Omega_\omega \circ (\vartheta_{\sigma_k(\omega)})^{-1} = \mathcal{P}_{X_{\sigma_k(\omega)}}, \quad \forall \omega \in A \cap \{ \sigma_k < \infty \} .
\]

Since \( \mathcal{P}(A) = 1, \) this last assertion implies that (6.9) equals

\[
\int_{\sigma_k < \infty} \mathcal{P}_{X_{\sigma_k(\omega)}}(C) \mathcal{P}(d\omega) .
\]

This concludes the proof. \( \square \)

### 6.3 A Solution Starting at 0 \( \in S \)

From now on, we shall denote by \( \mathcal{P}_0 \) the law of \( (X_t) \) (defined in (1.3)) so that we have now the complete set of laws \( \{ \mathcal{P}_x : x \in S \}. \) Obviously \( \mathcal{P}_0 \) starts at 0. We prove now that \( \mathcal{P}_0 \) is a solution of the \( (C^1(S), L) \)-martingale problem. Recall the sequence \( (T_n)_{n \geq 2} \) of independent random variables considered in (1.2). For each \( k \in \mathbb{N} \) define the process \( (X^k_t) \) as

\[
X^k_t = \begin{cases} 
1/k, & 0 \leq t < T_k, \\
1/(k-1), & T_k \leq t < T_k + T_{k-1} , \\
\vdots & \vdots \\
1/2, & \sum_{n=3}^k T_n \leq t < \sum_{n=2}^k T_n , \\
1, & t \geq \sum_{n=2}^k T_n .
\end{cases}
\]
for all \( t \geq 0 \). Clearly, the law of \((\mathcal{X}_t^k)\) is \(\mathbb{P}_1/k\). Also, observe that \((\mathcal{X}_t^k)\) is related to \((\mathcal{X}_t)\) by
\[
\mathcal{X}_t^k = \mathcal{X}_{S_k+t}, \quad \forall t \geq 0, \quad \text{where} \quad S_k := \sum_{n=k+1}^{\infty} T_n.
\]
In particular, for all \( t \geq 0 \),
\[
f(\mathcal{X}_t^k) \xrightarrow{a.s.} f(\mathcal{X}_t) \quad \text{and} \quad \mathcal{L} f(\mathcal{X}_t^k) \xrightarrow{a.s.} \mathcal{L} f(\mathcal{X}_t), \quad \text{as} \quad k \uparrow \infty,
\]
for all \( f \in C^1(S) \). In fact, thanks to the right continuity of the trajectories of \((\mathcal{X}_t)\), the previous convergences are pointwise convergences in the probability one event where \(S_1 < +\infty\) and \(T_m > 0\) for all \( m \geq 2 \). Fix an arbitrary \( f \in C^1(S) \), a continuous function \(G: S^m \rightarrow \mathbb{R}\) and a finite set of times \(0 \leq s_1 < \cdots < s_m \leq s < t\). In virtue of Remark 6.2, we have
\[
E \left[ G(\mathcal{X}_{s_1}^k, \ldots, \mathcal{X}_{s_m}^k) \left\{ f(\mathcal{X}_s^k) - f(\mathcal{X}_s) - \int_s^t \mathcal{L} f(\mathcal{X}_r) dr \right\} \right] = 0,
\]
for all \( k \geq 1 \). Letting \( k \uparrow \infty \) in (6.15) and using (6.14) we get
\[
E \left[ G(\mathcal{X}_{s_1}, \ldots, \mathcal{X}_{s_m}) \left\{ f(\mathcal{X}_t) - f(\mathcal{X}_s) - \int_s^t \mathcal{L} f(\mathcal{X}_r) dr \right\} \right] = 0.
\]
We have thus shown that \(\mathbb{P}_0\) is a solution of the \((C^1(S), \mathcal{L})\)-martingale problem.

### 6.4 Uniqueness Starting at 0 \( \in S \)

In this subsection we prove the uniqueness result that we used in Sect. 5. Let \(\sigma\) stand for the exit time from \(0 \in S\), i.e.
\[
\sigma := \inf \{ t \geq 0 : \mathcal{X}_t \neq 0 \}.
\]
Clearly, \(\sigma_k \downarrow \sigma\) pointwise. Notice that \(\sigma\) is not a \((\mathcal{G}_t)\)-stopping time.

**Proposition 6.5** There exists a unique probability measure \(\mathbb{P}\) on \((D(\mathbb{R}^+, S), \mathcal{G}_\infty))\) such that \(\mathbb{P}\{X_0 = 0, \sigma = 0\} = 1\) and
\[
f(\mathcal{X}_t) - \int_0^t \mathcal{L} f(\mathcal{X}_s) ds, \quad t \geq 0
\]
is a \(\mathbb{P}\)-martingale for every \(f \in D_0\).

Existence is, of course, a consequence of Lemma 5.1. Nevertheless, it follows from the conclusion of the previous subsection that \(\mathbb{P}_0\) fulfills all the requirements. In order to show uniqueness we first improve the result obtained in Lemma 6.4.

**Proposition 6.6** Let \(\mathbb{P}\) be a solution of the \((D_0, \mathcal{L})\)-martingale problem starting at \(0 \in S\). If \(\mathbb{P}\{\sigma = 0\} = 1\) then
\[
\mathbb{P}\{\vartheta_{\sigma_k} \in \mathcal{C}\} = \mathbb{P}_{1/k}(\mathcal{C}) , \quad \forall k \geq 1 \text{ and } \mathcal{C} \in \mathcal{G}_\infty.
\]

**Proof** We start showing that
\[
\mathbb{P}\{\sigma_m < \infty, \forall m \in \mathbb{N}\} = 1.
\]
Let us denote
\[
\mathcal{A} := \{\sigma_m < \infty, \forall m \in \mathbb{N}\} = \{\sigma_1 < \infty\}.
\]
Since $P_{1/n}(A) = 1$ for any $n \in \mathbb{N}$ then applying equation (6.7) for $C = A$ and using observation (6.6) we get
\[ P\{\vartheta_{\sigma_k} \in A, \, \sigma_k < \infty\} = P\{\sigma_k < \infty\}, \forall k \in \mathbb{N}. \]

But $\sigma_k + \sigma_1 \circ \vartheta_{\sigma_k} = \sigma_1$ and so $\{\vartheta_{\sigma_k} \in A, \, \sigma_k < \infty\} = A$. Using this observation in the last displayed equation we get
\[ P(A) = P\{\sigma_k < \infty\}, \forall k \in \mathbb{N}. \]

Since $\{\sigma_k < \infty\} \uparrow \{\sigma < \infty\}$ then, letting $k \uparrow \infty$ in the previous equation, we get $P(A) = P\{\sigma < \infty\}$ which equals one by assumption.

As second step, we prove that
\[ P\{X_{\sigma_m} = 1/m, \forall m \in \mathbb{N}\} = 1. \quad (6.19) \]

For it, consider the events
\[ B_n = \{X_0 = 1/n \text{ and } X_{\sigma_m} = 1/m \text{ for all } 1 \leq m \leq n\}, \quad n \in \mathbb{N} \]
and $B := \bigcup_{n \in \mathbb{N}} B_n$. Since $P_{1/n}(B_n) = 1$ for all $n \geq 1$, then, for all $k \in \mathbb{N}$, we have
\[ P_{X_{\sigma_k}(\omega)}(B) = 1, \quad \forall \omega \in \{\sigma_k < \infty\}. \]

Applying (6.7) for $C = B$ along with this last observation we get
\[ P\{\vartheta_{\sigma_k} \in B, \sigma_k < \infty\} = P\{\sigma_k < \infty\} = 1, \forall k \in \mathbb{N}. \]

We used (6.18) in the last equality. Therefore,
\[ P\{\vartheta_{\sigma_k} \in B \text{ and } \sigma_k < \infty, \text{ for all } k \geq 1\} = 1. \quad (6.20) \]

Now (6.19) follows from (6.20), assumption $P\{X_0 = 0, \sigma = 0\} = 1$ and the following observation
\[ \{X_0 = 0, \sigma = 0, \forall k \geq 1, \vartheta_{\sigma_k} \in B, \sigma_k < \infty\} \subseteq \{X_{\sigma_m} = 1/m, \forall m \in \mathbb{N}\} \]
To prove this inclusion, fix some $\omega$ in the event of the left hand side and fix an arbitrary $m' \in \mathbb{N}$. Since $\sigma_k(\omega) \downarrow \sigma(\omega) = 0$ then $X_{\sigma_k}(\omega) \rightarrow X_0(\omega) = 0$ as $k \uparrow \infty$ and so
\[ \exists k' \in \mathbb{N} \text{ such that } X_{\sigma_k'}(\omega) < 1/m'. \quad (6.21) \]

On the other hand, $\vartheta_{\sigma_k}(\omega) \in B$ for all $k \in \mathbb{N}$ and so $\exists n' \in \mathbb{N}$ such that
\[ \vartheta_{\sigma_k}(\omega) \in B_{n'}. \quad (6.22) \]

In virtue of (6.21) and (6.22) we necessarily have
\[ m' < n' \leq k' \]
because
\[ 1/k' \leq X_0 \circ \vartheta_{\sigma_k'}(\omega) \overset{(6.22)}{=} 1/n' = X_0 \circ \vartheta_{\sigma_k'}(\omega) \overset{(6.21)}{<} 1/m'. \]

From (6.22) it follows that
\[ X_{\sigma_m} \circ \vartheta_{\sigma_k'}(\omega) = 1/m, \forall 1 \leq m \leq n'. \]

Since $m' < n'$ in particular we have
\[ X_{\sigma_m'} \circ \vartheta_{\sigma_k'}(\omega) = 1/m'. \]
But $X_{\sigma^m} \circ \partial_{\sigma^k}(\omega) = X_{\sigma^m}(\omega)$ since $m' < k'$ and so $X_{\sigma^m}(\omega) = 1/m'$. This concludes the proof of the desired inclusion.

Finally, the desired result follows from (6.19) and (6.7).

\[ \qed \]

**Proof of Proposition 6.5** Let $\mathcal{P}$ be a probability satisfying the stated assumptions and let $E$ and $E_{1/k}$ stand for expectation with respect to $\mathcal{P}$ and $\mathcal{P}_{1/k}$ respectively. Fix an arbitrary $n \in \mathbb{N}$ some $0 \leq t_1 < t_2 < \cdots < t_n$ and a bounded continuous function $F : S^n \to \mathbb{R}$. In virtue of Proposition 6.6 we have

$$E\left[F(\sigma_{\sigma^k+t_1}, \ldots, \sigma_{\sigma^k+t_n})\right] = E_{1/k}\left[F(\sigma_{t_1}, \ldots, \sigma_{t_n})\right], \quad \forall k \in \mathbb{N}.$$ 

But $(\sigma_{\sigma^k+t_1}, \ldots, \sigma_{\sigma^k+t_n}) \Rightarrow (\sigma_{t_1}, \ldots, \sigma_{t_n})$ $\mathcal{P}$-a.s. as $k \to \infty$, because of (line above (6.21)), and so

$$E\left[F(\sigma_{t_1}, \ldots, \sigma_{t_n})\right] = \lim_{k \to \infty} E_{1/k}\left[F(\sigma_{t_1}, \ldots, \sigma_{t_n})\right].$$

This guarantees the desired uniqueness.

\[ \qed \]

### 6.5 Proof of Proposition 2.1

In light of Remark 6.2 and Lemma 6.3, in order to conclude the proof of Proposition 2.1, it remains to prove that $\mathcal{P}_0$ is the unique solution of the $(C^1(S), \mathcal{L})$-martingale problem starting at $0 \in S$.

Observe that $f, g \in C^1(S)$ for all $f, g \in C^1(S)$. We shall make use of the *carré du champ* corresponding to $(C^1(S), \mathcal{L})$:

$$\Gamma(f, g) := \mathcal{L}(fg) - g\mathcal{L}f - f\mathcal{L}g,$$

for every $f, g \in C^1(S)$.

Clearly, $\Gamma(f, g)$ turns out to be continuous for each $f, g \in C^1(S)$. Since $\mathcal{L}$ acts as a derivative at $0 \in S$ we have

$$\Gamma(f, g)(0) = 0, \quad \forall f, g \in C^1(S). \quad (6.23)$$

Recall definition of $(M^f_t)$ given in (6.1) for each $f \in C^1(S)$.

**Lemma 6.7** Let $\mathcal{P}$ be any solution of the $(C^1(S), \mathcal{L})$-martingale problem. For all $f, g \in C^1(S)$, the process

$$M^f_tM^g_t - \int_0^t \Gamma(f, g)(X_s)ds, \quad t \geq 0,$$

is a $\mathcal{P}$-martingale with respect to $(\mathcal{G}_t)$.

**Proof** Fix some $f, g \in C^1(S)$. Denote

$$V^f_t := \int_0^t \mathcal{L}f(X_s)ds \quad \text{and} \quad V^g_t := \int_0^t \mathcal{L}g(X_s)ds, \quad t \geq 0,$$

so that, for all $t \geq 0$,

$$M^f_t + V^f_t = f(X_t) \quad \text{and} \quad M^g_t + V^g_t = g(X_t).$$

By multiplying these equalities we get

$$M^f_tM^g_t + V^f_tV^g_t + M^f_tV^g_t + V^f_tM^g_t = (fg)(X_t). \quad (6.24)$$

By using

$$(fg)(X_t) = M^f_tV^g_t + \int_0^t \mathcal{L}(fg)(X_s)ds, \quad t \geq 0,$$

\[ \square \]
along with
\[ V_t^f V_t^g = \int_0^t V_s^f dV_s^g + \int_0^t V_s^g dV_s^f, \quad t \geq 0 , \]
in equality (6.24) we get
\[
M_t^f M_t^g + M_t^f V_t^g + V_t^f M_t^g \\
= M_t^{fg} + \int_0^f \mathcal{L}(fg)(X_s)ds - \int_0^t V_s^f dV_s^g - \int_0^t V_s^g dV_s^f.
\] (6.25)
If we denote, for all \( t \geq 0 \),
\[
M_1^f := M_t^f V_t^g - \int_0^f M_s^f dV_s^g \quad \text{and} \quad M_2^g := M_t^g V_t^f - \int_0^f M_s^g dV_s^f,
\] (6.26)
then equality (6.25) can be rewritten as
\[
M_t^f M_t^g + M_1^f + M_2^g = M_t^{fg} + \int_0^t \Gamma(f, g)(X_s)ds.
\] (6.27)
By assumption, \((M_t^{fg})\) is a \( \mathcal{P} \)-martingale. In addition, in virtue of [17, Theorem 1.2.8], \((M_1^f)\) and \((M_2^g)\) are also \( \mathcal{P} \)-martingales. Therefore the desired result follows from (6.27).
\(\square\)

We now use observation (6.23) to prove that \(0 \in S\) is an instantaneous state for any solution starting at 0.

**Lemma 6.8** For any solution \( \mathcal{P} \) of the \((C^1(S), \mathcal{L})\)-martingale problem starting at 0 \( \in S \) we have \( \mathcal{P} \{ \sigma = 0 \} = 1 \).

*Proof* Let \( \mathcal{P} \) be a probability satisfying the assumptions. Define \( f : S \to \mathbb{R} \) as the inclusion function i.e. \( f(x) = x \), for \( x \in S \). Clearly \( f \in C^1(S) \) and so
\[
M_t := X_t - \int_0^f \mathcal{L}(f)(X_s)ds, \quad t \geq 0
\] (6.28)
is a \( \mathcal{P} \)-martingale. Since \( \sigma_k \) is a stopping time then it follows from Lemma 6.7 that
\[
(M_t \wedge \sigma_k)^2 - \int_{0}^{t \wedge \sigma_k} \Gamma(f, f)(X_s)ds, \quad t \geq 0 ,
\]
is a \( \mathcal{P} \)-martingale. In particular, for all \( t \geq 0 \) we have
\[
E \left[ (M_t \wedge \sigma_k)^2 \right] = E \left[ \int_0^{t \wedge \sigma_k} \Gamma(f, f)(X_s)ds \right], \quad \forall k \in \mathbb{N} ,
\] (6.29)
(since \( M_0 = 0 \), \( \mathcal{P} \)-a.s.) where \( E \) represents the expectation with respect to \( \mathcal{P} \). By the bounded convergence theorem, letting \( k \uparrow \infty \) in (6.29) we get
\[
E \left[ (M_t \wedge \sigma)^2 \right] = E \left[ \int_0^{t \wedge \sigma} \Gamma(f, f)(X_s)ds \right], \quad \forall t \geq 0 .
\] (6.30)
Since \( \{s < \sigma\} \subseteq \{X_s = 0\} \), the right hand side in the above equation equals
\[
E[t \wedge \sigma] \Gamma(f, f)(0)
\]
which vanishes as noticed in observation (6.23). Therefore, from (6.30) we conclude that
\[
\mathcal{P} \{ M_t \wedge \sigma = 0 , \; \forall t \geq 0 \} = 1 .
\]
Using this fact in (6.28) we get that, \( \mathcal{P} \)-a.s.,

\[ X_{t \land \sigma} = \int_0^{t \land \sigma} (\mathcal{L} f)(X_s) \, ds = \frac{1}{2} (t \land \sigma), \quad \forall t \geq 0. \]

But, for any \( t > 0 \), we have on \( \{ t < \sigma \} \) that

\[ X_{t \land \sigma} = X_t = 0 \neq \frac{1}{2} (t \land \sigma). \]

Hence \( \mathcal{P} \{ t < \sigma \} = 0, \forall t > 0 \) and we are done. \( \square \)

It follows from Lemma 6.8 and Proposition 6.5 that \( \mathcal{P}_0 \) is the only solution of the \((C^1(S), \mathcal{L})\)-martingale problem.

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