Thermodynamic stability of a cosmological SU(2)-weak gauge field

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Abstract

The CGF cosmology is a complete theory of cosmology from the electroweak transition onward. It is semi-classical. At leading order the only matter is dark matter — a cosmological SU(2)-weak gauge field (the CGF). Ordinary matter is a subleading correction from fluctuations around the classical state. The CGF is periodic in imaginary time. It acts as thermal bath for the fluctuations of the Standard Model fields. Here, the initial thermal state of the SU(2) gauge field fluctuations is constructed and shown to be thermodynamically stable. This is a warm-up for (1) constructing the initial thermal state of all the fluctuations in order to calculate its time evolution and (2) showing that initial state to be thermodynamically stable in order to show that the CGF cosmology is physically natural.

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1 Introduction

The CGF cosmology is a complete theory of the Standard Model cosmological epoch, from the electroweak transition onward [1, 2, 3]. The theory has no free parameters and assumes no physical laws beyond the Standard Model and General Relativity. All of cosmology is given by the time evolution of a uniquely determined highly symmetric semi-classical initial state in the period leading up to the electroweak transition. The CGF universe in the leading order, classical approximation contains only a cosmological SU(2)-weak gauge field (the CGF). The CGF is the dark matter. The relatively small amount of ordinary matter in the universe is a higher order correction to the dark matter universe from the fluctuations of the Standard Model fields around the classical CGF.

The CGF cosmology is completely determined by four assumptions.

1. The universe is governed by the Standard Model and General Relativity (with cosmological constant). Nothing beyond the Standard Model, nothing beyond the known laws of physics, is assumed.

2. The universe is a 3-sphere.

3. The state of the universe is invariant under a Spin(4) symmetry group that acts on the 3-sphere as SO(4) and on the Standard Model fields such that the SU(2)-weak doublets transform as spinors.

4. The initial energy in the Standard Model fields is $> 10^{107}$ in natural units.
The Spin(4) symmetry and the initial energy completely determine the classical initial condition. The only nontrivial Spin(4)-symmetric Standard Model field is the SU(2)-weak gauge field, the CGF. It is a classical solution of the Yang-Mills equation of motion given by an elliptic function of the complex time. It is periodic in real time, oscillating anharmonically in the quartic Yang-Mills action. It is also periodic in imaginary time. The periodicity in imaginary time defines a temperature. The initial state of the fluctuations is given by correlation functions that respect the imaginary time periodicity. The CGF acts as a thermal bath for the fluctuations of the Standard Model fields.

Here, the initial thermal state of the SU(2)-weak gauge field fluctuations is constructed. This is done by constructing the 2-point function in the gaussian approximation. The \(n\)-point functions are Wick contractions of the 2-point function. Then the thermal state is shown to be thermodynamically stable by a combination of mathematical proof and numerical evidence.

The space of fluctuations of the SU(2) gauge field around the classical solution is decomposed under the Spin(4) symmetry group. The fluctuations in each irreducible representation satisfy, in the gaussian approximation, an equation of motion that is a linear ordinary differential equation (ode) second order in the time variable. The coefficients of the ode are analytic in the time variable. A certain property \(P\) of the ode is proved to imply the existence of a canonical change of variables mapping the thermal state of the gauge field fluctuations to an equilibrium state of an ordinary harmonic oscillator. So property \(P\) implies thermodynamic stability. Strong numerical evidence is given that property \(P\) holds and thus that the initial thermal state of the gauge field fluctuations is thermodynamically stable. Thermodynamic stability means that the cosmological initial condition is robust against small fluctuations of the SU(2) gauge field. Calculations are shown in the Supplemental Material [4]. The numerical calculations are done in SageMath [5] using the mpmath arbitrary-precision floating-point arithmetic library [6].

## 2 Spin(4)-symmetric CGF

Space is a 3-sphere. Let \(S^3\) be the unit 3-sphere in \(\mathbb{R}^4\) with O(4)-symmetric metric \(\hat{g}_{ij}(\hat{x})\) and volume 3-form \(\hat{\epsilon}_{ijk}(\hat{x})\). Let \(\gamma_i(\hat{x})\) be the Spin(4)-symmetric Dirac matrices on \(S^3\) and let \(\hat{\nabla}_i\) be the Spin(4)-symmetric covariant derivative on spinors (and tensors).

\[
\hat{\gamma}_i \hat{\gamma}_j = -\frac{1}{4} \hat{g}_{ij} - \frac{1}{2} \hat{\epsilon}_{ijk} \gamma_k \quad \hat{\nabla}_i \hat{\gamma}_j = 0 \quad [\hat{\nabla}_i, \hat{\nabla}_j] = \epsilon_{ijk} \hat{\gamma}_k
\]  
(2.1)

The O(4)-symmetric space-time metric (in \(c = 1\) units) is

\[
ds^2 = R(\hat{t})^2 (-d\hat{t}^2 + \hat{g}_{ij}(\hat{x}) d\hat{x}^i d\hat{x}^j) \]

\(\hat{t}\) is conformal cosmological time. \(R(\hat{t})\) is the radius of the spatial 3-sphere at conformal time \(\hat{t}\). Co-moving time \(t\) is given by \(dt = R(\hat{t}) d\hat{t}\).

The SU(2)-weak gauge bundle over the 3-sphere is identified with the spinor bundle. The general Spin(4)-symmetric gauge field in unitary gauge is

\[
D_0 = \partial \hat{t} \quad D_i = \hat{\nabla}_i + \hat{b}(\hat{t}) \hat{\gamma}_i
\]

\[
F_{0j} = [D_0, D_j] = \frac{db}{d\hat{t}} \hat{\gamma}_j \quad F_{ij} = [D_i, D_j] = (1 - \hat{b}^2) \hat{\epsilon}_{ijk} \hat{\gamma}_k
\]  
(2.3)
The Yang-Mills action is
\[
\frac{1}{\hbar} S_{\text{gauge}} = \int \frac{1}{2g^2} \text{tr}(-F_{\mu\nu}F^{\mu\nu}) \sqrt{-g} \, d^4x
\]  
(2.4)

\( g \) is the SU(2) coupling constant of the Standard Model. The Yang-Mills action is conformally invariant so is independent of the space-time scale \( \hat{R} \). The action of the Spin(4)-symmetric gauge field is
\[
\frac{1}{\hbar} S_{\text{gauge}} = \text{Vol}(S^3) \frac{3}{g^2} \int \left[ -\frac{1}{2} \left( \frac{d\hat{b}}{dt} \right)^2 + \frac{1}{2} (\hat{b}^2 - 1)^2 \right] dt \quad \text{Vol}(S^3) = 2\pi^2
\]  
(2.5)

The equation of motion is
\[
\frac{d^2\hat{b}}{dt^2} + 2\hat{b} (\hat{b}^2 - 1) = 0
\]  
(2.6)

The dimensionless energy
\[
E_{\text{CGF}} = \frac{1}{2} \left( \frac{d\hat{b}}{dt} \right)^2 + \frac{1}{2} (\hat{b}^2 - 1)^2
\]  
(2.7)

is conserved. The solution (up to time translation) is the elliptic function
\[
\hat{b}(\tau) = \frac{1}{\epsilon} k \text{cn}(z, k) \quad E_{\text{CGF}} = \frac{1}{8\epsilon^4} \quad z = \frac{1}{\epsilon} \hat{\tau} \quad k^2 = \frac{1}{2} + \epsilon^2
\]  
(2.8)

This is the initial cosmological gauge field, prior to the electroweak transition. The observed flatness of the present universe requires \( E_{\text{CGF}} > 10^{107}, \epsilon < 10^{-27} \) [2].

3 The Jacobi elliptic function \( \text{cn}(z, k) \)

\( \text{cn}(z, k) \) is a Jacobi elliptic function [7, Chapter 22], [8, 8.14-15]. It is analytic in \( z \) (with poles). It satisfies
\[
\text{cn}'' = (k^2 - k'^2) \text{cn} - 2k^2 \text{cn}^3 \quad \text{cn}'^2 = (1 - \text{cn}^2)(k'^2 + k^2 \text{cn}^2)
\]  
\[
k^2 + k'^2 = 1
\]  
(3.1)

\( k^2 \) is called the parameter, \( k'^2 \) the complementary parameter, \( k \) and \( k' \) the modulus and complementary modulus. We assume \( 0 < k, k' < 1 \). We are especially interested in \( k^2 = \frac{1}{2} + \epsilon^2, k'^2 = \frac{1}{2} - \epsilon^2 \) with \( \epsilon \ll 1 \). The Taylor series at 0 is
\[
\text{cn}(z) = 1 - \frac{1}{2} z^2 + O(z^4)
\]  
(3.2)

The reflection symmetries are
\[
\text{cn}(\bar{z}) = \overline{\text{cn}(z)} \quad \text{cn}(-z) = \text{cn}(z)
\]  
(3.3)

\( \text{cn}(z, k) \) is doubly periodic in \( z \).
\[
\text{cn}(z) = \text{cn}(z + 4K) = \text{cn}(z + 4K'i) = \text{cn}(z + 2K + 2K'i)
\]  
\[
K = K(k) \quad K' = K(k')
\]  
(3.4)
\( K \) and \( K' \) are the complete elliptic integrals of the first kind. The half-periods are

\[
\text{cn}(z, k) = \text{cn}(z + 2K, k) = \text{cn}(z + 2K', k) = -\text{cn}(z, k) \tag{3.5}
\]

The poles and residues are

\[
\text{cn}(z, k) \sim \frac{(-1)^{m+n+1}i^{k-1}}{z - z_{m,n}} \quad z_{m,n} = 2mK + (2n + 1)K'i \quad m, n \in \mathbb{Z} \tag{3.6}
\]

The poles are shown in Figure 1. The zeros are located by the identity

\[
\text{cn}(z + K + K'i) = -\frac{iK'k^{-1}}{\text{cn}(z)} \tag{3.7}
\]

4 Periodicity in imaginary time as temperature

The CGF \( \dot{b}(\hat{t}) \) oscillates anharmonically with period \( \Delta \hat{t} = 4K \epsilon \). The period in comoving time is \( \Delta t = 4K \epsilon R(\hat{t}) \). The CGF is also periodic in imaginary time with period \( \Delta t = 4K' \epsilon R(\hat{t})i \), defining a temperature \( T_{\text{CGF}}(\hat{t}) \) by

\[
\frac{\hbar}{k_B T_{\text{CGF}}(\hat{t})} = 4K' \epsilon R(\hat{t}) \tag{4.1}
\]

The initial state of the fluctuations is determined by the periodicity in imaginary time. In the path integral formulation of the Standard Model, the CGF is a classical trajectory in the phase space of SU(2) gauge theory. The classical trajectory analytically continues to an analytic trajectory in the complexified phase space. The initial thermal state of the fluctuations is defined by the path integral over the paths periodic in imaginary time.

This is a familiar picture when the classical real time trajectory is invariant under time translation. Here the classical CGF is not invariant under time translation. Moreover, the analytic continuation in complex time is obstructed by poles of the classical trajectory. It must be proved that the thermal state of the fluctuations is independent of the choice of periodic path in imaginary time over which the functional integral is performed.
5 Stability

After the CGF cosmological initial condition was proposed in [1], the stability of the CGF was investigated in [9]. Stability is a crucial physical requirement. An instability would render implausible that the initial condition could result from earlier cosmological developments. A stable initial condition is robust against small fluctuations — the essential condition of physical naturalness. However, the stability investigated in [9] was stability of the classical gauge field under small classical perturbations. Classical stability is not the relevant physical stability condition for the cosmological initial condition. The initial condition is a semi-classical thermal quantum state, i.e. a state concentrated near a classical trajectory that is periodic in imaginary time. The physical stability condition is thermodynamic stability. The thermal state has to be constructed and then shown to be thermodynamically stable.

Thermodynamic stability is the condition that the gaussian approximation to the path integral should be well defined — the quadratic approximation to the imaginary time action should be bounded below. Equivalently, the gaussian approximation to the quantum mechanical density matrix should be positive definite. Here, thermodynamic stability is shown by constructing a canonical transformation to an equilibrium thermal state of an ordinary harmonic oscillator.

6 Quadratic term in the action

The gauge field fluctuations are the perturbations \( B_i(\hat{x}) \) of the classical solution,
\[
\hat{D}_i = \hat{\nabla}_i + \hat{b}\gamma_i + B_i, \quad B_i(\hat{x}) = B_i^t(\hat{x})\gamma_j(\hat{x})
\]
modulo the infinitesimal gauge transformations
\[
B_i^\text{gauge} = \hat{\nabla}_i v + \hat{b}[\gamma_i, v] \quad v(\hat{x}) = v^i(\hat{x})\gamma_j(\hat{x})
\]
The gaussian path integral is constructed from the quadratic term in the action (2.4). Change time variable from conformal time \( \hat{t} = \frac{\hat{t}}{\epsilon} \)
\[
\hat{b}(\hat{t}) = b(z) \quad b(z) = k \text{cn}(z, k)
\]
and define two linear operators on the perturbations,
\[
\Gamma B_i = \epsilon_i^{jk}[\gamma_j, B_k] \quad *\hat{\nabla}B_i = \epsilon_i^{jk}\hat{\nabla}_jB_k
\]
After some algebra, the quadratic term in the action is written [4]
\[
\frac{1}{\hbar}S_2 = \frac{1}{\epsilon g^2} \int -\text{tr} \left( -\partial_{\hat{t}}B^i\partial_{\hat{t}}B_i + B^iK(z)B_i \right) \sqrt{-g}d^3\hat{x} \, dz
\]
\[
K(z) = b(z)^2K_2 + b(z)K_1 + K_0
\]
\[
K_2 = \Gamma^2 - \Gamma \quad \frac{1}{\epsilon}K_1 = \Gamma(*\hat{\nabla}) + (*\hat{\nabla})\Gamma \quad \frac{1}{\epsilon^2}K_0 = (*\hat{\nabla})^2 + \Gamma
\]
The operators \( \Gamma \) and \( *\hat{\nabla} \) are Spin(4)-invariant. When the space of perturbations \( B_i^t(\hat{x}) \) is decomposed under Spin(4), the operator \( K(z) \) becomes block diagonal. The gaussian quantum field theory becomes a discrete sum of finite quantum mechanical systems.
Identify $S^3$ with the group SU(2). Then Spin(4) is SU(2)$_L \times$SU(2)$_R$ acting by left and right multiplication on SU(2). Write the irreducible representations of SU(2) in the usual way $j = 0, 1/2, 1, 3/2, \ldots$ with dim$(j) = 2j + 1$. The irreducible representations of Spin(4) are the tensor products $(j_L, j_R)$ of an SU(2)$_L$ irreducible with an SU(2)$_R$ irreducible. The space of functions on $S^3$ decomposes under Spin(4) as the representation $\bigoplus j (j,j)$. Identify the tangent and cotangent spaces of $S^3$ with the Lie algebra of SU(2)$_L$ which is the representation $(1,0)$. The space of gauge field fluctuations decomposes as
\[
\{ B^j_l(\hat{x}) \} = \bigoplus_{j_R} (1 \otimes 1 \otimes j_R, j_R) = \bigoplus_{j_L,j_R} \mathbb{C}^{N(j_L,j_R)} \otimes (j_L,j_R) \tag{8.1}
\]
The multiplicity $N(j_L,j_R)$ is 0, 1, 2, or 3. The space of infinitesimal gauge transformations decomposes as
\[
\{ v^j_l(\hat{x}) \} = \bigoplus_{j_R} (1 \otimes j_R, j_R) = \bigoplus_{j_L,j_R} \mathbb{C}^{N_{\text{gauge}}(j_L,j_R)} \otimes (j_L,j_R) \tag{8.2}
\]
The multiplicity $N_{\text{gauge}}(j_L,j_R)$ is 0 or 1. The multiplicity of physical degrees of freedom is $N_{\text{phys}} = N - N_{\text{gauge}}$. The representations with $N_{\text{phys}} > 0$ fall into the five subsets listed in Table 1.

### Table 1: The irreducible representations with $N_{\text{phys}} > 0$. 

| $j_L - j_R$ | $(j_L, j_R)$ | $N$ | $N_{\text{gauge}}$ | $N_{\text{phys}}$ |
|-------------|--------------|-----|-------------------|-----------------|
| 1           | (0, 0)       | 1   | 0                 | 1               |
| 2           | ($\frac{1}{2}, \frac{1}{2}$) | 2   | 1                 | 1               |
| 3$_j$       | ($j - \frac{1}{2}, j - \frac{1}{2}$) | \[\frac{3}{2} \leq j\] | 3   | 1 | 2 |
| 2$_j$       | ($j, j - 1$) | \[\frac{3}{2} \leq j\] | 2   | 1 | 1 |
|             | ($-j - 1, -j$) | \[j \leq -\frac{3}{2}\] | 2   | 1 | 1 |
| 1$_j$       | ($j + \frac{1}{2}, j - \frac{3}{2}$) | \[\frac{3}{2} \leq j\] | 1   | 0 | 1 |
|             | ($-j - \frac{3}{2}, -j + \frac{1}{2}$) | \[j \leq -\frac{3}{2}\] | 1   | 0 | 1 |

8 Five ODEs

The fluctuation $B^j_l(\hat{x})$ breaks up into a sum of degrees of freedom in the irreducible representations.
\[
B = \sum_{j_L, j_R} C_{j_L,j_R} q_{j_L,j_R} \quad q_{j_L,j_R} \in \mathbb{C}^{N(j_L,j_R)} \otimes (j_L,j_R) \tag{8.1}
\]
The normalization constants $C_{j_L,j_R}$ are chosen so that
\[
\frac{1}{\epsilon g^2} \int -\text{tr} \left( B^l B_l \right) \sqrt{-g} d^3 \hat{x} = \sum_{j_L,j_R} \frac{1}{2} q^l_{j_L,j_R} q_{j_L,j_R} \tag{8.2}
\]
Each $q_{j_L,j_R}$ is an independent degrees of freedom in the quadratic action (6.5) governed by an action

$$\frac{1}{\hbar}S = \int dz \left( -\frac{1}{2} \frac{dq}{dz} \frac{dq}{dz} + \frac{1}{2} q(z) K(z) q(z) \right) \quad K(z)^t = K(z)$$

(8.3)

where $K(z)$ is an $N \times N$ matrix acting on the factor $\mathbb{C}^N$ in (8.1). $K(z)$ is a symmetric matrix because the linear operator $K(z)$ in the quadratic Yang-Mills action (6.5) is symmetric. The equation of motion is the ode

$$\frac{d^2 q}{dz^2} + K(z) q(z) = 0$$

(8.4)

When $N_{\text{gauge}} = 1$ there will be a nonzero solution

$$\left( \frac{d^2}{dz^2} + K(z) \right) w_{\text{gauge}}(z) = 0 \quad w_{\text{gauge}}(z) \in \mathbb{C}^N$$

(8.5)

The matrix $K(z)$ and gauge solution $w_{\text{gauge}}(z)$ are shown below for the five sets of representations in Table 1. Their calculation is shown in [4]. The method is:

1. The expansion of the triple tensor product in (7.1) is written

$$j_1 \otimes j_2 \otimes j_3 = \bigoplus_j \mathbb{C}^{N(j)} \otimes J \quad j_1 = j_2 = 1 \quad j_3 = j_R$$

(8.6)

The Spin(4)-invariant operators $\hat{\nabla}$ and $\Gamma$ defined in (6.4) are expressed as

$$\Gamma = C_{12} - 2 \quad *\hat{\nabla} - \Gamma = 2C_{23} - j_3(j_3 + 1)$$

(8.7)

where $C_{12}$ is the Casimir operator on the factor $j_1 \otimes j_2$, $C_{23}$ the Casimir on $j_2 \otimes j_3$, and $C_3$ the Casimir on $j_3$.

2. $C_{12}$ and $C_{23}$ act as matrices on $\mathbb{C}^{N(j)}$ but they do not commute. Each can be diagonalized in a canonical basis, but not simultaneously. The unitary matrix $U$ that translates between the two diagonalizations is given by Wigner 6-j symbols or Racah W-coefficients.

3. The Casimir eigenvalues and the matrix $U$ are combined to find the matrices $K(z)$.

4. The gauge solution $w_{\text{gauge}}(z)$ is the unique solution of (8.5) linear in $b(z)$.

The local physics in the CGF cosmology is expressed in terms of the scale $a = \epsilon R$. The physical length scale of a fluctuation in the representation $(j_L,j_R)$ is $a/p$ where

$$p = 2(C_{j_L} + C_{j_R})^{1/2} \epsilon$$

(8.8)

The eigenvalue of the dimensionful physical laplacian on fluctuations is

$$\frac{p^2}{a^2} = \frac{4(C_{j_L} + C_{j_R})}{R^2}$$

(8.9)

The numerical data for the series $3j$, $2j$, $1j$ is parametrized by $\epsilon$, $p$ instead of $\epsilon$, $j$. 

8
ode 1 \( N = 1, \ N_{\text{gauge}} = 0 \quad (j_L, j_R) = (0, 0) \)
\[
K(z) = 6b(z)^2 - 2\epsilon^2 \tag{8.10}
\]

ode 2 \( N = 2, \ N_{\text{gauge}} = 1 \quad (j_L, j_R) = (\frac{1}{2}, \frac{1}{2}) \quad \sigma = \frac{\epsilon}{\sqrt{2}} \)
\[
K(z) = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} b(z)^2 + \begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix} \sigma b(z) + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} 2\sigma^2
\]
\[
w_{\text{gauge}}(z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} b(z) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sigma \tag{8.11}
\]

ode 3, \( N = 3, \ N_{\text{gauge}} = 1 \quad (j_L, j_R) = (j - \frac{1}{2}, j - \frac{1}{2}) \quad j \geq \frac{3}{2} \)
\[
p = \epsilon \sqrt{4j^2 - 1} \quad \sigma = \epsilon \sqrt{\frac{2}{3}} \sqrt{j^2 - \frac{1}{4}} = \frac{p}{\sqrt{6}} \quad \alpha = \sqrt{2} \sqrt{\frac{j^2 - \frac{1}{4}}{j^2 + \frac{1}{4}}}
\]
\[
K(z) = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} b(z)^2 + \begin{pmatrix} 0 & -6 & 0 \\ -6 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sigma b(z) + \begin{pmatrix} \alpha^2 & 0 & \alpha \\ 0 & \alpha^2 + 1 & 0 \\ \alpha & 0 & 1 \end{pmatrix} 2\sigma^2 \tag{8.12}
\]
\[
w_{\text{gauge}}(z) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} b(z) + \begin{pmatrix} 1 \\ 0 \\ -\alpha \end{pmatrix} \sigma
\]

ode 2, \( N = 2, \ N_{\text{gauge}} = 1 \quad (j_L, j_R) = \begin{cases} (j, j - 1) & j \geq \frac{3}{2} \\ (j - 1, -j) & j \leq -\frac{3}{2} \end{cases} \)
\[
p = 2j\epsilon \quad \sigma = j\epsilon = \frac{p}{2} \quad \alpha = \sqrt{1 - \frac{1}{j^2}}
\]
\[
K(z) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} b(z)^2 + \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} \sigma b(z) + \begin{pmatrix} \alpha^2 & \alpha \\ \alpha & 1 \end{pmatrix} 2\sigma^2 \tag{8.13}
\]
\[
w_{\text{gauge}}(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} b(z) + \begin{pmatrix} 1 \\ -\alpha \end{pmatrix} \sigma
ode 1

\[ N = 1, \quad N_{\text{gauge}} = 0 \quad (j_L, j_R) = \begin{cases} 
(j + \frac{1}{2}, j - \frac{3}{2}) & j \geq \frac{3}{2} \\
(-j - \frac{3}{2}, -j + \frac{1}{2}) & j \leq -\frac{3}{2}
\end{cases} \]

\[ p = \epsilon \sqrt{4j^2 + 3} \quad \sigma = j \epsilon = \frac{\sqrt{p^2 - 3\epsilon^2}}{2} \quad \alpha = \sqrt{1 + \frac{1}{4j^2}} \]

\[ K(z) = 4\sigma b(z) + 4\alpha^2 \sigma^2 \tag{8.14} \]

9 Time-translation zero-mode

First consider the Spin(4)-invariant sector \((j_L, j_R) = (0, 0)\). The \((0, 0)\) perturbation is governed by ode 1. The infinitesimal time translation is a zero-mode,

\[ \left( \frac{d^2}{dz^2} + K(z) \right) b'(z) = 0 \tag{9.1} \]

Time translation is an exact symmetry of the Yang-Mills theory, so the path integral over the \((0, 0)\) perturbations must extend to a path integral over all the time translations of \(b(z)\). These form a circle — the periodic imaginary time trajectory.

Change variable,

\[ \tilde{q}(z) = \frac{q(z)}{b'(z)} \tag{9.2} \]

The equation of motion and action become

\[ \frac{1}{b'(z) \frac{d}{dz}} \left( b'(z)^2 \frac{d\tilde{q}}{dz} \right) = 0 \quad S = \int dz \left( -b'(z)^2 \left( \frac{d\tilde{q}}{dz} \right)^2 \right) \tag{9.3} \]

Now change time variable,

\[ \frac{d\tilde{z}}{dz} = \frac{1}{b'(z)^2} \tag{9.4} \]

The equation of motion and action become

\[ \frac{d^2\tilde{q}}{d\tilde{z}^2} = 0 \quad S = \int d\tilde{z} \left[ -\left( \frac{d\tilde{q}}{d\tilde{z}} \right)^2 \right] \tag{9.5} \]

The Spin(4)-invariant fluctuations are equivalent, in the gaussian approximation, to the fluctuations of an equilibrium thermal state of a free particle moving in a circle. This is a thermodynamically stable state.

10 Classical mechanics analytic in complex time \(z\)

For the sectors \((j_L, j_R) \neq (0, 0)\) transverse to the zero-mode, a formalism is developed for the quantum mechanical path integral for a quadratic hamiltonian \(H(z)\) that is analytic in the time \(z\). The formalism is used to construct the thermal state on the fluctuation degrees of freedom \(q(z)\). Thermodynamic stability implies a certain “property \(P\)” of the matrix-valued function \(K(z)\). Conversely, when property \(P\) is satisfied
q(z) becomes canonically equivalent to an ordinary harmonic oscillator at finite temperature, which is thermodynamically stable. Property P is equivalent to thermodynamically stability. Property P is then verified numerically for each of the four remaining ODEs.

10.1 First-order phase-space formalism

Introduce the phase-space degree of freedom

\[ Q = \begin{pmatrix} q \\ p \end{pmatrix} \]  

a rank \(2N\) block vector. \(p\) is the momentum conjugate to \(q\). The classical equation of motion becomes the first-order differential equation

\[ \frac{dQ}{dz} + A(z)Q(z) = 0 \quad A(z) = \begin{pmatrix} 0 & -1 \\ K(z) & 0 \end{pmatrix} \]  

i.e.,

\[ \frac{dq}{dz} = p(z) \quad \frac{dp}{dz} = -K(z)q(z) \]  

The first-order differential equation (10.2) has locally analytic solutions away from the poles of \(K(z)\). Global solutions are multi-valued in \(z\) because of monodromy around the poles.

10.2 Path-dependent classical propagator

Let \(C\) be a path in the complex \(z\) plane avoiding the poles. For points \(z_1, z_2\) on \(C\), the path-dependent classical propagator \(P_C(z_2, z_1)\) is the integral of

\[ \left( \frac{d}{dz} + A(z) \right) P_C(z, z_1) = 0 \quad P_C(z_1, z_1) = 1 \]  

along the path from \(z_1\) to \(z_2\). The propagator \(P_C(z_2, z_1)\) is a \(2N \times 2N\) complex matrix that depends only on the homotopy class of the path from \(z_1\) to \(z_2\). For any three points \(z_1, z_2, z_3\) on \(C\),

\[ P_C(z_3, z_1) = P_C(z_3, z_2)P_C(z_2, z_1) \quad P_C(z_2, z_1)^{-1} = P_C(z_1, z_2) \]  

\(K(z) = K(z)^t\) implies

\[ A(z)^t \Omega + \Omega A(z) = 0 \quad \Omega = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]  

so the propagator is a complex symplectic matrix

\[ P_C(z_2, z_1)^t \Omega P_C(z_2, z_1) = \Omega \quad P_C(z_2, z_1) \in \text{Sp}(2N, \mathbb{C}) \]  

The solution of the equation of motion (10.2) along \(C\) is

\[ Q(z) = P_C(z, z_1)Q(z_1) \]
10.3 Gauge symmetry

When there is a gauge solution \( w_{\text{gauge}}(z) \in \mathbb{C}^N \) then

\[
W_{\text{gauge}}(z) = \begin{pmatrix} w_{\text{gauge}}(z) \\ w'_{\text{gauge}}(z) \end{pmatrix}
\] (10.9)

is a solution of the first-order equation of motion (10.2). The \( \Omega \)-complement \( W_{\text{gauge}}(z)^\perp \) is the subspace of \( \mathbb{C}^{2N} \)

\[
W_{\text{gauge}}(z)^\perp = \{ W \in \mathbb{C}^{2N} : W_{\text{gauge}}(z)^t \Omega W = 0 \} \] (10.10)

\( \Omega \) is antisymmetric so

\[
W_{\text{gauge}}(z) \in W_{\text{gauge}}(z)^\perp \] (10.11)

The physical phase-space at time \( z \) is the quotient space

\[
V_{\text{phys}}(z) = W_{\text{gauge}}(z)^\perp / \mathbb{C}W_{\text{gauge}}(z) \] (10.12)

The physical degrees of freedom live in \( V_{\text{phys}}(z) \otimes (j_L, j_R) \). The propagator preserves \( W_{\text{gauge}}(z) \) and it preserves \( \Omega \),

\[
P_C(z_2, z_1)^t \Omega P_C(z_2, z_1) = \Omega \quad W_{\text{gauge}}(z_2) = P_C(z_2, z_1)W_{\text{gauge}}(z_1) \] (10.13)

so the propagator acts as a linear map from \( V_{\text{phys}}(z_1) \) to \( V_{\text{phys}}(z_2) \),

\[
P_C(z_2, z_1) : V_{\text{phys}}(z_1) \rightarrow V_{\text{phys}}(z_2) \] (10.14)

11 Quantum mechanics analytic in complex time \( z \)

Quantization makes \( q \) and \( p \) operators on Hilbert space, i.e. rank \( N \) vectors whose entries are operators. The canonical commutation relations

\[
[p_b, q^a] = i \delta^a_b \quad [q_b, q^a] = 0 \quad [p_b, p^a] = 0 \] (11.1)

are expressed by the matrix equation

\[
(QQ')^t - QQ' = \Omega \] (11.2)

where the matrix transpose does not change the operator ordering. For example,

\[
(p q^t)^t \delta_{ab}^a - (q p^t)^t \delta_{ab}^a = p_b q^a - q^a p_b = [p_b, q^a] = i \delta^a_b \] (11.3)

11.1 Path-dependent time evolution

The hamiltonian depends analytically on the complex time \( z \).

\[
H(z) = \frac{1}{2} \dot{b}^i p_i + \frac{1}{2} \dot{q}^i K(z) q^i \] (11.4)

The time evolution of the state vector \( \psi(z) \) is given by the Schrödinger equation.

\[
\frac{d\psi}{dz} = iH(z)\psi(z) \] (11.5)
Again, for \( z_1, z_2 \) two points on a path \( C \) in the complex \( z \) plane avoiding the poles, the path-dependent time evolution operator \( U_C(z_2, z_1) \) is constructed by integrating

\[
\left( \frac{d}{dz} - iH(z) \right) U_C(z, z_1) = 0 \quad U_C(z_1, z_1) = 1
\]

along the path from \( z_1 \) to \( z_2 \). Again, \( U_C(z_2, z_1) \) depends only on the homotopy class of the path from \( z_1 \) to \( z_2 \). For any three points \( z_1, z_2, z_3 \) on \( C \),

\[
U_C(z_3, z_1) = U_C(z_3, z_2)U_C(z_2, z_1) \quad U_C(z_2, z_1)^{-1} = U_C(z_1, z_2)
\]

(11.7)

The solution \( \psi(z) \) of the Schrödinger equation along \( C \) is

\[
\psi(z) = U_C(z, z_1)\psi(z_1)
\]

(11.8)

If \( H(z) \) were constant then \( U_C(z_2, z_1) \) would be independent of the path

\[
U_C(z_2, z_1) = e^{i(z_2-z_1)H}
\]

(11.9)

### 11.2 Path-dependent path integral

For \( C \) a path from \( z_1 \) to \( z_2 \) the time evolution operator along \( C \) is given by the phase-space path integral

\[
U_C(z_2, z_1) = \int \mathcal{D}Q \ e^{iS_C[Q]/\hbar}
\]

(11.10)

using the phase-space action

\[
\frac{1}{\hbar} S_C[Q] = \int_C dz \ \frac{1}{2i} Q' \Omega \left( \frac{d}{dz} + A \right) Q = \int_C dz \ \left( -p(z)^t \frac{dq}{dz} + H(z) \right)
\]

(11.11)

Integrating out \( p(z) \) gives the ordinary path-integral with the action (8.3)

\[
\int \mathcal{D}p \ e^{iS_C[Q]/\hbar} = e^{iS_C[q]/\hbar}
\]

(11.12)

### 11.3 Operator insertions

Suppose \( C \) is a path from \( z_1 \) to \( z_2 \). For \( z \) in \( C \), the insertion of \( Q(z) \) in the path integral is

\[
\int \mathcal{D}Q \ e^{iS_C[Q]/\hbar} Q(z) = U_C(z_2, z) Q U_C(z, z_1)
\]

(11.13)

The Schwinger-Dyson equation of the path integral is the equation of motion

\[
\left( \frac{d}{dz} + A(z) \right) \int \mathcal{D}Q \ e^{iS_C[Q]/\hbar} Q(z) = 0
\]

(11.14)
Equivalently,
\[
\left( \frac{d}{dz} + A(z) \right) \left[ U_C(z_2, z) Q U_C(z, z_1) \right] = \\
= U_C(z_2, z) \left( - [iH(z), Q] + A(z) Q \right) U_C(z, z_1) = 0
\]  
(11.15)

Integrating the equation of motion along \( C \),
\[
\int \mathcal{D}Q \, e^{iS_c[Q]/\hbar} \, Q(z') = \mathcal{P}_C(z', z) \int \mathcal{D}Q \, e^{iS_c[Q]/\hbar} \, Q(z)
\]  
(11.16)

12 Imaginary time path integral

Choose a real time \( t \) not in the set \( 2KZ \), i.e. such that \( 2mK < t < 2(m+1)K \) for some integer \( m \). Let \( C_t \) be the straight vertical path through \( t \). \( C_t \) avoids the poles since \( t \neq 2mK \). Let \( C'_t \) be the path from \( t \) to \( t + 4K'i \). The expectation values in the thermal state are constructed in two steps.

1. Perform the gaussian path integral on the path \( C'_t \) with periodic boundary conditions, \( Q(t + 4K'i) = Q(t) \).

2. Then integrate over the time-translation zero-mode.

The result will be independent of the choice of \( t \) (shown in section 19.3 below). Suppose the first step produces a stable state. Then the underlying time translation symmetry implies that all the time translated gaussian integrals are also stable. The second step is then an integral of stable gaussian integrals, so the resulting thermal state is stable. Stability of the thermal state follows from stability of the gaussian integral produced in step 1.

The two-point expectation values determine all the gaussian expectation values. The gaussian two-point expectation values are expressed by the matrix
\[
\mathcal{G}_t(z_2, z_1) = \langle Q(z_2) \, Q(z_1) \rangle_t
\]
\[ z_1 = t + \tau_1 i \quad z_2 = t + \tau_2 i \quad 0 \leq \tau_1 \leq \tau_2 \leq 4K'
\]  
(12.1)

The expectation value \( \langle \cdot \rangle_t \) is given by the path integral with periodic boundary conditions \( Q(t + 4K'i) = Q(t) \) which gives the operator trace.
\[
\langle Q(z_2) \, Q(z_1) \rangle_t = \frac{1}{Z_t} \int \mathcal{D}Q \, e^{iS_{C'_t}[Q]/\hbar} \, Q(z_2) \, Q(z_1)
\]
\[ = \frac{1}{Z_t} \text{tr} \left[ U_{C'_t}(t + 4K'i,z_2) Q U_{C'_t}(z_2,z_1) Q' U_{C'_t}(z_1,t) \right]
\]  
(12.2)

\( Z_t \) is the normalizing constant
\[
Z_t = \int \mathcal{D}Q \, e^{iS_{C'_t}[Q]/\hbar} = \text{tr} \, U_{C'_t}(t + 4K'i,t)
\]  
(12.3)
The expectation values obey the equation of motion in both variables,
\[ G_t(z_2, z_1) = \mathcal{P}_C(t, t) G_t(z_1, t) \mathcal{P}_C(t, t) \]  
so the gaussian expectation values are all determined by one matrix, \( G(t, t) \).

Periodicity is the condition
\[ G_t(t + 4K'i, z_1) = G_t(z_1, t) \]  
By (12.4) this is
\[ \mathcal{M}_i(t) G_t(t, t) = G_t(t, t) \mathcal{M}_i(t) = \mathcal{P}_C(t + 4K'i, t) \]  
\( \mathcal{M}_i(t) \) is the imaginary period monodromy matrix at time \( t \).

The canonical commutation relations (11.2) imply
\[ G_t(t, t) - G_t(t, t) = \Omega \]  
Combining (12.6) and (12.7) gives
\[ (\mathcal{M}_i(t) - 1) G_t(t, t) = \Omega \]  
Define
\[ \mathcal{H}(t) = \Omega (\mathcal{M}_i(t) - 1) \]  
\( \mathcal{H}(t) \) is a well defined quadratic form on the physical phase space \( \mathcal{V}_{\text{phys}}(t) \) because, if there is a gauge solution \( \mathcal{W}_{\text{gauge}}(z) \), then it has the same periodicities as \( b(z) \). In particular, \( \mathcal{W}_{\text{gauge}}(t + 4K'i) = \mathcal{W}_{\text{gauge}}(t) \) so \( (\mathcal{M}_i(t) - 1) \mathcal{W}_{\text{gauge}}(t) = 0 \). If \( \mathcal{H}(t) \) is invertible on \( \mathcal{V}_{\text{phys}}(t) \),
\[ G_t(t, t) = \mathcal{H}(t)^{-1} \]  
If \( \mathcal{H}(t) \) fails to be invertible on \( \mathcal{V}_{\text{phys}}(t) \), then there is a solution of the equation of motion that is periodic in imaginary time modulo gauge symmetry. This would be an accidental physical zero-mode. It would be necessary to go beyond the gaussian approximation to test for stability.

Here the strong meaning of stability is taken, stability in the gaussian approximation, allowing neither instabilities nor accidental zero modes.

### 13 Complex conjugation and operator adjoints

Use the following notation for complex conjugation and the operator adjoint:
\[ \bar{z} = \text{the complex conjugate of a complex number } z \]
\[ \bar{O} = \text{the adjoint of an operator } O \]
\[ M^\dagger = \bar{M}^t = \text{the adjoint of a matrix } M \text{ of operators or complex numbers} \]
\( b(z) \) is real on the real axis, \( \bar{b}(z) = b(\bar{z}) \), so \( K(z) = K(z)^\dagger \) is real on the real axis
\[ \bar{K}(z) = K(\bar{z}) = K(z)^\dagger \]  
The representations \((j_L, j_R)\) in the decomposition are all real, \( j_L - j_R \in \mathbb{Z} \), so
\[ \bar{q} = q \quad \bar{p} = q \quad \bar{Q} = Q \quad Q^\dagger = Q^t \]  
15
so the hamiltonian is real (self-adjoint) on the real axis
\[ \overline{H(z)} = H(\bar{z}) = H(z)^\dagger \] (13.3)

So the time evolution operator given by (11.6) satisfies, for any path \( C \),
\[ \overline{U_C(z_2, z_1)} = U_C(z_2, z_1)^\dagger = U_{\bar{C}}(\bar{z}_2, \bar{z}_1) = U_{\bar{C}}(\bar{z}_1, \bar{z}_2) \] (13.4)

\( \mathcal{A}(z) \) is real on the real axis because \( K(z) \) is,
\[ \overline{\mathcal{A}(z)} = \mathcal{A}(\bar{z}) \] (13.5)

Therefore, for any path \( C \),
\[ \overline{P_C(z_2, z_1)} = P_C(\bar{z}_2, \bar{z}_1) \] (13.6)

In particular, for \( C_t \) the vertical path through \( t \),
\[ \overline{P_{C_t}(z_2, z_1)} = P_{C_t}(\bar{z}_2, \bar{z}_1) = P_{C_t}(\bar{z}_2, \bar{z}_1) \] (13.7)

Imaginary time periodicity implies
\[ P_{C_t}(z_2, z_1) = P_{C_t}(z_2 + 4K' i, z_1 + 4K' i) \] (13.8)

so the imaginary period monodromy matrix satisfies
\[ \overline{M_i(t)} = P_{C_t}(t + 4K' i, t) = P_{C_t}(t - 4K' i, t) = P(t, t + 4K' i) = M_i(t)^{-1} \] (13.9)

Then, by the symplectic property (10.7),
\[ M_i(t)^\dagger \Omega = \Omega M_i(t) \] (13.10)

It follows that \( H(t) \) defined in (12.9) is hermitian.
\[ H(t)^\dagger = (M_i(t)^\dagger - 1) \Omega = (\Omega M_i(t) - \Omega) = H(t) \] (13.11)

### 14 Property P

14.1 Heisenberg picture

Write \( \mathbb{R} \) for the path along the real time axis. The real time evolution operator is unitary
\[ U_{\mathbb{R}}(t_2, t_1)^\dagger = U_{\mathbb{R}}(t_2, t_1)^{-1} = U_{\mathbb{R}}(t_1, t_2) \] (14.1)

The Heisenberg picture operator \( Q(t) \) is determined by
\[ \psi_2^\dagger(t) Q \psi_1(t) = \psi_2^\dagger(0) Q(t) \psi_1(0) \] (14.2)

for arbitrary states \( \psi_{1,2}(0) \). So
\[ Q(t) = U_{\mathbb{R}}(t, 0)^\dagger Q U_{\mathbb{R}}(t, 0) \] (14.3)

and
\[ Q(t)^\dagger = U_{\mathbb{R}}(t, 0)^\dagger Q^t U_{\mathbb{R}}(t, 0) = Q(t)^\dagger \] (14.4)
14.2 Stability requires Property P

The matrix of operators
\[ Q(t) \mathcal{Q}(t)^\dagger = \mathcal{Q}(t) Q(t)^\dagger \quad (14.5) \]
is hermitian and positive so the expectation value in a stable state must be hermitian and positive.
\[ \mathcal{G}_t(t, t) = \langle \mathcal{Q}(t) \mathcal{Q}(t)^\dagger \rangle_t = \mathcal{G}_t(t, t)^\dagger \quad \mathcal{G}_t(t, t) > 0 \quad (14.6) \]
so \( \mathcal{G}_t(t, t)^{-1} = \mathcal{H}(t) \) must be hermitian and positive
\[ \mathcal{H}(t)^\dagger = \mathcal{H}(t) \quad \mathcal{H}(t) > 0 \quad (14.7) \]
\( \mathcal{H}(t) \) is given by equation (12.9). It has already been shown to be hermitian, equation (13.11). Stability requires

Property P \( \mathcal{H}(t) = \Omega(\mathcal{M}_i(t) - 1) > 0 \) \( (14.8) \)
as hermitian form on the physical phase space \( \mathcal{V}_{\text{phys}}(t) \).

15 Path-independent time evolution in region \( R_m \)

For each integer \( m \) form the region \( R_m \subset \mathbb{C} \) shown in Figure 2, cutting out from the complex plane a set of half-infinite horizontal lines containing all the poles,
\[ R_m = \mathbb{C} - \bigcup_{n \in \mathbb{Z}} \{ t + (2n + 1)K' i : (2m + 1)K \leq t \} \]
\[ - \bigcup_{n \in \mathbb{Z}} \{ t + (2n + 1)K' i : t \leq 2mK \} \quad (15.1) \]
\( R_m \) is simply connected so time evolution within \( R_m \) is path-independent:
\[ U_m(z_2, z_1) = U_C(z_2, z_1) \quad z_1, z_2 \in R_m \quad C \subset R_m \]
\[ \mathcal{P}_m(z_2, z_1) = \mathcal{P}_C(z_2, z_1) \quad \mathcal{P}_m(z_2, z_1) = \mathcal{P}_C(z_2, z_1) \quad z_1, z_2 \in R_m \quad C \subset R_m \quad (15.2) \]
Solutions of the equation of motion are single-valued in \( R_m \) but are discontinuous across the cuts. The path-independent propagator in \( R_m \) satisfies
\[
\mathcal{P}_m(z, z) = 1 \quad \mathcal{P}_m(z_2, z_1)^{-1} = \mathcal{P}_m(z_1, z_2)
\]
\[
\mathcal{P}_m(z_3, z_2)\mathcal{P}_m(z_2, z_1) = \mathcal{P}_m(z_3, z_1)
\]
(15.3)
Translation by \( 4K'i \) takes \( R_m \) to \( R_m \), Translation by \( 2K + 2K'i \) takes \( R_m \) to \( R_{m+1} \). Translation by \( 4K \) takes \( R_m \) to \( R_{m+2} \). So the periodicities imply
\[
\mathcal{P}_m(z_2, z_1) = \mathcal{P}_m(z_2 + 4K'i, z_1 + 4K'i)
\]
\[= \mathcal{P}_{m+1}(z_2 + 2K + 2K'i, z_1 + 2K + 2K'i) \]
\[= \mathcal{P}_{m+2}(z_2 + 4K, z_1 + 4K) \]
(15.4)
\[
\mathcal{P}_m(z_2, z_1) = \mathcal{P}_m(z_2, z_1)
\]
(15.5)

16 Property P is independent of \( t \)

Now prove property \( P \) is independent of \( t \). That is, prove that \( \mathcal{P}(t) \) satisfies property \( P \) for all \( t \in \mathbb{R} - 2K\mathbb{Z} \) iff it satisfies property \( P \) for any \( t \in \mathbb{R} - 2K\mathbb{Z} \). So property \( P \) is a property of the ode, of the matrix function \( K(z) \). The proof is in two steps. Suppose
\[
2mK < t < 2(m + 1)K \quad m \in \mathbb{Z}
\]
(16.1)

1. If \( t' \) is in the same vertical strip, \( 2mK < t' < 2(m + 1)K \), then the imaginary period monodromy matrices
\[
\mathcal{M}_i(t) = \mathcal{P}_m(t + 4K'i, t) \quad \mathcal{M}_i(t') = \mathcal{P}_m(t' + 4K'i, t')
\]
(16.2)
are related by
\[
\mathcal{M}_i(t') = \mathcal{P}_m(t, t')^{-1}\mathcal{M}_i(t)\mathcal{P}_m(t, t')
\]
(16.3)
\( \mathcal{P}_m(t, t') \) is a real matrix so
\[
\mathcal{H}(t') = \Omega(\mathcal{M}_i(t') - 1) = \Omega\mathcal{P}_m(t, t')^{-1}(\mathcal{M}_i(t) - 1)\mathcal{P}_m(t, t')
\]
\[= \mathcal{P}_m(t, t')^{-1}\Omega(\mathcal{M}_i(t) - 1)\mathcal{P}_m(t, t')
\]
\[= \mathcal{P}_m(t, t')^{-1}\mathcal{H}(t)\mathcal{P}_m(t, t')
\]
(16.4)
Therefore \( \mathcal{H}(t') > 0 \) iff \( \mathcal{H}(t) > 0 \).

2. Suppose \( t' = t + 2K \). Then
\[
\mathcal{M}_i(t) = \mathcal{P}_m(t + 4K'i, t) = \mathcal{P}_m(t + 4K'i, t + 2K'i)\mathcal{P}_m(t + 2K'i, t)
\]
\[= \mathcal{P}_m(t, t - 2K'i)\mathcal{P}_m(t + 2K'i, t)
\]
\[
\mathcal{M}_i(t') = \mathcal{P}_{m+1}(t' + 4K'i, t') = \mathcal{P}_m(t + 2K + 4K'i, t + 2K)
\]
\[= \mathcal{P}_m(t + 2K'i, t - 2K'i)
\]
\[= \mathcal{P}_m(t + 2K'i, t)\mathcal{P}_m(t, t - 2K'i)
\]
(16.5)
so
\[ M_i(t') - 1 = \mathcal{P}_m(t - 2K'i, t)\mathcal{P}_m(t, t + 2K'i) - 1 \]
\[ = \mathcal{P}_m(t - 2K'i, t) (1 - M_i(t)) \mathcal{P}_m(t, t + 2K'i) \]
\[ = \mathcal{P}_m(t, t + 2K'i) ^\dagger \mathcal{H}(t) \mathcal{P}_m(t, t + 2K'i) \tag{16.6} \]

so
\[ \overline{\mathcal{H}(t')} = \Omega \left( M_i(t') - 1 \right) \]
\[ = -\Omega \mathcal{P}_m(t - 2K'i, t) (1 - M_i(t)) \mathcal{P}_m(t, t + 2K'i) \]
\[ = \mathcal{P}_m(t, t + 2K'i) ^\dagger \Omega (M_i(t) - 1) \mathcal{P}_m(t, t + 2K'i) \]
\[ = \mathcal{P}_m(t, t + 2K'i) ^\dagger \mathcal{H}(t) \mathcal{P}_m(t, t + 2K'i) \]
\[ \mathcal{H}(t') > 0 \iff \overline{\mathcal{H}(t')} > 0 \] so \( \mathcal{H}(t') > 0 \iff \mathcal{H}(t) > 0 \).

Together the two steps imply that Property \( P \) is independent of \( t \).

17 Property \( P \) implies stability

Now Property \( P \) is proved to imply existence of a canonical equivalence between the imaginary time path integral and that of an ordinary time-independent harmonic oscillator, which is the finite temperature equilibrium state of the ordinary harmonic oscillator, which is manifestly stable.

17.1 Spectrum of \( M_i(t) \)

Again suppose \( t \) is in the vertical strip \( 2mK < t < 2(m + 1)K \). Leave implicit the dependence on \( t \).

\[ \mathcal{P}(z) = \mathcal{P}_m(z, t) \quad M_i = M_i(t) = \mathcal{P}(4K'i) \quad \mathcal{H} = \mathcal{H}(t) \quad \mathcal{V}_{phys} = \mathcal{V}_{phys}(t) \tag{17.1} \]

Suppose Property \( P \) is satisfied, so \( \mathcal{H} \) is a positive definite hermitian form on \( \mathcal{V}_{phys} \). \( M_i \) is self-adjoint with respect to the positive definite hermitian form \( \mathcal{H} \) by (13.10).

\[ M_i ^\dagger \mathcal{H} = \Omega M_i \Omega (M_i - 1) = \Omega M_i (M_i - 1) \mathcal{H} = \mathcal{H} M_i \tag{17.2} \]

Therefore there is a \( \mathcal{H} \)-orthonormal basis of \( \mathcal{V}_{phys} \) consisting of eigenvectors \( W_a^\prime \) of \( M_i \) with real eigenvalues \( \lambda_a \)

\[ M_i W_a^\prime = \lambda_a W_a^\prime \quad W_a^\prime ^\dagger \mathcal{H} W_a^\prime = \delta_{a,b} \]
\[ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2N_{phys}} \tag{17.3} \]

\( \mathcal{V}_{phys} \) decomposes into \( \mathcal{H} \)-orthogonal eigenspaces \( \mathcal{V}_\lambda \)

\[ \mathcal{V}_{phys} = \bigoplus_\lambda \mathcal{V}_\lambda \tag{17.4} \]

None of the eigenvalues \( \lambda_a \) can equal 1 because \( M_i - 1 \) is invertible on \( \mathcal{V}_{phys} \).

The complex conjugate of the eigenvalue equation,

\[ \overline{M_i} W_a^\prime = \overline{\lambda_a} W_a^\prime \tag{17.5} \]
Define a linear operator

\[ \mathcal{H} = \mathcal{H} b \]

so \( b \) is a positive hermitian form so the last equation implies all \( \lambda_a > 0 \).

If \( 1 / \lambda_a \neq \lambda_b \) then \( \mathcal{W}_b \) is \( \mathcal{H} \)-orthogonal to \( \mathcal{W}_b' \).

\[ 0 = \mathcal{W}_a^\dagger \mathcal{H} \mathcal{W}_b = \mathcal{W}_a^\dagger \Omega (\lambda_a - 1) \mathcal{W}_b \]

and \( \lambda_b \neq 1 \) so

\[ \lambda_a \lambda_b \neq 1 \implies \mathcal{W}_a^\dagger \mathcal{W}_b = 0 \]

Let

\[ \mathcal{V}_\text{phys}^+ = \bigoplus_{\lambda > 1} \mathcal{V}_\lambda \quad \mathcal{V}_\text{phys}^- = \bigoplus_{\lambda < 1} \mathcal{V}_\lambda \]

\[ \dim \mathcal{V}_\text{phys}^+ = \dim \mathcal{V}_\text{phys}^- = N_{\text{phys}} \]

Define a linear operator \( \omega \) on \( \mathcal{W}_a^+ \) by letting \( a \) range over \( \{1, \ldots, N_{\text{phys}}\} \), i.e. the \( \lambda_a > 1 \), and letting

\[ \mathcal{W}_a^+ = \sqrt{\lambda_a - 1} \mathcal{W}_a \quad \mathcal{W}_a^- = \mathcal{W}_a^\dagger = \mathcal{W}_a^- \quad \omega a = \frac{1}{4K'} \ln \lambda_a \quad \omega a^+ = \omega a \mathcal{W}_a^+ \]

The \( \mathcal{W}_a^+ \) form a basis for \( \mathcal{V}_\text{phys}^+ \) and the \( \mathcal{W}_a^- \) form a basis for \( \mathcal{V}_\text{phys}^- \). The linear operator \( \omega \) on \( \mathcal{W}_a^+ \) is diagonal with eigenvalues \( \omega_a \) in this basis. The operator \( \mathcal{W}_a^\dagger = \omega_a \mathcal{W}_a^+ \) in this basis, writing \( \beta = 4K' \),

\[ \Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \mathcal{M}_1 = \begin{pmatrix} e^{\beta \omega} & 0 \\ 0 & e^{-\beta \omega} \end{pmatrix} \quad \mathcal{H} = \begin{pmatrix} 0 & 1 - e^{-\beta \omega} \\ e^{\beta \omega} - 1 & 0 \end{pmatrix} \]

These are identical to the \( \Omega \), \( \mathcal{M}_1 \), and \( \mathcal{H} \) for an ordinary time-independent harmonic oscillator with frequencies \( \omega_a \) in equilibrium at inverse temperature \( \beta \) in the phase space basis of creation and destruction operators \( \mathcal{V}_\text{phys}^+ \) and \( \mathcal{V}_\text{phys}^- \).

The spectrum of frequencies \( \omega_a \) is independent of \( t \) by the same arguments that gave the \( t \) independence of property \( \mathbf{P} \).
17.2 Equivalence to a time-independent oscillator

Property $P$ is now used to construct a canonical equivalence between $Q(z)$ and an ordinary time-independent harmonic oscillator. Let

$$J = \frac{i \ln \mathcal{M}_i}{4K'}$$

$$\mathcal{J}W_a^+ = i \omega_a W_a^+ \quad \mathcal{J}W_a^- = -i \omega_a W_a^-$$

so $\mathcal{J}$ is real

$$\overline{\mathcal{J}W_a} = \mathcal{J} \overline{W_a}$$

And

$$0 = \mathcal{J}^\dagger \mathcal{H} + \mathcal{H} \mathcal{J} = \mathcal{J}^\dagger \Omega (\mathcal{M}_i - 1) + \Omega (\mathcal{M}_i - 1) \mathcal{J}$$

so $\mathcal{J}$ is an infinitesimal symplectic transformation.

Define

$$\mathcal{R}(z) = e^{-z \mathcal{J}} \mathcal{P}(z)^{-1} = \mathcal{M}_i^{z/4K'i} \mathcal{P}(z)^{-1}$$

Recall that $\mathcal{P}(z) = \mathcal{P}_m(z,t)$. $\mathcal{R}(z)$ has three essential properties:

1. $\mathcal{R}(z)$ is symplectic.

$$\mathcal{R}(z)\dagger \Omega \mathcal{R}(z) = \Omega$$

2. $\mathcal{R}(z)$ is periodic in imaginary time.

$$\mathcal{R}(z + 4K'i) = \mathcal{R}(z)$$

3. $\mathcal{R}(z)$ is real on the real axis.

$$\overline{\mathcal{R}(z)} = e^{-z \mathcal{J}} \overline{\mathcal{P}(z)^{-1}} = e^{-z \mathcal{J}} \mathcal{P}(\bar{z})^{-1} = \mathcal{R}(\bar{z})$$

Make the canonical transformation

$$Q_{\text{HO}}(z) = \begin{pmatrix} q_{\text{HO}} \\ p_{\text{HO}} \end{pmatrix}(z) = \mathcal{R}(z) \mathcal{Q}(z)$$

The first-order equation of motion becomes time-independent.

$$\left( \frac{d}{dz} + \mathcal{J} \right) Q_{\text{HO}}(z) = 0$$

The imaginary time phase-space action becomes time-translation invariant.

$$\frac{1}{\hbar} S_C'(Q) = \int_{C'} dz \frac{1}{2i} Q_{\text{HO}}^t \Omega \left( \frac{d}{dz} + \mathcal{J} \right) Q_{\text{HO}}$$

The real symplectic transformation

$$\begin{pmatrix} v_a \\ w_a \end{pmatrix} = \sqrt{e^{2\omega_a} - 1} \begin{pmatrix} \omega_a & \omega_a \\ i & -i \end{pmatrix} \begin{pmatrix} W_a^+ \\ W_a^- \end{pmatrix}$$

$$\begin{pmatrix} v_a \\ w_a \end{pmatrix}^t \Omega \begin{pmatrix} v_b \\ w_b \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \delta_{a,b}$$

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brings $\mathcal{J}$ to the form

$$\mathcal{J} \begin{pmatrix} v_a \\ w_a \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ \omega_a^2 & 0 \end{pmatrix} \begin{pmatrix} v_a \\ w_a \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ \omega^2 & 0 \end{pmatrix}$$

(17.28)

So $q_{ho}$ is an ordinary time-independent harmonic oscillator satisfying the second-order equation of motion

$$\frac{d^2q_{ho}}{dz^2} + \omega^2 q_{ho} = 0$$

(17.29)

The time-dependent canonical transformation (17.24) is a change of variables in the phase-space path integral over paths $Q(z)$ periodic in imaginary time. So the imaginary time path integral is equivalent to that of an ordinary time-independent harmonic oscillator giving an equilibrium thermal density matrix which is thermodynamically stable.

18 Numerical evidence for Property P

Property P is the condition that the hermitian matrix $\mathcal{H}(t)$ is positive definite on the physical phase space $\mathcal{V}_{phys}(t)$ for some $t \notin \{2mK\}$ and therefore for all such $t$.

$$\mathcal{H}(t) = \Omega(M_i(t) - 1) > 0$$

(18.1)

The thermal state for a given value of $\epsilon$ is stable if and only if property P holds for all $(j_L, j_R) \neq (0, 0)$, i.e. for ode 2, 3$j$, 2$j$, and 1$j$ for all values of the parameter $j$.

I do not know how to prove property P for a given ode. Instead I assemble strong numerical evidence that property P holds for ode 2, 3$j$, 2$j$, and 1$j$ for all values of $j$ for all $\epsilon$ in the range $0 \leq \epsilon \leq 1/\sqrt{2}$ which is $1/2 \leq k^2 \leq 1$.

The numerical calculations are done at $t = K$ because the function $\text{cn}(K + \tau i, k)$ has symmetry properties that allow for more robust numerical integration. For each ode, the strategy is to check property P numerically on a finite sample of points in the parameter space. For each point in the sample, property P is found to hold. Then $M_i(t)$ is diagonalized and the minimum frequency $\omega_{\text{min}}$ is calculated,

$$0 < \omega_{\text{min}} = \min\{\omega_a\}$$

(18.2)

the numerical evidence shows that $\omega_{\text{min}}$ is bounded away from 0 throughout the parameter space so $\mathcal{H}(t)$ is positive everywhere. Property P holds.

The numerical calculations are shown in [4].

ode 2

The only parameter in ode 2 is $\epsilon$. $N_{\text{phys}} = 1$ so there is only one frequency, $\omega_{\text{min}} = \omega_1$. Property P is tested for a discrete set of values of $\epsilon$. At each $\epsilon$ property P holds and $\omega_1 = \epsilon$ to the numerical precision of the calculation. So property P holds for all $\epsilon$. The identity $\omega_1 = \epsilon$ suggests that ode 2 can be integrated analytically.

odes 3$j$, 2$j$, 1$j$

Figures 3–8 show the numerical data, plotting $\omega_{\text{min}}/p$ against $p$. The calculations are done only for $j > 3/2$ because the each ode is invariant under $j \rightarrow -j$, $z \rightarrow z + 2K'i$ since $\text{cn}(z + 2K'i, k) = -\text{cn}(z, k)$.

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The first graph for each ode shows \((p, \omega_{\text{min}}/p)\) at various \(\epsilon\) for the lowest values of \(j\), \(j = 3/2, 2, 5/2, \ldots\). The second graph shows the asymptotic limit \(\epsilon \to 0\) with \(p\) fixed. The small \(j\) data matches on to the asymptotic regime. The two graphs combined show that \(\omega_{\text{min}}/p\) is bounded away from 0 for all \(\epsilon\), certainly for the physically interesting values \(\epsilon < 10^{-27}\). So property \(P\) holds.

Figure 9 clarifies the \(\epsilon \to 0\) limit at fixed \(j\) for ode \(1_j\). For each of the three ode series,

\[
\omega_{\text{min}} \underset{\epsilon \to 0}{\longrightarrow} A\sigma
\]

for some constant \(A\). The parameters \(\sigma\) and \(p\) are related by

\[
\frac{\sigma^2}{p^2} = \begin{cases} \frac{1}{5} & \text{ode 3}_j \\ \frac{1}{4} & \text{ode 2}_j \\ \frac{j^2}{4j^2+3} & \text{ode 1}_j \end{cases}
\]

so

\[
\frac{\omega_{\text{min}}^2}{\omega_{\text{min,asymp}}^2} \underset{\epsilon \to 0}{\longrightarrow} \begin{cases} 1 & \text{ode 3}_j \\ 1 & \text{ode 2}_j \\ \frac{j^2}{4j^2+3} & \text{ode 1}_j \end{cases}
\]

For ode \(1_j\) the \(\epsilon \to 0\) limit at fixed \(j\) does not agree with the asymptotic limit holding \(p\) fixed. This is an artifact of the parametrization by \(p\).

19 Zero-mode integral

19.1 Observables

The classical solution \(b(z)\) defines a trajectory \(z \mapsto B(z)\) in the complexified phase space of the SU(2) gauge theory on \(S^3\). The periodicities of the classical solution imply that \(B(z)\) is a function on the complex plane modulo the lattice of periods, the complex torus

\[\mathbb{T} = \mathbb{C}/\mathbb{L} \quad \mathbb{L} = \{m(2K + 2K') + n(2K - 2K') : m, n \in \mathbb{Z}\}\]

The poles of \(B(z)\) are at \(z \in \mathbb{L} \pm K'i\). Away from the poles, \(B(z)\) takes values in the complex phase space. \(B(z)\) can be defined at the poles by adding points at infinity to the complex phase space.

The spaces \(V_{\text{phys}}(z)\) are the infinitesimal perturbations of \(B(z)\) modulo the gauge symmetry. Identifying the torus \(\mathbb{T}\) with the classical trajectory, the spaces \(V_{\text{phys}}(z)\) form a vector bundle over \(\mathbb{T}\). An observable — a linear function of the gauge field fluctuations — is an analytic section \(F\) of the dual vector bundle,

\[F(z) \in V_{\text{phys}}(z)^*\]

which is nonsingular away from the poles of \(B(z)\). The observable operator is

\[O(z) = F(z) Q(z) = F_a(z) Q^a(z)\]

\(z\) is playing two roles here. It is the complex time in \(O(z)\) and \(Q(z)\). In \(F(z)\), it is the complex coordinate on the classical trajectory in phase space.
Figure 3: The curves are ordered by increasing $\epsilon$ from left to right. The dots are $j = 3/2, 2, 5/2, \ldots$. The connecting lines are interpolations.

Figure 4: The open dots show the asymptotic limit $\epsilon \to 0$ with $p$ fixed. The closed dots are the $\epsilon = .01$ data.
Figure 5: The curves are ordered by increasing $\epsilon$ from left to right. The dots are $j = 3/2, 2, 5/2, \ldots$. The connecting lines are interpolations.

Figure 6: The open dots show the asymptotic limit $\epsilon \to 0$ with $p$ fixed. The closed dots are the $\epsilon = .01$ data.
Figure 7: The curves are ordered by increasing $\epsilon$ from left to right. The dots are $j = 3/2, 2, 5/2, \ldots$. The connecting lines are interpolations.

Figure 8: The open dots show the asymptotic limit $\epsilon \to 0$ with $p$ fixed. The closed dots are the $\epsilon = .01$ data.
Figure 9: The approach to the asymptotic limit at small $p$. 

$\omega_{\text{min}}^2/p^2$
The imaginary time gaussian 2-point expectation value of observables
\[ O_1(z) = \mathcal{F}_1(z) \mathcal{Q}(z) \quad O_2(z) = \mathcal{F}_2(z) \mathcal{Q}(z) \] (19.4)
is
\[
\langle O_2(z_2) O_1(z_1) \rangle_t = \mathcal{F}_2(z_2) \langle \mathcal{Q}(z_2) \mathcal{Q}(z_1)^\dagger \rangle_{t_1} \mathcal{F}_1(z_1)^\dagger
\]
\[
= \mathcal{F}_2(z_2) \mathcal{P}_m(z_2, t) \mathcal{G}(t, t) \mathcal{P}_m(t, z_1)^\dagger \mathcal{F}_1(z_1)^\dagger
\] (19.5)
\[ t = t_1 = t_2 \quad 2mK < t < 2(m+1)K \] (19.6)
The vertical path \( C \) through \( t \) is in the simply connected region \( R_m \) so \( \mathcal{P}_C(z, t) = \mathcal{P}_m(z, t) \). The 2-point function has periodicity properties
\[
\langle O_2(z_2) O_1(z_1) \rangle_t = \langle O_2(z_2 + 4K'i) O_1(z_1 + 4K'i) \rangle_t
\]
\[
= \langle O_2(z_2 + 2K + 2K'i) O_1(z_1 + 2K + 2K'i) \rangle_{t+2K}
\] (19.7)
\[
\langle O_2(t + 4K'i) O_1(z_1) \rangle_t = \langle O_1(z_1) O_2(t) \rangle_t
\]

19.2 Integrate over time translations

Expectation values in the thermal state are the integrals of the gaussian expectation values over the time-translation zero-mode. The thermal 2-point expectation value is
\[
\langle O_2(z_2) O_1(z_1) \rangle = \frac{1}{4K'} \int_0^{4K'} d\tau \langle O_2(z_2 + \tau i) O_1(z_1 + \tau i) \rangle_t
\] (19.8)
Change variable to \( z_0 = z_1 + \tau i \). Recall that \( C'_t \) is the vertical path from 0 to \( 4K'i \).
\[
\langle O_2(z_2) O_1(z_1) \rangle = \frac{1}{4K'} \int_{C'_t} dz_0 \langle O_2(z_{21} + z_0) O_1(z_0) \rangle_t \quad z_{21} = z_2 - z_1
\] (19.9)
Recall that \( z_1 \) and \( z_2 \) are both on \( C'_t \) with \( 0 \leq \tau_1 \leq \tau_2 \leq 4K' \), so \( z_{21} = \tau_2 i \) with \( 0 \leq \tau_2 \leq 4K' \). Later we will analytically continue the thermal expectation value to the complex \( z_{21} \) plane in order to construct the real time 2-point expectation values.

19.3 The thermal state is independent of \( t \)

Given that \( z_{21} = \tau_2 t \), the integrand in (19.9) is a nonsingular analytic function of \( z_0 \) in the vertical strip \( 2mK < t_0 < 2(m+1)K \). For any \( C'_t \) in the same vertical strip,
\[
\int_{C'_t} dz_0 \langle O_2(z_{21} + z_0) O_1(z_0) \rangle_t = \int_t^{t'} - \int_{t+4K'i}^{t'+K'i} \langle O_2(z_{21} + z_0) O_1(z_0) \rangle_t = 0
\] (19.10)
because the integrand is periodic under \( z_0 \to z_0 + 4K'i \). So the rhs of (19.9) is constant in \( t \) within each vertical strip. The periodicity \( z \to z + 2K + 2K'i \) gives
\[
\frac{1}{4K'i} \int_{C'_t} dz_0 \langle O_2(z_{21} + z_0) O_1(z_0) \rangle_t
\]
\[ = \frac{1}{4K'i} \int_{C'_t} dz_0 \langle O_2(z_{21} + z_0 + 2K + 2K'i) O_1(z_0 + 2K + 2K'i) \rangle_t
\] (19.11)
\[ = \frac{1}{4K'i} \int_{C'_{t+2K}} dz_0 \langle O_2(z_{21} + z'_0) O_1(z'_0) \rangle_t
\]
so the rhs of (19.9) is the same on neighboring strips. So it is independent of $t$. The thermal state is well defined, independent of the choice of $t$.

19.4 KMS condition

The last periodicity identity in (19.7) gives

$$\langle O_2(z_2) O_1(z_1) \rangle = \frac{1}{4K'} \int_0^{4K'} d\tau \langle O_2(z_2 + \tau i) O_1(z_1 + \tau i) \rangle_t$$

$$= \frac{1}{4K'} \int_0^{4K'} d\tau \langle O_1(z_1 + \tau i + 4K'i) O_2(z_2 + \tau i) \rangle_t$$

which is the KMS condition

$$\langle O_2(z_2) O_1(z_1) \rangle = \langle O_1(z_1 + 4K'i) O_2(z_2) \rangle$$  \hspace{1cm} (19.13)

20 Analytic continuation to real time

The euclidean signature formulation of thermodynamic stability should imply real time stability. As a step towards checking this, the analytic continuation to real time of the imaginary time two-point function is constructed, leaving for later the study of its large time asymptotic behavior.

Fix $m = 0$ and simply-connected region $R_0$. For every $t$ in the interval $0 < t < 2K$ the integrand in (19.9) is analytic and nonsingular in both $z_0$ and in $z_{21}$ as long as $0 < t + t_{21} < 2K$. Thus for each such $t$ there is an analytic continuation of the two-point expectation value to the strip $-t < t_{21} < 2K - t$. The periodic paths $C'_t$ for $0 < t < 2K$ are all deformable to each other, so all of these analytic continuations agree on the overlaps. So $\langle O_2(z_{21}) O_1(0) \rangle$ is analytic in the strip $-2K < t_{21} < 2K$.

To analytically continue beyond this strip, let $C''_t$ for $t \geq 2K$ be the periodic path within $R_0$ from $t$ to $t + 4K'i$ shown in Figure 10. For $t \leq -2K$, let $C''_{-t}$ be the reflection of $C''_{t}$ through the imaginary axis, $C''_{-t} = -C''_t$. All the periodic paths $C'_t$, $-2K < t < 2K$,
Figure 11: The domain of analyticity of $\langle \mathcal{O}_2(z_{21}) \mathcal{O}_1(0) \rangle$. Real time is the dotted line.

and $C''_t$, $2K \leq |t|$, are deformable to each other. Then, for $2K \leq |t|$, 

$$
\langle \mathcal{O}_2(z_2) \mathcal{O}_1(z_1) \rangle = \frac{1}{4K'i} \int_{C''_t} dz_0 \langle \mathcal{O}_2(z_{21} + z_0) \mathcal{O}_1(z_0) \rangle_t
$$

is analytic in $z_{21}$ for 

$$-2K - t < t_{21} < 2K - t \quad \text{and} \quad \tau_{21} \notin 2K'Z$$

because these are the conditions that $z_0 + z_{21} \in R_0$ for all $z_0$ in $C''_t$. Again, all the locally analytic constructions agree on the overlaps by contour deformation.

Therefore $\langle \mathcal{O}_2(z_2) \mathcal{O}_1(z_1) \rangle$

1. is time-translation invariant, 

$$
\langle \mathcal{O}_2(z_2) \mathcal{O}_1(z_1) \rangle = \langle \mathcal{O}_2(z_{21}) \mathcal{O}_1(0) \rangle \quad \text{where} \quad z_{21} = z_2 - z_1 = t_{21} + \tau_{21}i
$$

2. is analytic in $z_{21}$ in the region 

$$-2K < t_{21} < 2K \quad \text{or} \quad |t_{21}| \geq 2K, \quad \tau_{21} \notin 2K'Z$$

shown in Figure 11.

3. satisfies the KMS periodicity condition 

$$
\langle \mathcal{O}_2(z_2) \mathcal{O}_1(z_1) \rangle = \langle \mathcal{O}_1(z_1 + 4K'i) \mathcal{O}_2(z_2) \rangle
$$

The analytic continuation to real time is from above the real axis for $t_{21} > 2K$ and from below the axis for $t_{21} < -K$, as shown in Figure 11.
21 Comments and questions

21.1 A geometric proof?

There ought to be a geometric proof of thermodynamic stability directly from the self-adjointness of the Yang-Mills hamiltonian. The natural mathematical setting is the complexification of the phase-space of the SU(2) Yang-Mills theory on $S^3$. The Spin(4)-invariant SU(2) gauge solution for each value of the Yang-Mills energy is an elliptic curve, a torus, in the complex phase-space. The Yang-Mills energy parametrizes the moduli space of elliptic curves. The collection of all the solutions comprises a map from the universal elliptic curve to the complex phase-space. A rigorous construction of this map requires adding points at infinity to the complex phase-space to accommodate the poles in the classical solution. The complex gauge field $B^a_l(\hat{x})$ at each point $\hat{x}$ lives in a complex 9-dimensional vector space. Perhaps it is enough to projectify each of these vector spaces so the gauge field at each $\hat{x}$ lives in $\mathbb{C}P^9$. Once the projectification of the complex phase-space is constructed, the Yang-Mills theory has to be extended to the projectification. If this can be done, it should be possible to pull back the hamiltonian structure to the universal curve and construct the thermal state.

21.2 Other natural states?

There is a certain arbitrariness in defining the thermal state by the imaginary time periodicity $z \rightarrow z + 4K'i$. Why not the state defined by the periodicity $z \rightarrow z + 2K + 2K'i$? Are these different states? Is there more than one natural thermodynamically stable state?

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[4] The accompanying Supplemental Material consists of a note, Calculations for Thermodynamic stability of a cosmological SU(2)-weak gauge field, and two SageMath notebooks performing algebraic and numerical calculations and making plots, along with printouts of the notebooks. The Supplemental Material is also available at physics.rutgers.edu/~friedan and cocalc.com/dfriedan/DM/SM.
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