1. INTRODUCTION

This paper is devoted to the Gagliardo-Nirenberg and the Caffarelli-Kohn-Nirenberg interpolation inequalities associated with Sobolev-Coulomb spaces for the (fractional) derivative $0 \leq s \leq 1$. These inequalities are relevant and useful in the study of Thomas-Fermi-Dirac-von Weizsäcker models of density functional theory [4, 22, 26], or of Hartree-Fock theory [11, 28, 10], or of Hardy-Lieb-Thirring type inequalities [28].
The first part of the paper is devoted to the following type inequality, for \( g \in C^\infty_c(\mathbb{R}^d) \),

\[
\|g\|_{L^\gamma(\mathbb{R}^d)} \leq C\|g\|_{W^{s,p}(\mathbb{R}^d)}^{\beta_1 p} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|g(x)|^q |g(y)|^q}{|x-y|^{d-\alpha}} \, dx \, dy \right)^{\frac{\beta_2}{q}},
\]

for \( 1 < \gamma < +\infty, 1 \leq p, q < +\infty, 0 \leq s \leq 1, 0 < \alpha < d, 0 < \beta_1, \beta_2 < +\infty \) under appropriate assumptions on these parameters, where \( C \) is a positive constant independent of \( g \). Here and in what follows, for any open set \( \Omega \subset \mathbb{R}^d \), we use the standard notation

\[
\|g\|_{W^{s,p}(\Omega)} = \left\{ \begin{array}{ll}
\left( \int_\Omega \int_\Omega \frac{|g(x) - g(y)|^p}{|x-y|^{d+sp}} \, dx \, dy \right)^{\frac{1}{p}} & \text{for } 0 < s < 1,

\left( \int_\Omega |\nabla g(x)|^p \, dx \right)^{1/p} & \text{for } s = 1,

\left( \int_\Omega |g(x)|^p \, dx \right)^{1/p} & \text{for } s = 0.
\end{array} \right.
\]

One can easily check by scaling that (1.1) holds only if the following two identities hold

\[
\beta_1 p + 2\beta_2 q = 1 \quad \text{and} \quad (d - sp)\beta_1 + (d + \alpha)\beta_2 = d/\gamma,
\]

Mathematically, inequalities of type (1.1) was first studied by Lions [31, 32]. In his pioneering work, he established

\[
\|g\|_{L^3(\mathbb{R}^3)} \leq C \|\nabla g\|_{L^2(\mathbb{R}^3)}^{\frac{1}{6}} \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|g(x)|^2 |g(y)|^2}{|x-y|} \, dx \, dy \right)^{\frac{1}{6}} \forall g \in C^\infty_c(\mathbb{R}^3),
\]

in used it to study of Hartree-Fock equations. In the spirit of the famous Gagliardo-Nirenberg’s inequalities [17, 44], (1.4) has been extended to a more general setting of the form (1.1). Inequality (1.1) is so far only known for \( p = 2, 0 < s \leq 1 \). More precisely, (1.1) has been previously established by Bellazzini, Frank, Visciglia [2, Proposition 2.1] in the case \( p = 2, q = 2, 0 < s < 1 \) (see also [34, (21)] for the case \( p = q = 2, \alpha = d - 2s \), by Mercuri, Moroz, Van Schaftingen [38] in the case \( p = 2 \) and \( s = 1 \), and by Bellazzini, Ghimenti, Mercuri, Moroz, and Van Schaftingen [3, Theorems 1.1 and 1.2] in the case \( p = 2 \) and \( 0 < s < 1 \). The case \( p = 2 \) is thus completely understood but only established very recently unless \( s = 0 \). The case \( p \neq 2 \) is open even for \( s = 1 \) and the case \( s = 0 \) has not been considered previously.

The first main result of this paper is on the Gagliardo-Nirenberg interpolation type inequalities involving the Coulomb term. We have

**Theorem 1.1.** Let \( d \geq 1, 0 \leq s \leq 1, 1 < \gamma < +\infty, 1 \leq p, q < +\infty, 0 < \alpha < d, \) and \( 0 < \beta_1, \beta_2 < +\infty \). Assume (1.3) and the following fact

\[
\beta_1 \gamma + \beta_2 \gamma \geq 1.
\]

Then (1.1) holds for all \( g \in L^1(\mathbb{R}^d) \) with compact support \( ^1 \).

**Remark 1.1.** Condition (1.5) is optimal, see Remark 2.2.

**Remark 1.2.** One can ask what happens in Theorem 1.1 if \( \beta_1 = 0 \) or \( \beta_2 = 0 \). In fact, Theorem 1.1 holds with \( \beta_2 = 0 \). However, when \( \beta_2 = 0 \), (1.1) is just the standard Sobolev inequality and there is nothing new. Concerning \( \beta_1 \), one can check that \( \beta_1 \) cannot be 0 since the assumption

\[
(d - sp)\beta_1 + (d + \alpha)\beta_2 = d/\gamma
\]

\(^1\)We use here the convention \(+\infty \leq +\infty\).
implies that if \( \beta_1 = 0 \) then
\[
\gamma \beta_2 = \frac{d}{(d+\alpha)} < 1 : \text{ which contradicts to (1.5).}
\]

**Remark 1.3.** Theorem 1.1 also holds for \( \gamma = 1 \). In this case, since
\[
\beta_1 \gamma + \beta_2 \gamma = 1 \quad \text{and} \quad (d - sp) \beta_1 + (d + \alpha) \beta_2 = d,
\]
it follows that \( p = q = 1, s = 0, \) and \( \alpha = d. \) Then (1.1) is again just the standard Gagliardo-Nirenberg inequality.

**Remark 1.4.** Theorem 1.1 also holds for \( \alpha = d. \) Nevertheless, the conclusion in this case just follows from the the standard Gagliardo-Nirenberg inequalities.

Theorem 1.1 is known for the case \( p = 2 \) with \( s > 0 \) as mentioned above. Nevertheless, the known assumptions are stated in a more involved manner than condition (1.5). More precisely, when \( p = 2, \) instead of (1.5), the conclusion of Theorem 1.1 was shown under the condition \( ((d+\alpha) - q(d-2s) \neq 0 \) and (1.6) below) or \( ((d + \alpha) - q(d - 2s) = 0 \) and (1.7) below), where
\[
\begin{align*}
2(\alpha + 2qs) & \leq \gamma \leq \frac{2d}{d-2s} \quad \text{if } 2s < d \text{ and } (d + \alpha) - q(d - 2s) > 0, \\
\frac{2d}{d-2s} & \leq \gamma \leq \frac{2(\alpha + 2qs)}{\alpha + 2s} \quad \text{if } 2s < d \text{ and } (d + \alpha) - q(d - 2s) < 0, \\
\frac{2(\alpha + 2qs)}{\alpha + 2s} & \leq \gamma < \infty \quad \text{if } 2s \geq d,
\end{align*}
\]
and
\[
\frac{\alpha(d - 2s)}{2d(\alpha + 2s)} \leq \beta_1 < +\infty, \quad 0 \leq \beta_2 \leq \frac{s(d - 2s)}{d(\alpha + 2s)}.
\]
The proof is then given to case by case separately. One can check that (1.5) is equivalent to these conditions (see Section 2.3). In this paper, we present a new approach to obtain Theorem 1.1 in which condition (1.5) appears very naturally.

We next extend Theorem 1.1 in the spirit of Caffarelli-Kohn-Nirenberg’s inequalities. Caffarelli, Kohn, and Nirenberg [9] (see also [8]) proved the following well-known inequality
\[
\| |x|^{\gamma} g \|_{L^{\gamma'}(\mathbb{R}^d)} \leq C \| |x|^\alpha \nabla g \|_{L^p(\mathbb{R}^d)} \| |x|^\beta \|_{L^s(\mathbb{R}^d)}^{1-a} \quad \text{for } g \in C^1_c(\mathbb{R}^d),
\]
under appropriate assumptions of the parameters. This family of inequalities has been extended by Nguyen and Squassina [42] (see also [43]) for fractional Sobolev spaces where the quantity \( \| |x|^\alpha \nabla u \|_{L^p(\mathbb{R}^d)} \) is replaced by
\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|g(x) - g(y)|^p |x|^\alpha |y|^\beta}{|x-y|^{d+sp}} \, dx \, dy
\]
under appropriate assumptions of the parameters. Previous results can be found in [1, 16, 37].

In this direction, we establish our second main result of this paper on the Caffarelli-Kohn-Nirenberg interpolation inequalities associated with the Coulomb term.

**Theorem 1.2.** Let \( d \geq 1, 0 \leq s \leq 1, 1 \leq \gamma', p, q < +\infty, 0 < \alpha < d, 0 < \beta_1, \beta_2 < +\infty, \gamma', \alpha_1, \alpha_2, \alpha_2, \alpha_2 \in \mathbb{R}. \) Set \( \alpha_1 = \alpha_1 + \alpha_1, \alpha_2, \alpha_2 + \alpha_2, \alpha_2, \) and define \( \sigma, \gamma \) by
\[
\sigma = \beta_1 p \alpha_1 + \beta_2 q \alpha_2 \quad \text{and} \quad (d - sp) \beta_1 + (d + \alpha) \beta_2 = d / \gamma.
\]
Assume that
\begin{equation}
\beta_1 p + 2\beta_2 q = 1, \quad \frac{1}{\gamma'} + \frac{\tau'}{d} = \frac{1}{\gamma} + \frac{\sigma}{d},
\end{equation}
\begin{equation}
\gamma \geq \gamma', \quad \gamma > 1,
\end{equation}
and, either
\begin{equation}
\beta_1 \gamma' + \beta_2 \gamma' \geq 1,
\end{equation}
or
\begin{equation}
\left( \frac{1}{p} (sp - d - \alpha_1 p) + \frac{1}{2q} (\alpha + d + \alpha_2 q) \right) \neq 0 \quad \text{and} \quad \beta_1 \gamma + \beta_2 \gamma > 1.
\end{equation}
Then, if either
\begin{equation}
\frac{1}{\gamma'} + \frac{\tau'}{d} > 0 \quad \text{and} \quad g \in L^1(\mathbb{R}^d) \quad \text{with compact support},
\end{equation}
or
\begin{equation}
\frac{1}{\gamma'} + \frac{\tau'}{d} < 0 \quad \text{and} \quad g \in L^1_{loc}(\mathbb{R}^d) \quad \text{with} \quad g = 0 \quad \text{in a neighborhood of 0},
\end{equation}
then it holds
\begin{equation}
\left( \int_{\mathbb{R}^d} |g|^\gamma' |x|^{\tau' \gamma'} \right)^{\frac{1}{\gamma'}} \leq C \|g\|_{W^{s,p,\alpha_1,\alpha_2}(\mathbb{R}^d)} \times \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|g(x)|^q |g(y)|^q |x|^\alpha |y|^\beta}{|x-y|^d} dx \; dy \right)^{\frac{1}{q}},
\end{equation}
where \( C > 0 \) is a constant independent of \( g \).

Here and in what follows, we denote, for \( t_1, t_2 \in \mathbb{R}, \; 1 \leq p < +\infty, \) and \( 0 \leq s \leq 1, \)
\begin{equation}
\|g\|_{W^{s,p,t_1,t_2}(\mathbb{R}^d)}^p = \begin{cases} 
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|g(x) - g(y)|^p |x|^{t_1 p} |y|^{t_2 p}}{|x-y|^{d+sp}} dx \; dy & \text{for} \; 0 < s < 1 \\
\int_{\mathbb{R}^d} |g(x)|^p |x|^{(t_1+t_2)p} dx & \text{for} \; s = 0, \\
\int_{\mathbb{R}^d} |\nabla g(x)|^p |x|^{(t_1+t_2)p} dx & \text{for} \; s = 1.
\end{cases}
\end{equation}

**Remark 1.5.** The case \( \alpha = d, \) corresponds to the usual Caffarelli-Kohn-Nirenberg inequalities, see [42] to compare the conditions satisfied by the parameters. The study of the best constant of such inequalities are in general highly non-trivial and of great interest, see e.g., [14, 15] and references therein.

As an application of Theorem 1.2, we derive a new family of one body interpolation inequalities and then use it to establish a new family of many body interpolation inequalities. These are our third main results of this paper and presented later in Section 4.

We now review briefly several known approaches for (1.1). One is based on the standard Gagliardo-Nirenberg inequality and the fractional chain rule, see e.g. [18], and standard interpolation inequalities, as in [3, 2]. Another approach is based on the Hardy-Lieb-Thirring’s many body interpolation inequalities, as in [34]. These approaches use essentially the fact that \( p = 2. \)

In this paper, we propose a different approach to establish Theorem 1.1 and Theorem 1.2. The proof of Theorem 1.1 follows closely the approach proposed by Nguyen [41] in his study Sobolev’s inequalities associated with non-local, non-convex functionals. The idea is first to establish the corresponding Poincaré inequality. One then uses a covering argument to derive from it an estimate in \( L^w_\gamma \) (\( L^\gamma \)-weak). The estimate in \( L^\gamma \) is then established via the estimate in \( L^w_\gamma \) and involves the truncation technique due to Maz’ya and a result of the theory of maximal (sharp) functions.
due to Fefferman and Stein. The proof of Theorem 1.2 uses similar arguments of Nguyen and Squassina [42] where the authors established the full range Caffarelli-Kohn-Nirenberg’s inequality for fractional Sobolev spaces. The idea is to first establish the conclusion of Theorem 1.2 under the assumption (1.12). The starting point is the corresponding Gagliardo-Nirenberg type inequality established in Theorem 1.1 (see Lemma 3.1). One then decomposes the space into annulus and applies appropriately the corresponding Gagliardo-Nirenberg type inequality. This idea has its roots from harmonic analysis when the decomposition is given for the frequency variables. The proof in the case (1.13) is derived from (1.12) using a scaling argument.

The organization of the paper is as follows. Section 2 and Section 3 are devoted to the proof of Theorem 1.1 and Theorem 1.2, respectively. In Section 2 (Section 2.3), we also derive a connection between our conditions in Theorem 1.1 with the known result forms. In Section 4, we use Theorem 1.2 to establish a new family of one body interpolation inequalities. Another main ingredient of this proof is the sharp (fractional) Hardy inequalities with the remainder due to Frank and Seiringer [16]. We then use this family of one body interpolation inequalities to establish a new family of many body interpolation type inequalities following the strategy of Lundholm, Nam, and Portmann [34]. These results are new even in the case $s = 1$ and $p \neq 2$ and their proofs are new even in the case $p = 2$.

2. Gagliardo-Nirenberg interpolation inequalities involving Coulomb terms - Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. The proof is in the spirit of the one given in [41] where the author established new Poincaré and Sobolev inequalities related to new characterizations of Sobolev spaces via non-local, non-convex terms in [39, 5, 40] (see also [6, 7, 35] for related topics). The ideas of the proof are as follows. We first derive a Poincaré inequality involving Coulomb terms, which is almost free in this context. The integrability desired is then established via Vitali’s covering lemma and the truncation method, which has its root in the work of Mazda [36]. This part also uses an interesting result of the theory of sharp functions due to Fefferman and Stein [13]. This section containing three subsections is organized as follows. Several useful lemmas are presented in the first subsection. The proof of Theorem 1.1 is then given in the second one. In the last subsection, we discuss other forms of the assumptions of Theorem 1.1.

2.1. Preliminaries. For $D$ a measurable set of $\mathbb{R}^d$ and $g \in L^1(D)$, denote $|D|$ its Lebesgue mesure and

$$(g)_D = \int_D g(y) \, dy := \frac{1}{|D|} \int_D g(y) \, dy.$$  

We begin with a simple but useful version of Poincaré inequality involving a Coulombian term.

**Lemma 2.1.** Let $d \geq 1$, $0 \leq s \leq 1$, $1 < \gamma < +\infty$, $1 \leq p \leq +\infty$, $0 < \alpha < d$, and $0 < \beta_1, \beta_2 < +\infty$, and assume (1.3), and let $B$ be an open ball or an open cube in $\mathbb{R}^d$. We have, for $u \in L^1(B)$,

$$\int_B |u - (u)_B| \, dx \leq \frac{C}{|B|^s} \|u\|_{W^{s,p}(B)}^{\beta_1} \left( \int_B \int_B \frac{|u(x)||u(y)|^q}{|x-y|^{d-\alpha}} \, dx \, dy \right)^{\frac{\beta_2}{q}},$$

where $C > 0$ is a constant independent of $u$ and $B$.

**Proof.** It is easy to check that

$$\int_B |u - (u)_B| \, dx \leq \int_B \int_B |u(x) - u(y)| \, dx \, dy \leq \frac{C}{|B|^{\frac{d-\alpha}{pq}}} \|u\|_{W^{s,p}(B)}$$

and

$$\int_B |u - (u)_B| \, dx \leq \int_B \int_B \frac{|u(x)||u(y)|^q}{|x-y|^{d-\alpha}} \, dx \, dy \leq \frac{C}{|B|^s} \|u\|_{W^{s,p}(B)}^{\beta_1} \left( \int_B \int_B \frac{|u(x)||u(y)|^q}{|x-y|^{d-\alpha}} \, dx \, dy \right)^{\frac{\beta_2}{q}}.$$
(2.2) \[ \int_B |u - (u)_B| \, dx \leq 2 \int_B |u| \, dx \leq \frac{2}{|B|^{\frac{1}{q}}} \left( \int_B |u(x)|^q \, dx \right)^{\frac{1}{q}} \]
\[ \leq \frac{C}{|B|^{\frac{d+\alpha}{2q\gamma}}} \left( \int_B \int_B |u(x)|^q |u(y)|^q \, dx \, dy \right)^{\frac{1}{q}}. \]

Here and in what follows in this proof, \( C \) denotes a positive constant independent of \( u \) and \( B \).

Since \( \beta_1, \beta_2 > 0 \) and \( p\beta_1 + 2q\beta_2 = 1 \) by (1.3), we have

\[ \int_B |u - (u)_B| = \left( \int_B |u - (u)_B| \right)^{p\beta_1} \left( \int_B |u - (u)_B| \right)^{2q\beta_2} \]
\[ \leq \frac{C}{|B|^{\frac{d-2\gamma}{d}}} \left( \int_B |u|^{\beta_1} \, dx \right)^{\frac{\beta_1}{\gamma}} \left( \int_B \int_B |u(x)|^q |u(y)|^q \, dx \, dy \right)^{\frac{\beta_2}{\gamma}}. \]

Since
\[ \frac{1}{\gamma} \frac{(d-\gamma)\beta_1}{d} + \frac{(d+\alpha)\beta_2}{d}, \]

the conclusion follows from (2.3).

We next present a variant of [41, Lemma 7], whose proof is based on Vitali’s covering lemma.

**Lemma 2.2.** Let \( d \geq 1, 1 < \gamma < +\infty \), and \( 0 < \beta_1, \beta_2 < +\infty \) be such that \( \beta_1 \gamma + \beta_2 \gamma \geq 1 \). Let \( h_1, h_2 \in L^1(\mathbb{R}^d) \), and let \( g \) be a measurable function defined in \( \mathbb{R}^d \). Assume that

\[ g(x) \leq \frac{1}{|B|^{\frac{1}{\gamma}}} \left( \int_B |h_1| \right)^{\frac{\beta_1}{\gamma}} \left( \int_B |h_2| \right)^{\frac{\beta_2}{\gamma}} \quad \text{for a.e.} \ x \in \mathbb{R}^d, \]

where the supremum is taken over all open balls (or open cubes) containing \( x \). Then

\[ t^\gamma |\{g > t\}| \leq C \|h_1\|_{L^1(\mathbb{R}^d)} \|h_2\|_{L^1(\mathbb{R}^d)} \quad \forall t > 0, \]

for some positive constant \( C \) depending only on \( \gamma, \beta_1, \beta_2 \), and \( d \).

**Proof.** We only consider where the case the supremum is taken over all open balls containing \( x \). The case where the supremum is taken over all open cubes containing \( x \) follows in the same lines and is omitted.

Fix \( t > 0 \). From (2.4), for a.e. \( y \in \{g > t\} \), there exists an open ball \( B_y \) containing \( y \) such that

\[ t \leq \frac{2}{|B_y|^{\frac{1}{\gamma}}} \left( \int_{B_y} |h_1| \, dx \right)^{\frac{\beta_1}{\gamma}} \left( \int_{B_y} |h_2| \, dx \right)^{\frac{\beta_2}{\gamma}}, \]

which yields

\[ |B_y| \leq \frac{2}{t^\gamma} \left( \int_{B_y} |h_1| \, dx \right)^{-\frac{\beta_1\gamma}{\gamma}} \left( \int_{B_y} |h_2| \, dx \right)^{-\frac{\beta_2\gamma}{\gamma}}. \]

By applying Vittali’s covering lemma to the set \( \{g > t\} \) and to the family of open ball \( B_y \), there exists a countable collection of mutually disjoint open balls \( (B_i) \) such that outside a set of zero measure, \(^2\)

\[ (2.6) \quad \{g > t\} \subset \cup_i 5B_i, \]

\(^2\)Here 5\( B_i \) denotes the open ball with the same center as \( B_i \) but 5 times radius.
Applying Lemma 2.3 below with $\tau = \beta_1 \gamma$ and $\eta = \beta_2 \gamma$, after noting that $\beta_1 \gamma + \beta_2 \gamma \geq 1$, we have

\[ |\{|g > t\}| \leq C \sum_{i=1}^{\infty} |B_i| \]

where

\[ |B_i| \leq \frac{C}{t^\gamma} \left( \int_{B_i} |h_1| \right)^{\beta_1 \gamma} \left( \int_{B_i} |h_2| \right)^{\beta_2 \gamma}. \]

which is the conclusion. \(\square\)

**Remark 2.1.** The proof of Lemma 2.1 and Lemma 2.2 also work for $\gamma = 1$. Nevertheless, these results are later only applied to the case $\gamma > 1$.

The following simple result, whose proof is omitted, is used in the proof of Lemma 2.2.

**Lemma 2.3.** For $\tau, \eta > 0$ with $\tau + \eta \geq 1$, we have

\[ \sum_{i=1}^{\infty} |a_i|^\tau |b_i|^\eta \leq \left( \sum_{i=1}^{\infty} |a_i| \right)^\tau \left( \sum_{i=1}^{\infty} |a_i| \right)^\eta \quad \text{for } a_i, b_i \in \mathbb{R}. \]

As a consequence of (2.5), one derives that $g \in L^\gamma_\omega(\mathbb{R}^d)$ ($L^\gamma$-weak) if $g$ is non-negative and satisfies (2.4). We next present a variant of Lemma 2.2 which deals with the $L^\gamma$-integrability of $g$ instead of $L^\gamma$-weak integrability. This variant inspired by \[41, Lemma 8\] is the main ingredient of the proof of Theorem 1.1. To this end, we first recall the definition of the dyadic maximal functions and the dyadic sharp maximal functions, see, e.g., \[45\].

**Definition 2.1.** Let $g \in L^1_{\text{loc}}(\mathbb{R}^d)$. The dyadic maximal function $M^\Delta g$ and the dyadic sharp maximal function $g^\sharp, \Delta$ are defined as follows

\[ (M^\Delta g)(x) := \sup_Q \int_Q |g| dy, \]

and

\[ g^\Delta(x) := \sup_Q \int_Q |g - (g_Q)| dy, \]

where the supremum is taken over all dyadic cubes $Q$ containing $x$.

The following definition is also useful.

**Definition 2.2.** For each $k \in \mathbb{Z}$ and a non-negative function $g$ defined in $\mathbb{R}^d$, define the truncation operator

\[ T_k(g)(x) = \begin{cases} 
10^{k+1} - 10^k & \text{if } x \in \{g > 10^{k+1}\}, \\
g - 10^k & \text{if } x \in \{10^k < g \leq 10^{k+1}\}, \\
0 & \text{if } x \in \{g \leq 10^k\}.
\end{cases} \]

We are ready to present a variant of Lemma 2.2 concerning $L^\gamma$-integrability.
Lemma 2.4. Let \( d \geq 1, 1 < \gamma < +\infty \), and \( 0 < \beta_1, \beta_2 < +\infty \) be such that
\[
\beta_1 \gamma + \beta_2 \gamma \geq 1.
\]
Let \( g \in L^1(\mathbb{R}^d) \) with \(|\{|g| > 0\}| < +\infty \), and set
\[
g_k = T_k(g) \quad \text{for } k \in \mathbb{Z}.
\]
Assume that there exist two sequences \((h_{1,k}), (h_{2,k}) \subset L^1(\mathbb{R}^d)\), and two non-negative functions \( h_1, h_2 \in L^1(\mathbb{R}^d) \) such that, for \( t > 0 \) and \( k \in \mathbb{Z} \),
\[
\left| \left\{ g_k^{\frac{\gamma}{\alpha}} > t \right\} \right| \leq \frac{1}{t^\gamma} \| h_{1,k} \|_{L^1(\mathbb{R}^d)} \| h_{2,k} \|_{L^1(\mathbb{R}^d)} \delta_{\beta_1, \gamma} \| h_{1,k} \|_{L^1(\mathbb{R}^d)} \| h_{2,k} \|_{L^1(\mathbb{R}^d)} \delta_{\beta_2, \gamma},
\]
and, for \( j = 1, 2 \),
\[
\sum_{k=1}^{\infty} |h_{j,k}| \leq h_j \text{ in } \mathbb{R}^d.
\]
Then \( g \in L^\gamma(\mathbb{R}^d) \) and
\[
\|g\|_{L^\gamma(\mathbb{R}^d)} \leq C \| h_1 \|_{L^1(\mathbb{R}^d)} \| h_2 \|_{L^1(\mathbb{R}^d)}
\]
for some positive constant \( C \) independent of \( g, h_1, \) and \( h_2 \).

Proof. Let \( 0 < b < 1, c > 0 \), and \( f \in L^1_{\text{loc}}(\mathbb{R}^d) \). We recall that, see, e.g., [45, Estimate (22), p.153],
\[
\left| \left\{ M^\Delta f > \alpha, f^{\frac{\gamma}{\alpha}} \leq c \alpha \right\} \right| \leq \frac{2d}{1-b} \left| \left\{ M^\Delta f > b \alpha \right\} \right| \forall \alpha > 0.
\]
Applying (2.13) with \( f = g_k, b = \frac{1}{10}, \alpha = 10^k \), and \( 0 < c < \frac{1}{2} \) (to be chosen later), we have
\[
\left| \left\{ M^\Delta g_k > 10^k \right\} \right| \leq c^{2d+1} \left| \left\{ M^\Delta g_k > 10^{k-1} \right\} \right| + \left| \left\{ g_k^{\frac{\gamma}{\alpha}} > c10^k \right\} \right|.
\]
This yields, for any \( m, n \in \mathbb{Z} \) with \( n \geq m + 1 \),
\[
\sum_{k=m}^{n} 10^{k\gamma} \left| \left\{ M^\Delta g_k > 10^k \right\} \right| \leq c^{2d+1} \sum_{k=m}^{n} 10^{k\gamma} \left| \left\{ M^\Delta g_k > 10^{k-1} \right\} \right| + \sum_{k=m}^{n} 10^{k\gamma} \left| \left\{ g_k^{\frac{\gamma}{\alpha}} > c10^k \right\} \right|.
\]
We first derive an lower bound for the LHS of (2.14). From the definition of \( g_k \) and the fact \( \{ M^\Delta g_k > 10^k \} \supset \{ g_k > 10^k \} \), we have, for \( n \geq m + 1 \),
\[
\sum_{k=m}^{n} 10^{k\gamma} \left| \left\{ M^\Delta g_k > 10^k \right\} \right| \geq C \int_{10^{m+1}}^{10^{n+2}} t^{\gamma-1} |\{|g| > t| dt.
\]
Here and in what follows in this proof, \( C > 0 \) denotes a constant independent of \( g, h_1, h_2, \) and \( k, m, n \).
We next derive an upper bound of the RHS of (2.14). By the theory of maximal functions, we deduce from the definition of \( g_k \) in (2.18) that, for \( n \geq m + 1 \),
\[
\sum_{k=m}^{n} 10^{k\gamma} \left| \left\{ M^\Delta g_k > 10^{k-1} \right\} \right| \leq C \sum_{k=m}^{n} \int_{\mathbb{R}^d} |g_k|^{\gamma} dx \leq C \int_{10_{m+1}}^{10^{n+2}} t^{\gamma-1} |\{|g| > t| dt.
\]
We also have
\[ (2.17) \sum_{m=1}^{n} 10^{k \gamma} \left\{ \left| g_k^\Delta \right| > c10^k \right\} \leq \frac{1}{c^\gamma} \sum_{m=1}^{n} \left\| h_1^k \right\|_{L^1(\mathbb{R}^d)}^{\beta_1 \gamma} \left\| h_2^k \right\|_{L^1(\mathbb{R}^d)}^{\beta_2 \gamma} \leq \frac{1}{c^\gamma} \left\| h_1 \right\|_{L^1(\mathbb{R}^d)}^{\beta_1 \gamma} \left\| h_2 \right\|_{L^1(\mathbb{R}^d)}^{\beta_2 \gamma}. \]

Plugging the estimates (2.15), (2.16) and (2.17) into (2.14) we obtain
\[ \int_{10^n+1}^{10^{n+2}} t^{\gamma-1} \left\{ \left| g \right| > t \right\} dt \leq C \left( 2c^{2+1} \int_{10^n}^{10^{n+2}} t^{\gamma-1} \left\{ \left| g \right| > t \right\} dt + \frac{1}{c^\gamma} \left\| h_1 \right\|_{L^1(\mathbb{R}^d)}^{\beta_1 \gamma} \left\| h_2 \right\|_{L^1(\mathbb{R}^d)}^{\beta_2 \gamma} \right). \]

Choosing c small enough so that \(Cc^{2+1} = \frac{1}{2}\), we have
\[ \int_{10^n+1}^{10^{n+2}} t^{\gamma-1} \left\{ \left| g \right| > t \right\} dt \leq \int_{10^n}^{10^{n+1}} t^{\gamma-1} \left\{ \left| g \right| > t \right\} dt + C \left\| h_1 \right\|_{L^1(\mathbb{R}^d)}^{\beta_1 \gamma} \left\| h_2 \right\|_{L^1(\mathbb{R}^d)}^{\beta_2 \gamma}. \]

Letting first \( n \to +\infty \), then \( m \to -\infty \), and noting that \( \left\{ \left| g \right| > 0 \right\} < +\infty \), we obtain
\[ \int_{\mathbb{R}^d} \left| g(x) \right|^\gamma dx \leq C \left\| h_1 \right\|_{L^1(\mathbb{R}^d)}^{\beta_1 \gamma} \left\| h_2 \right\|_{L^1(\mathbb{R}^d)}^{\beta_2 \gamma}, \]
which implies the conclusion. The proof is complete. \( \square \)

### 2.2. Proof of Theorem 1.1

The proof follows from Lemma 2.1, Lemma 2.2, and Lemma 2.4 as in [41]. Set
\[ (2.18) \quad g_k = T_k(\left| g \right|) \quad \text{for} \quad k \in \mathbb{Z}, \]
where \( T_k \) is given by Definition 2.2. Define, for \( x \in \mathbb{R}^d \),
\[ h_1(x) = \begin{cases} \left| g(x) \right|^p & \text{for} \quad s = 0, \\ \int_{\mathbb{R}^d} \frac{\left| g(x) - g(y) \right|^p}{|x - y|^{d+sp}} dy & \text{for} \quad s \in (0, 1), \\ |\nabla g(x)|^p & \text{for} \quad s = 1, \end{cases} \]
\[ h_{1,k}(x) = \begin{cases} \left| g_k(x) \right|^p & \text{for} \quad s = 0, \\ \int_{\mathbb{R}^d} \frac{\left| g_k(x) - g_k(y) \right|^p}{|x - y|^{d+sp}} dy & \text{for} \quad s \in (0, 1), \\ |\nabla g_k(x)|^p & \text{for} \quad s = 1, \end{cases} \]
\[ h_2(x) := \left| g(x) \right|^q \int_{\mathbb{R}^d} \frac{\left| g(y) \right|^q}{|x - y|^{d-\alpha}} dy, \quad \text{and} \quad h_{2,k}(x) := \left| g(x) \right|^q \int_{\mathbb{R}^d} \frac{\left| g_k(y) \right|^q}{|x - y|^{d-\alpha}} dy. \]

We claim that, for \( j = 1, 2 \),
\[ (2.19) \quad \sum_{k \in \mathbb{Z}} \left| h_{j,k} \right| \leq h_j \quad \text{and} \quad h_j \in L^1(\mathbb{R}^d). \]

We admit Claim (2.19) and continue the proof.

By Lemma 2.1, we have
\[ g_k^{\Delta \Delta}(x) \leq C \sup_Q \frac{1}{|Q|^\gamma} \left( \int_Q |h_1^k|^{\beta_1} \right) \left( \int_Q |h_2^k|^{\beta_2} \right), \]
where the supremum is taken over all cubes \( Q \) containing \( x \in \mathbb{R}^d \). Applying Lemma 2.2, we obtain, for \( k \in \mathbb{Z} \),
\[ \left| \left\{ g_k^\Delta > t \right\} \right| \leq \frac{C}{t^\gamma} \left\| h_1 \right\|_{L^1(\mathbb{R}^d)}^{\beta_1 \gamma} \left\| h_2 \right\|_{L^1(\mathbb{R}^d)}^{\beta_2 \gamma} \quad \text{for} \quad t > 0. \]
The conclusion now follows from Lemma 2.4.

It remains to prove Claim (2.19). We first establish Claim (2.19) with \( j = 1 \). Claim (2.19) with \( j = 1 \) is clear for \( s = 0 \) and \( s = 1 \). Claim (2.19) with \( j = 1 \) in the case \( s \in (0, 1) \) follows from the fact \( p \geq 1 \) and
\[
(2.20) \quad \sum_{k \in \mathbb{Z}} |g_k(x) - g_k(y)| \leq |g(x) - g(y)|,
\]
which can be proved as follows.

We first deal with the case \( |g(x)| \neq 0 \) and \( |g(y)| \neq 0 \). Let \( m, n \in \mathbb{Z} \) be such that
\[
10^m < |g(x)| \leq 10^{m+1} \quad \text{and} \quad 10^n < |g(y)| \leq 10^{n+1}.
\]
Without loss of generality, one might assume that \( |g(y)| \geq |g(x)| \) and this in turn implies \( n \geq m \).

We have, for \( k \in \mathbb{Z} \),
\[
g_k(x) = \begin{cases} 
0 & \text{for } k \geq m + 1, \\
|g(x)| - 10^m & \text{for } k = m, \\
10^{k+1} - 10^k & \text{for } k < m,
\end{cases}
\]
and
\[
g_k(y) = \begin{cases} 
0 & \text{for } k \geq n + 1, \\
|g(y)| - 10^n & \text{for } k = n, \\
10^{k+1} - 10^k & \text{for } k < n.
\end{cases}
\]
This yields, if \( n \geq m + 1 \),
\[
\sum_{k \in \mathbb{Z}} |g_k(x) - g_k(y)| = \sum_{k=-\infty}^{m} |g_k(x) - g_k(y)| + \sum_{k=m+1}^{n} |g_k(y)|
\]
\[
= |g(x)| - 10^{m+1} + \sum_{k=m+1}^{n-1} (10^{k+1} - 10^k) + |g(y)| - 10^n = |g(y)| - |g(x)| \leq |g(x) - g(y)|,
\]
and, if \( n = m \),
\[
\sum_{k \in \mathbb{Z}} |g_k(x) - g_k(y)| = |g_m(x) - g_m(y)| = |g(y)| - |g(x)| \leq |g(x) - g(y)|.
\]
Hence Assertion (2.20) is proved in this case. We next deal with Assertion (2.20) in the case \( |g(x)| = 0 \) or \( |g(y)| = 0 \). This follows from the fact, for \( z \in \mathbb{R}^d \),
\[
(2.21) \quad \sum_{k \in \mathbb{Z}} |g_k(z)| = |g(z)|.
\]
Claim (2.19) with \( j = 2 \) is just a consequence of (2.21) and the fact \( q \geq 1 \).

The proof is complete. \( \square \)

**Remark 2.2.** It is worth noting that (1.1) is false if (1.5) does not hold, which shows the optimality of (1.5). Indeed, let \( a \in \mathbb{R}^d \setminus \{0\} \) and \( \eta \in C^\infty(\mathbb{R}^d) \setminus \{0\} \) be such that \( 0 \leq \eta \leq 1 \), \( \eta = 1 \) in \( B_{1/8} \), and \( \text{supp } \eta \subset B_{1/4} \). For \( m \in \mathbb{N} \), define
\[
v_{m,a}(x) = \sum_{k=1}^{m} \eta(x + ka) \quad \text{for } x \in \mathbb{R}^d.
\]
Then \( v_{m,a} \in C_c^\infty(\mathbb{R}^d) \), and for \( |a| \to \infty \) we have
\[
\|v_{m,a}\|_{L^\gamma} \geq C m^{1/\gamma},
\]
\[
\|v_{m,a}\|_{W^{s,p}(\mathbb{R}^d)} \leq C m^{\beta_1}, \quad \text{and} \quad \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|v_{m,a}(x)|^q |v_{m,a}(y)|^q}{|x-y|^{d-\alpha}} \, dx \, dy \right)^{\beta_2} \leq C m^{\beta_2}.
\]
Here $C$ denotes a positive constant independent of $m$. Thus if (1.1) holds then $m \leq Cm^{\beta_1 \gamma + \beta_2 \gamma}$. This proves the optimality of (1.5).

2.3. Other forms of the assumptions of the Gagliardo-Nirenberg interpolation inequalities. In this section, we give another form of condition (1.5), which is often found in the literature. We begin with (compare with (1.6) for $p = 2$)

**Lemma 2.5.** Let $d \geq 1, 0 \leq s \leq 1, 1 \leq \gamma, p, q < \infty, 0 < \alpha < d$, and $0 \leq \beta_1, \beta_2 < +\infty$, and assume (1.3), and $(d + \alpha)p - 2q(d - sp) \neq 0$. Then (1.5) is equivalent to the fact

$$
\begin{align*}
&\begin{cases}
\frac{p(\alpha + 2qs)}{\alpha + sp} \leq \gamma \leq \frac{pd}{d - sp} & \text{if } sp < d \text{ and } (d + \alpha)p - 2q(d - sp) > 0, \\
\frac{pd}{d - sp} \leq \gamma \leq \frac{p(\alpha + 2qs)}{\alpha + sp} & \text{if } sp < d \text{ and } (d + \alpha)p - 2q(d - sp) < 0, \\
\frac{p(\alpha + 2qs)}{\alpha + sp} \leq \gamma < \infty & \text{if } sp \geq d.
\end{cases}
\end{align*}
$$

(2.22)

**Proof.** Since $(d + \alpha)p - 2q(d - sp) \neq 0$, it follows from (1.3) that

$$
\beta_1 = \frac{\gamma(d + \alpha) - 2qd}{\gamma(p(d + \alpha) - 2q(d - sp))} \quad \text{and} \quad \beta_2 = \frac{pd - \gamma(d - sp)}{\gamma(p(d + \alpha) - 2q(d - sp))}.
$$

(2.23)

We then have

$$
\beta_1 \gamma + \beta_2 \gamma - 1 = \frac{1}{p(d + \alpha) - 2q(d - sp)} [\gamma(d + \alpha) - 2qd + pd - \gamma(d - sp) - p(d + \alpha) + 2q(d - sp)]
$$

$$
= \frac{\alpha + sp}{p(d + \alpha) - 2q(d - sp)} \left[ \gamma - \frac{p(\alpha + 2qs)}{\alpha + sp} \right].
$$

Since $\beta_1, \beta_2 > 0$, we derive the equivalence of (1.5) and (2.22). \qed

We next establish (compare with (1.7) for $p = 2$)

**Lemma 2.6.** Let $d \geq 1, 0 \leq s \leq 1, 1 \leq \gamma, p, q < \infty, 0 < \alpha < d$, and $0 \leq \beta_1, \beta_2 < +\infty$, and assume (1.3), and $(d + \alpha)p - 2q(d - sp) = 0$. Then (1.5) holds iff the following two conditions hold

$$
\frac{\alpha(d - sp)}{pd(\alpha + sp)} \leq \beta_1 < +\infty, \quad 0 \leq \beta_2 \leq \frac{s(d - sp)}{d(\alpha + sp)}.
$$

(2.24)

**Proof.** Since $(d + \alpha)p - 2q(d - sp) = 0$, it follows that $sp < d$ and $q = \frac{p(d + \alpha)}{2d(sp)}$. By (1.3), we have

$$
(d - sp)\beta_1 + (d + \alpha)\beta_2 = \frac{d}{\gamma} \quad \text{and} \quad \beta_1 + \frac{d + \alpha}{d - sp}\beta_2 = \frac{1}{p}.
$$

This yields

$$
\gamma = \frac{pd}{d - sp}.
$$

We then have

$$
\beta_1 \gamma + \beta_2 \gamma - 1 = \frac{\beta_1 pd}{d - sp} - \frac{\beta_1 pd}{d + \alpha} + \frac{d}{d + \alpha} - 1 = \frac{\beta_1 (\alpha + sp)pd}{(d - sp)(d + \alpha)} - \frac{\alpha}{d + \alpha}.
$$

Therefore $\beta_1 \gamma + \beta_2 \gamma \geq 1$ if and only if $\beta_1 \geq \frac{\alpha(d - sp)}{pd(\alpha + sp)}$. Since $\beta_1 + \frac{d + \alpha}{d - sp}\beta_2 = \frac{1}{p}$, we derive that $\beta_1 \geq \frac{\alpha(d - sp)}{pd(\alpha + sp)}$ if and only if (2.24) holds. \qed
3. CAFFARELLI-KOHN-NIRENBERG INTERPOLATION TYPE INEQUALITIES INVOLVING COULOMB TERMS - PROOF OF THEOREM 1.2

The main goal of this section is to prove Theorem 1.2. We closely follow the techniques introduced by Nguyen and Squassina in [42] (see also [43]) to derive our results. We begin with a consequence of Theorem 1.1.

Lemma 3.1. Let \( d \geq 1, 0 \leq s \leq 1, 1 < \gamma < +\infty, 1 \leq \gamma', p, q < +\infty, 0 < \alpha < d, \) and \( 0 < \beta_1, \beta_2 < +\infty. \) Assume that (1.3) and (1.5) hold, and
\[
\gamma \geq \gamma'.
\]
Let \( \lambda > 0 \) and \( 0 < r < R, \) and set
\[
D_{\lambda} := \{ x \in \mathbb{R}^d : \lambda r < |x| < \lambda R \}.
\]
We have, for \( g \in L^1(D_{\lambda}), \)
\[
\left( \int_{D_{\lambda}} |g - (g)_{D_{\lambda}}|^{\gamma'} \right)^{1/\gamma'} \leq \frac{C}{\lambda^2} \|g\|_{W^{s,p}(D_{\lambda})}^{\beta p} \left( \int_{D_{\lambda}} \int_{D_{\lambda}} \frac{|g(x)|^q |g(y)|^q}{|x-y|^{d-\alpha}} \, dx \, dy \right)^{\beta_2},
\]
for some positive constant \( C \) independent of \( \lambda \) and \( g. \)

Proof. Using (1.3), by scaling we can assume that \( \lambda = 1. \) Lemma 3.1 is now a consequence of Theorem 1.1. \( \square \)

In the next two subsections, we present the proof (i) and (ii) of Theorem 1.2, respectively.

3.1. Proof of (i) of Theorem 1.2. We are ready to give the proof of Theorem 1.2. We closely follow the strategy in [42]. We only consider the case \( 0 < s < 1, \) the proof in general case follows similarly and is omitted.

The proof is divided into two steps.

- Step 1: We establish (i) of Theorem 1.2 assuming (1.12).
- Step 2: We establish (i) of Theorem 1.2 assuming (1.13) and \( \gamma' < \gamma. \)

It is clear that Assertion (i) then follows from Steps 1 and 2.

We now proceed Steps 1 and 2.

Step 1: We establish (i) of Theorem 1.2 assuming (1.12).

Set
\[
A_k := \{ x \in \mathbb{R}^d : 2^k \leq |x| < 2^{k+1} \}.
\]
By Lemma 3.1, we derive from (1.11) that
\[
\left( \int_{A_k} |g - (g)_{A_k}|^{\gamma'} \right)^{1/\gamma'} \leq \frac{C}{2^k \gamma} \left( \int_{A_k} \int_{A_k} \frac{|g(x) - g(y)|^p}{|x-y|^{d+sp}} \, dx \, dy \right)^{\beta_1} \times \left( \int_{A_k} \int_{A_k} \frac{|g(x)|^q |g(y)|^q}{|x-y|^{d-\alpha}} \, dx \, dy \right)^{\beta_2}.
\]

Here and in what follows in the proof of Theorem 1.2, \( C \) denotes a positive constant independent of \( g \) and \( k \) (and also independent of \( m, \) and \( n, \) which appear later). Since
\[
2^{\tau' \gamma' k} \int_{A_k} |g|^{\gamma'} \leq C 2^{(\tau' \gamma' + d)k} \int_{A_k} |g - (g)_{A_k}|^{\gamma'} + C 2^{(\tau' \gamma' + d)k} \left( \int_{A_k} g \right)^{\gamma'},
\]
using condition (1.10) and the definition of $\sigma$, we derive that

$$
\int_{A_k} \left| g^{\gamma'} |x|^{\tau' \gamma} \right| dx \leq C 2^{(\gamma' \tau' + d)k} \left\| f_{A_k} \right\|^{\gamma'}
$$

$$
+ C \left( \int_{A_k} \int_{A_k} \frac{|g(x) - g(y)|^p |x|^{\alpha_1,1p} |y|^{\alpha_1,2p}}{|x-y|^{d+sp}} \, dx \, dy \right)^{\gamma' \beta_1} \times \left( \int_{A_k} \int_{A_k} \frac{|g(x)|^q |g(y)|^q |x|^{\alpha_2,1q} |y|^{\alpha_2,2q}}{|x-y|^{d-\alpha}} \, dx \, dy \right)^{\gamma' \beta_2}.
$$

Let $m, n \in \mathbb{Z}$ be such that $m \leq n - 2$. Summing (3.2) with respect to $k$ from $m$ to $n$, we get

$$
\int_{\{m < |x| < 2^{n+1}\}} \left| g^{\gamma'} |x|^{\tau' \gamma'} \right| dx \leq C \sum_{k=m}^{n} 2^{(\gamma' \tau' + d)k} \left\| f_{A_k} \right\|^{\gamma'}
$$

$$
+ C \sum_{k=m}^{n} \left( \int_{A_k} \int_{A_k} \frac{|g(x) - g(y)|^p |x|^{\alpha_1,1p} |y|^{\alpha_1,2p}}{|x-y|^{d+sp}} \, dx \, dy \right)^{\gamma' \beta_1} \times \left( \int_{A_k} \int_{A_k} \frac{|g(x)|^q |g(y)|^q |x|^{\alpha_2,1q} |y|^{\alpha_2,2q}}{|x-y|^{d-\alpha}} \, dx \, dy \right)^{\gamma' \beta_2}.
$$

Applying Lemma 2.3, we derive from (3.3) that

$$
\int_{\{m < |x| < 2^{n+1}\}} \left| g^{\gamma'} |x|^{\tau' \gamma'} \right| dx \leq C \sum_{k=m}^{n} 2^{(\gamma' \tau' + d)k} \left\| f_{A_k} \right\|^{\gamma'}
$$

$$
+ C \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|g(x) - g(y)|^p |x|^{\alpha_1,1p} |y|^{\alpha_1,2p}}{|x-y|^{d+sp}} \, dx \, dy \right)^{\gamma' \beta_1} \times \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|g(x)|^q |g(y)|^q |x|^{\alpha_2,1q} |y|^{\alpha_2,2q}}{|x-y|^{d-\alpha}} \, dx \, dy \right)^{\gamma' \beta_2}.
$$

We next estimate the first term of the RHS of (3.4). We have, as in (3.1),

$$
\left\| f_{A_k} - f_{A_{k+1}} \right\|^{\gamma'} \leq C \frac{2^k}{2^{\gamma'}} \left( \int_{A_k \cup A_{k+1}} \int_{A_k \cup A_{k+1}} \frac{|g(x) - g(y)|^p}{|x-y|^{d+sp}} \, dx \, dy \right)^{\gamma' \beta_1}
$$

$$
\times \left( \int_{A_k \cup A_{k+1}} \int_{A_k \cup A_{k+1}} \frac{|g(x)|^q |g(y)|^q}{|x-y|^{d-\alpha}} \, dx \, dy \right)^{\gamma' \beta_2}.
$$

With $c := 2/(1 + 2^\gamma \tau' + d) < 1$ (since $\gamma' \tau' + d > 0$), we have $c 2^\gamma \tau' + d > 1$. We derive from (3.5) that

$$
2^{(\gamma' \tau' + d)k} \left\| f_{A_k} \right\|^{\gamma'} \leq c 2^{(\gamma' \tau' + d)(k+1)} \left\| f_{A_{k+1}} \right\|^{\gamma'}
$$

$$
+ C \left( \int_{A_k \cup A_{k+1}} \int_{A_k \cup A_{k+1}} \frac{|g(x) - g(y)|^p |x|^{\alpha_1,1p} |y|^{\alpha_1,2p}}{|x-y|^{d+sp}} \, dx \, dy \right)^{\gamma' \beta_1}
$$

$$
\times \left( \int_{A_k \cup A_{k+1}} \int_{A_k \cup A_{k+1}} \frac{|g(x)|^q |g(y)|^q |x|^{\alpha_2,1q} |y|^{\alpha_2,2q}}{|x-y|^{d-\alpha}} \, dx \, dy \right)^{\gamma' \beta_2}.
$$
\begin{equation}
\left(\int_{A_k \cup A_{k+1}} \int_{A_k \cup A_{k+1}} \frac{|g(x)|^q |g(y)|^q |x|^{\alpha_2 q_1}|y|^{\alpha_2 q_2}}{|x-y|^{d-\alpha}} \, dx \, dy \right)^{\gamma' \beta_2} \times \left(\int_{A_k \cup A_{k+1}} \int_{A_k \cup A_{k+1}} \frac{|g(x)|^q |g(y)|^q |x|^{\alpha_2 q_1}|y|^{\alpha_2 q_2}}{|x-y|^{d-\alpha}} \, dx \, dy \right)^{\gamma' \beta_2}.
\end{equation}

Since \( g \) has a compact support, we derive that, for large \( n \),

\begin{equation}
\sum_{k=m}^{n} 2^{(\gamma' + d)k} \left| \int_{A_k} g \right|^{\gamma'} \leq C \sum_{k=m}^{n} \left( \int_{A_k \cup A_{k+1}} \int_{A_k \cup A_{k+1}} \frac{|g(x) - g(y)|^p |x|^{\alpha_1 p_1}|y|^{\alpha_1 p_2}}{|x-y|^{d+sp}} \, dx \, dy \right)^{\gamma' \beta_1} \times \left( \int_{A_k \cup A_{k+1}} \int_{A_k \cup A_{k+1}} \frac{|g(x)|^q |g(y)|^q |x|^{\alpha_2 q_1}|y|^{\alpha_2 q_2}}{|x-y|^{d-\alpha}} \, dx \, dy \right)^{\gamma' \beta_2}.
\end{equation}

Applying Lemma 2.3 and letting \( m \to -\infty \), we obtain

\begin{equation}
\sum_{k \in \mathbb{Z}} 2^{(\gamma' + d)k} \left| \int_{A_k} g \right|^{\gamma'} \leq C \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|g(x) - g(y)|^p |x|^{\alpha_1 p_1}|y|^{\alpha_1 p_2}}{|x-y|^{d+sp}} \, dx \, dy \right)^{\gamma' \beta_1} \times \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|g(x)|^q |g(y)|^q |x|^{\alpha_2 q_1}|y|^{\alpha_2 q_2}}{|x-y|^{d-\alpha}} \, dx \, dy \right)^{\gamma' \beta_2}.
\end{equation}

Combining (3.4) and (3.8) and letting \( n \to +\infty \), \( m \to -\infty \), we obtain (i) of Theorem 1.2. The proof of Step 1 is complete.

**Step 2:** We establish (i) of Theorem 1.2 assuming (1.13) and \( \gamma' < \gamma \).

Since \( \frac{1}{p}(sp - d - \alpha_1 p) + \frac{1}{2q}(\alpha + d + \alpha_2 q) \neq 0 \), by scaling, without loss of generality, one might assume that

\begin{equation}
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|g(x) - g(y)|^p |x|^{\alpha_1 p_1}|y|^{\alpha_1 p_2}}{|x-y|^{d+sp}} \, dx \, dy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|g(x)|^q |g(y)|^q |x|^{\alpha_2 q_1}|y|^{\alpha_2 q_2}}{|x-y|^{d-\alpha}} \, dx \, dy = 1.
\end{equation}

It then suffices to prove that

\begin{equation}
\left\| \left| \cdot \right|^\gamma g \right\|_{L^{\gamma'}(\mathbb{R}^d)} \leq C.
\end{equation}

Let \( \beta_{1,1}, \beta_{1,2} \) and \( \beta_{2,1}, \beta_{2,2} \) close to \( \beta_1 \) and \( \beta_2 \) respectively and non-negative be determined later such that \( \beta_{j,1}p + 2\beta_{j,2}q = 1 \), which is equivalent to

\[ p(\beta_{j,1} - \beta_1) + 2q(\beta_{j,2} - \beta_2) = 0. \]

Define \( \sigma_j, \gamma_j, \gamma'_j \), and \( \tau_j \) for \( j = 1, 2 \), as follows

\[ \sigma_j = \beta_{j,1}p\alpha_1 + \beta_{j,2}q\alpha_2, \]

\[ \gamma_j = (d-sp)\beta_{j,1} + (d+\alpha)\beta_{j,2} = \frac{d}{\gamma_j}. \]

\[ \gamma'_j = \gamma_j, \quad \tau'_j = \sigma_j. \]

We have

\begin{equation}
\frac{1}{\gamma_j} + \frac{\tau'_j}{d} - \frac{1}{\gamma_j} - \frac{\tau'}{d} = \frac{1}{\gamma_j} + \frac{\sigma_j}{d} - \frac{1}{\gamma} - \frac{\sigma}{d} = \frac{1}{d} \left( (d-sp+p\alpha_1)(\beta_{j,1} - \beta_1) + (d+\alpha+q\alpha_2)(\beta_{j,2} - \beta_2) \right).
\end{equation}
Since $2q(d - sp + p\alpha_1) \neq p(d + \alpha + q\alpha_2)$ and $\beta_1, \beta_2 > 0$, one can choose positive $\beta_{j,1}$ and $\beta_{j,2}$ close to $\beta_1$ and $\beta_2$ such that

\begin{equation}
(3.12) \quad p(\beta_{j,1} - \beta_1) + 2q(\beta_{j,2} - \beta_2) = 0,
\end{equation}

\begin{equation}
(3.13) \quad \frac{1}{\gamma_2} + \frac{\tau_j'}{d} < \frac{1}{\gamma_1} + \frac{\tau_j'}{d} < \frac{1}{\gamma_1} + \frac{\tau_j'}{d}.
\end{equation}

Since $\beta_{j,1}$ and $\beta_{j,2}$ are close to $\beta_1$ and $\beta_2$ and $\beta_1 \gamma + \beta_2 \gamma > 1$, we have

\begin{equation}
(3.14) \quad \beta_{j,1} \gamma_j' + \beta_{j,2} \gamma_j' = \beta_{j,1} \gamma_j + \beta_{j,2} \gamma_j > 1 \text{ for } j = 1, 2,
\end{equation}

and

\begin{equation}
(3.15) \quad \gamma_j' = \gamma_j > \gamma' \text{ for } j = 1, 2.
\end{equation}

Combining (3.13) and (3.15) yields

\begin{equation}
(3.16) \quad \| | \cdot |'g \|_{L^{\gamma'}(\mathbb{R}^d \setminus B_1)} \leq C \| | \cdot |'g \|_{L^{\gamma_1'}(\mathbb{R}^d \setminus B_1)},
\end{equation}

and

\begin{equation}
(3.17) \quad \| | \cdot |'g \|_{L^{\gamma'}(B_1)} \leq C \| | \cdot |'g \|_{L^{\gamma_1'}(B_1)}.
\end{equation}

On the other hand, applying Step 1, we have, by (3.9),

\begin{equation}
(3.18) \quad \| | \cdot |'g \|_{L^{\gamma_1'}(\mathbb{R}^d)} \leq C \quad \text{and} \quad \| | \cdot |'g \|_{L^{\gamma_1'}(\mathbb{R}^d)} \leq C.
\end{equation}

Combining (3.9), (3.16), and (3.17) yields (3.10). The proof of Step 2 is complete.

The proof of (i) of Theorem 1.2 is complete. \qed

3.2. Proof of (ii) of Theorem 1.2. The proof of (ii) of Theorem 1.2 is similar to that of (i). We only mention briefly the proof of (ii) assuming (1.12). We also have (3.4). To estimate the RHS of (3.4), one just needs to note that, instead of (3.6), with $\hat{\varepsilon} := (1 + 2^{\gamma_1' + d})/2 < 1$ (since $\gamma' \tau' + d < 0$),

\begin{equation}
2^{\gamma_1'} \int_{A_{k+1}} g \gamma' \leq \hat{\varepsilon} 2^{\gamma_1'} \int_{A_k} g \gamma'
\end{equation}

\begin{equation}
+ C \left( \int_{A_k \cup A_{k+1}} \int_{A_k \cup A_{k+1}} \frac{|g(x) - g(y)|^p \varphi_{a_1,b}(x,y)}{|x - y|^{d + sp}} dxdy \right)^{\gamma_1' \beta_1}
\end{equation}

\begin{equation}
\times \left( \int_{A_k \cup A_{k+1}} \int_{A_k \cup A_{k+1}} \frac{|g(x)|^q |g(y)|^q \varphi_{a_2,b}(x,y)}{|x - y|^{d - \alpha}} dxdy \right)^{\gamma_1' \beta_2}.
\end{equation}

Summing with respect to $k$, we also obtain (3.8). The conclusion now follows from (3.4) and (3.8).

4. Hardy-Lieb-Therring Inequalities

In this section, as an application of Theorem 1.2, we establish the following family of many body Hardy-Lieb-Therring inequalities.
Theorem 4.1. Let \( d \geq 1 \) and \( N \geq 1 \), \( 0 < s \leq 1 \) and \( 2 \leq p < \infty \) be such that \( sp < d \). Given \( \psi \in C^\infty_c(\mathbb{R}^{dN}) \) with \( \int_{\mathbb{R}^{dN}} |\psi(X)|^p \, dX = 1 \), define

\[
\rho \psi(x) := \sum_{i=1}^N \int_{\mathbb{R}^d} \frac{\psi(x_i \, x, x_i \, x_i^L)}{|x_i - x_i|^{d+sp}} \, dx_i \, dy_i
\]

where we have denoted, for \( X = (x_1, \ldots, x_N) \) with \( x_i \in \mathbb{R}^d \),

\[
X_i^R = (x_1, \ldots, x_{i-1}) \quad \text{and} \quad X_i^L = (x_{i+1}, \ldots, x_N) \quad \text{for } 1 \leq i \leq N.
\]

Then there exists a positive constant \( C \) depending only on \( s, p, d \) (\( C \) is independent of \( N \)) such that

\[
E(\psi) + \sum_{1 \leq i < j \leq N} \int_{\mathbb{R}^{dN}} \frac{|\psi(X)|^p \, dX}{|x_i - x_j|^{sp}} \geq C \int_{\mathbb{R}^d} |\rho \psi(x)|^{1+\frac{d}{sp}} \, dx.
\]

Here, for \( 0 < s < 1 \),

\[
E(\psi) := \sum_{i=1}^N \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}^d} \frac{|\nabla_x \psi(x_i \, x, x_i \, x_i^L)|^p \, dx_i \, dy_i}{|x_i - x_i|^{sp}} - C_{d,s,p} \int_{\mathbb{R}^d} \frac{|\psi(x_i \, x, x_i \, x_i^L)|^p \, dx_i}{|x_i|^{sp}} \right] \, dX_i^R \, dX_i^L,
\]

and, for \( s = 1 \),

\[
E(\psi) := \sum_{i=1}^N \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}^d} \frac{|\nabla_x \psi(x_i \, x, x_i \, x_i^L)|^p \, dx_i}{|x_i|^{sp}} - C_{d,s,p} \int_{\mathbb{R}^d} \frac{|\psi(x_i \, x, x_i \, x_i^L)|^p \, dx_i}{|x_i|^{sp}} \right] \, dX_i^R \, dX_i^L.
\]

The constant \( C_{d,s,p} \) in Theorem 4.1 is defined by, for \( 0 < s < 1 \),

\[
C_{d,s,p} := 2 \int_0^1 r^{sp-1} \left| 1 - r^\frac{d-2}{sp} \right|^p \, d \Phi_{d,s,p}(r),
\]

where

\[
\Phi_{d,s,p} = \begin{cases} \mathbb{S}^{d-2} \int_0^1 \frac{(1-t)^\frac{d-2}{sp}}{(1-2rt+r^2)^\frac{d-2}{sp}} \, dt & \text{for } d \geq 2, \\ \frac{1}{(1-r)^{1+ps}} + \frac{1}{(1+r)^{1+ps}} & \text{for } d = 1, \end{cases}
\]

and

\[
C_{d,1,p} = \left( \frac{n-p}{p} \right)^p.
\]

When \( s = 1, p = 2 \), the following many body inequality was derived by Lieb and Thirring [29, 30] in order to give a simpler proof of the stability of non-relativistic matter first given by Dyson and Lenard [21].

\[
E(\psi) + \sum_{i=1}^N \int_{\mathbb{R}^d} \rho \psi(x_i) \, dx_i \geq C_{LT} \int_{\mathbb{R}^d} \rho \psi \frac{\Delta \psi}{\rho} \, dx,
\]

where \( \psi \in H^1(\mathbb{R}^{dN}) \), anti-symmetric and normalized in \( L^2(\mathbb{R}^{dN}) \). \( \rho \psi \) is same as in (4.1), and \( C_{LT} > 0 \) is a constant independent of \( \psi \) and \( N \). It is easy to see that, (4.6) is no longer true if \( \psi \) is not anti-symmetric e.g. if \( \psi(x_1, \ldots, x_N) = u(x_1) \ldots u(x_N) \), which is a typical state of boson. In [33], Lundholm, Portmann and Solovej noticed that Lieb-Thirring type inequalities still hold true for
particles without any symmetry assumptions and therefore in particular for bosons, provided that the anti-symmetry assumption is replaced by a sufficiently strong repulsive interaction between particles. More precisely, they established (4.3) for \( s = 1, \ p = 2 \) in the absence of the inverse square potential \( \frac{1}{|x|^2} \). Subsequently, Lundholm, Nam and Portmann [34] established an improved version of this inequality. In fact, they proved (4.3) for \( s > 0 \) and \( p = 2 \). Our approach of proving Theorem 4.1 is as follows. First using Theorem 1.2 and a (fractional) Hardy inequality due to Frank and Seiringer [16], we derive Proposition 4.1 below. Then we follow the strategy of [34] to derive Theorem 4.1 from Proposition 4.1.

To prove Theorem 4.1, we will establish the following Hardy-Lieb-Thirring inequality.

**Proposition 4.1.** Let \( d \geq 1 \), \( 0 < s \leq 1 \) and \( p \geq 2 \) be such that \( sp < d \). Then there exist \( C > 0 \) such that for any \( u \in \mathcal{C}^\infty_0 (\mathbb{R}^d) \) we have

\[
F(u)^{1 - \frac{sp}{d}} \times \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^p |u(y)|^p \, dx \, dy}{|x - y|^{sp}} \right)^{\frac{sp}{d}} \geq C \int_{\mathbb{R}^d} |u|^{\frac{p(d + sp)}{d}} \, dx,
\]

where

\[
F(u) := \begin{cases} 
\left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p \, dx \, dy}{|x - y|^{d + sp}} - C_{d,s,p} \int_{\mathbb{R}^d} |u(x)|^p \, dx \right) & \text{for } 0 < s < 1, \\
\int_{\mathbb{R}^d} |\nabla u(x)|^p \, dx - C_{d,1,p} \int_{\mathbb{R}^d} |u(x)|^p \, dx & \text{for } s = 1.
\end{cases}
\]

The proof of Proposition 4.1 contains two main ingredients. The first one is Theorem 1.2 and the second one is a (fractional) Hardy inequality due to Frank and Seiringer [16].

The rest of this section containing two subsection is organized as follows. We prove Proposition 4.1 and Theorem 4.1 in the first subsection and the second subsection, respectively.

**4.1. Proof of Proposition 4.1.** Let \( u \in \mathcal{C}^\infty_0 (\mathbb{R}^d) \) and define \( \varphi(x) := |x|^{(d - sp)/p} u(x) \). We first consider the case \( 0 < s < 1 \). We have, see [16, Theorem 1.2],

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p}{|x - y|^{N + sp}} \, dx \, dy - C_{d,s,p} \int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{sp}} \, dx \geq c_p \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\left| \varphi(x) - \varphi(y) \right|^p |x|^{-d - sp} |y|^{-d - sp}}{|x - y|^{d + sp}} \, dx \, dy.
\]

where \( c_p := \min_{0 < r < \frac{d}{2}} ((1 - r)p - r p + p r p - 1) \).

We apply Theorem 1.2 to \( \varphi \) with

\[
p = q \geq 2, \quad \gamma' = (d + sp) \frac{p}{d}, \quad \alpha = d - sp, \quad \beta_1 = \frac{d - sp}{p(d + sp)}, \quad \beta_2 = \frac{s}{d + sp}, \quad \tau' = -\frac{d - sp}{p}, \quad \alpha_{1,1} = \alpha_{1,2} = -\frac{d - sp}{2p}, \quad \text{and} \quad \alpha_{2,1} = \alpha_{2,2} = -\frac{d - sp}{p}.
\]

We then have

\[
\alpha_1 = \alpha_{1,1} + \alpha_{1,2} = -\frac{d - sp}{p}, \quad \alpha_2 = \alpha_{2,1} + \alpha_{2,2} = -\frac{2(d - sp)}{p}, \quad \sigma = \beta_1 \rho \alpha_1 + \beta_2 \rho \alpha_2 = -(d - sp)/p = \tau', \quad \frac{1}{\gamma} = \frac{1}{d} [(d - sp) \beta_1 + (d + \alpha) \beta_2] = \frac{d}{p(d + sp)} s = \frac{1}{\gamma'}.
\]
One can check that (1.10) holds and $\frac{1}{p} + \frac{r'}{d} = \frac{2p}{d(d + sp)} > 0$.

We then obtain

\begin{equation}
\left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\varphi(x) - \varphi(y)|^p |x|^{-d-sp} |y|^{-d-sp} |x - y|^{-2}}{|x - y|^{d + sp}} dx dy\right)^{1 - \frac{sp}{d}} \times \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\varphi(x)^p |\varphi(y)|^p |x|^{-p(d - sp)} |y|^{-p(d - sp)} dx dy}{|x - y|^{sp}}\right)^{\frac{2p}{d}} \geq C \int_{\mathbb{R}^d} \left| x \right|^{-d - sp} \frac{\varphi(x)}{\left| x \right|^{\frac{d + sp}{d}}} dx,
\end{equation}

Putting $\varphi = |x|^{-d - sp} u$ in (4.10) and then using (4.9), we derive (4.8).

The proof in the case $0 < s < 1$ is complete.

The proof in the case $s = 1$ follows similarly. In this case, instead of (4.9), one has, see [16, Remark 2.5] (see also [20, Theorem 1]),

\begin{equation}
\int_{\mathbb{R}^d} |\nabla u(x)|^p dx - C_{d,1,p} \int_{\mathbb{R}^d} \frac{|u(x)|^p dx}{|x|^{sp}} \geq c_p \int_{\mathbb{R}^d} \frac{|\varphi(x)|^p |x|^{-p(d - sp)} dx}{|x|^{sp}}.
\end{equation}

The rest of the proof is almost unchanged and is omitted. \hfill \Box

4.2. Proof of Theorem 4.1. The proof of Theorem 4.1 is based on Proposition 4.1 and the following three lemmas. The first one is

**Lemma 4.2.** Let $d \geq 1$, $N \geq 1$, $0 < s < 1$, $1 \leq p < +\infty$, $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\psi$ be a measurable function defined in $\mathbb{R}^{dN}$. We have

\begin{equation}
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(\rho \psi)^\frac{1}{p}(x) - (\rho \psi)^\frac{1}{p}(y)|x|^{\alpha_1 p} |y|^{\alpha_2 p}}{|x - y|^{d + sp}} dx dy \leq \sum_{i=1}^N \int_{\mathbb{R}^{d(N-1)}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\psi(X_i^R, x_i, X_i^L) - \psi(X_i^R, y_i, X_i^L)|^p |x_i|^{\alpha_1 p} |y_i|^{\alpha_2 p}}{|x_i - y_i|^{d + sp}} dx_i dy_i dX_i^R dX_i^L.
\end{equation}

Here $\rho \psi$ is defined by (4.1), and $X_i^R$ and $X_i^L$ are given by (4.2).

For $s = 1$, we use the following lemma.

**Lemma 4.3.** Let $d \geq 1$, $N \geq 1$, $1 \leq p < +\infty$, $\alpha_1 \in \mathbb{R}$ and $\psi$ be a smooth function defined in $\mathbb{R}^{dN}$. We have

\begin{equation}
\int_{\mathbb{R}^d} \left| \nabla (\rho \psi)^\frac{1}{p}(x) \right|^p |x|^{\alpha_1 p} dx \leq \sum_{i=1}^N \int_{\mathbb{R}^{dN}} |\nabla_{x_i} \psi(X)|^p |x_i|^{\alpha_1 p} dX.
\end{equation}

Here $\rho \psi$ is defined by (4.1).

Inequality (4.13) was first discovered by Hoffman–Ostenhof in [19], $p = 2$ and $\alpha_1 = \alpha_2 = 0$. Later on, in [28, Lemma 8.4], (4.12), (4.13) was proved for $p = 2$ and $\alpha_1 = \alpha_2 = 0$. The proofs in the general cases stated here follows in the same spirit. For the convenience of the proof, we give the proof in Appendix A.

The third lemma used in the proof of Theorem 4.1 is
**Lemma 4.4.** Let $d \geq 1$ and $1 \leq p < +\infty$. For every $0 < \gamma < d$ and for every $\psi \in L^p(\mathbb{R}^{dN})$ with $\int |\psi(X)|^p dX = 1$, we have

$$\int_{\mathbb{R}^{dN}} \sum_{1 \leq i < j \leq N} \frac{|\psi(X)|^p}{|x_i - x_j|^\gamma} dX \geq \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\rho \psi(x) \rho \psi(y)}{|x - y|^\gamma} dxdy - C_{LO} \int_{\mathbb{R}^d} |\rho \psi(x)|^{1+\frac{\gamma}{d}} dx$$

for a constant $C_{LO} > 0$ depending only on $d$, $\gamma$ and $p$.

Inequality (4.14) was first studied in [23, 24] for the case $\gamma = 1$, $d = 3$ and $p = 2$ and known under the name the Lieb-Oxford inequality for homogeneous potential. Subsequently, it was derived in [25, Lemma 5.3] for $\gamma = 1$, $d = 2$, $p = 2$ and in [34, Lemma 16] for the case $0 < \gamma < d$ and $p = 2$. Interestingly, the proof of Lemma 4.4 in the assumption stated there does not differ much from that of [25] or [34] in the case $p = 2$, and is given in Appendix B for the completeness.

We are ready to give

**Proof of Theorem 4.1.** We only consider the case $0 < s < 1$, the proof in general case follows similarly and is omitted. We have

$$\sum_{i=1}^N \int_{\mathbb{R}^{dN}} \frac{|\psi(X)|^p}{|x_i|^{sp}} dX = \int_{\mathbb{R}^d} \frac{\rho \psi(x)}{|x|^{sp}} dx.$$  

Applying Lemma 4.2 and using (4.15), we obtain

$$\sum_{i=1}^N \int_{\mathbb{R}^{d(N-1)}} \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\psi(X_i^R, x_i, X_i^L) - \psi(X_i^R, y_i, X_i^L)|^p}{|x_i - y_i|^{d+sp}} dx_idy_i ight] dX_i^RdX_i^L$$

$$\geq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(\rho \psi(x))^\frac{1}{p} - (\rho \psi(y))^\frac{1}{p}}{|x - y|^{d+sp}} dxdy - C_{d,s,p} \int_{\mathbb{R}^d} \frac{\rho \psi(x)}{|x|^{sp}} dx.$$

By (4.14), we obtain, for $0 < \varepsilon < 1$,

$$\varepsilon \sum_{1 \leq i < j \leq N} \int_{(\mathbb{R}^d)^N} \frac{|\psi(X)|^p dX}{|x_i - x_j|^{sp}} \geq \frac{\varepsilon}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\rho \psi(x) \rho \psi(y) dxdy}{|x - y|^{sp}} - C_{LO} \varepsilon \int_{\mathbb{R}^d} |\rho \psi(x)|^{1+\frac{\gamma}{d}} dx.$$

Combining (4.16) and (4.17) yields

$$\sum_{i=1}^N \int_{(\mathbb{R}^d)^{N-1}} \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\psi(X_i^R, x_i, X_i^L) - \psi(X_i^R, y_i, X_i^L)|^p}{|x_i - y_i|^{d+sp}} dx_idy_i ight] dX_i^RdX_i^L$$

$$\geq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(\rho \psi(x))^\frac{1}{p} - (\rho \psi(y))^\frac{1}{p}}{|x - y|^{d+sp}} dxdy - C_{d,s,p} \int_{\mathbb{R}^d} \frac{\rho \psi(x)}{|x|^{sp}} dx.$$
Since, by Young’s inequality after noting that \( sp/d < 1 \),

\[
\frac{\varepsilon}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\rho \psi(x) \rho \psi(y) \, dx \, dy}{|x-y|^sp} - C_{LO} \varepsilon \int_{\mathbb{R}^d} |\rho \psi(x)|^{1+\frac{sp}{d}}.
\]

By choosing \( \varepsilon \) sufficiently so that \( C \varepsilon^{\frac{sp}{d}} - \varepsilon C_{LO} > 0 \) (this can be done since \( sp < d \)), we derive from (4.18) and (4.19) the conclusion. The proof is complete. \( \square \)

**APPENDIX A. PROOF OF LEMMA 4.2 AND 4.3**

First we consider \( s = 1 \), i.e., we establish Lemma 4.3. In this case (4.13) is a direct consequence of the following estimate which is proved by using Hölder’s inequality with exponents \( p, \frac{d}{sp-1} \).

\[
|\nabla_x \psi(x)|^p \leq (\rho \psi(x))^{p-1} \sum_{i=1}^{N} \int_{\mathbb{R}^d(N-1)} |\nabla_x \psi(X_i^R, x, X_i^L)|^p \, dX_i^R \, dX_i^L.
\]

We next consider \( 0 < s < 1 \), i.e., we prove Lemma 4.2. We have, by (4.1),

\[
(\rho \psi)^{\frac{1}{p}} (x) - (\rho \psi)^{\frac{1}{p}} (y) = \left( \sum_{i=1}^{\frac{N}{d(N-1)}} |\psi(X_i^R, x, X_i^L)|^p \, dX_i^R \, dX_i^L \right)^{\frac{1}{p}} - \left( \sum_{i=1}^{\frac{N}{d(N-1)}} |\psi(X_i^R, y, X_i^L)|^p \, dX_i^R \, dX_i^L \right)^{\frac{1}{p}}.
\]

By Minkowski’s inequality,

\[
\text{the RHS of (A.1)} \leq \sum_{i=1}^{\frac{N}{d(N-1)}} |\psi(X_i^R, x, X_i^L) - \psi(X_i^R, y, X_i^L)|^p \, dX_i^R \, dX_i^L.
\]
Combining (A.1) and (A.2) yields
\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|(\rho \psi)_{\hat{1}}(x) - (\rho \psi)_{\hat{1}}(y)|^p |x|^{\alpha_1 p} |y|^{\alpha_2 p}}{|x - y|^{d+sp}} dxdy \\
\leq \sum_{i=1}^{N} \int_{\mathbb{R}^{d(N-1)}} \int_{\mathbb{R}^d} \frac{\left|\psi(X_i^R, x, X_i^L) - \psi(Y_i^R, y, X_i^L)\right|^p |x|^{\alpha_1 p} |y|^{\alpha_2 p}}{|x - y|^{d+sp}} dxdy X_i^R dX_i^L,
\]
which is the conclusion.

\[\square\]

**Appendix B. Proof of Lemma 4.4**

Let \( \chi_R \) denote the characteristic function of the ball \( \overline{B}(0,R) \). Then the following identity, due to Fefferman-de la Llave [12] (see also, [27, Theorem 9.8]) holds
\[
(B.1) \quad \frac{1}{|x - y|^\gamma} = c_{d,\gamma} \int_0^\infty \int_{\mathbb{R}^d} \chi_R(x-z) \chi_R(y-z) \frac{dzdR}{R^{d+\gamma+1}}.
\]
for some positive constant \( c_{\gamma,d} \) depending only on \( \gamma \) and \( d \). Denote
\[
f_R(z) = \int_{\mathbb{R}^d} \rho \psi(x) \chi_R(x-z) dx \quad \text{for} \quad z \in \mathbb{R}^d.
\]
It follows from (B.1) that
\[
(B.2) \quad \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\rho \psi(x) \rho \psi(y)}{|x - y|^\gamma} dxdy = c_{d,\gamma} \int_0^\infty \int_{\mathbb{R}^d} f_R^2(z) \frac{dzdR}{R^{d+\gamma+1}},
\]
Using (B.1), we also have
\[
(B.3) \quad \int_{\mathbb{R}^d} |\psi(X)|^p \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|^\gamma} dX
\]
\[
= c_{d,\gamma} \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\psi(X)|^p \sum_{1 \leq i < j \leq N} \chi_R(x_i - z) \chi_R(x_j - z) \frac{dX dz dR}{R^{d+\gamma+1}}.
\]
We have
\[
(B.4) \quad \sum_{1 \leq i < j \leq N} \chi_R(x_i - z) \chi_R(x_j - z) = \frac{1}{2} \left( \sum_{i=1}^{N} \chi_R(x_i - z) \right)^2 - \frac{1}{2} \sum_{i=1}^{N} \chi_R(x_i - z)
\]
\[
= \frac{1}{2} \left( \sum_{i=1}^{N} \chi_R(x_i - z) - f_R(z) \right)^2 + f_R(z) \sum_{i=1}^{N} \chi_R(x_i - z) - \frac{1}{2} f_R^2(z) - \frac{1}{2} \sum_{i=1}^{N} \chi_R(x_i - z).
\]
and
\[
(B.5) \quad \int_{\mathbb{R}^d} |\psi(X)|^p \sum_{i=1}^{N} \chi_R(x_i - z) dX = \sum_{i=1}^{N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d(N-1)}} \chi_R(x_i - z) |\psi(X_i^R, x_i, X_i^L)|^p dx_i dX_i^R dX_i^L
\]
\[
= \int_{\mathbb{R}^d} \chi_R(x - z) \rho \psi(x) dx = f_R(z)
\]
From (B.4) we obtain
(B.6) \[ \int_{\mathbb{R}^d} |\psi(X)|^p \sum_{1 \leq i < j \leq N} \chi_R(x_i - z) \chi_R(x_j - z) dX \]
\[
\geq \int_{\mathbb{R}^d} |\psi(X)|^p \left( f_R(z) \sum_{i=1}^N \chi_R(x_i - z) - \frac{1}{2} f_R^2(z) - \frac{1}{2} \sum_{i=1}^N \chi_R(x_i - z) \right). \]

We then derive from (B.5) and the fact \( \int_{\mathbb{R}^d} |\psi(X)|^p dX = 1 \)

(B.7) \[ \int_{\mathbb{R}^d} |\psi(X)|^p \sum_{1 \leq i < j \leq N} \chi_R(x_i - z) \chi_R(x_j - z) dX \geq \frac{1}{2} f_R^2(z) - \frac{1}{2} f_R(z). \]

Plugging (B.7) into (B.3), using (B.2) and the fact \( f_R(z) \geq \min\{f_R(z), f_R^2(z)\} \), we obtain

(B.8) \[ \int_{\mathbb{R}^d} |\psi(X)|^p \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|^\gamma} dX \]
\[
\geq \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\rho \psi(x) \rho \psi(y)}{|x - y|^\gamma} dxdy - \frac{c_{d, \gamma}}{2} \int_0^{\infty} \int_{\mathbb{R}^d} \min\{f_R(z), f_R^2(z)\} \frac{dz dR}{R^{d+\gamma+1}}. \]

Let \( \rho^* \) denote the Hardy-Littlewood maximal function of \( \rho \psi \), i.e.,

(B.9) \[ \rho^*(z) := \sup_{R>0} \int_{B(z, R)} \rho \psi(x) dx = |B_1|^{-1} \sup_{R>0} \frac{f_R(z)}{R^d}. \]

For \( R^* > 0 \) and \( z \in \mathbb{R}^d \), we have

\[
\int_0^{\infty} \min\{f_R^2(z), f_R(z)\} \frac{dR}{R^{d+\gamma+1}} \leq \int_0^{R^*} f_R^2(z) \frac{dR}{R^{d+\gamma+1}} + \int_{R^*}^{\infty} f_R(z) \frac{dR}{R^{d+\gamma+1}} 
\leq \int_0^{R^*} \left( |B_1| R^d \rho^*(z) \right) ^2 \frac{dR}{R^{d+\gamma+1}} + \int_{R^*}^{\infty} |B_1| R^d \frac{dR}{R^{d+\gamma+1}} 
= \frac{|B_1|^2}{d-\gamma} (R^*)^{-d-\gamma} \left( \rho^*(z) \right) ^2 + \frac{|B_1|}{\gamma} (R^*)^{-\gamma} \rho^*(z), \]

Taking \( R^* = (|B_1| \rho^*(z))^{-1} \), we derive that

(B.10) \[ \int_0^{\infty} \min\{f_R^2(z), f_R(z)\} \frac{dR}{R^{d+\gamma+1}} \leq \frac{d}{\gamma(d-\gamma)} |B_1|^{1+\frac{\gamma}{d-\gamma}} \rho^*(z)^{1+\frac{\gamma}{d-\gamma}} \] for \( z \in \mathbb{R}^d \).

By the theory of maximal functions, see e.g. [45],

\[ \int_{\mathbb{R}^d} \rho^*(z)^{1+\frac{\gamma}{d-\gamma}} dz \leq C \int_{\mathbb{R}^d} \rho \psi(z)^{1+\frac{\gamma}{d-\gamma}} dz, \]

it follows from (B.10) that

(B.11) \[ \int_{\mathbb{R}^d} \int_0^{\infty} \min\{f_R^2(z), f_R(z)\} \frac{dR}{R^{d+\gamma+1}} \leq C_{LO} \int_{\mathbb{R}^d} \rho \psi(z)^{1+\frac{\gamma}{d-\gamma}} dz, \]

for some positive constant \( C_{LO} \) depending only on \( d, \gamma \).

The conclusion now follows from (B.8) and (B.11). The proof is complete. \( \square \)
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