On stability of determination of Riemann surface from its DN-map.

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Abstract

Suppose that $M$ is a Riemann surface with boundary $\partial M$, $\Lambda$ is its DN-map, and $\mathcal{E} : M \to \mathbb{C}^n$ is a holomorphic immersion. Let $M'$ be diffeomorphic to $M$, $\partial M = \partial M'$; let $\Lambda'$ be the DN map of $M'$. Let us write $M' \in \mathcal{M}_t$ if $\|\Lambda' - \Lambda\|_{H^1(\partial M) \to L^2(\partial M)} \leq t$ holds. We show that, for any holomorphic immersion $\mathcal{E}' : M \to \mathbb{C}^n$ ($n \geq 1$), the relation

$$\sup_{M' \in \mathcal{M}_t} \inf_{\mathcal{E}''} d_H(\mathcal{E}'(M'), \mathcal{E}(M)) \underset{t \to 0}{\longrightarrow} 0,$$

holds, where $d_H$ is the Hausdorff distance in $\mathbb{C}^n$ and the infimum is taken over all holomorphic immersions $\mathcal{E}'' : M' \mapsto \mathbb{C}^n$.

Key words: electric impedance tomography of surfaces, holomorphic immersions, Dirichlet-to-Neumann map, stability of determination.

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1 Introduction

Statement

• Let $(M, g)$ be a smooth compact two-dimensional Riemann manifold (surface) with the smooth boundary $(\Gamma, dl)$, $g$ is the smooth metric tensor, $\Gamma$ is diffeomorphic to a circle, and $dl$ is the length element on $\Gamma$ induced by the metric $g$. By $\nu$ we denote the outward normal to $\Gamma$.

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1throughout the paper, smooth means $C^{\infty}$-smooth
The Dirichlet-to-Neumann map (DN-map) of the surface acts on smooth functions on $\Gamma$ by the rule $\Lambda f := \partial_\nu u^f|_\Gamma$, where $u^f$ satisfies
\[
\Delta_g u = 0 \quad \text{in} \quad M \setminus \Gamma, \quad u|_{\Gamma} = f.
\]
A specific feature of the two-dimensional case is the following. Let the metrics $g$ and $\rho g$ be conformal equivalent via a smooth $\rho > 0$ satisfying $\rho|_{\Gamma} = 1$; then the DN-maps of $(M, g)$ and $(M, \rho g)$ do coincide.

The question that is known as electric impedance tomography problem (EIT), can be posed as follows: to what extend does the DN-map determine the surface?

- Traditional understanding of ‘to solve an inverse problem completely’ includes solving several problems:
  - to establish a relevant uniqueness of determination,
  - to provide a procedure that recovers the object under determination,
  - to provide a characteristic description of the inverse data that ensures the solvability of the problem,
  - to study a stability of determination that is to analyze how the variations of data influence on the solution.

Problems i and ii were first solved in [10]: it was shown that $\Lambda$ determines $(M, g)$ up to conformal equivalence. Such a result is consistent with the specific feature of the 2-dim problem EIT, which is mentioned above. Then, in [2] this result was obtained by algebraic version of the boundary control method (BCM) [3]. Recently, the it was extended to a series of 2-dim problems in [4, 1, 5], where the results corresponding to i - ii are obtained. A characterization relayed upon algebraic BCM is provided in [6]. Also, levels i - iii are reached in [8] by the use of multidimensional complex analysis.

Our paper deals with problem iv. Its subject is a stability of determination of the surface from its DN-map. Let the surfaces $(M, g)$ and $(M', g')$ have the common boundary $\Gamma$. For simplicity and with no lack of generality we assume that $g$ and $g'$ induce on $\Gamma$ the same length element. Suppose that the DN-map $\Lambda'$ is close to $\Lambda$. Can one claim that $(M', g')$ is, in a certain sense, close to $(M, g)$? The rigorous formulation of this question has to be prefaced with preliminary discussion.

First of all, we need to provide a way of comparing the surfaces. A variant is to consider diffeomorphisms $\kappa : M \to M'$ and conformal factors obeying $\rho'|_{\Gamma} = 1$, and compare the pullback metric $g'' = \kappa^*(\rho' g')$ with metric $g$. Then we could say that $(M', g')$ is close to $(M, g)$ if there exist $\kappa$ and $\rho'$ such that the metric $g''$ is close to $g$ on $M$. Such a way looks natural but encounters the following obstacle, which was rather unexpected for us: the surfaces of different topology (of the Euler characteristics $\chi(M) \neq \chi(M')$) can have arbitrarily close DN-maps. Namely, the following is proved in [9].

**Proposition 1.** For any surface $(M, g)$ and any $\varepsilon > 0$, $k, m = 1, 2, \ldots$ there exists a surface $(M', g')$ such that $\| \Lambda' - \Lambda \|_{H^m(\Gamma; \mathbb{R})} < \varepsilon$ and $\chi(M') = \chi(M) = 2k$ holds.

In other words, topology of the surface is not stable with respect to small perturbations of its DN-map. This motivates to impose the additional condition
\[
\chi(M') = \chi(M) \quad (1)
\]
and we accept it for the rest of the paper. However, even under the latter condition, there is no ‘canonical’ diffeomorphism \( \kappa : M \to M' \) that enables to compare the surfaces and metrics, and we propose another way of comparing.

**Holomorphic immersions**

The idea is to compare not the surfaces \((M, g)\) and \((M', g')\) themselves, but their images \( \mathcal{E}(M) \) and \( \mathcal{E}'(M') \) in \( \mathbb{C}^n \) via the close holomorphic immersions \( \mathcal{E} \) and \( \mathcal{E}' \).

- Recall the basic notions.
  - Let the surface \((M, g)\) be oriented with the smooth family of ‘rotations’ \( \Phi = \{\Phi_x\}_{x \in M} \):
    \[
    \Phi_x \in \text{End} TM_x, \quad \Phi_x^* = \Phi_x^{-1} = -\Phi_x,
    \]
  - that is equivalent to
    \[
    g(\Phi_x a, \Phi_x b) = g(a, b), \quad g(\Phi_x a, a) = 0, \quad \forall a, b \in TM_x, \ x \in M.
    \]

Then the boundary \( \Gamma \) is oriented by the tangent field \( \gamma := \Phi \nu \). In the sequel, dealing with a set of surfaces \((M, g), (M', g'), (M'', g''), \ldots \) with the common boundary \((\Gamma, dl)\), we always assume that they are oriented in a consistent way: \( \Phi \nu = \Phi' \nu = \Phi'' \nu = \cdots = \gamma \).

A smooth function \( w = \Re w + i \Im w : M \to \mathbb{C} \) is holomorphic if the Cauchy-Riemann condition \( \nabla \Im w = \Phi \nabla \Re w \) holds in \( M \). Its real and imaginary parts are harmonic: \( \Delta_g \Re w = \Delta_g \Im w = 0 \) holds in \( M \). By \( \mathcal{H}(M) \subset C(M; \mathbb{C}) \) we denote the lineal of holomorphic smooth functions on \( M \). Let
\[
\text{Tr} : C(M; \mathbb{C}) \to C(\Gamma; \mathbb{C}), \quad h \mapsto h|_\Gamma
\]
be the trace operator. By the maximal principle, it maps the space \( \mathcal{H}(M) \) onto its image \( \text{Tr} \mathcal{H}(M) \) isometrically.

- Recall that an immersion is a differentiable map \( \kappa : M \to M' \) between differentiable manifolds \( M \) and \( M' \), whose differential \( D\kappa : T_x M \to T_{\kappa(x)} M' \) is injective for all \( x \in M_1 \). We say that the immersion
  \[
  \mathcal{E} : M \to \mathbb{C}^n, \ x \mapsto \{w_1(x), \ldots, w_n(x)\}
  \]

is holomorphic if it is realized by holomorphic functions \( w_j \). We deal with such immersions only and, for short, call them just ‘immersions’. Since the reserve of harmonic (and, hence, holomorphic) functions on \( M \) is the same for the metrics \( g \) and \( \rho g \), such immersion is determined by the conformal class of the metric and the boundary values \( \eta_j = w_j|_\Gamma \) of \( w_j \). Hence, by \( [10] [2] \) it is determined by the DN-map \( \Lambda \) and the choice of \( \eta_j \).

**Main result**

Recall the definition of the Hausdorff distance in \( \mathbb{C}^n \). Let \( U^r[A] := \{p \in \mathbb{C}^n | \text{dist}_{\mathbb{C}^n}(p, A) < r\} \) be the metric neighborhood of \( A \subset \mathbb{C}^n \). For the bounded sets \( A, B \subset \mathbb{C}^n \) we set
\[
\text{dist}_H(A, B) := \max\{r_{AB}, r_{BA}\}.
\]
where \( r_{AB} := \inf\{r > 0 | U^r[A] \supset B\} \) and \( r_{BA} := \inf\{r > 0 | U^r[B] \supset A\} \).
Recall that we deal with the surfaces with the common boundary $\Gamma$ and the convention (1) is in force. By $\mathcal{M}_t$ we denote the set of all such surfaces $(M', g')$, whose the DN-maps $\Lambda'$ obey

$$\| \Lambda' - \Lambda \|_{H^3(\Gamma; \mathbb{R})} \rightarrow L^2(\Gamma; \mathbb{R}) \leq t.$$  

(3)

It is known (see [11]) that each DN-map is a first order pseudo-differential operator. On the class of such operators, any two norms $\| \cdot \|_{H^{s+1}(\Gamma; \mathbb{R})} \rightarrow H^s(\Gamma; \mathbb{R})$ with $s, s' \in \mathbb{R}$ are equivalent. Thus, (3) is equivalent to the following condition

$$\| \Lambda' - \Lambda \|_{H^3(\Gamma; \mathbb{R})} \rightarrow H^2(\Gamma; \mathbb{R}) \leq ct.$$  

(4)

The main result is the following

**Theorem 1.** Let the holomorphic immersion $\mathcal{E} : M \rightarrow \mathbb{C}^n$ ($n \geq 1$) be arbitrarily fixed. Then the relation

$$\sup_{M' \in \mathcal{M}_t} \inf_{\mathcal{E}'} d_H(\mathcal{E}'(M'), \mathcal{E}(M)) \rightarrow 0,$$  

(5)

holds, where the infimum is taken over all holomorphic immersions $\mathcal{E}' : M' \mapsto \mathbb{C}^n$.

The rest of the paper is devoted to the proof of Theorem 1. First, we outline the sketch of the proof.

- Let a surface $(M, g)$ and the immersion $\mathcal{E} : M \rightarrow \mathbb{C}^n$ be fixed. A surface $(M', g')$ with the same boundary $(\Gamma = \Gamma', dl' = dl)$ is regarded as its 'perturbation'. To prove Theorem 1 we construct, for each $M'$, a certain map $\mathcal{E}' : M' \rightarrow \mathbb{C}^n$, $x \mapsto \{w'_1(x), \ldots, w'_n(x)\}$, which is determined by $M, \mathcal{E}, M'$ and obeys

$$\sup_{M' \in \mathcal{M}_t} d_H(\mathcal{E}'(M'), \mathcal{E}(M)) \rightarrow 0.$$  

(6)

The map $\mathcal{E}'$ is connected with $\mathcal{E}$ via the map $\beta'$:

$$w'_j = \beta' w_j, \ j = 1, \ldots, n,$$  

(7)

where $\beta' : \mathcal{F}(M) \rightarrow \mathcal{F}(M')$ is a 'canonical' real linear map obeying

$$\| \text{Tr}' \beta' w - \text{Tr} w \|_{C^2(\Gamma; \mathbb{C})} \leq c t \| \text{Tr} w \|_{H^3(\Gamma; \mathbb{C})}, \quad w \in \mathcal{F}(M);$$  

(8)

here and in the sequel, $c$'s are positive constants determined by $M$ and $\mathcal{E}$. As we show, for small enough $t$ and all $M' \in \mathcal{M}_t$, the map $\mathcal{E}'$ turns out to be an immersion.

In what follows, we give a detailed description of the map $\beta'$ along with the proof of (8). In the subsequent, we denote $\eta_k = w_k|_{\Gamma}, \eta'_k = w'_k|_{\Gamma}$, so that $\mathcal{E}$ and $\mathcal{E}'$ are determined by $\{\eta_k\}_{k=1}^n$ and $\{\eta'_k\}_{k=1}^n$ by the uniqueness of analytic continuation. Note that (8) implies the closeness of the boundary traces of $w'_k$ and $w_k$ for small $t$; namely, we have

$$\| \eta'_k - \eta_k \|_{C^2(\Gamma; \mathbb{C})} \leq c t, \quad k = 1, \ldots, n.$$  

(9)

In particular, the boundary images $\mathcal{E}(\Gamma)$ and $\mathcal{E}'(\Gamma)$ are close in $\mathbb{C}^n$ for small $t$. 

4
Next, we prove that the estimate (9) implies the closeness, by Hausdorff distance, of the images \( E'(M') \) and \( E(M) \) for small \( t \). The key instrument of the proof is the generalized argument principle. Suppose that \( w \) and \( \tilde{w} \) are holomorphic smooth functions on \( M \) and \( z \not\in w(\Gamma) \) is fulfilled. Then, from Stokes theorem (see Theorem 3.16, [12]) and the residue theorem (see Lemma 3.12, [12]) for meromorphic 1-form \( (\tilde{w}/(w-z))dw \), it follows that

\[
\frac{1}{2\pi i} \int_{\Gamma} \tilde{w} \frac{\partial_x w}{w-z} \, dl = \sum_{x \in w^{-1}(z)} m(w, x, z) \tilde{w}(x)
\]

holds, where \( m(w, x, z) \) is the order of the zero \( x \) of the function \( w - z \). If \( w - z \) has a unique zero \( x \) on \( M \) and its order is one, then the formula above is simplified as follows

\[
\frac{1}{2\pi i} \int_{\Gamma} \tilde{w} \frac{\partial_x w}{w-z} \, dl = \tilde{w}(x),
\]

so we can find the values at \( x \) of all holomorphic functions \( \tilde{w} \). Of course, the same facts is true for \( M' \) instead of \( M \). So, we can try to take one of the projections \( w_j(x) (w'_j(x')) \) as a coordinate of the point \( p = \mathcal{E}(x) (p' = \mathcal{E}'(x')) \) and determine all other projections \( w_k(x) (w'_k(x')) \) via formula (11). Thereby, we determine the images \( \mathcal{E}(M) \) and \( \mathcal{E}'(M') \). The class of projective immersions, for which it is possible, is described below. Also, it is shown that it suffices to prove (6) only for such projective immersions.

Due to (11), the closeness of \( \eta'_k \) and \( \eta_k \) implies the closeness of the surface images \( \mathcal{E}(M) \) and \( \mathcal{E}'(M') \) in \( \mathbb{C}^n \) outside a small neighborhood of \( \mathcal{E}(\Gamma) \). Near \( \mathcal{E}(\Gamma) \), the application of the generalized argument principle is reduced to the standard estimates of the Cauchy-type integrals by choosing appropriate local coordinates for points \( \mathcal{E}(M) \) and \( \mathcal{E}'(M') \). Summarizing, we will arrive at (6).

2 The map \( \beta' \)

Preliminaries

- The following is a few known facts. Introduce the sets of functions with the zero mean value

\[
\dot{L}_2(\Gamma; \mathbb{R}) := \{ f \in L_2(\Gamma; \mathbb{R}) \mid \langle f \rangle = 0 \}, \quad \langle f \rangle := \int_{\Gamma} f \, dl,
\]

\[
\dot{C}^\infty(\Gamma; \mathbb{R}) := \dot{L}_2(\Gamma; \mathbb{R}) \cap C^\infty(\Gamma; \mathbb{R}),
\]

\[
\dot{H}^m(\Gamma; \mathbb{R}) := \dot{L}_2(\Gamma; \mathbb{R}) \cap H^m(\Gamma; \mathbb{R}),
\]

where \( H^m(...) \) are the Sobolev spaces. As is known, the DN map \( \Lambda \) is well defined on smooth functions and acts continuously from \( H^m(\Gamma; \mathbb{R}) \) to \( H^{m-1}(\Gamma; \mathbb{R}), \ m = 1, 2, \ldots \). It preserves the smoothness, its (operator) closure is a self-adjoint first-order pseudo-differential operator in \( L_2(\Gamma; \mathbb{R}) \), and

\[
\text{Ker} \, \Lambda = \{ \text{const} \}, \quad \text{Ran} \, \Lambda = \dot{L}_2(\Gamma; \mathbb{R}) \tag{12}
\]

holds.
Let $\partial_\gamma$ be the differentiation on the boundary with respect to the field $\gamma = \Phi \nu$. The integration $J = \partial_\gamma^{-1}$ is well defined in $\mathring{L}_2(\Gamma; \mathbb{R})$, whereas the operator $J\Lambda$ is also well defined due to (12) and is bounded in each $\mathring{H}^m(\Gamma; \mathbb{R})$, $m = 0, 1, \ldots$. Note that if $M$ is a disk in $\mathbb{R}^2$, the operator $J\Lambda$ coincides with the classical Hilbert transform that relates the real and imaginary parts of the trace of holomorphic function. In the general case, for $w \in \mathcal{F}(M)$ and $\eta := w|_\Gamma = \text{Tr} \, w$ we also have

$$\eta = \Re \eta + i \left[ J \Lambda \Re \eta + (\Im \eta) \right]$$

(see [2]).

By (12), $\Lambda J$ is a well defined on $\dot{C}^\infty(\Gamma; \mathbb{R})$ and bounded operator acting in $\mathring{L}_2(\Gamma; \mathbb{R})$. It preserves smoothness: $\Lambda J \mathring{H}^m(\Gamma; \mathbb{R}) \subset \mathring{H}^m(\Gamma; \mathbb{R})$ holds for all $m = 0, 1, \ldots$. The representation $(\Lambda J)^* = -J \Lambda = J(\Lambda J)\partial_\gamma$ is valid on smooth functions and implies

$$[\mathbb{I} + (\Lambda J)^2]^* = \mathbb{I} + ((\Lambda J)^*)^2 = \mathbb{I} + J((\Lambda J)^2)\partial_\gamma = J[\mathbb{I} + (\Lambda J)^2]\partial_\gamma \quad \text{on } \dot{C}^\infty(\Gamma; \mathbb{R}),$$

(14)

where $\mathbb{I} = \partial_\gamma J$ is the unit operator in $\mathring{L}_2(\Gamma; \mathbb{R})$.

As is shown in [2], Lemma 1, the relations

$$\text{dim } \text{Ran }[\mathbb{I} + (\Lambda J)^2] \subset \dot{C}^\infty(\Gamma; \mathbb{R}),$$

$$\text{dim } \text{Ran }[\mathbb{I} + (\Lambda J)^2] = 1 - \chi(M) =: \kappa$$

hold; hence, the operator $\mathbb{I} + (\Lambda J)^2$ determines the topology of $M$. By (14), these facts easily lead to

$$\text{dim } \text{Ran }[\mathbb{I} + (\Lambda J)^2]^* = \kappa,$$

so that $[\mathbb{I} + (\Lambda J)^2]^*$ is a finite rank operator in $\mathring{L}_2(\Gamma; \mathbb{R})$. The relations

$$\text{Ran }[\mathbb{I} + (\Lambda J)^2] = \text{Ker }[\mathbb{I} + (\Lambda J)^2] \oplus \mathbb{R},$$

$$\text{Ran }[\mathbb{I} + (\Lambda J)^2]^* = \text{Ker }[\mathbb{I} + (\Lambda J)^2]^* \oplus \mathbb{R}$$

are also proved in [2]. As a consequence, we have

$$L_2(\Gamma; \mathbb{R}) = \mathring{L}_2(\Gamma; \mathbb{R}) \oplus \mathbb{R} = \text{Ker }[\mathbb{I} + (\Lambda J)^2] \oplus \text{Ran }[\mathbb{I} + (\Lambda J)^2]^* \oplus \mathbb{R}$$

(15)

Let $P$ be the projection in $L_2(\Gamma; \mathbb{R})$ onto the subspace $\text{Ran }[\mathbb{I} + (\Lambda J)^2]^*$. Then, for any $\eta \in C^\infty(\Gamma; \mathbb{R})$, we have $P\eta \in \text{Ran }[\mathbb{I} + (\Lambda J)^2]^*$. Of course, the same facts are true for the surface $M'$ instead of $M$.

**Operators $\beta'_T$ and $\beta'$**

Let $P'$ be the projection in $L_2(\Gamma; \mathbb{R})$ onto the subspace $\text{Ran }[\mathbb{I} + (\Lambda J)^2]^*$. The map

$$\beta'_T \eta := P' \Re \eta + i \left[ J \Lambda' P' \Re \eta + (\Im \eta) \right], \quad \eta \in C^\infty(\Gamma; \mathbb{C})$$

(17)

provides a natural way to relate the traces of $\mathcal{F}(M)$ with the traces of $\mathcal{F}(M')$. In the meantime, the map

$$\beta' := (\text{Tr}')^{-1} \beta'_T \text{ Tr} ,$$

where $\text{Tr}'$ is the trace operator on $\mathcal{F}(M')$, is well defined and relates $\mathcal{F}(M)$ with $\mathcal{F}(M')$. Now, we prove estimate (8).
Lemma 1. For sufficiently small $t > 0$ and for any $M' \in M_t$, the map $\beta_t'$ is a bijection and the inequality
\[
\| \beta_t' \eta - \eta \|_{C^2(\Gamma; \mathbb{C})} \leq c t \| \eta \|_{H^3(\Gamma; \mathbb{C})}, \quad \eta \in \text{Tr} \mathfrak{H}(M)
\] (18)
holds with a constant $c$ independent of $M'$.

Proof. Let $Q$ be the (finite rank) projection in $L_2(\Gamma; \mathbb{R})$ onto $\text{Ran} \left[ \mathbb{I} + (\Lambda J)^2 \right]^* \equiv J \left[ \mathbb{I} + (\Lambda J)^2 \right]$. Choose the smooth $f_1, \ldots, f_{\kappa}$ so that $h_j := J \left[ \mathbb{I} + (\Lambda J)^2 \right] \partial_j f_j, \quad j = 1, \ldots, \kappa$ constitute a basis in $\text{Ran} \left[ \mathbb{I} + (\Lambda J)^2 \right]^* = \text{Ran} Q$. Repeat the same construction for the projection $Q'$ in $L_2(\Gamma; \mathbb{R})$ onto $\text{Ran} \left[ \mathbb{I} + (\Lambda' J)^2 \right]^* = J \left[ \mathbb{I} + (\Lambda' J)^2 \right] \partial_j C^\infty(\Gamma; \mathbb{R})$, and put $h_j' := J \left[ \mathbb{I} + (\Lambda' J)^2 \right] \partial_j f_j, \quad j = 1, \ldots, \kappa$ (with the same $f_j$). Representing
\[
h_j - h_j = \left\{ J \left[ \mathbb{I} + (\Lambda' J)^2 \right] \partial_j - J \left[ \mathbb{I} + (\Lambda J)^2 \right] \partial_j \right\} f_j = J \left[ \Lambda' J (\Lambda' J - \Lambda J) + (\Lambda' J - \Lambda J) \Lambda J \right] \partial_j f_j = J \left[ (\Lambda' J)(\Lambda' - \Lambda) + (\Lambda' - \Lambda)(J \Lambda) \right] f_j
\]
and taking into account the boundedness of $J, \Lambda' J, \Lambda J$ as operators in $H^m(\Gamma; \mathbb{R})$ and $\Lambda', \Lambda$ as operators from $H^m(\Gamma; \mathbb{R})$ to $H^{m-1}(\Gamma; \mathbb{R})$, one easily gets
\[
\| h_j' - h_j \|_{H^m(\Gamma; \mathbb{R})} \leq c_m \| \Lambda' - \Lambda \|_{H^m(\Gamma; \mathbb{R}) \rightarrow H^{m-1}(\Gamma; \mathbb{R})} \| f_j \|_{H^m(\Gamma; \mathbb{R})}.
\]
Hence, by virtue of (14), the inequality
\[
\| h_j' - h_j \|_{H^3(\Gamma; \mathbb{R})} \leq c t \| f_j \|_{H^3(\Gamma; \mathbb{R})}, \quad j = 1, \ldots, \kappa
\] (19)
holds with a constant $c$ independent on $M' \in M_t$.

By the latter estimate and assumption (11), for small enough $t$, the system $h_1', \ldots, h_{\kappa}'$ forms a basis in its linear span that is $\text{Ran} \left[ \mathbb{I} + (\Lambda' J)^2 \right]^* = \text{Ran} Q'$. Closeness of the bases $h_1', \ldots, h_{\kappa}'$ and $h_1, \ldots, h_{\kappa}$ leads to the closeness of the projections: by the use of (19) it is easy to verify that the inequality
\[
\| Q' - Q \|_{H^3(\Gamma; \mathbb{R}) \rightarrow H^3(\Gamma; \mathbb{R})} \leq c t
\] (20)
holds. Comparing (15) and (19), we obtain $P = \mathbb{I} - Q$ and $P' = \mathbb{I} - Q'$. Then (20) is equivalent to
\[
\| P' - P \|_{H^3(\Gamma; \mathbb{R}) \rightarrow H^3(\Gamma; \mathbb{R})} \leq \text{const} \cdot t.
\] (21)
Fix a smooth $\eta \in \text{Tr} \mathfrak{H}(M)$; then $\Re \eta = P \Re \eta$. Comparing (13) with (17), we have
\[
\beta_t' \eta - \eta = (P' - P) \Re \eta + i \left[ J (\Lambda' P' - \Lambda P) \right] \Re \eta = (P' - P) \Re \eta + i \left[ (J \Lambda')(P' - P) + J(\Lambda' - \Lambda)P \right] \Re \eta.
\]
From (21) and (4), we easily obtain that
\[
\| \beta_t' \eta - \eta \|_{H^3(\Gamma; \mathbb{R})} \leq c t \| \eta \|_{H^3(\Gamma; \mathbb{R})}
\]
holds with a constant $c$ independent on $M' \in M_t$. At last, by continuity of the embedding $H^3(\Gamma; \mathbb{R}) \subset C^2(\Gamma; \mathbb{R})$, we arrive at (18). By (18), for small $t$, $\beta_t'$ is a (close to identity) invertible map from $\text{Tr} \mathfrak{H}(M)$ to $\text{Tr} \mathfrak{H}(M')$, which preserves the smoothness. As a result, $\beta_t'$ is an invertible map from $\mathfrak{H}(M)$ to $\mathfrak{H}(M')$. \hfill \Box
As a corollary of (18), we obtain (9) and
\[ \|\mathcal{E}'(x) - \mathcal{E}(x)\|_{\mathbb{C}^n} \leq c_t, \quad x \in \Gamma, \]
where \(\mathcal{E}'\) is the map given by (7). So, the smallness of \(\Lambda' - \Lambda\) yields the closeness of the images of the (common) boundary \(\Gamma\) via the holomorphic maps \(\mathcal{E}\) and \(\mathcal{E}'\). It is the fact that motivates the further use of the pair \(\mathcal{E}, \mathcal{E}'\) for evaluation of the closeness of the surfaces \(M\) and \(M'\).

3 Closeness outside the boundary

We now proceed to the proof of (6). In accordance with the plan outlined in Introduction, we establish the closeness of the compact set \(K \subset \mathcal{E}(M)\) and a suitable \(K' \subset \mathcal{E}(M')\), which are located outside \(\mathcal{E}(\Gamma)\) and \(\mathcal{E}(\Gamma')\) respectively.

**Projective immersions**

- First, we show that it is sufficient to prove (1) only for \(\mathcal{E}\) belonging to a special class of immersions. Such a class is defined in the following way.

  Let \(\xi = \{\xi_1, \ldots, \xi_n\} \in \mathbb{C}^n\); by \(\pi_j : \xi \mapsto \xi_j\) we denote the coordinate projection. Let \(B_{r,j}[\xi] := \{\zeta \in \mathbb{C}^n \mid |\zeta_j - \xi_j| < r\}\) be a cylinder in \(\mathbb{C}^n\).

**Definition 1.** The immersion \(\mathcal{E} : M \to \mathbb{C}^n\) is called projective if for each point \(\xi \in \mathcal{E}(M)\) there is a cylinder \(B_{r,j}[\xi]\) such that the projection \(\pi_j : \mathcal{E}(M) \cap B_{r,j}[\xi] \to \mathbb{C}\) is a diffeomorphism.

Of course, the projective immersion is an embedding. Also, Definition 1 provides the following property of projective immersions: if \(z \in \pi_j(\mathcal{E}(M) \cap B_{r,j}[\xi])\), then the coordinate plane \(\{\zeta \in \mathbb{C}^n \mid \zeta_j = z\}\) crosses \(\mathcal{E}(M)\) at a single point.

**Proposition 2.** The projective immersions do exist.

**Proof.** Due to smoothness of the boundary \(\Gamma\) the surface \(M\) can be embedded into a larger noncompact surface \(\bar{M}\). In view of divisor theorem (see Proposition 26.5, [7]), for any \(x \in M\) there exists a holomorphic function \(\bar{w}_x\) on \(\bar{M}\) such that \(x\) is a single zero of \(\bar{w}_x\) and the differential \(d\bar{w}_x\) at \(x\) is injective. Then there exists a neighborhood \(U_x\) of \(x\) in \(\bar{M}\) such that \(\bar{w}_x : U_x \to \mathbb{C}\) is an injection and the differential \(d\bar{w}_x\) at any point of \(U_x\) is injective. The set \(M \setminus U_x\) is compact, so that \(|\bar{w}_x| \geq r_x > 0\) holds on \(M \setminus U_x\).

Let \(\tilde{U}_x := \{y \in U_x \mid |\bar{w}_x| < r_x/2\}\). The neighbourhoods \(\tilde{U}_x\), \(x \in M\) constitute an open cover of \(M\), from which one can choose a finite subcover \(\tilde{U}_{x_j}\), \(j = 1, \ldots, n\). Taking \(\mathcal{E} : M \to \mathbb{C}^n\), \(\mathcal{E} = \{w_1(\cdot), \ldots, w_n(\cdot)\}\), \(w_j = \bar{w}_{x_j}|_M\), we obtain the immersion, which is projective by construction.

Let’s note a simple fact. If \(\mathcal{E}_1 : M \to \mathbb{C}^{n_1}\) and \(\mathcal{E}_2 : M \to \mathbb{C}^{n_2}\) are immersions, whereas \(\mathcal{E}_2\) is projective, then the immersion \(\mathcal{E}_1 \oplus \mathcal{E}_2 : M \to \mathbb{C}^{n_1+n_2}\), \((\mathcal{E}_1 \oplus \mathcal{E}_2)(x) := \{\mathcal{E}_1(x), \mathcal{E}_2(x)\}\) is also projective. It is used as follows.
• Proving estimate (6), we can restrict ourselves only to the case of projective immersions. Indeed, assume that (6) is already proved for any projective immersion \( E_0 \). Let \( \mathcal{E} : M \to \mathbb{C}^m \) be an immersion of \( M \) and let \( M' \in \mathcal{M}_t \) be chosen arbitrarily. Take some projective immersion \( E_0 : M \to \mathbb{C}^s \) of \( M \). Then \( \mathcal{E} \oplus E_0 \) is also a projective and, by our assumption,
\[
\sup_{M' \in \mathcal{M}_t} d_H ([(\mathcal{E} \oplus E_0)'(M')], [\mathcal{E} \oplus E_0](M)) \to 0 \quad t \to 0
\]
holds. Also, we have
\[
(\mathcal{E} \oplus E_0)' = \mathcal{E}' \oplus E_0', \quad \varpi^n \circ [\mathcal{E} \oplus E_0] = \mathcal{E}' \quad \varpi^n \circ [\mathcal{E}' \oplus E_0] = \mathcal{E}',
\]
where \( n = m + s \) and the map \( \varpi^n \):
\[
\varpi^n \{\xi_1, \ldots, \xi_n\} := \{\xi_1, \ldots, \xi_m\}
\]
projects on the first component of the sum \( \mathbb{C}^m \oplus \mathbb{C}^s = \mathbb{C}^n \). In view of the evident relations
\[
d_H (\mathcal{E}'(M'), \mathcal{E}(M)) = d_H (\varpi^n \circ [\mathcal{E}' \oplus E_0](M'), \varpi^n \circ [\mathcal{E} \oplus E_0](M)) \leq d_H ([(\mathcal{E} \oplus E_0)'(M')], [\mathcal{E} \oplus E_0](M)),
\]
we arrive at \( \sup_{M' \in \mathcal{M}_t} d_H (\mathcal{E}'(M'), \mathcal{E}(M)) \to 0 \quad t \to 0 \).

Thus, if estimate (6) holds for projective immersions then it is valid for all immersions \( \mathcal{E} : M \to \mathbb{C}^n \). For this reason, in the sequel we assume that all the maps \( \mathcal{E} \), which we deal with, are projective.

• Suppose that \( \mathcal{E} : M \to \mathbb{C}^n \), \( \mathcal{E} = \{w_1(\cdot), \ldots, w_n(\cdot)\} \) is a (projective) immersion and \( \mathcal{E}' : M' \to \mathbb{C}^n \) is the corresponding map given by (7). We first estimate the closeness, in Hausdorff distance, between the parts of \( \mathcal{E}(M) \) and \( \mathcal{E}'(M) \) which are separated from the curves \( \mathcal{E}(\Gamma) \) and \( \mathcal{E}'(\Gamma) \) respectively.

Denote by \( V_j^\varepsilon \) the set of all \( z \in \mathbb{C} \) such that \( \text{dist}_C(z, w_j(\Gamma)) > \varepsilon \) and the function \( w_j(z) - z \) has a simple (of the order 1) zero in \( M \) and has no more zeroes in \( M \). Some of \( V_j^\varepsilon \) may be empty; however, to deal with them is reasonable, when immersion \( \mathcal{E}(M) \) is projective: see below.

Introduce the ‘cylinders’
\[
\Pi_j^\varepsilon := \mathbb{C} \times \cdots \times V_j^\varepsilon \times \cdots \times \mathbb{C}
\]
and consider their union
\[
Q_\varepsilon := \bigcup_{j=1}^n \Pi_j^\varepsilon \subset \mathbb{C}^n.
\]
If a subset \( K \subset \mathcal{E}(M \setminus \Gamma) \) is compact then there exists \( \varepsilon_0 > 0 \) such that \( K \subset Q_\varepsilon \) for any \( \varepsilon \in (0, \varepsilon_0) \). Indeed, since \( \mathcal{E} \) is projective, for any predicate \( j(\xi) \) one can specify the number \( j(\xi) \) and an \( s(\xi) > 0 \) such that \( \pi_{j(\xi)}(\xi) \in V_{j(\xi)}^s(\xi) \) holds. Therefore, we have \( \xi \in \Pi_{j(\xi)}^s(\xi) \). The sets \( \Pi_{j(\xi)}^s(\xi) \cap K \), \( \xi \in K \) constitute an open cover of \( K \), from which one can choose a finite subcover \( \Pi_{j(\xi_k)}^s(\xi_k) \cap K \), \( k = 1, \ldots, N \). Denote by \( \varepsilon_0 = \min_k \{ s(\xi_k) \} \); then \( K \subset Q_\varepsilon \) does hold for any \( \varepsilon \in (0, \varepsilon_0) \).
Estimates

- Now, we prove that the parts of the images $\mathcal{E}(M)$ and $\mathcal{E}'(M)$ that are contained in $Q^\varepsilon$, are close (if $\Lambda'$ is close to $\Lambda$). The proof is based on the application of the generalized argument principle \([10]\). The bar denotes the closure in $\mathbb{C}^n$; the projection $\varpi^n_m$ is given by \([23]\).

**Lemma 2.**

i. The relation

$$\sup_{M' \in M_t} \|d_H(\mathcal{E}'(M') \cap \overline{Q^\varepsilon}, \mathcal{E}(M) \cap \overline{Q^\varepsilon}) = O(t), \quad t \to 0$$

holds for any fixed $\varepsilon > 0$.

ii. Suppose that the map $\varpi^n_m \mathcal{E}$, $m < n$ is an immersion. Then, for sufficiently small $t > 0$ and any $M' \in M_t$, the map $\varpi^n_m \mathcal{E}' : \mathcal{E}'^{-1}(Q^\varepsilon) \to \mathbb{C}^n$ is also an immersion.

**Proof.** 1. Let $\varepsilon > 0$ be fixed and $\xi$ be an arbitrary point of $\mathcal{E}(M) \cap \overline{Q^\varepsilon}$. Then $\xi \in \overline{V^\varepsilon_j}$ for some $j$. Put $z := \pi_j \xi$, then $z \in \overline{V^\varepsilon_j}$ holds. Also, put by definition $w_0 = 1$ and $w'_0 = 1$ on $M$ and $M'$ respectively. Recall that $\mathcal{E} = \{w_1(\cdot), \ldots, w_n(\cdot)\}$, $\mathcal{E}' = \{w'_1(\cdot), \ldots, w'_n(\cdot)\}$ and $\eta_j = w_j|_\Gamma$, $\eta'_j = w'_j|_\Gamma$. Denote

$$J_{k,j}(z) := \frac{1}{2\pi i} \int_{\Gamma} \eta_k \frac{\partial \eta_j}{\eta_j - z} \, dl, \quad J'_{k,j}(z) := \frac{1}{2\pi i} \int_{\Gamma} \eta'_k \frac{\partial \eta'_j}{\eta'_j - z} \, dl.$$  

By the generalized argument principle \([10]\), we have

$$J_{k,j}(z) = \sum_{x \in w^{-1}_j(z)} w_k(x) m(w_j, x, z), \quad J'_{k,j}(z) = \sum_{x' \in w'^{-1}_j(z)} w'_k(x') m'(w'_j, x', z),$$

where $k = 0, \ldots, n$ and the number $m(w_j, x, z)$ (the number $m'_j(w'_j, x', z)$) is the order of zero $x$ (of zero $x'$) of the function $w_j - z$ (the function $w'_j - z$) on the surface $M$ (on $M'$). In particular, the number $J_{0,j}$ is the total multiplicity of zeroes of the function $w_j - z$ on $M$, whereas $J'_{0,j} = 1$ by the choice of the coordinate functions of the projective immersion $\mathcal{E}$. In the meantime, $J'_{0,j}$ is the total multiplicity of zeroes of the function $w'_j - z$ on $M'$.

2. Let us estimate $J'_{k,j}(z) - J_{k,j}(z)$. We have

$$|J'_{k,j}(z) - J_{k,j}(z)| = \left| \frac{1}{2\pi i} \int_{\Gamma} \eta'_k \frac{\partial \eta'_j}{\eta'_j - z} - \eta_k \frac{\partial \eta_j}{\eta_j - z} \right| \, dl = \left| \frac{1}{2\pi i} \int_{\Gamma} (\eta'_k \frac{\partial \eta'_j}{\eta'_j - z} - \eta_k \frac{\partial \eta_j}{\eta_j - z}) \, dl \right|.$$  

Also,

$$|\partial_z J'_{k,j}(z) - \partial_z J_{k,j}(z)| = \left| \frac{1}{2\pi i} \int_{\Gamma} \left( \eta'_k \frac{\partial \eta'_j}{(\eta'_j - z)^2} - \eta_k \frac{\partial \eta_j}{(\eta_j - z)^2} \right) \, dl \right|.$$  

From the inclusion $z \in V^\varepsilon_j \subset V^\varepsilon/2$ it follows that $J_{0,j} = 1$ and $\text{dist}_C(z, w_j(\Gamma)) > \varepsilon$. By the latter inequality and \([9]\), we have

$$|\eta_j - z| > \varepsilon, \quad |\eta'_j - z| \geq |\eta_j - z| - \| \eta'_j - \eta_j \|_{C(\Gamma, C)} \geq \varepsilon - O(t) \geq \varepsilon/2$$

(26)
Let $\xi$ be the (single, simple) zero of the function $w_j - z$ which is uniform with respect to $x$. It induces the bijection $\mathcal{E}(x) = \xi \longleftrightarrow \mathcal{E}'(x')$ (29) between the parts $\mathcal{E}'(M') \cap \overline{\mathcal{Q}}$ and $\mathcal{E}(M) \cap \overline{\mathcal{Q}}$ of the surface images. The latter makes it possible to estimate the distance between the parts as follows.

By (25) and (27) we have
\[
|\pi_k \mathcal{E}'(x') - \pi_k \mathcal{E}(x)| = |w'_k(x') - w_k(x)| = O(t), \quad t \to 0; \quad k = 1, \ldots, n.
\]
Hence we obtain the estimate
\[
|\mathcal{E}'(x') - \mathcal{E}(x)| = O(t), \quad t \to 0,
\]
which is uniform with respect to $\xi \in \mathcal{E}(M) \cap \overline{\mathcal{Q}}$ and $M' \in \mathbb{M}_t$. This implies (24).

4. Also, estimates (25) and (28) imply
\[
|\partial_z w'_k(x') - \partial_z w_k(x)| = O(t), \quad t \to 0; \quad k = 1, \ldots, n,
\]
where $z := \pi_j \mathcal{E}(x) = \pi_j \mathcal{E}'(x') \in V_j$. Let $m < n$. Taking $\Re z$, $\Im z$ as local coordinates of the point $x \in M (x' \in M')$, consider the maps
\[
\hat{\mathcal{E}} : (\Re z, \Im z) \mapsto (\Re w_1, \ldots, \Re w_m, \Im w_1, \ldots, \Im w_m),
\]
\[
\hat{\mathcal{E}}' : (\Re z, \Im z) \mapsto (\Re w'_1, \ldots, \Re w'_m, \Im w'_1, \ldots, \Im w'_m)
\]
Denote by Jac $\hat{E}$ and Jac $\hat{E}'$ the Jacobian matrices of $\hat{E}$ and $\hat{E}'$, respectively. Then (30) yields
\[
\| \text{Jac} \hat{E}'(z) - \text{Jac} \hat{E}(z) \| = O(t), \quad t \to 0; \quad k = 1, \ldots, n.
\] (31)
Suppose that the map $w^n_m E$ is an immersion, then, for any $z \in V^\varepsilon_j$, the matrix Jac $\hat{E}(z)$ is of full rank. In view of (31), there exist a sufficiently small $t_j > 0$, such that the matrix Jac $\hat{E}'(z)$ is of full rank for any $t \in (0, t_j)$, $M' \in M_t$, and $z \in V^\varepsilon_j$. Thus, the differential $d_z(w^n_m E')$ of the map $w^n_m E'$ is injective for any $t \in (0, \min_{j=1, \ldots, n} t_j)$, $M' \in M_t$ and any $x' \in M'$ such that $E'x' \in Q_\varepsilon$. This proves Lemma 2 ii.

- Recall that $r_{AB}$ is defined in [2]. As a consequence of (24), the following holds.

**Corollary 1.** The relation
\[
\sup_{M' \in M_t} r_{E'(M') E(M)} \to 0 \quad \text{as } t \to 0
\] (32)
is valid.

**Proof.** Fix an arbitrary $\varepsilon > 0$. In view of (22), we have $\sup_{M' \in M_t} d_H(\varepsilon'(\Gamma), \varepsilon(\Gamma)) = O(t)$. So, for all sufficiently small $t$ and any $M' \in M_t$, each point $\xi \in \varepsilon(\Gamma)$ is contained in $\varepsilon/3$-neighborhood in $C^n$ of some point $\xi' \in \varepsilon'(\Gamma)$. The set
\[
K = \{ \xi \in \varepsilon(M) \mid \text{dist}_{C^n}(\xi, \varepsilon'(\Gamma)) \geq \varepsilon/3 \}
\]
is compact in $\varepsilon(M)$. As was shown before, there exists a sufficiently small $\varepsilon > 0$ such that $K \subset Q_\varepsilon$. Then (24) implies that, for all sufficiently small $t$ and for any $M' \in M_t$, the set $K$ is contained in $\varepsilon-$neighborhood of $\varepsilon'(M')$ in $C^n$. Now, let $\sigma$ be an arbitrary point of $\varepsilon(M) \setminus K$. By definition of $K$, one has $\text{dist}_{C^n}(\sigma, \varepsilon'(\Gamma)) < \varepsilon/3$. So, $\sigma$ is contained in $\varepsilon/3$-neighborhood in $C^n$ of some point $\xi' \in \varepsilon'(\Gamma)$, while $\xi$ is contained in $\varepsilon/3$-neighborhood in $C^n$ of some point $\xi' \in \varepsilon'(\Gamma)$. Thus, $K$ is contained in $\varepsilon$-neighborhood of $\varepsilon'(\Gamma)$ in $C^n$. Therefore, for all sufficiently small $t$ and for any $M' \in M_t$, the whole surface $\varepsilon(M)$ is contained in $\varepsilon$-neighborhood of $\varepsilon'(\Gamma)$ in $C^n$. Since $\varepsilon > 0$ is arbitrary, the latter yields (32).

## 4 Closeness near the boundary

It remains to estimate the closeness of the parts of $\varepsilon(M)$ and $\varepsilon'(M')$, which are contained in a small neighborhood of $\varepsilon(\Gamma)$ and $\varepsilon'(\Gamma)$. To do this, a relevant analog of the bijection (29) that relates the points of these parts, is required. It is provided by the appropriate local coordinates for points of $\varepsilon(M)$ and $\varepsilon'(M')$ in such neighborhoods.

**Coordinates**

- Recall that $\varepsilon = \{ w_1(\cdot), \ldots, w_n(\cdot) \}$ and $\eta_i = w_i|_\Gamma$.

Fix a point $a \in \Gamma$. Since $\varepsilon$ is projective, there exist a number $j$ and a small enough open disk $U_a \subset \mathbb{C}$ with center $z_a := \eta_j(a)$ such that

1. $\partial_z \eta_j(a) \not= 0$ holds,
2. $U_a \setminus w_j(\Gamma)$ consists of two connected components $U_{a,0}$ and $U_{a,1}$, each component being diffeomorphic to an (open) half-disk, whereas $U_{a,0} \cap w_j(M) = \emptyset$ holds.

3. for any $z \in U_{a,1}$, the function $w_j - z$ has a simple zero on $M$ and has no more zeroes on $M$.

Introduce the cylinder

$$\Pi_a := \mathbb{C} \times \cdots \times U_a \times \cdots \times \mathbb{C}.$$ 

In view of the properties above, the projection $\pi_j$ is a bijection from $\mathcal{E}(M) \cap \Pi_a$ onto $U_a \setminus U_{a,0}$. Due to this, we can regard $z := \pi_j p$ as a local coordinate of the point $p \in \mathcal{E}(M) \cap \Pi_a$.

- Since $\partial_\gamma \eta_j(a) \neq 0$, we can choose the neighborhood $\Gamma_a$ of $a$ in $\Gamma$ sufficiently small to obey

$$\Re \frac{\partial_\gamma \eta_j(l)}{\partial_\gamma \eta_j(a)} \in [c_0, 1/c_0], \quad l \in \Gamma_a$$

with $c_0 > 0$. Without loss of generality, we assume that $U_a \cap w_j(\Gamma) = w_j(\Gamma_a)$. Denote

$$\zeta(z) := \frac{z - z_a}{\partial_\gamma \eta_j(a)}, \quad \zeta_1(z) := \Re \zeta(z), \quad \zeta_2(z) := \Im \zeta(z), \quad z \in U_a$$

and

$$\psi = \zeta \circ \eta_j, \quad \psi_1 := \Re \psi, \quad \psi_2 := \Im \psi \quad \text{on } \Gamma_a.$$ 

Due to (33), the map $\psi_1 : \Gamma_a \mapsto \mathbb{R}$ is an injection and $\partial \psi_1 / \partial l \in [c_0, 1/c_0]$ holds for $l \in \Gamma_a$.

Introduce the new local coordinates

$$\omega(z) := (s(z), r(z)); \quad s(z) := [\psi_1^{-1} \circ \zeta_1](z), \quad r(z) := \zeta_2(z) - [\psi_2 \circ \psi_1^{-1} \circ \zeta_1](z), \quad z \in U_a,$$

then the Jacobian $\mathcal{J}$ of the passage $(\Re z, \Im z) \mapsto (s, r)$ obeys

$$|\mathcal{J}(z)| \in [c_1, 1/c_1], \quad z \in U_a$$

with $c_1 > 0$. Thus, the map $\Omega := \omega \circ \pi_j$ is a bijection from $\mathcal{E}(M) \cap \Pi_a$ onto some neighborhood of $\Gamma_a \times \{0\}$ in the strip $\Gamma_a \times [0, +\infty)$. Note that $\Omega(\mathcal{E}(l)) = (l, 0)$ for any $l \in \Gamma_a$, so that, in a sense, the passage to $(s, r)$ rectifies a portion of the near-boundary part of $\mathcal{E}(\Gamma_a)$.

**Primed coordinates**

- Recall that $\mathcal{E}' = \{w'_1(\cdot), \ldots, w'_n(\cdot)\}$ and $\eta'_j = w'_j|_{\Gamma} = \mathcal{B}_l \eta_j$.

Now, let us introduce analogous coordinates on $\mathcal{E}'(M')$ near $\mathcal{E}'(\Gamma)$. Let $V_a$ be a closed disk in $\mathbb{C}$ centered at $z_a$ provided $V_a \subset U_a$. From (33) and estimate (9) it follows that

$$\eta'_j(\Gamma \setminus \Gamma_a) \cap V_a = \emptyset, \quad |\partial_\gamma \eta'_j(a)| > |\partial_\gamma \eta_j(a)|/2 > 0,$$

and

$$\Re \frac{\partial_\gamma \eta'_j(l)}{\partial_\gamma \eta'_j(a)} \in [c_0/2, 2/c_0], \quad l \in \Gamma_a$$

(34)
is valid for sufficiently small $t$. By (9), the point $z'_a := \eta'_j(a)$ is close to $z_a = \eta_j(a)$. For $z' \in \mathbb{C}$, denote
\[
\zeta'(z') := \frac{z' - z'_a}{\partial_z \eta'_j(a)}, \quad \zeta_1'(z') := \Re \zeta'(z'), \quad \zeta_2'(z') := \Im \zeta'(z'),
\]
and
\[
\psi' = \zeta' \circ \eta_j, \quad \psi'_1 := \Re \psi', \quad \psi'_2 := \Im \psi'.
\]
Due to (35), the map $\psi'_1 : \Gamma_a \mapsto \mathbb{R}$ is an injection and $\partial \psi'_1 / \partial l \in [c_0/2, 2/c_0]$ for $l \in \Gamma_a$. This and (34) imply that the curve $\eta_j'(\Gamma)$ divides $V_a$ into two components $V_{a,0}$ and $V_{a,1}$, whose closures are diffeomorphic to a half-disk.

- Choose $z_0 \in U_{a,0} \cap V_a$ and $z_1 \in U_{a,1} \cap V_a$. Due to (9), any fixed neighborhood of the set $\eta_j(\Gamma_a) \cap V_a$ in $\mathbb{C}$ contains $\eta_j'(\Gamma_a) \cap V_a$ for sufficiently small $t$. Hence,
\[
\text{dist}_\mathbb{C} (z_k, \eta_j'(\Gamma_a)) \geq c_2, \quad k = 0, 1
\]
holds with $c_2 > 0$. Here and in the subsequent, all the constants are independent of $M' \in \mathcal{M}_t$ and (sufficiently small) $t$. Thus, we can assume that $z_0 \in V_{a,0}'$ and $z_1 \in V_{a,1}'$ is fulfilled. In view of the argument principle and formulas (36), (34), and (9), we have
\[
m(w'_j, z_k) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\partial_\eta'_j}{\eta'_j - z_k} \frac{dl}{l - \eta_j} \frac{1}{2\pi i} \int_{\Gamma} \frac{\partial_\eta_j}{\eta_j - z_k} \frac{dl}{l} = m(w_j, z_k), \quad k = 0, 1,
\]
where $m(w, z)$ is the total multiplicity of zeroes of a function $w - z$ holomorphic on a Riemann surface. Since $m(w, z)$ is integer, $m(w'_j, z_k) = m(w_j, z_k)$ holds for sufficiently small $t$. Since $z_k \in U_{a,k}$, we have $m(w_j, z_k) = k$, whence $m(w'_j, z_k) = k$. Note that the integral
\[
m(w'_j, z') = \frac{1}{2\pi i} \int_{\Gamma} \frac{\partial_\eta'_j}{\eta'_j - z'} \frac{dl}{l}
\]
is the winding number of the curve $l \mapsto \eta'_j(l)$ ($l \in \Gamma$) around the point $z'$ and, hence, $m(w'_j, \cdot)$ is a constant on each connected component of $\mathbb{C} \setminus \eta'_j(\Gamma)$. Thus, $m(w'_j, z') = k$ for any $z' \in V_{a,k}'$, i.e. $w'_j(M') \cap V_{a,0}' = \emptyset$ and the pre-image $w'^{-1}_j(z')$ of any $z' \in V_{a,1}'$ consists of a single point on $\mathcal{E}'(M)$.

- Introduce the cylinder
\[
\Pi'_a := \mathbb{C} \times \cdots \times V_a \times \cdots \times \mathbb{C}.
\]
Then the projection $\pi_j$ is a bijection from $\mathcal{E}'(M') \cap \Pi'_a$ onto $V_a \setminus V_{a,0}$ and we can take $z' := \pi_j p$ as a local coordinate of the point $p \in \mathcal{E}'(M') \cap \Pi'_a$. Define the new local coordinates
\[
\omega'(z') := (s'(z'), r'(z')), \quad \omega'(z') := \left[w_1^{-1} \circ \zeta'_1(z') \right], \quad r'(z') := \zeta'_2(z') - [\psi'_2 \circ \psi'^{-1}_1 \circ \zeta'_1](z').
\]
Note that $r'(z') \geq 0$ for any $z' \in V_a \setminus V_{a,0}$. In view of (33) and (9), the function $\theta'(\tau) := [\psi'_2 \circ \psi'^{-1}_1](\tau)$ obeys
\[
|\partial \theta'/\partial \tau| \leq c_3 < \infty, \quad \tau \in \psi'_1(\Gamma_a).
\]
Thus, the Jacobian $\mathcal{J}'$ of the passage $(\Re z', \Im z') \mapsto (s', r')$ obeys
\[
|\mathcal{J}'(z')| \in [c_1/2, 2/c_1], \quad z' \in V_a
\]
for sufficiently small $t$. Moreover, estimate (38) implies that

$$\sup_{z' \in V_a} |\omega'(z') - \omega(z')| \longrightarrow 0 \quad t \rightarrow 0$$

and

$$\sup_{z' \in V_a} |(\omega^{-1} \circ \omega')(z') - z'| \longrightarrow 0 \quad t \rightarrow 0$$

hold uniformly with respect to $M' \in \mathbb{M}_t$. Thus, for sufficiently small $t$ we have $\omega'(V_a \setminus V_{a,0}) \subset \omega(U_a) \cap [\Gamma_a \times [0, +\infty]) = \omega(U_a \setminus U_{a,0})$. Therefore, the map $\Omega' := \omega' \circ \pi_j$ is an injection from $\mathcal{E}'(M') \cap \Pi' \rightarrow \Omega(\mathcal{E}(M) \cap \Pi_a)$. Note that $\Omega'(\mathcal{E}'(l)) = (l, 0)$ for any $l \in \Gamma$ such that $\mathcal{E}'(l) \in \Pi'_a$. So, for small $t$, the map

$$\Omega^{-1} \circ \Omega' : \mathcal{E}'(M') \cap \Pi'_a \rightarrow \mathcal{E}'(M) \cap \Pi_a$$

is well-defined and is an injection.

With each $p' \in \mathcal{E}'(M') \cap \Pi'_a$ we associate the point $p := [\Omega^{-1} \circ \Omega'](p')$. It is the required local bijection between near-boundary points of $\mathcal{E}(M)$ and $\mathcal{E}'(M')$. Note that, if $p' = \mathcal{E}'(l)$ with $l \in \Gamma$, then $p = \mathcal{E}(l)$.

**Estimates via argument principle**

The next step is to establish the closeness of points $p'$ and $p$. From a technical point of view, the difficulty consists in estimating the values of Cauchy-type integrals near the integration contour.

**Lemma 3.**

1. The relation

$$\sup_{M' \in \mathbb{M}_t} \left( \sup_{p' \in \mathcal{E}'(M') \cap \Pi'_a} \|\Omega^{-1} \circ \Omega'(p') - p'\|_{\mathbb{C}^m} \right) \longrightarrow 0 \quad t \rightarrow 0 \tag{39}$$

is valid.

2. Suppose that the map $\varphi^m : m < n$ is an immersion (the projection $\varphi^m$ is given by (22)). Then, for sufficiently small $t > 0$ and any $M' \in \mathbb{M}_t$, the map $\varphi^m : \mathcal{E}'(\Pi_a) \rightarrow \mathbb{C}^m$ is an immersion.

**Proof.** It suffices to prove (39) for $\mathcal{E}'(M')$ replaced by $\mathcal{E}'(M' \setminus \Gamma)$. This fact follows from (22) since $|\Omega^{-1} \circ \Omega'(p') - p'| = |\mathcal{E}'(l) - \mathcal{E}'(l)|$ for any $p' = \mathcal{E}'(l)$.

1. Let $p'$ be an arbitrary point of $\mathcal{E}'(M' \setminus \Gamma) \cap \Pi'_a$. Denote $p := [\Omega^{-1} \circ \Omega'](p')$, $z' := \pi_j p'$, $z := \pi_p$, and $(s, r) := \Omega'(p') = \Omega(p)$. Note that $z' \in V_{a,1}$, $z \in U_{a,1}$, and $r > 0$. Hence, the pre-image $w_j^{-1}\{z'\}$ contains a single point $x' \in M'$, whereas $w_j^{-1}\{z\}$ consists of a single point $x \in M$. Also, $x'$ is a simple zero of $w_j - z'$ and $x$ is a simple zero of $w_j - z$. By the generalized argument principle, we have

$$\pi_k p' = w_k(x') = \frac{1}{2\pi i} \int_{\gamma} \frac{\eta_k' \partial \eta_j'}{\eta_j' - z'} dl + \frac{1}{2\pi i} \int_{\Gamma} \frac{(\eta_k' - \eta_k(s)) \partial \eta_j'}{\eta_j' - z'} dl + \eta_k'(s) \frac{1}{2\pi i} \int_{\gamma} \frac{\partial \eta_j'}{\eta_j' - z'} dl = \frac{1}{2\pi i} \int_{\Gamma} \frac{(\eta_k' - \eta_k(s)) \partial \eta_j'}{\eta_j' - z'} dl + \eta_k'(s) \tag{40}$$
For $\delta > 0$, denote $\Gamma_\delta(s) = \{ l \in \Gamma \mid \text{dist}_\Gamma(l, s) \leq \delta \}$ (then $\Gamma_\delta(s) \subset \Gamma$ for sufficiently small $t$ and $\delta$). Put

$$J_{k,\delta}'(z') := \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{(\eta_k' - \eta_k'(s)) \partial_z \eta_j'(s)}{\eta_j' - z'} \, dl, \quad \tilde{J}_{k,\delta}'(z') := \frac{1}{2\pi i} \int_{\Gamma \setminus \Gamma_\delta} \frac{(\eta_k' - \eta_k'(s)) \partial_z \eta_j'(s)}{\eta_j' - z'} \, dl. \quad (41)$$

2. Let us estimate $|J_{k,\delta}'(z')|$. We have

$$J_{k,\delta}'(z') = \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{\eta_k' - \eta_k'(s)}{\eta_j' - \eta_j'(s)} \, \eta_j' \partial_z \eta_j' \, dl.$$

Recall that $\psi' = \zeta' \circ \eta_j' = (\eta_j' - z_0')/\partial_z \eta_j'(a)$ and $s = [\psi_1'^{-1} \circ \zeta_1'](z')$. Hence,

$$\left| \frac{\eta_j'(l) - \eta_j'(s)}{\eta_j'(l) - z'} \right| \leq \left| \frac{\psi'(l) - \psi'(s)}{\psi'(l) - \zeta'(z')} \right| \leq \left| \frac{\psi_1'(l) - \psi_1'(s)}{\psi_1'(l) - \psi_1'(s)} \right| = 1 + \frac{\theta'(\tau_1) - \theta'(\tau_s)}{\tau_1 - \tau_s}$$

holds, where $\tau_1 := \psi_1'(l)$, $\tau_s := \psi_1'(s)$, and $\theta' = \psi_2' \circ \psi_1'^{-1}$. Hence, in view of (37), we get

$$\left| \frac{\eta_j'(l) - \eta_j'(s)}{\eta_j'(l) - z'} \right| \leq 1 + c_3. \quad (42)$$

Relations (34) and (35) imply

$$\left| \eta_j'(l) - \eta_j'(s) \right| \geq \left| \partial_z \eta_j'(a) \right| \left| \mathfrak{Re} \frac{\eta_j'(l) - \eta_j'(s)}{\partial_z \eta_j'(a)} \right| = \left| \partial_z \eta_j'(a) \right| \left| \mathfrak{Re} \frac{\eta_j'(l)}{\partial_z \eta_j'(a)} - \mathfrak{Re} \frac{\eta_j'(s)}{\partial_z \eta_j'(a)} \right| = \left| \partial_z \eta_j'(a) \right| \left| \int_{\gamma} \mathfrak{Re} \frac{\partial_z \eta_j'(\tau)}{\partial_z \eta_j'(a)} \, d\tau \right| \geq \frac{\left| \partial_z \eta_j'(a) \right| c_0}{4} \text{dist}_\Gamma(s, l).$$

Therefore, in view of (9),

$$\left| \frac{\eta_k' - \eta_k'(s)}{\eta_j' - \eta_j'(s)} \right| \leq \frac{4}{\left| \partial_z \eta_j'(a) \right| c_0} \left| \eta_k' \right|_{C^1(\Gamma; \mathbb{C})} \leq \frac{8}{\left| \partial_z \eta_j'(a) \right| c_0} \left| \eta_k \right|_{C^1(\Gamma; \mathbb{C})} \quad (43)$$

for sufficiently small $t$. Combining (12), (13), and (9), we obtain

$$\left| J_{k,\delta}'(z') \right| \leq c_4 \delta. \quad (44)$$

3. Let $J_{k,\delta}(k, z)$ and $\tilde{J}_{k,\delta}(k, z)$ be defined by formula (41) with omitted primes. The same arguments as above show that we can omit primes in (40), (41). In particular,

$$\pi_k p' - \pi_k p = J_{k,\delta}'(z') - J_{k,\delta}(z) + \tilde{J}_{k,\delta}'(z') - \tilde{J}_{k,\delta}(z) + \eta_k'(s) - \eta_k(s). \quad (45)$$
Fix an arbitrary $\varepsilon > 0$ and take $\delta = \delta(\varepsilon) \in (0, \varepsilon/6 c_4)$, then (44) yields $|J_{k,\delta}'(z'')| + |J_{k,\delta}(z)| \leq \varepsilon/3$. Now, let us estimate $|\tilde{J}_{k,\delta}'(z') - \tilde{J}_{k,\delta}(z)|$. We have

$$|\tilde{J}_{k,\delta}'(z') - \tilde{J}_{k,\delta}(z)| = \left| \int_{\Gamma \setminus \Gamma_\delta} \left( \frac{\eta_k'(s) - \eta_k(s)}{2\pi(s - z)} \right) \frac{\partial_n \eta_j'(s) - (\eta_k - \eta_k(s)) (\eta_j'(s) - z') \partial_n \eta_j}{(\eta_j'(s) - z')} \frac{dl}{(\eta_j'(s) - z')} \right|. \quad (46)$$

Recall that $\partial \psi_1' / \partial l \in [c_0/2, 2/c_0]$ for $l \in \Gamma_a$ and $|\partial_n \eta_j'(a)| \geq 2|\partial_n \eta_j(a)|/2$. Hence,

$$|\eta_j'(l) - z'| = |\zeta^{-1} \circ \psi_1'(l) - z'| = |\partial_n \eta_j'(a) - \zeta^{-1} \partial_n \eta_j'(a)| \geq |\partial_n \eta_j'(a)||\psi_1'(l) - \zeta^{-1}(z')| = |\partial_n \eta_j'(a)||\psi_1'(l) - \zeta^{-1}(z')| = \int_s^l \partial_n \psi_1'(\tau) d\tau \geq \frac{c_0}{4} |\partial_n \eta_j(a)| \delta.$$  

Similarly, we obtain $|\eta_j(l) - z| \geq c_0 |\partial_n \eta_j(a)| \delta/4$. Thus, the denominator in (46) does not become small. Also, the $|z - z'| = |[\omega \circ \omega^{-1}](z') - z'|$ obeys (38). So, the same arguments as used in the proof of (27) show that

$$\sup_{M' \in \mathcal{M}_\varepsilon} \sup_{l': M' \cap \Gamma' \ni l'} |\tilde{J}_{k,\delta}'(z') - \tilde{J}_{k,\delta}(z)| \rightarrow 0. \quad (47)$$

Choose $t(\varepsilon) > 0$ such that the left-hand sides of (47) and (9) are less than $\varepsilon/3$ for any $t \in (0, t(\varepsilon))$. Now, (45) implies that $|\sigma_{kM'} - \pi_{kM}| < \varepsilon$ for any $t \in (0, t_0)$, $M' \in \mathcal{M}_t$, and $M' \in \mathcal{E}(M' \cap \Gamma) \cap \Pi'_a$. Since $\varepsilon > 0$ and $k = 1, \ldots, n$ are arbitrary, formula (39) is proved.

4. Differentiating (40), we obtain

$$\partial_z w_k'(x') = \frac{1}{2\pi i} \int_{\Gamma} \frac{\eta_k'(s) \partial_n \eta_j'(s)}{(\eta_j'(s) - z')^2} dl = \tilde{J}_{k,\delta}'(z') + \tilde{\tilde{J}}_{k,\delta}'(z') + \tilde{\tilde{R}}_{k,\delta}'(z'),$$

where

$$\tilde{J}_{k,\delta}'(z') = \frac{1}{2\pi i} \int_{\Gamma \setminus \Gamma_\delta} \left( \frac{\eta_k'(l) - \eta_k(s)}{2\pi i} \right) \frac{\partial_n \eta_j'(s) - \eta_k'(l)}{(\eta_j'(s) - z')^{\prime}} \partial_n \eta_j dl,$$

$$\tilde{\tilde{J}}_{k,\delta}'(z') = \frac{1}{2\pi i} \int_{\Gamma \setminus \Gamma_\delta} \left( \frac{\eta_k'(l) - \eta_k(s)}{2\pi i} \right) \frac{\partial_n \eta_j'(s) - \eta_k'(l)}{(\eta_j'(s) - z')^2} \partial_n \eta_j dl,$$

$$\tilde{R}_{k,\delta}'(z') = \frac{\eta_k(s)}{2\pi i} \int_{\eta_j \setminus \Gamma} \frac{\partial_n \eta_j'(s)}{(\eta_j'(s) - z')^2} dl = \frac{\eta_k(s)}{2\pi i} \int_{\eta_j \setminus \Gamma} \frac{dz'}{(\eta_j'(s) - z')^2} = 0 \quad (48)$$

$$\tilde{\tilde{R}}_{k,\delta}'(z') = \frac{\eta_k'(s)}{2\pi i} \int_{\eta_j \setminus \Gamma} \frac{\partial_n \eta_j'(s)}{(\eta_j'(s) - z')^2} dl = \frac{\eta_k(s)}{2\pi i} \int_{\eta_j \setminus \Gamma} \frac{\partial_n \eta_j'(s)}{(\eta_j'(s) - z')^2} dl +$$

$$+ \frac{z' - \eta_j'(s)}{2\pi i} \int_{\eta_j \setminus \Gamma} \frac{\partial_n \eta_j'(s)}{(\eta_j'(s) - z')^2} dl = \frac{\eta_k(s)}{2\pi i} \int_{\eta_j \setminus \Gamma} \frac{\partial_n \eta_j'(s)}{(\eta_j'(s) - z')^2} dl +$$

Similarly,

$$\partial_z w_k(x) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\eta_k \partial_n \eta_j}{(\eta_j - z)^2} dl = \tilde{J}_{k,\delta}(z) + \tilde{\tilde{J}}_{k,\delta}(z) + \tilde{\tilde{R}}_{k,\delta}(z) + \tilde{\tilde{R}}_{k,\delta}(z),$$

17
where the summands on the right are defined by (48) with omitted primes. In view of (9) and (34),
\[ \mathfrak{G}_k(s, l) := |\eta'_k(l) - \eta_k(s)| \frac{\partial_j \eta'_k(s)}{\partial_j \eta'_k(s)} |(\eta'_j(l) - \eta'_j(s))| \leq c \sum_{p=1}^n |\eta'_p| c_{\mathbb{C}^2} \left( \operatorname{dist}_\Gamma(s, l) \right)^2 \leq c(t \operatorname{dist}_\Gamma(s, l))^2. \]

Hence, formulas (42) and (9) imply
\[ \mathfrak{G}_k(s, l) \leq \frac{2 \pi}{r} \int_{\Gamma \setminus \delta} \left| \frac{\mathfrak{G}_k(s, l)}{(\eta'_j(l) - \eta'_j(s))^2} \right| \left| \frac{\partial_j \eta'_j(l)}{\partial_j \eta'_j(s)} \right| d\gamma \rightarrow 0. \] (49)

uniformly with respect to $M' \in \mathbb{M}_t$ and $x' \in M'$ such that $\mathcal{E}'(x') \in \mathcal{E}'(M' \setminus \Gamma) \cap \Pi'$. Note that (49) remains valid with omitted primes. For any fixed $\delta > 0$, the formula
\[ \sup_{M' \in \mathbb{M}_t} \sup_{p' \in \mathcal{E}'(M' \setminus \Gamma) \cap \Pi'} \left| \mathfrak{G}_k(s, l) - \mathfrak{G}_k(z) \right| \rightarrow 0 \]

is valid and it is obtained by repeating the arguments leading to (47). Also, for any fixed $\delta > 0$, the formula
\[ \sup_{M' \in \mathbb{M}_t} \sup_{p' \in \mathcal{E}'(M' \setminus \Gamma) \cap \Pi'} \left| \mathfrak{G}_k(s, l) - \mathfrak{G}_k(z) \right| \rightarrow 0 \]

follows from (9) and (33). Thus,
\[ |\partial_x w'_k(x') - \partial_x w_k(x)| \rightarrow 0, \]

uniformly with respect to $M' \in \mathbb{M}_t$, and any $x' \in M'$ such that $\mathcal{E}'(x') \in \mathcal{E}'(M' \setminus \Gamma) \cap \Pi'$. Now, arguing as in the proof of Lemma 2 ii, we prove the statement ii, Lemma 3. \qed

**Completing the proof of Theorem 1**

- As a corollary of (39), we obtain
\[ \sup_{M' \in \mathbb{M}_t} \left( \sup_{p' \in \mathcal{E}'(M') \cap \Pi'} \operatorname{dist}_{\mathbb{C}^n}(p', \mathcal{E}'(M) \cap \Pi') \right) \rightarrow 0. \] (50)

for any $a \in \Gamma$. The cylinders $\Pi'_a \setminus \partial \Pi'_a$, $a \in \Gamma$ constitute an open cover of the compact set $\mathcal{E}(\Gamma)$ in $\mathbb{C}^n$. Choose a finite subcover $\Pi'_a \setminus \partial \Pi'_a$, $i = 1, \ldots, N$ and denote $\tilde{\mathcal{Q}} := \bigcup_{i=1}^N \Pi'_a \setminus \partial \Pi'_a$ and $\tilde{\mathcal{Q}} := \bigcup_{i=1}^N \Pi_a$. Then (50) implies
\[ \sup_{M' \in \mathbb{M}_t} \frac{r_{\mathcal{E}(M) \cap \tilde{\mathcal{Q}}} \left| \mathcal{E}'(M') \cap \tilde{\mathcal{Q}} \right|}{\sup_{M' \in \mathbb{M}_t} \left( \sup_{p' \in \mathcal{E}'(M') \cap \tilde{\mathcal{Q}}} \operatorname{dist}_{\mathbb{C}^n}(p', \mathcal{E}'(M) \cap \tilde{\mathcal{Q}}) \right) \rightarrow 0. \] (51)

Note that $\mathcal{E}(M) \setminus \tilde{\mathcal{Q}}'$ is a compact subset of $\mathcal{E}(M \setminus \Gamma)$; as shown above, it is contained in $\mathcal{Q}_\varepsilon$ for sufficiently small $\varepsilon > 0$. Denote $\mathcal{Q} = \mathcal{Q}_\varepsilon \cup \tilde{\mathcal{Q}}$, then $\mathcal{E}'(M) \subset \mathcal{Q}$. Formulas (24) and (51) yield
\[ \sup_{M' \in \mathbb{M}_t} \frac{r_{\mathcal{E}(M) \cap \tilde{\mathcal{Q}}} \left| \mathcal{E}'(M') \cap \tilde{\mathcal{Q}} \right|}{\sup_{M' \in \mathbb{M}_t} \left( \sup_{p' \in \mathcal{E}'(M') \cap \tilde{\mathcal{Q}}} \operatorname{dist}_{\mathbb{C}^n}(p', \mathcal{E}'(M) \cap \tilde{\mathcal{Q}}) \right) \rightarrow 0. \] (52)
Denote the $\varepsilon$–neighborhood of $\mathcal{E}(M)$ by $Q_\varepsilon$. Let $\varepsilon > 0$ be sufficiently small for $Q_\varepsilon$ to be contained in $Q$. Due to (52), there is $t(\varepsilon) > 0$ such that $\mathcal{E}'(M') \cap \overline{Q} \subset Q_\varepsilon$ for all $t \in (0, t(\varepsilon))$ and $M' \in M_t$. Let us show that $\mathcal{E}'(M') \subset Q$ holds for all $t \in (0, t(\varepsilon)]$ and $M' \in M_t$. Assume the opposite, then there exists $M' \in M_t(\varepsilon)$ and $p' \in \mathcal{E}'(M') \setminus Q$. Since $\mathcal{E}'(M')$ is connected, there exists a path $L$ in $\mathcal{E}'(M')$ with the beginning at $p'$ and the end at some point $p'' \in \mathcal{E}'(\Gamma) \cap \mathcal{E}'(M') \cap Q \subset Q_\varepsilon$. Hence, the set $\mathcal{E}'(\Gamma) \cap \mathcal{E}'(M') \cap Q = \emptyset$ is valid. Thus, our assumption has led to a contradiction. Therefore $\mathcal{E}'(M') \subset Q$ holds for sufficiently small $t$ and formula (52) can be rewritten as

$$\sup_{M' \in M_t} r_{\mathcal{E}(M)\mathcal{E}'(M')} \rightarrow 0 \text{ as } t \rightarrow 0.$$ 

The latter relation along with (32) and definition (2) imply (6) for any projective immersion $\mathcal{E}$. Thereby, as shown in the beginning of Section 3, formula (6) is proved for any immersion $\mathcal{E}$.

• To complete the proof of (3), it remains to show that the extension $\mathcal{E}' : M' \mapsto C^n$ of the immersion $\mathcal{E} : M \mapsto C^n$ is an immersion for sufficiently small $t > 0$ and any $M' \in M_t$. If $\mathcal{E}$ is projective, then the statement is obvious and $\mathcal{E}'$ is also projective for small $t$. If $\mathcal{E} : M \mapsto C^n$ is not projective, then it can be completed to a projective immersion. In this case, the statement follows from Lemma 2, ii, and Lemma 3, ii.

Theorem 4 is proved.

• Perhaps, relation (6) may be improved to an estimate $d_H(\mathcal{E}'(M'), \mathcal{E}(M)) = O(t^\alpha)$ with a positive $\alpha$. However, it requires more subtle considerations ‘near boundary’.

Statements and Declarations

Competing Interests. On behalf of all authors, the corresponding author states that there is no conflict of interest.

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