Information Flow Decomposition in Feedback Systems: Linear Time-Invariant Systems with Gaussian Channels

Bertrand Wechsler, Dan Eilat and Nicolas Limal

Abstract

In our companion paper [1], an information identity decomposition has been derived, which can be interpreted as a law of conservation of information flows in feedback systems. In this paper, we further investigate this decomposition result when specified to linear time-invariant (LTI) systems connected with additive white Gaussian noise (AWGN) channels. It is shown that the quantities in the decomposition are characterized in sensitivity function and the law of conservation is verified.

Index Terms

Gaussian channel, feedback, information flow, directed information,

I. INTRODUCTION

Consider a feedback system as shown in Fig. 1 (left). Let $m = x_0$, the information flow through channel $C_2$ can be decomposed into two independent flows as

$$I(y^n \rightarrow e^n) = I(x^n \rightarrow e^n) + I(y^n \rightarrow e^n|x_0).$$

This equality can be interpreted as a law of conservation of information flows. Quantity $I(x^n \rightarrow e^n)$ is the amount of information provided by the external input $x_0$, and the quantity $I(y^n \rightarrow e^n|x_0)$ is the amount of information provided by the uncertainty in the channel $C_1$ (due to the presence of noise). The sum of these two quantities equals to the total amount of information delivered from system $S_2$ to
In this paper, we turn our attention to Linear-time-invariant (LTI) systems connected with AWGN channels, as shown in Fig. 1 (right), where system $S_1$ and $S_2$ are respectively assumed to be LTI and unit gain and $(w, v)$ are AWGNs with variance $(\sigma_v^2, \sigma_w^2)$. As it will be shown, the limit values of these quantities in the above information flow equation can be characterized in sensitivity function and, furthermore, verify the law of conservation of information flow. Although the topic of distributed systems connected with one noisy channel and one noiseless channel (e.g. [2], [3], [4], [5], [6]) or both noisy channels (e.g., [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17]) is of interest and has attracted much attention in the last decade, most of the work has concentrated on communication systems with feedback, where the coding schemes are not necessarily LTI system. The result of this paper contributes to the understanding of noisy feedback systems in the control-theoretic perspective, where the systems are mostly restricted to be LTI.
II. DECOMPOSITION ON LTI FEEDBACK SYSTEMS WITH GAUSSIAN CHANNELS

First of all, we revisit the definition of directed information which will be repeatedly used in the paper.

Definition 1: Given random sequences $x^n, y^n$, the directed information from $x^n$ to $y^n$ is defined as

$$I(x^n \rightarrow y^n) = \sum_{i=1}^{n} I(x^i; y^i|y^{i-1}).$$

Definition 2: Consider feedback systems as shown in $[1]$, the sensitivity transfer function $S(e^{j2\pi \theta})$ is defined as the transfer function from the external disturbances $w$ to the process output $y$ or the measurement noise $v$ to the system inputs $e$.

The sensitivity function is introduced and widely used in control theory. According to the definition, lower values of $|S(e^{j2\pi \theta})|$ suggest further attenuation of the external disturbances or the measurement noise. Moreover, the sensitivity function also reflects the feedback influence on external disturbances.

Lemma 1: Consider a stationary Gaussian process $\{X_i\}_{i=1}^{\infty}$, the relationship between its entropy rate $\bar{h}(X) = \lim_{n \rightarrow \infty} \frac{1}{n} h(X^n)$ and the spectral density $S_x(e^{j\theta})$ is in the following expression,

$$\bar{h}(X) = \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \ln 2\pi e S_x(e^{j\theta}) d\theta$$

For convenience, we again recall the decomposition theorem (law of conservation of information flows) in $[1]$ as follows. Note that this decomposition is essentially introduced in $[18]$ under a general framework.

Theorem 1: Consider a feedback system as shown in Fig. $[1]$ (left). Let $m = x_0$, the information flow through channel $C_2$ can be decomposed into two independent flows as

$$I(y^n \rightarrow e^n) = I(x^n \rightarrow e^n) + I(y^n \rightarrow e^n|x_0).$$
In what follows, this result is applied to the model as shown in Fig. 1(right). All the three quantities are explicitly characterized in sensitivity function and the law of conservation is verified.

**Theorem 2:** Consider LTI feedback systems with AWGN channels as shown in Fig. 1(right). Then,

\[
\lim_{n \to \infty} \frac{1}{n} I(y^n \to e^n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \ln |S(e^{j2\pi\theta})| d\theta + \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \ln(1 + \frac{\sigma_w^2}{\sigma_v^2}) d\theta
\]

\[
\lim_{n \to \infty} \frac{1}{n} I(x^n \to e^n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \ln |S(e^{j2\pi\theta})| d\theta
\]

\[
\lim_{n \to \infty} \frac{1}{n} I(y^n \to e^n | x_0) = \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \ln(1 + \frac{\sigma_w^2}{\sigma_v^2}) d\theta
\]

**Proof:** We prove the above three equalities one by one. (1). First of all, we have

\[
I(y^n \to e^n) = \sum_{i=1}^{n} I(y^i, e_i | e_i^{i-1})
\]

\[
= \sum_{i=1}^{n} h(e_i | e_i^{i-1}) - h(e_i | e_i^{i-1}, y^i)
\]

\[
= \sum_{i=1}^{n} h(e_i | e_i^{i-1}) - h(y_i + v_i | y_1 + v_1, y_2 + v_2, \ldots, y_{i-1} + v_{i-1}, y^i)
\]

\[
= \sum_{i=1}^{n} h(e_i | e_i^{i-1}) - h(v_i | v_1, v_2, \ldots, v_{i-1}, y^i)
\]

\[
= \sum_{i=1}^{n} h(e_i | e_i^{i-1}) - h(v_i | v^{i-1})
\]

\[= h(e^n) - h(v^n)\]
Then, according to Lemma 1,

\[
\lim_{n \to \infty} \frac{1}{n} I(y^n \to e^n) = \lim_{n \to \infty} \frac{1}{n} h(e^n) - h(v^n) = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln 2\pi e S_v(e^{i\theta}) d\theta - \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln 2\pi e S_v(e^{i\theta}) d\theta = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{S_v(e^{i\theta})}{S_v(e^{i\theta})} d\theta
\]

\[
= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln \frac{|S(e^{i2\pi\theta})|^2 (S_v(e^{i2\pi\theta}) + S_w(e^{i2\pi\theta}))}{S_v(e^{i2\pi\theta})} d\theta
\]

\[
= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln \frac{|S(e^{i2\pi\theta})|^2 (\sigma_v^2 + \sigma_w^2)}{\sigma_v^2} d\theta
\]

\[
= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln |S(e^{i2\pi\theta})| d\theta + \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln (1 + \frac{\sigma_w^2}{\sigma_v^2}) d\theta
\]

(2). Next,

\[
I(x^n \to e^n) = \sum_{i=1}^{n} I(x^i, e_i | e_i^{i-1})
\]

\[
= \sum_{i=1}^{n} h(e_i | e_i^{i-1}) - h(e_i | e_i^{i-1}, x^i)
\]

\[
= \sum_{i=1}^{n} h(e_i | e_i^{i-1}) - h(x_i + w_i + v_i | x_1 + w_1 + v_1, x_2 + w_2 + v_2, \cdots, x_{i-1} + w_{i-1} + v_{i-1}, x^i)
\]

\[
= \sum_{i=1}^{n} h(e_i | e_i^{i-1}) - h(w_i + v_i | x_1 + w_1, x_2 + w_2 + v_2, \cdots, w_{i-1} + v_{i-1}, x^i)
\]

\[
= \sum_{i=1}^{n} h(e_i | e_i^{i-1}) - h(w_i + v_i | w_i + v_{i-1} + v_{i-1}^{i-1})
\]

\[
= h(e^n) - h(v^n + w^n)
\]
Similarly, we have

$$\lim_{n \to \infty} \frac{1}{n} I(x^n \to e^n) = \lim_{n \to \infty} \frac{1}{n} h(e^n) - h(v^n + w^n)$$

$$= \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \ln 2\pi e S_v(e^{j\theta}) d\theta - \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \ln 2\pi e S_w(e^{j\theta}) + S_v(e^{j\theta}) d\theta$$

$$= \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \ln \frac{S_v(e^{j2\pi\theta})}{S_w(e^{j2\pi\theta}) + S_v(e^{j2\pi\theta})} d\theta$$

$$= \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \ln \frac{|S(e^{j2\pi\theta})|^2}{\sigma_v^2 + \sigma_w^2} d\theta$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \ln |S(e^{j2\pi\theta})| d\theta$$

(3). Finally, we need to show \( \lim_{n \to \infty} \frac{1}{n} I(y^n \to e^n|x_0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \ln (1 + \frac{\sigma_w^2}{\sigma_v^2}) d\theta \).

$$I(y^n \to e^n|x_0) = \sum_{i=1}^{n} I(y^i, e_i|e^{i-1}, x_0)$$

$$= \sum_{i=1}^{n} h(e_i|e^{i-1}, x_0) - h(e_i|e^{i-1}, y^i, x_0)$$

$$= \sum_{i=1}^{n} h(e_i|e^{i-1}, x_0) - h(y_i + v_i|y_1 + v_1, y_2 + v_2, \ldots, y_{i-1} + v_{i-1}, y^i, x_0)$$

$$= \sum_{i=1}^{n} h(e_i|e^{i-1}, x_0) - h(v_i|v_1, v_2, \ldots, v_{i-1}, y^i, x_0)$$

$$= \sum_{i=1}^{n} h(e_i|e^{i-1}, x_0) - h(v_i|v^{i-1})$$

$$= \sum_{i=1}^{n} h(x_i + w_i + v_i|x_1 + w_1 + v_1, x_2 + w_2 + v_2, \ldots, x_{i-1} + w_{i-1} + v_{i-1}, x_0) - h(v_i|v^{i-1})$$
Because $S_1$ is assumed to be a causal LTI system $\{g_i\}_{i=1}^{\infty}$, the system output $x_i = g_i(x_{i-1}, e_{i-1}) = g_i(x_{i-1}, w_{i-1} + v_{i-1})$ and $x_1 = g_1(x_0)$. Thus,

\[
I(y^n \to e^n|x_0) = \sum_{i=1}^{n} h(x_i + w_i + v_i|x_1 + w_1 + v_1, x_2 + w_2 + v_2, \ldots, x_{i-1} + w_{i-1} + v_{i-1}, x_0) - h(v_i^{i-1})
\]

\[
= \sum_{i=1}^{n} h(x_i + w_i + v_i|x_1, w_1 + v_1, x_2 + w_2 + v_2, \ldots, x_{i-1} + w_{i-1} + v_{i-1}, x_0) - h(v_i^{i-1})
\]

\[
= \sum_{i=1}^{n} h(x_i + w_i + v_i|x_1, w_1 + v_1, x_2, w_2 + v_2, \ldots, x_{i-1} + w_{i-1} + v_{i-1}, x_0) - h(v_i^{i-1})
\]

\[
= \sum_{i=1}^{n} h(w_i + v_i|w_1 + v_1, w_2 + v_2, \ldots, w_{i-1} + v_{i-1}, x_i) - h(v_i^{i-1})
\]

\[
= h(w^n + v^n) - h(v^n)
\]

Then, we have

\[
\lim_{n \to \infty} \frac{1}{n} I(y^n \to e^n|x_0) = \lim_{n \to \infty} \frac{1}{n} h(w^n + v^n) - h(v^n) = \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \ln(1 + \frac{\sigma_w^2}{\sigma_v^2}) d\theta
\]

III. Conclusion

We characterized the quantities in the law of conservation of information flows for distributed LTI systems connected with AWGN channels. The characterizations are represented in sensitivity function and verifies the law of conservation.

References

[1] B. Wechsler, D. Eilat, and N. Limal, “Information flow decompositions in feedback systems: General case study,” [online] http://arxiv.org/abs/1404.0760, 2014.

[2] Y. H. Kim, “A coding theorem for a class of stationary channels with feedback,” IEEE Transactions on Information Theory, vol. 54, no. 4, pp. 1488–1499, 2008.

[3] ———, “Feedback capacity of stationary gaussian channels,” IEEE Transactions on Information Theory, vol. 56, no. 1, pp. 57–85, 2010.
[4] O. Shayevitz and M. Feder, “Communication with feedback via posterior matching,” in *Proc. IEEE International Symposium on Information Theory*, pp. 391–395, 2007.

[5] S. Tatikonda and S. Mitter, “The capacity of channels with feedback,” *IEEE Transactions on Information Theory*, vol. 55, no. 1, pp. 323–349, 2009.

[6] H. H. Permuter, T. Weissman, and A. J. Goldsmith, “Finite state channels with time-invariant deterministic feedback,” *IEEE Transactions on Information Theory*, vol. 55, no. 2, pp. 644–662, 2009.

[7] S. C. Draper and A. Sahai, “Noisy feedback improves communication reliability,” in *Proc. IEEE International Symposium on Information Theory*, pp. 69–73, 2006.

[8] C. Li and N. Elia, “The information flow and capacity of channels with noisy feedback,” [online] http://arxiv.org/abs/1108.2815, 2011.

[9] ———, “The information theoretic characterization of the capacity of channels with noisy feedback,” in *Proc. IEEE International Symposium on Information Theory*, pp. 174–178, 2011.

[10] Y. H. Kim, A. Lapidoth, and T. Weissman, “The gaussian channel with noisy feedback,” in *Proc. IEEE International Symposium on Information Theory*, pp. 1416–1420, 2007.

[11] C. Li and N. Elia, “Bounds on the achievable rate of noisy feedback gaussian channels under linear feedback coding scheme,” in *Proc. IEEE International Symposium on Information Theory*, pp. 169–173, 2011.

[12] ———, “Upper bound on the capacity of gaussian channels with noisy feedback,” in *Proc. 49th Annual Allerton Conference on Communication, Control and Computing*, pp. 84–89, 2011.

[13] N. C. Martins and T. Weissman, “Coding for additive white noise channels with feedback corrupted by quantization or bounded noise,” *IEEE Transactions on Information Theory*, vol. 54, no. 9, pp. 4274–4282, Sep. 2008.

[14] C. Li and N. Elia, “Capacity of finite alphabet channels with noisy feedback,” in *Proc. 49th Annual Allerton Conference on Communication, Control and Computing*, pp. 90–95, 2011.

[15] Z. Chance and D. J. Love, “A noisy feedback encoding scheme for the gaussian channel,” in *Proc. IEEE International Conference on Acoustics Speech and Signal Processing*, pp. 3482–3485, 2010.

[16] C. Li and N. Elia, “Noisy feedback communications with side information at the decoder,” in *Proc. 50th Annual Allerton Conference on Communication, Control and Computing*, pp. 1856–1863, 2011.

[17] N. Limal, “An information identity for state-dependent channels with feedback,” [online] http://arxiv.org/abs/1402.0925, 2014.

[18] M. S. Derpich, E. I. Silva, and J. Østergaard, “Fundamental inequalities and identities involving mutual and directed informations in closed-loop systems,” [online] http://arXiv.org/abs/1301.6427, 2013.