State Sensitivity Evaluation within UD based Array Covariance Filters

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Abstract

This technical note addresses the UD factorization based Kalman filtering (KF) algorithms. Using this important class of numerically stable KF schemes, we extend its functionality and develop an elegant and simple method for computation of sensitivities of the system state to unknown parameters required in a variety of applications. For instance, it can be used for efficient calculations in sensitivity analysis and in gradient-search optimization algorithms for the maximum likelihood estimation. The new theory presented in this technical note is a solution to the problem formulated by Bierman et al. in [1], which has been open since 1990s. As in the cited paper, our method avoids the standard approach based on the conventional KF (and its derivatives with respect to unknown system parameters) with its inherent numerical instabilities and, hence, improves the robustness of computations against roundoff errors.

Index Terms

Array algorithms, Kalman filter, filter sensitivity equations, UD factorization.

I. INTRODUCTION

Linear discrete-time stochastic state-space models, with associated Kalman filter (KF), have been extensively used in practice. Application of the KF assumes a complete a priori knowledge of the state-space model parameters, which is a rare case. As mentioned in [2], the classical way of solving the problem of uncertain parameters is to use adaptive filters where the model parameters are estimated together with the dynamic state. This requires determination of sensitivities of the system state to unknown parameters. Other applications with similar requirements arise, for instance, in the field of optimal input design [3], [4] etc.

Straight forward differentiation of the KF equations is a direct approach to compute the state sensitivities to unknown parameters. This leads to a set of vector equations, known as the filter sensitivity equations and a set of matrix equations, known as the Riccati-type sensitivity equations. The main disadvantage of the standard approach is the problem of numerical instability of the conventional KF (see, for instance, discussion in [5]). The alternative approach can be found in, so-called, square-root (SR) algorithms developed to deal with the problem of the numerical instability. Among all existing implementation methods the array SR filters are currently the most preferable for practical implementations. Such methods have the property of better conditioning and reduced dynamical range. They also imply utilization of numerically

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stable orthogonal transformations for each recursion step that leads to more robust computations (see [6] Chapter 12 for an extended explanation). Since the appearance of the KF in 1960s a number of array SR filters have been developed for the recursive KF [7]–[12] and smoothing formulas [13]–[16]. Recently, some array SR methods have been designed for the $H^\infty$ estimation in [17]. Here, we deal with the array UD factorization based filters; see [18]. They represent an important class of numerically stable KF implementation methods.

For the problem of the sensitivities computation, the following question arises: would it be possible to update the sensitivity equations in terms of the variables that appear naturally in the mentioned numerically favored KF algorithms? The first attempt to answer this question belongs to Bierman et al. In [1] the authors proposed an elegant method that naturally extends the square-root information filter (SRIF), developed by Dyer and McReynolds [19], on the case of the log likelihood gradient (Log LG) evaluation. Later this approach was generalized to the class of covariance-type filters in [20]. However, the problem of utilizing the UD based KF algorithms to generate the required quantities has been open since 1990s. In this technical note we propose and justify an elegant and simple solution to this problem. More precisely, we present a new theory that equips any array UD based KF algorithm with a means for simultaneous computation of derivatives of the filter variables with respect to unknown system parameters. As in [1], this avoids implementation of the conventional KF (and its direct differentiation with respect to unknown system parameters) because of its inherent numerical instability and, hence, we improve the robustness of computations against roundoff errors. The new results can be used, e.g., for efficient calculations in sensitivity analysis and in gradient-search optimization algorithms for the maximum likelihood estimation of unknown system parameters.

II. PROBLEM STATEMENT AND THE CONVENTIONAL KALMAN FILTER

Consider the discrete-time linear stochastic system

$$x_k = Fx_{k-1} + Gw_k, \quad k \geq 0,$$

$$z_k = Hx_k + v_k$$

where $x_k \in \mathbb{R}^n$ and $z_k \in \mathbb{R}^m$ are, respectively, the state and the measurement vectors; $k$ is a discrete time, i.e. $x_k$ means $x(t_k)$. The process noise, $\{w_k\}$, and the measurement noise, $\{v_k\}$, are Gaussian white-noise processes, with covariance matrices $Q \geq 0$ and $R > 0$, respectively. All random variables have known mean values, which we can take without loss of generality to be zero. The noises $w_k \in \mathbb{R}^q$, $v_k \in \mathbb{R}^m$ and the initial state $x_0 \sim \mathcal{N}(0, \Pi_0)$ are taken from mutually independent Gaussian distributions.

The associated KF yields the linear least-square estimate, $\hat{x}_{k|k-1}$, of the state vector $x_k$ given the measurements $Z_{k-1}^k = \{z_1, \ldots, z_{k-1}\}$ that can be computed as follows [6]:

$$\hat{x}_{k+1|k} = F\hat{x}_{k|k-1} + K_{p,k}e_k, \quad e_k = z_k - H\hat{x}_{k|k-1}$$

where $\hat{x}_{0|0} = 0$, $e_k \sim \mathcal{N}(0, R_{e,k})$ are innovations of the KF, $K_{p,k} = \mathbf{E}\{\hat{x}_{k+1|k}e_k^T\}$ and $K_{p,k} = K_kR_{e,k}^{-1}$, $K_k = FP_{k|k-1}H^T$, $R_{e,k} = R + HP_{k|k-1}H^T$. The error covariance matrix

$$P_{k|k-1} = \mathbf{E}\{(x_k - \hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})^T\}$$

satisfies the difference Riccati equation

$$P_{k+1|k} = FP_{k|k-1}F^T + GQG^T - K_{p,k}R_{e,k}K_{p,k}^T, \quad P_{0|0} = \Pi_0 > 0.$$
In practice, the matrices characterizing the dynamic model are often known up to certain parameters. Hence, we move to a more complicated problem. Assume that system (1–2) is parameterized by a vector of unknown system parameters \( \theta \in \mathbb{R}^p \) that needs to be estimated. This means that the entries of the matrices \( F \in \mathbb{R}^{n \times n}, \ G \in \mathbb{R}^{n \times q}, \ H \in \mathbb{R}^{m \times n}, \ Q \in \mathbb{R}^{q \times q}, \ R \in \mathbb{R}^{m \times n} \) and \( \Pi_0 \in \mathbb{R}^{n \times n} \) are functions of \( \theta \). For the sake of simplicity we will suppress the corresponding notations below, i.e. instead of \( F(\theta), \ G(\theta), \ H(\theta) \) etc. we will write \( F, \ G, \ H \) etc.

Solving the parameter estimation problem by the method of maximum likelihood requires maximization of the likelihood function (LF) with respect to unknown system parameters. It is often done by using the gradient approach\(^1\) where the computation of the likelihood gradient (LG) is necessary. For the state-space system (1–2), the LF and LG evaluation demands an implementation of the KF and, so-called, “differentiated” KF to determine the sensitivities of the system state to the unknown system parameters, as explained, for instance, in \([1, 4, 20]\).

More precisely, the LG computation leads to a set of \( p \) filter sensitivity equations for computing \( \partial x_{k|k-1}/\partial \theta \) and a set of \( p \) matrix Riccati-type sensitivity equations for computing \( \partial P_{k|k-1}/\partial \theta \). Our goal is to avoid the direct differentiation of the conventional KF equations because of their inherent numerical instability. Alternatively, we are going to apply a numerically favored array UD based filter. However, for the task of sensitivities computation, we have to augment this numerical scheme with a procedure for robust evaluation of the derivatives of the UD filter variables with respect to unknown system parameters.

### III. UD Based Array Covariance Filter

**Notations to be used:** Let \( D \) denotes a diagonal matrix; \( U \) and \( L \) are, respectively, unit upper and lower triangular matrices; \( \bar{U} \) and \( \bar{L} \) are, respectively, strictly upper and lower triangular matrices. We use Cholesky decomposition of the form \( A = A^{T/2}A^{1/2} \), where \( A^{1/2} \) is an upper triangular matrix. For convenience we will write \( A^{-1/2} = (A^{1/2})^{-1}, \ A^{-T/2} = (A^{-1/2})^T \) and \( \partial A/\partial \theta_i \) implies the partial derivative of the matrix \( A \) with respect to the \( i \)th component of \( \theta \) (we assume that the entries of \( A \) are differentiable functions of a parameter \( \theta \)). Besides, we use modified Cholesky decomposition of the form \( A = U_A D_A U_A^T \).

The first UD based filter was developed by Bierman [18]. Later, Jover and Kailath [21] have presented the advantageous array form for the UD filters. In this technical note we use the UD based array covariance filter (UD-aCF) from [7] p. 261]. First, one have to set the initial values \( \hat{x}_{0|1} = 0, \ P_{0|1} = \Pi_0 > 0 \) and use the modified Cholesky decomposition to compute the factors \( \{U_\Pi_0, D_\Pi_0\}, \{U_R, D_R\}, \{U_Q, D_Q\} \). Then, we recursively update the \( \{U_{P_{k+1|k}}, D_{P_{k+1|k}}\} \) as follows \((k = 1, \ldots, N)\): given a pair of the pre-arrays \( \{A_k, D_k\} \)

\[
A_k^T = \begin{bmatrix} GU_Q & FU_{P_{k|k-1}} & 0 \\ 0 & HU_{P_{k|k-1}} & U_R \end{bmatrix}, \quad D_k = \text{diag}\{D_Q, D_{P_{k|k-1}}, D_R\}, \quad (3)
\]

apply the modified weighted Gram-Schmidt (MWGS) orthogonalization [22] of the columns of \( A_k \) with respect to the weighting matrix \( D_k \) to obtain a pair of the post-arrays \( \{\tilde{A}_k, \tilde{D}_k\} \)

\[
\tilde{A}_k = \begin{bmatrix} U_{P_{k+1|k}} & K_{p,k}U_{R_e,k} \\ 0 & U_{R_e,k} \end{bmatrix}, \quad \tilde{D}_k = \text{diag}\{D_{P_{k+1|k}}, D_{R_e,k}\}, \quad (4)
\]

\(^1\)More about computational aspects of maximum likelihood estimation, different gradient-based nonlinear programming methods and their applicability to maximum likelihood estimation could be found in [4].
such that
\[ A_k^T = \tilde{A}_k B_k^T \quad \text{and} \quad A_k^T D_k A_k = \tilde{A}_k \tilde{D}_k \tilde{A}_k^T \]
where \( D_k \in \mathbb{R}^{(n+m+q) \times (n+m+q)} \), \( A_k \in \mathbb{R}^{(n+m+q) \times (n+m)} \), \( B_k \in \mathbb{R}^{(n+m+q) \times (n+m)} \) is the MWGS transformation that produces the block upper triangular matrix \( A_k \in \mathbb{R}^{(n+m) \times (n+m)} \) and diagonal matrix \( D_k \in \mathbb{R}^{(n+m) \times (n+m)} \).

The state estimate can be computed as follows:
\[ \hat{x}_{k+1|k} = F \hat{x}_{k|k-1} + (K_{p,k} U_{R_e,k}) \tilde{e}_k \]
where \( \tilde{e}_k = U_{R_e,k}^{-1} e_k \), \( e_k = z_k - H \hat{x}_{k|k-1} \).

**Remark 1:** We note that the parenthesis in (5) are used to indicate the quantities that can be directly read off from the post-arrays.

Instead of the conventional KF, which is known to be numerically unstable, we wish to utilize stable UD-aCF filter presented above to compute the Log LF:
\[ \mathcal{L}_\theta (Z_1^N) = -\frac{N m}{2} \ln(2\pi) - \frac{1}{2} \sum_{k=1}^{N} \{ \ln \left( \det R_{e,k} \right) + \tilde{e}_k^T R_{e,k}^{-1} \tilde{e}_k \} \]
where \( Z_1^N = \{ z_1, \ldots, z_N \} \) is \( N \)-step measurement history and the innovations, \( \{ e_k \}, e_k \sim \mathcal{N}(0, R_{e,k}) \), are generated by the discrete-time KF.

One can easily obtain the expression for Log LF (6) in terms of the UD-aCF variables:
\[ \mathcal{L}_\theta (Z_1^N) = -\frac{N m}{2} \ln(2\pi) - \frac{1}{2} \sum_{k=1}^{N} \{ \ln \left( \det D_{R_e,k} \right) + \tilde{e}_k^T D_{R_e,k}^{-1} \tilde{e}_k \} \].

Let \( \theta = [\theta_1, \ldots, \theta_p] \) denote the vector of parameters with respect to which the likelihood function is to be differentiated. Then from (7), we have
\[ \frac{\partial \mathcal{L}_\theta (Z_1^N)}{\partial \theta_i} = -\frac{1}{2} \sum_{k=1}^{N} \left\{ \frac{\partial \ln \left( \det D_{R_e,k} \right)}{\partial \theta_i} + \frac{\partial \left[ \tilde{e}_k^T D_{R_e,k}^{-1} \tilde{e}_k \right]}{\partial \theta_i} \right\} \]
where \( i = 1, \ldots, p \).

Taking into account that the matrix \( D_{R_e,k} \) is diagonal and using Jacobi’s formula, we obtain the expression for the Log LG evaluation in terms of the UD-aCF variables \( (i = 1, \ldots, p) \):
\[ \frac{\partial \mathcal{L}_\theta (Z_1^N)}{\partial \theta_i} = -\frac{1}{2} \sum_{k=1}^{N} \{ \text{tr} \left[ \frac{\partial D_{R_e,k}}{\partial \theta_i} D_{R_e,k}^{-1} \right] + 2 \frac{\partial \tilde{e}_k^T}{\partial \theta_i} D_{R_e,k}^{-1} \tilde{e}_k \}
\]
\[ -\tilde{e}_k^T D_{R_e,k}^{-2} \frac{\partial D_{R_e,k}}{\partial \theta_i} \tilde{e}_k \].

Our goal is to compute Log LF (7) and Log LG (8) by using the UD-aCF variables. As can be seen, the elements \( \tilde{e}_k \) and \( D_{R_e,k} \) are readily available from UD based filter (3)–(5). Hence, our aim is to explain how the last two terms, i.e. \( \partial \tilde{e}_k / \partial \theta_i \) and \( \partial D_{R_e,k} / \partial \theta_i \), \( i = 1, \ldots, p \), can be computed using quantities available from the UD-aCF algorithm.
IV. MAIN RESULTS

In this section, we present a simple and convenient technique that naturally augments any array UD based filter for computing derivatives of the filter variables. To begin constructing the method, we note that each iteration of the UD based implementation has the following form: given a pair of the pre-arrays \( \{A, D_w\} \), compute a pair of the post-arrays \( \{U, D_\theta\} \) by means of the MWGS orthogonalization, i.e.

\[
A^T = U B^T \quad \text{and} \quad A^T D_w A = U D_\theta U^T
\]

(9)

where \( A \in \mathbb{R}^{r \times s} \), \( r > s \) and \( B \in \mathbb{R}^{r \times s} \) is the MWGS transformation that produces the block upper triangular matrix \( U \in \mathbb{R}^{s \times s} \). The diagonal matrices \( D_w \in \mathbb{R}^{r \times r} \), \( D_\theta \in \mathbb{R}^{s \times s} \) satisfy \( B^T D_w B = D_\theta \) and \( D_w > 0 \) (see \cite{18} Lemma VI.4.1] for an extended explanation).

**Lemma 1:** Let entries of the pre-arrays \( A, D_w \) in (9) be known differentiable functions of a parameter \( \theta \). Consider the transformation in (9). Given the derivatives of the pre-arrays \( A'_\theta \) and \( (D_w)'_\theta \), the following formulas calculate the corresponding derivatives of the post-arrays:

\[
U'_\theta = U (\bar{L}_0^T + \bar{U}_0 + \bar{U}_2) D^{-1}_\theta \quad \text{and} \quad (D_\theta)'_\theta = 2D_0 + D_2
\]

(10)

where the quantities \( \bar{L}_0, D_0, \bar{U}_0 \) are, respectively, strictly lower triangular, diagonal and strictly upper triangular parts of the matrix product \( B^T D_w A'_\theta U^{-T} \). Besides, \( D_2 \) and \( \bar{U}_2 \) are diagonal and strictly upper triangular parts of the product \( B^T(D_w)'_\theta B \), respectively.

**Proof:** For the sake of simplicity we transpose the first equation in (9) to obtain \( A = BL \). As shown in \cite{21}, the matrix \( B \) can be represented in the form of \( B = D_w^{1/2} TD_\theta^{1/2} \) where \( T \) is the matrix with orthonormal columns, i.e. \( T^T T = I \), and \( I \) is an identity matrix. Next, we define \( B^+ = D_\theta^{-1/2} T^T D_w^{1/2} \) and note that \( B^+ B = I \). The mentioned matrices \( B, B^+ \) exist since the \( D_w, D_\theta \) are invertible. Indeed, the diagonal matrix \( D_w \) is a positive definite matrix, i.e. \( D_w > 0 \), and \( D_\theta \) satisfies \( B^T D_w B = D_\theta \) (see \cite{18} Lemma VI.4.1] for further details).

Multiplication of both sides of the equality \( A = BL \) by the matrix \( B^+ \) and, then, their differentiation with respect to \( \theta \) yield

\[
B^+ A'_\theta + (B^+)'_\theta A = L'_\theta.
\]

Therefore, \( B^+ A'_\theta L^{-1} + (B^+)'_\theta AL^{-1} = L'_\theta L^{-1} \) or

\[
B^+ A'_\theta L^{-1} + (B^+)'_\theta B = L'_\theta L^{-1}.
\]

(11)

Now, we consider the product \( (B^+)'_\theta B \) in (11). For any diagonal matrix \( D_\beta \), we have \( (D_\beta^{-1/2})'_\theta = -D_\beta^{-1/2} (D_\beta^{-1/2})'_\theta D_\beta^{-1/2} \), and \( (D_\beta)'_\theta = 2D_\beta^{1/2} (D_\beta^{-1/2})'_\theta \).

Taking into account that \( B^+ = D_\theta^{-1/2} T^T D_w^{1/2} \), we further obtain

\[
(B^+)'_\theta B = \left( (D_\beta^{-1/2})'_\theta T^T D_w^{1/2} + D_\beta^{-1/2} (T'_\theta)^T D_w^{1/2} + D_\beta^{-1/2} T^T (D_w^{1/2})'_\theta D_w^{-1/2} T D_\beta^{-1/2} \right)
\]

\[
= \left( (D_\beta^{-1/2})'_\theta D_\beta^{1/2} + D_\beta^{-1/2} (T'_\theta)^T T D_\beta^{1/2} + D_\beta^{-1/2} T^T (D_w^{1/2})'_\theta D_w^{-1/2} T D_\beta^{1/2} \right)
\]
Thus, the above formula and an equality $T = D_{w}^{1/2} B D_{\beta}^{-1/2}$ yield

$$D_{\beta}^{-1/2} T (D_{w}^{1/2})' \bar{D}_{w}^{-1/2} = \frac{1}{2} D_{\beta}^{-1} B^T (D_{w})' A B.$$

Furthermore, we show that the term $(T_\theta)^T$ required in (12) is a skew symmetric matrix. For that, we differentiate both sides of the formula $T^T T = I$ with respect to $\theta$ and arrive at $(T_\theta)^T T + T^T T_\theta = 0$, or in the equivalent form

$$(T_\theta)^T T = -((T_\theta)^T T)^T.$$

The latter implies that the matrix $(T_\theta)^T T$ is skew symmetric and can be presented as a difference of two matrices, i.e. $(T_\theta)^T T = \bar{U}_1^T - \bar{U}_1$ where $\bar{U}_1$ is a strictly upper triangular matrix.

Final substitution of $(T_\theta)^T T = \bar{U}_1^T - \bar{U}_1$ and (13) into (12) and, then, into (11) yields

$$L_\theta L_\theta^{-1} = B^+ A_\theta L_\theta^{-1} + D_{\beta}^{-1} \left[ -\frac{1}{2} (D_{\beta})' + D_{\beta}^{1/2} (\bar{U}_1^T - \bar{U}_1) D_{\beta}^{1/2} + \frac{1}{2} B^T (D_{w})' B \right].$$

Next, from $B^+ = D_{\beta}^{-1/2} T^T D_{w}^{1/2}$ and $T = D_{w}^{1/2} B D_{\beta}^{-1/2}$, we derive $B^+ = D_{\beta}^{-1} B^T D_w$. Further multiplication of both sides of (14) by $D_{\beta}$ yields

$$D_{\beta} L_\theta L_\theta^{-1} = B^T D_\theta A_\theta L_\theta^{-1} - \frac{1}{2} (D_{\beta})' + D_{\beta}^{1/2} (\bar{U}_1^T - \bar{U}_1) D_{\beta}^{1/2} + \frac{1}{2} B^T (D_{w})' B.$$

Now, let us discuss equation (15) in details. We note that the term $B^T D_w A_\theta L_\theta^{-1}$ is a full matrix. Hence, it can be represented in the form of $B^T D_w A_\theta L_\theta^{-1} = \bar{L}_0 + D_0 + \bar{U}_0$ where $\bar{L}_0$, $D_0$ and $\bar{U}_0$ are, respectively, strictly lower triangular, diagonal and strictly upper triangular parts of $B^T D_w A_\theta L_\theta^{-1}$. Next, the matrix product $B^T (D_{w})' B$ in (15) is a symmetric matrix and, hence, it has the form $B^T (D_{w})' B = U_2^T + D_2 + U_2$ where $D_2$ and $U_2$ are, respectively, diagonal and strictly upper triangular parts of $B^T (D_{w})' B$. Thus, equation (15) can be represented in the following form:

$$D_{\beta} L_\theta L_\theta^{-1} = \frac{\bar{L}_0 + D_0 + \bar{U}_0}{B^T D_w A_\theta L_\theta^{-1}} - \frac{1}{2} (D_{\beta})' + D_{\beta}^{1/2} (U_1^T - U_1) D_{\beta}^{1/2}. $$
\[ + \frac{1}{2} \left( \hat{U}_2^T + D_2 + \bar{U}_2 \right). \]

Next, we note that the left-hand side matrix in (16), i.e. the matrix \( D_\beta L_\beta^{-1} \), is a strictly lower triangular (since \( L \) is a unit lower triangular matrix). Hence, the matrix on the right-hand side of (15) should also be a strictly lower triangular matrix. In other words, the strictly upper triangular and diagonal parts of the matrix on the right-hand side of (16) should be zero. Hence, the formulas
\[(D_\beta)^\prime_\theta = 2D_0 + D_2 \quad \text{and} \quad D_\beta^{1/2} \hat{U}_1 D_\beta^{1/2} = \hat{U}_0 + \frac{1}{2} \bar{U}_2. \]

Clearly, the first equation in (17) is exactly the second formula in (16). Eventually, the substitution of both formulas in (17) into (16) validates the first relation in (10). More precisely, it results in \( D_\beta L_\beta^{-1} = \hat{L}_0 + \hat{U}_0^T + \bar{U}_2^T \), and the latter formula means that \( \hat{L}_0 = D_\beta^{-1} (\hat{L}_0 + \hat{U}_0^T + \bar{U}_2^T) \) where \( \hat{L} \) stands for \( U^T \). This completes the proof of Lemma 1.

We see that the proposed computational procedure utilizes only the pre-arrays \( A, D_w \), their derivatives with respect to the unknown system parameters, the post-arrays \( \hat{U}, D_\beta \) and the MWGS orthogonal transformation in order to compute the derivatives of the post-arrays.

V. UD FILTER SENSITIVITY EVALUATION

Further, we suggest a general computation scheme that naturally extends any array UD based filtering algorithm to the above-mentioned derivative evaluation. We stress that our method allows the filter and Riccati-type sensitivity equations to be updated in terms of stable array UD filters.

To illustrate the proposed approach, we apply Lemma 1 to the UD-aCF algorithm presented in Section III with \( r = m + n + q, s = m + n \), and the following pre-, post-arrays from (3), (4):
\[ D_w = \mathcal{D}_k, \quad A = \mathcal{A}_k \quad \text{and} \quad D_\beta = \tilde{\mathcal{D}}_k, \quad \hat{U} = \tilde{\mathcal{A}}_k. \]

A. Summary of Computations. Extended UD based KF scheme.

Step 0. Set a current value of \( \hat{\theta} \).

Step 1. Evaluate \( \hat{F} = F |_{\hat{\theta}}, \hat{G} = G |_{\hat{\theta}}, \hat{H} = H |_{\hat{\theta}}, \)
\[ \hat{Q} = Q |_{\hat{\theta}}, \hat{R} = R |_{\hat{\theta}}, \tilde{\Pi}_0 = \Pi_0 |_{\hat{\theta}}; \]
\[ \frac{\partial \hat{F}}{\partial \theta_i}, \frac{\partial \hat{G}}{\partial \theta_i}, \frac{\partial \hat{H}}{\partial \theta_i}, \frac{\partial \hat{H}}{\partial \hat{\theta}_i} = \frac{\partial \hat{F}}{\partial \theta_i}, \frac{\partial \hat{G}}{\partial \theta_i}, \frac{\partial \hat{H}}{\partial \theta_i}, \frac{\partial \hat{H}}{\partial \hat{\theta}_i}; \]
\[ \frac{\partial \hat{Q}}{\partial \theta_i}, \frac{\partial \hat{R}}{\partial \theta_i}, \frac{\partial \hat{R}}{\partial \hat{\theta}_i} = \frac{\partial \hat{Q}}{\partial \theta_i}, \frac{\partial \hat{R}}{\partial \theta_i}, \frac{\partial \hat{R}}{\partial \hat{\theta}_i}; \]

Step 2. Set the initial conditions: \( P_{0|0} = \hat{\Pi}_0, \ x_{0|0} = 0 \) and 
\[ \frac{\partial P_{0|0}}{\partial \theta_i} = \frac{\partial \hat{\Pi}_0}{\partial \theta_i}, \frac{\partial x_0}{\partial \theta_i} = 0. \]

Step 3. Use the modified Cholesky decomposition to find 
\[ \{U_{\hat{\Pi}_0}, D_{\hat{\Pi}_0}\}, \{U_{\hat{R}}, D_{\hat{R}}\}, \{U_{\hat{Q}}, D_{\hat{Q}}\} \]
and 
\[ \{\frac{\partial U_{\hat{\Pi}_0}}{\partial \theta_i}, \frac{\partial D_{\hat{\Pi}_0}}{\partial \theta_i}\}, \frac{\partial U_{\hat{R}}}{\partial \theta_i}, \frac{\partial D_{\hat{R}}}{\partial \theta_i}\}, \frac{\partial U_{\hat{Q}}}{\partial \theta_i}, \frac{\partial D_{\hat{Q}}}{\partial \theta_i}\}; \]

Step 4. For \( t_k, k = 0, \ldots, N - 1, \) do
Step 5. Apply a numerically stable array UD based filter:

Step 6. Form the pre-arrays $A$, $D_w$ by using the matrices from Step 1;

Step 7. Compute the post-arrays $U$, $D_\beta$ using the MWGS algorithm. Save $B$, $U$, $D_\beta$;

Step 8. Compute the state estimate $\hat{x}_{k+1|k}$ according to (5).

Step 9. Apply the designed derivative computation method (for each $\theta_i : i = 1, \ldots, p$):

Step 10. Form the derivatives $\frac{\partial A}{\partial \theta_i}, \frac{\partial D_w}{\partial \theta_i}$ by using the matrices from Steps 1 and 3;

Step 11. Calculate $B^T D_w \frac{\partial A}{\partial \theta_i} U^{-T}$ (use the quantities from Steps 6 and 10);

Step 12. Split it into $\tilde{L}_0(i), D_0(i)$ and $\tilde{U}_0(i)$.

Save $\{\tilde{L}_0(i), D_0(i), \tilde{U}_0(i)\}$;

Step 13. Calculate $B^T \frac{\partial D_w}{\partial \theta_i} B$ (use the quantities from Steps 7, 10);

Step 14. Split it into $\tilde{U}_2^T(i), D_2(i)$ and $\tilde{U}_2(i)$.

Save $\{D_2(i), \tilde{U}_2(i)\}$;

Step 15. Find $\frac{\partial U}{\partial \theta_i} = U \left[ \tilde{L}_0^T(i) + \tilde{U}_0(i) + \tilde{U}_2(i) \right] D_\beta^{-1}$ (use Steps 7, 12, 14);

Step 16. Evaluate $\frac{\partial D_\beta}{\partial \theta_i} = 2D_0(i) + D_2(i)$ (use the saved values from Steps 12, 14);

Step 17. Evaluate the state sensitivity as

$$\frac{\partial \hat{x}_{k+1|k}}{\partial \theta_i} = \frac{\partial (F \hat{x}_{k|k-1})}{\partial \theta_i} + \frac{\partial (K_{p,k} U_{R_e,k})}{\partial \theta_i} \tilde{e}_k + (K_{p,k} U_{R_e,k}) \frac{\partial \tilde{e}_k}{\partial \theta_i};$$

Step 18. End.

**Remark 2:** The new approach naturally extends any UD based KF implementation on the filter sensitivities evaluation. Additionally, this allows Log LF (7) and Log LG (8) to be computed simultaneously. Hence, such methods are ideal for simultaneous state estimation and parameter identification.

**VI. Numerical Examples**

A. **Simple Test Problem**

First, we would like to check our theoretical derivations presented in Lemma 1. To do so, we apply the proposed UD based computational scheme to the following simple test problem.
Table I

| Steps 5, 6 | We are given \( A = \begin{bmatrix} \theta^5/20 & \theta^4/8 \\ \theta^4/8 & \theta^3/3 \\ \theta^3/6 & \theta^2/2 \end{bmatrix} \) and \( D_w = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \theta^2 & 0 \\ 0 & 0 & \theta^3 \end{bmatrix} \). Hence, \( A|_{\theta=2} = \begin{bmatrix} 8/5 & 2 \\ 2 & 8/3 \\ 4/3 & 2 \end{bmatrix} \), \( D_w|_{\theta=2} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix} \). |
| --- | --- |
| Step 7. | Compute the post-arrays \( U = \begin{bmatrix} 1.0000 & 0.7160 \\ 0 & 1.0000 \end{bmatrix} \), \( D_{\beta} = \begin{bmatrix} 0.1672 & 0 \\ 0 & 68.4444 \end{bmatrix} \) where \( B = \begin{bmatrix} 0.1662 & 2.0000 \\ -0.1004 & 2.0000 \end{bmatrix} \). |
| Steps 9, 10. | We are given \( A_{\theta} = \begin{bmatrix} \theta^4/4 & \theta^4/2 \\ \theta^3/2 & \theta^2 \\ \theta^2/2 & \theta \end{bmatrix} \), \( (D_{\omega})_{\beta} = \begin{bmatrix} 1 & 0 \\ 0 & 2\theta \\ 0 & 0 \end{bmatrix} \). So, \( A_{\theta}|_{\theta=2} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \), \( (D_{\omega})_{\beta}|_{\theta=2} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \). |
| Steps 11, 12. | Compute \( B^T D_w A_{\theta} U^{-T} \). Save \( L_0(1) = \begin{bmatrix} 25.6693 & 0 \\ 0 & 0 \end{bmatrix} \), \( D_0(1) = \begin{bmatrix} 0.3216 & 0 \\ 0 & 90.6667 \end{bmatrix} \), \( \bar{U}_0(1) = \begin{bmatrix} 0 & 1.1359 \\ 0 & 0 \end{bmatrix} \). |
| Steps 13, 14. | Evaluate \( B^T (D_{\omega})_{\beta} B \). Save \( D_2(1) = \begin{bmatrix} 0.1799 & 0 \\ 0 & 80.4444 \end{bmatrix} \), \( \bar{U}_2(1) = \begin{bmatrix} 0 & -1.1359 \\ 0 & 0 \end{bmatrix} \). |
| Step 15, 16. | Finally, the results that we are looking for are \( U'_{\theta}|_{\theta=2} = \begin{bmatrix} 0 & 0.3750 \\ 0 & 0 \end{bmatrix} \), \( (D_{\beta})'_{\theta}|_{\theta=2} = \begin{bmatrix} 0.8231 & 0 \\ 0 & 261.7778 \end{bmatrix} \). |

**Example 1:** For the given pre-arrays

\[
A = \begin{bmatrix} \theta^5/20 & \theta^4/8 \\ \theta^4/8 & \theta^3/3 \\ \theta^3/6 & \theta^2/2 \end{bmatrix} \quad \text{and} \quad D_w = \begin{bmatrix} \theta & 0 & 0 \\ 0 & \theta^2 & 0 \\ 0 & 0 & \theta^3 \end{bmatrix},
\]

compute the derivatives of the post-arrays \( U'_{\theta}, (D_{\beta})'_{\theta} \), say, at \( \theta = 2 \) where \( U, D_{\beta} \) comes from (9).

Since the pre-arrays \( A \) and \( D_w \) are fixed by (18), we skip Steps 0–3 in the above-presented computational scheme. We also skip steps 8 and 17, because we do not need to compute the state estimate (and its derivatives) in this simple test problem. Besides, the unknown parameter \( \theta \) is a scalar value, i.e. \( p = 1 \). Next, we show the detailed explanation of only one iteration step, i.e. we set \( k = 1 \) in Step 4. The obtained results are summarized in Table 1. As follows from (9), we have \( A^T D_w A = U D_{\beta} U^T \). Thus, the derivatives of both sides of the latter formula must also agree. We compute the norm \( \left\| (A^T D_w A)'_{\theta=2} - (U D_{\beta} U^T)'_{\theta=2} \right\|_2 = 5.68 \times 10^{-14} \). This confirms the correctness of the calculation and the above theoretical derivations presented in Lemma 1.

**B. Application to the Maximum Likelihood Estimation of Unknown System Parameters**

The computational approach presented in this paper can be used for the efficient evaluation of the Log LF and its gradient required in gradient-search optimization algorithms for the maximum likelihood estimation of unknown system parameters. To demonstrate this, we apply the proposed UD based algorithm to the problem from aeronautical equipment engineering.

**Example 2:** Consider a simplified version of the instrument error model for one channel of
The Inertial Navigation System (INS) of semi-analytical type given as follows [23]:

\[
\begin{bmatrix}
\Delta v_x(t_{k+1}) \\
\beta(t_{k+1}) \\
m_{Ax}(t_{k+1}) \\
n_{Gy}(t_{k+1})
\end{bmatrix}
= 
\begin{bmatrix}
1 & -\tau g & \tau & 0 \\
\tau/a & 1 & 0 & \tau \\
0 & 0 & b_1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\Delta v_x(t_k) \\
\beta(t_k) \\
m_{Ax}(t_k) \\
n_{Gy}(t_k)
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
0 \\
a_1 \\
0
\end{bmatrix}
w(t_k),
\]

where \(w(t_k) \sim N(0, 1)\), \(v(t_k) \sim N(0, 0.01)\), \(x_0 \sim N(0, I_4)\) and subscripts \(x, y, A, G\) denote “axis \(Ox\)”, “axis \(Oy\)”, “Accelerometer”, and “Gyro”, respectively. The \(\Delta v_x\) is the random error in reading velocity along axis \(Ox\) of a gyro-stabled platform (GSP), \(\beta\) is the angular error in determining the local vertical, \(m_{Ax}\) is the accelerometer reading random error, and \(n_{Gy}\) is the gyro constant drift rate. The quantities \(a_1 \approx H_1 \sqrt{2\gamma_1 \tau}\) and \(b_1 \approx 1 - \gamma_1 \tau\) depend on the unknown parameter \(\gamma_1\) that needs to be estimated.

In our simulation experiment we assume that the true value of \(\gamma_1\) is \(\gamma_1^* = 2 \cdot 10^{-4}\). We compute the negative Log LF and its gradient by the proposed array UD based scheme and then compare the results to those produced by the classical, i.e. the conventional KF approach. The computations are done on the interval \([10^{-5}, 4 \cdot 10^{-4}]\) with the step \(10^{-5}\). The outcomes of these experiments are illustrated by Fig. 1. As can be seen from Fig. 1(b), all algorithms for the gradient evaluation produce exactly the same result and give the same zero point. Besides, the zero point coincides with the minimum point of the negative Log LF; see Fig. 1(a). Furthermore, it is readily seen that the obtained maximum likelihood estimate, \(\hat{\gamma}_1^{MLE}\), coincides with the true value \(\gamma_1^* = 2 \cdot 10^{-4}\). All these evidences substantiate our theoretical derivations in Sections IV, V.

C. Ill-Conditioned Test Problems

Although we learnt from the previous examples that both methods, i.e. the conventional KF approach and the proposed array UD based scheme, produce exactly the same results, numerically they no longer agree. To illustrate this, we consider the following ill-conditioned test problem.
TABLE II
EFFECT OF ROUNDOFF ERRORS IN ILL-CONDITIONED TEST PROBLEMS

| $\delta$ | $\theta^*$ | “differentiated” KF technique | array SR based approach | array UD based approach |
|---------|-----------|-------------------------------|------------------------|------------------------|
| $10^{-2}$ | 7         | 6.9984, 0.1243, 1.4438         | 6.9984, 0.1243, 1.4438  | 6.9984, 0.1243, 1.4438  |
| $10^{-3}$ | 7         | 7.0035, 0.1233, 1.4096         | 6.9996, 0.1227, 1.4011  | 7.0012, 0.1227, 1.4011  |
| $10^{-4}$ | 7         | 7.5116, 1.1953, 10.8291        | 7.0083, 0.1111, 1.2794  | 7.0083, 0.1111, 1.2794  |
| $10^{-5}$ | 7         | 5.4700, 5.1658, 72.0857        | 6.9696, 0.1274, 1.4706  | 6.9966, 0.1274, 1.4706  |
| $10^{-6}$ | 7         | 4.0378, 12.1748, 157.6825      | 6.9979, 0.1266, 1.4581  | 6.9981, 0.1264, 1.4555  |

Example 3: Consider the state-space model (1)-(2) with \{F, G, H, \Pi_0, Q, R\} given by

\[
F = I_3, G = 0, R = I_2\delta^2\theta^2, \Pi_0 = I_3\theta^2, H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 + \delta \end{bmatrix}
\]

where $\theta$ is an unknown system parameter, that needs to be estimated. To simulate roundoff we assume that $\delta^2 < \epsilon_{\text{roundoff}}$, but $\delta > \epsilon_{\text{roundoff}}$ where $\epsilon_{\text{roundoff}}$ denotes the unit roundoff error.

When $\theta = 1$, Example 3 coincides with well-known ill-conditioned test problem from [7]. The difficulty to be explored is in the matrix inversion. As can be seen, although rank $H = 2$, for any fixed value of the parameter $\theta \neq 0$, the matrices $R_{e,1} = R + H\Pi_0H^T$ and $(R_{e,1})'_\theta$ are ill-conditioned in machine precision, i.e. as $\delta \to \epsilon_{\text{roundoff}}$. This leads to a failure of the conventional KF approach and, as a result, destroys the entire computational scheme. So, we plan to observe and compare performance of the “conventional KF technique” and our new “array UD based approach” in such a situation. Additionally, we compare these techniques with the “array SR based approach” presented in [20]. Also, we have to stress that Example 3 represents numerical difficulties only for the covariance-type implementations. As a result, the SRIF based method [1] can not be used in our comparative study because of its information-type. This follows from [5] where numerical insights of various KF implementations are analyzed and discussed at large.

In order to judge the quality of each above-mentioned computational technique, we conduct the following set of numerical experiments. Given the “true” value of the parameter $\theta$, say $\theta^* = 7$, the system is simulated for 1000 samples for various values of the problem conditioning parameter $\delta$. Then, we use the generated data to solve the inverse problem, i.e. to compute the maximum likelihood estimates by the above-mentioned three different approaches. The same gradient-based optimization method with the same initial value of $\theta^{(0)} = 1$ is applied in all estimators. More precisely, the optimization method utilizes the negative Log LF and its gradient that are calculated by the examined techniques. All methods are run in MATLAB with the same precision (64-bit floating point). We performed 250 Monte Carlo simulations and report the posterior mean for $\theta$, the root mean squared error (RMSE) and the mean absolute percentage error (MAPE).

Having carefully analyzed the obtained results presented in Table II we conclude the following. When $\delta = 10^{-2}$, all the considered approaches produce exactly the same result. The posterior means from all the estimators are all close to the “true” value, which is equal to 7. The RMSE and MAPE are equally small. Hence, we conclude that all three methods work similarly well.

\footnote{Computer roundoff for floating-point arithmetic is often characterized by a single parameter $\epsilon_{\text{roundoff}}$, defined in different sources as the largest number such that either $1 + \epsilon_{\text{roundoff}} = 1$ or $1 + \epsilon_{\text{roundoff}}/2 = 1$ in machine precision.}
in this well-conditioned situation. However, as \( \delta \to \epsilon_{\text{roundoff}} \), which corresponds to growing ill-conditioning, the conventional KF approach degrades much faster than the array UD and SR based approaches. Already for \( \delta = 10^{-4} \), the RMSE [MAPE] of the classical technique (i.e. the “differentiated” KF) is \( \approx 11 \) times [\( \approx 8.5 \) times] greater that of the RMSE [MAPE] from the array UD and SR methods. For \( \delta \leq 10^{-5} \), the output generated by the “differentiated” KF is of no sense because of huge errors. On the other hand, the new array UD based scheme and the previously proposed array SR method work robustly, i.e. with small errors, until \( \delta = 10^{-6} \), inclusively. Besides, we may note that both the array UD and SR approaches work equally well. Indeed, the RMSE and MAPE of these techniques are approximately the same. Moreover, their results change slightly for the above \( \delta \)'s (see the last two panels in Table II).

In conclusion, the “differentiated” KF performs markedly worse compared to the proposed array UD scheme (and previously known array SR approach). Moreover, both the UD and SR based techniques work equally well on the considered test problems. This creates a strong argument for their practical applications. However, further investigation and comparative study have to be performed to provide a rigorous theoretical and numerical analysis of the existing methods for the filter sensitivity computations. These are open questions for a future research. Here, we just mention that the derived UD based method requires utilization of the MWGS orthogonal transformation while the array SR based approach uses any orthogonal rotation. Thus, the latter technique is more flexible in practice, but might produce less accurate answers for some problems because of its flexibility.

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