Three Revolutions in The Kernel Are Worse Than One

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An example is constructed of a purely unrectifiable measure $\mu$ for which the singular integral associated to the kernel $K(z) = \frac{\bar{z}}{z^2}$ is bounded in $L^2(\mu)$. The singular integral fails to exist in the sense of principal value $\mu$-almost everywhere. This is in sharp contrast with the results known for the kernel $\frac{1}{z}$ (the Cauchy transform).

1 Introduction

Let $B(z, r)$ denote the closed disc in $\mathbb{C}$ centred at $z$ with radius $r > 0$. A finite Borel measure $\mu$ is said to be one-dimensional if $H^1(\text{supp}(\mu)) < \infty$, and there exists a constant $C > 0$ such that $\mu(B(z, r)) \leq Cr$ for any $z \in \mathbb{C}$ and $r > 0$.

For a kernel function $K : \mathbb{C}\setminus\{0\} \to \mathbb{C}$, and a finite measure $\mu$, we define the singular integral operator associated to $K$ by

$$T_\mu(f)(z) = \int_{\mathbb{C}} K(z - \xi)f(\xi)d\mu(\xi), \text{ for } z \notin \text{supp}(\mu).$$

A well-known problem in harmonic analysis is to determine geometric properties of $\mu$ from regularity properties of the operator $T_\mu$, see for instance the monograph of David and Semmes [2]. This article concerns the question of characterizing those
functions $K$ with the following property:

Let $\mu$ be a one-dimensional measure. Then

\[ \| T_\mu(1) \|_{L^\infty(\mathbb{C} \setminus \text{supp}(\mu))} < \infty \text{ implies that } \mu \text{ is rectifiable.} \] (\ast)

The class of functions $K$ for which (\ast) holds does not change if one replaces the condition $\| T_\mu(1) \|_{L^\infty(\mathbb{C} \setminus \text{supp}(\mu))} < \infty$ with the boundedness of $T_\mu$ as an operator in $L^2(\mu)$, see for instance [10]. A one-dimensional measure $\mu$ is rectifiable if $\text{supp}(\mu)$ can be covered (up to an exceptional set of $H^1$ measure zero) by a countable union of rectifiable curves. A measure $\mu$ is purely unrectifiable if its support is purely unrectifiable, that is, $H^1(\Gamma \cap \text{supp}(\mu)) = 0$ for any rectifiable curve $\Gamma$.

David and Léger [5] proved that the Cauchy kernel $\frac{1}{z}$ has property (\ast). As is remarked in [1], the proof in [5] extends to the case when the Cauchy kernel is replaced by either its real or imaginary part, that is, $\Re(\frac{z}{|z|^2})$ or $\Im(\frac{z}{|z|^2})$. Recently in [1], Chousionis, Mateu, Prat, and Tolsa extended the result of [5] and showed that kernels of the form $\frac{\Re(z)^k}{|z|^{k+1}}$ have property (\ast) for any odd positive integer $k$. Both of these results use the Melnikov–Menger curvature method.

On the other hand, Huovinen [4] has shown that there is a purely unrectifiable Ahlfors–David (AD)-regular set $E$ for which the singular integral associated to the kernel $\frac{\Re(z)}{|z|^2} - \frac{\Im(z)^3}{|z|^4}$ is bounded in $L^2(H^1|_E)$. In fact, an essentially stronger conclusion is proved that the principal values of the associated singular integral operator exist $H^1$-a.e. on $E$. Huovinen takes advantage of several non-standard symmetries and cancellation properties in this kernel to construct his very nice example.

The result of this article is that a weakened version of Huovinen’s theorem holds for a very simple kernel function. Indeed, it is perhaps the simplest example of a kernel for which the Menger curvature method fails to be directly applicable. From now on, we shall fix

\[ K(z) = \frac{\bar{z}}{z^2}, z \in \mathbb{C} \setminus \{0\}. \] (1.1)

We prove the following result.

**Theorem 1.1.** There exists a one-dimensional purely unrectifiable probability measure $\mu$ with the property that $\| T_\mu(1) \|_{L^\infty(\mathbb{C} \setminus \text{supp}(\mu))} < \infty$.

In other words, the kernel $K$ in (1.1) fails to satisfy property (\ast). At this point, we would also like to mention Huovinen’s thesis work [3], regarding the kernel function.
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It is proved that if \( \lim \inf_{r \to 0} \frac{\mu(B(z,r))}{r} \in (0, \infty) \) \( \mu \)-a.e. (essentially the AD-regularity of \( \mu \)), then the \( \mu \)-almost everywhere existence of \( T_\mu(1) \) in the sense of principal value implies that \( \mu \) is rectifiable. This result was proved by building upon the theory of symmetric measures, developed by Mattila [7], and Mattila and Preiss [9]. Unfortunately the measure in Theorem 1.1 does not satisfy the AD-regularity condition. In view of Huovinen’s work it would be of interest to construct an AD-regular measure supported on an unrectifiable set for which the conclusion of Theorem 1.1 holds. We have not been able to construct such a measure (yet).

For the measure \( \mu \) constructed in Theorem 1.1, we show that \( T_\mu(1) \) fails to exist in the sense of principal value \( \mu \)-almost everywhere. Thus the two properties of \( L^2(\mu) \) boundedness of the operator \( T_\mu \), and the existence of \( T_\mu(1) \) in the sense of principal value, are quite distinct for this singular integral operator.

2 Notation

- Let \( m_2 \) denote the two-dimensional Lebesgue measure normalized so that \( m_2(B(0,1)) = 1 \). We let \( m_1 \) denote the one-dimensional Lebesgue measure.
- A collection of squares are essentially pairwise disjoint if the interiors of any two squares in the collection do not intersect. Throughout the article, all squares are closed.
- We shall denote by \( C \) and \( c \) large and small absolute positive constants. The constant \( C \) should be thought of as large (at least 1), while \( c \) is to be thought of as small (smaller than 1).
- For \( a > 1 \), the disc \( aB \) denotes the concentric enlargement of a disc \( B \) by a factor of \( a \).
- We define the \( \mathcal{H}^1 \)-measure of a set \( E \) by
  \[
  \mathcal{H}^1(E) = \sup_{\delta > 0} \inf \left\{ \sum_j r_j : E \subset \bigcup_j B(x_j, r_j) \text{ with } r_j \leq \delta \right\}.
  \]
- For \( z \in \mathbb{C} \) and \( r > 0 \), we define the annulus \( A(z,r) = B(z,r) \setminus B(z,r/2) \).
- The set \( \text{supp}(\mu) \) denotes the closed support of \( \mu \).

3 A reflectionless measure

Let us make the key observation that allows us to prove Theorem 1.1.

Lemma 3.1. Let \( z \in \mathbb{C} \), \( r > 0 \). For any \( \omega \in B(z,r) \),

\[
\int_{B(z,r)} K(\omega - \xi) d m_2(\xi) = 0.
\]

\[\square\]
Proof. Without loss of generality, we may set \( z = 0 \) and \( r = 1 \). If \(|\omega| < |\xi|\), then
\[
K(\omega - \xi) = \frac{\omega - \xi}{\xi^2} \sum_{\ell = 0}^{\infty} (\ell + 1) \left( \frac{\omega}{\xi} \right)^{\ell}.
\]
So whenever \( t > |\omega| \), we have \( \int_{\partial B(0,t)} K(\omega - \xi) \, dm_1(\xi) = 0 \). (This follows merely from the fact that \( \int_{\partial B(0,t)} \xi^k \, dm_1(\xi) = 0 \) whenever \( k, \ell \in \mathbb{Z} \) satisfy \( k \neq \ell \).) On the other hand, if \(|\xi| < |\omega|\), then
\[
K(\omega - \xi) = \frac{\omega - \xi}{\omega^2} \sum_{\ell = 0}^{\infty} (\ell + 1) \left( \frac{\xi}{\omega} \right)^{\ell}.
\]
Therefore, if \( t < |\omega| \), then
\[
\int_{\partial B(0,t)} K(\omega - \xi) \, dm_1(\xi) = 2\pi \left[ t \frac{\omega}{\omega^2} - 2 \frac{t^3}{\omega^3} \right] = \frac{2\pi}{\omega^3} (t|\omega|^2 - 2t^3).
\]
Since \( \int_0^{|\omega|} (t|\omega|^2 - 2t^3) \, dt = 0 \), the desired conclusion follows. \( \blacksquare \)

The next lemma will form the basis of the proof of the non-existence of \( T_m(1) \) in the sense of principal value.

Lemma 3.2. There exists a constant \( \tilde{c} > 0 \) such that for any disc \( B(z, r) \), and \( \omega \in \partial B(z, r) \),
\[
\left| \int_{A(\omega, r) \cap B(z, r)} K(\omega - \xi) \frac{dm_2(\xi)}{r} \right| \geq \tilde{c}.
\]

Proof. By an appropriate translation and rescaling, we may assume that \( B(z, r) = B(i, 1) \), and \( \omega = 0 \). Making reference to Figure 1 above, we split the domain of integration into three regions, \( I = \{ \xi \in A(0, 1) : \arg(\xi) \in \left[ \frac{\pi}{6}, \frac{5\pi}{6} \right] \} \), \( II = \{ \xi \in A(0, 1) \cap B(i, 1) : \arg(\xi) \in [0, \frac{\pi}{6}] \} \), and \( III = \{ \xi \in A(0, 1) \cap B(i, 1) : \arg(\xi) \in \left[ \frac{5\pi}{6}, \pi \right] \} \). The regions \( II \) and \( III \) are respectively the right and left grey shaded regions in Figure 1. Note that \( \Im K(-\xi) < 0 \) if \( \arg(\xi) \in \left[ \frac{\pi}{6}, \frac{2\pi}{3} \right] \), and \( \Im K(-\xi) > 0 \) if \( \arg(\xi) \in \left[ 0, \frac{\pi}{6} \right] \cup \left[ \frac{2\pi}{3}, \pi \right] \). Furthermore, note that
\[
\int_I \Im K(-\xi) \, dm_2(\xi) = \frac{1}{\pi} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \Im (e^{-2\theta i}) t \, d\theta \, dt = 0.
\]
But \( \int_{II, III} \Im K(-\xi) \, dm_2(\xi) = 2 \int_{II} \Im K(-\xi) \, dm_2(\xi) > 0 \). Therefore, by setting \( \tilde{c} = 2 \int_{II} \Im K(-\xi) \, dm_2(\xi) \), the lemma follows. \( \blacksquare \)
4 Packing squares in a disc

Fix $r, R \in (0, \infty)$ such that $r < \frac{R}{16}$ and $\frac{R}{r} \in \mathbb{N}$.

**Lemma 4.1.** One can pack $\frac{R}{r}$ pairwise essentially disjoint squares of side length $\sqrt{\pi \frac{r}{R}}$ into a disc of radius $R(1 + 4\sqrt{\frac{r}{R}})$.

**Proof.** We may assume that the disc is centred at the origin. Consider the square lattice with mesh size $\sqrt{\pi \frac{r}{R}}$. Label those squares that intersect $B(0, R)$ as $Q_1, \ldots, Q_M$. These squares are contained in $B(0, R(1 + 4\sqrt{\frac{r}{R}}))$. Since $M r R = \sum_{j=1}^{M} m_2(Q_j) > m_2(B(0, R)) = R^2$, we have that $M > \frac{R}{r}$. By throwing away $M - \frac{R}{r}$ of the least desirable squares, we arrive at the desired collection. 

**Lemma 4.2.** Consider a disc $B(z, R)$. Let $Q_1, \ldots, Q_{R/r}$ be the collection of squares contained in $B(z, R(1 + 4\sqrt{\frac{T}{R}}))$ found in Lemma 4.1. Then $m_2(B(z, R) \triangle \bigcup_{j=1}^{R/r} Q_j) \leq Cr^{1/2}R^{3/2}$.

**Proof.** Since $m_2(B(z, R)) = m_2\left(\bigcup_{j=1}^{R/r} Q_j\right) = R^2$, the property follows from the fact that both sets are contained in $B(z, R(1 + 4\sqrt{\frac{T}{R}}))$.

5 The construction of the sparse Cantor set $E$

Let $r_0 = 1$, and choose $r_j, j \in \mathbb{N}$, to be a sequence which tends to zero quickly. Assume that $r_j < \frac{r_{j-1}}{100}$, $\frac{r_j}{r_j+1} \in \mathbb{N}$, and $\frac{r_j}{r_{j+1}} \in \mathbb{N}$ for all $j \geq 1$.

Several additional requirements will be imposed on the decay of $r_j$ over the course of the following analysis, and we make no attempt to optimize the conditions.

It will be convenient to let $s_{n+1} = 4\sqrt{\frac{n+1}{r_n}}$ for $n \in \mathbb{Z}_+$.

First define $\tilde{B}_1^{(0)} = B(0, 1)$. Given the $n$-th level collection of $\frac{1}{r_n}$ discs $\tilde{B}_j^{(n)}$ of radius $r_n$, we construct the $(n + 1)$-st generation according to the following procedure:
Fig. 2. The picture shows a single disc $B^{(n)}_j$ of radius $(1+s_{n+1})r_n$. The grey shaded squares are the squares $Q^{(n+1)}_\ell$ of sidelength $\sqrt{\pi r_n r_{n+1}}$ formed by applying Lemma 4.1 to the disc $\widetilde{B}^{(n)}_j$ of radius $r_n$. The boundary of the disc $\widetilde{B}^{(n)}_j$ is the dashed circle. Deep inside each square $Q^{(n+1)}_\ell$ is the disc $B^{(n+1)}_\ell$ of radius $(1+s_{n+2})r_{n+1}$.

Fix a disc $\widetilde{B}^{(n)}_j$. Apply Lemma 4.1 with $R = r_n$ and $r = r_{n+1}$ to find $\frac{r_n}{r_{n+1}}$ squares $Q^{(n+1)}_\ell$ of side length $\sqrt{\pi r_{n+1} r_n}$ that are pairwise essentially disjoint, and contained in $(1+s_{n+1}) \cdot \widetilde{B}^{(n)}_j$. Let $z^{(n+1)}_\ell$ be the centre of $Q^{(n+1)}_\ell$, and set $\widetilde{B}^{(n+1)}_\ell = B(z^{(n+1)}_\ell, r_{n+1})$. This procedure is carried out for each disc $\widetilde{B}^{(n)}_j$ from the $n$-th level collection. There are a total of $\frac{1}{r_{n+1}}$ discs $\widetilde{B}^{(n+1)}_\ell$ in the $(n+1)$-st level. See Figure 2 above.

The above construction is executed for each $n \in \mathbb{Z}_+$.

Now, set $B^{(n)}_j = (1+s_{n+1})\widetilde{B}^{(n)}_j$. Define $E^{(n)} = \bigcup_j B^{(n)}_j$. We shall repeatedly use the following properties of the construction:

(a) $\bigcup_j Q^{(n+1)}_\ell \subset E^{(n)}$, for all $n \geq 0$.
(b) $B^{(n)}_j \subset Q^{(n)}_\ell$ for each $n \geq 1$. Moreover, $\text{dist}(B^{(n)}_j, \partial Q^{(n)}_\ell) \geq \frac{1}{2} \sqrt{r_{n-1} r_n}$.
(c) $\text{dist}(B^{(n)}_j, B^{(n)}_k) \geq \frac{1}{2} \sqrt{r_{n-1} r_n}$, whenever $j \neq k$, $n \geq 0$.

Property (a) is immediate. To see property (b), merely note that $\text{dist}(B^{(n)}_j, \partial Q^{(n)}_j) = \frac{\sqrt{r_{n-1} r_n}}{2} - (1+s_{n+1})r_n \geq \frac{1}{2} \sqrt{r_{n-1} r_n}$. For property (c), we shall use induction. If $n = 0$, then the claim is trivial. Using (b), the claimed estimate is clear if $Q^{(n)}_j$ and $Q^{(n)}_k$ have been
created by an application of Lemma 4.1 in a common disc \( \tilde{B}^{|n-1|} \). Otherwise, the squares are born out of applying Lemma 4.1 to different discs at the \((n - 1)\)-st level, and those parent discs are already separated by \( \frac{1}{2}\sqrt{r_{m-2}r_{m-1}} \).

Courtesy of properties (a) and (b), we see that \( E^{(n+1)} \subset E^{(n)} \) for each \( n \geq 0 \). Set \( E = \bigcap_{n \geq 0} E^{(n)} \). Each \( z \in E^{(n)} \) is contained in a unique disc \( B^{(n)}_j \) (or square \( Q^{(n)}_j \)) which we shall denote by \( B^{(n)}(z) \) (respectively \( Q^{(n)}(z) \)).

If \( m \geq n \geq 0 \), then \( E \cap B^{(n)}_j \) is covered by the \( \frac{m}{r_m} \) discs \( B^{(m)}_j \) that are contained in \( B^{(n)}_j \), each of which has radius \((1 + s_{m+1})r_m \leq 2r_m\). Therefore \( \mathcal{H}^1(E \cap B^{(n)}_j) \leq 2r_n \). Taking \( n = 0 \) yields \( \mathcal{H}^1(E) \leq 2 \).

## 6 The measure \( \mu \)

Define \( \mu^{(n)}_j = \frac{1}{r_n} \chi_{B^{(n)}_j}, m_2 \). Set \( \mu^{(n)} = \sum_j \mu^{(n)}_j \). Then \( \text{supp}(\mu^{(n)}) \subset E^{(n)} \), and \( \mu^{(n)}(\mathbb{C}) = 1 \) for all \( n \). Therefore, there exists a subsequence of the sequence of measures \( \mu^{(n)} \) that converges weakly to a measure \( \mu \), with \( \mu(\mathbb{C}) = 1 \) and \( \text{supp}(\mu) \subset E \).

The following three properties hold:

(i) \( \text{supp}(\mu^{(m)}) \subset \bigcup_j B^{(n)}_j \) whenever \( m \geq n \),

(ii) \( \mu^{(m)}(B^{(n)}_j) = r_n \) for \( m \geq n \), and

(iii) there exists \( C_0 > 0 \) such that \( \mu^{(n)}(B(z, r)) \leq C_0 r \) for any \( z \in \mathbb{C}, r > 0 \) and \( n \geq 0 \).

Properties (i) and (ii) follow immediately from the construction of \( E^{(n)} \). To see the third property, note that since \( \mu^{(n)} \) is a probability measure, the property is clear if \( r \geq 1 \). If \( r < 1 \), then \( r \in (r_{m+1}, r_m) \) for some \( m \in \mathbb{Z}^+ \). If \( m \geq n \), then \( B(z, r) \) intersects at most one disc \( B^{(n)}_j \). Then \( \mu^{(m)}(B(z, r)) = \frac{1}{r_n} m_2(B(z, r) \cap \tilde{B}^{(m)}_j) \leq \frac{r^2}{r_n} \leq r \). Otherwise \( m < n \). In this case, note that since the discs \( B^{(m+1)}_j \) are \( \frac{1}{2}\sqrt{r_m r_{m+1}} \) separated, \( B(z, r) \) intersects at most \( 1 + C\left(\frac{r}{\sqrt{r_m r_{m+1}}}\right)^2 \) discs \( B^{(m+1)}_j \). Hence, by property (ii), we see that

\[
\mu^{(n)}(B(z, r)) = \sum_j \mu^{(m)}(B(z, r) \cap B^{(m+1)}_j) \leq \left[1 + C\left(\frac{r}{\sqrt{r_m r_{m+1}}}\right)^2\right] r_{m+1},
\]

which is at most \( Cr \).

The weak convergence of a subsequence of \( \mu^{(n)} \) to the measure \( \mu \), along with property (iii), yields that \( \mu(B(z, r)) \leq C_0 r \) for any disc \( B(z, r) \). We shall henceforth refer to this property by saying that \( \mu \) is \( C_0 \)-nice. We have now shown that \( \mu \) is one-dimensional.

Notice that we also have \( \mathcal{H}^1(E) \geq \frac{1}{C_0} \mu(E) > 0 \).
7 The boundedness of $T_\mu(1)$ off the support of $\mu$

As a simple consequence of the weak convergence of $\mu^{(n)}$ to $\mu$, the property that $\|T_\mu(1)\|_{L^\infty(C \setminus \text{supp}(\mu))} < \infty$ will follow from the following proposition.

**Proposition 7.1.** Provided that $\sum_{n \geq 1} \sqrt{s_n} < \infty$, there exists a constant $C > 0$ so that the following holds:

Suppose that $\text{dist}(z, \text{supp}(\mu)) = \varepsilon > 0$. Then for any $m \in \mathbb{Z}_+$ with $r_m < \frac{\varepsilon}{4}$,

$$\left| \int_{C} K(z - \xi) \, d\mu^{(m)}(\xi) \right| \leq C.$$ 

To begin the proof, fix $r_m$ with $r_m < \frac{\varepsilon}{4}$. Let $z^* \in \text{supp}(\mu)$ with $\text{dist}(z, z^*) = \varepsilon$. For any $\xi \in \text{supp}(\mu)$, $B^{(m)}(\xi) \cap \text{supp}(\mu^{(m)}) \neq \emptyset$, so $\text{dist}(z, \text{supp}(\mu^{(m)})) \geq \varepsilon - (1 + s_{m+1})r_m \geq \frac{\varepsilon}{2}$.

Now, let $q$ be the least integer with $r_q \leq \varepsilon$ (so $m \geq q$). Then by property (ii) of the previous section,

$$\int_{B^{(q)}(z^*)} |K(z, \xi)| \, d\mu^{(m)}(\xi) \leq \frac{2}{\varepsilon} \mu^{(m)}(B^{(q)}(z^*)) = \frac{2r_q}{\varepsilon} \leq 2. \quad (7.1)$$

The crux of the matter is the following lemma.

**Lemma 7.2.** There exists $C > 0$ such that for any $n \in \mathbb{Z}_+$ with $1 \leq n \leq q$,

$$\left| \int_{B^{(n-1)}(z^*), B^{(n)}(z^*)} K(z - \xi) \, d\mu^{(m)}(\xi) \right| \leq C\sqrt{s_n} + C\sqrt{\frac{\varepsilon}{r_{n-1}}}. \quad \square$$

For the proof of Lemma 7.2, we shall require the following simple comparison estimate.

**Lemma 7.3.** Let $z_0 \in \mathbb{C}$, and $\lambda > 0$. Fix $r, R \in (0, 1]$ with $100r \leq R$. Suppose that $\nu_1$ and $\nu_2$ are Borel measures, such that $\text{supp}(\nu_1) \subset Q(z_0, \sqrt{\pi Rr}) = Q$, $\text{supp}(\nu_2) \subset B(z_0, 2r) = B$, and $\nu_1(\mathbb{C}) = \nu_2(\mathbb{C})$. Then, for any $z \in \mathbb{C}$ with $\text{dist}(z, Q) \geq \lambda \sqrt{rR}$, we have

$$\left| \int_{Q} K(z - \xi) \, d\nu_1(\xi) - \int_{B} K(z - \xi) \, d\nu_2(\xi) \right| \leq \frac{1}{\lambda^2} \int_{Q} \frac{C \sqrt{Rr}}{|z - \xi|^2} \, d\nu_1(\xi) + \frac{1}{\lambda^2} \int_{B} \frac{Cr}{|z - \xi|^2} \, d\nu_2(\xi). \quad \square$$

**Proof.** The left hand side of the inequality can be written as

$$\left| \int_{Q} [K(z - \xi) - K(z - z_0)] \, d(\nu_1 - \nu_2)(\xi) \right|.$$
But, under the hypothesis on \( z \), we have that \(|K(z - \xi) - K(z - z_0)| \leq \frac{C|z - \xi|^2}{m_2(z)}|z - \xi|^2 r_n r_{n-1} \) for any \( \xi \in Q \).

Plugging this estimate into the integral and taking into account the supports of \( v_1 \) and \( v_2 \), the inequality follows.

**Proof of Lemma 7.2.** Write

\[ A = \{ j : B_j^{(n)} \neq B^{(n)}(z^*) \text{ and } B_j^{(n)} \subset B^{(n-1)}(z^*) \}. \]

First suppose that \( \text{dist}(z, Q_j^{(n)}) \geq \frac{1}{4} r_{n-1} r_n \) for \( j \in A \). Then the hypothesis of Lemma 7.3 are satisfied with \( v_1 = \chi_{Q_j^{(n)}} m_2, v_2 = \chi_{B_j^{(n)}} \mu^{(m)}, R = r_{n-1}, r = r_n, \) and \( z_0 = z_{Q_j^{(n)}} \). Thus

\[
\left| \int_{Q_j^{(n)}} K(z - \xi) \frac{dm_2(\xi)}{r_{n-1}} - \int_{B_j^{(n)}} K(z - \xi) d\mu^{(m)}(\xi) \right| \\
\leq \int_{Q_j^{(n)}} \frac{C\sqrt{r_{n-1} r_n}}{|z - \xi|^2} \frac{dm_2(\xi)}{r_{n-1}} + \int_{B_j^{(n)}} \frac{Cr_n d\mu^{(m)}(\xi)}{|z - \xi|^2}.
\]

Now suppose that \( j \in A \) and \( \text{dist}(z, Q_j^{(n)}) \leq \frac{1}{4} r_{n-1} r_n \). Since \( \text{dist}(z, Q_j^{(n)}) \geq \text{dist}(z^*, Q_j^{(n)}) - \text{dist}(z, z^*) \geq \frac{1}{2} r_{n-1} r_n - \epsilon \), we must have that \( \epsilon \geq \frac{1}{2} r_{n-1} r_n \). But as \( \text{dist}(z, \text{supp}(\mu^{(m)})) \geq \epsilon \), and \( \mu^{(m)}(B_j^{(n)}) = r_n \), we have the following crude bound

\[
\left| \int_{Q_j^{(n)}} K(z - \xi) \frac{dm_2(\xi)}{r_{n-1}} - \int_{B_j^{(n)}} K(z - \xi) d\mu^{(m)}(\xi) \right| \\
\leq \frac{C}{r_{n-1}} \sqrt{m_2(Q_j^{(n)})} + 2 \frac{\mu^{(m)}(B_j^{(n)})}{\epsilon} \leq C_n.
\]

(Here it is used that \( \int_A |K(\xi)| dm_2(\xi) \leq C \sqrt{m_2(A)} \) for any Borel measurable set \( A \subset \mathbb{C} \) of finite \( m_2 \)-measure.)

At most four of the essentially pairwise disjoint squares \( Q_j^{(n)}, j \in A \), can satisfy \( \text{dist}(z, Q_j^{(m)}) \leq \frac{1}{4} r_{n-1} r_n \) (and it can only happen at all if \( n = q \)). Therefore, by summing (7.2) and (7.3) over \( j \in A \) in the cases when \( \text{dist}(z, Q_j^{(m)}) \geq \frac{1}{4} r_{n-1} r_n \) and \( \text{dist}(z, Q_j^{(n)}) \leq \frac{1}{4} r_{n-1} r_n \) respectively, we see that the quantity

\[
\left| \int_{\cup_{j \in A} Q_j^{(n)}} K(z - \xi) \frac{dm_2(\xi)}{r_{n-1}} - \int_{B^{(n-1)}(z^*) \setminus B^{(n)}(z^*)} K(z - \xi) d\mu^{(m)}(\xi) \right|,
\]

is no greater than a constant multiple of

\[
\int_{B(x, 2 r_{n-1}) \setminus B(x, \frac{1}{4} r_{n-1} r_n)} \frac{r_n}{r_{n-1}} \frac{dm_2(\xi)}{|z - \xi|^2} + \int_{B(x, \frac{1}{4} r_{n-1} r_n)} \frac{r_n d\mu^{(m)}(\xi)}{|z - \xi|^2} + s_n.
\]
The first term here is bounded by \( C \sqrt{\frac{r_n}{r_{n-1}}} \log(\frac{r_{n-1}}{r_n}) \leq C_n \log(\frac{1}{s_n}) \leq C \sqrt{s_n} \). Since \( \mu^{(m)} \) is \( C_0 \)-nice, we bound the second term by

\[
Cr_n \int_{1/\sqrt{r_{n-1}}}^{1} \frac{dr}{r^2} \leq Cr_n \frac{1}{\sqrt{r_n r_{n-1}}} \leq C s_n.
\]

We now wish to estimate \( \int_{\bigcup_{j \in A} Q_j} K(z - \xi) \frac{dm_2(\xi)}{r_{n-1}} \). With a slight abuse of notation, write \( \tilde{B}^{(n-1)}(z^*) = B_j^{(n-1)} \) if \( z^* \in B_j^{(n-1)} \). Then

\[
\left| \int_{\tilde{B}^{(n-1)}(z^*)} K(z - \xi) \frac{dm_2(\xi)}{r_{n-1}} - \int_{\bigcup_{j \in A} Q_j} K(z - \xi) \frac{dm_2(\xi)}{r_{n-1}} \right|
\]

is bounded by \( \frac{C}{r_{n-1}} \left( m_2(\tilde{B}^{(n-1)}(z^*) \triangle \bigcup_{j \in A} Q_j) \right)^{1/2} \). By Lemma 4.2, this quantity is no greater than \( \frac{C}{r_{n-1}} \sqrt{r_{n-1}^{3/2} + r_n r_{n-1}} \leq C \sqrt{s_n} \).

It remains to employ the reflectionless property (Lemma 3.1). Since \( z \in (1 + \frac{\epsilon}{r_{n-1}}) B^{(n-1)}(z^*) \), we use Lemma 3.1 to infer that

\[
\left| \int_{\tilde{B}^{(n-1)}(z^*)} K(z - \xi) \frac{dm_2(\xi)}{r_{n-1}} \right| = \left| \int_{(1 + \frac{\epsilon}{r_{n-1}}) B^{(n-1)}(z^*) \setminus \tilde{B}^{(n-1)}(z^*)} K(z - \xi) \frac{dm_2(\xi)}{r_{n-1}} \right|
\]

This quantity is bounded by \( \frac{C}{r_{n-1}} \left( m_2((1 + \frac{\epsilon}{r_{n-1}}) B^{(n-1)}(z^*) \setminus \tilde{B}^{(n-1)}(z^*)) \right)^{1/2} \leq C \sqrt{s_n + \frac{\epsilon}{r_{n-1}}} \). The lemma follows.

With Lemma 7.2 in hand, we may complete the proof of Proposition 7.1. First write

\[
\int_C K(z - \xi) d\mu^{(m)}(\xi) = \int_{B^{(n)}(z^*)} K(z - \xi) d\mu^{(m)}(\xi) + \sum_{n=1}^{q} \int_{B^{(n)}(z^*) \setminus B^{(n)}(z^*)} K(z - \xi) d\mu^{(m)}(\xi)
\]

(7.4)

Next note that that \( \frac{\epsilon}{r_{n-1}} \leq 1 \) if \( n = q \), and \( \sqrt{\frac{\epsilon}{r_{n-1}}} \leq s_n \) for \( 1 \leq n < q \). As \( n \geq 1 \sqrt{s_n} < \infty \), it follows from Lemma 7.2 that the sum appearing in the right hand side of (7.4) is bounded in absolute value independently of \( q \), \( m \) and \( \epsilon \). The remaining term on the right hand side of (7.4) has already been shown to be bounded in absolute value, see (7.1).

8 \( T_\mu(1) \) fails to exist in the sense of principal value \( \mu \)-almost everywhere

We now turn to consider the operator in the sense of principal value. The primary part of the argument will be the following lemma.
Lemma 8.1. Provided that $n$ is sufficiently large, there exists a constant $c_0 > 0$ such that for any disc $B_j^{(n)}$, and $z \in \mathbb{C}$ satisfying
\[
\text{dist}(z, \partial B_j^{(n)}) \leq c_0 r_n,
\]
\[
\left| \int_{A(z, r_n)} K(z - \xi) d\mu(\xi) \right| \geq c_0.
\]
\[\square\]

Before proving the lemma, we deduce from it that $T_\mu(1)$ fails to exist in the sense of principal value for $\mu$-almost every $z \in \mathbb{C}$. To this end, we set $F = \{ z \in E : z \in (1 - c_0)B_j^{(m)}(z) \text{ for all but finitely many } n \}$. It suffices to show that $\mu(F) = 0$.

First note that, with $F_n = \{ z \in E : z \in (1 - c_0)B_j^{(m)}(z) \text{ for all } m \geq n \}$, we have $F \subset \bigcup_{n \geq 0} F_n$, so it suffices to show that $\mu(F_n) = 0$ for all $n$.

To do this, note that for each $m \geq 0$, at most $\left( 1 - c_0 \right) r_m + C \sqrt{r_m / r_{m+1}}$ squares $Q_{\ell}^{(m+1)}$ can intersect $(1 - c_0)B_j^{(m)}$. Thus
\[
\mu\left( \bigcup_{\ell} \left\{ B_{\ell}^{(m+1)} : B_{\ell}^{(m+1)} \cap (1 - c_0)B_j^{(m)} \neq \emptyset \right\} \right) \leq (1 - c_0) r_m + C \sqrt{r_m / r_{m+1}}
\]
\[
= (1 - c_0) \mu(B_j^{(m)}) + C s_{m+1} r_m \leq \left( 1 - \frac{c_0}{2} \right) \mu(B_j^{(m)}),
\]
where the last inequality holds provided that $m$ is sufficiently large. But then, as long as $n$ is large enough, this inequality may be iterated to yield
\[
\mu\left( \{ z \in E : z \in (1 - c_0)B_j^{(n+k)}(z) \text{ for } k = 1, \ldots, m \} \right) \leq (1 - \frac{c_0}{2})^m.
\]

Hence $\mu(F_n) = 0$.

In preparation for proving Lemma 8.1, we make the following claim.

Claim 8.2. Let $n \in \mathbb{Z}_+$. For any disc $B_j^{(n)}$, and $z \in \mathbb{C}$, we have
\[
\left| \int_{A(z, r_n) \cap B_j^{(n)}} K(z - \xi) d(\mu - \frac{m_2}{r_n}) (\xi) \right| \leq C s_{n+1}.
\]
\[\square\]

Proof. To derive this claim, first suppose that a square $Q_{\ell}^{(n+1)} \subset A(z, r_n)$. Then from a crude application of Lemma 7.3 (see (7.2)), we infer that
\[
\left| \int_{Q_{\ell}^{(n+1)}} K(z - \xi) d(\mu - \frac{m_2}{r_n}) (\xi) \right| \leq C \sqrt{r_n r_{n+1} r_{n+1}^2} \leq C \left( \frac{r_{n+1}}{r_n} \right)^{\frac{3}{2}}.
\]
If it instead holds that \( Q^{(n+1)}_i \cap \partial A(z, r_n) \neq \emptyset \), then we have the blunt estimate

\[
\left| \int_{Q^{(n+1)}_i \cap A(z, r_n)} K(z - \xi) d(\mu - \frac{m_2}{r_n})(\xi) \right| \leq \frac{2}{r_n} \left[ \mu(Q^{(n+1)}_i) + \frac{m_2(Q^{(n+1)}_i)}{r_n} \right],
\]

which is bounded by \( \frac{Cr_{n+1}}{r_{n+1}} \). There are most \( \frac{r_n}{r_{n+1}} \) squares \( Q^{(n+1)}_i \) contained in \( A(z, r_n) \), and no more than \( C \sqrt{\frac{r_n}{r_{n+1}}} \) squares \( Q^{(n+1)}_i \) can intersect the boundary of \( A(z, r_n) \).

On the other hand, the set \( \tilde{A} \) consisting of the points in \( A(z, r_n) \cap B_z(n) \) not covered by any square \( Q^{(n+1)}_i \) has \( m_2 \)-measure no greater than \( Cr_{n+1}^{1/2}r_n^{3/2} \) (see Lemma 4.2). Thus

\[
\int_{\tilde{A}} |K(z - \xi)| \frac{d m_2(\xi)}{r_n} \leq \frac{2m_2(\tilde{A})}{r_n} \leq C_{s, n+1}.
\]

Bringing these estimates together establishes Claim 8.2. ■

Let us now complete the proof of Lemma 8.1

**Proof of Lemma 8.1.** Note that \( \int_{A(z, r_n) \cap B_z(n)} K(z - \xi) \frac{d m_2(\xi)}{r_n} \) is a Lipschitz continuous function in \( C \), with Lipschitz norm at most \( \frac{C}{r_n} \). Thus, we infer from Lemma 3.2 that there is a constant \( c_0 > 0 \) such that

\[
\left| \int_{A(z, r_n) \cap B_z(n)} K(z - \xi) \frac{d m_2(\xi)}{r_n} \right| \geq \frac{c_0}{2},
\]

whenever \( \text{dist}(z, \partial B_z(n)) \leq c_0 r_n \). But now we apply Claim 8.2 to deduce that for all such \( z \),

\[
\left| \int_{A(z, r_n) \cap B_z(n)} K(z - \xi) d\mu(\xi) \right| \geq \frac{c_0}{2} - C_{s, n+1} \text{ (the only part of the support of } \mu \text{ that } A(z, r_n) \text{ intersects is contained in } B_z(n) \text{). The right hand side here is at least } \frac{c_0}{4} \text{ for all sufficiently large } n. \quad \square
\]

9 The set \( E \) is purely unrectifiable

We now show that \( E \) is purely unrectifiable, that is, \( \mathcal{H}^1(E \cap \Gamma) = 0 \) for any rectifiable curve \( \Gamma \). The proof that follows is a simple special case of the well-known fact that any set with zero lower \( \mathcal{H}^1 \)-density is unrectifiable (one can in fact say much more, see for instance [6]).

First notice that for each \( z \in \mathbb{C} \) and \( n \geq 1 \), \( B(z, \frac{1}{4} \sqrt{r_n r_{n-1}}) \) can intersect at most one of the discs \( B_z(n) \). Hence

\[
\mathcal{H}^1(E \cap B(z, \frac{1}{4} \sqrt{r_n r_{n-1}})) \leq 2r_n.
\]

A rectifiable curve \( \Gamma \) can be covered by discs \( B(z_j, \frac{1}{4} \sqrt{r_n r_{n-1}}), j = 1, \ldots, N \), the sum of whose radii is at most \( \ell(\Gamma) \).
Thus $\mathcal{H}^1(E \cap \Gamma) \leq \sum_{j=1}^{N} \mathcal{H}^1(E \cap B(z_j, \frac{1}{4} \sqrt{r_n r_{n-1}})) \leq 2 \sum_{j=1}^{N} r_n$. But $\sum_{j=1}^{N} \frac{1}{4} \sqrt{r_n r_{n-1}} \leq \ell(\Gamma)$, and so $\mathcal{H}^1(\Gamma \cap E) \leq 8 \sqrt{r_n} \ell(\Gamma)$, which tends to zero as $n \to \infty$ (the sequence $\sqrt{s_n}$ is summable).

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