UNIFORM NON–AMENABILITY OF FREE BURNSIDE GROUPS

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Abstract. The aim of the present note is to show that free Burnside groups of sufficiently large odd exponent are non–amenable in a certain strong sense, more precisely, their left regular representations are isolated from the trivial representation uniformly on finite generating sets. This result is applied to the solution of a strong version of the von Neumann – Day problem concerning amenability of groups without non–abelian free subgroups. As another consequence, we obtain that the above–mentioned groups are of uniform exponential growth. This answers a question of de la Harpe [12].

1. Introduction

Let us consider the left regular representation of a finitely generated group $G$ on the Hilbert space $L^2(G)$. For any finite generating set $X$ of $G$, we define $\alpha(G, X)$ as the maximal $\varepsilon \geq 0$ such that for any vector $v \in L^2(G)$ of norm $\|v\| = 1$, there exists an element $x \in X$ satisfying the inequality

$$\|xv - v\| \geq \varepsilon.$$ 

It is easy to check that the existence of a finite generating set $X$ of $G$ such that $\alpha(G, X) > 0$, implies the inequality $\alpha(G, Y) > 0$ for any other finite generating set $Y \subseteq G$. Thus it is natural to consider the quantity

$$\alpha(G) = \inf_X \alpha(G, X),$$

where the infimum is taken over all finite generating sets of $G$.

Recall that a group $G$ is called amenable if there exists a finitely additive measure $\mu$ on the set of all subsets of $G$ which is invariant under the left action of $G$ on itself and satisfies $\mu(G) = 1$. One of the most interesting characterizations of amenable groups was obtained by Hulaniski [13] in terms of the left regular representations. In the case of finitely generated groups it can be formulated as follows.

**Theorem 1.1** (Hulaniski, [13]). A finitely generated group $G$ is amenable if and only if $\alpha(G, X) = 0$ for some (and hence for any) finite generating set $X$ of $G$.

In particular, we have $\alpha(G) = 0$ for any amenable group $G$. The question whether the equality $\alpha(G) = 0$ is equivalent to the amenability of $G$ was open.
until recent time. The first examples of non–amenable finitely generated groups \( G \) satisfying \( \alpha(G) = 0 \) have been constructed in [20].

**Definition 1.2.** If \( \alpha(G) > 0 \) for a finitely generated group \( G \), we say that \( G \) is **uniformly non–amenable**.

Recall that a group is said to be elementary if it contains a cyclic subgroup of finite index. In the paper [24], Shalom proved that any residually finite non–elementary hyperbolic group is uniformly non–amenable. In fact, this is true for every non–amenable hyperbolic group, not necessarily residually finite (see Example 3.4 below). Essentially all uniformly non–amenable groups known up to now are hyperbolic and, in particular, they contain a non–abelian free subgroup. This raises the following natural question, which is a stronger version of the so called von Neumann–Day problem (see [9]).

**Question 1.1.** Does any uniformly non–amenable group contain a non–abelian free subgroup?

The main goal of this note is to show that the answer is negative and can be obtained in the same way as the solution of the classical von Neumann–Day problem. More precisely, we show that (non–cyclic) free Burnside groups of sufficiently large odd exponent are uniformly non–amenable. In contrast, we note that finitely generated torsion groups which are not amenable nor uniformly non–amenable are constructed in [20]. Our main result is the following.

**Theorem 1.3.** There exists an integer \( N > 0 \) such that any free Burnside group \( B(m, n) \) of odd exponent \( n > N \) and rank \( m \geq 2 \) is uniformly non–amenable.

**Corollary 1.4.** There exists a finitely generated uniformly non–amenable group without non–abelian free subgroups.

It should be mentioned that the first examples of non–amenable groups without free subgroups were constructed by Ol’shanskii in [15]. These groups are torsion with unbounded orders of elements. After that, in [21], Adian proved that the free Burnside groups \( B(m, n) \) are non–amenable whenever \( m \geq 2, n \) is odd, and \( n \geq 665 \). We use Adian’s result to prove our Theorem 1.1.

The notion of uniform non–amenability is interesting, in particular, in relation with exponential growth rates of finitely generated groups. Recall that the growth function \( \gamma_X^G : \mathbb{N} \to \mathbb{N} \) of a group \( G \) with respect to a finite generating set \( X \) is defined by

\[
\gamma_X^G(n) = |\{ g \in G : |g|^X \leq n \}|
\]

The **exponential growth rate** of \( G \) with respect to \( X \) is the number

\[
\omega(G, X) = \lim_{n \to \infty} \sqrt[n]{\gamma_X^G(n)}
\]

(the above limit always exists). The quantity

\[
\omega(G) = \inf_X \omega(G, X)
\]

is called a **minimal exponential growth rate** of \( G \) (the infimum is taken over all finite generating sets of \( G \)). One says that the group \( G \) has **exponential growth** if
\( \omega(G, X) > 1 \) for some (or, equivalently, for any) finite generating subset \( X \) of \( G \). If \( \omega(G) > 1 \), then the group \( G \) is said to have \textit{uniform exponential growth}.

This notion comes from geometry; in particular, if \( G \) is a fundamental group of a compact Riemannian manifold of unit diameter, then \( \log \omega(G) \) is a lower bound for the topological entropy of the geodesic flow of the manifold \[10\].

The first example of a group having non-uniform exponential growth was recently constructed by J. Wilson \[25\]. On the other hand, there are many examples of classes of groups which are known to have uniformly exponential growth, for example:

(a) non–elementary hyperbolic groups \[15\];

(b) free products with amalgamations \( G \ast_{A=H} H \) satisfying the condition \( (|G : A| - 1)(|H : B| - 1) \geq 2 \) and \( \text{HNN}-\text{extensions} \ G \ast_A \) associated with a monomorphism \( \phi_1, \phi_2 : A \to G \), where \( |G : \phi_1(A)| + |G : \phi_2(A)| \geq 3 \) (see \[3\]);

(c) one–relator groups of exponential growth \[3\];

(d) solvable groups of exponential growth \[23\] (the particular case of polycyclic groups was considered independently in \[3\]); more generally, any elementary amenable group of exponential growth has uniform exponential growth \[22\];

(e) linear groups of exponential growth \[7\].

In the survey \[12\], de la Harpe asked the following:

**Question 1.2.** Do the free Burnside groups of sufficiently large odd exponent have uniform exponential growth?

Here we answer the question positively. In \[24\], Shalom observed that every uniformly non–amenable finitely generated group has uniform exponential growth. From this and Theorem 1.1, we obtain immediately

**Corollary 1.5.** If \( m \geq 2, n \) is odd and large enough, then the group \( B(m, n) \) has uniform exponential growth.

It is worth to note, that in some similar sense uniform non–amenability of free Burnside groups of large odd exponent is claimed in the paper \[4\]. However, in \[4\] authors uses an unproved result about free subgroups of \( B(m, n) \). This result was announced by Ivanov in \[14\], but the complete proof have never been published.

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### 2. Constructing free Burnside groups

In this section we recall the construction of presentations of free Burnside groups and describe shortly the main properties used in this paper. For details we refer to the books \[1\], \[19\]. Our discussion heavily depends on results from \[19\] (for otherwise our paper would be unreasonably long). In fact, we will not obtain any new facts about Burnside groups by analyzing the geometric structure of van Kampen diagrams; we will only show how to combine certain lemmas from \[19\] in order to obtain the result we need. Our main goal here is to prove Theorem 2.7.
Given an alphabet $A$, we denote by $|W|$ the length of a word $W$ over $A$. For two words $U, V$ over $A$ we write $U \equiv V$ to express letter–by–letter equality. Finally if $A$ is a generating set of a group $G$, we write $U = V$ whenever two words $U$ and $V$ over $A^{\pm 1}$ represent the same elements of $G$; we identify the words over $A$ and the elements of $G$ represented by them.

Recall that $B(m,n)$, the free Burnside group of exponent $n$ and rank $m$, is the free group in the variety defined by the low $X^n = 1$. Throughout this paper we will assume that $n$ is odd and large enough, and $m \geq 2$; all lemmas and theorems are formulated under these assumptions.

The group $B(m,n)$ can be defined by the presentation

\[(2.1) \quad B(A,n) = \langle A : R = 1, R \in \bigcup_{i=1}^\infty \mathcal{R}_i \rangle,\]

where $A = \{a_1, \ldots, a_n\}$ and the sets of relations $\mathcal{R}_i$ are constructed as follows [19]. We put $\mathcal{R}_0 = \emptyset$. By induction, suppose that we have already defined the set of relations $\mathcal{R}_{i-1}$, $i \geq 1$. Denote by $G(i-1)$ the group with the presentation $\langle A : R = 1, R \in \mathcal{R}_i \rangle$.

For $i \geq 1$, a word $X$ over the alphabet $A$ is called simple in the rank $i-1$, if it is not conjugated to a power of a shorter word in the group $G(i-1)$ and is not conjugated to a power of a period of rank $k \leq i-1$ in the group $G(i-1)$. Let us denote by $\mathcal{X}_i$ a certain maximal subset of words satisfying the following conditions.

1) $\mathcal{X}_i$ consists of words of length $i$ which are simple in the rank $i-1$.

2) If $A, B \in \mathcal{X}_i$ and $A \neq B$, then $A$ is not conjugated to $B$ or $B^{-1}$ in the group $G(i-1)$.

Each word from $\mathcal{X}_i$ is called a period of rank $i$. We introduce the additional relations $\mathcal{S}_i = \{A^n : A \in \mathcal{X}_i\}$ and set $\mathcal{R}_i = \mathcal{R}_{i-1} \cup \mathcal{S}_i$.

Now we are going to list some results needed for the sequel.

**Lemma 2.1** ([19], Theorem 19.4). Any element of $B(m,n)$ is conjugated to a period of a certain rank.

**Lemma 2.2** ([19], Theorem 19.5). The centralizer of every nontrivial element of $B(m,n)$ is a cyclic subgroup of order $n$.

From Lemma 2.2 we have immediately

**Corollary 2.3.** Suppose that $a, b$ are two elements of $B(m,n)$ such that $[a, b] \neq 1$. Then $[a^i, b] \neq 1$ whenever $a^i \neq 1$.

**Proof.** Suppose that $[a^i, b] = 1$ and $a^i \neq 1$. By Lemma 2.2 the centralizer of $a^i$ is cyclic. As $a, b \in C(a^i)$, we have $a = z^{m_1}$ and $b = z^{m_2}$ for a certain $z \in B(m,n)$. This contradicts to $[a, b] \neq 1$. \qed

In the following two lemmas the additional parameters $d$ and $\sigma$ appear (the parameter $\sigma$ here corresponds to $100\zeta^{-1}$ from [19]). Exact values of these parameters are not important for us (in fact, $d, \sigma << n$, but the only inequality we need here
is $\sigma < \frac{n}{2}$). It is sufficient to know that there exists parameters $d$ and $\sigma$ such that the following two lemmas hold.

**Lemma 2.4.** Let $C$ be a period of a certain rank, $V \equiv C^k$, where $\sigma < k \leq \frac{1}{2}n$. Suppose that an element $W$ does not commute with $V$ and has minimal length among elements of the double coset $\langle C^k \rangle W \langle C^k \rangle$. We also assume that $[C^k, W]$ is conjugated to $A^i$, where $A$ is a period of a certain rank and $|l| \leq \frac{1}{2}n$. Then we have $|l| \leq \sigma$ and the pair $[C^k, W], C^k$ is conjugated to the pair $(A^l, B)$, where $|B| < d|A|$.

*On the proof.* The lemma is a simplification of Lemma 25.21 from [19]. We have to note that the presentation (2.1) satisfies the condition $R$ (there are no relations of the second type at all, see [19, Ch. 8] for definitions and details).

**Lemma 2.5.** Suppose that $l$ is an integer and $A, B \in B(m, n)$ such that

1) $A$ is a period of a certain rank;

2) $|A, B| \neq 1$ in $B(m, n)$;

3) $|l| \leq \sigma$ and $|B| < d|A|$.

Then there exists integer $s$ such that the pair $(BA^s, B)$ is conjugated to a pair $(F, T)$, where $F$ is a period of a certain rank, $|F, T| \neq 1$, and $|T| \leq 3F$.

*On the proof.* The proof of this lemma is a part of the proof of Lemma 27.3 from [19]. Namely in order to prove the above assertion we have to repeat (word for word) the last 5 paragraphs of the proof of Lemma 27.3 from [19].

**Lemma 2.6.** For every odd large enough $n \in \mathbb{N}$, there exist $h, k \in \mathbb{N}$ with the following property. Suppose that $F$ is a period of a certain rank, $T$ is an element of $B(m, n)$ such that $[F, T] \neq 1$, and $|T| \leq 3F$. Then the elements

$$b_1 = F^k TF^{k+2} \cdots TF^{k+2h-2}$$

and

$$b_1 = F^{k+1} TF^{k+3} \cdots TF^{k+2h-1}$$

form the basis of the free Burnside group of exponent $n$.

*Proof.* Up to notation the statement of the lemma can easily be obtained from Lemma 39.4 [19]. We only notice that our constants $n, k,$ and $h$ correspond to constants $n_0, n$ and $h$ from [19] respectively. \(\square\)

Now we are ready to prove the main result of this section.

**Theorem 2.7.** For any odd large enough $n$ and $m \geq 2$, there exists $M \in \mathbb{N}$ having the following property. Let $a, b$ be two non–commuting elements of $B(m, n)$. Then there exist elements $u, v \in \langle a, b \rangle$ such that $\{u, v\}$ is the basis of the free Burnside subgroup of exponent $n$ and the lengths of elements $u, v$ with respect to $\{a, b\}$ satisfy

$$|u|_{\{a, b\}} < M, \quad |v|_{\{a, b\}} < M.$$

*Proof.* By Lemma 2.4, we have $a = P^{-1}C^kP$, for some element $P$ and a period $C$ of a certain rank. Without loss of generality we can assume that $0 < k' < \frac{n}{2}$.
Recall that $\sigma < \frac{n}{2}$. Clearly there is a number $i \in \mathbb{N}$ such that
\begin{equation}
(2.2) \quad i < \frac{n}{2}
\end{equation}
and $\sigma < ik' < \frac{n}{2}$. Thus we have
\begin{equation}
(2.3) \quad a^i = P^{-1}C^k P,
\end{equation}
where
\begin{equation}
(2.4) \quad \sigma < k < \frac{n}{2}.
\end{equation}
Using Corollary 2.3 we note that $[C^k, P] = P[a^i, b]P^{-1} \neq 1$.

Denote by $W_0$ the element $PbP^{-1}$ and by $W$ the shortest element in the double coset $(C^k)W_0(C^k)$. Evidently $W = C^kk_1W_0C^kk_2$ for some $k_1, k_2$ satisfying the inequality
\begin{equation}
(2.5) \quad \max\{|k_1|, |k_2|\} < \frac{n}{2}.
\end{equation}
Since $W = Pa^{k_1}b^{-1}a^{k_2}P^{-1}$, we obtain the following inequality by using (2.5) and (2.3)
\begin{equation}
(2.6) \quad |P^{-1}WP|_{(a,b)} \leq 2i + k_1 + k_2 + 1 < 2n.
\end{equation}

It is clear that $[C^k, W] \neq 1$. By Lemma 2.1 $[C^k, W]$ is conjugated to $A^l$, where $A$ is a period of a certain rank, i.e.,
\begin{equation}
(2.7) \quad [C^k, W] = Q^{-1}A^lQ
\end{equation}
for some element $Q$. Set
\begin{equation}
(2.8) \quad B = Q^{-1}C^k Q.
\end{equation}

Applying Lemma 2.4 we obtain $|l| \leq \sigma$ and $|B| \leq d|A|$. Note that all conditions of Lemma 2.5 are satisfied for $A$ and $B$. Therefore, there is integer $s$ such that the pair $(BA^k, B)$ is conjugated to a pair $(F, T)$, where $F$ is a period of a certain rank, $|F, T| \neq 1$, and $|T| \leq 3|F|$. According to Lemma 2.6 there exists $b_1, b_2 \in \langle F, T \rangle$ such that $b_1, b_2$ freely generate the subgroup $\langle b_1, b_2 \rangle$ and
\[ \max\{|b_1|_{\langle F, T \rangle}, |b_2|_{\langle F, T \rangle}\} \leq N, \]
where $N$ depends on $k$ and $h$ only (it is easy to calculate the lengths of $b_1$ and $b_2$ exactly, but this is not our goal here). Therefore, there exists $x_1, x_2 \in \langle BA^s, B \rangle$ such that $x_1, x_2$ freely generate the subgroup $\langle x_1, x_2 \rangle$ and
\[ \max\{|x_1|_{\langle BA^s, B \rangle}, |x_2|_{\langle BA^s, B \rangle}\} \leq N. \]

Clearly we can assume $ls < \frac{n}{2}$. Passing to generators $A$ and $B$, we obtain
\[ \max\{|x_1|_{\langle A, B \rangle}, |x_2|_{\langle A, B \rangle}\} \leq N \left( \frac{n}{2} + 1 \right). \]

Therefore taking into account (2.7) and (2.8), we obtain the following estimates for the elements $y_i = Qx_iQ^{-1}, i = 1, 2$,
\[ \max\{|y_1|_{\langle C^s, W \rangle}, |y_2|_{\langle C^s, W \rangle}\} \leq 4 \max\{|x_1|_{\langle A, B \rangle}, |x_2|_{\langle A, B \rangle}\} \leq N(2n + 4). \]

Finally we set $z_i = P^{-1}y_iP$ for $i = 1, 2$. Combining (2.4), (2.6), and (2.8), we obtain
\[ \max\{|z_1|_{\langle a, b \rangle}, |z_2|_{\langle a, b \rangle}\} \leq 2n(2n + 4)N. \]
As the pair \((z_1, z_2)\) is conjugated to \((x_1, x_2)\), the elements \(z_1, z_2\) form the basis of free Burnside group of exponent \(n\). To conclude the proof it suffices to set \(M = 2n(2n + 4)N\). □

**Remark.** It should be noted that Ivanov announced the following stronger result in [14]. Given any \(n\) odd and large enough, and \(m \geq 2\), there exist words \(u(x, y), v(x, y)\) over the alphabet \(\{x, y\}\) such that if \(a, b\) are two non–commuting elements of \(B(m, n)\), then \(u(a, b), v(a, b)\) generate freely a free Burnside subgroup of \(B(m, n)\). Unfortunately the proof of this result has never been written.

### 3. Sufficient conditions for uniform non–amenability

**Definition 3.1.** Suppose that \(G\) is a group with a given finite set of generators \(X\), \(Y\) a subset of \(G\). The depth of \(Y\) with respect to \(X\) is defined by

\[
\text{depth}_X(Y) = \max_{y \in Y} |y|x.
\]

**Lemma 3.2.** Suppose that \(G\) is a group, \(H\) is a subgroup of \(G\), \(X\) and \(Y\) are finite generating sets of \(G\) and \(H\) respectively. Then we have

\[
\alpha(G, X) \geq \frac{\alpha(H, Y)}{\text{depth}_X(Y)}.
\]

**Proof.** In order to prove \((3.1)\), we have to show that for any vector \(f \in L^2(G)\) of norm one, there exists an element \(x \in X^{\pm 1}\) such that

\[
\|xf - f\| \geq \frac{\alpha(H, Y)}{\text{depth}_X(Y)}.
\]

As usual we denote by \(Hs\) the right coset representing by \(s\). Let us fix a (unique) element \(s\) in each right coset and denote by \(S\) the obtained system of representatives. Given a function \(f \in L^2(G)\), for every \(s \in S\), we introduce a new function

\[
f_s(g) = \begin{cases} f(g), & \text{if } g \in Hs, \\ 0, & \text{if } g \notin Hs. \end{cases}
\]

If \(f_s \neq 0\), we set

\[
\tilde{f}_s = \frac{f_s}{\|f_s\|}.
\]

Obviously the norm of \(\tilde{f}_s\) is equal to 1 whenever \(f_s \neq 0\). It is also clear that

\[
f = \sum_{s \in S} f_x.
\]

Further, to each \(f_s, s \in S\), we assign a function \(h_s \in L^2(H)\) by the rule

\[
h_s(g) = \tilde{f}_s(gs).
\]

Note that \(\|h_s\| = \|\tilde{f}_s\| = 1\). Therefore, by definition of \(\alpha(H, Y)\), there is an element \(y \in Y^{\pm 1}\) such that

\[
\|yh_s - h_s\| \geq \alpha(H, Y).
\]
On the other hand, we have
(3.5) \[ \| y\tilde{f}_s - \tilde{f}_s \| = \sqrt{\sum_{g \in H_s} (\tilde{f}_s(y^{-1} g) - \tilde{f}_s(g))^2} = \sqrt{\sum_{g \in H} (h_s(y^{-1} g) - h_s(g))^2} = \| yh_s - h_s \| . \]

Combining (3.3), (3.4), and (3.5) yields
(3.6) \[ \| yf_s - f_s \| \geq \alpha(H, Y) \| f_s \|. \]

Now we observe that the supports of the functions \((yf_s - f_s)\) are pairwise disjoint. Thus \(f_s, s \in S\), are pairwise orthogonal and we obtain
(3.7) \[ \| yf - f \| = \left\| y \sum_{s \in S} f_s - \sum_{s \in S} f_s \right\|^2 \geq \alpha(H, Y) \sum_{s \in S} \| f_s \|^2 = \alpha(H, Y). \]

Assume that there exists a function \(f \in L^2(G)\) that does not satisfy (3.2), i.e.,
(3.8) \[ \| xf - f \| < \frac{\alpha(H, Y)}{\text{depth}_X(Y)} \]
for every \(x \in X^{\pm 1}\). According to the above–mentioned arguments, there exists an element \(y \in Y\) satisfying the inequality (3.7). Suppose that \(y = x_d \ldots x_2 x_1\), where \(x_d, \ldots, x_2, x_1 \in X^{\pm 1}\) and \(d \leq \text{depth}_X(Y)\). Denote by \(w_i\) the \(i\)–th suffix of the word \(x_d \ldots x_2 x_1\), i.e., the word \(x_i \ldots x_2 x_1\), and set \(w_0 = 1\) for convenience. The inequality (3.8) implies
\[ \| yf - f \| = \left\| \sum_{i=0}^{d-1} (w_{i+1} f - w_i f) \right\| \leq \sum_{i=0}^{d-1} \| x_{i+1} w_i f - w_i f \| = \sum_{i=0}^{d-1} \| x_{i+1} f - f \| < d \cdot \frac{\alpha(H, Y)}{\text{depth}_X(Y)} \leq \alpha(H, Y). \]
This contradicts to (3.7). The theorem is proved. □

**Corollary 3.3.** Suppose that \(H\) is a non–amenable group with a given set of generators \(Y\), \(G\) is a finitely generated group. Assume that for every finite generating set \(X\) of \(G\) there exists an embedding \(\iota : H \to G\) such that
\[ \text{depth}_X(\iota(Y)) \leq D, \]
where the constant \(D\) is independent of \(X\). Then the group \(G\) is uniformly non–amenable.

**Example 3.4.** Any infinite hyperbolic group is uniformly non–amenable. Indeed, by the result of Koubi [15], we can apply the Corollary 3.3 for \(H = F_2\), the free group of rank 2.
Proof of Theorem 1.1. Recall that the group $B(2, n)$ is non–amenable for all odd $n > 665$. To prove the theorem it is sufficient to refer to Theorem 2.7 and Corollary 3.3.

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