A DUAL FORMULA FOR THE NONCOMMUTATIVE TRANSPORT DISTANCE

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ABSTRACT. In this article we study the noncommutative transport distance introduced by Carlen and Maas and its entropic regularization defined by Becker and Li. We prove a duality formula that can be understood as a quantum version of the dual Benamou–Brenier formulation of the Wasserstein distance in terms of subsolutions of the Hamilton–Jacobi–Bellmann equation.

INTRODUCTION

The theory of optimal transport [Vil03; Vil09] has experienced rapid growth in recent years with applications in diverse fields across pure and applied mathematics. Along with this growth came a lot of interest in extending the methods of optimal transport beyond the scope of its original formulation as an optimization problem for the transport cost between two probability measures.

One such extension deals with “quantum spaces”, where the probability measures are replaced by density matrices or density operators. Most of the work dealing with quantum optimal transport in this sense can be grouped into one of the following two categories. The first approach (see e.g. [CM14; CM17; MM17; CGT18; RD19; BV20; CM20]) relies on a noncommutative analog of the Benamou–Brenier formulation [BB00] of the Wasserstein distance for probability measures on Euclidean space

\[ W_2^2(\mu, \nu) = \inf \left\{ \int_0^1 \int_{\mathbb{R}^n} |v_t|^2 \, d\rho_t \, dt : \rho_0 = \mu, \rho_1 = \nu, \dot{\rho}_t + \nabla \cdot (\rho_t v_t) = 0 \right\}. \]

This approach has proven fruitful in applications to noncommutative functional inequalities, similar in spirit to the heuristics known as Otto calculus [CM17; CM20; DR20; WZ20].

The second approach (see e.g. [NGT15; GMP16; Pey+19; Duv20; Pal+20; DPT21]) seeks to find a suitable noncommutative analog of the Monge–Kantorovich formulation [Kan42] of the Wasserstein distance via couplings (or transport plans):

\[ W_p^p(\mu, \nu) = \inf \left\{ \int_{X \times X} d^p(x, y) \, d\pi(x, y) : (\text{pr}_1) \# \pi = \mu, (\text{pr}_2) \# \pi = \nu \right\}. \]

This approach also allows to consider a quantum version of Monge–Kantorovich problem for arbitrary cost functions. So far, possible connections between these two approaches in the quantum world stay elusive.

The focus of this article lies on the noncommutative transport distance \( \mathcal{W} \) introduced in the first approach. More precisely, we prove a dual formula that is a noncommutative analog of the expression of the classical \( L^2 \)-Wasserstein distance
in terms of subsolutions of the Hamilton–Jacobi equation [OV00; BGL01]

\[
W_2^2(\mu, \nu) = \frac{1}{2} \inf \left\{ \int_{\mathbb{R}^n} u_1 \, d\mu - \int_{\mathbb{R}^n} u_0 \, d\nu : \dot{u}_t + \frac{1}{2} |\nabla u_t|^2 \leq 0 \right\}.
\]

This result yields a noncommutative version of the dual formula obtained independently by Erbar, Maas and the author [EMW19] and Gangbo, Li and Mou [GLM19] for the Wasserstein-like transport distance on graphs. In fact, we prove a dual formula that is not only valid for the metric \( W \), but also for the entropic regularization recently introduced by Becker–Li [BL21].

With the notation introduced in the next section, the main result of this article reads as follows.

**Theorem.** Let \( \sigma \in M_n(\mathbb{C}) \) be an invertible density matrix and \((P_t)\) an ergodic quantum Markov semigroup on \( M_n(\mathbb{C}) \) that satisfies the \( \sigma \)-DBC. The entropic regularization \( W_\varepsilon \) of noncommutative transport distance induced by \((P_t)\) satisfies the following dual formula:

\[
\frac{1}{2} W_2^2(\rho_0, \rho_1) = \sup \{ \tau(A(1)\rho_1 - A(0)\rho_0) \mid A \in HJB_1^\varepsilon \}. \]

Here \( HJB_1^\varepsilon \) stands for the set of all Hamilton–Jacobi–Bellmann subsolutions, a suitable noncommutative variant of solutions of the differential inequality

\[
\dot{u}(t) + \frac{1}{2} |\nabla u(t)|^2 - \varepsilon \Delta u(t) \leq 0.
\]

Other metrics similar to \( W \) also occur in the literature, most notably the one called the “anticommutator case” in [CGT18; Che+20; BL21]. In [Wir18; CM20], a class of such metrics was studied in a systematic way, and our main theorem applies in fact to this wider class of metrics. For the anticommutator case, this duality formula was obtained before in [Che+20].

There are still some very natural questions left open. For one, we do not discuss the existence of optimizers. While for the primal problem this follows from a standard compactness argument, this question is more delicate for the dual problem, even when dealing with probability densities on discrete spaces instead of density matrices, and one has to relax the problem to obtain maximizers (see [GLM19, Sections 6–7]).

Another interesting direction would be to extend the duality result from matrix algebras to infinite-dimensional systems. While a definition of the metric \( W \) for quantum Markov semigroups on semi-finite von Neumann algebras is available [Hor18; Wir18], the problem of duality seems to be much harder to address. Even for abstract diffusion semigroups, the best known result only shows that the primal distance is the upper length distance associated with the dual distance and leaves the question of equality open [AES16, Proposition 10.11].

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1. Setting and basic definitions

In this section we introduce basic facts and definitions about quantum Markov semigroups that will be used later on. In particular, we review the definition of the
noncommutative transport distance from [CM17] and its entropic regularization introduced in [BL21]. Our notation mostly follows [CM17; CM20].

Let $M_n(\mathbb{C})$ denote the complex $n \times n$ matrices and let $\mathcal{A}$ be a unital *-subalgebra of $M_n(\mathbb{C})$. Let $\mathcal{A}_h$ denote the self-adjoint part of $\mathcal{A}$. We write $\tau$ for the normalized trace on $M_n(\mathbb{C})$ and $\mathcal{H}_A$ for the Hilbert space formed by equipping $\mathcal{A}$ with the GNS inner product

$$\langle \cdot, \cdot \rangle_{\mathcal{H}_A} : \mathcal{A} \times \mathcal{A} \to \mathbb{C}, (A, B) \mapsto \tau(A^*B).$$

The adjoint of a linear operator $\mathcal{K} : \mathcal{H}_A \to \mathcal{H}_A$ is denoted by $\mathcal{K}^\dagger$.

We write $\mathcal{G}(\mathcal{A})$ for the set of all density matrices on $\mathcal{A}$, that is, all positive elements $\rho \in \mathcal{A}$ with $\tau(\rho) = 1$. The subset of invertible density matrices is denoted by $\mathcal{G}_+(\mathcal{A})$.

A quantum Markov semigroup (QMS) on $\mathcal{A}$ is a family $(P_t)_{t \geq 0}$ of linear operators on $\mathcal{A}$ that satisfy the following conditions:

- $P_t$ is unital and completely positive for every $t \geq 0$,
- $P_0 = \text{id}_\mathcal{A}$, $P_{s+t} = P_sP_t$ for all $s, t \geq 0$,
- $t \mapsto P_t$ is continuous.

We consider a QMS $(P_t)$ on $\mathcal{A}$ which extends to a QMS on $M_n(\mathbb{C})$ satisfying the $\sigma$-DBC for some density matrix $\sigma \in \mathcal{G}_+(\mathcal{A})$, that is,

$$\tau((P_tA)^*B\sigma) = \tau(A^*(P_tB)\sigma)$$

for $A, B \in \mathcal{A}$. Let $\mathcal{L}$ denote the generator of $(P_t)$, that is, the linear operator on $\mathcal{A}$ given by

$$\mathcal{L}(A) = \lim_{t \searrow 0} \frac{P_tA - A}{t}.$$

We further assume that $(P_t)$ is ergodic (or primitive), that is, the kernel of $\mathcal{L}$ is one-dimensional.

By Alicki’s theorem [Ali76, Theorem 3], [CM17, Theorem 3.1] there exists a finite set $\mathcal{J}$, real numbers $\omega_j$ for $j \in \mathcal{J}$ and $V_j \in M_n(\mathbb{C})$ for $j \in \mathcal{J}$ with the following properties:

- $\tau(V_j^*V_k) = \delta_{jk}$ for $j, k \in \mathcal{J}$,
- $\tau(V_j) = 0$ for $j \not\in \mathcal{J}$,
- $\{V_j \mid j \in \mathcal{J}\} = \{V_j^* \mid j \in \mathcal{J}\}$,
- $\sigma V_j \sigma^{-1} = e^{-\omega_j} V_j$ for $j \in \mathcal{J}$

such that

$$\mathcal{L}(A) = \sum_{j \in \mathcal{J}} \left( e^{-\omega_j/2} V_j^* [A, V_j] - e^{\omega_j/2} [A, V_j] V_j^* \right)$$

for $A \in \mathcal{A}$.

The numbers $\omega_j$ are called Bohr frequencies of $\mathcal{L}$ and are uniquely determined by $(P_t)$.

The matrices $V_j$ are not uniquely determined by $(P_t)$ and $\sigma$, but in the following we will fix a set $\{V_j \mid j \in \mathcal{J}\}$ that satisfies the preceding conditions.

Let

$$\mathcal{H}_{\mathcal{A}, \mathcal{J}} = \bigoplus_{j \in \mathcal{J}} \mathcal{H}_{\mathcal{A}}^{(j)},$$

where $\mathcal{H}_{\mathcal{A}}^{(j)}$ is a copy of $\mathcal{H}_{\mathcal{A}}$ for $j \in \mathcal{J}$. We write $\partial_j$ for $[V_j, \cdot]$ and

$$\nabla : \mathcal{H}_{\mathcal{A}} \to \mathcal{H}_{\mathcal{A}, \mathcal{J}}, \nabla(A) = (\partial_j(A))_{j \in \mathcal{J}}.$$
We write div for the adjoint of $-\nabla$, that is,

$$\text{div} = -\sum_{j \in J} \partial_j^\dagger.$$  

For $X \in A_+$ and $\alpha \in \mathbb{R}$ define

$$[X]_\alpha : \mathcal{H}_A \to \mathcal{H}_A, \quad [X]_\alpha(A) = \int_0^1 e^{\alpha(s-1/2)} X^s A X^{1-s} \, ds.$$  

Given $\bar{\alpha} = (\alpha_j)_{j \in J}$, we define

$$[X]_{\bar{\alpha}} : \mathcal{H}_{A,J} \to \mathcal{H}_{A,J}, \quad ([X]_{\alpha_j} A_j)_{j \in J} \mapsto ([X]_{\alpha_j} A_j)_{j \in J}.$$  

Now let $\bar{\omega} = (\omega_j)_{j \in J}$ with the Bohr frequencies $\omega_j$ of $L$. For $\varepsilon \geq 0$ we write $C_{\varepsilon}(\rho_0, \rho_1)$ for the set of all pairs $(\rho, V)$ such that $\rho \in H^1([0,1]; \mathcal{G}_+(A))$ with $\rho(0) = \rho_0$, $\rho(1) = \rho_1$, $V \in L^2([0,1]; \mathcal{G}_A,J)$ and

$$\dot{\rho}(t) + \text{div} V(t) = \varepsilon \mathcal{L}_1 \rho(t)$$

for a.e. $t \in [0,1]$.  

We define a metric $\mathcal{W}_\varepsilon$ on $\mathcal{G}_+(A)$ by

$$\mathcal{W}_\varepsilon^2(\rho_0, \rho_1) = \inf_{(\rho, V) \in C_{\varepsilon}(\rho_0, \rho_1)} \int_0^1 \langle V(t), [\rho(t)]^\varepsilon \mathcal{L}_1 V(t) \rangle \, dt.$$  

A standard mollification argument shows that the infimum can equivalently be taken over $(\rho, V) \in C_{\varepsilon}(\rho_0, \rho_1)$ with $\rho \in C^\infty([0,1]; \mathcal{G}_+(A))$.  

For $\varepsilon = 0$, this is the noncommutative transport distance $\mathcal{W}$ introduced in [CM17] (as distance function associated with a Riemannian metric on $\mathcal{G}(A)_+$), and for $\varepsilon > 0$, this is the entropic regularization of $\mathcal{W}$ introduced in [BL21].  

By a substitution one can reformulate the minimization problem for $\mathcal{W}_\varepsilon$ in such a way that the constraint becomes independent from $\varepsilon$. For that purpose define the relative entropy of $\rho \in \mathcal{G}_+(A)$ with respect to $\sigma$ by

$$D(\rho \| \sigma) = \tau(\rho (\log \rho - \log \sigma))$$

and the Fisher information of $\rho \in \mathcal{G}_+(A)$ by

$$\mathcal{I}(\rho) = \langle [\rho]_{\varepsilon} \nabla (\log \rho - \log \sigma), \nabla (\log \rho - \log \sigma) \rangle_{\mathcal{H}_{A,J}}.$$  

According to [BL21, Theorem 1], one has

$$\mathcal{W}_\varepsilon^2(\rho_0, \rho_1) = \inf_{(\rho, W) \in C_{\varepsilon}(\rho_0, \rho_1)} \int_0^1 \langle (W(t), [\rho(t)]_{\varepsilon}^{-1} W(t) + \varepsilon^2 \mathcal{I}(\rho(t)) ) \rangle dt$$

$$+ 2\varepsilon (D(\rho(0) \| \sigma) - D(\rho_0 \| \sigma)).$$

The metric $\mathcal{W}$ is intimately connected to the relative entropy and therefore well-suited to study its decay properties along the quantum Markov semigroup. For other applications, variants of the metric $\mathcal{W}$ have also proven useful (e.g. [CGT18; Che+20]), for which the operator $[\rho]_{\varepsilon}$ is replaced. A systematic framework of these metrics has been developed in [Wir18; CM20]. It can be conveniently phrased in terms of so-called operator connections.

Let $H$ be an infinite-dimensional Hilbert space. A map $\Lambda : B(H)_+ \times B(H)_+ \to B(H)_+$ is called an operator connection [KA80] if

- $A \leq C$ and $B \leq D$ imply $\Lambda(A,B) \leq \Lambda(C,D)$,
- $C \Lambda(A,B) C \leq \Lambda(CAC, CBC)$,
- $A_n \succ A, B_n \succ B$ imply $\Lambda(A_n, B_n) \succ \Lambda(A,B)$.  

For example, for every $\alpha \in \mathbb{R}$ the map
\[
\Lambda_\alpha : (A, B) \mapsto \int_0^1 e^{\alpha(s^{-1/2})} A^s B^{1-s} \, ds
\]
is an operator connection.

It can be shown that every operator connection $\Lambda$ satisfies
\[
U^* \Lambda(A, B) U = \Lambda(U^* A U, U^* B U)
\]
for $A, B \in B(H)_+$ and unitary $U \in B(H)$ [KA80, Section 2]. Embedding $\mathbb{C}^n$ into $H$, one can view $A, B \in M_n(\mathbb{C})$ as bounded linear operators on $H$, and the unitary invariance of $\Lambda$ ensures that $\Lambda(A, B)$ does not depend on the embedding of $\mathbb{C}^n$ into $H$.

For $X \in \mathcal{A}$ define
\[
L(X) : \mathcal{H}_A \to \mathcal{H}_A, \quad A \mapsto X A
\]
\[
R(X) : \mathcal{H}_A \to \mathcal{H}_A, \quad A \mapsto A X.
\]

With this notation we can write
\[
\rho_\Lambda = \Lambda(L(\rho), R(\rho)).
\]

Since $L(X)$ and $R(X)$ commute, we have
\[
\Lambda(L(X), R(X)) A = \sum_{k,l=1}^n \Lambda(\lambda_k, \lambda_l) E_k A E_l
\]
for $X \in \mathcal{A}_+$ and $A \in \mathcal{H}_A$, where $(\lambda_k)$ are the eigenvalues of $X$ and $E_k$ the corresponding spectral projections.

More generally let $\tilde{\Lambda} = (\Lambda_j)_{j \in J}$ be a family of operator connections and define
\[
\rho_\Lambda = \Lambda_j(L(\rho), R(\rho)),
\]
\[
[\rho]_{\Lambda} = \bigoplus_{j \in J} [\rho]_{\Lambda_j}.
\]

Then one can define a distance $W_{\tilde{\Lambda}}$ by
\[
W_{\tilde{\Lambda}}(\rho_0, \rho_1)^2 = \inf_{(\rho, V) \in \mathcal{C}(\rho_0, \rho_1)} \int_0^1 \langle [\rho(t)]^{-1} V(t), V(t) \rangle_{\mathcal{H}_A} \, dt.
\]

If $\Lambda_j = \Lambda_{\omega_j}$ as above, then we retain the original metric $W_\varepsilon$, while for $\Lambda_j(A, B) = \frac{1}{2}(A + B)$ (and $\varepsilon = 0$) one obtains the distance studied in [CGT18; Che+20].

Later we will make the additional assumption that $\Lambda_j(A, B) = \Lambda_j(B, A)$. It follows from the representation theorem of operator means [KA80] that the class of metrics $W_{\tilde{\Lambda}}$ with $\tilde{\Lambda}$ subject to this symmetry condition is exactly the class of metrics satisfying Assumptions 7.2 and 9.5 in [CM20].

For technical reasons, it can be useful to allow for curves of density matrices that are not necessarily invertible. For this purpose, we make the following convention: If $\mathcal{K} : \mathcal{H}_A \to \mathcal{H}_A$ is a positive operator and $V \in \mathcal{H}_A$, we define
\[
\langle V, \mathcal{K}^{-1} V \rangle_{\mathcal{H}_A} = \begin{cases} 
\langle \mathcal{K} V, V \rangle_{\mathcal{H}_A} & \text{if } V \in (\ker \mathcal{K})^\perp, \mathcal{K} V = V,
\infty & \text{otherwise}.
\end{cases}
\]
Since \((\ker K)^\perp = \text{ran}K\) and \(K\) is injective on \((\ker K)^\perp\), the element \(W\) in this definition exists and is unique. Moreover, this convention is clearly consistent with the usual definition if \(K\) is invertible.

Alternatively, as a direct consequence of the spectral theorem, \(\langle V, K^{-1}V \rangle_{A,J}\) can be expressed as

\[
\langle V, K^{-1}V \rangle_{A,J} = \sum_{k=1}^{m} \frac{1}{\lambda_k} |\langle V, W_k \rangle_{A,J}|^2,
\]

where \(\lambda_1, \ldots, \lambda_m\) are the eigenvalues of \(K\) and \(W_1, \ldots, W_m\) an orthonormal basis of corresponding eigenvectors.

**Lemma 1.1.** If \(K_n: \mathcal{H}_{A,J} \to \mathcal{H}_{A,J}, n \in \mathbb{N}\), are positive invertible operators that converge monotonically decreasing to \(K\), then

\[
\langle V, K^{-1}V \rangle_{A,J} = \sup_{\delta > 0} \langle V, (K_n + \delta)^{-1}V \rangle_{A,J}
\]

for all \(V \in \mathcal{H}_{A,J}\).

**Proof.** From the spectral expression it is easy to see that

\[
\langle V, K^{-1}V \rangle_{A,J} = \sup_{\delta > 0} \langle V, (K_n + \delta)^{-1}V \rangle_{A,J}
\]

and the same for \(K_n\) replaced by \(K\). Moreover, since \(K_n \searrow K\), we have \((K_n + \delta)^{-1} \nearrow (K + \delta)^{-1}\). Thus

\[
\langle V, K^{-1}V \rangle_{A,J} = \sup_{\delta > 0} \sup_{n \in \mathbb{N}} \langle V, (K_n + \delta)^{-1}V \rangle_{A,J}
\]

Since \(\langle V, K^{-1}V \rangle_{A,J}\) is monotonically increasing, this settles the claim. \(\square\)

Write \(\mathcal{C}_+(\rho_0, \rho_1)\) for the set of all pairs \((\rho, V)\) such that \(\rho \in H^1([0,1]; \mathcal{S}(A))\) with \(\rho(0) = \rho_0, \rho(1) = \rho_1\), \(V \in L^2([0,1]; \mathcal{H}_{A,J})\) and

\[
\rho(t) + \text{div}V(t) = \varepsilon \dot{Z}^+ \rho(t)
\]

for a.e. \(t \in [0,1]\). The only difference to the definition of \(\mathcal{C}_+(\rho_0, \rho_1)\) is that \(\rho(t)\) is not assumed to be invertible.

**Proposition 1.2.** For \(\rho_0, \rho_1 \in \mathcal{S}_+(A)\) we have

\[
W_{A,x}^2(\rho_0, \rho_1) = \inf_{(\rho, V) \in \mathcal{C}_+(\rho_0, \rho_1)} \int_0^1 \langle V(t), [\rho(t)]^{-1}_A V(t) \rangle dt.
\]

**Proof.** It suffices to show that every curve \((\rho, V) \in \mathcal{C}_+(\rho_0, \rho_1)\) can be approximated by curves in \(\mathcal{C}_+(\rho_0, \rho_1)\) such that the action integrals converge.

For that purpose let

\[
\rho^\delta: [0,1] \to \mathcal{S}_+(A), t \mapsto \begin{cases} (1-t)\rho_0 + t1_A & \text{if } t \in [0,\delta], \\
(1-\delta)(1-2\delta)^{-1}(t-\delta) + \delta1_A & \text{if } t \in (\delta,1-\delta), \\
t\rho_1 + (1-t)1_A & \text{if } t \in [1-\delta,1]. \end{cases}
\]
Since \((P_1)\) is assumed to be ergodic, for \(t \in [0, \delta]\) there exists \(V^\delta(t) = \nabla U^\delta(t)\) such that
\[
\rho^\delta(t) + \text{div} V^\delta(t) = 1 - \rho_0 + \text{div} V^\delta(t) = \varepsilon(1 - t) \mathcal{L}^t \rho_0 = \varepsilon \mathcal{L}^t \rho^\delta(t)
\]
and
\[
\|V^\delta(t)\|_{\mathcal{B}_{A,J}} \leq C\|\varepsilon(1 - t) \mathcal{L}^t \rho_0 - 1 + \rho_0\|_{\mathcal{B}_{A,J}}
\]
with a constant \(C > 0\) (depending only on the spectral gap of \(\mathcal{L}\)). In particular, it is bounded independent of \(\delta\).

Moreover, if \(\lambda\) is the smallest eigenvalue of \(\rho_0\), which is strictly positive by assumption, then \(\rho^\delta(t) \geq ((1 - t)\lambda + t)1_A \geq \lambda 1_A\).

Thus
\[
\int_0^\delta \langle V^\delta(t), [\rho^\delta(t)]^{-1} V^\delta(t) \rangle_{\mathcal{B}_{A,J}} dt \leq \int_0^\delta \langle V^\delta(t), [\lambda 1_A]^{-1} V^\delta(t) \rangle_{\mathcal{B}_{A,J}} dt 
\]
\[
\leq ||[\lambda 1_A]^{-1}|| \int_0^\delta \|V^\delta(t)\|_{\mathcal{B}_{A,J}}^2 dt 
\]
\[
\to 0
\]
as \(\delta \to 0\). Similarly one can show
\[
\lim_{\delta \to 0} \int_{1-\delta}^1 \langle V^\delta(t), [\rho^\delta(t)]^{-1} V^\delta(t) \rangle_{\mathcal{B}_{A,J}} dt = 0.
\]

By the same argument as above, for a.e. \(t \in (\delta, 1-\delta)\) there exists a unique gradient \(W^\delta(t)\) such that
\[
\text{div} W^\delta(t) = -\frac{2\delta \varepsilon}{1 - 2\delta} \mathcal{L}^t \rho((1 - 2\delta)^{-1}(t - \delta))
\]
and
\[
\|W^\delta(t)\|_{\mathcal{B}_{A,J}} \leq \frac{2\delta \varepsilon}{1 - 2\delta} \|\mathcal{L}^t \rho((1 - 2\delta)^{-1}(t - \delta))\|_{\mathcal{B}_{A,J}}.
\]

Since \(\rho \in H^1([0, 1]; \mathfrak{C}(A)) \subset C([0, 1]; \mathfrak{C}(A))\), the norm on the right side is bounded independent of \(\delta\), so that
\[
\|W^\delta(t)\|_{\mathcal{B}_{A,J}} \leq \tilde{C} \delta
\]
with a constant \(\tilde{C} > 0\) independent of \(\delta\). As \(\rho^\delta(t) \geq \delta 1_A\) for \(t \in (\delta, 1-\delta)\), this implies
\[
\int_\delta^{1-\delta} \langle W^\delta(t), [\rho^\delta(t)]^{-1} W^\delta(t) \rangle_{\mathcal{B}_{A,J}} dt \leq \frac{1}{\delta} \int_\delta^{1-\delta} \langle W^\delta(t), [1_A]^{-1} W^\delta(t) \rangle_{\mathcal{B}_{A,J}} dt 
\]
\[
\leq \tilde{C} ||[1_A]^{-1}|| \delta 
\]
\[
\to 0
\]
as \(\delta \to 0\).

With
\[
V^\delta(t) = \frac{1}{1 - 2\delta} V((1 - 2\delta)^{-1}(t - \delta)) + W^\delta(t)
\]
we have
\[
\rho^\delta(t) + \text{div} V^\delta(t) = \varepsilon \mathcal{L}^t \rho^\delta(t).
\]
Furthermore,
\[
\int_{\delta}^{1-\delta} \left\langle V \left( \frac{t-\delta}{1-2\delta} \right), \left[ \rho(t) \right]^{-1}_A V \left( \frac{t-\delta}{1-2\delta} \right) \right\rangle_{\mathcal{H}_{A,J}} dt \\
= \frac{1}{1-\delta} \int_{\delta}^{1-\delta} \left\langle V \left( \frac{t-\delta}{1-2\delta} \right), \left[ \rho \left( \frac{t-\delta}{1-2\delta} + \frac{\delta}{1-\delta} \right) \right]^{-1}_A V \left( \frac{t-\delta}{1-2\delta} \right) \right\rangle_{\mathcal{H}_{A,J}} dt \\
= \frac{1-2\delta}{1-\delta} \int_0^1 \left\langle V(s), \left[ \rho(s) + \frac{\delta}{1-\delta} \right]^{-1}_A V(s) \right\rangle_{\mathcal{H}_{A,J}} ds,
\]
where we used the substitution \( s = (1-2\delta)^{-1}(t-\delta) \) in the last step.

By Lemma 1.1 and the monotone convergence theorem we obtain
\[
\lim_{\delta \to 0} \int_{\delta}^{1-\delta} \left\langle V \left( \frac{t-\delta}{1-2\delta} \right), \left[ \rho(t) \right]^{-1}_A V \left( \frac{t-\delta}{1-2\delta} \right) \right\rangle_{\mathcal{H}_{A,J}} dt = \int_0^1 \left\langle V(t), \left[ \rho(t) \right]^{-1}_A V(t) \right\rangle_{\mathcal{H}_{A,J}} dt.
\]
Together with the convergence result for \( \mathcal{W}^\delta \) from above, this implies
\[
\int_{\delta}^{1-\delta} \left\langle \mathcal{W}^\delta(t), \left[ \rho(t) \right]^{-1}_A \mathcal{W}^\delta(t) \right\rangle_{\mathcal{H}_{A,J}} dt \to \int_0^1 \left\langle \mathcal{W}(t), \left[ \rho(t) \right]^{-1}_A \mathcal{W}(t) \right\rangle_{\mathcal{H}_{A,J}} dt.
\]
Altogether we have shown
\[
\lim_{\delta \to 0} \int_0^1 \left\langle \mathcal{W}^\delta(t), \left[ \rho(t) \right]^{-1}_A \mathcal{W}^\delta(t) \right\rangle_{\mathcal{H}_{A,J}} dt = \int_0^1 \left\langle \mathcal{W}(t), \left[ \rho(t) \right]^{-1}_A \mathcal{W}(t) \right\rangle_{\mathcal{H}_{A,J}} dt. \quad \square
\]

### 2. Real subspaces

Since the proof of the main result relies on convex analysis methods for real Banach spaces, we need to identify suitable real subspaces for our purposes. For \( \mathcal{A} \) this is simply \( \mathcal{A}_0 \), but for \( \mathcal{H}_{A,J} \) this is less obvious and will be done in the following.

For \( j \in \mathcal{J} \) denote by \( j^\ast \) the unique index in \( \mathcal{J} \) such that \( V_j^* = V_j^\ast \). Let \( \tilde{\mathcal{H}}_{A,J}^{(j)} \) be the linear span of \( \{ X \partial_j A \mid A, X \in \mathcal{A} \} \), and define a linear map \( J: \tilde{\mathcal{H}}_{A,J}^{(j)} \to \tilde{\mathcal{H}}_{A,J}^{(j)} \) by
\[
J(X \partial_j A) = \partial_{j^\ast}(A^*)X^*.
\]
By the product rule, \( (\partial_j A)X \) also belongs to \( \tilde{\mathcal{H}}_{A,J}^{(j)} \) and \( J((\partial_j A)X) = X^*\partial_{j^\ast}(A^*) \).

**Lemma 2.1.** The map \( J \) is anti-unitary.

**Proof.** For \( A, B, X, Y \in \mathcal{A} \) we have
\[
\langle J(X \partial_j A), J(Y \partial_j B) \rangle_{\mathcal{H}_{A,J}} = \tau(X(AV_j - V_A)(V_j^*B^* - B^*V_j^*)Y^*) \\
= \tau((B^*V_j^* - V_j^*B^*)Y^*X(V_jA - AV_j)) \\
= \langle Y \partial_j B, X \partial_j A \rangle.
\]
Let
\[
\mathcal{H}_{A,J}^{(h)} = \{ V \in \bigoplus_{j \in \mathcal{J}} \tilde{\mathcal{H}}_{A,J}^{(j)} \mid J(V_j) = V_j^\ast \}.
\]
By the previous lemma, \( \mathcal{H}_{A,J}^{(h)} \) is a real Hilbert space.
Lemma 2.2. Let \((\Lambda_j)_{j \in J}\) be a family of operator connections such that \(\Lambda_j^* (A, B) = \Lambda_j (B, A)\) for all \(j \in J\). If \(A \in \mathcal{A}_h\) and \(\rho \in \mathcal{S}(A)\), then \(\nabla A, [\rho]_A \nabla A \in \mathcal{H}^{(h)}_{A,J}\).

Proof. For \(\nabla A\) the statement follows directly from the definitions. For \([\rho]_A \nabla A\) first note that

\[
J \Lambda (L(\rho), R(\rho)) = \Lambda (R(\rho), L(\rho)) J
\]
as a consequence of the spectral representation \((*).\)

Thus

\[
J ([\rho]_A \partial_j A) = J \Lambda_j (L(\rho), R(\rho)) \partial_j A
= \Lambda_j (R(\rho), L(\rho)) J \partial_j A
= \Lambda_j^* (L(\rho), R(\rho)) \partial_j A.
\]

\(\square\)

3. Duality

In this section we prove the duality theorem announced in the introduction. Our strategy follows the same lines as the proof in the commutative case in [EMW19]. It crucially relies on the Rockefellar–Fenchel duality theorem quoted below. Throughout this section we fix a quantum Markov semigroup with generator \(\mathcal{L}\) satisfying the \(\sigma\)-DBC for some \(\sigma \in \mathcal{S}_+(A)\) and a family \((\Lambda_j)_{j \in J}\) of operator connections such that \(\Lambda_j^* (A, B) = \Lambda_j (B, A)\) for all \(j \in J\).

We need the following definition for the constraint of the dual problem. Here and in the following we write

\[
\langle V, W \rangle_\rho = \langle V, [\rho]_A W \rangle_{\mathcal{H}^{(h)}_{A,J}}
\]

for \(V, W \in \mathcal{S}_{A,J}\) and \(\rho \in \mathcal{A}_+\).

Definition 3.1. A function \(A \in H^1((0, T); \mathcal{A}_h)\) is said to be a Hamilton–Jacobi–Bellmann subsolution if for a.e. \(t \in (0, T)\) we have

\[
\tau((\dot{A}(t) + \epsilon \mathcal{L} A(t)) \rho) + \frac{1}{2} \|\nabla A(t)\|_{\rho}^2 \leq 0 \quad \text{for all } \rho \in \mathcal{S}(A).
\]

The set of all Hamilton–Jacobi–Bellmann subsolutions is denoted by \(\text{HJB}_{\mathcal{A},\epsilon}\).

Our proof will establish equality between the primal and dual problem, but before we begin, let us show that one inequality is actually quite easy to obtain.

Proposition 3.2. For all \(\rho_0, \rho_1 \in \mathcal{S}_+(A)\) we have

\[
\inf_{A \in \text{HJB}_{\mathcal{A},\epsilon}} \sup_{\mathcal{L} \in \text{CE}(\rho_0, \rho_1)} \tau(A(1) \rho_1 - A(1) \rho_0) \leq \frac{1}{2} \frac{1}{(\rho, V) \in \text{CE}(\rho_0, \rho_1)} \int_0^1 \langle V(t), [\rho(t)]_A^{-1} V(t) \rangle_{\mathcal{H}^{(h)}_{A,J}} dt.
\]
Proof. For \( A \in \text{HJB}_{\Lambda,\varepsilon} \) and \((\rho, \mathbf{V}) \in \text{CE}_{\varepsilon}(\rho_0, \rho_1)\) we have

\[
\tau(A(1)\rho - A(0)\rho_0) = \int_0^1 \tau(\dot{A}(t)\rho(t) + A(t)\dot{\rho}(t)) \, dt \\
\leq - \int_0^1 (\varepsilon\tau((\mathcal{L}A(t))\rho(t)) + \frac{1}{2}\|\nabla A(t)\|_{\rho(t)}^2) \, dt \\
+ \int_0^1 ((\nabla A(t), \mathbf{V}(t))_{\mathcal{B}_A} + \varepsilon\tau(A(t)\mathcal{L}^\dagger \rho(t))) \, dt \\
= \int_0^1 ([\rho(t)]^{1/2}_A \nabla A(t), [\rho(t)]^{-1/2}_A \mathbf{V}(t))_{\mathcal{B}_A} \, dt \\
- \frac{1}{2} \int_0^1 ([\rho(t)]^{1/2}_A \nabla A(t), [\rho(t)]^{1/2}_A \nabla A(t))_{\mathcal{B}_A} \, dt \\
\leq \frac{1}{2} \int_0^1 (\mathbf{V}(t), [\rho(t)]^{-1}_A \mathbf{V}(t))_{\mathcal{B}_A} \, dt,
\]

where we used \( A \in \text{HJB}_{\Lambda,\varepsilon} \) and \((\rho, \mathbf{V}) \in \text{CE}_{\varepsilon}(\rho_0, \rho_1)\) for the first inequality and Young’s inequality for the second inequality. \(\square\)

To prove actual equality, our crucial tool is the Rockefellar–Fenchel duality theorem (see e.g. [Vil03, Theorem 1.9]), which we quote here for the convenience of the reader. Recall that if \( E \) is a (real) normed space, the Legendre–Fenchel transform \( F^* \) of a proper convex function \( F : E \to \mathbb{R} \cup \{\infty\} \) is defined by

\[
F^* : E^* \to \mathbb{R} \cup \{\infty\}, \quad F^*(x^*) = \sup_{x \in E} \langle x^*, x \rangle - F(x).
\]

**Theorem 3.3.** Let \( E \) be a real normed space and \( F, G : E \to \mathbb{R} \cup \{\infty\} \) proper convex functions with Legendre–Fenchel transforms \( F^*, G^* \). If there exists \( z_0 \in E \) such \( F \) is continuous at \( z_0 \) and \( F(z_0), G(z_0) < \infty \), then

\[
\sup_{z \in E} (-F(z) + G(z)) = \min_{z^* \in E^*} (F^*(z^*) + G^*(-z^*)).
\]

Before we state the main result, we still need the following useful inequality.

**Lemma 3.4.** For any operator connection \( \Lambda \) the map

\[
f_\Lambda : A_{++} \to B(\mathfrak{H}_\Lambda), \ A \mapsto [A]_\Lambda
\]

is smooth and its Fréchet derivative satisfies

\[
df_\Lambda(B)A \geq f_\Lambda(A)
\]

for \( A, B \in A_{++} \) with equality if \( A = B \).

**Proof.** Smoothness of \( f_\Lambda \) is a consequence of the representation theorem of operator connections [Theorem 3.4][KA80]. For the claim about the Fréchet derivative first note that \( f_\Lambda \) is concave [KA80, Theorem 3.5]. Therefore \( d^2 f_\Lambda(X)[Y, Y] \leq 0 \) for all \( X \in A_{++} \) and \( Y \in A_h \) by [Han97, Proposition 2.2].

The fundamental theorem of calculus implies

\[
df_\Lambda(A) - df_\Lambda(B))(A - B) = \int_0^1 d^2 f_\Lambda(tA + (1 - t)B)[A - B, A - B] \, dt \\
\leq 0.
\]
Theorem 3.5 (Duality formula). For \( \rho_0, \rho_1 \in \mathcal{S}_+(A) \) we have
\[
\frac{1}{2} W_{\mathcal{K}, \varepsilon}(\rho_0, \rho_1)^2 = \sup \{ \tau(A(1)\rho_1) - \tau(A(0), \rho_0) : A \in \mathcal{H}_{\mathcal{K}, \varepsilon} \} \\
= \sup \{ \tau(A(1)\rho_1) - \tau(A(0), \rho_0) : A \in \mathcal{H}_{\mathcal{K}, \varepsilon} \cap C^\infty([0, 1]; A) \}.
\]

Proof. The second inequality follows easily by mollifying. We will show the duality formula for Hamilton–Jacobi subsolutions in \( H^1 \). For this purpose we use the Rockefellar–Fenchel duality formula from Theorem 3.3.

Let \( E \) be the real Banach space
\[
H^1([0, 1]; \mathcal{H}_A^{(h)}) \times L^2([0, 1]; \mathcal{H}_A^{(h)}).
\]

By the theory of linear ordinary differential equations, the map
\[
H^1([0, 1]; \mathcal{H}_A^{(h)}) \to \mathcal{H}_A^{(h)} \times L^2([0, 1]; \mathcal{H}_A^{(h)}), \ A \mapsto (A(0), \dot{A} + \varepsilon \mathcal{L} A)
\]
is a linear isomorphism.

Thus the dual space \( E^* \) can be isomorphically identified with
\[
\mathcal{H}_A^{(h)} \times L^2([0, 1]; \mathcal{H}_A^{(h)}) \times L^2([0, 1]; \mathcal{H}_A^{(h)}),
\]
via the dual pairing
\[
\langle (A, V), (B, C, W) \rangle = \tau(A(0)B) + \int_0^1 \tau((\dot{A}(t) + \varepsilon \mathcal{L} A(t))C(t)) \, dt \\
+ \int_0^1 \langle V(t), W(t) \rangle_{\mathcal{H}_A, \mathcal{J}} \, dt.
\]

Define functionals \( F, G : E \to \mathbb{R} \cup \{ \infty \} \) by
\[
F(A, V) = \begin{cases} 
-\tau(A(1)\rho_1) + \tau(A(0)\rho_0) & \text{if } V = \nabla A, \\
\infty & \text{otherwise},
\end{cases}
\]
\[
G(A, V) = \begin{cases} 
0 & \text{if } (A, V) \in \mathcal{D}, \\
\infty & \text{otherwise}.
\end{cases}
\]

Here
\[
\mathcal{D} = \{(A, V) : \tau((\dot{A}(t) + \varepsilon \mathcal{L} A(t))\rho) + \frac{1}{2}\|V(t)\|^2_\rho \leq 0 \text{ for all } t \in [0, 1], \rho \in \mathcal{S}(A)\}.
\]

It is easy to see that \( F \) and \( G \) are convex. Moreover, for \( A_0(t) = -t1_A \) and \( V_0 = 0 \) we have \( V_0 = \nabla A_0 \), hence \( F(A_0, V_0) = 0 \), and
\[
\tau((\dot{A}_0(t) + \varepsilon \mathcal{L} A_0(t))\rho) + \frac{1}{2}\|V_0(t)\|^2_\rho = -1
\]
for all \( t \in [0, 1] \), \( \rho \in \mathcal{S}(A) \), hence \( G(A_0, V_0) = 0 \). Furthermore, \( G \) is clearly continuous at \( (A_0, V_0) \).
Moreover,
\[
\sup_{(A, V) \in \mathcal{E}} (-F(A, V) - G(A, V)) = \sup_{A \in \text{HJB}_{\mathcal{E}}(\rho_0, \rho_1)} (\tau(A(1)\rho_1) - \tau(A(0)\rho_0)).
\]

Let us calculate the Legendre transforms of $F$ and $G$, keeping in mind the identification of $E^*$. For $F$ we obtain
\[
F^*(B, C, W) = \sup_{(A, V) \in \mathcal{E}} \left\{ \langle (A, V), (B, C, W) \rangle - F(A, V) \right\} = \sup_A \left\{ \tau(A(0)B) + \int_0^1 \tau((\dot{A}(t) + \varepsilon \mathcal{L}A(t))C(t)) dt + \int_0^1 \langle \nabla A(t), W(t) \rangle_{\mathcal{H}_{A,t}} dt + \tau(A(1)\rho_1) - \tau(A(0)\rho_0) \right\}.
\]

Since the last expression is homogeneous in $A$, we have $F^*(B, C, W) = \infty$ unless
\[
-\tau(A(1)\rho_1) + \tau(A(0)(\rho_0 - B)) = \int_0^1 \tau((\dot{A}(t) + \varepsilon \mathcal{L}A(t))C(t)) dt + \int_0^1 \langle \nabla A(t), W(t) \rangle_{\mathcal{H}_{A,t}} dt
\]
for $A \in H^1([0, 1] ; \mathcal{H}_A^{(h)}).$

This implies $C_0 = -(\rho_0 - B)$ and $C_1 = -\rho_1$ and
\[
\dot{C}(t) + \text{div} W(t) = \varepsilon \mathcal{L}^t C(t).
\]

Thus
\[
F^*(B, C, W) = \begin{cases} 0 & \text{if } (-C, -W) \in \mathcal{CE}''_A(\rho_0 - B, \rho_1), \\ \infty & \text{otherwise}. \end{cases}
\]

Here $\mathcal{CE}''_A(\rho_0 - B, \rho_1)$ denotes the set of all pairs $(X, U) \in H^1((0, 1) ; \mathcal{H}_A^{(h)}) \times L^2((0, 1) ; \mathcal{H}_A^{(h)})$ satisfying $X(0) = \rho_0 - B$, $X(1) = \rho_1$ and
\[
\dot{X}(t) + \text{div} U(t) = \varepsilon \mathcal{L}^t X(t).
\]

The difference to the definitions of $\mathcal{CE}$ (or $\mathcal{CE}'$) and $\mathcal{CE}''$ is that we do not make any positivity or normalization constraints. Note however that if $(X, U) \in \mathcal{CE}''(\rho_0 - B, \rho_1)$, then
\[
\frac{d}{dt} \tau(X(t)) = \tau(\varepsilon \mathcal{L}^t X(t) - \text{div} U(t)) = 0
\]
so that $\tau(X(t)) = \tau(\rho_1) = 1$ (and $\tau(B) = 0$).

Now let us turn to the Legendre transform of $G$. We have
\[
G^*(B, C, W) = \sup_{(A, V) \in \mathcal{E}} \left\{ \langle (A, V), (B, C, W) \rangle - G(A, V) \right\} = \sup_{(A, V) \in \mathcal{D}} \left\{ \tau(A(0)B) + \int_0^1 \tau((\dot{A}(t) + \varepsilon \mathcal{L}A(t))C(t)) dt + \int_0^1 \langle V(t), W(t) \rangle_{\mathcal{H}_{A,t}} dt \right\}.
\]
Since $(A, V) \in \mathcal{D}$ implies $(A + X, V) \in \mathcal{D}$ for all $X \in \mathcal{A}$, we have $G^*(B, C, V) = \infty$ unless $B = 0$. Moreover, it follows from the definition of $\mathcal{D}$ that $G^*(0, C, W) = \infty$ unless $C(t) \geq 0$ for a.e. $t \in [0, 1]$.

For $B = 0$ we have

$$G^*(0, C, W) = \sup_{(A, V) \in \mathcal{D}} \left\{ \int_0^1 (\tau((\dot{A}(t) + \varepsilon L(A(t))C(t)) + \langle V(t), W(t) \rangle_{\mathcal{H}_A, J}) dt \right\}$$

$$\leq \sup_{(A, V) \in \mathcal{D}} \left\{ -\int_0^1 \frac{1}{2} \|V(t)\|^2_{C(t)} dt \
+ \int_0^1 \langle (|C(t)|^{-1/2} V(t), [C(t)]^{-1/2} W(t))_{\mathcal{H}_A, J} dt \right\}$$

$$\leq \frac{1}{2} \int_0^1 \langle (|C(t)|^{-1/2} W(t), W(t))_{\mathcal{H}_A, J} dt.$$

We will show next that the inequalities are in fact equalities. Let $C^\delta = C + \delta$ and $V^\delta(t) = [C(t)^\delta]^{-1} W(t)$. Moreover, let $f_j = f_{A, \delta}$ with the notation from Lemma 3.4.

Since

$$\mathcal{H}_A^{(h)} \rightarrow \mathbb{R}, B \mapsto \sum_{j \in J} \langle (df_j(C^\delta(t))B)V^\delta_j(t), V^\delta_j(t) \rangle_{\mathcal{H}_A}$$

is a bounded linear map that depends continuously on $t$, there exists a unique continuous map $X^\delta: [0, 1] \rightarrow \mathcal{H}_A^{(h)}$ such that

$$\tau(BX^\delta(t)) = \sum_{j \in J} \langle (df_j(C^\delta(t))B)V^\delta_j(t), V^\delta_j(t) \rangle_{\mathcal{H}_A}$$

for every $B \in \mathcal{H}_A^{(h)}$ and $t \in [0, 1]$.

Let

$$A^\delta: [0, 1] \rightarrow \mathcal{A}_k, A^\delta(t) = -\frac{1}{2} \int_0^t X^\delta(s) ds.$$

We claim that $(A^\delta, V^\delta) \in \mathcal{D}$. Indeed,

$$\tau(\dot{A}^\delta(t) \rho) = -\frac{1}{2} \sum_{j \in J} \langle (df_j(C^\delta(t))\rho)V^\delta_j(t), V^\delta_j(t) \rangle_{\mathcal{H}_A}$$

$$\leq -\frac{1}{2} \sum_{j \in J} \langle \rho_{\mathcal{A}}, V^\delta_j(t), V^\delta_j(t) \rangle_{\mathcal{H}_A}$$

$$= -\frac{1}{2} \|V^\delta(t)\|^2_{\rho},$$

where the inequality follows from Lemma 3.4. Note that we have equality for $\rho = C^\delta(t)$.

In particular, for $\rho = C(t)$ we obtain

$$\tau(\dot{A}^\delta(t)C(t)) + \langle V^\delta(t), W(t) \rangle_{\mathcal{H}_A, J} \leq \frac{1}{2} \langle [C(t)]^{-1} W(t), W(t) \rangle_{\mathcal{H}_A, J}.$$. 
On the other hand,
\[
\tau(\dot{A}^\delta(t) C(t)) = \frac{1}{2} \sum_{j \in \mathcal{J}} \langle (df_j(C^\delta(t)))(C^\delta(t) - \delta)\rangle V_j^\delta(t), V_j^\delta(t) \rangle_{\mathcal{B}_A} \\
\quad \geq -\frac{1}{2} \langle [C^\delta(t)]_A V^\delta(t), V^\delta(t) \rangle_{\mathcal{B}_A} + \frac{1}{2} \langle [\delta]_A V^\delta(t), V^\delta(t) \rangle_{\mathcal{B}_A,J} \\
\quad \geq -\frac{1}{2} \langle V^\delta(t), W(t) \rangle_{\mathcal{B}_A,J},
\]
where we again used Lemma 3.4 for the first inequality.

Put together, we have
\[
\frac{1}{2} \langle [C(t)]_A^{-1} W(t), W(t) \rangle_{\mathcal{B}_A,J} \leq \tau(\dot{A}^\delta(t) C(t)) + \langle V^\delta(t), W(t) \rangle_{\mathcal{B}_A,J} \\
\quad \leq \frac{1}{2} \langle [C(t)]_A^{-1} W(t), W(t) \rangle_{\mathcal{B}_A,J},
\]
and
\[
\int_0^1 (\tau(\dot{A}^\delta(t) C(t)) + \langle V^\delta(t), W(t) \rangle_{\mathcal{B}_A,J}) \, dt \to \frac{1}{2} \int_0^1 \langle [C(t)]_A^{-1} W(t), W(t) \rangle_{\mathcal{B}_A,J} \, dt
\]
follows from the monotone convergence theorem.

Hence
\[
G^*(0, C, W) = \frac{1}{2} \int_0^1 \langle [C(t)]_A^{-1} W(t), W(t) \rangle_{\mathcal{B}_A,J} \, dt
\]
if \( C(t) \geq 0 \) for a.e. \( t \in [0, 1] \). Together with the formula for \( F^* \), we obtain
\[
\inf_{(B, C, W) \in \mathcal{E}^*} (F^*(-B, -C, -W) + G^*(B, C, W))
\]
\[
= \inf_{(\rho, \mathcal{V}) \in \mathcal{C}_E(\rho_0, \rho_1)} \frac{1}{2} \int_0^1 \langle [\rho(t)]_A^{-1} \mathcal{V}(t), \mathcal{V}(t) \rangle_{\mathcal{B}_A,J} \, dt
\]
\[
= \frac{1}{2} W_{A,c}^2(\rho_0, \rho_1),
\]
where the last equality follows from Proposition 1.2.

An application of the Rockefellar–Fenchel theorem yields the desired conclusion.

\[\square\]

References

[AES16] Luigi Ambrosio, Matthias Erbar, and Giuseppe Savaré. Optimal transport, Cheeger energies and contractivity of dynamic transport distances in extended spaces. In: Nonlinear Anal. 137 (2016), pp. 77–134.

[Ali76] Robert Alicki. On the detailed balance condition for non-Hamiltonian systems. In: Rep. Math. Phys. 10.2 (1976), pp. 249–258.

[BB00] Jean-David Benamou and Yann Brenier. A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem. In: Numer. Math. 84.3 (2000), pp. 375–393.

[BGL01] Sergey G. Bobkov, Ivan Gentil, and Michel Ledoux. Hypercontractivity of Hamilton-Jacobi equations. In: J. Math. Pures Appl. (9) 80.7 (2001), pp. 669–696.

[BL21] Simon Becker and Wuchen Li. Quantum Statistical Learning via Quantum Wasserstein Natural Gradient. In: Journal of Statistical Physics 182.1 (2021).
[BV20] Yann Brenier and Dmitry Vorotnikov. On optimal transport of matrix-valued measures. In: *SIAM J. Math. Anal.* 52.3 (2020), pp. 2849–2873.
[CGT18] Yongxin Chen, Tryphon T. Georgiou, and Allen Tannenbaum. Matrix optimal mass transport: a quantum mechanical approach. In: *IEEE Trans. Automat. Control* 63.8 (2018), pp. 2612–2619.
[Che+20] Yongxin Chen, Wilfrid Gangbo, Tryphon T. Georgiou, and Allen Tannenbaum. On the matrix Monge-Kantorovich problem. In: *European J. Appl. Math.* 31.4 (2020), pp. 574–600.
[CM14] Eric A. Carlen and Jan Maas. An analog of the 2-Wasserstein metric in non-commutative probability under which the fermionic Fokker-Planck equation is gradient flow for the entropy. In: *Comm. Math. Phys.* 331.3 (2014), pp. 887–926.
[CM17] Eric A. Carlen and Jan Maas. Gradient flow and entropy inequalities for quantum Markov semigroups with detailed balance. In: *J. Funct. Anal.* 273.5 (2017), pp. 1810–1869.
[CM20] Eric A. Carlen and Jan Maas. Non-commutative Calculus, Optimal Transport and Functional Inequalities in Dissipative Quantum Systems. In: *J. Stat. Phys.* 178.2 (2020), pp. 319–378.
[DPT21] Giacomo De Palma and Dario Trevisan. Quantum Optimal Transport with Quantum Channels. In: *Annales Henri Poincaré* (2021).
[DR20] N. Datta and C. Rouzé. Relating Relative Entropy, Optimal Transport and Fisher Information: A Quantum HWI Inequality. In: *Ann. Henri Poincaré* (2020).
[Duv20] Rocco Duvenhage. Quadratic Wasserstein metrics for von Neumann algebras via transport plans. In: *arXiv e-prints* 2012.03564 (2020).
[EMW19] Matthias Erbar, Jan Maas, and Melchior Wirth. On the geometry of geodesics in discrete optimal transport. In: *Calc. Var. Partial Differential Equations* 58.1 (2019), Art. 19, 19.
[GLM19] Wilfrid Gangbo, Wuchen Li, and Chenchen Mou. Geodesics of minimal length in the set of probability measures on graphs. In: *ESAIM Control Optim. Calc. Var.* 25 (2019), Paper No. 78, 36.
[GMP16] François Golse, Clément Mouhot, and Thierry Paul. On the mean field and classical limits of quantum mechanics. In: *Comm. Math. Phys.* 343.1 (2016), pp. 165–205.
[Han97] Frank Hansen. Operator convex functions of several variables. In: *Publ. Res. Inst. Math. Sci.* 33.3 (1997), pp. 443–463.
[Hor18] David F. Hornshaw. $L^2$-Wasserstein distances of tracial $W^*$-algebras and their disintegration problem. In: *arXiv e-prints* 1806.01073 (June 2018).
[KA80] Fumio Kubo and Tsuyoshi Ando. Means of positive linear operators. In: *Math. Ann.* 246.3 (1980), pp. 205–224.
[Kan42] Leonid Kantorovich. On the translocation of masses. In: *C. R. (Doklady) Acad. Sci. URSS (N.S.)* 37 (1942), pp. 199–201.
[MM17] Markus Mittenzwieg and Alexander Mielke. An entropic gradient structure for Lindblad equations and couplings of quantum systems to macroscopic models. In: *J. Stat. Phys.* 167.2 (2017), pp. 205–233.
[NGT15] Lipeng Ning, Tryphon T. Georgiou, and Allen Tannenbaum. On matrix-valued Monge-Kantorovich optimal mass transport. In: IEEE Trans. Automat. Control 60.2 (2015), pp. 373–382.

[OV00] Felix Otto and Cédric Villani. Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality. In: J. Funct. Anal. 173.2 (2000), pp. 361–400.

[Pal+20] Giacomo De Palma, Milad Marvian, Dario Trevisan, and Seth Lloyd. The quantum Wasserstein distance of order 1. In: arXiv e-prints 2009.04469 (2020).

[Pey+19] Gabriel Peyré, Lénaïc Chizat, François-Xavier Vialard, and Justin Solomon. Quantum entropic regularization of matrix-valued optimal transport. In: European J. Appl. Math. 30.6 (2019), pp. 1079–1102.

[RD19] Cambyse Rouzé and Nilanjana Datta. Concentration of quantum states from quantum functional and transportation cost inequalities. In: J. Math. Phys. 60.1 (2019), pp. 012202, 22.

[Vil03] Cédric Villani. Topics in optimal transportation. Vol. 58. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2003, pp. xvi+370.

[Vil09] Cédric Villani. Optimal transport. Old and new. Vol. 338. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 2009, pp. xxii+973.

[Wir18] Melchior Wirth. A Noncommutative Transport Metric and Symmetric Quantum Markov Semigroups as Gradient Flows of the Entropy. In: arXiv e-prints 1808.05419 (Aug. 2018).

[WZ20] Melchior Wirth and Haonan Zhang. Complete gradient estimates of quantum Markov semigroups. In: arXiv e-prints 2007.13506 (2020).