Attractor metadynamics in terms of target points in slow-fast systems: adiabatic versus symmetry protected flow in a recurrent neural network

Hendrik Wernecke and Claudius Gros

Abstract

In dynamical systems with distinct time scales the time evolution in phase space may be influenced strongly by the fixed points of the fast subsystem. Orbits then typically follow these points, performing in addition rapid transitions between distinct branches on the time scale of the fast variables. As the branches guide the dynamics of a system along the manifold of former fixed points, they are considered transiently attracting states and the intermittent transitions between branches correspond to state switching within transient-state dynamics. A full characterization of the set of former fixed points, the critical manifold, tends to be difficult in high-dimensional dynamical systems such as large neural networks. Here we point out that an easily computable subset of the critical manifold, the set of target points, can be used as a reference for the investigation of high-dimensional slow-fast systems. The set of target points corresponds in this context to the adiabatic projection of a given orbit to the critical manifold. Applying our framework to a simple recurrent neural network, we find that the scaling relation of the Euclidean distance between the trajectory and its target points with the control parameter of the slow time scale allows to distinguish an adiabatic regime from a state that is effectively independent from target points.

1. Introduction

Coexisting fixed point attractors such as place cells [1] are commonly assumed to strongly influence cognitive processing in the brain, either alone [2] or in conjunction with feed–forward processing, with the latter being the case for the episodic memory [3]. A system characterized by a single fixed point attractor could however not be functional on its own, as it would depend on additional mechanisms to reset the dynamics. It is hence interesting, that neural activity characterized by transitions between multiple meta–stable attractors [4] has been discovered in the olfactory system of zebrafish [5] and in the gustatory cortex [6]. Similar transient state dynamics [7] is also found in resting state networks in low–frequency contributions of human fMRI data [8], where it enables processes that are associated with cognitive tasks even in the complete absence of external stimuli [9]. Resting–state brain networks also show complex spatio–temporal dynamics, in terms of transitions between states characterized by high and low functional connectivities, which resemble transiently existing attractor structures [10]. Such state–dependent fluctuations may play an important role in task–related brain computations [11] such as the interaction of motion and sensation (cf [12]).

Dynamics involving switching transitions between transiently stable states has been addressed in the contexts of semantic learning in autonomously active networks [13, 14], within reservoir computing [15] and in networks dominated by heteroclinic orbits [16]. In case of the latter, periodic orbits are formed when the dynamics follows heteroclinic connections between saddle points encoding information in the different states, which however exist only for symmetry–invariant networks. The details of the internal dynamics are on the
other hand more difficult to analyze for the case of reservoir computing [17], for which large random network layouts are generically used.

It is well known that transiently attracting states are present in slow–fast systems [18–20], i.e. in dynamical systems with distinct time scales. The fixed points of the fast subsystem, which are formally destroyed by the slower subsystem, form in this case a manifold that appears under several distinct names in the literature, such as slow manifold [18] and critical manifold [19]. This manifold corresponds to transiently stable states, and it has a pronounced influence on the overall dynamics, which takes place mostly in the close vicinity to this manifold.

The switching dynamics of low dimensional slow–fast dynamical systems can be addressed analytically by singular perturbation theory [19, 21], which has been used to investigate a number of relaxation oscillators [22], such as the van–der–Pol oscillator and the FitzHugh–Nagumo model for the dynamics of a single neuron [20, 23, 24]. Further applications of singular perturbation theory include the study of small Hopfield–type neural networks [25], the analysis of the motion on the critical manifold [26], of the detailed influence of different time scales [27], and of the scaling behavior close to bifurcations [28, 29].

It is generally accepted that a complete hierarchy of time scales [30] is necessary to describe the different tasks and activity levels in the brain. Examples are working memory, short–term and long–term memory and a range of distinct types of neural plasticity [31]. Functional hierarchy in recurrent neural networks for motion tasks [32] is another phenomenon emerging from a separation of time scales.

1.1. Adiabatic fixed points
We will start this study by considering attractor ruins [33] in slow–fast systems, i.e. systems that compose of a fast subsystem and an additional slow subsystem, both being coupled. Attractor ruins are the remnants of points that are fixed points in the fast subsystem, but stop being fixed points (i.e., they get ‘destroyed’ or ‘ruined’) by the interaction with the additional slow process (or by noise in noisy setups).

Such states can be found in literature under various names, such as attractor relics [33, 34], transiently attracting states [34] and ghosts [35]. Attractor ruins are to be found on the critical manifold and are hence transiently attracting, but not asymptotically stable. They will strongly influence the dynamics of a system whenever the coupling to the perturbation is weak.

Attractor ruins are defined on the level of the full system, being however embedded in the phase space of the fast (decoupled) variables. Hence, as it has been recognized previously [33, 34], the evolution of attractor ruins can be cast in terms of a metadynamics of attractors, which is in turn important for the modeling of cognitive processes like memory or decision making.

The aim of this study is to investigate the effect of attractor ruins on the dynamics in the overall system. Therefore we present an efficient measure to determine the influence of the attractor ruins and distinguish different dynamical regimes by that.

Furthermore, we introduce here the generic term adiabatic fixed point (AFP), denoting all fixed points of the fast system when the slow variables are fixed but otherwise arbitrary. All AFP form, by definition, the critical manifold. We note, however, that there are stable and unstable AFP.

1.2. A reference manifold on the critical manifold
Most of the stable adiabatic fixed points have in general nothing to do with the long–term behavior of the system. We will define in this context the unique set of target points as a subset of AFP corresponding to a given trajectory. We shall concentrate on trajectories that evolve on an attractor of the full system, i.e., on periodic or chaotic orbits of the full system that are reached after prolonged times. For an arbitrary locus on such a trajectory, a target point will be defined as the corresponding point that would be approached under the time evolution with fixed slow variables.

As the shape of the manifold of target points only depends on the fast subsystem and the trajectory it corresponds to, this manifold can be used as a reference manifold for the motion of the full system. Since slow variables in real world applications are not infinitesimally slow, the distance between the trajectory and the respective target points, being larger than zero, is an important measure for the assessment of the influence a target point and hence the critical manifold as a whole exerts on the dynamics. The critical manifold is both uniquely defined and numerically accessible; the manifold of target points, being a subset of the critical manifold, is also uniquely defined, as it corresponds to a certain trajectory. Therefore we will use the manifold of target points in this approach as a low dimensional reference manifold to characterize the corresponding dynamics in a potentially high dimensional phase space.

1.3. Adiabatic and non–adiabatic regimes
Analyzing the distance between a given orbit and its associated target points we discriminate two qualitatively different dynamical regimes:
• In the adiabatic regime the dynamics is effectively influenced by target points, with the average distance of the trajectory to the target points scaling with the ratio of the slow and the fast time scale. We emphasize that the dynamics of the slow subsystem does not vanish in the adiabatic regime. The term relates rather to a regime of parameters, for which the trajectory follows the target points (which are adiabatic fixed points) closely, interseeded with jumps between different branches of the critical manifold. The corresponding different branches of the manifold of target points represent different transiently attracting states.

• In the non–adiabatic regime the dynamics effectively decouples from the target points, with the distance to the target points being in essence invariant with respect to changes of the relative time scale.

One may discriminate equivalently between regimes in which a perturbation expansion in the slow time scale converges or diverges, respectively [19]. Our approach aims on the other hand to develop tools suitable for the numerical study of high–dimensional slow–fast systems. We find that the study of target points, which are straightforward to determine numerically, allows in this context to distinguish between the adiabatic and the non–adiabatic regime.

1.4. Three–site recurrent neural networks
In the second part of this study we apply our framework to a network of continuous–time rate–encoding neurons, which has been shown previously to exhibit non–trivial dynamical states [36]. As we are mainly interested in how AFP influence the intrinsic behavior of the network, we restrict ourselves to autonomously active networks, i.e. networks without external input. A separation between the time scale of the membrane potential and an intrinsic neural parameter, the threshold, is present in this system.

For concreteness we study a three–neuron network allowing for an in–depth understanding of the resulting dynamical states. Computing the transiently attracting states (as connected sets of target points) and the distance measures quantifying the influence of target points on the actual dynamics, leads to the characterization of two distinct dynamical regimes, i.e. the adiabatic and the non–adiabatic regime. We conclude with an analysis of the transition between these two regimes and of the role of target points for chaotic motion.

2. Theory—Attractor metadynamics in slow–fast dynamical systems

We consider with

\[ x_i = f_i(x, b), \quad \dot{b}_j = \epsilon_{b} g_j(x, b) \quad (1) \]

\[ N + M \text{ dimensional slow–fast dynamical systems, where we have denoted with } x = x(t) = (x_i(t))_{1 \leq i \leq N} \text{ the fast variables and with } b = b(t) = (b_j(t))_{1 \leq j \leq M} \text{ the slow variables. The slow time scale is set by } \epsilon_{b}, \text{ which is also the ratio of time scales.} \]

The set of fixed points in the fast subsystem, i.e. \( \frac{dx}{dt} = 0 \) for given and fixed \( b \), is of special interest for the analysis of slow–fast dynamical systems. The entirety of these points, which are the intersections of the nullclines \( \frac{dx_i}{dt} = 0 \) of the fast subsystem [37], are generally termed slow manifold [18] or critical manifold [19]. Please note that the term slow manifold is used in different contexts and fields and also with different definitions as already pointed out by Lorenz [38] (cf 18, 19, 21). In the context of this work we will refer to the set of fixed points in the fast subsystem as critical manifold, as this nomenclature is widely used in the field. Here we constrain ourselves to a fast subsystem with only fixed point attractors. This is for instance the case if the fast subsystem is a gradient system, e.g. when the fast dynamics is derived from a generating functional [36, 39]. The different branches of the critical manifold are therefore exclusively composed of isolated fixed points. They are often well characterized, for real–world systems, in terms of their physical and/or neurobiological properties [8].

2.1. Adiabatic fixed points (AFP) and target points
In physics terminology the limit \( \epsilon_{b} \to 0 \), i.e. when the slow subsystem is infinitely slow, is termed the adiabatic limit, such as in the Born–Oppenheimer approximation [40], where the slow movement of the atomic nuclei can be treated as parametric variables when addressing the relatively fast dynamics of the electrons. In the case \( \epsilon_{b} = 0 \) the configuration \( b \) of the slow subsystem is constant and can be treated as parameters. We will hence use the term adiabatic fixed point (AFP) for the fixed points of the fast subsystem. Note that the set of all AFP is equivalent to the critical manifold [19] of the system.

Considering a generic state \( \{x_0, b_0\} \) as the starting point of a solution \( x(t')|_{b=b_0} \) of equation (1) for fixed slow variables \( b = b_0 \) we define with
the fast component of the target point \( \{x_f(x_{0b}, b_0), b_0\} \) corresponding to the respective starting point. Here we take \( x_0 = x(t_0) \) and \( b_0 = b(t_0) \) to be the fast and respectively slow components of a point on a trajectory \( \{x(t), b(t)\} \) in the full system, as parameterized by the time \( t = t_0 \). The mapping (2) is unique for a given pair \( (x_0, b_0) \), depending furthermore not on \( \epsilon_b \), which is the ratio of the time scales. All target points are AFP and a set of target points corresponding to a given trajectory is a subset of the critical manifold. This set of target points can thus be used as a reference manifold that allows to analyze the dynamics of the overall system relative to the critical manifold.

The relation of target points, trajectory and critical manifold is sketched schematically in figure 1(b). We also present in table 1 (cf appendix A) a pseudo–code for the numerical computation of AFP and target points.

2.2. Kinetic energy of phase space evolution

Target points can be computed via a straightforward evolution of the equations of motion of the fast subsystem for fixed slow variables \( b \). Unstable adiabatic fixed points may be found [35], in addition, by minimizing the kinetic energy

\[
q_x = \frac{\dot{x}^2}{2} = \frac{1}{2} \sum_{i=1}^{N} \dot{x}_i^2
\]

of the flow in the fast subsystem. A minimum with \( q_x = 0 \) corresponds to a fixed point of the fast subsystem, regardless of its stability. This is used to compute AFP numerically (cf appendix A).

The zeros of \( q_x \) are generically not minima of the kinetic energy \( q = (\dot{x}^2 + b^2)/2 \) of the full system, for which also the flow in the slow subsystem, \( b \), needs to vanish. The same holds for slow points, which are defined by a vanishing \( q_s \) and small values of \( q \), see [35].

2.3. Attractor metadynamics

The set of target points \( \{x_f(t), b(t)\} \), where

\[
x_f(t) \equiv x_f(x(t), b(t))
\]

corresponds to the mapping (2) of a trajectory \( \{x(t), b(t)\} \) onto the critical manifold. As mentioned before, we shall only consider trajectories on attractors for this study. The set of target points hence has the same dimensionality as the respective attractor, e.g. one dimensional for a limit cycle or fractal for a chaotic attractor (cf section 3.1). In typical slow–fast systems the embedding dimension of the critical manifold is usually much lower than the dimension of the full system. The dimension of a set of target points corresponding to a given trajectory, being a subset of the critical manifold, is thus even lower, reducing therefore the complexity of the analysis.

One defines as attractor metadynamics the time evolution of the target points equation (4), which can be continuous or characterized by jumps between distinct sets of target points. Distinct branches of the critical manifold can often be classified in real world applications, e.g. in the neurosciences when using a slow feature analysis [41]. States on the same branch of the critical manifold are then lumped together, with distinct branches
corresponding to different objects, such as ‘chair’ and ‘table’. Target points continuously connected, compare figure 1, can then be considered a set of states corresponding to the same object and hence as a transiently attracting state. The mapping (4) therefore implies that the flow moves from one transiently attracting state to another whenever the respective target point jumps from one branch of the critical manifold to the next. This classification is considered important especially in the neurosciences, where attracting states guide decision making, memory storage and recognition [4].

2.4. Distance to target points
As a measure of the influence a given target point exerts on the trajectory we will consider the Euclidean distance $d(t)$ between the fast components of the trajectory $\{x(t), b(t)\}$ and the corresponding target points $\{x_t(t), b(t)\}$, see equation (4). The corresponding time–average $\langle d \rangle$ has been defined in (5) for the case of a limit cycle with period $\tau$, with a corresponding straightforward generalization for chaotic attractors.

The average distance $\langle d \rangle$ vanishes, per construction, in the adiabatic limit $\epsilon_b \to 0$. For finite $\epsilon_b > 0$ we find, on the other hand, two regimes, with $\langle d \rangle$ scaling with $\epsilon_b$ in the adiabatic regime, but not in the non–adiabatic regime.

Apart from the average distance $\langle d \rangle \to 0$, one can also evaluate the probability density function $\rho(d)$ of the distances over a given attractor. The corresponding cumulative distribution function of distances

$$P(d) = \int_0^d du \rho(u)$$

can then be used to characterize regimes. As a trajectory stays close to the target points for prolonged time spans in the adiabatic regime one finds contributions at small distances $d \sim \epsilon_b \ll 1$ for the cumulative distribution. A second contribution comes from the jumps between the different branches of target points at larger distances $d \gg \epsilon_b$ (usually $d \sim 1$). However, in the non–adiabatic case only the latter contribution, at larger distances, exists, as the system comes close to the target points only occasionally. A lack of contributions at small distances $d \sim \epsilon_b$ is an evidence that the dynamics is in the non–adiabatic regime.

3. Results

3.1. Three–neuron system
To investigate the effect of target points on the overall dynamics, we consider for this study a small neural network (cf figure 2) of three rate–encoding continuous–time point neurons [36]. Networks of three neurons have already served as model systems in different contexts such as modeling pacemaker circuits [42], the stomatogastric ganglion in lobster [43] or neural motifs [44]. Here we chose the three–neuron layout to study the fundamental properties of AFP and target points, as it allows for a full investigation of its non–trivial dynamics and phase diagram.

The fast subsystem corresponds in this case to the time evolution

$$\dot{x}_i = -x_i + \sum_{j=1}^N w_{ij} y_j,$$

Figure 2. Scheme of the three–neuron system. The second neuron (top) is exciting the other two neurons (solid arrows) and vice versa. The first and the third neuron (bottom) are coupled by an inhibitory connection $w_{13} < 0$ (dashed arrow).
of the membrane potential $x_i$, with $w_{ij} > 0$ and $w_{ji} < 0$ denoting excitatory and inhibitory connection $j \to i$ respectively. The corresponding firing rate $y_i \in [0, 1]$, the neural activity, is a sigmoidal function of the respective membrane potential:

$$y_i = \frac{1}{1 + \exp(a_y(b_i - x_i))}.$$  

We have used constant gains $a_y \equiv a = 6$, whereas the threshold $b_i = b_i(t)$ is adapting slowly [45] on extended time scales $1/\epsilon_b \gg 1$,

$$\dot{b}_i = \epsilon_b 2a((y_i - 1/2).$$  

In the context of this model the ratio of times scales $\epsilon_b$ is also called the adaption rate of the slow variables $b_i$. The time evolution equation (9) of the slow variables attempts to drive the dynamics towards $y_i \to 1/2$ as $b \to 0$, which is the fixed point of the full system (cf appendix B).

The state of a neuron is hence described by the tuple $\{x_i(t), b_i(t)\}$. For the particular three–neuron system sketched in figure 2, the dynamics of the fast variables $x_i$ is given by [33]

$$\dot{x}_1 = -x_1 + y_2 + w_{13}y_1, \quad \dot{x}_2 = -x_2 + y_1 + y_3, \quad \dot{x}_3 = -x_3 + y_1 + y_3 + w_{13}y_1,$$

with $w_{13} = w_{31} < 0$ being inhibitory. All remaining synaptic weights are unity $w_{12} = w_{21} = w_{23} = w_{32} = 1$ and thus excitatory connections. For the results presented here we computed the numerical solution of the ODE system equations (9) and (10) performing a fourth order Runge–Kutta integration algorithm with Fehlberg tableau [46] using step size $dt = 10^{-2}$. For computing the AFW we used a minimization algorithm with a BFGS strategy [47, 48] provided by the dlib optimization library [49] for the C++ programming language.

3.1.1. Symmetries

The network shown in figure 2 is symmetric under the exchange $1 \leftrightarrow 3$ of the first and the third neuron, a reflection symmetry. For the special case $w_{13} = -1$ the additional C3 rotational symmetry $x \to x$ and $b \to \tilde{b}$, with

$$\tilde{x}_1 = -x_1, \quad \tilde{x}_2 = x_3 + x_1, \quad \tilde{x}_3 = x_3 - x_1$$

$$\tilde{b}_1 = -b_3, \quad \tilde{b}_2 = b_1 + 1, \quad \tilde{b}_3 = b_2 - 1$$

is present. Equation (11) can be verified by inspecting equations (9) and (10). This leads to

$$\tilde{y}_1 = 1 - y_3, \quad \tilde{y}_2 = y_1, \quad \tilde{y}_3 = y_2.$$  

Applying this iteration three times yields the identity transformation, equation (11) is hence equivalent to a C3 symmetry.

3.1.2. Dynamics of the three–neuron system

Before analyzing the transiently attracting states of the three–neuron system we briefly discuss here the possible types of long–term dynamics in the network, i.e. attractors in the phase space of the full system, for different cases of the inhibitory weight $w_{13}$ and of the time scale difference $\epsilon_b$.

a. C3 symmetry Due to the C3 symmetry of the three–neuron system for $w_{13} = -1$ discussed in section 3.1.1, we find a traveling wave solution where all neurons show the same activities $x_i(t) = x(t - \theta_i)$ with period $\tau$, albeit with distinct phases $\theta_i$, shifted respectively by $\tau/6$. One can easily prove that this type of motion is always an exact solution in the case of $w_{13} = -1$ (cf appendix C).

In the top panel of figure 3 we show an example of the traveling waves solution for $w_{13} = -1$, $\epsilon_b = 8 \cdot 10^{-4}$ with green/violet/yellow encoding $y_1/y_2/y_3$. Time has been rescaled with respect to $1/\epsilon_b$. For a further quantification we define the dimensionless relative phase shift

$$\Delta \tau_3 = \Delta T_{13}/\tau$$

between the first and the third neuron, where $\Delta T_{13}$ is the time difference of two consecutive intersections of $y_1$ and $y_2$ with the Poincaré plain $y_2 = 1/2, \tilde{y}_1 < 0$ (cf top panel of figure 3). The traveling waves solution is characterized hence by $\Delta \tau_3 = 1/3$.

The traveling waves motion is an exact solution for all $\epsilon_b$, but stable only for $\epsilon_b \geq 6 \cdot 10^{-5}$. For all $\epsilon_b > 6 \cdot 10^{-5}$ the motion is hence symmetry protected [50] in the sense that the relative phase shift (13) is independent of $\epsilon_b$, viz $\Delta \tau_3 \equiv 1/3$.

In the top panel of figure 4 we show the trajectory for $\epsilon_b = 10^{-5}$, viz in the phase where the traveling wave solution is not anymore stable. The activity levels of the first and the third neuron now approach mutually either
Figure 3. Firing rates of the three neurons for $\epsilon_b = 8 \cdot 10^{-4}$ in the $C_3$ symmetric case $w_{13} = -1$ (indicated by the magenta bullet in figure 5) over rescaled time. (top) Green/violet/yellow denote $y_1/y_2/y_3$; the relative phase shift $\Delta T_{13} = \Delta T_{13}/\tau = 1/3$ is the ratio of the time shift $\Delta T_{13}$ between two consecutive Poincaré sections $y_i = 1/2$, $y_i < 0$ of the first and third neuron, and the oscillation period $\tau$. (bottom) $y_1$–component of the target points (red) and of the trajectory (green), with the arrows indicating the behavior of the system in the adiabatic limit $\epsilon_b \rightarrow 0$ (cf figure 1).

Figure 4. Firing rates of the three neurons for $\epsilon_b = 10^{-5}$ in the $C_3$ symmetric case $w_{13} = -1$ (indicated by the cyan square in figure 5) over rescaled time. (top) green/violet/yellow denote $y_1/y_2/y_3$, (bottom) the $y_1$–component of the target points (red) and of the trajectory $y_1$ (green). The system performs an anti–phase flip–flop oscillation of the first and the third neuron, $\Delta_{13} = 1/2$. 
full or low activity states, \( y_1 \approx 1 \) and \( y_2 \approx 0 \) respectively, while the second neuron stays mostly at \( y_2 \approx 1/2 \) (half active). The relative phase shift is \( \Delta_{13} = 1/2 \), which is the maximal possible value.

The system thus flips between the two states \( y = (y_1, y_2, y_3) \approx (0, 1/2, 1) \) and \( y \approx (1, 1/2, 0) \). The flipping time of this flip–flop oscillation scales inversely with the adaption rate \( \epsilon_b \).

b. No C3 symmetry In figure 5 we outline the distinct dynamical regimes observed for the general case of arbitrary inhibitory weights close to the symmetric case. The relative phase shift \( \Delta_{13} \) is color–coded and shown as a function of the adaption rate \( \epsilon_b \) and of the inhibitory weight \( w_{13} \).

On the horizontal center axis \( w_{13} = -1 \), which represents the case of a C3 symmetric system, the symmetry protected traveling wave solution with \( \Delta_{13} = 1/3 \) occurs for \( \epsilon_b > 6 \cdot 10^{-3} \) (cf figure 3) and the flip–flop oscillation \( \Delta_{13} = 1/2 \) for \( \epsilon_b < 6 \cdot 10^{-3} \) (cf figure 4).

Far off the symmetric axis only two possible states exist: either the system performs a flip–flop oscillation (\( \Delta_{13} = 1/2 \) and \( w_{13} = -1.15 \)) or an in–phase oscillation (\( \Delta_{13} = 0 \) and \( w_{13} > -0.85 \)). In the latter case all three neurons fire synchronously and switch between a state \( y \approx (1, 1, 1) \) of high activity and a state \( y \approx (0, 0, 0) \) of low activity.

These two main types of motion follow directly from the topology of the network illustrated in figure 2. For relatively strong inhibition \( w_{13} < -1 \) the first and the third neuron suppress each other mutually, they can therefore not be active at the same time. A dominance of inhibition thus leads to a flip–flop oscillation. Weak inhibition \( w_{13} > -1 \) allows on the other side both the first and the third neuron to be active at the same time, and therefore also the second neuron, leading to an in–phase oscillation of all three neurons.

The situation is more complicated in the inner region of \( -1.15 < w_{13} < -0.85 \) (cf figure 5), where inhibitory and excitatory weights are similar in magnitude. The relative phase shift \( \Delta_{13} \) varies continuously from its maximal value 1/2 to its minimal value 0, when increasing \( w_{13} \) for \( \epsilon_b > 6 \cdot 10^{-3} \), taking the value \( \Delta_{13} = 1/3 \) right at the symmetry point \( w_{13} = -1 \), where the traveling wave solution, illustrated in figure 3 exists.

Below, we will discuss further horizontal and vertical cuts through parameter space, indicated by the dashed gray lines in figure 5, as well as the occurrence of chaotic attractors for certain parameter values to be found in the area that is green shaded in figure 5.

In figure 6 we present the sketch of the behavior along the previously mentioned two cuts of the phase diagram for \( \epsilon_b = 10^{-3} \) and \( \epsilon_b = 8 \cdot 10^{-4} \) (cf figure 5). For low adaption \( \epsilon_b = 10^{-3} \) as shown in figure 6(a) we only find adiabatic behavior: either anti–phase oscillations (i), (iii) or in–phase oscillations (v). Close to the symmetric transition point \( w_{13} = -1 \) both types of attractors can be found depending on the initial conditions. As mentioned before, the anti–phase oscillation corresponds to a switching between the two anti–symmetric transiently attracting states \( y \approx [0, 1/2, 1] \) and \( y \approx [1, 1/2, 0] \). In the case of the in–phase oscillation this is a periodic switching between the symmetric transiently attracting states \( y \approx [0, 0, 0] \) and \( y \approx [1, 1, 1] \).

For low adaption rate we thus find a direct transition between the two different types of attractors of the adiabatic regime. With a larger adaption \( \epsilon_b = 8 \cdot 10^{-4} \) in figure 6(b) at the transition from strong inhibition to weak inhibition the traveling waves regime appears. The panels in figure 6(b) depict five examples in the transition...
from anti-phase (i), via traveling waves (ii)–(iv), to in-phase oscillation (v). In the extreme cases of $w_{13} = -1.1$ and $w_{13} = -0.92$ we can guess once again that the firing rate is shaped by the anti-symmetric and the symmetric transiently attracting states respectively. In contrast to the case of lower adaption here we find that the transition between the in-phase and the anti-phase oscillation of the adiabatic regime happens via the non-adiabatic regime when passing the region of symmetry protection close to the symmetric case $w_{13} = -1$.

### 3.2. Transiently attracting states in the three-neuron system

In the bottom panels of figures 3 and 4 we present the firing rate $y_1$ of the first neuron (green) and the $y_1$ component of the corresponding target point (red bullets and line), with the adiabatic evolution indicated by the black arrows.

- For $\epsilon_0 = 8 \times 10^{-4}$, the non-adiabatic case shown in figure 3, we find that the set of target points splits into four branches. The jumps between the distinct branches are however not directly visible in the original trajectory, which is almost completely decoupled of the dynamics of the target points.

- For $\epsilon_0 = 10^{-3}$, a case from the adiabatic regime illustrated in figure 4, we observe on the other hand only two branches of target point manifolds. For extended time spans the trajectory follows closely the critical manifold, jumping however periodically between the two distinct branches.

One observes in figure 4, that the orbit needs a certain time, a delay, to leave a given branch of the target point manifold. This is a phenomenon typical for systems with multiple time scales [28].

### 3.2.1. Stable and unstable adiabatic fixed points

In figure 7, which illustrates the dynamics along the horizontal cut through the phase diagram for $w_{13} = -1$, the adiabatic fixed points for five different values of the adaption rate are shown. Here we have included all adiabatic fixed points, including the unstable ones, which we have found by minimizing equation (3). Each of the five insets shows a sketch of the trajectory (green), target points (red), other stable AFP (blue) and saddle AFP (gray) in the activity $y_1$ of the first neuron (cf bottom panels of figures 3, 4). Each of these insets serves as an example of the five different phases that can be distinguished by the shape and stability of the AFP. The whole non-adiabatic regime with phase shift $\Delta_{13} = 1/3$ is marked by the yellow bar, while the adiabatic regime with $\Delta_{13} = 1/2$ is indicated by a brown bar. The range of occurrence in the adaption rate $\epsilon_0$ of the four non-adiabatic phases is indicated by the gray bars included in the yellow bar.

Starting at relatively high adaption $\epsilon_0 = 10^{-2}$ in figure 7 panel (v) we find only stable AFP of which all act as target points and the attractor metadynamics therefore is continuous. Decreasing the adaption rate $\epsilon_0$ (cf panels (iv) to (i)) we observe the occurrence of saddle type AFP and stable AFP which are no targets to the trajectory.

In panels (i)–(iii) of figure 7 one can find saddle-node bifurcations, in which stable (blue) and saddle type (gray) AFP merge. The saddle–node bifurcation points originate from a cusp bifurcation [51] on the critical manifold, with the slow variables as bifurcation parameters. The adaption rate $\epsilon_0$ determines at which point the trajectory crosses the cusp bifurcation. Thus one either finds a continuous manifold of target points or jumps.
This change in the structure of the corresponding target points with the adaption rate also affects the attractor metadynamics, which becomes discontinuous as well.

- For all adaption rates \( \epsilon_b > 6 \cdot 10^{-5} \) we find stable traveling waves solutions with a relative phase difference \( \Delta_{13} = 1/3 \), for which the trajectory is only marginally influenced by the transiently attracting states. This is the non-adiabatic regime (yellow bar).

- For lower adaption rates \( \epsilon_b < 6 \cdot 10^{-5} \) we find that both the trajectory and the attractor metadynamics perform flip–flop oscillation with a relative phase difference \( \Delta_{13} = 1/2 \). Here, in the adiabatic regime (brown bar), the motion follows strictly the metadynamics of the target points with some delay at the jumps, as mentioned earlier.

We emphasize that the transition between the adiabatic and non-adiabatic regime occurs at a remarkably low value of the adaption rate \( \epsilon_b \approx 6 \cdot 10^{-3} \). The traveling wave solution being stable for \( \epsilon_b > 6 \cdot 10^{-3} \) is hence strongly protected by the C3 symmetry of the system. Off symmetry, i.e. for \( w_{13} \neq -1 \), the transition between the adiabatic and the non-adiabatic regime may become continuous (compare figure 5), and shifts to higher values in \( \epsilon_b \). Along the symmetric axis \( w_{13} = -1 \) we find a small region in the adaption rate \( \epsilon_b \) close to the transition, as marked by stripes in figure 7, where attractors from both the adiabatic and the non-adiabatic regime coexist. Starting close to an attractor from the non-adiabatic regime we can trace the non-adiabatic regime decreasing the adaption rate \( \epsilon_b \) in small steps. This procedure succeeds up to a certain value of \( \epsilon_b \), depending on the precision of the computation and the step size in \( \epsilon_b \), before ending up in an attractor of the adiabatic regime. Vice versa we can start from the adiabatic regime and trace it increasing \( \epsilon_b \). This means we can observe hysteresis in the bistable transition region between the regimes.

### 3.2.2. Evolution of the average distance between trajectory and target points

Figure 8 shows the average distance \( \langle d \rangle \), as defined by equation (5), of the trajectory to the target points. Shown are horizontal cuts through the phase diagram figure 7, along \( w_{13} = -1.1, -1, -0.92 \). In the absence of C3 symmetry, the separation of time scales between fast and slow variables starts to fail for larger values of the adaption rate, \( \epsilon_b > 10^{-3} \), with the distance \( \langle d \rangle \) becoming independent of \( \epsilon_b \).

An adiabatic regime is always observed for low adaption rates \( \epsilon_b \). It is however clearly evident from figure 8 that the adiabatic regime is pushed down for the symmetric case \( w_{13} = -1 \), by more than one order of
magnitude. States protected by symmetry operations, in our case the traveling wave solution illustrated in figure 3, can be exceedingly stable.

The discontinuous transition between the two regimes in the symmetric case \( w_{13} = -1 \) is confirmed by both figure 5 and figure 8. There is a transition region in \( \epsilon_b \), where two arbitrarily close initial conditions can show fundamentally different dynamics being in the adiabatic and the non–adiabatic regime, respectively (cf figure 7 and the related discussion). On the two sides of this transition region, only one of these two different kinds of dynamics exists. Lacking any intermediate kind of dynamics, this abrupt transition is remarkable since the shape of the AFP manifold does not depend on the adaption rate \( \epsilon_b \) at all. We thus stress that the distance to the reference manifold of target points, which is unique for given parameter values and initial conditions, reveals a qualitative change in the dynamics.

3.2.3. Statistics of the distance between trajectory and target points

For the same parameters as used in figures 3 and 4 we present in figure 9 the cumulative distribution function \( P(d) \) of the distance \( d \) between trajectory and the corresponding target point, as defined by equation (6), for the adiabatic case \( (\epsilon_b = 10^{-5}, w_{13} = -1, \text{cya}n) \), the non–adiabatic case \( (\epsilon_b = 8 \cdot 10^{-2}, w_{13} = -1, \text{magenta}) \), and for the chaotic attractor \( (\epsilon_b = 10^{-1}, w_{13} = -0.9799, \text{light green}; \text{compare figure 10}) \). For the non–adiabatic case there are only contributions close to unity \( \langle d \rangle = 0.89 \). Contributions from significantly smaller distances are observable in addition both for the adiabatic case \( \langle d \rangle = 0.16 \) and for the chaotic attractor \( \langle d \rangle = 0.25 \). The latter one has contributions also for intermediate distances \( d \).

Figure 8. Average distance \( \langle d \rangle \) of the trajectory to the corresponding targets, as defined by equation (5). Shown are horizontal cuts through the phase diagram, figure 5, along \( w_{13} = -1.1 \) (gray), \(-1\) (yellow) and \(-0.92\) (brown). The regime is non–adiabatic, when the distance becomes independent of \( \epsilon_b \), as it happens for larger values of the adaption rate. The system is adiabatic, conversely, when the distance scales with \( \epsilon_b \), which happens for the symmetric case \( w_{13} = -1 \) only for low adaption rates.

Figure 9. The cumulative distribution function \( P(d) \) of the distance \( d \) between trajectory and the corresponding target point, as defined by equation (6), for the adiabatic case \( (\epsilon_b = 10^{-5}, w_{13} = -1, \text{cyan}) \), the non–adiabatic case \( (\epsilon_b = 8 \cdot 10^{-2}, w_{13} = -1, \text{magenta}) \), and for the chaotic attractor \( (\epsilon_b = 10^{-1}, w_{13} = -0.9799, \text{light green}; \text{compare figure 10}) \). For the non–adiabatic case there are only contributions close to unity \( \langle d \rangle = 0.89 \). Contributions from significantly smaller distances are observable in addition both for the adiabatic case \( \langle d \rangle = 0.16 \) and for the chaotic attractor \( \langle d \rangle = 0.25 \). The latter one has contributions also for intermediate distances \( d \).
For the non–adiabatic case (magenta) we find that there are only contributions in a narrow region $d \in [0.7, 1.0]$ close to unity, resulting in turn from small residual variations of distance. For the adiabatic case (cyan) we observe substantial contributions both close to unity, reflecting the jumps between different target branches, and extended contributions from small distances $d < 10^{-2}$, which result from the evolution close to the AFP manifold. The difference between the adiabatic and the non–adiabatic regime shows up prominently in the statistics of the distances between trajectory and target points.

3.2.4. Target points corresponding to chaotic motion

For a small region in the adiabatic regime $e_9 < 3 \cdot 10^{-4}$ and near the symmetric case, $w_{13} \in [-1, -0.93]$, we find patches in the parameter space that exhibit chaotic motion. An example is shown in figure 10 for $e_9 = 10^{-5}$ and $w_{13} = -0.970 9$. The trajectory and the corresponding target points are shown both as a time series (left panel) and projected to the $y_1$–$y_2$ plane in phase space. Applying a recently developed test for chaos, based on the cross–distance scaling of initially nearby trajectories [52], we find that this attractor is indeed chaotic.

Except for some overshooting, the trajectory shows a similar behavior to the adiabatic motion presented in figure 4, approaching the target point manifold and staying then close to it as long as it remains stable.

The manifold of target points has a highly non–trivial structure for the chaotic motion, in contrast to the piece–wise smooth and periodic structure observed for the case of limit cycle attractors, as observed e.g. in figure 7. A visual inspection indicates (see the inset in the right–hand panel of figure 10) that the phase space projection of the target points forms a fractal structure. We did not attempt to directly compute the fractal dimension of the manifold of target points shown in figure 10, as this is computationally highly demanding. Chaotic behavior, on the other hand, is typically linked to the presence of fractals in the dynamical behavior [53].

Speaking in terms of transiently attracting states this means that in case of chaotic motion we cannot describe the motion as switching between two transiently attracting states. But rather we find that there must be infinitely many branches of transiently attracting states forming a fractal set. One may nevertheless cluster these fractal sets into two broad classes, corresponding to small and large $y_2$, respectively. A possible interpretation of that behavior includes a chaotic motion of the overall system driving the dynamics along the critical manifold, which itself does not have a fractal structure. Thus the irregular motion near the critical manifold results in a fractal set of target points. We emphasize that the target points themselves are always stable fixed points of the fast subsystem, for any values of the slow variables.

In order to compare the chaotic motion more precisely to the adiabatic and the non–adiabatic regimes, we have included in figure 9 the cumulative distribution function of distances (light green line) for the chaotic attractor. Besides the contribution close to unity due to the large distance jumps between different AFP branches, it reveals contributions at small distances $d < 10^{-2}$, which are significant also for the adiabatic case (cf to cyan line). In this respect the chaotic attractor is close to the adiabatic regime. We find however additional medium size contributions $d \approx 0.3$ that result from smaller jumps within the fractal set of target branches.

The chaotic dynamics thus goes through an infinite series of transient states, trying to stay close to the critical manifold (itself having a simple, not fractal geometry), which is also confirmed by the distribution of distances to the target points. We do not observe chaotic exploration of the phase space, i.e. long detours leading away from the target points as one could imagine for chaotic dynamics. Due to the relatively small adapation rate the chaotic
dynamics is guided along different branches of the target manifold—similar to the regular adiabatic motion presented in figure 4.

Keeping on the other hand in mind that the inhibitory weight \( w_{13} = -0.970913 \) is only slightly off the symmetric case, it is to be expected that the shape of the critical manifold only changes smoothly in \( w_{13} \). Therefore we would expect the same qualitative properties for the dynamics, i.e. the trajectory moving mostly close to branches of the target point manifold, as in the symmetric case presented in figure 4. But the origin of the chaotic dynamics is thus not obvious.

4. Discussion and conclusions

We have proposed here that the study of adiabatic fixed points and transiently attracting states, which are sets of target points, are useful when trying to understand complex slow–fast systems. As these can be realized by adding an additional slow component to an attractor network, the target points have a well defined physical function representing, e.g., as cognitive states in a neural network. Both the location in phase space and the topology of the manifold of target points can be easily evaluated without explicit knowledge of the full set of fixed points in the fast subsystem. The mapping of a given trajectory onto the corresponding set of target points is, by definition, unique and does not depend on the time evolution on the slow time scale. Thus target points qualify as a unique reference manifold for the overall dynamics. Transitions between smooth subsections of the manifold of target points correspond generically to transitions between well defined biological, physical or cognitive states.

We have shown, analyzing a three–site network of adapting rate–encoding neurons, that system symmetries may stabilize peculiar solutions which effectively decouple from the dynamics of target points. This decoupling shows up also in the statistics of the Euclidean distance between the trajectory and the respective target points. The distance distribution can be used furthermore to classify states in terms of the relevance of the reference manifold.

We applied our analysis both to limit cycle and to chaotic dynamics. The detailed analysis of the latter phenomenon is a possible subject of future work. We find, somewhat surprisingly, that a fractal set of target points may guide chaotic behavior.

The above–described phenomenon is not to be confused with the case when the attractor of the fast subsystem is more complicated than a fixed point, e.g. limit cycles or chaotic attractors. In future studies the above described method for distinguishing dynamical regimes by means of target (fixed) points could be generalized to arbitrary target attractors. The key implication then will be to generalize the distance measurement between a trajectory and the corresponding attractor.

Our approach assumes a separation of time scales to be present, breaking down once the time scales for the slow and for the fast subsystem approach each other. The ability to treat the system analytically is however not needed.

We believe, in conclusion, that the study of target points could provide additional insights especially for slow–fast dynamical systems with a relatively high number of degrees of freedom. These points can serve as a unique reference to investigate the overall dynamics of a system and are easily computable. It would be interesting to follow the possibility to extend the here proposed approach to non–autonomous systems, such as the modulated neural networks processing cognitive stimuli.

Acknowledgments

The authors acknowledge the financial support from the German research foundation (DFG). H W acknowledges the support from Stiftung Polytechnische Gesellschaft Frankfurt am Main.

Appendix A. Computing adiabatic fixed points and target points

For an alternative view we present in table 1 a short pseudo–code description that explains, how to effectively compute AFP and target points. If more complex critical manifolds would be present, e.g. a limit cycle or a strange attractor, one could generalize the concept (2) to include these.
Appendix B. Fixpoints of the three–neuron system

The six dimensional three–neuron system equations (10), (9) has the only fixed point

\[ x_2 = 1, \quad x_3 = x_5 = (1 + w_{13})/2, \quad b_i = x_i. \]  

(B1)

From that it follows that the neurons are half–active \( y_i = 1/2 \) at the fixed point. This fixed point is generally stable for large values of \( \epsilon_b \), undergoing a supercritical Hopf bifurcation when \( \epsilon_b \) becomes smaller. This occurs, for \( a = 6 \) and \( w_{13} = -1 \), at \( \epsilon_b = 1/36 \approx 0.0278 \).

Appendix C. Traveling waves

One can prove that in the case of symmetric coupling \( w_{13} = -1 \) a traveling waves ansatz solves the system. This solution shows a relative phase shift of exactly \( \Delta_{13} = \frac{\pi}{3} \) between the first and the third neuron, where \( \tau \) is the period of the solution. We make an ansatz for the solution

\[ x_1(t) = x(t - \theta), \quad x_2(t) = 1 + x(t), \quad x_3(t) = x(t + \theta), \]

\[ b_1(t) = b(t - \theta), \quad b_2(t) = 1 + b(t), \quad b_3(t) = b(t + \theta) \]  

(C1)

with two periodic functions \( x(t), b(t) \) of period \( \tau \) and an arbitrary shift \( \theta \). The corresponding firing rates of the neurons therefore are given by

\[ y_1(t) = y(t - \theta), \quad y_2(t) = y(t), \quad y_3(t) = y(t + \theta), \]

(C2)

where the notation for the periodic function \( y(t) = y(x(t), a, b(t)) \) is used. From this we get to the resulting equations of motion for the membrane potential

\[ \hat{x}(t) = -x(t) + y(t + \theta) - y(t + 2\theta) \]

\[ \hat{\theta}(t) = -x(t) + y(t - \theta) + y(t + \theta) - 1 \]

\[ \hat{\theta}(t) = -x(t) + y(t - \theta) - y(t - 2\theta). \]  

(C3)

This leads to the condition \( y(t) = 1 - y(t \pm 3\theta) \) that has to hold when solving equation (C3). Assuming further that \( x(t + \frac{\pi}{6}) = -x(t) \) and \( b(t + \frac{\pi}{2}) = -b(t) \), one can solve equation (C3) for the constant phase shift \( \theta = \frac{\pi}{6} \). Therefore the phase shift \( \Delta_{13} \) between the first and the third neurons is exactly \( \Delta_{13} = 2\theta/\tau = 1/3 \).

ORCID iDs

Hendrik Wernecke @ https://orcid.org/0000-0002-0187-3079
Bulcsú Sándor @ https://orcid.org/0000-0003-4887-7859
Claudius Gros @ https://orcid.org/0000-0002-2126-0843

References

[1] Moser E I, Kropff E and Moser M-B 2008 Annual Review of Neuroscience 31 69–89
[2] Hopfield J 1982 Proceedings of the National Academy of Sciences 79 2554
[3] Wills T J, Lever C, Cacucci F, Burgess N and O’Keefe J 2005 Science 308 673
[4] Rabinovich M, Huerta R and Laurent G 2008 Science 321 48
[5] Niessing J and Friedrich R W 2010 Nature 465 47
[6] Miller P and Katz D B 2010 The Journal of Neuroscience 30 2559
[7] Gros C 2007 New J. Phys. 9 109
[8] Deco G and Jirsa V K 2012 The Journal of Neuroscience 32 3366
[9] Deco G, Jirsa V K and McIntosh A R 2011 Nat. Rev. Neurosci. 12 43
[10] Zalesky A, Fornito A, Cocchi L, Gollo L L and Breakspear M 2014 Proc. Natl Acad. Sci. USA 111 10341
[11] Ritter P, Jirsa V K, McIntosh A R and Breakspear M 2015 Frontiers in Computational Neuroscience 9 77
[12] Martin L, Sándor B and Gros C 2016 Frontiers in Neurorobotics 10 12
[13] Gros C and Kaczor G 2009 Logic Journal of IGPL 18 686
[14] Gros C 2009 Cognitive Computation 1 77
[15] Lukoševičius M and Jaeger H 2009 Computer Science Review 3 127
[16] Neves F S and Timme M 2012 Phys. Rev. Lett. 109 018701
[17] Jaeger H 2001 German National Research Center for Information Technology GMD Technical Report 148 13
[18] Berglund N and Gentz B 2006 Noise-Induced Phenomena in Slow–Fast Dynamical Systems: A Sample-Paths Approach (London: Springer-Verlag)
[19] Kuehn C 2015 Multiple time Scales Dynamics Vol 191 (Berlin: Springer)
[20] Zhikevich E M 2007 Dynamical Systems in Neuroscience (Cambridge, MA: MIT press)
[21] Verhulst F 2007 Nonlinear Dyn. 50 747
[22] Szamolyan P and Wachsmuth M 2004 J. Differ. Equs. 200 69
[23] Gros C 2015 Complex and Adaptive Dynamical Systems: A Primer (Berlin: Springer)
[24] Rocsoreanu C, Georgescu A and Giurgiteanu N 2012 The FitzHugh-Nagumo Model: Bifurcation and Dynamics Vol 10 (Dordrecht: Springer Science & Business Media)
[25] Zheng Y and Bao L 2014 Communications in Nonlinear Science and Numerical Simulation 19 1591
[26] Verhulst F and Bakri T 2006 Journal of the Indonesian Mathematical Society 2006 1–16
[27] Kuehn C 2012 Phys. Rev. E 85 026103
[28] Kuehn C 2008 Journal of Physics A 42 045101
[29] Berglund N 2000 Prog. Theor. Phys. Suppl. 139 325
[30] Kielb SJ, Daunizeau J and Friston K J 2008 PLoS Comput Biol 4 e1000209
[31] Tetzlaff C, Kolodziejski C, Markelic I and Wörgötter F 2012 Biol. Cybern. 106 715
[32] Yamashita Y and Tani J 2008 PLoS Comput Biol 4 e1000180
[33] Linkerhand M and Gros C 2013 Sci. Rep. 3 2042
[34] Gros C, Linkerhand M and Walther V 2014 Artificial Neural Networks and Machine Learning–ICANN 2014 (Berlin: Springer) pp 65–72
[35] Sussillo D and Barak O 2013 Neural Comput. 25 626
[36] Markovic D and Gros C 2012 Neural Comput. 24 523
[37] Simony J 1967period. Polytch. Electr. Eng. 11 309
[38] Lorenz E N 1992 J. Atmos. Sci. 49 2449
[39] Markovic D and Gros C 2010 Phys. Rev. Lett. 105 068702
[40] Born M and Oppenheimer R 1927 Ann. Phys., Lpz. 389 457
[41] Franzuius M, Wilbert N and Wiskott L 2011 Neural Comput. 23 2289
[42] Prinz A A, Bucher D and Marder E 2004 Nat. Neurosci. 7 1345
[43] Miller P and Silverston A J 1982 Journal of Neurophysiology 48 1416
[44] Morgan R J and Soltesz I 2008 Proceedings of the National Academy of Sciences 105 6179
[45] Triesch J 2005 Artificial Neural Networks: Biological Inspirations–ICANN 2005 (Berlin: Springer) 65–70
[46] Fehlberg E 1969 NASA Technical Report 315 NASA TR–315
[47] Broyden C G 1970 JMA J. Appl. Math. 6 76
[48] Shanno D F 1970 Math. Comput. 24 647
[49] "d1l1b: C++ optimization library", http://dlb.net/optimization.html, accessed: 08. 02. 2016.
[50] Pollmann F, Berg E, Turner A M and Oshikawa M 2012 Phys. Rev. E 85 075125
[51] Guckenheimer J and Holmes P 2013 Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields Vol 42 (New York: Springer Science & Business Media)
[52] Wernecke H, Sándor B and Gros C 2017 Sci. Rep. 7 1087
[53] Tel J and Gruiz M 2006 Chaotic Dynamics: An Introduction Based on Classical Mechanics (Cambridge: Cambridge University Press) p 393