Tension Perturbations of Black Brane Spacetimes

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Abstract

We consider black-brane spacetimes that have at least one spatial translation Killing field that is tangent to the brane. A new parameter, the tension of a spacetime, is defined. The tension parameter is associated with spatial translations in much the same way that the ADM mass is associated with the time translation Killing field. In this work, we explore the implications of the spatial translation symmetry for small perturbations around a background black brane. For static charged black branes we derive a law which relates the tension perturbation to the surface gravity times the change in the horizon area, plus terms that involve variations in the charges and currents. We find that as a black brane evaporates the tension decreases. We also give a simple derivation of a first law for black brane spacetimes. These constructions hold when the background stress-energy is governed by a Hamiltonian, and the results include arbitrary perturbative stress-energy sources.

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1. Introduction and Summary of Results

In this paper, we introduce a new gravitational charge, the spacetime tension. The idea of the tension of a spacetime is simple. A particle-type object, like a billiard ball, has a rest mass. When the mass becomes large and the self-gravity of the object is important, as for a star, there are gravitational contributions to the total mass of the system. The mass $M$ of an asymptotically flat spacetime is defined as a boundary integral of the long range gravitational field, and can be constructed as a conserved charge associated with the asymptotic time translation Killing vector. Now, consider an elastic band, rather than a ball. This has tension as well as mass. Suppose that the elastic band is either infinite or wrapped around an $S^1$ factor of the spacetime. If the band becomes self-gravitating, then there will be a gravitational contribution to the tension. In analogy with mass, the tension $\mu$ of a spacetime is the charge associated with an asymptotic spatial translation Killing vector which is tangent to the band.$^3$

The tension of a spacetime arises as an extension of the usual ADM gravitational charges that are derived from the variation of the gravitational Hamiltonian. Let $\mathcal{H}_t$ be the Hamiltonian that generates flow along a timelike vector field. The lapse and shift appear as Lagrange multipliers in $\mathcal{H}_t$. Regge and Teitelboim $^4$, $^5$ computed the variation of $\mathcal{H}_t$ for asymptotically flat spacetimes. Evaluated on solutions, it is given by the sum of boundary integrals at spacelike infinity, multiplied by the Lagrange multipliers. For asymptotically flat spacetimes, the asymptotic symmetry group is the Poincaré group. They demonstrated that if the Lagrange multipliers are chosen to be asymptotic to the generators of the Poincaré group, then the variation of $\mathcal{H}_t$ is the sum of the variation of the gravitational four-momentum and angular momentum, each multiplied by the corresponding generator. For example, the variation in the ADM mass $\delta M$ is the coefficient of an asymptotic time translation.

If we now consider a spacetime that is asymptotically flat cross an $S^1$, then the asymptotic symmetry group has an additional spatial translation Killing vector $\frac{\partial}{\partial x}$. Therefore in the Regge-Teitelboim construction the variation of $\mathcal{H}_t$ contains an additional linear momentum term $\delta P^x$. The key step in defining the spacetime tension is to note that because

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$^3$ After the first version of this paper appeared, work of Townsend and Zamaklar $^6$ appeared which also defined the spacetime ‘fvztension, using the covariant techniques of $^2$. It is straightforward to check that the two expressions for the definition of tension agree. Recently, one of us has used spinor techniques to prove a positivity result about gravitational tension $^7$. 

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of the covariance of the theory, one can just as well write down a Hamiltonian function $\mathcal{H}_x$ which generates flow along a spacelike vector field, asymptotic to $\frac{\partial}{\partial x}$. The variation in the gravitational tension $\delta\mu$ is then defined by computing the variation in $\mathcal{H}_x$ when the Lagrange multiplier is asymptotic to $\frac{\partial}{\partial x}$. This construction will be given in Section 2 below.

We will also be interested in properties of perturbations to the gravitational mass and tension of $p$-brane spacetimes. When a spacetime has a symmetry, then often one can prove useful relations that hold for perturbations around the spacetime. The most famous example of this is the first law of black hole mechanics \[3\], which holds if the background spacetime has a stationary Killing field $\xi^a$. In addition to $\xi^a$, a black $p$-brane spacetime may have one or more spatial translation Killing fields $X^a$ tangent to the brane. It is natural to ask whether there is another law of black brane mechanics that relates variations in the tension $\delta\mu$, to variations in the geometrical properties of the horizon.

In order to address this, we start in Section 2 by deriving a general integral identity, or constraint, which holds on solutions to the linearized Einstein equation coupled to matter fields. The derivation assumes that the background stress-energy comes from a matter Hamiltonian. Perturbative stress-energy sources are included, and these need not have a Hamiltonian description. For definiteness, when writing out explicit formulae, we assume that the background stress-energy comes from a $(p+1)$-form abelian gauge field, which generally arise in supergravity theories and couple top-branes. This general construction is then used to prove three results concerning perturbations of black $p$-branes.

First, to illustrate the techniques, in Section 3 we give a simple proof of the first law as it applies to black branes. For example, let the background spacetime be a static black 2-brane which is electrically charged with respect to the 3-form gauge potential $A_{abc}$. Let the directions tangent to the 2-brane be compact. Then using the static Killing field $\xi^a$, our general construction gives the first law

\[
\delta M = \frac{\kappa}{8\pi} \delta A + \Omega \delta J - \frac{3}{2\pi} A_{ab} \delta Q^{ab} + \int_V (\xi^a \delta T_{(s)}^{ta} + 4! A_{tc} \delta j^{tcd}).
\] (1)

Here $\kappa$ is the surface gravity, $A$ is the horizon area, $\Omega$ is the angular velocity of the horizon, and $J$ is the angular momentum of the spacetime. $\delta Q^{ab} = 4! \int_{\partial V} N d\sigma \delta F^{cabe}$ is the variation of the electric charge, where $F = dA$. The volume integral gives the contribution of perturbative sources $\delta T^{(s)}_{(a)}$ and charged currents $\delta j^{abc}$.
Evaluating (1) in the case that the background spacetime has no horizons and no gauge fields gives

$$\delta M = \int_V \xi^a \delta T^{(s)}_{ta},$$  \hspace{1cm} (2)$$

That is, $\delta M$ is equal to integral of the local energy density, which agrees with our Newtonian intuition about mass.

Second, in Section 4 we prove a statement analogous to equation (1) for variations in the tension $\delta \mu$ of black branes. $\delta \mu$ is first defined by a boundary integral which depends on $X^a$, the background metric, and the metric perturbation. The functional dependence of $\delta \mu$ on $X^a$ is similar to the dependence of $\delta M$ on $\xi^a$. The main result of this paper is to derive a relation for $\delta \mu$ which is analogous to equation (1). Consider again the example of a static charged, black 2-brane, and that the brane wraps a 2-torus. Assume that the spacetime has a spatial translation Killing vector $X^a = \frac{\partial}{\partial x}$, which is tangent to an $S^1$ wrapped by the brane. Then using $X^a$ in our general construction gives the relation

$$\delta \mu = -\frac{A}{8\pi L_H} \delta \kappa - \frac{3}{2\pi} A_{xab} \delta Q^{xab} + \int_{V_x} \left[ -X^a \delta T^{(s)}_{x} + \frac{3}{2\pi} A_{bcx} \delta J^{bcx} \right]$$  \hspace{1cm} (3)$$

Here $L_H$ is the length of the $S^1$, to which $X^a$ is tangent, at the horizon. $V_x$ is a subsurface of an asymptotically flat slice which has $x = constant$. Using a Komar-Smarr relation, this can be rewritten as

$$\delta \mu = \frac{\kappa}{8\pi(n-2)L_H} \delta A + \text{gauge field and source terms.}$$  \hspace{1cm} (4)$$

As for $\delta M$, we can gain physical understanding of the quantity $\delta \mu$ by specializing (3) to the case of perturbations off a spacetime with no horizons and no gauge fields. Then

$$\delta \mu = -\int V X^a \delta T^{(s)}_{x}$$  \hspace{1cm} (5)$$

i.e., $\delta \mu$ is minus the integral of the $x$-component of the pressure of the stress-energy. This is part of the justification of calling the boundary integral the tension. Note that the momentum associated with the Killing field $X^a$ is a different physical quantity; in this same example, $\delta P^x = \int_{\Sigma} X^a \delta T^{(s)}_{ta}$.

We use the term “tension” rather than “pressure” because test branes whose dynamics are governed by the area action have positive tension. Note that the tension is associated with a Killing field tangent to the brane, and is not a radial pressure as in a star. If the
two-brane has a second translational Killing field $Y^a$, then there is an analogous statement for variations in the tension associated with the y-direction.

Third, we compute the variation in the tension $\delta \mu$ when a charged test brane moves in a background magnetic field. Then $\delta \mu$ has contributions from the perturbation to the enclosed current, and from the variation of the magnetic field on the boundary, as well as the perturbed stress-energy. The resulting statement looks like a gravitational-Ampere’s Law. This is worked out in Section (4.2).

We close with some remarks about the relation of this work to some other work which has used Hamiltonian techniques. References [7] and [8] studied the linearization instability of Einstein-Maxwell and Einstein-Yang-Mills on spacetimes with compact spatial sections. They prove that linearization instability occurs when the spacetime has a Killing field. If the space is non-compact, there is no linearization instability. Instead, their approach, which we employ here, yields an identity on solutions to the linearized equations. The identity takes the form of a gauss’s law constraint, with a boundary term. When the Killing field of the spacetime is time translation, then the constraint is the first law (see Section 3).

Reference [9] studies Einstein-Yang-Mills theory on spacetimes with non-compact spatial sections. They use a Hamiltonian $h$ for Einstein-Yang Mills theory which includes an appropriate boundary term at infinity. Evaluated on solutions only the boundary term of $h$ is nonzero, and this is interpreted as the energy of the spacetime. The perturbations of $h$ are computed and this yields the First Law. By contrast, in this paper as in our previous work [10][11], we start with the field equations. The relations (1) and (3) are identities, or constraints, on solutions to the linearized equations. The challenge is to understand the geometrical meaning of the boundary terms. We argue that a particular boundary integral can be identified as the tension of the spacetime.

The definitions and properties of the gravitational charges in asymptotically flat spacetimes have been extensively studied. A nice feature of our approach is that one is not limited to asymptotically flat boundary conditions. The constraint(s) must be true for perturbations about any background spacetime which has Killing vector(s), such as Reissner-Nordstrom-deSitter or Ernst spacetimes. For example, consider a black hole in deSitter. Let $V$ be a spacelike slice bounded by the black hole and deSitter horizons. In this region there is a static Killing field $\xi^a$. Using $x^a$ in our general construction gives the identity $\kappa_{ds} \delta A_{ds} = \kappa_{bh} \delta A_{bh}$, where $\kappa_{bh}, \kappa_{ds}$ are the surface gravities of the black hole and deSitter horizons, and $\delta A_{bh}, \delta A_{ds}$ the change in the areas of the horizons. The interesting thing
is that the boundary term which gives $\delta M$ on an asymptotically flat boundary becomes $\kappa ds \delta A ds$ on the deSitter horizon.

This paper is organized as follows. In section 2 we derive general constraint relation on perturbations for a foliation of the spacetime by either spacelike or timelike surfaces. This relation is applied to prove the first law in section 3, and then to derive the constraint on $\delta \mu$ in section 4. We show that the $\delta \mu$ constraint simplifies to equation (3) when the background and the perturbations are static. Some geometrical properties of the horizon, related to the existence of a spatial translation Killing field, are derived. Section 5 contains concluding remarks. Appendix A contains some details of the Hamiltonian decomposition of the three-form gauge field Lagrangian. In Appendix B we show that a $p$-form gauge potential is constant on a bifurcate Killing horizon.

2. Integral Constraints on Perturbations

Since the idea of the derivation is simple, while the calculations are detailed, we first outline the idea. The Hamiltonian techniques in [10] [11] [9] and the covariant techniques in [12] [6] [13] are made use of. Consider the Einstein Lagrangian $R$ coupled to matter fields which are described by a Lagrangian, for example $L_M = -F^2$. Let the spacetime be foliated by spacelike slices, with timelike normal vector field $n = -N dt$. One can then construct the Hamiltonian which generates flow along the time direction $\xi^a = \partial / \partial t$. The Hamiltonian $H_{tot}$ for the coupled Einstein-Maxwell system is pure constraint, namely the sum of the Einstein constraints and Gauss’s law, multiplied by lagrange multipliers. On solutions $H_{tot} = 0$. Perturbative solutions linearized about a background satisfy $\delta H_{tot} = 0$, where the variation is with respect to all the dynamical fields “$p$” and “$q$”. The latter equation can be rewritten in terms of the adjoint operator $\delta H_{tot}^\dagger$ and a total derivative. That is, one can integrate the equation over a volume $V$ contained in a spatial slice, and integrate by parts. This yields a volume integral of the adjoint, plus an integral over the boundary $\partial V$. But the adjoint operator generates the Hamiltonian flow, and so Hamilton’s equations imply that on solutions the volume integral is simply the lie derivative along the direction $\xi^a$, of the $p$’s and $q$’s. If $\xi^a$ is a Killing vector for the background then the lie derivatives vanish and the result is greatly simplified. The resulting statement is an identity, or constraint, which must hold on solutions to the linearized equations. If the background spacetime is a black brane, this construction gives the first law [14]. To include perturbative sources $\delta S$, one simply starts with the linearized equation $\delta H_{tot} = \delta S$. 5
Next consider foliating the spacetime with slices that have a *spacelike* normal vector field. This defines a Hamiltonian flow along a spacelike field \( X^a \). The construction of \( \mathcal{H}_{\text{tot}}, \delta \mathcal{H}_{\text{tot}} \) and \( \delta \mathcal{H}^\dagger_{\text{tot}} \) is much the same as for the decompositon based on a timelike normal. One just has to be careful about various minus signs. Again the conclusion is that if \( X^a \) is a Killing field, then perturbations about the background spacetime obey an integral constraint. The integration volume is now a Lorentzian slice, so its boundary includes initial and final spacelike surfaces, plus the boundaries at infinity and along a horizon if present. In general there are fluxes through the initial and final slices. However, if the spacetime and the perturbations are static, the general statement can be reduced to the constraint (3) on an asymptotically flat spatial slice.

The difference between the constraints derived by slicing the spacetime with a timelike normal field, or with a spacelike normal, is similar to the difference between Gauss’ Law and Ampere’s Law in Maxwell theory. Start with Maxwell’s equations in covariant form, \( \nabla_a F^{ab} = J^b \). Then the difference between the derivation of the two laws is just the choice of splitting spacetime into space plus time, or into space plus a smaller spacetime, that is, taking the free index “\( b \)” to be a time direction or a space direction respectively. Gauss’s Law is a true constraint, whereas Ampere’s law holds only if the electric field is time independent. Even so, Ampere’s law is still a useful relation. In Gauss’s law, the source is the time component of the charged current; in Ampere’s law, the spatial components contribute. Similarly, in the gravitational construction the conserved charges \( Q^{tab} \) contribute to the mass, while the \( Q^{xab} \) contribute to the tension. In general this will include purely spatial components of the charged current. Having outlined the idea, we now proceed with the details of the calculation.

### 2.1. The \( d + 1 \) Split

Let \( M \) denote a \( d + 1 \)-dimensional manifold with metric \( g_{ab} \) of signature \((- + \cdots +)\). Let \( \nabla_a \) be the derivative operator compatible with \( g_{ab} \), \( \nabla_a g_{bc} = 0 \). Let \( \Sigma_w \) be a family of \( d \) dimensional submanifolds with constant coordinate \( w \), possibly defined just in some subset of \( M \). Let \( n = n \cdot n N dw \) be the unit normal to \( \Sigma_w \), where \( n \cdot n = n^a n_a = \pm 1 \) if \( n_a \) is spacelike or timelike respectively. We want to rewrite the Einstein equation in terms of \( d \) dimensional quantities. Here we keep track of various minus signs that distinguish between a spacelike and a timelike foliation. The details are omitted; the steps for the usual 3 + 1 decomposition can be found in standard texts [15], [16].
The metric $g_{ab}$ induces a metric $s_{ab}$ on $\Sigma_w$

$$g_{ab} = s_{ab} + (n \cdot n)n_an_b, \quad n_as^ab = 0 \quad (6)$$

and let $D_as_{cd} = 0$. The lapse $N$ and shift $N^a$ are defined by the decomposition

$$\frac{\partial}{\partial w} = W^a = Nn^a + N^a, \quad n^aN^a = 0$$

We will take the Einstein Lagrangian to be $L_G = +R$. The momentum conjugate to $s_{ab}$ is

$$\pi^{ab} = (n \cdot n)\sqrt{\vert s \vert}(Ks^{ab} - K^{ab}) \quad (7)$$

where $K_{ab} = s^c_{\ a}\n^c_{\ b}$ is the extrinsic curvature of $\Sigma_w$. The projection of a tensor onto the submanifold $\Sigma_w$ will be denoted by $\tilde{A}^b \equiv s^b_{\ c}A^c$. When no ambiguity arises we will drop the "tilde", for example, from their definitions, the shift, extrinsic curvature, and momenta fields are tangent to $\Sigma$.

By considering a spacetime decomposition as introduced above one finds that the components $G_{ab}n^b$ of the Einstein tensor only involve first derivatives in the $w$ coordinate. When $W^a$ is timelike then the components $G_{ab}n^b = 8\pi T_{ab}n^b$ are known as the Einstein constraint equations. Here we will refer to these equations as constraints whether $W^a$ is timelike or spacelike. Explicitly,

$$\begin{align*}
(n \cdot n)H_G &= 16\pi T_{ab}n^a n^b \\
(n \cdot n)H_G^b &= 16\pi s^{bc}T_{cd}n^d
\end{align*} \quad (8)$$

where

$$\begin{align*}
(n \cdot n)H_G &= 2G_{ab}n^a n^b = -(n \cdot n)R^{(d)} + \frac{1}{|s|}(\pi^2 - \pi^{ab}\pi_{ab}) \\
(n \cdot n)H_G^b &= H_G^b = 2s^{bc}G_{cd}n^d = -2(n \cdot n)D_a(|s|^{-\frac{1}{2}}\pi^{ab})
\end{align*} \quad (9)$$

The Hamiltonian for Einstein gravity is pure constraint. Variation of the Hamiltonian density

$$\mathcal{H}_G = \sqrt{|s|}(NH_G + N_bH_G^b) \quad (10)$$

yields the vacuum Einstein equations. In the Hamiltonian variation, "$q$" is $s_{ab}$ and "$p$" is $\pi^{ab}$. $N, N^b$ are Lagrange multipliers.

In this paper, we consider General Relativity coupled to matter fields whose stress energy is derived from a Lagrangian $L_M$. We will assume that the matter $L_M$ can be transformed to Hamiltonian form. The particular examples we work out have total Lagrangian
\( L = R - F^2 \) with \( F = dA \). The matter Hamiltonian \( H_M \) may contain additional constraint equations \( C = 0 \), for example, Gauss’ Law. The total Hamiltonian for the coupled system is

\[
(N, N^a, \alpha) \cdot \mathcal{H}_{\text{tot}}(g, \pi, A, p) = \sqrt{|s|[N(H_G + H_M) + N_b(H_G^b + H_M)]} + \alpha C
\]

(11)

where \( \alpha \) is a Lagrange multiplier for the matter constraints. Equations (8) and \( C = 0 \) imply that on solutions to the field equations,

\[
(N, N^a, \alpha) \cdot \mathcal{H}_{\text{tot}}(g, \pi, A, p) = 0
\]

(12)

2.2. Derivation of Constraints

The derivation of constraints on perturbations \([10] [11]\) which we use is based upon the fact that the gravitational Hamiltonian density is a sum of constraints, (12). This previous work includes an analysis of the Einstein equation with a fluid stress energy, applied to cosmological spacetimes. The fluid sources are described by a stress-energy tensor, and not derived from a Lagrangian; typically the fluid is described by a density, pressure, and velocity, rather than some more fundamental set of fields. In this paper we apply the same constraint vector construction for Einstein gravity coupled to matter fields which are described by a Lagrangian. The derivation is similar to calculations of linearization instability \([7] [8] [17]\). The linearization instability work was done for compact manifolds without boundary. Here we work on non-compact manifolds, and the boundary terms are of particular interest, since they are the mass, horizon area, and tension variations.

Gauss’ Law \( \nabla_a F^{abc} = J^{bct} \) is a nice, linear constraint, which can be rewritten in integral form. The following construction extracts a similar Gauss’ Law type statement from the linearized Einstein constraints, in the case that the background spacetime has a Killing field. Let \((\bar{s}_{ab}, \bar{\pi}^{ab}, \bar{A}_{abc}, \bar{p}^{abc})\) be a solution to the field equations, where \( p^{abc} = 8\sqrt{|s|}NF^{abcw} \) is the momentum conjugate to \( A_{abc} \) (see Appendix ). Consider j perturbations about this solution,

\[
q' s : s_{ab} = \bar{s}_{ab} + \epsilon h_{ab} , \ A_{abc} = \bar{A}_{abc} + \epsilon \delta A_{abc} \\
p' s : \pi^{ab} = \bar{\pi}^{ab} + \epsilon \delta \pi^{ab} , \ p^{abc} = \bar{p}^{abc} + \epsilon \delta p^{abc}
\]

(13)

where \( \epsilon \ll 1 \). The result is derived by finding a judicious linear combination of the linearized constraints. The idea is very simple, and to display this fact, we streamline the notation: Define the variation \( \delta \) by \( \delta f = \frac{\delta f}{\delta s_{ab}} \cdot h_{ab} + \frac{\delta f}{\delta \pi^{ab}} \cdot \delta \pi^{ab} + \frac{\delta f}{\delta A_{bcd}} \cdot \delta A_{bcd} + \frac{\delta f}{\delta p^{bcd}} \cdot \delta p^{bcd} \).
Let $F, \beta^a$ be an arbitrary function and vector on $\Sigma_w$, and consider the linear combination of constraints $(F, \beta^a, A_{wbc}) \cdot \mathcal{H}_{\text{tot}}(g, \pi, A, p) = 0$. The perturbative fields, which we have divided above into $\delta q$'s and $\delta p$'s, are solutions to the linearized constraints

$$(F, \beta^a, A_{wbc}) \cdot \delta \mathcal{H}_{\text{tot}} \cdot (\delta q, \delta p) = 0.$$ (14)

Rewrite (14) in terms of the adjoint operator and a total derivative,

$$(\delta q, \delta p) \cdot \delta \mathcal{H}^*_{\text{tot}} \cdot (F, \beta^a, A_{wbc}) + D_a B^a = 0$$ (15)

where $B^a$ is a function of the $\delta q, \delta p$, the Lagrange multipliers, and of course the background spacetime.

Therefore if $F, \beta^a$ are solutions to the differential equation $\delta \mathcal{H}^*_{\text{tot}} \cdot (F, \beta^a, A_{wbc}) = 0$, then any perturbation about the background spacetime must satisfy the Gauss’s Law type constraint

$$D_a B^a = 0.$$ (16)

When do solutions for $F, \beta^a$ exist? Hamilton’s equations are $(\dot{s}, -\dot{\pi}, \dot{A}, -\dot{p}) = \mathcal{H}^*_{\text{tot}} \cdot (F, \beta^a, A_{wbc})$, where $\dot{f}$ is the lie derivative of $f$ along $V^a$,

$$V^a = Fh^a + \beta^a$$ (17)

So the requirement that (16) hold for all perturbations about the background is that the lie derivative along $V^a$ of all the $q$'s and $p$'s is zero, that is, $V^a$ must be a Killing vector.

The boundary term vector is the sum of a gravitational piece and a contribution from the matter fields, $B^a = B^a_G + B^a_M$, where

$$B^a_G = F(D^a h - D_b h^{ab}) - h D^a F + h^{ab} D_b F + \frac{\beta^b}{\sqrt{|s|}}(\pi^{cd} h_{cd} s^a_b - 2\pi^{ac} h_{bc} - 2\delta \pi^a_b)$$ (18)

$$B^a_M = -\frac{1}{\sqrt{|s|}} A_w \delta p^a + 4F \tilde{F}^{ab} \delta \tilde{A}_b + \frac{2}{\sqrt{|s|}} \beta^{[a} p^{b]} \delta \tilde{A}_b , \ 1 - \text{form}$$

$$B^a_M = -\frac{3}{\sqrt{|s|}} \tilde{A}_{bcw} \delta p^{abc} + 8F \tilde{F}^{abcd} \delta \tilde{A}_{bcd} + \frac{2}{\sqrt{|s|}} \beta^{[a} p^{bcd]} \delta \tilde{A}_{abc} , \ 3 - \text{form}$$ (19)

Here $h = h_{ab} s^{ab}$ and all indices are raised and lowered using $s_{ab}$. See Appendix A for some details of the Hamiltonian decomposition of $F^2$. 

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It is simple to include perturbative sources $\delta T_{ab}^{(s)}$, i.e., stress energy which does not come from a Lagrangian. Equation (14) is replaced by

$$(F, \beta^a, A_{wbc}) \cdot \delta H_{tot} \cdot (\delta q, \delta p) = \delta S$$

where $\delta S \equiv 16\pi (n \cdot n) \delta T^b_b V^b n_a - 4! A_{bcw} \delta J^{bcw}$, where the charged current is defined by

$$\nabla_a F^{abcd} = J^{bcd}$$

It follows that when $V^a = F n^a + \beta^a$ is a Killing vector, then $D_a B^a = \delta S$. In integral form, this is

$$\int_\Sigma \sqrt{|s|} \delta S = \int_{\partial \Sigma} da_c B^c$$

(22) is the constraint on perturbations about a spacetime with a Killing field, when the background matter fields come from a Lagrangian. Why is this last condition needed? Consider the Einstein equation with a fluid stress-energy, as for cosmological spacetimes. In a $d+1$ split, the form of the Einstein constraints (8) still holds, but Hamilton’s equation for $\dot{\pi}$ becomes $\frac{\delta H_{tot}}{\delta \pi^{ab}} = -\pi^{ab} + 8\pi N T_{f_1}^{ab}$. One finds that the vector $V^a$ must satisfy a set of equations different from Killings equation, for a constraint of the form (22) to hold on perturbations [10]. $V^a$ was referred to as an integral constraint vector in that work, and the Freidman-Robertson-Walker spacetimes turn out to have constraint vectors. Tod [18] has shown that there exists a maximal set of solutions for $V^a$ when $\Sigma$ can be locally embedded in a space of constant curvature, and that in $3+1$ dimensions this is related to the existence of three-surface twistors.

### 3. First Law

In this section we use (22) to give a simple proof of the First Law for black branes.

Let $M$ be a black brane spacetime with a bifurcate Killing horizon, and let $V$ be an asymptotically flat spacelike slice with timelike normal $n^a$. Assume that the directions tangent to the brane are compact. For the non-compact case one must specify additional boundary conditions at infinity along the brane. A simple example is the two-parameter family of $M2$-brane spacetimes [19]

$$ds^2 = f^{-2/3}(-Hdt^2 + dx^2 + dy^2) + f^{1/3} \left( \frac{dr^2}{H} + \delta_{ij}dx^i dx^j \right)$$

$$A_{txy} = -\frac{\tilde{q}}{r^6 + q} + \frac{\tilde{q}}{q}$$

(23)
where \( f(r) = 1 + q/r^6 \), \( H(r) = 1 - q/r^6 \). The parameters are related by \( q = q_0 \sinh^2 \delta \), \( \tilde{q} = q_0 \sinh \delta \cosh \delta \). \( i, j = 1, \ldots, 7 \) are the transverse coordinates. The extremal, supersymmetric case occurs when \( q_0 = 0 \).

First consider a slice \( V_0 \) which intersects the horizon \( \mathcal{H} \) on the bifurcation surface. For example, the bifurcation surface of the \( M2 \)-brane \( \mathcal{G} \) has topology \( S^7 \times T^2 \). Assume that the spacetime has a stationary Killing field \( \xi^a = F^a + \beta^a \) and substitute \( \xi^a = V^a \) in the constraint relation \( \mathcal{G} \). Using the asymptotically flat boundary conditions, the boundary terms at infinity simplify, and have standard interpretations as the change in the mass and the momentum, see e.g., [20]. The gravitational boundary term \( \mathcal{G} \) becomes

\[
-16\pi \delta M = \int_{\partial V_{\infty}} da_c F(D^c h - D_b h^{cb}) - h D^c F + h_b^c D^b F
\]

\[
16\pi \Omega \delta J_M = -\Omega \int_{\partial V_{\infty}} da_c \frac{2\phi^{b} \delta \pi^{c}_b}{\sqrt{|s|}}
\]

The terms \(-h D^c F + h_b^c D^b F\) do not contribute to the boundary around an asymptotically flat black hole, but they may contribute in a black brane spacetime which includes compact dimensions. For example, if the spacetime has topology of a compact manifold \( K \) cross \( R^n \), then \( F \) may depend on the coordinates \( z^i \) on \( K \). From the lower \( N \)-dimensional point of view, the \( d + 1 \) dimensional \( \delta M \) would break up into an \( N \)-dimensional \( \delta M^{(N)} \) plus contributions from the variations in moduli fields.

Suppose that the background spacetime has no magnetic charge, \( \int_{\partial V_{\infty}} \tilde{F}_{abcd} = 0 \). Then the integrand \( da_d F \tilde{F}^{dabc} \delta \tilde{A}_{abc} \) vanishes at infinity. So the contribution to the boundary term at infinity from the gauge field is

\[
< A_{tab} \delta Q^{tab} > = 3 \int_{\partial V_{\infty}} da_c A_{tab} \frac{\delta p^{cab}}{\sqrt{|s|}}
\]

\[
16\pi \delta J_M = \int_{\partial V_{\infty}} da_c \frac{2}{\sqrt{|s|}} \phi^{[c} p^{abe]} \delta \tilde{A}_{abe}
\]

Here \( Q^{tab} \) are the electric charges, \( \delta Q^{tab} = 3 \int_{(\partial \Sigma)} da_c \frac{\delta p^{cab}}{\sqrt{|s|}} (= 4! \int_{\Sigma} \delta J^{tab}) \). The last equality just reminds the reader of the relation between the electric field and charged sources, if the electric field was generated by smooth sources.

\( J_M \) is the gauge contribution to the angular momentum. If the gauge potential is constant on the boundary at infinity, then \(< A_{tab} \delta Q^{tab} > = A_{tab} \delta Q^{tab} \); otherwise the charge term in the First Law is an average over the internal space.
On the bifurcation surface of the horizon $\xi^a$ vanishes, and the gravitational boundary term is

$$\int_{\partial V_0} da_c (-h D^c F + \ell^{cb} D_b F) = 2\kappa \delta A$$

(26)

where $\kappa$ is the surface gravity and $A$ is the area of $\partial V_0$.

When $\mathcal{H}$ is a Killing horizon then it can be shown that for the one-form gauge potential, $A_a \xi^a = \text{constant}$ on all of $\mathcal{H}$, see e.g., [13] [12]. We have not been able to generalize this result to higher-form potentials. However, when the horizon has a bifurcation surface then since $\xi^a = 0$, also $A_{abc} \xi^a = 0$ there, as was used in [21] for black strings. Then all gauge terms vanish on the horizon, and (22) gives the First Law,

$$\delta M = \frac{\kappa \delta A}{8\pi} + \Omega(\delta J^G + \delta J^M) - \frac{3}{2\pi} < A_{bct} \delta Q^{bct} > + \int_{V_0} \frac{(-\delta S)}{16\pi}$$

(27)

Here $\frac{\delta S}{16\pi} = \xi^a n^b \delta T_{ab} + \frac{3}{2\pi} A_{bct} \delta J^{bct}$. (27) assumes that there is no magnetic charge and that the spatial slice $V_0$ intersects $\mathcal{H}$ at the bifurcation surface. The perturbations are arbitrary, i.e., not necessarily stationary.

In the magnetic case, there is a boundary contribution from $\tilde{F}^{abcd} \delta \tilde{A}_{bcd}$. This is because the gauge potential is defined in patches, so when using Stokes theorem to change the volume term to a boundary term, there are contributions from the boundaries of different patches. In the spherically symmetric case, $ds^2 = m_{ab} dx^a dx^b + r^2 d\Omega_4^2$, and $F = Q_B \omega_4$, where $\omega_4$ is the volume form on the unit four-sphere, this boundary term gives a contribution proportional to $\delta Q_B$: the integral factors into $Q_B V_I \int dr d\alpha d\beta d\gamma (\delta A_{N}^{\alpha} - \delta A_{N}^{S})$, where $\alpha, \beta, \gamma$ are coordinates on the $S^3$ boundary of the $N$ and $S$ patches, and $V_I$ is the volume of the remaining internal space. The integral over the $S^3$ is $\delta Q_B$, but it would seem that one needs to know the actual metric functions to evaluate the remaining integral $dr$, even in this simple case. It would be interesting to know if the magnetic contribution could be evaluated in general.

### 3.1. Extension to General Slices

The First Law (27) was proved for arbitrary perturbations, but restricted to the case when the spatial slice intersects $\mathcal{H}$ on the bifurcation surface. It is clearly more difficult to evaluate the boundary term on a general cross section of the horizon, since $\xi^a$ does not vanish, and the momentum of the spatial slice $\pi^{ab}$ is also nonzero. However, we can use the divergence free property of the ICV boundary term as follows. Let the family of
spacelike slices \( \{ V_w \} \) be asymptotically flat, and let \( V_0 \) be a slice which intersects \( \mathcal{H} \) at the bifurcation surface. When there are no fluid sources, \( (22) \) becomes \( \int_{\partial V_w} d\sigma_c B^c = 0. \) The boundary term vector \( B^a \) depends on the geometry of the spatial slice \( V_w \) in the background, which we will indicate by "\( \epsilon \)" (see \( (13) \) ). First consider the vacuum case. We have shown that for any \( w \) the contribution to \( (22) \) from the boundary at infinity is \(-16\pi\delta M + \Omega \delta J \). On \( V_0 \), the boundary term at \( \mathcal{H} \) gives \( 2\kappa \delta A_0 \). Since all the \( V_w \) share the same boundary at infinity, \( (22) \) implies that the horizon boundary terms have the same value for all \( w \),

\[
I(V_w; \epsilon) \equiv \int_{\mathcal{H}_w} B^c d\sigma_c = 16\pi\delta M = 2\kappa\delta A_0
\]

(28)

Sorkin and \cite{12} have shown that even with time dependent perturbations, the variation in the expansion \( \theta \) on the horizon is zero through linear order. This implies that \( \delta A_0 = \delta A_w \), i.e., that \( \delta A \) is the same on each slice. Therefore \( (28) \) implies that \( I(V_T; \lambda) = \kappa \delta A_T \), since \( \kappa \) is constant over the horizon.

For Einstein-Maxwell, there is an additional contribution to the common boundary term at infinity of \(-4A_t \delta Q \). What about the gauge term contribution at the horizon? For a one-form potential, \cite{13},\cite{12} have proven that \( A_a \xi^a \) vanishes on a killing horizon. In appendix B we show that \( A_{abc} \xi^a = 0 \) on a bifurcate killing horizon; it would be interesting to know if this was true with just the assumption of a Killing horizon. Therefore, the total boundary term on the horizon \( V_0 \) is still \( 2\kappa \delta A \). As in the vacuum case, the inner boundary contribution is fixed at \( 2\kappa \delta A \).

4. Constraint on Tension Variations

Let the background spacetime have a spatial translation Killing field \( X^a \). We want to work out the implications of the general constraint relation \( (22) \) when the Killing field in the construction is taken to be \( X^a = \frac{\partial}{\partial x} \) and the \( d \)-dimensional slices \( \Sigma \) are surfaces of constant \( x \). The unit normal \( \hat{x}_a = L\nabla_a x \) is spacelike, and \( s_{ab} \) is Lorentzian. (For this decomposition we will write the normal as \( \hat{x}_a \) to distinguish from the timelike normal \( n_a \) used in the last section.) So

\[
g_{ab} = s_{ab} + \hat{x}_a \hat{x}_b , \quad \hat{x} \cdot \hat{x} = +1
\]

(29)

As in Section 3, we will assume that the directions tangent to the brane are compact. Therefore, we take the spacetime to have topology of \( \mathbb{R}^N \times K \), where \( K \) is a compact
manifold. For example, for an $M2$-brane, $K$ could be an $S^1$ and $N = 10$, or $K$ could be a 2-torus cross a four (real) dimensional Kahler manifold, with $N = 5$. We also assume that the spacetime is transverse asymptotically flat, i.e., is asymptotically flat in the noncompact dimensions. The boundary of $\Sigma$ at infinity is the product of a time interval and the $(d-2)$-dimensional boundary of a $(d-1)$-dimensional spatial volume $V_x$, $\partial \Sigma^\infty = \Delta t \times \partial V_x^\infty$. The notation $V_x$ indicates that $x = \text{constant}$ in this volume, since it is a subset of $\Sigma$. The rate of fall off of metric perturbations is $h_{ab} \to O(r^{-N+3})$ as $r \to \infty$.

We define the variation in the tension of the spacetime $\delta \mu$ to be to be minus the gravitational boundary term (18) evaluated at infinity, divided by $\Delta t$. With $X^a = F\hat{x}^a + \beta^a$,

\[
16\pi \delta \mu \equiv -\frac{1}{\Delta t} \int_{\partial \Sigma^\infty} dt da_b [F(D^a h - D_b h^{ab}) - h D^a F + h^{ab} D_b F]
\]

\[
+ \frac{\beta^b}{\sqrt{|s|}} (\pi^{cd} h_{cd} s^a_b - 2\pi^{ac} h_{bc} - 2\delta \pi^a_b)]
\]

The transverse asymptotically flat boundary conditions imply that the integrand at infinity is independent of time, so the integral over $t$ factors out, and $\int dt/\Delta t = 1$.

4.1. Perturbations off Minkowski Spacetime

We start by considering the integral constraint (22) when the background is flat $(d+1-p)$-dimensional spacetime, cross a $p$-dimensional torus. This will give a physical interpretation of the gravitational boundary term in the weakly gravitating limit. Let $\delta T_a^{(s)b}$ be perturbative sources which are localized in the transverse directions. The spacetime volume in (22) is $\Sigma = \Delta t \times R^{d-p} \times I^{p-1} = \Delta t \times V_x$. The integration over the time interval $\Delta t$ cancels. Then

\[
16\pi \delta \mu \equiv -\int_{\partial V_x^\infty} da_c (D^c h - D_b h^{cb}) = 16\pi \int_{V_x} dV (-\delta p_x)
\]

where $\delta p_x = \delta T_a^{(s)b}\hat{x}^a\hat{x}_b$

The volume integral is minus the $x$-component of the pressure of the source, i.e., it is the $x$-component of the tension. Therefore it is reasonable to interpret the boundary term at infinity as the spacetime tension. More precisely, $\delta \mu$ is the integral of the tension. This is (minus) the force per unit $(d-1)$-dimensional area, times the area, i.e., $\delta \mu$ is the $x$-momentum flux. The term tension rather than pressure is being used here since test $p$-branes governed by the area action have positive tension. We emphasize that the tension is in a direction tangent to the source, and is not a radial pressure as in a star.
4.2. Background Magnetic fields

One contribution to $\delta M$ is the variation of the electric charge. This occurs because $\delta M$ is associated with the time translation symmetry of the background, and the Lagrange multiplier of $\delta Q$ is the time component of the gauge potential. When the ICV construction is done using a spatial translation symmetry of the background, then there is a contribution from the spatial component of the field strength instead. Consider the case when a test brane moves in a static background magnetic field, which was studied for a test two-brane in flat spacetime in [22]. Assume that there is no background electric charge. Without gravity, this is the situation for Ampere’s Law. For example, suppose that there is a background gauge potential $A_{xyz}(x^i)$, and a test two-brane that lies in the $x - y$ plane and moves in the $z$-direction. The brane generates a current $\delta J^{xyz}$, which in turn induces a change in the magnetic field $\delta F^{cxyz}$ satisfying $\int_{\partial D} da_c \delta F^{cxyz} = \int_D dv \delta J^{xyz}$. Including the gravitational field, then a perturbative current $\delta J^{abc}$ generates a perturbation to the gauge field, and this causes a variation in the gravitational fields. Equation (22) becomes

$$16\pi \delta \mu = -4! \int_{\partial V_x} da_b N A_{cdx} \delta F^{bcdx} + \int_{V_x} \left[ -16\pi X^a \delta T_a^{(s)b} \dot{x}_b + 4! A_{bcx} \delta J^{bcx} \right]$$

(32)

In the example above, there is a gauge field boundary term from a nonzero $A_{xyz} \delta F^{xyz}$, rather than from a background electric charge.

4.3. Perturbations off Black Branes

Now we are ready to study perturbations of the spacetime tension in a black brane spacetime. Assume that the spacetime has a static Killing field $\xi^a$, as well as the spatial translation killing vector $X^a$, so that the horizon is a Killing horizon. For simplicity we will specialize to the case where the slicing can be chosen such that $X^a = L \dot{x}^a$, $g_{xx} = X^a X_a = L^2$. Let $L_H$ and $L_\infty$ equal length of the $S^1$, to which $X^a$ is tangent, at the horizon and at infinity respectively. Let $A$ equal to the area of the cross section where the spatial slice $V$ intersects the horizon, and let $A_x$ be the area of the intersection of the subsurface $V_x$ with $\mathcal{H}$, then

$$A = L_H A_x .$$

(33)

We will also assume that $X^a$ is tangent to the horizon, $X^a \xi_a = 0$ on $\mathcal{H}$.

First consider the contribution to (22) from the boundary on the horizon. In a neighborhood of the horizon it is useful to write the metric in terms of null coordinates. Let
$k^a = \frac{\partial}{\partial x^a}$ and $q^a = \frac{\partial}{\partial \xi^a}$ be null and geodesic, where $k^a$ a geodesic generator of the horizon and $\lambda$ is an affine parameter. On $\mathcal{H}$, $\xi^a = \kappa \lambda k^a$. Using these basis vectors the metric is

$$s_{ab} = \gamma_{ab} - \xi_a q_b - \xi_b q_a$$

(34)

where $\gamma_a^b \xi^a = 0$, $\gamma_a^b q^a = 0$ and $k_a k^a = q_a q^a = 0$. We normalize $q^a$ by $\xi^a q_a = -1$ on $\mathcal{H}$. Our assumptions on $X^a$ are

$$X^a = L \hat{x}^a, X^a = \gamma^a_b X^b \quad \text{on } \mathcal{H}.$$  

(35)

The gravitational boundary term (18) depends on derivatives of $L$. On the horizon, expand the gradient of the norm $X \cdot X$ in the above basis. Since $\xi^a, X^a$ are Killing vectors and $\xi^a$ is null, the expansion is of the form

$$\nabla^a (X_b X^b) = 2L \nabla^a L = -2\nu \xi^a + \gamma^a_b C^b$$

(36)

To evaluate the boundary term on the horizon, we will see that one only needs the form of the expansion (36) and not the values of $\nu$ and $C^a$. The area element on $\mathcal{H}$ is

$$da_b = -k_b \sqrt{\gamma} d\lambda dy,$$

where $dy$ indicates integration over all the spatial coordinates except $x$. Substituting (36) into (18), the integrand is

$$-k_a B^a = -\frac{1}{\kappa \lambda} \xi_a B^a$$

$$= -\frac{L}{\kappa \lambda} [(\xi_a D_b h^{ab} - \xi_a D^a h) + \frac{1}{L}(\mu \xi_a \xi_b h^{ab} + \xi_a h^{ab} C_b)]$$

(37)

To evaluate this, we make the gauge choices $\delta \xi^a = 0$ and $\delta q^a = 0$. These are consistent conditions when the spacetime is static. Since the hypersurface is fixed, we also have that $\delta n_a = 0$. The inner product conditions which define (34) hold in the background and the perturbed spacetime. Using these conditions, one finds that on $\mathcal{H}$, $\xi^a h_{ab} = 0$. So the integrand on the horizon reduces to

$$-k_a B^a = -\frac{L}{\kappa \lambda} [(D_a (\xi_b h^{ab}) - \xi_a D^a h)]$$

(38)

where we have used the fact that $\xi$ is a killing vector.

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4 When the background spacetime is stationary, the Killing field which generates the horizon is a linear combination of the time translation and rotational Killing vectors $\xi^a = t^a + \Omega \phi^a$. Then a superior choice of gauge conditions for evaluating the boundary term are $\delta t^a = 0, \delta \phi^a = 0$ [16].

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The spacetime volume $\Sigma$ in (22) is a slice of constant $x$. $\partial \Sigma_x$ is the sum of initial and final timelike slices, a null boundary over part of the horizon, and a timelike boundary at infinity. At infinity the spacetime approaches the direct product of asymptotically flat Minkowski spacetime cross a (static) compact space $K$. The integration at infinity reduces to an integral over the spatial boundary times a time interval $\Delta t$, where $t$ is the asymptotic killing time parameter. In general, the initial and final slices will contribute to relation (22), which therefore is a time-dependent flux balance statement. There does not appear to be much one can say about this general case. However, if the background spacetime and the perturbations are static, then we can extract an interesting statement, which we now turn to.

4.4. Static Perturbations: Reduction to a Spacelike Slice

In this section, we assume that the perturbations are static, as well as the background. This allows reduction of the integral constraint on $\delta \mu$ to a statement on a spacelike slice $V_x$. The perturbations need not be translationally invariant. We choose $\Sigma$ as follows. Pick an asymptotically flat spacelike slice, which can intersect the horizon at an arbitrary cross section. Choose $(\partial \Sigma)_i$, the initial spacelike boundary of $\Sigma$, to be a volume contained in this slice, with inner boundary $R_{\text{in}}$ and outer boundary $R_{\text{out}}$. We will eventually let $R_{\text{in}}$ go to the horizon and $R_{\text{out}}$ to spatial infinity. The static Killing field $\xi^a = \frac{\partial}{\partial t}$ is timelike in this volume. Define the spacetime volume $\Sigma$ by lie dragging $(\partial \Sigma)_i$ an amount $\Delta t$ along the flow of $\xi^a$ to $(\partial \Sigma)_f$. Then for static perturbations, the integral of the boundary term over the final time slice is equal in magnitude to the integral over the initial time slice, and so these terms cancel in the integration over $\partial \Sigma$.

The outer boundary of $\Sigma$ includes an integration over an interval $\Delta t$. Take the inner boundary to the horizon, which includes integrating over an interval of affine parameter $\Delta \lambda$. On $\mathcal{H}$, $\xi^a = \frac{\partial}{\partial s} = \kappa \lambda \frac{\partial}{\partial \lambda}$. Flowing along $\xi^a$ in a neighborhood of the horizon, one has $d\lambda = \frac{\partial \lambda}{\partial s} ds = \kappa \lambda ds$. Therefore, upon lie dragging the initial spatial surface by $\Delta t = \Delta s$,

$$\int^{f}_{i} \frac{d\lambda}{\lambda} = \kappa \Delta t$$  \hspace{1cm} (39)

Now we can evaluate the boundary term (37) on the horizon. For static perturbations, $\xi^a D_a h = 0$. Using (35) Killings equation, and the gauge condition $h_{ab} \xi^a = 0$ on $\mathcal{H}$,

$$D_a (h^{ab} \xi_b) = D_c [h^{ab}_a \xi_b (\gamma^c \xi^a - \xi^a q^c - \xi^a q^c)]$$

$$= -\xi^c D_c (h_{ab} \xi^a q^b) - q^c D_c (h_{ab} \xi^a q^b)$$  \hspace{1cm} (40)
The first term vanishes for static perturbations, and the second term is proportional to the change in the surface gravity: From the definition of \( \kappa \), and the gauge condition \( \delta q^a = 0 \),

\[ 2\delta\kappa \propto q^c \nabla_c (h_{ab} \xi^a \xi^b). \]

Then implies that

\[ 2\delta\kappa = -D_a (h^{ab} \xi_b) \quad (41) \]

Combining (38), (39), and (41) we can evaluate the boundary term on the horizon, in the case that the perturbations are static. We find

\[ -\int \sqrt{\gamma} d\lambda d\gamma k_a B^a = -2\delta\kappa \frac{A}{L_H} \Delta t \quad (42) \]

where we have used equation (33). Note that the time interval \( \Delta t \) will cancel an identical contribution from the boundary term at infinity, so one is left with a statement on a spacelike slice. This is what one expects for the static case; the spacetime integral reduces to a statement which is independent of which time.

The gauge field boundary integrand is \( A_{bcx} \delta p^{bcd} da_d \). If the brane carries electric charge and is static, then only the tangent components of the gauge potential are nonzero. Precisely, suppose that \( A_{bcx} = \Sigma_i f_i \xi_{[b} u_{c]} \). Then since \( da_d \sim \xi_d \) the gauge boundary term vanishes on \( H \).

Then using (42), equation (22) becomes

\[ \delta\mu = -\frac{A\delta\kappa}{8\pi L_H} - \frac{3}{2\pi} < A_{bcx} \delta Q^{bcx} >_x - \int_{V_x} \frac{1}{16\pi} \delta S(\vec{X}) \quad (43) \]

where

\[ \frac{1}{16\pi} \delta S(\vec{X}) = N \delta T^a_b X^b \hat{x}_a - \frac{3}{2\pi} A_{bcx} \delta J^{bcx}, \quad (44) \]

\( L_H \) is defined in equation (33) and where \( \delta\mu \) is given by

\[ 16\pi \delta\mu = -\int_{\partial V_x^\infty} da_c B^c_G = \int_{\partial V_x^\infty} da_c [L(D^c h - D_b h^{cb}) - hD^c L + h^{cb} D_b L] . \quad (45) \]

The brackets \( < ** >_x \) indicate integration over internal dimensions, with \( x \) is held constant,

\[ < A_{xab} \delta Q^{xab} >_x = 3 \int_{\partial V_x^\infty} da_c A_{xab} \delta p^{xac} \frac{\delta p^{xbc}}{\sqrt{|s|}}. \]

If the gauge field goes to a constant at infinity, then \( < A_{bcx} \delta Q^{bcx} >_x = A_{bcx}^\infty \delta Q^{bcx}. \)

Equation (43) is the main result of this paper. It relates variations in the spacetime tension to variations in the horizon geometry, the gauge fields, and sources. The derivation
of (43) assumes that the spacetime and the perturbations are static, the brane is electric, the Killing field $X^a$ is orthogonal to the surfaces of constant $x$, and that $X^a$ is tangent to the horizon.

The $A\delta \kappa$ term can be replaced with a $\kappa \delta A$ term by using a Smarr-Komar type relation, as follows. For a solution to the Einstein equation in $n$ dimensions with static Killing vector $\xi^a$, 

$$-\frac{1}{2} \int_{\partial V} dS_{ab} \nabla^b \xi^a = 8\pi \int_V dV (T^a_b - g^a_b \frac{T}{n-2}) \xi^b \eta_a,$$

$\rightarrow 8\pi \frac{n-3}{n-2} \int \rho$ in the non-relativistic limit. This implies the normalization for the Komar mass, $8\pi \frac{n-3}{n-2} M \equiv -\frac{1}{2} \int_{\partial V} dS_{ab} \nabla^b \xi^a$.

Applying this Stokes relation when there is a boundary at a horizon, and with gauge fields gives

$$8\pi \frac{n-3}{n-2} M = \kappa A - 6 < A_{bet} Q^{bet} > + 8\pi \int_V L \sqrt{q} (T^a_b - g^a_b \frac{T}{n-2}) \xi^b \eta_a$$

In equation (46), and in the next few formulae, we will need to distinguish between the volume elements on $V$ and $V_x$. These will be written as $L \sqrt{q}$ and $\sqrt{q}$ respectively.

For simplicity of presentation we will now specialize to the case where there are no gauge fields, $A_{abc} = 0$. It is straightforward to keep track of all the gauge field contributions, but the resulting expression is rather lengthy. Instead we will focus on the gravitational and stress-energy contributions. Varying (46) and using the First Law gives

$$A \delta \kappa = -\frac{\kappa \delta A}{n-2} - \frac{8\pi}{(n-2)} \int_V (\delta T^a_b \xi^b \eta_a - \xi^c \eta_c \delta T)L \sqrt{q}$$

Therefore

$$\delta \mu = \frac{\kappa \delta A}{L_H 8\pi (n-2)} - \int_{V_x} \sqrt{q} \left( N \delta T^a_b X^b \dot{x}_a + \frac{L}{L_H (n-2)} \int dx (\delta T^a_b \xi^b \eta_a - \xi^c \eta_c \delta T) \right)$$

Comparing the First Law (27) and (48) shows that in general $\delta M$ and $\delta \mu$ are independent physical quantities.

Actually, we should compare the mass per unit length to $\delta \mu$ so that the quantities have the same units, since in defining $\delta \mu$ we divided by the length of the time interval $\Delta t$. Since we are fixing boundary conditions at infinity, it makes sense to regard the length $L_\infty$ as fixed, and let $M = \frac{M}{L_\infty}$. When $T_{ab} = 0$,

$$\delta \tilde{M} = \frac{\kappa \delta A}{8\pi L_\infty}, \quad \delta \mu = \frac{\kappa \delta A}{(n-2)8\pi L_H}$$

So under classical processes the tension increases; under Hawking evaporation, the black brane unstresses.
4.5. Horizon Geometry

The surface gravity and area are purely geometrical properties of a Killing horizon. The definition of surface gravity, and the fact that it is constant, depend on the fact that the horizon is generated by a Killing field. Similarly, one can ask if the existence of the spacelike Killing field $X^a$ implies any geometrical properties of the horizon? Let us recall how the notion of the surface gravity arises. On the horizon, $\xi \cdot \xi = 0$. The gradient of the norm defines the surface gravity, $\nabla^a (\xi \cdot \xi) = -2\kappa \xi^a$. The gradient does not have terms of the form $\gamma_{ab} W^c$ since the vorticity of $\xi^a$, $\nabla_b [b \xi_c]$, vanishes when projected onto the horizon. These properties imply that that $\kappa$ is a constant on $H$ when $R_{ab} \xi^a W^b = 0$ for all $W^a$ which are tangent to $H$. (So $\kappa$ is constant in vacuum and Einstein-Maxwell black holes, for example.) $\kappa$ then appears in the horizon boundary term, because the boundary integrand involves the derivative of the norm of the constraint vector.

We continue to assume equation (35), that $X^a$ is hypersurface orthogonal and tangent to the horizon. Define the vorticity of $X^a$ by $w_{ab} = \nabla_{[a} X_{b]} = \nabla_a X_b$. The vorticity projected onto the horizon is $\hat{w}_{ab} = \gamma^c_a \gamma^d_b w_{cd}$. (34) defines the rate of change of the norm of $X^a$, which is analogous to the definition of surface gravity. The last term in (36) vanishes if $\hat{w}_{ab} = 0$. While the vorticity of the null generator $\xi^a$ necessarily vanishes on $H$, this is certainly not the case for a generic Killing field. For example, a rotational Killing vector for a spherically symmetric black hole has nonzero projected vorticity. However, it is true in cases of interest for a spatial translation –e.g. in (23), the static black M2-brane solutions.

If $\hat{w}_{ab} = 0$, then (36) becomes

$$\nabla^a (X \cdot X)|_H = 2L \nabla^a L = -2\nu \xi^a.$$ (50)

We will show that with this assumption, then (i) the norm $X \cdot X$ is constant on $H$, (ii) there is a simple formulae for the coefficient $\nu$ in terms of scalar quantitites, and that (iii) $2\nu \kappa = R_{ab} X^a X^b$ on $H$.

Let $D_a$ satisfy $D_a \gamma_{bc} = 0$. Then $D_c \gamma^a_b X^b = \gamma^m_c \gamma^a_n \nabla_m X^n + (X \cdot q) \gamma^m_c \gamma^a_n \nabla_m \xi^n = \hat{w}^a_c$, since the projected vorticity of $\xi^a$ vanishes on $H$. Therefore $D_b (X \cdot X) = -2X^a \hat{w}_{ab} = 0$, i.e., the norm is constant on $H$. In general the norm is a non-zero constant; for extremal branes wrapped around the $x$-direction, $X \cdot X = \xi \cdot \xi$ which is zero on $H$.

The coefficient $\nu$ in (50) which describes the rate of change of the norm is given by $\nu = X^a q^b \nabla_b X_a$. This is analogous to $\kappa = \xi^a q^b \nabla_b \xi_a = \xi^a \xi^b \nabla_a q_b$. The surface gravity can also be computed by the formulae $\kappa^2 = \lim \frac{1}{2} \nabla_a \xi_b \nabla^a \xi^b = \lim (\xi \cdot \xi)^{-1} (\xi^a \nabla_a \xi_b)(\xi^c \nabla_c \xi^b)$.
where the limit is taken as the ratio approaches the horizon. The latter expressions do not contain the basis vector $q^a$.

Using a derivation similar to the arguments in [13] [16], we next show that (35) and $\hat{w}_{ab} = 0$ imply

$$\nu^2 = \lim \frac{1}{2} \frac{X \cdot X}{\xi \cdot \xi} \nabla_a X_b \nabla^a X^b = \lim \frac{(X^a \nabla_a X_b)(X^c \nabla_c X^b)}{\xi \cdot \xi}.$$  \hspace{1cm} (51)

Since $X^a$ is a Killing field,

$$3X_{[a}w_{bc]}X^{[a}w^{bc]} = X \cdot X w_{bc}w^{bc} + \frac{1}{2} \nabla_c (X \cdot X) \nabla^c (X \cdot X).$$  \hspace{1cm} (52)

Equation (35) states that $X^a$ is hypersurface orthogonal, in which case Froebenius’ Theorem states that $X_{[a}w_{bc]} = 0$. (52) becomes

$$-(X \cdot X) w_{ab} w^{ab} = \frac{1}{2} \nabla_c (X \cdot X) \nabla^c (X \cdot X).$$  \hspace{1cm} (53)

Evaluating this on the horizon with (50) implies that $w_{ab} w^{ab} = 0$. Now, the Froebenius condition implies that the left hand side of (52) is zero, and so is its gradient. The second term on the right hand side approaches $-2\nu^2 \xi \cdot \xi$ on the horizon, but its gradient is nonzero. Therefore dividing (52) by $\xi \cdot \xi$ and taking the limit as the horizon is approached gives $0 = \lim X \cdot X w_{ab} w^{ab} (\xi \cdot \xi)^{-1} - 2\nu^2$, which is the desired result. The second form in (51) follows from using (53).

Lastly we show that $\hat{w}_{ab} = 0$, $2\nu \kappa = R_{ab} X^a X^b$ on $\mathcal{H}$. As just noted, on $\mathcal{H}$ $w_{ab} w^{ab} = 0$. Since $X^a$ is a Killing field, $-R_{ab} X^a X^b = X^a \nabla_c \nabla^c X_a = \frac{1}{2} \nabla_c \nabla^c (X \cdot X) - w_{ab} w^{ab} = \frac{1}{2} \nabla_c \nabla^c (X \cdot X) = -\xi^a q^b \nabla_a \nabla_b (X \cdot X) = -q^b \nabla_b (X \cdot X)(\xi^a \xi^c \nabla_a q_b) = \nu \kappa$, where we have used the fact that $X \cdot X$ is constant on $\mathcal{H}$. So $\nu = 0$ in a vacuum spacetime, as long as the surface gravity is non-zero.

5. Concluding Remarks

Expressions such as (27) for $\delta M$ let us make contact with our Newtonian intuition, since (27) states that one contribution to $\delta M$ is an integral over $\delta T_{tt}$, though $\delta M$ is defined even when $T_{ab} = 0$. The idea of the construction in this paper, is to relate the integral

\[ w_{ab} w^{ab} = \hat{w}_{ab} \hat{w}^{ab} - 4\gamma^{bc}(\omega_{bc} q^c)(\omega_{ef} \xi^f) - 2(\omega_{bc} q^b \xi^c)^2 \]  \hspace{1cm} (27)

On $\mathcal{H}$, $\gamma^{bc}(\omega_{ef} \xi^f) = 0$. Therefore if $\hat{w}_{ab} = 0$ on $\mathcal{H}$, then $\omega_{bc} q^b \xi^c = 0$ and the only possible nonzero components of the vorticity are $\gamma^{ab} q^c \omega_{bc}$. 

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of other components of the stress energy to boundary integrals, possibly with a horizon present. Foliating the spacetime by \( x = \text{constant} \) surfaces in the Hamiltonian construction, and using the Killing vector \( X^a = \frac{\partial}{\partial x} \) relates the integral of \( \delta T_{xx} \) to a boundary term. Using this boundary term, \( \delta \mu \) is still defined even when \( T_{ab} = 0 \).

There is a tension \( \mu_{(i)} \) and a constraint on \( \delta \mu_{(i)} \) for each symmetry \( \frac{\partial}{\partial w^i} \). The expressions have different forms for directions in which a charged brane is wrapped, and directions which are not wrapped. If \( \frac{\partial}{\partial x} \) is tangent to the gauge potential, then \( \delta Q_{abx} \) enters the constraint \( \delta \mu_{(x)} \), whereas if \( \frac{\partial}{\partial w} \) is a symmetry direction but is not tangent to the gauge potential, there is no contribution from the gauge field to \( \delta \mu_{(w)} \). We see from comparing the first law (27) to (18) or (13) that in general \( \delta M \) and \( \delta \mu \) are independent physical quantities. For example, perturbative sources contribute differently to the mass and the tension. In order to derive constraints which have the same form as the first law, we assumed that the spacetime has a spacelike and a timelike Killing vector. Analysis of \( \delta \mu \) could alternatively be done for a cosmological spacetime which has spatial isometries, but is not static. This might provide just such a contrast as is generally useful for our own instruction, and our neighbors’ entertainment.

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**Appendix A. The Hamiltonian Formulation of** \( L_M = -F_{abcd}F^{abcd} \) **in Curved Spacetime**

Let the total Lagrangian be \( L = L_G + L_M = R - F^2 \), where \( F_{abcd} = 4\nabla_{[a}A_{bcd]} \). The metric is split as in (3). The field coordinate is \( \tilde{A}_{abc} = s^m_a s^n_b s^c_c - l A_{mln} \) and the momentum conjugate to \( \tilde{A}_{abc} \) is \( p^{abc} = \frac{\delta L}{\delta \dot{A}_{abc}} = -8N \sqrt{|s|} \tilde{F}^{wabc} = -8(n \cdot n) \sqrt{|s|} \tilde{F}_{dabc} n_d \). Carrying out the standard Legendre transform, the Hamiltonian density is

\[
\sqrt{|s|} H_M = N \left[ \frac{(n \cdot n)}{16 \sqrt{|s|}} p_{abc} p^{abc} + \sqrt{|s|} \tilde{F}^2 \right] + \frac{1}{2} N^d \tilde{F}_{dabc} p^{abc} - 3 \tilde{A}_{bcw} \partial_a p^{abc} \quad (A.1)
\]
Hamilton’s equations are

\[
\dot{A}_{abc} = \frac{\delta \sqrt{|s|} H_M}{\delta p^{abc}} = -\left(\frac{n \cdot n}{8 \sqrt{|s|}}\right) p_{abc} + \frac{1}{2} N^d \tilde{F}_{dabc} + 3 \partial_{[a} A_{bc]} \varepsilon
\]

\[
\dot{p}^{abc} = -\frac{\delta \sqrt{|s|} H_M}{\delta \dot{A}_{abc}} = 8 \sqrt{|s|} D_d (N \tilde{F}^{dabc}) + D_d (N [p^{abc}])
\]

\[
0 = \frac{\delta \sqrt{|s|} H_M}{\delta A_{bcv}} = -3 \partial_a p^{abc}
\]

In the ICV construction one uses

\[
\frac{\delta \sqrt{|s|} H_M}{\delta s_{ab}} = \frac{(n \cdot n) N}{16 \sqrt{|s|}} \left[ \frac{1}{2} p^{cde} p_{cde} s^{ab} - 3 p^{acd} p_{cd} \right] + N \sqrt{|s|} \left( \frac{1}{2} \tilde{F}^{2} s^{ab} + 4 \tilde{F}^{a c d e} \tilde{F}_{c d e} \right)
\]

With these ingredients, one finds the boundary term (19) when performing the variations to arrive at (15).

We also record the corresponding expressions for the one-form gauge potential, which serves to completely define the boundary term (19). Let \( L_M = -F^2 \), where \( F_{ab} = 2 \nabla_{[a} A_{b]} \). Then \( p^a = 4(n \cdot n) \sqrt{|s|} n^c F_{cb} s^{ab} = -4N \sqrt{|s|} F^{wa} \) is the electromagnetic momentum conjugate to \( \tilde{A}_a \). The Hamiltonian density is

\[
\sqrt{|s|} H_M = N \left[ -\frac{(n \cdot n)}{8 |s|} p_a p^a + F_{ab} F^{ab} - N_b p^a F_a^b + A_w \partial_a p^a \right]
\]

And Hamilton’s equations are

\[
\dot{A}_a = \frac{\delta H_M}{\delta p^a} = -\left(\frac{n \cdot n}{4 \sqrt{|s|}}\right) p_a - \tilde{F}_{ab} N^b + \partial_a A_w
\]

\[
\dot{p}^a = -\frac{\delta \sqrt{|s|} H_M}{\delta A_a} = \sqrt{|s|} D_b 4 N \tilde{F}^{ba} - 2 D_b p^b N^a
\]

\[
0 = \frac{\delta H_M}{\delta A_w} = -\partial_i p^i
\]

Using these expressions, one finds the one-form boundary term in (19) when working out (15). In this case, the charge term in the First Law (27), is replaced by \( \frac{1}{4 \pi} A_t \delta Q \), where \( \delta Q = \int_{\partial V} N d a_c F^{c t} = \int_V J^t, \; \nabla_a F^{ab} = J^b \). The gauge field contribution to the angular momentum is \( \delta J_M = -\frac{1}{16 \pi} \int_{\partial V} \frac{d a}{\sqrt{|s|}} (2 p^a \phi^b) \delta A_b \)
Appendix B. Constancy of $p$-form gauge potential on $\mathcal{H}$

Let $\mathcal{H}$ be a bifurcate killing horizon, with null generator $\xi$. One can choose a gauge such that $\mathcal{L}_\xi A = 0$, where $A$ is the $p$-form gauge potential. Let $s$ be a coordinate along an integral curve of $\xi$, $\xi = \frac{\partial}{\partial s}$. If $A$ is a one-form, then $\mathcal{L}_\xi (A \cdot \xi) = 0$, which implies that $A \cdot \xi$ is independent of $s$ along the curve. Then on $\mathcal{H}$, $A \cdot \xi = 0$, since it is zero on the bifurcation surface.

Next let $A$ be a two-form, and define the one-form $u_a = A_{ab} \xi^b$. Then $\mathcal{L}_\xi u = 0$, and $\mathcal{L}_\xi u \cdot u = 0$. So the norm $u \cdot u$ is a constant independent of $s$ on each integral curve of $\xi$. Since $u \cdot u = 0$ on the bifurcation surface, it follows that everywhere on $\mathcal{H}, u \cdot u = 0$. But now the inner product is positive definite, since the metric can be written as $g_{ab} = \bar{\gamma}_{ab} - \xi_a q_b - \xi_b q_a$ (similar to (34)). So $u \cdot u = (\bar{\gamma}_{ab} - \xi_a q_b - \xi_b q_a) A^a_{\xi^c} A^b_{\xi^d} = \bar{\gamma}_{ab} u^a u^b$. Therefore $u_a = 0$ on $\mathcal{H}$. The case for a three-form gauge potential proceeds in the same way.
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