Evaluating linear response in active systems with no perturbing field

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Abstract – We present a method for the evaluation of time-dependent linear response functions for systems of active particles propelled by a persistent (colored) noise from unperturbed simulations. The method is inspired by the Malliavin weights sampling method proposed by Warren and Allen (Phys. Rev. Lett., 109 (2012) 250601) for out-of-equilibrium systems of passive Brownian particles. We illustrate our method by evaluating two linear response functions for a single active particle in an external harmonic potential. As an application, we calculate the time-dependent mobility function and an effective temperature, defined through the Einstein relation between the self-diffusion and mobility coefficients, for a system of many active particles interacting via a screened Coulomb potential. We find that this effective temperature decreases with increasing persistence time of the self-propulsion. Initially, for not too large persistence times, it changes rather slowly, but then it decreases markedly when the persistence length of the self-propelled motion becomes comparable with the particle size.

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Introduction. – One is often interested in the time-dependent response of a many-particle system to an external perturbation. In particular, long-time limits of functions describing responses to weak, time-independent perturbations give linear susceptibilities and linear transport coefficients. For systems in thermal equilibrium, calculating these time-dependent linear response functions is relatively easy since they are related, via fluctuation-dissipation relations, to time-dependent correlation functions evolving with unperturbed dynamics [1,2].

Fluctuation-dissipation relations are, in general, not valid for non-equilibrium systems. In particular, they are not valid for systems under an external drive (e.g., sheared systems) or systems driven internally (e.g., systems consisting of self-propelled/active objects). Thus, in principle, in order to calculate linear response in non-equilibrium systems one has to run simulations at a finite value of the perturbation and approximate the susceptibility by a finite difference (with due care given to the perturbation being weak enough so that one is in the linear response regime). Notably, due to the lack of time-translational invariance, evaluating time-dependent response functions requires performing many independent simulations in order to obtain statistically significant results. This makes such calculations computationally very expensive.

Fortunately, at least for some cases, methods have been developed which allow one to calculate linear response functions from simulations of unperturbed non-equilibrium systems. Chatelain [3] and Ricci-Tersenghi [4] introduced closely related methods to calculate linear response functions in aging Ising spin systems evolving with Monte Carlo dynamics. These methods were re-derived within a more general approach and compared with other no-field methods by Corberi et al. [5]. Berthier [6] derived a similar method for aging glass-forming fluids evolving with Monte Carlo dynamics. Finally, Warren and Allen [7] presented a general approach for calculating linear response functions in systems of interacting particles evolving with continuous-time Brownian dynamics. Warren and Allen placed their method (and earlier approaches of refs. [3,4,6]) in the context of Malliavin weight sampling used in quantitative finance to evaluate price sensitivities of derivative securities (“Greeks”) [8].

In this letter we present a method for calculating time-dependent linear response functions for a class of
non-equilibrium active systems from simulations without any perturbing field. The particles comprising these systems move under the influence of self-propulsion. We model the self-propulsion as a persistent (colored) noise. The presence of a finite persistence time of the noise requires a non-trivial generalization of the Malliavin weight sampling method derived by Warren and Allen.

One important motivation for our method is to model active microrheology experiments [9] which are used to describe mechanical properties of active biological systems. These experiments monitor frequency-dependent linear response functions which can be obtained from the time-dependent ones via Fourier transforms. More generally, our method opens the way to efficient calculations of sensitivities to changes of external parameters/conditions of stochastic processes evolving under the influence of colored noises.

We apply our method to a model system consisting of active particles with the self-propulsion evolving according to the Ornstein-Uhlenbeck process. Originally, we introduced this system [10] as a continuous-time version of a Monte Carlo model proposed by Berthier [11]. The same model was independently introduced by Maggi et al. [12]. It has recently been studied by Fodor et al. [13] and termed the active Ornstein-Uhlenbeck particles (AOUPs) model. We note that an approximate mapping has been proposed [14] between the AOUP system and the standard active Brownian particles model [15].

A single-component AOUP system is characterized by three parameters: the number density, single-particle effective temperature (which determines the long-time diffusion coefficient of an isolated particle) and persistence time of the self-propulsion. In the limit of vanishing persistence time an AOUP system becomes equivalent to a thermal Brownian system at the temperature equal to the single-particle effective temperature. Interestingly, Fodor et al. found that for a range of persistence times an AOUP system can be approximated by an equilibrium system with an effective, persistence time-dependent potential [12,14].

In the following, we state the main result, illustrate it using two analytically solvable examples, present a non-trivial application, and close with some discussion. The derivation of the main result is outlined in the appendix.

Main result. — To simplify the notation, we will discuss a single self-propelled particle evolving under the influence of an external force. The generalization to a system of many interacting self-propelled particles is straightforward. We will write the equations of motion in the form used in ref. [16], which is more consistent with equations of motion used by Fodor et al. [13] than the original equations of motion of ref. [10],

\[
\begin{align*}
\dot{x} &= \xi_0^{-1} (F(x) + f), \\
\tau_p \dot{f} &= -f + \eta.
\end{align*}
\]

Here \(F(x)\) is the external force, \(f\) is the self-propulsion, \(\xi_0\) is the friction coefficient of an isolated particle, \(\tau_p\) is the persistence time of the self-propulsion, and \(\eta\) a Gaussian white noise with zero mean and variance \(\langle \eta(t)\eta(t') \rangle_{\text{noise}} = 2\xi_0^2 \tau_p \delta(t - t')\), where \(\langle \ldots \rangle_{\text{noise}}\) denotes averaging over the noise distribution, and \(\tau_p^{-1}\) is the single-particle effective temperature. Without the external force, the long-time motion of the particle evolving according to eqs. (1), (2) is diffusive with the diffusion coefficient \(D_0 = \tau_p^{-1}/\xi_0\) (we use the system of units such that the Boltzmann constant is equal to 1).

The problem that we want to address can be formulated as follows. Let us assume that at \(t = 0\) the system is in the stationary state and then the external force changes, \(F(x) \to F_\lambda(x)\). We would like to evaluate the linear response of a function of the particle’s position \(\Phi(x)\), to this change. In other words, we are interested in \(\frac{\partial}{\partial \lambda} \langle \Phi(x(t)) \rangle_\lambda\). Here \(\langle \ldots \rangle_\lambda\) denotes averaging for the system prepared at \(t = 0\) in the steady state corresponding to force \(F(x)\), and then evolving for \(t > 0\) under the influence of modified force \(F_\lambda(x)\). All the derivatives with respect to \(\lambda\) are calculated at \(\lambda = 0\), corresponding to the unperturbed evolution. In the following, \(\langle \ldots \rangle\) denotes the unperturbed, steady-state average.

The main result of this letter is that \(\frac{\partial}{\partial \lambda} \langle \Phi(x(t)) \rangle_\lambda\) can be evaluated as a weighted average over unperturbed dynamics,

\[
\frac{d}{d\lambda} \langle \Phi(x(t)) \rangle_\lambda = \langle \Phi(x(t)) (q(t) + p(t)) \rangle_0 + \tau_p \langle \Phi(x(t)) q(t) \rangle_0.
\]

In eq. (3), \(q(t)\) and \(p(t)\) are Malliavin-like weighting functions that evolve according to the following equations of motion,

\[
\begin{align*}
\dot{q} &= \frac{1}{2\xi_0^2 \tau_p} \frac{\partial F_\lambda(x)}{\partial \lambda} \eta, \\
\dot{p} &= \frac{1}{2\xi_0^2 \tau_p} \tau_p (F(x) + f) \frac{\partial^2 F_\lambda(x)}{\partial x \partial \lambda} \eta.
\end{align*}
\]

with initial conditions \(q(0) = 0 = p(0)\).

We note that by taking the \(\tau_p \to 0\) limit while keeping \(\tau_p^{-1}\) constant, the equations of motion (1), (2) become equivalent to the Langevin equation describing a Brownian particle moving under the influence of an external force with thermal noise determined by \(T^{-1} = \tau_p^{-1}\) Correspondingly, in the same limit both the weight \(p(t)\) and the second average on the right-hand side of eq. (3) vanish, and our main result, eq. (3), becomes equivalent to the main result of Warren and Allen, eq. (3) of ref. [7].

Examples. — We consider a self-propelled particle in a harmonic potential. The equations of motion read

\[
\begin{align*}
\dot{x} &= \xi_0^{-1} (-kx + f), \\
\tau_p \dot{f} &= -f + \eta.
\end{align*}
\]

In the first example, we perturb the system by a constant force \(\lambda_1\). This amounts to the substitution
perturbation was turned on, d(\langle x(t) \rangle_{\lambda_1})/d\lambda_1. The constant force shifts the average position of the particle away from the center of the harmonic potential, and the most interesting linear response is the change of the average position of the self-propelled particle at time t after the perturbation was turned on, d(\langle x(t) \rangle_{\lambda_1})/d\lambda_1.

With the constant force perturbation \partial F_{\lambda_1}/\partial \lambda_1 = 1 and \partial^2 F_{\lambda_1}/\partial x \partial \lambda_1 = 0, and thus the equation of motion for weight q_1(t) reads

\[ \dot{q}_1 = \left(2\xi_0^2 T_{\text{eff}}^{\text{sp}}\right)^{-1} \eta \]

(8)

and weight p_1(t) vanishes. Equations (6)–(8) can be integrated,

\[ x(t) = x(0)e^{-kt/\xi_0} + \frac{f(0)}{k/\xi_0 - 1/\tau_p} \left( e^{-t/\tau_p} - e^{-kt/\xi_0} \right) + \tau_p^{-1} \int_0^t dt' e^{-k(t-t')/\xi_0} \int_0^{t'} dt'' e^{-(t''-t')/\tau_p} \eta(t''), \]

(9)

\[ q_1(t) = \left(2\xi_0^2 T_{\text{eff}}^{\text{sp}}\right)^{-1} \int_0^t dt' \eta(t'). \]

(10)

To calculate \langle x(t)q_1(t) \rangle_{\lambda_1}, we need to evaluate the weighted averages \langle x(t)q_1(t) \rangle_{\lambda_1} and \langle \dot{x}(t)q_1(t) \rangle_{\lambda_1}. Using eqs. (9), (10) we obtain

\[ \langle x(t)q_1(t) \rangle_{\lambda_1} = \frac{1 - e^{-kt/\xi_0}}{k} - \frac{e^{-t/\tau_p} - e^{-kt/\xi_0}}{k - \xi_0/\tau_p}, \]

(11)

\[ \tau_p \langle \dot{x}(t)q_1(t) \rangle_{\lambda_1} = \frac{e^{-t/\tau_p} - e^{-kt/\xi_0}}{k - \xi_0/\tau_p}. \]

(12)

Thus,

\[ \frac{d\langle x(t) \rangle_{\lambda_1}}{d\lambda_1} = \langle x(t)q_1(t) \rangle_{\lambda_1} + \tau_p \langle \dot{x}(t)q_1(t) \rangle_{\lambda_1} = \frac{1 - e^{-kt/\xi_0}}{k}. \]

(13)

which agrees with the result obtained by solving and then averaging the perturbed equations of motion. In fig. 1 we compare results of numerical simulations of eqs. (6)–(8) with analytical formulas (11)–(13). On the scale of the figure, the simulation results are indistinguishable from the analytical predictions.

We note that although for the constant force perturbation weighting function p_1(t) vanishes, to get the linear response we need to include the term \tau_p \langle \dot{x}(t)q_1(t) \rangle_{\lambda_1}, which implies that even in this case our method is different from that of Warren and Allen [7].

In the second example, we perturb the force constant in eq. (6), \( k \to k - \lambda_2 \). The change of the force constant does not change the average position of the self-propelled particle. Instead, it changes the spatial extent of the particle’s steady-state distribution, and the most interesting linear response function measures the change of the average square position of the particle at time \( t \) after the perturbation was turned on, \( d(\langle x^2(t) \rangle)_{\lambda_2}/d\lambda_2 \). In this case,\n
\[ \lim_{t \to \infty} \frac{d(\langle x(t) \rangle_{\lambda_1})}{d\lambda_1} = \frac{1}{k}. \]

(17)

\[ \lim_{t \to \infty} \frac{d(\langle x^2(t) \rangle_{\lambda_2})}{d\lambda_2} = \frac{T_{\text{eff}}^{\text{sp}} (1 + 2kT_\eta/\xi_0)}{k^2 (1 + kT_\eta/\xi_0)^2}. \]

(18)
ponent of its position changes linearly with time. This allows us to define mobility coefficient $\mu$,

$$\langle \alpha_1(t) \rangle_\lambda \sim \mu \lambda \text{ for } t \gg 1.$$  \hspace{1cm} (22)

In general, $\mu$ depends on the density, the single-particle effective temperature and the persistence time. Its inverse is the single-particle friction coefficient, $\xi = 1/\mu$.

We evaluated the mobility function for a $d = 3$ dimensional system of $N = 1372 \text{ AOUFs interacting via a screened Coulomb potential, } V(r) = A \exp(-\kappa(r-S))/r$, with $A = 475T_{\text{eff}}^{sp} \kappa$ and $\kappa \sigma = 24$, at number density $N \sigma^3/V = 0.51$. The parameters were chosen in such a way that in the limit of vanishing persistence time the present system becomes equivalent to a colloidal system that we investigated in the past [17]. In the following we use reduced units, with $\sigma$ being the unit of length and $\sigma^2 \kappa \sigma$ being the unit of time. We note that the system we considered is rather dense and its steady-state structure factor, for all persistence times investigated, for small wave vectors is approximately constant and small. This suggests that this system does not undergo a phase separation into a dilute and dense components, at least for the persistence times investigated.

The most direct application of the approach presented here would be to run an un-perturbed simulation and, starting at $t = 0$, to monitor the weighting function $q_{1\alpha}(t)$, which evolves according to equation of motion

$$\dot{q}_{1\alpha} = (2T_{\text{eff}}^{sp})^{-1} \eta_{1\alpha},$$  \hspace{1cm} (23)

where $\eta_{1\alpha}$ is the $\alpha$ component of the noise acting on the self-propulsion of particle $i$. Then, to get the response function one would need to evaluate $\langle \alpha_1(t) q_{1\alpha}(t) \rangle_{1\alpha}$.

In practice, it is advantageous to monitor $3N$ weighting functions corresponding to all particles and all Cartesian directions, and to average over time origins. This results in the following expression for the mobility function:

$$\chi(t) = \frac{1}{dN N_{t_0}} \sum_{\alpha, i} \left[ \langle \alpha_1(t) \rangle_{1\alpha}(t + t_0) - q_{1\alpha}(t_0) \right] + \tau_p \langle \dot{\alpha}_i(t) \rangle_{1,\alpha}(t_0).$$  \hspace{1cm} (24)

where $N_{t_0}$ is the number of time origins.

We should emphasize at this point that it is averaging over time origins that makes it possible to efficiently calculate the response function from a single unperturbed trajectory. For a system in thermal equilibrium, the mobility function $\chi(t)$ is simply related to the mean-square displacement (MSD),

$$T\chi(t) = (2d)^{-1} \langle (r_i(t) - r_i(0))^2 \rangle_{eq}.$$  \hspace{1cm} (25)

In the long-time limit the MSD grows as $2dDt$, with $D$ being the self-diffusion coefficient. Combining the definition of the mobility (22) and the asymptotic time dependence of the MSD we get the Einstein relation, $T\mu = D$.  

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Grzegorz Szamel

Fig. 2: (Color online) The response of the average square position to a change in the force constant, $d\langle x^2(t) \rangle_{\lambda\lambda}/d\lambda$. Symbols: results of numerical simulations of eqs. (6), (7) and (14), (15) for $k = 1$ and $\tau_p = 3$. Circles: $\langle x^2(t) \rangle_{p2}(t)$; triangles: $\langle x^2(t) \rangle_{p2}(t)$; squares: $\tau_p \langle x^2(t) \rangle_{p2}(t)$; diamonds: complete response. The lines show the analytical predictions.
and MSD/6 (dashed lines). Thin lines show \( \tau_p = 0.01 \) and thick lines show \( \tau_p = 1 \). The figure shows that the effective temperature based on the Einstein relation, \( T_{\text{eff}}^E = D/\mu \), is close to \( T_{\text{sp}}^\text{eff} \) for \( \tau_p = 0.01 \) and is notably smaller than \( T_{\text{sp}}^\text{eff} \) for \( \tau_p = 1 \).

This relation is, in general, not valid outside of thermal equilibrium.

The Einstein relation can, however, be used to define

\[
T_{\text{eff}}^E = D/\mu.
\]

Here the superscript “E” indicates that \( T_{\text{eff}}^E \) is defined through the Einstein relation. This effective temperature depends on the density, the single-particle effective temperature and the persistence time, and in general it is different from \( T_{\text{sp}}^\text{eff} \), except in the low density or vanishing persistence time limits. We showed previously that for a sheared Brownian system \( T_{\text{eff}}^E \) determines the density distribution in a slowly varying external potential beyond linear response [18]. It would be interesting to investigate whether this is also true for an AOUPs system.

In fig. 3 we compare mobility \( T_{\text{eff}}^E \chi(t) \) and the MSD/6 for two values of the persistence time, \( \tau_p = 0.01 \) and \( \tau_p = 1 \). For short times these two functions are notably different. In fact, it can be shown that in the short-time limit \( \chi(t) \) grows linearly with time, whereas the MSD grows quadratically with time. In the long-time limit both functions grow linearly with time. It can be seen the long-time limits of \( T_{\text{eff}}^E \chi(t) \) and MSD/6 are very close for \( \tau_p = 0.01 \) and markedly different for \( \tau_p = 1 \). This implies that for \( \tau_p = 0.01 \) the effective temperature \( T_{\text{eff}}^E \) is close to \( T_{\text{sp}}^\text{eff} \), whereas for \( \tau_p = 1 \) these two temperatures are different.

In fig. 4(a) we show the persistence time dependence of the effective temperature defined through the Einstein relation. For a range of persistence times \( T_{\text{eff}}^E \) changes rather slowly. Then, \( T_{\text{eff}}^E \) starts changing more rapidly and it reaches approximately \( T_{\text{eff}}^E/2 \) when the persistence length of the active motion, \( l_p \sim \sqrt{T_{\text{eff}}^E/\tau_p/\xi_0} \), becomes comparable to the range of the potential.

Interestingly, the dependence of the effective temperature defined through the Einstein relation on the persistence time of the self-propulsion is the opposite of that obtained by Levis and Berthier [19] for the system of self-propelled hard disks evolving with Monte Carlo dynamics. Other evidence of a non-universal dependence of the properties of active systems on the departure from thermal equilibrium was noted earlier in ref. [16].

In fig. 4(b) we show the \( \tau_p \)-dependence of the simplest structural quantity, the pair distribution function \( g(r) \) [1,2]. Even for the shortest persistence time investigated, \( \tau_p = 0.001 \), the height of the first peak of \( g(r) \) is markedly different from its value for the thermal Brownian system.

The results shown in fig. 4 are consistent with results obtained by Fodor et al. [13]. They found that for a range of persistence times the main effect of the departure from equilibrium is a renormalization of the interaction potential. This has a profound influence on the local structure but does not change the effective temperature defined through the Einstein relation.

**Discussion.** – We have presented here a method to calculate linear response functions for a class of self-propelled systems from un-perturbed simulations. Our approach generalizes the Malliavin weights method to systems evolving under the influence of a persistent noise. The method
can be easily applied to calculate the response of an AOPUs system to a periodic-in-space time-independent potential [19]. This will allow us to study the wave vector dependence of effective temperatures. We would also like to investigate the response to an externally imposed shear flow. This will allow us to study the linear viscoelastic properties of active systems. Finally, it would be interesting to investigate whether the approach presented here could be generalized to calculate directly the frequency-dependent response, i.e., the response to a perturbation periodic in time [9], and to calculate directly linear response functions for systems of active Brownian particles, without relying on the mapping procedure proposed in ref. [14].

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Appendix: derivation of the main result. – To evaluate the dependence of \( \langle \Phi(x;t) \rangle_\lambda \) on \( \lambda \) we follow the strategy inspired by sect. 2 of ref. [8]. Specifically, we write \( \langle \Phi(x;t) \rangle_\lambda \) as

\[
\langle \Phi(x;t) \rangle_\lambda = \int dx_N df_N \cdots dx_0 df_0 \Phi(x_N, f_N|x_{N-1}, f_{N-1}; \Delta t) \times \cdots P_A(x_1, f_1|x_0, f_0; \Delta t) P_{ss}(x_0, f_0),
\]

(A.1)

where, for \( 1 < i \), \( P_A(x_i, v_i|x_{i-1}, f_{i-1}; \Delta t) \) is the transition probability over a small interval \( \Delta t = t/N \), corresponding to the evolution equations (1), (2) with modified force \( F_\lambda \),

\[
P_A(x_i, f_i|x_{i-1}, f_{i-1}; \Delta t) = \delta (x_i - x_{i-1} - (F_\lambda(x_{i-1}) + f_{i-1}) \Delta t/\xi_0) \times \exp \left( -\frac{(f_i - f_{i-1} + f_{i-1}\Delta t/\xi_0)^2}{4\pi\xi_0 T^{ss}_F/\tau^2_p} \right),
\]

(A.2)

\( P_A(x_i, v_i|x_{i-1}, f_{i-1}; \Delta t) \) is the transition probability with the un-modified force \( F \), and \( P_{ss}(x_0, f_0) \) is the steady-state distribution for the un-perturbed system (i.e., the system evolving under the influence of force \( F \)).

At this point it is convenient to change the integration variables for \( 0 < i < N \) from \( (x_i, f_i) \) to \( (x_i, w_i) \), where \( w_i = F_\lambda(x_i) + f_i \). Next, one differentiates both sides of eq. (A.1) with respect to \( \lambda \). Then, after some transformations, one changes the variables back to the original variables and one obtains the following equation:

\[
\frac{d}{d\lambda} \langle \Phi(x;t) \rangle_\lambda = \int dx_N df_N \cdots dx_0 df_0 \Phi(x_N)
\]

\[
\times \left[ \frac{(f_N - f_{N-1} + f_{N-1}\Delta t/\tau_p)}{2\xi_0 T^{ss}_F/\tau^2_p} \right] \left( 1 - \frac{\Delta t}{\tau_p} \right) \frac{\partial F_\lambda(x_{N-1})}{\partial \lambda} \Delta t + \frac{\partial^2 F_\lambda(x_{i-1})}{\partial \xi_0 \partial \tau_p} \left( \frac{\partial F_\lambda(x_{i-1})}{\partial \lambda} \right) \Delta t \right.
\]

\[
\left. + \frac{\partial F_\lambda(x_{i-1})}{\partial \xi_0} \frac{\partial F_\lambda(x_{i-1})}{\partial \tau_p} \left( \frac{\partial F_\lambda(x_{i-1})}{\partial \lambda} \right) \Delta t \right]
\]

\[
+ \frac{(f_i - f_0 + f_0\Delta t/\tau_p) \partial F_\lambda(x_0)}{2\xi_0 T^{ss}_F/\tau^2_p} \] \times P(x_N, f_N|x_{N-1}, f_{N-1}; \Delta t) \cdots P(x_1, f_1|x_0, f_0; \Delta t) \times P_{ss}(x_0, f_0).
\]

(A.3)

Then, one uses an identity which follows from the time independence of the steady-state distribution,

\[
\int dx_N df_N \cdots dx_0 df_0 \Phi(x_N) = \Phi(x_{N-1})
\]

\[
\times \left[ \frac{(f_N - f_{N-1} + f_{N-1}\Delta t/\tau_p)}{2\xi_0 T^{ss}_F/\tau^2_p} \right] \frac{\partial F_\lambda(x_{N-1})}{\partial \lambda} \Delta t \times \cdots P(x_N, f_N|x_{N-1}, f_{N-1}; \Delta t) \cdots P(x_1, f_1|x_0, f_0; \Delta t) \times P_{ss}(x_0, f_0).
\]

(A.4)

Combining eqs. (A.3) and (A.4) one arrives at the final equation

\[
\frac{d}{d\lambda} \langle \Phi(x;t) \rangle_\lambda = \int dx_N df_N \cdots dx_0 df_0 \Phi(x_N)
\]

\[
\times \left[ \frac{(f_N - f_{N-1} + f_{N-1}\Delta t/\tau_p) \partial F_\lambda(x_{N-1})}{2\xi_0 T^{ss}_F/\tau^2_p} \right] \frac{\partial F_\lambda(x_{N-1})}{\partial \lambda} \Delta t + \frac{\partial^2 F_\lambda(x_{i-1})}{\partial \xi_0 \partial \tau_p} \left( \frac{\partial F_\lambda(x_{i-1})}{\partial \lambda} \right) \Delta t \right]
\]

\[
+ \frac{(f_i - f_0 + f_0\Delta t/\tau_p) \partial F_\lambda(x_0)}{2\xi_0 T^{ss}_F/\tau^2_p}
\]

\[
\times P(x_N, f_N|x_{N-1}, f_{N-1}; \Delta t) \cdots P(x_1, f_1|x_0, f_0; \Delta t) \times P_{ss}(x_0, f_0).
\]

(A.5)

In eqs. (A.3)–(A.5) \( P(x_i, f_i|x_{i-1}, f_{i-1}; \Delta t) \) is the transition probability corresponding to the un-perturbed
evolution. It has the same form as the transition probability (A.2) but the force is the unperturbed force $F(x_{i-1})$.

Assuming that the $\Delta t \to 0$ limit can be taken, we get our main result, eq. (3), with weights $q(t)$ and $p(t)$ evolving according to eqs. (4), (5).

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