Secants, Bitangents, and Their Congruences

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Abstract  A congruence is a surface in the Grassmannian Gr(1, P^3) of lines in projective 3-space. To a space curve C, we associate the Chow hypersurface in Gr(1, P^3) consisting of all lines which intersect C. We compute the singular locus of this hypersurface, which contains the congruence of all secants to C. A surface S in P^3 defines the Hurwitz hypersurface in Gr(1, P^3) of all lines which are tangent to S. We show that its singular locus has two components for general enough S: the congruence of bitangents and the congruence of inflectional tangents. We give new proofs for the bidegrees of the secant, bitangent and inflectional congruences, using geometric techniques such as duality, polar loci and projections. We also study the singularities of these congruences.

1 Introduction

The aim of this article is to study subvarieties of Grassmannians which arise naturally from subvarieties of complex projective 3-space P^3. We are mostly interested in threefolds and surfaces in Gr(1, P^3). These are classically known as line complexes and congruences. We determine their classes in the Chow ring of Gr(1, P^3) and their singular loci. Throughout the paper, we use the phrase ‘singular points of a congruence’ to simply refer to its singularities as a subvariety of the Grassmannian.
Gr(1, P³). In older literature, this phrase refers to points in P³ lying on infinitely many lines of the congruence; nowadays, these are called fundamental points.

The Chow hypersurface \( \text{CH}_3(C) \subset \text{Gr}(1, \mathbb{P}^3) \) of a curve \( C \subset \mathbb{P}^3 \) is the set of all lines in \( \mathbb{P}^3 \) that intersect \( C \), and the Hurwitz hypersurface \( \text{CH}_1(S) \subset \text{Gr}(1, \mathbb{P}^3) \) of a surface \( S \subset \mathbb{P}^3 \) is the Zariski closure of the set of all lines in \( \mathbb{P}^3 \) that are tangent to \( S \) at a smooth point. Our main results are consolidated in the following theorem.

**Theorem 1.1.** Let \( C \subset \mathbb{P}^3 \) be a nondegenerate curve of degree \( d \) and geometric genus \( g \) having only ordinary singularities \( x_1, x_2, \ldots, x_s \) with multiplicities \( r_1, r_2, \ldots, r_s \). Let \( \text{Sec}(C) \) denote the locus of secant lines to \( C \), then the singular locus of \( \text{CH}_3(C) \) is \( \text{Sec}(C) \cup \bigcup_{i=1}^s \{ L \in \text{Gr}(1, \mathbb{P}^3) : x_i \in L \} \), the bidegree of \( \text{Sec}(C) \) is

\[
\left( \frac{1}{2}(d-1)(d-2) - g - \sum_{i=1}^s \frac{1}{2} r_i(r_i-1), \frac{1}{2}d(d-1) \right),
\]

and the singular locus of \( \text{Sec}(C) \), when \( C \) is smooth, consists of all lines that intersect \( C \) with total multiplicity at least 3.

Let \( S \subset \mathbb{P}^3 \) be a general surface of degree \( d \) with \( d \geq 4 \). If \( \text{Bit}(S) \) denotes the locus of bitangents to \( S \) and \( \text{Infl}(S) \) denotes the locus of inflectional tangents to \( S \), then the singular locus of \( \text{CH}_1(S) \) is \( \text{Bit}(S) \cup \text{Infl}(S) \), the bidegree of \( \text{Bit}(S) \) is

\[
\left( \frac{1}{2}d(d-1)(d-2) - g, \frac{1}{2}d(d-2)(d-3)(d+3) \right),
\]

the bidegree of \( \text{Infl}(S) \) is \( (d(d-1)(d-2), 3d(d-2)) \), and the singular locus of \( \text{Infl}(S) \) consists of all lines that are inflectional tangents at at least two points of \( S \) or intersect \( S \) with multiplicity at least 4 at some point.

The bidegree of \( \text{Infl}(S) \) also appears in [22, Prop. 4.1]. The bidegrees of \( \text{Bit}(S) \), \( \text{Infl}(S) \), and \( \text{Sec}(C) \), for smooth \( C \), already appear in [2], a paper to which we owe a great debt. Nevertheless, we give new, more geometric, proofs not relying on Chern class techniques. The singular loci of \( \text{Sec}(C) \), \( \text{Bit}(S) \), and \( \text{Infl}(S) \) are partially described in Lemma 2.3, Lemma 4.3, and Lemma 4.6 in [2].

Using duality, we establish some relationships of the varieties in Theorem 1.1.

**Theorem 1.2.** If \( C \) is a nondegenerate smooth space curve, then the secant lines of \( C \) are dual to the bitangent lines of the dual surface \( C^\vee \) and the tangent lines of \( C \) are dual to the inflectional tangent lines of \( C^\vee \).

Congruences and line complexes have been actively studied both in the 19th century and in modern times. The study of congruences goes back to Kummer [20], who classified those of order 1; the order of a congruence is the number of lines in the congruence that pass through a general point in \( \mathbb{P}^3 \). The Chow hypersurfaces of space curves were introduced by Cayley [4] and generalized to arbitrary varieties by Chow and van der Waerden [5]. Many results from the second half of the 19th century are detailed in Jessop’s monograph [16]. Hurwitz hypersurfaces and further generalizations known as higher associated or coisotropic hypersurfaces are studied in [11, 19, 28]. Catanese [3] shows that Chow hypersurfaces of space curves
and Hurwitz hypersurfaces of surfaces are exactly the self-dual hypersurfaces in the Grassmannian \( \text{Gr}(1, \mathbb{P}^3) \). Ran [25] studies surfaces of order 1 in general Grassmannians and gives a modern proof of Kummer’s classification. Congruences play a role in algebraic vision and multi-view geometry, where cameras are modeled as maps from \( \mathbb{P}^3 \) to congruences [24]. The multidegree of the image of several of those cameras is computed by Escobar and Knutson in [9].

In Sect. 2 we collect basic facts about the Grassmannian \( \text{Gr}(1, \mathbb{P}^3) \) and its subvarieties. Section 3 studies the singular locus of the Chow hypersurface of a space curve and computes its bidegree. Section 4 describes the singular locus of a Hurwitz hypersurface and Sect. 5 uses projective duality to calculate the bidegree of its components. In Sect. 6 we connect the intersection theory in \( \text{Gr}(1, \mathbb{P}^3) \) to Chow and Hurwitz hypersurfaces. Finally, Section 7 analyzes the singular loci of secant, bitangent, and inflectional congruences.

This article provides complete solutions to Problem 5 on Curves, Problem 4 on Surfaces, and Problem 3 on Grassmannians in [29].

2 The Degree of a Subvariety in \( \text{Gr}(1, \mathbb{P}^3) \)

In this section, we provide the geometric definition for the degree of a subvariety in \( \text{Gr}(1, \mathbb{P}^3) \). An alternative approach, using coefficients of classes in the Chow ring, can be found in Sect. 6. For information about subvarieties of more general Grassmannians, we recommend [1].

The Grassmannian \( \text{Gr}(1, \mathbb{P}^3) \) of lines in \( \mathbb{P}^3 \) is a 4-dimensional variety that embeds into \( \mathbb{P}^3 \) via the Plücker embedding. In particular, the line in 3-space spanned by the distinct points \((x_0 : x_1 : x_2 : x_3), (y_0 : y_1 : y_2 : y_3) \in \mathbb{P}^3 \) is identified with the point \((p_{0,1} : p_{0,2} : p_{0,3} : p_{1,2} : p_{1,3} : p_{2,3}) \in \mathbb{P}^3 \), where \( p_{i,j} \) is the minor formed of \( i \)th and \( j \)th columns of the matrix \([a_{01} x_1 x_2 x_3] \). The Plücker coordinates \( p_{i,j} \) satisfy the relation

\[
p_{0,1}p_{2,3} - p_{0,2}p_{1,3} + p_{0,3}p_{1,2} = 0.
\]

Moreover, every point in \( \mathbb{P}^3 \) satisfying this relation is the Plücker coordinates of some line. Dually, a line in \( \mathbb{P}^3 \) is the intersection of two distinct planes. If the planes are given by the equations \( a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 = 0 \) and \( b_0x_0 + b_1x_1 + b_2x_2 + b_3x_3 = 0 \), then the minors \( q_{i,j} \) of the matrix \([a_{01} a_{02} a_{03}] \) are the dual Plücker coordinates and also satisfy \( q_{0,1}q_{2,3} - q_{0,2}q_{1,3} + q_{0,3}q_{1,2} = 0 \). The map given by \( p_{0,1} \mapsto q_{2,3}, p_{0,2} \mapsto -q_{1,3}, p_{0,3} \mapsto q_{1,2}, p_{1,2} \mapsto q_{0,3}, p_{1,3} \mapsto -q_{0,2}, \) and \( p_{2,3} \mapsto q_{0,1} \) allows one to conveniently pass between these two coordinate systems.

A line complex is a threefold \( \Sigma \subset \text{Gr}(1, \mathbb{P}^3) \). For a general plane \( H \subset \mathbb{P}^3 \) and a general point \( v \in H \), the degree of \( \Sigma \) is the number of points in \( \Sigma \) corresponding to a line \( L \subset \mathbb{P}^3 \) such that \( v \in L \subset H \). For instance, if \( C \subset \mathbb{P}^3 \) is a curve, then the Chow hypersurface \( \text{CH}_0(C) := \{ L \in \text{Gr}(1, \mathbb{P}^3) : C \cap L \neq \emptyset \} \) is a line complex. A general plane \( H \) intersects \( C \) in \( \deg(C) \) many points, so there are \( \deg(C) \) many lines in \( H \) that pass through a general point \( v \in H \) and intersect \( C \); see Fig. [1]. Hence, the degree of the Chow hypersurface is equal to the degree of the curve.

A congruence is a surface \( \Sigma \subset \text{Gr}(1, \mathbb{P}^3) \). For a general point \( v \in \mathbb{P}^3 \) and a general plane \( H \subset \mathbb{P}^3 \), the bidegree of a congruence is a pair \((\alpha, \beta)\), where the order \( \alpha \) is
the number of points in $\Sigma$ corresponding to a line $L \subset \mathbb{P}^3$ such that $v \in L$ and the class $\beta$ is the number of points in $\Sigma$ corresponding to lines $L \subset \mathbb{P}^3$ such that $L \subset H$.

For instance, consider the congruence of all lines passing through a fixed point $x$. Given a general point $v$, this congruence contains a unique line passing through $v$, namely the line spanned by $x$ and $v$. Given a general plane $H$, we have $x \notin H$, so this congruence does not contain any line that lies in $H$. Hence, the set of lines passing through a fixed point is a congruence with bidegree $(1, 0)$. A similar argument shows that the congruence of lines lying in a fixed plane has bidegree $(0, 1)$.

The degree of a curve $\Sigma \subset \text{Gr}(1, \mathbb{P}^3)$ is the number of points in $\Sigma$ corresponding to a line $L \subset \mathbb{P}^3$ that intersects a general line in $\mathbb{P}^3$. Equivalently, it is the number of points in the intersection of $\Sigma$ with the Chow hypersurface of a general line. For instance, the set of all lines in $\mathbb{P}^3$ that lie in a fixed plane $H \subset \mathbb{P}^3$ and contain a fixed point $v \in H$ forms a curve in $\text{Gr}(1, \mathbb{P}^3)$. This curve has degree 1, because a general line has a unique intersection point with $H$ and there is a unique line passing through this point and $v$. In other words, this curve is a line in $\text{Gr}(1, \mathbb{P}^3)$.

Finally, the degree of a zero-dimensional subvariety is simply the number of points in the variety.

### 3 Secants of Space Curves

This section describes the singular locus of the Chow hypersurface of a space curve. For a curve with mild singularities, we also compute the bidegree of its secant congruence.

A curve $C \subset \mathbb{P}^3$ is defined by at least two homogeneous polynomials in the coordinate ring of $\mathbb{P}^3$, and these polynomials are not uniquely determined. However, there is a single equation that encodes the curve $C$. Specifically, its Chow hypersurface $\text{CH}_0(C) := \{ L \in \text{Gr}(1, \mathbb{P}^3) : C \cap L \neq \emptyset \}$ is determined by a single polynomial in the Plücker coordinates on $\text{Gr}(1, \mathbb{P}^3)$. This equation, known as the Chow form...
of $C$, is unique up to rescaling and the Plücker relation. For more on Chow forms; see [6].

**Example 3.1 ([6 Prop. 1.2]).** The twisted cubic is a smooth rational curve of degree 3 in $\mathbb{P}^3$. Parametrically, this curve is the image of the map $\nu_3 : \mathbb{P}^1 \to \mathbb{P}^3$ defined by $(s : t) \mapsto (s^3 : s^2t : st^2 : t^3)$. The line $L$, which is determined by the two equations $a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 = 0$ and $b_0x_0 + b_1x_1 + b_2x_2 + b_3x_3 = 0$, intersects the twisted cubic if and only if there exists a point $(s : t) \in \mathbb{P}^1$ such that

$$a_0s^3 + a_1s^2t + a_2st^2 + a_3t^3 = 0 = b_0s^3 + b_1s^2t + b_2st^2 + b_3t^3.$$  

The resultant for these two cubic polynomials, which can be expressed as a determinant of an appropriate matrix with entries in $\mathbb{Z}[a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3]$, vanishes exactly when they have a common root. It follows that the line $L$ meets the twisted cubic if and only if

$$0 = \det \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 \end{bmatrix} = -\det \begin{bmatrix} q_{0,1} & q_{0,2} & q_{0,3} \\ q_{0,2} & q_{0,3} + q_{1,2} & q_{1,3} \\ q_{0,3} & q_{1,3} & q_{2,3} \end{bmatrix},$$  

where $q_{i,j}$ are the dual Plücker coordinates. Hence, the Chow form of the twisted cubic is

$$q_{0,3}^3 + q_{0,3}q_{1,2}^2 - 2q_{0,2}q_{0,3}q_{1,3} + \delta_{0,1}q_{1,3}^2 + q_{0,2}q_{2,3} - \delta_{0,1}q_{1,2}q_{2,3}.$$  

We next record a technical lemma. If $I_X$ is the saturated homogeneous ideal defining the subvariety $X \subset \mathbb{P}^n$, then the tangent space $T_x(X)$ at the point $x \in X$ can be identified with $\{y \in \mathbb{P}^n : \sum_{i=0}^n \frac{\partial f}{\partial x_i}(x)y_i = 0 \text{ for all } f(x_0, x_1, \ldots, x_n) \in I_X\}$.

**Lemma 3.2.** Let $f : X \to Y$ be a birational finite surjective morphism between irreducible projective varieties and let $y \in Y$. The variety $Y$ is smooth at the point $y$ if and only if the fibre $f^{-1}(y)$ contains exactly one point $x \in X$, the variety $X$ is smooth at the point $x$, and the differential $d_x f : T_x(X) \to T_y(Y)$ is an injection.

**Proof.** First, suppose that $Y$ is smooth at the point $y$. Since $Y$ is normal at the point $y$, the Zariski Connectedness Theorem [21 Sect. III.9.V] proves that the fibre $f^{-1}(y)$ is a connected set in the Zariski topology. As $f$ is a finite morphism, its fibres are finite and we deduce that $f^{-1}(y) = \{x\}$. If $Y_0$ is the open set of smooth points in $Y$ and let $X_0 := f^{-1}(Y_0)$, then Zariski’s Main Theorem [21 Sect. III.9.I] implies that the restriction of $f$ to $X_0$ is an isomorphism of $X_0$ with $Y_0$. In particular, we have that $x \in X_0 \subset X$ is a smooth point. Moreover, Theorem 14.9 in [13] shows that the differential $d_x f$ is injective.

For the other direction, suppose that $f^{-1}(y) = \{x\}$ for some smooth point $x \in X$ with injective differential $d_x f$. Let $Y_1$ be an open neighbourhood of $y$ containing points in $Y$ with one-element fibres and injective differentials. Combining Lemma 14.8 and Theorem 14.9 in [13] produces an isomorphism of $X_1 := f^{-1}(Y_1)$ with $Y_1$. Since $x \in X_1$ is smooth, we conclude that $y \in Y_1 \subset Y$ is smooth. \qed
When the curve $C$ has degree at least two, the set of lines that meet it in two points forms a surface $\text{Sec}(C) \subset \text{Gr}(1, \mathbb{P}^3)$ called the secant congruence of $C$. More precisely, $\text{Sec}(C)$ is the closure in $\text{Gr}(1, \mathbb{P}^3)$ of the set of points corresponding to a line in $\mathbb{P}^3$ which intersects the curve $C$ at two smooth points. A line meeting $C$ at a singular point might not belong to $\text{Sec}(C)$, even though it has intersection multiplicity at least two with the curve; see Remark 3.4.

The following theorem is the main result in this section.

**Theorem 3.3.** Let $C \subset \mathbb{P}^3$ be an irreducible curve of degree at least 2. If $\text{Sing}(C)$ denotes the singular locus of the curve $C$, then the singular locus of the Chow hypersurface for $C$ is $\text{Sec}(C) \cup \bigcup_{v \in \text{Sing}(C)} \{L \in \text{Gr}(1, \mathbb{P}^3) : x \in L\}$.

**Proof.** We first show that the incidence variety $\Phi_C := \{(v, L) : v \in L \subset C \times \text{Gr}(1, \mathbb{P}^3)\}$ is smooth at the point $(v, L)$ if and only if the curve $C$ is smooth at the point $v \in C$. Let $f_1, f_2, \ldots, f_k \in \mathbb{C}[x_0, x_1, x_2, x_3]$ be generators for the saturated homogeneous ideal of $C$ in $\mathbb{P}^3$. Consider the affine chart of $\mathbb{P}^3 \times \text{Gr}(1, \mathbb{P}^3)$ where $x_0 \neq 0$ and $p_{01} \neq 0$. We may assume that $v = (1 : \alpha : \beta : \gamma)$ and the line $L$ is spanned by the points $(1 : 0 : a : b)$ and $(0 : 1 : c : d)$. We have that $v \in L$ if and only if the line $L$ is given by the row space of matrix

$$
\begin{bmatrix}
1 & \alpha & \beta & \gamma \\
0 & 1 & c & d
\end{bmatrix}
= \begin{bmatrix}
1 & \alpha & 0 & \beta - \alpha c & \gamma - \alpha d \\
0 & 1 & c & d
\end{bmatrix},
$$

which is equivalent to $a = \beta - \alpha c$ and $b = \gamma - \alpha d$. Hence, in the chosen affine chart, $\Phi_C$ can be written as

$$\{(\alpha, \beta, \gamma, a, b, c, d) : f_i(1, \alpha, \beta, \gamma) = 0 \text{ for } 1 \leq i \leq k, a = \beta - \alpha c, b = \gamma - \alpha d\}.
$$

As $\dim \Phi_C = 3$, it is smooth at the point $(v, L)$ if and only if its tangent space has dimension three or, equivalently, the Jacobian matrix

$$
\begin{bmatrix}
\frac{\partial f_i}{\partial x_0}(1, \alpha, \beta, \gamma) & \frac{\partial f_i}{\partial x_1}(1, \alpha, \beta, \gamma) & \frac{\partial f_i}{\partial x_2}(1, \alpha, \beta, \gamma) & 0 & 0 & 0 \\
\frac{\partial f_i}{\partial x_1}(1, \alpha, \beta, \gamma) & \frac{\partial f_i}{\partial x_2}(1, \alpha, \beta, \gamma) & \frac{\partial f_i}{\partial x_3}(1, \alpha, \beta, \gamma) & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-\alpha c & 1 & 0 & -1 & 0 & -\alpha \\
-\alpha d & 0 & 1 & 0 & -1 & -\alpha
\end{bmatrix}
$$

has rank four. We see that this Jacobian matrix has rank four if and only if the Jacobian matrix of $C$ has rank two, in which case $v \in C$ is smooth. Therefore, we deduce that $\Phi_C$ is smooth at the point $(v, L)$ exactly when $C$ is smooth at the point $v$.

By Lemma 14.8 in [13], the projection $\pi : \Phi_C \to \text{CH}_0(C)$ defined by $(v, L) \mapsto L$ is finite; otherwise $C$ would contain a line contradicting our assumptions. Moreover, the general fibre of $\pi$ has cardinality 1 because the general line $L \in \text{CH}_0(C)$ intersects $C$ in a single point. Hence, $\pi$ is birational. Applying Lemma 3.2 shows that $\text{CH}_0(C)$ is smooth at $L$ if and only if $\pi^{-1}(L) = \{(v, L)\}$ where $v \in C$ is a smooth
point and the differential \( d_{(v, L)} \pi \) is injective. Using our chosen affine chart, we see that the differential \( d_{(v, L)} \pi \) sends every element in the kernel of the Jacobian matrix to its last four coordinates. This map is not injective if and only if the kernel contains an element of the form \([\ast \ast \ast 0 0 0 0]^T \neq 0\). Such an element belongs to the kernel if and only if it is equal to \([\lambda \ c\lambda \ d\lambda \ 0 0 0 0]^T\) for some \(\lambda \in \mathbb{C} \setminus \{0\}\) and
\[
\frac{\partial f_1}{\partial x_1}(1, \alpha, \beta, \gamma) + c \frac{\partial f_1}{\partial x_2}(1, \alpha, \beta, \gamma) + d \frac{\partial f_1}{\partial x_3}(1, \alpha, \beta, \gamma) = 0
\]
for all \(1 \leq i \leq k\). Hence, for a smooth point \(v \in C\), the differential \( d_{(v, L)} \pi \) is not injective if and only if \(L\) is the tangent line of \(C\) at \(v\). Since we have that \(|\pi^{-1}(L)| = 1\) if and only if \(L\) is not a secant line and all tangent lines to \(C\) are contained in \(Sec(C)\), we conclude that \(CH_0(C)\) is smooth at \(L\) if and only if \(L \notin Sec(C)\) and \(L\) meets \(C\) at a smooth point.

**Remark 3.4.** Local computations show that the secant congruence of \(C\) generally does not contain all lines through singular points of \(C\). To be more explicit, let \(x \in C\) be an ordinary singularity: the point \(x\) is the intersection of \(r\) branches of \(C\) with \(r \geq 2\), and the \(r\) tangent lines of the branches at \(x\) are pairwise different. We claim that a line \(L\) intersecting \(C\) only at the point \(x\) is contained in \(Sec(C)\) if and only if \(L\) lies in a plane spanned by two of the \(r\) tangent lines at \(x\). The union of all those lines forms the tangent star of \(C\) at \(x\); see [17][27].

Suppose that \(x = (1 : 0 : 0 : 0)\) and \(L \in Sec(C)\) intersects the curve \(C\) only at the point \(x\). The line \(L\) must be the limit of a family of lines \(L_t\) that intersect \(C\) at two distinct smooth points. Without loss of generality, the line \(L\) is not one of the tangent lines of the curve \(C\) at the point \(x\) and each line \(L_t\) intersects at least two distinct branches of \(C\). Since there are only finitely many branches, we can also assume that each line \(L_t\) in the family intersects the same two branches of the curve \(C\). These two branches are parametrized by \((1 : f_1(s) : f_2(s) : f_3(s))\) and \((1 : g_1(s) : g_2(s) : g_3(s))\) with \(f_i(0) = 0 = g_j(0)\) for \(1 \leq i, j \leq 3\). It follows that tangent lines to these branches are spanned by \(x\) and \((1 : f_1'(0) : f_2'(0) : f_3'(0))\) or \((1 : g_1'(0) : g_2'(0) : g_3'(0))\). Parametrizing intersection points, we see that the line \(L_t\) intersects the first branch at \((1 : f_1(\varphi(t)) : f_2(\varphi(t)) : f_3(\varphi(t)))\) and the second branch at \((1 : g_1(\psi(t)) : g_2(\psi(t)) : g_3(\psi(t)))\) where \(\varphi(0) = 0 = \psi(0)\). Hence, the Plücker coordinates for \(L_t\) are
\[
\left(\frac{g_1(\psi(t)) - f_1(\varphi(t))}{t} : \frac{g_2(\psi(t)) - f_2(\varphi(t))}{t} : \ldots : \frac{f_2(\varphi(t))g_3(\psi(t)) - f_3(\varphi(t))g_2(\psi(t))}{t}\right) .
\]
Taking the limit as \(t \to 0\), we obtain the line \(L\) with Plücker coordinates
\[
(g_1'(0)\psi(0) - f_1'(0)\varphi(0) : g_2'(0)\psi(0) - f_2'(0)\varphi(0) : g_3'(0)\psi(0) - f_3'(0)\varphi(0)) .
\]
This line is spanned by the point \(x\) and
\[
(1 : g_1'(0)\psi(0) - f_1'(0)\varphi(0) : g_2'(0)\psi(0) - f_2'(0)\varphi(0) : g_3'(0)\psi(0) - f_3'(0)\varphi(0)) .
\]
so it lies in the plane spanned by the two tangent lines. From this computation, we also see that all lines passing through \(x\) and lying in the plane spanned by the tangent lines can be approximated by lines that intersect both of the branches at points different from \(x\). For this, one need only choose \(\varphi(t) = \lambda t\) and \(\psi(t) = \mu t\) for all possible \(\lambda, \mu \in \mathbb{C} \setminus \{0\}\).

Using Chern classes, Proposition 2.1 in [2] calculates the bidegree of the secant congruence of a smooth curve. We give a geometric description of this bidegree and extend it to curves with ordinary singularities.

**Theorem 3.5.** If \(C \subset \mathbb{P}^3\) is a nondegenerate irreducible curve of degree \(d\) and genus \(g\) having only ordinary singularities \(x_1, x_2, \ldots, x_s\) with multiplicities \(r_1, r_2, \ldots, r_s\), then the bidegree of the secant congruence \(\text{Sec}(C)\) is

\[
\left(\left(\frac{d}{2}\right)^2 - g - \sum_{i=1}^{s} \binom{r_i}{2}\right).
\]

**Proof.** Let \(H \subset \mathbb{P}^3\) be a general plane. The intersection of \(H\) with \(C\) consists of \(d\) points. Any two of these points define a secant line lying in \(H\); see Fig. 2. Hence,

Fig. 2 The class of the secant congruence

there are \(\binom{d}{2}\) secant lines contained in \(H\), which gives the class of \(\text{Sec}(C)\).

To compute the order of \(\text{Sec}(C)\), let \(v \in \mathbb{P}^3\) be a general point. Projecting away from \(v\) defines a rational map \(\pi_v : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2\). Set \(C' := \pi_v(C)\). The map \(\pi_v\) sends a line passing through \(v\) and intersecting \(C\) at two points to a simple node of the plane curve \(C'\); see Fig. 6. Moreover, every ordinary singularity of \(C\) is sent to an ordinary singularity of \(C'\) with the same multiplicity, and the plane curve \(C'\) has the same degree as the space curve \(C\). As the geometric genus is invariant under birational transformation, it also has the same genus; see [14, Theorem II.8.19]. Thus, the genus-degree formula for plane curves [26, p. 54, Eq. (7)] shows that the genus of \(C\) is equal to \(\left(\frac{d}{2}\right)^2 - \sum_{i=1}^{s} \binom{r_i}{2}\) minus the number of secants of \(C\) passing through \(v\). \(\square\)
Remark 3.6. If $C \subset \mathbb{P}^3$ is a curve of degree at least 2 that is contained in a plane, then its secant congruence consists of all lines in that plane and has bidegree $(0, 1)$.

Problem 5 on Curves in [29] asks to compute the dimension and bidegree of $\text{Sing}(\text{CH}_0(C))$. When $C$ is not a line, Theorem 3.3 establishes that $\text{Sing}(\text{CH}_0(C))$ is 2-dimensional. For completeness, we also state its bidegree explicitly.

Corollary 3.7. If $C \subset \mathbb{P}^3$ is an irreducible curve of degree $d \geq 2$ and geometric genus $g$ having only ordinary singularities $x_1, x_2, \ldots, x_s$ with multiplicities $r_1, r_2, \ldots, r_s$, then the bidegree of $\text{Sing}(\text{CH}_0(C))$ equals $(\left(\frac{d-1}{2}\right) - g - \sum_{i=1}^{s} (\left(\frac{r_i}{2}\right)) + s, (\frac{d}{2}))$ if $C$ is non-degenerate, and $(s, 1)$ if $C$ is contained in a plane.

Proof. The bidegree of each congruence \{ $L \in \text{Gr}(1, \mathbb{P}^3) : x_i \in L$ \} is $(1, 0)$. Hence, combining Theorem 3.3, Theorem 3.5 and Remark 3.6 proves the corollary.

\[\square\]

4 Bitangents and Inflections of a Surface

This section describes the singular locus of the Hurwitz hypersurface of a surface in $\mathbb{P}^3$. For a surface $S \subset \mathbb{P}^3$ that is not a plane, the Hurwitz hypersurface $\text{CH}_1(S)$ is the Zariski closure of the set of all lines in $\mathbb{P}^3$ that are tangent to $S$ at a smooth point. Its defining equation in Plücker coordinates is known as the Hurwitz form of $S$; see [28].

In analogy with the secant congruence of a curve, we associate two congruences to a surface $S \subset \mathbb{P}^3$. Specifically, the Zariski closure in $\text{Gr}(1, \mathbb{P}^3)$ of the set of lines tangent to a surface $S$ at two smooth points forms the bitangent congruence;

\[\text{Bit}(S) := \{ L \in \text{Gr}(1, \mathbb{P}^3) : x, y \in L \subset T_x(S) \cap T_y(S) \text{ for distinct smooth points } x, y \in S \}\.\]

The inflectional locus associated to $S$ is the Zariski closure in $\text{Gr}(1, \mathbb{P}^3)$ of the set of lines that intersect the surface $S$ at a smooth point with multiplicity at least 3;

\[\text{Infl}(S) := \{ L \in \text{Gr}(1, \mathbb{P}^3) : \text{L intersects S at a smooth point with multiplicity at least 3}\}\.\]

A general surface of degree $d$ in $\mathbb{P}^3$ is a surface defined by a polynomial corresponding to a general point in $\mathbb{P}(\mathbb{C}[x_0, x_1, x_2, x_3]_d)$. For a general surface, the inflectional locus is a congruence. However, this is not always the case, as Remark 5.8 demonstrates.

In parallel with Sect. 3 the main result in this section describes the singular locus of the Hurwitz hypersurface of $S$.

Theorem 4.1. If $S \subset \mathbb{P}^3$ is an irreducible smooth surface of degree at least 4 which does not contain any lines, then we have $\text{Sing}(\text{CH}_1(S)) = \text{Bit}(S) \cup \text{Infl}(S)$.
Proof. We first show that the incidence variety

\[ \Phi_S := \{(v, L) : v \in L \subset T_v(S) \} \subset S \times \text{Gr}(1, \mathbb{P}^3) \]

is smooth. Let \( f \in \mathbb{C}[x_0, x_1, x_2, x_3] \) be the defining equation for \( S \) in \( \mathbb{P}^3 \). Consider the affine chart in \( \mathbb{P}^3 \times \text{Gr}(1, \mathbb{P}^3) \) where \( x_0 \neq 0 \) and \( p_{0,1} \neq 0 \). We may assume that \( v = (1 : \alpha : \beta : \gamma) \) and the line \( L \) is spanned by the points \((1 : 0 : a : b)\) and \((0 : 1 : c : d)\). In this affine chart, \( S \) is defined by \( g_0(x_1, x_2, x_3) := f(1, x_1, x_2, x_3) \). As in the proof of Theorem 3.3, we have that \( v \in L \) if and only if \( a = \beta - \alpha c \) and \( b = \gamma - \alpha d \). For such a pair \((v, L)\), we also have that \( L \subset T_v(S) \) if and only if \((0 : 1 : c : d) \in T_v(S)\).

Setting \( g_1 := \frac{\partial g_0}{\partial x_1} + c \frac{\partial g_0}{\partial x_2} + d \frac{\partial g_0}{\partial x_3} \), we have \( L \subset T_v(S) \) if and only if \( g_1(\alpha, \beta, \gamma) = 0 \). Hence, in the chosen affine chart, \( \Phi_S \) can be written as

\[ \{ (\alpha, \beta, \gamma, a, b, c, d) : g_j(\alpha, \beta, \gamma) = 0 \text{ for } 0 \leq j \leq 1, a = \beta - \alpha c, b = \gamma - \alpha d \} \]

As \( \dim \Phi_S = 3 \), it is smooth at the point \((v, L)\) if and only if its tangent space has dimension three or, equivalently, its Jacobian matrix

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\alpha & 0 & -1 & 0 \\
-\alpha & 0 & 0 & -1 \\
\end{bmatrix}
\]

has rank four. Since \( S \) is smooth, we deduce that this Jacobian matrix has rank four, so \( \Phi_S \) is also smooth.

Since \( S \) does not contain any lines, all fibres of the projection \( \pi : \Phi_S \to \text{CH}_1(S) \) defined by \((v, L) \mapsto L \) are finite, so Lemma 14.8 in [13] implies that \( \pi \) is finite. Moreover, the general fibre of \( \pi \) has cardinality 1, so \( \pi \) is birational. Applying Lemma 3.2 shows that \( \text{CH}_1(S) \) is smooth at the point \((v, L)\) if and only if the fibre \( \pi^{-1}(L) \) consists of one point \((v, L)\) and the differential \( d_{(v,L)}\pi \) is injective. In particular, we have \( |\pi^{-1}(L)| = 1 \) if and only if \( L \) is not a bitangent. It remains to show that the differential \( d_{(v,L)}\pi \) is injective if and only if \( L \) is a simple tangent of \( S \) at \( v \). Using our chosen affine chart, we see that the differential \( d_{(v,L)}\pi \) projects every element in the kernel of the Jacobian matrix on its last four coordinates. This map is not injective if and only if the kernel contains an element of the form \( [\ast \ast 0 0 0 0]^T \neq 0 \). Such an element belongs to the kernel if and only if it is equal to \( [\lambda \ c \lambda \ d \lambda \ 0 0 0]^T \) for some \( \lambda \in \mathbb{C} \setminus \{0\} \) and \( g_1(\alpha, \beta, \gamma) = 0 = g_2(\alpha, \beta, \gamma) \) where \( g_2 := \frac{\partial g_1}{\partial x_1} + c \frac{\partial g_1}{\partial x_2} + d \frac{\partial g_1}{\partial x_3} \).

Parametrizing the line \( L \) by

\[ \ell(s, t) := (s : s\alpha + t : s\beta + tc : s\gamma + td) \]

for \((s : t) \in \mathbb{P}^1\) shows that the line \( L \) intersects the surface \( S \) with multiplicity at least 3 at \( v \) if and only if \( f(\ell(s, t)) \) is divisible by \( t^3 \). This is equivalent to the conditions
that $g_1(\alpha, \beta, \gamma) = \frac{\partial}{\partial t} [f(\ell(s, t))] |_{(1, 0)} = 0$ and $g_2(\alpha, \beta, \gamma) = \frac{\partial^2}{\partial t^2} [f(\ell(s, t))] |_{(1, 0)} = 0$. □

Remark 4.2. If $S$ is a surface of degree at most 3 and the line $L$ is bitangent to $S$, then $L$ is contained in $S$. Indeed, if $L$ is not contained in $S$, then the intersection $L \cap S$ consists of at most 3 points, counted with multiplicity, so $L$ cannot be a bitangent. On the other hand, when the degree of $S$ is at least four, the hypothesis that $S$ does not contain any lines is relatively mild. For example, a general surface of degree at least 4 in $\mathbb{P}^3$ does not contain a line; see [32].

5 Projective Duality

This section uses projective duality to compute the bidegrees of the components of the singular locus of the Hurwitz hypersurface of a surface in $\mathbb{P}^3$, and to relate the secant congruence of a curve to the bitangent congruence of its dual surface.

Let $\mathbb{P}^n$ be the projectivization of the vector space $\mathbb{C}^{n+1}$. If $(\mathbb{P}^n)^*$ denotes the projectivization of the dual vector space $(\mathbb{C}^{n+1})^*$, then the points in $(\mathbb{P}^n)^*$ correspond to hyperplanes in $\mathbb{P}^n$. Given a projective subvariety $X \subset \mathbb{P}^n$, a hyperplane in $\mathbb{P}^n$ is tangent to $X$ at a smooth point $x \in X$ if it contains the embedded tangent space $T_x(X) \subset \mathbb{P}^n$. The dual variety $X^\vee$ is the Zariski closure in $(\mathbb{P}^n)^*$ of the set of all hyperplanes in $\mathbb{P}^n$ that are tangent to $X$ at some smooth point.

Example 5.1. If $V$ is a linear subspace of $\mathbb{C}^{n+1}$ and $X := \mathbb{P}(V)$, then the dual variety $X^\vee$ is the set of all hyperplanes containing $\mathbb{P}(V)$, which is exactly the projectivization of the orthogonal complement $V^\perp \subset (\mathbb{C}^{n+1})^*$ with respect to the nondegenerate bilinear form $(x, y) \mapsto \sum_{i=0}^n x_i y_i$. In particular, $X^\vee$ is not the projectivization of $V^*$, and $(\mathbb{P}^n)^\vee = \emptyset$.

Remark 5.2. The dual of a line in $\mathbb{P}^2$ is a point, and the dual of a plane curve of degree at least 2 is again a plane curve. The dual of a line in $\mathbb{P}^3$ is a line, and the dual of a curve in $\mathbb{P}^3$ of degree at least 2 is a surface. The dual of plane in $\mathbb{P}^3$ is a point and the dual of a surface in $\mathbb{P}^3$ of degree at least 2 can be either a curve or a surface.

From our perspective, the key properties of dual varieties are the following. If $X$ is irreducible, then its dual $X^\vee$ is also irreducible; see [11, Proposition I.1.3]. Moreover, the Biduality Theorem shows that, if $x \in X$ is smooth and $H \in X^\vee$ is smooth, then $H$ is tangent to $X$ at the point $x$ if and only if the hyperplane in $(\mathbb{P}^n)^*$ corresponding to $x$ is tangent to $X^\vee$ at the point $H$; see [11] Theorem I.1.1. In particular, any irreducible variety $X \subset \mathbb{P}^n$ is equal to its double dual $(X^\vee)^\vee \subset \mathbb{P}^n$; again see [11] Theorem I.1.1.

The next lemma, which relates the number and type of singularities of a plane curve to the degree of its dual variety, plays an important role in calculating the bidegrees of the bitangent and inflectional congruences. A point $v$ on a planar curve
C is a simple node or a cusp if the formal completion of \( \mathcal{O}_{C_v} \) is isomorphic to \( \mathbb{C}[z_1, z_2]/(z_1^2 + z_2^2) \) or \( \mathbb{C}[z_1, z_2]/(z_1^3 + z_2^3) \) respectively; see Fig. 3. Both singularities have multiplicity 2; nodes have two distinct tangents and cusps have a single tangent.

**Lemma 5.3** (Plücker’s formula [7] Example 1.2.8). If \( C \subset \mathbb{P}^2 \) is an irreducible curve of degree \( d \) with exactly \( \kappa \) cusps, \( \delta \) simple nodes, and no other singularities, then the degree of the dual curve \( C^\vee \) is \( d(d - 1) - 3\kappa - 2\delta \).

**Proof (Sketch).** Let \( f \in \mathbb{C}[x_0, x_1, x_2] \) be the defining equation for \( C \) in \( \mathbb{P}^2 \), so we have \( \text{deg}(f) = d \). To begin, assume that \( C \) is smooth. The degree of its dual \( C^\vee \subset (\mathbb{P}^2)^* \) is the number of points of \( C^\vee \) lying on a general line \( L \subset (\mathbb{P}^2)^* \). By duality, the degree equals the number of tangent lines to \( C \) passing through a general point \( y \in \mathbb{P}^2 \). Such a tangent line at the point \( v \in C \) passes through the point \( y \) if and only if 
\[ g := y_0 \frac{\partial f}{\partial x_0}(v) + y_1 \frac{\partial f}{\partial x_1}(v) + y_2 \frac{\partial f}{\partial x_2}(v) = 0. \]
Hence, the degree of \( C^\vee \) is the number of points in \( V(f, g) \); the vanishing set of \( f \) and \( g \). Since \( \text{deg}(g) = d - 1 \), this finite set contains \( d(d - 1) \) points.

If \( C \) is singular, then the degree of \( C^\vee \) is the number of lines that are tangent to \( C \) at a smooth point and pass through the general point \( y \). Those smooth points are contained in the set \( V(f, g) \), but all of the singular points also lie in \( V(f, g) \). The curve \( V(g) \) passes through each node of \( C \) with intersection multiplicity two and through each cusp of \( C \) with intersection multiplicity 3. Therefore, we conclude that \( \text{deg}(C^\vee) = d(d - 1) - 3\kappa - 2\delta \). \( \square \)

Using Lemma 5.3, we can compute the degree of the Hurwitz hypersurface of a smooth surface; this formula also follows from Theorem 1.1 in [28].
Proposition 5.4. For an irreducible smooth surface $S \subset \mathbb{P}^3$ of degree $d$ with $d \geq 2$, the degree of the Hurwitz hypersurface $CH_1(S)$ is $d(d - 1)$.

Proof. Let $H \subset \mathbb{P}^3$ be a general plane and $v \in H$ be a general point. The degree of $CH_1(S)$ is the number of tangent lines $L$ to $S$ such that $v \in L \subset H$. Bertini’s Theorem [13, Theorem 17.16] implies that the intersection $S \cap H$ is a smooth plane curve of degree $d$. The degree of $CH_1(S)$ is the number of tangent lines to $S \cap H$ passing through the general point $v$; see Fig. 4. By definition, this is equal to the degree of the dual plane curve $(S \cap H) \vee$, so Lemma 5.3 shows $\deg(CH_1(S)) = d(d - 1)$.

Using Lemma 5.3 we can also count the number of bitangents and inflectional tangents to a general smooth plane curve.

Proposition 5.5. A general smooth irreducible curve in $\mathbb{P}^2$ of degree $d$ has exactly $\frac{1}{2}d(d - 2)(d - 3)(d + 3)$ bitangents and $3d(d - 2)$ inflectional tangents.

Proof. Let $C \subset \mathbb{P}^2$ be a general smooth irreducible curve of degree $d$. A bitangent to $C$ corresponds to a node of $C^\vee$, and an inflectional tangent to $C$ corresponds to a cusp of $C^\vee$; see Fig. 5 and [12, pp. 277–278]. Lemma 5.3 shows that $C^\vee$ has degree $d(d - 1)$. Let $\kappa$ and $\delta$ be the number of cusps and nodes of $C^\vee$, respectively. Applying Lemma 5.3 to the plane curve $C^\vee$ yields

$$d = \deg(C) = \deg((C^\vee) \vee) = d(d - 1)(d(d - 1) - 1) - 3\kappa - 2\delta.$$ 

The dual curves $C$ and $C^\vee$ have the same geometric genus; see [31, Proposition 1.5]. Hence, the genus-degree formula [26, p. 54, Eq. (7)] gives

$$\frac{1}{2}(d - 1)(d - 2) = \text{genus}(C) = \text{genus}(C^\vee) = \frac{1}{2}(d(d - 1) - 1)(d(d - 1) - 2) - \kappa - \delta.$$
Solving this system of two linear equations in $\kappa$ and $\delta$, we obtain $\kappa = 3d(d-2)$ and $\delta = \frac{1}{2}d(d-2)(d-3)(d+3)$.

The next result is the main theorem in this section and solves Problem 4 on Surfaces in [29]. The bidegrees of the bitangent and the inflectional congruence for a general smooth surface appear in [2, Proposition 3.3], and the bidegree of the inflectional congruence also appears in [22, Proposition 4.1].

**Theorem 5.6.** Let $S \subset \mathbb{P}^3$ be a general smooth irreducible surface of degree $d$ with $d \geq 4$. The bidegree of $\text{Bit}(S)$ is $\left(\frac{1}{2}d(d-1)(d-2)(d-3)\right)$, and the bidegree of $\text{Infl}(S)$ is $(d(d-1)(d-2), 3d(d-2))$.

**Proof.** For a general plane $H \subset \mathbb{P}^3$, Bertini’s Theorem [13, Theorem 17.16] implies that the intersection $S \cap H$ is a smooth plane curve of degree $d$. By Proposition 5.5, the number of bitangents to $S$ contained in $H$ is $\frac{1}{2}d(d-2)(d-3)(d+3)$, which is the class of $\text{Bit}(S)$. Similarly, the number of inflectional tangents to $S$ contained in $H$ is $3d(d-2)$, which is the class of $\text{Infl}(S)$.

It remains to calculate the number of bitangents and inflectional lines of the surface $S$ that pass through a general point $y \in \mathbb{P}^3$. Following the ideas in [23, p. 230], let $f \in \mathbb{C}[x_0, x_1, x_2, x_3]$ be the defining equation for $S$ in $\mathbb{P}^3$, and consider the polar curve $C \subset S$ with respect to the point $y$; the set $C$ consists of all points $x \in S$ such that the line through $y$ and $x$ is tangent to $S$ at the point $x$; see Fig. 5. The condition that the point $x$ lies on the curve $C$ is equivalent to saying that the point $y$ belongs to $T_x(S)$. As in the proof for Lemma 5.3, we have $C = V(f, g)$ where 

$$g := y_0 \frac{\partial f}{\partial x_0} + y_1 \frac{\partial f}{\partial x_1} + \cdots + y_3 \frac{\partial f}{\partial x_3}.$$ 

Thus, the curve $C$ has degree $d(d-1)$.

Projecting away from the point $y$ gives the rational map $\pi_y: \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$. Restricted to the surface $S$, this map is generically finite, with fibres of cardinality $d$, and is ramified over the curve $C$. If $C'$ is the image of $C$ under $\pi_y$, then a bitangent to the surface $S$ that passes through $y$ contains two points of $C$ and these points are mapped to a simple node in $C'$; see Fig. 6. All of these nodes in $C'$ have two distinct tangent lines because no bitangent line passing through $y$ is contained in a bitangent plane.
that is tangent at the same two points as the line; the bitangent planes to \( S \) form a 1-dimensional family, so the union of bitangent lines they contain is a surface in \( \mathbb{P}^3 \) that does not contain the general point \( y \).

We claim that the inflectional lines to \( S \) passing through the point \( y \) are exactly the tangent lines of \( C \) passing through \( y \). The line between a point \( x \in S \) and the point \( y \) is parametrized by the map \( \ell: \mathbb{P}^1 \to \mathbb{P}^3 \) which sends the point \((s : t) \in \mathbb{P}^1 \) to the point \((sx_0 + ty_0 : sx_1 + ty_1 : sx_2 + ty_2 : sx_3 + ty_3) \in \mathbb{P}^3 \). It follows that this line is an inflectional tangent to \( S \) if and only if \( f(\ell(s,t)) \) is divisible by \( t^3 \). This is equivalent to the conditions that \( \frac{\partial}{\partial t} f(\ell(s,t)) \bigg|_{(1,0)} = 0 \) and \( \frac{\partial^2}{\partial t^2} f(\ell(s,t)) \bigg|_{(1,0)} = 0 \), which means that \( x \in C \) and \( y_0 \frac{\partial g}{\partial \sigma_0} + y_1 \frac{\partial g}{\partial \sigma_3} + \cdots + y_3 \frac{\partial g}{\partial \sigma_3} = 0 \), or in other words \( y \in T_{\ell}(C) \). Therefore, the inflectional lines to \( S \) passing through \( y \) are the tangents to \( C \) passing through \( y \), and are mapped to the cusps of \( C' \); again see Fig. 6.

Since the bitangent and inflectional lines to \( S \) passing through \( y \) correspond to nodes and cusps of \( C' \), it suffices to count the number \( \kappa' \) of cusps and the number \( \delta' \) of simple nodes in the plane curve \( C' \). We subdivide these calculations as follows.

\( \kappa' = d(d - 1)(d - 2) \): From our parametrization of the line through points \( x \in S \) and \( y \), we see that this line is an inflectional tangent to \( S \) if and only if \( x \in V(f,g,h) \) where \( h := y_0 \frac{\partial g}{\partial \sigma_0} + y_1 \frac{\partial g}{\partial \sigma_3} + \cdots + y_3 \frac{\partial g}{\partial \sigma_3} \). Since \( \deg(h) = d - 2 \) and \( S \) is general, the set \( V(f,g,h) \) consists of \( d(d - 1)(d - 2) \) points.

\( \text{deg}((C')^\vee) = \text{deg}(S') \): By duality, the degree \( d' \) of the curve \((C')^\vee\) is the number of tangent lines to \( C' \subset \mathbb{P}^2 \) passing through a general point \( z \in \mathbb{P}^2 \). The preimage of \( z \) under the projection \( \pi_2 \) is a line \( L \subset \mathbb{P}^3 \) containing \( y \); see Fig. 6. Hence, \( d' \) is the number of tangent lines to \( C \) intersecting \( L \) in a point different from \( y \). For every line \( T \) that is tangent to \( C \) at a point \( x \) and intersects the line \( L \), it follows that the pair \( L \) and \( T \) spans the tangent plane of \( S \) at the point \( x \). On the other hand, given any plane \( H \) which is tangent to \( S \) at the point \( x \) and contains \( L \), we deduce that \( x \) must lie on the polar curve \( C \) and \( H \) is spanned by \( L \) and the tangent line

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**Fig. 6** A secant projecting onto a node and a tangent projecting to a cusp
to $C$ at $x$, so this tangent line intersects $L$. Therefore, $d'$ is the number of tangent planes to $S$ containing $L$, which is the degree of the dual surface $S'$.

$\text{deg}(S') = d(d-1)^2$: By duality, the degree of $S'$ is the number of tangent planes to the surface $S$ containing a general line, or the number of tangent planes to $S$ containing two general points $y, z \in \mathbb{P}^3$. Thus, this is the number of intersection points of the two polar curves of $S$ determined by $y$ and $z$, which is the cardinality of the set $V(f, g, \bar{g})$ where \( \bar{g} := z_0 \frac{\partial f}{\partial x_0} + z_1 \frac{\partial f}{\partial x_1} + \cdots + z_3 \frac{\partial f}{\partial x_3} \). Since $\text{deg}(\bar{g}) = d - 1$, we conclude that $\text{deg}(S') = d(d-1)^2$.

Finally, both the surface $S$ and the point $y$ are general, so Lemma 5.3 implies that $d(d-1)^2 = \text{deg}(C' \cap \Sigma) = \text{deg}(C') \text{deg}(C') - 1 - 3d(d-1)(d-2) - 2\delta'$. Since $\text{deg}(C') = \text{deg}(C) = d(d-1)$, we have $\delta' = \frac{1}{2}d(d-1)(d-2)(d-3)$. \(\square\)

We end this section by proving that the secant locus of an irreducible smooth curve is isomorphic to the bitangent congruence of its dual surface via the natural isomorphism between $\text{Gr}(1, \mathbb{P}^3)$ and $\text{Gr}(1, (\mathbb{P}^3)^*)$. A subvariety $\Sigma \subset \text{Gr}(1, \mathbb{P}^3)$ is sent under this isomorphism to the variety $\Sigma' \subset \text{Gr}(1, (\mathbb{P}^3)^*)$ consisting of the dual lines $L'$ for all $L \in \Sigma$. For every congruence $\Sigma \subset \text{Gr}(1, \mathbb{P}^3)$ with bidegree $(\alpha, \beta)$, the bidegree of $\Sigma'$ is $(\beta, \alpha)$.

**Theorem 5.7.** If $C \subset \mathbb{P}^3$ is a nondegenerate irreducible smooth curve, then we have $\text{Sec}(C) = \text{Bit}(C')$, the inflectional lines of $C'$ are dual to the tangent lines of $C$, and $\text{Infl}(C') \subset \text{Bit}(C')$.

**Proof.** We first show that $\text{Sec}(C) = \text{Bit}(C')$. Consider a line $L$ that intersects $C$ at two distinct points $x$ and $y$, but is equal to neither $T_x(C)$ nor $T_y(C)$. Together the line $L$ and $T_y(C)$ span a plane corresponding to a point $a \in C'$. Similarly, the span of the lines $L$ and $T_x(C)$ corresponds to a point $b \in C'$. Without loss of generality, we may assume that both $a$ and $b$ are smooth points in $C'$. By the Biduality Theorem, the points $a, b \in C'$ must be distinct with tangent planes corresponding to $x$ and $y$. Thus, the line $L'$ is tangent to $C'$ at the points $a, b$, and $\text{Sec}(C) \subset \text{Bit}(C')$. To establish the other inclusion, let $L'$ be a line that is tangent to $C'$ at two distinct smooth points $a, b \in C'$. The tangent planes at the points $a, b$ correspond to two points $x, y \in C$. If $x \neq y$, then $(L')^\perp$ is the secant to $C$ through these two points. If $x = y$, then the Biduality Theorem establishes that $(L')^\perp$ is the tangent line of $C$ at $x$. In either case, we see that $\text{Bit}(C') \subset \text{Sec}(C')$, so $\text{Sec}(C) = \text{Bit}(C')$.

For the second part, let $L$ be an inflectional line at a smooth point $a \in C'$. A point $y \in L' \setminus C$ corresponds to a plane $H$ such that $L = T_y(C') \cap H$, so the line $L$ is also an inflectional line to the plane curve $C' \cap H \subset H$. Regarding $L$ as a subvariety of the projective plane $H$, its dual variety is a cusp on the plane curve $(C' \cap H)^\perp \subset H^*$; see Fig. 3. If $\pi_\gamma : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2 \cong H^*$ denotes the projection away from the point $y$, then we claim that $(C' \cap H)^\perp$ equals $\pi_\gamma(C)$; for a more general version see [15, Proposition 6.1]. Indeed, a smooth point $z \in \pi_\gamma(C)$ is the projection of a point of $C$ whose tangent line does not contain $y$. Together this tangent line and the point $y$ span a plane such that its dual point $w$ is contained in the curve $C' \cap H$. Thus, the tangent line $T_z(\pi_\gamma(C))$ equals $\pi_\gamma(w^*)$; the latter is the line in $H^*$ dual to the
point \( w \in H \). In other words, we have \((\pi_0(C))^\vee \subset C^\vee \cap H\). Since both curves are irreducible, this inclusion must be an equality. Hence, when considering \( L \) in the projective plane \( H \), its dual point is a cusp of \( \pi_0(C) \). It follows that \( L^\vee \) is the tangent line \( T_x(C) \), where \( x \in C \) is the point corresponding to the tangent plane \( T_y(C^\vee) \); see Fig. 6. Reversing these arguments shows that the dual of a tangent line to \( C \) in the projective plane \( H \) is contained in \( \text{Sec}(x) \), we conclude that \( \text{Infl}(C^\vee) \subset \text{Bit}(C^\vee) \). \( \Box \)

**Proof of Theorem 1.2.** This result is a restatement of Theorem 5.7.

**Remark 5.9.** For a curve \( C \subset \mathbb{P}^3 \) with dual surface \( C^\vee \subset (\mathbb{P}^3)^* \), Theorem 20 in [19] establishes that \( \text{CH}_0(C) = \text{CH}_1(C^\vee) \). Combined with Theorem 5.7 we see that the singular locus of the Hurwitz hypersurface \( \text{CH}_1(C^\vee) \), for smooth \( C \), has just one component, namely the bitangent congruence.

**Remark 5.10.** For a surface \( S \subset \mathbb{P}^3 \) with dual surface \( S^\vee \subset (\mathbb{P}^3)^* \), Theorem 20 in [19] also establishes that \( \text{CH}_1(S) = \text{CH}_1(S^\vee) \). If both \( S \) and \( S^\vee \) have mild singularities, then the proof of Lemma 5.1 in [2] shows that \( \text{Bit}(S)^\perp = \text{Bit}(S^\vee) \).

### 6 Intersection Theory on \( \text{Gr}(1, \mathbb{P}^3) \)

In this section, we recast the degree of a subvariety in \( \text{Gr}(1, \mathbb{P}^3) \) in terms of certain products in the Chow ring.

Consider a smooth irreducible variety \( X \) of dimension \( n \). For each \( j \in \mathbb{N} \), the group \( Z^j(X) \) of codimension-\( j \) cycles is the free abelian group generated by the closed irreducible subvarieties of \( X \) having codimension \( j \). Given a variety \( W \) of codimension \( j-1 \) and a nonzero rational function \( f \) on \( W \), we have the cycle \( \text{div}(f) := \sum_{Z} \text{ord}_Z(f)Z \) where the sum runs over all subvarieties \( Z \) of \( W \) with codimension 1 in \( W \) and \( \text{ord}_Z(f) \in \mathbb{Z} \) is the order of vanishing of \( f \) along \( Z \). The group of cycles rationally equivalent to zero is the subgroup generated by the cycles \( \text{div}(f) \) for all codimension-\( (j-1) \) subvarieties \( W \) of \( X \) and all nonzero rational functions \( f \) on \( W \). The Chow group \( A^j(X) \) is the quotient of \( Z^j(X) \) by the subgroup of cycles rationally equivalent to zero. We typically write \([Z]\) for the class of a subvariety \( Z \) in the appropriate Chow group. Since \( X \) is the unique subvariety of codimension 0, we see that \( A^0(X) \cong \mathbb{Z} \). We also have \( A^1(X) \cong \text{Pic}(X) \). Crucially, the direct sum \( A^j(X) := \bigoplus_{j=0}^{\infty} A^j(X) \) forms a commutative \( \mathbb{Z} \)-graded ring called the **Chow ring** of \( X \). The product arises from intersecting cycles: for subvarieties \( V \) and \( W \) of \( X \) having codimension \( j \) and \( k \) and intersecting transversely, the product \([V][W] \in A^{j+k}(X) \) is the sum of the irreducible components of \( V \cap W \). More generally, intersection theory aims to construct an explicit cycle to represent the product \([V][W]\).
Example 6.1. The Chow ring of \(\mathbb{P}^n\) is isomorphic to \(\mathbb{Z}[H]/(H^{n+1})\) where \(H\) is the class of a hyperplane. In particular, any subvariety of codimension \(d\) is rationally equivalent to a multiple of the intersection of \(d\) hyperplanes.

To a given a vector bundle \(\mathcal{E}\) of rank \(r\) on \(X\), we associate its Chern classes \(c_i(\mathcal{E}) \in A^i(X)\) for \(0 \leq i \leq r\); see [30]. When \(\mathcal{E}\) is globally generated, these classes are represented by degeneracy loci; the class \(c_{r+1-j}(\mathcal{E})\) is associated to the locus of points \(x \in X\) where \(j\) general global sections of \(\mathcal{E}\) fail to be linearly independent. In particular, \(c_r(\mathcal{E})\) is represented by the vanishing locus of a single general global section. Given a short exact sequence \(0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}''\to 0\) of vector bundles, the Whitney Sum Formula asserts that \(c_r(\mathcal{E}) = \sum_{i+j=k} c_i(\mathcal{E}') c_j(\mathcal{E}'')\); see [10] Theorem 3.2]. Moreover, if \(\mathcal{E}^* := \mathcal{H}om(\mathcal{E}, O_X)\) denotes the dual vector bundle, then we have \(c_i(\mathcal{E}^*) = (-1)^i c_i(\mathcal{E})\) for \(0 \leq i \leq r\); see [10] Remark 3.2.3.

Example 6.2. Given nonnegative integers \(a_1, a_2, \ldots, a_n\), consider the vector bundle \(\mathcal{E} := \mathcal{O}_{\mathbb{P}^n}(a_1) \oplus \mathcal{O}_{\mathbb{P}^n}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(a_n)\). Since each \(\mathcal{O}_{\mathbb{P}^n}(a_i)\) is globally generated, the Chern class \(c_1(\mathcal{O}_{\mathbb{P}^n}(a_i))\) is the vanishing locus of a general homogeneous polynomial \(C[a_0, x_1, \ldots, x_n]\) of degree \(a_i\), so \(c_1(\mathcal{O}_{\mathbb{P}^n}(a_i)) = a_iH\) in \(A^*\mathbb{P}^n\). Hence, the Whitney Sum Formula implies that \(c_n(\mathcal{E}) = \prod_{i=1}^{n} c_1(\mathcal{O}(a_i)) = \prod_{i=1}^{n} (a_iH)\).

Example 6.3. If \(\mathcal{T}_{\mathbb{P}^n}\) is the tangent bundle on \(\mathbb{P}^n\), then we have the short exact sequence \(0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1) \oplus (n+1) \to \mathcal{T}_{\mathbb{P}^n} \to 0\); see [14] Example 8.20.1. The Whitney Sum Formula implies that \(c_1(\mathcal{T}_{\mathbb{P}^n}) = (n+1)c_1(\mathcal{O}_{\mathbb{P}^n}(1)) - c_1(\mathcal{O}_{\mathbb{P}^n}) = (n+1)H\) and \(c_2(\mathcal{T}_{\mathbb{P}^n}) = c_2(\mathcal{O}_{\mathbb{P}^n}(1) \oplus (n+1)) = (n+1)H^2\).

Example 6.4. Let \(Y \subset \mathbb{P}^n\) be a smooth hypersurface of degree \(d\). If \(\mathcal{T}_Y\) is the tangent bundle of \(Y\), then we have the exact sequence \(0 \to \mathcal{T}_Y \to \mathcal{T}_{\mathbb{P}^n}|_Y \to \mathcal{O}(d)|_Y \to 0\); see [14] Proposition 8.20. Setting \(h := H|_Y\) in \(A^*(Y)\), the Whitney Sum Formula implies that \(c_1(\mathcal{T}_Y) = c_1(\mathcal{T}_{\mathbb{P}^n}|_Y) - c_1(\mathcal{O}(d)|_Y) = (n+1)h - dh = (n+1 - d)h\) and \(c_2(\mathcal{T}_Y) = c_2(\mathcal{T}_{\mathbb{P}^n}|_Y) - c_1(\mathcal{T}_Y)c_1(\mathcal{O}(d)|_Y) = ((n+1) - (n+1 - d))h^2\).

We next focus on the Chow ring of \(\text{Gr}(1, \mathbb{P}^3)\); see [1, 30]. Fix a complete flag \(v_0 \in L_0 \subset H_0 \subset \mathbb{P}^3\) where the point \(v_0\) lies in the line \(L_0\), and the line \(L_0\) is contained in the plane \(H_0\). The Schubert varieties in \(\text{Gr}(1, \mathbb{P}^3)\) are the following subvarieties:

\[
\Sigma_0 := \text{Gr}(1, \mathbb{P}^3), \quad \Sigma_1 := \{L : L \cap L_0 \neq \emptyset\} \subset \text{Gr}(1, \mathbb{P}^3), \\
\Sigma_{1,1} := \{L : L \subset H_0\} \subset \text{Gr}(1, \mathbb{P}^3), \quad \Sigma_2 := \{L : v_0 \in L\} \subset \text{Gr}(1, \mathbb{P}^3), \\
\Sigma_{2,1} := \{L : v_0 \in L \subset H_0\} \subset \text{Gr}(1, \mathbb{P}^3), \quad \Sigma_{2,2} := \{L_0\} \subset \text{Gr}(1, \mathbb{P}^3).
\]

The corresponding classes \(\sigma_i := [\Sigma_i]\), called the Schubert cycles, form a basis for the Chow ring \(A^*(\text{Gr}(1, \mathbb{P}^3))\); see [8] Theorem 5.26. Since the sum of the subscripts gives the codimension, we have

\[
A^0(\text{Gr}(1, \mathbb{P}^3)) \cong \mathbb{Z}\sigma_0, \quad A^1(\text{Gr}(1, \mathbb{P}^3)) \cong \mathbb{Z}\sigma_1, \quad A^2(\text{Gr}(1, \mathbb{P}^3)) \cong \mathbb{Z}\sigma_{1,1} \oplus \mathbb{Z}\sigma_2, \\
A^3(\text{Gr}(1, \mathbb{P}^3)) \cong \mathbb{Z}\sigma_{2,1}, \quad A^4(\text{Gr}(1, \mathbb{P}^3)) \cong \mathbb{Z}\sigma_{2,2}.
\]
To understand the product structure, we use the transitive action of $GL(4, \mathbb{C})$ on $Gr(1, \mathbb{P}^3)$. Specifically, Kleiman’s Transversality Theorem [18] shows that, for two subvarieties $V$ and $W$ in $Gr(1, \mathbb{P}^3)$, a general translate $U$ of $V$ under the $GL(4, \mathbb{C})$-action is rationally equivalent to $V$ and the intersection of $U$ and $W$ is transversal at the generic point of any component of $U \cap W$. Hence, we have $[V][W] = [U \cap W]$. To determine the product $\sigma_1 \sigma_2$, we intersect general varieties representing these classes: $\sigma_1$ consists of all lines $L$ contained in a fixed plane $H_0$, and $\sigma_2$ is all lines $L$ containing a fixed point $v_0$. Since a general point does not lie in a general plane, we see that $\sigma_1 \sigma_2 = 0$. Similar arguments yield all products:

$$
\begin{align*}
\sigma_1^2 &= \sigma_2, \\
\sigma_2 &= \sigma_2, \\
\sigma_1 \sigma_2 &= 0, \\
\sigma_1 \sigma_1 &= \sigma_2, \\
\sigma_1 &= \sigma_2, \\
\sigma_2 &= \sigma_2 + \sigma_1.
\end{align*}
$$

The degree of a subvariety in $Gr(1, \mathbb{P}^3)$, introduced in Sect.2, can be interpreted as certain coefficients of its class in the Chow ring. Geometrically, the order $\alpha$ of a surface $X \subset Gr(1, \mathbb{P}^3)$ is the number of lines in $X$ passing through the general point $v_0$. Since we may intersect $X$ with a general variety representing $\sigma_2$, it follows that $\alpha$ equals the coefficient of $\sigma_2$ in $[X]$. Similarly, the class $\beta$ of $X$ is the coefficient of $\sigma_1$ in $[X]$, the degree of a threefold $\Sigma \subset Gr(1, \mathbb{P}^3)$ is the coefficient of $\sigma_1$ in $[\Sigma]$, and the degree of a curve $C \subset Gr(1, \mathbb{P}^3)$ is the coefficient of $\sigma_{2,1}$ in $[C]$.

The degree of a subvariety in $Gr(1, \mathbb{P}^3)$ also has a useful reinterpretation via Chern classes of tautological vector bundles. Let $\mathcal{S}$ denote the tautological subbundle, the vector bundle whose fibre over the point $W \in Gr(1, \mathbb{P}^3)$ is the 2-dimensional vector space $W \subset \mathbb{C}^4$. Similarly, let $\mathcal{Q}$ be the tautological quotient bundle whose fibre over $W$ is $\mathbb{C}^4/W$. Both $\mathcal{S}$ and $\mathcal{Q}$ are globally generated; $H^0(Gr(1, \mathbb{P}^3), \mathcal{S}) \cong (\mathbb{C}^4)^*$ and $H^0(Gr(1, \mathbb{P}^3), \mathcal{Q}) \cong \mathbb{C}^4$; see [11] Proposition 0.5]. A global section of $\mathcal{S}$ corresponds to a nonzero map $\phi: \mathbb{C}^4 \to \mathbb{C}$, where its value at the point $W$ is $\phi|_W: W \to \mathbb{C}$. The Chern class $c_2(\mathcal{S})$ is represented by the vanishing locus of $\phi$, so we have $c_2(\mathcal{S}) = \sigma_{1,1} = c_2(\mathcal{S})$. For two general sections $\phi, \psi: \mathbb{C}^4 \to \mathbb{C}$ of $\mathcal{S}$, the Chern class $c_1(\mathcal{S})$ is represented by the locus of points $W$ where $\phi|_W$ and $\psi|_W$ fail to be linearly independent or $W \cap \ker(\phi) \cap \ker(\psi) \neq \{0\}$. Generality ensures that $\ker(\phi) \cap \ker(\psi)$ is a 2-dimensional subspace of $\mathbb{C}^4$, so $c_1(\mathcal{S}) = -c_1(\mathcal{S}) = \sigma_{1,1}$. Similarly, a global section of $\mathcal{Q}$ corresponds to a point $v \in \mathbb{C}^4$; its value at $W$ is simply the image of the point in $\mathbb{C}^4/W$. Thus, $c_2(\mathcal{Q})$ is represented by the locus of those $W$ containing $v$, which is $\sigma_2$. Two global sections of $\mathcal{Q}$ are linearly dependent at $W$ when the 2-dimensional subspace of $\mathbb{C}^4$ spanned by the points intersects $W$ nontrivially, so $c_1(\mathcal{Q}) = \sigma_1$. Finally, for a surface $X \subset Gr(1, \mathbb{P}^3)$ with $[X] = \alpha \sigma_2 + \beta \sigma_{1,1}$, we obtain

$$
\begin{align*}
c_2(\mathcal{Q}) [X] &= \sigma_1(\alpha \sigma_2 + \beta \sigma_{1,1}) = \alpha \sigma_2, \\
c_2(\mathcal{S}) [X] &= \sigma_{1,1}(\alpha \sigma_2 + \beta \sigma_{1,1}) = \beta \sigma_2,
\end{align*}
$$

so computing the bidegree is equivalent to calculating the products $c_2(\mathcal{Q}) [X]$ and $c_2(\mathcal{S}) [X]$ in the Chow ring.

We close this section with three examples demonstrating this approach.
Example 6.5. Given a smooth surface $S$ in $\mathbb{P}^3$, we recompute the degree of $\text{CH}_1(S)$; compare with Proposition 5.4. Theorem 9 in [19] implies that this degree equals the degree $\delta_1(S)$ of the first polar locus $M_1(S) = \{ x \in S : y \in T_x S \}$, where $y$ is a general point of $\mathbb{P}^3$ (this locus is the polar curve in the proof of Theorem 5.6). Letting $T_S$ be the tangent bundle of $S$, Example 14.4.15 in [10] shows that $\delta_1(S) = \deg(3h - c_1(T_S))$. Hence, Example 6.4 gives $\delta_1(S) = \deg(3h - h(3 + 1 - d)) = (d - 1) \deg(h)$. Since $S$ is a degree $d$ surface, the degree of the hyperplane $h$ equals $d$, so $\delta_1(S) = d(d - 1)$.

Example 6.6 (Problem 3 on Grassmannians in [29]). Let $S_1, S_2 \subset \mathbb{P}^3$ be general surfaces of degree $d_1$ and $d_2$, respectively, with $d_1, d_2 \geq 4$. To find the number of lines bitangent to both surfaces, it suffices to compute the cardinality of $\text{Bit}(S_1) \cap \text{Bit}(S_2)$. Theorem 5.6 establishes that, for all $1 \leq i \leq 2$, we have $[\text{Bit}(S_i)] = \alpha_i \sigma_2 + \beta_i \sigma_{1,1}$ where $\alpha_i := \frac{1}{4} d_i(d_i - 1)(d_i - 2)(d_i - 3)$ and $\beta_i := \frac{1}{2} d_i(d_i - 2)(d_i - 3)(d_i + 3)$. It follows that $[\text{Bit}(S_1) \cap \text{Bit}(S_2)] = [\text{Bit}(S_1)][\text{Bit}(S_2)] = (\alpha_1 \alpha_2 + \beta_1 \beta_2) \sigma_{2,2}$, so the number of lines bitangent to $S_1$ and $S_2$ is
\[
\frac{1}{4} d_1(d_1 - 1)(d_1 - 2)(d_1 - 3)d_2(d_2 - 1)(d_2 - 2)(d_2 - 3) + \frac{1}{4} d_1(d_1 - 2)(d_1 - 3)(d_1 + 3)d_2(d_2 - 2)(d_2 - 3)(d_2 + 3).
\]

Example 6.7. Let $S \subset \mathbb{P}^3$ be a general surface of degree $d_1$ with $d_1 \geq 4$, and let $C \subset \mathbb{P}^3$ be a general curve of degree $d_2$ and geometric genus $g$ with $d_2 \geq 2$. To find the number of lines bitangent to $S$ and secant to $C$, it suffices to compute the cardinality of $\text{Bit}(S) \cap \text{Sec}(C)$. Theorem 5.6 and Theorem 5.5 imply that
\[
[\text{Bit}(S)] = \frac{1}{4} d_1(d_1 - 1)(d_1 - 2)(d_1 - 3) \sigma_2 + \frac{1}{2} d_1(d_1 - 2)(d_1 - 3)(d_1 + 3) \sigma_{1,1},
\]
\[
[\text{Sec}(C)] = \left( \frac{1}{4} (d_2 - 1)(d_2 - 2) - g \right) \sigma_2 + \frac{1}{2} d_2(d_2 - 1) \sigma_{1,1}.
\]
It follows that $[\text{Bit}(S) \cap \text{Sec}(C)] = [\text{Bit}(S)][\text{Sec}(C)] = \gamma \sigma_{2,2}$ where
\[
\gamma := \frac{1}{4} d_1(d_1 - 1)(d_1 - 2)(d_1 - 3)((d_2 - 1)(d_2 - 2) - 2g) + \frac{1}{4} d_1(d_1 - 2)(d_1 - 3)(d_1 + 3)d_2(d_2 - 1),
\]
so the number of lines bitangent to $S$ and secant to $C$ is $\gamma$.

7 Singular Loci of Congruences

This section investigates the singular points of the secant, bitangent, and inflectional congruences. We begin with the singularities of the secant locus of a smooth irreducible curve.

Proposition 7.1. Let $C$ be a nondegenerate smooth irreducible curve in $\mathbb{P}^3$. If $L$ is a line that intersects the curve $C$ in three or more distinct points, then the line $L$ corresponds to a singular point in $\text{Sec}(C)$. 

Proof. The symmetric square $C^{(2)}$ is the quotient of $C \times C$ by the action of the symmetric group $\mathfrak{S}_2$, so points in this projective variety are unordered pairs of points on $C$; see [13] pp. 126–127. The map $\sigma : C^{(2)} \to \text{Sec}(C)$, defined by sending $\{x,y\}$ to the line spanned by the points $x$ and $y$ if $x \neq y$ or to the tangent line $T_x(C)$ if $x = y$, is a birational morphism. Since $|L \cap C| \geq 3$, the fibre $\sigma^{-1}(L)$ is a finite set containing more than one element. Hence, $\sigma^{-1}(L)$ is not connected and the Zariski Connectedness Theorem [21, Sect. III.9.V] proves that $\text{Sec}(C)$ is singular at $L$. \qed

Lemma 7.2. If $f \in \mathbb{C}[z,w]$ satisfies $f(z,w) = -f(w,z)$, then the linear form $z - w$ divides the power series $f$.

Proof. We write the formal power series $f$ as a sum of homogeneous polynomials $f = \sum_{i \in \mathbb{N}} f_i$. Since we have $f(z,w) + f(w,z) = 0$, it follows that, in each degree $i$, we have $f_i(z,w) + f_i(w,z) = 0$. In particular, we see that $f_i(w,w) = 0$. If we consider $f_i(w,z)$ as a polynomial in the variable $z$ with coefficients in $\mathbb{C}[w]$, it follows that $w$ is a root of $f_i$. Thus, we conclude that $z - w$ divides $f_i$ for all $i \in \mathbb{N}$. \qed

Theorem 7.3. Let $C$ be a nondegenerate smooth irreducible curve in $\mathbb{P}^3$. If a point in Sec$(C)$ corresponds to a line $L$ that intersects $C$ in a single point $x$, then the intersection multiplicity of $L$ and $C$ at $x$ is at least 2. Moreover, the line $L$ corresponds to a smooth point of Sec$(C)$ if and only if the intersection multiplicity is exactly 2.

We thank Jenia Tevelev for help with the following proof.

Proof. Suppose the line $L$ intersects the curve $C$ at the point $x$ with multiplicity 2. Without loss of generality, we may work in the affine open subset with $x_3 \neq 0$, and we assume that $x = (0 : 0 : 0 : 1)$ and $L = V(x_1,x_2)$. Since $C$ is smooth, there is a local analytic isomorphism $\varphi$ from a neighbourhood of the origin in $\mathbb{A}^1$ to a neighbourhood of the point $x$ in $C$. The map $\varphi$ will have the form $\varphi(z) = (\varphi_0(z), \varphi_1(z), \varphi_2(z))$ for some $\varphi_0, \varphi_1, \varphi_2 \in \mathbb{C}[z]$. We have $\varphi_0'(0) \neq 0$ and $\varphi_1'(0) = \varphi_2'(0) = 0$ because $L$ is the tangent to the curve $C$ at $x$. After making an analytic change of coordinates, we may assume that $\varphi(z) = (z, \varphi_1(z), \varphi_2(z))$. As $L$ is a simple tangent, at least one of $\varphi_1$ and $\varphi_2$ must vanish at 0 with order exactly 2. Hence, we may assume that $\varphi_1(z) = z^2 + z^3 f(z)$ and $\varphi_2(z) = z^2 g(z)$ for some $f,g \in \mathbb{C}[z]$. The line spanned by the distinct points $\varphi(z)$ and $\varphi(w)$ on the curve $C$ is given by the row space of the matrix

$$
\begin{bmatrix}
z & z^2 + z^3 f(z) & z^2 g(z) \\
w & w^2 + w^3 f(w) & w^2 g(w)
\end{bmatrix}.
$$

The Plücker coordinates are skew-symmetric power series, so Lemma 7.2 implies that they are divisible by $z - w$. In particular, if $f(z) = \sum_i a_i z^i$, then we have $p_{0,3} = z - w$,

$$
p_{0,1} = z(w^2 + w^3 f(w)) - w(z^2 + z^3 f(z)) = -zw(z - w)\left(1 + \sum_i a_i \sum_{j=0}^{i+1} w^j z^{i+1-j}\right),
$$

$$
p_{1,3} = z^2 + z^3 f(z) - w^2 - w^3 f(w) = (z - w)\left(z + w + \sum_i a_i \sum_{j=0}^{i+2} z^j w^{i+2-j}\right).$$
The symmetric square \((\mathbb{A}^1)^{(2)}\) of the affine line \(\mathbb{A}^1\) is a smooth surface isomorphic to the affine plane \(\mathbb{A}^2\); see [13] Example 10.23. Consider the map \(\varphi: (\mathbb{A}^1)^{(2)} \to \text{Sec}(C)\) defined by sending the pair \((z, w)\) of points in \(\mathbb{A}^1\) to the line spanned by the points \(\varphi(z)\) and \(\varphi(w)\) if \(z \neq w\) or to the tangent line of \(C\) at \(\varphi(z)\) if \(z = w\). In other words, the map \(\varphi\) sends \((z, w)\) to \((-zw + h_1(z, w) : \frac{p_0 z}{z - w} : 1 : \frac{p_1 z}{z - w} : z + w + h_2(z, w) : \frac{p_2}{z - w}\) where

\[
h_1(z, w) := -zw \sum_i a_i \sum_j w^j z^{d - j + 1} \quad \text{and} \quad h_2(z, w) := \sum_i a_i \sum_j z^j w^{2 - j}.
\]

Since the forms \(zw\) and \(z + w\) are local coordinates of \((\mathbb{A}^1)^{(2)}\) in a neighbourhood of the origin, we conclude that \(\varphi\) is a local isomorphism and \(\text{Sec}(C)\) is smooth at the point corresponding to \(L\).

Suppose the line \(L\) intersects the curve \(C\) at the point \(x\) with multiplicity at least 3. It follows that the line \(L\) is contained in the Zariski closure of the set of lines that intersect \(C\) in at least three points or that intersect \(C\) in two points, one with multiplicity at least 2. By Proposition 7.1 and Lemma 2.3 in [2], we conclude that the line is singular in \(\text{Sec}(C)\).

**Corollary 7.4.** Let \(C\) be a nondegenerate smooth irreducible curve in \(\mathbb{P}^3\). If the line \(L\) corresponds to a point in \(\text{Sec}(C)\), then \(L\) corresponds to a singular point of \(\text{Sec}(C)\) if and only if one of the following three conditions is satisfied:

- the line \(L\) intersects the curve \(C\) in 3 or more distinct points,
- the line \(L\) intersects the curve \(C\) in exactly 2 points and \(L\) is the tangent line to one of these two points,
- the line \(L\) intersects the curve \(C\) at a single point with multiplicity at least 3.

**Proof.** Combine Proposition 7.1, Lemma 2.3 in [2], and Theorem 7.3. □

Analogously, we want to describe the singularities of the inflectional locus \(\text{Infl}(S)\) and the bitangent locus \(\text{Bit}(S)\) of a surface \(S \subset \mathbb{P}^3\).

**Theorem 7.5.** If \(S \subset \mathbb{P}^3\) is an irreducible smooth surface of degree at least 4 which does not contain any lines, then the singular locus of \(\text{Infl}(S)\) corresponds to lines which either intersect \(S\) with multiplicity at least 3 at two or more distinct points, or intersect \(S\) with multiplicity at least 4 at some point.

**Proof.** We consider the incidence variety

\[
\Psi := \{(x, L) : L \text{ intersects } S \text{ at } x \text{ with multiplicity } 3\} \subset S \times \text{Gr}(1, \mathbb{P}^3).
\]

The projection \(\pi: \Psi \to \text{Infl}(S)\), defined by \((x, L) \mapsto L\), is a surjective morphism. Since \(S\) does not contain any lines, all fibres of \(\pi\) are finite and Lemma 14.8 in [13] implies that the map \(\pi\) is finite. Moreover, the general fibre of \(\pi\) has cardinality one, so \(\pi\) is birational. To apply Lemma 3.2, we need to examine the singularities of \(\Psi\) and the differential of \(\pi\).
Let \( f \in \mathbb{C}[x_0, x_1, x_2, x_3] \) be the defining equation for \( S \) in \( \mathbb{P}^3 \). Consider the affine chart in \( \mathbb{P}^3 \times \text{Gr}(1, \mathbb{P}^3) \) where \( x_0 \neq 0 \) and \( p_{0.1} \neq 0 \). We may assume \( x = (1 : \alpha : \beta : \gamma) \) and the line \( L \) is spanned by the points \( (1 : 0 : a : b) \) and \( (0 : 1 : c : d) \). In this affine chart, \( S \) is defined by \( g_0(x_1, x_2, x_3) := f(1, x_1, x_2, x_3) \). As in the proof of Theorem 3.3, we have \( x \in L \) if and only if \( a = \beta - \alpha c \) and \( b = \gamma - \alpha d \). Parametrizing the line \( L \) by \( \ell(s, t) := (s : s\alpha + t : s\beta + tc : s\gamma + td) \) for \( (s : t) \in \mathbb{P}^1 \) shows that \( L \) intersects \( S \) with multiplicity at least \( m \) at \( x \) if and only if \( f(\ell(s, t)) \) is divisible by \( t^m \). This is equivalent to

\[
\frac{\partial}{\partial t} [f(\ell(s, t))] \bigg|_{(1, 0)} = \frac{\partial^2}{\partial t^2} [f(\ell(s, t))] \bigg|_{(1, 0)} = \cdots = \frac{\partial^{m-1}}{\partial t^{m-1}} [f(\ell(s, t))] \bigg|_{(1, 0)} = 0.
\]

Setting \( g_k := [\frac{\partial g_0}{\partial x_1} + c \frac{\partial g_0}{\partial x_2} + d \frac{\partial g_0}{\partial x_3}]^k g_0 \) for \( k \geq 1 \), the incidence variety \( \Psi_S \) can be written on the chosen affine chart as

\[
\{(\alpha, \beta, \gamma, a, b, c, d) : g_k(\alpha, \beta, \gamma) = 0 \text{ for } 0 \leq k \leq 2, a = \beta - \alpha c, b = \gamma - \alpha d\}.
\]

As \( \dim \Psi_S = 2 \), it is smooth at the point \((x, L)\) if and only if its tangent space has dimension 2 or, equivalently, its Jacobian matrix

\[
\begin{bmatrix}
\frac{\partial g_0}{\partial x_1}(\alpha, \beta, \gamma) & \frac{\partial g_0}{\partial x_2}(\alpha, \beta, \gamma) & \frac{\partial g_0}{\partial x_3}(\alpha, \beta, \gamma) & 0 & 0 & 0 & 0 \\
\frac{\partial g_2}{\partial x_1}(\alpha, \beta, \gamma) & \frac{\partial g_2}{\partial x_2}(\alpha, \beta, \gamma) & \frac{\partial g_2}{\partial x_3}(\alpha, \beta, \gamma) & 0 & 0 & \frac{\partial g_0}{\partial x_2}(\alpha, \beta, \gamma) & \frac{\partial g_0}{\partial x_3}(\alpha, \beta, \gamma) \\
\frac{\partial g_3}{\partial x_1}(\alpha, \beta, \gamma) & \frac{\partial g_3}{\partial x_2}(\alpha, \beta, \gamma) & \frac{\partial g_3}{\partial x_3}(\alpha, \beta, \gamma) & 0 & 0 & \frac{\partial g_2}{\partial x_2}(\alpha, \beta, \gamma) & \frac{\partial g_2}{\partial x_3}(\alpha, \beta, \gamma) \\
-c & 1 & 0 & -1 & 0 & -\alpha & 0 \\
-d & 0 & 1 & 0 & -1 & 0 & -\alpha
\end{bmatrix}
\]

has rank five. Since \( S \) is smooth, the first 2 and the last 2 rows of the Jacobian matrix are linearly independent. If \( \Psi_S \) is singular at \((x, L)\), then the third row is a linear combination of the others; specifically, there exist scalars \( \lambda, \mu \in \mathbb{C} \) such that \( \frac{\partial g_2}{\partial x_1}(\alpha, \beta, \gamma) = \lambda \frac{\partial g_1}{\partial x_1}(\alpha, \beta, \gamma) + \mu \frac{\partial g_0}{\partial x_1}(\alpha, \beta, \gamma) \) for \( 1 \leq j \leq 3 \). It follows that

\[
g_3(\alpha, \beta, \gamma) = \lambda g_2(\alpha, \beta, \gamma) + \mu g_1(\alpha, \beta, \gamma) = 0.
\]

Thus, the line \( L \) intersects the surface \( S \) at the point \( x \) with multiplicity at least 4 if \( \Psi_S \) is singular at \((x, L)\).

It remains to show that the differential \( d_{(x, L)} \Psi : T_{(x, L)} \Psi_S \rightarrow T_L (\text{Infl}(S)) \) is not injective if and only if the line \( L \) intersects the surface \( S \) at the point \( x \) with multiplicity at least 4. The differential \( d_{(x, L)} \Psi \) sends every element in the kernel of the Jacobian matrix to its last four coordinates. This map is not injective if and only if the kernel contains an element of the form \([* * * 0 0 0 0]^T \neq 0\). Such an element belongs to the kernel if and only if it equals \([\lambda, c\lambda, d\lambda, 0, 0, 0, 0]^T\) for some \( \lambda \in \mathbb{C} \setminus \{0\} \) and

\[
g_1(\alpha, \beta, \gamma) = g_2(\alpha, \beta, \gamma) = g_3(\alpha, \beta, \gamma) = 0.
\]

This shows that the line \( L \) intersects the surface \( S \) at the point \( x \) with multiplicity at least 4 if and only if \( d_{(x, L)} \Psi \) is not injective.

Finally, the fibre \( \Psi_S^{-1}(L) \) consists of more than one point if and only if \( L \) intersects \( S \) with multiplicity at least 3 at two or more distinct points, so Lemma 3.2 completes the proof. \( \Box \)
Proof of Theorem 1.1. The first part related to the curve $C$ is an amalgamation of Theorem 3.3, Theorem 3.5, Theorem 7.3, and Corollary 7.4. Similarly, the second part related to the surface $S$ is an amalgamation of Theorem 4.1, Theorem 5.6, and Theorem 7.5. □

Proposition 7.6. Let $S \subset \mathbb{P}^3$ be a general irreducible surface of degree at least 4. If $L$ is a line that is tangent to $S$ at three or more distinct points, then the line $L$ corresponds to a singular point of $\text{Bit}(S)$.

Proof. As in the proof of Proposition 7.1, the symmetric square $S^{(2)}$ is the quotient of $S \times S$ by the action of the symmetric group $S_2$. The projection $\sigma$ from
$$\left\{\{(x,y),L\} : x \neq y, x,y \in L \subset T_x(S) \cap T_y(S)\right\} \subset S^{(2)} \times \text{Gr}(1,\mathbb{P}^3)$$
ono onto $\text{Bit}(S)$, defined by sending the pair $(\{x,y\},L) \mapsto L$ is a birational morphism. The fibre $\sigma^{-1}(L)$ is a finite set containing more than one element if $L$ is tangent to $S$ in at least three distinct points. Hence, $\sigma^{-1}(L)$ is not connected and the Zariski Connectedness Theorem [21, Sect. III.9.V] proves that $\text{Bit}(S)$ is singular at $L$. □

We do not yet have a full understanding of points in $\text{Bit}(S)$ for which the corresponding lines have an intersection multiplicity greater than 4 at a point of $S$. We know that a line $L$ that is tangent to the surface $S$ at exactly two points corresponds to a smooth point in $\text{Bit}(S)$ if and only if the intersection multiplicity of $L$ and $S$ at both points is exactly 2. Moreover, given a line $L$ that is tangent to $S$ at a single point, the intersection multiplicity of $L$ and $S$ at this point is at least 4, and the line $L$ corresponds to a smooth point of $\text{Bit}(S)$ when the multiplicity is exactly four; see [2, Lemma 4.3]. To complete this picture, we make the following prediction.

Conjecture 7.7. Let $S \subset \mathbb{P}^3$ be a general irreducible surface of degree at least 4. If a point in the bitangent congruence $\text{Bit}(S)$ corresponds to a line $L$ that is tangent to $S$ at a single point $x$ such that the intersection multiplicity of $L$ and $S$ at $x$ is at least 5, then $L$ corresponds to a singular point of $\text{Bit}(S)$.

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References

1. Enrique Arrondo: Subvarieties of Grassmannians, Lecture Note Series, Dipartimento di Matematica Univ. Trento 10 (1996) [www.mat.ucm.es/~arrondo/trento.pdf]
2. Enrique Arrondo, Marina Bertolini, and Cristina Turrini: A focus on focal surfaces, *Asian J. Math.* 5 (2001) 535–560.
3. Fabrizio Catanese: Cayley forms and self-dual varieties, *Proc. Edinb. Math. Soc. (2)* 57 (2014) 89–109.
4. Arthur Cayley: On a new analytical representation of curves in space, *The Quarterly Journal of Pure and Applied Mathematics* 3 (1860) 225–236.
5. Wei Liang Chow and Bartel L. van der Waerden: Zur algebraischen Geometrie. IX., *Math. Ann.* 113 (1937) 692–704.
6. John Dalbec and Bernd Sturmfels: Introduction to Chow forms, in *Invariant methods in discrete and computational Geometry* (Curacao 1995), 37–58, Kluwer Acad. Publ., Dordrecht, 1995.
7. Igor Dolgachev: *Classical algebraic geometry: a modern view*, Cambridge University Press, Cambridge, 2012.
8. David Eisenbud and Joe Harris: *3264 and all that*, Cambridge University Press, Cambridge, 2016.
9. Laura Escobar and Allen Knutson: The multidegree of the multi-image variety, in *Combinatorial Algebraic Geometry* (eds. G.G.Smith and B.Sturmfels), to appear.
10. William Fulton: *Intersection Theory*, Second edition, A Series of Modern Surveys in Mathematics, Springer-Verlag, Berlin, 1998.
11. Israel M. Gelfand, Mikhail M. Kapranov, and Andrei V. Zelevinsky, *Discriminants, resultants and multidimensional determinants*, Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1994.
12. Phillip Griffiths and Joe Harris: *Principles of algebraic geometry*, Pure and Applied Mathematics, Wiley-Interscience, John Wiley & Sons, New York, 1978.
13. Joe Harris: *Algebraic geometry, a first course*, Graduate Texts in Mathematics 133, Springer-Verlag, New York, 1992.
14. Robin Hartshorne: *Algebraic geometry*, Graduate Texts in Mathematics 52, Springer-Verlag, New York-Heidelberg, 1977.
15. Audun Holme: The geometric and numerical properties of duality in projective algebraic geometry, *Manuscripta Math.* 61 (1988) 145–162.
16. Charles Minshall Jessop: *A treatise on the line complex*, Cambridge University Press, Cambridge, 1903.
17. Kent W. Johnson, Immersion and embedding of projective varieties, *Acta Math.* 140 (1978) 49–74.
18. Steven L. Kleiman: The transversality of a general translate, *Compositio Math.* 28 (1974) 287–297.
19. Kathlén Kohn: Coisotropic Hypersurfaces in the Grassmannian, [arXiv:1607.05932 [math.AG]].
20. Ernst Kummer: Über die algebraischen Strahlensysteme, insbesondere über die der ersten und zweiten Ordnung, *Abhandlungen der Königlichen Akademie der Wissenschaften zu Berlin* (1866) 1–120.
21. David Mumford: *The Red Book of Varieties and Schemes*, Lecture Notes in Math. 1358, Springer-Verlag, Berlin, 1988.
22. Sylvain Petitjean: The complexity and enumerative geometry of aspect graphs of smooth surfaces, in *Algorithms in algebraic geometry and applications* (Santander, 1994), 317–352, Progr. Math. 143, Birkhäuser, Basel, 1996.
23. Rugni Piene: Some formulas for a surface in $\mathbb{P}^3$, in *Algebraic geometry (Proc. Sympos., Univ. Troms, Troms, 1977)*, 196–235, Lecture Notes in Math. 678, Springer, Berlin, 1978.
24. Jean Ponce, Bernd Sturmfels, and Matthew Trager: Congruences and concurrent lines in multi-view geometry, *Adv. in Appl. Math.* 88 (2017) 62–91.
25. Ziv Ran: Surfaces of order 1 in Grassmannians, *J. Reine Angew. Math.* 368 (1986) 119–126.
26. John G. Semple and Leonard Roth: *Introduction to Algebraic Geometry*, Oxford, at the Clarendon Press, 1949.
27. Aron Simis, Bernd Ulrich, and Wolmer V. Vasconcelos: Tangent star cones, *J. reine angew. Math.* 483 (1997) 23–59.
28. Bernd Sturmfels: The Hurwitz form of a projective variety, *J. Symbolic Comput.* **79** (2017) 186–196.
29. Bernd Sturmfels: Fitness, apprenticeship, and polynomials, in *Combinatorial Algebraic Geometry* (eds. G.G.Smith and B.Sturmfels), to appear [arXiv:1612.03539](http://arxiv.org/abs/1612.03539)
30. Zach Teitler: An informal introduction to computing with Chern classes (2004) [bepress.com/zach_teitler/2/](http://bepress.com/zach_teitler/2/)
31. Evgueni A. Tevelev, *Projective Duality and Homogeneous Spaces*, Encyclopaedia of Mathematical Sciences 133. Springer-Verlag, Berlin, 2005.
32. Bartel L. van der Waerden: *Zur algebraischen Geometrie II*, *Math. Ann.* **108** (1933) 253–259.