Geometric Methods for Stochastic Dynamical Systems *

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1 Introduction

Noisy fluctuations are ubiquitous in complex systems. They play a crucial or delicate role in the dynamical evolution of gene regulation, signal transduction, biochemical reactions, among other systems. Therefore, it is essential to consider the effects of noise on dynamical systems. It has been a challenging topic to have better understanding of the impact of the noise on the dynamical behaviors of complex systems. See [Arnold (2003), Duan (2015)] for more information.

1.1 Stochastic dynamical systems

A dynamical system may be thought as evolution of mechanical ‘particles’, and is then described by a system of ordinary differential equations

$$\frac{dx}{dt} = f(x),$$

(1)

where $f$ is often called a vector field. In natural science and applied science, the dynamical systems are used to describe the evolution of complex phenomena [Meiss(2017), Wiggins (2003), Guckenheimer & Holmes (1983)]. However, dynamical systems are often influenced by random factors in the environment, such as in the systems of the propagation of waves through random media, stochastic particle acceleration, signal detection, and optimal control with fluctuating constraints. Indeed, a small random disturbance may have an unexpected effect on the whole dynamical system. A stochastic dynamical system is a dynamical system with noisy components or under random influences [Arnold (2003), Duan (2015)] and thus may be modeled by a stochastic differential equation

$$\frac{dx}{dt} = f(x) + \text{noise}.$$  

(2)

Usually, we use two kinds of noise. One is Gaussian noise, and the other is non-Gaussian noise. Gaussian noise is modeled by Brownian motion $B_t$, while non-Gaussian noise is expressed via Lévy process (especially $\alpha$-stable Lévy motion $L^{\alpha, \beta}_t$) [Applebaum (2009), Sato (1999)]. Dynamical systems driven by Gaussian noise have been widely studied, but in some complex systems, the random influences or stochastic processes are non-Gaussian. For example, during the regulation of gene expression, transcriptions of DNA from genes and translations into proteins take place in a bursty, intermittent, unpredictable manner [Holloway & Spirov (2017), Kumar et al. (2015), Dar et al. (2012), Ozbudak et al. (2002), Blake et al. (2003)].

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Sanchez & Golding (2013). It is more suitable to model these processes by dynamical systems with non-Gaussian Lévy noise.

In order to better describe the impact of noise on the dynamical systems, we consider some geometric methods. These include most probable phase portraits, mean phase portraits, invariant manifolds, and slow manifolds. These geometric tools may help us characterize the impact of noise on the dynamical systems vividly from the geometric perspective.

2 Stochastic dynamical systems

Consider a stochastic dynamical system in $n$-dimensional Euclidean space $\mathbb{R}^n$, either with (Gaussian) Brownian noise
\[ dX_t = f(X_t)dt + \sigma(X_t)B_t, \quad X_0 = x_0, \quad (3) \]
or (non-Gaussian) Lévy noise
\[ dX_t = f(X_t)dt + \sigma(X_t)L_{\alpha,\beta}^t, \quad X_0 = x_0, \quad (4) \]
where $\sigma$ is noise intensity. Let us briefly recall the definition of Brownian motion and Lévy motion.

2.1 Brownian motion

Definition (Karatzas & Shreve (1991)) A Brownian motion $B_t$ is a stochastic process defined on a probability space $\Omega$ equipped with probability $\mathbb{P}$, with the following properties:
(i) $B_0 = 0$, almost surely;
(ii) $B_t$ has independent increments;
(iii) $B_t$ has stationary increments with normal distribution: $B_t - B_s \sim \mathcal{N}(t-s,0)$, for $t > s$;
(iv) $B_t$ has continuous sample paths, almost surely.

2.2 Lévy motion

Definition (Duan (2015)) On a sample space $\Omega$ equipped with probability $\mathbb{P}$, a scalar asymmetric stable Lévy motion $L_{t}^{\alpha,\beta}$, with the non-Gaussianity index $\alpha \in (0,2)$ and the skewness index $\beta \in [-1,1]$, is a stochastic process with the following properties:
(i) $L_{0}^{\alpha,\beta} = 0$, almost surely;
(ii) $L_{t}^{\alpha,\beta}$ has independent increments;
(iii) $L_{t}^{\alpha,\beta}$ has stationary increments with stable distribution: $L_{t}^{\alpha,\beta} - L_{s}^{\alpha,\beta} \sim S_{\alpha}((t-s)^{\frac{\beta}{2}}, \beta, 0)$, for $t > s$;
(iv) $L_{t}^{\alpha,\beta}$ has stochastically continuous sample paths, i.e., for every $s$, $L_{t}^{\alpha,\beta} \rightarrow L_{s}^{\alpha,\beta}$ in probability (i.e., for all $\delta > 0$, $\mathbb{P}(|L_{t}^{\alpha,\beta} - L_{s}^{\alpha,\beta}| > \delta) \rightarrow 0$), as $t \rightarrow s$.

The jump measure, which describes jump intensity and size for sample paths, for the asymmetric Lévy motion $L_{t}^{\alpha,\beta}$ is [Applebaum (2009), Duan (2015)],
\[ \nu_{\alpha,\beta}(dy) = \frac{C_1}{|y|^{1+\alpha}} I_{\{0<y<\infty\}}(y) + \frac{C_2}{|y|^{1+\alpha}} I_{\{-\infty<y<0\}}(y) dy, \quad (5) \]
with $C_1 = \frac{H_{\alpha}(1+\beta)}{2}$ and $C_2 = \frac{H_{\alpha}(1-\beta)}{2}$. When $\alpha = 1$, $H_{\alpha} = \frac{2}{\pi}$; when $\alpha \neq 1$, $H_{\alpha} = \frac{\alpha(1-\alpha)}{\Gamma(2-\alpha)\cos(\frac{\pi\alpha}{2})}$. 

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Especially for \( \beta = 0 \), this is the symmetric stable Lévy motion, which is usually denoted by \( L_\alpha^t \equiv L_\alpha^0 \). More especially, for \( \alpha = 2, \beta = 0 \), this is the well-known Brownian motion \( B_t \).

3 Geometric methods for stochastic dynamical systems

As in the geometrical approaches for deterministic dynamical systems \cite{Meiss2017,Arnold1988,Arnold1989}, we consider geometric concepts and methods for stochastic dynamical systems qualitatively. These include phase portraits and invariant manifolds.

We now introduce some geometric methods for stochastic dynamical systems in this section.

3.1 Most probable phase portraits

The Fokker-Planck equation for the stochastic differential equation (3) or (4) describes the time evolution of the probability density \( p(x,t) \equiv p(x,t|x_0,0) \) for the solution process \( X_t \) with initial condition \( X_0 = x_0 \). It is a linear deterministic partial differential equation [Duan (2015)]

\[
p_t = A^* p, \quad p(x,0) = \delta(x - x_0),
\]

where \( A^* \) is the adjoint operator of the generator \( A \) for this stochastic differential equation, and \( \delta \) is the Dirac delta function.

As the solution of the Fokker-Planck equation, the probability density function \( p(x,t) \) is a surface in the \((x,t,p)\)-space. At a given time instant \( t \), the maximizer \( x_m(t) \) for \( p(x,t) \) indicates the most probable (i.e., maximal likely) location of this orbit at time \( t \). The orbit traced out by \( x_m(t) \) is called a most probable orbit starting at \( x_0 \). Thus, the deterministic orbit \( x_m(t) \) — also denoted by \( x_m(x_0,t) \) — follows the top ridge of the surface in the \((x,t,p)\)-space as time goes on.

**Definition** (Most probable equilibrium point) A most probable equilibrium point (state) is a point (state) which either attracts or repels all nearby points (states). When it attracts all nearby points (states), it is called a most probable **stable** equilibrium point (state), while if it repels all nearby points (states), it is called a most probable **unstable** equilibrium point (state).

The most probable phase portrait \cite{Duan2015,Cheng2016} for a stochastic dynamical system is the state space with representative most probable orbits including equilibrium states. It is a deterministic geometric object, which describes which sample orbit is most probable (maximal likely).

3.2 Mean phase portraits

Given the probability density function \( p(x,t) \equiv p(x,t|x_0,0) \) as the solution of the Fokker-Planck equation (6). The mean orbit starting at an initial point \( x_0 \) in state space is defined as

\[
\bar{x}(x_0,t) = \int_{\mathbb{R}} \xi p(\xi,t|x_0,0) d\xi.
\]

We also give the definition of a mean equilibrium point (state).

**Definition** (Mean equilibrium point) A mean equilibrium point (state) is a point (state) which either attracts or repels all nearby points (states). When it attracts all nearby points (states), it is called a mean **stable** equilibrium point (state), while if it repels all nearby points (states), it is called a mean **unstable** equilibrium point (state).
The mean probable phase portrait is the state space with representative mean orbits including mean equilibrium states. It is also a deterministic geometric object, which describes the orbits in the mean sense, starting in state space.

3.3 Invariant manifolds

Invariant geometry structures play an important role in our understanding of dynamical systems. Invariant manifolds to nonlinear systems just as eigenspace to linear systems. With a better understanding of stochastic invariant manifolds, we may have a new insight to understand stochastic dynamical systems.

We recall some definitions.

**Definition (Random set)** A random set for a random dynamical \( \varphi \) in \( \mathbb{R}^n \) is a collection \( M = M(\omega) \), \( \omega \in \Omega \), satisfies the following property:

(i) \( M(\omega) \) is a nonempty closed set, \( M(\omega) \subset \mathbb{R}^n \), \( \forall \omega \in \Omega \);

(ii) \( V_{\delta}(\omega) \triangleq \inf_{y \in M(\omega)} d(x, y) \) is a scale random variable \( \forall x \in \mathbb{R}^n \).

**Definition (Random invariant set)** An invariant set for a random dynamical \( \varphi \) is a random set \( M \) satisfies:

\[
\varphi(t, \omega, M(\omega)) = M(\theta_t \omega), \quad \forall t \in \mathbb{R} \text{ and } \omega \in \Omega.
\]

**Definition (Random invariant manifold)** If a random invariant set \( M \) for a random dynamical \( \varphi \) can be represented by a graph of a Lipschitz mapping:

\[
\gamma^\ast(\omega, \cdot) : H^+ \rightarrow H^-, \quad \xi \in \mathbb{R}^n.
\]

With direct sum decomposition \( H^+ \oplus H^- = \mathbb{R}^n \), satisfies that \( M(\omega) = (\xi, \gamma^\ast(\omega, \xi)) \), \( \xi \in H^+ \), then \( M \) is called a Lipschitz invariant manifold.

For more information about random invariant manifolds, see [Arnold (2003), Duan et al. (2003), Schmalfuss & Schneider (2008)].

3.4 Slow manifolds

When a system has a slow component and a fast component, we call it slow-fast system. For example, temperature and molecular motion, climate and weather, are slow and fast components in a body and in the climate system, respectively. In this kind of systems, there are two time scales, and usually, we are more interested in the slow dynamics. But we still like to take the fast dynamics into account, even when the fast evolution is less important to our modeling purpose. We thus represent fast components in terms of slow components as slow manifolds. For more information about slow manifolds, see [Schmalfuss & Schneider (2008), Fu et al. (2013), Ren et al. (2015a), Ren et al. (2015b), Constable et al. (2013)].

Consider a stochastic slow-fast system:

\[
\begin{cases}
\dot{x} = Ax + f(x, y), x(0) = x_0 \in \mathbb{R}^n, \\
\dot{y} = \frac{1}{\varepsilon} By + \frac{1}{\sqrt{\varepsilon}} g(x, y) + \frac{\sigma}{\sqrt{\varepsilon}} \dot{W}_t,
\end{cases}
\]

where \( A \) and \( B \) are matrices, \( \varepsilon \) is a small positive parameter measuring slow and fast scale separation, \( f \) and \( g \) are nonlinear Lipschitz continuous functions with Lipschitz constant \( L_f \) and \( L_g \) respectively, \( \sigma \) is a noise intensity constant, and \( W_t : t \in \mathbb{R} \) is a two-sided \( \mathbb{R}^m \) valued Brownian motion on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \).

Under the exponential dichotomy and gap condition, there exists a random slow manifold \( \tilde{M}^\varepsilon(\omega) = (\xi, \tilde{h}^\varepsilon(\xi, \omega)) : \xi \in \mathbb{R}^n \) with \( \tilde{h}^\varepsilon \) can be expressed by Liapunov-Perron equation:

\[
\tilde{h}^\varepsilon(\xi, \omega) = \frac{1}{\varepsilon} \int_{-\infty}^{0} e^{-\frac{s}{\varepsilon}} g(X(s, \omega, \xi), Y(s, \omega, \xi) + \sigma \eta^\varepsilon(\theta_s \omega)) ds
\]
Then the slow system is
\[ \dot{x} = Ax + f(x, \tilde{h}^\varepsilon(x, \Theta_t \omega) + \varepsilon \tilde{h}_1^\varepsilon(x, \omega) + \sigma \eta(\psi_x \omega)) \] (10)
Moreover, \( \tilde{h}^\varepsilon(\xi, \omega) = \tilde{h}_1^\varepsilon(\xi, \omega) + \varepsilon \tilde{h}_1^\varepsilon(\xi, \omega) \) is an approximation or a first order truncation of \( h^\varepsilon(\xi, \omega) \). Thus, we have an approximate slow system
\[ \dot{x} = Ax + f(x, \tilde{h}^\varepsilon(x, \Theta_t \omega) + \sigma \eta(\psi_x \omega)) \] (11)
Based on the reduced slow system, we can conduct parameter estimation [Ren et al. (2015b)], data assimilation [Zhang et al. (2017), Qiao et al. (2018)], and stochastic bifurcation [He et al. (2018)].

4 Applications

In this section, we will discuss some applications associated with the geometric methods for stochastic dynamical systems.

4.1 Transition phenomena

In contrast to deterministic situation, it is natural to consider transitions among metastable states in stochastic dynamical systems. That is to say, a transition orbit is likely to occur between two stable equilibrium states, when a system is under random fluctuations.

Transition phenomena occur in gene regulation, tumor cell density, climate change, parametric oscillator, Briggs-Rauscher chemical reaction, and predator-prey systems. By the geometric methods for stochastic dynamical systems, we may qualitatively demonstrate these noise-induced transition phenomena.

In genetic regulatory systems, specific protein concentrations play an important role in cell life. The low concentration and high concentration of specific protein correspond to different cell activities. Recent studies [Xu et al. (2013), Zheng et al. (2016), Wang et al. (2018)] have recognized that Lévy motion can induce switches between different protein concentrations. Multiple phenotypic states often arise in a single cell with different gene expression states. The transitions between two phenotypic states due to stochastic fluctuations have been verified in [Ge et al. (2015)].

In population dynamics, determining the amount of predation which a prey population can sustain without endangering its survival is an important problem [Horsthemke & Lefever (2006)]. The problem is to find out the best strategy for a good management of biological resources. We may also think in the opposite direction, driving a prey population to extinction by sufficient predation. In this case the extinction problem does not only find applications in pest control but also in the medical sciences.

4.2 Stochastic bifurcation

Although bifurcation studies for deterministic dynamical systems have a long history, the stochastic bifurcation investigation is still in its early stage. One reason for this slow development in stochastic bifurcation is due to the lack of appropriate phase portraits, in contrast to deterministic dynamical systems.

Stochastic bifurcations have been observed in a wide range of nonlinear systems in physical science and engineering [Deco & Martí (2007), Bashkirtseva et al. (2018), Bogatenco & Semenov (2018)]. Some works about stochastic bifurcation are analytical studies of invariant measures, together with their spectra and supports. We recently have used most probable phase portraits to detect stochastic bifurcation [Wang et al. (2018)].
5 Conclusion

Stochastic dynamical systems are widely arise as mathematical models in biology, physics, chemistry and engineering. In this chapter, we have reviewed several geometric methods for stochastic dynamical systems and their applications. By most probable phase portraits and mean phase portraits, we may better understand stochastic dynamical behaviors such as transition orbits and stochastic bifurcation. Based on the reduced systems on slow manifolds, we may estimate parameters, detect stochastic bifurcation, and conduct data assimilation on lower dimensional slow systems.

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