Varadhan’s formula, conditioned diffusions, and local volatilities

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Abstract

Motivated by marginals-mimicking results for Itô processes via SDEs and by their applications to volatility modeling in finance, we discuss the weak convergence of the law of a hypoelliptic diffusions conditioned to belong to a target affine subspace at final time, namely $L(Z_t|Y_t=y)$ if $X_t=(Y_t,Z_t)$. To do so, we revisit Varadhan-type estimates in a small-noise regime, studying the density of the lower-dimensional component $Y$.

The application to stochastic volatility models include the small-time and, for certain models, the large-strike asymptotics of the Gyöngy–Dupire’s local volatility function, the final product being asymptotic formulae that can (i) motivate parameterizations of the local volatility surface and (ii) be used to extrapolate local volatilities in a given model.

1 Introduction

Consider an $n$-dimensional diffusion process given by the solution to

$$dX_t = b(X_t) dt + \sum_{j=1}^d \sigma_j(X_t) dW^j, \quad X_0 = x_0 \in \mathbb{R}^n.$$ (1.1)

Applications from finance suggest a splitting of the state space, say $X = (Y,Z) \in \mathbb{R}^l \times \mathbb{R}^{n-l} \cong \mathbb{R}^n$, where $Y$ is the process of main interest (for instance, price or log-price of an asset) and $Z$ some auxiliary process (for instance, stochastic volatility, possibly multi-dimensional). There is a massive amount of literature concerning $p_t$, the probability distribution function of $X_t$ at small times $t$. In the elliptic case, that is when $\text{span}\{\sigma_1,\ldots,\sigma_d\} = \mathbb{R}^n$, such investigations go back to Varadhan (“$2t \log p_t(x,y) \sim d^2(x,y)$”) and then Molchanov for full expansions (away from cut-loci). The hypoelliptic situation (assuming the strong Hörmander condition, $\text{Lie}\{\sigma_1,\ldots,\sigma_d\} = \mathbb{R}^n$) was then studied by Azencott, Ben Arous, Bismut, Leandre,... (the distance function $d$ must then be interpreted as control distance associated to the diffusion vector fields.)

Only recently, similar results where obtained for $f_t$, the pdf of $Y_t$ (a marginal of $X_t$), see [12, 13]. Under a set of new conditions (notably what the authors call non-focality), they obtain various asymptotic expansions of $f_t$. In the present note we continue and complement these investigations, together with some novel applications towards the behaviour of local volatility.

Our main results are:

(i) a Varadhan formula for $f_t$ in the short time limit, which is seen to be valid in great generality (without the need to check non-focality) and then (ii) a limit theorem for $Z$ conditioned on some value of $Y$. The limit here may again be short time or more generally small noise. In fact, the small noise situations poses new difficulties (for instance, in a strictly hypoelliptic setting Varadhan’s formula may fail!) but then offers new applications: indeed, contribution (iii) of this paper is concerned with a class of stochastic...
Theorem 1.1. (i) Let $X_t = (Y_t, Z_t)$ as above. Under a strong Hörmander condition, $Y_t$ admits a density $f_t$ for $t > 0$ and the following Varadhan type formula holds: for every $y \in \mathbb{R}^d$

$$\lim_{t \to 0} t \log f_t(y) = -\inf_{\{x = (y,z) : z \in \mathbb{R}^{n-1}\}} \Lambda(x) =: \Lambda(N_y)$$

where $2\Lambda(x) = d^2(x_0, x)$ is the squared control distance associated to $\{\sigma_1, \ldots, \sigma_d\}$ and $N_y = \{x = (y,z) : z \in \mathbb{R}^{n-1}\}$.

(ii) Under a further technical assumption (always satisfied in the elliptic case!)

$$\mathcal{L}(Z_t | Y_t = y) \Rightarrow \delta_{z^*(y)} \quad \text{as } t \downarrow 0$$

(in the sense of weak convergence of probability measures) provided there exists a unique minimizer for the problem $(y, z^*(y)) := \arg\min_{z \in \mathbb{R}^{n-1}} \Lambda(x)$.

While the above theorem is clearly useful (it implies for instance, short time asymptotics for local volatility; on a technical level, we remove the ellipticity requirement from [7]), it does not lend itself to understand spatial asymptotics.

Theorem 1.2. (i) Write $X_t^\varepsilon = (Y_t^\varepsilon, Z_t^\varepsilon)$. Under a strong Hörmander condition, and a further technical assumption (which is always satisfied in the elliptic case or also when $b_0 = 0$), the following Varadhan type formula holds for the density $f_t^\varepsilon$ of $Y_t^\varepsilon$: for every $t > 0$ and $y \in \mathbb{R}^d$

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log f_t^\varepsilon(y) = -\Lambda_t(N_y).$$

where, as before $N_y = \{x = (y,z) : z \in \mathbb{R}^{n-1}\}$. (Although the action $\Lambda$ is still given in terms of a variational problem, cf. [13], it has no interpretation as point-to-subspace distance.)

(ii) Under the same assumptions as above we have, for fixed $y$ and $t > 0$,

$$\mathcal{L}(Z_t^\varepsilon | Y_t^\varepsilon = y) \Rightarrow \delta_{z_t^\varepsilon(y)} \quad \text{as } \varepsilon \downarrow 0$$

provided there exists a unique minimizer for the problem $(y, z_t^\varepsilon(y)) := \arg\min_{z \in \mathbb{R}^{n-1}} \Lambda_t(x)$.

The case $l = n$ in Theorem 1.1(ii) (“from $x_0 \in \mathbb{R}^n$ to $x \in \mathbb{R}^n$”) is covered by the result of Molchanov [32] for elliptic diffusions, and more recently by Bailieu [2] in the hypoelliptic setting. Besides the more general framework of small noise systems that we consider in Theorem 1.2 in (ii) the final target set for the process $X$ is an affine subspace instead of a single point (“from $x_0$ to $N_y$”, restoring the point-to-point situation when $l = n$).

Also, the results mentioned above are given on compact manifolds, while we work with $\mathbb{R}^n$-valued processes, and need to rely on some non-trivial tail bounds.

Following the well-known projection results [22] [14] for Itô SDEs, we then have the following corollaries of Theorem 1.2 to local volatility:

Theorem 1.3. (i) [Local volatility, short time behavior] In a generic stochastic volatility model $(Y, Z)$ (where $Y$ denotes log-price and $Z$ stochastic volatility)

$$\sigma^2_{loc}(t, y) = \text{E}[(Z_t)^2 | Y_t = y] \to z^*(y)^2 \quad \text{as } t \to 0.$$  

(1.3)

4By Brownian scaling, the short time setting $t \to 0$ falls into this setting by taking $t = \varepsilon^2$, $b_\varepsilon = \varepsilon b$ and $x_0 = x_0$. 

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Here $z^* = z^*(y)$ is the “most likely” arrival point, computed as $\arg\min_{z \in \mathbb{R}} \Lambda^{SV}(y, z)$ where $\Lambda^{SV}$ is the action associated to the stochastic volatility model under consideration. (Explicit computations depend on the concrete model.)

(ii) [Local volatility ‘wings’ in the Stein–Stein model] In the Stein–Stein model\(^5\) (where $Z$ follows an Ornstein–Uhlenbeck process, see \(4.14\)) small noise asymptotics lead to

$$
\lim_{y \to \pm \infty} \frac{\sigma^2_{loc}(t, y)}{|y|} = \lim_{y \to \pm \infty} \frac{1}{|y|} \mathbb{E}[(Z_t)^2 | Y_t = y] = c^\pm_t > 0
$$

where the constants $c^\pm_t$ are given explicitly in terms of the model parameters.

As we will discuss in Section 4 under some special parameter configuration of the Stein–Stein model, the explicit expression of the constants $c^\pm_t$ in (1.4) turns out to be consistent with known results from moment explosion for affine models\(^6\).

In analogy with the large-strike behavior of implied volatility\(^5\)\(^6\), the linear asymptotic behavior of the local variance in Theorem 1.3(ii) is likely to hold in wide classes of stochastic volatility models (the same result is indeed known to hold for the Heston model, see\(^11\), based on affine principles). On the one hand, the knowledge of an explicit spatial asymptotics for the local volatility can motivate the choice of the functional forms that are used to smooth out and/or extrapolate a local volatility surface calibrated to market data. Already in use among practitioners, SVI-type parameterizations of the local variance, cf.\(^19\) Section 4), are compatible with the asymptotics in (1.4). On the other hand, a robust implementation of the local volatility surface is the basis for a Monte-Carlo evaluation of exotic options under local volatility; once this step is achieved, the comparison of the prices of volatility-sensitive products (cliquets, barriers) under stochastic volatility and the ‘projected’ local volatility model is often used by option trading desks in order to quantify the bias due to the use of different volatility dynamics, entering as an important step in the assessment of volatility model risk. Theorem 1.3 allows to extrapolate the local volatility function with explicit formulae in extreme regions, where the implementation of Dupire’s formula typically suffers from numerical instabilities. For a discussion of implications of these results relevant to practitioners, see\(^11\).

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1.1 Small noise systems

Standing assumption throughout this paper is that the vector fields $\sigma_1, \ldots, \sigma_d$ and the one parameter family $b_\varepsilon$ are smooth ($C^\infty$) functions: postponing any precise set of assumptions to the following sections, let us mention here that our main results are stated under a boundedness assumption on $b_\varepsilon$ and the $\sigma_j$, and then extended to a class of 2-dim diffusions (stochastic volatility models) with unbounded coefficients. We assume that

$$
x^\varepsilon_0 \to x_0 \in \mathbb{R}^n \quad \text{as } \varepsilon \downarrow 0,
$$

and that $b_\varepsilon$ converges to some limit vector field $b_0 \in C^\infty$

$$
b_\varepsilon \to b_0 \quad \text{as } \varepsilon \downarrow 0
$$

uniformly on compact sets of $\mathbb{R}^n$.

\(^5\)This result was announced in\(^11\).

\(^6\)When the correlation parameter between the log-price $Y$ and the instantaneous volatility $Z$ is not null, the Stein–Stein model is also known as Schöbel and Zhu model, see\(^34\).
Under assumptions (1.3) and (1.8), it is known that the process $X^\varepsilon$ satisfies a Large Deviation Principle (LDP) on the path space $C([0, T]; \mathbb{R}^n)$ as $\varepsilon \downarrow 0$ (for a nice recent summary about large deviation principles for small-noise diffusions, see Baldi and Caramellino [4] and the references therein). The deviations of $X^\varepsilon$ are driven by the solutions of the limiting controlled differential system

$$
dd x_j^\varepsilon = b_\varepsilon(x_j^\varepsilon)dt + \sum_{j=1}^d \sigma_j(x_j^\varepsilon)dh^\varepsilon_{\varepsilon}, \quad x_0 = 0,
$$

where $h \in H_T \subset C([0, T]; \mathbb{R}^n)$, and for any $t \leq T$, $H_t$ denotes the Cameron-Martin (Hilbert) space of absolutely continuous functions with derivative in $L^2([0, t]; \mathbb{R}^n)$, equipped with the norm $||h||^2_{H} := ||h||^2_{L^2} = \int_0^t |h_s|^2 ds$. Following the typical terminology in large deviations theory, for every $t \leq T$ we define the action function $\Lambda_t : \mathbb{R}^n \to [0, \infty)$ by

$$
\Lambda_t(x) = \inf \left\{ \frac{1}{2} |h|^2_{H} : h \in K^x_t \right\}, \quad x \in \mathbb{R}^n,
$$

with the convention $\inf \emptyset = \infty$, where

$$
K^x_t = \{ h \in H_t : \varphi^h_{\varepsilon}(x_0) = x \}
$$

is the set of controls steering the trajectories of the system (1.7) from the point $x_0$ to the point $x$ in time $t$. Following standard terminology, we call minimizing control any control $h_0 \in K^x_t$ realizing the infimum in (1.8), namely such that $\frac{1}{2} |h_0|^2_{H} = \Lambda_t(x)$. Some properties of $\Lambda_t$ are presented in Lemma 2.7 below. For every fixed $t > 0$, the LDP for the family of finite dimensional random variables $\{X^\varepsilon_t\}_\varepsilon$ reads

$$
\lim_{\varepsilon \to 0} \sup \varepsilon^2 \log \mathbb{P}(X^\varepsilon_t \in C) \leq -\Lambda_t(C); \quad \lim_{\varepsilon \to 0} \inf \varepsilon^2 \log \mathbb{P}(X^\varepsilon_t \in G) \geq -\Lambda_t(G),
$$

for every closed set $C$ and open set $G$ in $\mathbb{R}^n$. Following a common convention in large deviations theory, we denote $\Lambda_t(E) = \inf_{x \in E} \Lambda_t(x)$.

The large deviations principle (1.9) is very general, and depends only on some mild Lipschitz conditions on the coefficients of the SDE. We will be concerned with the situation where the fixed-time distribution of $X^\varepsilon$ possesses a density: as it is common in the field of hypoelliptic heat kernel asymptotics [29, 30, 5], we assume that strong Hörmander condition holds at all points:

$$
\text{(sH) Lie}(\sigma_1, \ldots, \sigma_d)_x := \text{span}\{\sigma_1, \ldots, \sigma_d; [\sigma_i, \sigma_j] : 1 \leq i, j \leq d; [\sigma_i, [\sigma_i, \sigma_m]] : 1 \leq i, l, m \leq d; \ldots \}_x = \mathbb{R}^n \quad \forall x \in \mathbb{R}^n,
$$

that is, the linear span of the $\sigma_1, \ldots, \sigma_d$ and all their Lie brackets is the full tangent space to $\mathbb{R}^n$ at $x_0$. It is a classical result (due to Hörmander, Malliavin) that the law of $X^\varepsilon_t$ admits a smooth density with respect to the Lebesgue measure on $\mathbb{R}^n$ for every $t > 0$.

In order to study the asymptotic behavior of the density of $X^\varepsilon_t$, we impose the convergence of the partial derivatives of the drift vector field $b_\varepsilon$: in addition to (1.6), for every multi-index $\alpha \in \{1, \ldots, n\}^k$

$$
\partial^\alpha_x b_\varepsilon \to \partial^\alpha_x b_0 \quad \text{as} \quad \varepsilon \downarrow 0
$$

uniformly on compact sets of $\mathbb{R}^n$, where $\partial^\alpha_x := \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}$. Furthermore, we assume that the families of norms $|b_\varepsilon|_\infty$ and $|\partial^\alpha_x b_\varepsilon|_\infty$ are uniformly bounded in $\varepsilon$, for every $\alpha$.

**The deterministic Malliavin matrix** $C_{x_0}(h)$. For every $t \in [0, T]$, the map $h \mapsto \varphi^h_{\varepsilon}(x_0)$ is differentiable (indeed, $C^\infty$) from $H$ into $\mathbb{R}^n$, see Bismut [3] Theorem 1.1]. Let us denote $D\varphi^h_{\varepsilon}(x_0) \in \text{Lin}(H, \mathbb{R}^n)$ its Fréchet
derivative at \( h \). On the other hand, for fixed \( h \), \( \varphi^h_t(x) \) is a diffeomorphism as a function of \( x \in \mathbb{R}^n \): we denote \( \Phi^h_t(x) \in \text{Lin}(\mathbb{R}^n, \mathbb{R}^n) \) its differential at \( x \). The method of variation of constants allows to express \( D\varphi^h_t(x_0)[k] \), the image of \( k \in H \) through the linear map \( D\varphi^h_t(x_0) \), via the representation formula

\[
D\varphi^h_t(x_0)[k] = \int_0^t \sum_{j=1}^d \Phi^h_t(x_0)\Phi^h_s(x_s)^{-1}\sigma_j(x_s)h_0^jsds, \quad x_s = \varphi^h_s(x_0).
\]

Following Bismut [8], and in analogy with the stochastic Malliavin matrix, we introduce the deterministic Malliavin covariance matrix \( C_{x_0}(h) \), whose entries are given by

\[
C_{x_0}(h)^{i,j} = \int_0^t \sum_{l=1}^d \left[ \Phi^h_t(x_0)\Phi^h_s(x_s)^{-1}\sigma_l(x_s) \right]^i \left[ \Phi^h_t(x_0)\Phi^h_s(x_s)^{-1}\sigma_l(x_s) \right]^j ds. \tag{1.12}
\]

It is a fundamental remark due to Bismut [8] Theorem 1.3] that \( D\varphi^h_t(x_0) \) has full rank \( n \) if and only if the matrix \( C_{x_0}(h) \) is invertible. The invertibility of \( C_{x_0}(h) \) is related to the non-degeneracy of the vector fields \( \sigma_j \); in the presence of a locally elliptic diffusion coefficient - which is the case for several financial applications - the following invertibility condition is useful, and easy to check:

**Lemma 1.4.** Let \( h \in \mathcal{K}_l^\varepsilon \). If there exists \( s \in [0,t] \) such that

\[
\text{span}[\sigma_1, \ldots, \sigma_d]_{x_s} = \mathbb{R}^n, \quad x_s = \varphi^h_s(x_0),
\]

then \( C_{x_0}(h) \) is invertible.

The proof of Lemma [1.4 is an easy linear-algebra exercise, see [8 Theorem 1.10] or [12 Proposition 2.1]. A sufficient condition for \( C_{x_0}(h) \) to be invertible for every \( h \neq 0 \), stronger than Hörmander ‘s condition, is given as condition (H2) in [8 Chap.1].

**Notation for densities.** We denote

\[
p_t^\varepsilon(\cdot) = p_t^\varepsilon(x), \quad x \in \mathbb{R}^n
\]

the density of \( X_t^\varepsilon \) and

\[
f_t^\varepsilon(\cdot) = f_t^\varepsilon(y), \quad y \in \mathbb{R}^l
\]

the density of the \( \mathbb{R}^l \)-valued projection

\[
Y_t^\varepsilon = \Pi_l X_t^\varepsilon := (X_t^{\varepsilon,1}, \ldots, X_t^{\varepsilon,l}) \in \mathbb{R}^l, \quad l \leq n.
\]

It is clear that

\[
f_t^\varepsilon(y) = \int_{\mathbb{R}^{n-l}} p_t^\varepsilon(y,z)dz, \tag{1.13}
\]

where \( p_t^\varepsilon(y,z) := p_t^\varepsilon((y,z)) \). Note that the (limiting) initial condition \( x_0 \) is fixed in the present discussion and, in contrast with the usual convention in heat kernel analysis, we do not write \( p_t(x_0,x) \) - including the initial condition in the symbol for the density - in order to avoid any confusion between initial and terminal points when writing \( p_t^\varepsilon(y,z) \) for \((y,z) \in \mathbb{R}^l \times \mathbb{R}^{n-l} \cong \mathbb{R}^n \).

Finally, we denote \( \| \cdot \| \) the infinity norm in \( \mathbb{R}^n \), and \( B_R(x) \) (resp. \( B_R^c(x) \)) the associated closed ball of radius \( R \) around \( x \) (resp. the complementary of the ball).

## 2 Theoretical main estimates

Ben Arous and Léandre [6, Section 3], showed that the asymptotics of the logarithm of the density for the small-noise problem \([1,2]\) as \( \varepsilon \to 0 \) might be governed by a different action function (what they call the “regular” action) defined by

\[
\Lambda_{R,t}(x) = \inf \left\{ \frac{1}{2} \| h \|^2_{H^2} : h \in \mathcal{K}_l^\varepsilon, C_{x_0}(h) \text{ is invertible} \right\}
\]

with the convention \( \inf \emptyset = \infty \).
\textbf{Theorem 2.1 (Ben Arous and Léandre \cite{5} revisited).} Consider
\[dX^\varepsilon = b_\varepsilon(X^\varepsilon_t)\,dt + \varepsilon\sigma(X^\varepsilon_t)\,dW, \quad X^\varepsilon_0 = x^\varepsilon_0\]
with \(x^\varepsilon_0 \to x_0\) as \(\varepsilon \to 0\), and \(b_\varepsilon \to b\) according to (1.11). Assume strong Hörmander condition (sH) at all points, and write \(p^*_t(x)\) for the density of \(X^\varepsilon_t\). Then

(i) the following estimates hold:
\[
\limsup_{\varepsilon \to 0} \varepsilon^2 \log p^*_t(x) \leq -\Lambda_t(x) \quad (2.2)
\]
and
\[
\liminf_{\varepsilon \to 0} \varepsilon^2 \log p^*_t(x) \geq -\Lambda_{R,t}(x) \quad (2.3)
\]
for every \(x \in \mathbb{R}^n\). In particular, if \(K^r_t\) is non-empty for some \(x\) and there exists a minimizing control \(h_0 \in K^r_t\) such that \(C_{x_0}(h_0)\) is invertible, then \(\Lambda_t(x) = \Lambda_{R,t}(x) < \infty\), so that
\[
\lim_{\varepsilon \to 0} \varepsilon^2 \log p^*_t(x) = -\Lambda_t(x). \quad (2.4)
\]

(ii) Assume there exists a minimizing control \(h_0 \in K^r_t\) with invertible Malliavin matrix \(C_{x_0}(h_0)\). Then, there exists an open neighborhood \(V\) of \(\mathbb{R}^t\) such that
\[
\limsup_{\varepsilon \to 0} \varepsilon^2 \log p^*_t(x) \leq -\Lambda_t(x) \quad (2.5)
\]
holds uniformly over \(x\) in compact sets contained in \(V\).

\textbf{Proof.} The proof is given in Appendix A.2. The statement with \(b_\varepsilon \equiv b_0\) and \(x^\varepsilon = x_0\), and without the uniform convergence in (2.5), is given in Theorem III.1 in \cite{6}.

\textbf{Remark 2.2.} Using the continuity of \(1 + |x|^2\) on \(H\), if one assumes \(C_{x_0}(h)\) to be invertible for \(h\) in a dense subset of \(K^r_t\) (for the strong topology), then \(\Lambda_t(x) = \Lambda_{R,t}(x)\) immediately follows from the definition of the two actions in (1.8) and (2.1). Under this assumption, (2.3) holds. In the end, the condition of invertibility of \(C_{x_0}(h)\) for all \(h\) in some \(K^r_t\) will be satisfied in our applications.

Some additional comments are in order.

\textbf{Comment 2.3.} (i) In light of Lemma 1.4 if \(\sigma_1, \ldots, \sigma_d\) span the whole \(\mathbb{R}^n\) at either \(x_0\) or \(x\), \(C_{x_0}(h)\) is invertible at every \(h \in K^r_t\).

(ii) Even when strong Hörmander condition is satisfied at all points, \(C_{x_0}(h)\) might not be invertible on some \(h\). But if \(b_0 \equiv 0\) on a neighborhood of \(x_0\), the two actions \(\Lambda_t\) and \(\Lambda_{R,t}\) coincide (see the discussion in \cite{6} Section 3), using results in [30, 28]). In this case, (2.4) holds.

(iii) In general, the two actions can be different. In [6] Section 1] an example on \(\mathbb{R}^2\) is given, where strong Hörmander condition is satisfied at all points, but the two actions \(\Lambda_t\) and \(\Lambda_{R,t}\) do not coincide. As a consequence, the classical Varadhan formula (2.4) does not hold everywhere.

The following tail bound will be useful in the proof of our main result in the next section.

\textbf{Proposition 2.4.} Let \(t > 0\) and \(y \in \mathbb{R}^t\) be fixed. Under the assumption of Theorem 2.1(i), we have, for every \(A > 0\), every \(\overline{\tau} \in \mathbb{R}^{n-1}\) and every \(k \in \mathbb{N}\)
\[
\limsup_{\varepsilon \to 0} \varepsilon^2 \log \int_{\{z \in \mathbb{R}^{n-1} : |z - \overline{\tau}| \geq A\}} |z|^k p^*_t(y, z)\,dz \leq -\Lambda_t(\{(y, z) : |z - \overline{\tau}| \geq A\}),
\]
where with the usual convention, \(\Lambda_t(E) = \inf_{x \in E} \Lambda_t(x)\).
Proof. Given in Appendix A.2.

Crucial for the applications, the optimal control problem (1.8) defining the action function \( \Lambda_t \) can be rephrased in terms of the Hamiltonian formalism. The following proposition provides necessary optimality conditions for the controls in \( \mathcal{K}_t^{(y,\cdot)} \) when \( y \) is fixed, in the spirit of Pontryagin’s maximum principle: as such, it appears as a generalization of the corresponding result in Bismut [8], from a point-to-point setting \((x_0 \in \mathbb{R}^n, \text{ to } x \in \mathbb{R}^n)\) to a point-to-subspace \((x_0 \in \mathbb{R}^n, \text{ to } N_y := (y, \cdot), y \in \mathbb{R}^l)\) setting. Let us introduce the Hamiltonian

\[
\mathcal{H}(x,p) = \langle b_0(x), p \rangle_{\mathbb{R}^n} + \frac{1}{2} \sum_{j=1}^{d} (\sigma_j(x), p)_n^2.
\]

**Proposition 2.5** (see Proposition 2 in [12]). Fix \( y \in \mathbb{R}^l \), and assume \( h_0 \in \mathcal{K}_t^{(y,\cdot)} \) is an optimal control for the problem

\[
\Lambda_t(N_y) = \inf \left\{ \frac{1}{2} \| h \|_H^2 : h \in \mathcal{K}_t^{(y,\cdot)} \right\} \quad N_y = (y, \cdot).
\]

Moreover, assume the deterministic Malliavin matrix \( C_{x_0}(h_0) \) is invertible. Then, there exists a unique \( p_0 \) such that \( \varphi_s^{h_0}(x_0) = x_s \) for all \( s \in [0, t] \), where \((x_s, p_s)_{s \leq t}\) solves the Hamiltonian ODEs

\[
\begin{pmatrix}
\dot{x}_s \\
\dot{p}_s
\end{pmatrix} = \begin{pmatrix}
\partial_{x_s} \mathcal{H}(x_s, p_s) \\
-\partial_{p_s} \mathcal{H}(x_s, p_s)
\end{pmatrix} \tag{2.6}
\]

subject to the (initial-, terminal- and transversality-) boundary conditions

\[
x_0 = x_0 \in \mathbb{R}^n, \quad x_t = (y, \cdot) \in \mathbb{R}^l \times \mathbb{R}^{n-l} \\
p_0 = p_0 \in \mathbb{R}^n, \quad p_t = (\cdot, 0) \in \mathbb{R}^l \times \mathbb{R}^{n-l}. \tag{2.7}
\]

Furthermore, the control \( h_0 \) is restored as

\[
\dot{h}_0^j(s) = (\sigma_j(x_s), p_s), \quad j = 1, \ldots, d
\]

and \( \Lambda_t(N_y) = \frac{1}{2} \| h_0 \|_H^2 \).

**Remark 2.6.** If \( x_t = (y, z_t) \) is the terminal value of the \( x \)-component of a solution to (2.4), then \( z_t \) is a minimizer of the map \( z \mapsto \Lambda_t(y, \cdot) \).

The following lemma summarizes some properties of the control system (1.7) and of the action \( \Lambda_t \) that will be extensively used throughout the paper.

**Lemma 2.7.** Assume the vector fields \( b_0 \) and \((\sigma_j)\) are Lipschitz continuous, with Lipschitz constant \( K \). Denote \( \varphi_s^{h}(x_0) \) the solution of the ODE (1.7) on the interval \([0, t]\), and \( \Lambda_t \) the action of the system as in (1.8). Then

(i) For every \( t \), the map \( h \rightarrow \varphi_s^{h}(x_0) \) is weakly continuous from \( H \) into \( \mathbb{R}^n \). Moreover, there exists a positive constant \( C(t, x_0, K) \), increasing in \( K \), such that

\[
\sup_{s \leq t} |\varphi_s^{h}(x_0)| \leq C(t, x_0, K) e^{C(t, x_0, K) \| h \|_H} \tag{2.8}
\]

for every \( h \in H \).

(ii) \( \Lambda_t \) is a good rate function of Large Deviations theory: that is, for every \( l \geq 0 \) the level sets \( \{ x : \Lambda_t(x) \leq l \} \) are compact. In particular, \( \Lambda_t \) is lower semi-continuous.

(iii) if \( \mathcal{K}_t^{\mathbb{R}} \neq \emptyset \), the infimum in (1.8) is attained: that is, there exists a minimizing control \( h_0 \in \mathcal{K}_t^{\mathbb{R}} \) such that \( \Lambda_t(x) = \frac{1}{2} \| h_0 \|_H^2 \).

Assume moreover that the vector fields are \( C_b^\infty \) (bounded with bounded derivatives). Then
(iv) If the \((\sigma_j)_j\) satisfy the strong Hörmander condition \((sH)\) at all \(x\), then \(K^x_t \neq \emptyset\) for every \(x\), and \(x \mapsto \Lambda_t(x)\) is finite on \(\mathbb{R}^n\).

(v) If there exists a minimizing control \(h_0 \in K^x_T\), \(x \in \mathbb{R}^n\), with invertible Malliavin matrix \(C_{x_0}(h_0)\), then there exists a neighborhood \(V\) of \(x\) such that \(\Lambda_t\) is continuous on \(V\).

**Proof.** (i). Weak continuity with respect to the control parameter is classical. See Bismut [8, Theorem 1.1] for the case of smooth vector fields: the case of Lipschitz continuous coefficients is handled analogously: in essence, the continuity property and estimate (2.8) follow from an application of Gronwall’s lemma. See also [4, proof of Lemma 2.5] for estimate (2.8). (ii) is a direct consequence of (i): use \(\{x : \Lambda_t(x) \leq l\} = \{\phi^h_t(x_0) : |h|^2_H \leq 2l\}\), and the latter set is compact since (weakly) continuous image of a (weakly) compact set. \((\mathbb{Z}^x)\) is the density of \(\Lambda_t\) induced by (1.7) with the Carnot-Carathéodory distance on a sub-Riemannian manifold (here: existence of minimizers with invertible Malliavin matrix. This statement is equivalent to well-known continuity of the linear map \(D\phi^h_t\) on this set. (iii) is a direct consequence of (i): indeed, assume \(\hat{h} \in K^x_t\). Then the infimum in (1.8) is in fact taken over the set \(\{h : \phi^h_t(x_0) = x, |h|^2_H \leq |\hat{h}|^2_H\}\), which is weakly compact; since the norm \(|\cdot|_H\) is weakly lower semi-continuous, \(\frac{1}{2}|\hat{h}|^2_H\) attains its minimum on this set. (iv). The non-emptiness of \(K^x_t\) (therefore, finiteness of \(\Lambda_t\)) under strong Hörmander condition is a classical result of controllability: see e.g. [23, Theorem 2, p. 106] for the affine control system with drift that we consider here. (v). In light of (ii), it is sufficient to prove that \(\Lambda_t\) is upper semi-continuous. Under the assumption of existence of a minimizing control \(h_0\) with invertible Malliavin matrix, upper semi-continuity is proven as in the second part of [9, Proposition 3.2]: as it is typical in the geometrical control setting, the key point is the implementation of an Implicit Function Theorem locally around \(\mathbb{Z}\), which is made possible by the fact that the linear map \(D_{h_0}\phi^h_t(x_0) \in \text{Lin}(H, \mathbb{R}^n)\) has full rank. ■

**Remark 2.8 (On point (v) of Lemma 2.7).** When \(b_0 \equiv 0\) in (1.7) and strong Hörmander condition \((sH)\) holds, it is classical that \(x \mapsto \Lambda_t(x)\) is finite and continuous on \(\mathbb{R}^n\), without any further assumption about the existence of minimizers with invertible Malliavin matrix. This statement is equivalent to well-known continuity of the Carnot-Carathéodory distance on a sub-Riemannian manifold (here: \(\mathbb{R}^n\) equipped with the control distance induced by (1.7) with \(b_0 \equiv 0\)). A standard proof, based on the small-time local controllability of driftless control systems, is provided in Bismut [8, Theorem 1.14]. For affine control systems with non-zero drift as (1.7), the continuity of \(\Lambda_t\) is not, in general, a consequence of Hörmander condition. In [11, Section 2] an example is provided, where strong Hörmander condition holds at all points, and the action function is not continuous.

**Remark 2.9.** If the function \(\Lambda_t\) is known to be continuous on \(\mathbb{R}^n\), estimate (2.8) in Theorem 2.1 holds uniformly over \(x\) in compact sets of \(\mathbb{R}^n\).

### 2.1 The conditioned diffusion

Denote \(Z^x_t := (X^x_t, l+1, \ldots, X^x_n)\) the projection of \(X^x_t\) over the last \(n-l\) components, so that

\[
X^x_t = (Y^x_t, Z^x_t).
\]

We write

\[
\mathcal{L}(Z^x_t|Y^x_t = y)
\]

for the law of \(Z^x_t\) conditional on \(Y^x_t\) being at level \(y \in \mathbb{R}^l\) at time \(t\). If \(f^x_t(y) > 0\), this is well-defined via

\[
\mathbb{E}[\varphi(Z^x_t)|Y^x_t = y] = \int_{\mathbb{R}^{n-l}} \varphi(z)g^x_t(z)dz
\]

for all \(\varphi \in C_b(\mathbb{R}^{n-l})\), where

\[
g^x_t(z) = g^x_{l,y}(z) := \frac{p^x_t(y, z)}{f^x_t(y)}
\]

(2.9)

is the density of \(Z^x_t\) conditional on \(Y^x_t = y\).

**Theorem 2.10.** Consider \(X^x = (Y^x, Z^x) \in \mathbb{R}^l \times \mathbb{R}^{n-l} \cong \mathbb{R}^n\) given by

\[
dX^x_t = b_x(X^x_t)dt + \varepsilon \sum_{j=1}^d \sigma_j(X^x_t)dW^j_t, \quad X^x_0 = x^x_0,
\]

(2.10)
with \( x_0 \to x_0 \) and \( b_z \to b_0 \) according to (1.11), and assume strong Hörmander condition \((sH)\) at all points. Fix \( y \in \mathbb{R}^l \) and \( t > 0 \), and set \( N_y = (y, \cdot) \). Assume that there exists a unique minimizing control \( z^*(y) \) for the problem

\[
(y, z^*) := \arg\min_{x \in N_y} \Lambda_t(x),
\]

(2.11)

and assume that for every \( z \) in a neighbourhood of \( z^*(y) \) there exists a minimizing control \( h_0 \in K_{t}^{(y,z)} \) with invertible deterministic Malliavin matrix \( C_{x_0}(h_0) \), as defined in (1.12). Then, \( f^*_t(y) > 0 \) and

\[
\mathcal{L}(Z_t^*|Y_t^* = y) \Rightarrow \delta_{z^*_t(y)} \quad \text{as } \varepsilon \downarrow 0
\]

in the sense of weak convergence of probability measures on \( \mathbb{R}^{n-l} \), i.e. for all \( \phi \in C_b(\mathbb{R}^{n-l}) \),

\[
E[\phi(Z_t^*)|Y_t^* = y] \to \phi(z^*_t(y)) \quad \text{as } \varepsilon \downarrow 0.
\]

(2.12)

Corollary 2.11 (Test functions with polynomial growth). Under the assumption of Theorem 2.10 assume \( \phi \) is continuous and has polynomial growth, that is \( \phi(z) \leq C(1 + |z|^k) \) for some \( C > 0 \) and \( k \in \mathbb{N} \), for all \( z \). Then

\[
E[\phi(Z_t^*)|Y_t^* = y] \to \phi(z^*_t(y)) \quad \text{as } \varepsilon \downarrow 0.
\]

holds.

Theorem 2.10 and Corollary 2.11 are proven at the end of this section.

Remark 2.12 (Extension to finitely many argmin’s). If there exist finitely many global minimizer \( z^{*,i} = z^{*,i}_t(y) \), \( i = 1, \ldots, N \), for the problem (2.11), assuming that \( C_{x_0}(h) \) is invertible for some minimizing control \( h_0 \in K_{t}^{(y,z)} \) for every \( z \) in a neighborhood of each \( z^{*,i} \), a modification of the arguments used in the proof of Theorem 2.10 allows to show that \( \mathcal{L}(Z_t^*|Y_t^* = y) \) converges to a law supported by the \( z^{*,i} \), i.e.

\[
\mathcal{L}(Z_t^*|Y_t^* = y) \Rightarrow \sum_{i=1}^N \alpha_i \delta_{z^{*,i}} \quad \text{as } \varepsilon \downarrow 0,
\]

for some \( (\alpha_i)_{i=1}^N \) with \( \alpha_i \geq 0 \) and \( \sum_{i=1}^N \alpha_i = 1 \), which means \( E[\phi(Z_t^*)|Y_t^* = y] \to \sum_{i=1}^N \alpha_i \phi(z^{*,i}) \) as \( \varepsilon \downarrow 0 \), for every \( \phi \in C_b \).

Remark 2.13 (Extension to the finite dimensional law). Under the hypotheses of Theorem 2.10 assume in addition that there exist a unique minimizing control \( h_0 \) in \( K_{t}^{(y,z)} \), that is

\[
K_{t}^{(y,z)} := \left\{ h \in K_{t}^{(y,z)} \mid \frac{1}{2} \| h \|^2_H = \Lambda_t(N_y) \right\} = \{ h_0 \}
\]

(2.13)

(in particular, \( (y, z^*_t(y)) = \varphi^{h_0}_t(x_0) \) is the unique minimizer of \( \Lambda_t \) on the set \( N_y \)). Then, for every \( 0 \leq t_1 < \cdots < t_n \leq t \),

\[
\mathcal{L}(Z_{t_1}, \ldots, Z_{t_n}|Y_t^* = y) \Rightarrow \delta_{(z_{t_1}, \ldots, z_{t_n})}
\]

where \( (y_s, z_s) := \varphi^{h_0}_s(x_0) \), \( s \leq t \), is the trajectory associated to the control \( h_0 \). The case of finitely many minimizing controls \( h^*_0 \) in \( K_{t}^{(y,z)}_{\min} \) gives rise to a limiting law supported by the \( (z^*_t, \ldots, z^*_{t_n}) \), with \( (y^*_s, z^*_s) = \varphi^{h^*_0}_s(x_0) \). Subject to a tightness estimate, the convergence of the finite dimensional law yields the convergence at the path level, namely \( \mathcal{L}(Z_t^*|Y_t^* = y) \Rightarrow \delta_z \) in the case of a unique minimizing path \( (y, z) = \varphi^{h_0}_t(x_0) \). In the point-to-point case \( l = n \), this result is proved in Molchanov [32] for elliptic diffusions and in Bailleul [2] in the hypoelliptic setting (see also Bailleul, Mesnager and Norris [3]).

In order to prove Theorem 2.10 we need a preliminary estimate on the marginal density \( f^*_t \).

---

10The existence of a minimizer follows from the lower semi-continuity and compactness of the level sets of the map \( z \mapsto \Lambda_L(y, z) \).
Proposition 2.14. Under the hypotheses of Theorem 2.10,

\[ \liminf_{\varepsilon \to 0} \varepsilon^2 \log f^\varepsilon_t(y) \geq -\Lambda_t(N_y), \]  

(2.14)

where with the usual convention, \( \Lambda_t(E) = \inf_{x \in E} \Lambda_t(x) \). In particular, \( f^\varepsilon_t(y) > 0 \) for \( \varepsilon \) small enough.

Proof. For simplicity, let us drop the explicit dependence on the (fixed) \( t > 0 \), and write \( \Lambda \) for \( \Lambda_t \), \( p^\varepsilon \) for \( p^\varepsilon_t \), etc. Also write \( z^* \) for \( z^*_t(y) \). We note that by definition, \( \Lambda(N_y) = \inf_t \Lambda_t(y, z) = \Lambda_y(z^*) \). Using Ben Arous and Léandre’s support theorem [6, Theorem II.1], the invertibility of \( C_{x_0}(h) \) for some \( h \in \mathcal{K}_t^{(y, z^*)} \) implies \( p^\varepsilon(y, z^*) > 0 \); the conditional density \( g^\varepsilon \) is then well-defined as in (2.9).

Let \( K \) be a neighborhood of \( z^* \) in \( \mathbb{R}^{n-I} \) such that, for every \( z \in K \), \( C_{x_0}(h) \) is invertible for some minimizing control \( h_0 \in \mathcal{K}_I^{(y, z)} \) (with no loss of generality, we may assume \( K \) to be compact). It follows from Theorem 2.1(\( t \)) that

\[ \Lambda(y, z) = \Lambda_R(y, z) < \infty, \quad \forall z \in K. \]

(2.15)

From point (v) in Lemma 2.1, \( \Lambda \) is continuous on a neighborhood of \((y, z^*)\): possibly making \( K \) smaller, we can assume

\[ \Lambda(y, z) - \Lambda(y, z^*) \leq \delta, \quad \forall z \in K, \]

for some fixed \( \delta > 0 \). It follows from estimate (2.22) in Theorem 2.1 that there exists \( \varepsilon_0 = \varepsilon_0(\delta) \) such that

\[ p^\varepsilon(y, z^*) \leq \exp \left( \frac{-(\Lambda(y, z^*) + \delta)}{\varepsilon^2} \right) \]

for every \( \varepsilon < \varepsilon_0 \). Analogously, for every \( z \in K \) there exists \( \varepsilon(z) = \varepsilon(z, \delta) \) such that

\[ p^\varepsilon(y, z) \geq \exp \left( \frac{-(\Lambda(y, z) - \delta)}{\varepsilon^2} \right) \]

for all \( \varepsilon < \varepsilon(z) \). It follows from these last two estimates that for every \( z \in K \),

\[ \frac{p^\varepsilon(y, z)}{p^\varepsilon(y, z^*)} \geq \exp \left( \frac{-(\Lambda(y, z) - \Lambda(y, z^*)) - 2\delta}{\varepsilon^2} \right) \geq \exp \left( -\frac{3\delta}{\varepsilon^2} \right) \]

(2.16)

for all \( \varepsilon < \varepsilon_0 \wedge \varepsilon(z) \). Now write

\[ f^\varepsilon_t = \int_{\mathbb{R}^{n-I}} p^\varepsilon(y, z) dz \geq \int_K p^\varepsilon(y, z) dz \]

\[ = p^\varepsilon(y, z^*) \lambda^{n-I}(K) \int_{K_y} \frac{p^\varepsilon(y, z)}{p^\varepsilon(y, z^*)} \lambda^{n-I}(K) \]

(2.17)

where \( \lambda^{n-I} \) is the Lebesgue measure on \( \mathbb{R}^{n-I} \). First applying Jensen’s inequality, then Fatou’s lemma, and using (2.10), one has

\[ \liminf_{\varepsilon \to 0} \varepsilon^2 \log \int_K p^\varepsilon(y, z) \lambda^{n-I}(K) \]

\[ \geq \liminf_{\varepsilon \to 0} \varepsilon^2 \log p^\varepsilon(y, z) \lambda^{n-I}(K) \]

\[ \geq \liminf_{\varepsilon \to 0} \left[ \varepsilon^2 \frac{p^\varepsilon(y, z)}{p^\varepsilon(y, z^*)} \right] \lambda^{n-I}(K) \]

\[
\geq \int_K \liminf_{\varepsilon \to 0} \left[ \varepsilon^2 \frac{p^\varepsilon(y, z)}{p^\varepsilon(y, z^*)} \right] \lambda^{n-I}(K) \]

\[
\geq -3\delta. \]

The strict positivity of \( f^\varepsilon_t \) is not, in general, a consequence of Hörmander condition. According to Ben Arous and Léandre’s support theorem [6, Theorem II.1], the density \( p^\varepsilon_t(x) \) of the full process \( X^\varepsilon_t \) is strictly positive at \( x \) if and only if there exists some \( h \in \mathcal{K}_I^x \) such that \( C_{x_0}(h) \) is invertible. While Hörmander condition ensures that \( \mathcal{K}_t^x \) is non-empty for every \( x \), the lack of controls with invertible Malliavin matrix cannot be excluded a priori.
Finally using the lower bound \(2.13\) for \(p^\varepsilon(y, z^*)\) and \(2.15\), it follows from \(2.14\) that
\[
\liminf_{\varepsilon \to 0} \varepsilon^2 \log f^\varepsilon(y) \geq \liminf_{\varepsilon \to 0} \varepsilon^2 \log p^\varepsilon(y, z^*) + \liminf_{\varepsilon \to 0} \varepsilon^2 \log \left( \frac{\lambda^{n-l}(K)}{\int_K p^\varepsilon_t(y, z^*)} \right)
\]
\[
\geq -\Lambda(y, z^*) - 3\delta.
\]
Since \(\delta\) was arbitrary, \(2.14\) is proved.

Remark 2.15. Note that we have nowhere used, in the proof of Proposition 2.14 the fact that the minimizer \(z^*_t(y)\) is unique, neither that there exist finitely many. Proposition 2.14 then holds under the weaker assumption that \(C_{\varepsilon_0}(h_0)\) is invertible on at least one minimizing control \(h_0 \in K_t^{(y, z^*)}\), for every \(z\) in a neighborhood of some global minimizer \(z^*\) of the map \(z \mapsto \Lambda_t(y, z).

Proof of Theorem 2.10. Let us drop the fixed \(t\) from the notation, and write \(\Lambda\) for \(\Lambda_t\), etc.

Step 1. We want to show that for all \(\phi \in C_b\), one has \(\int_{R_n-1} \phi(z) g^\varepsilon(z) dz - \phi(z^*)| \to 0\) as \(\varepsilon \to 0\), with \(z^* := z^*_t(y)\). Consider \(\eta^{(2)} > 0\) such that the map \(z \mapsto \Lambda(y, z)\) is continuous on \(B_{\eta^{(2)}}(z^*)\) (by Lemma 2.7, such a \(\eta^{(2)}\) exists) and such that estimate \((2.5)\) of Theorem 2.1 holds on \(B_{\eta^{(2)}}(z^*)\). Let \(\delta > 0\), and consider \(\eta^{(1)} = \eta^{(1)}_\delta\) such that \(0 < \eta^{(1)} < \eta^{(2)}\) and \(osc\(\phi, \eta^{(1)}\) \leq \delta\). We have
\[
\left| \int_{R_n-1} \phi(z) g^\varepsilon(z) dz - \phi(z^*) \right| \leq \int_{R_n-1} |\phi(z) - \phi(z^*)| g^\varepsilon(z) dz
\]
\[
= \int_{|z-z^*| \leq \eta^{(1)}} |\phi(z) - \phi(z^*)| g^\varepsilon(z) dz + \int_{|z-z^*| > \eta^{(1)}} |\phi(z) - \phi(z^*)| g^\varepsilon(z) dz
\]
\[
\leq \delta + 2|\phi|_\infty \int_{|z-z^*| > \eta^{(1)}} g^\varepsilon(z) dz,
\]
and our aim is to show that the last integral converges to 0 as \(\varepsilon \to 0\).

We have
\[
\int_{|z-z^*| > \eta^{(1)}} g^\varepsilon(z) dz \leq \int_{\eta^{(1)} < |z-z^*| < \eta^{(2)}} g^\varepsilon(z) dz + \int_{|z| \geq \eta^{(2)}} g^\varepsilon(z) dz := I_1^\varepsilon + I_2^\varepsilon.
\]

Step 2 \((I_1^\varepsilon \to 0)\). Set
\[
a_\delta := \inf \{\Lambda(y, z) - \Lambda(y, z^*) : |z - z^*| \geq \eta^{(1)}\}.
\]
By the lower semi-continuity of \(z \mapsto \Lambda(y, z)\) and the uniqueness of the minimizer \(z^*,\) one has \(a_\delta > 0\). Let now \(\delta_1\) be such that \(0 < \delta_1 < a_\delta/4\); on the one hand, since estimate \((2.5)\) in Theorem 2.1 is uniform over compacts, we know there exists \(\varepsilon_0 = \varepsilon_0(\delta_1, \eta^{(2)})\) such that
\[
p^\varepsilon(y, z) \leq \exp \left( \frac{-\Lambda(y, z) + \delta_1}{\varepsilon^2} \right)
\]
for all \(z \in B_{\eta^{(2)}}(z^*)\) and \(\varepsilon < \varepsilon_0\). On the other hand, it follows from estimate \((2.13)\) in Proposition 2.14 that
\[
f^\varepsilon(y) \geq \exp \left( \frac{-\Lambda(y, z^*) - \delta_1}{\varepsilon^2} \right)
\]
for all \(\varepsilon < \varepsilon_0\). Putting these two estimates together and using the definition of \(a_\delta\), it follows
\[
g^\varepsilon(z) \leq \exp \left( \frac{2\delta_1 - (\Lambda(y, z) - \Lambda(y, z^*))}{\varepsilon^2} \right) \leq \exp \left( \frac{-a_\delta}{2\varepsilon^2} \right)
\]
for all \(z\) such that \(\eta^{(1)} < |z - z^*| < \eta^{(2)}\) and \(\varepsilon < \varepsilon_0\). Therefore, we have
\[
I_1^\varepsilon \leq \exp \left( \frac{-a_\delta}{2\varepsilon^2} \right) \lambda^{n-l}(B_{\eta^{(2)}}(z^*)),
\]
for all \( \varepsilon < \varepsilon_0 \), where \( \lambda^{n-l} \) is the Lebesgue measure on \( \mathbb{R}^{n-l} \). For every choice of \( \delta \) and \( \eta^{(2)} \), the right hand side can be made arbitrarily small taking \( \varepsilon \) small enough.

Step 3 (\( I_2^* \to 0 \)). As done in Step 2, notice that \( a^{(2)} := \inf \{ \Lambda(y, z) - \Lambda(y, z^*) : |z - z^*| \geq \eta^{(2)} \} > 0 \). Since

\[
I_2^* = \int_{|z - z^*| \geq \eta^{(2)}} g^*(z) dz = \frac{1}{f^*(y)} \int_{|z - z^*| \geq \eta^{(2)}} p^*(y, z) dz,
\]

it follows from (2.14) and Proposition 2.4 that

\[
\limsup_{\varepsilon \to 0} \varepsilon^2 \log I_2^* \leq -\liminf_{\varepsilon \to 0} \varepsilon^2 \log f^*(y) + \limsup_{\varepsilon \to 0} \varepsilon^2 \log \int_{|z - z^*| \geq \eta^{(2)}} p^*(y, z) dz
\]

\[
\leq \Lambda(y, z^*) - \inf \{ \Lambda(y, z) : |z - z^*| \geq \eta^{(2)} \}
\]

\[
\leq \Lambda(y, z^*) - \Lambda(y, z^*) - a^{(2)} = -a^{(2)} < 0.
\]

The last inequality clearly implies that \( I_2^* \) vanishes as \( \varepsilon \to 0 \). ■

Proof of Corollary 2.11. Let \( \phi, R > 0 \), be a bounded continuous function that coincides with \( \phi \) on \( B_R(0) \). Assume \( R \) is fixed, but large enough so that \( B_R(0) \) contains \( z^*_1(\varepsilon) =: z^* \) and the compact set \( \{ z : \Lambda(y, z) \leq \Lambda(y, z^*) + 1 \} \). We have (dropping the fixed index \( t \) from the notation),

\[
|\mathbb{E}[\phi(Z^\varepsilon)|Y^\varepsilon = y] - \phi(z^*)| = |\mathbb{E}[\phi(Z^\varepsilon)1_{|Z^\varepsilon| \leq R}|Y^\varepsilon = y] - \phi(z^*)| + \mathbb{E}[\phi(Z^\varepsilon)1_{|Z^\varepsilon| > R}|Y^\varepsilon = y] - \phi(z^*)| + C \int_{|z| \geq R} (1 + |z|^k) p^*(y, z) dz
\]

By Theorem 2.10, the first term tends to \( |\phi_R(z^*) - \phi(z^*)| = |\phi(z^*) - \phi(z^*)| = 0 \) as \( \varepsilon \to 0 \). The second term can be bounded as in Step 3 of the proof of Theorem 2.10 that is

\[
\limsup_{\varepsilon \to 0} \varepsilon^2 \log \int_{|z| \geq R} (1 + |z|^k) p^*(y, z) dz \leq -\liminf_{\varepsilon \to 0} \varepsilon^2 \log f^*(y) + \limsup_{\varepsilon \to 0} \varepsilon^2 \log \int_{|z| \geq R} (1 + |z|^k)p^*(y, z) dz
\]

\[
\leq \Lambda(y, z^*) - \inf \{ \Lambda(y, z) : |z| \geq R \}
\]

\[
\leq \Lambda(y, z^*) - \Lambda(y, z^*) - 1 = -1.
\]

where we have used Proposition 2.4 and estimate (2.14) in the second step, and the choice of \( R \) to conclude. The last inequality implies that the left hand side vanishes as \( \varepsilon \to 0 \). ■

Note that corollary 2.11 can be straightforwardly extended to the case of finitely many minimizers (described in Remark 2.12).

3 Varadhan’s formula for marginal densities

As a by-product of the estimates presented so far (Theorem 2.1, Proposition 2.4 and Proposition 2.14), it is possible to show that a Varadhan-type formula holds for the density of the projected diffusion \( Y^\varepsilon \). Proposition 2.14 provides the lower bound. In the following theorem, the case \( l = n \) recovers the classical Varadhan’s formula [36], or rather Léandre’s extension [29, 30] to the hypoelliptic setting.

Theorem 3.1. Consider \( X^\varepsilon = (Y^\varepsilon, Z^\varepsilon) \) the strong solution to

\[
dX^\varepsilon_t = b_z(X^\varepsilon_t) dt + \varepsilon \sum_{j=1}^d \sigma_j(X^\varepsilon_t) dW^j_t, \quad X^\varepsilon_0 = x^\varepsilon_0,
\]

with \( x^\varepsilon_0 \to x_0 \), and assume strong Hörmander condition (sH) at all points. Fix \( y \in \mathbb{R}^l \) and \( t > 0 \). Then
(i) if \( b_\varepsilon \to 0 \) as \( \varepsilon \downarrow 0 \) in the sense of (1.11), the density \( f^\varepsilon_t \) of \( Y^\varepsilon_t \) satisfies
\[
\lim_{\varepsilon \to 0} \varepsilon^2 \log f^\varepsilon_t(y) = -\Lambda_t(N_y),
\]
where \( \Lambda_t \) is defined by (1.8) with \( b_0 \equiv 0 \).

(ii) If \( b_\varepsilon \to b_0 \neq 0 \), assume that \( C_{x_0}(h_0) \) is invertible for at least a minimizing control \( h_0 \in K^{(y,z)}_t \), for every \( z \) in a neighborhood of \( z^* \), where \( z^* \) is some (not necessarily strict, nor unique) global minimizer of the map \( z \mapsto \Lambda_t(y, z) \). Then, estimate (3.1) holds.

Consider a stochastic financial model \( X = (Y, Z) \in \mathbb{R} \times \mathbb{R}^{n-1} \), where \( Y \) models the log-price of a financial asset, and \( Z \) its (possibly multi-dimensional) stochastic volatility. The small-noise asymptotic behavior of the logarithm of the density of \( Y \) translates into the leading order term of the implied volatility of European options (in the corresponding asymptotic regime: see for example Gatheral et al. [18] for an implementation of this approach in the small-maturity case, for one-dimensional models).

**Remark 3.2.** Point (i) of Theorem 3.1 covers the small-time case: setting \( b_\varepsilon = \varepsilon^2 b \) and \( x_0^\varepsilon = x_0 \), Brownian scaling yields \( X^\varepsilon_t \sim X_{\varepsilon^2 t} \), where \( X_t = (Y_t, Z_t) \) is the solution to (1.11), hence (3.1) is equivalent to the small-time Varadhan formula
\[
\lim_{t \to 0} t \log f_t(y) = -\Lambda_1(N_y)
\]
with \( f_t \) the pdf of \( Y_t \).

**Comparison with the marginal density expansions of Deuschel et al. [12].** A sufficient condition for (3.1) to hold is given as condition (ND) in Deuschel et al. [12], see their Definition 2.7. This condition appears as a generalized “not in cut-locus” condition from sub-Riemannian geometry, and actually allows to derive a full expansion for the marginal density \( f^\varepsilon_t(y) \) as \( \varepsilon \to 0 \), of the form
\[
f^\varepsilon_t(y) = \frac{1}{(2\pi \varepsilon)^{\frac{n}{2}}} e^{-\frac{\Lambda(N_y)}{\varepsilon^2}} e^{-\frac{\Lambda(y)}{\varepsilon}} (c_0 + o(\varepsilon)),
\]
expansion of which (3.1) captures only the leading-order exponential term \( e^{-\frac{\Lambda(N_y)}{\varepsilon^2}} \) (we refer to [12, Theorem 2.8]) for an account of the additional term \( \tilde{\Lambda}(y)/\varepsilon \). In addition to the first order condition of invertibility of the deterministic Malliavin matrix along the minimizing controls in \( K^{(y,-)}_t \), their condition (ND) requires to check also a second-order condition corresponding to the non-degeneracy of the minimizers (in the sense of the strict positivity of the Hessian of the map \( h \mapsto \frac{1}{2} |h|^2_{\tilde{H}} \)), interpreted as the geometric non-focality of \( x_0 \) for the arrival subspace \( N_y \). Theorem 3.1 above precisely tells that this second-order condition is not necessary in order to establish the asymptotic behaviour of the density on the log-scale as in (3.1).

**Proof of Theorem 3.1** (ii) Let us establish the upper bound
\[
\limsup_{\varepsilon \to 0} \varepsilon^2 \log f^\varepsilon_t(y) \leq -\Lambda_t(N_y).
\]
From the hypotheses in (ii) and Theorem 2.1 we know that there exists \( \eta > 0 \) such that estimate (2.5) holds uniformly on \( B_\eta(z^*) \). Write (omitting the fixed index \( t \))
\[
\int_{\mathbb{R}^{n-1}} p^\varepsilon(y, z) dz = \int_{B_\eta(z^*)} p^\varepsilon(y, z) dz + \int_{B_\eta(z^*)^C} p^\varepsilon(y, z) dz.
\]
It follows from estimate (2.5) that for every \( \delta > 0 \) we can find \( \varepsilon_0 = \varepsilon_0(\delta, \eta) \) such that
\[
p^\varepsilon(y, z) \leq \exp \left( -\frac{\Lambda(y, z) + \delta}{\varepsilon^2} \right)
\]
(3.4)
for all \( z \in B_\eta(z^*) \) and \( \varepsilon < \varepsilon_0 \). For such values of \( \varepsilon \), one has
\[
\int_{B_\eta(z^*)} p_\varepsilon(y, z) dz \leq e^{\frac{2\delta}{\varepsilon^2}} \int_{B_\eta(z^*)} e^{-\frac{\Lambda(y, z^*)}{\varepsilon^2}} dz \leq e^{\frac{2\delta}{\varepsilon^2}} e^{-\frac{\Lambda(y, z^*)}{\varepsilon^2}} \lambda^{
u - 1}(B_\eta(z^*))
\]
where the last inequality trivially follows from \( \Lambda(y, z) \geq \Lambda(y, z^*) \). Therefore,
\[
\limsup_{\varepsilon \to 0} \varepsilon^2 \log \int_{B_\eta(z^*)} p_\varepsilon(y, z) dz \leq -\Lambda(y, z^*) + \delta.
\]
Now, using Proposition 2.14 one has
\[
\limsup_{\varepsilon \to 0} \varepsilon^2 \log \int_{B_\eta(z^*)} p_\varepsilon(y, z) dz = -\inf\{\Lambda(y, z') : z' \in B_\eta(z^*)\} \leq -\Lambda(y, z^*).
\]
Therefore, taking \( \limsup_{\varepsilon \to 0} \varepsilon^2 \log f_\varepsilon(y) = \limsup_{\varepsilon \to 0} \varepsilon^2 \log \int_{\mathbb{R}^n} p_\varepsilon(y, z) dz \leq \max(-\Lambda(y, z^*) + \delta, -\Lambda(y, z^*)) = -\Lambda(y, z^*) + \delta.
\]
Since \( \delta \) was arbitrary, the right hand side can be improved to \( -\Lambda(y, z^*) = -\Lambda(N_y) \), as claimed.

The lower bound \( \liminf_{\varepsilon \to 0} \varepsilon^2 \log f_\varepsilon(y) \geq -\Lambda_1(N_y) \) was already obtained in Proposition 2.14 (see Remark 2.16 for the case where the minimizer \( z^* \) is not unique).

(i) Under strong Hörmander condition, together with \( b_0 = \lim_{\varepsilon \to 0} b_\varepsilon \equiv 0 \), the function \( \Lambda_1 \) is continuous on \( \mathbb{R}^n \) (see Remark 2.8), and the two functions \( \Lambda \) and \( \Lambda_{R_1} \) coincide everywhere (see Comment 2.3(ii)). On the one hand, thanks to the continuity of \( \Lambda_1 \), estimate 2.5 in Theorem 2.1 holds uniformly over \( x \) in compact sets of \( \mathbb{R}^n \) (and not only locally around \( (y, z^*) \)). In the proof above, the upper bounds \( \limsup_{\varepsilon \to 0} \varepsilon^2 \log f_\varepsilon(y) \leq -\Lambda_1(N_y) \) only relies on estimate 2.5 and Proposition 2.4, and can be proven exactly as done above. On the other hand, identity 2.15 in Proposition 2.14 holds, and the proof of Proposition 2.14 can be rerun with no modifications, leading to the lower bound \( \liminf_{\varepsilon \to 0} \varepsilon^2 \log f_\varepsilon(y) \geq -\Lambda_1(N_y) \), which proves (3.1). ■

4 Applications: asymptotics of local volatilities

In this section we focus on the stochastic volatility model
\[
\begin{align*}
\frac{dY_t}{\varepsilon} &= \frac{1}{2} Z^2_t dt + Z_t dB_t^1, \quad Y_0 = 0, \\
\frac{dZ_t}{\varepsilon} &= \beta(Z_t) dt + \alpha(Z_t) dB_t^2, \quad Z_0 \neq 0
\end{align*}
\]
where the process \( Y \) models the log-value of a financial asset, and \( Z \) its stochastic volatility. Setting \( S_t = S_0 e^{Y_t} \) leads to the familiar “Black-Scholes with stochastic volatility” dynamics \( dS_t = S_t Z_t dB_t^1 + \rho dB_t^2 \). Here \( B_t = (B_t^1, B_t^2) \) is a two-dimensional Brownian motion with correlated components, in short \( dB^1, dB^2 = \rho dt \) for some \( \rho \in (-1, 1) \). \( B \) can be obtained from a two-dimensional standard Brownian motion \( W \) by setting \( B = \sqrt{T} W \), where \( \sqrt{T} \) is a (any) choice of the square root\(^{12} \) of the correlation matrix \( \Gamma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \).

Let us note straight away that the diffusion vector fields in (4.1) read
\[
\sigma_1(z) = \begin{pmatrix} z \sqrt{T}_{11} \\ \alpha(z) \sqrt{T}_{21} \end{pmatrix}, \quad \sigma_2(z) = \begin{pmatrix} z \sqrt{T}_{12} \\ \alpha(z) \sqrt{T}_{22} \end{pmatrix}.
\]
While it is clear that the couple \( \sigma_1(z), \sigma_2(z) \) spans \( \mathbb{R}^2 \) at every \( z \neq 0 \) such that \( \alpha(z) \neq 0 \) (recall \( \sqrt{T} \) is invertible under the assumption \( \rho \neq \pm 1 \)), this condition degenerates on the set \( \{ z = 0 \} \). The model (4.1) then naturally fits into the non-elliptic framework.

**Example 4.1.** A relevant parametric choice of the drift term in (4.1) is given by the affine function \( \beta(z) = a + b z \), which has the typical mean-reverting form \( \beta(z) = |b|(a/|b| - z) \) when \( b < 0 \).

\(^{12}\)A typical choice in this setting is provided by the Cholesky decomposition \( \sqrt{T} = C \sqrt{T}C \), with \( \sqrt{T} = \begin{pmatrix} \rho & \sqrt{T - \rho^2} \\ 0 & \sqrt{T - \rho^2} \end{pmatrix} \).
4.1 Extension of main results to Stochastic Volatility models

Postponing for a moment the precise assumptions on the coefficients $\alpha, \beta$, let us first describe the different types of asymptotic problems that can arise in the applications. The following class of small-noise equations embeds both small-time and (in some cases) space-asymptotic problems:

\[
\begin{align*}
    dY_t^\varepsilon &= -\frac{1}{2} \varepsilon^0 (Z_t^\varepsilon)^2 dt + \varepsilon Z_t^\varepsilon dB_t^1, \quad Y_0^\varepsilon = 0, \\
    dZ_t^\varepsilon &= \beta_\varepsilon (Z_t^\varepsilon) dt + \varepsilon (Z_t^\varepsilon) dB_t^2, \quad Z_0^\varepsilon = z_0^\varepsilon.
\end{align*}
\]

Here, $\theta \geq 0$ is a parameter that depends on the asymptotic regime under consideration (we will have $\theta \in \{0, 2\}$ in our applications),

\[z_0^\varepsilon \to z_0 \quad \text{as} \quad \varepsilon \to 0\]

and, analogously to (4.11),

\[\beta_\varepsilon \to \beta_0 \quad \text{as} \quad \varepsilon \to 0\]

for some limiting function $\beta_0$, in the sense of uniform convergence on compact sets of $\mathbb{R}$ together with the derivatives of any order. We also assume that the sequence of norms $|\partial^k \beta_\varepsilon|_\infty$ is uniformly bounded in $\varepsilon$, for every $k \geq 0$. The associated limiting controlled system reads

\[
\begin{align*}
    d\varphi_t^{(1)} &= -\frac{1}{2} 1_{\varepsilon=0} (\varphi_t^{(2)})^2 dt + \varphi_t^{(2)} d(\sqrt{\Gamma} h_t^{(1)}), \quad \varphi_0^{(1)} = 0, \\
    d\varphi_t^{(2)} &= \beta_0 (\varphi_t^{(2)}) dt + \alpha (\varphi_t^{(2)}) d(\sqrt{\Gamma} h_t^{(2)}), \quad \varphi_0^{(2)} = z_0;
\end{align*}
\]

where $h = (h^1, h^2)$ is a two-dimensional control and $\Gamma$ the correlation matrix in (4.1). Let us denote

\[\Lambda_t^{SV} (y, z) = \inf \left\{ \frac{1}{2} \| h \|_H^2 : h \in \mathcal{K}_t^{(y,z)} \right\}, \quad (y, z) \in \mathbb{R}^2\]

the action of the system (4.3), where $\mathcal{K}_t^{(y,z)} = \{ h : (\varphi_0^{(1)}, \varphi_0^{(2)}) = (0, z_0), (\varphi_t^{(1)}, \varphi_t^{(2)}) = (y, z) \}$.

We assume that the coefficients $\beta_\varepsilon, \beta_0$ and $\alpha$ satisfy:

(SV) $\beta_\varepsilon, \beta_0, \alpha : \mathbb{R} \to \mathbb{R}$ are smooth and Lipschitz functions, with $\alpha(0) \neq 0$.

The application of Theorem 2.10 to the system (4.3) is a priori not justified, because of the lack of global boundedness for the coefficients of the SDE (and their derivatives). In this respect, let us note that, even if a boundedness assumption were in force for $\alpha$ and $\beta_\varepsilon$, the two-dimensional system (4.3) would still not have bounded coefficients (because of the terms $z$ and $-\frac{1}{2} z^2$ in the equation for the $Y$ component). Nevertheless, one can exploit the structure of the SDE, together with the Lipschitz condition in (SV) (rather mild in this setting), in order to extend our main result on the asymptotics of conditional expectations.

**Theorem 4.2 (Small noise asymptotics of local volatility: the general case).** Assume condition (SV) on the coefficients $\beta_\varepsilon, \beta_0$ and $\alpha$, and denote $(Y^\varepsilon, Z^\varepsilon)$ the unique strong solution to (4.3) with $\theta \geq 0$. Fix $t > 0$ and $y \in \mathbb{R}$, and assume there exist a unique $z^* = z^*_t(y)$ minimizing the action $\Lambda_t^{SV}$, defined in (4.6), on the set $N_y = \{ y, \cdot \}$, and that there are finitely many minimizing controls in $\mathcal{K}_t^{(y,z^*)}$. If (at least) one of the following conditions is satisfied:

1. $z_0 > 0$ and $\alpha(z_0) \neq 0$;
2. $y \neq 0$;

then, for all functions $\phi \in C(\mathbb{R}^{n-1})$ with polynomial growth,

\[
E[\phi(Z_t^\varepsilon)|Y_t^\varepsilon = y] \to \phi \left( z_t^*(y) \right) \quad \text{as} \quad \varepsilon \downarrow 0.
\]
In particular,
\[ \sigma_{loc}(t, y)^2 = \mathbb{E}[(Z_{x}^t)^2 | Y_t^x = y] \rightarrow (z_0^*(y))^2 \quad \text{as} \quad \varepsilon \downarrow 0. \] (4.8)

Finally, if there exist finitely many minimizers \( z^*_i \) for \( \Lambda^V_\varepsilon \) on \( N_y \) (each one associated with finitely many minimizing controls \( b_0 \in K_{\varepsilon}^{(y, z^*_i)} \)), and one of Conditions (i) and (ii) is satisfied, then the limits in (4.7) and (4.8) are replaced with \( \sum_{i=1}^{N} a_i \phi(z^*_i) \), respectively \( \sum_{i=1}^{N} a_i (z^*_i)^2 \), for some weights \( a_i \geq 0 \) such that \( \sum_{i=1}^{N} a_i = 1 \).

**Proof.** Let \( b_0^R, b_0^R, \) and \( \alpha^R, R > 0 \), be smooth and bounded extensions of \( b_0|_{B_R(0)}, b_0|_{B_R(0)} \), and \( \alpha|_{B_R(0)} \) respectively (see the Appendix A.1 for a precise definition of \( b_0^R, b_0^R \) and \( \alpha^R \)), and denote \( (Y_\varepsilon^x, R, Y_\varepsilon^x, R) \) the unique strong solution to (4.3) when \( b_0 \) and \( \alpha \) are replaced with \( b_0^R \) and \( \alpha^R \). Also denote \( \Lambda^R_\varepsilon \) the action function associated to the limiting deterministic system (4.5) with coefficients \( b_0^R, \alpha^R \). Set
\[ \tau^R_\varepsilon = \inf\{s \geq 0 : |Z_{x}^s| > R\}. \]

On the event \( \{\tau^R_\varepsilon > t\} \), by the pathwise uniqueness for the second equation in (4.3), \( (Z_{x}^s)^{\tau^R_\varepsilon} \) is indistinguishable from \( (Z_{x}^s)^{\tau^R_\varepsilon} \). In addition,
\[ Y_t^\varepsilon = -\frac{1}{2} \int_0^t (Z_s^\varepsilon)^2 ds + \varepsilon \int_0^t Z_s^\varepsilon dB_s^1 = -\frac{1}{2} \int_0^t (Z_s^\varepsilon, R)^2 ds + \varepsilon \int_0^t Z_s^\varepsilon, R dB_s^1 = Y_t^{\varepsilon, R}, \quad \text{a.s.} \]

Note that the assumption \( \alpha(0) \neq 0 \) guarantees that strong Hörlmander condition is satisfied at all points for the SDE (4.3): indeed, some simple calculations show that
\[ [\sigma_1, \sigma_2](z) = -\det(\sqrt{\Gamma}) \begin{pmatrix} \alpha(z) \\ 0 \end{pmatrix}, \]

therefore the vectors
\[ \sigma_1(0) = \begin{pmatrix} 0 \\ \alpha(0)\sqrt{\Gamma_{21}} \end{pmatrix}; \quad \sigma_2(0) = \begin{pmatrix} 0 \\ \alpha(0)\sqrt{\Gamma_{22}} \end{pmatrix}; \quad [\sigma_1, \sigma_2](0) = -\det(\sqrt{\Gamma}) \begin{pmatrix} \alpha(0) \\ 0 \end{pmatrix} \]
span the full \( \mathbb{R}^2 \), under the assumption \( \det(\Gamma) = 1 - \rho^2 \neq 0 \).

Denoting \( |\phi|_{\infty, R} := \sup_{|z| \leq R} |\phi(z)| \), one has
\[ |E[\phi(Z_t^\varepsilon)|Y_t^\varepsilon = y] - \phi(z^*)| \leq E[|\phi(Z_t^\varepsilon) - \phi(z^*)|1_{\tau_R^\varepsilon > t}|Y_t^\varepsilon = y] \]
\[ + E[|\phi(Z_t^\varepsilon) - \phi(z^*)|1_{\tau_R^\varepsilon \leq t, |z| \leq R}|Y_t^\varepsilon = y] \]
\[ + E[|\phi(Z_t^\varepsilon, R) - \phi(z^*)|1_{\tau_R^\varepsilon \leq t, |z| > R}|Y_t^\varepsilon = y] \]
\[ \leq E[|\phi(Z_t^\varepsilon, R) - \phi(z^*)|1_{\tau_R^\varepsilon \leq t}|Y_t^{\varepsilon, R} = y] + 2|\phi|_{\infty, R} \mathbb{P}(\tau_R^\varepsilon \leq t|Y_t^\varepsilon = y) \]
\[ + C \int_{|z| > R} (1 + |z|^k) \frac{p^y(y, z)}{f^y(y)} dz \]
\[ \leq E[|\phi(Z_t^\varepsilon, R) - \phi(z^*)|^21_{Y_t^{\varepsilon, R} = y}]^{1/2} + 2|\phi|_{\infty, R} \mathbb{P} \left( \sup_{s \leq t} |Z_s^R| \geq R|Y_t^\varepsilon = y \right) \]
\[ + C \int_{|z| > R} (1 + |z|^k) \frac{p^y(y, z)}{f^y(y)} dz \]
\[ =: \epsilon_1(\varepsilon, R) + \epsilon_2(\varepsilon, R) + \epsilon_3(\varepsilon, R). \]

where we have used Hölder’s inequality in the last step. We will now show that taking \( R \) large enough, but fixed, the right hand side tends to zero as \( \varepsilon \to 0 \).

\( \epsilon_1 \): Lemma A.1 in Appendix A.1 establishes that \( R \) can be chosen such that \( z^* \) is also the unique minimum point of the function \( z \mapsto \Lambda^R(y, z) \). Assume that Condition (i) is satisfied. The vector fields \( \sigma_1(z), \sigma_2(z) \) defined in (4.2) span the whole \( \mathbb{R}^2 \) at the starting point \( z_0 \); this is enough to establish, see Lemma A.4, that the covariance matrix \( C_{(0, z_0)}(h) \) is invertible for all \( h \). On the other hand, assume \( z_0 = 0 \) and that Condition (ii) is satisfied.
From the continuity of $\alpha$ and the condition $\alpha(0) \neq 0$ in (SV), $\alpha$ is bounded away from zero in a neighbourhood $V$ of $z_0 = 0$. A simple inspection of the limiting (truncated) controlled system

$$
d\varphi_t^{(1,R)} = -\frac{1}{2}(\varphi_t^{(2,R)})^2 dt + \varphi_t^{(2,R)} d(\sqrt{\Gamma}h)^1_t, \quad \varphi_0^{(1,R)} = 0
$$

$$
d\varphi_t^{(2,R)} = \beta_0^R(\varphi_t^{(2,R)}) dt + \alpha^R(\varphi_t^{(2,R)}) d(\sqrt{\Gamma}h)^2_t, \quad \varphi_0^{(2,R)} = z_0
$$

shows that $\varphi_s^{(2,R)} = 0$ for all $s \in [0,t]$ implies $y = \varphi_t^{(1,R)} = 0$. Therefore, the second coordinate of the controlled path $z_s = \varphi_s^{(2,R)}$ must cross the set $V$, contained in the elliptic region, in order to have $(y_s, z_s) \in N_y$ with $y \neq 0$. Lemma 14 then allows to conclude that $C_{(0,z_0)}(h)$ is invertible for every $h \in K_t(y,\cdot)$. In both cases, the hypotheses of Theorem 2.10 are satisfied, and this theorem yields $\epsilon(v) \rightarrow |\phi(z^*) - \phi(z^*)| = 0$ as $v \rightarrow 0$.

$\epsilon_2$: Any optimal control $h_0 \in K_t(y,z^*)$ satisfies $\frac{1}{2}|h_0|^2_H = \Lambda_t(y,z^*) = \Lambda_0(N_y)$. Estimate 2.8 in Lemma 2.7 then implies

$$
\sup_{s \leq t} |\varphi^{(2)}_s(h_0)| \leq C e^{\frac{C}{h_0}|h_0| H} = C e^{C \sqrt{2\Lambda_0(N_y)}} := \tilde{R}
$$

where $C = C(t, z_0, K)$ is the constant defined in Lemma 2.7 and $K$ is a constant Lipschitz constant for $\beta_0$ and $\alpha$. It follows from Theorem 2.10 and the subsequent Remark 2.13 that the law of $(Z^*_s, s \leq t)$ conditional on $Y^*_t = y$ converges weakly to a law supported by the finitely many paths $\{\varphi^{(2)}_s(h_0) : h_0 \in K_{min}^{(y,z^*)}\}$. Taking $R > \tilde{R} + 1$, it is clear that

$$
P \left( \sup_{s \leq t} |Z^*_s| \geq R |Y^*_t = y \right) \rightarrow 0, \quad \text{as } v \rightarrow 0,
$$

therefore $\epsilon_2(v, R) \rightarrow 0$ as well.

$\epsilon_3$: The integral term in $\epsilon_3(v, R)$ also appears in the proof of Corollary 2.14 and can be bounded exactly as done there.

The proof in the case of finitely many minimizers $z^*_1$ goes through the same steps, using in 4.10 the fact that $\{z^*_1\}$ is also the set of global minimizers of $z \mapsto \Lambda_t^S(y, z)$ when $R$ is large enough (see Lemma A4.1 in the Appendix).

### 4.2 Small-time behavior and Berestycki, Busca and Florent [7] asymptotics of efficient volatility revisited

The short time behavior of the local volatility function obtained from the projection of stochastic volatility was addressed by Berestycki, Busca and Florent [7, section 5], who use local volatility as an intermediate step in the computation of the implied volatility of European options. Using Theorem 4.2, one can give a generalization of their result, formulated for stochastic volatility models with bounded and uniformly elliptic coefficients, to hypoelliptic models with unbounded coefficients.

**Theorem 4.3.** Assume that the coefficients $\beta, \alpha$ in [4.1] are smooth and Lipschitz with $\alpha(0) \neq 0$ and $\alpha(Z_0) \neq 0$, and consider the unique strong solution $(Y, Z)$ to [4.1], with $Z_0 \neq 0$. Let $\Lambda_t^{SV}(y, z)$ be the action of the system [1.0] when $b_0 \equiv 0, \theta = 2$ and $z_0 = Z_0$, i.e.

$$
d\varphi_t^{(1)} = \varphi_t^{(2)} d(\sqrt{\Gamma}h)^1_t, \quad \varphi_0^{(1)} = 0
$$

$$
d\varphi_t^{(2)} = \alpha(\varphi_t^{(2)}) d(\sqrt{\Gamma}h)^2_t, \quad \varphi_0^{(2)} = Z_0
$$

$$
\Lambda_t^{SV}(y, z) = \inf \left\{ \frac{1}{2} |h|^2_H : (\varphi_t^{(1)}, \varphi_t^{(2)}) = (0, Z_0); (\varphi_t^{(1)}, \varphi_t^{(2)}) = (y, z) \right\}
$$

Fix $y \in \mathbb{R}$ and assume there exists a unique $z^* = z^*(y)$ minimizing the function $\Lambda_t^{SV}$ on the set $N_y = (y, \cdot)$, and that the set of minimizing controls in $K_t(y,z^*)$ is finite. Then

$$
\sigma_{i\infty}^2(t, y) = E[(Z_t)^2 | Y_t = y] \rightarrow z^*(y)^2 \quad \text{as } t \rightarrow 0.
$$

(4.10)
In the presence of finitely many minimizers \( z_i^*(y), i = 1, \ldots, N \) (each one associated with finitely many minimizing controls in \( \mathcal{K}_i(y, z_i^*) \)), convergence holds towards a convex combination of the \( z_i^*(y)^2 \):

\[
\sigma^2_{\text{loc}}(t, y) \to \sum_{i=1}^{N} a_i z_i^*(y)^2 \quad \text{as } t \to 0. \tag{4.11}
\]

**Proof.** For every \( \varepsilon > 0 \), the process \((Y_t^\varepsilon, Z_t^\varepsilon)_{t \geq 0} := (Y_t^{\varepsilon z_0}, Z_t^{\varepsilon z_0})_{t \geq 0} \) has the same law as the solution of the SDE with \( \varepsilon = \varepsilon^2 \beta(x); \quad \theta = 2; \quad z_0^* = Z_0, \quad \forall \varepsilon > 0. \)

The functions \( \alpha, \beta_z \) and \( \beta_0 = \lim_{\varepsilon \to 0} \beta_z \equiv 0 \) clearly satisfy assumption (SV). Therefore, the hypotheses of Theorem 4.2 together with Condition \((i)\), are satisfied, and Theorem 4.3 follows. 

**Comments on heat kernel expansions and the Laplace method.** Starting from a small-time heat kernel expansion

\[
p_t(x) = \frac{1}{2\pi t} e^{-\frac{\Lambda(x)}{2t}} (c_0(x) + O(t; x)) \quad \text{as } t \downarrow 0, \tag{4.12}
\]

and assuming that for a fixed \( y \), the map \( z \mapsto \Lambda(y, z) \) has a unique minimizer \( z_y^* \) such that \( \partial_{zz} \Lambda(y, z_y^*) > 0 \), an heuristic application of the Laplace method yields:

\[
\sigma_{\text{loc}}(t, y)^2 = \mathbb{E} \left[ Z_t^2 \mid Y_t = y \right] = \frac{\int z^2 p_t(y, z)dz}{\int p_t(y, z)dz} = \frac{\int z^2 e^{-\frac{\Lambda(y, z)}{2t}} (c_0(y, z) + O(t; x))dz}{\int e^{-\frac{\Lambda(y, z)}{2t}} (c_0(y, z) + O(t; x))dz} \tag{4.13}
\]

\[
\approx \frac{\left( z_y^* \right)^2 e^{-\frac{\Lambda(y, z_y^*)}{2}} c_0(y, z_y^*)}{\sqrt{2\pi t} \left( \partial_{zz} \Lambda(y, z_y^*) \right)^{-1/2}}, \quad \text{as } t \downarrow 0
\]

in agreement with (4.10). Of course, here we have plugged the expansion (4.12), which is know to hold uniformly on compact sets (that do not intersect the cut-locus, see [5]), and integrated on the whole space, neglecting the residual tail contributions to the integrals in (4.13). The condition \( \partial_{zz} \Lambda(y, z_y^*) > 0 \), typical from Laplace asymptotics, is the finite-dimensional analogue of the second-order ‘non-focality’ condition in Deuschel et al. [12], Definition 2.7. As pointed out after Remark 2.2, we do not rely here on such a non-degeneracy assumption.

Moreover, the message of Theorems 4.2 and 4.3 is that the asymptotic behaviour of the logarithm of the density is enough to establish the leading order term of the local volatility function. On the other hand, when a full heat kernel expansion is available as in (4.12), the Laplace method allows to provide higher-order terms in (4.13); this approach is followed by Henry-Labordère [23] Chap. 6], relying on an ellipticity assumption.

### 4.3 Asymptotic slopes of local volatility in the Stein-Stein model

In the Stein–Stein model [35] (see Schöbel and Zhu [34] for the correlated case \( \rho \neq 0 \)), the volatility process follows an Ornstein-Uhlenbeck process:

\[
dY_t = -\frac{1}{2} Z_t^2 dt + Z_t dB^1_t, \quad Y_0 = 0, \\
dZ_t = (a + bZ_t) dt + cdB^2_t, \quad Z_0 = z_0 > 0, \tag{4.14}
\]

with \( a, b \in \mathbb{R}, c > 0 \) and \( d(B^1, B^2)_t = \rho dt \) with \( \rho \in (-1, 1) \). The typical mean-reverting form of the drift coefficient is obtained when \( a \geq 0 \) and \( b < 0 \). In the following, we consider \( b < 0 \) and \( \rho \leq 0 \) (the typical configuration in Equity markets) in order to streamline the computations, but this restriction is not essential.
Setting $Y_\varepsilon := \varepsilon^2 Y_t$, $Z_\varepsilon := \varepsilon Z_t$, the rescaled variables are seen to satisfy the small-noise problem

\[
\begin{align*}
    dY_\varepsilon^\varepsilon &= -\frac{1}{2}(Z_\varepsilon^\varepsilon)^2 dt + \varepsilon Z_\varepsilon^\varepsilon dW_t, \\
    dZ_\varepsilon^\varepsilon &= (a\varepsilon + bZ_\varepsilon^\varepsilon) dt + \varepsilon c dW_t, \\
    Y_0^\varepsilon &= 0,
\end{align*}
\]  

(4.15)

which belongs to the general class [13] with $\theta = 0$, $\beta_\varepsilon(z) = az + bz$, $\alpha(z) = c$. Note that $z_0^\varepsilon := \varepsilon z_0 \to 0$ as $\varepsilon \to 0$, that is, we are in a situation where the limiting starting point $x_0 = (y_0, z_0) = (0, 0)$ belongs to the sub-elliptic set $\{z = 0\}$.

The Hamiltonian system associated to the Stein–Stein model was solved in [13]. For every $y \neq 0$, the solution of the ODEs (2.6) subject to the boundary conditions

\[
x_0 = (0, 0); \quad x_t = (y, \cdot), \quad y \neq 0
\]

\[
p_t = (\cdot, 0)
\]

reveal the existence of two minimizing controls $h_0^\pm \in K_t^{\{y,\cdot\}}$, yielding the two (symmetric) arrival points $\varphi_{t}\pm = x_t^\pm = (y, z_t^\pm(y)) \in N_y$. Full details about explicit computations are found in [13] Section 5.2; the two minimizers $z_t^\pm(y)$ are given by

\[
z_t^\pm(y) = \pm \frac{q(y)c^2t}{r_1} \sin(r_1)
\]  

(4.16)

where

\[
q(y) = \frac{2}{c} \left[ \frac{2r_1y}{t^3 \left( (c^2(2p-1) - 2pcb)(2r_1 - \sin(2r_1)) + 2pcr_1(1 - \cos(2r_1))/t \right) } \right]^{1/2}
\]

(4.17)

with $\tilde{b} = b + \rho cp$ and $p = p(\tau, y)$, where

\[
p(r, y) = \frac{1}{2(1 - \rho^2)} \left[ \left( 1 + 2\rho \frac{b}{c} \right) + \text{sign}(y) \sqrt{ \left( 1 + 2\rho \frac{b}{c} \right)^2 + 4(1 - \rho^2) \left( \frac{b^2}{c^2} + \frac{r^2}{c^2t^2} \right) } \right],
\]

(4.18)

and $\tau = \tau(y)$ is the smallest strictly positive solution to

\[
r \cos r - t(b + \rho cp(r, y)) \sin r = 0.
\]

The equation (4.19) appears when imposing the transversality condition $p_t = (\cdot, 0)$ from (2.7).

**Remark 4.4.** It is not difficult to show that $\tau$ is the unique solution of equation (4.19) in a bounded interval $I$, which is independent of the model parameters. In practice, $\tau$ can be found using some simple root-finding procedure (such as bisection or Newton method).

Applying Theorem 4.2 and the scaling leading to (4.15), we are able to show that the local variance $\sigma_{loc}^2(t, y)$ in the Stein-Stein model is asymptotically linear for large values of $|y|$, in a similar spirit to Lee’s moment formula [31] for the implied volatility (see also the subsequent refinements in [20]).

**Theorem 4.5 (Local volatility ‘wings’ in the Stein-Stein model).** Denote $z_t(y)$ the common absolute value of $z_t^\pm(y)$ in (4.16). The local volatility in the correlated Stein–Stein model (4.14) satisfies, for any $t > 0$,

\[
\lim_{y \to \pm \infty} \frac{\sigma_{loc}^2(t, y)}{|y|} = \lim_{y \to \pm \infty} \frac{1}{|y|} E[(Z_t)^2 | Y_t = y] = (z_t(\pm 1))^2 = \left( \frac{q^2 t}{\tau} \sin(\tau) \right)^2,
\]

(4.20)

with $q = q(\pm 1)$ and $\tau = \tau(\pm 1)$ according to the sign of $y$ in (4.19), where $\tau(y)$ is the smallest strictly positive solution of equation (4.19) and the function $q$ is given in (4.14) above.

Note that the value of the limit in (4.20) does not depend on the initial volatility $z_0$, nor on the parameter $a$.
Comment 4.6. The asymptotic formula (4.20) can be used to patch the numerical evaluation of the local volatility from Dupire’s formula [15], typically affected by numerical instabilities in the region |y| >> 1. The use of (4.20), together with the evaluation of $\sigma_{loc}(t,y)$ in a region where numerics can be trusted (say a fixed, or adaptive, bounded domain in the (t, y)-plane) leads to a robust and globally defined local volatility surface, that can then serve as the basis for a Monte-Carlo evaluation of exotic option prices, with important consequences for model risk management. An analogous result for the asymptotic slopes of the local variance in the Heston model [23] was given in [11] (where the result (4.20) for the Stein-Stein model was announced), basing on previous work [24]. Note that the analysis in [11] is based on an implementation of the saddle-point method, and is hence justified. The functions $\beta_{z}(z) = az + bz \to bz =: \beta_{z}(z)$ and $\alpha(z) = cz$ clearly satisfy assumption (SV). Since the starting point is $(y_0, z_0) = (0, 0)$ and the arrival subspaces are $N_{\pm 1} = (\pm 1, \cdot)$, we are in case (ii) of Theorem 4.2 and the claim follows.

Proof of Theorem 4.3. Setting $\varepsilon^2 = 1/|y|$ and using the change of variable $Y_{t}^{\varepsilon} = \varepsilon^2 Y_{t}$, $Z_{t}^{\varepsilon} = \varepsilon Z_{t}$ that leads to (4.15), we have the identity

$$\lim_{y \to \pm \infty} \frac{\sigma^2_{loc}(t,y)}{|y|} = \lim_{y \to \pm \infty} \frac{1}{|y|} \mathbb{E}[Z_t^2 | Y_t = y] = \lim_{\varepsilon \to 0} \mathbb{E}[(Z_{t}^{\varepsilon})^2 | Y_{t}^{\varepsilon} = \pm 1].$$

The last limit above exists, and is equal to the right hand side of (4.20), if the application of Theorem 4.2 is justified. The functions $\beta_{z}(z) = az + bz \to bz =: \beta_{z}(z)$ and $\alpha(z) = cz$ clearly satisfy assumption (SV). Since the starting point is $(y_0, z_0) = (0, 0)$ and the arrival subspaces are $N_{\pm 1} = (\pm 1, \cdot)$, we are in case (ii) of Theorem 4.2 and the claim follows.

4.3.1 Consistency with the Heston model and moment explosion

Some basic Itô calculus shows that when $a = 0$, the Stein–Stein model (4.12) can be obtained as an instance of the Heston model. More precisely, consider a Heston model for the couple log-price/instantaneous variance $(Y_{t}^{H}, V_{t})$:

$$dY_{t}^{H} = -\frac{1}{2} V_{t} dt + \sqrt{V_{t}} dB_{1}^{H}, \quad Y_{0}^{H} = 0,$$

$$dV_{t} = (q + \kappa V_{t}) dt + \xi \sqrt{V_{t}} dB_{2}^{H}, \quad V_{0} = v_{0},$$

where $\kappa < 0; \xi, v_{0} > 0$ and $B_{1}^{H}, B_{2}^{H}$ are two Brownian motions with correlation $\rho$. When the parameters of the Heston model are given by

$$q = c^2, \quad \kappa = 2b, \quad \xi = 2c, \quad v_{0} = z_{0}^2,$$

then the couple $(Y_{t}, Z_{t}^{2})$ has same law as $(Y_{t}^{H}, V_{t})$, for every $t > 0$ (the identity in law actually holds for the entire processes; see [21 Section 2.4] for details). It follows that $\sigma_{loc}(t,y)^2 = \mathbb{E}[Z_{t}^2 | Y_{t} = y] = \mathbb{E}[V_{t} | Y_{t}^{H} = y] =: \sigma_{loc}^{Heston}(t,y)^2$ under the particular parameter configuration (4.22).

As pointed out in Comment 4.6 above, the local variance is known to be asymptotically linear also in the Heston model (with general parameters): from [11] Theorem 1:

$$\lim_{y \to \pm \infty} \frac{\sigma_{loc}^{Heston}(t,y)^2}{|y|} = \frac{2R_2(s_{\pm})}{s_{\pm}(s_{\pm} - 1)R_1(s_{\pm})},$$

where $s_{+} = s_{+}(t) := \sup\{s > 0 : \mathbb{E}[e^{sY_{t}^{H}}] < \infty\}$ and $s_{-} = s_{-}(t) := \inf\{s < 0 : \mathbb{E}[e^{sY_{t}^{H}}] < \infty\}$ are the upper resp. the lower critical exponents of $Y_{t}$, and

$$R_1(s) = Tc^2 s(s - 1) \left[ c^2(2s - 1) - 2pc(spc + b) \right] - 2(spc + b) \left[ c^2(2s - 1) - 2pc(spc + b) \right] + 4pc \left[ c^2 s(s - 1) - (spc + b)^2 \right];$$

$$R_2(s) = 2c^2 s(s - 1) \left[ c^2 s(s - 1) - (spc + b)^2 \right].$$
Comparing equations (4.19) and (4.28), it follows from Lemma 4.7 that for every proof.

For every solution of the equation $\kappa < 0$, the inverse of the map $s \mapsto \rho \xi s$ with the set of (infinitely many) solutions to the equation $R \tan R = \frac{t}{\kappa + \rho \xi s}$ is the smallest positive solution of the equation $R \cos R - \frac{t}{2} (\kappa + \rho \xi s) \sin R = 0$. (4.28)

**Proof.** For every $k \in \mathbb{N}^+$, the equation

$$\arctan \left( \frac{2R}{t(\kappa + \rho \xi s(R))} \right) + k \pi = R$$

(4.29)

has a unique root $R_k > 0$. Applying the tangent function to both sides, it is seen that \{ $R_k : k \in \mathbb{N}^+$ \} coincides with the set of (infinitely many) solutions to the equation $\frac{2R}{t(\kappa + \rho \xi s(R))} = \tan(R)$, which is equation (4.28). Using $\kappa < 0$, $\rho \leq 0$ and $s(R) \geq 0$, it is easy to see that the smallest positive solution to (4.28) is contained in the interval $(\pi/2, \pi)$. On the other hand, using $\arctan \in (-\pi/2, \pi/2)$, it is clear that $R_k \in (\pi/2, \pi)$ while $R_k \notin (\pi/2, \pi)$ for $k > 1$; it then follows that the smallest positive root of (4.28) coincides with the unique root $R_1$ of (4.29) with $k = 1$. Setting $s_1 = s(R_1)$, i.e. $R_1 = \frac{t}{2} \sqrt{-\Delta(s_1)}$, it then holds that $s_1$ is the unique positive solution to

$$\arctan \left( \frac{\sqrt{-\Delta(s)}}{t(\kappa + \rho \xi s)} \right) + \pi = \frac{t}{2} \sqrt{-\Delta(s)}$$

which is equation (4.28), therefore $s_1 = s_+$. Conversely, if $s_+$ denotes the unique root of (4.28), then $R(s_+) = R_1$, and the claim is proved. \qed

Now consider the particular Heston parameterization in (4.22). Plugging $\kappa = 2b$ and $\xi = 2c$ into (4.27) shows that the function $r \mapsto s(r)$ in Lemma 4.7 coincides with the function $r \mapsto p(2r, 1)$, with $p$ defined in (4.18). Then comparing equations (4.19) and (4.28), it follows from Lemma 4.7 that

$$2R(s_+) = t \sqrt{-\Delta(s_+)} = t \sqrt{e^{2s_+(s_+ - 1)} - (b + \rho \xi s_+)^2} = \pi(1),$$

(4.30)

\footnote{In its turn, the negative exponent $s_-(t)$ is the unique solution of equation (4.20) on $(-\infty, s_1)$, where $s_1$ is the negative root of $-\Delta(s)$.

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The two constants obtained by different methods are the same. Denote
\[ A^2_{\text{Hest}} := \frac{2R_2(s_+)}{s_+(s_+ - 1)\bar{\sigma}(s_+)}; \quad A^2_{\text{StSt}} := \left( \frac{qc^2t}{\tau} \sin(\tau) \right)^2 \]
the two asymptotic local variance slopes as defined resp. in (4.30) and (4.32), where we denote \( \tau = \bar{\tau}(1) \) and \( q = q(1) \). Plugging \( \kappa = 2b \) and \( \xi = 2c \) into \( A^2_{\text{Hest}} \), after some straightforward simplifications one obtains
\[ A^2_{\text{Hest}} = \frac{4[c^2s_+(s_+ - 1) - \bar{b}^2]}{Ts_+(s_+ - 1)[c^2(2s_+ - 1) - 2\rho c]\bar{b} - (2s_+ - 1)b + 2\rho cs_+(s_+ - 1)} \]
with \( \bar{b} = b + \rho c s_+ \). Substituting \( p = p(\tau, 1) = s_+ \) inside the expression for \( q \) in (4.17), \( A^2_{\text{StSt}} \) reads
\[ A^2_{\text{StSt}} = 4c^2\tau^2 \frac{\sin(\tau)^2}{t(c^2(2s_+ - 1) - 2\rho c\bar{b})(\tau^2 - \tau \sin(\tau) \cos(\tau)) + 2\rho c^2 \sin(\tau)^2} \]
\[ = 4c^2\tau^2 \left[ (tc^2(2s_+ - 1) - 2\rho \bar{b}) \frac{\tau^2 - \tau \sin(\tau) \cos(\tau)}{\sin(\tau)^2} + 2\rho c^2 \right]^{-1} \]
Repeatedly applying (4.19), one has
\[ \frac{\tau^2 - \tau \sin(\tau) \cos(\tau)}{\sin(\tau)^2} = \left( \frac{\tau}{\sin(\tau)} \right)^2 - \tau \frac{\cos(\tau)}{\sin(\tau)} = \left( \frac{tb}{\cos(\tau)} \right)^2 - tb \]
\[ = (tb)^2 \left( 1 + \frac{\tau^2}{(tb)^2} \right) - tb = (tb)^2 + \tau^2 - tb \]
where we have used the identity (4.30) in the last step. Using (4.32) and again \( \tau^2 = t^2c^2s_+(s_+ - 1) - t^2\bar{b}^2 \) from (4.30), after some straightforward simplifications it follows from (4.31) that
\[ A^2_{\text{StSt}} = \frac{4c^2\tau^2}{t^2c^2 \left[ Ts_+(s_+ - 1)(c^2(2s_+ - 1) - 2\rho \bar{b}) - (2s_+ - 1)b + 2\rho cs_+(s_+ - 1) \right]} \]
\[ = \frac{4[c^2s_+(s_+ - 1) - \bar{b}^2]}{Ts_+(s_+ - 1)(c^2(2s_+ - 1) - 2\rho \bar{b}) - (2s_+ - 1)b + 2\rho cs_+(s_+ - 1)}, \]
and the proof that \( A^2_{\text{StSt}} = A^2_{\text{Hest}} \) is complete.

We insist that our proof of consistency of the two local variance slopes \( A^2_{\text{Hest}} \) and \( A^2_{\text{StSt}} \) is valid for negative, non-zero correlation. In the context of implied volatility expansions, a similar consistency result was obtained by [21] p. 187-192, but only in the zero-correlation case.

4.3.2 Numerical tests

In the Heston model, the local volatility \( \sigma^2_{\text{Hest}}(t, y) \) can be evaluated using the classical inversion of characteristic functions within the computation of Call price derivatives in the Dupire’s formula \( \sigma^2_{\text{loc}}(t, y)^2 = \frac{\partial^2 C(t, K)}{\partial y^2} \bigg|_{K=S_{t,e}^y} \). This gives a way of computing the local volatility in the Stein–Stein model with \( a = 0 \), simply coinciding with a Heston local volatility, when the Heston parameters are given by (4.22). In Figures 4 and 2 we invert the Heston characteristic functions after a shift of the integration contour in the complex plane into an appropriate saddle-point, following the procedure described in [11], in order to obtain a stable implementation of the local volatility function for large values of \( |y| \). The two figures illustrate the convergence result in Theorem 4.5 for the two regions \( y < 0 \) and \( y > 0 \): the blue line represents the ratio \( y \mapsto \frac{\sigma^2_{\text{loc}}(y, t)}{y \times \mathbf{A}(\pi/2, 1)^2} \).
Figure 1: Illustration of the convergence result in Theorem 4.5 for the Stein–Stein model in the case $y < 0$ (left ‘wing’ of the local volatility). The blue line shows the value of the function $y \mapsto \sigma^2_{\text{loc}}(y, t)/(y \times z_t(-1)^2)$, with $z_t(-1)^2$ the theoretical asymptotic value in (4.20), which must converge to 1 as $y \to -\infty$. Model parameters: $a = 0, b = -0.5, c = 0.4, z_0 = 0.244, \rho = -0.75$.

which must tend to 1 as $y \to \pm\infty$. The empirical asymptotic behavior is in good agreement with formula (4.20), for both the wings: as one expects for a space-asymptotic result, the speed of convergence worsens with increasing maturity (i.e. as the density of the process gradually spreads out).

Note that the adaptive shift of the integration contour into the saddle-point allows to efficiently evaluate $\sigma^2_{\text{loc}}(y, t)$ for large values of $|y|$, but while this procedure is (i) relatively time consuming in comparison to any explicit formula, in particular if used on-the-fly inside a Monte-Carlo simulation and (ii) limited to models allowing for an explicit evaluation of Fourier transforms, the analysis behind Theorem 4.5 can a priori be extended to other models.

A Technical proofs

A.1 Localization

Here we define the precise localization procedure for the SDE (4.3) that is used in Theorem 4.2 and state the lemma about the localization of the action that is used therein. Consider a family of truncation functions $\psi_R \in C_b^\infty(\mathbb{R}; \mathbb{R})$, $R > 0$, such that

$$
\psi_R(x) = \begin{cases} 
  x & \text{if } |x| \leq R \\
  R + 1 & \text{if } |x| \geq R + 1
\end{cases}
$$

and set

$$
\beta^R_\varepsilon(x) = b_\varepsilon(\psi_R(x)); \quad \beta^R_0 = \beta_0(\psi_R(x)); \quad \alpha^R = \alpha(\psi_R(x)); \quad (A.1)
$$
so that for every $R$, the truncated vector fields $\beta_R^R, \beta_0^R$ and $\alpha^R$ coincide with the original ones on the ball $B_R(0)$, but they remain bounded on $\mathbb{R}^n$ (uniformly in $\varepsilon$ in the case of $\beta_R^B$, thanks to assumption (4.4)). If $K$ is a common Lipschitz constant for $b_0$ and $\alpha$, then

$$\text{Lip}(\beta_0^R) \leq K; \quad \text{Lip}(\alpha^R) \leq K,$$

for all $R$ (A.2)

and it is clear that $\alpha^R(0) = \alpha(0) \neq 0$ under assumption (SV). It follows that strong Hörmander condition (sH) holds at all points also for the controlled system with truncated coefficients

$$d\dot{y}_t^R(h) = -\frac{1}{2} \Delta_{\varepsilon=0} \left( z_t^R(h) \right)^2 dt + z_t^R(h) d\mathbf{W}_t, \quad y_0^R(h) = 0,$$

$$d\dot{z}_t^R(h) = \beta_0^R(z_t^R(h)) dt + \alpha^R(z_t^R(h)) d\mathbf{W}_t, \quad z_0^R(h) = z_0$$

where $\mathbf{W} = \sqrt{\Gamma} h$. Denote $\Lambda^R_t(x) = \inf \{ \frac{1}{2} \| h \|_H^2 : h \in K^+_t(R) \} , x \in \mathbb{R}^2$, the action of the system (A.3), with $K^+_t(R) = \{ h \in H : (y_0^R, z_0^R) = x_0, (y_t^R, z_t^R) = x \}$.

**Lemma A.1.** Assume $\beta_0$ and $\alpha$ are Lipschitz continuous on $\mathbb{R}^n$. Denote $\varphi(h) = (y(h), z(h))$ the solution to the ODE (4.3), and $\Lambda^SV_t$ the related action as in (4.6). For every $R > 0$, define $b_0^R$ and $\alpha^R$ according to (A.1), the corresponding ODE solution $\varphi^R(h) = (y^R(h), z^R(h))$ as in (A.3), and the related action $\Lambda^R_t$. Let $y \in \mathbb{R}^1$ and $t > 0$ be fixed as in Theorem 4.2. Then, if at least one of Conditions (i) and (ii) of Theorem 4.2 is satisfied:

(a) for every $R > 0$, there exists $\mathbf{T}(R) > 0$ such that

$$\Lambda^R_t(y, z) = \Lambda^SV_t(y, z) \quad \forall \ |z| \leq R.$$
(b) There exists $\hat{R} > 0$ such that the maps $z \mapsto \Lambda_t^{SV}(y, z)$ and $z \mapsto \Lambda_t^{R}(y, z)$ attain their (common) global minimum at the same points:

$$\{z \in \mathbb{R}^{n-I} : \Lambda_t^{SV}(y, z) = \Lambda_t^{SV}(N_y)\} = \{z \in \mathbb{R}^{n-I} : \Lambda_t^{R}(y, z) = \Lambda_t^{R}(N_y)\}.$$ 

**Proof.** Arguing as in the proof of Theorem 4.2 if at least one of Conditions (i) or (ii) is satisfied, the deterministic Malliavin matrix $C(0, z_0)(h)$ is invertible for all $h \in \mathcal{K}_t^{(y,z)}$; it then follows from Lemma 2.7(i) that $\Lambda_t$ is continuous on an open set of $\mathbb{R}^2$ containing the line $N_y = (y, \cdot)$.

Let us prove (a). Fix $R > 0$, and define $\mathcal{H}_R = \left\{h \in H : \frac{1}{2} |h|^2_H \leq \sup_{|z| \leq R} \Lambda_t^{SV}(x) \right\}$. Since $\Lambda_t^{SV}$ is finite and continuous around $N_y$, $\sup_{|z| \leq R} \Lambda_t^{SV}(y, z)$ is finite and $\mathcal{H}_R$ is bounded in $H$. Denote $\mathcal{K}_{\min}$ the set of minimizing controls in $\mathcal{K}_r$. It is clear that, for every $z \in B_{R}(0)$, $\mathcal{K}_{\min}^{(y,z)} \subset \mathcal{H}_R$, hence $\Lambda_t^{SV}(y, z) = \inf \left\{\frac{1}{2} |h|^2_H : h \in \mathcal{K}_{i}^{(y,z)} \cap \mathcal{H}_R\right\}$, and $\{y\} \times B_{R}(0) \subset \{\varphi_t(h) : h \in \mathcal{H}_R\}$. Setting

$$C \exp \left(C \sup_{h \in \mathcal{H}_R} |h^{(2)}| \right) \leq C \exp \left(C \left(2 \sup_{|z| \leq R} \Lambda_t^{SV}(y, z)\right)^{1/2}\right) := \mathcal{R}(R)$$

where $C = C(t, z_0, K)$ is the constant defined in Lemma 2.7(i) and $K$ is a Lipschitz constant for $\beta_0$ and $\alpha$, it follows from estimate (2.3) that $\sup_{s \leq t} |z_s(h)| \leq \mathcal{R}(R)$ for every $h \in \mathcal{H}_R$. Therefore, for such $h$ the trajectory $s \mapsto z_s(h)$, $s \leq t$, remains in the region where the vector fields $h_0$, $h_1$, $h_2$, and $h_3$ coincide: from the uniqueness of solutions for the second ODE in (1.3), it follows that $z_s(R) = z_s$ for all $s \in [0, t]$. Since $y(h)$ only depends on $z(h)$, one also has $y^R(y, h) = y_s(h)$ for all $s \leq t$, hence $\varphi_t^R(h) = \varphi_t(h)$ for all $h \in \mathcal{H}_R$. In particular, $\varphi_t^R(h_0) = \varphi_t(h_0)$ for $h_0 \in \mathcal{K}_{\min}^{(y,z)}$, and this establishes

$$\Lambda_t^R(y, z) \leq \frac{1}{2} |h_0|^2 = \Lambda_t^{SV}(y, z), \quad \forall z \in B_{R}(0). \tag{A.4}$$

On the other hand, $\Lambda_t^R(y, z) = \frac{1}{2} |h_0|^2$ for some $h_0 \in \mathcal{K}_t^{(y,z)}(\mathcal{R}(R))$: from (A.4), $h_0 \in \mathcal{H}_R$, therefore $\varphi_t(h_0) = \varphi_t^R(h_0) = (y, z)$ by the uniqueness argument above. This implies $h_0 \in \mathcal{K}_t^{(y,z)}$ and $\Lambda_t^{SV}(y, z) \leq \frac{1}{2} |h_0|^2 = \Lambda_t^R(y, z)$, and (a) is proved.

Let us now prove (b): note that estimate (2.3) in Lemma 2.7 together with the bound (A.2) on the Lipschitz constants of $\beta_0^R$ and $\alpha^R$, implies

$$\sup_{s \leq t} |z_s(h)| \leq C \exp \left(C |h|_H\right), \quad \forall R \tag{A.5}$$

where again $C = C(t, z_0, K)$. Set $R' = C \exp \left(C \left(2 \Lambda_t^{SV}(N_y) + 1\right)^{1/2}\right)$. Estimate (A.5) yields the inclusion

$$\{z \in \mathbb{R}^n : \Lambda_t^{SV}(y, z) \leq \Lambda_t^{SV}(N_y) + 1\} \subset B_{R'}(0),$$

because for every $z$ on the left hand side there exists $h_0$ with $\frac{1}{2} |h_0|^2 = \Lambda_t^{SV}(y, z) \leq \Lambda_t^{SV}(N_y) + 1$ such that $z = z(h_0)$. In particular, all the points $z^*$ are contained in $B_{R'}(0)$, where $z^*$ is a global minimizer of the map $z \mapsto \Lambda_t^{SV}(y, z)$. Now setting $\hat{R} = \mathcal{R}(R')$, point (a) of the current lemma entails that $\Lambda_t^R$ and $\Lambda_t^{SV}$ coincide on $\{y\} \times B_{R'}(0)$; therefore they - trivially - attain their common minimum $\Lambda_t^{SV}(N_y)$ on $\{y\} \times B_{R'}(0)$ at the same points. One has to make sure that $\Lambda_t^R$ is strictly above the level $\Lambda_t^{SV}(N_y)$ outside $\{y\} \times B_{R'}(0)$: but (A.5) ensures that $z^R = z \notin B_{R'}(0)$ can be reached only along controls $h$ such that $\frac{1}{2} |h|^2_H > \Lambda_t^{SV}(N_y) + 1$, hence $\Lambda_t^R(y, z) \geq \Lambda_t^{SV}(N_y) + 1$ for $z \notin B_{R'}(0)$, and this allows to conclude. ■

### A.2 Proofs of Theorem 2.1 and Proposition 2.4

The following lemma plays a key role. The proof is given under weak Hörmander condition at the starting point $x_0$:

$$\text{span}\{\sigma_1, \ldots, \sigma_d; [\sigma_0, \sigma_i] : 1 \leq i \neq j \leq d; \{[\sigma_i, [\sigma_i, \sigma_m]] : 1 \leq i, l, m \leq d; \ldots \}\}_{|x_0} = \mathbb{R}^n \tag{A.6}$$
that is, the linear span of $\sigma_1, \ldots, \sigma_d$ and all the Lie brackets of $b_0, \sigma_1, \ldots, \sigma_d$ is the whole $\mathbb{R}^n$ at $x_0$. Notice that under assumptions (1.5) and (1.11), one has
\[ [b_\varepsilon, \sigma_j]_x^0 = [b_\varepsilon, \sigma_j]_{x_0} + o(1), \quad [\sigma_j, \sigma_k]_x^0 = [\sigma_j, \sigma_k]_{x_0} + o(1) \]
hence (wH) also holds when $b_\varepsilon$ and $x_0$ are replaced with $b_\varepsilon$ and $x_0^\varepsilon$, when $\varepsilon$ is small enough. It is then classical that $X_t^\varepsilon$ admits a smooth density $p_t^\varepsilon$ for all $t > 0$ under condition (wH).

**Lemma A.2.** Let $X^\varepsilon$ be the solution to (1.2). Assume weak Hörmander condition (wH) at $x_0$. Then, for any $t > 0$, any $x \in \mathbb{R}^n$ and $q \in (0,1)$ there exist a constant $C_q$ and positive integers $N_1(q) = N_1(q, n)$ and $N(q) = N(q, n, x_0)$ such that
\[ p_t^\varepsilon(x) \leq C_q(1 + R^{-N_1(q)}) \varepsilon^{-N(q)} \mathbb{P}(|X_t^\varepsilon - x| \leq R)^q \]
for every $R > 0$. The constant $C_q$ also depends on $t$ and on the bounds on the derivatives of $b_0$ and the $(\sigma_j)_j$.

**Proof.** Given in section A.3 \(\blacksquare\)

**Proof of Theorem 2.1.** Let us write $\Lambda$ for the action function throughout this proof (dropping the fixed index $t$ from the notation). The following estimate is obtained with standard uniform large deviations arguments (see also Léandre [29, section 2]): for every $R > 0$, one has
\[ \limsup_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}(|X_t^\varepsilon - x| \leq R) \leq -\Lambda(B_R(x)) \]
uniformly over $x$ in compact sets.

Let first prove (2.2). $x$ is fixed; taking $\limsup_{\varepsilon \to 0} \varepsilon^2 \log$ in estimate (A.7) and applying (A.8), one has
\[ \limsup_{\varepsilon \to 0} \varepsilon^2 \log p_t^\varepsilon(x) \leq q \limsup_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}(|X_t^\varepsilon - x| \leq R) \leq -q\Lambda(B_R(x)). \]
Now taking the limit $R \to 0$, $\Lambda(B_R(x)) = \inf_{y \in B_R(x)} \Lambda(y) \to \Lambda(x)$ by the lower semi-continuity of $\Lambda$. Since $q < 1$ was arbitrary, (2.2) follows.

Let us now prove (2.3). Under the assumption in Theorem 2.1(ii), it follows from Lemma 2.7(iii) that $\Lambda$ is continuous on an open neighborhood $V$ of $x$, hence uniformly continuous on compact sets contained in $V$. Fix a compact ball $B \subset V$ and $\delta > 0$. We can find $R = R_{B, \delta}$ such that the closed $R$-neighborhood of $B$, $B^R = \cup_{y \in B} B_{R}(y)$, is contained in $V$ and moreover $\text{Osc}(\Lambda, B_R(y)) \leq \delta$ for all $y \in B$. In particular,
\[ \Lambda(B_R(y)) = \inf_{z \in B_R(y)} \Lambda(z) \geq \Lambda(y) - \delta \]
for all $y \in B$. Now taking $\limsup_{\varepsilon \to 0} \varepsilon^2 \log$ in estimate (A.7), and applying (A.8) and (A.9), it follows that for any $q \in (0,1)$
\[ \limsup_{\varepsilon \to 0} \varepsilon^2 \log p_t^\varepsilon(y) \leq q \limsup_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}(|X_t^\varepsilon - y| \leq R) \leq -q\Lambda(B_R(y)) \leq -q(\Lambda(y) - \delta) \]
where the limit holds uniformly over $y \in B$. Since $q < 1$ and $\delta$ were arbitrary, the right hand side can be improved to $\Lambda(y)$, and (2.3) follows.

The lower bound (2.3) is actually estimate (3.5) in Ben Arous and Léandre [6, Theorem III.1]. Their proof can be adapted to the case where $b$ and $x_0$ depend on $\varepsilon$, under the convergence conditions (1.3) et (1.11). The statement about $\Lambda = \Lambda_R$ in Theorem 2.1(i) is obvious from the definitions of the two actions, and (2.4) is a direct consequence of (2.2) and (2.3) \(\blacksquare\)

**Proof of Proposition 2.4.** It follows from Lemma A.2 that, for every $q, q' \in (0,1)$ and $R$ small enough,
\[ \int_{|z| \geq A} |z|^k p_t^\varepsilon(y, z) dz \leq C_q R^{-N_1(q)} \varepsilon^{-N(q)} \int_{|z| \geq A} |z|^k \mathbb{P}(|X_t^\varepsilon - (y, z)| \leq R)^q dz \]
\[ = C_q R^{-N_1(q)} \varepsilon^{-N(q)} \int_{|z| \geq A} |z|^k \mathbb{P}(|X_t^\varepsilon - (y, z)| \leq R)^{q'} \mathbb{P}(|X_t^\varepsilon - (y, z)| \leq R)^{(1-q')} dz. \]

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Since
\[ P(|X_t^\varepsilon - (y, z)| \leq R) \leq P(|Y_t^\varepsilon - y| \leq R, |Z_t^\varepsilon - z| \leq R) \]
one has
\[ \int_{|z-\bar{z}| \geq A} |z|^k \rho(y, z)dz \leq C_q R^{-N_1(q)} \varepsilon^{-N(q)} \times P(|Y_t^\varepsilon - y| \leq R, |Z_t^\varepsilon - \bar{z}| \geq A - R) \]
\[ \times \int_{|z-\bar{z}| \geq A} |z|^k P(|Z_t^\varepsilon - z| \leq R)^{q(1-q')}dz. \quad (A.10) \]

The integral on the right hand side of (A.10) can be bounded as follows
\[ \int_{|z-\bar{z}| \geq A} |z|^k P(|Z_t^\varepsilon - z| \leq R)^{q(1-q')}dz \leq \int_{\mathbb{R}^{n-1}} |z|^k P(|Z_t^\varepsilon| \geq |z| - R)^{q(1-q')}dz. \quad (A.11) \]

Note that the random variable $|Z_t^\varepsilon|$ has moment of all orders uniformly bounded in $\varepsilon$ (for so does $|X_t^\varepsilon|$): precisely, for any $T > 0$ and $r > 0$ there exists a constant $C_r = C_{r,T}$ such that $\sup_{t \leq T} E(|Z_t^\varepsilon|)^r \leq C_r$ for all $t \leq T$. Then, from Markov’s inequality
\[ \sup_{\varepsilon \leq 1} P(|Z_t^\varepsilon| \geq |z| - 2R) \leq \frac{C_r}{(|z| - R)^r} \]
for all $z$ such that $|z| - R > 0$ and all $r > 0$. The exponent $r$ can be chosen such that
\[ \sup_{\varepsilon \leq 1} P(|Z_t^\varepsilon| \geq |z| - R)^{q(1-q')} \leq \frac{C^{(1)}}{|z|^{k+n-1+1}}. \]
for all $|z|$ larger than, say, $1 + R$, for some positive constant $C^{(1)} = C^{(1)}_{r,T}$. It follows that for any choice of $q, q' \in (0, 1)$, the integral on the right hand side of (A.11) is convergent, and uniformly bounded in $\varepsilon$.

Finally taking log, multiplying by $\varepsilon^2$ and taking $\limsup_{\varepsilon \to 0}$ in (A.10), we have
\[ \limsup_{\varepsilon \to 0} \varepsilon^2 \log \int_{|z-\bar{z}| \geq A} |z|^k \rho(y, z)dz \leq \limsup_{\varepsilon \to 0} \varepsilon^2 \log P(|Y_t^\varepsilon - y| \leq R, |Z_t^\varepsilon - \bar{z}| \geq A - R) \]
\[ \leq -qq' \inf \{ \Lambda_t(y', z') : |y' - y| \leq R, |z' - \bar{z}| \geq A - R \}. \]

As $R \downarrow 0$, the right hand side tends to $-qq' \inf \{ \Lambda_t(y, z') : |z' - \bar{z}| \geq A \}$: since $q, q' < 1$ were arbitrary, we obtain the claim. ■

### A.3 Proof of Lemma [A.2]

Throughout this section, we denote $X = (X_t; t \geq 0)$ the strong solution of the SDE
\[ X_t = x_0 + \int_0^t B(X_s)ds + \sum_{j=1}^d \int_0^t A_j(X_s)dW_s^j, \quad t \geq 0 \]
(A.13)
where $B, A_j \in \mathcal{C}^\infty_b (\mathbb{R}^n; \mathbb{R}^n)$ for all $j$. For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_l) \in \{1, \ldots, n\}^l$, we denote $|\alpha| = l$ and $\partial_\alpha = \partial_\alpha^{(\alpha_1, \ldots, \alpha_l)}$. Setting
\[ |f|_k = \sup_{|\alpha| \leq k, y \in \mathbb{R}^n} |\partial_\alpha f(y)| \]
for smooth real valued functions $f$, we denote $|B|_k = 1 + \sum_{i=1}^n |B_i|_k$ and $|A|_k = 1 + \sum_{i,j} |A_{ij}|_k$.

**Some elements of Malliavin calculus.** Following the standard notation in [33], we denote $\mathbb{D}^{k,p}$ the domain of the $k$-th order Malliavin derivative, and $\mathbb{D}^{\infty} = \cap_{k \geq 1} \cap_{p \geq 1} \mathbb{D}^{k,p}$. It is classical, see [33], that $X_t$ is a smooth random variable in Malliavin’s sense for every $t$, that is $X_t \in \mathbb{D}^{\infty}$. Denoting $D_r X_t = (D_1^r X_t, \ldots, D_d^r X_t)$, $r \in [0, t]$
the \((d\text{-dimensional})\) Malliavin derivative of \(X_t\), the \(k\)-th order derivative is obtained by iterating the operator: 
\[ D_{i_1, \ldots, i_k} X_t := D_{i_1} \cdots D_{i_k} X_t, \text{ for every } (i_1, \ldots, i_k) \in \{1, \ldots, d\}^k. \]
It is well-known that the random variables \(D_{i_1, \ldots, i_k} X_t\) have finite moments of any order: the following lemma gives an explicit estimate on the \(L^p\) norms, in terms of the bounds on \(A\) and \(B\) and their derivatives, and will be useful in what follows.

**Lemma A.3** (Lemma 2.1 and Corollary 1 in [10]). For every \(k \geq 1\) and \(p > 1\) there exist positive integers \(\gamma, \gamma'\) and a positive constant \(C\), all depending on \(k, p\) but not on the bounds on \(B\) and \(A\) and their derivatives, such that, for any \(t > 0\)
\[
\sup_{r_1, \ldots, r_k \leq t} \mathbb{E} \left[ |D_{i_1, \ldots, i_k} X_t|^p \right] \leq C_{k, p} \left( t^{1/2} |B_k| + |A_k| \right)^{\gamma'} e^{\gamma' (t |B|+t^{1/2}|A|)^{p} + o(t^{1/2}|A|^{p})}
\]
for all \(i = 1, \ldots, m\) and \((j_1, \ldots, j_k) \in \{1, \ldots, d\}^k\). Moreover,
\[
||\phi(X_t)||_{k,p} \leq C_{k,p} |\phi|_k \left( 1 + (t \vee t^k)^{1/2} \right) \left( t^{1/2} |B_k| + |A_k| \right)^{\gamma'} e^{\gamma' (t |B|+t^{1/2}|A|)^{p}}
\]
for any \(\phi \in C^\infty(\mathbb{R}^n)\).

The notion of non-degeneracy for (Malliavin-)differentiable random variables \(F \in D^{1,2}\) is understood in the sense of the (stochastic) Malliavin covariance matrix
\[
\langle \gamma_F \rangle_{i,j} = \int_0^t \sum_{l=1}^d D_k^i F^l \, D_k^j F^l \, ds, \quad i, j = 1, \ldots, d.
\]
The fundamental tool in order estimate the density of random variables with invertible covariance Matrix is the notion of non-degeneracy for (Malliavin-)differentiable random variables \(F \in D^{1,2}\) is understood in the sense of the (stochastic) Malliavin covariance matrix

**Proposition A.4** (Integration by parts formula of the Malliavin calculus; see [33]). Let \(F = (F^1, \ldots, F^d) \in D^\infty\). Assume that \(\gamma_F\) is invertible a.s. and moreover \(\mathbb{E}[\det(\sigma_F)^{-p}] < \infty\) for all \(p \geq 1\). Let \(G \in D^\infty\) and \(\phi \in C^\infty_{\text{pol}}(\mathbb{R}^m)\). Then, for any \(k \geq 1\) and any multi-index \(\alpha = (\alpha_1, \ldots, \alpha_k) \in \{1, \ldots, m\}^k\) there exists a random variable \(H_\alpha(F,G) \in D^\infty\) such that
\[
\mathbb{E}[\partial_\alpha \phi(F)G] = \mathbb{E}[\phi(F)H_\alpha(F,G)],
\]
where the \(H_\alpha(F,G)\) are recursively defined by
\[
H_{\alpha}(F,G) = H_{(\alpha_k)}(F,H_{(\alpha_1, \ldots, \alpha_{k-1})}(F,G)), \quad H_{(i)}(F,G) = \sum_{j=1}^m \delta \left( \gamma_F^{-1} \right)_{i,j} DF^j
\]
where \(\delta\) denotes the adjoint operator of \(D\).

The key ingredient required to apply the integration by parts is an estimate of the \(L^p\) norms of the Malliavin weights \(H_\alpha\). The following theorem, proved in [10], provides explicit bounds in terms of the bounds on \(A\) and \(B\) and their derivatives.

**Theorem A.5** (Theorem 2.3 in [10]). For every \(k \geq 1\), there exist a positive constant \(C_k\) and positive integers \(a_k, b_k, c_k\) and \(r_k\), all possibly depending also on \(n\) and \(d\), such that for any multi-index \(\alpha \in \{1, \ldots, n\}^k\), any \(F \in D^\infty\) and any \(t > 0\),
\[
||H_\alpha(X_t,G)||_2 \leq C_k (1 + t^k) ||G||_2 e^{\mathbb{E}[(\gamma_t)^{-r_k}]} \times (|B_k| + |A_k|)^{r_k} e^{\gamma (t |B|+|A|+t^{1/2}|A|)^{p}}.
\]

Let us go back to equation (A.13). If the stochastic integral in (A.13) is intended in Stratonovich sense, the drift coefficient \(B\) is replaced by \(A_0 = B - \frac{1}{2} \sum_{j=1}^d \sum_{k=1}^n A_j^k \partial_x A_j\). If we assume that the vector field \((A_0, A_j)\) satisfy the weak Hörmander condition at \(x_0\)
\[
\text{span}\{A_1, \ldots, A_d; [A_0, A_i] : 1 \leq i \leq d; [A_i, A_j] : 1 \leq i, j \leq d; [A_i, [A_t, A_m]] : 1 \leq i, l, m \leq d; \ldots \} \mid_{x_0} = \mathbb{R}^n,
\]
(A.15)
then, denoting $V_{2o}^L$ the vector space spanned by the Lie brackets of length smaller or equal to $L$ in (A.15), and setting\[3\]
\[\mathcal{V}_L(x_0, v; A_0, A) = \sum_{V \in V_{2o}^L} \langle v, V v \rangle^2 \quad v \in \mathbb{R}^n, \tag{A.16}\]
it follows from (A.15) that there exists some $L \geq 1$ such that $\mathcal{V}_L(x_0, v; A_0, A) = 0 \Rightarrow v = 0$. In other words,

\[\mathcal{V}_L(x_0; A_0, A) = \inf_{|v|=1} \mathcal{V}_L(x, v; A_0, A) > 0\]

for some $L \geq 1$. Under condition (wH) at $x_0$, the Malliavin covariance matrix $\gamma_{X_t}$ satisfies the fundamental estimate of Kusuoka and Stroock [27, Corollary 3.25]: for every $T > 0$ and $r > 0$, there exist a constant $C_r = C_r(T)$ and an integer $N(L,n)$ such that

\[\mathbb{E}[\det(\gamma_t)^{-1/r}]^1/r \leq \frac{C_r}{t^{nL} \mathcal{V}_L(x_0; A_0, A)^{N(L,n)}} \tag{A.17}\]

for all $t \in (0,T]$.

We are now ready to provide the following

**Proof of Lemma (A.2)** Let $1_{\{B_R/2(0)\}} \leq \varphi_R \leq 1_{\{B_R(0)\}}$ be a $C^\infty$ function. We can construct $\varphi_R$ so that $|\varphi_R|^k \leq C_k(1+R^{-k})$ for some constant $C_k$ (eventually depending on the dimension $n$). Define $\varphi_R(y) := \varphi_R(y-x)$ and consider the Fourier transform of $\hat{p}_{t,R}^\alpha \varphi_R$, that is (up to a constant factor)

\[\hat{p}_{t,R}^\alpha(\xi) := \mathbb{E}[e^{i\langle \xi, X_t^\alpha \rangle} \varphi_R(X_t^\alpha)].\]

Since the function $y \to \hat{p}_{t,R}^\alpha(y) \varphi_R(y)$ is $C^\infty$ and compactly supported, $\hat{p}_{t,R}^\alpha$ is integrable and we can use Fourier inversion in order to write

\[\hat{p}_{t,R}^\alpha(x) = \hat{p}_{t,R}^\alpha(x) \varphi_R(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\langle \xi, x \rangle} \hat{p}_{t,R}^\alpha(\xi) d\xi. \tag{A.18}\]

On the one hand, it is clear that

\[|\hat{p}_{t,R}^\alpha| \leq \mathbb{P}(|X_t^\alpha - x| \leq R). \tag{A.19}\]

On the other hand, using $\partial_{x_1}^k \ldots \partial_{x_n}^k e^{i\langle \xi, x \rangle} = i^k n! \prod_{j=1}^n \xi_j^j e^{i\langle \xi, x \rangle}$, and applying Theorem (A.5) we have

\[\left|\left(1 + \prod_{j=1}^n \xi_j^j \right) \hat{p}_{t,R}^\alpha(\xi) \right| = \left|\left(1 + \prod_{j=1}^n \xi_j^j \right) \mathbb{E}[e^{i\langle \xi, X_t^\alpha \rangle} \varphi_R(X_t^\alpha)] \right| \leq 1 + \left|\mathbb{E}[\partial_\alpha e^{i\langle \xi, X_t^\alpha \rangle} \varphi_R(X_t^\alpha)]\right| = 1 + \left|\mathbb{E}[e^{i\langle \xi, X_t^\alpha \rangle} H_{\alpha}(X_t^\alpha, \varphi_R(X_t^\alpha))]\right| \leq 1 + \left|\left[H_{\alpha}(X_t^\alpha, \varphi_R(X_t^\alpha))\right]_2 \right| \tag{A.20}\]

where $\alpha = ((1)^1, \ldots, (n)^k)$. Using Lemma (A.3) and the fact that the norms $\{|b|_k\}_{k \geq 0}$ are bounded in $\varepsilon$, it follows that there exist some $\varepsilon_0 > 0$ such that

\[\|\varphi_R(X_t^\alpha)\|_{nk,2k^{n+k+1}} \leq C_k \left(1 + (t \vee t^k)^{1/2} \right) \left(t^{1/2} |b_k| + \varepsilon |\sigma|_k \right)^{\gamma^\prime} e^{k^\gamma(t \vee t^k + t^{1/2} |\sigma|_k)} (1 + R^{-nk}) \leq C_k^{(1)}(1 + R^{-nk}) \quad \tag{A.21}\]

for every $\varepsilon < \varepsilon_0$, for some constant $C_k^{(1)} = C_k^{(1)}(t, |b_k|_k, |\sigma|_k)$.

---

\[1\text{The sum in (A.16) is in fact taken over a finite number of generating brackets, and it can alternatively be written using the notation introduced in the Appendix of [27].}\]
When the SDE (1.2) is written in Stratonovich form, the drift \( b_\varepsilon \) is replaced by \( \overline{b}_\varepsilon = b_\varepsilon - \varepsilon^2 \frac{1}{2} \sum_{j=1}^d \sum_{k=1}^n \sigma_{jk} \partial_{x_k} \sigma_{ij} \). Noting that \( \mathcal{V}_L(x_0^\varepsilon, v; \overline{b}_\varepsilon, \varepsilon \sigma) \) contains terms propositional to \( \varepsilon \) (coming from the brackets \( [b_\varepsilon, \varepsilon \sigma_j] x_0^\varepsilon = \varepsilon [b_\varepsilon, \sigma_j] x_0 \) + \( o(1) \)) and terms proportional to \( \varepsilon^2 \) (coming from the brackets \( [\varepsilon \sigma_j, \varepsilon \sigma_k] x_0^\varepsilon \)), for \( \varepsilon \) small enough one has

\[
\mathcal{V}_L(x_0^\varepsilon, v; \overline{b}_\varepsilon, \varepsilon \sigma) \geq \frac{\varepsilon}{2} \mathcal{V}_L(x_0, v; b_0, \sigma).
\]

Therefore, \( \mathcal{V}_L^\varepsilon(x_0) := \text{inf}_{|v|=1} \mathcal{V}_L(x_0^\varepsilon, v; \overline{b}_\varepsilon, \varepsilon \sigma) \geq \frac{\varepsilon}{2} \mathcal{V}_L(x_0; b_0, \sigma) \). Under condition (wH), there exist some \( L \geq 1 \) such that \( \mathcal{V}_L(x_0; b_0, \sigma) > 0 \). Then, it follows from estimate (A.17) that for every \( r > 0 \) and \( t > 0 \) there exist \( \varepsilon_1 \), a function of time \( C_r(t) \) and an integer \( N(L, n) \) such that

\[
\mathbb{E}[\det(\gamma_i^r)^{-1}] \leq C_r(t) \varepsilon^{-r N(L, n)}
\]

for every \( \varepsilon < \varepsilon_1 \), where \( \gamma_i^r \) is the Malliavin covariance matrix of \( X_t^i \).

Now, it follows from (A.21), (A.22) and Theorem A.3 that

\[
\|H_\alpha(X_t^i, \varphi_R(X_t^i))\|_2 \leq C_k^{(2)} (1 + R^{-nk}) \varepsilon^{-N_k^i} \tag{A.23}
\]

for every \( \varepsilon < \varepsilon_0 \land \varepsilon_1 \), for some constant \( C_k^{(2)} \) also depending on \( (t, |b_0|_k, |\sigma|_k) \), with \( N_k^i = r_k N(L, n) \) and \( r_k \) is given in Theorem A.3. Plugging (A.23) into (A.20), one obtains a polynomial estimate for \( \tilde{p}_R^i \), namely

\[
\tilde{p}_R^i(\xi) \leq C_k^{(3)} (1 + R^{-nk}) \varepsilon^{-N_k^i} \left( 1 + \left( \prod_{j=1}^n \xi_j \right)^k \right)^{-1-q} \tag{A.24}
\]

Finally, using (A.19) and applying (A.21) and (A.24), for every \( q \in (0, 1) \) we can write

\[
p_i^q(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i(\xi, x)} \tilde{p}_R^i(\xi) d\xi = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i(\xi, x)} \tilde{p}_R^i(\xi)^q \tilde{p}_R^i(\xi)^{1-q} d\xi \leq \mathbb{P}(|X_t^i - x| \leq R)^q \left( C_k^{(3)} (1 + R^{-nk}) \varepsilon^{-N_k^i} \right)^{1-q} \text{Re} \int_{\mathbb{R}^n} \frac{1}{1 + \left( \prod_{j=1}^n \xi_j \right)^{1-q}} d\xi.
\]

Choosing \( k = k^* \) large enough (but only dependent on \( n \) and \( q \)), the last integral is convergent. Then, the final claim is proved, once we have set \( N_1(q) = \lceil nk^*(1 - q) \rceil \) and \( N(q) = \lceil N_k^i(1 - q) \rceil \).

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