Torons and black hole entropy

Miguel S. Costa\footnote{M.S.Costa@damtp.cam.ac.uk} and Malcolm J. Perry\footnote{malcolm@damtp.cam.ac.uk}

D.A.M.T.P.
University of Cambridge
Cambridge CB3 9EW
UK

Abstract

We consider a supersymmetric system of D-5-branes compactified on $T^4 \times S^1$ with a self-dual background field strength on $T^4$ and carrying left-moving momentum along $S^1$. The corresponding supergravity solution describes a 5-dimensional black hole with a regular horizon. The entropy of this black hole may be explained in terms of the Landau degeneracy for open strings stretching between different branes. In the gauge theory approximation this D-5-brane system is described by a super Yang-Mills theory with a t’Hooft twist. By choosing a supersymmetric branch of the theory we obtain perfect agreement with the entropy formula. The result relies on the number of massless torons associated with the gauge field components that obey twisted boundary conditions.
1 Introduction

One of the most attractive features of superstring theory as an unifying theory is that it naturally incorporates gravity and gauge theories. In other words, the dynamics of the background spacetime are determined by a given supergravity theory, and the dynamics for the massless modes associated with the solitonic objects (D-branes [1, 2, 3], solitonic 5-brane [4] and intersecting configurations [5, 6]) are determined by a given worldvolume gauge theory. A remarkable example of the interplay between these different aspects of superstring theory is the explanation of the statistical origin of the Bekenstein-Hawking entropy for supersymmetric black holes with a regular horizon and carrying Ramond-Ramond charge [7, 8, 9, 10, 11, 12]. These black holes may be studied in the low string coupling limit by identifying them with an excited D-brane system. The degeneracy of BPS states is then given by the number of massless open string states that reproduce the excited D-brane system. This argument, however, requires an assumption on which open string states are the relevant ones for the black hole entropy counting [8]. A more precise treatment involves considering the supersymmetric gauge theory describing the D-brane system, taking into account the interactions between different massless modes on the branes [12].

In a recent paper [13], we have addressed the problem of black hole entropy counting for a supersymmetric D-brane system with magnetic condensates on the constituent D-branes’ worldvolume theory. The agreement with the semi-classical entropy formula was shown by considering the Landau degeneracy for open strings stretching between different branes [14]. In this paper we will explore the existence of these Landau levels from the gauge theory point of view. In particular, we shall see that the Landau levels arise within the gauge theory context as torons, i.e. instantons on a torus. The matching with the semi-classical entropy formula uses non-trivial results about the dimension of the space of Θ-functions on $T^4$ [15, 16]. Also, in the gauge theory approximation we have control on the interactions between the different massless fields on the D-branes, and no assumption regarding which string states contribute to the entropy is necessary.

Our work provides another example of the interplay between string theory, black-hole physics and gauge theories. We will use old results of gauge theories on compact spaces with twisted boundary conditions on the fields [17, 18, 19, 20]. This work constituted an early attempt to understand quark confinement and has been placed in the context of D-brane physics to describe the worldvolume
theory of D-branes carrying magnetic fluxes \cite{21, 22}.

We start by describing the D-brane system studied in this paper. We will consider a system of coincident D-5-branes on $T^4 \times S^1$ associated with a 5-dimensional black hole. Each D-5-brane will have a constant (anti)self-dual field strength on $T^4$ and will carry momentum along $S^1$. The corresponding gauge theory describing the massless modes of this D-brane system is given by the compactification of $D = 10$ super Yang-Mills theory to 6 dimensions \cite{3} (we will ignore the Born-Infeld corrections to the action \cite{23}). To derive our D-brane system we start with the supersymmetric configuration of D-2-branes on $T^4$ intersecting at $SU(2)$ angles \cite{24}. We consider $N_i$ D-2-branes, which will be called of type $i$, and place them with respect to the coordinate system $x^2, \ldots, x^5$ on $T^4$ according to (following the notation of \cite{13})

$$2_i : \ X^2, X^4, \quad \phi_i^3 = \tan \theta_i \ X^2, \quad \phi_i^5 = \pm \tan \theta_i \ X^4, \quad (1.1)$$

for $i = 1, \ldots, n$, i.e. we have $n$ different types of D-2-branes. The angles $\theta_i$ obey the quantisation conditions

$$\tan \theta_i = \frac{q_i R_3}{\bar{p}_i R_2} = \frac{\bar{q}_i R_5}{p_i R_4}. \quad (1.2)$$

In other words, the type $i$ D-2-branes are wrapped on the $(p_i, q_i)$ and $(\bar{p}_i, \bar{q}_i)$ cycles of $T^4 = T^2 \times T^2$.

We proceed by performing T-duality transformations along the $x^1, x^3$- and $x^5$-directions. The resulting type $i$ D-5-branes are described by

$$5_i : \ X^1, X^2, X^3, X^4, X^5, \quad 2\pi \alpha' G_{23}^{(i)} = \tan \theta_i = \frac{q_i}{p_i R_2 R_3} \frac{\alpha'}{R}, \quad (1.3)$$

$$2\pi \alpha' G_{45}^{(i)} = \pm \tan \theta_i = \frac{\bar{q}_i}{\bar{p}_i R_4 R_5} \frac{\alpha'}{R},$$

where $G_{\alpha\beta}^{(i)}$ is the field strength of the type $i$ D-5-branes. It is (anti)self-dual on the subspace $T^4$. For each $i$ we have $N_i$ D-5-branes winding $p_i$ and $\bar{p}_i$ times in the $x^2$- and $x^4$-directions, respectively. They also carry $N_i q_i \bar{p}_i$ and $N_i \bar{q}_i p_i$ units of D-3-brane charge, as well as $N_i q_i \bar{q}_i$ units of D-string charge. This follows from the fact that each type $i$ D-5-brane carries fluxes $2\pi q_i \bar{p}_i$ and $2\pi \bar{q}_i p_i$ in the $x^2 x^3$- and $x^4 x^5$ 2-torus, respectively. We may now excite the D-brane system while
preserving some supersymmetry by allowing these D-branes to carry all together $N$ units of left-moving momentum along the $x^1$-direction,

$$ P = \frac{N}{R_1} . $$

(1.4)

Our configuration preserves $1/8$ of the spacetime supersymmetry and therefore $1/4$ of the D-5-branes’ worldvolume supersymmetry. Through this paper we will just consider the self-dual field strength case. The analysis for the antiself-dual case is identical.

We begin in section 2 by writing the supergravity solution that is associated with our D-brane system. We shall then derive the Bekenstein-Hawking entropy formula in terms of quantised charges. In section 3 we shall see how this entropy arises from the Landau degeneracy for strings stretching between different branes. We will be brief since the results are essentially the same as those presented in [13]. In section 4 we shall study the gauge theory describing our D-brane system. The main result will be that there are several massless torons whose degeneracy exactly match the entropy formula. This matching arises due to the fact that these torons are described in terms of $\Theta$-functions on $T^4$. These functions form a complex linear space whose dimension equals the number of Landau levels.

## 2 Supergravity solution

The supergravity solution describing the long range fields of the D-5-brane system discussed in the introduction may be obtained by following essentially the same steps. We start with the supergravity solution describing $n$ D-branes intersecting at $SU(2)$ angles [25, 26] and use the supergravity T-duality rules [27]. The resulting metric and dilaton field are

$$ ds^2 = H^{\frac{1}{2}} \left[ H^{-1} \left( -dt^2 + dx_1^2 + \frac{\alpha}{r^2}(dx_1 - dt)^2 \right) + \tilde{H}^{-1} ds^2(T^4) + ds^2(\mathbb{E}^4) \right] , $$

$$ e^{2(\phi - \phi_\infty)} = H \tilde{H}^{-2} , $$

(2.1)

where

$$ H = 1 + \sum_i \frac{\mu_i}{r^2} + \sum_{i<j} \frac{\mu_i \mu_j}{r^4} \sin^2 (\theta_i - \theta_j) , $$
\begin{equation}
\tilde{H} = 1 + \sum_i \frac{\mu_i}{r^2} \cos^2 \theta_i ,
\end{equation}

with \( r \) the radial coordinate on the 4-dimensional Euclidean space \( \mathbb{E}^4 \). The \( i \) and \( j \) indices in the sums run from 1 to \( n \). The constants \( \mu_i \) determine the ADM mass of the D-5-branes, the angles \( \theta_i \) determine the D-5-branes' field strength and the constant \( \alpha \) the left-moving momentum carried by all the D-5-branes along the \( x^1 \)-direction. If \( \theta_i = 0 \) for all \( i \) we obtain the D-5-brane solution and if \( \theta_i = \pi/2 \) the D-string solution. If \( \theta_i = \theta_j \) for all \( i \) and \( j \) we have enhancement of supersymmetry as our solution breaks \( 1/2 \) of the spacetime supersymmetry (1/2 due to the D-5-branes and the other 1/2 due to the momentum modes along the \( x^1 \)-direction).

From the worldvolume gauge theory perspective this fact follows because the non-vanishing worldvolume field strength is on the \( U(1) \) center of the gauge group and the non-linear realization of the worldvolume supersymmetry may be used to cancel the variation of the gaugino field under worldvolume supersymmetry transformations \[28\]. In this case the horizon area vanishes. Note also that for \( n = 2 \) with \( \theta_1 = 0 \) and \( \theta_2 = \pi/2 \) we recover the D-5-brane/D-string configuration used by Callan and Maldacena \[8\].

The solution \eqref{eq:2.1} may be reduced to five-dimensions giving a black hole solution with a regular horizon. From the area of the horizon we can determine the Bekenstein-Hawking entropy

\begin{equation}
S_{BH} = \frac{A_H}{4G_N^{(5)}} = \frac{A_3}{4G_N^{(5)}} \sqrt{\alpha \left( \sum_{i<j} \mu_i \mu_j \sin^2 (\theta_i - \theta_j) \right)} ,
\end{equation}

where \( G_N^{(5)} \) is the five-dimensional Newton constant and \( A_3 \) is the unit 3-sphere volume.

Next, we want to write the black hole entropy in terms of the microscopic quantities that define our D-brane system \[8\]. The mass of \( N_i \) D-5-branes defined by \eqref{eq:1.3} is \[8\]

\begin{equation}
M_{5,D} = N_i \frac{R_1}{g_0 a^3} \sqrt{(p_i R_2 R_3)^2 + (q_i \alpha')^2} \sqrt{\left( \bar{p}_i R_4 R_5 \right)^2 + (\bar{q}_i \alpha')^2} ,
\end{equation}

while the corresponding ADM mass that is obtained from the compactified supergravity solution is

\begin{equation}
M_{5,D} = 2 \mu_i \left( \frac{16 \pi G_N^{(5)}}{A_3} \right)^{-1} .
\end{equation}
The left-moving momentum carried by all the D-5-branes may also be related to the constant $\alpha$ in the solution (2.1) by

$$\alpha = \frac{4G_N(5)N}{\pi R_1}. \quad (2.6)$$

Using the conditions in (1.3) for the angles $\theta_i$ and writing the Newton coupling constant in terms of string theory quantities we obtain the following expression for the black hole entropy

$$S = 2\pi \sqrt{N \sum_{i<j} N_i N_j n_{ij} \bar{n}_{ij}}, \quad (2.7)$$

where

$$n_{ij} = |p_j q_i - p_i q_j|, \quad \bar{n}_{ij} = |\bar{p}_j \bar{q}_i - \bar{p}_i \bar{q}_j|. \quad (2.8)$$

The numbers $n_{ij}$ and $\bar{n}_{ij}$ appear in the entropy formula because of the existence of Landau levels in the $x^2, x^3$- and $x^4, x^5$-directions for open strings with ends on the type $i$ and $j$ D-5-branes. In the gauge theory picture they will correspond to the number of massless torons associated to such pair of D-5-branes.

### 3 String Theory description

In this section we derive the entropy formula (2.1) by studying the excitations of open strings ending on the D-5-branes. The analysis entirely parallels that of [13], therefore we will be brief.

The open strings describing the excitations of our D-brane system may be divided in two main sectors. The first one is associated with strings with both ends on the type $i$ D-5-branes. The corresponding bosonic massless modes are associated with a gauge field $A_{\alpha}$ ($\alpha = 0, ..., 5$) and 4 scalar fields $\phi_m$ ($m = 1, ..., 4$). We shall assume that these excitations do not contribute to the entropy formula. This assumption will be justified in the next section by using the gauge theory description of our D-brane system since the interactions between massless fields on the branes are taken into account.

The second sector is associated with open strings with ends on different type $i$ and $j$ D-5-branes. The corresponding massless modes are described by torons on $T^4$ and will be responsible for the entropy of the system. These strings obey Neummann boundary conditions at both ends in the $x^0, x^1$-directions and Dirichlet boundary conditions at both ends in the transverse $x^6, x^7, x^8, x^9$-directions.
Defining the complex world-sheet scalar fields \( Z_k = X^{2k} + iX^{2k+1} \) for \( k = 1, 2 \), the boundary conditions along these directions are

\[
\begin{align*}
\partial_\sigma Z_k &= -i \tan \theta_i \partial_\tau Z_k , \quad \sigma = 0, \\
\partial_\sigma Z_k &= -i \tan \theta_j \partial_\tau Z_k , \quad \sigma = \pi.
\end{align*}
\]

(3.1)

The corresponding mode expansion is [14]

\[
Z_k = z_k + i \left[ \sum_{n=1}^{\infty} a_{n-\epsilon}^k \phi_{n-\epsilon}(\tau, \sigma) - \sum_{n=0}^{\infty} a_{n+\epsilon}^{k\dagger} \phi_{n-\epsilon}(\tau, \sigma) \right],
\]

(3.2)

where

\[
\phi_{n-\epsilon} = \frac{1}{\sqrt{|n-\epsilon|}} \cos \left[ (n-\epsilon)\sigma + \theta_i \right] e^{-i(n-\epsilon)\tau},
\]

(3.3)

with \( \epsilon = (\theta_i - \theta_j)/\pi \) and \( n \) an integer. Note that without loss of generality we may assume that \( \epsilon \) lies between 0 and 1. The real operators \( a_{n-\epsilon}^k, a_{n+\epsilon}^{k\dagger} \) and \( a_{n-\epsilon}^{k\dagger}, a_{n+\epsilon}^k \) are annihilation and creation operators, respectively.

The zero modes \( z_k \) and \( z_k^{\dagger} \) are non-commuting variables. In terms of the \( x^\hat{\alpha} \)-coordinates (\( \hat{\alpha} = 2, 3, 4, 5 \)) the corresponding non-vanishing commutators are

\[
[x^2, x^3] = [x^4, x^5] = i \frac{\pi}{\tan \theta_i - \tan \theta_j}.
\]

(3.4)

As a consequence, the background gauge fields obey a Dirac quantisation condition (which follows from the flux quantisation), and for an appropriate choice the zero modes \( x^2 \) and \( x^4 \) can only take the values

\[
\begin{align*}
x^2_r &= \frac{r}{n_L^{ij}} L , \quad L = p_i p_j R_2 , \\
x^4_{\bar{r}} &= \frac{\bar{r}}{\bar{n}_{L}^{ij}} \bar{L} , \quad \bar{L} = \bar{p}_i \bar{p}_j R_4 ,
\end{align*}
\]

(3.5)

where \( n_L^{ij} \) and \( \bar{n}_{L}^{ij} \) are given in (2.8) and we have assumed that \( p_i \) and \( p_j \) (\( \bar{p}_i \) and \( \bar{p}_j \)) are co-prime. If this is the case, the system has periodicity \( L \) and \( \bar{L} \) along the \( x^2 \)- and \( x^4 \)-directions, respectively. Thus, the degeneracy of any string state is \( n_L^{ij} \bar{n}_{L}^{ij} \), i.e. the string spectrum is found by acting with creation operators on the degenerated ground states

\[
|x^2_r, x^4_{\bar{r}}\rangle , \quad r = 1, \ldots, n_L^{ij} , \quad \bar{r} = 1, \ldots, \bar{n}_{L}^{ij}.
\]

(3.6)
If \( p_i \) and \( p_j \) (\( \bar{p}_i \) and \( \bar{p}_j \)) are not co-prime, i.e. if there is an integer \( l \) (\( \tilde{l} \)) such that \( p_i = lp'_i \) and \( p_j = lp'_j \) (\( \bar{p}_i = l\bar{p}'_i \) and \( \bar{p}_j = l\bar{p}'_j \)) with \( p'_i \) and \( p'_j \) (\( \bar{p}'_i \) and \( \bar{p}'_j \)) co-prime, then the zero modes \( x^2 \) and \( x^4 \) can only take the values (see [13] for details)

\[
x^2_r = \frac{r}{n^L_{ij}} L', \quad L' = lp'_i p'_j R_2 ,
\]

\[
x^4_r = \frac{\tilde{r}}{n^L_{ij}} \tilde{L}', \quad \tilde{L}' = l\tilde{p}'_i \tilde{p}'_j R_4 ,
\]

where

\[
 n^L_{ij} = |p'_j q_i - p'_i q_j|, \quad \bar{n}^L_{ij} = |\bar{p}'_j \bar{q}_i - \bar{p}'_i \bar{q}_j| .
\]  

(3.8)

In this case there are \( n^L_{ij} \bar{n}^L_{ij} \) Landau levels but each one is itself \( \tilde{l}\tilde{l} \) times degenerated. Thus, the ground states

\[
 |x^2_{r,s}, x^4_{\tilde{r},\tilde{s}}⟩, \quad r = 1, ..., n^L_{ij}, \quad s = 1, ..., l , \quad \tilde{r} = 1, ..., \bar{n}^L_{ij}, \quad \tilde{s} = 1, ..., \bar{l} ,
\]  

(3.9)

again have degeneracy \( n^L_{ij} \bar{n}^L_{ij} \).

In order to find the massless modes associated with the previous open strings we have to consider the worldsheet fermionic fields. The spectrum is essentially similar to the one presented in [24] as we are considering the T-dual configuration. We shall use the notation of [13] for the worldsheet fields mode expansion. Consider first the NS sector of the theory. The zero energy is \( E_0 = -\frac{1}{2} + \epsilon \), where \( \epsilon \) was defined in (3.2). The spectrum of the low-lying bosonic states is given by

\[
 \Psi^{k+}_{\frac{1}{2}+\epsilon} \left( a^1_\epsilon \right)^{m_1} \left( a^2_\epsilon \right)^{m_2} |x^2_r, x^4_{\tilde{r}}⟩ , \quad \alpha' M^2 = \epsilon (m_1 + m_2) ,
\]  

(3.10)

\[
 \Psi^{k+}_{\frac{1}{2}+\epsilon} \left( a^1_\epsilon \right)^{m_1} \left( a^2_\epsilon \right)^{m_2} |x^2_r, x^4_{\tilde{r}}⟩ , \quad \alpha' M^2 = \epsilon (2 + m_1 + m_2) ,
\]  

(3.11)

\[
 \Psi^{m+}_{\frac{1}{2}} \left( a^1_\epsilon \right)^{m_1} \left( a^2_\epsilon \right)^{m_2} |x^2_r, x^4_{\tilde{r}}⟩ , \quad \alpha' M^2 = \epsilon (1 + m_1 + m_2) ,
\]  

(3.12)

where the states \( |x^2_r, x^4_{\tilde{r}}⟩ \) correspond to the Landau levels (3.6) (or (3.9) if that is the case). The real fermionic creation operators \( \Psi^{k+}_{\frac{1}{2}-\epsilon} \) and \( \Psi^{k+}_{\frac{1}{2}+\epsilon} \) for \( k = 1, 2 \) correspond to the two complex fermionic worldsheet fields in the \( T^4 \) directions. The fermionic creation operators \( \Psi^{m+}_{\frac{1}{2}} \) for \( m = 1, ..., 4 \) correspond to the four transverse directions to the D-5-branes. This spectrum should be doubled because strings are oriented. The massless states in the NS sector are obtained by taking
\( m_1 = m_2 = 0 \) in the states (3.10). These contribute with precisely \( 4n^{ij}_L \bar{n}^{ij}_L \) massless bosonic excitations for each pair of D-5-branes of type \( i \) and \( j \).

We now turn to the Ramond sector of the theory. The corresponding massless states form a representation of the 6-dimensional Dirac algebra, transforming as a spinor under \( SO(1,5) \). A basis for the vacuum states is

\[
| s_1, s_2, s_3, x^2_\perp, x^4_\perp \rangle, \quad (3.13)
\]

where \( s_i = \pm \). The GSO projection removes half of the states. The fact that physical states are annihilated by the zero mode of the supersymmetry generator removes a further half of these states. In the frame \( p_0 = p_1 \) the latter condition becomes \( s_1 = + \). The vacuum states may then be taken to be

\[
| +, \pm, \pm, x^2_\perp, x^4_\perp \rangle, \quad (3.14)
\]

Considering both orientations of the strings we end up with \( 4n^{ij}_L \bar{n}^{ij}_L \) massless fermionic excitations for each pair of D-5-branes of type \( i \) and \( j \). The fermionic partners of the massive tower of states (3.10-12) are then given by

\[
\Psi^{1\dagger}_\epsilon \Psi^{2\dagger}_\epsilon \left( a^{1\dagger}_\epsilon \right)^{m_1} \left( a^{2\dagger}_\epsilon \right)^{m_2} | +, \pm, \pm, x^2_\perp, x^4_\perp \rangle, \quad \alpha' M^2 = \epsilon (m_1 + m_2), \quad (3.15)
\]

\[
\Psi^{1k}_\epsilon \Psi^{2k}_\epsilon \left( a^{1\dagger}_\epsilon \right)^{m_1} \left( a^{2\dagger}_\epsilon \right)^{m_2} | +, \pm, \mp, x^2_\perp, x^4_\perp \rangle, \quad \alpha' M^2 = \epsilon (1 + m_1 + m_2), \quad (3.16)
\]

To count the entropy associated with the excited D-brane system representing our black hole we just have to realize that we have a gas of \( 4 \sum_{i<j} N_i N_j n^{ij}_L \bar{n}^{ij}_L \) massless bosonic and fermionic species carrying all together the left-moving momentum \( P = \frac{N}{R_1} \). For \( N \gg \sum_{i<j} N_i N_j n^{ij}_L \bar{n}^{ij}_L \) the corresponding entropy is given by the formula (2.7). If the previous condition does not hold our black hole is represented by a single D-5-brane of each type wrapped \( N \) times around the \( x^1 \)-direction [29]. For \( n = 2 \), i.e. for a system with just two D-5-branes, we have a gas with \( 4n^{12}_L \bar{n}^{12}_L \) bosonic and fermionic species and the momentum carried by these strings is quantised in units of \( (N_1 N_2 R_1)^{-1} \). To carry the left-moving momentum \( P = \frac{N}{R_1} \) the system is at the level \( N' = N_1 N_2 N \) and we obtain the correct result for \( NN_1 N_2 N' \gg n^{12}_L \bar{n}^{12}_L \). For \( n > 2 \), the \( N \) units of momentum along the \( x^1 \)-direction have to be distributed according to

\[
\frac{N}{R} = \sum_{i<j} \left( \frac{1}{N_i N_j R_1} \sum_k k n_k \right), \quad (3.18)
\]
where the last sum is over the $4n_{L}^{ij}n_{L}^{ij}$ bosonic and fermionic species, $k$ is the number of quanta of momentum and $n_{k}$ the corresponding occupation number. We expect the entropy formula (2.7) to remain valid in the limit $NN_{i}N_{j} > n_{L}^{ij}n_{L}^{ij}$.

4 Gauge Theory description

In this section we will study the gauge theory describing the massless modes that live on our D-brane system. We shall consider the case $N_{i} = 1$ for every $i = 1, ..., n$. The generalisation for arbitrary $N_{i}$, either for $N_{i}$ D-5-branes of the type $i$ on top of each other, or for a single D-5-brane of each type wrapping $N_{i}$ times around the $x^{1}$-direction, is straightforward. We will comment on this later.

The D-brane dynamics is determined by the Born-Infeld action [23] and the corresponding, and still not well understood, non-abelian generalisation [30]. We will work in the linear approximation where the action describing a system of D-5-branes is given by the dimensional reduction to 6 dimensions of the $D = 10 \ U(N)$ super Yang Mills action ($N$ is still to be specified). Supersymmetry of our field configuration guarantees that the number of massless particles remains unchanged in both descriptions (although there are corrections in the normalisation of the particles’ quantised momentum [22]). The action for the bosonic sector of $D = 10 \ U(N)$ super Yang Mills reduced to 6-dimensions is (we take $g_{YM} = 1$)

$$S = -\frac{1}{4} \int d^{6}x \ tr \left\{ (G^{\alpha\beta})^{2} + 2(\partial_{\alpha}\phi_{m} + i[B_{\alpha}, \phi_{m}])^{2} - [\phi_{m}, \phi_{n}]^{2} \right\}, \quad (4.1)$$

where

$$G^{\alpha\beta} = \partial_{\alpha}B_{\beta} - \partial_{\beta}B_{\alpha} + i[B_{\alpha}, B_{\beta}], \quad (4.2)$$

with $\alpha, \beta = 0, ..., 5$ and $m, n = 1, ..., 4$. The fields $B_{\alpha}$ and $\phi_{m}$ are in the adjoint representation of $U(N)$, i.e. they take values in the $U(N)$ Lie algebra which we take to be hermitian. The action (4.1) is invariant under the local gauge transformations

$$B_{\alpha} \rightarrow [\Omega]B_{\alpha} \equiv \Omega B_{\alpha} \Omega^{-1} - i\Omega \partial_{\alpha} \Omega^{-1}, \quad (4.3)$$

$$\phi_{m} \rightarrow [\Omega]\phi_{m} \equiv \Omega \phi_{m} \Omega^{-1},$$

where $\Omega \in U(N)$.

Our D-brane system is composed by $n$ D-5-branes, each one with a constant self-dual field strength on the 4-torus in the $x^{\hat{\alpha}}$-directions ($\hat{\alpha} = 2, ..., 5$) and
wrapped in the $x^2$- and $x^4$-directions with winding numbers $p_i$ and $\bar{p}_i$, respectively. In order to construct a $U(N)$ bundle over this $T^4$, the gauge potential $B_\alpha$ and the scalar fields $\phi_m$ have to satisfy the boundary conditions\cite{17,18}

\begin{align}
B_\alpha(x^\beta + L_\beta) &= [\Omega_\beta]B_\alpha(x^\beta) \equiv \Omega_\beta B_\alpha(x^\beta)\Omega^{-1}_\beta - i\Omega_\beta \partial_\alpha \Omega^{-1}_\beta , \\
\phi_m(x^\beta + L_\beta) &= [\Omega_\beta]\phi_m(x^\beta) \equiv \Omega_\beta \phi_m(x^\beta)\Omega^{-1}_\beta ,
\end{align}

(4.4)

where $L_\alpha = 2\pi R_\alpha$. Physically this means that the fields are periodic over $T^4$ up to gauge transformations. The $\Omega_\alpha$'s are called multiple transition functions. Consistency of the boundary conditions (4.4) requires that the $\Omega_\alpha$'s obey the periodicity conditions

$$
\Omega_\alpha(x^\beta + L_\beta)\Omega_\beta(x)\Omega^{-1}_\alpha(x^\beta + L_\alpha) = 1 \equiv \text{center of } U(N) ,
$$

(4.5)

and that $\Omega_\alpha(x)$ is independent of $x^\alpha$. If we perform a gauge transformation $\Omega$ the fields $B_\alpha$ and $\phi_m$ will change according to (4.3) while the $\Omega_\alpha$'s transformation rules are

$$
\Omega_\alpha \rightarrow \Omega(x^\alpha = L_\alpha)\Omega_\alpha \Omega^{-1}(x^\alpha = 0) .
$$

(4.6)

The point now is that we can have a non-trivial bundle with the fields $B_\alpha$ and $\phi_m$ being very simple while the structure of the solution is absorbed in the $\Omega_\alpha$'s \cite{18}.

Several examples of non-trivial bundles of this type have been presented in the literature to explain the gauge theory for D-brane systems carrying flux in the $U(1)$ part of the theory \cite{21,22,31,32} (see also \cite{33} for a case where space in not compactified). The general philosophy is to decompose the condition (4.3) in its $U(1)$ and $SU(N)/Z_N$ parts. The $U(1)$ flux will induce a 't Hooft twist and the old results for $SU(N)/Z_N$ bundles on a torus may be used \cite{17,18,19,20}.

Once we have specified the boundary conditions (4.4) for the $U(N)$ theory describing the system of $n$ D-5-branes, we may study the corresponding spectrum and analyse a supersymmetric branch of the theory with a maximum number of massless particles. We shall consider the simplest case $p_i = \bar{p}_i = 1$ for all $i$, and leave the more general case for an appendix. Let us start by writing some formulae that will be useful in both cases. We will expand the action (4.1) around

\^3\text{Note that } B_\alpha(x^\beta + L_\beta) \text{ is short for } B_\alpha(x^0, x^1, ..., x^\beta + L_\beta, ..., x^4) \text{ and similarly for the other fields.}
a given gauge field background $B^0_\alpha$ solving the classical equations of motion, i.e. we write the gauge potential and corresponding field strength as

$$B_\alpha = B^0_\alpha + A_\alpha,$$

$$G_{\alpha\beta} = G^0_{\alpha\beta} + F_{\alpha\beta},$$

with

$$G^0_{\alpha\beta} = \partial_\alpha B^0_\beta - \partial_\beta B^0_\alpha + i[B^0_\alpha, B^0_\beta],$$

$$F_{\alpha\beta} = D_\alpha A_\beta - D_\beta A_\alpha + i[A_\alpha, A_\beta],$$

where $D_\alpha \equiv \partial_\alpha + i[B^0_\alpha, \cdot]$. In the present case the background field strength is constant, diagonal and only the $x^{\hat{\alpha}}$-components ($\hat{\alpha} = 2, \ldots, 5$) are non-vanishing. After some algebra the following action may be obtained [20]

$$S = \frac{-1}{4} \int d^6x \ tr \left\{ -2A^\alpha D^2 A_\alpha - 4iA^\hat{\alpha} [G^0_{\hat{\alpha}\hat{\beta}}, A^\hat{\beta}] - 2\phi^m D^2 \phi_m ight.$$  
$$+ 2i(D_\alpha A_\beta - D_\beta A_\alpha)[A^\beta, A^\alpha] + 4i\phi^m D_\alpha [\phi_m, A^\alpha]$$

$$- [A_\alpha, A_\beta]^2 - 2[A_\alpha, \phi_m]^2 - [\phi_m, \phi_n]^2 \right\}$$

where the field $A_\alpha$ satisfies the background gauge fixing condition $D_\alpha A^\alpha = 0$.

4.1 $p_i = \bar{p}_i = 1$ case

We have explained in the introduction that the integers $p_i$ and $\bar{p}_i$ are the winding numbers of the type $i$ D-5-brane in the $x^2$- and $x^4$-directions, respectively. Hence, for the case considered in this subsection we have $n$ coincident D-5-branes, each one singly wrapped on $T^4$. The gauge group is $U(n)$ and it is broken to $U(1)^n$ by the background field strength (see eqn. [1.3])

$$G^0_{23} = \frac{2\pi}{L_2 L_3} \text{diag} (q_1, \ldots, q_n),$$

$$G^0_{45} = \frac{2\pi}{L_4 L_5} \text{diag} (\bar{q}_1, \ldots, \bar{q}_n).$$

We shall see that some of the particles coming from this symmetry breaking process turn out to be massless. They are described by torons on the $T^4$ directions and are responsible for the black hole entropy.
Each D-5-brane carries $U(1)$ fluxes $2\pi q_i$ and $2\pi \bar{q}_i$ in the $x^2x^3$ and $x^4x^5$ 2-torus, respectively. These will induce a non-trivial twist in the $SU(n)/Z_n$ part of the theory. Our bundle is defined by the following multiple transition functions

$$\Omega_{\hat{\alpha}} = \exp \left( -\pi i \sum_i n^i_{\hat{\alpha} \hat{\beta}} x^\hat{\beta} T_i \right), \quad (4.11)$$

where $(T_i)_{ab} = \delta_{ia}\delta_{ib}$ is the $n \times n$ matrix in the $U(n)$ Lie algebra whose only non-vanishing element is the $i$-th diagonal entry and

$$n^i_{\hat{\alpha} \hat{\beta}} = \begin{pmatrix} 0 & q_i & 0 & 0 \\ -q_i & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{q}_i \\ 0 & 0 & -\bar{q}_i & 0 \end{pmatrix} \quad (4.12)$$

is the twist tensor due to the $i$-th D-5-brane. The periodicity conditions (4.5) hold because $q_i$ and $\bar{q}_i$ are integers.

The boundary conditions for the gauge potential $B_\alpha$ are now seen to be

$$B_\alpha(x^\hat{\alpha} + L_\hat{\beta}) = \Omega_{\hat{\beta}} B_\alpha(x^\hat{\beta}) \Omega_{\hat{\beta}}^{-1} + \frac{\pi}{L_{\hat{\alpha}}} \sum_i n^i_{\hat{\beta} \hat{\alpha}} T_i, \quad (4.13)$$

$$B_\sigma(x^\hat{\alpha} + L_\hat{\beta}) = \Omega_{\hat{\beta}} B_\sigma(x^\hat{\beta}) \Omega_{\hat{\beta}}^{-1},$$

where $\sigma = 0, 1$ and $\Omega_{\hat{\beta}}$ is given by (4.11). These boundary conditions are solved by the background field

$$B^0_\alpha = -\pi \sum_i n^i_{\hat{\alpha} \hat{\beta}} x^\hat{\beta} T_i, \quad (4.14)$$

with field strength

$$G^0_{\hat{\alpha} \hat{\beta}} = 2\pi \sum_i \frac{n^i_{\hat{\alpha} \hat{\beta}}}{L_{\hat{\alpha}} L_{\hat{\beta}}} T_i, \quad (4.15)$$

which is, as expected, the same as in equation (4.10). The fluctuating fields $A_\alpha$ will now obey the same boundary conditions as the fields $\phi_m$ in (4.4).

We have found the background gauge field for our theory, i.e. our vacuum state, and the boundary conditions on the quantum fluctuations around this background. The next step is to expand the fields in the $U(n)$ Lie algebra:

$$A_\alpha = \sum_i a^i_\alpha T_i + \sum_{ij} b^{ij}_\alpha e_{ij}, \quad (4.16)$$

$$\phi_m = \sum_i c^i_m T_i + \sum_{ij} d^{ij}_m e_{ij},$$
where as before $(T_{i})_{ab} = \delta_{ia}\delta_{ib}$ and $(e_{ij})_{ab} = \delta_{ia}\delta_{jb}$. The fields are taken to be hermitian, therefore $a^{i}_{\alpha}$ and $c^{i}_{m}$ are real while $b^{ij}_{\alpha} = (b^{ij}_{\alpha})^{*}$ and $d^{ij}_{m} = (d^{ij}_{m})^{*}$. Expanding the action (4.9) in terms of these field components and keeping only the quadratic terms we obtain

\[
S = -\frac{1}{2} \int d^{6}x \left\{ \sum_{i} \left( a^{i}_{\alpha} M_{0} a^{i}_{\alpha} + c^{i}_{m} M_{0} c^{i}_{m} \right) + 2 \sum_{i<j} \left( (b^{ij}_{\alpha})^{*} M_{ij} b^{ij}_{\alpha} \right) \\
+ (b^{ij}_{\alpha})^{*} \left[ \delta_{\dot{\alpha}\dot{\beta}} M_{ij} - 4\pi J^{ij}_{\dot{\alpha}\dot{\beta}} \right] b^{ij}_{\beta} + (d^{ij}_{m})^{*} M_{ij} d^{ij}_{m} \right) + O(3) \right\} ,
\]

where \(\sigma = 0,1\) and the operators $M_{0}$ and $M_{ij}$ are given by

\[
M_{0} = \left( \frac{1}{i} \partial_{\alpha} \right)^{2} = \partial_{0}^{2} - \partial_{1}^{2} - (\partial_{\dot{\alpha}})^{2} ,
\]

\[
M_{ij} = \partial_{0}^{2} - \partial_{1}^{2} + \left( \frac{1}{i} \partial_{\alpha} - \pi J^{ij}_{\dot{\alpha}\dot{\beta}} x^{\dot{\beta}} \right)^{2} ,
\]

with

\[
J^{ij}_{\dot{\alpha}\dot{\beta}} = \frac{n^{i}_{\dot{\alpha}\dot{\beta}} - n^{j}_{\dot{\alpha}\dot{\beta}}}{L_{\dot{\alpha}} L_{\dot{\beta}}} \equiv \frac{n^{ij}_{\dot{\alpha}\dot{\beta}}}{L_{\dot{\alpha}} L_{\dot{\beta}}}
\]

\[
= \begin{pmatrix}
0 & \frac{q_{-q_{1}}}{L_{2} L_{3}} & 0 & 0 \\
\frac{q_{-q_{1}}}{L_{2} L_{3}} & 0 & 0 & 0 \\
0 & 0 & \frac{q_{-q_{2}}}{L_{4} L_{5}} & 0 \\
0 & 0 & \frac{q_{-q_{2}}}{L_{4} L_{5}} & 0
\end{pmatrix} \equiv \begin{pmatrix}
0 & f^{ij} & 0 & 0 \\
-f^{ij} & 0 & 0 & 0 \\
0 & 0 & 0 & f^{ij} \\
0 & 0 & -f^{ij} & 0
\end{pmatrix}
\]

a self-dual tensor on $T^{4}$. The boundary conditions on the fields $A_{\alpha}$ and $\phi_{m}$ now read

\[
a^{i}_{\alpha}(x^{\dot{\beta}} + L_{\dot{\beta}}) = a^{i}_{\alpha}(x^{\dot{\beta}}) ,
\]

\[
b^{ij}_{\alpha}(x^{\dot{\beta}} + L_{\dot{\beta}}) = \exp \left( -\pi i n^{ij}_{\beta\dot{\gamma}} x^{\dot{\gamma}} L_{\dot{\gamma}} \right) b^{ij}_{\alpha}(x^{\dot{\beta}}) ,
\]

\[
c^{i}_{m}(x^{\dot{\beta}} + L_{\dot{\beta}}) = c^{i}_{m}(x^{\dot{\beta}}) ,
\]

\[
d^{ij}_{m}(x^{\dot{\beta}} + L_{\dot{\beta}}) = \exp \left( -\pi i n^{ij}_{\beta\dot{\gamma}} x^{\dot{\gamma}} L_{\dot{\gamma}} \right) d^{ij}_{m}(x^{\dot{\beta}}) .
\]
Having found the boundary conditions and quadratic operators for all the field components we can proceed by identifying the classical configurations associated with each of these fields. In other words, we will solve the classical field equations neglecting interaction terms. We shall then choose an appropriate supersymmetric branch of the theory with a maximum number of massless fluctuations around the vacuum. To choose this branch of the theory we will consider the interaction terms in the Lagrangian. In subsection 4.1.1 below we shall study the spectrum of the $M_{ij}$ operator acting on $T^4$. These results were presented in [20] and are reviewed here for the sake of completeness. In subsection 4.1.2 we shall describe the classical configurations associated with each field and the corresponding particle excitations. These results will allow us to choose a supersymmetric branch of the theory and to compute the entropy of the corresponding excited D-brane system.

4.1.1 $M_{ij}$ operator on $T^4$

In order to find the spectrum of the $M_{ij}$ operator on $T^4$ for given $i$ and $j$, we define the complex coordinates

$$z = (z_1, z_2) = \frac{1}{\sqrt{2}} (x_2 - ix_3, x_4 - ix_5) ,$$

(4.21)

In these coordinates the worldvolume gauge field is written as

$$A_z = (A_{z_1}, A_{z_2}) = \frac{1}{\sqrt{2}} (A_2 + iA_3, A_4 + iA_5) .$$

(4.22)

Without loss of generality we assume that $f^{ij} \geq 0$ (see eqn. (4.19)) and define the positive definite hermitian form

$$H^{ij}(z, w) = 2 (z_1 f^{ij} w_1 + z_2 f^{ij} w_2) = w^\dagger h^{ij} z ,$$

(4.23)

where $h^{ij} = 2 \text{diag}(f^{ij}, f^{ij})$. We define the operators

$$a^{ij} = (a^{ij}_1, a^{ij}_2) , \quad a^{ij}_k = \frac{1}{i} (\partial_{z_k} + \pi f^{ij} z_k) ,$$

(4.24)

$$(a^{ij})^\dagger = \left( (a^{ij}_1)^\dagger, (a^{ij}_2)^\dagger \right) , \quad (a^{ij}_k)^\dagger = \frac{1}{i} (\partial_{\bar{z}_k} - \pi f^{ij} \bar{z}_k) ,$$

where $k = 1, 2$. The part of the $M_{ij}$ operator in (4.18) acting on $T^4$ may now be written as

$$M_{ij} = \left( \frac{1}{i} \partial_{\bar{a}} - \pi F_{\tilde{a}\tilde{\beta}} a^{ij}_\beta \right)^2 = \sum_k \{ a^{ij}_k, (a^{ij}_k)^\dagger \} ,$$

(4.25)
and we have
\[ [a_{ij}^*, (a_{ij}^*)^\dagger] = 2\pi f^{ij}\delta_{kl} \, . \] (4.26)
Thus, the operators \( a^{ij} \) and \((a^{ij})^\dagger\) may be seen as annihilation and creation operators, respectively. The ground state is defined by
\[ a^{ij}|0\rangle = 0 \, , \] (4.27)
with eigenvalue \( \lambda_0 = 4\pi f^{ij} \). The excited states are then
\[ |m_1m_2\rangle = \frac{((a_{ij}^*)^m_1((a_{ij}^*)^m_2)^{m_2}}{\sqrt{m_1!m_2!}}|0\rangle \, , \] (4.28)
with
\[ \lambda_{m_1m_2} = 4\pi f^{ij}(1 + m_1 + m_2) \, . \] (4.29)
The ground state wave function \( \chi_0(z, \bar{z}) = \langle z, \bar{z}|0\rangle \) satisfies
\[ a^{ij}\chi_0(z, \bar{z}) = 0 \Rightarrow \left(\partial_{z_k} + \pi f^{ij}z_k\right)\chi_0 = 0 , \quad k = 1, 2 \, , \] (4.30)
with \( \chi_0 \) satisfying the boundary conditions specified below. Writing
\[ \chi_0(z, \bar{z}) = \exp\left(-\frac{\pi}{2}H(z, \bar{z})\right)f(z, \bar{z}) \, , \] (4.31)
equation (4.30) becomes \( f(z, \bar{z}) = f(z) \), i.e. \( f \) is holomorphic. The wave functions \( \chi_{m_1m_2}(z, \bar{z}) \) for the excited states may be obtained by acting with the creation operators on the ground state wave function.

The operators \( M_{ij} \) in the action (4.17) act on the fields \( b_{\alpha}^{ij} \) and \( d_{\alpha}^{ij} \), both satisfying the same boundary conditions. The dependence of these fields on \( T^4 \) will be determined by the ground state wave function \( \chi_0(z, \bar{z}) \) and the excited states wave functions \( \chi_{m_1m_2}(z, \bar{z}) \). Hence, the function \( \chi_0(z, \bar{z}) \) obeys the following boundary conditions
\[ \chi_0(x^\beta + L_{\beta}) = \exp\left(-\pi in^{ij}_{\beta\gamma}\frac{x^\gamma}{L_{\gamma}}\right)\chi_0(x^\beta) \, . \] (4.32)
Note that for some function \( \chi_0(z, \bar{z}) \) obeying these boundary conditions and using equation (4.28) it follows that the functions \( \chi_{m_1m_2}(z, \bar{z}) \) also obey the same boundary conditions. We now define the vector
\[ q = (q_1, q_2) , \quad q_1 = \frac{1}{\sqrt{2}}(m_2L_2 - im_3L_3) \, , \] (4.33)
\[ q_2 = \frac{1}{\sqrt{2}}(m_4L_4 - im_5L_5) \, , \]
taking values on the $T^4$ lattice. The condition (4.32) reads

$$f(z + q) = \alpha(q) \exp \left( \frac{\pi}{2} H(q, q) + \pi H(z, q) \right) f(z) ,$$  

where

$$\alpha(q) = \exp \left( \pi i \sum_{\hat{\alpha} < \hat{\beta}} m_{\hat{\alpha}} n_{\hat{\alpha} \hat{\beta}} m_{\hat{\beta}} \right).$$  

(4.35)

The holomorphic functions $f$ satisfying (4.34) are called in the mathematical literature $\Theta$-functions of type $(H,\alpha)$ [15, 16]. They form a complex linear space $L(H,\alpha)$ of dimension

$$|Pf(n^{ij})| = \frac{1}{8} \epsilon_{\hat{\alpha}\hat{\beta}\hat{\delta}} n^{ij}_{\hat{\alpha}\hat{\beta}} n^{ij}_{\hat{\alpha}\hat{\delta}} = |(q_i - q_j)(\bar{q}_i - \bar{q}_j)| = n^{ij}_L \bar{n}^{ij}_L .$$  

(4.36)

Thus, there are $n^{ij}_L \bar{n}^{ij}_L$ orthogonal ground state wave functions that we write as

$$\chi^r_0(z,\bar{z}) , \quad r = 1, \ldots, n^{ij}_L \bar{n}^{ij}_L .$$  

(4.37)

This is the origin of the Landau degeneracy in the gauge theory description of our D-brane system. For a given $i,j$ pair each of these functions will be associated with a massless toronic excitation of the D-brane system, giving exactly the entropy formula (2.7).

### 4.1.2 Particle spectrum

In this subsection we shall describe the particle excitations associated with each field in the expansion (4.16). In other words, we will consider the free Lagrangian in (4.17). Consider first the gauge fields $a^i_\alpha$ associated with the remaining $U(1)^n$ gauge freedom. The classical equations of motion are solved by the zero modes of the operator $M_0$ in (4.17)

$$a^i_\alpha = \sum_p e_\alpha(p) \exp \left( ip_\beta x^{\beta} \right) , \quad p^2 = 0 .$$  

(4.38)

The gauge condition $D_\alpha A^\alpha = 0$ becomes for these field components the usual Lorentz gauge condition $\partial_\alpha a^i_\alpha = 0$. We can further impose the Coulomb gauge condition $A_0 = 0$. Thus, the fields $a^i_\alpha$ describe massless vector gauge particles in 6 dimensions corresponding to $4n$ degrees of freedom. The analysis for the $c^i_m$ scalar fields is similar. These fields describe $4n$ massless scalar particles.
We now turn to the fields corresponding to the non-diagonal elements of the Lie algebra. Consider first the $d_{ij}^m$ components of the scalar fields $\phi_m$. The classical equations for these fields are solved by

$$d_{ij}^m = \sum_{m_1m_2} \chi_{m_1m_2}^r \left( \mathcal{D}_{ij}^m \right)^r_{m_1m_2} (x^\sigma),$$  \hspace{1cm} (4.39)

where $\sigma = 0, 1$ and $(\mathcal{D}_{ij}^m)_{m_1m_2} (x^\sigma)$ is a complex conformal field satisfying the equation of motion

$$(\Box - \lambda_{m_1m_2}) \left( \mathcal{D}_{ij}^m \right)_{m_1m_2} (x^\sigma) = 0,$$  \hspace{1cm} (4.40)

i.e.

$$(\mathcal{D}_{ij}^m)^r_{m_1m_2} = \sum_{p_\sigma} \left( e_{ij}^m \right)^r_{m_1m_2} (p_\sigma) \exp (ip_\sigma x^\sigma), \quad p_\sigma^2 = -\lambda_{m_1m_2}. \hspace{1cm} (4.41)$$

Thus, for each pair $m_1, m_2$ we have 8 massive particles with mass given by $m^2 = \lambda_{m_1m_2}$ and described by a two dimensional field theory. These fields correctly reproduce the string excitations (3.12) if we make the identification [22]

$$\epsilon_{\alpha', \alpha} = \theta_i - \theta_j \pi_{\alpha', \alpha} \equiv \frac{\tan \theta_i - \tan \theta_j}{\pi \alpha'} = 4\pi f^{ij}. \hspace{1cm} (4.42)$$

The role that the Born-Infeld description of the D-brane dynamics may have in solving this discrepancy was discussed in [22].

We proceed by considering the $b_{ijz}^k$ components of the gauge field $A_\alpha$. Start by writing the quadratic operator acting on the $b_{ijz}^k$ fields in terms of the $z$ coordinates defined in (4.21-22) [20]

$$S \sim - \int d^6x \sum_{i<j} \left\{ (b_{ijz_k}^k)^* \left[ M_{ij} - 4\pi f^{ij} \right] b_{ijz_k}^k + (b_{ijz_k}^k)^* \left[ M_{ij} + 4\pi f^{ij} \right] b_{ijz_k}^k \right\}. \hspace{1cm} (4.43)$$

Hence, the spectrum of the operator $(\delta_{\beta\alpha} M_{ij} - 4\pi i J_{\beta\alpha}^{ij})$ in (4.17) is

$$\lambda_{m_1m_2}^- = \lambda_{m_1m_2} - 4\pi f^{ij}, \quad b_{ijz_k}^k = \chi_{m_1m_2}^r \quad (k = 1, 2), \hspace{1cm} (4.44)$$

$$\lambda_{m_1m_2}^+ = \lambda_{m_1m_2} + 4\pi f^{ij}, \quad b_{ijz_k}^k = \chi_{m_1m_2}^r \quad (k = 1, 2).$$

For simplicity we consider the cases $b_{ijz_k}^k = 0$ or $b_{ijz_k}^k = 0$ separately. In the former case the classical equations of motion for the $b_{ijz_k}^k$ fields are solved by

$$b_{ijz_k}^k = \sum_{m_1m_2} \chi_{m_1m_2}^r \left( B_{ijz_k}^r \right)^{m_1m_2} (x^\sigma), \quad (k = 1, 2), \hspace{1cm} (4.45)$$
with
\[ (\mathcal{B}^{ij}_{z_k})_{m_1m_2}^r = \sum_{p_\sigma} (e^{ij}_{z_k})_{m_1m_2}^r (p_\sigma) \exp (ip_\sigma x^\sigma), \quad p_\sigma^2 = -\lambda_{m_1m_2}^- . \] (4.46)

In the gauge \( A_0 = 0 \), the background gauge condition \( D_i A_i = 0 \) becomes
\[ \partial_1 b^{ij}_1 + ia^{ij}_k b^{ij}_k = 0 . \] (4.47)

The fields \( b^{ij}_1 \) obey the same equations of motions as the scalar fields \( a^{ij}_m \). Hence
\[ b^{ij}_1 = \sum_{rm_1m_2} \chi^r_{m_1m_2} (\mathcal{B}^{ij}_{1})_{m_1m_2}^r (x^\sigma) , \] (4.48)

with
\[ (\mathcal{B}^{ij}_{1})_{m_1m_2}^r = \sum_{p_\sigma} (e^{ij}_{1})_{m_1m_2}^r (p_\sigma) \exp (ip_\sigma x^\sigma), \quad p_\sigma^2 = -\lambda_{m_1m_2}^- . \] (4.49)

The gauge condition (4.47) becomes
\[ p_1 (e^{ij}_{1})_{m_1m_2}^r + 2\pi f^{ij} \left[ \sqrt{m_1} + 1 (e^{ij}_{z_1})_{m_1+1,m_2}^r + \sqrt{m_2} + 1 (e^{ij}_{z_2})_{m_1,m_2+1}^r \right] = 0 , \] (4.50)

completely determining the fields \( b^{ij}_1 \). Note that since \( \lambda_{m_1m_2} = \lambda^{-1}_{m_1+1,m_2} = \lambda^{-1}_{m_1,m_2+1} \) a given excitation \( b^{ij}_{z_k} \) induces a \( b^{ij}_1 \) field with the correct mass. Thus, the complex conformal fields \( \mathcal{B}^{ij}_{z_k} \) describe for each pair \( m_1m_2, 4n_L^{ij} \bar{n}_L^{ij} \) particles with mass given by \( m^2 = \lambda_{m_1m_2}^- \). With the identification (4.42) we correctly reproduce the spectrum associated with the string states (3.10). Note that if all excitations are in their vacuum state but the \( m_1 = m_2 = 0 \) toron then \( b^{ij}_1 = 0 \) and the field configuration (4.43) describes \( 4n_L^{ij} \bar{n}_L^{ij} \) massless torons. These are the particle-like excitations responsible for the bosonic contribution to the entropy formula as they will allow us to define a supersymmetric branch of the theory.

In the case \( b^{ij}_{z_k} = 0 \), the complex conformal fields \( \mathcal{B}^{ij}_{z_k} \) associated with the field \( b^{ij}_{z_k} \) are given by
\[ (\mathcal{B}^{ij}_{z_k})_{m_1m_2}^r = \sum_{p_\sigma} (e^{ij}_{z_k})_{m_1m_2}^r (p_\sigma) \exp (ip_\sigma x^\sigma), \quad p_\sigma^2 = -\lambda^+_{m_1m_2} . \] (4.51)

The gauge condition \( D_i A_i = 0 \) becomes
\[ \partial_1 b^{ij}_1 + i (a^{ij}_k)^+ b^{ij}_k = 0 , \] (4.52)

and the fields in (4.44) are determined by
\[ p_1 (e^{ij}_{1})_{m_1m_2}^r + \sqrt{m_1} (e^{ij}_{z_1})_{m_1+1,m_2}^r + \sqrt{m_2} (e^{ij}_{z_2})_{m_1,m_2+1}^r = 0 , \] (4.53)

These fields correctly reproduce the string states (3.11).
4.1.3 Supersymmetric branch

Now that we have found the different particle excitations associated with the fields leaving on the D-branes we can choose a supersymmetric configuration describing the excited D-brane system. We will consider the case when we have $N_i$ D-5-branes of a given type $i$ and comment later on the case when we have a single brane of such type wrapped in the $x^1$-direction with winding number $N_i$. We have to realize that the fields $a^i_\alpha$ and $c^m_i$ are in the adjoint representation of $U(N_i)$ and the fields $b^{ij}_\alpha$ and $d^{ij}_m$ in the fundamental representation of $U(N_i) \times U(\tilde{N}_j)$. In other words, the labels $i$ and $ij$ now carry some group theory indices. In order to find a supersymmetric branch of the theory with a maximum number of massless particles consider first the commutator term $[A_\alpha, \phi_m]^2$ in the action (4.9). The fields $c^m_i$ contribute with the following mass term to the $b^{ij}_\alpha$ fields (we are being schematic here and dropping $U(N_i)$ group theory indices)

$$S \sim -\int d^6x \sum_{i<j} \left( b^{ij}_\alpha \right)^\dagger \left( c^i - c^j \right)^2 b^{ij}_\alpha .$$

(4.54)

Physically this means that if the scalars $c^m_i$ are excited the D-5-branes will oscillate in the transverse space directions, giving a mass term to the strings stretching between different branes and therefore to the $b^{ij}_\alpha$ fields. Inversely, if there are massless toronic excitations due to the $b^{ij}_\alpha$ fields, there will be mass terms for the massless scalars and the branes become bounded, i.e. we have a bound state [12]. Our excited D-brane system is described by the latter picture.

Next consider the scalar fields $d^{ij}_m$. Since these are massive fields they are not relevant for the entropy counting. They break all the supersymmetry. Thus, all the components of the scalar fields $\phi_m$ are in their vacuum state. The remaining massless excitations correspond to the massless gauge bosons $a^i_\alpha$ and the massless torons $b^{ij}_\alpha$.

Having ruled out from our counting the scalar fields $\phi_m$, we have to impose supersymmetry of our field configuration. This means that not all the degrees of freedom in the $a^i_\alpha$ and $b^{ij}_\alpha$ fields will be independent. We start by imposing the conditions for our fields to carry left-moving momentum in the $x^1$-direction

$$b^{ij}_{zk} = \sum_r \lambda^r_\alpha \left( B^{ij}_{zk} \right)_0 (x^-) , \quad b^{ij}_{0k} = b^{ij}_0 = 0 ,$$

$$a^i_\alpha = a^i_\alpha (x^\beta, x^-) , \quad a^i_0 = a^i_1 = 0 ,$$

(4.55)
where \( x^- = \frac{1}{\sqrt{2}}(x^0 - x^1) \). Note that we are working in the gauge \( A_0 = 0 \), and further imposed the condition \( a_1^i = 0 \). With the fields as in (4.55) we have (recall that \( G_{\alpha\beta} = G^0_{\alpha\beta} + F_{\alpha\beta} \))

\[
G_{01} = F_{01} = 0 \quad , \quad G_{0\alpha} = F_{0\alpha} = -F_{1\dot{\alpha}} = -G_{1\dot{\alpha}} .
\]

The supersymmetry variation of the gaugino field (in 10D spinor language) is given by

\[
\delta \Psi = -\frac{1}{4} G_{\alpha\beta} \Gamma^{\alpha\beta} \epsilon - \frac{1}{2} \left( \partial_\alpha \phi_m + i [B_\alpha, \phi_m] \right) \Gamma^{\alpha m} \epsilon - \frac{i}{4} \left[ \phi_m, \phi_n \right] \Gamma^{mn} \epsilon ,
\]

where the last two terms vanish for our field configuration. The contribution of the field strength components in (4.56) to \( \delta \Psi \) vanishes if \( (\Gamma^0 - \Gamma^1) \epsilon = 0 \). This condition breaks 1/2 of the worldvolume supersymmetry. To preserve 1/4 of the supersymmetry, note that the background field strength is self-dual on \( T^4 \). Thus, by requiring self-duality of the field strength \( F_{\dot{\alpha}\dot{\beta}} \) associated with the fluctuating fields, we can cancel the contribution of \( G_{\dot{\alpha}\dot{\beta}} \) to \( \delta \Psi \). This follows by imposing the condition \( \Gamma^{2345} \epsilon = \epsilon \) on the spinor \( \epsilon \). We remark that in order to vanish the contribution of \( G^0_{\dot{\alpha}\dot{\beta}} \) to \( \delta \Psi \) we need this condition on the spinor \( \epsilon \) because \( G^0_{\dot{\alpha}\dot{\beta}} \) is not in the \( U(1) \) center of the group and therefore the non-linear realization of the worldvolume supersymmetry transformations can not be used [26]. Thus, we are bound to conclude that a different condition on \( F_{\dot{\alpha}\dot{\beta}} \) would most likely break further supersymmetry which is not desirable.

We were not able to solve the self-duality condition on \( F_{\dot{\alpha}\dot{\beta}} \) exactly. However, we solved this condition to second order in the fields. Starting with the free theory, i.e. neglecting interactions, the field configuration describing our excited D-brane system is

\[
b^{ij}_{zk} = \sum_r \chi^r_0 \left( B^{ij}_{zk} \right)_0 (x^-) , \quad b^{ij}_{0z} = b^{ij}_{1z} = 0 ,
\]

\[
a^i_{\dot{\alpha}} = 0 , \quad a^i_0 = a^i_1 = 0 .
\]

Thus the fields \( a^i_\alpha \) are in their vacuum state. The field strength \( F_{\alpha\beta} \) may be written as

\[
F_{\alpha\beta} = D_\alpha A_\beta - D_\beta A_\alpha + i [A_\alpha, A_\beta] \equiv F^{(F)}_{\alpha\beta} + i [A_\alpha, A_\beta] .
\]

\(^4\text{Alternatively we may write } a^i_\alpha = a^i_\alpha(x^-), \text{ i.e. we neglect the massive Kaluža-Klein modes arising from the compactification of our theory to } 1 + 1 \text{ dimensions, and the condition } a^i_1 = 0 \text{ follows from the Lorentz gauge condition } \partial_\alpha a^i_\alpha = 0 \text{ in the Coulomb gauge } a^i_0 = 0.\)
Equation (4.30) may be used to show that the free toronic excitations are self-dual on $T^4$, i.e. $F^{(F)}_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} F^{(F)}_{\gamma\delta}$. Note that we could have $a^i_{\alpha} = a^i_{\bar{\alpha}}(x^-)$ and the free theory would still be self-dual. We will comment on this later. In order to have the self-duality condition for the full $F^{\hat{\alpha}\hat{\beta}}$ field, i.e. considering the commutator term in (4.59), we impose the condition
\[
A^{(2)}_\beta = (D^2)^{-1} D_\bar{\alpha} T_{\bar{\alpha} \beta},
\]
where $T_{\bar{\alpha} \beta}$ is quadratic in the free fields in (4.58) and it is an antiself-dual tensor on $T^4$. In components
\[
T^{ij}_{\bar{\alpha} \beta} = -i \sum_{k \neq i,j} \left[ (b_{ik}^{a\hat{\alpha}} b_{kj}^{\hat{\beta}} - b_{jk}^{a\hat{\beta}} b_{ki}^{\hat{\alpha}}) - \frac{1}{2} \epsilon_{\bar{\alpha}\beta\gamma\delta} \left( b_{ik}^{\hat{\alpha}} b_{\gamma}^{\delta} - b_{\delta}^{\hat{\alpha}} b_{\gamma}^{\delta} \right) \right].
\]

Notice that the $i, j, k$ labels contain group theory indices. The $T^{ij}_{\bar{\alpha} \beta}$ components are in the fundamental representation of $U(N_i) \times U(N_j)$ and give corrections to the fields $b^{i\bar{a}}_{\alpha}$. The $T^{i\bar{a}}_{\hat{a} \bar{\alpha}} \equiv T^{i\bar{a}}_{\hat{a} \bar{\alpha}}$ components are in the adjoint representation of $U(N_i)$ and give corrections to the fields $a^{i\bar{a}}_{a\bar{\alpha}}$. We may proceed iteratively to guarantee self-duality to any order in the free fields in (4.58). It is probably not surprising that we were not able to show exactly that our field configuration preserves some supersymmetry. The problem is that the super Yang Mills action does not reproduce the correct interactions between the fields on the D-branes.

If we had a well defined non-Abelian Born-Infeld action, it could be that the corresponding torons would be modified (this is in fact expected in order to have a perfect agreement with the string theory spectrum [23]). Also, both actions above are an approximation to slow varying fields, i.e. the derivatives of the field strength are neglected. These facts may be preventing an exact solution of the problem.

In conclusion, the field configuration (4.58) is described quantum mechanically by the left moving sector of a conformal field theory with $4 \sum_{i<j} N_i N_j n^{ij}_L n^{ij}_L$ massless bosonic particles. Supersymmetry requires that we have the same number of fermionic particles. Thus, we have correctly reproduced the number of degrees of freedom required to explain the entropy formula. Two final remarks are in order: Firstly, as noted in the previous paragraph we could have excited the $a^i_{\alpha}$ fields in the free theory. In the interacting theory we can still excite the diagonal elements of these fields (more generally a maximal set of $N_i$ commuting fields). The self-duality condition on the field strength could still be satisfied by an appropriate redefinition of $T^{ij}_{\bar{\alpha} \beta}$ in (4.61). However, we could not excite
more than these $4N_i$ particles. The corresponding contribution to the entropy is subleading and the result remains the same ($N_iN_j \gg N_k$). Secondly, the same result as in the string theory description follows if we have a single D-5-brane of each type wrapped $N_i$ times along the $x^1$-direction. In this case we would have $4\sum_{i<j} n_{L_i}^{ij} n_{L_j}^{ij}$ bosonic and fermionic torons with momentum quantised in units of $(N_iN_jR_1)^{-1}$ for each $i, j$ pair.

5 Conclusion

In this paper we have studied a 5-dimensional black hole associated with an excited D-brane system described be several coincident D-5-branes with a self-dual field strength on a 4-torus and carrying left-moving momentum along a common string. We have analysed the spectrum of this system either from the string theory perspective and the gauge theory perspective. Both descriptions give the same answer for the excited D-brane system entropy matching the Bekenstein-Hawking entropy formula. These results arise from rather different mathematical approaches. In the string theory picture the entropy area law is reproduced by counting the Landau levels for strings stretching between different branes, while in the gauge theory picture by counting the number of massless torons. In the latter description the number of torons arises from highly non-trivial results on $\Theta$-functions on $T^4$. As such, our work provides another example of the fascinating interplay between string theory and gauge theory. Also, it is quite interesting that old results which constituted an attempt to understand quark confinement and non-perturbative aspects of QCD, are now placed in the context of black hole physics.

There are several unresolved problems that should fit in the general scheme of things. Firstly, once the non-abelian Born-Infeld action is known one may expect to find the corresponding BI-torons. These should yield a perfect agreement with the string theory spectrum [22]. It may also prove to be useful to show exactly the supersymmetry of our system. Secondly, although we have not solved the number theory problem (3.18) we expect it to reproduce the correct entropy formula. Finally, another problem is to generalise the results of these paper to the near-extreme black hole case. This has been done in [13] for a different case.
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Appendix: $p_i \neq 1 \neq \bar{p}_i$ case

In this appendix we shall describe the modifications required to treat the general case when the winding numbers of the type $i$ D-5-branes are different from unity. Consider first the case $p_i \neq 1$ but $\bar{p}_i = 1$. A D-brane with winding number $p_i$ in a compact direction is described by a $U(p_i)$ gauge theory broken to $U(1)^{p_i}$ by Wilson lines or non-trivial boundary conditions satisfied by the fields in the compact direction [2, 34]. Since we have $n$ D-5-branes on top of each other we start with $U(\sum_i p_i)$ gauge group broken to $\prod_i U(p_i)$ by the background gauge field. Each D-5-brane wraps around the $x^2$-direction $p_i$ times, breaking each $U(p_i)$ factor to $U(1)^{p_i}$. The general case $\bar{p}_i \neq 1$ may be treated similarly. We would have to consider the group $U(p_i\bar{p}_i)$ for each D-5-brane. Since this case does not bring any new feature we will describe below the simpler case $\bar{p}_i = 1$.

As in subsection 4.1 we start by writing the background field strength

\[ G_{23}^0 = \frac{2\pi}{L_2 L_3} \left( \frac{q_1, \ldots, q_n}{p_1, \ldots, p_n} \right), \]

\[ G_{45}^0 = \frac{2\pi}{L_4 L_5} \left( q_1, \ldots, q_n \right). \]

Each D-5-brane carries a flux $2\pi q_i$ and $2\pi p_i \bar{q}_i$ in the $x^2x^3$ and $x^4x^5$ 2-torus, respectively. The multiple transition functions for a $U(\sum_i p_i)$ bundle of this type are

\[ \Omega_{\tilde{\alpha}} = \text{Diag} \left( \Omega_{\tilde{\alpha}}^{(1)}, \ldots, \Omega_{\tilde{\alpha}}^{(n)} \right), \]

where the $\Omega_{\tilde{\alpha}}^{(i)}$ are the $p_i \times p_i$ matrices [22]

\[ \Omega_2^{(i)} = \Omega_2^{(i)} V_i, \quad \Omega_4^{(i)} = \Omega_4^{(i)}, \]

\[ \Omega_3^{(i)} = \Omega_3^{(i)} U_i^{q_i}, \quad \Omega_5^{(i)} = \Omega_5^{(i)}, \]

with

\[ U_i = \text{diag} \left( e^{2\pi i p_i^{-1}} , \ldots, e^{2\pi i p_i^{-1}} \right), \]
\[ V_i = \begin{pmatrix} 0 & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}, \]  
(A.4)

\[ \Omega_{\alpha}^{(i)} = \exp \left( -\pi i n_{\alpha\beta}^i \frac{x^\beta}{L_\beta} 1_i \right). \]

The twist tensor \( n_{\alpha\beta}^i \) is given by

\[ n_{\alpha\beta}^i = \begin{pmatrix} 0 & \frac{q_i}{p_i} & 0 & 0 \\ -\frac{q_i}{p_i} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{q}_i \\ 0 & 0 & -\bar{q}_i & 0 \end{pmatrix}, \]  
(A.5)

and \( 1_i \) is the \( i \)-dimensional unit matrix. The consistency conditions (4.5) for the \( \Omega_\alpha \)'s may be seen to hold. The boundary conditions (4.4) are now solved by the background field

\[ B_0^\alpha = \text{Diag} \left( B_0^{0(1)}, \ldots, B_0^{0(n)} \right), \]  
(A.6)

with

\[ B_0^{0(i)} = -\pi n_{\alpha\beta}^i \frac{x^\beta}{L_\beta} 1_i. \]  
(A.7)

The corresponding field strength is given by (A.1).

We proceed by expanding the fluctuating fields in the \( U(\sum_i p_i) \) Lie algebra. As before we write

\[ A_\alpha = \sum_i a_\alpha^i T_i + \sum_{ij} b_\alpha^{ij} e_{ij}, \]

\[ \phi_m = \sum_i c_m^i T_i + \sum_{ij} d_m^{ij} e_{ij}, \]  
(A.8)

but now \( T_i \) and \( e_{ij} \) are \( p_i \times p_i \) and \( p_i \times p_j \) matrices, respectively. Thus, the fields \( a_\alpha^i \) and \( c_m^i \) are in the adjoint representation of \( U(p_i) \) and the fields \( b_\alpha^{ij} \) and \( d_m^{ij} \) \( (i < j) \) in the fundamental representation of \( U(p_i) \otimes U(\bar{p}_j) \) (note that \( b_\alpha^{ij} = (b_\alpha^{ji})^\dagger \) and \( d_m^{ij} = (d_m^{ji})^\dagger \)). With the observation that the \( i \) and \( j \) labels contain group

\[ ^5 \text{There is a possible confusion with the notation here. We are considering the case } \bar{p}_i = 1 \text{ for all } i. \text{ Thus, we mean by } U(\bar{p}_j) \text{ the anti-fundamental representation of } U(p_j). \]
theory indices, the result of expanding the action in terms of the field components is the same as in (4.17). The quadratic operators acting on the fields are also as in (4.18), but the self-dual tensor $J^{ij}_{\alpha \beta}$ is now given by

$$J^{ij}_{\alpha \beta} = \frac{n^{ij}_{\alpha \beta} - n^{ij}_{\alpha \bar{\beta}}}{L^2} \equiv \left( \begin{array}{ccc} 0 & 0 & n^{ij}_{\alpha \beta} \\ \frac{p^i p^j L^2 L_3}{p^i p^j L_2 L_3} & 0 & 0 \\ 0 & 0 & \frac{n^{ij}_{\alpha \bar{\beta}}}{L_4 L_5} \end{array} \right) .$$  \hspace{1cm} (A.9)

The boundary conditions on the $a^i_\alpha$ fields are

$$(a^i_\alpha)_{ab}(x^2 + L_2) = (a^i_\alpha)_{a+1,b+1}(x^2) ,$$

$$(a^i_\alpha)_{ab}(x^3 + L_3) = e^{2\pi i q_i a \frac{a + 1}{p_i}} (a^i_\alpha)_{ab}(x^3) ,$$  \hspace{1cm} (A.10)

$$(a^i_\alpha)_{ab}(x^5 + L_\beta) = (a^i_\alpha)_{ab}(x^5) , \quad \beta = 4, 5 ,$$

where $a, b = 1, \ldots, p_i$. The fields $c^i_m$ obey similar boundary conditions. The fields $b^ij_\alpha$ satisfy the following boundary conditions

$$(b^ij_\alpha)_{ab}(x^2 + L_2) = \exp \left( -\pi i n^ij_\bar{\alpha} \frac{x^2}{p^i p^j L_3} \right) (b^ij_\alpha)_{a+1,b+1}(x^2) ,$$

$$(b^ij_\alpha)_{ab}(x^3 + L_3) = \exp \left( -\pi i n^ij_\bar{\alpha} x^2 \frac{2A_{ab}}{p^i p^j L_2} \right) (b^ij_\alpha)_{ab}(x^3) ,$$  \hspace{1cm} (A.11)

$$(b^ij_\alpha)_{ab}(x^5 + L_\beta) = \exp \left( -\pi i n^ij_\bar{\alpha} \frac{x^5}{L_\beta} \right) (b^ij_\alpha)_{ab}(x^5) , \quad \beta = 4, 5 ,$$

where

$$A_{ab} = L_2 \frac{p^i q_i (a - 1) - p^i q_j (b - 1)}{p^i q_i - p^i q_j} ,$$  \hspace{1cm} (A.12)

$a = 1, \ldots, p_i$ and $b = \bar{1}, \ldots, \bar{p}_i$. The fields $d^ij_m$ obey similar boundary conditions.

We proceed by studying the spectrum of the $M_{ij}$ operator on $T^4$ subjected to the boundary conditions (A.11) and the massless particle spectrum of our gauge theory.
The crucial difference between this case and the one presented in the subsection 4.1, is that the boundary conditions (A.11) relate different field components. The analysis of the spectrum of the $M_{ij}$ operator in terms of the creation and annihilation operators goes through as before. The only difference is that the ground state wave function is defined by

$$\chi_0^{a\bar{b}}(z, \bar{z}) = \exp \left(-\frac{\pi}{4} H^{ij}(z - \bar{z}, z_0 + \bar{z}_0) - \frac{\pi}{2} H^{ij}(z + z_0, z + \bar{z}_0)\right) g(z), \quad (A.13)$$

where $z_0 = \frac{1}{\sqrt{2}} (A_{ab}, 0)$ and the $a, \bar{b}$ indices are appropriate for a given field component $(b^{ij}_a)_{ab}$ with $A_{ab}$ given as in (A.12). For such a field component the boundary conditions on the function $(\chi_0^{a\bar{b}})$ (that describes the dependence on $T^4$) are

$$\chi_0^{a\bar{b}}(x^2 + p_ip_j L_2) = \chi_0^{a\bar{b}}(x^2), \quad \chi_0^{a\bar{b}}(x^3 + L_3) = \chi_0^{a\bar{b}}(x^3), \quad \chi_0^{a\bar{b}}(x^{\hat{\beta}} + L_{\hat{\beta}}) = \chi_0^{a\bar{b}}(x^{\hat{\beta}}), \quad \hat{\beta} = 4, 5. \quad (A.14)$$

Thus, apart from the $A_{ab}$ shift in the $x^2$-direction the problem is similar to the previous case. We just have to realize that we have a periodicity $p_ip_j L_2$ along the $x^2$-direction instead of $L_2$. We are assuming for now that $p_i$ and $p_j$ are co-prime. Defining the vector

$$q = (q_1, q_2), \quad q_1 = \frac{1}{\sqrt{2}} (m_2 p_i p_j L_2 - i m_3 L_3), \quad (A.15)$$

$$q_2 = \frac{1}{\sqrt{2}} (m_4 L_4 - i m_5 L_5),$$

and using equation (A.13) the condition (A.14) reads

$$g(z + q) = \alpha(q) \exp \left(\frac{\pi}{2} H^{ij}(q, q) + \pi H^{ij}(z + z_0, q)\right) g(z), \quad (A.16)$$

where $\alpha(q)$ is given by (1.33) with $n_{a\bar{b}}^{ij}$ defined as in (A.3). Using equation (1.34) we see that $g(z)$ is just a shifted $\Theta$-function, i.e. $g(z) = f(z + z_0)$. These functions form a complex linear space of dimension

$$|P f(n^{ij})| = |(p_j q_i - p_i q_j)(\bar{q}_i - \bar{q}_j)| = n_L^{ij} \bar{n}_L^{ij}. \quad (A.17)$$

Thus there are $n_L^{ij} \bar{n}_L^{ij}$ orthogonal ground state wave functions obeying the boundary condition (A.14).
Particle spectrum

The particle excitations associated with the $a^i_\alpha$ and $c^i_m$ fields correspond to the usual wrapped D-brane spectrum \[2, 34\]. It is a $p$-fold degenerate spectrum with the momentum along the $x^2$-direction quantised in units of $(p_i R_2)^{-1}$. We will concentrate only on the massless torons associated with the $b_{ij}^{ij}$ fields as these are the responsible for the entropy formula (all the $d_m^{ij}$ excitations are massive as before).

We start by considering a given $ab$ component of the $b^{ij}_a$ field that gives rise to the massless torons

$$
(b^{ij}_{z_k})_{ab} = \sum_r (\chi^r_0)_{ab} \left(B^{ij}_{z_k}\right)_{0} (x^\sigma) , \quad (k = 1, 2) ,
$$

(A.18)

$$
(b^{ij}_{z_k})_{ab} = (b^{ij}_0)_{ab} = (b^{ij}_1)_{ab} = 0 ,
$$

where as it will be explained below we have not written any $U(p_i) \otimes U(\tilde{p}_j)$ index in the conformal field $\left(B^{ij}_{z_k}\right)_{0} (x^\sigma)$. The function $(\chi^r_0)_{ab}$ satisfies the boundary conditions (A.14). Even though this function is defined for $x^2 \in [0, p_i p_j L_2]$ the field $\left(b^{ij}_{z_k}\right)_{ab}$ takes values only in the range $x^2 \in [0, L_2]$. Using the first condition in equation (A.11) it may be seen that the conformal fields $\left(B^{ij}_{z_k}\right)_{0} (x^\sigma)$ are in fact the same for all $\left(b^{ij}_{z_k}\right)_{ab}$ components. The only difference is the shift $z_0 = \frac{1}{\sqrt{2}}(A_{ab}, 0)$ in the functions $(\chi^r_0)_{ab}$. This is just a consequence that the branes are wrapped and it means that the $U(p_i) \otimes U(\tilde{p}_j)$ field components in $b^{ij}_a$ are not independent. Thus as in the subsection 4.1.2 we have $4n^{ij}_L \bar{n}^{ij}_L$ massless bosonic torons associated with the $b^{ij}_a$ field.

The analysis is now entirely similar to the one presented before, therefore it will not be repeated here. As a final remark, note that we have assumed that $p_i$ and $p_j$ are co-prime. If this is not the case, i.e. if $p_i = lp'_i$ and $p_j = lp'_j$ with $p'_i$ and $p'_j$ co-prime the ground state wave function $(\chi_0)_{ab}$ will obey the boundary conditions (A.14) with $p_i p_j L_2$ replaced by $lp'_i p'_j L_2$ and with $n^{ij}_{23} = -n^{ij}_{32}$ replaced by $p'_i q_i - p'_j q_j$. The Landau degeneracy for the $(\chi_0)_{ab}$ functions will then be

$$
n^{ij}_L \bar{n}^{ij}_L = |(p'_i q_i - p'_j q_j)(\bar{q}_i - \bar{q}_j)| \ .
$$

(A.19)

However, when writing the fields $(b^{ij}_{z_k})_{ab}$ as in (A.18) and imposing the first condition in equation (A.11) we find that there are $l$ independent conformal fields $\left(B^{ij}_{z_k}\right)_{0} (s = 1, \ldots, l)$ for each Landau level, i.e. each Landau level is itself degenerated. More precisely, these independent conformal fields arise in the following
components of the fields

\[(b^{ij}_{2k})_{a\bar{b}}, \quad (b^{ij}_{2k})_{a+1,\bar{b}}, \ldots, \quad (b^{ij}_{2k})_{a+l-1,\bar{b}}.\]  \hspace{1cm} (A.20)

This is in perfect agreement with the string theory picture described in section 3 [13].

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