Collective behavior in the system of self propelling particles with nonholonomic constraints.

V.L. Kulinskii

Department for Theoretical Physics, Odessa National University, 2 Dvoryanskaya St., 65026 Odessa, Ukraine

V.I. Ratushnaya, A.V. Zvelindovsky, D. Bedeaux

Colloid and Interface Science group, LIC, Leiden University,
P.O. Box 9502, 2300 RA Leiden, The Netherlands

We consider the dynamics of systems of self propelling particles with nonholonomic constraints. A continuum model for a discrete algorithm used in works by T. Vicsek et al. is proposed. For a case of planar geometry the finite flocking behavior is obtained. The circulation of the velocity field is found not to be conserved as a consequence of the nonholonomicity. The stability of ordered motion with respect to noise is discussed. An analogy with the kinetics of charges in superconductors is noted.

PACS numbers: 05.65.+b, 47.32.-y, 87.10.+e

The emergence of ordered structures in dynamic systems is a long standing problem in physics. Generally speaking one deals with dynamic phase transitions governed either by external or internal noise. Most interesting is the arising of ordered motion caused by the internal dynamics of the system. Recently, there has been a growing interest in studying the collective behavior in systems of self-propelling particles (SPP). One may distinguish systems of two types:

The first one is for systems of “unintelligent” particles interacting via real physical forces produced by the background in which they are moving (micelles, bacteria, etc). Here the driving forces are caused by the gradients of chemical factors (concentrations, chem. potential etc.) or physical factors (light, potential and dissipative fields etc.), which influence the motion. In particular, the earth magnetic field is vital for the orientation during long distance migration of biological species like birds or turtles [1, 2]. It is clear that the absence of conservation of translational and angular momentum is a direct consequence of these external factors [3].

The second class is formed by systems of particles, which interact via nonholonomic constraints imposed on their velocities. We would like to stress that the nonholonomicity of such systems clearly expresses the “intelligent” nature of the particles, since such constraints need instant exchange of information (visual or any other sensorial) between the particles and their environment. This explains the coherent motion and arising ordered patterns in the dynamics of systems like crowds, traffic or flocking. Usual potential gradients or other physical forces are not relevant for the collective behavior, though the particles need some physical source of energy to sustain the constraints. The physical origin of the nonholonomicity is the force which acts on the particle due to its interaction with the background (earth, air, liquid substrate etc.). Since we are not interested in the dynamic degrees of freedom of the background we lose this information. Such loss of dynamic information leads to the breaking of the conservation of the (angular) momentum and effectively to the nonholonomic constraints. In general it means that the system with such constraints is not closed and therefore its dynamics is not Hamiltonian, though it does not mean that the energy dissipates. Rather a redistribution among the dynamic degrees of freedom takes place. But the very form of the constraint is determined by the “intellect” of a particle which uses the information about the environment and moves accordingly. Note that the numerical algorithm used in Ref. [4] modelled that kind of systems. For shortness we call it the Czirók-Vicsek automaton or algorithm (CVA).

There are also modifications of this algorithm, which differ from the CVA by inclusion of potential interparticle forces [5], external regular and stochastic fields [6]. We will consider CVA as the minimal model for the collective behavior since the main cause for the self-organization in the system is the nonholonomicity of CVA dynamic rule.

To our knowledge none of the hydrodynamic models proposed earlier is aimed to reflect the essence of the CVA - its conservative and distinctly nonholonomic character. To a great extent such models are modifications of the Navier-Stokes equation. Such an approach is certainly valid for microorganisms floating in a medium. It is, however, hardly adequate for the collective behavior of “intelligent” boids, e.g. birds or drivers in traffic flow [7], when the individual behavior is determined rather by the instant exchange of information with the environment than by the action of some interparticle forces. The key point here is the nonholonomicity of the system. In addition all the models based on the modifications of the Navier-Stokes equation like Ref. [8, 9] include additional phenomenological terms which generate the “spontaneous” transition to the state of ordered motion. Such terms are added much in formal analogy with Landau’s theory of equilibrium continuous phase transitions and the notion of an order parameter but any firm base for
such an analogy is not given. Thus the ordered state introduced in such a model is rather an ad hoc assumption than the natural consequence of the underlying interparticle interactions. The interpretation of the viscous term for such systems is also completely unclear.

In Ref. [8] it was noted that the CVA can be considered as the dynamic $XY$-model. It was concluded also that the cause of the ordering is the convective term. The $XY$-model is a Hamiltonian dynamic system. The latter property is the key point since for Hamiltonian dynamic flow there is an ergodic measure. Due to nonholonomic character the dynamic rule of the CVA even in the static limit $|\mathbf{v}| \to 0$, i.e. when there is no any transfer flow term like $\mathbf{v} \cdot \nabla$, has no canonical Hamiltonian form. The general fact is that such systems do not have an invariant measure with respect to the dynamic flow. In fact the ordered state in the CVA appears as the fixed point (attractor) of its averaging dynamic rule. Note that from the point of view of the theory of dynamic systems the compactness of the phase space (i.e. the space of positions and velocities) is very important so the question about influence of the boundary conditions usually used in simulations on the ordering the system also needs discussion.

From such a nonholonomic point of view the appearance of ordered motion in the CVA and similar dynamic systems is a trivial consequence of their nonholonomicity. Indeed, the breaking of conservation of the (angular) momentum is due to nonholonomic constraints which as has been mentioned above mean that the system is not closed.

Here we discuss a hydrodynamic model which can be considered as the continuous analogue of the discrete dynamical automaton proposed in Ref. [4] for the SPP system. It manifestly takes into account all local conservation laws for the number of particles and the kinetic energy. The self propelling force and the frictional force are assumed to balance each other.

The algorithm used in Ref. [4] corresponds to the following equation of motion of a particle:

$$\frac{d}{dt} \mathbf{v}_i = \mathbf{\omega}_i \times \mathbf{v}_i, \quad (1)$$

where $\mathbf{\omega}_i$ is the "angular velocity" of $i$-th boid, which depends on what happens in the neighborhood. It is assumed also that the number of particles is conserved. Like algorithm in Ref. [4], Eq. (1) distinctly expresses the conservation of the kinetic energy.

From a physical point of view it is natural that the hydrodynamic model corresponding to the CVA is based on the following equations:

$$\frac{d}{dt} \int_V n(r,t) dV = 0, \quad (2)$$

$$\frac{d}{dt} \int_V n \mathbf{v}^2 dV = 0, \quad (3)$$

where $n(r,t)$ and $\mathbf{v}(r,t)$ are the number density and the Eulerian velocity. The volume $V$ moves along with the velocity field. The first condition is the conservation of number of particles. As usual we can rewrite this condition in the differential form:

$$\frac{\partial n}{\partial t} + \text{div} (n \mathbf{v}) = 0. \quad (4)$$

The second constraint Eq. (3) means that the kinetic energy of a Lagrange particle is conserved, i.e.

$$\frac{d}{dt} \int_V n \mathbf{v}^2 dV = \int_V \mathbf{v}^2 \left( \frac{d n}{dt} + n \text{div} \mathbf{v} \right) dV$$

$$+ \int_V n \frac{d \mathbf{v}^2}{dt} dV = 0. \quad (5)$$

The first integrand vanishes due to the conservation of the particle number. As a consequence the second integral also vanishes for an arbitrary choice of the volume $V$ which in view of the natural condition $n \geq 0$ leads to:

$$\frac{d}{dt} |\mathbf{v}(r,t)|^2 = 0. \quad (6)$$

This implies that a pseudovector field $\mathbf{\omega}$ exists such that

$$\frac{d}{dt} \mathbf{v} (r,t) = \mathbf{\omega} (r,t) \times \mathbf{v} (r,t). \quad (7)$$

This equation, which is the continuous analogue of Eq. (1), has now been derived from the conservation of particle number and kinetic energy, Eq. (2) and Eq. (3).

We will model $\mathbf{\omega}$ by:

$$\mathbf{\omega} (r,t) = \int K (r-r') n(r',t) \text{rot} \mathbf{v}(r',t) dr', \quad (8)$$

which has the proper pseudovector character and heuristically may be considered as the continual analog of the CVA dynamic rule. There are other possible choices like:

$$\mathbf{\omega} (r,t) = \int \tilde{K} (r-r') \nabla n(r',t) \times \mathbf{v}(r',t) dr', \quad (9)$$

and combinations of the two. The averaging kernels $K$ and $\tilde{K}$ should naturally decrease with the distance in realistic models. We concentrate our discussion on the case Eq. (8). Equations (4) and (7) obviously have the uniform ordered motion with a constant both density and velocity as trivial solution. One may consider Eq. (7) as the equation of motion of a charge in a magnetic field, where $\mathbf{v}$ is the charge velocity and $\mathbf{\omega}$ is proportional to the magnetic field. One may even include an "electric field" and dissipative (collision) terms in Eq. (7):

$$\frac{d}{dt} \mathbf{v} (r,t) = \mathbf{f} + \mathbf{\omega} (r,t) \times \mathbf{v} (r,t) - \xi \mathbf{v} (r,t), \quad (10)$$
where $\xi^{-1}$ is the mean free time. We will not consider such an extended model here since algorithm used in [4] is distinctly conservative. Further we exploit the analogy of Eq. (7) and the equation of motion for charges in superconductors (see e.g. Ref. [10]). In our model a current density corresponding to the particle velocity at some point $\mathbf{r}$ depends on "magnetic field" $\mathbf{a}$, i.e. "vector potential", at all neighboring points $\mathbf{r}'$ within some region of coherence. Therefore the relations (8) or (9) can be considered as a corresponding nonlocal relations between the current density and the vector potential in nonlocal Pippard’s electrodynamics of the superconductors. The situation in the system under consideration is more complex. In electrodynamics of superconductors the external magnetic field is the main cause of the vortical motion of charges since their own magnetic field is negligibly small and does not lead to formation of the ordered motion. In our case the "magnetic field" $\mathbf{a}$ itself depends on the motion of "charges", i.e. particles, and vice versa, which leads to a nonlinearity of the system. Depending on the parameters one can expect either the direct current state since the system is conservative or vortical states like Meissner currents or Abrikosov vortices.

Using such an analogy let us find the conditions for the existence of the stationary vortical states.

We can rewrite Eq. (7) in the following form:

$$\frac{\partial \mathbf{a}}{\partial t} + \nabla \times \mathbf{W} = \mathbf{v} \times \mathbf{W},$$

where $\mathbf{W}(\mathbf{r}, t) = \mathbf{v} \times \mathbf{a}$. Thus it follows that if $\mathbf{W}(\mathbf{r}, t)$ is equal to 0, then $\frac{\partial \mathbf{a}}{\partial t} = 0$ and therefore $\mathbf{a} = \mathbf{v} \times \mathbf{W}$ is independent of the time. Such states are naturally interpreted as stationary translational $\mathbf{a} = 0$ or rotational $\mathbf{a} \neq 0$ regimes of motion. For the model (8) together with $\mathbf{W} = 0$ we get the integral equation:

$$\int K(\mathbf{r} - \mathbf{r}') \mathbf{v}(\mathbf{r}') d\mathbf{r}' = \mathbf{v}(\mathbf{r}),$$

which determines such states. Equation (12) gives stationary vortical motion, represented by the vector field with $|\mathbf{v}| = \text{const}$. From here it follows that the vorticity of the velocity field is an eigenstate of the integral operator with $n(\mathbf{r})$ as the corresponding weight factor. It should be noted that these stationary states do not exhaust all stationary vortical states, since in general $\nabla \mathbf{v}^2 \neq 0$.

We further scale $K$ by multiplying with some $n^*$ and similarly scale the density by dividing by $n^*$. Furthermore we restrict our discussion to the simple case of a planar geometry with averaging kernel in [5] as $\delta$-functional:

$$K(\mathbf{r} - \mathbf{r}') = s \delta(\mathbf{r} - \mathbf{r}'), \quad s = \pm 1.$$  \hspace{1cm} (13)

We will call this the local model. For this case Eqs. (7) and (11) take the form:

$$\frac{\partial n}{\partial t} + \text{div} (n \mathbf{v}) = 0, \quad \text{(14)}$$

$$\frac{d}{dt} \mathbf{v} = \mathbf{s} n \mathbf{rot} \mathbf{v} \times \mathbf{v}. \quad \text{ (15)}$$

These equation (15) can be obtained as a special case of the corresponding one in Ref. [3]. In our work the foundation of the terms is given. We do not use general symmetry arguments to take all terms of a certain symmetry into account irrespective of their physical meaning. The corresponding local model with $\delta$-kernel for (9) may be identified with rotor chemotaxis (i.e. caused by chemical field) force introduced in Ref. [11] if one takes into account simple linear relation between field of food concentration and the number density of boids, which is surely valid for low concentrations of food and bacteria.

The parameter $s$ of the local model (15) distinguishes different physical situations concerning the microscopic nonholonomical constraint. To see this we find the stationary radially symmetric solutions of (14) and (15). As usual, we search for the solutions of the form $n = n(r)$, $\mathbf{v} = v_\varphi(r) \hat{e}_\varphi$. The continuity equation (14) is satisfied trivially. Substituting this into Eq. (15) we finally obtain:

$$v_\varphi(r) = \frac{C_{st}}{2\pi r} \exp \left( s \int_{r_0}^r \frac{1}{n(r')} dr' \right), \quad \text{ (16)}$$

where $r_0$ is the cut-off radius of the vortex. This is the core of the vortex. The constant $C_{st}$ is determined by the circulation of the core $\oint \mathbf{v} d\mathbf{l} = C_{st}$. The spatial character of the solution strongly depends on the parameter $s$. If $s = -1$ the infinitely extended distributions for $n(r)$ are allowed, e.g. $n(r) \propto r^{-\alpha}$, $\alpha > 0$. They lead to localized vortices with exponential decay of angular velocity. If $s = +1$ only compact distributions, i.e. $n(r) \equiv 0$ outside some compact region, are consistent with the finiteness of the total kinetic energy i.e. they corresponds to finite number of particles $\int n dV < \infty$.

As an example we may give:

$$n(r) = \begin{cases} \sqrt{\frac{\pi}{R^2}}, & r_0 < r < R, \\ 0, & \text{otherwise}. \end{cases} \quad \text{(17)}$$

Substituting Eq. (17) into Eq. (16) one obtains:

$$v_\varphi = \frac{C_{st}}{2\pi r} \exp \left[ 2 \sqrt{\frac{R}{r_0}} \left( \sqrt{1 - \frac{r}{R}} - \text{arctanh} \sqrt{1 - \frac{r}{R}} \right) \right] \bigg|_{r_0}^r. \quad \text{(18)}$$

The corresponding component of the velocity $\mathbf{v} = v_\varphi \hat{e}_\varphi$ for such a case is shown on Fig. 1.

Taking the rotation on both sides of Eq. (15) one obtains the following equation for the vorticity in a case of
FIG. 1: Velocity $V^*(r/R) = 2\pi R v_s(r)/C_s\cdot$ in the vortex of the local model with $n(r)$ given by Eq. (17) at $r_0/R = 1/3$.

The circulation is defined by

$$C(t) = \oint_{S} \mathbf{v} \cdot d\mathbf{a} = \int_{S} \text{rot} \mathbf{v} \cdot d\mathbf{S}, \quad (21)$$

where the integration contour and the corresponding surface area move along with the velocity field. The time derivative of the circulation is:

$$\frac{d}{dt} C = \int_{S} \text{rot} \frac{d\mathbf{v}}{dt} \cdot d\mathbf{S}. \quad (22)$$

Thus the circulation does not conserve in contrast to the ideal fluid model, where the microscopic interactions are of holonomic character.

The total momentum $\mathbf{P} = \int n \mathbf{v} \, dV$ does not conserve too. For the local model we can write:

$$\frac{d}{dt} \mathbf{P} = \int s n^2 \text{rot} \mathbf{v} \times \mathbf{v} \, dV. \quad (23)$$

From Eq. (23) it follows that the damping of vortical part of the velocity leads to the formation of the state of uniform motion with $\mathbf{P} = \text{const}$.

Here we consider the influence of noise on the stability of the flow with respect to the vortical perturbation. It is clear that instability with respect to such perturbations drives the system to disordered state. Inclusion of stochastic noise can be done in a way analogous to that used in Ref. [4]:

$$\mathbf{v}(r, t) = \mathbf{v}_0(r, t) + \delta \mathbf{v}(r, t), \quad \text{where} \quad \omega_0 = sn \text{rot} \mathbf{v}$$

is the same contribution as before and $\delta \omega$ is the stochastic contribution. These fluctuations lead to fluctuations of the density and velocity fields. Replacing $\partial n/\partial t$ by an average value $1/\tau$ plus a fluctuating contribution $\delta L(t)$ in Eq. (20) one obtains for the above described local model with $s = +1$:

$$\frac{d}{dt} C = - \left( \frac{1}{\tau} + \delta L \right) C, \quad (24)$$

The simplest model for the noise is the Gaussian white noise approximation:

$$\langle \delta L(t) \delta L(t') \rangle = 2 \Gamma \delta(t - t'). \quad (25)$$

The stochastic equation (24) has the solution:

$$C(t) = C_0 \exp \left( -\frac{t}{\tau} \right) \exp (-W(t)), \quad (26)$$

where $W(t) = \int_0^t \delta L(t') \, dt'$ is the Wiener process [13].

Averaging over the realization of the stochastic process we get the averaged evolution of the vorticity:

$$\langle C(t) \rangle = C_0 \exp \left( -\frac{t}{\tilde{\tau}} \right), \quad \tilde{\tau} = \frac{\tau}{1 - \Gamma}, \quad (27)$$

where $\tilde{\tau}$ is the relaxation time of the circulation in the system. For large enough noise $\tau\Gamma > 1$ the system becomes unstable. For $\tau\Gamma < 1$ the system is stable and the circulation decays to zero. When $\tau\Gamma$ increases to 1 the relaxation time $\tilde{\tau}$ goes to infinity, a result similar to critical slowing down near the critical point. These estimates are modified when other terms, which have been neglected, are taken into account but we believe that qualitatively the obtained results remain unchanged.

It should be noted that the average kernels can also contain noise contributions. In view of the above results for the character of stationary states and vorticity relaxation, which depends on the sign of parameter $s$, this case needs a more thorough investigation.
In conclusion we have constructed a continuum SPP model with particle number and kinetic energy conservation. We found in 2D that vortical solutions exist for the model and that they show a finite flocking behavior. These solutions, which qualitatively reproduce some results of Ref. [4], were obtained without imposing any boundary conditions on the velocity field. The non-holonomic constrains were found to lead to a circulation which was not conserved. The influence of noise on the stability of the system was discussed.

ACKNOWLEDGEMENTS

Vladimir Kulinskii thanks NWO (Nederlandse Organisatie voor Wetenschappelijk Onderzoek) for a grant, which enabled him to visit Dick Bedeaux’s group at the Leiden University.

* Electronic address: koul@paco.net

[1] K.J. Lohmann C.M.F. Lohmann, L.M. Ehrhart, D.A. Bagley, T. Swing, Nature, 428, 909 (2004).
[2] W.W. Cochran, H. Mouritsen, M. Wikelski, Science, 304, 405 (2004).
[3] W. Ebeling, Physica A, 314, 92 (2002).
[4] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen and O. Shochet, Phys. Rev. Lett., 75, 1226 (1995); A. Czirók, T. Vicsek, Physica A, 281, 17 (2000).
[5] G. Grégoire, H. Chaté, Phys. Rev. Lett., 92, 025702 (2004).
[6] S. Hubbard, P. Babak, S. Th. Sigurdsson, K. G. Magnusson, Ecol. Model., 174, 359 (2004).
[7] D. Helbing, Rev. Mod. Phys., 73, 1067 (2001).
[8] J. Toner, Y. Tu, Phys. Rev. E, 58, 4828 (1998).
[9] S. Ramaswamy, R.A. Simha, Phys. Rev. Lett., 89, 058101 (2002).
[10] A. A. Abrikosov, Fundamentals of the theory of metals, (Amsterdam: Elsevier Science Publ. Co., 1988).
[11] A. Czirók, E. Ben-Jacob, I. Cohen, T. Vicsek, Phys. Rev. E 54, 1791 (1996).
[12] L.D. Landau and E.M. Lifshitz, Fluid mechanics, (Pergamon Press, 1959).
[13] N.G. van Kampen, Stochastic processes in physics and chemistry, (North Holland Publ. Co. Amsterdam, New York Oxford, 1981).