Numerical Caseology by Lagrange Interpolation for the 1D Neutron Transport Equation in a Slab

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Abstract — Here, we are concerned with a new, highly precise, numerical solution to the one-dimensional neutron transport equation based on Case’s analytical, singular eigenfunction expansion (SEE). While a considerable number of numerical solutions currently exist, understandably, because of its complexity even in one dimension, there are only a few truly analytical solutions to the neutron transport equation. In 1960, Case introduced a consistent theory of the SEE for a variety of idealized transport problems and forever changed the landscape of analytical transport theory. Several numerical methods, including the Fx method, were based on the theory. What is presented is yet another, called the Lagrange order N method (LNM) featuring the simplicity and precision of the Fx method, but for a more convenient and natural Lagrangian polynomial basis.

Keywords — Singular eigenfunctions, slab geometry, Lagrange interpolation, Gauss quadrature.

I. INTRODUCTION

Because of full-range orthogonality, Case’s method, commonly known as Caseology, produces an elegant analytical form of solution to the monoenergetic one-dimensional (1D) neutron transport equation by following the Fourier approach for solving partial differential equations. In particular, a complete set of basis functions to the transport equation through separation of variables defines the eigenmodes of the homogeneous equation. The general solution is then expanded in the complete set and orthogonality gives the expansion coefficients. The curiosity of the expansion is that the operator spectrum consists of a finite number of discrete and a continuum set of eigenmodes.

Many individuals contributed to the development of singular eigenfunction (SE) solutions to various transport equations. Davison, commonly acknowledged as the originator of the approach, worked out some of the early proofs of concept in the mid-1940s. Other contributors include Lafore and Millot, Wigner, and Van Kampen, who was the first to apply the theory to plasmas. But it was Case’s 1960 paper that brought the theory together in a mathematically consistent fashion to analytically solve the most fundamental problems in transport theory. The method’s popularity peaked in the 1960s and 1970s with the appearance of Case and Zweifel’s classic book, Linear Transport Theory, which was essentially the second volume of Case, de Hoffman, and Placzek’s Introduction to Neutron Diffusion. During that time, numerous papers appeared on Caseology in many fields of physics, including of course, nuclear reactor theory, acoustics, radiative transfer, rarefied gas dynamics, kinetic theory, traffic flow, and graphics, to name several.

The value of Caseology is mainly theoretical to enable the solution of the transport equation to be as close to a closed-form exact solution as possible. For monoenergetic neutron transport with isotropic scattering, exclusively considered here, there are only a handful of problems that lend themselves to closed form; i.e., primarily transport in an infinite medium and a half-space. A second advantage of Caseology is the foundation it provides to develop highly precise numerical solutions, such as the Fx method. To the author’s knowledge, the numerical solution to be presented is different from those found in the literature. When expressed as the full-range singular eigenfunction expansion (SEE), the solution for the slab is no longer in a closed form unlike for infinite media. Its
evaluation involves iteration through auxiliary $X$- and $Y$-functions, as in invariant imbedding or a Fredholm integral equation, which solve nonlinear coupled integral equations. The same is true for the half-space to determine the Case $X$-function (or equivalently Chandrasekhar’s $H$-function); but unlike the slab, these functions have closed-form integral representations. Our goal is to avoid evaluation of the auxiliary nonlinear integral equations for the slab and remain true to the original form of the full-range SEE. While SEE is not the only way to find analytical solutions, in comparison to Weiner-Hopf, invariant imbedding, Laplace and Fourier transforms, arguably this author believes it to be the most mathematically pleasing.

A full-range approach coupled to Lagrange interpolation will determine the numerical solution for the slab. One may find the approach algebraically intimidating, but the motivation is clear. Full range not only avoids the auxiliary nonlinear integral equations, but also avoids half-range orthogonality, requiring knowledge of solutions to singular integral equations (SIEs). The identical approach was taken by Siewert\(^8\) who cleverly devised the FN method based on a full-range scheme to construct a spectral expansion through collocation. A subsequent version bypassed Case’s method altogether by expressing the transport equation for the exiting flux directly as SIEs.

Since the FN method is one of the most successful methods of solving the monoenergetic 1D neutron and coherent radiative transfer equations on the planet, one might ask: Why should an additional numerical solution be of interest? First, the Lagrange order $N$ method (LNM) expresses the SEE in a new form by decoupling the solution of the SIEs for the continuum coefficients from the discrete. Second, the required Lagrange polynomial basis is a natural choice that accommodates a half-range numerical formulation and seamlessly couples to Gauss Quadrature (GQ).

The LNM gives excellent agreement with other methods, such as the Response Matrix/Discrete Ordinates Method (RM/DOM), Adding and Doubling Method (ADM), the Double PN method (DPN), and the Matrix Riccati Equation Method (MREM) published by the author.\(^9–12\) Best results, however, seem to be for continuous source distributions, in particular an isotropic source in predominantly scattering media, though acceptable benchmark precision is also found for the perpendicular beam.

A word on the perspective of what is about to be presented. Unlike recent efforts of the author (RM/DOM, DPN, ADM, and MREM) to solve the 1D transport equation with anisotropic scattering, LNM is limited to isotropic scattering for several reasons. First, obviously, is that it is always easier to demonstrate a new numerical method for isotropic scattering. The second reason is one of simplicity in calculating the discrete eigenvalues. The determination of the discrete eigenvalues for anisotropic scattering kernels, whose number is generally unknown, is difficult because of the potential variability of scattering kernels. This becomes particularly challenging when absorption is small as the eigenvalues shed from the continuum spectrum and can become indistinguishable from unity. So we consider only isotropic scattering to emphasize the new formulation. At some later time, however, the theory for anisotropic scattering could be attempted.

We begin by representing Case’s solution in terms of the discrete eigenfunctions and SIEs for an isotropically scattering slab. The known incoming flux entering the slab surfaces provide two coupled singular equations for the expansion coefficients. By adding and subtracting, the equations uncouple to give a combination of just the continuum coefficients in terms of the discrete. Superposition eliminates the dependence on the discrete coefficients, enabling the continuum coefficients to be determined independently from the discrete. The discrete coefficients come from orthogonality requiring integration over the continuum coefficients already found by combining Lagrange interpolation with GQ.

II. THEORY

II.A. SIEs for Continuum Expansion Coefficients

Our focus is the solution to the following transport equation for a slab of width $\Delta$, measured in mean free paths (mfps):

$$\left[ \mu \frac{\partial}{\partial \nu} + 1 \right] \psi(x, \mu) = c \frac{1}{2} \int d\mu' \psi(x, \mu')$$

(1a)

with entering flux $\psi(\mu)$ (the source) at the near surface ($x = 0$) and no entering flux at the far surface ($x = \Delta$):

$$\psi(0, \mu) = \psi(\mu)$$

(1b)

$$\psi(\Delta, -\mu) = 0$$

(1c)

for $0 \leq \mu \leq 1$. Case’s method gives the full-range SEE solution\(^1\) for the homogeneous transport equation as

$$\psi(x, \mu) = a_0 + \phi_{0+}(\mu) e^{-\psi/\nu_0} + a_0 - \phi_{0-}(\mu) e^{\psi/\nu_0} + \int_{-1}^{1} d\psi e^{-\psi/\nu_0} \phi_0(\mu) A(\nu)$$

(2a)
with SEs,
\[
\phi_{0+}(\mu) = \frac{c\nu_0}{2} \frac{1}{v_0 - \mu} \quad (2b)
\]
\[
\phi_v(\mu) = \frac{c\nu}{2} P \frac{1}{v - \mu} + \lambda(v) \delta(v - \mu) \quad (2c)
\]
where \( P \) indicates the principal value under an integral, and
\[
\lambda(v) = 1 - \frac{c\nu_0}{2} \log \left[ \frac{v + 1}{v_0 - 1} \right] . \quad (2d)
\]
The SEs satisfy symmetries:
\[
\phi_{0\pm}(\mu) = \phi_{0+}(\pm\mu) \quad (3a)\]
\[
\phi_v(\mu) = \phi_{-v}(\pm\mu) , \quad (3b)\]
where \( v_0 \) is the discrete eigenvalue that satisfies the dispersion relation
\[
\lambda(v_0) = 1 - \frac{c\nu_0}{2} \log \left[ \frac{v_0 + 1}{v_0 - 1} \right] = 0 . \quad (3c)
\]
The expansion coefficients \( a_{0+}, a_{0-} \), and \( A(v) \) are explicitly determined from the orthogonality of the SEs:
\[
\int_{-1}^{1} d\mu \omega_v(\mu) \phi_v(\mu) = N_v \delta(v - v') \quad (4a)
\]
\[
N_v = \pm \nu \left[ \lambda(v) + \left( \frac{c\nu}{2} \right)^2 \right] \quad (4b)\]
\[
\int_{-1}^{1} d\mu \phi_{0\pm}(\mu)^2 = \pm N_{0+} = \pm \left[ \frac{c\nu_0}{2} \left( \frac{c}{1 - 1/v_0^2} \right) - 1 \right] , \quad (4c)\]
but will be found by an alternative procedure to be outlined.

Introducing the boundary conditions (BCs) at the two free surfaces for \( 0 \leq \mu \leq 1 \) into Eq. (2a) for \( x = 0 \), gives
\[
\psi(\mu) = a_{0+} \phi_{0+}(\mu) + a_{0-} \phi_{0+}(-\mu) + \frac{1}{0} d\mu \phi_v(\mu) A^+(v) + \frac{1}{0} d\mu \phi_v(-\mu) A^-(v) \quad (5a)
\]
and for \( x = \Delta \),
\[
0 = a_{0+} e_{0-} \phi_{0+}(-\mu) + a_{0-} e_{0+} \phi_{0+}(\mu) + \frac{1}{0} d\nu \left[ e_{v-} \phi_v(-\mu) A^+(v) + e_{v+} \phi_v(\mu) A^-(v) \right] , \quad (5b)
\]
with
\[
A^{\pm}(v) = A(\pm v) \quad (5c)
\]
\[
e_{0+} = e^{\pm \Delta/v_0} \quad (5d)
\]
\[
e_{v+} = e^{\pm \Delta/v} . \quad (5e)\]
By adding and subtracting Eqs. (5a) and (5b), there results, after changing \( \mu \) to \( v \),
\[
\psi(v) = [\phi_{0+}(v) \pm e_{0-} \phi_{0+}(-v)] b^+_0 + \frac{1}{0} d\nu' \left[ e_{v-} \phi_v(-v) ] B^+(v') \right] , \quad (6a)
\]
where the discrete and continuum coefficients conveniently combine as
\[
b^+_0 = a_{0+} \pm e_{0+} a_{0-} \quad (6b)
\]
\[
B^{\pm}(v) = A^+(v) \pm e_{v+} A^-(v) ; \quad (6c)\]
and after transposition of the first term, the following two uncoupled SIEs result for \( 0 \leq v \leq 1 \):
\[
\lambda(v) B^+(v) + \frac{c}{2} \int_0^1 d\nu' \frac{v'}{v' - v} B^+(v') + \pm \frac{c}{2} \int_0^1 d\nu' \frac{v'}{v' + v} e_{v-} B^+(v') = \psi(v) - u^+_0(v) b^+_0 \quad (7a)\]

with
\[
u^+_0(v) = \phi_{0+}(v) \pm e_{0-} \phi_{0+}(-v) . \quad (7b)\]
The horizontal line on the integral symbol indicates a principal value integration.

From linearity, we can further split the SIEs into
\[ \lambda(v)B_1^+(v) + \frac{c}{2} \int_0^v \frac{d\nu}{v^2 - v'} B_1^+(v') + \sum_{j=1}^{N} \frac{P_N(v_j)}{P_N'(v_j)(v - v_j)} B_0^+(v) = f_1^+(v) \]

(8a)\textsuperscript{±}

and re-sum to give

\[ B_1^+(v) = B_0^+(v) - b_0^+ B_2^+(v) . \]

(8b)\textsuperscript{±}

Note the linear dependence separates the unknown discrete coefficients \( b_0^+ \) from the equally unknown continuum coefficients in \( B_1^+(v) \). Equation (8a)\textsuperscript{±} is a key feature of our formulation since it eliminates the need to know \( b_0^+ \) to find \( B_1^+(v) \), which is not at all intuitive. Now on to solve Eqs. (8a)\textsuperscript{±}.

II.B. Lagrange Interpolation for \( B_1^+(v) \)

From the Lagrange interpolation at \( N \) nodes \( v_j \),

\[ B_1^+(v) = \sum_{j=1}^{N} B_0^+ l_j(v) \]

\[ = \sum_{j=1}^{N} B_0^+ \left[ \frac{P_N(v)}{P_N'(v_j)(v - v_j)} \right] , \]

(9a)\textsuperscript{±}

where \( v_j, j = 1, \ldots, N \) are the zeros of the shifted Legendre polynomial \( P_N(v_j) \),

\[ P_N(v_j) = P_N(2v_j - 1) = 0 , \]

(9b)\textsuperscript{±}

and its derivative is

\[ P_N'(v_j) = 2P_N(z)_{z=2v_j-1} . \]

(9c)\textsuperscript{±}

\( P_N(2v - 1) \) is the standard Legendre polynomial. Substituting Eq. (9a)\textsuperscript{±} into the first integral of Eq. (8a)\textsuperscript{±} and using GQ for the second, since it is not a principal value integral, gives

\[ \int_0^v \frac{d\nu}{v^2 - v'} B_1^+(v') = f_1^+(v) \]

\[ = -2 \left\{ \frac{v_j}{v_j - v} Q_N(2v_j - 1) - v Q_N(2v - 1) \right\} , \]

(10a)\textsuperscript{±}

\( Q_N(v) \) is the Legendre function of the second kind of order \( N \). Note that Lagrange interpolation combined with GQ is quite convenient if the interpolation abscissae are identical to the quadrature abscissae as is true here. This is the second key feature of LNM. In addition, the advantage of the interpolation is that the principal value integration in the first integral is treated exactly, giving rise to the \( Q \)-functions in \( I(v, v_j) \).

Finally, letting \( v = v_m \), and \( m = 1, \ldots, N \) leads to the following set of linear equations for Eqs. (10a)\textsuperscript{±}, which when solved give \( B_m^+ \), for \( m = 1, \ldots, N, i = 1, 2 \):

\[ \sum_{j=1}^{N} \left\{ \frac{\lambda(v_m) - c I(v_m, v_m)}{2P_N'(v)} \delta_{jm} + \frac{1}{2} \left[ (1 - \delta_{jm}) I(v_m, v_j) \pm \omega_j \frac{v_j}{v_j + v_m} \right] B_j^+ \right\} B_j^+ \]

\[ = f_1^+(v_m) . \]

(11)\textsuperscript{±}

Thus, from Eq. (6c)\textsuperscript{±}, the continuum coefficients \( A^+ (v_m) \) can be found; but, as will be shown, this will be unnecessary.

II.C. Determination of Discrete Coefficients \( b_0^+ \)

From the analytical expression for the flux given by Eq. (2a) written here again,

\[ \psi(x, \mu) = a_{0+} \phi_{0+}(\mu)e^{-x/\nu_0} + a_{0-} \phi_{0-}(\mu)e^{x/\nu_0} + \frac{1}{\nu_0} \int_{-1}^{1} dv e^{x/\nu_0} \phi_v(\mu) A(v) , \]

(12)

with \( \mu \) replaced by \(-\mu\) at the near surface, \( x = 0 \), and with \( x = \Delta \) at the far surface, one finds

\[ \psi(0, -\mu) = a_{0+} \phi_{0+}(-\mu) + a_{0-} \phi_{0-}(\mu) + \frac{1}{\nu_0} \int_{-1}^{1} dv e^{-x/\nu_0} \phi_v(\mu) A^+(v) + \phi_v(\mu) A^-(v) \]

(13a)\textsuperscript{±}

and
\[ \psi(\Delta, \mu) = a_0 + \phi_0^+(\mu) e_0^+ + a_{0-} \phi_{0-}(\mu) e_0^- + \frac{1}{\nu} \int d\nu [e_{\nu-} \phi_{\nu-}(\mu)] A^+(\nu) + e_{\nu+} \phi_{\nu+}(\mu) A^-(\nu) \]  

(13b)

over the full-range \(-1 \leq \mu \leq 1\). Then adding and subtracting,

\[ \psi(0, -\mu) \pm \psi(\Delta, \mu) = [\phi_{0+}-(-\mu) \pm e_{0+} \phi_{0+}(\mu)] b_0^+ + \frac{1}{\nu} \int d\nu [\phi_{\nu-}(\mu) \pm e_{\nu-} \phi_{\nu-}(\mu)] B^+(\nu) , \]  

(14a) \quad ^\sharp

and from orthogonality by projection over \(\mu \phi_{0-}(\mu)\),

\[- N_0 b_0^+ = \frac{1}{\nu} \int d\nu \psi(0, -\mu) [\psi(0, -\mu) \pm \psi(\Delta, \mu)] , \]  

(14b) \quad ^\sharp

using Eqs. (4). Next, separating the integration into half-ranges,

\[- N_0 b_0^+ = \frac{1}{\nu} \int d\nu \psi(0, -\mu) [\psi(0, -\mu) \pm \psi(\Delta, \mu)] - \frac{1}{\nu} \int d\nu \phi_{\nu+}(\mu) \psi(\mu) , \]  

(15) \quad ^\sharp

where we have introduced the BC [Eqs. (1b) and (1c)] into the second integral. When Eq. (14a)\(^\sharp\) is introduced into the first integral,

\[- N_0 b_0^+ = - J_0^+ + \frac{1}{\nu} \int d\nu \phi_{\nu+}(\mu) \psi(\mu) \left[ \phi_{\nu+}(\mu) \pm e_{\nu+} \phi_{nu+}(\mu) \right] b_0^+ + \frac{1}{\nu} \int d\nu [\phi_{\nu+}(\mu) \pm e_{\nu+} \phi_{\nu+}(\mu)] B^+(\nu) \right] , \]  

(16a) \quad ^\sharp

or more compactly,

\[- N_0 b_0^+ = J_0^+ - T_2^+ b_0^+ - \frac{1}{\nu} \int d\nu T_2^+ (\nu) [B_0^+ (\nu) - b_0^+ B_0^+ (\nu)] , \]  

(16b) \quad ^\sharp

with

\[ T_2^+ (\nu) = \frac{1}{\nu} \int d\nu \phi_{\nu+}(\mu) [\phi_{\nu+}(\mu) \pm e_{\nu+} \phi_{\nu+}(\mu)] , \]  

(16c) \quad ^\sharp

and substituting Eq. (8b)\(^\sharp\) for \(B_1^+(\nu)\). Five half-range integrals are required in Eqs. (16c), (16d)\(^\sharp\), (16e)\(^\sharp\), all of which can be done analytically once \(\psi(\mu)\) is specified. Thus, for \(T_1^+\):

\[ T_1^+ = T_{11} \pm e_{\nu} T_{12} \]  

(17a) \quad ^\sharp

\[ T_{11} = \frac{1}{\nu} \int d\nu \phi_{\nu+}(\mu) \psi(\mu) \left[ \frac{(c\nu_0)}{2} \ln \frac{v_0 + 1}{v_0} - \frac{1}{v_0 + 1} \right] \]  

(17b)

\[ T_{12} = \frac{1}{\nu} \int d\nu \phi_{\nu+}(\mu) \psi(\mu) \left[ 1 - c\nu_0 \ln \frac{v_0 + 1}{v_0} \right] . \]  

(17c)

Similarly, for \(T_2^+(\nu)\):

\[ T_2^+(\nu) = T_{21}(\nu) \pm e_{\nu} T_{22}(\nu) \]  

(18a) \quad ^\sharp

\[ T_{21}(\nu) = \frac{1}{\nu} \int d\nu \phi_{\nu+}(\mu) \psi(\mu) \left[ \frac{(c\nu_0)}{2} \ln \frac{v_0 + 1}{v_0} + \frac{c\nu_0}{2} \ln \frac{v + 1}{v} \right] + \right. \]  

(18b)

\[ T_{22}(\nu) = \frac{1}{\nu} \int d\nu \phi_{\nu+}(\mu) \psi(\mu) \left[ 1 - \frac{(c\nu_0)}{2} \ln \frac{v_0 + 1}{v_0} + \frac{c\nu_0}{2} \ln \frac{1 - v}{v} \right] . \]  

(18c)

Finally, solving for \(b_0^\pm\) in Eq. (16b)\(^\sharp\) by introducing GQ gives

\[ b_0^\pm = \left[ N_{0+} - T_1^+ - \sum_{j=1}^{N} \omega_j T_{2j}^+ B_{2j} \right]^{-1} \times \]  

(19) \quad ^\sharp

\[ \left[ J_0^+ - \sum_{j=1}^{N} \omega_j T_{2j}^+ B_{2j} \right] , \]  

(19) \quad ^\sharp
with $B^\pm_j, j = 1, \ldots, N, i = 1, 2$ known from Eqs. (11). We have found all the necessary coefficients to evaluate Eq. (2a), but first we must arrange Eq. (2a) appropriately.

For completeness, projecting Eq. (14a) over $\mu \phi_0(\mu)$ gives the alternative expression for $b_0^\pm$

$$
\begin{align*}
    b_0^\pm &= \left[ \pm e_0 - N_0 - T_3^\pm + \frac{1}{0} dv T_4^\pm(v) B^\pm_2(v) \right]^{-1} \\
    &\times \left[ \int_0^1 dv T_4^\pm(v) B^\pm_1(v) - J_0 \right],
\end{align*}
$$

where

$$
J_0 = -\frac{1}{0} d\mu \mu \phi_0(\mu) \psi(\mu)
$$

with

$$
\tau_0^\pm(\mu) = \phi_0(\mu) + e_0 - \phi_0(\mu) \quad (21b)
$$

and

$$
\tau^\pm(v, \mu) = \phi_0(\mu) + e_0 - \phi_0(\mu) \quad (21c)
$$

When we introduce the expressions for $\psi(\pm \mu)$ into Eqs. (21b,c) and subsequently into Eqs. (21a) for $0 \leq \mu \leq 1$, the result is

$$
\psi(0, -\mu) \pm \psi(\Delta, \mu) = \tau_0^\pm(\mu)b_0^\pm + I_1^\pm(\mu) + I_2^\pm(\mu) \pm e_0 - \phi_0(\mu)B^\pm(\mu), \quad (22a)
$$

where the integrals are

$$
I_1^\pm(\mu) = \int_0^1 dv \frac{v}{v + \mu} B^\pm(v) \quad (22b)
$$

$$
I_2^\pm(\mu) = \int_0^1 dv \frac{v}{v - \mu} e_0 - B^\pm(v) \quad (22c)
$$

For $I_2^\pm(\mu)$, we can add and subtract $e_0 - \lambda(\mu)$ in the integrand to remove the singularity to write,

$$
I_2^\pm(\mu) = \int_0^1 dv \frac{v}{v + \mu} B^\pm(v) + e_0 - \lambda(\mu)B^\pm(\mu) - u_0^\pm(\mu)b_0^\pm. \quad (23a)
$$

Conveniently, from Eq. (7a), the principal value integral in Eq. (23a) is

$$
\int_0^1 dv \frac{v}{v + \mu} B^\pm(v) = -\frac{1}{2} \int_0^1 dv \frac{v}{v + \mu} e_0 - B^\pm(v) + \lambda(\mu)B^\pm(\mu) - u_0^\pm(\mu)b_0^\pm.
$$

(23b)

Note that all integrals in Eqs. (20) can be analytically evaluated when $\psi(\mu)$ is known.

### III. Final Expressions for the Exiting Flux

From Eq. (14a),

$$
\psi(0, -\mu) \pm \psi(\Delta, \mu) = \tau_0^\pm(\mu)b_0^\pm + I_1^\pm(\mu) + I_2^\pm(\mu) \pm e_0 - \phi_0(\mu)B^\pm(\mu) \quad (21a)
$$

When introduced into Eq. (23a), we find with some algebra:

$$
\psi(0, -\mu) \pm \psi(\Delta, \mu) = \psi(\mu)e_\mu - \tau_0^\pm(\mu) - u_0^\pm(\mu)b_0^\pm + \frac{1}{2} \int_0^1 dv \frac{v}{v + \mu} [1 - e_\mu e_\nu] B^\pm(v) \pm e_\mu \lambda(\mu)B^\pm(\mu) - u_0^\pm(\mu)b_0^\pm.
$$

(24)
Finally, simply adding and subtracting Eq. (24) and dividing by 2 gives the individual exiting angular flux distributions $\psi(0, -\mu)$ and $\psi(\Delta, \mu)$ for $0 \leq \mu \leq 1$:

$$\psi(0, -\mu) = \frac{1}{2} \left[ \left( t_0^+ (\mu) - u_0^+ e_{\mu-} \right) b_0^+ + \left( t_0^- (\mu) + u_0^- e_{\mu-} \right) b_0^- \right]$$

and

$$\psi(\Delta, \mu) = \frac{1}{2} \left[ \left( t_0^+ (\mu) - u_0^+ e_{\mu-} \right) b_0^+ - \left( t_0^- (\mu) + u_0^- e_{\mu-} \right) b_0^- \right]$$

where all singularities have been resolved. The final step is to evaluate all integrals with GQ since the integrands are known.

Note that the first term on the right side of Eq. (25b) is the uncollided flux leaving the far surface.

IV. EXPRESSIONS FOR INTERIOR FLUX

For completeness, the coefficients $a_{0+}$, $a_{0-}$, and $A^\pm (v)$ come from Eqs. (6b) and (6c) as

$$a_{0+} = \frac{1}{2} \left[ b_0^+ + b_0^- \right], \quad a_{0-} = \frac{e_0}{2} \left[ b_0^+ - b_0^- \right]$$

and

$$A^+ (v) = \frac{1}{2} [B^+ (v) + B^- (v)],$$

$$A^- (v) = \frac{\epsilon_{\mu-}}{2} [B^+ (v) - B^- (v)].$$

Removing the singularity and integrating over the delta function in Eq. (29c) gives
\[ I_2^\pm (\mu) = \frac{c}{2} \int_0^1 dv \left[ \frac{e^{-(\Delta-x)/\nu} - e^{-(\Delta-x)/\mu}}{v - \mu} \right] B^\pm (v) + \]
\[ + e^{-(\Delta-x)/\nu} \frac{c}{2} \int_0^1 dv \frac{v}{v - \mu} B^\pm (v) + \lambda(v)e^{-(\Delta-x)/\mu}B^\pm (v). \]

(30)\textsuperscript{\pm}

As for the previous exiting distributions, we replace the principal value integral by Eq. (23b)\textsuperscript{\pm}:

\[ \frac{c}{2} \int_0^1 dv \frac{v}{v - \mu} B^\pm (v) = - \pm \frac{c}{2} \int_0^1 dv \frac{v}{v + \mu} e_{\nu \nu} B^\pm (v) + \psi(\mu) - \lambda(\mu)B^\pm (\mu) - u_0^\pm (\mu) b_0^\pm, \]

(31a)\textsuperscript{\pm}

and therefore,

\[ I_2^\pm (\mu) = \frac{c}{2} \int_0^1 dv \left[ \frac{e^{-(\Delta-x)/\nu} - e^{-(\Delta-x)/\mu}}{v - \mu} \right] B^\pm (v) - \]
\[ - e^{-(\Delta-x)/\mu} \frac{c}{2} \int_0^1 dv \frac{v}{v + \mu} e_{\nu \nu} B^\pm (v) + \psi(\mu)e^{-(\Delta-x)/\mu} - \]
\[ - u_0^\pm (\mu)e^{-(\Delta-x)/\mu} b_0^\pm. \]

(31b)\textsuperscript{\pm}

Introducing the two integrals into Eq. (29a)\textsuperscript{\pm} gives

\[ T^\pm (\mu) = \frac{c}{2} \int_0^1 dv \frac{v}{v + \mu} \left[ e^{-x/\nu} - e^{-(\Delta-x)/\mu} e^{-x/\nu} \right] B^\pm (v) + \]
\[ + \frac{c}{2} \int_0^1 dv \left[ e^{-x/\nu} - e^{-(\Delta-x)/\mu} e^{-x/\nu} \right] B^\pm (v) + \]
\[ + \psi(\mu)e^{-(\Delta-x)/\mu} - \pm u_0^\pm (\mu)e^{-(\Delta-x)/\mu} b_0^\pm. \]

(32)\textsuperscript{\pm}

Thus, for Eq. (28a)\textsuperscript{\pm},

\[ T^\pm (x, \mu) = \psi(x, -\mu) \pm \psi(\Delta - x, \mu) \]
\[ = \pm \psi(\mu)e^{-(\Delta-x)/\mu} + \left[ e^\pm_0 (x, \mu) - \pm u_0^\pm e^{-(\Delta-x)/\mu} \right] b_0^\pm + \]
\[ + \frac{c}{2} \int_0^1 dv \frac{v}{v + \mu} \left[ 1 - e^{-\Delta-x)/\mu} e^{-\Delta-x)/\nu} \right] B^\pm (v) + \]
\[ + \frac{c}{2} \int_0^1 dv \left[ e^{-\Delta-x)/\nu} - e^{-(\Delta-x)/\mu} e^{-\Delta-x)/\nu} \right] B^\pm (v). \]

(33)\textsuperscript{\pm}

Note that the first term on the second line of Eq. (33)\textsuperscript{\pm} is the uncollided flux. Finally, adding and subtracting Eq. (33)\textsuperscript{\pm} and dividing by 2 gives the angular fluxes at \( x \) and \( \Delta - x \) in all directions:

\[ \psi(x, -\mu) = \frac{1}{2} \left[ T^+(x, \mu) + T^-(x, \mu) \right] \]

(34a)

and

\[ \psi(\Delta - x, \mu) = \frac{1}{2} \left[ T^+(x, \mu) - T^- (x, \mu) \right]. \]

(34b)

V. NUMERICAL IMPLEMENTATION

Numerical implementation first requires \( v_0 \) from a high-precision Newton-Raphson solver applied to Eq. (3c). Apparently, high precision for the transport eigenvalue \( v_0 \) is crucial to the numerical results to follow. One can show that for \( c \) less than 0.05, any purely numerical method to determine \( v_0 \) will lose significance without extended precision, including the present method. Neither extended precision nor computer algebra are considered in this work in order to maintain reader accessibility to the numerical methods presented. For this reason, only \( c \geq 0.1 \) will be considered, and we otherwise defer to RM/DOM, DPN, ADM, and MREM. Next, the zeros of the modified Legendre polynomials come from expressing the recurrence for Legendre polynomials in matrix form and determining the matrix eigenvalues (zeros) when \( P_N \) is set to zero. The Legendre functions are determined from recurrence or infinite series according to the stability of the recurrence relation for \( Q_N \). The Legendre polynomials come from their stable recurrence relation. Finally, matrix inversions are computed by LU Lower/Upper matrix decomposition.

If you have followed my publications and presentations over the past 20 years, you may find it odd that I have not used Wynn-epsilon (W-e) acceleration\textsuperscript{13} to further accelerate precision in quadrature order. The reason is that a high quadrature order is required for LNM as is and there is little, though some slight advantage of W-e, but not worth the computational effort.

To demonstrate the extreme precision (greater than nine digits) that LNM can achieve, we compare results with the RM/DOM of Ref. 9 for several sample cases.

VI. AN ISOTROPIC SOURCE

The first comparison is for an isotropically distributed source on the near surface:
\[ \psi(0, \mu) = \psi(\mu) = 1. \]  (35)

Tables I through V each display the benchmark from RM/DOM and include the discrepant digits, at most by one digit, underlined in bold as calculated by LNM.

Table I shows a high precision demonstration of the exiting flux variation [\(\mu\) negative for \(x = 0\) and \(\mu\) positive for \(x = \Delta\)] with slab thickness \(\Delta\) and \(c = 0.9\). There are only three missed digits by one unit in the ninth place over the entire range of thicknesses considered. The quadrature order for LNM is 600 and for RM/DOM about 400. Thus, LNM gives nearly nine-place precision verified by comparison with RM/DOM. The time of computation for LNM is 17s and it is about 5s for RM/DOM on a 2.6-MHz Dell Precision PC.

Table II gives a demonstration for the variation \(c\) for a one mfp slab. Here, one observes the degradation of LNM because of its inability to fully capture \(v_0\) as \(c\) nears 0.1. For \(c = 0.1\), only six-place precision is achievable for \(N = 2500\), which is high for a 1D transport calculation. For the remainder of the table (of quadrature order 600) only four digits are missed by one unit in the ninth place.

Table III shows the flux for several interior points for a 10 mfp slab with \(c = 0.9\) from Eqs. (33) and (34). Again, in comparison with RM/DOM all but two entries agree to all nine places for a quadrature order of 600.

VII. AN EXPONENTIAL AND BEAM SOURCE

Now consider the normalized exponentially distributed source:

\[ \psi(0, \mu) = \psi(\mu) = \frac{e^{-\mu}}{1 - e^{-1}}. \]  (36)

For this case, we have no standard of comparison other than comparison to increasing quadrature and expecting convergence.

Table IV, for exiting flux, shows that the exponential source requires more effort than the isotropic source to achieve only six-digit precision, which is certainly adequate for a benchmark, but not an extreme benchmark.

The beam source,

\[ \psi(0, \mu) = \psi(\mu) = \delta(v - \mu_0), \]  (37)

is the most challenging for LNM. The primary difficulty is the numerical representation of the delta function on the right side of Eq. (8a). Ideally, the delta function should be carried along analytically in any theoretical manipulation, but apparently LNM, as presented, does not straightforwardly allow this. Theoretically, LNM solves SIEs to construct the flux from the expansion coefficients but not for the exiting flux directly. So we start at a more basic construction rather than at the solution itself, which is most likely why we have difficulties with the delta function since LNM does not require its integration.

Thus, we adopt the following formal/numerical representation:

\[ \delta(v - \mu_0) = \sum_{l=0}^{\infty} \frac{2l + 1}{2} P_l(v)P_l(\mu_0), \]  (38a)

which leads to

\[ \delta(v - \mu_0) = 2\delta((2v - 1) - (2\mu_0 - 1)) \]
\[ = \sum_{l=0}^{\infty} (2l + 1)P_l^2(v)P_l^2(\mu_0). \]  (38b)

If we truncate at \(N\),

\[ \delta(v - \mu_0) \approx \sum_{l=0}^{N} (2l + 1)P_l^2(v)P_l^2(\mu_0), \]  (39a)

then from the Darboux formula,\(^{14}\)

\[ \delta(v - \mu_0) \approx (N + 1) \left[ \frac{P_{N+1}^N(v)P_N^N(\mu_0) - P_N^N(v)P_{N+1}^N(\mu_0)}{v - \mu_0} \right], \]  (39b)

and for \(v = v_j\) and if \(\mu_0 = 1\), since \(P_N^N(v_j) = 0\) and \(P_N^N(1) = 1\):

\[ \delta(v_j - 1) \approx (N + 1) \left[ \frac{P_{N+1}^N(v_j)}{v_j - 1} \right]. \]  (39c)

In Table V, we have LNM results for increasing \(N\) showing convergence. The highlighted digits are in disagreement with RM/DOM. We observe that at best we get five-place precision confirming the difficulty of delta function sources. For this reason, RM/DOM, as well as ADM and DPN, where the delta function can be treated analytically, are more appropriate for the beam, though five-place precision, as found here, is generally adequate for a benchmark.
### Table I

**Exiting Flux Variation with $\Delta$ for $c = 0.9$ for an Isotropic Source**

| $\mu/\Delta$ | 0.5 | 1  | 2  | 4  | 8  | 16 |
|-------------|-----|----|----|----|----|----|
| -1.000E+00 | 1.690394663E-01 | 2.674103351E-01 | 3.616488728E-01 | 4.081422059E-01 | 4.18431584E-01 | 4.149474557E-01 |
| -8.000E-01 | 2.011521450E-01 | 3.078331616E-01 | 4.008507470E-01 | 4.419247595E-01 | 4.835786343E-01 | 4.85815135E-01 |
| -6.000E-01 | 2.477252111E-01 | 3.609294346E-01 | 4.474820691E-01 | 4.815092434E-01 | 5.24081109E-01 | 5.32458269E-01 |
| -4.000E-01 | 3.200999631E-01 | 4.311708733E-01 | 5.027117752E-01 | 5.429560312E-01 | 5.88949693E-01 | 5.915740025E-01 |
| -2.000E-01 | 4.373716248E-01 | 5.194020596E-01 | 5.969195941E-01 | 5.980846993E-01 | 6.387453519E-01 | 6.37722280E-01 |
| 0.000E+00  | 5.916234075E-01 | 6.353633692E-01 | 6.683181756E-01 | 7.19136727E-01 | 8.10085681E-01 | 8.30322255E-01 |
| 6.000E-01  | 2.584017054E-01 | 1.815479981E-01 | 2.00623523E-01 | 2.34030753E-01 | 2.85572196E-01 | 3.15322751E-04 |
| 2.000E-01  | 4.107668192E-01 | 2.675980096E-01 | 1.449056449E-01 | 1.86467943E-02 | 2.34850861E-02 | 2.65211599E-02 |
| 4.000E-01  | 5.612862521E-01 | 3.664802865E-01 | 1.90791557E-01 | 2.34850861E-02 | 3.02125924E-02 | 3.609294436E-01 |
| 6.000E-01  | 6.589098500E-01 | 4.589432518E-01 | 2.45252116E-01 | 3.15322751E-04 | 4.149474557E-01 | 5.82172384E-04 |
| 8.000E-01  | 7.214073554E-01 | 5.332715234E-01 | 3.022759024E-01 | 4.149474557E-01 | 5.82172384E-04 | 1.85717238E-04 |
| 1.000E+00  | 7.653830029E-01 | 5.916250896E-01 | 3.565007218E-01 | 4.149474557E-01 | 5.82172384E-04 | 2.40155892E-04 |

### Table II

**Exiting Flux Variation with $\Delta = 1$ for an Isotropic Source**

| $\mu/c$    | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 | 0.99 |
|------------|-----|-----|-----|-----|-----|------|
| -1.000E+00 | 1.504005472E-02 | 5.124571442E-02 | 9.919198554E-02 | 1.66016888E-01 | 2.674103351E-01 | 3.329218166E-01 |
| -8.000E-01 | 1.751883083E-02 | 5.955471562E-02 | 1.148847489E-01 | 1.91824165E-01 | 3.078331616E-01 | 3.825130896E-01 |
| -6.000E-01 | 2.092893876E-02 | 7.089686442E-02 | 1.361760310E-01 | 2.262501843E-01 | 3.609294436E-01 | 4.471117240E-01 |
| -4.000E-01 | 2.586159747E-02 | 8.702979248E-02 | 1.659221624E-01 | 2.732166735E-01 | 4.311708733E-01 | 5.311207558E-01 |
| -2.000E-01 | 3.353614972E-02 | 1.129234854E-01 | 2.087387645E-01 | 3.370946868E-01 | 5.190420596E-01 | 6.317828084E-01 |
| 0.000E+00  | 5.12873072E-02  | 1.628491358E-01 | 2.904739780E-01 | 4.423922290E-01 | 6.35363632E-01 | 7.447435638E-01 |
| 2.000E-01  | 8.028514627E-03 | 2.866250776E-02 | 5.854426708E-02 | 1.045295474E-01 | 1.85479981E-01 | 2.34850861E-01 |
| 4.000E-01  | 1.813781599E-02 | 4.764831692E-02 | 9.066270008E-02 | 1.568687208E-01 | 2.675898009E-01 | 3.440071157E-01 |
| 6.000E-01  | 9.550016120E-02 | 1.294549833E-01 | 1.774911944E-01 | 2.494526695E-01 | 3.668402865E-01 | 4.589432518E-01 |
| 8.000E-01  | 2.988610467E-01 | 3.295973198E-01 | 3.720890262E-01 | 4.342645153E-01 | 5.32715234E-01 | 5.99341293E-01 |
| 1.000E+00  | 3.792294807E-01 | 4.073575782E-01 | 4.460583899E-01 | 5.024632348E-01 | 5.916250896E-01 | 6.509775491E-01 |
### TABLE III

Interior Flux for $c = 0.9$ and $\Delta = 10$ for an Isotropic Source

| $\mu/N$ | 0 | $\Delta/4$ | $\Delta/2$ | $3\Delta/4$ | $\Delta$ |
|---------|---|------------|------------|-------------|---------|
| $-1.000E+00$ | 4.14934693E-01 | 1.05959641E-01 | 2.81953178E-01 | 6.90627717E-03 | 0.00000000E+00 |
| $-8.000E-01$ | 4.47413169E-01 | 1.13837349E-01 | 3.03308371E-02 | 7.55929891E-03 | 0.00000000E+00 |
| $-6.000E-01$ | 4.85843736E-01 | 1.22977311E-01 | 3.27955781E-02 | 8.29640490E-03 | 0.00000000E+00 |
| $-4.000E-01$ | 5.32452288E-01 | 1.33714425E-01 | 3.56747807E-02 | 9.14508935E-03 | 0.00000000E+00 |
| $-2.000E-01$ | 5.91608056E-01 | 1.46514811E-02 | 3.90998530E-02 | 1.01191695E-02 | 0.00000000E+00 |
| $0.000E+00$ | 6.83768947E-01 | 1.62052126E-01 | 4.32369136E-02 | 1.12708200E-02 | 0.00000000E+00 |
| $0.000E+00$ | 1.00000000E+00 | 1.62052126E-01 | 4.32369136E-02 | 1.12708200E-02 | 0.00000000E+00 |
| $-2.000E-01$ | 2.06652335E-01 | 5.48192403E-02 | 1.44394438E-02 | 2.69345739E-03 | 0.00000000E+00 |
| $-4.000E-01$ | 1.00000000E+00 | 2.06652335E-01 | 5.48192403E-02 | 1.44394438E-02 | 2.69345739E-03 |
| $-6.000E-01$ | 1.00000000E+00 | 2.06652335E-01 | 5.48192403E-02 | 1.44394438E-02 | 2.69345739E-03 |
| $-8.000E-01$ | 1.00000000E+00 | 2.06652335E-01 | 5.48192403E-02 | 1.44394438E-02 | 2.69345739E-03 |
| $-1.000E+00$ | 1.00000000E+00 | 2.06652335E-01 | 5.48192403E-02 | 1.44394438E-02 | 2.69345739E-03 |

### TABLE IV

Exiting Flux Variation with $N$ for Exponential Source: $\Delta = 1$, $c = 0.99$

| $\mu/N$ | 600 | 800 | 1000 | 1200 | 1400 | 1600 | 2000 |
|---------|-----|-----|-----|-----|-----|-----|-----|
| $-1.000E+00$ | 3.324640E-01 | 3.324640E-01 | 3.324640E-01 | 3.324640E-01 | 3.324640E-01 | 3.324640E-01 | 3.324640E-01 |
| $-8.000E-01$ | 3.823575E-01 | 3.823575E-01 | 3.823575E-01 | 3.823575E-01 | 3.823575E-01 | 3.823575E-01 | 3.823575E-01 |
| $-6.000E-01$ | 4.476441E-01 | 4.476441E-01 | 4.476441E-01 | 4.476441E-01 | 4.476441E-01 | 4.476441E-01 | 4.476441E-01 |
| $-4.000E-01$ | 5.333980E-01 | 5.333980E-01 | 5.333980E-01 | 5.333980E-01 | 5.333980E-01 | 5.333980E-01 | 5.333980E-01 |
| $-2.000E-01$ | 6.394156E-01 | 6.394156E-01 | 6.394156E-01 | 6.394156E-01 | 6.394156E-01 | 6.394156E-01 | 6.394156E-01 |
| $0.000E+00$ | 7.871976E-01 | 7.871976E-01 | 7.871976E-01 | 7.871976E-01 | 7.871976E-01 | 7.871976E-01 | 7.871976E-01 |
| $0.000E+00$ | 2.306036E-01 | 2.306036E-01 | 2.306036E-01 | 2.306036E-01 | 2.306036E-01 | 2.306036E-01 | 2.306036E-01 |
| $2.000E-01$ | 3.383456E-01 | 3.383456E-01 | 3.383456E-01 | 3.383456E-01 | 3.383456E-01 | 3.383456E-01 | 3.383456E-01 |
| $4.000E-01$ | 4.410439E-01 | 4.410439E-01 | 4.410439E-01 | 4.410439E-01 | 4.410439E-01 | 4.410439E-01 | 4.410439E-01 |
| $6.000E-01$ | 5.285819E-01 | 5.285819E-01 | 5.285819E-01 | 5.285819E-01 | 5.285819E-01 | 5.285819E-01 | 5.285819E-01 |
| $8.000E-01$ | 5.963301E-01 | 5.963301E-01 | 5.963301E-01 | 5.963301E-01 | 5.963301E-01 | 5.963301E-01 | 5.963301E-01 |
| $1.000E+00$ | 6.484839E-01 | 6.484839E-01 | 6.484839E-01 | 6.484839E-01 | 6.484839E-01 | 6.484839E-01 | 6.484839E-01 |
VIII. CONCLUSION

A new numerical neutron transport solution, called LNM, for a 1D slab with isotropic scattering is established for a surface source. The solution features Lagrange interpolation in combination with Gauss Legendre quadrature. The method seems stable, at least for an isotropic source, giving what is believed to be the first ever 1D slab benchmark to nearly nine places. The LNM approach is quite revolutionary as it relatively simply evaluates Case’s elegant solution without applying other than full-range orthogonality. A remaining issue is how to effectively treat the beam source analytically. One can approximate the delta function as its formal Legendre expansion, but only five digits of precision result. Thus, one cannot consider this a true extreme benchmark until the beam source is resolved, which will require an alternative procedure for the expansion coefficients (Ref. 15). However, there is no escaping that resolution of a benchmark for \( c < 0.1 \) will require multiprecision arithmetic.

I leave you with a thought expressed by Professor Norman McCormick, a noted transport theorist who is an expert in Caseology, regarding SEEs (Ref. 16):

...It seems safe to state that for someone interested only in solving practical nuclear engineering problems, eigenfunction expansions will have little appeal. If, however, one also seeks a thorough mathematical understanding of those problems, such a study may be quite rewarding and may provide excellent training in applied mathematics.

Dedication

This work is dedicated to my dear colleague, friend, and inspiration, Noel R. Corngold.

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