Spectral singularities of a general point interaction

Ali Mostafazadeh

Department of Mathematics, Koç University, 34450 Sarıyer, Istanbul, Turkey
E-mail: amostafazadeh@ku.edu.tr

Received 19 May 2011, in final form 25 July 2011
Published 22 August 2011
Online at stacks.iop.org/JPhysA/44/375302

Abstract
We study the problem of locating spectral singularities of a general complex point interaction with a support at a single point. We also determine the bound states, examine the special cases where the point interaction is $\mathcal{P}$-, $\mathcal{T}$- and $\mathcal{P}\mathcal{T}$- symmetric, and explore the issue of the coalescence of spectral singularities and bound states.

PACS number: 03.65.-w

(Some figures in this article are in colour only in the electronic version)

1. Introduction

In elementary courses on quantum mechanics we learn that the condition that observables are Hermitian operators\(^1\) ensures the reality of their spectrum. This is an indispensable requirement for the applicability of the quantum measurement theory \([1]\), for the points of the spectrum correspond to readings of a measuring device. Surprisingly, the obvious fact that the reality of the spectrum of an operator does not necessarily mean that it is Hermitian did not play much of a role in our understanding of quantum mechanics until the recent discovery of a large class of non-Hermitian $\mathcal{P}\mathcal{T}$-symmetric Schrödinger operators with a real spectrum \([2]\). This led to a great deal of excitement among some theoretical physicists. At first, it looked as if we could use these kinds of non-Hermitian operators to obtain a quantum theory that differed (if not generalized or even replaced) the standard quantum mechanics \([3, 4]\). But soon it became clear that the former theory is an equivalent representation of the latter \([5–7]\). The key was to note that not only the spectrum but the expectation values, that represented all the physical quantities, must be real. It turns out that the reality of expectation values is a stronger condition than the reality of the spectrum. In fact, it implies that the operator must be Hermitian with respect to the inner product used to compute the expectation values \([5]\). This in turn brings up the possibility of the use of non-standard inner products in quantum mechanics \([8, 3, 9, 4, 5]\). In short, we can use non-Hermitian operators as observables of a (unitary)\(^2\)

\(^1\) Throughout this paper, we follow von Neumann’s convention of identifying the terms ‘Hermitian’ and ‘self-adjoint’ \([1]\).
quantum system provided that we Hermitize them by defining the physical Hilbert space appropriately. In trying to implement this procedure to a delta-function potential with a complex coupling constant [10], it was noted that the standard computational techniques [5] for determining the inner product (or the corresponding metric operator) of the physical Hilbert space failed when the coupling constant was purely imaginary. This marked the occurrence of an intriguing mathematical phenomenon known as a spectral singularity [11, 12]. In [13] we explored this phenomenon for a point interaction consisting of two delta-function potentials with complex coupling constants. In [14, 15] we provided the physical meaning of spectral singularities and outlined their possible realizations and applications in optics. This was followed by further study of the physical implications of spectral singularities [16]. The purpose of this paper is to examine spectral singularities of a general point interaction with support at a single point.

First we give the definition of the point interaction in question.

Let \( \psi \) be a solution of the time-independent Schrödinger equation

\[
-\psi''(x) = k^2 \psi(x), \quad \text{for} \quad x \in \mathbb{R} \setminus \{0\},
\]

and \( \psi_- \) and \( \psi_+ \) be respectively the restrictions of \( \psi \) to the sets of negative and positive real numbers, i.e.

\[
\psi_{\pm}(x) := \psi(x) \quad \text{for} \quad \pm x > 0. \quad (2)
\]

We also set \( \psi_{\pm}(0) := \lim_{\epsilon \to 0^\pm} \psi_{\pm}(\epsilon) \) and introduce the two-component wavefunction

\[
\Psi_{\pm}(x) := \begin{pmatrix} \psi_{\pm}(x) \\ \psi'_{\pm}(x) \end{pmatrix} \quad \text{for} \quad \pm x \geq 0. \quad (3)
\]

Then we can define the point interaction of interest by imposing the matching condition

\[
\Psi_+(0) = B \Psi_-(0), \quad B := \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{C}. \quad (4)
\]

Depending on the choice of the coupling constants \( a, b, c, d \), this interaction may display \( \mathcal{P}, \mathcal{T} \) or \( \mathcal{PT} \)-symmetry. The effects of \( \mathcal{PT} \)-symmetry on the spectrum of this class of point interactions have been studied in [17]. Here, we consider the problem of locating spectral singularities of these interactions. We will also study the corresponding bound states (eigenvalues with square-integrable eigenfunctions) using a different approach than the one pursued in [17].

The problem of finding spectral singularities of the above point interaction turns out not to be as trivial as that of a complex delta-function potential [6] and not as complicated as that of a pair of complex delta-function potentials [13]. It is nevertheless exactly solvable and provides a useful toy model to examine the nature of spectral singularities and the effects of symmetries on them. For example it allows for the examination of the possibility of the coalescence of two spectral singularities.

### 2. Spectral singularities and bound states

There are a couple of equivalent descriptions of the notion of spectral singularity for a Schrödinger operator \( H \) defined on the real line [12]. The most straightforward of these is in terms of the coalescence of the so-called Jost solutions of the eigenvalue equation, \( H \psi = k^2 \psi \). These are the solutions \( \psi_{k\pm} \) fulfilling the asymptotic boundary conditions: \( \psi_{k\pm} \to e^{\pm ikx} \) as \( x \to \pm \infty \). More specifically, given a real number \( k \) we say that \( k^2 \) is a spectral singularity of \( H \), if \( \psi_{k-} \) and \( \psi_{k+} \) are proportional.
As discussed in [13], both the spectral singularities and bound states correspond to zeros of the $M_{22}$ entry of the transfer matrix $M$ of the system. For the point interaction given by (4),

$$\psi_{\pm}(x) = A_{\pm}e^{ikx} + B_{\pm}e^{-ikx},$$

and $M$ is the $2 \times 2$ matrix satisfying

$$\begin{pmatrix} A_+ \\ B_+ \end{pmatrix} = M \begin{pmatrix} A_- \\ B_- \end{pmatrix}.$$  \hspace{1cm} (6)

Combining (3)–(6), we find

$$M = N^{-1}BN = \frac{-i}{2k} \begin{pmatrix} -bk^2 + i(a + \delta)k + \epsilon & bk^2 + i(a - \delta)k + \epsilon \\ -bk^2 + i(a - \delta)k - \epsilon & bk^2 + i(a + \delta)k - \epsilon \end{pmatrix},$$

where

$$N := \begin{pmatrix} 1 & 1 \\ ik & -ik \end{pmatrix}.$$  \hspace{1cm} (7)

Because spectral singularities are given by real zeros of $M_{22}$, according to (7), they correspond to real $k$ values for which

$$bk^2 + i(a + \delta)k - \epsilon = 0.$$  \hspace{1cm} (8)

Bound states are given by the complex solutions of this equation that have a positive imaginary part [13].

Recall that the transfer matrix for a piecewise continuous scattering potentials has unit determinant [13]. For the point interaction, we consider

$$\det M = \det B = \delta \alpha - bc.$$  \hspace{1cm} (9)

Therefore, for the point interactions that are obtained as ‘limits’ of a sequence of piecewise continuous functions, $\delta \alpha - bc = 1$. We will call the point interactions violating this condition *anomalous point interactions*. 

In order to examine the solutions of (8) we consider the following cases.

**Case I.** $b \neq 0$: in this case (8) gives

$$k = -i(\mu \pm \sqrt{\mu^2 - \nu}),$$

where

$$\mu := \frac{a + \delta}{2b}, \quad \nu := \frac{c}{b}.$$  \hspace{1cm} (11)

Therefore a spectral singularity appears whenever

$$\text{Re}(\mu \pm \sqrt{\mu^2 - \nu}) = 0,$$  \hspace{1cm} (12)

and a bound state, with a possibly complex energy, $k^2 = -(\mu \pm \sqrt{\mu^2 - \nu})^2$, exists if

$$\text{Re}(\mu \pm \sqrt{\mu^2 - \nu}) < 0.$$  \hspace{1cm} (13)

The following are some notable special cases.

(I.a) $\delta = -\alpha$ (i.e. $\text{tr } B = 0$): then $\mu = 0$, $k = \pm \sqrt{\nu}$ and conditions (12) and (13) become $\nu \in \mathbb{R}^+$ and $\text{Im}(\pm \sqrt{\nu}) < 0$, respectively.

(I.b) $c = 0$: then $\nu = 0$, $k = -2i\mu$, we have a spectral singularity if $\mu$ is imaginary and a bound state if $\text{Re}(\mu) < 0$.

(I.c) $\mu^2 = \nu$: then the left-hand side of (8) has a double root, namely $k = -i\mu$. This corresponds to a spectral singularity, if $\text{Re}(\mu) = 0$ and $\nu \in \mathbb{R}^-$. It gives a bound state of energy $k^2 = -\mu^2 = -\nu$, if $\text{Re}(\mu) < 0$. 


Case II. $b = 0$: in this case, we consider the following two possibilities.

(II.a) $\alpha + \delta = \text{tr } B \neq 0$: then $k = -i e / (\alpha + \delta)$, and we have a spectral singularity if $\text{Re}(e / (\alpha + \delta)) = 0$ and a bound states if $\text{Re}(e / (\alpha + \delta)) < 0$. A concrete example is the delta-function potential with a possibly complex coupling constant $\delta$ which corresponds to the choice: $\alpha = \delta = 1$, $b = 0$ and $\epsilon = 3$. These imply $e / (\alpha + \delta) = 3/2$. Therefore, the system has a spectral singularity if $\delta$ is purely imaginary and a bound state if $\text{Re}(\delta) < 0$, as noted in [10, 6].

(II.b) $\alpha + \delta = \text{tr } B = 0$: then the condition of the existence of a spectral singularity or a bound state, namely $M_{22} = 0$, implies that $\epsilon = 0$. In this case $B = \alpha \sigma_3$ and $M = \alpha \sigma_1$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

In particular, $M$ is independent of $k$, $M_{22}$ vanishes identically, and the interaction is anomalous for $\alpha \neq \pm i$.

3. Coalescing spectral singularities and bound states

Consider case I of the preceding section, i.e. $b \neq 0$. Suppose that $\mu_r := \text{Re}(\mu) \leq 0$ and $\nu = (1 + \frac{1}{2})\mu^2$, where $\epsilon \in [-1, 1]$. Then (10) takes the form

$$k = -i \left(1 + \frac{\sqrt{-\epsilon}}{2}\right) \mu,$$

and we find

$$\text{Re}(k) = \begin{cases} 
\left(1 + \frac{\sqrt{\epsilon}}{2}\right) \mu_i & \text{for } -1 \leq \epsilon < 0, \\
\mu_i & \text{for } \epsilon = 0, \\
\mu_i \pm \frac{\sqrt{\epsilon} |\mu|}{2} & \text{for } 0 < \epsilon \leq 1, 
\end{cases}$$

$$\text{Im}(k) = \begin{cases} 
- \left(1 + \frac{\sqrt{\epsilon}}{2}\right) \mu_r & \text{for } -1 \leq \epsilon < 0, \\
- \mu_r & \text{for } \epsilon = 0, \\
- \mu_r \pm \frac{\sqrt{\epsilon} |\mu|}{2} & \text{for } 0 < \epsilon \leq 1, 
\end{cases}$$

where $\mu_i := \text{Im}(\mu)$.

According to (16), because $\mu_r \leq 0$ the system has spectral singularities or bound states. Specifically, if $\mu_r < 0$, then $\text{Im}(k) > 0$ and for $\epsilon \in [-1, 0)$ there are a pair of bound states that coalesce into a single bound state at $\epsilon = 0$ with energy $-\mu^2$. This marks an exceptional point [18]. For $\epsilon \in (0, 1]$, the system acquires a pair of bound states, provided that $|\mu_i| < 2 \mu_r$. If $|\mu_i| \sqrt{\epsilon} = -2 \mu_r$, then there will be a bound state and a spectral singularity. If $|\mu_i| \sqrt{\epsilon} > -2 \mu_r$, then there will be a single bound state as well as a solution of the Schrödinger equation that grows exponentially as $x \to \pm \infty$.

Figure 1 shows the plots of $\text{Re}(k)$ and $\text{Im}(k)$ for the case that $\mu = -1 + 4i$. As $\epsilon$ changes from $-1$ to 1, the two bound states coalesce at $\epsilon = 0$ and then split into another pair of bound states that survive as $\epsilon$ ranges between 0 and $\frac{1}{2}$. For $\epsilon = \frac{1}{2}$, one of these turns into a spectral singularity, and for $\epsilon \in (\frac{1}{2}, 1)$ it turns into a non-normalizable solution of the Schrödinger equation. The latter corresponds to the part of the graph of $\text{Im}(k)$ that appears below the $\epsilon$-axis.

A similar scenario holds for the case that $\mu_r = 0$ and $\mu_i \neq 0$. Then for $\epsilon \in [-1, 0)$, the system has a pair of spectral singularities that coalesce at $\epsilon = 0$. For $\epsilon \in (0, 1]$ the resulting
Figure 1. Graphs of $\text{Re}(k)$ (the thin blue curve) and $\text{Im}(k)$ (the thick purple curve) as a function of $\epsilon \in [-1, 1]$ for $\mu = -1 + 4i, \nu = (1 + \frac{\epsilon}{4})\mu^2$. The dashed (grey) line is the graph of $\epsilon = \frac{1}{4}$. As $\epsilon$ increases starting from $\epsilon = -1$, the initial pair of bound states coalesce at $\epsilon = 0$ and then splits into another pair of bound states. One of these only survives until $\epsilon$ reaches the critical value $1/4$ at which it turns into a spectral singularity and then disappears from the spectrum.

Figure 2. Graphs of $\text{Re}(k)$ (the thin blue curve) and $\text{Im}(k)$ (the thick purple curve) as a function of $\epsilon \in [-1, 1]$ for $\mu = 2i$ and $\nu = -(4 + \epsilon)$. As $\epsilon$ increases starting from $\epsilon = -1$, the initial pair of spectral singularities coalesces at $\epsilon = 0$. For $\epsilon > 0$, the system has a bound state and no spectral singularities.

(second-order) spectral singularity turns into a bound state with $k = (1 + i\frac{\sqrt{\epsilon}}{2})\mu$. There also appears a solution of the Schrödinger equation that grows exponentially as $x \to \pm \infty$. Figure 2 shows this behavior for $\mu = 2i$.

At $\epsilon = 0$ both the coalescing bound states and spectral singularities correspond to a repeated root of $M_{22}$, equivalently a second-order pole of (the singular eigenvalue of) the $S$-matrix [14].
4. \(\mathcal{P}, \mathcal{T}\) and \(\mathcal{PT}\)-symmetries

In this section, we examine the consequences of imposing \(\mathcal{P}, \mathcal{T}\) and \(\mathcal{PT}\)-symmetries on the point interaction (4) and their spectral singularities and bound states.

4.1. \(\mathcal{P}\)-symmetry

Let \(\mathcal{P}\) be the parity (reflection) operator acting in the space of all differentiable complex-valued functions \(\psi : \mathbb{R} \to \mathbb{C}\). Then for all \(x \in \mathbb{R}\), we have \((\mathcal{P}\psi)(x) := \psi(-x)\) and \((\mathcal{P}\psi')(x) = -\psi'(-x)\). Therefore, in terms of the two-component wavefunctions \(\Psi := (\psi, \psi')\), we have
\[
(\mathcal{P}\Psi)(x) = \sigma_3 \Psi(-x). \tag{17}
\]
We say that the point interaction (4) is \(\mathcal{P}\)-invariant (has \(\mathcal{P}\)-symmetry) if
\[
(\mathcal{P}\Psi_+)(0) = \mathbf{B}(\mathcal{P}\Psi_-)(0). \tag{18}
\]
We can use this relation and (17) to obtain the following simple expression for \(\mathcal{P}\)-invariance of the point interaction (4):
\[
\mathbf{B}\sigma_3\mathbf{B} = \sigma_3. \tag{19}
\]
In terms of the entries of \(\mathbf{B}\), this is equivalent to
\[
a^2 - bc = 1, \quad \varnothing = \pm \alpha, \quad b(a - \varnothing) = c(a - \varnothing) = 0. \tag{20}
\]
Next, consider the problem of spectral singularities and bound states for \(\mathcal{P}\)-invariant point interactions.

(Case I) \(b \neq 0\): then \(\varnothing = \alpha, \epsilon = (a^2 - 1)/b, \mu = a/b, \nu = (a^2 - 1)/b^2, \mu^2 - \nu = 1/b^2\) and \(k = -i(a \pm 1)/b\). Therefore, a spectral singularity exists if \(\text{Re}((a \pm 1)/b) < 0\), and a bound states arises if \(\text{Re}((a \pm 1)/b) < 0\).

(Case II.a) \(b = 0\) and \(a + \varnothing \neq 0\): then \(a = \varnothing = \pm 1, k = \mp i\epsilon/2\), a spectral singularity appears if \(\text{Re}(\epsilon) = 0\), and a bound state exists if \(\text{Re}(\mp \epsilon) > 0\).

(Case II.b) \(b = 0\) and \(a + \varnothing = 0\): then \(\varnothing = -a = \mp 1\), the interaction is anomalous and spectral singularities and bound states exist for \(k \in \mathbb{R}^+\) and \(\text{Im}(k) > 0\), respectively.

4.2. \(\mathcal{T}\)-symmetry

We identify the time-reversal operator \(\mathcal{T}\) as the operator that acts on complex-valued functions \(\psi : \mathbb{R} \to \mathbb{C}\) according to \((\mathcal{T}\psi)(x) := \psi(x)^*\). The point interaction (4) is the time-reversal invariant (has \(\mathcal{T}\)-symmetry), if
\[
(\mathcal{T}\Psi_+)(0) = \mathbf{B}(\mathcal{T}\Psi_-)(0), \tag{21}
\]
where the action of \(\mathcal{T}\) on a two-component wavefunction \(\Psi\) is defined componentwise. It is easy to see that this relation is equivalent to the requirement that \(\mathbf{B}\) is a real matrix, i.e. \(a, b, \epsilon\) and \(\varnothing\) must be real. In this case, we can summarize the conditions for the existence of spectral singularities and bound states as follows.

(Case I) \(b \neq 0\): Then a spectral singularity exists provided that \(\mu = 0\) and \(\nu \in \mathbb{R}^+\). In terms of the entries of \(\mathbf{B}\), these relations take the form of \(\varnothing = -a\) and \(\epsilon/b \in \mathbb{R}^+\), respectively. The \(k\) value associated with this spectral singularity is \(k = \sqrt{\nu} = \sqrt{\epsilon/b}\).

Similarly, a bound state with \(k = \sqrt{\nu - \mu^2 - i\mu}\) exists, if \(\mu < 0\) and \(\nu - \mu^2 > 0\). The latter conditions can also be expressed as \((a + \varnothing)/b < 0\) and \(4bc - (a + \varnothing)^2 > 0\), respectively.
Expressing this relation in terms of the entries of $B = a\sigma_3$, this is equivalent to $PT$. Therefore, the follows:

$\epsilon = b = 0$, and $a + \sigma_0 \neq 0$: then no spectral singularities exist and a bound state with $k = -i\epsilon/(a + \sigma_0)$ is present provided that $\epsilon/(a + \sigma_0) < 0$.

$\epsilon = b = 0$, and $a + \sigma_0 = 0$: then $M_{22} = 0$ implies $\epsilon = 0$. As a result, the interaction is anomalous, and spectral singularities and bound states exist for $k \in \mathbb{R}^+$ and $\text{Im}(k) > 0$, respectively.

4.3 \(PT\)-symmetry

We say that the point interaction (4) is $PT$-invariant or $PT$-symmetric if

$$(PT\Psi_+)(0) = B(PT\Psi_-)(0),$$

(22)

This is equivalent to

$$B^*\sigma_3B = \sigma_3.$$

(23)

Expressing this relation in terms of the entries of $B$ and solving the resulting equations yield

$$a = \sqrt{1 + \epsilon e^\alpha}, \quad b = \epsilon_2 e^{i(\alpha + \delta)/2}, \quad \epsilon = \epsilon_2 e^{i(\alpha + \delta)/2}, \quad \delta = \sqrt{1 + \epsilon e^\alpha} e^{i\delta},$$

(24)

where $\epsilon_2 = \pm 1$, $b, c \in [0, \infty)$, $\alpha, \delta \in [0, 2\pi)$ and

$$\epsilon_1 = \begin{cases} +1 & \text{if } bc > 1, \\ \pm 1 & \text{if } bc < 1. \end{cases}$$

(25)

Therefore, the $PT$-symmetric point interactions are determined by

$$B = e^{i(\alpha + \delta)/2}\left(\begin{array}{cc} \sqrt{1 + \epsilon e^\alpha} e^{i(\alpha - \delta)/2} & \epsilon_1 e_2 b \\ \epsilon_2 c & \sqrt{1 + \epsilon e^\alpha} e^{-i(\alpha + \delta)/2} \end{array}\right).$$

(26)

In particular, $\text{det}B = e^{i(\alpha + \delta)}$, and the interaction is anomalous unless $\alpha + \delta$ is an integer multiple of $2\pi$.

The conditions for the existence of spectral singularities and bound states are as follows:

(Case I) $b \neq 0$: a straightforward consequence of (24) is that both the parameters $\mu$ and $v$ take real values. More specifically, we have

$$\mu = \frac{\sqrt{1 + \epsilon e^\alpha} \cos \left(\frac{\alpha - \delta}{2}\right)}{\epsilon_1 \epsilon_2 b}, \quad v = \frac{\epsilon_1 c}{b}, \quad \mu^2 - v = \frac{(1 + \epsilon e^\alpha) \cos^2 \left(\frac{\alpha - \delta}{2}\right) - c^2}{b^2}.$$  

(27)

Now, if we impose the condition of the presence of spectral singularities (12), we find $\mu = 0$ and $v > 0$. In view of (27), these imply that $\epsilon_1 = +1$ and $\delta = \alpha + (2\ell + 1)\pi$, where $\ell$ is an arbitrary integer. Therefore, the $PT$-symmetric point interactions that have a spectral singularity (with $k = \sqrt{\mu}$) are given by

$$B = e^{i\alpha}\left(\begin{array}{cc} \sqrt{1 + \epsilon e^\alpha} & \epsilon e_2 b \\ \epsilon_2 c & -\sqrt{1 + \epsilon e^\alpha} \end{array}\right),$$

(28)

where $\epsilon = (-1)^\ell \epsilon_2 = \pm 1$. Similarly, we can check that the system has a bound state provided that $\mu < 0$. This is equivalent to $\epsilon_1 \epsilon_2 \cos[(\alpha - \delta)/2] < 0$. In this case there are three possibilities.

(1) $\mu^2 - v > 0$: then there is a pair of bound states with real and negative energies $k^2 = -(2\mu^2 - v \pm \sqrt{\mu^2 - v})$.

(2) $\mu^2 - v = 0$: this corresponds to an exceptional point [18], where there is single bound state with a real and negative energy $k^2 = -\mu^2$. 

7
(3) $\mu^2 - v < 0$: then there is a pair of bound states with complex-conjugate energies $k^2 = v - 2\mu^2 \pm i\mu\sqrt{v - \mu^2}$.

(Case II.a) $b = 0$ and $a + d \neq 0$: in this case there are no spectral singularities, but a bound state exists provided that $\epsilon_2 \cos[(\alpha - \delta)/2] < 0$. It has a real and negative energy given by $k^2 = -\frac{1}{2}c^2 \sec^2[(\alpha - \delta)/2]$.

(Case II.b) $b = 0$ and $a + d = 0$: then the condition $M_{22} = 0$ implies $c = 0$,

$$B = \begin{pmatrix} e^{ia} & 0 \\ 0 & e^{-ia} \end{pmatrix},$$

the interaction is anomalous unless $\alpha = \frac{\pi}{4}$ or $\frac{3\pi}{4}$, a spectral singularity arises for all $k \in \mathbb{R}$ and there is a bound state for all $k \in \mathbb{C}$ with $\text{Im}(k) > 0$.

5. Concluding remarks

In this paper, we present a complete solution for the problem of spectral singularities of a general point interaction with a support at a single point. We also examine the consequences of the presence of $\mathcal{P}$-, $\mathcal{T}$- and $\mathcal{PT}$-symmetries. Unlike the case of complex delta-function potential and double-delta-function potential, for a generic point interaction that we consider, the function whose zeros give the spectral singularities and bound states is a quadratic polynomial in $k$. This in turn implies the possibility of having a spectral singularity or a bound state that is related to a second-order zero of this polynomial. For the case of a bound state, this corresponds to a degeneracy or exceptional point. The latter leads to a well-known type of geometric phase [18]. The analogy with coalescing spectral singularities calls for a thorough examination of the geometric phase problem for systems supporting second- and higher order spectral singularities.

Acknowledgments

This work has been supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK) in the framework of the project no 110T611 and the Turkish Academy of Sciences (TÜBA).

References

[1] von Neumann J 1996 Mathematical Foundations of Quantum Mechanics (Princeton, NJ: Princeton University Press)
[2] Bender C M and Boettcher S 1998 Phys. Rev. Lett. 80 5243
[3] Bender C M, Brody D C and Jones H F 2002 Phys. Rev. Lett. 89 270401
Bender C M, Brody D C and Jones H F 2004 Phys. Rev. Lett. 92 119902 (erratum)
[4] Bender C M 2007 Rep. Prog. Phys. 70 947
[5] Mostafazadeh A 2010 Int. J. Geom. Meth. Mod. Phys. 7 1191
Mostafazadeh A 2008 arXiv:0810.5643
[6] Mostafazadeh A 2010 Phys. Scr. 82 038110
[7] Mostafazadeh A 2003 J. Phys. A: Math. Gen. 36 7081
Mostafazadeh A 2004 Czech J. Phys. 54 1125
[8] Mostafazadeh A 2002 J. Math. Phys. 43 205
Mostafazadeh A 2002 J. Math. Phys. 43 2814
Mostafazadeh A 2002 J. Math. Phys. 43 3944
[9] Mostafazadeh A and Batal A 2004 J. Phys. A: Math. Gen. 37 11645
[10] Mostafazadeh A 2006 J. Phys. A: Math. Gen. 39 13495
[11] Naimark M A 1954 *Trudy Moscov. Mat. Obsc.* 3 181 (in Russian)
Naimark M A 1960 *Am. Math. Soc. Transl.* (2) 16 103 (Engl. transl.)
Kemp R R D 1958 *Can. J. Math.* 10 447
Schwartz J 1960 *Commun. Pure Appl. Math.* 13 609
[12] Guseinov G Sh 2009 *Pramana J. Phys.* 73 587
[13] Mostafazadeh A and Mehri-Dehnavi H 2009 *J. Phys. A: Math. Theor.* 42 125303
[14] Mostafazadeh A 2009 *Phys. Rev. Lett.* 102 220402
[15] Mostafazadeh A 2009 *Phys. Rev.* A 80 032711
Mostafazadeh A 2011 *Phys. Rev.* A 83 045801
[16] Longhi S 2009 *Phys. Rev.* B 80 165125
Longhi S 2010 *Phys. Rev.* A 81 022102
Samsonov B F 2010 arXiv:1007.4421
[17] Albeverio S, Fei S-M and Kurasov P 2002 *Lett. Math. Phys.* 59 227
[18] Heiss W D, Müller M and Rotter I 1998 *Phys. Rev.* E 58 2894
Heiss W D 2004 *J. Phys. A: Math. Gen.* 37 2455
Stehmann T, Heiss W D and Scholtz F G 2004 *J. Phys. A: Math. Gen.* 37 7813
Dembovski C, Dietz B, Gräf H-D, Harney H L, Heine A, Heiss W D and Richter A 2004 *Phys. Rev.* E 69 056216
Mailybaev A A, Kirillov O N and Seyranian A P 2005 *Phys. Rev.* A 72 014104
Mehri-Dehnavi H and Mostafazadeh A 2008 *J. Math. Phys.* 49 082105