QUANTUM PARTICLE BEHAVIOR IN CLASSICALLY SINGULAR SPACETIMES

D. A. KONKOWSKI
Department of Mathematics, U.S. Naval Academy
Annapolis, Maryland, 21012, USA
E-mail: dak@usna.edu

T.M. HELLIWELL
Department of Physics, Harvey Mudd College
Claremont, California, 91711, USA
E-mail: helliwell@HMC.edu

We review the classical and quantum singularity structure of a broad class of spacetimes with asymptotically power-law behavior near the origin. Quantum considerations “heal” a large class of scalar curvature singularities.

1. Introduction

The question addressed in this review is: What happens if instead of classical particle paths (time-like and null geodesics) one uses quantum mechanical particles to identify singularities? The answer for any asymptotically power-law space-time is given. This conference proceeding is based on articles by the authors and by K. Lake.

2. Types of Singularities

2.1. Classical Singularities

A classical singularity is indicated by incomplete geodesics or incomplete paths of bounded acceleration in a maximal spacetime. Since, by definition, a spacetime is smooth, all irregular points (singularities) have been excised; a singular point is a boundary point of the spacetime. There are three different types of singularity: quasi-regular, non-scalar curvature and scalar curvature. Whereas quasi-regular singularities are topological, curvature singularities are indicated by diverging components of the Riemann tensor when it is evaluated in a parallel-propagated orthonormal frame carried along a causal curve ending at the singularity.

2.2. Quantum Singularities

A spacetime is QM (quantum-mechanically) nonsingular if the evolution of a test scalar wave packet, representing the quantum particle, is uniquely determined by the initial wave packet, manifold and metric, without having to put boundary conditions at the singularity. Technically, a static ST (spacetime) is QM-singular if the spatial portion of the Klein-Gordon operator is not essentially self-adjoint on $C_0^\infty(\Sigma)$ in $L^2(\Sigma)$ where $\Sigma$ is a spatial slice.
3. Asymptotically Power-Law Spacetimes
We consider a class of spacetimes that can be written in power-law metric form in the limit of small $r$,

$$ds^2 = -r^\alpha dt^2 + r^\beta dr^2 + C^{-2}r^\gamma d\theta^2 + r^\delta(dz + A d\theta)^2$$ (1)

where $\beta, \gamma, \delta, C, A$ are constant parameters and the variables have the usual ranges. We are particularly interested in the metrics at small $r$, because we suppose that if the spacetime has a classical curvature singularity (and nearly all of these do), it occurs at $r = 0$. We can eliminate $\alpha$ by rescaling $r$ which results in two separate metric types:

- **Type I:**
  $$ds^2 = r^\beta (-dt^2 + dr^2) + C^{-2}r^\gamma d\theta^2 + r^\delta(dz + A d\theta)^2 \quad \alpha \neq \beta + 2.$$ (2)

- **Type II:**
  $$ds^2 = -r^{\beta+2} dt^2 + r^\beta dr^2 + C^{-2}r^\gamma d\theta^2 + r^\delta(dz + A d\theta)^2 \quad \alpha = \beta + 2.$$ (3)

4. Classical Singularity Analysis
Except for isolated values of $\beta, \gamma, \delta, C, A$ all of these power-law spacetimes have diverging scalar polynomial invariants if and only if $\beta > -2$.

4.1. **Type I Spacetimes**
Lake has shown that in Type I STs $r = 0$ is timelike, naked and at a finite affine distance if and only if $\beta > -1$, implying that there is a classical singularity at $r = 0$ if and only if $\beta > -1$.

4.2. **Type II Spacetimes**
Likewise, Lake has shown that in Type II STs $r = 0$ is null, naked and at a finite affine distance and thus is a classical singularity for all $\beta > -2$.

5. Quantum Singularity Analysis
To study the quantum particle propagation in these spacetimes (for simplicity, we take $A = 0$), we use massive scalar particles described by the Klein-Gordon equation and the "limit point - limit circle" criterion of Weyl. This means that, in

*If $\alpha = \beta = \gamma = \delta = 0, C \neq 1$ indicates a quasi-regular singularity (a disclination) and $A \neq 0$ indicates a quasi-regular singularity (a dislocation) (see, e.g., Konkowski and Helliwell).
particular, we study the radial equation in a one-dimensional Schrödinger form with a ‘potential’ and determine the number of solutions that are square integrable. If we obtain a unique solution, without placing boundary conditions at the location of the classical singularity, we can then say that the Klein-Gordon operator is essentially self-adjoint and the spacetime is QM-nonsingular.

5.1. Type I Spacetimes

There is a quantum singularity "bowl" in parameter space for these metrics. The bowl is bounded by (1) a bottom which is formed from a $\beta = -2$ base plane and (2) the sides which are composed of (a) two vertical planes with $\gamma + \delta = 6$ and $\gamma + \delta = -2$ and (b) two tilted planes with $\delta = \beta + 2$ and $\gamma = \beta + 2$. Points within the bowl are QM singular; points outside the bowl are QM non-singular.

5.2. Type II Spacetimes

Type II STs are globally hyperbolic; the wave operator in this case must be essentially self-adjoint, so these spacetimes contain no quantum singularities. It is easy to verify this conclusion directly by checking the essential self-adjointness of the wave operator using the "limit point - limit circle" technique.

6. Conclusions

A large class of classically singular asymptotically power-law spacetimes has been shown to be quantum mechanically non-singular. Invoking an energy condition (e.g., weak or strong) can eliminate more singular spacetimes, but no choice completely eradicates them.

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