Abstract

We study the simplicial complex that arises from non-attacking rook placements on a subclass of Ferrers boards that have \(a_i\) rows of length \(i\) where \(a_i > 0\) and \(i \leq n\) for some positive integer \(n\). In particular, we will investigate enumerative properties of their facets, their homotopy type, and homology.

1 Introduction

A simplicial complex \(\Delta\) on a finite set \(X\) is a collection of subsets of \(X\) closed under inclusion. A chessboard complex is the collection of all non-attacking rook positions on an \(m \times n\) chessboard. It is clear that this is a simplicial complex as the removal of one rook from an admissible rook placement yields another admissible rook placement. Notice that a placement of \(i + 1\) rooks corresponds to a simplex of dimension \(i\).

Chessboard complexes first appeared in the 1979 thesis of Garst [10] concerning Tits coset complexes. By setting \(G = S_n\) and \(G_i = \{\sigma | \sigma(i) = i\}\) for \(i = 1, \ldots, m \leq n\), Garst obtained the chessboard complex \(M_{m,n} = \Delta(G; G_1, \ldots, G_m)\). Here, \(\Delta(G; G_1, \ldots, G_m)\) is the simplicial complex whose vertices are the cosets of the subgroups and whose facets have the form \(\{gG_1, \ldots, gG_m\}\), for \(g \in G\).

The chessboard complex later appeared in a paper by Björner, Lovász, Vrećica, and Živaljević [3] where they gave a bound on the connectivity of the chessboard complex and conjectured that their bound was sharp. This conjecture was shown to be true by Shareshian and Wachs [14]. In that same paper, Shareshian and Wachs also showed that if the chessboard met certain criteria, then it contained torsion in its homology.

In this paper, we will be studying the topology of the simplicial complex that arises from non-attacking rook placements on triangular boards. The triangular board \(\Psi_{a_n,\ldots,a_1}\) is a left justified board with \(a_i > 0\) rows of length \(i\) for \(1 \leq i \leq n\). In other words, given a positive integer \(n\), the triangular board \(\Psi_{a_n,\ldots,a_1}\) is the Ferrers board associated with the partition \(\pi = (n^{a_n}, \ldots, 1^{a_1})\) with \(a_i > 0\), see Figure 1. The squares of the triangular board will be labeled \((i,j)\) for \(i \leq j\) where \(i\) represents the columns (numbered left to right) while \(j\) represents the rows (labeled bottom to top). Motivated by results obtained using [4], we begin by showing that for \(a_i\) large enough, \(\Sigma(\Psi_{a_n,\ldots,a_1})\), the simplicial complex associated with rook placements on \(\Psi_{a_n,\ldots,a_1}\), is a pure complex that is vertex decomposable.

Next, we study the other extremal case. The Stirling complex \(\text{St}(n)\), originally defined by Ehrenborg and Hetyei [3], is the simplicial complex formed by rook placements on the board \(\Psi_{1,1,\ldots,1}\) with \(n\) rows, see Figure 2. It is a known fact that the \(f\)-vector of \(\text{St}(n)\) is given by...
\[ f_i = S(n+1, n+1-i) \] for \( i = 1, \ldots, (n-1) \) where \( S(n, i) \) denotes the Stirling number of the second kind, see [15]. However, this complex is not pure. We begin the study of \( \text{St}(n) \) by enumerating its facets via generating functions and then use Discrete Morse theory to study its topology. We end the paper with some open questions and concluding remarks.

2 Topological Tools

For an introduction to combinatorial topology, basic definitions, and results, we refer the reader to the books by Jonsson [11] and Kozlov [12].

Definition 2.1. For a family \( \Delta \) of sets and a set \( \sigma \) of \( \Delta \), the link \( \text{lk}_\Delta(\sigma) \) is the family of all \( \tau \in \Delta \) such that \( \tau \cap \sigma = \emptyset \), and \( \tau \cup \sigma \in \Delta \). The deletion \( \text{del}_\Delta(\sigma) \) is the family of all \( \tau \in \Delta \) such that \( \tau \cap \sigma = \emptyset \).

Definition 2.2. A simplicial complex \( \Delta \) is vertex decomposable if

1. Every simplex (including \( \emptyset \) and \( \{\emptyset\} \)) is vertex decomposable.
2. \( \Delta \) is pure and contains a 0-cell \( x \) – a shedding vertex – such that \( \text{del}_\Delta(x) \) and \( \text{lk}_\Delta(x) \) are both vertex decomposable.

Showing that a simplicial complex is vertex decomposable is useful in determining the topology of complex as can be seen in the following theorem.

Theorem 2.3. [11, Theorems 3.33 and 3.35] Let \( \Delta \) be a simplicial complex of dimension \( d \). If the complex \( \Delta \) is vertex decomposable, then \( \Delta \) is homotopy equivalent to a wedge of spheres of dimension \( d \).

We recall the following definitions and theorems from discrete Morse theory. See [8, 9, 12] for further details.

Definition 2.4. A partial matching in a poset \( P \) is a partial matching in the underlying graph of the Hasse diagram of \( P \), that is, a subset \( M \subseteq P \times P \) such that \( (x, y) \in M \) implies \( x < y \) and each \( x \in P \) belongs to at most one element of \( M \). For \( (x, y) \in M \) we write \( x = d(y) \) and \( y = u(x) \), where \( d \) and \( u \) stand for down and up, respectively.

Definition 2.5. A partial matching \( M \) on \( P \) is acyclic if there does not exist a cycle

\[ z_1 > d(z_1) < z_2 > d(z_2) < \cdots < z_n > d(z_n) < z_1, \]

in \( P \) with \( n \geq 2 \), and all \( z_i \in P \) distinct. Given a partial matching, the unmatched elements are called critical. If there are no critical elements, the acyclic matching is perfect.

We now state the main result from discrete Morse theory. For a simplicial complex \( \Delta \), let \( \mathcal{F}(\Delta) \) denote the poset of faces of \( \Delta \) ordered by inclusion.

Theorem 2.6. Let \( \Delta \) be a simplicial complex. If \( M \) is an acyclic matching on \( \mathcal{F}(\Delta) - \{\emptyset\} \) and \( k_i \) denotes the number of critical \( i \)-dimensional cells of \( \Delta \), then the complex \( \Delta \) is homotopy equivalent to a CW complex \( \Delta_k \) which has \( k_i \) cells of dimension \( i \).
For us it will be convenient to work with reduced discrete Morse theory, that is, we include the empty set.

**Corollary 2.7.** Let $\Delta$ be a simplicial complex and let $M$ be an acyclic matching on $\mathcal{F}(\Delta)$. Then the complex $\Delta$ is homotopy equivalent to a CW complex $\Delta_k$ which has $k_0 + 1$ cells of dimension 0 and $k_i$ cells of dimension $i$ for $i > 0$.

In particular, if the matching in Corollary 2.7 is perfect, then $\Delta_k$ is contractible. Also, if the matching has exactly one critical cell then $\Delta_k$ is a $d$-sphere where $d$ is the dimension of this cell.

Given a set of critical cells of differing dimension, in general it is difficult to conclude that the CW complex $\Delta_k$ is homotopy equivalent to a wedge of spheres. See Kozlov [13] for a non-trivial example. However, when some critical cells are facets, it may be possible to say more as seen in the following theorem.

**Theorem 2.8.** Let $M$ be a Morse matching on $\mathcal{F}(\Delta)$ with $k_i$ critical cells of dimension $i$. Assume that there are no critical cells of dimension less than $j$ and that all critical cells of dimension $j$ are facets. Then the complex $\Delta$ is homotopy equivalent to the wedge

$$(\mathbb{S}^j)^{k_j} \vee X,$$

where $X$ is a CW complex consisting of one point and $k_i$ i-dimensional cells for $i > j$.

**Proof.** The complex $\Delta$ without the critical cells of dimension $j$ is a CW complex $X$ consisting of one point and $k_i$ i-dimensional cells for $i > j$. Since every face of dimension less than $j$ have been matched, the boundaries of the $j$-dimensional critical cells contract to a point. Since all of these critical cells are maximal, they can be independently added back into the complex. \qed

Kozlov [13] gives a more general sufficient condition on an acyclic Morse matching for the complex to be homotopy equivalent to a wedge of spheres enumerated by the critical cells.

We are interested in finding an acyclic matching on the face poset of the Stirling complex. The Patchwork Theorem [12] gives us a way of constructing one.

**Theorem 2.9.** Assume that $\varphi: P \to Q$ is an order-preserving poset map, and assume that there are acyclic matchings on the fibers $\varphi^{-1}(q)$ for all $q \in Q$. Then the union of these matchings is itself an acyclic matching on $P$.

### 3 Triangular Boards

Let $\Sigma(\Psi_{a_n,\ldots,a_1})$ denote the simplicial complex formed by all non-attacking rook placements on the triangular board. Note, we call a rook placement maximal if no other rook can be added to the placement, that is, every square on the board is attacked.

**Theorem 3.1.** Let $a_1 \geq 1$, $a_n \geq n$, and $a_i \geq i - 1$ for all $i = 2, \ldots, n - 1$. Then the simplicial complex $\Sigma(\Psi_{a_n,\ldots,a_1})$ is vertex decomposable.

We note that this theorem does not extend further in general. A cursory glance at complexes $\Sigma(\Psi_{a_n,\ldots,a_1})$ where $a_i \geq i - 1$ for $i = 1, \ldots, n$, (for example, $\Sigma(\Psi_{2,1,0})$) we see we have a non-pure complex which, in general, is not even non-pure vertex decomposable. In addition, loosening our
conditions to allow \( a_i \geq i - 2 \) for \( i = 1, \ldots, n - 1 \) and \( a_n \geq n \) allows complexes such as \( \Sigma(\Psi_{4,0,0}) \) which is a torus \([3]\).

In order to prove Theorem 3.1, we need the following lemmas. Recall that the squares of the first column of \( \Psi_{a_n, \ldots, a_1} \) are labeled \((1,1), \ldots, (1,p)\) where \( p = \sum_{i=1}^{n} a_i \). Let \( V_j \) denote a collection of the top \( j \) elements of the first column that is, \( V_j = \{(1, p-j+1), \ldots, (1,p)\} \) for \( j = 1, \ldots, p \) and let \( V_0 = \emptyset \).

**Lemma 3.2.** Consider \( \Psi_{a_n, \ldots, a_1} \) with \( a_1 \geq 1, a_n \geq n, \) and \( a_i \geq i - 1 \) for all \( i = 2, \ldots, n - 1 \) Then for \( j = 0, \ldots, p - 1 \) the simplicial complex \( \text{del}_\Sigma(\Psi_{a_n, \ldots, a_1})(V_j) \) is pure of dimension \( n - 1 \).

**Proof.** We have two cases to consider.

Let \( j = 0 \). Then \( \text{del}_\Sigma(\Psi_{a_n, \ldots, a_1})(V_0) = \Sigma(\Psi_{a_n, \ldots, a_1}) \). Any facet of \( \Sigma(\Psi_{a_n, \ldots, a_1}) \) comes from some maximal rook placement on \( \Psi_{a_n, \ldots, a_1} \). Any maximal rook placement must cover the rectangular board \( n \times a_n \). Since \( a_n \geq n \), this requires exactly \( n \) rooks, one in each of the \( n \) columns. Since every column contains a rook, the entire board \( \Psi_{a_n, \ldots, a_1} \) is covered.

Let \( 1 \leq j \leq p - 1 \). Here, \( \text{del}_\Sigma(\Psi_{a_n, \ldots, a_1})(V_j) \) is the Ferrers board with the top \( j \) entries in the first column deleted. Any facet of this board must cover the \( a_n \times n - 1 \) rectangular sub-board created by rows \( p - a_n + 1, \ldots, p \) and columns \( 2, 3, \ldots, n \). Since \( a_n \geq n - 1 \), this requires exactly \( n - 1 \) rooks to cover, one in each of the columns \( 2, 3, \ldots, n \). The first column will contain at least one square (namely \((1,1)\)) that is not covered by any of these \( n - 1 \) rooks. Thus by placing a rook in the first column, we see that any facet of \( \text{del}_\Sigma(\Psi_{a_n, \ldots, a_1})(V_j) \) contains \( n \) rooks.

Since this simplicial complex is pure, it is natural to ask about its topology.

**Lemma 3.3.** If \( a_i \geq i \) for all \( i = 1, \ldots, n \), then \( \Sigma(\Psi_{a_n, \ldots, a_1}) \) is vertex decomposable.

**Proof.** We will proceed by induction on \( n \), the length of the largest row.

For \( n = 1 \), we have a \( 1 \times a_1 \) chessboard which is clearly vertex decomposable. Now assume \( \Sigma(\Psi_{a_k, \ldots, a_1}) \) is vertex decomposable and consider \( \Sigma(\Psi_{a_{k+1}, \ldots, a_1}) \). We maintain our labeling of the vertices using \( p = \sum_{i=1}^{k+1} a_i \). We note that \( \Sigma(\Psi_{a_{k+1}, \ldots, a_1}) \) is pure by Lemma 3.2.
We claim that the square \((1, p)\) is a shedding vertex of \(\Sigma(\Psi_{a_k+1, \ldots, a_1})\). First, \(\text{lk}_\Sigma(\Psi_{a_k+1, \ldots, a_1})(1, p)\) is the set of all faces on the Ferrers board \(\Psi_{a_k+1, \ldots, a_1}\) where we delete the \(p\)th row (i.e. top row) and the first column of \(\Psi_{a_k+1, \ldots, a_1}\). That is,

\[
\text{lk}_\Sigma(\Psi_{a_k+1, \ldots, a_1})(1, p) = \Sigma(\Psi_{a_k+1, a_k, \ldots, a_2}).
\]

Note that the largest row is now length \(k\). Since \(a_i \geq i - 1\) and \(a_k+1 - 1 \geq k\) the link of \((1, p)\) is vertex decomposable by our induction hypothesis.

We now must show that \(\text{del}_\Sigma(\Psi_{a_k+1, \ldots, a_1})(1, p)\) is vertex decomposable by showing that \((1, p - 1)\) is a shedding vertex which begins a recursive process.

At the \(j\)th iteration, we need to show that \(\Delta_j = \text{del}_\Sigma(\Psi_{a_k+1, \ldots, a_1})(V_j)\) is vertex decomposable by showing that \((1, p - j)\) is a shedding vertex. Suppose row \(p - j\) has length \(\ell\). First, \(\text{lk}_{\text{del}_\Delta_j}(1, p - j)\) is the set of all rook placements on the Ferrers board \(\Psi_{a_k+1, \ldots, a_1}\) with row \(p - j\) and the first column deleted. This board has \(a_i\) rows of length \(i - 1\) for \(i = 2, \ldots, \ell + 1, \ldots, k + 1\) and \(a_{\ell+1} - 1\) rows of length \(\ell\). That is

\[
\text{lk}_{\text{del}_\Delta_j}(1, p - j) = \Sigma(\Psi_{a_k+1, \ldots, a_{\ell+1}-1, \ldots, a_2}).
\]

Once again as \(a_i \geq i - 1\) and \(a_{\ell+1} - 1 \geq \ell\), this board is vertex decomposable by our induction hypothesis.

Similarly, the vertex decomposability of \(\text{del}_\Delta_j((1, p - j))\) remains undetermined and we proceed to another iteration of this process.

At the \(p\)th and final step of this process we test the link and deletion of \((1, 1)\) on the board \(\Psi_{a_k+1, \ldots, a_2} \cup \{(1, 1)\}\), where \((1, 1)\) forms its own row and column. This board remains pure by Lemma 3.2. Moreover, \(\Delta_p(1, 1) = \text{del}_\Delta_p((1, 1)) = \Sigma(\Psi_{a_k+1, \ldots, a_2})\) which is vertex decomposable by our induction hypothesis, verifying vertex decomposability by moving backwards through our deletions. Thus \(\Sigma(\Psi_{a_k+1, \ldots, a_1})\) is vertex decomposable.

\textbf{Proof of Theorem 3.1.} Proceed by induction on \(n\) with base case \(n = 2\). Apply the above proof and Lemma 3.3.

Since \(\Sigma(\Psi_{a_n, \ldots, a_1})\) is vertex decomposable for \(a_i \geq i - 1, i = 1, \ldots, n - 1\) and \(a_n \geq n\), we know by Theorem 2.3 that it will be homotopy equivalent to a wedge of spheres of dimension \(n - 1\) or contractible. The number of spheres can be computed by finding the reduced Euler characteristic which is the alternating sum of the \(f\)-vector. Let \(\ell(i)\) denote the length of column \(i\) in \(\Psi_{a_n, \ldots, a_1}\) with \(\ell(i) = \sum_{j=1}^{n} a_j\).

\textbf{Theorem 3.4.} The \(f\)-vector of \(\Sigma(\Psi_{a_n, \ldots, a_1})\) is given by

\[
f_i = \sum_{S \subseteq \binom{[n]}{i+1}} \prod_{j=0}^{i} (\ell(s_j) - j),
\]

where \(S = \{s_0 > s_1 > \cdots > s_i\}\).

\textbf{Proof.} Let \(S\) be the collection of \(i + 1\) columns where the rooks are placed. Notice that \(\ell(s_k) \leq \ell(s_{k+1})\). Therefore, placing a rook in column \(s_k\) removes a possible location to place a rook in
columns \( s_{k+1}, \ldots, s_i \). Thus, there are \( \prod_{j=0}^{i} (\ell(s_j) - j) \) ways to place \( i + 1 \) rooks on these \( i + 1 \) columns. The result follows by summing over all subsets of \( i + 1 \) columns. \qedhere

Theorem 3.1 mirrors the result of Ziegler [16] which says that the (rectangular) chessboard complex \( M_{n,m} \) is vertex decomposable if \( n \geq 2m - 1 \). That is, extending a triangular board, like extending a rectangular board, far enough allows one to conclude that the associated complex is vertex decomposable.

4 Facets of the Stirling Complex

We now turn our attention to the Stirling complex. Recall the Stirling complex \( \text{St}(n) \) is equal to the simplicial complex associated to valid rook placements on the triangular board of size \( n, \Psi_{1, \ldots, 1} \). It is clear that the Stirling complex is not pure. In this section, we will enumerate the facets of the Stirling complex in each dimension.

The \( f \)-vector of the Stirling complex is given by Stirling numbers of the second kind, that is, faces of the Stirling complex are in bijection with partitions. This is done using the map \( R \) where any placement of \( k \) non-attacking rooks gets mapped to the partition \( A \) where if a rook occupies the square \((i,j)\) then \( i \) and \( j + 1 \) are in the same block of the partition \( A \), see [15, Corollary 2.4.2]. In what follows, we show that facets of the Stirling complex are in bijection with a particular subset of partitions.

**Definition 4.1.** Let \( B \) and \( C \) be two disjoint nonempty subsets of \([n]\) such that \( \min(B \cup C) \in B \). Then \( B \) and \( C \) are intertwined if \( \max(B) > \min(C) \). We say a partition \( P \) is intertwined if every pair of blocks in \( P \) is intertwined.

The idea of intertwined partitions first appeared with the use of the intertwining number of a partition in [6] where they were used to provide a combinatorial interpretation for \( q \)-Stirling numbers of the second kind. The following definitions are from [6]. For two integers \( i \) and \( j \), define
the interval \( \text{int}(i, j) \) to be the set
\[
\text{int}(i, j) = \{ n \in \mathbb{Z} : \min(i, j) < n < \max(i, j) \}.
\]

**Definition 4.2.** For two disjoint nonempty subsets \( B \) and \( C \) of \([n]\), define the intertwining number \( \iota(B, C) \) to be
\[
\iota(B, C) = |\{(b, c) \in B \times C : \text{int}(b, c) \cap (B \cup C) = \emptyset\}|.
\]

These two ideas are connected as can be seen in the following proposition.

**Proposition 4.3.** Let \( B \) and \( C \) be two disjoint nonempty subsets of \([n]\) such that \( \min(B \cup C) \in B \). Then \( B \) and \( C \) are intertwined if and only if \( \iota(B, C) \geq 2 \).

**Proof.** (\( \Rightarrow \)) Suppose \( \min(B) < \min(C) < \max(B) \). Let \( b_0 \) be the maximum element of \( B \) such that \( \min(B) \leq b_0 < \min(C) \). Then \( \text{int}(b_0, \min(C)) \cap (B \cup C) = \emptyset \). Let \( c_1 \) be the maximum element of \( C \) such that \( c_1 < \max(B) \) and \( b_1 \) be the minimum element of \( B \) such that \( c_1 < b_1 \). Then \( \text{int}(b_1, c_1) \cap (B \cup C) = \emptyset \). Therefore, the intertwining number is at least 2.

(\( \Leftarrow \)) Suppose \( \max(B) < \min(C) \). Clearly, for \( b \in B \) and \( c \in C \), \( \text{int}(b, c) \cap (B \cup C) \neq \emptyset \) unless \( b = \max(B) \) and \( c = \min(C) \). Thus, the intertwining number is 1. \( \square \)

The bijection between facets of the Stirling complex and intertwined partitions can now be verified.

**Theorem 4.4.** The set of maximal rook placements with \( k \) rooks on a triangular board \( \Psi_{1,\ldots,1} \) of size \( n \) is in bijection with intertwined partitions of \( n+1 \) into \( n+1-k \) blocks.

**Proof.** We first show that a maximal rook placement gives rise to an intertwined partition. Let \( P = (P_1, P_2, \ldots, P_{n+1-k}) \), be a partition of \( n+1 \) into \( n+1-k \) blocks such that there exists two blocks, \( P_i \) and \( P_j \), that are not intertwined. Then, without loss of generality, \( M_i < m_j \) where \( M_i \) is the maximal element of \( P_i \) and \( m_j \) is the minimal element of \( P_j \). We claim there is no maximal placement of \( k \) non-taking rooks, \( R_k \), such that \( R(R_k) = P \).

As \( M_i \) is the maximal element of \( P_i \), there is no rook in the \( M_i \)th column of our triangular board for any rook placement. For otherwise, \( M_i \) would not be the maximal element of \( P_i \) as any rook in this column would force our map \( R \) to place \( M_i \) and \( r \) in the same block for some \( r > M_i \). As \( m_j \) is the minimal element of \( P_j \), there is no rook in row \( m_j - 1 \) of our triangular board for any rook placement. For otherwise, \( m_j \) would not be the minimal element of \( P_j \) as any rook in this row would force \( R \) to place \( m_j \) and \( s \) in the same block for some \( s < j \). Since we have a vacant row and column we may place a rook in position \((M_i, m_j - 1)\). So the placement is not maximal.

We now show that an intertwined partition gives rise to a maximal rook placement. Suppose \( R_k \) is not a maximal rook placement. We consider \( R(R_k) = Q \). As \( R_k \) not maximal, there exists a square \((i, j)\) where we may place a rook. As there is no rook in column \( i \), this implies that \( i \) is the maximal element of some block \( Q_i \) in \( Q \). Similarly, no rook in row \( j \) implies that \( j + 1 \) is the minimal element of some block \( Q_j \) in \( Q \). Hence, \( Q \) contains two blocks that are not intertwined. \( \square \)

We now count the number of partitions with intertwined blocks.

**Theorem 4.5.** The number of partitions of \([n]\) into \( k \) intertwined blocks is given by
\[
I(n, k) = (k - 1)! \sum_{i=k-1}^{n-k} S(i, k-1) \cdot S(n-i, k).
\]

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Figure 3: This diagram denotes the intertwined partition \{\{1, 6\}, \{2, 3, 5, 7\}, \{4, 8\}\} or the intertwined partition \{\{1, 5\}, \{2, 3, 6\}, \{4, 8\}\} depending on how the edges from the first part are connected to the edges of the second part.

Proof. Let \(1 \leq i \leq n\). Let \(P\) be a partition of \([i]\) into \(k - 1\) blocks. This can be done in \(S(i, k - 1)\) ways. Next, let \(Q\) be a partition the remaining \(n - i\) elements into \(k\) blocks which can be done in \(S(n - i, k)\) ways. In order to combine these two partitions into a single partition of \([n]\) into \(k\) intertwined blocks, ignore the block containing \(i + 1\) in \(Q\) and pair each block of \(P\) to exactly one unique block in \(Q\). This can be done in \((k - 1)!\) ways.

We need to check that any such partition is intertwined. Let \(B\) and \(C\) be blocks that contain an element of \(\{1, 2, \ldots, i\}\) with \(\min(B \cup C) \in B\). Each of these, by construction, will also contain at least one element from \(\{i + 2, i + 3, \ldots, n\}\). Thus, we have \(\max(B) > i + 1 > \min(C)\). Thus \(B\) and \(C\) are intertwined by definition. Now suppose \(B\) is a block containing an element of \(\{1, 2, \ldots, i\}\) and \(C\) is the block containing \(i + 1\). Again, by construction we have \(\max(B) > i + 1 = \min(C)\).

Lastly, we must show that every partition with intertwined blocks can be obtained in this way. Let \(P\) be such a partition and let \(i\) be the largest minimal element from the blocks. Suppose that \(i = \min(C)\). For any block \(B \neq C\) we have \(\min(B \cup C) \in B\). Since \(B\) and \(C\) are intertwined we have \(\max(B) > \min(C) = i\). Also, by construction we have \(\min(B) < \min(C)\). Thus, every block other than \(C\) has elements from \(1, 2, \ldots, i - 1\) and from \(i + 1, \ldots, n\).

Now our result follows by summing over \(i\). \(\square\)

**Corollary 4.6.** The number of maximal rook placements of size \(n - k\) on a triangular board \(\Psi_{1, \ldots, 1}\) of size \(n\) is given by

\[
F_{n-k}^n = k! \sum_{i=k}^{n-k} S(i, k) \cdot S(n + 1 - i, k + 1).
\]

Using this corollary, we can write the generating function for the facets of the Stirling complex. It is interesting to note that the generating function obtained is the product of the generating function for \(S(n, k)\), a shift of the generating function for \(S(n, k + 1)\), and \(k!\).

**Corollary 4.7.** The generating function for \(F_{n-k}^n\) is given by

\[
\sum_{n \geq 0} F_{n-k}^n x^n = \frac{k! \cdot x^{2k}}{\left(\prod_{i=1}^{k} (1 - ix)\right)^2 \cdot (1 - (k + 1)x)}.
\]
Proof. We have

\[
\sum_{n \geq 0} P_{n-k} x^n = k! \sum_{n \geq 0} \sum_{i=0}^{n} S(i, k) \cdot S(n + 1 - i, k + 1) x^n
\]

\[
= k! \left( \sum_{n \geq 0} S(n, k) x^n \right) \cdot \left( \sum_{n \geq 0} S(n + 1, k + 1) x^n \right)
\]

\[
= k! \left( \frac{x^k}{\prod_{i=1}^{k} (1 - ix)} \right) \cdot \left( \frac{x^k}{\prod_{i=1}^{k+1} (1 - ix)} \right)
\]

\[
= \frac{k! \cdot x^{2k}}{\left( \prod_{i=1}^{k} (1 - ix) \right)^2 \cdot (1 - (k + 1)x)}.
\]

\]

5 Topology of the Stirling complex

In this section we examine the topology of the Stirling complex. Work on this has been done by Barmak [1] where he showed that the Stirling complex St(n) is $\lceil \frac{n+3}{2} \rceil$-connected. Our technique uses discrete Morse theory by defining poset maps and creating a Morse matching using the Patchwork theorem. We are able to show that Barmak’s connectivity bound is sharp in the case when $n$ is even and further give a partial description of its homotopy type.

For a positive integer $n$, let $P$ be the following poset on the set $\{2, 3, 4, \ldots, 2n\}$. The even integers have the order

$2n < 2(1) < 2(n - 1) < 2(2) < \cdots < 2(n - k) < 2(k + 1) < 2(n - k - 1) < \cdots < 2[n/2],$
while the odd integers have the cover relations \( k_i < k_{i+1} \) where \( k_{i+1} = k_i + (-1)^{n+i+1} \cdot 2i \) and \( k_1 = 2[n/2] + 1 \). The evens and odds are not comparable, see Figure 4(a).

Using \( P \), we define a total order \( Q \) on the squares of the triangular board \( \Psi_{1,...,1} \). For \((i,j) \in [n] \times [n] \) with \( i \leq j \), \((i,j) \prec Q (k,\ell) \) if \( j - i < \ell - k \) in the standard order. If \( j - i = \ell - k \) then \((i,j) \prec Q (k,\ell) \) if \( i + j < P \ell + k \).

Informally, this order is obtained by starting on the largest diagonal and alternating upper-right to lower-left from the outside to the middle. We continue on the next diagonal alternating from the middle to the outside. The next diagonal moves again from the outside to the middle, etc., see Figure 4(b).

Let \( Q_1 \) be the sub-chain of \( Q \) consisting of the lowest elements \((n,n) < (1,1) < (n-1,n-1) < \cdots < \left( \left\lceil \frac{n}{2} \right\rceil , \left\lceil \frac{n}{2} \right\rceil \right) \) adjoined with a maximal element \( \hat{1}_{Q_1} \). We define the map \( \varphi \) from the face poset of \( \text{St}(n) \) to the poset \( Q_1 \). For \( x \in \text{St}(n) \), let

\[
\varphi(x) = \begin{cases} 
(i,i), & \text{if } (i,i) \text{ is the smallest in } Q_1 \text{ such that } \langle (i,i) \rangle \\
(1,i),(2,i),\ldots,(i-1,i),(i,i+1),(i,i+2),\ldots,(i,n) \notin x, & \text{otherwise.}
\end{cases}
\]

**Lemma 5.1.** The map \( \varphi : \mathcal{F}(\text{St}(n)) \rightarrow Q_1 \) is an order-preserving poset map.

**Proof.** Let \( x,y \in \text{St}(n) \) with \( x \subseteq y \). Suppose \( \varphi(x) = (i,i) \). That is, \( (i,i) \) is the smallest ordered pair such that \( (1,i),(2,i),\ldots,(i-1,i),(i,i+1),(i,i+2),\ldots,(i,n) \) are not elements of \( x \). Since \( y \) contains \( x \), \( \varphi(y) \) can be no smaller than \( (i,i) \). Therefore, \( \varphi(x) \leq \varphi(y) \). Suppose \( \varphi(x) = \hat{1}_{Q_1} \). Again, since \( y \) contains \( x \), we have \( \varphi(y) = \hat{1}_{Q_1} \) also. \( \square \)

**Lemma 5.2.** For \((i,i) \prec Q_1 \hat{1}_{Q_1} \), the collection \( \{(x,x \cup \{(i,i)\}) : (i,i) \notin x \in \varphi^{-1}(i,i) \} \) is a perfect acyclic matching on the fiber \( \varphi^{-1}(i,i) \).

**Proof.** Suppose \( \varphi(x) = (i,i) \) and \( (i,i) \notin x \). Since \( (1,i),(2,i),\ldots,(i-1,i),(i,i+1),(i,i+2),\ldots,(i,n) \) are all not in \( x \), \( u(x) = x \cup \{(i,i)\} \) is a valid rook placement in \( \text{St}(n) \). Also, \( \varphi(u(x)) = (i,i) \). Suppose \( \varphi(x) = (i,i) \) and \( (i,i) \in x \). It is clear that \( d(x) = x - \{(i,i)\} \) is a valid rook placement. Also, removing the element \( (i,i) \) will not affect the mapping under \( \varphi \). Therefore, \( \varphi(d(x)) = (i,i) \).

Finally, this matching is clearly acyclic since the same element is either added or removed from a placement. \( \square \)

Using the Patchwork theorem, we have an acyclic matching on \( \mathcal{F}(\text{St}(n)) \) whose only critical cells are the elements of the fiber \( \Gamma = \varphi^{-1}(\hat{1}_{Q_1}) \). From the definition of the function \( \varphi \), the following is clear.

**Lemma 5.3.** The rook placement \( x \) is an element of \( \Gamma \) if for each \( i \in [n] \), there is a \( j \neq i \) such that \((i,j) \in x \) or \((j,i) \in x \).

Therefore, we must find an acyclic matching on the fiber \( \Gamma \). We do this in a similar fashion as before. Let \( Q_2 = (Q - Q_1) \cup \{\hat{1}_{Q_2}\} \), that is, the subchain of \( Q \) that starts at \((\lceil n/2 \rceil, \lceil n/2 \rceil + 1)\) adjoined with a new maximal element \( \hat{1}_{Q_2} \). Let \( \psi : \Gamma \rightarrow Q_2 \) be a function such that for \( x \in \Gamma \),

\[
\psi(x) = \begin{cases} 
(i,j), & \text{if } (i,j) \text{ is the smallest in } Q_2 \text{ such that } \langle (i,j) \rangle \\
(\ell,i),(j,k) \in x \text{ for some } \ell,k \in [n] \text{ and } x \cup \{(i,j)\} \in \text{St}(n), & \text{otherwise.}
\end{cases}
\]
Lemma 5.4. The function \( \psi \) is an order-reversing poset map.

Proof. Let \( x, y \in \Gamma \) such that \( x \subset y \). Suppose \( \psi(x) = (i, j) \). Then \((\ell, i), (j, k)\) are elements of \( x \) and therefore \( y \). Now, \( \psi(y) \) could be smaller but will be no bigger than \((i, j)\). Thus \( \psi(x) \preceq \psi(y) \). If \( \psi(x) = \hat{1}_{Q_2} \) then clearly \( \psi(x) \preceq \psi(y) \). If \( \psi(y) = \hat{1}_{Q_2} \), then for all \((i, j)\) either \((\ell, i)\) or \((j, k)\) are not in \( y \). Then these elements are not in \( x \) either. Thus, \( \psi(x) = \hat{1}_{Q_2} \).

For use with the Patchwork theorem, this order-reversing map is sufficient because we could compose \( \psi \) with the natural order-reversing map from \( Q_2 \) to its dual \( Q_2^! \) and get an order preserving map.

Define a matching on the fiber \( \psi^{-1}((i, j)) \) by the collection

\[
\{(x, y) : x, y \in \psi^{-1}((i, j)), y = x \cup \{(i, j)\}\}.
\]

The matching on each fiber is clearly acyclic. This will not, in general, be a perfect matching.

We are now able to say something about the topology of the Stirling complex. Recall that the Durfee square of a Ferrers board is the largest contiguous square sub-board. For example, the Durfee square of the board in Figure 2 is of size 3.

Theorem 5.5. The Stirling complex \( \text{St}(n) \) is homotopy equivalent to a CW complex with no cells of dimension \( k \) for \( k < \lceil n/2 \rceil - 1 \) and for \( k \geq n - 1 \).

Proof. We see from Lemma 5.3 that without at least \( \lceil n/2 \rceil \) rooks, a placement \( x \) cannot get mapped by \( \varphi \) to \( \hat{1}_{Q_1} \) and will therefore not be critical. Also, it is clear that there cannot be a rook placement with greater than \( n \) rooks. The single placement with \( n \) rooks will necessarily get sent by \( \varphi \) to \((n, n)\). Thus, any possible critical rook placement will have no more than \( n - 1 \) rooks.

Theorem 5.6. The Stirling complex \( \text{St}(2n) \) is homotopy equivalent to a wedge of \( n! \) spheres of dimension \( n - 1 \) with a space \( X \) where \( X \) is \((n - 1)\)-connected.

Proof. We can see from Corollary 4.6 that for a triangular board \( \Psi_{1, \ldots, 1} \) of size \( 2n \), there are \( n! \) facets using \( n \) rooks. In fact, these are precisely the placements that fit inside the Durfee square of the triangular board. These will all clearly be mapped to \( \hat{1}_{Q_1} \) by \( \varphi \). Also, since every position in the Durfee square has coordinates \((i, j)\) for \( 1 \leq i \leq n \) and \( n \leq j \leq 2n \), \((\ell, i)\) and \((j, k)\) cannot be elements of the placement. Thus, these will all turn be mapped to \( \hat{1}_{Q_2} \) by \( \psi \) and are therefore critical.

We now must show that no other placement with \( n \) rooks will be critical. Let \( x \) be a rook placement with \( n \) rooks that are not all contained in the Durfee square and suppose \( \varphi(x) = \hat{1}_{Q_1} \). We first show that \( \psi(x) \neq \hat{1}_{Q_2} \). From Corollary 4.6 we know that \( x \) is not a facet and so there exists a position \((i, j)\) that is not attacked. This means that column \( i \) and row \( j \) do not contains any rooks. However, since \( \varphi(x) = \hat{1}_{Q_1} \), the positions \((i, i)\) and \((j, j)\) must be attacked. This implies that \((\ell, i)\) and \((j, k)\) are elements of \( x \). Also, since \((i, j)\) was previously not attacked, \( x \cup \{(i, j)\} \) is a face of \( \text{St}(n) \).

Therefore, suppose \( \psi(x) = (i, j) \). We claim that \((i, j) \not\in x \). Since \( \varphi(x) = \hat{1}_{Q_1} \), we know that all of the \( 2n \) positions of the form \((m, m)\) will be attacked exactly once. Thus, if \((i, j), (\ell, i), \) and \((j, k)\) are all elements of \( x \) for \( \ell, i, j, k \) all distinct, then \((i, i)\) and \((j, j)\) would be attacked twice. We therefore need to show that \( u(x) = x \cup \{(i, j)\} \) is also mapped by \( \psi \) to \((i, j)\).

Suppose \( \psi(u(x)) = (s, t) \prec_{Q_2} (i, j) \). That is, \((w, s)\) and \((t, y)\) are elements of \( u(x) \) for some \( w \) and \( y \) and \( u(x) \cup \{(s, t)\} \in \text{St}(n) \). If both of these are also elements of \( x \), then we would have
ψ(x) = (s, t). Since x and u(x) only differ by one element, we cannot have both elements outside x. Therefore, we know that only one of these elements is not in x. Due to the symmetry of the boards we can assume, without loss of generality, that (w, s) /∈ x and (t, y) ∈ x. That is, (w, s) = (i, j) and so ψ(u(x)) = (s, t) = (j, t). We already know that (j, k) ∈ x ⊂ u(x) so u(x) ∪ \{(j, t)\} is not an element of St(2n) unless t = k.

Therefore, we now suppose that ψ(u(x)) = (s, t) = (j, k). That is, (w, j) = (i, j) and (t, y) = (k, y) are both elements of u(x). Since (k, y) is not equal to (i, j) then we have (k, y) ∈ x. From above, we also have (j, k) ∈ x. This implies that the position (k, k) is attacked twice in x which is a contradiction. Therefore, ψ(u(x)) = (i, j).

To see that these n! critical cells form a wedge of spheres, let x be a rook placement contained within the Durfee square. From Theorem 5.5, we know that for w ⊂ x, w will be matched. Also, since x is a facet, there is no placement above it. Therefore, using Theorem 2.8, we can conclude that these critical cells form a wedge of n! spheres of dimension n − 1.

The following corollaries follow directly from Theorem 5.6.

**Corollary 5.7.** The (n − 1)st reduced Betti number of the Stirling complex St(2n) is n!.

**Corollary 5.8.** The Stirling complex St(2n) is exactly (n − 2)-connected.

6 Further questions

1. Using the program Macaulay2, we have been able to compute the reduced homology of the Stirling complex up to St(8), see Table 1. The Morse matching presented in this paper gives the correct number of critical cells for these first eight complexes. Is this matching maximal? If so, is there a way to count the other critical cells?

2. Can anything more be said about the homotopy type of St(n)? In particular, what is the complex X from Theorem 5.6? The numerical computations of homology died at St(9). Coincidentally, this is the first instance where the Durfee square of the board has size 5. Shareshian and Wachs [14] showed that this was the first square board whose chessboard complex contained torsion in its bottom non-vanishing homology. Is there torsion in St(9)?

3. Looking at the data in Table 1, we conjecture that Barmak’s connectivity bound is also sharp in the case when n is odd.

| n | β̂₀ | β̂₁ | β̂₂ | β̂₃ | β̂₄ | β̂₅ |
|---|---|---|---|---|---|---|
| 1 | 0 |   |   |   |   |   |
| 2 | 1 |   |   |   |   |   |
| 3 | 0 | 1 |   |   |   |   |
| 4 | 0 | 2 |   |   |   |   |
| 5 | 0 | 0 | 9 |   |   |   |
| 6 | 0 | 0 | 6 | 15 |   |   |
| 7 | 0 | 0 | 0 | 58 | 8 |   |
| 8 | 0 | 0 | 0 | 24 | 292 | 1 |

Table 1: The reduced Betti numbers of the Stirling complex through St(8).
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