On decomposability of 4-ary distance 2 MDS codes, double-codes, and $n$-quasigroups of order 4

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Abstract

A subset $S$ of $\{0, 1, \ldots, 2t-1\}^n$ is called a $t$-fold MDS code if every line in each of $n$ base directions contains exactly $t$ elements of $S$. The adjacency graph of a $t$-fold MDS code is not connected if and only if the characteristic function of the code is the repetition-free sum of the characteristic functions of $t$-fold MDS codes of smaller lengths.

In the case $t = 2$, the theory has the following application. The union of two disjoint $(n, 4^{n-1}, 2)$ MDS codes in $\{0, 1, 2, 3\}^n$ is a double-MDS-code. If the adjacency graph of the double-MDS-code is not connected, then the double-code can be decomposed into double-MDS-codes of smaller lengths. If the graph has more than two connected components, then the MDS codes are also decomposable. The result has an interpretation as a test for reducibility of $n$-quasigroups of order 4.

Keywords: MDS codes, $n$-quasigroups, decomposability, reducibility, frequency hypercubes, latin hypercubes

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1 Introduction

We consider the subsets $S$ of $\{0, 1, \ldots, q-1\}^n$, where $q \geq 4$ is even, with the following property: every line in each of $n$ base directions contains exactly $q/2$ elements of $S$. We call such objects $q/2$-fold MDS codes. This paper establish a connection between the connectivity of a $q/2$-fold MDS code and its decomposability. More accurately, by the example of $q = 4$, we prove that the adjacency graph of a $q/2$-fold MDS code is not connected if and only if the characteristic

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function of the code is the repetition-free sum of the characteristic functions of $q/2$-fold MDS codes of smaller lengths.

$q/2$-Fold MDS codes are very natural objects of study; they can be considered as a partial case of (strongly defined) frequency hypercubes, an $n$-dimensional generalization of frequency squares (questions of connectivity for a partial type of frequency squares were considered in [4]). Nevertheless, our research is motivated by studying the 4-ary distance 2 MDS codes or, equivalently, the $n$-quasigroups of order 4.

A distance 2 MDS code is decomposable if it can be represented as a “concatenation” (see (8) in Section 5) of MDS codes of smaller length. The goal of this work is to prove the following test for decomposability of 4-ary distance 2 MDS codes.

Let $C$ and $C'$ be two disjoint MDS codes in $\{0,1,2,3\}^n$. Assume that the adjacency graph of their union (2-fold MDS code) has more than the minimal (1 or 2) and less than the maximal ($2^{n-1}$) number of connected components. Then the MDS codes $C$ and $C'$ are decomposable. Note that if $C'' = \pi C$ where $\pi$ is a permutation of type $(a,b)(c,d)$ of the alphabet symbols in one coordinate, then there are at least 2 connected components. Otherwise the minimal number of components is 1. If the adjacency graph of $C \cup C'$ has $2^{n-1}$ connected components, then $C$ and $C'$ belong to the class of semilinear MDS codes (see Section 5).

In particular, this test means that we cannot get a “new” code if we combine parts of two disjoint 4-ary distance 2 MDS codes $C_1$ and $C_2$ (see Theorem 5-4 for the details). So, this “switching” method, which works well, for example, for constructing 1-perfect binary codes with nontrivial properties (see e.g. [8]), cannot provide something interesting in the case of 4-ary distance 2 MDS codes.

Since there is a one-to-one correspondence between $q$-ary distance 2 MDS codes of length $n+1$ and $n$-quasigroups of order $q$ (the value arrays of $n$-quasigroups are also known as latin $n$-cubes, an $n$-dimensional generalization of latin squares), we can interpret the results in terms of $n$-quasigroups of order 4 (Section 6). The decomposability, or reducibility, of $n$-quasigroups is a natural concept; for arbitrary order it was considered, for example, in [2, 3].

The mention of 1-perfect binary codes above is not an accident. There are concatenation constructions of such codes [6, 9] based on distance 2 MDS codes, or $n$-quasigroups. Moreover, as shown in [1], any 1-perfect binary code of length $m$ and rank $\leq \min\{\text{rank}+2\}$ is described by a collection of distance 2 MDS codes in $\{0,1,2,3\}^{(m+1)/4}$ (the rank is the dimension of the code linear span; the minimal rank of a 1-perfect binary code is $m - \log_2(m + 1)$). So, the properties of distance 2 MDS codes are closely related to properties of some 1-perfect codes.

Concepts closely related with 4-ary distance 2 MDS codes are the concepts of a double-code and a double-MDS-code (i.e., 2-fold MDS codes in $\{0,1,2,3\}^n$). Double-codes and double-MDS-codes have many useful properties, which are discussed in Section 3. Studying MDS codes, we
can think that a double-MDS-code is the union of two disjoint 4-ary distance 2 MDS codes and a double-code is a part of a double-MDS-code closed with respect to adjacency. In fact, there are double-MDS-codes, as well as $q/2$-fold MDS codes, that are not splittable into distance 2 MDS codes, see [5], and the class of all double-MDS-codes can be considered independently.

In Section 2 we give main definitions and notations. In particular, we define the concept of a double-MDS-code, which is a set with properties of the union of two disjoint distance 2 MDS codes. In Section 3 we prove some preliminary results. In Section 4 we prove the theorem on the decomposition of double-MDS-codes into prime double-MDS-codes and show how to generalize the result to $q/2$-fold MDS codes. In Sections 5 and 6 we discuss the decomposability of distance 2 MDS codes and $n$-quasigroups. In the Appendix we prove some auxiliary lemmas about functions with separable arguments, which are used in Sections 3 and 4.

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2 Basic notations and definitions

Let $\Sigma \triangleq \{0, 1, 2, 3\}$ and $\Sigma^n$ be the set of words of length $n$ over the alphabet $\Sigma$. Denote $[n] \triangleq \{1, \ldots, n\}$. For $\bar{x} = (x_1, x_2, \ldots, x_n)$ we use the following notation:

$$
\bar{x}[k] \triangleq (x_1, \ldots, x_{k-1}, y, x_{k+1}, \ldots, x_n),
\bar{x}[k_1,k_2,\ldots,k_s] \triangleq \bar{x}[k_1][y_1][k_2][y_2] \ldots [k_s][y_s].
$$

A set of four elements of $\Sigma^n$ that differ in only one ($i$th) coordinate is called a line ($i$-line) of $\Sigma^n$. Let $\mathcal{E}_i(\bar{x})$ denote the $i$-line that contains $\bar{x} \in \Sigma^n$. If $S \subset \Sigma^n$, then

$$
\mathcal{E}_i(S) \triangleq \bigcup_{\bar{x} \in S} \mathcal{E}_i(\bar{x})
$$

(the union of the $i$-lines through the points of $S$) and

$$
\mathcal{F}_{i,j;\bar{x}} S \triangleq \{(b, c) \in \Sigma^2 : \bar{x}^{[i,j]}[b, c] \in S\}
$$

(the cut of $S$ in the “$i, j$-plane” through $\bar{x}$).

A set $C \subset \Sigma^n$ is called a 4-ary distance 2 MDS code (of length $n$) or $(n, 2)_4$ MDS code if each line of $\Sigma^n$ contains exactly one element of $C$. A function $g : \Sigma^n \to \Sigma$ is called an $n$-quasigroup of order 4 if for each $i \in [n]$ and $y, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \in \Sigma$ there exists $x_i = g^{(i)}(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \in \Sigma$ such that $y = g(x_1, \ldots, x_n)$. Clearly, the function $g^{(i)}$ is also an $n$-quasigroup of order 4. For the rest of the paper we omit the words “4-ary distance 2”
and “of order 4” because we consider only MDS codes and \( n \)-quasigroups with such parameters. The following one-to-one correspondence between MDS codes and \( n \)-quasigroups is obvious and well known.

**Proposition 2-1.** A set \( C \subset \Sigma^n \) is an \((n,2)_4\) MDS code if and only if \( C = \{(\bar{x}, g(\bar{x})) \mid \bar{x} \in \Sigma^{n-1}\} \) for some \((n-1)\)-quasigroup \( g \).

The following statements are also well known and easy to prove.

**Proposition 2-2.**

(a) The superposition \( g(\bar{x}[i][f(\bar{y})]) \) of an \( n \)-quasigroup \( g \) and an \( m \)-quasigroup \( f \) is an \((n+m-1)\)-quasigroup.

(b) If \( g \) is an \( n \)-quasigroup and \( i \in [n] \), then its inversion in the \( i \)th position \( f^{(i)} \) is an \( n \)-quasigroup too.

(c) If \( g \) is an \( n \)-quasigroup and \( a \in \Sigma \), then the set

\[
M_a \triangleq \{\bar{x} \in \Sigma^n \mid g(\bar{x}) = a\}
\]

is an MDS code.

(d) A 1-quasigroup \( p : \Sigma \to \Sigma \) is a permutation of \( \Sigma \).

A set \( S \subset \Sigma^n \) is called a double-code if each line of \( \Sigma^n \) contains zero or two elements from \( S \). A double-code \( S \subset \Sigma^n \) is called double-MDS-code if each line of \( \Sigma^n \) contains exactly two elements from \( S \). If a double-code is a subset of some double-MDS-code, then we call it complementable. If a double-code is complementable, nonempty, and cannot be split into more than one nonempty double-codes, then we call it prime.

**Remark.** The union of two disjoint \((n,2)_4\) MDS codes is always a double-MDS-code. The converse statement does not hold for \( n \geq 3 \) (see e.g. [5]).

![Figure 1:](image)

**Example 2-3.** Figure 1 shows all double-codes in \( \Sigma^2 \) up to permutations of rows and columns. The double-codes a)-d) are complementable and e) is not. The double-codes c) and d) are double-MDS-codes. The double-codes b) and d) are prime.
3 Preliminary statements

Proposition 3-1. (a) If $S \subset \Sigma^n$ is a double-MDS-code, then its supplement $\Sigma^n \setminus S$ is a double-MDS-code.
(b) A double-code $S \subset \Sigma^n$ is a double-MDS-code if and only if $|S| = |\Sigma^n|/2 = 2^{2n-1}$.

Proof. (a) follows from the definition of a double-MDS-code. (b) is obvious if we consider the partition of $\Sigma^n$ into $i$-lines where $i \in [n]$ is fixed. □

For arbitrary subset $S \subseteq \Sigma^n$ we define the adjacency graph $G(S)$ with vertex set $S$, where two vertices are adjacent if and only if they differ in exactly one coordinate.

The following proposition gives a natural treatment of a complementable double-code in terms of connected components of the adjacency graph of a double-code that includes the given double-code.

Proposition 3-2. Let $S$ be a complementable double-code and $S_0$ be an arbitrary subset of $S$; then
(a) $S_0$ is a double-code if and only if $G(S_0)$ is a union of connected components of $G(S)$;
(b) $S_0$ is a prime double-code if and only if $G(S_0)$ is a connected component of $G(S)$.

Proof. The graph $G(S)$ has an edge between $S_0$ and $S \setminus S_0$ if and only if there is a line that has nonempty intersections with both $S_0$ and $S \setminus S_0$. Now, (a) follows from the definitions of double-codes and connected components of a graph. (b) can be easily derived from (a). □

Corollary 3-3. Assume that prime double-codes $C$ and $C'$ are included in the same complementable double-code. Then $C = C'$ or $C \cap C' = \emptyset$.

The following simple proposition will be used in Sections 5 and 6.

Proposition 3-4. Let $S$ be a double-MDS-code and let $\gamma$ be the number of prime double-codes included in $S$. (a) If $G(S)$ is a bipartite graph, then $S$ includes exactly $2^\gamma$ different MDS codes.
(b) Otherwise, $S$ does not include an MDS code.

Proof. (a) By Proposition 3-2(b), $\gamma$ is the number of connected components in $G(S)$. A part of the bipartite graph $G(S)$ is an MDS code by the definition. So, the number of the MDS codes that $S$ includes equals the number of the ways of choosing a part of the bipartite graph $G(S)$, i.e., $2^\gamma$.
(b) Assume that a double-MDS-code $S$ includes an MDS code $C$. Then, by the definition, $S \setminus C$ also is an MDS code. So, the graphs $G(C)$ and $G(S \setminus C)$ do not contain edges, and hence the graph $G(S)$ is bipartite. □
Let \( S \subset \Sigma^n \) and \( i \in [n] \); then we denote
\[
\underline{i}S \triangleq \mathcal{E}_i(S) \setminus S
\]
(see Fig. 1 for example).

**Proposition 3-5.** Let \( S, S' \subset \Sigma^n \) be double-codes and \( i, i' \in [n] \). Then
(a) \( S \cap \underline{i}S = \emptyset \);
(b) \( \underline{i} \underline{i}S = S \);
(c) \(|S| = |\underline{i}S|\);
(d) \( S \subseteq S' \) if and only if \( \underline{i}S \subseteq \underline{i}S' \);
(e) if \( S \cup S' \) is a double-code, then \( \underline{i}(S \cup S') = \underline{i}S \cup \underline{i}S' \);
(f) \( S \) is a double-MDS-code if and only if \( \underline{i}S \) is a double-MDS-code;
(g) \( S \) is a double-MDS-code if and only if \( \underline{i}S = \Sigma^n \setminus S \);
(h) \( S \) is prime if and only if \( \underline{i}S \) is a prime double-code;
(i) if \( S \) is prime, then either \( \underline{i}S = \underline{i}S' \) or \( \underline{i}S \cap \underline{i}S' = \emptyset \);
(j) if \( S \) is complementable, then \( \underline{i} \underline{i}S = \underline{i} \underline{i}S' \);
(k) \( S \) is a double-MDS-code if and only if \( |S| > 0 \) and \( \underline{j}S = \underline{j}S' \) for each \( j, j' \in [n] \).

**Proof.** (a) is clear. The set \( \mathcal{E}_i(S) = \mathcal{E}_i(\underline{i}S) = S \cup \underline{i}S \) can be partitioned into \( i \)-lines. Each line of the partition has two elements from \( S \) and the other two from \( \underline{i}S \). Now (b) and (c) are also obvious.

(d) Suppose, \( S \subseteq S' \). Then \( \mathcal{E}_i(S) \subseteq \mathcal{E}_i(S') \). Each line \( \mathcal{E}_i(\vec{x}) \), where \( \vec{x} \in \mathcal{E}_i(S) \), contains two elements from \( S \) and the other two from \( \underline{i}S \). They also are elements of \( S' \) and \( \underline{i}S' \) respectively. So, each element from \( \underline{i}S \) is in \( \underline{i}S' \). The converse statement is proved in the same way.

(e) It is easy to see that each \( i \)-line \( \mathcal{E} \)
- either has the same intersection with both \( S \) and \( S' \)
- or is disjoint with \( S \) or \( S' \).
In any case, \( \mathcal{E} \cap \underline{i}(S \cup S') = \mathcal{E} \cap (\underline{i}S \cup \underline{i}S') \). Since \( \Sigma^n \) is the union of \( i \)-lines, the statement is proved.

(f) follows from (a), (c), and Proposition 3-1(a,b).

(g) Assume the double-code \( S \) is complementable. First we will show that \( \underline{i}S \) is a double-code. Let \( \mathcal{E}_j(\vec{x}) \) be an arbitrary line, where \( j \in [n] \) and \( \vec{x} = (x_1, \ldots, x_n) \in \Sigma^n \). If \( j = i \), then \(|\mathcal{E}_j(\vec{x}) \cap S| = |\mathcal{E}_j(\vec{x}) \cap \underline{i}S| \in \{0, 2\} \). If \( j \neq i \). It is clear that \( \mathcal{F}_{i,j;S} \) is a double-code in \( \Sigma^2 \) and \( \mathcal{F}_{i,j;S} \cap \underline{i}S = \underline{i} \mathcal{F}_{i,j;S} \). Furthermore, the fact that \( S \) is complementable implies that \( \mathcal{F}_{i,j;S} \) is complementable too. It is easy to check (see Fig. 1(a-d)) that \( \underline{i} \mathcal{F}_{i,j;S} \) is a double-code.
Consequently, \( |\mathcal{E}_j(\bar{x}) \cap \set{x}S| = |\mathcal{E}_2(x_i, x_j) \cap \set{i}F_{i,j;x}S| \in \{0, 2\} \) and \( \set{i}S \) is a double-code by the definition.

Since \( S \) is a complementable double-code, there is a double-MDS-code \( S'' \supseteq S \). By (f), the set \( \set{i}S'' \) is a double-MDS-code. By (d), we have \( \set{i}S \subseteq \set{i}S'' \). Consequently, the double-code \( \set{i}S \) is complementable.

Similarly, if \( \set{i}S \) is a complementable double-code, then \( S \) is.

(h) By (g), we may assume that \( S \) and \( \set{i}S \) are complementable double-codes. Let \( S \) be non prime, i.e., \( S = S_1 \cup S_2 \), where \( S_1 \) and \( S_2 \) are disjoint nonempty double-codes. Double-codes \( S_1 \) and \( S_2 \) are complementable by the definition; \( \set{i}S_1 \) and \( \set{i}S_2 \) are also complementable double-codes by (g). The sets \( \mathcal{E}_i(S_1) \) and \( \mathcal{E}_i(S_2) \) are disjoint. Therefore, \( \set{i}S_1 \) and \( \set{i}S_2 \) are disjoint and the double-code \( \set{i}S = \set{i}S_1 \cup \set{i}S_2 \) is not prime. This proves that if \( \set{i}S \) is prime, then \( S \) is prime. Similarly, the converse also holds.

(i) Let \( S \subseteq S'' \), where \( S'' \) is a double-MDS-code. It follows from (d) and (f) that \( \set{i}S \subseteq \set{i}S'' = \Sigma^n \set{i}S'' \). On the other hand, \( \set{i'}S \subseteq \set{i'}S'' = \Sigma^n \set{i'}S'' \). By (h), the sets \( \set{i}S \) and \( \set{i'}S \) are prime double-codes; by Corollary 3-3, they are either coincident or disjoint.

(j) It is enough to check that for each \( \bar{x} \in \Sigma^n \) it holds \( \set{i'}S_{i,i';\bar{x}} = \set{i'}S_{i,i';\bar{x}} \), where \( S_{i,i';\bar{x}} = S \cap \{ \bar{x}[i',j][b,c] \mid b,c \in \Sigma \} \). Equivalently, \( \set{i}F_{i,i';\bar{x}}S = \set{i}F_{i,i';\bar{x}}S \) for all \( \bar{x} \in \Sigma^n \). The last can be checked directly, taking into account that \( F_{i,i';\bar{x}}S \) is a complementable double-code (see Fig. 1a-d).

(k) We first note that the condition \( \set{i}S = \set{i'}S \) for all \( j,j' \in [n] \) is equivalent to the condition \( \set{i}F_{i,j';\bar{x}}S = \set{i}F_{i,j';\bar{x}}S \) for all different \( i,j \in [n] \) and \( \bar{x} \in \Sigma^n \). Since \( F_{i,j';\bar{x}}S \) is a double-code, it is straightforward (see Fig. 1) that the last condition is equivalent to

\[
F_{i,j;x}S = \emptyset \quad \text{or} \quad F_{i,j;x}S \text{ is a double-MDS-code.} \quad (1)
\]

Only if: If \( S \) is a double-MDS-code, then (1) holds automatically.

If: Suppose, by contradiction, that \( S \) is not a double-MDS-code. Then there exist \( \bar{v} = (v_1, \ldots, v_n) \) and \( \bar{z} = (z_1, \ldots, z_n) \in \Sigma^n \) such that \( \mathcal{E}_1(\bar{v}) \cap S \neq \emptyset \) and \( \mathcal{E}_1(\bar{z}) \cap S = \emptyset \). Consider the sequence \( \bar{v} = \bar{v}^0, \bar{v}^1, \ldots, \bar{v}^n = \bar{z} \), where \( \bar{v}_j = (z_1, \ldots, z_j, v_{j+1}, \ldots, v_n) \). Note that \( \bar{v}^{j-1} \) and \( \bar{v}^{j} \) coincide in all positions may be except the \( j \)th one. There exists \( j \in [n] \) such that \( \mathcal{E}_1(\bar{v}^{j-1}) \cap S \neq \emptyset \) and \( \mathcal{E}_1(\bar{v}^{j}) \cap S = \emptyset \). Then \( |\mathcal{F}_{i,j';\bar{v}^j}S| \) contradicts to (1) with \( i = 1 \) and \( \bar{x} = \bar{v}^j \).

Let \( S \) be a double-MDS-code in \( \Sigma^n \) and let \( R \subseteq S \) be a prime double-code. We say that \( i \) and \( i' \) from \( [n] \) are equivalent (and write \( i \sim i' \)) if \( \set{i}R = \set{i'}R \). In Corollary 3-8 below we will show that the equivalence \( \sim \) does not depend on the choice of \( R \). Let \( K_1 = \{ i_{1,1}, i_{1,2}, \ldots, i_{1,n_1} \} \), \( K_2 = \{ i_{2,1}, i_{2,2}, \ldots, i_{2,n_2} \} \), \ldots, \( K_k = \{ i_{k,1}, i_{k,2}, \ldots, i_{k,n_k} \} \) be the equivalence classes of \( \sim \). By Proposition 3-5(k), we have the following:
Proposition 3-6. The double-MDS-code $S$ is prime if and only if $k = 1$.

Denote by $\{0,1\}_e^n$ the set of the even-weight (i.e., with even number of ones) elements of $\{0,1\}^n$. Denote by $\bar{e}_j$ the word in $\{0,1\}^n$ with the only 1 in the $j$th position. Let the sets $R_{\bar{y}}$, where $\bar{y} \in \{0,1\}^n$, be inductively defined by the equalities

$$R_0 \triangleq R \quad \text{and} \quad R_{\bar{y} \oplus e_j} \triangleq \\setminus_j R_{\bar{y}}.$$ 

As follows from Proposition 3-5(b,j), the sets $R_{\bar{y}}$ are well defined. The next proposition is a corollary of Proposition 3-5.

Proposition 3-7.

(a) For each $\bar{y} \in \{0,1\}^n$ the set $R_{\bar{y}}$ is a prime double-code.

(b) For each $\bar{y} \in \{0,1\}^n$ the equality $\setminus_{i'} R_{\bar{y}} = \setminus_{i''} R_{\bar{y}}$ holds if and only if $i' \sim i''$.

(c) For each $\bar{y}, \bar{z} \in \{0,1\}^n$ either $R_{\bar{y}} = R_{\bar{z}}$ or $R_{\bar{y}} \cap R_{\bar{z}} = \emptyset$.

(d) $S = \bigcup_{\bar{y} \in \{0,1\}_e^n} R_{\bar{y}}$.

Proof. (a) follows from Proposition 3-5(h).

(b) Let $\bar{y} = \bar{e}_{j_1} \oplus \ldots \oplus \bar{e}_{j_w}$. Then $R_{\bar{y}} = \setminus_{j_w} \ldots \setminus_{j_1} R_{\bar{y}}$. By Proposition 3-5(b,j), we have

$$\setminus_{i'} R_{\bar{y}} = \setminus_{i''} R_{\bar{y}} \iff \setminus_{i'} \setminus_{j_w} \ldots \setminus_{j_1} R = \setminus_{i''} \setminus_{j_w} \ldots \setminus_{j_1} R$$

$$\iff \setminus_{j_w} \ldots \setminus_{j_1} i'R = \setminus_{j_w} \ldots \setminus_{j_1} i'' R \iff \setminus_{i'} R = \setminus_{i''} R \iff i' \sim i''.$$ 

(c,d) Let again $R_{\bar{y}} = \setminus_{j_w} \ldots \setminus_{j_1} R$. We have $R \subseteq S$. It follows by induction from Proposition 3-5(d,f) that $R_{\bar{y}} \subseteq S$ if $\bar{y} \in \{0,1\}_e^n$ and $R_{\bar{y}} \subseteq \Sigma^n \setminus S$ otherwise. So, (c) holds by Proposition 3-1(a) and Corollary 3-3. Further, the union $\bigcup_{\bar{y} \in \{0,1\}_e^n} R_{\bar{y}}$ is a subset of $S$ and, by the definition, is a complementable double-code. On the other hand, by Proposition 3-5(e), the set

$$\setminus_{i} \bigcup_{\bar{y} \in \{0,1\}_e^n} R_{\bar{y}} = \bigcup_{\bar{y} \in \{0,1\}_e^n} \setminus_{i} R_{\bar{y}} = \bigcup_{\bar{y} \in \{0,1\}_e^n} R_{\bar{y} \oplus e_i} = \bigcup_{\bar{y} \in \{0,1\}_{odd}} R_{\bar{y}}$$

does not depend on $i$ and, by Proposition 3-5(k), the set $\bigcup_{\bar{y} \in \{0,1\}_e^n} R_{\bar{y}}$ is a double-MDS-code. Therefore, it coincides with $S$.

Corollary 3-8. The equivalence $\sim$ does not depend on the choice of the prime double-code $R \subseteq S$.

Proof. It follows from Proposition 3-7(d) and Corollary 3-3 that for each prime double-code $R' \subseteq S$ there exists $\bar{y} \in \{0,1\}_e^n$ such that $R' = R_{\bar{y}}$. Proposition 3-7(b) completes the proof.

Corollary 3-9. Let $S, S', S'' \subseteq \Sigma^n$ be double-MDS-codes and $S_0$ be a double-code.

(a) If $S_0 \subseteq S \cap S'$, then $S = S'$.

(b) If $S_0 \subseteq S \setminus S''$, then $S = \Sigma^n \setminus S''$. 

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Proof. Let \( R \subseteq S_0 \) be a prime double-code. Then \( R \subseteq S \) and \( R \subseteq S' \). By Proposition 3-7(d), we have \( S = \bigcup_{y \in \{0,1\}_\text{even}}^n R_y \) and \( S' = \bigcup_{y \in \{0,1\}_\text{even}}^n R_y \). So, (a) is proved; taking \( S' = \Sigma^n \setminus S'' \), we get (b).

Let \( \sigma = \chi_S : \Sigma^n \to \{0,1\} \) be the characteristic function of the double-MDS-code \( S \) and let for each \( j \in [k] \)
\[
\sigma_j(y_1, \ldots, y_{n_j}) = \sigma(\bar{y}^{[j,1,\ldots,j,n_j]}_j(y_1, \ldots, y_{n_j})
\]
be its subfunction with the set of arguments that correspond to the class \( K_j \).

**Proposition 3-10.** For any nonequivalent \( i', i'' \in [n] \), any \( \bar{x} \in \Sigma^n \), and any \( a', a'' \in \Sigma \) it holds
\[
\sigma(\bar{x}) \oplus \sigma(\bar{x}^{i'}_j|a') \oplus \sigma(\bar{x}^{i''}_j|a'') \oplus \sigma(\bar{x}^{i',i''}_j|a', a'') = 0.
\]

*Proof.* Let \( \bar{y} \in \{0,1\}^n_\text{even} \) be such that \( R_{\bar{y}} \cap \mathcal{E}_{i'}(\bar{x}) \neq \emptyset \) (by Proposition 3-7(d), such \( \bar{y} \) exists). Let us consider the double-MDS-code \( S_2 = \mathcal{F}_{i',i'';\bar{x}} S \subseteq \Sigma^2 \) and the prime double-code \( R_{\bar{y}}^2 = \mathcal{F}_{i',i'';\bar{x}} R_{\bar{y}} \subseteq S_2^2 \). Since, by Proposition 3-7(b), \( \mathcal{E}_{i''} R_{\bar{y}} \neq \emptyset \), we find by Proposition 3-5(i) that \( \mathcal{E}_{i''} R_{\bar{y}} \cap \emptyset = \emptyset \) and, consequently, \( \emptyset \mathcal{E}_{i''} R_{\bar{y}} \subseteq \emptyset \). Therefore, \( R_{\bar{y}}^2 \) corresponds to the case b) of Fig. 1 and \( S_2^2 \) corresponds to the case c), up to permutations of the rows and the columns. For this case, it is easy to check that
\[
\chi_{S_2}(b', b'') \oplus \chi_{S_2}(a', b'') \oplus \chi_{S_2}(b', a'') \oplus \chi_{S_2}(a', a'') = 0 \quad \forall b', b'', a', a'' \in \Sigma.
\]
The statement follows from the obvious identity
\[
\sigma(\bar{x}^{i',i''}_j|a', c'') = \chi_{S_2}(c', c'') \quad \forall c', c'' \in \Sigma.
\]

In the proofs of the following statements we will use the results and notation of the Appendix on functions with separable arguments.

**Proposition 3-11.** For each \( \bar{x} \) from \( \Sigma^n \) it holds
\[
\sigma(\bar{x}) \equiv \bigoplus_{j=1}^k \sigma_j(x_{i_{j,1}}, x_{i_{j,2}}, \ldots, x_{i_{j,n_j}}) \oplus \sigma_0, \quad \text{where } \sigma_0 = (k-1)\sigma(\bar{0}).
\]

*Proof.* By the criterion of Lemma A-1 of the Appendix, Proposition 3-10 means that \( \sigma \) has \( \{K_1, \ldots, K_k\}\)-separable arguments, i.e.,
\[
\sigma(\bar{x}) \equiv \bigoplus_{j=1}^k f_j(x_{i_{j,1}}, x_{i_{j,2}}, \ldots, x_{i_{j,n_j}})
\]
for some functions \( f_j : \Sigma^{n_j} \to \{0,1\} \). Then,
\[
\sigma(\bar{x}) \oplus \sigma(\bar{0}) \equiv \bigoplus_{j=1}^k \left( f_j(x_{i_{j,1}}, x_{i_{j,2}}, \ldots, x_{i_{j,n_j}}) \oplus f_j(\bar{0}) \right).
\]
Setting \( x_i = 0 \) for all \( i \notin K_j \), we have

\[
\sigma_j(x_{i,j,1}, x_{i,j,2}, \ldots, x_{i,j,n_j}) \oplus \sigma(\bar{0}) \equiv f_j(x_{i,j,1}, x_{i,j,2}, \ldots, x_{i,j,n_j}) \oplus f_j(\bar{0}).
\] (4)

Substituting (4) to (3) proves the statement.

**Proposition 3-12.** For each \( j \in [k] \) the function \( \sigma_j \) is the characteristic function of a prime double-MDS-code.

**Proof.** For each \( j \in [k] \) the function \( \sigma_j \) is a subfunction of \( \sigma \) and, consequently, the characteristic function of some double-MDS-code \( S_j \). It remains to prove that \( S_j \) is prime. The idea is to show that the \( \tilde{=}_j \) equivalence of the indexes from \( K_j \) yields the \( \tilde{=} \) equivalence of the indexes from \([n_j]\).

We first observe the following straightforward fact:

\((*)\) if \( R_1 \subset \Sigma^{n_1}, \ldots, R_k \subset \Sigma^{n_k} \) are double codes,

\[
R \triangleq R_1 \times \ldots \times R_k,
\] (5)

and \( i \in [n_1] \), then \( R \) is a double-code and \( \backslash_i R = (\backslash_1 R_1) \times R_2 \times \ldots \times R_k \).

Without loss of generality assume that \( K_1 = \{1, \ldots, n_1\} \), \( K_2 = \{n_1 + 1, \ldots, n_1 + n_2\} \), and so on. Then, from (2) we derive that \( S \supseteq S_1 \times \ldots \times S_k \) or \( \Sigma^n \setminus S \supseteq S_1 \times \ldots \times S_k \). For each \( j \in [k] \) we choose a prime double-code \( R_j \subseteq S_j \). Then, the double-code \( R \) defined as in (5) is included in \( S \) or \( \Sigma^n \setminus S \), and for any equivalent (in the sense of \( \tilde{=}_j \)) \( i \) and \( i' \) we have \( \backslash_i R = \backslash_{i'} R \). From (*) we derive that for each \( i, i' \in K_1 \) we have \( \backslash_i R_1 = \backslash_{i'} R_1 \), i.e., \( i \tilde{=} i' \). Thus, by Proposition 3-6, \( S_1 \) is a prime double-MDS-code. The same is true of \( S_j \) for every \( j \in [k] \). \( \square \)

## 4 Decomposition of double-MDS-codes

**Theorem 4-1.** (a) The characteristic function \( \chi_S \) of a double-MDS-code \( S \) has a unique representation in the form

\[
\chi_S(\bar{x}) = \bigoplus_{j=1}^{k} \chi_{S_j}(\bar{x}_j) \oplus \sigma_0 \quad \text{where}
\] (6)

\begin{itemize}
  \item \( k \in [n] \),
  \item \( \bar{x}_1, \ldots, \bar{x}_k \) are disjoint collections of variables from \( \bar{x} \), \( \bar{x}_j \triangleq (x_{i,j,1}, \ldots, x_{i,j,n_j}) \),
  \item for each \( j \in [k] \) the set \( S_j \subseteq \Sigma^{n_j} \) is a prime double-MDS-code and \( \bar{0} \in S_j \),
  \item \( \sigma_0 \in \{0, 1\} \).
\end{itemize}
(b) $S$ is a union of $2^{k-1}$ equipotent prime double-codes; 
$\Sigma^n \setminus S$ is a union of $2^{k-1}$ equipotent prime double-codes.

(c) If $k \geq 2$ and the adjacency graph $G(S)$ is bipartite, then for each $j \in [k]$ the graphs $G(S_j)$ and $G(\Sigma^n \setminus S_j)$ are also bipartite.

Proof. (a) is a corollary of Propositions 3-11 and 3-12. The uniqueness of the representation (6) follows from Lemma A-4 of the Appendix.

(b) Without loss of generality we assume that the variables are arranged in such a way that 
\[ \bar{x} = (\bar{x}_1, \ldots, \bar{x}_k). \]
Then,
\[ S = \bigcup_{(\gamma_1, \ldots, \gamma_k) \in \{0,1\}^k} S_{\gamma_1}^1 \times \ldots \times S_{\gamma_k}^k, \quad \Sigma^n \setminus S = \bigcup_{(\gamma_1, \ldots, \gamma_k) \in \{0,1\}^k} S_{\gamma_1}^0 \times \ldots \times S_{\gamma_k}^0 \]
where $S_j^0 \triangleq S_j$ and $S_j^1 \triangleq \Sigma^n \setminus S_j$. For all $(\gamma_1, \ldots, \gamma_k) \in \{0,1\}^k$ the adjacency graph $G(\gamma_1, \ldots, \gamma_k) \triangleq G(S_{\gamma_1}^1 \times \ldots \times S_{\gamma_k}^k) = G(S_{\gamma_1}^0) \times \ldots \times G(S_{\gamma_k}^0)$ is connected and has the degree $n = n_1 + \ldots + n_k$, because the graphs $G(S_{\gamma_1}^1), \ldots, G(S_{\gamma_k}^1)$ are connected and have the degrees $n_1, \ldots, n_k$. Consequently, $G(\gamma_1, \ldots, \gamma_k)$ is a connected component of $G(S)$ (if $\gamma_1 \oplus \ldots \oplus \gamma_k \oplus \sigma_0 = 0$) or $G(\Sigma^n \setminus S)$ (if $\gamma_1 \oplus \ldots \oplus \gamma_k \oplus \sigma_0 = 1$). Moreover, the cardinality of $G(\gamma_1, \ldots, \gamma_k)$ equals $|S_1| \cdot \ldots \cdot |S_k|$ and does not depend on $(\gamma_1, \ldots, \gamma_k)$. Proposition 3-2 completes the proof of (b).

(c) Let $k \geq 2$. It is easy to see that fixing the arguments $\bar{x}_1, \ldots, \bar{x}_{j-1}, \bar{x}_{j+1}, \ldots, \bar{x}_k$ we can obtain the function $\chi_{S_j}(\bar{x}_j)$, as well as $\chi_{\Sigma^n \setminus S_j}(\bar{x}_j)$, in the right part of (6). Consequently, $\chi_{S_j}$ and $\chi_{\Sigma^n \setminus S_j}$ are subfunctions of $\chi_{S}$. Thus, $G(S_j)$ and $G(\Sigma^n \setminus S_j)$ are subgraphs of $G(S)$, which proves the statement. \[ \square \]

Corollary 4-2. If a double-MDS-code $S$ is not prime, then $\chi_{S}$ is the sum of the characteristic functions of prime double-MDS-codes of smaller lengths. Moreover, if $G(S)$ has $K$ connected components, then the number of the summands is $1 + \log_2 K$.

Remark 4-3. (On the general $q$-valued case.) The results above can be generalized to the arbitrary even size of the alphabet $\Sigma = \{0,1, \ldots, q-1\}$. A set $S \subset \Sigma^n$ is called a $q/2$-fold MDS code ($q/2$-code) if each line of $\Sigma^n$ contains $q/2$ (respectively, 0 or $q/2$) elements from $S$. The concepts of complementable and prime $q/2$-codes are defined as for double-codes. If we replace double-codes and double-MDS-codes by, respectively, $q/2$-codes and $q/2$-fold MDS codes in Theorem 4-1 and Corollary 4-2, then the statements will hold as well. Indeed, all the proofs, without essential changes, are valid for the $q$-valued case. It should only be noted that for each even $q$ there are exactly one non-prime $q/2$-fold MDS code in $\Sigma^2$ (see Fig. 1(c) for the case $q = 4$) and exactly one “non-MDS” prime $q/2$-code in $\Sigma^2$ (Fig. 1(b)), up to equivalence. So, it is easy to check that all the simple statements on the $q/2$-codes in $\Sigma^2$ that are used in the proofs (Propositions 3-5(g,j,k) and 3-10) are valid for the $q$-valued case.


## 5 Decomposition of \((n, 2)_4\) MDS codes

The next theorem gives a representation of \((n, 2)_4\) MDS codes, which is based on the decomposition of double-MDS-codes presented in Theorem 4-1.

A double-MDS-code \(S\) in \(\Sigma^n\) is called **linear** if its characteristic function \(\chi_S\) can be represented in the form

\[
\chi_S(x_1, \ldots, x_n) = \chi_1(x_1) \oplus \cdots \oplus \chi_n(x_n)
\]

for some functions \(\chi_1, \ldots, \chi_n : \Sigma \to \Sigma\), which are, clearly, the characteristic functions of double-MDS-codes in \(\Sigma^1\). There is only one double-MDS-code in \(\Sigma^1\) up to permutation of the symbols of \(\Sigma\). So, there is only one linear double-MDS-code in \(\Sigma^n\) up to permutations of the alphabet symbols in each coordinate.

An MDS code \(C\) is called **semilinear** if \(C \subset S\) for some linear double-MDS-code \(S\). Since in this case \(C\) is a part of the bipartite graph \(G(S)\), it is not difficult to describe the class of semilinear MDS codes. The number of such codes of length \(n\) is \(3^n2^{2n-1}+2n+3n^{-1}\) (see e.g. [7]).

**Theorem 5-1.** Let \(S\) be a double-MDS-code in \(\Sigma^n\) and \(C \subset S\) be an \((n, 2)_4\) MDS code. Then

\[
C = \{(x_1, \ldots, x_n) | (g_1(\tilde{x}_1), \ldots, g_k(\tilde{x}_k)) \in B_C\}, \quad (7)
\]

\[
C = \{(x_1, \ldots, x_n) | (\tilde{x}_j, y_j) \in C_j, \ j = 1, \ldots, k; \ (y_1, \ldots, y_k) \in B_C\}, \quad (8)
\]

where \(\tilde{x}_j = (x_{i_1,1}, \ldots, x_{i_j,n_j})\), the set \(B_C\) is a semilinear \((k, 2)_4\) MDS code, the set \(C_j\) is a \((n_j + 1, 2)_4\) MDS code, the mapping \(g_j : \Sigma^{n_j} \to \Sigma\) is a \(n_j\)-quasigroup, \(j = 1, \ldots, k\), and \(k, n_j, i_j, s\) are specified by Theorem 4-1. Moreover, all the parameters except \(B_C\) depend only on \(S\) and do not depend on \(C \subset S\).

**Proof.** It is easy to see that (7) and (8) are equivalent if \(C_j = \{ (\tilde{z}, g_j(\tilde{z})) | \tilde{z} \in \Sigma^{n_j} \}. \) So, it is enough to show only (7).

If \(k = 1\), then the statement is obvious. Assume that \(k > 1\). By Theorem 4-1(c), the graphs \(G(S_j)\) and \(G(\Sigma^{n_j}\setminus S_j)\) are bipartite (\(S_j\) are specified in Theorem 4-1(a)). Then, for each \(j \in [k]\) we can easily define an \(n_j\)-quasigroup \(g_j\) such that its set of 1s and 0s is \(S_j\) (more accurately, define the set of 0s of \(g_j\) as a part of \(G(S_j)\), and the set of 1s as the other part; the set of 2s as a part of \(G(\Sigma^{n_j}\setminus S_j)\), and the set of 3s as the other part); i.e.,

\[
\chi_{S_j}(\tilde{x}_j) \equiv \chi_{\{0, 1\}}(g_j(\tilde{x}_j)). \quad (9)
\]

Let the linear double-MDS-code \(D \subset \Sigma^k\) be defined by the equality

\[
\chi_D(y_1, \ldots, y_k) = \chi_{\{0, 1\}}(y_1) \oplus \cdots \oplus \chi_{\{0, 1\}}(y_k) \oplus \sigma_0. \quad (10)
\]

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Using (9) and (10), we can rewrite (6) in the following way:

\[ S = \{(x_1, \ldots, x_n) | (g_1(\bar{x}_1), \ldots, g_k(\bar{x}_k)) \in D\}. \]

If \( B \subset D \) is an MDS code, then the set

\[ \{(x_1, \ldots, x_n) | (g_1(\bar{x}_1), \ldots, g_k(\bar{x}_k)) \in B\} \subset S \]

is also an MDS code. The double-code \( D \) has \( 2^{2^{k-1}} \) MDS-code-subsets (all these MDS codes are semilinear). Then, \( 2^{2^{k-1}} \) different MDS-code-subsets of \( S \) are represented in the form (7).

On the other hand, by Theorem 4-1(b), the set \( S \) is the union of \( 2^{k-1} \) prime double-codes.

By Proposition 3-4(a), there are exactly \( 2^{2^{k-1}} \) subsets of \( S \) that are MDS codes. Therefore, all these MDS codes have the representation (7) and the code \( C \) is one of them (with \( B = B_C \)). □

We say that an \((n, 2)_4\) MDS code \( C \) is decomposable if there are \( m \in \{2, \ldots, n - 2\} \), an \( m\)-quasigroup \( g' \), an \((n - m)\)-quasigroup \( g'' \), and a permutation \( \sigma : [n] \rightarrow [n] \) such that

\[ C = \{(x_1, \ldots, x_n) \in \Sigma^n | g'(x_{\sigma(1)}, \ldots, x_{\sigma(m)}) = g''(x_{\sigma(m+1)}, \ldots, x_{\sigma(n)})\}. \]

Taking into account Proposition 2-1, we can say that a decomposable MDS code can be represented as a “concatenation” of MDS codes of smaller lengths.

**Corollary 5-2.** (a) If \( 2 < k < n \) or \( k = 2, n_1 > 1, n_2 > 1 \), then the MDS code \( C \) is decomposable.

(b) If \( k = n \), then \( C \) is a semilinear MDS code.

**Example 5-3.** Let \( \pi = (01)(23) \) and \( S = C \cup C' \), where \( C' = \{(\pi(x_1), x_2, \ldots, x_n) | (x_1, \ldots, x_n) \in C\} \). Then \( S \) is a non-prime double-MDS-code, \( k \geq 2, n_1 = 1 \). The lengths of the codes \( B_C, C_1, \ldots, C_k \) in (8) are smaller than \( n \) (in this case, \( C \) and \( C' \) are decomposable) if and only if \( 2 < k < n \).

We say that two sets \( C, C' \subseteq \Sigma^n \) are isotopic if there are permutations \( \pi_1, \ldots, \pi_n : \Sigma \rightarrow \Sigma \) such that

\[ (x_1, \ldots, x_n) \in C \iff (\pi_1(x_1), \ldots, \pi_n(x_n)) \in C'. \]

The following theorem means that we cannot get a “new” MDS code if we combine parts of two disjoint MDS codes \( C_1 \) and \( C_2 \), i.e., the resulting code can be obtained as semilinear, or as isotopic to \( C_1 \) and \( C_2 \), or it can be composed from MDS codes of smaller lengths.

**Theorem 5-4.** Let \( C_1 \) and \( C_2 \) are disjoint MDS codes. Suppose an MDS code \( C_{\text{new}} \) is a subset of \( C_1 \cup C_2 \). Then there are only three possibilities:

1. \( C_{\text{new}} \) is isotopic with \( C_1 \) and \( C_2 \),
2. \( C_{\text{new}} \) is decomposable,
3. \( C_{\text{new}} \) is semilinear.
Proof. Since the MDS codes $C_1$ and $C_2$ are disjoint, the set $S \triangleq C_1 \cup C_2$ is a double-MDS-code.

Consider the representation (7) for the code $C_{\text{new}} \subset S$. By Corollary 5-2, it is enough to consider the case $k = 2$, $\{n_1, n_2\} = \{1, n - 1\}$. W.l.o.g. we assume that (7) has the form

$$C = \{(x_1, \ldots, x_n) | (g_1(x_1), g_2(x_2, \ldots, x_n)) \in B_C\}$$

with $C = C_{\text{new}}$. By Theorem 5-1, this equation also holds for any MDS code $C \subset S$. By Proposition 2-1, we have an equivalent equation

$$C = \{(x_1, \ldots, x_n) | f_C(g_1(x_1)) = g_2(x_2, \ldots, x_n)\}$$

for any MDS code $C \subset S$, where 1-quasigroup $f_C$ is defined by $B_C = \{(y, f_C(y)) | y \in \Sigma\}$. Since $g_1, f_C : \Sigma \to \Sigma$ are permutations, any two MDS codes that are subsets of $S$ are isotopic. 

6 Decomposition of $n$-quasigroups of order 4

In this section we derive two conditions guaranteeing that an $n$-quasigroup can be represented as a superposition of $n_j$-quasigroups with $n_j < n$. Note that, taking into account the one-to-one correspondence between $n$-quasigroups and MDS-codes (Proposition 2-1), the following two theorems are closely related with Theorem 5-1.

If $f$ is an $n$-quasigroup and $B \subseteq \Sigma$, then we denote

$$M_B(f) \triangleq \{\bar{x} \in \Sigma^n | g(\bar{x}) \in B\}.$$ 

**Theorem 6-1.** Let $g$ be an $n$-quasigroup. Assume $\Sigma = \{a, b, c, d\}$ and $S = M_{\{a,b\}}(g)$. Then

$$g = g_0(g_1(\bar{x}_1), \ldots, g_k(\bar{x}_k)), \quad (11)$$

where $\bar{x}_j = (x_{i_{j,1}}, \ldots, x_{i_{j,n_j}})$, the mappings $g_0, g_1, \ldots, g_k$ are $k$-, $n_0$-, $\ldots$, $n_k$-quasigroups, and $k, n_j, i_{j,s}$ are specified by Theorem 4-1.

**Proof.** If $k = 1$, then the statement is obvious. Suppose $k > 1$. As in the proof of Theorem 5-1, we get

$$S = \{(x_1, \ldots, x_n) | (g_1(\bar{x}_1), \ldots, g_k(\bar{x}_k)) \in D\}$$

for some linear double-MDS-code $D \in \Sigma^k$.

Let $g_0$ be a $k$-quasigroup such that $M_{\{a,b\}}(g_0) = D$. Then the mapping

$$f(\bar{x}) = g_0(g_1(\bar{x}_1), \ldots, g_k(\bar{x}_k)) \quad (12)$$

is an $n$-quasigroup such that $M_{\{a,b\}}(f) = S$. 

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We claim that:

(*) the number of ways to choose \(g_0\) equals \(2^k\);

(***) the number of \(n\)-quasigroups \(f\) such that \(M_{\{a,b\}}(f) = S\) equals \(2^k\).

By Theorem 6-1, each of the sets \(M_{\{a,b\}}(f) = S\), \(M_{\{c,d\}}(f) = \Sigma^n \setminus S\), \(M_{\{a,b\}}(g_0) = D\), \(M_{\{c,d\}}(g_0) = \Sigma^k \setminus D\) is the union of \(2^{k-1}\) different prime double-codes. The number of ways to choose \(g_0\) equals the number of ways to choose an MDS code \(M_{\{a\}}(g_0) \subset D\) multiplied by the number of ways to choose an MDS code \(M_{\{c\}}(g_0) \subset \Sigma^k \setminus D\), i.e., \(2^{2k-1} \cdot 2^{2k-1} = 2^k\) (see Proposition 3-4(a)). The claim (*) is proved. Similarly, (***) is also true.

So, we conclude that all the \(n\)-quasigroups \(f\) such that \(M_{\{a,b\}}(f) = S\) (\(g\) is one of them) have the representation (12).

We say that an \(n\)-quasigroup \(g\) is reducible if it can be represented as a superposition of \(n_j\)-quasigroups where \(n_j < n\). We say that an \(n\)-quasigroup \(g\) is semilinear if the corresponding MDS code \(\{(\vec{x}, q(\vec{x})) \mid \vec{x} \in \Sigma^n\}\) is semilinear.

**Corollary 6-2.** Assume the conditions of Theorem 6-1 holds. (a) If \(1 < k < n\), then the \(n\)-quasigroup \(g\) is reducible. (b) If \(k = n\), then the \(n\)-quasigroup \(g\) is semilinear.

**Proof.** (a) is straightforward. (b) From the description of \(g_0\) (in the proof of Theorem 6-1) we derive that it is semilinear. Since, in the case \(k = n\), the \(1\)-quasigroups \(g_j, j \in [n]\), are just permutations, \(g\) is also semilinear.

The next theorem interpret Corollary 6-2 in terms of the \(n\)-quasigroup that is inverse to \(g\) in some (say, \(n\)th) argument.

**Theorem 6-3.** Let \(h\) be an \(n\)-quasigroup, \(\{a, b, c, d\} = \Sigma\), and \(Q\) be the set of \(n\)-quasigroups \(f\) such that \(f|_{\Sigma^{n-1} \times \{a,b\}} \equiv h|_{\Sigma^{n-1} \times \{a,b\}}\). If \(2 < |Q| < 2^{2n-1}\), then the \(n\)-quasigroup \(h\) is reducible. If \(|Q| = 2^{2n-1}\), then \(h\) is semilinear.

**Proof.** Assume \(f \in Q\) and \(f^{(n)}\) denotes the \(n\)-quasigroup that is inverse to \(f\) in \(n\)th argument. Then, the \(as\) and the \(bs\) of \(f^{(n)}\) coincide with respectively \(as\) and the \(bs\) of \(h^{(n)}\), i.e., \(M_{\{a\}}(f^{(n)}) = M_{\{a\}}(h^{(n)})\) and \(M_{\{b\}}(f^{(n)}) = M_{\{b\}}(h^{(n)})\). Let \(S \triangleq M_{\{a,b\}}(h^{(n)})\). Therefore, \(|Q|\) is the number of ways to choose the \(cs\) of \(f^{(n)}\), i.e., the number of MDS codes \(C \subset \Sigma^n \setminus S\). So, by Theorem 4-1(b) and Proposition 3-4(a), we have \(|Q| = 2^{2k-1}\). Since by the condition of the theorem \(2 < |Q| < 2^{2n-1}\) (or \(|Q| = 2^{2n-1}\)), we have \(1 < k < n\) (respectively, \(k = n\)). Then, by Corollary 6-2, the \(n\)-quasigroup \(h^{(n)}\) is reducible (respectively, semilinear). It is straightforward that \(h^{(n)}\) is reducible (semilinear) if and only if \(h\) is.
A On functions with separable arguments

In the Appendix, we will consider the functions with separable arguments, i.e., the functions that can be represented as the sum of functions of smaller arity depending on mutually disjoint collections of arguments of the original function. We will prove a criterion for a function to have separable arguments and will show that a function has a unique canonical representation as such the sum.

Let $\Sigma$ be an arbitrary set that contains 0. Let $n$ be a natural number, $K = \{K_1, \ldots, K_k\}$, where $\emptyset \neq K_j \subseteq [n]$, be a partition of the set $[n]$, and $K_j = \{i_{j,1}, \ldots, i_{j,n_j}\}$, where $n_j = |K_j|$, $j \in [k]$. Let $(\Gamma, \oplus)$ be an Abelian group. We say that a function $f : \Sigma^n \to \Gamma$ has $K$-separable arguments if

$$f(\bar{x}) = f_1(\bar{x}_K_1) \oplus \ldots \oplus f_k(\bar{x}_K_k),$$

where $f_j : \Sigma^{n_j} \to \Gamma$, $\bar{x} \triangleq (x_1, \ldots, x_n)$, and $\bar{x}_K_j \triangleq (x_{i_{j,1}}, \ldots, x_{i_{j,n_j}})$. If $|K| > 1$, then we say that $f$ has separable arguments. We say that a function $f : \Sigma^n \to \Gamma$ has non-separable arguments if (13) implies $|K| = 1$.

Lemma A-1. A function $f : \Sigma^n \to \Gamma$ has $K$-separable arguments if and only if for each $i', i''$ that belong to different elements of $K$, for each $\bar{x} \in \Sigma^n$ and $a', a'' \in \Sigma$ it holds

$$f(\bar{x}) \oplus f(\bar{x}^{[i'][a']}) \oplus f(\bar{x}^{[i''][a'']}) = 0. \quad (14)$$

Proof. Assume (14) holds. Let $P = \{p_1, \ldots, p_m\} \subseteq [n]$, $Q = \{q_1, \ldots, q_r\} \subseteq [n]$, and each $K_j$ is disjoint with at least one of $P$ and $Q$. Then, by (14), for each $\bar{x} = \{x_1, \ldots, x_n\} \in \Sigma^n$ we have

$$\bigoplus_{s=1}^{m} \bigoplus_{t=1}^{r} \left( f \left( \bar{x}^{[p_1, \ldots, p_{s-1}, q_{t-1}]} \right) \right) \oplus \ldots \oplus \left( f \left( \bar{x}^{[p_1, \ldots, p_{s-1}, q_{t-1}]} \right) \right) \oplus \ldots \oplus \left( f \left( \bar{x}^{[p_1, \ldots, p_{s-1}, q_{t-1}]} \right) \right) = 0.

Collecting similar terms we get

$$f(\bar{x}) \oplus f(\bar{x}^{[p_1, \ldots, p_m]}) \oplus \ldots \oplus f(\bar{x}^{[q_1, \ldots, q_r]}) \oplus \ldots \oplus f(\bar{x}^{[p_1, \ldots, p_m, q_1, \ldots, q_r]}) = 0. \quad (15)$$

Without loss of generality we can assume that $\bar{x} = (\bar{x}_{K_1}, \bar{x}_{K_2}, \ldots, \bar{x}_{K_k})$, i.e., $(i_{1,1}, \ldots, i_{1,n_1}, i_{2,1}, \ldots, i_{2,n_2}, \ldots, i_{k,n_k}) = (1, \ldots, n)$ (recall that $\bar{x}_{K_j} = (x_{i_{j,1}}, \ldots, x_{i_{j,n_j}})$). By (15), it holds

$$\bigoplus_{j=2}^{k} f(\bar{x}_{K_1}, \ldots, \bar{x}_{K_{j-1}}, 0, 0 \ldots 0) \oplus f(0 \ldots 0, \bar{x}_{K_j}, 0 \ldots 0)$$
⊕ f(\bar{x}_{K_1}, \ldots, \bar{x}_{K_{j-1}}, \bar{x}_{K_j}, 0 \ldots 0) = 0.

Collecting similar terms we get

\[ f(\bar{x}) = \bigoplus_{j=1}^{k} f(0, \ldots, 0, \bar{x}_{K_j}, 0, \ldots, 0) \ominus (k - 1)f(\bar{0}) \]

and the function f has K-separable arguments by the definition.

The inverse statement is straightforward. □

**Lemma A-2.** If a function \( f : \Sigma^n \to \Gamma \) has K-separable arguments, then the functions \( f_1, \ldots, f_k \) such that

\[ f(\bar{x}) \equiv \bigoplus_{i=1}^{k} f_i(\bar{x}_{K_i}) \]

are uniquely defined up to constant summand.

**Proof.** Let

\[ f(\bar{x}) \equiv \bigoplus_{i=1}^{k} f_i(\bar{x}_{K_i}) \equiv \bigoplus_{i=1}^{k} g_i(\bar{x}_{K_i}). \]

Let \( j \) be fixed. Setting \( \bar{x}_{K_i} = (0, \ldots, 0) \) for \( i \neq j \), we get

\[ g_j(\bar{x}_{K_j}) \ominus f_j(\bar{x}_{K_j}) \equiv \bigoplus_{i \neq j} f_i(0, \ldots, 0) \ominus \bigoplus_{i \neq j} g_i(0, \ldots, 0) \equiv \text{const} \in \Gamma. \]

□

If \( K = \{K_i\}_{i=1}^{k} = \{K_1, \ldots, K_k\} \) and \( L = \{L_j\}_{j=1}^{l} \) are two partitions of \([n]\), then we define by \( K \wedge L \) the partition \( \{K_i \cap L_j\}_{i=1}^{k} \wedge_{j=1}^{l} \setminus \{\emptyset\} \).

**Lemma A-3.** If the arguments of a function \( f : \Sigma^n \to \Gamma \) are K- and L-separable, then they are \((K \wedge L)\)-separable.

**Proof.** Let

\[ f(\bar{x}) \equiv \bigoplus_{i=1}^{k} f_i(\bar{x}_{K_i}) \equiv \bigoplus_{j=1}^{l} g_j(\bar{x}_{L_j}). \]  

(16)

For each \( j \) from 1 to \( l \) we define the function \( g'_j(\bar{x}_{K_i \cap L_j}, \ldots, \bar{x}_{K_k \cap L_j}) \triangleq g_j(\bar{x}_{L_j}) \), which differs from \( g_j \) by the appropriate permutation of the arguments, and the functions \( h_{i,j}(\bar{x}_{K_i \cap L_j}) \triangleq \)
From (16) for each i from 1 to k we have
\[ f_i(\bar{x}_{K_i}) \oplus \bigoplus_{i' \neq i} f_{i'}(\bar{0}) = \bigoplus_{j=1}^{l} g_j'(\bar{0}, \ldots, \bar{0}, \bar{x}_{K_i \cap L_j}, \bar{0}, \ldots, \bar{0}), \]
\[ f_i(\bar{x}_{K_i}) = \bigoplus_{j=1}^{l} h_{i,j}(\bar{x}_{K_i \cap L_j}) \oplus \text{const}, \]
and
\[ f(\bar{x}) = \bigoplus_{i=1}^{k} \bigoplus_{j=1}^{l} h_{i,j}(\bar{x}_{K_i \cap L_j}) \oplus \text{const}. \]

Since \( K_i \cap L_j = \emptyset \Rightarrow h_{i,j}(\bar{x}_{K_i \cap L_j}) = \text{const} \), the lemma is proved. \( \square \)

**Lemma A-4.** The decomposition
\[ f(\bar{x}) = \bigoplus_{i=1}^{k} f_i(\bar{x}_{K_i}) \]

of a function \( f: \Sigma^n \rightarrow \Gamma \) into functions \( f_i \) with non-separable arguments is unique up to constant summands in \( f_i \).

**Proof.** Let \( \mathcal{K} \) be the set of partitions \( K \) of \([n]\) for which the function \( f \) has \( K \)-separable arguments. By Lemma A-3, \( (\mathcal{K}, \land) \) is a semilattice. As we can see from the proof of Lemma A-3, only the least element of this semilattice corresponds to decomposition into functions with non-separable arguments. By Lemma A-2 these functions are unique up to constant summand. \( \square \)

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