Loop Representations

Bernd Brügmann

Max-Planck-Institute of Physics, 80805 München, Germany
bruegman@iws170.mppmu.mpg.de

Abstract

The loop representation plays an important role in canonical quantum gravity because loop variables allow a natural treatment of the constraints. In these lectures we give an elementary introduction to (i) the relevant history of loops in knot theory and gauge theory, (ii) the loop representation of Maxwell theory, and (iii) the loop representation of canonical quantum gravity.
1 Introduction

1.1 Generalities

The task of theoretical physics is to find an adequate mathematical description of physical ideas, and there is always a tension between real world experiments and the mathematical structures modelling them. In quantum gravity this tension appears with a twist, since there is no direct experimental evidence to be explained. Guided by the hypothesis that there exists a unified description of nature, we try to find a theory that could describe phenomena that belong at the same time to the domain of quantum theory and general relativity.

It is generally agreed upon that such phenomena exist even though they have not been observed. An example could be matter near the big bang singularity, and predictions like that of Hawking radiation for quantum fields near a black hole horizon are expected to eventually be valid limiting cases of a full theory of quantum gravity. But there are widely varying opinions on how one should go about constructing a theory of quantum gravity.

If we examine the theories that we want to combine closely, they are not flawless to begin with. One can for example argue that the measurement process of quantum mechanics is not sufficiently explained, while classical general relativity does not allow for quantum matter. In fact, we hope to remedy such problems in the more complete theory of quantum gravity.

It is not the case that the lack of experimental data does not allow us to pick the right theory out of a host of possibilities. The most important fact to remember about quantum gravity is that to date there does not exist a single model for a quantum theory of general relativity that is (i) self-consistent and (ii) contains as a special case a reasonable approximation to the observed physical world. There exist many research programs to construct quantum gravity, but all are incomplete even by their own criteria. However, some interesting partial results have been obtained in certain approaches, and here we will focus our attention on one such approach.

We consider the program of canonical quantization of general relativity. Although as incomplete as other approaches, there has been some recent progress initiated by the discovery of a new set of canonical variables for general relativity by Ashtekar [As86]. The reason for our discussion of loop representations is that the loop representation of Rovelli and Smolin [RoSm88] figures prominently in the quantum theory based on the Ashtekar variables.

There are many facets to canonical quantum gravity. In particular, there is much more to the program of canonical quantization starting from the Ashtekar variables than the loop representation, and the Ashtekar variables are in addition very interesting for the classical theory (see this volume). The goal of this paper is to demonstrate that similarly there is more to the concept of a loop representation than that it is a useful technique in canonical quantum gravity. To this end we will try to paint a coherent picture — drawing on knot theory, gauge theory and canonical quantum gravity — of why loops find a natural place in canonical quantum gravity.

There exist several excellent reviews of canonical quantum gravity, the Ashtekar variables, and the loop representation (which we will point out as we go along). Our emphasis will be on a complete and up to date development of the main ideas behind loop representations rather than on technical details, in the hope that this way the motivation for the loop representation becomes apparent and the reader may answer the question, “Why loops?”
1.2 What we mean by canonical quantum gravity

To put our discussion of quantum gravity into perspective, let us first attach a few labels to what we mean and imply by the term 'canonical quantum gravity' in these lectures. A much more thorough background to what quantum gravity could mean can be found in the articles of Isham in this volume.

1. Starting point is standard general relativity, i.e. the theory of a Lorentzian metric $g_{\mu\nu}(x)$ on a four dimensional differentiable manifold $\mathcal{M}$ defined by the Einstein-Hilbert action,

$$S[g] = \int_{\mathcal{M}} d^4x \sqrt{-g} R,$$

where $R$ is the Ricci scalar and $g$ the determinant of $g_{\mu\nu}(x)$. There are all sorts of reasons why one might want to consider discrete spaces instead of $\mathcal{M}$, or strings instead of points, or actions modified by higher order curvature terms, or other alterations. Whenever we refer to general relativity, we mean standard general relativity.

2. Even though there currently is a great deal interest in theories in dimension unequal four, here we consider general relativity in four spacetime dimensions if we do not state otherwise. In fact, one of the reasons why loop representations are of interest is that they are tailored to the physically observed number of four dimensions.

3. Other approaches to quantum gravity, in particular path integral quantization and quantum cosmology, are based on an Euclideanization. Since Euclidean quantum gravity cannot in general be extended to Lorentzian quantum gravity, it may be an advantage that canonical quantization and the loop representation are Lorentzian (although they are simple to Euclideanize).

4. Since we consider a canonical formulation, we only allow spacetimes that can be split into space and time, $\mathcal{M} = \pm \times \mathbb{R}$. The three-manifold $\Sigma$ is assumed to be compact for simplicity. Such a choice for $\mathcal{M}$ excludes the possibility of topology change, which can be accommodated, for example, in path integral quantization.

5. Our goal is an inherently non-perturbative formulation, for which canonical quantization is well-suited. This has to be contrasted with standard field theoretic methods, which are based on perturbation theory using Feynman diagrams obtained from path integrals. Simply put, perturbation theory for general relativity fails at two loop, and the characteristic of general relativity, e.g. diffeomorphism invariance, seem to make a non-perturbative approach necessary. Let us also point out that a non-perturbative, canonical formulation looks and is in fact very different from perturbation theory. Many of the unconventional features that we encounter in the loop representation are not specific to the loop representation but to the canonical approach in general.

6. We consider gravity without matter, only at one point will we introduce a cosmological constant. Matter can be incorporated into the loop representation, but we will only briefly comment on it in section 5.1. The attitude in particle physics is that gravity can be treated by the same methods that are successful for the other interactions, the only difference being the energy scale. The most elegant scheme along these lines is certainly string theory, although gravity is introduced via gravitons, which is a concept from perturbation theory. The presence of matter
is thought to be essential. A common philosophy among relativists is that we can learn something about the deep conceptual issues in quantum gravity already in the absence of matter. Matter is thought to cloud some of these issues. But for questions like the issue of time in quantum gravity, we expect that we do have to include matter.

7. Canonical quantization of the type considered here is a very conservative approach. No new structures are postulated, like strings or supersymmetry. The idea is to use new techniques rather than new concepts as long as they are not forced upon us, i.e. to stay as close as possible to conventional quantum mechanics.

8. A loop is a map from the circle into a manifold. While mathematically the same objects as closed strings in string theory or loops in Feynman diagrams, loops play a completely different role in the loop representation. Also, their algebraic structure is not what is called a loop group, although a group of loops can be defined.

1.3 Outline

Loop representations have their origin in gauge theories, in fact, any Yang-Mills theory admits formally a loop representation. Let us summarize how a connection to general relativity is established. Canonical Yang-Mills theory is characterized by a gauge constraint, $G$, while general relativity in the usual metric variables is characterized by a diffeomorphism constraint $D$, which generates three-dimensional diffeomorphisms in $\Sigma$ and the Hamiltonian constraint $H$, which essentially specifies how $\Sigma$ is imbedded in $M$. In the canonically quantized theory the constraints are imposed as operator equations on the states $\psi$.

While quantum gravity in terms of the metric variables and Yang-Mills theory in terms of a connection are disjoint in both variables and invariances, quantum gravity in the Ashtekar variables is formulated in terms of a connection and all three constraints are present. The relation of quantum gravity to Yang-Mills theory is summarized in the following diagram:

$$
\begin{align*}
\hat{G}\psi &= 0 & \text{YMT} \\
\hat{D}\psi &= 0 & \text{QG} \\
\hat{H}\psi &= 0 & \text{QG in Ashtekar variables}
\end{align*}
$$

The idea behind the loop representation is to choose a representation of the quantum algebra of observables on a state space that contains functionals $\psi[\gamma]$ of loops $\gamma$ (as opposed to $\psi[A]$ or $\psi[g]$). The point is that the constraints can be treated more easily in the loop representation than in other representations: (i) The gauge constraint can be solved by gauge invariant variables (the Wilson loops) already on the classical level. (ii) The diffeomorphism constraint can be solved by considering states that are knot invariants, i.e. states that only depend on the diffeomorphism equivalence class of a loop. And (iii), the Hamiltonian constraint, also known as the Wheeler-DeWitt equation, has non-trivial solutions in terms of loops with intersections. This is, in a nutshell, the message we want to clarify. The loop representation of quantum gravity has also very interesting features beyond the constraints, which we will mention only briefly.

In section 2, we give a naturally very brief history of loops in mathematics (knot theory) and physics (gauge theory), and we argue why loop states can be expected to be useful in quantum gravity. In section 3, we discuss Maxwell theory as an example for
a theory that possesses a complete formulation in the loop representation. In section 4, we introduce the loop representation for quantum gravity, and discuss various steps of the program of canonical quantization in the loop representations. In section 5, we conclude with a few comments on loop representations in general and on the status of the loop representation for general relativity.

2 History of Loops

For the purpose of motivating the loop representation of quantum gravity, we first introduce loops in the context of knot theory and gauge theory. A starting point for these two topics can be found in the early work by Faraday and Gauss.

Before entering the discussion, let us define loops. First, we define a path \( \mu \) as a continuous, piecewise smooth map \( \mu : [s, t] \to \Sigma \) from an interval into the three-manifold \( \Sigma \). We usually choose \( \Sigma = \mathbb{R}^3 \). A loop is a closed path (figure 1), \( \alpha : [0, 1] \to \Sigma \) such that \( \alpha(0) = \alpha(1) \). Equivalently, \( \alpha \) is a map from the circle into the three manifold.

We denote the path from \( \alpha(s) \) to \( \alpha(t) \) along \( \alpha \) by \( \alpha_{s}^{t} \). The parametrization implies an orientation along the loop, e.g. \( \alpha(0) \) lies on \( \alpha_{s}^{t} \) only if \( s > t \). The inverse \( \alpha^{-1} \) of a loop \( \alpha \) is defined by reversing the parametrization,

\[
\alpha^{-1}(s) = \alpha(1 - s). 
\]

(3)

If two loops \( \alpha \) and \( \beta \) intersect at a point, i.e. \( \alpha(s) = \beta(t) \) for some \( s \neq t \), then we can define a combined loop \( \gamma = \alpha_{s}^{t} \beta_{t}^{s} \), as the loop obtained by first going around \( \alpha \) from \( \alpha(s) \) to \( \alpha(t) \) and then around \( \beta \) from \( \beta(t) \) to \( \beta(s) \). If the point of combination is clear from the context we may just write \( \gamma = \alpha \beta \). For example, if \( \alpha(1) = \beta(0) \),

\[
\gamma(s) = \begin{cases} 
\alpha(2s) & \text{if } 0 \leq s < 1/2 \\
\beta(2s - 1) & \text{if } 1/2 \leq s \leq 1 
\end{cases}
\]

(4)

where we have made the new parametrization explicit. Paths are combined analogously, usually at their endpoints. Finally, we also consider multiloops, which are unordered collections of loops. Given two loops \( \alpha \) and \( \beta \), we denote the multiloop \( \eta : S^1 \times S^1 \to \Sigma \) containing \( \alpha \) and \( \beta \) once by \( \eta = \alpha \cup \beta \) (\( = \beta \cup \alpha \)).

2.1 Loops in the Work of Faraday and Gauss

During the years of 1821–32, when M. Faraday was working on electrodynamics, he also developed the concept that the electromagnetic forces are transmitted through a force field, and that this force field has physical reality [Be74]. The competing point of view...
on electromagnetic forces at that time was that of 'action at a distance', in particular that no additional physical effect comes between charges and their relative forces. The idea to use force fields as intermediaries was not new, but fields were considered to be just a useful mathematical tool. As it often happens when a mathematical construction captures the essentials of a physical phenomenon, it becomes part of our intuition about what is actually physically present. We are justified to think of the electric field as 'real' because we can, for example, store energy in it or compute its propagation in electromagnetic waves. (At a deeper level of 'reality' we will have to deal with QED.)

What is relevant here is that Faraday also noted that in the absence of sources the field lines have to close on themselves, i.e. to form loops, since only then the electric field is divergence free,

$$D_a E^a(x) = 0. \quad (5)$$

Furthermore, we can argue that the elementary excitation of the electric field is based on loops. We cannot construct a divergence free vector field with support on only a point, but it works one dimension up. Of course, we usually consider electric fields with three-dimensional support, but gauge invariance requires only loops, not three-spaces.

This simple fact has applications in modern gauge theories through the so-called Wilson loops, which are gauge invariant variables defined through the parallel transport of spinors around loops. We will discuss the role played by loops in gauge theory in section 2.3.

What may be less known is that electromagnetism also lead to the study of the second type of invariance that we want to consider, namely diffeomorphism invariance. On January 22, 1833, C.F. Gauss found the answer to the following problem \[Ga1833\], which reads like a recent textbook problem, although the implications are quite subtle. What is the work done on a magnetic pole which is moved on a closed curve in the presence of a current loop? Notice that there are two loops in the problem, say $\alpha$ and $\beta$, which we assume to be non-intersecting (see figure 1). The answer can be expressed in terms of what is now known as the Gauss linking number,

$$gl(\alpha, \beta) = \frac{1}{4\pi} \int ds \int dt \epsilon_{abc} \dot{\alpha}^a(s) \dot{\beta}^b(t) \left( \frac{\alpha^c(s) - \beta^c(t)}{|\alpha(s) - \beta(t)|^3} \right). \quad (6)$$

Although not obvious when written this way, $gl(\alpha, \beta)$ is an integer. The Gauss linking number counts (with an appropriate sign) how often two loops are linked, or equivalently, how often one loop winds around the other. If there is no linking, then $gl(\alpha, \beta) = 0$.

The key point is that the Gauss linking number does not change under small, smooth deformations of the loops, that is, the Gauss linking number is invariant under diffeomorphisms. Gauss himself found it quite remarkable that an integral as in (6) has this property. The equivalence classes of loops under diffeomorphism (connected to the identity) is the topic of a branch of mathematics called knot theory.

Historically, the relation between loops and diffeomorphism invariance was developed before loops found their way into gauge theories. We therefore make a few comments on knot theory in section 2.2 and then briefly discuss loops in gauge theories in section 2.3.

### 2.2 Knot Theory

Knot theory studies the equivalence classes of loops without intersections under diffeomorphisms that are connected to the identity (e.g. \[Ka91\]). For a single loop the equivalence class is called a knot class, equivalence classes of multiloops are also called
Recall that a diffeomorphism is a $C^\infty$ map between manifolds that is one-to-one, onto, and has a $C^\infty$ inverse. An example for what a diffeomorphism can do with the unknot, which is the trivial knot, is shown in figure 2. A diffeomorphism can deform the unknot quite arbitrarily, but it cannot tie a trefoil knot in the unknot. The simple intuitive reason is that in a continuous transformation from one case to the other, two lines would cross, but a one-to-one and onto map cannot produce intersections or take them apart.

One could say that knot theory was founded in 1877 when P.G. Tait formulated the program of classifying all knots. To this end he introduced knot diagrams. A knot diagram is a non-degenerate projection of a knot that lives in three dimensions onto a two dimensional plane plus a prescription whether an intersection in the plane came from an under crossing or over crossing of two lines in three dimensions (see figure 3). One can show that there always exists a projection that is non-degenerate in that no more than two different points are mapped into one. For example, we are not allowed to project the unknot ’sideways’ onto a line segment.

The problem of classifying knots in three dimensions can be shown to be equivalent to classifying knot diagrams in two dimensions. An important tool to establish the equivalence of two knot diagrams are the Reidermeister moves. As a first step Tait compiled tables of inequivalent knot diagrams. Figure 4 shows a copy of Tait’s original table from 1884 [Ta1877] for the first seven orders of knottiness. Such tables are even today an important source for examples.

As an aside, the first application of knot classes to physics appeared in the work of Lord Kelvin in 1869 [Th1869] (which prompted Tait to start his investigations). He proposed that atoms are ’smoke ring vertices in the ether’. The stability and variety of atoms are to be directly related to the equivalence classes of loops under diffeomorphism. The spectral lines would be created by vibrations of the loop. Today we know about transmutations of elements, and those could have been discussed in terms of line crossings (see below). For about thirty years Kelvin’s theory of ’topological matter’
Figure 3: a) Projecting a knot onto a knot diagram; b) the four crossings in a knot diagram

was taken seriously, and for example Maxwell concluded that it accommodated more features of atomic physics than other models.

Returning to the classification problem, another important tool are knot invariants. A knot invariant is a functional on the space of loops that assigns to loops in the same knot class the same number, i.e. $\psi[\eta] = \psi[\{\eta\}]$. If $\psi$ is a knot invariant, then

$$\alpha \sim \beta \Rightarrow \psi[\alpha] = \psi[\beta].$$

(7) Therefore, if $\psi[\alpha] \neq \psi[\beta]$ then the loops are not equivalent. The Gauss linking number is an example for a link invariant.

The hard part is to construct the inverse. Indeed, one of the central, unsolved problems of knot theory is to find a complete set of knot invariants, $\{\psi_i\}$, such that

$$\psi_i[\alpha] = \psi_i[\beta] \quad \forall i \quad \Rightarrow \quad \alpha \sim \beta.$$ (8)

However, a complete, indirect classification is possible via the complement of loops in $\Sigma$. Also, there exist algorithms to generate all knot classes.

The most important (and most complete) knot invariants arise in the study of knot polynomials. Knot polynomials were introduced by Alexander in 1928 \cite{Al28}. A knot polynomial $P_q(\gamma)$ assigns to each knot diagram of a loop $\gamma$ a Laurent polynomial in a complex variable $q$ such that

(i) $P_q(\gamma)$ is a knot invariant, \hspace{1cm} (9)

(ii) $P_q(\text{unknot}) = 1$, \hspace{1cm} (10)

(iii) The skein relations are satisfied. \hspace{1cm} (11)

In the skein relations, or crossing change formulas, one considers three knot diagrams that differ only at one crossing. The three different possibilities are an over crossing, $c_+$, an under crossing, $c_-$, and no crossing, $c_\times$ (see figure 3). If intersections are allowed, one includes the intersections, $c_\times$. Since a reflection at a plane is a diffeomorphism which is not connected to the identity, $c_+$ and $c_-$ are inequivalent.

For example, the Alexander-Conway polynomial $A_q(\gamma)$ is uniquely determined by the skein relation

$$A_q(c_+) - A_q(c_-) = qA_q(c_\times).$$

(12)
Figure 4: “The first seven orders of knottiness”, Tait 1884.
One can show that such crossing changes are sufficient to reduce any knot diagram to the unknot by recursion. As a simple exercise the reader may show that

\[ A_q(\text{trefoil}) = 1 + q^2. \]  

While the knot polynomial \( A_q(\gamma) \) allows one to distinguish a large number of knots, it is not complete. For example, the mirror images of the trefoil lead to the same polynomial.

### 2.3 Gauge Theory

The canonical formulation of non-abelian gauge theories and the application of Wilson loops is a very interesting topic, and it is reviewed by Loll in this volume. Here we collect only a minimal set of definitions and facts that fit our discussion of the loop representation.

The canonically conjugate phase space variables of gauge theory are a configuration variable, the connection one-form \( A_a^i(x) \), and a momentum variable, the densitized triad \( E_{bj}(x) \), on \( \Sigma \). Here \( a, b, ... = 1, 2, 3 \) are (co-)tangent space indices of \( \Sigma \), \( i, j, ... \) are the indices of the internal gauge group. For matrices in the Lie algebra of the gauge group we write for example \( A_a^i = A_i^a \tau_i \), where \( \tau_i \) are the generators of the group (in the fundamental representation). The Poisson algebra is

\[
\begin{align*}
\{ A_a^i(x), A_b^j(y) \} &= 0, \\
\{ E_{ai}(x), E_{bj}(y) \} &= 0, \\
\{ A_a^i(x), E_{bj}(y) \} &= \delta_b^a \delta_{ij} \delta^3(x, y).
\end{align*}
\]

(14)

(15)

Gauge invariance implies the presence of a constraint in the canonical formalism, the Gauss constraint

\[ G^i(x) \equiv D_a E_{ai}(x) = 0, \]  

(16)

where \( D_a \) is the covariant derivative constructed from \( A_a \).

There are different ways to deal with the gauge constraint. For example, we can choose to perform a gauge-fixing or not, and we can choose to solve the constraint in the classical theory or in the quantum theory. Given a gauge fixing, one has to check whether the final result depends on the gauge or not, and there may be ambiguities in the quantum theory. In principle, these problems do not appear if one can find a gauge invariant formulation, that is if one is able to solve the constraints classically.

Solving the constraint classically is referred to as reduced phase space quantization, the alternative is to impose the constraints in the quantum theory as in Dirac quantization. In general, the result is not the same (e.g. [RoTa89]).

The idea leading to loops is to give a reduced phase space formulation of gauge theories in terms of Wilson loops [Ma62, Wi74, Po79, GaTr80, Mi83] (and Loll in this volume). A Wilson loop, \( h[\gamma, A] \), is the trace of the holonomy of \( \gamma \) and \( A \),

\[
\begin{align*}
h[\gamma, A] &= \text{tr} U_\gamma(A), \\
U_\gamma(A) &= P \exp \int_0^1 ds \dot{\gamma}^a(s) A_a(\gamma(s)),
\end{align*}
\]

(17)

(18)

where \( U_\gamma(A) \) is the matrix for parallel transport of spinors around the loop. The \( P \) denotes path ordering, i.e. for some one parameter family of matrices \( M(s) \),

\[ P \exp \int_0^1 ds M(s) = 1 + \int_0^1 ds M(s) + \int_0^1 ds \int_0^s dt M(t)M(s) + \ldots, \]

(19)

such that the products of matrices are always ordered according to the size of the parameter. (There is no factorial factor.)
The Wilson loops are gauge invariant since under a gauge transformation \( U \rightarrow gUg^{-1} \) and \( \text{tr}gUg^{-1} = \text{tr}U \). This is the reason why we consider loops and not paths. What makes the Wilson loops important is that not only are they gauge invariant, but in a sense they span the space of all gauge invariant functionals of the connection. There are reconstruction theorems, for example for \( SU(N) \) \[114, 115\], of the type that if the trace of the holonomy \( h[\gamma, A] \) is known for a given \( A \) and for all \( \gamma \), then \( A \) is determined up to gauge.

Starting point for the loop representations is that instead of elementary variables \( A_i^a \) and \( E^{bj} \) we can choose the loop variables

\[
T^0[\gamma] = \text{tr}U_\gamma(A), \quad T^1[\gamma]^a(s) = \text{tr}U_{\gamma^s}(A)E^a(\gamma(s)),
\]

where the \( T^1 \) variables are obtained by inserting the matrix \( E^a \) at the parameter \( s \) into the parallel transport around \( \gamma \) (see also section 4.2). Classically, the choice of variables is equivalent modulo gauge for \( SU(N) \). The question is whether loop variables offer any advantages over the conventional approaches in the quantum theory.

We could now give the definition of a loop representation for a gauge theory, but since our main objective is the loop representation of quantum gravity, let us first complete the picture of how quantum gravity relates to knot theory and gauge theory.

### 2.4 Quantum Gravity

#### 2.4.1 Before 1984.

Let us summarize the situation in knot theory, gauge theory, and general relativity up to the year 1984 from the perspective of what is important for the loop representation of quantum gravity:

- **Knot theory:**
  The characteristic invariance is diffeomorphism invariance. The project is to classify all knots. The status is that generating and labeling all knots is well understood, but a deep insight into what a complete set of knot invariants could be is missing.

- **Gauge theory:**
  The characteristic invariance is gauge invariance. The project is to perform a reduced phase space quantization. The status is that for non-abelian gauge theories it is not known how to treat the equations of motions, and around 1980 most people in the field give up on using loop variables (with the notable exception of the group around Gambini, see references).

- **General relativity:**
  The characteristic invariance is space-time diffeomorphism invariance, represented in the canonical formalism by the diffeomorphism and Hamiltonian constraints. The project is canonical quantization. The status is that canonical quantization is incomplete, in particular because one does not know how solve the constraints in the metric variables.

Notice that at this time the three topics are completely unrelated. While diffeomorphism invariance plays a central role in both knot theory and general relativity, knots play no role in general relativity. And general relativity is not a gauge theory in the sense explained in section 2.3.
What we are leading up to is, of course, that in the following years a fruitful combination of all these ideas became possible. While the problems mentioned above under status have not been solved, there has been some progress.

2.4.2 After 1984.

Let us now summarize some of the recent developments:

- Jones 1985 [Jo85]
  After Alexander, no new knot polynomials had been found until the discovery by Jones of the Jones polynomial, \( J_q(\gamma) \). The important point is that his techniques allow a systematic investigation into the space of knot invariants. Roughly speaking, each representation of the braid group that carries a Markov trace defines a knot invariant.

- Ashtekar 1986 [As86, As87, As91]
  Via a canonical transformation from the metric variables, Ashtekar constructs a new set of canonical variables. These are an \( SL(2, C) \) connection \( A^a_i \) and a conjugate momentum \( E_{bj} \), the same type of variables as in a non-abelian gauge theory. There are as before a spatial diffeomorphism constraint \( D \) and a Hamiltonian constraint \( H \), and in addition there is a Gauss constraint \( G \) (since \( g^{ab} = E^{ai} E_{bj} \), which is invariant under internal gauge transformations).

  One advantage of the Ashtekar variables is that the constraints are considerably simpler as in the metric variables (at least as far as solving the constraints is concerned, see below). The price to be paid is that there is an additional constraint, and that the Ashtekar connection is complex, and suitable reality conditions have to be imposed to recover real general relativity.

  The perhaps most important feature of the new variables is that in the Ashtekar formalism the kinematics of general relativity is imbedded in that of a non-abelian gauge theory for the Ashtekar connection. We have a standard gauge constraint plus the diffeomorphism and Hamiltonian constraint. This imbedding allows us to import ideas from Yang-Mills theory into general relativity, the loop representation being the prime example.

- Rovelli and Smolin 1988 [RoSm88, RoSm90]
  Rovelli and Smolin took the idea to use Wilson loops as gauge invariant variables one step further and constructed a representation of the operator algebra of quantum gravity on wavefunctions that are functionals of loops, \( \psi[\eta] \). After all that has been said, it should be obvious that loops have a natural role to play in canonical quantum gravity. Let us consider the constraints:

  1. We can solve the gauge constraint,

     \[ G = 0, \tag{22} \]

     on the classical level as in the reduced phase space formulation of gauge theory by using the loop variables \( T^0 \) and \( T^1 \), \( [24, 21] \), for the Ashtekar connection.

  2. We can solve the diffeomorphism constraint of the quantum theory,

     \[ \hat{D}\psi[\eta] = 0, \tag{23} \]

     by imposing that \( \psi[\eta] \) is a knot invariant. Indeed, since a finite diffeomorphism \( f \) has a natural representation on the support of the wavefunctions,
\[(f \cdot \psi)[\eta] = \psi[f^{-1} \circ \eta],\] equation (23) for the generator of diffeomorphisms \(\hat{D}\) implies that \(\psi[\eta]\) must be a knot invariant, \(\psi[\eta] = \psi[\{\eta\}]\).

This formal solution of the constraint has to be compared with the metric representation. In that case one can impose that the wavefunctions are functionals of equivalence classes \(\{g\}\) of metrics under diffeomorphisms (called three-geometries), \(\psi[g] = \psi[\{g\}]\). However, one of the major problems with canonical quantization based on the metric variables is the complicated structure of the space of three geometries. In the case of the loop representation, the situation is simpler since the knot invariants can be easily labeled. A similar simplification occurs for the hydrogen atom, where the energy angular-momentum basis \(\psi(n, l, m)\) has many advantages over the position basis \(\psi(x)\).

3. A priori it is not clear at all whether the Hamiltonian constraint,
\[
\hat{H}\psi[\eta] = 0, \tag{24}
\]
can be solved in the loop representation. One reason why the metric representation is incomplete is precisely that we do not know how to factor-order, regularize and solve the Wheeler-DeWitt equation. The encouraging fact in the loop representation is that there exist at least some trivial solutions to the Wheeler-DeWitt equation, while for the metric variables not a single solution to the full equation had been known. As noted by Rovelli and Smolin based on (JaSm88), the Hamiltonian constraint trivially annihilates all loop functionals that have support only on non-intersecting loops.

At this point, the loop representation seems to be a promising approach to canonical quantum gravity. The Gauss and diffeomorphism constraints can be solved formally by loop techniques, and the Wheeler-DeWitt equation is at least not completely incompatible with loop functionals.

- Witten 1989 [Wi89]
Witten showed that the Jones polynomial, which is tied to two dimensions through the knot diagrams, can be computed as the vacuum expectation value of the Wilson loops in Chern-Simons field theory, which is a topological field theory in three dimensions:
\[
J_q(\gamma) \sim \langle h[\gamma, A_{CS}] \rangle_{CS}, \tag{25}
\]
where \(A_{CS}\) is the Chern-Simons connection. There are many important aspects to this work, here we want to focus on the new link between invariants in two and three dimensions. While Witten’s proof is abstract in the sense that he does not directly evaluate the path integral that defines the expectation value (and mathematicians are still struggling to make his arguments rigorous), one can show that in a perturbation theory based on the path integral, the leading terms satisfy the skein relation that define the Jones polynomial [Sm89, GuMaMi90]. In this expansion one obtains knot invariants in analytic form analogously to the integral \(\int\) for the Gauss linking number, as opposed to the recursion formulas for knot diagrams.

The above results are very interesting and far reaching for many more reasons than just the canonical quantization of gravity. One of the reasons why the author finds the loop representation appealing is that we can tie together all we have said about knot theory and the loop representation, and actually use knot theory (beyond formal solution of the diffeomorphism constraint) to construct non-trivial solutions to the Wheeler-DeWitt equation:
One can generalize the Jones polynomial to the case of non-intersecting loops. The action of the Hamiltonian constraint on the analytic knot in variants arising in the perturbation theory of Chern-Simons theory can be computed explicitly. In fact, the second coefficient $a_2(\gamma)$ of the Alexander-Conway polynomial is annihilated by the Hamiltonian constraint. As explained in section 4.6, the non-triviality of this solution lies in the fact that because of the use of intersecting loops it is not simultaneously annihilated by the determinant of the metric.

Further understanding of the structure of the space of solutions can be gained by considering a transformation between the connection representation and the loop representation. If there exists a suitable transform, then the Jones polynomial (and not just a few coefficients) is a solution to the Hamiltonian constraint with a non-vanishing cosmological constant. This one solution for a cosmological constant leads to several solutions for vanishing cosmological constant.

This concludes the motivational part why the loop representation is natural in quantum gravity. In the remainder of this lecture we discuss some details of the loop representation in the context of an example, Maxwell theory, and for quantum gravity.

### 3 Example for Complete Loop Representation: Maxwell Theory

The loop representation of Maxwell theory is an ideal example for an introduction to loop representations. First of all, in this simple case the program of canonical quantization can be rigorously completed in the loop representation. Furthermore, the standard formulation in terms of Fock space is well understood, and we can compare the two representations and obtain a physical interpretation for the loop operators. In fact, the two representations turn out to be equivalent, since there exists a faithful transform between them.

The earliest work on a loop variable formulation for a $U(1)$ gauge theory is that of Mandelstam [Ma62], and by Gambini and Trias [GaTr80, GaTr83]. We will follow closely Ashtekar and Rovelli [AsRo92], where the loop representation is constructed in a form and with techniques that are directly related to the loop representation of quantum gravity.

#### 3.1 Bargmann Representation of Maxwell Theory

We consider a pure $U(1)$ gauge theory on $R^3$ with a flat background metric. The canonical variables defining the phase space are a one-form $A_a(x)$ and an electric field $E^a(x)$ such that $\{A_a(x), E^b(y)\} = \delta^a_b \delta^3(x,y)$. There is one first class constraint,

$$D_a E^a(x) = 0. \quad (26)$$

By fixing the gauge, $D^a A_a(x) = 0$, we pass to the reduced phase space which is parametrized by divergence free fields $A^T_a(x)$ and $E^T_a(x)$.

Such transverse fields are conveniently described in the momentum representation. Let us introduce canonical variables $q_j(k)$ and $p_j(k)$, $j = 1, 2$, on momentum space with the only non-vanishing Poisson bracket $\{q_m(-k), p_n(k')\} = \delta_{mn} \delta^3(k, k')$. In terms of complex polarization vectors $m_a(k)$, $m_a k^a = m_a m^a = 0$ and $m_a \bar{m}^a = 1$, we can write
the transverse variables as
\[
A^T_a(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k e^{ik\cdot x} (q_1(k)m_a(k) + q_2(k)\bar{m}_a(k)),
\]
\[
E^T_a(x) = -\frac{1}{(2\pi)^{3/2}} \int d^3k e^{ik\cdot x} (p_1(k)m_a(k) + p_2(k)\bar{m}_a(k)).
\]

In order to make a direct transition from the Fock representation to the loop representation possible, we have to work with Bargmann coordinates, \(\zeta\) and \(\bar{\zeta}\), defined by
\[
\zeta_j(k) = \frac{1}{\sqrt{2}}(|k|q_j(k) - ip_j(k)).
\]

The non-vanishing Poisson bracket is now
\[
\{\zeta_m(k), \bar{\zeta}_n(k')\} = i|k|\delta_{mn}\delta^3(k, k').
\]

The one dimensional analog of these variables are the Bargmann variables \(z\) and \(\bar{z}\) for the simple harmonic oscillator, \(z = \frac{1}{\sqrt{2}}(\omega q - ip)\). The Bargmann variables \(\zeta\) and \(\bar{\zeta}\) are directly related to the positive frequency part of the transverse fields \(A^T_a\) and the negative frequency part of \(E^T_a\),
\[
+ A^T_a(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{|k|} e^{ik\cdot x} (\zeta_1(k)m_a(k) + \zeta_2(k)\bar{m}_a(k)),
\]
\[
- E^T_a(x) = -\frac{i}{(2\pi)^{3/2}} \int \frac{d^3k}{|k|} e^{ik\cdot x} (\zeta_1(-k)m_a(k) + \bar{\zeta}_2(-k)\bar{m}_a(k)).
\]

The quantum theory can be formulated in the Bargmann representation, where states are holomorphic functionals \(\Psi[\zeta] \equiv \Psi[\zeta_1, \zeta_2]\) of the Bargmann variables. We represent the operators corresponding to the Bargmann variables via
\[
\hat{\zeta}_j(k)\Psi[\zeta] = \zeta_j(k)\Psi[\zeta],
\]
\[
\hat{\bar{\zeta}}_j(k)\Psi[\zeta] = -i\hbar|k|\frac{\delta}{\delta\zeta_j(k)}\Psi[\zeta].
\]

which implies that, as required, the canonical commutation relations are satisfied.

Again, compare with the harmonic oscillator where \(\hat{z}\Psi(z) = z\Psi(z)\) and \(\hat{\bar{z}}\Psi(z) = -i\hbar\omega(d/dz)\Psi(z)\).

To complete the mathematical setup, we have to specify an inner product that turns the space of states into an Hilbert space. This poses in general a non-trivial problem, see below. Here, an inner product exists and is uniquely determined by the reality conditions \(\hat{\zeta}_j(k) = \hat{\bar{\zeta}}_j^\ast(k)\). This inner product is the unique, Poincaré invariant inner product of the Bargmann representation,
\[
\langle \Phi[\zeta]|\Psi[\zeta]\rangle = \int d\mu(\zeta, \bar{\zeta})\Phi[\zeta]\overline{\Psi[\zeta]},
\]
\[
d\mu(\zeta, \bar{\zeta}) = \prod_j d\zeta_j(k) \wedge d\bar{\zeta}_j(k) \exp \left( -\frac{1}{2\hbar} \int \frac{d^3k}{|k|} |\zeta_j(k)|^2 \right).
\]

For the harmonic oscillator, the measure is of the type \(dz \wedge d\bar{z}e^{-z\bar{z}}\).

The Bargmann representation has the usual interpretation in terms of Fock states. The vacuum \(|0\rangle\) and the one photon state \(|k, \epsilon\rangle\) for momentum \(k\) and helicity \(\epsilon\) are given via \(\Psi[\zeta] = \langle \zeta | \Psi \rangle\) by
\[
|0\rangle : \quad \Psi_0[\zeta] = 1,
\]
\[
|k, \epsilon\rangle : \quad \Psi_{k, \epsilon}[\zeta] = \epsilon^\epsilon(k).
\]
The multiplication operator \( \hat{\varsigma}_j(k) \) acts as creation operator in the Fock basis, the derivative operator \( \hat{\varsigma}_j(k) \) as annihilation operator. For example, the one photon state is created by action with \( \hat{\varsigma}_j(k) \) on the vacuum, \( \Psi_{k,\epsilon}[\varsigma] \equiv \hat{\varsigma}_j(k) \Psi_0[\varsigma] \).

### 3.2 Loop Representation of Maxwell Theory

Let us consider the same \( U(1) \) gauge theory as in the preceding section, but now we choose loop variables instead of \( \zeta \) and \( \xi \). Because the gauge group is abelian, the loop variable \( T^0 [20] \) simplifies, and a simpler choice than \( T^1 [21] \) is possible. We define

\[
 h[\alpha] = \exp \int_{\alpha} ds^a A_a^T, \tag{39}
\]

\[
 E[f] = \int_{\mathbb{R}^3} d^3 x f_a + E^{Ta}, \tag{40}
\]

where \( \alpha \) is a loop and \( f_a \) a one-form. The loop variable \( h[\alpha] \) is the abelian holonomy and takes the place of \( T^0[\alpha] \). The analog of \( T^a[\alpha](s) = \text{tr} U_a E^a(\alpha(s)) \) is \( h[\alpha] + E^a(\alpha(s)) \), but since for \( U(1) \) gauge invariance no trace is necessary, we can use the loop-independent (smeared) variable \( E[f] \). (Note that \( \text{tr} E^a = 0 \) for \( SU(N) \).)

Our choice of loop variables highlights the fact that there are different possibilities to construct loop variables. In fact, we could as well start with \( h[\alpha] \) and \( h[\alpha] + E[f] \) as our elementary variables. Of course, the interpretation of the momentum operators is then different, and not as nicely related to the Fock representation. The choice of the negative frequency connection for \( h[\alpha] \) and the positive frequency electric field is necessary, since other frequency splittings are not consistent. Still another choice is to use anti-self-dual connections and self-dual electric fields, as we actually do in gravity.

Notice that the loop variables \( h[\alpha] \) and \( E[f] \) are in two ways overcomplete. First of all, not each label \( \alpha \) and \( f \) corresponds to different variables. Two different loops \( \alpha \) and \( \beta \) may lead to \( h[\alpha] = h[\beta] \) \( \forall A \), e.g. if \( \beta = \alpha \eta^{-1} \) for some path \( \eta \). And \( f_a \) and \( f_a + \partial_a g \) give the same \( E[f] \). In addition, there is the non-linear identity

\[
 h[\alpha] h[\beta] = h[\alpha \eta \beta \eta^{-1}] \tag{41}
\]

for any path \( \eta \) that connects \( \alpha \) and \( \beta \); we also write \( h[\alpha] h[\beta] = h[\alpha \# \beta] \). The latter type of identities play an important role since they differ for different gauge groups (compare [22] for \( SL(2, \mathbb{C}) \)).

The Poisson bracket algebra is (using the same simplectic structure as before)

\[
 \{ h[\alpha], h[\beta] \} = 0, \quad \{ E[f], E[g] \} = 0, \tag{42}
\]

\[
 \{ h[\alpha], E[f] \} = \left( \oint_{\alpha} ds^a f_a \right) h[\alpha]. \tag{43}
\]

Quantization in the loop representation is based on a space of states \( \psi[\alpha] \) which are functionals of loops. How can we represent the operators \( h[\alpha] \) and \( E[f] \)? Since the space of loops is not identical with the classical configuration space, which is the space of connections, the representation will not be the ‘usual’ one in terms of multiplication and derivative operators. Perhaps one should ask the question of how to find a representation with a different emphasis: Is there any chance at all that there exists a sensible representation for our unconventional choice of elementary variables?

Rovelli and Smolin suggested in their work on quantum gravity [RoSm90] that a transform between connection functionals and loop functionals can be used to derive
the loop representation from the connection representation. An analog in quantum mechanics is the Fourier transform between the position and the momentum representation, which allows us to transfer both states and operators, e.g.

$$\psi(k) = \int dx e^{ikx} \psi(x),$$

(44)
$$ik \leftrightarrow \frac{d}{dx}.\quad (45)$$

The Rovelli-Smolin loop transform is

$$\psi[\gamma] = \int d\mu(A) h[\gamma, A] \psi[A].$$

(46)

Given a functional $\psi[A]$ in the connection representation, the integration over all connections (modulo gauge) leaves only the loop dependence in $h[\gamma, A]$, and the result is a loop functional. The problem in general, and in particular for the $SL(2, C)$ connections of general relativity, is that the measure $d\mu(A)$ is not known. Without a definition of the measure, the loop transform is completely formal.

For Maxwell theory, however, we do have a measure explicitly available. We therefore can compute the transform from states in the Bargmann representation to the loop representation via

$$\psi[\gamma] = \int d\mu(\zeta, \bar{\zeta}) h[\gamma] \Psi[\zeta],$$

(47)

where the measure is defined in (36). An explicit calculation leads to the following result for the transform of a Bargmann state, and this result is of central importance for the relation between the two representations. Ashtekar and Rovelli show that

$$\psi[\gamma] = \Psi[F_j(\gamma, k)],$$

(48)

where $F_j(\gamma, k)$ is obtained from the Fourier transform of the so-called formfactor $F^a(\gamma, x)$,

$$F^a(\gamma, x) = \int ds \gamma^a(s) \delta^3(x, \gamma(s)), \quad (49)$$

$$= \frac{1}{2\hbar(2\pi)^{3/2}} \int d^3k e^{ik \cdot x} (F_1(\gamma, k)m^a(k) + F_2(\gamma, k)\bar{m}^a(k)).$$

(50)

In words, given a loop $\gamma$ we compute the number $\psi[\gamma]$ for a state in the loop representation that was obtained via the transform of a state $\Psi[\zeta]$ in the Bargmann representation by evaluating $\Psi[\zeta]$ for $\zeta = F_j(\gamma, k)$.

The name ‘form factor’ is appropriate since according to

$$\oint \gamma ds f_a = \int ds \gamma^a(s) f_a(\gamma(s)) = \int d^3x F^a(\gamma, x) f_a(x),$$

(51)

the factor $F^a(\gamma, x)$ allows us to separate the dependence on the loop from the dependence on the one-form. In the loop transform, the loop is combined with the connection (expressed in terms of the Bargmann coordinates) in precisely the form (51), and after the integration over one-forms in the loop transform, only the loop dependence remains. Therefore it is plausible that, as in (48), the transform of an arbitrary state $\Psi[\zeta]$ depends on the loop only through the form factor.
Since we can explicitly compute the transform of any state, we can define an operator \( \hat{O}_L \) in the loop representation by the transform of \( \hat{O}_B \Psi[\zeta] \) in the Bargmann representation. For some operators, however, we do not have to be able to perform an explicit calculation. In fact,

\[
\hat{h}_L[\alpha]\psi[\gamma] := \int d\mu h[\gamma](\hat{h}_B[\alpha]\Psi)[\zeta] = \int d\mu(\hat{h}_B^*[\alpha]h[\gamma])\Psi[\zeta] = \int d\mu h[\alpha]h[\gamma]\Psi[\zeta] = \psi[\alpha\#\gamma].
\]

All we have to know in this calculation is that \( \hat{O}_B \) is self-adjoint with respect to the measure, and we have to know an operator \( \hat{O}_L \) such that

\[
\hat{O}_L h[\alpha] = \hat{O}_B h[\alpha].
\]

The resulting representation is defined by

\[
\hat{h}[\alpha]\psi[\gamma] = \psi[\alpha\#\gamma],
\]

\[
\hat{E}[f]\psi[\gamma] = \left(ih \oint_\gamma ds^a f_a\right)\psi[\gamma].
\]

The \( \hat{h}[\alpha] \) operator acts by adding the loop \( \alpha \) to the argument of the loop state, and the \( \hat{E}[f] \) acts by multiplying the loop state with the loop integral of \( f_a \).

Several remarks on the construction of the loop representation via the Rovelli-Smolin loop transform are in order.

1. Ashtekar and Rovelli prove that the transform is faithful. Since no information is lost in the transition from the Bargmann representation to the loop representation, we have that the loop representation is equivalent to the Fock representation.

2. It is easy to check that we have correctly represented the classical Poisson algebra (42,43), in particular that \([\hat{h}[\alpha],\hat{E}[f]] = i\hbar\{h[\alpha],E[f]\}\). We can therefore define the loop representation without ever introducing the Bargmann representation.

3. The loop representation for quantum gravity (but also for other gauge groups) can be constructed by assuming that the loop operators are self-adjoint, since for the loop operators we do know transfer relations (57) [RoSm90]. Furthermore, one finds that the operators so obtained have the correct commutator algebra. Consequently, while the loop representation can be motivated by a formal transform, the definition of the loop representation is independent of that transform.

Since for Maxwell theory we know a faithful transform, we can answer the question what the translation of the physical interpretation of the theory in terms of photons is in the loop representation. We find with \( \psi[\gamma] = \langle \gamma | \psi \rangle \) that

\[
|0\rangle : \quad \psi_0[\gamma] = 1,
\]

\[
|k, \epsilon\rangle : \quad \psi_{k,\epsilon}[\gamma] = F_{\epsilon}(\gamma, k).
\]

Hence we have discovered a formulation of Maxwell theory in which the elementary quantum excitations of the electric field are based on loops, just as we have argued...
for Faraday’s picture of closed field lines. Is this result helpful for our intuition about physics? Even though the two representations are equivalent, and the loop representation for Maxwell theory is structurally appealing, the photon picture is much closer to how we think about experiments. However, for a non-abelian gauge theory and in the absence of a background metric (so that for example the split into positive and negative frequency parts is not possible), there does not exist a Fock representation. Nevertheless, it is quite possible that a loop representation still exists, and in this case, the loop picture may become a part of our intuition (see section 5.2).

4 Loop Representation of Canonical Quantum Gravity

4.1 Algebraic Quantization

Before we introduce the loop representation for canonical quantum gravity, it is appropriate to recall the program of algebraic quantization and point out the choices that lead to the loop representation. Our discussion of Maxwell theory already serves as an example for algebraic quantization, but the framework described below is designed with the kind of generality appropriate for general relativity.

The program of canonical quantization of constraint systems, in its algebraic form due to Dirac [Di65] and Ashtekar [As91], can be summarized as follows. Given is the classical phase space with the Poisson algebra of classical observables, which are functions from the phase space into the complexes, and constraints C.

→ Choose a set S of elementary variables which is complete in that it coordinatizes the phase space, and which is closed under the Poisson bracket.

  • These variables will become the elementary operators of the quantum theory. For a particle on the line we can choose \( S = \{1, q, p\} \). On the one hand, S has to be large enough such that the elements of S coordinatize the phase space. On the other hand, S has to be small enough so that we are able to consistently impose the non-commutative structure of the quantum theory. In particular, if S consists of all polynomials in the canonical variables, then there does not exist an irreducible representation of the quantum algebra (van Hove theorem, [Ho51]).

  • We do not have to choose the canonical variables, for instance in gauge theory we can choose the loop variables instead.

→ Elevate the elementary variables to operators of the quantum theory, i.e. form the free algebra \( \mathcal{A} \) of elements of S, \( f \mapsto \hat{f} \forall f \in S \), such that

\[
[\hat{f}, \hat{g}] = i\hbar \{f, g\} \quad \forall f, g \in S, \tag{62}
\]

\[
\hat{f}\hat{g} + \hat{g}\hat{f} = 2\hat{f}g \quad \forall f, g : fg \in S. \tag{63}
\]

  • The relation (62) is the reason why we call the procedure 'quantization'. The classical Poisson algebra of commuting observables determines a non-commutative algebra of quantum operators.

  • Anti-commutation relations like (63) arise if the set S is overcomplete, since if \( fg \in S \) we need a unique definition of \( fg \mapsto \hat{fg} \). More general relations are possible. The loop variables are overcomplete in that way, compare (41, 69).

For a particle on the circle, e.g. \( S = \{1, \cos \phi, \sin \phi, p\} \), \( \cos^2 \phi + \sin^2 \phi = 1 \).

→ Choose a linear representation of \( \mathcal{A} \) on a vector space \( V \).
• The vector space is called the space of states, \( V = \{ \psi \} \). Again, we have a choice, and for the loop representation we choose \( V = \{ \psi[\eta] \} \), and we choose a particular representation, e.g. as for Maxwell theory in (5.8, 59). Notice that in the infinite-dimensional case we do not have a Stone-von-Neumann theorem guaranteeing unitary equivalence of representations. Hence, a different choice of elementary variables and representation can in general lead to inequivalent quantum theories. In other words, different choices can describe different physical systems, e.g. free electrons and electrons in a superconductor. In the case of the loop representation there are indications that the loop transform may relate the different representations to each other.

→ Represent the constraints,

\[ C \mapsto \hat{C}. \] (64)

• While the preceding steps can be performed for a judicious choice of variables, typically it is in the representation of the constraints that serious problems arise. First of all, it may not be trivial to express the constraints in terms of the elementary variables, say in terms of loop variables. Second, we have to define a factor ordering and usually also a regularization.

→ Solve the constraints, i.e. find the space of solutions, \( V_{\text{sol}} \subset V \), on which

\[ \hat{C}\psi = 0. \] (65)

• The space of solutions is sometimes called the space of physical states, but we prefer to reserve the predicate 'physical' to states in the Hilbert space of solutions. Canonical quantization calls for the construction of a Hilbert space of states, and states in \( V_{\text{sol}} \) may not be normalizable.

→ Define the Hilbert space \( H_{\text{phys}} \) of physical states. To this end we have to find an inner product on a subset of \( V_{\text{sol}} \) such that suitable \( \ast \)-relations on \( \mathcal{A} \) are satisfied.

• For quantum gravity we do not have a criterion like Poincaré invariance that allows us to pick out an unique inner product. There is a conjecture that if we know a complete set of \( \ast \)-relations, for example, if we know which operators are supposed to be self-adjoint (i.e. observables), then the inner product is determined uniquely. Such a criterion suffices in most examples, Maxwell theory included. Its merit is background independence, and therefore it can be applied in quantum gravity. There the main problem is existence rather then uniqueness \[ \text{[Re93]} \].

• One often introduces an inner product at an earlier stage in the quantization program before the constraints have been solved (see Hajicek on pre-quantization in this volume). This may be important for the regularization of the constraint operators and the solution of the constraints. However, such an inner product is in general not related to the inner product of \( H_{\text{phys}} \).

→ Define observables of the quantum theory, define the measurement process, give an interpretation, make predictions.

• The previous step has completed the mathematical setup of the theory. Note that for quantum gravity, not even the mathematical part has been completed (the analog is true for other approaches to quantum gravity). This step summarizes all the really important, physical questions, and it is kept so short only because in the case of quantum gravity so little is known about it.
Now that a concrete proposal for a quantization program has been put on the table, let us quickly point out what such a program cannot be. As a matter of principle, there cannot exist a program into which we feed a classical theory and by turning a crank or letting a computer run we produce the corresponding, unique quantum theory. The reason is that quantum theory is the fundamental theory of which the classical world is a special limit. One and the same classical theory may be obtained as the limit of inequivalent quantum theories. Only if we specify information external to the classical theory, then we can hope to find a unique quantum theory.

A quantization program is similar to a computer program that requires additional input at various places in order to run at all. Relevant input for the loop representations is the choice of loop variables and the choice of loop states. As already mentioned, different choices may lead to different quantum theories, and only in experiments can we find out whether our theory describes the model that we had in mind.

In this sense the program of algebraic quantization of general relativity is more like a recipe that is justified only if it is successful. If the steps of quantization listed above can be taken, even excluding the ‘physical’ one, then this would constitute a major success. If algebraic quantization in this form fails, then we have not, of course, shown that quantization of general relativity is not possible.

Notice that we have resisted the temptation to number the steps. An obvious possibility is that they may have to be arranged in a different order, or more seriously that they cannot be separated as shown. For example, we may need an inner product to make the discussion of the constraints rigorous. Or we may have to solve the conceptual, physical problems like the meaning of time in quantum gravity (see Isham in this volume) before we can even formulate a quantization procedure.

After these remarks of caution about the nature of the proposed quantization procedure, let us list the steps that we will discuss for quantum gravity in the loop representation in the next sections. We choose loop variables as our elementary variables, and we can form the quantum algebra $A$ (section 4.2). We define a representation of the loop operators on the space of loop functionals such that the relations (62, 63) for the commutators and Poisson brackets are satisfied (section 4.2). We define the constraints in the loop variables (section 4.3) and express the constraint operators as differential operators on loop functionals (section 4.4). We explain how certain analytical knot invariants (section 4.5) can give rise to solutions to the constraints (section 4.6). Finally, in section 5 we give a critical appraisal of the status of the loop representation.

### 4.2 Loop Variables

The Rovelli-Smolin loop variables for general relativity are ([RoSm90, As91], compare section 2.3)

\[
T[\gamma] = \text{tr} U_\gamma, \tag{66}
\]

\[
T^a[\gamma](s) = \text{tr} U^a_{\gamma(s)} E^a(\gamma(s)), \tag{67}
\]

\[
T^{ab}[\gamma](s, t) = \text{tr} U^a_{\gamma(s)} U^b_{\gamma(t)} E^a(\gamma(s)) E^b(\gamma(t)), \tag{68}
\]

\[\vdots\]

where $U_\gamma$ is the parallel transport for the $SL(2, C)$ Ashtekar connection and $E^a$ its conjugate momentum. Each $T^n$ variable is characterized by $n$ insertions of the momenta into the trace of the holonomy (see figure 3).

The loop variables are overcomplete. Since for any $SL(2, C)$ matrices $A$ and $B,$
\[
\text{tr} AB = \text{tr} AB + \text{tr} AB^{-1},
\]
we have for example that
\[
T[\alpha]T[\beta] = T[\alpha\beta] + T[\alpha\beta^{-1}],
\]
if \(\alpha\) and \(\beta\) intersect so that \(\alpha\beta\) exists. This identity is the Mandelstam identity (also called spinor identity) for \(SL(2, C)\), compare \(\text{(41)}\) for \(U(1)\). We will incorporate the spinor identity into the quantum theory via the anti-commutation relations, but in principle one can remove the overcompleteness by a different choice of variables \([\text{Lo91}]\).

The Poisson algebra has the structure
\[
\{T^m, T^n\} \sim T^{m+n-1},
\]
where \(\sim\) means that the bracket can be expressed in terms of linear combinations and integrals of loop variables of the given order. The algebra of the \(T^0\) and \(T^1\) closes:
\[
\{T^\alpha[\alpha](s), T^\beta[\beta](t)\} = \int du \delta^d(u) \delta^d(\gamma(s), \eta(u)) \frac{1}{2\pi} \delta^d(\alpha, \beta(u)) \delta^d(\beta, \alpha(u))
\]
where \(\tilde{u}\) is the parameter of the intersection of the combined loop. Notice that the combination of loops occurs with a sign opposite to \(\text{(69)}\) and cannot be further simplified.

Initially, we define the space of states to be the space of functionals of multiloops. The definition of the loop operators \(\hat{T}^n\) can be motivated via the Rovelli-Smolin transform \(\text{(46)}\) as explained in section 3.2. We define
\[
\hat{T}^\gamma[\gamma](s)\psi[\eta] = \psi[\gamma \cup \eta],
\]
where \(\gamma\) operator acts by adding a loop into the multiloop argument of the loop state. The \(\hat{T}^\alpha[\alpha](s)\psi[\eta]\) operator gives a non-zero result only if the loop \(\eta\) in the argument of the state intersects \(\gamma\) at \(\gamma(s)\). In this case the result is a linear combination of the loop state evaluated for the two different reroutings of the loop through the intersection. In a graphical short hand,
For \( n \geq 1 \), \( \hat{T}^{a...b}[\gamma](s,...,t)\psi[\eta] \) is non-zero only if the loops \( \gamma \) and \( \eta \) intersect in all the distinguished points along \( \gamma \), and the result is a linear combination of all the possible reroutings of the two loops through the intersection. For example, in the action of \( \hat{T}^2 \) there appear (with correct signs) the four terms

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{loop_diagram.png}
\end{array}
\]

One can check that the relations \([ \ , \ ] = i\hbar\{ \ , \ \}, \ (62)\), are indeed satisfied for the operators \( \hat{T}^0 \) and \( \hat{T}^1 \)!

Furthermore, we have to require that the action of the \( \hat{T}^n \) is consistent with identities for the \( T_n \). This gives rise to several properties of the loop states \( \psi \). First of all, \( \psi[\eta] \) has to be invariant under reparametrizations of \( \eta \). For any loops \( \alpha \) and \( \beta \) and any path \( \mu \), we require

\[
\psi[\alpha\beta] = \psi[\beta\alpha]\quad\text{and}\quad\psi[\alpha\mu^{-1}] = \psi[\alpha].
\]

In addition there are anti-commutation relations \([33]\) for the non-linear identity \([39]\) where \( \alpha \) and \( \beta \) intersect. We have for all \( \eta \) that

\[
\begin{align*}
(\hat{T}[\alpha]\hat{T}[\beta] + \hat{T}[\beta]\hat{T}[\alpha])\psi[\eta] &= 2\psi[\alpha \cup \beta \cup \eta], \\
2\hat{T}[\alpha]\hat{T}[\beta]\psi[\eta] &= 2(\hat{T}[\alpha\beta] + \hat{T}[\alpha\beta^{-1}])\psi[\eta] = 2(\psi[\alpha\beta \cup \eta] + \psi[\alpha\beta^{-1} \cup \eta]).
\end{align*}
\]

Hence we have to impose on the space of states the spinor identity

\[
\psi[\alpha \cup \beta] = \psi[\alpha\beta] + \psi[\alpha\beta^{-1}],
\]

As an immediate consequence we obtain that \( \psi[\gamma_0 \cup \eta] = 2\psi[\eta] \) for \( \gamma_0 \) the zero loop, and \( \psi[\eta^{-1}] = \psi[\eta] \).

There also are identities among \( T^0 \) and \( T^1 \) variables (and similarly for higher orders), but these imply that the space of states contains only the zero state, \( \psi \equiv 0 \) \([Br93a]\).

However, in a natural regularization of the \( \hat{T}^0, \hat{T}^1 \) algebra (using strips, see below) such identities are absent, and the spinor identity \([74]\) is the only consequence of the anticommutation relations.

### 4.3 Constraints from Loop Variables

How can we define the constraints, which are given to us in terms of the Ashtekar variables, in terms of loop variables? The basic observation is that local variables in terms of \( A \) and \( E \) at a point \( x \) can be obtained from the non-local loop variables in the limit that a loop is shrunk to a point. Considering that local and non-local field theories are fundamentally different, the limit of shrinking loops is expected to be non-trivial.

We choose a coordinate system in a neighborhood of \( x \), and we consider a family of loops, \( \gamma_{xab}^\delta \), of size \( \delta \) at the point \( x \) in the \( a-b \)-coordinate plane (figure 6) such that

\[
\begin{align*}
\gamma_{xab}^\delta(s) &= x + \delta \gamma_{ab}(s), \\
\gamma_{ab}(0) = \gamma_{ab}(1) &= 0, \\
\sigma^{cd}(\gamma_{ab}) &= \delta_a^c \delta_b^d,
\end{align*}
\]
where $\gamma_{ab}$ is a planar loop at the origin with unit area and area element $\sigma^{cd}$, i.e. $\gamma_{ab}^\delta$ has area $\delta^2$. In the limit $\delta \to 0$, $\gamma_{ab}^\delta$ reduces to the zero loop, $\gamma_{ab}^0(x) = x \forall s$.

The parallel transport around infinitesimal loops $\gamma_{ab}^\delta$ gives rise to the curvature $F$ of $A$ at $x$.

$$U_{\gamma_{ab}^\delta} = 1 + \delta^2 F_{ab}(x) + O(\delta^3),$$

(83)

where $F_{ab} = \partial_a A_b^i - \partial_b A_a^i + \epsilon^{ijk} A_a^j A_b^k$. This is just the equation we need for the transition from the loop variables $T^1$ and $T^2$ to the constraints $D$ and $H$.

Indeed, in the Ashtekar variables the vector constraint $C$ and the Hamiltonian constraint $H$ are ([As91] or Giulini in this volume)

$$C_a(x) = \text{tr} E^b F_{ab}(x),$$

(84)

$$H(x) = \text{tr} E^a E^b F_{ab}(x).$$

(85)

The vector constraint generates diffeomorphisms up to gauge. (As an aside, compare the canonical formulation in the metric variables, e.g. Beg in this volume, where $H$ is more complicated, in particular due to the presence of a potential term.)

In order to make the transition from $T^a = \text{tr} E^a U$ to $C_a = \text{tr} E^b F_{ab}$, we only have to shrink the loop. To be precise,

$$C_a(x) = \lim_{\delta \to 0} \frac{1}{\delta^2} T^b [\gamma_{ab}^\delta](0),$$

(86)

$$H(x) = \lim_{\delta \to 0} \frac{1}{\delta^2} T^b [\gamma_{ab}^\delta](\delta^2, 1),$$

(87)

where the summation over indices is for $a \neq b$.

Quantization in the loop representation elevates the loop variables to the loop operators, and since we know how to express the constraints in the loop variables, we obtain immediately the constraints in the loop representation via

$$C_a(T^1) \rightarrow \hat{C}_a = C_a(\hat{T}^1),$$

(88)

$$H(T^2) \rightarrow \hat{H} = H(\hat{T}^2).$$

(89)

The question is whether the limit of shrinking loops can actually be computed for the loop operators, and with what result.

The natural differential operator in the context of shrinking loops turns out to be the area derivative $\Delta_{ab}(s)$ of loop functionals. A loop functional is called area differentiable if the following limit exists independently of the choice of small loops $\gamma_{ab}^\delta$ and transforms like a two-form under coordinate transformations:

$$\Delta_{ab}(s)\psi[\eta] = \lim_{\delta \to 0} \frac{\psi[\eta_{s \circ \gamma_{ab}^\delta}] - \psi[\eta]}{\delta^2},$$

(90)
where the combination of the loops \( \eta \) and \( \gamma^\delta_{xab} \) occurs at \( x \). Since we are considering general relativity, one has to show that a background independent definition of the area derivative exists. This can be shown for small parallelograms built from the integral curves of two commuting vector fields \([\text{BrPu93}]\). The beginning of a rigorous calculus on the space of loop functionals can be found in the work of Tavares \([\text{Ta93}]\).

The area derivative satisfies

\[
\Delta_{ab}(s) = \Delta_{[ab]}(s),
\]

\[
\frac{\delta}{\delta \eta^a(s)} = i \eta^b(s) \Delta_{ab}(s),
\]

\[
\Delta_{ab}(s) \text{tr} U_{\gamma} = \text{tr} U_{\gamma} F_{ab}(\gamma(s)).
\]

The area derivative is antisymmetric as expected from its definition in terms of an area element \((91)\). The ordinary functional derivative is a special, one-directional case of the area derivative \((92)\). And the area derivative inserts \( F \) into the Wilson loop as expected for the limit of shrinking loops \((93)\).

### 4.4 Constraint Operators

In general terms, the area derivative is defined by a deformation of the loop argument of a loop functional that consists of appending a small loop. And the loop operators insert loops into the argument of loop functionals in the specific way defined in \((74,75)\). The perhaps surprising result is that the terms appearing in the definition of the constraints can be combined into area derivatives. In this calculation we have to make use of the spinor identity for loop states, \((79)\).

For the smeared diffeomorphism constraint \( \hat{D}(v) = \int d^3x v^a(x) C_a(x) \) we find that

\[
\hat{D}(v) \psi[\eta] = \lim_{\delta \to 0} \frac{1}{\delta^2} \int d^3x v^a(x) T^b_{[\eta^\delta_{xab}](0)}
\]

\[
= \int ds v^a(\eta(s)) \frac{\delta}{\delta \eta^a(s)} \psi[\eta],
\]

which is the natural generator of diffeomorphisms along the vector field \( v^a \), and where we have used \((12)\).

The unregulated result for the smeared Hamiltonian constraint \( \hat{H}(N) = \int d^3N(x) H(x) \) is

\[
\hat{H}_{\text{unreg}}(N) \psi[\eta] = \int ds \int dt \delta^3(\eta(s), \eta(t)) N(\eta(s)) \frac{\delta}{\delta \eta^a(s)} \psi[\eta^a(t) \eta^b(\eta(s)) \Delta_{ab}(s, \eta^e(t)) \psi[\eta^e(t) \cup \eta^e(t)]]). \quad (96)
\]

The second argument of the area derivative indicates on which of the elements of the multiloop the area derivative acts. The result is non-zero only if \( \eta(s) = \eta(t) \). This occurs for \( s = t \), but also for \( s \neq t \) if \( \eta \) intersects itself. If \( s = t \), then we expect a zero result since then we have the antisymmetrized product of two identical tangent vectors because the area derivative is antisymmetric in its indices, \((11)\). If there is an intersection, then the action of \( \hat{H} \) is to split the loop \( \eta \) at the intersection into a multiloop (figure \([\text{fig}]\)), and to act with the area derivative on one of the component loops.

Notice that the Hamiltonian constraint operator has to be regulated. Since the three-dimensional delta distribution \( \delta^3(\eta(s), \eta(t)) \) depends only on two parameters, there is a \( \delta^1(0) \) divergence. There are different ways known to regulate the constraints, all of which share the same problem that some sort of background dependence is introduced.

Let us comment on the point-splitting regularization most often used in practice \([\text{JaSm88}, \text{Br93c}]\). We pick a regulator \( f_\epsilon(x, y) \) such that \( \lim_{\epsilon \to 0} f_\epsilon(x, y) = \delta^3(x, y) \), and
replace the delta distribution by the regulator. The resulting operator, $\hat{H}_\epsilon$, is finite for $\epsilon > 0$. We perform all calculations for $\epsilon > 0$ to leading order in $\epsilon$, which is at order $1/\epsilon$ and corresponds to $\delta^4(0)$. We define the regularized Hamiltonian constraint operator by a multiplicative renormalization as $\hat{H} = \lim_{\epsilon \to 0} \epsilon \hat{H}_\epsilon$.

Background dependence enters since in $f_\epsilon(x, y)$ we measure the separation of $x$ and $y$ by a background metric, e.g. a common choice is $f_\epsilon(x, y) = \frac{3}{4\sqrt{\pi}} \Theta(\epsilon - |x - y|)$. A background dependence breaks diffeomorphism invariance and is therefore unacceptable in quantum gravity. However, the leading order term in the Hamiltonian constraint at intersections is background independent. The $s = t$ terms vanish because of antisymmetry at order $1/\epsilon^2$, but there is a background dependent contribution at $1/\epsilon$, which however vanishes for diffeomorphism invariant states satisfying a certain regularity conditions. Therefore, it is possible to discuss solutions to the constraints in a background independent way, but there still remain some problems regarding the regularization of the constraint algebra.

4.5 Analytic Knot Invariants

How can we find the simultaneous space of solutions to both the diffeomorphism constraint and the Hamiltonian constraint of quantum gravity? The problem is that very little is known about the Hamiltonian constraint in the loop representation as a differential operator. As a first step we therefore consider a particular class of knot invariants and examine the action of the Hamiltonian constraint on such knot invariants [BrGaPu92a]. At the time of writing, there exists only one class of knot invariants for which we can compute the area derivative, namely the analytic knot invariants that appear as coefficients of knot polynomials in perturbative Chern-Simons theory.

The action of Chern-Simons theory is

$$S_{CS}[A] = \int_{\Sigma} d^3 x \, \epsilon^{abc} \text{tr}(A_a \partial_b A_c + \frac{2}{3} A_a A_b A_c),$$

where we choose for simplicity a $SU(2)$-valued one-form $A_a(x)$. At this point there is no relation between the Chern-Simons connection and the Ashtekar connection, the Chern-Simons connection serves only as a calculational tool. The vacuum expectation value of the Wilson loop in Chern-Simons field theory is

$$\langle h[\gamma, A] \rangle = \int D A \, h[\gamma, A] \exp \left( \frac{ik}{4\pi} S_{CS}[A] \right),$$

where $k \in \mathbb{Z}$ is the coupling constant.

We have two ways to evaluate $\langle h[\gamma, A] \rangle$. Witten showed that

$$\frac{\langle h[\gamma, A] \rangle}{\langle h[\gamma_0, A] \rangle} = e^{-\omega(\gamma) J_q(\gamma)}$$

Figure 7: A loop with one self-intersection.
where \( \gamma_0 \) denotes the unknot, \( q = \exp \frac{2\pi i}{k+2} \), \( c = c(k) \), \( w(\gamma) \) is the writhe of \( \gamma \), and \( J_q(\gamma) \) is the Jones polynomial. The right hand side is also known as the Kauffman bracket [Ka91]. The Jones polynomial is defined by the skein relations (compare (12))

\[
q J_q(c_+) - q^{-1} J_q(c_-) = (q^\frac{1}{k} - q^{-\frac{1}{k}}) J_q(c_{\infty}).
\]  

(100)

The Jones polynomial is a knot invariant, or in knot diagram language, the Jones polynomial is invariant under ambient isotopy. The writhe \( w(\gamma) \) is defined as the sum over the crossings in a knot diagram counting +1 for \( c_+ \) and -1 for \( c_- \). The writhe is not a knot invariant, but only a regular isotopy invariant (as is the Kauffman bracket).

The reason is that a projection of a knot from three into two dimensions may introduce arbitrary numbers of crossings. For example, depending on the projection we can obtain from an unknotted line segment a line without or with twist, which are equivalent under ambient isotopy, but inequivalent under regular isotopy. The corresponding crossing change formulas for \( J_q(\gamma) \) and \( \langle h[\gamma, A] \rangle \) are shown in figure 8. The projection dependence of \( \langle h[\gamma, A] \rangle \) is known as framing dependence. The framing of a loop can be defined in three dimensions by replacing the loop by a strip, which changes under twists (figure 8).

We can also obtain a perturbation expansion for \( \langle h[\gamma, A] \rangle \) by inserting for \( h[\gamma, A] \) the expansion [GuMaMi90]. The result is

\[
\frac{\langle h[\gamma, A] \rangle}{\langle h[\gamma_0, A] \rangle} = c_0(\gamma) + c_1(\gamma) \frac{1}{k} + c_2(\gamma) \frac{1}{k^2} + \ldots.
\]  

(101)

where the coefficients \( c_i(\gamma) \) are known to be regular isotopy invariants related to the coefficients of the Jones polynomial by Witten’s result. The point is that the \( c_i(\gamma) \) are expressed as multiple integrals along the loop like the Gauss linking number \( gl(\alpha, \beta) \), (6). We have that up to constant factors

\[
c_0(\gamma) = 1, \quad c_1(\gamma) = gl(\gamma, \gamma), \quad c_2(\gamma) = (c_1(\gamma))^2 + \rho(\gamma).
\]  

(102)

(103)

(104)

\( gl(\gamma, \gamma) \) is called the Gauss self linking number. Despite appearance, \( gl(\alpha, \beta) \) in (6) is finite for \( \alpha = \beta \), but it depends on the coordinates, which is another sign for framing dependence. If we assign to \( \gamma \) a framed loop \( \gamma^f \), then we can define the framed self linking number for the strip formed by \( \gamma \) and \( \gamma^f \) by \( gsl^f(\gamma) = gl(\gamma, \gamma^f) \). The second coefficient, however, contains a framing independent term \( \rho(\gamma) \), which is a true knot invariant related to the second coefficient of the Alexander-Conway polynomial \( a_2(\gamma) \), \( \rho \sim a_2 + \frac{1}{12} \). \( \rho(\gamma) \) is the sum of a three-fold and a four-fold integral along \( \gamma \), but its precise form is not important here, since we are not going to perform an explicit calculation with it. Each coefficient \( c_i(\gamma) \) can be shown to contain a framing independent piece, and these constitute the class of knot invariants on which we want to study the action of the Hamiltonian constraint.
4.6 Knot Invariants as Solutions to the Constraints

The history of solutions to the constraints in the loop representation begins with the discovery in [JaSm88] that in the connection representation any $\psi[A] = h[\gamma, A]$ for some loop $\gamma$ without intersections is a solution to the Wheeler-DeWitt equation. (Such states are clearly not diffeomorphism invariant.) The simple reason is as in (96) that two linearly independent tangent vectors are required at one point for a non-zero result due to the presence of an antisymmetrization.

In fact, the operator corresponding to the determinant of the three-metric, $\sqrt{-g}$, gives zero if there are not three linearly independent tangents at one point, which for smooth loops can only happen at a triple intersection (figure 9). Since the metric itself is not an observable of the theory, one might think that one does not have to worry about its degeneracy on the space of solutions. However, the determinant of the metric typically appears in matter couplings, and it should therefore not be zero. Also, the Hamiltonian for a non-zero cosmological constant $\Lambda$ is

$$\hat{H}_\Lambda = \hat{H} + \Lambda \sqrt{-g}.$$  \hspace{1cm} (105)

States without intersections are therefore solutions to the Wheeler-DeWitt equation for arbitrary cosmological constant. We consider this as an argument that the sector of states for non-intersecting loops is degenerate.

Consequently, solutions to the Hamiltonian constraint for intersecting loops were constructed in the connection representation [JaSm88, Hu89, BrPu91]. However, in [BrPu91] it is shown that all such solutions are necessarily annihilated by $\sqrt{-g}$. Since in the connection representation we do not know a general strategy to solve the diffeomorphism constraint anyway, it is natural to look for non-degenerate solutions to the Hamiltonian constraint in the loop representation. One reason why non-degenerate solutions can be found in the loop representation is that the loop representation is based on the opposite factor ordering of the $A$ and $\hat{E}$ then the one in which $h[\gamma, A]$ leads to solutions.

Let us summarize the situation in the loop representation for the coefficients $c_i(\gamma)$ [BrGaPu92a]. $c_0$ is a non-degenerate solution, but not an interesting one. Let us consider loop states with support on loops $\gamma = \gamma_1 \gamma_2 \gamma_3$, $\gamma_i$ without self-intersections (figure 9), which have one triple intersection with three linearly independent tangent vectors (the generic case in three dimensions). We find that
\begin{array}{|c|c|c|}
\hline
\psi[\gamma] & \hat{H}\psi[\gamma] & \sqrt{-g}\psi[\gamma] \\
\hline
\gamma_1(\gamma) & 0 & 0 \\
\gamma_1(\gamma_1 \cup \gamma_2 \cup \gamma_3) & 0 & \text{nonzero} \\
\rho(\gamma) & 0 & \text{nonzero!} \\
\hline
\end{array}

\(c_1\) (which is not a knot invariant), is non-degenerate but not annihilated by \(\hat{H}\). For the more symmetric loop as shown, \(\gamma_1\) is a degenerate solution. But a non-trivial and non-degenerate solution to both constraints is the second coefficient of the Alexander-Conway polynomial. In \cite{BaGaGrPu93}, the same is shown for a part of \(c_3\).

So, using a particular factor ordering and regularization, we can find interesting solutions to both constraints among the coefficients \(c_i(\gamma)\). The mere fact that solutions can be found is a success of the loop representation, since as already mentioned, in the traditional variables not a single solution to the full Wheeler-DeWitt equation had been known.

However, it is still completely unclear what kind of structure the \textit{complete} space of solutions might have. The following argument provides at least some insight into what this structure could be. In the connection representation there is exactly one solution known to all constraints \cite{Ko90}, and that in the factor ordering that corresponds to the loop representation and for non-vanishing cosmological constant:

\[
\psi_\Lambda[A] = \exp\left(-\frac{6}{\Lambda} S_{CS}[A]\right),
\]

where \(A\) is now the Ashtekar connection. This state is gauge invariant for appropriate values of \(\Lambda\) and diffeomorphism invariant since \(S_{CS}[A]\) is 'topological'. Since

\[
\hat{H}_\Lambda = \epsilon^{ijk} \frac{\delta}{\delta A^i_a} \frac{\delta}{\delta A^j_b} F_{k}^b + \frac{\Lambda}{6} \epsilon^{ijk} \epsilon_{abc} \frac{\delta}{\delta A^i_a} \frac{\delta}{\delta A^j_b} \frac{\delta}{\delta A^k_c},
\]

\[
\frac{\delta}{\delta A^k_c} \exp\left(-\frac{6}{\Lambda} S_{CS}[A]\right) = \frac{3}{\Lambda} \epsilon^{cde} F_{de}^k \exp\left(-\frac{6}{\Lambda} S_{CS}[A]\right),
\]

we immediately have by differentiating only once that

\[
\hat{H}_\Lambda \psi_\Lambda[A] = 0.
\]

In other words, property (108) of the Chern-Simons action makes it possible to choose the coefficient in the exponent of \(\psi_\Lambda[A]\) such that the contributions to \(\hat{H}_\Lambda\) from \(\hat{H}\) and the cosmological constant term cancel.

The idea in \cite{BrGaPu92b} is to consider the loop transform \([\mathcal{H}]\) of \(\psi_\Lambda[A]\),

\[
\psi_\Lambda[\gamma] = \int d\mu[A] h[\gamma,A] \psi_\Lambda[A].
\]

If the loop transform exists, then \(\psi_\Lambda[\gamma]\) is a solution to all the constraints in the loop representation by construction. In general, we cannot compute the transform since we do not know the measure. Let us assume for the moment, that for the transform of \(\psi_\Lambda[A]\) we can use the measure of Chern-Simons theory. Then

\[
\psi_\Lambda[\gamma] = \langle h[\gamma,A] \rangle_{CS} \sim e^{-w(\gamma)} J_q(\gamma),
\]

for \(k = -24/(\pi i \Lambda)\), is a solution to all constraints.

There are several obvious problems with this construction. In gravity, the internal group is \(SL(2,\mathbb{C})\) and not \(SU(2)\) (in terms of \(A^i_a\), in the former case \(A^i_a\) is complex, in the latter it is real). Furthermore, \(\Lambda\) is complex (and takes discrete values). Another
problem is that we want to allow loops with intersections, but we can in fact construct an extension of the Jones polynomial to intersecting loops \[BrGaPu92b\].

While there is no proof that (111) does or does not make sense, it hints at a very interesting relation between quantum gravity and topological field theory. Furthermore, from the single solution (111) for \(\Lambda \neq 0\) we can derive a whole tower of solutions for \(\Lambda = 0\) by the following simple argument \[BrGaPu93\]. Consider

\[
\hat{H}_\Lambda \psi_\Lambda[\gamma] = (\hat{H} + \Lambda \sqrt{-g})(c_0 + c_1 \Lambda + c_2 \Lambda^2 + \ldots) = 0,
\]

where we have absorbed the factor between \(\Lambda\) and \(k\) in the \(c_i\). Since this equation holds for all \(\Lambda \neq 0\), we conclude order for order that

\[
\hat{H} c_0 = 0, \quad (113)
\]

\[
\hat{H} c_1 + \sqrt{-g} c_0 = 0, \quad (114)
\]

\[
\hat{H} c_2 + \sqrt{-g} c_1 = 0, \quad (115)
\]

The action of \(\sqrt{-g}\) is simpler to compute than that of \(\hat{H}\). Since \(c_2 = c_1^2 + a_2 + \frac{1}{12}\) and \(\hat{H} c_2 + \sqrt{-g} c_1 = 0\) we arrive at a simple proof of \(\hat{H} a_2 = 0\) for loops as above.

From the structure of the series, we can guess that at each order a part of the coefficients \(c_i\) has to be annihilated by \(\hat{H}\). Recently it has been argued that indeed \(\hat{H} e^{-w(\gamma)} = 0\) \[GaPu93\], and therefore \(\hat{H} J_{q(\Lambda)} = 0\). This explains why each coefficient contains a diffeomorphism invariant part that is a solution to the Hamiltonian constraint (recall that \(w(\gamma)\) is not diffeomorphism invariant).

5 Discussion

5.1 Simple loop representations

The main motivation for our discussion of the loop representation comes from quantum gravity. Before we discuss the status of the loop representation for quantum gravity in the next section, let us at least briefly comment on the loop representation of simpler models.

1. Maxwell theory.
   As we have seen in section 3, the loop representation is equivalent to the Fock representation and as complete. Here we can gain some intuition about the physical meaning of loops.

2. Yang-Mills theory (see Loll in this volume).
   As mentioned in section 2.3, loop variables play a natural role in non-abelian gauge theories both on the lattice and in the continuum. The mathematical problems that effectively stopped the continuum approach in the beginning of the 1980’s are now being adressed by results stimulated by the loop representation of quantum gravity, see below. On the lattice, we have a complete formulation, and it is well suited for numerical Hamiltonian lattice gauge theory. For example, one can compute the glue ball state and its mass in the high temperature regime of pure 2+1 \(SU(2)\) lattice gauge theory, i.e. numerically solve the eigenvalue problem for the Hamiltonian (here not a constraint), \(\hat{H} \psi[\gamma] = E\psi[\gamma]\) \[Br91\] (see also \[GaLeTr89\]). We can introduce matter into the loop representation by including paths that carry fermions at their ends. Again, on the lattice we can perform numerical computation, see \[GaSe93\] for \(SU(2)\) with fermions in 3+1 dimensions.
3. Linearized gravity.
Similar to the situation in Maxwell theory, the loop representation is able to reproduce the Fock representation of gravitons [AsRoSm91], and hence the elementary excitations of the gravitational fields are based on loops. Furthermore, one can actually show that the loop representation of quantum gravity, although incomplete, contains the singular limit of gravitons [IwRo93].

4. 2+1 gravity.
The loop representation can be constructed explicitly [AsHuRoSaSm89]. Many of the problems regarding regularization are absent, notice for example that in (96) there appears only $\delta^2(\gamma(s), \gamma(t))$. Together, the diffeomorphism and the Hamiltonian constraint generate homotopy transformations (that can take intersections apart and transform the unknot to the trefoil, figure 2) [BrVa93]. An important point is that the loop transform in its naive form is degenerate [Ma93], but there exists a non-degenerate generalization [AsLo93].

5. Classical limit in the loop representation — weaves.
On the kinematic level, i.e. ignoring the Hamiltonian constraint, so-called weave states allow us to define a classical metric [AsRoSm92]. A weave is a multiloop obtained by sprinkling loops randomly into the manifold. Under appropriate conditions, weave states are the eigenstates of a smeared, diffeomorphism invariant metric operator, and the classical metric arises as its eigenvalue on scales large in comparison to the density of the sprinkling. Remarkably, in this context we can derive that physics becomes discrete at the Planck scale. Weaves can accommodate, for example, the classical black hole solution [Ze93].

5.2 Status of the Loop Representations for Quantum Gravity
We have seen in some detail how various steps of the program of algebraic quantization of quantum gravity can be performed in the loop representation. Let us collect the main negative and positive points about this approach, the negative ones first:

- The program is incomplete. As long as the program is incomplete, it is totally unclear whether any parts of it will be part of the 'final' theory of quantum gravity. It does not help that all other approaches to quantum gravity are incomplete, too (see for example Isham in this volume). Whenever the claim of progress in full quantum gravity is raised, so far it is only valid with respect to a particular program. More on the positive side, neither has it been shown that the loop representation must necessarily fail.

- The main reason why the loop representation is incomplete is that we do not have an inner product, in particular we do not know how to obtain a complete set of observables that could lead us to an inner product. The absence of an inner product implies, for example, that all we have are examples for $\psi[\gamma] \in V_{sol}$, not for $\psi[\gamma] \in H_{phys}$.

- The construction of the loop representation ignores the reality conditions. The Ashtekar variables are complex, and certain reality conditions are imposed to obtain real general relativity. We do not know how to impose the reality conditions in the loop representation, although there may be an analog to holomorphicity of $\psi[A]$ in the connection representation. Also, the issue of reality becomes intertwined with that of self-adjointness and the inner product. Therefore, at the level of our discussion we deal with a quantum theory of complex general relativity.
– Regularization is not complete. We know how to regularize the $\hat{T}^0-\hat{T}^1$ algebra (using strips for $\hat{T}^1$, e.g. [Ro91]), and the constraints are reasonably well understood in terms of a point-splitting regularization. What is missing is a regularization of the full $\hat{T}^n$ algebra, and of the constraint algebra. Without progress on regularization we cannot hope to decide, for example, whether there are anomalies.

– The loop representation has not lead to a breakthrough regarding the interpretation of quantum gravity, say of observables or the issue of time.

There are, however, also several positive aspects of the loop representation approach:

+ The loop representation is natural for the treatment of the constraints. As argued in depth, the loop representation is well-adapted not only to gauge and diffeomorphism invariance, but also the Wheeler-DeWitt equation seems to become tractable in the loop representation.

+ There exist loop representations for many models that are simpler than full quantum gravity in 3+1 dimensions, that can be called complete to a varying degree (section 5.1). This fact gives us confidence that at least the general idea behind the loop representation is sound. Of course, there are many good ideas that cease to be valid in the case of quantum gravity. Therefore the previous point is important.

+ There has been progress on several mathematical aspects related to loops:

1. There are two, not in any obvious way related ways to extend the space of loop states to distributions on the space of loops. [AsIs92] and [BaGaGr93]. As usual in quantum field theory, we expect such states rather than just functions on configuration space to be relevant.

2. A differential calculus for extended loop variables has been developed [Ta93] that allows one to give rigorous meaning to heuristic constructions of operators like the area derivative.

3. A diffeomorphism invariant measure has been constructed on a completion of the space of connections modulo gauge [AsLe93], see also [Ba93]. This measure may lead to a rigorous definition of the loop representation via the transform.

4. There are new knot invariants that are more powerful then the knot polynomials, the Vassiliev invariants [Va90]. These can be characterized as the coefficients of knot polynomials (e.g. $a_2$ corresponds directly to a Vassiliev invariant), which they include as a special case [Bi93]. For the construction of the Vassiliev invariants, intersecting loops are essential. Before the Vassiliev invariants became known, knots with intersections were not studied in knot theory, but given the importance of Vassiliev invariance in knot theory, the application of loops with intersections in quantum gravity has gained additional justification.

+ There are new ideas about quantum gravity physics that have been introduced, or concretized, by the loop representation. Although these ideas refer mostly to limiting situations of full quantum gravity, this is where our intuition originates, and if as often assumed new conceptual ideas are needed for the quantization of gravity, it is good to know that the loop representation produces such ideas. As mentioned in the context of weaves [AsRoSm92], in the classical approximation to full quantum gravity in the loop representation, the discrete structure of space time...
at the Planck scale can be derived. Furthermore, loops allow one to construct diffeomorphism invariant observables that measure the area of a surface by counting the number of intersections of a loop with this surface (which is diffeomorphism invariant). A prediction of such a framework is that area is quantized.

5.3 Conclusion

As it is often the case with a theory, it is a matter of taste and interests whether one feels that the pro outweighs the contra — especially with a theory as remote from the 'real world' as quantum gravity. Let us draw our conclusion.

We believe that the loop representation is an interesting proposal about how to solve some of the long-standing problems of canonical quantum gravity, in particular solving the constraint equations. While far from complete, the loop representation offers promise for the future.

Here we have focused on three issues. For Maxwell theory we have argued that in the loop representation we can replace the physical picture of photons by that of elementary excitations based on loops, both being equivalent. For full quantum gravity we have shown how to find states that solve all the constraints in the loop representation, something not possible in the traditional approach in terms of metric variables.

The one aspect of the loop representation that arguably is the most important one is the following. The loop representation is not a strange idea unrelated to physics found in an appendix to an obscure theory called canonical quantum gravity. Rather the loop representation is a fine example for the surprisingly fruitful interplay between three, initially unrelated theories: knot theory, gauge theory, and quantum gravity.

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References

No attempt was made to cover all the literature related to the loop representation, but let us point out to the non-specialist some of the references that can serve as entry points to the literature. For a guide to the issues of quantum gravity in general see Isham in this volume. For a review of the loop representation in Yang-Mills theory see Loll in this volume. For a list of references on canonical quantum gravity in the Ashtekar variables in general and the loop representation in particular see [Br93b]. Taken together, the latter two sources give a rather complete overview of all the work related to the loop representation.

For a complete and authoritative review of quantum gravity in the Ashtekar variables see the book by Ashtekar [As91]. There are also review papers by Rovelli [Ro91] and Smolin [Sm91] that cover the loop representation. For a self-contained introduction to the application of knot theory to quantum gravity along the lines of section 4 see Pullin [Pu93]. For a recent account of knot theory in physics see Kauffman [Ka91].
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