Background-independent quantization and the uncertainty principle

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Abstract
It is shown that polymer quantization leads to a modified uncertainty principle similar to that motivated by string theory and non-commutative geometry. When applied to quantum field theory on general background spacetimes, corrections to the uncertainty principle acquire a metric dependence. For Friedmann–Robertson–Walker cosmology this translates to a scale factor dependence which gives a large effect in the early Universe.

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1. Introduction
There is a belief that if quantum gravity effects are taken into account, then the uncertainty principle of quantum mechanics is modified. This is based on arguments that make use of the existence of a fundamental length scale in quantum gravity, and the possibility of black hole formation at sufficiently large densities. Candidate theories of quantum gravity such as string theory and non-commutative spacetime models provide a more concrete realization of this possibility.

An early suggestion that gravity has the possibility of affecting the uncertainty principle was given by Mead [1], who re-considered the Heisenberg microscope experiment with gravitational interaction. The basic argument does not require quantum gravity and can be made using Newtonian gravity along with special relativity: consider a probe photon of frequency $\omega$ and a particle of mass $m$ that is to be observed. The acceleration of the particle due to gravity when the probe is within a distance $\Delta x$ is $a = G\omega / (\Delta x)^2$, where $G$ is Newton’s constant. As a result its gravitationally induced displacement in a time interval $\Delta t$ is

$$\Delta x \gtrsim a(\Delta t)^2 \sim G\omega \sim G\Delta p,$$

where the last step follows because the particle’s momentum is uncertain by an amount given by the probe momentum. Thus including gravity suggests an additional contribution to
position uncertainty that is proportional to the momentum uncertainty. A more refined general relativistic view of this gedanken experiment gives a similar result.

Beyond gedanken experiments, modified uncertainty relations can be derived from candidate theories of quantum gravity that come with a fundamental length scale. For example, in string theory a basic observation is that the mass of a string is proportional to its length. Strings therefore expand in size when probed at sufficiently high energies. This suggests an additional uncertainty in the position of a string which is proportional to the uncertainty in its momentum $\Delta p$. This indicates a modification to the uncertainty relation of type [2, 3]

$$\Delta x \geq \frac{1}{2\Delta p} + kl_p^2 \Delta p,$$

where $l_p$ is the Planck length and $k$ is a dimensionless constant. The second term in expression (2) is of the same form as in equation (1). The generalized uncertainty relation (2) can also be seen to arise from a theory-independent consideration of particle emission from a black hole [4]; the argument uses the observation that the uncertainty in the horizon radius is proportional to the emitted particle’s mass.

A further observation follows from combining the relation $\Delta E \sim \Delta x$ for a string with the usual time–energy uncertainty relation. This leads to an effective non-commutativity of spacetime as first observed by Yoneya [2, 5]:

$$\Delta x \Delta t \geq l_p^2.$$  

The realization that string theory suggests a non-commutativity of spacetime has been a motivation for postulating non-commutative spacetime theories for the purpose of generalizing quantum field theory and for developing models for quantum gravity. There are many approaches for developing this idea, but perhaps the main one involves a dynamics-independent replacement of a classical manifold by a non-commutative one [6–8]. In this approach the commutative algebra of smooth complex-valued functions with specified fall-off conditions is replaced by a non-commutative algebra, and much work has gone into what algebra this should be. A related approach is to use modified Heisenberg [9] and $\kappa$-deformed Poincaré algebras [10] as a basis for constructing quantum theory. This approach has been applied to obtain corrections to atomic spectra [11].

In this work we consider an alternative approach based on polymer quantization[12]. In this mathematically well-defined quantization, the Hilbert space which is used for representing physical variables as operators is significantly different from the usual $L_2(\mathbb{R})$ (the space of square-integrable functions). Unlike the approaches mentioned earlier, there is no deformation of the algebra of observables; rather the Hilbert space is such that the momentum operator is realized only indirectly as a function of translation operators. A direct consequence of this feature is that the quantization comes with a fixed mass or length scale. This scale does not arise from gravitational interaction as in the case of gedanken experiment. However, its presence is crucial in modifying the uncertainty principle.

It has been noted in [12] that the uncertainty principle receives polymer corrections. Our formulation, however, provides a direct comparison with the general form (2), and an extension to the field theory where metric dependence of the correction terms becomes manifest. This property may have interesting applications in the cosmological settings where polymer corrections become significant in the early Universe. We note further that when combined with the time–energy uncertainty relation, polymer quantization provides a hint of space non-commutativity.
2. Polymer quantization and the uncertainty relation

Polymer quantization is a theory-independent procedure that requires an additional mass scale besides Planck’s constant $\hbar$. As such it may be viewed as a generalization of quantum mechanics where the ‘Schrödinger’ or ‘long wavelength’ limit is recovered if a suitably defined dimensionless parameter is small. (For example, in polymer quantization of a harmonic oscillator of mass $m$ and frequency $\omega$ such a parameter is $g = m \omega / M^\star$, where $M^\star$ denotes the polymer mass scale.) More generally such a parameter could involve not just coupling constants in a Hamiltonian but also parameters in a wavefunction such as the width of a Gaussian wave packet.

The following summary of polymer quantization follows that given in [13]. We start with the basis states

$$\psi_{x_0}(x) = \begin{cases} 1, & x = x_0 \\ 0, & x \neq x_0. \end{cases} \tag{4}$$

The polymer Hilbert space is the Cauchy completion of the linear span of these basis states in the inner product

$$\langle \psi_x, \psi_{x'} \rangle = \delta_{x,x'}, \tag{5}$$

where the quantity on the right-hand side is the Kronecker delta rather than the Dirac delta. This Hilbert space is non-separable and inequivalent to $L^2(\mathbb{R})$ [14–16]. Intuitively, constructing a single nonzero $L^2(\mathbb{R})$ state using the polymer basis states would require an uncountable superposition and thus leads to a non-normalizable state in the polymer Hilbert space. Conversely, any state in the polymer Hilbert space has support on at most countably many points and thus will represent the null state in $L^2(\mathbb{R})$.

Next we define the action of the basic quantum operators. The position operator $\hat{x}$ acts by multiplication,

$$(\hat{x} \psi)(x) = x \psi(x), \tag{6}$$

and its domain contains the linear span of the basis states (4). The translation operators $\hat{U}_\lambda$, $\lambda \in \mathbb{R}$, act by

$$(\hat{U}_\lambda \psi)(x) = \psi(x + \lambda) \tag{7}$$

and are clearly unitary. The parameter $\lambda$ is dimensionful and defines the polymer mass scale. Formulae (6) and (7) are identical to those in $L^2(\mathbb{R})$. However, in $L^2(\mathbb{R})$, the action of $\hat{U}_\lambda$ is weakly continuous in $\lambda$, and there exists a densely defined self-adjoint momentum operator $\hat{p}$ such that $\hat{p} = -i[\hbar, \hat{U}_\lambda]_{\lambda=0} = i \partial_x$ and $\hat{U}_\lambda = e^{i\lambda \hat{p}}$. By contrast, in the polymer Hilbert space the action of $\hat{U}_\lambda$ is not weakly continuous in $\lambda$, and hence the basic momentum operator does not exist.

The states in the polymer Hilbert space can be described as points in a certain compact space, the (Harald) Bohr compactification of the real line, and the operators introduced above can be described in terms of a representation of the Weyl algebra associated with the classical position and momentum variables [14–16]. There also exists a ‘dual’ quantization in which a momentum operator and a family of translation operators in the momenta exist, but there is no basic position operator [17]. These mathematical structures will however not be used in the rest of the paper.

As the basic momentum operator cannot be defined, any phase space function containing the classical momentum $p$, most importantly the Hamiltonian, has to be quantized in an indirect way. With a length scale $\lambda > 0$, we can define the operators

$$\hat{p}_\lambda = \frac{1}{2i\lambda}(\hat{U}_\lambda - \hat{U}^\dagger_\lambda), \tag{8a}$$
\[ \hat{p}_2^2 = \frac{1}{4\lambda^2} (2 - \hat{U}_{2\lambda} - \hat{U}^\dagger_{2\lambda}). \]  

(8b)

In \( L_2(\mathbb{R}) \), the \( \lambda \to 0 \) limit in (8) would give the usual momentum and momentum-squared operators \( i\partial_x \) and \( -\partial_x^2 \). In the polymer Hilbert space the \( \lambda \to 0 \) limit does not exist, and \( \lambda \) is regarded as a fundamental length scale. The Hamiltonian operator that corresponds to the classical Hamiltonian \( H = \frac{1}{2} p^2 + V(x) \) is then

\[ \hat{H} = \frac{1}{8\lambda^2} (2 - \hat{U}_{2\lambda} - \hat{U}^\dagger_{2\lambda}) + \hat{V}, \]

(9)

where \( \hat{V} \) is assumed so regular that \( \hat{V} \psi \) can be defined by pointwise multiplication, \( (\hat{V} \psi)(x) = V(x) \psi(x) \). One expects the polymer dynamics to be well approximated by the Schrödinger dynamics in an appropriate regime, and certain results to this effect are known [12, 18–20].

Let us now turn to the derivation of the uncertainty relation. We apply the general result that for two operators \( \hat{A} \) and \( \hat{B} \),

\[ (\Delta_1 \hat{A})^2 (\Delta_1 \hat{B})^2 \geq \frac{1}{4} |\langle [\hat{A}, \hat{B}] \rangle|^2, \]

(10)

where \( (\Delta_1 \hat{A})^2 \equiv \langle \hat{A} - \langle \hat{A} \rangle \rangle^2 \) and \( (\Delta_1 \hat{B})^2 \equiv \langle \hat{B} - \langle \hat{B} \rangle \rangle^2 \) are the standard uncertainties in the measurement of operators in a given state. This identity of course holds irrespective of the quantization scheme.

Let us denote the polymer basis states as \( |\psi_x \rangle = |x \rangle \). A general normalized state can then be expressed as

\[ |\Psi \rangle = \sum_{k \in \mathbb{Z}} c(x_k) |x_k \rangle, \quad \sum_{k \in \mathbb{Z}} c(x_k)^* c(x_k) = 1. \]

(11)

It follows from the definition of the momentum operator (8) that

\[ \langle [\hat{x}, \hat{p}_2] \rangle = \frac{1}{2i} \sum_{k \in \mathbb{Z}} c(x_k)^* [c(x_k + \lambda) + c(x_k - \lambda)], \]

(12)

\[ \langle \hat{p}_2^2 \rangle = \frac{1}{\lambda^2} \sum_{k \in \mathbb{Z}} c(x_k)^* [2c(x_k) - c(x_k + \lambda) - c(x_k - \lambda)]. \]

(13)

Combining these two formulae with the normalization condition (11) gives the exact expression

\[ \langle [\hat{x}, \hat{p}_2] \rangle = \frac{1}{2} \left[ 1 - \frac{\lambda^2}{2} \langle \hat{p}_2^2 \rangle \right]. \]

(14)

It is interesting to note that the rhs vanishes for the states that satisfy \( \langle \hat{p}_2^2 \rangle = 2/\lambda^2 \). Expanding for small \( \lambda \) gives

\[ \Delta x \Delta p_x \geq \frac{1}{2} \left[ 1 - \frac{\lambda^2}{2} \langle \hat{p}_2^2 \rangle + O(\lambda^4) \right]. \]

(15)

These results hold for any normalizable state. For states such that \( \langle \hat{p}_2 \rangle = 0 \), the last equation becomes

\[ \Delta x \Delta p_x \geq \frac{1}{2} \left[ 1 - \frac{\lambda^2}{2} (\Delta p_x)^2 + O(\lambda^4) \right]. \]

(16)

We note the following concerning this result. (i) It is similar to (2), but with the important difference that the sign of the correction term is negative definite. Thus the uncertainty decreases due to the presence of the scale \( \lambda \). Due to the sign of the correction, there is some tension between Mead’s argument and the exact result we have found. However the former
is just an argument, and it uses gravity as an important input, whereas our result is exact and does not depend even on the Hamiltonian. (ii) It is general in the sense that it does not rely on uniform sampling where one assumes \( x_{k+1} - x_k = \lambda \). (iii) It is kinematic, whereas the string argument requires the dynamical input that the size of a string at high energy depends on its energy. (iv) The simplest form of the momentum operator depends only on \( \lambda \) through the Hermitian combination \( \left( \hat{U}_\lambda - \hat{U}^\dagger_\lambda \right) / i \), up to an arbitrary scaling of \( \lambda \). Our result holds for this class of momentum operators. (However, as is usual in quantum theory, there can be a variety of operators corresponding to classical phase space functions that depend on additional arbitrary constants. Such operators may lead to dependence of the coefficient of \( (\Delta_1 p)^2 \) in the uncertainty relation (16) on the additional parameters, and may change the sign of this term.)

In Schrödinger quantization minimum uncertainty states are the Gaussian states; therefore, it is instructive to see the polymer corrections for the Gaussian state

\[
|\Psi(x_0, p_0; \sigma)\rangle = \frac{1}{N} \sum_{k \in \mathbb{Z}} e^{-(x_k - x_0)^2/2\sigma^2} e^{-ip_0 x_k} |x_k\rangle,
\]

where \( N \) is the normalization factor. Using actions of the operators \( \hat{x}, \hat{p}_\lambda \) on this state, computation of the rhs of (10) gives

\[
\Delta x \Delta p \geq \frac{1}{4} e^{-\lambda^2/2\sigma^2} \cos^2 \lambda p_0,
\]

where we have chosen uniform sampling and approximated sums by integrals. (The expectation values can be computed without this approximation by using the Poisson resummation formula, as used for example in [21].) In the regime where \( \lambda p_0 \ll 1 \), and \( \lambda^2/4\sigma^2 \ll 1 \), and using the fact that \( \langle \hat{p}_\lambda \rangle \approx p_0 \) and \( \Delta p_\lambda \approx 1/2\sigma^2 \), equation (18) gives

\[
\Delta x \Delta p \geq \frac{1}{2} \left[ 1 - \frac{\lambda^2}{2} \left( \langle \hat{p}_\lambda \rangle^2 + (\Delta p_\lambda)^2 \right) + \mathcal{O}(\lambda^4) \right],
\]

which is the same as equation (15).

3. Uncertainty relation in polymer field theory

We consider in this section a generalization to field theory [22] of the quantization described above. Our aim is to demonstrate how the uncertainty relation acquires a metric dependence. Let us focus on scalar field with the canonical variables \((\phi(x, t), P(x, t))\) on a background

\[
dx^2 = -dt^2 + q_{ab} \, dx^a \, dx^b,
\]

where \( q_{ab} \) is the spatial metric. The classical polymer variables we consider are

\[
\phi_f \equiv \int d^3 x \sqrt{q} \, f(x) \phi(x), \quad U_\lambda \equiv \exp \left( \frac{i\lambda P}{\sqrt{q}} \right),
\]

where the smearing function \( f(x) \) is a scalar. Since \( P \) transforms as a density under coordinate transformations, the \( \sqrt{q} \) factor in the second definition is required to make the argument of the exponent a scalar. The parameter \( \lambda \) is now a spacetime constant with the dimensions of \((\text{mass})^{-2}\). These variables satisfy the Poisson algebra

\[
[\phi_f, U_\lambda] = i \lambda f U_\lambda.
\]

Now having seen how the metric enters in the definition of these variables, let us specialize to a Friedman–Robertson–Walker spacetime with the spatial metric \( q_{ab} = a(t)^2 q^0_{ab} \), where \( a(t) \) is the scale factor and the fiducial metric \( q^0_{ab} = \delta_{ab} \). To keep it compatible with homogeneity,
we set the smearing function \( f(x) = 1 \). Employing a standard box normalization to regulate the spatial integration in (21) gives the reduced variables
\[
\phi_f = V_0 a^3 \phi, \quad U_s = \exp(i \lambda P / a^3),
\] (23)
where \( V_0 = \int d^3x \sqrt{q_{ij}} \) is the fiducial volume. The Poisson brackets of the reduced variables are the same as those in equation (22) with \( f = 1 \). Quantization proceeds as before by realizing this Poisson algebra as a commutator algebra on the polymer Hilbert space:
\[
\hat{\phi}_f |\mu\rangle = \mu |\mu\rangle, \quad \hat{U}_s |\mu\rangle = |\mu + \lambda\rangle.
\] (24)
The scale-dependent momentum is now
\[
P_s = \frac{a^3}{2\lambda} (U_s - U_s^d),
\] (25)
where we note the non-trivial scale factor dependence. To derive the uncertainty relations we again choose a Gaussian state peaked at the phase space values \((\phi_0, P_0)\):
\[
|\Psi\rangle = \frac{1}{N} \sum_{k \in \mathbb{Z}} c_k |\lambda_k\rangle, \quad c_k \equiv e^{-\frac{1}{2} \phi_0^2 / \sigma^2} e^{-i P_0 \phi_0 V_0},
\] (26)
where \( \phi_0 = \lambda k / V_0 a^3 \) is an eigenvalue of the scalar field operator derived from \( \hat{\phi}_f \) in equation (24). The scalar configuration points \( \lambda_k \) are chosen such that the Gaussian profile is well sampled. Here we choose a uniform sampling. This state gives the expectation value
\[
\langle \hat{U}_s \rangle = e^{i \Theta} e^{-\Theta^2 / 4 \Sigma^2},
\] (27)
where
\[
\Theta \equiv \lambda P_0 a^{-3}, \quad \Sigma \equiv \sigma V_0 P_0
\] (28)
are dimensionless variables. By computing the rhs of equation (10) for the Gaussian state (26), we arrive at the scale factor-dependent uncertainty relation
\[
V_0^2 (\Delta \phi)^2 (\Delta P_s)^2 \geq \frac{1}{2} e^{-\Theta^2 / 2 \Sigma^2} \cos^2 \Theta.
\] (29)
This is very similar to equation (18); the \( V_0 \) factor on the rhs is present due to the reduced classical Poisson bracket \([\phi, P] = 1 / V_0\).

Computing the lhs of equation (10) explicitly for the Gaussian state, we find that the expectation values of the smeared field operators are
\[
\langle \hat{\phi}_f \rangle = \phi_0 V_0 a^3, \quad \langle \hat{\phi}_f^2 \rangle = \phi_0^2 V_0^2 a^6 + \frac{1}{2} V_0^2 a^6 \sigma^2,
\] (30)
which give the field fluctuation
\[
(\Delta \phi)^2 \equiv \frac{1}{V_0^2 a^6} (\langle \hat{\phi}_f^2 \rangle - \langle \hat{\phi}_f \rangle^2) = \frac{\sigma^2}{2}.
\] (31)
Similarly, the expectation value of the field momentum operators are
\[
\langle \hat{P}_s \rangle = \frac{a^3}{\lambda} e^{-\Theta^2 / 4 \Sigma^2} \sin \Theta, \quad \langle \hat{P}_s^2 \rangle = \frac{a^6}{2 \lambda^2} (1 - e^{-\Theta^2 / 2 \Sigma^2} \cos 2 \Theta),
\] (32)
\[
\Delta P^2_s \equiv \langle \hat{P}_s^2 \rangle - \langle \hat{P}_s \rangle^2 = \frac{a^6}{2 \lambda^2} (1 - e^{-\Theta^2 / 2 \Sigma^2})(1 + e^{-\Theta^2 / 2 \Sigma^2} \cos 2 \Theta).
\] (33)
Thus the lhs of the uncertainty relation is
\[
V_0^2 (\Delta \phi)^2 (\Delta P_s)^2 = \frac{\Sigma^2}{4 \Theta^2} (1 - e^{-\Theta^2 / 2 \Sigma^2})(1 + e^{-\Theta^2 / 2 \Sigma^2} \cos 2 \Theta).
\] (34)
The scale factor dependence in this equation appears through the variable $\Theta$. For large volumes ($a \to \infty$) or a vanishing discreteness scale ($\lambda \to 0$), we have $\Theta \to 0$. Therefore it is useful to expand the right-hand side of equation (34) for small $\Theta$. This leads to

$$V_0(\Delta \phi)(\Delta P_\lambda) = \frac{1}{2} - \frac{\Theta^2}{4} \left( 1 + \frac{1}{2\Sigma^2} \right) = \frac{1}{2} \left[ 1 - \lambda^2 \left( \frac{1}{2a^6} \right)^2 \right] + O(\lambda^4).$$

(35)

where we have used the small $\Theta$ expansion of (32),

$$\langle \hat{P}_\lambda^2 \rangle = \frac{a^6}{\lambda^2} \Theta^2 \left( 1 + \frac{1}{2\Sigma^2} \right) + O(\Theta^4).$$

(36)

We note the following concerning this application to cosmology. (i) With the identifications $a \to 1$, $P_\lambda \to p_\lambda$ and $\phi \to x$, we note that equation (35) becomes the same as the rhs of equation (15). This means that with polymer corrections up to order $\lambda^2$, the Gaussian states remain the minimum uncertainty states. However, this is no longer the case if the next to leading order corrections are included, as is apparent from a comparison of the rhs of equation (29) and equation (34). (ii) The $\lambda^2$ correction in equation (35) has a remarkable form, namely that the first term inside the square brackets is the classical energy density, and the second is its fluctuation. Thus the uncertainty relation acquires corrections proportional to the energy density for the massless scalar field. (iii) It also follows from equation (35) that in the early universe, the correction to the uncertainty principle was much larger. The $\Theta \ll 1$ expansion of equations (29), (34) of course breaks down for $\Theta \sim 1$ and is not reliable also for sufficiently large values of $\Theta$, where the approximation of quantum field on a fixed background breaks down. Nevertheless, it is evident that the modifications to the uncertainty principle are significant in a certain epoch. (We note that this result is similar to that obtained using a deformed Heisenberg algebra realized in the $L_2(\mathbb{R})$ Hilbert space [23]; the only common feature with our approach is a fundamental length scale.)

Our results have direct consequences for the spectrum of cosmological fluctuations. Since the quantization method comes with a scale, there will be $\lambda$-dependent corrections at least to the scale invariance of this spectrum. Specifically, in the standard inflationary power spectrum, the Bunch–Davies vacuum is a Gaussian state in Schrödinger quantization. It would have to be replaced by a suitable state in the polymer Hilbert space, with its concomitant correction to the uncertainty principle and departure from Gaussianity, as described briefly in (i) above. How this works out in detail is currently under investigation [24].

4. Summary

We have seen that polymer quantization gives a modified uncertainty relation that resembles the one coming from heuristic arguments involving gravity, from string theory, and from generalized commutators of position and momentum. The common feature in all these approaches is a length scale in addition to $\hbar$. However it is only in the polymer approach that the quantization method itself comes with a scale due to the choice of Hilbert space, and in this sense the modifications are independent of the theory.

More generally we have seen that if polymer quantization is applied to field theory on a given background, the uncertainty relation acquires an explicit metric dependence. This dependence is dramatically illustrated for the case of the Friedmann–Robertson–Walker background, where we see that the correction dominates at early times. This feature could lead to interesting consequences in the physics of early universe. It would also be of interest to see what happens for other backgrounds, especially the black hole, to see how the presence of an event horizon affects the relation. We finally note that Yoneya’s general arguments [2]...
that $\Delta p$ corrections to the $\Delta x\Delta p$ uncertainty relation give rise to a position–time uncertainty relation $\Delta x \Delta t \geq \hbar/2$ apply equally well in the setting we have discussed. Therefore polymer quantization gives an indication of spacetime non-commutativity. This is another intriguing aspect that deserves further study.

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