HODGE THEORY AND UNITARY REPRESENTATIONS, IN THE EXAMPLE OF $SL(2,\mathbb{R})$.

WILFRED SCHMID AND KARI VILONEN

Dedicated to David Vogan, on the occasion of his sixtieth birthday

In our paper [4] we formulated a conjecture on unitary representations of reductive Lie groups. We are currently working towards a proof; the technical difficulties are formidable. It has been suggested that an explicit description in the case of $SL(2,\mathbb{R})$ would be helpful. The unitary representations of $SL(2,\mathbb{R})$ have been worked out in great detail, of course, but even in this special case our construction of the inner product in terms of the $D$-module realization is not obvious.

We begin with a quick summary of our conjecture in the general case of a reductive, linear, connected Lie group $G_R$, with maximal compact subgroup $K_R$. We let $G$ and $K$ denote the complexifications. The complex group $G$ contains a unique compact real form $U_R$ such that $U_R \cap G_R = K_R$. Then $U_R$ acts transitively on the flag variety $X$ of $G$, and $K$ acts with finitely many orbits. The points of $X$ correspond to Borel subalgebras $b$ of the Lie algebra $g$ of $G$. The quotients $b = b/[b, b]$ constitute the fibers of a flat vector bundle. Since $X$ is simply connected, we can think of $h$ as a fixed vector space. This is the “universal Cartan”, and is acted on by the “universal Weyl group” $W$. Its dual $h^*$ contains the “universal root system” $\Phi$ and the system of positive roots $\Phi^+$, chosen so that $[b, b]$ becomes the direct sum of the negative root spaces; $h^*$ also contains the “universal weight lattice” $\Lambda$. Further notation: lower case Fraktur letters, such as $g_R, k_R, g$, refer to the Lie algebras of $G_R, K_R, G$, etc.

Via the Harish Chandra isomorphism, $h^*$ parameterizes the characters $\chi_\lambda$ of the center of the universal enveloping algebra, with $\chi_\lambda = \chi_\mu$ if and only if $\mu = w\lambda$ for some $w \in W$. We shall say that a Harish Chandra module $M$ has a real infinitesimal character if it is of the form $\chi_\lambda$ with $\lambda \in \mathbb{R} \otimes \mathbb{Z} \Lambda$. David Vogan, many years ago, pointed out that to understand the irreducible unitary representations of $G_R$ it suffices to treat the case of real infinitesimal character [3]. Let then $M_\lambda$ be an irreducible Harish Chandra module with real infinitesimal character $\chi_\lambda$. Since $\lambda$ is determined only up to the Weil group action, we may and shall assume that $\lambda$ is dominant, i.e.,

\[(\alpha, \lambda) \geq 0 \quad \text{for all } \alpha \in \Phi^+.\]

To determine whether or not $M_\lambda$ underlies an irreducible unitary representation, one needs to know if it carries a nonzero $g_R$-invariant hermitian form $(\cdot, \cdot)_{g_R}$ – this

1991 Mathematics Subject Classification. Primary 22E46, 22D10, 58A14; Secondary 32C38.

Key words and phrases. Representation theory, Hodge theory.

The first author was supported in part by NSF grant DMS-1300185.

The second author was supported in part by NSF grants DMS-1402928, DMS-1069316, and the Academy of Finland.

1
question has a simple answer, see below – and, when that is the case, if $(\ , \ )_{\mathfrak{g}_K}$ has a definite sign.

Vogan and his coworkers \[5\] made the important observation that the condition of having a real infinitesimal character ensures the existence of a nonzero $\mathfrak{u}_\mathbb{R}$-invariant hermitian form $(\ , \ )_{\mathfrak{g}_\mathbb{R}}$. If both types of hermitian forms exist, they are explicitly related: the Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$ then acts also on the Harish Chandra module $M_\lambda$, and after suitable rescaling of the hermitian forms,

$$(v_1, v_2)_{\mathfrak{g}_\mathbb{R}} = (\theta v_1, v_2)_{\mathfrak{g}_\mathbb{R}}.$$ \[2\]

The $\mathfrak{u}_\mathbb{R}$-invariant form is easier to deal with, both computationally and from a geometric point of view. At the same time the action of $\theta$ on $M_\lambda$ can be described quite concretely. Thus, if one understands the hermitian form $(\ , \ )_{\mathfrak{g}_\mathbb{R}}$, one can decide if $M_\lambda$ is unitarizable.

As usual we write $\rho$ for the half sum of the positive roots. We let $\mathcal{D}$ denote the sheaf of linear differential operators, with algebraic coefficients, on the flag variety $X$; here $X$ is equipped with the Zariski topology. The sheaf of algebras $\mathcal{D}$ can be twisted by $G$-equivariant line bundles, and more generally, by any $\lambda \in \mathfrak{h}^*$. It is convenient to parameterize the twists so that $\mathcal{D}_\lambda$, for $\lambda \in \Lambda + \rho$, acts on sections of the $G$-equivariant line bundle $L_{\lambda - \rho} \to X$ with Chern class $\lambda - \rho \in \Lambda \cong H^2(X, \mathbb{Z})$; for arbitrary $\lambda \in \mathfrak{h}$ one then defines $\mathcal{D}_\lambda$ by a process of analytic continuation. The sheaves $\mathcal{D}_\lambda$ are $G$-equivariant, in the Zariski sense locally isomorphic to $\mathcal{D}$, and every $\zeta \in \mathfrak{g}$ acts as an infinitesimal automorphism and thus defines a global section of $\mathcal{D}_\lambda$. Note that $\mathcal{D}_\rho = \mathcal{D}$, and that $\mathcal{D}_{-\rho}$ acts on sections $L_{-2\rho}$ = canonical bundle of $X$.

The Beilinson-Bernstein construction\[1\] realizes the irreducible Harish Chandra module $M_\lambda$, with $\lambda$ real and dominant as in \[1\], as the space of global sections

$$M_\lambda = H^0(X, \mathcal{M}_\lambda)$$ \[3\]
of an irreducible, $K$-equivariant sheaf of $\mathcal{D}_\lambda$-modules $\mathcal{M}_\lambda$ – or for short, an irreducible, $K$-equivariant $\mathcal{D}_\lambda$-module. Then $\mathfrak{g}$ acts on $M_\lambda = H^0(X, \mathcal{M}_\lambda)$ via the inclusion $\mathfrak{g} \hookrightarrow H^0(X, \mathcal{D}_\lambda)$. The correspondence between the Harish Chandra module $M_\lambda$ and the “Harish Chandra sheaf” $\mathcal{M}_\lambda$ extends functorially to all Harish Chandra modules of finite length, with infinitesimal character $\lambda$. Irreducible Harish Chandra sheaves $\mathcal{M}_\lambda$ are easy to describe: they arise from direct images, in the category of $\mathcal{D}_\lambda$-modules, under the embedding $j : Q \hookrightarrow X$ of a $K$-orbit $Q$ in $X$, applied to a $K$-equivariant “twisted local system” $\mathcal{C}_{Q, \lambda}$ on $Q$, with twist $\lambda - \rho$. A formal, general definition of $\mathcal{C}_{Q, \lambda}$ would lead too far; in the special case of $G_\mathbb{R} = SL(2, \mathbb{R})$ we describe it implicitly in \[20\] below, where its generating section will be denoted by $\sigma_0^{\lambda - \rho}$. In any case, the tensor product $\mathcal{O}_Q \otimes \mathcal{C}_{Q, \lambda}$ has the structure of a $\mathcal{D}_{Q, \lambda}$-module on the $K$-orbit $Q$, and the direct image $j_* (\mathcal{O}_Q \otimes \mathcal{C}_{Q, \lambda})$ of a Harish Chandra sheaf: a $K$-equivariant $\mathcal{D}_\lambda$-module on $X$. In general the direct image is not irreducible, but it contains a unique irreducible $\mathcal{D}_\lambda$-submodule\[2\] and

$$M_\lambda = \text{unique irreducible submodule of } j_* (\mathcal{O}_Q \otimes \mathcal{C}_{Q, \lambda}).$$ \[4\]

\[1\] A more detailed summary of the Beilinson-Bernstein construction of Harish Chandra modules can be found in \[2\].

\[2\] Since we assumed $G_\mathbb{R}$, and hence also $K$, to be connected, any $\mathcal{D}_\lambda$-subsheaf of a Harish Chandra sheaf is automatically $K$-equivariant and is therefore also a Harish Chandra sheaf.
The realization \( \text{\textdagger} \) of irreducible Harish Chandra sheaves is unique. It almost sets up a bijection between irreducible Harish Chandra modules \( M_\lambda \), with the parameter \( \lambda \) of the infinitesimal character as in \( \text{\textdagger} \), and \( K \)-equivariant, irreducible local systems \( \mathbb{C}_{Q,\lambda} \), with twist \( \lambda - \rho \), on \( K \)-orbits \( Q \subset X \) – the qualifier “almost” is necessary because when \( \lambda \) is singular, certain irreducible Harish Chandra sheaves have no nonzero global sections. This phenomenon explains why the classification of irreducible Harish Chandra modules with \textit{regular infinitesimal character} looks simpler than that of irreducible Harish Chandra modules with \textit{singular infinitesimal character}.

We shall not attempt to summarize Saito’s theory of mixed Hodge modules here. Rather, we shall state the relevant facts, which apply to all members of the category of “geometrically constructible” Harish Chandra sheaves \( \mathcal{M}_\lambda \) – this includes in particular the sheaves obtained by the standard \( \mathcal{D} \)-module operations applied to \( \mathcal{D}_\lambda \)-modules of the type \( j_* (\mathcal{O}_Q \otimes \mathbb{C}_{Q,\lambda}) \) and their \( \mathcal{D}_\lambda \)-subsheaves. A mild generalization of Saito’s theory\( \text{\textdagger} \) puts three additional structures on each object \( \mathcal{M}_\lambda \). First of all, the \textit{weight filtration}, a functorial, finite increasing filtration

\[
\begin{align*}
0 & \subset W_0 \mathcal{M}_\lambda \subset W_1 \mathcal{M}_\lambda \subset \cdots \subset W_k \mathcal{M}_\lambda \subset \cdots \subset W_n \mathcal{M}_\lambda = \mathcal{M}_\lambda
\end{align*}
\]

by \( \mathcal{D}_\lambda \)-subsheaves, with completely reducible quotients \( W_k \mathcal{M}_\lambda / W_{k-1} \mathcal{M}_\lambda \) which are themselves objects in the category of Harish Chandra sheaves. Secondly, the \textit{Hodge filtration}, a typically infinite, increasing filtration

\[
\begin{align*}
0 & \subset F_0 \mathcal{M}_\lambda \subset \cdots \subset F_p \mathcal{M}_\lambda \subset F_{p+1} \mathcal{M}_\lambda \subset \cdots \subset \mathcal{M}_\lambda = \cup_{p \geq 0} F_p \mathcal{M}_\lambda
\end{align*}
\]

by \( \mathcal{O}_X \)-coherent, \( K \)-equivariant, \( \mathcal{O}_X \)-submodules. This is a good filtration in the sense of \( \mathcal{D} \)-module theory: let \( (\mathcal{D}_\lambda)_d \subset \mathcal{D}_\lambda \) denote the \( \mathcal{O}_X \)-subsheaf of differential operators of degree at most \( d \); then

\[
\begin{align*}
(\mathcal{D}_\lambda)_d F_p \mathcal{M}_\lambda & \subseteq F_{p+d} \mathcal{M}_\lambda, \quad \text{with equality holding if } p \gg 0.
\end{align*}
\]

The third ingredient, the \textit{polarization} on any irreducible Harish Chandra sheaf \( \mathcal{M}_\lambda \), is a nontrivial \( \mathcal{D}_\lambda \times \overline{\mathcal{D}}_\lambda \)-bilinear pairing

\[
\begin{align*}
P : \mathcal{M}_\lambda \times \overline{\mathcal{M}}_\lambda & \longrightarrow \mathcal{C}^{-\infty}(X_{\mathbb{R}}).
\end{align*}
\]

Here \( \mathcal{C}^{-\infty}(X_{\mathbb{R}}) \) refers to the sheaf of distributions on \( X \), considered as \( C^\infty \) manifold, and \( \overline{\mathcal{M}}_\lambda \) is the complex conjugate of \( \mathcal{M}_\lambda \), viewed as \( \overline{\mathcal{D}}_\lambda \)-module on \( X \), equipped with the complex conjugate algebraic structure.

Morphisms in the category of mixed Hodge modules preserve both filtrations strictly: if \( T : \mathcal{M} \to \mathcal{N} \) is a morphism, then \( T(F_p \mathcal{M}) = (T \mathcal{M}) \cap (F_p \mathcal{N}) \), and analogously for the weight filtration. We should also mention Saito’s normalization of the indexing of the two filtrations. Going back to \( \text{\textdagger} \), the Hodge filtration on the sheaf \( \mathcal{O}_Q \otimes \mathbb{C} \mathbb{C}_{Q,\lambda} \) on \( Q \) is trivial, in the sense that \( F_0 (\mathcal{O}_Q \otimes \mathbb{C} \mathbb{C}_{Q,\lambda}) = \mathcal{O}_Q \otimes \mathbb{C} \mathbb{C}_{Q,\lambda} \) and \( F_{-1} (\mathcal{O}_Q \otimes \mathbb{C} \mathbb{C}_{Q,\lambda}) = 0 \). As a sheaf on \( Q \) it is irreducible and has weight equal to \( \dim Q \). The process of direct image shifts the lowest index of the Hodge filtration to \( a = \text{codim} Q \), and puts the weights into degrees \( \geq \dim Q \), with the irreducible subsheaf \( \mathcal{M}_\lambda \) having weight equal to \( \dim Q \).

The polarization leads to a geometric description of the \( u_{\mathbb{R}} \)-invertible hermitian form on any irreducible Harish Chandra module \( M_\lambda \). Let \( \omega \) denote the – unique, up to scaling – \( U_{\mathbb{R}} \)-invariant measure on \( X_{\mathbb{R}} \). Like any smooth measure on a compact \( C^\infty \) manifold it can be integrated against any distribution. When \( M_\lambda \) is realized

\[\text{\textdagger} 3\]Without the assumption of an underlying rational structure, which Saito requires.
as the space of global sections of the corresponding Harish Chandra sheaf \( \mathcal{M}_\lambda \) as in \( \text[(3)]{3} \), then

\[
(s_1, s_2)_{u_R} = \int_{X_{\mathbb{A}}} P(s_1, \tilde{s_2}) \omega \quad \text{for } s_1, s_2 \in H^0(X, \mathcal{M}_\lambda),
\]
does indeed define a \( u_R \)-invariant hermitian form, as can be checked readily \( \text[(4)]{4} \). The Cartan involution \( \theta \) acts on \( X \) and on the set of \( K \)-orbits in \( X \). If \( \theta \) fixes a particular \( K \)-orbit \( Q \), then it acts on the twisted local systems on \( Q \), and if it fixes also the twisted local system \( \mathcal{C}_{Q, \lambda} \) as in \( \text[(4)]{4} \), then it acts on the sections of the direct image \( j_* (\mathcal{O}_Q \otimes_{\mathcal{C}} \mathcal{C}_{Q, \lambda}) \) and of its unique irreducible subsheaf \( \mathcal{M}_\lambda \). In this sense, the action of \( \theta \) on the Harish Chandra module \( M_\lambda \) – which relates \( (\cdot, \cdot)_{u_R} \) to \( (\cdot, \cdot)_{g_R} \) as in \( \text[(2)]{2} \) – is visible geometrically.

Via the global section functor the Hodge and weight filtrations induce filtrations on \( M_\lambda \), the space of global sections of the Harish Chandra sheaf \( \mathcal{M}_\lambda \), whether or not the latter is irreducible:

\[
\begin{align*}
0 \subset W_0 M_\lambda & \subset W_1 M_\lambda \subset \cdots \subset W_k M_\lambda \subset \cdots \subset W_n M_\lambda = M_\lambda, \\
0 \subset F_0 M_\lambda & \subset \cdots \subset F_p M_\lambda \subset F_{p+1} M_\lambda \subset \cdots \subset M_\lambda = \cup_{p \geq a} F_p M_\lambda.
\end{align*}
\]

The \( W_k M_\lambda \) are Harish Chandra submodules of \( M_\lambda \), and the \( F_p M_\lambda \) are finite dimensional, \( K \)-invariant subspaces. In the irreducible case the weight filtration collapses as was mentioned earlier, \( M_\lambda \) in \( \text[(3)]{3} \) has weight dim \( Q \), and the lowest index in the Hodge filtration is \( a = \text{codim} \ Q \). We can now state our conjecture. It asserts that if \( M_\lambda \) is irreducible, the \( u_R \)-invariant hermitian form is nondegenerate on each \( F_p M_\lambda \), and

\[
(-1)^{p-a} (s, s)_{u_R} > 0 \quad \text{for all nonzero } s \in F_p M_\lambda \cap (F_{p-1} M_\lambda)^\perp.
\]

Whenever \( M_\lambda \) also admits a \( g_R \)-invariant hermitian form it would be related to \( u_R \)-invariant one via \( \text[(2)]{2} \), and the resulting hermitian form \( (\cdot, \cdot)_{g_R} \) would then have a definite sign if and only if \( M_\lambda \) is unitarizable.

The significance of the conjecture is discussed in \( \text[(4)]{4} \). While it does not amount to a description of the unitary dual of \( G_R \) in terms of representation parameters, it puts the study of the irreducible unitary representation into a functorial context.

We now turn to the example of \( SL(2, \mathbb{R}) \). It is conjugate to \( SU(1, 1) \) under an inner automorphism of \( SL(2, \mathbb{C}) \), and various formulas have a simpler appearance for \( SU(1, 1) \). Thus we suppose \( G = SL(2, \mathbb{C}) \),

\[
G_R = SU(1, 1) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \bar{\alpha} \end{pmatrix} \bigg| \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 - |\beta|^2 = 1 \right\},
\]

\[
K_R = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} \bigg| \alpha \in \mathbb{C}, \quad |\alpha| = 1 \right\}, \quad K = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \bigg| \alpha \in \mathbb{C}^* \right\},
\]

and \( U_R = SU(2) \). These groups act on the flag variety of \( G \),

\[
X = \mathbb{P}^1 = \mathbb{C} \cup \{ \infty \},
\]

by linear fractional transformations, and \( K \) acts with three orbits, namely \( \{ 0 \} \), \( \{ \infty \} \), and \( \mathbb{C}^* \). In the notation of \( \text[(12)]{12} \text{(13)} \), \( K \) acts on \( \mathbb{C}^* \) by \( \alpha^2 \), so \( \mathbb{C}^* \) admits two irreducible \( K \)-equivariant local systems, corresponding to the trivial and the nontrivial character of the (component group of the) generic isotropy group \( \{ \pm 1 \} \).

This is true both in the scalar – i.e., non-twisted – and twisted case. Since \( K \) is
connected, the two point orbits admit only the trivial irreducible $K$-equivariant local system. The Cartan involution,
\[
\theta = \text{conjugation by } \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},
\]
is inner, it preserves each of the three orbits and the $K$-equivariant local systems on them. Thus all irreducible Harish Chandra modules with real infinitesimal character admit both $\mathfrak{u}_R$- and $\mathfrak{g}_R$-invariant hermitian forms.

The dual $\mathfrak{h}^*$ of the universal Cartan can be identified with $\mathbb{C}$ so that $\Lambda \cong \mathbb{Z}$, $\Phi \cong \{ \pm 2 \}$, and $\rho \cong 1$. With this identification an infinitesimal character $\chi_\lambda$ is real in the earlier sense if and only if $\lambda \in \mathbb{R}$, and $\lambda \in \mathbb{R}$ is dominant if and only if $\lambda \geq 0$.

The standard $SL_2$-triple
\[
e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
spans $\mathfrak{g}$ over $\mathbb{C}$, with $\mathfrak{t}$ spanned by $h$, and satisfies the conjugation relations $\bar{e}_+ = e_-$, $\bar{h} = -h$. The elements of this triple operate on the sheaf of algebraic functions on $\mathbb{C} \cup \{ \infty \}$ by infinitesimal left translation. One computes readily that via this action,
\[
e_+ \cong -\frac{d}{dz}, \quad e_- \cong z^2 \frac{d}{dz}, \quad h \cong -2z \frac{d}{dz}.
\]

For example, $e_+$ acts on $f(z)$ by the derivative with respect to $t$, at the origin, of $f(\exp(-t e_+)z) = f(z - t)$, resulting in the formula $(e_+ f)(z) = -z \frac{df}{dz}(z)$; the other cases are treated similarly.

The $G$-equivariant line bundle $\mathcal{L}_2$ coincides with the tangent bundle of $\mathbb{P}^1$, so we can identify (14) with a basis of the space of global sections of $\mathcal{L}_2$. However, for notational reasons, we choose the new symbols $\sigma_2, \sigma_0, \sigma_{-2}$, corresponding to $e_+, h, e_-$, in that order. Then $\sigma_2$ vanishes to second order at $\infty$, $\sigma_{-2}$ vanishes to second order at $0$, and $\sigma_0$ has first order zeros at both $0$ and $\infty$, and these are the only zeroes in each case. Moreover,
\[
e_+ \sigma_2 = 0, \quad h \sigma_2 = 2\sigma_2, \quad e_- \sigma_2 = -\sigma_0 = \frac{1}{2} z \sigma_2, \\
e_+ \sigma_0 = -2\sigma_2 = -z^{-1} \sigma_0, \quad h \sigma_0 = 0, \quad e_- \sigma_0 = 2 \sigma_{-2} = -z \sigma_0, \\
e_+ \sigma_{-2} = \sigma_0 = -\frac{1}{2} z^{-1} \sigma_{-2}, \quad h \sigma_{-2} = -2 \sigma_{-2}, \quad e_- \sigma_{-2} = 0,
\]
as can be read off from (15). The $U_R$-invariant measure on $\mathbb{P}^1$ is
\[
\omega = (1 + |z|^2)^{-2} dz d\bar{z}.
\]
The coefficient of $dz d\bar{z}$ in this formula can be interpreted as the squared length of $\frac{dz}{dz}$ with respect to the $U_R$-invariant hermitian metric, or equivalently, the squared length of $\sigma_2$ relative to the $U_R$-invariant hermitian metric on the line bundle $\mathcal{L}_2$.

Thus
\[
\|\sigma_2\| = \frac{1}{1 + |z|^2} , \quad \|\sigma_0\| = \frac{2|z|}{1 + |z|^2} , \quad \|\sigma_{-2}\| = \frac{|z|^2}{1 + |z|^2}
\]
describes the length, as measured by the $U_R$-invariant metric on $\mathcal{L}_2$, of the three sections $\sigma_2, \sigma_0, \sigma_{-2}$.

As was mentioned already, there exist two irreducible $K$-invariant local systems, with twist $\lambda - \rho$, on the $K$-orbit $\mathbb{C}^*$, corresponding to the trivial and the nontrivial
character of the generic isotropy subgroup \( \{ \pm 1 \} \) of \( K \). The corresponding Harish Chandra sheaves can be realized as

\[
\mathcal{M}_{C^\ast, \lambda, \text{even}} = \{ f(z) \frac{\lambda_{-1}}{\sigma_0^{\lambda_{-1}}} \mid f \in \mathbb{C}(z) \},
\]

\[
\mathcal{M}_{C^\ast, \lambda, \text{odd}} = \{ f(z) z^{1/2} \frac{1}{\sigma_0^{1/2}} \mid f \in \mathbb{C}(z) \}.
\]

These are Zariski-locally defined algebraic functions, multiplied by the “section” \( \sigma_0^{\lambda_{-1}} \) of the formal power \( L_2^\lambda \), either on \( \mathbb{C}^\ast \) (in the even case), where the section is well defined, or its twofold cover (in the odd case). As such they are naturally \( D_\lambda \)-modules on \( \mathbb{C}^\ast \), and then, via the direct image functor corresponding to the open embedding \( \mathbb{C}^\ast \subset \mathbb{C} \cup \{ \infty \} \), on all of \( \mathbb{C} \cup \{ \infty \} \). How \( D_\lambda \) acts is not so relevant for us, but the action of \( g \subset \Gamma \mathcal{D}_\lambda \) is. That action is given by the product rule, with \( g \) acting on \( f(z) \) or \( z^{1/2} f(z) \) according to the formulas (10), and on the formal powers of \( \sigma_0 \) according to (17). Since \( \sigma_0 \) has first order zeroes at 0 and \( \infty \),

\[
f(z) \frac{\lambda_{-1}}{\sigma_0^{\lambda_{-1}}} \sim f(z) z^{\frac{\lambda_{-1}}{2}} \text{ near the origin, and}
\]

\[
f(z) \frac{1}{\sigma_0^{1/2}} \sim f(z) z^{-\frac{1}{2}} \text{ near } \infty.
\]

In particular, \( \mathcal{M}_{C^\ast, \lambda, \text{even}} \) is reducible if and only if \( \lambda \) is an odd integer, whereas \( \mathcal{M}_{C^\ast, \lambda, \text{odd}} \) reduces if and only if \( \lambda \) is even.

We recall that the sheaves (20), when restricted to \( \mathbb{C}^\ast \), are irreducible and have weight one, which is the dimension of \( \mathbb{C}^\ast \). That remains correct for these sheaves on all of \( \mathbb{C} \cup \{ \infty \} \) when they are irreducible:

\[
\begin{align*}
W_0 \mathcal{M}_{C^\ast, \lambda, \text{even}} &= 0 \quad \text{and} \quad W_1 \mathcal{M}_{C^\ast, \lambda, \text{even}} = \mathcal{M}_{C^\ast, \lambda, \text{even}} \quad \text{if } \lambda \notin 2\mathbb{Z} + 1, \\
W_0 \mathcal{M}_{C^\ast, \lambda, \text{odd}} &= 0 \quad \text{and} \quad W_1 \mathcal{M}_{C^\ast, \lambda, \text{odd}} = \mathcal{M}_{C^\ast, \lambda, \text{odd}} \quad \text{if } \lambda \notin 2\mathbb{Z}.
\end{align*}
\]

In the reducible case,

\[
\begin{align*}
W_0 \mathcal{M}_{C^\ast, 2m+1, \text{even}} &= 0, & W_1 \mathcal{M}_{C^\ast, 2m+1, \text{even}} &= \mathcal{O}_P (L_{2m}), \\
W_0 \mathcal{M}_{C^\ast, 2m+1, \text{even}} &= \mathcal{M}_{C^\ast, 2m+1, \text{even}}; & W_1 \mathcal{M}_{C^\ast, 2m+1, \text{even}} &= \mathcal{O}_P (L_{2m-1}), \\
W_0 \mathcal{M}_{C^\ast, 2m, \text{odd}} &= 0, & W_1 \mathcal{M}_{C^\ast, 2m, \text{odd}} &= \mathcal{O}_P (L_{2m-1}), \\
W_0 \mathcal{M}_{C^\ast, 2m, \text{odd}} &= \mathcal{M}_{C^\ast, 2m, \text{odd}}.
\end{align*}
\]

To justify these descriptions of the weight filtrations one should notice that \( \sigma_0^m \) can be viewed as a section of \( L_2^m = L_{2m} \), and \( z^{1/2} \sigma_0^{1/2} \) as a meromorphic section of \( L_1 \). The quotients \( \text{gr}_{W, 2} \mathcal{M}_{C^\ast, 2m+1, \text{even}} \) and \( \text{gr}_{W, 2} \mathcal{M}_{C^\ast, 2m, \text{odd}} \) are Harish Chandra sheaves supported on \( \{ 0, \infty \} \). We shall discuss these later.

The Hodge filtration for the sheaves (20) starts at level \( a = 0 \), since that is the codimension. In general the Hodge filtration of the direct image under an open embedding is governed by the – not necessarily integral – order of poles. The case of \( \mathbb{C}^* \hookrightarrow \mathbb{C} \), and analogously for \( \mathbb{C}^* \hookrightarrow \mathbb{C} \cup \{ \infty \} \), is especially simple: poles of order \( \leq 1 \) have Hodge level 0, those of order \( \leq 2 \) have Hodge level 1, and so forth. Thus, in view of (20) and (21), for \( n \in \mathbb{Z} \) and \( p \geq 0 \),

\[
z^n \frac{\lambda_{-1}}{\sigma_0^{\lambda_{-1}}} \in F_p \mathcal{M}_{C^\ast, \lambda, \text{even}} \iff -\frac{\lambda_{-1}}{2} - p \leq n \leq \frac{\lambda_{-1}}{2} + p,
\]

\[
z^{n+1/2} \frac{1}{\sigma_0^{1/2}} \in F_p \mathcal{M}_{C^\ast, \lambda, \text{odd}} \iff -\frac{\lambda_{-1}}{2} - p \leq n + \frac{1}{2} \leq \frac{\lambda_{-1}}{2} + p,
\]

for all \( n \in \mathbb{Z} \) and \( p \geq 0 \). In the reducible case, there is a connection between the induced Hodge filtrations on the quotient sheaves and the intrinsic Hodge filtrations on the quotients; this, too, will be described later.
We now turn to the polarizations of the sheaves $\mathcal{M}_{C^*,2n+1,\text{even}}$, $\mathcal{M}_{C^*,2n+1,\text{odd}}$ and the resulting hermitian forms on their spaces of global sections. On the $K$-orbit $C^*$ these sheaves are always irreducible, and the only possible hermitian pairing of the type (8) on $C^*$ is, up to scaling,

$$P \left( f(z) \sigma_0^{\lambda/2}, g(z) \sigma_0^{\lambda/2} \right) = f(z) g(z) \| \sigma_0 \|^{\lambda-1},$$

which is a real analytic function, and thus distribution, on $C^*$. This is correct in both cases, if we take $f, g \in C(z)$ in the case of $\mathcal{M}_{C^*,2n+1,\text{even}}$, and $f, g \in \mathbb{Z}/2\mathbb{C}(z)$ in the case of $\mathcal{M}_{C^*,2n+1,\text{odd}}$. (27) makes the last factor on the right explicit. The two sheaves were defined as the $D_{\lambda}$-module direct image under the open embedding $C^* \hookrightarrow \mathbb{C} \cup \{ \infty \}$ which, as always in the case of open embeddings, coincides with the $O$-module direct image. The general theory ensures that

$$f(z) g(z) \| \sigma_0 \|^{\lambda-1}$$

makes sense as global distribution on $\mathbb{C} \cup \{ \infty \}$, for all global sections $f(z) \sigma_0^{\lambda/2}$, $g(z) \sigma_0^{\lambda/2}$, provided the sheaf in question is irreducible.

To see how this works out in the current setting, applied to the spaces of global sections $M_{C^*,2n+1,\text{even}}$, $M_{C^*,2n+1,\text{odd}}$ of the two sheaves, we note that

$$M_{C^*,\lambda,\text{even}} \text{ has basis } \{ z^n \sigma_0^{\lambda/2} | n \in \mathbb{Z} \}, \text{ and }$$

$$M_{C^*,\lambda,\text{odd}} \text{ has basis } \{ z^n \sigma_0^{\lambda/2} | n \in \mathbb{Z} + 1/2 \}.$$  

For reasons of radial symmetry we only need to consider the integral of the expression (26) over $\mathbb{C} \cup \{ \infty \}$ against the $U_{\mathbb{R}}$-invariant measure $\omega$ when $f$ and $g$ are the same basis element. With the convention of (27), with $n$ denoting either an integer or a true half integer, and using (18), (19), we find

$$\left( z^n \sigma_0^{\lambda/2}, z^n \sigma_0^{\lambda/2} \right)_{uk} = \int_{\mathbb{C} \cup \{ \infty \}} |z|^{2n} \| \sigma_0 \|^{\lambda-1} \omega =$$

$$= \int_{\mathbb{C} \cup \{ \infty \}} \frac{4 |z|^{2n+\lambda-1}}{(1+|z|^2)^{\lambda+1}} \, dz \, d\bar{z} = 8 \pi \int_0^\infty \frac{r^{2n+\lambda} \, dr}{(1+r^2)^{\lambda+1}} = 4 \pi \int_0^\infty \frac{u^{n+(\lambda-1)/2} \, du}{(1+u)^{\lambda+1}}.$$

This integral converges if and only if $-\lambda+1/2 < n < (\lambda+1)/2$. Since the integrand is positive, the integral has a strictly positive value in the range of convergence.

To continue the integral meromorphically beyond the range of convergence one uses integration by parts. Formally, for $s, t \in \mathbb{R},$

$$s \int_0^\infty \frac{u^{s-1} \, du}{(1+u)^t} = t \int_0^\infty \frac{u^s \, du}{(1+u)^{t+1}}.$$  

Applying this identity with $s = n+(\lambda+1)/2$ and $t = \lambda+1$, one finds that the integral changes sign and becomes strictly negative for $-\lambda+1/2 < n < -\lambda+1/2$. The same argument, with $-n$ substituted for $n$, shows that the integral is also strictly negative for $(\lambda+1)/2 < n < (\lambda+3)/2$. Then the pattern continues: when $n$ is negative and decreased by one, or if $n$ is positive and increased by one, the integral changes sign. Poles occur when $\lambda$ is an odd integer in the even case, or an even integer in the odd case – i.e., exactly when the module becomes reducible.
That is what must happen, of course; the polarization is well defined only in the irreducible range. The preceding discussion can be summarized succinctly in terms of the Hodge filtration: with \( \epsilon \) referring to either the even or the odd parity, in the irreducible range,

\[
\tag{30}
\begin{align*}
s \in F_0 M_{C\ast, \lambda, \epsilon} & \implies \text{the integral defining } (s, s)_{u_k} \text{ converges} \\
s \in F_p M_{C\ast, \lambda, \epsilon} \cap (F_p M_{C\ast, \lambda, \epsilon})^\perp, s \neq 0 & \implies (-1)^{p} (s, s)_{u_k} > 0;
\end{align*}
\]

cf. \([24]\). The second statement is the assertion of our conjecture in the case of the open \( K \)-orbit \( C^\ast \).

The change in sign is directly related to a change in the weight filtration. Let \( \lambda_0 > 0 \) be reduction point – i.e., a positive odd or even integer, depending on whether the parity is even or odd. In terms of the basis \([27]\), with the same convention of letting \( n \) refer to an integer or true half integer, depending on the parity,

\[
\tag{31}
W_1 M_{C\ast, \lambda_0, \epsilon} \text{ has basis } \{ z^n \sigma_0 z^{-1} \mid -(\lambda - 1) \leq 2n \leq \lambda - 1 \}.
\]

Thus, as the parameter \( \lambda \) crosses the reduction point \( \lambda_0 \) going from left to right, the sign of \((z^n \sigma_0, z^n \sigma_0)_{u_k}\) remains the same if and only if \(z^n \sigma_0 \in W_1 M_{C\ast, \lambda_0, \epsilon}\). This is one instance of a general fact. As the parameter for a family of induced representations crosses a reduction point, the sign changes of \((\cdot, \cdot)\) are governed by the weight filtration – that is the assertion of the Jantzen conjecture proved by Beilinson-Bernstein \([1]\). The jumps of the Hodge filtration at the reduction point line up with the weight filtration, to produce exactly the sign changes predicted by our conjecture.

When \( \lambda = m > 0 \) is a positive integer and \( \epsilon \) denotes the opposite parity, i.e., the parity of \( m + 1 \), the Harish Chandra module \( W_1 M_{C\ast, m, \epsilon} \) has dimension \( m \), and the integral \([25]\) converges for all basis elements: this is the usual description of the positive definite \( U_R \)-invariant inner product on the irreducible \( m \)-dimensional representation. For \( \lambda = 0 \) and \( \epsilon \) odd, \( W_1 M_{C\ast, m, \epsilon} \) reduces to zero. The corresponding sheaf \( W_1 M_{C\ast, 0, \text{odd}} \) is the one and only irreducible Harish Chandra sheaf for \( G_R = SU(1, 1) \) without nonzero sections.

The two singleton orbits \( \{0\}, \{\infty\} \) are related by an outer automorphism of \( G_R = SU(1, 1) \). Thus it is only necessary to discuss the orbit \( \{0\} \). Since \( K \) fixes the origin, it must act on the geometric fiber of any \( \mathcal{D}_\lambda \)-module supported at \( \{0\} \), and that forces an integral twisting parameter:

\[
\tag{32}
\lambda = m \in \mathbb{Z}_{\geq 0}.
\]

In the untwisted case, the only irreducible \( \mathcal{D} \)-module supported at the origin in \( C \) is the one generated by the “holomorphic delta function”,

\[
\tag{33}
\mathcal{D}_C \delta_0 = \mathbb{C}[z, z^{-1}] / \mathbb{C}[z].
\]

Thus \( \delta_0 \cong z^{-1} \), and the \( SL_2 \)-triple \([16]\) acts according to the formulas

\[
\tag{34}
\begin{align*}
h \delta_0 &= 2\delta_0, \\
e_+ \delta_0 &= 0,
\end{align*}
\]

with \( e_+ \) acting freely, by normal differentiation. The section \( \sigma_2 \) of \( \mathcal{L}_2 \) in \([16]\) is nonzero at the origin and is \( K \)-invariant, and this leads to a description of the sheaf \( M_{\{0\}, m} \), or equivalently, to its space of global sections \( M_{\{0\}, m} \),

\[
\tag{35}
M_{\{0\}, m} \text{ has basis } \{ (\frac{d^n}{dz^n} \delta_0) \sigma_2^{(m-1)/2} \mid n \geq 0 \};
\]
here $\sigma_2^{(m-1)/2}$ can be viewed as a section of $C_{m-1}$, a section that is regular and nonzero except at $\infty$. The $SL_2$-triple $e_+, h, e_-$ acts by the product rule, on $\frac{d^n}{dz^n} \delta_0$ according to (10) and (34), and on $\sigma_2^{(m-1)/2}$ according to (17). In particular,

$$
e_+ \left( \frac{d^n}{dz^n} \delta_0 \right) \sigma_2^{(m-1)/2} = - \left( \frac{d^{n+1}}{dz^{n+1}} \delta_0 \right) \sigma_2^{(m-1)/2},$$

$$h \left( \frac{d^n}{dz^n} \delta_0 \right) \sigma_2^{(m-1)/2} = (2n + m + 1) \left( \frac{d^{n+1}}{dz^{n+1}} \delta_0 \right) \sigma_2^{(m-1)/2},$$

$$e_- \left( \frac{d^n}{dz^n} \delta_0 \right) \sigma_2^{(m-1)/2} = n(n + 1) \left( \frac{d^{n-1}}{dz^{n-1}} \delta_0 \right) \sigma_2^{(m-1)/2},$$

(36)

as can be checked readily.

The inclusion $\{0\} \hookrightarrow C \cup \{\infty\}$ is a very special case of a closed embedding. In general the $D$-module direct image of an irreducible module under a closed embedding remains irreducible, so the weight filtration collapses. The effect of closed embeddings on the Hodge filtration also has a simple description: the Hodge index is increased by the order of normal derivative. In the case of $M_{\{0\},m}$ this means

$$W_0 M_{\{0\},m} = M_{\{0\},m},$$

(37)

$$F_p M_{\{0\},m} \text{ has basis } \{ \left( \frac{d^n}{dz^n} \delta_0 \right) \sigma_2^{(m-1)/2} \mid 0 \leq n \leq p - 1 \},$$

because the weight equals the dimension of the support, and the Hodge filtration starts at the codimension of the support.

The polarization pairs $\delta_0$ and $\overline{\delta_0}$ into $\delta_{R,0}$, the delta function in the usual sense on $C \cong \mathbb{R}^2$. It also pairs $\sigma_2^{(m-1)/2}$ and its complex conjugate into $\|\sigma_2^{(m-1)/2}\|^2 = \|\sigma_2\|^{m-1} = (1 + |z|^2)^{-m+1}$, as follows from (19). Thus

$$\left( \frac{d^n}{dz^n} \delta_0 \right) \sigma_2^{(m-1)/2}, \overline{\left( \frac{d^n}{dz^n} \delta_0 \right) \sigma_2^{(m-1)/2}} \right)_{u_{k \ell}} = \int_{C \cup \{\infty\}} \frac{d^k}{dz^k} \frac{d^\ell}{dz^\ell} \|\sigma_2^{(m-1)/2}\|^2 \delta_{R,0} \omega$$

(38)

$$= \int_{C \cup \{\infty\}} \frac{d^k}{dz^k} \frac{d^\ell}{dz^\ell} (1 + |z|^2)^{-m-1} \delta_{R,0} dz d\bar{z} = \frac{d^k}{dz^k} \frac{d^\ell}{dz^\ell} (1 + |z|^2)^{-m-1} \bigg|_{z=0}$$

vanishes unless $k = \ell$, in which case

$$\left( \frac{d^n}{dz^n} \delta_0 \right) \sigma_2^{(m-1)/2}, \overline{\left( \frac{d^n}{dz^n} \delta_0 \right) \sigma_2^{(m-1)/2}} \right)_{u_{k \ell}} = (-1)^k k! \prod_{j=1}^k (m + j).$$

(39)

That, of course, is consistent with our conjecture in this particular instance.

References

[1] Alexander Beilinson and Joseph Bernstein, *A proof of the Jantzen conjectures*, Advances in Soviet Math. 16 (1993), 1–50.

[2] Henryk Hecht, Dragan Miličić, Wilfried Schmid, and Joseph A. Wolf, *Localization and standard modules for real semisimple Lie groups. I. The duality theorem*, Invent. Math. 90 (1987), 297–332.

[3] Anthony W. Knapp, *Representation Theory of Semisimple Groups, An Overview Based on Examples*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, N.J, 2001. Reprint of the 1986 original.

[4] Wilfried Schmid and Kari Vilonen, *Hodge theory and unitary representations of reductive Lie groups*, Frontiers of mathematical sciences, Int. Press, Somerville, MA, 2011, pp. 397–420.

[5] David A. Vogan, *Signatures of hermitian forms and unitary representations. Slides of a talk at the Utah conference on Real Reductive Groups, 2009*, [http://www.math.utah.edu/realgroups/conference/conference-slides.html]
DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MA 02138
E-mail address: schmid@math.harvard.edu

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, IL 60208, also DEPARTMENT OF MATHEMATICS, HELSINKI UNIVERSITY, HELSINKI, FINLAND
E-mail address: vilonen@northwestern.edu, kari.vilonen@helsinki.fi