Planar matrices and arrays of Feynman diagrams: poles for higher $k$

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Abstract

Planar arrays of tree diagrams were introduced as a generalization of Feynman diagrams that enable the computation of biadjoint amplitudes $m_n^{(k)}$ for $k > 2$. In this follow-up work, we investigate the poles of $m_n^{(k)}$ from the perspective of such arrays. For general $k$, we characterize the underlying polytope as a Flag Complex and propose a computation of the amplitude-based solely on the knowledge of the poles, whose number is drastically less than the number of the full arrays. As an example, we first provide all the poles for the cases $(k, n) = (3, 7), (3, 8), (3, 9), (3, 10), (4, 8)$ and $(4, 9)$ in terms of their planar arrays of degenerate Feynman diagrams. We then implement simple compatibility criteria together with an addition operation between arrays and recover the full collections/arrays for such cases. Along the way, we implement hard and soft kinematical limits, which provide a map between the poles in kinematic space and their combinatoric arrays. We use the operation to give a proof of a previously conjectured combinatorial duality for arrays in $(k, n)$ and $(n - k, n)$. We also outline the relation to boundary maps of the hypersimplex $\Delta_{k,n}$ and rays in the tropical Grassmannian $\text{Tr}(k, n)$.

Supplementary material for this article is available online

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(Some figures may appear in colour only in the online journal)

1. Introduction

The Cachazo–He–Yuan (CHY) formulation provides a direct window into the scattering amplitudes of a wide range of Quantum Field Theories, by expressing them as a localized integral over the moduli space of punctures of $\mathbb{C}P^1$ [1–5]. Such formulation was generalized by Cachazo, Early, Mizera, and one of the authors (CEGM), who extended it to configuration spaces over $\mathbb{C}P^{k-1}$, or equivalently to the Grasmanian $\text{Gr}(k, n)$ modulo rescalings [6]. This unveiled a beautiful connection to tropical geometry, revealing that the CEGM amplitudes (for the generalized biadjoint scalar theory, $m_n^{(k)}$) can be computed either from a CHY formula or by more geometrical methods [6, 7]. In particular, the full amplitude $m_n^{(k)}(\|\|)$ can be obtained as the volume of the positive Tropical Grassmannian $\text{TrG}^+(k, n)$ viewed as a polyhedral fan. The CEGM amplitudes are also closed related to the cluster algebras [7–16], positroid subdivisions [17–19], and new stringy canonical forms [20–22]. The case $k = 4$ is also especially interesting due to its connection with the symbol alphabet of $\mathcal{N} = 4$ SYM [8, 23–25].

Based on the application of metric tree arrangements for parameterizing $\text{TrG}(3,n)$ [26], Borges and Cachazo introduced a diagrammatic description of the biadjoint amplitude $m_n^{(3)}(\|\|)$, as a sum over such arrangements instead of single
Feynman diagrams [27]. This was then extended to the case $k = 4$ by Cachazo, Gimenez, and the authors [28]: In this case, the building blocks are not collections but arrays (matrices) of planar Feynman diagrams. Each entry $M_{ij}$ of the matrix is a planar Feynman diagram concerning the canonical ordering $\{1, \ldots, n\} [i, j]$. We endow the diagram $M_{ij}$ with a tree metric $d_{ij}^{\text{tree}}$ which corresponds to the distance between leaves $k$ and $l$ according to the diagram $M^{ij}$ and require that it defines a completely symmetric tensor,

$$\tau_{ijkl} := d_{ij}^{\text{tree}} \text{permutation invariant}[i, j, k, l]. \quad (1.1)$$

The contribution to the biadjoint amplitude is obtained by defining the function

$$F(M) := \frac{1}{4!} \sum_{i,j,k,l} s_{ijkl} \tau_{ijkl}, \quad (1.2)$$

where $s_{ijkl}$ are generalized kinematic invariants [6], namely totally symmetric tensors satisfying generalized on-shell condition $s_{\mu \ldots} = 0$ and momentum conservation:

$$\sum_{j,k,l} s_{ijkl} = 0 \quad \forall i. \quad (1.3)$$

Using the Schwinger parametrization of the propagators, identifying Schwinger parameters to the independent internal lengths $f_i$ under the compatibility constraints (1.1), one can compute the contribution of an array $M$ to the amplitude via

$$R(M) = \int_{\Delta} d^{3(n-5)} f_i \times e^{F(M)}, \quad (1.4)$$

where $\Delta$ is the domain where all internal lengths are positive. See [27, 28] for more details. If we define by $J(\alpha)$ the set of all planar arrays for the ordering $\alpha$, the biadjoint amplitude for two orderings is then [6]

$$m^{(4)}(\alpha | \beta) = \sum_{M \in J(\alpha) \cap J(\beta)} R(M). \quad (1.5)$$

In this follow-up note, we introduce a new representation of the poles of this amplitude, for the most general case $\alpha = \beta = \mathbb{I}_n$, and also discuss the general $k$ setup. We then explain how the amplitude can be recovered from such poles when they are understood from planar arrays of Feynman diagrams. This new representation corresponds to collections/arrays of degenerate ordinary Feynman diagrams and can easily be translated to kinematic invariants in terms of $s_{\ldots}$. This way, we get the all the poles for $(k, n) = (3, 7), (3, 8), (3, 9), (4, 7), (4, 8)$, and $(4, 9)$, which agree with those obtained in [21] from the stringy canonical form construction. Conversely, as explained in section 3, we will also explain a method to get all the new presentations of poles as degenerate arrays by applying hard limits to kinematic invariants (in the sense defined in [28, 29]). We also provide a map for which any such array defines a ray satisfying tropical Plücker relations and hence lies inside the Tropical Grassmannian polytope $\text{TrG}(k, n)$.

In the supplementary data, we provide the explicit degenerate planar arrays for the cases $(k, n) = (3, 6), (3, 7), (3, 8), (3, 9), (3, 10), (4, 7), (4, 8)$, and $(4, 9)$. Those of $(3, 10)$ are obtained by translating the poles given in [21] but one in principle can also get $(3, 10)$ planar collections of Feynman diagrams first using the bootstrap method given in [28] and then degenerate them to get all poles in their new representations.

Given two degenerated planar rays $V_1, V_2$ as poles, we derive and prove criteria to check compatibility in terms of their Feynman diagram components. We show that this criterion can be translated to the weak separation condition studied in the mathematical literature, see e.g. [18, 30–32]. We then provide an operation of addition which is equivalent to the Minkowski sum of the corresponding tropical vectors. Using this operation we can reconstruct the full facets, e.g. the planar collections and arrays previously obtained in [27, 28]. More generally we can construct a graph of compatibility relations, where poles correspond to vertices and facets correspond to maximal cliques. This gives a realization of our polytope as a Flag Complex, as observed long ago for the original construction of $\text{TrG}(3, 6)$ [33]. We provide a Mathematica notebook as supplementary data to implement this algorithm for all $(k, n)$ belonging to $(3, 6) – (3, 10)$ and $(4, 7) – (4, 9)$.

This paper is organized as follows. In section 2 we construct poles as one-parameter arrays of degenerate Feynman diagrams and explain how to sum them to obtain higher dimensional objects, such as the full arrays of [28]. We focus on $k = 3, 4$. We then provide the compatibility criteria for general $k$ and give a Mathematica implementation of the compatibility graphs. In section 3 we present both a kinematic and combinatoric description of the soft and hard limits of the planar arrays of Feynman diagrams. We use it to construct the arrays corresponding to our poles and further prove the general implementation of Grassmannian duality conjectured in [28]. In the discussion, we outline future directions as well as a relation with the matroid subdivision of the hypersimplex $\Delta(k, n)$.

2. From full arrays to poles and back

Here we initiate the study of higher $k$ poles from the perspective of collections of tree diagrams. Recall that these correspond to vertices of the dual polytope (up to certain redundancies we will review), or to facets of the positive geometry associated with stringy canonical forms [20]. From the perspective of the collections, we shall find that the vertices are, in a precise sense, collections of $k = 2$ poles. The full collections of cubic diagrams, studied in [28], correspond to convex facets of the $(k, n)$ polytope and can be expressed as (Minkowski) sums of poles. We start the illustration with $k = 3$ and then the general cases.

2.1. $k = 3$ Planar collections, poles and compatibility criteria

An illustrative example is the case of the bipyramidal facet appearing in $(3, 6)$ [34]. This is a $k = 3$ example but the discussion will readily extend to arbitrary $k$. The description of the bipyramid in terms of a collection has been done in [27, 28]. From there, we recall
We denote each planar diagram in the collection vector by $T_{bip}^i$, i.e. $C_{bip} = \{T_{bip}^1 \ldots T_{bip}^6\}$. Note that $T_{bip}^i$ does not contain the $i$th label. We have labeled by $x, y, z, u, v, w > 0$ six internal distances which are independent solutions to compatibility conditions,

$$\pi_{jk} = d_{jk}^{(i)} = d_{ji}^{(i)},$$

imposed on the distances $d_{jk}^{(i)}$ from leaf $j$ to $k$ according to the graph $T_{bip}^i$.

A vertex is given by three or more simultaneous degenerations. By performing each of the two valid degenerations of the edge, that is $x \to 0$ or $w \to 0$, we arrive at the following vertices:

$$C_{T_{1234}} = \{\}

That this collection has the geometry of a bipyramid is seen as follows. We define its six faces by each of the allowed degeneration of metric tree arrangement. These correspond to the hyperplanes

$$Z_1: x = 0, \quad Z_2: y = 0, \quad Z_3: z = 0, \quad Z_4: w = 0, \quad Z_5: u \equiv y - z + w = 0, \quad Z_6: v \equiv x - z + w = 0,$$

in $\mathbb{R}^4$. We can project the planes into three dimensions by imposing an inhomogenous constraint, e.g. $x + y + z + w = 1$, leading to the picture of figure 1. The region where all the distances in (2.1) are strictly positive corresponds to the interior of the bipyramid. The notation for the vertices of the figure, e.g. $t_{134}, t_{3456}, t_{5612}, R, \tilde{R}$, will become natural in a moment. Two faces that meet at an edge correspond to two simultaneous degenerations. For instance the edge $\{R, t_{1234}\}$ is given by setting $y = 0$ and $z = 0$, leading to the two-parameter collection:

$$C_{R, t_{1234}} = \{\}

We say that two vertices are compatible if they are connected by an edge, which can be either in the boundary or in the interior of a facet. For readers familiar with the notion of tropical hyperplanes, an edge emerges when the sum of the two independent solutions of the tropical hyperplane equations (e.g. two different vertices in $TrG$) is also a solution, see [7].

Inspection of the facets of $(3, 6)$ show that they are all convex and hence any two vertices in a facet are compatible. In the context of planar arrays of Feynman diagrams, we will prove below that this is a general fact for all $(k, n)$. This is also motivated by the original work [33], where it was argued that the polytope $(3, 6)$ can be characterized as a Flag Complex, which we can introduce as follows:
Definition 2.1. The Flag Complex associated with a graph $G$ is a simplicial complex (a collection of simplices) such that each simplex is spanned by a maximal collection of pairwise compatible vertices in $G$.

As a graph, the bipyramid corresponds to a four-dimensional simplex in the sense that it is spanned by the five vertices $\{R, \bar{R}, t_{1234}, t_{3456}, t_{5612}\}$, which are pairwise compatible, i.e., connected by an edge. However, the edge $\{R, \bar{R}\}$ is contained inside the bipyramid, which is equivalent to the geometrical fact that the simplex degenerates from four to three dimensions, as we explain below. The three-dimensional object drawn in Figure 1 is what we interpret as a facet of the polytope. In general, any simplex of the Flag Complex associated with $(k, n)$ is indeed a geometrical facet, and can be embedded in $(k - 1)n - k - 1 - 1$ dimensions.

Following the guidelines from Tropical Geometry it is convenient to interpret each vertex as a ray in the space of metrics embedded in $\mathbb{R}^k$. Explicitly, for a compatible collection (2.2) and a vertex, $\pi_{ijk}$ depends on a single parameter $x > 0$, and indeed we can write it as a ray $\pi_{ijk} \sim x\pi_{ijk}$. The equivalence $\sim$ here means that we are modding out by the shift

$$\pi_{ijk} \sim \pi_{ijk} + w_i + w_j + w_k$$

for arbitrary $w$. This is a redundancy characteristic of tropical hyperplanes, e.g. [26]. An edge then corresponds to the Minkowski sum of two vertices. For instance, the edge $\{R, t_{1234}\}$ corresponds to a plane in $\mathbb{R}^k$ given by

$$\pi_{ijk}^{R,t_{1234}}(x, y) \sim w \pi_{ijk} + x \pi_{ijk}^{t_{1234}}, \quad w, x > 0.$$  

(2.7)

We can now justify our notation for the vertices $\{R, \bar{R}, t_{1234}, t_{3456}, t_{5612}\}$, namely argue for a correspondence between the vertices and kinematic poles. Indeed, the relation (2.7) can be expressed in terms of generalized $k = 3$ kinematic invariants. Under the support of $k = 3$ momentum conservation

$$\sum_{i,j} s_{ijk} = 0, \quad \forall i,$$  

(2.8)

the $\sim$-symbol turns into an equality:

$$\mathcal{F}(\{R, t_{1234}\}) := \frac{1}{6} \sum_{ijk} s_{ijk} \pi_{ijk}^{R,t_{1234}}(x, y)$$

$$= xR + y t_{1234},$$  

(2.9)

where we used the definitions

$$t_{1234} = s_{123} + s_{124} + s_{134} + s_{234},$$  

(2.10)

$$R = R_{t_{1234}, t_{3456}} = t_{1234} + s_{345} + s_{346}.$$  

(2.11)

Assuming that we know the collections (2.5), we can argue that (2.7) indeed defines an edge of the polytope, meaning it is associated with a compatible collection of Feynman diagrams

7 One can also understand this redundancy by realizing $\sum_{k=1}^{n} d_{ijk}(x) = 0$ on the support of $k = 3$ momentum conservation (2.8), which means the contribution of a planar collection of Feynman diagram to the amplitude like (2.9) keeps the same.

8 The ‘only if’ part is easily seen to follow from a known fact of $k = 2$ (applied to each of the components $T_i$): if the sum of two vertices $d_{ij}(x, y) \sim x V_i^x + y V_i^y$ corresponds to a Feynman diagram (i.e. to a tropical hyperplane or a line in $\text{TrG}(2, n)$) then the two vertices $V_i^x, V_i^y$ are compatible as $k = 2$ poles.

9 Now, an important property of $k = 2$ poles is the following: given a set of poles that are pairwise compatible, e.g. $\{s_{12}, s_{13}, s_{56}\}$, then such poles are compatible simultaneously, meaning that there exists a $k = 2$ Feynman diagram that includes them all. Considering this together with theorem 2.1, we can further state the following

Theorem 2.1. Two one-parameter collections $C^X$ and $C^Y$ are compatible if and only if their respective $k = 2$ components $T_i^X$ and $T_i^Y$ are compatible for all $i$. Now, a Feynman diagram with all three poles.

Figure 1. Bipyramid projected into three dimensions.
Corollary 2.1. A set of collections \( \{C^k\} \) which is pairwise compatible is also simultaneously compatible.

For collections, simultaneous compatibility means that the Minkowski sum of the corresponding vertices also leads to a collection. This is best explained with our example: using theorem 2.1 we can easily check that the vertices in \( \{R, \tilde{R}, t_{1234}, t_{3456}, t_{5612}\} \) are compatible in pairs. This then implies that the Minkowski sum

\[
\pi_{ijk}^{\text{bipyramid}}(\alpha') \sim (\alpha' + \alpha^2) V_{ijk}^R + (\alpha^3 + \alpha^2) V_{ijk}^{\tilde{R}} + (\alpha^4 + \alpha^2) V_{ijk}^{t_{1234}} + (\alpha^5 + \alpha^2) V_{ijk}^{t_{3456}} + (\alpha^6 + \alpha^2) V_{ijk}^{t_{5612}},
\]

will also be associated with an arrangement of Feynman diagrams. As we anticipated, this has the topology of a four-dimensional simplex (in a projective sense). However, it degenerates to three dimensions due to the identity

\[
V_{ijk}^R + V_{ijk}^{\tilde{R}} + V_{ijk}^{t_{1234}} + V_{ijk}^{t_{3456}} + V_{ijk}^{t_{5612}},
\]

or, in terms of generalized kinematic invariants,

\[
R + \tilde{R} = t_{1234} + t_{3456} + t_{5612}.
\]

This implies that the Minkowski sum (2.12) can be rewritten as

\[
\pi_{ijk}^{\text{bipyramid}}(\alpha') \sim (\alpha' + \alpha^2) V_{ijk}^R + (\alpha^3 + \alpha^2) V_{ijk}^{\tilde{R}} + (\alpha^4 + \alpha^2) V_{ijk}^{t_{1234}} + (\alpha^5 + \alpha^2) (V_{ijk}^{t_{3456}} - V_{ijk}^R),
\]

which indeed lives in three dimensions and spans the bipyramid facet. We can obtain a more familiar parametrization of the facet, as given in [27]. First, let us project again the Minkowski sum into kinematic invariants, i.e.

\[
\mathcal{F}(\mathcal{M}) = \sum_{i<j<k<l} s_{ijk} d_{ij}^{(ij)},
\]

where

\[
R_{1234567} = R_{12345} + R_{12346} + R_{12347} + R_{12356} + R_{12357} + R_{12367} + R_{12456} + R_{12457} + R_{12467} + R_{13456} + R_{13457} + R_{13467} + R_{14567} - R_{1234567},
\]

and

\[
\sum_{J \subseteq \{1,2,3,4,5\}} s_J = 1.
\]

The previous approach can be extended to the case \( k > 3 \). Planar arrays of Feynman diagrams for \( k = 4 \) and higher were defined in [28] as rank \( k - 2 \) objects, and involve the natural generalization of the compatibility condition (2.2).

Using the second bootstrap approach introduced there, one can obtain matrices of planar cubic Feynman diagrams for \( (k, n) = (4, 7) \) starting from collections of \( (k, n) = (3, 6) \). A few interesting features arise for poles of \( k = 4 \). Again, let us examine a particular planar array of \( (4, 7) \) as shown by the symmetric matrix in figure 3 to illustrate the construction of the poles. This array can be obtained from a \( (3, 7) \) collection via the duality procedure of [28], which we review in the next section. Recall that in this case, the metric compatibility conditions are given by (1.1) and the kinematic function reads

\[
\mathcal{F}(\mathcal{M}) = \sum_{i<j<k<l} s_{ijk} d_{ij}^{(ij)} + z(R_{1234567} - R_{12345} - R_{12346} - R_{12347} - R_{12356} - R_{12357} - R_{12367} - R_{12456} - R_{12457} - R_{12467} - R_{13456} - R_{13457} - R_{13467} - R_{14567} - R_{1234567}),
\]

where

\[
R_{1234567} = h_{12345} + s_{1237} + s_{1236},
\]

\[
R_{12345} = h_{12345},
\]

\[
\sum_{J \subseteq \{1,2,3,4,5\}} s_J = 1.
\]

hence such a column must be a collection, i.e. corresponds to \( k = 3 \). For \( (3, 6) \) such collections can only be bipyramids (as

\[
x + w - z = \alpha^1 + \alpha^3
\]

are also positive. These are nothing but the \( u, v > 0 \) conditions that we have started with and recovered our description of the collection \( C^{bipyramid} \), equation (2.1), whereas \( \mathcal{F}(\text{bipyramid}) \) agrees with that given in [27]. Our compatibility criteria thus allowed us to translate back a description in terms of the vertices (2.12) (a Minkowski sum) to a description in terms of the full collection of cubic diagrams.

2.2. Planar arrays and \( k > 3 \) poles

The previous approach can be extended to the case \( k > 3 \). Planar arrays of Feynman diagrams for \( k = 4 \) and higher were defined in [28] as rank \( k - 2 \) objects, and involve the natural generalization of the compatibility condition (2.2).

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\[
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\]

where

\[
R_{1234567} = h_{12345} + s_{1237} + s_{1236},
\]

\[
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\sum_{J \subseteq \{1,2,3,4,5\}} s_J = 1.
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\[
\mathcal{F}(\mathcal{M}) = \sum_{i<j<k<l} s_{ijk} d_{ij}^{(ij)} + z(R_{1234567} - R_{12345} - R_{12346} - R_{12347} - R_{12356} - R_{12357} - R_{12367} - R_{12456} - R_{12457} - R_{12467} - R_{13456} - R_{13457} - R_{13467} - R_{14567} - R_{1234567}),
\]

where

\[
R_{1234567} = h_{12345} + s_{1237} + s_{1236},
\]

\[
R_{12345} = h_{12345},
\]

\[
\sum_{J \subseteq \{1,2,3,4,5\}} s_J = 1.
\]
the example of the previous section) or simplices (if they have four boundaries) [33]. In fact, for our example, we can write the array $\mathcal{M}$ as a ‘collection of collections’ as shown in figure 4.

Each entry represents a $(3, 6)$ collection, namely a vector of cubic diagrams. They are labeled by kinematic poles as in the previous section, where collection $C^{(i)}$ contains labels $\{1, \ldots, 7\} \cap \{i\}$. Besides the standard degenerations (boundaries) of each $C^{(i)}$, we have depicted some internal boundaries in yellow. These arise from the external boundaries of another collection, say $C^{(j)}$, through the compatibility condition (1.1). For instance, translating the bipyramid boundaries (2.3) to the new variables we used in (2.18) we see that collection $C^{(5)}$ has the following six boundaries:

$$C^{(5)}: x = 0, y = 0, w = 0, q = 0, w + x - y = 0, q + x - y = 0,$$

(2.22)

which depend only on four variables $\{x, y, w, q\}$ instead of six, as expected for a bipyramid living in three dimensions. Now, we further consider the following (external) degenerations of collections $C^{(1)}$ and $C^{(3)}$,

$$C^{(1)}: -p + w - y + z = 0, \quad z = 0$$

$$C^{(3)}: p = 0,$$

which altogether induce the plane $w = y$ as a new degeneration, depicted by the internal yellow plane bisecting the bipyramid $C^{(5)}$ in (4). The intersection of this plane with its external faces $q = q + x - y = 0$ in (2.22) induces a new ray in (3, 6), obtained as the midpoint of the vertices $t_{1234}$ and $t_{6712}$, labelled as $t_{1234} + t_{6712}$ in (4). In the other collections $C^{(i)}$, the induced $(3, 6)$ rays can be further labeled in the same way and lead to the $(4, 7)$ ray we denote $W$:

$$\mathcal{M}^W = \{s_{345} + s_{567}, s_{345} + s_{567}, R_{67,45,12}, R_{12,67,35}, t_{6712}, t_{1234} + t_{6712}\},$$

(2.23)

while the kinematic function (2.18) becomes

$$\mathcal{F}(\mathcal{M}^W) = xW_{1234567} = x \left( \sum a (s_{a567} + s_{a345} + s_{3467}) \right).$$

(2.24)

In (2.23) the sum of vertices must be understood in the sense

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**Figure 3.** A particular planar array of $(4, 7)$. Note that the omitted labels in the Feynman diagram entries can be deduced by their planarity.
of the previous section. That is, we consider the line $xV_{abc}^{437} + yV_{abc}^{234}$, which belongs to the $k = 3$ polytope since $i_{0712}$ and $i_{1234}$ are compatible (in fact, they appear together in a bipyramid). The vertex $i_{0712}$ corresponds to its midpoint $x = y$. Addition of (collections of) Feynman diagrams is done as in figure 2, for instance:

\[ C_{i_{0712}}(x) = \{ \begin{array}{c} \begin{array}{c} x \\ 3 \\ 6 \\ 4 \\ 6 \\ 4 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \end{array} \end{array} \} \]

\[ C_{i_{1234}}(x) = \{ \begin{array}{c} \begin{array}{c} x \\ 3 \\ 6 \\ 4 \\ 6 \\ 4 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \end{array} \end{array} \} \]

\[ C_{i_{1234}}(x) + C_{i_{0712}}(x) = \{ \begin{array}{c} \begin{array}{c} x \\ 3 \\ 6 \\ 4 \\ 6 \\ 4 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \end{array} \end{array} \} \] (2.25)

which can also be written more compactly as

\[ C_{i_{0712}} = \{ s_{43}, s_{43}, 0, 0, s_{43}, s_{43} \}, \]

\[ C_{i_{1234}} = \{ s_{67}, s_{67}, s_{67}, s_{67}, s_{67}, 0, 0 \}, \]

\[ C_{i_{1234}} + C_{i_{0712}} = \{ s_{67} + s_{43}, s_{67} + s_{43}, s_{67}, s_{67}, s_{67}, s_{43}, s_{43} \}. \] (2.26)

(Note that $s_{43}$ and $s_{67}$ are compatible, thus their sum also belongs to the $k = 2$ polytope.) Hence each of the entries of (2.23) can be represented as a column, and $M^W$ can be written as a $7 \times 7$ matrix with a single internal distance parameter $x$. This is precisely our original array of cubic diagrams (3) after the degenerations have been imposed.

We have learned that the compatibility condition (1.1) can lead to particular boundary structures as in (2.23). Because of this the vertex $W$ of (4, 7) is not only decomposed in terms of vertices of (3, 6) but also certain internal rays (midpoints) of the (3, 6) polytope. It would be interesting to classify the kind of internal rays that can appear in this decomposition.

### 2.2.1. Compatibility criteria for general $k$

We now present the criteria for compatibility of poles, which easily extends from the case $k = 3$ in the previous part of this section to general $k$. A vertex of the $(k, n)$ polytope, realized as a ray in the space of metrics will be determined by a completely symmetric array of rank $k - 2$. From now on we will denote such array as $V_{i_{k-2}}^{k,n}$ e.g. a vector for $k = 3$ and a matrix for $k = 4$, where each component is a planar Feynman diagram.

Such an array can be organized by ‘columns’ we define by $T_{i_{k-2}}^{(i)} := V_{i_{k-2}}^{(i)}$, so we write:

\[ V_{i_{k-2}}^{(k,n)} = [T^{(1)}, \ldots, T^{(n)}] \] (2.27)

where $T^{(i)}$ are rays in $(k - 1, n - 1)$, which can also be written as arrays of Feynman diagrams. In the previous example this corresponds to our decomposition (2.23), where $V_{i_{k-2}}^{(k,n)} = M^W$ and $T^{(i)}$ are $k = 2$ rays.

In the $k = 2$ polytope two rays $d_j(x)$ and $d'_j(y)$ are compatible (their sum corresponds to a Feynman diagram) if and only if their respective kinematic poles are compatible.

For general $k$ this means that the components $d_j^{i_{k-2}}(x)$ and $d_j^{i_{k-2}}(y)$ are compatible if, and only if, the diagrams $V_{i_{k-2}}^{(k,n)}$ and $V_{i_{k-2}}^{(k,n)}$ are compatible in the $k = 2$ sense. Furthermore, two rays represented by arrays $V_{i_{k-2}}^{(k,n)}$ and $V_{i_{k-2}}^{(k,n)}$ are compatible when all their diagrams are. This implies the following extension of theorem 2.1

**Theorem 2.2.** Two vertex arrays $V_{i_{k-2}}^{(k,n)}$ and $V_{i_{k-2}}^{(k,n)}$ are compatible if and only if their components $T^{(i)}$ and $T^{(i)}$ are compatible as rays of $(k - 1, n - 1)$, for all $i = 1, \ldots, n$.

Assuming we know all possible poles, this provides an efficient criterion for checking their compatibility and, repeating the Minkowski sum construction of the previous section, constructing the facets of the polytope. This criterion has already appeared in the math literature in the context of boundary maps of the hypersimplex $\Delta_{k,n}$, e.g. [18, 30, 31], we will establish the precise equivalence in section 4.1.

**Corollary 2.2.** Two vertex arrays $V_{i_{k-2}}^{(k,n)}$ and $V_{i_{k-2}}^{(k,n)}$ are compatible if and only if their entries $T_{i_{k-2}}^{(i)}$ and $T_{i_{k-2}}^{(i)}$ are compatible for all $\{i_1, i_2, \ldots, i_{k-2}\} \subset \{1, 2, \ldots, n\}$. The proof is straightforward.

The full list of poles can be obtained by degenerations of just a few known facets of the polytope, given in our previous work [28], or more simply from singularities in the moduli space through the $\mathbb{C}P^{k-1}$ scattering equations of [34], as done in [21]. Once the full list of poles is known as a combination of kinematic invariants, we can easily translate them to arrays.
using the procedure described in the next section. In an attached notebook, we construct the full facets for $k = 3$ and $k = 4$ starting from the kinematic poles and provide a simple implementation of theorems 2.1 and 2.2 to check the compatibility relations of any pair of poles.

### 2.3. Summary of the results

We list the number of poles and the number of pairs of compatible poles for $(3, 6) - (3, 9)$ and for $(4, 7) - (4, 9)$ in tables 1 and 2 respectively. What is more, we recover all the planar arrays of Feynman diagrams using the compatibility rules, whose number is also summarized in the tables. Using the $(3, 10)$ poles given in [21], we also predict that the number of $(3, 10)$ planar collections of Feynman diagrams should be 11 187 660. We see the number of poles is much less than that of planar GFDs. The minimal number of poles for a $(k, n)$ planar array of Feynman diagrams is $(k-1)(n-k-1)$ and we list the maximal number in the tables. We see that a $(4, 9)$ planar matrix of Feynman diagrams could have 461 poles, much larger than the minimal number 12.

### 3. Soft/hard limits and duality

In [29], a kinematic hard limit has been introduced, based on the Grassmannian duality of the generalized amplitude $m^{(k)}_{ij}$ and its soft limit. Up to the momentum conservation ambiguities, we can define it as (take e.g. particle label 1)

$$s_{\ldots} \rightarrow \tau s_{\ldots}, \quad \text{with } \tau \rightarrow \infty.$$ (3.1)

In this section, we give a combinatoric description of such soft and hard kinematic limits in terms of the arrays. This provides a method for constructing the array of a pole only from the knowledge of the corresponding kinematic invariant.

Very nicely, this will also provide a geometric interpretation to the combinatorial soft and hard limits discussed in [28] for facets, e.g. full collections and arrays. At the same
time, it leads to proof of the combinatorial duality proposed there for such facets. Let us use \( k = 4 \) as an example to introduce the connection between kinematics and combinatorics. A pole is described in terms of generalized kinematic invariants by the function

\[
\mathcal{F}(V) = \frac{1}{4!} \sum_{i,j,l,m} s_{ijkl} \delta_{ij}^{(m)}(x) = \frac{1}{4!} \sum_{i,j,l,m} s_{ijkl} V_{ijkl},
\]

(3.2)

Let us first consider the soft limit on label 1, i.e. \( s_1 \rightarrow \tau s_1 \), extracting the leading order as \( \tau \rightarrow 0 \). The pole then becomes

\[
\mathcal{F}(V) \rightarrow \mathcal{F}^{\text{soft}}(V) = \frac{1}{4!} \sum_{i,j,l,m} s_{ijkl} \tilde{\delta}_{ij}^{(m)}(x) = \mathcal{F}(\tilde{V})
\]

(3.3)

where \( s_{ijkl} \) with \( i, j, l, m \neq 1 \) correspond to hard kinematic invariants, they satisfy generalized momentum conservation (1.3) for \( n - 1 \) labels. Also, \( \delta_{ij}^{(m)}(x) \) corresponds to the metric of \( V^{\text{im}}_1 \). The latter is obtained by removing column 1 and row 1 in the matrix of Feynman diagrams \( V^{\text{im}} \). Furthermore, the restriction \( i, j, l, m \) in \( \delta_{ij}^{(m)}(x) \) means particle \( 1 \) can be removed in the Feynman diagrams of \( V^{\text{im}}_1 \). The object \( \tilde{V}^{\text{im}}_1 \) is also a one-parameter array for \( k = 4 \), i.e. corresponds to \((4, n - 1)\). It is precisely the combinatorial soft limit of \( V^{\text{im}} \) in the sense of [28].

The hard limit proceeds in a similar fashion. On label 1, take \( s_1 \rightarrow \tau s_1 \), and extract the leading terms as \( \tau \rightarrow \infty \). The result can be written as

\[
\mathcal{F}(V) \rightarrow \mathcal{F}^{\text{hard}}(V) = \frac{1}{3!} \sum_{j,k,l} \tilde{s}_{jkl} \tilde{d}_{ij}^{(l)}(x) = \mathcal{F}(T^{(1)}),
\]

(3.4)

where \( d_{ij}^{(l)} = d_{ij}^{(1)}(x) \) and \( \tilde{s}_{jkl} = s_{jkl} \) are now interpreted as \( k = 3 \) kinematic invariants since they satisfy \( \sum_{kl} \tilde{s}_{jkl} = 0 \). Thus, the kinematic hard limit of the collection \( V^{\text{h}} \) can be obtained from its column \( T^{(1)} \), and corresponds to the \( k = 3 \) collection defined as \( \mathcal{C}^{(1)} := \{T^{(1)}y\} = \mathcal{V}^{\text{h}} \). As discussed, \( T^{(1)} \) gives an array of \((3, n - 1)\), and indeed turns out to be the combinatorial hard limit of \( V^{\text{h}} \) in the sense of [28].

This has a direct application for constructing an array given certain kinematics. Take for instance the pole \( W \) in (4, 7), equation (2.24)

\[
W_{1234567} = \sum_{a} (s_{34567} + s_{345} + s_{3467}).
\]

(3.5)

Under the hard limit in e.g. particle 7

\[
W_{1234567} \rightarrow \sum_{a} \delta_{56} + \delta_{345} + \delta_{346} = R_{12,34,56}.
\]

(3.6)

Thus, from (3.4), \( R_{12,34,56} \) corresponds to the valuation of the column \( T^{(1)} \) in the array of \( W \), i.e. \( \mathcal{F}(T^{(1)}) \), and can be used interchangeably. Indeed, by applying the hard limits in all the labels one recovers the array

\[
W_{1234567} \rightarrow \{s_{345} + s_{346}, s_{345} + s_{346}, R_{67,45,12}, R_{26,67,35}, t_{6712} + t_{1234}, R_{12,34,57}, R_{12,34,56}\}
\]

(3.7)

which justifies the notation (2.23). Of course, each of the elements here is indeed a column, which can be constructed by applying yet another hard limit, e.g.

\[
t_{1234} + t_{6712} \rightarrow \{s_{67} + s_{43}, s_{67} + s_{43}, s_{67}, s_{67}, s_{43}, s_{43}\}
\]

(3.8)

which is (2.26). Thus, the matrix associated with \( W_{1234567} \) is obtained by a consecutive double hard limit. The fact that the matrix \( \mathcal{M}^{(a)l}_{ij} \) obtained so is symmetric corresponds to the statement that the two hard limits commute.

For a general vertex of the \((k, n)\) polytope, of the form (2.27), we have

\[
\mathcal{F}^{\text{soft}}(V) = \mathcal{F}(\tilde{V})
\]

\[
\mathcal{F}^{\text{hard}}(V) = \mathcal{F}(T^{(1)})
\]

(3.9)

where \( T^{(1)} \) is a vertex of \((k - 1, n - 1)\) and \( \tilde{V} \) is a vertex of \((k, n - 1)\):

\[
\tilde{V} := [T^{(2)}, \ldots, T^{(n)}]_{\text{removed}}.
\]

(3.10)

This provides a kinematic interpretation of the combinatorial hard \((k\text{-reducing})\) and soft \((k\text{-preserving})\) operations for general \( k, n \). Also, given \( \mathcal{F}(V) \), the array \( V_{6, \ldots, k - 2} \) can be constructed by applying \( k - 2 \) consecutive hard limits in labels \( i_1, \ldots, i_{k - 2} \) and identifying the resulting \( k = 2 \) Mandelstam with a Feynman diagram as in (2.25) and (2.26).

3.1. Duality

It is known that the \((k, n)\) polytope admits a dual description as a \((n - k, n)\) polytope, induced by Grassmannian duality \( G(k, n) \sim G(n - k, n) \). In the moduli space, this identification was shown to imply the relation \( m_n^{(k)} = m_n^{(n-k)} \) [34, 29]. In the context of the polytope (i.e. kinematic space), the identification is true for facets, edges, vertices, etc. Indeed, a duality for planar arrays of Feynman diagrams was conjectured in [28] and relates arrays in \((k, n)\) and \((n - k, n)\). It is such that

\[
\mathcal{F}(\mathcal{M}^{(k,n)}) = \mathcal{F}(\mathcal{M}^{(n-k,n)})
\]

(3.11)

under appropriate relabeling. Furthermore, both collections \( \mathcal{M}^{(k,n)} \) and \( \mathcal{M}^{(n-k,n)} \) have the same boundaries arising as degenerations and hence lead to the same contribution to the biadjoint amplitude \( m_n^{(k)} \).

We now describe and provide a proof of the duality. Let us first consider the case of vertices and then promote it to facets via the sum procedure of the previous section. Two
vertices in \((k, n)\) and \((n - k, n)\) are defined as duals when they satisfy
\[
\mathcal{F}(\mathcal{V}^{(k,n)}) = \mathcal{F}(\mathcal{V}^{*(n-k,n)}),
\] (3.12)
i.e. they are kinematically the same.

Of course, the previous definition requires relabeling the kinematic invariants. Focusing on the case \((4, 7) \sim (3, 7)\) to illustrate this, the relabeling is \(s_{abcd} \sim s_{efg}\) where \([a, b, c, d, e, f, g] = \{1, \ldots, 7\}\). Suppose now \(a = 1\). The hard limit in label 1 of \(s_{abcd}\)
\[
s_{abcd} \rightarrow \hat{s}_{bcd},
\] (3.13)
while under the soft limit
\[
s_{efg} \rightarrow s_{efg}.
\] (3.14)
But \([b, c, d, e, f, g] = \{2, \ldots, 7\}\) and hence \(\hat{s}_{bcd} \sim s_{efg}\) under the duality \((3, 6) \sim (3, 6)\). This can be repeated while replacing \(s_{abcd}\) by any linear combination of kinematic invariants, in particular by the one given by \(\mathcal{F}(\mathcal{V})\). The conclusion can be nicely depicted by the diagram:
\[
\begin{array}{c}
\mathcal{F}(\mathcal{V}^{(k,n)}) \\
\text{(relabel)}
\end{array}
\begin{array}{c}
\text{hard} \\
\mathcal{F}_1^{\text{hard}}(\mathcal{V}^{(k,n)}) \\
\text{(relabel)} \\
\text{soft}
\end{array}
\begin{array}{c}
\mathcal{F}(\mathcal{V}^{*(n-k,n)}) \\
\mathcal{F}_1^{\text{soft}}(\mathcal{V}^{*(n-k,n)})
\end{array}
\] (3.15)
But from (3.9) we conclude that \(\mathcal{F}(T^{(i)}) \sim \mathcal{F}(\mathcal{V}_i^*)\). That is, the \((k - 1, n - 1)\) array \(T^{(i)}\), which is the first column of \(\mathcal{V}_i\), is dual to the \((n - k, n - 1)\) array \(\mathcal{V}_i^*\), which corresponds to \(\mathcal{V}^*_i\) with the first components removed. Repeating the steps for all other labels proves the following:

**Theorem 3.1.** Let \(\mathcal{V}^{*(n-k,n)}\) be the dual ray to \(\mathcal{V}^{(k,n)} = [T^{(1)}, \ldots, T^{(n)}]\), that is \(\mathcal{F}(\mathcal{V}) = \mathcal{F}(\mathcal{V}^*)\) under appropriate relabelings. Then the hard limit \(T^{(i)}\) is dual to the soft limit \(\mathcal{V}_i^*\) for all \(i = 1, \ldots, n\).

Since soft and hard limits always reduce the number of labels \(n\), this theorem can be iterated to check whether two given rays are duals.

This criterion was conjectured for facets in [28]. To prove the criteria for two dual facets, say \(\mathcal{M}^{(k,n)}\) and \(\mathcal{M}^{*(n-k,n)}\), we resort to the construction in section 2. According to the characterization 2.1 of the polytope, facets or full collections are maximal sums of poles. Using the notation of equation (2.12) for the space of compatible metrics for an array, we can write
\[
\mathcal{V}_i^* = \sum_l \alpha_l^i \mathcal{V}_i^*, \quad \alpha_l^i > 0,
\] (3.16)
or simply, using the addition of compatible Feynman diagrams
\[
\mathcal{M}^{\text{h}_l, \ldots, -h_{l-1}}_i = \sum_l \alpha_l^i \mathcal{V}_l^{\text{h}_l, \ldots, -h_{l-1}}_i, \quad \alpha_l^i > 0.
\] (3.17)
Now let us define the following object:
\[
\mathcal{M}^{\text{h}_l, \ldots, -h_{l-1}}_{i-l} = \sum_l \alpha_l^i \mathcal{V}_l^{\text{h}_l, \ldots, -h_{l-1}}, \quad \alpha_l^i > 0.
\] (3.18)

The duality relation (3.11) follows from (3.12) together with the linearity of the map \(\mathcal{F}\). Of course, this definition requires that the vertices \(\mathcal{V}^*_l\) can be added. We now prove

**Theorem 3.2.** The set \(\{\mathcal{V}^*_l\}\) is a maximally compatible collection of vertices (a clique) of \((n - k, n)\). Hence \(\mathcal{M}^*\) is a facet, the dual facet of \(\mathcal{M}\).

**Proof.** It suffices to show that if two vertices, say \(\mathcal{V}\) and \(\mathcal{W}\), are compatible, so are their duals \(\mathcal{V}^*\) and \(\mathcal{W}^*\). This follows from induction in \(n\) (the case \(n = 4\) being trivial): If \(\mathcal{V}\) and \(\mathcal{W}\) are compatible in \((k, n)\), it is easy to see that their combinatorial soft limits \(\hat{\mathcal{V}}\) and \(\hat{\mathcal{W}}\) are compatible in \((k, n - 1)\), for all \(i\). Then, using the induction hypothesis we find that their duals, \((\hat{\mathcal{V}})^*_i\) and \((\hat{\mathcal{W}})^*_i\) are compatible. From theorem 3.1 these are the hard limits of \(\mathcal{V}^*_i\) and \(\mathcal{W}^*_i\) for all \(i\). It then follows from theorem 2.2 that \(\mathcal{V}^*_i\) and \(\mathcal{W}^*_i\) are themselves compatible as \((k - n, n)\) arrays, which completes the induction.

Having successfully characterized the dual facet of \(\mathcal{M}\) by equation (3.18) we can now prove the extension of theorem 3.1 for facets. For this, let us simply denote by \(\hat{\mathcal{M}}_i\) the combinatorial soft limit of \(\mathcal{M}\) in particle \(i\), which indeed defines a facet of the \((k, n - 1)\) polytope. From (3.17), it is easy to see that the soft limit is
\[
\hat{\mathcal{M}}_i = \sum_l \alpha_l^i \mathcal{V}_i^*.
\] (3.19)
On the other hand the combinatorial hard limit of \(\mathcal{M}^*\) (given by (3.18)) in particle \(i\) is
\[
T^{(i)}_{\mathcal{M}^*} = \sum_l \alpha_l^i T^{(i)}_{\mathcal{M}^*},
\] (3.20)
where \(\mathcal{M}^* = [T^{(i)}_{\mathcal{M}^*}, \ldots, T^{(n)}_{\mathcal{M}^*}]\) etc. As each \(\mathcal{V}_i^*\) is dual to \(T^{(i)}_{\mathcal{M}^*}\), we conclude that \(T^{(i)}_{\mathcal{M}^*}\) is the dual facet to \(\hat{\mathcal{M}}_i\). This proves the duality criteria for facets, first proposed in [28].

**4. Discussion**

In [28], a combinatorial bootstrap was introduced for obtaining the collections corresponding to facets of \((k, n)\) with \(k > 4\). In a nutshell, for \(k = 4\) one writes a candidate symmetric array of Feynman diagrams using as a set of columns the facets of \((3, n - 1)\), i.e.
\[
\mathcal{M} = \{C_1, \ldots, C_n\}.
\] (4.1)
Then, one checks whether the corresponding metric \(d_{ijkl}^{(2)}\) can be imposed to be symmetric, in which case a facet of \((4, n)\) is found. In this work, we have explored the representation (4.1) for vertices \(\mathcal{V}\) of \((k, n)\). However, in section 2.2 we have discovered that in this case the columns \(T^{(i)}\) (which play the role of the \(C\) in (4.1)) are not necessarily vertices of...
(k − 1, n − 1) but rather certain internal rays in the polytope. It would be interesting to classify which kind of internal rays can appear, as a way of implementing the combinatorial bootstrap more efficiently at the level of vertices. Interestingly, for all the examples explored in this work, we were able to check a weaker version: a compatible collection \( \{ T^{(1)}_V, \ldots, T^{(n)}_V \} \) for which all the \( T^{(k)}_V \) are vertices in \((k − 1, n − 1)\) is indeed a vertex of \((k, n)\). For instance, for \((4, 8)\) we obtain 98 poles given by all possible compatible collections of poles of \((3, 7)\) (also including the trivial column).

Despite these subtleties, we found there is a universal way to translate any poles in terms of generalized kinematics as degenerated arrays and vice versa, check their compatibility relations, and reconstruct the full planar arrays of Feynman diagrams for any \( k \geq 3 \).

This research opens up a myriad of intriguing mathematical avenues for further investigation. Primarily, our exploration in this paper delves into the poles of the partial amplitude \( m^{(\hat{n})}_{\hat{I}}[\hat{I}] \) in alignment with the concept of global planarity. Notably, in [35, 36], there is the introduction of diverse forms of partial amplitudes grounded in the idea of local planarity. This encompasses the usage of generalized color orderings to elaborate on the color-dressed generalized biadjoint amplitudes. An exciting prospect would be the application of our current methodology to probe into these broader partial amplitudes, emphasizing their poles manifested as degenerate arrays and their ensuing compatibility criteria.

Since the initial unveiling of CEGM amplitudes in [6], the field has witnessed considerable advancements. Prominent among these are the exploration of generalized soft theorems [29, 37], the discernment of the semi-locally triple splitting amplitudes [38], and the unveiling of certain factorization channels [39, 40]. Furthermore, comprehensive characterizations of the scattering equations have been presented in [34, 41–45] within the original CEGM integrals in configuration spaces over \( \mathbb{CP}^{n−1} \). Delving into the intricate relationships between the poles and these inherent properties of the amplitudes promises to be a captivating avenue of research.

Our findings also suggest potential intersections with positroid subdivisions [17–19], intricate combinatorial structures that provide a richer context to the dynamics of the amplitudes. Let us end up this paper by a brief discussion on the relation between the higher \( k \) poles to boundary map from the \((k, n)\) hypersimplex. This offers an intriguing perspective that may act as a stepping stone for subsequent inquiries in this domain.

### 4.1. Relation to boundary map from the \((k, n)\) hypersimplex

The hard limit we have introduced here can be understood as a map \((k, n) \rightarrow (k − 1, n − 1)\) for any ray in \( \text{TrG}(k, n) \). Early introduced a basis of planar poles, corresponding to certain matroid subdivisions of the hypersimplex \( \Delta(k, n) \) [18, 19, 32]. Then, there is a natural boundary restriction \( \partial^{(1)} \Delta_{k,n} = \Delta_{k−1,n−1} \) that can be applied to characterize such matroid subdivisions. When acting on Early’s planar basis, we will now argue that the boundary restriction agrees with our hard kinematical limit. Since the action of the boundary restriction on a generic pole is the linear extension of the action on the basis, this means that the hard limit effectively implements the boundary map in general.

We construct the planar basis as follows: consider the hypersimplex \( \Delta_{k,n} \) defined by

\[
x_1 + \ldots + x_n = k, \quad x_j \in [0, 1].
\]

Let \( I \subset \{1, \ldots, n\} \) be a subset of \( k \) elements, \( |I| = k \). The vertex \( e_I \in \mathbb{R}^n \) of the hypersimplex corresponds to \( x_i = 1 \) for \( i \in I \), with all the other \( x_j = 0 \). The boundary \( \partial^{(1)} \Delta_{k,n} \) is obtained by setting \( x_i = \hat{1} \), i.e. it becomes the hypersimplex \( \Delta_{k−1,n−1} \) given by \( \sum_j x_j = k − 1 \). Note that this decreases the value of \( k \) and \( n \) exactly as the hard limit introduced in section 3. Focusing on the boundary \( x_i = 1 \), let us denote its vertices by \( e^{(i)} = e_i \) where \( I \subset \{2, \ldots, n\} \) with \(|I| = k − 1 \).

Positroid subdivisions can be obtained from the level function \( h(x) : \Delta_{k,n} \rightarrow \mathbb{R} \) studied in [18, 19]. This is piece-wise linear in the hypersimplex and its curvature is localized on certain hyperplanes defining the subdivision. Early’s basis is in correspondence with the subdivisions arising in the set of functions

\[
h(x − e), \quad J \text{ non-consecutive}.
\]

Since there are \( n \) consecutive subsets \( I \), the number of such functions is \( \binom{n}{k} = n \), namely the number of independent kinematics for \((k, n)\). To obtain the explicit planar basis in kinematic space we take the linear combination (up to an overall normalization)

\[
\eta_I = \sum_{|I| = k} h(e_I − e_j)s_j.
\]

(One can check that by virtue of momentum conservation \( \eta_I = 0 \) for a consecutive subset \( J [18] \)). Now we show that the hard limit as constructed in section 3 has precisely the same effect as the boundary map restriction \( \partial^{(1)} \) of the subdivision \( h(x − e) \). Taking the hard limit in label 1 we obtain:

\[
\eta_I \rightarrow \sum_{|I| = k−1} h(e_{Ii} − e_j)s_{ij} = \sum_{|I| = k−1} h(e^{(i)} − e_j)s_{ij}.
\]

Recall that the hard limits \( s_{ij} \) are interpreted here as the kinematic invariants for \((k − 1, n − 1)\), i.e. a system that does not include particle label 1. From (4.5) we conclude that the function \( h(x − e) \) that defines de subdivision is restricted to the domain \( x \in \partial^{(1)} \Delta_{k,n} \) after the hard limit is taken. Hence the boundary restriction of the subdivision is equivalent to the hard limit on the basis \( \eta_I \), as we wanted to show.

---

12 In this notation a set of independent kinematic invariants (which is nonplanar, i.e. does not entirely correspond to the poles of \( m^{(\hat{n})}_{\hat{I}}[\hat{I}] \)), is generated by \( s_j \) with \(|I| = k \). Recall that we further have \( n \) momentum conservation constraints, in this notation \( \sum_{j = 1}^{n} s_{ij} = 0 \) for all \( j = 1, \ldots, n \).
Very nicely, the hard limit of $\eta_j$ also gives an element of the planar basis $\hat{\eta}_j$ for $(k - 1, n - 1)$. For instance, if $J$ is of the form $J = (1J)$, it clear from (4.5) that we obtain $\eta_{j} \rightarrow \hat{\eta}_j$. A similar analysis can be done for general $J$ and recovers the general rule given in [18] for the boundary map.

A future direction is to further elucidate the relation between the compatibility formulation using Steinman relations/Weak Separation [18, 19] and the compatibility criteria implemented here in the context of planar arrays of degenerate Feynman diagrams.

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References

[1] Fairlie D and Roberts D 1972 Durham preprint (PRINT-72-24401972) Dual models without tachyons—a new approach https://inspirehep.net/literature/1116351
[2] Fairlie D B 2009 A coding of real null four-momenta into world-sheet coordinates Adv. Math. Phys. 2009 284689
[3] Cachazo F, He S and Yuan E Y 2014 Scattering equations and Kawai–Lewellen–Tye orthogonality Phys. Rev. D 90 065001
[4] Cachazo F, He S and Yuan E Y 2014 Scattering of massless particles in arbitrary dimensions Phys. Rev. Lett. 113 171601
[5] Dolan L and Goddard P 2014 Proof of the formula of Cachazo, He and Yuan for Yang–Mills tree amplitudes in arbitrary dimension J. High Energy Phys. JHEP05(2014)010
[6] Cachazo F, Early N, Guevara A and Mizera S 2019 Scattering equations: from projective spaces to tropical Grassmannians J. High Energy Phys. JHEP06(2019)039
[7] Drummond J, Foster J, Gürdogan O and Kalousios C 2020 Tropical Grassmannians, cluster algebras and scattering amplitudes J. High Energy Phys. JHEP04(2020)146
[8] Henke N and Papathanasiou G 2020 How tropical are seven- and eight-particle amplitudes? J. High Energy Phys. JHEP08(2020)005
[9] Drummond J, Foster J, Gürdogan O and Kalousios C 2021 Tropical fans, scattering equations and amplitudes J. High Energy Phys. JHEP11(2021)071
[10] Drummond J, Foster J, Gürdogan O and Kalousios C 2021 Algebraic singularities of scattering amplitudes from tropical geometry J. High Energy Phys. JHEP04(2021)002
[11] Speyer D and Williams L K 2003 The tropical totally positive Grassmannian arXiv:math/0312297
[12] Arkani-Hamed N, He S and Lam T 2021 Cluster configuration spaces of finite type SIGMA 17 092
[13] Arkani-Hamed N, He S, Lam T and Thomas H 2023 Binary geometries, generalized particles and strings, and cluster algebras Phys. Rev. D 107 066015
[14] Gates S J, Hazel Mak S N, Spradlin M and Volovich A 2021 Cluster superalgebras and stringy integrals arXiv:2111.08186
[15] Henke N and Papathanasiou G 2021 Singularities of eight- and nine-particle amplitudes from cluster algebras and tropical geometry J. High Energy Phys. JHEP10(2021)007
[16] Arkani-Hamed N, He S, Salvatori G and Thomas H 2022 Causal diamonds, cluster polytopes and scattering amplitudes J. High Energy Phys. JHEP11(2022)049
[17] Lukowski T, Parisi M and Williams L K 2002 The positive tropical Grassmannian, the hypersimplex, and the $m = 2$ amplituhedron arXiv:2002.06164
[18] Early N 2020 Planar kinematic invariants, matroid subdivisions and generalized Feynman diagrams arXiv:1912.13513
[19] Early N 2019 From weakly separated collections to matroid subdivisions arXiv:1910.11522
[20] Arkani-Hamed N, He S and Lam T 2021 Stringy canonical forms J. High Energy Phys. JHEP02(2021)069
[21] He S, Ren L and Zhang Y 2020 Notes on polytopes, amplitudes and boundary configurations for Grassmannian string integrals J. High Energy Phys. JHEP04(2020)140
[22] He S, Li Z, Raman P and Zhang C 2020 Stringy canonical forms and binary geometries from associahedra, cyclohedra and generalized permutohedra J. High Energy Phys. JHEP10(2020)054
[23] Arkani-Hamed N, Lam T and Spradlin M 2021 Non-perturbative geometries for planar $N=4$ SYM amplitudes J. High Energy Phys. JHEP03(2021)065
[24] Ren L, Spradlin M and Volovich A 2021 Symbol alphabets from tensor diagrams J. High Energy Phys. JHEP12(2021)079
[25] He S, Li Z and Yang Q 2021 Truncated cluster algebras and Feynman integrals with algebraic letters J. High Energy Phys. JHEP12(2021)110
[26] He S, Li Z and Yang Q 2022 J. High Energy Phys. JHEP05(2022)075 (https://doi.org/10.1007/JHEP05(2022)075)
[27] Herrmann S, Jensen A, Joswig M and Sturmfels B 2009 How to draw tropical planes Electron. J. Combinatorics 16 6
[28] Borges F and Cachazo F 2020 Generalized planar feynman diagrams; collections J. High Energy Phys. JHEP11(2020)164
[29] Cachazo F, Guevara A, Umbert B and Zhang Y 2019 Planar matrices and arrays of feynman diagrams arXiv:1912.09422
[30] García-Sepúlveda D and Guevara A 2019 A soft theorem for the tropical Grassmannian arXiv:1909.05291
[31] Speyer D and Williams L K 2020 The positive dresssian equals the positive tropical Grassmannian arXiv:2003.10231
[32] Arkani-Hamed N, Lam T and Spradlin M 2021 Positive configuration space Commun. Math. Phys. 384 909–54
[33] Early N 2020 Weighted blade arrangements and the positive tropical Grassmannian arXiv:2005.12305
[34] Speyer D and Sturmfels B 2004 The tropical Grassmannian Adv. Geom. 4 389–411
[35] Cachazo F and Rojas J M 2020 Notes on biadjoint amplitudes, Trop G(3, 7) and X(3, 7) scattering equations J. High Energy Phys. JHEP04(2020)176
[36] Cachazo F, Early N and Zhang Y 2022 Color-dressed generalized biadjoint scalar amplitudes: local planarity arXiv:2212.11243
[37] Cachazo F, Early N and Zhang Y 2023 Generalized color orderings: CEGM integrands and decoupling identities arXiv:2304.07351
[38] Abhishek M, Hegde S, Jatkar D P and Saha A P 2021 Double soft theorem for generalized biadjoint scalar amplitudes SciPost Phys. 10 036
[38] Cachazo F, Early N and Giménez Umbert B 2022 Smoothly splitting amplitudes and semi-locality J. High Energy Phys. JHEP08(2022)252

[39] Early N 2022 Factorization for generalized biadjoint scalar amplitudes via matroid subdivisions arXiv:2211.16623

[40] Cachazo F and Early N 2022 Biadjoint scalars and associahedra from residues of generalized amplitudes arXiv:2204.01743

[41] Cachazo F, Umbert B and Zhang Y 2020 Singular solutions in soft limits J. High Energy Phys. JHEP05(2020)148

[42] Agostini D, Brysiewicz T, Fevola C, Kühne L, Sturmfels B, Telen S and Lam T 2023 Likelihood degenerations Adv. Math. 414 108863

[43] Sturmfels B and Telen S 2012 Likelihood equations and scattering amplitudes arXiv:2012.05041

[44] Cachazo F and Early N 2021 Minimal kinematics: an all k and n peek into Trop^+ G(k, n) SIGMA 17 078

[45] Cachazo F and Early N 2010 Planar kinematics: cyclic fixed points, mirror superpotential, k-dimensional catalan numbers, and root polytopes arXiv:2010.09708