Zero Mode Problem of Liouville Field Theory

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Abstract

We quantise canonical free-field zero modes \( p, q \) on a half-plane \( p > 0 \) both, for the Liouville field theory and its reduced Liouville particle dynamics. We describe the particle dynamics in detail, calculate one-point functions of particle vertex operators, deduce their zero mode realisation on the half-plane, and prove that the particle vertex operators act self-adjointly on a Hilbert space \( L^2(\mathbb{R}_+) \) on account of symmetries generated by the \( S \)-matrix. Similarly, self-adjointness of the corresponding vertex operator of Liouville field theory in the zero mode sector is obtained by applying the Liouville reflection amplitude, which is derived by the operator method.

\textit{Keywords:} Conformal field theory; Liouville theory; Hamiltonian reduction; Liouville particle dynamics; Zero modes; Half-plane quantisation;
\textit{PACS:} 11.10. EF; 11.10. Kk; 11.10. Lm; 11.25. Hf

1 Introduction

Although Liouville field theory was intensively studied in different areas of mathematics and physics, its quantum mechanical picture remained incomplete with respect to some fundamental questions. But the rather simple form of the general solution \cite{1}, its free field representations \cite{2-6} and several already obtained exact quantum results \cite{4-7} are promising indications for a complete description. The unsolved problem we are referring to is the quantisation of the canonical free-field zero modes \( p, q \) on a half-plane \( p > 0 \) \cite{4,7} which describe the vacuum of regular periodic Liouville field theory. This half-plane problem is a consequence of the Liouville dynamics.

In this paper, we quantise these canonical zero modes, and demonstrate, in particular, the self-adjoint action of vertex operators on a Hilbert space \( L^2(\mathbb{R}_+) \) by taking into consideration symmetries generated by an \( S \)-matrix. We have in view to complete our previous work on periodic Liouville theory \cite{7} in order to be able to calculate correlation functions.
Moreover, solving this zero mode problem would allow us, as well, to develop further the quantisation of the $SL(2,\mathbb{R})/U(1)$ black hole model treated in ref. [8]. In fact such investigations are fundamental for the quantisation of gauged WZNW theories. Our calculations are based on an exact canonical operator formalism in a Hilbert space. However, we do not follow here our previous suggestion [7] to use a coherent state formalism which considers translation and dilatation symmetry of the half-plane.

We investigate the zero modes by using a canonical map from a plane to a half-plane. Such a map naturally arises for particle dynamics in an exponential potential by passing from the particle phase-space coordinates to their asymptotic (in- or out-) variables. The particle approach seems to be useful because Liouville field theory can be reduced to a mechanical model, and much is already known about Liouville particle dynamics [3], [9]-[11]. But the quantisation on the half-plane was not sufficiently discussed so far. In particular, the self-adjoint action of $q$-exponentials on $L^2(\mathbb{R}_+)$ is not understood and the use of Liouville vertex operators for the calculation of Liouville correlation functions is therefore not yet fully justified. The detailed treatment of Liouville particle dynamics will prove to be very helpful to understand the zero mode structure of Liouville field theory.

In Section 2 we summarise the free field representation of the periodic Liouville theory and define the zero mode problem. Here we describe the classical and quantum mechanical reduction of the Liouville theory to particle dynamics, and deliver the canonical transformation to the free-field zero modes on the half-plane. In Section 3 we elaborate the quantum particle dynamics in the exponential potential, use the Møller matrix to get the quantum realisation of the relevant zero mode operators, calculate matrix elements of the mechanical Liouville vertex operator and extract from them the particle vertex operator in terms of the half-plane zero modes. Here a smooth continuation in the coupling has to be performed of the matrix elements which have to be considered as generalised functions. The self-adjoint action of vertex operators on the Hilbert space $L^2(\mathbb{R}_+)$ can be realised both, for the particle case and the zero mode sector of field theory, by using symmetries of the vertices under reflection. The reflection amplitude of the Liouville field theory is derived by the operator method. It is surprisingly identical to that deduced in [12] by a symmetry of a 3-point correlation function suggested in [13] on account of path-integral considerations. We summarise the results and conclude. Some details are given in Appendices.
2 Zero modes of periodic Liouville theory

To obtain a useful definition of the zero mode problem, we report the canonical free-field parametrisation of the periodic Liouville theory, reduce it to Liouville particle dynamics, classically and quantum mechanically, and observe that the particle phase-space coordinates are asymptotically related to the free-field zero modes on the half-plane by means of a canonical transformation which is the classical analogue of the Møller matrix.

2.1 Periodic Liouville field theory

Solutions of the Liouville equation

\[ \varphi_{\tau\tau}(\tau, \sigma) - \varphi_{\sigma\sigma}(\tau, \sigma) + \frac{4m^2}{\gamma} e^{2\gamma \varphi(\tau, \sigma)} = 0 \]  

have the useful free-field representation \[2, 3\]

\[ e^{-\gamma \varphi(\tau, \sigma)} = e^{-\gamma \phi(\tau, \sigma)} \left( 1 + m^2 \int \int dy \, d\bar{y} \, G(z, \bar{z}; y, \bar{y}) e^{2\gamma \phi(y)} e^{2\gamma \phi(\bar{y})} \right), \]  

where \( \gamma \) and \( m \) are positive constants, \( z = \tau + \sigma \) and \( \bar{z} = \tau - \sigma \) light-cone coordinates, \( \phi(\tau, \sigma) = \phi(z) + \bar{\phi}(\bar{z}) \) is a chirally decomposed free field, and \( G(z, \bar{z}; y, \bar{y}) \) the Green’s function of the D’Alembert operator \( \partial^2_{zz} \). For periodic boundary conditions \( \varphi(\tau, \sigma + 2\pi) = \varphi(\tau, \sigma) \) the range of integration in (2) is \((y, \bar{y}) \in [0, 2\pi] \times [0, 2\pi]\), the parametrising free field \( \phi(\tau, \sigma) \) is also periodic and it can be expanded in Fourier modes

\[ \phi(\tau, \sigma) = q + \frac{p}{2\pi} \tau + \frac{i}{\sqrt{4\pi}} \sum_{n \neq 0} \frac{1}{n} \left( a_n e^{-in(\tau+\sigma)} + \bar{a}_n e^{-in(\tau-\sigma)} \right). \]  

Canonical Poisson brackets for the zero modes and oscillators

\[ \{ p, q \} = 1, \quad \{ a_m, a_n \} = i m \delta_{m+n,0} = \{ \bar{a}_m, \bar{a}_n \} \]  

define relation (2) as a canonical transformation between the Liouville and the free field.

For periodic boundaries the Green’s function \( G \) of (2) can be expressed by \[6\]

\[ G(z, \bar{z}; y, \bar{y}) = \theta_{\gamma p} (z - y) \theta_{\gamma p} (\bar{z} - \bar{y}), \]  

where

\[ \theta_{\gamma p}(z - y) = \frac{e^{\frac{\pi}{2\gamma} (z-y)}}{2 \sinh \frac{\pi}{2\gamma}} \]
is the Green’s function which inverts the operator \( \partial_z \) on functions \( A(z) \) with the monodromy 
\[ A(z + 2\pi) = e^{\gamma p} A(z), \]
for \( p \neq 0 \). \( \epsilon(z) \) is the stair-step function. Its differentiation gives 
the periodic \( \delta \)-function. The point \( p = 0 \) is singular and will be excluded from (6). One 
can show that \( p > 0 \) (or \( p < 0 \)) covers the class of all regular periodic Liouville fields, 
and in order to avoid double counting of the solution \( \psi \) the zero modes are assumed to live on the half-plane \( p > 0 \) only. This dynamical restriction requires however additional investigations for the quantisation \[7\]. The Hilbert space of the non-zero modes \( a_n, \bar{a}_n \) is 
the usual Fock space, whereas the Hilbert space of the free-field zero modes is defined by 
wave functions \( \psi(p) \in L^2(\mathbb{R}_+) \). Unfortunately, the standard coordinate operator in the 
momentum representation \( \hat{q} = i\hbar \partial_p \) is not self-adjoint on \( L^2(\mathbb{R}_+) \), and the operators for the 
exponentials \( e^{\pm \gamma \hat{q}} \) which appear in (2) have to be suitably defined for applications. This 
hermiticity problem is related to the missing translation symmetry on the half-plane \( p > 0 \) in ‘\( p \)-direction’. But we are going to show that \( S \)-matrix transformations allow to ‘restore’ 
this symmetry by replacing \( L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_1) \), where self-adjointness of \( e^{\pm \gamma \hat{q}} \) is obvious.

2.2 Reductions of Liouville theory to particle dynamics

We can reduce the periodic Liouville theory to a mechanical Liouville model by different 
methods. Since Liouville vacuum configurations are described by a homogeneous field 
\( \partial_\sigma \phi(\tau, \sigma) = 0 \), the belonging Liouville field is a time-dependent coordinate only \( \phi(\tau, \sigma) = Q(\tau) \) and the Liouville equation \[1\] reduces to the mechanical Liouville model

\[
\ddot{Q}(\tau) + \frac{4m^2}{\gamma} e^{2\gamma\phi(\tau)} = 0.
\] (7)

The reduced Hamiltonian becomes

\[
H = \frac{1}{4\pi} \left( P^2 + 4\omega^2 e^{2\gamma\phi} \right),
\] (8)

where \( P = 2\pi \dot{Q} \) is the canonical conjugated momentum, and the parameter \( \omega = 2\pi m / \gamma \). 
The chiral energy-momentum tensor \( T(z) = (\partial_z \phi)^2 - \frac{1}{\gamma} \partial^2_{zz} \phi \) is then a positive constant 
\( T(z) = c^2 \), and its free-field representation \( T(z) = \phi'^2(z) - \frac{1}{\gamma} \phi''(z) \) leads to \( \phi'(z) = c \), 
which requires vanishing oscillator modes \( a_n = 0 = \bar{a}_n \). As a consequence, the Liouville 
solution \( \psi(\tau) \) reduces to the general solution of the particle equation \[7\]

\[
e^{-\gamma\phi(\tau)} = e^{-\gamma(q + \pi \omega)} + \frac{\omega^2}{p^2} e^{\gamma(q + \pi \omega)}.
\] (9)
It is obvious that the mechanical Liouville model can be obtained also vice versa from Liouville theory by Hamiltonian reduction with the second class constraints $a_n = 0 = \bar{a}_n$, and that the reduced Hamiltonian (8) is canonically related by (9) to the free form

$$H = \frac{p^2}{4\pi}. \quad (10)$$

It is worth mentioning that the above result can be obtained directly from $SL(2,\mathbb{R})$ WZNW theory [14, 7] or its homogeneous form, the $SL(2,\mathbb{R})$ particle model [11], by a suitable Hamiltonian reduction. If we identify the gauge invariant part of the $SL(2,\mathbb{R})$ field, the $g_{12}(\tau)$ of [7], with the particle vertex function $V(\tau) = e^{-\gamma Q(\tau)}$, from the $SL(2,\mathbb{R})$ particle equations of motion

$$\ddot{g}_{\alpha\beta}(\tau) = \frac{\gamma^2}{\pi} H g_{\alpha\beta}(\tau) \quad (11)$$

one reads off the gauge invariant equation

$$\ddot{V}(\tau) = \frac{\gamma^2}{\pi} H V(\tau), \quad (12)$$

which is equivalent to (7). Its integration reproduces the general solution (9) by using (10) in the form

$$p = \sqrt{4\pi H}. \quad (13)$$

It might be interesting to discuss the relic of the conformal symmetry of Liouville theory

$$\varphi(z, \bar{z}) \rightarrow \varphi(f(z), \bar{f}(\bar{z})) + \frac{1}{2\gamma} \log f'(z) \bar{f}'(\bar{z}). \quad (14)$$

For $\sigma$-independent Liouville fields the functions $f(z)$ and $\bar{f}(\bar{z})$ can be linear only, and with the parametrisation $f(z) = az + b$, $\bar{f}(\bar{z}) = a\bar{z} + b$ [14] reduces to

$$Q(\tau) \rightarrow Q(\tau + b), \quad Q(\tau) \rightarrow Q(a\tau) + \frac{\log a}{\gamma}, \quad a > 0. \quad (15)$$

These transformations are symmetries of the dynamical equations (7) respectively (12), and they transform the zero modes $p, q$ of (9) as

$$p, q \rightarrow (p + \frac{pb}{2\pi}), \quad (p, q) \rightarrow (a p, q + \frac{\log a}{\gamma}). \quad (16)$$

The time translations are obviously generated by the free Hamiltonian (10), whereas the remaining dilatation are not canonical anymore, since they transform the action by $S \rightarrow a S$ which is not a Noether symmetry.

In Appendix A, for amuse, we show that the mechanical Liouville model has a nice relativistic free-particle interpretation.
2.3 Asymptotics of mechanical Liouville model

The mechanical Liouville model becomes asymptotically a free theory. The general solution \( q(\tau) = q + \frac{p^2}{2\pi} \) is the free-particle solution. Therefore, \( p \) and \( q \) can be interpreted as the \textit{in}-variables of the Liouville particle dynamics. Similarly we introduce \textit{out}-variables by the behaviour of (9) as \( \tau \to +\infty \). The \textit{in}- and \textit{out}-variables are related by

\[
P_{\text{out}} = -p, \quad Q_{\text{out}} = -q + \frac{2}{\gamma} \log \frac{p}{\omega},
\]

which combines the reflection of \( p, q \) with a canonical map. It is an important observation that the general solution (9) is invariant under the transformation (18). Quantum mechanically this symmetry will be generated by the \( S \)-matrix of the particle theory. Note that in Liouville field theory \textit{in}– and \textit{out}–fields [2] are related by the special Möbius transformation \( A \to -(m^2 A)^{-1}, \quad \bar{A} \to -(m^2 \bar{A})^{-1} \), which keeps invariant the general solution

\[
\varphi(\tau, \sigma) = \log \frac{A'(\tau + \sigma)\bar{A}'(\tau - \sigma)}{[1 + m^2 A(\tau + \sigma)A(\tau - \sigma)]^2}.
\]

Due to (9), the phase space coordinates \( P, Q \) at \( \tau = 0 \) and the asymptotic variables \( p, q \) are related by

\[
e^{-\gamma Q} = e^{-\gamma q} + \frac{\omega^2}{p^2} e^{\gamma q}, \quad P = -p \tanh \left( \gamma q + \log \frac{\omega}{p} \right).
\]

This is a canonical and invertible map of the plane \((P, Q)\) onto the half-plane \((p, q)\). Its inversion gives (13) and provides \( q \) as a function of \( P, Q \). The transformation can be simplified for the coordinate \( \tilde{q} = q + \frac{1}{\gamma} \log \frac{\omega}{p} \)

\[
e^{-\gamma Q} = \frac{2\omega}{p} \cosh \gamma \tilde{q}, \quad 2\omega \sinh \gamma \tilde{q} = -P e^{-\gamma Q};
\]

\[
P = -p \tanh \gamma \tilde{q}, \quad p = \sqrt{P^2 + 4\omega^2 e^{2\gamma Q}},
\]

and summarised by the generating function \( F(Q, \tilde{q}) = \frac{2\omega}{\gamma} e^{\gamma Q} \sinh \gamma \tilde{q} \) of ref. [10].

Properly speaking, the half-plane problem becomes visible already on the \((P, Q)\)-plane, where the Hamiltonian (8) continuously relates the positive \textit{in}– and negative \textit{out}–momenta. This is illustrated in Fig.1. Each phase-space trajectory of the full \((P, Q)\)-plane
is one-to-one mapped by (21) to a corresponding line $p = \text{const}$ on the upper half-plane $(p, q)$ of Fig. 2. We observe a ‘non-degeneracy’ on the phase-space $(P, Q)$, which quantum mechanically will be described by the non-degeneracy of the Hamiltonian spectrum.

Unfortunately, the map (20) is non-linear and its quantum realisation becomes non-trivial. But there is a limiting procedure which will simplify the quantum calculations. For this purpose we introduce an additional free Hamiltonian $H_0 = P^2/4\pi$ and consider the Hamiltonian flows $U_{H_0}(\tau)$ and $U_H(\tau)$ generated by $H_0$ and $H$ respectively

$$U_{H_0}(\tau) (P, Q) = (P, Q + \frac{P\tau}{2\pi}), \quad U_H(\tau) (P, Q) = (P(\tau), Q(\tau)).$$

(22)

Here $P(\tau), Q(\tau)$ describe the solution of Hamiltonian equations with the initial data $P, Q$. The composition of the two flows $U(\tau) = U_H(\tau) \cdot U_{H_0}(-\tau)$ yields

$$U(\tau) (P, Q) = (P(\tau), Q(\tau) - \frac{P(\tau)\tau}{2\pi}),$$

(23)

and because of (17) we obtain the canonical map of the $(P, Q)$-plane onto the half-plane $(p, q)$ by the asymptotic relation

$$\lim_{\tau \to -\infty} U(\tau) (P, Q) = (p, q).$$

(24)

Quantum mechanically this map is mediated by the Møller matrix.

Note that $U(\tau)$ defines a canonical transformation of the coordinates $P, Q$ for any $\tau$. We also mention that (16) yields $G(z, \bar{z}; y, \bar{y}) \to 0$ as $\tau \to -\infty$, and the parametrising free field (3) is therefore an asymptotic $in$-field of the Liouville theory. The Liouville zero mode sector is so consistently given by the $in$-variables of the mechanical model.
2.4 Reduction of quantum Liouville field theory

We only sketch the reduction of quantum Liouville theory to quantum particle dynamics.

The free field representation (2)-(6) was used in [6] for an algebraic construction of the
Liouville vertex operator \( (e^{\lambda \varphi(\tau,\sigma)})_{\text{op}} \). With the definition of the Liouville operator field
\( \hat{\varphi}(\tau,\sigma) = \partial_\lambda (e^{\lambda \varphi(\tau,\sigma)})_{\text{op}} |_{\lambda=0} \) it was shown that the operator Liouville equation and the
canonical commutation relations remain valid, although the free-field description of the
Liouville operators is deformed as compared with the classical analogue. Both, the oscillator
and zero mode operators contribute jointly to these deformations and yield conformal
invariant, local and anomaly-free operator structures. These deformations also guarantee,
e.g., that the 4-point function of the 25-dimensional non-critical string becomes consistently
the critical Shapiro-Virasoro amplitude in 26-dimensional Minkowski space-time.

The mechanical Liouville model is obtained by Hamiltonian reduction of the Liouville
field theory with respect to the constraints \( a_n = 0 = \bar{a}_n \). One can simulate such a reduction
quantum mechanically by calculating oscillator vacuum matrix elements of vertex operators

\[
\hat{V}_\lambda(\tau) = \langle 0 | (e^{\lambda \varphi(\tau,\sigma)})_{\text{op}} | 0 \rangle,
\]

where the vacuum state \( | 0 \rangle \) is annihilated by the oscillators \( \hat{a}_n, \hat{\bar{a}}_n \) for \( n > 0 \).

This matrix element is obviously an operator in the zero mode sector, and one can
prove its \( \sigma \)-independence. Then the Liouville operator equation simply reduces to a particle
operator equation. But we should point out here that we have not disentangled the
oscillator and zero mode contributions by this reduction, since the constraints \( a_n = 0 = \bar{a}_n \)
are of second class. The result will be nevertheless useful.

For the application it will be sufficient to treat the special Liouville vertex operator for
\( \lambda = -\gamma \) only. Its explicit form is derived in Appendix B as

\[
\hat{V}(\tau) = e^{-\frac{\gamma \pi}{4}\tau} e^{-\gamma^2 \tau} + \omega_\alpha^2 \frac{e^{\frac{2\gamma \pi}{4}\tau}}{\bar{p}} \frac{\Gamma \left( -i \frac{2\gamma \pi}{4\pi} \right)}{\Gamma \left( i \frac{2\gamma \pi}{4\pi} \right)} e^{\gamma \hat{q}} \frac{\Gamma \left( -i \frac{2\gamma \pi}{4\pi} \right)}{\Gamma \left( i \frac{2\gamma \pi}{4\pi} \right)} \frac{e^{\frac{2\gamma \pi}{4}\tau}}{\bar{p}},
\]

where

\[
\omega_\alpha = \frac{2\pi m_\alpha \Gamma(1 + 2\alpha)}{\gamma} \quad \text{and} \quad m_\alpha^2 = \frac{\sin 2\pi \alpha}{2\pi \alpha} m^2.
\]

Both, the deformed parameter \( \omega_\alpha \) (B.8) and the \( \Gamma \)-functions (B.6) are due to oscillator as
well as zero mode contributions, and (26) has, of course, the classical limit (9).
3 Quantisation and self-adjoint vertex operators

In this section we quantise the mechanical Liouville model, construct the operators $\hat{p}$ and $e^{\pm \gamma \hat{q}}$ by means of the Møller matrix, calculate matrix elements of particle vertex operators and describe their realisation in terms of the zero mode operators $\hat{p}, \hat{q}$. Moreover, we shall investigate the self-adjoint action of vertex operators on the Hilbert space $L^2(\mathbb{R}_+)$ using $S$-matrix properties, and derive, in particular, the Liouville reflection amplitude by means of operator methods.

3.1 Møller matrix and operators of asymptotic zero modes

For the quantisation of the canonical map of the $(P,Q)$-plane onto the half-plane $(p,q)$ we introduce a Møller matrix \cite{15}

$$\hat{U} = \lim_{\tau \to -\infty} \hat{U}(\tau),$$

which is given by the asymptotics of the unitary operator

$$\hat{U}(\tau) = e^{\frac{i}{\hbar} \hat{H}_\tau} e^{-\frac{i}{\hbar} \hat{H}_0 \tau}.$$ (29)

The operators corresponding to the two phase-spaces are then related by

$$\hat{p} = \hat{U} \hat{P} \hat{U}^+, \quad e^{\pm \gamma \hat{q}} = \hat{U} e^{\pm \gamma \hat{Q}} \hat{U}^+.$$ (30)

To simplify the following calculations it is convenient to use the notation

$$x = \gamma Q - \log \frac{\alpha}{m} \quad \text{and} \quad \alpha = \frac{\hbar \gamma^2}{4\pi}.$$ (31)

We write the Liouville and free Hamiltonian operators in the $x$-representation

$$\hat{H} = \hbar \alpha (-\partial_x^2 + e^{2x}) \quad \text{and} \quad \hat{H}_0 = -\hbar \alpha \partial_x^2,$$ (32)

which act on the Hilbert space $L^2(\mathbb{R})$. The eigenstates of the Hamiltonian $\hat{H}$ are the Kelvin functions \cite{3} discussed in Appendix C. In ‘bracket’ notation these eigenstates will be designed by $|\Psi_k\rangle$, and for the eigenstates of $\hat{H}_0$ we use $|k\rangle (-\infty < k < +\infty)$. Both eigenstates have the same normalisation $\langle k|k'\rangle = \delta(k - k') = \langle \Psi_k|\Psi_{k'}\rangle$ and the same eigenvalue $E = \hbar \alpha k^2 (= p^2 / 4\pi)$. But here we have to emphasise that the spectrum of $\hat{H}_0$
is degenerated ($|k\rangle \neq |-k\rangle$), whereas the spectrum of $\hat{H}$ is not degenerated ($|\Psi_k\rangle = |\Psi_{-k}\rangle$); $|\Psi_k\rangle$ is a complete set already for $k > 0$ (see C3). This describes the quantum mechanical half-plane situation which we have discussed in section 2.3 classically.

For the exponential operator (29) we have so a 'mixed' spectral decomposition

$$\hat{U}(\tau) = \int_0^{+\infty} dk \int_{-\infty}^{+\infty} dk' |\Psi_k\rangle U(k,k';\tau) \langle k'|,$$  \hspace{1cm} (33)

with $U(k,k';\tau) = e^{i\alpha(k^2-k'^2)\tau} \langle \Psi_k|k'\rangle$. In Appendix C it is shown that the matrix element $\langle \Psi_k|k'\rangle$ is divergent and it has to be considered as a generalised function. Its regularised form $\langle \Psi_k|k'\rangle_\varepsilon$ is given by C14.

With C14 and the well-known formulas of scattering theory

$$\lim_{t \to -\infty} \frac{e^{i(E-E')t}}{E - E' \pm i\varepsilon} = -2\pi i \theta(\pm\varepsilon) \delta(E - E'), \quad E = \alpha k^2,$$  \hspace{1cm} (34)

the regularised kernel of C13 has the limit

$$\lim_{\tau \to -\infty} U_\varepsilon(k,k';\tau) = a_k \theta(k') \delta(k - k').$$  \hspace{1cm} (35)

According to C6-C9, $a_k$ is the coefficient of the 'left-moving' wave of the eigenfunction $\Psi_k(x)$ as $x \to -\infty$ with $|a_k| = 1$; and $|\Psi_0\rangle = a_k |\Psi_k\rangle$ can be called therefore the in-eigenstate of the Hamiltonian $\hat{H}$ [16]. The Møller matrix then becomes

$$\hat{U} = \int_0^{+\infty} dk |\Psi_0\rangle \langle k|,$$  \hspace{1cm} (36)

where the integration takes into consideration the non-degenerated spectrum of the Hamiltonian $\hat{H}$ with $k > 0$. The Hermitean conjugated to the Møller matrix is obtained by similar calculations

$$\hat{U}^+ = \lim_{\tau \to -\infty} e^{\frac{i}{\hbar} \hat{H}_0 \tau} e^{-\frac{i}{\hbar} \hat{H}_0 \tau} = \int_0^{+\infty} dk |\Psi_k\rangle \langle k|.$$  \hspace{1cm} (37)

It is important to note that the operator $\hat{U}(\tau)$ is unitary for finite $\tau$, but in the limit $\tau \to -\infty$ only a 'one-side' unitarity condition remains

$$\hat{U} \hat{U}^+ = \hat{I}, \quad \hat{U}^+ \hat{U} = \int_0^{+\infty} dk |\Psi_k\rangle \langle k|.$$  \hspace{1cm} (38)

This non-unitarity is a consequence of the different spectra of $\hat{P}$ and $\hat{p}$.
The transformation (30) formally preserves hermiticity, and with (36), (37) the spectral decomposition of $\hat{p}$ on the half-line follows

$$\hat{p} = \hbar \gamma \int_{0}^{+\infty} dk \langle \Psi_{k}^{-} | k \rangle \langle k | \Psi_{k}^{-} \rangle.$$  

(39)

The momentum operator $\hat{p}$ is so self-adjoint, $|\Psi_{k}^{-}\rangle$ is its eigenstate with eigenvalue $p = \hbar \gamma k > 0$, and using again (36), (37) one obtains the useful relations

$$\hat{p} \hat{U} = \hat{U} \hat{p}, \quad \hat{U}^{+} \hat{p} = \hat{P} \hat{U}^{+}.$$  

(40)

In the $x$-representation the operators $\hat{P}$ and $e^{\pm \gamma \hat{Q}}$ are

$$\hat{P} = -i \hbar \gamma \partial_{x}, \quad e^{-\gamma \hat{Q}} = \frac{m}{\alpha} e^{-x}, \quad e^{\gamma \hat{Q}} = \frac{\alpha}{m} e^{x},$$  

(41)

and they satisfy the commutation relations $[\hat{P}, e^{\pm \gamma \hat{Q}}] = \mp i \hbar \gamma e^{\pm \gamma \hat{Q}}$. Multiplying these relations on the left and right by $\hat{U}$ and $\hat{U}^{+}$ respectively, from (40) we obtain a result consistent with canonical quantisation

$$[\hat{p}, e^{\pm \gamma \hat{q}}] = \mp i \hbar \gamma e^{\pm \gamma \hat{q}}.$$  

(42)

We are also able to calculate the matrix elements $\langle \Psi_{k}^{-} | e^{\pm \gamma \hat{q}} | \Psi_{k'}^{-}\rangle$. Using (30), (36)-(37), (41) and the orthonormality of $|\Psi_{k}^{-}\rangle$, they are given by

$$\langle \Psi_{k}^{-} | e^{\pm \gamma \hat{q}} | \Psi_{k'}^{-}\rangle = \left( \frac{\alpha}{m} \right)^{\pm 1} \langle k | e^{\pm \hat{x}} | k' \rangle.$$  

(43)

The matrix elements on the right hand side can be obtained by means of the eigenstate $|k\rangle$ in the $x$-representation, $\sqrt{2\pi} \langle x | k \rangle = e^{ikx}$, as

$$\langle k | e^{\pm \hat{x}} | k' \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx e^{-i(k-k' \pm i)x} = \delta(k-k' \pm i).$$  

(44)

The δ-function with complex argument is defined on analytical test function in the standard way [17], and we will use

$$\int_{0}^{+\infty} dk' \delta(k-k' \pm i) \psi(k') = \psi(k \pm i).$$  

(45)

Since $|\Psi_{k}^{-}\rangle$ is an eigenstate of $\hat{p}$, let us come back to the initial zero mode variables and identify $\sqrt{\hbar \gamma} |p\rangle = |\Psi_{k}^{-}\rangle$ for $p = \hbar \gamma k$. The eigenstates $|p\rangle$ are normalised $\langle p | p' \rangle = \delta(p-p')$.
and they define the $p$-representation by wave functions $\psi(p) = \langle p | \Psi \rangle \in L^2(\mathbb{R}^+_+)$, which relates $\psi(p)$ with wave functions of the $Q$-representation $\Psi(x)$

$$\psi(p) = \int_{-\infty}^{+\infty} dx \ \Psi^*_k(x) \Psi(x), \quad \text{for} \quad p = h\gamma k.$$  \hfill (46)

This transformation is unitary because the eigenstates $\Psi_k(x)$ are complete. Then $\hat{p}\psi(p) = p\psi(p)$ and the operators $e^{\pm \gamma \hat{q}}$ act due to (43), (44) as

$$e^{-\gamma \hat{q}} \psi(p) = \frac{m}{\alpha} \psi(p - ih\gamma), \quad e^{\gamma \hat{q}} \psi(p) = \frac{\alpha}{m} \psi(p + ih\gamma). \hfill (47)$$

These relations define the action of the zero mode operators in the Hilbert space $L^2(\mathbb{R}^+_+)$. But we have still to discuss the hermiticity properties of the exponential $q$-operators, which will be done in detail in section 3.3.

Note that any operator of $F(p)$ given by a spectral decomposition like (39)

$$F(\hat{p}) = \int_{0}^{+\infty} dp \ |p\rangle \ F(p) \langle p|$$

satisfies relations similar to (40), $F(\hat{p}) \hat{U} = \hat{U} F(\hat{P})$ or $\hat{U}^+ F(\hat{p}) = F(\hat{P}) \hat{U}^+$, and as a consequence the multiplication rule holds

$$F(\hat{p}) e^{\pm \gamma \hat{q}} = e^{\pm \gamma \hat{q}} F(\hat{p} \mp ih\gamma). \hfill (49)$$

This relation was used for the algebraic construction of Liouville vertex operators in [4,6].

### 3.2 Vertex operators and their matrix elements

The Heisenberg equations derived from the Hamilton operator $\hat{H} = \hbar \alpha(-\partial^2_{xx} + e^{2x})$ provide the operator equation $\dddot{\hat{x}}(\tau) + 4\alpha^2 e^{2\hat{x}(\tau)} = 0$, which has similarly as in Liouville field theory the classical form (7). However the particle vertex operator

$$\hat{V}(\tau) = e^{-\gamma \hat{Q}(\tau)} = e^{\frac{i}{\hbar} \hat{H}_x} e^{-\gamma \hat{Q} e^{-\frac{i}{\hbar} \hat{H}_x}}$$

satisfies a deformed equation

$$\frac{1}{2} \left( \dddot{\hat{V}}(\tau) \hat{V}(\tau) + \dot{\hat{V}}(\tau) \ddot{\hat{V}}(\tau) \right) - \dot{\hat{V}}(\tau)^2 + 2\alpha^2 \hat{V}(\tau)^2 = 4m^2, \hfill (51)$$

as compared with the corresponding classical one $\dddot{V}(\tau)V(\tau) - \dot{V}(\tau)^2 = 4m^2$. One observes the ordering prescription for the term $\dddot{V}(\tau)V(\tau)$ and a quantum correction proportional to
$\alpha^2$. Even the quantum version of the gauge invariant linear equation (12) has a quantum deformation

$$\tilde{V}(\tau) = \frac{g^2}{2\pi} (\hat{H}\tilde{V}(\tau) + \tilde{V}(\tau)\hat{H}) + \alpha^2 \tilde{V}(\tau).$$

(52)

This parallels again Liouville theory, where the (anti-) chiral operators satisfy quantum mechanically deformed Schrödinger equations [5, 6].

We are able to calculate matrix elements of vertex operators of the mechanical model (Please, note that we also use the notation $V$ for $V^{-1}$!)

$$V_{2b}(k, k'; \tau) = \langle \Psi^{-}_k | e^{2b\gamma\hat{Q}(\tau)} | \Psi^{-}_k \rangle.$$  

(53)

By (31) and (C.10), Eq. (53) becomes

$$V_{2b}(k, k'; \tau) = \left(\frac{\alpha}{m}\right)^{2b} \int_{-\infty}^{\infty} dx K_{ik}(e^x) e^{2bx} K_{ik'}(e^x).$$

(54)

But the integral is well defined for $b > 0$ only, and it diverges for $b \leq 0$. Eq. (53) defines so a generalised function which is a kernel of the vertex operator, and in order to calculate it we need its smooth continuation from positive to negative $b$. Substituting (C.2) into (54) and integrating over $u = e^x$, the integral splits into a product of two integrals of the type (C.13) and gives with the notation

$$\kappa = \frac{k + k'}{2}, \quad \rho = \frac{k - k'}{2},$$

(55)

for $b > 0$ the result [9]

$$V_{2b} = \left(\frac{\alpha}{m}\right)^{2b} e^{4i\alpha \rho \epsilon} 4^{b-i\rho} \frac{\Gamma(b + i\kappa) \Gamma(b - i\kappa)}{\Gamma(i\rho + i\kappa) \Gamma(i\rho - i\kappa)} \frac{\Gamma(b + i\rho) \Gamma(b - i\rho)}{4\pi \Gamma(2b)}.$$  

(56)

In order to obtain the vertex function for negative $b$ one needs a smooth continuation of (56) as a generalised function, which was not done before. Since $\kappa > |\rho|$ ambiguities can arise at $b = -n$, $n = 0, 1, 2, \ldots$. Near these points if $b = -n + \epsilon$ (56) behaves like

$$\frac{1}{\pi} \frac{\epsilon}{(k - k')^2 + \epsilon^2}.$$  

(57)

We have shown in Appendix D that this generalised function has for holomorphic test functions the following smooth continuation from positive to negative values of $\epsilon$

$$\frac{1}{\pi} \frac{\epsilon}{(k - k')^2 + \epsilon^2} + \delta(k - k' + i\epsilon) + \delta(k - k' - i\epsilon).$$

(58)
The continuation of \((58)\) to negative values of \(b\) so creates a pair of \(\delta\)-functions with complex arguments each time \(b\) passes a negative integer value, and for \(b = -|b|\) results

\[
V_{-2|b|} = V_{-2|b|}^+ (\kappa, \rho; \tau) + \left( \frac{m}{\alpha} \right)^{2|b|} \sum_{l=0}^{|b|} C_{2|b|}^l \exp^{-4\alpha(b-l)\kappa\tau} \frac{\Gamma(-|b| + i\kappa) \Gamma(-|b| - i\kappa)}{4^l \Gamma(-|b| + l + i\kappa) \Gamma(-|b| + l - i\kappa)} \delta[2\rho - 2i(|b| - l)]
\]

\[+ C_{2|b|}^l \exp^{4\alpha(|b|-l)\kappa\tau} \frac{\Gamma(-|b| + i\kappa) \Gamma(-|b| - i\kappa)}{4^{|b|-l} \Gamma(|b| - l + i\kappa) \Gamma(|b| - l - i\kappa)} \delta[2\rho + 2i(|b| - l)]. \quad (59)\]

Here \(V_{-2|b|}^+ (\kappa, \rho; \tau)\) is defined by the r.h.s of \((56)\) where \(b\) is replaced by \(-|b|\), \([|b|]\) is the integer part of \(|b|\), and

\[
C_{2|b|}^l = \prod_{j=0}^{l-1} \frac{2|b| - j}{j + 1}. \quad (60)
\]

In particular, the function \(V_{-2|b|}^+ (\kappa, \rho; \tau)\) vanishes at half-integer \(|b|\) due to the pole of \(\Gamma(-2|b|)\), and it creates at integer \(|b|\) the terms proportional to \(\delta(\rho)\), so that for \(2|b| = n\) one has

\[
V_{-n}(\kappa, \rho; \tau) = \left( \frac{m}{\alpha} \right)^n \sum_{l=0}^{n} C_{n}^l e^{-2(n-2l)\kappa\tau} \prod_{j=0}^{l-1} \frac{1}{4\kappa^2 + (n - 2j)^2} \delta[2\rho - i(n - 2l)], \quad (61)
\]

where the \(C_{n}^l\) are now binomial coefficients.

Let us consider the matrix element of the vertex operator \(\hat{V}(\tau)\) which corresponds to the case \(n = 1\) of \((61)\)

\[
V(k, k'; \tau) = \frac{m}{\alpha} \left( e^{-\alpha(k+k')\tau} \delta(k - k' - i) + \frac{e^{\alpha(k+k')\tau}}{4kk'} \delta(k - k' + i) \right). \quad (62)
\]

From this expression and \((57)\) one can easily read off the structure of the particle vertex operator as

\[
\hat{V}(\tau) = e^{-\frac{\omega}{4\tau}} \exp^{-\frac{\omega}{4\tau} \frac{\hat{\eta}^q}{\hat{\eta}}} + \omega^2 \frac{e^{\frac{\omega}{4\tau}}}{\hat{p}} \exp^{\frac{\omega}{4\tau} \frac{\hat{\eta}^q}{\hat{\eta}}} \frac{e^{\frac{\omega}{4\tau}}}{\hat{p}}. \quad (63)
\]

We observe here an ordering similar to the one of the reduced quantum Liouville operator \((20)\), and by \((49)\) one can easily check that the operator \((63)\), indeed, satisfies both the operator equations \((51)\) and \((52)\).

One can show that the vertex operator \(\hat{V}_{-n}\) is the \(n\)-th power of \((63)\), however, for positive or non-integer negative \(2b\) the operator \(\hat{V}_{2b}\) read off from \((56)\) or the first term of \((59)\) respectively is given (using Fourier transformations) by an infinite series of \(q\)-exponentials, much as in quantum Liouville field theory \((6)\).
3.3 Self-adjointness of half-plane operators

We have to ask whether (63), and (26), are physical operators with Hermitean action on the Hilbert space $L^2(\mathbb{R}_+)$. Since asymptotically for $\tau \to \pm \infty$ only one term survives one is tempted to demand hermiticity for each term of (63) separately. But a proof of hermiticity for $e^{\gamma \hat{q}}$ would require very special boundary conditions on holomorphic functions $\psi(p)$ at $\text{Re} \ p = 0$. This mathematically by itself interesting but still unsolved problem will be further discussed in Appendix E.

However, the vertex operator (63) is expected to become self-adjoint as a whole on account of its symmetry under transformations given by the $S$-matrix \[ \hat{S} = \hat{P} S(p). \] (64)

For the particle model $S(p)$ is the multiplicative reflection amplitude \[ (C.11) \] with $p = \hbar \gamma k$ and $\hat{P}$ the parity operator $\hat{P} \psi(p) = \psi(-p)$. It is easy to see that (64) replaces the in-coming first term of (63) by the out-going second one, and vice versa, so that the particle vertex operator remains invariant. That means $\hat{S} \hat{V}(\tau) \hat{S}^{-1}$ is identical to (63). This also holds in general as one can see from (56) and (59). Note that the $S$-matrix is just the quantum version of the symmetry transformation (18). $\hat{S}$ also maps the Hilbert space of the in-fields $L^2(\mathbb{R}_+)$ onto $L^2(\mathbb{R}_-)$ for the out-fields, which for the wave functions $\psi(p) \in L^2(\mathbb{R}_+)$, $\tilde{\psi}(p) \in L^2(\mathbb{R}_-)$ is given by $\tilde{\psi}(-p) = S(p)\psi(p)$. The last relation is defined by (46) using \[ (C.10), (C.11) \] and $\Psi_k^*(x) = d_{-k}^* K_{-ik} S_k \Psi_k^*(x)$.

Due to these properties one can extend the definition of the matrix element of the vertex operator from the Hilbert space of the half-plane $L^2(\mathbb{R}_+)$ to that of the plane $L^2(\mathbb{R})$

$\int_0^\infty dp \ \psi_+^*(p) \hat{V}(\tau) \psi_+(p) = \frac{1}{2} \int_{-\infty}^{\infty} dp \ \Psi^*(p) \hat{V}(\tau) \Psi(p).$ (65) 

Here $\Psi(p) = \psi(p)$ for $p > 0$, $\Psi(p) = \tilde{\psi}(p) = S(-P)\psi(-p)$ for $p < 0$, and $\Psi(p) \in L^2(\mathbb{R})$ satisfies $\hat{S} \Psi(p) = \Psi(p)$. Self-adjointness of the operator (63) obviously holds on $L^2(\mathbb{R})$ for holomorphic wave functions $\Psi(p)$. As a consequence, self-adjointness of $\hat{V}(\tau)$ on $L^2(\mathbb{R}_+)$ requires for $\psi(p)$ a holomorphic extension to the negative half-line so that $\psi(-p) = \tilde{\psi}(-p) = S(p)\psi(p)$. Such functions are given by $\psi(p) = d^*(p)f(p)$, where $d(p)$ is defined by \[ (C.10) \] with $p = \hbar \gamma k$ and $f(p)$ is an even holomorphic function $f(-p) = f(p)$. The integral (63) is well-defined at the singular point of $\hat{V}(\tau)$ $p = 0$, since $\psi(0) = 0$ on account of $d(0) = 0$. 

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We can finally ask the question whether the Liouville vertex operator in the vacuum sector $\hat{V}(\tau)$ is invariant under the $S$-matrix transformation $\hat{S}_L = \hat{P} S_L(p)$ which exchanges the in-coming with the deformed out-going zero mode part. We have solved the equation

$$\hat{S}_L^{-1}\hat{V}(\tau)S_L = \hat{V}(\tau)$$

and found exactly the deformed reflection amplitude of

$$S_L(p) = -\left(\frac{4\pi m a^2}{\Gamma(1 + 2\alpha) \sin 2\pi \alpha}\right)^{\frac{2p}{\pi}} \frac{\Gamma\left(i\frac{2p}{4\pi \alpha}\right)}{\Gamma\left(-i\frac{2p}{4\pi \alpha}\right)} \frac{\Gamma\left(i\frac{2p}{2\pi}\right)}{\Gamma\left(-i\frac{2p}{2\pi}\right)}. \tag{67}$$

This result motivates one to look for a deeper understanding of the path-integral based conjectures of 3-point functions by means of the operator approach. But for a proof of self-adjointness the knowledge of the full Liouville $S$-matrix is required. The operator equation for the Liouville $S$-matrix and its solution will be discussed elsewhere.

## 4 Summary and conclusions

We investigated in detail Liouville particle dynamics which describes Liouville field theory in the zero mode sector. Half-plane zero modes have been quantised and proved to be self-adjoint on account of hidden symmetries of vertex operators under $S$-matrix transformation. In order to obtain matrix elements of particle vertex operators, a method was developed to continue generalised functions smoothly in a coupling parameter.

We learned from particle dynamics to treat quantum Liouville field theory in the zero mode sector, calculated as an interesting result the reflection amplitude by the operator method, and used it for the proof of self-adjointness of corresponding vertex operators.

The analytical properties discovered for the particle matrix elements are expected to be useful to relate results which have been derived alternatively by the operator and path-integral approaches, so testing the conjectured Liouville 3-point functions.

### Acknowledgements

We thank T. Curtright, D. Fairlie, C. Thorn and C. Zachos for bringing their work about Liouville particle dynamics to our attention, and C. Ford, M. Reuter, W. Rühl, G. Savvidy, R. Seiler, and J. Teschner for discussions. G.J. is grateful to DESY Zeuthen for hospitality. His research was supported by grants from the DFG, INTAS, RFBR and GAS.
A Relation with a relativistic free-particle

Here we give a relativistic free-particle interpretation of the Liouville particle model. For this purpose we introduce a 2-dimensional Minkowskian manifold with coordinates $X^0 = T$, $X^1 = X$ and the conformal metric tensor

\[ g_{\mu\nu}(T, X) = e^{2X} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  

(A.1)

In this space-time the Lagrangian of a relativistic particle

\[ L = -m_0 e^X \sqrt{T^2 - X^2} \]  

(A.2)

leads to the mass-shell condition $P_T^2 - P_X^2 = m_0^2 e^{2X}$. In the gauge $T = P_T \tau$ one obtains a Liouville particle action

\[ S = \int \left( P_X dX - \frac{1}{2} (P_X^2 + m_0^2 e^{2X}) d\tau \right). \]  

(A.3)

The scalar curvature calculated for the metric (A.1) vanishes and one can pass to flat coordinates $Y^\pm = \pm e^{X\pm T}$, for which (A.2) simply becomes the relativistic free Lagrangian

\[ L = -m_0 \sqrt{\dot{Y}_+ \dot{Y}_-}, \]  

(A.4)

and it defines straight-line trajectories

\[ P_- Y_+ - P_+ Y_- = M. \]  

(A.5)

Here $P_\pm$ are light-cone components of the relativistic momentum ($P_+ P_- = m_0^2$) and $M$ is the boost. Rewriting these trajectories in $T, X$ coordinates and taking into account the gauge fixing condition $T = P_T \tau$, one gets

\[ e^{-X(\tau)} = \frac{P_+}{M} e^{-P_T \tau} + \frac{P_-}{M} e^{P_T \tau}. \]  

(A.6)

This result reproduces the general solution (9), and we have got a map of the solutions (9) to trajectories of a free relativistic particle (A.5). But since $Y_+ > 0$ and $Y_- < 0$ these trajectories cover only one ‘quarter’ of the space-time $Y^\mu$. Note that the time translation symmetry of (15) corresponds to Lorentz transformations of $Y^\mu$ coordinates.
B Reduced vertex operator

We take the operator form of the Liouville vertex function (2) from [6] (adapting the notation), calculate the normal-ordered oscillator matrix element (25) for $\lambda = -\gamma$

$$\hat{V}(\tau) = : e^{-\gamma(\hat{q} + i\frac{\hat{p}}{2})} \left( 1 + m_\alpha^2 \int_0^{2\pi} \int_0^{2\pi} dy d\bar{y} \, G_\alpha(z, \bar{z}; y, \bar{y}) e^{2\gamma(\hat{q} + \frac{i\gamma}{4} y + \frac{i\gamma}{4} \bar{y})} \right) :.$$  \hspace{1cm} (B.1)

and use the ordering prescription

$$: e^{2\alpha q} \, A(p) := e^{a \hat{q}} \, A(\hat{p}) e^{a \hat{q}}.$$  \hspace{1cm} (B.2)

The index $\alpha$ symbolises the quantum deformations, where

$$m_\alpha^2 = \frac{\sin \pi \alpha}{\pi \alpha} \, m^2, \quad \text{with} \quad \alpha = \frac{\hbar \gamma^2}{4\pi}.$$  \hspace{1cm} (B.3)

$G_\alpha(z, \bar{z}; y, \bar{y})$ is the Green’s function [3], [4] deformed as

$$\sinh^2 \frac{\gamma p}{2} \rightarrow \sinh^2 \left( \frac{\gamma p}{2} \right) + \sin^2 \left( \frac{\hbar \gamma^2}{4} \right),$$  \hspace{1cm} (B.4)

and multiplied with the short distance factor $f_\alpha(z - y) \, f_\alpha(\bar{z} - \bar{y})$ given by

$$f_\alpha(x) = \left( 4 \sin^2 \frac{x}{2} \right)^\alpha.$$  \hspace{1cm} (B.5)

We should mention here that the short distance factors are a consequence of conformal invariance and the deformations (B.3) and (B.4) are due to locality. With the integral [20]

$$\int_0^{2\pi} dx \, \left( \sin \frac{x}{2} \right)^{2\alpha} e^{\frac{\alpha x}{2}} e^{-\frac{\gamma x}{2}} = \frac{2^{2-2\alpha} \pi \, \Gamma(1 + 2\alpha) \, e^{\frac{\gamma x}{2}}}{\Gamma(1 + \alpha + i\frac{\gamma p}{2\pi}) \, \Gamma(1 + \alpha - i\frac{\gamma p}{2\pi})},$$  \hspace{1cm} (B.6)

and the identity

$$\Gamma \left( 1 - \alpha - i\frac{\gamma p}{2\pi} \right) \Gamma \left( 1 - \alpha + i\frac{\gamma p}{2\pi} \right) \Gamma \left( 1 + \alpha - i\frac{\gamma p}{2\pi} \right) \Gamma \left( 1 + \alpha + i\frac{\gamma p}{2\pi} \right) \hspace{1cm} (B.7)$$

$$= \frac{(\pi \alpha)^2 + (\frac{2p}{\gamma})^2}{\sinh^2 \frac{2p}{\gamma} + \sin^2 \frac{\pi \alpha}{\gamma}},$$

we arrive at (26), where

$$\omega_\alpha = \frac{2\pi m_\alpha \Gamma(1 + 2\alpha)}{\gamma}.$$  \hspace{1cm} (B.8)

contains all $p$-independent contributions, and it has the classical limit $\frac{2\pi m}{\gamma}$ of [3].
C Eigenstates of mechanical Liouville Hamiltonian

The solutions of the stationary Schrödinger equation

\[-\Psi_k''(x) + e^{2x} \Psi_k(x) = k^2 \Psi_k(x),\]  

are given by Kelvin (modified Bessel) functions \(\Psi_k(x) = c_k K_{ik}(e^x)\) \[3\] (see also Eqs. (2.16)-(2.22) of \[9\]), where \(c_k\) is a normalisation coefficient. The Kelvin functions are real \(K^*_k(u) = K_{-ik}(u) = K_{ik}(u)\) and have the useful integral representation \[20\]

\[K_{ik}(u) = \frac{1}{2} \int_{-\infty}^{+\infty} dy e^{-u \cosh y} e^{iky}.\]  

The eigenstates \(\Psi_k(x)\) vanish rapidly for \(x \to \infty\) and oscillate as \(x \to -\infty\). The spectrum \(E = \hbar\alpha k^2\) is non-degenerated, but the zero energy point \(k = 0\) has to be excluded from the spectrum since the corresponding wave function diverges as \(x \to -\infty\), and for the eigenstates \(\Psi_k(x)\) we take \(k > 0\) only.

The normalisation and completeness conditions

\[
\int_{-\infty}^{+\infty} dx \, \Psi_k(x)\Psi_{k'}(x) = \delta(k - k'), \quad \int_0^{+\infty} dk \, \Psi_k(x)\Psi_k(x') = \delta(x - x')
\]

are valid \[21\] if

\[c_k = \frac{1}{\pi} \sqrt{2 \kappa \sinh \pi \kappa}.\]  

It is known from scattering theory that the solutions of the stationary Schrödinger equation give asymptotically complete information about a scattering process \[16\]. The asymptotics of \(K_\nu(u)\) for \(u \to 0\) \[20\]

\[K_\nu(u) \to -\frac{\pi}{2} \left[ \frac{2^{-\nu} u^\nu}{\sin \pi \nu \Gamma(1 + \nu)} - \frac{2^\nu u^{-\nu}}{\sin \pi \nu \Gamma(1 - \nu)} \right]\]  

delivers for the wave functions

\[
\Psi_k(x) \to a_k' \frac{e^{ikx}}{\sqrt{2\pi}} + a_k e^{-ikx} \sqrt{2\pi}, \quad \text{as} \quad x \to -\infty.
\]

The coefficient of the out-going wave

\[a_k = -i \frac{2^{ik}}{\Gamma(1 - ik) \sqrt{\pi \kappa \sinh \pi \kappa}}\]  

(C.7)
is a phase due to the relation
\[ \Gamma(1 + ik)\Gamma(1 - ik) = \frac{\pi k}{\sinh \pi k}. \] (C.8)

The eigenstate \( \Psi_k^-(x) = a_k \Psi_k(x) \) has therefore the asymptotics
\[ \Psi_k^-(x) \to \frac{e^{ikx}}{\sqrt{2\pi}} + S_k \frac{e^{-ikx}}{\sqrt{2\pi}}, \quad \text{as} \quad x \to -\infty. \] (C.9)

They satisfy obviously the normalisation and completeness conditions \((C.3)\), and are given by
\[ \Psi_k^-(x) = d_k K_{ik}(e^x), \quad \text{with} \quad d_k = \sqrt{\frac{2}{\pi}} \frac{2^{ik}}{\Gamma(-ik)}. \] (C.10)

With \((C.8)\) one easily finds for the reflection amplitude \[9\]
\[ S_k = 2^{2ik} \frac{\Gamma(ik)}{\Gamma(-ik)} = \frac{d_k}{d_k^\ast}. \] (C.11)

The momentum representation of the eigenstates is defined by the Fourier transform of the Kelvin function
\[ \langle \Psi_k | k' \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx e^{ik'x} \Psi_k(x). \] (C.12)

This integral diverges and it has to be considered as a generalised function. We regularise \((C.12)\) by a factor \(e^{\epsilon x}\), and using \((C.2)\) as well as the integral \[20\]
\[ \int_{-\infty}^{+\infty} dy e^{iky} \cosh^{\mu} u = \frac{2^{\mu-1}}{\Gamma(\mu)} \Gamma \left( \frac{\mu + ik}{2} \right) \Gamma \left( \frac{\mu - ik}{2} \right) \quad \text{for} \quad \text{Re} \mu > 0, \] (C.13)

we get the regularised momentum representation of the eigenstate \( \Psi_k(x) \) \[10\]
\[ \langle \Psi_k | k' \rangle_\epsilon = c_k \frac{2^{\epsilon + ik'}}{4\sqrt{2\pi}} \Gamma \left( \frac{\epsilon + ik' - ik}{2} \right) \Gamma \left( \frac{\epsilon + ik' + ik}{2} \right), \] (C.14)

which we shall use in Eq. \((35)\).

## D Continuation of integral operators

Since the features of the mechanical Liouville model and the free-particle dynamics on the half-line \( x < 0 \) are similar, it is sufficient to describe the procedure of analytical continuation used in section 3.2 for the free-particle case only. With the required boundary condition \( \psi(0) = 0 \) the Hamilton operator \( \hat{H} = -\hbar^2 \partial_{xx}^2 \) has eigenstates \( \psi_k(x) = \sqrt{2/\pi} \sin kx \), with
eigenvalues $\hbar^2 k^2$. This spectrum is non-degenerated like in Liouville particle dynamics and we choose $k > 0$.

The calculation of matrix elements of exponentials of the coordinate operator

$$V_\beta(k, k') = \langle \psi_k | e^{i\beta x} | \psi_{k'} \rangle = \frac{2}{\pi} \int_{-\infty}^{0} dx \, e^{i\beta x} \sin kx \sin k'x \quad (D.1)$$

so simplifies as compared with Eqs. (D.1). The operator $e^{i\beta x}$ is bounded for $\beta > 0$ and we get

$$V_\beta(k, k') = \frac{1}{\pi} \left( \frac{\beta}{(k - k')^2 + \beta^2} - \frac{\beta}{(k + k')^2 + \beta^2} \right). \quad (D.2)$$

But for $\beta < 0$ the integral (D.1) diverges and $V_\beta(k, k')$ has to be considered again as a generalised function which is a kernel of the operator $e^{i\beta x}$ in the $k$-representation. So we need in $\beta$ a smooth continuation of (D.2) acting on test functions $\psi(k)$. This continuation does not exist for arbitrary $\psi(k)$. Since $k$ and $k'$ are positive, the second term of (D.2) is regular for any $\beta$ and has a trivial smooth continuation, whereas a naive continuation of the first term is singular at $\beta = 0$. Indeed, the integral

$$I_\beta(\psi; k) = \frac{1}{\pi} \int_{0}^{\infty} dk' \frac{\beta}{(k' - k)^2 + \beta^2} \psi(k'), \quad (D.3)$$

has a discontinuity at $\beta = 0$

$$\lim_{\beta \to \pm 0} I_\beta(\psi; k) = \pm \psi(k). \quad (D.4)$$

To construct a smooth continuation of the integral (D.3) we split it into a sum of two integrals $I_\beta = I_\beta^{(+)} + I_\beta^{(-)}$,

$$I_\beta^{(\pm)}(\psi; k) = \pm \frac{1}{2\pi i} \int_{0}^{\infty} dk' \frac{\psi(k')}{k' - (k \pm i\beta)}, \quad (D.5)$$

and choose a class of test functions $\psi(k)$ which are holomorphic on the half-plane Re $k > 0$ and vanish as Re $k \to \infty$. We introduce the Cauchy integrals

$$J_\beta^{(\pm)}(\psi; k) = \pm \frac{1}{2\pi i} \int dz \frac{\psi(z)}{z - (k \pm i\beta)}, \quad (D.6)$$

with the integration contours given in Fig. 4.
The Cauchy integrals are well defined for $a > \beta > 0$, where $a$ is a characteristic parameter of the contour. In this case the integrands of \( (D.6) \) have poles at $z = k \pm i\beta$, and $J^{(\pm)} = \psi(k \pm i\beta)$. $J^{(\pm)}$ is then given by

$$I^{(\pm)}(\psi; k) = \pm \frac{1}{2\pi} \int_0^a d\eta \frac{\psi(i\eta)}{-k \pm i(\eta - \beta)} \pm \frac{1}{2\pi i} \int_0^\infty d\xi \frac{\psi(\xi \pm ia)}{\xi - k \pm i(a - \beta)} + \psi(k \pm i\beta). \quad (D.7)$$

This form of $I^{(\pm)}(\psi; k)$ obviously has a smooth continuation from positive ($0 < \beta < a$) to negative values of $\beta$. Let us denote the continuation of \( (D.7) \) at $\beta = -a$ by $\tilde{I}^{(\pm)}(\psi; k)$. Since at $\beta = -a$ the contour integrals \( (D.6) \) vanish, the first two terms of $\tilde{I}^{(\pm)}$ coincide with $I^{(\pm)}_{\beta}$ and one gets

$$\tilde{I}^{(\pm)}_{\beta}(\psi; k) = I^{(\pm)}_{\beta}(\psi; k) + \psi(k \mp ia). \quad (D.8)$$

Thus we have the following smooth set of integral operators $\tilde{I}^{(\pm)}_{\beta}$:

$$\tilde{I}^{(\pm)}_{\beta}(\psi; k) = \begin{cases} I^{(\pm)}_{\beta}(\psi; k) & \text{for } \beta > 0, \\ \psi(k) & \text{for } \beta = 0, \\ I^{(\pm)}_{\beta}(\psi; k) + \psi(k + i\beta) + \psi(k - i\beta) & \text{for } \beta < 0. \end{cases} \quad (D.9)$$

The kernel \( (D.2) \) for negative $\beta$ therefore becomes

$$V^{(\pm)}_{\beta}(k, k') = \frac{1}{\pi} \left( \frac{\beta}{(k - k')^2 + \beta^2} - \frac{\beta}{(k + k')^2 + \beta^2} \right) + \delta(k - k' + i\beta) + \delta(k - k' - i\beta). \quad (D.10)$$

This defines the smooth continuation which is applied for the calculation of the matrix elements of the vertex operators of the Liouville particle model in section 3.2.
E About hermiticity of $q$-exponentials

Observables which are not generators of symmetry transformations can be at most Hermitian \[22\]. This holds in particular for the coordinate operator $i\hbar \partial_x$ on $L^2(\mathbb{R}_+)$, and we ask whether the operator $e^{-\gamma \hat{q}}$ (or $e^{\gamma \hat{q}}$) and therefore the vertex \[23\] can be Hermitian.

Eq. \[17\] requires for the wave function $\psi(p) \in L^2(\mathbb{R}_+)$ an analytical continuation on the complex half-plane $z = p + i\xi$. Assuming $\psi(z) \to 0$ as $p \to \infty$, the set of wave functions $\psi(p)$, $\mathcal{D}_0$, is dense in $L^2(\mathbb{R}_+)$. But the operator $e^{-\gamma \hat{q}}$ might be Hermitian on a subset $\mathcal{D} \in \mathcal{D}_0$ only, and for any $\psi_1(p)$, $\psi_2(p) \in \mathcal{D}$ hermiticity requires

$$\int_0^{+\infty} dp \; \psi_2^*(p) \psi_1(p - i\hbar \gamma) - \int_0^{+\infty} dp \; \psi_2^*(p + i\hbar \gamma) \psi_1(p) = 0. \tag{E.1}$$

If $\psi(p) \in \mathcal{D}$ then $\psi^*(p) \in \mathcal{D}$, and $[\psi(p - i\hbar \gamma)]^* = \psi^*(p + i\hbar \gamma)$. So $F(z) = \psi_2^*(z) \psi_1(z - i\hbar \gamma)$ is analytical for $p > 0$ and $\oint dz \; F(z) = 0$ over the contour of Fig. 3.

For $\epsilon \to 0$ and $A \to \infty$ the contributions of (1,2) and (3,4) to the contour integral coincide with the first respectively second term of \[17\]. Since the line (2, 3) does not contribute, $\oint dz \; F(z) = 0$ gives for a Hermitian action of $e^{-\gamma q}$ on the Hilbert space the condition

$$\lim_{\epsilon \to 0} \int_0^{\hbar \gamma} d\xi \; \psi_2^*(i\xi + \epsilon) \psi_1(i\xi - i\hbar \gamma + \epsilon) = 0. \tag{E.2}$$

This requires for $\psi(z)$ a definite boundary behaviour on the half-plane. In case we would assume regularity at the boundary $p = 0$, $\lim_{\epsilon \to 0} \psi(\epsilon - i\xi) = f(\xi)$, the hermiticity condition \[17\] becomes

$$\int_0^{\hbar \gamma} d\xi \; f_2^*(\xi) f_1(\hbar \gamma - \xi) = 0. \tag{E.3}$$

This hermiticity condition is obviously satisfied by the periodic functions

$$f(\xi) = \sum_{n > 0} c_n e^{-\frac{2\pi i n \xi}{\hbar \gamma}}, \tag{E.4}$$
but they are not invariant under time evolution $\psi(p) \mapsto e^{-i\frac{p^2}{4m^2}} \psi(p)$, which requires

$$\int_{0}^{h\gamma} d\xi \ f_{2}^{*}(\xi) f_{1}(h\gamma - \xi)e^{i\frac{\xi z}{2\pi}} = 0. \quad (E.5)$$

For $f_{1}(\xi) = f_{2}(\xi) = f(\xi)$, then $f(\xi) = 0$, at least, on a half of the interval $\xi \in (0, h\gamma)$.

From this result one easily obtains properties for the wave functions $\psi(p)$, if we transform by

$$\zeta = \frac{z - 1}{z + 1} \quad (E.6)$$

the half-plane $\text{Re} \ z > 0$ onto the interior of the unit disk $|\zeta| < 1$, the boundary $\text{Re} \ z = 0$ to the unit circle $|\zeta| = 1$, and exclude the point $\zeta = 1$. A theorem [23] tells us that if an analytical function $f(\zeta)$ is bounded for $|\zeta| < 1$, and $f(\zeta) \to 0$ as $|\zeta| \to 1$ on a continuous part of $|\zeta| = 1$, then $f(\zeta)$ is identically zero, i.e., $\psi(p) = 0$.

We conclude that the hermiticity condition (E.2) has a solution only if the analytical functions $\psi(z)$ do not have a regular limit at the boundary $\text{Re} \ z = 0$, and cannot be continued either out of the half-plane. Such analytical functions are called ‘functions with natural boundaries’ [23], but so far we do not have the solution for such a function $\psi(p)$.

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