Screening in Two Dimensional Nonabelian Vortex Systems

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ABSTRACT

We study charge screening in a system of two dimensional nonabelian vortices, at finite temperature. Such vortices are generated after an $SO(3)$ global symmetry group is spontaneously broken to a discrete subgroup $\mathbb{Q}_8$, where the latter is isomorphic to the nonabelian group of quaternions. Poisson-Boltzmann like equations are derived for the various inter-vortex potentials, and the solutions to lowest order in a small fugacity expansion, are shown to behave much like those in abelian or Coulomb charge systems. The consequences for the phase structure of the system are discussed, where the fugacities associated to the thermal production of the various types of nonabelian vortices, are shown to play a key role.

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1 Introduction

Vortex defects in two and three spatial dimensions, have many interesting properties which have found a wide variety of applications in two notable areas, condensed matter physics [1], and cosmology [2]. Indeed it is well known [3] that the occurrence of vortices in 3-dimensional condensed systems such as superfluid helium IV, play an important role in understanding the physical properties of these systems.

Vortex defects in any dimension are characterized by $\pi_1(M)$, the fundamental group of the vacuum manifold $M$ of the theory. The type of vortex defects that occur in for example helium IV are ‘abelian’ in the sense that they are characterized by an abelian fundamental group, isomorphic to the group of integers $\mathbb{Z}$. In 2-spatial dimensions, abelian vortices have particularly simple interactions which allows one to write down the grand canonical partition function for thermal pair creation to all orders in the vortex fugacity in terms of the so called Coulomb gas model [4]. The pioneering work of Berezinsky [5], and Kosterlitz and Thouless [6], into their statistical mechanical properties, showed that a gas of such vortices underwent a novel kind of phase transition at some critical temperature $T_c$. A simple physical picture emerged where for $T < T_c$, vortices and antivortices form a medium of bound pairs which subsequently dissociates into free vortices and antivortices for $T > T_c$. These results when applied to approximately 2-dimensional systems such as helium IV thin films, lead Kosterlitz and Nelson [7] to predict a universal jump in the superfluid density at $T_c$, which was later experimentally verified [8].

In a previous publication [9], we presented the results of some preliminary investigations into the nature of interactions of nonabelian vortices in a particular 2-dimensional model. In this paper we shall focus on the problem of screening of the forces between nonabelian vortices, again in the context of the simple model considered in [9]. Ordinary abelian charge screening is the mechanism behind the Kosterlitz-Thouless phase transition [1]. Although there are a number of ways of exhibiting this screening, for example by exploiting the map between the vortex gas partition function and that of
sine-Gordon theory in 2-dimensions [10], perhaps the most illuminating is to derive a so called Poisson-Boltzmann (P-B) equation [11], satisfied by the linearly screened potential between a test vortex and antivortex placed in the gas. One of the goals of the present paper is to derive a P-B equation for the various inter-vortex potentials for the case of nonabelian vortices. In the model under consideration, there are basically 2 species or types of vortices in the nonabelian system at finite temperature. We shall show that to first order in a small fugacity expansion, the P-B equations effectively become those of a pair of ‘abelian’ P-B equations, one for each vortex type. However because of the nonabelian nature of the vortices in question, the system is not simply reducible into two non-interacting ‘Coulomb’ systems to this order. In fact, because of the nonabelian fundamental group $\pi_1(M)$, we find that the coupling constants of the two Coulomb systems are constrained.

The structure of the paper is as follows. In Section 2 we give a brief review of the 2-dimensional field theory model [12] which describes nonabelian vortices associated with $\pi_1(M) \cong \mathbb{Q}_8$, and which was studied further in [9]. Vortices corresponding to this fundamental group may have some relevance to certain liquid crystals [13]. In section 3, as an introduction to Poisson-Boltzmann equations and to fix notation, we show how to derive the P-B equation for a Coulomb gas, which is known to be physically equivalent to a system of abelian vortices. Sections 4 and 5 are concerned with deriving the same for the nonabelian vortex system defined in section 2, and some conclusions are drawn about the possible phase structure of the system.

## 2 A model of two dimensional nonabelian vortices

Perhaps the simplest model [12] in which nonabelian vortex defects occur, is that in which spontaneous symmetry breaking occurs in two spatial dimensions, with an order parameter $\Phi$ characterizing a system whose total energy $E$ we may choose to be

$$E = \int d^2 x \, \frac{1}{2} \text{Tr} \left[ g^{ab} \partial_a \Phi \partial_b \Phi \right] + V(\Phi)$$

(2.1)
\[ V(\Phi) = \frac{\lambda}{4} \text{Tr} \Phi^4 + \frac{\lambda'}{4} (\text{Tr} \Phi^2)^2 + \frac{\rho}{3} \text{Tr} \Phi^3 - \frac{1}{2} \mu_0^2 \text{Tr} \Phi^2 \] (2.2)

where \( \Phi \) is a scalar field transforming in the five dimensional representation of the symmetry group \( G = SO(3) \), i.e. it is a traceless \( 3 \times 3 \) symmetric matrix.

When \( \rho \neq 0 \), \( V \) has three isolated minima and the unbroken symmetry group is \( U(1) \). This case is not particularly interesting since there are no stable vortex defects produced. This is a result of a topological theorem which states that if the first homotopy group \( \pi_1(G/H) \) is trivial, where \( H \) is the unbroken symmetry group, then the vortices are unstable, and indeed in this case \( \pi_1(SO(3)/U(1)) \simeq \mathbb{I} \). If \( \rho = 0 \) however, \( SO(3) \) is broken down to a discrete subgroup \( \mathbb{IK} \) isomorphic to \( \mathbb{Z}_4 \) the additive group of order 4, and in this case there is a degenerate set of minima of \( V \) which lie on the ellipse \( \varphi_1^2 + \varphi_2^2 + \varphi_1 \varphi_2 = \mu_0^2/(\lambda + \lambda') \), where the latter two fields are the independent eigenvalues of the diagonalized field \( \Phi \). The fundamental group of the manifold of vacuum states of the theory defined in eqs.\((2.1)\) and \((2.2)\), is given by

\[ \pi_1(SO(3)/\mathbb{IK}) \simeq \mathbb{Q}_8 \] (2.3)

\( \mathbb{Q}_8 \) being a nonabelian discrete group of order 8, (isomorphic to the group of quaternions). It is generated by elements \( i, j, k, -i, -j, -k \) and \( -\mathbb{I} \), with

\[ i^2 = j^2 = k^2 = -\mathbb{I} \]

\[ ij = k, \quad jk = i, \quad ki = j \] (2.4)

where \( i, j, k \) define a basis of quaternions. Hence there are three types of nonabelian vortices in this model corresponding to \( i, j \) and \( k \), (the elements \( -i, -j \) and \( -k \) refer to the corresponding anti-vortices, whilst \( -\mathbb{I} \) actually defines a \( \mathbb{Z}_2 \) vortex which is abelian in nature since \( -\mathbb{I} \) commutes with all the other elements in \( \mathbb{Q}_8 \) ). The vortices which are produced by this symmetry breaking are guaranteed to be stable by topological arguments.

In ref.[9], the following ansatz was used to minimize (numerically) the energy \( E \)

\[ \Phi(r, \theta) = \mathcal{G}(\theta) \Phi_{\text{diag}}(r) \mathcal{G}^{-1}(\theta), \] (2.5)
where the diagonal matrix $\Phi_{\text{diag}} = (\varphi_1, \varphi_2, -(\varphi_1 + \varphi_2))$. The $SO(3)$ group elements $G(\theta)$ satisfy the nonabelian vortex boundary conditions

$$G(\theta + 2\pi) = G(\theta) h$$  \hspace{1cm} (2.6)$$

where $h$ is an element of the group $\mathbb{K}$, and $(r, \theta)$ are polar coordinates centered on the vortex core. Explicit solutions for the functions $\varphi_1, \varphi_2$ and $G$ pertinent to describing vortices of various types, were given in [9]. Moreover, it was also shown that the ansatz in eq.(2.5) was stable to perturbations that preserve the boundary conditions, eq.(2.6). In this sense, lowest energy vortices are correctly described by eq.(2.5).

For the energy of the vortex to be finite, $\Phi(r, \theta)$ must tend asymptotically as $r \to \infty$, to a minima of the potential $V(\Phi)$. In addition, $\Phi(r, \theta)$ should vanish sufficiently quickly as $r \to 0$. Classical field configurations $\Phi$ with these properties were obtained by numerical minimization of the energy $E$ and the reader is referred to [9] for further details.

Having mentioned the stability of vortices described by eq.(2.5) to perturbations that maintain the homotopy classes, one still has to be cautious before concluding that such vortices will be stable if produced thermally in a system at finite temperature. This is because dissociation can occur between one vortex type and another, depending on the relative size of the chemical potentials involved. From a group theoretic point of view, this process of dissociation can be understood simply as a consequence of group multiplication, e.g. the relation $ij = k$ in eq.(2.4) implies that a $k$ type vortices could decay into an $i$ and $j$ type. In [9] it was shown that the chemical potentials controlling the concentration of the various types of vortices, depends on the single real parameter $a = \frac{<\varphi_1>}{<\varphi_2>}$, which essentially fixes a point on the ellipse of vacuum states in this model. As the value of this parameter is varied, either one of $i, j$ or $k$ type vortices are unstable to dissociation into the other two. For definiteness, we will restrict $a$ to be in a range such that $k$ types are unstable to dissociation into those of $i$ and $j$. (In passing, it was proved in [9] that the abelian vortices corresponding to the element $-\mathbb{I}$ always dissociates into vortex-antivortex pairs for any choice of $a$, so
we will ignore these in the remain of the paper). To summarize, kinematic constraints simplify the study of nonabelian vortices at finite temperature, at least in the model under consideration, by allowing us to focus on just two types of vortices corresponding to particular non-commuting elements of $Q_8$.

It is the purpose of this paper to try and understand the screening properties of the $(i \text{ and } j)$ type nonabelian vortices, by deriving Poisson-Boltzmann (P-B) like equations for the corresponding inter-vortex potentials. Since the P-B equations are more familiar in the context of screening in systems of abelian vortices, in the next section we shall review their derivation, and the consequences they have for Kosterlitz-Thouless type phase transitions.

3 Coulomb gas system

It is well known that a system of abelian vortices in 2-dimensions can be mapped onto that of the so called Coulomb gas [4]. Abelian means that the fundamental group associated with spontaneous symmetry breaking is itself abelian. Such vortices occur when $U(1)$ symmetry is broken e.g. in superfluids, superconductors etc. [1]. Because of the mapping between systems of Coulomb charges and abelian vortices, the notions of charge and vorticity are interchangeable, and we shall often use both sets of terminology in what follows. Consider then, a Coulomb system of $N$ charged particles, with charges $\pm q$, and let $N_+$ ( $N_-$) be the total number of positive (negative) charges with $N = N_+ + N_-$. Moreover, let us restrict ourselves to neutral Coulomb gases, i.e. those in which $N_+ = N_-$. The total energy of such a configuration is given by

$$H_N = \sum_{a \neq b=1}^N \frac{1}{2} q_a q_b U(r_{ab}) - N \mu$$

(3.1)

where in eq.(3.1), $U(r_{ab})$ is the electrostatic energy of charge $q_a$ due to charge $q_b$ and $r_{ab} = |r_a - r_b|$ is the distance between those charges. The chemical potential $\mu$ is a sum of the electrostatic self-energy $U(0)$ and non-electrostatic contribution, $E_c$ or the
core energy.

$$\mu = -\frac{1}{2} U(0) + E_c$$  \hspace{1cm} (3.2)

The partition function of the system is

$$Z = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\{q_a; \sum q_a = 0\}} \prod_{a=1}^N \int \frac{d^2 r_a}{\pi \xi^2} e^{-H_N/T}$$  \hspace{1cm} (3.3)

where $$\pi \xi^2$$ is a phase-space division factor and $$\xi$$ is a length scale typically the size of a Coulomb particle. In eq.(3.3) and subsequently, we set the Boltzmann constant $$k_B = 1$$. For a system of interacting abelian vortices, $$\xi$$ corresponds to the core radius. The restriction $$\sum_{\{q_a; \sum q_a = 0\}}$$ on the sum over charges in eq.(3.3) follows from the assumption that we are considering a neutral system. Introducing two infinitesimal test charges $$\delta q_+, \delta q_-$$ into the system at fixed positions $$\mathbf{r}_+$$ and $$\mathbf{r}_-$$, we can approximate the partition function of this modified system as follows

$$Z(\delta q, \mathbf{r}_+, \mathbf{r}_-) = 1 + \int d^2 r \ e^{-\frac{1}{2} \delta q_+ \delta q_- V_L(r)/T} \delta^2(\mathbf{r} - (\mathbf{r}_+ - \mathbf{r}_-))$$

$$+ \int \frac{d^2 r}{\pi \xi^2} e^{-[\frac{1}{2} \delta q_+ q_+ V_L(\mathbf{r}_+-\mathbf{r}_+)/T + \frac{1}{2} V_L(0)+E_c]/T - [\frac{1}{2} \delta q_- q_- V_L(\mathbf{r}_-\mathbf{r}_+)/T + \frac{1}{2} V_L(0)+E_c]/T}$$

$$+ \int \frac{d^2 r}{\pi \xi^2} e^{-[\frac{1}{2} \delta q_+ q_- V_L(\mathbf{r}_+-\mathbf{r}_+)/T + \frac{1}{2} V_L(0)+E_c]/T - [\frac{1}{2} \delta q_- q_+ V_L(\mathbf{r}_-\mathbf{r}_+)/T + \frac{1}{2} V_L(0)+E_c]/T}$$

$$+ \ldots$$  \hspace{1cm} (3.4)

where in eq.(3.4), we have only kept terms at most linear in the fugacity $$z = e^{-E_c/T}$$, which is taken to be small.

Since the two test charges introduced into the Coulomb system are not thermally created, there will be no chemical potentials appearing in the second term of eq.(3.4). $$V_L(\mathbf{r})$$ is the linearly screened potential at $$\mathbf{r}$$ which in the absence of background charges, is simply the unscreened Coulomb potential depending logarithmically on distance between charges. The interaction between the test charges $$\delta q_+$$ and $$\delta q_-$$ and the background charges $$q_+$$ and $$q_-$$ which are thermally created, depends on the core energy $$E_c$$ and the self-energy $$V_L(0)$$. We define $$\delta q_+ = -\delta q_- = -\delta q$$, $$q_+ = -q_- = q$$ and the density of positive(negative) background Coulomb gas charges $$n_F^{(\pm)}$$ as

$$n_F^{(\pm)} = \frac{e^{-\frac{1}{2} V_L(0)-E_c}/T}{\pi \xi^2} = \frac{z}{\pi \xi^2} e^{-V_L(0)/2T}$$  \hspace{1cm} (3.5)
then we may rewrite the partition function as

\[
Z(\delta q, r_+, r_-) = 1 + \int d^2 r \, e^{\frac{1}{2}(\delta q)^2 V_L(r)/T} \delta^2(r - (r_+ - r_-))
\]

\[
+ n_F^+ \int \frac{d^2 r}{\pi \xi^2} \, e^{-\frac{1}{2}\delta q V_L(r_+)} + \frac{1}{2}\delta q V_L(r_-)/T
\]

\[
+ n_F^- \int \frac{d^2 r}{\pi \xi^2} \, e^{\frac{1}{2}\delta q V_L(r_+)} - \frac{1}{2}\delta q V_L(r_-)/T + ...
\]

(3.6)

To obtain the linear screening potential at the point \( r_0 = (r_+ - r_-) \), we apply techniques which are familiar in field theory. First introduce a source term or test charge into the generating function or partition function (as we have done above). Then, differentiating the new partition function with respect to these sources gives us a correlation function which is proportional to \( V_L \),

\[
\delta q V_L(r_+ - r_-) = T \frac{\partial \ln Z(\delta q, r_+, r_-)}{\partial \delta q} = \frac{T}{Z(\delta q, r_+, r_-)} \frac{\partial Z(\delta q, r_+, r_-)}{\partial \delta q}
\]

(3.7)

Thus,

\[
\delta q V_L(r_+ - r_-) = \frac{T}{Z} \left\{ \delta q V_L(r_+ - r_-) e^{\frac{1}{2}(\delta q)^2 V_L(r_+ - r_-)/T} \right. \]

\[
+ \frac{q n_F^+}{2T} \int d^2 r \, [-V_L(r - r_+) + V_L(r - r_-)] e^{-\frac{1}{2}\delta q V_L(r_+) - \frac{1}{2}\delta q V_L(r_-)/T} \]

\[
+ \frac{q n_F^-}{2T} \int d^2 r \, [V_L(r - r_+) - V_L(r - r_-)] e^{\frac{1}{2}\delta q V_L(r_+) - \frac{1}{2}\delta q V_L(r_-)/T} + ...
\]

(3.8)

Since we assume small fugacity \( z \) and shall ignore nonlinear terms in \( \delta q \) (by taking \( \delta q \) sufficiently small), one can approximate \( 1/Z \) by unity \( (Z = 1 + O(z)) \). To obtain the Poisson-Boltzmann equation [1, 11], we apply the two dimensional Laplacian \( \nabla^2_{r_0} \) with respect to the relative position of the test charges \( r_0 = (r_+ - r_-) \) to both sides of eq.(3.8)

\[
\delta q \nabla^2_{r_0} V_L(r_+ - r_-) \approx \delta q \nabla^2_{r_0} V_0(r_+ - r_-) e^{\frac{1}{2}(\delta q)^2 V_L(r_+ - r_-)/T}
\]

\[
+ \frac{q n_F^+}{2} \int d^2 r \, \nabla^2_{r_0} [-V_0(r - r_+) + V_0(r - r_-)] e^{-\frac{1}{2}\delta q V_L(r_+) - \frac{1}{2}\delta q V_L(r_-)/T} + \frac{1}{2}\delta q V_L(r_+) - \frac{1}{2}\delta q V_L(r_-)/T
\]

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\[ + \frac{q n_F}{2} \int d^2 r \nabla^2_{r_0} [V_0(r - r_+ - V_0(r - r_-)] e^{\frac{1}{2} \delta q q V_L(r - r_+)/T} - \frac{1}{2} \delta q q V_L(r - r_-)/T + \ldots \]  

\[(3.9)\]

where \(V_0(r)\) is the lowest order (unscreened) approximation of \(V_L(r)\). Since \(V_0(r)\) is the interaction potential of charge particles in two spatial dimensions, it is proportional to the logarithmic function, so that

\[\nabla^2_{r_0} V_0(r - r_+) = -2\pi \alpha \delta^2(r - r_+) \tag{3.10}\]

where \(\alpha\) is a constant with dimensions of temperature, and is the effective coupling constant of the Coulomb gas. Thus we obtain the Poisson-Boltzmann equation as

\[\delta q \nabla^2 V_L(r_+ - r_-) = -2\pi \alpha \delta q \delta^2(r_+ - r_-) + \frac{2\pi q n_F}{2} e^{\frac{1}{2} \delta q q [V_L(r_+ - r_-) - V_L(0)]/T} - \frac{2\pi q n_F}{2} e^{-\frac{1}{2} \delta q q [V_L(r_+ - r_-) - V_L(0)]/T} \tag{3.11}\]

where \(n_F = n^+_F + n^-_F\) is the number density of free charges. Next, we redefine \(V_L(r_+ - r_-)\) as the separation energy between a test charge and background charge,

\[E_{Sep} = V_L(r_+ - r_-) - V_L(0) \rightarrow V_L(r_+ - r_-) \tag{3.12}\]

where \(V_L(0)\) is the energy of zero separation between test charge and background charge. Finally, dropping the \(\tilde{\cdot}\) (tilde) and expanding the exponentials to lowest order in \(\delta q\) we obtain the Poisson-Boltzmann equation corresponding to an abelian (in the sense defined earlier) Coulomb gas

\[\delta q \nabla^2 V_L(r_+ - r_-) = -2\pi \alpha \delta q \delta^2(r_+ - r_-) + \frac{\pi \alpha \delta q q^2 n_F}{T} V_L(r_+ - r_-)\]  



Solutions to eq.(3.12) have been discussed in the literature [1,11] in the context of charge unbinding in the Coulomb system. It is clear that when \(n_F \neq 0\), the potential \(V_L\) is screened, and decreases exponentially with the separation between a pair of opposite charges (or vortex-antivortex pair in the case of abelian vortices). Consequently, the pairs become unstable to dissociation into free charges, and so ‘unbinding’ of the
charge or vorticity occurs. This in turn implies of course, a non-vanishing density of free charges \( n_F \) so is a consistent physical picture. The alternative possibility is that \( n_F = 0 \), in which case the P-B equation for \( V_L \) implies that the latter is logarithmically decreasing with the pair separation, giving rise to a medium of bound neutral pairs. The question of whether \( n_F \) is vanishing or not depends on the temperature \( T \) of the system, i.e. \( n_F = 0 \) if \( T < T_c \), and \( n_F \neq 0 \) if \( T > T_c \), where the critical temperature \( T_c = \frac{\alpha}{4}(1 - 2\pi z) \) define a line of values as \( z \) is varied [5,6] . \( T_c \) marks the position of the so called Kosterlitz-Thouless phase transition of the Coulomb gas [6] . Perhaps the simplest way of obtaining the leading order (in \( z \)) approximation of \( T_c \), (i.e. \( T_c = \frac{\alpha}{4} \)) is by the method of self-consistent screening [7]. One introduces the so called screening length \( \omega(z, T) \), which for an infinite system is related to the density of free charges \( n_F \) by

\[
n_F(z,T) = \frac{T}{\pi \alpha \omega^2(z,T)} \tag{3.13}
\]

where it is clear from eq.(3.13) that \( \omega \) depends on the fugacity \( z \) and \( T \). At the same time, since the linearly screened potential at the origin \( V_L(0) = \ln(\frac{\omega}{\xi}) \), eqs.(3.5) and (3.13) together provide a consistent set of equations determining the value of \( \omega \). There are two self-consistent solutions namely for \( T > T_c \), \( \omega \) is finite (and \( V_L \) exponentially decays) whilst for \( T < T_c \) the screening length is infinite and the potential remains logarithmic, with \( T_c = \frac{\alpha}{4} \) [7]. Alternatively, one can deduce the dependence of \( \lambda \) on \( z \) and \( T \), by utilising the well known map of the Coulomb gas onto the sine-Gordon model [10], and exploiting the known renormalization properties of the latter. This method also gives the next to leading order form of \( T_c \) given above. In section 5, we shall return to the problem of solving the Poisson-Boltzmann equation for systems of nonabelian vortices or ‘charges’, which we shall introduce in the next section.

## 4 System of nonabelian vortices.

We now wish to turn our attentions to a system of two dimensional nonabelian vortices of type \( i \) and \( j \), (i.e. field configurations satisfying the ansatz of eq.(2.5) with twisted
boundary conditions as in eq.(2.6) with $h = i$ and $j$ respectively.) To construct the Poisson-Boltzmann equation for the nonabelian vortex system, we first have to define the infinitesimal charge of a nonabelian vortex. From our ansatz eq.(2.5), the field configuration $\Phi$ depends on the $SO(3)$ group elements $G_\alpha$, which represent rotations about the $x, y,$ or $z$ axes. Therefore, the infinitesimal charges are defined correspondingly as rotations through an infinitesimal angle $\epsilon \theta$ about those axes. For example, the classical field configuration $\Phi$ of an infinitesimal test charge of type $i$ is

$$\Phi_{\epsilon i} = G_{\epsilon i} \Phi_{\text{diag}} G_{\epsilon i}^{-1}$$  

where

$$G_{\epsilon i} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\frac{\epsilon \theta}{2}) & \sin(\frac{\epsilon \theta}{2}) \\ 0 & -\sin(\frac{\epsilon \theta}{2}) & \cos(\frac{\epsilon \theta}{2}) \end{pmatrix}$$

with similar definitions for the vortices of type $j$ and $k$. One can compute the infinitesimal nonabelian vorticity or nonabelian charge of these classical field configurations, by for example calculating the form of the self energy in each case and comparing it to the abelian case. One finds

$$\delta q_i = -\delta q_{-i} = \frac{\epsilon}{\sqrt{2}}(\varphi_1 + 2 \varphi_2)$$

$$\delta q_j = -\delta q_{-j} = \frac{\epsilon}{\sqrt{2}}(2 \varphi_1 + \varphi_2)$$

$$\delta q_k = -\delta q_{-k} = \frac{\epsilon}{\sqrt{2}}(\varphi_1 - \varphi_2)$$

(4.3)

with infinitesimal parameter $\epsilon$. The corresponding finite nonabelian charges are

$$q_i = -q_{-i} = \frac{1}{\sqrt{2}}(\varphi_1 + 2 \varphi_2)$$

$$q_j = -q_{-j} = \frac{1}{\sqrt{2}}(2 \varphi_1 + \varphi_2)$$

$$q_k = -q_{-k} = \frac{1}{\sqrt{2}}(\varphi_1 - \varphi_2)$$

(4.4)

It should be noted that the “charges” defined in eq.(4.4) have dimensions of $\sqrt{\text{energy}}$, compared to the dimensionless ones introduced in section 3.
Having discussed how to define the notion of infinitesimal charge for nonabelian vortices, we are now in a position to consider the ‘nonabelian’ charge-unbinding scheme via the Poisson-Boltzmann description. Let \((N_i - N_j, N_i + N_j)\) be the total number of \(i\) and \(j\) type (anti) vortices present. As usual we define \(N_i = N_{+i} + N_{-i}\) and \(N_j = N_{+j} + N_{-j}\) and again restrict ourselves to neutral systems i.e. to configurations having equal numbers of vortices and antivortices of the same type, so \(N_{+i} = N_{-i}\) similarly for \(j\) types.

Under these circumstances, the partition function of the system is

\[
Z = \sum_{N_i=0}^{\infty} \sum_{N_j=0}^{\infty} \frac{1}{N_i! N_j!} \prod_{a_i=1}^{N_i} \prod_{a_j=1}^{N_j} \sum_{q_a_i=0}^{\Sigma a_i} \sum_{q_a_j=0}^{\Sigma a_j} \frac{1}{N_i! N_j!} \prod_{a_i=1}^{N_i} \prod_{a_j=1}^{N_j} \sum_{q_a_i=0}^{\Sigma a_i} \sum_{q_a_j=0}^{\Sigma a_j} \int \frac{d^2r_{a_i}}{\pi \xi^2} \int \frac{d^2r_{a_j}}{\pi \xi^2} e^{-H(N_i,N_j)/T} \quad (4.5)
\]

Generally, one can write down the hamiltonian of the nonabelian system as

\[
H(N_i, N_j) = \sum_{a_i \neq b_i}^{N_i} \frac{1}{2} q_a_i q_b_i U_{ii}(r_{a_i} - r_{b_i}) + \sum_{a_j \neq b_j=1}^{N_j} \frac{1}{2} q_a_j q_b_j U_{jj}(r_{a_j} - r_{b_j}) + \sum_{a_i, b_j=1}^{N_i, N_j} \frac{1}{2} q_a_i q_b_j U_{ij}(r_{a_i} - r_{b_j}) - \mu_i N_i - \mu_j N_j \quad (4.6)
\]

The quantity \(U_{ii}(r_{a_i} - r_{b_i})\) is the interaction energy due to the charges \(q_{a_i}, q_{b_i}\) of the \(i\) type vortices amongst themselves, with similar definitions for \(U_{jj}\) and \(U_{ij}\). The chemical potentials \(\mu_i\) and \(\mu_j\) are

\[
\mu_i = -1 \frac{1}{2} U_{ii}(0) + E_i^c \\
\mu_j = -1 \frac{1}{2} U_{jj}(0) + E_j^c \quad (4.7)
\]

where \(E_i^c\) and \(E_j^c\) are the corresponding core energies of \(i\) and \(j\) type vortices respectively, explicit expression for which may be found in [9]. We define the fugacities corresponding to the thermal creation of vortices of type \(i\) and \(j\) respectively

\[
z_i = e^{-E_i^c/T}, \quad z_j = e^{-E_j^c/T}. \quad (4.8)
\]

Now we introduce two infinitesimal test charges \(\delta q_i\) and \(\delta q_{-i}\) into our system at the points \(r_{ai}\) and \(r_{-ai}\) where we remind the reader that only the vortices of type \(i\) and \(j\) need be considered if we restrict the parameter \(a\) in the range \((-2 < a < -\frac{1}{2})\).
Then, expanding the partition function denoted by $Z_i(\delta q_i, r_{ei}, r_{-ei})$ for the modified system in the presence of such test charges to linear order in the fugacities (which are again taken to be small), we obtain

$$Z_i(\delta q_i, r_{ei}, r_{-ei}) = 1 + \int d^2r \ e^{-\frac{1}{T} \delta q_i \delta q_{-i} V_{ii}^0(r)/T} \delta^2(r - (r_{ei} - r_{-ei}))$$

where the subscript $i$ on $Z$ indicates the partition function in the presence of an infinitesimal $i$ type vortex and $V_{ii}^0$ is the linearly screened potential between a vortex-antivortex pair of type $i$. The interaction energy $E_{ii}(r - r_{ei}) = \delta q_i q_i U_{ii}(r - r_{ei})$ with similar definitions for the other interaction energies appearing in eqs. (4.9) in terms of the interaction potentials $U_{jj}$ and $U_{ij}$. According to our ansatz for the classical field configuration $\Phi$ of a nonabelian vortex configuration, is

$$\Phi = G \Phi_{diag} G^{-1}$$

and the energy of such a configuration can be approximated by the kinetic energy term in eq.(2.1)

$$E \approx \int d^2r \ Tr \left[ g^{ab} \partial_a \Phi \partial_b \Phi \right]$$

Therefore, it is clear that the unscreened interaction energy $E_{eii}^0(r_i - r_{ei})$ between a background charge $q_i$ and infinitesimal charge $\delta q_i$ at points $r_i$ and $r_{ei}$ is given by

$$\Phi_{eii}(r, r_i, r_{ei}) = G_{ei} G_i \Phi_{diag} G_i^{-1} G_{ei}^{-1}$$

$$E_{eii}^0(r_i - r_{ei}) = \int d^2r \ Tr \left[ g^{ab} \partial_a \Phi_{eii} \partial_b \Phi_{eii} \right]$$

$$= \int d^2r \left\{ 2\epsilon^2 (\varphi_1 + 2 \varphi_2)^2 [\nabla_r A(r - r_{ei})]^2 \\
+ 4\epsilon (\varphi_1 + 2 \varphi_2)^2 \nabla_r A(r - r_{ei}) \cdot \nabla_r A(r - r_i) \right\}$$
\[
+2(\varphi_1 + 2 \varphi_2)^2[\nabla_r A(r - r_i)]^2 \}
= 2(\delta q_i)^2 V_{0}^{ii}(0) + 4\delta q_i q_i V_{0}^{ii}(r_i - r_{\epsilon i}) + 2(q_i)^2 V_{0}^{ii}(0)
\tag{4.12}
\]

where \(\delta q_i = \epsilon q_i\), \(A(r) = \text{Im} \left[ \ln(re^{i\theta}) \right]\) and

\[
V_{0}^{ii}(r_i - r_{\epsilon i}) = \int d^2r \left[ \nabla_r A(r - r_i) \cdot \nabla_r A(r - r_i) \right]
\tag{4.13}
\]
is the unscreened interaction potential between two \(i\)-type vortic es. Furthermore,

\[
V_{0}^{ii}(0) = \int d^2r \left[ \nabla_r A(r - r_i) \right]^2
= \int d^2r \left[ \nabla_r A(r - r_i) \right]^2
\tag{4.14}
\]

are the self-energy of the background and test vortic es at the points \(r\) and \(r_{\epsilon i}\) respectively. Since the test charge \(\delta q_i\) is not thermally created, we should subtract it's self-energy term \(2(\delta q_i)^2 V_{0}^{ii}(0)\) from the right hand side of eq.(4.12). Henceforth, we shall take for the form of the interaction energy \textit{including} the effects of screening, \(E_{ji}\), the following

\[
E_{ji} = 4\delta q_i q_i V_{L}^{ii}(r_i - r_{\epsilon i}) + 2q_i^2 V_{L}^{ii}(0)
\tag{4.15}
\]

( This is analogous to the replacement of the unscreened logarithmic Coulomb potential by \(V_{L}\) in the previous section.) Similarly,

\[
E_{-ji} = -4\delta q_i q_i V_{L}^{ii}(r_i - r_{\epsilon i}) + 2q_i^2 V_{L}^{ii}(0)
\]

\[
E_{ji-i} = -4\delta q_i q_i V_{L}^{ii}(r_i - r_{\epsilon i}) + 2q_i^2 V_{L}^{ii}(0)
\]

\[
E_{-ji-i} = 4\delta q_i q_i V_{L}^{ii}(r_i - r_{\epsilon i}) + 2q_i^2 V_{L}^{ii}(0)
\tag{4.16}
\]

Next we shall consider interaction energies between an \(i\) type test vortex and a background vortex of type \(j\). This now involves 2 possible orientations of the group elements \(G_{\epsilon i}\) and \(G_j\), since the latter elements are non-commuting, but a straight forward calculation gives

\[
\Phi_{\epsilon ij} = G_{\epsilon i} G_j \Phi_{\text{diag}} G_j^{-1} G_{\epsilon i}^{-1}, \quad \Phi_{ji i} = G_j G_{\epsilon i} \Phi_{\text{diag}} G_{\epsilon i}^{-1} G_j^{-1}
\]

\[
13
\]
\[ E_{\epsilon ij}^0 = \frac{1}{2} \left\{ \int d^2 r \text{ Tr} \left[ g^{ab} \partial_a \Phi_{\epsilon ij} \partial_b \Phi_{\epsilon ij} \right] + \int d^2 r \text{ Tr} \left[ g^{ab} \partial_a \Phi_{\epsilon ji} \partial_b \Phi_{\epsilon ji} \right] \right\} \]

\[
= \int d^2 r \left\{ (2 \epsilon^2 (\varphi_1 + 2 \varphi_2)^2 - 3 \epsilon^2 \varphi_2 (2 \varphi_1 + \varphi_2) \sin^2[A(r - r_j)] \right) \right]\]

\[
+ \left( 2 (2 \varphi_1 + \varphi_2)^2 - 3 \epsilon^2 \varphi_2 (2 \varphi_1 + \varphi_2) \sin^2[A(r - r_{\epsilon i})] \right) \right) \right] \}

(4.17)

whilst for a background anti-vortex of type \( j \) we have

\[
\Phi_{\epsilon i - j} = G_{\epsilon i} G_j^{-1} \Phi_{\text{diag}} G_j G_{\epsilon i}^{-1}, \quad \Phi_{- j i} = G_j^{-1} G_{\epsilon i} \Phi_{\text{diag}} G_{\epsilon i}^{-1} G_j
\]

\[ E_{\epsilon i - j}^0 = \int d^2 r \text{ Tr} \left[ g^{ab} \partial_a \Phi_{\epsilon i j} \partial_b \Phi_{\epsilon i j} \right] \]

\[ = E_{\epsilon i j}^0 \]  

(4.18)

From eqs. (4.17), (4.18) one can see that in fact, there are no interaction terms between the two different type of charges \( \delta q_i \) and \( q_j \), which would have corresponded to terms linear in \( \epsilon \). The terms present can be interpreted as the self energies of the \( i \) type test charge and the \( j \) type background charge. It should be noticed here that the self energy of the test charge \( \delta q_i \) has been modified by the \( \sin^2 \) terms. Since the test charge is not thermally created, all such self energy terms should be subtracted from \( E_{\epsilon ij} \) as stated previously.

As is shown in appendix A, the modification terms themselves do indeed take the form of divergent self energies of the test charge \( \delta q_i \), so we are justified in dropping them from the expressions for the interaction energies. Hence finally we have

\[ E_{\epsilon ij} = 2 q_j^2 V_L^{jj}(0) = E_{-ei j} = E_{ei j} = E_{-ei j} \]

(4.19)

Similarly, one can check that

\[ E_{\epsilon jj} = 4 \delta q_j q_j V_L^{jj}(r - r_{\epsilon j}) + 2 q_j^2 V_L^{jj}(0) \]

\[ E_{- ej j} = -4 \delta q_j q_j V_L^{jj}(r - r_{- ej}) + 2 q_j^2 V_L^{jj}(0) \]

\[ E_{ej j} = -4 \delta q_j q_j V_L^{jj}(r - r_{ej}) + 2 q_j^2 V_L^{jj}(0) \]

\[ E_{- ej j} = 4 \delta q_j q_j V_L^{jj}(r - r_{- ej}) + 2 q_j^2 V_L^{jj}(0) \]  

(4.20)
and
\[ E_{ij} = 2q_i^2 V_L^{ii}(0) = E_{-ij} = E_{e_j-i} = E_{-e_j-i} \]  (4.21)

The density of positive(negative) background nonabelian charges is defined as follows
\[
n_i^{(+)} = \frac{e^{-(4V^{ii}_L(0)-E_L)/T}}{\pi \xi^2} n_i^{(-)} = \frac{z_i}{\pi \xi^2} e^{-4V^{ii}_L(0)/T}
\]

(4.22)

Substituting the expressions for the total energies \( E \) of the various configuration as calculated above in terms of the linearly screened potentials, we obtain for the partition function of eq.(4.9)
\[
Z_i(\delta q_i, r_{ei}, r_{-ei}) = 1 + \int d^2 r \ e^{-\frac{1}{2} \delta q_i \delta q_{-i} V_L^{ii}(r) \delta^2 (r - (r_e - r_{ei}))} \\
+ \frac{z_i^+}{\pi \xi^2} \int d^2 r \ e^{-[4\delta q_i q_i V_L^{ii}(r-r_{ei})-4\delta q_i q_{-i} V_L^{ii}(r-r_{-ei})+4q_i^2 V_L^{ii}(0)]/T} \\
+ \frac{z_i^-}{\pi \xi^2} \int d^2 r \ e^{-4q_i^2 V_L^{ii}(0)/T} + ...
\]

(4.23)

Rewriting the partition function in terms of the free charge densities, we obtain
\[
Z_i = 1 + \int d^2 r \ e^{\frac{1}{2} (\delta q_i)^2 V_L^{ii}(r) \delta^2 (r - (r_{ei} - r_{-ei}))} \\
+ n_i^+ \int d^2 r \ e^{-4\delta q_i q_{-i} V_L^{ii}(r-r_{et})-4\delta q_i q_{-i} V_L^{ii}(r-r_{et})+4q_i^2 V_L^{ii}(0)]/T} \\
+ n_i^- \int d^2 r \ e^{-4q_i^2 V_L^{ii}(0)/T} + ...
\]

(4.24)

5 P-B equations for nonabelian vortices

Before we compute the Poisson-Boltzmann equations of the system, we shall consider the relationship between the two different types of nonabelian vortex charge. The
energy density (again dropping the potential term as explained earlier) of a single $i$ type vortex is written as

$$E_i = \text{Tr} \left[ g^{ab} \partial_a \Phi_i \partial_b \Phi_i \right]$$

which, up to core corrections, can be approximated by

$$E_i = \frac{1}{2 r_2^2} \left( (\varphi_1 + 2 \varphi_2) \nabla_r A[r - r_1] \right)^2$$

at distance scales greater than the typical vortex core size $\xi$. In eq.(5.2) we have considered a single vortex of type $i$ located at the point $r_1$ and the field configuration $\Phi_i$ is given by

$$\Phi_i = G_i \Phi_{\text{diag}} G_{-i}$$

as described earlier.

Similarly, for a single $j$-type vortex at the point $r_2$,

$$E_j = \text{Tr} \left[ g^{ab} \partial_a \Phi_j \partial_b \Phi_j \right]$$

$$E_j = \frac{1}{2 r_2^2} \left( (2 \varphi_1 + \varphi_2) \nabla_r A[r - r_2] \right)^2$$
From table 1, it is clear that we can transform $G_i$ to $G_j$ by using the constant transformation matrix $U$, so we can rewrite $\Phi_j$ in terms of $G_i$ as follows

$$\Phi_j = U G_i U^{-1} \Phi_{\text{diag}} U^{-1} G_j U$$  \hfill (5.5)

Thus we could have obtained the energy density of the $j$-type vortex, by substituting using eq.(5.5) into the self-energy of an $i$-type vortex. Since we know that $U$ is unitary, by the property of the trace which is invariant under the similarity transformation

$$\text{Tr}[B] = \text{Tr}[UBU^{-1}]$$

we then conclude that

$$\text{Tr}\left[g^{ab} \partial_a \Phi_j \partial_b \Phi_j\right] = \text{Tr}\left[g^{ab} \partial_a \Phi_i \partial_b \Phi_i\right]_{\varphi_1 + \varphi_2}$$  \hfill (5.6)

which is evident from comparing $E_i$ with $E_j$. It follows that the infinitesimal charges $\delta q_i$ and $\delta q_j$ transform into one another by simply interchanging $\varphi_1$ and $\varphi_2$ in $\Phi_{\text{diag}}$. This is sufficient to show that both infinitesimal charges $\delta q_i$ and $\delta q_j$ depend on the same infinitesimal rotation parameter $\epsilon$.

To find the linear screening potential of the system with both types of infinitesimal charge $\delta q_i$ and $\delta q_j$ present, we simply use the chain rule by differentiating the partition function first with respect to $\epsilon$ and apply the following relations

$$\delta q_i = \frac{\epsilon}{\sqrt{2}}(\varphi_1 + 2 \varphi_2)$$

$$\frac{\partial \delta q_i}{\delta \epsilon} = \frac{1}{\sqrt{2}}(\varphi_1 + 2 \varphi_2) = q_i$$

$$\delta q_j = \frac{\epsilon}{\sqrt{2}}(2 \varphi_1 + \varphi_2)$$

$$\frac{\partial \delta q_j}{\delta \epsilon} = \frac{1}{\sqrt{2}}(2 \varphi_1 + \varphi_2) = q_j$$  \hfill (5.7)

For later use, we shall need to consider the case when infinitesimal test charges of type $i$ and $j$ are simultaneously present in the system. For this system with two different test charges $\delta q_i$ and $\delta q_j$ at $r_{i\epsilon}$ and $r_{ ej}$, the lowest order partition function can be reduced
to

\[ Z_{ij}(\delta q_i, \delta q_j, r_{ei}, r_{ej}) = 1 + \int d^2r \, e^{-\frac{1}{2} \delta q_i \delta q_j V_L^{ij}(r)/T} \delta^2(r - (r_{ei} - r_{ej})) \]

\[ + z_i \int d^2r \, e^{-[E_{ei} + E_{ej}]/T} \]
\[ + z_i \int d^2r \, e^{-[E_{ei} + E_{ej}]/T} \]
\[ + z_j \int d^2r \, e^{-[E_{ej} + E_{ej}]/T} \]
\[ + z_j \int d^2r \, e^{-[E_{ej} + E_{ej}]/T} + ... \] (5.8)

where the subscript \(ij\) on \(Z\) in eq.(5.8) denotes the fact that both \(i\) and \(j\) type test charges are present. Substituting the expressions for the total energy \(E\) from the previous section, we get

\[ Z_{ij}(\delta q_i, \delta q_j, r_{ei}, r_{ej}) = 1 + \int d^2r \, e^{-\frac{1}{2} \delta q_i \delta q_j V_L^{ij}(r)/T} \delta^2(r - (r_{ei} - r_{ej})) \]

\[ + n_i^+ \int d^2r \, e^{-4 \delta q_i \delta q_i V_L^{ii}(r-r_{ei})/T} \]
\[ + n_i^- \int d^2r \, e^{4 \delta q_i \delta q_i V_L^{ii}(r-r_{ei})/T} \]
\[ + n_j^+ \int d^2r \, e^{4 \delta q_j \delta q_j V_L^{jj}(r-r_{ej})/T} \]
\[ + n_j^- \int d^2r \, e^{-4 \delta q_j \delta q_j V_L^{jj}(r-r_{ej})/T} + ... \] (5.9)

The linearly screened potential at \((r_{ei} - r_{ej})\) is given by

\[ \delta q_i V_L^{ii}(r_{ei} - r_{ej}) = T \frac{\partial \ln Z_{ij}}{\partial (\delta q_i)} = \frac{T}{Z_{ij}} \frac{\partial Z_{ij}}{\partial (\delta q_i)} \] (5.10)

and

\[ \delta q_j V_L^{jj}(r_{ej} - r_{ei}) = T \frac{\partial \ln Z_{ij}}{\partial (\delta q_j)} = \frac{T}{Z_{ij}} \frac{\partial Z_{ij}}{\partial (\delta q_j)} \] (5.11)

Now we are in a position to construct a Poisson-Boltzmann equation for the linearly screened potentials between various test charges. In the first case, two test charge will be taken to be of the same type, whilst in the second case we will take different types.

For the same type of test charges, say \(\delta q_i\) and \(\delta q_{-i}\), the linearly screened potential at \((r_{ei} - r_{-ei})\) is given by

\[ \delta q_i V_L^{ii}(r_{ei} - r_{-ei}) = T \frac{\partial \ln Z_i}{\partial (\delta q_i)} = \frac{T}{Z_i} \frac{\partial Z_i}{\partial (\delta q_i)} \] (5.12)
Since we use the two infinitesimal charges $\delta q_i$, $\delta q_j$, we can ignore the higher order terms and therefore approximate $1/Z_i$ by $1$ ($Z_i = 1 + O(z_i, z_j) + O(\delta q_i^2) + O(\delta q_j^2)$). Hence we find

$$\delta q_i V_{L}^{ii}(r_i - r_{-i}) = \delta q_i V_{L}^{ii}(r_i - r_{-i}) e^{\tilde{z}(\delta q_i)^2 V_{L}^{ii}(r_i - r_{-i})/T}$$

$$- \frac{4q_i n_i^+}{T} \int d^2r \left[ V_{L}^{ii}(r - r_{ei}) - V_{L}^{ii}(r - r_{-ei}) \right] e^{-4\delta q_i q_i V_{L}^{ii}(r - r_{ei}) - V_{L}^{ii}(r - r_{-ei})/T}$$

$$- \frac{4q_i n_i^-}{T} \int d^2r \left[ -V_{L}^{ii}(r - r_{ei}) + V_{L}^{ii}(r - r_{-ei}) \right] e^{-4\delta q_i q_i [-V_{L}^{ii}(r - r_{ei}) + V_{L}^{ii}(r - r_{-ei})]/T}$$

(5.13)

To lowest order $V_{L}^{ii}(r) = V_{0}^{ii}(r) + O(z_i, z_j)$ where the unscreened potential $V_{0}^{ii}(r)$ varies logarithmically with distance. Applying the Laplacian operator $\nabla^2_{r_0}$ to both sides of eq.(5.13) where $r_0$ is the relative position of test charges $r_0 = r_{ei} - r_{-ei}$, we get

$$\delta q_i \nabla^2_{r_0} V_{L}^{ii}(r_i - r_{-i}) = \delta q_i \nabla^2_{r_0} V_{L}^{ii}(r_i - r_{-i}) e^{4\tilde{z}(\delta q_i)^2 V_{L}^{ii}(r_i - r_{-i})}$$

$$- \frac{4q_i n_i^+}{T} \int d^2r \nabla^2_{r_0} \left[ V_{L}^{ii}(r - r_{ei}) - V_{L}^{ii}(r - r_{-ei}) \right] e^{-4\delta q_i q_i [V_{L}^{ii}(r - r_{ei}) - V_{L}^{ii}(r - r_{-ei})]/T}$$

$$- \frac{4q_i n_i^-}{T} \int d^2r \nabla^2_{r_0} \left[ -V_{L}^{ii}(r - r_{ei}) + V_{L}^{ii}(r - r_{-ei}) \right] e^{-4\delta q_i q_i [-V_{L}^{ii}(r - r_{ei}) + V_{L}^{ii}(r - r_{-ei})]/T}$$

(5.14)

Since, as mentioned, the unscreened potential $V_{0}^{ii}(r_{ei} - r_{-ei})$ is proportional to the logarithm function, it follows that

$$\nabla^2_{r_0} V_{0}^{ii}(r - r_{ei}) = -2\pi \delta^2(r - r_{ei})$$

(5.15)

Hence, using eqs.(5.14), (5.13) we find a Poisson-Boltzmann equation satisfied by $V_{L}^{ii}(r_{ei} - r_{-ei})$

$$\delta q_i \nabla^2_{r_0} V_{L}^{ii}(r_{ei} - r_{-ei}) = -2\pi \delta q_i \delta^2(r_{ei} - r_{-ei})$$

$$+ 8\pi q_i n_i^+ e^{4\delta q_i q_i [V_{L}^{ii}(r_{ei} - r_{-ei}) - V_{L}^{ii}(0)]/T}$$

$$- 8\pi q_i n_i^- e^{-4\delta q_i q_i [V_{L}^{ii}(r_{ei} - r_{-ei}) - V_{L}^{ii}(0)]/T}$$

(5.16)

Again as in the abelian case we redefine

$$V_{L}^{ii}(r_{ei} - r_{-ei}) - V_{L}^{ii}(0) \rightarrow \tilde{V}_{L}^{ii}(r_{ei} - r_{-ei})$$

(5.17)
We get finally, (dropping \(\tilde{\text{\(\tau\)}}\) 
\[
\nabla^2_{r_0} V_{ij}^L(r_{ei} - r_{-ei}) = -2\pi \delta^2(r_{ei} - r_{-ei}) - \frac{32\pi \alpha_i n_i}{T} V_{ij}^L(r_{ei} - r_{-ei}) 
\tag{5.18}
\]
where \(n_i^+ + n_i^- = n_i\) and \(\alpha_i = q_i^2\) is the analogue of the quantity \(\alpha\) introduced in section 2.

Similarly, for the two test charges of \(j\) and \(-j\) type, we can derive a P-B equation for the corresponding linearly screened potential \(V_{ij}^L(r_{ej} - r_{-ej})\)
\[
\nabla^2_{r_0} V_{ij}^L(r_{ej} - r_{-ej}) = -2\pi \delta^2(r_{ej} - r_{-ej}) - \frac{32\pi \alpha_j n_j}{T} V_{ij}^L(r_{ej} - r_{-ej}) 
\tag{5.19}
\]
with \(n_j^+ + n_j^- = n_j\) and \(\alpha_j = q_j^2\) It should be remembered that in deriving the P-B eqs.(5.16) and (5.19), we have used the property, that \(V_{ij}^L\) is vanishing at lowest order, as discussed in section 4 immediately following eqs.(4.18). This is why \(V_{ij}^L\) plays no role in the P-B equations derived above. A consistency condition on this property of the screened potential can be derived by considering a P-B equation for \(V_{ij}^L\) itself, and showing that the solution to lowest order is indeed vanishing. To derive such a P-B equation, we consider a simultaneous configuration of different types of infinitesimal test charges \(\delta q_i, \delta q_j\). From (5.10) we obtain
\[
\delta q_i V_{ij}^L(r_{ei} - r_{ej}) = \frac{T}{q_i} \frac{\partial \ln Z_{ij}}{\partial \epsilon} 
= -\delta q_i V_{ij}^L(r_{ei} - r_{ej}) e^{-\frac{1}{2} \delta q_i \delta q_j} V_{ij}^L(r_{ei} - r_{ej}) / T 
\tag{5.20}
\]
with \(\delta q_i V_{ij}^L(r_{ei} - r_{ej})\) 
\[
V_{ij}^L(r) = V_0^{ij}(r) + O(z_i, z_j) 
\tag{5.21}
\]
where as we have shown in Appendix A, $V_{ij}^0(r)$ is in fact a divergent self energy contribution to the test charge, independent of $r$. Applying the Laplacian operator $\nabla^2_{r_0}$ to both sides of eq.(5.20), where $r_0$ is the relative position of the test charges $r_0 = r_{\epsilon i} - r_{\epsilon j}$, we obtain

$$
\delta q_i \nabla^2_{r_0} V_{ij}^L (r_{\epsilon i} - r_{\epsilon j}) = -\delta q_i \nabla^2_{r_0} V_{ij}^L (r_{\epsilon i} - r_{\epsilon j}) e^{-\frac{1}{2} \delta q_i \delta q_j V_{ij}^L (r_{\epsilon i} - r_{\epsilon j}) / T} - \frac{4q_i^2 n_i^+}{q_j T} \int d^2 r \left[ \nabla^2_{r_0} V_{ij}^L (r - r_{\epsilon i}) \right] e^{-4 \delta q_i \delta q_j V_{ij}^L (r - r_{\epsilon i}) / T} + \frac{4q_i^2 n_i^-}{q_j T} \int d^2 r \left[ \nabla^2_{r_0} V_{ij}^L (r - r_{\epsilon i}) \right] e^{4 \delta q_i \delta q_j V_{ij}^L (r - r_{\epsilon i}) / T} + \frac{4q_j^2 n_j^+}{q_j T} \int d^2 r \left[ \nabla^2_{r_0} V_{ij}^L (r - r_{\epsilon j}) \right] e^{-4 \delta q_i \delta q_j V_{ij}^L (r - r_{\epsilon j}) / T} - \frac{4q_j^2 n_j^-}{q_j T} \int d^2 r \left[ \nabla^2_{r_0} V_{ij}^L (r - r_{\epsilon j}) \right] e^{4 \delta q_i \delta q_j V_{ij}^L (r - r_{\epsilon j}) / T} + ... \tag{5.22}
$$

From the properties of the lowest order potentials $V_{0i}^i, V_{0j}^j$ and $V_{0ij}$ discussed above, the P-B equation becomes

$$
\delta q_i \nabla^2_{r_0} V_{ij}^L (r_{\epsilon i} - r_{\epsilon j}) = \frac{8\pi q_i^2 n_i^+}{q_j} e^{-4 \delta q_i \delta q_j V_{ij}^L (0) / T} - \frac{8\pi q_i^2 n_i^-}{q_j} e^{4 \delta q_i \delta q_j V_{ij}^L (0) / T} + \frac{8\pi q_j^2 n_j^+}{q_j} e^{-4 \delta q_i \delta q_j V_{ij}^L (0) / T} + \frac{8\pi q_j^2 n_j^-}{q_j} e^{4 \delta q_i \delta q_j V_{ij}^L (0) / T} \tag{5.23}
$$

For neutral system, we know that $n_i^+ = n_i^-$, then expanding the exponential terms out to the lowest order in fugacities, we get the final form of the P-B equation for $V_{ij}^L$

$$
\nabla^2_{r_0} V_{ij}^L (r_{\epsilon i} - r_{\epsilon j}) = \frac{32\pi \alpha_i n_i}{q_j T} V_{ij}^L (0) - \frac{32\pi \alpha_j n_j}{T} V_{ij}^L (0) \tag{5.24}
$$

A similar equation can be derived for $V_{ji}^L$ which is given by the r.h.s. of eq.(5.24) but with the subscripts $i$ and $j$ interchanged. Notice that an important difference between...
eq.(5.24) and the previous two P-B equations, (5.18) and (5.19) is the absence of the (fugacity independent) delta function term on the r.h.s.. Indeed the r.h.s. of eq.(5.24) is simply constant in the variable $r_0^3$. Before moving on to discuss the solutions and consequences of these equations, it is clear that the role of the potential $V_{ij}^L$ can be ignored, to the order in which we work.

A comparison of eqs.(5.18) and (5.19) to that of the P-B equation derived in section 2 for the abelian Coulomb gas, shows that the potentials $V_{ii}^L$ and $V_{jj}^L$ are essentially Coulombic in nature with the free charge density and fugacity $n_F$ and $z$ replaced by the same quantities with subscripts $i, j$ appropriate for the system of nonabelian vortices. Consequently, the solutions of eqs.(5.18) and (5.19) are exactly of the type described in section 3, with the replacement of fugacity $z$ by $z_i, z_j$ and screening length $\omega$ by $\omega_i(z_i,T)$ and $\omega_j(z_j,T)$. Furthermore we see that there will in principle be two lines of critical temperatures $T_{ci}$ and $T_{cj}$ given by

$$T_{ci} = \frac{\alpha_i}{4}(1 - 2\pi z_i)$$

$$T_{cj} = \frac{\alpha_j}{4}(1 - 2\pi z_j)$$

(5.25)

It is perhaps worth commenting at this point, that the two critical temperatures given in eqs.(5.25) for vanishing fugacities $z_i$ and $z_j$, are in agreement with those one could have obtained through heuristic free energy arguments first discussed in [6]. If we imagine increasing the temperature of the system from $T = 0$, eqs.(5.25) indicate that either the $i$ followed by the $j$ type vortices dissociate or vice versa depending on the relative magnitude of the critical temperatures in eq.(5.25). Note that since the quantities $\alpha_i$ and $\alpha_j$ can be re-expressed in terms of the vacuum values of the fields $\varphi_1$ and $\varphi_2$

$$\alpha_i = \frac{1}{2} < \varphi_1 + 2 \varphi_2 >^2$$

Thus the vanishing of $V_{ij}^L$ found previously to lowest order in fugacity is consistent with the P-B equation for $V_{ij}^L$.
\[
\alpha_j = \frac{1}{2} \langle \varphi_2 + 2\varphi_1 \rangle^2
\]

There is a constraint on \(\alpha_i\) and \(\alpha_j\) since the vacuum values of \(\varphi_1, \varphi_2\) must lie on the ellipse \(\varphi_1^2 + \varphi_2^2 + \varphi_1\varphi_2 = \mu_0^2/(\lambda + \lambda')\). This constraint in terms of \(\alpha_i\) and \(\alpha_j\) is

\[
\alpha_i + \alpha_j - \sqrt{\alpha_i\alpha_j} = \frac{3\mu_0^2}{2(\lambda + \lambda')}
\]

which shows that the values of \(\sqrt{\alpha_i}, \sqrt{\alpha_j}\) also parameterize an ellipse. It is clear from this that there exists an ellipse of critical temperatures \(T_{c_i}^i\) and \(T_{c_j}^j\), as the fugacities \(z_i\) and \(z_j\) are taken to zero,

\[
T_{c_i}^i + T_{c_j}^j - \sqrt{T_{c_i}^i T_{c_j}^j} = \frac{3\mu_0^2}{2(\lambda + \lambda')}
\]

This curious phase structure is clearly a consequence of the degeneracy present in the vacuum manifold which as we have seen in section 2, is an essential feature of the nonabelian vortices considered in this paper.

In conclusion, we have investigated in this paper lowest order screening in a system of nonabelian vortices in 2-dimensions, through a detailed study of Poisson-Boltzmann like equations for the various inter-vortex potentials. Remarkably, we have seen that such equations are similar in form to those of an abelian system of vortices, the latter being physically equivalent to a Coulomb gas. This feature might be a consequence of the simple model of nonabelian vortices we studied and perhaps also to the fact that we considered only the lowest non-trivial order in a small fugacity expansion. It would certainly be interesting to verify if these features persist in other models of nonabelian vortices, involving different homotopy groups. In the present model, we find a Kosterlitz-Thouless type phase structure in the underlying system, i.e. both \(i\) and \(j\) type vortices undergo K-T like phase transitions at some particular critical temperatures. What is interesting, and a direct consequence of the nonabelian nature of the system, is the connection between these critical temperatures, as evident in eq. (5.28).

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Acknowledgements

The work of S.T. was supported by the Royal Society of Great Britain.

Appendix A

Self Energy Modifications

In this section we wish to show that the term $V_M$. 

$$V_M = \int d^2z \sin^2[A(z - z_2)] \partial_z A(z - z_1) \partial_z A(z - z_1) \quad (A.1)$$

(see eqs.(4.17) and (4.18)), can indeed be interpreted as modifications to the divergent self energy of test charges placed in the system, as claimed in section 4. In eqn. (A.1) $z_1, z_2$ are the positions of the $i$ type test charge and $j$ type background charge respectively. Rather than compute $V_M$ by integrating, we instead consider $\nabla^2 V_M$ since this quantity appears in the corresponding P-B equation. One finds

$$\partial_{z_1} \partial_{z_1} V_M = \partial_{z_1} \int d^2z \sin^2[A(z - z_2)] 
\times \left[ \partial_{z_1} \partial_{z} A(z - z_1) \partial_z A(z - z_1) + \partial_z A(z - z_1) \partial_{z_1} \partial_{z} A(z - z_1) \right]$$

$$= \partial_{z_1} \int d^2z \sin^2[A(z - z_2)] \left[ \partial_{z_1} \delta^2(z - z_1) \frac{1}{z - z_1} + \partial_{z_1} \delta^2(z - z_1) \frac{1}{z - z_1} \right. 
\left. + \delta^2(z - z_1) \delta^2(z - z_1) + \frac{1}{|z - z_1|^4} \right] \quad (A.2)$$

where we remind the reader that $A(z - z_1) = \text{Im}(\ln(z - z_1))$

One can see that all terms in eq.(A.2) are manifestly singular except the last term, and hence may be subtracted from the Hamiltonian of the system as discussed in section 4. Let us concentrate then on the remaining term,

$$\int d^2z \frac{\sin^2[A(z - z_2)]}{|z - z_1|^4} \quad (A.3)$$
By a change of variables

\[
\sin^2[A(z' + \omega_i)] = \frac{1}{2} - \frac{1}{4} \left[ \left( \frac{z' + \omega_i}{z' + \bar{\omega}_i} \right) + \left( \bar{z}' + \bar{\omega}_i \right) \right] \tag{A.4}
\]

we obtain

\[
\int \frac{d^2 z}{|z - z_1|^4} \left\{ \frac{1}{2} - \frac{1}{4} \left[ \left( \frac{z - z_2}{z - \bar{z}_2} \right) + \left( \bar{z} - \bar{z}_2 \right) \right] \right\} = \int \frac{d^2 z}{|z - z_1|^4} \left\{ \frac{1}{2} - \frac{1}{4} \left[ 2\text{Re}[zz] + 2\text{Re}[z_2 z_2] - 4\text{Re}[zz_2] \right] \right\} \tag{A.5}
\]

Next, we wish to consider the behaviour of the integral as \( z \to z_1 \). In this respect we define

\[
z' = z - z_1 = r' e^{i\theta} \]

\[
\Delta = z_1 - z_2 = d \cos \beta + i d \sin \beta \tag{A.6}
\]

so that we get for the integral in eq. (A.3)

\[
\int \frac{d^2 r'}{r'^4} \left\{ \frac{1}{2} - \frac{1}{4} \left[ \frac{2 r'^2 \cos 2\theta + 2 d^2 \cos 2\beta - 4 r'd \cos(\theta + \beta)}{(r' + d)^4} \right] \right\} \tag{A.7}
\]

which has divergent contributions when \( z \to z_1 \). In fact the \( 1/r'^4 \) term in this expression is exactly what we would expect to obtain if we act on the (usual) logarithmically divergent self energy of a single vortex in 2-dimensions, with \( \nabla^2 \).

Having shown that \( \partial_{z_1} \partial_{\bar{z}_1} V_M \) gives rise to divergent terms we can conclude that contributions like \( V_M \) can be interpreted as self energy modifications to the test charges, and hence should be subtracted from the effective hamiltonian of our system.
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