Algebraic analysis on scalar generalized Verma modules of Heisenberg parabolic type I.: \( A_n \)-series

Libor Křižka, Petr Somberg

Abstract

In the present article, we combine some techniques in the harmonic analysis together with the geometric approach given by modules over sheaves of rings of twisted differential operators (D-modules), and reformulate the composition series and branching problems for objects in the Bernstein-Gelfand-Gelfand parabolic category \( \mathcal{O}^p \) geometrically realized on certain orbits in the generalized flag manifolds. The general framework is then applied to the scalar generalized Verma modules supported on the closed Schubert cell of the generalized flag manifold \( G/P \) for \( G = SL(n+2, \mathbb{C}) \) and \( P \) the Heisenberg parabolic subgroup, and the algebraic analysis gives a complete classification of \( \mathfrak{g}'_r \)-singular vectors for all \( \mathfrak{g}'_r = \mathfrak{sl}(n-r+2, \mathbb{C}) \subset \mathfrak{g} = \mathfrak{sl}(n+2, \mathbb{C}) \), \( n-r > 2 \). A consequence of our results is that we classify \( SL(n-r+2, \mathbb{C}) \)-covariant differential operators acting on homogeneous line bundles over the complexification of the odd dimensional CR-sphere \( S^{2n+1} \) and valued in homogeneous vector bundles over the complexification of the CR-subspheres \( S^{2(n-r)+1} \).

Keywords: Generalized Verma modules, composition series, branching laws, sheaf of rings of twisted differential operators, D-modules, covariant differential operators, CR-sphere.

2010 Mathematics Subject Classification: 53A30, 22E47, 33C45, 58J70.

Contents

Introduction

1 D-modules on generalized flag manifolds
   1.1 Sheaves of rings of twisted differential operators
   1.2 Generalized flag manifolds

2 Generalized Verma modules
   2.1 Geometric realization of generalized Verma modules
   2.2 Algebraic Fourier transform
   2.3 Singular vectors in generalized Verma modules

3 \( A_n \)-series of Lie algebras with Heisenberg parabolic subalgebras
   3.1 Representation theoretical conventions
   3.2 The branching problem for the pair \( (\mathfrak{g}, \mathfrak{g}'_r) \)
   3.3 Embedding of \( \mathfrak{g} \) into the Weyl algebras \( \mathbb{A}_n^p \) and \( \mathbb{A}_n^{p*} \)
   3.4 Algebraic analysis on generalized Verma modules and singular vectors
   3.5 General structure of singular vectors
   3.6 Examples

4 CR-equivariant differential operators

Appendix A The Fischer decomposition for \( \mathfrak{sl}(n, \mathbb{C}) \)
Introduction

Let \( g' \subset g \) be a pair of reductive Lie algebras of reductive Lie groups \( G' \subset G \), \( P \subset G \) and \( P' \subset G' \) their parabolic subgroups, \( P' = P \cap G' \), \( p \) and \( p' \) the parabolic Lie algebras of \( P \) and \( P' \), respectively. The subject of the present paper is the study of the \( g' \)-composition series of objects in the Bernstein-Gelfand-Gelfand (BGG for short) parabolic category \( \mathcal{O}^p \) associated to the pair \((g, p)\). The question of composition series for an object in \( \mathcal{O}^p \) and a reductive Lie subalgebra \( g' \) can be rephrased in terms of the existence of singular vectors or homomorphisms between generalized Verma modules, and belongs to branching problems in representation theory. A systematic approach to such problems was initiated in \cite{16, 17, 13, 22, 21}. Despite the fact that the branching problem can have a continuous spectrum, \cite{20}, there is a wide class of discrete multiplicity free branching problems, \cite{22}. This property is fulfilled by our example discussed in the second half of the article.

There are two, mutually dual, motivations for writing the present article. The first motivation is to develop a tool which allows a complete classification scheme for a wide class of modules of interest in the BGG parabolic category \( \mathcal{O}^p \). Secondly, we would like to get an explicit description of \( G' \)-covariant differential operators acting between homogeneous vector bundles on generalized flag manifolds \( G/P \) and \( G'/P' \). The \( G' \)-covariant differential operators differentiate sections of a homogeneous vector bundle on \( G/P \) and then restrict them on the submanifold \( G'/P' \subset G/P \). The existence of covariant differential operators is guaranteed by the discrete decomposability for the branching problem in the BGG parabolic category \( \mathcal{O}^p \), cf. \cite{16, 19} for a general theory or \cite{21} Section 3 for a review. The multiplicity-free condition of the branching problem then assures uniqueness of covariant differential operators.

In \cite{21} Introduction, two problems were formulated. For \( g' \subset g \) as above and \( M^g_{\mathbf{p}}(\lambda) \) the generalized Verma \( g \)-module induced from an irreducible finite-dimensional \( \mathbf{p} \)-module \( V_{\lambda} \) with highest weight \( \lambda \), find the generators (or precise positions) of irreducible \( g' \)-submodules in \( M^g_{\mathbf{p}}(\lambda) \) and find the associated Jordan-Hölder series for \( g' \). Every irreducible \( g' \)-submodule of an object in \( \mathcal{O}^p \) contains a singular vector and vice versa, every singular vector generates a finite length \( g' \)-submodule. In particular, the singular vectors are responsible for the description of composition series in the Grothendieck group \( K(\mathcal{O}^p) \) of the BGG parabolic category \( \mathcal{O}^p \).

A class of simple examples related to orthogonal Lie algebras \( g, g' \) and parabolic subalgebras with commutative nilradicals is discussed in \cite{22, 21}. The technique to find the \( g' \)-singular vectors is based on the geometric realization of generalized Verma modules as twisted \( \mathcal{D} \)-modules supported on the closed (point) Schubert cell in \( G/P \). Then the complicated algebro-combinatorial problem characterizing the singular vectors and thus the branching problem is converted, by algebraic Fourier transform, to a system of partial differential equations acting on the algebra of polynomials on the opposite nilradical. This step allows to determine completely its space of solutions.

The aim of our article is twofold. We follow the seminal work of Beilinson and Bernstein, \cite{3}, and generalize the approach and results achieved in \cite{22, 21}, towards the realization and characterization of \( (g, N) \)-modules in the BGG parabolic category \( \mathcal{O}^p \) as \( N \)-equivariant twisted \( \mathcal{D} \)-modules supported on \( N \)-orbits in \( G/P \) \((N \subset P \) is the nilpotent subgroup of \( P \)). The situation in \cite{22, 21} is then recovered by considering the generalized Verma \( g \)-modules supported on the closed point \( N \)-orbit \( \{eP\} \subset G/P \). In a forthcoming work, we employ the partial algebraic Fourier transform and treat the branching problems analogous to ones formulated above for twisted generalized Verma \( g \)-modules. The language of \( \mathcal{D} \)-modules is an indispensable tool to deal with the parabolic subalgebras with non-commutative nilradicals, and allows a natural implementation of the symmetrization map between the universal enveloping algebra and the symmetric algebra associated to the pair \((g, p)\). We would like to remark that even the construction of non-standard homomorphisms between generalized Verma modules in the case \( g' = g \) is a difficult task, not known in general. In the second part of our article, we exploit the results of the first part and give a complete classification of singular vectors (mentioned in the first paragraph of the Introduction) in a particularly important CR (CR stands for Cauchy-Riemann or Complex-Real) case of generalized flag manifolds and their CR flag submanifolds. This means that we consider the case of complex Lie algebras \( g = \mathfrak{sl}(n + 2, \mathbb{C}) \), \( g' = \mathfrak{sl}(n - r + 2, \mathbb{C}) \), such that the nilradicals of their parabolic subalgebras
are isomorphic to Heisenberg Lie algebras and the objects in \( O^p \) are the scalar generalized Verma modules.

The content of our article is as follows. In Section 1, we briefly recollect the notions of sheaves of rings of twisted differential operators \( D^X_\lambda \) and equivariant \( D^X_\lambda \)-modules. Focusing on the case of a generalized flag manifold \( X = G/P \), we describe the homomorphism from \( g \) into \( \Gamma(X, D^X_\lambda) \) for a general pair \((g, p)\). In Section 2, we discuss the geometric realization of generalized Verma modules \( M^G_\lambda(\lambda) \) as \( D^X_\lambda \)-modules supported on the closed \( N \)-orbit \( \{ e P \} \) in \( G/P \). An important device is the symmetrization map from the polynomial algebra \( S(\mathfrak{u}) \) to the universal enveloping algebra \( U(\mathfrak{u}) \) of the opposite nilradical \( \mathfrak{u} \), and the algebraic Fourier transform of \( M^G_\lambda(\lambda) \). In the last part of this section we introduce the key structure of \( g' \)-singular vectors, and explain our approach leading to the complete classification results in particular examples. In Section 3, we apply all steps developed in Section 1 and Section 2 to the simplest examples going beyond the Hermitian symmetric spaces (characterized by parabolic subalgebras with commutative nilradicals). Namely, we focus on the first in the row, and as for the applications the most important, example of \( A_n \)-series of Lie algebras and their parabolic subalgebras with Heisenberg nilradicals together with 1-dimensional inducing representations. Using the tool of harmonic analysis, conveniently organized by the Fischer decomposition of the symmetric algebra of \( \mathfrak{u} \) with respect to \( \mathfrak{sl}(n-r, \mathbb{C}) \oplus \mathfrak{sl}(r, \mathbb{C}), n-r > 2 \), we carry on the complete classification and explicit description of \( g' = \mathfrak{sl}(n-r+2, \mathbb{C}) \)-singular vectors. In Section 4, we discuss the construction of \( G' \)-covariant differential operators on the complexification of the CR-sphere. Finally, in Appendix A, we summarize for reader’s convenience the structure of the Fischer decomposition for the action of \( \mathfrak{sl}(n, \mathbb{C}) \) on the polynomial algebra \( \mathbb{C}[\mathbb{C}^n]^* \oplus \mathbb{C}^n] \).

Let us emphasize that due to the length of the article we decided to present here just the results describing the composition and branching problems in the Grothendieck group \( K(O^p) \) of the BGG parabolic category \( O^p \). The finer property, determined by the Jordan-Hölder series, is intimately related to the factorization properties of special polynomials responsible for the structure of singular vectors (cf. \[21\] for examples related to the Gegenbauer polynomials) and will appear in a forthcoming work. The cases of other series of simple Lie algebras with Heisenberg parabolic subalgebras as well as the cases of twisted Verma modules supported on the other \( N \)-orbits are also the subject of our forthcoming work.

To summarize our achievements, the results in the present article contribute both to the pure representation theoretical problems on the locus of reducibility (and their generalizations for a class of reductive Lie subalgebras) of generalized Verma modules (cf. \[9,2\] for the formulation of Gyoja’s conjectures), and to applications related to the classification and explicit construction of covariant differential operators for CR-geometries and their CR-subgeometries.

## 1 \( D \)-modules on generalized flag manifolds

### 1.1 Sheaves of rings of twisted differential operators

As it follows from the seminal work of Beilinson and Bernstein (\[3\]), representation theory of Lie groups and algebras can be studied with the aid of the geometry of \( D \)-modules on generalized flag manifolds. In this section we review several basic notations and introduce some conventions useful to our further considerations. More detailed information on \( D \)-modules can be found in \[14,15,11\].

Let \( X \) be a complex manifold, \( O_X \) the sheaf of holomorphic functions and \( \Theta_X \) the sheaf of holomorphic vector fields on \( X \).

**Definition 1.1.** A sheaf of rings of twisted differential operators on \( X \) is a sheaf of rings \( D \) on \( X \), equipped with an increasing exhausting filtration \( \{ F_m D \}_{m \in \mathbb{Z}} \) of \( D \) (\( F_m D = 0 \) for \( m < 0 \)) and a morphism of sheaves of rings \( i : O_X \rightarrow D \) satisfying

1. the constant sheaf \( \mathbb{C}_X \) is contained in the center of \( D \), i.e. \( D \) is a \( \mathbb{C}_X \)-algebra,
2. \( i : O_X \rightarrow F_0 D \) is an isomorphism of sheaves of rings,
The constant sheaf of algebras\(\mathcal{O}_X\)-module defined by \(i(\sigma(a)(f)) = [a, i(f)]\) induces the isomorphism \(\varphi_{\mathcal{F}}: \mathcal{F} \rightarrow \mathcal{F}\) of left \(\mathcal{O}_X\)-modules, where \(\mathcal{F}\) is the symmetric algebra of \(\mathcal{O}_X\) over \(\mathcal{O}_X\).

The simplest example of a sheaf of rings of twisted differential operators on \(X\) is the sheaf of rings of differential operators \(\mathcal{D}_X\). If \(\mathcal{D}\) is a sheaf of rings of twisted differential operators on \(X\) and \(L\) an invertible \(\mathcal{O}_X\)-module, then \(L \otimes_{\mathcal{O}_X} \mathcal{D} \otimes_{\mathcal{O}_X} L^{-1}\) is a sheaf of rings of twisted differential operators on \(X\) as well.

There are two basic operations on \(\mathcal{D}\)-modules called the inverse image and the direct image. We shall briefly introduce just the operation of the direct image which will be important later on. Let \(f: X \rightarrow Y\) be a morphism of complex manifolds and let \(\mathcal{D}\) be a sheaf of rings of twisted differential operators on \(Y\). Then we denote by \(f^!\mathcal{D}\) the sheaf of rings of twisted differential operators on \(X\) given by the pull-back of \(\mathcal{D}\), in the case \(\mathcal{D} = \mathcal{D}_Y\) we have \(f^!\mathcal{D}_Y \simeq \mathcal{D}_X\). Denoting \(\mathcal{M}\) a left \(f^!\mathcal{D}\)-module, its direct image is the left \(\mathcal{D}\)-module defined by \(f_* (\mathcal{D}_{Y^!} \otimes_{f^!\mathcal{D}} \mathcal{M})\), where the \((f^{-1}\mathcal{D}, f^!\mathcal{D})\)-bimodule \(\mathcal{D}_{Y^!} \otimes_{f^!\mathcal{D}} \mathcal{M}\) is given by

\[
\mathcal{D}_{Y^!} \otimes_{f^!\mathcal{D}} \mathcal{M} = f^{-1}\mathcal{D} \otimes_{f^{-1}\mathcal{O}_Y} \omega_{X/Y}.
\]

Here \(\omega_{X/Y} = \omega_X \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\omega_Y\), and \(\omega_X, \omega_Y\) are the canonical sheaves of \(X, Y\), respectively.

If \(X\) carries an action of a complex Lie group \(G\), then there is a notion of \(G\)-equivariant sheaves of rings of twisted differential operators on \(X\). We will not give a precise definition of this structure here, because we will not need it in its full generality.

Let \(X\) be a homogeneous space for a complex Lie group \(G\). Then we get a principal \(H\)-bundle \(p: G \rightarrow X\), where \(H\) is the stabilizer of a point in \(X\). We denote by \(\mathfrak{g}\) and \(\mathfrak{h}\) the Lie algebras of the Lie groups \(G\) and \(H\), respectively, and by \(\text{Hom}_H(\mathfrak{h}, \mathbb{C})\) the vector space of \(H\)-equivariant homomorphisms from \(\mathfrak{h}\) to \(\mathbb{C}\). The left action of \(G\) on \(X\) induces the Lie algebra homomorphism

\[
L_X: \mathfrak{g} \rightarrow \Gamma(X, \Theta_X).
\]

The constant sheaf of algebras \(\mathfrak{g}_X\) on \(X\) naturally acts on \(\mathcal{O}_X\) and by the left multiplication on \(U(\mathfrak{g})_X\), where \(U(\mathfrak{g})\) is the universal enveloping algebra of \(\mathfrak{g}\) and \(U(\mathfrak{g})_X\) the corresponding constant sheaf on \(X\). Then \(\mathfrak{g}_X\) naturally acts on \(U_X(\mathfrak{g}) = \mathcal{O}_X \otimes_{\mathbb{C}} U(\mathfrak{g})_X\), and extends to the action of \(U(\mathfrak{g})_X\) on \(U_X(\mathfrak{g})\). Because \(\mathcal{O}_X\) acts on \(U_X(\mathfrak{g})\) as well, \(U_X(\mathfrak{g})\) is the sheaf of rings on \(X\).

Any element \(\lambda \in \text{Hom}_H(\mathfrak{h}, \mathbb{C})\) gives rise to the 1-dimensional representation \(\mathbb{C}_\lambda\) of \(\mathfrak{h}\), defined by

\[
A v = \lambda(A)v, \quad A \in \mathfrak{h}, \ v \in \mathbb{C}.
\]

On the other hand, any \(H\)-equivariant homomorphism \(\lambda: \mathfrak{h} \rightarrow \mathbb{C}\) yields the \(G\)-equivariant morphism \(f_\lambda: \mathcal{V}_X(\mathfrak{h}) \rightarrow \mathcal{V}_X(\mathbb{C})\) of \(\mathcal{O}_X\)-modules associated to the representations \(\mathfrak{h}\) and \(\mathbb{C}\) of \(H\). Since \(\sum_{s \in \mathcal{V}_X(\mathfrak{h})} U_X(\mathfrak{g})(s - f_\lambda(s))\) is a two-sided ideal of \(U_X(\mathfrak{g})\), we obtain a \(G\)-equivariant sheaf of rings of twisted differential operators

\[
\mathcal{D}_X(\lambda) = U_X(\mathfrak{g})/\sum_{s \in \mathcal{V}_X(\mathfrak{h})} U_X(\mathfrak{g})(s - f_\lambda(s))
\]

on \(X\), and any \(G\)-equivariant sheaf of rings of twisted differential operators on \(X\) is isomorphic to \(\mathcal{D}_X(\lambda)\) for a uniquely determined \(\lambda \in \text{Hom}_H(\mathfrak{h}, \mathbb{C})\).

There is an alternative description of \(\mathcal{D}_X(\lambda)\) given as follows. We start with the sheaf of rings of differential operators \(\mathcal{D}_G\) on \(G\) equipped with the natural structure of an \(H\)-equivariant (with respect to the right action of \(H\) on \(G\)) sheaf of rings of twisted differential operators, and consider the left \(\mathcal{D}_G\)-module \(\mathcal{D}_G(\mathfrak{h}, \lambda)\) given by

\[
\mathcal{D}_G(\mathfrak{h}, \lambda) = \mathcal{D}_G/\sum_{A \in \mathfrak{h}} \mathcal{D}_G(R_G(A) + \lambda(A)), \quad \lambda \in \text{Hom}_H(\mathfrak{h}, \mathbb{C}),
\]
for \( R_G : \mathfrak{g} \to \Gamma(G, \Theta_G) \) the Lie algebra homomorphism induced by the right action of \( G \) on itself. Then we get
\[
D_X(\lambda) \simeq (p_* D_G(h, \lambda))^H,
\]
where \((p_* D_G(h, \lambda))^H\) means the subsheaf of \( H \)-invariants of the direct image \( p_* D_G(h, \lambda) \).

Since \( D_X(\lambda) \) is a \( G \)-equivariant sheaf of rings of twisted differential operators on \( X \), we obtain the Lie algebra homomorphism
\[
\alpha_{D_X(\lambda)} : \mathfrak{g} \to \Gamma(X, D_X(\lambda))
\]
described in the following way. For any \( Y \in \mathfrak{g} \), we define the morphism \( \varphi_Y : D_G(h, \lambda) \to D_G(h, \lambda) \) of left \( D_G \)-modules by
\[
P \mapsto PL_G(Y)|_U, \quad P \in D_G(U).
\]
Because \([L_G(Y), R_G(A)] = 0\) for all \( A \in \mathfrak{g} \), the morphism \( \varphi_Y \) gives rise to the \( H \)-equivariant morphism \( \psi_Y : D_G(h, \lambda) \to D_G(h, \lambda) \) of left \( D_G \)-modules. Then we set
\[
\alpha_{D_G(h, \lambda)}(Y) = \psi_Y(u)
\]
with \( u = 1 \) mod \( \sum A \in \mathfrak{h} D_G(R_G(A) + \lambda(A)) \). Furthermore, if we denote by \( D_X(\lambda)^{op} \) the sheaf of rings opposite to \( D_X(\lambda) \), then \( D_X(\lambda)^{op} \) is also a \( G \)-equivariant sheaf of rings of twisted differential operators on \( X \) and we have
\[
D_X(\lambda)^{op} \simeq D_X(2\rho - \lambda),
\]
where the element \( \rho \in \text{Hom}_H(h, \mathbb{C}) \) is given by
\[
\rho(A) = -\frac{1}{2} \text{tr}_{\mathfrak{g}/h} \text{ad}(A), \quad A \in \mathfrak{h}.
\]

### 1.2 Generalized flag manifolds

Let \( G \) be a complex semisimple Lie group, \( H \) be a maximal torus of \( G \) and \( \mathfrak{g} \) and \( \mathfrak{h} \) be the Lie algebras of the Lie groups \( G \) and \( H \), respectively. Then \( \mathfrak{h} \) is a Cartan subalgebra of \( \mathfrak{g} \). We denote by \( \Delta \) the root system of \( \mathfrak{g} \) with respect to \( \mathfrak{h} \), \( \Delta^+ \) the positive root system in \( \Delta \) and \( \Pi \subset \Delta^+ \) the set of simple roots. Furthermore, we associate to the positive root system \( \Delta^+ \) the nilpotent Lie subalgebras
\[
\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \quad \text{and} \quad \bar{\mathfrak{n}} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha},
\]
and the solvable Lie subalgebras
\[
\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \quad \text{and} \quad \bar{\mathfrak{b}} = \mathfrak{h} \oplus \bar{\mathfrak{n}}
\]
of \( \mathfrak{g} \). The Lie algebras \( \mathfrak{b} \) and \( \bar{\mathfrak{b}} \) are the (standard and opposite standard) Borel subalgebras of \( \mathfrak{g} \).

Let us consider a subset \( \Sigma \) of \( \Pi \) and denote by \( \Delta_\Sigma \) the root subsystem in \( \mathfrak{h}^* \) generated by \( \Sigma \). Then the standard parabolic subalgebra \( \mathfrak{p} \) of \( \mathfrak{g} \) associated to \( \Sigma \) is defined by
\[
\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u},
\]
where the reductive Levi factor \( \mathfrak{l} \) of \( \mathfrak{p} \) is defined through
\[
\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_\Sigma} \mathfrak{g}_\alpha
\]
and the nilradical $u$ of $\mathfrak{p}$ and the opposite nilradical $\mathfrak{n}$ are given by

$$u = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta^+_\Sigma} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{n} = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta^+_\Sigma} \mathfrak{g}_{-\alpha}. \quad (1.16)$$

We say that a weight $\lambda \in \mathfrak{h}^*$ is $\mathfrak{p}$-dominant integral provided $\frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{N}_0$ for all $\alpha \in \Sigma$, and we denote the set of all $\mathfrak{p}$-dominant integral weight by $\Lambda^+(\mathfrak{p})$. Furthermore, let $P$ be a parabolic subgroup of $G$ with the Lie algebra $\mathfrak{p}$. Then $p : G \to G/P$ is a principal $P$-bundle and $X = G/P$ is called the generalized flag manifold associated of $G$.

We define the subgroups $\mathfrak{u}$ and $\mathfrak{n}$ of $G$ to be the image of $\mathfrak{n}$ and $\mathfrak{p}$ under the exponential map $\exp : \mathfrak{g} \to G$, respectively. Moreover, the mapping

$$\exp : \mathfrak{u} \to \mathfrak{u} \quad (1.17)$$

is a diffeomorphism and $\mathfrak{u}$ is a closed nilpotent subgroup of $G$. Therefore, for $g \in G$ the mapping $f_g : \mathfrak{u} \to X$ given by

$$f_g(n) = p(gn) = gnP \quad (1.18)$$

is an open embedding, and the generalized flag manifold $X$ is covered by open subsets $U_g = p(g\mathfrak{u}) \subset X$, i.e. we have

$$X = \bigcup_{g \in G} U_g. \quad (1.19)$$

Hence, we get the atlas $\{(U_g, u_g)\}_{g \in G}$ on $X$, where $u_g : U_g \to \mathfrak{u}$ is defined by

$$u_g^{-1} = f_g \circ \exp. \quad (1.20)$$

Now, let $s_g : U_g \to G$ be the local section of the principal $P$-bundle $p : G \to X$ defined through

$$s_g(x) = g f_g^{-1}(x) = g s_e(g^{-1} x), \quad x \in U_g, g \in G. \quad (1.21)$$

The local section $s_g : U_g \to G$ gives the trivialization of the principal $P$-bundle $p : G \to X$ over $U_g$. Therefore, we get the isomorphism

$$\begin{array}{ccc}
\varphi_g : p^{-1}(U_g) & \longrightarrow & U_g \times P \\
p \downarrow & & \downarrow p_1 \\
U_g & \overset{id_{U_g}}\longrightarrow & U_g \\
\end{array} \quad (1.22)$$

of principal $P$-bundles over $U_g$, where the mapping $\varphi_g : p^{-1}(U_g) \to U_g \times P$ is defined by

$$\varphi_g(h) = (p(h), s_g(p(h))^{-1} h) \quad \text{for all} \quad h \in p^{-1}(U_g). \quad (1.23)$$

Denoting by $\boxtimes$ the outer tensor product, we obtain an isomorphism

$$\Phi_g : D_G|_{p^{-1}(U_g)} \iso D_X \times P|_{p^{-1}(U_g)} \iso D_X|_{U_g} \boxtimes D_P \quad (1.24)$$

of $P$-equivariant sheaves of rings of twisted differential operators and consequently the isomorphism

$$(p_* D_G(p, \Lambda)|_{U_g})^P \iso D_X|_{U_g} \boxtimes \Gamma(p_* D_P / \sum_{A \in \mathfrak{p} \cap \mathfrak{p}_+} D_P(R_P(A) + \lambda(A)))^P \quad (1.25)$$

of sheaves of rings of twisted differential operators on $U_g$. Since $D_P / \sum_{A \in \mathfrak{p} \cap \mathfrak{p}_+} D_P(R_P(A) + \lambda(A))$ is isomorphic to $\mathcal{O}_P$ as an $\mathcal{O}_P$-module, we have the isomorphism

$$j_{s_g} : D_X(\lambda)|_{U_g} \iso (p_* D_G(p, \Lambda)|_{U_g})^P \iso D_X|_{U_g} \quad (1.26)$$
of sheaves of rings of twisted differential operators on $U_g$.

**Proposition 1.2.** Let $\lambda \in \text{Hom}_P(p, \mathbb{C})$. Then the Lie algebra homomorphism

$$\alpha_{\mathcal{D}_X(\lambda)} : g \to \Gamma(X, \mathcal{D}_X(\lambda))$$

is given by

$$(j_{s_g} \circ \alpha_{\mathcal{D}_X(\lambda)})(Y) = \pi_g^\lambda(Y),$$

where $\pi_g^\lambda(Y) \in \mathcal{D}_X(U_g)$ is defined by

$$(\pi_g^\lambda(Y)f)(x) = (L_X(Y)f)(x) + \lambda((\text{Ad}(s_g(x)^{-1})Y)_p)f(x)$$

for all $x \in V$, $f \in \mathcal{O}_X(V)$ and $V \subset U_g$.

**Proof.** For $Y \in g$, we define the morphism $\varphi_Y : \mathcal{D}_G \to \mathcal{D}_G$ of left $\mathcal{D}_G$-modules by

$$P \mapsto P\mathcal{L}_G(Y)|_U$$

for all $P \in \mathcal{D}_G(U)$. Since we have $[L_G(Y), R_G(A)] = 0$ for all $A \in g$, the morphism $\varphi_Y$ gives rise to a $P$-equivariant morphism $\psi_Y : \mathcal{D}_G(p, \lambda) \to \mathcal{D}_G(p, \lambda)$ of left $\mathcal{D}_G$-modules. Then we have $\alpha_{\mathcal{D}_X(\lambda)}(Y) = \psi_Y(u) \in \Gamma(X, \mathcal{D}_X(\lambda)) \simeq \Gamma(X, p, \mathcal{D}_G(p, \lambda))^P$, where $u = 1 \mod \sum_{A \in p} \mathcal{D}_G(R_G(A) + \lambda(A))$.

In the next step we compute $j_{s_g}(\psi_Y(u)|_{U_g}) \in \mathcal{D}_X(U_g)$. Let $\varphi_g : p^{-1}(U_g) \to U_g \times P$ be the isomorphism of principal $P$-bundles over $U_g$ given by (1.23). Then the corresponding element $\Phi_g(L_G(Y)|_{p^{-1}(U_g)}) \in \mathcal{D}_X|_{U_g} \boxtimes \mathcal{D}_P$ is given by the formula

$$\Phi_g(L_G(Y)|_{p^{-1}(U_g)})f = ((L_G(Y)|_{p^{-1}(U_g)})(f \circ \varphi_g) \circ \varphi_g^{-1}$$

for all $f \in \mathcal{O}_{X \times P}(U_g \times P)$. Hence we can write

$$(\Phi_g(L_G(Y))f)(x, p) = \frac{d}{dt}|_{t=0} f\left(\exp(-tY).x, s_g(\exp(-tY).x)^{-1}\exp(-tY)s_g(x)p\right)$$

$$= \frac{d}{dt}|_{t=0} f(\exp(-tY).x, p) + \frac{d}{dt}|_{t=0} f(x, s_g(\exp(-tY).x)^{-1}\exp(-tY)s_g(x)p).$$

The first term can be rewritten as

$$\frac{d}{dt}|_{t=0} f(\exp(-tY).x, p) = ((L_X(Y)|_{U_g} \boxtimes 1_p)f)(x, p).$$

For the second term we can write

$$\frac{d}{dt}|_{t=0} f(x, s_g(\exp(-tY).x)^{-1}\exp(-tY)s_g(x)p) = \frac{d}{dt}|_{t=0} f(x, pg(t)),$$

where

$$g(t) = p^{-1}s_g(\exp(-tY).x)^{-1}\exp(-tY)s_g(x)p$$

$$= p^{-1}s_c(g^{-1}\exp(-tY).x)^{-1}g^{-1}\exp(-tY)gs_c(g^{-1}x)p.$$ 

Further, if $g \in \mathcal{U}P$, then there exist uniquely determined elements $\mathcal{U}(g) \in \mathcal{U}$ and $P(g) \in P$ such that $g = \mathcal{U}(g)P(g)$. Since $g(t) \in P$, we can write

$$g(t) = P(p^{-1}s_c(g^{-1}\exp(-tY).x)^{-1}g^{-1}\exp(-tY)gs_c(g^{-1}x)p)$$

$$= p^{-1}P(s_c(g^{-1}\exp(-tY).x)^{-1}g^{-1}\exp(-tY)gs_c(g^{-1}x)p)$$

$$= p^{-1}P(g^{-1}\exp(-tY)gs_c(g^{-1}x)p)$$

$$= p^{-1}P(s_c(g^{-1}x)^{-1}g^{-1}\exp(-tY)gs_c(g^{-1}x)p)$$

$$= p^{-1}P(s_g(x)^{-1}\exp(-tY)s_g(x)p).$$
This gives
\[ \frac{d}{dt} \big|_{t=0} g(t) = -\text{Ad}(p^{-1})(\text{Ad}(s_g(x)^{-1})Y)_p, \]

where \( \text{Ad}(s_g(x)^{-1})Y = (\text{Ad}(s_g(x)^{-1})Y)_\mathfrak{g} + (\text{Ad}(s_g(x)^{-1})Y)_\mathfrak{p} \) is the decomposition of the element \( \text{Ad}(s_g(x)^{-1})Y \) with respect to \( \mathfrak{g} = \mathfrak{g} \oplus \mathfrak{p} \). Therefore, we obtain
\[ \frac{d}{dt} |_{t=0} f(x, s_g(\exp(-tY))x, -1 \exp(-tY)s_g(x)) = -(R_P(\text{Ad}(p^{-1})(\text{Ad}(s_g(x)^{-1})Y)_p)f(x, p). \]

But since the operator \( Q \in \mathcal{D}_G(p^{-1}(U_g)) \) defined by
\[ (\Phi_g(Q)f)(x, p) = (R_P(\text{Ad}(p^{-1})(\text{Ad}(s_g(x)^{-1})Y)_p)f(x, p) + \lambda(\text{Ad}(p^{-1})(\text{Ad}(s_g(x)^{-1})Y)_p)f(x, p) \]
belongs to the ideal \( \sum_{A \in \mathfrak{p}} \mathcal{D}_G(R_G(A) + \lambda(A)) \), the \( P \)-equivariance of \( \lambda: \mathfrak{p} \to \mathbb{C} \) gives
\[ (\Phi_g(L_X(Y)|_U_g)f)(x, p) = ((L_X(Y)|_U_g \otimes 1_P)f)(x, p) + \lambda(\text{Ad}(p^{-1})(\text{Ad}(s_g(x)^{-1})Y)_p)f(x, p) \]
\[ = ((L_X(Y)|_U_g \otimes 1_P)f)(x, p) + \lambda((\text{Ad}(s_g(x)^{-1})Y)_p)f(x, p) \]
\[ + (\Phi_g(Q)f)(x, p). \]

If we put altogether, we obtain
\[ (j_{s_g}(\psi_Y(u)|_\mathfrak{g})f)(x) = (L_X(Y)f)(x) + \lambda((\text{Ad}(s_g(x)^{-1})Y)_p)f(x) \]
for all \( f \in \mathcal{O}_X(U_g) \), and the proof is complete. \( \square \)

Now, let us recall the atlas \( \{ (U_g, u_g) \}_{g \in G} \) on \( X \) defined in (1.20), and let \( (f_1, f_2, \ldots, f_n) \) be a basis of \( \mathfrak{g} \). Then for \( x \in U_g \) we have
\[ u_g(x) = \sum_{i=1}^n u^i_g(x)f_i, \] (1.30)

the functions \( u^i_g: U_g \to \mathbb{C} \) are called the coordinate functions on \( U_g \).

**Theorem 1.3.** Let \( \lambda \in \text{Hom}_P(\mathfrak{p}, \mathbb{C}) \). Then we have
\[ \pi^\lambda_g(Y) = -\sum_{i=1}^n \left[ \frac{\text{ad}(u_g(x))e^{\text{ad}(u_g(x))}}{e^{\text{ad}(u_g(x))} - \text{id}_\mathfrak{g}} (e^{-\text{ad}(u_g(x))}\text{Ad}(g^{-1})Y)_\mathfrak{p} \right] \frac{\partial}{\partial u^i_g} + \lambda(e^{-\text{ad}(u_g(x))}\text{Ad}(g^{-1})Y)_\mathfrak{p} \] (1.31)
for all \( Y \in \mathfrak{g} \), where \([X]_i\) denotes the \( i \)-th coordinate of \( X \in \mathfrak{g} \) with respect to the basis \((f_1, f_2, \ldots, f_n)\) of \( \mathfrak{g} \). In particular, we have
\[ \pi^\lambda_e(Y) = -\sum_{i=1}^n \left[ \frac{\text{ad}(u_e(x))}{e^{\text{ad}(u_e(x))} - \text{id}_\mathfrak{g}} Y \right] \frac{\partial}{\partial u^i_e} \] for \( Y \in \mathfrak{g} \) (1.32)

and
\[ \pi^\lambda_e(Y) = \sum_{i=1}^n [\text{ad}(u_e(x))]_i \frac{\partial}{\partial u^i_e} + \lambda(Y) \] for \( Y \in \mathfrak{l} \). (1.33)

**Proof.** Let \( \{(U_g, u_g)\}_{g \in G} \) be the atlas on \( X \) defined by the formula (1.20). Assuming \( Y \in \mathfrak{g} \), we can write by (1.21)
\[ \text{Ad}(s_g(x)^{-1})Y = \text{Ad}(\exp(-u_g(x))g^{-1})Y = \text{Ad}(\exp(-u_g(x)))\text{Ad}(g^{-1})Y. \]
Hence we get a category relying on the notation introduced in Section 1.2. The Bernstein-Gelfand-Gelfand (BGG for short) algebraic definition of a class of representations called generalized Verma modules, generated by the Borel subalgebra $O_b$, can be rewritten as follows. We have, for $V \subset U_g$ and $f \in \mathcal{O}_X(V)$,

$$(L_X(Y)f)(x) = \frac{d}{dt}|_{t=0} f(\exp(-tY).x) = \frac{d}{dt}|_{t=0} (f \circ u_g^{-1})(u_g(\exp(-tY).x))$$

$$= \sum_{i=1}^{n} \frac{du_g^i(\exp(-tY).x)}{dt}|_{t=0} \frac{\partial}{\partial u_g^i}(f).$$

If we denote $o = eP$, then we can write

$$\frac{d}{dt}|_{t=0} u_g(\exp(-tY).x) = \frac{d}{dt}|_{t=0} u_g(\exp(-tY)s_g(x).o)$$

$$= \frac{d}{dt}|_{t=0} u_g(s_g(x)s_g(x)^{-1}\exp(-tY)s_g(x).o)$$

$$= \frac{d}{dt}|_{t=0} u_g(gs_e(g^{-1}.x)s_g(x)^{-1}\exp(-tY)s_g(x).o)$$

$$= \frac{d}{dt}|_{t=0} \exp^{-1}(U(s_e(g^{-1}.x)s_g(x)^{-1}\exp(-tY)s_g(x)))$$

$$= \frac{d}{dt}|_{t=0} \exp^{-1}(s_e(g^{-1}.x)U(s_g(x)^{-1}\exp(-tY)s_g(x)))$$

$$= \frac{d}{dt}|_{t=0} \exp^{-1}(\exp(u_g(x))) \exp(-t(\Ad(s_g(x)^{-1})Y)_{\pi}).$$

The Baker-Campbell-Hausdorff formula for the nilpotent group $\mathfrak{u}$ gives

$$\exp^{-1}(\exp(X)\exp(tZ)) = X + \frac{\exp(X)\exp(tZ)}{e^{\exp/X} - \text{Id}_{\mathfrak{u}}} tZ + t^2 g(t),$$

where $g(t)$ is a $\mathfrak{u}$-valued polynomial in $t$, for all $X, Z \in \mathfrak{u}$ and $t \in \mathbb{R}$, and so we obtain

$$\frac{d}{dt}|_{t=0} u_g(\exp(-tY).x) = - \frac{\exp(u_g(x))e^{\exp(u_g(x))}}{e^{\exp(u_g(x))} - \text{Id}_{\mathfrak{u}}} (\Ad(s_g(x)^{-1})Y)_{\pi}.$$

Hence we get

$$L_X(Y)|_{U_g} = - \sum_{i=1}^{n} \left[ \frac{\exp(u_g(x))e^{\exp(u_g(x))}}{e^{\exp(u_g(x))} - \text{Id}_{\mathfrak{u}}} (e^{-\exp(u_g(x))}\Ad(g(x)^{-1})Y)_{\pi} \right] \frac{\partial}{\partial u_g^i}$$

and thus we are done. \hfill \Box

## 2 Generalized Verma modules

We remind the algebraic definition of a class of representations called generalized Verma modules, relying on the notation introduced in Section 1.2. The Bernstein-Gelfand-Gelfand (BGG for short) category $\mathcal{O}$ is the full subcategory of the category of $U(\mathfrak{g})$-modules whose objects are finitely generated $U(\mathfrak{g})$-modules, $\mathfrak{h}$-semisimple and locally $\mathfrak{n}$-finite. Let $\mathfrak{h}$ be a parabolic subalgebra of $\mathfrak{g}$ containing the Borel subalgebra $\mathfrak{b}$, and its reductive Levi factor, cf. \cite{12}. The BGG parabolic category $\mathcal{O}^\mathfrak{p}$ is the full subcategory of $\mathcal{O}$ whose objects are locally $\mathfrak{l}$-finite. The category $\mathcal{O}^\mathfrak{p}$ contains the full subcategories $\mathcal{O}^\mathfrak{p}_{\chi^\mathfrak{p}}$ whose objects have generalized infinitesimal character $\chi^\mathfrak{p}$ in the Harish-Chandra parametrization, see \cite{4} \cite{12} for the detailed exposition.

We denote by $K(\mathcal{O}^\mathfrak{p})$ the Grothendieck group of the abelian category $\mathcal{O}^\mathfrak{p}$ defined as the restricted product $K(\mathcal{O}^\mathfrak{p}) = \prod_{\{\lambda\} \in \mathfrak{h}^*/W} K(\mathcal{O}^\mathfrak{p}_{\chi^\mathfrak{p}})$, where $W$ is the Weyl group of $\mathfrak{g}$ with the affine
action on \(h^*\) and \(\prod\) denotes the restricted product in which all up to a countable number of components are zero.

Let \(V_\lambda\) denote a finite-dimensional irreducible \(\mathfrak{g}\)-module with highest weight \(\lambda\) regarded as \(\mathfrak{p}\)-module with trivial action of its nilradical \(u\), where \(\lambda\) is a \(\mathfrak{p}\)-dominant weight \(\lambda \in \Lambda^+ (\mathfrak{p})\). The generalized Verma module \(M^\mathfrak{g}_\lambda (\lambda)\) is defined by

\[
M^\mathfrak{g}_\lambda (\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V_\lambda,
\]

and \(M^\mathfrak{g}_\lambda (\lambda)\) is called to be of scalar type provided \(V_\lambda\) is a 1-dimensional \(\mathfrak{p}\)-module. The highest weight modules \(M^\mathfrak{g}_\lambda (\lambda)\) are objects in \(\mathcal{O}^\mathfrak{p}\), and any irreducible \(U(\mathfrak{g})\)-module in \(\mathcal{O}^\mathfrak{p}\) is the quotient of some \(M^\mathfrak{g}_\lambda (\lambda)\) by its maximal submodule. Let us notice that in our conventions for the Harish-Chandra parametrization of maximal ideals in the center \(Z\) of the universal enveloping algebra \(U(\mathfrak{g})\), the trivial \(U(\mathfrak{g})\)-module has \(Z\)-infinitesimal character \(\chi_{\rho_b}\) and \(M^\mathfrak{g}_\lambda (\lambda)\) has \(Z\)-infinitesimal character \(\chi_\lambda + \rho_b\). Here \(\rho_b \in h^*\) stands for the Weyl vector, i.e. \(\rho_b\) is the half-sum of positive roots of \(\mathfrak{g}\).

### 2.1 Geometric realization of generalized Verma modules

In the present section we rely on the notation of Section 1.1 and Section 1.2. Let us consider an \(N\)-orbit \(Q \in \mathcal{O}^\mathfrak{p}\) in the generalized flag manifold \(X\) and denote by \(i: Q \hookrightarrow X\) its embedding into \(X\). Further, let \(\tau\) be an irreducible object in the category \(\text{Con}(i^! \mathcal{D}_X^\mathfrak{X}, N)\) of \(N\)-equivariant integrable \(i^! \mathcal{D}_X^\mathfrak{X}\)-connections, i.e. the category of \(N\)-equivariant \(i^! \mathcal{D}_X^\mathfrak{X}\)-modules which are locally free \(\mathcal{O}_\mathcal{Q}\)-modules of finite rank. Here \(i^! \mathcal{D}_X^\mathfrak{X}\) denotes the pull-back of \(\mathcal{D}_X^\mathfrak{X}\) and \(\mathcal{D}_X^\mathfrak{X} = \mathcal{D}_X (\lambda + \rho)\). Then to a given pair \((Q, \tau)\) we attach an \(N\)-equivariant regular holonomic \(\mathcal{D}_X^\mathfrak{X}\)-module \(\mathcal{I}(Q, \tau)\) defined as the maximal regular holonomic extension of \(i^! (\mathcal{D}_X^\mathfrak{X} \otimes_{i^! \mathcal{D}_X^\mathfrak{X}} \tau)|_{X \setminus \partial Q}\) for \(\partial Q = \mathcal{Q} \setminus Q\), see [14].

In the present article we are mostly interested in the generalized Verma modules, related to the closed \(N\)-orbit \(X_e = \{ eP \}\). The case of other orbits realizing twisted generalized Verma modules, will be discussed elsewhere. For the closed embedding

\[
i_e: X_e \hookrightarrow X
\]

we have \(i_e^! \mathcal{D}_X^\mathfrak{X} \simeq \mathcal{D}_{X_e}\) as \(N\)-equivariant sheaves of rings of twisted differential operators on \(X_e\). Since \(\mathcal{O}_{X_e}\) is an irreducible object in the category \(\text{Con}(i^! \mathcal{D}_X^\mathfrak{X}, N)\) and \(X_e\) is a closed orbit, we get \(\mathcal{I}(X_e, \mathcal{O}_{X_e}) = i_e^! (\mathcal{D}_{X_e}^\mathfrak{X} \otimes_{i^! \mathcal{D}_X^\mathfrak{X}} \mathcal{O}_{X_e})\).

**Proposition 2.1.** Let \(\lambda \in \text{Hom}_{\mathfrak{g} (\mathfrak{p}, \mathbb{C})}\). Then we have

\[
\Gamma (X, \mathcal{I}(X_e, \mathcal{O}_{X_e})) \simeq M^\mathfrak{g}_\lambda (\lambda - \rho)
\]

as \((\mathfrak{g}, N)\)-modules.

**Proof.** Since \(i_e: X_e \to X\) is a closed embedding, we obtain

\[
\mathcal{I}(X_e, \mathcal{O}_{X_e}) = i_e^! (\mathcal{D}_{X_e}^\mathfrak{X} \otimes_{i^! \mathcal{D}_X^\mathfrak{X}} \mathcal{O}_{X_e})
\]

Further, from the definition of the \((i_e^{-1} \mathcal{D}_{X_e}^\mathfrak{X}, i_e^! \mathcal{D}_X^\mathfrak{X})\)-bimodule \(\mathcal{D}_{X_e}^\mathfrak{X}\) we have

\[
\mathcal{D}_{X_e}^\mathfrak{X} = i_e^{-1} \mathcal{D}_X^\mathfrak{X} \otimes_{i^! \mathcal{D}_X^\mathfrak{X}} \omega_{X_e/X}
\]

where \(\omega_{X_e/X} = \omega_{X_e} \otimes_{i_e^! \mathcal{O}_{X_e}} i_e^{-1} \omega_{X_e}^{-1}\), \(\omega_X\) and \(\omega_{X_e}\) are the canonical sheaves of \(X\) and \(X_e\), respectively. As a consequence of \(\omega_{X_e/X} \simeq \mathcal{O}_{X_e}\), we can write

\[
\Gamma (X, \mathcal{I}(X_e, \mathcal{O}_{X_e})) = \Gamma (X, i_e^! (\mathcal{D}_{X_e}^\mathfrak{X} \otimes_{i^! \mathcal{D}_X^\mathfrak{X}} \mathcal{O}_{X_e}))
\]

\[
= \Gamma (X_e, \mathcal{D}_{X_e}^\mathfrak{X} \otimes_{i^! \mathcal{D}_X^\mathfrak{X}} \mathcal{O}_{X_e})
\]

\[
= \Gamma (X_e, (i_e^{-1} \mathcal{D}_X^\mathfrak{X} \otimes_{i^! \mathcal{D}_X^\mathfrak{X}} \omega_{X_e/X}) \otimes_{i^! \mathcal{D}_X^\mathfrak{X}} \mathcal{O}_{X_e})
\]

\[
\simeq \mathcal{D}_{X_e}^\mathfrak{X} \otimes_{\mathcal{O}_{X_e}} \mathbb{C},
\]
where \( a = eP \). Therefore, we need to know the geometric fiber of the right \( \mathcal{O}_X \)-module \( \mathcal{D}_X^\lambda \) at the point \( o \). The left action of \( U(\mathfrak{g}) \) on \( \mathcal{D}_X^\lambda \otimes_{\mathcal{O}_X,o} \mathbb{C} \) is given through the mapping
\[
U(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_X^\lambda) \to \mathcal{D}_X^\lambda_o,
\]
which is a homomorphism of associative \( \mathbb{C} \)-algebras. From (1.4) we have
\[
\mathcal{D}_X^\lambda \simeq U_X(\mathfrak{g})/\sum_{s \in \mathcal{V}_X(\mathfrak{p})} (s - f_{\lambda + \rho}(s)) U_X(\mathfrak{g}),
\]
hence we get
\[
\mathbb{C} \otimes_{\mathcal{O}_X,o} \mathcal{D}_X^\lambda_o \simeq U(\mathfrak{g})/\sum_{A \in \mathfrak{p}} (A - (\lambda + \rho)(A)) U(\mathfrak{g}).
\]
Furthermore, if we use the isomorphism \( \mathcal{D}_X^\lambda_o \simeq (\mathcal{D}_X^{-\lambda})^{op} \) of associative \( \mathbb{C} \)-algebras, we obtain
\[
\mathcal{D}_X^\lambda_o \otimes_{\mathcal{O}_X,o} \mathbb{C} \simeq (\mathcal{D}_X^{-\lambda})^{op} \otimes_{\mathcal{O}_X,o} \mathbb{C} \simeq \mathbb{C} \otimes_{\mathcal{O}_X,o} \mathcal{D}_X^{-\lambda} \simeq \mathbb{C}_{-\lambda + \rho} \otimes_{U(\mathfrak{p})} U(\mathfrak{g}).
\]
Hence \( \mathbb{C}_{-\lambda + \rho} \otimes_{U(\mathfrak{p})} U(\mathfrak{g}) \) has a natural right \( U(\mathfrak{g}) \)-module structure. The left \( U(\mathfrak{g}) \)-module structure on \( \mathcal{D}_X^\lambda_o \otimes_{\mathcal{O}_X,o} \mathbb{C} \) is given as follows. Since the diagram
\[
\begin{array}{ccc}
U(\mathfrak{g}) & \rightarrow & \mathcal{D}_X^\lambda_o \\
\tau & \downarrow & \downarrow \\
U(\mathfrak{g})^{op} & \rightarrow & (\mathcal{D}_X^{-\lambda})^{op}
\end{array}
\]
is commutative, where \( \tau: U(\mathfrak{g}) \rightarrow U(\mathfrak{g})^{op} \) is the homomorphism of associative \( \mathbb{C} \)-algebras uniquely determined by \( \tau(A) = -A \) for all \( A \in \mathfrak{g} \), we get
\[
a(1 \otimes b) = 1 \otimes b \tau(a)
\]
for all \( a, b \in U(\mathfrak{g}) \). It is easy to see that it is a highest weight module with highest weight \( \lambda - \rho \). Moreover, it is also free \( U(\mathfrak{g}) \)-module of rank one with the free generator \( 1 \otimes 1 \), thus the generalized Verma module \( M^\lambda_{\mathfrak{g}}(\lambda - \rho) \). \( \square \)

**Lemma 2.2.** The mapping
\[
\tau_o: \mathcal{D}_{X,o} \otimes_{\mathcal{O}_{X,o}} \mathbb{C} \rightarrow \mathcal{D}_{X,o}/\mathcal{D}_{X,o}m_o
\]
defined by
\[
\tau_o(Q \otimes v) = vQ \mod \mathcal{D}_{X,o}m_o
\]
for \( Q \in \mathcal{D}_{X,o} \) and \( v \in \mathbb{C} \), where \( m_o \) is the maximal ideal of \( \mathcal{O}_{X,o} \), is an isomorphism of left \( \mathcal{D}_{X,o} \)-modules.

**Proof.** For \( Q \in \mathcal{D}_{X,o} \) and \( f \in \mathcal{O}_{X,o} \), we have
\[
\tau_o(Qf \otimes 1) = Qf \mod \mathcal{D}_{X,o}m_o.
\]
On the other hand, since \( Qf \otimes 1 = Q \otimes f(o) \), we obtain
\[
\tau_o(Q \otimes f(o)) = f(o)Q \mod \mathcal{D}_{X,o}m_o.
\]
But \( f = f(o) + (f - f(o)) \) implies \( \tau_o(Qf \otimes 1) = \tau_o(Q \otimes f(o)) \). This means that the mapping \( \tau_o \) is well-defined.
As for the injectivity of \( \tau_o \), the condition \( \tau_o(Q \otimes 1) = 0 \) implies \( Q = \sum_{i=1}^{m} Q_i g_i \), where \( Q_i \in \mathcal{D}_{X,o} \) and \( g_i \in \mathfrak{m}_o \). Therefore, we may write

\[
Q \otimes 1 = \sum_{i=1}^{m} (Q_i g_i \otimes 1) = \sum_{i=1}^{m} (Q_i \otimes g_i(o)) = 0.
\]

The surjectivity of the mapping \( \tau_o \) is obvious. The proof is complete.

Let us consider the local chart \((U_e, u_e)\) on \(X\) given by (1.20). Since the mapping \( u_e: U_e \to \overline{u} \) is a diffeomorphism, it induces the isomorphism

\[
\Psi_{u_e}: \mathcal{D}_X(U_e) \to \Gamma(\overline{u}, \mathcal{D}_{\overline{u}})
\]

of associative \(\mathbb{C}\)-algebras. Let \((x_1, x_2, \ldots, x_n)\) be linear coordinate functions on \(\overline{u}\), then we get the coordinate functions \((u_e^1, u_e^2, \ldots, u_e^n)\) on \(U_e\) defined by

\[
u^i_e = x_i \circ u_e
\]

for \(i = 1, 2, \ldots, n\). Then we have

\[
\Psi_{u_e}(u^i_e) = x_i \quad \text{and} \quad \Psi_{u_e}(\partial_{u^i_e}) = \partial_{x_i}
\]

for \(i = 1, 2, \ldots, n\).

We denote by \(\mathcal{A}_\mathbb{R}^g\) the Weyl algebra of the complex vector space \(\overline{u}\) (see Section 2.2 for the definition) and by \(I_e\) the left ideal of \(\mathcal{A}_\mathbb{R}^g\) generated by polynomials on \(\overline{u}\) vanishing at the point 0. Let us note that the Weyl algebra \(\mathcal{A}_\mathbb{R}^g\) is contained in \(\Gamma(\overline{u}, \mathcal{D}_{\overline{u}})\).

**Lemma 2.3.** The mapping

\[
\sigma_o: \mathcal{A}_\mathbb{R}^g/I_e \to \mathcal{D}_{X,o}/\mathcal{D}_{X,o}\mathfrak{m}_o
\]

defined by

\[
P \mod I_e \mapsto \Psi_{u_e}^{-1}(P) \mod \mathcal{D}_{X,o}\mathfrak{m}_o
\]

is an isomorphism of left \(\mathcal{A}_\mathbb{R}^g\)-modules.

**Proof.** The canonical mapping \(\mathcal{A}_\mathbb{R}^g \to \mathcal{D}_{X,o}\) induces the mapping \(I_e \to \mathcal{D}_{X,o}\mathfrak{m}_o\), hence \(\sigma_o\) is well-defined. Now, let us assume that \(P \in \mathcal{D}_{X,o}\), then we can write \(P\) in the form

\[
P = \sum_{\gamma \in \mathbb{N}_0^n} (\partial_{a_1})^{\gamma_1}(\partial_{a_2})^{\gamma_2} \cdots (\partial_{a_n})^{\gamma_n} a_\gamma,
\]

where \(a_\gamma \in \mathcal{O}_{X,o}\). Thus, if we define \(Q \in \mathcal{A}_\mathbb{R}^g\) by

\[
Q = \sum_{\gamma \in \mathbb{N}_0^n} (\partial_{a_1})^{\gamma_1}(\partial_{a_2})^{\gamma_2} \cdots (\partial_{a_n})^{\gamma_n} a_\gamma(o),
\]

then we get \(\sigma_o(Q \mod I_e) = P \mod \mathcal{D}_{X,o}\mathfrak{m}_o\). The injectivity of the mapping \(\sigma_o\) is obvious. \(\square\)

By Proposition 2.1, the generalized Verma module \(M^g_\lambda(\lambda - \rho)\) is isomorphic to \(\mathcal{D}^g_{X,o} \otimes_{\mathcal{O}_{X,o}} \mathbb{C}\). If we use the isomorphism \(j_{s_e}: \mathcal{D}^g_{X,U_e} \to \mathcal{D}_{X,U_e}\) of sheaves of rings of twisted differential operators on \(U_e\) given by (1.20), we obtain the \(U(\mathfrak{g})\)-isomorphism

\[
\varphi_\lambda: M^g_\lambda(\lambda - \rho) \to \mathcal{D}_{X,o} \otimes_{\mathcal{O}_{X,o}} \mathbb{C}
\]

uniquely determined by

\[
\varphi_\lambda(1 \otimes 1) = 1 \otimes 1.
\]
The structure of a left $U(g)$-module on $D_{X,o} \otimes_{O_{X,o}} \mathbb{C}$ follows from the composition
\[ U(g) \to \Gamma(X, D^X_{\lambda}) \to D^X_{\lambda}(U_e) \to D_X(U_e) \to D_{X,o}, \]  \hspace{1cm} (2.13)
which is a homomorphism of associative $\mathbb{C}$-algebras. By Lemma 2.2 and Lemma 2.3 we obtain the isomorphism of $U(g)$-modules
\[ \Phi_\lambda: M^g_\rho(\lambda - \rho) \to A^g_\rho/I_e, \]  \hspace{1cm} (2.14)
uniquely determined by
\[ \Phi_\lambda(1 \otimes 1) = 1 \mod I_e. \]  \hspace{1cm} (2.15)

The structure of a left $U(g)$-module on $A^g_\mu$ is induced by the homomorphism of associative $\mathbb{C}$-algebras
\[ \pi_\lambda: U(g) \to \Gamma(X, D^X_{\lambda}) \to D^X_{\lambda}(U_e) \to D_X(U_e) \to \Gamma(\mathfrak{u}, D_\mathfrak{u}), \]  \hspace{1cm} (2.16)
where $\pi_\lambda(U(g)) \subset A^g_\mu$ as it follows from Theorem 1.3.

It the next Theorem we introduce coordinate-free description of a linear surjective mapping from the symmetric algebra $S(\mathfrak{u})$ to the universal enveloping algebra $U(\mathfrak{u})$ of the Lie algebra $\mathfrak{u}$, completely characterized to be the identity map on $\mathfrak{u}$. Hereby we realize $\Phi_\lambda$ in an explicit way. This result will be used in the subsequent Section 3 for the construction of both singular vectors and equivariant differential operators as elements of $U(\mathfrak{u})$, where $\mathfrak{u}$ the opposite nilradical of the parabolic subalgebra, see (1.10).

**Theorem 2.4.** Let $(f_1, f_2, \ldots, f_n)$ be a basis of $\mathfrak{u}$, $(x_1, x_2, \ldots, x_n)$ be the corresponding linear coordinate functions on $\mathfrak{u}$, and let $\beta: S(\mathfrak{u}) \to U(\mathfrak{u})$ be the symmetrization map defined by
\[ \beta(f_1 f_2 \ldots f_k) = \frac{1}{k!} \sum_{\sigma \in S_k} f_{\sigma(1)} f_{\sigma(2)} \cdots f_{\sigma(k)} \]  \hspace{1cm} (2.17)
for all $k \in \mathbb{N}$ and $f_1, f_2, \ldots, f_k \in \mathfrak{u}$. Then
\[ \Phi_\lambda(\beta(f_{i_1} f_{i_2} \ldots f_{i_k}) \otimes 1) = (-1)^k \partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} \mod I_e, \]  \hspace{1cm} (2.18)
where $\partial_i = \partial_{x_i}$, for all $k \in \mathbb{N}$ and $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\}$.

**Proof.** For $f \in \mathfrak{u}$, we get from Theorem 1.3
\[ \pi_\lambda(f) = -\sum_{i=1}^n \sum_{k=0}^\infty g_k(f) \partial_{x_i}, \]
where
\[ g_k(f) = \alpha_k \sum_{m_1, \ldots, m_k} x_{m_1} \ldots x_{m_k} ([\text{ad}(f_{m_1}) \ldots \text{ad}(f_{m_k})(f)]_i \] for the Bernoulli numbers $\alpha_k$, determined by the generating series
\[ \frac{z}{e^z - 1} = \sum_{k=0}^\infty \alpha_k z^k \]
for $z \in \mathbb{C}$ satisfying $0 < |z| < 1$. We shall prove the claim by induction. Because $g_0(f_{i_1}) = \delta_{i_1}^1$, we can write
\[ \Phi_\lambda(\beta(f_{i_1}) \otimes 1) = \pi_\lambda(f_{i_1}) \mod I_e = -\sum_{i=1}^n (g_0(f_{i_1}) \partial_i + g_1(f_{i_1}) \partial_i) \mod I_e \]
\[ = -\partial_{i_1} - \alpha_1 \sum_{i=1}^n [\text{ad}(f_{i_1})(f_i)]_i \mod I_e \]
\[ = -\partial_{i_1} - \alpha_1 \text{tr}_\mathfrak{u}(\text{ad}(f_{i_1})) \mod I_e = -\partial_{i_1} \mod I_e. \]
Let us now assume that (2.18) holds for some \( k \in \mathbb{N} \). The formula

\[
\beta(f_{i_1} \cdots f_{i_{k+1}}) = \frac{1}{k+1} \sum_{j=1}^{k+1} f_{i_j} \beta(f_{i_1} \cdots \hat{f}_{i_j} \cdots f_{i_{k+1}})
\]

allow us to write

\[
(k+1) \Phi(\beta(f_{i_1} f_{i_2} \cdots f_{i_{k+1}}) \otimes 1) = \sum_{j=1}^{k+1} \pi(f_{i_j}) \Phi(\beta(f_{i_1} \cdots \hat{f}_{i_j} \cdots f_{i_{k+1}}) \otimes 1) \mod I_e
\]

\[
= (-1)^k \sum_{j=1}^{k+1} \pi(f_{i_j}) \partial_{i_1} \cdots \partial_{i_{k+1}} \mod I_e
\]

\[
+ (-1)^k \sum_{j=1}^{k+1} [\pi(f_{i_j}), \partial_{i_1} \cdots \partial_{i_{k+1}}] \mod I_e
\]

so that we get

\[
\Phi(\beta(f_{i_1} f_{i_2} \cdots f_{i_{k+1}}) \otimes 1) = (-1)^{k+1} \partial_{i_1} \cdots \partial_{i_{k+1}}
\]

\[
+ \frac{(-1)^k}{(k+1)!} \sum_{\sigma \in S_{k+1}} [\pi(f_{i_{\sigma(1)}}, \partial_{i_{\sigma(2)}} \cdots \partial_{i_{\sigma(k+1)}}) \mod I_e.
\]

Therefore, it is enough to show that

\[
\sum_{\sigma \in S_r} [\pi(f_{j_{\sigma(1)}}, \partial_{j_{\sigma(2)}}) \cdots \partial_{j_{\sigma(r)}}] \mod I_e = 0
\]

for all \( r \geq 2 \). Because we have

\[
\sum_{\sigma \in S_r} [\pi(f_{j_{\sigma(1)}}, \partial_{j_{\sigma(2)}}) \cdots \partial_{j_{\sigma(r)}}] = \sum_{k=0}^{\infty} \sum_{n=1}^{n} \sum_{\sigma \in S_r} [\partial_{j_{\sigma(2)}}, g_k(f_{j_{\sigma(1)}})] \cdots \partial_{j_{\sigma(r)}}, \partial_{i_k}
\]

for \( r \geq 2 \), it is clearly sufficient to prove

\[
\sum_{k=0}^{\infty} \sum_{n=1}^{n} \sum_{\sigma \in S_r} [\partial_{j_{\sigma(2)}}, g_k(f_{j_{\sigma(1)}})] \cdots \partial_{j_{\sigma(r)}}, \partial_{i_k} \mod I_e = 0
\]

(2.19)

for all \( r \geq 2 \). Again, we shall prove this fact by induction on \( r \). If \( r = 2 \), we can rewrite the left hand side of (2.19) into the form

\[
\sum_{k=0}^{\infty} \sum_{n=1}^{n} \sum_{\sigma \in S_2} (\partial_{i_1} \partial_{j_{\sigma(2)}}, g_k(f_{j_{\sigma(1)}})] - [\partial_{i_1}, g_k(f_{j_{\sigma(1)}})] \mod I_e
\]

\[
= \sum_{n=1}^{n} \sum_{\sigma \in S_2} (\partial_{i_1} \partial_{j_{\sigma(2)}}, g_k(f_{j_{\sigma(1)}})] - [\partial_{i_1}, g_k(f_{j_{\sigma(1)}})] \mod I_e
\]

where we used the fact that \( g_k(f_j) \) is a polynomial of degree \( k \) in the variables \( \{x_1, x_2, \ldots, x_n\} \) and \( g_0(f) = \delta_i^j \). In addition, we have

\[
\sum_{n=1}^{n} \sum_{\sigma \in S_2} \partial_{i_1} \partial_{j_{\sigma(2)}}, g_k(f_{j_{\sigma(1)}})] = \alpha_1 \sum_{n=1}^{n} \sum_{\sigma \in S_2} \partial_{i_1} \partial_{j_{\sigma(2)}}, \partial_{j_{\sigma(1)}}] = 0
\]
and
\[
\sum_{i=1}^{n} \sum_{\sigma \in S_2} \sum_{e_{i,2}} \sum_{f_{i,2}} \sum_{g_{i,2}} [\partial_i, \{\partial_j, g_{i,j}(f_{j,1})\}] = \alpha_2 \sum_{i=1}^{n} \sum_{\sigma \in S_2} \sum_{e_{i,2}} \sum_{f_{i,2}} \sum_{g_{i,2}} (\text{ad}(f_i) \text{ad}(f_{j,2}) (f_{j,1}))i - (\text{ad}(f_{j,2}) \text{ad}(f_{j,1}) (f_i))i
\]
\[
= -\alpha_2 \sum_{i=1}^{n} \sum_{\sigma \in S_2} \sum_{e_{i,2}} \sum_{f_{i,2}} \sum_{g_{i,2}} \text{ad}(f_{j,2}) \text{ad}(f_{j,1}) (f_i)i
\]
\[
= -\alpha_2 \sum_{\sigma \in S_2} \text{tr}(\text{ad}(f_{j,2}) \text{ad}(f_{j,1})) = 0,
\]
since \(\text{ad}(f_{j,2}) \text{ad}(f_{j,1})\) is a nilpotent mapping on \(\mathfrak{h}\). Now, let us assume that (2.19) holds for all \(r \in \{2, 3, \ldots, r_0 - 1\}\), then the left hand side (2.19) reduces to
\[
(-1)^{r_0 - 1} \sum_{k=0}^{r_1} \sum_{i=1}^{n} \sum_{\sigma \in S_{r_0}} [\partial_i, \{\partial_j, g_{i,j}(f_{j,1})\} \ldots] \mod I_e
\]
\[
= (-1)^{r_0 - 1} \sum_{i=1}^{n} \sum_{\sigma \in S_{r_0}} [\partial_i, \{\partial_j, g_{i,j}(f_{j,1})\} \ldots] \mod I_e
\]
\[
= \alpha_0 (-1)^{r_0} (r_0 - 1)! \sum_{i=1}^{n} \sum_{\sigma \in S_{r_0}} (\text{ad}(f_{j,1}) \ldots \text{ad}(f_{j,2}) \text{ad}(f_{j,1}) (f_i))i \mod I_e
\]
\[
= \alpha_0 (-1)^{r_0} (r_0 - 1)! \sum_{\sigma \in S_{r_0}} \text{tr}(\text{ad}(f_{j,1}) \ldots \text{ad}(f_{j,1})) \mod I_e = 0.
\]

The proof is complete. \(\square\)

### 2.2 Algebraic Fourier transform

In this section we recall the Fourier transform of modules over Weyl algebra and use it later to convert the algebraic characterization of certain vectors in modules in the BGG parabolic category \(\mathcal{O}\) into characterization by a system of linear partial differential equations.

Let \(V\) be a finite-dimensional complex vector space, regarded as a complex algebraic variety \((V^{\text{alg}}, \mathcal{O}_{V^{\text{alg}}})\). The associative \(\mathbb{C}\)-algebra \(A_V = \Gamma(V^{\text{alg}}, \mathcal{D}_{V^{\text{alg}}})\) is called the Weyl algebra. Since the canonical morphism \(i: V \to V^{\text{alg}}\) of topological spaces induces an injective morphism \(A_V \to \Gamma(V, \mathcal{D}_V)\) of associative \(\mathbb{C}\)-algebras, the Weyl algebra \(A_V\) can be regarded as a subalgebra of \(\Gamma(V, \mathcal{D}_V)\) (cf. Section 1.1). We have
\[
A_V \simeq S(V^*) \otimes_{\mathbb{C}} S(V),
\]
(2.20)

where \(S(V^*) \simeq \mathbb{C}[V] = \Gamma(V^{\text{alg}}, \mathcal{O}_{V^{\text{alg}}})\) and we regard \(S(V) \simeq \mathbb{C}[V^*]\) as the \(\mathbb{C}\)-algebra of constant coefficient differential operators.

Let \((x_1, x_2, \ldots, x_n)\) be linear coordinate functions on \(V\) and let \((y_1, y_2, \ldots, y_n)\) be the dual linear coordinate functions on \(V^*\). Then there is a canonical isomorphism
\[
\mathcal{F}: A_V \to A_{V^*},
\]
(2.21)
of associative \(\mathbb{C}\)-algebras given by
\[
\mathcal{F}(x_i) = -\partial_{y_i}, \quad \mathcal{F}(\partial_{x_i}) = y_i
\]
(2.22)
for \(i = 1, 2, \ldots, n\). Let us note that the definition does not depend on the choice of linear coordinates on \(V\).
Let $M$ be a left $A_V$-module. The Fourier transform $\hat{M}$ of $M$ has the same underlying vector space as $M$, and the left $A_{V^*}$-module structure is given by

$$Pu = \mathcal{F}^{-1}(P)u$$

for all $u \in \hat{M}$ and $P \in A_{V^*}$. The Fourier transform induces equivalences of categories

$$\hat{\psi}: \text{Mod}(A_V) \to \text{Mod}(A_{V^*}), \quad \hat{\tau}: \text{Mod}_{f}(A_V) \to \text{Mod}_{f}(A_{V^*}),$$

where $\text{Mod}_{f}(A_V)$ is the category of finitely generated $A_V$-modules.

**Lemma 2.5.** Let $I$ be a left ideal of $A_V$ and let $M$ be a left $A_{V^*}$-module of the form

$$M = A_{V^*}/I.$$  

Then the Fourier transform $\hat{M}$ is isomorphic to the left $A_{V^*}$-module $\hat{M}$ of the form

$$\hat{M} = A_{V^*}/\mathcal{F}(I),$$

where the isomorphism

$$\varphi: \hat{M} \to \hat{M}$$

is given by

$$Q \mod \mathcal{F}(I) \mapsto \mathcal{F}^{-1}(Q) \mod I$$

for all $Q \in A_V$.

**Proof.** The morphism $\psi: A_{V^*} \to \hat{M}$ of left $A_{V^*}$-modules defined by

$$\psi(Q) = \mathcal{F}^{-1}(Q) \mod I$$

for all $Q \in A_{V^*}$ is clearly surjective. It is also injective, because $\psi(Q) = 0$ implies $Q \in \mathcal{F}(I)$, since $\mathcal{F}$ is an isomorphism of associative $\mathbb{C}$-algebras, and therefore $\ker \psi \subset \mathcal{F}(I)$. The opposite inclusion is trivial, hence $\psi$ induces the isomorphism $\varphi: \hat{M} \to \hat{M}$. □

For a left $A_{V^*}$-module $\hat{M}$ we consider a system of linear partial differential equations for a single unknown element $u \in \hat{M}$ in the form

$$P_1u = P_2u = \cdots = P_ku = 0,$$

where $k \in \mathbb{N}$ and $P_1, P_2, \ldots, P_k \in A_{V^*}$. We denote by $\text{Sol}(P_1, P_2, \ldots, P_k; M)$ the complex vector space of solutions of the system (2.29).

**Lemma 2.6.** Let $I$ be a left ideal of $A_V$ and let $M$ be a left $A_{V^*}$-module of the form $M = A_{V^*}/I$. Let us consider elements $P_1, P_2, \ldots, P_k \in A_{V^*}$ for $k \in \mathbb{N}$. Then the mapping

$$\tau: \text{Sol}(P_1, P_2, \ldots, P_k; M) \to \text{Sol}(\mathcal{F}(P_1), \mathcal{F}(P_2), \ldots, \mathcal{F}(P_k); \hat{M})$$

given by

$$Q \mod I \mapsto \mathcal{F}(Q) \mod \mathcal{F}(I)$$

is an isomorphism of complex vector spaces.

**Proof.** First of all we show that $\tau$ maps into $\text{Sol}(\mathcal{F}(P_1), \mathcal{F}(P_2), \ldots, \mathcal{F}(P_k); \hat{M})$. For an element $Q \mod I \in \text{Sol}(P_1, P_2, \ldots, P_k; M)$, we get

$$\mathcal{F}(P_i)\tau(Q \mod I) = \mathcal{F}(P_i)\mathcal{F}(Q) \mod \mathcal{F}(I) = P_iQ \mod I \mod \mathcal{F}(I) = 0 \mod \mathcal{F}(I)$$

for $i = 1, 2, \ldots, k$, where we used the fact that $P_iQ \in I$ and so $\mathcal{F}(P_iQ) \in \mathcal{F}(I)$ for $i = 1, 2, \ldots, k$. Then it is easy to see that the mapping

$$\tilde{\tau}: \text{Sol}(\mathcal{F}(P_1), \mathcal{F}(P_2), \ldots, \mathcal{F}(P_k); \hat{M}) \to \text{Sol}(P_1, P_2, \ldots, P_k; M)$$

defined by the formula

$$Q \mod \mathcal{F}(I) \mapsto \mathcal{F}^{-1}(Q) \mod I,$$

is the inverse mapping to $\tau$, and so $\tau$ is an isomorphism. The proof is complete. □
2.3 Singular vectors in generalized Verma modules

This section contains the definition of the key concept of singular vectors. The complete collection of singular vectors in generalized Verma modules in the BGG parabolic category $O^p$, or in particular, in a given block $O^p_{\lambda \mu}$, fully encodes the structure of its morphisms. \[12\]

Let us consider a complex reductive Lie algebra $g'$ of a complex reductive Lie algebra $g$ together with its parabolic subalgebra $p \subset g$ such that $p' = p \cap g'$ is a parabolic subalgebra of $g'$. We recall that the formula $\Phi(2.34)$ associates to any element $\lambda \in \text{Hom}_p(p, \mathbb{C})$ the mapping $\Phi(2.34)$, which induces the isomorphism of $\Gamma'$-modules

$$M^p_\gamma(\lambda - \rho)_{\gamma'} \cong \text{Sol}(g, g', p; A^p_{\gamma}/I_\gamma).$$

Here $M^p_\gamma(\lambda - \rho)_{\gamma'}$ denotes the subspace in the scalar generalized Verma module $M^p_\gamma(\lambda - \rho)$ (cf. $2.1$) annihilated by the nilradical $u'$ of $p'$, and the elements in $M^p_\gamma(\lambda - \rho)_{\gamma'}$ are called $g'$-singular vectors. The right hand side of (2.32) is defined by

$$\text{Sol}(g, g', p; A^p_{\gamma}/I_\gamma) = \text{Sol}((\pi_\lambda(X); X \in u'); A^p_{\gamma}/I_\gamma).$$

Since the Levi factor $\Gamma'$ of $p'$ is a reductive Lie algebra, we have $\Gamma' = \gamma' \oplus [\gamma', \gamma']$ with $\gamma'$ the center of $\Gamma'$, and $\gamma'$ acts on any finite-dimensional irreducible $\Gamma'$-module $V$ by a character $\mu: \gamma' \to \mathbb{C}$ fulfilling $Xv = \mu(X)v$ for all $X \in \gamma'$, $v \in V$.

Let $V \subset M^p_\gamma(\lambda - \rho)_{\gamma'}$ be a finite-dimensional irreducible $\Gamma'$-module, where the action of $\gamma'$ on $V$ is given by a character $\mu: \gamma' \to \mathbb{C}$. Then we have

$$\Phi^V_\lambda(V) \subset \text{Sol}((\pi_\lambda(X), \pi_\lambda(Y) - \mu(Y); X \in u', Y \in \gamma'); A^p_{\gamma}/I_\gamma).$$

The application of Fourier transform, see (2.30), implies

$$\tau(\Phi^V_\lambda(V)) \subset \text{Sol}((\pi_\lambda(X), \pi_\lambda(Y) - \mu(Y); X \in u', Y \in \gamma'); A^p_{\gamma}/\mathcal{F}(I_\gamma))$$

where

$$\text{Sol}(g, g', p; A^p_{\gamma}/\mathcal{F}(I_\gamma)) = \text{Sol}((\pi_\lambda(X), \pi_\lambda(Y) - \mu(Y); X \in u', Y \in \gamma'); A^p_{\gamma}/\mathcal{F}(I_\gamma)).$$

is an $\Gamma'$-submodule in the polynomial ring $\mathbb{C}[\pi'] \simeq A^p_{\gamma}/\mathcal{F}(I_\gamma)$ given by the solution of a system of partial differential equations.

A $g'$-singular vector is a generator of a $g'$-submodule in $M^p_\gamma(\lambda - \rho)$, and its existence is equivalent to a discrete component in the branching problem $M^p_\gamma(\lambda)|_{\gamma'}$. The results in our article classify discrete components in the branching problem for a class of generalized Verma modules regarded as objects in the BGG parabolic category $O^p$ and a class of $g'$-compatible parabolic subalgebras $p$. The condition of $g'$-compatibility for parabolic subalgebras guarantees that the decompositions are discrete and thus realized by singular vectors, \[10\] [20]. Let us recall that a parabolic subalgebra $p \subset g$ is $g'$-compatible provided there exists a hyperbolic element $E' \in g'$ such that $p$ is the direct sum of eigenspaces of ad($E'$) with non-negative integral eigenvalues. This then implies that $G^p$ is a closed submanifold of $G$ and a geometric argument inspired by $D$-module theory gives the discrete decomposability of all modules in $O^p$ with respect to $g'$.

Assuming that $p = l \oplus u$ is a $g'$-compatible parabolic subalgebra of $g$ and $V_\lambda$ a finite-dimensional irreducible $\Gamma$-module with highest weight $\lambda$, we consider the induced $\Gamma'$-module structure on the symmetric algebra $S(U/\overline{(U \otimes g')})$ and define for any finite-dimensional irreducible $\Gamma'$-module $V_\mu'$ with highest weight $\mu$ the multiplicity function

$$m(\lambda, \mu) = \dim_{\mathbb{C}} \text{Hom}_\Gamma(V_\mu', V_\lambda \otimes S(U/\overline{(U \otimes g')})).$$

Then $m(\lambda, \mu) < \infty$ for all $\mu \in \Lambda^+(p')$, and in the Grothendieck group $K(O^p)$ of the BGG parabolic category $O^p$ holds

$$M^p_\gamma(\lambda)|_{\gamma'} \simeq \bigoplus_{\mu \in \Lambda^+(p')} m(\lambda, \mu) M^p_\gamma(\mu)$$

for a generalized Verma module $M^p_\gamma(\lambda)$. We shall use (2.38) in Section 3.2 below to produce in a particular case of our interest a characterization of the generators of $g'$-submodules.
3 $A_n$-series of Lie algebras with Heisenberg parabolic subalgebras

In the present section we implement and apply the previous general exposition to the case of scalar generalized Verma modules for the pair given by the classical complex Lie algebra $A_{n+1}$ and its parabolic subalgebra with the nilradical isomorphic to the Heisenberg Lie algebra, and its complex Lie subalgebra $A_{n-r+1}$ $(n-r \geq 1)$ together with compatible parabolic subalgebra whose nilradical is isomorphic to the Heisenberg Lie algebra of less dimension. In the Dynkin diagrammatic notation, this type of parabolic subalgebra is determined by omitting the first and the last simple nodes in the diagram.

Let us briefly indicate the roadmap for the present section. Following Section 1.2 we set our representation theoretical conventions and use Theorem 1.3 to describe the embedding of the Lie algebra $\mathfrak{sl}(n+2,\mathbb{C})$ into the Weyl algebra $A^G_{n+1}$. As a $\mathfrak{g}$-module we take the left $A^G_{n+1}$-module given as the quotient of $A^G_{n+1}$ by the left ideal of $A^G_{n+1}$ generated by polynomials vanishing at the origin of $\mathfrak{u}$, and apply to it the algebraic Fourier transform (2.21). This converts the former task of finding $\mathfrak{sl}(n-r+2,\mathbb{C})$-singular vectors into the problem of solving a system of partial differential equations, realized in $A^G_{n-r+1}$, and acting on the polynomial algebra $\mathbb{C}[\mathbb{R}]$. The main technical device allowing the complete classification of the space of this system of partial differential equations is the Fischer decomposition, describing the $(\mathfrak{sl}(n-r,\mathbb{C}) \oplus \mathfrak{sl}(r,\mathbb{C}))$-module structure on $\mathbb{C}[\mathfrak{g}^{-1}]$, see Appendix A.

3.1 Representation theoretical conventions

In the rest of the article we consider the complex semisimple Lie group $G = \text{SL}(n+2,\mathbb{C})$, $n \in \mathbb{N}$, and its Lie algebra $\mathfrak{g} = \mathfrak{sl}(n+2,\mathbb{C})$. The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is given by diagonal matrices

$$\mathfrak{h} = \{ \text{diag}(a_1, a_2, \ldots, a_{n+2}); a_1, a_2, \ldots, a_{n+2} \in \mathbb{C}, \sum_{i=1}^{n+2} a_i = 0 \}.$$  (3.1)

For $i = 1, 2, \ldots, n+2$ we define $\varepsilon_i \in \mathfrak{h}^*$ by $\varepsilon_i(\text{diag}(a_1, a_2, \ldots, a_{n+2})) = a_i$. Then the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is $\Delta = \{ \varepsilon_i - \varepsilon_j; 1 \leq i \neq j \leq n+2 \}$. The root space $\mathfrak{g}_{\varepsilon_i - \varepsilon_j}$ is the complex linear span of $\varepsilon_{ij}$, the $(n+2 \times n+2)$-matrix such that $(\varepsilon_{ij})_{kl} = \delta_{ik}\delta_{jl}$. The positive root system is $\Delta^+ = \{ \varepsilon_i - \varepsilon_j; 1 \leq i < j \leq n+2 \}$, in which the set of simple roots is $\Pi = \{ \alpha_1, \alpha_2, \ldots, \alpha_{n+1} \}$, $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i = 1, 2, \ldots, n+1$. Then the fundamental weights are $\omega_i = \sum_{j=1}^{n+2} \varepsilon_j$, $i = 1, 2, \ldots, n+1$. The Lie subalgebras $\mathfrak{b}$ and $\mathfrak{u}$ defined by the linear span of positive and negative root spaces together with the Cartan subalgebra are called the standard Borel subalgebra and the opposite standard Borel subalgebra of $\mathfrak{g}$, respectively. The subset $\Sigma = \{ \alpha_2, \alpha_3, \ldots, \alpha_n \}$ of $\Pi$ generates the root subsystem $\Delta_\Sigma$ in $\mathfrak{h}^*$, and we associate to $\Sigma$ the standard parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ by $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$. The reductive Levi factor $\mathfrak{l}$ of $\mathfrak{p}$ is defined through

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_\Sigma} \mathfrak{g}_\alpha.$$  (3.2)

and the nilradical $\mathfrak{u}$ of $\mathfrak{p}$ and the opposite nilradical $\mathfrak{u}$ are

$$\mathfrak{u} = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_\Sigma^+} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{u} = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_\Sigma^+} \mathfrak{g}_{-\alpha},$$  (3.3)

respectively. We define the $\Sigma$-height $ht_\Sigma(\alpha)$ of $\alpha \in \Delta$ by

$$ht_\Sigma(\sum_{i=1}^{n+1} \alpha_i) = a_1 + a_{n+1},$$  (3.4)

so $\mathfrak{g}$ is a $\mathbb{Z}$-graded Lie algebra with respect to the grading given by $\mathfrak{g}_i = \bigoplus_{\alpha \in \Delta, ht_\Sigma(\alpha) = i} \mathfrak{g}_\alpha$ for $0 \neq i \in \mathbb{Z}$, and $\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta, ht_\Sigma(\alpha) = 0} \mathfrak{g}_\alpha$. Moreover, we have $\mathfrak{u} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, $\mathfrak{u} = \mathfrak{g}_2 \oplus \mathfrak{g}_1$ and $\mathfrak{l} = \mathfrak{g}_0$. 

18
The basis $(f_1, \ldots, f_n, g_1, \ldots, g_n, c)$ of the root spaces in the opposite nilradical $\mathfrak{p}$ is given by

$$f_i = \begin{pmatrix} 0 & 0 & 0 \\ 1_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad g_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1_i^T & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

(3.5)

where the only non-trivial Lie brackets are $[f_i, g_i] = -c$ for all $i = 1, 2, \ldots, n$. Analogously, the basis $(d_1, \ldots, d_n, e_1, \ldots, e_n, a)$ of the root spaces in $u$ is given by

$$d_i = \begin{pmatrix} 0 & 1_i^T & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1_i \\ 0 & 0 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

(3.6)

where $[d_i, e_i] = a$ for all $i = 1, 2, \ldots, n$. The Levi subalgebra $\mathfrak{l}$ of $\mathfrak{p}$ is the linear span of

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{2}{n} I_n & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad h_A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

(3.7)

where $A \in M_{n \times n}(\mathbb{C})$ satisfies $\text{tr} A = 0$. Moreover, the elements $h_1$ and $h_2$ form a basis of the center $\mathfrak{z}(\mathfrak{l})$ of $\mathfrak{l}$.

Finally, the parabolic subgroup $P$ of $G$ with the Lie algebra $\mathfrak{p}$ is defined by

$$P = \left\{ \begin{pmatrix} a & x^T & c \\ 0 & A & y \\ 0 & 0 & b \end{pmatrix} : a, b, c \in \mathbb{C}, x, y \in \mathbb{C}^n, A \in M_{n \times n}(\mathbb{C}), ab \det(A) = 1 \right\}.$$  

(3.8)

Any character $\lambda \in \text{Hom}_R(\mathfrak{p}, \mathbb{C})$ is given by

$$\lambda = \lambda_1 \omega_1 + \lambda_2 \omega_{n+1}$$

(3.9)

for some $\lambda_1, \lambda_2 \in \mathbb{C}$, where $\omega_1, \omega_{n+1} \in \text{Hom}_R(\mathfrak{p}, \mathbb{C})$ are equal to $\omega_1, \omega_{n+1} \in \mathfrak{h}^*$ and regarded as trivially extended to $\mathfrak{p} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta_E} \mathfrak{g}_\alpha) \oplus \mathfrak{u}$. Then the vector $\rho \in \text{Hom}_R(\mathfrak{p}, \mathbb{C})$ defined by the formula (1.11) is given by

$$\rho = \frac{n+1}{2} \omega_1 + \frac{n+1}{2} \omega_{n+1}.$$  

(3.10)

For $r \in \{0, 1, \ldots, n - 1\}$ we define an injective Lie algebra homomorphism

$$i_r : \mathfrak{sl}(n - r + 2, \mathbb{C}) \rightarrow \mathfrak{sl}(n + 2, \mathbb{C})$$

(3.11)

by

$$\begin{pmatrix} a & x^T & b \\ y & A & u \\ c & v^T & d \end{pmatrix} \mapsto \begin{pmatrix} a & x^T & 0 & b \\ y & A & 0 & u \\ 0 & 0 & 0 & 0 \\ c & v^T & 0 & d \end{pmatrix},$$

(3.12)

where $a, b, c, d \in \mathbb{C}, x, y, u, v \in \mathbb{C}^{n-r}, A \in M_{n-r \times n-r}(\mathbb{C})$ and $a + d + \text{tr} A = 0$. Then we regard the semisimple Lie subalgebra $\mathfrak{g}_r'$ of $\mathfrak{g}$ to be the image of the mapping $i_r$. We set $\mathfrak{p}'_r = \mathfrak{g}_r' \cap \mathfrak{p}$, so that $\mathfrak{p}'_r$ is a parabolic subalgebra of $\mathfrak{g}_r'$ and $\mathfrak{p}'_r = \mathfrak{l}'_r \oplus \mathfrak{u}'_r$ with $\mathfrak{l}'_r = \mathfrak{g}_r' \cap \mathfrak{l}, \mathfrak{u}'_r = \mathfrak{g}_r' \cap \mathfrak{u}$. Finally, we obtain $\mathfrak{l}'_r = \mathfrak{z}(\mathfrak{l}'_r) \oplus [\mathfrak{l}'_r, \mathfrak{l}'_r]$ for $[\mathfrak{l}'_r, \mathfrak{l}'_r] = \mathfrak{g}_r' \cap [\mathfrak{l}, \mathfrak{l}]$, and $h'_1, h'_2$, where

$$h'_1 = h_1, \quad h'_2 = h_2 + h_A,$$

(3.13)

for $A_r$ given by

$$A_r = \begin{pmatrix} -2c & -2(n-r)I_{n-r} & 0 \\ 0 & 2 \mathfrak{i}_r \end{pmatrix},$$

(3.14)

form a basis of the center $\mathfrak{z}(\mathfrak{l}'_r)$ of $\mathfrak{l}'_r$. 
3.2 The branching problem for the pair \((g, g')\)

This section is a short digression in which we explain and determine, as a consequence of character formulas, qualitative properties of the branching problem for the pair \((g, g')\) and the scalar generalized Verma module \(M^\mathcal{G}_\mu(\lambda)\) (see (2.11)) induced from a character \(\lambda \in \text{Hom}_P(p, \mathbb{C})\) (see (3.9)).

Let us denote by \(\mathbb{C}_\lambda\) the 1-dimensional \(t\)-module given by a highest weight \(\lambda = \lambda_1 \omega_1 + \lambda_2 \omega_{n+1}\). The induced \(\mathcal{L}_\lambda\)-module on the symmetric algebra \(S(\mathcal{U}/\mathcal{U}')\), which is a free commutative \(\mathbb{C}\)-algebra generated by the \(2\)-dimensional complex vector space \(\mathcal{U}/\mathcal{U}'\), is isomorphic to the direct sum of 1-dimensional \(\mathcal{L}_\lambda\)-modules. Then it follows that in the branching problem for the pair \((g, g')\) and the scalar generalized Verma \(g\)-module \(M^\mathcal{G}_\mu(\lambda)\) appear only scalar generalized Verma \(g'\)-modules \(M^\mathcal{G}_{\mu'}(\mu)\), their multiplicities (cf. (2.37)) are given by

\[
m(\lambda, \mu) = \dim \text{Hom}_{\mathcal{L}_\mu}(\mathbb{C}_\mu, \mathbb{C}_\lambda |_{\mathcal{L}_\mu}) \otimes \mathcal{S}(\mathcal{U}/\mathcal{U}'),
\]

and are equal to \((a+b+1)\) provided the highest weight \(\mu\) of \(\mathcal{L}_\mu\) is equal to \((\lambda_1 - a) \omega_1 + (\lambda_2 - b) \omega_{n-r+1}\) for \(a, b \in \mathbb{N}_0\), and zero otherwise.

In the Grothendieck group \(K(\mathcal{O}^p)\) of the Bernstein-Gelfand-Gelfand parabolic category \(\mathcal{O}^p\) (cf. (2.35)) holds

\[
M^\mathcal{G}_\mu(\lambda_1 \omega_1 + \lambda_2 \omega_{n+1}) |_{\mathcal{L}_\mu'} \simeq \bigoplus_{a, b \in \mathbb{N}_0} (a+b+1) M^\mathcal{G}_{\mu'}((\lambda_1 - a) \omega_1 + (\lambda_2 - b) \omega_{n-r+1}).
\]

In the rest of the article we construct, among others, the generators of \(g'\)-submodules on the right hand side of (3.16).

3.3 Embedding of \(g\) into the Weyl algebras \(A^\mathcal{G}_\mathcal{U}\) and \(A^\mathcal{G}_{\mathcal{U}'}\).

Let us denote by \((\hat{x}_1, \ldots, \hat{x}_n, \hat{y}_1, \ldots, \hat{y}_n, \hat{z})\) the linear coordinate functions on \(\mathcal{U}\) with respect to the basis \((f_1, \ldots, f_n, g_1, \ldots, g_n, c)\) of the opposite nilradical \(\mathcal{U}\), and by \((x_1, \ldots, x_n, y_1, \ldots, y_n, z)\) the dual linear coordinate functions on \(\mathcal{U}'\). Then the Weyl algebra \(A^\mathcal{G}_\mathcal{U}\) is generated by

\[
\{\hat{x}_1, \ldots, \hat{x}_n, \hat{y}_1, \ldots, \hat{y}_n, \hat{z}, \partial_{\hat{x}_1}, \ldots, \partial_{\hat{x}_n}, \partial_{\hat{y}_1}, \ldots, \partial_{\hat{y}_n}, \partial_{\hat{z}}\}
\]

and the Weyl algebra \(A^\mathcal{G}_{\mathcal{U}'}\) is generated by

\[
\{x_1, \ldots, x_n, y_1, \ldots, y_n, z, \partial_{x_1}, \ldots, \partial_{x_n}, \partial_{y_1}, \ldots, \partial_{y_n}, \partial_{z}\}.
\]

The coordinate functions \((u^1_c, u^2_c, \ldots, u^{2n+1}_c)\) on \(U_c\) are defined by

\[
u^i_c(x) = \sum_{i=1}^n u^i_c(x) f_i + \sum_{i=1}^n u^{2n+1}_c(x) g_i + u^{2n+1}_c(x) c
\]

for all \(x \in U_c\).

Now we apply Theorem 1.3 to \(\lambda \in \text{Hom}_P(p, \mathbb{C})\) and find an explicit realization of \(g\) as a Lie subalgebra of \(A^\mathcal{G}_\mathcal{U}\). Thus, we get a homomorphism

\[
\pi_\lambda = \Psi_{u_c} \circ \pi^{\lambda+\rho}_c : U(g) \rightarrow A^\mathcal{G}_\mathcal{U}
\]

of associative \(\mathbb{C}\)-algebras. Further, we define the Fourier transform

\[
\mathcal{F} : A^\mathcal{G}_\mathcal{U} \rightarrow A^\mathcal{G}_\mathcal{U},
\]

by (2.22) with respect to the generators (3.17) and (3.18). Since \(\mathcal{F}\) is an isomorphism of associative \(\mathbb{C}\)-algebras, the composition

\[
\hat{\pi}_\lambda = \mathcal{F} \circ \pi_\lambda,
\]

(3.22)
gives the homomorphism

\[ \hat{\pi}_\lambda : U(\mathfrak{g}) \to A^\mathfrak{g}_n. \]  

(3.23)

doing associative \( \mathbb{C} \)-algebras.

Let us introduce the notation

\[ E_x = \sum_{j=1}^n x_j \partial x_j, \quad E_z = z \partial_z, \quad E_y = \sum_{j=1}^n y_j \partial y_j \]  

(3.24)

and

\[ E_{\hat{x}} = \sum_{j=1}^n \hat{x}_j \partial_{\hat{x}_j}, \quad E_{\hat{z}} = \hat{z} \partial_{\hat{z}}, \quad E_{\hat{y}} = \sum_{j=1}^n \hat{y}_j \partial_{\hat{y}_j} \]  

(3.25)

for the Euler homogeneity operators, and

\[ \Box = \sum_{j=1}^n \partial x_j \partial y_j, \quad q = \sum_{j=1}^n \hat{x}_j \hat{y}_j, \quad \hat{q} = \sum_{j=1}^n \hat{x}_j \hat{y}_j \]  

(3.26)

for the Laplace operator and quadratic polynomials. In the following Theorem we retain the notation of previous sections.

**Theorem 3.1.** Let \( \lambda \in \text{Hom}_R(\mathfrak{p}, \mathbb{C}) \). Then the embedding of \( \mathfrak{g} \) into \( A^\mathfrak{g}_n \) and \( A^\mathfrak{g}_n^* \) is given by

1) \[
\begin{align*}
\pi_{\lambda}(f_i) &= -\partial x_i + \frac{1}{2} \hat{y}_i \partial \hat{z}, \\
\pi_{\lambda}(g_i) &= -\partial y_i - \frac{1}{2} \hat{x}_i \partial \hat{z}, \\
\pi_{\lambda}(c) &= -\partial z
\end{align*}
\]  

(3.27)

for \( i = 1, 2, \ldots, n \);

2) \[
\begin{align*}
\hat{\pi}_{\lambda}(f_i) &= -x_i - \frac{1}{2} z \partial y_i, \\
\hat{\pi}_{\lambda}(g_i) &= -y_i + \frac{1}{2} z \partial x_i, \\
\hat{\pi}_{\lambda}(c) &= -z
\end{align*}
\]  

(3.28)

for \( i = 1, 2, \ldots, n \);

3) \[
\begin{align*}
\pi_{\lambda}(h_1) &= E_x + E_y + 2E_z + \lambda_1 + \lambda_2 + n + 1, \\
\pi_{\lambda}(h_2) &= (1 + \frac{\lambda}{n}) E_x - (1 + \frac{\lambda}{n}) E_y + \lambda_1 - \lambda_2, \\
\pi_{\lambda}(h_A) &= -\sum_{i,j=1}^n a_{ij} (x_i \partial x_j - y_j \partial y_i)
\end{align*}
\]  

(3.29)

for all \( A \in M_{n \times n}(\mathbb{C}) \) satisfying \( \text{tr} A = 0 \);

4) \[
\begin{align*}
\hat{\pi}_{\lambda}(h_1) &= -E_x - E_y - 2E_z + \lambda_1 + \lambda_2 - (n + 1), \\
\hat{\pi}_{\lambda}(h_2) &= -(1 + \frac{\lambda}{n}) E_x + (1 + \frac{\lambda}{n}) E_y + \lambda_1 - \lambda_2, \\
\hat{\pi}_{\lambda}(h_A) &= \sum_{i,j=1}^n a_{ij} (x_i \partial x_j - y_j \partial y_i)
\end{align*}
\]  

(3.30)

for all \( A \in M_{n \times n}(\mathbb{C}) \) satisfying \( \text{tr} A = 0 \);
5) \[
\pi_\lambda(d_i) = \dot{z}\partial_y + \dot{x}_1(E_x + \frac{1}{2}E_y + \lambda_1 + \frac{i}{2}(n + 1)) - \frac{1}{2}q(\partial_y, -\frac{1}{2}\dot{x}_1\partial_z), \\
\pi_\lambda(e_i) = -\dot{z}\partial_x + \dot{y}_i(E_y + \frac{1}{2}E_x + \lambda_2 + \frac{i}{2}(n + 1)) - \frac{1}{2}q(\partial_x, \frac{i}{2}\dot{y}_i\partial_z), \\
\pi_\lambda(a) = \dot{z}(E_x + E_y + E_z + \lambda_1 + \lambda_2 + n + 1) + \frac{1}{2}q(E_x - E_y + \lambda_1 - \lambda_2 + \frac{i}{2}\dot{z}\partial_z)
\]
for \(i = 1, 2, \ldots, n;\)

6) \[
\tilde{\pi}_\lambda(d_i) = -y_i\partial_z + \partial_x(E_x + \frac{1}{2}E_y - \lambda_1 + \frac{1}{2}(n - 1)) - \frac{1}{2}(y_i + \frac{1}{2}z\partial_x)\Box, \\
\tilde{\pi}_\lambda(e_i) = x_i\partial_y + \partial_y(E_y + \frac{1}{2}E_x - \lambda_2 + \frac{1}{2}(n - 1)) - \frac{1}{2}(x_i - \frac{1}{2}z\partial_y)\Box, \\
\tilde{\pi}_\lambda(a) = \partial_x(E_x + E_y + E_z - \lambda_1 - \lambda_2 + n) - \frac{1}{2}(E_x - E_y - \lambda_1 + \lambda_2 - \frac{i}{2}\partial_z)\Box
\]
for \(i = 1, 2, \ldots, n.\)

**Proof.** We have from Theorem [1.3]

\[
\pi_\lambda^{\lambda + \rho}(Y) = -\sum_{i=1}^{2n+1} \left[ \frac{\text{ad}_{u_i(x)}^{\text{ad}_{u_i}} Y}{e^{\text{ad}_{u_i}} - \text{id}_\Pi} \right] \partial_u_i
\]
for all \(Y \in \Pi.\) Since \(\text{ad}_{u_i(x)}^{2} Y = 0,\) the expansion of the exponential implies

\[
\pi_\lambda^{\lambda + \rho}(Y) = -\sum_{i=1}^{2n+1} \left[ Y - \frac{1}{2} \text{ad}_{u_i(x)} Y \right] \partial_u_i
\]
and the first item follows by (3.30). Similarly, we get from Theorem [1.3]

\[
\pi_\lambda^{\lambda + \rho}(Y) = \sum_{i=1}^{2n+1} [\text{ad}_{u_i(x)} Y]_i \partial_u_i + (\lambda + \rho)(Y)
\]
for all \(Y \in \Pi.\) The statement of the third item then follows from (3.30). Finally, we have

\[
\pi_\lambda^{\lambda + \rho}(Y) = -\sum_{i=1}^{2n+1} \left[ \frac{\text{ad}_{u_i(x)} e^{\text{ad}_{u_i}(x)}}{e^{\text{ad}_{u_i}} - \text{id}_\Pi} (e^{-(\text{ad}_{u_i(x)} Y))_p \right] \partial_u_i + (\lambda + \rho)((e^{-(\text{ad}_{u_i(x)} Y))_p)
\]
for all \(Y \in \mathfrak{g}.\) Since \(e^{-(\text{ad}_{u_i(x)} Y))_p = (e^{-(\text{ad}_{u_i(x)} Y))_p + (e^{-(\text{ad}_{u_i(x)} Y))_p,\) we can write

\[
(e^{-(\text{ad}_{u_i(x)} Y))_p = e^{-\text{ad}_{u_i(x)} Y} - (e^{-(\text{ad}_{u_i(x)} Y))_p = e^{-\text{ad}_{u_i(x)} Y} - e^{-\text{ad}_{u_i(x)} Y} - Y)_p
\]
for \(Y \in \mathfrak{p}.\) Therefore, we have

\[
\frac{\text{ad}_{u_i(x)} e^{\text{ad}_{u_i}(x)}}{e^{\text{ad}_{u_i}} - \text{id}_\Pi} (e^{-\text{ad}_{u_i(x)} Y))_p = -\text{ad}_{u_i(x)} Y - \text{ad}_{u_i(x)} (e^{-\text{ad}_{u_i(x)} Y} - Y)_p
\]
for \(Y \in \mathfrak{p}.\) Furthermore, since \(\mathfrak{g}\) is \(2|-\)graded, \(\text{ad}_{u_i(x)}^{2} Y = 0\) for all \(Y \in \mathfrak{p},\) we obtain

\[
\frac{\text{ad}_{u_i(x)} e^{\text{ad}_{u_i}(x)}}{e^{\text{ad}_{u_i}} - \text{id}_\Pi} (e^{-\text{ad}_{u_i(x)} Y))_p = -\text{ad}_{u_i(x)} Y - \text{ad}_{u_i(x)} (e^{-\text{ad}_{u_i(x)} Y} - Y)_p - (e^{-\text{ad}_{u_i(x)} Y} - Y)_p
\]

\[
+ \frac{1}{2} \text{ad}_{u_i(x)} (e^{-\text{ad}_{u_i(x)} Y} - Y)_p - \frac{1}{12} \text{ad}_{u_i(x)}^{2} (e^{-\text{ad}_{u_i(x)} Y} - Y)_p.
\]
Putting all ingredients together, we conclude
\[
\frac{\text{ad}_{u_e(x)} e^{\text{ad}_{u_e(x)}}}{e^{\text{ad}_{u_e(x)}} - 1} ((e - \text{ad}_{u_e(x)}(Y)) = \frac{1}{2} (\text{ad}_{u_e(x)}(\text{ad}_{u_e(x)}(Y)) + \frac{1}{4} \text{ad}_{u_e(x)}(\text{ad}_{u_e(x)}^2(\text{ad}_{u_e(x)} Y)) + \frac{1}{24} \text{ad}_{u_e(x)}(\text{ad}_{u_e(x)}^2(\text{ad}_{u_e(x)} Y)) \text{p}
\]
for \( Y \in \mathfrak{p} \), and the fifth item follows by (3.19).

The computation of the Fourier transform of all operators is straightforward. \( \square \)

### 3.4 Algebraic analysis on generalized Verma modules and singular vectors

In what follows the generators \( x_1, \ldots, x_n, y_1, \ldots, y_n, z \) of the symmetric algebra \( S(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}] \) have the grading \( \deg(x_i) = \deg(y_j) = 1 \) for \( i = 1, 2, \ldots, n \) and \( \deg(z) = 2 \). Thus we regard \( \mathbb{C}[\mathfrak{g}] \) as a graded commutative \( \mathbb{C} \)-algebra.

Let us recall the existence of a canonical isomorphism of left \( \mathfrak{g} \)-modules
\[
\mathbb{C}[\mathfrak{g}] \cong \mathfrak{g} / (I_e).
\] (3.33)

Let \( \mu : \mathfrak{g} \to \mathbb{C} \) be a character of \( \mathfrak{g} \). Then we have
\[
\hat{\pi}_\lambda(h'_1)R = \mu(h'_1)R, \quad \hat{\pi}_\lambda(h'_2)R = \mu(h'_2)R
\] (3.34)
for \( R \in \text{Sol}(\mathfrak{g}, \mathfrak{g'}, \mu ; \mathbb{C}[\mathfrak{g}])^T \), which is by Theorem 3.1 equivalent to
\[
(E_x' + E_y' + E_x'' + E_y'' + 2E_z)R = (\lambda_1 + \lambda_2 - (n + 1) - \mu(h'_1))R, \quad \text{for } R \in \text{Sol}(\mathfrak{g}, \mathfrak{g'}, \mu ; \mathbb{C}[\mathfrak{g}])^T.
\] (3.35)

Here we introduced the Euler homogeneity operators
\[
E_x' = \sum_{j=1}^{n-r} x_j \partial x_j, \quad E_x'' = \sum_{j=n-r+1}^{n} x_j \partial x_j, \quad E_y' = \sum_{j=1}^{n-r} y_j \partial y_j, \quad E_y'' = \sum_{j=n-r+1}^{n} y_j \partial y_j.
\] (3.37)

This gives a restriction on \( \mu \) for which \( \text{Sol}(\mathfrak{g}, \mathfrak{g'}, \mu ; \mathbb{C}[\mathfrak{g}])^T \) is a non-zero vector space, because the Euler homogeneity operators \( E_x', E_y', E_x'', E_y'' \) and \( E_z \) acting on \( \mathbb{C}[\mathfrak{g}] \) have eigenvalues in \( \mathbb{N}_0 \). Thus, we may assume
\[
(E_x' + E_y' + E_x'' + E_y'' + 2E_z)R = mR,
\] (3.38)
\[
((n - r + 2)(E_x' - E_y') + (n - r)(E_x'' - E_y''))R = tR
\] (3.39)
for some \( m \in \mathbb{N}_0 \) and \( t \in \mathbb{Z} \). Consequently, we shall apply the solution operators
\[
\hat{\pi}_\lambda(d_i)R = 0 \quad \text{and} \quad \hat{\pi}_\lambda(e_i)R = 0
\] (3.40)
for \( i = 1, 2, \ldots, n - r \) to polynomials \( R \) of the form
\[
R = \sum_{k=0}^{m} R_{m-2k} z^k,
\] (3.41)
where \( R_{m-2k} \in \mathbb{C}[(g_{-1})^\ast] \) satisfy
\[
(E_x' + E_y' + E_x'' + E_y'')R_{m-2k} = (m - 2k)R_{m-2k},
\] (3.42)
\[
((n - r + 2)(E_x' - E_y') + (n - r)(E_x'' - E_y''))R_{m-2k} = tR_{m-2k},
\] (3.43)
and obtain the recurrence relations

\[-(k + 1)y_iR_{m-2k-2} + \partial_{x_i}(E_x + \frac{1}{2}k - \lambda_1 + \frac{1}{2}(n - 1))R_{m-2k} - \frac{1}{\lambda_i}R_{m-2k} - \frac{1}{\lambda_i}R_{m-2k+2} = 0\]  

(3.44)

and

\[(k + 1)x_iR_{m-2k-2} + \partial_{y_i}(E_y + \frac{1}{2}k - \lambda_2 + \frac{1}{2}(n - 1))R_{m-2k} - \frac{1}{\lambda_i}R_{m-2k} + \frac{1}{\lambda_i}R_{m-2k+2} = 0\]  

(3.45)

for \(k = 0, 1, \ldots, \lfloor \frac{m}{2} \rfloor \), where \(R_{m-2k} = 0\) for \(k < 0\) and for \(k > \lfloor \frac{m}{2} \rfloor \). In particular, for \(k = 0\) we get

\[-y_iR_{m-2} + \partial_{x_i}(E_x - \lambda_1 + \frac{1}{2}(n - 1))R_{m} - \frac{1}{\lambda_i}R_{m} = 0,\]

(3.46)

\[x_iR_{m-2} + \partial_{y_i}(E_y - \lambda_2 + \frac{1}{2}(n - 1))R_{m} - \frac{1}{\lambda_i}R_{m} = 0,\]

(3.47)

which implies

\[\partial_{x_i}(E_x - \lambda_1 + \frac{1}{2}(n - 1))R_m \in (y_i),\]

(3.48)

\[\partial_{y_i}(E_y - \lambda_2 + \frac{1}{2}(n - 1))R_m \in (x_i),\]

(3.49)

for \(i = 1, 2, \ldots, n - r\), where \((x_i)\) and \((y_i)\) are the ideals of \(\mathbb{C}[(\mathfrak{g}-1)^*]\) generated by \(x_i\) and \(y_i\), respectively.

Beside the structure of recurrence relations (3.44) and (3.45) we see that \(R\) is uniquely determined by \(R_m\). Therefore, we can define a linear mapping

\[\text{Sol}(\mathfrak{g}, \mathfrak{g}', \mathfrak{p}; \mathbb{C}[\mathfrak{r}^*])_{\mu}^x \rightarrow \mathbb{C}[(\mathfrak{g}-1)^*],\]

(3.50)

\[R \mapsto R_m,\]

which is injective and \(\mu\)-equivariant. Since \(\text{Sol}(\mathfrak{g}, \mathfrak{g}', \mathfrak{p}; \mathbb{C}[\mathfrak{r}^*])_{\mu}^x\) is a completely reducible \(\mu\)-module, any irreducible \(\mu\)-submodule is contained in an isotypical component. Therefore, we can restrict to the case when \(R_m\) is in an isotypical component of the representation \(\mu\) on \(\mathbb{C}[(\mathfrak{g}-1)^*]\).

Furthermore, the Fischer decomposition (cf. Appendix A) implies the isomorphism of vector spaces

\[\varphi: \mathbb{C}[\mathfrak{r}^*] \xrightarrow{\sim} \bigoplus_{(a, b, c, d) \in \mathbb{N}_0^4} \mathbb{C}[q', q'', z] \otimes \mathbb{C} \mathcal{H}_{a, b} \otimes \mathbb{C} \mathcal{H}_{c, d},\]

(3.51)

where

\[q' = \sum_{j=1}^{n-r} x_j y_j, \quad q'' = \sum_{j=n-r+1}^{n} x_j y_j\]

(3.52)

are quadratic polynomials, \(\mathcal{H}_{a, b}\) and \(\mathcal{H}_{c, d}\) are the spaces of \((a, b)\)-homogeneous polynomials in the variables \((x_1, \ldots, x_{n-r}, y_1, \ldots, y_{n-r})\) and \((c, d)\)-homogeneous polynomials in the variables \((x_{n-r+1}, \ldots, x_n, y_{n-r+1}, \ldots, y_n)\) harmonic for

\[\square' = \sum_{j=1}^{n-r} \partial_{x_j} \partial_{y_j} \quad \text{and} \quad \square'' = \sum_{j=n-r+1}^{n} \partial_{x_j} \partial_{y_j},\]

(3.53)

respectively. Consequently, we have

\[\varphi \circ \square \circ \varphi^{-1} | \mathbb{C}[q', q'', z] \otimes \mathbb{C} \mathcal{H}_{a, b} \otimes \mathbb{C} \mathcal{H}_{c, d} = Q_{a+b, c+d} \otimes 1 \otimes 1,\]

(3.54)

where

\[Q_{\alpha, \beta} = q' \partial^2_{q'} + (n - r + \alpha) \partial_{q'} + q'' \partial^2_{q''} + (r + \beta) \partial_{q''}\]

(3.55)
for \( \alpha, \beta \in \mathbb{N}_0 \), and

\[
\partial_x, \varphi^{-1}((C[q', q'', z] \otimes_{c} \mathcal{H}'_{a, b} \otimes_{c} \mathcal{H}''_{c, d}) = y_i(\varphi^{-1} \circ (\partial q' \otimes 1 \otimes 1) \circ \varphi),
\]

\[
\partial_y, \varphi^{-1}((C[q', q'', z] \otimes_{c} \mathcal{H}'_{a, b} \otimes_{c} \mathcal{H}''_{c, d}) = x_i(\varphi^{-1} \circ (\partial q' \otimes 1 \otimes 1) \circ \varphi)
\]

for \( i = 1, 2, \ldots, n - r \).

**Lemma 3.2.** Let us assume \( n - r > 2 \). Then the isotypical components of the \( \mathfrak{l}' \)-module \( \mathbb{C}[\mathfrak{u}^*] \) are of the form

\[
\bigoplus_{\ell=0}^{\min(c_a, d_0)} \varphi^{-1}((C[q', q'', z] \otimes_{c} \mathcal{H}'_{a, b} \otimes_{c} \mathcal{H}''_{c, d} - \ell, d_0 - \ell))
\]

for \( a_0, b_0, c_0, d_0 \in \mathbb{N}_0 \), where \( C[q', q'', z] \otimes_{c} \mathcal{H}'_{a, b} \otimes_{c} \mathcal{H}''_{c, d} \) is the subspace of polynomials of degree \( \ell \). These \( \mathfrak{l}' \)-isotypical components are of the highest weight \( \mu_1 \omega_1 + a_0 \omega_2 + b_0 \omega_n - r + \mu_2 \omega_{n-r+1} \), where

\[
\mu_1 = \lambda_1 - \frac{1}{2}(n + 1) - 2a_0 - c_0,
\]

\[
\mu_2 = \lambda_2 - \frac{1}{2}(n + 1) - 2b_0 - d_0.
\]

**Proof.** Since the mapping

\[
\varphi: \mathbb{C}[\mathfrak{u}^*] \xrightarrow{\sim} \bigoplus_{(a, b, c, d) \in \mathbb{N}^4} C[q', q'', z] \otimes_{c} \mathcal{H}'_{a, b} \otimes_{c} \mathcal{H}''_{c, d}
\]

is an isomorphism of vector spaces, we can define the structure of an \( \mathfrak{l}' \)-module on the vector space \( \bigoplus_{(a, b, c, d) \in \mathbb{N}^4} C[q', q'', z] \otimes_{c} \mathcal{H}'_{a, b} \otimes_{c} \mathcal{H}''_{c, d} \), so that \( \varphi \) is an \( \mathfrak{l}' \)-equivariant mapping. For the action of \( [\mathfrak{l}', \mathfrak{l}'] \) we get

\[
\varphi \circ \hat{\pi}_\lambda(h_{A'}) \circ \varphi^{-1} = 1 \otimes \hat{\pi}_\lambda(h_{A'}) \otimes 1,
\]

where \( h_{A'} \in [\mathfrak{l}', \mathfrak{l}'] \). Because \( \mathcal{H}'_{a, b, 0} \) is an irreducible \( [\mathfrak{l}', \mathfrak{l}'] \)-module with the highest weight \( a_0 \omega_2 + b_0 \omega_{n-r} \) and the highest weight vector \( (a_0 + b_0) \cdot (a_0, b_0, 0) \) (cf. Appendix [A]), we obtain that the isotypical component of \( [\mathfrak{l}', \mathfrak{l}'] \) with highest weight \( a_0 \omega_2 + b_0 \omega_{n-r} \) is

\[
\bigoplus_{(c, d) \in \mathbb{N}^2} C[q', q'', z] \otimes_{c} \mathcal{H}'_{a, b, 0} \otimes_{c} \mathcal{H}''_{c, d}.
\]

The generators of the center \( \mathfrak{z}(\mathfrak{l}'_c) \) act by

\[
\varphi \circ \hat{\pi}_\lambda(h_{i_1}) \circ \varphi^{-1} = -(1 \otimes (E'_{x} + E'_{y}) \otimes 1 - 1 \otimes 1 \otimes (E''_{x} + E''_{y})
\]

\[
- 2(q' \partial q' + q'' \partial q'' + E_x) \otimes 1 \otimes 1 + \lambda_1 + \lambda_2 - (n + 1)
\]

and

\[
\varphi \circ \hat{\pi}_\lambda(h_{i_2}) \circ \varphi^{-1} = -(1 + \frac{2}{n-r}) (1 \otimes (E'_x - E'_y) \otimes 1) - 1 \otimes 1 \otimes (E''_x - E''_y) + \lambda_1 - \lambda_2.
\]

Therefore, we have

\[
(\varphi \circ \hat{\pi}_\lambda(h_{i_1}) \circ \varphi^{-1}) v = -(a_0 + b_0 + c + d + 2\ell - \lambda_1 - \lambda_2 + n + 1)v,
\]

\[
(\varphi \circ \hat{\pi}_\lambda(h_{i_2}) \circ \varphi^{-1}) v = -((1 + \frac{2}{n-r}) (a_0 - b_0) + c - d - \lambda_1 + \lambda_2)v
\]

for \( v \in C[q', q'', z] \otimes_{c} \mathcal{H}'_{a, b, 0} \otimes_{c} \mathcal{H}''_{c, d} \), which implies that the center \( \mathfrak{z}(\mathfrak{l}'_c) \) acts by a character on the subspace

\[
\bigoplus_{\ell=0}^{\min(c_a, d_0)} C[q', q'', z] \otimes_{c} \mathcal{H}'_{a, b, 0} \otimes_{c} \mathcal{H}''_{c, d} - \ell, d_0 - \ell
\]
of the vector space $\bigoplus_{(c,d)\in\mathbb{N}_0^2} \mathbb{C}[q', q'', z] \otimes \mathbb{C} \mathcal{H}_{a_0,b_0} \otimes \mathbb{C} \mathcal{H}_{c,d}''$. \hfill \Box

Lemma 3.2 yields the restrictions on the rank of the Lie subalgebra $\mathfrak{g}'_r = \mathfrak{sl}(n-r+2, \mathbb{C})$, because for $n-r = 2$ or $n-r = 1$ the decomposition of $\mathbb{C}[\mathfrak{r}]$ on $\mathfrak{l}'$-isotypical components differs from (5.55). So we shall focus on $n-r > 2$ and omit these two special cases from our discussion. Furthermore, by Lemma 3.2 we see that the isotypical components of the $\mathfrak{l}'$-module $\mathbb{C}[(g_{-1})^*]$ are uniquely determined by $a_0, b_0, c_0, d_0 \in \mathbb{N}_0$ and the highest weight of the corresponding isotypical component is

$$\left(\lambda_1 - \frac{1}{2}(n+1) - 2a_0 - c_0\right)\omega_1 + a_0\omega_2 + b_0\omega_{n-r} + \left(\lambda_2 - \frac{1}{2}(n+1) - 2b_0 - d_0\right)\omega_{n-r+1}. \quad (3.61)$$

Let us consider

$$\varphi(R_m) \in \bigoplus_{\ell=0}^{\min\{c_0,d_0\}} \mathcal{P}_e \otimes \mathbb{C} \mathcal{H}_{a_0,b_0} \otimes \mathbb{C} \mathcal{H}_{c_0-d_0,e_0-d_0'} \quad (3.62)$$

where $\mathcal{P}_e \subset \mathbb{C}[q', q'']$ is the subspace of polynomials of degree $e$ and $a_0 + b_0 + c_0 + d_0 = m$, $(n-r+2)(a_0 - b_0) + (n-r)(c_0 - d_0) = t$ as a consequence of (5.55) and (3.69). The notation

$$a_1(a,b,c,d) = a + c - \lambda_1 + \frac{1}{2}(n-1), \quad (3.63)$$

$$a_2(a,b,c,d) = b + d - \lambda_2 + \frac{1}{2}(n-1) \quad (3.64)$$

for $a, b, c, d \in \mathbb{N}_0$, turns the conditions (3.65) and (3.69) into

$$\partial_{x_1}(E_x - \lambda_1 + \frac{1}{2}(n-1))R_m = a_1(a_0,b_0,c_0,d_0)\partial_{x_1}R_m, \quad (3.65)$$

$$\partial_{y_1}(E_y - \lambda_2 + \frac{1}{2}(n-1))R_m = a_2(a_0,b_0,c_0,d_0)\partial_{y_1}R_m \quad (3.66)$$

for $i = 1, 2, \ldots, n-r$. Therefore, we get four mutually exclusive cases:

1) $a_0 \neq 0$, $b_0 \neq 0$, $a_1(a_0,b_0,c_0,d_0) = 0$, $a_2(a_0,b_0,c_0,d_0) = 0$;
2) $a_0 \neq 0$, $b_0 = 0$, $a_1(a_0,b_0,c_0,d_0) = 0$;
3) $a_0 = 0$, $b_0 \neq 0$, $a_2(a_0,b_0,c_0,d_0) = 0$;
4) $a_0 = 0$, $b_0 = 0$.

In what follows, we present a particular discussion of the construction of $R$ for all these possibilities. First of all, we shall start with two preparatory technical Lemmas.

**Lemma 3.3.** Let $\partial_{y_1}(E_y - \lambda_2 + \frac{1}{2}(n-1))R_m = 0$ for $i = 1, 2, \ldots, n-r$, then

$$R = \sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \frac{z^k}{2^kk!} \Box^k R_m, \quad (3.67)$$

where $R_m$ satisfies

$$y_i \Box R_m = \partial_{x_1}(E_x - \lambda_1 + \frac{1}{2}(n-1))R_m \quad (3.68)$$

for $i = 1, 2, \ldots, n-r$.

**Proof.** First of all, we shall prove by induction that $R_{m-2k} = \frac{1}{2^kk!}\Box^k R_m$ and $\partial_{y_1}(E_y - \lambda_2 + \frac{1}{2}(n-1))R_{m-2k} = -k R_{m-2k}$ for $i = 1, 2, \ldots, n-r$ and all $k \in \mathbb{N}_0$. This statement holds for $k = 0$. Let us assume that $R_{m-2k} = \frac{1}{2^kk!}\Box^k R_m$ and $\partial_{y_1}(E_y - \lambda_2 + \frac{1}{2}(n-1))R_{m-2k} = -k R_{m-2k}$ for $k = 0, 1, \ldots, k_0$. Then (3.65) for $k = k_0$ gives

$$(k_0 + 1) x_i R_{m-2k_0-2} - \frac{1}{2} k_0 \partial_{y_1} R_{m-2k_0} - \frac{1}{2} x_i \Box R_{m-2k_0} + \frac{1}{2} \partial_{y_1} \Box R_{m-2k_0+2} = 0,$$

which is equivalent to

$$x_i (k_0 + 1) R_{m-2k_0-2} - \frac{1}{2} \Box R_{m-2k_0} - \frac{1}{2} \partial_{y_1} (k_0 R_{m-2k_0} - \frac{1}{2} \Box R_{m-2k_0+2}) = 0.$$
Since $k_0 R_{m-2k_0} - \frac{1}{2} \Box R_{m-2k_0+2} = 0$, we obtain

$$R_{m-2k_0-2} = \frac{1}{2^{k_0+1}} \Box R_{m-2k_0} - \frac{1}{2^{k_0+1} (k_0+1)!} \Box^{k_0+1} R_m$$

and

$$\partial_{y_1}(E_y - \lambda_2 + \frac{1}{2}(n-1)) R_{m-2k_0-2} = \frac{1}{2^{k_0+1}} \partial_{y_1}(E_y - \lambda_2 + \frac{1}{2}(n-1)) \Box R_{m-2k_0} - \frac{1}{2^{k_0+1} (k_0+1)!} \Box^{k_0+1} R_m$$

$$= \frac{1}{2^{k_0+1}} \Box \partial_{y_1}(E_y - 1 - \lambda_2 + \frac{1}{2}(n-1)) R_{m-2k_0} - \frac{1}{2^{k_0+1} (k_0+1)!} \Box^{k_0+1} R_m$$

$$= \frac{1}{2^{k_0+1}} (-k_0 - 1) \Box R_{m-2k_0} - \frac{1}{2^{k_0+1} (k_0+1)!} \Box^{k_0+1} R_m$$

where we used $[\Box, E_y] = \Box$ in $A^{\alpha}_m$.

Now the formulas $2k R_{m-2k} = \Box R_{m-2k+2}$ and $2(k+1) R_{m-2k-2} = \Box R_{m-2k}$ reduce (3.44) into

$$y_i \Box R_{m-2k} = \partial_{x_i}(E_x - \lambda_1 + \frac{1}{2}(n-1)) R_{m-2k}$$

for $k = 0, 1, \ldots, \lfloor \frac{m}{2} \rfloor$. Let us assume (3.69) holds for some $k \in \{0, 1, \ldots, \lfloor \frac{m}{2} \rfloor \}$. Then the application of the Laplace operator $\Box$ to (3.69), together with the relations $[\Box, \partial_{x_1}] = 0$, $[\Box, y_1] = \partial_{x_1}$, and $[\Box, E_x] = \Box$ in $A^{\alpha}_m$, gives

$$y_i \Box^2 R_{m-2k} = \partial_{x_i}(E_x - \lambda_1 + \frac{1}{2}(n-1)) R_{m-2k},$$

and the equality $2(k+1) R_{m-2k-2} = \Box R_{m-2k}$ implies

$$y_i \Box R_{m-2k-2} = \partial_{x_i}(E_x - \lambda_1 + \frac{1}{2}(n-1)) R_{m-2k-2}.$$

Therefore, the system of equations (3.69) for $k = 0, 1, \ldots, \lfloor \frac{m}{2} \rfloor$ reduces to the single equation

$$y_i \Box R_m = \partial_{x_i}(E_x - \lambda_1 + \frac{1}{2}(n-1)) R_m$$

for $R_m$ and we are done. \hfill \Box

**Lemma 3.4.** Let $\partial_{x_i}(E_x - \lambda_1 + \frac{1}{2}(n-1)) R_m = 0$ for $i = 1, 2, \ldots, n-r$. Then

$$R = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^k k^2}{2^{k} k!} \Box^k R_m,$$

(3.70)

where $R_m$ satisfies

$$x_i \Box R_m = \partial_{y_1}(E_y - \lambda_2 + \frac{1}{2}(n-1)) R_m,$$

(3.71)

for $i = 1, 2, \ldots, n-r$.

**Proof.** The proof can be given either by similar argument as in the previous Lemma 3.3 or we can use the fact that the system of equations (3.44) can be obtained from the system of equations (3.43) by the substitution $x_i \rightarrow y_i$, $y_i \rightarrow x_i$, $z \rightarrow -z$ and $\lambda_1 \rightarrow \lambda_2$. \hfill \Box

**Case 1** Let us assume

$$\phi(R_m) \in \bigoplus_{\ell=0}^{\min\{c_0,d_0\}} P_\ell \otimes \mathcal{H}'_{a_0,b_0} \otimes \mathcal{H}''_{c_0-d_0-\ell},$$

(3.72)

where $a_1(a_0,b_0,c_0,d_0) = 0$, $a_2(a_0,b_0,c_0,d_0) = 0$ and $a_0 \neq 0$, $b_0 \neq 0$. Then we have $\partial_{x_i}(E_x - \lambda_1 + \frac{1}{2}(n-1)) R_m = 0$ and $\partial_{y_1}(E_y - \lambda_2 + \frac{1}{2}(n-1)) R_m = 0$ for $i = 1, 2, \ldots, n-r$. Hence, from Lemma 3.4, we get

$$R = R_m,$$

(3.73)
where \( R_m \) satisfies
\[
\square R_m = 0. \tag{3.74}
\]

By the isomorphism (3.51) and the formula (3.54), we can rewrite the equation (3.74), into the form
\[
(Q_{a_0+b_0, c_0+d_0+2\ell} \otimes 1 \otimes 1) \varphi(R_m)_\ell = 0 \tag{3.75}
\]
for \( \ell = 0, 1, \ldots, \min\{c_0, d_0\} \), where \( \varphi(R_m)_\ell \in \mathcal{P}_\ell \otimes \mathbb{C} \mathcal{H}'_{a_0, b_0} \otimes \mathcal{H}''_{c_0+\ell, d_0-\ell} \) and
\[
\varphi(R_m) = \sum_{\ell=0}^{\min\{c_0, d_0\}} \varphi(R_m)_\ell. \tag{3.76}
\]

As we have
\[
Q_{a_0+b_0, c_0+d_0+2\ell} = Q_{-c_0-d_0+\lambda_1+\lambda_2-n+1, c_0+d_0-2\ell}, \tag{3.77}
\]
where we used \( a_1(a_0, b_0, c_0, d_0) = 0 \) and \( a_2(a_0, b_0, c_0, d_0) = 0 \), we get
\[
\varphi(R) \in \bigoplus_{\ell=0}^{\min\{c_0, d_0\}} S_{\ell, c_0+d_0}^{\lambda-\rho} \otimes \mathbb{C} \mathcal{H}'_{a_0, b_0} \otimes \mathcal{H}''_{c_0-\ell, d_0-\ell} \tag{3.78}
\]
where the subspace \( S_{\ell, c_0+d_0}^{\lambda-\rho} \) of \( \mathcal{P}_\ell \) is defined by
\[
S_{\ell, s}^{\lambda-\rho} = \{ u \in \mathcal{P}_\ell; Q_{-s+\lambda_1+\lambda_2-n+1, s-2\ell} u = 0 \} \tag{3.79}
\]
for \( s, \ell \in \mathbb{N}_0 \).

**Case 2** Let us now suppose that
\[
\varphi(R_m) \in \bigoplus_{\ell=0}^{\min\{c_0, d_0\}} \mathcal{P}_\ell \otimes \mathbb{C} \mathcal{H}'_{a_0, b_0} \otimes \mathcal{H}''_{c_0-\ell, d_0-\ell}, \tag{3.80}
\]
where \( a_1(a_0, 0, c_0, d_0) = 0 \) and \( a_0 \neq 0 \). Then we have \( \partial_{x_i}(E_x - \lambda_1 + \frac{1}{2}(n-1)) R_m = 0 \) for \( i = 1, 2, \ldots, n-r \), and so by Lemma 3.3
\[
R = \sum_{k=0}^{\lfloor \frac{n}{2k} \rfloor} \frac{(-1)^k x^k}{2^k k!} \square^k R_m, \tag{3.81}
\]
where \( R_m \) satisfies
\[
x_i \square R_m = a_2(a_0, 0, c_0, d_0) \partial_{y_i} R_m \tag{3.82}
\]
for \( i = 1, 2, \ldots, n-r \). Using (3.57), we can write (3.82) as
\[
\square R_m = a_2(a_0, 0, c_0, d_0)(\varphi^{-1} \circ (\partial_{y'} \otimes 1 \otimes 1) \circ \varphi) R_m. \tag{3.83}
\]
The combination of (3.51) and (3.54) rewrites the equation (3.83) into the form
\[
(Q_{a_0, c_0+d_0-2\ell} \otimes 1 \otimes 1) \varphi(R_m)_\ell = a_2(a_0, 0, c_0, d_0)(\partial_{y'} \otimes 1 \otimes 1) \varphi(R_m)_\ell \tag{3.84}
\]
for \( \ell = 0, 1, \ldots, \min\{c_0, d_0\} \). As we have
\[
Q_{a_0, c_0+d_0-2\ell} - a_2(a_0, 0, c_0, d_0) \partial_{y'} = Q_{-c_0-d_0+\lambda_1+\lambda_2-n+1, c_0+d_0-2\ell}, \tag{3.85}
\]
due to \(a_1(a_0, 0, c_0, d_0) = 0\), we get

\[
\varphi(R_m) \in \bigoplus_{\ell=0}^{\min\{c_0, d_0\}} S^{\lambda - \rho}_{\ell, c_0 + d_0} \otimes \mathcal{H}_0^{\prime} \otimes \mathcal{H}_0^{\prime \prime} \otimes \mathcal{H}_{c_0 - \ell, d_0 - \ell} \quad (3.86)
\]

for the subspace \(S^{\lambda - \rho}_{\ell, c_0 + d_0}\) of \(\mathcal{P}_c\) defined by (3.79). We shall now prove by induction

\[
\Box^k R_m = k! \left( a_2(a_0, 0, c_0, d_0) \right) (\varphi^{-1} \circ (\partial_{q'}^k \otimes 1 \otimes 1) \circ \varphi) R_m \quad (3.87)
\]

for all \(k \in \mathbb{N}_0\), which evidently holds for \(k = 0\) and \(k = 1\). We may write

\[
[\varphi \circ \Box \circ \varphi^{-1}, \partial_{q'}^k \otimes 1 \otimes 1] = -k (\partial_{q'}^{k+1} \otimes 1 \otimes 1) \quad (3.88)
\]

for \(k \in \mathbb{N}_0\), where we used (3.54). The previous formula and the induction hypothesis lead to

\[
\Box^{k+1} R_m = k! \left( a_2(a_0, 0, c_0, d_0) \right) \Box (\varphi^{-1} \circ (\partial_{q'}^k \otimes 1 \otimes 1) \circ \varphi) R_m
\]

\[
= k! \left( a_2(a_0, 0, c_0, d_0) \right) (\varphi^{-1} \circ (\partial_{q'}^k \otimes 1 \otimes 1) \circ \varphi) R_m
\]

\[
= k! \left( a_2(a_0, 0, c_0, d_0) \right) (\varphi^{-1} \circ (\partial_{q'}^k \otimes 1 \otimes 1) \circ \varphi) R_m
\]

\[
= (k + 1)! \left( a_2(a_0, 0, c_0, d_0) \right) \varphi^{-1} \circ (\partial_{q'}^k \otimes 1 \otimes 1) \circ \varphi) R_m,
\]

where we used (3.83). Denoting

\[
T^{\lambda - \rho}_{a_0 + c_0, d_0} = \sum_{k=0}^{\frac{m}{2}} \left( -\frac{1}{2k} \left( a_2(a_0, 0, c_0, d_0) \right) (z \partial_{q'})^k \right)
\]

we get

\[
\varphi(R) \in \bigoplus_{\ell=0}^{\min\{c_0, d_0\}} T^{\lambda - \rho}_{a_0 + c_0, d_0} S^{\lambda - \rho}_{\ell, c_0 + d_0} \otimes \mathcal{H}_0^{\prime} \otimes \mathcal{H}_0^{\prime \prime} \otimes \mathcal{H}_{c_0 - \ell, d_0 - \ell} \quad (3.90)
\]

**Case 3)** We now assume

\[
\varphi(R_m) \in \bigoplus_{\ell=0}^{\min\{c_0, d_0\}} \mathcal{P}_\ell \otimes \mathcal{H}_0^{\prime} \otimes \mathcal{H}_0^{\prime \prime} \otimes \mathcal{H}_{c_0 - \ell, d_0 - \ell} \quad (3.91)
\]

where \(a_2(0, b_0, c_0, d_0) = 0\) and \(b_0 \neq 0\). Then we have \(\partial_{y_i} (E_y - \lambda_2 + \frac{1}{2}(n - 1)) R_m = 0\) for \(i = 1, 2, \ldots, n - r\), and so by Lemma 3.3, we get

\[
R = \sum_{k=0}^{\frac{m}{2}} \frac{z^k}{2k!} \Box^k R_m, \quad (3.92)
\]

where \(R_m\) satisfies

\[
y_i \Box R_m = a_1(0, b_0, c_0, d_0) \partial_{x_i} R_m \quad (3.93)
\]

for \(i = 1, 2, \ldots, n - r\). Using (3.56), we can rewrite (3.93) as

\[
\Box R_m = a_1(0, b_0, c_0, d_0) (\varphi^{-1} \circ (\partial_{q'}^k \otimes 1 \otimes 1) \circ \varphi) R_m. \quad (3.94)
\]

29
The isomorphism (3.51) and the formula (3.54) reduce the equation (3.94) into
\[ (Q_{b_0,c_0}+d_0-2\ell \otimes 1 \otimes 1)\varphi(R_m)_\ell = a_1(0,b_0,c_0,d_0)(\partial_{q'} \otimes 1 \otimes 1)\varphi(R_m)_\ell, \]  
for \( \ell = 0, 1, \ldots, \min\{c_0,d_0\} \). As we have
\[ Q_{b_0,c_0}+d_0-2\ell - a_1(0,b_0,c_0,d_0)\partial_{q'} = Q_{b_0-a_1(0,b_0,c_0,d_0),c_0+d_0-2\ell} = Q_{c_0-d_0+a_1+2-n+1,c_0+d_0-2\ell}, \]  
where we used \( a_2(0,b_0,c_0,d_0) = 0 \), we get
\[ \varphi(R_m) \in \bigoplus_{\ell=0}^{\min\{c_0,d_0\}} S^{\lambda-\rho}_{\ell,c_0+d_0} \otimes \mathcal{H}_0^{\prime}_{b_0} \otimes \mathcal{H}''_{c_0-\ell,d_0-\ell}, \]  
for the subspace \( S^{\lambda-\rho}_{\ell,c_0+d_0} \) of \( \mathcal{P}_\ell \) defined by (3.79). We shall prove by induction that
\[ \Box^{k+1}R_m = k!\left(\frac{a_1(0,b_0,c_0,d_0)}{k}\right)\Box^{k}(\varphi^{-1} \circ (\partial_{q'}^k \otimes 1 \otimes 1) \circ \varphi)R_m \]  
for all \( k \in \mathbb{N}_0 \), which evidently holds for \( k = 0 \) and \( k = 1 \). Using (3.88) and the induction hypothesis, we obtain
\[ \Box^{k+1}R_m = k!\left(\frac{a_1(0,b_0,c_0,d_0)}{k}\right)^2(\varphi^{-1} \circ (\partial_{q'}^{k+1} \otimes 1 \otimes 1) \circ \varphi)R_m \]  
\[ = k!\left(\frac{a_1(0,b_0,c_0,d_0)}{k}\right)^2(\varphi^{-1} \circ (\partial_{q'}^k \otimes 1 \otimes 1) \circ \varphi)R_m \]  
\[ = k!\left(\frac{a_1(0,b_0,c_0,d_0)}{k}\right)^2(\varphi^{-1} \circ (\partial_{q'}^{k+1} \otimes 1 \otimes 1) \circ \varphi)R_m \]  
\[ = (k+1)!\left(\frac{a_1(0,b_0,c_0,d_0)}{k+1}\right)^2(\varphi^{-1} \circ (\partial_{q'}^k \otimes 1 \otimes 1) \circ \varphi)R_m, \]  
where we used (3.83). Denoting
\[ T^{\lambda-\rho}_{\ell,c_0+b_0+d_0} = \sum_{k=0}^{\lfloor \frac{c_0}{2} \rfloor} \frac{1}{2^k} \left(\frac{a_1(0,b_0,c_0,d_0)}{k}\right)^2(\varphi^{-1} \circ (\partial_{q'}^k \otimes 1 \otimes 1) \circ \varphi)^k, \]  
we have
\[ \varphi(R_m) \in \bigoplus_{\ell=0}^{\min\{c_0,d_0\}} T^{\lambda-\rho}_{\ell,c_0+b_0+d_0} \otimes \mathcal{H}_0^{\prime}_{b_0} \otimes \mathcal{H}''_{c_0-\ell,d_0-\ell}. \]  

**Case 4** Finally, let us suppose that
\[ \varphi(R_m) \in \bigoplus_{\ell=0}^{\min\{c_0,d_0\}} \mathcal{P}_\ell \otimes \mathcal{H}_0^{\prime}_{b_0} \otimes \mathcal{H}''_{c_0-\ell,d_0-\ell}. \]  

Using (3.50) and (3.57), we can rewrite (3.49) and (3.47) into the form
\[ -R_{m-2} + a_1(0,0,c_0,d_0)(\varphi^{-1} \circ (\partial_{q'} \otimes 1 \otimes 1) \circ \varphi)R_m - \frac{1}{2}\Box R_m = 0, \]  
\[ R_{m-2} + a_2(0,0,c_0,d_0)(\varphi^{-1} \circ (\partial_{q'} \otimes 1 \otimes 1) \circ \varphi)R_m - \frac{1}{2}\Box R_m = 0, \]  
and their sum yields
\[ \Box R_m = (a_1(0,0,c_0,d_0) + a_2(0,0,c_0,d_0))(\varphi^{-1} \circ (\partial_{q'} \otimes 1 \otimes 1) \circ \varphi)R_m. \]
The isomorphism \(3.51\) and the formula \(3.104\) allow to rewrite \(3.104\) into
\[
(Q_{0,c_0+d_0-2\ell} \otimes 1 \otimes 1)\varphi(R_m)_{\ell} = (a_1(0,0,c_0,d_0) + a_2(0,0,c_0,d_0))((\partial q' \otimes 1 \otimes 1))\varphi(R_m)_{\ell}
\] (3.105)
for \(\ell = 0,1,\ldots,\min\{c_0,d_0\}\). As we have
\[
Q_{0,c_0+d_0-2\ell} - (a_1(0,0,c_0,d_0) + a_2(0,0,c_0,d_0))\partial q' = Q-c_0-d_0+\lambda_1+\lambda_2-\ell+1,c_0+d_0-2\ell,
\] (3.106)
we get
\[
\varphi(R_m) \in \bigoplus_{\ell=0}^{\min\{c_0,d_0\}} S_{c_0+d_0}^{\lambda-\rho} \otimes \mathcal{H}_{0,0} \otimes \mathcal{H}_{c_0-\ell,d_0-\ell}^\prime.
\] (3.107)
where the subspace \(S_{c_0+d_0}^{\lambda-\rho}\) of \(\mathcal{P}_\ell\) is defined by \(3.79\). We shall prove by induction that
\[
\square R_{m-2k} = (a_1(0,0,c_0,d_0) + a_2(0,0,c_0,d_0) - k)(\varphi^{-1} \circ (\partial q' \otimes 1 \otimes 1) \circ \varphi)R_{m-2k}
\] (3.108)
and
\[
\varphi(R_{m-2k}) \in \bigoplus_{\ell=k}^{\min\{c_0,d_0\}} \mathcal{P}_{\ell-k} \otimes \mathcal{H}_{0,0} \otimes \mathcal{H}_{c_0-\ell,d_0-\ell}^\prime.
\] (3.109)
for all \(k \in \mathbb{N}_0\). The claim holds for \(k = 0\). The recurrence relation \(3.44\) for \(k = k_0\) and the induction hypothesis imply
\[
(k_0 + 1)R_{m-2k_0-2} = \frac{1}{2}(a_1(0,0,c_0,d_0) - a_2(0,0,c_0,d_0))((\varphi^{-1} \circ (\partial q' \otimes 1 \otimes 1) \circ \varphi)R_{m-2k_0}
\]
\[
- \frac{1}{4}(a_1(0,0,c_0,d_0) + a_2(0,0,c_0,d_0) - k_0 + 1)(\varphi^{-1} \circ (\partial q' \otimes 1 \otimes 1) \circ \varphi)R_{m-2k_0+2},
\] (3.110)
which means that
\[
\varphi(R_{m-2k_0-2}) \in \bigoplus_{\ell=k_0+1}^{\min\{c_0,d_0\}} \mathcal{P}_{\ell-k_0-1} \otimes \mathcal{H}_{0,0} \otimes \mathcal{H}_{c_0-\ell,d_0-\ell}^\prime.
\] (3.111)
By \(3.44\) and \(3.45\) for \(k = k_0 + 1\), we have
\[
- (k_0 + 2)R_{m-2k_0-4} + (a_1(0,0,c_0,d_0) - \frac{1}{2}(k_0 + 1))(\varphi^{-1} \circ (\partial q' \otimes 1 \otimes 1) \circ \varphi)R_{m-2k_0-2}
\]
\[
- \frac{1}{4}\square R_{m-2k_0-2} - \frac{1}{4}(a_1(0,0,c_0,d_0) + a_2(0,0,c_0,d_0) - k_0)(\varphi^{-1} \circ (\partial q' \otimes 1 \otimes 1) \circ \varphi)R_{m-2k_0} = 0
\] (3.112)
and
\[
(k_0 + 2)R_{m-2k_0-4} + (a_2(0,0,c_0,d_0) - \frac{1}{2}(k_0 + 1))(\varphi^{-1} \circ (\partial q' \otimes 1 \otimes 1) \circ \varphi)R_{m-2k_0-2}
\]
\[
- \frac{1}{4}\square R_{m-2k_0-2} + \frac{1}{4}(a_1(0,0,c_0,d_0) + a_2(0,0,c_0,d_0) - k_0)(\varphi^{-1} \circ (\partial q' \otimes 1 \otimes 1) \circ \varphi)R_{m-2k_0} = 0,
\] (3.113)
which implies
\[
\square R_{m-2k_0-2} = (a_1(0,0,c_0,d_0) + a_2(0,0,c_0,d_0) - k_0 - 1)(\varphi^{-1} \circ (\partial q' \otimes 1 \otimes 1) \circ \varphi)R_{m-2k_0-2}.
\] (3.114)
As it follows from \(3.108\) and \(3.109\), the recurrence relations \(3.44\) and \(3.45\) are equivalent to
\[
(k + 1)R_{m-2k} = \frac{1}{2}(a_1(0,0,c_0,d_0) - a_2(0,0,c_0,d_0))((\varphi^{-1} \circ (\partial q' \otimes 1 \otimes 1) \circ \varphi)R_{m-2k}
\]
\[
- \frac{1}{4}(a_1(0,0,c_0,d_0) + a_2(0,0,c_0,d_0) - k + 1)(\varphi^{-1} \circ (\partial q' \otimes 1 \otimes 1) \circ \varphi)R_{m-2k+2}
\] (3.115)
for } k = 0, 1, \ldots, \lfloor \frac{m}{2} \rfloor {, so we get by induction

\[
R_{m-2k} = \frac{\alpha_k}{2^k} (\varphi^{-1} \circ (\partial_q^k \otimes 1 \otimes 1) \circ \varphi) R_m,
\]

where } \alpha_k \in \mathbb{C} { satisfy the following recurrence relation

\[
(k + 2)\alpha_{k+2} = (c_0 - d_0 - \lambda_1 + \lambda_2)\alpha_{k+1} - (c_0 + d_0 - \lambda_1 - \lambda_2 + n - 1 - k)\alpha_k
\]

with } \alpha_0 = 1 \text{ and } \alpha_{-1} = 0. \text{ The notation}

\[
T_{c_0, d_0}^{\lambda-\rho} = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{\alpha_k}{2^k} (z \partial_q)^k
\]

allows to write

\[
\varphi(R) \in \bigoplus_{\ell=0}^{\min\{c_0, d_0\}} T_{c_0, d_0}^{\lambda-\rho} S_{\ell, c_0+d_0}^{\lambda-\rho} \otimes_{\mathbb{C}} \mathcal{H}_{c_0-\ell, d_0-\ell}'.
\]

\[3.5 \quad \text{General structure of singular vectors}\]

In this subsection we summarize and organize the results achieved in preceding sections into particular statements describing the structure of singular vectors. Let us recall the isomorphism

\[
\tau \circ \Phi_{\lambda+\rho} : M_p^g(\lambda)^{\ell'} \xrightarrow{\sim} \text{Sol}(g, g', p; A_{\pi}^0/I_{\pi}) \xrightarrow{\sim} \text{Sol}(g, g', p; \mathbb{C}[\pi^*])^F
\]

of } \ell'-\text{modules. Since } M_p^g(\lambda)^{\ell'} \text{ is a completely reducible } \ell'-\text{module, we can find its decomposition into the isotypical components. As a consequence of the previous considerations we see that the isotypical components can be uniformly written as}

\[
\bigoplus_{\ell=0}^{\min\{c_0, d_0\}} \varphi^{-1}(T_{a_0+b_0+d_0}^{\lambda} S_{\ell, c_0+d_0}^{\lambda} \otimes_{\mathbb{C}} \mathcal{H}_{a_0+b_0}^{\lambda} \otimes_{\mathbb{C}} \mathcal{H}_{c_0-\ell, d_0-\ell}) \subset \text{Sol}(g, g', p; \mathbb{C}[\pi^*])^F,
\]

where } a_0, b_0, c_0, d_0 \in \mathbb{N}_0 { and the subspace } S_{\ell, s}^{\lambda} \subset \mathcal{P}_\ell { is defined by}

\[
S_{\ell, s}^{\lambda} = \{u \in \mathcal{P}_\ell; Q_{-s+\lambda_1+\lambda_2+s-2\ell} u = 0\}.
\]

The differential operator } T_{r_1, r_2}^{\lambda} : \mathbb{C}[q', q'', z] \to \mathbb{C}[q', q'', z] \text{ has the form}

\[
T_{r_1, r_2}^{\lambda} = \sum_{k=0}^{\lfloor \frac{r_1+r_2}{2} \rfloor} \frac{\alpha_k}{2^k} (z \partial_q)^k,
\]

where } \alpha_k \in \mathbb{C} { satisfy the following recurrence relation

\[
(k + 2)\alpha_{k+2} = (r_1 - r_2 - \lambda_1 + \lambda_2)\alpha_{k+1} - (r_1 + r_2 - \lambda_1 - \lambda_2 - 2 - k)\alpha_k
\]

with } \alpha_0 = 1 \text{ and } \alpha_{-1} = 0. \text{ We denote this isotypical component by } \mathcal{V}_{a_0, b_0, c_0, d_0}^{\lambda}.

Let us introduce the generating function for the collection } \{\alpha_k\}_{k \in \mathbb{N}_0} \text{.}

\[
g(w) = \sum_{k=0}^{\infty} \alpha_k w^k,
\]

Then the recurrence relation } (3.124) \text{ is equivalent to the first order linear differential equation}

\[
((1 - w^2)\partial_w + (r_1 + r_2 - \lambda_1 - \lambda_2 - 2)w - (r_1 - r_2 - \lambda_1 + \lambda_2))g(w) = 0
\]
for \( g(w) \) with \( g(0) = 1 \), whose unique solution is given by
\[
g(w) = (1 + w)^{r_1 - \lambda_1 - 1}(1 - w)^{r_2 - \lambda_2 - 1}
\] (3.127)

with \( g(0) = 1 \).

For the reader’s convenience, we split our discussion according to the integrality of the inducing weight \( \lambda \in \text{Hom}_P(\mathfrak{g}, \mathbb{C}) \).

**Theorem 3.5.** Let us suppose that \( \lambda_1, \lambda_2 \in N_0 \), and let \( n - r > 2 \).

i) If \( r > 0 \), then we have
\[
\tau \circ \Phi_{\lambda+\rho}: M^\mu_p(\lambda)^{u_r} \xrightarrow{\sim} \bigoplus_{c_0=0 \atop d_0=0}^{\lambda_1} \bigoplus_{c_0=0 \atop d_0=0}^{\lambda_2} \bigoplus_{c_0=0 \atop d_0=0}^{\lambda} V_{\lambda_1+1-c_0,\lambda_2+1-d_0,c_0,d_0}^{\lambda} \oplus \bigoplus_{c_0=0 \atop d_0=0}^{\lambda_1} V_{\lambda_1+1-c_0,0,c_0,d_0}^{\lambda} \oplus \bigoplus_{c_0=0 \atop d_0=0}^{\lambda_2} V_{0,\lambda_2+1-d_0,c_0,d_0}^{\lambda} \oplus \bigoplus_{c_0=0 \atop d_0=0}^{\lambda} V_{0,0,c_0,d_0}^{\lambda}.
\] (3.128)

ii) If \( r = 0 \), then we have
\[
\tau \circ \Phi_{\lambda+\rho}: M^\mu_p(\lambda)^{u_r} \xrightarrow{\sim} \mathcal{H}_{0,0} \oplus \mathcal{H}_{\lambda_1+1,0} \oplus \mathcal{H}_{0,\lambda_2+1} \oplus \mathcal{H}_{\lambda_1+1,\lambda_2+1} \oplus P_\lambda \mathcal{H}_{0,0},
\] (3.129)
where \( P_\lambda = T^{\lambda}_{\lambda_1+\lambda_2+n+1,\lambda_1+\lambda_2+n+1}(q^{\lambda_1+\lambda_2+n+1}) \).

**Proof.** The isotypical component \( V_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \) appears in the decomposition if and only if \( a, b, c, d \in N_0 \) satisfy the constraints in Case 1 up to Case 4, i.e. \( a \neq 0, b \neq 0, a + c - \lambda_1 - 1 = 0, b + d - \lambda_2 - 1 = 0 \) in Case 1, \( a \neq 0, b = 0, a + c - \lambda_1 - 1 = 0 \) in Case 2, \( a = 0, b \neq 0, b + d - \lambda_2 - 1 = 0 \) in Case 3, and \( a = 0, b = 0 \) in Case 4.

**Theorem 3.6.** Let us suppose that \( \lambda_1 \in N_0 \) and \( \lambda_2 \notin N_0 \), and let \( n - r > 2 \).

i) If \( r > 0 \), then we have
\[
\tau \circ \Phi_{\lambda+\rho}: M^\mu_p(\lambda)^{u_r} \xrightarrow{\sim} \bigoplus_{c_0=0 \atop d_0=0}^{\lambda_1} \bigoplus_{c_0=0 \atop d_0=0}^{\lambda_2} \bigoplus_{c_0=0 \atop d_0=0}^{\lambda} V_{\lambda_1+1-c_0,0,c_0,d_0}^{\lambda} \oplus \bigoplus_{c_0=0 \atop d_0=0}^{\lambda_1} V_{\lambda_1+1-c_0,0,c_0,d_0}^{\lambda} \oplus \bigoplus_{c_0=0 \atop d_0=0}^{\lambda_2} V_{0,\lambda_2+1-d_0,c_0,d_0}^{\lambda} \oplus \bigoplus_{c_0=0 \atop d_0=0}^{\lambda} V_{0,0,c_0,d_0}^{\lambda}.
\] (3.130)

ii) If \( r = 0 \), then we have
\[
\tau \circ \Phi_{\lambda+\rho}: M^\mu_p(\lambda)^{n} \xrightarrow{\sim} \begin{cases} \mathcal{H}_{0,0} \oplus \mathcal{H}_{\lambda_1+1,0} & \text{for } \lambda_1 + \lambda_2 + n \notin N_0, \\ \mathcal{H}_{0,0} \oplus \mathcal{H}_{\lambda_1+1,0} \oplus P_\lambda \mathcal{H}_{0,0} & \text{for } \lambda_1 + \lambda_2 + n \in N_0 \\ \mathcal{H}_{0,0} \oplus \mathcal{H}_{\lambda_1+1,0} \oplus P_\lambda \mathcal{H}_{0,0} \oplus P^1_\lambda \mathcal{H}_{-\lambda_2-2,n} & \text{for } \lambda_1 + \lambda_2 + n \in N_0 \end{cases}
\] (3.131)
where \( P_\lambda = T^{\lambda}_{\lambda_1+\lambda_2+n+1,\lambda_1+\lambda_2+n+1}(q^{\lambda_1+\lambda_2+n+1}) \) and \( P^1_\lambda = T^{\lambda}_{\lambda_1+1,\lambda_1+\lambda_2+n+1}(q^{\lambda_1+\lambda_2+n+1}) \).

**Proof.** The structure of the proof is identical as in Theorem 3.5

**Theorem 3.7.** Let us suppose that \( \lambda_1 \notin N_0 \) and \( \lambda_2 \in N_0 \), and let \( n - r > 2 \).

i) If \( r > 0 \), then we have
\[
\tau \circ \Phi_{\lambda+\rho}: M^\mu_p(\lambda)^{u_r} \xrightarrow{\sim} \bigoplus_{c_0=0 \atop d_0=0}^{\lambda_1} \bigoplus_{c_0=0 \atop d_0=0}^{\lambda_2} \bigoplus_{c_0=0 \atop d_0=0}^{\lambda} V_{0,\lambda_2+1-d_0,c_0,d_0}^{\lambda} \oplus \bigoplus_{c_0=0 \atop d_0=0}^{\lambda} V_{0,0,c_0,d_0}^{\lambda}.
\] (3.132)
ii) If \( r = 0 \), then we have

\[
\tau \circ \Phi_{\lambda + \rho} : M_p^g(\lambda)^{\mu_r} \quad \cong \quad \begin{cases} 
\mathcal{H}_{0,0} \oplus \mathcal{H}_{0,\lambda_2+1} & \text{for } \lambda_1 + \lambda_2 + n \notin \mathbb{N}_0, \\
\mathcal{H}_{0,0} \oplus \mathcal{H}_{0,\lambda_2+1} \oplus P_\lambda \mathcal{H}_{0,0} & \text{for } \lambda_1 + \lambda_2 + n \in \mathbb{N}_0, \\
\mathcal{H}_{0,0} \oplus \mathcal{H}_{0,\lambda_2+1} \oplus P_\lambda \mathcal{H}_{0,0} \oplus P^2 \mathcal{H}_{0,-\lambda_1-n} & \text{and } -\lambda_2 - n \notin \mathbb{N}, \\
\mathcal{H}_{0,0} \oplus \mathcal{H}_{0,\lambda_2+1} \oplus P_\lambda \mathcal{H}_{0,0} \oplus P^2 \mathcal{H}_{0,-\lambda_1-n} & \text{and } -\lambda_2 - n \in \mathbb{N},
\end{cases}
\]

(3.133)

where \( P_\lambda = T_{\lambda_1+\lambda_2+n+1, \lambda_1+\lambda_2+n+1}^\lambda (q^{\lambda_1+\lambda_2+n+1}) \) and \( P^2_\lambda = T_{\lambda_1+\lambda_2+n+1, \lambda_2+1}^\lambda (q^{\lambda_1+\lambda_2+n+1}) \).

Proof. The structure of the proof is identical as in Theorem 3.5. \( \square \)

**Theorem 3.8.** Let us suppose that \( \lambda_1, \lambda_2 \notin \mathbb{N}_0 \), and let \( n - r > 2 \).

i) If \( r > 0 \), then we have

\[
\tau \circ \Phi_{\lambda + \rho} : M_p^g(\lambda)^{\mu_{r-1}} \quad \cong \quad \bigoplus_{c_0=0}^{\infty} \bigoplus_{d_0=0}^{\infty} \mathcal{V}_{0,0,c_0,d_0}^\lambda.
\]

(3.134)

ii) If \( r = 0 \), then we have

\[
\tau \circ \Phi_{\lambda + \rho} : M_p^g(\lambda)^{\mu_r} \quad \cong \quad \begin{cases} 
\mathcal{H}_{0,0} & \text{for } \lambda_1 + \lambda_2 + n \notin \mathbb{N}_0, \\
\mathcal{H}_{0,0} \oplus P_\lambda \mathcal{H}_{0,0} & \text{for } \lambda_1 + \lambda_2 + n \in \mathbb{N}_0,
\end{cases}
\]

(3.135)

where \( P_\lambda = T_{\lambda_1+\lambda_2+n+1, \lambda_1+\lambda_2+n+1}^\lambda (q^{\lambda_1+\lambda_2+n+1}) \).

Proof. The structure of the proof is identical as in Theorem 3.5. \( \square \)

For \( r = 0 \), it is straightforward to identify in previous theorems the singular vectors corresponding to standard and non-standard homomorphisms of generalized Verma modules, respectively, cf. [23].

### 3.6 Examples

In this subsection we illustrate the general results given in Theorem 3.5, Theorem 3.6, Theorem 3.7, and Theorem 3.8 and write down explicit formulas for homomorphisms between generalized Verma modules in several examples for \( r = 0 \) and \( r = 1 \).

We shall start with \( r = 0 \). Let \( v_\lambda \) and \( v_\mu \) be the highest weight vectors of \( M_p^g(\lambda) \) and \( M_p^g(\mu) \), respectively. A homomorphism

\[
\varphi : M_p^g(\mu) \rightarrow M_p^g(\lambda)
\]

(3.136)
of generalized Verma modules is uniquely determined by \( \varphi(v_\mu) \in M_p^g(\lambda) \). For \( n > 2 \), we have in the Poincaré-Birkhoff-Witt basis of \( U(\mathfrak{h}) \):

1. If \( \lambda = \lambda_1 \omega_1 + \lambda_2 \omega_{n+1}, \mu = (\lambda_1 - 1) \omega_1 + (\lambda_2 - 1) \omega_{n+1}, \lambda_1 + \lambda_2 + n = 0 \), then the singular vector

\[
\varphi(v_\mu) = (\sum_{i=1}^{n} f_ig_i + \frac{1}{2}(\lambda_1 - \lambda_2 + n) c) v_\lambda,
\]

(3.137)
in \( M_p^g(\lambda) \) of graded homogeneity two induces a homomorphism of scalar generalized Verma modules, cf. Section 3.1 for the notation of root spaces.
We do not know a direct proof of this observation. orthogonal Lie algebras and their compatible conformal para obolic subalgebras with commutative

The examples (1) and (2) suggest the following remarkable conjectural factorization property of

(3) If \( \lambda = 0 \), \( \mu = -2 \omega_1 + \omega_2 + \omega_n - 2 \omega_{n+1} \), then Theorem 3.5 implies that

\[
\varphi(v_{\mu}) = (f_1 g_n)v_{\lambda} 
\] (3.139)

induces a homomorphism from the vector valued generalized Verma module \( M^G_{p_1}(\mu) \) to \( M^G_{p_1}(\lambda) \).

We do not know a direct proof of this observation.

Now we pass to \( r = 1 \). Denoting \( v'_\mu \) the highest weight vector of \( M^G_{p_1}(\mu) \), a \( g'_1 \)-homomorphism

\[
\varphi: M^G_{p_1}(\mu) \to M^G_{p_1}(\lambda) 
\] (3.141)

of generalized Verma modules is again uniquely determined by \( \varphi(v'_\mu) \in M^G_{p_1}(\lambda) \). For \( n > 3 \), we have in the Poincaré-Birkhoff-Witt basis of \( U(\mathfrak{p}) \):

(1) If \( \lambda = \lambda_1 \omega_1 + \lambda_2 \omega_{n+1}, \mu = (\lambda_1 - a)\omega_1 + \lambda_2 \omega_n, a \in \mathbb{N}_0 \), then the singular vector of graded homogeneity \( a \) in \( M^G_{p_1}(\lambda) \),

\[
\varphi(v'_\mu) = f^a_n v_{\lambda}, 
\] (3.142)

induces a \( g'_1 \)-homomorphism of scalar generalized Verma modules.

(2) If \( \lambda = \lambda_1 \omega_1 + \lambda_2 \omega_{n+1}, \mu = \lambda_1 \omega_1 + (\lambda_2 - a) \omega_n, a \in \mathbb{N}_0 \), then the singular vector of graded homogeneity \( a \) in \( M^G_{p_1}(\lambda) \),

\[
\varphi(v'_\mu) = g^a_n v_{\lambda}, 
\] (3.143)

induces a \( g'_1 \)-homomorphism of scalar generalized Verma modules.

(3) If \( \lambda = \lambda_1 \omega_1 + \lambda_2 \omega_{n+1}, \mu = (\lambda_1 - a - 1)\omega_1 + (\lambda_2 - 1) \omega_n, a \in \mathbb{N}_0 \), then the singular vector of graded homogeneity \( a + 2 \) in \( M^G_{p_1}(\lambda) \),

\[
\varphi(v'_\mu) = (\sum_{i=1}^{n-1} g_i f_i + (\lambda_1 - a) c - \frac{\lambda_1 + \lambda_2 + n - 1 - a}{a+1} g_n f_n) f^a_n v_{\lambda}, 
\] (3.144)

induces a \( g'_1 \)-homomorphism of scalar generalized Verma modules.

(4) If \( \lambda = \lambda_1 \omega_1 + \lambda_2 \omega_{n+1}, \mu = (\lambda_1 - 1)\omega_1 + (\lambda_2 - a - 1) \omega_n, a \in \mathbb{N}_0 \), then the singular vector of graded homogeneity \( a + 2 \) in \( M^G_{p_1}(\lambda) \),

\[
\varphi(v'_\mu) = (\sum_{i=1}^{n-1} f_i g_i - (\lambda_2 - a) c - \frac{\lambda_1 + \lambda_2 + n - 1 - a}{a+1} f_n g_n) g^a_n v_{\lambda}, 
\] (3.145)

induces a \( g'_1 \)-homomorphism of scalar generalized Verma modules.

We notice that for \( r = 0 \), the analogues of scalar valued singular vectors for the pair of orthogonal Lie algebras and their compatible conformal parabolic subalgebras with commutative nilradicals were constructed in [13] [21].
4 CR-equvariant differential operators

A CR (a shorthand notation for Cauchy-Riemann or Complex-Real) structure is given by a real differential manifold $M$ together with a complex distribution $H$, i.e. a complex vector subbundle of the complexified tangent bundle $TM_{\mathbb{C}}$ such that

1. $[H, H] \subset H$, i.e. $H$ is integrable,
2. $H \cap \overline{H} = \{0\}$, i.e. $H$ is almost Lagrangian.

A CR-density on $M$ is a section of the complex line bundle $(\Lambda^{n,0} H^*)^\lambda \otimes (\Lambda^0,n H^*)^{\lambda'}$, defined for all $\lambda, \lambda' \in \mathbb{C}$ such that $\lambda = \lambda' \in \mathbb{Z}$. Manifolds endowed with CR-structure emerge on the boundaries of strictly pseudo-convex domains in $\mathbb{C}^{n+1}$, and invariants of integrable strictly pseudo-convex CR-structures are used to obtain an expression for the asymptotic expansion of the Bergman kernel in $\mathbb{C}^{n+1}$. For the introduction and various aspects of CR-structure, we refer to [24, 10, 7, 5, 6, 1] and references therein.

Our results are directly related to the homogeneous model of CR-structure, which is the real form of the complex flag manifold, $G/P$ compatible parabolic subgroups $\mathfrak{p}$, where $\mathfrak{p}'$, $\mathfrak{p}$-equivariant differential operators acting on principal series representations for $\mathfrak{g}'$, $\mathfrak{g}'$-equivariant differential operators acting on principal series representations for $G$ and $G'$ supported on $G/P$ and $G'/P'$ ($G'/P' \subset G/P$), respectively. There is a contravariant bijection

$$\text{Hom}_{(\mathfrak{g}', \mathfrak{n}')}((\mathfrak{m}_{\mathfrak{p}}^{\mathfrak{g}}(\mathfrak{w})), (\mathfrak{m}_{\mathfrak{p}}^{\mathfrak{g}}(\mathfrak{w}))) \simeq \text{Hom}_{\text{Diff}}((\text{Ind}_{\mathfrak{g}}^{\mathfrak{g}'}(V^*), \text{Ind}_{\mathfrak{g}}^{\mathfrak{g}'}(W^*))), \quad (4.1)$$

where the subscript Diff denotes the homomorphisms given by differential operators. The generalized flag manifolds $G/P$ and $G'/P'$ are complexifications of the real flag manifolds $G_R/P_R \simeq S^{2n+1}$ and $G'_R/P'_R \simeq S^{2(n-r)+1}$ $(G_R/P_R \subset G/R$, respectively. Thus the $G'$-covariant differential operators in (4.1) induce, by restriction to the real locus of the complex generalization of flag manifolds, $G'_R$-equivariant differential operators acting between sections of homogeneous vector bundles on $S^{2n+1}$ and $S^{2(n-r)+1}$, respectively.

It is rather straightforward to transfer the singular vectors in Theorem 3.5, Theorem 3.6, Theorem 3.7 and Theorem 3.8 to covariant differential operators acting on the non-compact model of induced representations, cf. [24] for similar questions related to parabolic subalgebras of Hermitean symmetric type.

Appendix A The Fischer decomposition for $\mathfrak{sl}(n, \mathbb{C})$

In this section we recall the Fischer decomposition proved by E. Fischer ([5]) in 1917 and apply it to one specific example.

Let $\mathcal{A}_V$ be the Weyl algebra of a finite-dimensional complex vector space $V$ (see Section 2.2 for the definition). Then $\mathbb{C}[V]$ is a left $\mathcal{A}_V$-module and

$$\mathcal{A}_V \simeq \mathbb{C}[V] \otimes \mathbb{C}[V^*], \quad (A.1)$$
where we regard $\mathbb{C}[V^\ast]$ as the $\mathbb{C}$-algebra of constant coefficient differential operators. Let us consider a polynomial $q \in \mathbb{C}[V]$ and a constant coefficient differential operator $P \in \mathbb{C}[V^\ast]$. We say that the pair $(q, P)$ forms a Fischer pair on $V$, if we have the decomposition
\[
\mathbb{C}[V] \simeq \mathbb{C}[q] \otimes \mathbb{C} \mathcal{H},
\]
where $\mathcal{H} = \ker P$ and $\mathbb{C}[q]$ is the $\mathbb{C}$-subalgebra of $\mathbb{C}[V]$ generated by $q$.

To construct some Fischer pair on $V$, let us consider linear coordinate functions $(x_1, x_2, \ldots, x_n)$ on $V$, which give us a canonical isomorphism $\mathbb{C}[V] \simeq \mathbb{C}[x]$, $x = (x_1, x_2, \ldots, x_n)$. We introduce the scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}[x]$ by
\[
\langle p(x), q(x) \rangle = p^n(\partial_x)q(x)|_{x=0},
\]
where $\partial_x = (\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n})$ and $p^n(x) = p(x^n)$. Now, let $q \in \mathbb{C}[x]$ be a homogenous polynomial of degree $n \in \mathbb{N}$, and let us consider the mapping $T_q : \mathbb{C}[x] \to \mathbb{C}[x]$ given by $T_q(p) = qp$. Then we have
\[
(T_q(p_1), p_2) = (p_1, q^*(\partial_x)p_2)
\]
for all $p_1, p_2 \in \mathbb{C}[x]$, hence $q^*(\partial_x) : \mathbb{C}[x] \to \mathbb{C}[x]$ is the adjoint mapping to $T_q$. Therefore, $\mathbb{C}[x]$ decomposes into two orthogonal subspaces $\text{im} T_q$ and $\ker q^*(\partial_x)$, and hence $(q, q^*(\partial_x))$ is a Fischer pair.

We use this decomposition to the following specific example. Let $\mathbb{C}^n$ and $(\mathbb{C}^n)^\ast$ be the standard representation and the standard dual representation of the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, $n \geq 2$, respectively. Then we have the induced representation of $\mathfrak{g}$ on $\mathbb{C}[(\mathbb{C}^n)^\ast \oplus \mathbb{C}^n]$. Using the canonical linear coordinate functions $y = (y_1, y_2, \ldots, y_n)$ on $\mathbb{C}^n$ and the dual linear coordinate functions $x = (x_1, x_2, \ldots, x_n)$ on $(\mathbb{C}^n)^\ast$, we get the isomorphism $\mathbb{C}[(\mathbb{C}^n)^\ast \oplus \mathbb{C}^n] \simeq \mathbb{C}[x,y]$. The induced representation of $\mathfrak{g}$ on $\mathbb{C}[x,y]$ is given by
\[
\pi(A) = \sum_{i,j=1}^n a_{ij}(x_i \partial_{x_j} - y_j \partial_{y_i})
\]
for all $A \in \mathfrak{g}$. Moreover, if we introduce on $\mathbb{C}[x,y]$ the scalar product by (A.3), then
\[
q = \sum_{i=1}^n x_i y_i \quad \text{and} \quad \Box = \sum_{i=1}^n \partial_{x_i} \partial_{y_i}
\]
form a Fischer pair, hence we have the decomposition
\[
\mathbb{C}[x,y] = \mathbb{C}[q] \otimes \mathbb{C} \mathcal{H},
\]
where $\mathcal{H} = \ker \Box$ and $\mathbb{C}[q]$ is the $\mathbb{C}$-subalgebra of $\mathbb{C}[x,y]$ generated by $q$. Since $q$ is a $\mathfrak{g}$-invariant polynomial and $\Box$ is a $\mathfrak{g}$-invariant differential operator, we obtain that $\mathbb{C}[q]$ and $\mathcal{H}$ are representations of $\mathfrak{g}$. Because the Euler homogeneity operators
\[
E_x = \sum_{i=1}^n x_i \partial_{x_i} \quad \text{and} \quad E_y = \sum_{i=1}^n y_i \partial_{y_i}
\]
satisfy $[\Box, E_x] = \Box$, $[\Box, E_y] = \Box$ and $[E_x, E_y] = 0$, we get the direct sum decomposition
\[
\mathcal{H} = \bigoplus_{(a,b) \in \mathbb{N}_0^2} \mathcal{H}_{a,b}
\]
into the common eigenspaces $\mathcal{H}_{a,b}$ of $E_x$ and $E_y$ with the eigenvalues $a$ and $b$, respectively. As $E_x$ and $E_y$ are also $\mathfrak{g}$-invariant differential operators, the space $\mathcal{H}_{a,b}$ is a representation of $\mathfrak{g}$. In fact, it is an irreducible representation of $\mathfrak{g}$, since $\mathcal{H}_{a,b}$ contains an irreducible representation of $\mathfrak{g}$ with the highest weight $\omega_1 + b \omega_{n-1}$ ($x_1^a y_n^b$ is the highest weight vector with respect to the Borel subalgebra of upper triangular matrices) and its dimension is equal to the dimension of $\mathcal{H}_{a,b}$. 

Acknowledgments

L. Křížka is supported by PRVOUK p47, P. Somberg acknowledges the financial support from the grant GA P201/12/G028. The second author is grateful to T. Kobayashi for sharing his ideas and experience.

References

[1] Andreas Čap and Jan Slovák, *Parabolic Geometries I: Background and General Theory*, Mathematical Surveys and Monographs, vol. 154, American Mathematical Society, Providence, 2009.

[2] Leticia Barchini, Anthony C. Kable, and Roger Zierau, *Conformally invariant systems of differential equations and prehomogeneous vector spaces of Heisenberg parabolic type*, Publ. Res. Inst. Math. Sci. **44** (2008), no. 3, 749–835.

[3] Alexander A. Beilinson and Joseph N. Bernstein, *Localisation de \( g \)-modules*, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), 15–18.

[4] Joseph N. Bernstein, Izrail M. Gelfand, and Sergei I. Gel’fand, *Structure of representations generated by vectors of highest weight*, Functional Anal. Appl. **5** (1971), no. 1, 1–8.

[5] Thomas P. Branson, Luigi Fontana, and Carlo Morpurgo, *Moser-Trudinger and Beckner-Onofri’s inequalities on the CR sphere*, Ann. of Math. (2) **177** (2013), no. 1, 1–52.

[6] David H. Collingwood and Brad Shelton, *A duality theorem for extensions of induced highest weight modules*, Pacific J. Math. **146** (1990), no. 2, 227–237.

[7] Charles L. Fefferman, *Monge-Ampere Equations, the Bergman Kernel, and Geometry of Pseudoconvex Domains*, Ann. of Math. (2) **103** (1976), no. 3, 395–416.

[8] Ernst Fischer, *Über die Differentiationsprozesse der Algebra*, J. Reine Angew. Math. **148** (1918), 1–78.

[9] Akihiko Gyoja, *Highest weight modules and b-functions of semi-invariants*, Publ. Res. Inst. Math. Sci. **30** (1994), no. 3, 353–400.

[10] Reese F. Harvey and H. Blaine Lawson, Jr., *On Boundaries of Complex Analytic Varieties, I*, Ann. of Math. (2) **102** (1975), no. 2, 223–290.

[11] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki, *\( D \)-Modules, Perverse Sheaves, and Representation Theory*, Progress in Mathematics, vol. 236, Birkhäuser, Boston, 2008.

[12] James E. Humphreys, *Representations of Semisimple Lie Algebras in the BGG Category \( \mathcal{O} \)*, Graduate Studies in Mathematics, vol. 94, American Mathematical Society, Providence, 2008.

[13] Andreas Juhl, *Families of Conformally Covariant Differential Operators, Q-Curvature and Holography*, Progress in Mathematics, vol. 275, Birkhäuser, Basel, 2009.

[14] Masaki Kashiwara, *Representaion theory and \( D \)-modules on flag varieties*, Astérisque **173-174** (1989), 55–109.

[15] ***, *\( D \)-modules and Microlocal Calculus*, Translations of Mathematical Monographs, vol. 217, American Mathematical Society, Providence, 2003.

[16] Toshiyuki Kobayashi, *Discrete decomposability of the restriction of \( A_q(\lambda) \) with respect to reductive subgroups and its applications*, Invent. Math. **117** (1994), no. 1, 181–205.
[17] ______., *Discrete decomposability of the restriction of \( A_q(\lambda) \) with respect to reductive subgroups. II: Micro-local analysis and asymptotic \( K \)-support*, Ann. of Math. (2) 147 (1998), no. 2, 709–729.

[18] ______., *Discrete decomposability of the restriction of \( A_q(\lambda) \) with respect to reductive subgroups. III: Restriction of Harish-Chandra modules and associated varieties*, Invent. Math. 131 (1998), no. 2, 229–256.

[19] ______., *Multiplicity-free Theorems of the Restrictions of Unitary Highest Weight Modules with respect to Reductive Symmetric Pairs*, Representation Theory and Automorphic Forms, Progress in Mathematics, vol. 255, Birkhäuser, Boston, 2007, pp. 45–109.

[20] ______., *Restrictions of generalized Verma modules to symmetric pairs*, Transformation Groups 17 (2012), no. 2, 523–546.

[21] Toshiyuki Kobayashi, Bent Ørsted, Petr Somberg, and Vladimír Souček, *Branching laws for Verma modules and applications in parabolic geometry. I*, (2013).

[22] Toshiyuki Kobayashi and Michael Pevzner, *Differential symmetry breaking operators. I-General theory and F-method. II-Rankin-Cohen Operators for Symmetric Pairs*, (2013).

[23] James Lepowsky, *A generalization of the Bernstein-Gelfand-Gelfand resolution*, J. Algebra 49 (1977), no. 2, 496–511.

[24] Noboru Tanaka, *A Differential Geometric Study on Strongly Pseudoconvex Manifolds*, Lectures in Mathematics, Kinokuniya Book-Store, Tokyo, 1975.

(L.Křižka) Mathematical Institute of Charles University, Sokolovská 83, 180 00 Praha 8, Czech Republic
*E-mail address: krizka@karlin.mff.cuni.cz*

(P.Somberg) Mathematical Institute of Charles University, Sokolovská 83, 180 00 Praha 8, Czech Republic
*E-mail address: somberg@karlin.mff.cuni.cz*