Higher fundamental forms of the conformal boundary of asymptotically de Sitter spacetimes

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Received 29 August 2022
Accepted for publication 18 November 2022
Published 30 November 2022

Abstract
We provide a partial characterization of the conformal infinity of asymptotically de Sitter spacetimes by deriving constraints that relate the asymptotics of the stress-energy tensor with conformal geometric data. The latter is captured using recently defined objects, called higher conformal fundamental forms. For the boundary hypersurface, these generalize to higher order the trace-free part of the second form.

Keywords: conformal fundamental forms, asymptotically de Sitter spacetimes, tractor calculus

1. Introduction

Spacetimes with positive cosmological constant \( \Lambda \) have attracted increasing attention in mathematical relativity in recent years, see e.g. [1–3]. This was motivated by the observational implications that the Universe is best described if \( \Lambda > 0 \) is included in the Einstein field equations [4]. Moreover, asymptotically de Sitter spacetimes are also used in the context of the de Sitter/conformal field theory (dS/CFT) correspondence [5] and in the conformal cyclic cosmology scenario [6], which relies on the positivity of the cosmological constant.

The natural way to study the asymptotic structure of a spacetime \((\mathcal{M}, g_{ab})\) is through the conformal Einstein field equations formalism, introduced in [7]. In this approach, one considers a
conformal extension (unphysical spacetime) \((M, g_{ab})\) of \((\tilde{M}, \tilde{g}_{ab})\)—a four-dimensional manifold with the boundary \(\Sigma\) such that \(\tilde{M}\) can be identified with the interior of \(M\) and the metric \(\tilde{g}_{ab}\) is singular on \(\Sigma\). Certain global problems associated with the solution \((\tilde{M}, \tilde{g}_{ab})\) of the Einstein field equations can then be studied in terms of the local analysis in the neighbourhood of \(\Sigma\).

Under appropriate conditions regarding the conformal rescaling of the matter fields one can show that the conformal Einstein field equations form a regular system of partial differential equations on the unphysical spacetime \((M, g_{ab})\), see e.g. [8] for the case where gravitational field is coupled to the Maxwell and Yang–Mills fields. It should be noted that if a non-vanishing cosmological constant is included in the system, the conformal boundary is an umbilic hypersurface. This is a prime example of the constraints on the conformal fundamental forms of \(\Sigma\), as such hypersurface has vanishing trace-free extrinsic curvature.

The other approach to describe asymptotically de Sitter spacetimes is based on the Fefferman–Graham power series expansion of the metric, which stems from the theory of conformal invariants of manifolds of arbitrary dimensions [9]. The same type of construction appeared earlier in a somewhat similar setting of solving Einstein field equations with matter fields and a positive cosmological constant [10]. The key feature of this approach is that the conformally rescaled bulk metric and the physical stress-energy tensor \(\tilde{T}_{ab}\) are expanded in terms of the geodesic distance to the conformal boundary and the Einstein field equations are solved order by order. An example of such procedure is given in [11] and [12] in the construction of asymptotically de Sitter aeons in the conformal cyclic cosmology model.

Our work here utilizes the tractor calculus in the study of conformal infinity of spacetimes with positive cosmological constant. It is an efficient and effective tool for studying conformal invariants and invariant operators in conformal geometry. The natural connection between tractor calculus and general relativity stems from the fact that in the study of asymptotic properties of spacetimes one considers their conformal extension, which focuses on the causal structure while abandoning the notion of distance. An extended discussion of the tractor calculus can be found in a review [13], while its application in general relativity can be found for example in [14–18], among other works.

The main result of this paper builds on the notion of conformal fundamental forms introduced in [19] and aims to utilize it in the setting of asymptotically de Sitter spacetimes with the stress-energy tensor \(\tilde{T}_{ab}\). We will focus on the most commonly assumed decay rates of \(\tilde{T}_{ab}\), and formulate the main result in terms of constraints that relate matter fields along the conformal boundary to its conformal fundamental forms \(\tilde{K}^{(d)}_{ab}\), \(d = 2, 3, 4, 5\). It turns out that this amounts to showing how the divergence of suitable projected part of the Cotton, Weyl, and Bach tensors on \(\Sigma\) are related to the asymptotics of the stress-energy tensor there.

It should be noted that, for a given decay rate of matter fields, the definition of conformal fundamental forms given in [19] can only be applied directly to derive the form of a certain number of those objects in terms of the intrinsic geometry of the conformal boundary alone—a contribution from \(\tilde{T}_{ab}\) would appear in them, when naively using the definition beyond this range. In principle, the theory could be extended to generate such objects to arbitrarily high order, and for any decay rate of matter fields. Here we avoid these difficulties and our conformal fundamental form will precisely match the ones from [19], modulo the sign of the norm of the normal vector to the conformal boundary and the asymptotic value of the scalar curvature (determined by the cosmological constant). This aligns well with the conformal treatment of asymptotically de Sitter spacetimes widely used in the literature and reveals conformal fundamental forms simply linked to the well-known Weyl, Cotton and Bach tensors.

The main result of this article has the form of the following theorem.
**Theorem 1.** Let \((\tilde{M}, \tilde{g}_{ab})\) be a four-dimensional asymptotically de Sitter spacetime with the stress-energy tensor \(\tilde{T}_{ab}\) and the positive cosmological constant \(\Lambda\) admitting a conformal extension \((M, g_{ab})\) with
\[
\tilde{g}_{ab} = \Omega^{-2}g_{ab}, \quad \tilde{T}_{ab} = \Omega^2 T_{ab}, \quad q \in \{0, 1, 2\},
\]
where \(T_{ab}\) is the (regular everywhere) unphysical stress-energy tensor. Then,
\[
(q - 2) n^b T_{ab} + n_a T = 0 \quad \text{on} \quad \Sigma,
\]
\[
\nabla_a [(q - 2) T_{ab} n^b + n_a T] - \nabla_b T_{ab} = 0 \quad \text{on} \quad \Sigma,
\]
where
\[
n_a n^a = -1 + \frac{2 H}{\sqrt{3} \Lambda} \Omega + O(\Omega^2),
\]

\(H\) is the mean curvature of \(\Sigma\), and \(T := g^{cd} T_{cd}\). Moreover, for \(q = 0, 2\) the \(q + 3\) conformal fundamental form \(K_{ab}^{(q+3)}\) of the conformal infinity \(\Sigma\) is related to the intrinsic trace-free part of \(T_{ab}\), i.e.
\[
K_{ab}^{(q+3)} = C(q, \Lambda) \, (T_{ab})^\perp \quad \text{on} \quad \Sigma,
\]
where \((\,)^\perp \rightarrow p\) denotes the (trace-free) projection on \(\Sigma\). If \(q \geq 1\), then the divergence of the fourth conformal fundamental form \(\tilde{K}_{ab}^{(4)}\) of \(\Sigma\) satisfies the following constraint,
\[
\nabla_a \tilde{K}_{ab}^{(4)} = \begin{cases} 
\left(\frac{4}{3}\right)^{3/2} \left(\frac{4}{3} \nabla_a T - \sqrt{\frac{4}{3} j_a}\right) & \text{for} \quad q = 1, \\
-\frac{2}{3} \Omega p^n (n^b T_{ab}) & \text{for} \quad q = 2
\end{cases}
\]
on \(\Sigma\), where \(\nabla_a\) is the induced Levi-Civita connection on \(\Sigma\) and \(j_a\) is defined by the expansion \(T_{ab} n^b = \Omega j_a + O(\Omega)\). The \(\tilde{K}_{ab}^{(4)}\) is otherwise undetermined by the local data on the conformal boundary \(\Sigma\) but a constraint
\[
T_{ab} = -n_a n_b T \quad \text{on} \quad \Sigma \quad \text{for} \quad q = 1
\]
arises.

It can be shown that the condition for the regularity of the conformal field equations from [20] is an example of applying (1.2) in the \(q = 0\) case. The constraint (1.7) has been previously derived with the use of different methods (see e.g. [2] and [3]). The fourth fundamental form of the conformal boundary cannot be determined locally by the stress-energy tensor, because (as we shall see) it is an image of the Dirichlet-to-Neumann map for the conformal Einstein field equations (viewed as boundary value problem)—in the spirit of [21].

**Remark 1.** The principal parts of the higher-order conformal fundamental forms discussed in this paper are given by the electric part of the ambient Weyl tensor \(C_{\alpha\beta\mu\nu} n^\alpha n^\nu\) and the projected part of the Cotton \(A_{\alpha\beta\mu} n^\alpha\) and Bach \(B_{\alpha\beta}\) tensors.

Strictly speaking, a spacetime \((\tilde{M}, \tilde{g}_{ab})\), as in theorem 1, should not be called asymptotically de Sitter in the case when \(q = 0\). Nevertheless, we include this case here as it naturally fits in the current application of tractor calculus to general relativity.

The structure of this article is as follows. In section 2 we discuss the definitions of asymptotically de Sitter spacetimes and their conformal extensions together with some basic concepts from the tractor calculus. The following section is then dedicated to the application of tractor calculus in the setting of asymptotically de Sitter spacetimes. Section 4 contains the derivation of constraints relating the matter fields to the conformal fundamental forms of the conformal...
boundary of asymptotically de Sitter spacetimes. The final section contains a discussion about a possible application of the results of this work.

1.1. Notation and conventions

We will work with an \( n \)-dimensional \( (n \geq 3) \) conformal manifold equipped with the equivalence class of metric tensors \( c \) of signature \( (1, n-1, 0) \), and then restrict our attention to \( n = 4 \) in section 4. The abstract index notation will be used throughout the paper with the lower case Latin letters \( a, b, c \ldots \) associated with tensors, and upper case ones \( A, B, C \ldots \) with tractors.

Any tensor \( T_{ab} \) can be decomposed into its symmetric and antisymmetric parts in accordance with the formula

\[
T_{ab} = T_{(ab)} + T_{[ab]},
\]

where

\[
T_{(ab)} := \frac{1}{2} (T_{ab} + T_{ba}), \quad T_{[ab]} := \frac{1}{2} (T_{ab} - T_{ba}),
\]

The trace-free part of \( T_{ab} \) will be denoted by \( \tilde{T}_{ab} \), i.e.

\[
\tilde{T}_{ab} := T_{ab} - \frac{1}{n} g_{ab} T,
\]

for some \( g_{ab} \in c \), whereas the trace-free symmetrized part of \( T_{ab} \) by \( T_{(ab)} \), i.e.

\[
T_{(ab)} := \frac{1}{n} g_{ab} T.
\]

The convention that we use for the Riemann tensor associated with a metric \( g_{ab} \) is as follows,

\[
[R_{abcd} = C_{abcd} + 2 \left( g_{[a}P_{b]d} + g_{d[b}P_{a]} \right),
\]

where \( C_{abcd} \) is the (fully trace-free) Weyl tensor and

\[
P_{ab} := \frac{1}{n-2} \left( R_{ab} - \frac{R}{n-1} g_{ab} \right)
\]

is the Schouten tensor. We will use \( J \) to denote its trace, i.e. \( J := g^{ab} P_{ab} \). Lastly, we have

\[
A_{abc} := 2 \nabla_{[a}P_{b]c}, \quad B_{ab} := -\nabla^c A_{abc} + P^{cd} C_{dabc},
\]

where \( A_{abc} \) and \( B_{ab} \) are Cotton and Bach tensors, respectively. It should be noted that the Bianchi identities imply

\[
(n - 3) A_{abc} = \nabla^d C_{abcd}.
\]

When working with a hypersurface (i.e. a codimension one embedded submanifold) \( \Sigma \) with normal vector field \( n^a \), we shall identify \( T^*\Sigma \) with the subbundle \( n^\perp \) of \( TM|\Sigma \) consisting of tangent vectors orthogonal to \( n^a \). Similarly, the hypersurface cotangent bundle \( T^*\Sigma \) will be identified with the annihilator of \( n^a \) in \( T^*M|\Sigma \). In this way we use the same abstract indices for \( T\Sigma \) and \( T^*\Sigma \) as we use for, respectively, \( TM \) and \( T^*M \). With this understood, the quantities intrinsic to \( \Sigma \) will be denoted by a bar. For example, given a metric \( g_{ab} \) on \( M \), \( g_{ab} \) denotes the metric induced on \( \Sigma \) (i.e. the first fundamental form) and \( \nabla_a \) is its Levi-Civita connection. Note
also that with these conventions $g_a^c$ (index raised using $g^bc$) gives, by a single contraction, the orthogonal projection from $\mathcal{TM}$ to $\mathcal{T}_\Sigma$. We will use superscript $\Sigma$ when working with the orthogonal projections of the ambient tensors to the hypersurface tensor bundles. e.g. $\mathcal{T}^\Sigma_{ab} := g_a^c g_b^d T_{cd}$. The abstract index $n$ will denote contraction with the normal vector $n^a$, e.g.

\begin{equation}
\mathcal{T}^\Sigma_{anbn} = g_a^c g_b^d T_{cfdh} n^f n^h,
\end{equation}

and $\tilde{}$ will be used when considering the trace-free part of the projection.

The second fundamental form $K_{ab}$ of $\Sigma$ is defined in terms of its normal vector $n^a$ and can be decomposed as follows,

\begin{equation}
K_{ab} := (\nabla_a n_b) = K_{ab} + \frac{H}{n-1} g_{ab}, \quad H := \pi^d K_{cd},
\end{equation}

where $\hat{K}_{ab}$ denotes the traceless part of $K_{ab}$ and $H$ will be called the mean curvature of $\Sigma$.

Lastly, we will use $\Sigma \equiv -1$ when an ambient quantity is evaluated on $\Sigma$, e.g.

\begin{equation}
n_a n^a \Sigma \equiv -1
\end{equation}

indicates that $n^a$ has a negative unit norm on this hypersurface.

2. Preliminaries

Let $(\tilde{M}, \tilde{g}_{ab})$ be a spacetime that satisfies the Einstein field equations with positive cosmological constant $\Lambda$,

\begin{equation}
\tilde{R}_{ab} - \frac{1}{2} \tilde{g}_{ab} \Lambda = \tilde{T}_{ab},
\end{equation}

where $\tilde{T}_{ab}$ is the stress-energy tensor. Central to this paper is the notion of the asymptotically de Sitter spacetime. We will work with the following definition.

**Definition 1.** A spacetime $(\tilde{M}, \tilde{g}_{ab})$ is asymptotically de Sitter if there exists a manifold $M$, with boundary $\Sigma$ and a metric $g_{ab}$, such that

- there is an embedding $\varphi : \tilde{M} \rightarrow M$ such that $\varphi(\tilde{M}) = M \setminus \Sigma$,
- the metric $g_{ab} \in \mathcal{C}$ is regular on $M$ and satisfies $\tilde{g}_{ab} = \Omega^{-2} g_{ab}$ (on $\tilde{M}$) for some smooth non-negative function $\Omega$ on $M$,
- $\Omega$ is a defining function of the boundary $\Sigma$, i.e. $\Sigma = \Omega^{-1}(0)$ and $d\Omega$ is nowhere zero on $\Sigma$,
- the stress-energy tensor $\tilde{T}_{ab}$ vanishes along $\Sigma$.

The boundary $\Sigma$ is often called the conformal infinity of $(\tilde{M}, \tilde{g}_{ab})$.

2.1. Tractor calculus

Here by a conformal manifold $(M, \mathcal{C})$ we mean a smooth manifold of dimension $n \geq 3$ equipped with an equivalence class $\mathcal{C}$ of metrics, where $g_{ab}, \tilde{g}_{ab} \in \mathcal{C}$ means that $\tilde{g}_{ab} = \Theta^2 g_{ab}$ for some smooth positive function $\Theta$. On a general conformal manifold $(M, \mathcal{C})$, there is no distinguished connection on $\mathcal{TM}$. But there is an invariant and canonical connection on a closely related bundle, namely the conformal tractor connection on the standard tractor bundle, see [22, 23].

Here we review the basic conformal tractor calculus on pseudo-Riemannian and conformal manifolds. See [13, 24] for more details. Unless stated otherwise, every calculation will be
done with the use of generic $g_{ab} \in \mathfrak{c}$. Hence, we will omit the superscript $g$ in the objects determined by this metric, e.g. $\nabla_a$ will be used instead of $\nabla^g_a$.

On any manifold $M$ of dimension $n$ the line bundle $\mathcal{K} := (\Lambda^n TM)^{\otimes 2}$ of volume densities is canonically oriented and thus one may take oriented roots of it: Given $w \in \mathbb{R}$ we set

$$\mathcal{E}[w] := \mathcal{K}^{\frac{1}{2w}},$$

and refer to this as the bundle of conformal densities. For any vector bundle $\mathcal{V}$, we write $\mathcal{V}[w]$ to mean $\mathcal{V} \otimes \mathcal{E}[w]$. For example, $\mathcal{E}_{(ab)}[w]$ denotes the symmetric second tensor power of the cotangent bundle tensored with $\mathcal{E}[w]$, i.e. $S^2T^*M \otimes \mathcal{E}[w]$ on $M$. When discussing bundles on $\Sigma$ a $\mathcal{E}$ symbol will be used, e.g. $\mathcal{E}^a$ is the tangent bundle of this hypersurface.

On a conformal structure there is a canonical section $g_{ab} \in \Gamma(\mathcal{E}_{(ab)}[2])$. This has the property that for each positive section $\sigma \in \Gamma(\mathcal{E}^+[1])$ (called a scale) $g_{ab} := \sigma^{-2}g_{ab}$ is a metric in $\mathfrak{c}$. Moreover, the Levi-Civita connection of $g_{ab}$ preserves $\sigma$ and therefore $g_{ab}$. Thus it makes sense to use the conformal metric to raise and lower indices, even when we have nominated a metric $g_{ab} \in \mathfrak{c}$ to split the tractor bundles and determine a Levi-Civita connection. It turns out that this simplifies many computations, and so (following the mentioned references) in this section we will do that without further mention.

Considering Taylor series for sections of $\mathcal{E}[1]$ one recovers the jet exact sequence at two-jets,

$$0 \to \mathcal{E}_{(ab)}[1] \to J^2\mathcal{E}[1] \to J^1\mathcal{E}[1] \to 0.$$  

(2.3)

Then given the conformal structure we have the orthogonal decomposition in trace-free and trace parts

$$\mathcal{E}_{ab}[1] = \mathcal{E}_{(ab)}[1] \oplus g_{ab} \cdot \mathcal{E}[-1].$$

(2.4)

Thus we can canonically quotient $J^2\mathcal{E}[1]$ by the image of $\mathcal{E}_{(ab)}[1]$ under $\iota$ (in (2.3)) to form the bundle $\mathcal{T}^*$, called the conformal contractor bundle.

Given a choice of metric $g_{ab} \in \mathfrak{c}$, the formula

$$\sigma \mapsto \frac{1}{n}[D\sigma]_g := \left(\begin{array}{cc} \sigma & \nabla a \sigma \\ -\frac{1}{n}(\Delta + J)\sigma \end{array}\right)$$

(2.5)

(where $\Delta$ is the Laplacian $\nabla^a\nabla_a$) gives a second-order differential operator on $\mathcal{E}[1]$ which is a linear map $J^2\mathcal{E}[1] \to \mathcal{E}[1] \oplus \mathcal{E}^a[1] \oplus \mathcal{E}[-1]$ that clearly factors through $\mathcal{T}^*$ and so determines an isomorphism

$$\mathcal{T}^* \xrightarrow{\sim} [\mathcal{T}^*]_g = \mathcal{E}[1] \oplus \mathcal{E}^a[1] \oplus \mathcal{E}[-1].$$

(2.6)

The tractor defined in (2.5) will be called the scale tractor corresponding to the scale $\sigma$ and denoted by $I_\sigma$, i.e.

$$I_\sigma := \frac{1}{n}D\sigma.$$  

(2.7)

In subsequent discussions, we will use (2.6) to split the tractor bundles without further comment. Thus, given $g_{ab} \in \mathfrak{c}$, an element $V^A$ of $\mathcal{E}^A$ may be represented by a triple $(\sigma, \mu_a, \rho)$, or equivalently by

$$V_A = \sigma Y_A + \mu_a Z^a_A + \rho X_A.$$  

(2.8)

The last display defines the algebraic splitting operators $Y: \mathcal{E}[1] \to \mathcal{T}^*$ and $Z: \mathcal{T}^* M[1] \to \mathcal{T}^*$ (determined by the choice $g_{ab} \in \mathfrak{c}$) which may be viewed as sections $Y_A \in \Gamma(\mathcal{E}[-1])$ and $Z_A^a \in \Gamma(\mathcal{E}_a[-1])$. We call these sections $X_A$, $Y_A$ and $Z_A^a$ tractor projectors.
It is straightforward to verify that the equation
\[ \nabla_{(a} \nabla_{b)} \sigma + P_{(ab)c} \sigma = 0 \]  
(2.9)
on conformal densities \( \sigma \in \Gamma(\mathcal{E}[1]) \) is conformally invariant. As it is overdetermined, solutions may not exist and indeed it is well known that positive ones are equivalent to vacuum-Einstein metrics in the conformal class: \( \sigma \in \Gamma(\mathcal{E}_+[1]) \) solves (2.9) is equivalent to \( P^i_{(ab)i} = 0 \), where \( \hat{g}_{ab} = \sigma^{-2} g_{ab} \). More generally, non-trivial solutions are non-vanishing on an open dense set, on which they determine a vacuum-Einstein metric \([13, 25]\). Thus (2.9) is sometimes called the almost Einstein equation.

Given a metric \( g_{ab} \in \mathfrak{e} \), the tractor connection is given by the formula
\[ \nabla^T_a \left( \begin{array}{c} \sigma \\ \mu_b \\ \rho \\ \end{array} \right) := \left( \begin{array}{c} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + P_{ab} \sigma + g_{ab} \rho \\ \nabla_a \rho - P_{ac} \rho \\ \end{array} \right) \]  
(2.10)
and the equation of parallel transport is equivalent to a prolongation of the almost Einstein equation (2.9), see \([13, 22]\) (and section 3 below). Thus, in particular, solutions \( V_A \) of \( \nabla^T_a V_A = 0 \) are equivalent to solutions \( \sigma \) of (2.9) with the explicit relations:
\[ \sigma = X^A V_A \quad \text{and} \quad V = I^A. \]

The tractor bundle is also equipped with a tractor metric \( h_{AB} \in \Gamma(\mathcal{E}_{(AB)}) \), defined using the mapping
\[ [V_A]_g = \left( \begin{array}{c} \sigma \\ \mu_a \\ \rho \\ \end{array} \right) \rightarrow \mu_a \rho + 2 \sigma \rho =: h(V, V) \]  
(2.11)
combined with the polarization identity. It can be checked that the tractor metric is preserved by \( \nabla^T_a \), i.e. \( \nabla^T_a h_{AB} = 0 \).

The curvature of the tractor connection \( \kappa_{abcd} \) can be recovered with the use of the following relation,
\[ 2 \nabla^T_a \nabla^T_b V^C = \kappa_{abcd} V^d \quad \text{for all} \quad V^C \in \Gamma(\mathcal{E}^A), \]  
(2.12)
and can be written in terms of tractor projectors as
\[ \kappa_{abcd} = A_{cab} X^C X_D - A_{cab} X_C Z^D + C_{abcd} Z^C Z^D. \]  
(2.13)

2.2. The scale singularity set and the normal tractor

Since metrics \( g_{ab} \in \mathfrak{e} \) correspond to section \( \sigma \in \Gamma(\mathcal{E}[1]) \) via \( g_{ab} = \sigma^{-2} g_{ab} \), points where \( \sigma \) vanishes are conformal singularities of \( g_{ab} \). It is elementary to verify that if the 'length squared' of the scale tractor, meaning \( \hat{F}_2 := h(I_a, I_a) \), is nowhere zero then the zero locus \( \Sigma = \mathcal{Z}(\sigma) \) of \( \sigma \) is a smoothly embedded separating hypersurface.

Alternatively, if we are considering a manifold with boundary, it can be the case that \( \Sigma = \mathcal{Z}(\sigma) \) is the boundary. Along such \( \mathcal{Z}(\sigma) \) there is a conformally invariant tractor analogue of the normal vector called the normal tractor \([22]\)—a section \( N_A \) of \( T|_\Sigma \) that is given in a metric \( g_{ab} \in \mathfrak{e} \) by the formula
\[ N_A = \left( \begin{array}{c} 0 \\ n_a \\ -\frac{n}{n-1} \\ \end{array} \right), \]  
(2.14)
where \( n_a \in \Gamma(\mathcal{E}_w[1]) \) and \( H \in \Gamma(\mathcal{E}[-1]) \) are the densities corresponding to the normal vector and the mean curvature of \( Z(\sigma) \), which will be defined in section 3 for asymptotically de Sitter spacetimes.

If \( I_\sigma^2 = \pm 1 \) along \( \Sigma \), then a rather nice feature is captured by the following proposition [13, 25]:

**Proposition 1.** Let \((M, c)\) be a pseudo-Riemannian conformal structure and suppose that a scale tractor \( I_\sigma \) has a scale singularity set \( \Sigma = Z(\sigma) \neq \emptyset \) and \( I_\sigma^2 = \pm 1 + \sigma^2 f \) for some smooth (weight \(-2\)) density \( f \). Then we have \( N = I_\sigma |_\Sigma \), where \( N_\sigma \) is the normal tractor.

In particular, this holds if the norm of the scale tractor is equal to a constant on the boundary \( \Sigma \) of asymptotically de Sitter spacetimes. This will be explored in section 3.

Along a hypersurface (or boundary) \( \Sigma \), the normal tractor can be used to introduce the tractor projection operator,

\[
\Pi_A^B = \delta_A^B \mp N_A N^B,
\]

with signs reflecting the cases \( N_A N^A = \pm 1 \). The image of \( \Pi_A^B \) is isomorphic to the hypersurface tractor bundle \( \mathcal{T} \), where the isomorphism is given by [13, 25]

\[
\begin{align*}
\mathcal{F}^A &= X^A, \\
\mathcal{Y}^A &= Y^A + Z^{ka} \frac{H}{n-1} n_a \pm \frac{H^2}{2 (n-1)^2} X^A, \\
Z^{ka} &= g^{ab} Z_{ab}. 
\end{align*}
\]

### 2.3. Elements of tractor calculus

We will use the symbol \( \mathcal{E}^\Phi[w] \) to denote any tractor bundle of weight \( w \). The operator \( D_A \) in (2.5) generalizes to a conformally invariant differential operator on sections of \( \mathcal{E}^\Phi[w] \) [22, 24]

\[
D_A : \mathcal{E}^\Phi[w] \to \mathcal{E}^\Phi[w-1],
\]

and is given by

\[
V \mapsto [D_A V]_g := \begin{pmatrix}
(n + 2w - 2) w V \\
(n + 2w - 2) \nabla_a V \\
- (\Delta V + wJV)
\end{pmatrix}
\]

in a metric \( g_{ab} \in c \). On the right-hand side of the display the \( \nabla_a \) is the Levi-Civita connection coupled with the tractor connection (including in the Laplacian). This is usually called the tractor-D, or Thomas-D, operator. It is often useful to use its rescaled version, denoted here by \( \tilde{D}_A \), i.e.

\[
V \mapsto [\tilde{D}_A V]_g := \begin{pmatrix}
w V \\
\nabla_a V \\
- \frac{1}{(n+2w-2)} (\Delta V + wJV)
\end{pmatrix},
\]

which is defined for \( w \neq 1 - \frac{n}{2} \). The Thomas-D operator may be combined with the scale tractor to produce a canonical degenerate Laplacian type differential operator [26], namely

\[
ID := I_\sigma^a D_A.
\]

This acts on any weighted tractor bundle, preserving its tensor type but lowering the weight,

\[
ID : \mathcal{E}^\Phi[w] \to \mathcal{E}^\Phi[w-1].
\]
It can be expanded in terms of a metric \( g_{ab} \in \mathfrak{e} \) to yield

\[
\text{ID} \overset{\varepsilon}{=} -\sigma \Delta + (n + 2w - 2) \left( \nabla^a \sigma \nabla_a - \frac{w}{n} (\Delta \sigma) \right) - \frac{2w}{n} (n + w - 1) \sigma f
\]

(2.22)
on \( \mathcal{E}^\Phi[w] \). Now if we calculate in the metric \( g_{ab} = \sigma^{-2} g_{ab} \), away from the zero locus of \( \sigma \), and trivialize the densities accordingly, then \( \sigma \) is represented by 1 in the trivialization, and we have

\[
\text{ID} \overset{\varepsilon}{=} - \left( \Delta + \frac{2w(n + w - 1)}{n} \right) f.
\]

(2.23)

On the other hand, looking again to (2.22), we see that ID degenerates along the conformal infinity \( \Sigma = \mathcal{Z}(\sigma) \) (assumed non-empty), and there the operator is first order. In particular, if the structure is asymptotically almost scalar constant in the sense that \( I^2 = \pm 1 + \sigma^2 f \) for some smooth (weight \(-2\)) density \( f \), then along \( \Sigma \)

\[
\text{ID} \overset{\varepsilon}{=} (n + 2w - 2) \delta_R,
\]

(2.24)

where \( \delta_R \) is the conformal Robin operator,

\[
\delta_R \overset{\varepsilon}{=} \nabla_n - \frac{H}{n - 1},
\]

(2.25)
of [27, 28] (twisted with the tractor connection).

Given a \( t_{ab} \in \Gamma (\mathcal{E}_{(ab)}[w]) \) with \( w \neq 2 - n \), \( 3 - n \) we can define a map \( p \) which inserts \( t_{ab} \) into a symmetric trace-free tractor \( T_{AB} \).

\[
p : \Gamma (\mathcal{E}_{(ab)}[w]) \rightarrow \Gamma (\mathcal{E}_{(AB)}[w - 2]), \quad w \neq 2 - n, 3 - n,
\]

(2.26)

which is given by

\[
p (t_{ab}) := Z^a b^b c^c t_{ab} - \frac{2}{n + w - 2} X(A Z_B \cdot c) + \frac{\nabla \cdot \nabla \cdot t + (n + w - 2) P \cdot t}{(n + w - 2) (n + w - 3)} X_A X_B.
\]

(2.27)

We have

\[
D^A T_{AB} = 0 = X^A T_{AB}.
\]

(2.28)

It can be seen that a general property of a symmetric tractor \( T_{AB...} \in \Gamma (\mathcal{E}_{(AB...)}[w]) \) with \( X^A T_{AB...} = 0 \) is that the tensor \( Z^a b^b ... T_{AB...} \) is conformally invariant. Therefore, the notion of extraction operator \( p^* \) can be defined for such tractors, i.e. let

\[
p^* : \Gamma (\mathcal{E}_{(AB...)}[w]) \rightarrow \Gamma (\mathcal{E}_{(ab...)}[w + v]),
\]

(2.29)

where \( \mathcal{E}^X \) denotes the tractor bundle whose sections vanish when contracted with \( X^A \) and \( v \) is the valence of \( T_{AB...} \). In particular,

\[
p^* (T_{AB...}) := Z^a b^b ... T_{AB...}, \quad T_{AB...} \in \Gamma (\mathcal{E}_{(AB...)}[w]),
\]

(2.30)

The crucial property of the operators \( p \) and \( p^* \) is that

\[
p^* \circ p = \text{Id},
\]

(2.31)
i.e. the insertion operator is the right-inverse of the extraction operator. (Such operators are often called differential splitting operators.)

In order to obtain a section of the \( \mathcal{E}^X \) bundle, from section of a generic tractor bundle, a tractor projection differential operator \( r \) can be used.
We will restrict ourselves to traceless tractors of valence 2, i.e.
\[
    r(T^{AB}) := T^{AB} - \frac{2}{w} \tilde{D}^{[A}(X_c T^{C][B])_b) + \frac{1}{w(w+1)} \tilde{D}^{[A}(\tilde{D})^{B])_b) (X_c X_d T^{CD})
\]
\[
\quad - \frac{8}{w(w+1)} X^{[A} \tilde{D}^{B)]_b (\tilde{D}_c (X_d T^{CD})) ,
\]
for \( w \neq 0, -1, -\frac{n}{2}, -1 - \frac{n}{2}, -2 - \frac{n}{2} \) and \( T_{AB} \in \Gamma(\mathcal{E}_{(AB)}_b[w]) \).

**Remark 2.** If \( n \geq 4 \) then the injection, extraction and projection operators have their hypersurface analogues \( \overline{\mathcal{P}}, \mathcal{P}^*, \) and \( \mathcal{P} \) which can be defined with the use of hypersurface tractor connection, the corresponding hypersurface tractor operators, and hypersurface tractor projectors (2.16). The notion of hypersurface operators can still be introduced in three dimensions with an additional Möbius structures over the conformal boundary.

### 3. The almost-Einstein-matter equation and its consequences

Let \((\tilde{M}, \tilde{g}_{ab})\) be an asymptotically de Sitter spacetime. After taking a divergence of the Einstein field equation (2.1) one arrives at the matter continuity equation,
\[
    \nabla^a \tilde{T}_{ab} = 0. \tag{3.1}
\]

We can utilize a decomposition of the Ricci tensor \( \tilde{R}_{ab} \) into the Schouten tensor \( \tilde{P}_{ab} \) and its trace \( \tilde{J} \),
\[
    \tilde{R}_{ab} = (n-2) \tilde{P}_{ab} + \tilde{J} \tilde{g}_{ab}, \tag{3.2}
\]
to obtain a trace-free part of the Einstein field equation (2.1),
\[
    \tilde{P}_{ab} = \frac{1}{n} \tilde{J} \tilde{g}_{ab} = \frac{1}{n-2} \left( \tilde{T}_{ab} - \frac{1}{n} \tilde{J} \tilde{g}_{ab} \right). \tag{3.3}
\]

Suppose that
\[
    \tilde{g}_{ab} = f^2 \tilde{g}_{ab}, \tag{3.4}
\]
where \( f \) is a positive smooth function, i.e. \( g_{ab} \) is in the conformal class of \( \tilde{g}_{ab} \). Then the transformation rule for the Schouten tensor reads
\[
    \tilde{P}_{ab} = P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \frac{1}{2} \tilde{g}_{ab} \Upsilon^2, \tag{3.5}
\]
where \( \Upsilon_a = \nabla_a \log f \). We will now use this to relate (3.3) with the almost Einstein equation (2.9). Let \( \sigma \) and \( \tilde{\sigma} \) be scales corresponding to \( g_{ab} \) and \( \tilde{g}_{ab} \) respectively, i.e.
\[
    \sigma^2 \tilde{g}_{ab} = g_{ab} = \sigma^2 \tilde{g}_{ab}, \tag{3.6}
\]
where \( g_{ab} \) is the conformal metric. When working in the scale \( \sigma \), the density \( \tilde{\sigma} \) is determined by \( f^{-1} \) and
\[
    \nabla_a \nabla_b \tilde{\sigma} = \tilde{\sigma} (\nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b). \tag{3.7}
\]
The Einstein field equation (3.3) can now be written as
\[
    \nabla_a \nabla_b \tilde{\sigma} + \tilde{\sigma} P_{ab} - \frac{1}{n} \tilde{g}_{ab} (\Delta \tilde{\sigma} + \tilde{\sigma} J) = \frac{\tilde{\sigma}}{n-2} \tilde{T}_{ab}, \tag{3.8}
\]
where
\[
    \tilde{T}_{ab} := \tilde{T}_{ab} - \frac{1}{n} \tilde{K}_{ab} \tilde{T} \tag{3.9}
\]
is the traceless part of the physical stress-energy tensor.

**Definition 2.** The scale $\tilde{\sigma}$ will be called the almost-Einstein-matter scale.

Let $\tau_{ab} \in \mathcal{E}_{ab}[-q]$ be the conformally weighted tensor, corresponding to $\tilde{T}_{ab}$, defined by

$$
\tau_{ab} := \sigma^{-q} \tilde{T}_{ab},
$$

where we assume that $q \in \{0, 1, 2\}$. In order to attribute a physical meaning to the parameter $q$ consider a scale $\eta$ corresponding to the regular metric $g_{ab}$ which defines the conformal extension of the spacetime $(\mathcal{M}, g_{ab})$ (cf definition 1). When working in the scale $\eta$, the almost-Einstein-matter scale $\tilde{\sigma}$ is characterized by the defining function of the conformal boundary $\Omega$, i.e. $\tilde{\sigma} = \Omega \eta$, so

$$
\tilde{T}_{ab} = \Omega^q T_{ab},
$$

where $T_{ab}$ is the unphysical stress-energy tensor ($T_{ab}$ and $\Omega$ are, respectively, $\tau_{ab}$ and $\tilde{\sigma}$ in the scale $\eta$). Hence, if $T_{ab}$ is regular everywhere, then $q$ characterizes the decay of the stress-energy tensor $\tilde{T}_{ab}$ when one approaches conformal infinity of the asymptotically de Sitter spacetime. Based on this observation the case $q = 0$ should be excluded from the analysis, as such matter fields do not vanish on the conformal boundary. Nevertheless, it fits naturally in the tractor calculus approach presented here, so we include it in the computations.

The traceless part of $\tilde{T}_{ab}$ can be related to the traceless part of $\tau_{ab}$ in the following way,

$$
\tilde{T}_{ab} = \tilde{\sigma}^q \left( \tau_{ab} - \frac{1}{n} g_{ab} \tau \right) = \tilde{\sigma}^q \tau_{ab},
$$

where $\tau = g^{cd} \tau_{cd} \in \mathcal{E}[-q - 2]$ is a density corresponding to the trace $\tilde{T}$. Equation (3.8) now reads

$$
\nabla_a \nabla_b \tilde{\sigma} + \tilde{\sigma} P_{ab} - \frac{1}{n} g_{ab} (\Delta \tilde{\sigma} + \tilde{\sigma} J) = \frac{\tilde{\sigma}^{q+1}}{n-2} \tau_{ab},
$$

and will be called the almost-Einstein-matter equation.

### 3.1. Prolongation of the almost-Einstein-matter equation

Let

$$
\rho := -\frac{1}{n} (\Delta \tilde{\sigma} + \tilde{\sigma} J).
$$

By taking two different contractions of the covariant derivative of (3.13) we obtain

$$
\Delta \nabla_a \tilde{\sigma} + \tilde{\sigma} \nabla_a P_{a}^c + P_{a}^b \nabla_b \tilde{\sigma} + \nabla_a \rho = \frac{q+1}{n-2} \tilde{\sigma}^{q+1} \nabla_b \tilde{\sigma} + \frac{\tilde{\sigma}^{q+1}}{n-2} \nabla_b \tilde{\sigma}^b, 
$$

$$
\nabla_a \Delta \tilde{\sigma} + \tilde{\sigma} \nabla_a J + J \nabla_a \tilde{\sigma} + n \nabla_a \rho = 0.
$$

After taking a difference, using the contracted Bianchi identity,

$$
\nabla_b P_{a}^b = \nabla_a J,
$$

and expressing the commutator $[\nabla_c, \Delta]$ in terms of the Ricci tensor one gets

$$
\nabla_a \rho - P_{a}^b \nabla_b \tilde{\sigma} = -\frac{\tilde{\sigma}^{q+1}}{(n-1)(n-2)} ((q+1) \tilde{\sigma}^{b} \nabla_b \tilde{\sigma} + \tilde{\sigma} \nabla_b \tilde{\sigma}^b).
$$
Hence, the second-order almost-Einstein-matter equation (3.13) is equivalent to the first-order system of three equations,
\[
\begin{align*}
\nabla_a \tilde{\sigma} - \mu_a &= 0, \\
\nabla_a \mu_b + \tilde{\sigma} \mathcal{P}_{ab} + \rho g_{ab} &= \frac{\tilde{\sigma}^{q+1}}{n-2} \tilde{\tau}_{ab}, \\
\nabla_a \rho - P_a^b \mu_b &= -\frac{\tilde{\sigma}^q}{(n-1)(n-2)} ((q+1) \tilde{\tau}_{ab}^b \mu_b + \tilde{\sigma} \nabla_b \tilde{\tau}_{ab}^b),
\end{align*}
\] (3.18)
in three variables \(\tilde{\sigma}, \mu_a\) and \(\rho\) (compare with the definition (2.10) of the tractor connection).

Due to the presence of matter fields, the r.h.s. of the above is non-zero, which has direct consequences for the derivative of the almost-Einstein-matter scale tractor \(I_\tilde{\sigma}\). Unlike in the standard picture presented in section 2, \(I_\tilde{\sigma}\) will no longer be parallel with respect to the tractor connection. This fact is explored in more detail below.

3.2. The almost-Einstein-matter scale tractor

Let
\[
I_\tilde{\sigma} := \begin{pmatrix}
\tilde{\sigma} \\
\nabla_b \tilde{\sigma} \\
-\frac{1}{n} (\Delta \tilde{\sigma} + J \tilde{\sigma})
\end{pmatrix}
\] (3.19)
This is the \(\tilde{\sigma}\)-scale tractor. In the scale \(\tilde{\sigma}\)
\[
I_\tilde{\sigma}^2 \equiv -\frac{\tilde{R}}{n(n-1)} = -\lambda + \frac{2 \tilde{\sigma}^{q+2} \tau}{n(n-1)(n-2)},
\] (3.20)
where
\[
\lambda := \frac{2 \Lambda}{(n-1)(n-2)}. \tag{3.21}
\]
If the matter fields are present, then the scale tractor will no longer be parallel, i.e.
\[
\nabla_a I_\tilde{\sigma} = \begin{pmatrix}
\tilde{\sigma}^q \\
\frac{n(n-1)}{(n-2)} (\tilde{\sigma} (n-1) \tilde{\tau}_{ab}) \\
-\frac{1}{n} (\Delta \tilde{\sigma} + J \tilde{\sigma})
\end{pmatrix},
\] (3.22)
which is a consequence of the almost-Einstein-matter equation (3.13) and its prolongation (3.18). Calculating the conformal transformation
\[
\nabla_b \tilde{\tau}_{ab}^b = \nabla_b \tilde{\tau}_{ab}^b + (n-2-q) \mathcal{Y}_{b} \tilde{\tau}_{ab}^b,
\] (3.23)
also verifies that the r.h.s. of (3.22) transforms as a tractor. Moreover, a direct calculation yields
\[
\nabla_a^2 I_\tilde{\sigma} = 2 I_\tilde{\sigma} \cdot \nabla_a I_\tilde{\sigma} = -\frac{2 \tilde{\sigma}^{q+2}}{(n-1)(n-2)} \nabla_b \tilde{\tau}_{ab}^b,
\] (3.24)
where (3.22) and (3.23) have been used. The continuity equation (3.1) implies
\[
0 = \tilde{\sigma}^{q+2} \left( \nabla_b \tilde{\tau}_{ab}^b + \frac{1}{n} \nabla_{a} \tau \right),
\] (3.25)
so outside of a zero locus of \(\tilde{\sigma}\) equation (3.24) agrees with the derivative of (3.20).
3.3. Trace-free second fundamental form $K_{ab}$ of $\Sigma$

The conformal infinity $\Sigma$ of the asymptotically de Sitter spacetime is an embedded hyper-surface of its conformal extension $(M, g_{ab})$. As the almost-Einstein-matter scale is a defining density of $\Sigma$, there is a natural notion of a normal vector associated with it. It can be used to construct the first two conformal fundamental forms of $\Sigma$.

Let

$$n_a := \frac{1}{\sqrt{\lambda}} \nabla \tilde{\sigma}.$$  \hspace{1cm} (3.26)

It can be seen that $n_a$ is a conformal density of weight 1, i.e. $n_a \in \Gamma(E_1)$. According to (3.20), the norm of this vector is as follows,

$$n_a n^a = -1 + \frac{2}{n \lambda} \tilde{\sigma} (\Delta + J) \tilde{\sigma} + O(\tilde{\sigma}^{q+2}).$$  \hspace{1cm} (3.27)

The first conformal fundamental form of $\Sigma$, its induced conformal metric $g_{ab}$, can now be defined as

$$g_{ab} := g_{ab} + n_a n_b.$$  \hspace{1cm} (3.28)

The extrinsic curvature of the conformal boundary (i.e. the second fundamental form) is given by

$$K_{ab} := g_{a[c} \nabla_b n_{d]} \Sigma,$$

i.e.

$$\nabla_a n_b \Sigma = K_{ab} - n_a \nabla_b \tilde{\sigma} \Sigma.$$  \hspace{1cm} (3.29)

However, since

$$\nabla_a n_b = \frac{1}{\sqrt{\lambda}} \nabla_b n_a \tilde{\sigma} = 0,$$

the $\nabla_a n_b$ can be expressed as

$$\nabla_a n_b = -\frac{1}{2} \nabla_a (n^b n_b) = -\frac{1}{\sqrt{\lambda}} n_a \rho + O(\tilde{\sigma}) = \frac{1}{n \sqrt{\lambda}} n_a \Delta \tilde{\sigma} + O(\tilde{\sigma}).$$  \hspace{1cm} (3.30)

We can use (3.32) to compute the mean curvature $H$ of $\Sigma$,

$$H := \tilde{\nabla}^{ab} K_{ab} = \tilde{\nabla}^{ab} \nabla_a n_b$$

$$= \left( \frac{1}{\sqrt{\lambda}} \Delta \tilde{\sigma} + \frac{1}{2} \nabla_a (n_a n^a) \right) \left| \Sigma \right. = \frac{n-1}{n \sqrt{\lambda}} \Delta \tilde{\sigma} \left| \Sigma \right. .$$  \hspace{1cm} (3.33)

As a result,

$$\nabla_a n_b \Sigma = K_{ab} - \frac{H}{n-1} n_a n_b = K_{ab} + \frac{H}{n-1} g_{ab}$$  \hspace{1cm} (3.34)

where the decomposition of $K_{ab}$ into its traceless part and the mean curvature $H$ has been used. Thus, we obtain

$$I_a \equiv \left( \begin{array}{c} 0 \\ \sqrt{\lambda} n_a \\ -\frac{\sqrt{\lambda}}{n-1} H \end{array} \right).$$  \hspace{1cm} (3.35)
which is the asymptotically de Sitter analogue of the normal tractor (2.14). The derivative of \( I_\tilde{a} \) evaluated on \( \Sigma \) reads
\[
\nabla_\alpha I_\tilde{a} \equiv \begin{pmatrix} 0 \\ \sqrt{\lambda} \left( \nabla_\alpha n_b - g_{ab} \frac{H}{n+1} \right) \\ -\sqrt{\lambda} \left( \frac{1}{n-1} \nabla_\alpha H + \nabla_{\alpha} P_{mn} \right) - n_\alpha \left( \nabla_\alpha P_{mn} - \sqrt{\lambda} P_{mn} \right) \end{pmatrix},
\]
(3.36)
where
\[
\nabla_\alpha \rho \equiv \sqrt{\lambda} \left( P_{mn} + \frac{1}{2} \tilde{K}_{ab} \tilde{K}^{ab} + \frac{1}{48} (q+1)(q+2) \tilde{\sigma}^q \right) \bigg|_{\Sigma}.
\]
(3.37)
(see [29, lemma 3.8] for a derivation of this identity without the matter fields). The hypersurface Codazzi equation
\[
\nabla_\alpha K_{bc} - \nabla_b K_{ac} = R^\top_{abcn}
\]
(3.38)
can be used to simplify (3.36). If we use a decomposition (1.13) of the Riemann tensor, then (3.38) reads
\[
\frac{1}{n-1} \nabla_\alpha H = \frac{1}{n-2} \nabla_\alpha \tilde{K}_a^b - P^\top_{mn},
\]
(3.39)
Ultimately, it can be computed that the projected part of (3.36) has the following form
\[
\rho_{ab} \nabla_\alpha I_\tilde{a} \equiv \sqrt{\lambda} \begin{pmatrix} 0 \\ \tilde{K}_{ab} \\ -\frac{1}{n-2} \nabla_\alpha \tilde{K}_c^b \end{pmatrix}.
\]
(3.40)
It follows that \( \tilde{K}_{ab} = p^* (\rho_{ab} \nabla_\alpha I_\tilde{a}) \) is conformally invariant, as is of course well known.

The normal component of \( \nabla_\alpha I_\tilde{a} \) reads
\[
\nabla_\alpha I_\tilde{a} \equiv \begin{pmatrix} 0 \\ \sqrt{\lambda} \nabla_\alpha n_b + n_\alpha \rho + P_{ab} \tilde{\sigma} \\ \nabla_\alpha \rho - \sqrt{\lambda} P_{mn} \end{pmatrix}
\equiv \sqrt{\lambda} \times \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \tilde{K}_{ab} \tilde{K}^{ab} \end{pmatrix}.
\]
(3.41)

4. The conformal fundamental forms of the conformal boundary \( \Sigma \)

We will move now to the discussion of four-dimensional asymptotically de Sitter spacetimes and derive constraints relating the conformal fundamental forms of its conformal infinity with the matter fields. Before doing so, we will discuss the more fundamental constraints which appear when the derivative of the almost Einstein-matter scale tractor is considered.
4.1. Constraints on the matter fields on \( \Sigma \)

The almost-Einstein-matter scale tractor \( I_5 \) can be used to derive the constraints on the matter fields on the conformal boundary \( \Sigma \). From (3.22) we have

\[
I_5 \cdot \nabla_a I_\bar{5} = \frac{\bar{\sigma} \bar{q}^{a+1}}{6} \left( (2 - q) r_a^b \nabla_b \bar{\sigma} - \bar{\sigma} \nabla_b \tilde{r}_a^b \right).
\]  

(4.1)

On the other hand,

\[
I_5 \cdot \nabla_a I_\bar{5} = \frac{1}{2} \nabla_a \bar{\sigma}^2 = \frac{1}{24} \nabla_a (\bar{\sigma}^{q+2} \sigma).
\]  

(4.2)

Therefore, from (4.1) and (4.2),

\[
\sqrt{\lambda} \left[ (q - 2) \tilde{r}_{am} + n_a \tau \right] = -\bar{\sigma} \nabla_b \tilde{r}_a^b.
\]  

(4.3)

After evaluating (4.3) at \( \Sigma \) we get

\[
(q - 2) \tilde{r}_{am}^\Sigma = 0, \quad (q - 2) \tilde{r}_{am} - \tau ^\Sigma = 0.
\]  

(4.4)

Moreover, taking a derivative of (4.3) and evaluating at \( \Sigma \) reads

\[
\begin{align*}
\bar{\gamma}_c b \nabla_b \left[ n_a (\tau - (q - 2) \tilde{r}_{am}) \right] + (q - 2) \nabla_c \tilde{r}_{am}^\Sigma + (q - 2) n_a \tilde{r}_a^b k_c b \Sigma = 0, \\
\nabla_a \left[ (q - 2) \tilde{r}_{am} + n_a \tau \right] - \nabla_b \tilde{r}_a^b \Sigma = 0.
\end{align*}
\]  

(4.6)

The first equation is trivially satisfied due to (4.4). The second equation can be decomposed into transversal and intrinsic components,

\[
\begin{align*}
\nabla_a \left[ (q - 1) \tilde{r}_{am} - \tau \right] + \tilde{r}_{am} k_c b \Sigma - \nabla_c \tilde{r}_{am} - \frac{H}{3} \left( (q - 2) \tilde{r}_{am} \right) \Sigma = 0, \\
\nabla_c b \tilde{r}_{am} - H \tilde{r}_{am}^\Sigma + \tilde{r}_a^b k_a b - (q - 1) \nabla_n \tilde{r}_{am}^\Sigma = 0,
\end{align*}
\]  

(4.7)

where (4.4) has been used.

**Remark 3.** In [20] a four-dimensional asymptotically de Sitter spacetime \((\bar{M}, \bar{g}_{ab})\) with \( \bar{g}_{ab} = \Omega^{-2} g_{ab} \) (cf definition 1) and the stress-energy tensor of a scalar fluid \( \phi \) of mass \( m \) is considered, i.e.

\[
\bar{T}_{ab} = \bar{\nabla}_a \phi \bar{\nabla}_b \phi - \bar{g}_{ab} \left[ \frac{1}{2} \left( \bar{\nabla}_c \phi \bar{\nabla}^c \phi + m^2 \phi^2 \right) + V(\phi) \right]
\]  

(4.8)

with \( V(\phi) = \mu \phi^3 + \phi^4 U(\phi) \), where \( \mu \) is a constant and \( V'(0) = 0 \). In terms of a new variable \( \psi = \Omega^{-1} \phi \) the stress-energy tensor (4.8) can be written as

\[
\bar{T}_{ab} = \psi^2 \left[ \frac{\Lambda}{3} n_a n_b + \frac{1}{2} g_{ab} \left( \frac{\Lambda}{3} - m^2 \right) \right] + O(\Omega)
\]  

(4.9)

where \( n_a \) is the unit normal vector of a conformal boundary. This corresponds to (3.11) with \( q = 0 \), and the constraint (4.4) in this case reduces to

\[
m^2 = \frac{2}{3} \Lambda,
\]  

(4.10)

which matches the condition for the regularity of the conformal field equations derived there.
4.2. Conformal fundamental forms

The starting point in the construction of conformal fundamental forms is at the second jet of the scale $\sigma$ (the almost-Einstein-matter equation operator). The key object will be denoted by $E_{ab}$ and is given by

$$E_{ab} := p^* \left( \hat{D}^a \hat{P}^b_\sigma \right) = \nabla_a \nabla_b \hat{\sigma} + \hat{\sigma} P_{ab} - \frac{1}{4} g_{ab} \left( \Delta \hat{\sigma} + \hat{\sigma} \right).$$

(4.11)

Indeed, this tensor field, which is smooth to the boundary, restricts to ($p$ times) the second fundamental form there, but on the interior gives the traceless part of the Schouten for $\hat{g}_{ab}$.

Due to (3.13) we know that $E_{ab}$ can be associated with the stress-energy tensor density in the following way,

$$E_{ab} = \frac{\hat{\sigma}^{q+1}}{2} \hat{\tau}_{ab}. \quad (4.12)$$

This observation will allow us to relate the conformal fundamental forms of $\Sigma$ to the matter fields on the conformal boundary. The immediate consequence of (4.12) and (3.40) is the fact that $\Sigma$ is an umbilic hypersurface for asymptotically de Sitter spacetimes ($q \geq 0$). More generally, the almost-Einstein-matter scale $\hat{\sigma}$ has the following properties:

- $\nabla^k E_{ab} \equiv 0$ for $k \leq q$;
- $(q + 3)$rd jet of $\hat{\sigma}$ on $\Sigma$ will involve the trace-free stress-energy tensor density $\tau_{ab}$ there;
- $(q + 4)$th and higher jets of $\hat{\sigma}$ on $\Sigma$ will involve (at least first) derivatives of the trace-free stress-energy tensor density $\tau_{ab}$ there;

Before moving forward, let us recall (from (3.20) and (3.22)) formulas for the almost Einstein scale tractor and its derivative in four dimensions. We have

$$\hat{I}^a_\beta = -\lambda + \frac{1}{12} \hat{\sigma}^{q+2} \tau$$

(4.13)

and

$$\nabla_a I_\beta = \frac{\hat{\sigma}^{q}}{6} \left( -\left( q + 1 \right) \hat{\tau}_{ab} \nabla_b \hat{\sigma} - \hat{\sigma} \nabla_b \hat{\tau}^b_{ab} \right). \quad (4.14)$$

In the sequel we will focus on $q = 0, 1, 2$. This is motivated by the fact that those values are most commonly used in the analysis of the conformal extensions of asymptotically de Sitter spacetimes (see e.g. [2, 3, 20, 30]). Moreover, this choice will allow us to focus on the first five conformal fundamental forms of $\Sigma$, which turn out to consist of projections of Weyl, Cotton and Bach tensors (plus the induced metric and extrinsic curvature), well-known objects in the conformal geometry.

Following [19], we will present a construction of conformal fundamental forms. The basic principle behind it is to consider normal derivatives of $E_{ab}$ adjusted in a way that makes the whole expression conformally invariant. To achieve this goal, we will consider two differential operators, the tractor Robin operator $\delta_R$ from (2.25) and the canonical degenerate Laplacian $I_D$, defined in (2.20), adjusted to act on trace-free tensorial densities. The former can be defined as

$$\delta_R t_{ab} := \hat{p}^* o \bar{r} o \hat{\delta_R} o p \left( t_{ab} \right) \quad (4.15)$$
for \( w \neq 3 \), where the operators \( p, p^* \) and \( r \) are defined in section 2. It can be verified that

\[
\delta_R t_{ab} = \mathfrak{g}_{ab}^d \left( \nabla_n t_{cd} - \frac{w-2}{3} H t_{cd} \right) - \frac{1}{3} \mathfrak{g}_{ab} \left( n^r \nabla_n t_{cd} - \frac{w-2}{3} H t_{cd} \right) \\
+ \frac{2}{w-3} \left( \mathfrak{g}_{ab} (\mathcal{S} \nabla_t \mathfrak{b}) \right) - \frac{1}{3} \mathfrak{g}_{ab} \mathfrak{r} t_{ac}^{(c)}
\]

\[
= \left( \nabla_n t_{ab} - \frac{w-2}{3} H t_{ab} + \frac{2}{w-3} \mathfrak{r} t_{ab}^{(a)} n \right)
\]

such that

\[
\delta_R : \mathcal{E}_{(ab)_{n}}[w] \rightarrow \mathcal{E}_{(ab)_{n}}[w-1] \quad \text{for} \quad w \neq 3,
\]

i.e. \( \delta_R t_{ab} \) takes values in a weight twisting of the trace-free part of the symmetric covariant submanifold two-tensors. The canonical degenerate Laplacian \( \mathcal{D} \) acting on the tensorial density \( t_{ab} \) can be defined as

\[
\mathcal{D} t_{ab} := p^* \circ r \circ \mathcal{D} \circ p (t_{ab}),
\]

or

\[
\mathcal{D} t_{ab} = 2(w-1) \sqrt{\lambda} \left( \nabla_n + (w-2) \rho \right) t_{ab} - \frac{2(w-2)}{w-3(w+2)} n (a \nabla \cdot t_b)_n \\
+ \frac{2}{w-3} \left( n \cdot \nabla (a b)_n + t_a \cdot \nabla b) n \right) \\
- \sigma \left( \Delta t_{ab} + (w-2) H t_{ab} + \frac{8}{w-3(w+2)} \nabla (a \nabla \cdot t_b)_n - 4 P (a \cdot t_b)_n \right),
\]

hence

\[
\mathcal{D} : \mathcal{E}_{(ab)_{n}}[w] \rightarrow \mathcal{E}_{(ab)_{n}}[w-1] \quad \text{for} \quad w \neq -2, 3.
\]

The formula for conformal fundamental forms can now be given in terms of applying appropriate power of the degenerate Laplacian \( \mathcal{D} \) and \( \delta_R \) to \( t_{ab} \).

**Definition 3.** Let \( i \in \{0, 1, 2, 3, 4\} \). A conformal fundamental form \( \mathcal{K}^{(i+2)}_{ab} \) can be defined as

\[
\mathcal{K}^{(i+2)}_{ab} := \begin{cases} 
\mathcal{K}_{ab} & \text{for} \quad i = 0, \\
\delta_R \circ (\mathcal{D})^{i-1} E_{ab} & \text{for} \quad 1 \leq i \leq 4.
\end{cases}
\]

The upper bound for \( i \) in definition 3 is dictated by the fact that the conformal weight of \( \mathcal{D}^3 E_{ab} \) is \(-2\), so the \( \mathcal{D} \) operator applied to this quantity will have a pole (because of the coefficients \( \frac{1}{\pi^2} \)).

Unlike in the current scenario, the definition of conformal fundamental forms from [19] relied on the fact that \( \mathcal{P}_R = \pm 1 + \mathfrak{O} (\sigma^2) \) (which can always be achieved after improving the scale, see e.g. [29, theorem 1.3]). In order to retain the same formulae for those objects, we will restrict ourselves to the discussion of the conformal fundamental forms up to \( \mathcal{K}^{(q+4)}_{ab} \) for the given value of the decay parameter \( q \). It can be verified that the higher-order tensors include contributions from the trace \( \tau \). The conformal fundamental forms up to \( \mathcal{K}^{(q+4)}_{ab} \) are given solely in terms of geometric quantities because of the fact that definition 3 involves taking a projected trace-free part (via \( \delta_R \)) of a differential operator acting on \( E_{ab} \). It also implies that \( \mathcal{K}^{(6)}_{ab} \) will contain derivatives of the Bach tensor. An object of this transverse order is not usually considered in the literature, so \( \mathcal{K}^{(6)}_{ab} \) will be excluded from the discussion presented here.
In the sequel we will use definition 3 together with relation (4.12) to derive constraints relating conformal fundamental forms of $\Sigma$ and the matter fields there. The first case has already been discussed above—we observed that the second fundamental form $K_{ab}$ vanishes when $q \geq 0$, i.e. $\Sigma$ is an umbilic hypersurface. It should be noted that by definition 3, the higher conformal fundamental forms will also be trace-free.

### 4.3. Third fundamental form—the Weyl tensor

To obtain the third jet of $\bar{\sigma}$ one needs to apply $\delta_R$ to $E_{ab}$, i.e.

$$K^{(3)}_{ab} := \delta_R \left( \sqrt{\lambda} \nabla \cdot n_b + \bar{\sigma} P_{ab} - \frac{1}{4} g_{ab} \left( \sqrt{\lambda} \nabla \cdot n^a + \bar{\sigma} J \right) \right). \quad (4.22)$$

We will do it in steps. Firstly, we have

$$E^\perp \cdot n_\Sigma = \sum^{(2)} \cdot n_\Sigma = 0,$$

from (3.34). So the formula for $K^{(3)}_{ab}$ reduces to

$$K^{(3)}_{ab} := \mathcal{T} \left( \nabla \cdot n E_{ab} + \frac{H}{3} E_{ab} \right) = \mathcal{T} (\nabla_\Sigma E_{ab}) + \frac{H}{3} \sqrt{\lambda} K_{ab},$$

where $K_{ab}$ will be set to zero later on ($\Sigma$ is umbilic). Moreover, we can directly use the definition (1.12) of the Riemann tensor and its decomposition into the Weyl and Schouten tensors (1.13) to compute the normal derivative of $E_{ab}$,

$$\nabla \cdot n E_{ab} \Sigma = \sum^{(3)} \cdot n_\Sigma + H \sum^{(2)} \cdot n_\Sigma = 0,$$

from (3.34) and (4.23). Hence,

$$\mathcal{T} (\nabla_\Sigma E_{ab}) \Sigma = \mathcal{T} \left( \sum^{(2)} \cdot n_\Sigma + \frac{H}{3} \sum^{(1)} \cdot n_\Sigma + \frac{1}{3} \sum^{(1)} \cdot n_\Sigma \sum^{(2)} \cdot n_\Sigma \right).$$

Thus we ultimately obtain the following,

$$K^{(3)}_{ab} := \sqrt{\lambda} \left( C^{ab}_{\Sigma} - \sum^{(1)} \cdot n_\Sigma \sum^{(2)} \cdot n_\Sigma \right),$$

where the r.h.s. of this expression (modulo constant) is called the Fialkow tensor [31]. It reduces to

$$K^{(3)}_{ab} := \sqrt{\lambda} C^{ab}_{\Sigma},$$

on totally umbilic hypersurfaces (meaning $K_{ab} \cdot n_\Sigma = 0$). Therefore, the third conformal fundamental form of a conformal boundary of asymptotically de Sitter spacetime is proportional to the electric part of the Weyl tensor.

To relate this to matter fields on $\Sigma$ one needs to apply $\delta_R$ to the right-hand side of (4.12), i.e.

$$\delta_R \left( \frac{1}{2} \bar{\sigma}^{q+1} \tau_{ab} \right) = - \sqrt{\lambda} (q + 1) \bar{\sigma}^{q+1} \tau_{ab} \Sigma.$$
The combination of (4.28) and (4.29) yields

\[ C_{nab}^\Sigma = \frac{1}{2} (q + 1) \bar{\sigma}^a \nabla b (\tau_{ab}) \mid _\Sigma, \]

which gives a non-trivial relation between the third conformal fundamental form of \( \Sigma \) and the stress-energy tensor when \( q = 0 \):

\[ C_{nab}^\Sigma = \frac{1}{2} \nabla b (\tau_{ab}) \quad \text{for} \quad q = 0. \]

For higher \( q \) we conclude that the third fundamental form \( C_{nab}^\Sigma \) must be zero along \( \Sigma \).

### 4.4. Fourth fundamental form—the Cotton tensor

The action of the canonical degenerate Laplacian \( ID \) (4.19) on \( E_{ab} \) simplifies to

\[ L_D E_{ab} = -\bar{\sigma} \left( \Delta E_{ab} - J E_{ab} - \frac{4}{3} \nabla (a \nabla \cdot E_b)_a - 4 P_{(a} \cdot E_{b)b} \right). \]

The fourth trace-free fundamental form then reads,

\[ \delta^{(4)}_{ab} := \delta_k \circ L_D E_{ab} = \sqrt{\bar{\lambda}} \nabla \bar{\lambda} \left( \Delta E_{ab} - J E_{ab} - \frac{4}{3} \nabla (a \nabla \cdot E_b)_a - 4 P_{(a} \cdot E_{b)b} \right) \]

\[ = -2 \lambda \nabla \bar{\lambda} \left( \bar{K}^d C_{abcd} \right), \]

where (4.11), the definition (1.12) of the Riemann tensor \( R_{abcd} \) in terms of a commutator of covariant derivatives acting on \( n^a \), together with its decomposition (1.13) and the Bianchi identity (1.16) have been used. The third equality can be obtained by noticing that the principal parts (the highest-order derivatives acting on \( n^a \)) cancel each other out. Hence, \( \delta^{(4)}_{ab} \) vanishes if \( \bar{K}_{ab} \equiv 0 \) and the constraints on the behaviour of the matter fields on the conformal boundary \( \Sigma \) for asymptotically de Sitter spacetimes can be obtained by applying \( \delta_k \circ ID \) to the stress-energy counterpart of \( E_{ab} \) from (4.12), as follows. If \( q = 0 \), then

\[ \nabla n \tau_{ab} + \frac{2}{3} H \tau_{ab} \Sigma \equiv 0, \]

where (4.4) and the normal component of (4.7) with \( \bar{K}_{ab} \Sigma \equiv 0 \) have been used, i.e.

\[ \tau_{an} \Sigma \equiv 0, \quad \tau \Sigma \equiv -2 \tau_{mn}, \quad \nabla n (\tau + \tau_{mn}) \Sigma \equiv \frac{4}{3} H \tau_{mn}. \]

In the case where \( q = 1 \) a simple condition

\[ \nabla (\tau_{ab}) \Sigma \equiv 0 \]

arises. After combining it with the constraint (4.4), we get

\[ \tau_{ab} \Sigma = -n_a n_b \tau, \]

which was also derived in the context of conformal Einstein field equations, see e.g. \([2, 3]\).

For \( q = 2 \) the constraint is trivial because the decay of matter fields is too fast to be captured by \( \delta_k \circ ID \). Thus our definition of \( \delta^{(4)}_{ab} \), as above, fails to give a meaningful constraint relating the geometry of \( \Sigma \) to the stress-energy tensor for asymptotically de Sitter spacetimes in this case. However, based on the results from [19] we will make use of the following choice for the fourth fundamental conformal form of \( \Sigma \), sometimes called (modulo constant) a hypersurface Bach tensor.
Definition 4. The fourth conformal fundamental form $\tilde{K}_{ab}^{(4)}$ can be defined as

$$\tilde{K}_{ab}^{(4)} := \lambda \left( \nabla^\top C_{\tau_n b c} - A^\top_{(a|n|b)} - \frac{1}{3} HC_{nab} \right).$$  \hspace{1cm} (4.38)

It can be checked that (4.38) has the desired properties, i.e. is trace-free, intrinsic to $\Sigma$ and has conformal weight $-1$. The latter property can be verified with the use of the following conformal transformation rules ($g_{ab} = f^2 g_{ab}$),

$$\tilde{A}_{(a|b)} = \frac{1}{f} \left( A_{(a|b)} - n_q Y^d C_{nab} - \nabla^\top C_{\tau_n ab} \right),$$

$$\frac{1}{f^3} HC_{nab} = \frac{1}{f} \left( \frac{H}{3} + n_d Y^d \right) C_{nab},$$

$$\nabla^\top C_{\tau_n b c} = \frac{1}{f} \left( \nabla^\top C_{\tau_n b c} - \nabla^\top C_{\tau_n b c} \right).$$ \hspace{1cm} (4.39)

The choice for the fourth conformal fundamental form from definition 4 has been motivated by considering the action of $\delta_R \circ \text{ID} E_{ab}$ computed in arbitrary dimension $\dim M = n$. With the assumption that $\Sigma$ is an umbilic hypersurface ($\tilde{K}_{ab} \equiv 0$, i.e. $q \geq 1$), one obtains the following,

$$\frac{\delta_R \circ \text{ID} E_{ab}}{n - 4} = \lambda \nabla^\top C_{\tau_n b c} + \lambda (n - 5) \left( A^\top_{\tau_n b c} + \frac{H}{n - 1} C_{\tau_n b c} \right),$$ \hspace{1cm} (4.40)

where the $\nabla^\top C_{\tau_n b c}$ term vanishes because $\tilde{K}_{ab} \equiv 0$ and the Codazzi equation (3.39). It should be noted that the fourth conformal fundamental form definition 4 does not, in dimension four, arise the construction which simply factors through the jets of $E_{ab}$. This is evident from our calculations here. Conceptually this shows that (and is happening because) in dimension four it is an image of the Dirichlet-to-Neumann map for the conformal Einstein field equations—cf our discussion in the Introduction and [21].

Since we assume that the Weyl tensor $C_{abcd}$ is smooth in the neighbourhood of the conformal boundary $\Sigma$, and is seen to vanish along $\Sigma$ for $q \geq 1$, the hypersurface divergence of $\tilde{K}_{ab}^{(4)}$ from definition 4 can be determined by the stress-energy tensor $\Sigma$ in that case.

Theorem 2. Let $(\tilde{M}, \tilde{g}_{ab})$ be a four-dimensional asymptotically de Sitter spacetime with the stress-energy tensor $\tilde{T}_{ab}$ and the conformal extension $(M, g_{ab})$ such that

$$\tilde{T}_{ab} = \delta^\top \tau_{ab}, \quad q = 1, 2.$$ \hspace{1cm} (4.41)

Then, the Weyl tensor and $\tau_{\tau_n}^\top$ vanish on the conformal boundary and

$$\nabla^\top K_{ab}^{(4)} \equiv \begin{cases} -\lambda^2 \tau_{ja} + \lambda^{1/2} \nabla_{j} \tau & \text{for } q = 1, \\ -\lambda^2 \tau_{\tau_n}^\top & \text{for } q = 2, \end{cases}$$ \hspace{1cm} (4.42)

where $j_a$ is defined via the expansion $\tau_{\tau_n}^\top = \tilde{\sigma} j_a + O(\tilde{\sigma})$ and $\tilde{K}_{ab}^{(4)} = -\lambda A_{\tau_n}^\top$.

Proof. We can use (2.12) and (2.13) to obtain the following,

$$Z_{\delta_{\lambda} C_{\tau_n} \nabla_{\tau_n}} \| \| = \tilde{\sigma} A_{\tau_n} + \sqrt{\lambda} C_{\tau_n b c n}$$

$$\hspace{1cm} = \sqrt{\lambda}(q + 1) \tilde{\sigma}^q \left( n_{[a} \tau_{b|c]} + \frac{1}{3} \tau q_{[a} g_{b|c]} - \frac{1}{3} \tau n_{[a} g_{b|c]} \right)$$

$$\hspace{1cm} + \tilde{\sigma}^{q+1} \left( \nabla_{[a} \tau_{b]} - \frac{1}{3} g_{[a} \nabla^d \tau_{b]} + \frac{1}{3} g_{[a} \nabla^d \tau_{b]} \right),$$ \hspace{1cm} (4.43)
where the second equality comes from considering the matter counterpart of $\nabla_a I_{\beta}$, i.e. the right-hand side of (4.14).

It is known that any timelike vector $n^a$ induces a decomposition of the Weyl tensor into its electric and magnetic parts, $C_{anbn}$ and $C^\top nabc$, respectively, which fully determine this tensor in four dimensions. Due to the assumption on $q$, the hypersurface $\Sigma$ is umbilic. In that case the Codazzi equation (3.39) and the constraint (4.30) imply $C^\top nabc \Sigma = 0$ and $C_{anbn} \Sigma = 0$. Hence, the whole Weyl tensor vanishes on the conformal boundary, i.e. $C_{abcd} \Sigma = 0$ for $q \geq 1$. Because $\Sigma$ is smooth, we can define a rescaled Weyl tensor $K_{abcd}$,

$$K_{abcd} := \frac{1}{\sigma} C_{abcd}, \tag{4.44}$$

which is regular on $M$. Equation (4.43) now implies

$$A_{anbn}^\top \Sigma = \sqrt{\lambda} K_{anbn}^\top, \tag{4.45}$$

where constraints (4.4) and (4.37) have been used.

To show the hypersurface divergence constraint, we can use the Bianchi identity and (4.43) again to get

$$\nabla^d K_{dabc} = \nabla^d \left( \frac{1}{\sigma} C_{dabc} \right) = \frac{1}{\sigma} A_{abc} - \frac{\sqrt{\lambda}}{\sigma^2} C_{nabc}$$

$$= \sqrt{\lambda} (q + 1) \bar{\sigma}^{-2} \left( n_{[b \tau_c]a} + \frac{1}{3} \tau_{a[nb\tau_c]} - \frac{1}{3} \tau_{n[a nb\tau_c]} \right)$$

$$+ \bar{\sigma}^{-1} \left( \nabla_{[b \tau_c]a} - \frac{1}{3} \nabla_{[b \tau_c]} n_{a} \nabla_{[b \tau_c]} \right), \tag{4.46}$$

which, after contracting with $n^a n^c$ and evaluating on the conformal boundary gives

$$\nabla^b K_{anbn}^\top \Sigma = \begin{cases} \frac{1}{\sqrt{\lambda}} \lim_{\bar{\sigma} \to 0} \left( \frac{\tau_{a}^\top}{\bar{\sigma}} \right) - \frac{1}{3} \nabla_a \tau - \frac{1}{3} \nabla_n \tau_{an}^\top & \text{for } q = 1, \\ \sqrt{\lambda} \tau_{an}^\top & \text{for } q = 2, \end{cases} \tag{4.47}$$

where constraints (4.4) and (4.37) have been used again. If $q = 1$, then $\tau_{an}^\top \Sigma = 0$ and

$$\tau_{an}^\top = \sigma j_a + O \left( \bar{\sigma}^2 \right), \tag{4.48}$$

so the ultimate form of (4.47) is

$$\nabla^b A_{anbn}^\top \Sigma = \begin{cases} \lambda j_a - \frac{\lambda}{3} \nabla_a \tau & \text{for } q = 1, \\ \lambda \tau_{an}^\top & \text{for } q = 2, \end{cases} \tag{4.49}$$

where (4.45) has been used.

4.5. Fifth fundamental form—the Bach tensor

Following definition 3 and the discussion afterwards, we can now compute the fifth conformal fundamental form of $\Sigma$ for $q \geq 1$. We have

$$K_{(5)} := \delta_R \circ \text{ID} E_{ab} = 6 \lambda^3 \tau^\top \left( B_{ab} \right) \quad \text{for } q \geq 1, \tag{4.50}$$

where the definition of the Riemann tensor (1.12) in the context of a commutator of covariant derivatives acting on the normal vector $n^a$ and the definition of the Bach tensor from (1.15)
have been used. As in the case of the $K_{ab}^{(4)}$, an important step in deriving the last equality is noticing that the highest-order derivatives acting on $n^a$ cancel each other out.

We will analyse the matter counterpart of $K_{ab}^{(5)}$ for two different values of the parameter $q$. If $q = 1$, then (4.11) implies

$$\delta R \circ \text{ID}^2 E_{ab} = 2 \lambda^2 \left\{ 9 \nabla_n \tau^\top_{ab} - 3 \mathfrak{R}_{ab} \nabla_n \tau_{nm} + 8 H \tau^\top_{ab} - \mathcal{H} \mathfrak{R}_{ab} \tau_{nm} \right\},$$

(4.51)

where (4.4) and (4.7) have been used. Therefore,

$$K_{ab}^{(5)} \equiv 2 \lambda^2 \left\{ 9 \nabla_n \tau^\top_{ab} - 3 \mathfrak{R}_{ab} \nabla_n \tau_{nm} + 8 H \tau^\top_{ab} - \mathcal{H} \mathfrak{R}_{ab} \tau_{nm} \right\} \text{ for } q = 1.$$  

(4.52)

For $q = 2$, $\delta R \circ \text{ID}^2$ applied to the right-hand side of (4.11) yields

$$\delta R \circ \text{ID}^2 E_{ab} = \delta R \circ \text{ID}^2 \left( \frac{1}{2} \sigma^3 \nabla_{ab} \right) \equiv -18 \lambda^2 \nabla^\top \left( \tau_{ab} \right),$$

(4.53)

where the constraint (4.4) has been taken into account ($\tau \equiv 0$). Ultimately,

$$K_{ab}^{(5)} \equiv -18 \lambda^2 \nabla^\top \left( \tau_{ab} \right) \text{ for } q = 2.$$  

(4.54)

The constraints derived above allowed us to relate the projected part of the Bach tensor with the matter fields on the conformal boundary $\Sigma$. In order to associate the other components of $B_{ab}$ with the stress-energy tensor on $\Sigma$ a commutator of the two Thomas-D operators can be used. The strategy is to apply it to the scale tractor $I_\bar{\gamma}$ and compute the resulting expression in two different ways, either by exploiting the form of $\nabla_a I_\bar{\gamma}$ given in terms of geometric objects or by using (4.14). Firstly, we have

$$D_{[\alpha} D_{\beta]} \left( I_\bar{\gamma} \right) \equiv 2 \nabla^\alpha X C_{\alpha} Z_{\beta} B_{\gamma \delta},$$

(4.55)

which can be obtained as a direct application of the definition of the Thomas-D operator from section 2. On the other hand,

$$D_{[\alpha} D_{\beta]} I_\bar{\gamma} = -X_{[\alpha} \left( \Delta - J \right) \left( 2 Z_{\beta]} \nabla_a I_\bar{\gamma} - X_{\beta]} \nabla^b \nabla_b I_\bar{\gamma} \right),$$

(4.56)

so after using (4.14) this expression can be written solely in terms of the stress-energy tensor density $\tau_{ab}$. After comparing it with (4.55) we get the following,

$$B_{an} \equiv 0, \quad B^\top_{an} \equiv \sqrt{\lambda} \left( \nabla_n \tau^\top_{an} + \frac{1}{3} \nabla_a \tau \right) \text{ for } q = 1,$$

(4.57)

and

$$B_{an} \equiv 0, \quad B^\top_{an} \equiv -\lambda \tau^\top_{an} \text{ for } q = 2,$$

(4.58)

where constraints (4.4) and (4.7) have been used.

4.6. Summary

The results of this section can be stated in the form of the following theorem.

**Theorem 3.** Let $(\tilde{M}, \tilde{g}_{ab})$ be a four-dimensional asymptotically de Sitter spacetime with the stress-energy tensor density $\tau_{ab}$ of weight $q \in \{0, 1, 2\}$. Then the conformal boundary $\Sigma$ is an umbilic hypersurface, and

$$\left( q - 2 \right) \tau_{an} + n_a \tau \equiv 0,$$

$$\nabla_n [\left( q - 2 \right) \tau_{an} + n_a \tau] - \nabla_b \tau^b \equiv 0,$$

(4.59)
where

\[ n_{a}n^{a} = -1 + \frac{2}{n\lambda} \tilde{\sigma} (\Delta + J) \tilde{\sigma} + O (\tilde{\sigma}^{q+2}) \sum = -1. \]  

(4.60)

Moreover, the constraints relating conformal fundamental forms of \( \Sigma \) to the matter fields there have the following form for the specific values of \( q \):

- \( q = 0 \)
  \[ C_{nab} \equiv 1 \frac{1}{2} \nabla^{	op} (\tau_{ab}) , \]
  \[ \nabla n_{\tau_{ab}}^{\top} + \frac{2}{3} H\tau_{ab} \equiv 0, \]  
  (4.61)

- \( q = 1 \)
  \[ C_{abcd} \equiv 0 , \quad \tau_{ab} \equiv -n_{a}n_{b}\tau , \]
  \[ \nabla_{a}A_{\nabla}^{b} \equiv \lambda_{a} - \frac{\sqrt{\lambda}}{3} \nabla_{a}\tau , \]
  \[ \nabla_{a} (B_{ab}) \equiv \frac{\sqrt{\lambda}}{3} \left( 9 \nabla n_{\tau_{ab}}^{\top} - 3 g_{ab} \nabla n_{\tau_{an}} + 8 H\tau_{ab} - Hg_{ab} \tau_{mn} \right) , \]  
  (4.62)

- \( q = 2 \)
  \[ C_{abcd} \equiv 0 , \]
  \[ \nabla_{a}A_{\nabla}^{b} \equiv \lambda_{a} \tau_{ab} , \]
  \[ \nabla_{a} (B_{ab} + 3 \lambda\tau_{ab}) \equiv 0 , \]  
  (4.63)

5. Conclusions

We have derived constraints relating the conformal fundamental forms of the conformal infinity of asymptotically de Sitter spacetime with its stress-energy tensor. As a result, projected parts of the Weyl and Bach tensors and a divergence of the Cotton tensor on the conformal boundary have been determined locally by the matter fields there. The natural application of this result is the study of matching of spacetimes in the conformal cyclic cosmology scenario, where the conformal infinity of the asymptotically de Sitter spacetime is identified with the Big Bang hypersurface of the second conformally extended solution of the Einstein field equations. Mimicking the procedure of joining spacetimes via the Darmois–Israel junction condition, the natural strategy to consider in this setting is the identification of the conformal fundamental forms of the conformal boundaries of asymptotically de Sitter and initial singularity spacetimes to obtain constraints on the matter content on the transition hypersurface. This will be considered elsewhere.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.
Acknowledgments

A R G gratefully acknowledges support from the Royal Society of New Zealand via Marsden Grants 16-UOA-051 and 19-UOA-008. J K would like to thank Paweł Nurowski for the encouragement to pursue the topic of this work. He acknowledges funding received from the Norwegian Financial Mechanism 2014-2021, project registration number UMO-2019/34/H/ST1/00636.

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