THE 2-HESSIAN AND SEXTACTIC POINTS
ON PLANE ALGEBRAIC CURVES

PAUL ALEKSANDER MAUGESTEN AND TORGUNN KAROLINE MOE

Abstract. In an article from 1865 Arthur Cayley claims that given a plane
algebraic curve there exists an associated 2-Hessian curve that intersects it in
its sextactic points. In this paper we fix an error in Cayley's calculations and
provide the correct defining polynomial for the 2-Hessian of a plane curve.
In addition, we provide a formula for the number of sextactic points on
cuspidal curves and tie this formula to the 2-Hessian.
Lastly, we consider the special case of rational curves, where the sextactic
points can be interpreted as the zeros of the Wronski determinant of the 2nd
Veronese embedding of the curve.

1. Introduction

Let \( C = V(F) \) be an algebraic curve of geometric genus \( g \) and degree \( d \), given by
a polynomial \( F \in \mathbb{C}[x,y,z]_d \), in the projective plane \( \mathbb{P}^2 \) over the complex numbers
\( \mathbb{C} \). In standard terms, the singular points of an irreducible curve \( C \) are the points
where the partial derivatives of \( F \) vanish, and a cusp is a unibranch singular point.
A non-singular point is referred to as a smooth point.

Given two curves \( C \) and \( C' \) and a point \( p \in C \cap C' \), let \((C \cdot C')_p\) denote the
intersection multiplicity of \( C \) and \( C' \) at \( p \). Moreover, for any point \( p \in C \), let \( m \)
denote its multiplicity, i.e. the intersection multiplicity of the curve and a generic
line at the point.

Now, given an irreducible curve \( C \) and fixed \( n \in \mathbb{N} \), consider curves, not neces-
sarily irreducible, of degree \( n \) in \( \mathbb{P}^2 \), and look at the intersections these curves have
with \( C \). With \( r(n) = \frac{1}{2} n(n + 3) \), for every smooth point \( p \in C \) there exists a curve
of degree \( n \) such that the local intersection multiplicity is equal to or bigger than
\( r(n) \) [Arn96]. Such a curve is called an osculating curve to \( C \) at \( p \). A smooth point
where the intersection multiplicity between a curve of degree \( n \) and \( C \) is strictly
bigger than \( r(n) \) is referred to as an \( n \)-Weierstrass point. In this case, the curve of
degree \( n \) is called a hyperosculating curve.

For \( n = 1 \) this comes down to tangent lines to \( C \) at smooth points \( p \in C \). The
tangent to \( C \) at \( p \) is denoted by \( T_p \), is given by the linear polynomial \( xF_x(p) + yF_y(p) + zF_z(p) \), and has the property that \((T_p \cdot C)_p \geq 1 \). The 1-Weierstrass points
are nothing but the inflection points, points where \((T_p \cdot C)_p \geq 2 \), and the order of
inflection is equal to \((T_p \cdot C)_p - 2 \).

The main focus of this paper is the case \( n = 2 \). For every smooth point \( p \in C \)
there exists a unique conic \( O_p \) such that \((O_p \cdot C)_p \geq 5 \). This conic is referred to
as the osculating conic to \( C \) at \( p \) [Cay59]. In particular, we look at points where
\((O_p \cdot C)_p > 5 \), i.e. the 2-Weierstrass points. The 2-Weierstrass points include
inflection points and a class of smooth points called sextactic points, first studied
in a general setting by Cayley in [Cay65].
In the case \( n = 1 \) it is well known that the Hessian curve of degree \( 3(d - 2) \), given by the polynomial

\[
H = H_1(F) = \begin{vmatrix}
F_{xx} & F_{xy} & F_{xz} \\
F_{yx} & F_{yy} & F_{yz} \\
F_{zx} & F_{zy} & F_{zz}
\end{vmatrix},
\]

intersects \( C \) in its inflection points and singular points. By abuse of notation we refer to both the Hessian curve and its defining polynomial as \( H \).

In [Cay65], Cayley presents a curve with similar properties; a curve of degree \( 12d - 27 \) that intersects \( C \) in its sextactic points, higher order inflection points, and its singular points. The first main result of this article is a correction of Cayley’s defining polynomial for this curve, referred to as the 2-Hessian of \( C \). See Section 2 for notation.

**Theorem 1.1.** Let \( C = V(F) \) be a plane curve of degree \( d \geq 3 \), with \( H \) the defining polynomial of the Hessian curve of \( C \). Then there exists a curve of degree \( 12d - 27 \) given as the zero set of

\[
H_2 = H_2(F) = (12d^2 - 54d + 57) \text{Jac}(F, H, \Omega) + (d - 2)(12d - 27) \text{Jac}(F, H, \Omega_F) - 20(d - 2)^2 \text{Jac}(F, H, \Psi),
\]

such that the intersection points between \( C \) and this curve are the singular points, the higher order inflection points, and the sextactic points of \( C \).

As in the case of the Hessian curve, we abuse notation and refer to both the 2-Hessian curve and its defining polynomial as \( H_2 \).

Note that a modern treatment of higher Hessians for smooth plane curves and generalized Hessians for smooth curves in \( \mathbb{P}^n \) that are complete intersections can be found in [Cuk97].

Weierstrass points of curves with respect to a linear system \( Q \) have been intensively studied, both classically in the case of smooth curves, and more recently for singular curves [BG97, DC08, Lak84, LW90, Pie77]. For a singular plane curve, the Weierstrass points with respect to a linear system \( Q \) are the singular points and the smooth Weierstrass points. Note that since we in this article we will restrict our study to Weierstrass points on plane curves with respect to linear systems \( Q \) of curves of degree \( n \), we interchangeably use the notation \( n \)- or \( Q \)-Weierstrass points.

To each \( Q \)-Weierstrass point \( p \) on a curve \( C \) it is possible to assign a so-called Weierstrass weight \( w_p(Q) \); in the case of plane curves by means of intersection multiplicities [Not99]. On the other hand, the sum of the Weierstrass weights can be computed through a generalization of the Brill–Segre formula [BG97]. Either way, this makes it possible to establish generalized Plücker formulas. In the case of sextactic points on smooth plane curves and curves with ordinary singularities, this has been done by Thorbergsson and Umehara [TU02, p. 90] and Coolidge [Coo31, Theorem 4, p. 280], respectively.

Another hot research topic the last 20 years is plane cuspidal curves, i.e. curves where all the singularities are cusps. Up to topological type, a cusp can be described by its multiplicity sequence, \( m \), defined as the multiplicities of the points above \( p \) in the minimal embedded resolution of the cusp, see [BK86, p. 503]. With this in mind, the question of how many and what kind of cusps a cuspidal curve can have naturally arises. This question has been studied partially to get an overview of plane curves, furthermore because the complement of some such curves is examples of log-general open surfaces. For a brief overview of this topic we refer to [Moe08, Moe13].
The second main result in this article exploits the simplicity of cuspidal curves and the progress in the study of Weierstrass points on singular curves in order to provide a formula for the number of sextactic points on plane cuspidal curves. Unfortunately, this result does not provide new answers to questions raised for cuspidal curves.

Before we state the result, note that in the case of a cusp $p$ on $C$, since it is unibranched of multiplicity $m$, there exists a unique line $T_p$ such that $(T_p \cdot C)_p > m$, referred to as the tangent line to $C$ at $p$. Similarly, applicable only to cusps where $(T_p \cdot C)_p = 2m$, there exists a, not necessarily unique, irreducible osculating conic $O_p$ to $C$ at $p$ (see Proof of Theorem 1.2 in Section 3).

**Theorem 1.2 (Sextactic point formula).** Let $C$ be a cuspidal curve of genus $g$ and degree $d \geq 3$. Let $p$ be a point on $C$, $m$ its multiplicity, $T_p$ the tangent line to $C$ at $p$, and $O_p$ an osculating conic to $C$ at $p$. Moreover, let $l = (T_p \cdot C)_p$ and $c = (O_p \cdot C)_p$. Let $I$ denote the set of inflection points and cusps on $C$ where $l \neq 2m$, and let $J$ denote the set of cusps on $C$ where $l = 2m$. Then the number of sextactic points $s$ on $C$, counted with multiplicity, is given by

$$s = 6(2d + 5g - 5) - \sum_I (4m + 4l - 15) - \sum_J (10m + c - 15).$$

**Remark 1.** The values $l$ and $c$ can frequently be determined simply by the multiplicity sequence of the cusp and the degree of the curve. See Lemmas 3.2 and 3.3. If the defining polynomial for the curve $C$ is known, $l$ and $c$ can be computed explicitly.

This article has the following structure. In Section 2 we fix Cayley’s polynomial for the 2-Hessian, and we explore some curves. In Section 3 we prove the formula from Theorem 1.2 for the number of sextactic points on cuspidal curves, and we apply this formula to examples. Moreover, we derive an associated formula with an apparent geometrical interpretation. In Section 4 we take a closer look at rational curves. In this case, the osculating conic to a curve at a smooth point can be calculated directly from the parametrization. Moreover, we show that the Weierstrass weight of a point corresponds to the order of zeros of the Wronski determinant of the 2nd Veronese embedding of the curve.

The figures in this articles are made with \[\text{[G12]}\]. Computations are made with \[\text{Maple}\] and programs are presented in \[\text{Mau17, Appendix B}\].

2. The 2-Hessian Curve

In this section we show that the polynomial in Theorem 1.1 on page 2 is the correct defining polynomial for the 2-Hessian of a plane curve, and apply it to examples. We start with some notation, mostly following Cayley’s original articles.

With $C$ and $F$ as before, and $p$ a point on the curve, let

$$D^F_p(x, y, z) = x\partial_x F(p) + y\partial_y F(p) + z\partial_z F(p),$$

$$D^2 F_p(x, y, z) = x^2 \partial_x^2 F(p) + y^2 \partial_y^2 F(p) + z^2 \partial_z^2 F(p) + 2xy\partial_x\partial_y F(p) + 2xz\partial_x\partial_z F(p) + 2yz\partial_y\partial_z F(p).$$

For the mixed second order partial derivatives of $F$, write

$$a = F_{xx}, b = F_{yy}, c = F_{zz}, f = F_{yz}, g = F_{xz}, h = F_{xy},$$

so that

$$H = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}.$$
In a similar fashion, let the mixed second order partial derivatives of $H$ be denoted by

$$a' = H_{xx}, b' = H_{yy}, c' = H_{zz}, f' = H_{yz}, g' = H_{xz}, h' = H_{xy}.$$  

Moreover, put

$$A = bc - f^2, B = ac - g^2, C = ab - h^2,$$

$$F = hg - af, G = hf - bg, H = fg - hc,$$

$$\Omega = (A, B, C, F, G, H) \cdot (a', b', c', 2f', 2g', 2h'),$$

$$\partial_2 \Omega_H = (\partial_2 A, \partial_2 B, \partial_2 C, \partial_2 F, \partial_2 G, \partial_2 H) \cdot (a', b', c', 2f', 2g', 2h'),$$

$$\partial_2 \Omega_F = (A, B, C, F, G, H) \cdot (\partial_2 a', \partial_2 b', \partial_2 c', 2\partial_2 f', 2\partial_2 g', 2\partial_2 h').$$

The expressions $\partial_2 \Omega_H, \partial_2 \Omega_F, \partial_2 \Omega_F$ and $\partial_2 \Omega_F$ are obtained by replacing $x$ with $y$ and $z$, respectively, in $\partial_2 \Omega_H$ and $\partial_2 \Omega_F$.

Lastly, let $\Psi$ denote the determinant

$$\Psi = -\begin{vmatrix} 0 & \partial_x H & \partial_y H & \partial_z H \\ \partial_x H & a & h & g \\ \partial_y H & h & b & f \\ \partial_z H & g & f & c \end{vmatrix}.$$  

### 2.1. The osculating conic.

Given a curve $C$, for any smooth point $p$ there exists a unique osculating conic $O_p$. Indeed, in the case of inflection points, this conic is the double tangent line, given by

$$(xF_x(p) + yF_y(p) + zF_z(p))^2.$$

Moreover, for a smooth point that is not an inflection point, the polynomial of the osculating conic is given by Cayley in [Cay59].

**Theorem 2.1 ([Cay59 377]).** Let $C$ be a plane curve of degree $d$ given by a polynomial $F$. If $p$ is a point on $C$ that is neither singular nor an inflection point, and $\Lambda = -3\Omega H + 4\Psi$, then the osculating conic $O_p$ to $C$ at $p$ is given by the defining polynomial

$$D^2 F_p = \left( \frac{1}{3} \frac{1}{H(p)} DH_p + \Lambda(p) D F_p \right) D F_p.$$  

For completion, we include the formal definition of a sextactic point.

**Definition 2.2.** Let $p$ be a smooth point that is not an inflection point on a curve $C$, and let $O_p$ be the osculating conic to $C$ at $p$. Then $p$ is called a **sextactic point** if

$$(O_p \cdot C)_p \geq 6.$$  

With $s = (O_p \cdot C)_p - 5$, a sextactic point $p$ is said to be of order $s$, or $s$-sextactic.

**Example 2.3.** Let $C$ be the nodal cubic curve given by

$$F = -x^3 - x^2 z + y^2 z.$$  

Now, choose $p = (-1 : 0 : 1)$, a smooth point on $C$. The osculating conic to $C$ at $p$ can be directly computed with [Mau17, Program B.2, pp. 67–68],

$$2x^2 + 3xz + y^2 + z^2.$$  

The intersection multiplicity of the conic and the curve at the point is 6, hence this is an example of a sextactic point and a hyperosculating conic. See Figure 1 on the following page.
2.2. The correct 2-Hessian. A defining polynomial of the 2-Hessian to a curve \( C \) first appears in [Cay65]. There is, however, an elementary computational error in the proof that makes Cayley’s version of the 2-Hessian incorrect. This mistake will be corrected in the following.

To state the defining polynomial of the 2-Hessian, one more shorthand notation is required. Given three polynomials \( F, G, H \), the determinant of the Jacobian matrix is denoted by \( \text{Jac}(F, G, H) \),

\[
\text{Jac}(F, G, H) = \begin{vmatrix}
F_x & F_y & F_z \\
G_x & G_y & G_z \\
H_x & H_y & H_z
\end{vmatrix}.
\]

With notation as above, the incorrect defining polynomial for the 2-Hessian by Cayley can be found in [Cay65, p. 556]:

\[
(12d^2 - 54d + 57)H \text{Jac}(F, H, \Omega_H) + (d - 2)(12d - 27)H \text{Jac}(F, H, \Omega_F) - 40(d - 2)^2 \text{Jac}(F, H, \Psi).
\]

Recall that the correct defining polynomial for the 2-Hessian to a curve \( C \) is given in [Theorem 1.1] by

\[
(12d^2 - 54d + 57)H \text{Jac}(F, H, \Omega_H) + (d - 2)(12d - 27)H \text{Jac}(F, H, \Omega_F) - 20(d - 2)^2 \text{Jac}(F, H, \Psi).
\]

Comparing these two polynomials, we notice that Cayley’s error affects only the coefficient of the last term. For many curves this term vanishes, which may explain why the flaw has gone unnoticed for more than 150 years (cf. [Cuk97]).

For completion we now dive into Cayley’s proof from [Cay65], point out the mistake he makes, and show that correct calculations lead to the defining polynomial in [Theorem 1.1]
Proof of [Theorem 1.1]. Cayley’s proof starts out with restrictions that arise when more than five points in the intersection between a conic and the curve $C$ coalesce. After a few pages of calculations, Cayley reaches a condition for the 2-Hessian in Section 17 on p. 552 of [Cay65] on the form

\begin{equation}
(15d^2 - 54d + 51)H \text{Jac}(F, \nabla, H)H + (30d - 54)(d - 2)H \text{Jac}(F, \nabla H, H) + (d - 2)^2 [9H^2 \partial \Omega - 45H\Omega \partial H + 40\Psi \partial H] = 0,
\end{equation}

where $\partial = (B\nu - C\mu)\partial_x + (C\lambda - A\nu)\partial_y + (A\mu - B\lambda)\partial_z$, with $\lambda, \mu$ and $\nu$ arbitrary constants and $A = F_x, B = F_y, C = F_z$. Moreover, $\nabla$ is not the usual gradient, but rather a function defined similarly to $\Psi$ by Cayley in Section 7 of [Cay65], while $\nabla H = (\partial_x \nabla H, \partial_y \nabla H, \partial_z \nabla H)$. Beware that Cayley uses slightly different notation, as he has $m := d$ and $U := H$.

Cayley’s mistake occurs as he attempts to simplify the last term of Equation (1), in Section 19 on p. 553 of [Cay65]. In the simplification Cayley introduces a variable $W$ in Section 18 and correctly states that

\[ W := H \partial \Omega - 5\Omega \partial H = -\frac{3}{4d - 9} \partial \text{Jac}(F, \Omega, H) - \frac{5d - 9}{4d - 9} \partial(\Omega H), \]

where $\partial = \lambda x + \mu y + \nu z$. Moreover, in Section 19, Cayley states that

\[ \Psi \partial H = \frac{1}{2} \frac{\partial}{4d - 9} \partial \text{Jac}(F, \Psi, H) + \frac{3}{2} \frac{(d - 2)}{4d - 9} H \partial \Psi. \]

Observing that $9H^2 \partial \Omega - 45H\Omega \partial H + 40\Psi \partial H = 9HW + 40\Psi \partial H$, Cayley rewrites the last term of Equation (1) and obtains

\begin{equation}
(2) \hspace{1cm} 9HW + 40\Psi \partial H = -\frac{9(5d - 9)}{4d - 9} H \partial(\Omega H) + \frac{60(d - 2)}{4d - 9} H \partial \Psi + \frac{\partial}{4d - 9} [-27H \text{Jac}(F, \Omega, H) + 40 \text{Jac}(F, \Psi, H)],
\end{equation}

where he makes the mistake of forgetting to multiply 40 by $\frac{1}{2}$.

The correct calculations yield

\begin{equation}
(3) \hspace{1cm} 9HW + 40\Psi \partial H = -\frac{9(5d - 9)}{4d - 9} H \partial(\Omega H) + \frac{60(d - 2)}{4d - 9} H \partial \Psi + \frac{\partial}{4d - 9} [-27H \text{Jac}(F, \Omega, H) + 20 \text{Jac}(F, \Psi, H)],
\end{equation}

where the coefficient of $\text{Jac}(F, \Psi, H)$ in the parenthesis is 20 as opposed to 40 in Equation (2).

Using the correct result from Equation (3), we manipulate Equation (1) along the same lines as Cayley in Section 20 of [Cay65], and obtain the condition

\begin{equation}
(4) \hspace{1cm} 3H \Pi + (d - 2)^2 \partial \{-27H \text{Jac}(F, \Omega, H) + 20 \text{Jac}(F, \Psi, H)\} = 0,
\end{equation}

where $\Pi$ is an expression that in Sections 21–25 of [Cay65] is simplified to

\[ \Pi = - (5d^2 - 18d + 17) \partial \text{Jac}(F, H, \Omega_{R}) - (5d - 9)(d - 2) \partial \text{Jac}(F, H, \Omega_{F}). \]

Thus, after throwing out the common factor $\partial$, Equation (4) becomes

\[ 3H \{- (5d^2 - 18d + 17) \text{Jac}(F, H, \Omega_{R}) - (5d - 9)(d - 2) \text{Jac}(F, H, \Omega_{F})\} + (d - 2)^2 \{-27H \text{Jac}(F, \Omega, H) + 20 \text{Jac}(F, \Psi, H)\} = 0. \]
Interchanging the last two rows of the determinants of the Jacobian matrices in the
last term changes their signs, and gives

\[
3H\left\{- (5d^2 - 18d + 17) \text{Jac}(F, H, \Omega_H) - (5d - 9)(d - 2) \text{Jac}(F, H, \Omega_F)\right\}
+ (d - 2)^2\{27H \text{Jac}(F, H, \Omega) - 20 \text{Jac}(F, H, \Psi)\} = 0.
\]

By the product rule \(\partial_x \Omega = \partial_x \Omega_H + \partial_x \Omega_F\) and likewise for \(y\) and \(z\), so
\[
\text{Jac}(F, H, \Omega) = \text{Jac}(F, H, \Omega_H) + \text{Jac}(F, H, \Omega_F).
\]

Using this, gathering terms and simplifying, we get the desired result.

\[\square\]

**Example 2.4.** Let \(C\) be the curve given by the defining polynomial
\[
F = x^4 - x^3y + y^3z.
\]
This curve has one cusp with multiplicity sequence \([3]\) and two simple inflection
points.

The defining polynomial for the 2-Hessian, computed with [Mau17, Program
B.3, p.69], is
\[
-4442639360x^3y^{18} + 33331979520x^2y^{19} - 11904278400xy^{20} + 1587237120y^{21}.
\]

The intersection points of \(H_2\) and \(C\) are
\[
p_1 = (0 : 0 : 1),
p_2 = \left(\frac{64}{24} : \frac{256}{3} : 1\right),
p_3 = \left(\frac{49}{24} + i\frac{\sqrt{7}}{24} : \frac{-637}{24} + i\frac{343\sqrt{7}}{48} : 1\right),
p_4 = \left(\frac{49}{24} - i\frac{\sqrt{7}}{24} : \frac{-637}{24} - i\frac{343\sqrt{7}}{48} : 1\right).
\]

The point \(p_1\) is the cusp, while \(p_2, p_3\) and \(p_4\) are sextactic points. With [Mau17,
Program B.2, pp. 67–68] we compute the osculating conics for the latter, and check
with [Map16] that
\[
(O_{p_2} \cdot C)_{p_2} = (O_{p_3} \cdot C)_{p_3} = (O_{p_4} \cdot C)_{p_4} = 6.
\]

A complete overview of this curve in terms of singularities, inflection points and
sextactic points, and intersections with associated curves, can be found in Table 1.

| Point \(p\) | \(m_p\) | \(\delta_p\) | \((T_p \cdot C)_p\) | \((O_p \cdot C)_p\) | \((H \cdot C)_p\) | \((H_2 \cdot C)_p\) |
|-----------|--------|--------|----------------|----------------|----------------|----------------|
| \((0 : 0 : 1)\) | [3]    | 3      | 4              | -              | 22             | 81             |
| \((8 : 16 : 1)\) |        | 0      | 3              | -              | 1              | 0              |
| \((0 : 1 : 0)\) |        | 0      | 3              | -              | 1              | 0              |
| \((\frac{64}{24} : \frac{256}{3} : 1)\) |        | 0      | 2              | 6              | 0              | 1              |
| \((\frac{49}{24} + i\frac{\sqrt{7}}{24} : \frac{-637}{24} + i\frac{343\sqrt{7}}{48} : 1)\) | | 0 | 2 | 6 | 0 | 1 |
| \((\frac{49}{24} - i\frac{\sqrt{7}}{24} : \frac{-637}{24} - i\frac{343\sqrt{7}}{48} : 1)\) | | 0 | 2 | 6 | 0 | 1 |

**Table 1.** Invariants and intersections for the rational cuspidal
quartic in Example 2.4

**Example 2.5.** In [AS09] Alwaleed and Sakai consider a family of smooth plane
quartics \(C_a\) of genus \(g = 3\), given by the polynomial
\[
F = x^4 + y^4 + z^4 + a(x^2y^2 + x^2z^2 + y^2z^2),
\]

where \(a\) is a parameter.
where \( a \neq -1, \pm 2 \). They study the 2-Weierstrass points of \( C_a \) by the Wronskian form and the order of zeros of this form.

Now, explicit calculation of the intersection between \( C_a \) and the 2-Hessian provides the same results. For example, in the case \( a = 14 \) we compute the 2-Hessian and find a total of 68 points in the intersection; eight 3-sextactic points and 60 1-sextactic points. The coordinates of the eight 3-sextactic points are

\[
\begin{align*}
p_1 &= \left( \frac{1}{2} + \frac{\sqrt{3}}{2} i : -\frac{1}{2} + \frac{\sqrt{3}}{2} i : 1 \right), \\
p_2 &= \left( \frac{1}{2} - \frac{\sqrt{3}}{2} i : -\frac{1}{2} - \frac{\sqrt{3}}{2} i : 1 \right), \\
p_3 &= \left( -\frac{1}{2} + \frac{\sqrt{3}}{2} i : \frac{1}{2} + \frac{\sqrt{3}}{2} i : 1 \right), \\
p_4 &= \left( -\frac{1}{2} - \frac{\sqrt{3}}{2} i : \frac{1}{2} - \frac{\sqrt{3}}{2} i : 1 \right), \\
p_5 &= \left( -\frac{1}{2} - \frac{\sqrt{3}}{2} i : -\frac{1}{2} + \frac{\sqrt{3}}{2} i : 1 \right), \\
p_6 &= \left( -\frac{1}{2} + \frac{\sqrt{3}}{2} i : -\frac{1}{2} - \frac{\sqrt{3}}{2} i : 1 \right), \\
p_7 &= \left( \frac{1}{2} - \frac{\sqrt{3}}{2} i : \frac{1}{2} + \frac{\sqrt{3}}{2} i : 1 \right), \\
p_8 &= \left( \frac{1}{2} + \frac{\sqrt{3}}{2} i : \frac{1}{2} - \frac{\sqrt{3}}{2} i : 1 \right).
\end{align*}
\]

At each of these points, computing the osculating conic and directly checking the intersection multiplicity in \([\text{Map16}]\) shows that \((O_{p_i} \cdot C)_{p_i} = 8\).

3. Sextactic point formulas

In this section we prove the formula for the number of sextactic points on a cuspidal curve in \([\text{Theorem 1.2}]\) on page \(2\). Moreover, we state a variation of this formula that reflects properties of the Hessian and 2-Hessian curves associated to the cuspidal curve.

3.1. Proof of \([\text{Theorem 1.2}]\) and applications. The key ingredient in our proof is a generalized Plücker formula by Ballico and Gatto in \([\text{BG97}]\) that gives a connection between invariants of \( C \) and the linear system \( Q \), and the total sum of Weierstrass weights of the \( Q \)-Weierstrass points on a curve. Note that this formula is simply a generalization of the Brill–Segre formula to the singular case. In our situation the proposition can be stated as follows.

**Proposition 3.1** (\([\text{BG97}]\) Proposition 3.4, p. 153). Let \( C \) be a projective, irreducible, cuspidal curve of geometric genus \( g \), let \( Q \) be a complete linear system of degree \( \deg Q \) and dimension \( r \), and let \( w_p(Q) \) denote the Weierstrass weight of \( C \) at \( p \) with respect to \( Q \). Then

\[
\sum_{p \in C} w_p(Q) = (r + 1)(\deg Q + rg - r).
\]

In the case of plane curves, to compute the Weierstrass weight \( w_p(Q) \) of a point \( p \in C \) with respect to a complete linear system \( Q \) of dimension \( r \), we use a technique by Notari from \([\text{Not99}]\) 24–26]. Assuming that \( C \) is cuspidal, Notari’s algorithm reduces to, for each point \( p \in C \), finding curves \( C_0, \ldots, C_r \) of degree \( n \) such that the intersection multiplicities at \( p \) are distinct. Subsequently, with \( h_i = (C \cdot C_i)_p \), the \( Q \)-Weierstrass weight of a unibranch point \( p \) can be expressed as

\[
w_p(Q) = \sum_{i=0}^{r} (h_i - i).
\]

To compute \( h_i \), we make use of the Puiseux parametrization of \( C \) at a point \( p \) \([\text{Fis01}]\) Cor. 7.7, p. 135], which ensures that a branch of \( C \) at \( p \) can be represented by

\[
(t^m : at^l + \cdots : 1),
\]

where \( m \) and \( l \) are as above, \( a \neq 0 \), and \( \cdots \) denotes higher order terms in \( t \).
Proof of Theorem 1.2. Let $C$ be a cuspidal curve of geometric genus $g$, and let $Q$ be the complete linear system of conics on $C$, with $\deg Q = 2d$ and $r = \dim Q = 5$. Hence, the right hand side of the formula in Proposition 3.1 reads

$$6(2d + 5g - 5).$$

For the left hand side of the formula, to compute the $Q$-Weierstrass weight of a point $p$, we proceed by considering two cases separately: The set of points for which $l \neq 2m$, and the set of points for which $l = 2m$.

The case $l \neq 2m$: This case includes the inflection points and some of the cusps, and coincides with the set $I$.

We choose the standard basis for $Q$ given by all degree 2 monomials in $x,y,z$, i.e.

$$x^2, y^2, z^2, yz, xz, xy,$$

and substitute the Puiseux parametrization of $C$ at $p$ into this basis. This gives

$$t^{2m}, a_2 t^{2l} + \cdots, 1, at^l + \cdots, t^m, at^{m+l} + \cdots .$$

By assumption $l \neq 2m$, hence the basis elements represent curves with distinct intersection multiplicities at $p$. Observing that

$$h_0 = 0, h_1 = m, h_2 = l, h_3 = 2m, h_4 = m + l, h_5 = 2l,$$

and inserting this into Equation (5) yields

$$w_p(Q) = \sum_{i=0}^{5} (h_i - i)$$

$$= 4m + 4l - 15.$$  

The case $l = 2m$: Note that this case includes all smooth points that are not inflection points and some of the cusps; the latter coincides with the set $J$.

First, as above, substitute the Puiseux parametrization of $C$ at $p$ into the standard basis. Since $l = 2m$, two of the orders of $t$ in the basis elements are equal,

$$t^{2m}, a_2 t^{2m} + \cdots, 1, a t^l + \cdots, t^m, a t^{3m} + \cdots .$$

If $p$ is a smooth point, with $m = 1$, there exists by Theorem 2.1 a unique irreducible conic $O_p$ that intersects $C$ at $p$ with intersection multiplicity $c$, where $c \geq 5$. If $p$ is a cusp on $C$, we may construct, by taking linear combinations of the basis elements, an irreducible conic that intersects $C$ at $p$ with intersection multiplicity $c$, for a $c \neq m, 2m, 3m, 4m$. This conic is not necessarily unique, but if there exists a family of such conics at $p$, it follows from explicit calculations with the Puiseux parametrization that $c$ is uniquely determined. Hence, for cusps where $l = 2m$, by an osculating conic to a curve at the cusp we refer to any member of this family.

In either case, we retrieve

$$h_0 = 0, h_1 = m, h_2 = 2m, h_3 = 3m, h_4 = 4m, h_5 = c.$$  

Thus, the $Q$-Weierstrass weight of $p$ is

$$w_p(Q) = 10m + c - 15.$$  

Note that if $p$ is smooth, then $w_p(Q)$ is equal to its type as a sextactic point, $s_p = c - 5$. The formula is valid even when $p$ is not sextactic, as in this case
$w_p(Q) = 0$. Thus, we get that the total number of sextactic points on $C$, counted with multiplicity, is

$$s = \sum_{p \in \text{Sext} C} s_p = \sum_{p \in \text{Sext} C} w_p(Q).$$

Putting this together while isolating $s$, we get

$$s = 6(2d + 5g - 5) - \sum_l (4m + 4l - 15) - \sum_j (10m + c - 15),$$

which is exactly what we wanted to show. □

Remark 2. Note that, under respective conditions, this formula coincides with the sextactic point formulas by Thorbergsson and Umehara [TU02, p. 90] and Coolidge [Coo31, Theorem 4, p. 280].

In the study of cuspidal curves, the curves are often given only by means of the degree $d$ and the multiplicity sequences of its cusps, $\mathbf{m} = [m, m_1, \ldots, 1]$. We now show that sometimes this information is enough to determine $l$ and $c$. In this case Theorem 1.2 can be applied directly, without knowledge of the defining polynomial and brute force calculations.

Lemma 3.2. With notation as above,

$$d \geq l = km + m_k \geq m + m_1$$

for some $k \geq 1$, with $m = m_1 = \ldots = m_{k-1}$.

Proof. Since $l$ represents the intersection of a curve and a line, it follows from Bézout’s theorem that $l \leq d$. Moreover, by [FZ96, Proposition 1.2, p. 440], $l = km + m_k$ for some $k \geq 1$, with $m = m_1 = \ldots = m_{k-1}$, and the last inequality follows. □

Lemma 3.3. With notation as above, in the case where $p$ is a cusp for which $l = 2m$,

$$2d \geq c = km + m_k > 2m$$

for some $k \geq 2$, with $m = m_1 = \ldots = m_{k-1}$.

Proof. Since $c$ represents the intersection of a curve and a conic, it follows from Bézout’s theorem that $c \leq 2d$. Moreover, since $O_p$ is irreducible, again by [FZ96, Proposition 1.2, p. 440], $c = km + m_k$ for some $k \geq 1$, with $m = m_1 = \ldots = m_{k-1}$. By construction $c \geq l$, and since $c \neq m, 2m, 3m, 4m$, it follows that $c > 2m$. □

Example 3.4. Consider the cuspidal quintic given by the defining polynomial

$$F = 27x^5 - 2x^2y^3 + 18x^2yz - y^4z + 2x^2y^2z^2 - x^2z^3.$$

A complete overview of this curve is given in Table 2, where the intersection multiplicities are computed explicitly. rational

This curve has four cusps and no inflection points. Note that this curve is particularly interesting, since it is, up to projective equivalence, the only known rational cuspidal curve with more than three cusps. Three of the cusps have identical properties; they have $m = 2$ and $l = 3$. The fourth cusp, with multiplicity sequence $[2, 4]$,
the number of sextactic points on this curve is 3.2. A corollary that ties everything together. As a corollary to Theorem 1.2 in this section we state a formula that reflects the intersection of a curve of degree $d$ with its 2-Hessian of degree $12d - 27$.

Corollary 3.5. Let $C$ be a cuspidal curve of genus $g$ and degree $d \geq 3$. Let $p$ be a point on $C$, $m$ its multiplicity, $T_p$ the tangent line to $C$ at $p$, and $O_p$ an osculating conic to $C$ at $p$. Moreover, let $l = (T_p \cdot C)_p$ and $c = (O_p \cdot C)_p$. Let $I$ denote the set of inflection points and cusps on $C$ where $l \neq 2m$, and let $J$ denote the set of cusps on $C$ where $l = 2m$. Finally, let $s$ denote the sextactic points on $C$, counted with multiplicity. Then the following equation holds:

$$d(12d - 27) = s + 24 \sum \delta_p + \sum_i (3m + 3l - 12) + \sum_j (7m + c - 12).$$

Before we prove Corollary 3.5 note that the formula could be interpreted as an application of Bézout’s theorem to $C$ and its 2-Hessian; the term $d(12d - 27)$ is simply the product of the degrees. The remaining terms are local in nature, and we claim in Conjecture 3.6 that these terms reflect a natural geometrical interpretation. We have verified that the conjecture holds for all rational cuspidal curves of degree 4 and 5; for examples, see the intersection tables for the curves appearing in this article. We emphasize that a similar result exists for the Hessian curve of a cuspidal curve [Moe13, Theorem 2.1.9, p.32].

Conjecture 3.6. The intersection multiplicity $(H_2 \cdot C)_p$ of a cuspidal curve $C$ and its 2-Hessian curve $H_2$ at any point $p$ is determined by the delta invariant $\delta$, the multiplicity $m$, the intersection with the tangent $l$, or the intersection with an osculating conic $c$. If $p$ is a point on $C$ such that $l \neq 2m$, then

$$(H_2 \cdot C)_p = 24\delta + 3m + 3l - 12.$$ 

If $p$ is a point on $C$ such that $l = 2m$, then

$$(H_2 \cdot C)_p = 24\delta + 7m + c - 12.$$ 

has $m = 2$ and $l = 4$, and by Lemma 3.3 we have $c = 7$. Hence, by Theorem 1.2 the number of sextactic points on this curve is

$$s = 6 \cdot (2 \cdot 5 + 5 \cdot 0 - 5) - 3 \cdot (4 \cdot 2 + 4 \cdot 3 - 15) - (10 \cdot 2 + 7 - 15)$$

$$= 30 - 15 - 12$$

$$= 3,$$

which agrees with the number of sextactic points on the curve described in Table 2.

Table 2. Invariants and intersections for the rational cuspidal quintic with four cusps in Example 3.4 on page 10.
Proof of Corollary 3.5. By substituting Clebsch’ formula for the genus of a plane curve \[ g = \frac{(d - 1)(d - 2)}{2} - \sum \delta_p, \]
into the formula from Theorem 1.2, we get
\[ s = 12d + 15(d - 1)(d - 2) - 30 \sum \delta_p - 30 - \sum_l (4m + 4l - 15) - \sum_J (10m + c - 15). \]
This reduces to
\[ s = 15d^2 - 33d - 30 \sum \delta_p - \sum_l (4m + 4l - 15) - \sum_J (10m + c - 15). \]
Moreover, we have the inflection point formula for cuspidal curves, explicitly stated in \[ \text{[Moe13, Theorem 2.1.8, p.32]}, \]
\[ v = 3d(d - 2) - \sum (6\delta_p + m + l - 3), \]
where \( v \) denotes the number of inflection points counted with multiplicity, and where the sum is taken over all cusps on \( C \). This can be rewritten
\[ 0 = 3d^2 - 6d - 6 \sum \delta_p - \sum_{l \in J} (m + l - 3), \]
By subtracting Equation (7) from Equation (6) and sorting terms, we reach the desired expression. \( \square \)

4. Sextactic points on rational curves

In this section we assume that \( C \) is a rational plane curve, i.e. \( g = 0 \) and \( C \) can be given by a parametrization
\[ \varphi(s, t) = (\varphi_0(s, t) : \varphi_1(s, t) : \varphi_2(s, t)), \text{ for } (s : t) \in \mathbb{P}^1. \]
We will exploit properties of this parametrization and show that the 2-Weierstrass points of a curve can be found in a natural way.

First in this section, we state a corollary to Theorem 1.2 for rational cuspidal curves, which is obtained by setting \( g = 0 \).

**Corollary 4.1.** Let \( C \) be a rational cuspidal curve of degree \( d \geq 3 \). Let \( p \) be a point on \( C \), \( m \) its multiplicity, \( T_p \) the tangent line to \( C \) at \( p \), and \( O_p \) an osculating conic to \( C \) at \( p \). Moreover, let \( l = (T_p \cdot C)_p \) and \( c = (O_p \cdot C)_p \). Let \( I \) denote the set of inflection points and cusps on \( C \) where \( l \neq 2m \), and let \( J \) denote the set of cusps on \( C \) where \( l = 2m \). Then the number of sextactic points \( s \) on \( C \), counted with multiplicity, is given by
\[ s = 6(2d - 5) - \sum_l (4m + 4l - 15) - \sum_J (10m + c - 15). \]

4.1. The osculating conic for rational curves. For a smooth point \( p \) that is not an inflection point on a rational curve, it is possible to compute the osculating conic \( O_p \) directly from the parametrization.
Theorem 4.2. Let \( C \) be a rational plane curve given by a parametrization \( \varphi(s,t) \), and let \( \omega(s,t) \) denote the determinant

\[
\omega(s,t) = \begin{vmatrix}
\frac{\partial^2 (\varphi_s^2)}{\partial s^2} & \frac{\partial^2 (\varphi_s^2)}{\partial s\partial t} & \frac{\partial^2 (\varphi_s^2)}{\partial t^2} & \frac{\partial^2 (\varphi_{s^2}s)}{\partial s^2} & \frac{\partial^2 (\varphi_{s^2}s)}{\partial s\partial t} & \frac{\partial^2 (\varphi_{s^2}s)}{\partial t^2} \\
\frac{\partial^2 (\varphi_s^2)}{\partial s\partial t} & \frac{\partial^2 (\varphi_s^2)}{\partial s^2} & \frac{\partial^2 (\varphi_s^2)}{\partial s\partial t} & \frac{\partial^2 (\varphi_{s^2}s)}{\partial s^2} & \frac{\partial^2 (\varphi_{s^2}s)}{\partial s\partial t} & \frac{\partial^2 (\varphi_{s^2}s)}{\partial t^2} \\
\frac{\partial^2 (\varphi_s^2)}{\partial s\partial t} & \frac{\partial^2 (\varphi_s^2)}{\partial s\partial t} & \frac{\partial^2 (\varphi_s^2)}{\partial s\partial t} & \frac{\partial^2 (\varphi_{s^2}s)}{\partial s^2} & \frac{\partial^2 (\varphi_{s^2}s)}{\partial s\partial t} & \frac{\partial^2 (\varphi_{s^2}s)}{\partial t^2} \\
\frac{\partial^2 (\varphi_s^2)}{\partial t^2} & \frac{\partial^2 (\varphi_s^2)}{\partial t^2} & \frac{\partial^2 (\varphi_s^2)}{\partial t^2} & \frac{\partial^2 (\varphi_{s^2}s)}{\partial s^2} & \frac{\partial^2 (\varphi_{s^2}s)}{\partial s\partial t} & \frac{\partial^2 (\varphi_{s^2}s)}{\partial t^2} \\
\frac{\partial^2 (\varphi_s^2)}{\partial s^2} & \frac{\partial^2 (\varphi_s^2)}{\partial s\partial t} & \frac{\partial^2 (\varphi_s^2)}{\partial t^2} & \frac{\partial^2 (\varphi_{s^2}s)}{\partial s^2} & \frac{\partial^2 (\varphi_{s^2}s)}{\partial s\partial t} & \frac{\partial^2 (\varphi_{s^2}s)}{\partial t^2} \\
\frac{\partial^2 (\varphi_s^2)}{\partial s^2} & \frac{\partial^2 (\varphi_s^2)}{\partial s\partial t} & \frac{\partial^2 (\varphi_s^2)}{\partial t^2} & \frac{\partial^2 (\varphi_{s^2}s)}{\partial s^2} & \frac{\partial^2 (\varphi_{s^2}s)}{\partial s\partial t} & \frac{\partial^2 (\varphi_{s^2}s)}{\partial t^2}
\end{vmatrix}.
\]

Then, for a smooth point \( p = \varphi(s_0, t_0) \), \( \omega(s_0, t_0) \in \mathbb{C}[x, y, z]_2 \) is the defining polynomial of the osculating conic \( \mathcal{O}_p \) to \( C \) at \( p \).

Proof. Let \( v_2(C) \subseteq \mathbb{P}^5 \) denote the image of \( C \) under the 2nd Veronese embedding of \( \mathbb{P}^2 \) to \( \mathbb{P}^5 \), such that

\[
v_2(C)(s,t) = (\varphi_0^2 : \varphi_1^2 : \varphi_2^2 : \varphi_0 \varphi_2 : \varphi_0 \varphi_1).
\]

With \((x_0 : x_1 : x_2 : x_3 : x_4 : x_5)\) denoting the coordinates of \( \mathbb{P}^5 \), consider the determinant \( \bar{\omega}(s,t) \),

\[
\bar{\omega}(s,t) = \begin{vmatrix}
\frac{\partial (x_0)}{\partial s} & \frac{\partial (x_1)}{\partial s} & \frac{\partial (x_2)}{\partial s} & \frac{\partial (x_3)}{\partial s} & \frac{\partial (x_4)}{\partial s} & \frac{\partial (x_5)}{\partial s} \\
\frac{\partial (x_0)}{\partial t} & \frac{\partial (x_1)}{\partial t} & \frac{\partial (x_2)}{\partial t} & \frac{\partial (x_3)}{\partial t} & \frac{\partial (x_4)}{\partial t} & \frac{\partial (x_5)}{\partial t} \\
\frac{\partial (x_0)}{\partial s^2} & \frac{\partial (x_1)}{\partial s^2} & \frac{\partial (x_2)}{\partial s^2} & \frac{\partial (x_3)}{\partial s^2} & \frac{\partial (x_4)}{\partial s^2} & \frac{\partial (x_5)}{\partial s^2} \\
\frac{\partial (x_0)}{\partial s\partial t} & \frac{\partial (x_1)}{\partial s\partial t} & \frac{\partial (x_2)}{\partial s\partial t} & \frac{\partial (x_3)}{\partial s\partial t} & \frac{\partial (x_4)}{\partial s\partial t} & \frac{\partial (x_5)}{\partial s\partial t} \\
\frac{\partial (x_0)}{\partial s\partial t} & \frac{\partial (x_1)}{\partial s\partial t} & \frac{\partial (x_2)}{\partial s\partial t} & \frac{\partial (x_3)}{\partial s\partial t} & \frac{\partial (x_4)}{\partial s\partial t} & \frac{\partial (x_5)}{\partial s\partial t} \\
\frac{\partial (x_0)}{\partial t^2} & \frac{\partial (x_1)}{\partial t^2} & \frac{\partial (x_2)}{\partial t^2} & \frac{\partial (x_3)}{\partial t^2} & \frac{\partial (x_4)}{\partial t^2} & \frac{\partial (x_5)}{\partial t^2}
\end{vmatrix}.
\]

For a smooth point \( v_2(C)(s_0, t_0) \), the linear polynomial \( \bar{\omega}(s_0, t_0) \) defines a unique osculating hyperplane to \( v_2(C) \) in \( \mathbb{P}^5 \), and this hyperplane corresponds to the osculating conic to \( C \) at \( p = \varphi(s_0, t_0) \), with defining polynomial \( \omega(s_0, t_0) \). \( \square \)

For a smooth point \( p \in C \), the osculating conic \( \mathcal{O}_p \) is unique, so for rational curves and points that are not inflection points, the conic from Theorem 4.2 and Cayley’s osculating conic from Theorem 2.1 obviously coincide, as can be seen in the following example.

Example 4.3. Consider the rational cuspidal quintic curve with parametrization

\[
\varphi(s,t) = \left(s^4 - \frac{1}{2}s^5 : s^3t^2 : -\frac{3}{2}st^4 + t^5\right),
\]

and defining polynomial

\[
F = 9xy^4 - 4y^5 - 24x^2y^2z + 48xy^3z - 16y^4z + 16x^3z^2.
\]
This curve has three cusps with multiplicity sequences [3], [2], and [2]. The osculating conic to $C$ at a smooth point $\varphi(s,t)$ of this curve, is given as

$$Ks^{10}t(15s^{12}z^{2} - 70s^{11}t^{2} + 270s^{11}t^{2} + 130s^{11}t^{2} - 432s^{10}t^{2}x + 324s^{10}t^{2}y - 1584s^{10}t^{2}y - 120s^{10}t^{2}y + 2610s^{9}t^{4}y^{2} + 55s^{9}t^{4}y^{2} - 6540s^{8}t^{5}y^{2} + 9900s^{8}t^{5}y^{2} - 5000s^{7}t^{5}y^{2} - 480s^{7}t^{5}y^{2} + 3630s^{7}t^{5}y^{2} - 1800s^{7}t^{5}y^{2} - 12150s^{4}t^{5}y^{2} + 324s^{3}t^{5}y^{2} + 450s^{3}t^{5}y^{2} + 7320s^{4}t^{5}y^{2} - 6030s^{3}t^{5}y^{2} - 3920s^{3}t^{5}y^{2} + 1080s^{2}t^{5}y^{2} + 2640s^{2}t^{5}y^{2} + 560s^{2}t^{5}y^{2} - 585st^{12}x^{2} - 480st^{12}x^{2} + 126t^{13}x^{2}).$$

where $K = -286773903360$. Now pick a smooth point on the curve, for example $p = (2,1) = (0 : -4 : 1)$, then the expression above becomes

$$K'(\frac{256}{3}x^{2} + \frac{131072}{27}x^{2} - \frac{131072}{27}y^{2} - \frac{606280}{27}y^{2} + \frac{5120}{9}xy - \frac{327680}{27}z^{2}),$$

where $K' = 123886326251520$.

Using Cayley's implicit formula from [Theorem 2.1], we get, up to the constant $K'$, the same defining polynomial for the osculating conic $O_{p}$ to $C$ at this point,

$$\frac{256}{3}x^{2} + \frac{131072}{27}x^{2} - \frac{131072}{27}y^{2} - \frac{606280}{27}y^{2} + \frac{5120}{9}xy - \frac{327680}{27}z^{2}.$$

4.2. The Weierstrass weight. We now show by direct computation that for rational curves, not necessarily cuspidal, the Weierstrass weight of points on $C$ with respect to the linear system of conics can be computed as the order of zeros of a homogeneous polynomial.

**Theorem 4.4.** Let $C$ be a rational plane curve with parametrization $\varphi(s,t)$, and let $\xi(s,t)$ denote the Wronski determinant

$$\xi(s,t) = \begin{vmatrix} \frac{\partial \varphi(s)}{\partial s} & \frac{\partial \varphi(t)}{\partial s} & \frac{\partial \varphi(s)}{\partial s} & \frac{\partial \varphi(t)}{\partial s} & \frac{\partial \varphi(s)}{\partial s} & \frac{\partial \varphi(t)}{\partial s} \\ \frac{\partial \varphi(s)}{\partial t} & \frac{\partial \varphi(t)}{\partial t} & \frac{\partial \varphi(s)}{\partial t} & \frac{\partial \varphi(t)}{\partial t} & \frac{\partial \varphi(s)}{\partial t} & \frac{\partial \varphi(t)}{\partial t} \\ \frac{\partial \varphi(s)}{\partial s} & \frac{\partial \varphi(t)}{\partial s} & \frac{\partial \varphi(s)}{\partial s} & \frac{\partial \varphi(t)}{\partial s} & \frac{\partial \varphi(s)}{\partial s} & \frac{\partial \varphi(t)}{\partial s} \\ \frac{\partial \varphi(s)}{\partial t} & \frac{\partial \varphi(t)}{\partial t} & \frac{\partial \varphi(s)}{\partial t} & \frac{\partial \varphi(t)}{\partial t} & \frac{\partial \varphi(s)}{\partial t} & \frac{\partial \varphi(t)}{\partial t} \\ \frac{\partial \varphi(s)}{\partial s} & \frac{\partial \varphi(t)}{\partial s} & \frac{\partial \varphi(s)}{\partial s} & \frac{\partial \varphi(t)}{\partial s} & \frac{\partial \varphi(s)}{\partial s} & \frac{\partial \varphi(t)}{\partial s} \\ \frac{\partial \varphi(s)}{\partial t} & \frac{\partial \varphi(t)}{\partial t} & \frac{\partial \varphi(s)}{\partial t} & \frac{\partial \varphi(t)}{\partial t} & \frac{\partial \varphi(s)}{\partial t} & \frac{\partial \varphi(t)}{\partial t} \end{vmatrix}.$$

Moreover, let $(s_{i}, t_{i})$ denote the distinct zeros of $\xi(s,t)$, $i \leq 6(2d - 5)$. Then the points $p_{i} = \varphi(s_{i}, t_{i})$ are the 2-Weierstrass points on $C$, and the Weierstrass weight $w_{p_{i}}(Q)$, is equal to the order of the zero of $\xi(s,t)$ corresponding to $(s_{i}, t_{i})$.

**Proof.** First observe that whenever $\xi(s,t)$ vanishes, the corresponding point on $v_{2}(C)$ is either singular, or there exists a hyperplane in $\mathbb{P}^{5}$ that is hyperosculating to $v_{2}(C)$. As before, this hyperplane corresponds to a hyperosculating conic to $C$ in $\mathbb{P}^{2}$, hence determining an inflection point or a sextactic point.

For a smooth curve, [Mir95] Chapter VII, Section 4, pp. 233–246] ensures that the multiplicity of a zero of $\xi(s,t)$ equals the Weierstrass weight of the corresponding point. Note that this is the same as the flattening points of the Veronese embedding, as described in [Arn90, 15]. This takes care of the smooth points. Alternatively, the below formula for singular points can be applied to smooth points.
In the case of singular points, we consider each branch separately, and perform a local computation. So choose one branch and perform a linear transformation on \( C \) so that the chosen branch of \( p \) corresponds to the parameter values \((s : t) = (1 : 0)\), and so that its tangent is \( y = 0 \). Moreover, by abuse of notation, observe that

\[
\xi(1, t) = \xi(t) = \begin{vmatrix}
\psi_2(C)(t) \\
\psi_2(C)'(t) \\
\psi_2(C)''(t) \\
\psi_2(C)^3(t) \\
\psi_2(C)^4(t) \\
\psi_2(C)^5(t)
\end{vmatrix}.
\]

Assume first that the chosen branch can be parametrized by \((t^m : at^l + \ldots : 1)\), with \( a \neq 0 \) and \( l \neq 2m \). Substituting this into \( \xi(t) \), computing the determinant, and comparing with the proof of Theorem 1.2 in Section 3.1, it follows that

\[
\text{ord}_t \xi(t) = 4m + 4l - 15 = \sum_{i=0}^{5} (h_i - i).
\]

If \( l = 2m \), first transform the branch of \( C \) at \( p \) so that it is given by the parametrization \((t^m : at^{2m} + \ldots : 1)\), where \( a \neq 0 \). Subsequently, by applying the Veronese embedding, consider the curve

\[
\rho(t) = (t^{2m} : a^2 t^{4m} + \ldots : 1 : t^m : at^{2m} + \ldots : at^{3m} + \ldots)
\]
in \( \mathbb{P}^5 \), which by a suitable linear transformation in \( \mathbb{P}^5 \) can be given by the parametrization

\[
\sigma(t) = (t^{2m} : a^2 t^{4m} + \ldots : 1 : t^m : bt^c + \ldots : at^{3m} + \ldots),
\]
for a \( b \neq 0 \) and \( c \neq m, 2m, 3m, 4m \). Then, for a parametrized curve \( \psi \) in \( \mathbb{P}^5 \), consider the determinant

\[
W_\psi(t) = \begin{vmatrix}
\psi(t) \\
\psi'(t) \\
\psi''(t) \\
\psi^{(4)}(t) \\
\psi^{(5)}(t)
\end{vmatrix}.
\]

Straight forward computations gives that the order of \( t \) in \( W_\psi(t) \) is \( 10m + c - 15 \). Obviously, \( \text{ord}_t W_\sigma(t) = \text{ord}_t W_\psi(t) = \text{ord}_t \xi(t) \), hence \( \text{ord}_t \xi(t) = 10m + c - 15 \). Moreover, notice that in this case the inverse image of the hyperplane \( x_4 = 0 \) under the linear transformation in \( \mathbb{P}^5 \) and the Veronese embedding corresponds to a conic in \( \mathbb{P}^2 \) that intersects the branch of \( C \) at \( p \) with intersection multiplicity \( h_5 = c \). Hence, we have that \( \text{ord}_t \xi(t) = \sum_{i=0}^{5} (h_i - i) \).

Performing a similar analysis of all branches of \( C \) at \( p \), and summing up, we reach \( \nu_p(Q) \).

**Remark 3.** Observe that the polynomial \( \xi \) is homogeneous in \( s, t \) of degree \( 6(2d - 5) \). Hence, by summing the Weierstrass weights, Theorem 4.4 provides another proof of Corollary 4.1.

We now look at two examples where we compare the Weierstrass weights found using either intersections or the order of zeros of the Wronskian determinant.

**Example 4.5.** Let \( C \) be the rational cuspidal quintic with parametrization

\[
\varphi(s, t) = (s^5 : s^5 t^2 : st^4 + t^5),
\]
and defining polynomial

\[ F = y^5 + 2x^2y^2z - x^3z^2 - xy^4. \]

| Point \( p \) | \( \overline{m}_p \) | \( \delta_p \) | \( (T_p \cdot C)_p \) | \( (O_p \cdot C)_p \) | \( (H \cdot C)_p \) | \( (H_2 \cdot C)_p \) |
|---------------|---------|--------|-----------|----------|---------|----------|
| \( (0 : 0 : 1) \) | \([3,2]\) | 4      | 5         | -        | 29      | 108      |
| \( (1 : 0 : 0) \) | \([2]\)  | 2      | 4         | 5        | 15      | 55       |
| \( (759375 : 3375 : 1) \) | 0      | 3      | -         | 1        | 0       | 0        |
| \( p_4 \) | 0      | 2      | 6         | 0        | 1       |          |
| \( p_5 \) | 0      | 2      | 6         | 0        | 1       |          |

Table 3. Invariants and intersections for the rational cuspidal quintic in Example 4.5 on page 15.

By explicit calculations, this curve has two cusps, one inflection point and two sextactic points, see Table 3. These are all the Weierstrass points for the linear system of conics on \( C \).

Computing the Wronski determinant gives

\[ \xi(s, t) = Ks^{17}t^{10}(192s^3 + 1680s^2t + 5275st^2 + 5250t^3), \]

where \( K = -535188929406566400 \).

Since \( C \) is cuspidal, each of the coordinates of the points on \( C \) correspond uniquely to a pair of parameters \((s : t)\). The cusps \( p_1 \) and \( p_2 \) correspond to \((0 : 1)\) and \((1 : 0)\), respectively, while the inflection point \( p_3 \), and the sextactic points \( p_4 \) and \( p_5 \) correspond to zeros of \( 192s^3 + 1680s^2t + 5275st^2 + 5250t^3 \). Thus, we conclude that \( w_{p_1}(Q) = 17 \), \( w_{p_2}(Q) = 10 \), and \( w_{p_3}(Q) = w_{p_4}(Q) = w_{p_5}(Q) = 1 \).

Using the information in Table 3 we verify this. For \( p_1 \), \( l \neq 2m \), so \( w_{p_1}(Q) = 4 \cdot 3 + 4 \cdot 5 - 15 = 17 \). For \( p_2 \), \( l = 2m \), and by Lemma 3.3 we have that \( c = 5 \), so \( w_{p_2}(Q) = 10 \cdot 2 + 5 - 15 = 10 \). The inflection point \( p_3 \) is simple, hence \( w_{p_3}(Q) = 4 \cdot 1 + 4 \cdot 3 - 15 = 1 \). The sextactic points have 2-Weierstrass weight equal to their sextactic type, \( w_{p_4}(Q) = w_{p_5}(Q) = 1 \).

Next, we consider a curve with cusps with the same multiplicity sequences as in Example 4.5 and we notice that the 2-Weierstrass points are different.

**Example 4.6.** Let \( C \) be the rational cuspidal quintic with parametrization

\[ \varphi(s, t) = (s^5 : s^3t^2 : t^5), \]

and defining polynomial

\[ F = x^3z^2 - y^5. \]

The cusps are the only 2-Weierstrass points of this curve.

| Point \( p \) | \( \overline{m}_p \) | \( \delta_p \) | \( (T_p \cdot C)_p \) | \( (O_p \cdot C)_p \) | \( (H \cdot C)_p \) | \( (H_2 \cdot C)_p \) |
|---------------|---------|--------|-----------|----------|---------|----------|
| \( (0 : 0 : 1) \) | \([3,2]\) | 4      | 5         | -        | 29      | 108      |
| \( (1 : 0 : 0) \) | \([2]\)  | 2      | 5         | -        | 16      | 57       |

Table 4. Invariants and intersections for the rational cuspidal quintic in Example 4.6.

The cusp with coordinates \( p_1 = (0 : 0 : 1) \) corresponds to the parameter value \((0 : 1)\), and its Weierstrass weight is \( w_{p_1}(Q) = 4 \cdot 3 + 4 \cdot 5 - 15 = 17 \). The cusp
with coordinates \( p_2 = (1 : 0 : 0) \) corresponds to the parameter value \((1 : 0)\), and its Weierstrass weight is \( w_{p_2}(Q) = 4 \cdot 2 + 4 \cdot 5 - 15 = 13 \).

Computing the Wronskian gives
\[
\xi(s, t) = -2809741879384473600000s^{17}t^{13},
\]
and again the orders of zeros of \( \xi \) equals the Weierstrass weight of the respective points.

**Remark 4.** Note that Example 4.6 provides an example of a cuspidal curve \( C_{m,l} \) of degree \( l \) with defining polynomial
\[
F = x^m z^l - y^l.
\]
For any \( l, m \in \mathbb{N} \), with \( m < l \) and \( \gcd(m, l) = 1 \), the curve \( C_{m,l} \) is cuspidal; bicuspidal when \( 1 < m < l - 1 \), and unicuspidal with an inflection point otherwise. It can be shown that these curves have no other \( n \)-Weierstrass points for \( 2 \leq n < l \).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OSLO, P.O. BOX 1053 BLINDERN, NO-0316
OSLO, NORWAY
E-mail address: paulamau@math.uio.no

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