MONODROMY OF THE $SL_2$ HITCHIN FIBRATION

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Abstract. We calculate the monodromy action on the mod 2 cohomology for $SL(2, \mathbb{C})$ Hitchin systems and give an application of our results in terms of the moduli space of semistable $SL(2, \mathbb{R})$ Higgs bundles.

Let $\Sigma$ be a Riemann surface of genus $g \geq 3$, and denote by $K$ the canonical bundle. An $SL(2, \mathbb{C})$ Higgs bundle as defined by Hitchin [7] and Simpson [8] is given by a pair $(E, \Phi)$, where $E$ is a rank 2 holomorphic vector bundle with $\text{det}(E) = \mathcal{O}_\Sigma$ and the Higgs field $\Phi$ is a section of $\text{End}_0(E) \otimes K$, for $\text{End}_0$ the bundle of traceless endomorphisms. A Higgs bundle is said to be semistable if for any subbundle $F \subset E$ such that $\Phi(F) \subset F \otimes K$ one has $\deg(F) \leq 0$. When $\deg(F) < 0$ we say the pair is stable.

Considering the moduli space $\mathcal{M}$ of $S$-equivalence classes of semistable $SL(2, \mathbb{C})$ Higgs bundles and the map $\Phi \mapsto \text{det}(\Phi)$, one may define the so-called Hitchin fibration [7]:

$$h : \mathcal{M} \to \mathcal{A} := H^0(\Sigma, K^2).$$

The moduli space $\mathcal{M}$ is homeomorphic to the moduli space of reductive representations of the fundamental group of $\Sigma$ in $SL(2, \mathbb{C})$ via non-abelian Hodge theory [7, 8, 4, 5]. The involution on $SL(2, \mathbb{C})$ corresponding to the real form $SL(2, \mathbb{R})$ defines an antiholomorphic involution on the moduli space of representations which, in the Higgs bundle complex structure, is the holomorphic involution $\sigma : (E, \Phi) \mapsto (E, -\Phi)$. In particular, the isomorphism classes of stable Higgs bundles fixed by the involution $\sigma$ correspond to $SL(2, \mathbb{R})$ Higgs bundles $(E = V \oplus V^*, \Phi)$, where $V$ is a line bundle on $\Sigma$, and the Higgs field $\Phi$ is given by

$$\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$$

for $\beta : V^* \to V \otimes K$ and $\gamma : V \to V^* \otimes K$.

We shall denote by $\mathcal{M}_{\text{reg}}$ the regular fibres of the Hitchin map $h$, and let $\mathcal{A}_{\text{reg}}$ be the regular locus of the base, which is given by quadratic differentials with simple zeros. From [7] one knows that the smooth fibres are tori of real dimension $6g-6$. There is a section of the fibration fixed by $\sigma$ and this allows us to identify each fibre with an abelian variety, in fact, a Prym variety. The involution $\sigma$ leaves invariant $\text{det}(\Phi)$ and so defines an involution on each fibre. The fixed points then become the elements of order 2 in the abelian variety. Hence, the points corresponding to $SL(2, \mathbb{R})$ Higgs bundles give a covering space of $\mathcal{A}_{\text{reg}}$. This covering space is determined by the action of $\pi_1(\mathcal{A}_{\text{reg}})$ on the first cohomology of the fibres.

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with \( \mathbb{Z}_2 \) coefficients. In this paper we study this action, and thus obtain information about the moduli space of \( SL(2, \mathbb{R}) \) representations of \( \pi_1(\Sigma) \). Our main result is the following:

**Theorem 1.** The monodromy action on the first mod 2 cohomology of the fibres of the Hitchin fibration is given by the group of matrices acting on \( \mathbb{Z}_2^{6g-6} \) of the form

\[
\begin{pmatrix}
I_{2g} & A \\
0 & \pi
\end{pmatrix},
\]

where

- \( \pi \) is the quotient action on \( \mathbb{Z}_2^{4g-5}/(1, \cdots, 1) \) induced by the permutation action of the symmetric group \( S_{4g-4} \) on \( \mathbb{Z}_2^{4g-5} \).
- \( A \) is any \((2g) \times (4g-6)\) matrix with entries in \( \mathbb{Z}_2 \).

Finally, we give an application of our result in terms of geometric properties of the moduli space of semistable \( SL(2, \mathbb{R}) \) Higgs bundles. In particular, we show the following:

**Corollary 2.** The number of connected components of the moduli space of semistable \( SL(2, \mathbb{R}) \) Higgs bundles is \( 2^{2g} + g \).

These connected components are known to be parametrized by the Euler class of the associated flat \( \mathbb{R}P^1 \) bundle. From our point of view this number \( k \) relates to the orbit of a subset of \( 1, 2, \ldots, 4g - 4 \) with \( 2g - 2k - 2 \) points under the action of the symmetric group. In a later work we shall extend this approach to the group \( SU(p, p) \).

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### 1. THE REGULAR FIBRES OF THE HITCHIN FIBRATION

Let us consider the Hitchin map \( h : \mathcal{M} \to \mathcal{A} \) given by \((E, \Phi) \mapsto \det(\Phi)\). From [7, Theorem 8.1] the map \( h \) is proper and surjective, and its regular fibres are abelian varieties. Moreover, for any \( a \in \mathcal{A} - \{0\} \) the fibre \( \mathcal{M}_a \) is connected [6, Theorem 8.1]. For any isomorphism class of \((E, \Phi)\) in \( \mathcal{M} \), one may consider the zero set of its characteristic polynomial

\[
\det(\Phi - \eta I) = \eta^2 + a = 0,
\]

where \( a = \det(\Phi) \in \mathcal{A} \). This defines a spectral curve \( S \) in the total space of \( K \), for \( \eta \) the tautological section of \( \rho^* K \) where \( \rho : S \to \Sigma \) is the projection. Note that the curve \( S \) is non-singular over \( \mathcal{A}_{reg} \), and the ramification points are given by the intersection of \( S \) with the zero section. The curve \( S \) has a natural involution \( \tau(\eta) = -\eta \) and thus we can define the Prym variety \( \text{Prym}(S, \Sigma) \) as the set of line bundles \( M \in \text{Jac}(S) \) which satisfy

\[
\tau^* M \cong M^*.
\]

**Proposition 3.** [7, Theorem 8.1] The fibres of \( \mathcal{M}_{reg} \) are isomorphic to \( \text{Prym}(S, \Sigma) \).
To see this, given a line bundle $L$ on $S$ one may consider the rank two vector bundle $E := \rho_* L$ and construct an associated Higgs bundle as follows. Given an open set $U \subset \Sigma$, multiplication by the tautological section $\eta$ gives a homomorphism

$$\eta : H^0(\rho^{-1}(U), L) \to H^0(\rho^{-1}(U), L \otimes \rho^* K).$$

By the definition of a direct image sheaf, we have

$$H^0(U, E) = H^0(\rho^{-1}(U), L).$$

Hence, equation (2) can then be seen as

$$\Phi : H^0(U, E) \to H^0(U, E \otimes K),$$

defining the Higgs field $\Phi \in H^0(\Sigma, \text{End}_0 E \otimes K)$. Note that the map $\Phi$ is traceless as it satisfies its characteristic equation, which by construction is $\eta^2 + a = 0$.

Any line bundle $M$ on the curve $S$ such that $\tau^* M \cong M^*$ satisfies $\text{Nm}(M) = 0$. From [1] one has that

$$\Lambda^2 E = \text{Nm}(L) \otimes K^{-1}.$$

For $K^{1/2}$ a choice of square root, $M = L \otimes \rho^* K^{-1/2}$ is in the Prym variety and hence the vector bundle $E$ has trivial determinant:

$$\Lambda^2 E = \text{Nm}(M \otimes \rho^* K^{1/2}) \otimes K^{-1} = \mathcal{O}.$$

2. A COMBINATORIAL APPROACH TO MONODROMY

The Gauss-Manin connection on the cohomology of the fibres of $\mathcal{M}_{\text{reg}} \to \mathcal{A}_{\text{reg}}$ defines the monodromy action for the Hitchin fibration. As each regular fibre is a torus, the monodromy is generated by the action of $\pi_1(\mathcal{A}_{\text{reg}})$ on $H^1(\text{Prym}(S, \Sigma), \mathbb{Z})$. The generators and relations of the monodromy action for hyperelliptic surfaces were studied from a combinatorial point of view by Copeland in [3, Theorem 1.1]. Furthermore, by [9, Section 4] one may extend these results to any compact Riemann surface:

**Theorem 4 ([3, 9]).** To each compact Riemann surface $\Sigma$ of genus greater than 2, one may associate a graph $\Gamma$ with edge set $E$ and a skew bilinear pairing $< e, e' >$ on edges $e, e' \in \mathbb{Z}[E]$ such that

1. the monodromy representation of $\pi_1(\mathcal{A}_{\text{reg}})$ acting on $H_1(\text{Prym}(S, \Sigma), \mathbb{Z})$ is generated by elements $\sigma_e$ labelled by the edges $e \in E$,
2. one can define an action of $\pi_1(\mathcal{A}_{\text{reg}})$ on $e' \in \mathbb{Z}[E]$ given by
   $$\sigma_e(e') = e' - < e', e >,$$
3. the monodromy representation of the action of $\pi_1(\mathcal{A}_{\text{reg}})$ on $H_1(\text{Prym}(S, \Sigma), \mathbb{Z})$ is a quotient of this module $\mathbb{Z}[E]$. 


In order to construct the graph $\bar{\Gamma}$ Copeland looks at the particular case of $\Sigma$ given by the non-singular compactification of the zero set of $y^2 = f(x) = x^{2g+2} - 1$. Firstly, by considering $\omega \in A$ given by
\[
\omega = (x - 2\zeta^2)(x - 2\zeta^4)(x - 2\zeta^6)(x - 2\zeta^8) \prod_{9 \leq j \leq 2g+2} (x - 2\zeta^j) \left( \frac{dx}{y} \right)^2,
\]
for $\zeta = e^{2\pi i/2g+2}$, it is shown in [3, Section 7] how interchanging two zeros of the differential provides information about the generators of the monodromy. Then, by means of the ramification points of the surface, a dual graph to $\bar{\Gamma}$ for which each zero of $\omega$ is in a face could be constructed. Copeland’s analysis extends to any element in $A_{reg}$ over a hyperelliptic curve [3, Section 23]. Moreover, by work of Walker [9, Section 4] the above construction can be done for any compact Riemann surface.

**Remark 5.** Following [3, Section 6], we shall consider the graph $\bar{\Gamma}$ whose $4g - 4$ vertices are given by the ramification divisor of $\rho: S \to \Sigma$, i.e., the zeros of $a = \det(\Phi)$.

As an example, for genus $g = 3, 5,$ and $10$, the graph $\bar{\Gamma}$ is given by:

![Graphs](image)

For $g > 3$ the graph $\bar{\Gamma}$ is given by a ring with 8 triangles next to each other, $2g - 6$ quadrilaterals and $4g - 4$ vertices. In this case, we shall label its edges as follows:

![Labeled Graph](image)

**Figure 1.**

Considering the lifted graph of $\bar{\Gamma}$ in the curve $S$ over $\Sigma$, Copeland could show the following:
Proposition 6. [3, Theorem 11.1] If $E$ and $F$ are respectively the edge and face sets of $\tilde{\Gamma}$, then there is an induced homeomorphism

$$\text{Prym}(S, M) \cong \frac{\mathbb{R}[E]}{\left(\mathbb{R}[F] + \frac{1}{2}\mathbb{Z}[E]\right)},$$

where the inclusion $\mathbb{R}[F] \subset \mathbb{R}[E]$ is defined by the following relations involving the boundaries of the faces:

$$\bar{x}_1 := \sum_{i=1}^{2g-2} l_i ; \quad \bar{x}_2 := \sum_{i=1}^{2g-2} u_i ;$$

$$\bar{x}_3 := \sum_{\text{even} \geq 6} u_i - \sum_{\text{odd} \geq 5} l_i + \sum_{i=1}^{2g+2} b_i ;$$

$$\bar{x}_4 := l_1 + l_3 - u_2 - u_4 + \sum_{\text{odd}} u_i - \sum_{\text{even}} l_i + \sum_{i=1}^{2g+2} b_i .$$

Note that $\mathbb{R}[F] \cap \frac{1}{2}\mathbb{Z}[E]$ can be understood by considering the following sum:

$$\bar{x}_3 + \bar{x}_4 + \bar{x}_1 - \bar{x}_2 = 2 \left( l_1 + l_3 - u_2 - u_4 + \sum_{i=1}^{2g+2} b_i \right) = 2\bar{x}_5 .$$

Although the summands above are not individually in $\mathbb{R}[F] \cap \frac{1}{2}\mathbb{Z}[E]$, when summed they satisfy

$$\bar{x}_5 \in \mathbb{R}[F] \cap \frac{1}{2}\mathbb{Z}[E] .$$

Remark 7. For $g = 2$ it is known that $\pi_1(A_{reg}) \cong \mathbb{Z} \times \pi_1(S^2_6)$, where $S^2_6$ is the sphere $S^2$ with 6 holes (e.g. [3, Section 6]).

3. THE FIXED POINTS OF $(E, \Phi) \mapsto (E, -\Phi)$

The direct image of the trivial bundle $O$ in $\text{Prym}(S, \Sigma)$ is given by $\rho_* O = O \oplus K$. So for $L = \rho^* K^{-1/2}$ one has

$$\rho_* \rho^* K^{-1/2} = K^{-1/2} \otimes \rho_* O = K^{-1/2} \oplus K^{1/2} .$$

It follows from Section 4 that the line bundle $O \in \text{Prym}(S, \Sigma)$ has an associated Higgs bundle $(K^{-1/2} \oplus K^{1/2}, \Phi_a)$, where the Higgs field $\Phi_a$ is obtained via Proposition 3:

$$\Phi_a = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix} , \quad \text{for} \quad a \in H^2(\Sigma, K^2) .$$

Note that the automorphism

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

conjugates $\Phi_a$ to $-\Phi_a$ and so the equivalence class of the Higgs bundle is fixed by the involution $\sigma$. Thus, this family of Higgs bundles defines an origin in the set of fixed points on each fibre.
An infinitesimal deformation of a Higgs bundle is given by \((A, \dot{\Phi})\) where \(A \in \Omega^0(\text{End}_0E)\) and \(\dot{\Phi} \in \Omega^0(\text{End}_0E)\). The holomorphic involution \(\sigma\) on \(\mathcal{M}\) induces an involution on the tangent space \(T\) of \(\mathcal{M}\) at a fixed point of \(\sigma\). Moreover, there is a natural symplectic form \(\omega\) defined on the infinitesimal deformations by

\[
\omega((A_1, \dot{\Phi}_1), (A_2, \dot{\Phi}_2)) = \int_\Sigma \text{tr}(\dot{A}_1 \dot{\Phi}_2 - \dot{A}_2 \dot{\Phi}_1).
\]

As the trace is invariant under \(\sigma\) and \(\sigma(\Phi_1) = -\Phi_1\), the induced involution on the tangent space maps \(\omega \mapsto -\omega\). It follows that the ±1-eigenspaces \(T_{\pm}\) of this involution are isotropic and complementary, and hence Lagrangian. Let us denote by \(Dh\) the derivative of the map \(h\), which maps the tangent spaces of \(\mathcal{M}\) to the tangent space of \(A\). As \(h\) is invariant under the involution \(\sigma\), the eigenspace \(T_-\) is contained in the kernel of \(Dh\). Since the derivative is surjective at a regular point, its kernel has dimension \(\dim(\mathcal{M})/2\) and thus it equals \(T_-\). Then, \(Dh\) is an isomorphism from \(T_+\) to the tangent space of the base. Since \(h\) is a proper submersion on the fixed point set, it defines a covering space. The tangent space to the identity in the Prym variety is acted as \(-1\) by the involution \(\sigma\) and as the Prym variety is connected, by exponentiation the action of \(\sigma\) on the regular fibres corresponds to \(x \mapsto -x\). Hence, the points of order two in the fibres of \(\mathcal{M}_{\text{reg}}\) over the regular locus \(\mathcal{A}_{\text{reg}}\) correspond to stable \(SL(2, \mathbb{R})\) Higgs bundles.

By Proposition 6 one may describe the Prym variety as \(\text{Prym}(S, \Sigma) \cong \mathbb{R}^{6g-6}/\wedge\), where

\[
\wedge := \frac{\mathbb{Z}/[E]}{\mathbb{R}[E] \cap \frac{1}{2}\mathbb{Z}[E]}.
\]

In particular, one has \(\wedge \cong H_1(\text{Prym}(S, \Sigma), \mathbb{Z})\). Let us denote by \(P[2]\) the elements of order 2 in \(\text{Prym}(S, \Sigma)\), which are equivalent classes in \(\mathbb{R}^{6g-6}\) of points \(x\) such that \(2x \in \wedge\). Then, \(P[2]\) is given by \(\frac{1}{2}\wedge\) modulo \(\wedge\) and as \(\wedge\) is torsion free,

\[
P[2] \cong \wedge/2\wedge \cong H_1(\text{Prym}(S, \Sigma), \mathbb{Z}_2).
\]

Moreover, \(H^1(\text{Prym}(S, \Sigma), \mathbb{Z}_2) \cong \text{Hom}(H_1(\text{Prym}(S, \Sigma), \mathbb{Z}), \mathbb{Z}_2)\) and thus

\[
H^1(\text{Prym}(S, \Sigma), \mathbb{Z}_2) \cong \text{Hom}(\wedge, \mathbb{Z}_2) \cong \wedge/2\wedge \cong P[2].
\]

The monodromy action on \(H^1(\text{Prym}(S, \Sigma), \mathbb{Z}_2)\) is equivalent to the action on \(P[2]\), the space of elements of order 2 in \(\text{Prym}(S, \Sigma)\). Note that over \(\mathbb{Z}_2\), the equations for \(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4\) and \(\bar{x}_5\) are equivalent to

\[
x_1 := \sum_{i=1}^{2g-2} l_i; \quad x_2 := \sum_{i=1}^{2g-2} u_i;
\]

\[
x_3 := l_1 + l_3 + u_2 + u_4 + \sum_{\text{even}} u_i + \sum_{\text{odd}} l_i + \sum_{i=1}^{2g+2} b_i;
\]

\[
x_4 := l_1 + l_3 + u_2 + u_4 + \sum_{\text{odd}} u_i + \sum_{\text{even}} l_i + \sum_{i=1}^{2g+2} b_i;
\]

\[
x_5 := l_1 + l_3 + u_2 + u_4 + \sum_{i=1}^{2g+2} b_i.
\]
Proposition 8. The space $P[2]$ is given by the quotient of $\mathbb{Z}_2[E]$ by the subspace generated by $x_1, x_2, x_4$ and $x_5$.

4. THE ACTION ON $\mathbb{Z}_2[E]$

It is convenient to consider $\mathbb{Z}_2[E]$ as the space of 1-chains $C_1$ for a subdivision of the annulus in Figure 1. The boundary map $\partial$ to the space $C_0$ of 0-chains (spanned by the vertices of $\Gamma$) is defined on an edge $e \in C_1$ with vertices $v_1, v_2$ as $\partial e = v_1 + v_2$. Let $\Sigma^{[4g-4]}$ be the configuration space of $4g-4$ points in $\Sigma$. Then, there is a natural map $p : A_{reg} \to \Sigma^{[4g-4]}$ which takes a quadratic differential to its zero set. Furthermore, $p$ induces the following maps

$$\pi_1(A_{reg}) \to \pi_1(\Sigma^{[4g-4]}) \to S_{4g-4},$$

where $S_{4g-4}$ is the symmetric group of $4g-4$ elements. Thus there is a natural permutation action on $C_0$ and Copeland’s generators in $\pi_1(A_{reg})$ map to transpositions in $S_{4g-4}$. Concretely, these generators are defined as transformations of $\mathbb{Z}_2[E]$ as follows. The action $\sigma_e$ labelled by the edge $e$ on another edge $x$ is

$$\sigma_e(x) = x + <x, e>e,$$

where $<\cdot, \cdot>$ is the intersection pairing. As this pairing is skew over $\mathbb{Z}$, for any edge $e$ one has $<e, e> = 0$. Let $G_1$ be the group of transformations of $C_1$ generated by $\sigma_e$, for $e \in E$. Then, one can see the following:

Proposition 9. The group $G_1$ acts trivially on $Z_1 = \ker(\partial : C_1 \to C_0)$.

Proof. Let us consider $a \in C_1$ such that $\partial a = 0$, i.e., the edges of $a$ have vertices which occur an even number of times. By definition, $\sigma_e \in G_1$ acts trivially on $a$ for any edge $e \in E$ non adjacent to $a$. Furthermore, if $e \in E$ is adjacent to $a$, then $\partial a = 0$ implies that an even number of edges in $a$ is adjacent to $e$, and thus the action $\sigma_e$ is also trivial on $a$. □

We shall give an ordering to the vertices in $\Gamma$ as in the figure below, and denote by $E' \subset E$ the set of dark edges:

![Figure 1: Annulus with vertices labeled and dark edges indicated.](image)

For $(i, j)$ the edge between the vertices $i$ and $j$, the set $E'$ is given by $e_{4g-4} := (4g-4, 1)$ together with the natural succession of edges $e_i := (i, i + 1)$ for $i = 1, \ldots, 4g-5$. 
Proposition 10. The reflections labelled by the edges in $E' \subset E$ generate a subgroup $S'_{4g-4}$ of $G_1$ isomorphic to the symmetric group $S_{4g-4}$.

Proof. To show this result, one needs to check that the following properties characterizing generators of the symmetric group apply to the reflections labelled by $E'$:

(i) $\sigma_{e_i}^2 = 1$ for all $i$,
(ii) $\sigma_{e_i}\sigma_{e_j} = \sigma_{e_j}\sigma_{e_i}$ if $j \neq i \pm 1$,
(iii) $(\sigma_{e_i}\sigma_{e_{i+1}})^3 = 1$.

By equation (6), it is straightforward to see that the properties (i) and (ii) are satisfied by $\sigma_{e_i}$ for all $e_i \in E$. In order to check (iii) we shall consider different options for edges adjacent to $e_i$ and $e_{i+1}$ when $e_i, e_{i+1} \in E'$. Let $c_1, c_2, \cdots, c_n \in E$ be the edges adjacent to $e_i$ and $e_{i+1}$, where $n$ may be 5, 6 or 7. Taking the basis $\{c_1, \cdots, c_n, e_i, e_{i+1}\}$, the action $\sigma_{e_i}\sigma_{e_{i+1}}$ is given by matrices $B$ divided into blocks in the following manner:

$$B := \begin{pmatrix}
I_n & 0 & 0 \\
\vdots & \vdots & \vdots \\
0 & 0 & 0 \\
n & a_1 & a_n \\
b & b_1 & b_n
\end{pmatrix},$$

where the entries $a_i$ and $b_j$ are 0 or 1, depending on the number of common vertices with the edges and their locations. Over $\mathbb{Z}_2$ one has that $B^3$ is the identity matrix and so property (iii) is satisfied for all edges $e_i \in E'$.

The subgroup $S'_{4g-4}$ preserves $\mathbb{Z}_2[E']$. Furthermore, the boundary $\partial : C_1 \to C_0$ is compatible with the action of $S'_{4g-4}$ on $C_1$ and $S_{4g-4}$ on $C_0$. Thus, from Proposition 10, there is a natural homomorphism

$$\alpha : G_1 \longrightarrow S_{4g-4},$$

which is an isomorphism when restricted to $S'_{4g-4}$. For $N := \ker \alpha$ one has

$$G_1 = N \ltimes S'_{4g-4}.$$  

Any element $g \in G_1$ can be expressed uniquely as $g = s \cdot h$, for $h \in N \subset G_1$ and $s \in S'_{4g-4} \subset G_1$. The group action on $C_1$ may then be expressed as

$$\left( h_1s_1 \right) \left( h_2s_2 \right) = h_1s_1h_2s_1^{-1}s_1s_2,$$

for $s_1, s_2 \in S'_{4g-4}$ and $h_1, h_2 \in N$. In order to understand the subgroup $N$, we define $E_0 := E - E'$ and let $\Delta_e \subset C_1$ denote the boundary of the square or triangle adjacent to the edge $e \in E_0$ in $\hat{G}$. Note that each edge $e \in E_0$ is contained in only one of such boundaries. We shall write $\hat{E}_0 := \{\Delta_e \text{ for } e \in E_0\}$. From Proposition 9 the boundaries $\Delta_e$ are acted on trivially by $G_1$.

By comparing the action labelled by the edges in $E_0$ with the action of the corresponding transposition in $S'_{4g-4}$ one can find out which elements are in $N$. We shall denote by $\sigma_{(i,j)}$ the action on $C_1$ labeled by the edge $(i, j)$. Furthermore, we consider $s_{(i,j)}$ the element in $S_{4g-4}$ such that $\alpha(\sigma_{(i,j)}) = \alpha(s_{(i,j)}) \in S_{4g-4}$, where $\alpha(s_{(i,j)})$ interchanges the vertices of $(i, j)$. Then, we have the following possibilities:
Theorem 11. The action of \( \sigma_e \) labelled by \( e \) is given by
\[
\sigma_e(x) = h_e \cdot s_e(x),
\]
where \( s_e \in S_{4g-4}' \) is the element which maps under \( \alpha \) to the transposition of the two vertices of \( e \), and the action of \( h_e \in N \) is given by
\[
h_e(x) = \begin{cases} 
  x & \text{if } e \in E', \\
  x+<e,x> & \text{if } e \in E_0.
\end{cases}
\]

Proof. For \( e \in E' \), one has \( \sigma_e \in S_{4g-4}' \). Furthermore, one can see that the action of \( \sigma_e \) labelled by \( e \in E_0 \) on an adjacent edge \( x \) is given by
\[
\sigma_e(x) = x+<e,x> \in \Delta_e.
\]
From Proposition 9 the boundary \( \Delta_e \) is acted on trivially by \( G_1 \) and thus the above action is given by \( h_e \cdot s_e(x) = h_e(\sigma_e(x) + \Delta_e) = h_e(x') + <e,x'> \in \Delta_e = \sigma_e(x) \).

Remark 12. Note that for \( e,e' \in E_0 \), the maps \( h_e \) and \( h_{e'} \) satisfy
\[
h_e h_{e'}(x) = x+<e',x'> \Delta_e' + <e,x> \Delta_e.
\]

In order to construct a representation for the action of \( \pi_1(\mathcal{A}_{reg}) \) on \( C_1 \), we shall begin by studying the image \( B_0 \) and the kernel \( Z_1 \) of \( \partial : C_1 \to C_0 \).

5. THE REPRESENTATION OF \( G_1 \)

For \( y = (y_1, \ldots, y_{4g-4}) \in C_0 \), we define the linear map \( f : C_0 \to \mathbb{Z}_2 \) by
\[
f(y) = \sum_{i=1}^{4g-4} y_i.
\]

Proposition 13. The image \( B_0 \) of \( \partial : C_1 \to C_0 \) is formed by elements with an even number of 1's, i.e., \( B_0 = \ker f \).

Proof. It is clear that \( B_0 \subset \ker f \). In order to check surjectivity we consider the edges \( e_i \in C_1 \) given by \( e_i = (i,i+1) \in E' \), for \( i = 1, \ldots, 4g-5 \). Given the elements \( R^k \in B_0 \) for \( k = 2, \ldots, 4g-4 \) defined as
\[
R^2 := \partial e_1 = (1,1,0,\ldots,0),
\]
\[
\vdots
\]
\[
R^k := R^{k-1} + \partial e_{k-1} = (1,0,\ldots,0,1,0,\ldots,0),
\]
one may generate any distribution of an even number of 1’s. Hence, \( B_0 = \text{span}\{R^k\} \) which is the kernel of \( f \).

**Proposition 14.** The dimensions of the image \( B_0 \) and the kernel \( Z_1 \) of \( \partial : C_1 \rightarrow C_0 \) are, respectively, \( 5g - 5 \) and \( 2g + 3 \).

**Proof.** From Proposition 13 one has that \( \dim(B_0) = \dim(\ker f) = 4g - 5 \). Furthermore, as \( \dim C_1 = \dim Z_1 + \dim B_0 \), the kernel \( Z_1 \) of \( \partial \) has dimension \( 2g + 3 \). \( \square \)

Note that \( x_1, x_2, x_4 \in Z_1 \) and \( x_5 \notin Z_1 \). From a homological viewpoint one can see that \( x_4 \) and \( \Delta_e \) for \( e \in E_0 \) form a basis for the kernel \( Z_1 \). We can extend this to a basis of \( C_1 \) by taking the edges

\[ \beta' := \{ e_i = (i, i + 1) \text{ for } 1 \leq i \leq 4g - 5 \} \subset E' , \]

whose images under \( \partial \) form a basis for \( B_0 \), and hence a basis for a complementary subspace \( V \) of \( Z_1 \). We shall denote by \( \beta := \{ \beta_0, \beta' \} \) the basis of \( C_1 \) where \( \beta_0 = \{ \tilde{E}_0, x_4 \} \).

In order to generate the whole group \( G_1 \), we shall study the action of \( S'_{4g-4} \) by conjugation on \( N \). Considering the basis \( \beta \) one may construct a matrix representation of the maps \( h_{e_i} \) for \( e_i \in E = \{ E_0, E' \} \).

**Proposition 15.** For \( e \in E \), the matrix \([h_e] \) associated to \( h_e \) in the basis \( \beta \) is given by

\[
[h_e] = \begin{pmatrix}
I_{2g+3} & A_e \\
0 & I_{4g-5}
\end{pmatrix},
\]

where the \((2g + 3) \times (4g - 5)\) matrix \( A_e \) satisfies one of the following:

- it is the zero matrix for \( e \in E' \),
- it has only four non-zero entries in the intersection of the row corresponding to \( \Delta_e \) and the columns corresponding to an adjacent edge of \( e \) for \( e \in E_0 - \{ u_5, l_6 \} \),
- it has three non-zero entries in the intersection of the row corresponding to \( \Delta_e \) and the columns corresponding to an adjacent edge of \( e \) for \( e = u_5, l_6 \).

**Proof.** As we have seen before, for \( e \in E_0 \) the map \( h_e \) acts as the identity on the elements of \( \beta_0 \). Furthermore, any edge \( e \in E_0 - \{ u_5, l_6 \} \) is adjacent to exactly four edges in \( \beta' \). In this case \( h_e \) has exactly four non-zero elements in the intersection of the row corresponding to \( \Delta_e \) and the columns corresponding to edges in \( \beta' \) adjacent to \( e \). In the case of \( e = u_5, l_6 \), the edge \( e \) is adjacent to exactly 3 edges in \( \beta' \) and thus \( h_{e} \) has only 3 non-zero entries. \( \square \)

Let us recall that \( S'_{4g-4} \) preserves the space spanned by \( E' \), hence also the subspace \( V \), and acts trivially on \( Z_1 \). In the basis \( \beta \) the action of an element \( s \in S'_{4g-4} \) has a matrix representation given by

\[
[s] = \begin{pmatrix}
I_{2g+3} & 0 \\
0 & \pi_s
\end{pmatrix},
\]

where \( \pi \) is the permutation action corresponding to \( s \). Hence, for \( f \in E_0 \) we may construct the matrix for a conjugate of \( h_f \) as follows:

\[
[s][h_f][s]^{-1} = \begin{pmatrix}
I_{2g+3} & 0 \\
0 & \pi_s
\end{pmatrix} \begin{pmatrix}
I_{2g+3} & A_f \\
0 & I_{4g-5}
\end{pmatrix} \begin{pmatrix}
I_{2g+3} & 0 \\
0 & \pi_s^{-1}
\end{pmatrix} = \begin{pmatrix}
I_{2g+3} & A_f \pi_s^{-1} \\
0 & I_{4g-5}
\end{pmatrix}.
\]
Proposition 16. The normal subgroup $N \subset G_1$ consists of all matrices of the form

$$H = \begin{pmatrix} I_{2g+3} & A \\ 0 & I_{4g-5} \end{pmatrix}$$

where $A$ is any matrix whose rows corresponding to $\Delta_e$ for $e \in E_0 - \{u_5, l_6\}$ have an even number of 1’s, the row corresponding to $x_4$ is zero and the rows corresponding to $\Delta_{u_5}, \Delta_{l_6}$ have any distribution of 1’s.

Proof. Given $e_i \in E_0 - \{u_5, l_6\}$, we know that the matrix $A_{e_i}$ has only four non-zero entries in the row corresponding to $\Delta_{e_i}$. Thus, for $g > 2$, there exist elements $s_1, s_2 \in S_{4g-4}'$ with associated permutations $\pi_1$ and $\pi_2$ such that the matrix $\tilde{A}_{e_i} := A_{e_i} \pi_1^{-1} + A_{e_i} \pi_2^{-1}$ has only two non-zero entries in the row corresponding to $\Delta_{e_i}$, given by the vector $R^5$ defined in the proof of Proposition 13. Furthermore, by Remark 12, we have

$$[s_1 h_{e_i} s_1^{-1} s_2 h_{e_i} s_2^{-1}] = \begin{pmatrix} I_{2g+3} & A_{e_i} \pi_1^{-1} \\ 0 & I_{4g-5} \end{pmatrix} \begin{pmatrix} I_{2g+3} & A_{e_i} \pi_2^{-1} \\ 0 & I_{4g-5} \end{pmatrix} = \begin{pmatrix} I_{2g+3} & \tilde{A}_{e_i} \\ 0 & I_{4g-5} \end{pmatrix}.$$ 

Considering different $s \in S_{4g-4}'$ acting on $s_1 h_{e_i} s_1^{-1} s_2 h_{e_i} s_2^{-1}$, one can obtain the matrices $\{A^k_{e_i}\}_{k=2}$ with $R^5$ as the only non-zero row corresponding to $\Delta_{e_i}$. Thus, by composing the elements of $N$ to which each $A^k_{e_i}$ corresponds, we can obtain any possible distribution of an even number of 1’s in the only non-zero row.

Similar arguments can be used for the matrices corresponding to $u_5, l_6$ which in this case may have any number of 1’s in the only non-zero row. Recalling Remark 12 we are then able to generate any matrix $A \in N$ as described in the proposition. \qed

From the decomposition of $G_1$ and the results obtained previously, we have the following theorem:

Theorem 17. The representation of $\sigma \in G_1$ in the basis $\beta$ is given by

$$[\sigma] = \begin{pmatrix} I & A \\ 0 & \pi \end{pmatrix},$$

where $\pi$ represents a permutation on $\mathbb{Z}^{4g-5}_2$ and $A$ is any matrix whose rows corresponding to $\Delta_e$ for $e \in E_0 - \{u_5, l_6\}$ have an even number of 1’s, the row corresponding to $x_4$ is zero and the rows corresponding to $\Delta_{u_5}$ and $\Delta_{l_6}$ have any distribution of 1’s.

6. THE MONODROMY ACTION OF $\pi_1(\mathcal{A}_{reg})$ ON $P[2]$

As seen previously, $P[2]$ can be obtained as the quotient of $C_1$ by the four relations $x_1, x_2, x_4$ and $x_5$. It is important to note that these relations are preserved by the action of $\pi_1(\mathcal{A}_{reg})$. Furthermore, $x_1, x_2, x_4 \in Z_1$ and $\partial x_5 = (1, 1, \cdots, 1) \in B_0$. Hence, we have the following maps

$$\mathbb{Z}_2^{2g} \cong \frac{Z_1}{< x_1, x_2, x_4 >} \to P[2] \to \frac{B_0}{< (1, 1, \cdots, 1) >} \cong \mathbb{Z}_2^{4g-6}. \quad (12)$$
The monodromy group $G$ is given by the action on the quotient $P[2]$ induced by the action of $G_1$ on $C_1$. Note that $x_1, x_2, x_4 = 0$ imply

$$
\Delta_{l_6} = \sum_{l_i \in E_0 - \{l_6\}} \Delta_{l_i}; \quad \Delta_{u_5} = \sum_{l_i \in E_0 - \{u_5\}} \Delta_{u_i}.
$$

Moreover, as $x_1 + x_2 + x_4 + x_5 = 0$, one can express the edge $u_6$ in terms of elements in $\beta' - \{u_6\}$. For $\tilde{\beta}_0 := \beta_0 - \{x_4, \Delta_{l_6}, \Delta_{u_5}\}$ and $\tilde{\beta}' := \beta' - \{u_6\}$, let $\tilde{\beta} := \{\tilde{\beta}_0, \tilde{\beta}'\}$. Then one obtains the following:

**Proposition 18.** The elements in $\tilde{\beta}_0$ generate $Z_4/\langle x_1, x_2, x_4 \rangle$ and $\tilde{\beta}'$ generates a complementary subspace in $P[2]$.

From the previous analysis, one has the following explicit description of the monodromy action on $P[2]$:

**Theorem 1.** The representation of $\sigma \in G$ in the basis $\tilde{\beta}$ is given by

$$
[\sigma] = \begin{pmatrix} I_{2g} & A \\ 0 & \pi \end{pmatrix},
$$

where

- $\pi$ is the quotient action on $\mathbb{Z}_2^{4g-5}/(1, \cdots, 1)$ induced by the permutation representation on $\mathbb{Z}_2^{4g-5}$,
- $A$ is any $(2g) \times (4g - 6)$ matrix with entries in $\mathbb{Z}_2$.

**Proof.** From the above analysis, the action of $G$ on $B_0/\langle \partial x_5 = (1,1,\cdots,1) \rangle$ is given by the quotient action of the symmetric group. Furthermore, replacing $\Delta_{u_5}$ and $\Delta_{l_6}$ by the sums in (13), one can use similar arguments to the ones in Proposition 16 to obtain any number of 1’s in all the rows of the matrix $A$. \qed

**Remark 19.** We have seen in Proposition 9 that the action of $G_1$ is trivial on $Z_1$. Moreover, the space $\mathbb{Z}_2[x_1, x_2, x_4, x_5]$ is preserved by the action of $G_1$, and thus one can see that the induced monodromy action on the $2g$-dimensional subspace $Z_1/\mathbb{Z}_2[x_1, x_2, x_4, x_5]$ is trivial. Geometrically, there are $2^{2g}$ sections of the Hitchin fibration given by choices of the square root of $K$. These sections meet each Prym in $2^{2g}$ points which also lie in $P[2]$. Since we can lift a closed curve by a section, these are acted on trivially by the monodromy.

7. AN APPLICATION

The bundle of $P[2]$ is a finite covering of $\mathcal{A}_{\text{reg}}$ of degree $2^{6g-6}$ and hence each of its connected components corresponds to a maximal integrable submanifold. Considering two points $p$ and $q$ which are in the same orbit under the monodromy action, there is a path in $\mathcal{A}_{\text{reg}}$ whose action connects them. The horizontal lift of the path in $\mathcal{A}_{\text{reg}}$ is a path in $P[2]$ which connects these two points. Hence $p$ and $q$ are in the same connected component of the fixed point submanifold of $\sigma : (V, \Phi) \mapsto (V, -\Phi)$. 
7.1. The orbits of the monodromy action. We shall study now the orbits \( G(s, x) \) of each \((s, x) \in P[2] \cong \mathbb{Z}_2^{2g} \oplus \mathbb{Z}_2^{4g-6} \) under the action of \( G \). Note that for \( g \in G \), its action on \((s, x)\) is given by

\[
g \cdot (s, x)^t = \begin{pmatrix} I & A \\ 0 & \pi \end{pmatrix} \begin{pmatrix} s \\ x \end{pmatrix} = \begin{pmatrix} s + Ax \\ \pi x \end{pmatrix}.
\]

Proposition 20. The action of \( G \) on \( P[2] \) has \( 2^{2g} + g - 1 \) different orbits.

Proof. The matrices \( A \) have any possible number of 1’s in each row, and so for \( x \neq 0 \) any \( s' \in \mathbb{Z}_2^{2g} \) may be written as \( s' = s + Ax \) for some \( A \). Hence, the number of orbits \( G(s, x) \) for \( x \neq 0 \) is determined by the number of orbits of the action in \( \mathbb{Z}_2^{4g-6} \) defined by \( \xi : x \rightarrow \pi x \). The map \( \xi \) permutes the non-zero entries of \( x \) and thus the orbits of this action are given by elements with the same number of 1’s.

From equation (12), the space \( \mathbb{Z}_2^{4g-6} \) can be thought of as vectors in \( \mathbb{Z}_2^{4g-4} \) with an even number of 1’s, modulo \((1, \ldots, 1)\). Thus, for \( x \in \mathbb{Z}_2^{4g-6} \) and \( x \neq 0 \), each orbit \( \xi_x \) is defined by a constant \( m \) such that \( x \) has \( 2m \) non-zero entries, for \( 0 < 2m \leq 4g - 4 \). Let us call \( \bar{x} \in \mathbb{Z}_2^{4g-6} \) the element defined by the constant \( \bar{m} \) such that \( 2\bar{m} = (4g - 4) - 2m \). With this notation, we can see that \( \bar{x} \) and \( x \) belong to the same equivalence class in \( \mathbb{Z}_2^{4g-6} \). Note that for \( m \neq g - 1, 2g - 2 \), the corresponding \( x \) is equivalent to \( \bar{x} \) under \( \xi \) and thus in this case there are \( g - 2 \) equivalent classes \( \xi_x \). Then, considering the equivalence class for \( m = g - 1 \), one has \( g - 1 \) different classes for the action of \( \xi \). From equation (15), the action of \( G \) on an element \((s, 0) \in P[2] \) is trivial. Thus, in this case one has \( 2^{2g} \) different orbits of \( G \). \( \square \)

Let us recall that the fixed point set in \( \mathcal{M} \) of the involution \( \sigma \) is given by the moduli space of semistable \( SL(2, \mathbb{R}) \) Higgs bundles. We shall now check that no connected component of the fixed point set of \( \sigma \) lies entirely over the discriminant locus of \( \mathcal{A} \).

7.2. Connected components of the fixed point set of \( \sigma \). We shall study the connected components corresponding to stable and strictly semistable \( SL(2, \mathbb{R}) \) Higgs bundles.

Note that for a stable \( SL(2, \mathbb{C}) \) Higgs bundle \((E, \Phi)\) whose isomorphism class is fixed by the involution \( \sigma : (E, \Phi) \mapsto (E, -\Phi) \), there is an automorphism \( \alpha : E \rightarrow E \) whose action by conjugation on \( \Phi \) is given by

\[
\alpha^{-1} \Phi \alpha \mapsto -\Phi.
\]

As \( \alpha^2 \) commutes with \( \Phi \), by \cite{7}, Proposition 3.15 it acts as \( \lambda^2 \) for some \( \lambda \in \mathbb{C}^* \). Furthermore, \( \alpha \) has constant eigenvalues \( \pm \lambda \) and thus \( E \) can be decomposed into the corresponding eigenspaces \( V \) and \( V^* \). Then, the Higgs field can be expressed as

\[
\Phi = \begin{pmatrix} a & \beta \\ \gamma & -a \end{pmatrix} \in H^0(\Sigma, \text{End}_0(E) \otimes K).
\]

From this decomposition and the action of \( \alpha \), one has that necessarily \( \lambda = \pm i \) and \( a = 0 \). In particular, \( \sigma \) acts on \( E \) via transformations of the form

\[
\pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\]

In the case of strictly semistable Higgs bundles, the following Proposition applies:
Proposition 21. Any point representing a strictly semistable Higgs bundle in $\mathcal{M}$ fixed by $\sigma$ is in the connected component of a Higgs bundle with zero Higgs field.

Proof. The moduli space $\mathcal{M}$ is the space of $S$-equivalence classes of semistable Higgs bundles. A strictly semistable $SL(2, \mathbb{C})$ Higgs bundle $(E, \Phi)$ is represented by $E = V \oplus V^*$ for a degree zero line bundle $V$, and

$$\Phi = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \text{ for } a \in H^0(\Sigma, K).$$

If $V^2$ is nontrivial, only automorphisms fix $\Phi$ so this is an $SL(2, \mathbb{R})$ Higgs bundle only for $\Phi = 0$, and corresponds to a flat connection with holonomy in $SO(2) \subset SL(2, R)$. If $V^2$ is trivial then the automorphism $(u, v) \mapsto (v, -u)$ takes $\Phi$ to $-\Phi$, corresponding to a flat bundle with holonomy in $\mathbb{R}^* \subset SL(2, \mathbb{R})$. By scaling $\Phi$ to zero this is connected to the zero Higgs field. The differential $a$ can be continuously deformed to zero by considering $ta$ for $0 \leq t \leq 1$. Hence, by stability of line bundles, one can continuously deform $(E, \Phi)$ to a Higgs bundle with zero Higgs field via strictly semistable pairs. $\square$

In the case of stable $SL(2, \mathbb{R})$ Higgs bundles we have the following result:

Proposition 22. Any stable $SL(2, \mathbb{R})$ Higgs bundle is in a connected component which intersects $\mathcal{M}_{reg}$.

Proof. Let $(E = V \oplus V^*, \Phi)$ be a stable $SL(2, \mathbb{R})$ Higgs bundle with $d := \deg(V) \geq 0$ and

$$\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \in H^0(\Sigma, \text{End}_0(E) \otimes K).$$

Stability implies that the section $\gamma \in H^0(\Sigma, V^{-2}K)$ is non-zero, and thus $0 \leq 2d \leq 2g - 2$. Moreover, the section $\beta$ of $V^2K$ can be deformed continuously to zero.

The section $\gamma$ defines a divisor $[\gamma]$ in the symmetric product $S^{2g-2-2d}\Sigma$. As this space is connected, one can continuously deform the divisor $[\gamma]$ to any $[\tilde{\gamma}]$ composed of distinct points. For $a \in \mathcal{A}_{reg}$ with zeros $x_1, \ldots, x_{4g-4}$, we may deform $[\gamma]$ to $[\tilde{\gamma}]$ given by the points $x_1, \ldots, x_n \in \Sigma$ for $n := 2g - 2 - 2d$, and such that $\tilde{\gamma}$ is a section of $U^{-2}K$ for some line bundle $U$.

The complementary zeros $x_{n+1}, \ldots, x_{4g-4}$ of $a$ form a divisor of $U^2K$. Any section $\tilde{\beta}$ with this divisor can be reached by continuously deforming $[\beta]$ from zero to the set $x_{n+1}, \ldots, x_{4g-4}$. Hence, we may continuously deform any stable pair $(V \oplus V^*, \Phi = \{\beta, \gamma\})$ to $(U \oplus U^*, \Phi = \{\tilde{\beta}, \tilde{\gamma}\})$ in $\mathcal{A}_{reg}$. $\square$

The above analysis establishes that the number of connected components of the fixed point set of the involution $\sigma$ on $\mathcal{M}$ is less than or equal to the number of orbits of the monodromy action in $P[2]$. From [7, Section 10], a flat $SL(2, \mathbb{R})$ Higgs bundle has an associated $\mathbb{RP}^1$ bundle whose Euler class $0 \leq k \leq g - 1$ is a topological invariant. In particular, $SL(2, \mathbb{R})$ Higgs bundles with different Euler class lie in different connected components of the fixed point set of $\sigma$. Moreover, for $k = g - 1$ one has $2^{2g}$ connected components corresponding to the so-called Hitchin components. Hence, the lower bound to the number of connected components of the fixed point set of the involution $\sigma$ is $2^{2g} + g - 1$. As this lower bound
equals the number of orbits of the monodromy action on the fixed points of \( \sigma \) on \( \mathcal{M}_{\text{reg}} \), one has that the closures of these orbits can not intersect. Hence, the number of connected components of the fixed points of the involution \( \sigma : (V, \Phi) \mapsto (V, -\Phi) \) on \( \mathcal{M}_{\text{reg}} \) is equal to the number of orbits of the monodromy action on the points of order two of the regular fibres \( \mathcal{M}_{\text{reg}} \), i.e. \( 2^{2g} + g - 1 \). Note that for \( \Phi = 0 \) there is a connected component (namely the moduli space of stable bundles) which does not intersect any non-singular fibres and hence does not appear in the description of the orbits of the monodromy action. From the above analysis, one has the following:

**Corollary 2.** The number of connected components of the moduli space of semistable \( SL(2, \mathbb{R}) \) Higgs bundles is \( 2^{2g} + g \).

The construction of the orbits of the monodromy action provides a decomposition of the \( 4g - 4 \) zeros of \( \det \Phi \) via the \( 2m \) non-zero entries in Proposition 20. An element \( M \) of order two in the Prym variety has the property that \( \tau^* M \cong M \). Considering the notation of Section 4, the distinguished subset of zeros correspond to the points in the spectral curve \( S \) where the action on the line bundle \( L \) is trivial.

**Remark 23.** While the monodromy action for the \( SL_2 \) Hitchin fibration was considered previously when studying the singular fibres [2], Theorem 7 and Corollary 2 provide information about the global topology of the moduli space by means of the regular fibres of the Hitchin fibration.

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