Superfield Hamiltonian quantization in terms of quantum antibrackets

(In respectful memory of Professor Raymond Stora)

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Abstract

We develop a new version of the superfield Hamiltonian quantization. The main new feature is that the BRST-BFV charge and the gauge fixing Fermion are introduced on equal footing within the sigma model approach, which provides for the actual use of the quantum/derived antibrackets. We study in detail the generating equations for the quantum antibrackets and their primed counterparts. We discuss the finite quantum antcanonical transformations generated by the quantum antibracket.

Keywords: Hamiltonian quantization, superfield, quantum antibracket

PACS numbers: 11.10.Ef, 11.15.Bt

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1 Introduction

In the present paper we develop further the Hamiltonian superfield quantization suggested in [1, 2, 3]. The main new feature is that the BRST-BFV charge and the gauge fixing fermion are introduced on equal footing within the sigma model approach [4, 5, 6, 7], which provides for the actual use of the quantum/derived antibrackets. We study in detail the generating equations for the quantum antibrackets and their primed counterparts, as well as the finite quantum anticanonical transformations generated by the quantum antibracket. In this connection we note also that the quantum antibrackets yield the effective means of representation of gauge fields via suitable BRST operator in the approach recently proposed [8].

As the BRST-BFV supersymmetry plays its fundamental role in the generalized Hamiltonian formalism for dynamical systems with first-class constraints [9], it appears as a quite natural idea to use the BRST-BFV superfields as to develop the sigma-model-like approach specific to the topological field theories. With these regards, we suggest a total superfield action as a sum of the two dual sigma models related to the BRST-BFV charge and to the gauge fixing Fermion, respectively. In this way, we reproduce the superfield covariant derivative in kinetic part of the Hamiltonian action, while the unitarizing Hamiltonian itself appears in the specific form of a supercommutator of the BRST-BFV charge and the gauge fixing Fermion. The latter general form of the unitarizing Hamiltonian is a characteristic feature of the generalized Hamiltonian formalism in its form invariant under reparametrizations of time. Then, the main observation is that the mentioned general form of the unitarizing Hamiltonian rewrites in a natural way entirely in terms of the two dual quantum antibrackets known to mathematicians as ”derived brackets”. That is a ”synthetic” object constructed of double supercommutators of its entries and the generating nilpotent Fermion. These derived brackets have been introduced by mathematicians in [10, 11] (for further discussion see also [12, 13, 14]), and then, independently, by physicists in [15, 16, 17]. These objects have very nice algebraic properties such as the generalized Jacobi relations and the modified Leibnitz rule. In terms of the dual quantum antibrackets, the unitarizing Hamiltonian splits additively into two commuting parts, which implies the respective multiplicative splitting as to the evolution operator. We study in detail the generating equations for all the quantum antibrackets and their primed counterparts as well.

2 Classical action for dual sigma models

Let

\[ z^A, \quad \varepsilon(z^A) := \varepsilon_A, \quad A = 1, ..., 2N, \]  

be a set of canonical pairs coordinate-momentum of a dynamical system with constraints.
Let us consider the two dual partial actions of the respective sigma models for superfields $z^A(t, \theta)$.

\[
\Sigma_\Omega := \int dt d\theta [V_A \theta \partial_t z^A (-1)^{\varepsilon_A} + \Omega], \quad (2.2)
\]
\[
\Sigma_\Psi := \int dt d\theta [V_A \partial_\theta z^A (-1)^{\varepsilon_A} - \Psi], \quad (2.3)
\]

Here in (2.2), (2.3) $t$ and $\theta$ is a Boson and Fermion time variable,

\[
\varepsilon(t) = 0, \quad \varepsilon(\theta) = 1, \quad (2.4)
\]

respectively;

\[
V_A = V_A(z), \quad \varepsilon(V_A) := \varepsilon_A, \quad (2.5)
\]

is a symplectic potential whose shifted vorticity determines the respective covariant symplectic metric

\[
\omega_{AB} := \partial_A V_B + \partial_B V_A (-1)^{(\varepsilon_A+1)(\varepsilon_B+1)}, \quad (2.6)
\]

$\Omega$ and $\Psi$ is a BRST-BFV charge and a gauge-fixing Fermion, respectively,

\[
\varepsilon(\Omega) = 1, \quad \{\Omega, \Omega\} = 0, \quad (2.7)
\]
\[
\varepsilon(\Psi) = 1, \quad \{\Psi, \Psi\} = 0, \quad (2.8)
\]

where a Poisson bracket is defined as usual with the contravariant symplectic metric $\omega^{AB}$ inverse to (2.6),

\[
\{F, G\} := F \leftarrow - \partial_A \omega^{AB} \partial_B G. \quad (2.9)
\]

Now, the complete superfield action is defined as a sum of the partial actions (2.2) and (2.3),

\[
\Sigma := \Sigma_\Omega + \Sigma_\Psi = \int dt d\theta [V_A D z^A (-1)^{\varepsilon_A} + Q], \quad (2.10)
\]

where

\[
D := \partial_\theta + \theta \partial_t, \quad \varepsilon(D) = 1, \quad D^2 = \frac{1}{2}[D, D] = \partial_t, \quad (2.11)
\]

is a superfield covariant derivative, and a total Fermion generator ("Hamiltonian"),

\[
Q := \Omega - \Psi, \quad \varepsilon(Q) = 1, \quad \{Q, Q\} = -2\{\Omega, \Psi\}, \quad (2.12)
\]

respectively. The complete action (2.10) yields the following superfield equation of motion

\[
D z^A - \{Q, z^A\} = 0. \quad (2.13)
\]

\textsuperscript{3}Here and in what follows we use the following normalization of Berezin’s integral $\int d\theta \theta = 1.$
Due to (2.11), (2.12), it follows immediately from (2.13)
\[ \partial_t z^A + \{H, z^A\} = 0, \]  
(2.14)

where
\[ H := -\frac{1}{2}\{Q, Q\} = \{\Omega, \Psi\}, \]  
(2.15)
is a complete Boson Hamiltonian. The Hamiltonian (2.15) provides for correct dynamical
description for general system with first-class constraints, being the time sector \((q^0, p_0)\) included
into the original set (2.1) as well, so that the time first-class constraint has the form
\[ T_0 := p_0 + H_0, \]  
(2.16)
with \(H_0(z)\) being an original Hamiltonian [18, 19].

Now, let us consider in short the supersymmetry properties of the superfield action (2.10).
It appears as a natural idea to consider the "BRST" variation of the superfield
\(z^A(t, \theta)\) in the form of a \(\theta\)-translation
\[ \delta z^A(t, \theta) := z^A \xrightarrow{\delta} \partial_\theta \mu. \]  
(2.17)

As far as a Fermion parameter \(\mu\) is a constant, the Jacobian of the transformation (2.17)
equals to one, because the derivative of the delta function of \(\theta - \theta'\) equals to one. On the other hand,
by choosing \(\mu\) in the form
\[ \mu := \frac{i}{\hbar} \int dt d\theta \ \delta Q(z(t, \theta)), \]  
(2.18)
with \(\delta Q\) being a "desired" functional variation of \(Q\) (do not confuse with (2.17)!)

we find the Jacobi
\[ J = 1 + \frac{i}{\hbar} \int dt d\theta \ \delta Q(z(t, \theta)) \]  
(2.19)
which reproduces the "desired" variation of the superfield action (2.10). Here in (2.19), we
have integrated by part over \(\theta\)
\[ \int d\theta \ \delta Q(z(t, \theta)) = - \int d\theta \ \delta Q = - \frac{i}{\hbar} \int dt d\theta \ \delta Q. \]  
(2.20)

As to the variation of kinetic part of the superfield action (2.10) under the variation (2.17) with
any \(\mu\), it has the form \((\omega_{AB} = \text{const}(z)\) )
\[ -\mu \int dt d\theta \ (\partial_\theta z^B) \omega_{BA} Dz^A (-1)^{\varepsilon_A} = -\mu \int dt d\theta \ (\partial_\theta z^B) (\omega_{BA} Dz^A (-1)^{\varepsilon_A} + \partial_B Q). \]  
(2.21)
where we have taken into account that

\[
\int d\theta \, (\partial_0 z^B) \partial_B Q = \int d\theta \, \partial_0 Q = 0.
\]

(2.22)

The expression in the second parentheses in the right-hand side in (2.21) is nothing else but the left-hand side of the superfield equations of motion for \( z^B(t, \theta) \). Thus we have shown that the kinetic part of the superfield action (2.10) is invariant under the variation (2.17) with any \( \mu \), at the extremals of the whole superfield action (2.10). On the other hand, the variation of the rest of the superfield action (2.10) under the variation (2.17) with any \( \mu \) vanishes trivially due to (2.22), while the \( \mu \) (2.18) yields the Jacobian (2.19) that reproduces exactly the "desired" functional variation of the superfield action (2.10) under the "desired" variation \( \delta Q \). As usual in the theories with weak global supersymmetries, the statements proven are sufficient to conclude that the superfield path integral constructed of the superfield action (2.10) is independent of the "desired" variations of the gauge fixing Fermion \( \Psi \). The latter conclusion can be confirmed in an independent way via the well-known proof within the component formalism (see Appendix).

3 Quantum dynamics and quantum antibrackets

As to the quantum description, we proceed with the operator valued form of the equations of motion (2.13), (2.14), and the Hamiltonian (2.15), where all Poisson brackets should be replaced by the respective commutators multiplied by \((i\hbar)^{-1}\),

\[
\{F, G\} \rightarrow (i\hbar)^{-1}[F, G], \quad [F, G] := FG - GF(-1)^{\varepsilon_F \varepsilon_G}.
\]

(3.1)

For the sake of simplicity, we assume in what follows below that

\[
(i\hbar)^{-1}\{z^A(t), z^B(t)\} = \omega^{AB} = \text{const}(z).
\]

(3.2)

Due to (3.1), the Hamiltonian (2.15) takes the operator valued form

\[
\mathcal{H} := (i\hbar)^{-1}[[\Omega, \Psi]].
\]

(3.3)

Then the equation of motion for an operator \( A(z) \) reads

\[
\partial_t A = (i\hbar)^{-2}[A, [\Omega, \Psi]].
\]

(3.4)

For any two operators \( F \) and \( G \), let us define their quantum \( \Omega \)-antibracket \([15]\)

\[
(F, G)_\Omega := \frac{1}{2}([F, [\Omega, G]] - (F \leftrightarrow G)(-1)^{(\varepsilon_F + 1)(\varepsilon_G + 1)}) = \begin{cases} 
[\Omega, F, G](-1)^{\varepsilon_F + 1} + \frac{1}{2}[\Omega, [F, G]](-1)^{\varepsilon_F}.
\end{cases}
\]

(3.5)
By definition (3.5) it follows an important formula

\[ [\Omega, (F,G)_{\Omega}] = [[\Omega, F], [\Omega, G]]. \tag{3.6} \]

These quantum antibrackets have very nice algebraic properties. First of all, we mention their Jacobi identity in a purely Boson sector

\[ 6(B, (B,B)_{\Omega})_{\Omega} = [[B, (B,B)_{\Omega}], \Omega], \quad \varepsilon(B) = 0. \tag{3.7} \]

Then we apply the differential polarization procedure. By choosing the Boson B in the form

\[ B = \alpha F + \beta G + \gamma H, \tag{3.8} \]

where \( F, G, H \) are any operators whose Grassmann parities coincide with the ones of the parameters \( \alpha, \beta, \gamma \), respectively, we then apply the operator

\[ \partial_\alpha \partial_\beta \partial_\gamma (-1)^{(\varepsilon_\alpha+1)(\varepsilon_\gamma+1)+\varepsilon_\beta+1}, \tag{3.9} \]

to the relation (3.7), to derive the general form of the Jacobi identity

\[ (F, (G,H)_{\Omega})_{\Omega}(-1)^{(\varepsilon_F+1)(\varepsilon_H+1)} + \text{cycle}(F,G,H) = -\frac{1}{2}[(F,G,H)_{\Omega}(-1)^{(\varepsilon_F+1)(\varepsilon_H+1)}, \Omega], \tag{3.10} \]

where

\[ (F,G,H)_{\Omega} = \frac{1}{3}(-1)^{(\varepsilon_F+1)(\varepsilon_H+1)}([F,(G,H)_{\Omega}](-1)^{\varepsilon_G+\varepsilon_F(\varepsilon_H+1)} + \text{cycle}(F,G,H)), \tag{3.11} \]

is the so-called quantum 3 - antibracket. The generalized Leibnitz rule for the quantum antibracket reads

\[ (FG,H)_{\Omega} = F(G,H)_{\Omega} - (F,H)_{\Omega}G(-1)^{\varepsilon_G(\varepsilon_H+1)} = \]

\[ = \frac{1}{2}([F,H][G,\Omega](-1)^{\varepsilon_H(\varepsilon_G+1)} + [F,\Omega][G,H](-1)^{\varepsilon_G}). \tag{3.12} \]

Now, let us turn again to the double commutator in the right-hand side in (3.4). It is a remarkable fact that \([16,17]\)

\[ [A, [\Omega, \Psi]] = \frac{2}{3}((A, \Psi)_{\Omega} + (A, \Omega)_{\Psi}), \tag{3.13} \]

together with\[4\]

\[ [\text{ad}_\Omega(\Psi), \text{ad}_\Psi(\Omega)]A := (\Psi, (\Omega, A)_{\Psi})_{\Omega} - (\Omega \leftrightarrow \Psi) = 0, \tag{3.14} \]

\[ 4\text{If one does not assume the nilpotency (2.8) as for the gauge Fermion } \Psi, \text{ then the right-hand side of the formula (3.14) becomes } -\frac{1}{4}[\Omega, (\frac{1}{2}[\Psi, \Psi], A)_{\Omega}] = -\frac{1}{4}\text{ad}(\Omega)\text{ad}_{\Omega}(\frac{1}{2}[\Psi, \Psi])A, \text{ while the formula (3.13) remains the same.} \]
which means in turn that the evolution operator

\[ \exp\{-\frac{\hbar}{i} t \mathcal{H}\}, \]  

(3.15)

does factorize into a product of the two commuting operators defined by the respective two terms in the right-hand side in (3.13), that is

\[ \exp\left\{-\left(\frac{\hbar}{i}\right)^{-2} t \frac{2}{3} \text{ad}_\Omega(\Psi)\right\}, \]  

(3.16)

and

\[ \exp\left\{-\left(\frac{\hbar}{i}\right)^{-2} t \frac{2}{3} \text{ad}_\Psi(\Omega)\right\}, \]  

(3.17)

each of which being a quantum anticanonical transformation generated by a quantum antibracket.

It follows from the first equation in (2.12) that the operator \( Q \) does satisfy the closed equation

\[ (\mathcal{G}, \mathcal{G})_Q = -\frac{\hbar}{i} Q, \]  

(3.18)

where the \( Q \)-quantum antibracket, \((F, G)_Q\), is defined in (3.5) with \( Q \) standing for \( \Omega \), while the \( G \) in the left-hand side in (3.18) is the total ghost number operator

\[ [\mathcal{G}, \Omega] = i\hbar \Omega, \quad [\mathcal{G}, \Psi] = -i\hbar \Psi. \]  

(3.19)

4 Generating operator for higher quantum antibrackets

Let us consider a chain of operators \( f_a \) and parameters \( \lambda^a \),

\[ \{f_a(z); \lambda^a|\varepsilon(F_a) = \varepsilon(\lambda^a) := \varepsilon_a, \quad a = 1, ..., n, \ldots\}, \]  

(4.1)

to define the Fermion nilpotent generating operator

\[ \Omega(\lambda) := \exp\{-F\} \Omega \exp\{F\}, \quad F := f_a \lambda^a, \]  

(4.2)

\[ \varepsilon(\Omega(\lambda)) := 1, \quad [\Omega(\lambda), \Omega(\lambda)] = 0. \]  

(4.3)

In terms of the generating operator (4.2) the \( n - \)th quantum antibracket is defined as (16)

\[ (f_{a_1}, ..., f_{a_n})_\Omega := -\Omega(\lambda) \overleftarrow{\partial}_{a_1} ... \overleftarrow{\partial}_{a_n}|_{\lambda=0} (-1)^{E_n}, \]  

(4.4)

where

\[ E_n := \sum_{k=0}^{[n/2]} \varepsilon_{a_{2k+1}}. \]  

(4.5)
In more detail, we have

\[ (f_{a_1}, \ldots, f_{a_n})_\Omega = -\left[ \cdots [\Omega, f_{a_1}], \ldots, f_{b_n} \right] S_{a_1 \ldots a_n}^{b_1 \ldots b_n}(-1)^{E_n}, \]

(4.6)

with the symmetrizer being defined as

\[ n!S_{a_1 \ldots a_n}^{b_1 \ldots b_n} := (\lambda^{b_n} \ldots \lambda^{b_1} \frac{\partial}{\partial a_1} \ldots \frac{\partial}{\partial a_n}), \quad \partial_a := \frac{\partial}{\partial \lambda^a}. \]

(4.7)

All the Jacobi relations for higher quantum antibrackets are accumulated in the nilpotency equation (4.3). In terms of the generating operator (4.2) together with the operator

\[ R_a(\lambda) := \exp\{-F\} \left( \exp\{F\} \frac{\partial}{\partial a} \right) \]

(4.8)

the following closed set of the generating equations holds

\[ \Omega(\lambda) \frac{\partial}{\partial a} = [\Omega(\lambda), R_a(\lambda)], \]

(4.9)

\[ R_a(\lambda) \frac{\partial}{\partial b} - (a \leftrightarrow b)(-1)^{\varepsilon_a \varepsilon_b} = [R_a(\lambda), R_b(\lambda)], \]

(4.10)

\[ \Omega(\lambda = 0) = \Omega, \quad R_a(\lambda = 0) = f_a. \]

(4.11)

In turn, these generating equations do imply further equations for primed quantum antibrackets defined as

\[ (f_{a_1}, \ldots, f_{a_n})'_\Omega := -\Omega(\lambda) \frac{\partial}{\partial a_1} \ldots \frac{\partial}{\partial a_n} (-1)^{E_n}. \]

(4.12)

In particular, for primed quantum 2-antibracket we get

\[ (f_a, f_b)'_\Omega = (R_a(\lambda), R_b(\lambda))_{\Omega(\lambda)} - \frac{1}{2} \Omega(\lambda), R_a(\lambda) \frac{\partial}{\partial b} + (a \leftrightarrow b)(-1)^{\varepsilon_a \varepsilon_b}(-1)^{\varepsilon_a}. \]

(4.13)

5 Finite quantum anticanonical transformations

In Section 3, we have already mentioned finite quantum anticanonical transformations (3.16), (3.17). Now we are in a position to present such transformations explicitly in their most general setting. Let \( \lambda \) be a boson parameter, \( \varepsilon(\lambda) = 0 \). Given an operator \( A \), define then the transformed operator as

\[ A' := \exp\{\lambda \text{ ad}_\Omega(\Psi)\}A, \]

(5.1)

to satisfy the equation

\[ \partial_\lambda A' = (\Psi, A')_{\Omega}, \quad A'(\lambda = 0) = A. \]

(5.2)
Its explicit solution has the form \[20\]

\[A' = \tilde{A}(\lambda) - \frac{1}{2} \int_{0}^{\lambda} d\lambda' \exp \left\{ \frac{\lambda - \lambda'}{2} \{\Omega, \Psi\} \right\} \{\Omega, [\Psi, \tilde{A}(\lambda')]\} \exp \left\{ -\frac{\lambda - \lambda'}{2} \{\Omega, \Psi\} \right\}, \quad (5.3)\]

where

\[\tilde{A}(\lambda) = \exp\{\lambda[\Omega, \Psi]\} A \exp\{-\lambda[\Omega, \Psi]\}. \quad (5.4)\]

By interchanging \(\Omega \leftrightarrow \Psi\) in (5.3), we get the transformation dual to (5.3). In this way, the operators (3.16), (3.17) are reproduced at

\[\lambda = -(ih)^{-2}t^2/3. \quad (5.5)\]

Also, notice that the quantum antibracket of the two transformed operators \(A'\) and \(B'\) satisfies the equation \[20\]

\[\partial_\lambda (A', B')_\Omega = (\Psi, (A', B')_\Omega)_\Omega + \frac{1}{2}((\Psi, A', B')_\Omega, \Omega), \quad (5.6)\]

that follows from (5.2) for \(A'\) and \(B'\), together with the Jacobi relation (3.10).

Explicit solution to the equation (5.6) has the form

\[(A', B')(\lambda)_\Omega = (A, B)_\Omega(\lambda) + \frac{1}{2} \int_{0}^{\lambda} d\lambda' \exp\{(\lambda - \lambda')ad_\Omega(\Psi)\}[(\Psi, A'(\lambda'), B'(\lambda'))_\Omega, \Omega], \quad (5.7)\]

where all primed operators are defined similarly to (5.1). For instance, the first term in the right-hand side in (5.7) is decoded as

\[\exp\{\lambda ad_\Omega(\Psi)\} (A, B)_\Omega, \quad (5.8)\]

not to be confused with (4.13)!

6 Finite transformations of general open group: integrating arbitrary involutions [15, 21]

Let

\[\{\phi_a | \varepsilon(\phi_a) := \varepsilon_a = \varepsilon(T_a)\}, \quad (6.1)\]

be a set of parameters of gauge transformations generated by the first-class constraints \(T_a\) encoded in the BRST-BFV operator \(\Omega\). Let us consider the general Lie equation for an operator valued transformation \(A_0 \rightarrow A(\phi), \quad (6.2)\)

\[A(\phi) \overset{\text{def}}{=} (ih)^{-1}[A(\phi), Y_a(\phi)], \quad (6.3)\]

\[A(\phi = 0) = A_0, \quad (6.4)\]

\[\partial_a := \frac{\partial}{\partial \phi^a}, \quad \varepsilon(Y_a) := \varepsilon_a. \quad (6.5)\]
Integrability of that equation requires
\[ Y_a \overset{\leftarrow}{\partial}_b - (a \leftrightarrow b)(-1)^{\varepsilon_a \varepsilon_b} = (i\hbar)^{-1}[Y_a, Y_b]. \quad (6.5) \]

We choose the operators \( Y_a \) in the form generated by the one \( \Omega \),
\[ Y_a(\phi) := (i\hbar)^{-1}[\Omega, \Omega_a(\phi)], \]
\[ \varepsilon(\Omega_a) := \varepsilon_a + 1. \quad (6.7) \]
The form (6.6) implies that
\[ [\Omega, A_0] = 0 \quad \Rightarrow \quad [\Omega, A(\phi)] = 0. \quad (6.8) \]
Then, the integrability (6.5) together with the choice (6.6) implies that
\[ A \overset{\leftarrow}{\partial}_a = (i\hbar)^{-2}(A, \Omega_a) + (i\hbar)^{-2}\frac{1}{2}[A, \Omega_a, \Omega](-1)^{\varepsilon_a}, \]
\[ \Omega_a \overset{\leftarrow}{\partial}_b - (a \leftrightarrow b)(-1)^{\varepsilon_a \varepsilon_b} = (i\hbar)^{-2}(\Omega_a, \Omega_b) - \frac{1}{2}(i\hbar)^{-1}[\Omega, \Omega_{ab}], \quad (6.10) \]
In its own turn, the integrability condition (6.10) requires further integrability conditions, and so on. It is a remarkable fact that all these subsequent integrability conditions are naturally accumulated in a single quantum master equation
\[ (S, S)_\Delta = i\hbar[\Delta, S], \quad \varepsilon(S) = 0, \quad (6.11) \]
where
\[ \Delta := \Omega + \eta^a \pi_a(-1)^{\varepsilon_a}, \quad \Delta^2 = 0, \quad (6.12) \]
with
\[ \pi_a, \quad \varepsilon(\pi_a) := \varepsilon_a, \quad (6.13) \]
being momenta canonically conjugated to \( \phi^a \),
\[ [\phi^a, \pi_b] = i\hbar \delta^a_b, \quad (6.14) \]
and
\[ \eta^a, \quad \varepsilon(\eta^a) := \varepsilon_a + 1, \quad (6.15) \]
being new ghost variables viewed as parameters. As we have by definition
\[ (S, S)_\Delta = [S, [\Delta, S]], \quad (6.16) \]
the master equation (6.11) implies
\[
[\Delta, (S, S)_\Delta] = 0 \quad \Rightarrow \quad [\Delta, S]^2 = 0. \quad (6.17)
\]

Now, let \( \mathcal{G} \) be the standard ghost number operator,
\[
[\mathcal{G}, \Omega] = i\hbar\Omega. \quad (6.18)
\]

Let us seek for a solution to the master equation (6.11) in the form of an \( \eta \)-power series expansion,
\[
S(\phi, \eta) = \mathcal{G} + \eta^a \Omega_a(\phi) + \frac{1}{2} \eta^b \eta^a (-1)^{\varepsilon_b} \Omega_{ab}(\phi) + \frac{1}{6} \eta^b \eta^a \eta^c (-1)^{\varepsilon_b + \varepsilon_c + \varepsilon_e} \Omega_{abc}(\phi) + \ldots + \frac{1}{n!} \eta^{a_n} \ldots \eta^{a_1} (-1)^{\varepsilon_n} \Omega_{a_1 \ldots a_n}(\phi) + \ldots, \quad (6.19)
\]
where
\[
\varepsilon_n := \sum_{k=1}^{[\frac{n}{2}]} \varepsilon_{a_{2k}} + \sum_{k=1}^{[\frac{n-1}{2}]} \varepsilon_{a_{2k-1}} \varepsilon_{a_{2k+1}}. \quad (6.20)
\]

The coefficient operators \( \Omega_{a_1 \ldots a_n} \) in the expansion (6.19) have the properties
\[
\varepsilon(\Omega_{a_1 \ldots a_n}) = \varepsilon_{a_1} + \ldots + \varepsilon_{a_n} + n, \quad (6.21)
\]
\[
[\mathcal{G}, \Omega_{a_1 \ldots a_n}] = -n i\hbar \Omega_{a_1 \ldots a_n}. \quad (6.22)
\]

In the zeroth and first orders in \( \eta \), the master equation (6.11) is satisfied identically. However, in the second order, it yields exactly (6.10). In the third order in \( \eta \) it yields
\[
(\partial_a \Omega_{bc} + \frac{1}{2} (i\hbar)^{-2} (\Omega_a, \Omega_{bc}) \Omega - \frac{1}{12} (i\hbar)^{-2} [[\Omega_{abc}, \Omega_c], \Omega]) (-1)^{\varepsilon_a \varepsilon_c} + \text{cycle}(a, b, c) = \nonumber
- (i\hbar)^{-3} (\Omega_a, \Omega_b, \Omega_c) \Omega (-1)^{\varepsilon_a \varepsilon_c} - \frac{2}{3} (i\hbar)^{-1} [\Omega_{abc}, \Omega], \quad (6.23)
\]
which is exactly the integrability condition to (6.10). A natural automorphism of the master equation (6.11) is given by
\[
S \rightarrow S' := \exp\{- (i\hbar)^{-2} [\Delta, \Xi]\} \; S \; \exp\{(i\hbar)^{-2} [\Delta, \Xi]\}, \quad (6.24)
\]
where \( \Xi \) is an arbitrary odd operator. For infinitesimal transformation we have
\[
\delta S = (i\hbar)^{-2} [S, \Delta] \Xi = (i\hbar)^{-2} \left( \frac{2}{3} ([S, \Xi] \Delta + (S, \Delta) \Xi) \right), \quad (6.25)
\]
\[
\delta_{21} S := [\delta_2, \delta_1] S = (i\hbar)^{-2} [S, [\Delta, \Xi_2]], \quad (6.26)
\]
\[
\Xi_{21} = (i\hbar)^{-2} (\Xi_2, \Xi_1) \Delta. \quad (6.27)
\]
If the transformation (6.24) acts transitively on the set of solutions to the master equation (6.11), then the general solution is
\[ S = \exp\{(i\hbar)^{-2}[\Delta, \Xi]\} \mathcal{G} \exp\{(i\hbar)^{-2}[\Delta, \Xi]\}. \] (6.28)

Now, let us consider the transformation
\[ S(\alpha) := \exp\{i\alpha F/\hbar\} S \exp\{-i\alpha F/\hbar\}, \] (6.29)
\[ \Delta(\alpha) := \exp\{i\alpha F/\hbar\} \Delta \exp\{-i\alpha F/\hbar\}, \] (6.30)
where \( \alpha \) is an even parameter, and \( F \) is an arbitrary even operator. If \( S \) and \( \Delta \) satisfy the master equation (6.11), then \( S(\alpha) \) and \( \Delta(\alpha) \) satisfy the transformed master equation
\[ (S(\alpha), S(\alpha))_{\Delta(\alpha)} = i\hbar[\Delta(\alpha), S(\alpha)]. \] (6.31)
If \( F \) is restricted to satisfy itself the master equation (6.11), i.e.
\[ (F, F)_{\Delta} = i\hbar[\Delta, F], \] (6.32)
then
\[ \Delta''(\alpha) + \Delta'(\alpha) = 0, \] (6.33)
and \( \Delta(\alpha) \) in (6.30) reduces to
\[ \Delta(\alpha) = \Delta + (i\hbar)^{-1}[\Delta, F](1 - \exp\{-\alpha\}). \] (6.34)
For \( F = S \), in particular, \( S \) satisfies the master equation (6.11) with \( \Delta \) replaced by \( \Delta(\alpha) \) in (6.34), where \( F \) is replaced by \( S \).

7 Discussion

The main result of the present consideration is that the dynamical evolution of an arbitrary dynamical system with first-class constraints is represented entirely in terms of the two dual quantum antibrackets related to the two nilpotent Fermion operators, the BRST-BFV charge and the gauge-fixing Fermion. Although in the standard BRST-BFV scheme there is no need to impose the nilpotency requirement as to the gauge Fermion, in the sigma model approach developed above that requirement should be imposed certainly on equal footing upon both the BRST-BFV charge and the gauge fixing Fermion. If one allows for the gauge-fixing Fermion operator to deviate from being nilpotent, then the closed character of the description in terms of the dual quantum antibrackets will be failed immediately. It should be noticed however that the standard properties of gauge invariance in the physical sector remain maintained in the latter case, as well.
Acknowledgments

I. A. Batalin would like to thank Robert Marnelius of Chalmers University (Ret.), Klaus Bering of Masaryk University and Igor Tyutin of Lebedev Institute for interesting discussions. The work of I. A. Batalin is supported in part by the RFBR grants 14-01-00489 and 14-02-01171. The work of P. M. Lavrov is supported by the Ministry of Education and Science of Russian Federation, project No 2014/387/122.

A Component formalism

Let $\theta$ be a Fermionic time (the BRST - parameter), and let $z^A(t, \theta) := z_0^A(t) + \theta z_1^A(t)$, $\varepsilon(z_0^A) := \varepsilon_A$, $\varepsilon(z_1^A) := \varepsilon_A + 1$, (A.1) be a component expansion to the superfield in the left-hand side of the first in (A.1). First, let us reproduce in terms of the component expansion (A.1) the above Jacobian (2.19). The $\theta$ - translation (2.17) takes the form

$$\theta \rightarrow \theta + \mu,$$

(A.2)

where the ”parameter” $\mu$ is chosen in the form of the functional (2.18),

$$\mu = \frac{i}{\hbar} \int dtd\theta \theta \delta Q(z(t, \theta)) = \frac{i}{\hbar} \int dt \delta Q(z_0(t)).$$

(A.3)

Here, the translation (A.2) induces the component variations

$$\delta z_0^A(t) = \mu z_1^A(t), \quad \delta z_1^A(t) = 0.$$ (A.4)

Due to the choice (A.3), the variations (A.4) yield the Jacobian

$$J = 1 + \int dt \frac{\delta}{\delta z_0^A(t)}(-1)^{\varepsilon_A} = 1 + \frac{i}{\hbar} \int dt \frac{\delta}{\delta z_0^A(t)} z_1^A(t)(-1)^{\varepsilon_A} =$$

$$= 1 - \frac{i}{\hbar} \int dtd\theta \frac{\delta}{\delta z_0^A(t)} \theta z_1^A(t) =$$

$$= 1 - \frac{i}{\hbar} \int dtd\theta \delta Q(z(t, \theta)) = (2.19).$$ (A.5)

Thus, we have reproduced exactly the Jacobian (2.19) within the component formalism.

By substituting the component form (A.1) into the superfield action (2.10), we get

$$\Sigma = \int dt \left[ \frac{1}{2} \omega_{BA}^B \partial_t z_0^A + \frac{1}{2} z_0^B \omega_{BA}^B z_1^A(-1)^{\varepsilon_A} + z_1^B \partial_B Q(z_0) \right], \quad \omega_{BA} = \text{const}(z_0, z_1).$$ (A.6)
Under the variation in the first in (A.4), the variation of kinetic part of the action $\Sigma$ (A.6) reads
\[ \int dt \mu z^B_1 (\omega_{BA} \partial_t z^A_0 + \partial_B z^A_1 \partial_A Q(z_0)). \] (A.7)

Here in (A.7), the second term does not contribute actually due to the nilpotency property
\[ (z^A_1 \partial_A)^2 = 0. \] (A.8)

The expression inside the parentheses in (A.7) is nothing else but the left-hand side of the motion equation for $z^B_0$ as to the action $\Sigma$ (A.6). Thus, the expression (A.7) is a component counterpart to (2.21).

Within the path integral
\[ Z =: \int Dz_0 Dz_1 \exp \left\{ \frac{i}{\hbar} \Sigma \right\}, \] (A.9)
by taking the Gaussian integral over $z_1$, one arrives at the expression
\[ Z = \int Dz_0 \sqrt{s \text{Det} (\omega)} \exp \left\{ \frac{i}{\hbar} \Sigma_0 \right\}, \] (A.10)
where the zero-th sector action $\Sigma_0$ is given by
\[ \Sigma_0 := \int dt \left[ \frac{1}{2} z^B_0 \omega_{BA} \partial_t z^A_0 - H_0 \right], \] (A.11)
where the Hamiltonian in the zero-th sector is given by
\[ H_0 := -\frac{1}{2} \{Q, Q\}(z_0). \] (A.12)

As usual, the Gaussian integral over $z_1$ is equivalent, up to the constant factor of $\sqrt{s \text{Det} (\omega)}$, to eliminating the $z_1$ component by resolving the classical equation for $z_1$,
\[ \omega_{BA} z^A_1 (-1)^{\varepsilon_A} + \partial_B Q(z_0) = 0, \] (A.13)
in the form
\[ z^A_1 = -\omega^{AB} \partial_B Q(z_0)(-1)^{\varepsilon_A}, \] (A.14)
which is equivalent to
\[ z^A_1 = \{Q, z^A_0\}. \] (A.15)

By substituting the latter into the component action (A.6), we get exactly the zero-th sector action (A.11).
Now, let us consider the infinitesimal BRST-BFV transformation,

\[ \delta z_0^A := \{ z_0^A, Q(z_0) \} \mu. \]  \hfill (A.16)

As far as a BRST- BFV parameter \( \mu \) is a constant, the action \( (A.11) \) and the measure in \( (A.10) \) are invariant under the transformation \( (A.16) \). Then let us choose in \( (A.16) \) the \( \mu \) in the form

\[ \mu := \frac{i}{\hbar} \int dt \delta Q(z_0), \]  \hfill (A.17)

where \( \delta Q \) is the desired functional variation of \( Q \). As the \( \mu \) \( (A.17) \) is not a *function* of the phase variables \( z_0(t) \), the action \( (A.11) \) remains invariant under the transformation \( (A.16), (A.17) \). However the functional \( (A.17) \) yields the following infinitesimal Jacobian to the latter transformation,

\[ J := 1 - \frac{i}{\hbar} \int dt \{ Q, \delta Q \}(z_0), \]  \hfill (A.18)

which is exactly the desired functional variation of the action \( (A.11) \). Thus, we have shown that the integral \( (A.9) \) remains stable under the desired functional variations \( \delta Q \).

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