Biharmonic maps into compact Lie groups
and integrable systems

Hajime URAKAWA

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Abstract. In this paper, the formulation of the biharmonic map equation in terms of
the Maurer-Cartan form for all smooth maps of a compact Riemannian manifold into
a compact Lie group \((G, h)\) with the bi-invariant Riemannian metric \(h\) is obtained.
Using this, all biharmonic curves into compact Lie groups are determined exactly, and
all the biharmonic maps of an open domain of \(\mathbb{R}^2\) equipped with a Riemannian metric
conformal to the standard Euclidean metric into \((G, h)\) are determined.

Key words: harmonic map, biharmonic map, compact Lie group, integrable system,
Maurer-Cartan form.

1. Introduction and statement of results

The theory of harmonic maps of a Riemann surface into Lie groups,
symmetric spaces or homogeneous spaces has been extensively studied in
connection with the integrable systems ([1], [2], [4], [5], [6], [8], [9], [16]).
Let us recall the theory of harmonic maps of a Riemann surface \(M\) into
a compact Lie group \(G\), briefly. A harmonic map is a critical map of the
energy functional defined by

\[
E(\psi) := \frac{1}{2} \int_M |d\psi|^2 g.
\]

For such a map \(\psi\), let \(\alpha\) be the pull back of the Maurer-Cartan form \(\theta\)
of \(G\) which is decomposed into the sum of the holomorphic part and the
antiholomorphic one as \(\alpha = \alpha' + \alpha''\). Then, it satisfies \(d\alpha = (1/2)[\alpha \wedge \alpha] = 0\)
(the integrability condition), and the harmonicity of \(\psi\) is equivalent to the
condition \(\delta \alpha = 0\). Introducing a parameter \(\lambda \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}\) as

\[
\alpha_\lambda := \frac{1}{2} (1 - \lambda) \alpha' + \frac{1}{2} (1 - \lambda^{-1}) \alpha''.
\]
both the harmonicity and the integrability condition are equivalent to
\[ d\alpha_{\lambda} + \frac{1}{2} [\alpha_{\lambda} \wedge \alpha_{\lambda}] = 0, \]
which implies that there exists an extended solution \( \Phi_{\lambda} : M \to G \) satisfying \( \Phi_{\lambda}^{-1} d\Phi_{\lambda} = \alpha_{\lambda} \) ([16]). Guest and Ohnita ([9]) showed that the loop group \( \Lambda G^C \) of \( G \) acts on the space of all harmonic maps of \( M \) into \( G \), and Uhlenbeck ([16]) showed that every harmonic map from the two-sphere into \( G \) is a harmonic map of finite uniton number, and Wood ([17]) determined explicitly harmonic maps of finite uniton numbers. On the other hand, the theory of biharmonic maps was initiated by Eells and Lemaire ([6]) and Jiang ([12]). A biharmonic map is a natural extension of harmonic map, and is a critical map of the bienergy functional defined by
\[ E_2(\psi) := \frac{1}{2} \int_M |\delta d\psi|^2 v_g = \frac{1}{2} \int_M |\tau(\psi)|^2 v_g, \]
where \( \tau(\psi) \) is the tension field of \( \psi \), and, by definition, \( \psi \) is harmonic if and only if \( \tau(\psi) \equiv 0 \).

In this paper, we study biharmonic maps of a compact Riemannian manifold \((M, g)\) into a compact Lie group \((G, h)\) with the bi-invariant Riemannian metric \( h \). For every \( C^\infty \) map \( \psi : (M, g) \to (G, h) \), let us consider again the pullback \( \alpha \) of the Maurer-Cartan form \( \theta \). We first will show that the biharmonicity condition for \( \psi \) is that
\[ \delta d\delta \alpha + \text{Trace}_g([\alpha, d\delta \alpha]) = 0 \]
(cf. Corollary 3.5) which is a natural extension of harmonicity. Due to this formula, we can determine all real analytic biharmonic curves into a compact Lie group \((G, h)\) in terms of the initial data \( F(0), F'(0) \) and \( F''(0) \), where \( F(t) = \alpha(\partial/\partial t) \) (cf. Section 4). We give a characterization of biharmonic maps of \((\mathbb{R}^2, \mu^2 g_0)\), where \( g_0 \) is the standard Euclidean metric on \( \mathbb{R}^2 \) and \( \mu \) is a positive real analytic function on \( \mathbb{R}^2 \) (cf. Sections 5, 6 and 7).

2. Preliminaries

In this section, we prepare general materials and facts on harmonic maps, biharmonic maps into Riemannian manifolds (cf. [6], [12], [13]). Let
(M, g) be an m-dimensional compact Riemannian manifold, and (N, h), an n-dimensional Riemannian manifold.

The energy functional on the space \( C^\infty(M, N) \) of all \( C^\infty \) maps of M into N is defined by

\[
E(\psi) = \frac{1}{2} \int_M |d\psi|^2 v_g,
\]

and for a compactly supported \( C^\infty \) one parameter deformation \( \psi_t \in C^\infty(M, N) \) \((-\epsilon < t < \epsilon)\) of \( \psi_0 = \psi \), the first variation formula is given by

\[
\left. \frac{d}{dt} \right|_{t=0} E(\psi_t) = -\int_M \langle \tau(\psi), V \rangle v_g,
\]

where \( V \) is a variation vector field along \( \psi \) defined by \( V = d/dt|_{t=0} \psi_t \) which belongs to the space \( \Gamma(\psi^{-1}TN) \) of sections of the induced bundle of the tangent bundle \( TN \) by \( \psi \). The tension field \( \tau(\psi) \) is defined by

\[
\tau(\psi) = -\delta(d\psi), \tag{2.1}
\]

where recall the definition \( \delta \alpha \) for a \( \psi^{-1}TN \)-valued 1-form \( \alpha \),

\[
\delta \alpha = -\sum_{i=1}^m (\nabla e_i \alpha)(e_i) = -\sum_{i=1}^m \{ \nabla(\alpha(e_i)) - \alpha(\nabla e_i e_i) \}.
\]

Here, \( \nabla, \nabla^h \) and \( \nabla \) are the Levi-Civita connections of \((M, g), (N, h)\), and the induced connections on the induced bundle \( \psi^{-1}TN \) from \( \nabla^h \), respectively.

For a harmonic map \( \psi : (M, g) \to (N, h) \), the second variation formula of the energy functional \( E(\psi) \) is

\[
\left. \frac{d^2}{dt^2} \right|_{t=0} E(\psi_t) = \int_M \langle J(V), V \rangle v_g
\]

where
\[ J(V) = \Delta V - R(V), \]
\[ \Delta V = \nabla^* \nabla V = - \sum_{i=1}^{m} \{ \nabla_{e_i} (\nabla_{e_i} V) - \nabla_{\nabla_{e_i} e_i} V \}, \]
\[ R(V) = \sum_{i=1}^{m} R^h(V, d\psi(e_i))d\psi(e_i). \]

Here, \( \nabla \) is the induced connection on the induced bundle \( \psi^{-1}TN \), and \( R^h \) is the curvature tensor of \( (N, h) \) given by \( R^h(U, V)W = [\nabla^h_U, \nabla^h_V]W - \nabla^h_{[U,V]}W \) \( (U, V, W \in \mathfrak{X}(N)) \). The bienergy functional is defined by

\[ E_2(\psi) = \frac{1}{2} \int_M |\delta \psi|^2 v_g = \frac{1}{2} \int_M |\tau(\psi)|^2 v_g, \quad (2.2) \]

and the first variation formula of the bienergy is given ([12]) by

\[ \frac{d}{dt} \bigg|_{t=0} E_2(\psi_t) = - \int_M \langle \tau_2(\psi), V \rangle v_g \quad (2.3) \]

where the bitension field \( \tau_2(\psi) \) is defined by

\[ \tau_2(\psi) = J(\tau(\psi)) = \Delta \tau(\psi) - \mathcal{R}(\tau(\psi)), \quad (2.4) \]

and a \( C^\infty \) map \( \psi : (M, g) \to (N, h) \) is called to be biharmonic if

\[ \tau_2(\psi) = 0. \quad (2.5) \]

The biharmonic maps are real analytic when both \( (M, g) \) and \( (N, h) \) are real analytic. This is because the solutions of non-linear elliptic partial differential equations are real analytic.

3. Determination of the bitension field

Now, assume that \( (N, h) \) is an \( n \)-dimensional compact Lie group with Lie algebra \( \mathfrak{g} \), and \( h \), the bi-invariant Riemannian metric on \( G \) corresponding to the \( \text{Ad}(G) \)-invariant inner product \( \langle , \rangle \) on \( \mathfrak{g} \). Let \( \theta \) be the Maurer-Cartan form on \( G \), i.e., a \( \mathfrak{g} \)-valued left invariant 1-form on \( G \) which is defined by \( \theta_y(Z_y) = Z, \) \( (y \in G, Z \in \mathfrak{g}) \). For every \( C^\infty \) map \( \psi \) of \( (M, g) \) into \( (G, h) \), let
us consider a $\mathfrak{g}$-valued 1-form $\alpha$ on $M$ given by $\alpha = \psi^* \theta$. Then it is well known (see for example, [4]) that

**Lemma 3.1** For every $C^\infty$ map $\psi : (M, g) \rightarrow (G, h)$,

\[ \theta(\tau(\psi)) = -\delta \alpha. \] (3.1)

Thus, $\psi : (M, g) \rightarrow (G, h)$ is harmonic if and only if $\delta \alpha = 0$.

Let $\{X_s\}_{s=1}^n$ be an orthonormal basis of $\mathfrak{g}$ with respect to the inner product $\langle , \rangle$. Then, for every $V \in \Gamma(\psi^{-1}TG)$,

\[
V(x) = \sum_{s=1}^n h_{\psi(x)}(V(x), X_s \psi(x))X_s \psi(x) \in T_{\psi(x)}G,
\]

\[
\theta(V)(x) = \sum_{s=1}^n h_{\psi(x)}(V(x), X_s \psi(x))X_s \in \mathfrak{g},
\] (3.2)

for all $x \in M$. Then, for every $X \in \mathfrak{X}(M)$,

\[
\theta(\nabla_X V) = \sum_{s=1}^n h(\nabla_X V, X_s) X_s
\]

\[ = \sum_{s=1}^n \{X h(V, X_s) - h(V, \nabla_X X_s)\} X_s \]

\[ = X(\theta(V)) - \sum_{s=1}^n h(V, \nabla_X X_s) X_s,
\] (3.3)

where we regarded a vector field $Y \in \mathfrak{X}(G)$ by $Y(x) = Y(\psi(x))$ ($x \in M$) to be an element in the space $\Gamma(\psi^{-1}TG)$ of smooth sections of $\psi^{-1}TG$. Here, let us recall that the Levi-Civita connection $\nabla^h$ of $(G, h)$ is given (cf. [13, Vol. II, p. 201, Theorem 3.3]) by

\[
\nabla^h_{X_t} X_s = \frac{1}{2}[X_t, X_s] = \frac{1}{2} \sum_{\ell=1}^n C_{ts}^\ell X_\ell,
\] (3.4)

where the structure constant $C_{ts}^\ell$ of $\mathfrak{g}$ is defined by $[X_t, X_s] = \sum_{\ell=1}^n C_{ts}^\ell X_\ell$, and satisfies
\[ C^\ell_{ts} = \langle [X_t, X_s], X_\ell \rangle = -\langle X_s, [X_t, X_\ell] \rangle = -C^s_{t\ell}. \quad (3.5) \]

Thus, we have by (3.4) and (3.5),

\[
\sum_{s=1}^{n} h(V, \nabla_X X_s) X_s = \frac{1}{2} \sum_{s,t=1}^{n} h\left(V, \sum_{\ell=1}^{n} h(\psi_* X, X_\ell) C^\ell_{ts} X_\ell\right) X_s
\]

\[
= -\frac{1}{2} \sum_{s,t,\ell=1}^{n} h(V, X_\ell) h(\psi_* X, X_\ell) C^s_{t\ell} X_s
\]

\[
= -\frac{1}{2} \sum_{t,\ell=1}^{n} h(V, X_\ell) h(\psi_* X, X_\ell)[X_t, X_\ell]
\]

\[
= -\frac{1}{2} \left[ \sum_{t=1}^{n} h(\psi_* X, X_t) X_t, \sum_{\ell=1}^{n} h(V, X_\ell) X_\ell \right]
\]

\[
= -\frac{1}{2} [\alpha(X), \theta(V)], \quad (3.6)
\]

which is because we have

\[ \alpha(X) = \theta(\psi_* X) = \sum_{t=1}^{n} h(\psi_* X, X_t) X_t, \quad (3.7) \]

and

\[ \theta(V) = \sum_{\ell=1}^{n} h(V, X_\ell) \theta(X_\ell) = \sum_{\ell=1}^{n} h(V, X_\ell) X_\ell. \quad (3.8) \]

Therefore, inserting (3.6) into (3.3), we obtain

**Lemma 3.2** For every \( C^\infty \) map \( \psi : (M, g) \to (G, h), \)

\[ \theta(\nabla_X V) = X(\theta(V)) + \frac{1}{2} [\alpha(X), \theta(V)], \quad (3.9) \]

where \( V \in \Gamma(\psi^{-1}TG) \) and \( X \in \mathfrak{X}(M). \)

We shall show

**Theorem 3.3** For every \( \psi \in C^\infty(M, G), \) we have
\[ \theta(\tau_2(\psi)) = \theta(J(\tau(\psi))) \]
\[ = -\delta d\delta \alpha - \text{Trace}_{g}([\alpha, d\delta \alpha]), \quad (3.10) \]

where \( \alpha = \psi^\ast \theta \).

Here, let us recall the definition:

**Definition 3.4** For two \( g \)-valued 1-forms \( \alpha \) and \( \beta \) on \( M \), we define a \( g \)-valued symmetric 2-tensor \([\alpha, \beta]\) on \( M \) by

\[ [\alpha, \beta](X, Y) := \frac{1}{2}\{[\alpha(X), \beta(Y)] + [\alpha(Y), \beta(X)]\}, \quad (X, Y \in \mathfrak{X}(M)) \quad (3.11) \]

and its trace \( \text{Trace}_{g}([\alpha, \beta]) \) by

\[ \text{Trace}_{g}([\alpha, \beta]) := \sum_{i=1}^{m} [\alpha, \beta](e_i, e_i). \quad (3.12) \]

Recall that the \( g \)-valued 2-form \([\alpha \wedge \beta]\) on \( M \) is given by

\[ [\alpha \wedge \beta](X, Y) := \frac{1}{2}\{[\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)]\}, \quad (X, Y \in \mathfrak{X}(M)). \quad (3.13) \]

Then, we have immediately by Theorem 3.3,

**Corollary 3.5** For every \( \psi \in C^\infty(M, G) \), we have (1) \( \psi : (M, g) \to (G, h) \) is harmonic if and only if

\[ \delta \alpha = 0. \quad (3.14) \]

(2) \( \psi : (M, g) \to (G, h) \) is biharmonic if and only if

\[ \delta d\delta \alpha + \text{Trace}_{g}([\alpha, d\delta \alpha]) = 0. \quad (3.15) \]

We give a proof of Theorem 3.3.

**Proof.** (The first step) We first show that, for all \( V \in \Gamma(\psi^{-1}TG) \),
\[\theta(\Delta V) = \Delta_g \theta(V) - \sum_{i=1}^{m} \left\{ \frac{1}{2} [e_i(\alpha(e_i)), \theta(V)] + [\alpha(e_i), e_i(\theta(V))] \\
+ \frac{1}{4} [\alpha(e_i), [\alpha(e_i), \theta(V)]] - \frac{1}{2} [\alpha(\nabla e_i e_i), \theta(V)] \right\}, \quad (3.16)\]

where \(\{e_i\}_{i=1}^{m}\) is a locally defined orthonormal frame field on \((M, g)\), and \(\Delta_g\) is the (positive) Laplacian of \((M, g)\) acting on \(C^\infty(M)\).

Indeed, we have by using Lemma 3.2 twice,

\[\theta(\Delta V) = - \sum_{i=1}^{m} \left\{ \theta(\nabla_{e_i} (\nabla V)) - \theta(\nabla_{e_i} e_i V) \right\}
= - \sum_{i=1}^{m} \left\{ e_i(\theta(V)) + \frac{1}{2} [\alpha(e_i), \theta(\nabla e_i V)] \\
- \nabla e_i e_i(\theta(V)) - \frac{1}{2} [\alpha(\nabla e_i e_i), \theta(V)] \right\}
= - \sum_{i=1}^{m} \left\{ e_i(\theta(V)) + \frac{1}{2} [\alpha(e_i), \theta(V)] \right\}
+ \frac{1}{2} \left[ \alpha(e_i), e_i(\theta(V)) + \frac{1}{2} \alpha(e_i), \theta(V) \right]
- \nabla e_i e_i(\theta(V)) - \frac{1}{2} [\alpha(\nabla e_i e_i), \theta(V)] \right\}
= - \sum_{i=1}^{m} \left\{ e_i(\theta(V)) - \nabla e_i e_i(\theta(V)) \right\}
- \sum_{i=1}^{m} \left\{ \frac{1}{2} e_i([\alpha(e_i), \theta(V)]) + \frac{1}{2} [\alpha(e_i), e_i(\theta(V))] \\
+ \frac{1}{4} [\alpha(e_i), [\alpha(e_i), \theta(V)]] - \frac{1}{2} [\alpha(\nabla e_i e_i), \theta(V)] \right\}. \quad (3.17)\]

Here, we have

\[e_i([\alpha(e_i), \theta(V)]) = [e_i(\alpha(e_i)), \theta(V)] + [\alpha(e_i), e_i(\theta(V))],\]
which we substitute into (3.17), and by definition of $\Delta_g$, we have (3.16).

(The second step) On the other hand, we have to consider

$$ -\sum_{i=1}^{m} R^h(V, \psi_* e_i) \psi_* e_i = -\sum_{i=1}^{m} R^h(L_{\psi(x)}^{-1} V, L_{\psi(x)}^{-1} \psi_* e_i)L_{\psi(x)}^{-1} \psi_* e_i. $$

(3.18)

Under the identification $T_eG \ni Z \leftrightarrow Z \in g$, we have

$$ T_eG \ni L^{-1}_{\psi(x)} \psi_* e_i \leftrightarrow \alpha(e_i) \in g, $$

(3.19)

$$ T_eG \ni L^{-1}_{\psi(x)} V \leftrightarrow \theta(V) \in g, $$

(3.20)

respectively. Because, we have

$$ L^{-1}_{\psi(x)} \psi_* e_i = \sum_{s=1}^{n} h(\psi_* e_i, X_s \psi(x)) X_s e $$

and

$$ \alpha(e_i) = \psi^* \theta(e_i) = \theta(\psi_* e_i) = \sum_{s=1}^{n} h(\psi_* e_i, X_s \psi(x)) \theta(X_s \psi(x)) $$

$$ = \sum_{s=1}^{n} h(\psi_* e_i, X_s \psi(x)) X_s, $$

(3.21)

which implies that (3.19). Analogously, we obtain (3.20).

Under this identification, the curvature tensor of $(G, h)$ is given as (see Kobayashi-Nomizu ([13, pp. 203–204])),

$$ R^h(X, Y)|_e = -\frac{1}{4} \text{ad}([X, Y]) \quad (X, Y \in g), $$

and then, we have

$$ \theta\left( -\sum_{i=1}^{m} R^h(V, \psi_* e_i) \psi_* e_i \right) = \frac{1}{4} \sum_{i=1}^{m} [[\theta(V), \alpha(e_i)], \alpha(e_i)] $$

$$ = \frac{1}{4} \sum_{i=1}^{m} [\alpha(e_i), [\alpha(e_i), \theta(V)]]. $$

(3.22)
(The third step) By (3.16) and (3.21), for \( V \in \Gamma(\psi^{-1}TG) \), we have

\[
\theta \left( \Delta V - \sum_{i=1}^{m} R^h_i(V, \psi e_i) \psi e_i \right) 
\]

\[
= \Delta g \theta(V) - \sum_{i=1}^{m} \left\{ 1/2 [e_i(\alpha(e_i)), \theta(V)] + [\alpha(e_i), e_i(\theta(V))] 
  \right. 
  \]

\[
+ 1/4 [\alpha(e_i), [\alpha(e_i), \theta(V)]] - 1/2 [\alpha(\nabla e_i e_i), \theta(V)] 
\right\} 
\]

\[
+ 1/4 \sum_{i=1}^{m} [\alpha(e_i), \alpha(e_i), \theta(V)] 
\]

\[
= \Delta g \theta(V) - \frac{1}{2} \sum_{i=1}^{m} e_i(\alpha(e_i)), \theta(V)] 
+ \sum_{i=1}^{m} [\alpha(e_i), e_i(\theta(V))] 
\]

\[
+ \frac{1}{2} \sum_{i=1}^{m} [\alpha(\nabla e_i e_i), \theta(V)] 
\]

\[
= \Delta g \theta(V) - \frac{1}{2} \left[ \sum_{i=1}^{m} (e_i(\alpha(e_i)) - \alpha(\nabla e_i e_i), \theta(V)] + \sum_{i=1}^{m} [\alpha(e_i), e_i(\theta(V))] \right] 
\]

\[
= \Delta g \theta(V) + \frac{1}{2} [\delta \alpha, \theta(V)] + \sum_{i=1}^{m} [\alpha(e_i), e_i(\theta(V))] \right] \] \quad (3.23)

(The fourth step) For \( V = \tau(\psi) \) in (3.22), since \( \theta(\tau(\psi)) = -\delta \alpha \), we have

\[
\theta(J(\tau(\psi))) = \Delta g \theta(\tau(\psi)) + \frac{1}{2} [\delta \alpha, \theta(\tau(\psi))] + \sum_{i=1}^{m} [\alpha(e_i), e_i(\theta(\tau(\psi))] 
\]

\[
= -\Delta g \delta \alpha - \frac{1}{2} [\delta \alpha, \delta \alpha] - \sum_{i=1}^{m} [\alpha(e_i), e_i(\delta \alpha)] 
\]

\[
= -\Delta g \delta \alpha - \sum_{i=1}^{m} [\alpha(e_i), e_i(\delta \alpha)] 
\]

\[
= -\Delta g \delta \alpha - \sum_{i=1}^{m} [\alpha(e_i), (d\delta \alpha)(e_i)] \right] \] \quad (3.24)
Then, (3.23) implies the desired (3.10).

4. Biharmonic curves from $\mathbb{R}$ into compact Lie groups

In this section, we consider the simplest case: $(M, g) = (\mathbb{R}, g_0)$ is the standard 1-dimensional Euclidean space, and $(G, h)$ is an $n$-dimensional compact Lie group with the bi-invariant Riemannian metric $h$.

4.1.

First, let $\psi : \mathbb{R} \ni t \mapsto \psi(t) \in (G, h)$, a $C^\infty$ curve in $G$. Then, $\alpha := \psi^*\theta$ is a $\mathfrak{g}$-valued 1-form on $\mathbb{R}$. So, $\alpha$ can be written at $t \in \mathbb{R}$ as

$$\alpha_t = F(t)dt,$$

where $F : \mathbb{R} \ni t \mapsto F(t) \in \mathfrak{g}$ is given by

$$F(t) = \alpha \left( \frac{\partial}{\partial t} \right) = \psi^* \theta \left( \frac{\partial}{\partial t} \right) = \theta \left( \psi_* \left( \frac{\partial}{\partial t} \right) \right).$$

(4.2)

Here, since

$$\psi'(t) := \psi_* \left( \frac{\partial}{\partial t} \right) = \sum_{s=1}^{n} h_{\psi(t)} \left( \psi_* \left( \frac{\partial}{\partial t} \right), X_s \psi(t) \right) X_s \psi(t),$$

(4.3)

we have

$$F(t) = \sum_{s=1}^{n} h_{\psi(t)} \left( \psi_* \left( \frac{\partial}{\partial t} \right), X_s \psi(t) \right) X_s,$$

(4.4)

so that we have the following correspondence:

$$T_{\psi(t)} G \ni L_{\psi(t)}^{-1} \psi'(t) = \sum_{s=1}^{n} h_{\psi(t)} \left( \psi'(t), X_s \psi(t) \right) X_s \leftrightarrow F(t) = \theta \left( \psi_* \left( \frac{\partial}{\partial t} \right) \right) \in \mathfrak{g}.$$

(4.5)

4.2.

We have that
\[ \delta \alpha = -F'(t), \quad (4.6) \]

since we have \( \delta \alpha = -e_1(\alpha(e_1)) = -e_1(F(t)) = -F'(t) \).

Therefore, we have \( \psi : (\mathbb{R}, g_0) \to (G, h) \) is harmonic if and only if

\[
\begin{align*}
\delta \alpha &= 0 \iff F' = 0 \\
&\iff \alpha = X \otimes dt \quad (\text{for some } X \in \mathfrak{g}) \\
&\iff \psi : \mathbb{R} \to (G, h), \text{ is a geodesic},
\end{align*}
\]

(4.7)

since

\[
F(t) = \theta(\psi'(t)) = L_{\psi(t)}^{-1} \psi'(t),
\]

(4.8)

we have

\[
\psi'(t) = L_{\psi(t)} X = X_{\psi(t)},
\]

(4.9)

for some \( X \in \mathfrak{g} \) which yields that

\[
\psi(t) = x \exp(tX).
\]

Therefore, any geodesic through \( \psi(0) = x \) is given by

\[
\psi(t) = x \exp(tX), \quad (t \in \mathbb{R})
\]

(4.10)

for some \( X \in \mathfrak{g} \).

On the other hand, we want to determine a biharmonic curve \( \psi : (\mathbb{R}, g_0) \to (G, h) \). By (4.6), we have

\[
\delta d \delta \alpha = -\frac{\partial^2}{\partial t^2} (-F'(t)) = F^{(3)}(t),
\]

(4.11)

and

\[
\text{Trace}_g[\alpha, d \delta \alpha] = \left[ \alpha \left( \frac{\partial}{\partial t} \right), d \delta \alpha \left( \frac{\partial}{\partial t} \right) \right] = [F(t), F''(t)],
\]

(4.12)

so by (4.9), (4.10), and (3.16) in Corollary 3.5, \( \psi : (\mathbb{R}, g_0) \to (G, h) \) is biharmonic if and only if
\[ F^{(3)} - [F(t), F''(t)] = 0. \] (4.13)

4.3.

For a \( C^\infty \) curve \( \psi : \mathbb{R} \to G \), let \( \psi(t) := \exp X(t) \), where \( X(t) \in \mathfrak{g} \). Then,

\[ F(t) = \theta \left( \psi_* \left( \frac{\partial}{\partial t} \right) \right), \quad \psi_* \left( \frac{\partial}{\partial t} \right) \in T_{\psi(t)} G, \] (4.14)

and by the following formula (cf. [10, p. 95])

\[ \exp_* X = L_{\exp X} e \circ \frac{1 - e^{-\text{ad} X}}{\text{ad} X} \quad (X \in \mathfrak{g}), \]

we have

\[ \psi_* \left( \frac{\partial}{\partial t} \right) = \exp_* X(t) X'(t) \]
\[ = L_{\exp X(t)} e \left( \sum_{n=0}^{\infty} \frac{(-\text{ad} X(t))^n}{(n+1)!} (X'(t)) \right). \] (4.15)

Since \( \theta \) is a left invariant 1-form, we have

\[ F(t) = \sum_{n=0}^{\infty} \frac{(-\text{ad} X(t))^n}{(n+1)!} (X'(t)). \] (4.16)

4.4.

The initial value problem

\[
\begin{align*}
F^{(3)}(t) &= [F(t), F''(t)], \\
F(0) &= B_0, \quad F'(0) = B_1, \quad F''(0) = B_2,
\end{align*}
\] (4.17)

for every \( B_i \in \mathfrak{g} \ (i = 0, 1, 2) \), has a unique solution \( F(t) \). Assume that \( X(t) \) is a real analytic curve in \( t \), and \( X(0) = 0 \). Then, \( F(t) \) is also real analytic in \( t \), and we can write as

\[ X(t) = \sum_{n=1}^{\infty} A_n t^n, \quad F(t) = \sum_{n=0}^{\infty} B_n t^n. \] (4.18)
By (4.16), we have
\[
F(t) = X'(t) + \frac{1}{2}[-X(t), X'(t)] + \frac{1}{6}[-X(t), [-X(t), X'(t)]]
+ \sum_{n=3}^{\infty} \frac{(-\text{ad} X(t))^n}{(n + 1)!}(X'(t)). \tag{4.19}
\]
Since \(X'(t) = \sum_{m=0}^{\infty} A_{m+1}(m+1)t^m\), we have
\[
\frac{1}{2}[-X(t), X'(t)] = -\frac{1}{2}[A_1, A_2]t^2 + O(t^3),
\]
and
\[
\frac{1}{6}[-X(t), [-X(t), X'(t)]] = O(t^3),
\]
so that we have
\[
F(t) = A_1 + 2A_2t + \left(3A_3 - \frac{1}{2}[A_1, A_2]\right)t^2 + O(t^3).
\]
Continuing this process, we have
\[
\begin{align*}
B_0 &= A_1, \\
B_1 &= 2A_2, \\
B_2 &= 3A_3 - \frac{1}{2}[A_1, A_2], \\
&\quad \vdots \\
B_n &= (n + 1)A_{n+1} + G_n(A_1, \ldots, A_n),
\end{align*}
\tag{4.20}
\]
where \(G_n(x_1, \ldots, x_n)\) is a polynomial in \((x_1, \ldots, x_n)\). Notice that for arbitrary given data \((B_0, B_1, B_2)\), all \(B_n\) \((n = 0, 1, \ldots)\) are determined, and by using (4.20), one can determine all \(A_n\) \((n = 1, 2, \ldots)\), uniquely. Therefore, by summarizing the above, we obtain

**Theorem 4.1** For every \(C^\infty\) curve \(\psi : \mathbb{R} \to G\), \(\psi(t) = \exp X(t)\) \((X(t) \in \mathfrak{g})\), and
\[ \alpha \left( \frac{\partial}{\partial t} \right) = F(t) = \sum_{n=0}^{\infty} \frac{(-\text{ad} \ X(t))^n}{(n+1)!} (X'(t)). \] (4.21)

(1) \( \psi : (\mathbb{R}, g_0) \to (G, h) \) is biharmonic if and only if
\[ F^{(3)}(t) = [F(t), F''(t)]. \] (4.22)

(2) The initial value problem
\[
\begin{cases}
F^{(3)}(t) = [F(t), F''(t)], \\
F(0) = B_0, \quad F'(0) = B_1, \quad F''(0) = B_2,
\end{cases}
\] (4.23)

has a unique solution \( F(t) \) for arbitrary given data \((B_0, B_1, B_2)\) in \( g \).

(3) Assume that \( \psi : (\mathbb{R}, g_0) \to (G, h) \) is a real analytic biharmonic curve with \( \psi(0) = e \). Then, \( \psi(t) \) is uniquely determined by \( F(0) = B_0, \ F'(0) = B_1, \) and \( F''(0) = B_2 \).

Example If \( G \) is abelian, let us consider a \( C^\infty \) curve \( \psi : \mathbb{R} \to G \) given by \( \psi(t) = \exp X(t) \). Then, \( F(t) = X'(t) \), and \( \psi : (\mathbb{R}, g_0) \to (G, h) \) is biharmonic if and only if \( F^{(3)}(t) = X^{(4)}(t) = 0 \). Then, \( X(t) = A_0 + A_1 t + A_2 t^2 + A_3 t^3 \). Thus, every biharmonic curve \( \psi : (\mathbb{R}, g_0) \to (G, h) \) with \( \psi(0) = e \) is given by
\[ \psi(t) = \exp(A_1 t + A_2 t^2 + A_3 t^3). \]

4.5.

Now we will solve the ODE (4.22) for a biharmonic isometric immersion \( \psi : (\mathbb{R}, g_0) \to G \) and a \( g \)-valued curve \( F(t) \) in the case of \( g = \mathfrak{su}(2) \). Let \( G = SU(2) \) with the bi-invariant Riemannian metric \( h \) which corresponds to the following \( \text{Ad}(SU(2)) \)-invariant inner product \( \langle , \rangle \) on
\[ \mathfrak{g} = \mathfrak{su}(2) = \{ X \in M(2, \mathbb{C}); X + \overline{X} = 0, \text{Tr}(X) = 0 \}, \]
\[ \langle X, Y \rangle = -2\text{Tr}(XY) \quad (X, Y \in \mathfrak{su}(2)). \]

If we choose
\[ X_1 = \begin{pmatrix} \frac{\sqrt{-1}}{2} & 0 \\ 0 & -\frac{\sqrt{-1}}{2} \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & \frac{\sqrt{-1}}{2} \\ \frac{\sqrt{-1}}{2} & 0 \end{pmatrix}, \]
then \( \{X_1, X_2, X_3\} \) is an orthonormal basis of \((\mathfrak{su}(2), \langle , \rangle)\), and satisfies the Lie bracket relations:

\[ [X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2. \]

Thus, the ODE (4.22) becomes

\[
\begin{align*}
    y_1^{(3)} &= y_2 y_3''' - y_3 y_2''', \\
    y_2^{(3)} &= y_3 y_1''' - y_1 y_3''', \\
    y_3^{(3)} &= y_1 y_2''' - y_2 y_1''',
\end{align*}
\]

which is equivalent to

\[ y^{(3)} = y \times y'', \]

where \( y := \langle y_1, y_2, y_3 \rangle \in \mathbb{R}^3 \), and \( a \times b \) stands for the vector cross product in \( \mathbb{R}^3 \). Notice here that \( g \) is non-abelian, but our equation (4.22) turns to the vector equation (4.26) depending on the time \( t \) of the Euclidean space \( \mathbb{R}^3 \) by identifying \( g \ni \sum_{i=1}^{3} y_i X_i \mapsto (y_1, y_2, y_3) \in \mathbb{R}^3 \).

Then, the ODE (4.25) can be solved as follows:

Let \( x(s) = \langle x_1(s), x_2(s), x_3(s) \rangle \) be a \( C^\infty \) curve in \( \mathbb{R}^3 \) with arc length parameter \( s \), and then

\[ y(s) = x'(s) = e_1(s). \]

Let \( \{e_1(s), e_2(s), e_3(s)\} \) be the Frenet frame field along \( x(s) \). Recall the Frenet-Serret formula:

\[
\begin{align*}
    e_1' &= \kappa e_2 \\
    e_2' &= -\kappa e_1 + \tau e_3 \\
    e_3' &= -\tau e_2
\end{align*}
\]

where \( \kappa \) and \( \tau \) are the curvature and torsion of \( x(s) \), respectively. Then, we have
\[
\begin{aligned}
\begin{cases}
y' &= \kappa e_2 \\
y'' &= -\kappa^2 e_1 + \kappa' e_2 + \kappa \tau e_3 \\
y''' &= -3\kappa \kappa' e_1 + (\kappa'' - \kappa^3 - \kappa \tau^2) e_2 + (2\kappa' \tau + \kappa \tau') e_3.
\end{cases}
\end{aligned}
\]  
(4.26)

Thus, (4.24) is equivalent to
\[
-3\kappa \kappa' e_1 + (\kappa'' - \kappa^3 - \kappa \tau^2) e_2 + (2\kappa' \tau + \kappa \tau') e_3 \\
= e_1 \times (-\kappa^2 e_1 + \kappa' e_2 + \kappa \tau e_3) \\
= -\kappa \tau e_2 + \kappa' e_3
\]  
(4.27)

which is equivalent to
\[
\begin{cases}
-3\kappa \kappa' &= 0 \\
\kappa'' - \kappa^3 - \kappa \tau^2 &= -\kappa \tau \\
2\kappa' \tau + \kappa \tau' &= \kappa'.
\end{cases}
\]  
(4.28)

Then, the first equation of (4.28) turns out that \((\kappa^2)' = 0\), that is, \(\kappa^2\) is constant, i.e., \(\kappa \equiv 0\), or \(\kappa \equiv \kappa_0 \neq 0\). In the case that \(\kappa \equiv 0\), the solution of (4.28), \(x(s)\), is a line in \(\mathbb{R}^3\).

For the case that \(\kappa \equiv \kappa_0 \neq 0\), the only solution of (4.24) is
\[
\begin{cases}
\kappa \equiv \kappa_0 \neq 0, \\
\tau \equiv \tau_0, \, \text{and} \\
\kappa_0^2 &= \tau_0(1 - \tau_0),
\end{cases}
\]  
(4.29)

and the unique solution of (4.25) is given by
\[
x(s) = \begin{pmatrix} x_1(s) \\ x_2(s) \\ x_3(s) \end{pmatrix} = \begin{pmatrix} a \cos \frac{s}{\sqrt{a^2+1}} + b \\ a \sin \frac{s}{\sqrt{a^2+1}} + b \\ \frac{s}{\sqrt{a^2+1}} + b \end{pmatrix}
\]  
(4.30)

for some positive constant \(a > 0\) and some constant \(b\). Thus, \(F(s)\) is given as follows:
\[
F(s) = x'(s) = \sum_{i=1}^{3} x_i'(s) X_i \\
= \left(-\frac{a}{\sqrt{a^2 + 1}} \sin \frac{s}{\sqrt{a^2 + 1}}\right) X_1 + \left(\frac{a}{\sqrt{a^2 + 1}} \cos \frac{s}{\sqrt{a^2 + 1}}\right) X_2 \\
+ \left(\frac{1}{\sqrt{a^2 + 1}}\right) X_3,
\]

(4.31)

for any constant \(a > 0\). Conversely, it is easy to see that every such \(F(s)\) in (4.31) is a solution of (4.22): \(F^{(3)} = [F(s), F''(s)]\).

**Remark** It is still difficult to determine \(X(t)\) to satisfy (4.21):

\[
F(t) = \sum_{n=0}^{\infty} \frac{(- \text{ad} X(t))^n}{(n + 1)!} (X'(t)),
\]

in the case of \(su(2)\).

5. **Biharmonic maps from an open domain in \(\mathbb{R}^2\)**

In this section, we consider a biharmonic map \(\psi : (\mathbb{R}^2, g) \supset \Omega \rightarrow (G, h)\). Here, we assume that \(G\) is a linear compact Lie group, i.e., \(G\) is a subgroup of the unitary group \(U(N) \subset GL(N, \mathbb{C})\) of degree \(N\) with a bi-invariant Riemannian metric \(h\) on \(G\). Let \(g\) be the Lie algebra of \(G\) which is a Lie subalgebra of the Lie algebra \(u(N)\) of \(U(N)\). The Riemannian metric \(g\) on \(\mathbb{R}^2\) is a conformal metric which is given by \(g = \mu^2 g_0\) with a \(C^\infty\) positive function \(\mu\) on \(\Omega\) and \(g_0 = dx \cdot dx + dy \cdot dy\), where \((x, y)\) is the standard coordinate on \(\mathbb{R}^2\).

Let \(\psi : \Omega \ni (x, y) \mapsto \psi(x, y) = (\psi_{ij}(x, y)) \in U(N)\) a \(C^\infty\) map. Let us consider

\[
\frac{\partial \psi}{\partial x} := \left(\frac{\partial \psi_{ij}}{\partial x}\right), \quad \frac{\partial \psi}{\partial y} := \left(\frac{\partial \psi_{ij}}{\partial y}\right).
\]

Then,

\[
A_x := \psi^{-1} \frac{\partial \psi}{\partial x}, \quad A_y := \psi^{-1} \frac{\partial \psi}{\partial y}
\]

(5.1)
are $g$-valued $C^\infty$ functions on $\Omega$. It is known that, for two given $g$-valued 1-forms $A_x$ and $A_y$ on $\Omega$, there exists a $C^\infty$ mapping $\psi : \Omega \to G$ satisfying the equations (5.1) if the integrability condition holds:

$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} + [A_x, A_y] = 0. \quad (5.2)$$

The pull back of the Maurer-Cartan form $\theta$ by $\psi$ is given by

$$\alpha := \psi^* \theta = \psi^{-1} d\psi = \psi^{-1} \frac{\partial \psi}{\partial x} dx + \psi^{-1} \frac{\partial \psi}{\partial y} dy
= A_x \, dx + A_y \, dy, \quad (5.3)$$

which is a $g$-valued 1-form on $\Omega$.

Recall that the codifferential $\delta \alpha$ of a $g$-valued 1-form $\alpha = A_x \, dx + A_y \, dy$, where $A_x = \psi^{-1}(\partial \psi/\partial x)$ and $A_y = \psi^{-1}(\partial \psi/\partial y)$, is given by

$$\delta \alpha = -\mu^{-2} \left\{ \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y \right\}. \quad (5.4)$$

Then, we have the following well known facts:

**Lemma 5.1** We have

$$\delta \alpha = -\mu^{-2} \left\{ \frac{\partial}{\partial x} \left( \psi^{-1} \frac{\partial \psi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \psi^{-1} \frac{\partial \psi}{\partial y} \right) \right\} \quad (5.5)$$

$$= -\mu^{-2} \left\{ \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right\}. \quad (5.6)$$

Therefore, the following three statements are equivalent:

(i) \( \psi : (\Omega, g) \to (G, h) \) is harmonic,

(ii) \( \delta \alpha = 0 \), \hspace{1cm} (5.7)

(iii) \( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} = 0 \). \hspace{1cm} (5.8)

Next, calculate the Laplacian $\Delta_g$ of $(\mathbb{R}^2, g)$ for $g = \mu^2 g_0$. We obtain
\[
\Delta_g = -\sum_{i,j=1}^{2} g^{ij} \left( \frac{\partial^2}{\partial x^i \partial x^j} - \sum_{k=1}^{2} \Gamma^k_{ij} \frac{\partial}{\partial x^k} \right)
= -\mu^{-2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right). \tag{5.9}
\]

Thus we have
\[
\delta d\delta \alpha = \Delta_g (\delta \alpha)
= \mu^{-2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left[ \mu^{-2} \left\{ \frac{\partial}{\partial x} \left( \psi^{-1} \frac{\partial \psi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \psi^{-1} \frac{\partial \psi}{\partial y} \right) \right\} \right]
= \mu^{-2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left[ \mu^{-2} \left\{ \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right\} \right]
= -\mu^{-2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\delta \alpha). \tag{5.10}
\]

On the other hand, by taking an orthonormal local frame field \(\{e_1, e_2\}\) of \((\mathbb{R}^2, g)\), as \(e_1 = \mu^{-1}(\partial/\partial x)\), \(e_2 = \mu^{-1}(\partial/\partial y)\), we have
\[
\text{Trace}_g ([\alpha, d\delta \alpha]) = \left[ \alpha(e_1), d\delta \alpha(e_1) \right] + \left[ \alpha(e_2), d\delta \alpha(e_2) \right]
= -\mu^{-2} \left[ A_x, \frac{\partial}{\partial x} \left( \mu^{-2} \left\{ \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right\} \right) \right]
- \mu^{-2} \left[ A_y, \frac{\partial}{\partial y} \left( \mu^{-2} \left\{ \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right\} \right) \right]
= \mu^{-2} \left[ A_x, \frac{\partial}{\partial x} (\delta \alpha) \right] + \mu^{-2} \left[ A_y, \frac{\partial}{\partial y} (\delta \alpha) \right]. \tag{5.11}
\]

By (5.10) and (5.11), we obtain
\[
\delta d\delta \alpha + \text{Trace}_g ([\alpha, d\delta \alpha])
= -\mu^{-2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\delta \alpha) + \mu^{-2} \left[ A_x, \frac{\partial}{\partial x} (\delta \alpha) \right] + \mu^{-2} \left[ A_y, \frac{\partial}{\partial y} (\delta \alpha) \right]
= -\mu^{-2} \left\{ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\delta \alpha) - \frac{\partial}{\partial x} [A_x, \delta \alpha] - \frac{\partial}{\partial y} [A_y, \delta \alpha] \right\}. \tag{5.12}
\]
where in the last equation in (5.11), we only notice that
\[
\frac{\partial}{\partial x}[A_x, \delta \alpha] + \frac{\partial}{\partial y}[A_y, \delta \alpha] \\
= \left[ \frac{\partial}{\partial x} A_x, \delta \alpha \right] + \left[ A_x, \frac{\partial}{\partial x}(\delta \alpha) \right] + \left[ \frac{\partial}{\partial y} A_y, \delta \alpha \right] + \left[ A_y, \frac{\partial}{\partial y}(\delta \alpha) \right] \\
= \left[ \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y, \delta \alpha \right] + \left[ A_x, \frac{\partial}{\partial x}(\delta \alpha) \right] + \left[ A_y, \frac{\partial}{\partial y}(\delta \alpha) \right] \\
= [-\mu^{-2} \delta \alpha, \delta \alpha] + \left[ A_x, \frac{\partial}{\partial x}(\delta \alpha) \right] + \left[ A_y, \frac{\partial}{\partial y}(\delta \alpha) \right] \\
= \left[ A_x, \frac{\partial}{\partial x}(\delta \alpha) \right] + \left[ A_y, \frac{\partial}{\partial y}(\delta \alpha) \right].
\]

Thus, we have

**Theorem 5.2** Let \( \Omega \) be an open subset of \( \mathbb{R}^2 \), \( g = \mu^2 g_0 \), a Riemannian metric conformal to the standard metric \( g_0 \) on \( \Omega \) with a \( C^\infty \) positive function \( \mu \) on \( \Omega \), and \( \psi : \Omega \to G \), a \( C^\infty \) map of \( \Omega \) into a compact linear Lie group \((G, h)\) with bi-invariant Riemannian metric \( h \). Then,

1. The 1-form \( \alpha \) satisfies \( d\alpha + (1/2)[\alpha \wedge \alpha] = 0 \) which is equivalent to
   \[
   \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} + [A_x, A_y] = 0. \tag{5.13}
   \]

2. The following three are equivalent:
   
   (i) \( \psi : (\Omega, g) \to (G, h) \) is harmonic,
   
   (ii) \( \delta \alpha = 0 \), \tag{5.14}
   
   (iii) \( \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y = 0. \tag{5.15} \)

3. The following three are equivalent:
   
   (i) \( \psi : (\Omega, g) \to (G, h) \) is biharmonic,
   
   (ii) \( \delta \delta \alpha + \text{Trace}_g([\alpha, d\delta \alpha]) = 0 \), \tag{5.16}
(iii) \( \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\delta \alpha) - \frac{\partial}{\partial x} [A_x, \delta \alpha] - \frac{\partial}{\partial y} [A_y, \delta \alpha] = 0. \) (5.17)

(4) Let us consider two \( g \)-valued 1-forms \( \beta \) and \( \Theta \) on \( \Omega \), defined by

\[
\beta := [A_x, \delta \alpha] dx + [A_y, \delta \alpha] dy,
\]

\[
\Theta := d\delta \alpha - \beta,
\]

respectively. Then, \( \psi : (\Omega, g) \rightarrow (G, h) \) is biharmonic if and only if

\[
\delta \Theta = 0.
\] (5.20)

Proof. (1) is clear. We see already (2) and (3). For (4), we only have to see that (5.17) is equivalent to

\[
0 = -\Delta_g (\delta \alpha) + \delta \beta = -\delta (d\delta \alpha - \beta) = -\delta \Theta
\] (5.21)

where

\[
\Theta := d\delta \alpha - \beta
\]

\[
= \frac{\partial}{\partial x} (\delta \alpha) dx + \frac{\partial}{\partial y} (\delta \alpha) dy - [A_x, \delta \alpha] dx - [A_y, \delta \alpha] dy
\]

\[
= \left\{ \frac{\partial}{\partial x} (\delta \alpha) - [A_x, \delta \alpha] \right\} dx + \left\{ \frac{\partial}{\partial y} (\delta \alpha) - [A_y, \delta \alpha] \right\} dy.
\] (5.22)

\[\square\]

6. Complexification of the biharmonic map equation

We use the complex coordinate \( z = x + iy \) \( (i = \sqrt{-1}) \) in \( \Omega \), and we put \( A_z = (1/2)(A_x - i A_y) \) and \( A_\bar{z} = (1/2)(A_x + i A_y) \) which are \( g^C \)-valued functions with \( A_\bar{z} = \bar{A_z} \). Then, it is well known that

\[
\frac{\partial}{\partial z} A_z + \frac{\partial}{\partial \bar{z}} A_\bar{z} = \frac{1}{2} \left\{ \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y \right\},
\]

\[
\frac{\partial}{\partial \bar{z}} A_\bar{z} - \frac{\partial}{\partial z} A_z + [A_z, A_\bar{z}] = \frac{i}{2} \left\{ \frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x + [A_x, A_y] \right\},
\] (6.1)
and also
\[ \alpha = A_x dx + A_y dy = A_z dz + A_\tau d\tau, \]
\[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial z \partial \tau}, \]
\[ \delta \alpha = -\mu^{-2} \left( \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y \right) = -2\mu^{-2} \left( \frac{\partial}{\partial z} A_z + \frac{\partial}{\partial z} A_\tau \right). \]  
(6.2)

Then, the condition (5.20) is equivalent to
\[ \delta \tilde{\Theta} = 0, \]  
(6.3)

where
\[ \tilde{\Theta} := \left\{ \frac{\partial}{\partial z} (\delta \alpha) - [A_z, \delta \alpha] \right\} dz + \left\{ \frac{\partial}{\partial z} (\delta \alpha) - [A_\tau, \delta \alpha] \right\} dz. \]  
(6.4)

The integrability condition (5.13) is equivalent to
\[ \frac{\partial}{\partial z} A_\tau - \frac{\partial}{\partial \tau} A_z + [A_z, A_\tau] = 0 \]  
(6.5)

7. Determination of biharmonic maps

In this section, we want to show how to determine all the biharmonic maps of \((\Omega, g)\) into a compact Lie group \((G, h)\) where \(g = \mu^2 g_0\) with a positive \(C^\infty\) function on \(\Omega\) and \(h\) is a bi-invariant Riemannian metric on \(G\). Our method to obtain all the biharmonic maps can be divided into three steps:

(The first step) We first solve the equation:
\[ \frac{\partial}{\partial z} B_z + \frac{\partial}{\partial z} B_\tau = 0 \]  
(7.1)

Notice that, if these \(B_z\) and \(B_\tau\) satisfy furthermore, the integrability condition
\[ \frac{\partial}{\partial z} B_\tau - \frac{\partial}{\partial z} B_z + [B_z, B_\tau] = 0, \]  
(7.2)
then, there exists a harmonic map \( \Psi : (\Omega, g) \to (G, h) \) such that

\[
\begin{align*}
\Phi^{-1} \frac{\partial \Psi}{\partial z} &= B_z, \\
\Phi^{-1} \frac{\partial \Phi}{\partial z} &= B_{\bar{z}},
\end{align*}
\]

(7.3)

and the converse is true.

(The second step) For such two \( g^C \)-valued functions \( B_z \) and \( B_{\bar{z}} \) on \( \Omega \) satisfying (7.1) not necessarily satisfying (7.2), we should detect two \( g^C \)-valued functions \( A_z \) and \( A_{\bar{z}} \) on \( \Omega \) satisfying that

\[
\begin{align*}
\frac{\partial}{\partial z} \left( -2\mu^2 \left( \frac{\partial A_z}{\partial z} + \frac{\partial A_{\bar{z}}}{\partial z} \right) \right) &- \left[ A_z, -2\mu^2 \left( \frac{\partial A_z}{\partial z} + \frac{\partial A_{\bar{z}}}{\partial z} \right) \right] = B_z, \\
\frac{\partial}{\partial \bar{z}} \left( -2\mu^2 \left( \frac{\partial A_z}{\partial z} + \frac{\partial A_{\bar{z}}}{\partial z} \right) \right) &- \left[ A_{\bar{z}}, -2\mu^2 \left( \frac{\partial A_z}{\partial z} + \frac{\partial A_{\bar{z}}}{\partial z} \right) \right] = B_{\bar{z}}, \\
\frac{\partial}{\partial z} A_{\bar{z}} - \frac{\partial}{\partial \bar{z}} A_z + [A_z, A_{\bar{z}}] &= 0.
\end{align*}
\]

(7.4)

(The third step) Finally, for the above \( g^C \)-valued functions \( A_z \) and \( A_{\bar{z}} \) on \( \Omega \) satisfying (7.4) and \( a \in G \), there exists a \( C^\infty \) mapping \( \psi : \Omega \to G \) satisfying that

\[
\begin{align*}
\psi(x_0, y_0) &= a, \\
\psi^{-1} \frac{\partial \psi}{\partial z} &= A_z, \\
\psi^{-1} \frac{\partial \psi}{\partial \bar{z}} &= A_{\bar{z}}.
\end{align*}
\]

(7.5)

Then, \( \psi : (\Omega, g) \to (G, h) \) is a biharmonic map due to (5.20), (6.1) and (7.4), and conversely, every biharmonic map \( \psi : (\Omega, g) \to (G, h) \) could be obtained in this way. To do the these procedures rigorously, let us define

**Definition 7.1**

(1) Let us define the four sets \( \Lambda, \Lambda_1, \Lambda_2, \) and \( \Lambda_0 \):

- Let \( \Lambda \) be the set of all \( g \)-valued two functions \( (A_x, A_y) \) on \( \Omega \), (or all \( g^C \)-valued two functions \( (A_z, A_{\bar{z}}) \) on \( \Omega \) with \( A_{\bar{z}} = \overline{A_z} \),
• let $\Lambda_1$, the set of $(A_x, A_y) \in \Lambda$ which satisfy the harmonic map equation (5.12) (or (7.1)),
• let $\Lambda_2$, the set of $(A_x, A_y) \in \Lambda$ which satisfy the biharmonic map equation (5.17) (or (6.1)), and
• let $\Lambda_0$, the set of $(A_x, A_y) \in \Lambda$ which satisfy the integrability condition (5.13), (or (6.3)), respectively.

(2) Let us define two sets $\Xi$ and $\Xi_1$:

• Let $\Xi$ be the set of all $g$-valued two real analytic functions $(B_x, B_y)$ on $\Omega$ (or $g^c$-valued two real analytic functions $(B_z, B_{\overline{z}})$ on $\Omega$ with $B_{\overline{z}} = B_z$), and
• let $\Xi_1$, the set of all $(B_x, B_y) = (B_z, B_{\overline{z}}) \in \Xi$ satisfying the harmonic map equation (7.1), respectively.

Definition 7.2 Let us define two $C^\infty$ mappings $\Phi_i$ ($i = 1, 2$) of $\Lambda$ into $\Xi$ by

$$\Phi_1(A_x, A_y) := \left( \frac{\partial}{\partial x} \left( -\mu^{-2} \left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right) \right) - \left[ A_x, -\mu^{-2} \left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right) \right], \right.$$  

$$\left. \frac{\partial}{\partial y} \left( -\mu^{-2} \left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right) \right) - \left[ A_y, -\mu^{-2} \left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right) \right] \right), \tag{7.6}$$

and also

$$\Phi_2(A_x, A_y) := \left( -\mu^{-2} \left( \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} \right) - \frac{\partial}{\partial y} [A_x, A_y] \right)$$

$$- \frac{\partial \mu^{-2}}{\partial x} \left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right) - \left[ A_x, -\mu^{-2} \left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right) \right],$$

$$- \mu^{-2} \left( \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} \right) - \frac{\partial}{\partial x} [A_x, A_y] \right)$$

$$- \frac{\partial \mu^{-2}}{\partial y} \left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right) - \left[ A_y, -\mu^{-2} \left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right) \right], \tag{7.7}$$

respectively.
Then, we obtain

**Theorem 7.3** Assume that $\Omega$ be a simply connected open domain in $\mathbb{R}^2$, and $\mu$ is a positive real analytic function on $\Omega$. Then, we have:

1. For every $(B_x, B_y) = (B_z, B_z) \in \Xi$ there exists $(A_x, A_y) = (A_z, A_z) \in \Lambda$ such that $\Phi_2(A_x, A_y) = (B_x, B_y)$ (or $\Phi_2(A_z, A_z) = (B_z, B_z)$). The solution $(A_x, A_y) = (A_z, A_z)$ is uniquely determined by the initial data $A_x(x_0, y), A_y(x_0, y), (\partial A_x/\partial x)(x_0, y)$ and $(\partial A_y/\partial x)(x_0, y), (x_0, y) \in \Omega$.

2. $\Phi_1 = \Phi_2$ on $\Lambda_0$.

3. $\Phi_1^{-1}(\Xi_1) = \Lambda_2$, and $\Phi_1(\Lambda_2 \cap \Lambda_0) = \Phi_2(\Lambda_2 \cap \Lambda_0) = \Xi_1$.

**Proof.** For (1), by definition of $\Phi_2$, that $\Phi_2(A_x, A_y) = (B_x, B_y)$ is equivalent to the following two equations:

\[
\frac{\partial^2 A_x}{\partial x^2} = -\frac{\partial^2 A_x}{\partial y^2} + \frac{\partial}{\partial y} [A_x, A_y] - \mu^2 \frac{\partial \mu^{-2}}{\partial x} \left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right) - \mu^2 \left[ A_x, -\mu^{-2} \left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right) \right] - \mu^2 B_x,
\]

and also

\[
\frac{\partial^2 A_y}{\partial x^2} = -\frac{\partial^2 A_y}{\partial y^2} + \frac{\partial}{\partial x} [A_x, A_y] - \mu^2 \frac{\partial \mu^{-2}}{\partial y} \left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right) - \mu^2 \left[ A_y, -\mu^{-2} \left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right) \right] - \mu^2 B_y.
\]

Notice that the system of (7.8) and (7.9) satisfies all the conditions of the theorem of Cauchy-Kovalevskaya when $n_i = 2$ ($i = 1, 2$) (cf. [7, p. 1305, 429 B], [14, p. 224], [11, p. 181]).

**Theorem 7.4** (Cauchy-Kovalevskaya) Let us consider the following Cauchy problem of unknown $N$ functions $u_i(t, x)$ ($i = 1, \ldots, N$) in $t$ and $x = (x_1, \ldots, x_m)$,
\[
\begin{align*}
\frac{\partial^{n_i} u_i}{\partial t^{n_i}} &= F_i(t, x, D^k_x D^p_x u_j) \quad (i = 1, \ldots, N), \\
\frac{\partial^k u_i}{\partial t^k}(t_0, x) &= \varphi^k_i(x) \quad (0 \leq k \leq n_i - 1; i = 1, \ldots, N),
\end{align*}
\tag{7.10}
\]

where, for \( p = (p_1, \ldots, p_m) \), \(|p| = p_1 + \cdots + p_m\), \( D^k_x D^p_x := (\partial^k / \partial t^k) \cdot (\partial |p| / \partial x_1^{p_1} \cdots \partial x_m^{p_m}) \) and in the right hand side of the first equation of (7.10), \( k \) and \( p \) satisfy
\[ k < n_j \quad \text{and} \quad k + |p| \leq n_j \quad (j = 1, \ldots, N). \]

Assume that each \( F_i \) and \( \varphi^k_i \) are real analytic functions. Then, there exists a real analytic solution \( u_i \) (\( i = 1, \ldots, N \)) of (7.10) and it is unique in the class of real analytic functions.

Then, for each \((B_x, B_y) \in \Xi\), there exists a real analytic solution \((A_x, A_y)\) of the Cauchy problem (7.8) and (7.9) with the initial condition:
\[
\begin{align*}
\left( \frac{\partial A_x}{\partial x} \right)(x_0, y) &= f_1(y), \quad A_x(x_0, y) = f_0(y), \\
\left( \frac{\partial A_y}{\partial x} \right)(x_0, y) &= g_1(y), \quad A_y(x_0, y) = g_0(y),
\end{align*}
\tag{7.11}
\]

and the real analytic solution \((A_x, A_y)\) is unique for real analytic functions \( f_i \) and \( g_i \) (\( i = 0, 1 \)). By taking this process at each point \((x_0, y_0)\) in \( \Omega \), we have a real analytic solution \((A_x, A_y)\) of (7.8) and (7.9) in an open neighborhood of \((x_0, y_0)\). Then, by the uniqueness theorem of the continuation of a real analytic function on a simply connected domain \( \Omega \), we have a solution \((A_x, A_y)\) of (7.8) and (7.9) on \( \Omega \). We have (1).

For (2), we have to see \( \Phi_1(A_x, A_y) = \Phi_2(A_x, A_y) \) for every \((A_x, A_y) \in \Lambda_0\), which follows from that
\[
\begin{align*}
\frac{\partial}{\partial x} \left( \mu^{-2} \left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right) \right)
&= \mu^{-2} \left( \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x \partial y} \right) + \mu^{-2} \left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right) \\
&= \mu^{-2} \left( \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial}{\partial y}[A_x, A_y] + \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right), \tag{7.12}
\end{align*}
\]
because of (5.13) and it is a similar for \((\partial/\partial y)(\mu^{-2}(\partial A_x/\partial x + \partial A_y/\partial y))\), so that we have (2).

For (3), due to (2), we only have to see \(\Phi_1^{-1}(\Xi_1) = \Lambda_2\) which is equivalent to that:

\[
\text{for all } (B_x, B_y) \in \Xi, \text{ exists a unique } (A_x, A_y) \in \Lambda_2 \text{ such that } \\
\Phi_1(A_x, A_y) = (B_x, B_y), \text{ and vice versa.}
\]

But, that \((B_x, B_y) = (B_z, B_{\bar{z}}) \in \Xi_1\) means that it satisfies the harmonic map equation (7.1). On the other hand, \(\Phi_1(A_x, A_y) = (B_x, B_y)\) means that \(\Phi_1(A_x, A_{\bar{z}}) = (B_z, B_{\bar{z}})\) which is equivalent to that the first two equations of (7.4) hold by definition of \(\Phi_1\), and notice here that \(\Phi_1(A_x, A_y) = (B_x, B_y)\) is equivalent to the two following equations

\[
\begin{align*}
\frac{\partial}{\partial x} \left( - \mu^{-2} \left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right) \right) - \left[ A_x, -\mu^{-2} \left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right) \right] &= B_x, \quad (7.13) \\
\frac{\partial}{\partial y} \left( - \mu^{-2} \left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right) \right) - \left[ A_y, -\mu^{-2} \left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right) \right] &= B_y, \quad (7.14)
\end{align*}
\]

which are also equivalent to

\[
\begin{align*}
\frac{\partial}{\partial z} \left( - 2\mu^{-2} \left( \frac{\partial A_z}{\partial z} + \frac{\partial A_{\bar{z}}}{\partial \bar{z}} \right) \right) - \left[ A_z, -2\mu^{-2} \left( \frac{\partial A_z}{\partial z} + \frac{\partial A_{\bar{z}}}{\partial \bar{z}} \right) \right] &= B_z, \quad (7.15) \\
\frac{\partial}{\partial \bar{z}} \left( - 2\mu^{-2} \left( \frac{\partial A_z}{\partial z} + \frac{\partial A_{\bar{z}}}{\partial \bar{z}} \right) \right) - \left[ A_{\bar{z}}, -2\mu^{-2} \left( \frac{\partial A_z}{\partial z} + \frac{\partial A_{\bar{z}}}{\partial \bar{z}} \right) \right] &= B_{\bar{z}}. \quad (7.16)
\end{align*}
\]

But, by inserting both (7.14) and (7.15) into

\[
\frac{\partial}{\partial \bar{z}} B_z + \frac{\partial}{\partial z} B_{\bar{z}} = 0, \quad (7.17)
\]

we obtain

\[
\begin{align*}
\frac{\partial^2}{\partial z \partial \bar{z}} \left( - 2\mu^{-2} \left( \frac{\partial A_z}{\partial z} + \frac{\partial A_{\bar{z}}}{\partial \bar{z}} \right) \right) - \frac{\partial}{\partial \bar{z}} \left[ A_{\bar{z}}, -2\mu^{-2} \left( \frac{\partial A_z}{\partial z} + \frac{\partial A_{\bar{z}}}{\partial \bar{z}} \right) \right] \\
+ \frac{\partial^2}{\partial z \partial \bar{z}} \left( - 2\mu^{-2} \left( \frac{\partial A_z}{\partial z} + \frac{\partial A_{\bar{z}}}{\partial \bar{z}} \right) \right) - \frac{\partial}{\partial z} \left[ A_z, -2\mu^{-2} \left( \frac{\partial A_z}{\partial z} + \frac{\partial A_{\bar{z}}}{\partial \bar{z}} \right) \right] \\
= 0,
\end{align*}
\]
which is just the biharmonic map equation for \((A_z, A_{\bar{z}})\): (6.1) \(\delta \tilde{\Theta} = 0\). By the same way, one can see also immediately \((A_x, A_y)\) satisfies the biharmonic map equation (5.20) if \((B_x, B_y)\) satisfies the harmonic map equation (5.15) by using Theorem 5.2, (5.6) and (5.22). Thus, we obtain \(\Phi_1^{-1}(\Xi_1) = \Lambda_2\) and (3).

**Remark** The solution \((A_x, A_y)\) in (1) of Theorem 7.3 can be chosen in such a way that they satisfy the integrability condition (5.13) at the initial value \((x_0, y)\),

\[
\frac{\partial A_y}{\partial x}(x_0, y) - \frac{\partial A_x}{\partial y}(x_0, y) + [A_x(x_0, y), A_y(x_0, y)] = 0, \tag{7.19}
\]

for each \(y\), i.e., the initial functions \(f_0, f_1\) and \(g_1\) may be chosen to satisfy that

\[
\frac{\partial A_x}{\partial y}(x_0, y) = g_1(y) + [f_0(y), f_1(y)]. \tag{7.20}
\]

Finally, we introduce a loop group formulation for biharmonic maps. We first consider a \(g^C\)-valued 1-forms

\[
\beta_\nu = \frac{1}{2} (1 - \nu)B_z dz + \frac{1}{2} (1 - \nu^{-1})B_{\bar{z}} d\bar{z} \tag{7.21}
\]

for a parameter \(\nu \in S^1\), which satisfy that

\[
d\beta_\nu + [\beta_\nu \wedge \beta_\nu] = 0 \quad (\forall \ \nu \in S^1), \tag{7.22}
\]

where for the definition of \([\beta_\nu \wedge \beta_\nu]\), see (3.13).

Next, we consider \(g^C\)-valued 1-forms

\[
\alpha_\nu = \frac{1}{2} (1 - \nu)A_z dz + \frac{1}{2} (1 - \nu^{-1})A_{\bar{z}} d\bar{z} \tag{7.23}
\]

which satisfy that
\begin{equation}
\begin{cases}
\frac{\partial}{\partial z}(\delta \alpha_\nu) - \left[\frac{1}{2}(1-\nu)A_z, \delta \alpha_\nu\right] = B_z, \\
\frac{\partial}{\partial z}(\delta \alpha_\nu) - \left[\frac{1}{2}(1-\nu)A_\tau, \delta \alpha_\nu\right] = B_\tau,
\end{cases}
\end{equation}

(7.24)

for each \(\nu \in S^1\). Here, the co-differentiation \(\delta \alpha_\nu\) of \(\alpha_\nu\) is given by

\[
\delta \alpha_\nu = -2\mu^{-2}\left(\frac{1}{2}(1-\nu)\frac{\partial}{\partial z}A_z + \frac{1}{2}(1-\nu^{-1})\frac{\partial}{\partial z}A_\tau\right).
\]

(7.25)

Then, the mapping \(\psi_\nu : \Omega \rightarrow G\) satisfying \(\psi_\nu^*\theta = \alpha_\nu\) is a biharmonic map of \((\Omega,g)\) into \((G,h)\) where \(g = \mu^2g_0\) for a positive \(C^\infty\) function \(\mu\) on \(\Omega\).

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Division of Mathematics
Graduate School of Information Sciences
Tohoku University
Aoba 6-3-09, Sendai, 980-8579
Japan

Current Address:
Institute for International Education
Tohoku University
Kawauchi 41, Aoba, Sendai, 980-8576
Japan
E-mail: urakawa@math.is.tohoku.ac.jp