A GEOMETRIC DESCRIPTION OF $m$-CLUSTER CATEGORIES

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ABSTRACT. We show that the $m$-cluster category of type $A_{n-1}$ is equivalent to a certain geometrically defined category of diagonals of a regular $nm+2$-gon. This generalises a result of Caldero, Chapoton and Schiffler for $m=1$. The approach uses the theory of translation quivers and their corresponding mesh categories. We also introduce the notion of the $m$-th power of a translation quiver and show how it can be used to realise the $m$-cluster category in terms of the cluster category.

INTRODUCTION

Let $n, m \in \mathbb{N}$ and let $\Pi$ be a regular $nm+2$-sided polygon. We show that a category $\mathcal{C}^m_{A_{n-1}}$ of diagonals can be associated to $\Pi$ in a natural way. The objects of $\mathcal{C}^m_{A_{n-1}}$ are the diagonals in $\Pi$ which divide $\Pi$ into two polygons whose numbers of sides are congruent to 2 modulo $m$, as considered in [PS]. A quiver $\Gamma^m_{A_{n-1}}$ can be defined on the set of such diagonals, with arrows given by a simple geometrical rule. It is shown that this quiver is a stable translation quiver in the sense of Riedtmann [Rie] with translation $\tau$ given by a certain rotation of the polygon. For a field $k$, the category $\mathcal{C}^m_{A_{n-1}}$ is defined as the mesh category associated to $(\Gamma^m_{A_{n-1}}, \tau)$.

Let $Q$ be a Dynkin quiver of type $A_{n-1}$, and let $D^b(kQ)$ denote the bounded derived category of finite dimensional $kQ$-modules. Let $\tau$ denote the Auslander-Reiten translate of $D^b(kQ)$, and let $S$ denote the shift. These are both autoequivalences of $D^b(kQ)$. Our main result is that $\mathcal{C}^m_{A_{n-1}}$ is equivalent to the quotient of $D^b(kQ)$ by the autoequivalence $\tau^{-1}S^m$. We thus obtain a geometric description of this category in terms of $\Pi$.

The $m$-cluster category $D^b(kQ)/\tau^{-1}S^m$ associated to $kQ$ was introduced in [Kel] and has also been studied by Thomas [Tho], Wraelsen [Wra] and Zhu [Zhu]. It is a generalisation of the cluster category defined in [CCS1] (for type $A$) and [BMRRT] (the general hereditary case). Keller has shown that it is a Calabi-Yau category of
We remark that such Calabi-Yau categories have also been studied in [KR].

Our definition is motivated by and is a generalisation of the construction of the cluster category in type $A$ given in [CCS1], where a category of diagonals of a polygon is introduced. The authors show that this category is equivalent to the cluster category associated to $kQ$. This can be regarded as the case $m = 1$ here. The aim of the current paper is to generalise the construction of [CCS1] to the diagonals arising in the $m$-divisible polygon dissections considered in [PS]. Note that Tzanaki [Tza] has also studied such diagonals. We also remark that a connection between the $m$-cluster category associated to $kQ$ and the diagonals considered here was given in [Tho].

We further show that if $(\Gamma, \tau)$ is any stable translation quiver, then the quiver $\Gamma^m$ with the same vertices but with arrows given by sectional paths in $\Gamma$ of length $m$ is again a stable translation quiver with translation given by $\tau^m$. If $(\Gamma, \tau)$ is taken to be the Auslander-Reiten quiver of the cluster category of a Dynkin quiver of type $A_{nm-1}$, we show that $\Gamma^m$ contains $\Gamma_{A_{n-1}}^m$ as a connected component. It follows that the $m$-cluster category is a full subcategory of the additive category generated by the mesh category of $\Gamma^m$.

Since $\Gamma$ is known to have a geometric construction [CCS1], our definition provides a geometric construction for the additive category generated by the mesh category of any connected component of $\Gamma^m$. We give an example to show that this provides a geometric construction for quotients of $D^b(kQ)$ other than the $m$-cluster category.

1. Notation and definitions

In [Tza], E. Tzanaki studied an abstract simplicial complex obtained by dividing a polygon into smaller polygons.

We recall the definition of an abstract simplicial complex. Let $X$ be a finite set and $\triangle \subseteq P(X)$ a collection of subsets. Assume that $\triangle$ is closed under taking subsets (i.e. if $A \in \triangle$ and $B \subseteq A$, then $B \in \triangle$). Then $\triangle$ is an abstract simplicial complex on the ground set $X$. The vertices of $S$ are the single element subsets of $\triangle$ (i.e. $\{A\} \in \triangle$). The faces are the elements of $\triangle$, and the facets are the maximal among those (i.e. the $A \in \triangle$ such that if $A \subseteq B$ and $B \in \triangle$, then $A = B$). The dimension of a face $A$ is equal to $|A| - 1$ (where $|A|$ is the cardinality of $A$). The complex is said to be pure of dimension $d$ if all its facets have dimension $d$.

Let $\Pi$ be an $nm + 2$-gon, $m, n \in \mathbb{N}$, with vertices numbered clockwise from 1 to $nm + 2$. We regard all operations on vertices of $\Pi$ modulo $nm + 2$. A diagonal $D$ is denoted by the pair $(i,j)$ (or simply by the pair $ij$ if $1 \leq i, j \leq 9$). Thus $(i,j)$ is the same as $(j,i)$. We call a diagonal $D$ in $\Pi$ an $m$-diagonal if $D$ divides $\Pi$ into an $(mj + 2)$-gon and an $(mn - j + 2)$-gon where $j = 1, \ldots, \left\lceil \frac{n-2}{2} \right\rceil$. Then Tzanaki defines the abstract simplicial complex $\triangle = \triangle_{A_{n-1}}^m$ on the $m$-diagonals of $\Pi$ as follows.

The vertices of $\triangle$ are the $m$-diagonals. The faces of $\triangle_{A_{n-1}}^m$ are the sets of $m$-diagonals which pairwise do not cross. They are called $m$-divisible dissections (of $\Pi$). Then the facets are the maximal collections of such $m$-diagonals. Each facet contains exactly $n - 1$ elements, so the complex $\triangle_{A_{n-1}}^m$ is pure of dimension $n - 2$.

The case $m = 1$ is the complex whose facets are triangulations of an $n + 2$-gon.
2. A STABLE TRANSLATION QUIVER OF DIAGONALS

To $\Delta = \Delta_{A_{n-1}}^m$ we associate a category along the lines of [CCS1]. As a first step, we associate to the simplicial complex a quiver, called $\Gamma_{A_{n-1}}^m$. The vertices of the quiver are the $m$-diagonals in the defining polygon $\Pi$, i.e. the vertices of $\Delta_{A_{n-1}}^m$.

The arrows of $\Gamma_{A_{n-1}}^m$ are obtained in the following way:

Let $D, D'$ be $m$-diagonals with a common vertex $i$ of $\Pi$. Let $j$ and $j'$ be the other endpoints of $D$, respectively $D'$. The points $i, j, j'$ divide the boundary of the polygon $\Pi$ into three arcs, linking $i$ to $j$, $j$ to $j'$ and $j'$ to $i$. (We usually refer to a part of the boundary connecting one vertex to another as an arc.) If $D, D'$ and the arc from $j$ to $j'$ form an $m + 2$-gon in $\Pi$ and if, furthermore, $D$ can be rotated clockwise to $D'$ about the common endpoint $i$, we draw an arrow from $D$ to $D'$ in $\Gamma_{A_{n-1}}^m$. (By this we mean that $D$ can be rotated clockwise to the line through $D'$.)

Note that if $D, D'$ are vertices of the quiver $\Gamma_{A_{n-1}}^m$, then there is at most one arrow between them.

Examples 2.4 and 2.5 below illustrate this construction.

We then define an automorphism $\tau_m$ of the quiver: let $\tau_m : \Gamma_{A_{n-1}}^m \rightarrow \Gamma_{A_{n-1}}^m$ be the map given by $D \mapsto D'$ if $D'$ is obtained from $D$ by an anticlockwise rotation through $\frac{2\pi}{m+2}$ about the centre of the polygon. Clearly, $\tau_m$ is a bijective map and a morphism of quivers.

![Figure 1](image.png)

**Figure 1.** The translation $\tau_m$, $\tau_m(14) = 92$, where $n = 4, m = 2$

**Definition 2.1.** (1) A translation quiver is a pair $(\Gamma, \tau)$ where $\Gamma$ is a locally finite quiver and $\tau : \Gamma \rightarrow \Gamma$ is an injective map defined on a subset $\Gamma'$ of the vertices of $\Gamma$ such that for any $X \in \Gamma_0$, $Y \in \Gamma'_0$, the number of arrows from $X$ to $Y$ is the same as the number of arrows from $\tau(Y)$ to $X$. The vertices in $\Gamma_0 \setminus \Gamma'_0$ are called projective. If $\Gamma'_0 = \Gamma_0$ and $\tau$ is bijective, $(\Gamma, \tau)$ is called a stable translation quiver.

(2) A stable translation quiver is said to be connected if it is not a disjoint union of two non-empty stable subquivers.

**Proposition 2.2.** The pair $(\Gamma = \Gamma_{A_{n-1}}^m, \tau_m)$ is a stable translation quiver.
Proof. By definition, \( \tau_m \) is a bijective map from \( \Gamma \) to \( \Gamma \), and \( \Gamma \) is a finite quiver. We have to check that the number of arrows from \( D \) to \( D' \) in \( \Gamma \) is the same as the number of arrows from \( \tau_m D' \) to \( D \). Since there is at most one arrow from one vertex to another, we only have to see that there is an arrow \( D \to D' \) if and only if there is an arrow \( \tau_m D' \to D \).

Assume that there is an arrow \( D \to D' \), and let \( i \) be the common vertex of \( D \) and \( D' \) in the polygon, \( D = (i,j), \, D' = (i,j + m) \). Then \( \tau_m D' = (i - m, j) \). In particular, \( j \) is the common vertex of \( D \) and \( \tau_m D' \). Furthermore, we obtain \( D \) from \( \tau_m D' \) by a clockwise rotation about \( j \), and these two \( m \)-diagonals form an \( m + 2 \)-gon together with an arc from \( i - m \) to \( i \); hence there is an arrow \( \tau_m D' \to D \). See Figure 2.

The converse follows with the same reasoning. \( \square \)

**Figure 2.** \( D \to D' \iff \tau_m D' \to D \)

**Proposition 2.3.** \( (\Gamma, \tau_m) \) is a connected stable translation quiver.

Proof. Note that every vertex of \( \Pi \) is incident with some element of any given \( \tau_m \)-orbit of \( m \)-diagonals: any \( m \)-diagonal is of the form \((i, i + km + 1)\) and

\[
\tau_m^{k-n}(i, i + km + 1) = (i + (n - k)m, i + nm + 1) = (i + (n - k)m, i - 1).
\]

Assume that \( \Gamma \) is the disjoint union of two non-empty stable subquivers. So there exist \( m \)-diagonals \( D = (i, j) \) and \( D' = (i', j') \) that cannot be connected by any path in \( \Gamma \). After rotating \( D' \) using \( \tau_m \) we can assume that \( i = i' \). By assumption, \( j' \neq j + rm \) for any \( r \). Without loss of generality, \( j < j' \). The diagonal \( D \) can be rotated clockwise about \( i \) to another \( m \)-diagonal \( D'' = (i, j'') \) such that \( j' = j'' + s \) with \( 0 < s < m \). Since \( D'' \) is an \( m \)-diagonal, the arc from \( i \) to \( j'' \), not including \( j' \), together with \( D'' \), bounds a \( (um + 2) \)-gon for some \( u \). But then the arc from \( i \) to \( j' \), including \( j'' \), together with the diagonal \( D' \), bound a \( (um + 2 + s) \)-gon where \( um + 2 < um + 2 + s < (u + 1)m + 2 \). Hence \( D' \) cannot be an \( m \)-diagonal. \( \square \)

In the examples below we draw the quiver associated to the complex \( \Delta_{A_{n-1}}^m \) in the standard way of Auslander-Reiten theory: the vertices and arrows are arranged so that the translation \( \tau_m \) is a shift to the left. We indicate it by dotted lines.
Example 2.4. Let \( n = 4, \ m = 1 \), i.e. \( \Pi \) is a 6-gon. The rotation group given by rotation about the centre of \( \Pi \) through \( k \times \frac{\pi}{3} \) degrees \((k = 0, \ldots, 5)\) acts on the facets of \( \triangle_{A_4} \). There are four orbits, \( \mathcal{O}_{\{13,14,15\}} \) of size 6, \( \mathcal{O}_{\{13,14,46\}} \) and \( \mathcal{O}_{\{13,36,46\}} \) of size 3 and \( \mathcal{O}_{\{13,15,35\}} \) with two elements, making a total of 14 elements.

The vertices of the quiver \( \Gamma_{A_4} \) are the nine 1-diagonals \( \{13, 14, 15, 24, 25, 26, 35, 36, 46\} \), and we draw the quiver as follows:

\[
\begin{array}{ccccccccc}
46 & \rightarrow & 15 & \rightarrow & 26 & \rightarrow & 13 \\
\uparrow & & \downarrow & & \downarrow & & \downarrow \\
13 & \rightarrow & 14 & \rightarrow & 25 & \rightarrow & 36 & \rightarrow & 14 \\
\end{array}
\]

Example 2.5. Let \( m = 2 \) and \( n = 4 \), i.e. \( \Pi \) is a 10-gon. The rotation group is given by the rotations about the centre of \( \Pi \) through \( k \times \frac{2\pi}{3} \) degrees \((k = 0, \ldots, 9)\) and acts on the facets of \( \triangle_{A_4} \). The orbits are \( \mathcal{O}_{\{14,16,18\}} \), \( \mathcal{O}_{\{14,18,47\}} \), \( \mathcal{O}_{\{18,38,47\}} \) and \( \mathcal{O}_{\{47,38,39\}} \) of size 10, and \( \mathcal{O}_{\{14,16,69\}} \), \( \mathcal{O}_{\{14,49,69\}} \) and \( \mathcal{O}_{\{29,38,47\}} \) of size 5, making a total of 55 elements. The vertices of \( \Gamma_{A_4} \) are the fifteen 2-diagonals \( \{14, 16, 18, 25, 27, 29, 36, 38, (3, 10), 47, 49, 58, (5, 10), 69, (7, 10)\} \) and the quiver is

\[
\begin{array}{ccccccccc}
69 & \rightarrow & 18 & \rightarrow & 3, 10 & \rightarrow & 25 & \rightarrow & 47 & \rightarrow & 69 \\
\uparrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
14 & \rightarrow & 36 & \rightarrow & 58 & \rightarrow & 7, 10 & \rightarrow & 29 & \rightarrow & 14 \\
\end{array}
\]

3. \( m \)-CLUSTER CATEGORIES

Let \( G \) be a simply laced Dynkin diagram with vertices \( I \). Let \( Q \) be a quiver with underlying graph \( G \), and let \( k \) be an algebraically closed field. Let \( kQ \) be the corresponding path algebra. Let \( D^b(kQ) \) denote the bounded derived category of finitely generated \( kQ \)-modules, with shift denoted by \( S \) and Auslander-Reiten translate given by \( \tau \). It is known that \( D^b(kQ) \) is triangulated, Krull-Schmidt and has almost-split triangles (see [Hap]). Let \( \mathbb{Z}Q \) be the stable translation quiver associated to \( Q \), with vertices \( (n, i) \) for \( n \in \mathbb{Z} \) and \( i \) a vertex of \( Q \). For every arrow \( \alpha : i \rightarrow j \) in \( Q \) there are arrows \( (n, i) \rightarrow (n, j) \) and \( (n, j) \rightarrow (n + 1, i) \) in \( \mathbb{Z}Q \), for all \( n \in \mathbb{Z} \). Together with the translation \( \tau \), taking \( (n, i) \) to \( (n - 1, i) \), \( \mathbb{Z}Q \) is a stable translation quiver. We note that \( \mathbb{Z}Q \) is independent of the orientation of \( Q \) and can thus be denoted \( \mathbb{Z}G \).

We recall the notion of the mesh category of a stable translation quiver with no multiple arrows (the mesh category is defined for a general translation quiver, but we shall not need that here). Recall that for a quiver \( \Gamma \), \( k(\Gamma) \) denotes the path category on \( \Gamma \), with morphisms given by arbitrary \( k \)-linear combinations of paths.

Definition 3.1. Let \( (\Gamma, \tau) \) be a stable translation quiver with no multiple arrows. Let \( Y \) be a vertex of \( \Gamma \) and let \( X_1, \ldots, X_k \) be all the vertices with arrows to \( Y \), denoted \( \alpha_i : X_i \rightarrow Y \). Let \( \beta_i : \tau(Y) \rightarrow X_i \) be the corresponding arrows from \( \tau(Y) \) to \( X_i \) \((i = 1, \ldots, k)\). Then the mesh ending at \( Y \) is defined to be the quiver consisting of the vertices \( Y, \tau(Y), X_1, \ldots, X_k \) and the arrows \( \alpha_1, \alpha_2, \ldots, \alpha_k \) and
$\beta_1, \beta_2, \ldots, \beta_k$. The mesh relation at $Y$ is defined to be

$$m_Y := \sum_{i=1}^{k} \beta_i \alpha_i \in \text{Hom}_{k\langle \Gamma \rangle}(\tau(Y), Y).$$

Let $J_m$ be the ideal in $k\langle \Gamma \rangle$ generated by the mesh relations $m_Y$ where $Y$ runs over all vertices of $\Gamma$.

Then the mesh category of $\Gamma$ is defined as the quotient $k\langle \Gamma \rangle / J_m$.

For an additive category $\varepsilon$, denote by ind $\varepsilon$ the full subcategory of indecomposable objects. Happel [Hap] has shown that ind $D^b(kQ)$ is equivalent to the mesh category of $\mathbb{Z}Q$, from which it follows that it is independent of the orientation of $Q$. Its Auslander-Reiten quiver is $\mathbb{Z}G$.

For $m \in \mathbb{N}$, we denote by $C^m_G$ the $m$-cluster category associated to the Dynkin diagram $G$, so

$$C^m_G = \frac{D^b(kQ)}{F_m},$$

where $Q$ is any orientation of $G$ and $F_m$ is the autoequivalence $\tau^{-1} \circ S^m$ of $D^b(kQ)$. This was introduced by Keller [Kel] and has been studied by Thomas [Tho], Wraalsen [Wra] and Zhu [Zhu]. It is known that $C^m_G$ is triangulated [Kel], Krull-Schmidt and has almost split triangles [BMRRT, 1.2.1.3]. Let $\varphi_m$ denote the automorphism of $\mathbb{Z}G$ induced by the autoequivalence $F_m$. The Auslander-Reiten quiver of $C^m_G$ is the quotient $\mathbb{Z}G / \varphi_m$, and ind $C^m_G$ is equivalent to the mesh category of $\mathbb{Z}G / \varphi_m$.

4. Coloured almost positive roots

Our main aim in the next two sections is to show that, if $G$ is of type $A_{n-1}$, then ind $C^m_G$ is equivalent to the mesh category $D^m_{A_{n-1}}$ of the stable translation quiver $\Gamma^m_{A_{n-1}}$ defined in the previous section. From the previous section we can see that it is enough to show that, as translation quivers, $\mathbb{Z}G / \varphi_m$ is isomorphic to $\Gamma^m_{A_{n-1}}$. In this section, we recall the discussion of $m$-diagonals and $m$-coloured almost positive roots in Fomin-Reading [FR].

4.1. $m$-coloured almost positive roots and diagonals. For $\Phi$ a root system, with positive roots $\Phi^+$ and simple roots $\alpha_1, \alpha_2, \ldots, \alpha_n$, let $\Phi^{m}_{\geq -1}$ denote the set of $m$-coloured almost positive roots (see [FR]). An element of $\Phi^{m}_{\geq -1}$ is either a $m$-coloured positive root $\alpha^k$ where $\alpha \in \Phi^+$ and $k \in \{1, 2, \ldots, m\}$ or a negative simple root $-\alpha_i$ for some $i$ which we regard as having colour 1 for convenience (it is thus also denoted $-\alpha_1^1$). Fomin-Reading [FR] show that there is a one-to-one correspondence between $m$-diagonals of the regular $nm + 2$-gon $\Pi$ and $\Phi^{m}_{\geq -1}$ when $\Phi$ is of type $A_{n-1}$. We now recall this correspondence.

Recall that $R_m$ denotes the anticlockwise rotation of $\Pi$ taking vertex $i$ to vertex $i - 1$ for $i \geq 2$, and vertex 1 to vertex $nm + 2$. For $1 \leq i \leq \frac{n}{2}$, the negative simple root $-\alpha_{2i - 1}$ corresponds to the diagonal $((i - 1)m + 1, (n - i)m + 2)$. For $1 \leq i \leq \frac{n-1}{2}$, the negative simple root $-\alpha_{2i}$ corresponds to the diagonal $(im + 1, (n - i)m + 2)$. Together, these diagonals form what is known as the $m$-snake; cf. Figure 3. For $1 \leq i \leq j \leq n$, there are exactly $m$ $m$-diagonals intersecting the diagonals labelled $-\alpha_i, -\alpha_{i+1}, \ldots, -\alpha_j$ and no other diagonals labelled with negative simple roots. These diagonals are of the form $D, R^1_m(D), \ldots, R^{m-1}_m(D)$ for some diagonal $D$, and $\alpha^k$ corresponds to $R^{k-1}_m(D)$ for $k = 1, 2, \ldots, m$, where $\alpha$ denotes the positive
root $\alpha_i + \cdots + \alpha_j$. For an $m$-coloured almost positive root $\beta^k$, we denote the corresponding diagonal by $D(\beta^k)$.

It is clear that, for $1 \leq i \leq \frac{n}{2}$, the coloured root $\alpha_{2i}^j$ corresponds to the diagonal $(im+1, (n+1-i)m+2)$. Also, the diagonals $D(-\alpha_i)$, for $i$ even, together with $D(\alpha_j)$, for $j$ odd, form a ‘zig-zag’ dissection of $\Pi$ which we call the opposite $m$-snake; cf. Figure 3.

![Figure 3. m-snake and opposite m-snake for $n = 6, m = 2$](image)

Let $I = I^+ \cup I^-$ be a decomposition of the vertices $I$ of $G$ so that there are no arrows between vertices in $I^+$ or between vertices in $I^-$; such a decomposition exists because $G$ is bipartite. For type $A_{n-1}$, we take $I^+$ to be the even-numbered vertices and $I^-$ to be the odd-numbered vertices.

Let $R_m : \Phi^m_{\geq -1} \to \Phi^m_{\geq -1}$ be the bijection introduced by Fomin-Reading [FR, 2.3]. This is defined using the involutions [FZ2] $\tau_{\pm} : \Phi_{\geq -1} \to \Phi_{\geq -1}$ given by

$$\tau_{\varepsilon}(\beta) = \begin{cases} \alpha & \text{if } \beta = -\alpha_i, \text{ for } i \in I^- \varepsilon, \\ \left(\prod_{i \in I^+} s_i\right)(\beta) & \text{otherwise}. \end{cases}$$

Then, for $\beta^k \in \Phi^m_{\geq -1}$, we have

$$R_m(\beta^k) = \begin{cases} \beta^{k+1} & \text{if } \alpha \in \Phi^+ \text{ and } k < m, \\ \left((\tau^- \tau^+)(\beta)\right)^1 & \text{otherwise}. \end{cases}$$

**Lemma 4.1** (Fomin-Reading). For all $\beta^k \in \Phi^m_{\geq -1}$, we have: $D(R_m(\beta^k)) = R_mD(\beta^k)$.

**Proof.** See the discussion in [FR, 4.1].

### 4.2. Indecomposable objects in the $m$-cluster category and $m$-diagonals.

Let $Q_{\text{alt}}$ denote the orientation of $G$ obtained by orienting every arrow to go from a vertex in $I^+$ to a vertex in $I^-$, so that the vertices in $I^+$ are sources and the vertices in $I^-$ are sinks.

For a positive root $\alpha$, let $V(\alpha)$ denote the corresponding $kQ_{\text{alt}}$-module, regarded as an indecomposable object in $D^b(kQ_{\text{alt}})$. Then it is clear from the definition that
the indecomposable objects in $C^m_G$ are the objects $S^{k-1}V(\alpha)$ for $k = 1, 2, \ldots, m$, $\alpha \in \Phi^+$, and $S^{-1}I_i$ for $I_i$ an indecomposable injective $kQ_{alt}$-module corresponding to the vertex $i \in I$ (all regarded as objects in the $m$-cluster category). Following Thomas [Tho] or Zhu [Zhu], we define $V(\alpha^k)$ to be $S^{k-1}V(\alpha)$ for $k = 1, 2, \ldots, m$, $\alpha \in \Phi^+$, and $V(-\alpha_i) = S^{-1}I_i$ for $i \in I$.

We have:

**Lemma 4.2** (Thomas, Zhu). For all $\beta^k \in \Phi^m_{\geq -1}$, $V(R_m\beta^k) \cong SV(\beta^k)$, where $S$ denotes the autoequivalence of $C^m_G$ induced by the shift on $D^b(kQ)$.

*Proof.* See [Tho, Lemma 2] or [Zhu, 3.8]. \qed

5. AN ISOMORPHISM OF STABLE TRANSLATION QUIVERS

From the previous two sections, we see that in type $A_{n-1}$ we have a bijection $D$ from $\Phi^m_{\geq -1}$ to the set of $m$-diagonals of $\Pi$ and a bijection $V$ from $\Phi^m_{\geq -1}$ to the objects of $\text{ind}C^m_{A_{n-1}}$ up to isomorphism, i.e. to the vertices of the Auslander-Reiten quiver of $C^m_{A_{n-1}}$. Composing the inverse of $D$ with $V$ we obtain a bijection $\psi$ from the set of $m$-diagonals of $\Pi$ to $\text{ind}C^m_{A_{n-1}}$.

**Lemma 5.1.** For every $m$-diagonal $D$ of $\Pi$, we have that

$$\psi(R_m(D)) \cong S\psi(D),$$

and therefore that

$$\psi(\tau_m(D)) \cong \tau(\psi(D)).$$

*Proof.* The first statement follows immediately from Lemmas 4.1 and 4.2. We can deduce from this that $\psi(\tau_m(D)) = \psi(R_m(D)) = S^m\psi(D)$ and thus obtain the second statement, since $S^m$ coincides with $\tau$ on every indecomposable object of $C^m_{A_{n-1}}$ by the definition of this category. \qed

It remains to show that $\psi$ and $\psi^{-1}$ are morphisms of quivers.

**Lemma 5.2.**

- For $1 \leq i \leq \frac{n-1}{2}$, there is an arrow in $\Gamma^m_{A_{n-1}}$ from $D(-\alpha_{2i-1})$ to $D(-\alpha_{2i})$.
- For $1 \leq i \leq \frac{n-1}{2}$, there is an arrow in $\Gamma^m_{A_{n-1}}$ from $D(-\alpha_{2i+1})$ to $D(-\alpha_{2i})$.
- For $1 \leq i \leq \frac{n}{2}$, there is an arrow in $\Gamma^m_{A_{n-1}}$ from $D(-\alpha_{2i})$ to $D(\alpha_{2i-1}^1)$.
- For $1 \leq i \leq \frac{n-2}{2}$, there is an arrow in $\Gamma^m_{A_{n-1}}$ from $D(-\alpha_{2i})$ to $D(\alpha_{2i+1}^1)$.

These are the only arrows amongst the diagonals $D(-\alpha_i)$ and $D(\alpha_j)$, for $1 \leq i,j \leq n-1$, with $j$ odd, in $\Gamma^m_{A_{n-1}}$.

*Proof.* We first note that, for $1 \leq i \leq \frac{n-1}{2}$, the diagonals corresponding to the negative simple roots $-\alpha_{2i-1}$ and $-\alpha_{2i}$, together with an arc of the boundary containing vertices $(i-1)m + 1, \ldots, im + 1$, bound an $m+2$-gon. The other vertex is numbered $(n-i)m + 2$. Furthermore, $D(-\alpha_{2i-1})$ can be rotated clockwise about the common endpoint $(n-i)m + 2$ to $D(-\alpha_{2i})$, so there is an arrow in $\Gamma^m_{A_{n-1}}$ from $D(-\alpha_{2i-1})$ to $D(-\alpha_{2i})$.

Similarly, for $1 \leq i \leq \frac{n-2}{2}$, the diagonals corresponding to the negative simple roots $-\alpha_{2i}$ and $-\alpha_{2i+1}$, together with an arc of the boundary containing vertices $(n-i-1)m + 2, \ldots, (n-i)m + 2$, bound an $m+2$-gon (with the other vertex being numbered $im + 1$), and $D(-\alpha_{2i+1})$ can be rotated clockwise about the common
endpoint $im + 1$ to $D(-\alpha_{2i})$, so there is an arrow in $\Gamma_{A_{n-1}}^m$ from $D(-\alpha_{2i+1})$ to $D(-\alpha_{2i})$.

We have observed that, for $1 \leq i \leq \frac{n}{2}$, the coloured root $\alpha_{2i-1}^1$ corresponds to the diagonal $(im + 1, (n + 1 - i)m + 2)$. Consideration of the $m + 2$-gon with vertices $(n - i)m + 2, \ldots, (n + 1 - i)m + 2$ and $im + 1$ shows that there is an arrow from $D(-\alpha_{2i})$ to $D(\alpha_{2i-1}^1)$. For $1 \leq i \leq \frac{n-2}{2}$, consideration of the $m + 2$-gon with vertices $im + 1, \ldots, (i + 1)m + 1$ and $(n - i)m + 2$ shows that there is an arrow from $D(-\alpha_{2i})$ to $D(\alpha_{2i-1}^1)$.

The statement that these are the only arrows amongst the diagonals considered is clear. \hfill \Box

The following follows from the well-known structure of the Auslander-Reiten quiver of $D^b(kQ)$.

**Lemma 5.3.**

- For $1 \leq i \leq \frac{n-1}{2}$, there is an arrow in the Auslander-Reiten quiver of $\mathcal{C}_{A_{n-1}}^m$ from $I_{2i-1}[1]$ to $I_{2i}[-1]$.
- For $1 \leq i \leq \frac{n-2}{2}$, there is an arrow from $I_{2i+1}[-1]$ to $I_{2i}[-1]$.
- For $1 \leq i \leq \frac{n}{2}$, there is an arrow from $I_{2i}[-1]$ to $P_{2i-1}$.
- For $1 \leq i \leq \frac{n+2}{2}$, there is an arrow from $I_{2i}[-1]$ to $P_{2i+1}$.

These are the only arrows amongst the vertices $I_i[-1]$ and $P_j$ for $1 \leq i,j \leq n-1$, with $j$ odd, in the Auslander-Reiten quiver of $\mathcal{C}_{A_{n-1}}^m$.

**Proposition 5.4.** The map $\psi$ from $m$-diagonals in $\Pi$ to indecomposable objects in $\mathcal{C}_{A_{n-1}}^m$ is an isomorphism of quivers.

**Proof.** Suppose that $D, E$ are $m$-diagonals in $\Pi$ and that there is an arrow from $D$ to $E$. Write $D = D(\beta^k)$ and $E = D(\gamma^l)$ for coloured roots $\beta^k$ and $\gamma^l$. Then $V := \psi(D) = V(\beta^k)$ and $W := \psi(E) = V(\gamma^l)$ are corresponding vertices in the Auslander-Reiten quiver of $\mathcal{C}_{A_{n-1}}^m$. Since there is an arrow from $D$ to $E$, there is an $m + 2$-gon bounded by $D$ and $E$ and an arc of the boundary of $\Pi$.

Since $D$ is an $m$-diagonal, on the side of $D$ not in the $m + 2$-gon, there is a $dm + 2$-gon bounded by $D$ and an arc of the boundary of $\Pi$ for some $d \geq 1$. Similarly, since $E$ is an $m$-diagonal, on the side of $E$ not in the $m + 2$-gon, there is an $em + 2$-gon bounded by $D$ and an arc of the boundary of $\Pi$, for some $e \geq 1$. It is clear that each of these polygons can be dissected by an $m$-snake such that, together with $D$ and $E$, we obtain a ‘zig-zag’ dissection $\chi$ of $\Pi$. Let $v$ be one of its endpoints. The other endpoint of the diagonal containing $v$ must be $v - m - 1$ or $v + m + 1$ (modulo $nm + 2$).

In the first case, we have that for some $t \in \mathbb{Z}$, $R^t_m(v) = 1$ and $R^t_m$ applied to $\chi$ is the $m$-snake. In the second case, we have that, for some $t \in \mathbb{Z}$, $R^t_m(v) = nm + 2$ and $R^t_m$ applied to $\chi$ is the opposite $m$-snake. It follows from Lemma 5.3 that there is an arrow from $R^t_m(V)$ to $R^t_m(W)$ in the Auslander-Reiten quiver of $\mathcal{C}_{A_{n-1}}^m$, and hence from $V$ to $W$.

Conversely, suppose that $V, W$ are vertices of the Auslander-Reiten quiver of $\mathcal{C}_{A_{n-1}}^m$ and that there is an arrow from $V$ to $W$. We can write $V = V(\beta^k)$ and $W = V(\gamma^l)$ for coloured roots $\beta^k$ and $\gamma^l$. Let $D := \psi^{-1}(V) = D(\beta^k)$ and let $E := \psi^{-1}(W) = D(\gamma^l)$. It is clear that $\tau^u(V) \cong I_{l+1}[-1]$ for some $i$ and some $u$. By Lemma 5.3, we must have that either $\tau^u(W) \cong I_{l+1}[-1]$ or $\tau^u(W) \cong P_{i+1}$. In the latter case we must have that $i$ is even. Note that $S^{uu}(V) \cong \tau^u(V)$ and
\( S^m(W) \cong \tau^m(W) \). It follows from Lemmas 5.1 and 5.2 that there is an arrow from \( R^m(D) \) to \( R^m(E) \) in \( \Gamma_{A_{n-1}}^m \), and thus from \( D \) to \( E \).

It follows that \( \psi \) is an isomorphism of quivers. □

**Proposition 5.5.** There is an isomorphism of translation quivers between the stable translation quiver \( \Gamma_{A_{n-1}}^m \) of \( m \)-diagonals and the Auslander-Reiten quiver of the \( m \)-cluster category \( C_{A_{n-1}}^m \).

**Proof.** This now follows immediately from Proposition 5.4 and Lemma 5.1. □

We therefore have our main result.

**Theorem 5.6.** The \( m \)-cluster category \( C_{A_{n-1}}^m \) is equivalent to the additive category generated by the mesh category of the stable translation quiver \( \Gamma_{A_{n-1}}^m \) of \( m \)-diagonals.

We remark that a connection between the \( m \)-cluster category and the \( m \)-diagonals has been given in [Tho]. In particular, Thomas gives an interpretation of \( \text{Ext} \)-groups in the \( m \)-cluster category in terms of crossings of diagonals. However, Thomas does not give a construction of the \( m \)-cluster category using diagonals.

6. The \( m \)-th Power of a Translation Quiver

In this section we define a new category in a natural way in which the \( m \)-cluster category \( C_{A_{n-1}}^m \) will appear as a full subcategory. We start with a translation quiver \( \Gamma \) and define its \( m \)-th power.

Let \( \Gamma \) be a translation quiver with translation \( \tau \).

Let \( \Gamma^m \) be the quiver whose objects are the same as the objects of \( \Gamma \) and whose arrows are the sectional paths of length \( m \). A path \( (x = x_0 \to x_1 \to \cdots \to x_{m-1} \to x_m = y) \) in \( \Gamma \) is said to be sectional if \( \tau x_{i+1} \neq x_{i-1} \) for \( i = 1, \ldots, m-1 \) (for which \( \tau x_{i+1} \) is defined) (cf. [Rin]). Let \( \tau^m \) be the \( m \)-th power of the translation, i.e. \( \tau^m = \tau \circ \tau \circ \cdots \circ \tau \) (\( m \) times). Note that the domain of the definition of \( \tau^m \) is a subset of the domain of the definition \( \Gamma_0^m \) of \( \tau \).

Recall that a translation quiver is said to be hereditary (see [Rin]) if:

- for any non-projective vertex \( z \), there is an arrow from some vertex \( z' \) to \( z \);
- there is no (oriented) cyclic path of length at least one containing projective vertices, and
- if \( y \) is a projective vertex and there is an arrow \( x \to y \), then \( x \) is projective.

The last condition is what we need to ensure that \( (\Gamma^m, \tau^m) \) is again a translation quiver:

**Theorem 6.1.** Let \( (\Gamma, \tau) \) be a translation quiver such that if \( y \) is a projective vertex and there is an arrow \( x \to y \), then \( x \) is projective. Then \( (\Gamma^m, \tau^m) \) is a translation quiver.

**Proof.** We prove the following statement by induction on \( m \):

Suppose that there is a sectional path

\[ x = x_0 \to x_1 \to \cdots \to x_m = y \]

in \( \Gamma \) and \( \tau^m y \) is defined. Then \( \tau^i x_i \) is defined for \( i = 0, 1, \ldots, m \) and there is a sectional path

\[ \tau^m y = \tau^m x_m \to \tau^{m-1} x_{m-1} \to \cdots \to \tau x_1 \to x = x_0 \]
in $\Gamma$. Furthermore, if the multiplicities of arrows between consecutive vertices in the first path are $k_1, k_2, \ldots, k_m$, the multiplicities of arrows between consecutive vertices in the second path are $k_m, k_{m-1}, \ldots, k_1$.

This is clearly true for $m = 1$, since $\Gamma$ is a translation quiver. Suppose it is true for $m - 1$, and that

$$x = x_0 \to x_1 \to \cdots \to x_m = y$$

is a sectional path in $\Gamma$. Since $\tau^{m-1}x_m$ is defined, we can apply induction to the sectional path,

$$x_1 \to x_2 \to \cdots \to x_m,$$

to obtain that $\tau^{i-1}x_i$ is defined for $i = 1, 2, \ldots, m$ and that there is a sectional path

$$\tau^{m-1}x_m \to \tau^{m-2}x_{m-1} \to \cdots \to x_1$$

in $\Gamma$, with multiplicities $k_2, k_3, \ldots, k_m$. As $\tau^m x_m$ is defined, $\tau^{m-1}x_m$ is not projective, and it follows that $\tau^{i-1}x_i$ is not projective for $i = 1, 2, \ldots, m$ by our assumption. Therefore $\tau^i x_i$ is defined for $i = 1, 2, \ldots, m$. For $i = 2, 3, \ldots, m$, there are $k_i$ arrows from $\tau^{i-1}x_i$ to $\tau^{i-2}x_{i-1}$. Therefore there are $k_i$ arrows from $\tau^{i-1}x_i$ to $\tau^{i-2}x_{i-1}$. Thus there are $k_i$ arrows from $\tau^i x_i$ to $\tau^{i-1}x_{i-1}$. As there are $k_i$ arrows from $x_0$ to $x_1$, there are $k_1$ arrows from $\tau x_1$ to $x_0$. If $\tau(\tau^i x_i) = \tau^{i+2}x_{i+2}$ for some $i$, then $x_i = \tau x_{i+2}$, contradicting the fact that $x_0 \to x_1 \to \cdots \to y$ is sectional. It follows that

$$\tau^m x_m \to \tau^{m-1}x_{m-1} \to \cdots \to x_0 = x$$

is a sectional path with multiplicities of arrows $k_1, k_2, \ldots, k_m$ as required.

It follows that the number of sectional paths with sequence of vertices $x_0, x_1, \ldots, x_m$ is less than or equal to the number of sectional paths with a sequence of vertices

$$\tau^m y = \tau^m x_m, \tau^{m-1}x_{m-1}, \ldots, \tau x_1, x_0 = x.$$

Suppose that

$$x = x'_0 \to x'_1 \to \cdots \to x'_m = y$$

is a sectional path from $x$ to $y$ with a different sequence of vertices. Then $x_i \neq x'_i$ for some $i$, $0 < i < m$. It follows that $\tau^i x_i \neq \tau^i x'_i$ and thus that the sectional path from $\tau^m y$ to $x$ provided by the above argument is also on a different sequence of vertices. Thus, applying the above argument to every sectional path of length $m$ from $x$ to $y$, we obtain an injection from the set of sectional paths of length $m$ from $x$ to $y$ to the set of sectional paths of length $m$ from $\tau^m y$ to $x$.

A similar argument shows that whenever there is a sectional path

$$\tau^m y = y_0 \to y_1 \to \cdots \to y_m = x$$

in $\Gamma$ with multiplicities $l_1, l_2, \ldots, l_m$, then $\tau^i y_i$ is defined for all $i$ and there is a sectional path

$$x \to \tau^{-1} y_{m-1} \to \cdots \to \tau^{-1} y_1 \to \tau^m y = y_0$$

in $\Gamma$ with multiplicities $l_m, l_{m-1}, \ldots, l_1$, and as above we obtain an injection from the set of sectional paths of length $m$ from $\tau^m y$ to $x$ to the set of sectional paths of length $m$ from $x$ to $y$.

Since $\Gamma$ is locally finite, the number of sectional paths of fixed length between two vertices is finite. It follows that the number of sectional paths of length $m$ from $x$ to $y$ is the same as the number of sectional paths of length $m$ from $\tau^m y$ to $x$. Hence $(\Gamma^m, \tau^m)$ is a translation quiver. □
We remark that the square of the translation quiver below, which does not satisfy the additional assumption of the theorem, is not a translation quiver:

\[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}
\]

\[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}
\]

Corollary 6.2. (1) Let \((\Gamma, \tau)\) be a hereditary translation quiver. Then \((\Gamma^m, \tau^m)\) is a translation quiver.

(2) Let \((\Gamma, \tau)\) be a stable translation quiver. Then \((\Gamma^m, \tau^m)\) is a stable translation quiver.

Proof. Part (1) is immediate from Theorem 6.1 and the definition of a hereditary translation quiver. For (2), note that if \((\Gamma, \tau)\) is stable, no vertex is projective, so \((\Gamma^m, \tau^m)\) is a translation quiver by Theorem 6.1. Since \(\tau\) is defined on all vertices of \(\Gamma\), so is \(\tau^m\). \(\square\)

We remark that the \(m\)-th power of a hereditary translation quiver need not be hereditary: there can be non-projective vertices \(z\) without any vertex \(z'\) such that \(z' \rightarrow z\). For example, consider the hereditary translation quiver below. It is clear that its square in the above sense has no arrows, but does have non-projective vertices.

\[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}
\]

\[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}
\]

However, we do have the following:

Proposition 6.3. Let \((\Gamma, \tau)\) be a translation quiver such that for any arrow \(x \rightarrow y\) in \(\Gamma\), \(x\) is projective whenever \(y\) is projective. Then the translation quiver \((\Gamma^m, \tau^m)\) has the same property.

Proof. We know by Theorem 6.1 that \((\Gamma^m, \tau^m)\) is a translation quiver. Suppose that

\[
x_0 = x \rightarrow x_1 \rightarrow \cdots \rightarrow x_m = y
\]

is a sectional path in \(\Gamma\) and that \(\tau^m x\) is defined, i.e. \(x\) is not projective in \((\Gamma^m, \tau^m)\). Then \(\tau x\) is defined, so \(x\) is not projective in \((\Gamma, \tau)\). Hence \(x_1, x_2, \ldots, x_m\) are not projective in \((\Gamma, \tau)\). Since there are arrows \(x_{i-1} \rightarrow x_i\) for \(i = 1, 2, \ldots, m\), there are arrows \(\tau x_i \rightarrow x_{i-1}\) and therefore arrows \(\tau x_{i-1} \rightarrow \tau x_i\) for \(i = 1, 2, \ldots, m\). Repeating this argument we see that \(\tau^m x_i\) is defined for all \(i\). In particular, \(\tau^m x_m\) is defined, so \(y = x_m\) is not projective in \((\Gamma^m, \tau^m)\), and we are done. \(\square\)

7. The \(m\)-Cluster Category in Terms of \(m\)-Th Powers

We consider the construction of Section 6 in the case where \(\Gamma\) is the quiver given by the diagonals of an \(N\)-gon \(\Pi\), i.e. \(\Gamma = \Gamma^1_{AN-3}\) as in Section 2. Here, we fix \(m = 1\), i.e. the vertices of the quiver are the usual diagonals of \(\Pi\), and there is an arrow from \(D\) to \(D'\) if \(D, D'\) have a common endpoint \(i\) so that \(D, D', \) together with the arc from \(j\) to \(j'\) between the other endpoints, form a triangle and \(D\) is rotated to \(D'\) by a clockwise rotation about \(i\). We will call this rotation \(\rho_i\). Furthermore, we have introduced an automorphism \(\tau_1\) of \(\Gamma\): \(\tau_1\) sends \(D\) to \(D'\) if \(D\) can be rotated to
$D'$ by an anticlockwise rotation about the centre of the polygon through $\frac{2\pi}{N}$. Then $\Gamma = \Gamma^1_{A_{N-3}}$ is a stable translation quiver (cf. Proposition 2.2).

The geometric interpretation of a sectional path of length $m$ from $D$ to $D'$ is given by the map $\rho^m_i$: $\rho^m_i$ sends the diagonal $D$ to $D'$ if $D$, $D'$ have a common endpoint $i$ and, together with the arc between the other endpoints $j, j'$, form an $m + 2$-gon, and if $D$ can be rotated to $D'$ with a clockwise rotation about the common endpoint.

Furthermore, the $m$-th power $\tau^m_1$ of the translation $\tau_1$ corresponds to an anticlockwise rotation through $\frac{2m\pi}{N}$ about the centre of the polygon. From that one obtains:

**Proposition 7.1.** 1) The quiver $(\Gamma^1_{A_{N-3}})^m$ contains a translation quiver of $m$-diagonals if and only if $N = nm + 2$ for some $n$.

2) $\Gamma^m_{A_{n-1}}$ is a connected component of $(\Gamma^1_{A_{nm-1}})^m$.

**Proof.** 1) Note that if $N \neq nm + 2$ (for some $n$), then $\Gamma^m_{A_{N-3}}$ contains no $m$-diagonals.

So assume that $N = nm + 2$ for some $n$. Let $\Gamma := \Gamma^1_{A_{N-3}} = \Gamma^1_{A_{nm-1}}$. We have to show that $\Gamma^m$ contains $Q := \Gamma^m_{A_{n-1}}$. Recall that the vertices of the quiver $\Gamma^m$ are the diagonals of an $nm + 2$-gon and that $Q$ is the quiver whose vertices are the $m$-diagonals of an $nm + 2$-gon. So the vertices of $Q$ are vertices of $\Gamma^m$.

We claim that the arrows between those vertices are the same for $Q$ and for $\Gamma^m$. In other words, we claim that there is a sectional path of length $m$ between $D$ and $D'$ if and only if $D$ can be rotated clockwise to $D'$ about a common endpoint and $D$ and $D'$, together with an arc joining the other endpoints, bound an $(m+2)$-gon.

Let $D \rightarrow D'$ be an arrow in $\Gamma^m$, where $D$ is the diagonal $(i,j)$ from $i$ to $j$. Without loss of generality, let $i < j$. The arrow $D \rightarrow D'$ in $\Gamma^m$ corresponds to a sectional path of length $m$ in $\Gamma$, $D \rightarrow D_1 \rightarrow \cdots \rightarrow D_{m-1} \rightarrow D_m = D'$. We describe such sectional paths. The first arrow is either $D = (i,j) \rightarrow (i,j+1)$ or $(i,j) \rightarrow (i-1,j)$, i.e. $D_1 = (i,j+1)$ or $D_1 = (i-1,j)$ (vertices taken mod $N$). In the first case, one then gets an arrow $D_1 = (i,j+1) \rightarrow (i+1,j+1)$ or $D_1 \rightarrow (i,j+2)$. Now $\tau(i,j+1) = (i,j)$, and since the path is sectional, we get that $D_2$ can only be the diagonal $(i,j+2)$.

Repeatedly using the above argument, we see that the sectional path has to be of the form

$$D = D_0 = (i,j) \rightarrow (i,j+1) \rightarrow (i,j+2) \rightarrow \cdots \rightarrow (i,j+m) = D_m = D'$$

where all vertices are taken mod $N$.

Similarly, if $D_1 = (i-1,j)$, then $D_2 = (i-2,j)$ and so on, and $D_m = (i-m,j) \pmod N$.

In particular, in the first case, the arrow $D \rightarrow D'$ corresponds to a rotation $\rho^m_i$ about the common endpoint $i$ of $D, D'$. In the second case, the arrow $D \rightarrow D'$
corresponds to $\rho_j^m$. In each case $D, D'$ and an arc between them bound an $(m+2)$-gon, so there is an arrow from $D$ to $D'$ in $Q$.

Since it is clear that every arrow in $Q$ arises in this way, we see that the arrows between the vertices of $Q$ and of the corresponding subquiver of $\Gamma^m$ are the same.

2) We know by Proposition 2.3 that $Q = \Gamma_{A_{n-1}}^m$ is a connected stable translation quiver. If there is an arrow $D \to D'$ in $\Gamma^m$ where $D$ is an $m$-diagonal, then $D'$ is an $m$-diagonal. Similarly, $\tau_1^m(D)$ is also an $m$-diagonal. $\square$

**Theorem 7.2.** The $m$-cluster category $\mathcal{C}_{A_{n-1}}^m$ is a full subcategory of the additive category generated by the mesh category of $(\Gamma_{A_{n+m-1}}^1)^m$.

**Proof.** This is a consequence of Proposition 7.1 and Theorem 5.6 $\square$

**Remark 7.3.** Even if $\Gamma$ is a connected quiver, $\Gamma^m$ need not be connected. As an example we consider the quiver $\Gamma = \Gamma_{A_5}^1$ and its second power $(\Gamma_{A_5}^1)^2$ pictured in Figures 4 and 5. The connected components of $(\Gamma_{A_5}^1)^2$ are $\Gamma_{A_2}^2$ and two copies of a translation quiver whose mesh category is equivalent to $\text{ind}D^b(A_3)/[1]$ (where $D^b(A_3)$ denotes the derived category of a Dynkin quiver of type $A_3$). We thus obtain a geometric construction of a quotient of $D^b(A_3)$ which is not an $m$-cluster category.

![Figure 4. The quiver $\Gamma_{A_5}^1$](image)

![Figure 5. The three components of $(\Gamma_{A_5}^1)^2$](image)

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