A Probabilistic Call-by-Need Lambda-Calculus
Extended Version

David Sabel
david.sabel@lmu.de
LMU Munich
Munich, Germany

Manfred Schmidt-Schauß
schauss@ki.cs.uni-frankfurt.de
Goethe-University Frankfurt
Frankfurt, Germany

Luca Maio
lmaio@campus.lmu.de
LMU Munich
Munich, Germany

ABSTRACT
To support the understanding of declarative probabilistic program-
ming languages, we introduce a lambda-calculus with a fair binary
probabilistic choice that chooses between its arguments with equal
probability. The reduction strategy of the calculus is a call-by-need
strategy that performs lazy evaluation and implements sharing by
recursive let-expressions. Expected convergence of expressions is
the limit of the sum of all successful reduction outputs weighted
by their probability. We use contextual equivalence as program se-
manics: two expressions are contextually equivalent if and only if
the expected convergence of the expressions plugged into an y pro-
gram context is always the same. We develop and illustrate tech-
niques to prove equivalences including a context lemma, two de-
derived criteria to show equivalences and a syntactic diagram-based
method. This finally enables us to show correctness of a large set
of program transformations with respect to the contextual equiva-
rence.

CCS CONCEPTS
• Theory of computation → Probabilistic computation;
Lambda calculus; Operational semantics.

KEYWORDS
semantics, lambda calculus, probabilistic programming, call-by-
need evaluation, program transformations, contextual equivalence

1 INTRODUCTION
Probabilistic programming aims at expressing probabilistic prob-
lems and models using programming language techniques. Purely
functional programming languages like Haskell allow expressing
programs in a formal, declarative high-level manner. Equational
reasoning and techniques for program transformations are available
for those languages. Lazy evaluation combined with sharing
results in call-by-need evaluation (see e.g. [2, 3] for call-by-
need lambda-calculi). It enables an efficient implementation of lazy
functional languages. In this paper, we combine both worlds and
thus investigate call-by-need functional languages, extended with
a probabilistic operator ⊕, such that s ⊕ t performs a fair choice
between programs s and t. Our hypothesis is, that such a lan-
guage supports the declarative construction of probabilistic pro-
grams and models, provides a large set of correct program trans-
formations, and allows to apply techniques from program transfor-
mations and program equivalence to them.

We present and develop an operational as well as a contextual
semantics of call-by-need evaluation in combination with a proba-
ability operator. The goal of the formal development is to enable us
proving correctness of a rich set of semantic equivalences, which
are reminiscent of denotational semantics We present a minimal
probabilistic call-by-need lambda-calculus to develop the notions,
explore the techniques, and figure out the core properties of such
a calculus. Thus we enrich the untyped lambda-calculus with re-
cursive let-bindings, to implement sharing, and with a fair binary
probabilistic choice ⊕.

As semantic equivalence, we define a Morris’ style contextual equiva-
rence [20], adapted to the setting of probability and expecta-
tion, i.e. two programs s, t are contextually equivalent if, and only if,
the programs C{s} and C{t} behave the same, where C is any pro-
gram context, i.e. any surrounding program. Since arbitrary con-
texts are permitted as a test, in a deterministic setting, observing
whether C{s} and C{t} terminate, usually suffices to discriminate
all obviously different programs while identifying as many pro-
grams as it makes sense; (for investigations in non-deterministic
functional languages see [12, 18, 19, 23]). In the probabilistic set-
ting, termination of programs depends on the concrete random ex-
ecution. In our approach we replace observing termination with
observing the expectation of termination (i.e. the limit of the sum
of the probabilities of all successful evaluations), where contextual
equivalence holds if this expected termination is the same for C{s}
and C{t}, again for any surrounding program context C.

Our operational semantics performs (randomized) execution of
an expression resulting in an expression (a weak head normal form
(WHNF) in the case of success). Thus, it does not evaluate an ex-
pression to a multi-distribution representing all possible outputs.
However, for the semantic equivalence, we collect all successful
executions such that, in principle, a multi-distribution could be
reconstructed. However, the core test in the notion of contextual
equivalence checks whether a WHNF is the result or not (and thus
in the probabilistic case it observes the expectation of a Bernoulli-
experiment). Contextual equivalence does not need the informa-
tion about the whole multi-distribution. Since the operational se-
manetics stops if a WHNF is reached, i.e. abstractions, the corre-
spanding multi-distribution would be based on WHNFs, and thus
in general, it is not possible to construct a (non-multi!) distribution
from it, since this would require to identify all contextually
equivalent WHNFs, which is undecidable.

For our program equivalence, our next goal is to show concrete
laws for program transformations, that usually occur as local com-
piler optimizations like garbage collection, partial evaluation, inlin-
ing, and also to prove algebraic laws of the ⊕-operator. We prove
these equivalences to emphasize that the notions are defined in
the right way. However, due to the quantification over all contexts,
establishing contextual equivalences is usually hard and requires
techniques. Thus, as a first step, we prove a novel context lemma
(see [17, 26] for some work on context lemmas in other calculi) in
Theorem 3.4, which shows that it is sufficient to take into account

the more specific class of reduction contexts $R$ to conclude contextual equivalence. To prove the context lemma, we had to restrict the formulation: equal expected convergence of $R[s]$ and $R[t]$ must also hold if the number of probabilistic evaluation steps is bounded by any fixed number. However, to make the context lemma applicable to expressions that have a different number of probabilistic evaluation steps, the formulation allows for a (fix) difference between the bound for $R[s]$ and the bound for $R[t]$.

Based on the context lemma, we prove the correctness of two criteria to establish contextual equivalences: one criterion (Proposition 4.10) establishes contextual equivalence by allowing to pre-evaluate the expressions, such that same reduction successors, on different (probabilistic) evaluation paths, can be combined before they are compared, while the other criterion (Proposition 4.4) requires the same probabilistic choices during the evaluation for the compared expressions. The former allows proving the correctness of algebraic laws in Corollary 4.11. The latter is a preparation for further correctness proofs of program transformations, that are used as local optimizations in compilers: here we apply a syntactic method to show correctness. This so-called diagram method [21–23, 28] computes all overlaps between transformation steps and reductions of the operational semantics, and joins these overlaps, such that the results are complete sets of diagrams. The diagrams are then used to inductively show the (expected) convergence equivalence. The diagram computation and the inductive construction are semi-automated by techniques and tools, from unification and term rewriting, that were developed in previous work [21, 22, 27] for deterministic and non-deterministic calculi. We show that they are transferable and still very useful for the probabilistic setting. In Theorem 5.5 our results on correctness of the given call-by-value and call-by-name semantics is defined analogous to our definition by observing the expected convergence (also called the probability of convergence). While also giving a general overview of probabilistic lambda-calculi, Dal Lago discusses in [13] operational semantics, contextual equivalence, expressive power and termination of a typed call-by-value calculus, expanding on Plotkin’s PCF. Dal Lago distinguishes between randomized lambda-calculi and Bayesian lambda-calculi. Our calculus is a randomized lambda-calculus since it performs random evaluation of the choice-operator.

Regarding a polymorphically typed call-by-value higher-order language with, probabilistic extensions (among others), in [4], a logical relation, CIU-equivalence, and contextual equivalence are shown to coincide. Their notion of contextual equivalence is analogous to ours since it compares the probabilities of termination, the coincidence of CIU-equivalence is similar to our context lemma, while adapted to the call-by-value setting.

Even more recent studies include [7] and [14], which both mainly focus on confluence and standardisation, but with different approaches. In [7], probability distributions and surface contexts are used to achieve confluence in their calculus.

In [14] different probabilistic choices can be shared or not-shared by using labels to make them a common or separate events – hence a probabilistic event lambda-calculus is introduced. With this decomposition of the probabilistic choice operator, they achieve confluence, with call-by-name and call-by-value variants. For our call-by-need calculus, choices are shared by default, but they can be duplicated if they occur below abstractions (since abstractions are not evaluated by the operational semantics).

We do not focus on confluence and related notions, since contextual equivalence does not require confluence (like other equivalence notions, like convertibility). However, our diagram-based proof technique is related to local confluence, where, however, the rewrite relations are mixed of the operational semantics and transformation steps.

**Outline.** In Section 2 we introduce the syntax, operational semantics, and contextual equivalence of the probabilistic lambda-calculus $\text{L}_{\text{need, @}}$. In Section 3 we prove the context lemma. In Section 4 we introduce a set of program transformations and prove two criteria to show the correctness of transformations. Most of the correctness proofs are obtained in Section 5 using the diagram method. In Section 6 we discuss extensions of the calculus. We conclude in Section 7. Due to space constraints, details are given in the appendix. Outputs of our automated tools and automated termination proofs can be found via https://p9471.gitlab.io/prob-need/.

## 2 THE CALCULUS $\text{L}_{\text{need, @}}$

### 2.1 Syntax and Operational Semantics

We define the syntax of the call-by-need lambda-calculus with a binary, probabilistic operator $\oplus$.

**Definition 2.1 (Syntax of Expressions and Environments).** Let $\text{Var}$ be an infinite, countable set of variables. We use $x, y, z, x_1, y_1, z_1$ for variables of $\text{Var}$. The syntax of expressions $s, t, r \in \text{Expr}$ and environments $\text{env} \in \text{Env}$ of the probabilistic call-by-need letrec-calculus...
The above chain... the main focus, which is the... the restriction to closed expressions (see [26, 28]).

The standard (call-by-need) reduction of the calculus $L_{\text{need}}$ defines the operational semantics where pure lambda expressions are evaluated using lazy evaluation and sharing (with 1let-bindings). The probabilistic operator is evaluated non-deterministically by choosing the left or the right expression.

**Definition 2.4.** The standard reduction $\overset{\text{	extit{sr}}}{\longrightarrow}$ of $L_{\text{need}}$ is defined as the union of the steps $\overset{\text{llet-e}}{\longrightarrow}$ and $\overset{\text{cp-e}}{\longrightarrow}$ (which is the union of $\overset{\text{cp-in}}{\longrightarrow}$ and $\overset{\text{cp-out}}{\longrightarrow}$), $\overset{\text{llet-in}}{\longrightarrow}$ and $\overset{\text{let-e}}{\longrightarrow}$ (which is the union of $\overset{\text{let-in}}{\longrightarrow}$ and $\overset{\text{let-out}}{\longrightarrow}$), and $\overset{\text{prob}}{\longrightarrow}$ (where the rules $\overset{\text{prob}}{\longrightarrow}$ are defined in Fig. 1 with label $(\sigma, a)$).

The transitive closure of $\overset{\text{sr}}{\longrightarrow}$ is denoted with $\overset{\text{sr}^+}{\longrightarrow}$ and the reflexive-transitive closure is denoted with $\overset{\text{sr}^*}{\longrightarrow}$. With $\overset{\text{sr}^*}{\longrightarrow}$ we denote the union of $\overset{\text{sr}^+}{\longrightarrow}$ and $\overset{\text{refl}}{\longrightarrow}$.

Reduction rule $(\text{sr, lbeta})$ is the sharing-variant of $\beta$-reduction where the argument is shared by a new binding. Rules $(\text{sr, cp-in})$ and $(\text{sr, cp-e})$ inline a binding, if it is needed and already evaluated (and hence is an abstraction). Rules $(\text{sr, lapp})$, $(\text{sr, llet-in})$, $(\text{sr, let-e})$ rearrange let-environments w.r.t. applications and nesting of let-expressions. Rules $(\text{sr, prob})$ and $(\text{sr, prob})$ evaluate $\oplus$-expression by choosing either the left or the right argument. We call these two rules also non-prob-reductions and all other reductions are called non-prob-reductions.

As usual in lazy functional programming languages, successfully evaluated expressions are identified with weak head normal forms.

**Definition 2.5 (Weak Head Normal Form, Evaluation).** Let $s$ be an expression. Then $s$ is a weak head normal form (WHNF) if it is an abstraction, or of the form $(\text{let env in } \lambda x.s')$. A sequence of reductions $\overset{\text{sr}^*}{\longrightarrow} t$ where $t$ is a WHNF is called an evaluation of $s$.

We write $\text{Eval}(s)$ for the set of all evaluations of expression $s$.

**Example 2.6.** Let $a, b, c, d$ be different abstractions (for instance, $a=x_1, b=x_2, c=x_3, d=x_4$, where $x_1=\lambda x_1.x_2, x_2, x_3, x_4$). There are two evaluations of the expression $\text{let } z=K @ K2 \text{ in } (z (a b)) (z c d)$. Due to sharing only $a$ and $d$ are possible results (plus some additional environment which is garbage), i.e. the evaluations end with $\text{let } z=K, x=x_1, y_1=\lambda x, y=\lambda z, d=x_2, \text{ in } a$ and $\text{let } z=K2, x=(a b), y_1, y_2, d=x_1, \text{ in } d$ resp. (the complete reduction sequences can be found in the appendix, Example A.1).

However, abstractions are not shared, but copied and thus if we shift the $\oplus$-operator under the $\lambda$, we get the expression $\text{let } z=\lambda x.\lambda y.x @ y \text{ in } z (a b) (z c d)$.
which now has four evaluations (see in the appendix, Example A.1) ending with all four possibilities:

- let \( z = \lambda x. \lambda y. x \mathbin{@} y, x_1 = a, y_1 = b, x = x_1, y = (z \mathbin{c.d}) \mathbin{\in} a \)
- let \( z = \lambda x. \lambda y. x \mathbin{@} y, x_1 = a, y_1 = b, x = x_1, y = (z \mathbin{c.d}) \mathbin{\in} b \)
- let \( z = \lambda x. \lambda y. x \mathbin{@} y, x = (z \mathbin{a.b}), x_1 = c, y_1 = d, x_1 = y_1 \mathbin{\in} c \)
- let \( z = \lambda x. \lambda y. x \mathbin{@} y, x = (z \mathbin{a.b}), x_1 = c, y_1 = d, y = y_1 \mathbin{\in} d \)

By inspecting the definition of reduction contexts and the standard reduction rules the following lemma can be verified:

**Lemma 2.7.** For every expression \( s \), there is either no standard reduction applicable (if \( s \) is a WHNF, or \( s \) is of the form \( R[x] \) where the shown occurrence of \( x \) is free), or there is exactly one standard reduction applicable (which is not a prob-reduction), or a \( \text{sr.prob} \)- and \( \text{sr.prob} \)-reduction are applicable, where the placed \( @ \)-expression is the same for both reductions. Thus the redex of the standard reduction is unique and standard reduction is deterministic up to prob-reductions.

**Corollary 2.8.** Each evaluation \( s \xrightarrow{\text{sr.a}_1} \cdots \xrightarrow{\text{sr.a}_n} s' \) of an expression \( s \) is uniquely determined by \( s \) and the subsequence of \( a_1, \ldots, a_n \) where all labels that are not prob nor probr are removed.

**Corollary 2.9.** For every expression \( s \), every evaluation of \( s \) is a finite sequence, and the set of all evaluations of \( s \) is countable.

**Definition 2.10.** For a reduction sequence \( s \xrightarrow{\text{sr.a}_1} \cdots \xrightarrow{\text{sr.a}_n} t \), with \( \text{PS}(s) = \{a_1, \ldots, a_n \} \), we denote the subsequence of labels \( a_1, \ldots, a_n \) which is derived from the sequence \( a_1, \ldots, a_n \) after removing all \( a_j \) with \( a_j \notin \{\text{probl, probr}\} \). We call the sequence the prob-sequence of reduction sequence.

To identify a single evaluation \( s \xrightarrow{\text{sr}^*} t \) in the set \( \text{Eval}(s) \), we write \( s \xrightarrow{\text{sr}^*} t \) in \( \text{Eval}(s) \), where \( L = \text{PS}(s) \rightarrow t \).

### 2.2 Contextual Equivalence

The defined operational semantics does not track the probability of different events where an event is a single evaluation together with the WHNF at the end of the evaluation. We now define a weighted reduction which keeps track of the probability.

**Definition 2.11 (Weighted Expressions and Reduction).** A weighted expression is a pair \( (p, s) \) where \( p \in \{0, 1\} \) is a rational number and \( s \) is an \( I_{\text{need},p} \)-expression. Let \( (p, s) \) be a weighted expression. A weighted standard reduction step \( \xrightarrow{\text{wr.a}} \) (or \( \xrightarrow{\text{wr.a}} \)) to make the rule explicit) on \( (p, s) \) is defined as follows:

- \((p, s) \xrightarrow{\text{wr.a}} (p, t) \) iff \( s \xrightarrow{\text{sr.a}} t \) and \( a \notin \{\text{probl, probr}\} \)

Again we use \( \xrightarrow{\text{wr.a}} \) and \( \xrightarrow{\text{wr.a}} \) for the reflexive-transitive, or transitive closure of \( \xrightarrow{\text{wr.a}} \).

An evaluation of a weighted expression \( (p, s) \) is a sequence \( (p, s) \xrightarrow{\text{wr.a}} (q, t) \) where \( t \) is a WHNF. Clearly, the evaluations of a weighted expression are countable. Again the sequence of labels \( a \) for each prob-reduction \( \xrightarrow{\text{wr.a}} \) together with the weighted expression \( (p, s) \) uniquely identifies an evaluation of \( (p, s) \).

**Definition 2.12 (Expected Convergence).** For a weighted expression \( (p, s) \) we denote with \( \text{Eval}(p, s) \) the set of evaluations of \( (p, s) \). A single evaluation in this set is noted as \( (p, s) \xrightarrow{\text{wr.a}} (q, t) \), i.e. if \( (p, s) \xrightarrow{\text{wr.a}} (q, t) \in \text{Eval}(p, s) \) then there is an evaluation \( (p, s) \xrightarrow{\text{wr.a}} (q, t) \) where \( (q, t) \) is the resulting weighted expression, \( t \) is a WHNF, and \( L = \text{PS}(s) \rightarrow t \). The expected convergence \( \text{ExCv}(p, s) \) of a weighted expression \( (p, s) \) is the (perhaps infinite) sum \( \text{ExCv}(p, s) = \sum (p, s) \xrightarrow{\text{wr.a}} (q, t) \in \text{Eval}(p, s) q \), and the expected convergence \( \text{ExCv}(s) \) of an expression \( s \) is \( \text{ExCv}(s) = \text{ExCv}(1, s) \). For \( s \) with \( \text{ExCv}(s) = q \), we also write \( s \xrightarrow{\text{q}} q \).

**Proposition 2.13.** Expected convergence is well-defined, i.e. for \( (p, s) \), the limit \( \sum (p, s) \xrightarrow{\text{wr.a}} (q, t) \in \text{Eval}(p, s) q \) always exists and is unique. In particular, its value is independent of the enumeration of the countable set \( \text{Eval}(p, s) \). In addition, \( \text{ExCv}(p, s) = p \cdot \text{ExCv}(s) \) follows, since \( p \) can be multiplied into the summands of the (infinite) sum. □

**Example 2.14.** The expression \( \text{let } x = (\lambda y. \text{x id}) \mathbin{@} K \text{ in } (\text{x id}) \) converges with expectation of 1. Both expressions \( \Omega = (\lambda x. (x x)) \) and \( \text{Bot} = (\text{let } x = x \text{ in } x) \) converge with expectation of 0 (and thus they do not converge). The expression \( p = \text{let } x = (\lambda y. y) \mathbin{@} K \text{ in } x \) has chances to converge and diverge. We obtain \( \text{ExCv}(p) = 1/2 \).

We compare the expected convergence with convergence tests for non-deterministic calculi (see e.g. [25] for an overview). An expression \( s \) may-converges iff there exists a reduction sequence from \( s \) to a WHNF. Thus this is exactly the same as, that there exists an evaluation. In terms of expected convergence: an expression \( s \)
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is may-convergent iff \((1,s) \downarrow_q^1\) with \(q > 0\). The negation of may-convergence is must-divergence. Thus we also know an expression \(s\) is must-divergent iff \((1,s) \downarrow_0\). Clearly, if \(s\) is must-divergent, then \(\text{ExcV}(p,s) = 0\) for every \(p \in \{0,1\}\). An expression \(s\) must-converges iff any reduction sequence starting from \(s\) is finite and ends with a WHNF. Expression \(s\) being must-convergent is not the same as \((1,s) \downarrow_1^1\): there are expressions that converge with expectation 1, but are not must-convergent, e.g. \(\lambda x. (\lambda y. (x y) \odot K) \in x \odot id\). However, if \(s\) is must-convergent then \((p,s) \downarrow_1 p\) for all \(p\).

An expression \(s\) is should-convergent iff for any \(t\) with \(s \rightsquigarrow_t^s t\), the expression \(t\) is may-convergent. Should-convergence does not imply convergence with expectation of 1, since there are should-convergent expressions where the expected convergence is strictly smaller than 1. An example is the should-convergent expression

\(s := \lambda x. (\lambda y. (x \odot (\lambda i. \text{cprob} i K \ (\text{gen} \ (i+1))) \in \text{gen} 2\)

where numbers, and if-then-else have to be encoded using Church numerals. We illustrate the executions of \(\text{cprob} \ i \ s_1 \ s_2\) and gen 2:

|\(\text{cprob} \ i \ s_1 \ s_2\): \(\frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2}\) |
|---|---|---|---|
|\(s_1\): \(\frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2}\) |
|\(s_2\): \(\frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2}\) |

In \(\text{cprob} \ i \ s_1 \ s_2\) is reached with probability \(1/2^i\) and \(s_1\) with probability \(1 - 1/2^i\); Hence \((1,s) \downarrow_q^1\) with \(q = \sum_{i=2}^{\infty} \frac{1}{i} \cdot (1/2^i) \cdot 2^i\) which is smaller than \(\sum_{i=2}^{\infty} \frac{1}{i} = 0.5\) and greater than \(\sum_{i=2}^{\infty} \frac{1}{i} \cdot 3/4 \cdot 2^i = 3/8 = 0.375\) hence \(0.375 < q < 0.5\). The exact sum is easily computed using geometric sums to \(q = 1/2 - 1/12 = 5/12\).

Since should-convergence implies may-convergence, should-convergence of \((1,s) \downarrow_1^1\) with \(q > 0\).

**Proposition 2.15.** If \((1,s) \downarrow_1^1\), then \(s\) is should-convergent.

**Proof.** Suppose that for some \(s\) with \(\text{ExcV}(s) = 1\), \(s\) is not should-convergent. Then there is some expression \(t\) with \(s \rightsquigarrow_t^s t\). But then \((1,s) \rightsquigarrow_t^s (p,t)\) for some \(p > 0\). Since \(\text{ExcV}(p,t) = 0\) due to must-divergence of \(t\), this contradicts the assumption that \(\text{ExcV}(s) = 1\).

Expression \(t\) contextually approximates \(s\) if we replace \(s\) (as a subprogram) by \(t\), then the expected convergence is not decreased. If \(s\) approximates \(t\) and \(t\) approximates \(s\), then \(s\) and \(t\) are contextually equivalent.

**Definition 2.16.** Contextual approximation \(s \approx_c\) on expressions of \(L_{\text{need}}\) is defined as follows. For \(s, t \in \text{Expr}, s \approx_c t\) holds iff for all contexts \(C: C[s] \downarrow_1^1 C[t] \downarrow_1^1 q \leq q^t\). Contextual equivalence \(\approx_c\) is the symmetrization of \(\approx_c\), i.e. \(s \approx_c t\) iff \(s \approx_c t\) and \(t \approx_c s\).

The restriction to start with weight 1, is no real restriction in the definition of contextual equivalence (see also Proposition 2.13):

**Remark 2.17.** The inequation \(s \approx_c t\) holds iff \(\forall p \in \{0,1\}, C \in C: \text{ExcV}(p,C[s]) \leq \text{ExcV}(p,C[t])\). The equation \(s \approx_c t\) holds iff \(\forall p \in \{0,1\}, C \in C: \text{ExcV}(p,C[s]) = \text{ExcV}(p,C[t])\).

**Lemma 2.18.** Contextual approximation is a precongruence, and contextual equivalence is a congruence. In addition, \(s \approx_c t\) implies that \(C[s] \downarrow_q^1 \equiv C[t] \downarrow_q^1\) for any context \(C\).

**Proof.** We show that \(s \approx_c t\) is a precongruence. The other part then follows obviously. We have to show that \(s \approx_c t\) is a preorder that is compatible with contexts. The relation \(s \approx_c s\) is obviously reflexive (as \(s \approx_c s\)). For transitivity, let \(r \approx_c s\) and \(s \approx_c t\), and \(C\) be a context with \(C[r] \downarrow_1 p\). From \(r \approx_c s\) we have that \(C[s] \downarrow_1 p\) with \(p^r \geq p\). Now \(s \approx_c t\) implies \(C[t] \downarrow_1 p^t\) with \(p^t \geq p^r\). Thus, in conclusion this shows \(C[t] \downarrow_1 p^t\) with \(p^t \geq p\) and thus \(r \approx_c t\). For proving compatibility with contexts, let \(s \approx_c t\) and \(C\) be a context. We have to show that for every context \(C\prime\), we have \(C'[s] \downarrow_1^1 \equiv C'[t] \downarrow_1^1\) for all \(C\prime\) with \(s \approx_c t\). But this follows from \(s \approx_c t\) since \(C'[C]\) is also a context.

**Example 2.19.** The inequations (1) \(K @ K2 \neq C\), (2) \((K @ K2) @ K2 \neq C\), (3) \(K @ K2 \neq K2\). The first inequation can be proved by the context \(C = (\ldots id)\). Then \(C[K] \downarrow 1\), but \(C[K @ K2] \downarrow 0.5\) and thus \(K @ K2 \neq C\). The second inequation can be proved by the same context: \(C[(K @ K2) @ K2] \downarrow 0.25\) and \(0.25 < 0.5\). Note that \(K @ K2 \neq K @ K2\) however holds. The third inequation can be proved using the context \(C = ([\ldots] id)\).

While refuting contextual equivalence is possible by providing a single context as counter-example, proving contextual equivalences requires to reason about all contexts. Hence, we develop techniques to enable such proofs.

## 3 CONTEXT LEMMA

The goal of this section is to show that observing expected convergence in reduction contexts is sufficient to conclude contextual equivalence. Such a result is usually called a context lemma. Our formulation of the context lemma is more special, and we require some preparation to introduce it. We first introduce multicontexts: these are contexts with several (or no) holes \(\_i\), where every hole occurs exactly once. We write a multicontext as \(C[\_1,\ldots,\_n]\), and if the expressions \(s_j\) for \(i = 1,\ldots,n\) are placed into the holes \(\_i\), then we denote the resulting expression as \(C[s_1,\ldots,s_n]\). For a multicontext \(C[\_1,\ldots,\_n]\), a hole \(\_j\) is a **reduction hole**, iff for all expressions \(s_j\), \(j = 1,\ldots,n\), the context \(C[s_1,\ldots,s_{i-1},\_j,s_{i+1},\ldots,s_n]\) is a reduction context. Note that if for a multicontext \(C[\_1,\ldots,\_n]\) and expressions \(s_1,\ldots,s_n\), \(C[s_1,\ldots,s_{i-1},\_j,s_{i+1},\ldots,s_n]\) is a reduction context, then there exists an index \(j\), such that hole \(\_j\) is a reduction hole of \(C[\_1,\ldots,\_n]\).

**Definition 3.1.** The **prob-length** of an evaluation \((s_\downarrow_k t)\) written \(s \downarrow_{k,\text{prob}}\) is the number of reductions in the evaluation, \(\text{PL}(s_\downarrow_k t) = |L|\). Let \(\text{Eval}(p,s,k)\) be the set of evaluations \((p,s) \downarrow_k t)\) with \(\text{PL}(p,s) \leq k\) and \(\text{Eval}(p,s,k) \leq k\). We also define \(\text{ExcV}(s,k)\) as the expected convergence of \(s\) w.r.t. evaluations of prob-length \(\leq k\).

It is obvious that \(\lim_{k \to \infty} \text{ExcV}(s,k) = \text{ExcV}(s)\).
Definition 3.2. For two expressions \( s, t \), we define \( s \equiv_R t \) iff for all reduction contexts \( R : \mathbb{R}[s] \rightarrow \mathbb{R}[t] \).

Assume the claim is false, i.e. let \( k \geq 0 \) such that \( \forall k \geq 0 : \exists d \geq 0 : \exists C[s, k] \leq \exists C[t, k + d] \). Then \( \exists C[s, k] \leq \exists C[t, k + d] \).

Equation (2) also implies for all \( d \geq 0 : \forall k > k_1 : \exists C[s, k + d] \leq \exists C[t, k + d] \). This shows for all \( k > k_1 \), \( \exists C[R[s], k] \leq \exists C[R[t], k + d] \). Let \( k \) be a multicontext with \( n \) holes. Then the inequality \( \exists C[s, 1, \ldots, n] \leq \exists C[t, 1, \ldots, n] \) holds.

Proof. First show:

\( \forall k \geq 0 : \exists d \geq 0 \) such that:

\[
\exists C[s, 1, \ldots, n], K \leq \exists C[t, 1, \ldots, n], K + d
\]

We use induction on the lexicographically ordered triple:

(i) the number \( K \).
(ii) the maximal length of the evaluations (with at most \( K \) prob-reduction steps) of \( C[s, 1, \ldots, n] \), where the measure is 0 if no such evaluation exists.

(Note that the measure is well-defined, since the set of evaluations with at most \( K \) prob-steps is finite.)

(iii) the number \( n \) of holes of the multicontext \( C \).

As a base case, assume that \( K \) is arbitrary, but \( C \) has no holes. Then the claim obviously holds.

Now assume that \( C \) has at least one hole. We consider evaluations of \( C[s, 1, \ldots, n] \) and take only evaluations into account that have \( k \leq K \) prob-reductions. We distinguish two cases:

Case 1: The maximal length of the evaluations of \( C[s, 1, \ldots, n] \) that use at most \( K \) prob-reductions is 0. Then either no such evaluation exists, or \( C[s, 1, \ldots, n] \) is a WHNF:

(1) If \( C[s, 1, \ldots, n] \) is not a WHNF, then \( \exists C[s, 1, \ldots, n], K = 0 \) and the claim holds.

(2) If \( C[s, 1, \ldots, n] \) is a WHNF, then there are subcases:

(a) No hole of \( C[s, 1, \ldots, n] \) is a reduction hole. Then \( C[s, 1, \ldots, n] \) is also a WHNF, \( \exists C[t, 1, \ldots, n], K = 1 \) and the claim holds.

(b) For some \( i \), hole \( \{ i \} \) is a reduction hole for context \( C \). Let \( C' = C[s, 1, \ldots, n] \) with \( n - 1 \) holes. By the induction assumption, we have \( D' \geq 0 \) such that \( \exists C[t, 1, \ldots, n], K + D' \) and by the assumption on \( s, t \) we have

\[
\exists C[s, 1, \ldots, n], K + D' \leq \exists C[t, 1, \ldots, n], K + D + D''
\]

since hole \( \{ i \} \) is a reduction hole and thus \( C[t, 1, \ldots, n], K + D + D'' \) are reduction contexts. Thus the claim holds for \( D = D' + D'' \).

Case 2: There is an evaluation of \( C[s, 1, \ldots, n] \) with prob-length \( k \leq K \) and of length \( n > 0 \). Then there are subcases:

(1) No hole of \( C \) is a reduction hole. Then we consider the first standard reduction step of the expression \( C[s, 1, \ldots, n] \) with \( k < K \) prob-reductions. Note that this step is either unique for all evaluations of \( C[s, 1, \ldots, n] \) (if it is a non-prob-reduction) or there may be two possible prob-steps. Hence we distinguish between non-prob and prob-steps:

- If it is a non-prob-reduction, then the reduction is unique, the number \( k \) remains untouched. Since no hole is in a reduction context, the reduction must be of the form \( C[s, 1, \ldots, n] \rightarrow C'[s', 1, \ldots, n'] \) where there is a mapping \( f \) such that \( s'_i = s(f(i)) \) and for any expressions \( r_1, \ldots, r_n \) the reduction \( C[r_1, \ldots, r_n] \rightarrow C'[r'_{1}, \ldots, r'_{n}] \) exists where \( r'_i = f(r_i) \).

Thus the reduction can also be done for \( r_i = t_i \) for \( i = 1, \ldots, n \). The expected convergence is not changed, i.e. \( \exists C[s, 1, \ldots, n], K = \exists C[t, 1, \ldots, n], K \) and for all \( K' \), \( \exists C[t, 1, \ldots, n], K' = \exists C[t, 1, \ldots, n], K' \) and we can use the induction hypothesis, since \( K \) remains the same, the length of a maximal evaluation using at most \( K' \) prob-steps is strictly smaller for \( C[s, 1, \ldots, n] \), and the precondition \( \forall k \exists d : \exists C[R[s], k] \leq \exists C[R[t], k + d] \) still holds. This shows that there exists \( D' \) such that \( \exists C[s, 1, \ldots, n], K = \exists C[t, 1, \ldots, n], K + D' \).

Hence the claim holds for \( D = D' \).

- If it is a prob-reduction, then there are two possibilities with probability measure 0.5:

  i) \( C[s, 1, \ldots, n] \rightarrow_{\text{prob}} C'[s', 1, \ldots, n'] \) and \( C[t, 1, \ldots, n] \rightarrow_{\text{prob}} C'[t', 1, \ldots, n'] \) and there is a mapping \( f \) such that \( s'_i = s(f(i)) \), and \( t'_i = t(f(i)) \) for \( i = 1, \ldots, n' \), and

  ii) \( C[s, 1, \ldots, n] \rightarrow_{\text{prob}} C'[s', 1, \ldots, n'] \) and \( C[t, 1, \ldots, n] \rightarrow_{\text{prob}} C'[t', 1, \ldots, n'] \) and there is a mapping \( f \) such that \( s'_i = s(f(i)) \), and \( t'_i = t(f(i)) \) for \( i = 1, \ldots, n' \).

Let us abbreviate the new expressions as \( r'_i = C'[s'_i, 1, \ldots, n'] \), \( r'_i = C'[t'_i, 1, \ldots, n'] \), \( r''_i = C'[s''_i, 1, \ldots, n''] \), \( r''_i = C'[t''_i, 1, \ldots, n''] \).

We apply the induction hypothesis twice: to \( r'_i \) and \( r''_i \) as well as to \( r'_i \) and \( r''_i \), where we use \( K = 1 \) instead of \( K \). Note that the precondition \( \forall k : \exists C[R[s], k] \leq \exists C[R[t], k + d] \) (resp.)
holds for \( k \geq 0 \), and since \( K - 1 < K \) the induction measure is strictly smaller. From the induction hypothesis we obtain that there exist \( D', D'' \) with \( \text{ExCv}(r'_u, K - 1) \leq \text{ExCv}(r'_u, K + D' - 1) \) and \( \text{ExCv}(r''_u, K - 1) \leq \text{ExCv}(r''_u, K + D'' - 1) \). Clearly, the following equations hold for any \( k' \):

\[
\text{ExCv}(C[s_1, \ldots, s_n], k') = 0.5(\text{ExCv}(r'_u, K - 1) + \text{ExCv}(r''_u, K - 1))
\]

\[
\text{ExCv}(C[t_1, \ldots, t_n], k') = 0.5(\text{ExCv}(r'_u, K - 1) + \text{ExCv}(r''_u, K - 1))
\]

Finally, this shows

\[
\text{ExCv}(C[s_1, \ldots, s_n], K) = 0.5(\text{ExCv}(r'_u, K - 1) + \text{ExCv}(r''_u, K - 1))
\]

\[
\leq 0.5(\text{ExCv}(r'_u, K + D' - 1)) + \text{ExCv}(r''_u, K + \max(D', D''))
\]

Thus the claim holds for \( D = \max(D', D'') \).

(2) A hole of \( C \) is a reduction hole. Then this hole is a reduction hole for \( s_1, \ldots, s_n \) as well as for \( t_1, \ldots, t_n \). Now the same reasoning as in Case 1 item 2 is valid.

The final reasoning is as follows: we apply Lemma 3.3 to Eq. (3). This shows \( \text{ExCv}(C[s_1, \ldots, s_n]) \leq \text{ExCv}(C[t_1, \ldots, t_n]) \).

\[\square\]

## 4 PROGRAM TRANSFORMATIONS

We define the notion of correct program transformations:

**Definition 4.1.** A program transformation \( \xrightarrow{\text{T}} \) is a binary relation on expressions. It is correct, iff \( \xrightarrow{\text{T}} \subseteq \xrightarrow{\text{c}} \). With \( \xrightarrow{\text{T}} \) and \( \xrightarrow{\text{T}+} \) we denote the reflexive-transitive and the transitive closures of \( \xrightarrow{\text{T}} \), and \( \xrightarrow{T+\Delta} \) denotes the union of \( \xrightarrow{\text{T}} \) and \( \xrightarrow{\text{T}+} \).

In Fig. 2 we define several program transformation. Some are generalizations of the standard reductions and can be used for partial evaluation. The rule (xch) exchanges a variable-to-variable-binding, the rule (ucp) means unique copying and inlines binding if the bound variable occurs once and not below a λ-binder. The rule (ge) performs garbage collection, i.e. it removes (parts of) let-environments that are unused. Rules (probld), (probassoc) and (probcomm) are algebraic laws (idempotence, associativity, commutativity) of the \( \oplus \)-operator. (probdist) shifts a \( \oplus \)-operation over another one, and (probreorder) reorders nested \( \oplus \)-operations. The transformation (probassoc) is not correct:

**Proposition 4.2.** The transformation probassoc is not correct.

**Proof.** Let \( w \) be a WHNF. Then \( s = \text{w} \odot (\Omega \odot \Omega) \)

\[
\xrightarrow{\text{probassoc}} (\text{w} \odot \Omega) \odot \Omega \equiv \text{t} \quad \text{but s} \not\in \text{t} \quad \text{and thus s} \not\in \text{t} \).
\]

Correctness of \( \xrightarrow{\text{I}} \beta \) follows with the context lemma, since applying it inside reduction context is always a \( \text{sr} \rightarrow \text{step} \).

**Proposition 4.3.** The transformation \( \xrightarrow{\text{I}} \beta \) is correct.

Proving correctness of other transformations requires more sophisticated techniques. In the remaining part of the section we provide two criteria for proving correctness of transformations. For a transformation \( \xrightarrow{T} \), we denote the closure of \( T \) inside all (surface, resp.) contexts.

**Proposition 4.4 (Correctness Criterion: Same Prob-Sequences).** Let \( \xrightarrow{T} \) be a program transformation and \( X \in \{\emptyset, R, S\} \). If for all \( s, t \) with \( s \xrightarrow{T} t \) the following holds:

\[
\text{for all } s \xrightarrow{\text{T}} t \text{ there exists } t' \xrightarrow{\text{T}} t \text{ such that } (i, s) \xrightarrow{T} \text{ and its inverse } t' \xrightarrow{T} t \}
\]

then \( \xrightarrow{T} \subseteq \xrightarrow{\text{c}} \) holds. Correctness of \( \xrightarrow{T} \) can be shown by applying the criterion for \( \xrightarrow{T} \) and its inverse \( t' \xrightarrow{T} t \) implies \( t \xrightarrow{T} t' \).

**Proof.** We use expressions without weights, but keep track of the prob-sequences: for \( s \in \text{Expr} \) and an evaluation \( s \xrightarrow{\text{I}} t \), the resulting probability of \( (1, s) \) is \( \frac{1}{L_0} \), where \( |L| \) is the length of \( L \).

We show \( \forall R \in \mathbb{R}, \forall k \geq 0 : \text{ExCv}(R[s], k) \leq \text{ExCv}(R[t], k) \). The context lemma (Theorem 3.4) with \( n = 1 \) and \( d = 0 \) for all reduction contexts) then shows \( \forall C \in \mathbb{C} : \text{ExCv}(C[s]) \leq \text{ExCv}(C[t]) \) and hence \( s \leq t \).

Let \( R[k] \) be arbitrary but fixed and \( \text{ExCv}(R[s], k) = q \), and let \( e \) be the set of evaluations of \( R[s] \) of prob-length \( k \). This set is finite and thus \( \text{ExCv}(R[s], k) = \sum_{e \in e} \frac{1}{|\text{probdist}|} \). Now apply Condition (4) of the claim to every \( s \xrightarrow{\text{T}} t \) is possible, for \( X \in \{\emptyset, \mathbb{R}, \mathbb{C}\} \) (surface contexts) include all reduction contexts. This results in a finite set of evaluations \( e' = \{t' \in \text{Eval}(k) \} \) of \( \text{ExCv}(R[k]) \). Since the prob-labels are kept (from \( L \) to \( L \)), \( |e'| = |e| \) and since also the prob-lengths are same, we have the prob-sequences: for \( s \) to \( t \) the claim is shown.

\[\square\]

**Definition 4.5.** A frontier is a set of words over \{prob, probr\} generated by starting with \( e \) (\( e \) denotes the empty string) and applying the following operation multiple times to the set: take a string \( r \) and replace it by two words \( r; \text{probl} \) and \( \text{probr} \).

For example \( e \), \( \{\text{probl}, \text{probr}\} \) and \{\text{probl}, \text{probl}, \text{probr}, \text{probr}\} \) are frontiers.

**Definition 4.6.** Let \( s \in \text{Expr}, d, 1 \geq 1 \) and \( F = \{l_1, \ldots, l_n\} \) be a frontier with \( |L| \leq d \) for all \( i \). Then a frontier-evaluation of \( s \) w.r.t. \( F, d \) is a subset of the form \( (l_1, q_1, s_1), \ldots, (l_n, q_n, s_n) \) such that for all \( i = 1, \ldots, n \) \( (l_i, q_i, s_i) \) with \( l_i = \text{probl}(1, s) \) \( \text{probr}(1, q_i, s_i) \) (i.e. there is a reduction sequence from \( s_i \) to \( s_i \) resulting in weight \( q_i \) that uses only prob-reductions and the prob-sequence is \( L_i \)). Let the multiset \( \{q_1, s_1, \ldots, q_n, s_n\} \) be the frontier evaluation result. In a frontier evaluation result, the sum of all probabilities is always 1.

We compare two multisets of frontier evaluation results \( A \) and \( B \) in order to reconstruct the contextual equivalence or preorder resp.

**Definition 4.7.** Let \( A, B \) be two frontier evaluation results. We define the following criteria:

**EqCr1** For every \( (q_1, s_1) \in A \) there is some \( (q', s') \in B \) with \( q \leq q' \). **EqCr2** For every \( (q_1, s_1) \in A \), let \( q_{s,A} \) be the sum of all \( q' \) such that \( (q', s') \) is an entry in \( A \), and let \( q_{s,B} \) be the sum of all \( q' \) such that \( (q', s') \) is an entry in \( B \), then the inequation \( q_{s,A} \leq q_{s,B} \) must hold.
**EqCr3** This is applicable in case that Ω may appear as expression s. For every (q, s) ∈ A, with s ∈ Ω, then for s, A be the sum of all q' such that (q', s) is an entry in A, and let q_{s,A} be the sum of all q' such that (q', s) is an entry in B. Then the inequation q_{s,A} ≤ q_{s,B} must hold.

**Example 4.8.** An example is the evaluation of \((s_1 ⊕ s_2) ⊕ (s_3 ⊕ s_3)\) w.r.t. \(\{\text{probl, obl, propl, obl, propl, obl, propl, obl, propl}\}\) which results in \(\{\text{probl, obl, 0.25, s_1}, \text{propl, obl, 0.25, s_2}, \text{propl, obl, 0.25, s_1}, \text{propl, obl, 0.25, s_2}\}\). The other side of the probdisr-rule is \(s_1 ⊕ (s_2 ⊕ s_3)\), which has a frontier evaluation result \((0.25, s_1), (0.25, s_2), (0.25, s_3)\), and satisfies criterion (EqCr2) of Definition 4.7 in both directions.

**Example 4.9.** Further examples for the criteria in Definition 4.7.

1. Let \(M_1 := \{(0.5, a), (0.5, \Omega)\}\) and \(M_2 := \{(0.7, a), (0.3, \Omega)\}\). Then for \(0.5, a\), the element \(0.7, a\) is sufficient to detect \(s_{c,e}\), for \(0.5, \Omega\), we obtain \(s_{c,e} ≥ 1\). This satisfies (EqCr3).
2. Let \(M_1 := \{(0.1, \Omega), (0.6, a), (0.3, b)\}\) and \(M_2 := \{(0.2, \Omega), (0.5, a), (0.3, b)\}\), no criterion is satisfied.

The idea of the criteria is to evaluate the two expressions in several ways and to get intermediate resulting sets that can be compared with the criteria. For example, consider probreorder: Evaluating the left-hand side to a frontier-evaluation result: \((s_1 ⊕ s_2) ⊕ (t_1 ⊕ t_2)\) results in \(\{(0.25, s_1), (0.25, s_2), (0.25, t_1), (0.25, t_2)\}\), and \((s_1 ⊕ t_1) ⊕ (s_2 ⊕ t_2)\) evaluates also to the same frontier-evaluation result. For probdisr the left-hand side \(r ⊕ (s ⊕ t)\) has a frontier-evaluation result \(\{(0.5, r), (0.25, s), (0.25, t)\}\), and from \(r ⊕ s\) or \(r ⊕ t\) we obtain \(\{(0.25, r), (0.25, r), (0.25, s), (0.25, t)\}\).

The criterion (EqCr3) is satisfied in all cases.

**Proposition 4.10 (Correctness Criterion: Same Distribution after Prob-reduction).** Let \(T\) be a program transformation. The following claim holds for \(X ∈ \{\text{C, S, R}\}\). Transformation \(T\) is included in \(s_{c,e}\), i.e., \(T \subseteq s_{c,e}\), if for all \(s, t\) with \(s \not= t\) the following holds: There is a frontier-evaluation result \(R_t\) of \(s\) and a frontier-evaluation result \(R_s\) of \(t\) such that criteria (EqCr1), (EqCr2), (EqCr3), holds. If \(R_t \not= R_s\), then correctness of \(T\) follows by symmetry.

**Proof.** We verify the preconditions of context lemma (Theorem 3.4) for \(n = 1\). Since \(X \not= T\) already covers \(T\)-steps in all reduction contexts, it suffices to show that there exists \(d ≥ 0\) such that all \(k ≥ 0\) we have \(X \not= C\), \(k \not= X(v, t + d)\). This holds, since the frontier-evaluations results satisfy criteria (EqCr1), (EqCr2), or (EqCr3) hold, where the difference \(d\) can be chosen as the maximum of the prob-reduction depth of the frontier-evaluations.

**Corollary 4.11.** The program transformations \(\{\text{probl, obl, propl, obl, propl, obl, propl, obl, propl}\}\), probcomm, probdisr, and probreorder are correct.

We also obtain inequations like \(s \not= s_{c,e} \not= s\), since this corresponds to the resulting sets \(\{(0.5, s), (0.5, \Omega)\}\) and \(\{(1, s)\}\).

**5 CORRECTNESS BY DIAGRAMS**

In this section we want to show that the remaining transformations in Fig. 2 are correct. We use the so-called diagram method [21–23, 28] to prove their correctness. Given a transformation \(T\), the method uses Proposition 4.4 to show correctness: for a step \(s \not= t\) (note that we work with the closure of \(T\) w.r.t. surface contexts) it shows that for every evaluation of \(s\) there is an evaluation of \(t\) and vice versa, where the prob-sequences are the same.

Base cases cover the cases that \(s\) (or \(t\), resp.) already is a WHNF, and show that \(t\) (or \(s\), resp.) can then be evaluated to a WHNF. In the general case, so-called forking or commuting diagrams are used. The forking diagrams are for constructing an evaluation for \(t\), from a given evaluation for \(s\). The commuting diagrams serve the same purpose for the other direction. A single forking diagram describes a fork overlap \(s' \not= s\) and \(t\) (using solid arrows) and how the pair \(s'\) and \(t\) can be joined using transformations and reductions (dashed arrows). A set of forking diagrams is complete if each fork overlap that occurs is covered by at least one of the diagrams. The base case and the forking diagrams are inductively
applied to construct an evaluation for \( t \), starting from a (given) evaluation for \( s \).

The commuting diagrams cover commuting overlaps \( \frac{s \rightarrow t}{r} \) (solid arrows) and show how the pair \( t, t' \) can be joined by transformations and standard reductions (written as dashed arrows). A set of commuting diagrams is complete if it covers all commuting overlaps. Such a set is used to (inductively) show that given an evaluation for \( t \), there is an evaluation for \( s \).

In both directions, the preservation of the prob-sequences has to be proved by inspecting the diagrams.

The diagrams abstract from the concrete expressions (they do not occur in the diagrams), since a single diagram usually represents infinitely many concrete overlaps of concrete expressions (and thus the complete set of diagrams is a finite representation of all concrete overlaps). Information that is kept is the labels of the transformations and reductions. Thus the (forking and commuting) diagrams are interpreted and used as (non-deterministic) rewrite rules on reduction sequences. The non-determinism reflects the missing information on the concrete expressions that are manipulated.

In earlier work (e.g. [23, 28]), diagrams were computed manually, but nowadays they are computed automatically by a tool, that unifies the left-hand sides of the standard reductions with left-(forking) and also right-hand sides (commuting) of the transformation, and then searches to show joinability (the unification algorithm is described in [27], the tool is described in [22]). If not stated otherwise, the diagrams in this paper were computed by this tool.

We prove the correctness of transformation \( \frac{s \rightarrow t}{r} \) by the diagram method. Forking and commuting diagrams for \( \frac{s \rightarrow t}{r} \) were computed by the automated tool. Additionally, unifying WHNFs with left or right-hand sides of transformation \( \frac{s \rightarrow t}{r} \), then applying the unifier to the other side of the rule, and then applying standard reductions to the obtained expressions allows to compute the base case automatically. We show the diagram and the base case in Fig. 3.

**Proposition 5.1.** The transformation \( \frac{s \rightarrow t}{r} \) is correct.

**Proof.** We first show \( \frac{s \rightarrow t}{r} \leq \frac{s \rightarrow t}{r} \). Let \( s, t \in \text{Expr} \) with \( s \rightarrow t \). We show that for every evaluation \( t \in \text{Eval}(s) \), there exists \( t \in \text{Eval}(t) \). The technique to show that any reduction sequence \( s \rightarrow t \) where \( s \) is a WHNF can be transformed in a finite number of steps into \( t \) by applying the forking diagrams. As a base case, we have that \( t \) is a WHNF, or can be reduced to a WHNF not changing the prob-sequence. Termination of transforming the reduction sequence can be shown by proving termination of the rewrite system induced by the diagrams: they are encoded as the term rewrite system (TRS) \( R_1 \) shown in Fig. 3 and a termination prover is applied to show innermost termination (in [21] this technique is explained in detail). \(^1\)

In \( R_1 \) the function symbol \( \text{S} \) represents any \( \frac{s \rightarrow t}{r} \) step, \( \text{SR} \) represents any \( \frac{s \rightarrow t}{r} \) step, and \( \text{SR} \) represents a step \( \frac{s \rightarrow t}{r} \) or \( \frac{s \rightarrow t}{r} \), and \( x \) is a variable. The termination prover TTT [9, 11, 30, 31] delivers the Knuth-Bendix-order (KBO) with weight function \( w(\text{S}l) = 0, w(\text{SR}) = 1 \) and precedence \( \text{S}l > \text{SR, SII} > \text{SII} \) and \( \text{SII} > c \) for any constant \( c \) proving innermost-termination. The diagrams show that prob-sequences are kept by the construction.

For \( \frac{s \rightarrow t}{r} \), we show that if \( s \rightarrow t \), then for every evaluation \( t \in \text{Eval}(t) \), there exists \( t \in \text{Eval}(s) \). We apply the commuting diagrams starting with an evaluation for \( t \) to derive an evaluation for \( s \). Termination of the rewriting on the sequences can be shown by proving innermost termination of the TRS \( R_2 \) shown in Fig. 3.\(^2\) Here \( \text{SIII} \) represents a \( \frac{s \rightarrow t}{r} \) step with \( a \in \{ llet, lapp \} \), \( \text{SR} \) represents a \( \frac{s \rightarrow t}{r} \) step with \( a \in \{ lapp, llet \} \), \( \text{Sapp} \) represents \( \frac{s \rightarrow t}{r} \) (and \( x \) is a variable). Applying TTT to \( R \) shows (innermost) termination using the lexicographic path order (LPO): \( \text{SIII} \succ \text{SR} \), \( \text{SR} \), \( \text{SIII} \), \( \text{SII} \), \( \text{SII} \) and \( \text{SII} \) are used. The diagrams also show that prob-sequences are kept by the construction.

Finally, Proposition 4.4 shows that \( \frac{s \rightarrow t}{r} \) is correct. \( \square \)

For transformation \( \frac{s \rightarrow t}{r} \), we computed the diagrams and the base case automatically, the results are shown in Fig. 4. where \( (cp) \) is split into \( (cpS) \) and \( (cpd) \), distinguishing whether the target of the copy-operation is inside surface context, or it is inside an abstraction. If we do not distinguish between \( (cpS) \) and \( (cpd) \), then termination of the diagrams cannot be proved.

**Proposition 5.2.** The transformation \( \frac{s \rightarrow t}{r} \) is correct.

**Proof.** We use Proposition 4.4 and show for all \( s, t \) with \( s \rightarrow t \): (1) For every \( t \in \text{Eval}(s) \), there exists \( t \in \text{Eval}(t) \). (2) For every \( t \in \text{Eval}(t) \), there exists \( t \in \text{Eval}(s) \).

For proving Item 1, first assume that a sequence \( s \rightarrow t \) is given, where \( s \) is a WHNF. This is transformed into a sequence \( t \rightarrow t' \) by applying the forking diagrams. Termination of this rewriting on the sequences can be shown automatically by proving innermost termination of the TRS \( R_1 \) shown in Fig. 4. Here \( \text{SR} \) represents an \( \frac{s \rightarrow t}{r} \) step, \( \text{S} \) represents an \( \frac{s \rightarrow t}{r} \) step. Termination prover TTT shows innermost termination of \( R_1 \) by using the KBO with weight function \( w(\text{SR}) = 0, w(\text{S}) = 1 \) and precedence \( \text{S}l > \text{SR, SII} > \text{SII} \) and \( \text{SII} > c \). After obtaining \( t \rightarrow t', t \rightarrow t' \), the base case for \( \frac{s \rightarrow t}{r} \) is applied which shows that \( t' \) must be a WHNF. Finally, we observe that the diagrams do not change the prob-sequences and/or -length.

For proving Item 2, first assume that a sequence \( s \rightarrow t \) is given, where \( t \) is a WHNF. This is transformed into a sequence \( t \rightarrow t' \) by applying the commuting diagrams. Termination of rewriting the sequences is shown automatically by proving innermost termination of the TRS \( R_2 \) shown in Fig. 4.
Forking diagrams:

\[
\begin{array}{c}
S,III \quad S,III \quad S,III \quad S,III \\
\begin{array}{ccc}
S,III & S,III & S,III \\
\text{lapp} & \text{lapp} & \text{lapp} \\
\text{llet} & \text{llet} & \text{llet} \\
\end{array}
\end{array}
\]

\[a \in \{\text{lapp, llet}\} \quad a \in \{\text{probl, probr}\}\]

Commuting diagrams:

\[
\begin{array}{c}
S,b \quad S,b \quad S,b \\
\begin{array}{ccc}
S,b & S,b & S,b \\
\text{sr} & \text{sr} & \text{sr} \\
\text{cp} & \text{cp} & \text{cp} \\
\end{array}
\end{array}
\]

\[b \in \{\text{lapp, llet}\} \quad a \in \{\text{probl, probr}\}\]

**Base cases:** Let \( s \xrightarrow{\text{cpS}} t \) if \( s \) is a WHNF, then \( t \) is a WHNF. If \( t \) is a WHNF then \( s \) is a WHNF or \( s \xrightarrow{\text{cp}} s' \) and \( s' \) is a WHNF.

**TRS \( R_1 \) for forking diagrams:**

\[
\begin{array}{c}
\text{SIII}(\text{SIII}(x)) \rightarrow x \\
\text{SIII}(\text{SR}(x)) \rightarrow \text{SR}(x) \\
\text{SIII}(\text{SR}(x)) \rightarrow \text{SR}(\text{SIII}(x)) \\
\text{SIII}(\text{SR}(x)) \rightarrow \text{SR}(\text{SIII}(x)) \\
\end{array}
\]

**TRS \( R_2 \) for commuting diagrams:**

\[
\begin{array}{c}
\text{Scep}(\text{SR}(x)) \rightarrow \text{SR}(\text{Scep}(x)) \\
\text{Scep}(\text{SR}(x)) \rightarrow \text{SR}(\text{Scep}(x)) \\
\text{Scep}(\text{SR}(x)) \rightarrow \text{SR}(\text{Scep}(x)) \\
\text{Scep}(\text{SR}(x)) \rightarrow \text{SR}(\text{Scep}(x)) \\
\end{array}
\]

**Figure 3:** Diagrams, Base Cases and TRSs for \((\text{III})\)

Here \( \text{SR} \) represents any \( \xrightarrow{\text{sra}} \)-reduction with \( a \neq \text{lbeta} \), \( \text{SRlbeta} \) represents a \( \xrightarrow{\text{sr}\text{lbeta}} \)-step, and \( \text{Scep} \) and \( \text{ScpS} \) represent \( \xrightarrow{\text{cpd}} \) and \( \xrightarrow{\text{cp}} \) steps. \( \text{TTT} \) or \( \text{AProve} \) [1, 8] are able to show innermost termination of \( R_2 \) \((\text{verifier CeTA} [5, 29]) \) can certify these proofs). After obtaining \( s \xrightarrow{\text{sr} \rightarrow} s' \) the base case for \( \xrightarrow{\text{Scep}} \) shows that \( s' \) can be standard reduced to a WHNF. The diagrams and base cases show that the prob-sequences are not changed.

The correctness proofs for \( \xrightarrow{\text{cpS}} \) and \( \xrightarrow{\text{ch}} \) are straightforward and we omit them (they are given in the appendix).

**Proposition 5.3.** The transformations \( \xrightarrow{\text{cpS}} \) and \( \xrightarrow{\text{ch}} \) are correct.

The diagrams and base cases for \((\text{gc})\) and \((\text{ucp})\) are in Fig. 5.

**Proposition 5.4.** The transformations \( \xrightarrow{\text{gc}} \) and \( \xrightarrow{\text{wcp}} \) are correct.

**Figure 4:** Diagrams and Base Cases for \((\text{cp})\)

**Proof.** The proof is analogous to the proof of Proposition 5.2, where the diagrams and the base cases for \((\text{ucp})\) and \((\text{gc})\) are applied. They show that prob-sequences are not changed. For proving termination of diagram application, we use automated termination techniques, where, however, we have to encode the transitive closure that occurs in the diagrams. In [21] they were encoded by integer rewrite systems and free integer variables to guess any number. We use a similar TRS where free variables occur on the right-hand sides and are interpreted as any constructor term. We encode numbers using the Peano-encoding. An adapted version of \( \text{AProve} \) and the certifier CeTA can handle those termination problems. For the forking diagrams, \( \text{AProve} \) shows innermost termination of the TRS \( R_1 \) in Fig. 5, where \( \text{SR} \) represents any \( \xrightarrow{\text{sr} \rightarrow} \)-reduction that is not an \( \xrightarrow{\text{sr} \rightarrow} \)-step, \( \text{SRlbeta} \) represents \( \xrightarrow{\text{sr} \rightarrow} \)-reductions. \( \text{Srug} \) represents the union of \( \xrightarrow{\text{sr} \rightarrow} \) and \( \xrightarrow{\text{sr} \rightarrow} \), \( k \) and \( x \) are variables, \( s \) represents the successor of Peano-numbers, \( W \) is used to generate the transformations from the guessed number. CeTA certifies the proof.

For the commuting diagrams the TRS \( R_2 \) shown in Fig. 5 is shown to be innermost terminating by \( \text{AProve} \) and the proof is
Forking diagrams:

Base cases: Let $s \xrightarrow{sucv} t$.

If $s$ is a WHNF, then $t$ is a WHNF.

If $t$ is a WHNF, then $s = a \xrightarrow{sr} sul, app, ac(probl, prob, lapp)$ where the arity of the former 3 constructors is 0 and for the latter two, it is 2. We assume that there is a case$_T$-operator for every type $T$, and that type $T$ has constructors $c_{Ti}, \ldots, c_{Tn}$. The syntax of $L_{\text{case, seq}}$ extends the syntax of $L_{\text{need, @}}$ as follows:

\[
s, t, r \in \text{Expr} ::= \ldots \ | \text{seq } s \ t \ | c_{Ti} \ s_1 \ldots s_{ar(c_{Ti})} \ | \text{case}_T \ s \ \text{of} \ \text{alt}_T^r \ \\
\text{alt}_T^r ::= \{ a_{1}, \ldots, a_{n_{c_{Ti}}} \} \ | \text{case}_T \ s \ \\
\text{alt}_{Ti,r} ::= c_{Ti} \ s_{1} \ldots s_{ar(c_{Ti})} -> s
\]

Contexts and surface contexts are extended to the new syntax: in contexts $C$ the hole can appear at any expression-position, and in surface contexts $S$, the hole is not inside the body of an abstraction. The $A$-contexts are extended as:

\[
A \in A ::= \ldots \ | \text{seq } A \ s \ | \text{case}_T \ A \ \text{of} \ \text{alt}_T s
\]

Reduction contexts are defined as before using the extended $A$-contexts. WHNFs are extended such that also $c_{Ti} \ s_1 \ldots s_n$, let $env$ in $c_{Ti} \ s_1 \ldots s_n$, and let $x_i = x_{i+1}^{m_{i+1} \cdot s_{ar(c_{Ti})}}$. $c_{Ti} \ s_1 \ldots s_n$ in $x_i$ are WHNFs. Standard reduction is defined by the rules already introduced in Fig. 1 (where $R$-contexts stem from the extended definition) and by the rules defined in Fig. 6 allowing to evaluate case- and seq-expressions. The rules $(sr, \text{seq})$ and $(sr, \text{case})$ adjust let-environments w.r.t. seq- and case-expressions.
(sr, case-c) \( R[\text{case}\_t c_t, x_1, \ldots, x_n; \text{of} \ldots; c_t, x_1 \rightarrow \ldots; x_n \rightarrow t; \ldots] \) \( \rightarrow R[\text{let} \{ x_i = s_i \}_{i=1}^n \text{ in } t] \)

(sr, case-in) \( \text{let} \{ x_i = s_i \}_{i=1}^n, x_m = c_t, x_1, \ldots, x_n, \text{env} \in \text{A}[\text{case}\_t x_1 \text{ of} \ldots; c_t, y_1, \ldots, y_n \rightarrow t; \ldots] \)

\( \rightarrow \text{let} \{ x_i = s_i \}_{i=1}^n, x_m = c_t, z_1, \ldots, z_n, \{ z_i = s_i \}_{i=1}^n, \text{env} \in \text{A}[\text{let} \{ y_i = z_i \}_{i=1}^n \text{ in } t] \)

(sr, case-e) \( \text{let} \{ x_i = \text{A}_t[x_i+1] \}_{i=1}^n, x_m = \text{A}_t[\text{case}\_z z_1 \text{ of} \ldots; c_t, y_1, \ldots, y_n \rightarrow t; \ldots] \}

\( \{ z_i = z_{j+1} \}_{j=1}^{k-1}, z_k = c_t, s_1, \ldots, s_n, \text{env} \in \text{A}[x_i] \)

\( \rightarrow \text{let} \{ x_i = \text{A}_t[x_i+1] \}_{i=1}^n, x_m = \text{A}_t[\text{let} \{ y_i = w_i \}_{i=1}^n \text{ in } t], \{ z_i = z_{j+1} \}_{j=1}^{k-1}, z_k = c_t, w_1, \ldots, w_n, \{ w_i = s_i \}_{j=2}^n, \text{env} \in \text{A}[x_i] \)

(sr, lcase) \( R[\text{case} \left( \text{let} \text{env} \in s \right) \text{ of } \text{alt}\_s] \rightarrow R[\text{let} \text{env} \in \text{case}\_t x_1 \text{ of } \text{alt}\_s] \)

(sr, seq-c) \( R[\text{seq} v \rightarrow t] \rightarrow R[t] \) if \( v \) is an abstraction or a constructor application

(sr, seq-in) \( \text{let} x_i = x_i+1 \}_{i=1}^n, x_m = c_t, x_1, \ldots, x_n, \text{env} \in \text{A}[\text{seq} x_1 \rightarrow t] \)

\( \rightarrow \text{let} \{ x_i = s_i \}_{i=1}^n, x_m = c_t, s_1, \ldots, s_n, \text{env} \in \text{A}[t] \)

(sr, seq-e) \( \text{let} \{ x_i = \text{A}_t[x_i+1] \}_{i=1}^n, x_m = \text{A}_t[\text{seq} z_1 \rightarrow t], \{ z_j = z_{j+1} \}_{j=1}^{k-1}, z_k = c_t, s_1 \ldots s_n, \text{env} \in \text{A}[x_1] \)

\( \rightarrow \text{let} \{ x_i = \text{A}_t[x_i+1] \}_{i=1}^n, x_m = \text{A}_t[t] \{ z_j = z_{j+1} \}_{j=1}^{k-1}, z_k = c_t, s_1 \ldots s_n, \text{env} \in \text{A}[x_1] \)

(sr, lseq) \( R[\text{seq} (\text{let} \text{env} \in s) \rightarrow t] \rightarrow R[\text{let} \text{env} \in \text{seq} s \rightarrow t] \)

**Figure 6: Standard Reduction-Rules for case and seq**

**Unions:**
- (case-c) (case-e) \& (case-x-in)
- (case) (case-c) \& (case-in) \& (case-e)
- (seq) (seq-c) \& (seq-in) \& (seq-e)
- (lacs) (lapp) \& (lcase) \& (lseq)
- (lill) (llet) \& (lacs)

**Weighted Standard Reduction.** Contextual preorder, contextual equivalence, and correctness of program transformations are defined analogously for the calculus \( L_{\text{need}, \text{app}} \) but now instantiated with the extended syntax and standard reduction. The context lemma also holds for the extended calculus: this can be verified by checking all cases for the extended syntax. The transformations in **Fig. 2** are also transformations in \( L_{\text{need}, \text{app}} \) (again using the extended syntax for all meta-expressions), and in **Fig. 7** additional transformations and unions are shown where we also extend the union (Ill).

**Correctness of the Transformations.** (Ibeta), (seq-c), (case-c) follows from the context lemma, (the arguments are analogous to Proposition 4.3). Correctness of (case-in), (case-e), (seq-in), (seq-e) can be obtained by combining other transformations (i.e. (case-c) or (seq-c), resp.) (cpdx), (abs), (gc) and (cpx)). Correctness of (abs) follows form the correctness of (ucp), since (abs) can be reversed by (ucp), Correctness of (probid), (propcomm), (probreorder), (probdistr) can be proved analogously as the proofs in the calculus \( L_{\text{need}, \text{app}} \). For the remaining transformations, the diagram-based method can be used to show correctness, where the diagrams have to be re-computed since there are more rules and an extended syntax. The diagrams computed are shown in the appendix (Figs. 10 and 11). In our automated tool, we had to restrict the case-expressions and types and data constructors, to Booleans, lists, and pairs (since the more general syntax using \( c_t, t \) for constructors of type \( T \) is not supported). However, the diagrams for the full syntax could be obtained by manually extending the cases of case-expressions and constructors (and inspecting them). This changed the diagrams, for instance, a sequence of two (cpx)-steps has to be replaced by a sequence of arbitrary many (cpx)-steps in the diagrams for (5, cpdx), since the number of steps depends on the arity of some constructor. Note also, that some diagrams of transformations are required, even if we can prove correctness of the transformation without diagrams: this is the case if the transformations occur in other diagrams. For all diagrams, termination of the induction proofs can be shown by transforming them into term rewrite systems (with free variables on the right-hand sides to encode transitive closures) and proving innermost termination using AProVE. The diagrams show that prob-sequences are preserved and thus the correctness of the transformations holds. The TRSs and the automated proofs are available from https://p9471.gitlab.io/pro-lneed/.

**7 CONCLUSION.**

We have introduced a call-by-need lambda-calculus with a binary operator for probabilistic computations. A small-step evaluation
that keeps track of the probabilities results in a semantics of programs that observes expected convergence in all program contexts. Based on this new notion of contextual equivalence we have developed techniques and tools to show equivalences and correctness of program transformations. We have applied them to prove the correctness of several transformations. We have discussed extensions of the calculus to make them a more realistic model of a probabilistic programming language and have sketched how to transfer our techniques and results to the extended language. Future work may take into account extensions with (polymorphic) typing and notions of equivalence, that restrict observations to data values (or even numbers) only. A goal may be to show that the program-calculus with its contextual semantics is a fully-abstract model w.r.t. usual probabilistic models of mathematics.

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A  LARGER EXAMPLES

Example A.1. We show the two evaluations for the expression

\[ \text{let } z = K \odot K2 \in (z \, (z \, a \, b)) \]

One evaluation is:

\[
\text{let } z = K \odot K2 \in (z \, (z \, a \, b)) \rightarrow
\text{let } z = K \in (z \, (z \, a \, b)) \rightarrow
\text{let } z = ((\text{let } x = (z \, a \, b)) \in \lambda y. x) \rightarrow
\]

Another evaluation is:

\[
\text{let } z = K \odot K2 \in (z \, (z \, a \, b)) \rightarrow
\text{let } z = K \in (z \, (z \, a \, b)) \rightarrow
\text{let } z = \lambda y. x \in ((\lambda y. x) \, (z \, c \, d)) \rightarrow
\]

Then there are the following continuations:

\[ (1) \]

\[
\text{let } z = \lambda x. \lambda y. x \in y \in (z \, (z \, a \, b)) \rightarrow
\text{let } z = \lambda x. \lambda y. x \in y \in ((\text{let } x = (z \, a \, b) \in \lambda y. x \in y) \rightarrow
\]

Now again, two continuations exist:

\[ (2) \]

\[
\text{let } z = \lambda x. \lambda y. x \in y \in (z \, (z \, a \, b)) \rightarrow
\text{let } z = \lambda x. \lambda y. x \in y \in ((\text{let } x = (z \, a \, b) \in \lambda y. x \in y) \rightarrow
\]

The four evaluations of the expression

\[ \text{let } z = \lambda x. \lambda y. x \in y \in (z \, (z \, a \, b)) \]

are as follows: all of them start with

\[
\text{let } z = \lambda x. \lambda y. x \in y \in (z \, (z \, a \, b)) \rightarrow
\text{let } z = \lambda x. \lambda y. x \in y \in ((\text{let } x = (z \, a \, b) \in \lambda y. x \in y) \rightarrow
\]

Then there are the following continuations:

\[ (1) \]

\[
\text{let } z = \lambda x. \lambda y. x \in y \in (z \, (z \, a \, b)) \rightarrow
\text{let } z = \lambda x. \lambda y. x \in y \in ((\text{let } x = (z \, a \, b) \in \lambda y. x \in y) \rightarrow
\]

Now again, two continuations exist:

\[ (2) \]

\[
\text{let } z = \lambda x. \lambda y. x \in y \in (z \, (z \, a \, b)) \rightarrow
\text{let } z = \lambda x. \lambda y. x \in y \in ((\text{let } x = (z \, a \, b) \in \lambda y. x \in y) \rightarrow
\]
We first show that applying
\[ \lambda x . y . x \otimes y = (a b) \], \( x_1 = c, y_1 = d \) in \ y \]

increased. Hence there are no infinite reduction sequences consistent with the focus: \[ \lambda x . y . x \otimes y = (a b) \], \( x_1 = c, y_1 = d \) in \ y \]

B PROOFS

Proposition 4.3. The transformation \( \beta \) is correct.

Proof. We use the context lemma (Theorem 3.4) for \( n = 1 \) (and where \( d = 0 \)). Let \( s \beta \rightarrow t \) and \( R \) be a reduction context. The structure of reduction contexts implies \( R[s] \stackrel{sr_1}{\rightarrow} R[t] \). Thus there is a reduction context \( \lambda x . y . x \otimes y = (a b) \), \( x_1 = c, y_1 = d \) in \ y \]

show by specifying a rewrite strategy (covering all possibilities) that then for every evaluation \( \lambda x . y . x \otimes y = (a b) \), \( x_1 = c, y_1 = d \) in \ y \]

B.1 A Manual Proof for (III)

We first show that applying (III) steps in arbitrary contexts terminates:

Lemma B.1. Every sequence of \( \beta \) -steps is finite.

Proof. Let the measure \( LM(s) \) be defined as the pair \( (#(1 . 1 . s), s) \), ordered lexicographically, where \( # \) is the number of occurrences of \( 1 \) in \( s \) and \( LM \) is a polynomial measure as \( LM(x) = 1 \) and \( LM(\lambda x . s) = LM(s) + 2LM(s) + LM(t) \). The definitions show that \( C \Rightarrow \beta \) and \( C \Rightarrow \beta \) -steps strictly reduce the number of let-expressions and thus the first component of the measure is strictly decreased. For the \( C \Rightarrow \beta \) -steps, the first component is not changed, but the measure \( LM(\cdot) \) is strictly decreased. Hence there are no infinite reduction sequences consisting only of \( \beta \) -steps.

Lemma B.2. Let \( s, t \) be expressions such that \( s \beta \rightarrow t \). We show that then for every evaluation \( s \beta \rightarrow t \), there exists an evaluation \( s \beta \rightarrow t \) in \( \beta \).

Proof. The base case is covered by the base cases for (III). In the general case, we consider a representation of a reduction sequence, i.e. \( s \beta \rightarrow t \) where \( s \) is a WHNF and to show by specifying a rewrite strategy (covering all possible instances) that it can be transformed in a finite number of steps into \( t \) where \( t \) is a WHNF. Then as a base case, we have that \( t \) is a WHNF, or can be reduced to a WHNF, or can be reduced to t, then shifting is finished.

The following strategy is used to transform the reduction sequence:

1. First let (III) be the smallest substring containing all \( \beta \) -reductions.
2. If (III) is empty, then we finish.
3. If the reduction sequence is of the form \( s \beta \rightarrow t \), then shifting is finished.
4. Otherwise, it is of the form \( s \beta \rightarrow t \), and we select the focus as \( s \beta \rightarrow t \), such that the current sequence is \( s \beta \rightarrow t \). Now the transformations take only place on the focus part:
   a. Repeat the following step until the rightmost element of the focus is a \( \beta \) -reduction, or there are no more \( \beta \) in the focus.
   b. Use one of the forsaking diagrams for (III) to shift the rightmost \( \beta \) -reduction in the focus to the right, thereby moving \( \beta \) -steps to the left.

5. Jump to Item 1

Now we check the property of the non-deterministic transformation algorithm. There are two cases:

- The start focus is \( s \beta \rightarrow t \) and a \( \beta \). The diagrams show that the iterated shift ends with \( s \beta \rightarrow t \). This shows that prob-reductions are moved without change.
- The start focus is \( s \beta \rightarrow t \). The situation is similar, however, the focus part may be a mix of \( \beta \) and \( \beta \) reductions. It is easy to see that the effect now is that more and more \( \beta \) are generated to the right of the focus. This stops after generating a finite number due to Lemma B.1.

As a summary, we see that our strategy will produce a final situation \( s' \beta \rightarrow t' \) and we can apply the base case of the diagrams.

Lemma B.3. Let \( s, t \) be expressions such that \( s \beta \rightarrow t \). Then for every evaluation \( s \beta \rightarrow t \), there exists an evaluation \( s \beta \rightarrow t \) in \( \beta \).

Proof. Let \( s, t \) be expressions with \( s \beta \rightarrow t \). We show that an evaluation of \( t \) can be transformed into an evaluation of \( s \) where the set of commuting diagrams is used as transformations (on the reduction sequence) and where we have to take into account the base cases.

We consider a representation of a sequence \( s \beta \rightarrow t \) where \( t \) is a WHNF. We show that it can be transformed in a finite number of steps into \( s \beta \rightarrow t \). The base case is that \( s \beta \rightarrow t \) is a WHNF, or can be reduced to a WHNF, or can be reduced to t, then shifting is finished.

The following strategy is used to transform the sequence:

1. First let (III) be the smallest substring containing all \( \beta \) -reductions.
2. If (III) is empty, then we finish.
3. If the reduction sequence is of the form \( s \beta \rightarrow t \), then shifting is finished.
Forking diagrams:

\[
\begin{align*}
\text{Commuting diagrams:} & \quad \begin{array}{c}
\begin{array}{c}
\text{Forking diagrams:}
\end{array}
\end{array} \\
\text{Base cases:} & \quad \begin{array}{c}
\begin{array}{c}
\text{Commuting diagrams:}
\end{array}
\end{array} \\
\end{align*}
\]

The diagrams and base cases for (cpx) are shown in Fig. 8. In the case of forking diagrams, the symbol SR is interpreted as an SR-step, while in the case of commuting diagrams it represents an SR-step. R is shown to be innermost terminating using TTT2. □

\[
\begin{align*}
\text{Base cases: if } & s \xrightarrow{\text{cpx}} s', \text{then } s \text{ is a WHNF iff } s' \text{ is a WHNF.} \\
\text{TRS } R \text{ for forking and commuting diagrams:} & \\
\text{Figure 9: Diagrams and Base Cases for (xch)} \\
\text{The diagrams and base cases for (xch) are shown in Fig. 9.} \\
\text{PROPOSITION B.5. The transformation (xch) is correct.} \\
\text{Proof. The proof is analogous to the proof of Proposition 5.2, where the diagrams and the bases cases for (xch) are applied. Termination of the diagram application is obvious, and also that prob-sequences are preserved.} \\
\text{PROPOSITION 5.3. The transformations } \xrightarrow{\text{cpx}} \text{ and } \xrightarrow{\text{xch}} \text{ are correct.} \\
\end{align*}
\]

\[
\begin{align*}
\text{C DIAGRAMS} \\
\text{Figs. 10 and 11 show the diagrams for transformations in the calculus } \caseseq_{\text{need},@}.
\end{align*}
\]

B.2 Correctness of (cpx) and (xch)

The diagrams and base cases for (cpx) are shown in Fig. 8.

PROPOSITION B.4. The transformation (cpx) is correct.

Proof. The proof is analogous to the proof of Proposition 5.2, where the diagrams and the bases cases for (cpx) are applied. One has to verify that prob-sequences are not changed by the diagrams and the base cases (which holds). We also have to verify that diagram application terminates. In both cases (forking diagrams and commuting diagrams), termination can be shown by proving (innermost) termination of the TRS R shown in Fig. 8. In the case of forking diagrams, the symbol SR is interpreted as an SR-step, while in the case of commuting diagrams it represents an SR-step. R is shown to be innermost terminating using TTT2. □
Forking diagrams:

Base cases: If \( \frac{s \rightarrow s'}{s\text{lacs}} \) then \( s' \) is a WHNF iff \( s \) is a WHNF.
If \( \frac{s \rightarrow s'}{s\text{let}} \) then: if \( s \) is a WHFN, then \( s' \) is a WHNF, and if \( s' \) is a WHNF, then \( s \) is a WHNF or \( s \rightarrow s'' \) where \( s'' \) is a WHNF.

(a) Diagrams for \( (\text{lll}) \)

Commuting diagrams:

Base cases: If \( \frac{s \rightarrow t}{s_{ep}} \), then if \( s \) is a WHNF, then \( t \) is a WHNF, and if \( t \) is a WHNF then either \( s \) is a WHNF or \( s \rightarrow s' \) and \( s' \) is a WHNF.

(b) Diagrams for \( (cp) \)

Commuting diagrams:

Base case: For \( s \rightarrow t, s \) is a WHNF iff \( t \) is a WHNF.

(c) Diagrams for \( (xch) \)

Forking diagrams:

Base cases: If \( \frac{s \rightarrow s'}{s_{ep}} \), then \( s' \) is a WHNF.

(d) Diagrams for \( (cpx) \)

Commuting diagrams:

Base cases: If \( s \rightarrow s' \) and \( s' \) is a WHNF, then \( s \) is a WHNF.

Figure 10: Diagrams for \( l_{\text{case, seq}} \)
Forking diagrams:

**Base cases:** If \( S_{abs} \rightarrow t \), then \( s \) is a WHNF iff \( t \) is a WHNF.

(a) Diagrams for \((abs)\)

(b) Diagrams for \((gc)\)

(c) Diagrams for \((ucp)\)

Commuting diagrams:

**Base cases:** If \( S_{gc} \rightarrow t \), then if \( s \) is a WHNF, then \( t \) is a WHNF; and if \( t \) is a WHNF, then \( s \) is a WHNF or \( s \rightarrow s' \) and \( s' \) is a WHNF.

Figure 11: Diagrams for \( L_{\text{need@}} \) (cont.)