Some general properties of unified entropies

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Basic properties of the unified entropies are examined. The consideration is mainly restricted to the finite-dimensional quantum case. Bounds in terms of ensembles of quantum states are given. Both the continuity in Fannes’s sense and stability in Lesche’s sense are shown for wide ranges of parameters. In particular, uniform estimates are obtained for the quantum Rényi entropies. Stability properties in the thermodynamic limit are discussed as well. It is shown that the unified entropies enjoy both the subadditivity and triangle inequality for a certain range of parameters. Non-decreasing of all the unified entropies under projective measurements is proved.

Keywords: Rényi entropy, Tsallis entropy, Fannes inequality, Lesche stability criterion, subadditivity, quantum measurement

I. INTRODUCTION

The concept of entropy is one of fundamentals in statistical physics and information theory. Entropic quantities are also interesting mathematical subjects with many facets. Developing statistical methods in various fields, some extensions of the Shannon and von Neumann entropies were found to be expedient. Most important of them are the Rényi entropy [35] and the entropy introduced independently by Havrda and Charvát [16] and by Tsallis [37]. Another class of one-parameter extensions of the Shannon entropy was considered in the paper [3]. Hu and Ye proposed the concept of unified entropy in both the classical and quantum regimes [17]. Up to a notation, this is actually the entropic form shortly discussed by Tsallis in the review [38]. All the above entropies are particular cases of the unified entropies. At the same time, some of so-called $f$-entropies [10] are not covered by the notion of unified entropy. Note that Petz’s quasi-entropies [27, 28] were emerged as the quantum counterpart of the Csiszár $f$-divergence.

There exist those general properties that are of primary importance in physical applications of entropic functionals. In the present paper, we analyze some of these properties with respect to the family of unified entropies, mainly in the finite-dimensional quantum regime. In particular, we consider simple bounds in terms of pure-state ensembles and convex combinations of density operators. The continuity in the sense of Fannes as well as the stability in the sense of Lesche are examined. Although the complete picture is not reached, few important facts are resolved here. We prove the subadditivity of the quantum unified entropy for a wide range of parameters. We also show that all the quantum unified entropies are non-decreasing under action of projective measurements.

The paper is organized as follows. In Section II, the main definitions are given. Some bounds in terms of ensembles of quantum states are established. In Section III, uniform estimates on the unified entropy for a wide range of parameter values are obtained. These relations can be treated as inequalities of Fannes’ type. Using these results, in Section IV we analyze the unified entropies with respect to the Lesche stability criterion. Both the stability and instability in the thermodynamic limit are considered as well. In Section V, the known result on the subadditivity of the quantum Tsallis entropy for $q > 1$ is naturally extended to some of the unified entropies. In Section VI, non-decreasing of the quantum unified entropy under projective measurements is established. It is also shown that measurements of more general type can decrease the unified entropy. Section VII concludes the paper.

II. DEFINITIONS AND TWO BOUNDS

At first, we briefly recall some required material. Let $\mathcal{L}_{s.a.}(\mathcal{H})$ be the space of Hermitian operators on $d$-dimensional Hilbert space $\mathcal{H}$. The set $\mathcal{L}_+(\mathcal{H})$ of positive semidefinite operators is a convex cone in $\mathcal{L}_{s.a.}(\mathcal{H})$. For any operator $X$, we put $|X|\in\mathcal{L}_+(\mathcal{H})$ as the positive square root of $X^*X$. The singular values $\sigma_j(X)$ of operator $X$ are then defined as the eigenvalues of $|X|$ [6]. For $q \geq 1$, the Schatten $q$-norm is defined by

$$
\|X\|_q := \left(\sum_{j=1}^d \sigma_j(X)^q\right)^{1/q}.
$$

Both the Rényi and Tsallis entropies are fruitful one-parametric generalizations of the Shannon entropy. For unifying treatment, we will denote the corresponding entropic index by $q$. Let $p = \{p_1, \ldots, p_m\}$ be a probability distribution. For $q > 0$ and $q \neq 1$, the Rényi $q$-entropy of this probability distribution is defined as [35]

$$
R_q(p) := \frac{1}{1-q} \ln \left(\sum_{i=1}^m p_i^q\right).
$$

For $q < 0$ and $q \neq -1$, the Rényi $q$-entropy is defined as [35]

$$
R_q(p) := \frac{1}{1-q} \ln \left(\sum_{i=1}^m p_i^{-q}\right).
$$
In the nonextensive statistical physics, the Tsallis $q$-entropy of the probability distribution $p$ is defined as

$$H_q(p) := \frac{1}{1-q} \left(\sum_{i=1}^{m} p_i^q - 1\right) = -\sum_{i=1}^{m} p_i^q \ln_q p_i,$$

where the $q$-logarithm $\ln_q x = (x^{1-q} - 1)/(1-q)$. Methods of nonextensive statistical mechanics have found use in many topics of physics and other sciences (see the works [14, 38] and references therein). In the limit $q \to 1$, both the above quantities coincide with the Shannon entropy $H_1(p) = -\sum_i p_i \ln p_i$. Another extension is the entropy of type $q$ [3] defined as

$$qH(p) := \frac{1}{q-1} \left[\left(\sum_{i=1}^{m} p_i^{1/q}\right)^q - 1\right],$$

where $q > 0$ and $q \neq 1$. The authors of the paper [17] proposed the notion of unified entropy

$$E^{(s)}_q(p) := \frac{1}{1-q} \left(\sum_{i=1}^{m} p_i^q - 1\right)$$

for $q > 0$, $q \neq 1$ and $s \neq 0$. This two-parametric entropic functional includes the entropies (3) and (4) as partial cases and the entropy (2) as the limiting case $s \to 0$ [17]. Taking $s = 1 - q'$, the entropy (5) is actually the entropic $q, q'$-form previously discussed by Tsallis [38]. This form was emerged in lines with the paper [21], where a nonextensive thermodynamic formalism for description of chaos is developed. Entropies of quantum states are important in various topics. The quantum Rényi entropy of density operator $\rho$ is defined as

$$R_q(\rho) := \frac{1}{1-q} \ln[\text{tr}(\rho^q)] ,$$

that can be rewritten as $R_q(\rho) = q(1-q)^{-1} \ln \|\rho\|_q$ for $q > 1$. The quantum Tsallis entropy of density operator $\rho$ is defined as

$$H_q(\rho) := \frac{1}{1-q} \text{tr}(\rho^q - \rho) ,$$

in particular, $H_q(\rho) = (1-q)^{-1} (\|\rho\|_q^q - 1)$ for $q > 1$. For $q = 1$, we have the von Neumann entropy $S(\rho) = -\text{tr}(\rho \ln \rho)$. Like the Rényi and Tsallis entropies, the expression (5) can be adopted for the quantum case. The quantum unified $(q, s)$-entropy of density operator $\rho$ is defined as

$$E^{(s)}_q(\rho) := \frac{1}{1-q} \left\{\text{tr}(\rho^q)^s - 1\right\}$$

for $q > 0$, $q \neq 1$ and $s \neq 0$, and as the quantum Rényi entropy $E^{(0)}_q(\rho) = R_q(\rho)$ for $s = 0$ [17]. Some properties of the quantum unified entropy were considered in [17]. In particular, lower and upper bounds on the entropy (8) are given for $1 \leq q, 1 \leq s$. We now obtain two results which complement the discussion of the paper [17] in this regard.

**Theorem II.1** Let $\{p_i, \langle \psi_i \rangle\}$, where $\langle \psi_i | \psi_i \rangle = 1$ and $\sum_i p_i = 1$, be an ensemble of pure states that gives rise to the density operator $\rho$, i.e.

$$\rho = \sum_i p_i | \psi_i \rangle \langle \psi_i | .$$

For $0 < q$ and $s \neq 0$ as well as for $0 < q < 1$ and $s = 0$, there holds

$$E^{(s)}_q(\rho) \leq E^{(s)}_q(p) .$$

**Proof.** We will assume that $q \neq 1$. Let $\lambda_j$ and $| \varphi_j \rangle$ denote eigenvalues and eigenstates of $\rho = \sum_j \lambda_j | \varphi_j \rangle \langle \varphi_j |$. The ensemble classification theorem says that [18]

$$\sqrt{p_i} | \psi_i \rangle = \sum_j u_{ij} \sqrt{\lambda_j} | \varphi_j \rangle$$

for some unitary matrix $[[u_{ij}]]$. It follows from [11] and $\langle \varphi_j | \varphi_k \rangle = \delta_{jk}$ that $p_i = \sum_j w_{ij} \lambda_j$, where $w_{ij} = u_{ij}^* u_{ij}$ are elements of a unistochastic matrix, i.e. $\sum_j w_{ij} = 1$ for all $j$ and $\sum_j w_{ij} = 1$ for all $i$. The function $x \mapsto x^q$ is concave for $0 < q < 1$ and convex for $1 < q$. Applying Jensen's inequality to this function, we obtain

$$\sum_i p_i^q = \sum_i \left(\sum_j w_{ij} \lambda_j\right)^{\frac{q}{n}} \left\{ \begin{array}{ll} \geq, & 0 < q < 1 \\ \leq, & 1 < q \end{array} \right\} \sum_i \sum_j w_{ij} \lambda_j^q = \sum_j \lambda_j^q$$

(12)
in view of unistochasticity of the matrix \([|w_{ij}|]\). The function \(y \mapsto y^s/s\) monotonically increases for \(s \neq 0\), whence

\[
\frac{1}{s} \left( \sum_i p_i y_i^s \right)^s \left\{ \begin{array}{c}
\geq 1, \quad 0 < q < 1 \\
\leq 1, \quad q < 1
\end{array} \right. \frac{1}{s} \left( \sum_j \lambda_j^q \right)^s.
\]

(13)

Since the factor \((1 - q)^{-1}\) is positive for \(q < 1\) and negative for \(1 < q\), the relations (13) are combined as

\[
\frac{1}{(1-q)s} \left( \sum_i p_i y_i^s \right)^s \geq \frac{1}{(1-q)s} \left( \sum_j \lambda_j^q \right)^s.
\]

(14)

Due to \(\sum_j \lambda_j^q = \text{tr}(\rho^q)\) and the definitions (8) and (9), the inequality (14) provides (10). In the case \(s = 0\), the quantum Rényi entropy (6) is dealt, the result (10) for 0

\[
\text{relative entropy were obtained in the paper [3]. As given in terms of difference distances, these bounds characterize}
\]

operators. Suppose that \(\omega \) is just concave, whence

\[
\rho = \sum_i p_i \omega_i,
\]

(15)

where \(\text{tr}(\omega_i) = 1\) and \(\sum_i p_i = 1\). For \(1 \leq q \leq 1\), the authors of the paper [17] also presented a lower bound on \(E^q(x)\) in terms of combinations of a kind (15). So we shall now discuss this inequality for the range \(0 < q < 1\).

**Theorem II.2** For \(0 < q < 1\), \(s \leq 1\) and any convex combination (15), there holds

\[
\sum_i p_i E^q(x_i) \leq E^q(x).
\]

(16)

**Proof.** We firstly assume that \(s \neq 0\). For any concave function \(f(x)\), the functional \(\text{tr}[f(X)]\) is also concave on \(X \in \mathcal{L}_{s.a.}(\mathcal{H})\) (for details, see section III in [32]). For \(0 < q < 1\), the function \(x \mapsto x^q\) is just concave, whence

\[
\text{tr}(\rho^q) \geq \sum_i p_i \text{tr}(\omega_i^q).
\]

(17)

The function \(y \mapsto y^s/s\) on \(y \in [0; \infty)\) is increasing for all \(s \neq 0\) and concave for \(s \leq 1\) (its second derivative \((s - 1)y^{s-2} \leq 0\)). Applying these two points step-by-step, we have

\[
\frac{1}{s} \left[ \text{tr}(\rho^q) \right]^s \geq \frac{1}{s} \left[ \sum_i p_i \text{tr}(\omega_i^q) \right]^s \geq \frac{1}{s} \sum_i p_i \left[ \text{tr}(\omega_i^q) \right]^s.
\]

(18)

for \(s \leq 1\). Multiplying the term (15) by \((1 - q)^{-1} > 0\) and using the definition (8), we obtain the claim (10). When \(s = 0\), the quantum Rényi entropy of order \(q > 1\) is not purely convex nor purely concave [19], even in the classical regime. In general, the inequality (10) can also be regarded as lower bounds on the quantum unified entropy in terms of convex combination of density operators. Thus, we can summarize the following. In a wide range of parameter values, the quantum unified entropy (8) enjoy concavity properties similarly to the von Neumann entropy. Since such properties are very important for entropic functionals, potential capabilities of the unified entropies become more clear with the above results.

### III. UNIFORM ESTIMATES

Any entropic quantity must have good properties as a state functional. First of all, we are interested in continuity. For the von Neumann entropy, the answer is given by the well-known Fannes inequality [11]. This inequality in terms of trace norm distance bounds from above a potential change of the von Neumann entropy. Fannes’ inequality has been extended to the Tsallis entropy [13, 40] and its partial sums [31]. Some bounds of Fannes' type on the quantum relative entropy were obtained in the paper [3]. As given in terms of difference distances, these bounds characterize
a continuity of the quantum relative entropy in the sense of Fannes. We shall examine the unified entropy from this viewpoint.

Recall some results related to the Tsallis entropy. In the considered question, both the classical and quantum regimes can be treated simultaneously. So we will formulate for the quantum one. The trace distance between two operators $X$ and $Y$ is defined as

$$D_{\text{tr}}(X, Y) := \frac{1}{2} \text{tr}|X - Y| = \frac{1}{2} \|X - Y\|_1 .$$

(19)

In the classical regime, the trace distance is replaced with the $l_1$-distance. The two complementary results are known for the Tsallis entropy. The first method of estimating with restriction $q \in [0; 2]$ is presented in the paper [12]. Let us denote $\eta_q(x) := (x^q - x) / (1 - q)$. For $q \in [0; 2]$ and $d = \dim(H)$, we have

$$|H_q(\rho) - H_q(\omega)| \leq (2\epsilon)^q \ln q d + \eta_q(2\epsilon)$$

(20)

provided that $2D_{\text{tr}}(\rho, \omega) = 2\epsilon \leq q^{1/(1-q)}$. On the interval $0 < 2\epsilon < q^{1/(1-q)}$, the quantity $\eta_q(2\epsilon)$ and hence the right-hand side of (20) monotonically increase. The second method uses probabilistic coupling techniques [40]. For $q > 1$, we then have

$$|H_q(\rho) - H_q(\omega)| \leq \epsilon^q \ln q (d - 1) + H_q(\epsilon, 1 - \epsilon) ,$$

(21)

where $\epsilon = D_{\text{tr}}(\rho, \omega)$ and $H_q(\epsilon, 1 - \epsilon) = (1 - q)^{-1} (\epsilon^q + (1 - \epsilon)^q - 1)$ denotes the binary entropy. By calculus, the right-hand side of (21) monotonically increases for $0 < \epsilon < (d - 1)/d$. The inequality (21) is sharp in the sense that the equality sometimes takes place [40]. Including the dimension $d$ explicitly, the above bounds are finite-dimensional. The upper bounds (20) and (21) can be used for obtaining upper bounds on the unified entropy for some parameter ranges. Our method is based on the following statement.

Lemma III.1 For $s \geq 1$ and $x, y \in [0; 1]$ as well as for $0 < s \leq 1$ and $x, y \in [1; \infty)$, there holds

$$|x^s - y^s| \leq s |x - y| .$$

(22)

For $0 < s \leq 1$ and $x, y \in [0; 1]$ as well as for $s \geq 1$ and $x, y \in [1; \infty)$, the reversed inequality holds. For $x, y \in [1; \infty)$, the following inequality also takes place:

$$|\ln x - \ln y| \leq |x - y| .$$

(23)

Proof. When $x < y$, the aim (22) and the reversed one are recast as $(y^s - x^s) \leq s (y - x)$ and $(y^s - x^s) \geq s (y - x)$ respectively. Here we write down

$$y^s - x^s = \int_x^y s^{t^s - 1} dt \leq \begin{cases} \leq, & t^s - 1 \leq 1 \\ \geq, & t^s - 1 \geq 1 \end{cases} \int_x^y s dt = s (y - x) .$$

(24)

The condition $t^s - 1 \leq 1$ is satisfied for $0 < s < 1$ and $t \geq 1$ as well as for $s > 1$ and $t \leq 1$. The condition $t^s - 1 \geq 1$ is satisfied for $0 < s < 1$ and $t \leq 1$ as well as for $s > 1$ and $t \geq 1$. The aim (23) is justified by

$$\ln y - \ln x = \int_x^y \frac{dt}{t} \leq \int_x^y dt = y - x$$

(25)

under the condition $t \geq 1$. ■

Note that the case $s = 1$ is the scope of the continuity bounds (20) and (21) for the Tsallis entropy. We do not take $q = 1$, since the von Neumann entropy itself is considered in the original Fannes inequality [11]. The main result of this section is formulated as follows.

Theorem III.2 Let $\rho$ and $\omega$ be density operators on $d$-dimensional Hilbert space $H$. For the parameter range

$$\{ (q, s) : 0 < q < 1, s \in (-\infty; -1] \cup [0; +1] \}$$

(26)

under the condition $2D_{\text{tr}}(\rho, \omega) = 2\epsilon \leq q^{1/(1-q)}$, there holds

$$|E_q^{(s)}(\rho) - E_q^{(s)}(\omega)| \leq (2\epsilon)^q \ln q d + \eta_q(2\epsilon) .$$

(27)

Define the factor $\kappa_s = d^{2(q-1)}$ for $s \in [-1; 0]$ and $\kappa_s = 1$ for $s \in [+1; +\infty)$. For the parameter range

$$\{ (q, s) : 1 < q, s \in [-1; 0] \cup [+1; +\infty) \}$$

(28)

and arbitrary $D_{\text{tr}}(\rho, \omega) = \epsilon$, there holds

$$|E_q^{(s)}(\rho) - E_q^{(s)}(\omega)| \leq \kappa_s \left[ \epsilon^q \ln q (d - 1) + H_q(\epsilon, 1 - \epsilon) \right] .$$

(29)
Proof. Let us put the quantities \( \xi = \text{tr}(\rho^q) \) and \( \zeta = \text{tr}(\omega^q) \). For the range (26), we have \( \xi \geq 1 \) and \( \zeta \geq 1 \) due to \( 0 < q < 1 \). For \( s \in (-\infty; -1] \), we put \( \nu = -s \geq +1 \) and then write

\[
|E_q^{(s)}(\rho) - E_q^{(s)}(\omega)| = \frac{1}{(1-q)\nu} \left| \frac{1}{\xi^\nu} - \frac{1}{\zeta^\nu} \right|.
\]

(30)

Using the inequality (22) with \( x = 1/\xi, y = 1/\zeta \) and \( \nu \) instead of \( s \), the formula (30) leads to

\[
|E_q^{(s)}(\rho) - E_q^{(s)}(\omega)| \leq \frac{1}{(1-q)} \left| \frac{\zeta - \xi}{\xi \zeta} \right| \leq \frac{1}{1-q} |\xi - \zeta| = |H_q(\rho) - H_q(\omega)|
\]

due to the definition (7). Putting \( x = \xi \) and \( y = \zeta \), we use the inequalities (22) for \( s \in (0; +1] \) and (23) for \( s = 0 \) (the Rényi entropy case) with the result

\[
|E_q^{(s)}(\rho) - E_q^{(s)}(\omega)| \leq \frac{\xi - \zeta}{1-q} = |H_q(\rho) - H_q(\omega)|.
\]

(32)

Combining the right-hand side of (33) and (32) with the bound (20) finally gives (27).

For the range (28), we have \( \xi \leq 1 \) and \( \zeta \leq 1 \) by \( q > 1 \). In addition, both the traces \( \xi = \text{tr}(\rho^q) \) and \( \zeta = \text{tr}(\omega^q) \) are not less than \( d^{1-q} \). For \( s \in [-1; 0] \), we take \( \nu = -s \in (0; +1] \) and write down

\[
|E_q^{(s)}(\rho) - E_q^{(s)}(\omega)| = \frac{1}{(q-1)\nu} \left| \frac{1}{\xi^\nu} - \frac{1}{\zeta^\nu} \right| \leq \frac{1}{(q-1)} \left| \frac{\zeta - \xi}{\xi \zeta} \right| \leq \frac{d^{2(1-q)-1}}{1-q} |\xi - \zeta| = d^{2(q-1)}|H_q(\rho) - H_q(\omega)|
\]

(33)

by (22) with \( x = 1/\xi, y = 1/\zeta \) and \( \nu \) instead of \( s \). When \( s = 0 \) (the Rényi entropy case), we use (23) and get the upper bound (33) as well. For \( +1 \leq s \), we merely apply the formula (22) with \( x = \xi, y = \zeta \) and obtain (32) again. Combining the relations (32) and (33) with (24) finally gives (29).

For \( 1 \leq q \) and \( +1 \leq s \), the authors of the paper [17] have presented the upper bound

\[
|E_q^{(s)}(\rho) - E_q^{(s)}(\omega)| \leq 2q(q-1)^{-1}D_{\Phi}(\rho, \omega).
\]

(34)

So the inequality (29) gives another bound for such values of \( q \) and \( s \). But the novel point is that the bound (29) is also valid for values \( 1 \leq q \) and \( s \in [-1; 0] \). In addition, for the parameter range (26) with \( 0 < q < 1 \), the upper bound is stated by the formula (27). Like the bounds (20) and (21), the new bounds (27) and (29) are related to finite dimensions solely. These bounds are also not sharp. Nevertheless, they show that the unified entropy is continuous in the sense of Fannes for the parameter ranges (26) and (28). Moreover, the above results lead to some conclusions on the stability.

IV. NOTES ON STABILITY PROPERTIES

The stability property is very important for application of some quantity in a practice. This issue was inspired by Lesche [22] who discussed the Rényi entropy. The Tsallis entropy itself [1] and its partial sums [31] were considered with respect to the stability property. The class of f-entropies is examined in the papers [4, 25]. In this section, we analyze the unified entropies from this viewpoint. The Lesche stability criterion can be posed as follows [11, 9]. Let \( \Phi(\rho) \) be a state functional. We assume its values to be bounded from above by the least bound \( \Phi_M \). The stability means that for every \( \delta > 0 \), there exists \( \varepsilon > 0 \) such that

\[
|\Phi(\rho) - \Phi(\omega)| \cdot \Phi_M^{-1} \leq \delta,
\]

(35)

whenever \( D_{\Phi}(\rho, \omega) \leq \varepsilon \). In the finite-dimensional case, the unified entropy is bounded from above for all real \( s \) and \( q > 0 \) [17]. In particular, for \( q \neq 1 \) and \( s \neq 0 \) we have

\[
E_q^{(s)}(\rho) \leq \max E_q^{(s)} = \frac{1}{(1-q)s} \left( d^{1-q)s} - 1 \right),
\]

(36)

and \( \max E_q^{(0)} = \ln d \). Using the bounds (27) and (29), for the parameter ranges (26) and (28) we obtain

\[
|E_q^{(s)}(\rho) - E_q^{(s)}(\omega)| \cdot (\max E_q^{(s)})^{-1} \leq F(\varepsilon).
\]

(37)
Here the function $F(\varepsilon)$ respectively denotes the right-hand sides of (24) and (28) multiplied by $(\max E_q(s))^{-1}$. This function obeys $F(0) = 0$ and monotonically increases with $\varepsilon \in (0; \varepsilon_0)$, where $2 \varepsilon_0 = q^{1/(1-q)}$ for (24), $\varepsilon_0 = (d - 1)/d$ for (28). Hence one defines some one-to-one correspondence between the intervals $[0; \varepsilon_0] \text{ and } [0; \delta(\varepsilon_0)]$. For each $\delta \in (0; \delta(\varepsilon_0))$ we herewith have $\varepsilon_\delta > 0$ such that $F(\varepsilon) \leq \delta$ whenever $\varepsilon \leq \varepsilon_\delta$. Combining this with the relation (37) implies the stability property of the unified entropy for the parameter ranges (26) and (28).

The case $s = 0$, when Rényi’s entropy is dealt, demands few remarks. It is a common opinion that the Rényi entropy of order $q$ is not stable for all $q \neq 1$ [1, 9, 25]. This important conclusion was obtained by Lesche [22] and developed by Abe [1]. Meanwhile, both the parameter ranges (26) and (28) include the Rényi entropy. No contradiction is really posed here. Indeed, the reasons of the papers [1, 22] assume the thermodynamic limit, i.e. an infinity of degrees of freedom. But the above bounds are just finite-dimensional. Dividing (27) and (29) by $\max E_q(0)$, the right-hand side of (38) contains the ratio $\ln d/\ln d = (1 - q)^{-1}d^{1-q}/\ln d$ for $0 < q < 1$, $s = 0$, and the one $d^{2(q-1)}/\ln d$ for $1 < q$, $s = 0$. They are both unbounded as $d \to \infty$. In this limit, no conclusions on the Rényi entropy can be got from the inequality (37). The stability of the Rényi entropy of order $q \neq 1$ does hold for a finite $d$, but does not in the thermodynamic limit. In contrast, the Tsallis entropy enjoys the following. When $0 < q < 1$, we have (20) and $F(\varepsilon) \to (2\varepsilon)^q$ in the limit $d \to \infty$; when $q > 1$, we have (24) and $F(\varepsilon) \to \varepsilon^q + (q - 1)H_q(\varepsilon, 1 - \varepsilon)$.

Since the thermodynamic limit is of great importance in statistical physics, we shall now discuss this issue for the unified entropies. We divide the relations (27) and (29) by $\max E_q(s)$ and take the limit $d \to \infty$. Some inspection shows that the stability in the thermodynamic limit holds for the case $1 \leq q$ and $+1 \leq s$. Taking $q \neq 1$ and combining (20) and (38), for this case we have

$$\frac{|E_q(s)(\rho) - E_q(s)(\omega)|}{\max E_q(s)} \leq \frac{(q - 1)s}{1 - d^{1-q}s} \left[\varepsilon^q \frac{(d - 1)^{1-q} - 1}{1 - q} + H_q(\varepsilon, 1 - \varepsilon)\right] \xrightarrow{d \to \infty} s\varepsilon^q + (q - 1)sH_q(\varepsilon, 1 - \varepsilon).$$

Rewriting the right-hand side of (38) as $s\left[1 - (1 - \varepsilon)^q\right] \leq sq\varepsilon$ by (22), the stability property is obvious from this relation. For remaining parts of the ranges (26) and (28), the right-hand side of (37) is not bounded as $d \to \infty$. So, no conclusions can be made with the inequality (37) for these parts.

Instabilities are usually observed by constructing an explicit example. We will adopt the two examples employed in the paper [22]. Consider the two commuting density operators defined as

$$\rho_0 = \text{diag}(1, 0, \ldots, 0), \quad \omega_0 = \text{diag}\left(1 - \varepsilon, \frac{\varepsilon}{d - 1}, \ldots, \frac{\varepsilon}{d - 1}\right),$$

in their eigenbasis. It is clear that $D_{tr}(\rho_0, \omega_0) = \varepsilon$. Further, for all $q > 0$ we have $\text{tr}(\rho_q^0) = 1$ and

$$\text{tr}(\omega_q^0) = (1 - \varepsilon)^q + \varepsilon^q(d - 1)^{1-q}.$$  

Note that the inequality (21) is saturated with the states (39). We claim that the unified entropy is not stable in the thermodynamic limit for $0 < q < 1$ and $s < 0$. Indeed, we have $(1 - q)s < 0$ and

$$\frac{|E_q(s)(\rho_0) - E_q(s)(\omega_0)|}{\max E_q(s)} = \frac{1 - \left[\text{tr}(\omega_q^0)^s\right]^s}{1 - d^{1-q}s} \xrightarrow{d \to \infty} 1$$

in view of $\text{tr}(\omega_q^0) \to \infty$ and $s < 0$. Hence the stability condition is clearly violated.

Consider now the two commuting density operators defined as

$$\rho_1 = \text{diag}\left(0, \frac{1}{d - 1}, \ldots, \frac{1}{d - 1}\right), \quad \omega_1 = \text{diag}\left(\varepsilon, \frac{1 - \varepsilon}{d - 1}, \ldots, \frac{1 - \varepsilon}{d - 1}\right),$$

where $D_{tr}(\rho_1, \omega_1) = \varepsilon$ as well. Here we get $\text{tr}(\rho_q^1) = (d - 1)^{1-q}$ and $\text{tr}(\omega_q^1) = \varepsilon^q + (1 - \varepsilon)^q(d - 1)^{1-q}$. This example allows to resolve a violation of the stability for $1 < q$ and $s < 0$. For $d \gg 1$, we can asymptotically take

$$\max E_q(s) \approx \frac{1}{(1 - q)^s} d^{(1-q)s}, \quad \text{tr}(\omega_q^1) \approx \varepsilon^q,$$

in view of $(1 - q)s > 0$ and $(1 - q) < 0$ respectively. Due to the relations (13), there holds

$$\frac{|E_q(s)(\rho_1) - E_q(s)(\omega_1)|}{\max E_q(s)} \approx \frac{|(d - 1)^{(1-q)s} - \varepsilon^qs|}{d^{(1-q)s}} \xrightarrow{d \to \infty} 1.$$
Thus, the stability condition is also violated in the thermodynamic limit for $1 < q$ and $s < 0$.

The above results can be summarized as follows. For $1 \leq q$ and $+1 \leq s$, the unified entropy is actually stable including the thermodynamic limit. For other parts of the parameter ranges (26) and (28), the stability in the finite-dimensional case holds as well. It is not insignificant that the Rényi $q$-entropy enjoys this property for $q \neq 1$. However, its stability fails in the thermodynamic limit. The same can be said for those parts of the ranges (26) and (28) that correspond to negative $s$. For all positive $q \neq 1$ and $s < 0$, the unified $(q,s)$-entropy is not stable in the thermodynamic limit. The case $q = 1$ should be separated, since it includes the Shannon and von Neumann entropies by definition. From the stability viewpoint, the unified entropies enjoy desired properties for $1 \leq q$ and $+1 \leq s$. This conclusion is useful, though we do not obtain the complete picture for all the adopted values of $q$ and $s$.

V. SUBADDITIVITY AND TRIANGLE INEQUALITY

Quantum systems of interest are usually made up of two (or more) distinct physical systems. Let the state of the composite system 'AB' be described by density operator $\rho_{AB}$ on the product space $\mathcal{H}_A \otimes \mathcal{H}_B$. Then density operators of subsystems 'A' and 'B' are obtained by the partial trace operation \[26\]. Namely, we have

$$\rho_A = \text{tr}_B(\rho_{AB}), \quad \rho_B = \text{tr}_A(\rho_{AB}),$$

(45)

where the partial traces are taken over $\mathcal{H}_B$ and $\mathcal{H}_A$ respectively. Hence we are interested in how used entropic and other measures may be changed by the partial trace operation. Except for states of a kind $\rho_{AB} = \rho_A \otimes \rho_B$, the question is sufficiently difficult to resolve. In the paper [23], this problem is discussed for those unitarily invariant norms that are multiplicative over tensor products. Some results have also been obtained for Uhlmann’s partial fidelities [30] and the Ky Fan norms \[33\]. For $k = 1, \ldots, d$, the Ky Fan $k$-norm of operator $X$ is defined by \[6\]

$$\|X\|_{(k)} := \sum_{j=1}^k \sigma_j^k(X),$$

(46)

where the singular values should be put in the decreasing order. The Ky Fan norms and the Schatten norms form especially important classes of unitarily invariant norms. A unitarily invariant norm $\| \cdot \|$ is a norm that satisfies $\|X\| = |||UXV|||$ for each $X$ and all unitary $U, V$ \[6\].

In the classical regime, the Shannon entropy enjoys the subadditivity property. The von Neumann entropy is also subadditive (see, e.g., section 11.3.4 in [26]), i.e.

$$S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B).$$

(47)

The subadditivity property was generally assumed to be true for the Wigner–Yanase entropy [24], until Hansen provided a counterexample [15]. Another counterexample was given in [36]. Meantime, if the bipartite state is pure then it is sufficient for the subadditivity. Other sufficient conditions for subadditivity of the Wigner–Yanase entropy are obtained in [8]. It is easy to verify that the Shannon entropy enjoys the strong subadditivity. The quantum analog holds for the von Neumann entropy, but the proof turns out to be hard enough (see, e.g., the recent review [19] and references therein).

In the classical regime, the Shannon entropy enjoys even the strong subadditivity for $q > 1$ [12]. In the finite-dimensional quantum case, we have the subadditivity for $q > 1$:

$$H_q(\rho_{AB}) \leq H_q(\rho_A) + H_q(\rho_B).$$

(48)

This result has been conjectured by Raggio [29] and later proved by Audenaert [4]. It turns out that the reasons of the paper \[4\] can be extended to many of the quantum unified entropies. The key point is an inequality with the Schatten $q$-norms of the reduced densities [45], namely (see theorem 1 in [4])

$$\|\rho_A\|_q + \|\rho_B\|_q \leq 1 + \|\rho_{AB}\|_q.$$  

(49)

Due to this result, we obtain the following statement.

**Theorem V.1** For $q > 1$ and $s \geq q^{-1}$, the quantum unified entropy is subadditive, that is

$$E_q^{(s)}(\rho_{AB}) \leq E_q^{(s)}(\rho_A) + E_q^{(s)}(\rho_B).$$

(50)

**Proof.** Following the paper \[4\], we introduce the two 2-dimensional vectors

$$x = \left(\frac{\|\rho_A\|_q}{\|\rho_B\|_q}\right), \quad z = \left(\frac{1}{\|\rho_{AB}\|_q}\right).$$

(51)
Consider the Ky Fan norms of these 2-vectors. First, we have max\(\{\|\rho_A\|_q, \|\rho_B\|_q\} \leq 1\), since \(q\)-norms of any density operator with trace one do not exceed one. Second, the formula \((55)\) takes place. For all \(k = 1, 2\), we then write

\[
\|x\|_{(k)} \leq \|z\|_{(k)}.
\] (52)

In other words, the vector \(x\) is weakly submajorized by \(z\). By Ky Fan’s dominance theorem (see, e.g., theorem IV.2.2 in the book [8]), we have \(||x|| \leq ||z||\) for each unitarily invariant norm. In particular, there holds \(||x||_p \leq ||z||_p\) for all the vector \(p\)-norms. The last relation can be rewritten as

\[
\left[\tr(\rho_A^q)\right]^{p/q} + \left[\tr(\rho_B^q)\right]^{p/q} \leq 1 + \left[\tr(\rho_{AB}^q)\right]^{p/q}.
\] (53)

Putting \(s = p/q\) and using the definition \((55)\), we see that the relation \((55)\) is equivalent to \((50)\). Because of \(p \geq 1\), the reasons hold for \(s \geq 1/q\). 

Thus, we have established the subadditivity of the quantum unified entropy in a wide range of parameter values. Namely, the entropy of a state of the whole system is smaller than the sum of the entropies of its reduced density matrices. In this regard, the notion of unified entropy concurs with the intuitive reason that a situation becomes more uncertain when only partial information is available. For product states of a kind \(\rho_{AB} = \rho_A \otimes \rho_B\), additive properties have been examined in the paper [17]. It was claimed there that the subadditivity of the form \((50)\) fails in

\[
\{(q, s) : 1 < q, s < 0\} \cup \{(q, s) : 0 < q < 1, 0 < s\}.
\] (54)

We have proved the subadditivity in the range \(\{(q, s) : 1 < q, q^{-1} \leq s\}\). Although we have incomplete picture of the subadditivity, the question is resolved for mostly utilized values of the parameters. Like the case of the von Neumann entropy, the subadditivity inequality \((50)\) leads to the triangle (or “Araki-Lieb”) inequality [2].

Theorem V.2  For \(q > 1\) and \(s \geq q^{-1}\), the quantum unified entropy satisfies the triangle inequality

\[
|E_q^{(s)}(\rho_A) - E_q^{(s)}(\rho_B)| \leq E_q^{(s)}(\rho_{AB}).
\] (55)

Proof. Let us put the reference system 'C' in such a way that the triple system 'ABC' is being in a pure state \(|\Psi_{ABC}\rangle\). This vector is a purification of given density operator \(\rho_{AB}\), namely [20]

\[
\rho_{AB} = \tr C|\Psi_{ABC}\rangle\langle \Psi_{ABC}|, \quad \rho_C = \tr AB|\Psi_{ABC}\rangle\langle \Psi_{ABC}|.
\] (56)

It follows from the Schmidt decomposition of \(|\Psi_{ABC}\rangle\) that the density matrices \(\rho_{AB}\) and \(\rho_C\) have the same non-zero eigenvalues (see, e.g., section 2.5 in [20]). Hence \(E_q^{(s)}(\rho_{AB}) = E_q^{(s)}(\rho_C)\). In a similar manner, we have \(E_q^{(s)}(\rho_A) = E_q^{(s)}(\rho_{BC})\). Combining these two points with the inequality

\[
E_q^{(s)}(\rho_{BC}) \leq E_q^{(s)}(\rho_C) + E_q^{(s)}(\rho_B)
\] (57)

finally gives \(E_q^{(s)}(\rho_A) - E_q^{(s)}(\rho_B) \leq E_q^{(s)}(\rho_{AB})\). By symmetry, we also have \(E_q^{(s)}(\rho_B) - E_q^{(s)}(\rho_A) \leq E_q^{(s)}(\rho_{AB})\). So the claim \((55)\) is provided. 

To sum up, we see the following. For \(q > 1\) and \(s \geq q^{-1}\), the quantum unified entropy enjoys the subadditivity properties similar to the von Neumann entropy. In particular, the triangle inequality holds. In general, these results may enough simplify the use of unified entropies for composite systems. It is important, because the partial trace operation is usually inevitable. Note that the above proofs are all restricted to the discrete case. As was pointed out in [8], some essential reasons of the result \((19)\) fail in the case of continuous distributions. At the same time, finite-dimensional setting is adequate for many problems of quantum information theory [26].

VI. ENTROPY AND PROJECTIVE MEASUREMENTS

One of physically important properties of the von Neumann entropy is related to the measurement process [20]. A quantum measurement is described by a set \(\{M_i\}\) of measurement operators. Let \(\rho\) be density operator of the quantum system right before the measurement. Separate terms of the sum

\[
\sum_i M_i \rho M_i^\dagger
\] (58)
are related to different outcomes of the measurement. Since the probability of \(i\)th outcome is equal to \(\text{tr}(M_i^\dagger M_i \rho)\), the completeness relation

\[
\sum_i M_i^\dagger M_i = \mathbb{1}
\]  

(59)

must be enjoyed \[26\]. So the operators \(M_i^\dagger M_i\) form a generalized resolution of the identity \(\mathbb{1}\) (or a ”positive operator-valued measure”). A standard projective measurement is described by the set \(\{P_j\}\) of mutually orthogonal projectors which form an orthogonal resolution of the identity. As a rule, projective measurements are easier to realize experimentally. In effect, projective measurements cannot decrease the von Neumann entropy (see, e.g., theorem 11.9 in \[26\]). In opposite, generalized measurements can decrease the entropy. We shall consider this issue for the quantum unified entropy. For given projective measurement \(\{P_j\}\), the output density operator is expressed as \[26\]

\[
\tilde{\rho} = \sum_j P_j \rho P_j .
\]  

(60)

In matrix analysis, an operation of such a kind is usually referred to as ”pinching” (see, e.g., section IV.2 in the book \[6\]). For given orthogonal resolution \(\{P_j\}\) and any operator \(X\) on \(\mathcal{H}\), we define its pinching

\[
C(X) = \sum_j P_j X P_j.
\]  

(61)

We are interested in the following question. What is a relation between the traces \(\text{tr}(\tilde{\rho}^q)\) and \(\text{tr}(\rho^q)\)? If \(\text{rank}(P_j) = 1\) for all \(j\), then the answer is simply formulated as

\[
\text{tr}(\tilde{\rho}^q) \begin{cases} 
\geq & 0 < q < 1 \\
\leq & 1 < q 
\end{cases} \text{tr}(\rho^q) .
\]  

(62)

The claim follows from Jensen’s inequality and the known fact that

\[
f(\langle j | X | j \rangle) \leq \langle j | f(X) | j \rangle
\]  

(63)

with \(X \in \mathcal{L}_{s.a.}(\mathcal{H})\), convex function \(f(x)\) and unit vector \(|j\rangle\). The question is not so obvious, when \(\text{rank}(P_j) \neq 1\) in general. For \(q \geq 1\), the answer directly follows from the pinching inequality. This useful inequality states that \[6\]

\[
|||C(X)||| \leq |||X||||
\]  

(64)

for each unitarily invariant norm \(||| \cdot |||\). Using the Schatten \(q\)-norm \((q \geq 1)\), we then obtain \(|||\tilde{\rho}|||_q \leq |||\rho|||_q\), whence

\[
\text{tr}(\tilde{\rho}^q) \leq \text{tr}(\rho^q)
\]  

(65)

due to the positivity of eigenvalues. For \(0 < q < 1\), the traces of interest do not related with unitarily invariant norms. Here some properties of operator convex functions are required.

Let \(x \mapsto f(x)\) be a function of real variable. We say \(f\) is ”operator convex” if

\[
f(\theta X + (1 - \theta) Y) \leq \theta f(X) + (1 - \theta) f(Y)
\]  

(66)

for all \(X, Y \in \mathcal{L}_{s.a.}(\mathcal{H})\) and for \(0 \leq \theta \leq 1\) \[4\]. The operator concavity implies the inequality with reversed sign. It is assumed that eigenvalues of a Hermitian operator lie in some interval \(I\) of the real axis. We take \(I = [0; +\infty)\) for positive semidefinite operators. For \(0 < q < 1\), we shall use the following statement.

**Lemma VI.1** Let \(I\) be an interval containing \(0\), and let \(f\) be an operator convex function on \(I\) such that \(f(0) \leq 0\). For each operator \(X \in \mathcal{L}_{s.a.}(\mathcal{H})\) with \(\text{spec}(X) \subset I\), there holds

\[
\text{tr}[f(C(X))] \leq \text{tr}[f(X)]
\]  

(67)

**Proof.** Various characterizations of operator convex functions can be found in chapter V of the book \[6\]. In particular (see theorem V.2.3 therein), for all projections \(P\) and Hermitian operators \(X\) with spectrum in \(I\) we have

\[
f(PX P) \leq Pf(X) P,
\]  

(68)

under the precondition on \(f\). All the operators \(P_j XP_j\) have mutually orthogonal supports, whence

\[
f(C(X)) = \sum_j f(P_j XP_j) \leq \sum_j P_j f(X) P_j.
\]  

(69)
Since any positive operator has the positive trace, we finally write
\[ \text{tr}[f(C(X))] \leq \text{tr}\left[\sum_j P_j f(X)\right] = \text{tr}[f(X)]. \] (70)

Here we used the linear and cyclic properties of the trace, \( P_j^2 = P_j \) and the completeness relation \( \sum_j P_j = 1 \).

Recall the two known results. The function \( x \mapsto x^q \) is operator concave on \( L_+(\mathcal{H}) \) for \( 0 \leq q \leq 1 \) and operator convex on \( L_+(\mathcal{H}) \) for \( 1 \leq q \leq 2 \). For instance, these results are respectively proved as theorem 4.2.3 and theorem 1.5.8 in [2]. Combining these points with the statement of Lemma VI.1, we obtain
\[ \text{tr}(\tilde{\rho}^q) \geq \text{tr}(\rho^q) \] (71)
for \( 0 < q < 1 \) and the inequality (65) for \( 1 \leq q \leq 2 \) solely. The main result of this section is formulated as follows.

**Theorem VI.2** Let \( \{P_j\} \) be an orthogonal resolution of the identity and let the density operator \( \tilde{\rho} \) be defined by the formula (60). For \( q > 0 \) and all real \( s \), there holds
\[ E_q^{(s)}(\tilde{\rho}) \geq E_q^{(s)}(\rho). \] (72)

**Proof.** We assume \( q \neq 1 \), since the case of von Neumann entropy is well known. The function \( y \mapsto y^q/s \) monotonically increases for \( s \neq 0 \). Combining this with the inequalities (71) and (65) gives
\[ \frac{1}{s} [\text{tr}(\tilde{\rho}^q)]^s \left\{ \begin{array}{ll} \geq 0 & 0 < q < 1 \\ \leq & 1 < q \end{array} \right\} \frac{1}{s} [\text{tr}(\rho^q)]^s. \] (73)

Since the factor \((1-q)^{-1}\) is positive for \( q < 1 \) and negative for \( q > 1 \), the relation (73) is reduced to
\[ \frac{1}{(1-q)s} [\text{tr}(\tilde{\rho}^q)]^s \geq \frac{1}{(1-q)s} [\text{tr}(\rho^q)]^s. \] (74)

By the definition (8), the relation (74) implies (72) for \( q > 0 \) and all \( s \neq 0 \). When \( s = 0 \), the quantum Rényi entropy (8) is dealt. Here we merely rewrite the relations (73) and (74) with the function \( y \mapsto \ln y \), which is increasing as well. So similar reasons provide the inequality (72) for this case.

Thus, we have proved non-decreasing of the quantum entropy (8) under the action of arbitrary projective measurements for all the considered values of parameters. It is easy to see that generalized measurements can decrease the quantum unified entropy. For the von Neumann entropy of a qubit, an example is presented in exercise 11.15 of the book [26]. This measurement is such that \( \rho_{00} = \rho_{00} + \rho_{11} \), \( \rho_{11} = 0 \). If the state \( \rho \) is both impure and diagonal in the measurement basis, then
\[ \text{tr}(\tilde{\rho}^q) = \rho_{00}^q \left\{ \begin{array}{ll} <, & 0 < q < 1 \\ >, & 1 < q \end{array} \right\} \text{tr}(\rho^q) = \rho_{00}^q + \rho_{11}^q, \] (75)
whence \( E_q^{(s)}(\tilde{\rho}) < E_q^{(s)}(\rho) \). For \( q > 0 \) and all real \( s \), the quantum unified \((q, s)\)-entropy can be decreased by a generalized measurement. In this regard, the unified entropies are similar to the von Neumann entropy.

**VII. CONCLUSIONS**

We have considered those properties of the unified entropies that are of primary importance in physical applications. The unified entropies have originally been proposed as a two-parameter extension of the Shannon and von Neumann entropies. It is not surprising that properties are dependent on the parameter values. Raised mathematical problems are also of some interest in their own rights. Some bounds in terms of quantum ensembles are derived. For certain ranges of parameters, the upper continuity bounds on the quantum unified entropy are established. The derived inequalities of Fannes type have lead to the stability property in the finite-dimensional case. Taking the thermodynamic limit, we have observed the parameter ranges corresponding to both the stability and its violation. For a wide range of parameters, the subadditivity and the triangle inequality are also proved. It turned out that all the unified entropies are non-decreasing under projective measurements. Overall, we have observed that the quantum unified \((q, s)\)-entropy enjoys most of the desired properties for \( q \geq 1 \) and \( s \geq 1 \). In particular, negative values of the parameter \( s \) leads to a stability violation, at least in the thermodynamic limit. The presented discussion can herewith be viewed as a supplement and development of the previously given facts about the unified entropies. The obtained results are also interesting as applications for some power methods of matrix analysis.
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