Cosmological evolution of warm dark matter fluctuations II: Solution from small to large scales and keV sterile neutrinos

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(Dated: February 5, 2013)

We solve the cosmological evolution of warm dark matter (WDM) density fluctuations within the analytic framework of Volterra integral equations presented in the accompanying paper [1]. In the absence of neutrinos, the anisotropic stress vanishes and the Volterra-type equations reduce to a single integral equation. We solve numerically this single Volterra-type equation both for DM fermions decoupling at thermal equilibrium and DM sterile neutrinos decoupling out of thermal equilibrium. We give the exact analytic solution for the density fluctuations and gravitational potential at zero wavenumber. We compute the density contrast as a function of the scale factor \(a\) for a relevant range of wavenumbers \(k\). At fixed \(a\), the density contrast turns to grow with \(k\) for \(k < k_c\) while it decreases for \(k > k_c\), where \(k_c \approx 1.6/\text{Mpc}\). The density contrast depends on \(k\) and \(a\) mainly through the product \(ka\) exhibiting a self-similar behavior. Our numerical density contrast for small \(k\) gently approaches our analytic solution for \(k = 0\). For fixed \(k < 1/(60 \text{ kpc})\), the density contrast generically grows with \(a\) while for \(k > 1/(60 \text{ kpc})\) it exhibits oscillations starting in the radiation dominated (RD) era which become stronger as \(k\) grows. We compute the transfer function of the density contrast for thermal fermions and for sterile neutrinos decoupling out of equilibrium in two cases: the Dodelson-Widrow (DW) model and a model with sterile neutrinos produced by a scalar particle decay. The transfer function grows with \(k\) for small \(k\) and then decreases after reaching a maximum at \(k = k_c\) reflecting the time evolution of the density contrast. The integral kernels in the Volterra equations are nonlocal in time and their falloff determine the memory of the past evolution since decoupling. We find that this falloff is faster when DM decouples at thermal equilibrium than when it decouples out of thermal equilibrium. Although neutrinos and photons can be neglected in the matter dominated (MD) era, they contribute to the Volterra integral equation in the MD era through their memory from the RD era.

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I. INTRODUCTION AND SUMMARY OF RESULTS

In an accompanying paper [1] we provided a framework to study the complete cosmological evolution of dark matter (DM) density fluctuations for DM particles that decoupled being ultrarelativistic during the radiation dominated era which is the case of keV scale warm DM (WDM). In this paper, we solve the evolution of DM density fluctuations following the framework developed in ref. [1].

The new framework presented in ref. [1] and here is generic for any type of DM and applies in particular to cold DM (CDM) too. The collisionless and linearized Boltzmann-Vlasov equations (B-V) for WDM and neutrinos in the presence of photons and coupled to the linearized Einstein equations are studied in detail in the presence of anisotropic stress with the Newtonian potential generically different from the spatial curvature perturbations.

In ref. [1] the full system of B-V equations for DM and neutrinos is recast as a system of coupled Volterra integral equations. (Ref. [18] has recently considered this issue in absence of anisotropic stress). These Volterra-type equations are valid both in the radiation dominated (RD) and matter dominated (MD) eras during which the WDM particles are ultrarelativistic and then nonrelativistic. This generalizes the so-called Gilbert integral equation only valid for nonrelativistic particles in the MD era.

We succeed to reduce the system of four Volterra integral equations for the density and anisotropic stress fluctuations of DM and neutrinos into a system of only two coupled Volterra equations.

In summary, the pair of partial differential Boltzmann-Vlasov equations in seven variables for DM and for neutrinos become a system of four Volterra linear integral equations on the density fluctuations $\Delta_{dm}(\eta, \vec{k})$, $\Delta_{\nu}(\eta, \vec{k})$ and anisotropic stress $\Sigma_{dm}(\eta, \vec{k})$, $\Sigma_{\nu}(\eta, \vec{k})$ for DM and neutrinos, respectively.

In addition, because we deal with linear fluctuations evolving on an homogeneous and isotropic cosmology, the Volterra kernel turns to be isotropic, independent of the $\vec{k}$ directions. As stated above, the $\vec{k}$ dependence factorizes out and we arrive to a final system of two Volterra integral equations in two variables: the modulus $k$ and the time that we choose to be as

$$y \equiv a(\eta)/a_{eq} \simeq 3200 \ a(\eta) .$$

We have thus considerably simplified the original problem: we reduce a pair of partial differential B-V equations on seven variables $\eta$, $\vec{q}$, $\vec{x}$ into a pair of Volterra integral equations on two variables: $\eta$, $k$.

The customary DM density contrast $\delta(\eta, \vec{k})$ is connected with the density fluctuations $\Delta_{dm}(\eta, \vec{k})$ by

$$\delta(\eta, \vec{k}) = \frac{\Delta_{dm}(\eta, \vec{k})}{\rho_{dm}[a_{eq} + a(\eta)]} , \quad a_{eq} \simeq \frac{1}{3200} ,$$

where $\rho_{dm}$ is the average DM density today.

It is convenient to define dimensionless variables as

$$\alpha \equiv \frac{k \ l_{fs}}{\sqrt{I_{4}^{dm}}} , \quad l_{fs} = \frac{2}{H_{0}} \ T_{d} \ \frac{T_{d}}{a_{eq} \ \Omega_{dm}} ,$$

where $l_{fs}$ stands for the free-streaming length [9, 10, 14], $T_{d}$ is the comoving DM decoupling temperature and $I_{4}^{dm}$ is the dimensionless square velocity dispersion given by

$$I_{n}^{dm} = \int_{0}^{\infty} Q^{n} \ f_{0}^{dm}(Q) \ dQ \ , \quad \text{while} \ f_{0}^{dm}(Q) \ \text{is normalized by} \ I_{2}^{dm} = 1 .$$
$Q$ is the dimensionless momentum $Q \equiv q/T_d$ whose typical values are of order one.

A relevant dimensionless rate emerges: the ratio between the DM particle mass $m$ and the decoupling temperature at equilibration,

$$\xi_{dm} \equiv \frac{m a_{eq}}{T_d} = 4900 \frac{m}{\text{keV}} \left( \frac{g_d}{100} \right)^{1/2},$$

$g_d$ being the effective number of UR degrees of freedom at the DM decoupling. Therefore, $\xi_{dm}$ is a large number provided the DM is non-relativistic at equilibration. For $m$ in the keV scale we have $\xi_{dm} \sim 5000$.

DM particles and the lightest neutrino become non-relativistic by a redshift

$$z_{trans} + 1 \equiv \frac{m}{T_d} \simeq 1.57 \times 10^7 \frac{m}{\text{keV}} \left( \frac{g_d}{100} \right)^{1/2} \text{ for DM particles , } z_{trans}^\nu = 34 \frac{m_\nu}{0.05 \text{ eV}} \text{ for the lightest neutrino} .$$

$z_{trans}$ denoting the transition redshift from ultrarelativistic regime to the nonrelativistic regime of the DM particles. The final pair of dimensionless Volterra integral equations take the form

$$\bar{\Delta}(y, \alpha) = C(y, \alpha) + B_{\xi}(y) \bar{\phi}(y, \alpha) + \int_0^y dy' \left[ G_\alpha(y, y') \bar{\phi}(y', \alpha) + G_\alpha^s(y, y') \bar{\sigma}(y', \alpha) \right], \quad (1.5)$$

$$\bar{\sigma}(y, \alpha) = C_\sigma(y, \alpha) + \int_0^y dy' \left[ I_{\alpha}^s(y, y') \bar{\sigma}(y', \alpha) + I_\alpha(y, y') \bar{\phi}(y', \alpha) \right], \quad (1.6)$$

with initial conditions $\bar{\Delta}(0, \alpha) = 1$ , $\bar{\sigma}(0, \alpha) = \frac{2}{3} I_\xi$ . This pair of Volterra equations is coupled with the linearized Einstein equations.

The kernels and the inhomogeneous terms in eqs. (1.5)-(1.9) are given explicitly by eqs. (2.7)-(2.11), (2.16)-(2.18) and (2.19)-(2.21). The arguments of these functions contain the dimensionless free-streaming distance $l(y, Q)$,

$$l(y, Q) = \int_0^y \frac{dy'}{\sqrt{1 + y'} \left[ y'^2 + (Q/\xi_{dm})^2 \right]} .$$

The coupled Volterra integral equations (1.5)-(1.6) are easily amenable to a numerical treatment.

During the RD era the gravitational potential is dominated by the radiation fluctuations (photons and neutrinos). The photons can be described in the hydrodynamical approximation (their anisotropic stress is negligible). The tight coupling of the photons to the electron/protons in the plasma suppresses before recombination all photon multipoles except $\Theta_0$ and $\Theta_1$. (The $\Theta_l$ stem from the Legendre polynomial expansion of the photon temperature fluctuations $\Theta(\eta, \vec{q}, \vec{k})$.)

$\Theta_0$ and $\Theta_1$ obey the hydrodynamical equations [2]

$$\frac{d\Theta_0}{d\eta} + k \Theta_1(\eta, \bar{\alpha}) = \frac{d\phi}{d\eta} , \quad (1.8)$$

$$\frac{d\Theta_1}{d\eta} - \frac{k}{3} \Theta_0(\eta, \bar{\alpha}) = \frac{k}{3} \phi(\eta, \bar{\alpha}) . \quad (1.9)$$

This is a good approximation for the purposes of following the DM evolution [2].

The photons gravitational potential is given in the RD and MD eras by (ref. [2] and Appendix A)

$$\phi(\eta, \vec{k}) = \psi(\eta, \vec{k}) = 3 \psi(0, \vec{k}) \frac{\sqrt{3}}{\kappa y} j_1 \left( \frac{\kappa y}{\sqrt{3}} \right) , \quad \kappa = k \eta^* , \quad \eta^* \equiv \sqrt{a_{eq} \frac{1}{H_0}} = 143 \text{ Mpc} , \quad (1.10)$$

where $j_1(x)$ is the spherical Bessel function of order one.

For redshift $z < 30000$ the kernel $G_\alpha(y, y')$ in eq.(1.5) simplifies as

$$G_\alpha(y, y') \mid_{y, y' > 0} \frac{\xi_{dm}}{2 I_\xi} \frac{y y'}{\sqrt{1 + y}} \Pi \left[ \alpha (s(y) - s(y')) \right] \quad \text{where},$$
\[ \Pi(x) = \int_0^\infty \frac{dQ}{Q} f_{DM}^0(Q) \sin(Qx) \quad \text{and} \quad s(y) = -\text{ArgSinh}\left(\frac{1}{\sqrt{y}}\right). \] (1.11)

In this regime \( z < 30000 \), \( y > 0.1 \), the anisotropic stress \( \tilde{\sigma}(y, \alpha) \) turns to be negligible and eqs. (1.5)-(1.6) becomes a single Volterra integral equation. In the MD era this equation takes the form

\[ \frac{\Delta(y, \alpha)}{y} = g(y, \alpha) + \frac{6}{\alpha} \int_{s(1)}^{s(y)} ds' \Pi[\alpha (s(y) - s')] \Delta(y', \alpha), \quad y \geq 1, \] (1.12)

where the inhomogeneous term \( g(y, \alpha) \) contains the memory from the previous times \( y < 1 \) of the RD era. When DM is non-relativistic the memory from the regime where DM was ultrarelativistic turns out to fade out as \( 1/\xi_{dm} \sim 0.0002 \) compared to the recent memory where DM is non-relativistic.

The falloff of the kernel \( \Pi[\alpha (s - s')] \) determines the memory in the regime where DM is non-relativistic. We find that this falloff is faster when DM decouples at thermal equilibrium than when it decouples out of thermal equilibrium (see fig. 3). This can be explained by the general mechanism of thermalization [27]: in the out of equilibrium situation the momentum cascade towards the ultraviolet is incomplete and there is larger occupation at low momenta and smaller occupation at large momenta than in the equilibrium distribution. Therefore, the out of equilibrium kernel \( \Pi(x) \) which is the Fourier transform eq. (1.11) of the freezeout momentum distribution exhibits a longer tail than the equilibrium kernel.

Neutrinos and photons can be neglected in the matter dominated era. However, they contribute to the Volterra integral equation in the MD era through the memory integrals over \( 0 < y' < 1 \), namely the memory of the RD era.

When the anisotropic stress \( \tilde{\sigma}(y, \alpha) \) is negligible, eqs. (1.5)-(1.6) reduce to a single Volterra integral equation for the DM density fluctuations \( \Delta_{dm}(y, \alpha) \) when the anisotropic stress \( \tilde{\sigma}(y, \alpha) \) is negligible. We find the solution of this single Volterra equation for a broad range of wavenumbers \( 0.1/\text{Mpc} < k < 1/5 \text{ kpc} \).

At zero wavenumber \( k = 0 \) the kernel of this Volterra equation vanishes and the DM fluctuations can be expressed explicitly in terms of the gravitational potential \( \phi \). The gravitational potential at \( k = 0 \) follows solely from the hydrodynamic equations for the radiation combined with the regularity requirement at \( k = 0 \) of the first linearized Einstein equation. Namely, the gravitational potential \( \phi \) is solely obtained from the radiation without specifying the sources of the DM and radiation fluctuations. Using this explicit and well known form of \( \phi \), (see e. g. ref. [2]) the DM fluctuations are obtained at \( \alpha = 0 \). The fact that the Einstein equations constrain their sources was first noticed in ref. [10] in a completely different context.

We depict in figs. 2 the normalized density contrast vs. \( y \) (the scale factor divided by \( a_{eq} \)) for thermal fermions and sterile neutrinos in the Dodelson-Widrow (DW) model [3] (both models yield identical density fluctuations for a given value of \( \xi_{dm} \)). Similar curves are obtained in the \( \chi \) model where sterile neutrinos are produced by the decay of a real scalar [21].

At fixed \( y \) we find that the density contrast grows with \( k < k_c \), while it decreases for \( k > k_c \), where \( k_c \approx 1.6/\text{Mpc} \). We find that the density contrast depends on \( \alpha \) and \( y \) mainly through the product \( \alpha y \) exhibiting a self-similar behavior. The density contrast curves computed numerically for small \( \alpha \) gently approach in the upper fig. 2 our analytic solution for \( \alpha = 0 \). For fixed \( \alpha < 1 \), the density contrast generically grows with \( y \) while for \( \alpha > 1 \) it exhibits oscillations starting in the RD era which become stronger as \( \alpha \) grows (see fig. 2). The density contrast becomes proportional to \( y \) (to the scale factor) at sufficiently late times. The larger is \( \alpha \), the later starts \( \delta(y, \alpha) \) to grow proportional to \( y \) (see fig. 2). Also, the larger is \( \alpha > 1 \), the later the oscillations remain.

We depict in fig. 3 the transfer function for thermal fermions and sterile neutrinos in the DW model and for sterile neutrinos decoupling out of equilibrium in the \( \chi \) model. The transfer function grows with \( k \) for small \( k \) and then decreases after reaching a maximum at \( k = k_c \).

We analyze in section 3 the system of two Volterra integral equations in the regimes where DM is in the transition from UR to NR and when DM is nonrelativistic. In sec. 4 we take the nonrelativistic limit of our system of Volterra integral equations in the MD era. This yields the Gilbert equation (plus extra terms). We find extra memory terms and different inhomogeneities arising from our system of Volterra equations.

In section 5 we consider the zero anisotropic stress case where the system of Volterra equations reduces to a single Volterra equation. The numerical solution for the DM fluctuations in a broad range of wavenumbers is presented and discussed, as well as the transfer function and the analytic solution for zero wavenumber.

We present in sec. 6 the distribution functions, main parameters and integral kernels for sterile neutrinos decoupling out of equilibrium and compare them to fermions decoupling with a Fermi-Dirac distribution. Finally, we present in sec. 7 the generalization of the Volterra integral equation for cold dark matter.

\[ \Pi(x) = \int_0^\infty Q dQ f_0^a(Q) \sin(Qx) \quad \text{and} \quad s(y) = -\text{ArgSinh}\left(\frac{1}{\sqrt{y}}\right). \] (1.11)
In the RD era where radiation fluctuations dominates the gravitational potential we derive in Appendix A a second order differential equation for the gravitational potential. We show that the solution of this differential equation is well approximated by the Bessel function of order one eq.(1.10).

We provide in Appendix B explicit and useful expressions for the free-streaming distance \(l(y,Q)\) [see eq.(1.7)] in the main relevant regimes.

II. THE VOLTERRA INTEGRAL EQUATIONS AND RELEVANT PHYSICAL SCALES

We recall here the pair of coupled Volterra integral equations derived in the accompanying paper \[1\] from the Boltzmann-Vlasov equations for DM and for neutrinos.

A. Density fluctuations and anisotropic stress fluctuations

In the companion paper \[1\] we defined dimensionless density fluctuations \( \bar{\Delta}_{dm}(y,\alpha) \) and \( \bar{\Delta}_{\nu}(y,\alpha) \) and dimensionless anisotropic stress fluctuations \( v(y,\alpha) \) factoring out the initial gravitational potential \( \psi(0,k) \) in order to obtain quantities independent of the \( k \) direction. These relevant quantities are expressed as

\[
\bar{\Delta}_{dm}(y,\alpha) = \int \frac{d^3Q}{4\pi} \xi(y,Q) f_0^{dm}(Q) \frac{\Psi_{dm}(y,\vec{Q},\vec{K})}{\psi(0,\vec{K})} , \quad \bar{\Delta}_{\nu}(y,\alpha) = \int \frac{d^3Q}{4\pi} Q f_0^{\nu}(Q) \frac{\Psi_{\nu}(y,\vec{Q},\vec{K})}{\psi(0,\vec{K})} ,
\]

\[
\phi(y,\vec{K}) = \hat{\phi}(y,\alpha) , \quad \psi(y,\vec{K}) = \hat{\psi}(y,\alpha) \quad \text{and} \quad \hat{\psi}(0,\alpha) = 1 ,
\]

\[
\sigma(y,\vec{K}) = \hat{\sigma}(y,\alpha) , \quad \hat{\sigma}(y,\alpha) = \hat{\sigma}(y,\alpha) - \hat{\psi}(y,\alpha) , \quad \hat{\sigma}(0,\alpha) = \hat{\sigma}(0,\alpha) - 1 .
\]

We then introduce in ref. \[1\] the combined density fluctuation \( \check{\Delta}(y,\alpha) \)

\[
\check{\Delta}(y,\alpha) = -\frac{1}{2 \bar{\xi}} \left[ \frac{1}{\xi_{dm}} \bar{\Delta}_{dm}(y,\alpha) + \frac{\check{R}_\nu(y)}{\check{I}_3} \bar{\Delta}_{\nu}(y,\alpha) \right] , \quad I_\xi = \frac{I_{3,\xi}}{\xi_{dm} + \check{R}_\nu(0) \simeq \check{R}_\nu(0) = 0.727 , \quad \check{\Delta}(0,\alpha) = 1 ,
\]

where \( \xi_{dm} \) is the ratio between the DM particle mass \( m \) and the physical decoupling temperature at equilibration redshift \( z_{eq} + 1 \simeq 3200 ,

\[
\xi_{dm} = \frac{m a_{eq}}{T_d} = 4900 \left( \frac{g_d}{100} \right)^{\frac{3}{2}} = 5520 \left( \frac{m_{keV}}{g_d N_{dm}} \right)^{\frac{3}{5}} .
\]

We use here the dimensionless wavenumbers \[14\]

\[
\kappa \equiv k \eta^* \quad \text{and} \quad \alpha \equiv \frac{2}{\xi_{dm}} \frac{\kappa}{\sqrt{a_{eq} \Omega_{dm}}} k \quad \text{where} \quad \eta^* = \frac{a_{eq}}{\Omega_M} \frac{1}{H_0} = 143 \text{ Mpc} .
\]

Using \( \check{\Delta}(y,\alpha) \) and \( \hat{\sigma}(y,\alpha) \) in ref. \[1\] allowed to reduce the system of four Volterra integral equations into a the following pair of Volterra integral equations:

\[
\check{\Delta}(y,\alpha) = C(y,\alpha) + B_\xi(y) \check{\Delta}(y,\alpha) + \int_0^y dy' \left[ \Gamma_{\alpha}(y,y') \check{\Delta}(y',\alpha) + \check{G}_{\alpha}(y,y') \check{\Delta}(y',\alpha) \right] ,
\]

\[
\hat{\sigma}(y,\alpha) = C_{\sigma}(y,\alpha) + \int_0^y dy' \left[ \check{I}^{\nu}_{\sigma}(y,y') \hat{\sigma}(y',\alpha) + \check{I}_{\alpha}(y,y') \check{\Delta}(y',\alpha) \right] ,
\]

with the initial conditions \[1\]

\[
\check{\Delta}(0,\alpha) = 1 \quad \hat{\sigma}(0,\alpha) = \frac{2}{3} I_\xi \simeq \frac{2}{3} \check{R}_\nu(0) .
\]

We have in eqs.2.5-2.6

\[
C(y,\alpha) = -\frac{1}{2 I_\xi} \left[ \frac{a(y,\alpha)}{\xi_{dm}} + \frac{\check{R}_\nu(y)}{\check{I}_3} a_{\nu r}(y,\alpha) \right] , \quad C_{\sigma}(y,\alpha) \equiv \frac{a_{\sigma}(y,\alpha)}{\xi_{dm}} + \frac{\check{R}_\nu(y)}{\check{I}_3} a_{\nu r \sigma}(y,\alpha) ,
\]

\[2.7\]
\[ B_\xi(y) = -\frac{1}{2 I_\xi} [y b_{dm}(y) + 4 R_\nu(y)] , \]

\[ G_\alpha(y, y') = -\frac{\kappa}{2 I_\xi \sqrt{1+y'}} \left[ \frac{1}{\xi_{dm}} N_\alpha(y, y') + \frac{R_\nu(y)}{I_3^3} N_{\alpha r}^r(y, y') \right] , \quad (2.8) \]

\[ G_\alpha^r(y, y') = -\frac{\kappa}{2 I_\xi \sqrt{1+y'}} \left[ \frac{1}{\xi_{dm}} N_\alpha^r(y, y') - \frac{R_\nu(y)}{2 I_3^3} N_{\alpha r}^r(y, y') \right] , \quad (2.9) \]

\[ I_\alpha(y, y') = \frac{\kappa}{\sqrt{1+y'}} \left[ \frac{1}{\xi_{dm}} U_\alpha(y, y') + \frac{R_\nu(y)}{I_3^3} U_{\alpha r}^r(y, y') \right] , \quad (2.10) \]

\[ I_\alpha^r(y, y') = \frac{\kappa}{\sqrt{1+y'}} \left[ \frac{1}{\xi_{dm}} U_\alpha^r(y, y') - \frac{R_\nu(y)}{2 I_3^3} U_{\alpha r}^r(y, y') \right] . \quad (2.11) \]

In eqs. (2.8)-(2.11) we can use \( I_\xi \simeq R_\nu(0) \). The DM integral kernels and inhomogeneity functions in eqs. (2.7)-(2.11) are given by

\[ a(y, \alpha) = \int_0^\infty Q^2 dQ \frac{\epsilon(y, Q)}{\xi_{dm}(y)} \left[ f_{b_{dm}}^0(Q) \epsilon_{\xi_{dm}}(Q) + \tilde{\phi}(0) \frac{d f_{dm}^0}{d \ln Q} \right] j_0 \left[ \alpha \frac{Q}{2} l(y, Q) \right] , \quad (2.12) \]

\[ y \xi_{dm} b_{dm}(y) = \int_0^\infty Q^2 dQ \frac{\epsilon(y, Q)}{\xi_{dm}(y)} f_{b_{dm}}^0(Q) \left[ 4 Q^2 + 3 (\xi_{dm}(y))^2 \right] , \quad (2.13) \]

\[ N_\alpha(y, y') = \int_0^\infty Q^2 dQ \frac{\epsilon(y, Q)}{\xi_{dm}(y)} \left[ \epsilon_{\xi_{dm}}(Q) \right] j_1 \left[ \alpha l_Q(y, y') \right] \left[ \epsilon(y', Q) + \frac{Q^2}{\epsilon(y', Q)} \right] , \quad (2.14) \]

\[ N_{\alpha r}^r(y, y') = -\int_0^\infty Q^2 dQ \frac{\epsilon(y, Q)}{\xi_{dm}(y)} \left[ \epsilon_{\xi_{dm}}(Q) \right] j_1 \left[ \alpha l_Q(y, y') \right] \epsilon(y, Q) \epsilon(y', Q) , \quad (2.15) \]

\[ a^r(y, \alpha) = \frac{3}{\kappa^2 \epsilon} \int_0^\infty Q^4 dQ \frac{\epsilon(y, Q)}{\xi_{dm}(y)} \left[ f_{b_{dm}}^0(Q) \epsilon_{\xi_{dm}}(Q) + \tilde{\phi}(0) \frac{d f_{dm}^0}{d \ln Q} \right] j_2 \left[ \frac{\alpha}{2} Q l(y, Q) \right] , \quad (2.16) \]

\[ U_\alpha(y, y') = -\frac{3}{5 \kappa^2 \epsilon} \int_0^\infty Q^4 dQ \frac{\epsilon(y, Q)}{\xi_{dm}(y)} \left[ \epsilon(y', Q) + \frac{Q^2}{\epsilon(y', Q)} \right] \left[ 2 j_1 \left[ \alpha l_Q(y, y') \right] - 3 j_3 \left[ \alpha l_Q(y, y') \right] \right] , \quad (2.17) \]

\[ U_{\alpha r}^r(y, y') = \frac{3}{5 \kappa^2 \epsilon} \int_0^\infty Q^4 dQ \frac{\epsilon(y, Q)}{\xi_{dm}(y)} \epsilon(y', Q) \left[ 2 j_1 \left[ \alpha l_Q(y, y') \right] - 3 j_3 \left[ \alpha l_Q(y, y') \right] \right] . \quad (2.18) \]

The function \( \tilde{c}_{dm}^0(Q) \) determines the initial conditions. We have for thermal initial conditions (TIC) and for Gilbert initial conditions (GIC)

\[ \tilde{c}_{dm}^0(Q) = \begin{cases} 1 & \text{for thermal initial conditions (TIC)}, \\ \frac{d \ln f_{dm}^0}{d \ln Q} & \text{for TIC}, \\ \frac{1}{2} d \ln f_{dm}^0 & \text{for Gilbert initial conditions (GIC)} . \end{cases} \quad (2.19) \]

The neutrino integral kernels in eqs. (2.7)-(2.11) and inhomogeneity functions are given by

\[ N_{\alpha r}^{ur}(y, y') = -8 I_3^3 j_1 [\kappa r(y, y')] , \]

\[ U_{\alpha r}^{ur}(y, y') = \frac{24 I_3^3}{5 \kappa^2 y_\epsilon} \left\{ 2 j_1 [\kappa r(y, y')] - 3 j_3 [\kappa r(y, y')] \right\} , \quad (2.20) \]
The Volterra integral equations (2.23)-(2.26) are coupled with the linearized Einstein equations derived in the accompanying paper \[1\].

\[
\left[(1 + R_0(y)) \frac{d}{dy} + 1 + \frac{1}{3}(\kappa y)^2\right] \ddot{\phi}(y, \alpha) = [1+R_0(y)] \ddot{\sigma}(y, \alpha) - \frac{1}{2} \xi_{dm} \ddot{\Delta}_m(y, \alpha) - \frac{R_r(y)}{2 I_3^2} \ddot{\Delta}_r(y, \alpha) - 2 R_r(y) \ddot{\Theta}_0(y, \alpha) .
\]

Here,

\[
R_0(y) = \frac{\rho_{dm}(y)}{\rho_r(y)} = \int_0^\infty Q^2 dQ \sqrt{y^2 + \frac{Q^2}{\xi_{dm}^2}} f_{dm}(Q) , \quad \rho_r(y) = \frac{\rho_r}{a^4(y)} \quad \text{and}
\]

\[
\begin{aligned}
R_0(y) &= \begin{cases} 
I_{d/3}^{dm} & 1 + \mathcal{O}(\xi_{dm}^2 y^2) \ , \quad \xi_{dm} y \lesssim 1 , \\
y + \frac{I_{d/4}^{dm}}{2 \xi_{dm}^2 y} + \mathcal{O}\left(\frac{1}{\xi_{dm}^4 y^2}\right) \ , \quad \xi_{dm} y \gtrsim 5 .
\end{cases}
\end{aligned}
\]

B. Relevant scales in the ultra-relativistic and non-relativistic DM regimes

The evolution of the DM fluctuations presented here is valid generically for DM particles that decouple at redshift \(z_d\), being ultrarelativistic in the RD era and become non-relativistic in the same RD era. That is, the evolution presented here is valid as long as \(\xi_{dm} \gg 1\) which is the case from eq. (2.23) provided DM decouples ultrarelativistically deep enough in the RD era.

The framework presented in this paper is general, valid for any DM particle, not necessarily in the keV scale. More precisely, the treatment presented here is valid for \(\xi_{dm} \gg 1\) and:

\[
1 \gg \frac{m}{T_{d,\text{phys}}} = \frac{m}{T_d} z_d = 3200 \frac{\xi_{dm}}{z_d},
\]

which implies \(z_d \gg 3200 \xi_{dm}\). The redshift at decoupling turns to be

\[
z_d + 1 = \frac{T_{d,\text{phys}}}{T_d} = 1.57 \times 10^{15} \frac{T_{d,\text{phys}}}{100 \text{ GeV}} \left(\frac{g_d}{100}\right)^{1/3}.
\]

where we used \(T_d = (2/g_d)^{1/3} T_{\text{cmb}}\) and \(T_{\text{cmb}} = 0.2348\) meV.

DM particles are ultra-relativistic (UR) for \(z \gtrsim z_{\text{trans}}\), \(z_{\text{trans}}\) being the redshift at the transition from ultra-relativistic to non-relativistic DM particles

\[
z_{\text{trans}} + 1 \equiv \frac{m}{T_d} \simeq 1.57 \times 10^7 \frac{m}{\text{keV}} \left(\frac{g_d}{100}\right)^{1/3}.
\]

Then, they become non-relativistic (NR) for \(z \lesssim z_{\text{trans}}\). In terms of the variable \(y\) [eq.(1.1)] the transition from UR to NR DM particles takes place around \(y \sim y_{\text{trans}}\) while decoupling happens well before \(y_{\text{trans}}\) by \(y \sim y_d\):

\[
y_{\text{trans}} = 1/\xi_{dm} \simeq 0.0002 , \quad y_d = 3200/z_d \simeq 2 \times 10^{-12} .
\]

Notice that modes that reenter the horizon by the UR-NR transition \(y \sim y_{\text{trans}}\), have from eqs. (2.30) and (2.40) of the accompanying paper ref. \[1\], wavenumbers

\[
k \sim \frac{1}{\eta \gamma_{\text{trans}}} \sim \frac{\xi_{dm}}{\eta^*} = 2 \sqrt{I_{d/4}^{dm}} \frac{1}{l_{fs}} \sim \frac{1}{l_{fs}} .
\]

That is, when DM particles become nonrelativistic the free-streaming length \(l_{fs}\) is of the order of the comoving horizon \[18\].
At decoupling, the covariant neutrino temperature, decoupling neutrino redshift and $y$ variable are,

$$T_d = 0.17 \times 10^{-3} \text{ eV} \quad , \quad z_d' \simeq 6 \times 10^9 \quad \text{and} \quad y_d' \simeq 0.5 \times 10^{-6} .$$

The lightest neutrinos become non-relativistic at a redshift

$$z_{\nu \text{trans}}' = 95 \frac{m_{\nu}}{0.05 \text{ eV}} \quad \text{and} \quad y_{\nu \text{trans}}' = 34 \frac{0.05 \text{ eV}}{m_{\nu}} .$$

Namely, neutrinos become non-relativistic in the MD era when their density as well as their fluctuations are negligible. Thus, we can treat the neutrinos as ultra-relativistic or neglect them.

The neutrino and photon fractions of the energy density are defined in general as

$$R_\nu(\eta) \equiv \frac{\rho_\nu(\eta)}{\rho(\eta)} = \frac{\Omega_\nu}{\Omega_\tau + a(\eta) \Omega_M} , \quad R_\gamma(\eta) \equiv \frac{\rho_\gamma(\eta)}{\rho(\eta)} = \frac{\Omega_\gamma}{\Omega_\tau + a(\eta) \Omega_M} ,$$

where $\rho_\nu(\eta), \rho_\gamma(\eta)$ and $\rho(\eta)$ stand for the neutrino, photon and total energy density, respectively. In the radiation dominated era $\Omega_\tau \gg a(\eta) \Omega_M$ and $R_\nu(\eta) + R_\gamma(\eta) = 1$. The neutrino fraction changes after neutrino decoupling when the cosmic temperature crosses the $e^+ - e^-$ threshold, that is [2],

$$R_\nu(\eta) = \begin{cases} 0.727 , & 4 \times 10^9 \lesssim z \lesssim 6 \times 10^9 \\ 0.41 , & 3200 \lesssim z \lesssim 4 \times 10^9 \\ 0 , & 0 \leq z \lesssim 3200 \end{cases} . \quad (2.28)$$

The quantity $I_\xi$ defined by eq. (2.22) is dominated by the neutrino piece $R_\nu(0)$ and takes the value

$$I_\xi \simeq R_\nu(0) = 0.727 . \quad (2.29)$$

In the MD dominated era both $R_\nu(\eta)$ and $R_\gamma(\eta)$ become very small and can be neglected.
Range of Validity | \( \epsilon(y, Q) \) | \( l(y, Q) \) |
|-----------------|---------------------|---------------------|
| UR DM particles | \( \xi_{dm} y \ll 1 \) | \( Q \) | \( \frac{\xi_{dm} y}{Q} \sqrt{1 + \frac{1}{\xi_{dm}}} \) |
| Transition regime from UR to NR DM particles | \( 0 < y < 10^{-6} \) | \( \sqrt{Q^2 + (\xi_{dm})^2 y^2} \) | \( \text{Arg Sinh} \left( \frac{\xi_{dm} y}{Q} \right) \) |
| NR DM particles | \( 0.01 < y < 3200 \) | \( \xi_{dm} y \) | \( -2 \text{ Arg Sinh} \left( \frac{1}{\sqrt{y}} \right) + \log \left( \frac{8 \xi_{dm}}{Q} \right) + \frac{1}{2} \frac{Q}{\xi_{dm}} - \frac{1}{8} \left( \frac{Q}{y \xi_{dm}} \right)^2 \left[ 3 y \sqrt{1 + y + y + 2} \right] \) |
| MD era | \( y \gg 1 \) | \( \xi_{dm} y \) | \( -\frac{2}{\sqrt{y}} + \log \left( \frac{8 \xi_{dm}}{Q} \right) + \frac{1}{2} \frac{Q}{\xi_{dm}} \) |

TABLE II: The different regimes ultra-relativistic (UR), transition and non-relativistic (NR) of the free-streaming distance \( l(y, Q) \). Notice that the second (third) formula for \( l(y, Q) \) is also valid in the first (fourth) formula for \( 0 < y < 10^{-6} \) \( (y \gg 1) \). In addition, the third formula of \( l(y, Q) \) for \( y \ll 1 \) matches for \( \xi_{dm} y \gg 1 \) with the asymptotic behaviour of the second formula for \( l(y, Q) \). The precise behaviours of \( l(y, Q) \) are derived in Appendix B and given by eqs.(B3)-(B8). When DM is UR \( l(y, Q) \) grows as the comoving horizon \( \eta^* y \) and thus free-streaming efficiently erases fluctuations. When DM becomes NR \( l(y, Q) \) grows much slower and free-streaming is inefficient to erase fluctuations.

We summarize in Table II the ranges of the redshift \( z \) and the variable \( y \) (the scale factor normalized to unity at equilibration) for the main events in the DM, neutrinos and the universe evolution.

The free-streaming distance \( l(y, Q) \) is expressed by eq. (2.7). \( l(y, Q) \) can be expressed in general in terms of elliptic integrals. In the present case where \( \xi_{dm} \sim 5000 \) we find in appendix B excellent approximations to \( l(y, Q) \) in terms of simple elementary functions. We display the free-streaming distance \( l(y, Q) \) and the particle energy \( \epsilon(y, Q) \) for the different regimes in Table II. It must be stressed that each of the four formulas displayed in Table II match with its neighboring expression as discussed in appendix B.

From eqs.(2.3) and (2.4) we obtain for the dimensionless variable \( \alpha \),

\[
\alpha = 58.37 \frac{\text{keV}}{m} \left( \frac{100}{g_d} \right)^{\frac{2}{3}} k\text{kpc} .
\]  

(2.30)

In terms of \( \alpha \), the primordial gravitational potential eq.(3.17) in the accompanying paper [1] becomes,

\[
\psi(0, \vec{\alpha}) = \frac{1.848}{\alpha^2} \left( \frac{\alpha}{\alpha_0} \right)^{\frac{2}{3}(n_s-1)} \left( \frac{\text{keV}}{m} \right)^3 \left( \frac{100}{g_d} \right)^3 (kpc)^3 G(\vec{\alpha}) ,
\]

(2.31)

where \( \alpha_0 = 1.167 \times 10^{-4} \text{ (keV/m)} \) \( (100/g_d)^{\frac{2}{3}} \) and

\[
< G(\vec{\alpha}) G^*(\vec{\alpha}') > = \delta(\vec{\alpha} - \vec{\alpha}') .
\]

III. FROM THE ULTRARELATIVISTIC TO THE NON-RELATIVISTIC REGIME OF THE DM IN THE VOLterra EQUATIONS

We investigate here the system of Volterra integral equations (2.5)-(2.6) first in the transition regime for DM and then in the non-relativistic DM regime.
A. Transition Regime

We consider here the coupled Volterra integral equations \[ \text{(2.5)-(2.6)} \] in the transition regime from ultrarelativistic to non-relativistic DM particles \( 0.5 \times 10^{-6} < y < 0.01 \) well inside the RD era where the neutrinos are ultrarelativistic and they have already decoupled.

The second entry of Table II for the one-particle energy \( \varepsilon(y, Q) \approx \sqrt{\xi_{dm}} y + Q^2 \) and the free-streaming length \( l(y, Q) \) applies now. Therefore, we have from eq.\[ \text{(B3)} \],

\[
l(y, Q) = \left[ 1 - \frac{3}{16} \left( \frac{Q}{\xi_{dm}} \right)^2 \right] \text{Arc Sinh} \left( \frac{\xi_{dm} y}{Q} \right) - \frac{1}{2} \left[ \left( 1 - \frac{3}{8} y \right) \sqrt{y^2 + \left( \frac{Q}{\xi_{dm}} \right)^2} - \frac{Q}{\xi_{dm}} \right] + O(y^3).
\]

These formulas are to be inserted in eqs.\[ \text{(2.12)-(2.18)} \] for \( a(y, \alpha) \), \( a^\sigma(y, \alpha) \), \( N_\alpha(y, y') \), \( N^\sigma_\alpha(y, y') \), \( U_\alpha(y, y') \) and \( U_\sigma^\sigma_\alpha(y, y') \).

B. Non-relativistic Regime

We write here the Volterra integral equation in the non-relativistic regime \( 3200 > y > 0.01 \).

The third entry of Table II for the one-particle energy \( \varepsilon(y, Q) \approx \xi_{dm} y \) and the free-streaming length \( l(y, Q) \) applies in this case. Notice that the difference of the free-streaming lengths which appears in the integrand of the kernel \( N_\alpha(y, y') \) eq.\[ \text{(2.14)} \] is now \( Q \)-independent because the DM particles are non-relativistic:

\[
l_Q(y, y') = \frac{Q}{2} \left[ l(y, Q) - l(y', Q) \right] = Q \left[ s(y) - s(y') \right],
\]
where we used eq. 1.18, neglected terms $O\left([Q/\xi_{dm}]^2 \log Q/\xi_{dm}\right)$ in the free-streaming length $l(y, Q)$ and

$$s(y) \equiv -\text{Arg Sinh}\left(\frac{1}{y}\right), \quad \frac{ds}{dy} = \frac{1}{2 y \sqrt{1 + y}}.$$  \quad (3.1)

In the non-relativistic regime the kernels $N_\alpha(y, y')$ and $N^\sigma_\alpha(y, y')$ in eqs. (2.14)-(2.15) both in the RD and the MD eras become,

$$N_\alpha(y, y') = -N^\sigma_\alpha(y, y') = (\xi_{dm})^2 y y' \int_0^\infty Q^2 dQ \frac{df_{0d}^m}{dQ} j_1 \{\alpha Q [s(y) - s(y')]\}.$$  \quad (3.2)

Integrating by parts $df_{0d}^m/dQ$ in the above integral leads to the simpler expression

$$N_\alpha(y, y') = -N^\sigma_\alpha(y, y') = -(\xi_{dm})^2 y y' \Pi(\alpha \{s(y) - s(y')\}) \quad \text{for} \quad y, y' > 0.01,$$  \quad (3.3)

where

$$\Pi(x) \equiv \int_0^\infty Q dQ \frac{f_{0d}^m(Q)}{\sin(Q x)}.$$  \quad (3.4)

That is, in the nonrelativistic regime the kernel $N_\alpha(y, y')$ becomes the Fourier transform of the zeroth order momentum distribution $f_{0d}^m(Q)$. Notice that

$$\Pi(0) = 0, \quad \Pi'(0) = 1,$$

where we used eqs. 1.13 and 3.3.

In a similar way we obtain for the kernels $U_\alpha(y, y')$ and $U^\sigma_\alpha(y, y')$ [given by eqs. 2.17, 2.18] in the nonrelativistic regime,

$$U_\alpha(y, y') = -U^\sigma_\alpha(y, y') = -\frac{3 y' y}{5 \kappa^2 y^3} \int_0^\infty Q^2 dQ \frac{df_{0d}^m}{dQ} \left\{2 j_1 \{\alpha Q [s(y) - s(y')]\} - 3 j_3 \{\alpha Q [s(y) - s(y')]\}\right\}.$$  \quad (3.5)

Upon integrating by parts this formula can be recast in the simpler form

$$U_\alpha(y, y') = -U^\sigma_\alpha(y, y') = \frac{3}{\kappa^2 y^3} y' \int_0^\infty Q^4 dQ \frac{df_{0d}^m}{dQ} \left\{j_0 \{\alpha [s(y) - s(y')]\} Q - j_1 \{\alpha [s(y) - s(y')]\} Q\right\}.$$  \quad (3.6)

where we used also the angular integrals in the Appendix B of the accompanying paper ref. 1.

When the DM particles become nonrelativistic the anisotropic stress $\tilde{\sigma}(y, \alpha)$ decreases fast as $1/(\kappa y)^2$. For $y \alpha \gtrsim 1$, then $1/(\kappa y)^2 < 10^{-6}$ the anisotropic stress can be neglected. Therefore, for $y \alpha \gtrsim 1$ the system of Volterra equations (2.16)-(2.17) reduces to a single Volterra equation for the density fluctuations $\tilde{\Delta}(y, \alpha)$.

In this nonrelativistic regime where $\kappa y \gtrsim 1$, $e(y, Q) \simeq \xi_{dm} y$, the inhomogeneous pieces $C(y, \alpha)$ and $C^\sigma(y, \alpha)$ from eqs. (2.22), (2.24), (2.25) and (2.27) become

$$C(y, \alpha) = \frac{y}{2 T_\xi} \int_0^\infty Q^2 dQ \left[f_{0d}^m(Q) \tilde{c}_{0d}(Q) + \tilde{\phi}(0) \frac{df_{0d}^m}{d\ln Q}\right] \left\{j_0 \left\{\frac{\alpha}{2} Q l^{NR}(y, Q)\right\} + \right.$$  

$$+ \frac{R_\alpha(y)}{T_\xi} \left[1 + 2 \tilde{\phi}(0)\right] j_0 \{\kappa r(y, 0)\},$$

$$C^\sigma(y, \alpha) = \frac{3}{(\kappa \xi_{dm})^2 y^3} \int_0^\infty Q^4 dQ \left[f_{0d}^m(Q) \tilde{c}_{0d}(Q) + \tilde{\phi}(0) \frac{df_{0d}^m}{d\ln Q}\right] \left\{j_2 \left\{\frac{\alpha}{2} Q l^{NR}(y, Q)\right\} \right.$$

$$- 6 \frac{R_\sigma(y)}{T_\xi} \left[1 + 2 \tilde{\phi}(0)\right] \frac{j_2 \{\kappa r(y, 0)\}}{\kappa^2 y^2},$$  \quad (3.7)

where we used eq. 1.18

$$l(y, Q) \simeq l^{NR}(y, Q) \equiv 2 s(y) + \log (8 \xi_{dm}/Q).$$  \quad (3.8)
The Volterra equation (2.3) - (2.6) at \( y \) involves the integral over all \( y' \) in the interval \( 0 < y' < y \). Namely, we need the kernel \( N\nu(y,y') \) for all \( y' \) in \( 0 < y' < y \). Therefore, in the nonrelativistic regime \( y > y_1 = 0.01 \) we need the kernels with mixed arguments, where \( y' \) belongs to the transition or to the ultrarelativistic regime \( (0 < y' < y_1) \).

We obtain from eq. (2.14) for the mixed kernel

\[
N\alpha(y,y') = \xi_{dm} y \int_0^\infty Q^2 \frac{d\phi_{dm}}{dQ} \frac{d\xi_{dm}}{dQ} \left\{ \frac{\alpha}{2} Q \left[ \log \left( \frac{8 \xi_{dm}}{Q} \right) - 2 \operatorname{ArgSinh} \left( \frac{1}{\sqrt{y'}} \right) - \operatorname{ArgSinh} \left( \frac{\xi_{dm} y'}{Q} \right) \right] \right\} \\
\times \left[ \varepsilon(y',Q) + \frac{Q^2}{\varepsilon(y',Q)} \right] \quad \text{for} \quad y > y_1 = 0.01 \quad \text{and} \quad y' < y_1 = 0.01 .
\]  

(3.7)

The kernel \( N\alpha(y,y') \) is proportional to \( (\xi_{dm})^2 \) when both \( y \) and \( y' \) are in the nonrelativistic regime [eq. (3.2)] while it is proportional to \( \xi_{dm} \) when \( y \) is in the nonrelativistic regime and \( y' \) is in the transition or ultrarelativistic regimes [eq. (5.7)]. Namely, in the nonrelativistic regime, the memory of the transition regime and ultrarelativistic regime fades as \( 1/\xi_{dm} \sim 0.0002 \).

In the MD dominated era \( y > 1 \), radiation (photons and neutrinos) can be neglected: \( R\nu(y) = R\gamma(y) = 0 \). Once neutrinos are negligible, the anisotropic stress \( \sigma(y,\kappa) \) becomes very small and can be neglected too. Therefore, we have for \( y > 1 \) dropping the neutrino contributions in eqs. (2.2), (2.5) and (2.7) - (2.8).

\[
\tilde{\Delta}_{dm}(y,\kappa) = -2 \xi_{dm} I_\xi \tilde{\Delta}(y,\kappa) , \quad \text{MD era}
\]  

(3.8)

\[
\tilde{\Delta}_{dm}(y,\alpha) = a(y,\alpha) + y \xi_{dm} b_{dm}(y) \tilde{\phi}(y,\alpha) + \kappa \int_0^y \frac{dy'}{\sqrt{1+y'}} N\alpha(y,y') \tilde{\phi}(y',\alpha) + \kappa \int_0^1 \frac{dy'}{\sqrt{1+y'}} N\alpha(y,y') \tilde{\sigma}(y',\alpha) .
\]

Notice that the integrals here cover the radiation dominated era \( 0 < y < 1 \). That is, the memory of the neutrinos and photons during the RD era is preserved in the MD era.

The linearized Einstein equation for the gravitational potential in the MD era \( \tilde{\phi}(y,\alpha) = \tilde{\psi}(y,\alpha) \) become from eq. (2.23)

\[
\left[ y(1+y) \frac{d}{dy} + 1+y + \frac{1}{3} (\kappa y)^2 \right] \tilde{\phi}(y,\alpha) = -\frac{1}{2 \xi_{dm}} \tilde{\Delta}_{dm}(y,\alpha) .
\]  

(3.9)

This equation can be solved as

\[
\tilde{\phi}(y,\alpha) = -\frac{1}{2 \xi_{dm} y} \int_0^y \frac{dy'}{1+y'} \beta_\kappa(y,y') \tilde{\Delta}_{dm}(y',\alpha) \quad \text{where} \quad \beta_\kappa(y,y') = \left( \frac{1+y}{1+y'} \right)^{\kappa/3} .
\]  

(3.10)

We compute in appendix A of the accompanying paper [1] this integral in the asymptotic regime \( \kappa y \gg 1 \). We find at leading order from eq.(A2) of ref. [1],

\[
\tilde{\phi}(y,\alpha) \sim \frac{1}{2 \xi_{dm} (\kappa y)^2} \tilde{\Delta}_{dm}(y,\alpha) ,
\]  

(3.11)

which corresponds to the Poisson’s law. This result applies for \( \kappa = \xi_{dm} \alpha/2 \gg 1 \).

The asymptotic expansion of the function \( b_{dm}(y) \) for large \( y \) follows expanding the integral representation eq. (2.11) in inverse powers of \( \xi_{dm} y \). We obtain after calculation,

\[
b_{dm}(y) \xi_{dm} y^{\gg 1} \sim 3 + \frac{5}{2} \frac{I_{4dm}^4}{(\xi_{dm} y)^2} + O \left[ \frac{1}{(\xi_{dm} y)^4} \right] .
\]  

(3.12)

where we used eq. (11.3).

In the MD era we can approximate the inhomogeneous term \( a(y,\alpha) \) in eq. (3.8) as

\[
a(y,\alpha) = a^{MD}(y,\alpha) = \frac{2 \xi_{dm} y}{\alpha} \int_0^\infty \frac{Q dQ}{[\ln R(y,Q)]} \left[ f_{0dm}^d(Q) \tilde{\phi}(Q) + \tilde{\phi}(0) \frac{d\phi_{dm}}{d\ln Q} \sin \left( \frac{\alpha}{2} Q I_{NR}(y,Q) \right) \right] .
\]  

(3.13)

In eq. (3.8) we can approximate the kernel \( N\alpha(y,y') \) for \( y' > 0.1 \) according to eq. (3.2) and change the integration variable from \( y' \) to \( s' \), defined as in eq. (5.1),

\[
s' = -\operatorname{ArgSinh} \left( \frac{1}{\sqrt{y'}} \right) , \quad y' = y(s') = \frac{1}{\sin^2 s'} , \quad ds' = \frac{1}{2 y' \sqrt{1+y'}} .
\]
Then eq. (3.8) becomes
\[ \bar{\Delta}_{dm}(y, \alpha) = g(y, \alpha) + \frac{6}{\alpha} \int_{s(1)}^{s(y)} ds' \Pi [\alpha (s(y) - s')] \bar{\Delta}_{dm}(y', \alpha) \]  
(3.14)
where \( \Pi(x) \) is given by eq. (3.3), we used eq. (3.11) and
\[ g(y, \alpha) = \frac{a^{MD}(y, \alpha)}{y} + \frac{\kappa}{y} \int_{0}^{1} \frac{dy'}{\sqrt{1 + y'}} \left[ N_\alpha(y, y') \bar{\phi}(y', \alpha) + N_\alpha^c(y, y') \bar{\sigma}(y', \alpha) \right]. \]  
(3.15)
The term proportional to \( \bar{\phi}(y, \alpha) \) in eq. (3.8) becomes negligible using eqs. (3.11) and (3.12):
\[ \xi_{dm} b_{dm}(y) \bar{\phi}(y, \alpha) \approx 1 \frac{\bar{\Delta}_{dm}(y, \alpha)}{(\kappa y)^2}. \]
Notice that \( a^{MD}(y, \alpha) \) is explicitly known. Once the Volterra equations (2.5)–(2.6) are solved from \( y = 0 \) till \( y = 1 \) we explicitly know \( g(y, \alpha) \) from eq. (3.11). Then, the Volterra equation (3.14) can be solved to find \( \bar{\Delta}_{dm}(y, \alpha) \) for \( y > 1 \).

By setting
\[ \bar{\Delta}_{dm}(y, \alpha) = y \ d_{dm}(y, \alpha), \]  
(3.16)
we obtain from the Volterra equation (3.14) the Gilbert-type equation for the density fluctuations valid when the DM particles are nonrelativistic. Notice that contrary to the original Gilbert equation only valid in the MD dominated era, our equation is valid for all \( y \gtrsim 0.01, \ z \lesssim 300000 \) well inside the RD era,
\[ d_{dm}(y, \alpha) = g(y, \alpha) + \frac{6}{\alpha} \int_{s(1)}^{s(y)} \frac{ds'}{\sin^2 s'} \Pi [\alpha (s(y) - s')] \ d_{dm}(y', \alpha). \]  
(3.17)
where \( g(y, \alpha) \) is given by eqs. (3.13) and (3.15).

The memory piece in eq. (3.15) turns to be smaller than the first term \( a^{MD}(y, \alpha)/y \) by an order of magnitude or so as shown by numerical calculations. In addition, for \( y > 1 \) (MD era) we can neglect the logarithmic dependence in \( Q \) present in \( i^{NR}(y, Q) \) eq. (3.6). We therefore have for \( a^{MD}(y, \alpha) \) in the MD era from eq. (3.13)
\[ a^{MD}(y, \alpha) = \frac{\xi_{dm} y}{z(y)} \int_{0}^{\infty} Q dQ \left[ f_{dm}^0(Q) \bar{c}_{dm}(Q) + \bar{\phi}(0) \frac{df_{dm}^0}{d \ln Q} \right] \sin[z(y) Q], \]
where \( s(y) \) is defined by eq. (3.1) and
\[ z(y) = \alpha \left[ s(y) + \frac{1}{2} \log (8 \xi_{dm}) \right]. \]
For thermal and Gilbert initial conditions eqs. (2.19) we can express \( a^{MD}(y, \alpha) \) in terms of the kernel \( \Pi(z) \) defined by eq. (3.3) as
\[ \frac{a^{MD}(y, \alpha)}{y} = -\xi_{dm} \left\{ \left[ \frac{\bar{\phi}(0) + 1}{2} \right] \frac{2 \Pi(z(y))}{z(y)} + \Pi'(z(y)) \right\} + j \left[ \frac{\Pi(z(y))}{z(y)} - \frac{1}{2} \Pi'(z(y)) \right] \].  
(3.18)
where \( j = 0 \) for thermal initial conditions and \( j = 1 \) for Gilbert initial conditions.

Eqs. (3.14) and (3.17) have the same kernel as the Gilbert equation in the DM era [11, 14] as it must be. The inhomogeneous term \( g(y, \alpha) \) differs to those in refs. [11, 14]: this is so because \( g(y, \alpha) \) takes into account the memory from the previous evolution of the fluctuations since the DM decoupling in the RD era considered here. In Appendix III C we derive a Gilbert-type equation in the MD era from eqs. (3.13), (3.15) and (3.17) valid in the MD era. The obtained Gilbert-type equation eq. (3.19) contains an inhomogeneous term corresponding to temperature perturbation initial conditions plus a memory term including the contributions from the RD era.
C. The Gilbert equation from the Volterra equation in the MD era.

Eq. (3.17) can be written as
\[
d_{dm}(v, \alpha) - \frac{6}{\alpha} \int_{v_1}^{v} dv' y(v') \Pi [\alpha (v - v')] d_{dm}(v', \alpha) = \xi_{dm} \left\{ [2 \phi(0) + 1] \left[ \frac{\Pi [\alpha (v + v_0)]}{\alpha (v + v_0)} + \frac{1}{2} \Pi' [\alpha (v + v_0)] \right] + g \left[ \frac{\Pi [\alpha (v + v_0)]}{\alpha (v + v_0)} - \frac{1}{2} \Pi' [\alpha (v + v_0)] \right] \right\} = M[y(v), \alpha],
\]
where \( g = 0 \) for thermal initial conditions (TIC) and \( g = 1 \) for Gilbert initial conditions (GIC),
\[
v = s + 1 = 1 - \text{Arg Sinh} \left( \frac{1}{\sqrt{y}} \right), \quad v_0 = \frac{1}{2} \log(8 \xi_{dm}) - 1 \simeq 4.288 \ldots + \frac{1}{2} \ln \left( \frac{m_{\text{keV}}}{100} \right), \quad v_1 = 1 + s(1) \quad \text{and} \quad M[y, \alpha] \]
stands for the memory term containing the contribution of the gravitational potential and anisotropic stress from the RD era
\[
M[y, \alpha] = \frac{\kappa}{y} \int_{0}^{1} \frac{dy'}{\sqrt{1 + y'}} \left[ N_{\alpha}(y, y') \phi(y', \alpha) + N_{\alpha}(y, y') \bar{\sigma}(y', \alpha) \right].
\]
On the other hand, the Gilbert equation in the MD era can be written as \((11, 14)\)
\[
\delta_{MD}(y, \alpha) = \frac{6}{\alpha} \int_{0}^{v(y)} dv' y_{MD}(v') \Pi [\alpha (v(y') - v')] \delta_{MD}[y(v'), \alpha] = I_L[\alpha y(v)],
\]
where \( L = G \) or \( T \). In the notation of ref. \((14)\) \( L = G \) or \( T \) corresponds to Gilbert or temperature perturbation initial conditions, respectively,
\[
I_G(z) = \frac{\Pi(z)}{z}, \quad I_T(z) = \frac{1}{3} \left[ \Pi(z) + 2 \frac{\Pi(z)}{z} \right] \quad y_{MD}(v) = \frac{1}{(1 - v)^2} \quad \text{and} \quad v_{MD}(y) = 1 - \frac{1}{\sqrt{y}}.
\]
We see comparing eqs. (3.19) and (3.21) that the inhomogeneous terms are different. The inhomogeneities in the Gilbert equation (3.21) contain the functions \( I_G(\alpha v) \) or \( I_T(\alpha v) \), while the inhomogeneities in the Volterra equation (3.19) contain \( I_T[\alpha (v + v_0)] \) for TIC and a linear combination of \( I_T[\alpha (v + v_0)] \) and \( I_G[\alpha (v + v_0)] \) for GIC. Namely, the argument \( v \) in the inhomogeneous terms containing the kernels \( \Pi \) and \( \Pi' \) is shifted by the quantity \( v_0 \) given by eq. (3.20) (A similar shift was noticed in ref. \((18)\)). In addition, the inhomogeneous term \( M[y, \alpha] \), memory of the RD era in the Volterra equation is necessarily absent in the Gilbert equation which only takes into account the MD era.

In summary, choosing TIC at decoupling in the Volterra system of equations yields TIC at \( y = 1 \) for the Gilbert equation. On the contrary, choosing GIC at decoupling in the Volterra system of equations yields a linear combination of TIC and GIC at \( y = 1 \) for the Gilbert equation. One can thus say that TIC are \textbf{stable} under the evolution of the fluctuations.

In order to complete the comparison of eq. (3.19) in the late MD era with the Gilbert equation (3.21), notice that
\[
v = s + 1 = 1 - \text{Arg Sinh} \left( \frac{1}{\sqrt{y}} \right) \quad \text{and} \quad y(v) = \frac{1}{1 - v} \quad \text{as in eq. (3.22)}.
\]
We conclude that the Volterra integral equation (3.19) in the MD era is very close although not identical to the Gilbert equation in the MD era eq. (3.21). The inhomogeneity in the Volterra integral equation (3.19) for TIC has a factor in front and its argument is shifted by the constant \( v_0 \) with respect to the usual inhomogeneity in the Gilbert equation (3.21). For GIC a linear combination of Gilbert and thermal initial conditions appear at \( y = 1 \) for the Gilbert equation. In addition, the term \( M[y, \alpha] \) containing the memory from the RD era is present in the Volterra integral equation (3.19) while such term is absent in the Gilbert equation (3.21).

IV. SOLVING THE VOLterra EQUATION FOR THE DM DENSITY FLUCTUATIONS (W/O ANISOTROPIC STRESS)

The formulation of the cosmological fluctuations evolution in terms of Volterra equations provides an efficient computational framework for both analytic and numerical treatment. In the following subsections we solve the Volterra equation for DM fluctuations in the absence of neutrinos, i.e. without anisotropic stress. First, we solve the Volterra equation numerically for a wide range of wavenumbers. Second, we find the analytic solution at zero wavenumber.
A. Numerical solution of the Volterra equation for a wide range of wavenumbers

In the absence of anisotropic stress the radiation fluctuations during the RD era can be treated in the fluid approximation. Neglecting the DM gravitational potential in the RD era, the gravitational potential is given in the fluid approximation by (see ref. [2] and Appendix A)

\[ \ddot{\phi}(y, \alpha) = \phi(y, \alpha) = \psi(y, \alpha) = 3 \sqrt{\frac{3}{\kappa}} \frac{\gamma_1}{\sqrt{3}} j_1 \left( \frac{\kappa y}{\sqrt{3}} \right) , \quad \ddot{\phi}(0, \alpha) = 1 , \]  

(4.1)

where \( \gamma_1(x) \) is the spherical Bessel function of order one. Notice that \( \lim_{x \to 0} \gamma_1(x)/x = 1/3 \).

In the MD era we can neglect the gravitational potential produced by the radiation and take as gravitational potential the one sourced by the DM fluctuations eq.(3.10). In the absence of anisotropic stress, the DM fluctuations \( \Delta_{dm}(y, \alpha) \) from eq.(3.8) obeys the Volterra equation:

\[ \Delta_{dm}(y, \alpha) = a(y, \alpha) + y \xi_{dm} \tilde{b}_{dm}(y) \tilde{\phi}(y, \alpha) + \kappa \int_0^y \frac{dy'}{\sqrt{1 + y'^2}} N_{\alpha}(y, y') \tilde{\phi}(y', \alpha) , \]  

(4.2)

where \( a(y, \alpha) \) is given by eq.(4.12) with \( \tilde{\phi}(0) = 1 \).

We present here the numerical solution of eq.(4.2) where we smoothly match the gravitational potentials given by eqs.(4.1) and (3.10). The full numerical analysis of the system of Volterra equations (2.5)-(2.6) including the anisotropic stress will be the subject of future work where we will also compare our approach with the numerical solution of the ODE hierarchy of B-V equations [3-4].

For \( y \gtrsim 0.01 \) the DM particles become nonrelativistic eq.(4.2) simplifies and takes the form of eq.(3.17)

\[ d_{dm}(y, \alpha) = h(y, \alpha) + \frac{6}{\alpha} \int_{s(1)}^{s(y)} \frac{ds'}{\sinh^2 s'} \Pi \left[ \alpha (s(y) - s') \right] d_{dm}(y(s'), \alpha) , \quad d(y, \alpha) \approx \frac{\Delta_{dm}(y, \alpha)}{y} , \]  

(4.3)

where \( l_{NR}(y, Q) \) is defined by eq.(3.6),

\[ h(y, \alpha) \equiv \frac{a_{MD}(y, \alpha)}{y} + \frac{\kappa}{y} \int_0^1 \frac{dy'}{\sqrt{1 + y'^2}} N_{\alpha}(y, y') \tilde{\phi}(y', \alpha) \]  

(4.4)

and we have neglected the memory piece from the DM fluctuations in the UR regime but kept the gravitational potential of the photons which is dominant. Eq.(4.3) is a closed integral equation of Volterra type that determines approximately \( d(y, \alpha) \). We have checked numerically that eq.(4.3) reproduces the solutions of the full Volterra equation (4.2) within a few percent.

From the numerical resolution of the Volterra equation (4.2) we find the normalized density contrast

\[ \tilde{\delta}(y, \alpha) \equiv \frac{\delta(y, \alpha)}{\delta(0, \alpha)} = -\frac{1}{2 \sqrt{3} \xi_{dm}} \tilde{\Delta}_{dm}(y, \alpha) + 1 , \quad \tilde{\delta}(0, \alpha) = -\frac{1}{\sqrt{3} \xi_{dm}} , \quad \tilde{\delta}(0, \alpha) = 1 . \]  

(4.5)

The density contrast \( \delta(y, \alpha) \) is given by eq.(1.2) and eq.(4.36) in ref. [1].

We depict in fig. 2 the logarithm of the absolute value of the normalized density contrast for fermions with \( \xi_{dm} = 5000 \) which corresponds to DM fermions in thermal equilibrium with \( m = 0.6736 \) keV and sterile neutrinos in the DW model with \( m = 1.685 \) keV (both models yield identical density fluctuations for a given value of \( \xi_{dm} \)). In both cases we used thermal initial conditions.

The density contrast generically grows with \( y \) for fixed \( \alpha < 1 \) while it exhibits oscillations starting in the RD era for \( \alpha > 1 \) which become stronger as \( \alpha \) grows (see fig. 2). As expected, the Jeans’ instability makes the density contrast proportional to \( y \) (to the scale factor) at sufficiently late times. The larger is \( \alpha \), the later the oscillations remain.

There exists a value \( \alpha = \alpha_c \simeq 0.1 \) determining the transition between two regimes. We separately display the plots corresponding to \( \alpha < \alpha_c \) and \( \alpha > \alpha_c \). We find that for \( \alpha < \alpha_c \) and fixed \( y \), \( \tilde{\delta}(y, \alpha) \) increases for increasing \( \alpha \) while the opposite happens for \( \alpha > \alpha_c \). Namely, \( \tilde{\delta}(y, \alpha) \) at fixed \( y \) decreases for increasing \( \alpha \). We see from fig. 2 that the curves for \( \log_{10} |\tilde{\delta}(y, \alpha)| \) vs. \( \log_{10} y \) keep bending for decreasing \( \alpha \to 0 \) towards the \( \alpha = 0 \) curve. [The \( \alpha = 0 \) curve is obtained analytically in eqs. (4.3) and (4.17) below].
In figs. 2 we see that for both $\alpha < \alpha_c$ and $\alpha > \alpha_c$ varying $\alpha$ shifts the curves $\tilde{\delta}(y, \alpha)$ vs. $y$ with respect to each other but keeping their form essentially unchanged. This property indicates that $\tilde{\delta}(y, \alpha)$ mainly depends on $\alpha$ and $y$ through the product $\alpha y$, namely in a selfsimilar manner.

We have computed $\tilde{\delta}(y, \alpha)$ in the $\chi$-model for sterile neutrinos and found curves quite similar to the thermal case fig. 2.

B. Analytic solution of the Volterra equation at zero wavenumber

At $\alpha = 0$ (that is, $k = 0$), the Volterra equation (4.2) (zero anisotropic stress) can be solved in close form since from eq. (2.14) its kernel vanishes: $N_{\alpha=0}(y, y') = 0$.

The inhomogeneous term $a(y, 0)$ in the Volterra equation (4.2) becomes using eq. (2.12)

$$a(y, 0) = -y \xi_{dm} b_{dm}(y) + \int_0^\infty Q^2 dQ \, \varepsilon(y, Q) f_0^{dm}(Q) \, \tilde{c}_{dm}^0(Q),$$

Therefore, the Volterra equation (4.2) at $\alpha = 0$ simply relates the DM density fluctuations in terms of the gravitational potential as

$$\tilde{\Delta}_{dm}(y, 0) = \int_0^\infty Q^2 dQ \, \varepsilon(y, Q) f_0^{dm}(Q) \, \tilde{c}_{dm}^0(Q) + y \xi_{dm} b_{dm}(y) \left[ \tilde{\phi}(y, 0) - 1 \right].$$

(4.7)

More explicitly, for thermal (TIC) and Gilbert (GIC) initial conditions eq.(2.19), $\tilde{\Delta}_{dm}(y, 0)$ takes the form

$$\tilde{\Delta}_{dm}(y, 0) = \begin{cases} y \xi_{dm} b_{dm}(y) \left[ \tilde{\phi}(y, 0) - \frac{3}{2} \right] & \text{for TIC} , \\ -2 \int_0^\infty Q^2 dQ \, \varepsilon(y, Q) f_0^{dm}(Q) + y \xi_{dm} b_{dm}(y) \left[ \tilde{\phi}(y, 0) - 1 \right] & \text{for GIC}. \end{cases}$$

(4.8)

We can obtain $\tilde{\phi}(y, 0)$ solving the hydrodynamic equations for the radiation fluctuations (1.8)-(1.9) together with the linearized Einstein equations for the gravitational potential in the $k \to 0$ limit

$$-k^2 \phi_r(\eta, \vec{k}) = 16 \pi G a^2(\eta) \rho_r(\eta) \left[ \Theta_{r,0}(\eta, \vec{k}) + \frac{3}{k} h(\eta) \Theta_{r,1}(\eta, \vec{k}) \right],$$

$$3 h(\eta) \frac{\partial \phi}{\partial \eta} + k^2 \phi(\eta, \vec{k}) + 3 h^2(\eta) \psi(\eta, \vec{k}) = -4 \pi G a^2(\eta) \left[ 4 \rho_r(\eta) \Theta_{r,0}(\eta, \vec{k}) + \rho_{dm}(\eta) \delta_{dm}(\eta, \vec{k}) \right],$$

(4.10)

where $\phi_r(\eta, \vec{k})$ stands for the radiation contribution to the gravitational potential and $\rho_{dm}(\eta) \delta_{dm}(\eta, \vec{k})$ for the DM fluctuations. Eq. (4.10) can be written in dimensionless variables as

$$y \left[ 1 + R_0(y) \right] \frac{d\tilde{\phi}}{dy} + \frac{1}{3} (k y)^2 \tilde{\phi}(y, \alpha) + [1 + R_0(y)] \tilde{\psi}(y, \alpha) = -2 \Theta_{r,0}(y, \alpha) - \frac{1}{2} R_0(y) \delta_{dm}(y, \alpha),$$

(4.11)

where $R_0(y)$ is defined in eqs. (2.24)-(2.25) and we used eq. (A3).

Since the left hand side of eq.(4.10) vanishes at $k = 0$ we have

$$\Theta_{r,1}(\eta, \vec{k}) \overset{k=0}{=} -\frac{k}{3 h(\eta)} \Theta_{r,0}(\eta, \vec{k}) + O(k^3).$$

(4.12)

We neglect radiation momenta higher than $l = 1$ thus neglecting the anisotropic stress and set $\psi(\eta, \vec{k}) = \phi(\eta, \vec{k})$.

We have from eq.(1.8) in the $k \to 0$ limit

$$\Theta_{r,0}(\eta, 0) - \phi(\eta, 0) = c$$

(4.13)

where $c$ is a constant.

The initial values $\psi(0, \vec{k})$ and $\Theta_{r,0}(0, \vec{k})$ are related by the $\eta \to 0$ limit of eq.(4.10) as

$$\psi(0, \vec{k}) = -2 \Theta_{r,0}(0, \vec{k}),$$

(4.14)
FIG. 2: The ordinary logarithm of the normalized density contrast \( \bar{\delta}(y, \alpha) \) vs. \( \log_{10} y \) from the numerical resolution of the Volterra equation (4.2) for DM fermions in thermal equilibrium with \( m = 0.6736 \) keV and for sterile neutrinos in the DW model with \( m = 1.685 \) keV. (Both models yield identical density fluctuations for a given value of \( \xi_{\text{dm}} \)). There is a value \( \alpha = \alpha_c \approx 0.1 \) determining the transition between two regimes. We separately display the plots corresponding to \( \alpha < \alpha_c \) and \( \alpha > \alpha_c \). We see that for \( \alpha < \alpha_c \) and fixed \( y \), \( \bar{\delta}(y, \alpha) \) increases for increasing \( \alpha \) while the opposite happens for \( \alpha > \alpha_c \). \( \bar{\delta}(y, \alpha = 0) \) is plotted from the analytic solution eq.(4.8)-(4.17) for TIC. We see that the different curves have essentially the same shape and are shifted from each other in an almost selfsimilar manner indicating that \( \bar{\delta}(y, \alpha) \) is mainly a function of \( \kappa y \). \( \bar{\delta}(y, \alpha) \) generically grows with \( y \) for fixed \( \alpha < 1 \) while it exhibits oscillations starting in the RD era for \( \alpha > 1 \) which become stronger as \( \alpha \) grows. \( \bar{\delta}(y, \alpha) \) becomes proportional to \( y \) at sufficiently late times. The larger is \( \alpha \), the later starts \( \delta(y, \alpha) \) to grow proportional to \( y \) and the later the oscillations remain.
up to small $1/\xi_{dm}$ corrections. We thus find from eqs. (4.13) and (4.14),
\begin{equation}
c = \frac{3}{2} \phi(0, 0) = 3 \Theta_{r,0}(0, 0) .
\end{equation}
Inserting eqs. (4.12) and (4.13) in eq. (1.9) yields
\begin{equation}
\frac{d}{d\eta} \left[ \frac{\Theta_{r,0}}{h(\eta)} \right] + 2 \Theta_{r,0}(\eta, 0) = c ,
\end{equation}
which in terms of the variable $y$ becomes
\begin{equation}
y \frac{d\Theta_{r,0}}{dy} + \left[ 3 - \frac{y}{2 R_0(y) + 1} \frac{dR_0(y)}{dy} \right] \Theta_{r,0}(y) = c .
\end{equation}
This first order differential equation can be resolved with the explicit solution
\begin{equation}
\Theta_{r,0}(y) = \Theta_{r,0}(0) \frac{2}{5 y^2} \left[ 3 y^3 - y^2 + 4 y + 8 \left( 1 - \sqrt{y + 1} \right) \right] ,
\end{equation}
up to small corrections of the order $1/\xi_{dm}$ because we set here $R_0(y) = y$ [see eqs. (2.24)–(2.25)].
Then, the gravitational potential follows from eqs. (4.13) and (4.15) and we recover the known expression \[2\]
\begin{equation}
\tilde{\phi}(y, 0) = \frac{\phi(y, 0)}{\phi(0, 0)} = \frac{3}{2} - \frac{\Theta_{r,0}(y)}{2 \Theta_{r,0}(0)} = \frac{1}{10 y^2} \left[ 9 y^3 + 2 y^2 - 8 y + 16 \left( \sqrt{y + 1} - 1 \right) \right] .
\end{equation}
For zero or small redshift, eq. (4.17) becomes
\begin{equation}
\tilde{\phi}(y, 0) \approx \frac{9}{10} + \frac{1}{5 y} + \mathcal{O} \left( \frac{1}{y^2} \right) .
\end{equation}
It must be noticed that the known expression eq. (4.17) for the superhorizon gravitational potential (see for example ref. \[2\]) follows here solely from the hydrodynamic equations for the radiation \[1.8\]–\[1.9\] combined with eq. (4.12).
[Eq. (4.12) follows from the first linearized Einstein equation (4.9) in the $k \rightarrow 0$ limit]. Namely, $\tilde{\phi}(y, 0)$ and $\Theta_{r,0}(y, 0)$ are obtained without specifying the sources of the DM and radiation fluctuations.

We can find the matter source of the superhorizon gravitational potential $\tilde{\phi}(y, 0)$ by inserting eq. (4.17) in the left hand side of eq. (4.10) for $\psi(\eta, \tilde{k}) = \phi(\eta, \tilde{k})$ and $k = 0$. We obtain using the dimensionless variable $y$
\begin{equation}
[1 + R_0(y)] \left[ y \frac{d}{dy} + 1 \right] \tilde{\phi}(y, 0) = - \left[ 2 + 2 R_0(y) - \frac{y}{2} \frac{dR_0(y)}{dy} \right] \Theta_{r,0}(y, 0) ,
\end{equation}
Contrasting eq. (4.19) with eq. (4.11) and using eq. (2.25) implies a DM source
\begin{equation}
\tilde{\delta}_{dm}(y, 0) = \left( 4 - \frac{d \ln R_0}{d \ln y} \right) \Theta_{r,0}(y, 0) = \begin{cases} 4 \Theta_{r,0}(y, 0) & \text{for } \xi_{dm} y \lesssim 1 , \\ 3 \Theta_{r,0}(y, 0) & \text{for } \xi_{dm} y \gtrsim 1 . \end{cases}
\end{equation}
That is, combining the linearized Einstein equations with the hydrodynamic equations for the radiation requires for consistency a precise relation between the dark matter and radiation fluctuations. This is a consequence of the fact that the Einstein equations constrain their sources as was first noticed in ref. \[16\] in a completely different context.

Inserting eq. (4.18) into eq. (4.8) yields for the DM density fluctuations today
\begin{equation}
\Delta_{dm}(y, 0) \approx \begin{cases} \frac{9}{5} - \frac{3}{5} y \left[ 1 + \mathcal{O} \left( \frac{1}{y} \right) \right] & \text{for TIC} , \\ \frac{23}{10} - \frac{3}{5} y \left[ 1 + \mathcal{O} \left( \frac{1}{y} \right) \right] & \text{for GIC} . \end{cases}
\end{equation}
The normalized density contrast eq.(4.5) becomes today and for zero wavenumber,

\[
\tilde{\delta}(y,0) \overset{y \gg 1}{=} \frac{\xi_{dm}}{10 I_{3}^{dm}} \begin{cases} 
9 \left[ 1 + \mathcal{O} \left( \frac{1}{y} \right) \right] & \text{for TIC} \\
\frac{23}{2} \left[ 1 + \mathcal{O} \left( \frac{1}{y} \right) \right] & \text{for GIC} 
\end{cases}, \tag{4.21}
\]

We depict \(\log_{10} |\tilde{\delta}(y,0)|\) vs. \(\log_{10} y\) in fig. 2 for TIC. Fig. 2 exhibits this constant behaviour in \(\tilde{\delta}(y,0)\) for large \(y\).

C. The transfer function for the density contrast

The transfer function at redshift \(z\) can be defined as the density contrast at redshift \(z \geq 0\) normalized by its initial value and then normalized by the whole expression at \(k = 0\). That is,

\[
T(y,\alpha) \equiv \frac{\tilde{\delta}(y,\alpha)}{\tilde{\delta}(y,0)} , \quad T(y,0) = 1.
\]

The transfer function today becomes

\[
T(\alpha) = \lim_{y \gg 1} T(y,\alpha) = \frac{10 I_{3}^{dm}}{9 \xi_{dm}} \tilde{\delta}(y,\alpha) \quad \text{for TIC} \quad \text{and} \quad T(\alpha) = \frac{20 I_{3}^{dm}}{23 \xi_{dm}} \tilde{\delta}(y,\alpha) \quad \text{for GIC} , \quad T(0) = 1 \tag{4.22},
\]

and we used eq.(4.21).
We plot in fig. 3 the zero redshift transfer function $T(\gamma)$ vs. $\gamma$ for $\xi_{dm} = 5000$ and TIC. The (red) solid line curve is for DM fermions decoupling in thermal equilibrium with $m = 0.6736$ keV and sterile neutrinos out of equilibrium in the DW model with $m = 1.685$ keV. The (blue) dotted line corresponds to sterile neutrinos out of equilibrium in the $\chi$-model with $m = 0.7203 \tau^{-1/4}$ keV. That is, $0.9365 < m/\text{keV} < 1.665$ [see eq. (5.1)]. The variable $\gamma$ is defined as

$$\gamma \equiv \alpha \sqrt{\frac{f_{dm}}{4}} \frac{1}{3}.$$ (4.23)

We find that the transfer functions have a single maximum at $\alpha_c$ consistent with the behaviour of $\delta(y, \alpha)$ in figs. 2

Notice that $T(\gamma)$ grows fast with $\gamma$ for $0 < \gamma < \gamma_c$ as we see from fig. 3. Previous calculations of the transfer function in refs. [12, 13] and with better precision in ref. [14] only exhibit the portion of $T(\gamma)$ where it decreases with $\gamma$.

The transfer function computed solely in the MD era monotonically decreases with $\gamma$ for growing $\gamma \geq 0$ [2, 11]. The new piece of $T(\gamma)$ increasing with $\gamma$ for $0 < \gamma < \gamma_c$ comes from the behaviour of the DM fluctuations in the RD era computed here. $\alpha_c$ and $\gamma_c$ correspond here to a wavenumber $k_c \simeq 1.6/\text{Mpc}$.

The transfer function for DM fermions decoupling in thermal equilibrium and for sterile neutrinos out of equilibrium in the $\chi$-model turn out to be very similar as seen from fig. 3.

**V. FERMIONS IN THERMAL EQUILIBRIUM AND STERILE NEUTRINOS OUT OF EQUILIBRIUM**

The sterile neutrino is a serious candidate for WDM [2, 6, 21]. The freezeout distribution of sterile neutrinos turns to be out of thermal equilibrium in most models [22]. We consider here two sterile neutrino models for illustration. The Dodelson-Widrow model (DW) [5] and the $\chi$ model of ref. [21].

The freeze-out DM distributions are given by

$$f_{0}^{DM}(Q) = \frac{f_0}{m e^{y/T} + 1} \ , \ f_0 \simeq 0.043 \text{ keV} \ , \ \chi \text{ model} : \ f_{0}^{\chi}(Q) = \tau f_{0}^{dm}(Q) , \ 0.035 \lesssim \tau \lesssim 0.35 ,$$ (5.1)

where $\tau$ is a coupling constant and the normalized DM distribution function for the $\chi$ model [5, 23] takes the form

$$f_{0}^{dm}(Q) = \frac{4}{3} \zeta(5) \sqrt{\pi} Q \sum_{n=1}^{\infty} \frac{e^{-nQ}}{n^{3/2}}.$$ (5.2)

The normalized DM distribution in the DW model [5] is identical to the normalized Fermi-Dirac distribution

$$f_{0}^{dm}(Q) = \frac{2}{3\zeta(3)} \frac{1}{e^{y} + 1}.$$ (5.3)

The simple formula eqs. (5.1) - (5.3) for the DW freeze-out distributions were given in [5] and are widely used in the literature. A more sophisticated freeze-out distribution is derived in ref. [24].

We plot in fig. 4 the normalized distribution functions $f_{0}^{dm}(Q)$ for fermions in thermal equilibrium (which is identical to the DW model) and for sterile neutrinos out of thermal equilibrium in the $\chi$ model.

We find from eqs. (2.3) and (5.1) the values for the parameters $\xi_{dm}$ and $N_{dm}$ in the three DM fermion models considered here:

$$\xi_{dm}^{FD} = 6721 (g_{dm})^{1/2} \left( \frac{m}{\text{keV}} \right)^{3/2} , \ \xi_{dm}^{DW} = 2355 (g_{dm})^{1/2} \frac{m}{\text{keV}} , \ \xi_{dm}^{\chi} = 6146 (g_{dm} \tau)^{1/2} \left( \frac{m}{\text{keV}} \right)^{3/2} ,$$

$$N_{dm}^{FD} = 1.805 , \ N_{dm}^{DW} = 0.07765 \ \frac{\text{keV}}{m} , \ N_{dm}^{\chi} = 1.380 \tau .$$ (5.4)

Notice that the out of thermal equilibrium distribution is larger than the equilibrium distribution for small momenta $Q \lesssim 2$ while the opposite happens for $Q \gtrsim 2$. This can be explained by the general mechanism of thermalization: the momentum cascade towards the ultraviolet [25]. The distributions out of equilibrium therefore display larger occupation at low momenta and smaller occupation at large momenta than the equilibrium distribution.
Thermal FD and DW

| Thermal FD and DW | $I_{dn}^{\text{FD}} = \frac{2}{3} \zeta(3) \left(1 - 2^{-n}\right) n! \zeta(n+1)$ | $I_{dn}^{\text{DW}} = \frac{4}{3} \zeta(5) \sqrt{\frac{\pi}{\Gamma\left(n + \frac{1}{2}\right)}} \zeta(n+3)$ |

**TABLE III:** The normalized momenta $I_{dn}^{\text{dm}}$ defined by eq. (1.3). Notice that $I_{2}^{\text{dm}} \equiv 1$.

FIG. 4: The ordinary logarithm of the normalized distribution functions $f_0^{\text{dm}}(Q)$ vs. $Q$ for fermions at thermal equilibrium (which is identical to the out of thermal equilibrium DW model) and for sterile neutrinos out of thermal equilibrium in the $\chi$ model.

We display in Table III the momenta $I_{dn}^{\text{dm}}$ defined by eq. (1.3) for the normalized DM distributions considered in this paper.

For fermions decoupling ultrarelativistically at thermal equilibrium (and in the DW model of sterile neutrinos) the normalized freezed out distribution function is given by eq. (5.3), and the kernel $\Pi(x)$ for the non-relativistic regime eq. (3.3) can be expressed as

$$
\Pi^{FD}(x) = \frac{4}{3} \frac{x}{\zeta(3)} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{(n^2 + x^2)^2}.
$$

This kernel decreases for large argument $x$ as

$$
\Pi^{FD}(x) \xrightarrow{x \to \infty} \frac{1}{3} \frac{1}{\zeta(3)} x^3 + O\left(\frac{1}{x^5}\right).
$$

For sterile neutrinos out of thermal equilibrium in the $\chi$ model the kernel $\Pi(x)$ for the non-relativistic regime eq. (5.3) can be expressed as

$$
\Pi^{\chi}(x) = \frac{\sqrt{2}}{3 \zeta(5)} \sum_{n=1}^{\infty} \frac{\sqrt{(n^2 + x^2)^{\frac{3}{2}} + 3 n x^2 - x^3}}{n^{\frac{3}{2}} (n^2 + x^2)^{\frac{3}{2}}}, \quad (5.5)
$$
This kernel decreases for large argument \( x \) as
\[
\Pi^{\chi}(x) \xrightarrow{x \to \infty} \frac{\sqrt{2} \zeta(5/2)}{3 \zeta(5)} \frac{1}{x^{3/2}} + O\left(\frac{1}{x^{5/2}}\right).
\]

We plot \( \Pi^{FD}(x) \) and \( \Pi^{\chi}(x) \) as functions of \( x \) in fig. 5.

\( \Pi^{\chi}(x) \) has a longer tail than \( \Pi^{FD}(x) \) due to the higher occupancy of the low \( Q \) modes in the out of equilibrium momentum distribution. The out of equilibrium kernel \( \Pi^{\chi}(x) \) therefore exhibits a longer memory than \( \Pi^{FD}(x) \).

![Graph showing \( \Pi(x) \) vs. \( x \) for Fermi-Dirac distribution and sterile neutrinos in the \( \chi \) model.](image)

**FIG. 5**: The kernel \( \Pi(x) \) defined by eq. (3.3) as a function of \( x \) for fermions in thermal equilibrium (which is identical to the DW model) and for sterile neutrinos out of thermal equilibrium in the \( \chi \) model.

### VI. VOLTERRA INTEGRAL EQUATIONS FOR COLD DARK MATTER

All the framework of this paper easily generalizes to cold dark matter (CDM), that is DM particles with mass beyond one GeV that decouples nonrelativistically. For CDM the parameter \( \xi_{dm} \) is much larger than for WDM. Typically,

\[
\xi_{cdm} = \frac{m_{aeq}}{T_d} = 10^{11} \frac{m}{100 \text{ GeV}} \frac{5 \text{ GeV}}{T_{d,\text{phys}}}.
\]

where we take 5 GeV as reference value for the physical decoupling temperature \( T_{d,\text{phys}} \) of CDM. Other values for \( T_{d,\text{phys}} \) appears in the literature according to the particle physics model chosen but \( \xi_{cdm} \) turns out to be very large in all cases (and much larger than for WDM where \( \xi_{wdm} \sim 5000 \)).

CDM decouples being (by definition) nonrelativistic thus we have at decoupling \( \varepsilon(y_d, Q) = \xi_{cdm} y_d \gg 1 \). In addition, we have for the ratio of cdm to radiation densities \( R_0(y) = y \) at all times after decoupling.

CDM decouples at thermal equilibrium with a normalized Boltzmann distribution function [9, 10]

\[
f_{0}^{cdm}(Q) = \sqrt{\frac{2}{\pi}} \frac{e^{-Q^2/(2 x)}}{x^{3/4}} \quad \text{where} \quad x \equiv \xi_{cdm} y_d \gg 1.
\]

For DM particles decoupling nonrelativistic we have instead of eq. (2.1) for DM decoupling ultrarelativistic,

\[
\tilde{\Delta}_{dm}(y, \kappa) = \xi_{cdm} y \int \frac{d^3 Q}{4 \pi} f_{dm}^{0}(Q) \frac{\Psi_{dm}(y, \vec{Q}, \vec{\kappa})}{\Psi(y_d, \vec{\kappa})} \Psi(y_d, \vec{\kappa})
\]

(6.1)
and therefore at the initial (decoupling) time $y = y_d$

$$\Delta_{dm}(y_d, \kappa) = x \int_0^\infty Q^2\,dQ\,f_{\!0D}^\!dm(Q)\,\bar{c}_{dm}(Q),$$  \hfill (6.3)

where we used the initial conditions for the BV distribution function discussed in ref.\cite{[1]}.

From eqs.\ref{2.19} and \ref{6.3} we find as initial CDM fluctuations for TIC

$$\Delta_{cdm}(y_d, \kappa) = \frac{1}{2} \xi_{cdm} y_d \int_0^\infty Q^3\,dQ\,\frac{df_{\!0D}^{\!cdm}}{dQ} = -\frac{3}{2} x.$$

For CDM the free-streaming distance is defined similarly to eq.\ref{1.7} as

$$\psi_{\!0D}(y_d, \bar{Q}, \bar{\kappa}) = \psi(y_d, \bar{\kappa}) \bar{c}_0 \, dm(\bar{Q}),$$

Thus

$$l_{\!cdm}(y_d) = \ln(\frac{4}{y_d}) + 2 s(y_d).$$  \hfill (6.4)

The free-streaming distance turns to be independent of $Q$ as it should be because CDM particles are very slow.

Let us consider zero anisotropic stress here, for simplicity. Thus, the evolution of the CDM fluctuations is given by the single Volterra equation \ref{4.2}. Since CDM particles are always nonrelativistic with $\xi_{cdm} y_d > 1$ the coefficients and kernel in eq.\ref{4.2} take the form

$$a_{\!cdm}(y_d, \alpha) = 2 \xi_{cdm} y_d \int_0^\infty Q\,dQ\left[ f_{\!0D}^{\!dm}(Q) \bar{c}_{cdm}(Q) + \frac{df_{\!0D}^{\!cdm}}{dQ} \right] \sin \left[ \frac{\alpha}{2} Q l(y_d) \right],$$

$$b_{\!cdm}(y_d) = 3,$$  \hfill (6.5)

where we used eqs.\ref{2.12}, \ref{2.13}, and \ref{3.2}. The Volterra integral equation for CDM takes thus a form similar to eq.\ref{4.2} for WDM

$$\Delta_{cdm}(y, \alpha) = a_{\!cdm}(y, \alpha) + 3 \xi_{cdm} y \bar{\phi}(y, \alpha) + \kappa \int_{y_d}^y \frac{dy'}{\sqrt{1 + y'}} N_{\alpha}(y, y') \bar{\phi}(y', \alpha)$$

and also

$$\Delta_{cdm}(y, \alpha) = a_{\!cdm}(y, \alpha) + 3 \xi_{cdm} y \bar{\phi}(y, \alpha) - 2 (\xi_{cdm})^2 \kappa y \int_{s(y_d)}^{s(y)} \frac{ds'}{\sinh^4 s'} N_{\alpha}(s(y) - s'(y)) \bar{\phi}(y') = \frac{1}{\sinh^2 s'},$$

At $y = y_d$, $l_{\!cdm}(y_d) = 0$ and eqs.\ref{6.3} and \ref{6.5} yield

$$a_{\!cdm}(y_d, \alpha) = \Delta_{cdm}(y_d, \kappa) - 3 x.$$

Therefore, the Volterra equation \ref{6.6} is identically satisfied at $y = y_d$ since $\bar{\phi}(y_d, \kappa) = 1$.

The whole section \ref{IV B} translates to the CDM case. Since for CDM $\xi_{cdm} y_d > 1$ eq.\ref{4.20} results for all $y$

$$\bar{\delta}_{cdm}(y, 0) = 3 \bar{\Theta}_{r,0}(y, 0),$$

as it must be.

\textbf{Acknowledgments}

We are grateful to D. Boyanovsky and C. Destri for useful discussions.
Appendix A: The gravitational potential in the RD era

During the RD era the gravitational potential \( \phi(\eta, \vec{\alpha}) \) is dominated by the radiation (photons and neutrino) fluctuations. Neglecting the anisotropic stress, the following equations relate the gravitational potential with the first two radiation momenta \([2, 3]\):

\[
-k^2 \phi(\eta, \vec{\alpha}) = 16 \pi G a^2(\eta) \rho_r(\eta) \left[ \Theta_{r,0}(\eta, \vec{\alpha}) + \frac{3}{k} h(\eta) \Theta_{r,1}(\eta, \vec{\alpha}) \right],
\]

\[
\frac{d\Theta_{r,0}}{d\eta} + k \Theta_{r,1}(\eta, \vec{\alpha}) = \frac{d\phi}{d\eta},
\]

\[
\frac{d\Theta_{r,1}}{d\eta} - \frac{k}{3} \Theta_{r,0}(\eta, \vec{\alpha}) = \frac{k}{3} \phi(\eta, \vec{\alpha}).
\]

Here \( \Theta_{r,0}(\eta, \vec{\alpha}) \) and \( \Theta_{r,1}(\eta, \vec{\alpha}) \) are the first two momenta of the radiation temperature field

\[
\Theta_{r,0}(\eta, \vec{\alpha}) = R_\gamma(\eta) \Theta_0(\eta, \vec{\alpha}) + R_\nu(\eta) N_0(\eta, \vec{\alpha}) \quad \text{and} \quad N_0(\eta, \vec{\alpha}) = \frac{1}{4 I_5} \bar{\Delta}_\nu(y, \kappa) \phi(0, \vec{\alpha}).
\]

The infinite hierarchy of equations arising from the Boltzmann-Vlasov equation for radiation and matter, has been truncated to the first two equations \([2, 3]\). \( h(\eta) = d \ln a/d\eta \) stands for the Hubble parameter and \( \rho_r(\eta) = \Omega_r \rho_c/a^4(\eta) \) for the radiation density.

Eliminating \( \Theta_{r,0}(\eta, \vec{\alpha}) \) among eqs.\( \text{[A1]} \) yields

\[
\frac{d}{d\eta} \left[ h(\eta) \Theta_{r,1}(\eta, \vec{\alpha}) \right] - k^2 \left[ \Theta_{r,1}(\eta, \vec{\alpha}) - k \frac{d\phi}{d\eta}(\eta, \vec{\alpha}) - \frac{k^2}{48 \pi G} \frac{d}{d\eta} \left[ \frac{\phi(\eta, \vec{\alpha})}{a^2(\eta) \rho_r(\eta)} \right] \right] = 0,
\]

\[
\frac{d\Theta_{r,1}}{d\eta} + h(\eta) \Theta_{r,1}(\eta, \vec{\alpha}) + k \left[ 1 - \frac{k^2}{16 \pi G a^2(\eta) \rho_r(\eta)} \right] \phi(\eta, \vec{\alpha}) = 0. \quad \text{(A2)}
\]

It is convenient to use the variable \( y \) defined in eq.\( \text{[A1]} \) instead of the conformal time \( \eta \). We find in terms of \( y \)

\[
a^2(\eta) \rho_r(\eta) = \frac{3}{8 \pi G} \frac{1}{\eta^2 y^2} \quad \text{(A3)}
\]

and eqs.\( \text{[A2]} \) read

\[
\frac{d\Theta_{r,1}}{dy} - \frac{1}{1 + y} \left( \frac{1}{y} + \frac{1}{2} + \frac{\kappa^2}{3} y \right) \Theta_{r,1}(y, \vec{\alpha}) = \frac{\kappa}{3 \sqrt{1 + y}} \left[ 1 + \frac{\kappa^2 y^2}{6} \right] \frac{dy}{dy} + \frac{\kappa^2 y^2}{3} \phi(y, \vec{\alpha}),
\]

\[
\frac{d\Theta_{r,1}}{dy} + \frac{1}{y} \Theta_{r,1}(y, \vec{\alpha}) = - \frac{\kappa}{3 \sqrt{1 + y}} \left[ 1 - \frac{\kappa^2 y^2}{6} \right] \phi(y, \vec{\alpha}). \quad \text{(A4)}
\]

Eliminating now \( \Theta_{r,1}(\eta, \vec{\alpha}) \) we have

\[
\frac{2}{y} \left( 1 + \frac{3}{4} y + \frac{\kappa^2 y^2}{6} \right) \Theta_{r,1}(y, \vec{\alpha}) + \frac{\kappa}{3} \sqrt{1 + y} \left( 1 + \frac{\kappa^2 y^2}{6} \right) \left[ 1 + \frac{dy}{dy} \right] \phi(y, \vec{\alpha}) = 0. \quad \text{(A5)}
\]

Taking the \( y \) derivative of this equation and replacing \( d\Theta_{r,1}(y, \vec{\alpha})/dy \) and \( \Theta_{r,1}(y, \vec{\alpha}) \) from eqs.\( \text{[A4]} \) and \( \text{[A5]} \) respectively, we get the second order differential equation for the gravitational potential \( \phi(y, \vec{\alpha}) \):

\[
\frac{d^2\phi}{dy^2} + \frac{2}{y} R_\kappa(y) \frac{d\phi}{dy} + \frac{2}{y^2} S_\kappa(y) \phi(y, \vec{\alpha}) = 0, \quad \text{(A6)}
\]

where,

\[
R_\kappa(y) = 1 + \frac{y}{4 (1 + y)} + \frac{\kappa^2 y^2/6}{1 + \kappa^2 y^2/6} + \frac{1 + \kappa^2 y^2/6}{1 + y + \kappa^2 y^2/6},
\]

\[
S_\kappa(y) = \frac{\kappa^2 y^2/6}{1 + \kappa^2 y^2/6} + \frac{1 + \frac{3}{4} y}{1 + \frac{3}{4} y + \kappa^2 y^2/6} - \frac{1 + \frac{y}{2} (1 - \kappa^2 y^2/6) - \kappa^4 y^2/36}{(1 + y) (1 + \kappa^2 y^2/6)} \quad \text{(A7)}
\]
FIG. 6: The gravitational potential $\ddot{\phi}(y, \alpha)$ and $\phi^0(y, \alpha)$ vs. $\log_{10} \zeta$ for $\alpha = 0.1, 1$ and $10$ defined by eqs. (A8) and (A9). We see that $\phi^0(\zeta)$ is a very good approximation to $\ddot{\phi}(y, \alpha)$ in the whole range of $\zeta$.

In the radiation dominated era and for $\kappa y \gg 1$ this equation reduces to

$$
\frac{d^2 \phi}{dy^2} + \frac{4}{y} \frac{d\phi}{dy} + \frac{1}{3} \kappa^2 \phi(y, \alpha) = 0
$$

whose solution in terms of Bessel functions is given by eq. (4.1).

We solve numerically eq. (A6) both in the radiation dominated and in the matter dominated eras. We plot in fig. 6 the normalized gravitational potential

$$
\ddot{\phi}(y, \alpha) = \frac{\phi(y, \alpha)}{\phi_{prim}(\alpha)} , \quad \ddot{\phi}(0, \alpha) = 1 ,
$$

as a function of $\log_{10} \zeta$ for $\alpha = 0.1, 1$ and $10$ as well as the function

$$
\phi^0(\zeta) = 3 \frac{j_1(\zeta)}{\zeta} , \quad \zeta = \frac{\kappa y}{\sqrt{3}} , \quad \phi^0(0) = 1 .
$$

We see from fig. 6 that the function $\phi^0(\zeta)$ is a very good approximation to $\ddot{\phi}(y, \alpha)$.

**Appendix B: The free-streaming length in the different regimes.**

We evaluate now the integral eq. (1.7)

$$
l(y, Q) = \int_0^y \frac{dy'}{\sqrt{(1 + y') \left[ y'^2 + (Q/\xi_{dm})^2 \right]}} ,
$$

(B1)
where

\[
2 \hat{p}^2 = 1 + \frac{1}{\sqrt{1 + (Q/\xi_{dm})^2}} \quad \frac{2}{\sin \varphi(y)} = \frac{1 + (Q/\xi_{dm})^2}{\sqrt{1 + y}} + \frac{\sqrt{1 + y}}{1 + (Q/\xi_{dm})^2},
\]

Taking into account that \(\xi_{dm} \sim 5000 \gg 1\) we can express this integral quite accurately in terms of elementary functions.

- For \(y < 0.01 \ll 1\) we can expand \(1/\sqrt{1 + y'}\) in eq. (B1) in powers of \(y'\) and obtain

\[
l(y, Q) = \xi_{dm} \int_0^y \frac{dy'}{\sqrt{Q^2 + (\xi_{dm})^2 y'^2}} \left[ 1 - \frac{1}{2} y' + \frac{3}{8} y'^2 + \mathcal{O}(y'^3) \right] = \left[ 1 - \frac{3}{16} \left( \frac{Q}{\xi_{dm}} \right)^2 \right] \text{Ar} \text{Sinh} \left( \frac{\xi_{dm} y}{Q} \right) - \frac{1}{2} \left[ \left( 1 - \frac{3}{8} y \right) \sqrt{y^2 + \left( \frac{Q}{\xi_{dm}} \right)^2} - \frac{Q}{\xi_{dm}} \right] + \mathcal{O}(y'^3). \tag{B3}
\]

Notice that in this range of \(y\) and for typical \(Q \sim 1\) the arguments in eq. (B3) can go from \(Q \gg \xi_{dm} y\) till \(Q \ll \xi_{dm} y\). In the case \(\xi_{dm} y \ll Q < \xi_{dm}\) this formula simplifies as

\[
l(y, Q) \approx \frac{\xi_{dm} y}{Q}.
\]

- For \(y > 0.01\) it is convenient to split the integral eq. (B1) in two pieces:

\[
l(y, Q) = l(\infty, Q) - \xi_{dm} \int_y^\infty \frac{dy'}{(1 + y')\left[Q^2 + (\xi_{dm})^2 y'^2\right]^2}, \tag{B4}
\]

where

\[
l(\infty, Q) = \xi_{dm} \int_0^\infty \frac{dy}{\sqrt{(1 + y)[Q^2 + (\xi_{dm})^2 y^2]}}.
\]

In order to obtain the asymptotic expansion of \(l(\infty, Q)\) for \(Q/\xi_{dm} \rightarrow 0\) it is convenient to change the integration variable as

\[
y = t - 1 - \left( \frac{Q}{2 \xi_{dm}} \right)^2 \frac{1}{t - 1},
\]

and \(l(\infty, Q)\) becomes

\[
l(\infty, Q) = \int_{1 + \frac{Q}{\xi_{dm}}}^\infty \frac{dt}{\sqrt{t (t - 1)}} \frac{1}{\sqrt{1 - \frac{(Q/\xi_{dm})^2}{4 t (t - 1)}}}. \tag{B5}
\]

Expanding the integrand of eq. (B5) in powers of \((Q/\xi_{dm})^2\) and integrating term by term yields the asymptotic expansion

\[
l(\infty, Q) \approx \left[ 1 - \frac{3}{16} \left( \frac{Q}{\xi_{dm}} \right)^2 \log \left( \frac{8 \xi_{dm}}{Q} \right) + \frac{1}{2} \frac{Q}{\xi_{dm}} + \frac{7}{16} \left( \frac{Q}{\xi_{dm}} \right)^2 + \mathcal{O} \left[ \left( \frac{Q}{\xi_{dm}} \right)^3 \right] \right] \tag{B6}
\]

The integral in the second term of eq. (B4) is evaluated by expanding the integrand in powers of \((Q/\xi_{dm})^2\)

\[
\xi_{dm} \int_y^\infty \frac{dy'}{(1 + y')\left[Q^2 + (\xi_{dm})^2 y'^2\right]^2} = \int_y^\infty \frac{dy'}{y' \sqrt{1 + y'}} - \frac{1}{2} \left( \frac{Q}{\xi_{dm}} \right)^2 \int_y^\infty \frac{dy'}{y'^3 \sqrt{1 + y'}} + \mathcal{O} \left[ \left( \frac{Q}{\xi_{dm}} \right)^3 \right] =
\]
\[ l(y, Q) = \left[ 1 - \frac{3}{16} \left( \frac{Q}{\xi_{dm}} \right)^2 \right] \log \left( \frac{8 \xi_{dm}}{Q} \right) - 2 \operatorname{Arg} \sinh \left( \frac{1}{\sqrt{y}} \right) + \frac{1}{2} \frac{Q}{\xi_{dm}} - \frac{2 \sqrt{u}}{8} \log \left( \frac{Q}{\xi_{dm}} \right)^2 + \mathcal{O} \left( \left( \frac{Q}{\xi_{dm}} \right)^3 \right). \] (B8)

For \( y \gg 1 \), eq. (B8) becomes

\[ l(y, Q) \simeq \log \left( \frac{8 \xi_{dm}}{Q} \right) + \frac{1}{2} \frac{Q}{\xi_{dm}} + \mathcal{O} \left( \left( \frac{Q}{\xi_{dm}} \right)^2 \log \left( \frac{8 \xi_{dm}}{Q} \right) \right). \]

We have thus derived the four entries for \( l(y, Q) \) in Table III.

Moreover, ref. [20] provides asymptotic expressions for the incomplete elliptic integral of first kind \( F(\varphi(y), k) \) valid for \( k^2 \ll 1 \) (therefore \( Q \ll \xi_{dm} \)) which are uniform in \( \varphi \). The first order asymptotic expression takes the form [20]

\[ F_1(\varphi(y), k) = \left[ 1 + \frac{1}{8} \left( 1 - \frac{1}{p} \right) \right] \log \left( \frac{\sqrt{u} + \sqrt{p}}{\sqrt{u} - \sqrt{p}} \right) + \frac{1}{2} \left( \sqrt{\frac{u}{p}} + \sqrt{\frac{p}{u}} \right) \log \left( \frac{1}{2} + \sqrt{\frac{Q}{\xi_{dm}}} \right) \]

\[ p \equiv 1 + \left( \frac{Q}{\xi_{dm}} \right)^2, \quad u \equiv 1 + y. \] (B9)

We display in fig. 1 \( l(y, Q) \) for \( Q = 0.1, 1 \) and 10. Large values of \( Q \) get suppressed in the integrals by the decrease of the distribution function \( f_{\theta}^{dm}(Q) \).

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