Aggregation for potentially infinite populations without continuity or completeness

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Abstract

We present an abstract social aggregation theorem. Society, and each individual, has a preorder that may be interpreted as expressing values or beliefs. The preorders are allowed to violate both completeness and continuity, and the population is allowed to be infinite. The preorders are only assumed to be represented by functions with values in partially ordered vector spaces, and whose product has convex range. This includes all preorders that satisfy strong independence. Any Pareto indifferent social preorder is then shown to be represented by a linear transformation of the representations of the individual preorders. Further Pareto conditions on the social preorder correspond to positivity conditions on the transformation. When all the Pareto conditions hold and the population is finite, the social preorder is represented by a sum of individual preorder representations. We provide two applications. The first yields an extremely general version of Harsanyi’s social aggregation theorem. The second generalizes a classic result about linear opinion pooling.

Keywords. Social aggregation; discontinuous preferences and comparative likelihood relations; incomplete preferences and comparative likelihood relations; infinite populations; Harsanyi’s social aggregation theorem; linear opinion pooling; partially ordered vector spaces.

JEL Classification. D60, D63, D70, D81, D83.

1 Introduction

Individuals may have incomplete values and beliefs\(^1\). They may be undecided which of two goods is preferable, or which of two events is more likely. The former is reflected, for example, in the literature developing multi-utility theory, and the latter in work on representing beliefs by sets of probability measures, and on decision theories that feature such sets. Individuals may also have discontinuous values and beliefs\(^2\). For example, they may see some goods as being infinitely more valuable than others (or, equivalently, see some goods as only having significance as tiebreakers). In addition, allowing individuals to regard some events as infinitesimally likely

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\(^1\)The literature on the topics of this paragraph is vast. References that are directly relevant to our approach are given in section 2.1.

\(^2\)The applications we discuss are either decision theoretic (values) or to do with comparative likelihood (beliefs). Each subject has its own family of continuity axioms, but we rarely need to specify these, and we rely on context to determine which family we are discussing.
provides a solution to a number of problems to do with conditional probability, decision theory, game theory and conditional preference.

A standard question is how to aggregate the values or beliefs of individuals to form a collective view. Here we also wish to allow the population to be infinite. One rationale is that even in a finite society, a decision maker having incomplete information about the values and beliefs of members of society may wish to model each individual as an infinite set of types. But the most obvious rationale comes from the much discussed problem of intergenerational equity. Here, the possibility that society will extend indefinitely into the future requires an infinite number of people. It is also commonly modelled by an infinite sequence of generations, each one with a social utility function; in such a model, the generations play the role of ‘individuals’ whose interests are to be aggregated.

Section 2 presents a family of abstract aggregation theorems. Each of possibly infinitely many individuals, and society, is equipped with a preorder on a given set. We assume that these preorders are represented by functions with values in partially ordered vector spaces, and whose product has convex range. The use of partially ordered vector spaces is explained and motivated in section 2.1; axiomatizations of such representations are given in sections 3.1 and 3.2. Roughly speaking, our main result shows that Pareto indifference holds if and only if the social preorder can be represented by a linear combination of the representations of the individual preorders. Further Pareto conditions correspond to positivity conditions on the linear mapping.

Section 3 illustrates the interest of the results with two applications, corresponding respectively to the aggregation of values and of beliefs. The first yields an extremely general version of the celebrated social aggregation theorem of Harsanyi (1955) that assumes only the central expected utility axiom of strong independence. The other shows that aggregate beliefs are given by linear pooling. To reiterate, unlike other results in the literature, these results hold without any completeness or continuity assumptions, and allow for an infinite population.

We end in section 4 with a discussion of related literature, but for now we acknowledge Mongin (1995) and De Meyer and Mongin (1995) for drawing attention to formal similarities between preference aggregation and opinion pooling, and emphasizing the usefulness of the convex range assumption we use below.

The proofs of our main theorems rely only on concepts from basic linear algebra, rather than any theorems from convex or functional analysis. All proofs are in the Appendix.

2 Main results

2.1 Representations in partially ordered vector spaces

We are going to consider representations of preorders with values in partially ordered vector spaces. We first recall the basic definitions, and give some examples illustrating this type of representation.

Recall that a preorder $\succeq$ is a binary relation that is reflexive and transitive; we write $\sim$ and $\succ$ for the symmetric and asymmetric parts of $\succeq$ respectively. Since preorders can be incomplete, we write $x \preceq y$ if neither $x \succeq y$ nor $y \succeq x$.

Let $(X, \succeq)$ and $(X', \succeq')$ be preordered sets. A function $f : X \to X'$ is increasing if $x \succeq y \Rightarrow f(x) \succeq' f(y)$; strictly increasing if $x \succeq y \Rightarrow f(x) \succeq' f(y)$ and $x \succ y \Rightarrow f(x) \succ' f(y)$; a represen-

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3In the context of values, see e.g. Zhou (1997), attributing the idea to Harsanyi (1967–68); for beliefs, see e.g. Herzberg (2013).

4The literature on this topic is largely shaped by the approaches of Ramsey (1928) (opposing impatience), and of Koopmans (1960) and Diamond (1965) (requiring impatience). Our results are compatible with both approaches.
ation of $\preceq$ if $x \succeq y \iff f(x) \succeq f(y)$; an order embedding if it is an injective representation; and an order isomorphism if it is a bijective representation.

A preordered vector space is a real vector space $V$ with a (possibly incomplete) preorder $\preceq_V$ that is linear in the sense that $v \succeq w \iff \lambda v + w \preceq_V \lambda v' + w$, for all $v, v', w \in V$ and $\lambda > 0$. Note that we allow vector spaces to have infinite dimension. A partially ordered vector space is a preordered vector space in which the linear preorder is a partial order; that is, it is antisymmetric. An ordered vector space is a partially ordered vector space in which the partial order is an order; that is, it is complete. When $L: V \to V'$ is a linear map between partially ordered vector spaces, $L$ is increasing if and only if it is positive, in the sense that $v \succeq_V 0 \implies Lv \succeq_{V'} 0$; and $L$ is strictly increasing if and only if it is strictly positive, meaning that $v \succeq_V 0 \implies Lv \succ_{V'} 0$ and $v \succeq_{V'} 0 \implies Lv \succ_{V'} 0$.

In this paper we will be exclusively concerned with representations with values in partially ordered vector spaces. We provide an axiomatic basis for using such representations for preference relations in Lemma and for comparative likelihood relations in Lemma, but for now we focus on examples. The set $\mathbb{R}$ of real numbers, with the usual ordering, is a simple example of a partially ordered vector space, so our representations include familiar real-valued ones. Allowing for arbitrary partially ordered vector spaces allows for natural representations of incomplete and discontinuous preorders, as the following examples illustrate.

**Example 1.** Consider bundles of three goods (say fame, love, and money) represented by points in $X = \mathbb{R}_+^3$. Let $V = \mathbb{R}^3$, with the following linear partial order:

$$(x_1, x_2, x_3) \succeq_V (y_1, y_2, y_3) \iff (x_1 \geq y_1, x_2 \geq y_2, x_3 \geq y_3),$$

and, if $x_1 = y_1$ and $x_2 = y_2$, then $x_3 \geq y_3$.

Let $f: \mathbb{R}_+^3 \to V$ be the function $f(x_1, x_2, x_3) = (\frac{2}{3}x_1 + \frac{1}{3}x_2, \frac{2}{3}x_1 + \frac{1}{3}x_2, x_3)$. Suppose person $A$’s preference relation $\succeq_A$ over simple lotteries over $\mathbb{R}_+^3$ is represented by expectations of $f$, thereby satisfying the expected utility axiom of strong independence. It violates the completeness axiom, because, for example, $(1, 0, 0) \succ_A (0, 1, 0)$, reflecting the fact that $A$ finds fame and love only roughly comparable. (Note that the incomparability is limited in the sense that two units of fame are preferred to one unit of love, and vice versa.) The preferences violate standard decision theoretic notions of continuity, such as the Archimedean axiom of Blackwell and Girshick (1954) and the mixture continuity axiom of Herstein and Milnor (1953), reflecting the fact that while $A$ sees money as valuable, she finds fame (likewise love) infinitely more valuable.

**Example 2.** Consider a sphere $S$ divided into open northern and southern hemispheres $S_+$ and $S_-$, and equator $S_0$. Let $\mu_+, \mu_-$, and $\mu_0$ be the uniform probability measures on $S_+$, $S_-$, and $S_0$, respectively. For every measurable set $A \subset S$ define $f(A) = (\mu_+(A \cap S_+), \mu_-(A \cap S_-), \mu_0(A \cap S_0)) \in V$, for $V = \mathbb{R}^3$ as in the previous example. This $f$ represents a likelihood preorder on the algebra $X$ of measurable subsets of $S$. The preorder is incomplete, since, for example, the hemispheres $S_+$ and $S_-$ are incomparable. (Here we have allowed the incomparability to be unlimited, in the sense that any positive-measure subsets of $S_+$ and $S_-$ are incomparable.) Moreover, the equator $S_0$, though more likely than the null set, is less likely than the interior of any spherical triangle, no matter how small. Correspondingly, the likelihood preorder violates standard continuity axioms for comparative likelihood, such as the Archimedean axiom C6 of Fine (1973), or the monotone continuity axiom of Villegas (1964) and its weakening C8 in Fine (1973). For both of these reasons, the likelihood preorder cannot be represented by a standard $([0, 1])$-valued probability measure.

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5 A linear preorder, in our sense, is sometimes called a vector preorder. See section for generalisation to vector spaces over ordered fields other than $\mathbb{R}$.

6 See however Remark for comments relevant to merely preordered vector spaces.

7 A preorder $\preceq$ on a convex set $X$ satisfies strong independence if for each $\alpha \in (0, 1)$, $x, y, z \in X$, $x \preceq y$ if and only if $\alpha x + (1 - \alpha)z \preceq \alpha y + (1 - \alpha)z$. 

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In Example 1, the domain $X = \mathbb{R}^n_+$ is a convex set and the representation $f$ is mixture preserving, i.e., $f(\alpha x + (1-\alpha)x') = \alpha f(x) + (1-\alpha)f(x')$ for all $x, x' \in X$, $\alpha \in [0,1]$. When $V = \mathbb{R}$, von Neumann-Morgenstern expected utility representations of preferences are the paradigmatic representations of this type. In Example 2, the domain $X$ is a Boolean algebra of sets, and the representation $f$ is a vector measure, meaning that it is finitely additive: $f(x \cup x') = f(x) + f(x')$ for disjoint $x, x' \in X$. When $V = \mathbb{R}$, probability measures are the paradigmatic representations of this type.

The applications of our aggregation theorems we provide in sections 3.1 and 3.2 involve mixture-preserving and vector-measure representations with values in arbitrary partially ordered vector spaces. The next examples illustrate how such representations generalize some standard ways of representing incomplete or discontinuous preferences.

**Example 3 (Multi-representation).** Suppose a preorder $\succsim$ on a set $X$ is represented in the following way. There is a family $\{f_i: i \in I\}$ of functions from $X$ to $\mathbb{R}$ such that $x \succsim x'$ if and only if $f_i(x) \geq f_i(x')$ for all $i \in I$. This can be rewritten as a representation by a single function $F: X \to \prod_{i \in I} \mathbb{R}$ when $\prod_{i \in I} \mathbb{R}$ is equipped with the product partial order, making it a partially ordered vector space, and $F$ is defined by $(F(x))_i = f_i(x)$. Note that if the $f_i$ are mixture preserving on convex $X$, then $F$ is also mixture preserving; and if the $f_i$ are probability measures on an algebra $X$, then $F$ is a vector measure.

Multi-representations like this are used in expected utility theory to represent the preferences of agents with incomplete values or in decision theory to represent the preferences of agents with incomplete beliefs and in probability theory to represent agents with incomplete beliefs.

**Example 4 (Lexicographic representation).** Suppose a preorder $\succsim$ on a set $X$ is represented in the following way. There is a finite vector $(f_1, \ldots, f_n)$ of functions from $X$ to $\mathbb{R}$ such that $x \succsim x'$ if and only if $f_j(x) = f_j(x')$ for all $j$ or $f_j(x) > f_j(x')$ for the least $j$ such that $f_j(x) \neq f_j(x')$. The $f_j$ can again be combined into a single function $F: X \to \mathbb{R}^n$, with $(F(x))_i = f_i(x)$. As in Example 3 if the $f_j$ are mixture-preserving functions or probability measures, then $F$ is a mixture-preserving function or a vector measure. Moreover, this $F$ represents $\succsim$ if we endow $\mathbb{R}^n$ with the ‘lexicographic’ order. To give the general picture, let $(\mathcal{J}, \succsim)$ be an ordered set, and let $\mathbb{R}^n_{\succsim}$ be the subspace of $\mathbb{R}^n$ whose members have well-ordered support. The lexicographic order $\succsim_{\text{lex}}$ on $\mathbb{R}^n_{\succsim}$ is defined by the property that $f \succsim_{\text{lex}} f'$ if and only if $f = f'$ or $f(j) > f'(j)$ for the $\succsim_{\text{lex}}$-least $j$ such that $f(j) \neq f'(j)$. This makes $\mathbb{R}^n_{\succsim}$ what Hausner and Wendel (1954) call a ‘lexicographic function space’. It is an ordered vector space. In the original example we can identify $\mathbb{R}^n$ with $\mathbb{R}^n_{\succsim}$, for $\mathcal{J} = \{1, 2, \ldots, n\}$ with the usual ordering.

Representations with values in lexicographic function spaces are used in expected utility theory to represent the preferences of agents with discontinuous values in decision and game theory.

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8Standard treatments of probability theory assume countable additivity, but the motivation for this further requirement is reduced given that we will not be assuming continuity axioms like Monotone Continuity.

9In general, for a family $\{(X_i, \succsim_i): i \in I\}$ of preordered sets, the product preorder $\succsim \prod X_i$ is defined by the condition that $(x_i)_{i \in I} \succsim \prod (y_i)_{i \in I}$ if and only if $x_i \succsim_i y_i$ for each $i \in I$.

10See e.g., Aumann (1962); Fishburn (1982); Seidenfeld et al. (1992); Shapley and Birnbaum (1998); Dubra et al. (2000); Hausken and Shapley (2001); Evrendilek (2006); Mansini and Mariotti (2008); Evrendilek (2010); Galaabaatar and Karni (2012, 2014, 2017).

11See e.g., Gilboa and Schmeidler (1981); Seidenfeld et al. (1993); Bewley (2002); Ghirardoto et al. (2003); Nau (2004); Gilboa et al. (2010); Ok. Ortoleva and Riefler (2012); Galaabaatar and Karni (2015).

12For entries to a vast literature, see Halpern (2003) and Stinchcombe (2013); on multi-representation of incomplete comparative likelihood preorders, discussed further in section 3.2 see Insua (1992) and Alon and Lehrer (2013).

13That is, $\mathbb{R}^n_{\succsim} = \{ f \in \mathbb{R}^n \mid \forall j \in \mathcal{J} \mid f(j) \neq 0 \}$ is well-ordered by $\succsim$.

14See Hausner (1954); Fishburn (1971); Blume, Brandenburger and Dekel (1981); Borie (2016); Hara, Ok and Riefler (2018); McCarthy et al. (2017).
theory to represent the preferences of agents with discontinuous beliefs and in probability theory to represent agents with discontinuous beliefs.

The constructions in these examples can be combined, yielding representations of possibly incomplete and discontinuous relations that take values in $\prod_{i \in I} \mathbb{R}_{\geq 0}$, a partially ordered vector space where each component has the lexicographic order and their product has the partial order. In fact, this construction is fully general: every partially ordered vector space can be order-embedded in such a space.

Finally, we note that our approach is also compatible with the use of representations with values in non-Archimedean ordered fields, such as non-standard models of the real-numbers. These provide natural ways of representing discontinuous relations; see Example below. We discuss this topic further in section 4.

Remark 5 (Preordered Vector Spaces). We focus on representations in partially ordered vector spaces, rather than merely preordered ones, because it seems desirable for objects that are ranked equally to be assigned the same value. But for some purposes it is useful to consider preordered vector spaces instead. Our results have implications for this more general case, insofar as any preordered vector space has a partially ordered vector space as a quotient. Lemma 18 and Theorem 13 illustrate some of the extensions this makes available.

2.2 Framework and axioms

We assume throughout that $X$ and $I$ are nonempty, possibly infinite, sets. For each $i \in I \cup \{0\}$, let $\succsim_i$ be a preorder on $X$. In applications, $I$ is typically a population, and for each $i$ in $I$, $\succsim_i$ expresses the values or beliefs of individual $i$, while $\succsim_0$ expresses those of the social observer.

Say that a family $\{f_i : X \rightarrow V_i\}_{i \in I}$ of functions with values in vector spaces is co-convex if their joint range $(f_i)_{i \in I}(X) \subset \prod_{i \in I} V_i$ is convex. Equivalently, the $f_i$ are co-convex if there is a surjective map $q$ from $X$ onto a convex set $\bar{X}$, and mixture-preserving functions $f_i : \bar{X} \rightarrow V_i$, such that $f_i = f_i \circ q$. Thus co-convexity is a modest generalisation of the assumption that $X$ is a convex set and the $f_i$ are mixture preserving.

The central assumption of our main theorems will be that for $i \in I \cup \{0\}$, the $\succsim_i$ have co-convex representations $f_i$ with values in partially ordered vector spaces $V_i$. In the special case in which $I$ is finite and each $V_i$ is the real numbers with the usual ordering, the requirement of co-convex representations was introduced by De Meyer and Mongin (1995).

Any preordered set can be shown to have a representation with values in some partially ordered vector space [McCarthy et al. 2013; Lemma A.6], so the question is the significance of co-convexity. We postpone this question, though, until sections 3.1 and 3.2 where we present contexts in which co-convex representations naturally arise.

We consider the following Pareto-style axioms.
Theorem 6. Assume that for \( i \in \mathbb{I} \cup \{0\} \) the \( \succeq_i \) have co-convex representations \( f_i : X \to V_i \) of the \( \succeq_i, \ i \in \mathbb{I} \). Set \( V := \prod_{i \in \mathbb{I}} V_i \) and \( f := (f_i)_{i \in \mathbb{I}} : X \to V \). We are interested in the question of whether \( \succeq_0 \) has a representation of the form \( Lf \), with \( L \) a linear map from \( V \) into some partially ordered vector space \( V \). When \( \mathbb{I} \) is finite, we can write
\[
Lf = \sum_{i \in \mathbb{I}} L_i f_i
\]
where \( L_i : V_i \to V \) is the \( i \)th component of \( L \) and \( Lf_1 \) is an additive representation. But when \( \mathbb{I} \) is infinite, \( L \) is not determined by its components; relatedly, the sum over \( i \) does not make sense.\(^{23}\) In any case, by endowing \( V \) with the product partial order \( \succeq_P \), we can consider the positivity of \( L \). Note that if \( L \) is positive, or strictly positive, then so is every \( L_i \), but if \( \mathbb{I} \) is infinite, the converse need not hold.

Since \( f_i \) and \( Lf_i \) are automatically co-convex, a necessary condition for the existence of a representation of \( \succeq_0 \) of the form \( Lf \) is that there exists some representation \( f_0 : X \to V_0 \) of \( \succeq_0 \) such that all the \( f_i \), including \( f_0 \), are co-convex. Our first theorem says that this necessary condition is also sufficient, and lays out the connection between the Pareto-style axioms and the positivity properties of \( L \).

2.3 Results

Suppose we are given co-convex representations \( f_i : X \to V_i \) of the \( \succeq_i, \ i \in \mathbb{I} \). Set \( V := \prod_{i \in \mathbb{I}} V_i \) and \( f := (f_i)_{i \in \mathbb{I}} : X \to V \). We are interested in the question of whether \( \succeq_0 \) has a representation of the form \( Lf \), with \( L \) a linear map from \( V \) into some partially ordered vector space \( V \). When \( \mathbb{I} \) is finite, we can write
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Theorem 6. Assume that for \( i \in \mathbb{I} \cup \{0\} \) the \( \succeq_i \) have co-convex representations \( f_i \).

(a) \( \text{P1} \) holds if and only if \( \succeq_0 \) has a representation of the form \( Lf \), with \( L \) linear.

(b) \( \text{P1} \) and \( \text{P2} \) hold if and only if \( \succeq_0 \) has a representation of the form \( Lf \), with \( L \) linear and positive.

(c) \( \text{P1} \) and \( \text{P3} \) hold if and only if \( \succeq_0 \) has a representation of the form \( Lf \), with \( L \) linear and strictly positive.

(d) \( \text{P1} \) and \( \text{P4} \) hold if and only if \( \succeq_0 \) has a representation of the form \( Lf \), with \( L \) linear and strictly positive, and every \( L_i \) an order embedding.

An obvious question is whether every representation \( f_0 : X \to V_0 \) of \( \succeq_0 \), co-convex with \( f_i \), has the form \( Lf \). Note that, if \( f_0 \) is a representation, then so is \( f_0 + b \), for any constant \( b \in V_0 \), but at most one of these will be of the form \( Lf \). So a more reasonable question is whether we can write every \( f_0 \) in the form \( Lf + b \). The following result gives an affirmative answer, subject to the following domain richness condition.

\[
\text{DR} \ V_i = \text{Span}(f_i(X) - f_i(X)).
\]

\(^{22}\)In other words, regardless of whether \( \mathbb{I} \) is finite, \( L_i := L_1 \), where \( L_1 \) is the natural embedding of \( V_i \) into \( V \).

\(^{23}\)Nonetheless, the linearity of \( L \) does entail a kind of finite additivity; see section 3.4 for further discussion in the context of Harsanyi’s theorem.
In other words, $f_i(X)$ is not contained in any proper affine subspace of $V_i$; equivalently, $V_i$ is the affine hull of $f_i(X)$.

**Theorem 7.** Assume that for $i \in \mathbb{I} \cup \{0\}$ the $\succsim_i$ have co-convex representations $f_i$. For the left-to-right directions of parts (b)–(d), assume DR.

(a) $P1$ holds if and only if $f_0 = Lf_1 + b$ for some $b \in V_0$ and $L$ linear.
(b) $P1$ and $P2$ hold if and only if $f_0 = Lf_2 + b$ for some $b \in V_0$ and $L$ linear and positive.
(c) $P1$ and $P3$ hold if and only if $f_0 = Lf_3 + b$ for some $b \in V_0$ and $L$ linear and strictly positive.
(d) $P1$ and $P4$ hold if and only if $f_0 = Lf_1 + b$ for some $b \in V_0$ and $L$ linear and strictly positive, with every $L_i$ an order embedding.

In the special case in which $\mathbb{I}$ is finite and $V_i = \mathbb{R}$ for each $i \in \mathbb{I} \cup \{0\}$ (so that $P4$ is vacuous), this is proved in De Meyer and Mongin (1995, Prop. 1) without assuming DR.

So far we have allowed the representations $f_i$ to have values in different partially ordered vector spaces. One might wish to construct a single partially ordered vector space $V$ in which all of the different preorders are represented, and in which the representation of $\succsim_0$ is essentially the sum of the representations of the $\succsim_i$. ‘Essentially’ is required here since we cannot literally sum over $\mathbb{I}$ when it is infinite. But say that a map $S : \prod_{i \in \mathbb{I}} V \rightarrow V$ extends summation if it restricts to the summation map $\bigoplus_{i \in \mathbb{I}} V \rightarrow V$; equivalently, each component $S_i$ of $S$ is the identity map on $V$.

**Theorem 8.** Assume that for $i \in \mathbb{I} \cup \{0\}$ the $\succsim_i$ have co-convex representations $f_i$. Then $P1$–$P4$ hold if and only if there exists a partially ordered vector space $V$, representations $g_i : X \rightarrow V$ of the $\succsim_i$, for $i \in \mathbb{I} \cup \{0\}$, and a linear map $S : \prod_{i \in \mathbb{I}} V \rightarrow V$ such that

(a) $g_0 = Sg_1$
(b) $S$ extends summation and is strictly positive on $\prod_{i \in \mathbb{I}} \text{Span}(g_i(X) - g_i(X))$
(c) $f_1, f_0, g_1,$ and $g_0$ are together co-convex.

If DR holds as well as $P1$–$P4$ we can further require $V = V_0$ and $g_0 = f_0$.

2.3.1 Domain assumptions

DR is a stronger domain assumption than is needed for Theorem 4(b)–(d) and the last claim in Theorem 8 and we adopt it for its simplicity. It cannot simply be dropped from these theorems, even when $\mathbb{I}$ is finite, as the following example shows.

**Example 9.** Let $X = \mathbb{R}$ and $\mathbb{I} = \{1, 2\}$. Let $\succsim_1$ and $\succsim_2$ be represented by $f_1(x) = x$ and $f_2(x) = -x$. Thus $V_i = \mathbb{R} \times \mathbb{R}, f_1(x) = (x, -x)$, and $\text{Span}(f_1(X) - f_1(X)) = \mathbb{R}(1, -1)$. Let $V_0 = \mathbb{R}$ with the ‘trivial’ linear partial order such that any two distinct elements are incomparable (thus

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24To interpret DR it is worth noting that, by Lemma 25 below, and taking into account the fact that $f_0(X)$ is convex, Span($f_0(X) - f_0(X)$) = \{X(f_0(x) - f_0(y)) : \lambda > 0, x, y \in X\}. Thus when the $\succsim_i$ are preference relations, so that the $V_i$ may be seen as utility spaces, DR says that any logically possible profile of utility differences (i.e. any element of $V_i$) is realized as the utility difference between some $x, y \in X$, at least up to scale. Indeed, DR is equivalent to the conjunction of the claims (i) that $V_i = \text{Span}(f_i(X) - f_i(X))$, so that none of the utility spaces $V_i$ is gratuitously large (in terminology introduced in section 2.5 this means that $f_i$ is pervasive); and (ii) that $\text{Span}(f_1(X) - f_1(X)) = \bigoplus_{i \in \mathbb{I}} \text{Span}(f_i(X) - f_i(X))$.

25In this special case, our methods give an alternative and arguably simpler DR-free proof of part (a) but they do not appear to help at all with DR-free proofs of the other parts. See however the discussion at the end of section 2.5.1

26Here positivity refers to the product partial order on $\prod_{i \in \mathbb{I}} V_i$, restricted to $\prod_{i \in \mathbb{I}} \text{Span}(g_i(X) - g_i(X))$.

27For example, for Theorem 4(b)–(d) DR could be replaced by the weaker assumption that Span($f_0(X) - f_2(X)$) contains the positive cone \{ $v \in V_i : v \succeq p$ \}; for part (d) and for the last part of Theorem 8 one could assume that Span($f_1(X) - f_2(X)$) contains both the positive cone and $\bigoplus_{i \in \mathbb{I}} V_i$. However, even these assumptions are stronger than necessary.
ordered vector spaces are unique. Say that a function \( f \) is convex, \( \geq_0 \) satisfies \([P1\,P2\,P4]\) but \([DR]\) fails. 

Suppose \( f_0 = Lf_1 + b \), with \( L : V_1 \to V_0 \) positive. We have \((1,0) \geq_0 0 \) and since \( L \) is positive, \( L((1,0)) = 0 \); and similarly, \( L((0,1)) = 0 \). Linearity of \( L \) implies \( L = 0 \), contradicting \( f_0 = Lf_1 + b \). Thus parts \([D]\) \([d]\) of Theorem 7 do not hold. Similarly for the last claim of Theorem 8 there is no representation \( g_1 \) of \( \geq_1 \) with values in \( V_0 \).

When \( I \) is infinite, \([DR]\) is perhaps surprisingly strong.

**Example 10.** Suppose that \( I = \mathbb{N} \), \( X = \prod_{i \in I} [0,1] \), \( V_i = \mathbb{R} \), \( f_i(x) = x_i \) for \( i \in I \). Then \( V_1 = \prod_{i \in I} \mathbb{R} \) is the space of sequences of real numbers, but \( \text{Span}(f_i(X) - f_i(X)) \) is the subspace of bounded sequences, so \([DR]\) fails. 

Theorem \([L^1\,L^2\,L^3]\) fail. As in Benci, Horsten and Wenmackers (2018a), choose a uniform probability measure \( \mu \) on \( \mathbb{N} \) with values in a non-Archimedean ordered field \( F \) that extends the reals; as such, it is an ordered real vector space. Let \( V_0 \) be the ordered real vector space of finite sequences (including infinitesimal) elements of \( F \); thus for every \( x \) in \( V_0 \) there is some natural number \( n \) with \( x \geq_0 n \). Note that \( \mu \) has values in \( V_0 \). Let \( f_0 : X \to V_0 \) map each element of \( X \) to its \( \mu \)-expectation; that is \( f_0(x) = \sum_{i \in I} \mu(\{i\}) x_i \). The \( \geq_0 \) so represented satisfies \([P1\,P2\,P4]\) (note that the \( \geq_i \) are complete, so \([P2]\) is vacuous). All the \( \geq_i \) satisfy strong independence, but \( \geq_0 \) violates both the Archimedean and mixture continuity axioms of expected utility.

Suppose \( f_0 = Lf_1 + b \) for some linear mapping \( V_1 \to V_0 \), implying \( b = 0 \). Consider the sequence \( v = (1,2,3,4,\ldots) \) in \( V_1 \), and for each natural number \( n \), the bounded sequence \( v_n = (1,2,\ldots,n-1,n,n,n,n,\ldots) \) By linearity of \( L \) we must have \( L(v_n) \geq_0 n - 1 \) (since \( f_0(\frac{1}{n} v_n) \) is infinitesimally close to 1). But \( v \geq_0 v_n \) for every natural number \( n \). So if \( L \) is positive, we must have \( L(v) \geq_0 n \) for every \( n \). But that is impossible, by construction of \( V_0 \). A similar argument shows that the last claim of Theorem 8 involving \([DR]\) also fails.

An interesting question therefore is how far \([DR]\) can be weakened in Theorems 7 and 8.

We plan to take up that question in other work, but for now we note that one can sometimes bypass \([DR]\) by allowing \( L \) in Theorem 7 to be defined only on a subspace of \( V_1 \) that contains \( f_1(X) \). For example, by adding constants to the \( f_i \), we can assume that \( f_1(X) \) is contained in \( Y := \text{Span}(f_1(X) - f_i(X)) \). Without any need for \([DR]\), Theorem 2(a)(b) hold if \( L \) is only required to be defined on \( Y \). Part \([d]\) also holds under the further assumption that \( Y \) contains \( \bigoplus_{i \in I} V_i \), which is needed for the components \( L_i \) to be defined. This last assumption is weaker than \([DR]\) when \( I \) is infinite, and is satisfied in Example 10. To illustrate, \( f_o \) in that example extends uniquely to a strictly positive linear \( L : Y \to V_0 \) that maps each bounded sequence to its \( \mu \)-expectation, with every \( L_i \) an order embedding, and we have \( f_0 = Lf_1 \). But the example shows that \( L \) cannot be extended to a positive linear map \( V_1 \to V_0 \).

We will not pursue domain questions any further. For simplicity, we use \([DR]\) to discuss the uniqueness of the representations discussed section 2.3, to which we now turn. But we avoid it in the applications we present in section 3.

### 2.4 Uniqueness

We first address the general question of to what extent representations with values in partially ordered vector spaces are unique. Say that a function \( f : X \to V \) is pervasive if \( V = \text{Span}(f(X)) \)
Lemma 11. Suppose given co-convex representations \( f : X \to V \) and \( g : X \to V' \) of a preorder \( \preceq \) on \( X \). Suppose that \( f \) and \( g \) are pervasive. Then there exists a unique linear order isomorphism \( L : V \to V' \) and unique \( b \in V' \) such that \( g = Lf + b \).

The following result explains the sense in which the type of representation of \( \preceq_0 \) discussed in Theorem 6 is unique.

Proposition 12. Assume that for \( i \in \mathbb{I} \cup \{0\} \) the \( \preceq_i \) have co-convex representations \( f_i \) such that DR holds. Suppose that \( L : V_i \to V \) and \( L' : V_i \to V' \) are linear maps to partially ordered vector spaces such that \( Lf_i \) and \( L'f_i \) represent \( \preceq_0 \). Then there is a unique linear order isomorphism \( M : L(V_i) \to L'(V_i) \) such that \( L' = ML \).

The proof establishes in part that \( Lf_i : X \to (L(V_i)) \) and \( L'f_i : X \to (L'(V_i)) \) are co-convex, pervasive representations of \( \preceq_0 \), so that Lemma XIII applies. Note that Proposition 12, unlike Theorem 6, assumes DR. Here is an example to illustrate why.

Example 13. Let \( X = \mathbb{R} \), let \( \mathbb{I} = \{1, 2\} \), and let \( \preceq_0, \preceq_1, \preceq_2 \) all equal the standard ordering on \( \mathbb{R} \). Let \( V_1, V_2 = \mathbb{R} \) with the standard ordering, and let \( f_1, f_2 : X \to \mathbb{R} \) both be the identity map. Thus \( V_1 = \mathbb{R} \times \mathbb{R} \) and DR fails. Let \( L : V_1 \to V : (x, y) \mapsto x + y \), and let \( L' : V_2 \to V' := V_1 \) be the identity map. Then \( Lf_1 \) and \( L'f_2 \) both represent \( \preceq_0 \), but there is no linear map \( M : V \to V' \) such that \( L' = ML \).

Next we establish that the \( L \) and \( b \) in Theorem 8 are unique.

Proposition 14. Assume that for \( i \in \mathbb{I} \cup \{0\} \) the \( \preceq_i \) have co-convex representations \( f_i \) such that DR holds. Then there at most one linear map \( L : V_1 \to V_0 \) and one \( b \in V_0 \) such that \( f_0 = Lf_1 + b \).

Here, DR cannot be dropped, even in the special case in which \( \mathbb{I} \) is finite and \( V_i = \mathbb{R} \) for each \( i \in \mathbb{I} \cup \{0\} \), by Fishburn [1984, Cor. 1].

In Theorem 8, given DR, we can choose \( g_0 \) to be a pervasive representation of \( \preceq_0 \). (Choose \( f_0 \) to be pervasive, using the construction before Lemma XIV and then apply the last statement of Theorem 8.) Thus we assume that \( g_0 \) and \( g_0' \) are pervasive in the following result.

Proposition 15. Assume that for \( i \in \mathbb{I} \cup \{0\} \) the \( \preceq_i \) have co-convex representations \( f_i \) such that DR holds. Suppose \( g_i : X \to V \) and \( S : \prod_{i \in \mathbb{I}} V \to V \) satisfy the conditions (a)(c) of Theorem 8 as do \( g_i' : X \to V' \) and \( S' : \prod_{i \in \mathbb{I}} V' \to V' \). Assume also that \( g_0 \) and \( g_0' \) are pervasive. Then there exists a unique linear order isomorphism \( L : V \to V' \) and unique constants \( b_i \in V' \) such that, for all \( i \in \mathbb{I} \cup \{0\} \), \( g_i' = Lg_i + b_i \). Moreover, \( S'(v) = LS((L^{-1}v_i)_{i \in \mathbb{I}}) \) for all \( v \in \bigoplus_{i \in \mathbb{I}} V' + \text{Span}(g_i'(X) - g_i'(X)) \).

3 Applications

We now present two applications in which co-convex representations naturally arise.
3.1 Preference aggregation

Let us assume that the $\succ_i$ are preference relations, with $\succ_0$ that of the social observer. In this context, co-convex representations arise naturally from the main expected utility axiom of strong independence (see note 7).

**Lemma 16.** Suppose $X$ is a convex set.

(i) A preorder on $X$ satisfies strong independence if and only if it has a mixture-preserving representation. The representation may be chosen to be pervasive.

(ii) Any family $\{f_i: X \to V_i\}_{i \in I}$ of mixture-preserving functions is co-convex.

Given that the $\succ_i$ satisfy strong independence, pervasive (and, by Lemma 11 essentially unique) mixture-preserving representation $f_i$ may be chosen for each of them, and these representations are automatically co-convex. The results of section 2 therefore apply. Here, for example, is a corollary of Theorem 8.

**Theorem 17.** Suppose $X$ is convex, and for all $i \in I \cup \{0\}$, $\succ_i$ satisfies strong independence. Assume $P1 - P3$. Then there exists a partially ordered vector space $V$, mixture-preserving representations $f_i: X \to V$ of the $\succ_i$, and a linear map $S: \prod_{i \in I} V \to V$ that extends summation, such that $f_0 = Sf_1$.

In this context, our results generalize the celebrated social aggregation theorem of [Harsanyi, 1955]. In his framework, the population $I$ is finite, and $X$ is the set of simple probability measures on a given set of social outcomes. In contrast, we allow $I$ to be infinite, and we allow $X$ to be an arbitrary convex set, which may in particular be infinite dimensional. This allows for a wide range of models of uncertainty. For example, in the setting of objective risk, $X$ may be any convex set of probability measures on a measurable space; in the setting of objective risk and subjective uncertainty, it may be the set of Savage acts when those are equipped with convex structure, as in for example Chirardoto et al. (2003) it may be a set of simple lotteries with nonstandard probabilities; or it may be an arbitrary mixture space, and hence isomorphic to a convex subset of a vector space as noted by [Hausner, 1954].

One version of Harsanyi’s result is that, if $P1 - P3$ hold, as well as strong independence, continuity and completeness for each $\succ_i$, then the social preorder can be represented by the sum of real-valued mixture-preserving individual utility functions. Here, Theorem 17 shows that

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29 For applications of Theorem 17 it may be useful to note that, in this context, DR is equivalent to the following domain richness condition, stated directly in terms of $X$ and the $\succ_i$:

**DR** Suppose given $x_i, y_i \in X$ and $\lambda_i > 0$, for each $i \in I$. Then there exist $z, w \in X$ and $\lambda > 0$ such that, for all $i \in I$, $w + \lambda \lambda_i x_i \succ_i v + \lambda \lambda_i y_i$.

(Heuristically: the difference in value for $i$ between $z$ and $w$ is $\lambda_i/\lambda$ times the difference between $x_i$ and $y_i$.)

30 The dimension of $X$ is, by definition, the dimension of the smallest affine space that contains it; equivalently, the dimension of the vector space $\text{Span}(X - X)$.

31 As is well known, however, allowing for subjective uncertainty is likely to lead to impossibility results; see the end of section 3.

32 For a discussion of the relationship between mixture spaces in the sense of Hausner and mixture sets in the sense of [Herstein and Milnor, 1954], see [Mongin, 2001]. Mongin constructs a natural map from each mixture set onto a convex set, and shows that it is an isomorphism if and only if the mixture set is a mixture space (or in his terminology ‘non-degenerate’).

33 For discussion of different variations of Harsanyi’s result, see e.g. [Weymark, 1993]. Note that Harsanyi’s result is usually stated in terms of a weighted sum of individual utility functions; but one can absorb the weights into the utility functions to get an unweighted sum. Of course, in both the weighted and unweighted versions, the representation of the social preorder is a linear transformation of the profile of individual utilities. Roughly speaking, Theorems 6 and 7 generalize the formulation with weighted sums, while Theorem 8 and its corollaries Theorems 17 and 19 generalize the unweighted version.
Lemma 18. (which is vacuous given completeness), and allowing the utility functions to be vector-valued. In light of the discussion after Example 4 and especially note 17 the conclusion of Theorem 17 may be made visibly closer to Harsanyi’s by taking utility values to be matrices of real numbers; Harsanyi’s conclusion is then the case where the matrices are one-by-one.

The only caveat is that, when $\mathbb{I}$ is infinite, the linear map $S$ used to combine the individual utility functions $f_i$ does not simply sum them up. However, we still get separability, or ‘finite additivity’. If $\mathbb{I} = \mathbb{J} \cup \mathbb{K}$, then $V_2 = V_3 \times V_6$; denoting restrictions in the obvious way, we have $Sf_1 = S_2f_2 + S_3f_3$. So much follows from the linearity of $S$; since $S$ also extends summation, $S_3$ is simply the summation map when $\mathbb{J}$ is finite.

Our results also yield fully additive representations in some important cases when $\mathbb{I}$ is infinite. Suppose, for example, that in the context of Theorem 17, each element of $X$ is a gamble in which only finitely many people from $\mathbb{I}$ have any chance to exist. For each $i \in \mathbb{I}$ and $x, x' \in X$, it is natural to suppose that $x \sim_i x'$ if $i$ is certain not to exist in either one; thus $f_i(x) = f_i(x')$. By subtracting a constant from $f_i$, we may assume that $f_i(x) = f_i(x') = 0$. (Note that renormalizing $f_i$ in this way changes neither the fact that $f_i$ represents $\succcurlyeq_j$ nor the fact that $Sf_i$ represents $\succeq_0$.)

The upshot of this construction is that, for each $x \in X$, $f_i(x) = 0$ for all but finitely many $i \in \mathbb{I}$. We can therefore write $Sf_i = \sum_{i \in \mathbb{I}} f_i$, a fully additive representation of the social preorder.

Suppose now that $X$ is a convex set of probability measures on a measurable space $Y$. It is natural to ask whether the representations constructed in Theorem 17 for example, can be written as integrals over $Y$, in the style of expected utility theory.

To make the question precise, suppose we are given a vector space $V$ and a separating vector space $V'$ of linear functionals $V \to \mathbb{R}$. A function $U : Y \to V$ is weakly $X$-integrable with respect to $V'$ if there exists $f : X \to V$ such that $f'_x \Lambda \circ U \, d\mu = \Lambda \circ f(\mu)$ for all $\Lambda \in V'$, $\mu \in X$. In particular, every $\Lambda \circ U$ must be Lesbesgue integrable against every $\mu \in X$. The Pettis or weak integral is defined by setting $f'_x \, U \, d\mu := f(\mu)$. When $f : X \to V$ can be written in this form, we say that $f$ is expected integral. The question, then, is whether the representations $f_i$, including $f_0$, can be chosen to be expected functions.

In the most common case, where $X$ is a convex set of finitely supported probability measures on a measurable space $Y$ with measurable singletons, there is a straightforward positive answer: any mixture-preserving, vector-valued function on $X$ is expected integral (independently of how $V'$ is chosen). However, in the general case, it turns out that we need to consider representations with values in preordered vector spaces (cf. Remark 5). Indeed, the following result (McCarthy et al. 2018, Lemma 4.3) contrasts with Lemma 17.

**Lemma 18.** Let $X$ be an arbitrary convex set of probability measures. A preorder on $X$ satisfies strong independence if and only if it has an expected integral (and not merely mixture-preserving) representation with values in a preordered (but not necessarily partially ordered) vector space.

The next result uses this to elaborate on Theorem 8.

**Theorem 19.** Suppose $X$ is a convex set of probability measures on a measurable space $Y$, and, for all $i \in \mathbb{I} \cup \{0\}$, $\succeq_i$ satisfies strong independence. Assume [P1] - [P4]. Then there exists a preordered vector space $V$ equipped with a separating vector space $V'$ of linear functionals; for $i \in \mathbb{I} \cup \{0\}$, functions $U_i : Y \to V$ that are weakly $X$-integrable with respect to $V'$, such that $\mu \mapsto f'_x U_i \, d\mu$ represents $\succeq_i$; and a linear map $S : \prod_{i \in \mathbb{I}} V \to V$ that extends summation such that $S \mu = SU_0$.

34 Endowing $V$ with the weak topology with respect to $V'$ makes it a locally convex topological vector space whose dual is $V'$ (Rudin [1993, 3.10]).
Thus when $I$ is finite, for example, without assuming continuity or completeness or interpersonal comparisons, we find that the social preorder is represented by the mapping $\mu \mapsto \int Y \sum_{i \in I} U_i \, d\mu$; in other words, by expected total utility.

### 3.2 Opinion pooling

Here we assume that $X$ is a Boolean algebra of events, understood as sets of states of nature, and the $\succ_i$ are comparative likelihood relations on $X$, expressing the beliefs of each individual, with $\succ_0$ expressing those of the social observer. So for events $A, B \in X$, $A \succ_i B$ has the interpretation that, according to $i$, $A$ is at least as likely as $B$.

Generalising ordinary probability measures, we will be considering representations by vector measures with values in partially ordered vector spaces. In Lemma 22(i) below, we give a necessary and sufficient condition for a preorder $\succeq$ on $X$ to be representable by a vector measure $f$. But in terms of standard axioms of comparative probability (see e.g. Alon and Lehrer, 2014), the existence of such a representation entails that $\succeq$ satisfies Reflexivity, Transitivity, and Generalized Finite Cancellation. Positivity and Non-Triviality are equivalent respectively to the further conditions that $f(A) \succeq_V 0$ for all $A \in X$ and that $f(A) \succ_V 0$ for some $A \in X$. We treat Positivity and Non-Triviality as natural but optional assumptions about $\succ$, rather than imposing these conditions on $f$. The main remaining axioms—Completeness and Monotone Continuity—are the ones which our use of vector measures is intended to avoid.

Our Pareto-style axioms $[P1][P4]$ may seem as plausible here as in the context of preference aggregation. However, consider

**Example 20.** Let $\mathbb{I} = \{1, 2\}$. A ball is going to be drawn randomly from an urn containing three balls, red, yellow and blue. Individual 1 privately observes that the ball is not red, and concludes $\{B\} \sim_1 \{R, Y\}$. Individual 2 privately observes that the ball is not yellow, and concludes $\{B\} \sim_2 \{R, Y\}$. The social observer, privy to each individual’s private information, will conclude $\{B\} \succ_0 \{R, Y\}$, contrary to $P1$.

The natural reply, though, is that $[P1][P4]$ are not designed for the problem in which the observer’s task is to aggregate the opinions represented by $(\succ_i, I_i)$, where $I_i$ is $i$’s private information. They are meant for circumstances in which the only data the observer has, or considers relevant, is the $\succ_i$, say when all private information is either hidden from the observer, or has been made common knowledge. Thus we assume that $[P1][P4]$ are applicable in at least an important range of cases; see Dietrich and List (2016) for related discussion.

Linear opinion pooling is the idea that society’s beliefs should be represented by a linear combination of individual beliefs with nonnegative coefficients, and goes back at least to Stone (1961). When each $\succ_i$ can be represented by a probability measure on $X$, linear pooling was axiomatized by McConway (1981). An alternative axiomatization using Pareto-style conditions was given in Mongin (1995) and De Meyer and Mongin (1997), applying the Lyapunov convexity theorem. Further references are given in section 4. Here we give linear pooling results that replace ordinary probability measures with vector measures. We will first extend the approach based on Lyapunov’s theorem. However, this requires a finite population, as well as some technical restrictions, so we will go on to explain an alternative approach, still using Pareto-style conditions, that mixes objective and subjective probability, in the spirit of Anscombe-Aumann decision theory.

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35 These were defined following Example 2.

36 In this special case, the linear combination is typically normalized to a convex combination.
3.2.1 Lyapunov

Our aggregation theorems apply if we assume that the $\succsim_i$ are represented by co-convex vector measures. This assumption follows under certain conditions from the Lyapunov convexity theorem. The following version is proved in Armstrong and Prikry [1981] (their Theorem 2.2, which allows X more generally to be an F-algebra).

**Theorem 21 (Lyapunov).** Suppose $X$ is a $\sigma$-algebra, and $\{f^i : X \to \mathbb{R}\}_{i \in I}$ is a finite family of finitely additive, bounded, non-atomic signed measures on $X$. Then the $f^i$ are co-convex.

With this in mind, we say that a vector measure $f : X \to V$ is admissible if $V$ is finite-dimensional, and, with respect to some (hence any) basis of $V$, each component of $f$ is bounded and non-atomic. To illustrate, these conditions are fulfilled by $f$ in Example 2. Then Lyapunov’s theorem ensures that $f(X)$ is convex; it also ensures that any finite number of admissible vector measures are co-convex.

We thus obtain (for example) this application of Theorem 21.

**Theorem 22.** Suppose that $\mathbb{I}$ is finite, and that, for $i \in \mathbb{I} \cup \{0\}$, $\succsim_i$ is a preorder on a $\sigma$-algebra $X$ that can be represented by an admissible vector measure $f^i : X \to V_i$. Suppose also that $[\mathbb{P}^1, \mathbb{P}^2]$ hold. Then there exists a finite dimensional partially ordered vector space $V$ and, for each $i \in \mathbb{I}$, a linear order embedding $L_i : V_i \to V$, such that the admissible vector measure $\sum_{i \in \mathbb{I}} L_i f^i$ represents $\succsim_0$.

So, under the stated assumptions, the social observer’s beliefs are given by a linear pooling of individual beliefs.

3.2.2 Convexification

Here we give an approach that allows the population to be infinite and that provides an axiomatic basis for the use of vector measures. It mixes objective and subjective probabilities in the style of Anscombe and Aumann [1963].

We proceed by embedding $X$ in a convex set $\overline{X}$. Here we only assume that $X$ is a Boolean algebra on a set $S$ of states of nature. Say that an extended event is a function $F : S \to [0, 1]$ that is constant on each cell of a finite partition $\mathcal{A} \subset X$ of $S$. We identify each $A \in X$ with the extended event $\chi_A$ given by the characteristic function of $A$. Let $\overline{X}$ be the set of extended events; it is a convex subset of the vector space of all functions $S \to \mathbb{R}$.

A preorder $\succsim$ on $\overline{X}$ can be understood as a comparative likelihood relation in the following way. Suppose that, for each $p \in [0, 1]$, a coin with bias $p$ is going to be tossed, independently of the events encoded in $X$; let $H_p$ be the event that it lands heads. Given $F \in \overline{X}$, constant on each cell of some partition $\{E_1, \ldots, E_n\} \subset X$ of $S$, we associate the event $H_F = E_1 H_{p_1} \lor \cdots \lor E_n H_{p_n}$, where $p_j$ is the value that $F$ takes on $E_j$. Thus $H_F$ occurs whenever nature selects some event $E_j$ from the partition, and, independently, the coin with bias $p_j$ lands heads. $F \succsim F'$ holds just in case $H_F$ is judged at least as likely as $H_{F'}$. So interpreted, $\succsim$ should appropriately take into account the preferences of an agent who gets a prize on heads.

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37 Each component of $f$ is automatically a finitely additive signed measure on $X$. Following Armstrong and Prikry [1981], a finitely additive signed measure $f^i$ is said to be non-atomic if for every $\epsilon > 0$ there is a finite partition $\{A_1, \ldots, A_k\}$ of $X$ such that for all $\epsilon$ the total variation $\|f^i(A_k)\|$ of $A_k$ under $f^i$ is less than $\epsilon$. Note that non-atomicity in this sense has nothing to do with the partial order on $V$.

38 At a cost in abstraction, we could avoid the need for infinitely many coins by considering a single sample from $[0, 1]$ with the uniform measure.

39 While emphasizing that our conceptual approach is not decision theoretic, $\succsim$ should match the preferences of an agent who gets a prize on heads.
account the objective probabilities of the coin tosses; we suggest that the appropriate constraint is strong independence.

The following proposition allows us to generate representations of likelihood relations on \( X \) by vector measures.

Lemma 23. Let \( X \) be a Boolean algebra and \( \overline{X} \) the set of extended events.

(i) If a preorder \( \succsim \) on \( \overline{X} \) satisfies strong independence, its restriction \( \succsim \) to \( X \) can be represented by a vector measure. Conversely, a preorder \( \succsim \) on \( X \) can be represented by a vector measure only if it arises in this way.

(ii) If a mixture-preserving function \( f: X \to V \), with \( V \) a vector space, satisfies \( f(\chi_\emptyset) = 0 \), its restriction \( f \) to \( X \) is a vector measure. Conversely, a function \( f: X \to V \) is a vector measure only if it arises in this way.

Lemma 23 in combination with Lemma 16 yields the following application of Theorem 6.

Theorem 24. Let \( X \) be a Boolean algebra, and \( \overline{X} \) the space of extended events. For \( i \in I \cup \{0\} \), let \( \succsim_i \) be a preorder on \( \overline{X} \) whose restriction to \( X \) is \( \succsim_i \). Assume that each \( \succsim_i \) satisfies strong independence, and that the \( \succsim_i \) together satisfy P1–P4.

Then there is for each \( i \in I \cup \{0\} \) a vector measure \( f_i: X \to V \) that represents \( \succsim_i \), and a strictly positive linear map \( L: V_I \to V_0 \), with each \( L_i \) an order embedding, such that \( f_0 = L f_I \).

Let us contrast the two routes to linear pooling. Convexification allows the population to be infinite, and avoids the restriction of the Lyapunov approach to \( \succsim_i \) with admissible representations. In addition, Lemma 23 provides necessary and sufficient conditions for a preorder \( \succsim \) on \( X \) to be represented by a vector measure, whereas no such result has been given for \( \succsim \) to be represented by an admissible vector measure. On the other hand, convexification requires extending \( X \) to \( \overline{X} \), along with Pareto and strong independence for the extended preorders. It is hard to see an objection to this extension of Pareto, but strong independence has its critics (even while remaining very popular) in the standard decision theoretic version of the Anscombe-Aumann framework. Those with similar doubts in our comparative likelihood framework (cf. note 39) might prefer the Lyapunov approach. Finally, for those who prefer to avoid objective probabilities, we note the possibility of imposing convex structure directly on \( X \); compare Ghirardoto et al. (2003).

4 Related literature

We conclude by relating our results to extant work.

In section 2.1 we noted that our use of partially ordered vector spaces generalises some other forms of representation. These include representations with values in a non-Archimedean ordered field \( F \), often used to model failures of continuity (see note 19 for references). Here, \( F \) is generally assumed to be an extension of the real numbers, and as such is an ordered real vector space (cf. Example 10). However, advocates of this approach may prefer to rework our results using \( F \), rather than \( \mathbb{R} \), as the basic field. This would mean interpreting such conditions as linearity, convexity, and strong independence using coefficients drawn from \( F \). We do not pursue this project, but the only results that we do not expect to extend almost verbatim are Theorems 19 and 22.

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40 We note, however, that if the \( \succsim_i \) satisfy P1–P4 and are extended to strongly independent preorders \( \succsim_i \) on \( \overline{X} \), it does not follow that the \( \succsim_i \) satisfy P1–P4. See e.g. Gilboa (2009) for discussion.

41 Thus, for example, a subset \( X \) of an \( F \)-vector space is convex over \( F \) if and only if, for all \( x, y \in X \) and all \( \alpha \in F \) with \( 0 < \alpha < 1 \), we have \( \alpha x + (1-\alpha)y \in X \).
As already noted, in the special case in which $I$ is finite and $V_i = \mathbb{R}$ for each $i \in I \cup \{0\}$, Theorem 7 is proved in De Meyer and Mongin (1995, Prop. 1), without requiring DR note that $P_4$ is vacuous under these assumptions. This is the first result we know of that emphasizes the usefulness of co-convexity in the context of social aggregation. They apply their result to preference aggregation to obtain Harsanyi’s theorem, and to opinion pooling using Lyapunov, in the special case in which each $\succsim_i$ is represented by a standard probability measure.

Zhou (1997) allows $I$ to be infinite, and considers the special case in which $X$ is a mixture set and the $f_i$ are mixture-preserving representations with values in $\mathbb{R}$ (i.e. von Neumann-Morgenstern expected utility functions) so that the $\succsim_i$ satisfy strong independence, continuity and completeness. Under these assumptions, his Theorems 1 and 2 are essentially our Theorem (a) and (b) respectively. He also raises the question of whether, in our terminology, DR can be dropped from Theorem (b) and gives an affirmative answer under yet further assumptions. Our examples in section 2.3.1 show that the answer is negative in general. But we emphasise that our Theorem 6 and the main part of Theorem 8 do not assume DR.

Despite the importance of the problem, the results of Danan et al. (2015) are the only ones we know of that generalize Harsanyi’s theorem by dropping completeness. They assume a finite population, and take $X$ to be the set of probability measures on a finite set of outcomes. They assume that the $\succsim_i$ satisfy strong independence and a slight strengthening of mixture continuity. It follows that the $\succsim_i$ have expected multi-utility representations (Shapley and Baucells, 1998; Dubra et al. 2004), and they present their results in such terms. Recall from Example 3 that any multi-representation can be re-interpreted as a representation in a partially ordered vector space. Our Theorem (a) and (b) say that, given $P_1$ or $P_1$ and $P_2$, the vector-valued representation of $\succsim_0$ is linearly related to those of the other $\succsim_i$. Danan et al. prove essentially this same result in their setting, and go slightly further in describing the components of the linear relation in terms of the original multi-representations. They also explain how these results extend to the case of an infinite population, generalizing Zhou’s results to allow for incompleteness (although, unlike Zhou, assuming a finite set of outcomes).

Our results differ from theirs in three main ways. First, we do not require continuity; one motivation for this is that in the absence of completeness, continuity is a more problematic assumption than it appears. Second, as emphasized in section 3.3, we allow for a much wider range of interpretations of the domain $X$. Third, while they only consider the axioms $P_1$ and $P_2$ we consider the effects of adding the strong Pareto-style axiom $P_3$, and in addition, the apparently novel $P_4$. This whole package of Pareto-style axioms is plausible, in our view, and essential to those of our results that are arguably the fullest generalizations of Harsanyi. Of course, our results do not wholly replace those of Danan et al. (2015), as they study details that arise specifically in the context of continuity and multi-utility representations. But we believe that it is important to consider the full range of Pareto-style conditions, even in that context; our results provide a framework for doing so, although we do not pursue it here.

As in Harsanyi’s theorem, the results of Zhou (1997), Danan et al. (2015) and this paper do not assume interpersonal comparisons. Immediately after the statement of his main theorem, though, Harsanyi (1955) introduces interpersonal comparisons and a form of anonymity, leading to the further conclusion that the social preorder can be represented by the sum of real-valued mixture-preserving individual utility functions that have been normalized to reflect interpersonal comparisons. Generalizations of this version of Harsanyi’s result, that assume interpersonal comparisons...
comparisons and anonymity at the outset but drop one or more of continuity, completeness, strong independence, and the requirement of a finite population, are given in (Fleurbaey (2009), Pivato (2013, 2014), and McCarthy et al. (2018)).

Generalizations of Harsanyi to the case where preferences are over Anscombe-Aumann or Savage acts, allowing individuals to have different beliefs and values, are well known to lead to impossibility results. Our treatment of opinion pooling separately from preference aggregation may therefore be seen as following the advice of, for example, Mongin (1998, pp. 352–3) to aggregate opinion prior to aggregating preferences.

The classic axiomatization of linear opinion pooling, when the population is finite and the $p_i$ are standard ([0,1]-valued) probability measures on a $\sigma$-algebra, was given in McConway (1981). An axiomatization using analogues of the Pareto-style conditions $[P1]$, $[P2]$ was given in Mongin (1995); see also De Meyer and Mongin (1995). Chambers (2007) extends that approach to axiomatize linear pooling for ordinal probabilities. Extending McConway’s approach, linear pooling results for finitely additive probability measures and an infinite population are given in Herzberg (2013) and Nielsen (2019); the latter giving an equivalent result using analogues of $[P1]$ and $[P2]$.

5 Proofs

We begin with some simple observations about convex sets, cones and vector preorders.

**Lemma 25.** Let $Z$ be a nonempty convex set. Then $\text{Span}(Z - Z) = \{ \lambda(z - z') \mid \lambda \in \mathbb{R}, \lambda > 0, z, z' \in Z \}$.

**Proof.** The right-hand side is clearly contained in and spans the left-hand side. It suffices to show that it equals its own span. It is clearly closed under scalar multiplication. To show that it is closed under addition, suppose given $\lambda(z - z')$ and $\mu(w - w')$ with $\lambda, \mu > 0$ and $z, z', w, w' \in Z$. Then it is easy to check that $\lambda(z - z') + \mu(w - w') = \nu(v - v')$, for $\nu := \lambda + \mu$, $v := \frac{\lambda}{\nu}z + \frac{\mu}{\nu}w$, and $v' := \frac{\lambda}{\nu}z' + \frac{\mu}{\nu}w'$. Note that $\nu > 0$ and that $v, v'$ are elements of $Z$, since it is convex.

Given a vector space $V$, recall that $C \subseteq V$ is a convex cone if $0 \in C$, $C + C \subseteq C$ and $\lambda C \subseteq C$ for all $\lambda > 0$. Clearly if $C, C' \subseteq V$ are convex cones, then so is $C + C'$. The following is well-known; see e.g. Ok (2007, G.1.3).

**Lemma 26.** Let $C \subseteq V$, where $V$ is a vector space. The binary relation $\succsim_V$ on $V$ defined by $v \succsim_V w \iff v - w \in C$ is a linear preorder if and only if $C$ is a convex cone. Conversely, any linear preorder on $V$ is of this form.

**Lemma 27.** Suppose $Z$ is a nonempty subset of a vector space $V$, and $\succsim$ is a preorder on $Z$ that satisfies strong independence. Let $C := \{ \lambda(x - y) \mid \lambda \in \mathbb{R}, \lambda > 0, x, y \in Z, x \succsim y \}$. Then $C$ is a convex cone in $V$. Equip $V$ with the linear preorder $\succsim_V$ defined by $v \succsim_V w \iff v - w \in C$. Then the inclusion $\iota: Z \rightarrow V$ represents $\succsim$.

**Proof.** It is clear that $0 \in C$, and that for $\lambda > 0$, $\lambda C \subseteq C$. To show that $C + C \subseteq C$, let $v, w \in C$. Then $v = \lambda(x - y)$ and $w = \mu(x' - y')$ for some $\lambda, \mu > 0, x, x', y, y' \in Z$ with $x \succsim y$ and $x' \succsim y'$.

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$[46]$ While it would be straightforward to add interpersonal comparisons and anonymity to the results of this paper when $I$ is finite, there is a well known incompatibility between full anonymity and Pareto when $I$ is infinite. This is avoided in Pivato (2014) by restricting to finite anonymity, and in McCarthy et al. (2018) by restricting to social lotteries in which only finitely many individuals have a chance of existing.

$[47]$ From among a very large body of literature, see e.g. Broome (1994), Mongin (1992, 1998), Mongin and Pivato (2013), and Pivato (2013).
We have

\[ v + w = (\lambda + \mu) \left( \frac{1}{\lambda + \mu} x + \frac{\mu}{\lambda + \mu} x' \right) - \left( \frac{1}{\lambda + \mu} y + \frac{\mu}{\lambda + \mu} y' \right) . \]

Let \( \alpha = \frac{1}{\lambda + \mu} \). Since \( \succsim \) satisfies strong independence, we have \( ax + (1 - \alpha)x' \succsim ay + (1 - \alpha)y' \) and \( ay + (1 - \alpha)x' \succsim ay + (1 - \alpha)y' \), and thus \( ax + (1 - \alpha)x' \succsim ay + (1 - \alpha)y' \). The displayed equation then shows that \( v + w \in C \), establishing that \( C \) is a convex cone.

We now show that for \( x, y \in Z \), \( x \succsim y \Leftrightarrow x \succsim V y \). Clearly \( x \succsim y \Rightarrow x - y \in C \Rightarrow x \succsim V y \). Conversely, suppose \( x \succsim V y \). Then \( x - y \in C \), hence there exist \( \lambda > 0 \), \( x', y' \in Z \) with \( x' \succsim y' \) and \( x - y = \lambda (x' - y') \). Letting \( \alpha = \frac{1}{\lambda + \mu} \), this rearranges to \( ax + (1 - \alpha)y' = ay + (1 - \alpha)x' \). Since \( x' \succsim y' \) and \( \succsim \) satisfies strong independence, we have \( ax + (1 - \alpha)x' \succsim ax + (1 - \alpha)y' = ay + (1 - \alpha)x' \). By strong independence again, we must have \( x \succsim y \).

Recall that we write \( 1_i : V_1 \to V_1 \) for the natural embedding of \( V_1 \) into \( V_1 \).

**Proof of Theorem 6.** First let us verify that, in each case, the right-hand side entails the left-hand side. So suppose we have a partially ordered vector space \( (V, \succsim V) \) and a linear \( L : V_1 \to V \) such that \( L f_i \) represents \( \succsim 0 \). Let \( f = L f_i \). Then suppose we have a partially ordered vector space \( (V, \succsim V) \) and a linear \( L : V_1 \to V \) such that \( L f_i \) represents \( \succsim 0 \). Let \( f = L f_i \). Then \( f_i \) represents \( \succsim 0 \), and \( f \) represents \( \succsim 0 \) as well. Therefore \( \succsim \) holds.

Suppose that \( L \) is strictly positive. If \( x \succsim_0 y \) for all \( i \in I \) and \( x \succsim_0 y \) for some \( j \in I \), then \( f_i(x) >_P f_i(y) \), whence \( L f_i(x) > V L f_i(y) \) and \( x \succsim_0 y \). Thus \( \succsim \) holds as well. Similarly, suppose that \( L \) is strictly positive. If \( x \succsim_0 y \) for all \( i \in I \) and \( x \succsim_0 y \) for some \( j \in I \), then \( f_i(x) >_P f_i(y) \), whence \( L f_i(x) > V L f_i(y) \) and \( x \succsim_0 y \). Thus \( \succsim \) holds as well.

As for \( (d) \), suppose that \( L \) is strictly positive and each \( L_i \) is an order embedding. Suppose that \( x \succsim_0 y \) for all \( i \in I \) and \( x \succsim_0 y \). This implies that \( f_i(x) \succsim V f_i(y) \) for all \( i \in I \setminus \{ j \} \) and \( f_j(x) \lambda V f_j(y) \). We can therefore write \( f_i(x) - f_i(y) = v_1 + v_2 \) with \( v_1 \succsim V 0 \) and \( v_2 = 1_j(f_j(x) - f_j(y)) \succ P 0 \). Let \( L \) be linearly independent, we have \( L(v_1) \succsim V 0 \); since \( L_i \) represents \( \succsim V_i \), we have \( L(v_2) \lambda V 0 \). Therefore \( L(f_i(x) - f_i(y)) = L(v_1) + L(v_2) \succsim V 0 \). This implies \( x \succsim_0 y \). This shows \( \succsim \) must hold as well.

Conversely, now, we show that the left-hand side entails the right in each case.

Assume \( \succsim_0 \). Define subsets of \( V_1 \):

\[ C_0 = \{ \lambda(f_i(x) - f_i(y)) \mid \lambda > 0, x, y \in X, x \succsim_0 y \} \]
\[ C_P = \{ v \mid v \in V_1, v \succsim_0 P 0 \} \]
\[ C = C_0 + C_P . \]

We first prove that these are convex cones. Let \( X' = (f_i, f_0)(X) \subset V_1 \times V_0 \). By the co-convexity assumption, \( X' \) is convex. Let \( \pi_1 \) and \( \pi_0 \) be the projections of \( V_1 \times V_0 \) onto \( V_1 \) and \( V_0 \) respectively. Let \( \succsim' \) be the preorder on \( X' \) represented by the restriction of \( \succsim_0 \) to \( X' \): \( x \succsim' y \iff \pi_0(x') \succsim V_1 \pi_0(y') \). This \( \succsim' \) satisfies strong independence, as in Lemma 16(ii) Define a subset of \( V_1 \times V_0 \):

\[ C'_0 = \{ \lambda(x' - y') \mid \lambda > 0, x', y' \in X', x' \succsim' y' \} . \]

By Lemma 27 \( C'_0 \) is a convex cone. Note that, for \( x' = (f_i, f_0)(x) \) and \( y' = (f_i, f_0)(y) \in X' \), we have

\[ x' \succsim' y' \iff \pi_0(x') \succsim V_1 \pi_0(y') \iff f_0(x) \succsim V_1 f_0(y) \iff x \succsim_0 y . \]

This shows that \( C_0 = \pi_1(C'_0) \), and therefore that \( C_0 \) is a convex cone. Meanwhile, \( C_P \) is a convex cone by Lemma 26. Therefore the sum \( C \) is also a convex cone.

Next we prove

\[ x \succsim_0 y \iff f_1(x) - f_2(y) \in C_0 . \]
The left-to-right direction is obvious from the definition of $C_0$. For the converse, we first show that $\pi_1$ is injective on $Y := \text{Span}(X' - X')$. Since $\pi_1$ is linear and $Y$ is a vector space, it suffices to show that if $\pi_1(v) = 0$ for some $v \in Y$, then $v = 0$. So suppose $\pi_1(v) = 0$. By Lemma $\ref{lemma1}$, we can write $v = \lambda(a' - b')$, with $a', b' \in X'$, $\lambda > 0$; by definition of $X'$, we can write $a' = (f_1(a), f_0(a))$ and $b' = (f_1(b), f_0(b))$ with $a, b \in X$. Then $\pi_1(v) = 0$ implies $f_1(a) = f_1(b)$; so, by $\ref{part1}$, $a \sim_0 b$, and hence $f_0(a) = f_0(b)$. Therefore $a' = b'$ and $v = 0$, establishing injectivity of $\pi_1$ on $Y$.

Now suppose $f_1(x) - f_1(y) \in C_0$. Let $x' = (f_1, f_0(x))$ and $y' = (f_1, f_0(y))$. Note that $\pi_1(x' - y') = f_1(x) - f_1(y) \in C_0$. Since $x' - y' \in Y$, $C_0 = \pi_1(C_0')$, $C_0 \subset Y$, and $\pi_1$ is injective on $Y$, we can conclude that $x' - y' \in C_0'$. By Lemma $\ref{lemma2}$, however, $x' - y' \in C_0'$ implies $x' \not\sim y'$, and by $\ref{part1}$, we obtain $x \not\sim_0 y$. This establishes the right-to-left direction in $\ref{part1}$.

With these preliminaries, we can now prove the left-to-right direction in part $\ref{part1}$. Assuming $\ref{part1}$, we define a partially ordered vector space $(V; \not\sim)$ as follows. We first let $L$ be the quotient of $V_1$ by the subspace $C_0 \cap -C_0$, and let $L$: $V_1$ $\rightarrow$ $V$ be the quotient map. $L(C_0)$ is a convex cone in $V$, so it defines a linear preorder $\not\sim$ on $V$, by Lemma $\ref{lemma3}$. Namely, we have

$$L(v) \not\sim L(w) \iff L(v - w) \in L(C_0) \iff v - w \in C_0.$$ 

It is clear from the fact that $C$ contains $C_0$ that $L$ is positive. It remains to prove that $L_1$ represents $\not\sim_0$. Suppose first that $x \not\sim_0 y$. Then $f_1(x) - f_1(y) \in C_0 \subset C$, so $L_1(f_1(x)) \not\sim L_1(f_1(y))$. Conversely, if $L_1(f_1(x)) \not\sim L_1(f_1(y))$, then $f_1(x) - f_1(y) \in C$. We may therefore write $f_1(x) - f_1(y) = v_0 + v_p$ for some $v_0 \in C_0, v_p \in C_P$. Solving this equation for $v_p$, we find $v_p \in \text{Span}(f_1(X - f_1(X)))$. Since $f_1(X)$ is convex, Lemma $\ref{lemma4}$ implies $v_p = \lambda(f_1(x') - f_1(y'))$ for some $\lambda > 0, x', y' \in X$. Since $v_p \in C_P$, $\ref{part2}$ implies $x' \not\sim_0 y'$, hence, by $\ref{part2}$, $v_p \in C_0$. This implies $f_1(x) - f_1(y) \in C_0$, and hence by $\ref{part2}$, again, $x \not\sim_0 y$.

For parts $\ref{part3}$ and $\ref{part4}$, we use the same $V$, $L$, and $\not\sim_V$ as in part $\ref{part1}$. For part $\ref{part3}$, it only remains to show that if $\ref{part1}$ and $\ref{part2}$ hold, then $L$ is strictly positive. As before, $L$ is positive, and $\not\sim_V$ is a partial order; because of this, it suffices to show that, for $v \in V, v \not\sim_0 0$ rules out $L_1(v) = 0$. Suppose on the contrary that $L_1(v) = 0$. Then, by the way $L$ was defined, we must have $v \in C \cap -C$. We can therefore write $v = v_0 + v_p$ with $v_0 \in C_0$ and $v_p \in C_P$. Since $v_0 \in C_0$, we can further write $v_0 = \lambda(f_1(x) - f_1(y))$ with $\lambda > 0, x \not\sim_0 y$. Rearranging, we find $f_1(y) - f_1(x) = \frac{1}{\lambda}(v + v_p)$. So if $v \not\sim_0 0$, we have $f_1(y) - f_1(x) \not\sim_0 0$. $\ref{part3}$ yields $y \not\sim 0 x$, a contradiction.

Finally, for part $\ref{part4}$, it suffices to show that, if $\ref{part1}$ and $\ref{part2}$ hold, then every $L_i$ is an order embedding: $v \not\sim V, 0 \Leftrightarrow L_i(v) \not\sim V, 0$. Since, as just established, $L$ is strictly positive, so is each $L_i$, and it remains to show that if $L_i(v) \not\sim V, 0$, then $v \not\sim V, 0$. Suppose therefore that $L_i(v) \not\sim V, 0$. That is, $L_1 v \not\sim V, 0$, so by $\ref{part3}$, $L_1 v \not\sim V, 0$. Since $C$ is an order cone, we can write $v = v_0 + v_p$ for some $v_0 \in C_0, v_p \in C_P$. We may further write $v_0 = \lambda(f_1(x) - f_1(y))$ for some $\lambda > 0, x \not\sim_0 y$. Rearranging, we find $f_1(y) - f_1(x) = \frac{1}{\lambda}(-1, v + v_p)$. Therefore, $f_1(y) \not\sim V, f_1(x)$ for all $j \in I \setminus \{i\}$. By $\ref{part2}$ and $\ref{part4}$ respectively, we will have $y \not\sim V, X$ (a contradiction) if $f_1(y) \not\sim V, f_1(x)$ or $f_1(y) \not\sim V, f_1(x)$; therefore, we must have $f_1(x) \not\sim V, f_1(y)$. Now, by choice of $v_0$ and $v_p, v = \lambda(f_1(x) - f_1(y)) + (v_p); \text{ both terms on the right are } \not\sim V, 0 \text{, so we find } v \not\sim V, 0, \text{ as desired.} \quad \square$

The proof of Theorem $\ref{theorem}$ rests on the following ‘abstract Harsanyi theorem’. 

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Theorem 28. Let $X$ be a nonempty set. Let $Y$ and $Z$ be vector spaces and let $f: X \to Y$, $g: X \to Z$ be co-convex. Then

$$g(x) = g(x') \iff f(x) = f(x') \quad \text{for all } x, x' \in X$$

if and only if $f = Lg + y_0$ for some linear $L: Z \to Y$ and some $y_0 \in Y$. Moreover, the restriction of $L$ to $\text{Span}(g(X) - g(X))$ is uniquely determined.

Proof. To take the last statement first, suppose that $f = Lg + y_0$ and also $f = L'g + y'_0$. Subtracting, we see that $L$ and $L'$ differ by the constant $y'_0 - y_0$ on $g(X)$, and therefore they are equal on $\text{Span}(g(X) - g(X))$.

For the first statement, it is clear that \((1)\) holds if $f$ is of the form $f = Lg + y_0$. For the converse, let $X' := \langle f, g(X) \rangle$; it is a convex set by assumption. Let $A := \text{Span}(X' - X')$; it is a linear subspace of $Y \times Z$. In light of Lemma 25 applied to $X'$, the condition \((1)\) is equivalent to the condition that $A$ contains no elements of the form $(y, 0)$ with $y \neq 0$. Since $A$ is a linear subspace, we find that $(y, z), (y', z) \in A \implies y = y'$.

$A$ is therefore the graph of a partial function $L$ from $Z$ to $Y$. By definition, the domain of $L$ is the projection of $A$ to $Z$, namely $\text{Span}(g(X) - g(X))$, and $L$ is characterized by the equation

$$A = \{(L(z), z) : z \in \text{Span}(g(X) - g(X))\}.$$ 

Also, $L$ is a linear function since $A$ is a linear subspace. Extend $L$ arbitrarily to a linear function from $Z$ to $Y$. Fix $(y, z) \in X'$ and set $y_0 = y - L(z)$. Then for any $x \in X$, we have $f(x) = L(g(x)) + y_0$. □

Proof of Theorem 7. Note that, if $Lf_1 + b$ represents $\succ_0$, then so does $L_f$. Thus the right-to-left direction in each part is a special case of the corresponding claim in Theorem 6.

For the left-to-right directions, assume for the remainder that \((1)\) holds. This implies $f_1(x) = f_1(x') \implies f_0(x) = f_0(x')$ for $x, x' \in X$. Co-convexity of the $f_i$ is equivalent to co-convexity of $f_1$ and $f_0$. Theorem 28 therefore yields a linear map $L: V_1 \to V_0$ and some $v \in V_1$ such that $f_0 = Lf_1 + b$, establishing part \(\text{(a)}\).

Let $v \in V_1$. By the co-convexity assumption, $f_1(X)$ is convex, so \(\text{DR}\) and Lemma 25 imply that there exist $x, y \in X$, $\lambda > 0$ such that $f_1(x) - f_1(y) = \lambda v$. Since $Lf_1$ represents $\succ_0$,

$$x \succ_0 y \iff Lv \succ_0 0.$$ 

We have $v \ succ p 0 \iff x \succ_0 i$ for every $i \in I$. If \(\text{DR}\) holds, $v \ succ p 0 \implies x \succ_0 y$; if \(\text{P2}\) holds $v \succ p 0 \implies x \succ y$. Using the displayed equivalence, if \(\text{P2}\) holds, $L$ is positive, and if \(\text{P2, P3}\) hold, $L$ is strictly positive. This establishes \(\text{(b)}\) \(\text{(c)}\).

Now assume \(\text{P1, P3}\) hold. By Theorem 1 \(\text{(d)}\), there exists a partially ordered vector space $(V', \succ_{V'})$ and a linear map $L': V_1 \to V'$ such that $L'f_1$ represents $\succ_0$, with every $L'_{\lambda}$ an order embedding. Let $v \in V_1$. By convexity of $f_1(X)$, \(\text{DR}\) and Lemma 25 again, there exist $x, y \in X$, $\lambda > 0$ such that $f_1(x) - f_1(y) = \lambda v$. Then $v \ succ_{V_1} 0 \iff L'_v \ succ_{V'} 0 \iff L'_f(x) \ succ_{V'} 0$. This shows that $L_1$ represents $\succ_{V_1}$. Since $V_0$ and $V_1$ are partially ordered, it also shows that $\ker L_4 = \{0\}$: $L_4v = 0 \iff L_4v \ sim_{V_0} 0 \iff v \ sim_{V_0} 0 \iff v = 0$. Therefore $L_4$ is injective, and $L_4$ is an order embedding, establishing \(\text{(d)}\). □

Proof of Theorem 8. For the right-to-left direction, suppose we are given $V$, $g$, and $S$ satisfying the three conditions \(\text{(a)}\) \(\text{(c)}\) or even just \(\text{(a)}\) and \(\text{(b)}\). Suppose $x, y \in X$ with $x \succ_i y$ for all $i \in I$.
Since \( g_i \) represents \( \succcurlyeq_i \), this implies \( g_l(x) - g_l(y) \succcurlyeq_P 0 \), where now \( \succcurlyeq_P \) is the product partial order on \( \prod_{i \in I} V_i \). Since \( g_l(x) - g_l(y) \in Y := \prod_{i \in I} \text{Span}(g_l(X) - g_l(X)) \), and \( S \) is assumed strictly positive on \( Y \), we find \( Sg_l(x) \succcurlyeq_V Sg_l(y) \), and since \( Sg_l \) represents \( \succcurlyeq_0 \), this implies \( x \succcurlyeq_0 y \). This establishes (3) and (4). If \( x >_i y \) for some \( i \in I \), then a similar argument yields \( x >_0 y \), and hence (5).

For (5), suppose \( x \succcurlyeq_j y \) for all \( i \in I \setminus \{j \} \) and \( x \notsucccurlyeq_j y \). Let \( l_j \) be the inclusion of \( V \) into \( \prod_{i \in I} V_i \) as the \( j \)th factor. Set \( v := g_j(x) - g_j(y) \), so that \( v \notsucccurlyeq_V 0 \). We may write \( g_l(x) - g_l(y) = v_P + 1 \), with \( v_P \succcurlyeq_P 0 \), and \( v_P, 1 \) \( \in Y \). Since \( S \) is strictly positive on \( Y \), we have \( S v_P \succcurlyeq_V 0 \), and since \( S \) extends summation we have \( S v_P + S 1 v \succcurlyeq_V 0 \). Therefore (since \( \succcurlyeq_V \) is a linear preorder) we have \( S v_P + S 1 v \succcurlyeq_V 0 \), and since \( Sg_l \) represents \( \succcurlyeq_0 \) we find \( x \succeq_0 y \), as desired.

Conversely, suppose that (3) and (4) hold. From Theorem (5) there is a strictly positive linear map \( L : V_i \to V \) with values in some partially ordered vector space \( V \), such that \( L f_l \) represents \( \succcurlyeq_0 \) and every \( L_i \) is an order embedding. This implies that \( g_i := L_i f_l \) is a representation of \( \succcurlyeq_i \) with values in \( V \) for each \( i \in I \). Together the \( g_i \) define \( g : X \to \prod_{i \in I} V_i \).

We now define the map \( S : \prod_{i \in I} V_i \to V \). The construction is illustrated by the following commutative diagram.

\[
\begin{array}{c}
X \\
\downarrow f_l \\
\prod_{i \in I} V_i \\
\downarrow L \\
\prod_{i \in I} V_i' \\
\downarrow s_1 \\
\prod_{i \in I} V_i \\
\downarrow s_2 \\
\bigoplus_{i \in I} V_i \\
\end{array}
\]

Here, let \( V_i' := L_i(V_i) \); it is a subspace of \( V \) order-isomorphic to \( V_i \). (Note that \( L_i \) is injective, since it is an order embedding.) The \( L_i \) together define an isomorphism \( (L_i) : \prod_{i \in I} V_i \to \prod_{i \in I} V_i' \). On the other hand, \( L \) is a map from \( V_i := \prod_{i \in I} V_i \) to \( V \). We therefore obtain a map \( S_1 : \prod_{i \in I} V_i' \to V \) by \( S_1 = L \circ (L_i)^{-1} \).

Next, we have the summation \( S_2 : \bigoplus_{i \in I} V_i \to V \). The domains of \( S_1 \) and \( S_2 \) are both subspaces of \( \prod_{i \in I} V_i \). The intersection of their domains is \( \bigoplus_{i \in I} V_i' \). Given \( v \in V_i^\prime \), \( L(1, L_i^{-1}(v)) = v \) (where here \( 1_i \) is the inclusion of \( V_i \) into \( \prod_{i \in I} V_i \)). This shows that \( S_1 \) and \( S_2 \) coincide on each \( V_i' \), and therefore on \( \bigoplus_{i \in I} V_i \). Therefore there exists a linear map \( S : \prod_{i \in I} V_i \to V \) that extends both \( S_1 \) and \( S_2 \).

By construction, \( g_l = (L_i) f_l \) and therefore \( S g_l = L f_l \). Therefore \( g_0 := S g_l \) represents \( \succcurlyeq_0 \), as required for part (a).

For part (b) it remains to show that \( S \) is strictly positive on \( Y \). Note that in the commutative diagram defining \( S \), the horizontal map \( (L_i) : \prod_{i \in I} V_i \to \prod_{i \in I} V_i \) is an order embedding, under which \( Y \) is the image of \( Y' := \prod_{i \in I} \text{Span}(f_i(X) - f_i(X)) \). Therefore \( S \) is strictly positive on \( Y' \) if and only if \( L \) is strictly positive on \( Y' \). But \( L \) is strictly positive on \( V \), which contains \( Y' \).

Part (c) is straightforward from the fact that \( f_1 \) and \( f_0 \) are co-convex and \( g \) and \( g_0 \) are linear transforms of \( f_1 \).

For the last claim of the theorem, given [DR] as well as [P1] and [P4], Theorem (d) provides a linear \( L : V_i \to V_0 \) such that \( f_0 = L f_i + b \) for some \( b \in V_0 \). Repeating the construction above, now with \( V = V_0 \), we obtain \( g_i \) and \( S \) satisfying conditions (b) and (c) but instead of (a) we have \( f_0 = L f_i + b = S g_i + b \). However, picking any \( i \in I \), we can replace \( g_i \) by \( g_i - b \) to obtain \( f_0 = S g_i \)

\[^{48}\text{Anticipating the proof of Theorem (d) we note that the following construction of the } g_i \text{ and } S \text{ does not depend on the } V_i \text{ or } V \text{ being partially ordered, rather than merely preordered.}\]
We have \( f(x) = f(x') \iff x \sim x' \iff g(x) = g(x') \). Since \( f \) and \( g \) are pervasive, Theorem 28 tells us that there is a unique linear \( L: V \to V' \) and \( b \in V' \) such that \( g = Lf + b \), and also a unique linear \( L': V' \to V \) and \( b' \in V \) such that \( f = L'g + b' \). (Note that here \( b \) and \( b' \) are uniquely determined as well as \( L \) and \( L' \).) Together we find \( f = L'Lf + (L'b + b') \).

We also have \( f = id_V f \), so the uniqueness statement in Theorem 28 implies that \( L'L \) is id\(_V\). A similar argument gives \( LL' = id_{V'} \), showing that \( L \) is bijective. It remains to show that \( L \) is an order isomorphism: for all \( v_1, v_2 \in V \), we want \( v_1 \preceq_V v_2 \iff Lv_1 \preceq_{V'} L v_2 \). By Lemma 25 and the fact that \( f \) is pervasive, there exist \( x_1, x_2 \in X \) and \( \lambda > 0 \) such that \( v_1 - v_2 = \lambda(f(x_1) - f(x_2)) \). We have \( v_1 \preceq_V v_2 \iff f(x_1) \preceq_V f(x_2) \iff x_1 \preceq x_2 \iff g(x_1) \preceq_{V'} g(x_2) \iff Lf(x_1) \preceq_{V'} Lf(x_2) \iff L(v_1) \preceq_{V'} L(v_2) \), as desired.

**Proof of Proposition 17.** \( \text{Span}(Lf_1(X) - Lf_2(X)) = \text{LSpan}(f_1(X) - f_2(X)) \) by the linearity of \( L \), and \( \text{Span}(f_1(X) - f_2(X)) = V \), by DR. Thus \( \text{Span}(Lf_1(X) - Lf_2(X)) = \text{Span}(V) = V \), showing that \( Lf_1: V_1 \to L(V_1) \) is pervasive. So too is \( Lf_2: V_2 \to L(V_2) \).

We next show that \( Lf_1 \) and \( Lf_2 \) are co-convex. Fix any \( x_1, x_2 \in X \) and \( \alpha \in [0,1] \). Since \( f_1(X) \) is convex, there exists \( x_3 \in X \) such that \( f_1(x_3) = \alpha f_1(x_1) + (1 - \alpha) f_1(x_2) \). Therefore \( Lf_1(x_3) = \alpha Lf_1(x_1) + (1 - \alpha) Lf_1(x_2) \), and similarly \( L'f_1(x_3) = \alpha L'f_1(x_1) + (1 - \alpha) L'f_1(x_2) \). Therefore \( (Lf_1, L'f_1)(x_3) = \alpha (Lf_1(x_1), L'f_1(x_1)) + (1 - \alpha) (Lf_1(x_2), L'f_1(x_2)) \). Thus \( Lf_1 \) and \( L'f_1 \) are co-convex.

By Lemma 11 there is a unique linear order isomorphism \( M: L(V_1) \to L(V_2) \) and \( b \in L'(V_1) \) such that \( Lf_1 = MLf_1 + b \). Suppose given \( v \in V_1 \); by DR and Lemma 25 there are \( x_1, x_2 \in X \) and \( \lambda > 0 \), such that \( v = \lambda(f_1(x_1) - f_1(x_2)) \). From the fact that \( Lf_1 = MLf_1 + b \) it follows that \( L'v = MLv \) as required.

**Proof of Proposition 18.** Suppose that \( f_0 = Lf_1 + b \) and also \( f_0 = L'f_1 + b' \), so we have to show \( L = L' \) and \( b = b' \). By DR and Lemma 25 we can write any \( v \in V_1 \) in the form \( v = \lambda(f_1(x_1) - f_1(x_2)) \), with \( x_1, x_2 \in X \) and \( \lambda > 0 \). Applying \( L \) or \( L' \) we find \( Lv = \lambda(f_0(x_1) - f_0(x_2)) = L'v \); therefore \( L = L' \). Moreover, we have \( b = f_0(x_1) - Lf_1(x_1) = f_0(x_1) - L'f_1(x_1) = b' \).

**Proof of Proposition 19.** First we claim that \( g_0 \) and \( g_0' \) are co-convex. Fix \( x_1, x_2 \in X \) and \( \alpha \in [0,1] \). Since \( f_1 \) and \( g_0 \) are co-convex, there exists \( x_3 \in X \) such that \( f_1(x_3) = \alpha f_1(x_1) + (1 - \alpha) f_1(x_2) \) and \( g_0(x_3) = \alpha g_0(x_1) + (1 - \alpha) g_0(x_2) \). Similarly there exists \( x_3' \in X \) such that \( f_1(x_3') = \alpha f_1(x_1) + (1 - \alpha) f_1(x_2) \) and \( g_0'(x_3') = \alpha g_0'(x_1) + (1 - \alpha) g_0'(x_2) \). By Theorem 3 \( f_1 \) holds. Since \( f_1(x_3') = f_1(x_3) \), we must, by DR, have \( g_0(x_3') = g_0(x_3) \). Therefore \( g_0(x_3') = \alpha g_0(x_1) + (1 - \alpha) g_0(x_2) \), establishing that \( g_0 \) and \( g_0' \) are co-convex.

Thus, by Lemma 11 there is a unique linear order isomorphism \( L: V \to V' \) and unique \( b_0 \in V' \) such that \( g_0 = Lg_0 + b_0 \).

By assumption, for each \( i \in I \), \( f_i \) and \( g_i \) are co-convex, and (by DR) \( f_i \) is pervasive. As described before Lemma 11 for any \( x_0 \in X \), \( g_i - g_i(x_0) \) is a pervasive map \( X \to \text{Span}(g_i(X) - g_i(x_0)) \). Since \( f_i \) is also pervasive, Lemma 11 gives a unique linear order isomorphism \( M_i: V_i \to \text{Span}(g_i(X) - g_i(x_0)) \) and unique \( c_i \in V \) such that \( g_i - g_i(x_0) = M_i f_i + c_i \). Thus \( g_i = M_i f_i + c_i + g_i(x_0) \). It is easy to verify from the uniqueness claim in Lemma 11 that both \( M_i \) and \( c_i + g_i(x_0) \) are uniquely determined by this equation, independently of \( x_0 \). As \( i \) varies these together define \( \alpha \in \prod_{i \in I} V \) and a linear order isomorphism \( M: V_1 \to \prod_{i \in I} \text{Span}(g_i(X) - g_i(x_0)) \) such that \( g_0 = M f_1 + a \) and \( M_{i_1} f_{i_1}(x) + c_{i_1} + g_i(x_0) \). Now fix \( i \in I \\ x \in X \). By DR and Lemma 25 there exist \( y, z \in X \) and \( \lambda \) such that \( \lambda(g_i(y) - g_i(z)) = \lambda(M_i f_i(y) - M_i f_i(z)) = M_1 f_1(x) - M_1 f_i(x) = 1_i (g_i(x) - a_i) \). By the last two terms \( 1_i \) is the inclusion of \( V_i \) into \( V_1 \). We therefore have \( (g_i(y) - g_i(z)) = (M_i f_i(y) - M_i f_i(z)) = M_1 f_1(x) = 1_i (g_i(x) - a_i) \). Where in the last two terms \( 1_i \) is the inclusion of \( V_i \) into \( V_1 \) as the \( i \)th factor. Applying \( S \) and using the fact that it extends summation we find \( g_0(y) - g_0(z) = Sg_i(y) - Sg_i(z) = (g_i(x) - a_i)/\lambda \).
A parallel argument gives $g_0' (y) - g_0'(z) = (g_0'(x) - a'_i')/\lambda$ for some $a'_i \in V'$. Since $g_0' = Lg_0 + b_0$, we have $g_0'(y) - g_0'(z) = Lg_0(y) - Lg_0(z)$; so in combination we find

$$Lg_0(x) - La_i = g'_0(x) - a'_i.$$  

(5)

Rearranging, $g'_i = Lg_i + (a'_i - La_i)$. We set $b_i = (a'_i - La_i) \in V'$. Note that by (5), $b_i = g'_0(x) - Lg_0(x)$ for every $x \in X$, and is uniquely determined by this equation.

For the last statement, suppose first that $v \in \bigcap_{i \in I} V'$. We have

$$LS((L^{-1}v_i)_{i \in I}) = L(\sum_{i \in I} L^{-1}(v_i)) = L\sum_{i \in I} -v_i = S'(v),$$

where every sum has finitely many non-zero terms. On the other hand, suppose $v \in \text{Span}(g'_0(X) - g'_i(X))$. Using Lemma 25 we can write $v$ in the form $v = \lambda (g'_0(x) - g'_i(y))$ with $x, y \in X$ and $\lambda > 0$. By equation (5), we have $L^{-1}v_i = \lambda (g_i(x) - g_i(y))$, so $S((L^{-1}v_i)_{i \in I}) = \lambda (g_0(x) - g_0(y))$ and then $LS((L^{-1}v_i)_{i \in I}) = \lambda (g'_0(x) - g'_0(y)) = S'v$.

Proof of Lemma 18 Suppose that $\succsim$ is a preorder on $X$. It is straightforward to check that $\succsim$ satisfies strong independence if it has a mixture-preserving representation. Conversely, if $\succsim$ satisfies strong independence, let $V$ be the vector space $\text{Span}X$ and $\iota : X \to V$ the inclusion. Taking $Z = X$ in Lemma 24, $\succsim$ and $\iota$ define a convex cone $C$ in $V$ and a linear preorder $\succsim_C$ such that $\iota$ then represents $\succsim$. Let $V'$ be the quotient of $V$ by the subspace $C \cap -C$, and let $L : V \to V'$ be the quotient map. Define a linear partial order $\succsim_{V'}$ on $V'$ by $L(v) \succsim_{V'} L(w) \iff v - w \in C$. It is clear that $L$ is a representation of $\succsim_{V'}$, hence $L\iota$ is a representation of $\succsim$, with values in the partially ordered vector space $V'$. To complete the proof of part (ii) it suffices to apply the construction of pervasive representations described before Lemma 11.

For part (iii) fix any $x, y \in X$ and $\alpha \in [0, 1]$. Let $z = \alpha x + (1 - \alpha) y$. From the fact that the $f_i$ are mixture-preserving, it follows that, for all $i$, $f_i(z) = \alpha f_i(x) + (1 - \alpha) f_i(y)$, and therefore $\alpha (f_i)_{i \in I}(x) + (1 - \alpha) (f_i)_{i \in I}(y) = (f_i)_{i \in I}(z)$, establishing that $(f_i)_{i \in I}(X)$ is convex.

Proof of Theorem 7 By Lemma 10 we can choose co-convex, mixture-preserving representations $F_i : X \to V_i$, for $i \in I \cup \{0\}$. We use these $F_i$ as the ‘$f_i$’ in Theorem 8 which then yields $V, g_i : X \to V$ and $S : \prod_{i \in I} V \to V$. For our current purposes we define $f_i := g_i$.

The only thing left to show is that the $g_i$ are mixture-preserving. This follows from the construction of the $g_i$ in Theorem 8 but here is a direct argument. Fix $x, y \in X$ and $\alpha \in [0, 1]$. By Theorem 8, $g_i$ and $F_i$ are co-convex. So there exists $z \in X$ such that $g_i(z) = \alpha g_i(x) + (1 - \alpha) g_i(y)$ and $F_i(z) = \alpha F_i(x) + (1 - \alpha) F_i(y)$. Since $F_i$ is mixture preserving, the second of these equations holds if and only if $F_i(z) = F_i(\alpha x + (1 - \alpha) y)$, equivalently if and only if $z \sim_i \alpha x + (1 - \alpha) y$. But then, since $g_i$ represents $\succsim_i$, $g_i(z) = g_i(\alpha x + (1 - \alpha) y)$, and therefore $g_i(\alpha x + (1 - \alpha) y) = g_i(x) + (1 - \alpha) g_i(y)$, as desired.

Proof of Theorem 7.2 Lemma 18 yields an expectation preserving representation $f_i : X \to V_i$ of each $\succsim_i$; suppose $f_i(\mu) = \int_X y_i \, d\mu$. Let $V_i = V_i/\sim_i$ be the partially ordered quotient of $V_i$, and $f_i : X \to V_i$ be the composition of $f_i$ with the quotient map; $f_i$ also represents $\succsim_i$. Since the $f_i$ are expectation preserving, they are mixture preserving, and it follows from Lemma 10 that they are co-convex; then the $f_i$ are co-convex as well. By Theorem 4, there is a partially ordered vector space $V$ and a strictly positive linear map $L : V_1 \to V$ such that $L(f_1)$ represents $\succsim_0$, and such that the components $L_i : V_i \to V$ are order embeddings. Now define a preorder $\succsim_V$ on $V := V_1$ by $x \sim_V y \iff L(x) \sim_L V y$, where $x, y \in V_1$. Let $L$ denote the set-theoretic identity map $V_1 \to V$. Since $L$ is strictly positive, so is $L$. Since each $L_i$ is an order-embedding, so is each $L_i$ (note $L$ is injective). Moreover, $L(f_1)$ represents $\succsim_0$. If $V'_i$ is the
given separating vector space of linear functionals on $V_i$, then $V' := \bigoplus_{i \in I} V'_i$ is a separating vector space of linear functionals on $V_i$, so on $V$; with respect to $V'$, we have $f_i(\mu) = \int_V u_i d\mu$. Because $L$ is the set-theoretic identity map we have $L f_i(\mu) = L \int_V u_i d\mu = \int_V L u_i d\mu$.

We now appeal to the proof of the left-to-right direction of Theorem 23(ii), the construction (which works even though $V_i$ and $V$ are now merely preordered vector spaces) takes as input $f_i$ and $L$ and yields $g_i : X \to V$ representing $\preceq_i$ (for $i \in I \cup \{0\}$) and $S : \bigoplus_{i \in I} V \to V$ extending summation such that $g_0 = S g$. Moreover, by construction $g_i = L_i f_i$ for $i \in I$, and $g_0 = L f_0$. We find $g_i(\mu) = L_i \int_V u_i d\mu = \int_V L_i u_i d\mu$, and, as already stated, $g_0(\mu) = \int_V L u_0 d\mu$. Defining $U_0 = L u_0$ and $U_i = L_i u_i$ for $i \in I$, it remains to show that $U_0 = S U_0$, i.e., that $L u_0 = S (L_i u_i)$. And indeed $S$ is constructed so that $S \circ (L_i) = L_i$. \hfill \Box

**Proof of Theorem 22** Since Lyapunov’s Theorem (Theorem 21) implies that the $f_i$ are co-convex, Theorem B yields the vector space $V$ and linear map $L$ with components $L_i$. The only additional points to be checked are that $V$ can be taken to be finite-dimensional and that $\sum_{i \in I} L_i f_i$ is an admissible vector measure.

On the first point, one can harmlessly replace $V$ by its subspace $L(V_i)$, since the latter contains the image of every $L_i$; since, under the hypotheses of the theorem, $V_i$ is finite-dimensional, so is $L(V_i)$. (Alternatively, $V$ as constructed in the proof of Theorem 6 will already be finite-dimensional.)

On the second point, given that the $f_i$ are admissible, it is straightforward to verify that their product $f : X \to V_i$ is admissible, and so then is the linear transform $L f = \sum_{i \in I} L_i f_i$ of $f$.

**Proof of Lemma 23.** We begin with part (ii). Given $\overline{f}$, we have to check that its restriction $f$ to $X$ is additive. For $A, B$ disjoint in $X$ we have

\[
\frac{1}{2} f(A \cup B) = \frac{1}{2} \overline{f}(\chi_A + \chi_B) = \frac{1}{2} \overline{f}(\chi_A) + \frac{1}{2} \overline{f}(\chi_B) = \frac{1}{2} \overline{f}(\chi_A) + \frac{1}{2} \overline{f}(\chi_B) = \frac{1}{2}(f(A) + f(B)).
\]

Hence $f(A \cup B) = f(A) + f(B)$. Here we have used the fact that $X$ is embedded into $\overline{X}$ by $A \mapsto \chi_A$, the assumption that $\overline{f}(\chi_0) = 0$, the mixture preservation property (twice), and again the embedding of $X$.

Conversely, suppose given a vector measure $f : X \to V$. We essentially define $\overline{f} : \overline{X} \to V$ by setting $\overline{f}(F) = \int_S F \, df$ for each $F \in \overline{X}$. Explicitly, suppose $F$ is constant on each cell of the partition $\{E_1, \ldots, E_n\} \subset X$ of $S$, taking value $p_j$ on $E_j$. Then set $\overline{f}(F) = \sum_{j=1}^n p_j f(E_j)$. This $\overline{f}$ is clearly mixture preserving, and the restriction of $\overline{f}$ to $X$ is $f$.

For part (i) suppose given $\preceq_i$ satisfying strong independence; by Lemma 16 it admits a mixture-preserving representation $\overline{f}_i : \overline{X} \to V$. Subtracting a constant, we can assume $\overline{f}_i(\chi_0) = 0$. Thus, by part (ii) $\overline{f}$ restricts to a vector measure $f$ on $X$, which automatically represents $\preceq_i$.

Conversely, if $\preceq_i$ can be represented by a vector measure $f : X \to V$, then, by part (ii) $\overline{f}$ extends to a mixture-preserving function $\overline{f} : \overline{X} \to V$; the preorder $\preceq_i$ on $\overline{X}$ represented by $\overline{f}$ satisfies strong independence, by Lemma 16 and its restriction to $X$ is $\preceq_i$. \hfill \Box

**Proof of Theorem 24** From Lemma 16 we get co-convex mixture-preserving representations $\overline{f}_i : \overline{X} \to V_i$ of each $\preceq_i$. Subtracting a constant in each case, we can assume $\overline{f}_i(\chi_0) = 0$.

Theorem B yields a strictly positive linear map $L : V_i \to V$, for some partially ordered vector space $V$, with every $L_i$ an order embedding, and such that $L \overline{f}_i$ represents $\preceq_i$. By Lemma 23(ii) the $\overline{f}_i$ restrict to representations $f_i$ of the $\preceq_i$ by vector measures. We recover the statement of the theorem by redefining $V_0 := V$ and $f_0 := L \overline{f}_i$. \hfill \Box
References

Alon, S., Lehrer, E., 2014. Subjective multi-prior probability: a representation of a partial likelihood relation. *Journal of Economic Theory* 151: 476–492.

Anscombe, F., Aumann, R., 1963. A definition of subjective probability. *The Annals of Mathematical Statistics* 34: 199–205.

Armstrong, T., Prikry, K., 1981. Liapounoff’s theorem for nonatomic, finitely-additive, bounded, finite-dimensional, vector-valued measures. *Transactions of the American Mathematical Society* 266: 499–514.

Aumann, R., 1962. Utility theory without the completeness axiom. *Econometrica* 30: 455–462.

Baucells, M., Shapley, L. 2008. Multiperson utility. *Games and Economic Behavior* 62: 329–347.

Benci, V., Horsten, L., Wenmackers, S., 2018a. Infinitesimal probabilities. *British Journal for the Philosophy of Science* 69:509–552.

Benci, V., Horsten, L., Wenmackers, S., 2018b. Non-Archimedean probability. arXiv:1106.1524 [math.PR]

Bewley, T., 2002. Knightian decision theory. Part I. *Decisions in Economics and Finance* 25(2): 79–110. [Originally appeared in 1986 as Cowles Foundation Discussion Paper No. 807, Yale University.]

Blackwell, D., Girshick, M., 1954. *Theory of Games and Statistical Decisions*. New York, John Wiley.

Blume, L., Brandenburger, A., Dekel, E., 1989. An overview of lexicographic choice under uncertainty. *Annals of Operations Research* 19: 231–246.

Blume, L., Brandenburger, A., Dekel, E., 1991a. Lexicographic probabilities and choice under uncertainty. *Econometrica* 59(1): 61–79.

Blume, L., Brandenburger, A., Dekel, E., 1991b. Lexicographic probabilities and equilibrium refinements. *Econometrica* 59(1): 81–98.

Borie, D., 2016. Lexicographic expected utility without completeness. *Theory and Decision* 81, 167–176.

Brandenburger, A., Friedenberg, A., Keisler, H. J., 2008. Admissibility in games. *Econometrica* 76(2): 307–352.

Brickhill, H., Horsten, L., 2018. Popper functions, lexicographical probability, and non-Archimedean probability. arXiv:1608.02850 [math.LO]

Broome, J., 1990. Bolker-Jeffrey expected utility theory and axiomatic utilitarianism. *Review of Economic Studies* 57: 477–502.

Chambers, C., 2007. An ordinal characterization of the linear opinion pool. *Economic Theory* 33: 457–474.

Danani, E., Gajdos, T., Tallon, J.-M., 2013. Aggregating sets of von Neumann-Morgenstern utilities. *Journal of Economic Theory* 148: 663–688.

Danani, E., Gajdos, T., Tallon, J.-M., 2015. Harsanyi’s aggregation theorem with incomplete preferences. *American Economic Journal: Microeconomics* 7(1): 61–69.

De Meyer, B., Mongin, P., 1995. A note on affine aggregation. *Economics Letters* 47: 177–183.

Diamond, P. 1965. The evaluation of infinite utility streams. *Econometrica* 33: 170–177.

Dietrich, F., List, C., 2016. Probabilistic opinion pooling. In A. Hájek and C. Hitchcock eds. *The Oxford Handbook of Philosophy and Probability*, Oxford University Press, 2016, 518–19.

Dubra, J., 2011. Continuity and completeness under risk. *Mathematical Social Sciences* 61: 80–81.

Dubra, J., Maccheroni, F., Ok, E., 2004. Expected utility theory without the completeness axiom. *Journal of Economic Theory* 115: 118–133.

Esvren, Ö., 2008. On the existence of expected multi-utility representations. *Economic Theory*
Evren, O., 2014. Scalarization methods and expected multi-utility representations. *Journal of Economic Theory* 151: 30–63.

Fine, T., 1973. *Theories of Probability*, New York, Academic Press.

Fishburn, P., 1971. A study of lexicographic expected utility. *Management Science* 17: 672–678.

Fishburn, P., 1982. *The foundations of expected utility*. Dordrecht, Reidel.

Fishburn, P., 1984. On Harsanyi’s utilitarian cardinal welfare theorem. *Theory and Decision* 17: 21–28.

Fleurbaey, M., 2009. Two variants of Harsanyi’s aggregation theorem. *Economics Letters* 105, 300–302.

Galaabaatar, T., Karni, E., 2012. Expected multi-utility representations. *Mathematical Social Sciences* 64: 242–246.

Galaabaatar, T., Karni, E., 2013. Subjective expected utility with incomplete preferences. *Econometrica* 81: 255–284.

Ghirardato, P., Maccheroni, F., Marinacci, M., Siniscalchi, M., 2003. A subjective spin on roulette wheels. *Econometrica* 71: 1897–1908.

Gilboa, I., 2009. *Theory of Decision under Uncertainty*, Cambridge University Press.

Gilboa, I., Maccheroni, F., Marinacci, M., Schmeidler, D., 2010. Objective and subjective rationality in a multiple prior model. *Econometrica* 78: 755–770.

Gilboa, I., Schmeidler, D., 1989. Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics* 18: 141–153.

Halpern, J., 2003. Reasoning about Uncertainty, MIT Press.

Halpern, J., 2010. Lexicographic probability, conditional probability, and nonstandard probability. *Games and Economic Behavior* 68: 155–179.

Hammond P., 1994a. Elementary non-Archimedean representations of probability for decision theory and games. In: Humphreys, P. (Ed.), *Patrick Suppes: Scientific Philosopher* vol. 1. Kluwer, Dordrecht, pp. 25–49.

Hammond, P., 1994b. Consequentialism, non-Archimedean probabilities, and lexicographic expected utility. *Operations Research* 93. A. Bachem, U. Derigs, M. Jünger and R. Schrader. Heidelberg, Physica: 219–250.

Hammond, P., 1999. Non-Archimedean subjective probabilities in decision theory and games. *Mathematical Social Sciences* 38(2): 139–156.

Hara, K., Ok, E., Riella, G., 2016. Coalitional expected multi-utility theory. [http://economics.mit.edu/files/12659](http://economics.mit.edu/files/12659)

Harsanyi, J., 1955. Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility. *Journal of Political Economy* 63, 309–21.

Harsanyi, J., 1967–1968. Games with incomplete information played by “Bayesian” players, I–III. *Management Science* 14: 159–182, 320–334, 486–502.

Hausner, M., 1954. Multidimensional utilities. In Thrall, R., Coombs, C. and Davis, R. eds. *Decision Processes* John Wiley.

Hausner, M., Wendel, J., 1952. Ordered vector spaces. *Proceedings of the American Mathematical Society* 3, 977–982.

Herzberg, F., 2009. Elementary non-Archimedean utility theory. *Mathematical Social Sciences* 58: 8–14.

Herzberg, F., 2015. Aggregating infinitely many probability measures. *Theory and Decision* 78: 319–337.

Herstein, I., Milnor, J., 1953. An axiomatic approach to measurable utility. *Econometrica* 21: 291–297.

Insua, D., 1992. On the foundations of decision making under partial information. *Theory and
Koopmans, T., 1960. Stationary ordinal utility and impatience. *Econometrica* 28: 287–309.

Manzini, P., Mariotti, M., 2008. On the representation of incomplete preferences over risky alternatives. *Theory and Decision* 65: 303–323.

McCarthy, D., Mikkola, K., 2018. Continuity and completeness of strongly independent preorders. *Mathematical Social Sciences* 93: 141–145.

McCarthy, D., Mikkola, K., Thomas, T., 2017a. Representation of strongly independent preorders by sets of scalar-valued functions. MPRA Paper No. 79284. [https://mpra.ub.uni-muenchen.de/79284/](https://mpra.ub.uni-muenchen.de/79284/)

McCarthy, D., Mikkola, K., Thomas, T., 2017b. Representation of strongly independent preorders by vector-valued functions. MPRA Paper No. 80806. [https://mpra.ub.uni-muenchen.de/80806/](https://mpra.ub.uni-muenchen.de/80806/)

McConway, K., 1981. Marginalization and linear opinion pools. *Journal of the American Statistical Association* 76(374): 410–414.

Mongin, P., 1995. Consistent Bayesian aggregation. *Journal of Economic Theory* 66: 313–351.

Mongin, P., 1998. The paradox of the Bayesian experts and state-dependent utility theory. *Journal of Mathematical Economics* 29: 331–61.

Mongin, P., 2001. A note on mixture sets in decision theory. *Decisions in Economics and Finance* 24: 59–69.

Nau, R., 2006. The shape of incomplete preferences. *The Annals of Statistics* 34: 2430–2448.

Ok, E., 2007. *Real Analysis with Economic Applications*, Princeton University Press.

Ok, E., Ortoleva, P., Riella, G., 2012. Incomplete preferences under uncertainty: indecisiveness in beliefs vs. tastes. *Econometrica* 80: 1791–1808.

Pivato, M., 2013. Risky social choice with incomplete or noisy interpersonal comparisons of well-being. *Social Choice and Welfare* 40: 123–139.

Pivato, M., 2014. Additive representation of separable preferences over infinite products. *Theory and Decision* 77, 31–83.

Ramsey, F., 1928. A mathematical theory of saving. *Economic Journal* 38: 543–559.

Rudin, W., 1991. *Functional Analysis*, 2nd edition. McGraw-Hill.

Seidenfeld, T., Schervish, M., Kadane, J., 1995. A representation of partially ordered preferences. *Annals of Statistics* 23: 2168–2217.

Shapley, L., Baccelli, M., 1998. Multiperson utility. University of California, Los Angeles (UCLA) Department of Economics Working Paper 779.

Stinchcombe, M., 2016. Objective and subjective foundations for multiple priors. *Journal of Economic Theory* 165: 263–291.

Stone, M., 1961. The opinion pool. *Annals of Mathematical Statistics* 32, 4: 1339–1342.

Villegas, C., 1964. On qualitative probability σ-algebras. *Annals of Mathematical Statistics* 35: 1787–1796.

Weymark, J., 1993. Harsanyi’s social aggregation theorem and the weak Pareto principle. *Social Choice and Welfare* 10: 209–221.

Weymark, J., 1995. Further remarks on Harsanyi’s social aggregation theorem and the weak
Pareto principle. *Social Choice and Welfare* 12: 87–92.

Zhou, L., 1997. Harsanyi’s utilitarianism theorems: general societies. *Journal of Economic Theory* 72: 198–207.

Zuber, S., 2016. Harsanyi’s theorem without the sure-thing principle: On the consistent aggregation of Monotonic Bernoullian and Archimedean preferences. *Journal of Mathematical Economics* 63: 78–83.