TWO CONSTRUCTIONS OF LOW-HIT-ZONE FREQUENCY-HOPPING SEQUENCE SETS

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ABSTRACT. In this paper, we present two constructions of low-hit-zone frequency-hopping sequence (LHZ FHS) sets. The constructions in this paper generalize the previous constructions based on m-sequences and d-form functions with difference-balanced property, and generate several classes of optimal LHZ FHS sets and LHZ FHS sets with optimal periodic partial Hamming correlation (PPHC).

1. Introduction

In communication systems, frequency-hopping spread spectrum and direct-sequence spread spectrum are two main spread-coding technologies. Frequency-hopping multiple-access (FHMA) is widely used in modern communication systems such as ultra-wideband, military communications, Bluetooth and so on [23]. Typically, in FHMA systems, when two or more users transmit information on the same frequency slot set at the same time, mutual interference will occur. From this aspect, Hamming correlation of frequency-hopping sequences (FHSs) is introduced to measure the degree of the mutual interference of the employed FHS set. Therefore, it is desirable to design FHSs with Hamming cross-correlations and out-of-phase Hamming autocorrelations as low as possible.

In general, the Hamming correlation of FHSs can be divided into three types: the periodic Hamming correlation (PHC), the aperiodic Hamming correlation, and the partial Hamming correlation. Although a lot of knowledge exists for the PHC (see for example [2, 3, 4, 5, 6, 29]), limited work has been done in partial Hamming correlation of FHSs. However, in the practical application scenarios, in order to

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reduce hardware complexity or minimize the synchronization time of receivers, the length of a correlation window is usually shorter than the period of the chosen FHSs. Consequently, in recent years, there has been tremendous interest in studying FHSs with the partial Hamming correlation.

In asynchronous FHMA systems, FHSs with low Hamming correlation throughout the whole period are required. Different from asynchronous FHMA systems, in a quasi-synchronous FHMA system, relative time delay between different users are restricted within a zone around the origin. FHS sets with low Hamming correlation within the fixed zone are called low-hit-zone FHS (LHZ FHS) sets. As the number of hits between difference sequences will always be very small as long as the relative delay does not exceed certain zone, LHZ FHS sets are more useful.

It is of great interest to construct optimal sets of LHZ FHSs. There have been a number of optimal LHZ FHS sets satisfying Peng-Fan-Lee bound [22] (see for example [1, 11, 15, 17, 18, 19, 25, 28]) and LHZ FHS sets with optimal partial Hamming correlation (PPHC) satisfying Niu-Peng-Liu-Liu bound [16] (see for example [9, 10, 13, 14, 24, 26, 28]). Now we give attention to the earlier constructions that are related to our work. In 2008, Ding and Yin [3] obtained an optimal FHS set by using the \((q-1)\)-decimated \(m\)-sequences. Afterwards, Zhou et al. gave two generic constructions of optimal FHS sets [29] employing \(d\)-form functions with difference-balanced property, and constructed strictly optimal FHS sets [30] by employing sequences with array structure. In 2017 and 2020, Han et al. [11, 9] constructed several classes of optimal sets of LHZ FHSs based on \(m\)-sequences and its decimated sequences. In 2019, Ouyang et al. [20] constructed three classes of strictly optimal FHS sets.

In this paper, we give two constructions of optimal FHS sets with low hit zone with respect to the Peng-Fan-Lee bound and the Niu-Peng-Liu-Liu bound. The first construction can be viewed as a unification and a generalization of the constructions in [11, 9, 20, 27, 29, 30]. In the first construction, we not only make corrections to Theorem 1 and Theorem 2 of [27] in Remark 4, but also generate several classes of optimal FHS sets with low hit zone with new parameters, which can be seen through Table 1. The second construction can be viewed as a generalization of the constructions in [11, 29, 27].

This paper is organized as follows. In Section 2, we introduce some basic notation and results of FHS set, \(d\)-form functions with difference-balanced property and sequences with array structure, which will be used in the following sections. In Section 3, the first construction of LHZ FHS sets is given. In Section 4, the second construction of LHZ FHS sets is given. Finally, concluding remarks are given in Section 5.

2. Preliminaries

In this section, we state some notation and basic facts about FHS sets, \(d\)-form functions with difference-balanced property and sequences with array structure, which will be used in the following sections.

2.1. Some notation fixed throughout this paper. For convenience, we adopt the following notation unless otherwise stated.

- \(q\) is a power of a prime \(p\);
- \(\mathbb{F}_q\) is the finite field of order \(q\);
- \(n, m\) are two integers with \(m|n\);
Table 1. The parameters of some existing LHZ FHS sets with optimal PPHC properties and our new ones.

| Parameters | Constraints | Reference |
|------------|-------------|-----------|
| \( (k_1L_1, M_1N_1, q, z_1, 1: W, \frac{m}{L_0} ) \) | \( \gcd(L_0 + 1, L_0) = 1, k_1L_1 + 1 = 1 (\text{mod} L_0) \) | [14] |
| \( (L_1, L_2, \rho_1, \rho_2, \rho_3, \min(L_1, L_2) - 1: W, \frac{m}{L_0} ) \) | \( \gcd(L_1, L_2) = 1, \rho_1(\frac{L_1}{\rho_1} - 1 + q)(\min(L_1, L_2) - 1) = \min(L_1, L_2) - 1, 0 \leq q \leq 1 \) | [24] |
| \( (\alpha(q^n - 1), s, q, t - 1, \omega \frac{m}{L_0} + \omega L_{m-1}^Z ) \) | \( s = q^n - 1, 2 \leq r_2 = \frac{\omega}{\gcd(\omega, r_1)} = t \text{ mod } t \) | [10] |
| \( (\{q - 1, \alpha(q^m - 1), q, q^m - 2, W, \frac{m}{L_0} ) \) | \( q_1, q_2 \) are two different prime powers satisfying \( \gcd(q_1, q_2 - 1) = 1, q_1 \geq q_2 \) | [28] |
| \( (\alpha(q^n - 1, 1: W, \frac{m}{L_0} ) \) | \( \alpha(q^n - 1, 1), q, q^n - 1 - 1: W, \frac{m}{L_0} ) \) | [28] |
| \( (\alpha(q^n - 1, p(q^n - 1), \alpha(q^n - 1, q^n - 1) - 1: W, \frac{m}{L_0} ) \) | \( \alpha(q^n - 1, p(q^n - 1), q^n - 1: W, \frac{m}{L_0} ) \) | [28] |

- \( \alpha \) is a primitive element of \( \mathbb{F}_q^n \);
- \( Q = \frac{T^n - 1}{q - 1} \);
- \( tr_{q^{m_1}}(x) = x + x^q + \cdots + x^{q^{m_1 - 1}} \) is the trace function from \( \mathbb{F}_{q^{m_1}} \) to \( \mathbb{F}_{q^{m_1 - 1}} \), where \( m_1, n_1 \) are two integers satisfying \( m_1 | n_1 \);
- \( \mathbf{0}_k = (0, \ldots, 0) \in \mathbb{F}_q^k \) is the zero vector over \( \mathbb{F}_q \) of length \( k \);
- \( a_0, a_1, \ldots, a_{k-1} \) are \( k \) elements in \( \mathbb{F}_q^m \) which are linearly independent over \( \mathbb{F}_q \), where \( 1 \leq k \leq m \);
- \( f: \mathbb{F}_q^n \rightarrow \mathbb{F}_q^m \), \( d \)-form function with difference-balanced property, where \( \gcd(q^m - 1, d) = 1 \);
- \( \lfloor h \rfloor \) denotes the smallest integer \( h \) satisfying \( h_2 \geq h_1 \);
- \( \lceil g \rceil \) denotes the largest integer \( g \) satisfying \( g_2 \leq g_1 \);
- \( \gcd(x, y) \) denotes the greatest common divisor of integers \( x \) and \( y \).

2.2. Bounds on the Hamming correlation of FHS set. In this subsection, we give some notation on FHS set and review some known lower bounds on the Hamming correlation of FHS set.

Let \( F = \{ f_0, f_1, \ldots, f_{r-1} \} \) be the frequency slot set of size \( r \). Let \( \mathcal{F} \) be the set of all FHSs of length \( L \) over \( F \). Any subset of \( \mathcal{F} \) is called an FHS set. Given any two sequences \( X = (x(t))^L_{t=0} \) and \( Y = (y(t))^L_{t=0} \), their periodic Hamming correlation (PHC) at time delay \( \tau \) is defined by

\[
H_{XY}(\tau) = \sum_{t=0}^{L-1} h[x(t), y(t+\tau)], \quad 0 \leq \tau \leq L - 1,
\]

where \( h[a, b] = 1 \) if \( a = b \) and 0 otherwise. And all the operations among the position indices are performed modulo \( L \).
Let $\mathcal{M}$ be a subset of $\mathcal{F}$ containing $M$ sequences. The maximum periodic Hamming autocorrelation $H_{am}(\mathcal{M})$, the maximum periodic Hamming cross-correlation $H_{cm}(\mathcal{M})$ and the maximum PHC $H_m(\mathcal{M})$ of $\mathcal{M}$ are defined as follows, respectively:

$$H_{am}(\mathcal{M}) = \max_{1 \leq \tau < L} \{H_{X,X}(\tau) : X \in \mathcal{M}\},$$

$$H_{cm}(\mathcal{M}) = \max_{0 \leq \tau < L} \{H_{X,Y}(\tau) : X,Y \in \mathcal{M}, X \neq Y\},$$

$$H_m(\mathcal{M}) = \max \{H_{am}(\mathcal{M}), H_{cm}(\mathcal{M})\}.$$

In 2004, Peng and Fan [21] established the following lower bound on the maximum PHC of FHS set.

**Lemma 2.1. (Peng-Fan Bound [21])** Let $\mathcal{M}$ be an FHS set of $M$ sequences of length $L$ over a frequency slot set of size $r$, then

$$H_m(\mathcal{M}) \geq \left\lceil \frac{(LM - r)L}{(LM - 1)r} \right\rceil. \quad (2)$$

Throughout this paper, we use $(L, M, r; H_m)$ to denote an FHS set containing $M$ FHSs of length $L$ over a frequency slot set of size $r$, with the maximum PHC $H_m$. And when the formula (2) in Lemma 2.1 is valid with equality, we say that it is an optimal FHS set.

Now we consider LHZ FHS set. Let $\mathcal{M}$ be a set of $M$ FHSs of length $L$ over the alphabet $F$ within LHZ $Z_H$ (called LHZ FHS set). The maximum periodic Hamming autocorrelation $H_{az}(\mathcal{M})$, the maximum periodic Hamming cross-correlation $H_{cz}(\mathcal{M})$ and the maximum PHC $H_{mz}(\mathcal{M})$ of $\mathcal{M}$ are defined as follows, respectively:

$$H_{az}(\mathcal{M}) = \max_{1 \leq \tau \leq Z_H} \{H_{X,X}(\tau) : X \in \mathcal{M}\},$$

$$H_{cz}(\mathcal{M}) = \max_{0 \leq \tau \leq Z_H} \{H_{X,Y}(\tau) : X,Y \in \mathcal{M}, X \neq Y\},$$

$$H_{mz}(\mathcal{M}) = \max \{H_{az}(\mathcal{M}), H_{cz}(\mathcal{M})\}. \quad (4)$$

In 2006, Peng et al. [22] established the following lower bound on the maximum PHC of LHZ FHS set.

**Lemma 2.2. (Peng-Fan-Lee Bound [22])** Let $\mathcal{M}$ be an LHZ FHS set with $M$ sequences of length $L$ over the frequency slot set with size $r$. Let $Z_H$ be the LHZ of $\mathcal{M}$ with regard to $H_{mz}(\mathcal{M})$. Then, we have

$$H_{mz}(\mathcal{M}) \geq \left\lceil \frac{(MZ_H + M - r)L}{(MZ_H + M - 1)r} \right\rceil. \quad (3)$$

It is clear that the bound (2) is just a special case of the bound (3) when $Z_H$ in the bound (3) equals $L - 1$. We use $(L, M, r, Z_H; H_{mz})$ to denote an LHZ FHS set containing $M$ FHSs of length $L$ over a frequency slot set of size $r$, with the maximum PHC $H_{mz}$ within the LHZ $Z_H$. And when the formula (3) in Lemma 2.2 is valid with equality, we say that it is an optimal LHZ FHS set.

The periodic partial Hamming correlation (PPHC) of two FHSs $X,Y \in \mathcal{M}$, for a correlation window length $W$ ($1 \leq W \leq L$) starting at $\omega$ ($0 \leq \omega \leq L - 1$) is defined by

$$H_{X,Y}(\omega|W; \tau) = \sum_{t=\omega}^{\omega+W-1} h[x(t), y(t+\tau)], \quad 0 \leq \tau \leq L - 1,$$
where \( h[a,b] = 1 \) if \( a = b \) and 0 otherwise. And all the operations among the position indices are performed modulo \( L \). In particular, when \( W = L \), the PPHC function defined in (1) is exactly the conventional periodic Hamming correlation defined in (1).

The maximum periodic partial Hamming autocorrelation \( H_{pa}(\mathcal{M};W) \) for a correlation window length \( W \), the maximum periodic partial Hamming cross-correlation \( H_{pc}(\mathcal{M};W) \) for a correlation window length \( W \) and the maximum PPHC \( H_{pm}(\mathcal{M};W) \) for a correlation window length \( W \) of \( \mathcal{M} \) are defined as follows, respectively:

\[
H_{pa}(\mathcal{M};W) = \max_{0 \leq \omega < L} \max_{1 \leq \tau < L} \{ H_{X,X}(\omega|W;\tau) : X \in \mathcal{M} \},
\]

\[
H_{pc}(\mathcal{M};W) = \max_{0 \leq \omega < L} \max_{0 \leq \tau < L} \{ H_{X,Y}(\omega|W;\tau) : X,Y \in \mathcal{M}, X \neq Y \},
\]

\[
H_{pm}(\mathcal{M};W) = \max \{H_{pa}(\mathcal{M};W), H_{pc}(\mathcal{M};W)\}.
\]

In 2012, Zhou et al. [30] established the following lower bounds on the maximum PPHC of FHS set for a correlation window length \( W \).

**Lemma 2.3.** ([30]) Let \( \mathcal{M} \) be an FHS set with \( M \) sequences of length \( L \) over the frequency slot set with size \( r \). Define \( I = \lfloor \frac{ML}{r} \rfloor \). Then for each window length \( 1 \leq W \leq L \), we have

\[
H_{pm}(\mathcal{M};W) \geq \left\lfloor \frac{W}{L} \cdot \frac{(LM - r)L}{(LM - 1)r} \right\rfloor,
\]

\[
H_{pm}(\mathcal{M};W) \geq \left\lfloor \frac{W}{L} \cdot \frac{2ILM - (I + 1)Ir}{(LM - 1)M - r} \right\rfloor.
\]

We use \((L,M,r,W;H_{pm})\) to denote an FHS set containing \( M \) FHSs of length \( L \) over a frequency slot set of size \( r \), with the maximum PPHC \( H_{pm} \) for a correlation window length \( W \). And when one of the bound in Lemma 2.3 is met for a given correlation window length \( W \) (\( 1 \leq W \leq L \)), we say that \( \mathcal{M} \) is an FHS set with optimal PPHC. Further, if one of the bound in Lemma 2.3 is met for arbitrary window length \( W \) (\( 1 \leq W \leq L \)), \( \mathcal{M} \) is said to be a strictly optimal FHS set.

Now we consider the PPHC of an LHZ FHS set. The maximum periodic partial Hamming autocorrelation \( H_{pa}(\mathcal{M};W) \) for a correlation window length \( W \), the maximum periodic partial Hamming cross-correlation \( H_{pc}(\mathcal{M};W) \) for a correlation window length \( W \) and the maximum PPHC \( H_{pmz}(\mathcal{M};W) \) for a correlation window length \( W \) within LHZ \( Z_H \) of \( \mathcal{M} \) are defined as follows, respectively:

\[
H_{pa}(\mathcal{M};W) = \max_{0 \leq \omega < L} \max_{1 \leq \tau \leq Z_H} \{ H_{X,X}(\omega|W;\tau) : X \in \mathcal{M} \},
\]

\[
H_{pc}(\mathcal{M};W) = \max_{0 \leq \omega < L} \max_{0 \leq \tau \leq Z_H} \{ H_{X,Y}(\omega|W;\tau) : X,Y \in \mathcal{M}, X \neq Y \},
\]

\[
H_{pmz}(\mathcal{M};W) = \max \{H_{pa}(\mathcal{M};W), H_{pc}(\mathcal{M};W)\}.
\]

In 2010, Niu et al. [16] derived the following lower bound on the maximum PPHC of LHZ FHS set.

**Lemma 2.4.** ([Niu-Peng-Liu-Liu [16]]) Let \( \mathcal{M} \) be an LHZ FHS set with \( M \) sequences of length \( L \) over the frequency slot set with size \( r \). Let \( Z_H \) be the LHZ of \( \mathcal{M} \) with regard to \( H_{pmz}(\mathcal{M};W) \) for a correlation window length \( W \) (\( 1 \leq W \leq L \)). Then, we have

\[
H_{pmz}(\mathcal{M};W) \geq \left\lfloor \frac{(MZ_H + M - r)W}{(MZ_H + M - 1)r} \right\rfloor.
\]
We use \((L, M, r, Z_H, W; H_{pmz})\) to denote an FHS set containing \(M\) FHSs of length \(L\) over a frequency slot set of size \(r\), with PPHC \(H_{pmz}\) within the low hit zone \(Z_H\) for a correlation window length \(W\). And when the formula (7) in Lemma 2.4 is valid with equality for a given correlation window length \(W\) \((1 \leq W \leq L)\), we say that \(\mathcal{M}\) is an \textbf{LHZ FHS set with optimal PPHC}. Further, if the formula (7) in Lemma 2.4 is met for arbitrary window length \(W\) \((1 \leq W \leq L)\), we say that \(\mathcal{M}\) is a \textbf{strictly optimal LHZ FHS set}.

2.3. \textit{d}-form functions with difference-balanced property. For the constructions of FHS sets in the sequel, we will give a brief introduction of \textit{d}-form functions with difference-balanced property in this subsection.

\textbf{Definition 2.5.} ([29]) Let \(n, m\) be two integers with \(m|n\) and \(d\) be an integer prime to \(q^m - 1\). A function \(f(x)\) from \(\mathbb{F}_{q^n}\) onto its subfield \(\mathbb{F}_{q^m}\) is called a \textit{d}-form function if

\[f(yx) = y^d f(x)\]

for any \(y \in \mathbb{F}_{q^m}\) and \(x \in \mathbb{F}_{q^n}\).

Note that \(d\)-form functions provide a rich source of functions with the following difference-balanced property.

\textbf{Definition 2.6.} ([29]) A function \(f(x)\) from \(\mathbb{F}_{q^n}\) onto \(\mathbb{F}_{q^m}\) is said to be difference-balanced, if for any \(\delta \in \mathbb{F}_{q^n} \setminus \{0, 1\}\), \(f(\delta x) - f(x)\) takes each nonzero element of \(\mathbb{F}_{q^m}\) \(q^n - m\) times and the zero element \(q^n - m - 1\) times, as \(x\) ranges over \(\mathbb{F}_{q^n}^*\).

So far, there are five types of \textit{d}-form functions from \(\mathbb{F}_{q^n}\) onto \(\mathbb{F}_{q^m}\) with difference-balanced property, which is described as follows.

(I): Functions which are surjective and \(\mathbb{F}_{q^m}\)-linear.

(II): Functions of the form

\[f(x) = \text{tr}_{q^n/q^m}(x^d),\]

where \(d\) is a positive integer prime to \(q^m - 1\).

(III): Functions of Helleseth-Gong type [7]

\[f(x) = \text{tr}_{q^n/q^m}(\sum_{i=0}^{N} u_i x^i (q^{2mi} + 1)^{i/2}),\]

where \(n = (2N + 1)m, 1 \leq s \leq 2N\) is an integer such that \(\gcd(s, 2N + 1) = 1, b_0 = u_0 = 1, b_i = (-1)^i\) and \(b_i = b_{2N+1-i}\) for \(i = 1, 2, \ldots, N, u_i = b_{2i}\) for \(i = 1, 2, \ldots, N, u_i = b_{2i}\) for \(i = 1, 2, \ldots, N\). Herein, all the indexes of \(b\) are taken mod \((2N + 1)\).

(IV): Functions of Lin type

\[f(x) = \text{tr}_{3^m/3}(x + x^s),\]

where \(q = 3, m = 2N + 1\) and \(s = 2 \times 3^N + 1\). The difference balance property of functions of this type was a conjecture of Lin [12] and proved by Hu et al. [8].

(V): Functions which are composites of functions of the previous types.

It is easy to see that \(f(x)\) in (I) is 1-form, \(f(x)\) in (II) is \(d\)-form, \(f(x)\) in (III) and (IV) is 1-form.

In this paper, the calculation of the Hamming correlation of the FHSs needs some equation systems over \(\mathbb{F}_q\) which are described in the following two Lemmas.
Identify $\mathbb{F}_{q^n}$ as the $m$-dimensional $\mathbb{F}_q$-vector space, then each element in $\mathbb{F}_{q^n}$ can be viewed as a vector over $\mathbb{F}_q$. Let $a_0, a_1, \ldots, a_{k-1}$ be $k$ elements in $\mathbb{F}_{q^n}$ which are linearly independent over $\mathbb{F}_q$, where $1 \leq k \leq m$, and $f$ be a $d$-form function from $\mathbb{F}_{q^n}$ to $\mathbb{F}_{q^m}$ with difference-balanced property, where $\gcd(d, q^m - 1) = 1$.

**Lemma 2.7.** ([30]) Let $b = (b_0, b_1, \ldots, b_{k-1}) \in \mathbb{F}_q^k$. For any $\delta \in \mathbb{F}_{q^n}\backslash\{0, 1\}$, the equation system

$$
\begin{align*}
\text{tr}_{q^m/q}(a_0(f(x) - f(\delta x))) &= b_0 \\
\text{tr}_{q^m/q}(a_1(f(x) - f(\delta x))) &= b_1 \\
& \vdots \\
\text{tr}_{q^m/q}(a_{k-1}(f(x) - f(\delta x))) &= b_{k-1}
\end{align*}
$$

has $q^n - k - 1$ solutions in $\mathbb{F}_{q^n}$ if $b = 0_k$, and $q^n - k$ solutions in $\mathbb{F}_{q^n}$ otherwise.

Let $l$ be a positive integer such that $l|(q^n - 1)$. Define $C_i^{(l,q^n)} = \alpha^i\langle\alpha^l\rangle$ for $i = 0, 1, \ldots, l-1$, where $\langle\alpha^l\rangle$ denotes the subgroup of $\mathbb{F}_{q^n}$ generated by $\alpha^l$. The cosets $C_i^{(l,q^n)} (0 \leq i < l)$ are called the cyclotomic classes of order $l$ in $\mathbb{F}_{q^n}$.

**Lemma 2.8.** ([30]) If $l|(q - 1)$ and $\gcd(l,n) = 1$. Let $C_0^{(l,q^n)}, \ldots, C_{l-1}^{(l,q^n)}$ be the cyclotomic classes of order $l$ in $\mathbb{F}_{q^n}$. Then for any $\delta \in \mathbb{F}_{q^n}\backslash\{0, 1\}$, the equation system

$$
\begin{align*}
\text{tr}_{q^m/q}(a_0(f(x) - f(\delta x))) &= 0 \\
\text{tr}_{q^m/q}(a_1(f(x) - f(\delta x))) &= 0 \\
& \vdots \\
\text{tr}_{q^m/q}(a_{k-1}(f(x) - f(\delta x))) &= 0
\end{align*}
$$

has $q^{n-k-1}$ solutions in each $C_i^{(l,q^n)}, 0 \leq i < l$.

### 2.4. Sequences with Array Structure

In [30], Zhou et al. generalized array structure to sequences over the vector space $\mathbb{F}_q^k$ and gave the definition as follows.

**Definition 2.9.** ([30]) Let $V = \{v(t)\}$ be a sequence of period $L = sQ$ over $\mathbb{F}_q^k$. $V$ is said to have array structure, if for any integer $1 \leq i \leq s - 1$, there exists $e_i \in \mathbb{F}_q$ such that

$$v(t + iQ) = e_i v(t) \text{ for } 0 \leq t \leq Q - 1,$$

where $e_i v(t)$ denotes the scalar multiplication of $e_i$ with $v(t)$.

In terms of the array structure property, we can arrange $V$ into the following array representation:

$$V = \begin{pmatrix}
\vdots & \vdots & \vdots \\
v(0) & v(1) & \cdots & v(Q - 1) \\
v(Q) & v(Q + 1) & \cdots & v(2Q - 1) \\
& \vdots & \vdots & \ddots & \vdots \\
v((s - 1)Q) & v((s - 1)Q + 1) & \cdots & v(sQ - 1) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
e_1 v(0) & e_1 v(1) & \cdots & e_1 v(Q - 1) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
e_{s-1} v(0) & e_{s-1} v(1) & \cdots & e_{s-1} v(Q - 1)
\end{pmatrix}.$$
3. The first construction of LHZ FHS sets

In this section, we shall present the first construction of LHZ FHS sets using \textit{d}-form functions with difference-balanced property. In this construction we generate some optimal sets of LHZ FHSs with new parameters, and make corrections (see Remark 4) to Theorems 1 and 2 of [27].

Let \( l \) and \( T \) be two positive integers with \( l|\( q^n - 1 \) \), \( gcd(l, n) = 1 \), \( T|\( q^n - 1 \) \), \( T \not| l \) and \( gcd(T, l) = d' \). Let \( \alpha \) be a primitive element of \( F_{q^n} \). Choose \( k \) elements \( a_0, a_1, \ldots, a_{k-1} \) in \( F_{q^n} \) which are linearly independent over \( F_q \), where \( 1 \leq k \leq m \), and \( f \) be a \( d' \)-form function from \( F_{q^n} \) to \( F_{q^n} \) with difference-balanced property, where \( gcd(d, q^n - 1) = 1 \). Define \( \beta_i = \alpha^{iT} \), \( i = 0, 1, \ldots, q^n-1 - 1 \). The first construction of FHS sets is defined by the sequence set

\[
S = \left\{ S_i : 0 \leq i < \frac{q^n - 1}{t} \right\}
\]

over \( F_q^k \), where \( S_i = \{s_i(t)\}_{t=0}^{\frac{q^n - 1}{t} - 1} \) and

\[
s_i(t) = (tr_{q^m/q}(a_0f(\beta_i\alpha^{It})), tr_{q^m/q}(a_1f(\beta_i\alpha^{It})), \ldots, tr_{q^m/q}(a_{k-1}f(\beta_i\alpha^{It}))).
\]

From the first construction, we give two classes of LHZ FHS sets with optimal PPHC in Theorems 3.1 and 3.3 and a class of strictly optimal LHZ FHS sets in Theorem 3.2. Meanwhile, we also give a class of optimal LHZ FHS sets in Corollary 1. In addition, we make corrections to Theorems 1 and 2 of [27] in Remark 4.

**Theorem 3.1.** Let \( l \) and \( T \) be two positive integers with \( l|\( q^n - 1 \) \), \( gcd(l, n) = 1 \), \( T|\( q^n - 1 \) \), \( T \not| l \) and \( gcd(T, l) = d' \). Then the FHS set \( S \) defined in (10) is a \((\frac{q^n-1}{t}, \frac{q^n-1}{q}, \frac{T}{d'} - 1, s\frac{q^n-1}{q}, \frac{q^n-k}{q-k})\) LHZ FHS set with optimal PPHC, where \( s \) is an integer satisfying \( 1 \leq s \leq \frac{t-1}{k} \).

**Proof.** It is clear that the sequence length of \( S \) is \( \frac{q^n-1}{t} \), the family size of \( S \) is \( \frac{q^n-1}{t} \) and the size of frequency slot set of \( S \) is \( q^k \). For each \( S_i, S_j \in S \) with \( 0 \leq i, j < \frac{q^n-1}{t} \), their periodic Hamming correlation (PHC) at time delay \( \tau \) with \( 0 \leq \tau < \frac{q^n-1}{t} \) can be defined as follows:

\[
H_{S_i, S_j}(\tau) = \left\{ t \left| s_j(t + \tau) - s_i(t) = 0_k, 0 \leq t < \frac{q^n-1}{l} \right. \right\}.
\]

By (11), we know that the PHC \( H_{S_i, S_j}(\tau) \) is equal to the number of solutions \( 0 \leq t < \frac{q^n-1}{l} \) to the equation system

\[
\begin{align*}
tr_{q^m/q} [a_0f(\delta_\tau \alpha^{It}) - f(\beta_i\alpha^{It})] &= 0 \\
tr_{q^m/q} [a_1f(\delta_\tau \alpha^{It}) - f(\beta_i\alpha^{It})] &= 0 \\
&\cdots \\
tr_{q^m/q} [a_{k-1}f(\delta_\tau \alpha^{It}) - f(\beta_i\alpha^{It})] &= 0
\end{align*}
\]

where \( \delta = \beta_{j-i} \alpha^{\tau t}, 0 \leq i, j < \frac{q^n-1}{t} \) and \( 0 \leq \tau < \frac{q^n-1}{t} \). Obviously, \( \delta \neq 0 \). From Lemma 2.8 we have

\[
H_{S_i, S_j}(\tau) = \begin{cases} \\
\frac{q^n-1}{t}, & \text{if } \delta = 1; \\
\frac{q^n-k-1}{t}, & \text{if } \delta \neq 1.
\end{cases}
\]
Moreover, we have, for any $0 \leq \tau < \frac{q^n - 1}{T}$ and $0 \leq i, j < \frac{q^n - 1}{T}$,

$$H_{S_i, S_j} = \frac{q^{n-k} - 1}{l}.$$  

With the parameters $i, j$ and $\tau$, we give the following two cases to calculate $H_{S_i, S_j}(\tau)$.

(I) **Case 1.** $0 \leq i, j < \frac{q^n - 1}{T}$, $i = j$ and $0 < \tau < \frac{q^n - 1}{T}$.

In this case, $\tau l \equiv 0 \pmod{(q^n - 1)}$ and $\delta \neq 1$. Thus, from (12) we have, for any $0 \leq i < \frac{q^n - 1}{T}$ and $0 < \tau < \frac{q^n - 1}{T}$,

$$H_{S_i, S_i}(\tau) = \frac{q^{n-k} - 1}{l}.$$  

(II) **Case 2.** $0 \leq i, j < \frac{q^n - 1}{T}$, $i \neq j$ and $0 \leq \tau < \frac{q^n - 1}{T}$.

In this case, let $\gcd(T_1, l) = d$, $T = T_1d'$ and $l = l_1d'$, then $\gcd(T_1, l_1) = 1$. Let $\tau = \tau_1T_1 + \tau_2$, $0 \leq \tau_1 < \frac{q^n - 1}{T_1}$ and $0 \leq \tau_2 < T_1$, then we have

$$H_{S_i, S_j}(\tau) = \begin{cases} 
\frac{q^{n-k} - 1}{l}, & \text{if } \tau = \frac{T}{d} - 1, \\
\frac{q^{n-k} - 1}{q-k}, & \text{otherwise.}
\end{cases}$$

Note that when $\tau_1 = 0$, there is no $i \neq j$ such that $j - i \equiv 0 \pmod{\frac{q^n - 1}{T}}$, and when $\tau_1 = 1$, there exits $i \neq j$ such that $(j - i) + l_1 \equiv 0 \pmod{\frac{q^n - 1}{T}}$. As a result, for any $0 \leq i, j < \frac{q^n - 1}{T}$,

$$H_{S_i, S_j}(\tau) = \frac{q^{n-k} - 1}{l}, \text{ if } 0 \leq \tau \leq \frac{T}{d} - 1.$$  

And there exists $0 \leq i', j' < \frac{q^n - 1}{T}$ and $i' \neq j'$ such that

$$H_{S_i, S_j}(\tau) = \begin{cases} 
\frac{q^{n-k} - 1}{l}, & \text{if } \tau = \frac{T}{d}, \\
\frac{q^{n-k} - 1}{q-k}, & \text{otherwise.}
\end{cases}$$

Combining the above case 1 and case 2, we have

$$H_{S_i, S_j}(\tau) = \frac{q^{n-k} - 1}{l}$$

for any $0 \leq i, j < \frac{q^n - 1}{T}$, $0 \leq \tau \leq \frac{T}{d} - 1$ if $i = j$ and for any $0 \leq i, j < \frac{q^n - 1}{T}$, $0 \leq \tau \leq \frac{T}{d} - 1$ if $i \neq j$.

Let $Q = \frac{q^n - 1}{q-1}$. Define

$$U_{i,j}(t) = S_j(t + \tau) - S_i(t),$$

where $0 \leq t < \frac{q^n - 1}{T}$, $0 \leq i, j < \frac{q^n - 1}{T}$, $0 \leq \tau \leq \frac{T}{d} - 1$ if $i = j$ and $0 \leq \tau \leq \frac{T}{d} - 1$ if $i \neq j$. Since for each $1 \leq s \leq \frac{q^n - 1}{q-1}$, there exists $e_s = \alpha^{i\omega s}Q \in \mathbb{F}_q$ such that

$$U_{i,j}(t + sQ) = e_sU_{i,j}(t).$$

Thus, we have $\{U_{i,j}(t)\}$ is a sequence of length $\frac{q^n - 1}{l}$ over $\mathbb{F}_q^k$ with array structure. By Lemma 2.8, we know that there are $\frac{q^{n-k} - 1}{q-1}$ 0k’s in $\{U_{i,j}(t)\}$. From the array representation of $\{U_{i,j}(t)\}$, we know every segment of $sQ$ consecutive elements from $\{U_{i,j}(t)\}$ contains 0k exactly $s\frac{q^{n-k} - 1}{q-1}$ times. Thus, for $0 \leq i, j < \frac{q^n - 1}{T}$, $0 \leq \omega < \frac{q^n - 1}{T}$, $0 \leq \tau \leq \frac{T}{d} - 1$ if $i = j$ and $0 \leq \tau \leq \frac{T}{d} - 1$ if $i \neq j$, the PPHC $H_{S_i, S_j}(\omega|sQ; \tau)$
is equal to the maximal number of $\theta_i$'s in segment of $sQ$ consecutive elements from \{${U_{ij}(t)}$\}. As a result, for any $0 \leq i, j < \frac{q^n-1}{T}$ and $0 \leq \omega < \frac{q^n-1}{T}$, we have

$$H_{S_i,S_j}(\omega|sQ;\tau) \leq \frac{s^{q^n-1} - 1}{q-1},$$

where $0 < \tau \leq \frac{T}{d} - 1$ if $i = j$ and $0 \leq \tau \leq \frac{T}{d} - 1$ if $i \neq j$. Therefore, $S$ is an LHZ FHS set with parameters $\left(\frac{q^n-1}{l}, \frac{q^n-1}{l}, q^k, \frac{T}{d} - 1, s\frac{q^n-1}{q-1}; s\frac{q^{n-k}-1}{q-1}\right)$.

Put the parameters into bound (7), we have

$$H_{pmz}(S; sQ) \geq s\frac{q^{n-k}-1}{q-1}.$$

Since $s\frac{q^{n-k}-1}{q-1}$ is a positive integer and

$$0 < \frac{s(d' - 1)q^{n+k} - s(d' - 1)q^n + sq^k - s}{(q-1)q^{n+k} - (d' + 1)(q-1)q^k} < 1,$$

we have

$$H_{pmz}(S; sQ) \geq s\frac{q^{n-k}-1}{q-1}.$$  

By Lemma 2.4, formulas (14) and (15), $S$ is an LHZ FHS set with optimal PPHC.

This completes the proof. \qed

**Remark 1.** In Theorem 3.1, let $m = n$, $f(x) = x$, $T = \frac{q^n-1}{q-1}$ and $a_i = \beta \alpha^i (\beta \in \mathbb{F}_q^n, 0 \leq i < k)$, then $S$ is exactly

$$\left(\frac{q^n-1}{l}, q-1, q^k, \frac{q^n-1}{q-1} - 1, s\frac{q^n-1}{q-1}; s\frac{q^{n-k}-1}{q-1}\right)$$

LHZ FHS set with optimal PPHC constructed by Han et al. in [9].

**Corollary 1.** Let $l$ and $T$ be two positive integers with $l||(q-1)$, gcd($l, n$) = 1, $T|(q^n-1)$, $T \nmid l$ and gcd($T, l$) = $d'$. Then the FHS set $S$ defined in (10) is an optimal LH FHS set with parameters $\left(\frac{q^n-1}{l}, \frac{q^n-1}{l}, q^k, \frac{T}{d'} - 1; \frac{q^n-1}{l}\right)$.

**Proof.** The result can be deduced by the special case $s = \frac{q^n}{l}$ from that in Theorem 3.1. \qed

**Remark 2.** In Corollary 1, let gcd($T, l$) = $d'$, $T = T'd'$, $l = l'd'$ and $T = \frac{q^n-1}{l}$, the FHS set $S$ defined in (10) is an optimal FHS set with parameters

$$\left(\frac{q^n-1}{l}, l_1, q^k, \frac{q^{n-k}-1}{l}\right).$$

- Specially, let $T = \frac{q^n-1}{l}$ and gcd($\frac{q^n-1}{l}, l$) = 1, then \{$\alpha^T : 0 \leq i < l$\} is a complete set of representatives for the cyclotomic classes of order $l$ in $\mathbb{F}_q^n$, and $S$ is exactly the optimal FHS set with parameters $\left(\frac{q^n-1}{l}, l, q^k, \frac{q^{n-k}-1}{l}\right)$ constructed by Zhou et al. in [29].
- Specially, if we let $l = q - 1$, $T = \frac{q^n}{q-1}$ and $k = 1$, then the FHS set $S$ is an optimal FHS set with parameters $\left(\frac{q^n}{q-1}, q-1, q; \frac{q^{n-k}-1}{q-1}\right)$ presented by Ding et al. [3].
Example 1. Let $q = 7, n = 2, m = 1, k = 1$, and $\alpha$ be a primitive element of $\mathbb{F}_7^*$ satisfying $\alpha^2 + 6\alpha + 3 = 0$. Let $f(x) = tr_{7/2}(x^5), T = 12, l = 3$, then $d' = \gcd(T, l) = 3$. Let $a_0 = 1$, then $S$ has 4 FHSs of length 16:

$S_0 = \{2, 6, 3, 3, 0, 3, 4, 6, 5, 1, 4, 4, 0, 4, 3, 1\}$;
$S_1 = \{0, 3, 4, 6, 5, 1, 4, 4, 0, 4, 3, 1, 2, 6, 3, 3\}$;
$S_2 = \{5, 1, 4, 4, 0, 4, 3, 1, 2, 6, 3, 3, 0, 3, 4, 6\}$;
$S_3 = \{0, 4, 3, 1, 2, 6, 3, 3, 0, 3, 4, 6, 5, 1, 4, 4\}$.

By computer experiments, for any $S_i, S_j \in S, 0 \leq i, j < 4, 0 < \tau \leq 3$ if $i = j$ and $0 \leq \tau \leq 3$ if $i \neq j$,

$$H_{S_i, S_j}(\tau) = 2.$$  

Thus, we have $S$ is an optimal LHZ FHS set with parameters $(16, 4, 7, 3, 2)$.

Theorem 3.2. Let $l$ and $T$ be two positive integers with $l|(q - 1), \gcd(l, n) = 1, T \not| (q^n - 1), T \equiv l$ and $d' = \gcd(T, l)$. If $d' = 1$ and $k = n - 1$, then the FHS set $S$ defined in (10) is a strictly optimal LHZ FHS set with parameters

$$\left(\frac{q^n - 1}{T}, \frac{q^n - 1}{T}, q^n, T - 1, W; \left\lceil \frac{W(q - 1)}{q^n - 1} \right\rceil \right),$$

where $W$ is any positive integer with $1 \leq W \leq \frac{q^n - 1}{T}$.

Proof. As in the proof of Theorem 3.1, we have that for any $0 \leq i, j < \frac{q^n - 1}{T}, \{U_{i,j}(t)\}$ is a sequence of length $\frac{q^n - 1}{T}$ over $\mathbb{F}_q$ with array structure. By Lemma 2.8, we know that there are $\frac{q^n - 1}{T}$ $\mathbf{0}_k$'s in $\{U_{i,j}(t)\}$ if $k = n - 1$. From the array representation of $\{U_{i,j}(t)\}$, we know every segment of $Q$ consecutive elements from $\{U_{i,j}(t)\}$ contains $\mathbf{0}_k$'s exactly one time. Thus, for $0 \leq i, j < \frac{q^n - 1}{T}, 0 \leq \omega < \frac{q^n - 1}{T}, 1 \leq W \leq \frac{q^n - 1}{T}, 0 < \tau \leq \frac{T}{\sigma} - 1$ if $i = j$ and $0 \leq \tau \leq \frac{T}{\sigma} - 1$ if $i \neq j$, the maximum PPHC $H_{S_i, S_j}(\omega|W; \tau)$ is equal to the maximal number of $\mathbf{0}_k$'s in segment of $W$ consecutive elements from $\{U_{i,j}(t)\}$. As a result, we have for any $0 \leq \omega < \frac{q^n - 1}{T}$ and $1 \leq W \leq \frac{q^n - 1}{T}$,

$$H_{S_i, S_j}(\omega|W; \tau) = \left\lceil \frac{W(q - 1)}{q^n - 1} \right\rceil,$$

where $0 \leq i, j < \frac{q^n - 1}{T}, 0 < \tau \leq \frac{T}{\sigma} - 1$ if $i = j$ and $0 \leq \tau \leq \frac{T}{\sigma} - 1$ if $i \neq j$. So $S$ is an LHZ FHS set with parameters

$$\left(\frac{q^n - 1}{T}, \frac{q^n - 1}{T}, q^n, \frac{T}{\sigma} - 1, W; \left\lceil \frac{W(q - 1)}{q^n - 1} \right\rceil \right).$$

Put the parameters into bound (7), we have

$$H_{pm_z}(S; W) \geq \frac{W(q - 1)}{q^n - 1} - \frac{[(d' - 1)q^n + 1](q^{n - 1} - 1)W}{(q^n - 1 - d')q^{n - 1}(q^n - 1)}.$$

Since $d' = 1$, we have

$$\frac{q - 1}{q^n - 1} > \frac{[(d' - 1)q^n + 1](q^{n - 1} - 1)W}{(q^n - 1 - d')q^{n - 1}(q^n - 1)}$$

for any $1 \leq W \leq \frac{q^n - 1}{T}$. From (17) we have when $d' = 1$,

$$H_{pm_z}(S; W) \geq \left\lceil \frac{W(q - 1)}{q^n - 1} \right\rceil.$$

By Lemma 2.4, formulas (16) and (18), $S$ is a strictly optimal LHZ FHS set. This completes the proof.
Remark 3.
- In Theorem 3.2, if we let $T = \frac{q^n-1}{l}$ and $gcd(\frac{q^n-1}{l}, l) = 1$, we get the strictly optimal FHS set with parameters $(\frac{q^n-1}{l}, l, q^{n-1}, W; \left\lfloor \frac{W(q^n-1)}{q^n-1} \right\rfloor)$ produced by Zhou et al. in [30].
- In Theorem 3.2, if we choose $q = p^m, n = 2, k = 1, T = \frac{p^{2m}-1}{l}$ and $gcd(\frac{p^{2m}-1}{l}, l) = 1$, we get the strictly optimal FHS set with parameters
\[
(\frac{p^{2m}-1}{l}, l, p^m, W; \left\lfloor \frac{W}{p^m+1} \right\rfloor)
\]
presented by Ouyang et al. in [20].
- In Theorem 3.2, let $q = p, m = n, k = m-1, T = \frac{p^{m}-1}{l}$ with $gcd(\frac{p^{m}-1}{l}, l) = 1$, then we get the strictly optimal FHS set with parameters
\[
(\frac{p^{m}-1}{l}, l, p^{m-1}, W; \left\lfloor \frac{W(p-1)}{p^m-1} \right\rfloor)
\]
presented by Ouyang et al. in [20].

Example 2. Let $q = 7, n = 2, m = 1, k = 1$, and $\alpha$ be a primitive element of $\mathbb{F}_{7^2}$ satisfying $\alpha^2 + 6\alpha + 3 = 0$. Let $f(x) = tr_{2/7}(x^5), T = 8, l = 3$, then $d' = gcd(T, l) = 1$. Let $a_0 = 1$, then $S$ has 6 FHSs of length 16:
- $S_0 = \{2, 3, 4, 1, 0, 1, 3, 5, 4, 3, 6, 0, 6, 4, 4\}$;
- $S_1 = \{3, 1, 6, 5, 0, 5, 1, 1, 4, 6, 1, 2, 0, 2, 6, 6\}$;
- $S_2 = \{1, 5, 2, 4, 0, 4, 5, 5, 6, 2, 5, 3, 0, 3, 2, 2\}$;
- $S_3 = \{5, 4, 3, 6, 0, 6, 4, 4, 2, 3, 4, 1, 0, 1, 3, 3\}$;
- $S_4 = \{4, 6, 1, 2, 0, 2, 6, 6, 3, 1, 6, 5, 0, 5, 1, 1\}$;
- $S_5 = \{6, 2, 5, 3, 0, 3, 2, 2, 1, 5, 2, 4, 0, 4, 5, 5\}$.

By computer experiments, for any $S_i, S_j \in S, 0 \leq i, j < 6, 0 \leq \omega < 16, 0 < \tau \leq 7$ if $i = j$ and $0 \leq \tau \leq 7$ if $i \neq j$,
\[
H_{S_i, S_j}(\omega|W; \tau) = \begin{cases} 
1, & 1 \leq W \leq 8; \\
2, & 9 \leq W \leq 16.
\end{cases}
\]

Thus, we have $S$ is a strictly optimal LHZ FHS set with parameters $(16, 6, 7, 7, W; \left\lfloor \frac{W}{8} \right\rfloor)$.

Remark 4. Let $m = n, f(x) = x$ and $a_i = \beta \alpha^i, \beta \in \mathbb{F}_{q^n}^*, 0 \leq i < k$, then the first construction in this paper is the same with Construction 1 in [27] based on the decimated $m$-sequences. In [27], Zhou et al. obtained the optimal LHZ FHS set with parameters
\[
(\frac{q^n-1}{l}, \frac{q^n-1}{l}, q^k, T-1, \frac{q^{n-k}-1}{l})
\]
in Theorem 1 and the strictly optimal LHZ FHS set with parameters
\[
(\frac{q^n-1}{l}, \frac{q^n-1}{l}, q^{n-1}, T-1, W, \left\lfloor \frac{W(q^n-1)}{q^n-1} \right\rfloor)
\]
in Theorem 2, where $T'(q^n-1)$. However, we have to point out that the results of Theorem 1 and Theorem 2 in [27] lack of the necessary condition $gcd(T, l) = 1$. The following Example is the counterexample.
Example 3. Let $q = 5, n = 3$, and $\alpha$ be the primitive element of $\mathbb{F}_5^*$ with $\alpha^3 = 4\alpha^2 + 3$. Then we can get an 5-ary $m$-sequence $s$ of degree 3 as:

$$s = \{3, 0, 4, 1, 3, 0, 3, 1, 1, 3, 4, 3, 4, 4, 1, 1, 0, 4, 2, 3, 2, 0, 4, 0, 3, 3, 1, 2, 3, 1, 0, 3, 2, 1, 0, 1, 2, 2, 1, 3, 1, 3, 3, 3, 2, 2, 0, 4, 1, 4, 0, 0, 3, 1, 1, 2, 4, 1, 2, 0, 1, 4, 2, 0, 2, 4, 4, 2, 1, 2, 1, 1, 4, 4, 0, 1, 3, 2, 3, 0, 0, 1, 0, 2, 2, 4, 3, 2, 4, 0, 2, 3, 4, 0, 4, 3, 4, 2, 4, 2, 2, 2, 3, 0, 2, 1, 4, 1, 0, 0, 2, 0, 4, 3, 1, 4\}.$$

If we choose $l = 2, T = 62, k = n - 1 = 2$ and $\gcd(T, l) = 2$. By the Construction 1 in [27] we can get an FHS set $\mathcal{S} = \{S_i = \{s_i(t), 0 \leq t < 62\}, i = 0, 1\}$ as follows:

$$S_0 = \{(3, 4), (4, 3), (3, 3), (3, 1), (1, 4), (4, 4), (4, 4), (4, 1), (1, 4), (4, 3), (3, 0), (0, 4), (4, 3), (3, 1), (1, 3), (3, 0), (0, 2), (2, 0), (0, 2), (2, 1), (1, 1), (1, 3), (3, 2), (2, 0), (0, 4), (4, 4), (4, 0), (0, 0), (0, 1), (1, 4), (4, 2), (2, 1), (1, 2), (2, 2), (2, 4), (4, 1), (1, 1), (1, 1), (1, 4), (4, 1), (1, 2), (2, 0), (0, 1), (1, 2), (2, 4), (4, 2), (2, 0), (0, 3), (3, 0), (0, 3), (3, 4), (4, 4), (4, 2), (2, 3), (3, 0), (0, 1), (1, 1), (1, 0), (0, 0), (0, 4), (4, 1), (1, 3), (3, 4), (4, 3), (3, 3), (3, 1), (1, 4), (4, 4), (4, 1), (1, 4), (4, 3), (3, 1), (0, 4), (4, 3), (3, 1), (1, 3), (3, 0), (0, 2), (2, 0), (0, 2), (2, 1), (1, 1), (1, 3), (3, 2), (2, 0), (0, 4), (4, 4), (4, 0), (0, 0), (0, 4), (4, 2)\}.$$

By computer experiments, for any $S_i, S_j \in \mathcal{S}, 0 \leq i, j < 2, 0 < \tau \leq 31$ if $i = j$ and $0 \leq \tau \leq 31$ if $i \neq j$,

$$H_{S_i, S_j}(\tau) = 2.$$

Moreover, we also have $H_{S_i, S_j}(32) = 62$. Thus, $\mathcal{S}$ is an optimal LHZ FHS set with parameters $(62, 2, 25, 31; 2)$, but not an optimal LHZ FHS set with parameters $(62, 2, 25, 61; 2)$ (i.e., optimal FHS set with parameters $(62, 2, 25; 2)$) of Theorem 1 in [27]. By computer experiments, for any $S_i, S_j \in \mathcal{S}, 0 \leq i, j < 2, 0 \leq \omega < 62, 0 < \tau \leq 31$ if $i = j$ and $0 \leq \tau \leq 31$ if $i \neq j$, we have

$$H_{S_i, S_j}(\omega|W; \tau) = \begin{cases} 1, & 1 \leq W \leq 31; \\ 2, & 32 \leq W \leq 62. \end{cases}$$

Thus, $\mathcal{S}$ is an LHZ FHS set with parameters $(62, 2, 25, 31, W; [\frac{W}{32}])$, but not a strictly optimal LHZ FHS set with parameters $(62, 2, 25, 61, W; [\frac{W}{32}])$ (i.e., strictly optimal FHS set with parameters $(62, 2, 25, W; [\frac{W}{32}])$ of Theorem 2 in [27].

**Theorem 3.3.** Let $T$ be a positive integer with $T|(q^n - 1)$, and let $l = q - 1$, $\gcd(l, n) = 1$, $T \nmid l$ and $d' = \gcd(T, l)$. Let $k = n - 1$, then the FHS set $\mathcal{S}$ defined in
Example 4. Let $q = 3, n = m = 5, k = 4$ and $\omega$ be a primitive element of $F_{3^5}$ satisfying $\omega^5 + 2\omega + 1 = 0$. Let $f(x) = x, T = 22, \omega = q - 1 = 2, d’ = gcd(T, l) = 2$. Let $(a_0, a_1, a_2, a_3) = (1, \alpha, \alpha^2, \alpha^3)$, then $S$ has $11$ FHSs of length $121$:

$S_0 = \{(2, 0, 0, 0), (0, 0, 1, 0), (1, 1, 0, 1), (0, 0, 1, 0), (1, 0, 2, 0), (2, 0, 1, 2), \cdots\};$

$S_1 = \{(2, 1, 0), (1, 0, 2, 0), (2, 0, 2, 0), (2, 2, 2, 0), (2, 1, 2, 0), (2, 0, 2, 0), \cdots\};$

$S_2 = \{(0, 1, 2, 0), (2, 0, 1, 1), (1, 1, 1, 0), (1, 0, 0, 0), (0, 0, 0, 2), \cdots\};$

$S_3 = \{(2, 0, 2, 0), (0, 2, 0, 0), (0, 0, 1, 2), (1, 2, 1, 0), (1, 0, 1, 1), (1, 1, 2, 0), \cdots\};$

$S_4 = \{(2, 0, 2, 1), (2, 1, 0, 1), (0, 1, 2, 0), (2, 2, 2, 0), (2, 1, 2, 0), (1, 0, 0, 2), \cdots\};$

$S_5 = \{(2, 1, 2, 1), (2, 1, 0, 2), (0, 2, 1, 2), (1, 2, 2, 1), (2, 2, 2, 1), (2, 1, 0, 0), \cdots\};$

$S_6 = \{(0, 2, 1, 1), (1, 1, 2, 0), (2, 2, 2, 0), (2, 0, 1, 0), (1, 0, 1, 2), (0, 1, 1, 2), \cdots\};$

$S_7 = \{(0, 0, 1, 2), (1, 2, 2, 0), (2, 0, 1, 1), (1, 1, 0, 1), (0, 1, 0, 1), (1, 0, 1, 2), \cdots\};$

$S_8 = \{(0, 0, 1, 0), (1, 0, 0, 0), (0, 0, 1, 2), (1, 2, 0, 0), (0, 0, 1, 1), (1, 1, 1, 0), \cdots\};$

$S_9 = \{(2, 1, 1, 2), (1, 2, 0, 2), (0, 2, 0, 1), (0, 1, 1, 1), (1, 2, 1, 0), (1, 1, 0, 1), \cdots\};$

$S_{10} = \{(0, 0, 1, 0), (1, 0, 1, 0), (1, 0, 1, 2), (1, 2, 1, 0), (1, 2, 1, 0), (1, 1, 2, 1), \cdots\};$

By computer experiments, for any $S_i, S_j \in S, 0 \leq i, j < 11, 0 \leq \omega < 121, 1 \leq W \leq 121, 0 < \tau \leq 10$ if $i = j$ and $0 \leq \tau \leq 10$ if $i \neq j$,

$$H_{S_i,S_j}(\omega|W;\tau) = 1.$$

Thus, we have $S$ is a $(121, 11, 81, 10, W; 1)$ LHZ FHS set.
4. The second construction of LHZ FHS sets

In this section, we present another construction of optimal LHZ FHS sets using d-form functions with difference-balanced property and give some new FHS sets.

As before, Let $T$ be a positive integer with $T | (q^n - 1)$ and $a$ be a primitive element of $\mathbb{F}_{q^n}$. Choose $k$ elements $a_0, a_1, \ldots, a_{k-1}$ in $\mathbb{F}_{q^m}$ which are linearly independent over $\mathbb{F}_q$, where $1 \leq k \leq m$, and $f$ be a $d$-form function from $\mathbb{F}_{q^n}$ to $\mathbb{F}_{q^m}$ with difference-balanced property, where $gcd(d, q^m - 1) = 1$. Define $\beta_i = \alpha^i T$, $i = 0, 1, \ldots, q^m - 1$. Let $\mathcal{V} = \{V_i = (v_i(0), v_i(1), \ldots, v_i(k-1)) : 0 \leq i < q^k\}$ be the set of all vectors of length $k$ over $\mathbb{F}_q$. The second construction of FHS sets is defined by the sequence set

$$S = \left\{ S_i : 0 \leq i < \frac{(q^n - 1)q^k}{T} \right\}$$

over $\mathbb{F}_q^k$, where $S_i = \{s_i(t)\}_{t=0}^{q^m-1}$ and

$$s_i(t) = \left(tr_{q^m/q}(a_0 f(\beta_i \alpha^t)) + v_{i_2}(0), \ldots, tr_{q^m/q}(a_{k-1} f(\beta_i \alpha^t)) + v_{i_2}(k-1)\right),$$

$$0 \leq i_1 < \frac{q^n}{T} - 1, 0 \leq i_2 < q^k, i = i_1 q^k + i_2.$$

**Theorem 4.1.** With the symbols above, the FHS set $S$ defined in (22) is an optimal LHZ FHS set with parameters $\left(q^n - 1, \frac{(q^n - 1)q^k}{T}, q^k, T - 1; q^{n-k}\right)$.

**Proof.** Clearly, $S$ contains $\frac{(q^n - 1)q^k}{T}$ sequences of length $q^n - 1$ over $\mathbb{F}_q^k$. For any $S_i, S_j \in S$ with $0 \leq i, j < \frac{(q^n - 1)q^k}{T}$, their periodic Hamming correlation $H_{S_i, S_j}(\tau)$ with $0 \leq \tau < \frac{q^n - 1}{T}$ is equal to the number of solutions $0 \leq t < q^n - 1$ to the equation system

$$\begin{align*}
tr_{q^m/q} \left[a_0 (f(\beta_i \alpha^t) - f(\beta_j \alpha^t))\right] &= v_{i_2}(0) - v_{j_2}(0) \\
tr_{q^m/q} \left[a_1 (f(\beta_i \alpha^t) - f(\beta_j \alpha^t))\right] &= v_{i_2}(1) - v_{j_2}(1) \\
\ldots \\
tr_{q^m/q} \left[a_{k-1} (f(\beta_i \alpha^t) - f(\beta_j \alpha^t))\right] &= v_{i_2}(k-1) - v_{j_2}(k-1)
\end{align*}$$

where $\delta = \beta_{j_1-i_1} \alpha^\tau$. Obviously, $\delta \neq 0$. From Lemma 2.7 and formula (23), it is easy to see that

$$H_{S_i, S_j}(\tau) = \begin{cases} q^n - 1, & \text{if } \delta = 1 \text{ and } V_{i_2} = V_{j_2}; \\
0, & \text{if } \delta = 1 \text{ and } V_{i_2} \neq V_{j_2}; \\
q^{n-k} - 1, & \text{if } \delta \neq 1, \text{ and } V_{i_2} = V_{j_2}; \\
q^{n-k}, & \text{if } \delta \neq 1 \text{ and } V_{i_2} \neq V_{j_2}.
\end{cases}$$

Moreover, we have, for any $0 \leq i, j < \frac{(q^n - 1)q^k}{T}$ and $0 \leq \tau < q^n - 1$,

$$\begin{align*}
\delta &= 1, \text{ if } (j_1 - i_1)T + \tau \equiv 0 \pmod{(q^n - 1)}; \\
\delta &\neq 1, \text{ if } (j_1 - i_1)T + \tau \not\equiv 0 \pmod{(q^n - 1)}.
\end{align*}$$

With the parameters $i, j$ and $\tau$, we give the following four cases to calculate $H_{S_i, S_j}(\tau)$.

**Case 1.** $V_{i_2} = V_{j_2}$, $i_1 = j_1$ and $0 < \tau < q^n - 1$.

In this case, $\delta \neq 1$. By formula (24), we have, for any $0 \leq i < \frac{(q^n - 1)q^k}{T}$ and $0 < \tau < q^n - 1$,

$$H_{S_i, S_j}(\tau) = q^{n-k} - 1.$$
(II) **Case 2.** \( V_{i_2} \neq V_{j_2}, i_1 = j_1 \) and \( 0 \leq \tau < q^n - 1 \).

In this case, if \( \tau = 0 \), we have \( \delta = 1 \), otherwise \( \delta \neq 1 \). From formula (24), we have

\[
H_{S_i, S_j}(\tau) = \begin{cases} 
0, & \tau = 0; \\
q^{n-k}, & \tau \neq 0.
\end{cases}
\]

(III) **Case 3.** \( V_{i_2} = V_{j_2}, i_1 \neq j_1 \) and \( 0 \leq \tau < q^n - 1 \).

In this case, from formulas (24) and (25), we have

\[
H_{S_i, S_j}(\tau) = \begin{cases} 
q^n - 1, & (j_1 - i_1)T + \tau \equiv 0 \pmod{q^n - 1}; \\
q^{n-k} - 1, & (j_1 - i_1)T + \tau \not\equiv 0 \pmod{q^n - 1}.
\end{cases}
\]

(IV) **Case 4.** \( V_{i_2} \neq V_{j_2}, i_1 \neq j_1 \) and \( 0 \leq \tau < q^n - 1 \).

In this case, from formulas (24) and (25), we have

\[
H_{S_i, S_j}(\tau) = \begin{cases} 
0, & (j_1 - i_1)T + \tau \equiv 0 \pmod{q^n - 1}; \\
q^{n-k}, & (j_1 - i_1)T + \tau \not\equiv 0 \pmod{q^n - 1}.
\end{cases}
\]

As a result, from the above four cases we have

\[
H_{S_i, S_j}(\tau) \leq q^{n-k}
\]

for any \( 0 \leq i, j < (q^n - 1)q^k, 0 < \tau \leq T - 1 \) if \( i = j \) and for any \( 0 \leq i, j < (q^n - 1)q^k, 0 \leq \tau \leq T - 1 \) if \( i \neq j \). Therefore, \( S \) is an LHZ FHS set with parameters \( \left(q^n - 1, \frac{(q^n - 1)q^k}{q^k - 1}, q^k, T - 1; q^{n-k}\right) \).

Put the parameters of \( S \) into bound (3), then we have

\[
H_{mz}(S) \geq \left\lceil \frac{q^n - q^{n-k} - 2}{q^{n+k} - q^k - 1} \right\rceil.
\]

Since \( q^{n-k} \) is a positive integer and

\[
0 < \frac{2q^n - q^{n-k} - 2}{q^{n+k} - q^k - 1} < 1,
\]

we have

\[
H_{mz}(S) \geq q^{n-k}.
\]

By Lemma 2.2, formulas (26) and (27), \( S \) is an optimal LHZ FHS set. This completes the proof. \( \square \)

**Remark 5.**

- In Theorem 4.1, if \( T = q^n - 1 \), \( S \) is the optimal FHS set with parameters \( (q^n - 1, q^k, q^k; q^{n-k}) \) constructed by Zhou et al. in [29].
- In Theorem 4.1, if \( T = \frac{q^n - 1}{q^k - 1} \), \( S \) is the optimal LHZ FHS set with parameters \( (q^n - 1, q^k(q - 1), q^k, \frac{q^n - 1}{q^k - 1} - 1; q^{n-k}) \) presented by Han et al. in [11].
- In the second construction, if we choose \( m = n, f(x) = x \) and \( a_i = \beta \alpha^i, \beta \in \mathbb{F}_{q^k}^* \), \( 0 \leq i < k \), \( S \) is exactly the optimal LHZ FHS set of Construction 2 in [27] based on \( m \)-sequence with the same parameters.

**Remark 6.** Although the second construction can’t give new parameters, from Theorem 4 in [29], it follows that we can get new FHSs with large linear complexity by choosing appropriate \( d \)-form functions with difference-balanced property, and produces many new optimal FHS sets that cannot be produced by the earlier construction.
Example 5. Let $q = 2, n = 12, m = 3, k = 2$, and $\alpha$ be a primitive element of $\mathbb{F}_{2^{12}}$ satisfying $\alpha^{12} + \alpha^7 + \alpha^6 + \alpha^3 + \alpha + 1 = 0$. Let $f(x) = \left( \text{tr}_{n/2}/q^3 \left( \left( \text{tr}_{q^{12}/q^6} (x) \right)^{\frac{3}{5}} \right) \right)^{3}$, $T = 1365$. Choose $a_0 = 1, a_1 = \alpha$, then $S$ has 12 FHSs of length 4095:

- $S_0 = \{(0, 0), (0, 1), (0, 1), (1, 0), (0, 0), (0, 1), (1, 1), (1, 0), (0, 1), (1, 1), (0, 1), \ldots\}$;
- $S_1 = \{(1, 0), (1, 1), (1, 1), (0, 0), (1, 0), (1, 1), (0, 0), (1, 1), (0, 1), (1, 1), \ldots\}$;
- $S_2 = \{(0, 1), (0, 0), (0, 0), (1, 1), (0, 1), (0, 0), (1, 0), (1, 1), (0, 0), (1, 0), (0, 0), \ldots\}$;
- $S_3 = \{(1, 1), (1, 0), (1, 0), (0, 1), (1, 1), (1, 0), (0, 0), (1, 1), (0, 0), (0, 1), (0, 0), \ldots\}$;
- $S_4 = \{(1, 1), (0, 0), (0, 0), (0, 1), (1, 0), (1, 1), (0, 0), (1, 1), (1, 1), (0, 0), (0, 0), \ldots\}$;
- $S_5 = \{(0, 1), (1, 1), (1, 0), (1, 0), (1, 0), (0, 1), (0, 0), (1, 1), (0, 1), (0, 1), (1, 0), \ldots\}$;
- $S_6 = \{(1, 0), (0, 0), (1, 0), (1, 1), (0, 0), (1, 0), (1, 1), (0, 0), (1, 0), (0, 1), (1, 0), \ldots\}$;
- $S_7 = \{(0, 0), (1, 0), (1, 1), (1, 1), (0, 0), (1, 0), (0, 1), (1, 1), (0, 0), (0, 0), (1, 1), \ldots\}$;
- $S_8 = \{(1, 1), (0, 0), (0, 0), (0, 1), (0, 0), (0, 0), (1, 0), (1, 0), (1, 0), (0, 1), (0, 1), \ldots\}$;
- $S_9 = \{(0, 1), (0, 0), (1, 0), (1, 0), (0, 0), (0, 0), (1, 0), (0, 0), (0, 0), (0, 1), (1, 1), \ldots\}$;
- $S_{10} = \{(1, 0), (0, 1), (0, 1), (0, 0), (0, 1), (1, 1), (1, 1), (0, 1), (0, 0), (0, 0), \ldots\}$;
- $S_{11} = \{(0, 0), (1, 1), (1, 1), (1, 1), (1, 0), (1, 1), (1, 1), (1, 0), (0, 1), (0, 1), (1, 0), \ldots\}$;

By computer experiments, for any $S_i, S_j \in S, 0 \leq i, j < 12, 0 < \tau \leq 1364$ if $i = j$ and $0 \leq \tau \leq 1364$ if $i \neq j$,

$$H_{S_i, S_j}(\tau) \leq 1024.$$  

Thus, we have $S$ is an optimal LHIZ FHS set with parameters

$$\left(4095, 12, 4, 1364; 1024\right).$$

From Theorem 4 in [29], we know that three of the FHSs in $S$ have linear complexity 144, and the others have linear complexity 145.

5. Concluding remarks

The presented constructions in this paper generalize the previous constructions ([29, 27, 11, 9, 20, 30]) based on $m$-sequence and $d$-form function with difference-balanced property, and generate several classes of optimal FHS sets with low hit zone. Moreover, from the first construction we make corrections to Theorems 1 and 2 in [27] and give a counterexample. Furthermore, the presented constructions are generic in the sense that any $d$-form function with difference-balanced property produces optimal FHS sets, and yield FHSs with large linear complexity by choosing appropriate $d$-form function with difference-balanced property.

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