KODAIRA VANISHING ON SINGULAR VARIETIES REVISITED

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Abstract. We correct the proof and slightly strengthen a Kodaira-type vanishing theorem for singular varieties originally due to Jaffe and the first author. Specifically, we show that if $L$ is a nef and big line bundle on a projective variety of characteristic zero, the $i$th cohomology of $L^{-1}$ vanishes for $i$ in a range determined by the depth and dimension of the singular locus.

1. Introduction

Some years ago, David Jaffe and the first author considered a version of Kodaira’s vanishing theorem for a singular variety, where the bound involves its depth and dimension of the singular locus [AJ89, Proposition 1.1]; a slight strengthening of this says:

Theorem 1.1. Let $X$ be a projective variety over an algebraically closed field of characteristic 0. Let $k \geq 0$ be an integer. Suppose $k < \operatorname{codim}(X_{\text{sing}})$ and $X$ satisfies Serre’s condition $S_{k+1}$. Then for any nef and big line bundle $L$ on $X$, we have

$$H^i(X, L^{-1}) = 0.$$ 

Unfortunately, the argument given in [AJ89] contains a gap, because it implicitly assumes that a general hyperplane section of an $S_{k+1}$ scheme remains $S_{k+1}$. The purpose of this note is to fix the gap. We use a different strategy involving Grothendieck duality. A small irony is that this is close to the original approach that Jaffe and the first author used, although this proof never made it to the final published version.

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2. Notation and conventions

We fix some terminology and notation.

(2.1) Given a projective scheme $X$ over a field, it admits a dualizing complex $\omega_X^\bullet \in D^+(\text{Coh}(X))$ with coherent cohomology sheaves [Har66]. For a normalized $\omega_X^\bullet$, the sheaf $\omega_X = h^{-\dim X}(\omega_X^\bullet)$ is called a dualizing sheaf.

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(2.2) For a complex $\mathcal{F}^\bullet$ of sheaves on $X$, $h^i(\mathcal{F}^\bullet)$ denotes its $i$-th cohomology sheaf and $H^i(\mathcal{F}^\bullet)$ denotes the hypercohomology $R^i\Gamma(\mathcal{F}^\bullet) = h^i(R\Gamma(\mathcal{F}^\bullet))$. We will use $H^i(\cdot)$ to denote the usual cohomology for sheaves. $R\text{Hom}(\mathcal{F}, \cdot)$ is the derived functor of $\text{Hom}(\mathcal{F}, \cdot)$.

(2.3) Let $X$ be a projective variety and given a line bundle $L$. We say $L$ is nef, if for any proper morphism $f : C \to X$ from a smooth projective curve over the base field, $\deg f^*L \geq 0$. We say $L$ is big, if for any resolution of singularities $\mu : X' \to X$, the pullback $\mu^*L$ is big on $X'$. Note that this is independent of $X'$.

(2.4) Let $X$ be a projective variety and $L$ a big line bundle. There exists a proper closed subset $V = V(L) \subset X$ with the property that if $Y \subset X$ is a subvariety not contained in $V$, then $L|_Y$ is also big ([Laz04, Corollary 2.2.11]).

3. Proof of theorem

We work over an algebraically closed field of characteristic zero.

We have a second quadrant spectral sequence

\[(3.1) \quad E_2^{i,j} = H^j(X, h^i(\omega_X^\bullet) \otimes L) \Rightarrow H^i+j(X, \omega_X^\bullet \otimes L).\]

The key technical result is

**Theorem 3.1.** Let $X$ be a projective variety and $L$ a nef and big line bundle on $X$. Then for any $i, j$ such that $j > 0$ and $i + j > -\text{codim}(X_{\text{Sing}})$, we have

\[E_{\infty}^{i,j} = 0.\]

Chasing the spectral sequence (3.1) more carefully, one can say something more.

**Corollary 3.2.** Let $X$ be a projective variety and $L$ a nef and big line bundle on $X$.

1. The differential $H^0(X, h^{-1}(\omega_X^\bullet) \otimes L) \to H^2(X, h^{-2}(\omega_X^\bullet) \otimes L)$ of (3.1) is zero.

2. Suppose $\dim X \leq 3$. Then for any $i, j$ such that $j > 0, i + j > -\text{codim}(X_{\text{Sing}})$, we have that

\[h^i(X, h^j(\omega_X^\bullet) \otimes L) = 0.\]

We start with some lemmas.

**Lemma 3.3.** Let $X$ be a projective variety and $L$ a nef and big line bundle on $X$. Then for $j > \dim(X_{\text{Sing}})$ and any $i \in \mathbb{Z}$,

\[H^i(X, h^j(\omega_X^\bullet) \otimes L) = 0.\]
Proof. Let $n = \dim X$. For $i > -n$, the coherent sheaf $h^i(\omega_X^*)$ is supported on $X_{\text{Sing}}$, so the vanishing is automatic. For $i = -n$, $h^{-n}(\omega_X^*) \cong \omega_X$. Take a resolution of singularities $f : Y \to X$. The natural morphism $f_*\omega_Y \to \omega_X$ is injective, because $f_*\omega_Y$ is torsion free and the map is generically isomorphic. Consider the exact sequence

$$0 \to f_*\omega_Y \to \omega_X \to Q \to 0,$$

where the quotient sheaf $Q$ is supported on $X_{\text{Sing}}$. By [EV84, Théorème 3.1], $H^j(X, f_*\omega_Y \otimes L) = 0$ for all $j > 0$ (The vanishing also follows from Grauert-Riemenschneider vanishing and Kawamata-Viehweg vanishing). So the assertion follows.

Lemma 3.4. If $X$ is a 2-dimensional projective variety and $L$ a nef and big line bundle on $X$. Then for any $i, j$ such that $j > 0$, $i + j > -\text{codim}(X_{\text{Sing}})$, we have

$$H^j(X, h^i(\omega_X^*) \otimes L) = 0.$$  

Proof. If $\text{codim}(X_{\text{Sing}}) = 1$, by Lemma 3.3

$$E_{i,j} = 0$$

if $j \geq 2$. By Grothendieck duality [Har66], $H^0(X, \omega_X^* \otimes L) = H^0(X, L^{-1}) = 0$. Therefore $E_{i,j} = 0$ for $i + j = 0$. This implies that $E_{2,1} = 0$.  

If $\text{codim}(X_{\text{Sing}}) = 2$, by Lemma 3.3 again,

$$E_{i,j} = 0$$

if $j > 0$.  

Proof of Theorem 3.1. We proceed by induction on $n = \dim X$. The $n = 2$ case is proved in Lemma 3.4. Fix a projective variety $X$ with $n \geq 3$ and a big and nef line bundle $L$ on $X$. Put $k = -(i + j)$, so $k < \text{codim}(X_{\text{Sing}})$. Suppose the statement is true for any projective varieties of dimension $n-1$. Fix a sufficiently ample Cartier divisor $H$ and $D \in |H|$ with the following properties:

(P1) $D$ is integral and does not contain a component of $X_{\text{sing}}$;
(P2) $L_D$ is big and nef;
(P3) $H^i(X, h^j(\omega_X^*) \otimes \mathcal{O}_X(H) \otimes L) = 0$ for all $i > 0$ and $j \in \mathbb{Z}$.

(P1) is by the Bertini theorem, and (P2) is by (2.4). (P3) is true because $h^j(\omega_X^*)$ are coherent and only finitely many of them are nonzero.

Consider the short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{O}_X(-H) \otimes L^{-1} \to L^{-1} \to i_*\mathcal{O}_D \otimes L^{-1} \to 0.$$  

By applying $R\text{Hom}(\cdot, \omega_X^*) = R\Gamma \circ R\mathcal{H}\text{om}(\cdot, \omega_X^*)$ and taking the cohomology, we get the exact sequence of vector spaces

$$\cdots \to h^{-k}(R\text{Hom}(i_*\mathcal{O}_D \otimes L^{-1}, \omega_X^*)) \to h^{-k}(R\text{Hom}(L^{-1}, \omega_X^*)) \to$$
Using the fact that $L$ is invertible and the Grothendieck duality for $i : D \hookrightarrow X$ [Har66], we arrive at

\begin{equation}
\cdots \to H^{-k}(\omega^* \otimes L \mid |D) \to H^{-k}(\omega^* \otimes L) \to H^{-k}(\omega^* \otimes \Theta_X(H) \otimes L) \to \cdots
\end{equation}

The hypercohomology above can be computed by a spectral sequence (3.1):

\[ E_p^q := H^q(D, h^{-k}\omega_D \otimes \otimes |D) \Rightarrow H^{p+q}(\omega^*_D \otimes L \mid |D). \]

Similarly we denote by $E^*\text{ for } H^*_X(\omega^* \otimes L)$ and $E^*\text{ for } H^*_X(\omega^* \otimes \Theta_X(H) \otimes L)$ respectively.

By (P3), $E^*$ degenerates at the second page to give an isomorphism

\[ H^{-k}(\omega^*_X \otimes \Theta_X(H) \otimes L) \cong E^{-k,0}_\infty \cong H^0(X, h^{-k}(\omega^*_X) \otimes \Theta_X(H) \otimes L). \]

(P1) implies that $\text{codim}(D_{\text{Sing}}) \geq \text{codim}(X_{\text{Sing}})$. By the induction hypothesis, we have for $q > 0$ and $p + q \geq -k$,

\[ E^p_\infty = 0. \]

By looking at the $q$-th graded pieces with respect to the natural filtration of the hypercohomology in (3.2), we deduce that for $q > 0$ and $p + q \geq -k$,

\[ E^p_\infty = 0, \]

as desired. \hfill \Box

Keep the notation introduced above in the rest of the section.

**Proof of Theorem 1.1.** By Grothendieck duality [Har66] and Theorem 3.1, we have

\[ H^k(X, L^{-1})^\vee \cong H^{-k}(\omega^* \otimes L) \cong E^{-k,0}_\infty \subseteq \cdots \subseteq E^{-k,0}_2 = H^0(X, h^{-k}(\omega^*_X) \otimes L). \]

Thanks to Serre’s condition $S_{k+1}$, $h^{-k}(\omega^*_X) = 0$; so the assertion follows. \hfill \Box

**Proof of Corollary 3.2.** (1) Consider the commutative diagram

\[
\begin{array}{ccc}
E^{-1,0}_\infty & \longrightarrow & H^0(X, h^{-1}(\omega^*_X) \otimes \Theta_X(H) \otimes L) \longrightarrow 0, \\
\downarrow & & \downarrow \\
H^0(X, h^{-1}(\omega^*_X) \otimes L) & \longrightarrow^g & H^0(X, h^{-1}(\omega^*_X) \otimes \Theta_X(H) \otimes L)
\end{array}
\]

where the first row is exact, coming from (3.2); and the left column map is an inclusion because

\[ E^{-1,0}_\infty \subseteq \cdots \subseteq E^{-1,0}_2 = H^0(X, h^{-1}(\omega^*_X) \otimes L). \]
We claim that the map $\alpha$ is injective for general $D$. In fact, if $D \in |H|$ is general such that $D$ does not contain any scheme defined by an associate prime of $h^{-1}(\omega_X^*)$, then we have the exact sequence

$$0 \to h^{-1}(\omega_X^*) \otimes L \to h^{-1}(\omega_X^*) \otimes \mathcal{O}_X(H) \otimes L \to h^{-1}(\omega_X^*) \otimes \mathcal{O}_D(H) \otimes L \to 0.$$ 

Thus $\alpha$ is injective. It then follows that

$$E_{\infty}^{1,0} \simeq H^0(X, h^{-1}(\omega_X^*) \otimes L),$$

which in turn implies that

$$H^0(X, h^{-1}(\omega_X^*) \otimes L) \to H^2(X, h^{-2}(\omega_X^*) \otimes L).$$

is zero.

(2) The case $\dim X = 2$ has been treated in Lemma 3.4. When $\dim X = 3$, we just show that $H^2(X, h^{-2}(\omega_X^*) \otimes L) = 0$ when $\text{codim}(X_{\text{Sing}}) = 1$. The other cases are left to the interested reader.

By Theorem 3.1, we deduce that the natural map $H^0(X, h^{-1}(\omega_X^*) \otimes L) \to H^2(X, h^{-2}(\omega_X^*) \otimes L)$ surjects. But by (1) above, the map is indeed zero; so the assertion follows.

\[ \Box \]

4. Comments

(4.1) By Serre’s normality criterion, Theorem 1.1 implies that

**Corollary 4.1** (Mumford [Mum67]). *Suppose that $X$ is a normal projective variety of dimension at least 2, defined over a field of characteristic zero, and that $L$ is an ample line bundle. Then $H^1(X, L^{-1}) = 0$.***

This statement was in fact the original inspiration for Theorem 1.1. In the same paper, Mumford goes on to give an easy two dimensional counterexample to this statement in positive characteristic. This was decade before Raynaud [Ray78] gave a rather different counterexample to the usual Kodaira vanishing for smooth projective varieties.

(4.2) The depth condition is essential for the validity of Theorem 1.1, while the condition on the dimension of the singular locus is not. Consider a projective variety $X$ with rational singularities. It is well-known that for any $i < \dim X$ and $L$ ample line bundle, the vanishing holds

$$H^i(X, L^{-1}) = 0.$$ 

To exhibit such a variety but with relatively small codimension of the singular locus, take a smooth projective variety $Y$ of dimension $n$ with the property that $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$, and fix a sufficiently ample line bundle $M$ on $Y$. Then the secant variety

$$\Sigma = \Sigma(Y, M) \subset \mathbb{P}(H^0(Y, M))$$
is $2n + 1$ dimensional with rational singularities (cf. [CS17]), and the singular locus is precisely $X \subset \Sigma$. Thus $\text{codim}(\Sigma_{\text{Sing}}) = n + 1 < 2n + 1$.

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