Constraint rescaling in refined algebraic quantisation: momentum constraint

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Abstract

We investigate refined algebraic quantisation within a family of classically equivalent constrained Hamiltonian systems that are related to each other by rescaling a momentum-type constraint. The quantum constraint is implemented by a rigging map that is motivated by group averaging but has a resolution finer than what can be peeled off from the formally divergent contributions to the averaging integral. Three cases emerge, depending on the asymptotics of the rescaling function: (i) quantisation is equivalent to that with identity scaling; (ii) quantisation fails, owing to nonexistence of self-adjoint extensions of the constraint operator; (iii) a quantisation ambiguity arises from the self-adjoint extension of the constraint operator, and the resolution of this purely quantum mechanical ambiguity determines the superselection structure of the physical Hilbert space. Prospects of generalising the analysis to systems with several constraints are discussed.

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1 Introduction

In a classical Hamiltonian system, a gauge symmetry is generated by constraint functions known as first class constraints: constraints whose Poisson brackets with each other and with the Hamiltonian are linear combinations of the constraints themselves. In the Dirac-Bergmann quantisation scheme the constraint functions are promoted into quantum constraint operators, and the physical quantum states are required to be annihilated by the quantum constraints [1, 2, 3].

To find physical quantum states, one may wish to start from a state that is not necessarily annihilated by the quantum constraints and average this state over the action generated by the quantum constraints [4, 5]. When the quantum constraints generate the action of a genuine Lie group, this group averaging proposal can be given a precise implementation in the framework known in physics as refined algebraic quantisation (RAQ) [6, 7, 8, 9] and in mathematics as Rieffel induction [10], with results on both uniqueness and generality of the resulting quantum theory [11, 12]. Case studies of specific quantum mechanical systems can be found in [13, 14, 15, 16, 17], applications to de Sitter invariant quantum field theory are considered in [4, 5, 18, 19] and applications to loop quantum gravity are considered in [20, 21].

A Lie group action generated by the quantum constraints is however a very special case: it can be expected to occur only when the Poisson brackets of the classical constraint functions form a Lie algebra, that is, close with structure coefficients that are constants on the phase space. In many systems of interest, including general relativity in both metric and connection formulations [22], the structure coefficients are nonconstant functions on the phase space. Further, given a system with at least two constraints and constant structure coefficients, redefining the constraints by an invertible linear map that is not constant on the phase space yields a classically equivalent system that can be arranged to have nonconstant structure functions. The distinction between structure constants (known as a closed gauge algebra) and nonconstant structure functions (known as an open gauge algebra) is hence not intrinsic to the true physical degrees of freedom but depends also on how the generators of the gauge transformations are chosen [23, 24, 25, 26, 27]. These considerations show that there would be considerable interest to extend the group averaging method to systems with open algebras.

A proposal for extending group averaging to open gauge algebras has been given by Shvedov [28], using the Becchi-Rouet-Stora-Tyutin (BRST) formalism [29, 30, 31, 32] and building on the previous work in [33, 34, 35, 36, 37, 38, 39], in particular on the Batalin-Marnelius inner product [37]. When the structure functions are constants, Shvedov’s proposal duly reduces to averaging over a Lie group in the measure adopted in [12]. To recover a full quantum theory, however, an averaging formula must be supplanted with
additional structure, including the state space on which the averaging acts and the sense in which the averaging converges. These issues have proven quite delicate already in the Lie group context when the group is not compact, despite the control provided by the Giulini-Marolf uniqueness theorem [12]; for example, the averaged states can turn out to have negative norm squared [15].

In this paper we address group averaging in refined algebraic quantisation in a class of systems related by rescaling a classical constraint [40]. We focus on a system with a single constraint, so that the gauge algebra is trivially closed regardless the scaling of the constraint. To avoid built-in topological complications in the classical theory, we take the phase space to be $T^*\mathbb{R}^2 \simeq \mathbb{R}^4$ and the constraint to be linear in one of the momenta, but we allow this momentum to be scaled by a nowhere-vanishing function of the coordinates. The classical reduced phase space is then just $T^*\mathbb{R} \simeq \mathbb{R}^2$, obtained by dropping the canonical pair whose momentum appears in the constraint. The main issue that remains in quantisation is then how to promote the classical constraint into an operator in terms of which the quantum gauge transformations and the averaging over these transformations can be defined.

We shall see that once the auxiliary Hilbert space structure is specified, the options to define the constraint operator depend on the asymptotics of the scaling function in the classical constraint. Three cases emerge:

(i) The constraint operator is essentially self-adjoint, and the quantisation is equivalent to the group averaging that arises when the scaling function is the constant function 1.

(ii) The constraint operator has no self-adjoint extensions, and we are unable to extract a notion of quantum gauge transformations, let alone a definition of averaging over them. No quantum theory is recovered.

(iii) There is an infinite quantisation ambiguity, arising from a choice in the self-adjoint extension of the constraint operator. Within a subclass of extensions parametrised by one smooth function of one variable, the superselection structure of the physical Hilbert space depends strongly on the choice of the extension, but the quantum theory is insensitive to the residual freedom in the scaling function.

The superselection sectors that emerge in case (iii) resemble closely those in refined algebraic quantisation of the Ashtekar-Horowitz-Boulware model [17]. However, whereas with the Ashtekar-Horowitz-Boulware model these sectors are determined by the potential term in the classical constraint, here the sectors are determined solely by a quantisation ambiguity.
We begin by introducing the classical system in section 2. Section 3 specifies the auxiliary structure for refined algebraic quantisation, establishing the conditions under which the scaling functions belong to cases (i)–(iii). Case (i) is briefly addressed in section 4. The main content of the paper is in the analysis of case (iii) in section 5. Section 6 presents a summary and concluding remarks. Appendix A reviews the relationship of group averaging and the BRST inner product for a system with a single constraint. The proofs of certain technical results are deferred to appendices B and C.

We set $\hbar = 1$. Complex conjugate is denoted by overline, except in appendix A where it is denoted by $^\ast$. In asymptotic analyses, $O(u)$ is such that $u^{-1}O(u)$ remains bounded as $u \to 0$, $o(u)$ is such that $u^{-1}o(u) \to 0$ as $u \to 0$ and $o(1) \to 0$ as $u \to 0$ [41].

2 Classical system: one momentum-type constraint

We consider a system with configuration space $\mathbb{R}^2 = \{ (\theta, x) \}$ and phase space $\Gamma = T^*\mathbb{R}^2 = \{ (\theta, x, p_\theta, p_x) \} \simeq \mathbb{R}^4$. The system has one constraint,

$$\phi := M(\theta, x)p_\theta,$$

where the real-valued function $M$ is smooth and nowhere vanishing. We may assume without loss of generality that $M$ is positive. We assume that there is no true Hamiltonian, although inclusion of a true Hamiltonian that only depends on $x$ and $p_x$ would be straightforward.

The constraint hypersurface $\phi = 0$ is $\Gamma_c = \{ (\theta, x, 0, p_x) \} \simeq \mathbb{R}^3$. The generator of gauge transformations on $\Gamma_c$ is the restriction of the Hamiltonian vector field of $\phi$,

$$X^+ := M(\theta, x)\partial_\theta.$$

As $X^+$ is nowhere vanishing, the constraint is regular in the sense of [3, 42]. The integral curves of $X^+$ have constant $x$ and $p_x$ and they connect any two given values of $\theta$. The reduced phase space is hence $\Gamma_{\text{red}} = \{ (x, p_x) \} \simeq \mathbb{R}^2$.

If we wish to view the gauge transformations as maps on $\Gamma_c$, rather than just as maps of individual initial points in $\Gamma_c$, a subtlety arises. The gauge transformation with the (finite) parameter $\lambda$ is the exponential map of $\lambda X^+$, $\exp(\lambda X^+)$. If $M$ satisfies

$$\int_{-\infty}^{0} \frac{d\theta}{M(\theta, x)} = \infty = \int_{0}^{\infty} \frac{d\theta}{M(\theta, x)}$$

for all $x$, then $X^+$ is a complete vector field, and the family $\{ \exp(\lambda X^+) \mid \lambda \in \mathbb{R} \}$ is a one-parameter group of diffeomorphisms $\Gamma_c \to \Gamma_c$ [43]. If (2.3) does not hold for all $x$, then $X^+$ is incomplete. It is still true that the action of $\exp(\lambda X^+)$ on any given initial
point in \( \Gamma_c \) is well defined for sufficiently small \( |\lambda| \); however, there are no values of \( \lambda \neq 0 \) for which both of \( \exp(\pm \lambda X^+) \) are defined as maps \( \Gamma_c \to \Gamma_c \), since at least one of them will try to move points past the infinity. It is this classical subtlety whose quantum mechanical counterpart will be at the heart of our quantisation results.

Finally, note that when \( M \) is the constant function \( 1 \), we have \( \phi = p_\theta \) and \( X^+ = \partial_\theta \), and the gauge transformation \( \exp(\lambda X^+) : \Gamma_c \to \Gamma_c \) is just the translation \( (\theta, x, p_x) \mapsto (\theta + \lambda, x, p_x) \). Other choices for \( M \) amount to rescaling the constraint of this prototype system by a positive function that may depend on both the gauge variable \( \theta \) and the non-gauge variable \( x \). We refer to \( M \) as the scaling function.

3 Refined algebraic quantisation: action of the gauge group

We wish to quantise the system in the refined algebraic quantisation (RAQ) framework as reviewed in [8]. In this section we specify the auxiliary structure and examine conditions under which the quantum constraint generates the action of a unitary group on the auxiliary Hilbert space. Textbook expositions of the requisite theory of self-adjoint operators are given in [44, 45] and a pedagogical introduction can be found in [46].

We take the auxiliary Hilbert space to be square integrable functions on the classical configuration space \( \mathbb{R}^2 = \{ (\theta, x) \} \), \( \mathcal{H}_{\text{aux}} := L^2(\mathbb{R}^2, d\theta dx) \). The inner product in \( \mathcal{H}_{\text{aux}} \) reads

\[
(\psi,\chi)_{\text{aux}} := \int_{\mathbb{R}^2} d\theta dx \overline{\psi(\theta,x)} \chi(\theta,x),
\]  

where the overline denotes complex conjugation.

We promote the classical constraint \( \phi (2.1) \) into a quantum constraint by the substitution \( p_\theta \mapsto -i\partial_\theta \) and a symmetric ordering, with the result

\[
\hat{\phi} := -i \left( M\partial_\theta + \frac{1}{2} (\partial_\theta M) \right).
\]  

We wish to obtain a family of operators \( \{ U(\lambda) \} \) by exponentiating \( \hat{\phi} \),

\[
U(\lambda) := \exp(i\lambda \hat{\phi}),
\]  

and to find an inner product on the physical Hilbert space by a suitable interpretation of the sesquilinear form

\[
(\psi,\chi)_{\text{ave}} := \int d\lambda \left( \psi, U(\lambda)\chi \right)_{\text{aux}}.
\]
In this section we consider (3.3).

The operator $\hat{\phi}$ (3.2) is symmetric on the dense linear subspace of smooth functions of compact support in $\mathcal{H}_{\text{aux}}$. If $\hat{\phi}$ has self-adjoint extensions on $\mathcal{H}_{\text{aux}}$, a choice of the self-adjoint extension in (3.3) defines $\{U(\lambda) \; | \; \lambda \in \mathbb{R}\}$ as a one-parameter group of unitary operators, and we can seek to implement (3.4) as the group averaging sesquilinear form in RAQ. We hence need to analyse the self-adjoint extensions of $\hat{\phi}$.

The existence of self-adjoint extensions of $\hat{\phi}$ is determined by the deficiency indices of $\hat{\phi}$, that is, the dimensions of the linear subspaces of $\mathcal{H}_{\text{aux}}$ satisfying $\hat{\phi}\psi = \pm i\psi$ [44, 45]. The solutions to the differential equation $\hat{\phi}\psi = \pm i\psi$ are

$$\psi_{\pm}(\theta, x) = \frac{F_{\pm}(x)}{\sqrt{M(\theta, x)}} \exp[\mp \sigma_x(\theta)], \quad (3.5)$$

where

$$\sigma_x(\theta) := \int_{0}^{\theta} \frac{d\theta'}{M(\theta', x)} \quad (3.6)$$

and the complex-valued functions $F_{\pm}$ are arbitrary. There are three qualitatively different cases, depending on the asymptotics of $\sigma_x(\theta)$ as $\theta \to \pm \infty$.

**Type I scaling functions.** Suppose that

$$\sigma_x(\theta) \to \pm \infty \quad \text{as} \quad \theta \to \pm \infty \quad \text{for a.e.} \quad x, \quad (3.7)$$

where “a.e.” stands for almost everywhere in the Lebesgue measure on $\mathbb{R}$. Then every nonzero $\psi_{\pm}$ (3.5) has infinite norm, for $\psi_+$ because of the behaviour at $\theta \to -\infty$ and for $\psi_-$ because of the behaviour at $\theta \to \infty$. The deficiency indices are $(0, 0)$ and $\hat{\phi}$ is essentially self-adjoint. The operator $U(\lambda)$ is unitary, and it acts on the wave functions by the exponential map of the vector field $X^+$ (2.2), where the wave functions are understood as half-densities (see for example Appendix C in [22]). Explicitly, we have

$$(U(\lambda)\psi)(\theta, x) = \frac{\sqrt{M(\sigma_x^{-1}(\sigma_x(\theta) + \lambda), x)}}{\sqrt{M(\theta, x)}} \psi(\sigma_x^{-1}(\sigma_x(\theta) + \lambda), x), \quad (3.8)$$

where the formula is well-defined for all $x$ except the set of measure zero (if non-empty) where (3.7) does not hold. The group multiplication law in the one-parameter group $\{U(\lambda) \; | \; \lambda \in \mathbb{R}\} \simeq \mathbb{R}$ is addition in $\lambda$. In the special case $M(\theta, x) = 1$, we recover the group of translations in $\theta$, $(U(\lambda)\psi)(\theta, x) = \psi(\theta + \lambda, x)$.

**Type II scaling functions.** Suppose that (3.7) holds either with the upper signs or with the lower signs but not both. If (3.7) holds for the upper signs, then every nonzero $\psi_-$ (3.5) has again infinite norm; however, any $F_+ \in L^2(\mathbb{R}, dx)$ whose support is in the set where (3.7) with the lower signs fails will give a square integrable $\psi_+$. The deficiency
indices are hence \((\infty, 0)\). Similarly, if (3.7) holds for the lower signs, the deficiency indices are \((0, \infty)\). \(\hat{\phi}\) has no self-adjoint extensions in either case, and (3.3) does not provide a definition of \(U(\lambda)\). At the level of formula (3.8), the problem is that \(\sigma_x^{-1}\) is not defined even for a.e. \(x\).

**Type III scaling functions.** Suppose that (3.7) holds with neither upper nor lower signs. Reasoning as with Type II above shows that the deficiency indices are \((\infty, \infty)\). \(\hat{\phi}\) has an infinity of self-adjoint extensions, and each of them defines \(\{U(\lambda) \mid \lambda \in \mathbb{R}\}\) as a one-parameter group of unitary operators. Formula (3.8) has again a problem in that \(\sigma_x^{-1}\) is not defined, but the self-adjoint extension of \(\hat{\phi}\) provides a rule by which the probability that is pushed beyond \(\theta = \pm \infty\) by (3.8) will re-emerge from \(\theta = \mp \infty\). The group \(\{U(\lambda) \mid \lambda \in \mathbb{R}\}\) may be isomorphic to either \(\mathbb{R}\) or \(U(1)\).

We are hence able to proceed only with Types I and III. In sections 4 and 5 we address the integral (3.4) for these two types.

### 4 RAQ for Type I scaling functions

For Type I scaling functions, the multiplication law in the group \(\{U(\lambda) \mid \lambda \in \mathbb{R}\} \simeq \mathbb{R}\) is addition in \(\lambda\). We hence take the range of integration in (3.4) to be the full real axis.

It is convenient to map \(\mathcal{H}_{aux}\) into \(\mathcal{H}_{aux} := L^2(\mathbb{R}^2, d\Theta dx)\) by the Hilbert space isomorphism

\[
\mathcal{H}_{aux} \rightarrow \mathcal{H}_{aux}, \\
\psi \mapsto \tilde{\psi}, \\
\tilde{\psi}(\Theta, x) := \sqrt{M(\sigma_x^{-1}(\Theta), x)} \psi(\sigma_x^{-1}(\Theta), x),
\]

where the last line is well defined for a.e. \(x\). Working in \(\mathcal{H}_{aux}\), the auxiliary inner product reads

\[
\left(\tilde{\psi}, \tilde{\chi}\right)_{aux} := \int_{\mathbb{R}^2} d\Theta dx \overline{\tilde{\psi}(\Theta, x)} \tilde{\chi}(\Theta, x),
\]

and the group averaging sesquilinear form takes the form

\[
\left(\tilde{\psi}, \tilde{\chi}\right)_{ave} := \int_{-\infty}^{\infty} d\lambda \left(\tilde{\psi}, \tilde{U}(\lambda)\tilde{\chi}\right)_{aux};
\]

where

\[
\left(\tilde{U}(\lambda)\tilde{\psi}\right)(\Theta, x) = \tilde{\psi}(\Theta + \lambda, x).
\]

The system has thus been mapped to that in which \(M\) is the constant function 1.
RAQ in $\tilde{H}_{\text{aux}}$ can now be carried out as for the closely related system discussed in Section IIB of [6]. We can choose smooth functions of compact support on $\mathbb{R}^2 = \{(\Theta, x)\}$ as the dense linear subspace of $\tilde{H}_{\text{aux}}$ on which (4.3) is well defined. The averaging projects out the $\Theta$-dependence of the wave functions, and the physical Hilbert space is $L^2(\mathbb{R}, dx)$. The technical steps are essentially identical to those in [6] and we will not repeat them here.

5  RAQ for Type III scaling functions

For Type III scaling functions, an attempt to classify the self-adjoint extensions of $\hat{\phi}$ would face two challenges. First, the sets in which the conditions (3.7) fail for the upper and lower signs can be arbitrary sets of positive measure. Second, even after these sets are fixed, the deficiency indices are $(\infty, \infty)$, and the self-adjoint extensions of $\hat{\phi}$ comprise only a subset of all maximal extensions of $\hat{\phi}$ [45]. We shall consider a subfamily of self-adjoint extensions of $\hat{\phi}$ that is small enough to allow the action of the gauge group to be written down in an explicit form, yet broad enough to contain situations where rigging maps of interesting structure can be extracted from the group averaging formula (3.4).

5.1 Subfamily of classical rescalings and quantum boundary conditions

We make two simplifying assumptions, one concerning the classical rescaling function and the other concerning the quantum self-adjointness boundary conditions.

First, we assume that (3.7) fails for all $x$ for both signs, so that the formula

$$N(x) := 2\pi \left( \int_{-\infty}^{\infty} \frac{d\theta'}{M(\theta', x)} \right)^{-1}$$

defines a function $N : \mathbb{R} \to \mathbb{R}_+$. It follows that we can map $\mathcal{H}_{\text{aux}}$ to $\mathcal{H}_c := L^2(I \times \mathbb{R}, d\omega dx)$, where $I = [0, 2\pi]$, by the Hilbert space isomorphism

$$\mathcal{H}_{\text{aux}} \to \mathcal{H}_c,$$

$$\psi \mapsto \psi_c,$$

$$\psi_c(\omega, x) := \sqrt{\frac{M(\tilde{x}^{-1}(\omega/N(x)), x)}{N(x)}} \psi(\tilde{x}^{-1}(\omega/N(x)), x),$$

where

$$\tilde{x}(\theta) := \int_{-\infty}^{\theta} \frac{d\theta'}{M(\theta', x)}.$$
The auxiliary inner product in $\mathcal{H}_c$ reads
\[
(\psi_c, \chi_c)_c := \int_{I \times \mathbb{R}} \psi_c(\omega, x) \chi_c(\omega, x) \, d\omega \, dx,
\]
and $\hat{\phi}$ (A.9) is mapped to
\[
\hat{\phi}_c := -iN(x) \partial_\omega.
\]
We work from now on in $\mathcal{H}_c$, dropping the subscript $c$ from the wave functions.

Second, we consider only those self-adjoint extensions of $\hat{\phi}_c$ (5.5) where the boundary conditions at $\omega = 0$ and $\omega = 2\pi$ do not couple different values of $x$. The self-adjointness analysis then reduces to that of the momentum operator on an interval [44], independently at each $x$. The domains of self-adjointness are
\[
D_\alpha := \{ \psi, \partial_\omega \psi \in \mathcal{H}_c \mid \psi(\cdot, x) \in ac(0, 2\pi) \text{ and } \psi(0, x) = e^{i2\pi\alpha(x)} \psi(2\pi, x), \forall x \},
\]
where $ac(0, 2\pi)$ denotes absolutely continuous functions of $\omega$ and the function $\alpha : \mathbb{R} \to \mathbb{R}$ specifies the phase shift between $\omega = 0$ and $\omega = 2\pi$ at each $x$.

Under these assumptions, the remaining freedom in the classical rescaling function $M : \mathbb{R}^2 \to \mathbb{R}^2$ is encoded in the function $N : \mathbb{R} \to \mathbb{R}_+$, while the remaining freedom in the self-adjoint extension of $\hat{\phi}_c$ (5.5) is encoded in the function $\alpha : \mathbb{R} \to \mathbb{R}$. Note that no smoothness assumptions about either function are needed at this stage.

The action of $U_c(\lambda) := \exp(i\lambda \hat{\phi}_c)$ takes now a simple form in a Fourier decomposition adapted to $D_\alpha$. We write each $\psi \in \mathcal{H}_c$ in the unique decomposition
\[
\psi(\omega, x) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} e^{i[n-\alpha(x)]\omega} \psi_n(x),
\]
where each $\psi_n$ is in $L^2(\mathbb{R}, dx)$. It follows that
\[
(\psi, \chi)_c = \sum_{n \in \mathbb{Z}} (\psi_n, \chi_n)_\mathbb{R},
\]
where $(\cdot, \cdot)_\mathbb{R}$ is the inner product in $L^2(\mathbb{R}, dx)$. The action of $U_c(\lambda)$ reads
\[
(U_c(\lambda)\psi)_n(x) = e^{iR_n(x)\lambda} \psi_n(x),
\]
where for each $n \in \mathbb{Z}$ the function $R_n : \mathbb{R} \to \mathbb{R}$ is defined by
\[
R_n(x) = [n - \alpha(x)] N(x).
\]
5.2 Test space, observables and rigging map candidates

Let $\Phi$ be the dense linear subspace of $\mathcal{H}_c$ where the states have the form (5.7) such that every $\psi_n$ is smooth with compact support and only finitely many of them are nonzero for each $\psi \in \Phi$. From (5.9) we see that $\Phi$ is invariant under $U_c(\lambda)$ for each $\lambda$. We adopt $\Phi$ as the RAQ test space of ‘sufficiently well-behaved’ auxiliary states.

Given $\mathcal{H}_c$, $\Phi$ and $U_c(\lambda)$, the RAQ observables are operators $A$ on $\mathcal{H}_c$ such that the domains of $A$ and $A^\dagger$ include $\Phi$, $A$ and $A^\dagger$ map $\Phi$ to itself and $A$ commutes with $U_c(\lambda)$ on $\Phi$ for all $\lambda$. We denote the algebra of the observables by $A_{\text{obs}}$.

The final ingredient in RAQ is to specify the rigging map $\eta : \Phi \rightarrow \Phi^*$, where the star denotes the algebraic dual, topologised by pointwise convergence. $\eta$ is antilinear, it must be real and positive in the sense that the properties

$$
\eta(f)[g] = \overline{\eta(g)[f]},
$$

$$
\eta(f)[f] \geq 0,
$$

hold for all $f, g \in \Phi$, and states in the image of $\eta$ must be invariant under the dual action of $U_c(\lambda)$. Finally, $\eta$ must intertwine with the representations of $A_{\text{obs}}$ on $\Phi$ and $\Phi^*$ in the sense that

$$
\eta(Af)[g] = \eta(f)[A^\dagger g],
$$

for all $A \in A_{\text{obs}}$ and $f, g \in \Phi$, where the left-hand side denotes the dual action of $\eta(Af) \in \Phi^*$ on $g \in \Phi$ and the right-hand side denotes the dual action of $\eta(f) \in \Phi^*$ on $A^\dagger g \in \Phi$. The physical Hilbert space $\mathcal{H}_{\text{RAQ}}$ is then the completion of the image of $\eta$ in the inner product

$$
(\eta(g), \eta(f))_{\text{RAQ}} := \eta(f)[g],
$$

and the properties of $\eta$ and $A_{\text{obs}}$ imply that $\eta$ induces an antilinear representation of $A_{\text{obs}}$ on $\mathcal{H}_{\text{RAQ}}$, with the image of $\eta$ as the dense domain [6, 12].

We seek a rigging map in the form

$$
\eta(f)[g] = \lim_{L \to \infty} \frac{1}{\rho(L)} \int_{-L}^L d\lambda \left( f, U_c(\lambda)g \right)_c
$$

$$
= \lim_{L \to \infty} \frac{1}{\rho(L)} \sum_{n \in \mathbb{Z}} \int_{-\infty}^\infty dx \int_{-L}^L d\lambda e^{iR_n(x)\lambda},
$$

where the last expression follows from (5.7) and (5.9) after interchanging sums and integrals, justified by the assumptions about $\Phi$. The function $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ has been included in order to seek a finite answer in cases where the limit would otherwise diverge.

The existence of the limit in (5.14) depends delicately on the zero sets and the stationary point sets of the functions $R_n$. In subsections 5.3 and 5.4 we introduce conditions that make the limit controllable.
5.3 \( N \) and \( \alpha \) smooth, \( \alpha \) with integer-valued intervals

We assume that \( \alpha \) and \( N \) are smooth. What will play a central role are the integer value sets of \( \alpha \) and the stationary point sets of the functions \( \{ R_n \mid n \in \mathbb{Z} \} \). To control the stationary point sets, we assume that \( R_n \) satisfy the following technical condition:

(a) The stationary point set of each \( R_n \) is either empty or the union of at most countably many isolated points, at most countably many closed intervals and at most two closed half-lines, such that any compact subset of \( \mathbb{R} \) contains at most finitely many of the isolated points and at most finitely many of the finite intervals.

To control the integer value set of \( \alpha \), we assume in this subsection the following condition:

(b) \( \alpha \) takes an integer value on at least one interval.

It follows from (b) that at least one \( R_n \) takes the value zero on an interval. Note that (a) and (b) include the special case where \( \alpha \) takes an integer value everywhere, and the very special case where this integer value is zero.

The group averaging formula (5.14) takes the form

\[
\eta(f)[g] = \lim_{L \to \infty} \frac{2L}{\rho(L)} \sum_{n \in \mathbb{Z}} \left( \int_{J_n} \, dx \, f_n(x) g_n(x) + \int_{\mathbb{R} \setminus J_n} \, dx \, f_n(x) g_n(x) \frac{\sin[LR_n(x)]}{LR_n(x)} \right),
\]

(5.15)

where \( J_n \subset \mathbb{R} \) is the union of all open intervals contained in the zero set of \( R_n \), that is, in the solution set of \( \alpha(x) = n \). Setting \( \rho(L) = 2L \), the second term in (5.15) vanishes by dominated convergence, and from the first term we obtain the map \( \eta_\infty : \Phi \to \Phi^* \),

\[
(\eta_\infty(f))[g] = \sum_{n \in \mathbb{Z}} \int_{J_n} \, dx \, f_n(x) g_n(x). 
\]

(5.16)

We have the following theorem.

**Theorem 5.1** \( \eta_\infty \) is a rigging map, with a nontrivial image.

**Proof.** All the rigging map axioms except the intertwining property (5.12) are immediate. We verify (5.12) in Appendix B. \( \square \)

Group averaging has thus yielded a genuine rigging map \( \eta_\infty \) after a suitable renormalisation. The Hilbert space \( \mathcal{H}_\infty \) is separable and carries a nontrivial representation of \( \mathcal{A}_{\text{obs}} \). Comparison of (5.8) and (5.16) shows that \( \mathcal{H}_\infty \) can be antilinearly embedded in \( \mathcal{H}_c \) as a Hilbert subspace, such that \( \eta_\infty \) extends into the (antilinear) projection \( L^2(\mathbb{R}, dx) \to L^2(J_n, dx) \) in each of the components in (5.7).
Note that the function $N$ does not appear in $\eta_\infty (5.16)$, and the discussion in Appendix B shows that the representation of $A_{\text{obs}}$ on the image of $\eta_\infty$ does not depend on $N$ either. The quantum theory has turned out completely independent of the remaining freedom in the rescaling function, even though the rescaling function may vary nontrivially over the sets $J_n$ that contribute in (5.16).

In the special case where $\alpha(x) = 0$ for all $x$, we have

$$\eta_\infty (f)[g] = (f_0, g_0)_\mathbb{R} .$$

(5.17)

Embedding $\mathcal{H}_\infty$ antilinearly as a Hilbert subspace of $\mathcal{H}_c$ as above, this means that $\eta_\infty$ extends into the (antilinear) projection to the $n = 0$ sector of $\mathcal{H}_c$. When $N$ is a constant function, $N(x) = N_0$ for all $x$, we can recover this extension of $\eta_\infty$ directly, without introducing a test space, by noticing that the quantum gauge group $\{ U_c(\lambda) \mid \lambda \in \mathbb{R} \} \simeq U(1)$ is then compact and taking the group averaging formula to read

$$\eta(f)[g] = \frac{N_0}{2\pi} \int_0^{2\pi/N_0} d\lambda \left( f, U_c(\lambda)g \right)_c ,$$

(5.18)

where the integration is over $U(1)$ exactly once. However, if $N$ is not constant, this shortcut is not available because the quantum gauge group is then still isomorphic to $\mathbb{R}$ rather than $U(1)$.

### 5.4 $N$ and $\alpha$ smooth and generic

In subsection 5.3 the quantum theory arose entirely from the integer value intervals of $\alpha$. We now continue to assume that $\alpha$ and $N$ are smooth, the technical stationary point condition (a) holds and $\alpha$ takes an integer value somewhere, but we take the integer value set of $\alpha$ to consist of isolated points. We first replace condition (b) by the following:

(b′) The integer value set of $\alpha$ is non-empty, at most countable and without accumulation points, and $\alpha$ has a nonvanishing derivative of some order at each integer value.

Second, we introduce the following notation for the zeroes of $R_n$. Let $p$ be the order of the lowest nonvanishing derivative of $\alpha$ (and hence also of $R_n$) at a zero of $R_n$. For odd $p$, we write the zeroes as $x_{pnj}$, where the last index enumerates the solutions with given $p$ and $n$. For even $p$, we write the zeroes as $x_{penj}$, where $\epsilon \in \{1, -1\}$ is the sign of the $p$th derivative of $\alpha$ and the last index enumerates the zeroes with given $p$, $\epsilon$ and $n$. Let $\mathcal{P}$ be the value set of the first index of the zeroes $\{ x_{pnj} \}$ and $\{ x_{penj} \}$. Given this notation, we assume:

(c) If $p \in \mathcal{P}$, then $\mathcal{P}$ contains no factors of $p$ smaller than $p/2$. 

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Before examining the group averaging formula (5.14) under these assumptions, we use the assumptions to define directly a family of rigging maps as follows. For each odd $p \in \mathcal{P}$ we define the map $\eta_p : \Phi \rightarrow \Phi^*$, and for each even $p \in \mathcal{P}$ and $\epsilon \in \{1, -1\}$ for which the set $\{x_{p\epsilon n_j}\}$ is non-empty, we define the map $\eta_{p\epsilon} : \Phi \rightarrow \Phi^*$, by the formulas

$$\eta_p(f)[g] = \sum_{nj} \frac{f_n(x_{pnj}) g_n(x_{pnj})}{|\alpha^p(x_{pnj}) N(x_{pnj})|^{1/p}}, \quad (5.19a)$$

$$\eta_{p\epsilon}(f)[g] = \sum_{nj} \frac{f_{p\epsilon n_j} g_{p\epsilon n_j}}{|\alpha^p(x_{p\epsilon n_j}) N(x_{p\epsilon n_j})|^{1/p}}. \quad (5.19b)$$

These maps are rigging maps, with properties given in the following theorem.

**Theorem 5.2**

1. Each $\eta_p$ and $\eta_{p\epsilon}$ is a rigging map, with a nontrivial image.

2. The representation of $\mathcal{A}_{\text{obs}}$ on the image of each $\eta_p$ and $\eta_{p\epsilon}$ is irreducible.

**Proof.**

1. All the rigging map axioms except the intertwining property (5.12) are immediate from (5.19). We verify (5.12) in Appendix B.

2. The proof is an almost verbatim transcription of that given for a closely similar system in Appendix C of [17]. We omit the details. ■

The rigging maps (5.19) thus yield a family of quantum theories, one from each $\eta_p$ and $\eta_{p\epsilon}$. Each of the Hilbert spaces is either finite-dimensional or separable and carries a nontrivial representation of $\mathcal{A}_{\text{obs}}$ that is irreducible on its dense domain. Functions $f \in \Phi$ whose only nonvanishing component $f_n$ is non-negative and is positive only near a single zero of $R_n$ provide the Hilbert spaces with a canonical orthonormal basis.

Proceeding as in Appendix C of [17], we see that the representation of $\mathcal{A}_{\text{obs}}$ on the image of each $\eta_p$ and $\eta_{p\epsilon}$ is not just irreducible but has the following stronger property, which one might call strong irreducibility: given any two vectors $v$ and $v'$ in the canonical orthonormal basis, there exists an element of $\mathcal{A}_{\text{obs}}$ that annihilates all the basis vectors except $v$ and takes $v$ to $v'$. The upshot of this is that the function $N$ plays little role in the quantum theory, despite appearing in the rigging map formulas (5.19). The Hilbert spaces and their canonical bases are determined by the function $\alpha$ up to the normalisation of the individual basis vectors, and the representation of $\mathcal{A}_{\text{obs}}$ is so ‘large’ that the normalisation of the individual basis vectors, determined by $N$, is of limited consequence. In particular,
the representation of $A_{\text{obs}}$ on any Hilbert space with dimension $n_0 < \infty$ is isomorphic to the complex $n_0 \times n_0$ matrix algebra, independently of $N$.

Now, we wish to relate these quantum theories to the group averaging formula (5.14), which takes the form

$$\eta(f)[g] = \lim_{L \to \infty} \frac{2}{\rho(L)} \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} dx \, f_n(x) g_n(x) \frac{\sin[LR_n(x)]}{R_n(x)}.$$  (5.20)

Note that the integral over $x$ in (5.20) is well defined because the zeroes of the denominator are isolated and the integrand does not diverge at them.

Suppose first that $\mathcal{P} = \{1\}$ and we set $\rho(L) = 2\pi$. The lemmas of Appendix C then show that (5.20) is well defined and equals $\eta_1(f)[g]$ provided the assumptions on $N$ are modestly strengthened, in particular to preclude any $R_n$ from taking a constant value on any interval.

Suppose then that $\mathcal{P} \neq \{1\}$ and we again set $\rho(L) = 2\pi$. Suppose further that the assumptions on $N$ are again modestly strengthened so that the conditions of Appendix C hold, and suppose that condition (c) above is strengthened to the following:

$$(c') \text{ If } p \in \mathcal{P}, \text{ then } \mathcal{P} \text{ contains no factors of } p.$$  

The lemmas of Appendix C then show that (5.20) contains contributions that diverge in the $L \to \infty$ limit; however, these divergences come in well-defined inverse fractional powers of $L$ such that the coefficient of each $L^{(p-1)/p}$ is proportional to $\eta_p(f)[g]$ for odd $p$ and to $\eta_p,1(f)[g] + \eta_p,-1(f)[g]$ for even $p$.

When $\mathcal{P} = \{1\}$, we may hence regard the rigging map $\eta_1$ as arising from (5.14) with only minor strengthening of our technical assumptions. When $\mathcal{P} \neq \{1\}$, we may regard the rigging maps $\eta_p$ and $\eta_p,1 + \eta_p,-1$ as arising from (5.14) by peeling off and appropriately renormalising the various divergent contributions, but only after strengthening the assumptions so that some generality is lost, and even then the two signs of $\epsilon$ are recovered only in a fixed linear combination but not individually.

We end with two technical comments. First, it may be possible to find assumptions that interpolate between those in subsections 5.3 and 5.4, allowing both a superselection sector that comes from integer-valued intervals of $\alpha$ and superselection sectors that come from isolated zeroes of $\alpha$. In formula (5.15), the task would be to provide a peeling argument in the $L$-dependence of the second term. In the observable analysis of Appendix B, the task would be to provide a peeling argument in the small $|s|$ behaviour of the integrands in (B.3b).

Second, our quantum theories arise from the integer value set of $\alpha$, both in subsection 5.3 and in subsection 5.4. Neither the averaging formulas nor the observable analysis of Appendix B suggest ways to proceed when $\alpha$ takes no integer values. In (5.20), the
challenge would be to recover from the oscillatory $L$-dependence a map that satisfies the positivity condition (5.11b). A similar oscillatory dependence on $\lambda$ presents the challenge in the observable formula (B.2).

6 Summary and discussion

In this paper we have investigated refined algebraic quantisation under rescalings of a single momentum-type constraint in a Hamiltonian system whose unreduced configuration space is $\mathbb{R}^2$. While such rescalings do not affect the classical reduced phase space, they do affect the options to find a rigging map by which the constraint is implemented in the quantum theory. We found that the rescalings fall into three cases, depending on the choice of the rescaling function. In case (i), the rescaled constraint operator is essentially self-adjoint in the auxiliary Hilbert space, and the quantisation is equivalent to that with identity scaling. In case (ii), the rescaled constraint operator has no self-adjoint extensions and no quantum theory is recovered. In case (iii), the rescaled constraint operator admits a family of self-adjoint extensions, and the choice of the extension has a significant effect on the quantum theory. In particular, the choice determines whether the quantum theory has superselection sectors.

Within case (iii), we analysed in full a subfamily of rescalings and self-adjoint extensions in which the superselection structures turned out to resemble closely that of the Ashtekar-Horowitz-Boulware model [17]. There are however two significant differences, one conceptual and one technical. Conceptually, the superselection sectors in the Ashtekar-Horowitz-Boulware model are determined by the classical potential function in the constraint, while in our system the superselection sectors are determined by a quantisation ambiguity that has no counterpart in the classical system. Technically, in our system it is ‘natural’ to consider a wider family of self-adjoint extensions than the family of potential functions considered in [17], and we duly found a wider set of quantum theories. In particular, while all the quantum theories in [17] have finite-dimensional Hilbert spaces, some of our quantum theories have separable Hilbert spaces, and some of them can even be realised as genuine Hilbert subspaces of the auxiliary Hilbert space.

Within those case (iii) theories that we analysed in full, we found the quantum theory to be insensitive to the remaining freedom in the rescaling function. We in particular discovered situations where the quantum gauge group is $\mathbb{R}$ for generic rescaling functions but reduces to $U(1)$ in the special case of a constant rescaling function: yet this difference between a compact and noncompact gauge group was irrelevant for the quantum theory, and the quantum theory coincided with that which is obtained with the compact gauge group by a projection into the $U(1)$-invariant subspace of the auxiliary Hilbert space.
The formalism of refined algebraic quantisation is thus here able to handle seamlessly the transition between a compact and a noncompact gauge group.

As our system has just one constraint, the quantum gauge transformations form an Abelian Lie group both before and after the constraint rescaling. In a system with more constraints, constraint rescalings can relate closed gauge algebras to open ones, and even among closed algebras they can change the underlying Lie group, in particular taking an Abelian Lie group to a non-Abelian one. Extending the analysis of this paper to more than one constraint via the BRST tools of [28] would hence raise a number of new issues. However, we emphasise that while the search for rigging maps in this paper used group averaging as the starting point, the nontrivial part in showing that a rigging map is actually recovered was in the action of the quantum gauge transformations on the observables, and in subsection 5.4 a direct analysis of these observables allowed us in fact to find more rigging maps than suggested by the group averaging. Should notions of averaging be difficult to generalise to rescalings with more than one constraint, it may hence well be sufficient to focus directly on the action of the quantum gauge transformations on the observables.

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A Appendix: BRST quantisation and the averaging proposal

In this appendix we review how the group averaging formula (3.4) for our system emerges from the BRST formalism, adopting the conventions of [28]. For detailed expositions of the BRST formalism we refer to [3, 31].

Domains of operators are unspecified throughout the appendix and Hermiticity considerations remain formal.

A.1 Classical BRST formalism

Let \( q \) and \( p \) denote respectively the coordinates \((\theta, x)\) and the momenta \((p_\theta, p_x)\) on the original phase space \( \Gamma \). The new canonical variables are the Lagrange multiplier \( \lambda \), the ghost \( C \), the antighost \( \bar{C} \) and their respective conjugate momenta \( \pi \), \( \bar{P} \) and \( P \). The ghost
number \( \text{gh}(\cdot) \) and Grassmann parity \( \epsilon(\cdot) \) of the variables are

\[
\begin{align*}
\text{gh}(q) &= \text{gh}(\lambda) = \text{gh}(p) = \text{gh}(\pi) = 0, & \epsilon(q) &= \epsilon(\lambda) = \epsilon(p) = \epsilon(\pi) = 0, \\
\text{gh}(\mathcal{C}) &= \text{gh}(\mathcal{P}) = 1, & \epsilon(\mathcal{C}) &= \epsilon(\mathcal{P}) = 1, \\
\text{gh}(\overline{\mathcal{C}}) &= \text{gh}(\overline{\mathcal{P}}) = -1, & \epsilon(\overline{\mathcal{C}}) &= \epsilon(\overline{\mathcal{P}}) = 1.
\end{align*}
\] (A.1)

All the bosonic variables are real-valued. Of the fermionic variables, we take the pair \((\mathcal{C}, \mathcal{P})\) to be real and the pair \((\overline{\mathcal{C}}, \overline{\mathcal{P}})\) purely imaginary [28]. The nonvanishing (graded) Poisson brackets are

\[
\{ \theta, p_{\theta} \} = \{ x, p_x \} = \{ \lambda, \pi \} = 1, \quad \text{(bosonic)} \] (A.2a)

\[
\{ \mathcal{C}, \mathcal{P} \} = \{ \overline{\mathcal{C}}, \overline{\mathcal{P}} \} = -i. \quad \text{(fermionic)} \] (A.2b)

We note in passing that the fermionic brackets (A.2b) are imaginary because of the fermionic reality conditions. If \( \mathcal{C} \) and \( \overline{\mathcal{C}} \) are instead chosen real and their conjugate momenta imaginary [3], the fermionic brackets (A.2b) must be taken real, with concomitant changes in the subsequent formulas; in particular, a Hermitian \((\cdot, \cdot)\)_{BRST} is then obtained by \( c = \pm i \) in (A.12) below. The fermionic reality convention does however not affect the content of the resulting quantum theory.

As the original Lagrange multiplier \( \lambda \) has become a phase space variable, the extended system has two constraints: the original constraint \( \phi \) (2.1) and the new constraint \( \pi \). The BRST generator \( \Omega \) has contributions from both constraints and reads

\[
\Omega := \phi \mathcal{C} - i\pi \mathcal{P}. \] (A.3)

\( \Omega \) is real and satisfies \( \{ \Omega, \Omega \} = 0. \)

### A.2 BRST quantisation

We choose a representation in which the wave functions depend on the bosonic coordinates \((\theta, x, \lambda)\) and the fermionic momenta \((\mathcal{P}, \mathcal{P})\). A wave function can be expanded in the fermionic variables as

\[
\Psi(\theta, x, \lambda, \mathcal{P}, \mathcal{P}) = \psi(\theta, x, \lambda) + \Psi^1(\theta, x, \lambda)\mathcal{P} + \Psi_1(\theta, x, \lambda)\mathcal{P} + \Psi^1_1(\theta, x, \lambda)\mathcal{P}^2. \] (A.4)
where $\psi$, $\Psi^1$, $\Psi_1$ and $\Psi_1^1$ are complex-valued. The action of the fundamental operators reads

\begin{align*}
\hat{\theta} \Psi &= \theta \Psi, & \hat{p}_\theta \Psi &= -i \frac{\partial \Psi}{\partial \theta}, & \text{(A.5a)} \\
\hat{x} \Psi &= x \Psi, & \hat{p}_x \Psi &= -i \frac{\partial \Psi}{\partial x}, & \text{(A.5b)} \\
\hat{\lambda} \Psi &= \lambda \Psi, & \hat{\pi} \Psi &= -i \frac{\partial \Psi}{\partial \lambda}, & \text{(A.5c)} \\
\hat{C} \Psi &= \frac{\partial \Psi}{\partial \mathcal{P}}, & \hat{P} \Psi &= \mathcal{P} \Psi, & \text{(A.5d)} \\
\hat{C} \Psi &= \frac{\partial \Psi}{\partial \mathcal{P}}, & \hat{P} \Psi &= \mathcal{P} \Psi, & \text{(A.5e)}
\end{align*}

where the superscript $l$ on the fermionic derivative indicates the left derivative. The (graded) commutators of the fundamental operators are equal to $i$ times the corresponding (graded) Poisson brackets (A.2):

\begin{align*}
\{ \hat{\theta}, \hat{p}_\theta \} = \{ \hat{x}, \hat{p}_x \} &= \{ \hat{\lambda}, \hat{\pi} \} = i, & \text{(bosonic)} & \text{(A.6a)} \\
\{ \hat{C}, \hat{\mathcal{P}} \} = \{ \hat{C}, \hat{\mathcal{P}} \} &= 1. & \text{(fermionic)} & \text{(A.6b)}
\end{align*}

The physical quantum states satisfy

\begin{align*}
\hat{\Omega} \Psi &= 0, & \text{(A.7a)} \\
\hat{N}_\mathcal{G} \Psi &= 0, & \text{(A.7b)}
\end{align*}

where the BRST operator $\hat{\Omega}$ and the ghost number operator $\hat{N}_\mathcal{G}$ are defined by

\begin{align*}
\hat{\Omega} := \hat{\phi} \hat{C} - i \hat{\pi} \hat{\mathcal{P}}, & & \text{(A.8)} \\
\hat{\phi} := -i \left( M \partial_\theta + \frac{1}{2} (\partial_\theta M) \right), & & \text{(A.9)} \\
\hat{N}_\mathcal{G} := \hat{\mathcal{P}} \hat{C} - \hat{\mathcal{P}} \hat{\mathcal{C}}. & & \text{(A.10)}
\end{align*}

If $X$ is any state, the transformation

$$
\Psi \mapsto \Psi' := \Psi + \hat{\Omega} X
$$

is called a BRST gauge transformation, and states related by a gauge transformation are called gauge-equivalent. As $[\hat{\Omega}, \hat{\Omega}] = 2(\hat{\Omega})^2 = 0$, a gauge transformation preserves the condition (A.7a), and if $X$ has ghost number $-1$, a gauge transformation also preserves the condition (A.7b). A gauge transformation in which $X$ has ghost number $-1$ hence takes physical states to physical states.
The BRST ‘inner product’ is the sesquilinear form

\[(\Psi, \Upsilon)_{\text{BRST}} := c \int d\lambda d\theta dx d\overline{P} dP \Psi^*(\theta, x, \lambda, \overline{P}, P) \Upsilon(\theta, x, \lambda, P, \overline{P}), \tag{A.12}\]

where * denotes complex conjugation and c is a nonzero constant that may a priori take complex values. This definition has a number of desirable properties that are independent of c. First, \((\cdot, \cdot)_{\text{BRST}}\) is compatible with the reality conditions of the classical fundamental variables, in the sense that \(\hat{C}\) and \(\hat{P}\) are antihermitian and all the other fundamental operators in (A.5) are Hermitian. Second, the BRST operator \(\hat{\Omega}\) is Hermitian with respect to \((\cdot, \cdot)_{\text{BRST}}\), which property is compatible with the reality of the classical BRST charge \(\Omega\): the only nontrivial ordering issue in \(\hat{\Omega}\) is that of the purely bosonic factor \(\hat{\phi}\) (A.9). Third, from the Hermiticity of \(\hat{\Omega}\) it follows that \((\cdot, \cdot)_{\text{BRST}}\) on physical states depends on the states only through their gauge-equivalence class.

If c is real, \((\cdot, \cdot)_{\text{BRST}}\) is Hermitian, but it fails to provide a genuine inner product because it is not positive definite. We shall comment on the choice of c below.

### A.3 Averaging

To connect the BRST quantisation to a formalism that only involves bosonic variables, it is not possible simply to drop all powers of the fermions from the quantum states since the fermionic integrations in (A.12) annihilate such states. There is however the option to start from states without fermions and evaluate \((\cdot, \cdot)_{\text{BRST}}\) on suitable gauge-equivalent states.

Suppose that \(\Psi\) and \(\Upsilon\) are physical states without fermions. The physical state conditions (A.7) imply that the states take the form

\[\Psi = \psi(\theta, x), \quad \Upsilon = \chi(\theta, x), \tag{A.13}\]

where the \(\lambda\)-independence follows from the BRST condition (A.7a). We wish to define a regularised sesquilinear form \((\cdot, \cdot)^r_{\text{BRST}}\) by

\[(\psi, \chi)^r_{\text{BRST}} := \left(\psi, \hat{V}\chi\right)_{\text{BRST}}, \tag{A.14}\]

where \(\hat{V} := \exp(\{\hat{\Omega}, \hat{K}\})\) and \(\hat{K}\) is a suitable operator with ghost number \(-1\). Note that as \(\chi\) and \(\hat{V}\chi\) are gauge-equivalent physical states, the right-hand side of (A.14) would be independent of \(\hat{K}\) if well defined for all \(\hat{K}\). \(\hat{K}\) is called the gauge-fixing fermion.

The usual procedure is to choose a Hermitian gauge-fixing fermion by \(\hat{K} = -\lambda\overline{P}\) \([3, 28, 35, 36, 37, 38]\). It follows that \(\{\hat{\Omega}, \hat{K}\} = -\lambda\hat{\phi} - \overline{P}P\). The integrations over the ghost momenta in (A.14) are elementary and we obtain

\[(\psi, \chi)^r_{\text{BRST}} = c \int d\lambda d\theta dx \psi^*(\theta, x)\exp(-\lambda\hat{\phi})\chi(\theta, x). \tag{A.15}\]
The constant $c$ is then chosen equal to 1. Finally, the quantisation of the pair $(\lambda, \pi)$ is understood in a sense that makes the spectrum of $\hat{\lambda}$ purely imaginary \cite{47}. The final formula for $(\cdot, \cdot)_{\mathrm{BRST}}^r$ reads

$$(\psi, \chi)_{\mathrm{BRST}}^r = \int d\mu d\theta dx \psi^*(\theta, x) \left[ \exp(i\mu\hat{\phi}) \chi \right](\theta, x), \quad (A.16)$$

where $\mu$ is real-valued. Formula (A.16) provides the candidate for a refined algebraic quantisation sesquilinear form for the purely bosonic system, and it is our starting point (3.4) in section 3.

An alternative is to choose the antihermitian gauge-fixing fermion $\hat{K} = i\hat{\lambda}\hat{P}$ \cite{35}. This choice makes $\hat{V}$ unitary, and integration over the ghosts yields

$$(\psi, \chi)_{\mathrm{BRST}}^r = -ic \int d\lambda d\theta dx \psi^*(\theta, x) \left[ \exp(i\lambda\hat{\phi}) \chi \right](\theta, x). \quad (A.17)$$

Choosing now $c = i$ and quantising the pair $(\lambda, \pi)$ in a way that keeps the spectrum of $\hat{\lambda}$ real, we again arrive at (A.16).

**B Appendix: Intertwining property of the rigging maps**

In this Appendix we verify that the rigging maps (5.16) and (5.19) have the intertwining property (5.12), completing the proof of Theorems 5.1 and 5.2. We follow the method introduced in Appendix B of \cite{17}.

To begin, we assume just that $\alpha$ and $N$ satisfy condition (a) of subsection 5.3. The fork between the remaining conditions of subsections 5.3 and 5.4 takes place after (B.3).

Let $A \in \mathcal{A}_{\text{obs}}$. Let $m$ and $n$ be fixed integers and let $f, g \in \Phi$ such that their only components in the decomposition (5.7) are respectively $f_m$ and $g_n$. As $U_c(\lambda)$ is unitary and commutes with $A^\dagger$, we have $(U_c(-\lambda)f, A^\dagger g)_c = (f, U_c(\lambda)A^\dagger g)_c = (f, A^\dagger U_c(\lambda)g)_c = (Af, U_c(\lambda)g)_c$. Using (5.8) and (5.9), the leftmost and rightmost expressions yield

$$\int dx \, e^{iR_m(x)\lambda} \frac{f_m(x)}{f_m(x)} (A^\dagger g)_m(x) = \int dx \, e^{iR_m(x)\lambda} \frac{(Af)_n(x)}{(Af)_n(x)} g_n(x). \quad (B.1)$$

We denote the intervals in which $R_q$ has no stationary points by $I_{qr}$, where the second index $r$ enumerates the intervals with given $q$. We similarly denote the intervals in which $R_q$ is constant by $\tilde{I}_{qr}$. We take these intervals to be open and inextendible, and we understand “interval” to include half-infinite intervals and the full real line.

On the left-hand side (respectively right-hand side) of (B.1), we break the integral over $x \in \mathbb{R}$ into a sum of integrals over $\{I_{mr}\}$ and $\{\tilde{I}_{mr}\}$ (\{I_{nr}\} and $\{\tilde{I}_{nr}\}$). By condition
(a) of subsection 5.3 and the assumptions about \( \Phi \), the sums contain at most finitely many terms.

Let \( R_{qr} \) be the restriction of \( R_q \) to \( I_{qr} \), and let \( R_{qr}^{-1} \) be the inverse of \( R_{qr} \). Changing the integration variable in each \( I_{mr} \) on the left-hand side to \( s := R_{mr}(x) \) and in each \( I_{nr} \) on the right-hand side to \( s := R_{nr}(x) \), we obtain

\[
\sum \int_{I_{mr}} dx \ e^{iR_m(x)\lambda} \overline{f_m(x)} \left( A^1 g \right)_m(x) + \int ds \ e^{i\lambda s} \sum_r \frac{\overline{f_m(A^1 g)_m}}{|R_m|} (R_{mr}^{-1}(s)) = \sum \int_{I_{nr}} dx \ e^{iR_n(x)\lambda} \overline{(Af)_n(x)} g_n(x) + \int ds \ e^{i\lambda s} \sum_r \left( \frac{\overline{(Af)_n(g_n)}}{|R_n|} \right) (R_{nr}^{-1}(s)) ,
\]

where for given \( s \) the sum over \( r \) on the left-hand side (respectively right-hand side) is over those \( r \) for which \( s \) is in the image of \( R_{mr} \) (\( R_{nr} \)).

We now regard each side of (B.2) as a function of \( \lambda \in \mathbb{R} \). On each side, the integral over \( s \) is the Fourier transform of an \( L^1 \) function and hence vanishes as \( |\lambda| \to \infty \) by the Riemann-Lebesgue lemma, whereas the sum over \( \hat{r} \) is a finite linear combination of imaginary exponentials and does not vanish as \( |\lambda| \to \infty \) unless identically zero. A peeling argument shows that (B.2) breaks into the pair

\[
\sum \int_{I_{mr}} dx \ e^{iR_m(x)\lambda} \overline{f_m(x)} \left( A^1 g \right)_m(x) = \sum \int_{I_{nr}} dx \ e^{iR_n(x)\lambda} \overline{(Af)_n}(x) g_n(x) , \quad (B.3a)
\]
\[
\int ds \ e^{i\lambda s} \sum_r \left( \frac{\overline{f_m(A^1 g)_m}}{|R_m|} \right) (R_{mr}^{-1}(s)) = \int ds \ e^{i\lambda s} \sum_r \left( \frac{\overline{(Af)_n(g_n)}}{|R_n|} \right) (R_{nr}^{-1}(s)) . \quad (B.3b)
\]

Suppose now that condition (b) of subsection 5.3 holds. A peeling argument shows that the \( \lambda \)-independent component of (B.3a) reads

\[
\eta_\infty(f)[A^1 g] = \eta_\infty(Af)[g] , \quad (B.4)
\]

where \( \eta_\infty \) is defined in (5.16). By linearity, (B.4) continues to hold for all \( f \) and \( g \) in \( \Phi \). \( \eta_\infty \) hence has the intertwining property (5.12). This completes the proof of Theorem 5.1.

Suppose then that conditions (b') and (c) of subsection 5.4 hold. Examination of the integrands in (B.3b) near \( s = 0 \) by the technique of Appendix B of [17] shows that

\[
\eta_p(f)[A^1 g] = \eta_p(Af)[g] , \quad (B.5a)
\]
\[
\eta_{pe}(f)[A^1 g] = \eta_{pe}(Af)[g] , \quad (B.5b)
\]

for all \( p \) and \( \epsilon \) for which the maps \( \eta_p \) and \( \eta_{pe} \) (5.19) are defined. By linearity, (B.5) continues to hold for all \( f \) and \( g \) in \( \Phi \). Each \( \eta_p \) and \( \eta_{pe} \) hence has the intertwining property (5.12). This completes the proof of Theorem 5.2.
### C Appendix: Lemmas on asymptotics

In this appendix we record two lemmas on asymptotics of integrals that occur in section 5.

**Lemma C.1** Let \( f \in C_0^\infty(\mathbb{R}) \), \( L > 0 \), \( p \in \{1, 2, \ldots\} \) and

\[
G_p(L) := \int_{-\infty}^{\infty} f(u) \frac{\sin(Lu^p)}{u^p} \, du. 
\]

As \( L \to \infty \),

\[
G_p(L) = \sum_{q=0}^{p-1} K_{p,q} f^{(q)}(0) L^{(p-1-q)/p} + O(L^{-1/p})
\]

where

\[
K_{p,q} = \frac{\sqrt{\pi} \, 2^{(q+1-p)/p} \Gamma(q+1) \Gamma(q+1)}{pq! \Gamma(3p-q-1)}.
\]

**Proof.** (Sketch.) We replace \( f(u) \) in (C.1) by its Taylor series about the origin, including terms up to \( u^{p-1} \), at the expense of an error that is \( O(L^{-1/p}) \). The terms in the Taylor series give respectively the terms shown in (C.2) plus an error that is \( O(L^{-1}) \). \( \blacksquare \)

Let \( f \in C_0^\infty(\mathbb{R}) \) and \( R \in C^\infty(\mathbb{R}) \). Let \( R \) have at most finitely many zeroes and at most finitely many stationary points, and let all stationary points of \( R \) be of finite order. Denote the zeroes of \( R \) by \( x_{pj} \), where \( p \in \{1, 2, \ldots\} \) is the order of the lowest nonvanishing derivative of \( R \) at \( x_{pj} \) and \( j \) enumerates the zeroes with given \( p \). For \( L > 0 \), let

\[
I(L) := \int_{-\infty}^{\infty} f(x) \frac{\sin[L R(x)]}{R(x)} \, dx.
\]

**Lemma C.2** As \( L \to \infty \),

\[
I(L) = \sum_{pj} I_{pj}(L) + o(1)
\]

where

\[
I_{pj}(L) = K_{p,0} \left( \frac{p!}{[R^{(p)}(x_{pj})]} \right)^{1/p} f(x_{pj}) L^{(p-1)/p} + \sum_{q=1}^{p-1} A_{pj,q} L^{(p-1-q)/p} + O(L^{-1/p})
\]

and the coefficients \( A_{pj,q} \) can be expressed in terms of derivatives of \( f \) and \( R \) at \( x_{pj} \).

**Proof.** (Sketch.) Lemma C.1 and the techniques of Section II.3 in [41] show that the contribution from a sufficiently small neighbourhood of \( x_{pj} \) is \( I_{pj} \) (C.6). The techniques in Section II.3 in [41] further show that the contributions from outside these small neighbourhoods are \( o(1) \). \( \blacksquare \)

Note that \( K_{1,0} = \pi \). This will be used to choose \( \rho(L) = 2\pi \) in (5.20).
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