A DIAGONAL ON THE ASSOCIAHEDRA

SAMSON SANEBLIDZE AND RONALD UMBLE

Abstract. Let \( C_*(K) \) denote the cellular chains on the Stasheff associahedra. We construct an explicit combinatorial diagonal \( \Delta : C_*(K) \to C_*(K) \otimes C_*(K) \); consequently, we obtain an explicit diagonal on the \( A_\infty \)-operad. We apply the diagonal \( \Delta \) to define the tensor product of \( A_\infty \)-(co)algebras in maximal generality.

1. Introduction

Let \( C_*(K) \) denote the cellular chains on the disjoint union of the Stasheff associahedra \( \{K_n\}_{n \geq 2} \). In this paper we construct an explicit combinatorial diagonal \( \Delta : C_*(K) \to C_*(K) \otimes C_*(K) \) based on a direct decomposition of the top dimensional cells of \( K \). This leads to an explicit diagonal on the \( A_\infty \)-operad and solves a long-standing problem. We apply the diagonal \( \Delta \) to define the tensor product of \( A_\infty \)-(co)algebras in maximal generality. We also include an appendix in which we define an associahedral set \( K \) and lift \( \Delta \) to a diagonal on the chain complex of \( K \).

We mention that Chapoton \cite{Chapoton1}, \cite{Chapoton2} constructed a diagonal on \( C_*(K) \) of the form \( \Delta : C_*(K_n) \to \bigoplus_{i+j=n} C_*(K_i) \otimes C_*(K_j) \), which coincides with the diagonal of Loday and Ronco \cite{LodayRonco} in dimension zero. Whereas Chapoton’s diagonal is primitive on generators, our diagonal is defined by geometrically decomposing the generators. Thus the two are totally different.

2. The Stasheff Associahedra

In his seminal papers of 1963, J. Stasheff \cite{Stasheff} constructs the associahedra \( \{K_{n+2}\}_{n \geq 0} \) as follows: Let \( K_2 = * \); if \( K_{n+1} \) has been constructed, let

\[
L_{n+2} = \bigcup_{r+s=n+3, 1 \leq k \leq n-s+3} (K_r \times K_s)_k
\]

and define \( K_{n+2} = CL_{n+2} \), i.e., the cone on \( L_{n+2} \). The associahedron \( K_{n+2} \) is an \( n \)-dimensional polyhedron, which serves as a parameter space for homotopy associativity in \( n+2 \) variables. The top dimensional face of \( K_{n+2} \) corresponds to a pair of level 1 parentheses enclosing all \( n+2 \) indeterminants; each component \( (K_{n-\ell+2} \times K_{\ell+1})_{i+1} \) of \( \partial K_{n+2} \) corresponds to a pair of level 2 parentheses enclosing \( \ell+1 \) indeterminants beginning with the \((i+1)^{st}\). We denote this parenthesization by

\[
d_{(i,\ell)} = (x_1 \cdots (x_{i+1} \cdots x_{i+\ell+1}) \cdots x_{n+2})
\]
and refer to the inner and outer parentheses as the first and last pair, respectively. Note that indices $i$ and $\ell$ are constrained by

$$\begin{cases} 0 \leq i \leq n, & i = 0 \\ 1 \leq \ell \leq n, & 1 \leq i \leq n \\ 1 \leq \ell \leq n + 1 - i, & 1 \leq i \leq n \end{cases}.$$ 

Thus, there is a one-to-one correspondence between $(n-1)$-faces of $K_{n+2}$ and parenthesizations $d_{i,\ell}$ of $n+2$ indeterminants.

Alternatively, $K_{n+2}$ can be realized as a subdivision of the standard $n$-cube $I^n$ in the following way: Let $\epsilon = 0, 1$. Label the endpoints of $K_3 = [0,1]$ via $\epsilon \mapsto d_{(\epsilon,1)}$. For $1 \leq i \leq n$, let $e^{n-1}_{i,0}$ denote the $(n-1)$-face $(x_1, \ldots, x_{i-1}, \epsilon, x_{i+1}, \ldots, x_n) \subset I^n$ and obtain $K_4$ from $K_3 \times I = I^2$ by subdividing the edge $e^{n-1}_{1,1}$ as the union of intervals $1 \times I_{0,1} \cup 1 \times I_{1,\infty}$. Label the edges of $K_4$ as follows: $e^{1}_{1,0} \leftrightarrow d_{(0,1)}$; $e^{2}_{1,0} \leftrightarrow d_{(2,1)}$; $1 \times I_{0,1} \leftrightarrow d_{(1,1)}$; and $1 \times I_{1,\infty} \leftrightarrow d_{(1,2)}$ (see Figure 1). Now for $0 \leq i \leq j \leq \infty$, let $I_{i,j}$ denote the subinterval $[2^i - 1, 2^{j+1} - 1] \subset I$, where $(2^{\infty} - 1)/2^\infty$ is defined to be 1. For $n > 2$, assume that $K_{n+1}$ has been constructed and obtain $K_{n+2}$ from $K_{n+1} \times I \approx I^n$ by subdividing the $(n-1)$-faces $d_{i,n-i} \times I$ as unions $d_{i,n-i} \times I_{0,i} \cup d_{i,n-i} \times I_{i,\infty}$, $0 < i < n$. Label the $(n-1)$-faces of $K_{n+2}$ as follows:

| Face of $K_{n+2}$ | Label |
|-------------------|-------|
| $e^{n-1}_{i,0}$   | $d_{(0,\ell)}$, $1 \leq \ell \leq n$ |
| $e^{n-1}_{n,1}$   | $d_{(n,1)}$ |
| $d_{(i,\ell)} \times I$ | $d_{(i,\ell)}$, $1 \leq \ell < n - i$, $0 < i < n - 1$ |
| $d_{(i,n-i)} \times I_{0,i}$ | $d_{(i,n-i)}$, $0 < i < n$ |
| $d_{(i,n-i)} \times I_{i,\infty}$ | $d_{(i,n-i+1)}$, $0 < i < n$ |

In Figure 2 we have labeled the 2-faces of $K_5$ that are visible from the viewpoint of the diagram.

Compositions $d_{(i_0,\ell_m)} \cdots d_{(i_2,\ell_2)} d_{(i_1,\ell_1)}$ denote a successive insertion of $m+1$ pairs of parentheses into $n+2$ indeterminants as follows: Given $d_{(i,\ell)} \cdots d_{(i,\ell)}$, $1 \leq r < m$, regard each pair of level 2 parentheses and its contents as a single indeterminant and apply $d_{(i_{r+1},\ell_{r+1})}$. Conclude by inserting a last pair enclosing everything. Note that each parenthesization can be expressed as a unique composition $d_{(i_0,\ell_n)} \cdots d_{(i_1,\ell_1)}$ with $i_{r+1} \leq i_r$ for $1 \leq r < m$, in which case the parentheses inserted by $d_{(i_{r+1},\ell_{r+1})}$ begin at or to the left of the pair inserted by $d_{(i_r,\ell_r)}$. Such compositions are said to have first fundamental form. Thus for $0 \leq k < n$, the $k$-faces of $K_{n+2}$ lie in one-to-one correspondence with compositions $d_{(i_{n-k},\ell_{n-k})} \cdots d_{(i_1,\ell_1)}$ in first fundamental form. The two extremes with $m$ pairs of parentheses inserted as far to the left and right as possible, are respectively denoted by

$$d_{(0,\ell_m)} \cdots d_{(0,\ell_1)} \quad \text{and} \quad d_{(i_0,\ell_{m-1-i_0})} \cdots d_{(i_1,\ell_0-i_1)}$$

where $i_0 = n + 1$ and $i_{r+1} < i_r$, $0 \leq r < m$. In particular, the $n$-fold compositions $d_{(0,1)} \cdots d_{(0,1)}$ and $d_{(1,1)} \cdots d_{(n,1)}$
denote the extreme full parenthesizations of $n + 2$ indeterminants. When $m = 0$, define $d_{(i_m, \ell_m)} \cdots d_{(i_1, \ell_1)} = Id$.

\begin{figure}[h]
\centering
\begin{tikzpicture}
\node at (0,0) {$K_4$};
\node at (-1,0) {$(0,0)$};
\node at (0,0) {$(0,1)$};
\node at (0,1) {$(1,1)$};
\node at (1,0) {$(1,0)$};
\node at (0,1) {$(2,1)$};
\node at (0,-1) {$(0,2)$};
\node at (0,-2) {$(1,0)$};
\draw (0,0) -- (0,1) -- (1,1) -- (1,0) -- (0,0);
\end{tikzpicture}
\caption{$K_4$ as a subdivision of $K_3 \times I$.}
\end{figure}

\begin{figure}[h]
\centering
\begin{tikzpicture}
\node at (0,0) {$K_5$};
\node at (-1,0) {$(1,0,0)$};
\node at (0,0) {$(0,0,1)$};
\node at (0,1) {$(0,0,1)$};
\node at (1,0) {$(0,1,0)$};
\node at (0,1) {$(1,1,0)$};
\node at (1,1) {$(2,1,0)$};
\node at (2,0) {$(2,2,0)$};
\node at (2,1) {$(2,2,1)$};
\node at (2,2) {$(2,2,2)$};
\node at (3,1) {$(2,3,1)$};
\draw (0,0) -- (0,1) -- (1,1) -- (1,0) -- (0,0);
\draw (0,1) -- (1,1) -- (1,2) -- (0,2) -- (0,1);
\draw (1,0) -- (1,1) -- (2,1) -- (2,0) -- (1,0);
\draw (1,1) -- (2,1) -- (2,2) -- (1,2) -- (1,1);
\draw (3,1) -- (2,3) -- (0,0);
\end{tikzpicture}
\caption{$K_5$ as a subdivision of $K_4 \times I$.}
\end{figure}

Alternatively, each face of $K_{n+2}$ can be represented as a planar rooted tree (PRT) with $n + 2$ leaves; its leaves correspond to indeterminants and its nodes correspond to pairs of parentheses. Let $T_{n+2}$ denote the PRT with $n + 2$ leaves attached to the root at a single node $N_0$, called the root node (see Figure 3). The leaves correspond to a single pair of parentheses enclosing all $n + 2$ indeterminants. Now given an arbitrary PRT $T$, consider a node $N$ of valence $r + 1 \geq 4$. Choose a neighborhood $U$ of $N$ that excludes the other nodes of $T$ and note that $T_r \subseteq U \cap T$. Labeling from left to right, index the leaves of $T_r$ from 1 to $r$ as in Figure 3. Perform an $(i, \ell)$-surgery at node $N$ in the following way: Remove leaves $i + 1, \ldots, i + \ell + 1$ of $T_r$, reattach them at a new node $N' \neq N$ and graft in a new branch connecting $N$ to $N'$ (see Figure 4). Now let $n \geq 1$. Given a parenthesization $d_{(i, \ell)}$ of $n + 2$ indeterminants, obtain the PRT $T_{n+2}^{(i, \ell)}$ from $T_{n+2}$ by performing an $(i, \ell)$-surgery at the root node $N_0$ as shown in Figure 4. Inductively, given a parenthesization $d_{(i_m, \ell_m)} \cdots d_{(i_1, \ell_1)}$ of $n + 2$ indeterminants expressed as a composition in first fundamental form, construct the corresponding PRT $T_{n+2}^{(i_1, \ell_1) \cdots (i_m, \ell_m)}$ as follows: Assume that $T_{n+2}^{(i_1, \ell_1) \cdots (i_r, \ell_r)}$ with nodes $N_0, \ldots, N_r$ has been constructed for some $1 \leq r < m$ and note that the root node $N_0$ has valence $n + 3 - \ell_1 - \cdots - \ell_r$. Perform an $(i_{r+1}, \ell_{r+1})$-surgery at
and obtain $T^{(i_1, \ell_1), \ldots, (i_{r+1}, \ell_{r+1})}$ containing a new node $N_{r+1}$ and a new branch connecting $N_0$ to $N_{r+1}$. Finally, define $T^{(i_1, \ell_1), \ldots, (i_m, \ell_m)} = T_{n+2}$ when $m = 0$ and obtain a one-to-one correspondence between $k$-faces of $K_{n+2}$, $0 \leq k \leq n$ and PRT’s $T^{(i_1, \ell_1), \ldots, (i_{n-k}, \ell_{n-k})}$ consisting of $n - k + 1$ nodes and $n + 2$ leaves. In particular, each vertex of $K_{n+2}$ corresponds to a planar binary rooted tree $T^{(i_1, \ell_1), \ldots, (i_{n}, \ell_{n})}_{n+2}$ (see Figure 5).

Now given a $k$-face $a_k \subseteq K_{n+2}$, $k > 0$, consider the two vertices of $a_k$ at which parentheses are shifted as far to the left and right as possible; we refer to these vertices as the minimal and maximal vertices of $a_k$, and denote them by $a_{k}^{\text{min}}$ and $a_{k}^{\text{max}}$, respectively. In particular, the minimal and maximal vertices of $K_{n+2}$ are
the origin and the vertex of \( I^n \) diagonally opposite to it, i.e.,
\[
K_{n+2}^{\text{min}} \leftrightarrow (0, 0, \ldots, 0) \quad \text{and} \quad K_{n+2}^{\text{max}} \leftrightarrow (1, 1, \ldots, 1);
\]
the respective binary trees in Figure 5 correspond to \( K_4^{\text{min}} \) and \( K_4^{\text{max}} \). Given a representation \( T_{n+2}^{(i_1, \ell_1), \ldots, (i_{n-k}, \ell_{n-k})} \) of \( a_k \), construct the minimal (resp., maximal) tree of \( a_k \) by replacing each node of valence \( r \geq 4 \) with the planar binary rooted tree representing \( K_{r-1}^{\text{min}} \) (resp., \( K_{r-1}^{\text{max}} \)). Note that \( a_k^{\text{min}} \) and \( a_k^{\text{max}} \) determine \( a_k \) since their convex hull is a diagonal of \( a_k \). But we can say more.

When a composition of face operators \( d_{(i_m, \ell_m)} \cdots d_{(i_1, \ell_1)} \) is defined we refer to the sequence of lower indices \( I = (i_1, \ell_1), \ldots, (i_m, \ell_m) \) as an admissible sequence of length \( m \); if \( d_{(i_m, \ell_m)} \cdots d_{(i_1, \ell_1)} \) has first fundamental form we refer to the the sequence \( I \) as a type I sequence of length \( m \). The set of all planar binary rooted trees
\[
Y_{n+2} = \{ T_{n+2}^I \mid I \text{ is a type I sequence of length } n \}
\]
is a poset with partial ordering defined as follows (cf. [6]): Say that \( T_{n+2}^{I_p} \leq T_{n+2}^{I_q} \) if there is an edge-path in \( K_{n+2} \) from vertex \( T_{n+2}^{I_p} \) to vertex \( T_{n+2}^{I_q} \) along which parentheses shift strictly to the right. This partial ordering can be expressed geometrically in terms of the following operation on trees: Let \( N_0 \) denote the root node of \( T_{n+2}^{I_0} \) and let \( N \) be a node joining some left branch \( L \) and a right branch or leaf \( R \) in \( T_{n+2}^{I_0} \). Let \( N_L \) denote the node on \( L \) immediately above \( N \). A right-shift through node \( N \) repositions \( N_L \) either at the midpoint of leaf \( R \) or midway between \( N \) and the node immediately above it. Then \( T_{n+2}^{I_p} \leq T_{n+2}^{I_q} \) if there is a right-shift sequence of planar binary rooted trees \( \{ T_{n+2}^{I_r} \}_{p \leq r \leq q} \), i.e., for each \( r < q \), tree \( T_{n+2}^{I_r} \) is obtained from \( T_{n+2}^{I_p} \) by a right-shift through some node in \( T_{n+2}^{I_r} \) (see Figure 6).

![Figure 6: A right-shift sequence of planar binary trees](image)

Let \( I' = (i'_1, \ell'_1), \ldots, (i'_m, \ell'_m) \) be an admissible sequence of length \( m > 0 \) and consider a node \( N' \) distinct from the root node \( N_0 \) in the PRT \( T_{n+2}^{I'} \). Let \( N \) denote the node immediately below \( N' \) and \( NN' \) denote the branch from \( N \) to \( N' \); we refer to the quotient space \( T_{n+2}^{I_p} = T_{n+2}^{I_p}/NN' \) as the \((N, N')\)-contraction of \( T_{n+2}^{I_p} \). Now given a type I sequence \( J' \) of length \( n \), consider the planar binary tree \( T_{n+2}^{J'} \) and let \( T_{n+2}^{J'} \) be the PRT obtained from \( T_{n+2}^{J'} \) by some sequence of \( k \) successive \((N, N')\)-contractions. The subposet \( Y_{n+2}^{J'} \subseteq Y_{n+2} \) of all planar binary rooted trees from which \( T_{n+2}^{J'} \) can be so obtained is exactly the poset of vertices of the \( k \)-face \( a_k \subseteq K_{n+2} \) represented by \( T_{n+2}^{J'} \). In this way, we may regard \( a_k \) as the geometric realization of \( Y_{n+2}^{J'} \) just as we regard a \( k \)-face of the standard \( n \)-simplex as the geometric realization of a \((k+1)\)-subset of a linearly ordered \((n+1)\)-set. In particular, \( K_{n+2} \) is the geometric realization of \( Y_{n+2} \).
We summarize the discussion above as a proposition:

**Proposition 1.** For $0 \leq k \leq n$, the following correspondences preserve combinatorial structure:

$$\{k\text{-faces of } K_{n+2}\} \leftrightarrow \begin{cases} (n-k)\text{-fold compositions of face operators in first fundamental form} \\
\text{Planar rooted trees with } n-k+1 \text{ nodes and } n+2 \text{ leaves} \\
\text{Subsets of planar binary rooted trees } Y^j_{n+2} \end{cases}$$

where $J$ is a type I sequence of length $n-k$.

3. A Diagonal $\Delta$ on $C_*(K_n)$

For notational simplicity, we suppress upper indices $q_1, \ldots, q_m$ in a composition $d_{(i_m, \ell_m)}^{q_m} \cdots d_{(i_2, \ell_2)}^{q_2} d_{(i_1, \ell_1)}^{q_1}$ when $q_{j+1} = q_j + 1$ for all $j \geq 1$; if, in addition, $q_1 = 1$, we suppress all $q$'s.

**Definition 1.** Let $m \geq 2$. A sequence of lower indices $I = (i_1, \ell_1), \ldots, (i_m, \ell_m)$ is admissible whenever the composition of face operators $d_{(i_m, \ell_m)}^{q_m} \cdots d_{(i_2, \ell_2)}^{q_2} d_{(i_1, \ell_1)}^{q_1}$ is defined. The sequence $I$ is a type I (resp. type II) sequence if $I$ is admissible and $i_k \geq i_{k+1}$ (resp. $i_k \leq i_{k+1} + \ell_{k+1}$) for $1 \leq k < m$. The empty sequence ($m = 0$) and sequences of length 1 ($m = 1$) are sequences of types I and II. A composition of face operators $d_{(i_m, \ell_m)}^{q_m} \cdots d_{(i_2, \ell_2)}^{q_2} d_{(i_1, \ell_1)}^{q_1}$ has first (resp. second) fundamental form if $(i_1, \ell_1), \ldots, (i_m, \ell_m)$ is a type I (resp. type II) sequence. When $m = 0$, the composition $d_{(i_m, \ell_m)}^{q_m} \cdots d_{(i_2, \ell_2)}^{q_2} d_{(i_1, \ell_1)}^{q_1}$ is defined to be the identity. An element $b = d_{(i_m, \ell_m)}^{q_m} \cdots d_{(i_2, \ell_2)}^{q_2} d_{(i_1, \ell_1)}^{q_1}(a)$ is expressed in first (resp. second) fundamental form as a face of $a$ if

$$d^{s_1} \cdots d^{s_m} d^{s_n} = (d \cdots d^{s_m}) \cdots (d \cdots d^{s_2}) (d \cdots d^{s_1}),$$

where $s_1 < s_2 < \cdots < s_m$ and each composition $d_{(i_j, \ell_j)}^{q_j} \cdots d_{(i_2, \ell_2)}^{q_2} d_{(i_1, \ell_1)}^{q_1}$ has first (resp. second) fundamental form.

Face operators satisfy the following relations:

$$\begin{align*}
d_{(i_p, \ell_p)}^{q_p} d_{(i_q, \ell_q)}^{p} & = d_{(i_q, \ell_q)}^{q_p} d_{(i_p, \ell_p)}^{p}, & p < q & \quad (1) \\
d_{(i_{q+1}, \ell_{q+1})}^{q+1} d_{(i_q, \ell_q)}^{q} & = d_{(i_q-i_{q+1}, \ell_q)}^{q} d_{(i_{q+1}, \ell_{q+1}+\ell_q)}^{q+1}, & i_{q+1} \leq i_q \leq i_{q+1} + \ell_{q+1} & \quad (2) \\
d_{(i_{q+1}, \ell_{q+1}+\ell_q)}^{q+1} d_{(i_q, \ell_q)}^{q} & = d_{(i_q-i_{q+1}, \ell_q)}^{q} d_{(i_{q+1}, \ell_{q+1}+\ell_q)}^{q+1}, & i_q < i_{q+1} & \quad (3) 
\end{align*}$$

Furthermore, every composition of face operators can be uniquely transformed into first or second fundamental form by successive applications of face relations (1) to (3). For example, when $n = 2$, the following five face operators relate $T_4 \in K_3^3$ to the edges of the pentagon $K_4$:

$$\begin{align*}
d_{(0,2)}(T_4) & \mapsto ( \bullet \bullet \bullet \bullet ) \in K_3 \times K_2 \\
d_{(1,2)}(T_4) & \mapsto ( \bullet \bullet \bullet \bullet \bullet ) \in K_3 \times K_2 \\
d_{(0,1)}(T_4) & \mapsto ( \bullet \bullet \bullet \bullet ) \in K_2 \times K_3 \\
d_{(1,1)}(T_4) & \mapsto ( \bullet \bullet \bullet \bullet \bullet ) \in K_2 \times K_3 \\
d_{(2,1)}(T_4) & \mapsto ( \bullet \bullet \bullet \bullet \bullet \bullet ) \in K_2 \times K_3.
\end{align*}$$
There are four compositions of face operators
\[ d_{(i_1,1)}^1 d_{(i_1,2)}^1 : K_4 \rightarrow K_3 \times K_2 \rightarrow K_2 \times K_2 \times K_2 \]
with \( 0 \leq i_1, i_2 \leq 1 \), and six compositions
\[ d_{(i_1,1)}^2 d_{(i_1,1)}^1 : K_4 \rightarrow K_2 \times K_3 \rightarrow K_2 \times K_2 \times K_2 \]
with \( 0 \leq i_1 \leq 2 \) and \( 0 \leq i_2 \leq 1 \), which pair off via relations (1) to (3) and relate \( T_4 \) to each of the five vertices of \( K_4 \):
\[
\begin{align*}
    d_{(0,1)}^2 d_{(0,1)}^1 (T_4) &= d_{(0,1)}^1 d_{(0,2)}^1 (T_4) \quad \mapsto \quad (((\bullet)\bullet)\bullet) \\
    d_{(0,1)}^2 d_{(1,1)}^1 (T_4) &= d_{(1,1)}^1 d_{(0,2)}^1 (T_4) \quad \mapsto \quad (((\bullet)(\bullet))\bullet) \\
    d_{(1,1)}^2 d_{(1,1)}^1 (T_4) &= d_{(0,1)}^1 d_{(1,2)}^1 (T_4) \quad \mapsto \quad ((\bullet)\bullet(\bullet)) \\
    d_{(1,1)}^2 d_{(2,1)}^1 (T_4) &= d_{(1,1)}^1 d_{(1,2)}^1 (T_4) \quad \mapsto \quad ((\bullet)(\bullet)\bullet) \\
    d_{(0,1)}^2 d_{(0,1)}^1 (T_4) &= d_{(0,1)}^1 d_{(0,2)}^1 (T_4) \quad \mapsto \quad ((\bullet)\bullet)\bullet \\
\end{align*}
\]
These relations encode the fact that when inserting two pairs of parentheses into a string of four variables either pair may be inserted first.

Given a type I sequence \((j_1, n_1 + 1), \ldots, (j_k, n_k + 1)\) and a sequence \((i_1, \ldots, i_{k+1})\) with \( 0 \leq i_1 \leq n_q, 1 \leq q \leq k + 1 \), define a function of two variables
\[
    j(q, r) = \begin{cases} 
    i_q + j_q + n_{q-1} + \cdots + n_r + q - r, & 1 \leq r < q \\
    i_q + j_q, & 1 \leq r = q
    \end{cases}
\]
for \( 1 \leq r \leq q \leq k + 1 \) (assuming \( j_{k+1} = 0 \)), and let
\[
    \beta = \max_{1 \leq r \leq q} \{ r \mid j(q, r) \leq j_{r-1} \}, \quad q > 1,
\]
(assuming \( j_0 = \infty \).)

**Definition 2.** For \( k \geq 0 \), let \( I_k = (j_1, n_1 + 1), (j_2, n_2 + 1), \ldots, (j_k, n_k + 1) \) be a type I sequence. If \( 1 \leq q \leq k + 1 \), the sign of the face \( b = d_{(i_1, \ell_q)}^1 (T_{n+2}^{I_k}) \in C_{n-k-1} (K_{n+2}) \) is defined to be \((-1)^{e_1 + e_2} \), where
\[
    e_1 = (i_q + 1) \ell_q + n_1 + \cdots + n_{q-1},
\]
\[
    e_2 = \begin{cases} 
    0, & 1 \leq \beta = q, \\
    (\ell_q - 1)(n_{q-1} + \cdots + n_{\beta}), & 1 \leq \beta < q
    \end{cases}
\]
and \( \beta \) is defined by (3.3).

Let us construct an explicit diagonal on associahedra in terms of compositions of face operators in first and second fundamental form; the formulas we obtain also determine a DG coalgebra structure on \((C_*(K_{n+2}), d)\).

We begin with an overview of the geometric ideas involved. Let \( 0 \leq q \leq n \) and let \( T_{n-q} \) be a type I sequence. The \( q \)-dimensional generator \( a_q = T_{n+2}^{I_q} \) is associated with a face of \( K_{n+2} \) corresponding to \( n + 2 \) indeterminants with \( n - q + 1 \) pairs of parentheses. Identify \( a_q \) with its associated face of \( K_{n+2} \) and consider the minimal and maximal vertices \( a_q^{\text{min}} \) and \( a_q^{\text{max}} \) of \( a_q \). Define the primitive terms of \( \Delta T_{n+2} \) to be
\[
    T_{n+2}^{\text{min}} \otimes T_{n+2} + T_{n+2} \otimes T_{n+2}^{\text{max}}.
\]
Let \( 0 < p < q = n \) and consider distinct \( p \)-faces \( b \) and \( b' \) of \( T_{n+2} \). Say that \( b \leq b' \) if there is a path of \( p \)-faces from \( b \) to \( b' \) along which parentheses shift strictly to the right. Now given a \( q \)-dimensional face \( a_q \) of \( T_{n+2} \) such that \( a_q^{\text{min}} \neq T_{n+2}^{I_q} \), there is a unique path of \( p \)-faces \( b_1 \leq b_2 \leq \cdots \leq b_r \) with minimal length such that
$T_{n+2}^{\min} = b_{n+2}^{\min}$ and $a_{n+2}^{\min} = b_{n}^{\max}$. Up to sign, we define the non-primitive terms of $\Delta T_{n+2}$ to be
\[
\sum \pm b_j \otimes a_q.
\]

To visualize this, consider the edge $d_{(1,2)}(T_4) \in C_1(K_4)$ whose minimal vertex is the point $(1, \frac{1}{2})$ (see Figure 7). The edges $d_{(0,2)}(T_4) \leq d_{(1,1)}(T_4)$ form a path of minimal length from $(0, 0)$ to $(1, \frac{1}{2})$. Consequently, $\Delta T_4$ contains the non-primitive terms $\{(\pm d_{(0,2)} \pm d_{(1,1)}) \otimes d_{(1,2)}\} (T_4 \otimes T_4)$.

Precisely, for $T_2 \in C_2(K_2)$ define $\Delta T_2 = T_2 \otimes T_2$; inductively, assume that the map $\Delta : C_*(K_{i+2}) \rightarrow C_*(K_{i+2}) \otimes C_*(K_{i+2})$ has been defined for all $i < n$. For $T_{n+2} \in C_n(K_{n+2})$ define
\[
\Delta T_{n+2} = \sum_{0 \leq p \leq p+q=n} (-1)^p d(i'_1, \ell'_1) \cdot \cdots \cdot d(i'_q, \ell'_q) (T_{n+2}) \otimes d(i_p, \ell_p) \cdot \cdots \cdot d(i_1, \ell_1) (T_{n+2})
\]
where
\[
\epsilon = \sum_{j=1}^{q} i'_j (\ell'_j + 1) + \sum_{k=1}^{p} (i_k + k + p + 1) \epsilon_k,
\]
and lower indices $((i_1, \ell_1), \ldots, (i_p, \ell_p); (i'_1, \ell'_1), \ldots, (i'_q, \ell'_q))$ range over all solutions of the following system of inequalities:

\[
(3.4) \begin{cases}
1 \leq i_j < i_{j-1} \leq n + 1 \\
1 \leq \ell_j \leq n + 1 - i_j - \ell_{(j-1)} \\
0 \leq i'_k \leq \min \{i'_r, i_{rk} - \ell'_{(r)(k)}\} \\
1 \leq \ell'_k = \epsilon_k - i'_k - \ell'_{(k-1)}
\end{cases}
\]

where
\[
\{\epsilon_1 < \cdots < \epsilon_q \} = \{1, \ldots, n\} \setminus \{i_1, \ldots, i_p\};
\]
\[
e_0 = \ell_0 = \ell'_0 = i_{p+1} = i'_{q+1} = 0;
\]
\[
i_0 = i'_0 = \epsilon_{q+1} = \ell_{(p+1)} = \ell'_{(q+1)} = n + 1;
\]
\[
\ell_{(u)} = \sum_{j=0}^{u} \ell_j \text{ for } 0 \leq u \leq p + 1;
\]
Theorem 1. The sign in $\Delta (p) \cdot \text{sgn} (b) \cdot \text{sgn} (a)$, using the fact that the cells of $K_{n+2}$ are products of cells $K_{i+2}$ with $i < n$.

Note that right-hand and left-hand factors in each component of $\Delta T_{n+2}$ are expressed in first and second fundamental form, respectively. In particular, the terms given by the extremes $p = 0$ and $p = n$ are the primitive terms of $\Delta T_{n+2}$:

$$\{d_{(0,1)} \cdots d_{(0,1)} \otimes 1 + 1 \otimes d_{(1,1)} \cdots d_{(n,1)}\} (T_{n+2} \otimes T_{n+2}).$$

The sign in $\Delta T_{n+2} = \sum (-1)^p b \otimes a$ is the product of five signs: $(-1)^p = \text{sgn} (b) \cdot \text{sgn} (a), \text{sgn} (a_1) \cdot \text{sgn} (a_2)$, where the face $b_0$ is obtained from $b$ by using the same $\epsilon$'s but all $i_k^p = 0$ (i.e., $i_k^p$ is replaced by $i_k^p - i_{k-1}^p$), the face $a_1$ is obtained from $a$ by using the same $i_k$'s but all $i_k = 1$, and $\text{sgn} (b_0, a_1)$ is the sign of the unshuffle $\{i_p < \cdots < i_1, \epsilon_1 < \epsilon_2 < \cdots < \epsilon_q\}$. Geometrically, $b_0$ and $a_1$ lie on orthogonal faces of the cube $I^n$ and are uniquely defined by the property that the canonical cellular projection $K_{n+2} \rightarrow I^n$ maps $b_0 \mapsto (x_1, \ldots, x_{i_1-1}, 0, \ldots, x_{i_q-1}, 0, \ldots, x_n)$ and $a_1 \mapsto (x_1, \ldots, x_{i_q-1}, 1, \ldots, x_{i_1-1}, 1, \ldots, x_n)$.

Example 1. We obtain by direct calculation:

For $T_3 \in C_1 (K_3)$:

$$\Delta T_3 = (d_{(0,1)} \otimes 1 + 1 \otimes d_{(1,1)}) (T_3 \otimes T_3).$$

For $T_4 \in C_2 (K_4)$:

$$\Delta T_4 = (d_{(0,1)}d_{(0,1)} \otimes 1 + 1 \otimes d_{(1,1)}d_{(2,1)} + d_{(0,2)} \otimes d_{(1,1)} + d_{(0,2)} \otimes d_{(1,2)} + d_{(1,1)} \otimes d_{(1,2)} - d_{(0,1)} \otimes d_{(2,1)}) (T_3 \otimes T_3).$$

For $T_5 \in C_3 (K_5)$:

$$\Delta T_5 = (d_{(0,1)}d_{(0,1)}d_{(0,1)} \otimes 1 + 1 \otimes d_{(1,1)}d_{(2,1)}d_{(3,1)} + d_{(0,3)} \otimes d_{(1,1)}d_{(2,2)} - d_{(1,1)} \otimes d_{(1,2)}d_{(3,1)} + d_{(0,1)}d_{(0,2)} \otimes d_{(1,2)} + d_{(0,3)} \otimes d_{(1,2)}d_{(2,1)} + d_{(0,1)}d_{(1,2)} \otimes d_{(1,3)} + d_{(0,3)} \otimes d_{(1,2)}d_{(2,1)} - d_{(1,2)} \otimes d_{(1,1)}d_{(2,2)} + d_{(0,2)}d_{(0,1)} \otimes d_{(1,2)} + d_{(0,2)}d_{(1,1)} \otimes d_{(1,2)} + d_{(0,2)}d_{(1,1)} \otimes d_{(2,1)} - d_{(0,2)} \otimes d_{(1,1)}d_{(3,1)} + d_{(1,1)}d_{(1,1)} \otimes d_{(1,3)} + d_{(0,2)}d_{(1,1)} \otimes d_{(1,3)} - d_{(0,2)} \otimes d_{(1,2)}d_{(3,1)} + d_{(0,1)}d_{(0,2)} \otimes d_{(1,1)})(T_4 \otimes T_5).$$

Our main result, stated as the following theorem, is proved in Section [4].

Theorem 1. For each $n \geq 0$, the map $\Delta : C_*(K_{n+2}) \rightarrow C_*(K_{n+2}) \otimes C_*(K_{n+2})$ defined above is a chain map.

Identify the sequence of cellular chain complexes $\{C_*(K_n)\}_{n \geq 2}$ with the $A_\infty$-operad $A_\infty$. Since $\Delta$ is extended multiplicatively on decomposable faces, we immediately obtain:

Corollary 1. The sequence of chain maps $\{\Delta : C_*(K_n) \rightarrow C_*(K_n) \otimes C_*(K_n)\}_{n \geq 2}$ induces a morphism of operads $A_\infty \rightarrow A_\infty \otimes A_\infty$. 

4. A Proof of Theorem 4

In this section we prove that the diagonal $\Delta$ defined in Section 3 is a chain map. We begin with some preliminaries. It will be convenient to rewrite face relation (3) as follows:

\[ d_{(i_q+1,\ell_q+1)}^{q+1} d_{(i_q,\ell_q)}^{q} = d_{(i_q-\ell_q+1,\ell_q)}^{q+1} d_{(i_q+1,\ell_q+1)}^{q}; \quad i_q > i_{q+1} + \ell_{q+1}. \quad \text{(3')}
\]

**Definition 3.** For $1 \leq k \leq m$, the $k^{th}$ left-transfer $\tau^k_\ell$ of a composition $d_{(i_m, \ell_m)} \cdots d_{(i_2, \ell_2)} d_{(i_1, \ell_1)}$ in first fundamental form is one of the following compositions:

(a) If $i_{k+1} + \ell_{k+1} \geq i_k$, apply face relation (2) to $d^{k+1} d^k$ and obtain

\[ \cdots d_{(i_k-i_{k+1}, \ell_k)}^{k} d_{(i_{k+1}, \ell_{k+1}+\ell_k)}^{k+1} \cdots ; \]

then successively apply face relation (1) to $d^{j+1} d^j$ for $j = k, k+1, \ldots, m-2$, and obtain

\[ \tau^k_\ell = d_{(i_k-i_{k+1}, \ell_k)}^{k} d_{(i_{k+1}, \ell_{k+1}+\ell_k)}^{k+1} \cdots d_{(i_1, \ell_1)}^{1}. \]

(b) If $i_{k+1} + \ell_{k+1} < i_k \leq i_{q+1} + \ell_{(q+1)} - \ell_{(k)}$ for some smallest integer $q > k$, successively apply face relation (3') to $d^{q+1} d^q$ for $j = k, k+1, \ldots, q$, and obtain

\[ \cdots d_{(i_q+1, \ell_q+1)}^{q+1} d_{(i_q, \ell_q)}^{q} \cdots d_{(i_k+1, \ell_{k+1})}^{k} d_{(i_k-\ell_k+1, \ell_k)}^{k+1} \cdots d_{(i_1, \ell_1)}^{1}. \]

Apply face relation (2) to $d^{q+1} d^q$; then successively apply face relation (1) to $d^{j+1} d^j$ for $j = q, q+1, \ldots, m-2$, and obtain

\[ \tau^k_\ell = d_{(i_k, \ell_k)}^{q} d_{(i_m, \ell_m)}^{m-1} \cdots d_{(i_k+1, \ell_{k+1})}^{k} d_{(i_{k-1}, \ell_{k-1})}^{k-1} \cdots d_{(i_1, \ell_1)}, \]

where $i = i_k - i_{q+1} + \ell_{(k)} - \ell_{(q)}$.

(c) Otherwise, successively apply face relation (3') to $d^{j+1} d^j$ for $j = k, k+1, \ldots, m-1$, and obtain

\[ \tau^k_\ell = d_{(i_m, \ell_m)}^{m} \cdots d_{(i_k+1, \ell_{k+1})}^{k} d_{(i_{k-1}, \ell_{k-1})}^{k-1} \cdots d_{(i_1, \ell_1)}^{1}, \]

where $i = i_k + \ell_{(k)} - \ell_{(m)}$.

**Definition 4.** For $1 \leq k \leq m$, the $k^{th}$ right-transfer $\tau^k_\ell$ of a composition $d_{(i_m, \ell_m)} \cdots d_{(i_2, \ell_2)} d_{(i_1, \ell_1)}$ in first fundamental form is one of the following compositions:

(a) If $i_p + \ell_{(p)} \leq i_k + \ell_{(k)} < i_{p-1} + \ell_{(p-1)}$ for some greatest integer $1 < p \leq k$, successively apply face relation (3') to $d^{j+1} d^j$ for $j = p-1, \ldots, 2, 1; j = p, \ldots, 3, 2; \ldots, j = k-1, k-2, \ldots, k-(p-1)$. Then apply face relation (2) to $d^{j+1} d^j$ for $j = k-p, \ldots, 2, 1$ and obtain

\[ \tau^k_\ell = d_{(i_m, \ell_m)} \cdots d_{(i_k+1, \ell_{k+1})}^{k} d_{(i_{p-1}-\ell_{p-1})}^{k-1} \cdots d_{(i_1, \ell_1)}, \]

where $\ell = \ell_{(k)} - \ell_{(p-1)}$. 


(b) Otherwise, successively apply face relation (2) to \( d^{j+1}d^i \) for \( j = k - 1, k - 2, \ldots, 2, 1 \), and obtain

\[
\tau^k_r = d_{(i_m, \ell_m)} \cdots d_{(i_{k+1}, \ell_{k+1})} d^{k-1}_{(i_{k-1} - i_k, \ell_{k-1})} \cdots d_{(i_1 - i_k, \ell_1)} d_{(i_k, \ell_k)}.
\]

Note that if \( I = (i_1, \ell_1), \ldots, (i_m, \ell_m) \) is a type \( I \) sequence and \( a = d_I \left( T_{n+2} \right) \in C_{n-m} \left( K_{n+2} \right) \), then for each \( k \), the \( k^{th} \) right transfer \( \tau^k_r \left( T_{n+2} \right) \) expresses \( a \) in first fundamental form as a face of some \((n-1)\)-face \( b \) of \( T_{n+2} \). The expressions \( \tau^1_r \left( T_{n+2} \right), \ldots, \tau^m_r \left( T_{n+2} \right) \) determine the \( m \) distinct \((n-1)\)-faces \( b \) containing \( a \).

There are analogous left and right transfers for compositions in second fundamental form.

**Definition 5.** For \( 1 \leq k \leq m \), the \( k^{th} \) left-transfer \( \tau^k_l \) of a composition \( d_{(i_m, \ell_m)} \cdots d_{(i_2, \ell_2)} d_{(i_1, \ell_1)} \) in second fundamental form is one of the following compositions:

(a) If \( i_{k+1} \leq i_k \), apply face relation (2) to \( d^{k+1}d^k \), then successively apply face relation (1) to \( d^{j+2}d^j \) for \( j = k, k + 1, \ldots, m - 2 \), and obtain

\[
\tau^k_l = d_{(i_k - i_{k+1}, \ell_k)} d^{m-1}_{(i_m, \ell_m)} \cdots d^{k+1}_{(i_{k+2}, \ell_{k+2})} d^{k}_{(i_{k+1} + \ell_{k+1}, \ell_k)} \cdots d_{(i_1, \ell_1)}.
\]

(b) If \( i_{k+1} > i_k \geq i_{q+1} \) for some smallest integer \( q > k \), successively apply face relation (3) to \( d^{q+1}d^q \) for \( j = k, k + 1, \ldots, q \). Apply face relation (2) to \( d^{q+1}d^q \), then successively apply face relation (1) to \( d^{j+2}d^j \) for \( j = q, q + 1, \ldots, m - 2 \), and obtain

\[
\tau^k_l = d_{(i_k - i_{q+1}, \ell_k)} d^{m-1}_{(i_m, \ell_m)} \cdots d^{q+1}_{(i_{q+2}, \ell_{q+2})} d^{q}_{(i_{q+1} + \ell_{q+1}, \ell_k)} \cdots d^{q+1}_{(i_{q+2}, \ell_{q+2})} d^{q}_{(i_{q+1} + \ell_{q+1}, \ell_k)} \cdots d_{(i_1, \ell_1)}.
\]

(c) Otherwise, successively apply face relation (3) to \( d^{j+1}d^j \) for \( j = k, k + 1, \ldots, m - 1 \), and obtain

\[
\tau^k_l = d^{m}_{(i_k, \ell_k)} d^{m-1}_{(i_m, \ell_m)} \cdots d^{k+1}_{(i_{k+2} + \ell_{k+2}, \ell_k)} d^{k}_{(i_{k+1} + \ell_{k+1}, \ell_k)} \cdots d_{(i_1, \ell_1)}.
\]

**Definition 6.** For \( 1 \leq k \leq m \), the \( k^{th} \) right-transfer \( \tau^k_r \) of a composition \( d_{(i_m, \ell_m)} \cdots d_{(i_2, \ell_2)} d_{(i_1, \ell_1)} \) in second fundamental form is one of the following compositions:

(a) If \( i_{p-1} < i_k \leq i_p \) for some greatest integer \( 1 < p \leq k \), successively apply face relation (3) to \( d^{j+1}d^j \) for \( j = p-1, \ldots, 2, 1 ; j = p, \ldots, 3, 2 \); \( j = k-1, k-2, \ldots, k-(p-1) \). Then apply face relation (2) to \( d^{j+1}d^j \) for \( j = k - p, \ldots, 2, 1 \) and obtain

\[
\tau^k_r = d_{(i_m, \ell_m)} \cdots d_{(i_{k+1}, \ell_{k+1})} d^{k}_{(i_{p-1} - i_p, \ell_{p-1})} d^{k-p+2}_{(i_1, \ell_1)} d^{k-p}_{(i_k - i_{k-1}, \ell_{k-1})} \cdots d^{k-p}_{(i_{p-1} - i_p, \ell_{p-1})} d^{k-p}_{(i_k + \ell_{k+1}, \ell_1)} d_{(i_1, \ell_1)}.
\]

(b) Otherwise, successively apply face relation (2) to \( d^{j+1}d^j \) for \( j = k-1, \ldots, 2, 1 \), and obtain:

\[
\tau^k_r = d_{(i_m, \ell_m)} \cdots d_{(i_{k+1}, \ell_{k+1})} d^{k}_{(i_{k-1} - i_k, \ell_{k-1})} d^{k}_{(i_1 - i_k, \ell_1)} d_{(i_1, \ell_1)}.
\]
Once again, if $I = (i_1, \ell_1), \ldots, (i_m, \ell_m)$ is a type II sequence and $a = d_t(T_{n+2}) \in C_{n-m} (K_{n+2})$, then for each $k$, the $k^{th}$ right-transfer $\tau^k_r(T_{n+2})$ expresses $a$ in second fundamental form as a face of some $(n-1)$-face $b$; the expressions $\tau^1_r(T_{n+2}), \ldots, \tau^m_r(T_{n+2})$ determine the $m$ distinct $(n-1)$-faces $b$ containing $a$. Thus two $(n-m)$-faces $a_1$ and $a_2$ expressed in first and second fundamental form, respectively, are contained in the same $(n-1)$-face $b$ if and only if there exist right transfers $\tau^k_r$ such that $a_j = \tau^k_r(T_{n+2}), j = 1, 2$, and $b = d_{(i,\ell)}(T_{n+2})$, where $d_{(i,\ell)}$ is the rightmost face operator common to $\tau^k_r$ and $\tau^k_r$. We state this formally in the following lemma:

Lemma 1. Let $0 \leq m_1, m_2 \leq n$ and assume that $a_1 = d_{(i_{m_1},\ell_{m_1})} \cdots d_{(i_1,\ell_1)}(T_{n+2})$ and $a_2 = d_{(i_{m_2},\ell_{m_2})} \cdots d_{(i_1,\ell_1)}(T_{n+2})$ are expressed in first and second fundamental form, respectively. Then $a_1$ and $a_2$ are contained in the same $(n-1)$-face $d_{(i,\ell)}(T_{n+2})$ if and only if there exist integers $k_j \leq m_j$ and greatest integers $1 \leq p_j < k_j$ for $j = 1, 2$, such that

$$i^j_{p_2} < i^j_{p_2},$$

and

$$i_{k_1} + \ell_{(k_1)} - \ell_{(p_1)} < i_{p_1}$$

Consider a solution $((i_1, \ell_1), \ldots, (\ell_p,\ell_p); (i'_1, \ell'_1), \ldots, (\ell'_p,\ell'_p))$ of system (3.1) and its related $(p, q)$-shuffle $\{i_p < \cdots < i_1, \epsilon_1 < \epsilon_2 < \cdots < \epsilon_q\}$. Given $1 \leq k \leq q + 1$, let $k_1 = k - 1, k_{j+1} = o'(t_{k_j})$ for $j \geq 1$ and note that $k_{j+1} < k_j$ for all $j$. For $0 \leq m < \ell'_k$, consider the following selection algorithm:

1. If $k = 1$ then $z = p + 1 - i'_1 - m$
2. Else $z = n + 2, j = 0$
3. Repeat
   1. $j \leftarrow j + 1$
   2. If $i^j_{k_j} < i^j_{k_j} + m$ then $i_z = \epsilon_{k_j} - i^j_{k_j} + i^j_{k} + m$
   3. If $i_{k_j} - \ell'_{(k_{j+1})} = i'_k + m$ then $z = t_{k_j}$
4. Until $z < n + 2$
5. Endif

It will be clear from the proof of Lemma 2 below that the selection algorithm eventually terminates.

Example 2. Let $p = q = 4$ and consider the following solution of system (3.1):

$$((7, 1), (6, 1), (4, 2), (2, 3); (0, 1), (1, 1), (1, 2), (0, 4)).$$

Then $t_1 = 5, t_2 = 4, t_3 = 3, t_4 = 4$ and the selection algorithm produces the following:

\[
\begin{array}{cccccccc}
 k & 1 & 2 & 3 & 3 & 4 & 4 & 4 \\
 m & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 3 \\
 z & 5 & 5 & 4 & 9 & 5 & 4 & 2 & 1
\end{array}
\]

The key to our proof that $\Delta$ is a chain map is given by our next lemma.
Lemma 2. Given $1 \leq k \leq q + 1$ and $0 \leq m < \ell'_k$, let $z$ be the integer given by the selection algorithm.

(a) $z > o(k)$
(b) $i_z + \ell(z) - \ell(o(k)) \geq \epsilon k$
(c) $i'_k + m = i_z - \ell'(o'(z))$
(d) If $o'(z) < r < k$, then $i'_k + m \leq i'_r$
(e) If $i'_k + m > \min_{o'(t_k) < r < k} \left\{ i'_r, i_t - \ell'(o'(t_k)) \right\}$, then

1. $z < t_k$ and $i_z + \ell(z) - \ell(o(k)) = \epsilon k$
2. $o(k) = \max_{r < k} \{ r \mid i_r > i_z + \ell(z) - \ell(r) \}$
3. $o'(z) = \max_{r < k} \{ r \mid i'_r < i'_k + m \}$

Proof:

(a) When $k = 1$, $z = (p + 1 + \epsilon_1) \ell'(1 - m) > p + 1 = o(1)$. For $k > 1$, first note that $i_{k_j} < \epsilon k_j < \epsilon k = i_{o(k)}$ for all $j \geq 1$, by the definition of $t_k$.

So the result follows whenever $i_z \leq i_{t_k}$ for some $j \geq 1$. If $i'_{k-1} < i'_k + m$, then $i_z = \epsilon - \ell(z) < \epsilon k - \ell(o(k)) = \epsilon k$. If $i'_k + m < i'_k + m$ for some $j \geq 2$ and $i'_k + m \leq i'_k$ for all $r < j$, where the latter follows from inequality (3) of system (6), then $i_z = \epsilon k + i'_k + m - i'_k < \epsilon k + i_{t_{j-1}} - i'_{k_j} = i_{t_{k+1}}$.

(b) When $k = 1$, indices $0 = i_p < \cdots < i_{p+2-\epsilon}$ are consecutive; hence $i_r + \ell_r = i_r + 1 + \ell_{r-1} + \ell_r - 1 \geq i_r + 1 + \ell_{r-1}$ for $p + 2 - \epsilon \leq r \leq p + 1$. Since $z \geq p + 2 - \epsilon$, we have $i_z + \ell(z) - \ell(o(k)) \geq i_{p+2-\epsilon} + \ell(p+2-\epsilon) - \ell(p+1-\epsilon) = \epsilon 1 - \ell(p+2-\epsilon) \geq \epsilon 1$. So consider $k > 1$. If $i'_{k-1} < i'_k + m$, then $e_{k-1} < \epsilon k < i_{o(k)}$

so that $e_k = i_z + z - o(k) < i_z + \ell(z) - \ell(o(k))$. If $i'_k \geq m$ and $z = t_k$, then $o(k-1) - o(k) = \epsilon k - \epsilon k + 1 - 1$ so that $i_z + \ell(z) - \ell(o(k)) = i_z + \ell(z) - \ell(o(k)) + \ell(o(k-1)) - \ell(o(k)) = \epsilon k + 1 - 1 = \epsilon k - 1$ by the definition of $t_k$.

If $i'_k \geq m$ then $e_{k-1} \leq i_z < i_{t_k} < \epsilon k$, so that $\ell(z) - \ell(t_{k+1}) \geq \epsilon z - t_{k+1} = i_{t_k} - i_z$, where equality holds iff $i_{t_k} = \cdots = i_z = 1$; hence $i_z + \ell(z) - \ell(o(k)) = i_z + \ell(z) - \ell(t_{k+1}) + \ell(o(k)) > e_{k-1}$ by the previous calculation.

If $i'_k \geq i'_k + m$ and $z = t_k$, then $i_z + \ell(z) - \ell(o(k)) = i_z + \ell(z) - \ell(o(k)) + \ell(o(k)) = \epsilon k + 1 - 1 = \epsilon k - 1$. Note that in general, $i_{t_{k+1}} + \ell(o(k)) = \epsilon k$ so that $i_{t_{k+1}} + \ell(o(k)) = \epsilon k$ for all $j \geq 1$.

So if necessary, continue in like manner until the desired $z$ is found at which point the result follows.

(c) This result follows immediately from the choice of $z$.

(d) Note that $k > 1$. If $i'_k < i'_k + m$, then $o'(z) = k - 1$ and the case is vacuous.

So assume that $i'_k + m \leq i'_k$. If either $i'_k + m = i_{t_{k+1}} - i'_{k+1}$ or $i'_k < i'_k + m$, then $o'(z) = o'(t_{k+1})$ and $i'_k \leq i'_{k+1}$ for $o'(z) < r < k$. Otherwise, $i'_k + m \leq \min \left\{ i'_k, i'_{k+1} \right\}$.
If either \( i'_k + m = i_{(k-1)} \) or \( i'_k < i'_k + m \), then \( o' (z) = o' (t_k) \) and \( i'_k \leq i'_k \) for \( o' (z) < r \leq k \). But \( i'_k \leq i'_k \) for \( k = o' (t_k) < r < k \), so the desired inequality holds for \( o' (z) < r < k \). Continue in this manner until \( i'_k + m \leq \min \{ i'_k, \ldots, i'_k \} \) for some \( j \), at which point the conclusion follows.

(e) If \( k = 1 \), \( o' (z) = 0 \) so that \( i'_k + m = i_k \) and \( i_k + \ell (z) \geq c_k \). Now \( i'_k + m > i_k \) by assumption, hence \( z < t_k \). But \( t_k \) is the smallest integer \( r \) such that \( i_r + \ell (r) > c_k \); therefore \( i_k + \ell (z) = c_k \), by (b). Let \( k > 1 \) and suppose that \( z \geq t_k \). Then \( i_k + t_k, \ell \), in which case \( o' (z) = o' (t_k) \). But by (c), \( i'_k + m = i_k - \ell (o' (z)) \leq t_k - \ell (o' (t_k)) \) and by (d), \( i'_k + m = i_k \) for \( o' (t_k) < r < k \), contradicting the hypothesis. Hence \( z < t_k \).

But \( t_k \) is the smallest possible integer such that \( i_k + t_k, \ell > c_k \), so (e.1) follows from (b). Results (e.2) and (e.3) are obvious.

Proof of Theorem II

Let \( n \geq 0 \). We show that if \( u \otimes v \) is a component of \( d^{\otimes 2} \Delta (T_{n+2}) \) or \( \Delta d (T_{n+2}) \), there is a corresponding component that cancels it, in which case \( (d^{\otimes 2} \Delta - \Delta d) (T_{n+2}) = 0 \). Let \( a' \cap a \) be a component of \( \Delta T_{n+2} \). We consider various cases.

Case I. Consider a component \( d^r (a') (a') \) to an expression in second fundamental form, i.e., if \( 1 \leq r \leq q + 1 \), reduce \( b' = d^r (a') (a') \) to an expression in second fundamental form, i.e., if \( 1 \leq r \leq q + 1 \), successively apply relation (1) to \( d^r (a') \) for \( j = q, r + 1 \) and \( r < q + 1 \). Reduce \( b' = d^r (a') (a') \) to an expression in second fundamental form, i.e., if \( 1 \leq r \leq q + 1 \), successively apply relation (1) to \( d^r (a') \) for \( j = q, r + 1 \) and \( r < q + 1 \).

Case Ia. Let \( i + \ell < l' \) and \( i'_k + i + \ell > i'_{3-1} \). As in the case of \( a' \) above, the inequalities for lower indices in an expression of \( b' \) as a face of \( T_{n+2} \) in first fundamental form are strict, but whose number now is increased by one. Obviously we will have that

\[
e_{k} = i'_k + i + \ell_{k} \geq i_{k}
\]

for certain \( 1 \leq k \leq p \). Apply the \( k^{th} \) left-transfer to obtain

\[
a = r_{k}^{i} (T_{n+2}) = d_{i}^{k} (a),
\]

where \( k = \alpha - 1 \), \( i = i_{k} - i_{\alpha} - l_{(\alpha-1)} - l_{(k)} \), \( \ell = k \) and \( \alpha \) is the smallest integer \( k \leq \alpha \leq p + 1 \) with \( i_{\alpha} + \ell (a) - l_{(k)} \geq i_{k} \). Then \( b' \otimes b \) is a component of \( \Delta T_{n+2} \) as well.

Case Ib. Let \( i + \ell < l' \) and \( i'_k + i + \ell = i'_{k-1} \). Apply the \( (k - 1)^{th} \) left-transfer to obtain

\[
b' = r_{k-1}^{i} (T_{n+2}) = d_{i}^{k-1} (a'),
\]

where \( i = \ell = l'_{k-1} \). Then \( c' \otimes b \) is also a component of \( \Delta T_{n+2} \).
Subcase i. If $1 \leq r \leq q$ and $i'_r + i \leq \min(i_r - \ell'_o(t_r), i'_j)$, $o'(t_r) < j < r$, apply the $(r + 1)^{th}$ left-transfer to obtain
\[ b' = \tau^{r+1}_r(T_{n+2}) = \delta^r_{(i', j)}(c'), \]
where $\tilde{r} = \alpha$, $\tilde{i} = i'_r - i'_o$, $\tilde{\ell} = \ell'_o - \ell$ and $\alpha$ is the smallest integer $r \leq \alpha \leq p + 1$ with $i'_\alpha \leq i'_r$. Although certain $i'$'s are increased by $i$, while $\ell'_o$ is reduced by $i$ to obtain $\ell$, it is straightforward to check that $c' \otimes a$ is also a component of $\Delta T_{n+2}$.

Subcase ii. Suppose that $1 \leq r \leq q+1$ and $i'_r + i > \min(i_r - \ell'_o(t_r), i'_j)$, $o'(t_r) < j < r$. In view of Lemma 2 (with $k = r$ and $m = i$) we have
\[ \epsilon_r = i_z + \ell(z) - \ell(o(r)) \text{ and } i'_r + i = i_z - \ell'_o(o(z)), \]
from which we also establish the equality
\[ \ell(z) - \ell(o(r)) = \ell(r-1) + \ell - \ell'_o(o(z)). \]
The hypotheses of Lemma 11 are satisfied by setting $k_1 = z$, $p_1 = o(r)$ and $k_2 = r$, $p_2 = o'(z)$. Hence, $b' \otimes a$ is a component of $\Delta d^r_{(i, j)}(T_{n+2})$ with $\tilde{i} = i_z$ and $\tilde{\ell} = \ell(z) - \ell(o(r))$.

Case II. Consider a component $1 \otimes d^r_{(i, j)}$ of $1 \otimes d$, where $i + \ell \leq \ell_r$, $i \geq 0$, $1 \leq \ell < \ell_r$, and $1 \leq r \leq p + 1$. Reduce $b = d^r_{(i, j)}(a)$ to the first fundamental form, i.e., if $1 \leq r \leq p$, successively apply relation (1) to $d^{r+1}d^r$ for $j = p, \ldots, r + 1$, and apply (2) to replace $d^r_{(i, j)}d^r_{(i, r)}$ by $d^r_{(i, x_r)}d^r_{(i_r, i)}$. Then if $i > 0$, successively apply (3) to $d^{r+1}d^r$, for $j = r - 1, \ldots, \beta$; if $r = p + 1$, successively apply (3) to $d^{r+1}d^r$ for $j = p, \ldots, \beta$, where $\beta$ is the greatest integer with $1 \leq \beta \leq r$ and $i_r + i + \ell_i - 1 + \cdots + \ell_{\beta} \leq i_{\beta - 1}$, that is $\beta$ is determined by $\frac{4}{3}$. Then for $\beta = r$, we have $i_r + i \leq i_{\beta - 1}$; and for $i > 0$, $1 \leq r \leq p + 1$,
\[ b = d^{r+1}_{(i_{p+1}, \ell_{p+1})} \cdots d^{r+2}_{(i_{r+1}, \ell_{r+1})} d^{r+1}_{r, r-1} d^{r+1}_{(i_{r}, \ell_{r-1})} \cdots d^{3+1}_{(i_{\beta+1}, \ell_{\beta+1})} d^{3+1}_{(i_{\beta+2}, \ell_{(\beta+2)})} \cdots d^1_{(i_z, \ell_z)}(T_{n+2}), \]
where $\tilde{i} = i_r + i + \ell_{r-1} - \ell_{(\beta-1)}$, and
\[ b = d^p_{(i_p, \ell_p)} \cdots d^{r+1}_{(i_{r+1}, \ell_{r+1})} d^{r+1}_{i_r, r-1} d^{r+1}_{r, \ell_{r-1}} d^{r-1}_{i_{r-1}, \ell_{r-1}} \cdots d^1_{(i_z, \ell_z)}(T_{n+2}), \]
for $i_i > 0$ and $1 \leq r \leq p + 1$.

Case IIa. Let $i > 0$ and $i_r + i + \ell_{r-1} - \ell_{(\beta-1)} < i_{\beta - 1}$. Once again, the inequalities for the lower indices in the expression of $b$ in first fundamental are strict inequalities as they were for $a$ but whose number now is increased by one. Obviously we will have that
\[ i_{\beta} = i_r + i + \ell_{r-1} - \ell_{(\beta-1)} = \epsilon_k \]
for certain $1 \leq k \leq q$. Apply the $k^{th}$ left-transfer to obtain
\[ a' = \tau^k_{(i, j)}(T_{n+2}) = \delta^k_{(i, j)}(b'), \]
where $\tilde{k} = \alpha - 1$, $\tilde{i} = i'_k - i'_o$, $\tilde{\ell} = \ell'_k$ and $\alpha$ is the smallest integer $1 \leq \alpha \leq q + 1$ with $i'_\alpha \leq i'_r$. Then $b' \otimes b$ is a component of $\Delta T_{n+2}$ as well.

Case IIb. Let $i > 0$ and $i_r + i + \ell_{r-1} - \ell_{(\beta-1)} = i_{\beta - 1}$. Apply the $(\beta - 1)^{th}$ left-transfer to obtain
\[ b = \tau^{\beta-1}_{(i, j)}(T_{n+2}) = \delta^{\beta-1}_{(i, j)}(c), \]
where $\tilde{i} = 0$ and $\tilde{\ell} = \ell_{\beta - 1}$. Then $a' \otimes c$ is also a component of $\Delta T_{n+2}$.
Case IIc. Let \( i = 0 \); there are two subcases:

Subcase i. If \( 1 \leq r \leq p \) and no integer \( 1 \leq k \leq q \) exists with \( r = t_k \), then
\[
e_k = i_r + \ell + \ell(i_r - 1) - \ell(o(k)){\ if \ }k > 0 \\
n\and \ i'_k = i_r - \ell(o'(r)).
\]

Apply the \((r + 1)^{th}\) left-transfer to obtain
\[
b = \tau^r_{\ell+1}(T_{n+2}) = d_{\ell,\ell}(c),
\]
where \( \tilde{\ell} = \alpha, \ \tilde{i} = \tilde{\ell} - \ell = \ell_r - \ell = \ell_{(r)} \) and \( \alpha \) is the smallest integer \( r < \alpha \leq p + 1 \) such that \( i_\alpha + \ell(\alpha) - \ell(i) \geq i_r \); namely, \( \alpha = r + 1 \) or \( \alpha = t_o'(r) \).

Now the required inequality for the \( i'_k \)'s could conceivably be violated for \( k > o'(r) \), but this is not so since it is easy to see that: (a) if \( r \) is not realized as \( t_k \) for some \( k > o'(r) \), then each \( z \) with \( \alpha < z < r \) is so; and (b) if \( r = t_k \) for some \( k > o'(r) \), while \( e_k = i_r + \ell + \ell(i_r - 1) - \ell(o(k)) \), then \( \alpha \) (and not \( r \)) serves as \( t_k \) for indices of face operators in expressions of \( a \) and \( c \); moreover, for either \( \alpha = r + 1 \) or \( \alpha = t_o'(r) \) (in which case \( i'_k \leq i'_o'(r) \), one has \( i'_k \leq \min_{o'(a)<j<k} \{i'_j, \ i_\alpha - \ell(o'(a))\} \) so that \( a' \otimes c \) is also a component of \( \Delta T_{n+2} \).

Subcase ii. If \( 1 \leq r \leq p + 1 \) and \( r = t_k \) for some \( 1 \leq k \leq q \), then
\[
e_k = i_r + \ell + \ell(i_r - 1) - \ell(o(k)) \quad \text{and} \quad i'_k = i_r - \ell(o'(r)),
\]
from which we establish the equality
\[
\ell + \ell(i_r - 1) - \ell(o(k)) = \ell'(k) - \ell(o'(r)).
\]

The hypotheses of Lemma \[H\] are satisfied by putting \( k_1 = r = t_k, \ p_1 = o(k) \) and \( k_2 = k, \ p_2 = o'(r) \) so that \( a' \otimes c \) is a component of \( \Delta d_{\ell,\ell}(T_{n+2}) \) with \( \tilde{i} = i_r \) and \( \tilde{\ell} = \ell + \ell(i_r - 1) - \ell(o(k)) \).

Case III. Let \( c' \otimes c \) be a component of \( \Delta d_{\ell,\ell}(T_{n+2}) \). Reduce \( c \) and \( c' \) to the first and second fundamental forms, respectively. According to Lemma \[H\] we have either \( \tilde{i} = i_r \) or \( \tilde{i} = i_{r_1} + \ell'(r_2) + \cdots + \ell'(r_{2^m} + \ell'(r_2 + 1) + \ell'(r_2 + 1) + \cdots + \ell'(r_2 + 1) \) for certain integers \( r_1, r_2, k_2 \), i.e., \( i_{r_1 + 1} = i_{r_1} \) or \( i_{r_1 + 1} + \ell'(r_2 + 1) = i_{r_2} \) Note that the shuffles under consideration prevent both cases from occurring simultaneously, and we obtain the situation dual to either Subcase ii of Case IIc, or to Subcase ii of Case Ic. Thus we obtain components \( c' \otimes a \) or \( a' \otimes c \) of \( \Delta T_{n+2} \) with \( d_{\ell,\ell}^1(a) = c \) or \( d_{\ell,\ell}^2(a') = c' \), respectively.

5. Application: Tensor Products of \( A_\infty \)-(co)algebras

In this section, we use the diagonal \( \Delta \) to define the tensor product of \( A_\infty \)-(co)algebras in maximal generality. We note that a special case was given by J. Smith [8] for certain objects with a richer structure than we have here. We also mention that Lada and Markl [9] defined an \( A_\infty \) tensor product structure on a construct different from the tensor product of graded modules.

We adopt the following notation and conventions: Let \( R \) be a commutative ring with unity; \( R \)-modules are assumed to be \( \mathbb{Z} \)-graded, tensor products and \( Hom's \) are defined over \( R \) and all maps are \( R \)-module maps unless otherwise indicated. If an \( R \)-module \( V \) is connected, \( \bigoplus V = V/V_0 \). The symbol \( 1 : V \to V \) denotes the identity map; the suspension and desuspension maps \( \uparrow \) and \( \downarrow \) shift dimension by \( +1 \) and \( -1 \), respectively. Define \( V^{\otimes 0} = R \) and \( V^{\otimes n} = V \otimes \cdots \otimes V \) with \( n > 0 \) factors; then \( TV = \oplus_n V^{\otimes n} \) and \( T^n V \) (respectively, \( T^n V \)) denotes the free tensor algebra (respectively, cofree tensor coalgebra) of \( V \). Given \( R \)-modules...
following quadratic relations:

**Proposition 2.** For each $V_1, \dots, V_n$, a permutation $\sigma \in S_n$ induces an isomorphism $\sigma : V_1 \otimes \cdots \otimes V_n \rightarrow V_{\sigma^{-1}(1)} \otimes \cdots \otimes V_{\sigma^{-1}(n)}$ by $\sigma(x_1 \cdots x_n) = \pm x_{\sigma^{-1}(1)} \cdots x_{\sigma^{-1}(n)}$, where $\pm$ is the Koszul sign. In particular, $\sigma_{2,n} = (1 3 \cdots (2n - 1) 2 4 \cdots 2n) : (A \otimes B)^{\otimes n} \rightarrow A^{\otimes n} \otimes B^{\otimes n}$ and $\sigma_{n,2} = \sigma_{2,n}^{-1}$ induce isomorphisms $(\sigma_{2,n})^* : \text{Hom}(A^{\otimes n} \otimes B^{\otimes n}, A \otimes B) \rightarrow \text{Hom} \left( (A \otimes B)^{\otimes n}, A \otimes B \right)$ and $(\sigma_{n,2})_* : \text{Hom}(A \otimes B, A^{\otimes n} \otimes B^{\otimes n}) \rightarrow \text{Hom}(A \otimes B, (A \otimes B)^{\otimes n})$. The map $\iota : \text{Hom}(U, V) \otimes \text{Hom}(U', V') \rightarrow \text{Hom}(U \otimes U', V \otimes V')$ is the canonical isomorphism. If $f : V^{\otimes r} \rightarrow V^{\otimes s}$ is a map, we let $f_{i,n-p+i} = 1^{\otimes i} \otimes f \otimes 1^{\otimes n-p-i} : V^{\otimes n} \rightarrow V^{\otimes n-p+q}$, where $0 \leq i \leq n - p$. The abbreviations $DGM$, $DGA$, and $DGC$ stand for differential graded $R$-module, $DG R$-algebra and $DG R$-coalgebra, respectively.

We begin with a review of $A_{\infty}$-(co)algebras paying particular attention to the signs. Let $A$ be a connected $R$-module equipped with operations $\{ \varphi^k \in \text{Hom}^k(A, A) \}_{k \geq 1}$. For each $k$ and $n \geq 1$, linearly extend $\varphi^k$ to $A^{\otimes n}$ via

$$
\sum_{i=0}^{n-k} \varphi_{i,n-k-i}^k : A^{\otimes n} \rightarrow A^{\otimes n-k+1},
$$

and consider the induced map of degree $-1$ given by

$$
\sum_{i=0}^{n-k} \left( \uparrow \varphi^k \downarrow \otimes^k \right)_{i,n-k-i} : (\uparrow A)^{\otimes n} \rightarrow (\uparrow A)^{\otimes n-k+1}.
$$

Let $BA = T^c (\uparrow A)$ and define a map $d_{BA} : BA \rightarrow BA$ of degree $-1$ by

$$
d_{BA} = \sum_{1 \leq k \leq n} \sum_{0 \leq i \leq n-k} \left( \uparrow \varphi^k \downarrow \otimes^k \right)_{i,n-k-i}.
$$

The identities $(-1)^{\lfloor n/2 \rfloor} \uparrow \otimes_t A^{\otimes n} = 1^{\otimes n}$ and $[n/2] + [(n+k)/2] = nk + [k/2]$ (mod 2) imply that

$$
d_{BA} = \sum_{1 \leq k \leq n} \sum_{0 \leq i \leq n-k} (-1)^{\lfloor (n-k)/2 \rfloor + i(k+1)} \uparrow \otimes^{n-k+1} \varphi^k_{i,n-k-i} \downarrow \otimes^{n}. \tag{5.2}
$$

**Definition 7.** $(A, \varphi^n)_{n \geq 1}$ is an $A_{\infty}$-algebra if $d_{BA}^2 = 0$.

**Proposition 2.** For each $n \geq 1$, the operations $\{ \varphi^n \}$ on an $A_{\infty}$-algebra satisfy the following quadratic relations:

$$
\sum_{0 \leq i \leq n-1} (-1)^{\ell(i+1)} \varphi^{n-\ell} \varphi^{\ell+1}_{i,n-\ell-1-i} = 0. \tag{5.3}
$$
Proof. For \( n \geq 1 \),

\[
0 = \sum_{1 \leq k \leq n, 0 \leq i \leq n-k} (-1)^{(n-k)/2}i^{(k+1)} n^{k} \uparrow \otimes n-k \downarrow \otimes n-k+1 \varphi_{i,n-k-i} \downarrow \otimes n
\]

\[
= \sum_{1 \leq k \leq n, 0 \leq i \leq n-k} (-1)^{n-k+i(k+1)} \varphi_{n-k-1}^{k} \uparrow \varphi_{i,n-k-i}^{i,n-k-i}
\]

\[
= -(-1)^{n} \sum_{0 \leq \ell \leq n-1, 0 \leq i \leq n-\ell-1} (-1)^{\ell+1} n^{\ell} \uparrow \otimes n-\ell \downarrow \otimes n \downarrow \otimes \psi_{\ell,\ell+1}^{i,n-\ell-1-i}.
\]

\[
\square
\]

It is easy to prove that

**Proposition 3.** If \((A, \varphi^n)_{n \geq 1}\) is an \( A_{\infty} \)-algebra, then \( (\tilde{B}A, d_{\tilde{B}A}) \) is a DGC.

**Definition 8.** Let \((A, \varphi^n)_{n \geq 1}\) be an \( A_{\infty} \)-algebra. The \( \text{tilde} \) bar construction on \( A \) is the DGC \( (\tilde{B}A, d_{\tilde{B}A}) \).

**Definition 9.** Let \( A \) and \( C \) be \( A_{\infty} \)-algebras. A chain map \( f = f^1 : A \to C \) is a map of \( A_{\infty} \)-algebras if there exists a sequence of maps \( \{ f^k \in \text{Hom}^{k-1}(A \otimes k, C) \}_{k \geq 2} \) such that

\[
\tilde{f} = \sum_{n \geq 1} \left( \sum_{k \geq 1} f^k \downarrow \otimes k \right) \uparrow \otimes n : \tilde{B}A \to \tilde{B}C
\]

is a DGC map.

Dually, consider a sequence of operations \( \{ \psi^k \in \text{Hom}^{k-2}(A, A \otimes k) \}_{k \geq 1} \). For each \( k \) and \( n \geq 1 \), linearly extend each \( \psi^k \) to \( A \otimes n \) via

\[
\sum_{i=0}^{n-1} \psi^k_{i,n-1-i} : A \otimes n \to A \otimes n+k-1,
\]

and consider the induced map of degree \(-1\) given by

\[
\sum_{i=0}^{n-1} \left( \downarrow \otimes k \psi^k \uparrow \right)_{i,n-1-i} : (\downarrow A) \otimes n \to (\downarrow A) \otimes n+k-1.
\]

Let \( \tilde{\Omega}A = T^n (\downarrow A) \) and define a map \( d_{\tilde{\Omega}A} : \tilde{\Omega}A \to \tilde{\Omega}A \) of degree \(-1\) by

\[
d_{\tilde{\Omega}A} = \sum_{n,k \geq 1} \left( \downarrow \otimes k \psi^k \uparrow \right)_{i,n-1-i},
\]

which can be rewritten as

\[
(5.4) \quad d_{\tilde{\Omega}A} = \sum_{n,k \geq 1} (-1)^{[n/2]+i(k+1)+k(n+1)} \uparrow \otimes n-1 \downarrow \otimes n \downarrow \otimes n \downarrow \otimes n \downarrow \otimes n
\]

**Definition 10.** \((A, \psi^n)_{n \geq 1}\) is an \( A_{\infty} \)-coalgebra if \( d_{\tilde{\Omega}A}^2 = 0 \).
Proposition 4. For each \( n \geq 1 \), the operations \( \{ \psi^k \} \) on an \( A_\infty \)-coalgebra satisfy the following quadratic relations:

\[
\sum_{0 \leq \ell \leq n-1} \sum_{0 \leq i \leq n-\ell-1} (-1)^{\ell(n+i+1)} \psi_{i,n-\ell-1-i}^{\ell+1} \psi_{i,n-\ell-1-i}^{n-\ell} = 0.
\]

Proof. The proof is similar to the proof of Proposition 2 and is omitted. \( \square \)

Again, it is easy to prove that

Proposition 5. If \((A, \psi^n)_{n \geq 1}\) is an \( A_\infty \)-coalgebra, then \((\Omega A, d_{\Omega A})\) is a DGA.

Definition 11. Let \((A, \psi^n)_{n \geq 1}\) be an \( A_\infty \)-coalgebra. The tilde cobar construction on \( A \) is the DGA \((\Omega A, d_{\Omega A})\).

Definition 12. Let \( A \) and \( B \) be \( A_\infty \)-coalgebras. A chain map \( g = g^1 : A \to B \) is a map of \( A_\infty \)-coalgebras if there exists a sequence of maps \( \{g^k \in \text{Hom}^{k-1} (A, B^{-\otimes k})\}_{k \geq 2} \) such that

\[
\tilde{g} = \sum_{n \geq 1} \left( \sum_{k \geq 1} \psi^k \right) \otimes g^k \uparrow : \Omega A \to \Omega B,
\]

is a DGA map.

The structure of an \( A_\infty \)-(co)algebra is encoded by the quadratic relations among its operations (also called “higher homotopies”). Although the “direction,” i.e., sign, of these higher homotopies is arbitrary, each choice of directions determines a set of signs in the quadratic relations, the “simplest” of which appears on the algebra side when no changes of direction are made; see \((5.1)\) and \((5.3)\) above. Interestingly, the “simplest” set of signs appear on the coalgebra side when \( \psi^n \), \( n \geq 1 \), i.e., the direction of every third and fourth homotopy is reversed. The choices one makes will depend on the application; for us the appropriate choices are as in \((5.2)\) and \((5.4)\).

Let \( A_\infty = \bigoplus_{n \geq 2} C_*(K_n) \) and let \((A, \varphi^n)_{n \geq 1}\) be an \( A_\infty \)-algebra with quadratic relations as in \((5.3)\). For each \( n \geq 2 \), associate \( e^{n-2} \in C_{n-2} (K_n) \) with the operation \( \varphi^n \) via

\[
e^{n-2} \mapsto (-1)^n \varphi^n
\]

and each codimension 1 face \( d_{(i,\ell)} \left( e^{n-2} \right) \in C_{n-3} (K_n) \) with the quadratic composition

\[
d_{(i,\ell)} \left( e^{n-2} \right) \mapsto \varphi^{n-\ell} \varphi_{i,n-\ell-1-i}^{\ell+1}.
\]

Then \((5.7)\) and \((5.8)\) induce a chain map

\[
\zeta_A : A_\infty \to \bigoplus_{n \geq 2} \text{Hom}^* \left( A^{\otimes n}, A \right)
\]

representing the \( A_\infty \)-algebra structure on \( A \). Dually, if \((A, \psi^n)_{n \geq 1}\) is an \( A_\infty \)-coalgebra with quadratic relations as in \((5.5)\), the associations

\[
e^{n-2} \mapsto \psi^n \text{ and } d_{(i,\ell)} \left( e^{n-2} \right) \mapsto \psi_{i,n-\ell-1-i}^{\ell+1} \psi_{i,n-\ell-1-i}^{n-\ell}
\]

induce a chain map

\[
\xi_A : A_\infty \to \bigoplus_{n \geq 2} \text{Hom}^* \left( A, A^{\otimes n} \right)
\]
representing the $A_\infty$-coalgebra structure on $A$. The definition of the tensor product is now immediate:

**Definition 13.** The tensor product of $A_\infty$-algebras $(A, \zeta_A)$ and $(B, \zeta_B)$ is given by

$$(A, \zeta_A) \otimes (B, \zeta_B) = (A \otimes B, \zeta_{A \otimes B}),$$

where $\zeta_{A \otimes B}$ is the sum of the compositions

$$C_*(K_n) \xrightarrow{\zeta \otimes B} \text{Hom}(A \otimes B)^{\otimes n}, A \otimes B)$$

$$\Delta_K \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
Note that the compositions in Definition 13 only use the operations $\psi^n$ and not the quadratic relations (5.5). Indeed, one can iterate an arbitrary family of operations $\{\psi^n\}$ as in Example 3 to produce iterated operations $\Psi^n : A^\otimes k \to (A^\otimes k)^{\otimes n}$ whether or not $(A, \psi)$ is an $A_\infty$-coalgebra. Of course, the $\Psi^n$’s define an $A_\infty$-coalgebra structure on $A^\otimes k$ whenever $d^2_{11}(A^\otimes k) = 0$, and we make extensive use of this fact in the sequel. Finally, since $\Delta_K$ is homotopy coassociative (not strict), the tensor product only iterates up to homotopy. In the sequel we always coassociate on the extreme left.

6. APPENDIX: ASSOCIATIONAL SETS

An associahedral set is a combinatorial object generated by Stasheff associahedra $K$ and equipped with appropriate face and degeneracy operators. Associahedral sets are similar in many ways to simplicial or cubical sets.

6.1. Singular associahedral sets. To motivate the notion of an associahedral set, we begin with a construction of singular associahedral sets, our universal example. Let $X$ be a topological space. Define the singular associahedral complex $\text{Sing}^K X$ as follows: Let

$$\text{Sing}^K X_{(n_1, n_2, \ldots, n_{k+2})} = \{\text{Continuous maps } K_{n_1+2} \times \cdots \times K_{n_{k+1}+2} \to X\},$$

where $\sum_{q=1}^{k+1} n_q = n - k, \ n_q \geq 0, 0 \leq k \leq n, n \geq j_1 \geq \cdots \geq j_k \geq j_{k+1} = 0$ and $K_{n_1+2} \times \cdots \times K_{n_{k+1}+2}$ is a Cartesian product of associahedra. Let

$$\delta_i^{(q, q)} : K_{n_1+2} \times \cdots \times K_{n_{j-1}+2} \times K_{j_q+1} \times \cdots \times K_{n_{q-1}+2} \times K_{n_q+2} \times \cdots \times K_{K_{n_{k+1}+2}},$$

be the map determined by $\delta_i^{(q, q)} = \delta_i^{(q, q)} \circ T_{(q, q)}$, where

$$\delta_i^{(q, q)} : K_{n_1+2} \times \cdots \times K_{j_q+1} \times K_{n_q-\ell_q+2} \times \cdots \times K_{n_{k+2}},$$

is the standard inclusion corresponding to the pair $(i_q, \ell_q)$, and

$$T_{(q, q)} : K_{n_1+2} \times \cdots \times K_{n_{j-1}+2} \times K_{j_q+1} \times \cdots \times K_{n_{q-1}+2} \times \cdots \times K_{n_{j-1}+2} \times \cdots \times K_{n_{j-1}+2} \times K_{n_{q-1}+2} \times \cdots \times K_{n_{k+1}+2},$$

is the permutation isomorphism in which $\beta$ is defined by (5.5). Let

$$\eta_q : K_{n_1+2} \times \cdots \times K_{n_{q-1}+2} \times K_{n_{q-1}+2} \times \cdots \times K_{n_{q-1}+2} \times \cdots \times K_{n_{q-1}+2} \times \cdots \times K_{n_{k+1}+2}$$

be the projection (cf. (3)). Then for $f \in (\text{Sing}^K X_{(n_1, n_2, \ldots, n_{k+2})}$, define

$$d^n_{(q, q)} : (\text{Sing}^K X_{n-k+2})_{(n_{k+1}, n_{k+2})} \to (\text{Sing}^K X_{n-k+1})_{(n_{k+1}, n_{k+2})},$$

where $\text{Sing}^K X_{n-k+2}$
with \( j(q, \beta) \) is defined in (3.1) and
\[
s^q_i: (\text{Sing}^K X_{n-k+2})^{(j_1, n_1), \ldots, (j_{k+1}, n_{k+1})} \rightarrow (\text{Sing}^K X_{n-k+3})^{(j_1, n_1), \ldots, (j_{q+1}, n_{q+1}), \ldots, (j_{k+1}, n_{k+1})},
\]
as compositions
\[
d^q_{(i_q, \ell_q)}(f) = f \circ s^q_{(i_q, \ell_q)} \text{ and } s^q_i(f) = \eta^q_i \circ f.
\]
Given the abstract definition below, one can easily check that \((\text{Sing}^K X, d_{(i_q, \ell_q)}, s^q_i)\) is an associahedral set.

6.2. Abstract associahedral sets.

**Definition 14.** An associahedral set is a graded set
\[
\mathcal{K} = \{ K^{(j_1, n_1), \ldots, (j_{k+1}, n_{k+1})} / n \geq j_1 \geq \cdots \geq j_{k+1} = 0, n_q \geq 0, n(k+1) = n-k \}_{0 \leq k \leq n},
\]

together with face and degeneracy operators defined for \( 1 \leq q \leq k+1 \):
\[
d^q_{(i_q, \ell_q)}: K^{(j_1, n_1), \ldots, (j_{k+1}, n_{k+1})}_{n-k+2} \rightarrow K^{(j_1, n_1), \ldots, (j_{q-1}, n_{q-1}), (j_{q}, \ell_{q}-1), (j_{q+1}, n_{q+1}), \ldots, (j_{k+1}, n_{k+1})}_{n-k+1}
\]

where \( j(q, \beta) \) and \( \beta \) are defined in (3.1) and (3.2), \( 0 \leq i_q \leq n_q; 1 \leq \ell_q \leq n_q; i_q + \ell_q \leq i_q + n_q + 1 \), and
\[
s^q_j: K^{(j_1, n_1), \ldots, (j_{k+1}, n_{k+1})}_{n-k+2} \rightarrow K^{(j_1, n_1), \ldots, (j_{q-1}, n_{q-1}), (j_{q}, n_q+1), (j_{q+1}, n_{q+1}), \ldots, (j_{k+1}, n_{k+1})}_{n-k+3}
\]

for \( 1 \leq j \leq n_q + 3 \), satisfying relations \((1)-(\beta)\) as well as
\[
d^p_{(i, \ell)} s^q_{j} = s^q_{j} + d^p_{(i, \ell)}; \quad p < q
\]
\[
d^p_{(i, \ell)} s^q_{j} = s^q_{j} + d^p_{(i, \ell)}; \quad p > q
\]
\[
d^q_{(i, \ell + 1)} s^q_{j} = s^q_{j} - d^q_{(i, \ell)}; \quad i < j \leq i + \ell + 2, \ell > 1
\]
\[
d^q_{(i, \ell)} s^q_{j} = s^q_{j} - d^q_{(i, \ell)}; \quad i \geq j, \ell \leq n_q
\]
\[
d^q_{(i, \ell)} s^q_{j} = 1; \quad (i, \ell) = (j+1, 1), 1 \leq j < n_q + 3
\]
\[
d^q_{(i, \ell)} s^q_{j} = 1; \quad (i, \ell) = (j+2, 1), 1 \leq j < n_q + 3
\]
\[
d^q_{(i, \ell)} s^q_{j} = 1; \quad (i, \ell) = (0, n_q + 1), j = n_q + 3
\]
\[
s^q_j s^q_{j'} = s^q_{j+1} s^q_{j'}; \quad p \neq q
\]
\[
s^q_j s^q_{j'} = s^q_{j+1} s^q_{j'}; \quad p = q, j \leq j'.
\]

Given an associahedral set \( \mathcal{K} \), let
\[
(C_*(\mathcal{K}), d) = \bigoplus C_{n-k}(K^{(j_1, n_1), \ldots, (j_{k+1}, n_{k+1})}_{n-k+2}, d^{n_1, \ldots, n_{k+1}}),
\]
where
\[
d^{n_1, \ldots, n_{k+1}} = \sum_{(i_q, \ell_q)} (-1)^{i_1 + i_2} d^q_{(i_q, \ell_q)},
\]
with \(\epsilon_q\) defined in (3.3); define the diagonal
\[
\Delta_K : C_\ast(K) \to C_\ast(K) \otimes C_\ast(K)
\]
on \(K^{(0,n)}\) by (3.4) and extend to other components of \(K^{(j_1;n_1), \ldots, (j_k+1;n_{k+1})}\) by the formal multiplicative rule with respect to indices \((n_1, \ldots, n_{k+1})\), i.e., by the same formulas as on a product cell \(K_{n_1+2} \times \cdots \times K_{n_{k+1}+2}\). Finally, set \(C_\ast^N(K) = C_\ast(K)/D\), where \(D\) is the submodule generated by the degeneracies; then \((C_\ast^N(K), d)\) is a chain complex equipped with a diagonal \(\Delta_K\) induced by \(\Delta\).

References

[1] F. Chapoton, Algèbres de Hopf des permutahédres, associahédres et hypercubes, Adv. in Math. 150, No. 2, (2000), 264-275.
[2] ———, Bigèbres différentielles graduées associées aux permutoèdres, associaèdres et hypercubes, preprint.
[3] T. Lada and M. Markl, Strongly homotopy Lie algebras, Communications in Algebra 23 (1995), 2147-2161.
[4] J.-L. Loday and M. Ronco, Hopf algebra of the planar binary trees, Adv. in Math. 139, No. 2 (1998), 293-309.
[5] P. May, “The Geometry of Iterated Loop Spaces,” SLNM 271, Springer-Verlag, Berlin, 1972.
[6] J. M. Pallo, Enumerating, Ranking and Unranking Binary Trees, The computer journal, 29 (1986), 171-175.
[7] S. Saneblidze and R. Umble, The biderivative and \(A_\infty\)-Hopf algebras, preprint.
[8] J. R. Smith, “Iterating the cobar construction,” Memoirs of the Amer. Math. Soc. 109, Number 524, Providence, RI, 1994.
[9] J. D. Stasheff, Homotopy associativity of \(H\)-spaces I, II, Trans. Amer. Math. Soc. 108 (1963), 275-312.

A. RAZMADZE MATHEMATICAL INSTITUTE, GEORGIAN ACADEMY OF SCIENCES, M. ALEKSIDZE ST., 1, 0193 TBILISI, GEORGIA
E-mail address: SANE@rmi.acnet.ge <mailto:SANE@rmi.acnet.ge>

DEPARTMENT OF MATHEMATICS, MILLERSVILLE UNIVERSITY OF PENNSYLVANIA, MILLERSVILLE, PA. 17551
E-mail address: ron.umble@millersville.edu <mailto:ron.umble@millersville.edu>