Approximation to probability density functions in sampling distributions based on Fourier cosine series

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1 Introduction

Fourier series and transforms are applied for many statistical purposes, because, in mathematical point of view, they have tractable property such as uniformly convergent in a closed interval and integration term by term and so on (Whittaker [10]). In particular, the substantial applications are the generalized representation (or estimation) of statistical curves like probability density functions (pdf) and the associated cumulative distribution functions (cdf). See for example, Kronmal and Tarter [5].

In sampling distribution theory, many approximations to pdfs or cdfs have been developed. The Edgeworth series of a pdf of a statistic is widely used and well-known as refinement of the central limit theorem and are comprised of Hermite polynomials as orthogonal polynomials with respect to the standard normal density function as the weight function. Also, Fourier series has trigonometric functions as orthogonal functions, but the target for the approximation is slightly different. The former can treat with the case that pdf’s support is not only unbounded but also bounded. The latter can only treat with the case of the bounded support and is needed periodic property of the pdf. The constraints for applying the Fourier series seem to be severe in the sense of obtaining the Fourier coefficients.

In this paper, we give the approximations to the pdf or the cdf based on the cosine Fourier series. In addition, we give three conditions to be needed. The first condition is that the pdf has a bounded support. The second condition is that it is both a piecewise smooth function and an even function. The third condition is assuming that the pdf has moments up to the required order. We remark that the first and second conditions are technical and the third is crucial. Moreover, we remark that there are many examples in sampling distributions: sample skewness, sample kurtosis, the Shapiro–Wilk test statistic, sample correlation coefficient, and so on.

This paper is organized as follows. In Section 2 we give a brief survey of Fourier cosine series. Two examples are illustrated in Sections 3 and 4. In Section 5 we consider the distribution of a sum of random variables uniformly distributed as the case where the pdf has an explicit expression. We also show the accuracy of the proposed method through the numerical experiments. In Section 6 we consider the case where the statistic has no explicit expression of pdf but has its moments up to required orders. The distribution of sample skewness $\sqrt{b_1}$ drawn from a normal population is illustrated. The pdfs are not given except for the sample of size $n = 3$ and 4, and the pdfs for $n = 3$ and 4 were given by
Fisher [3] and McKay [6], respectively. Here, we make mention of the approximations to the distribution of $\sqrt{b_1}$ in view of sample of size $n$. When $n$ is large, $\sqrt{b_1}$ is asymptotically normally distributed with mean zero and variance $n/6$ (Thode [9]). When $n$ is moderate, D’Agostino’s [1] transformation, that is a fit of Johnson SU curve, works well. When $n$ is at most 25, Mulholland [7] has arrived at approximations, although mathematical expressions of the pdf’s are very complicated.

Based on the Fourier cosine series, we provide concrete approximations to the sampling distribution, in particular, when $n$ is small. We also give percentiles of $\sqrt{b_1}$.

2 Fourier series of probability density functions

Let $T_n = T_n(X_1, X_2, \ldots, X_n)$ be a statistic having a probability density function $f_n(x)$, where $(X_1, X_2, \ldots, X_n)$ is a random sample of size $n$. We assume that the following conditions are satisfied:

1. The $f_n(x)$ has a bounded support $[-A_n, A_n]$, where $A_n > 0$.

2. The $f_n(x)$ is a piecewise smooth function and also an even function.

From Condition 1 and 2, $f_n(x)$ has a Fourier cosine series within $[-A_n, A_n]$. That is,

$$f_n(x) = \frac{a_{n,0}}{2} + \sum_{k=1}^{\infty} a_{n,k} \cos \frac{k\pi}{A_n}x,$$  

where the Fourier cosine coefficients $a_{n,k}$ are given by

$$a_{n,0} = \frac{1}{A_n} \int_{-A_n}^{A_n} f_n(x) \, dx = \frac{1}{A_n},$$

$$a_{n,k} = \frac{1}{A_n} \int_{-A_n}^{A_n} f_n(x) \cos \frac{k\pi}{A_n}x \, dx.$$  

In addition, we assume that moments of $T_n$ about the origin with the requisite order exist:

3. For any $j$, there exists $\mu'_{n,j} = \int_{-\infty}^{\infty} x^j f_n(x) \, dx < \infty$.

From Condition 3 and $\cos x = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} x^{2j}$, the coefficient $a_{n,k}$ is

$$a_{n,k} = \frac{1}{A_n} \int_{-A_n}^{A_n} f_n(x) \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} \left( \frac{k\pi}{A_n} \right)^{2j} x^{2j} \, dx$$

$$= \frac{1}{A_n} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} \left( \frac{k\pi}{A_n} \right)^{2j} \mu'_{n,2j},$$  

where integration term by term is applied in [3].

The cumulative distribution function $F_n(x) = \Pr \{T_n < x\}$ is also obtained by

$$F_n(x) = \frac{1}{2} \left( \frac{x}{A_n} + 1 \right) + \sum_{k=1}^{\infty} \frac{a_{n,k} A_n}{k\pi} \sin \frac{k\pi}{A_n} x.$$

The Fourier cosine series [11] of $f_n(x)$ yields the following facts:
• Approximations
\[ \tilde{f}_n^{(K)}(x) = \frac{a_{n,0}}{2} + \sum_{k=1}^{K} a_{n,k} \cos \frac{k\pi}{A_n} x \]  
(4) and
\[ \tilde{F}_n^{(K)}(x) = \frac{1}{2} \left( \frac{x}{A_n} + 1 \right) + \sum_{k=1}^{K} a_{n,k} \frac{A_n}{k\pi} \sin \frac{k\pi}{A_n} x \]  
(5)
are the best approximations in the sence that they minimize
\[
\int_{-A_n}^{A_n} \left( \tilde{f}_n^{(K)}(x) - f_n(x) \right)^2 \, dx \quad \text{and} \quad \int_{-A_n}^{A_n} \left( \tilde{F}_n^{(K)}(x) - F_n(x) \right)^2 \, dx,
\]
respectively (Whittaker [10]).

• From (5), we obtain the percentile \( x_\alpha \) \((0 < \alpha < 1)\). It is calculated by solving the nonlinear equation \( \tilde{F}_n^{(K)}(x) = \alpha \) such as Newton’s method with an initial value \( x_{0.5} = 0 \).

• If \( f_n(x) \) is given, we can directly obtain the coefficients \( a_{n,k} \) from (2). It is not given, but if the moments \( \mu'_{n,j} \) of requisite order are known, the approximation such as (4) can be obtained from (3).

We denote
\[ \tilde{a}_{n,k}^{(J)} = \frac{1}{A_n} \sum_{j=0}^{J} \frac{(-1)^j}{(2j)!} \left( \frac{k\pi}{A_n} \right)^{2j} \mu'_{n,2j} \]
as an approximation to the Fourier cosine coefficient (2) and
\[ \tilde{f}_n^{(K,J)}(x) = \frac{a_{n,0}}{2} + \sum_{k=1}^{K} \tilde{a}_{n,k}^{(J)} \cos \frac{k\pi}{A_n} x, \]
\[ \tilde{F}_n^{(K,J)}(x) = \frac{1}{2} \left( \frac{x}{A_n} + 1 \right) + \sum_{k=1}^{K} \tilde{a}_{n,k}^{(J)} \frac{A_n}{k\pi} \sin \frac{k\pi}{A_n} x, \]
as approximations to (4) and (5), respectively. We note that it depends on the sample of size \( n \) in order to choose \( K \) and \( J \) appropriately.

3 Distribution of a sum of random variables uniformly distributed

Random variables \( X_1, X_2, \ldots, X_n \) are mutually independent and uniformly distributed on the interval \([-\frac{1}{2}, \frac{1}{2}]\). Let \( f_n(x) \) be the probability density function of the sum \( T_n = X_1 + X_2 + \cdots + X_n \). Then, using the relation
\[ f_n(x) = \int_{-\infty}^{\infty} f_{n-1}(x-t) f_1(t) \, dt = \int_{x-1/2}^{x+1/2} f_{n-1}(t) \, dt \quad (n \geq 2), \]
\[ f_1(x) = \begin{cases} 1 & (-\frac{1}{2} \leq x \leq \frac{1}{2}) \\ 0 & \text{(otherwise)} \end{cases} \]

we have, for any \(-\frac{n}{2} + i \leq x \leq -\frac{n}{2} + i + 1 (i = 0, 1, \ldots, n - 1),\)

\[ f_n(x) = \frac{1}{(n-1)!} \sum_{j=0}^{i} (-1)^j \binom{n}{j} \left( x + \frac{n}{2} - j \right)^{n-1}, \]

\[ f_n(x) = 0 \text{ (otherwise)}. \] That is, \( f_n(x) \) is not only a smooth function but also an even function.

It has the bounded support \([-\frac{n}{2}, \frac{n}{2}]\). Moments \(\mu'_{n,j}\) are given by the following recurrence formula:

\[ \mu'_{n,2j} = \sum_{k=0}^{j} \frac{(2j)_{j}}{(2k+1)4^k \mu'_{n-1,2j-2k}}. \]

**Example** For \( n = 4, \) the probability density function is

\[ f_4(x) = \begin{cases} \frac{1}{6} (x + 2)^3 & (-2 \leq x \leq -1) \\ \frac{1}{6} \left\{ (x + 2)^3 - 4(x + 1)^3 \right\} & (-1 \leq x \leq 0) \\ \frac{1}{6} \left\{ (x + 2)^3 - 4(x + 1)^3 + 6x^3 \right\} & (0 \leq x \leq 1) \\ \frac{1}{6} \left\{ (x + 2)^3 - 4(x + 1)^3 + 6x^3 - 4(x - 1)^3 \right\} & (1 \leq x \leq 2) \\ 0 & \text{(otherwise)} \end{cases} \]

Even moments are

\[ \mu'_{4,2j} = \frac{8 \cdot 4^j - 1}{(1 + 2j)(2 + 2j)(3 + 2j)(4 + 2j)} \]

and odd moments are all 0.

The Fourier cosine series of \( f_4(x) \) in \([-2, 2]\) is

\[ f_4(x) = \frac{1}{4} + \sum_{k=1}^{\infty} a_{4,k} \cos \frac{k\pi}{2} x, \]

and the coefficients are given by

\[ a_{4,k} = \frac{128}{\pi^4 k^4} \sin^4 \left( \frac{k\pi}{4} \right). \]

The cumulative distribution function is

\[ F_4(x) = \frac{1}{2} + \frac{x}{4} + \sum_{k=1}^{\infty} \frac{2a_{n,k}}{k\pi} \sin \frac{k\pi}{2} x. \]

Table II shows that

\[ \max_{0 \leq k \leq K} \left| a_{n,k}^{(J)} - a_{n,k} \right| \]

for given \( n = 2(2)12 \) and selected values of \( K \) and \( J \). For any cases, the approximate coefficient \( a_{n,k}^{(J)} \) coincide with the exact coefficient \( a_{n,k} \) up to the sixth decimal places.
Tables 2 and 3 show the approximate Fourier cosine coefficients. For example, we have

\[
\hat{f}_4^{(8,35)}(x) = 2.5 \times 10^{-1} + (3.28511 \times 10^{-1}) \cos\left(\frac{\pi x}{2}\right) + (8.21279 \times 10^{-2}) \cos(\pi x) \\
\quad + (4.0557 \times 10^{-3}) \cos\left(\frac{3\pi x}{2}\right) + (1.0306 \times 10^{-14}) \cos(2\pi x) \\
\quad + (5.25618 \times 10^{-4}) \cos\left(\frac{5\pi x}{2}\right) + (1.01392 \times 10^{-3}) \cos(3\pi x) \\
\quad + (1.36823 \times 10^{-4}) \cos\left(\frac{7\pi x}{2}\right) - (5.73436 \times 10^{-10}) \cos(4\pi x)
\]

and

\[
\hat{F}_4^{(8,35)}(x) = \frac{1}{2} \left(\frac{u}{2} + 1\right) + (2.09137 \times 10^{-1}) \sin\left(\frac{\pi x}{2}\right) + (2.61421 \times 10^{-2}) \sin(\pi x) \\
\quad + (8.60646 \times 10^{-4}) \sin\left(\frac{3\pi x}{2}\right) + (1.64026 \times 10^{-15}) \sin(2\pi x) \\
\quad + (6.69238 \times 10^{-5}) \sin\left(\frac{5\pi x}{2}\right) + (1.07581 \times 10^{-4}) \sin(3\pi x) \\
\quad + (1.24434 \times 10^{-5}) \sin\left(\frac{7\pi x}{2}\right) - (4.56326 \times 10^{-11}) \sin(4\pi x).
\]

Table 4 shows percentiles \(x_\alpha\) of the sum \(T_n\) for specified \(\alpha\). For example, for \(n = 4\) and \(\alpha = 0.99\), solving

\[
\hat{F}_4^{(8,35)}(x) = 0.99,
\]

we have \(x_{0.99} = 1.3002\). We confirm that

\[
\int_{-2}^{x_{0.99}} f_4(x) \, dx = 0.990006.
\]

Table 1: \(\max_{0 \leq k \leq K} |a_{n,k}^{(J)} - a_{n,k}|\)

| \(n\) | \(K\) | \(J\) | \(\max_{0 \leq k \leq K} |a_{n,k}^{(J)} - a_{n,k}|\) |
|---|---|---|---|
| 2 | 8 | 35 | \(3.61470 \times 10^{-7}\) |
| 4 | 8 | 35 | \(5.73436 \times 10^{-10}\) |
| 6 | 8 | 30 | \(1.13062 \times 10^{-7}\) |
| 8 | 8 | 30 | \(1.58854 \times 10^{-9}\) |
| 10 | 7 | 25 | \(6.93824 \times 10^{-10}\) |
| 12 | 6 | 20 | \(1.02801 \times 10^{-9}\) |
Table 2: Fourier coefficients $\hat{a}_{2k}^{(35)}$, $\hat{a}_{4k}^{(35)}$, $\hat{a}_{6k}^{(30)}$ ($k = 0, 1, \ldots, 8$)

| $k$  | $\hat{a}_{2k}^{(35)}$ | $\hat{a}_{4k}^{(35)}$ | $\hat{a}_{6k}^{(30)}$ |
|------|---------------------|---------------------|---------------------|
| 0    | 1.                  | $5. \times 10^{-1}$ | $3.3333 \times 10^{-1}$ |
| 1    | $4.05285 \times 10^{-1}$ | $3.28511 \times 10^{-1}$ | $2.52759 \times 10^{-1}$ |
| 2    | $2.18614 \times 10^{-16}$ | $8.21279 \times 10^{-2}$ | $1.06633 \times 10^{-1}$ |
| 3    | $4.50316 \times 10^{-2}$ | $4.0557 \times 10^{-3}$ | $2.21901 \times 10^{-2}$ |
| 4    | $1.64157 \times 10^{-13}$ | $1.0306 \times 10^{-14}$ | $1.66614 \times 10^{-3}$ |
| 5    | $1.62114 \times 10^{-2}$ | $5.25618 \times 10^{-4}$ | $1.61766 \times 10^{-5}$ |
| 6    | $1.83356 \times 10^{-11}$ | $1.01392 \times 10^{-3}$ | $-7.88258 \times 10^{-15}$ |
| 7    | $8.27112 \times 10^{-3}$ | $1.36823 \times 10^{-4}$ | $2.14845 \times 10^{-6}$ |
| 8    | $-3.6147 \times 10^{-7}$ | $-5.73436 \times 10^{-10}$ | $2.61465 \times 10^{-5}$ |

Table 3: Fourier coefficients $\hat{a}_{8,k}^{(30)}$, $\hat{a}_{10,k}^{(25)}$, $\hat{a}_{12,k}^{(20)}$ ($k = 0, 1, \ldots, 8$)

| $k$  | $\hat{a}_{8,k}^{(30)}$ | $\hat{a}_{10,k}^{(25)}$ | $\hat{a}_{12,k}^{(20)}$ |
|------|---------------------|---------------------|---------------------|
| 0    | $2.5 \times 10^{-1}$ | $2. \times 10^{-1}$ | $1.66667 \times 10^{-1}$ |
| 1    | $2.03319 \times 10^{-1}$ | $1.69572 \times 10^{-1}$ | $1.45271 \times 10^{-1}$ |
| 2    | $1.0792 \times 10^{-1}$ | $1.02664 \times 10^{-1}$ | $9.58308 \times 10^{-2}$ |
| 3    | $3.57613 \times 10^{-2}$ | $4.34405 \times 10^{-2}$ | $4.72705 \times 10^{-2}$ |
| 4    | $6.74499 \times 10^{-3}$ | $1.23309 \times 10^{-2}$ | $1.70558 \times 10^{-2}$ |
| 5    | $6.00653 \times 10^{-4}$ | $2.18691 \times 10^{-3}$ | $4.34421 \times 10^{-3}$ |
| 6    | $1.64487 \times 10^{-5}$ | $2.13837 \times 10^{-4}$ | $7.38603 \times 10^{-4}$ |
| 7    | $3.52694 \times 10^{-8}$ | $9.08017 \times 10^{-6}$ | $-$ |
| 8    | $1.58854 \times 10^{-9}$ | $-$ | $-$ |

Table 4: Percentiles $x_\alpha$ of the sum $T_n$

| $n$  | 0.900 | 0.950 | 0.975 | 0.990 | 0.995 | 0.999 |
|------|-------|-------|-------|-------|-------|-------|
| 2    | 0.5528 | 0.6838 | 0.7768 | 0.8571 | 0.8993 | 0.9649 |
| 4    | 0.7534 | 0.9534 | 1.1198 | 1.3002 | 1.4114 | 1.6063 |
| 6    | 0.9170 | 1.1663 | 1.3759 | 1.6097 | 1.7618 | 2.0536 |
| 8    | 1.0556 | 1.3457 | 1.5916 | 1.8694 | 2.0527 | 2.4120 |
| 10   | 1.1781 | 1.5039 | 1.7815 | 2.0971 | 2.3067 | 2.7232 |
| 12   | 1.2889 | 1.6469 | 1.9532 | 2.3028 | 2.5355 | 2.9964 |
4 Distribution of a sample skewness drawn from normal population

Let \((X_1, X_2, \ldots, X_n)\) be a random sample of size \(n\) drawn from normal population. Let \(\sqrt{b_1}\) be a sample skewness defined by

\[
\sqrt{b_1} = \frac{m_3}{m_2^{3/2}}, \quad m_r = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^r \quad (r = 2, 3), \quad \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i
\]

and \(f_n(x)\) be the probability density function.

Dalen [2] showed that a range of \(\sqrt{b_1}\) is

\[-A_n \leq \sqrt{b_1} \leq A_n, \quad A_n = \frac{n - 2}{\sqrt{n - 1}}.\]

Obviously, \(f_n(x)\) is a smooth and an even function on \([-A_n, A_n]\). Geary [4] gave a recurrence formula of \(f_n(x)\) as

\[
f_n(x) = \frac{(n-1)n^{1/2}}{B\left(\frac{1}{2}, \frac{n-2}{2}\right)} \int_{-1}^{1} f_{n-1}(\sigma_{n-1}(x, z)) (1 - z^2)^{(n-7)/2} \, dz,
\]

where

\[
\sigma_{n-1}(x, z) = \left\{ \sqrt{n - 1} x - 3z + (n + 1)z^3 \right\} n^{-1/2} (1 - z^2)^{-3/2},
\]

but the analytical expression of \(f_n(x)\) is still unknown. The moments of \(\sqrt{b_1}\) are well-known through a recurrence relation, obtained by Muholland [7],

\[
\mu'_{n+2s} = \frac{(n + 1)^s}{n^s (\frac{n}{2})_{3s}} \sum_{j=0}^{s} \frac{(2s)_{2j}}{2j} \frac{\mu'_{n,2s-2j}}{(n + 1)^{j}}
\]

\[
\times \sum_{i=0}^{2j} \frac{(2j)_i}{i!} 3^{2j-i} (1 - n)^i \left( \frac{1}{2} \right)^{j+i} \left( \frac{n - 1}{2} \right)^{3s-j-i},
\]

where \((a)_m\) is a Pochhammer symbol defined by

\[
(a)_m = a(a + 1) \cdots (a + m - 1) \quad (m \geq 1), \quad (a)_0 = 1.
\]

From (1), the Fourier cosine series is

\[
f_n(x) = \frac{a_{n,0}}{2} + \sum_{k=1}^\infty a_{n,k} \cos \frac{k\pi}{A_n} x
\]

and

\[
a_{n,0} = \frac{1}{A_n}, \quad a_{n,k} = \frac{1}{A_n} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} \left( \frac{k\pi}{A_n} \right)^{2j} \mu'_{n,2j}.
\]

Using the integration term by term in (8), the cumulative distribution function \(F_n(x) = \text{Pr}\{\sqrt{b_1} < x\}\) is

\[
F_n(x) = \frac{1}{2} \left( \frac{x}{A_n} + 1 \right) + \sum_{k=1}^\infty a_{n,k} \frac{A_n}{k\pi} \sin \frac{k\pi}{A_n} x.
\]
Tables 5, 6 and 7 show the approximate Fourier cosine coefficients $\hat{a}_{n,k}^{(12,50)} (k = 0, \ldots, 12)$ for $n = 4(2)22$. Especially, we show the approximations to $f_6(x)$ and $F_6(x)$ as

\[
\hat{f}_6^{(12,50)}(x) = 2.79508 \times 10^{-1} + (3.08052 \times 10^{-1}) \cos \left( \frac{1}{4} \sqrt{5}\pi x \right) \\
+ (5.75070 \times 10^{-2}) \cos \left( \frac{1}{2} \sqrt{5}\pi x \right) + (1.17190 \times 10^{-2}) \cos \left( \frac{3}{4} \sqrt{5}\pi x \right) \\
- (5.99392 \times 10^{-3}) \cos \left( \sqrt{5}\pi x \right) + (6.78429 \times 10^{-3}) \cos \left( \frac{5}{4} \sqrt{5}\pi x \right) \\
+ (7.71527 \times 10^{-3}) \cos \left( \frac{3}{2} \sqrt{5}\pi x \right) + (6.95419 \times 10^{-3}) \cos \left( \frac{7}{4} \sqrt{5}\pi x \right) \\
+ (1.62249 \times 10^{-4}) \cos \left( 2\sqrt{5}\pi x \right) + (2.38820 \times 10^{-5}) \cos \left( \frac{9}{4} \sqrt{5}\pi x \right) \\
+ (6.33581 \times 10^{-4}) \cos \left( \frac{5}{2} \sqrt{5}\pi x \right) + (3.10573 \times 10^{-3}) \cos \left( \frac{11}{4} \sqrt{5}\pi x \right) \\
+ (1.7351 \times 10^{-3}) \cos \left( 3\sqrt{5}\pi x \right)
\]

and

\[
\hat{F}_6^{(12,50)}(x) = \frac{1}{2} \left( \sqrt{5}x \right)^4 + 1 + (1.75408 \times 10^{-1}) \sin \left( \frac{1}{4} \sqrt{5}\pi x \right) \\
+ (1.63725 \times 10^{-2}) \sin \left( \frac{1}{2} \sqrt{5}\pi x \right) + (2.22431 \times 10^{-3}) \sin \left( \frac{3}{4} \sqrt{5}\pi x \right) \\
- (8.53250 \times 10^{-4}) \sin \left( \sqrt{5}\pi x \right) + (7.72609 \times 10^{-4}) \sin \left( \frac{5}{4} \sqrt{5}\pi x \right) \\
+ (7.32192 \times 10^{-4}) \sin \left( \frac{3}{2} \sqrt{5}\pi x \right) + (5.65684 \times 10^{-4}) \sin \left( \frac{7}{4} \sqrt{5}\pi x \right) \\
+ (1.15483 \times 10^{-5}) \sin \left( 2\sqrt{5}\pi x \right) + (1.51096 \times 10^{-6}) \sin \left( \frac{9}{4} \sqrt{5}\pi x \right) \\
+ (3.60767 \times 10^{-5}) \sin \left( \frac{5}{2} \sqrt{5}\pi x \right) + (1.60767 \times 10^{-4}) \sin \left( \frac{11}{4} \sqrt{5}\pi x \right) \\
+ (8.233 \times 10^{-5}) \sin \left( 3\sqrt{5}\pi x \right)
\]

Figure illustrates a graph of $y = \hat{f}_6^{(12,50)}(x)$ and a histogram of $\sqrt{b_1}$ ($10^6$ replications).
Table 5: Fourier coefficients $\widehat{a}_{n,k}^{(50)}$  $(n = 4, 6, 8, 10)$

| $k$ | $n = 4$   | $n = 6$   | $n = 8$   | $n = 10$   |
|-----|-----------|-----------|-----------|------------|
| 0   | $8.66025 \times 10^{-1}$ | $5.59017 \times 10^{-1}$ | $4.40959 \times 10^{-1}$ | $3.75000 \times 10^{-1}$ |
| 1   | $1.76257 \times 10^{-1}$ | $3.08052 \times 10^{-1}$ | $3.12106 \times 10^{-1}$ | $2.97971 \times 10^{-1}$ |
| 2   | $7.26283 \times 10^{-2}$ | $5.75070 \times 10^{-2}$ | $1.18300 \times 10^{-1}$ | $1.55996 \times 10^{-1}$ |
| 3   | $5.5174 \times 10^{-2}$  | $1.17190 \times 10^{-2}$ | $3.20582 \times 10^{-2}$ | $6.04639 \times 10^{-2}$ |
| 4   | $3.80806 \times 10^{-2}$ | $-5.99392 \times 10^{-3}$ | $6.29481 \times 10^{-3}$ | $1.95825 \times 10^{-2}$ |
| 5   | $3.28048 \times 10^{-2}$ | $6.78429 \times 10^{-3}$ | $1.54291 \times 10^{-3}$ | $5.48500 \times 10^{-3}$ |
| 6   | $2.57761 \times 10^{-2}$ | $7.1527 \times 10^{-3}$  | $4.84745 \times 10^{-4}$ | $1.49687 \times 10^{-3}$ |
| 7   | $2.32437 \times 10^{-2}$ | $6.95419 \times 10^{-3}$ | $-5.28008 \times 10^{-4}$ | $2.86985 \times 10^{-4}$ |
| 8   | $1.94759 \times 10^{-2}$ | $1.62249 \times 10^{-4}$ | $-2.89091 \times 10^{-4}$ | $5.48249 \times 10^{-5}$ |
| 9   | $1.79941 \times 10^{-2}$ | $2.38820 \times 10^{-5}$  | $6.47841 \times 10^{-4}$ | $7.54897 \times 10^{-5}$ |
| 10  | $1.56490 \times 10^{-2}$ | $6.33581 \times 10^{-4}$ | $1.01858 \times 10^{-3}$ | $2.55678 \times 10^{-5}$ |
| 11  | $1.46777 \times 10^{-2}$ | $3.10573 \times 10^{-3}$ | $6.95872 \times 10^{-4}$ | $-7.05180 \times 10^{-5}$ |
| 12  | $1.33900 \times 10^{-2}$ | $1.73510 \times 10^{-3}$ | $3.76670 \times 10^{-4}$ | $-5.51200 \times 10^{-5}$ |

Table 6: Fourier coefficients $\widehat{a}_{n,k}^{(50)}$  $(n = 12, 14, 16, 18)$

| $k$ | $n = 12$   | $n = 14$   | $n = 16$   | $n = 18$   |
|-----|-------------|-------------|-------------|-------------|
| 0   | $3.31662 \times 10^{-1}$ | $3.00463 \times 10^{-1}$ | $2.76642 \times 10^{-1}$ | $2.57694 \times 10^{-1}$ |
| 1   | $2.81214 \times 10^{-1}$ | $2.65297 \times 10^{-1}$ | $2.50979 \times 10^{-1}$ | $2.38295 \times 10^{-1}$ |
| 2   | $1.75649 \times 10^{-1}$ | $1.85442 \times 10^{-1}$ | $1.89288 \times 10^{-1}$ | $1.89688 \times 10^{-1}$ |
| 3   | $8.64878 \times 10^{-2}$ | $1.06921 \times 10^{-1}$ | $1.21861 \times 10^{-1}$ | $1.32313 \times 10^{-1}$ |
| 4   | $3.61873 \times 10^{-2}$ | $5.33942 \times 10^{-2}$ | $6.92058 \times 10^{-2}$ | $8.27429 \times 10^{-2}$ |
| 5   | $1.34095 \times 10^{-2}$ | $2.39311 \times 10^{-2}$ | $3.56438 \times 10^{-2}$ | $4.73779 \times 10^{-2}$ |
| 6   | $4.55324 \times 10^{-3}$ | $9.85304 \times 10^{-3}$ | $1.69758 \times 10^{-2}$ | $2.52441 \times 10^{-2}$ |
| 7   | $1.43298 \times 10^{-3}$ | $3.79153 \times 10^{-3}$ | $7.58376 \times 10^{-3}$ | $1.26687 \times 10^{-2}$ |
| 8   | $4.17297 \times 10^{-4}$ | $1.37582 \times 10^{-3}$ | $3.21053 \times 10^{-3}$ | $6.04348 \times 10^{-3}$ |
| 9   | $1.25392 \times 10^{-4}$ | $4.76811 \times 10^{-4}$ | $1.29802 \times 10^{-3}$ | $2.75976 \times 10^{-3}$ |
| 10  | $3.65441 \times 10^{-5}$ | $1.59551 \times 10^{-4}$ | $5.04694 \times 10^{-4}$ | $1.21325 \times 10^{-3}$ |
| 11  | $1.38971 \times 10^{-6}$ | $5.02895 \times 10^{-5}$ | $1.89440 \times 10^{-4}$ | $5.15815 \times 10^{-4}$ |
| 12  | $-1.40240 \times 10^{-6}$ | $1.49690 \times 10^{-5}$ | $6.87622 \times 10^{-5}$ | $2.12808 \times 10^{-4}$ |
Table 7: Fourier coefficients $a_{n,k}^{(50)} \ (n = 20, 22)$

| $k$ | $n = 20$          | $n = 22$          |
|-----|-------------------|-------------------|
| 0   | $2.42161 \times 10^{-1}$ | $2.29129 \times 10^{-1}$ |
| 1   | $2.27078 \times 10^{-1}$ | $2.17130 \times 10^{-1}$ |
| 2   | $1.88097 \times 10^{-1}$ | $1.85373 \times 10^{-1}$ |
| 3   | $1.39342 \times 10^{-1}$ | $1.43837 \times 10^{-1}$ |
| 4   | $9.38340 \times 10^{-2}$ | $1.02652 \times 10^{-1}$ |
| 5   | $5.83684 \times 10^{-2}$ | $6.82124 \times 10^{-2}$ |
| 6   | $3.39847 \times 10^{-2}$ | $4.26601 \times 10^{-2}$ |
| 7   | $1.87134 \times 10^{-2}$ | $2.53303 \times 10^{-2}$ |
| 8   | $9.82378 \times 10^{-3}$ | $1.43802 \times 10^{-2}$ |
| 9   | $4.94778 \times 10^{-3}$ | $7.84948 \times 10^{-3}$ |
| 10  | $2.40299 \times 10^{-3}$ | $4.13869 \times 10^{-3}$ |
| 11  | $1.13008 \times 10^{-3}$ | $2.11578 \times 10^{-3}$ |
| 12  | $5.16361 \times 10^{-4}$ | $1.05205 \times 10^{-3}$ |

Figure 1: Graph of $y = f_0^{(12,50)}(x)$ and a histogram of $\sqrt{b_1} \ (10^6$ riplications)
Here, we show the case that \( n = 20 \) (See also Figure 2):

\[
\hat{F}_{20}^{(12,50)}(x) = 1.21081 \times 10^{-1} + (2.27078 \times 10^{-1}) \cos \left( \frac{1}{18} \sqrt{19\pi x} \right)
+ (1.88097 \times 10^{-1}) \cos \left( \frac{1}{9} \sqrt{19\pi x} \right) + (1.39342 \times 10^{-1}) \cos \left( \frac{1}{6} \sqrt{19\pi x} \right)
+ (9.38340 \times 10^{-2}) \cos \left( \frac{2}{9} \sqrt{19\pi x} \right) + (5.83684 \times 10^{-2}) \cos \left( \frac{5}{18} \sqrt{19\pi x} \right)
+ (3.39847 \times 10^{-2}) \cos \left( \frac{1}{3} \sqrt{19\pi x} \right) + (1.87134 \times 10^{-2}) \cos \left( \frac{7}{18} \sqrt{19\pi x} \right)
+ (9.82378 \times 10^{-3}) \cos \left( \frac{4}{9} \sqrt{19\pi x} \right) + (4.94778 \times 10^{-3}) \cos \left( \frac{1}{2} \sqrt{19\pi x} \right)
+ (2.40299 \times 10^{-3}) \cos \left( \frac{5}{9} \sqrt{19\pi x} \right) + (1.13008 \times 10^{-3}) \cos \left( \frac{11}{18} \sqrt{19\pi x} \right)
+ (5.16361 \times 10^{-4}) \cos \left( \frac{2}{3} \sqrt{19\pi x} \right)
\]

and

\[
\hat{F}_{20}^{(12,50)}(x) = \frac{1}{2} \left( \sqrt{\frac{19x}{18}} + 1 \right) + (2.98484 \times 10^{-1}) \sin \left( \frac{1}{18} \sqrt{19\pi x} \right)
+ (1.23623 \times 10^{-1}) \sin \left( \frac{1}{9} \sqrt{19\pi x} \right) + (6.10528 \times 10^{-2}) \sin \left( \frac{1}{6} \sqrt{19\pi x} \right)
+ (3.08352 \times 10^{-2}) \sin \left( \frac{2}{9} \sqrt{19\pi x} \right) + (1.53445 \times 10^{-2}) \sin \left( \frac{5}{18} \sqrt{19\pi x} \right)
+ (7.44524 \times 10^{-3}) \sin \left( \frac{1}{3} \sqrt{19\pi x} \right) + (3.51399 \times 10^{-3}) \sin \left( \frac{7}{18} \sqrt{19\pi x} \right)
+ (1.61412 \times 10^{-3}) \sin \left( \frac{4}{9} \sqrt{19\pi x} \right) + (7.22627 \times 10^{-4}) \sin \left( \frac{1}{2} \sqrt{19\pi x} \right)
+ (3.15863 \times 10^{-4}) \sin \left( \frac{5}{9} \sqrt{19\pi x} \right) + (1.35040 \times 10^{-4}) \sin \left( \frac{11}{18} \sqrt{19\pi x} \right)
+ (5.65611 \times 10^{-5}) \sin \left( \frac{2}{3} \sqrt{19\pi x} \right).
\]

Tables 8 and 9 show the upper tail probabilities \( 1 - \hat{F}_{20}^{(12,50)}(x) \) and percentiles \( x_\alpha \) for specified \( \alpha = 0.900, 0.950, 0.975, 0.990, 0.995, 0.999 \), respectively, where \( x_\alpha \) is a solution of \( \hat{F}_{20}^{(12,50)}(x_\alpha) = \alpha \). In these tables, the numbers with the underline point out where the decimal places are different from the result of Mulholland [7].

5 Conclusion remarks

We gave the approximation to the sampling distributions based on the Fourier cosine series with their coefficients constructed by higher order moments. We illustrated that proposed approximations have worked well for the case of the sum of the uniform distributed random
Figure 2: Graph of $y = \hat{f}_{20}^{(12,50)}(x)$ and a histogram of $\sqrt{b_1}$ (10^6 replications)

Table 8: Upper tail probability $1 - \hat{F}_n^{(12,50)}(x)$ of $\sqrt{b_1}$

| $x$ | $n = 4$ | $n = 6$ | $n = 8$ | $n = 10$ | $n = 12$ |
|-----|---------|---------|---------|----------|---------|
| 0.1 | 0.4178  | 0.4332  | 0.4311  | 0.4276   | 0.4240  |
| 0.2 | 0.3604  | 0.3705  | 0.3648  | 0.3579   | 0.3511  |
| 0.3 | 0.3083  | 0.3123  | 0.3028  | 0.2932   | 0.2839  |
| 0.4 | 0.2631  | 0.2583  | 0.2464  | 0.2352   | 0.2244  |
| 0.5 | 0.2207  | 0.2093  | 0.1967  | 0.1849   | 0.1736  |
| 0.6 | 0.1818  | 0.1653  | 0.1544  | 0.1427   | 0.1316  |
| 0.7 | 0.1451  | 0.1279  | 0.1194  | 0.1082   | 0.0979  |
| 0.8 | 0.1103  | 0.0986  | 0.0907  | 0.0808   | 0.0717  |
| 0.9 | 0.0776  | 0.0756  | 0.0676  | 0.0594   | 0.0517  |
| 1.0 | 0.0459  | 0.0567  | 0.0496  | 0.0430   | 0.0368  |
| 1.1 | 0.0162  | 0.0414  | 0.0359  | 0.0307   | 0.0258  |
| 1.2 | –       | 0.0291  | 0.0257  | 0.0217   | 0.0179  |
| 1.3 | –       | 0.0193  | 0.0181  | 0.0150   | 0.0123  |
| 1.4 | –       | 0.0118  | 0.0124  | 0.0103   | 0.0083  |
| 1.5 | –       | 0.0063  | 0.0082  | 0.0069   | 0.0056  |
| 1.6 | –       | 0.0026  | 0.0051  | 0.0046   | 0.0037  |
| 1.7 | –       | 0.0006  | 0.0030  | 0.0029   | 0.0024  |
| 1.8 | –       | –       | 0.0016  | 0.0018   | 0.0015  |
| 1.9 | –       | –       | 0.0008  | 0.0010   | 0.0009  |
| 2.0 | –       | –       | 0.0003  | 0.0006   | 0.0005  |
Table 9: Upper percentiles $x_\alpha$ of $\sqrt{b_1}$

| $n$  | 0.900 | 0.950 | 0.975 | 0.990 | 0.995 | 0.999 |
|------|-------|-------|-------|-------|-------|-------|
| 4    | 0.8305| 0.9869| 1.0697| 1.1210| 1.1379| 1.1513|
| 6    | 0.7945| 1.0412| 1.2392| 1.4299| 1.5306| 1.6707|
| 8    | 0.7652| 0.9977| 1.2080| 1.4524| 1.6046| 1.8668|
| 10   | 0.7275| 0.9539| 1.1595| 1.4075| 1.5785| 1.9065|
| 12   | 0.6931| 0.9100| 1.1091| 1.3532| 1.5255| 1.8818|
| 14   | 0.6622| 0.8699| 1.0614| 1.2985| 1.4683| 1.8315|
| 16   | 0.6347| 0.8337| 1.0176| 1.2467| 1.4121| 1.7739|
| 18   | 0.6102| 0.8012| 0.9778| 1.1988| 1.3595| 1.7133|
| 20   | 0.5881| 0.7721| 0.9419| 1.1543| 1.3108| 1.6520|
| 22   | 0.5681| 0.7459| 0.9097| 1.1130| 1.2637| 1.6045|

Variables and of sample skewness with small $n$. In a future work, we will derive an approximate distribution function of sample kurtosis under normality, because the recurrence relations of moments like (7) are given by Nakagawa et al. [8].

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