ON EFFECTIVE CONDUCTIVITY
OF FLAT RANDOM TWO-PHASE MODELS

S.A. Bulgadaev

Landau Institute for Theoretical Physics
Chernogolovka, Moscow Region, Russia, 142432

An approximate functional equation for effective conductivity $\sigma_{\text{eff}}$ of systems with a finite maximal scale of inhomogeneities is deduced. An exact solution of this equation is found and its physical meaning is discussed. A two-phase randomly inhomogeneous model is constructed by a hierarchical method and its effective conductivity at arbitrary phase concentrations is found in the mean-field-like approximation. These expressions satisfy all necessary symmetries, reproduce the known formulas for $\sigma_{\text{eff}}$ in weakly inhomogeneous case and coincide with two recently found partial solutions of the duality relation. It means that $\sigma_{\text{eff}}$ even of the two-phase randomly inhomogeneous system may be a nonuniversal function and can depend on some details of the structure of the inhomogeneous regions. The percolation limit is briefly discussed.

PACS: 73.61.-r, 75.70.Ak

The electrical transport properties of the classical inhomogeneous systems have an important practical interest. The simplest problem in this region is a finding of the effective conductivity $\sigma_{\text{eff}}$ of an isotropic inhomogeneous (randomly or regularly) two-phase system, which is a mixture of two phases with different conductivities $\sigma_i (i=1,2)$. Despite of its relative simplicity only a few general results have been obtained so far. In case of weakly inhomogeneous isotropic medium

\[ \sigma_{\text{eff}} = \langle \sigma \rangle \left( 1 - \frac{\langle \sigma^2 \rangle - \langle \sigma \rangle^2}{2\langle \sigma \rangle^2} \right). \]  

(1)

For two-phase system $\langle \sigma \rangle = x\sigma_1 + (1 - x)\sigma_2$, $\langle \sigma^2 \rangle - \langle \sigma \rangle^2 = 4x(1 - x)(\sigma_+ - \sigma_-)^2$, where $x$ is a concentration of the first phase, $\sigma_- = (\sigma_1 - \sigma_2)/2$, and (1) takes the form

\[ \sigma_{\text{eff}} = \sigma_+ \left( 1 + 2\epsilon z - (1 - 4\epsilon^2)z^2/2 \right), \]  

(1')

where $\sigma_+ = (\sigma_1 + \sigma_2)/2$, $z = \sigma_+/\sigma_+ (-1 \leq z \leq 1)$, $\epsilon = x - 1/2$.

Another general formula is the dilute limit of the Maxwell–Garnett formula

\[ \sigma_{\text{eff}} = \sigma_1 (1 - 2(1 - x)z), \]  

(2)

where $1 - x \ll 1$ is a small concentration of the second phase and a round form of the inclusions of this phase is suggested.
Then, the exact Keller – Dykhne formula for $\sigma_{\text{eff}}$ in systems with equal concentrations of both phases and the exact dual relation, connecting effective conductivities at adjoint concentrations $x$ and $1-x$, have been obtained [2, 3]

$$
\sigma_{\text{eff}}(x, \sigma_1, \sigma_2)\sigma_{\text{eff}}(1-x, \sigma_1, \sigma_2) = \sigma_1 \sigma_2
$$

(3)
The exact Keller – Dykhne formula follows from (3) at $x = x_c = 1/2$

$$
\sigma_{\text{eff}}(x_c, \sigma_1, \sigma_2) = \sqrt{\sigma_1 \sigma_2}.
$$

(4)

This formula is very simple and universal, because it does not depend on inhomogeneous structure and takes place for regular inhomogeneous two-phase systems [4, 5] as well as for slightly nonregular inhomogeneous systems [5].

Of course, a formula for the effective conductivity at arbitrary phase concentrations has the main interest in this problem. One such approximate formula for $\sigma_{\text{eff}}$ has been obtained many years ago in the so called effective medium (EM) approximation [6]

$$
\sigma_{\text{eff}}(\epsilon, \{\sigma\}) = \sigma + \left(2\epsilon z + \sqrt{1 - z^2 + 4(\epsilon z)^2}\right).
$$

(5)

where $\{\sigma\} = (\sigma_1, \sigma_2)$. Though it corresponds to the weakly inhomogeneous case it turns out to be a good approximation, when $\sigma_i \neq 0$ [7].

It was shown recently that the duality relation (3) together with boundary conditions and some assumptions about the properties of $\sigma_{\text{eff}}$ allow to find an explicit form of $\sigma_{\text{eff}}$ at arbitrary $x$ and two such expressions were found [8, 9]. In this paper we will represent two randomly inhomogeneous models, having their effective conductivities just of these two forms. Another important question appears naturally in this problem: is a formula for the effective conductivity may be nonuniversal even in the two-phase case. At the end of the paper we will briefly discuss some peculiarities of the percolation limit.

We start our investigation with a general discussion of the averaging procedure for obtaining $\sigma_{\text{eff}}$. It is easy to see that the effective conductivity will depend on a scale $l$ of a region over which an averaging is done. This takes place due to the possible existence of different characteristic scales in the inhomogeneous medium. In the most general case there will be a whole spectrum of these characteristic scales. This spectrum can be very different: from discrete finite till continuous infinite, and is defined by the structure of the inhomogeneities of the system. For this reason it can depend on the phase concentration $x$. Suppose, for the simplicity, that the randomly inhomogeneous structure of our system has the scale spectrum with a maximal scale $l_m(x)$, which is finite for all $x$ in the region $1 \geq x > 1/2$ (or $1-x$ in the region $0 \leq 1-x < 1/2$). Let us assume that we know an exact formula for $\sigma_{\text{eff}}(x, \{\sigma\})$ of this system, which
is applicable from scales $l > l_m$. It means that this formula for $\sigma_{eff}(x,\{\sigma}\})$ takes place after the averaging over regions with a mean size $l \geq l_m$ and does not change for all larger scales $l \gg l_m$. Now consider a square lattice with the squares of length $l_L \gg l_m$ and suppose that they have the effective conductivities corresponding to different values of the concentrations $x_1$ and $x_2$ with equal probabilities $p = 1/2$ (see Fig.1).

After the averaging over the scales $l \gg l_L$ one must obtain on much larger scales $l \gg l_L$ the same effective conductivity, but corresponding to another concentration $x = (x_1 + x_2)/2$. This is possible due to the similar random structure of different squares and due to the conjectures that (1) in this model there are only two important scales: a maximum of the maximal characteristic scales (we suppose here that $l_1 \sim l_2 \sim l_m(x)$, where $l_m(x_i) \equiv l_i$ $(i = 1, 2)$) and the lattice square size $l_L \gg l_i$, (2) the averaging procedures over these scales do not correlate (or weakly correlate) between themselves. Thus, for a compatibility, all concentrations must be out of small region around critical concentration $x_c$, where $l_i$ or $l_m(x)$ can be very large. We will call further this set of the conjectures the finite maximal scale averaging approximation (FMSA approximation). It can be considered as some nontrivial modification of the EM approximation and can be implemented for systems with compact inhomogeneous inclusions with finite $l_m$.

From the other side the effective conductivity on scales $l \gg l_L$ must be determined by the universal Keller – Dykhne formula (4). Thus we obtain the next functional equation for the effective conductivity, connecting $\sigma_{eff}(x,\{\sigma}\})$ at different concentrations,

$$\sigma_{eff}(x,\{\sigma}\}) = \sqrt{\sigma_{eff}(x_1,\{\sigma}\})\sigma_{eff}(x_2,\{\sigma}\}), \quad x = (x_1 + x_2)/2. \quad (6)$$

It must be supplemented by the boundary conditions

$$\sigma_{eff}(1,\{\sigma}\}) = \sigma_1, \quad \sigma_{eff}(0,\{\sigma}\}) = \sigma_2, \quad (6')$$

The equation (6) can be considered as a generalization of the duality relation (3), the latter being a particular case of (6) at $x_1 + x_2 = 1$. It follows from
that for \( z \neq 1 \) and due to the exactness of the duality relation it really works at all concentrations \( x \) except maybe of small region near \( x \geq x_c \) and \( \sigma_2 = 0 \) \((z = 1)\) (see below a discussion of the percolation limit). It is easy to see also that the approximate formula (2) for \( \sigma_{eff}(x, \{\sigma\}) \) satisfies equation (6). Moreover, one can find an exact solution of this equation. It has an exponential form with a linear function of \( x \) (or \( \epsilon \))

\[
\sigma_{eff}(x, \{\sigma\}) = \sigma_1 \exp(ax + b),
\]

where the constants \( a, b \) can be determined from the boundary conditions

\[
a = -b, \quad \exp b = \sigma_2 / \sigma_1.
\]

(7′)

Substituting these coefficients into (7) one obtains

\[
\sigma_{eff}(x, \{\sigma\}) = \sigma_1 (\sigma_2 / \sigma_1)^{(1-x)} = \sigma_1 x \sigma_2^{1-x}.
\]

(8)

The solution (8) satisfies all required symmetry relations and exactly coincides with the case (a) from [8, 9].

It is interesting to note that the form (8) means that in the FMSA approximation one has effectively the averaging of \( \ln \sigma \) since it can be represented as

\[
\ln \sigma_{eff} = \langle \ln \sigma \rangle = x \ln \sigma_1 + (1-x) \ln \sigma_2.
\]

(9)

This was noted already in [8] for the case of equal concentrations \( x = 1/2 \). The analogous result has been obtained later for random system in the theory of two-dimensional weak localization [11]. One can check that (8) reproduces in the weakly inhomogeneous limit the universal Landau – Lifshitz expression (1).

In the low concentration limit of the second phase it gives

\[
\sigma_{eff}(x, z) = \sigma_1 (1 + (1-x) \log \frac{1 - z}{1 + z} + ...), \quad 1 - x \ll 1,
\]

(10)

what coincides with (2) in the weakly inhomogeneous case. Note that the expansion (10) contains the coefficients diverging in the limit \( |z| \to 1 \). Such behaviour of the coefficients denote the existence of a possible singularity in this limit, where the FMSA approximation is not applicable (see below a discussion of this percolation limit).

Now we will construct a hierarchical model of flat isotropic randomly inhomogeneous two-phase system, using the composite method introduced above, and find its effective conductivity \( \sigma_{eff}(x, \{\sigma\}) \).

Let us consider a simple square lattice with the squares consisting of a random layered mixture of two conducting phase with constant conductivities \( \sigma_i \) \((i = 1, 2)\) and the corresponding concentrations \( x \) and \( 1 - x \). A schematic picture of such square is given in Fig.2.

The layered structure of the squares means that the squares have some preferred direction, for example along the layers. Let us suppose that the directions of different squares are randomly oriented (parallelly or perpendicularly) relatively to the external electric field, which is directed along \( x \) axis. In order
Fig. 2. a) An elementary square of the model with a vertical orientation, the dotted regions denote layers of the second phase; b) a lattice of the model, the small lines on the squares denote their orientations.

For the system to be isotropic the probabilities of the parallel and perpendicular orientations of squares must be equal or (what is the same) the concentrations of the squares with different orientations must be equal $p_{||} = p_{\perp} = 1/2$.

This structure can appear, for example, on the small macroscopic scales, when a random medium is formed as a result of the stirring of the two-phase mixture. The corresponding averaged parallel and perpendicular (or serial) conductivities of squares $\sigma_{||}(x)$ and $\sigma_{\perp}(x)$ are defined by the following formulas

$$
\sigma_{||}(x) = x\sigma_1 + (1-x)\sigma_2 = \sigma_+ (1 + 2\epsilon z),
$$

$$
\sigma_{\perp}(x) = \left(\frac{x}{\sigma_1} + \frac{1-x}{\sigma_2}\right)^{-1} = \sigma_+ \frac{1 - z^2}{1 - 2\epsilon z},
$$

(11)

Thus we have obtained the hierarchical representation of random medium (in this case a two-level one). On the first level it consists from some regions (two different squares) of the random mixture of the two layered conducting phases with different conductivities $\sigma_1$ and $\sigma_2$ and arbitrary concentration. On the second level this medium is represented as a random parquet constructed from two such squares with different conductivities $\sigma_{||}$ and $\sigma_{\perp}$, depending nontrivially on concentration of the initial conducting phases, and randomly distributed with the same probabilities $p_i = 1/2$ (Fig.2). This representation allows us to divide the averaging process into two steps (firstly averaging over each square and then averaging over the lattice of squares) and implement on the second step the exact formula (4). This can be considered as some modification of the FMSA approximation. As a result one obtains for the effective conductivity of the introduced random two-phase model the following formula, which is applicable for arbitrary concentration

$$
\sigma_{\text{eff}}(\epsilon, \{\sigma\}) = \sigma_+ f(\epsilon, z), \quad f(\epsilon, z) = \sqrt{1 - z^2} \frac{1 + 2\epsilon z}{1 - 2\epsilon z}^{1/2},
$$

(12)

This function has all necessary properties, satisfies equation (2) and coincides with the second form from [8, 9].

5
It is interesting to compare this formula with the known general formulas. 
(a) In case of small concentration of the first phase \( x \ll 1 \) one gets
\[
\sigma_{\text{eff}}(x, \{\sigma\}) \simeq \sigma_2 \left(1 + \frac{2xz}{1 - z^2}\right).
\] (13)
It follows from (13) that an addition of small part of the first higher conducting phase increases an effective conductivity of the system as it should be.
(b) In the opposite case of small concentration of the second phase \( 1 - x \ll 1 \) one obtains
\[
\sigma_{\text{eff}}(x, \{\sigma\}) \simeq \sigma_1 \left(1 - \frac{2(1 - x)z}{1 - z^2}\right),
\] (14)
i.e. an addition of the phase with smaller conductivity decreases \( \sigma_{\text{eff}} \). It is worth to note that both these expressions for arbitrary values of the conductivities \( \sigma_1 \) and \( \sigma_2 \) differ from equation (2) and coincide with it only in the weakly inhomogeneous case \( z \ll 1 \). It must be not surprising because a form of the inclusions of the second phase in this model has completely different, layered, structure. In the low concentration expansion one can see again that the divergencies appear in the limit \( |z| \rightarrow 1 \).
(c) In case of almost equal phase concentrations \( x = 1/2 + \epsilon, \; \epsilon \ll 1 \) one obtains
\[
\sigma_{\text{eff}}(\epsilon, \{\sigma\}) \simeq \sigma_+ \sqrt{1 - z^2} (1 + 2\epsilon z).
\] (15)
The Keller–Dykhne formula (3) is reproduced for equal concentrations.

One must note that at the same time the formula (12) does not satisfy the equation (6) except of the trivial case \( x_1 = x_2 \).

For a comparison of the different expressions of the effective conductivity the plots of \( f(\epsilon, z) \) in the EM approximation, in FMSA approximation and of the hierarchical model in FMSA-like approximation were constructed (see [8, 9]). It follows from these plots that all derived above formulas for \( \sigma_{\text{eff}} \), despite of their various functional forms, differ from each other weakly for \( z \lesssim 0.8 \) due to very restrictive boundary conditions (6') and the exact Keller-Dykhne value. This range of \( z \) corresponds approximately to the ratio \( \sigma_2/\sigma_1 \sim 10^{-1} \). For the smaller ratios a difference between these functions become distinguishable.

Now let us consider in the more details the derived formulas for \( \sigma_{\text{eff}}(\epsilon, \{\sigma\}) \) in case when \( \sigma_2 \rightarrow 0(z \rightarrow 1) \). It is clear that for regularly inhomogeneous medium one can always construct such distribution of the conducting phase that \( \sigma_{\text{eff}}(\epsilon, 1) \) will differ from zero for all \( 1/2 \geq \epsilon > -1/2 \). But in the case of randomly inhomogeneous medium the limit \( \sigma_2 \rightarrow 0 \) is equivalent to the well known percolation problem [12, 13]. In terms of \( z \) it corresponds to the limit \( z \rightarrow 1 \) and is also similar to the superconducting limit \( \sigma_1 \rightarrow \infty \). Strictly speaking, an implementation of the duality transformation (3) is not obvious in this case. However, if one supposes that the dual symmetry relation (3) fulfills in this limit too due to a continuity then it follows from (3) that
\[
\sigma_{\text{eff}}(\epsilon)\sigma_{\text{eff}}(-\epsilon) = 0.
\] (16)
The relation (16) does not contradict to the known basic results of the percolation theory that \( \sigma_{\text{eff}}(\epsilon) = 0 \) for \( \epsilon \leq 0 \) and \( \sigma_{\text{eff}}(\epsilon) \neq 0 \) for \( \epsilon > 0 \). Moreover, one can show that in this case

\[
\sigma_{\text{eff}}(\epsilon) = \begin{cases} 
0, & \epsilon \leq 0, \\
2\sigma_a, & \epsilon > 0,
\end{cases}
\]  

(17)

where \( \sigma_a \) is the odd part of \( \sigma_{\text{eff}}(\epsilon) \). It means that a behaviour of \( \sigma_{\text{eff}}(\epsilon) \) in the percolation theory is completely determined by its odd part. It is known from the experimental and numerical results that in the percolation limit the effective conductivity \( \sigma_{\text{eff}} \) have a nonanalytical behaviour near the percolation edge \( x_c = 1/2 \) (or at small \( \epsilon > 0 \))

\[
\sigma_{\text{eff}}(\epsilon) \sim \sigma_1(1 - x_c)^t \sim \sigma_1\epsilon^t,
\]

(18)

where a critical exponent of the conductivity \( t \) is slightly above 1 and can be represented in the form \( t = 1 + \delta \). Since the values of this exponent found by the numerical calculations are confined to be in the interval \((1,10 - 1,4)\) [13], then \( \delta \) have to be small and belongs to the interval \((0.1 - 0.4)\). It follows from (17) that the same behaviour must have \( \sigma_a \). It means that at small \( \epsilon \) there is some crossover on \( z \) under \( z \to 1 \) from a regular (analytical) behaviour to a singular one. At the moment an exact form of this crossover is unknown.

From the formulas obtained above, one gets always \( \sigma_{\text{eff}} \to 0 \) in the limit \( \sigma_2 \to 0 \), except the region near \( x = 1 \). It means that all these formulas, obtained in FMSA approximation, are not valid in the limit \( \sigma_2 \to 0 \). This is confirmed by the appearance of the divergencies in the limit \( z \to 1 \). This fact is a consequence of the made approximation. For example, in case of the model of the layered squares this is due to the "closing" (or "locking") effect of the layered structure in the adopted approximation in the limit \( \sigma_2 \to 0 \). In order to obtain a finite conductivity in this model above threshold concentration \( x_c \) one needs to take into account the correlations between adjacent squares. It is easy to show that near the threshold an effective conductivity is determined by random conducting clusters formed out of the crossing random layers from neighbouring elementary squares. As is well known, the mean size of these clusters diverges near the percolation threshold [12] [13] and for this reason the FMSA approximation cannot be applicable for the description of \( \sigma_{\text{eff}} \) in the region \( z \to 1 \) and \( x \leq 1/2 \). It follows from our results that EM approximation overestimates \( \sigma_{\text{eff}} \), whereas both other formulas underestimate it in this region. We hope to investigate the percolation limit in detail in the subsequent papers.

Thus, though both formulae for the effective conductivity obtained above have the various functional forms, they satisfy all symmetries, including the dual symmetry and all necessary inequalities, and reproduce the general formulae for \( \sigma_{\text{eff}} \) in the weakly inhomogeneous case. These results allow us to make a conjecture that \( \sigma_{\text{eff}} \) even of the two-phase randomly inhomogeneous systems may be a nonuniversal function and can depend on some details of the structure of the randomly inhomogeneous regions. The obtained formulae can be considered as the regular ones, since they are applicable only for systems with \( \sigma_i \neq 0 \),
when there are no singularities connected with a percolation problem. The introduced composite method of the construction of the model random medium can be generalized on the heterophase systems with arbitrary number of phases and on the other ways of determination of the effective intermediate conducting boxes. It can be done for various types of boxes as well as for different numbers of the possible types of the boxes. It is clear that then one will have to use instead of (4) another formula. The constructed models and obtained expressions can be used for the modelling and description of some real composite systems.

The author thanks referees for useful remarks. This work was supported by RFBR grants # 2044.2003.2 and 02-02-16403.

References

[1] Landau L.D., Lifshitz E.M., *Electrodynamics of condensed media*, Moscow, 1982 (in Russian).

[2] Keller J.B., *J.Math.Phys.*, 5 (1964) 548.

[3] Dykhne A.M., *ZhETF*, 59 (1970) 110 (in Russian).

[4] Emetz Yu.P., *ZhETF*, 96 (1989) 701 (in Russian).

[5] Ovchinnikov Yu.N., Dyugaev A., *ZhETF*, 117 (2000) 1013; Balagurov B.Ya., *ZhETF*, 117 (2000) 1561, 118 (2001) 665 (in Russian).

[6] Bruggeman D.A.G., *Ann.Physik*, 24(1935) 636; Landauer R., *J.Appl.Phys.*, 23 (1952) 779.

[7] Kirkpatrick S., *Phys.Rev.Lett.*, 27 (1971) 1722.

[8] Bulgadaev S.A., Preprint ITP 02-12-02 (2002), [cond-mat/0212104](http://arxiv.org/abs/cond-mat/0212104)

[9] Bulgadaev S.A., *Phys.Lett.*, A313 (2003) 106.

[10] Fel L.G., Machavariani V.Sh., Khalatnikov I.M. and Bergman D.J., *J.Phys.*, A33 (2000) 6669.

[11] Anderson P.W., Thouless D.J., Abrahams E. and Fisher D.S., *Phys.Rev.*, B22 (1980) 3519.

[12] Kirkpatrick S., *Rev.Mod.Phys.*, 45 (1973) 574.

[13] Shklovskii B.I., Efros A.L., *Electronic Properties of Doped Semiconductors*, Vol.45, Springer Series in Solid State Sciences, (Springer Verlag, Berlin) 1984.