MEROMORPHIC FUNCTIONS WITH TWO COMPLETELY INVARIANT DOMAINS

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Dedicated to the memory of Professor I. N. Baker

Abstract. We show that if a meromorphic function has two completely invariant Fatou components and only finitely many critical and asymptotic values, then its Julia set is a Jordan curve. However, even if both domains are attracting basins, the Julia set need not be a quasicircle. We also show that all critical and asymptotic values are contained in the two completely invariant components. This need not be the case for functions with infinitely many critical and asymptotic values.

1. Introduction and main result

Let $f$ be a meromorphic function in the complex plane $\mathbb{C}$. We always assume that $f$ is not fractional linear or constant. For the definitions and main facts of the theory of iteration of meromorphic functions we refer to a series of papers by Baker, Kotus and Lü [2, 3, 4, 5], who started the subject, and to the survey article [8]. For the dynamics of rational functions we refer to the books [7, 11, 21, 25].

A completely invariant domain is a component $D$ of the set of normality such that $f^{-1}(D) = D$. There is an unproved conjecture (see [4, p. 608], [8, Question 6]) that a meromorphic function can have at most two completely invariant domains. For rational functions this fact easily follows from Fatou’s investigations [15], and it was first explicitly stated by Brolin [10, §8]. Moreover, if a rational function has two completely invariant domains, then their common boundary is a Jordan curve on the Riemann sphere, and each of the domains coincides with the basin of attraction of an attracting or superattracting fixed point, or of an attracting petal of a neutral fixed point with multiplier 1; see [15, p. 300-303] and [10]. All critical values of $f$ are contained in the completely invariant domains.

In this paper we extend these results to a class of transcendental meromorphic functions in $\mathbb{C}$. This class $S$ consists of meromorphic functions with finitely many critical and asymptotic values. Let $A = A(f)$ be the set of critical and asymptotic values. Let $A = A(f)$ be the set of critical and asymptotic values. We also call the elements of $A$ singular values of $f$. For $f \in S$ the map

$$ f : \mathbb{C} \setminus f^{-1}(A) \to \mathbb{C} \setminus A $$

is a covering. By $J = J(f) \subset \mathbb{C}$ we denote the Julia set of $f$.

Baker, Kotus and Lü [3 Theorem 4.5] proved that functions of the class $S$ have at most two completely invariant domains. Cao and Wang [12 Theorem 1] have shown that if a function in the class $S$ has two completely invariant domains, then

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its Fatou set is the union of these domains. We complement these results with the following

**Theorem.** Let \( f \) be a function of the class \( S \), having two completely invariant domains \( D_j, j = 1, 2 \). Then

(i) each \( D_j \) is the basin of attraction of an attracting or superattracting fixed point, or of a petal of a neutral fixed point with multiplier 1,

(ii) \( A(f) \subset D_1 \cup D_2 \),

(iii) each \( D_j \) contains at most one asymptotic value, and if \( a \) is an asymptotic value and \( 0 < \epsilon < \text{dist}(a, A\{a\}) \), then the set \( \{ z : |f(z) - a| < \epsilon \} \) has only one unbounded component,

(iv) \( J \cup \{ \infty \} \) is a Jordan curve in \( \overline{\mathbb{C}} \).

A simple example of a meromorphic function of class \( S \) with two completely invariant domains is \( f(z) = \tan z \), for which the upper and lower half-planes are completely invariant, and each of these half-planes is attracted to one of the two petals of the fixed point \( z = 0 \).

More examples will be given later in \( \S 3 \).

In the case that \( f \) is rational and both \( D_1 \) and \( D_2 \) are attracting or superattracting basins, Sullivan [27, Theorem 7] and Yakobson [28] proved that \( J \) is a quasicircle. Steinmetz [26] extended this result to the case that both completely invariant domains are basins of two petals attached to the same neutral fixed point. We will construct examples of transcendental functions in \( S \) for which \( D_1 \) and \( D_2 \) are attracting basins, or basins of petals attached to the same neutral fixed point, but where \( J \) is not a quasicircle; see Examples 1 and 2 in \( \S 3 \).

On the other hand, Keen and Kotus [16, Corollary 8.2] have shown that for the family \( f_\lambda(z) = \lambda \tan z \) there exists a domain \( \Omega \) containing \( (1, \infty) \) such that \( f_\lambda \) has two completely invariant attracting basins and \( J(f_\lambda) \) is a quasicircle for \( \lambda \in \Omega \). Meromorphic functions for which the Julia set is contained in a quasicircle were also considered by Baker, Kotus and Lü [2, \S 5].

Baker [1] proved that an entire function \( f \) can have at most one completely invariant component of the set of normality, and that such a domain contains all critical values. Eremenko and Lyubich [14, \S 6] proved that a completely invariant domain of an entire function also contains all asymptotic values of a certain type, namely those associated with direct singularities of \( f^{-1} \). On the other hand, Bergweiler [9] constructed an entire function with a completely invariant domain \( D \), and such that some asymptotic value belongs to the Julia set \( J = \partial D \). Example 3 in \( \S 3 \) shows that meromorphic functions with two completely invariant components of the set of normality can have asymptotic values on their Julia sets. So (ii) does not hold for general meromorphic functions, without the assumption that \( f \in S \).

2. Proof of the Theorem

We shall need the following result of Baker, Kotus and Lü [4] Lemmas 4.2 and 4.3 which does not require that \( f \in S \). Here and in the following all topological notions are related to \( \mathbb{C} \) unless \( \overline{\mathbb{C}} \) is explicitly mentioned.

**Lemma 1.** Let \( f \) be meromorphic with two completely invariant components \( D_1 \) and \( D_2 \) of the set of normality. Then \( D_1 \) and \( D_2 \) are simply-connected and \( J = \partial D_1 = \partial D_2 \). In particular, \( J \) is a connected subset of \( \mathbb{C} \).
Proof of the Theorem. As the result is known for rational $f$, we assume that our $f$ is transcendental.

Statement (i) follows from the classification of dynamics on the Fatou set for meromorphic functions of the class $S$, given in [51 Theorems 2.2 and 2.3], [8 Theorem 6], and [21 p. 3252].

To prove (ii), we consider for $j = 1, 2$ the finite sets $A_j = A \cap D_j$. Let $\Gamma_j$ be a Jordan curve in $D_j$ which separates $A_j$ from $\partial D_j$. Let $G_j$ be the Jordan regions bounded by the $\Gamma_j$. Let $G = \mathbb{C}\setminus(G_1 \cup G_2)$ be the doubly connected region in $\mathbb{C}$ bounded by $\Gamma_1$ and $\Gamma_2$. Notice that $G$ contains the Julia set $J$.

We define $\gamma_j = f^{-1}(\Gamma_j)$. Then

\[ f : \gamma_j \to \Gamma_j, \quad j = 1, 2, \]

are covering maps.

We claim that each $\gamma_j \subset \mathbb{C}$ is a single simple curve tending to infinity in both directions, which means that $\gamma_j \cup \{\infty\}$ is a Jordan curve in $\overline{\mathbb{C}}$.

To prove the claim, we fix $j$ and consider the full preimages $H_j = f^{-1}(G_j)$ and $F_j = f^{-1}(D_j \setminus \overline{G_j})$. Then $D_j = F_j \cup H_j \cup \gamma_j$. The boundary of each component of $F_j$ contains a component of $\gamma_j$, and this gives a bijective correspondence between components of $F_j$ and components of $\gamma_j$.

We notice that $H_j$ is connected. Indeed, by complete invariance of $D_j$, we have $H_j \subset D_j$, so every two points $z_1$ and $z_2$ in $H_j$ can be connected by a curve $\beta$ in $D_j$, so that $\beta$ does not pass through the critical points of $f$. The image $f(\beta)$ of this curve begins and ends in $G_j$, and does not pass through the critical values of $f$. By a small perturbation of $\beta$ we achieve that $f(\beta)$ does not pass through asymptotic values. Using the fact that

\[ f : D_j \setminus f^{-1}(A_j) \to D_j \setminus A_j, \]

is a covering and that $A_j \subset G_j$, we can deform $\beta$ into a curve in $H_j$ which still connects $z_1$ and $z_2$. This proves that the $H_j$ are connected.

It follows that $H_j$ is unbounded, as it contains infinitely many preimages of a generic point in $G_j$.

It is easy to see that the boundary of each component $F'_j$ of $F_j$ intersects the Julia set.

For each component $F'_j$ of $F_j$, the intersection $\partial F'_j \cap \overline{H_j}$ is a component $\gamma'_j$ of $\gamma_j$. This component $\gamma'_j$ divides the plane into two parts, one containing $H_j$ and the other containing $F'_j$. We conclude that every component of $\gamma_j$ is unbounded, because $H_j$ is unbounded, and $\partial F_j$ intersects the Julia set which is unbounded and connected by Lemma 1. (A similar argument for unboundedness of each component of $\gamma_j$ was given in [41].)

For every component $\gamma'_j$ of $\gamma_j$, the component of $\mathbb{C} \setminus \gamma'_j$ that contains $F'_j$ intersects the Julia set. Since the Julia set is connected by Lemma 1, we conclude that $F_j$ and $\gamma_j$ are connected. So the map $f$ is a universal covering by a connected set $\gamma_j$, for each $j = 1, 2$. Thus $\gamma_1 \cup \{\infty\}$ and $\gamma_2 \cup \{\infty\}$ are Jordan curves in $\overline{\mathbb{C}}$ whose intersection consists of the single point $\infty$. This proves our claim.

As a corollary we obtain that the point $\infty$ is accessible from each $D_j$, and so all poles of all iterates $f^n$ are accessible from each $D_j$. (This fact was established in [41].)
Next we note that the set \( \gamma_1 \cup \gamma_2 \cup \{ \infty \} \) separates the sphere into three simply connected regions. We denote by \( W \) that region whose boundary in \( \mathbb{C} \) is \( \gamma_1 \cup \gamma_2 \cup \{ \infty \} \). Then
\[
(2) \quad f^{-1}(G) = W,
\]
in particular, \( W \) contains the Julia set \( J \).

To complete the proof of (ii) we choose an arbitrary point \( w \in J \) and show that \( w \) is neither a critical value nor an asymptotic value.

Fix an arbitrary point \( w_1 \in \Gamma_1 \). The preimage \( f^{-1}(w_1) \) consists of infinitely many points \( a_k \in \gamma_1 \), which we enumerate by all integers in a natural order on \( \gamma_1 \). Let \( \phi_k \) be the branches of \( f^{-1} \) such that \( \phi_k(w_1) = a_k \). We find a simple curve \( \Delta \) from \( w_1 \) to some point \( w_2 \in \Gamma_2 \) such that \( \Delta \setminus \{ w_1, w_2 \} \) is contained in \( G \setminus \{ w \} \), and such that all branches \( \phi_k \) have analytic continuation along \( \Delta \) to the point \( w_2 \). We define
\[
G' = G \setminus \Delta \subset \mathbb{C}.
\]
The full preimage \( f^{-1}(\Delta) \) consists of infinitely many disjoint simple curves \( \delta_k \) starting at the points \( a_k \) and ending at some points \( b_k \in \gamma_2 \). The open curves \( \delta_k \setminus \{ a_k, b_k \} \) are disjoint from \( \gamma_1 \cup \gamma_2 \).

For every integer \( k \), let \( Q_k \) be the Jordan region bounded by \( \delta_k, \delta_{k+1} \), the arc \( (a_k, a_{k+1}) \) of \( \gamma_1 \) and the arc \( (b_k, b_{k+1}) \) of \( \gamma_2 \). Then \( f \) maps \( Q_k \) into \( G' \), and \( f(\partial Q_k) \subset \partial G' \). So
\[
(3) \quad f : Q_k \to G'
\]
is a ramified covering, continuous up to the boundary. Furthermore, the boundary map is a local homeomorphism. As each point of \( \Gamma_1 \setminus \{ w_1 \} \) has only one preimage on \( \partial Q_k \), we conclude that \( \Phi_k \) is a homeomorphism. Now it follows that the restriction \( f : (b_k, b_{k+1}) \to \Gamma_2 \setminus \{ w_2 \} \) is a homeomorphism and
\[
W = \bigcup_{k=-\infty}^{\infty} Q_k \cup \delta_k \setminus \{ a_k, b_k \}.
\]
It follows that there are no critical points over \( w \), so \( w \) is not a critical value.

If \( w \) were an asymptotic value, there would be a curve \( \alpha \) in \( W \) which tends to infinity, and such that \( f(z) \to w \) as \( z \to \infty \), \( z \in \alpha \). But this curve \( \alpha \) would intersect infinitely many of the curves \( \delta_k \), so its image \( f(\alpha) \) would intersect \( \Delta \) infinitely many times, which contradicts the assumption that \( f(\alpha) \) tends to \( w \).

This completes the proof of (ii). The proof actually shows that \( f : W \to G \) is a universal covering, a fact which we will use later.

To prove (iii), let us assume that \( D_1 \) contains two asymptotic values, or that \( \{ z \in D_1 : |f(z) - a| < \epsilon \} \) has two unbounded components for some asymptotic value \( a \in D_1 \). Then there exists a curve \( \alpha \subset D_1 \), tending to infinity in both directions, such that \( f(z) \) has limits as \( z \to \infty \), \( z \in \alpha \), in both directions, where these limits are the two asymptotic values in the first case, and where both limits are equal to \( a \) in the second case, but the two tails of the curve \( \alpha \) are in different components of \( \{ z \in D_1 : |f(z) - a| < \epsilon \} \). Now one of the regions, say \( R \), into which \( \alpha \) partitions the plane does not intersect the Julia set \( J \) (because \( J \) is connected by Lemma 3), and thus \( R \subset D_1 \). We want to conclude that \( f \) has a limit as \( z \to \infty \) in \( R \).
To do this, we choose an arbitrary point \( b \in D_2 \) and consider the function \( g(z) = (f - b)^{-1} \) which is holomorphic and bounded in \( D_1 \). This function has limits when \( z \to \infty \), \( z \in \alpha \), so by a theorem of Lindelöf [22], these limits coincide and \( g \) has a limit as \( z \to \infty \) in \( R \). This proves (iii).

To prove (iv), we distinguish several cases, according to the dynamics of \( f \) in each \( D_j \).

1. Suppose first that both \( D_1 \) and \( D_2 \) are basins of attraction of attracting or superattracting points. Then we choose the curves \( \Gamma_j \) as above, but with the additional property that \( f(\Gamma_j) \subset G_j \), that is \( f(\Gamma_j) \cap \overline{G} = \emptyset \). To achieve this, we denote by \( z_j \) the attracting or superattracting fixed point in \( D_j \), choose \( G_j \) to be the open hyperbolic disc centered at \( z_j \), of large enough hyperbolic radius, so that \( A_j \subset G_j \), and put \( \Gamma_j = \partial G_j \). Then the \( G_j \) are \( f \)-invariant, and moreover \( f(G_j) \subset G_j \) for \( j = 1, 2 \), because \( f \) is strictly contracting the hyperbolic metric in \( D_j \). It follows that the closure of \( W = f^{-1}(G) \) is contained in \( G \). Let \( h \) be the hyperbolic metric in \( G \), and \( |f'(z)|_h \) the infinitesimal length distortion by \( f \) at the point \( z \in W \) with respect to \( h \). By the Theorem of Pick [21, Theorem 2.11] there exists \( K > 1 \) such that

\[
|f'(z)|_h \geq K, \quad z \in W.
\]

Now we consider successive preimages \( W_n = f^{-n}(W) \). Note that \( \infty \notin A(f) \) by (ii) which implies that the components of \( f^{-1}(\gamma_j) \) are bounded for \( j = 1, 2 \). We deduce that every component of \( W_n \) is a Jordan domain whose boundary consists of two cross-cuts, one of \( D_1 \) and the other of \( D_2 \). These crosscuts meet at two poles of \( f^n \). It follows from [11] that the diameter (with respect to the metric \( h \)) of every component of \( W_n \) is at most \( CK^{-n} \), where \( C > 0 \) is a constant. Now we notice that

\[
J = \bigcap_{n=1}^{\infty} \overline{W_n}
\]

and prove that every point \( z \in J \) is accessible both from \( D_1 \) and \( D_2 \).

The accessibility of poles of the iterates \( f^n \) was already noticed before. Now we assume that \( z \) is not a pole of any iterate. Let \( V_n \) be the component of \( W_n \) that contains \( z \). Then \( V_1 \supset V_2 \supset \ldots \). The intersection \( V_k \cap D_j \) is connected (its relative boundary with respect to \( D_j \) is a cross-cut in \( D_j \)), so one can choose a sequence \( z_{k,j} \in V_k \cap D_j \) and connect \( z_{k,j} \) with \( z_{k+1,j} \) by a curve \( \ell_{k,j} \) in \( V_k \cap D_j \). The union of these curves gives a curve in \( D_j \) which tends to \( z \).

The proof of (iii) in the attracting case is completed by an application of Schoenflies’ theorem that if each point of a common boundary of two domains on the sphere is accessible from both domains then this common boundary is a Jordan curve [23].

2. To prove (iv) in the remaining cases, suppose, for example, that \( D_1 \) is the domain of attraction of a petal associated with a neutral fixed point \( a \). We need several lemmas.

Lemma 2. There exists a Jordan domain \( G_1 \) with the properties \( \overline{G_1} \subset D_1 \cup \{a\} \), \( f(\overline{G_1}) \subset G_1 \cup \{a\} \), \( A_1 \subset G_1 \), and \( G_1 \) is absorbing, that is for every compact \( K \subset D_1 \) there exists a positive integer \( n \) such that \( f^n(K) \subset G_1 \).
Proof. It is well known (see, e. g. [21 §10]) that there exists a domain $G_1$ having all properties mentioned except possibly $A_1 \subset G_1$. Such a domain is called an attracting petal.

Let $P$ be an attracting petal. Choose a point $z_0 \in D_1$ and let $r > 0$ be so large that the open hyperbolic disc $B(z_0, r)$ of radius $r$ centered at $z_0$ contains $A_1$, and $z_1 = f(z_0) \in B(z_0, r)$. Then put

$$G'_1 = P \cup \left( \bigcup_{k=0}^{\infty} B(z_k, r) \right), \quad z_k = f^k(z_0).$$

Then $G'_1$ is absorbing because the petal $P$ is absorbing. Notice that for every neighborhood $V$ of $a$, all but finitely many discs $B(z_k, r)$ are contained in $V$. This easily follows from the comparison of the Euclidean and hyperbolic metrics near $a$, or, alternatively, from the local description of dynamics near a neutral fixed point with multiplier 1. It is easy to see that $f(G'_1) \subset G'_1 \cup \{a\}$. Now we fill the holes in $G'_1$: let $X$ be the unbounded component of $\overline{G'_1}$ and $G_1 = \mathbb{C} \setminus X$. It is easy to see that $G_1$ is a Jordan domain (its boundary is a union of arcs of hyperbolic circles which is locally finite, except at the point $a$, plus some boundary arcs of the petal). \quad $\square$

Now we fix the following notations till the end of the proof of the Theorem. If $D_j$ is a basin of an attracting or superattracting fixed point, let $G_j$ be the Jordan region constructed in the first part of the proof of (iv). If $D_j$ is a basin of a petal, let $J_j$ be the region from Lemma 2. We define $\Gamma_j = \partial G_j$. This is a Jordan curve in $D_j$ or in $D_j \cup \{a\}$ which encloses all singular values in $D_j$.

Next we define $G = \mathbb{C} \setminus (\overline{G'_1} \cup \overline{G'_2})$. This region is simply connected in the case that both $D_1$ and $D_2$ are basins of two petals associated with the same fixed point, and doubly connected in all other cases. If $G$ is doubly connected, we make a simple cut $\delta$ disjoint from the set $A$ of singular points, as in the proof of (ii), to obtain a simply connected region $G' = G \setminus \delta$. If $G$ is simply connected we set $G' = G$. All branches of $f^{-n}$ are holomorphic in $G'$. Let $\gamma_j = f^{-1}(\Gamma_j)$.

**Lemma 3.** There exists a repelling fixed point $b \in J$ which is accessible from both $D_1$ and $D_2$ by simple curves $\beta_j$ which begin at some points of $\Gamma_j$ and do not intersect $G_j$, and which satisfy $f(\beta_j) \cap \overline{G'} = \beta_j$, for $j = 1, 2$.

**Proof.** We use the notation introduced before the statement of the Lemma. Fix one of the components $Q$, of $f^{-1}(G')$, such that $\overline{Q} \subset G'$. Let $\phi$ be the branch of $f^{-1}$ which maps $G'$ onto $Q$. Then $\phi$ has an attracting fixed point $b \in Q$. Let $z_0 \in \Gamma_1$ and $z_1 = \phi(z_0) \in \partial Q$. We connect $z_0$ and $z_1$ by a simple curve $\beta$ in $(G' \setminus Q) \cap D_1$. Such a curve exists because $z_1 \in \gamma_1$, and the component of $\mathbb{C} \setminus \gamma_1$ that contains $G_1$ is completely contained in $D_1$.

Now

$$\beta_1 = \bigcup_{k=1}^{\infty} \phi^k(\beta)$$

is a curve in $D_1$ tending to $b$ which satisfies $f(\beta_1) \cap \overline{G'} = \beta_1$. Similarly a curve $\beta_2$ in $D_2$ is constructed. \quad $\square$

Now, if $G$ is doubly connected, we set $G'' = G \setminus (\beta_1 \cup \beta_2 \cup \{b\})$. If $G$ is simply connected then $G'' = G$. Then $G''$ is a simply connected region which contains no singular values of $f$. Let $\{\phi_k\}_{k \in \mathbb{N}}$ be the set of all branches of $f^{-1}$ in $G''$. These
branches map $G''$ onto Jordan regions $T_k \subset G''$. These regions $T_k$ are of two types: the regions of the first type are contained in $G''$ with their closures, while the regions of the second type have common boundary points with $G''$.

We claim that there are only finitely many regions of the second type. To study these regions $T_k$, we first observe that the full preimage of $\Gamma_j$ is a curve $\gamma_j$ which can have at most one point in common with $\Gamma_j$, namely the neutral fixed point on $\Gamma_j$. Thus the region $W = f^{-1}(G)$ bounded by $\gamma_1$ and $\gamma_2$ is a simply connected region contained in $G$, and the boundary $\partial W$ has at most two common points with $\partial G$, namely the neutral fixed points. The full preimage of the cross-cut

$$\alpha = \beta_1 \cup \beta_2 \cup \{b\}$$

can be constructed in Lemma 2 consists of countably many disjoint curves $\alpha_k \subset \overline{W}$. Each $\alpha_k$ connects a point on $\gamma_1$ to a point on $\gamma_2$. One of the $\alpha_k$, say $\alpha_1$, is contained in $\alpha$ while all others are disjoint from $\alpha$. Thus our regions $T_k$ are curvilinear rectangles, similar to the $Q_k$ used in the proof of (ii). In particular, they cluster only at $\infty$ so that only finitely many of them are of the first type.

It is easy to see that every region of the second type has on its boundary exactly one of the following points: a neutral fixed point or the repelling fixed point $\gamma_k$. The proof that the points of the type d) are accessible is similar to the argument in the case that both $D_1$ and $D_2$ are attracting basins: we will show that each such
point $z$ can be surrounded by a nested sequence of Jordan curves whose diameter tends to zero.

Indeed, each point $z$ of the class d) can be obtained as a limit

$$z = \lim_{n \to \infty} (\phi_{k_1} \circ \phi_{k_2} \circ \ldots \circ \phi_{k_n})(w),$$

where $w \in G''$. For a point $z$ of the category d), the sequence $k_1, k_2, \ldots$ is uniquely defined. We will call this sequence the itinerary of $z$. Let us consider the domains

$$T_n(z) = (\phi_{k_1} \circ \phi_{k_2} \circ \ldots \circ \phi_{k_n})(G''),$$

in other words, $T_n(z)$ is that component of $f^{-n}(G'')$ which contains $z$. The boundary of $T_n(z)$ is a Jordan curve which intersects the Julia set at a finite set of points of categories a)-c). The complementary arcs of these points are cross-cuts of $D_1$ and $D_2$. Thus, to show that $z$ is accessible from $D_1$ and $D_2$, it is enough to show that the diameter of $T_n(z)$ tends to zero as $n \to \infty$. Let $z \in J$ be a point of category d), and $k_1, k_2, \ldots$ its itinerary. Then the sequence $k_1, k_2, \ldots$ cannot have an infinite tail consisting of the branch numbers of the second type. Indeed, the iterates of any branch of the second type converge to a boundary point $x$ of $G''$ (a neutral fixed point or the point $b$). In this case, $z$ will be a preimage of $x$.

Since the itinerary does not stabilize on a branch number of the second type, we can use (5) and (6) to conclude that $\text{diam } T_n(z) \to 0$.

This completes the proof. 

\section{Examples}

\textbf{Example 1.} Let

$$g(z) = \frac{1}{1 + a \cos \sqrt{z}}$$

where $0 < a < \frac{1}{5}$. Then there exists $b < 0$ such that

$$f(z) = \frac{g(z + b) - g(b)}{g'(b)}$$

has a parabolic fixed point at zero, with two completely invariant parabolic basins attached to it. Moreover, $f \in S$ and the Julia set of $f$ is a Jordan curve, but not a quasicircle.

\textit{Verification.} Note that $g$ has no poles on the real axis. We have

$$g'(z) = \frac{a \sin \sqrt{z}}{\sqrt{z}(1 + a \cos \sqrt{z})^2}$$

and

$$g''(z) = -\frac{a^2(\cos \sqrt{z})^2 - a \cos \sqrt{z} - 2a^2}{z(1 + a \cos \sqrt{z})^3} - \frac{a \sin \sqrt{z}}{z\sqrt{z}(1 + a \cos \sqrt{z})^2}.$$  

It follows that

$$\lim_{x \to -\infty} g''(x)x \cos \sqrt{x} = -\frac{1}{4a} < 0$$

so that $g''(x) > 0$ if $x$ is negative and of sufficiently large modulus. On the other hand, $g''(0) = \frac{a(5a - 1)}{2(a + 1)^2} < 0$.

Thus there exists $b \in (-\infty, 0)$ with $g''(b) = 0$ and $g''(x) > 0$ for $x < b$. 

The critical points of \( g \) are given by \((k\pi)^2\) where \( k \in \mathbb{N} \), and \( g \) has a maximum there for odd \( k \) and a minimum for even \( k \). It follows that \( g'(x) > 0 \) for \( x < \pi^2 \) and thus in particular for \( x \leq b \). Thus \( f \) has the critical points \((k\pi)^2 - b\), with corresponding critical values

\[
d_{\pm} = \frac{(1 \pm a)^{-1} - g(b)}{g'(b)}.
\]

Moreover, \( f \) has the asymptotic value \( c = -g(b)/g'(b) \), which is also a Picard exceptional value of \( f \), and no other asymptotic values. Thus \( A(f) = \{c, d_+, d_-\} \).

Next we note that \( f(0) = 0 \), \( f'(0) = 1 \) and \( f''(0) = 0 \). Since \( f'(x) = g'(x+b)/g'(b) \), we have \( 0 < f'(x) < 1 \) for \( x < 0 \). It follows from the mean value theorem that if \( x < 0 \), then \( x < f(x) < 0 \). Thus \((-\infty, 0)\) lies in a parabolic basin \( U \) attached to the parabolic point \( b \). In particular, \( U \) contains the value \( c = -g(b)/g'(b) \) which is a Picard exceptional value of \( f \). We note that \( f^{-1}(D(c,r)) \) is connected for sufficiently small \( r > 0 \), and thus \( U \) is completely invariant.

Since \( f''(0) = 0 \) there is at least one parabolic basin \( V \) different from \( U \) attached to the parabolic point \( 0 \). As \( f \) has a completely invariant domain, every component of the set of normality is simply connected. Thus \( V \) is simply connected. Now \( V \) must contain a singularity of \( f^{-1} \). Thus \( V \) contains one of the critical values \( d_+ \) and \( d_- \), and in fact a corresponding critical point \( \xi = (k\pi)^2 - b \). Since \( f''(\xi) \in \mathbb{R} \cap V \) and \( f''(\xi) \to 0 \) as \( n \to \infty \), and since \( V \) is simply connected and symmetric with respect to the real axis, we conclude that \((0, \xi) \subset V \). Since \( f((0, \infty)) \subset (0, d_-) = f(\pi^2 - b) \) we conclude that the positive real axis is contained in \( V \).

We now show that \( V \) is completely invariant. Suppose that \( W \) is a component of \( f^{-1}(V) \) with \( W \neq V \). Since \( W \) contains no critical points of \( f \), and \( V \) contains no asymptotic values, there exists a branch \( \varphi \) of \( f^{-1} \) which maps \( V \) to \( W \). This functions \( \varphi \) can be continued analytically to any point in \( \overline{\mathbb{C}} \setminus \{c\} \). By the monodromy theorem, \( \varphi \) extends to a a meromorphic function from \( \overline{\mathbb{C}} \setminus \{c\} \) to \( \mathbb{C} \). But this implies that \( f \) is univalent, a contradiction.

It follows from part (iv) of our Theorem that the Julia set of \( f \) is a Jordan curve. On the other hand we note that if \( w = u + iv \) with \( |v| < T \), then \((\text{Im}(w^2))^2 = (2uv)^2 \leq 4T^2u^2 < 4T^2(u^2 - v^2) + 4T^4 = 4T^2\text{Re}(w^2) + 4T^4 \). It follows that if \( 4T^2\text{Re}z \leq (\text{Im}z)^2 - 4T^4 \), then \( |\cos \sqrt{z}| \geq \sinh T \) and thus \( z \in U \), if \( T \) is large enough. Thus the Julia set of \( f \) is contained in the domain \( \{z \in \mathbb{C} : 4T^2\text{Re}z > (\text{Im}z)^2 - 4T^4\} \) if \( T \) is large enough. This implies that it is not a quasicircle.

**Remark.** It seems that \( g'' \) has only one negative zero. But since we have not proved this, we have just defined \( b \) to be the smallest zero of \( g'' \). The values \( a \) and \( b \) are related by

\[
a = \frac{\sqrt{b} \cos \sqrt{b} - \sin \sqrt{b}}{\sqrt{b} + \sqrt{b} \sin^2 \sqrt{b} - \sin \sqrt{b} \cos \sqrt{b}}
\]

For example, if \( b = -1 \), then \( a = 0.16763487 \ldots \), \( g(b) = 0.764166 \ldots \) and \( 1/g'(b) = 16.083479 \ldots \) so that

\[
f(z) = 16.083479 \left( \frac{1}{1 + 0.16763487 \cos \sqrt{z} - 1} - 0.764166 \right).
\]
Example 2. Let $g$ and $f$ be as in Example 1 and let $\alpha > 1$. Then there exists $\alpha_0 = \alpha_0(a) > 1$ such that if $1 < \alpha < \alpha_0$, then $f_\alpha(z) = \alpha f(z)$ has two completely invariant attracting basins.

Verification. It is not difficult to see that if $\alpha$ is sufficiently close to 1, then $f_\alpha$ does indeed have two attracting fixed points $\xi_+ > 0$ and $\xi_- < 0$, with $\xi_\pm \to 0$ as $\alpha \to 1$. The verification that their immediate attracting basins are completely invariant is analogous to that in Example 1.

Remark. We consider again the case $b = -1$. Then $f_\alpha$ has the form

$$f_\alpha(z) = \beta \left( \frac{1}{1 + 0.16763487 \cos \sqrt{z - 1}} - 0.764166 \right)$$

with $\beta > 16.083479 \ldots$. For $\beta = 26.712615 \ldots$ the positive attracting fixed point coincides with the critical point $1 + \pi^2$ and thus is superattracting.
MEROMORPHIC FUNCTIONS WITH TWO COMPLETELY INVARIANT DOMAINS

Figure 3. The black region is the superattracting basin of the function $f$ from Example 2, with $b = -1$ and $\alpha$ chosen such that $1 + \pi^2$ is a superattracting fixed point. The range shown is $-100 < \Re z < 300, |\Im z| < 100$.

Example 3. Let

$$g(z) = \sum_{k=0}^{\infty} \frac{1}{a_k - z}, \quad \text{where} \quad 0 < a_0 < a_1 < \ldots, \quad \sum_{k=0}^{\infty} \frac{1}{a_k} < \infty.$$ 

Both the upper and lower half-plane are $g$-invariant, and $g(x) \to 0$ as $x \to -\infty$ along the negative ray, so 0 is an asymptotic value. Evidently, the second derivative $g''$ changes sign on $(a_0, a_1)$, so there exists $c \in (a_0, a_1)$ such that $g''(c) = 0$. Then the function

$$f(z) = g(z + c) - g(c)g'(c) = z + O(z^3), \quad z \to 0$$

has a neutral fixed point with two petals at 0. It follows that the Julia set $J(f)$ coincides with the real line, and thus $0 \in J(f)$.

To get an example $f$ where the upper and lower half-plane are superattracting basins, we note that $g$ can be chosen such that $g'$ has a non-real zero $\tau$, and with $a = \Im \tau / \Im g(\tau)$ and $b = \Re \tau - a \Re g(\tau)$ the function $f(z) = a g(z) + b$ satisfies $f(\tau) = \tau$ and $f'(\tau) = 0$, as well as $f(\overline{\tau}) = \overline{\tau}$ and $f'(\overline{\tau}) = 0$.

Example 4. For $a = -3.7488381 - 1.3843391i$ the function $f(z) = a \tan z / \tan a$ has fixed points $\pm a$ of multiplier 1. The Julia set is a Jordan curve by our theorem, but clearly not a quasicircle.

Example 5. For $a = 1/(1 - \tanh^2 1) = 2.3810978$ the function

$$f(z) = a \tan z - a \tan i + i$$

has the fixed point $i$ of multiplier 1 and the attracting fixed point $-3.1864112i$. Again the Julia set is a Jordan curve, but not a quasicircle.
Figure 4. The parabolic basins of the function from Example 4. The range shown is $|\text{Re } z| < 5, |\text{Im } z| < 2$.

Figure 5. The parabolic basin of the function from Example 5 is shown in white, the attracting one in black. The range shown is $|\text{Re } z| < 5, |\text{Im } z| < 2$.

**Example 6.** Our final example has two completely invariant half-planes, but unlike $\tan z$, it has no asymptotic values. Another feature of this example is that it has minimal possible growth among the functions of class $S$, namely

$$T(r, f) = O((\log r)^2), \quad r \to \infty,$$

where $T$ is the Nevanlinna characteristic. Langley [18, 19] proved that meromorphic functions with the property $T(r, f) = o((\log r)^2)$ have infinitely many singular values.

Let $h$ be the branch of the arccosine which maps the 4-th quadrant $Q_4 = \{z : \text{Re } z > 0, \text{Im } z < 0\}$ onto the half-strip $H = \{z : \text{Re } z \in (0, \pi/2), \text{Im } z > 0\}$.

Let $g$ be the conformal map of a rectangle $R = \{z : \text{Re } z \in (0, \pi/2), \text{Im } z \in (0, a)\}$ with $a > 0$ onto $Q_4$, such that $g(\pi/2) = 0$ and $g(\pi/2 + ia) = \infty$, and $g(ia) > g(0) > 0$. By the Reflection Principle, $g$ has an analytic continuation to
the half-strip $H$ and maps this half-strip into the left half-plane. It is easy to see that $g$ is an elliptic function.

The composite function $f = g \circ h$ maps the positive ray into itself, and applying the reflection again we conclude that it maps the right half-plane into itself. The boundary values on the imaginary axis belong to the imaginary axis, so by another reflection $f$ extends to a meromorphic function in the plane. We see that both the right and left half-plane are completely invariant.

The function $f$ has 4 critical values, $\pm g(ia)$ and $\pm g(0)$, two in the right half-plane and two in the left half-plane.

To estimate the growth of $f$ is it enough to notice that $\arccos z = i \log z + O(1)$ as $z \to \infty$ in the lower half-plane and in the upper half-plane. Taking into account that $g$ is an elliptic function we obtain (7).

Our function $f$ satisfies the differential equation

$$(1 - z^2)(f')^2 = c(f^2 - p^2)(f^2 - q^2),$$

where $p = g(ia)$, $q = g(0)$ and $c$ is a real constant.

A similar differential equation was considered by Bank and Kaufman [6]; see also [17, 20].

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