From Fermat’s Last Theorem
To the Quantum Computer

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Abstract

Despite of an active work of many researchers in the theory of quantum computations, this area still saves some mysterious charm. It is already an almost common idea, that maybe many fashionable current projects will fade in future, but some absolutely unpredictable applications appear instead. Why such optimistic predictions are legal here, despite of an extreme difficulty to suggest each one new promising quantum algorithm or realistic “industrial” application? One reason — is very deep contents of this area. It maybe only an extremely unlucky occasion, if such a fundamental thing won’t supply us with some bright insights and serious new applications. A sign of such nontrivial contents of a theory — are unexpected links between different branches of our knowledge. In the present paper is mentioned one such link — between application of Weyl quantization in the theory of quantum computations and abstract mathematical constructions born in mid of XIX century due to unsuccessful tries to prove Fermat’s last theorem.

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1 Introduction

The title of the paper is partially inspired by an old lecture presented at 1929 in Liverpool by John E. Littlewood and reprinted in [1]. Let us briefly recollect ideas of this work: unsuccessful tries to prove Fermat’s last theorem encourage introduction of new ideas of special abstract classes of numbers: “ideals” and generalization of such ideas to other areas of mathematics.

From the point of view of Littlewood some constructions developed during such tries have encouraged to introduce a mathematical notion of a function as some class. He criticized an “old” idea to consider a function \( y = y(x) \) as some method of calculation of a value \( y \) of the function from an argument \( x \) using series of operations instead of the “new” idea to define the function simply as a class of ordered pairs \((x, y)\) where all \( x \) are different.

One question discussed in the mentioned lecture was the specific determinism related with such definition of a function: if an evolution of any system instead of \( S = S(t) \) is treated as a class of pairs \((S, t)\), then such a model already does not look like an “evolution” and rather resembles a “historical record” or adopted to description to some mechanical deterministic evolution without any options.

Of course the definition of a function as a class of pair is a standard thing then and now, but the “old” definition should not be rejected, because in works of Turing, Post and Church written just few years after this Littlewood’s lecture the “old” idea of a function and “series of operations” was used as a basis of the theory of recursion and later it was developed to the modern computer science. Let us denote such definition an operational.

The operational approach also related with the description of some “stand-alone” function (c.f. algorithm) vs a definition using the set of arguments and the values. It also raises some new questions hardly expressed in another approach. One such question is universality.
If we are trying to express functions as a set of operations, it is reasonable to consider a question, if the operations are powerful enough. For the definition with a set of pairs, such a question is some kind of tautology, because it is possible to choose any class of pairs we want, and it could be considered as yet another demonstration of an elegance of the “new” definition, but the problem is that such a definition often could hardly be applied to the real world. It is especially clear for functions with an infinite (or very big finite) domain, then instead of a short string like \( y(x) = x^2 \) it is suggested to consider a table like \{ (0, 0), (1, 1), (2, 4), (3, 9), (4, 16), \ldots \}.

From such a constructive point of view the operational definition of a function is preferable and so the question about an accessible set of basic operations is reasonable. Here the set of basic operations is considered as an universal, if it is possible to represent any function. For the infinite set of arguments it is more difficult to explain such idea, but for the purposes of given paper it is enough to consider finite sets.

It is also useful to consider a physical analogue of such a question: is it possible to suggest some set of an elementary universal operations for modeling of the arbitrary physical process? Such an idea was discussed for example by Richard Feynman at 1981 in his lecture at PhysComp’81 conference in MIT [2].

The quantum mechanics is a fundamental theory about our world, so it was reasonable to give an answer about the universal set of operations using some simple quantum mechanical models [2, 3, 4, 5].

It is interesting, that some constructions used in the theory of quantum computation also have a close relation with the algebraic ideas developed due to the tries to prove Fermat’s last theorem at mid XIX century and discussed in Littelwood’s work. These ideas are briefly recollected in Sec. 2 together with related algebraic constructions like the group algebras, Clifford algebras, etc. In Sec. 3 it is discussed a relation between Lie algebras and universality in quantum computation. It is used for a special construction of the universal set of gates using Lie and Clifford algebras discussed in Sec. 1. Such universal elements may be applied to an array of two-dimensional quantum systems, the qubits. It has an analogue with the binary logic and arithmetic. An application of the similar ideas of universality to the higher-dimensional quantum system based on ideas of Weyl quantization is represented in Sec. 4. Constructions used in this section are close related with algebraic ideas discussed in Sec. 2. In Sec. 4 are discussed some other areas of quantum computation linked with such an algebraic approach to the discrete mathematics.

2 Complex, algebraic, Clifford numbers and all that

Fermat’s last theorem declares impossibility to resolve an equation \( x^l + y^l = z^l \) for natural numbers \( X,Y,Z \) and \( l > 2 \). For \( l = 2 \) there are infinite amount of natural solutions of the equation. For example it is possible to choose \( X = a^2 - b^2, Y = 2ab, Z = a^2 + b^2 \) for an arbitrary natural \( a > b \). It may be checked directly, but more useful to derive those expressions with application of complex numbers.

Let \( z = a + bi \) is a complex number and \( \|z\| = a^2 + b^2 \) is the square of the norm. Because \( \|z\|^2 = \|z\|^2 \) and \( z^2 = (a^2 - b^2) + 2ab i \), it is possible to rewrite it as \( (a^2 + b^2)^2 = (a^2 - b^2)^2 + (2ab)^2 \) and it corresponds to the definitions of \( X,Y \) and \( Z \) used above.

Let us not discuss in details neither the history of proof of Fermat’s last theorem [9], nor the theory of algebraic numbers [10] and simply describe some useful constructions introduced due to these tries.

It is possible to write the norm of a complex number as \( a^2 + b^2 = (a + bi)(a - bi) \), but for an arbitrary sum of two odd powers \( (l = 2k + 1) \) it is possible to write (again using complex numbers)

\[
a^l + b^l = (a + b)(a + \zeta b)(a + \zeta^2 b) \cdots (a + \zeta^{l-1} b), \quad \zeta = \sqrt{-1} = e^{2\pi i/l},
\]

it follows from a more general solution for arbitrary \( l \)

\[
a^l - b^l = (a - b)(a - \zeta b)(a - \zeta^2 b) \cdots (a - \zeta^{l-1} b), \quad \zeta = \sqrt{-1} = e^{2\pi i/l}.
\]

It may be rewritten as

\[
a^l + (-1)^{l-1} b^l = (a + b)(a + \zeta b)(a + \zeta^2 b) \cdots (a + \zeta^{l-1} b),
\]

or

\[
a^l + b^l = (a - \nu b)(a - \nu^2 b)(a - \nu^3 b) \cdots (a - \nu^{2l-1} b), \quad \nu = \sqrt{\zeta} = e^{\pi i/l},
\]
using the substitutions \( b \to -b \) or \( b \to \nu b \) respectively.

It was already discussed above, how “complex-integer” numbers like \( a + bi \), \( a, b \in \mathbb{Z} \) may help with solution of a quadratic equation with integer coefficients, but it is useful also to introduce more general algebraic integer numbers like \( \sum a_k \zeta^k \) [10].

At 1843 E. Kummer and 1847 G. Lamé suggested to use such numbers for a proof of Fermat’s last theorem for any prime power \( l \). It was generalizations of Euler’s proof for \( l = 3 \) and, roughly, an idea was related with Eq. (1) describing two different decompositions of the same number (as the power of \( z \) and as the product Eq. (11)), but such a thing is impossible for usual natural numbers there each number may be expressed as an unique product of the prime numbers. The problem with such a proof was found soon by P. Dirichlet, E. Kummer, J. Liouville and related with non-uniqueness of decomposition of the sums \( n_k \zeta^k \) introduced by Kummer and Lamé (\( \zeta^l = 1 \)) for some \( l \) [10].

Really there are some subtleties non-relevant for present consideration, for example for any \( l \): \( 1 + \zeta + \cdots + \zeta^{l-1} = 0 \) and so some sum vanishes. For prime \( l \) it is enough to exclude only \( \zeta^{l+1} \), but here the theory of algebraic numbers is considered only as some intermediate step.

Really, let us consider \( \zeta \) not as some complex number, but as an element of some abstract cyclic group \( \mathbb{Z}_l \) generated by the powers of \( \zeta \) with the property \( \zeta^l = 1 \). For any \( l \) it is possible to consider the group algebra described by formal series \( \sum a_k \zeta^k \) of such elements with naturally defined laws of addition and multiplication [11]. Example of representation of such algebra is the algebra of diagonal \( l \times l \) matrices generated by matrix \( V \) defined below by Eq. (20).

The group algebra described above is a commutative algebra. Let us return again to the case \( l = 2 \). Earlier it was described the representation of a sum of squares as \( a^2 + b^2 = (a + bi)(a - bi) \), but it is also possible to rewrite that using a representation of the imaginary unit as the real \( 2 \times 2 \) matrix \( i \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) (realification) and so the decomposition of \( a^2 + b^2 \) may be considered as some matrix equation.

It is also possible to write the quadratic form not as a product of two different terms, but as the full square using other \( 2 \times 2 \) matrices \( e_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \), \( e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \); \( a^2 + b^2 = (a e_1 + b e_2)^2 \). Such representation is possible because \( e_1^2 = e_2^2 = 1 \), \( e_1 e_2 = -e_2 e_1 \) and so \( (a e_1 + b e_2)^2 = a^2 + a b e_1 e_2 + b a e_2 e_1 + b^2 = a^2 + b^2 \). It is already the noncommutative algebra.

A generalization of such an equation for the arbitrary number of terms

\[
(a_1 e_1 + a_2 e_2 + \cdots + a_n e_n)^2 = a_1^2 + a_2^2 + \cdots + a_n^2 \tag{5}
\]

is equivalent with the definition of Clifford algebra \( \mathbb{C}(n) \) [12, 13]

\[
e_i e_j + e_j e_i = 2 \sigma_{ij}. \tag{6}
\]

Similar noncommutative algebraic version of condition for Fermat’s last theorem is the equation

\[
(a_1 f_1 + a_2 f_2)^l = a_1^l + a_2^l, \tag{7}
\]

where \( f \) are elements of some noncommutative algebra.

Let us show, that two elements of an algebra with the property

\[
f_1^l = f_2^l = 1, \quad f_1 f_2 = \zeta f_2 f_1 \quad (\zeta = e^{2 \pi i/l}), \tag{8}
\]

satisfy to the necessary equation. Really

\[
(a_1 f_1 + a_2 f_2)^l = (a_1 + a_2 f_2 f_1^{-1}) (a_1 + a_2 f_2 f_1^{-1}) f_1 \cdots (a_1 + a_2 f_2 f_1^{-1}) f_1 = (a_1 + a_2 f_2 f_1^{-1}) \cdots (a_1 + a_2 f_2 f_1^{-1}) (f_1)^l = a_1^l + (-1)^{l-1} a_2^l (f_1 f_2)^{-l} = a_1^l + a_2^l.
\]

It is also possible to satisfy the equation

\[
(a_1 f_1 + a_2 f_2 + \cdots + a_n f_n)^l = a_1^l + a_2^l + \cdots + a_n^l \tag{9}
\]

using the noncommutative algebra with \( n \) generators \( f_i \) and the relations [15]

\[
f_i^l = 1, \quad f_i f_j = \zeta f_j f_i, \quad i < j. \tag{10}
\]
3 Universal quantum gates and Lie algebras

Let us return to the theory of computation and universal sets of operations. It was already mentioned an idea to apply a similar theory to the physical systems and find some set of universal operations \[2, 3, 4, 5, 6\].

Here is discussed a simple model with the quantum world described as a finite-dimensional Hilbert space (the complex vector space \(\mathbb{C}^n\) with Hermitian scalar product) and unitary operators on this space \((n \times n\) complex matrices \(U\) with the property \(UU^* = 1\)). In the theory of quantum computations the matrices are called the quantum gates. It is also used an abstract operation of the composition of such systems described as the tensor product \(\mathbb{C}^m \otimes \mathbb{C}^n \cong \mathbb{C}^{mn}\).

For such a model the question about universality may be reformulated as a necessity to find some set of unitary matrices (the quantum gates) \(\{U_\mu\}\) with possibility to express any unitary transformation \(U\) as a product of the matrices (gates) from the set \(U = U_{\mu_1}U_{\mu_2} \cdots U_{\mu_k}\). For a finite set \(\{U_\mu\}\) the index \(\mu\) is simply a natural number. In such a case it is impossible to represent any matrix \(U\) precisely using a finite number of terms \(U_{\mu_k}\), but it is enough to consider the possibility to approximate any matrix with an arbitrary accuracy \(U \approx U_{\mu_1}U_{\mu_2} \cdots U_{\mu_k}\). Sometime it is called the universality in approximate sense.

The group of unitary matrices is Lie ("smooth") group. It was found, that the Lie algebra of the Lie group is a convenient tool for the theory of universality \[6, 7\]. The idea uses correspondence between the operations like addition \(a + b\) and Lie bracket \([a, b]\) for elements of Lie algebras with operations \(AB\) and \(ABA^{-1}B^{-1}\) for Lie group.

So instead of elements \(\{U_\mu\}\) of Lie group \(SU(n)\) it is possible to consider elements \(\{u_\mu\}\) of Lie algebra \(su(n)\) and the notion of universality should be adopted to the Lie algebra.

If a set of elements \(\{u_\mu \in su(n)\}\) may generate the full algebra using additions and commutators, then the set is called universal.

Using such a set of elements and the map \(U_\mu = \exp(\tau u_\mu)\) with small \(\tau\) it is possible to construct the universal set of operators \(\{U_\mu \in SU(n)\}\) \[6, 7\].

It is also possible to consider \(\tau\) as a continuous parameter and to use the family of gates \(\{U_\mu(\tau)\}\) for construction of the "strictly" universal set. It should be mentioned, that an element of Lie algebra corresponds to Hamiltonian used for construction of the gate \(H_\mu = \tau u_\mu\) and in such a case the parameter \(\tau\) corresponds to the time. It follows directly from the solution of the Schrödinger equation with time-independent Hamiltonian

\[
\dot{\psi} = -iH \psi \implies \psi(t) = \exp(-iHt) \psi.
\]

4 Qubits

Let us consider a simplest example with two-dimensional Hilbert spaces \(\mathcal{H}_2\). Two vectors of a basis are usually denoted as \(|0\rangle\) and \(|1\rangle\), i.e.

\[
|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathcal{H}_2.
\]

Such an abstract quantum system with two states are usually called the qubit. A quantum gate, i.e. an unitary matrix \(U \in U(2)\) may be expressed as

\[
U = e^{i\varphi}(a_0 + a_1i\sigma_1 + a_2i\sigma_2 + a_3i\sigma_3), \quad a_k \in \mathbb{R}, \quad a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1,
\]

where

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

(11)

are three Pauli matrices and \(e^{i\varphi}\) is an unessential phase multiplier.

It can be checked directly, \(\sigma_1\) corresponds to the classical NOT gate: \(\sigma_1|0\rangle = |1\rangle, \sigma_1|1\rangle = |0\rangle\) and it explains the idea to consider unitary matrices as analogues of classical gates.

It is convenient to use Pauli matrices also as elements of the Lie algebra and there is the simple expression

\[
\exp(i\sigma_k \tau) = \cos(\tau) + i\sigma_k \sin(\tau), \quad k = 1, 2, 3.
\]

(12)
A quantum state of \( n \) such systems is described as the tensor product
\[
\mathcal{H} = \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_2
\]
and has the dimension \( 2^n \). Basic vectors for such states may be denoted as \(|0 \ldots 00\rangle, |0 \ldots 01\rangle, \ldots, |1 \ldots 11\rangle\). A general element of such Hilbert space may be written as
\[
|v\rangle = v_{0 \ldots 00} |0 \ldots 00\rangle + v_{0 \ldots 01} |0 \ldots 01\rangle + \cdots + v_{1 \ldots 11} |1 \ldots 11\rangle.
\]
It is convenient sometime to consider it as a binary decomposition of indexes of vectors \( v \) and elements of basis.

Quantum gates are described by \( 2^n \times 2^n \) unitary matrices \( SU(2^n) \) and the action of such matrix \(|v\rangle = U|v\rangle\) may be rewritten as
\[
v'_{i_1 i_2 \ldots i_n} = \sum_{j_1 j_2 \ldots j_n = 0}^1 U^{j_1 j_2 \ldots j_n}_{i_1 i_2 \ldots i_n} v_{j_1 j_2 \ldots j_n},
\]
Here set of indexes like \( i_1 i_2 \ldots i_n \) is again simpler to compare with binary decomposition of some number.

But it is possible also to consider an action of some “one-gate” \( U \in U(2) \) on a singular qubit with index \( k \)
\[
v'_{i_1 \ldots i_k \ldots i_n} = \sum_{j_k = 0}^1 U^{j_k}_{i_k} v_{i_1 \ldots j_k \ldots i_n},
\]
or an action of some “two-gate” \( U \in U(4) \) on two qubits with indexes \( k, l \)
\[
v'_{i_1 \ldots i_k \ldots i_l \ldots i_n} = \sum_{j_k, j_l = 0}^1 U^{j_k j_l}_{i_k i_l} v_{i_1 \ldots j_k j_l \ldots i_n},
\]
and similarly with any \( k \)-gate, \( k \leq n \).

It is also convenient to use the basis of \( 2^n \times 2^n \) complex matrices expressed via \( 4^n \) different tensor products of Pauli matrices together with the unit matrix \( \sigma_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) \[16\] \[17\]
\[
\sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_n}, \quad i_1, i_2, \ldots, i_n = 0, \ldots, 3.
\]
In such a notation the \( k \)-gate is a sum of terms with non-unit elements \( \sigma_i \) \( (i = 1, 2, 3) \) only in \( k \) given positions.

Already for such a simple case with the qubits systems it is clear, that a complexity of the models is very high, for example for the composition of 10 qubits Hilbert space has dimension \( 2^{10} = 1024 \), for 20 qubits — \( 2^{20} = 1048576 \), etc. Dimension of the space of unitary matrices is even bigger: \( 4^n \), e.g. for 20 qubits such a matrix contains \( 1048576 \times 1048576 = 1099511627776 \) complex numbers.

So question about the universal set of elements is actual here. Very important results here are related with universality for sets of two-gates \[16\] \[17\] \[18\].

Together with proofs of existence for such sets of gates it is useful to know some constructive algorithms and have possibility to decompose or approximate some matrix or estimate of complexity of some class of gates. From such a point of view the method of construction of an universal set of gates based on mechanical testing of completeness of the commutator algebra may be not very convenient. It is more useful, then the universal set of gates has some clear algebraic structure.

In \[19\] was suggested to use Clifford algebras for construction of the universal sets of quantum gates. It is especially convenient due to interesting and useful relation between the structure of Clifford algebra \[16\] \[17\] and the product operator formalism \[16\] \[17\]. Really, generators of Clifford algebra for even dimension \( \mathfrak{Cl}(2n) \) satisfying Eq. \[16\] may be expressed using Pauli matrices as \[16\]
\[
\begin{align*}
\sigma_{2k} &= \sigma_0 \otimes \cdots \otimes \sigma_{n-k} \otimes \sigma_1 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3, \\
\sigma_{2k+1} &= \sigma_0 \otimes \cdots \otimes \sigma_{n-k} \otimes \sigma_2 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3,
\end{align*}
\]
(19)
where $k = 0, \ldots, n-1$ and $\sigma_0$ is $2 \times 2$ unit matrix. Clifford algebra $\mathcal{C}(2n)$ coincides with the algebra of $2^n \times 2^n$ complex matrices, the expressions for generators Eq. (19) have clear product structure and it is convenient for theory of quantum computations (both usual and fermionic case) [18].

The elements Eq. (19) do not correspond to two-gates, but it is possible to consider elements $e_{j,j+1} \equiv [e_j, e_{j+1}] = 2e_j e_{j+1}$. Such elements of “second order” correspond to one-gates for $j = 2k$ and two-gates for $j = 2k+1$ and generate Lie algebra $\text{so}(2n)$ represented by all possible bi-products $e_{jk} \equiv e_j e_k = [e_j, e_k]/2$. It does not produce an universal set of gates, because dimension of such an algebra is $(2n-1)n < 4^n - 1 = \text{dim SU}(2^n)$.

It should be mentioned, that it is enough to add only two extra gates to produce an universal set: it should be one initial element $e_j$ and an arbitrary product with three or four elements, for example it may be two elements $e_0$ and $e_0 e_1 e_2$ [19]. Second element has a third (or fourth) order and this property is important for discussions on the fermionic quantum computation [18, 20], but in the product operator representation both elements may be chosen as one-gates:

$$e_0 = 1 \otimes \cdots \otimes 1 \otimes \sigma_1, \quad e_0 e_1 e_2 = 1 \otimes \cdots \otimes \sigma_1 \otimes 1.$$ 

5 Universality for $l \geq 2$ and Weyl quantization

The theory of quantum gates represented here may be quite general and common, but it seems lack of some habitual attributes of the quantum mechanics. Where is Heisenberg uncertainty relation, the coordinates and momenta, the wave-particle duality and all that? Of course here are represented the discrete models, but for significant amount of qubits dimension of Hilbert space becomes very big and so there is some hope to consider an analogue of a continuous limit.

A very convenient tool for such a problem is Weyl representation of Heisenberg commutation relations and some other methods related with Weyl quantization. It is discussed below.

Let us first instead of two-dimensional Hilbert space consider finite-dimensional one with arbitrary dimension $l \geq 2$ and denote it as $\mathcal{H}_l$. It is possible without a big problem to generalize the most properties described above. The compound systems may be described as the $l^n$-dimensional tensor product like with Eq. (13)

$$\mathcal{H} = \mathcal{H}_l \otimes \cdots \otimes \mathcal{H}_l,$$ 

an action of a matrix $U \in \text{SU}(l^n)$ may be represented as

$$v'_{i_1 i_2 \ldots i_n} = \sum_{j_1, j_2, \ldots, j_n=0}^{l-1} U^{j_1 j_2 \ldots j_n}_{i_1 i_2 \ldots i_n} v_{j_1 j_2 \ldots j_n},$$ 

and actions of $k$-gates for $k < n$ also may be described using same formulae, as for qubits, but with indexes range $(0, \ldots, l-1)$ instead of $(0, 1)$.

The more nontrivial thing is to introduce an analogue of Pauli matrices, but it also exists and in addition provides some passage to the continuous limit mentioned above.

Let us consider usual Heisenberg commutation relation $(\hbar = 1)$

$$[p, q] = pq - qp = -i,$$ 

where $p, q$ are operators of momentum and coordinate. Let us consider two families of operators

$$U^a = \exp(iap), \quad V^\beta = \exp(i\beta q).$$ 

Using Campbell-Hausdorff formula for the formal operator series (for operators with zero third-order commutators like for $p, q$)

$$\exp(a + b) = \exp(a) \exp(b) \exp\left(-\frac{1}{2}[a, b]\right),$$
it is possible to write
\[ U^{\alpha} V^{\beta} = \exp(i\alpha \beta) V^{\beta} U^{\alpha}. \] (24)

It is Weyl system \[22\] or Weyl representation of Heisenberg commutation relations.

Weyl relation Eq. (24) is even more general, than Heisenberg one \[21\], e.g. it is applied to compact operators instead of \( p, q \) and so widely used in many areas of the quantum theory. But in the present paper these relations are used, because they work for finite dimensional Hilbert spaces and this application was discussed already in initial Weyl work at 1927 (first English translation at 1931 \[14\]).

Let us find two \( l \times l \) unitary matrices with the property similar with Eq. (24)
\[ U V = \zeta V U. \] (25)

It can be shown that such matrices are really exist for \( \zeta^l = 1 \) and may be written as \[14\]
\[
U = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
1 & 0 & 0 & \ldots & 0
\end{pmatrix},
V = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & \zeta & 0 & \ldots & 0 \\
0 & 0 & \zeta^2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \zeta^{l-1}
\end{pmatrix}.
\] (26)

So Weyl relations work also for the finite-dimensional case. Really the “shift” and “clock” matrices Eq. (26) were introduced even earlier, at 1882–84 in works of J. J. Sylvester \[23\].

It should be mentioned, that due to Eq. (25) the matrices are satisfying to “operator Fermat’s theorem” discussed above, i.e.
\[ (a V + b U)^l = a^l + b^l. \] (27)

The matrices widely used in the theory of quantum computation after reintroducing for the theory of quantum error correction \[24, 25\]. The matrices also may be very useful for the theory of universal quantum gates for the higher dimensional quantum systems \[28\], there it has analogue with the application of Clifford algebras discussed above. For \( l = 2 \) the matrices coincide with \( \sigma_1 \) and \( \sigma_3 \). In papers about quantum computer applications the Weyl pair \( U, V \) is often called generalized Pauli matrices with yet another notation \( X, Z \).

Really such approach corresponds to some discrete analogue of Weyl quantization. Let us discuss it in more details. In Weyl quantization \[14, 29\] any function \( f(p, q) \) with two real arguments \( p, q \) and with Fourier image \( \tilde{f}(\alpha, \beta) \) described by expression
\[
f(p, q) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp(i\alpha p + i\beta q) \tilde{f}(\alpha, \beta) \, d\alpha \, d\beta
\] (28)
is associated with the operator \( f \) defined as
\[
f = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp(i\alpha p + i\beta q) \tilde{f}(\alpha, \beta) \, d\alpha \, d\beta.
\] (29)

Using Hausdorff formula and the definition Eq. \[29\] it can be rewritten as
\[
f = \int_{-\infty}^{+\infty} \exp(-i\alpha \beta/2) \tilde{f}(\alpha, \beta) U^{\alpha} V^{\beta} \, d\alpha \, d\beta.
\] (30)

For finite case, there the integrals should be changed to sums, the Eq. (30) would correspond to a decomposition of some matrices as a sum with product of different integer powers of matrices \( U, V \) Eq. \[26\], like
\[ \sum_{k,j} f_{k,j} U^k V^j \] (up to nonsignificant complex multiplier like \( \zeta^{kj/2} \)). Such decomposition is really always exist
and unique, because matrices $U^{kj}V^j$ ($k, j = 0, \ldots, l-1$) produce basis in space of all $l \times l$ complex matrices. It is also possible to use natural norm on space of matrices $\|A, B\| = \text{Tr}(AB^*)/l$ to make the basis orthonormal.

It should be only mentioned, that matrices used in such decomposition are not Hermitian. It is some difference with two-dimensional case and Pauli matrices. It is possible to use following method to resolve such problem: if there is some matrix $A$, then it is possible to consider two Hermitian matrices $A + A^*$ and $i(A - A^*)$ instead of it. So instead of Lie algebra $\mathfrak{su}(l)$ it is possible to consider Lie algebra $\mathfrak{sl}(l, \mathbb{C})$ of all complex matrices with trace zero.

Note: There is yet another way to represent Hermitian matrix using basis generated by Weyl pair. Instead of $U^jV^k$ it is possible to use “90° rotated” matrices $\Xi$.[26, 27]

$$\zeta^{k,j/2}U^jV^k \Xi,$$

where $\Xi$ is reflection matrix defined as $\Xi |n\rangle = |l - n - 1\rangle$. Such decomposition was used in a representation of quantum computation in phase space with a discrete Wigner function [28], and formally very similar with ideas described below, but in applications with Lie algebras matrix $\Xi$ in such products “spoils” some equations used in proof of universality and so this representation is not used here.

In a non-Hermitian case an universal set of elements may be based on arbitrary complex matrices $M_k$ if the matrices generate full Lie algebra of traceless matrices using sums and commutator. It is enough to consider unitary gates

$$G_k^l = e^{i(M_k + M^*_l)}, \quad G_k^l = e^{i(M_k - M^*_l)} \tau.$$

Due to such a method it is possible to use the non-Hermitian matrix basis like $U^jV^k$ without a special care. It is only necessary to prove, that all such products may be generated using only commutators. It is not very difficult, because the commutators are proportional to the products $[U^aV^b, U^cV^d] = (\zeta^{bc} - \zeta^{ad})U^a+c\tau^b+\tau^d$ and it is only necessary to be careful with commuting elements [28].

Here was only discussed an example with one system, but similar methods may be used for the composition, using an analogue with Clifford algebras. Let us denote

$$\tau_1 = U, \quad \tau_2 = \zeta^{(l-1)/2}UV, \quad \tau_3 = V,$$

where the complex multiplier $\zeta^{(l-1)/2}$ is used for normalization $\tau_2^l = 1$. So here are analogues of all three Pauli matrices with properties

$$\tau_1 \tau_2 = \zeta \tau_2 \tau_1, \quad \tau_1 \tau_3 = \zeta \tau_3 \tau_1, \quad \tau_2 \tau_3 = \zeta \tau_3 \tau_2, \quad \tau^f_j = 1.$$

Using these matrices it is possible to write an analogue of Eq. [10]

$$f_{2k} = \mathcal{R}_{0(n-k-1)} \otimes \cdots \otimes \mathcal{R}_{0(n-k-1)} \otimes \mathcal{R}_{n-k-1} \otimes \cdots \otimes \mathcal{R}_{n-k-1},$$

$$f_{2k+1} = \mathcal{R}_{0(n-k-1)} \otimes \cdots \otimes \mathcal{R}_{0(n-k-1)} \otimes \mathcal{R}_{n-k-1} \otimes \cdots \otimes \mathcal{R}_{n-k-1},$$

where $k = 0, \ldots, n-1$ and $\mathcal{R}_0$ is $l \times l$ unit matrix.

It can be checked directly, that the elements are satisfying to Eq. [10] and it is also possible to construct the full Lie algebra $\mathfrak{sl}(l^2, \mathbb{C})$ using these $2n$ elements and so the construction described by Eq. [32] produces the set of universal quantum gates [28].

It should be mentioned, that here again may be used the construction with only one- and two-gates, if to consider the set with $2n$ elements

$$f_0, \quad f_k f_{k+1}^* \quad (k = 0, \ldots, 2n - 2)$$

and to use two exponential formulas Eq. [32].
It is also possible to consider the elements \( f_j \) Eq. (35) from point of view of Weyl quantization. Despite of the specific form such elements may be directly derived from a general theory if instead of the canonical commutator form \( [14] \), i.e. \( 2n \times 2n \) symplectic matrix

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & \ldots \\
-1 & 0 & 0 & 0 & \ddots \\
0 & 0 & 0 & 1 & \ddots \\
0 & 0 & -1 & 0 & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots 
\end{pmatrix}
\]

(37)

to use the special (non-canonical) form \( [15] \)

\[
\begin{pmatrix}
0 & 1 & 1 & 1 & 1 & \ldots \\
-1 & 0 & 1 & 1 & 1 & \ddots \\
-1 & -1 & 0 & 1 & 1 & \ddots \\
-1 & -1 & -1 & 0 & 1 & \ddots \\
-1 & -1 & -1 & -1 & 0 & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots 
\end{pmatrix}
\]

(38)

Here the elements of the commutator form \( c_{ij} \in \{+1, -1, 0\} \) correspond to the relations \( f_i f_j = \zeta^{c_{ij}} f_j f_i \).

It should be mentioned what such “\( \zeta \)-commuting” elements are also quite common in the theory of quantum algebras \( [30, 31] \), but this theory has a bit different prerequisites based on applications of so called \( R \)-matrix for Yang-Baxter equation, the soliton theory etc. \( [32, 33] \) and so the new constructions related with the theory of quantum computations seem quite promising.

6 Quantum computation in action

It was considered an approach to the quantum computation more similar with an initial idea of universal physical operations \( [2, 4, 34] \). Maybe such an approach more close to the ideas of a quantum control \( [35, 36] \), because the universality was discussed without necessity of the relation with “traditional” computing tasks. Such a distinction may be a bit formal, but it should be emphasized what the quick growth of an interest to the area of the quantum computation after about fifteen years of a latent development was related with such a typically “arithmetic” task as the factorization of numbers by a quantum computer using P. Shor algorithm \( [37] \) or quantum error correction codes \( [24, 25, 38] \).

In previous sections it was considered, how to exploit the whole set of states and transformations using only some basic operations. For composition of few quantum systems the spaces grow exponentially with respect to the number of elements. It was shown that it is possible anyway to use only Hamiltonians for transformations of each elements (one-gates) together with Hamiltonians of pairwise interactions (two-gates) to construct an universal set. It is clear, that the number of such gates grows linearly with the number of elements.

So, despite of an universality of such a set of gates, the number of gates in a product used for a presentation of the general element of SU(\( l^n \)) may be exponential. Due to it, not only the universal gates are necessary, but also some special sets and constructions for a particular task. Here is not discussed the general theory of quantum computations, and currently there are lot of papers and monographs like \( [39, 40] \). Only some selected point are mentioned below.

It is interesting to compare general ideas of construction of some new mathematical structures for resolution of particular tasks. It was already mentioned above, what abstract models, like the algebraic integer numbers
were constructed for research of some equation with natural numbers. Such construction used some extension of the idea of integer number\textsuperscript{10}.

The theory of quantum computation uses similar ideas, because for the research of computational problems, defined usually as some operations with numbers or other finite models like Boolean algebras, are used some "continuous" algebraic extensions of such models. Even if technologically the large-scale quantum computers, satisfying to all subtleties used in such abstract theories, would never be build, they are already provide a great impact to the both fields of pure mathematics and the quantum theory.

Similarly with the ideas of Lamé and Kummer about generalisation of natural numbers, the quantum computer science instead of natural numbers 0, 1, \cdots or Boolean values 0, 1 (\{false,true\}) uses formal series like \(\alpha_0|0\rangle + \alpha_1|1\rangle\), \(\alpha_k \in \mathbb{C}\).

Instead of a discrete finite set of functions with integer or boolean values it has used a continuous group, that also may be represented as a formal series. To show it, let us express usual "classical" functions with notations more habitual in the quantum information science (cf\textsuperscript{41,42}).

It was already mentioned, that \(\sigma_x\) corresponds to a NOT gate in the usual computation. Let us denote a matrix with all zeros except one unit in the position \(a_{ij}\) as \(|i\rangle\langle j|\), so the NOT gate may be represented as the sum of two such matrices \(|0\rangle\langle 1| + |1\rangle\langle 0|\). Similarly an arbitrary function on \(\mathbb{Z}_d = \{0, \ldots, l - 1\}\) may be written as \(M_f = \sum_{k=0}^{l-1} |f(k)\rangle\langle k|\).

Note: Only for the reversible function \(f\) the matrix \(M_f\) is orthogonal, but here is a simple trick to associate a reversible function \(F\) with any irreversible one, the function is defined on pairs of numbers \(F: (x, y) \mapsto (x, f(x) + y \mod l)\) and so produces for a pair \((x, 0)\) the pair of values \((x, f(x))\) (this idea is widely used in the theory of quantum computation, there it is denoted as \(|x\rangle\langle y| \mapsto |x\rangle\langle f(x) + y \mod l|\)\textsuperscript{39,13,41}). Formally such a function may be considered as defined on \(\mathbb{Z}_d\).

A complex matrix may be expressed as the similar sum \(U = \sum_{j,k=0}^{l-1} U_{jk} |j\rangle\langle k|\) and unitary matrices are analogue of reversible functions, because for a reversible function \(M_f\) corresponds to an unitary (orthogonal) matrix with \(l\) units and \(l^2 - l\) zeros.

In such a correspondence with classical computations Weyl pair of matrices \(U, V\) also have interesting properties. Say elements of Hilbert space representing natural numbers, i.e. the computational basis \(|0\rangle, |1\rangle, \cdots, |l - 1\rangle\), are eigenvectors of the matrix \(V\) with eigenvalues \(\zeta^k\): \(V|k\rangle = \zeta^k|k\rangle\) and the matrix \(U\) corresponds to a cyclic shift of the elements \(|k\rangle = |k + 1 \mod l\rangle\).

Eigenvectors of the matrix \(U\) may be expressed as
\[
|\tilde{k}\rangle = \sum_{j=0}^{l-1} \zeta^{kj} |j\rangle, \quad k = 0, \ldots, l - 1, \quad \zeta^{kj} = \exp(2\pi i k j / l),
\]
and have an analogue with a basis in momentum space. Transition between the computational basis \(|k\rangle\) and the momentum basis \(|\tilde{k}\rangle\) may be represented by a matrix \(F\) with indexes \(F_{kj} = \zeta^{kj}\), i.e.
\[
F = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & \zeta & \zeta^2 & \ldots & \zeta^{l-1} \\
1 & \zeta^2 & \zeta^4 & \ldots & \zeta^{2l-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \zeta^{l-1} & \zeta^{2l-2} & \ldots & \zeta^{(l-1)(l-1)}
\end{pmatrix}.
\]

The operator \(F\) is called the discrete (or quantum) Fourier transform and the most principal quantum algorithms are just based on application of such transform\textsuperscript{35} and possibility of fast implementation using the special set of quantum gates\textsuperscript{46}. It is some demonstration of the notion, that not only an universal set is necessary, but the special gates for the fast implementations of the specific transformations.

Another natural area for application of Weyl pair \(U, V\) is the quantum error correction codes, there the products and bases like \(U^k V^l\) can be directly used for construction of such codes\textsuperscript{24,25}. Such situation maybe not so unexpected, because the theory of quantum error correction codes “borrows” some part from the classical
one and so was related with an extensive branch of the discrete mathematics. Idea of stabilizer codes has connected that area with specific commuting relation and, finally, with Weyl pair. So specific constructions like Galois fields and algebraic numbers may be naturally interlaced with the quantum mechanical ideas. It is relevant not only to the theory of quantum error correction codes, for example similar ideas may be applied to construction of mutually unbiased bases used for the security of quantum communications and the theory of quantum measurements. For such applications there are also interesting analogues with the discrete Wigner function defined in and briefly mentioned in Sec. in relation with alternative decomposition Eq.

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