DYNAMICS FOR THE DIFFUSIVE LESLIE-GOWER MODEL WITH DOUBLE FREE BOUNDARIES

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Abstract. In this paper we investigate a free boundary problem for the diffusive Leslie-Gower prey-predator model with double free boundaries in one space dimension. This system models the expanding of an invasive or new predator species in which the free boundaries represent expanding fronts of the predator species. We first prove the existence, uniqueness and regularity of global solution. Then provide a spreading-vanishing dichotomy, namely the predator species either successfully spreads to infinity as $t \to \infty$ at both fronts and survives in the new environment, or it spreads within a bounded area and dies out in the long run. The long time behavior of $(u, v)$ and criteria for spreading and vanishing are also obtained. Because the term $v/u$ (which appears in the second equation) may be unbounded when $u$ nears zero, it will bring some difficulties for our study.

1. Introduction. Prey-predator systems (or consumer-resource systems) are basic differential equation models for describing the interactions between two species with a pair of positive-negative feedbacks. The classical Leslie-Gower prey-predator model is ([18])

\[
\begin{cases}
\frac{du}{dt} = u(a - u) - buv, \\
\frac{dv}{dt} = \mu v(1 - v/u),
\end{cases}
\]

where $a$, $b$ and $\mu$ are positive constants, $u(t)$ and $v(t)$ represent the population densities of prey and predator, respectively. In this model, the prey is assumed to grow in logistic patterns. It is known that this system has a globally asymptotically stable equilibrium $(\frac{a}{1+b}, \frac{\mu}{1+b})$.

The diffusive Leslie-Gower prey-predator model with homogeneous Neumann boundary conditions takes the form

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Leslie-Gower prey-predator model with double free boundaries can be written as:

\[
\begin{aligned}
&u_t - u_{xx} = u(a - u) - buv, \quad t > 0, \ x \in \Omega, \\
v_t - dv_{xx} = \mu v(1 - v/u), \quad t > 0, \ x \in \Omega, \\
\frac{\partial u}{\partial v} = \frac{\partial u}{\partial v} = 0, \quad t > 0, \ x \in \partial \Omega, \\
u(0, x) = u_0(x) > 0, \ v(0, x) = v_0(x) > 0, \ x \in \Omega,
\end{aligned}
\]

(P)

where \(\Omega\) is a bounded and smooth domain of \(\mathbb{R}^N\). The global stability of \((0, 0, 0, 0)\) for the problem (P) had been studied by many authors, see \([3, 5, 20]\) for example.

In the problem (P), it is assumed that the habitats of prey and predator are the same and fixed, and no flux through the boundary. However, in some situations, predator and/or prey will have a tendency to emigrate from the boundary to obtain their new habitat, i.e., they will move outward along the unknown curve (free boundary) as time increases. The spreading and vanishing of multiple species is an important content in understanding ecological complexity. In order to study the spreading and vanishing phenomenon, many mathematical models have been established.

In this paper we only consider the one-dimensional case and assume that the prey distributes in the whole line \(\mathbb{R}\), the predator exists initially in a bounded interval and invades into the new environment from two sides. In such a situation the diffusive Leslie-Gower prey-predator model with double free boundaries can be written as:

\[
\begin{aligned}
&u_t - u_{xx} = u(a - u) - buv, \quad t > 0, \ x \in \mathbb{R}, \\
v_t - dv_{xx} = \mu v(1 - v/u), \quad t > 0, \ g(t) < x < h(t), \\
v(t, x) = v(t, x) = 0, \quad t \geq 0, \ x \not\in (g(t), h(t)), \\
g'(t) = -\beta v_x(t, g(t)), \quad t \geq 0, \\
h'(t) = -\beta v_x(t, h(t)), \quad t \geq 0, \\
u(0, x) = u_0(x), \quad x \in \mathbb{R}, \\
v(0, x) = v_0(x), \quad -h_0 \leq x \leq h_0, \\
g(0) = -h_0, \ h(0) = h_0,
\end{aligned}
\]

(1.1)

where \(a, b, d, h_0, \mu \) and \(\beta\) are given positive constants. The initial functions \(u_0(x)\), \(v_0(x)\) satisfy:

\[
\begin{aligned}
&u_0 \in C_b(\mathbb{R}), \quad u_0 > 0 \ \text{in} \ \mathbb{R}; \\
v_0 \in W^p_b((-h_0, h_0)), \quad v_0(\pm h_0) = 0, \ v_0 > 0 \ \text{in} \ (-h_0, h_0),
\end{aligned}
\]

(1.2)

where \(p > 3, C_b(\mathbb{R})\) is the space of continuous and bounded functions in \(\mathbb{R}\). The free boundary conditions \(g'(t) = -\beta v_x(t, g(t)), h'(t) = -\beta v_x(t, h(t))\) are the Stefen type, and the deduction can refer to \([1]\) and \([32]\). The positive constant \(\beta\) is called the moving parameter. When the free boundary conditions are \(g'(t) = -\beta_g v_x(t, g(t)), h'(t) = -\beta_h v_x(t, h(t))\) and \(\beta_g \neq \beta_h\), there will be some different phenomena and it will be discussed in our further work. The logistic equation with double free boundaries and different moving parameters had been studied in \([39]\).

Because the term \(v/u\) may be unbounded when \(u\) nears zero, it will bring some difficulties for the study. We will use some new methods and carefully estimates to deal with the “bad term \(v/u\)” and get the satisfactory results.

Before stating our results, we first give some notations. Set:

\[
g^* = -\beta v_x'(h_0), \ h^* = -\beta v_x'(h_0), \ \mathbb{R}_+ = (0, \infty), \ \mathbb{R}_+ = [0, \infty).
\]
Then $g^* \leq 0$ and $h^* \geq 0$. In order to facilitate the writing, we denote
\[ \Lambda = \{a, b, d, \mu, \beta, \alpha, p\} \]
with $0 < \alpha < 1 - 3/p$. For the given interval $I \subset \mathbb{R}_+$, we set
\[ I \times [g(t), h(t)] = \bigcup_{t \in I} \{t\} \times [g(t), h(t)], \quad I \times (g(t), h(t)) = \bigcup_{t \in I} \{t\} \times (g(t), h(t)). \]

Our first main result is

**Theorem 1.1.** For the given $(u_0, v_0)$ satisfying (1.2), the problem (1.1) admits a unique global solution $(u, v, g, h)$ and for any given $0 < \gamma < 1$ we have
\[ u \in C_b(\mathbb{R}_+ \times \mathbb{R}) \cap C^{1+\frac{3}{2}, 2+\gamma}(\mathbb{R}_+ \times \mathbb{R}), \quad v \in C^{1+\frac{4}{2}, 2+\gamma}(\mathbb{R}_+ \times [g(t), h(t)]), \]
with $0 < u \leq \max\{a, \max u_0(x)\} := A$ in $\mathbb{R}_+ \times \mathbb{R},$
\[ 0 < v \leq \max\{A, \max v_0(x)\} := B \text{ in } \mathbb{R}_+ \times (g(t), h(t)), \]
\[ g, h \in C^{1+\frac{1}{2}, 2}(\mathbb{R}_+), \quad g'(t) < 0, \quad h'(t) > 0 \text{ in } \mathbb{R}_+. \]
Moreover, for any given $0 < T < \infty$ and $0 < \alpha < 1 - 3/p,$
\[ v \in W^{1, 2}_p((0, T) \times (g(t), h(t))) \hookrightarrow C^{1+\frac{2}{p}, 1+\alpha}([0, T] \times [g(t), h(t)]), \]
and
\[ \|v\|_{W^{1, 2}_p((0, T) \times (g(t), h(t)))} + \|g, h\|_{C^{1+\frac{2}{p}, [0, T]}} \leq C, \]
where the positive constant $C$ depends only on $T, \Lambda, h_0, g^*, h^*, \|u_0\|_{L^\infty(\mathbb{R})}$, and $\|v_0\|_{W^{2}_p([h_0, h_0])}$.

Our second main result is the following spreading-vanishing dichotomy and criteria for spreading and vanishing.

**Theorem 1.2.** Assume $b < 1$. Let $(u, v, g, h)$ be the unique global solution of (1.1). Then the following alternative holds:

Either
(i) Spreading: $g_\infty = -\infty$, $h_\infty = \infty$ and
\[ \lim_{t \to \infty} u(t, x) = \frac{a}{1+b}, \quad \lim_{t \to \infty} v(t, x) = \frac{a}{1+b} \]
uniformly in any compact subset of $\mathbb{R}$,

Or
(ii) Vanishing: $h_\infty - g_\infty \leq \pi \sqrt{d/\mu}$ and
\[ \lim_{t \to \infty} \max_{g(t) \leq x \leq h(t)} v(t, x) = 0, \]
\[ \lim_{t \to \infty} u(t, x) = a \quad \text{uniformly on the compact subset of } \mathbb{R}. \]

Moreover,
(iii) If $h_0 \geq \frac{1}{2} \pi \sqrt{d/\mu}$, then $g_\infty = -\infty$, $h_\infty = \infty$ for all $\beta > 0$;
(iv) If $h_0 < \frac{1}{2} \pi \sqrt{d/\mu}$, then there exist $\beta_* \geq \beta > 0$, such that $g_\infty = -\infty$ and $h_\infty = \infty$ for $\beta > \beta^*$, while $h_\infty - g_\infty \leq \pi \sqrt{d/\mu}$ for $\beta \leq \beta_*$ or $\beta = \beta_*$. 

Diffusive Leslie-Gower Model with Free Boundaries
Parameter $b$ is the predation rate. The assumption $b < 1$ in Theorem 1.2 means that the predation rate is not too large. If we use $u(a - ku) - buv$ and $\mu v (1 - mv/u)$ instead of $u(a - u) - buv$ and $\mu v (1 - v/u)$, respectively, we can find another value $b^* > 0$ such that when $b < b^*$, the corresponding conclusions hold.

Free boundary problems of the prey-predator models had been investigated systematically by many authors, please refer to [32] (with double free boundaries), [22, 23, 26, 27, 36] (with homogeneous Dirichlet, Neumann or Robin boundary conditions at the left side and free boundary at the right side), and [37] (the prey distributes in the whole space $\mathbb{R}^N$, while the predator exists initially in a ball and invades into the new environment).

There were many related works for the classical Lotka-Volterra type competition models. Authors of [8, 10, 38] investigated a competition model in which the invasive species exists initially in a ball and invades into the new environment, while the resident species distributes in the whole space $\mathbb{R}^N$. In [13, 31, 38], two competition species are assumed to spread along the same free boundary at the right side and with homogeneous Dirichlet (Neumann, Robin) boundary conditions at the left side. Especially, the growth rates permit sign-changing in [38]. For the heterogeneous time-periodic environments, authors of [28] investigated the case with sign-changing growth rates.

The classical Lotka-Volterra type competition systems and prey-predator systems with different free boundaries had been studied in [14, 29, 30, 35].

In the absence of $v$, the problem (1.1) reduces to a free boundary problem for $u$ considered in the pioneer work [7]. In this relatively simpler situation a spreading-vanishing dichotomy is known, and when spreading happens, the spreading speed has been determined through a semi-wave problem involving a single equation. More general results in this direction can be found in [2, 4, 6, 9, 16, 24, 25], where [2] concerns with a nonlocal reaction term, [4, 25] considers time-periodic environment, [6] studies space-periodic environment, [9, 16] investigates more general reaction terms. Particularly, in [24, 25] the growth rates are allowed to change signs.

Free boundary problems of reaction diffusion equations and systems with advection had been studied by many authors, refer to [11, 12, 15, 19, 21, 34, 40].

This paper is organized as follows. In Section 2 we shall prove Theorem 1.1 and give the uniform estimates of $v$ and $g'$, $h'$ when $h_\infty - g_\infty < \infty$. Section 3 is concerned with the long time behaviors of $(u, v)$, and Section 4 deals with the criteria governing spreading and vanishing. Theorem 1.2 is the direct consequence of Theorems 3.1, 3.2 and 4.6.

At last we mention that for the free boundary problem of Holling-Tanner prey-predator model with double free boundaries

\[
\begin{cases}
  u_t - u_{xx} = u(a - u) - buv/(m + u), & t > 0, \ x \in \mathbb{R}, \\
  v_t - dv_{xx} = \mu v (1 - v/u), & t > 0, \ g(t) < x < h(t), \\
  v(t, x) \equiv 0, & t \geq 0, \ x \notin (g(t), h(t)), \\
  g'(t) = -\beta v_x(t, g(t)), \ h'(t) = -\beta v_x(t, h(t)), & t \geq 0, \\
  u(0, x) = u_0(x), & x \in \mathbb{R}, \\
  v(0, x) = v_0(x), & -h_0 \leq x \leq h_0, \\
  g(0) = -h_0, \ h(0) = h_0,
\end{cases}
\]

the methods used here are valid and the corresponding results are still true.
2. Existence, uniqueness, regularities and estimates of global solution. The main aim of this section is to prove Theorem 1.1 and give the uniform estimates of \(v \) and \(g', h' \) when \(h_\infty - g_\infty < \infty \). We first state a lemma which can be proved by the same way as that of [32, Lemma 3.1] and the details will be omitted.

**Lemma 2.1.** (Comparison principle) Let \(c \) and \(T_0 \) be two positive constants, \(g_i, h_i \in C^1([0,T_0]) \) and \(g_i(t) < h_i(t) \) in \([0,T_0] \), \(i = 1, 2 \). Let \(v_i \in C^{1,2}((0,T_0) \times (g_i(t),h_i(t))) \) and \(v_{ix} \in C([0,T_0] \times [g_i(t),h_i(t)]) \), \(i = 1, 2 \). Assume that \((v_1,g_1,h_1)\) and \((v_2,g_2,h_2)\) satisfy

\[
\begin{align*}
&v_{1t} - dv_{1xx} \leq \mu v_1(1 - cv_1), \quad 0 < t < T_0, \quad g_1(t) < x < h_1(t), \\
v_1(t,g_1(t)) = 0, \quad g_1(t) \geq -\beta v_{1x}(t,g_1(t)), \quad 0 < t < T_0, \\
v_1(t,h_1(t)) = 0, \quad h_1(t) \leq -\beta v_{1x}(t,h_1(t)), \quad 0 < t < T_0
\end{align*}
\]

and

\[
\begin{align*}
v_{2t} - dv_{2xx} \geq \mu v_2(1 - cv_2), \quad 0 < t < T_0, \quad g_2(t) < x < h_2(t), \\
v_2(t,g_2(t)) = 0, \quad g_2(t) \leq -\beta v_{2x}(t,g_2(t)), \quad 0 < t < T_0, \\
v_2(t,h_2(t)) = 0, \quad h_2(t) \geq -\beta v_{2x}(t,h_2(t)), \quad 0 < t < T_0
\end{align*}
\]

respectively. If \(g_1(0) \geq g_2(0), h_1(0) \leq h_2(0) \), \(v_1(0,x) \geq 0 \) on \([g_1(0),h_1(0)]\), \(v_1(0,x) \leq v_2(0,x)\) on \([g_1(0),h_1(0)]\), then we have

\[
\begin{align*}
g_1(t) \geq g_2(t), \quad h_1(t) \leq h_2(t) \quad \text{on } [0,T_0), \\
v_1(t,x) \leq v_2(t,x) \quad \text{on } [0,T_0) \times [g_1(t),h_1(t)].
\end{align*}
\]

**Proof of Theorem 1.1.** We first use the contraction mapping theorem to get the existence and uniqueness of local solution, and then extend the local solution to a global one by making some suitable estimates. At last, take advantage of the \(L^p\) and Schauder theories we show the regularities and estimates. The proof will be divided into three steps.

**Step 1.** Local existence and uniqueness. The idea of this part comes from [30] and [32]. Let

\[
y = \frac{2x - g(t) - h(t)}{h(t) - g(t)},
\]

\[
w(t,y) = u(t,1/2[(h(t) - g(t))y + h(t) + g(t)]),
\]

\[
z(t,y) = v(t,1/2[(b(t) - g(t))y + h(t) + g(t)]).
\]

Then (1.1) is equivalent to

\[
\begin{align*}
&w_t - \rho^2(t)w_{yy} - \zeta(t,y)w_y = w(a - w - bz), \quad t > 0, \quad y \in \mathbb{R}, \\
w(0,y) = w_0(h_0y) := w_0(y), \quad y \in \mathbb{R}, \\
z_t - \frac{d\rho^2(t)}{2}z_{yy} - \zeta(t,y)z_y = \mu z(1 - z/\rho), \quad t > 0, \quad -1 < y < 1, \\
z(t,\pm 1) = 0, \quad t \geq 0, \\
z(0,y) = z_0(h_0y) := z_0(y), \quad -1 \leq y \leq 1, \\
g'(t) = -\beta p(t)z_y(t,-1), \quad t \geq 0, \\
h'(t) = -\beta p(t)z_y(t,1), \quad t \geq 0, \\
g(0) = -h_0, \quad h(0) = h_0.
\end{align*}
\]
where \( \rho(t) = \frac{2}{h(t) - g(t)}, \ \zeta(t, y) = \frac{h'(t) + g'(t)}{h(t) - g(t)} \). For \( 0 < T \leq \frac{h_0}{2 + |h'(0)|} \), we denote \( I_T = [0, T] \times [-1, 1] \), and

\[
\begin{align*}
\mathcal{D}_1 &= \{ z \in C(I_T) : \ z(0, y) = z_0(y), \ z(t, \pm 1) = 0, \ ||z - z_0||_{C(I_T)} \leq 1 \}, \\
\mathcal{D}_2 &= \{ g \in C^1([0, T]) : \ g(0) = -h_0, \ g'(0) = g^*, \ ||g' - g^*||_{C([0, T])} \leq 1 \}, \\
\mathcal{D}_3 &= \{ h \in C^1([0, T]) : \ h(0) = h_0, \ h'(0) = h^*, \ ||h' - h^*||_{C([0, T])} \leq 1 \}.
\end{align*}
\]

Clearly, \( \mathcal{D}_T = \mathcal{D}_1 \times \mathcal{D}_2 \times \mathcal{D}_3 \) is a closed convex set of \( C(I_T) \times [C^1([0, T])]^2 \).

Next, we shall apply the contraction mapping theorem to show the existence and uniqueness result. Due to the choice of \( T \), we see that, for \( (g, h) \in \mathcal{D}_1 \times \mathcal{D}_2 \),

\[
|g(t) + h_0| + |h(t) - h_0| \leq T(||g'||_{C([0, T])} + ||h'||_{C([0, T])}) \leq h_0/2,
\]

which implies

\[
h(t) - g(t) \geq h_0.
\]

Given \( (z, g, h) \in \mathcal{D}_T \), we set \( z = 0 \) in \([0, T] \times \{(-\infty, -1] \cup [1, \infty)\}\) and substitute \((z(t, y), g(t), h(t))\) into (2.1). Then (2.1) is a Cauchy problem of \( w \). The standard theory (cf. [17]) guarantees that the problem (2.1) admits a unique solution \( w \in C([0, T] \times \mathbb{R}) \). As \( u_0(x) > 0 \) in \([-2h_0, 2h_0]\), by use of the structure of \( \mathcal{D}_T \) and the continuity of \( w \), we can find a \( T_2 > 0 \) depending on \( a, b, h_0, g^*, h^* \), \( T_1 \) and \( u_0(x) \) such that

\[
\frac{1}{2} \min_{[-2h_0, 2h_0]} u_0(x) > 0
\]

provided \( 0 < T \leq T_2 \).

Substituting this known function \( w(t, y) \) into (2.2) and taking advantage of the \( L^p \) theory and Sobolev’s imbedding theorem we have that the problem (2.2) admits a unique solution, denoted by \( \tilde{z}(t, y) \), and \( \tilde{z} \in W_{p, 2}^1(I_T) \cap C^{\frac{1-\alpha}{2}}(I_T) \),

\[
\begin{align*}
\|\tilde{z}\|_{W_{p, 2}^1(I_T)} &\leq C_1, \\
\|\tilde{z}\|_{C^{\frac{1-\alpha}{2}}(I_T)} &\leq C_1,
\end{align*}
\]

for some constant \( C_1 > 0 \), depending on \( d, \mu, p, \alpha, h_0, g^*, h^* \), \( \|u_0\|_{W_{p, 2}^1((-h_0, h_0))} \) and \( \frac{1}{2} \min_{[-2h_0, 2h_0]} u_0(x) \).

Define

\[
\tilde{g}(t) = -h_0 - \beta \int_0^t \rho(\tau)\tilde{z}_y(\tau, -1)\,d\tau, \quad \tilde{h}(t) = h_0 + \beta \int_0^t \rho(\tau)\tilde{z}_y(\tau, 1)\,d\tau.
\]

Then \( \tilde{g}', \tilde{h}' \in C^{\frac{1}{2}}([0, T]) \), and

\[
\begin{align*}
\|\tilde{g}'\|_{C^{\frac{1}{2}}([0, T])}, &\quad \|\tilde{h}'\|_{C^{\frac{1}{2}}([0, T])} \leq C_2,
\end{align*}
\]

where \( C_2 \) depends on \( C_1, \beta, h_0, g^* \) and \( h^* \).

Now we define a mapping \( \mathcal{F} : \mathcal{D}_T \to C(I_T) \times [C^1([0, T])]^2 \) by

\[
\mathcal{F}(z, g, h) = (\tilde{z}, \tilde{g}, \tilde{h}).
\]

Clearly, \((z, g, h) \in \mathcal{D}_T \) is a fixed point of \( \mathcal{F} \) if and only if \((w, z, g, h) \) solves (2.1)-(2.3). Similar to the arguments in the proof of [32, Theorem 2.1], it can be shown that \( \mathcal{F} \) maps \( \mathcal{D}_T \) into itself and is a contraction mapping on \( \mathcal{D}_T \) when \( 0 < T \ll 1 \), where \( T \) depends only on \( \Lambda, h_0, g^*, h^* \), \( \|u_0\|_{L^\infty(\mathbb{R})} \), \( \|\tilde{z}\|_{W_{p, 2}^1((-h_0, h_0))} \) and \( \frac{1}{2} \min_{[-2h_0, 2h_0]} u_0(x) \).

Thus \( \mathcal{F} \) has a unique fixed point \((z, g, h) \) in \( \mathcal{D}_T \) by the contraction mapping theorem, and \((z, g, h) \in C^{\frac{1-\alpha}{2}}([0, T]) \times [C^1([0, T])]^2 \). That is, (2.1)-(2.3) have a unique
solution \((w, z, g, h)\). By using of the \(L^p\) theory firstly, and the Schauder theory secondly, we can show that
\[
z \in C^{1+\frac{\alpha}{2}, 2+\alpha}([\tau, T] \times [-1, 1]), \quad w \in C^{1+\frac{\alpha}{2}, 2+\alpha}([\tau, T] \times [-L, L])
\]
\[
g, h \in C^{1+\frac{\alpha}{2}}([\tau, T]),
\]
for any given \(0 < \tau < T\) and \(L > 0\) (cf. [25, Theorem 2.1]). Moreover, \(w, z > 0\) by the maximum principle, and \(z_y(t, -1) > 0, z_y(t, 1) < 0\) by the Hopf boundary lemma, the latter imply that \(g'(t) < 0, h'(t) > 0\) in \((0, T]\). Therefore, the problem (1.1) admits a unique solution \((u, v, g, h)\) and
\[
(u, v, g, h) \in C([0, T] \times \mathbb{R}) \times C^{1+\frac{\alpha}{2}, 1+\alpha}([0, T] \times [g(t), h(t)]) \times [C^{1+\frac{\alpha}{2}}([0, T])]^2.
\]

**Step 2.** Global existence. We extend the solution \((u, v, g, h)\) of (1.1) to the maximal time interval \([0, T_0]\) and show that \(T_0 = \infty\).

Firstly, by the maximum principle,
\[
0 < u \leq \max\{a, \max u_0(x)\} := A \quad \text{in} \quad [0, T_0] \times \mathbb{R},
\]
\[
0 < v \leq \max\{A, \max v_0(x)\} := B \quad \text{in} \quad [0, T_0] \times (g(t), h(t)).
\]

Using (2.2), (2.3) and the Hopf boundary lemma we have that \(g'(t) < 0\) and \(h'(t) > 0\) in \([0, T_0]\).

Assume on the contrary that \(T_0 < \infty\). Let \((\bar{v}, \bar{g}, \bar{h})\) be the unique solution of the following free boundary problem
\[
\begin{align*}
\bar{v}_t - d\bar{v}_{xx} &= \mu \bar{v}, \quad t > 0, \quad \bar{g}(t) < x < \bar{h}(t), \\
\bar{v}(t, \bar{g}(t)) &= \bar{v}(t, \bar{h}(t)) = 0, \quad t \geq 0, \\
\bar{g}'(t) &= -\beta \bar{v}(t, \bar{g}(t)), \quad t \geq 0, \\
\bar{h}'(t) &= -\beta \bar{v}(t, \bar{h}(t)), \quad t \geq 0, \\
\bar{v}(0, x) &= v_0(x), \quad -h_0 \leq x \leq h_0, \\
g(0) &= -h_0, \quad \bar{h}(0) = h_0.
\end{align*}
\]

In view of Lemma 2.1,
\[
g(t) \geq \bar{g}(t) \geq \bar{g}(T_0), \quad h(t) \leq \bar{h}(t) \leq \bar{h}(T_0) \quad \text{in} \quad [0, T_0).
\]

Because \(u\) satisfies
\[
\begin{align*}
u_t - d\nu_{xx} &\geq u(a - bB - u), \quad 0 < t < T_0, \quad x \in \mathbb{R}, \\
u_x(t, 0) &= 0, \quad 0 \leq t < T_0, \\
u(0, x) &= u_0(x) > 0, \quad x \in \mathbb{R},
\end{align*}
\]

we have
\[
\min_{[0, T_0] \times [\bar{g}(T_0), \bar{h}(T_0)]} u(t, x) = \delta(T_0) > 0.
\]

Notice that \([g(t), h(t)] \subset [\bar{g}(T_0), \bar{h}(T_0)]\) for all \(t \in [0, T_0]\), we can find a positive constant \(K(T_0)\) such that
\[
\mu v(1 - v/u) \leq K(T_0) \quad \text{in} \quad [0, T_0] \times [g(t), h(t)].
\]

Similar to the proof of [32, Lemma 2.1] we can show that there exists a positive constant \(M(T_0)\), which depends only on \(\Lambda, K(T_0)\), \(\min v'_0(x)\) and \(\max v'_0(x)\), such that \(g'(t) \geq -M(T_0), h'(t) \leq M(T_0)\) in \([0, T_0]\).
Let $0 < \alpha < 1 - 3/p$ and
\[
\Lambda(T_0) = \{\alpha, p, \bar{g}(T_0), \bar{h}(T_0), K(T_0), M(T_0)\}.
\]
Applying the $L^p$ theory to (2.2) and the embedding theorem we have
\[
z \in W^{1,2}_p((0, T_0) \times (-1, 1)) \hookrightarrow C^{1+\frac{1+p}{2}}_{t\times\alpha}([0, T_0] \times [-1, 1]),
\]
and there exists a positive constant $C_1(T_0)$ depending only on $\Lambda(T_0)$ and $d$ such that
\[
\|z\|_{C^{1+\frac{1+p}{2}, \alpha}([0, T_0] \times [-1, 1])} \leq C_1(T_0).
\]
Thus, by use of (2.3), $g, h \in C^{1+\frac{1+p}{2}}([0, T_0])$ and
\[
\|g, h\|_{C^{1+\frac{1+p}{2}, \alpha}([0, T_0])} \leq C_2(T_0),
\]
where $C_2(T_0)$ depends on $\Lambda(T_0)$, $C_1(T_0)$ and $\beta$. Take advantage of the Schauder theory to (2.2) we have
\[
z \in C^{1+\frac{1+p}{2}\cdot 2+\alpha}([T_0/2, T_0 - \varepsilon] \times [-1, 1]), \ \forall 0 < \varepsilon < T_0/2,
\]
and there exists a positive constant $C_3(T_0)$, which depends only on $C_1(T_0)$ and $C_2(T_0)$, but not on $\varepsilon$, such that
\[
\|z\|_{C^{1+\frac{1+p}{2}, \alpha}([T_0/2, T_0 - \varepsilon] \times [-1, 1])} \leq C_3(T_0), \ \forall 0 < \varepsilon < T_0/2.
\]
Then, in view of (2.3), we have $g, h \in C^{1+\frac{1+p}{2}}([T_0/2, T_0 - \varepsilon])$ and
\[
\|g, h\|_{C^{1+\frac{1+p}{2}, \alpha}([T_0/2, T_0 - \varepsilon])} \leq C_4(T_0), \ \forall 0 < \varepsilon < T_0/2,
\]
where $C_4(T_0)$ depends on $C_2(T_0)$ and $C_3(T_0)$.

Therefore, $v(t, \cdot) \in C^2([g(t), h(t)])$ for any $T_0/2 \leq t < T_0$, and
\[
\|v(t, \cdot)\|_{C^2([g(t), h(t)])} = \|g(t)\|_{\bar{g}(\Pi)} + \|g'(t)\|_{\bar{g}'(\Pi)} + \|h(t)\|_{\bar{h}(\Pi)} + \|h'(t)\|_{\bar{h}'(\Pi)} \leq C_5(T_0), \ \forall \tau \leq t < T_0.
\]
Repeating the discussion of Step 1 we can find a positive constant $T$ depending only on $a, b, d, \mu, \beta, \alpha, p, A, C_2(T_0)$ and $\delta(T_0)$ which was given by (2.4), such that the solution of (1.1) with initial time $T - T/2$ can be extended uniquely to the time $T_0 - T/2 + T$. But this contradicts the definition of $T_0$.

**Step 3.** The regularity and estimate (1.3) can be proved by the similar way to those of [29, Theorem 2.2] and [33, Theorem 1.2], and the details are omitted here. The proof is finished. 

Let $(u, v, g, h)$ be the unique global solution of (1.1). As $g'(t) < 0$ and $h'(t) < 0$, we can define the limits $\lim_{t \to \infty} g(t) = g_\infty \geq -\infty$ and $\lim_{t \to \infty} h(t) = h_\infty \leq \infty$.

At last, we shall give the uniform estimates of $v$ and $g$, $h'$ when $h_\infty - g_\infty < \infty$.

To this purpose, we first state a proposition.

**Proposition 1.** Let $d, m, \theta, k, T$ be constants and $d, m, \theta > 0, k, T \geq 0$. For any given $\varepsilon, L > 0$, there exist $T_\varepsilon > T$ and $l_\varepsilon > \max \{L, \frac{\pi}{2} \sqrt{d/m}\}$, such that when the function $w \in C^{1,2}((T, \infty) \times (-l_\varepsilon, l_\varepsilon))$ and satisfies $w \geq 0$,
\[
\begin{cases}
w_{t} - dw_{xx} \geq (\leq) w(m - \theta w), & t > T, \ -l_\varepsilon < x < l_\varepsilon, \\
w(T, x) > 0, & -l_\varepsilon < x < l_\varepsilon,
\end{cases}
\]
and for $t > T$, $w(t, \pm l_\varepsilon) \geq (\leq) k$ if $k > 0$, while $w(t, \pm l_\varepsilon) \geq (\leq) 0$ if $k = 0$, we must have
\[
w(t, x) > m/\theta - \varepsilon \ (w(t, x) < m/\theta + \varepsilon), \ \forall t \geq T_\varepsilon, \ x \in [-L, L].
\]
This implies
\[
\liminf_{t \to \infty} w(t, x) \geq m/\theta - \varepsilon \left( \limsup_{t \to \infty} w(t, x) < m/\theta + \varepsilon \right) \quad \text{uniformly on } [-L, L].
\]

Proof. This proposition can be proved by the same way as that of [32, Proposition B.1] and the details are omitted here. \(\square\)

**Theorem 2.2.** Assume \(b < 1\). Let \((u, v, g, h)\) be the unique global solution of (1.1). Then, for any given \(L > 0\), there exists a positive constant \(\sigma(L) > 0\), which does not depend on \(\beta\), such that \(u(t, x) \geq \sigma(L)\) in \([0, \infty) \times [-L, L]\).

Proof. It is easy to see from the first equation of (1.1) that \(\limsup_{t \to \infty} \max_{x \in \mathbb{R}} u(t, x) \leq a\).

For any given \(\varepsilon > 0\), there exists \(T_\varepsilon \gg 1\) such that \(u(t, x) \leq a + \varepsilon\) for all \(t \geq T_\varepsilon\) and \(x \in \mathbb{R}\). Then \(v\) satisfies
\[
\begin{cases}
  v_t - dv_{xx} \leq \mu v [1 - v/(a + \varepsilon)], & t \geq T_\varepsilon, \ g(t) < x < h(t), \\
  v(t, x) = 0, & t \geq T_\varepsilon, \ x \notin (g(t), h(t)).
\end{cases}
\]

Let \(w(t)\) be the unique positive solution of
\[w'(t) = \mu w [1 - w/(a + \varepsilon)], \quad t > T_\varepsilon; \quad w(T_\varepsilon) = \max_{[g(T_\varepsilon), h(T_\varepsilon)]} v(T_\varepsilon, x).\]

Then \(\lim_{t \to \infty} w(t) = a + \varepsilon\), and \(v(t, x) \leq w(t)\) for all \(t \geq T_\varepsilon, x \in \mathbb{R}\) by the comparison principle. Therefore,
\[
\limsup_{t \to \infty} \max_{x \in \mathbb{R}} v(t, x) \leq a
\]
by the arbitrariness of \(\varepsilon\).

Take \(0 < \omega, \varepsilon \ll 1\) such that \(a - b(a + \omega) > 0\). Then there exists \(T > 0\) such that
\[v(t, x) \leq a + \omega, \quad \forall t \geq T, \ x \in \mathbb{R}.
\]

Let \(T_\varepsilon\) and \(l_\varepsilon\) be given by Proposition 1 with \(d = \theta = 1, m = a - b(a + \omega), k = 0\). It is clear that
\[
\begin{cases}
  u_t - u_{xx} \geq u(a - b(a + \omega) - u), & t \geq T, \ x \in [-l_\varepsilon, l_\varepsilon], \\
  u(t, \pm l_\varepsilon) > 0, & t \geq T.
\end{cases}
\]

Take advantage of Proposition 1 it arrives at \(\liminf_{t \to \infty} u(t, x) \geq a - b(a + \omega) - \varepsilon\) uniformly on \([-L, L]\). The arbitrariness of \(\varepsilon\) and \(\omega\) imply that \(\liminf_{t \to \infty} u(t, x) \geq a(1 - b) > 0\) uniformly on \([-L, L]\). Notice that \(u(t, x) > 0\) on \([0, \infty) \times [-L, L]\), we can find a constant \(\sigma(L) > 0\) such that
\[
u(t, x) \geq \sigma(L) \quad \text{in } [0, \infty) \times [-L, L].
\]

The proof is finished. \(\square\)

**Theorem 2.3.** Assume \(b < 1\). Let \((u, v, g, h)\) be the unique global solution of (1.1). If \(h_\infty - g_\infty < \infty\), then there exists a positive constant \(C\) depending only on \(\Lambda, g_\infty\) and \(h_\infty\) such that
\[
\|v(t, \cdot)\|_{C^1([g(t), h(t)])} \leq C, \quad \forall t \geq 0; \quad \|g', h'|_{C^{\infty}/2([0, \infty))} \leq C. \quad (2.5)
\]

Proof. The condition \(h_\infty - g_\infty < \infty\) implies \(g_\infty > -\infty\) and \(h_\infty < \infty\). By Theorem 2.2, there exists a positive constant \(\sigma > 0\), which does not depend on \(\beta\), such that \(u(t, x) \geq \sigma\) in \([0, \infty) \times [g_\infty, h_\infty]\). Follow the proof of [25, Theorem 2.1] step by step we can get the estimate (2.5). The details are omitted here. \(\square\)
3. Longtime behavior of \((u,v)\). This section concerns with the limits of \(u(t,x)\), and \(v(t,x)\) as \(t \to \infty\).

**Case 1.** \(h_\infty - g_\infty < \infty\). In this case we shall prove that if \(b < 1\) then

\[
\lim_{t \to \infty} \max_{g(t) \leq x \leq h(t)} v(t,x) = 0, \quad (3.1)
\]

and

\[
\lim_{t \to \infty} u(t,x) = a \quad \text{uniformly on the compact subset of } \mathbb{R}. \quad (3.2)
\]

For this purpose, we first give a proposition.

**Proposition 2.** Let \(d, C, \mu \) and \(\eta_0 \) be positive constants, \(w \in W^{1,2}_p((0,T) \times (0,\eta(t)))\) for some \(p > 1\) and any \(T > 0\), and \(w_x \in C([0,\infty) \times (0,\eta(t))], \eta \in C^1([0,\infty))\). If \((w,\eta)\) satisfies

\[
\begin{aligned}
  w_t - dw_{xx} &\geq -Cw, & t > 0, & 0 < x < \eta(t), \\
  w &\geq 0, & t > 0, & x = 0, \\
  w & = 0, & \eta'(t) &\geq -\mu w_x, & t > 0, & x = \eta(t), \\
  w(0,x) & = w_0(x) \geq 0, & x \in (0,\eta), \\
  \eta(0) & = \eta_0,
\end{aligned}
\]

and

\[
\lim_{t \to \infty} \eta(t) = \eta_\infty < \infty, \quad \lim_{t \to \infty} \eta'(t) = 0,
\]

\[
\|w(t,\cdot)\|_{C^1([0,\eta(t)])} \leq M, \quad \forall t > 1
\]

for some constant \(M > 0\). Then

\[
\lim_{t \to \infty} \max_{0 \leq x \leq \eta(t)} w(t,x) = 0.
\]

**Proof.** Firstly, by the maximum principle we have \(w(t,x) > 0\) for \(t > 0\) and \(0 < x < \eta(t)\). Follow the proof of [31, Theorem 2.2] word by word we can prove this proposition and the details are omitted here.

**Theorem 3.1.** (Vanishing) Suppose \(b < 1\). Let \((u,v,g,h)\) be the unique global solution of \((1.1)\). If \(h_\infty - g_\infty < \infty\), then \((3.1)\) and \((3.2)\) hold.

**Proof.** Notice Theorem 2.3, in view of Proposition 2 we can get \(\lim_{t \to \infty} \max_{t \to \infty} 0 \leq x \leq h(t)} v(t,x) = 0\) directly. Similarly, \(\lim_{t \to \infty} \max_{t \to \infty} 0 \leq x \leq 0}} v(t,x) = 0\). Thus \((3.1)\) holds. The proof of \((3.2)\) is standard and we shall omit the details. The interested readers can refer to the proof of [32, Theorem 4.2].

**Case 2.** \(h_\infty - g_\infty = \infty\). In this case we shall prove that if \(b < 1\) then

\[
\lim_{t \to \infty} u(t,x) = \frac{a}{1+b}, \quad \lim_{t \to \infty} v(t,x) = \frac{a}{1+b} \quad (3.3)
\]

uniformly in any compact subset of \(\mathbb{R}\). To do this we first show a proposition which alleges that \(h_\infty - g_\infty = \infty\) implies \(g_\infty = -\infty\) and \(h_\infty = \infty\).

**Proposition 3.** Assume \(b < 1\). Let \((u,v,g,h)\) be the unique global solution of \((1.1)\). If \(h_\infty - g_\infty = \infty\), then \(g_\infty = -\infty\) and \(h_\infty = \infty\).
Proof. Assume on the contrary that $h_\infty < \infty$. Then the condition $h_\infty - g_\infty = \infty$ implies $g_\infty = -\infty$. There exists $T \gg 1$ such that

$$h_0 - g(T) > \pi \sqrt{d/\mu}.$$  

Similar to the proof of Theorem 2.2, we can find a constant $\sigma = \sigma(T) > 0$ such that $u(t,x) \geq \sigma$ in $[T,\infty) \times [g(T), h_\infty]$. Then $v$ satisfies

$$\begin{cases}
  v_t - dv_{xx} \geq \mu v (1 - v/\sigma) , & t > T, \ g(T) < x < h(t), \\
  v \geq 0 , & t \geq T, \ x = g(T), \\
  v = 0 , & t \geq T, \ x = h(t).
\end{cases}$$

Let $(w, \eta)$ be the unique global solution of

$$\begin{cases}
  w_t - dw_{xx} = \mu w (1 - w/\sigma) , & t > T, \ g(T) < x < \eta(t), \\
  w = 0 , & t \geq T, \ x = g(T), \\
  w = 0 , & t \geq T, \ x = \eta(t), \\
  w(T,x) = v(T,x) , & g(T) \leq x \leq \eta(T), \\
  \eta(T) = h(T).
\end{cases}$$

The comparison principle yields $h(t) \geq \eta(t)$ in $[T,\infty)$ and $v(t,x) \geq w(t,x)$ in $[T,\infty) \times [g(T), \eta(t)]$. Since

$$\eta(T) - g(T) = h(T) - g(T) > h_0 - g(T) > \pi \sqrt{d/\mu},$$

by use of [24, Lemma 3.2] we have that $\lim_{t \to \infty} \eta(t) = \infty$, which implies $h_\infty = \infty$. A contradiction is obtained and so $h_\infty = \infty$. Similarly, we can show that $g_\infty = -\infty$. The proof is complete.

**Theorem 3.2.** (Spreading) Assume $b < 1$. Let $(u,v,g,h)$ be the unique global solution of (1.1). If $g_\infty = -\infty$, $h_\infty = \infty$, then (3.3) holds.

Proof. This proof is similar to that of [32, Theorem 4.3]. For the completeness we shall give the details.

Firstly,

$$\limsup_{t \to \infty} \max_{x \in \mathbb{R}} u(t,x) \leq a, \quad \limsup_{t \to \infty} \max_{x \in \mathbb{R}} v(t,x) \leq a, \quad (3.4)$$

see the proof of Theorem 2.2.

Chosen $L \gg 1$ and $0 < \omega, \varepsilon \ll 1$ such that $a - b(a + \omega) > 0$. Let $l_\varepsilon$ be given by Proposition 1 with $d = \theta = 1$, $m = a - b(a + \omega)$, $k = 0$. Using (3.4), we can choose a $T_1 > 0$ such that

$$v(t,x) \leq a + \omega, \quad \forall t \geq T_1, \ x \in [-l_\varepsilon, l_\varepsilon].$$

Then $u$ satisfies

$$\begin{cases}
  u_t - u_{xx} \geq u[a - b(a + \omega) - u], & t \geq T_1, \ x \in [-l_\varepsilon, l_\varepsilon], \\
  u(t, \pm L) > 0 , & t \geq T_1.
\end{cases}$$

In view of Proposition 1 we have $\liminf_{t \to \infty} u(t,x) \geq a - b(a + \omega) - \varepsilon$ uniformly on $[-L,L]$. In consideration of the arbitrariness of $L$, $\omega$ and $\varepsilon$, it follows that

$$\liminf_{t \to \infty} u(t,x) \geq a - ba := \mathcal{g}_1$$

uniformly on the compact subset of $\mathbb{R}$. (3.5) By our assumption, $\mathcal{g}_1 > 0$. 


Thus \( u_{t_2} = \frac{1}{\omega_1} \) be given by Proposition 1 with 
\( m = \mu, \theta = \mu/(\omega_1 - \omega) \) and \( k = 0 \). According to (3.5) and \( \gamma \to \infty, h = \infty \), there is \( T_2 > 0 \) such that \( u \geq \omega_1 \) and \( g(t) < -\epsilon, h(t) > \epsilon \) for \( t \geq T_2 \). Consequently, \( v \) satisfies

\[
\begin{cases}
  v_t - dv_{xx} \geq \mu v [1 - v/(\omega_1 - \omega)], & t \geq T_2, \ x \in [-\epsilon, \epsilon], \\
  v(t, \pm \epsilon) \geq 0, & t \geq T_2.
\end{cases}
\]

Similar to the above,

\[
\liminf_{t \to \infty} v(t, x) \geq \omega_1 \quad \text{uniformly on the compact subset of} \ R. \quad (3.6)
\]

Clearly, \( a - b\omega_1 > 0 \). Take \( L \gg 1, 0 < \omega, \epsilon \ll 1 \) with \( a - b(\omega_1 - \omega) > 0 \). Let \( l_\epsilon \) be given by Proposition 1 with

\[
d = \theta = 1, \quad m = a - b(\omega_1 - \omega), \quad k = A := \max\{a, \max u_0(x)\}.
\]

By virtue of (3.6) we can find a \( T_3 > 0 \) such that

\[
v(t, x) \geq \omega_1, \quad \forall \ t \geq T_3, \ x \in [-\epsilon, \epsilon].
\]

Thus \( u \) satisfies

\[
\begin{cases}
  u_t - u_{xx} \leq u[a - b(\omega_1 - \omega) - u], & t \geq T_3, \ x \in [-\epsilon, \epsilon], \\
  u(t, \pm \epsilon) \leq A, & t \geq T_3.
\end{cases}
\]

The same as the above,

\[
\limsup_{t \to \infty} u(t, x) \leq a - b\omega_1 := \bar{\omega}_1 \quad \text{uniformly on the compact subset of} \ R. \quad (3.7)
\]

Given \( L \gg 1, 0 < \omega, \epsilon \ll 1 \). Let \( l_\epsilon \) be determined by Proposition 1 with

\[
m = \mu, \quad \theta = \mu/(\bar{\omega}_1 + \omega), \quad k = B := \max\{A, \max u_\theta(x)\}.
\]

where \( A \) is given by the above. Thanks to (3.7), there exists \( T_4 > 0 \) such that

\[
u(t, x) \leq \bar{\omega}_1 + \omega, \quad \forall \ t \geq T_4, \ x \in [-\epsilon, \epsilon].
\]

Consequently, \( v \) satisfies

\[
\begin{cases}
  v_t - dv_{xx} \leq \mu v [1 - v/(\bar{\omega}_1 + \omega)], & t \geq T_4, \ x \in [-\epsilon, \epsilon], \\
  v(t, \pm \epsilon) \leq B, & t \geq T_4.
\end{cases}
\]

The same as the above,

\[
\limsup_{t \to \infty} v(t, x) \leq \bar{\omega}_1 \quad \text{uniformly on the compact subset of} \ R.
\]

Repeating the above procedure, we can find two sequences \( \{\omega_i\} \) such that, for all \( i \),

\[
\begin{cases}
  a_i \leq \liminf_{t \to \infty} u(t, x) \leq \limsup_{t \to \infty} u(t, x) \leq \bar{\omega}_i, \\
  a_i \leq \liminf_{t \to \infty} v(t, x) \leq \limsup_{t \to \infty} v(t, x) \leq \bar{\omega}_i
\end{cases}
\]

uniformly in the compact subset of \( R \). Moreover, these sequences can be determined by the following iterative formulas:

\[
a_1 = a - b\omega, \quad \bar{\omega}_i = a - b\omega_i, \quad \bar{\omega}_{i+1} = a - b\bar{\omega}_i, \quad i = 1, 2, \ldots.
\]

The direct calculation yields

\[
\bar{\omega}_1 = a(1 - b + b^2), \quad \bar{\omega}_2 = a(1 - b + b^2 - b^3), \quad \bar{\omega}_2 = a(1 - b + b^2 - b^3 + b^4).
\]
Using the inductive method we have the following expressions:
\[ g_i = a(1 - b + b^2 - \cdots + b^{i-2} - b^{i-1}), \quad \tilde{a}_i = a(1 - b + b^2 - \cdots + b^{2i}), \quad i \geq 3. \]
Because of \( 0 < b < 1 \), one has
\[
\lim_{i \to \infty} \tilde{a}_i = \lim_{i \to \infty} \frac{a}{(1 + b)}.
\]
This fact combined with (3.8) allows us to derive (3.3).

4. The criteria governing spreading and vanishing, proof of Theorem 1.2.
This section concerns with the criteria governing spreading and vanishing. We first give a necessary condition for vanishing.

**Theorem 4.1.** Assume \( b < 1 \). Let \((u, v, g, h)\) be the unique global solution of (1.1). If \( h_\infty - g_\infty < \infty \), then
\[
h_\infty - g_\infty \leq \pi \sqrt{d/\mu}. \tag{4.1}
\]
Hence, \( h_0 \geq \frac{\pi}{2} \sqrt{d/\mu} \) implies \( h_\infty - g_\infty = \infty \) due to \( h'(t) - g'(t) > 0 \) for \( t > 0 \).

**Proof.** The condition \( h_\infty - g_\infty < \infty \) implies that \( \lim_{t \to \infty} \|v(t, \cdot)\|_{C([g(t), h(t)])} = 0 \), \( \lim_{t \to \infty} u(t, x) = a \) uniformly in the compact subset of \( \mathbb{R} \) (Theorem 3.1). We assume \( h_\infty - g_\infty > \pi \sqrt{d/\mu} \) to get a contradiction. For any given \( 0 < \varepsilon < 1 \), there exists \( T \gg 1 \) such that
\[
u(t, x) \geq a - \varepsilon, \quad \forall \ t \geq T, \ x \in [g_\infty, h_\infty],
\]
\[
h(T) - g(T) > \pi \sqrt{d/\mu}.
\]
Let \( w \) be the unique solution of
\[
\begin{align*}
w_t - dw_{xx} &= \mu w [1 - w/(a - \varepsilon)], \quad t > T, \ g(T) < x < h(T), \\
w(t, g(T)) &= w(t, h(T)) = 0, \quad t \geq T, \\
w(T, x) &= v(T, x), \quad g(T) \leq x \leq h(T).
\end{align*}
\]
As \( v \) satisfies
\[
\begin{align*}
v_t - dv_{xx} &\geq \mu v[1 - v/(a - \varepsilon)], \quad t > T, \ g(T) < x < h(T), \\
v(t, g(T)) &\geq 0, \quad v(t, h(T)) \geq 0, \quad t \geq T,
\end{align*}
\]
the comparison principle gives \( w \leq v \) in \([T, \infty) \times [g(T), h(T)]\). Since \( h(T) - g(T) > \pi \sqrt{d/\mu} \), it is well known that \( w(t, x) \to \theta(x) \) as \( t \to \infty \) uniformly on \([g(T), h(T)]\), where \( \theta(x) \) is the unique positive solution of
\[
\begin{align*}
-d\theta_{xx} &= \mu \theta [1 - \theta/(a - \varepsilon)], \quad \theta(T) < x < h(T), \\
\theta(T) &= \theta(h(T)) = 0.
\end{align*}
\]
Hence, \( \lim_{t \to \infty} v(t, x) \geq \lim_{t \to \infty} w(t, x) = \theta(x) > 0 \) in \([g(T), h(T)]\). This is a contradiction to (3.1), and hence (4.1) holds. The proof is complete.

If \( b < 1 \), by Theorem 4.1 and Proposition 3 we see that \( h_0 \geq \frac{\pi}{2} \sqrt{d/\mu} \) implies \( g_\infty = -\infty \) and \( h_\infty = \infty \) for all \( \beta > 0 \).
Now we discuss the case \( h_0 < \frac{\pi}{2} \sqrt{d/\mu} \).
Lemma 4.2. Let \((u, v, g, h)\) be the unique global solution of (1.1). If \(h_0 < \frac{\pi^2}{2\sqrt{d/\mu}}\), then there exists \(\beta_0 > 0\), depending on \(d, h_0, \mu\) and \(v_0(x)\), such that \(g_\infty > -\infty\), \(h_\infty < \infty\) provided \(\beta \leq \beta_0\).

Proof. Obviously, \(\lambda_1 = \frac{d}{4h_0^2} \pi^2\) and \(\phi(x) = \sin \frac{\pi(x+h_0)}{2h_0}\) are the principal eigenvalue and the corresponding positive eigenfunction of the following problem

\[
\begin{cases}
-\phi_{xx} = \lambda \phi, \quad -h_0 < x < h_0, \\
\phi(\pm h_0) = 0,
\end{cases}
\]

and there exists \(k > 0\) such that

\[x \phi'(x) \leq k \phi(x) \quad \text{in} \quad [-h_0, h_0].\]

The condition \(h_0 < \frac{\pi^2}{2\sqrt{d/\mu}}\) implies \(\lambda_1 > \mu\). Let \(0 < \varepsilon, \rho < 1\) and \(K > 0\) be constants, which will be determined. Set

\[s(t) = 1 + 2\varepsilon - \varepsilon e^{-\rho t}, \quad \eta(t) = h_0 s(t), \quad t \geq 0,
\]

\[w(t, x) = Ke^{-\rho t} \phi(x/s(t)), \quad t \geq 0, \quad -\eta(t) \leq x \leq \eta(t).
\]

Clearly, \(w(t, \pm \eta(t)) = 0\). Similar to the calculations in the proof of [24, Lemma 3.4] ([25, Lemma 5.3]), we can show that there exists \(\delta > 0\) such that

\[w_t - dw_{xx} - \mu w > 0, \quad \forall t > 0, \quad -\eta(t) < x < \eta(t)
\]

for all \(0 < \varepsilon, \rho \leq \delta\) and all \(K > 0\). Fixed \(0 < \varepsilon, \rho \leq \delta\), then

\[w(0, x) = K\phi(x/(1+\varepsilon)) \geq v_0(x) \quad \text{in} \quad [-h_0, h_0]
\]

provided \(K \gg 1\). For these fixed \(0 < \varepsilon, \rho \leq \delta\) and \(K \gg 1\), remember \(\phi'(-h_0) > 0\) and \(\phi'(h_0) < 0\), we can find a \(\beta_0: 0 < \beta_0 \ll 1\) such that, for all \(0 < \beta \leq \beta_0\),

\[-h_0 \varepsilon \rho \leq -\beta \frac{1}{s(t)} K\phi'(-h_0), \quad h_0 \varepsilon \rho \geq -\beta \frac{1}{s(t)} K\phi'(h_0), \quad \forall t \geq 0.
\]

This implies

\[-\eta'(t) \leq -\beta w_x(t, -\eta(t)), \quad \eta'(t) \geq -\beta w_x(t, \eta(t)), \quad \forall t \geq 0.
\]

Because of \(v\) satisfies

\[v_t - dv_{xx} - \mu v < 0, \quad \forall t > 0, \quad g(t) < x < h(t),
\]

by the comparison principle we conclude

\[g(t) \geq -\eta(t), \quad h(t) \leq \eta(t), \quad \forall t \geq 0,
\]

and so \(g_\infty \geq -\eta(\infty) = -(1+2\varepsilon)h_0, \quad h_\infty \leq \eta(\infty) = (1+2\varepsilon)h_0\). The proof is complete. \(\square\)

Lemma 4.3. Let \(C\) be a positive constant. For any given positive constants \(h_0, L\), and any function \(\bar{v}_0 \in W^2_0(-h_0, h_0)\) with \(p > 1, \bar{v}_0(\pm h_0) = 0\) and \(\bar{v}_0 > 0\) in \((-h_0, h_0)\), there exists \(\beta^0 > 0\) such that when \(\beta \geq \beta^0\) and \((\bar{v}, \bar{g}, \bar{h})\) satisfies

\[
\begin{cases}
\bar{v}_t - \bar{v}_{xx} \geq -C\bar{v}, \quad t > 0, \quad \bar{g}(t) < x < \bar{h}(t), \\
\bar{v} = 0, \quad \bar{g}'(t) \leq -\beta \bar{v}_x, \quad t \geq 0, \quad x = \bar{g}(t), \\
\bar{v} = 0, \quad \bar{h}'(t) \geq -\beta \bar{v}_x, \quad t \geq 0, \quad x = \bar{h}(t), \\
\bar{v}(0, x) = \bar{v}_0(x), \quad -h_0 \leq x \leq h_0, \\
\bar{g}(0) = -h_0, \quad \bar{h}(0) = h_0,
\end{cases}
\]

(4.2)

we must have \(\lim_{t \to \infty} \bar{g}(t) < -L, \lim_{t \to \infty} \bar{h}(t) > L\).
Proof. Follow the proof of [27, Lemma 3.2] step by step and use the comparison principle, we can prove the conclusion. The details are omitted here. □

Lemma 4.4. Assume that \( b < 1 \). Let \((u, v, g, h)\) be the unique global solution of (1.1). If \( h_0 < \frac{\sigma}{\sqrt{d/\mu}} \), then there exists \( \beta^0 > 0 \) such that \( h_\infty - g_\infty > \pi \sqrt{d/\mu} \) for all \( \beta \geq \beta^0 \).

Proof. Write \((u^\beta, v^\beta, g^\beta, h^\beta)\) in place of \((u, v, g, h)\) to clarify the dependence of the solution of (1.1) on \( \beta \). Assume on the contrary that there exists \( \{\beta_n\} \) with \( \beta_n \to \infty \). In view of Theorem 2.2, there exists a positive constant \( \sigma > 0 \) such that \( w_\beta(t, x) \geq \sigma \) in \([0, \infty) \times [-\pi \sqrt{d/\mu}, \pi \sqrt{d/\mu}]\) for all \( n \). Therefore \( \mu(1 - \frac{w_\beta}{\mu^n}) \geq -C \) in \([0, \infty) \times [-\pi \sqrt{d/\mu}, \pi \sqrt{d/\mu}]\) for some positive constant \( C \) and all \( n \), and so
\[
v_\beta^n - dv_\beta^n \geq -Cv_\beta^n, \quad t > 0, \quad g_\beta^n(t) < x < h_\beta^n(t).
\]
According to Lemma 4.3, there exists \( \beta^0 > 0 \) such that \( h_\beta^n - g_\beta^n > \pi \sqrt{d/\mu} \) for all \( \beta \geq \beta^0 \). Since \( \beta_n \to \infty \), we get a contradiction and the proof is complete. □

Making use of Lemmas 4.2 and 4.4, we can prove the following theorem by the same manner as that of [32, Theorem 5.2] and the details will be omitted.

Theorem 4.5. Assume that \( b < 1 \). Let \((u, v, g, h)\) be the unique global solution of (1.1). If \( h_0 < \frac{\sigma}{\sqrt{d/\mu}} \), then there exist \( \beta^* \geq \beta_\ast > 0 \), depending on \( a, b, d, \mu, g_0(x), v_0(x) \) and \( h_0 \), such that \( g_\infty = -\infty \) and \( h_\infty = \infty \) when \( \beta > \beta^* \), and \( h_\infty - g_\infty \leq \pi \sqrt{d/\mu} \) when \( \beta \leq \beta_\ast \) or \( \beta = \beta^* \).

Now we state the criteria for spreading \((g_\infty = -\infty, h_\infty = \infty)\) and vanishing \((h_\infty - g_\infty < \infty)\).

Theorem 4.6. Assume that \( b < 1 \). Let \((u, v, g, h)\) be the unique global solution of (1.1).

(i) If \( h_0 \geq \frac{\sigma}{\sqrt{d/\mu}} \), then \( g_\infty = -\infty \) and \( h_\infty = \infty \) for all \( \beta > 0 \);
(ii) If \( h_0 < \frac{\sigma}{\sqrt{d/\mu}} \), then there exist \( \beta^* \geq \beta_\ast > 0 \), such that \( g_\infty = -\infty \) and \( h_\infty = \infty \) for \( \beta > \beta^* \), while \( h_\infty - g_\infty \leq \pi \sqrt{d/\mu} \) for \( \beta \leq \beta_\ast \) or \( \beta = \beta^* \).

Theorem 4.6 is the direct consequence of Theorems 4.1 and 4.5.
Theorem 1.2 is the direct consequence of Theorems 3.1, 3.2 and 4.6.

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