SL(2, C) GROUP ACTION ON COHOMOLOGICAL FIELD THEORIES

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ABSTRACT. We introduce the SL(2, C) group action on a partition function of a Cohomological field theory via the certain Givental’s action. Restricted to the small phase space we describe the action via the explicit formulae on a CohFT genus g potential. We prove that applied to the total ancestor potential of a simple elliptic singularity the action introduced coincides with the transformation of Milanov–Ruan changing the primitive form (cf. [MR]). Finally using the results obtained we find explicitly the genus zero Gromov–Witten potentials of the orbifolds \(\mathbb{P}^1_{4,4,2}\) and \(\mathbb{P}^1_{6,3,2}\).

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1. INTRODUCTION

Cohomological field theories (CohFT for brevity) were introduced in the early 90s by Kontsevich and Manin in [KM]. They appeared to play an important role in many different subjects of mathematics - they are key objects in the mirror symmetry conjecture [FJR], integrable hierarchies [DZ, FSZ] and even geometry of the moduli space of curves [PPZ]. An important tool to work with CohFTs that is used in all the aspects listed is Givental’s action. However in some cases (and in singularity theory in particular) it does not give the feeling of the initial object geometry where another - SL(2, C) action is defined naturally. Based on the examples coming from singularity theory we introduce analytically the SL(2, C) action on the genus zero
part of an arbitrary CohFT and write it in terms of Givental’s action to act on the higher genera of the CohFT too.

1.1. Cohomological Field Theory. Denote by $\overline{M}_{g,k}$ the moduli space of stable genus $g$ curves with $k$ marked points. Let $V = \langle e_1, \ldots, e_n \rangle$ be a finite dimensional vector space with a non-degenerate scalar product $\eta$. A Cohomological field theory on the state space $(V, \eta)$ is a system of linear maps

$$\Lambda_{g,k} : V^\otimes k \to H^*(\overline{M}_{g,k}, \mathbb{C})$$

for all $g, k$ such that $\overline{M}_{g,k}$ exists and is non-empty. It is required to satisfy certain axioms that arise naturally from the geometry of the moduli space of curves. Let $\psi_l \in H^*(\overline{M}_{g,k}), 1 \leq l \leq k$ be Mumford–Morita–Miller classes. The genus $g$ correlators of the CohFT are the following numbers:

$$\langle \tau_{d_1}(e_{\alpha_1}) \cdots \tau_{d_k}(e_{\alpha_k}) \rangle_g := \int_{\overline{M}_{g,k}} \Lambda_{g,k}(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k}) \psi_{d_1}^{\alpha_1} \cdots \psi_{d_k}^{\alpha_k}.$$

Denote by $F_g$ the generating function of the genus $g$ correlators, called genus $g$ potential of the CohFT:

$$F_g := \sum_{\text{Aut}(\alpha, d)} \langle \tau_{d_1}(e_{\alpha_1}) \cdots \tau_{d_k}(e_{\alpha_k}) \rangle_g t^{d_1, \alpha_1} \cdots t^{d_k, \alpha_k}.$$

It depends on the formal variables $t^{k,\alpha}$ for $1 \leq \alpha \leq n = \dim(V)$ and $k \geq 0$.

The restriction of $F_g$ to $t^{k,\alpha} = 0$, $k \geq 1$ is called restriction to the small phase space. Introduce the notation:

$$F_g := F_g \big|_{t^{k,\alpha} = 0, k \geq 1}.$$

We have $F_g = F_g(t^{0,1}, \ldots, t^{0,n})$. These generating functions will be called genus $g$ small phase space potentials.

It is useful to assemble the correlators into a generating function called partition function of the CohFT:

$$Z := \exp \left( \sum_{g \geq 0} \hbar^{g-1} F_g \right).$$

1.2. Frobenius manifold of a CohFT. The notion of Frobenius manifold was introduced by B. Dubrovin in 90s. It provides a generalization of the flat structures of K. Saito introduced in the early 80s and is a crucial step towards the Saito–Givental CohFT of a singularity.

The structure of a Frobenius manifold $M$ is defined by the so-called Frobenius potential $F^M \in \mathbb{C}[[t^1, \ldots, t^n]]$ that is subject to the non-linear system of PDE’s called WDVV equation. Our interest in it comes from the fact that every unital CohFT
\( \Lambda_{g,k} \) defines a (formal) Frobenius manifold. Namely the generating function \( F_0 \) restricted to the small phase space is a solution to the WDVV equation and considered as Frobenius potential it defines a certain Frobenius structure:

\[
F^M(t^1, \ldots, t^n) := F_{g=0} |_{t^0,_{\alpha}=0, \, k\geq 1} = F_0,
\]

where we associate \( t^\alpha \) on the LHS with the \( t^{0,\alpha} \) on the RHS.

The formalism of Dubrovin allows us to consider the action of \( A \in \text{SL}(2, \mathbb{C}) \) analytically on the Frobenius manifold of a CohFT and to extend it to the higher genera via the technique of Givental.

**Example 1** (Appendix C in [D]). Consider the potential of the 3-dimensional Frobenius manifold:

\[
F(t) = \frac{1}{2}(t^1)^2 t^3 + \frac{1}{2} t^1 (t^2)^2 - \frac{(t^2)^4}{16} \gamma(t^3).
\]

The WDVV equation on \( F \) is equivalent to the Chazy equation on \( \gamma \):

\[
\gamma''' = 6\gamma\gamma'' - 9(\gamma')^2.
\]

It is a well-known property of the Chazy equation that for \( A \in \text{SL}(2, \mathbb{C}) \) the function \( \gamma^A \) solves the Chazy equation too:

\[
\gamma^A(t) := \gamma \left( \frac{at + b}{ct + d} \right) - \frac{c}{2(ct + d)}.
\]

As the consequence of this property of the Chazy equation we get an \( \text{SL}(2, \mathbb{C}) \) action on the space of 3-dimensional Frobenius manifolds.

1.3. **Definition of the \( \text{SL}(2, \mathbb{C}) \) action.** Consider the action of \( A \in \text{SL}(2, \mathbb{C}) \) on the variable \( t^{0,n} \):

\[
A \cdot t^{0,n} = \frac{a t^{0,n} + b}{ct^{0,n} + d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

We would like to “quantize” this action on the variable \( t^{0,n} \) to the action \( \hat{A} \) on the CohFT partition function. We require it to satisfy the following conditions:

- \( \hat{A} \) acts on the space of partition functions. Namely \( \hat{A} \cdot Z \) is a partition function of some CohFT.
- By the action \( \hat{A} \) the variable \( t^{0,n} \) is transformed by \( t^{0,n} \rightarrow A \cdot t^{0,n} \).

Note that the first requirement of the quantization rules means also that \( \hat{A} \) should act on the space of WDVV solutions too. Namely \( (\hat{A} \cdot Z)_{g=0} \) should solve WDVV.

The potential of a Frobenius manifold can be written in coordinates as:

\[
F(t) = F(t^1, \ldots, t^n) = \frac{1}{2}(t^1)^2 t^n + \frac{1}{2} t^1 \sum_{i=2}^{n-1} t^i t^{n+1-i} + H(t^2, \ldots, t^n),
\]
for some function \( H \) depending on \( t^2, \ldots, t^n \) only.

**Definition 1.1.** Let \( F \) be a Frobenius potential written as in (1). For \( A \in SL(2, \mathbb{C}) \) define the function:

\[
F^A(t) := \frac{1}{2}(t^1)^2 t^n + \frac{1}{2} t^1 \sum_{i=2}^{n-1} t^i t^{n+1-i} + \frac{c}{8(ct^n + d)}(t^2 t^{n-1} + \ldots + t^{n-1} t^2)^2 \\
+ (ct^n + d)^2 H \left( \frac{t^2}{ct^n + d}, \ldots, \frac{t^{n-1}}{ct^n + d}, \frac{at^n + b}{ct^n + d} \right).
\]

In what follows we shows that \( \hat{A} \cdot F := F^A \) can be considered as the quantization of \( SL(2, \mathbb{C}) \) action in genus zero.

1.4. **Givental’s action.** A. Givental introduced in [G] two group actions on the space of partition functions of the CohFT’s. These are now known as the actions of the lower-triangular and upper-triangular Givental groups or \( S \)- and \( R \)-actions respectively.

Let \( \Lambda_{g,k} \) be a CohFT on \((V, \eta)\). Consider

\[
r(z) \in \text{Hom}(V, V) \otimes \mathbb{C}[z], \quad s(z) \in \text{Hom}(V, V) \otimes \mathbb{C}[z^{-1}],
\]

such that \( r(z) + r(-z)^* = 0 \) and \( s(z) + s(-z)^* = 0 \) (where the star means dual w.r.t. \( \eta \)). Following Givental define:

\[
\hat{R} := \exp(\widehat{r(z)}), \quad \hat{S} := \exp(\widehat{s(z)}),
\]

where \( \widehat{r(z)} \) and \( \widehat{s(z)} \) are certain differential operators. The importance of Givental’s action is given by the following proposition.

**Proposition 1.1.** Differential operators \( \hat{R} \) and \( \hat{S} \) obtained by the quantization of the Givental group elements \( R \) and \( S \) act on the space of partition functions of the Cohomological Field Theories.

We can also consider the action of the Givental’s group element \( R \) on the Frobenius manifold \( M \) with the potential \( F(t) \). Doing this we act on the CohFT partition function \( Z \) such that \( F = [\hbar^{-1}] Z \) and take again \( \hat{R} \cdot F := [\hbar^{-1}] \left( \hat{R} \cdot Z \right) \). The function \( \hat{R} \cdot F \) will be potential of some Frobenius manifold that we will denote by \( \hat{R} \cdot M \).

1.5. **\( SL(2, \mathbb{C}) \) group action on the genus \( g \) potential of the CohFT.** Let \( F(t) = F(t^1, \ldots, t^n) \) be the potential of a Frobenius manifold \( M \). In what follows for \( s \in M \) we denote by \( M \mid_{t=s} \) a neighborhood in \( M \) of the point \( s \). The structure of the Frobenius manifold \( M \) at the point \( s \) is given by the expansion of \( F(t) \) at \( t = s \). The first theorem of this paper states:
Theorem 1. Let $A = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}(2, \mathbb{C})$.

a. The function $F^A(t)$ defined in (2) is a solution to the WDVV equation,

b. Let $F$ be analytic at $t = (0, \ldots, 0, A \cdot \tau) \in M$ for some $\tau$ and $M^A$ be the Frobenius manifold defined by $F^A$. Then we have an isomorphism:

$$M^A \mid_{t=(0,\ldots,0,\tau)} \cong \hat{R}^\sigma \cdot M \mid_{t=(0,\ldots,0,A\cdot\tau)}$$

where

$$\hat{R}^\sigma(z) := \exp \left( \begin{array}{ccc} 0 & \ldots & \sigma \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{array} \right) z, \quad \sigma := -c(c\tau + d).$$

Proof. The statements of Propositions 2.1, 4.1 and 4.2 sum up to the proof of the theorem. $\square$

This theorem sets up the bridge between the analytical action of $\text{SL}(2, \mathbb{C})$ on the genus 0 part of a CohFT and the particular Givental’s action on a CohFT partition function. Let $F_g(t) = F_g(t^1, \ldots, t^n)$ be the genus $g$ small phase space potentials of a CohFT. For $g \geq 1$ consider:

$$F_g^A(t) := (ct^1 + d)^2 - 2g F_g \left( \frac{t^1}{ct^1 + d}, \ldots, \frac{t^n}{ct^n + d}, \frac{at^n + b}{ct^n + d}, \delta_{1,g} \log \left( \frac{ct^n + d}{c\tau + d} \right) \right),$$

The following theorem extends the $\text{SL}(2, \mathbb{C})$ action to the higher genera.

Theorem 2. Let $F_g(t)$ be analytic at $p_1 := (0, \ldots, 0, A \cdot \tau)$. Let $Z_{p_1}$ and $(F_g^A)_{p_2}$ be expansions of $\exp(\sum_{g \geq 0} h^g F_g)$ and $F_g^A$ at the points $p_1$ and $p_2 := (0, \ldots, 0, \tau)$ respectively. Then

$$[h]^{g-1} \log \left( \hat{R}^\sigma \cdot Z_{p_1} \right)(\hat{t}) = (c\tau + d)^{2-2g} \left( F_g^A(t) \right)_{p_2}$$

where $\hat{t} = (t^1, (c\tau + d) t^2, \ldots, (c\tau + d) t^{n-1}, (c\tau + d)^2 t^n)$.

This theorem is proved in subsection 5.3.

1.6. $\text{SL}(2, \mathbb{C})$ group action on the total ancestor potential of the singularity. In singularity theory the structure of a Frobenius manifold appears naturally on the base space of a singularity unfolding. It was defined in the early 80s by Kyoji Saito [S]. Given a singularity $W(x)$ one of the main objects of Saito’s theory needed to build up the Frobenius manifold is the so–called primitive form of $W(x)$. The choice of a primitive form for the fixed singularity is generally not unique.

Given a primitive form $\zeta$ of the singularity $W(x)$ one can construct much more than just a Frobenius manifold. Namely one can construct a particular CohFT, called
nowadays Saito–Givental’s CohFT. The partition function of this CohFT is called the total ancestor potential of the singularity $W(x)$ and is denoted by $A_{\zeta}(h,t)$.

For the simple elliptic singularities using certain Givental’s action T.Milanov and Y.Ruan gave in [MR] the formula connecting $A_{\zeta_1,s}$ and $A_{\zeta_2,s'}$ with a two different primitive forms $\zeta_1, \zeta_2$ of the same singularity $W(x)$.

In [BT] the authors proposed certain $SL(2,\mathbb{C})$ action in the form of (2) to write down explicitly the effect of the primitive form change on the Frobenius manifold potential. However this action is written in a completely different form comparing to the formula of Milanov–Ruan and was not extended to the total ancestor potential. Using Theorem 1 we show that the two approaches agree.

**Theorem (Theorem 8).** Let $F$ be the Frobenius manifold potential of a simple elliptic singularity $W(x)$. Then the formula of Milanov–Ruan connecting the total ancestor potentials of $W(x)$ with two different primitive forms is equivalent to the $SL(2,\mathbb{C})$ action on $F$ given by formula (2).

This theorem allows one to use the quasi–modularity of the total ancestor potential of a simple–elliptic singularity and certain mirror symmetry theorem to compute explicitly the genus zero Gromov–Witten potentials of the orbifolds $\mathbb{P}^{1,3,3}_{2}$, $\mathbb{P}^{1,4,4,2}$ and $\mathbb{P}^{1,6,3,2}$. The first one was already computed by Sakate and Takahashi in [ST], while the explicit expression for the other two is new.

1.7. **Organization of the paper.** In section 2 we give an analytical approach to the $SL(2,\mathbb{C})$ action on the genus 0 part of the CohFT. We recall basic facts about Givental’s action in section 3. In section 4 we define an analog of the $SL(2,\mathbb{C})$ action that is later approved by means of graphical calculus to coincide with the analytical approach and also with the action of $\hat{R}^{\sigma}$ as above. We compute $\hat{R}^{\sigma}$ action on a CohFT higher genera data and prove Theorem 2 in section 5. In section 6 we write the $SL(2,\mathbb{C})$ action in terms of Givental’s group action only. We discuss singularity theory applications in section 7 giving the proof of Theorem 8. Using this singularity theory result we compute the genus zero Gromov–Witten potentials of the orbifolds $\mathbb{P}^{1,4,4,2}$ and $\mathbb{P}^{1,6,3,2}$.

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2. **Analytical quantization of $SL(2,\mathbb{C})$ action**

In this section we develop an analytical approach to the quantization of the $SL(2,\mathbb{C})$ action on the CohFT via the genus zero data of it — Frobenius manifold.
2.1. Frobenius manifolds. Let $M$ be a domain in $\mathbb{C}^n$. Assume its tangent space $\mathcal{T}_M$ to be endowed with the constant non-degenerate bilinear form $\eta$,

$$\eta : \mathcal{T}_M \times \mathcal{T}_M \to \mathbb{C}.$$ 

Let $t^1, \ldots, t^n$ be coordinates on $M$. We associate the basis of $\mathcal{T}_M$ with the vectors $\partial/\partial t^i$ and consider $\eta_{pq}$ as components of $\eta$ in this basis.

Consider the function $F(t) = F(t^1, \ldots, t^n)$ on $M$, represented by a convergent power series in $t^1, \ldots, t^n$. The function $F(t)$ is said to satisfy WDVV equation if for every fixed $1 \leq i, j, k, l \leq n$ holds:

$$\sum_{p,q} \frac{\partial^3 F}{\partial t^i \partial t^j \partial t^p} \eta^{pq} \frac{\partial^3 F}{\partial t^q \partial t^k \partial t^l} = \sum_{p,q} \frac{\partial^3 F}{\partial t^i \partial t^k \partial t^p} \eta^{pq} \frac{\partial^3 F}{\partial t^q \partial t^j \partial t^l},$$

where $\eta^{pq} = (\eta^{-1})^{p,q}$.

Using the function $F$ define an algebra structure on $\mathcal{T}_M$. Let $c^k_{ij}(t)$ be the structure constants of the multiplication $\circ : \mathcal{T}_M \times \mathcal{T}_M \to \mathcal{T}_M$ defined by $c^k_{ij}(t) := \sum_p c_{ijp}(t)\eta^{pk}$, where

$$c_{ijk}(t) := \frac{\partial^3 F}{\partial t^i \partial t^j \partial t^k}, \quad 1 \leq i, j, k \leq n,$$

and $\eta^{ij} := \sum_{p,q} \eta_{pq} \delta^{pi} \delta^{qj}$. The structure constants $c^k_{ij}(t)$ define a commutative algebra structure by the construction, while the associativity is equivalent to the WDVV equation on $F(t)$.

Assume in addition that $F(t)$ is such that $\partial/\partial t^1$ is the unit of the algebra. Therefore we have:

$$\eta_{ij} = \frac{\partial^3 F}{\partial t^1 \partial t^i \partial t^j}.$$ 

Then $\eta_{ij}$ together with $c^k_{ij}(t)$ define the Frobenius algebra structure:

$$\eta \left( \frac{\partial}{\partial t^i} \circ \frac{\partial}{\partial t^j} \circ \frac{\partial}{\partial t^k} \right) = \eta \left( \frac{\partial}{\partial t^1} \circ \frac{\partial}{\partial t^i} \circ \frac{\partial}{\partial t^k} \right).$$

Note that different points of the Frobenius manifold $M$ give generically different potentials $F$.

**Definition 2.1.** The data $\eta$ and $\circ$ satisfying conditions as above define a Frobenius manifold structure on $M$. The function $F$ is called Frobenius potential of $M$ and the coordinates $t$ – flat coordinates.

**Remark 2.1.** Usually the potential of a Frobenius manifold is assumed to satisfy certain quasi–homogeneity condition w.r.t. a specially chosen vector field called Euler vector field. We do not make such an assumption here, working with a larger class of the Frobenius manifolds.
Sometimes we are given first the function $F$ satisfying WDVV equation without any underlying space $M$ and holomorphicity property. In these occasions $F$ could anyway define a Frobenius manifold structure that is called sometimes formal. We will drop this word assuming it to be clear from the context.

**Definition 2.2.** Two Frobenius manifold $M_1$ and $M_2$ are called (locally) isomorphic if there is a diffeomorphism $\phi : M_1 \to M_2$ such that for some fixed $t \in M_1$ and $\phi(t) \in M_2$ holds:
- $\phi$ is linear conformal transformation of the metrics of $M_1$ and $M_2$,
- the differential of $\phi$ is an isomorphism of the algebras $T_tM_1$ and $T_{\phi(t)}M_2$.

In this case we write $M_1 \mid _t \cong M_2 \mid _{\phi(t)}$.

In what follows we assume the scalar product of the Frobenius manifold to have a particular form: $\eta_{ij} = \delta_{i+j,n+1}$. Note however that our results are easily translated to the arbitrary choice of $\eta$. Choosing appropriate coordinates $F$ can be written as follows:

$$F(t) = F(t^1, \ldots, t^n) = \frac{1}{2} t^n + \frac{1}{2} \sum_{i=2}^{n-1} t^i t^{n+1-i} + H(t^2, \ldots, t^n),$$

for some function $H(t^2, \ldots, t^n)$ that does not depend on $t^1$. Motivated by this presentation of the potential we introduce the notation for the summands of $F^A$ defined in (2).

**Notation 2.1.** Consider the formula (2). We adopt the notation:

- $H$–term of $F^A := (ct^n + d)^2 H \left( \frac{t^2}{ct^n + d}, \ldots, \frac{t^{n-1}}{ct^n + d}, \frac{a t^n + b}{ct^n + d} \right)$
- $Q$–term of $F^A := \frac{c}{8(ct^n + d)} \left( \sum_{k \neq 1} t^k t^{n+1-k} \right)^2$.

Hence $F^A = H$–term + $Q$–term + cubic terms.

It was noted by Dubrovin (cf. Appendix B in [D]) that there is a non-trivial symmetry of the WDVV equation such that for $F(t)$ as above the function $F^I(t)$ solves WDVV too:

$$F^I(t^I) = (t^n)^{-2} \left( F(t) - \frac{1}{2} t^1 \sum_{i=0}^{n} t^i t^{n+1-i} \right).$$

where for $1 < \alpha < n$:

$$\hat{t}^1 := \frac{1}{2} \frac{\sum_{i} t^i t^{n+1-i}}{t^n}, \hat{t}^\alpha := \frac{t^\alpha}{t^n}, \hat{t}^n := -\frac{1}{t^n}.$$
We will call it “Inversion transformation” and write:

\[ F^I(\hat{t}) = \hat{I} \cdot F(t). \]

2.2. Analytical quantization via composition. Consider \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C}) \).

The action of \( A \) on the variable \( t^n \) can be decomposed as follows.

\[ A \cdot t^n = \frac{at^n + b}{ct^n + d} = T_2 \cdot S_{c^2} \cdot I \cdot T_1 \cdot t^n, \]

where we have:

- \( T_1 \) is one more shift \( t^n \rightarrow t^n + \frac{a}{c} \),
- \( S_{c^2} \) is the scaling \( t^n \rightarrow c^2 t^n \),
- \( T_2 \) is the shift \( t^n \rightarrow t^n + \frac{d}{c} \),
- \( I : t^n \rightarrow -\frac{1}{t^n} \).

The shifts \( T_i \) can be easily quantized to the operators on the space of the WDVV equation solutions because linear change of variables preserves the WDVV equation. More complicated is the quantization of the scaling \( S_{c^2} \). Let \( F \) be a Frobenius manifold potential. The operator \( \hat{S}_c \) acts in the following way.

\[ \hat{S}_c \cdot F(t) = \frac{1}{2} (t^1)^2 t^n + \sum_{p=2}^{n-1} t^p t^{n+1-p} + c H(t^2, \ldots, t^{n-1}, ct^n). \]

Finally, the action of \( I \) is quantized by the Inversion transformation \( \hat{I} \).

**Definition 2.3.** Define the action \( \hat{A} \) on the space of WDVV solutions:

\[ \hat{A} = \hat{T}_2 \cdot \hat{S}_c \cdot \hat{I} \cdot \hat{S}_{c^{-1}} \cdot \hat{T}_1. \]

**Proposition 2.1.** For the \( \hat{A} \) as above we have:

- The action \( \hat{A} \) satisfies the quantization condition in genus 0.
- The action \( \hat{A} \) agrees with the formula (2) up to quadratic terms:

\[ \hat{A} \cdot F = F^A + \text{quadratic terms}. \]

**Proof.** The quantization presented coincides with action of \( A \) on \( t^n \) by the construction and also \( \hat{A} \) acts on the space of WDVV solutions as the composition of operators acting of the space of WDVV solutions.

For the second part note that applying \( \hat{S}_c \cdot \hat{I} \cdot \hat{S}_{c^{-1}} \) to \( H(t^2, \ldots, t^n) \) both \( S_c \) and \( S_{c^{-1}} \) add their factors that cancel out. Hence for the \( H \)-terms the proposition follows. For
the Q-terms we have:

\[ Q \text{- terms of } \hat{S}_c \cdot \hat{I} \cdot \hat{S}_{c-1} \cdot \hat{T}_1 = \frac{c \left( \sum_{p \neq 1, n} t_p t^{n-p} \right)^2}{ct_n}, \]

where the factor \( c \) comes twice because of the quantization rule of \( \hat{S}_c \). In the denominator because \( t^n \) is multiplied by \( c \) and in the nominator because the function \( H \) in the presentation (1) is multiplied by \( c \). At the same time by the definition of \( \hat{I} \) the action of \( \hat{S}_{c-1} \cdot \hat{T}_1 \) does not affect these terms.

Applying the shift \( t^n \rightarrow t^n + \frac{d}{t} \) we get exactly the Q-term of \( F^A \).

This proves a-part of Theorem 1.

2.3. Example: Gromov–Witten theory of \( \mathbb{P}^1_{2,2,2,2} \). Consider the so-called theta–constants \( \vartheta_2(\tau), \vartheta_3(\tau), \vartheta_4(\tau) \) that are the values at \( z = 0 \) of the Jacobi theta functions \( \vartheta_k(z, \tau) \):

\[ \vartheta_2(\tau) = \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2}, \quad \vartheta_3(\tau) = \sum_{n=-\infty}^{\infty} q^n, \]

\[ \vartheta_4(\tau) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}, \quad q := \exp(\pi \sqrt{-1} \tau). \]

Let \( X^\infty_k(\tau) \) and \( X^\infty_k(q) \) be the logarithmic derivatives of the theta–constants:

\[ X^\infty_k(\tau) := 2 \frac{\partial}{\partial \tau} \log \vartheta_k(\tau), \quad X^\infty_k(q) := \frac{1}{\pi \sqrt{-1}} X^\infty_k \left( \frac{\tau}{\pi \sqrt{-1}} \right). \]

The functions \( \vartheta_k(\tau) \) and \( X^\infty_k(\tau) \) are examples of the modular and quasi–modular forms.

**Definition 2.4.** Let \( \Gamma \subset \text{SL}(2, \mathbb{Z}) \) be a finite index subgroup. Let \( k \in \mathbb{N}_{\geq 0} \) and \( f(\tau) \) — a holomorphic function on \( \mathbb{H} \).

- \( f(\tau) \) is called a modular form of weight \( k \) if it satisfies the following condition:

  \[ \frac{1}{(ct+d)^k} f \left( \frac{a \tau + b}{ct+d} \right) = f(\tau) \text{ for any } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \]

- \( f(\tau) \) is called a quasi-modular form of weight \( k \) and depth \( m \) if there are functions \( f_0(\tau), \ldots, f_m(\tau) \), holomorphic in \( \mathbb{H} \) s.t.:

  \[ \frac{1}{(ct+d)^k} f \left( \frac{a \tau + b}{ct+d} \right) = \sum_{l=0}^{m} f_k(\tau) \left( \frac{c}{ct+d} \right)^k \text{ for any } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \]

It’s clear that every modular form is quasi–modular too with all functions \( f_1, \ldots, f_m \) identically zero.
Proposition 2.2 (cf. [Z]). The functions \((\vartheta_2(\tau))^2, (\vartheta_3(\tau))^2, (\vartheta_4(\tau))^2\) are modular forms of weight 1 on \(\Gamma(2) = \{ A \in SL(2, \mathbb{Z}) \mid A \equiv 1 \, \text{mod} \, 2 \} \).

Frobenius manifold potential of the Gromov–Witten theory of the orbifold \(\mathbb{P}^1_{2,2,2,2}\) was found in [ST] by I.Satake and A.Takahashi. It reads:

\[
F_{0}^{\mathbb{P}^1_{2,2,2,2}}(t_0, t_1, t_2, t_3, t_4, \tau) = \frac{1}{2} t_0^2 \tau + \frac{1}{4} t_0 \left( \sum_{i=1}^{4} t_i^2 \right) + \frac{1}{8} (t_1 t_2 t_3 t_4) (X_3^\infty(q) - X_4^\infty(q))
- \frac{1}{192} \left( \sum_{i=1}^{4} t_i^4 \right) (4X_2^\infty(q) + X_3^\infty(q) + X_4^\infty(q)) - \frac{1}{32} \left( \sum_{i<j} t_i^2 t_j^2 \right) (X_3^\infty(q) + X_4^\infty(q)).
\]

The modularity of \((\vartheta_k)^2\) allows us to consider the action of \(\Gamma(2)\) on the coefficients of the function \(F_{0}^{\mathbb{P}^1_{2,2,2,2}}\). The (a)–part of Theorem 1 extends this action to the action on the Frobenius manifold potential.

Proposition 2.3. Let \(A \in \Gamma(2)\), consider the action of it by formula (2). Then we have:

\[
\left( F_{0}^{\mathbb{P}^1_{2,2,2,2}} \right)^A = F_{0}^{\mathbb{P}^1_{2,2,2,2}}.
\]

Proof. It’s easy to see from the proposition above that the functions \(X_k^\infty(\tau)\) are quasi–modular of weight 2. Hence the coefficient \([t_1 t_2 t_3 t_4]F_{0}^{\mathbb{P}^1_{2,2,2,2}}\), considered as the function in \(\tau\) is modular of weight 2 on \(\Gamma(2)\) and the coefficients \([t_i^4]F_{0}^{\mathbb{P}^1_{2,2,2,2}}, [t_i^2 t_j^2]F_{0}^{\mathbb{P}^1_{2,2,2,2}}\) are quasi–modular of weight 2 on \(\Gamma(2)\). Some easy computations show that by the transformation \(\tau \rightarrow A \cdot \tau\) this quasi–modularity gives exactly additional summands annihilating the \(Q\)–terms of the function \(\left( F_{0}^{\mathbb{P}^1_{2,2,2,2}} \right)^A\).

\(\square\)

It’s important to note that taking \(A \in SL(2, \mathbb{Z})\) such that \(A \not\in \Gamma(2)\) the genus zero potential of \(\mathbb{P}^1_{2,2,2,2}\) is transformed differently. For example the following formulae are straightforward from the definition of \(X_k^\infty(\tau)\):

\[
X_2^\infty(\tau + 1) = X_2^\infty(\tau), \quad X_3^\infty(\tau + 1) = X_4^\infty(\tau), \quad X_4^\infty(\tau + 1) = X_3^\infty(\tau),
\]

\[
X_2\left( -\frac{1}{\tau} \right) = \tau^2 X_4(\tau) + \tau, \quad X_3\left( -\frac{1}{\tau} \right) = \tau^2 X_3(\tau) + \tau, \quad X_4\left( -\frac{1}{\tau} \right) = \tau^2 X_2(\tau) + \tau.
\]

Hence taking \(A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\) or \(A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\) we get \(\left( F_{0}^{\mathbb{P}^1_{2,2,2,2}}(t) \right)^A = F_{0}^{\mathbb{P}^1_{2,2,2,2}}(\hat{t})\) for \(\hat{t}\) differing from \(t\) by a permutation of the variables \(t_1, \ldots, t_4\). Hence the Frobenius manifold obtained by such an action is still isomorphic to the initial one. We can not expect such a behaviour for an arbitrary \(A \in SL(2, \mathbb{C})\).
2.4. Example: Hurwitz–Frobenius manifolds. Consider the space of meromorphic functions $\lambda : C \to \mathbb{P}^1$ on the compact genus $g$ Riemann surface $C$. Fix the pole orders of $\lambda$ to be $k := \{k_1, \ldots, k_m\}$:

$$\lambda^{-1}(\infty) = \{\infty_1, \ldots, \infty_m\}, \quad \infty_p \in C,$$

so that locally at $\infty_p$ we have $\lambda(z) = z^{k_p}$.

Such meromorphic functions define the ramified coverings of $\mathbb{P}^1$ by $C$ with the ramification profile $k$ over $\infty$. Assume further that $\lambda$ has only simple ramification points at $P_i \in \mathbb{P}^1 \setminus \{0\}$. On the space of the pairs $(C, \lambda)$, considered up to a certain equivalence, B. Dubrovin introduced in [D, Lecture 6] a Frobenius manifold structure that is now known under the name Hurwitz–Frobenius manifold and is denoted by $H_{g,k}$.

When $g = 1$ the ramified covering $\lambda$ is written via the elliptic functions and one of the parameters of it (and hence of the Hurwitz–Frobenius manifold) is $\tau \in \mathbb{H}$, that stands for the modulus of an elliptic curve. For $k = \{2, 2, 2, 2\}$ it has the following form:

$$\lambda(z) = \sum_{i=1}^{4} \left( \wp(z - a_i, \tau)u_i + \frac{1}{2} \wp'(z - a_i, \tau)s_i \right) + c,$$

where $\wp(z, \tau)$ is the Weierstrass function and $a_i, u_i, s_i, c$ are complex parameters. The corresponding Frobenius potential is also written in terms of a certain quasi–modular forms (see [P2]). One can consider the Frobenius manifold structure on $H_{1,k}$ at different points $p_1$ and $p_2$. Because one of the parameters of $p_1$ and $p_2$ is the modulus of the corresponding elliptic curve, it’s natural for two to be connected by a certain $\text{SL}(2, \mathbb{C})$ action.

Given a Frobenius potential $F_1(t)$ encoding the algebra structure at $p_1$ one can consider $F_2(t) := F_1(t + p_2 - p_1)$. However such a shift applied to the function $f(z)$, holomorphic in $\mathbb{H}$ (like for example the functions $X_k^\infty(\tau)$), reduces drastically the domain of the holomorphy. In order to keep the domain of holomorphy big, one should apply not the Taylor series shift, but the following action instead (Z):

$$f(z) \to f \left( \frac{\tau_0 - \bar{\tau}_0 z}{1 - z} \right).$$

This action can be realized by the composition of the rescaling and the $\text{SL}(2, \mathbb{C})$ action developed above.

3. Cohomological field theories and Givental’s action

We briefly recall some basic facts of the CohFTs and Givental’s action on the partition function of it. We introduce Givental’s action in two forms – via the symplectic formalism and in the infinitesimal form. The first one is used in the last section and
is more natural from the point of view of singularity theory. The second approach is more explicit and allows us to make explicit computations via the graphical calculus introduced in [DBSS].

3.1. **Cohomological Field Theory axioms.** Let \((V, \eta)\) be a finite–dimensional vector space with a non–degenerate bilinear form on it. Consider a system of linear maps

\[ \Lambda_{g,k} : V^\otimes k \to H^*(\mathcal{M}_{g,k}) \]

defined for all \(g,k\) such that \(\mathcal{M}_{g,k}\) exists. It is called Cohomological field theory on \((V, \eta)\) if it satisfies the following axioms.

**A1:** \(\Lambda_{g,k}\) is equivariant w.r.t. the \(S_k\)–action permuting the factors in the tensor product and the numbering of marked points in \(\mathcal{M}_{g,k}\).

**A2:** For the gluing morphism \(\rho : \mathcal{M}_{g_1,k_1+1} \times \mathcal{M}_{g_2,k_2+1} \to \mathcal{M}_{g_1+g_2,k_1+k_2}\) we have:

\[ \rho^* \Lambda_{g_1+g_2,k_1+k_2} = (\Lambda_{g_1,k_1+1} \cdot \Lambda_{g_2,k_2+1}, \eta^{-1}), \]

where we contract with \(\eta^{-1}\) the factors of \(V\) that correspond to the node in the preimage of \(\rho\).

**A3:** For the gluing morphism \(\sigma : \mathcal{M}_{g,k+2} \to \mathcal{M}_{g+1,k}\) we have:

\[ \sigma^* \Lambda_{g+1,k} = (\Lambda_{g,k+2}, \eta^{-1}), \]

where we contract with \(\eta^{-1}\) the factors of \(V\) that correspond to the node in the preimage of \(\sigma\).

In this paper we further assume that the CohFT \(\Lambda_{g,k}\) is unital — there is a fixed vector \(\mathbf{1} \in V\) called unit such that the following axioms are satisfied.

**U1:** For every \(a, b \in V\) we have: \(\eta(a, b) = \Lambda_{0,3}(\mathbf{1}, a, b)\).

**U2:** Let \(\pi : \mathcal{M}_{g,k+1} \to \mathcal{M}_{g,k}\) be the map forgetting the last marking, then:

\[ \pi^* \Lambda_{g,k}(a_1 \otimes \cdots \otimes a_n) = \Lambda_{g,k+1}(a_1 \otimes \cdots \otimes a_k \otimes \mathbf{1}). \]

In what follows we will denote the CohFT just by \(\Lambda\) rather than \(\Lambda_{g,k}\) when there is no ambiguity.

The correlators of an arbitrary CohFT satisfy the following equations: Dilaton equation:

\[ \left\langle \tau_1(e_1) \prod_{k=1}^l \tau_{d_k}(e_{i_k}) \right\rangle_g = (2g - 2 + l) \left\langle \prod_{k=1}^l \tau_{d_k}(e_{i_k}) \right\rangle_g. \]  \hspace{1cm} (6)

and String equation:

\[ \left\langle \tau_0(e_1) \prod_{k=1}^l \tau_{d_k}(e_{i_k}) \right\rangle_g = \sum_{m=1, d_m \neq 0} \left\langle \prod_{k \neq m} \tau_{d_k}(e_{i_k}) \cdot \tau_{d_m-1}(e_{i_m}) \right\rangle_g. \]  \hspace{1cm} (7)
Example 2. Let $V$ be 1-dimensional vector space generated by $e_1$. The map
$$\alpha_{g,k} : V^\otimes k \to 1 \in H^*(\overline{M}_{g,k})$$
satisfies the axioms of the unital CohFT and is called trivial CohFT.

3.2. Symplectic geometry of Givental’s action. For every CohFT $\Lambda_{g,k}$ on the vector space $(V, \eta)$ Givental introduced in [G] the following symplectic formalism. Consider the space of formal Laurent series $H := V((z))$ with the symplectic structure $\Omega$:
$$\Omega(f(z), g(z)) := \text{res}_{z=0} \eta(f(-z), g(z)) \, dz, \quad f, g \in H.$$ 
There is a natural polarization $H = H_q \oplus H_p$ for $H_q = V[z]$ and $H_p = z^{-1}V[[z^{-1}]]$. Together with the symplectic form $\Omega$ the space $H$ can be identified with the cotangent bundle $T^*H_q$.

Let the vectors $\partial_k$ for $1 \leq k \leq n$ build up the basis of $V$ and $dt_k$ constitute the dual basis w.r.t. $\eta$. For any $f \in H$ define $p_{k,i}$ and $q^{i}_{k}$ to be Darboux coordinates on $H$ by the following equation:
$$f(z) = \sum_{k=0}^{\infty} \sum_{i=1}^{n} \left( q^{i}_{k} \partial_{k} z^k + p_{k,i} dt_i (-z)^{-k-1} \right).$$
In the coordinates $\{p_{k,i}, q^{i}_{k}\}$ the symplectic form takes the canonical form $\Omega = \sum_{i,k} dp_{k,i} \wedge dq^{i}_{k}$.

3.2.1. Quantization of quadratic Hamiltonians. Let $A(z)$ be an infinitesimal symplectic operator on $H$:
$$\Omega(A \cdot f, g) + \Omega(f, A \cdot g) = 0.$$ 
Givental associates to it a quadratic Hamiltonian $P(A)$ by the following equation:
$$P(A)(f) = \frac{1}{2} \Omega(A \cdot f, f), \quad \forall f \in H.$$ 
Quadratic Hamiltonian $P$ can be quantized to a differential operator $\hat{P}$ by the following Weyl quantization rules:
$$\hat{1} = 1, \quad \hat{\partial}_k = \sqrt{h} \partial_{q^{i}_k}, \quad \hat{q}^{i}_k = \frac{q^{i}_k}{\sqrt{h}},$$ 
$$(p_{k,i} p_{l,j})^\wedge = h \frac{\partial^2}{\partial q^{i}_k \partial q^{j}_l}, \quad (p_{k,i} q^{j}_l)^\wedge = q^{i}_k \partial_{q^{j}_l}, \quad (q^{i}_k q^{j}_l)^\wedge = \frac{1}{h} q^{i}_k q^{j}_l.$$ 
This allows one to define the quantization of $\exp(A(z))$ by the formula:
$$\exp(A(z))^\wedge := \exp(\hat{P}(A)).$$
3.2.2. CohFT partition function as the function on $\mathcal{H}_q$. Let $Z(t)$ be the partition function of the CohFT $\Lambda$. We can consider $Z$ as the function of $\mathcal{H}_q$ by applying the so-called dilaton shift coordinate transformation:

$$t^{d,i} = q^i_d + \delta_{d,1}\delta_{i,1}.$$ 

Because of this identification the action of $\hat{R}$ arising from the quantized quadratic Hamiltonian $P(r)$ can be applied to the partition function $Z$ too.

The quantization procedure is of big importance for us because of the following theorem essentially due to Givental.

**Theorem 3 ([K][L][FSZ]).** Let $R = R(z) \in \text{Hom}(V,V)[[z]]$ be such that $R^*(-z)R(z) = 1$. Then the quantized operator $\hat{R}$ acts on the space of partition functions of the cohomological field theories.

The following proposition due to Givental gives explicit formula for a $\hat{R}$–action in the symplectic setting.

**Proposition 3.1 (Proposition 7.3 in [G]).** Let $R = R(z) \in \text{Hom}(V,V)[[z]]$ be such that $R^*(-z)R(z) = 1$. The action of $\hat{R}$ on $F(q) \in \mathbb{C}[[q]]$ reads:

$$\hat{R} \cdot F(q) := \left(\exp\left(\frac{\hbar}{2} \sum_{k,l} \eta(\partial^a, V^a_{k,l}\partial^b)\partial_{q^a_k}\partial_{q^b_l}\right)F(q)\right)|_{q \rightarrow \frac{R^{-1}q}{R(z)}}.$$ 

for the operators $V^a_{k,l} \in \text{Hom}(V)$ defined by:

$$\sum_{k,l=0}^{\infty} V^a_{k,l}(-z)^k(-w)^l = \frac{R^*(z)R(w) - 1}{z+w}.$$ 

3.3. Inifinitesimal version of Givental’s action. In this subsection we introduce Givental’s group action on the partition function of a CohFT via the inifinitesimal action computed by Y.P. Lee [L]. Let $\Lambda_{g,k}$ be a unital CohFT on $(V,\eta)$ with the unit $e_1 \in V$.

3.3.1. Upper–triangular group. Consider

$$r(z) = \sum_{l \geq 1} r_l z^l \in \text{Hom}(V,V) \otimes \mathbb{C}[z],$$ 

such that $r(z) + r(-z)^* = 0$ (where the star means dual w.r.t. $\eta$). Following Givental, we define:

$$\hat{R} := \exp(\sum_{l=1}^{\infty} \widehat{r_l} z^l),$$
where
\[
\hat{r}_{l}z^{l} := -(r_{l})_{1}^{\alpha} \frac{\partial}{\partial t_{l+1,\alpha}} + \sum_{d=0}^{\infty} t^{d,\beta} (r_{l})_{\beta}^{\alpha} \frac{\partial}{\partial t_{d+l,\alpha}} + \frac{\hbar}{2} \sum_{i+j=l-1} (-1)^{i+1} (r_{l})_{\alpha,\beta} \frac{\partial^{2}}{\partial t_{i,\alpha} t_{j,\beta}},
\]
and \((r_{l})_{\alpha,\beta} = (r_{l})_{\sigma}^{\eta,\sigma,\beta}\).

**Definition 3.1.** The action of the differential operator \(\hat{R}\) on the partition function of the CohFT is called Givental’s \(\hat{R}\)–action or upper–triangular Givental’s group action.

**3.3.2. Lower–triangular group.** Consider
\[
s(z) = \sum_{l \geq 1} s_{l}z^{-l} \in \text{Hom}(V, V) \otimes C[z^{-1}],
\]
such that \(s(z) + s(-z)^{*} = 0\). Following Givental, we define:
\[
\hat{S} := \exp(\sum_{l=1}^{\infty} s_{l}z^{-l}),
\]
where
\[
\sum_{l=1}^{\infty} (s_{l}z^{-l})^{*} = -(s_{1})_{1}^{\eta} \frac{\partial}{\partial t_{0,\alpha}} + \frac{1}{\hbar} \sum_{d=0}^{\infty} (s_{d+2})_{1,\alpha} t^{d,\alpha} + \sum_{d=0}^{\infty} (s_{1})_{\beta}^{\alpha} t_{d+1,\beta}^{d,\alpha} + \frac{1}{2\hbar} \sum_{d_{1},d_{2},\alpha_{1},\alpha_{2}} (-1)^{d_{1}} (s_{d_{1}+d_{2}+1})_{\alpha_{1},\alpha_{2}} t_{d_{1},\alpha_{1}} t_{d_{2},\alpha_{2}}.
\]

In what follows we also need the \(\hat{S}_{0}\) action, which has to be treated exclusively. It is actually clear that for \(s_{0} \in \text{Hom}(V, V)\) such that \(s_{0} + s_{0}^{*} = 0\) the action of \(S_{0} := \exp(s_{0})\) defined by:
\[
\tilde{t}^{\alpha} = (S_{0})_{\beta}^{\alpha} t^{\beta}
\]
preserves the WDVV equation. We will comment more on the \(S_{0}\) action on the full CohFT partition function later.

**Definition 3.2.** The action of the differential operator \(\hat{S}\) on the partition function of the CohFT is called Givental’s \(\hat{S}\)–action or lower–triangular Givental’s group action.

The action of the differential operators \(\hat{R}\) and \(\hat{S}\) as above are indeed just another language of the same action introduced via the quantization of quadratic Hamiltonians. However there is a small conventional difference in the two definitions that will play an important role in the last section of this paper.
Proposition 3.2. Let $R = \exp\left(\sum_k r_k z^k\right)$ be an element of the upper-triangular group of Givental. Then symplectic form of the action (8) is equivalent to the infinitesimal form of the action of $\tilde{R} := \exp\left(-\sum_k r_k z^k\right)$.

Proof. The operator $\hat{R}$ can be decomposed as follows (see Section 2 in [DbSS]):

\begin{equation}
\exp\left(\sum_k r_k z^k\right) = \exp\left(\sum_{d \geq 0, l \geq 1} t^{d,\mu}(r_l)^{\mu} \frac{\partial}{\partial t^{d+l,\mu}}\right) \exp\left(\sum_{k,l \geq 0} (V_{k,l})^{\mu\nu} \frac{\partial^2}{\partial t^k,\mu \partial t^l,\nu}\right),
\end{equation}

where $V_{k,l}$ is defined by:

$$
\sum_{k,l} V_{k,l} z^k w^l = \frac{h \exp (-r(-z)) \exp (r(w)) - 1}{2 (z + w)}.
$$

It’s not hard to see that the first multiple in equation (9) corresponds to the (dilaton-shifted) change of variables and the second multiple is exactly certain quadratic differential operator.

The differential operator of equation (9) defined by $V_{k,l}$ is easily matched with the action of the operator $V_{k,l}^s$ in the formula (8). Note however that these two are related by a sign change: $V_{k,l}^s = -V_{k,l}$.

Another difference of the formulas (9) and (8) is use of the inverse operator $R^{-1}$ in the latter one. Because for an infinitesimal version the inverse is obtained by changing the sign we get the proposition. □

3.4. Givental graph. We briefly introduce here the construction of [DbSS] for the graphical computation of the Givental’s $R$–action on a CohFT partition function. From now on let us consider particular upper–triangular group element $R = \exp(\sum r_l z^l)$ acting on the partition function of the fixed unital CohFT $\Lambda_{g,k}$.

Let $\gamma$ be a connected graph with leaves and oriented edges. Introduce the decoration of it:

3.4.1. Leaves. Leaves are decorated either with the vector:

$$
L := \exp\left(\sum_{l=1}^{\infty} r_l z^l\right) \left(\sum_{d=0}^{\infty} \sum_{\alpha=1}^{n} e^{\alpha} t^{d,\alpha} z^d\right) \in V[[z]],
$$

or with the vector:

$$
L_0 := -z \left(\exp\left(\sum_{l=1}^{\infty} r_l z^l\right) - 1\right) (e_1) \in V[[z]].
$$

\footnote{See also [DbOSS] for the more general framework.}
3.4.2. **Edges.** Let every edge be oriented and consist of two leaves. The decoration of the edge is given by \( \mathcal{E} \in V[[z]] \otimes V[[w]] \). The factor of \( V \) with the variable \( z \) corresponds to the input leaf and the factor with the variable \( w \) — to the output leaf of the edge. For \( \tilde{\mathcal{E}} \in \text{Hom}(V, V)[z, w] \) given by:

\[
\tilde{\mathcal{E}} := -\hbar \exp(-r(z)) \exp(r(w)) - 1 \frac{z + w}{z + w},
\]

define \( \mathcal{E} \) by the equality:

\[
\mathcal{E} = \tilde{\mathcal{E}} \eta.
\]

3.4.3. **Vertices.** The vertex with the valence \( n \) is decorated with:

\[
\mathcal{V}[n] := \sum_{g \geq 0} \hbar^{g-1} \mathcal{V}_g[n],
\]

where the tensor \( \mathcal{V}_g[n] \in (V^*[z])^\otimes n \) is defined by its values on the basis:

\[
\mathcal{V}_g[n] (e_{\alpha_1}z^{d_1} \otimes \cdots \otimes e_{\alpha_n}z^{d_n}) := \langle \tau_{d_1}(\alpha_1) \cdots \tau_{d_n}(\alpha_n) \rangle_g.
\]

The correlators taken are those of the CohFT fixed.

3.4.4. **Contraction of tensors.** Let \( \Gamma \) be the set of all connected graphs. Introduce an orientation of its edges. Denote by \( V(\gamma) \) the set of vertices of the graph \( \gamma \) and for \( v \in V(\gamma) \) denote by \( H(v) \) the set of half-edges and leaves adjacent to \( v \). Denote by \( \mathcal{V}_v \) the decoration of the vertex \( v \). Denote by \( D(h) \) for \( h \in H(v) \) the decoration of \( h \). It is either \( \mathcal{E}, \mathcal{L}_0 \) or \( \mathcal{E}_v \).

**Theorem 4** (Section 2 in [DbSS]).

\[
\log(\hat{R} \cdot Z) = \sum_{\gamma \in \Gamma} \frac{1}{\text{Aut}(\gamma)} \prod_{v \in V(\gamma)} \mathcal{V}_v \left( \bigotimes_{h \in H(v)} D(h) \right).
\]

**Remark 3.1.** The orientation of the graph edges does not appear to contribute to the contraction of tensors — \( \mathcal{V}_v \). However it plays important role in the automorphisms counting.

Note that the decoration of the graph has to be consistent in order to give non-zero contribution to the the formula above. For example contracting the tensors at the particular vertex only one summand \( \mathcal{V}_g \) turns out to give a non-zero value. We will use this fact later computing the \( R \)-action explicitly.
3.5. Givental’s action on the small phase space. Let $Z':=\hat{R} \cdot Z$. We are interested in the functions:

$$F_g' = [\hbar^g - 1] \log (\hat{R} \cdot Z) |_{t^k \geq 1, \alpha = 0},$$

where $[\hbar^g - 1]$ means taking the coefficient of $\hbar^g - 1$ in the formal series. Recall that the action of $\hat{R}$ is given by the exponentiation of the differential operator $\hat{r}(z)$. To compute $F_g'$ it is not enough to consider the operator $\hat{r}(z)$ restricted to the small phase space. The first problem is that in $\hat{R}$ the differential operator $\hat{r}$ can act on itself — Givental’s action $r(z)$ introduces higher order variables and can “eat” them by a subsequent differentiation. The second problem is that only the derivatives of $\hat{F}_g$ and not $F_g$ are used by a Givental’s action.

The second problem can not be resolved, but the first one can be simplified when applying Givental graph calculus. To compute $Z'$ on the small phase space we should consider the restriction of the decoration $\mathcal{L}$ to the small phase space.

**Proposition 3.3.** Let $Z':=\hat{R} \cdot Z$. In order to compute the restriction of $Z'$ to the small phase space it is enough to consider the graphs with the leaves decorated by

$$\mathcal{L}_s = \mathcal{L} |_{t^k, \alpha = 0, k \geq 1}.$$

**Proof.** It’s clear from formula (9) and the definition of the graphical calculus that these are indeed the leaves, that “introduce” the variables of a $R$–transformed partition function. □

4. **Givental’s quantization of the SL(2, C) action**

The expression of $\hat{A}$ given in Proposition 2.1 motivates considering the following “composition” action $\hat{A}_C$ defined in a similar way. Define:

$$\hat{A}_C := \hat{S}^{-\beta} \cdot \hat{S}_0^c \cdot \hat{R}^1 \cdot \hat{S}_0^c \cdot \hat{S}_0^\alpha,$$

where $\alpha = \frac{a+1}{c}$ and $\beta = -\frac{d+1}{c}$, $S^a = S^a(z) = \exp(s^a z^{-1})$, $S_0^a$ are Givental’s $S$–actions for:

$$s^a := \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & 0 & \vdots \\ a & \ldots & 0 \end{pmatrix}, \quad S_0^c := \begin{pmatrix} c & \ldots & 0 \\ \vdots & I_{n-2} & \vdots \\ 0 & \ldots & c^{-1} \end{pmatrix},$$

and $R^1$ is the Givental’s $R$–action “corresponding” to the Inversion of Dubrovin. Explicitly this $R$–action was found in [DbSS]:
Theorem 5 (Theorem 3.1 in [DbSS]). The Inversion transformation on the Frobenius manifold with potential \( F \) coincides with Givental’s \( R \)-action given by:

\[
R^1(z) = \exp\left( \begin{pmatrix} 0 & \ldots & 1 \\ 0 & \ddots & 0 \\ 0 & \ldots & 0 \end{pmatrix} z \right).
\]

Namely, \( \hat{R}^1 \)-transformed Frobenius structure of \( F \) at the point \((0, \ldots, 0, 1)\) coincides with the Frobenius structure of \( F^I \) at the point \((0, \ldots, 0, -1)\).

Note that unlike the “analytical” Inversion \( \hat{I} \) that we have in the formula for \( \hat{A} \), Givental’s analog \( \hat{R}^1 \) also applies certain shift of variables on both sides. In order for two quantizations to agree these additional shifts have to be absorbed.

Proposition 4.1. The action \( \hat{A}_G \) agrees with the action \( \hat{A} \).

Proof. It is clear by construction that both \( \hat{A}_G \) and \( \hat{A} \) act similarly. We only have to show that the shifts of the coordinates applied on both sides agree.

Let \( t^s \) and \( t^e \) be the coordinates before and respectively after the operator \( \hat{A} \) is applied. Let \( t^{A1} \) and \( t^{BI} \) be the coordinates after and respectively before the inversion operator \( I \). The action of \( \hat{A} \) changes the coordinates in the following way.

\[
S_{c^{-1}} \cdot T_1 : t^{BI} = \frac{t^s}{c} + \frac{a}{c}, \quad I : t^{BI} = -\frac{1}{t^{A1}}, \quad T_2 \cdot S_c : t^{A1} = c(t^e + \frac{d}{c}).
\]

On the \( \hat{A}_G \) side \( t^s \) and \( t^e \) are the coordinates before and respectively after \( \hat{A}_G \) is applied. Let \( \tilde{t}^{A1} \) and \( \tilde{t}^{BI} \) be the coordinates after and respectively before \( \hat{R}^1 \) is applied.

\[
S_0^{-1} \cdot S_\alpha : \tilde{t}^{BI} = \frac{t^s}{c} + \frac{a + 1}{c}, \quad S_\beta \cdot S_0 : \tilde{t}^{A1} = c(t^e + \frac{d + 1}{c}).
\]

Due to the theorem above the coordinates \( \tilde{t}^{A1} \) and \( t^{A1} \) are related by:

\[
-1 + \tilde{t}^{A1} = t^{A1}.
\]

This approves the shift \( \beta \). In order to approve the shift \( \alpha \) it’s enough to check that

\[
A \cdot \beta = \frac{a\beta + b}{c\beta + d} = \alpha.
\]

Corollary 4.1. \( \hat{A}_G \)-transformed Frobenius structure of \( F \) at the point \((0, \ldots, 0, \frac{a+1}{c})\) coincides with the Frobenius structure of \( F^A \) at the point \((0, \ldots, 0, -\frac{d+1}{c})\).
Note that the decomposition of the form \((10)\) does not allow one to make a statement about a more general \(R\)-action because the actions of the lower- and upper-triangular groups do not commute. Therefore we are not allowed to multiply the matrices \(S_0\) and \(R^1\) to get a new \(R\)-action.

Another disappointment of the action \(\hat{A}_C\) is that the choice of the points of the Frobenius structure expansion on both sides is very restrictive due to the corollary above. This restriction cannot be relaxed while we use the Givental’s analogue of the Inversion – the action of \(\hat{R}^1\). The following proposition makes more general statement without a special choice of the Frobenius structure expansion points.

**Proposition 4.2.** For every \(A \in \text{SL}(2, \mathbb{C})\) and \(s = (0, \ldots, 0, \tau) \in \mathbb{C}^n\) such that \(F(t)\) is analytic at \(A \cdot s = (0, \ldots, 0, A \cdot \tau)\) consider the upper–triangular group element \(R^\sigma:\)

\[
R^\sigma(z) = \exp(r^\sigma z), \quad r^\sigma := \begin{pmatrix}
0 & \ldots & \sigma \\
\vdots & 0 & \vdots \\
0 & \ldots & 0
\end{pmatrix}, \quad \sigma = -c(c \tau + d).
\]

Then we have:

- The action \(\hat{A}\) on \(F(t)\) given by \((2)\) coincides with the action of \(\hat{R}^\sigma(z)\) up to the rescaling:

\[
F^A \rightarrow (c \tau + d)^2 F^A, \quad t^n \rightarrow (c \tau + d)^2 t^n, \quad t^k \rightarrow (c \tau + d) t^k, \quad 1 < k < n.
\]

- The action of \(\hat{R}^\sigma\) identifies Frobenius manifold structure of \(F^A\) at the point \((0, \ldots, 0, \tau)\) with the \(\hat{R}^\sigma\)-transformed Frobenius manifold structure of \(F\) at the point \((0, \ldots, 0, A \cdot \tau)\).

Proposition 4.2 together with Proposition 4.1 complete the proof of Theorem 1. While the formulation of Proposition 4.2 is enough for the isomorphism of the Frobenius manifolds given by \(F^A\) and \(\hat{R}^\sigma F\), we consider in more details the rescaling used in the next sections where it will play an important role.

The rest of this section is devoted to the proof of Proposition 4.2.

### 4.1. Graphical calculus of \(\hat{R}^\sigma\) in genus 0

From now on we consider a particular \(R\)-action given by \(R^\sigma\) on the CohFT whose Frobenius structure is given by the potential \(F\). In this subsection fix the notation:

\[
R^\sigma(z) = \exp(r^\sigma z), \quad r^\sigma := \begin{pmatrix}
0 & \ldots & \sigma \\
\vdots & 0 & \vdots \\
0 & \ldots & 0
\end{pmatrix}, \quad \sigma \in \mathbb{C}\setminus\{0\}.
\]

We further assume \(F\) to be given in the form \((1)\). In particular we assume \(\eta_{i,j} = \delta_{i+j,n+1}\). To simplify the formulae introduce the notation:
Notation 4.1. For $1 \leq \alpha_i \leq n$ define:

$$\langle \tau_{d_1}(\alpha_1) \ldots \tau_{d_k}(\alpha_k) \rangle_g := \langle \tau_{d_1}(e_{\alpha_1}) \ldots \tau_{d_k}(e_{\alpha_k}) \rangle_g.$$ 

We also denote with the capital $G$ subscript the correlators of Givental-transformed partition function. Namely for $\langle \cdot \rangle_g$ — correlators of $Z$ we denote by $\langle \cdot \rangle^G_g$ the correlators of $\hat{R}^\sigma \cdot Z$:

$$\left\langle \prod_k \tau_{d_p}(\alpha_p) \right\rangle^G_g := \prod_p \frac{\partial}{\partial \theta_{d_p,\alpha_p}} \left[ h^{g-1} \log \left( \hat{R}^\sigma \cdot Z \right) \right]_{t=0}.$$ 

For $e_\alpha \in V$ define $\bar{e}_\alpha \in V$ to be the vector such that $\eta(e_\alpha, \bar{e}_\alpha) = 1$. With our choice of $\eta$ it is equivalent to:

$$\bar{\alpha} := n + 1 - \alpha \quad \text{for} \quad 1 \leq \alpha \leq n.$$ 

We classify the graphs giving non–trivial contribution to the action of $R^\sigma$ in the following proposition.

Proposition 4.3. The only decorated graphs giving non-trivial contribution to the Frobenius structure of $\hat{R}^\sigma \cdot Z$ are those from the following two series:

with the vertices decorated by $\mathcal{V}_{g=0}[\cdot]$.

Notation 4.2. We will denote by

$$\langle \tau_0(\alpha_1) \ldots \tau_0(\alpha_l)(\tau_0(n))^{k_1} \rangle^G_{\mathcal{H}}$$
the genus 0 correlators of \( \hat{R} \cdot Z \) obtained by graph calculus from the first (one-vertex) series above and by
\[
\langle \tau_0(\alpha) \tau_0(\bar{\alpha}) \tau_0(\beta) \tau_0(\bar{\beta})(\tau_0(n))^k \rangle^G_Q
\]
the correlators obtained from the second series above.

Proof of the proposition 4.3. Note that the only non-zero element of \((r^\sigma)^{ij}\) is \((r^\sigma)^{11} = \sigma\). Therefore there is a unique choice for an edge decoration:
\[
\mathcal{E} = -(r^\sigma)^{11} e_1 \otimes e_1.
\]
At the same time \(r^\sigma(e_\alpha) \neq 0 \iff \alpha = n\) and for the leaves decoration we have:
\[
\mathcal{L} = \sum_{d \geq 0} \sigma e_1 t^{d,n} z^{d+1} + \sum_{d=0}^{\infty} \sum_{\alpha=1}^{n} e_\alpha t^{d,\alpha} z^{d}, \quad \mathcal{L}_0 = 0.
\]
Restricting ourselves to the Frobenius structure defined by the CohFT \( \hat{R}^\sigma \cdot Z \) we consider only genus zero graphs decorated with (recall Proposition 3.3):
\[
\mathcal{L}_s = \sigma e_1 t^{0,n} z^1 + \sum_{\alpha=1}^{n} e_\alpha t^{0,\alpha} z^0, \quad \mathcal{L}_0 = 0.
\]
Hence the graphs giving non-trivial contribution are genus 0 graphs obtained by \(\mathcal{E}\)-gluing from the following two types of graphs:
\[
\begin{align*}
\text{\begin{tikzpicture}
edge[->](a1)(a2); \node (a2) at (0,0) {$e_1$}; \node (a1) at (-1,0) {$e_\alpha$}; \node (a3) at (1,0) {$e_{\bar{\alpha}}$}; \end{tikzpicture}} & \{ \text{k times} \} & \begin{tikzpicture}
edge[->](a1)(a2); \node (a2) at (0,0) {$z r(e_n)$}; \node (a1) at (-1,0) {$e_1$}; \node (a3) at (1,0) {$e_{\bar{\alpha}}$}; \node (a4) at (2,0) {$e_{i_1}$}; \node (a5) at (-2,0) {$e_{i_2}$}; \node (a6) at (0,-1) {$z r(e_n)$}; \node (a7) at (0,-2) {$z r(e_n)$}; \node (a8) at (0,-3) {$z r(e_n)$}; \end{tikzpicture}
\end{align*}
\]
However the gluing is constituted by the \(e_1 \otimes e_1\)-decorated edges only. Because of the Dilaton equation we arrive at the series presented in the statement of the proposition.

Corollary 4.2. Frobenius potential \(F^G\) of the CohFT \( \hat{R}^\sigma \cdot Z \) reads:
\[
F^G(t) = \left( \frac{(t^1)^2}{2} + t^1 \sum_{p=2}^{n-1} t^p t^p + \sum_{\alpha} \frac{(t^n)^k \prod_{i} t^{\alpha_i} k! \text{Aut}(\{\alpha\})}{\prod \tau_0(\alpha)^k \prod \tau_0(\bar{\alpha})} \right)^G_H
\]
\[
+ \sum_{\alpha, \beta, k} \frac{t^{\alpha \beta} t^{\bar{\alpha} \bar{\beta}} (t^n)^k}{k! \text{Aut}(\alpha, \beta, \bar{\alpha}, \bar{\beta})} \langle \tau_0(\alpha) \tau_0(\bar{\alpha}) \tau_0(\beta) \tau_0(\bar{\beta}) \tau_0(n)^k \rangle^G_Q,
\]
where the summations are taken over \(\alpha_i, \alpha, \beta \neq 1\).
Proof. Indeed the $R$-action does not change the algebra structure at $t = 0$. Hence the third order terms of $F^G$ coincide with the third order terms of $F$. We do not consider lower order terms (that are indeed different) because they do not change the Frobenius structure. The summands written obviously come from the graphical calculus of the Givental graphs and the proposition above.

However the correlators $\langle \cdot \rangle^G_H$ and $\langle \cdot \rangle^G_Q$ are allowed to have $\tau_0(e_1)$ insertions both contributing to $F^G$ as the coefficients of the monomials containing $t_1$. But such monomials could only appear in the third order terms of the Frobenius potential that are preserved by the $R$-action. □

Proposition 4.4.

$$\langle \tau_0(\alpha_1) \ldots \tau_0(\alpha_N) (\tau_0(n))^q \rangle^G_H = \sum_{p+k=q} \frac{q!}{p!k!} \binom{N + k + p - 3}{k} \langle \tau_0(\alpha_1) \ldots \tau_0(\alpha_N) (\tau_0(e_1))^p \rangle \cdot \sigma^k.$$ 

Proof. Recall that $r(e_n) = \sigma e_1$. From the graphical calculus we have:

$$\langle \tau_0(\alpha_1) \ldots \tau_0(\alpha_N) (\tau_0(n))^q \rangle^G_H = \sum_{p+k=q} \binom{q}{k} \sigma^k \cdot \{ \text{k times} \} \begin{array}{c} e_{\alpha_1} \\ e_{\alpha_N} \\ e_n \end{array} \{ \text{p times} \} \begin{array}{c} e_1 \\ z e_1 \\ z e_1 \\ z e_1 \\ e_n \end{array}$$

where the decoration of the vertices on the right hand side is given by the partial derivatives of the function $F$. In terms of correlators the equation above reads:

$$\langle \tau_0(\alpha_1) \ldots \tau_0(\alpha_N) (\tau_0(n))^q \rangle^G_H = \sum_{p+k=q} \frac{q!}{k!p!} \langle \tau_0(\alpha_1) \ldots \tau_0(\alpha_N) (\tau_0(n))^p (\tau_0(1))^k \rangle \cdot \sigma^k.$$ 

Using the Dilaton equation we get:

$$\langle \tau_0(\alpha_1) \ldots \tau_0(\alpha_N) (\tau_0(n))^q \rangle^G_H = q! \sum_{p+k=q} \frac{(N + k + p - 3)!}{p!k!(N + p - 3)!} \langle \tau_0(\alpha_1) \ldots \tau_0(\alpha_N) (\tau_0(n))^p \rangle \cdot \sigma^k.$$ 

The proposition follows. □
**Definition 4.1.** For any $1 \leq \alpha, \beta \leq n$ define:

$$\text{Aut}_2(\alpha, \beta) := \begin{cases} 8 & \text{if } \alpha = \beta = \bar{\alpha} = \bar{\beta}, \\ 1 & \text{if all } \alpha, \beta, \bar{\alpha}, \bar{\beta} \text{ are different,} \\ 2 & \text{otherwise}. \end{cases}$$

**Proposition 4.5.** For $\alpha, \beta \neq 1, n$ we have:

$$\left\langle \tau_0(\alpha)\tau_0(\beta)\tau_0(\bar{\alpha})\tau_0(\bar{\beta}) (\tau_0(n))^k \right\rangle^G_Q = -\frac{k!|\text{Aut}(\alpha, \beta)|}{|\text{Aut}_2(\alpha, \beta)|} \sigma^{k+1}.$$

**Proof.** From graphical calculus (keeping in mind Remark 3.1) we have:

$$\left\langle \tau_0(\alpha)\tau_0(\beta)\tau_0(\bar{\alpha})\tau_0(\bar{\beta}) (\tau_0(n))^k \right\rangle^G_Q = \sum_{l=0}^{k} \sum_{m_1, \ldots, m_e, m_s \geq 0} \frac{(-1)^{l+1} \sigma^{k+1}}{m! m_e! m_s! \prod_{i=1}^l m_i!}$$

where we denote $|m| := m_s + m_e + \sum_{i=1}^l m_i$ and assume $0! = 1$.

Using Dilaton equation and writing in correlators the equation above reads:

$$\left\langle \tau_0(\alpha)\tau_0(\bar{\alpha})\tau_0(\beta)\tau_0(\bar{\beta}) (\tau_0(n))^k \right\rangle^G_Q = \frac{k!|\text{Aut}(\alpha, \beta)|}{|\text{Aut}_2(\alpha, \beta)|} \sigma^{k+1} \sum_{p=1}^{k+1} (-1)^p \binom{k + 1}{p}.$$

**4.2. From $\hat{R}^c \cdot Z$ to $F^A_0$.** Following the method of [DbSS] we prove Proposition 4.2.

**Proof of Proposition 4.2.** Consider the potential $F^A$. We show explicitly that the coefficients of its series expansion coincide with the correlators obtained by Givental’s group action.
Let \((0, \ldots, 0, \tau)\) be a point on the Frobenius manifold of \(F^A\). Introduce the constants:
\[
\gamma := \frac{a\tau + b}{c\tau + d}, \quad \mu := \tau.
\]

**H-correlators.** Write the series expansion of the function \(H\) (recall the formula (2)) of \(F^A\) at the point \((0, \ldots, 0, \gamma)\). We have:
\[
K := (ct^n + d)^2 \left( t^2 \frac{t}{ct^n + d}, \ldots, t^{n-1} \frac{at^n + b}{ct^n + d}, \frac{at^n + b}{ct^n + d} \right)
\]
\[
= (ct^n + d)^2 \sum_{p, \alpha_1, \ldots, \alpha_N} \frac{t^{\alpha_1} \ldots t^{\alpha_N}}{|\text{Aut}(\alpha)|p!} \left( \frac{at^n + b}{ct^n + d} - \gamma \right)^p \left( \frac{H_{\alpha_1 \ldots \alpha_N, np}}{(ct^n + d)^N} \right),
\]
where we denote
\[
H_{\alpha_1 \ldots \alpha_N, np} := \frac{\partial^{N+p} H(t^2, \ldots, t^n)}{\partial t^{\alpha_1} \ldots \partial t^{\alpha_N} (\partial t^n)^p} \bigg|_{t=(0, \ldots, 0, \gamma)}.
\]

Let \(\tilde{t}^n = t^n - \mu\). The function rewrites:
\[
K = \sum_{p, \alpha_1, \ldots, \alpha_N} \frac{t^{\alpha_1} \ldots t^{\alpha_N}}{|\text{Aut}(\alpha)|p!} \left( \tilde{t}^n \right)^p \frac{H_{\alpha_1 \ldots \alpha_N, np}}{(ct^n + d)^{N+p-2}}
\]
\[
= \sum_{p, \alpha_1, \ldots, \alpha_N} \frac{t^{\alpha_1} \ldots t^{\alpha_N}}{|\text{Aut}(\alpha)|p!} \left( \tilde{t}^n \right)^p \frac{1}{(ct^n + d)^{N+p-2}} \frac{H_{\alpha_1 \ldots \alpha_N, np}}{(ct^n + d)^{N+p-2}}
\]
\[
= \left( ct^n + d \right)^2 \sum_{p, \alpha_1, \ldots, \alpha_N} \frac{\tilde{t}^{\alpha_1} \ldots \tilde{t}^{\alpha_N}}{|\text{Aut}(\alpha)|p!} \left( \tilde{t}^n \right)^p \frac{H_{\alpha_1 \ldots \alpha_N, np}}{(1 - \sigma \tilde{t}^n)^{N+p-2}}
\]
\[
= \left( ct^n + d \right)^2 \sum_{k \geq 0} \sum_{N+p \geq 3} \sum_{\alpha_1, \ldots, \alpha_N} \binom{N+k+p-3}{k} \frac{\tilde{t}^{\alpha_1} \ldots \tilde{t}^{\alpha_N}}{|\text{Aut}(\alpha)|p!} \left( \tilde{t}^n \right)^{p+k} H_{\alpha_1 \ldots \alpha_N, np} \left( \sigma \right)^k
\]

Using Proposition 4.4 we get in new variables exactly one of the higher than three order summands of the Frobenius potential written in the Corollary 4.2.
**Q-correlators.** In an analogous way we write:

\[
\frac{c}{8(\epsilon t^n + d)} (t^2 t^{n-1} + \ldots + t^{n-1} t^2)^2 =
\]

\[
\frac{c}{8(\epsilon t^n + \epsilon t + d)} (t^2 t^{n-1} + \ldots + t^{n-1} t^2)^2 =
\]

\[
(c + d)^2 \frac{-\sigma}{8(1 - \sigma t^n)} (\hat{t}^2 \hat{t}^{n-1} + \ldots + \hat{t}^{n-1} \hat{t}^2)^2 =
\]

\[
= -(c + d)^2 \sum_{k \geq 0} 1 \left| \text{Aut}_2(\alpha, \beta) \right| \hat{t}^{\alpha} \hat{t}^{\beta} (\sigma)^{k+1} (\hat{t}^n)^k.
\]

Using Proposition 4.5 we recognize here the second higher than three order sum-
mand of the Frobenius potential of Corollary 4.2. \[\square\]

Note that the rescaling applied in the proof above can be written as the Givental’s
S-action. In what follows we will adopt the notation \(S_0^A\) for it:

\[
S_0^A := \begin{pmatrix} 1 & \ldots & 0 \\ \vdots & (c + d) I_{n-2} & \vdots \\ 0 & \ldots & (c + d)^2 \end{pmatrix}.
\]

Applied to the potential of a Frobenius manifold \(F(t)\), \(S_0^A\) acts by the change of vari-
ables. However this would change also the pairing of the Frobenius manifold, hence
some additional factors are needed:

\[
S_0^A \cdot F = (c + d)^{-2} \left( \frac{1}{2} \hat{t}^1 \hat{t}^n + \frac{1}{2} \hat{t} \sum_{k=2}^{n-1} \hat{t}^k \hat{t}^{n+1-k} + H(\hat{t}^2, \ldots, \hat{t}^n) \right),
\]

where \(\hat{t} = S_0^A t\) and we assume \(F\) to be written in the form of equation (11).

**Corollary 4.3.** Denote by \(F_\sigma^A\) and \(F_{A\cdot\tau}\) the local expansions of \(F^A\) and \(F\) at the points
\((0, \ldots, 0, \tau)\) and \((0, \ldots, 0, A \cdot \tau)\) respectively. For \(\sigma := -c(c + d)\), \(\sigma' := -c/(c + d)\) and
\(S_0^A\) as above holds:

\[
F_\tau^A = \left( S_0^A \right)^{-1} \cdot \hat{R}^\sigma \cdot F_{A\cdot\tau},
\]

\[
F_\tau^A = \hat{R}^\sigma' \cdot \left( S_0^A \right)^{-1} \cdot F_{A\cdot\tau}.
\]

**Proof.** The first equation is the Givental action form of Proposition 4.2 and was proved
above. We only have to prove the second equation. Slightly altering the proof of the
proposition above we write:

\[
\frac{c}{8(ct^n + d)}(t^2t^{n-1} + \ldots + t^{n-1}t^2)^2 = \\
\frac{-\sigma'}{8(1 - \sigma't^n)}(t^2t^{n-1} + \ldots + t^{n-1}t^2)^2 =
\]

\[
= -\sum_{k\geq 0} \frac{1}{|\text{Aut}_2(\alpha, \beta)|} t^\alpha t^\beta t^\beta (\sigma')^{k+1} (\tilde{t}^n)^k,
\]

and

\[
K := (ct^n + d)^2 H \left( \frac{t^2}{ct^n + d}, \ldots, \frac{t^{n-1}}{ct^n + d}, \frac{at^n + b}{ct^n + d} \right) \]

\[
= \sum_{p, \alpha_1, \ldots, \alpha_N} t^{\alpha_1} \ldots t^{\alpha_N} (\tilde{t}^n)^p \frac{1}{|\text{Aut}(\alpha)|p! (ct + d)^p (ct + d)^{N+p-2}} \frac{H_{\alpha_1, \ldots, \alpha_N, n^p}}{(1 + c(ct + d)^{1-n^2})^{N+p-2}}
\]

\[
= \sum_{k\geq 0} \sum_{N+p\geq 3} \sum_{\alpha_1, \ldots, \alpha_N} \binom{N + k + p - 3}{k} t^{\alpha_1} \ldots t^{\alpha_N} (\tilde{t}^n)^{p+k} H_{\alpha_1, \ldots, \alpha_N, n^p} (\sigma')^k.
\]

Hence by rescaling the variables in the function $K$ only we get the equality with the local expansion $F^A_A$. But this is exactly what we need because the action of $S_0$ does not change cubic terms of the potential that build up the $Q$–term computed above.

\[\square\]

5. Graphical Calculus of $\hat{R}^\sigma$ in Higher Genera

In this section we prove Theorem 2 using graphical calculus. Let $R^\sigma$ and $r^\sigma$ be as in Subsection 4.1 with $\sigma \in \mathbb{C}\setminus\{0\}$, a complex parameter.

5.1. Graphs of the $R^\sigma$ action in genus $g \geq 1$. We start by analyzing the graphs contributing to the graphical calculus of Givental’s $R^\sigma$ action in higher genera.

**Proposition 5.1.** Let $Z$ be the partition function of the fixed CohFT $\Lambda_{g,k}$. The graphs contributing to the genus $g$ potential of $R^\sigma Z$ restricted to the small phase space are the following:
where \( p, q \geq 0, m_i \geq 0, \sum_{i=1}^{p} m_i + p = k \geq 1. \)

**Notation 5.1.** We will denote the first series by \( \Gamma_{p,q;\alpha} \) and the second series by \( \Gamma_{0,k,p'} \).

**Proof.** As in Proposition 4.3 the only possible decorations of leaves restricted to the small phase space (recall Proposition 3.3) are:

\[
\mathcal{L}_s = \sigma e_1 t^{0,n} z_1 + \sum_{\alpha=1}^{n} e_{\alpha} t^{0,\alpha} z_1, \quad \mathcal{L}_0 = 0.
\]

The edges are decorated by:

\[
\mathcal{E} = -(r^\sigma)^{11} e_1 \otimes e_1.
\]

Consider all possible genus \( g \) graphs with the decoration given. Contraction of leaves in the vertices are given by the following correlators that can be simplified using the Dilaton equation:

\[
\left< \left( \tau_0(e_1) \right)^k (\tau_1(e_1)) \prod_{i=1}^{N} \tau_0(\alpha_i) \right>_{\tilde{g}} = \left< \left( \tau_0(e_1) \right)^k \prod_{i=1}^{N} \tau_0(\alpha_i) \right>_{\tilde{g}}.
\]

It is clear from the axioms of the unital CohFT that the correlator on the right hand side vanishes unless \( k = 0 \) or \( \tilde{g} = 0 \). Taking graphs with only one vertex we get exactly the first series of the proposition.

In order to build genus \( g \) graph having more than one vertex we have to use vertices with at least two half-edges decorated by \( e_1 \). By the reasoning above such vertices need to be of genus 0. At the same time we know that in genus 0 the correlators with more than two \( \tau_0(e_1) \) insertions vanish. Hence we can only build the genus 1 graph using the vertices of the lower genus.

Due to the reasoning of Proposition 4.3 the second series represents an arbitrary decoration of such graphs. \( \square \)
Remark 5.1. According to the decoration $\mathcal{V}_k$ assigned to the vertex its contraction with the half-edges could give the contribution to the correlators of any genus. It is clear however from the proof of the proposition that the graphs from $\Gamma^g_{p,q,\alpha}$ contribute to arbitrary genus $g$ with the vertex decorated by $\mathcal{V}_g$ only and the graphs from $\Gamma^0_{k,p}$ contribute to the genus 1 only with all the vertices decorated by $\mathcal{V}_0$ only.

In order to use graphical calculus we have to compute the automorphisms of the graphs presented above. It is not a problem at all for the series $\Gamma^g_{p,q,\alpha}$ but some additional analysis is needed for the graphs from $\Gamma^0_{k,p}$.

5.1.1. Graphs counting. By setting $a_i = m_i + 1$ we associate to every graph $\gamma \in \Gamma^0_{k,p}$ the ordered sequence of numbers such that $a_1 + \cdots + a_p = k$ and $k \geq a_i \geq 1$. Obviously it is defined up to cyclic shift. Such partitions of $k$ are called cyclic compositions.

Definition 5.1. For $\gamma \in \Gamma^0_{k,p}$ define by $\text{Cy}(\gamma)$ corresponding cyclic composition $(1 + m_1) + \cdots + (1 + m_p) = k$.

Definition 5.2. For $p, k \in \mathbb{N}$ and $p \leq k$ define:

$$\left\langle \frac{k}{p} \right\rangle := |\Gamma^0_{k,p}|, \quad \left\langle \frac{k}{p} \right\rangle_{\text{prim}} := \# \{ \gamma \in \Gamma^0_{k,p} \mid |\text{Aut}(\gamma)| = 1 \}.$$

These numbers are called cyclic compositions and primitive cyclic compositions numbers respectively (cf. [KR, FS]).

Proposition 5.2 (Theorem 2 in [KR]). The number of cyclic compositions is equal to:

$$\left\langle \frac{n}{p} \right\rangle = \frac{1}{n} \sum_{a \mid \gcd(n,p)} \varphi(a) \left( \frac{n/a}{p/a} \right),$$

where $\varphi$ is Euler’s totient function.

We adopt the proof of [KR] to compute the number of primitive cyclic compositions.

Proposition 5.3. The number of primitive cyclic compositions is equal to:

$$\left\langle \frac{n}{p} \right\rangle_{\text{prim}} = \frac{1}{n} \sum_{a \mid \gcd(n,p)} \mu(a) \left( \frac{n/a}{p/a} \right),$$

where $\mu(a)$ is M"{o}bius function.

Proof. Consider the generating function:

$$PC(z, u) := \sum \left\langle \frac{k}{l} \right\rangle_{\text{prim}} z^k u^l.$$
It can be expressed (Appendix A.4 in [FS]):

\[
PC(z, u) = \sum_{k \geq 1} \frac{\mu(k)}{k} \log \left( 1 - \frac{u^k z^k}{1 - z^k} \right)^{-1}.
\]

Expanding further we get:

\[
PC(z, u) = \sum_{k \geq 1} \frac{\mu(k)}{k} \left( -\sum_{m \geq 1} \frac{z^{km}}{m} + \sum_{m \geq 1} \frac{(1 + u^k)z^{km}}{m} \right)
\]

\[
= \sum_{k \geq 1} \sum_{m \geq 1} \frac{\mu(k)}{km} z^{km} ((1 + u^k) - 1).
\]

Setting \( l = km \) one gets the formula. □

**Lemma 5.1.**

\[
\sum_{p=1}^{q} \sum_{\gamma \in \Gamma_{q,p}^\circ} \frac{(-1)^p}{\text{Aut}(\text{Cy}(\gamma))} = -\frac{1}{q}
\]

**Proof.** All automorphisms of \( \text{Cy}(\gamma) \) for \( \gamma \in \Gamma_{q,p}^\circ \) are given by the shifts, which correspond to the rotations of the graph \( \gamma \). Therefore we can write:

\[
\sum_{\gamma \in \Gamma_{q,p}^\circ} \frac{1}{\text{Aut}(\text{Cy}(\gamma))} = \sum_{d \mid p, d \mid q} \frac{1}{d} \left( \frac{q/d}{p/d} \right)_{\text{prim}}.
\]

Using this and (13) we write:

\[
\sum_{p=1}^{q} \sum_{\gamma \in \Gamma_{q,p}^\circ} \frac{(-1)^p}{\text{Aut}(\text{Cy}(\gamma))} = \sum_{p=1}^{q} (-1)^p \sum_{d \mid p, d \mid q} \frac{1}{q} \sum_{a \mid \gcd(q/d, p/d)} \mu(a) \left( \frac{q/ad}{p/ad} \right)
\]

Changing the summation by introducing \( l = ad \) we get:

\[
\sum_{p=1}^{q} \sum_{\gamma \in \Gamma_{q,p}^\circ} \frac{(-1)^p}{\text{Aut}(\text{Cy}(\gamma))} = \frac{1}{q} \sum_{p=1}^{q} (-1)^p \sum_{l \mid \gcd(q,p)} \left( \frac{q/l}{p/l} \right) \sum_{b \mid l} \mu(b).
\]

Due to:

\[
\sum_{d \mid n} \mu(d) = \begin{cases} 
1 & \text{if } n = 1 \\
0 & \text{if } n > 1.
\end{cases}
\]

we conclude:

\[
\sum_{p=1}^{q} \sum_{\gamma \in \Gamma_{q,p}^\circ} \frac{(-1)^p}{\text{Aut}(\text{Cy}(\gamma))} = \frac{1}{q} \sum_{p=1}^{q} (-1)^p \left( \frac{q}{p} \right) = -\frac{1}{q}.
\]

□
5.2. From $\hat{R}^c \cdot Z$ to $F_g^A$. From graphical calculus we get:

**Proposition 5.4.** The correlators of the genus $g$ potential of $R^c \cdot Z$ read:

$$
\langle \tau_0(\alpha_1) \ldots \tau_0(\alpha_N) (\tau_0(e_n))^q \rangle_g^G =
\sum_{p+k=q} \frac{q!}{p!} \left( N + q + 2g - 3 \right) \langle \tau_0(\alpha_1) \ldots \tau_0(\alpha_N) (\tau_0(e_n))^p \rangle_g \cdot \sigma^k - \delta_{g,1} \frac{\sigma^q}{q}.
$$

Proof. We compute the contribution of the graphs listed in Proposition 5.1 (see also the remark after it). For $g \geq 2$ these are only $\gamma \in \Gamma_{k,p}$. The contribution of these graphs reads:

$$
\langle \tau_0(\alpha_1) \ldots \tau_0(\alpha_N) (\tau_0(e_n))^q \rangle_g =
\sum_{p+k=q} \frac{q!}{p!} \langle \tau_0(\alpha_1) \ldots \tau_0(\alpha_N) (\tau_0(e_1))^p \rangle_g \cdot \sigma^k.
$$

Using the Dilaton equation we get:

$$
\langle \tau_0(\alpha_1) \ldots \tau_0(\alpha_N) (\tau_0(e_n))^q \rangle_g =
q! \sum_{p+k=q} \frac{(N + k + p + 2g - 3)!}{p! (N + p + 2g - 3)!} \langle \tau_0(\alpha_1) \ldots \tau_0(\alpha_N) (\tau_0(e_n))^p \rangle_g \cdot \sigma^k.
$$

For $g = 1$ we get the contribution as above with the additional contribution of the graph $\gamma^0 \in \Gamma_{k,p}^0$. Note that the relation between $\text{Aut}(\gamma^0)$ and $\text{Aut}(\text{Cy}(\gamma^0))$ is given by:

$$
\text{Aut}(\gamma^0) = \text{Aut}(\text{Cy}(\gamma^0)) \prod_{i=1}^p m_i!.
$$

Using the Dilaton equation for the graph $\gamma^0$ the factor $\prod_{i=1}^p m_i!$ cancels outs and the contribution of $\gamma^0 \in \Gamma_{k,p}^0$ for $g = 1$ reads:

$$
\langle \tau_0(\alpha_1) \ldots \tau_0(\alpha_N) (\tau_0(e_n))^q \rangle_1 =
q! \sum_{p+k=q} \frac{(N + k + p + 1)!}{p! (N + p + 1)!} \langle \tau_0(\alpha_1) \ldots \tau_0(\alpha_N) (\tau_0(e_n))^p \rangle_1 \cdot \sigma^k
$$

$$
+ \sigma^q \sum_{k=1}^q \sum_{\gamma \in \Gamma_{q,k}} \frac{(-1)^k}{\text{Aut}(\text{Cy}(\gamma))}.
$$

Because of the graphs counting formula (14) the proposition follows. □

We illustrate the graph computations of the proposition above in the Figure 1 on page 33.
We do not decorate the half–edges denoting by the black dot the gluing of two \((e_1, e_1)\) half–edges by \(r^{11}\). Summing up the numbers beneath the graphs one gets the contribution \(-\sigma^q/q\) as in Equation (15).
5.3. **Proof of Theorem 2** Let \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C}) \). With the help of graphical calculus we give the formulae for the action of \( A \) on the higher genus potential of the CohFT partition function.

**Proof.** We show explicitly that the coefficients of \( \tilde{F}_g^A(t) \) series expansion coincide with the correlators obtained by Givental’s group action.

As in the proof of Proposition 4.2 consider the constants:

\[
\gamma := \frac{a \tau + b}{c \tau + d}, \quad \mu := \tau.
\]

Write the series expansion of the function \( F_g^A \) at the point \((0, \ldots, 0, \gamma)\). We have:

\[
\mathcal{K} := (ct^n + d)^{2-2g} F_g \left( \frac{t^2}{ct^n + d}, \ldots, \frac{t^{n-1}}{ct^n + d}, \frac{at^n + b}{ct^n + d} \right)
\]

\[
= (ct^n + d)^{2-2g} \sum_{|\text{Aut}(\alpha)|} \frac{t^{\alpha_1} \cdots t^{\alpha_N}}{p!} \left( \frac{at^n + b}{ct^n + d} - \gamma \right)^p \frac{(F_g)_{\alpha_1 \cdots \alpha_N, np}}{(ct^n + d)^N}.
\]

Let \( \tilde{t}^n = t^n - \mu \). The series expansion rewrites:

\[
\mathcal{K} = \sum_{p, \alpha_1, \ldots, \alpha_N} \frac{t^{\alpha_1} \cdots t^{\alpha_N}}{|\text{Aut}(\alpha)|} \left( \frac{\tilde{t}^n}{p!} \right)^p \left( \frac{(F_g)_{\alpha_1 \cdots \alpha_N, np}}{(ct^n + d)^p} \right).
\]

Taking \( \tilde{t}^{\alpha_k} := \tilde{t}^{\alpha_k} / (ct^n + d) \), \( \tilde{t}^n := \tilde{t}^n / (ct^n + d)^2 \) and \( \sigma = -c(ct^n + d) \) we rewrite:

\[
\mathcal{K} = (ct^n + d)^{2-2g} \sum_{p, \alpha_1, \ldots, \alpha_N} \frac{\tilde{t}^{\alpha_1} \cdots \tilde{t}^{\alpha_N}}{|\text{Aut}(\alpha)|} \left( \frac{\tilde{t}^n}{(1 - \sigma \tilde{t}^n)^{N+p+2g-2}} \right)^p \frac{(F_g)_{\alpha_1 \cdots \alpha_N, np}}{(ct^n + d)^N}.
\]

Using equation (15) we get in the new variables exactly the Givental-transformed genus \( g \) potential for \( g \geq 2 \). For the genus 1 potential \( F_1^A \) we get also the additional term:

\[
- \log \left( \frac{ct^n + d}{ct^n + d} \right) = - \log \left( 1 + c(ct^n + d) \tilde{t}^n \right) = - \sum_{k=1}^{\infty} \frac{\sigma^k}{k} (\tilde{t}^n)^k.
\]
This coincides with formula (15) of $R^\sigma$–transformed genus 1 potential correlators obtained by graphical calculus. \qed

Extend the action of $S_0^A$ to the partition function $Z = (\sum_{g \geq 0} h^{g-1} F_g(t))$ by the following:

\begin{equation}
\hat{S}_0^A \cdot Z(h, t) = Z \left( ((ct+d)^2 h, \tilde{t}) \right), \quad \tilde{t} = S_0^A t.
\end{equation}

Similarly to Corollary 4.3 one can show the following corollary.

**Corollary 5.1.** Denote by $(F^A_g)^\tau$ and $(F_g)^{A\cdot\tau}$ the local expansions of $F^A_g$ and $F_g$ at the points $(0, \ldots, 0, \tau)$ and $(0, \ldots, 0, A\cdot\tau)$ respectively. For $\sigma := -c(ct+d)$, $\sigma' := -c/(ct+d)$ and $S_0^A$ as in (11) holds:

\[
(F^A_g)^\tau = (\hat{S}_0^A)^{-1} \cdot \hat{R}^\sigma \cdot (F_g)^{A\cdot\tau},
\]

\[
(F^A_g)^\tau = \hat{R}^{\sigma'} \cdot (\hat{S}_0^A)^{-1} \cdot (F_g)^{A\cdot\tau}.
\]

**Proof.** This is obtained by some easy computation similar to the proof of Corollary 4.3 \qed

6. **COORDINATE–FREE FORM OF THE SL(2, C) ACTION**

Theorems 1 and 2 make use of the functions expanded at the certain points. We would like to get rid of this special requirement. The statement of the above mentioned theorems can be rewritten as:

\[
\hat{S}_0^A \cdot \hat{S}^\tau \cdot F^A_g = \hat{R}^\sigma \cdot \hat{S}^{A\cdot\tau} \cdot F_g,
\]

for

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \sigma = -c(ct+d),
\]

with $S_0^A$, $S^\tau$ and $S^{A\cdot\tau}$ defined in Section 4 and above. This equation suggests a Givental analogue of the SL(2, C)–action to be defined by:

\[
\hat{A}_G := (\hat{S})^{-1} \cdot (\hat{S}_0^A)^{-1} \cdot \hat{R}^\sigma \cdot \hat{S}^{A\cdot\tau}
\]

\[
= \hat{S}^{-\tau} \cdot (\hat{S}_0^A)^{-1} \cdot \hat{R}^\sigma \cdot \hat{S}^{A\cdot\tau}.
\]

It’s easy to check by hands that SL(2, C) action defined in the analytic form (2) and (3) is indeed a group action. However this is not clear on the Givental’s side. In what follows we show that the action $\hat{A}_G$ forms a group. This is an interesting result because the upper–triangular and lower–triangular actions form two different groups, whose elements do not commute, while the action $\hat{A}_G$ makes use of both.
**Proposition 6.1.** For any $A \in \text{SL}(2, \mathbb{C})$ the following formula holds:

$$\hat{A}_G \cdot (A^{-1})^{\wedge} = \text{Id}.$$ 

**Proof.** Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C})$ then $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Denote by $\sigma_A$ and $\sigma'_A$ the numbers:

$$\sigma_A := -c(c\tau_0 + d) \quad \text{and} \quad \sigma'_A := -c/(c\tau_0 + d).$$

Let $A \cdot \tau_0 = \tau_1$. Then we have:

$$\sigma_A = -\sigma'_{A^{-1}}, \quad \sigma'_A = -\sigma_{A^{-1}}, \quad S_0^A = \left(S_0^{A^{-1}}\right)^{-1}.$$ 

Using these equalities together with Corollary 4.3 the composition of Givental’s actions reads:

$$\hat{A}_G \cdot (A^{-1})^{\wedge} = \hat{S}^{-\tau_0} \left(\hat{S}_0^A\right)^{-1} \cdot \hat{R}^{\sigma_A} \cdot \hat{S}^{A \tau_0} \cdot \hat{S}^{-\tau_1} \left(\hat{S}_0^{A^{-1}}\right)^{-1} \cdot \hat{R}^{\sigma'_{A^{-1}}} \cdot \hat{S}^{A^{-1} \tau_1} = \hat{S}^{-\tau_0} \left(\hat{S}_0^A\right)^{-1} \cdot \hat{R}^{\sigma_A} \cdot \left(S_0^A\right)^{-1} \cdot \hat{R}^{\sigma'_{A^{-1}}} \cdot \hat{S}_0^A = \hat{S}^{-\tau_0} \left(\hat{S}_0^A\right)^{-1} \cdot \hat{R}^{\sigma_A} \cdot \hat{R}^{\sigma'_{A^{-1}}} \cdot \left(S_0^A\right)^{-1} = \text{Id}. $$

$$\square$$

**Proposition 6.2.** Let $A, B \in \text{SL}(2, \mathbb{C})$ then $\hat{A}_G \cdot \hat{B}_G = \hat{C}_G$ for $C = BA$.

**Proof.** Let $A = \begin{pmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{21} \\ b_{21} & b_{22} \end{pmatrix}$ and $\tau_1 = A \cdot \tau_0$. Then we can write:

$$\hat{A}_G \cdot \hat{B}_G = S^{-\tau_0} \left(S_0^A\right)^{-1} \cdot R^{\sigma_A} \left(S_0^B\right)^{-1} R^{\sigma_B} S^{B \tau_1}.$$ 

Commuting $(S_0^B)^{-1}$ with $R^{\sigma_A}$ as in Corollary 4.3 and observing that $S_0^B S_0^A = S_0^C$:

$$\hat{A}_G \cdot \hat{B}_G = S^{-\tau_0} \left(S_0^C\right)^{-1} \cdot R^{\sigma_A + \sigma_B} S^{C \tau_0},$$

for $\tilde{\sigma} = \sigma_A (b_{21} \tau_1 + b_{22})^2$. Some easy but long computations give the needed equality $\tilde{\sigma} + \sigma_B = \sigma_C$. 

$$\square$$

6.1. **The calibration of $F^A$.** For an arbitrary Givental’s lower-triangular group element $S(z) = 1 + \sum_{k \geq 1} S_{-k} z^k$ the action of $\hat{S}$ on the partition function can be written as (see [G, L]):

$$\hat{S} \cdot \mathcal{Z}(\mathbf{t}) = \mathcal{Z}(\mathbf{t}(\mathbf{t})) + \frac{1}{2} W(\mathbf{t}),$$
for the certain function $W(t)$, quadratic in $t$ and the certain linear change of variables $\tilde{t}(t)$. Let $q^k$, $k \geq 0$ be infinite set of vectors defined in coordinates by $(q^k)_l = t^{k,l} - \delta_{k,1}$. We have:

\[
\tilde{t}^{k,l} = \left( \sum_{p=k}^{\infty} S_{p-k} q^p \right)^l,
\]

where the expression under the summation is understood as the action of the operator on the vector and the superscript $l$ takes the corresponding coordinate of the vector obtained. In particular restricted to the small phase space we get $\tilde{t}^{0,l} = t^{0,l} - (S_1)_m^l 1$, where $1 = (1, \ldots, 1)$.

The function $W$ is defined by:

\[
W(t) := \sum_{i,j} (W_{i,j} q^i, q^j), \quad \sum_{i,j} W_{i,j} z^{-i} w^{-j} = \frac{S(z) S^*(w) - 1}{z^{-1} + w^{-1}}.
\]

Note that the action of $\hat{S}$ does not change the Frobenius structure of the CohFT - quadratic terms do not affect the structure constants of the algebra. But this quadratic terms of the Frobenius potential play an important role in particular applications (cf. [G, DZ]). They give the so-called calibration of the Frobenius manifold.

**Proposition 6.3.** The action of $\hat{A}_G$ on the calibration of $F_0$ is given by the addition of the following terms:

\[
(\hat{A}_G \cdot F_0)(t) = F_0^A(t) + (A \cdot \tau - \tau)(t^1)^2.
\]

**Proof.** In the definition of $\hat{A}_G$ it was indeed the purpose of the introduction of the $S$–action to apply certain change of variables by $S^{-\tau_0}$ and $S^{A\tau_0}$. Let’s compute the quadratic form $W$ for both. It’s easy to see that in both cases $W_{i,j}$ is only non–zero when $i = 0$ and $j = 0$. We have:

\[
W_{0,0}^\alpha = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \alpha & \cdots & 0 \end{pmatrix} \quad \Rightarrow \quad W^\alpha(t) = \alpha (t^1)^2,
\]

where $\alpha$ is either $A \cdot \tau_0$ or $-\tau_0$. Hence the actions of $S^{A\tau_0}$ and $S^{-\tau_0}$ add quadratic term $(A \cdot \tau_0 - \tau_0)(t^1)^2$. The action of $S_0^A$ does not rescale the coordinate $t^1$ and we only have to take care of the $R^\sigma$–action.

Generally a $R$–action does not change any lower than cubic terms of the potential. However in the formula of $\hat{A}_G$ we apply $R^\sigma$ to the $S$–transformed partition function. In particular $\hat{R}^\sigma$ can act on the quadratic term added by $S^{A\tau_0}$. This is clear however that in our particular case this does not happen on the level of Frobenius manifold giving higher order corrections only. \qed
This section is devoted to the connection of the $\text{SL}(2, \mathbb{C})$ action developed above to the action on the total ancestor potential of a simple elliptic singularity given by [MR]. We do not give all the details of Saito’s (Frobenius) structures of the singularity and total ancestor potential referring the interested reader to [G] and [H].

7.1. Frobenius manifold of the hypersurface singularity. Let $W(x) = W(x_1, \ldots, x_N)$ define an isolated singularity at $0 \in \mathbb{C}^N$. The unfolding of the singularity is the following function:

$$W(x, s) := W(x) + \sum_{i=1}^{\mu} \phi_i(x)s_i,$$

where $\mu$ is the Milnor number of $W(x)$ and the functions $\phi_i(x) \in \mathcal{O}_{\mathbb{C}^N, 0}$ generate the Milnor algebra of the singularity: $\langle \phi_1, \ldots, \phi_n \rangle = \mathcal{O}_{\mathbb{C}^N, 0}/\langle \partial_{x_1}W, \ldots, \partial_{x_N}W \rangle$. We assume further that $\phi_1 = 1$.

Consider the base space $S \subset \mathbb{C}^\mu$ of the singularity unfolding: $(s_1, \ldots, s_\mu) \in S$. For some volume form $\omega = f(x, s)dx_1 \ldots dx_N$ define:

$$c_{ijk}(s) := \int_{\Gamma_\epsilon} \frac{\partial_{x_i}W(x, s)\partial_{x_j}W(x, s)\partial_{x_k}W(x, s)}{\partial_{x_1}W(x, s) \ldots \partial_{x_N}W(x, s)} \omega, \quad \eta_{ij}(s) := c_{1ij}(s),$$

and $\Gamma_\epsilon = \Gamma_\epsilon(s)$ is supported on $\{ x \in \mathbb{C}^N ||\partial_{x_1}W(x, s)|| = \cdots = ||\partial_{x_N}W(x, s)|| = \epsilon \}$ for some $\epsilon$ small enough.

Defined in this way $\eta_{ij}$ will be structure constants of a non–degenerate bilinear form on $T_S$ (cf. [AGV]). As in Section 2 the multiplication can be defined on $T_S$ by the structure constants $c_{ij}^k(s) := c_{ijp}(s)\eta^{pk}(s)$. This multiplication together with $\eta$ define a Frobenius algebra structure. However special choice of the volume form is needed to make $\eta$ flat and get the Frobenius manifold structure.

**Theorem 6 ([S]).** There is a choice of the volume form $\omega$ such that the pairing $\eta$ is flat and defines together with $c_{ij}^k$ a structure of the Frobenius manifold (of dimension $\mu$) on $S$.

The volume form as above is called a primitive form of Saito and its existence is another complicated question (cf. [SM] and [H]). The choice of the primitive form is generally not unique. In what follows we will denote by $t = t(s)$ the coordinates on $S$ that are flat for $\eta$.

We will denote by $M_{W, \zeta}$ the Frobenius manifold structure of $W(x)$ with the choice of the primitive form $\zeta$. 

7.2. Total ancestor potential. Let $F(t) = F(t^1, \ldots, t^\mu)$ be a potential of the Frobenius manifold $M_{W,\xi}$. Consider the change of variables $u^\nu = u^\nu(t)$ for $1 \leq \nu \leq \mu$ such that
\[ c^k_{ij}(u) = \Delta_i \delta_{i,j} \delta_{j,k}, \quad 1 \leq i, j, k \leq n, \]
for some functions $\Delta_i = \Delta_i(u)$. Such coordinates $u$ are called canonical. Denote by $\Psi$ the transformation matrix:
\[ \Psi = \Psi(t) : (\partial/\partial t^i) \rightarrow (\partial/\partial u^i). \]
The point $\tau \in M_{W,\xi}$ is called semi–simple if $\Psi(\tau)$ is non–degenerate.
Canonical coordinates $u^i(t)$ as the functions of the flat coordinates are found as the critical values of the singularity unfolding (written in flat coordinates):
\[ u^i(t) = W(P_i, t), \quad 0 = \frac{\partial W(x, t)}{\partial x_1} |_{x = P_i} = \cdots = \frac{\partial W(x, t)}{\partial x_n} |_{x = P_i}. \]
In what follows define $U := \text{diag}(u^1, \ldots, u^\mu)$.

7.2.1. Lefschetz thimbles. Let $W(x, s)$ be the unfolding of a fixed isolated quasi–homogeneous singularity $W(x) = W(x_1, \ldots, x_N)$ with the Milnor number $\mu$. Fix some positive $\rho$, $\delta$ and $\nu$. Let $B^N_\rho \subset \mathbb{C}^N$, $B^1_\delta \subset \mathbb{C}$ and $B^\mu_\nu \subset \mathbb{C}^\mu$ be respectively the balls of radii $\rho$, $\delta$ and $\nu$ centered at the origin. Consider the space:
\[ X_{\rho,\delta,\nu} := (B^N_\rho \times B^1_\delta \times B^\mu_\nu) \cap \varphi^{-1} (B^1_\delta \times B^\mu_\nu) \subset \mathbb{C}^N \times \mathbb{C}^\mu. \]
Taking $\rho$ such that $X_{0,0,0}$ is intersected transversally by $\partial B^N_\rho$ for all $r : 0 < r \leq \rho$ and $\delta, \nu$ such that $X_{\lambda,\lambda}$ is intersected transversally by $\partial B^N_\rho$ for all $(\lambda, s) \in B^1_\delta \times B^\mu_\nu$ we get the following proposition.

**Proposition 7.1 ([AGV]).** There is $D \subset B^1_\delta \times B^\mu_\nu$ such that for $X' := X_{\rho,\delta,\nu} \setminus \varphi^{-1}(\partial B^N_\rho)$ the map $\varphi : X' \rightarrow (B^1_\delta \times B^\mu_\nu) \setminus D$ is a locally trivial fibration with a generic fibre homeomorphic to a bouquet of $\mu$ spheres of dimension $N - 1$.

For $m \in \mathbb{R}$ consider the space $X_m^\nu \subset X_{\rho,\delta,\nu}$ defined by:
\[ X_m^\nu := \{ (x, s) \in X' \mid \text{Re } (W(x, s)/z) \leq -m \}. \]
Because of the proposition above and exact sequence of the pair we get the following isomorphisms:
\[ H_N(X', X_m^\nu) \cong H_{N-1}(X_{\lambda,\lambda}) \cong \mathbb{Z}^\mu. \]
Define the cycles\(^2\)
\[ A_{s,z} \in \lim_{m \to \infty} H_N(X', X_m^\nu; \mathbb{C}) \cong \mathbb{C}^\mu. \]
\(^2\) These cycles have first appeared in [G2] and used later for example in [MR] in the following form. For $(\mathbb{C}^N)_m := \{ x \in \mathbb{C}^N | \text{Re } (W(s, x)/z) \leq -m \}$. Define $A \in \lim_{m \to \infty} H_N(\mathbb{C}^N, (\mathbb{C}^N)_m; \mathbb{C}) \cong \mathbb{C}^\mu$. However such a definition should be considered more like a notation based on the fact that $X'^\nu$ is contractible while the similarity of $X_m^\nu$ and $(\mathbb{C}^N)_m$ is clear.
Slightly obusing the notation in what follows we denote by $A_1, \ldots, A_\mu$ the basis of the relative homology group above.

7.2.2. Oscillatory integrals. Having built the cycles $A_k$ we are going to integrate the function $\exp(W(x,s)/z)$ against them. Considered as the functions of $s$ the stationary phase asymptotic of these integrals by taking $z \to 0$ reads:

$$\int_{A_k} \exp(W(x,s)/z) \zeta \sim e^{u_k/z} \frac{e^{\mu}}{\sqrt{\Delta_k}} (1 + R_1 z + R_2 z^2 + R_3 z^3 + \ldots) (e_k), \quad z \to 0,$$

where $u_k, \Delta_k$ are as in the previous subsection, $e_k$ is a basis vector from $T_s$ and the matrices $R_p$ on the RHS are functions of $s$. We define $R_s := 1 + R_1 z + R_2 z^2 + R_3 z^3 + \ldots$. It’s not hard to check this is indeed an upper–triangular Givental’s group element. Note that the matrix $R_s$ is only defined for a semi–simple $s \in S$.

7.2.3. Total ancestor potential. Consider the formal coordinates $u^p := \{u^{d,p}\}$ for a fixed $p \text{ s.t. } 1 \leq p \leq \mu$ and all integer $d \geq 0$. Define:

$$D_{KdV}^{(p)}(\hbar, u^p) := \exp \left( \sum_{n \geq 1} \frac{\hbar^{n-1}}{n!} \int_{\gamma_{p,n}} \prod_{i=1}^n \left( \sum_{d \geq 0} u^{d,p}(\psi_i)^d + \psi_i \right) \right).$$

This is the partition function of the trivial CohFT with the Dilaton shift applied.

**Definition 7.1.** The partition function $A_s(\hbar, t)$ formally defined as follows is called total ancestor potential.

$$A_s(\hbar, t) := \hat{\Psi}_s \hat{R}_s \prod_{p=1}^\mu D_{KdV}^{(p)}(\hbar \Delta_p, u^p \sqrt{\Delta_p}),$$

where $\hat{R}_s$ acts as the Givental’s upper triangular group element, $\hat{\Psi}_s$ acts by the change of variables $u^{d,i} := (\Psi(s))^{i,t_{d,j}}$.

In the formula above it’s very important to distinguish the dependance of the total ancestor potential on $t$ and on $s$. The first one is a formal variable, while $s$ stands for a choice of the point of $S$ around which the structure is defined. This is best seen on the connection of the total ancestor potential to the Frobenius manifold $M_{W,\zeta}$. The function

$$F_s(t) := [\hbar^{-1}] \log A_s(\hbar, t)|_{t^0, d=0, d \geq 1}$$

is the potential of the Frobenius manifold structure of $M_{W,\zeta}$ in the neighborhood of the point $s$.

Hence the variables $t^{0,p}$ correspond indeed to the flat coordinates discussed at the start of this section while for $s$, considered as the parameter, it’s even irrelevant to speak about the flatness.
7.3. Choice of the primitive form. All the elements building up both total ancestor potential and Frobenius manifold of the singularity $W(x)$ depend heavily on the choice of the primitive form. Let $W(x)$ be one of the following polynomials:

\[
\begin{align*}
    W_1(x) &= x^3 + x_2^3 + x_3^3 + \sigma x_1 x_2 x_3, \\
    W_2(x) &= x_1^2 x_3 + x_1 x_2^3 + x_3^2 + \sigma x_1 x_2 x_3, \\
    W_3(x) &= x_1^3 x_3 + x_2^3 + x_3^2 + \sigma x_1 x_2 x_3,
\end{align*}
\]

that depend also on the additional complex parameter $\sigma$. These polynomials $W(x) = W_\sigma(x)$ defines so-called simple--elliptic singularities. Considering $E_\sigma := \{W_\sigma(x) = 0\} \subset \mathbb{P}^2(c_1, c_2, c_3)$ for some $c_i$ one gets a family of elliptic curves $E_\sigma \to \mathbb{C} \setminus D$, where $D := \{\sigma \in \mathbb{C} \mid E_\sigma \text{ is singular}\}$.

Let $\Omega_\sigma$ be a holomorphic volume form on $E_\sigma$ and $A_\sigma \in H_1(E_\sigma, \mathbb{C})$--flat family of cycles. The primitive form for simple elliptic singularities was found by K. Saito (Examples in [S, Paragraph 3]):

\[
\zeta = \frac{d^3 x}{\pi A(\sigma)} \quad \text{for} \quad \pi A(\sigma) := \int_{E_\sigma} \Omega_\sigma.
\]

The period $\pi A(\sigma)$ is a solution to the Picard-Fuchs equation and is fixed by the choice of a particular $A_{\sigma_0} \in H_1(E_{\sigma_0})$ for some fixed $\sigma_0$.

This property was explored differently in [MR] and [BT] to consider different primitive forms of the fixed simple elliptic singularity.

7.3.1. Analytical action on the 3–dimensional Frobenius manifolds. Let $\gamma(t)$ be a Chazy equation solution $\gamma''' = 6\gamma\gamma'' - 9(\gamma')^2$. It’s easy to see that for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{C})$ the following function is solution to Chazy equation too.

\[
\gamma^A(t) := \frac{\det(A)}{(ct+d)^2} \gamma\begin{pmatrix} at+b \\ ct+d \end{pmatrix} - \frac{c}{2(ct+d)}. \]

Hence for any $A \in \text{GL}(2, \mathbb{C})$ we get the potential of the 3–dimensional Frobenius manifold (recall Example 1):

\[
F^A(t^1, t^2, t^3) := \frac{1}{2} (t^1)^2 t^3 + \frac{1}{2} t^1 (t^2)^2 - \frac{(t^2)^4}{16} \gamma^A(t^3).
\]

However it’s clear that for

\[
A' := \begin{pmatrix} a/\det(A) & b \\ c/\det(A) & d \end{pmatrix} \in \text{SL}(2, \mathbb{C})
\]
the potentials $F^A$ and $F^{A'}$ differ by the following rescaling:

$$\det(A) F^{A'} \left( \frac{t^1}{\det(A)^{1/2}}, t^2, \det(A)t^3 \right) = F^A(t^1, t^2, t^3).$$

In [BT] the authors proposed the following $GL(2, \mathbb{C})$ action as the model for the primitive form change. For any $\tau_0 \in \mathbb{H}$ and $\omega_0 \in \mathbb{C}^*$ define:

$$A^{(\tau_0, \omega_0)} := \begin{pmatrix} \bar{\tau}_0 & \omega_0 \tau_0 \\ \frac{4\pi \omega_0 \text{Im}(\tau_0)}{\omega} & \omega \end{pmatrix}. $$

For any $\tau_0 \in \mathbb{H}$ and $\omega_0 \in \mathbb{C}^*$ we have $\det(A^{(\tau_0, \omega_0)}) = 1/(2\pi \sqrt{-1})$ and because of the reasoning above we can consider unique $SL(2, \mathbb{C})$ action defined by $(\tau_0, \omega_0)$. Fixing some particular solution to Chazy equation $\gamma^\infty(t)$ consider the Frobenius manifold potential

$$F^{(\tau_0, \omega_0)}(t^1, t^2, t^3) := \frac{1}{2}(t^1)^2 t^3 + \frac{1}{2}(t^2)^2 - \frac{(t^3)^4}{16} \gamma^{(\tau_0, \omega_0)}(t^3),$$

where $\gamma^{(\tau_0, \omega_0)}$ is obtained from $\gamma^\infty$ by the $SL(2, \mathbb{C})$–action of $(\tau_0, \omega_0)$.

The action of $A^{(\tau_0, \omega_0)}$ on the space of 3–dimensional Frobenius manifolds can be considered as the model for the change of the primitive form of a simple elliptic singularity (cf. Section 2.5 in [BT]). We are going to compare it with the completely different approach of Milanov–Ruan.

7.3.2. Approach of Milanov–Ruan. From now on we fix $n := \mu$, the Milnor number of the singularity. Let $e_1, \ldots, e_n$ be the basis vectors of $T_s S$ at $s = \{s^1, \ldots, s^n\}$. For any $A \in SL(2, \mathbb{C})$ let the linear operator $J : T_s S \rightarrow T_s S$ be defined by:

$$J(e_1) = e_1, \quad J(e_n) = (cs^n + d)^2 e_n, \quad J(e_p) = (cs^n + d)e_p, \quad 1 < p < n, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. $$

Let $q = \{q^i_d\}$ be as in Section 3.2 with the upper index corresponding to the vector number in the basis fixed above. Introduce the action of $J$ on $q$ by:

$$J : q^i_d \rightarrow \sum_{i=1}^{n} J^i_d q^i_d, \quad \forall d \geq 0.$$ 

Consider Givental’s upper–triangular group element $X_v(z)$ defined by:

$$(X_v(z))^i_d = \delta^i_d + -z \frac{c}{cs^n + d} \delta_i n \delta^i_d.$$
Theorem 7 (Theorem 4.4 in [MR]). Consider total ancestor potentials of a simple elliptic singularity with the primitive forms $\zeta_1$ and $\zeta_2$. Then there is $A \in \text{SL}(2, \mathbb{C})$ such that the corresponding total ancestor potentials are connected by the following transformation:

$$A_{\zeta_1, A \cdot \mathbf{s}}(\hbar, \mathbf{q}) = (\tilde{X}_s \cdot A_{\zeta_2, \mathbf{s}})((c\tau + d)^2 \hbar, J\mathbf{q}),$$

where the action of $\tilde{X}_s$ is applied as in Proposition (3.1), $\mathbf{s} = (0, \ldots, 0, \tau)$ and $A \cdot \mathbf{s} = (0, \ldots, 0, A \cdot \tau)$.

In [MR] in order to fix particular primitive form the authors choose particular solution to the Picard–Fuchs equation. In this way it’s very hard to present particular $\text{SL}(2, \mathbb{C})$ matrix of the theorem above.

7.4. Equivalence of the approaches. Comparing the total ancestor potential formula of Milanov–Ruan with the $\text{SL}(2, \mathbb{C})$ action developed in this paper we get the theorem.

Theorem 8. Let $F$ be the Frobenius manifold potential of a simple elliptic singularity. Then the transformation of Milanov–Ruan is equivalent to the $\text{SL}(2, \mathbb{C})$ action on $F$ given by (2).

Proof. Consider particular pair of primitive forms $\zeta_1$ and $\zeta_2$ with the corresponding $\text{SL}(2, \mathbb{C})$–matrix $A$ as in Theorem 7. We show that the transformation of Milanov-Ruan acts in the same way as Givental’s form of our $\text{SL}(2, \mathbb{C})$ action.

Denote by $\hat{J}$ the rescaling action $(\hbar, \mathbf{q}) \rightarrow ((c\tau + d)^2 \hbar, J\mathbf{q})$. Note that its action is the same as $S_0^A$ action of Corollary 4.3. Recall $R^\sigma$–action of Section 4.1. Because of its particular form we have:

$$R^\sigma(z) = \exp(r^\sigma z) = 1 + r^\sigma z.$$

By Corollary 4.3 we know that for $\sigma := -c(c\tau + d)$ the following equality holds:

$$\hat{R}^\sigma \cdot \hat{S}_0^A = \hat{J} \cdot \tilde{X}_s.$$

Hence the transformation of Milanov–Ruan is represented by the $\text{SL}(2, \mathbb{C})$ action we consider. We only have to compare the points of the Frobenius manifolds on both sides. Using Theorem 7, Theorem 1 and Proposition 3.2 we write:

$$F_{\zeta_1, A \cdot \mathbf{s}} = \left(\hat{R}^\sigma\right)^{-1} \hat{S}_0^A F_{\zeta_2, \mathbf{s}} = \hat{R}^{-\sigma} \hat{S}_0^A F_{\zeta_2, \mathbf{s}} = \hat{S}_0^A \tilde{X}_s F_{\zeta_2, \mathbf{s}}.$$

This completes proof of the theorem. □

Corollary 7.1. Two approaches of [MR] and [BT] for the action changing the primitive form of the simple elliptic singularity coincide.

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3Only one particular case of simple elliptic singularity $P_8$ was considered in [MR]. However the technique used is extended in a straightforward way to all other cases too.
This result shows that taking all the possible primitive forms for a fixed simple elliptic singularity there are only two “geometrically” different Frobenius manifolds — $F_\zeta$ and $I \cdot F_\zeta$ (where $I$ is the Inversion transformation of Dubrovin). All the other Frobenius manifolds fixed by other choices of the primitive forms are obtained from these two by taking linear changes of the variables.

7.5. **Application to the Gromov–Witten theory.** The change of the primitive form for a given singularity $W(x)$ is of big importance in Mirror symmetry. For $W(x)$ defining a hypersurface simple–elliptic singularity it was proved by Milanov–Ruan and Milanov–Shen (cf. [MR, MS1]) that for a special choice of the primitive form $\zeta_\infty$ the total ancestor potential of $W$ with this primitive form coincides with the partition function of the (orbifold) Gromov–Witten theory of $X$, for $X$ one of $\mathbb{P}^{3,3,3}, \mathbb{P}^{4,4,2}, \mathbb{P}^{6,3,2}$. Another amazing result of Milanov–Ruan later extended by Milanov–Shen is the following theorem proved via the analysis of the oscillator y integrals:

**Theorem 9** (Theorem 4.2 in [MR], Proposition 3.4 in [MS2]). For any choice of the primitive form $\zeta$ for a hypersurface simple–elliptic singularity there are $A \in \text{SL}(2, \mathbb{Z})$ such that:

$$(\hat{X}_s \cdot A \cdot s) (\frac{(c\tau + d)^2}{\hbar}, Jq) = A \cdot \zeta_\infty (h, q),$$

where the action of $\hat{X}_s$ is applied as in Proposition (3.1), $s = (0, \ldots, 0, \tau)$ and $A \cdot s = (0, \ldots, 0, A \cdot \tau)$.

Namely, comparing to Theorem 7 the primitive form on the both sides of the equality above is the same. Such transformtaions were called modular.

Applied to the total ancestor potential with the primitive form $\zeta_\infty$ the partition functions of the Gromov–Witten theories of the elliptic orbifolds were proved to be quasi–modular forms (recall the definition from Section 2) in [MR, MS2]. We give the quasi–modularity result mentioned in a simplified — genus 0 version.

**Corollary 7.2** (Corollary 6.7 in [MR], Corollary 1.3 in [MS2]). Consider the genus zero potential $F^X(t)$ of an elliptic orbifold $X$. Let $f_k(t^n)$ be its coefficients in $t^1, \ldots, t^{n-1}$:

$$f_k(t^n) = \left[ t^{k_1} \cdots t^{k_p} \right] F(t), \quad k = \{ k_1, \ldots, k_p \},$$

for a different index set $k$. Then the functions $f_k(t^n)$ are quasi–modular forms.

Because we have identified the transformation of Theorem 7 with the analytical $\text{SL}(2, \mathbb{C})$ action on the Frobenius manifold potential we can make the following state-ment:

**Proposition 7.2.** Consider the Frobenius manifold potential $F^X(t)$ of an elliptic orbifold $X$. Let $f_k(t^n)$ be its coefficients in $t^1, \ldots, t^{n-1}$. Then holds:
(a) $f_k$ are quasi–modular of weight 2 if

$$ \frac{\partial^4}{\partial t^{k_1} \partial t^{k_2} \partial t^{k_3} \partial t^{k_4}} \left( \frac{\partial F^X(t)}{\partial t^k} \right)^2 \neq 0, $$

(b) $f_k$ is modular of weight $|k| - 2$ otherwise.

**Proof.** The proposition follows immediately by comparing the coefficients in $t^2, \ldots, t^{n-1}$ of the equality $(F^X)^A = F^X$. In particular the $Q$–term has to be annihilated by some $f_k$, that hence turn out to be quasi–modular. □

The genus 0 potentials of the elliptic orbifolds are not polynomial — the functions $f_k$ in the theorem above have infinite Fourier series. However their Fourier series can be computed up to any order needed (using for example the reconstruction theorems 3.1 and 4.2 in [IST]). It’s well known that if two functions $g_1(\tau)$ and $g_2(\tau)$ are modular, then only finite number of terms (given by the Stein bound) in their Fourier expansions are to be identified to check if $g_1(\tau) \equiv g_2(\tau)$. Because of Proposition 7.2 we know that majority of functions $f_k$ in the theorem above are indeed modular and we can apply the Stein bound argument.

For those $f_k$ that are not modular amazing theorem of Kaneko–Zagier ([KZ]) can be applied, giving that any quasi–modular form $g(\tau)$ of weight 2 can be expressed as the product of the second Eisenstein series $E_2(\tau)$ and a weight 0 modular form. Hence we can apply the Stein bound argument again. Using the computer program we have computed the genus 0 potentials of the elliptic orbifolds in the closed form using this method. The expressions are too big and therefore we put them on the webpage [B1].

Using the explicit characterization of the ring of modular forms, similar computations were done by Y.Shen and J.Zhou in [SZ]. They have computed explicitly some of the correlators of the Gromov–Witten theory of the elliptic orbifolds, that are enough to reconstruct all genera data. Our result agree, however they do not give a closed formula for the genus 0 potential and we do not have a formula for the higher genera potentials.

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