THE RESEARCH ON ROTATIONAL SURFACES IN PSEUDO EUCLIDEAN 4-SPACE WITH INDEX 2

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Abstract. In this study, we define a brief description of the hyperbolic and elliptic rotational surfaces using a curve and matrices in 4-dimensional semi-Euclidean space with index 2. That is, we provide different types of rotational matrices, which are the subgroups of $M$ by rotating a selected axis in $E^4$. Also, we choose two parameter matrices groups of rotations and we give the matrices of rotation corresponding to the appropriate subgroup in 4-dimensional semi-Euclidean space. Therefore, we generate surfaces of rotation using Killing vector fields in $E^4_2$ and we give the Gaussian curvature and the mean curvature of the surfaces of rotation.

1. Introduction

From the past to the present many studies have been done that deal with rotational surfaces from algebraic and geometric aspects. The rotational surfaces are parametrized with the help of the Killing vector field. Therefore, the different types of matrices of rotations which are the subgroups of a manifold corresponding to rotation about a chosen axis in the arbitrary 4D-space are expressed. Hence, the two parameter matrices groups of rotations can be chosen and the matrices of rotation corresponding to the appropriate subgroup in arbitrary 4D-space are expressed. To mention briefly for the publications taken as reference related to the subject studied. In [1], the geometric quantities associated with the concept of surfaces and the indicatrix of a surface are discussed in four-dimensional Galilean space by the authors. In [2], the brief description of rotational surfaces are given using a curve and matrices in 4-dimensional (4D) Galilean space. Also, choosing two parameter matrices groups of rotations, the matrices of rotation corresponding to the appropriate subgroup in Galilean 4-space, the rotated surfaces are expressed by the authors. In [3, 4], the authors gave magnetic rotated surfaces in lightlike cone $Q^2 \subset E^3_1$. Furthermore, the conditions being geodesic on rotational surface generated by magnetic curve are expressed with the help of Clairaut’s theorem. In [5], the representation formulas of non-null curves are expressed in semi-Euclidean 4-space $E^4_2$ and some certain results of describing the non-null normal curve are presented in $E^4_2$. In [7, 8], the rotational surfaces are studied by different authors in Minkowski 4-space. In [9], the some issues of displaying two-dimensional surfaces in 4D space are examined by authors. In [14], the translation surface in the case being harmonic surface are mainly studied, the necessary and sufficient conditions of being semi-parallel surfaces by considering semi-parallel condition given

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by the authors. In [15], the surfaces of revolution are characterized in the three dimensional pseudo-Galilean space.

2. Preliminaries

Let \( E_4^3 \) denote the 4-dimensional pseudo-Euclidean space with signature \((2, 4)\), that is, the real vector space \( \mathbb{R}^4 \) endowed with the metric \( \langle \cdot, \cdot \rangle_{E_4^3} \) which is defined by

\[
\langle \cdot, \cdot \rangle_{E_4^3} = -dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2,
\]

where \((x_1, x_2, x_3, x_4)\) is a standard rectangular coordinate system in \( E_4^3 \).

Recall that an arbitrary vector \( v \in E_4^3 \setminus \{0\} \) can have one of three characters: it can be space-like if \( g(v, v) > 0 \) or \( v = 0 \), time-like if \( g(v, v) < 0 \) and null if \( g(v, v) = 0 \) and \( v \neq 0 \).

The norm of a vector \( v \) is given by \( \| v \| = \sqrt{g(v, v)} \) and two vectors \( v \) and \( w \) are said to be orthogonal if \( g(v, w) = 0 \). An arbitrary curve \( x(s) \) in \( E_4^3 \) can locally be space-like, time-like or null.

A space-like or time-like curve \( x(s) \) has unit speed, if \( g(x', x') = \pm 1 \).

Let \((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4), (z_1, z_2, z_3, z_4)\) be any three vectors in \( E_4^3 \). The pseudo Euclidean cross product is given as

\[
x \wedge y \wedge z = \begin{pmatrix} -i_1 & -i_2 & i_3 & i_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix},
\]

where \( i_1 = (1, 0, 0, 0), i_2 = (0, 1, 0, 0), i_3 = (0, 0, 1, 0), i_4 = (0, 0, 0, 1) \).

The pseudo-Riemannian sphere \( S_2^3(m, r) \) centered at \( m \in E_4^3 \) with radius \( r > 0 \) is defined by

\[
S_2^3(m, r) = \{ x \in E_4^3 : \langle x - m, x - m \rangle = r^2 \}.
\]

The pseudo-hyperbolic space \( H_1^3(m, r) \) centered at \( m \in E_4^3 \) with radius \( r > 0 \) is defined by

\[
H_1^3(m, r) = \{ x \in E_4^3 : \langle x - m, x - m \rangle = -r^2 \}.
\]

The pseudo-Riemannian sphere \( S_3^3(m, r) \) is diffeomorphic to \( \mathbb{R}^2 \times S^1 \) and the pseudo-hyperbolic space \( H_1^3(m, r) \) is diffeomorphic to \( S^1 \times \mathbb{R}^2 \). The hyperbolic space \( H_1^3(m, r) \) is given by

\[
H_1^3(m, r) = \{ x \in E_4^3 : \langle x - m, x - m \rangle = -r^2, x_1 > 0 \}.
\]

Let \( \Psi : M \to E_4^3 \) be an isometric immersion of oriented pseudo-Riemannian submanifold \( M \) into \( E_4^3 \). Henceforth, a submanifold in \( E_4^3 \) always means pseudo-Riemannian. Let \( \nabla \) be the Levi-Civita connection of \( E_4^3 \) and \( \nabla \) be the induced connection on \( M \). Also, for any vector fields \( X, Y \) tangent to \( M \), we get the Gaussian formula

\[
\nabla_X Y = \nabla_X Y + h(X, Y),
\]

where \( h \) is the second fundamental form which is symmetric in \( X \) and \( Y \). For a unit normal vector field \( \xi \), the Weingarten formula is defined by

\[
\nabla_X \xi = -A_\xi X + D_\xi X,
\]
where $A_\xi$ is the Weingarten map or the shape operator with respect to $\xi$ and $D$ is the normal connection. The Weingarten map $A_\xi$ is a self-adjoint endomorphism of $TM$ which cannot be diagonalized generally. It is known that $h$ and $A_\xi$ are related by

\begin{equation}
(2.5) \quad \langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.
\end{equation}

The covariant derivative $\tilde{\nabla} h$ of the second fundamental form $h$ is given by

\begin{equation}
(2.6) \quad \tilde{\nabla}_X h(Y, Z) = \nabla^\perp_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),
\end{equation}

where $\nabla^\perp$ indicates the linear connection induced on the normal bundle $T^\perp M$. Also, Codazzi equation is given by

\begin{equation}
(2.7) \quad \tilde{\nabla}_X h(Y, Z) = \tilde{\nabla}_Y h(X, Z).
\end{equation}

Let $e_1, e_2, ..., e_m$ be a local orthonormal frame field in $E^m_4$ such that $e_1, e_2, ..., e_n$ are tangent to $M^n$ and $\{e_{n+1}, ..., e_m\}$ are normal to $M^n$. Let $w_1, w_2, ..., w_m$ be the coframe of $e_1, e_2, ..., e_m$. We'll make use of the following convention on the ranges $1 \leq i, j, ..., \leq n, n+1 \leq s, t, ..., \leq 4, 1 \leq A, B, ..., \leq 4$. Also, $w_A(e_B) = \delta_{AB}$ and the pseudo-Riemannian metric on $E^m_4$ is given by

\begin{equation}
(2.8) \quad ds^2 = \sum_{i=1}^{n} \varepsilon_A w_A^2; \varepsilon_A = \langle e_A, e_A \rangle = \pm 1.
\end{equation}

Let $w_A$ be the dual 1-form of $e_A$ defined by $w_A X = \langle e_A, X \rangle$. Also, the connection forms $w_{AB}$ are defined by

\begin{equation}
(2.9) \quad de_A = \sum \varepsilon_B w_{AB} e_B; w_{AB} + w_{BA} = 0.
\end{equation}

After, the structure equations of $E^4_2$ are written as follows

\begin{equation}
(2.10) \quad dw_A = \sum_B \varepsilon_B w_{AB} \wedge w_B; dw_A = \sum_C \varepsilon_C w_{AC} \wedge w_CB.
\end{equation}

The canonical forms $\{w_A\}$ and the connection forms $\{w_{AB}\}$ restricted to $M^n$ are also indicated by the same symbols. Also, we get

\[ w_s = 0, s = n+1, ..., 4 \]

and since $w_s$ are zero forms on $M^n$, there are symmetric tensor $h^s_{ij}$ by Cartan’s lemma such

\begin{equation}
(2.11) \quad w_{is} = \sum_j \varepsilon_j h^s_{ij} w_j; h^s_{ij} = h^s_{ji}.
\end{equation}

The mean curvature vector $H$ of $M^n$ in $E^m_4$ is given by

\begin{equation}
(2.12) \quad H = \frac{1}{2} \sum_{s=n+1}^{m} \varepsilon_s \sum_{i=1}^{n} \varepsilon_j h^s_{ij} e_s.
\end{equation}

Also, the covariant differentiation of $e_i$ is given by

\[ de_i = \sum_A \varepsilon_A w_i A e_A \text{ or } \tilde{\nabla}_e_i e_j = \sum_B \varepsilon_B w_j B (e_i) e_B, \]

\[ [6, 10, 11]. \]

**Definition 1.** [10], A one-parameter group of Diffeomorphisms of a manifold $M$ is a regular map $\psi : M \times \mathbb{R} \rightarrow M$, such that $\psi_t(x) = \psi(x, t)$, where
1. \( \psi_t : M \rightarrow M \) is a Diffeomorphism
2. \( \psi_0 = \text{id} \)
3. \( \psi_{s+t} = \psi_s \circ \psi_t \).

This group is attached with a vector field \( W \) given by \( \frac{\partial}{\partial t} \psi_t(x) = W(x) \), and the group of Diffeomorphisms is said to be as the flow of \( W \).

**Definition 2.** If a one-parameter group of isometries is generated by a vector field \( W \), then this vector field is called a Killing vector field, \([10]\).

**Definition 3.** Let \( W \) be a vector field on a smooth manifold \( M \) and \( \psi_t \) be the local flow generated by \( W \). For each \( t \in \mathbb{R} \), the map \( \psi_t \) is Diffeomorphism of \( M \) and given a function \( f \) on \( M \), we consider the Pull-back \( \psi_t^*f \). We define the Lie derivative of the function \( f \) as to \( W \) by

\[
L_W f = \lim_{t \to 0} \left( \psi_t f - f \right) = \frac{d}{dt} \psi_t f \bigg|_{t=0}.
\]

Let \( g_{xy} \) be any pseudo-Riemannian metric, then the derivative is given as

\[
L_W g_{xy} = g_{xz}W^z_y + g_{zy}W^z_x.
\]

In Cartesian coordinates in Euclidean spaces where \( g_{xy,z} = 0 \), and the Lie derivative is given by

\[
L_W g_{xy} = g_{xz}W^z_y + g_{zy}W^z_x.
\]

In \([10]\), the vector \( W \) generates a Killing field if and only if

\[
L_W g = 0.
\]

### 3. The Surfaces of Rotation in \( E^4_2 \)

In this chapter, we provides a description of surfaces of rotation in \( E^4_2 \). Here, we have used the metric \((2.1)\). Therefore, we will provide different types of matrices of rotations, which are the subgroups of \( M \) by rotated a selected axis in \( E^4 \). Hence, we will choose two parameter matrices groups of rotations. In particular, we have defined a brief description of rotational surfaces in four dimensional \( E^4_2 \) and we give the rotational matrices corresponding to the appropriate subgroup in \( E^4_2 \). Hence, we generate the rotational surfaces.

The rotation matrices are replaced by Lorentz transformation as follows

\[
M^T g T = g,
\]

where \( M^T \) is the transpose, \( g \) is the metric matrix of \( E^4_2 \) and for the metric \((2.1)\).

Let’s obtain the set of all \( 4 \times 4 \) type matrices satisfying \((3.1)\). The Lorentz group is a subgroup of the Diffeomorphisms group in \( E^4_2 \).

**Theorem 1.** Let the pseudo-Euclidean group be a subgroup of the Diffeomorphisms group in \( E^4_2 \) and let \( W \) be vector field which generate the isometries. Then, the killing vector field associated with the metric \( g \) is given as

\[
W(\xi, \eta, \vartheta, \varphi) = a(\eta \partial \xi + \xi \partial \eta) + b(\vartheta \partial \eta + \varphi \partial \vartheta) + c(\vartheta \partial \xi + \xi \partial \vartheta) + d(\eta \partial \vartheta + \varphi \partial \eta) + e(\vartheta \partial \eta - \eta \partial \vartheta) + f(\xi \partial \varphi - \vartheta \partial \xi),
\]

where \( a, b, c, d, e, f \in \mathbb{R}_0^+ \).
Let $W$ be the vector which generate the isometries in $E^2_3$. We can write as the following the general vector field;

\[(3.3)\]

\[W(\xi, \vartheta, \eta) = W^1(\xi, \vartheta, \eta)\partial\xi + W^2(\xi, \vartheta, \eta)\partial\vartheta + W^3(\xi, \vartheta, \eta)\partial\eta + W^4(\xi, \vartheta, \eta),\]

where $W^j$ are real functions (for $j = 1, 2, 3, 4$). Also, by using definition 2 and definition 3, the expression of the (3.3) is

\[(3.4)\]

\[W^1_\xi = W^2_\vartheta = W^3_\eta = W^4 = 0,\]

\[(3.5)\]

\[W^1_\vartheta + W^2_\xi = 0; W^1_\vartheta - W^3_\xi = 0; W^1_\eta - W^4_\xi = 0,\]

\[(3.6)\]

\[W^2_\vartheta - W^3_\xi = 0, W^2_\eta - W^4_\xi = 0, W^3_\eta + W^4_\xi = 0,\]

first, we will obtain the function $W^1$, then from (3.4) and (3.5) we write

\[(3.7)\]

\[W^1_\vartheta + W^2_\xi = 0.\]

then differentiating with respect to $\vartheta$ in the previous equation (3.7), we have

\[(3.8)\]

\[W^1_{\vartheta\vartheta} + W^2_{\xi\vartheta} = 0\]

and then differentiating with respect to $\vartheta$ in the equations $W^1_\vartheta - W^3_\xi = 0$, we obtain

\[(3.9a)\]

\[W^1_{\vartheta\vartheta} - W^3_{\xi\vartheta} = 0,\]

and then differentiating with respect to $\eta$ in the equations $W^1_\eta - W^4_\xi = 0$, we obtain

\[(3.9b)\]

\[W^1_{\eta\eta} - W^4_{\xi\eta} = 0\]

and from (3.4) we get $W^2_{\vartheta\vartheta}, W^4_{\xi\eta}, W^3_{\xi\eta} = 0$. From (3.8) and (3.9a), (3.9b) which gives $W^1_{\vartheta\vartheta}, W^1_{\eta\eta}, W^1_{\vartheta\vartheta} = 0$. Therefore, the function $W^1$ can be written as follows

\[(3.10)\]

\[W^1(\vartheta, \eta) = f^1_1(\vartheta, \eta)\vartheta + g^1_1(\vartheta, \eta),\]

\[(3.11)\]

\[W^1(\vartheta, \eta) = f^1_2(\vartheta, \eta)\eta + g^1_2(\vartheta, \eta),\]

\[W^1(\vartheta, \eta) = f^1_3(\eta, \vartheta)\eta + g^1_3(\eta, \vartheta),\]

where $f^1_i, g^1_i \in C^\infty, i = \{1, 2, 3\}$. From (3.10) and since $W^1_{\eta\eta} = 0$ we get

\[W^1_{\eta\eta}(\vartheta, \eta) = f^1_{1\eta\eta}(\vartheta, \eta)\vartheta + g^1_{1\eta\eta}(\vartheta, \eta) = 0,\]

this means $f^1_{1\eta\eta}(\vartheta, \eta) = g^1_{1\eta\eta}(\vartheta, \eta) = 0$. Thus, we can write the equations $f^1_1(\vartheta, \eta)$ and $g^1_1(\vartheta, \eta)$ as follows

\[f^1_1(\vartheta, \eta) = h^1_1(\vartheta, \eta) + m^1_1(\vartheta),\]

\[g^1_1(\vartheta, \eta) = h^1_1(\vartheta, \eta) + m^2_1(\vartheta).\]

Furthermore, since $W^1_{\vartheta\vartheta} = 0$ we can choose the functions $h_1, h^*_1, m_1, m_2$ as

\[h^*_1(\vartheta) = a^*_1 \vartheta + b_1, h^*_1(\vartheta) = a^*_1 \vartheta + b_2,\]

\[m^*_1(\vartheta) = c_1 \vartheta + d_1, m^*_1(\vartheta) = c^*_2 \vartheta + d_2.\]

Furthermore, substituting this equation into (3.10), we have

\[(3.13)\]

\[W^1(\vartheta, \eta) = ((a^*_1 + b_1) \eta + c_1 \vartheta + d_1) \vartheta + ((a^*_2 \vartheta + b_2) \eta + c^*_2 \vartheta + d_2).\]
Similarly, by making the necessary algebraic operations, the following component equations are obtained, respectively.

\[ W^2(\xi, \vartheta, \eta) = ((a_2 \vartheta + b) \eta + x_1^2 \xi + y_1^2) \xi + (a_1^* \vartheta + b^*) \eta + x_2^2 \vartheta + y_2^2; \]
\[ W^3(\varrho, \xi, \eta) = ((a_3 \eta + d) \xi + x_1^3 \eta + y_1^3) \varrho + (a_2^* \eta + d^*) \xi + x_2^3 \eta + y_3^2; \]
\[ W^4(\xi, \vartheta, \varrho) = ((a_4 \vartheta + e) \varrho + x_1^4 \xi + (a_3^* \vartheta + e^*) \varrho + x_2^4 \vartheta + y_4^2; \]

where \( a_i, a_i^*, x_i^j, b, d, e, b^*, d^*, e^* \in \mathbb{R}; i, j \in I. \)

If we assume arbitrary constants as

\[ a_i = x_i^1 = c_1 = b_1 = b = e = a_i^* = y_i^2 = d_2 = 0; i \in \{1, 2, 3, 4\} \]

then we obtain

\[ W^1(\varrho, \vartheta, \eta) = d_1 \varrho + b_2 \eta + c_2 \varrho; W^2(\xi, \vartheta, \eta) = y_1^2 \xi + b^* \eta + x_2^2 \vartheta; \]
\[ W^3(\varrho, \xi, \eta) = y_1^3 \varrho + d^* \xi + x_3^3 \eta; W^4(\xi, \vartheta, \varrho) = y_1^4 \xi + e^* \vartheta + x_4^2 \varrho. \]

Furthermore, by using the equations (3.4), (3.5) and (3.6), we write

\[ y_1^2 = -c_2 = f; d_1 = d^* = c; b_2 = y_1^2 = \alpha; \]
\[ x_2^2 = y_3^3 = b; b^* = x_4^2 = d; e^* = -x_3^2 = e. \]

Hence, the vector fields \( W^1, W^2, W^3, W^4 \) are given by

\[ W^1(\varrho, \vartheta, \eta) = c \varrho + a \eta - f \varrho; W^2(\xi, \vartheta, \eta) = f \xi + d \eta + b \vartheta; \]
\[ W^3(\varrho, \xi, \eta) = b \varrho + c \xi - e \eta; W^4(\xi, \vartheta, \varrho) = a \xi + e \vartheta + d \varrho. \]

By using the equation (3.14) into the equation (3.3), we have

\[ W(\xi, \vartheta, \eta) = (c \varrho + a \eta - f \varrho) \partial \xi + (f \xi + d \eta + b \vartheta) \partial \varrho + (b \varrho + c \xi - e \eta) \partial \vartheta \]
\[ + (a \xi + e \vartheta + d \varrho) \partial \eta; \]
\[ W(\xi, \vartheta, \eta) = a (\eta \partial \xi + \xi \partial \eta) + b (\vartheta \partial \varrho + \varrho \partial \vartheta) + c (\vartheta \partial \xi + \xi \partial \vartheta) \]
\[ + d (\varrho \partial \varrho + \varrho \partial \eta) + e (\vartheta \partial \eta - \eta \partial \vartheta) + f (\xi \partial \varrho - \varrho \partial \xi), \]

where \( a, b, c, d, e, f \in \mathbb{R}^+ \).

Theorem 2. Let \( W(\xi, \vartheta, \eta) \) be the killing vector field and let \( \gamma = (f_1, f_2, f_3, f_4) \) be a curve in \( E_2^3 \), then the surfaces of rotation are given as follows

1. For the rotations \( \Omega_1 = \varrho \partial \xi + \xi \partial \varrho \) and \( \Omega_4 = \eta \partial \varrho + \varrho \partial \eta \), the hyperbolic surface of rotation is given as

\[ S_{14}(x, \alpha, s) = \left( \begin{array}{c} f_1 \cosh x + f_3 \sinh x, f_2 \cosh \alpha + f_4 \sinh \alpha, \\ f_1 \sinh x + f_3 \cosh x, f_2 \sinh \alpha + f_4 \cosh \alpha \end{array} \right) \]

and for the curve \( \gamma(s) = (f_1(s), 0, 0, f_4(s)) \) the Gaussian curvature \( K \) and the mean curvature vector \( H \) of the rotational surface \( S_{14}(x(t), \alpha(t), s) \) are given as

\[ K = \frac{(f_1 f_4 - f_1 f_3)^2 (\dot{x} \dot{\alpha})^2}{f_4^2 \dot{x}^2 - f_1^2 \dot{\alpha}^2} + \frac{(f_1 f_4 \dot{x}^2 - f_1 f_3 \dot{\alpha}^2) (f_1 f_4 - f_1 f_3)}{f_4^2 \dot{x}^2 - f_1^2 \dot{\alpha}^2}, \]
\[ H = \frac{f_1 f_4 (\dot{x} \dot{\alpha} + \dot{x} \dot{\alpha})}{2 \sqrt{f_4^2 \dot{x}^2 - f_1^2 \dot{\alpha}^2}} + \frac{f_1 (f_1 \dot{x} \dot{\alpha}^2 - f_1 f_3 \dot{\alpha})}{2 \sqrt{f_4^2 \dot{x}^2 - f_1^2 \dot{\alpha}^2}} e_3 + \frac{(f_1 f_4 - f_1 f_3 \dot{\alpha})}{2 \sqrt{f_4^2 \dot{x}^2 - f_1^2 \dot{\alpha}^2}} e_4 \]

where
Let by using the equations (3.15), we will find 4 of rotation \( \Omega \).

(2) For the rotations \( \Omega_2 = \eta \partial \xi + \xi \partial \eta \) and \( \Omega_3 = \partial \vartheta \partial \varphi + \varphi \partial \vartheta \), the hyperbolic surface of rotation is given as

\[
S_{23}(y, z, s) = \left( f_1 \cosh y + f_4 \sinh y, f_2 \cosh z + f_3 \sinh z, \right) \]

and for the curve \( \gamma(s) = (f_1(s), f_2(s), 0, 0) \) the Gaussian curvature \( K \) and the mean curvature vector \( H \) of the rotational surface \( S_{23}(y(t), z(t), s) = (f_1 \cosh y, f_2 \cosh z, f_3 \sinh z, f_4 \sinh y) \) are given as

\[
K = -\left( \frac{(f_1 f_y + f_4 f_z)^2(y_z)}{(f_1 f_y^2 + f_4 f_z^2)(f_1 f_y + f_4 f_z)} \right); \quad H = \left( \frac{f_1 f_y f_z (y_z)}{2 \sqrt{f_1^2 + f_y^2}} e_3 + \frac{f_1 f_y f_z^2 + f_4 f_z f_y^2 - f_1 f_y f_z}{2 \sqrt{f_1^2 + f_y^2}} e_4 \right),
\]

where

\[
e_3 = \left( f_2 \sinh y, f_1 \sinh z f_2 \cosh y, f_1 \cosh y \right) ; \quad e_4 = \left( f_4 \cosh y, f_2 \sinh y, f_1 \cosh y \right) \]

(3) For the rotations \( \Omega_5 = \xi \partial \varphi - \varphi \partial \xi \) and \( \Omega_6 = \vartheta \partial \eta - \eta \partial \vartheta \), the elliptic surface of rotation is given as

\[
S_{56}(\beta, \theta, s) = \left( f_1 \cos \beta + f_2 \sin \beta, f_3 \cos \theta + f_4 \sin \theta, \right) \]

and for the curve \( \gamma(s) = (0, f_2(s), 0, 0) \) the Gaussian curvature \( K \) and the mean curvature vector \( H \) of the rotational surface \( S_{56}(\beta(t), \theta(t), s) = (f_2 \sin \beta, f_2 \cos \beta, f_4 \sin \theta, f_4 \cos \theta) \) are given as

\[
K = -\left( \frac{(f_1 f_2 f_y + f_2 f_4 f_z)^2(y_z)}{(f_1 f_2 f_y^2 + f_2 f_4 f_z^2)(f_1 f_2 f_y + f_2 f_4 f_z)} \right) ; \quad H = \frac{f_1 f_2 (\beta \theta - \theta \beta)}{2 \sqrt{f_1^2 - f_2^2}} e_3 + \frac{f_1 \beta f_2 (f_2 f_4 f_z^2 - f_1 f_2 f_4 f_z^2)}{2 \sqrt{f_1^2 - f_2^2}} e_4,
\]

where

\[
e_3 = \left( f_2 \cos \beta, f_2 \sin \beta, f_4 \sin \theta \right) ; \quad e_4 = \left( f_1 \sin \beta, f_1 \cos \beta, f_2 \sin \theta \right) \quad (-\infty < x, y, z, \alpha, \beta, \theta < \infty, s \in I) \text{ and } f_i \in C^\infty.
\]

Proof. Let \( W(\xi, \varphi, \vartheta, \eta) = \sigma \Omega_2 + \beta \Omega_3 + \epsilon \Omega_4 + \sigma \Omega_6 + f \Omega_5 \) be the killing vector field. Hence, we can give vector fields generating the rotations as follows

\[
(3.15)(a) \quad \Omega_1 = \partial \xi \xi + \xi \partial \eta; \quad \Omega_2 = \eta \partial \xi + \xi \partial \eta; \quad \Omega_3 = \partial \vartheta \partial \varphi + \varphi \partial \vartheta;
\]

\[
(3.15)(b) \quad \Omega_4 = \eta \partial \varphi + \varphi \partial \eta; \quad \Omega_5 = \xi \partial \varphi - \varphi \partial \xi; \quad \Omega_6 = \vartheta \partial \eta - \eta \partial \vartheta,
\]

by using the equations (3.15), we will find \( 4 \times 4 \) matrices of hyperbolic and elliptic by rotating \( \Omega_i, i \in I \).

a) Hyperbolic matrices: we give some one-parameter hyperbolic matrices groups of rotation \( \Omega_i, i = 1, 2, 3, 4 \).

1) For \( \Omega_1 = \partial \xi \xi + \xi \partial \eta \), we write the vector field

\[
(3.16) \quad \Lambda_{\Omega_1} = \left[ \begin{array}{cc}
0 & \xi \\
\eta & 0 
\end{array} \right]^T,
\]
then, the previous equation can be given as follows

\[(3.17) \Delta_{\Lambda_1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \]

from definition 1, by using the differential equation \( \frac{d}{dx} \psi_w(x) = W(x) \) we have

\[
\Pi_w(x) = e^{\Delta_{\Lambda_1} x} = I_{4\times4} + \Delta_{\Lambda_1} x + \frac{(\Delta_{\Lambda_1} x)^2}{2!} + \ldots
\]

\[(3.18) \Pi_{\Omega_1}(x) = \begin{bmatrix} \cosh x & 0 & \sinh x & 0 \\ 0 & 1 & 0 & 0 \\ \sinh x & 0 & \cosh x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \]

2) For \( \Omega_2 = \eta \partial \xi + \xi \partial \eta \), we write the vector field

\[(3.19) \Lambda_{\Omega_2} = [\eta \ 0 \ 0 \ \xi]^T, \]

then, the previous equation can be given as follows

\[(3.20) \Delta_{\Lambda_2} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \]

from definition 1, by using the differential equation \( \frac{d}{dx} \psi_u(y) = W(y) \) we have

\[
\Pi_u(y) = e^{\Delta_{\Lambda_2} u}(y) = I_{4\times4} + \Delta_{\Lambda_2} y + \frac{(\Delta_{\Lambda_2} y)^2}{2!} + \ldots
\]

\[(3.21) \Pi_{\Omega_2}(y) = \begin{bmatrix} \cosh y & 0 & 0 & \sinh y \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh y & 0 & 0 & \cosh y \end{bmatrix}. \]

3) For \( \Omega_3 = \vartheta \partial \vartheta + \vartheta \partial \vartheta \), we write the vector field given as

\[(3.22) \Lambda_{\Omega_3} = [0 \ \vartheta \ \vartheta \ 0]^T, \]

then, the previous equation can be given as follows

\[
\Delta_{\Lambda_3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.
\]

Now, from definition 1 we can say that the one-parameter group of homomorphism \( \psi_z(\xi, \vartheta, \vartheta, \varsigma) \) is expressed by \( \psi'_z(\xi) = \psi^\xi \psi_z(\xi) \). So, we find \( \psi_z(\xi) = e^{\psi_z \xi} \) and calculating the matrix exponential, we have

\[
\Delta_v(z) = e^{\Delta_{\Lambda_3} z}(z) = I_{4\times4} + \Delta_{\Lambda_3} z + \frac{(\Delta_{\Lambda_3} z)^2}{2!} + \ldots
\]
THE RESEARCH ON ROTATIONAL SURFACES IN PSEUDO EUCLIDEAN 4-SPACE WITH INDEX 2

(3.23) \( \Pi_{\Omega_3}(z) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh z & \sinh z & 0 \\ 0 & \sinh z & \cosh z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \).

Similarly for \( \Omega_4 = \eta \partial \theta + \varphi \partial \eta \), we get

\[
\Pi_{\Omega_4}(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh \alpha & 0 & \sinh \alpha \\ 0 & 0 & 1 & 0 \\ 0 & \sinh \alpha & 0 & \cosh \alpha \end{bmatrix},
\]

and for \( \Omega_5 = \xi \partial \varphi - \varphi \partial \xi \) and \( \Omega_6 = \vartheta \partial \eta - \eta \partial \vartheta \), we obtain two one-parameter matrix groups of rotation.

b) Elliptic matrices: we give some one-parameter elliptic matrices groups of rotation \( \Omega_5 \) and \( \Omega_6 \)

\[
\Pi_{\Omega_5}(\beta) = \begin{bmatrix} \cos \beta & \sin \beta & 0 & 0 \\ -\sin \beta & \cos \beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \Pi_{\Omega_6}(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \theta & \sin \theta \\ 0 & 0 & -\sin \theta & \cos \theta \end{bmatrix}.
\]

Now if we want to express surfaces of rotation generated by two hyperbolic and elliptic subgroups, the sub-algebra of the lie algebra of the Lorentz group can be obtained, then we can write the closed subgroups of Lorentz group. Hence, two parameter subgroups of \( SO(4, 2) \) are obtain, and two parameter subgroups that fix some axis of rotation can be expressed. Therefore, we can write 2D sub-algebras, and therefore we need to obtain two vectors. In this context, by using Poisson bracket of two vectors \( X = \sum_{i=1}^{n} X^i \partial_i, Y = \sum_{i=1}^{n} Y^i \partial_i \) defined by

\[
[X, Y] = \sum_{i=1}^{n} \sum_{j=1}^{n} (X^j \partial_j Y^i - Y^j \partial_j X^i) \partial_i,
\]

we can write the following expressions

\[
\begin{align*}
[\Omega_1, \Omega_2] &= \Omega_6; [\Omega_1, \Omega_3] = \Omega_5; [\Omega_1, \Omega_5] = \Omega_3; [\Omega_1, \Omega_6] = \Omega_2; \\
[\Omega_2, \Omega_4] &= \Omega_5; [\Omega_2, \Omega_6] = \Omega_4; [\Omega_3, \Omega_4] = \Omega_6; \\
[\Omega_5, \Omega_3] &= \Omega_1; [\Omega_4, \Omega_5] = \Omega_4; [\Omega_5, \Omega_4] = \Omega_2; [\Omega_5, \Omega_6] = \Omega_3
\end{align*}
\]

then these Poisson brackets are not in \( Sp\{\Omega_1, \Omega_2\} \) excluding \( Sp\{\Omega_1, \Omega_4\}, Sp\{\Omega_2, \Omega_3\} \) and \( Sp\{\Omega_5, \Omega_6\} \). Therefore, these are not closed sub-algebra. Also,

\[
[\Omega_1, \Omega_4] = [\Omega_2, \Omega_3] = [\Omega_5, \Omega_6] = 0,
\]

\( \{\Omega_1, \Omega_4\}, \{\Omega_2, \Omega_3\}, \{\Omega_5, \Omega_6\} \) are the closed sub-algebra and we can think \( \{\Omega_1, \Omega_4\}, \{\Omega_2, \Omega_3\}, \{\Omega_5, \Omega_6\} \) as basis. Thus, abelian subgroups of \( SO(2, 2) \) can be expressed. Then, \( \Omega_1, \Omega_4 \) and \( \Omega_2, \Omega_4 \) generate abelian sub-algebras being hyperbolic. Therefore, we can write matrices \( \Pi_{\Omega_1}(x)\Pi_{\Omega_4}(\alpha) \) and \( \Pi_{\Omega_2}(y)\Pi_{\Omega_3}(z) \) being the rotational groups of matrices. Hence, these subgroups don’t fix any axis and so it is not a rotation about any axis. First, for the rotations \( \Omega_1 \) and \( \Omega_4 \), the matrices of rotations of this surface can be written as \( \Pi_{\Omega_1}(x)\Pi_{\Omega_4}(\alpha) \). We are interested in taking a planar curve \( \gamma \) with parameter as follows

(3.24) \( \gamma(s) = (f_1(s), f_2(s), f_3(s), f_4(s)), s \in I \).
and rotating it with 2D subgroup of isometry. Hence, the surface of revolution $S_{14}$ around $\Pi_{\Omega_1}(x)$ and $\Pi_{\Omega_4}(\alpha)$ can be parametrized as follows

\[(3.25a) \quad S_{14}(x, \alpha, s) = \Pi_{\Omega_1}(x) \cdot \Pi_{\Omega_4}(\alpha). \]

\[
\begin{bmatrix}
  f_1(s) \\
  f_2(s) \\
  f_3(s) \\
  f_4(s)
\end{bmatrix} = 
\begin{bmatrix}
  f_1 \cosh x + f_3 \sinh x, \\
  f_2 \cosh \alpha + f_4 \sinh \alpha, \\
  f_1 \sinh x + f_3 \cosh x, \\
  f_2 \sinh \alpha + f_4 \cosh \alpha
\end{bmatrix},
\]

where for $i \in \{1, 2, 3, 4\}$, $f_i$ are smooth functions and $-\infty < x, \alpha < \infty, s \in I$. Now, we consider the following rotational surface

\[(3.25b) \quad S_{14}(x(t), \alpha(t), s) = (f_1 \cosh x(t), f_4 \sinh \alpha(t), f_1 \sinh x(t), f_4 \cosh \alpha(t)), \]

where $f_1$ and $f_4$ are nonzero smooth functions and the curve $\gamma(s) = (f_1(s), 0, f_4(s))$ lies on the $\xi \eta$–plane. For the rotational surface (3.25b) we have the parametrizations:

\[
\begin{align*}
S_{s1}^{14}(x, \alpha, s) &= (f_1^\prime \cosh x, f_4^\prime \sinh \alpha, f_1^\prime \sinh x, f_4^\prime \cosh \alpha) ; \\
S_{s3}^{14}(x, \alpha, s) &= (f_1^\prime \prime \cosh x, f_4^\prime \sinh \alpha, f_1^\prime \sinh x, f_4^\prime \cosh \alpha) ; \\
S_{t1}^{14}(x, \alpha, s) &= (f_1^\prime \cosh x, f_4 \dot{\alpha} \cosh \alpha, f_1^\prime \cosh x, f_4 \dot{\alpha} \cosh \alpha) ; \\
S_{t4}^{14}(x, \alpha, s) &= (f_1^\prime \cosh x, f_4 \dot{\alpha} \cosh \alpha, f_1^\prime \cosh x, f_4 \dot{\alpha} \cosh \alpha) ; \\
S_{t4}^{14}(x, \alpha, s) &= (f_1^\prime \cosh x, f_4 \dot{\alpha} \cosh \alpha, f_1^\prime \cosh x, f_4 \dot{\alpha} \cosh \alpha) ; \\
\end{align*}
\]

and

\[
\langle S_{s1}^{14}, S_{s3}^{14} \rangle = -f_{11}^2 + f_{44}^2 > 0; \langle S_{t1}^{14}, S_{t4}^{14} \rangle = f_{11}^2 \alpha^2 - f_{44}^2 \alpha^2 < 0.
\]

Therefore, we choose the following moving frame $e_1, e_2, e_3, e_4$, such that $e_1, e_2$ are tangent to $S_{14}$ and $e_3, e_4$ are normal to $S_{14}$. Also, we write as

\[
e_1 = \frac{\begin{bmatrix}
  f_1 \dot{x} \sinh x, \\
  f_4 \dot{\alpha} \cosh \alpha, \\
  f_1 \dot{x} \cosh x, \\
  f_4 \dot{\alpha} \sinh \alpha
\end{bmatrix}}{\sqrt{f_{11}^2 + f_{44}^2}}; \\
\]

\[
e_2 = \frac{\begin{bmatrix}
  f_1 \cosh x, \\
  f_4 \sinh \alpha, \\
  f_1^\prime \sinh x, \\
  f_4 \dot{\alpha} \cosh \alpha
\end{bmatrix}}{\sqrt{f_{11}^2 + f_{44}^2}}; \\
\]

\[
e_3 = \frac{\begin{bmatrix}
  f_1 \cosh x, \\
  f_4 \sinh \alpha, \\
  f_1^\prime \sinh x, \\
  f_4 \dot{\alpha} \sinh \alpha
\end{bmatrix}}{\sqrt{f_{11}^2 + f_{44}^2}}; \\
\]

Then, we can easily get

\[
\varepsilon_1 = (e_1, e_1) = -1; \varepsilon_2 = (e_2, e_2) = 1; \varepsilon_3 = (e_3, e_3) = -1; \varepsilon_4 = (e_4, e_4) = 1.
\]

By using (2.9), (2.10), (2.11), (2.12), we obtain the following coefficients of the second fundamental form $h$ and the connection forms

\[
h_{11}^3 = \frac{f_1 f_4 (\dot{x} \alpha + \dot{\alpha})}{\sqrt{f_{11}^2 + f_{44}^2}}, h_{12}^3 = \frac{(f_1^\prime f_4 - f_1 f_4^\prime) \dot{x} \alpha}{\sqrt{f_{11}^2 + f_{44}^2}}, h_{32}^3 = 0; \\
\]

\[
h_{11}^4 = \frac{f_1 f_4 \dot{\alpha}^2 - f_1^\prime f_4 x^2}{\sqrt{f_{11}^2 + f_{44}^2}}, h_{12}^4 = \frac{(f_1^\prime f_4^\prime - f_1 f_4 \dot{\alpha}^2) \dot{x} \alpha}{\sqrt{f_{11}^2 + f_{44}^2}}, h_{22}^4 = 0; \\
\]
and from (2.12) the mean curvature vector $H$ of the rotational surface $S^{14}$ is

$$H = \left\{ \frac{f_1 f_4 (\dot{x} \dot{z} + \dot{z} \dot{x})}{2 \sqrt{f_1^2 \dot{x}^2 - f_1^4}} \right\}_c^e + \left\{ \frac{(f_1 f_4')^2 - f_1 f_4''}{2 \sqrt{f_1^2 + f_4^2}} \right\}_c^e.$$ 

The Gaussian curvature $K$ of the rotational surface $S^{14}$ is obtained as

$$K = \sum_{s=3}^4 \varepsilon_s [h_{ij}] = \left( \frac{f_1 f_4}{f_1^2 \dot{x}^2 - f_1^4} \right)^2 (\dot{x} \dot{a})^2 + \left( \frac{f_1 f_4 \omega^2}{-f_1^2 \dot{x}^2} - f_1 f_4'' \right) \left( \frac{f_1 f_4'}{f_1^2 + f_4^2} \right).$$

Secondly, for the rotations $\Omega_2$ and $\Omega_3$, by using the curve $\gamma(s)$, the surface of rotation $S_{22}$ around $\Pi_{\Omega_2}(y)$. $\Pi_{\Omega_3}(z)$ is given as follows

$$(3.26a) \quad S_{22}(y, z, s) = \Pi_{\Omega_2}(y). \Pi_{\Omega_3}(z), \quad \begin{bmatrix} f_1(s) \\ f_2(s) \\ f_3(s) \\ f_4(s) \end{bmatrix} = \begin{bmatrix} f_1 \cosh y + f_1 \sinh y, \\ f_2 \cosh z + f_2 \sinh y, \\ f_2 \cosh z + f_3 \sinh z, \\ f_1 \sinh y + f_3 \cosh y \end{bmatrix},$$

where $-\infty < z, y < \infty, s \in I$. Now, we consider the following the surface of rotation $S_{22}$ where $f_1$ and $f_2$ are non-zero smooth functions and the curve $\gamma(s) = (f_1(s), f_2(s), 0, 0)$ lies on the $\xi \rho$-plane. For (3.26b) we have the parametrizations

$$S_{22}^1(y, z, s) = (f_1 \dot{y} \sinh y, f_2 \dot{z} \cosh z, f_2 \dot{z} \cosh z, f_1 \dot{y} \cosh y);$$

$$S_{22}^2(y, z, s) = \begin{bmatrix} f_1(\dot{y} \sinh y + y^2 \cosh y), f_2(\dot{z} \cosh z + z^2 \sinh z), f_1(\dot{y} \cosh y + y^2 \sinh y) \end{bmatrix};$$

$$S_{22}^3(y, z, s) = (f_1 \cosh y, f_2 \cosh z, f_2 \sinh z, f_1 \sinh y);$$

$$S_{22}^4(y, z, s) = (f_1 \dot{y} \sinh y, f_2 \dot{z} \cosh z, f_2 \dot{z} \cosh z, f_1 \dot{y} \cosh y).$$

and

$$\langle S_{22}^1, S_{22}^2 \rangle = -f_1^2 - f_2^2 < 0; \langle S_{22}^3, S_{22}^4 \rangle = f_2^2 \dot{z} + f_1^2 \dot{y} > 0.$$ 

Hence, the following moving frame $e_1, e_2, e_3, e_4$ can be chosen, such that $e_1, e_2$ are tangent to $S_{22}$ and $e_3, e_4$ are normal to $S_{22}$, we obtain as follows

$$e_1 = \frac{(f_1 \dot{y} \sinh y, f_2 \dot{z} \cosh z, f_2 \dot{z} \cosh z, f_1 \dot{y} \cosh y)}{\sqrt{f_2^2 \dot{z} + f_1^2 \dot{y}}};$$

$$e_2 = \frac{(f_1 \dot{y} \cosh y, f_2 \dot{z} \cosh z, f_2 \dot{z} \sinh z, f_1 \dot{y} \sinh y)}{\sqrt{f_1^2 + f_2^2}};$$

$$e_3 = \frac{(f_2 \dot{z} \sinh y, f_1 \dot{y} \cosh z, f_2 \dot{z} \cosh z, f_1 \dot{y} \cosh y)}{\sqrt{f_2^2 \dot{z} + f_1^2 \dot{y}}};$$

$$e_4 = \frac{(f_2 \dot{z} \cosh y, f_1 \dot{y} \cosh z, f_2 \dot{z} \sinh z, f_1 \dot{y} \sinh y)}{\sqrt{f_2^2 \dot{z} + f_1^2 \dot{y}}}.$$

Also, we have $\varepsilon_{1,3} = 1; \varepsilon_{2,4} = -1$. For the equations (2.9), (2.10), (2.11), (2.12), the following coefficients of the second fundamental form $h$ and the connection forms
are obtained as

\[
\begin{align*}
\h_{11}^3 &= \frac{f_1 f_2 (\dot{y} \ddot{z} + \ddot{y} \dot{z})}{\sqrt{f_2^2 z + f_1^2 y}}; \h_{12}^3 = \frac{(f_1 f_2 + f_1^2 f_2) \dot{y} \ddot{z}}{\sqrt{f_2^2 z + f_1^2 y}}; \h_{22}^3 = 0; \\
\h_{11}^4 &= \frac{-f_1 \dot{f}_2 \ddot{y}^2 - f_1 f_2 \dot{z}^2}{\sqrt{f_1^2 + f_2^2}}; \h_{12}^4 = \frac{-f_1 \dot{f}_2 \ddot{y}^2 - f_1 f_2 \dot{z}^2}{\sqrt{f_1^2 + f_2^2}}; \h_{22}^4 = 0;
\end{align*}
\]

and from (2.12) the Gaussian curvature \( K \) and the mean curvature vector \( H \) of the rotational surface \( S^2 \) are obtained as follows

\[
\begin{align*}
H &= \frac{f_1 f_2 (\dot{y} \ddot{z} + \ddot{y} \dot{z})}{2 \sqrt{f_2^2 z + f_1^2 y}} e_3 + \frac{f_1 f_2 \dot{y}^2 + f_1 f_2 \dot{z}^2 - f_1 f_2 \dot{f}_2 - f_1 f_2 \dot{f}_2}{2 \sqrt{f_1^2 + f_2^2}} e_4, \\
K &= \frac{-(f_1 f_2 + f_1^2 f_2) (\dot{y} \ddot{z})^2}{f_1^2 + f_2^2} - \frac{f_1 f_2 \dot{y}^2 + f_1 f_2 \dot{z}^2}{f_1^2 + f_2^2} \left( f_1 f_2 + f_1 f_2 \right).
\end{align*}
\]

Also, \( \Omega_5 \) and \( \Omega_6 \) generate abelian sub-algebra being elliptic. Therefore, we can write matrix \( \Pi_{\Omega_5}(\beta) \Pi_{\Omega_6}(\theta) \) being the rotational group of matrices. This subgroup doesn’t fix any axis and so it is not a rotation about any axis. For the rotations \( \Omega_5 \) and \( \Omega_6 \), the matrices of rotations of this surface can be written as \( \Pi_{\Omega_5}(\beta) \Pi_{\Omega_6}(\theta) \), by using a planar curve \( \gamma \) with \( s \) parameter the surface of rotation \( S_{56} \) around \( \Pi_{\Omega_5}(\beta) \Pi_{\Omega_6}(\theta) \) can be parametrized as follows

\[
(3.27a) \quad S_{56}(\beta, \theta, s) = \Pi_{\Omega_5}(\beta) \Pi_{\Omega_6}(\theta), \quad \begin{bmatrix} f_1(s) \\ f_2(s) \\ f_3(s) \\ f_4(s) \end{bmatrix} = \begin{bmatrix} f_1 \cos \beta + f_2 \sin \beta, \\ -f_1 \sin \beta + f_2 \cos \beta, \\ f_3 \cos \theta + f_4 \sin \theta, \\ -f_3 \sin \theta + f_4 \cos \theta \end{bmatrix},
\]

where \(-\infty < \beta, \theta < \infty, s \in \mathbb{I}\). Now, we consider the following rotational surface

\[
(3.27b) \quad S_{56}(\beta(t), \theta(t), s) = (f_2 \sin \beta, f_2 \cos \beta, f_4 \sin \theta, f_4 \cos \theta),
\]

where \( f_2 \) and \( f_4 \) are non-zero smooth functions and the curve \( \gamma(s) = (0, f_2(s), 0, f_4(s)) \) lies on the \( \rho \eta \)-plane. From (3.27b) we have the parametrizations

\[
\begin{align*}
S_{56}^s(\beta, \theta, s) &= (f_2^s \sin \beta, f_2^s \cos \beta, f_4^s \sin \theta, f_4^s \cos \theta); \\
S_{56}^s(\beta, \theta, s) &= (f_2^s \sin \beta, f_2^s \cos \beta, f_4^s \sin \theta, f_4^s \cos \theta); \\
S_{56}^s(\beta, \theta, s) &= \left( f_2^s \beta \cos \beta, -f_2^s \beta \sin \beta, f_4^s \beta \cos \theta, -f_4^s \beta \sin \theta \right); \\
S_{56}^s(\beta, \theta, s) &= \left( f_2^s \beta \cos \beta, -f_2^s \beta \sin \beta, f_4^s \beta \cos \theta, -f_4^s \beta \sin \theta \right); \\
S_{56}^s(\beta, \theta, s) &= \left( f_2(\beta \cos \beta - \beta^2 \sin \beta), -f_2(\beta \sin \beta + \beta^2 \cos \beta), \\ f_4(\beta \cos \theta - \beta^2 \sin \theta), -f_4(\beta \sin \theta + \beta^2 \cos \theta) \right);
\end{align*}
\]

and

\[
\left< S_{s_{56}}, S_{s_{56}}^s \right> = -f_2^2 + f_4^2 > 0; \left< S_{t_{56}}, S_{t_{56}}^s \right> = -f_2^2 \beta^2 + f_4^2 \theta < 0.
\]
For the following moving frame $e_1, e_2, e_3, e_4$, we say that $e_1, e_2$ are tangent to $S^{36}$ and $e_3, e_4$ are normal to $S^{36}$. Therefore, we get

$$e_1 = \frac{(f_3 f_4 \cos \beta, -f_3 \dot{f}_4 \sin \beta, f_3 \dot{f}_4 \cos \theta, -f_3 \dot{f}_4 \sin \theta)}{\sqrt{-f_3^2 \beta^2 + f_3^2 \theta^2}};$$

$$e_2 = \frac{(f_4' \sin \beta, f_4' \cos \beta, f_4' \sin \theta, f_4' \cos \theta)}{\sqrt{-f_4''^2 + f_4'^2}};$$

$$e_3 = \frac{(-f_4 \dot{f}_4 \cos \beta, f_4 \dot{f}_4 \sin \beta, -f_4 \dot{f}_4 \cos \theta, f_4 \dot{f}_4 \sin \theta)}{\sqrt{-f_4^2 \beta^2 + f_4^2 \theta^2}};$$

$$e_4 = \frac{(f_4' \sin \beta, f_4' \cos \beta, f_4' \sin \theta, f_4' \cos \theta)}{\sqrt{-f_4''^2 + f_4'^2}};$$

and we also get $\varepsilon_{1,4} = -1; \varepsilon_{3,4} = 1$. By considering (2.9), (2.10), (2.11), (2.12), we obtain the following coefficients of the second fundamental form $h$ and the connection forms

$$h_{11}^3 = \frac{f_4 f_2 \left(\dot{\beta} - \ddot{\beta} \right)}{\sqrt{-f_2^2 \beta^2 + f_2^2 \theta^2}}; h_{12}^3 = \frac{(f_2 f_4 - f_2 f_4') \dot{\beta} \theta}{\sqrt{-f_2^2 \beta^2 + f_2^2 \theta^2}}; h_{22}^3 = 0;$$

$$h_{11}^4 = \frac{f_4 f_2 \beta^2 - f_2 f_4 \theta^2}{\sqrt{-f_2^2 \beta^2 + f_2^2 \theta^2}}; h_{22}^4 = \frac{(-f'' f_4 + f'_4 f'_4)}{-f_2^2 + f_4^2}; h_{44}^4 = 0;$$

and from (2.12) the mean curvature vector $H$ of the rotation surface $S^{36}$ is

$$H = -\frac{f_4 f_2 \left(\dot{\beta} - \ddot{\beta} \right)}{2 \sqrt{-f_2^2 \beta^2 + f_2^2 \theta^2}} e_3 + \frac{(f_4 f_2 \beta^2 - f_2 f_4 \theta^2 + f'' f_4 - f'_4 f'_4)}{2 \sqrt{-f_2^2 + f_4^2}} e_4;$$

and the Gaussian curvature $K$ of the rotation surface $S^{36}$ is obtained as

$$K = -\frac{(f_4 f_2 \beta^2 - f_2 f_4 \theta^2) \dot{\beta} \theta^2}{-f_2^2 \beta^2 + f_2^2 \theta^2} - \frac{(-f'' f_4 + f'_4 f'_4) (f_4 f_2 \beta^2 - f_2 f_4 \theta^2)^2}{-f_2^2 + f_4^2}.$$

\[ \square \]

**Example 1.** We consider the surfaces of rotation given as follows

1. For the curve $\gamma(s) = (s + \sinh s, 0, 0, s + \cosh s)$ the hyperbolic surface of rotation is given as
   
   $$S_1(x, \alpha, s) = \left( \begin{array}{c} (s + \sinh s) \cosh x, (s + \cosh s) \sinh \alpha, \\ (s + \sinh s) \sinh x, (s + \cosh s) \cosh \alpha \end{array} \right);$$

2. For the curve $\gamma(s) = (s \cosh s, s \sinh s, 0, 0)$ the hyperbolic surface of rotation is given as
   
   $$S_2(y, z, s) = (s \cosh s \cosh y, s \sinh s \cosh z, s \sinh s \sinh z, s \cosh s \cosh y);$$

3. For the curve $\gamma(s) = (0, ax^2 \sin s, 0, ax^2 \cos s)$ the elliptic surface of rotation is given as
   
   $$S_3(\beta, \theta, s) = \left( ax^2 \sin \sin \beta, ax^2 \sin \cos \beta, ax^2 \cos \sin \theta, ax^2 \cos \cos \theta \right); a, c \in \mathbb{R}.$$
4. Conclusion

In this paper, we gave different types of matrices of rotation which are the subgroups of the manifold $M$ corresponding to rotation about a chosen axis in $E^4$. Hence, we used two parameter matrices groups of rotations and we gave the matrices of rotation corresponding to the appropriate subgroup of the $E^4_2$ and we defined a brief description of rotational surfaces using a curve and matrices in $E^4_2$. Furthermore, we examined the special rotated surfaces generated by these matrices of rotation in $E^4_2$ and we expressed some certain results of describing the surface obtaining Killing vector field in $E^4_2$ in detail. Also, we gave the Gaussian curvature and the mean curvature of the surfaces of rotation.

The authors are currently working on the properties of these rotated surfaces with a view to devising suitable metric in $E^4_2$ by adapting the type of conservation laws considered in the paper. In our future studies, we will study geodesics on the rotational surface obtained in $E^4_2$. Also, the physical terms such as specific energy and specific angular momentum will be examined with the help of the conditions obtained by using the Clairaut’s theorem on these special surfaces.

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7. Conflicts of interest statement

The authors have NO affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

8. Declarations

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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