Separable functors for the category of Doi-Hopf modules. Applications

S. Caenepeel
University of Brussels, VUB
Faculty of Applied Sciences,
Pleinlaan 2
B-1050 Brussels, Belgium

Bogdan Ion*
Department of Mathematics,
Princeton University,
Fine Hall, Washington Road
Princeton, NJ 08544-1000, USA

G. Militaru†
University of Bucharest
Faculty of Mathematics
Str. Academiei 14
RO-70109 Bucharest 1, Romania

Shenglin Zhu‡
Institute of Mathematics
Fudan University
Shanghai 200433, China

Abstract

We prove a Maschke type Theorem for the category of Doi-Hopf modules. In fact, we give necessary and sufficient conditions for the functor forgetting the $C$-coaction to be separable. This leads to a generalized notion of integrals. Our results can be applied to obtain Maschke type Theorems for Yetter-Drinfel'd modules, Long dimodules and modules graded by $G$-sets. Existing Maschke type Theorems due to Doi and the authors are recovered as special cases.

0 Introduction

One of the key results in classical representation theory is Maschke’s Theorem, stating that a finite group algebra $kG$ is semisimple if and only if the characteristic of $k$ does not divide the order of $G$. Many similar results (we call them Maschke type Theorems) exist in the literature, and the problem is always to find a necessary and sufficient condition for a certain object $O$ to be reducible or semisimple. These objects occur in several disciplines in mathematics, they can be for example groups, affine algebraic groups, Lie groups, locally compact groups, or Hopf algebras, and, in a sense, this last example contains the previous ones as special cases. The general idea is to consider representations of $O$. Roughly stated, $O$ is reducible if and only if all representations of $O$ are completely reducible, and this comes to the fact that any monomorphism between two representations splits.

The idea behind the proof of a Maschke type Theorem is then the following: one applies a functor to the category of representations of $O$, forgetting some of the structure, and in such a way that we obtain objects in a more handable category, for instance vector spaces over a field or modules over a commutative ring. Then the strategy is to look for a deformation of a splitting map of a

*Tempus visitor at UIA
†Research supported by the bilateral project “Hopf algebras and co-Galois theory” of the Flemish and Romanian governments.
‡Research supported by the “FWO Flanders research network WO.011.96N”
monomorphism in the “easy” category in such a way that it becomes a splitting in the category of representations. In a Hopf algebraic setting, the tool that is applied to find such a deformation is often called an integral.

In fact there is more: the deformation turns out to be functorial in all cases that are known, and this can be restated in a categorical setting: the Maschke type Theorem comes down to the fact that the forgetful functor is separable in the sense of \( \mathcal{C} \). A separable functor \( F : \mathcal{C} \to \mathcal{D} \) has more properties: information about semisimplicity, injectivity, projectivity of objects in \( \mathcal{D} \) yields information about the corresponding properties in \( \mathcal{C} \).

At first sight, it seems to be a very difficult problem to decide whether a functor is separable. However, if a functor \( F \) has a right adjoint \( G \), then there is an easy criterium for the separability of \( F \): the unit \( \rho : 1 \to GF \) has to be split (see \([28]\)) and this criterium will play a crucial role in this paper.

The classical “Hopf algebraic Maschke Theorem” of Larson and Sweedler (\([21]\)) tells us that a finite dimensional Hopf algebra is semisimple if and only if there exists an integral that is not annihilated by the augmentation map. Several generalizations to categories of (generalized) Hopf modules have been presented in the literature, see Cohen and Fishman (\([7\] and \([8]\)), Doi (\([11\] and \([12]\)), Ştefan and Van Oystaeyen (\([34]\)) and the authors (\([4]\)). The results in all these papers can be reformulated in terms of separable functors, in fact they give sufficient conditions for a forgetful functor to be separable. An unsatisfactory aspect is that these conditions are sufficient, but not necessary.

In this paper, we return to the setting of \([4]\): we consider a Doi-Hopf datum \((H,A,C)\), and the category of so-called Doi-Hopf modules \(C\mathcal{M}(H)_A\) consisting of modules with an algebra action and a coalgebra coaction. As explained in previous publications (\([2\], \([3\], . . . \)) \(C\mathcal{M}(H)_A\) unifies modules, comodules, Sweedler’s Hopf modules, Takeuchi’s relative Hopf modules, graded modules, modules graded by \(G\)-sets, Long dimodules and Yett–Drinfel’d modules. We consider the functor \( F \) forgetting the \( C \)-coaction, and we give necessary and sufficient conditions for this functor to be separable.

To this end, we study natural transformations \( \nu : GF \to 1 \), and the clue result is the following: the natural transformation \( \nu \) is completely determined if we know the map \( \nu_{C \otimes A} : C \otimes C \otimes A \to C \otimes A \).

We recall that \( C \otimes A \) plays a special role in the category of Doi-Hopf modules (although it is not a generator). Conversely, a map \( C \otimes C \otimes A \to C \otimes A \) in the category of Doi-Hopf modules can be used to construct a natural transformation, provided it suffices two additional properties. The next step is then to show that the natural transformation splits the unit \( \rho \) if and only if the map \( \nu_{C \otimes A} \) satisfies a certain normalizing condition. We obtain a necessary and sufficient condition for the functor \( F \) to be separable, and this condition can be restated in several ways. Actually we prove that the \( k \)-algebra \( V \) consisting of all natural transformations \( \nu : GF \to 1 \) is isomorphic to five different \( k \)-algebras, consisting of \( k \)-linear maps satisfying certain properties. One of these algebras, named \( V_4 \), consists of right \( C^* \)-linear maps \( \gamma : C \to \text{Hom}(C, A) \) that are centralized by a left and right \( A \)-actions. We have called these maps \( \gamma \) \( A \)-integrals, because they are closely related to Doi’s total integrals (see Section 3.1). The normalized \( A \)-integrals (also called total \( A \)-integrals) are then right units in \( V_4 \), and our main Theorem takes the following form: the forgetful functor is separable if and only if there exists a total \( A \)-integral \( \gamma : C \to \text{Hom}(C, A) \).

Our technique can also be applied to the right adjoint \( G \) of the functor \( F \) (see Section 2.2), and this leads to the notion of dual \( A \)-integral. In Section 3 we give several applications and examples: we explain how the results of \([1], \([11\] and \([12]\) are special cases (Sections 3.1 and 3.2), and apply our results to some Hopf module categories that are special cases of the Doi-Hopf module category: Yett–Drinfel’d modules, Long dimodules and modules graded by \( G \)-sets (see Sections 3.3 and 3.6).

In the Yett–Drinfel’d case, this leads to a generalization of the Drinfel’d double in the case of...
an infinite dimensional Hopf algebra, using Koppinen’s generalized smash product \cite{17} and the results of \cite{2}. Finally, if $C$ is finitely generated and projective, then we find necessary and sufficient conditions for the extension $A \to A\# C^*$ to be separable (Section 3.4). In the situation where $C = H$, we recover some existing results of Cohen and Fischman \cite{7}, Doi and Takeuchi \cite{14}, and Van Oystaeyen, Xu and Zhang \cite{37}.

1 Preliminary results

Throughout this paper, $k$ will be a commutative ring with unit. Unless specified otherwise, all modules, algebras, coalgebras, bialgebras, tensor products and homomorphisms are over $k$. $H$ will be a bialgebra over $k$, and we will extensively use Sweedler’s sigma-notation. For example, if $(C,\Delta_C)$ is a coalgebra, then for all $c \in C$ we write

$$\Delta_C(c) = \sum c_{(1)} \otimes c_{(2)} \in C \otimes C.$$  

If $(M,\rho_M)$ is a left $C$-comodule, then we write

$$\rho_M(m) = \sum m_{< -1 >} \otimes m_{< 0 >},$$

for $m \in M$. $CM$ will be the category of left $C$-comodules and $C$-colinear maps. For a $k$-algebra $A$, $MA$ (resp. $AM$) will be the category of right (resp. left) $A$-modules and $A$-linear maps.

The dual $C^* = \text{Hom}(C, k)$ of a $k$-coalgebra $C$ is a $k$-algebra. The multiplication on $C^*$ is given by the convolution

$$\langle f \ast g, c \rangle = \sum \langle f, c_{(1)} \rangle \langle g, c_{(2)} \rangle,$$

for all $f, g \in C^*$ and $c \in C$. $C$ is a $C^*$-bimodule. The left and right action are given by the formulas

$$c^{\ast} \cdot c = \sum \langle c^{\ast}, c_{(2)} \rangle c_{(1)} \quad \text{and} \quad c \cdot c^{\ast} = \sum \langle c^{\ast}, c_{(1)} \rangle c_{(2)}$$ \hspace{1cm} (1)

for $c^{\ast} \in C^*$ and $c \in C$. This also holds for $C$-comodules. For example if $(M,\rho_M)$ is a left $C$-comodule, then it becomes a right $C^*$-module by

$$m \cdot c^{\ast} = \sum \langle c^{\ast}, m_{< -1 >} \rangle m_{< 0 >},$$

for all $m \in M$ and $c^{\ast} \in C^*$. If $C$ is projective as a $k$-module, then a $k$-linear map $f : M \to N$ between two left $C$-comodules is $C$-colinear if and only if it is right $C^*$-linear.

An algebra $A$ that is also a left $H$-comodule is called a left $H$-comodule algebra if the comodule structure map $\rho_A$ is an algebra map. This means that

$$\rho_A(ab) = \sum a_{< -1 >}b_{< -1 >} \otimes a_{< 0 >}b_{< 0 >} \quad \text{and} \quad \rho_A(1_A) = 1_H \otimes 1_A$$

for all $a, b \in A$.

Similarly, a coalgebra that is also a right $H$-module is called a right $H$-module coalgebra if

$$\Delta_C(c \cdot h) = \sum c_{(1)} \cdot h_{(1)} \otimes c_{(2)} \cdot h_{(2)}$$

and

$$\varepsilon_C(c \cdot h) = \varepsilon_C(c)\varepsilon_H(h),$$

for all $c \in C$, $h \in H$.

We recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is called fully faithful if the maps

$$\text{Hom}_\mathcal{C}(M, N) \to \text{Hom}_\mathcal{D}(FM, FN)$$

are isomorphisms for all objects $M, N \in \mathcal{C}$.  

1.1 Doi-Hopf modules

Let \( H \) be a bialgebra, \( A \) a left \( H \)-comodule algebra and \( C \) a right \( H \)-module coalgebra. We will always assume that \( C \) is flat as a \( k \)-module. Following [3], we will call the threetuple \((H,A,C)\) a Doi-Hopf datum. A right-left Doi-Hopf module is a \( k \)-module \( M \) that has the structure of right \( A \)-module and left \( C \)-comodule such that the following compatibility relation holds

\[
\rho_M(ma) = \sum m_{<1>} \cdot a_{<-1>} \otimes m_{<0>} a_{<0>},
\]

for all \( a \in A, m \in M \). \( C^\mathcal{M}(H)_A \) will be the category of right-left Doi-Hopf modules and \( A \)-linear, \( C \)-colinear maps. In [3], induction functors between categories of Doi-Hopf modules are studied. It follows from [3, Theorem 1.3] that the forgetful functor \( F : C = C^\mathcal{M}(H)_A \rightarrow \mathcal{M}_A \) has a right adjoint

\[
G : \mathcal{M}_A \rightarrow C^\mathcal{M}(H)_A
\]

given by

\[
G(M) = C \otimes M
\]

with structure maps

\[
(c \otimes m) \cdot a = \sum c \cdot a_{<-1>} \otimes ma_{<0>}
\]

\[
\rho_{C \otimes M}(c \otimes m) = \sum c_{(1)} \otimes c_{(2)} \otimes m
\]

for any \( c \in C, a \in A \) and \( m \in M \). It is easy to see that the unit \( \rho : 1_C \rightarrow GF \) of the adjoint pair \((F,G)\) is given by the \( C \)-coaction \( \rho_M : M \rightarrow M \otimes C \) on any Doi-Hopf module \( M \). The counit \( \delta : FG \rightarrow 1_C \) is given by

\[
\delta_N : C \otimes N \rightarrow N, \quad \delta_N(c \otimes n) = \varepsilon(c)n
\]

for any right \( A \)-module \( N \).

\( A \) is a right \( A \)-module, so \( G(A) = C \otimes A \) and \( GFG(A) = C \otimes C \otimes A \) are Doi-Hopf modules. For later use, we give the action and coaction explicitly:

\[
(c \otimes b)a = \sum ca_{<-1>} \otimes ba_{<0>}
\]

\[
\rho(c \otimes b) = \sum c_{(1)} \otimes c_{(2)} \otimes b
\]

\[
(c \otimes d \otimes b)a = \sum ca_{<-2>} \otimes da_{<-1>} \otimes ba_{<0>}
\]

\[
\rho(c \otimes d \otimes b) = \sum c_{(1)} \otimes c_{(2)} \otimes d \otimes b
\]

\( \rho \) is the unit of the adjoint pair \((F,G)\), and therefore the coaction \( \rho_{C \otimes A} : C \otimes A \rightarrow C \otimes C \otimes A \) is \( A \)-linear and \( C \)-colinear.

Now assume that \( H \) is a Hopf algebra. Then we can also consider the category \( A^\mathcal{M}(H)^C \) of left \( A \)-modules that are also right \( C \)-comodules, with the additional compatibility relation (see [3])

\[
\rho(am) = \sum a_{<0>} m_{<0>} \otimes m_{<1>} S(a_{<-1>})
\]

Now the forgetful functor \( G' : A^\mathcal{M}(H)^C \rightarrow \mathcal{M}^C \) has a left adjoint \( F' \) given by \( F'(M) = M \otimes A \) and (see [3])

\[
a(m \otimes b) = m \otimes ab
\]

\[
\rho'(m \otimes b) = \sum m_{<0>} b_{<0>} \otimes m_{<1>} S(b_{<-1>})
\]
It follows in particular that $C \otimes A = F(C) \in \mathcal{A} \mathcal{M}(H)^C$, and this makes $C \otimes A$ into a left $A$-module and a right $C$-comodule.

The algebra $C^*$ is a left $H$-module algebra; the $H$-action is given by the formula

$$\langle h \cdot c^*, c \rangle = \langle c^*, c \cdot h \rangle$$

for all $h \in H$, $c \in C$ and $c^* \in C^*$. The smash product $A \# C^*$ is equal to $A \otimes C^*$ as a $k$-module, with multiplication defined by

$$(a \# c^*)(b \# d^*) = \sum a_{<0>} b \# c^* \cdot (a_{<-1>} \cdot d^*),$$

(12)

for all $a, b \in A$, $c^*, d^* \in D^*$. Recall that we have a natural functor $P : \mathcal{C} \mathcal{M}(H)_A \to \mathcal{M}_A \# C^*$ sending a Doi-Hopf module $M$ to itself, with right $A \# C^*$-action given by

$$m \cdot (a \# c^*) = \sum \langle c^*, m_{<-1>} \rangle m_{<0>} a$$

(13)

for any $m \in M$, $a \in A$ and $c^* \in C^*$. $P$ is fully faithful if $C$ is projective as a $k$-module, and $P$ is an equivalence of categories if $C$ is finitely generated and projective as a $k$-module (cf. [13]).

In [17], Koppinen introduced the following version of the smash product: $\#(C, A)$ is equal to $\text{Hom}(C, A)$ as a $k$-module, with multiplication given by the formula

$$(f \cdot g)(c) = \sum f\left(c_{(1)}\right)_{<0>} g\left(c_{(2)} \cdot f\left(c_{(1)}\right)_{<-1>}\right)$$

(14)

An easy computation shows that $\#(C, A)$ is an associative algebra with unit $u_A \circ \varepsilon_C$. The natural map $i : A \# C^* \to \text{Hom}(C, A)$ given by

$$i(a \# c^*)(c) = \langle c^*, c \rangle a$$

is an algebra morphism which is an isomorphism if $C$ is finitely generated and projective as a $k$-module. We obtain the following diagram of algebra maps

$$\begin{array}{ccc}
A & \xrightarrow{i_A} & A \# C^* & \xrightarrow{i C^*} & C^* \\
& \downarrow i & & \downarrow i & \\
& \#(C, A) & & \end{array}$$

where $i_A(a) = a \# \varepsilon$, $i C^* = 1_A \# c^*$ for any $a \in A$, $c^* \in C^*$. It follows that $\text{Hom}(C, A)$ can be viewed as an $(A, A \# C^*)$-bimodule, where the actions are given by the restriction of scalars. For further use, we write down the actions explicitly

$$(a \cdot f)(c) = (i(a \# \varepsilon) \cdot f)(c) = \sum a_{<0>} f\left(c \cdot a_{<-1>}\right)$$

(15)

$$(f \cdot a)(c) = f(c)a$$

(16)

$$(f \cdot c^*)(c) = \sum f\left(c_{(1)}\right)_{<0>} (c^*, c_{(2)} \cdot f\left(c_{(1)}\right)_{<-1>})$$

(17)

for any $f \in \text{Hom}(C, A)$, $a \in A$, $c^* \in C^*$ and $c \in C$.

Let $N$ be a left $A$-module. Then we can make $\text{Hom}(C, N)$ into a left $\#(C, A)$-module as follows

$$(\alpha \cdot g)(c) = \sum \alpha(c_{(1)})_{<0>} g\left(c_{(2)} \cdot \alpha(c_{(1)})_{<-1>}\right)$$

(18)
for any $\alpha \in \#(C, A)$ and $g \in \Hom(C, N)$. By restriction of scalars, $\Hom(C, N)$ becomes a left $A$-module. For a right $A$-module $N$, $\Hom(C, N)$ becomes a right $A$-module in the usual way:

$$(f \cdot a)(c) = f(c)a$$

for any $f \in \Hom(C, N)$, $a \in A$ and $c \in C$. An easy verification shows that $\Hom(C, N)$ is an $A$-bimodule if $N$ is an $A$-bimodule. We will apply this to the $A$-bimodule $N = \#(C, A)$. In this case the left and right $A$-action on $\Hom(C, N) = \Hom(C, \Hom(C, A))$ take the form

$$(a \cdot \gamma)(c) = \sum a_{<0>} \cdot \gamma(c \cdot a_{<-1>}) \quad (18)$$

$$(\gamma \cdot a)(c) = \gamma(c) \cdot a \quad (19)$$

for any $a \in A$, $\gamma \in \Hom(C, \#(C, A))$ and $c \in C$.

We have several functors between the categories $C\mathcal{M}(H)_A$, $\mathcal{M}_A$, $\mathcal{M}_A \#(C, A)$ and $\mathcal{M}_A \#C^*$.

\[
\begin{array}{ccc}
C\mathcal{M}(H)_A & \xrightarrow{V} & \mathcal{M}_A \#(C, A) \\
\downarrow \scriptstyle R & & \downarrow \scriptstyle \mathcal{M}_A \#C^*
\end{array}
\]

Here $R$ is the restriction of scalars functor, $V(M) = M$ with the induced $\#(C, A)$-action

$$m \cdot f = \sum m_{<0>} f(m_{<-1>}) \quad (20)$$

for all $m \in M$, $f \in \Hom(C, A)$. $V$ is fully faithful if $C$ is a projective $k$-module. Also observe that $P = RV$, where $P$ is defined in (13). We will also write $T = R(\bullet \otimes_A \#(C, A))$. For example, the right $A$ and $C^*$-action on $P(C \otimes A)$ are given by the formulas

$$(c \otimes b) \cdot a = \sum c \cdot a_{<-1>} \otimes ba_{<0>} \quad \text{and} \quad (c \otimes b) \cdot c^* = c^* \cdot c \otimes b$$

Moreover, with the action

$$a \cdot (c \otimes b) = c \otimes ab,$$

$C \otimes A$ becomes an $(A, A \#C^*)$-bimodule.

### 1.2 Separable functors

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and $F : \mathcal{C} \to \mathcal{D}$ a covariant functor. Observe that we have two covariant functors

$$\Hom_{\mathcal{C}}(\bullet, \bullet) : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Sets} \quad \text{and} \quad \Hom_{\mathcal{D}}(F(\bullet), F(\bullet)) : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Sets}$$

and a natural transformation

$$\mathcal{F} : \Hom_{\mathcal{C}}(\bullet, \bullet) \to \Hom_{\mathcal{D}}(F(\bullet), F(\bullet))$$

Recall from [27] that $F$ is called separable if $\mathcal{F}$ splits, this means that we have a natural transformation $\mathcal{P} : \Hom_{\mathcal{D}}(F(\bullet), F(\bullet)) \to \Hom_{\mathcal{C}}(\bullet, \bullet)$ such that $\mathcal{P} \circ \mathcal{F}$ is the identity natural transformation of $\Hom_{\mathcal{C}}(\bullet, \bullet)$. 

6
The terminology is motivated by the fact that a ring extension $R \to S$ is separable (in the sense of [10]) or right semisimple (in the sense of [16]) if and only if the restriction of scalars functor $R\mathcal{M} \to S\mathcal{M}$ is separable.

If the functor $F$ is separable, then we have the following version of Maschke’s Theorem (cf. [27, Prop. 1.2]): if $\alpha : M \to N$ in $\mathcal{C}$ is such that $F(\alpha)$ splits or co-splits in $\mathcal{D}$, then $f$ splits or co-splits in $\mathcal{C}$.

Now suppose that $F : \mathcal{C} \to \mathcal{D}$ has a right adjoint $G$, and write $\rho : 1_{\mathcal{C}} \to GF$ and $\delta : FG \to 1_{\mathcal{D}}$ for the unit and counit of this adjunction. Then we have the following result (see [28] and [29]):

**Theorem 1.1** Let $G : \mathcal{D} \to \mathcal{C}$ be a right adjoint of $F : \mathcal{C} \to \mathcal{D}$.

1) $F$ is separable if and only if $\rho$ splits, this means that there is a natural transformation $\nu : GF \to 1_{\mathcal{C}}$ such that $\nu \circ \rho$ is the identity natural transformation of $\mathcal{C}$, or $\nu_M \circ \rho_M = I_M$ for all $M \in \mathcal{C}$.

2) $G$ is separable if and only if $\delta$ co-splits, this means that there is a natural transformation $\theta : FG \to 1_{\mathcal{D}}$ such that $\delta \circ \theta$ is the identity natural transformation of $\mathcal{D}$.

As we already remarked, a separable functor reflects split sequences. However, not every functor that reflects split sequences is separable: it suffices to take a nonseparable finite field extensions $k \to l$. The restriction of scalars functor reflects split sequences, but is not separable.

### 2 Separability Theorems for Doi-Hopf modules

#### 2.1 The forgetful functor

In this Section, we will give necessary and sufficient conditions for the forgetful functor $F : C\mathcal{M}(H)_A \to \mathcal{M}_A$ to be separable. This will lead to generalized notions of integral and separability idempotent; in Section 3, we will discuss special cases, and these will explain our terminology. We will recover some existing results, and obtain some new applications. We will always assume that $C$ is flat as a $k$-module, this is of course no problem if we work over a field $k$.

In the sequel, we will write $C = C\mathcal{M}(H)_A$, $F$ will be the forgetful functor, and $G$ its right adjoint.

Having Theorem 1.1 in mind, we will first study the natural transformations $\nu : GF \to 1_{\mathcal{C}}$. Recall that such a natural transformation consists of the following data: for every $M \in \mathcal{C}$, we have an $A$-linear, $C$-colinear map $\nu_M : C \otimes M \to M$ satisfying the following naturality condition: for any map $f : M \to N$ in $C\mathcal{M}(H)_A$, we have a commutative diagram

\[
\begin{array}{ccc}
C \otimes M & \xrightarrow{\nu_M} & M \\
\downarrow {I_C \otimes f} & & \downarrow f \\
C \otimes N & \xrightarrow{\nu_N} & N 
\end{array}
\]

From Lemma 2.2, it will follow that the natural transformations $\nu : GF \to 1_{\mathcal{C}}$ form a set $V$. Actually $V$ is a $k$-algebra; the addition and scalar multiplication are given by the formulas

$$(\nu + \nu')_M = \nu_M + \nu'_M \text{ and } (x\nu)_M = x\nu_M$$

for all $x \in k$. The multiplication is defined as follows: for two natural transformations $\nu$ and $\nu'$, we define $\nu \cdot \nu' = \nu' \circ \rho \circ \nu$. In general, $V$ does not have a unit, but if $\nu$ is a splitting of $\rho$, then $\nu$ is a right unit. We will first give descriptions of $V$ as a $k$-module, and come back to the algebra structure afterwards.
Lemma 2.1 Let $H$ be a Hopf algebra, and $(H,A,C)$ a Doi-Hopf datum. With notation as above, consider a natural transformation $\nu : GF \to 1_C$. We use $\nu$ also as a notation for $\nu = \nu_{C \otimes A} : C \otimes C \otimes A \to C \otimes A$. $\nu$ then satisfies the following properties:

\begin{align*}
\nu(c \otimes d \otimes ba) &= b\nu(c \otimes d \otimes a) \quad (21) \\
\sum (d^{(2)} \otimes 1_A)(\varepsilon_C \otimes \rho_A)\nu(c \otimes d^{(1)} \otimes 1) &= \nu(c \otimes d \otimes 1) \quad (22)
\end{align*}

for all $a,b \in A$ and $c,d \in C$.

Proof For any $b \in A$, consider the map

$f_b : C \otimes A \to C \otimes A, \quad f_b(c \otimes a) = b(c \otimes a) = c \otimes ba$

for all $a,b \in A$ and $c \in C$ (see (10)). It is easy to check that $f_b$ is a morphism in $C$, and, from the naturality of $\nu$, it follows that we have a commutative diagram

\[
\begin{array}{ccc}
C \otimes C \otimes A & \xrightarrow{\nu} & C \otimes A \\
I_C \otimes f_b & & f_b \\
C \otimes C \otimes A & \xrightarrow{\nu} & C \otimes A
\end{array}
\]

and (21) is equivalent to the commutativity of this diagram.

If $M$ is a $k$-module, then $C \otimes A \otimes M$ can be made into a Doi-Hopf module in a natural way, by using the $A$-action and $C$-coaction on $C \otimes A$. From the naturality of $\nu$, it follows that

$$\nu_{C \otimes A \otimes M} = \nu \otimes I_M \quad (23)$$

Indeed, for every $m \in M$, the map $g_m : C \otimes A \to C \otimes A \otimes M$ given by

$$g_m(c \otimes a) = c \otimes a \otimes m$$

is in the category of Doi-Hopf modules $C$, and we have a commutative diagram

\[
\begin{array}{ccc}
C \otimes C \otimes A & \xrightarrow{\nu} & C \otimes A \\
I_C \otimes g_m & & g_m \\
C \otimes C \otimes A \otimes M & \xrightarrow{\nu_{C \otimes A \otimes M}} & C \otimes A \otimes M
\end{array}
\]

and we find (23).

Now let $C = C$ considered only as a $k$-module. Then the right $C$-coaction $\rho' : C \otimes A \to C \otimes A \otimes C$ (see (11)) is a morphism in $C$, and, using the naturality of $\nu$ and (23), we find a commutative diagram

\[
\begin{array}{ccc}
C \otimes C \otimes A & \xrightarrow{\nu} & C \otimes A \\
I_C \otimes \rho' & & \rho' \\
C \otimes C \otimes A \otimes C & \xrightarrow{\nu \otimes I_C} & C \otimes A \otimes C
\end{array}
\]
Write \( \nu(c \otimes d \otimes 1_A) = \sum_i c_i \otimes a_i \), and apply the diagram to \( c \otimes d \otimes 1_A \), with \( c, d \in C \). We obtain

\[
\sum \nu(c \otimes d_1 \otimes 1_A) \otimes d_2 = \sum c_{i(1)} \otimes a_{i<0>} \otimes c_{i(2)} S(a_{i<-1>})
\]

Applying \( \varepsilon_C \otimes I_A \otimes I_A \) to both sides, and then switching the two factors, we find

\[
\sum d_2 \otimes (\varepsilon_C \otimes I_A)(\nu(c \otimes d_1 \otimes 1_A)) = \sum c_i S(a_{i<-1>}) \otimes a_{i<0>}
\]

Now we apply \( \rho_A \) to the second factor of both sides. Using the fact that \( \rho_A \circ (\varepsilon_C \otimes I_A) = \varepsilon_C \otimes \rho_A \), we obtain

\[
\sum d_2 \otimes (\varepsilon_C \otimes \rho_A)(\nu(c \otimes d_1 \otimes 1_A)) = \sum c_i S(a_{i<-1>}) \otimes a_{i<0>} \otimes a_{i<0>}
\]

We now find (22) after we let the second factor act on the first one.

Observe that (21) means that \( \nu \) is left \( A \)-linear. From (21), (22) and (5), it follows that

\[
\nu(c \otimes d \otimes a) = a \nu(c \otimes d \otimes 1) = \sum (d_2 \otimes m)((\varepsilon_C \otimes I_A)\nu(c \otimes d_1 \otimes 1))
\]

We will now prove that the natural transformation \( \nu \) is completely determined by \( \nu_{C \otimes A} \).

**Lemma 2.2** With notations as in Lemma 2.1, we have

\[
\nu_M(c \otimes m) = \sum m_{<0>}(\varepsilon_C \otimes I_A)\nu(c \otimes m_{<-1>} \otimes 1)
\]

for every Doi-Hopf module \( M \), \( m \in M \) and \( c \in C \).

**Proof** First take \( N = C \otimes M \), where \( M \) is a right \( A \)-module. For any \( m \in M \), the map \( g_m : C \otimes A \to C \otimes M \), \( g(c \otimes a) = c \otimes ma \) is a morphism in \( C \), so we have a commutative diagram

\[
\begin{array}{ccc}
C \otimes C \otimes A & \xrightarrow{\nu_{C \otimes A}} & C \otimes A \\
I_C \otimes g_m & & \downarrow g_m \\
C \otimes C \otimes A & \xrightarrow{\nu_{C \otimes M}} & C \otimes A
\end{array}
\]

Evaluating this diagram at \( c \otimes d \otimes 1_A \), we obtain

\[
\nu_{C \otimes M}(c \otimes d \otimes m) = g_m(\nu_{C \otimes A}(c \otimes d \otimes 1_A)) = \sum (d_2 \otimes m)((\varepsilon_C \otimes I_A)\nu(c \otimes d_1 \otimes 1_A))
\]

Now consider an arbitrary Doi-Hopf module \( M \in C \). The coaction \( \rho_M : C \to C \otimes M \) is a morphism in \( C \), and we have a commutative diagram

\[
\begin{array}{ccc}
C \otimes M & \xrightarrow{\nu_M} & M \\
I_C \otimes \rho_M & & \downarrow \rho_M \\
C \otimes C \otimes M & \xrightarrow{\nu_{C \otimes M}} & C \otimes M
\end{array}
\]
We apply the diagram to \( c \otimes m \). Using (26), we obtain
\[
\rho_M(\nu_M(c \otimes m)) = \nu_{C \otimes M}(\sum c \otimes m_{<1>} \otimes m_{<0>}) = \sum (m_{<1>} \otimes m_{<0>})(\varepsilon_C \otimes I_A)\nu(c \otimes m_{<2>} \otimes 1_A)
\]
and (27) follows after we apply \( \varepsilon_C \otimes I_M \) to both sides.  

Now let \( V_1 \) be the \( k \)-module consisting of all maps \( \nu \in \text{Hom}_C(C \otimes C \otimes A, C \otimes A) \) satisfying (21) and (22). \( \nu \in V_1 \) is called normalized or splitting if
\[
\sum \nu(c(1) \otimes c(2) \otimes a) = c \otimes a
\]
for all \( c \in C \) and \( a \in A \). Recall that \( V \) is the \( k \)-module consisting of all natural transformations \( \nu : GF \to 1_C \). With these notations, we have the following result:

**Theorem 2.3** Let \( H \) be a Hopf algebra, and \( (H, A, C) \) a Doi-Hopf datum. Then the \( k \)-modules \( V \) and \( V_1 \) are isomorphic, and normalized elements in \( V_1 \) correspond to splittings of the unit \( \rho : 1_C \to GF \).

Consequently \( F \) is a separable functor if and only if there exists a normalized element in \( V_1 \).

**Proof** For \( \nu \in V_1 \), we define \( g(\nu) = \nu \) as follows: for every \( M \in \mathcal{C} \), \( \nu_M : C \otimes M \to M \) is given by
\[
\nu_M(c \otimes m) = \sum m_{<0>}(\varepsilon_C \otimes I_A)\nu(c \otimes m_{<1>} \otimes 1_A)
\]
(cf. Lemma 2.2). We have to show that \( \nu \) is indeed a natural transformation. It suffices to show that \( \nu_M \) is right \( A \)-linear and left \( C \)-colinear, and that the naturality condition is satisfied.

a) \( \nu_M \) is right \( A \)-linear. For all \( m \in M \), \( c \in C \) and \( a \in A \), we have
\[
\begin{align*}
\nu_M(c \otimes m)a &= \sum m_{<0>}(\varepsilon_C \otimes I_A)\nu(c \otimes m_{<1>} \otimes 1)a \\
&= \sum m_{<0>}((\varepsilon_C \otimes I_A)\nu(c \otimes m_{<1>} \otimes 1)a) \\
&= \sum m_{<0>}((\varepsilon_C \otimes I_A)\nu(c \cdot a_{<2>} \otimes m_{<1>} \cdot a_{<1>} \otimes a_{<0>})) \\
&= \sum (m_{<0>}a_{<0>})(\varepsilon_C \otimes I_A)(\nu(c \cdot a_{<2>} \otimes m_{<1>} \cdot a_{<1>} \otimes 1)) \\
&= \nu_M(\sum c \cdot a_{<1>} \otimes ma_{<0>})
\end{align*}
\]
where we used the fact that \( \varepsilon_C \otimes I_A : C \otimes A \to A \) is left and right \( A \)-linear.

b) \( \nu_M \) is left \( C \)-colinear. Write
\[
\nu(c \otimes d \otimes 1) = \sum_i c_i \otimes a_i
\]
From the fact that \( \nu \) is left \( C \)-colinear, it follows that
\[
\sum c_{(1)} \otimes \nu(c_{(2)} \otimes d \otimes 1) = \rho(\nu(c \otimes d \otimes 1)) = \sum_i c_{(1)} \otimes c_{i(2)} \otimes a_i
\]
Applying \( I_C \otimes \varepsilon_C \otimes I_A \) to both sides, we obtain
\[
\sum c_{(1)} \otimes (\varepsilon_C \otimes I_A)(\nu(c_{(2)} \otimes d \otimes 1)) = \sum_i c_i \otimes a_i = \nu(c \otimes d \otimes 1)
\]
We temporarily introduce the following notation, for \( m \in M, \ c \in C \) and \( a \in A \):

\[
(I \otimes m)(c \otimes a) = c \otimes ma
\]

Now

\[
\rho_M(\nu_M(c \otimes m)) = \rho_M \left( \sum m_{<0>}((\varepsilon_C \otimes I_A)\nu(c \otimes m_{<1>} \otimes 1_A)) \right)
\]

\[
= \sum \rho_M(m_{<0>})\rho_A \left( (\varepsilon_C \otimes I_A)\nu(c \otimes m_{<1>} \otimes 1_A) \right)
\]

\[
= \sum (m_{<1>} \otimes m_{<0>})\rho_A \left( (\varepsilon_C \otimes I_A)\nu(c \otimes m_{<2>} \otimes 1_A) \right)
\]

\[
= \sum (I \otimes m_{<0>})\nu(c \otimes m_{<1>} \otimes 1_A)
\]

\[
= \sum (c(1) \otimes m_{<0>}) \cdot \left( (\varepsilon_C \otimes I_A)(\nu(c(2) \otimes m_{<1>} \otimes 1_A)) \right)
\]

\[
= \sum c(1) \otimes \nu_M(c(2) \otimes m)
\]

\[
= (I_C \otimes \nu_M)(\rho_M(c \otimes m))
\]

c) We now prove the naturality condition: let \( \alpha : M \to N \) is a morphism in \( C \). Then

\[
\nu_N(c \otimes \alpha(m)) = \sum \alpha(m_{<0>})((\varepsilon_C \otimes I_A)\nu(c \otimes \alpha(m_{<1>} \otimes 1))
\]

(\( \alpha \) is left \( C \) colinear) \[
= \sum \alpha(m_{<0>})(\varepsilon_C \otimes I_A)\nu(c \otimes m_{<1>} \otimes 1)
\]

(\( \alpha \) is right \( A \) linear) \[
= \sum \alpha(m_{<0>}((\varepsilon_C \otimes I_A)\nu(c \otimes m_{<1>} \otimes 1))
\]

\[
= \alpha(\nu_M(c \otimes m))
\]

Now define \( f : V \to V_1 \) by

\[
f(\nu) = \nu_{C \otimes A}
\]

\( \nu \)From Lemma 2.1, it follows that \( f(\nu) \in V_1 \), and from Lemma 2.2 that \( g \circ f = I_V \). \( \nu \)From (24), it follows that \( f \circ g = I_1 \).

Finally, if \( \nu \) is normalized, then for \( M \in C \) and \( m \in M \), we have

\[
\nu_M(\sum m_{<1>} \otimes m_{<0>}) = \sum m_{<0>}((\varepsilon_C \otimes I_A)\nu(m_{<2>} \otimes m_{<1>} \otimes 1_A))
\]

\[
= \sum m_{<0>}((\varepsilon_C \otimes I_A)(m_{<1>} \otimes 1_A)) = m
\]

and \( \rho \) is split by \( \nu \). The other statements of the Theorem are immediate consequences. \( \square \)

**Remark 2.4** Observe that the antipode \( S \) of \( H \) is used in the construction of \( f : V \to V_1 \), but not in the construction of \( g : V_1 \to V \). So if \( H \) is only a bialgebra, then we are still able to define a homomorphism \( g : V_1 \to V \), and we can still conclude that the existence of a normalized element in \( V_1 \) is a sufficient condition for the separability of the forgetful functor \( F \). We will come back to this situation in Section 2.3.

The \( k \)-module \( V_1 \) can be rewritten in at least four different ways. Consider the following \( k \)-modules \( V_2, V_3, V_4 \) and \( V_5 \).

2) \( V_2 \) consists of all maps \( \lambda : C \otimes C \to C \otimes A \) satisfying the following properties:

\[
\sum (d(2) \otimes 1_A)(c \otimes d(1)) = \lambda(c \otimes d)
\]

(30)

\[
\sum a_{<0>}(c \cdot a_{<2>} \otimes d \cdot a_{<1>}) = \lambda(c \otimes d) a
\]

(31)

\[
\sum c(1) \otimes \lambda(c(2) \otimes d) = \rho_{C \otimes A}(\lambda(c \otimes d))
\]

(32)
for all $c, d \in C$. $\lambda \in V_2$ is called normalized if
\[
\sum \lambda(c_{(1)} \otimes c_{(2)}) = c \otimes 1
\]  
(33)
for all $c \in C$.

\textbf{3)} $V_3$ consists of all maps $\theta : C \otimes C \to A$ satisfying the following properties:
\[
\theta(c \otimes d)a = \sum a_{<0>} \theta(c \cdot a_{<-2>} \otimes d \cdot a_{<-1>})
\]  
(34)
\[
\sum c_{(1)} \otimes \theta(c_{(2)} \otimes d) = \sum d_{(2)} \theta(c \otimes d_{(1)})_{<-1>} \otimes \theta(c \otimes d_{(1)})_{<0>}
\]  
(35)
for all $c, d \in C$ and $a \in A$. $\theta \in V_3$ is called normalized, or a \textit{coseparability idempotent} if
\[
\sum \theta(c_{(1)} \otimes c_{(2)}) = \eta_A(\varepsilon_C(c))
\]  
(36)
for all $c \in C$ (cf. Example \[2.11\]).

\textbf{4)} $V_4$ consists of all maps $\gamma : C \to \text{Hom}(C, A)$ satisfying the following properties:
\[a \mapsto \gamma = \gamma \mapsto a\]  
(37)
for any $a \in A$, or, equivalently,
\[
\sum a_{<0>} \cdot \gamma(c \cdot a_{<-1>}) = \gamma(c) \cdot a
\]  
(38)
for all $a \in A$ and $c \in C$, or,
\[
\sum a_{<0>} \gamma(c \cdot a_{<-2>})(da_{<-1>}) = (\gamma(c)(d))a
\]  
(39)
for all $a \in A$ and $c, d \in C$.

\textbf{b)} For all $c, d \in C$, we have
\[
\sum c_{(1)} \otimes \gamma(c_{(2)})(d) = \sum d_{(2)} \gamma(c)(d_{(1)})_{<-1>} \otimes \gamma(c)(d_{(1)})_{<0>}
\]  
(40)
which means that $\gamma$ is right $C^*$-linear, where the right $C^*$-actions on $C$ and $\text{Hom}(C, A)$ are given by the formulas \[\square\] and \[\Diamond\]. An element of $V_4$ is called normalized if
\[
\sum \gamma(c_{(1)})c_{(2)} = \varepsilon(c)1_A
\]  
(42)
for all $c \in C$.

\textbf{5)} $V_5$ consists of all $A$-bimodule maps $\psi : C \otimes A \to \text{Hom}(C, A)$ satisfying the additional condition
\[
\sum c_{(1)} \otimes \psi(c_{(2)} \otimes 1_A)(d) = \sum d_{(2)} \psi(c \otimes 1_A)(d_{(1)})_{<-1>} \otimes \gamma(c)(d_{(1)})_{<0>}
\]  
(43)
for all $c, d \in C$. If $C$ is projective as a $k$-module, then a map $\psi \in V_5$ is nothing else than an $(A, A\#C^*)$-bimodule map. $\psi \in V_5$ is called normalized if
\[
\sum \psi(c_{(1)} \otimes 1_A)c_{(2)} = \varepsilon(c)1_A
\]  
(44)
for all $c \in C$. 

\[\square\] looks artificial, but it has a natural interpretation in the case where $C$ is projective as a $k$-module. Then \[\square\] is equivalent to
\[
\sum (c^*, c_{(1)}) \gamma(c_{(2)})(d) = \sum (c^*, d_{(2)} \gamma(c)(d_{(1)})_{<-1>}) \gamma(c)(d_{(1)})_{<0>}
\]  
(41)
Proposition 2.5 Let $H$ be a bialgebra, and $(H, A, C)$ a Doi-Hopf datum. Then the $k$-modules $V_1, \ldots, V_5$ are isomorphic, and normalized elements correspond to normalized elements.

Proof 1) We define $f_1 : V_1 \to V_2$ and $g_1 : V_2 \to V_1$ by

$$f_1(\nu) = \lambda \text{ and } g_1(\lambda) = \nu$$

with

$$\lambda(c \otimes d) = \nu(c \otimes d \otimes 1) \ ; \ \nu(c \otimes d \otimes a) = a\lambda(c \otimes d) \quad (45)$$

It is straightforward to see that (31) is equivalent to right $A$-linearity of $\nu$, and (32) to left $C$-co-linearity of $\nu$, and (22) to (31). (21) follows from (45). It is also clear that $f_1$ and $g_1$ are each others inverses.

2) $f_2 : V_2 \to V_3$ and $g_2 : V_3 \to V_2$ are defined by

$$f_2(\lambda) = \theta \text{ and } g_2(\theta) = \lambda$$

with

$$\theta(c \otimes d) = (\epsilon_C \otimes I_A)(\lambda(c \otimes d))$$

$$\lambda(c \otimes d) = \sum c_{(1)} \otimes \theta(c_{(2)} \otimes d)$$

We first show that $f_2$ is well-defined: if $\lambda \in V_2$, then $\theta = f_2(\lambda)$ satisfies (34) and (35). We apply $\epsilon_C \otimes I_A$ to both sides of (31). Writing $\lambda(c \otimes d) = \sum c_i \otimes a_i$, we obtain for the right hand side:

$$(\epsilon_C \otimes I_A)(\lambda(c \otimes d) a) = (\epsilon_C \otimes I_A)(\sum c_i a_{<-1>} \otimes a_i a_{<0>})$$

$$= \sum \epsilon_C(c_i) a_i a = \theta(c \otimes d) a$$

For all $a \in A$, $c, d \in C$, with again $\lambda(c \otimes d) = \sum c_i \otimes a_i$, we also have

$$(\epsilon_C \otimes I_A)(a \lambda(c \otimes d)) = \sum \epsilon(c_i) a a_i = a\theta(c \otimes d)$$

so the left hand side amounts to

$$\sum a_{<0>} \theta(ca_{<-2>} \otimes da_{<-1>})$$

and (34) follows.

Now apply $I_C \otimes \epsilon_C \otimes I_A$ to both sides of (32). This gives

$$\sum c_{(1)} \otimes \theta(c_{(2)} \otimes d) = ((I_C \otimes \epsilon_C \otimes I_A) \circ \rho_{C \otimes A} \circ \lambda)(c \otimes d)$$

$$= \lambda(c \otimes d) \quad (33)$$

$$= \sum d_{(2)}(\epsilon_C \otimes \rho_A)\lambda(c \otimes d_{(1)})$$

$$= \sum d_{(2)}(\theta(c \otimes d_{(1)})_{<-1>} \otimes (\theta(c \otimes d_{(1)})<0> \quad (33)$$

proving (33).

Conversely, if $\theta \in V_3$, then $\lambda = g_2(\theta) \in V_2$. Indeed,

$$\sum d_{(2)}(\epsilon_C \otimes \rho_A)\lambda(c \otimes d_{(1)}) = \sum d_{(2)}\rho_A(\theta(c \otimes d_{(1)}))$$

$$= \sum d_{(2)}(\theta(c \otimes d_{(1)})_{<-1>} \otimes (\theta(c \otimes d_{(1)})<0> \quad (35)$$

$$= \sum c_{(1)} \otimes \theta(c_{(2)} \otimes d) \lambda(c \otimes d)$$

13
proving (30). Furthermore, 
\[ \sum_{\alpha} a_{<0>}^{\alpha} \lambda(c_{<-2>}^{\alpha} \otimes da_{<-1>}) = \sum_{\gamma} c_{(1)}^{\gamma} a_{<-3>}^{\gamma} \otimes a_{<0>}^{\gamma} \theta(c_{(2)}^{\gamma} c_{<-2>}^{\gamma} \otimes da_{<-1>}) \]
\[ = \sum_{\gamma} c_{(1)}^{\gamma} a_{<-1>}^{\gamma} \otimes \theta(c_{(2)}^{\gamma} \otimes d)a_{<0>}^{\gamma} \]
proving (31). Finally, 
\[ \sum_{\gamma} c_{(1)}^{\gamma} \otimes \lambda(c_{(2)}^{\gamma} \otimes d) = \sum_{\gamma} c_{(1)}^{\gamma} \otimes c_{(2)}^{\gamma} \otimes \theta(c_{(3)}^{\gamma} \otimes d) \]
\[ = \rho_{C \otimes A}(\lambda(c \otimes d)) \]
proving (32). It is clear that \( g_{3} \) and \( f_{3} \) are each other's inverses.

3) \( f_{3} : V_{3} \to V_{4} \) and \( g_{3} : V_{4} \to V_{3} \) are defined by 
\[ f_{3}(\theta) = \gamma \text{ and } g_{3}(\gamma) = \theta \]
with
\[ \gamma(c)(d) = \theta(c \otimes d) \]
It can be checked easily that (34) is equivalent to (37), and (35) to (40), so \( f_{3} \) and \( g_{3} \) are well-defined. Obviously \( g_{3} = f_{3}^{-1} \).

4) \( f_{4} : V_{4} \to V_{5} \) and \( g_{4} : V_{5} \to V_{4} \) are defined by 
\[ f_{4}(\gamma) = \psi \text{ and } g_{4}(\psi) = \gamma \]
with
\[ \psi(c \otimes a) = a \cdot \gamma(c) \text{ and } \gamma(c) = \psi(c \otimes 1_{A}) \]
Let us show that \( f_{4} \) is well-defined: \( f_{4}(\gamma) = \psi \in V_{5} \) if \( \gamma \in V_{4} \). \( \psi \) is left \( A \)-linear since 
\[ \psi(a \cdot (c \otimes b)) = \psi(c \otimes ab) \]
\[ = ab \cdot \gamma(c) \]
\[ = a \cdot (\psi(c \otimes b)) \],
right \( A \)-linear since 
\[ \psi((c \otimes b) \cdot a) = \psi(\sum c \cdot a_{<-1>} \otimes ba_{<0>}) \]
\[ = \sum ba_{<0>} \cdot \gamma(c \cdot a_{<-1>}) \]
\[ = b \cdot (a^{-1}) \gamma(c) \]
\[ = b \cdot (\gamma^{-1})(c) \]
\[ = b \cdot (\gamma(c) \cdot a) \]
\[ = \psi(c \otimes b) \cdot a, \]
and right \( C^{*} \)-linear since 
\[ \psi((c \otimes b) \cdot c^{*}) = \psi(c \cdot c^{*} \otimes b) \]
\[ = b \cdot \gamma(c \cdot c^{*}) \]
\[ = b \cdot (\gamma(c) \cdot c^{*}) \]
\[ = \psi(c \otimes b) \cdot c^{*}. \]
Conversely, if \( \psi \in V_5 \), then \( \gamma = g_4(\psi) \in V_4 \). Indeed, \( \gamma \) is right \( C^\ast \)-linear, since

\[
\gamma(c \leftarrow c^\ast) = \psi(c \leftarrow c^\ast \otimes 1_A) = \psi((c \otimes 1_A) \cdot c^\ast) = \psi(c \otimes 1_A) \cdot c^\ast = \gamma(c) \cdot c^\ast.
\]

For all \( a \in A \), \( c \in C \) we have

\[
(a \rightarrow \gamma)(c) = \sum a_{<0>} \cdot \gamma(c \cdot a_{<-1>}) = \sum a_{<0>} \cdot \psi(c \cdot a_{<-1>} \otimes 1_A)
\]

(\( \psi \) is left \( A \)-linear)

\[
= \sum \psi(c \cdot a_{<-1>} \otimes a_{<0>}) = \psi((c \otimes 1_A) \cdot a)
\]

(\( \psi \) is right \( A \)-linear)

\[
= \psi(c \otimes 1_A) \cdot a = (\gamma \leftarrow a)(c).
\]

proving that \( \gamma \) is centralized by the left and right actions of \( A \). It is also clear that \( f_4 \) and \( g_4 \) are each others inverses, and this finishes the proof of the fact that the \( V_i \) are isomorphic as \( k \)-modules. We leave it to the reader to prove that the \( f_i \) and \( g_i \) send normalized elements to normalized elements.

\( \square \)

Combining Theorem 2.3 and Proposition 2.5, we conclude that the (normalized) elements of each of the five vector space \( V_i \) can be used as tools to construct a splitting of \( \rho : 1_C \rightarrow GF \), and, following the philosophy of the introduction, then can be called integrals. We have reserved this terminology for the elements \( \gamma \in V_4 \), because they are closely related to Doi’s total integrals (see Section 3.2).

**Definition 2.6** A map \( \gamma \in V_4 \), that is, a \( k \)-linear map \( \gamma : C \rightarrow \text{Hom}(C,A) \) satisfying (37) and (40) is called an \( A \)-integral for the Doi-Hopf datum \( (H,A,C) \). If \( \gamma \) is normalized (that is, \( \gamma \) satisfies (42)), then we call \( \gamma \) a total \( A \)-integral.

Our results can be summarized in the following Theorem, which is the main result of this paper.

**Theorem 2.7 (Maschke’s Theorem for Doi-Hopf modules)**

Let \( H \) be a bialgebra, and \( (H,A,C) \) a Doi-Hopf datum. If there exists a total \( A \)-integral \( \gamma : C \rightarrow \text{Hom}(C,A) \) (or, equivalently, a normalized element in \( V_i \) (\( i = 1, \ldots, 5 \))), then the forgetful functor \( F : C\mathcal{M}_A \rightarrow \mathcal{M}_A \) is separable. The converse holds if \( H \) is a Hopf algebra.

In this situation, the following assertions hold:

1) If a morphism \( u : M \rightarrow N \) in \( C\mathcal{M}_A \) has a retraction (resp. a section) in \( \mathcal{M}_A \), then it has a retraction (resp. a section) in \( C\mathcal{M}_A \).

2) If \( M \in C\mathcal{M}_A \) is semisimple (resp. projective, injective) as a right \( A \)-module, then \( M \) is also semisimple (resp. projective, injective) as an object in \( C\mathcal{M}_A \).

As we have remarked, the natural transformations not only form a \( k \)-module, but even a \( k \)-algebra. The isomorphisms \( f \) and \( f_i \) then define an algebra structure on each of the vector spaces \( V_i \). The multiplication can be described in a natural way for \( i = 3 \) and \( i = 4 \), and this is what we will do.
next.
On $C^\text{cop} \otimes C$, we define an $H$-coaction

$$(c \otimes d) \cdot h = c \otimes d \cdot h$$

and this coaction makes $C^\text{cop} \otimes C$ into a right $H$-module coalgebra, so that we can consider the $k$-algebra $\#(C^\text{cop} \otimes C, A)$ (see (14)).

**Lemma 2.8** $V_3$ is a $k$-subalgebra of $\#(C^\text{cop} \otimes C, A)$. Normalized elements in $V_3$ are right units of $V_3$, and consequently they are idempotents.

**Proof** Applying (35), we find the following formula for the multiplication of $\theta$, $\theta' \in V_3$:

$$((\theta \bullet \theta')(c \otimes d)) \cdot a = \sum \theta(c(3) \otimes d) a_{<0>} \theta'(c(1) \cdot a_{<2>} \otimes c(2) \cdot a_{<1>})$$

Using (46), we can check easily that $\theta \bullet \theta'$ satisfies (34) and (35). Indeed, for all $c, d \in C$ and $a \in A$, we have

and

$$\sum d(2)((\theta \bullet \theta')(c \otimes d(1)))_{<1>} \otimes ((\theta \bullet \theta')(c \otimes d(1)))_{<0>}$$

Using (46), it follows immediately that normalized elements in $V_3$ are right units.

**Lemma 2.9** $V_4$ is a $k$-subalgebra of $\text{Hom}(C^\text{cop}, \#(C, A))$, and $f_3 : V_3 \to V_4$ is a $k$-algebra isomorphism.

**Proof** Left as an exercise to the reader.

Now write $h = g \circ g_1 \circ g_2 : V_3 \to V$.

**Theorem 2.10** $h$ is a $k$-algebra homomorphism. If $H$ is a Hopf algebra, then $V$, $V_3$ and $V_4$ are isomorphic as $k$-algebras.

**Proof** We have to show that $h$ is multiplicative. From the definitions of $g$, $g_1$ and $g_2$, we easily find that $h(\theta) = \nu$ with

$$\nu_M(c \otimes m) = \sum m_{<0>} \cdot \theta(c \otimes m_{<1>})$$

(47)
for all $M \in \mathcal{C}$. Now take $\theta' \in V_3$ and write $h(\theta') = \nu'$. From (46), we find that
\[
(h(\theta \cdot \theta'))_M(c \otimes m) = \sum m_{<0>} \theta(c(3) \otimes m_{<-1>}) \theta'(c(1) \otimes c(2))
\]
By definition $\nu \cdot \nu' = \nu' \circ \rho \circ \nu$, and
\[
\begin{align*}
(h \cdot (\nu \cdot \nu'))_M(c \otimes m) &= \nu'_M(\rho_M(\nu_M(c \otimes m))) \\
&= \nu'_M(\sum c(1) \otimes \nu_M(c(2) \otimes m)) \\
&= \sum \nu_M(c(3) \otimes m_{<0>}) \cdot \theta'(c(1) \otimes c(2)) \\
&= \sum m_{<0>} \theta(c(3) \otimes m_{<-1>}) \theta'(c(1) \otimes c(2))
\end{align*}
\]
and it follows that $h$ is multiplicative.

\[\square\]

**Example 2.11** Take $H = A = k$. The $k$-algebra $V_3$ then consists of maps $\theta \in (C \otimes C)^*$ satisfying
\[
\sum c(1) \otimes \theta(c(2) \otimes d) = \sum d(2) \otimes \theta(c \otimes d(1))
\]
for all $c, d \in C$. The multiplication on $V_3$ is given by convolution, and the normalized elements in $V_3$ are nothing else then the coseparability idempotents in the sense of Larson \[18\]. Thus $C$ is a coseparable $k$-coalgebra if and only if the forgetful functor $C \mathcal{M} \to k \text{-mod}$ is separable. If $k$ is a field, then this property is also equivalent to $C \otimes L$ being a cosemisimple coalgebra over $L$ for any field extension $L$ of $k$ (see \[4, Theorem 4.3\]). This is in fact the dual of the well-known result that a $k$-algebra $A$ is separable over $k$ if and only if $A \otimes L$ is a semisimple $L$-algebra for any field extension $L$ of $k$.

We also remark that a method to construct coseparable coalgebras was recently developed in \[5\].

Using an FRT type Theorem, one can construct coseparability idempotents from solutions of the so-called separability equation
\[
R_{12}^2 R_{23}^3 = R_{23}^3 R_{13}^1 = R_{13}^1 R_{12}^3
\]
where $R \in \text{End}(M \otimes M)$, with $M$ a finite dimensional vector space.

**Remark 2.12** Starting with a normalized element $\nu \in V_1$, we can construct a splitting $\nu$ of $\rho$, and this $\nu$ is given by (28). Of course one can also construct $\nu$ starting from a normalized element in any of of the other four $V_i$. Let us do this explicitely for $i = 4$: take a total $A$-integral $\gamma \in V_4$. Thus $\gamma : C \to \#(C, A)$, and the relation between $\gamma$ and the corresponding $\nu \in V_1$ is the following:
\[
\gamma(c)(d) = (\varepsilon_C \otimes I_A)\nu(c \otimes d \otimes 1_A)
\]
and consequently (28) can be rewritten as
\[
\nu_M(c \otimes m) = \sum m_{<0>} \gamma(c)(m_{<-1>}) = m \cdot \gamma(c)
\]
(see (20) for the definition of the right $\#(C, A)$-action on $M$). From this formula, we easily deduce the following result.
Proposition 2.13 Let $H$ be a bialgebra. For a normalized $A$-integral $\gamma : C \rightarrow \#(C,A)$, the following statements are equivalent:

1) $\gamma$ is normalized;
2) for all $c \in C$, $a \in A$, we have
   \[ \sum (c(2) \otimes a) \triangleleft \gamma(c(1)) = c \otimes a \] (49)
3) for all $M \in C$ and $m \in M$, we have
   \[ \sum m_{<0>} \triangleleft \gamma(m_{<-1>}) = m \] (50)

Proof 1) $\Rightarrow$ 3) From (48) and the fact that $\nu$ splits $\rho$, we find immediately that
   \[ m = \nu_M(\rho_M(m)) = \sum m_{<0>} \triangleleft \gamma(m_{<-1>}) \]
3) $\Rightarrow$ 2) follows after we apply 3) with $M = C \otimes A$.
2) $\Rightarrow$ 1) taking $a = 1_A$, we find
   \[ \sum c(3) \cdot \gamma(c(1))(c(2))_{<-1>} \otimes \gamma(c(1))(c(2))_{<0>} = c \otimes 1_A \]
and (42) follows after we apply $\varepsilon \otimes I$. $\square$

2.2 The induction functor

In [3], the authors called an element $z = \sum c_i \otimes a_i \in C \otimes A$ an integral for the Doi-Hopf datum $(H,A,C)$ if

\[ az = za \]
for all $a \in A$, or, equivalently, the map $\theta_A : A \rightarrow C \otimes A$ given by $\theta_A(a) = za$ is left and right $A$-linear. From now on, we will call such a $z$ a dual $A$-integral for $(H,A,C)$. A dual $A$-integral is called normalized if $\sum \varepsilon(c_i)a_i = 1_A$. It was shown in [3] that the existence of dual $A$-integrals is connected to the fact that the forgetful functor $F$ is Frobenius, which means that $G$ is at once a left and right adjoint of $F$. Now we will show that the existence of a normalized dual $A$-integral is equivalent to the separability of the functor $G$.

Theorem 2.14 Let $(H,A,C)$ be a Doi-Hopf datum, and let $W$ and $W_1$ be the $k$-modules consisting of respectively all natural transformations $\theta : 1_{\mathcal{M}_A} \rightarrow FG$, and all dual $A$-integrals. Then we have an isomorphism $f : W \rightarrow W_1$. The counit $\delta$ of the adjoint pair $(F,G)$ is cosplit by $\delta \in W$ if and only if the corresponding $A$-integral $f(\delta)$ is normalized. Consequently the induction functor $G : \mathcal{M}_A \rightarrow C\mathcal{M}_A$ is separable if and only if there exists a normalized dual integral $z \in C \otimes A$.

Proof We define $f : W \rightarrow W_1$ as follows: $f(\theta) = \theta_A(1_A)$. We have to show that $z = \theta_A(1_A) = \sum c_i \otimes a_i \in W_1$. From the fact that $\theta_A$ is right $A$-linear, it follows that

\[ \theta_A(a) = \sum c_i \cdot a_{<-1>} \otimes a_i a_{<0>} \]
for all $a \in A$. Let $N$ be an arbitrary right $A$-module. For $n \in N$, we define $\phi_n : A \rightarrow N$ by $\phi_n(a) = na$, for all $a \in A$. Using the naturality of $\theta$, we obtain the following commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\theta_A} & C \otimes A \\
\downarrow{\phi_n} & & \downarrow{1_C \otimes \phi_n} \\
N & \xrightarrow{\theta_N} & C \otimes N
\end{array}
\]
Hence \( \theta_N \circ \phi_n = (I_C \otimes \phi_n) \circ \theta_A \). Applying the diagram to \( a \in A \) we find
\[
\theta_N(na) = \sum c_i \cdot a_{<1>} \otimes \phi_n(a_i a_{<0>}) \\
= \sum c_i \cdot a_{<1>} \otimes na_i a_{<0>} \\
= (\sum c_i \otimes na_i) \cdot a
\]
and it follows that
\[
\theta_N(n) = \sum c_i \otimes na_i
\]
Taking \( N = A \) and \( n = a \) we find
\[
\sum c_i \otimes aa_i = \sum c_i \cdot a_{<1>} \otimes a_i a_{<0>}
\]
and this shows that \( z \) is a dual \( A \)-integral.

Now define \( g : W_1 \to W \) by \( g(\sum_i c_i \otimes a_i) = \theta \), with
\[
\theta_N(n) = \sum_i c_i \otimes na_i
\]
for all \( N \in M_A \), and \( n \in N \). Using the fact that \( z = \sum_i c_i \otimes a_i \) is a dual \( A \)-integral, it follows easily that \( \theta_N \) is right \( A \)-linear. The naturality of \( \theta \) can also be verified easily: for any right \( A \)-linear map \( \alpha : N \to N' \), we have
\[
(I_C \otimes \alpha)(\theta_N(a)) = \sum_i c_i \otimes \alpha(na_i) \\
\sum_i c_i \otimes \alpha(n) a_i = \theta_{N'}(\alpha(n))
\]
From the definition of \( f \) and \( g \), it follows immediately that \( f \circ g = I_{W_1} \), and \((51)\) implies \( g \circ f = I_W \). The statement about normalized elements is easy to verify, and the rest of the Theorem then follows immediately from Theorem \ref{1.1}.

**Remark 2.15** \( W \) is a \( k \)-algebra. The multiplication is given by the formula
\[
\theta \circ \theta' = \theta' \circ \delta \circ \theta
\]
The corresponding multiplication on \( W_1 \) is then the following:
\[
(\sum_i c_i \otimes a_i) \cdot (\sum_j c'_j \otimes a'_j) = \sum_{i,j} c'_j \otimes \varepsilon(c_i) a_i a'_j
\]
Obviously normalized dual \( A \)-integrals are left units, and idempotents for this multiplication.

**Example 2.16** We return to situation considered in Example \ref{2.11}. From Theorem \ref{2.14}, it follows immediately that the induction functor \( C \otimes \bullet : M_k \to CM \) is separable if and only if there exists a \( c \in C \) such that \( \varepsilon_C(c) = 1 \). In particular, if \( C \) has a group-like element, then the functor \( C \otimes \bullet \) is separable, and this was pointed out already in \[23\].
If \( C \) is a nonzero coalgebra over a field \( k \), then \( C \otimes \bullet \) is separable: take \( c \neq 0 \in C \), then \( c = \sum \varepsilon(c_{(1)}) c_{(2)} \), and there exists a \( d \in C \) with \( \varepsilon(d) \neq 0 \).
2.3 Bialgebras and Relative separability

In Theorem 2.3, we have seen that the $k$-modules $V_1, \ldots, V_5$ are isomorphic to $V$ if $H$ is a Hopf algebra. Now assume that $H$ is a bialgebra, not necessarily with an antipode. In this Section, we will give a description of the $V_i$ in terms of natural transformations. To this end, we first introduce a relative version of separability.

Let $(F, G)$ be a pair of adjoint functors between two categories $\mathcal{C}$ and $\mathcal{D}$. As usual, we write $\rho : 1_{\mathcal{C}} \to GF$ for the unit of the adjunction. Let $B$ be a third category such that we have functors $T : B \to \mathcal{D}$ and $P : \mathcal{C} \to B$, and a natural transformation $\chi : TF \to P$:

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
G & \downarrow & \\
\mathcal{B} & \xleftarrow{T} & \mathcal{D} \\
\end{array}
$$

We call the functor $F$ separable relative to $(B, P, T, \chi)$ if there exists a natural transformation $\phi : PG \to T$ such that

$$\chi \circ \phi(F) \circ P(\rho) = 1_P \quad (52)$$

the identity natural transformation on $P$.

If the functor $P$ is fully faithful, then relative separability implies separability. Indeed, separability is equivalent to the splitting of the unit $\rho$, and, because of the fact that $P$ is fully faithful, this is equivalent to the fact that $P(\rho) : P \to PGF$ splits, and this happens if $P$ is separable relative to $(B, P, T, \chi)$.

Now we consider the following situation: $\mathcal{C} = \mathcal{C}^\mathcal{M}_A$, $\mathcal{D} = \mathcal{M}_A$, $F$ the functor forgetting the coaction, $B = \mathcal{M}_A^\#C^*$, and the functors $P$ and $T$ defined as in Section 1.1. $P$ is fully faithful if $C$ is projective as a $k$-module, and we have the following diagram:

$$
\begin{array}{ccc}
\mathcal{C}^\mathcal{M}(H)_A & \xrightarrow{F} & \mathcal{M}_A \\
G & \downarrow & \\
\mathcal{M}_A^\#C^* & \xleftarrow{T} & \mathcal{M}_A \\
\end{array}
$$

We define $\chi : TG \to P$ as follows: $\chi_M : M \otimes_A \#(C, A) \to M$ is given by

$$\chi_M(m \otimes f) = m \circ f$$

for all $M \in \mathcal{M}_A$, $m \in M$ and $f \in \text{Hom}(C, A)$.

**Theorem 2.17** Let $H$ be a bialgebra, and $(H, A, C)$ a Doi-Hopf datum. With notations as above, the $k$-module $V'$ consisting of all natural transformations $\phi : PG \to T$ is isomorphic to $V_5'$, the $k$-module consisting of all $(A, A^\#C^*)$-bimodule maps $\psi : C \otimes A \to \text{Hom}(C, A)$. If $C$ is projective as a $k$-module, then $V' \cong V_5'$ is isomorphic to each of the $V_i$, and, consequently, $F$ is separable relative to $(B, P, T, \chi)$ if and only if there exists a normalized integral in one of the $V_i$'s.
Proof We will construct an isomorphism $g' : V'_5 \to V'$. Let $\psi : C \otimes A \to \text{Hom}(C, A)$ be an integral in $V'_5$. We define

$$\phi = g'(\psi) : PG \to T$$

by

$$\phi_M : C \otimes M \to M \otimes_A \#(C, A), \quad \phi_M(c \otimes m) = m \otimes \psi(c \otimes 1_A)$$

for all $M \in \mathcal{M}_A$, $c \in C$ and $m \in M$. We will prove that $\phi$ is a natural transformation from $PG$ to $T$. First, $\phi_M$ is right $A\#C^*$-linear. For all $a \in A$, $c \in C$, $c^* \in C^*$ and $m \in M$ we have

$$\phi_M((c \otimes m) \cdot a) = \sum \phi_M(c \cdot a_{<1>} \otimes m \cdot a_{<0>})$$

$$= \sum m \cdot a_{<0>} \otimes A \psi(c \cdot a_{<1>} \otimes 1_A)$$

$$= \sum m \otimes_A a_{<0>} \cdot \psi(c \cdot a_{<1>} \otimes 1_A)$$

$$= m \otimes_A (a \cdot \psi)(c \otimes 1_A)$$

$$= m \otimes_A \psi(c \otimes a)$$

$$= m \otimes_A \psi(c \otimes 1_A) \cdot a$$

$$= \phi_M(c \otimes m) \cdot a$$

and

$$\phi_M((c \otimes m) \cdot c^*) = \phi_M(c \cdot c^* \otimes m)$$

$$= m \otimes_A \psi(c \cdot c^* \otimes 1_A)$$

$$= m \otimes \psi(c \otimes 1_A) \cdot c^*$$

$$= \phi_M(c \otimes m) \cdot c^*$$

The naturality condition can be verified as follows. Take $M, N \in \mathcal{M}_A$ and a right $A$-linear map $f : M \to N$. Then

$$(f \otimes_A I_{\#(C, A)}) \circ \phi_M(c \otimes m) = (f \otimes_A I_{\#(C, A)})(m \otimes_A \psi(c \otimes 1_A))$$

$$= f(m) \otimes_A \psi(c \otimes 1_A)$$

$$= \phi_N(c \otimes f(m))$$

$$= \phi_N \circ (I_C \otimes f).$$

$f' : V' \to V'_5$ is defined as follows: for any natural transformation $\phi : PG \to T$, we take

$$f'(|\phi|) = |\psi| = \phi_A : C \otimes A \to A \otimes_A \#(C, A) \cong \#(C, A)$$

We have to show that $\psi$ is an integral in $V'_5$. $\psi$ is right $A\#C^*$-linear by definition. The next thing to show is that $\psi$ is left $A$-linear. Let $N \in \mathcal{M}_A$ and $n \in N$, and consider the map

$$\varphi_n : A \to N; \quad \varphi_n(a) = na$$

for all $a \in A$. It is clear that $\varphi_n$ is right $A$-linear, and, by the naturality of of $\phi$, we have the following commutative diagram

$$\begin{array}{ccc}
C \otimes A & \xrightarrow{\phi_A} & A \otimes_A \#(C, A) \\
I_C \otimes \varphi_n \downarrow & & \varphi_n \otimes I_{\#(C, A)} \downarrow \\
C \otimes N & \xrightarrow{\phi_N} & N \otimes_A \#(C, A)
\end{array}$$
Applying the diagram to $c \otimes a \in C \otimes A$, we obtain
\[(\varphi_n \otimes I_{#(C,A)})(\phi_A(c \otimes a)) = \phi_N(c \otimes na),\]
for all $c \in C, a \in A$. Taking $a = 1$, we find
\[\phi_N(c \otimes n) = n \triangleleft \psi(c \otimes 1).\]
If we apply $(53)$ for $N = A$ and $n = b \in A$ we get
\[b \cdot \psi(c \otimes a) = \psi(c \otimes ba),\]
and this proves that $\psi$ is left $A$-linear.

It is easy to see that $f'$ and $g'$ are each other’s inverses. Let $\phi = g'(\psi)$. Then $\phi_A(c \otimes a) = a\psi(c \otimes 1_A) = \psi(c \otimes a)$, so $\phi_A = f'(g'(\psi)) = \psi$, and $f' \circ g' = I_{V_{3}}$.

Conversely, take $\phi \in V'$, and write $\tilde{\phi} = (g' \circ f')(\phi)$. Then
\[\tilde{\phi}_N(c \otimes n) = n \otimes \phi_A(c \otimes 1_A) = (\varphi_n \otimes I_{#(C,A)})(\phi_A(c \otimes 1_A)) = \phi_N(c \otimes n)\]
and this proves that $g' \circ f' = I_{V'}$.

Finally $\chi \circ \phi(F) \circ P(\rho) = 1_P$ if and only if
\[\chi_M \circ \phi_{C \otimes M} \circ \rho_M = I_M\]
for all $M \in C \mathcal{M}_A$, and this condition in fact means that
\[\sum m_{<0>} \triangleleft \psi(m_{<-1>} \otimes 1_A) = m\]
for all $m \in M$, and this equivalent to $M$ being normalized, by Proposition 2.13.

3 Applications and examples

In this Section, we will make clear how Theorem 2.7 generalizes some existing versions of Maschke’s Theorem. We will also give some new applications and examples.

3.1 Previous types of integrals for Doi-Hopf modules

Let $(H, A, C)$ be a Doi-Hopf datum, where $H$ is a Hopf algebra with a bijective antipode $S$, and $C$ is projective as a $k$-module. According to [3], an integral is an $(A, A \# C^*)$-linear map
\[\psi : C \otimes A \rightarrow C^* \otimes A\]
where the $(A, A \# C^*)$-bimodule structure on $C^* \otimes A$ is given by the following formulas:
\[b \cdot (c^* \otimes a) = c^* \otimes ba\]
\[(c^* \otimes a) \cdot b = \sum (c^*, ? \cdot S^{-1}(b_{<-1>}) \otimes ab_{<0>})\]
\[(c^* \otimes a) \cdot d^* = c^* \cdot d^* \otimes a\]
for all $c^*, d^* \in C^*$ and $a, b \in A$. Such an integral is called normalized if

$$\sum (c_{(2)} \otimes a) \cdot (c_{(1)})^{[0]} \cdot (c_{(1)})^{-1} = c \otimes a$$

for all $c \in C$, $a \in A$ (we denoted $\psi(c \otimes 1_A) = \sum c^{-1} \otimes c^{[0]}$).

If $\psi : C \otimes A \rightarrow C^* \otimes A$ is a normalized integral in the sense of [4] then $\gamma : C \rightarrow \text{Hom}(C, A)$, defined by $\gamma(c) = i \circ f \circ \psi(c \otimes 1_A)$, for all $c \in C$, is a normalized $A$-integral in our sense. Here $f : C^* \otimes A \rightarrow A \# C^*$ is the $(A, A \# C^*)$-bimodule map given by the formula

$$f(c^* \otimes a) = \sum a_{<0>} \# a_{<-1>} \cdot c^*$$

A straightforward computation shows that $f$ is $(A, A \# C^*)$-linear. Let us check that $\gamma$ is normalized. For all $c \in C$ and $a \in A$, we have that

$$\sum (c_{(2)} \otimes a) \cdot \gamma(c_{(1)}) = \sum c_{(3)} \cdot \gamma(c_{(1)}) \cdot (c_{(2)})_{<-1>} \otimes a \cdot (c_{(1)})_{<0>} \cdot (c_{(2)})_{<0>}$$

and it follows from [43] that $\gamma$ is normalized.

Thus, all the examples of normalized integrals given in [4] provide examples of normalized $A$-integrals in our new sense. For example, let $(k, k, C)$ be the Doi-Hopf datum where $C = M_n(k)$, is the $n \times n$ matrix coalgebra, that is, $C$ is the dual of the $n \times n$ matrix algebra $M_n(K)$. Let $\{e_{ij}, e_{ij} | 1 \leq i, j \leq n\}$ be the canonical dual basis for $C$, and let $\mu = (\mu_{ij})$ be an arbitrary $n \times n$-matrix. Then the map $\gamma_{\mu} : C \rightarrow C^*$ defined by

$$\gamma_{\mu}(e_{ij}) = \sum_{k=1}^n \mu_{kj} e_{ki}$$

is a $k$-integral map for $(k, k, C)$. Furthermore, $\gamma_{\mu}$ is normalized if and only if $\text{Tr}(\mu) = 1$ (see Example 2.3 of [4]).

### 3.2 Relative Hopf modules

#### 3.2.1 Doi’s total integrals and the forgetful functor

Let $H$ be a projective Hopf algebra and $A$ a left $H$-comodule algebra. $H$ is an $H$-module coalgebra, and the category $^H\mathcal{M}(H)_A$ is nothing else then the category of relative Hopf modules $^H\mathcal{M}_A$. We recall that a total integral $\varphi : H \rightarrow A$ is left $H$-colinear map such that $\varphi(1_H) = 1_A$ (see [11]).

Several results about the separability of the forgetful functor $F : ^H\mathcal{M}_A \rightarrow \mathcal{M}_A$ have appeared in the literature. Doi [11, Theorem 1.7] proved that the following condition is sufficient:

- $H$ is commutative and $\varphi : H \rightarrow A$ is a total integral such that $\varphi(H) \subseteq Z(A)$.

In [12, Theorem 1] the following two sufficient conditions for $F$ to be separable are given:

- $H$ is involutory (i.e. $S^2 = 1_H$), $\varphi(H) \subseteq Z(A)$ and $\varphi(hk) = \varphi(kh)$ for all $h, k \in H$;
- $A$ is faithful as a $k$-module and $\varphi(H) \subseteq k$.

and in this paper Doi asks if we can prove a Maschke type Theorem (in our language: a separable functor Theorem) for relative Hopf modules under more general assumptions. Theorem 2.7 gives an answer this problem:
Corollary 3.1 Let $H$ be a projective $k$-Hopf algebra, and $A$ a left $H$-comodule algebra. Then the following statements are equivalent:
1) the forgetful functor $F: H\mathcal{M}_A \rightarrow \mathcal{M}_A$ is separable;
2) there exists a normalized $A$-integral $\gamma: H \rightarrow \text{Hom}(H, A)$.

We will now investigate the relation between Doi’s total integrals and our total $A$-integrals. This will explain our terminology, and we will also prove that the forgetful functor is separable if and only if there exists a total integral $\varphi$ such that the image of $\rho \circ \varphi$ is contained in the center of $H \otimes A$. We will see that this last condition is implied by Doi’s conditions mentioned above (see Corollaries 3.4 and 3.5).

Proposition 3.2 Let $H$ be a $k$-projective bialgebra and $A$ be a left $H$-comodule algebra. If $\gamma: H \rightarrow \text{Hom}(H, A)$ is a (normalized) $A$-integral for $(H, A, H)$ then
$$\varphi_\gamma: H \rightarrow A, \quad \varphi_\gamma(h) := \gamma(h)(1_H),$$
for all $h \in H$, is a total integral in the sense of Doi [11].

Proof $\varphi_\gamma$ is right $H^*$-linear since
$$\varphi_\gamma(h\leftarrow h^*) = \gamma(h\leftarrow h^*)(1_H)$$
$$= (\gamma(h) \cdot h^*)(1_H)$$
$$= \sum \gamma(h)(1_H)\langle 0, h^*, \gamma(h)(1_H)\rangle$$
$$= (\gamma(h)(1_H)) \cdot h^*$$
$$= \varphi_\gamma(h) \cdot h^*$$
for all $h \in H$, $h^* \in H^*$. Furthermore, if $\gamma$ is normalized, then we obtain from (12)
$$\varphi_\gamma(1_H) = \gamma(1_H)(1_H) = \varepsilon(1_H)1_A = 1_A.$$
proving that $\varphi_\gamma$ is a total integral.

Conversely, let $\varphi: H \rightarrow A$ be a (total) integral, and define
$$\gamma^\varphi: H \rightarrow \text{Hom}(H, A), \quad \gamma^\varphi(h)(k) = \varphi(hS(k)), \quad (55)$$
for all $h, k \in H$. We will now present some necessary and sufficient condition for $\gamma^\varphi$ to be a (normalized) $A$-integral in $V_4$.

Theorem 3.3 Let $A$ be a left $H$-comodule algebra and $\varphi: H \rightarrow A$ an integral. If
$$\rho(\varphi(H)) \subseteq Z(H \otimes A), \quad (56)$$
the center of the tensor product of $H$ and $A$, then the map $\gamma^\varphi$ defined by (55) is an $A$-integral for $(H, A, H)$.
Conversely, if $H$ is projective as a $k$-module, and $\gamma^\varphi$ is an $A$-integral, then (56) holds.
Finally, $\varphi$ is a total integral if and only if $\gamma^\varphi$ is normalized $A$-integral.
Proof First we remark that (56) is equivalent to
\[(g \otimes 1_A)(\varphi(h)) = (\varphi(h))(g \otimes 1_A)\] (57)
for all \(g, h \in H\) and
\[\varphi(h) \subset Z(A)\] (58)
Now assume that (56) holds. Then for any \(h, g \in H, h^* \in H^*\) we have
\[(\gamma^\varphi(h) \cdot h^*)(g) = \sum \gamma^\varphi(h)(g(1))<0\langle h^*, g(2)\rangle\gamma^\varphi(h)(g(1)<_{-1})\]
\[= \sum \varphi(hS(g(1))<0\langle h^*, g(2)\rangle\varphi(hS(g(1))<_{-1})\]
(by assumption) \[= \sum \varphi((hS(g(1)))(2))\langle h^*, g(2)\rangle(hSg(1)(1))\]
\[(\varphi \text{ is } H\text{-colinear}) = \sum \varphi((hS(g(1)))(2))\langle h^*, (hSg(1)(1))g(2)\rangle\]
\[= \sum \varphi(h(2)S(g(1))\rangle h^*, h(1))\]
\[= \varphi((h \leftarrow h^*)S(g))\]
\[= \gamma^\varphi(h \leftarrow h^*)(g).\]
So \((\gamma^\varphi(h) \cdot h^*) = \gamma^\varphi(h \leftarrow h^*), \text{ for any } h \in H, h^* \in H^*, \text{ and } \gamma^\varphi \text{ is right } H^*\text{-linear.}\)
Now for any \(a \in A, g, h \in H,\)
\[(a \rightarrow \gamma^\varphi)(h)(g) = \sum (a_{<0>} \cdot \gamma^\varphi(ha_{<-1}>))(g)\]
\[= \sum a_{<0}>[\gamma^\varphi(ha_{<-2}>)(ga_{<-1}>)]\]
\[= \sum a_{<0>}[\gamma^\varphi(ha_{<-2}>S(a_{<-1>}S(g))]\]
\[= a\varphi(hS(g))\]
\[= a\gamma^\varphi(h)(g)\]
(by assumption) \[= \gamma^\varphi(h)(g)a\]
\[= (\gamma^\varphi \leftarrow a)(h)(g).\]
and \(a \rightarrow \gamma^\varphi = \gamma^\varphi \leftarrow a \text{ for any } a \in A.\) This proves that \(\gamma^\varphi\) is an \(A\)-integral.
Conversely, assume that \(H\) is projective as a \(k\)-module and that \(\gamma^\varphi\) is an \(A\)-integral. Then for any \(h \in H, h^* \in H^*\), we have
\[\gamma^\varphi(h \leftarrow h^*) = \gamma^\varphi(h) \leftarrow h^*\]
Thus for any \(g, h \in H, h^* \in H^*,\)
\[\sum \varphi(h(2))\langle h^*, h(1)g\rangle = \sum \varphi(h(2)g(2)S(g(3))\langle h^*, h(1)g(1)\rangle\]
\[= \sum \varphi((hg(1)) \leftarrow h^*)S(g(2))\]
\[(\gamma^\varphi \text{ is integral}) = \sum \gamma^\varphi(hg(1)) \leftarrow h^*)(g(2))\]
(by definition of right \(H^*\)-module) \[= \sum [\gamma^\varphi(hg(1))(g(1))]<0\langle h^*, g(3)[\gamma^\varphi(hg(1))(g(2))]<_{-1}\rangle\]
\[= \sum [\varphi(h)]<0\langle h^*, g[\varphi(h)]<_{-1}\rangle\]
\[= \sum \varphi(hg(1))\langle h^*, gh(1)\rangle.\]
or
\[ \sum \varphi(h(2)) \langle h^*, h(1)g \rangle = \sum \varphi(hg(1)) \langle h^*, gh(1) \rangle \]
for any \( h, g \in H, \ h^* \in H^* \). Using the fact that \( H \) is projective as a \( k \)-module, we obtain
\[ \sum h(1)g \otimes \varphi(h(2)) = \sum gh(1) \otimes \varphi(h(2)) \]
and (57) follows from the fact that \( \varphi \) is \( H \)-colinear.
For all \( a \in A \), we have that \( a \mapsto \gamma^\varphi = \gamma^\varphi \mapsto a \). Therefore we find for any \( h \in H \) that
\[ \varphi(h)a = \gamma^\varphi(h)(1)a = (\gamma^\varphi(h) \cdot a)(1) = (\gamma^\varphi \mapsto a)(h)(1) = (a \mapsto \gamma^\varphi)(h)(1) = \sum [a_{<0>} \gamma^\varphi(ha_{<-1>})](1) = \sum a_{<0>} \gamma^\varphi(ha_{<-2>})(a_{<-1>}) = \sum a_{<0>} \varphi(ha_{<-2>}S(a_{<-1>})) = a \varphi(h). \]
So \( \varphi(h) \in Z(A) \) for any \( h \in H \), and (58) follows. As we remarked earlier, (57,58) imply (56).
It is routine to check that for any left \( H \)-comodule map \( \varphi : H \to A \)
\[ \varphi \gamma^\varphi = \varphi \]
So if \( \gamma^\varphi \) is a normalized integral then \( \varphi \) is total by Proposition 3.2. Finally, if \( \varphi \) is a total integral, then
\[ \gamma^\varphi(h(1))(h(2)) = \varphi(h(1)Sh(2)) = \varepsilon(h) \varphi(1) = \varepsilon(h) 1_A \]
for all \( h \in H \), and \( \gamma^\varphi \) is normalized.

**Corollary 3.4** Let \( \varphi : H \to A \) be left \( H \)-colinear. If
\[ \varphi(gh) = \varphi(hS^2(g)) \text{ and } \varphi(H) \subseteq Z(A) \]
for all \( g, h \in H \), then \( \gamma^\varphi \) is an \( A \)-integral.

**Proof** It follows immediately from our assumptions that
\[ \varphi(h) = \varphi(h1_H) = \varphi(1_H S^2 h) = \varphi(S^2 h) \] (59)
for all \( h \in H \). Applying \( \rho \), and using the fact that \( \varphi \) is \( H \)-colinear, we obtain
\[ \sum h(1) \otimes \varphi(h(2)) = \sum S^2(h(1)) \otimes \varphi(S^2(h(2))) = \sum S^2(h(1)) \otimes \varphi(h(2)) \] (60)
and
\[ \sum h(1)S^2(g) \otimes \varphi(h(2)) = \sum S^2(h(1)g) \otimes \varphi(h(2)) = \sum S^2(h(1))S^2(g(1)) \otimes \varphi(h(2)g(2)S(g(3))) \]
Thus, we find for all \( h \in H \) that
\[
\sum h_{(1)} S^2(g) \otimes \varphi(h_{(2)}) = \sum S(g) h_{(1)} \otimes \varphi(h_{(2)})
\]
and
\[
\sum S(g) h_{(1)} \otimes \varphi(h_{(2)}) = \sum S(g(3)) h_{(1)} S^2(g(2)) S(g(1)) \otimes \varphi(h_{(2)})
\]
\[
= \sum S(g(3)) S^2(g(2)) h_{(1)} S(g(1)) \otimes \varphi(h_{(2)})
\]
\[
= \sum h_{(1)} S(g) \otimes \varphi(h_{(2)}).
\]

This proves that \((g \otimes 1_A) \rho \varphi(h) = \rho \varphi(h)(g \otimes 1_A)\). Now observe that \( \varphi(H) \subseteq Z(A) \), and
\[
\rho \left( \varphi(H) \right) \subseteq Z(H \otimes A)
\]
and it follows from Theorem 3.3 that \( \gamma^\varphi \) is an integral. \( \square \)

As a special case, we recover the following result of Doi [24]:

**Corollary 3.5** Let \( H \) be a Hopf algebra, \( A \) a left \( H \)-comodule algebra, and \( \varphi : H \rightarrow A \) be a (total) integral. Assume that one of the following two conditions holds:
1) \( H \) is involutory (i.e. \( S^2 = I_H \)), \( \varphi(H) \subseteq Z(A) \) and \( \varphi(hk) = \varphi(kh) \) for all \( h, k \in H \);
2) The antipode of \( H \) is bijective and \( \varphi(H) \subseteq H \).

Then \( \gamma^\varphi \) is a (normalized) \( A \)-integral, and the forgetful functor \( H\mathcal{M}_A \rightarrow \mathcal{M}_A \) is separable.

**Proof** 1) follows immediately from Corollary 3.4.

2) Using the fact that \( \varphi \) is left \( H \)-colinear, we find for all \( h \in H \) that
\[
\sum h_{(1)} \otimes \varphi(h_{(2)}) = 1_H \otimes \varphi(h)
\]
It is obvious that \( \rho \varphi(H) \subseteq Z(H \otimes A) \), and the result follows again from Corollary 3.4. \( \square \)
3.2.2 Hopf Galois extensions and the induction functor

Let $H$ be a Hopf algebra, and $A$ be a left $H$-comodule algebra. Recall that the subalgebra of coinvariants is given by the formula

$$B = A^{coH} = \{a \in A \mid \rho(a) = 1 \otimes a\}$$

It is well-known that we have an adjoint pair of functors (see e.g. [3])

$$\bullet \otimes_B A : \mathcal{M}_B \to H \mathcal{M}_A ; (\bullet)^{coH} : H \mathcal{M}_A \to \mathcal{M}_B$$

Let $\delta$ be the counit of this adjunction. We know that $H \otimes A$ is a relative Hopf module, and, by definition, $A/B$ is a Hopf Galois extension if the map $\delta_{H \otimes A} : A \otimes_B A \to H \otimes A$ is an isomorphism. This adjunction map is given by the formula

$$\delta_{H \otimes A}(a \otimes b) = \sum a_{-1} \otimes a_{<0>} b$$

Here we have to be careful with the structure maps on $H \otimes A$. They are not given by (61-62), but by

$$(h \otimes b) \cdot a = \sum h \otimes ba \quad (61)$$

$$\rho'(h \otimes b) = \sum h_{(1)} b_{-1} \otimes h_{(2)} \otimes b_{<0>} \quad (62)$$

It is well-known that the isomorphism

$$f : H \otimes A \to H \otimes A ; \quad f(h \otimes a) = \sum ha_{-1} \otimes a_{<0>}$$

translates the structures (61-62) into (5-6), and consequently $A/B$ is a Hopf Galois extension if and only if the map

$$f \circ \delta_{H \otimes A} = \delta'_{H \otimes A} : A \otimes_B A \to H \otimes A ; \quad \delta'_{H \otimes A}(a \otimes b) = \sum b_{-1} \otimes ab_{<0>}$$

is an isomorphism. $\delta'_{H \otimes A}$ is a map in $H \mathcal{M}_A$ if we give $H \otimes A$ the usual structures (5-6). We now have the following result:

**Theorem 3.6** Let $H$ be a Hopf algebra, and $A$ a left $H$-comodule algebra, and $B = A^{coH}$. If $A$ is a separable extension of $B$, then the induction functor $H \otimes \bullet : \mathcal{M}_A \to H \mathcal{M}_A$ is separable. The converse property holds if $A/B$ is a Hopf Galois extension.

**Proof** We have remarked already that $\delta'_{H \otimes A}$ is right $A$-linear, and it can be verified easily that it is also left $A$-linear. Let $e = \sum e^1 \otimes e^2$ be a separability idempotent in $A \otimes_B A$. Then $e$ has the following properties:

$$\sum ae^1 \otimes e^2 = \sum e^1 \otimes e^2 a \quad \text{and} \quad \sum e^1 e^2 = 1_A$$

for all $a \in A$. Consider $z = \sum e^2_{<0>} \otimes e^1_{<0>} = \delta'_{H \otimes A}(e) \in H \otimes A$. From the fact that $\delta'_{H \otimes A}$ is left and right $A$-linear, it follows immediately that $az = za$, so $z$ is a dual $A$-integral in the sense of Section 2.2. Also the normalization condition follows easily, since

$$\sum e(e_{<0>})e^1 e_{<0>} = \sum e^1 e^2 = 1_A$$

and the first statement follows from Theorem 2.14.

If $A/B$ is a Hopf Galois extension, then we proceed in a similar way, but using the inverse of $\delta'_{H \otimes A}$.

**Remark 3.7** In the literature we often find the category $\mathcal{M}(H)_A$ instead of $H \mathcal{M}_A$, where $A$ is now a right $H$-comodule algebra. Adapting our results, we find that the functor $\bullet \otimes H : \mathcal{M}_A \to \mathcal{M}_A^H$ is separable if $A/A^{coH}$ is separable, and the converse holds if $A$ is a Hopf Galois extension of $A^{coH}$.
3.3 Classical integrals

Let $H$ be a Hopf algebra, and assume that $H$ is flat as a $k$-module. Recall from [35] that $\varphi \in H^*$ is called a left integral on $H^*$ if $h^* \cdot \varphi = \langle h^*, 1_H \rangle \varphi$ for all $h^* \in H^*$, or, equivalently, if $\varphi : H \to k$ is left (or right) $H$-colinear.

Now suppose that there exists a (total) $k$-integral $\gamma$ for $(k, k, H)$, or, equivalently, a map $\theta : H \otimes k \to k$ in $V_3$ (see Example 2.11). The map $\varphi = i(\gamma) \in H^*$ defined by

$$\langle \varphi, h \rangle = \theta(h \otimes 1_H) = \gamma(h)(1_H)$$

is a left integral. Conversely, if $\varphi \in \bigcap_{H^*}$, the $k$-module consisting of classical integrals on $H$, then $p(\varphi) = \gamma : H \to H^*$ given by

$$\gamma(h)(g) = \varphi(hS(g))$$

is a $k$-integral. This can be proved directly, but it also follows from Theorem 3.3. So we have maps

$$i : \bigcap_{H^*} \to V_3 \cong V_4 \quad \text{and} \quad p : V_4 \cong V_3 \to \bigcap_{H^*}$$

and it can be seen easily that $p$ is a left inverse of $i$.

Surprisingly, $i$ is not an isomorphism. To see this, let $H$ be a finite dimensional Hopf algebra over a field $k$. It is well-known that $\dim_k(\bigcap_{H^*}) = 1$. On the other hand, a $k$-integral $\gamma$ is nothing else then a right $H^*$-linear map $\gamma : H \to H^*$. Now if $t \in H$ is a left integral, then $H = t \leftarrow H^*$ (35), hence $\dim_k(V_4) = \dim_k(HOM_{H^*}(H, H^*)) = n$, and this shows that the space of $k$-integrals $V_4$ is larger than the classical $\bigcap_{H^*}$.

$\text{Im}(i)$ is the subspace of $V_3$ consisting of all $\theta \in V_3$ satisfying $(i \circ p)(\theta) = \theta$, or

$$\theta(h \otimes k) = \theta(hS(k) \otimes 1_H)$$

for all $h, k \in H$. This condition is equivalent to

$$\sum \theta(hl(1) \otimes kl(2)) = \theta(h \otimes k)\varepsilon(l) \quad (63)$$

for all $h, k, l \in H$. (35) means in fact that $\theta$ is right $H$-linear. In terms of the corresponding integral maps $\gamma \in V_4$, (35) can be rewritten as follows:

$$h \leftarrow \gamma = \gamma \leftarrow h \quad (64)$$

for all $h \in H$, where the actions of $H$ on $\text{Hom}(H, H^*)$ are given by

$$(h \leftarrow \gamma)(g)(k) = \sum h_{(3)} \gamma(g(h_{(1)}))(kh_{(2)})$$

$$(\gamma \leftarrow h)(g)(k) = \gamma(g)(kh)$$

for all $h, g, k \in H$. Our results can be summarized as follows.

**Theorem 3.8** Let $H$ be a flat Hopf algebra over $k$, and consider the Doi-Hopf datum $(k, k, H)$. There exist $k$-linear maps

$$i : \bigcap_{H^*} \to V_3 \cong V_4 \quad \text{and} \quad p : V_4 \cong V_3 \to \bigcap_{H^*}$$

such that $p \circ i = I_{\bigcap_{H^*}}$. Furthermore, the image of $i$ in $V_3$ (resp. $V_4$) is the submodule consisting of maps that satisfy [35] (resp. [64]).
3.4 The finite case

Let \((H, C, A)\) be a Doi-Hopf datum such that \(H\) is a Hopf algebra and \(C\) is finitely generated and projective over \(k\). Then \(\mathcal{M}(H)_A \cong \mathcal{M}_{A\#C^*}\) and the forgetful functor \(F : \mathcal{M}(H)_A \to \mathcal{M}_A\) is isomorphic to the restriction of scalars functor \(R : \mathcal{M}_{A\#C^*} \to \mathcal{M}_A\). Moreover, \(i : A\#C^* \to \text{Hom}(C, A)\) is an isomorphism of algebras and of \((A, A\#C^*)\)-bimodules. Therefore an \(A\)-integral \(\gamma\) can be viewed as a right \(C^*\)-module map \(\gamma : C \to A\#C^*\) which is centralized by the action of \(A\) on \(\text{Hom}(C, A\#C^*)\). Recall that \(A\#C^*\) is a right \(C^*\)-module after restriction of scalars via the map \(i : C^* \to A\#C^*\). Identifying \(A\#C^* \cong \text{Hom}(C, A)\), we find that \(\text{Hom}(C, A\#C^*)\) is an \(A\)-bimodule (see [8,[9]). The left and right action of \(A\) on \(\text{Hom}(C, A\#C^*)\) are given by

\[
(a \cdot \gamma)(c) = \sum (a_{<0>} \# \varepsilon) \gamma(c \cdot a_{<-1>}) \\
(\gamma \cdot a)(c) = \gamma(c)(a \# \varepsilon)
\]

for all \(a \in A\), \(\gamma \in \text{Hom}(C, A\#C^*)\) and \(c \in C\).

\(i\)From Theorem 2.3 we obtain the following necessary and sufficient condition for the separability of the extension \(A \to A\#C^*\).

**Corollary 3.9** Let \((H, C, A)\) be a Doi-Hopf datum, with \(H\) a Hopf algebra, and \(C\) finitely generated and projective as a \(k\)-module. The following statements are equivalent:

1) the extension \(A \to A\#C^*\) is separable;
2) there exists a normalized \(A\)-integral \(\gamma : C \to A\#C^*\).

In this situation, if \(A\) is semisimple artinian, then \(A\#C^*\) is semisimple artinian also.

**Remarks 3.10**

1) In Corollary 3.3 of [4] we proved that if there exists \(\gamma : C \to A\#C^*\) a normalized \(A\)-integral then the extension \(A \to A\#C^*\) is right semisimple. As any separable extension is a semisimple extension, the above Corollary improves Corollary 3.3 of [4] and also gives us the converse.

2) Consider the particular case \(C = H\). Then \(A\) is an \(H^*\)-module algebra and we get a necessary and sufficient condition for the extension \(A \to A\#H^*\) to be separable.

Several results connected to the separability of this extension \(A \to A\#H^*\) have appeared in the literature.

In the first place, if \(H\) is finitely dimensional and cosemisimple, then the extension \(A \to A\#H^*\) is separable (see [3, Theorem 4]).

Secondly, Propositions 1.3 and 1.5 of [37] give necessary and sufficient condition for the extension \(A \to A\#H^*\) to be separable in the cases where \(A\) is an \(H\)-Galois extension and \(H\) contains a cocommutative integral.

Finally, Theorem 3.14 of [14] can also be connected to our results: \(A\#H^*\) is a right \(H^*\)-comodule algebra via \(I_A \otimes \delta_{H^*}\), and this makes the extension \(A \to A\#H^*\) an \(H^*\)-Galois extension. Now

\[
(A\#H^*)^A := \{a\#h^* | (b\#\varepsilon)(a\#h^*) = (a\#h^*)(b\#\varepsilon) \text{ for all } b \in A\}
\]

is a right \(H^*\)-module algebra, by the Miyashita-Ulbrich action, and therefore a left \(H\)-comodule algebra, since \(H\) is finitely generated projective. From [14, Theorem 3.14], we obtain that the extension \(A \to A\#H^*\) is separable if and only if there exists a total integral \(\varphi : H \to (A\#H^*)^A\). Disadvantages of this approach are the lack of control that we have on the space \((A\#H^*)^A\), and also the fact that the \(H\)-coaction coming from the Miyashita-Ulbrich action is not very handable.

3) Let \(C = H\), and assume that \(H\) is finite dimensional over a field \(k\). Let \(t\) be a right integral in
\[ H \text{ and } \Lambda \text{ a right integral in } H^*. \text{ Then } H \text{ is free and cyclic as a right } H^*-\text{module with basis } \{t\}; \]

\[ H = t \rightarrow H^* \text{ (see } [31] \text{ or } [35]) \]. The inverse of the map

\[ H^* \rightarrow H : h^* \mapsto t \leftarrow h^* \]

is the map

\[ H \rightarrow H^* : h \mapsto S^{-1}(h) \rightarrow \Lambda \]

Thus right \( H^* \)-linear maps \( \gamma : H \rightarrow A \# H^* \) correspond to elements \( s = \sum a_i \otimes h_i^* \in A \otimes H^* \).

From Corollary 3.9, it follows that the separability of the extension \( A \rightarrow A \# H^* \) is equivalent to the existence of an element \( s = \sum a_i \otimes h_i^* \in A \otimes H^* \) satisfying the two following conditions.

\[
\sum a_{<0>} a_{<0>} \otimes \langle h_i^*, l(1) a_{<2>} \rangle l(2) a_{<1>} a_{<1>} a_{<1>} S^{-1}(a_{<3>}) = \sum a_{<0>} a \otimes \langle h_i^*, l(1) \rangle l(2) a_{<1>} a_{<1>}
\]

\[
\sum \langle h_i^*, h(2) \rangle (S^{-1}(h(1)) \rightarrow \Lambda, h(3) a_{<1>} a_{<1>}) a_{<0>} = \varepsilon(h) 1_A
\]

for all \( a \in A, l, h \in H \).

Now take \( H = A = C \) and consider the Doi-Hopf datum \((H, H, H)\), with \( H \) a finitely generated and projective Hopf algebra. The smash product \( \mathcal{H}(H) = H \# H^* \) is usually called the Heisenberg double of \( H \). Applying Corollary 3.11, we find the following necessary and sufficient conditions for \( H \rightarrow \mathcal{H}(H) \) to be separable.

**Corollary 3.11** Let \( H \) be a finitely generated and projective Hopf algebra over \( k \). The following statements are equivalent:

1) the extension \( H \rightarrow \mathcal{H}(H) \) is separable;
2) there exists a normalized \( H \)-integral \( \gamma : H \rightarrow \mathcal{H}(H) \).

### 3.5 Yetter-Drinfel’d modules

In this section \( H \) will be a \( k \)-flat Hopf algebra with bijective antipode \( S \). Recall that a right-left Yetter Drinfel’d module \( M \) is a \( k \)-module, which is at once a left \( H \)-comodule and a right \( H \)-module, such that the following compatibility relation holds:

\[
\sum m_{<1>} h(1) \otimes m_{<0>} h(2) = \sum S^{-1}(h(3)) m_{<1>} h_{<1>} \otimes m_{<0>} h(2)
\]

(65)

for all \( h \in H \) and \( m \in M \). The category of right-left Yetter-Drinfel’d modules and \( H \)-linear \( H \)-colinear maps will be denoted by \( ^H \text{YD}_H \).

In [3] it is shown that there is a category isomorphism

\[
^H \text{YD}_H \cong ^H \mathcal{M}(H \otimes H^{op})_H
\]

The left \( H \otimes H^{op} \)-coaction and right \( H \otimes H^{op} \)-action on \( H \) are given by the formulas

\[
\rho(h) = \sum h_{(1)} \otimes S^{-1}(h_{(3)}) \otimes h_{(2)}
\]

(66)

\[
l \cdot (h \otimes k) = klh
\]

(67)

for all \( h, k, l \in H \). From now on, we will identify the categories \(^H \text{YD}_H \) and \(^H \mathcal{M}(H \otimes H^{op})_H \) using the isomorphism from [2].

We recall some basic facts about the Drinfel’d double, as introduced in [15]. Our main references are [22] and [31]. Let \( H \) be a finitely generated projective Hopf algebra. Then the antipode
\textit{S} is bijective (see [32]), and the Dixmier–Dinfeld double \( \mathcal{D}(H) = H \rtimes H^* \) is defined as follows: \( \mathcal{D}(H) = H \otimes H^* \) as a \( k \)-module, with multiplication, comultiplication, counit and antipode given by the formulas

\[
(h \rtimes f)(h' \rtimes f') = \sum h_{(2)} h' \rtimes f \ast (f', S^{-1} h_{(3)} \triangleright h_{(1)}) \\
\Delta_{\mathcal{D}(H)}(h \rtimes f) = \sum (h_{(1)} \rtimes f_{(2)}) \otimes (h_{(2)} \rtimes f_{(1)}) \\
\varepsilon_{\mathcal{D}(H)} = \varepsilon_H \otimes \varepsilon_{H^*} \\
S_{\mathcal{D}(H)}(h \rtimes f) = \sum f_{(2)} \triangleright (S_H h_{(1)}) \rtimes (S(h_{(2)}) \triangleright f_{(1)})
\]

for \( h, h' \in H \) and \( f, f' \in H^* \).

Consider the Doi-Hopf datum \( (H \otimes H^\text{op}, H, H) \), with structures given by (66-67). \( H^* \) is a left \( H \otimes H^\text{op} \)-module algebra, with

\[
\langle (h \otimes k) \triangleright h^*, l \rangle = \langle h^*, klh \rangle
\]

for all \( h, k, l \in H \) and \( h^* \in H^* \), and in [2], it is shown that the Dixmier–Dinfeld double is the smash product

\( \mathcal{D}(H) = H \# H^* \)

For a Hopf algebra \( H \) that is not necessarily finitely generated projective, consider Koppinen’s smash product \( \mathcal{D}(H) = \#(H, H) \). If \( H \) is finitely generated and projective, then \( \mathcal{D}(H) \cong \mathcal{D}(H) \) (see [36] for a similar result, where it was proved that \( \mathcal{D}(H) \cong \text{End}(H^*) \)). We can therefore view \( \mathcal{D}(H) \) as a generalization of the Dixmier–Dinfeld double to the case of infinite dimensional Hopf algebras. The structure of \( \mathcal{D}(H) \) is the following: as a \( k \)-module, \( \mathcal{D}(H) = \text{End}(H) \), and the multiplication is given by the formula

\[
(f \cdot g)(h) = \sum f(h_{(1)})(2) g(S^{-1}(f(h_{(1)})(3)) h_{(2)} f(h_{(1)})(1))
\]

\( \mathcal{D}(H) \) is a right \( H^* \)-module via

\[
(f \cdot h^*)(h) = \sum f(h_{(1)})(2) h^*, S^{-1}(f(h_{(1)})(3)) h_{(2)} f(h_{(1)})(1)
\]

and \( \text{Hom}(H, \mathcal{D}(H)) \) is an \( H \)-bimodule via the formulas

\[
(g \triangleright \gamma)(h)(l) = \sum g_{(3)} \gamma(S^{-1}(g_{(5)}) h_{(1)}) (S^{-1}(g_{(4)}) l g_{(2)})
\]

and

\[
(\gamma \triangleright g)(h)(l) = \gamma(h)(l) g
\]

for any \( g, h, l \in H \), and \( \gamma : H \to \mathcal{D}(H) \).

We can now apply the Maschke Theorem [2,4] to the Doi-Hopf datum \( (H \otimes H^\text{op}, H, H) \). An \( A = H \)-

\text{integral for} \( (H \otimes H^\text{op}, H, H) \) will be called a \textit{quantum} \( H \)-\textit{integral}. This can be reformulated as follows.

**Definition 3.12** Let \( H \) be a Hopf algebra with a bijective antipode. A \textit{quantum} \( H \)-\textit{integral} is a right \( H^* \)-module map \( \gamma : H \rightarrow \mathcal{D}(H) \) which is centralized by the left and right \( H \)-action, that is

\[
g \triangleright \gamma = \gamma \triangleright g
\]

for all \( g \in H \). \( \gamma \) is a total \textit{quantum} \( H \)-\textit{integral} if

\[
\sum \gamma(h_{(1)}) h_{(2)} = \varepsilon(h)1_H
\]

for all \( h \in H \).

32
From Theorem 2.4, we obtain immediately the following version of Maschke’s Theorem for Yetter-Drinfel’d modules:

**Corollary 3.13** Let $H$ be a Hopf algebra. Assume that the antipode is bijective, and that $H$ is projective as a $k$-module. Then the following statements are equivalent:

1) the forgetful functor $F : \mathcal{YD}_H \to \mathcal{M}_H$ is separable;
2) there exists a total quantum $H$-integral $\gamma : H \to \mathcal{D}(H)$.

If $H$ is finite, then we obtain the following result:

**Corollary 3.14** Let $H$ be a finitely generated and projective Hopf algebra. The following statements are equivalent:

1) The extension $H \to D(H)$ is separable;
2) there exists a total quantum $H$-integral $\gamma : H \to \mathcal{D}(H) \cong D(H)$.

Let $H$ be a finite dimensional Hopf algebra over a field of characteristic zero. A classical result of Larson and Radford [10] tells us that $H$ is semisimple if and only if $H$ is cosemisimple. We will now show that the space of quantum $H$-integrals measures how far semisimple Hopf algebras are from cosemisimple Hopf algebras, if we work over an arbitrary field. First recall from [26] that $D(H)$ is semisimple if and only if $H$ is semisimple and cosemisimple.

**Corollary 3.15** Let $H$ be a finite dimensional semisimple Hopf algebra over a field $k$. The following statements are equivalent:

1) $H$ is cosemisimple;
2) there exists a total quantum $H$-integral $\gamma : H \to \mathcal{D}(H) \cong D(H)$.

**Proof** If $H$ is cosemisimple, then $D(H)$ is semisimple, and therefore a separable extension of the field $k$. But then $D(H)$ is separable over $H$ (see for example [26], Lemma 1.1), and the first implication follows from Corollary 3.14.

Conversely, if there exists total quantum $H$-integral, then $H \to D(H)$ is separable, and it follows that $D(H)$ is semisimple, since $H$ is semisimple. From Radford’s results (26), it follows that $H$ is cosemisimple. \qed

**Remark 3.16** If $H$ is finite dimensional, then $H$ is free of rank one as a right $H^*$-module, generated by a nonzero right integral $t \in H$. The existence of a total quantum $A$-integral $\gamma : H \to D(H)$ is then equivalent to the existence of an element $z(= \gamma(t)) \in D(H)$ satisfying two properties. These properties are the translations of the facts that $\gamma$ is centralized by the left and right action of $H$ and that $\gamma$ is normalized. Unfortunately, these properties cannot be written down in an elegant and transparent way.

### 3.6 Long’s category of dimodules

Consider the Doi-Hopf datum $(H, A = H, C = H)$, where, $A = H$ is a left $H$-comodule algebra via $\Delta$ and $C = H$ is a right $H$-module coalgebra with the trivial $H$-action, that is $g \cdot h = g \varepsilon(h)$, for all $g, h \in H$. The compatibility relation for Doi-Hopf modules now takes the form

$$\rho(mh) = \sum m_{<-1>} \otimes m_{<0>} h$$
for all $m \in M$ and $h \in H$. If $H$ is commutative and cocommutative, then we obtain $H$-\textit{dimodules} in the sense of Long \cite{20}, and in this situation, dimodules coincide with Yetter-Drinfel’d modules. Dimodules and dimodule algebras have been a basic tool in the study of the generalizations of the Brauer-Wall group, and, recently, the third author investigated the relation between the category of dimodules and certain nonlinear equation (see \cite{24}).

The category of Long dimodules will be denoted by $H\mathcal{L}_H$. If $H$ is finitely generated and projective, then $H\mathcal{L}_H \cong \mathcal{M}_{H\otimes H^*}$. For a group $G$, a $kG$-dimodule is a $G$-graded module $M$, on which the group $G$ acts in such a way that the homogeneous components are themselves $G$-modules:

$$g \cdot M_h \subset M_h$$

for all $g, h \in G$.

It is easy to see that the multiplication on the Koppinen smash product $\#(H, H) = \text{End}(H)$ is nothing else then the convolution

$$(f \cdot g)(h) = (f * g)(h) = \sum f(h_1)g(h_2)$$

for any $f, g \in \text{End}(H)$ and $h \in H$. Furthermore, the right $H^*$-action on $\#(H, H) = \text{End}(H)$ is given by

$$(f \cdot h^*)(h) = \sum f(h_1)(h^*, h_2) = f(h^* \cdot h)$$

for all $f \in \text{End}(H)$, $h^* \in H^*$, and $h \in H$, and the $H$-bimodule structure of $\text{Hom}(H, \text{End}(H))$ is trivial:

$$(g \cdot \gamma)(h)(l) = g\gamma(h)(l) \quad \text{and} \quad (\gamma \cdot g)(h)(l) = \gamma(h)(l)g$$

Applying Theorem \ref{2.7}, we now find a Maschke Theorem for dimodules:

**Corollary 3.17** For a projective Hopf algebra over a commutative ring $k$, the following statements are equivalent:

1) the forgetful functor $F: H\mathcal{L}_H \to \mathcal{M}_H$ is separable;

2) there exists an $H^*$-linear map $\gamma: H \to \text{End}(H)$ such that $\text{Im}(\gamma(h)) \subset Z(H)$ and $\sum \gamma(h_1)(h_2) = \varepsilon(h)1_H$, for all $h \in H$.

### 3.7 Modules graded by $G$-sets

Let $G$ be a group, $X$ a right $G$-set and $H = kG$ a group algebra. Then the grouplike coalgebra $C = kX$ is a right $kG$-module coalgebra, and a left $kG$-comodule algebra $A$ is nothing else than a $G$-graded $k$-algebra (see for example \cite{23}). In this case the category of Doi-Hopf module $kX\mathcal{M}(kG)_A$ is the category $\text{gr}-(G, X, A)$ of right $X$-graded $A$-modules (see \cite{20}). An object in this category is a right $A$-module such that $M = \bigoplus_{x \in X} M_x$ and $M_x A_g \subset M_{xg}$, for all $g \in G$ and $x \in X$.

Applying Theorem \ref{2.7}, we obtain the following result. Observe that, in the case where $X = G$, the second statement of the next Theorem can be found in \cite{27}.

**Proposition 3.18** Let $G$ be a group, $X$ a right $G$-set and $A$ be a $G$-graded $k$-algebra. Then the map $\gamma: kX \to \text{Hom}(kX, A)$ given by the formula

$$\gamma(x)(y) = \delta_{x,y}1_A$$

for all $x, y \in X$, is a normalized $A$-integral.

Consequently, the forgetful functor $F: \text{gr}-(G, X, A) \to \mathcal{M}_A$ is separable.
Proof Let \( \{ x, p_x \mid x \in X \} \) be the canonical dual basis of \( kX \). For all \( x, y, z \in X \) we have
\[
\gamma(x \leftarrow p_y)(z) = \gamma((p_y, x)x)(z) = \delta_{x,y} \delta_{x,z} 1_A
\]
and
\[
(\gamma(x \cdot p_y)(z) = \sum \gamma(x)(z)_{<0>} (p_y, z \cdot \gamma(x)(z)_{<-1>}) = \delta_{x,z} \delta_{y,z} 1_A
\]
and this shows that \( \gamma \) is right \((kX)^*\)-linear. Now, let \( g \in G \) and \( a_g \in A_g \) (this means that \( \rho_A(a_g) = g \otimes a_g \)). Then
\[
(a_g \rightarrow \gamma)(x)(y) = (a_g \cdot \gamma(x \cdot g))(y) = a_g \gamma(x \cdot g)(y \cdot g) = \delta_{x,g,y} a_g = \delta_{x,y} a_g
\]
and
\[
(\gamma \leftarrow a_g)(x)(y) = (\gamma(x) \cdot a_g)(y) = \gamma(x)(y)a_g = \delta_{x,y} a_g
\]
and \( \gamma \) is centralized by the action of \( A \). Hence \( \gamma \) is a normalized \( A \)-integral. \( \square \)

We now focus attention to the right adjoint
\[
G = kX \otimes \bullet : \mathcal{M}_A \rightarrow \text{gr}(G, X, A)
\]
of the forgetful functor \( F \). This case is interesting because, in the particular situation where \( X = G \) and \( A \) is a strongly \( G \)-graded algebra, the separability of the functor \( G \) is equivalent to the separability of \( A \) as an \( A_1 \)-algebra.

From Theorem 2.14, we obtain the following result, which was already shown using other methods in [27, Theorem 3.6] (if \( X = G \)) and in [23, Sec. 4] (for general \( X \)).

Corollary 3.19 Let \( G \) be a group, \( X \) a right \( G \)-set and \( A \) a \( G \)-graded \( k \)-algebra. The following statements are equivalent:
1) the functor \( G = kX \otimes \bullet : \mathcal{M}_A \rightarrow \text{gr}(G, X, A) \) is separable;
2) there exists a normalized dual \( A \)-integral \( z = \sum_i x_i \otimes a_i \in kX \otimes A \), that is
\[
\sum_i a_i = 1_A
\]
and
\[
\sum x_i \otimes b_g a_i = \sum x_i \cdot g \otimes a_i b_g
\]
for all \( g \in G \) and \( b_g \in A_g \).

We will further investigate the second condition of Corollary 3.19. First recall that a \( G \)-subset \( X' \) of \( X \) is a subset \( X' \subset X \) such that \( x' \cdot g \in X' \), for all \( x' \in X', \ g \in G \).
Corollary 3.20 Let G be a group, X a right G-set and A be a G-graded K-algebra.
1) Let \( X' \) be a finite G-subset of X such that \( \#(X') \) is invertible in k. Then the functor \( G = kX \otimes \bullet : \mathcal{M}_A \rightarrow \text{gr-(G, X, A)} \) is separable.
2) If A is strongly graded, and the functor \( G : \mathcal{M}_A \rightarrow \text{gr-(G, X, A)} \) is separable, then there exists a finite G-subset \( X' \) of X.

Proof 1) We have a dual total A-integral of \( kX \otimes A \), namely

\[
z = \frac{1}{\#(X')} \sum_{x' \in X'} x' \otimes 1_A
\]

2) Let \( 0 \neq z = \sum_{i=1}^{n} x_i \otimes a_i \), with \( x_i \in X \) and \( a_i \in A \), be dual integral. We take \( n \) as small as possible. Then \( a_i \neq 0 \), for any \( 1 \leq i \leq n \). We claim that \( \{x_1, \ldots, x_n\} \) is a G-subset of X. It suffices to show that \( x_i \cdot g \in \{x_1, \ldots, x_n\} \), for any \( i \in \{1, \ldots, n\} \) and \( g \in G \).

Since A is strongly graded, we can find finite sets \( \{b_1, \ldots, b_m\} \subset A_g, \{b'_1, \ldots, b'_m\} \subset A_{g^{-1}} \) such that

\[
\sum_{j=1}^{m} b_j b'_j = 1
\]

For all \( j \), we have, using the fact that \( z \) is a dual integral:

\[
\sum_{i=1}^{n} x_i \otimes b_j a_i = \sum_{i=1}^{n} x_i \cdot g \otimes a_i b_j
\]

and

\[
\sum_{i=1}^{n} x_i \cdot g \otimes a_i = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i \cdot g \otimes (a_i b_j b'_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} b_j a_i b'_j
\]

and the statement follows from the fact that \( X \) is a basis of \( kX \otimes A \) as a right \( A \)-module, and the fact that \( a_i \neq 0 \).

In particular, we have the following result for a field of characteristic zero.

Corollary 3.21 Let \( k \) be a field of characteristic zero, \( G \) a group, \( X \) a right G-set, and \( A \) a strongly G-graded \( k \)-algebra. The following statements are equivalent:
1) The functor \( G = kX \otimes \bullet : \mathcal{M}_A \rightarrow \text{gr-(G, X, A)} \) is separable.
2) there exists a finite G-subset \( X' \) of X;
3) There exists \( x \in X \) with finite G-orbit \( \mathcal{O}(x) \).

Remarks 3.22 1) In the second part of Corollary 3.21, we need the assumption that \( A \) is strongly graded. An easy counterexample is the following: let \( X \) be any G-set without finite orbit, and \( A \) an arbitrary \( k \)-algebra. We give \( A \) the trivial G-grading. Then for any \( x \in X \), \( x \otimes 1_A \) is a normalized dual integral.
2) Let \( X = G = \mathbb{Z} \), and \( A \) a strongly G-graded \( k \)-algebra. Then the functor \( G : \mathcal{M}_A \rightarrow \text{gr-A} \) is not separable.
3) Let \( X = G \), where \( G \) acts on \( G \) by conjugation, that is \( g \cdot h = h^{-1}gh \), for all \( g, h \in G \). The corresponding Doi-Hopf modules will be called right G-crossed A-modules, and we will call denote the category of right G-crossed A-modules by \( \text{gr-(G, G, A)} = ^G\mathcal{C}_A \). \( z = 1_G \otimes 1_A \) is a dual normalized A-integral, hence the functor \( F : \mathcal{M}_A \rightarrow ^G\mathcal{C}_A \) is separable. If \( A = kG \), then the category \( ^G\mathcal{C}_A \) is just the category of crossed G-modules defined by Whitehead (see [38]).
Now take $X = G$, where the action is the usual multiplication of $G$. Then the category $\text{gr-}(G, G, A)$, also denoted by $\text{gr-}A$ is the category of $G$-graded $A$-modules, and we obtain the following properties:

1) if $F : \mathcal{M}_A \to \text{gr-}A$ is separable then $G$ is finite;
2) if $G$ is finite and $\# G$ is invertible in $k$, then $F$ is a separable functor.

In particular, for a strongly graded $G$-algebra $A$ (i.e., a $kG$-Hopf Galois extension of $A_1$), we obtain a necessary and sufficient condition for the extension $A_1 \subset A$ to be separable, using Theorem 3.6 and Corollary 3.19. Compare this to [27, Proposition 2.1].

If $A$ is a strongly graded $G$-algebra with $G$ a finite group and $\# G$ invertible in $k$, then $A$ is a separable extension of $A_1$.

In the next example we will construct a strongly graded $G$-algebra $A$ such that $A/A_1$ is a separable extension, with $\# G$ not invertible in $k$.

**Example 3.23** Let $k = \mathbb{Z}_2$ and $C_2 = \{1, c\}$ the cyclic group of order two. On the matrix ring $A = M_2(\mathbb{Z}_2)$, we consider the following grading (see [8]): $A_1$ and $A_c$ are the vector spaces with respectively

\[
\begin{align*}
&\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\} \text{ and } \\
&\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}
\end{align*}
\]

as $k$-basis. Then $A$ is a strongly $G$-graded ring (even a crossed product), and

\[
z = 1 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + c \otimes \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}
\]

is a normalized dual $A$-integral (this can be proved by an easy computation on the basis elements). Hence $A$ is a separable extension of $A_1$, and, obviously, $\# C_2 = 2 = \text{char}(k)$.

We end this section pointing out that, over a field $k$ of characteristic zero, and for a strongly $G$-graded $k$-algebra $A$, the extension $A_1 \subset A$ is separable if and only if $G$ is a finite group.

**Acknowledgement** The authors thank Paul Taylor for his kind permission to use the ”diagrams” software.

**References**

[1] E. Abe, Hopf Algebras, Cambridge University Press, Cambridge, 1977.

[2] S. Caenepeel, G. Militaru, and Shenglin Zhu, Crossed modules and Doi-Hopf modules, *Israel J. Math.* 100 (1997), 221-247.

[3] S. Caenepeel, G. Militaru and Shenglin Zhu, Doi-Hopf modules, Yetter-Drinfel’d modules and Frobenius type properties, *Trans. Amer. Math. Soc.* 349 (1997), 4311-4342.

[4] S. Caenepeel, G. Militaru and Shenglin Zhu, A Maschke type theorem for Doi-Hopf modules and applications, *J. Algebra* 187 (1997), 388-412.

[5] S. Caenepeel, Bogdan Ion, G. Militaru and M. Stânciulescu, Notes on the separability equation, preprint 1997.

[6] S. Caenepeel and Ş. Raianu, Induction functors for the Doi-Koppinen unified Hopf modules, in *Abelian groups and Modules*, p. 73-94, Kluwer Academic Publishers, Dordrecht, 1995.

[7] M. Cohen and D. Fischman, Hopf Algebra Actions, *J. Algebra* 100 (1986), 363-379.
[8] M. Cohen and D. Fischman, Semisimple Extensions and Elements of Trace 1, *J. Algebra* **149** (1992), 419-437.

[9] S. Dăscălescu, Bogdan Ion, C. Năstăsescu and J. Rios Montes, Comodule algebra structures of matrix rings: Gradings, Preprint 1998.

[10] F. DeMeyer, E. Ingraham, Separable algebras over commutative rings, *Lecture Notes in Math.* **181**, Springer Verlag, Berlin, 1971.

[11] Y. Doi, Algebras with total integrals, *Comm. in Algebra* **13**(1985), 2137-2159.

[12] Y. Doi, Hopf extensions of algebras and Maschke type theorems, *Israel J. Math.* **72** (1990), 99-108.

[13] Y. Doi, Unifying Hopf modules, *J. Algebra* **153** (1992), 373-385.

[14] Y. Doi, M. Takeuchi, Hopf-Galois extensions of algebras, the Miyashita-Ulbrich action and Azumaya algebras, *J. Algebra* **121** (1989), 488-516.

[15] V. G. Drinfel’d, Quantum groups, *Proc. Int. Cong. Math.*, Berkeley, **1** (1986), 789–820.

[16] K. Hirata, K. Sugano, On semisimple and separable extensions over noncommutative rings, *J. Math. Soc. Japan* **18** (1966), 360-373.

[17] M. Koppinen, Variations on the smash product with applications to group-graded rings, *J. Pure Appl. Algebra* **104** (1995), 61-80.

[18] R. G. Larson, Coseparable coalgebras, *J. Pure Appl. Algebra* **3** (1973), 261-267.

[19] R. G. Larson, D. E. Radford, Finite dimensional cosemisimple Hopf algebras in characteristic 0 are semisimple, *J. Algebra* **117** (1988), 267-289.

[20] F. W. Long, The Brauer group of dimodule algebras, *J. Algebra* **31** (1974), 559-601.

[21] R.G. Larson, M.E. Sweedler, An associative orthogonal bilinear form for Hopf algebras, *Amer. J. Math.* **91** (1969), 75-93.

[22] S. Majid, Physics for algebrists: non-commutative and non-cocommutative Hopf algebras by a bycrossproduct construction, *J. Algebra* **130** (1990), 17-64.

[23] G. Militaru, Functors for relative Hopf modules. Applications, *Rev. Roumaine Math. Pures Appl.* **41** (1996), 451-512.

[24] G. Militaru, The Long dimodule category and nonlinear equations, *Bull. Belgian Math. Soc.-Simon Stevin*, to appear.

[25] S. Montgomery, Hopf algebras and their actions on rings, American Mathematical Society, Providence, 1993.

[26] C. Năstăsescu, Ş. Raianu and F. van Oystaeyen, Modules graded by G-sets, *Math. Z.* **203** (1990), 605-627.

[27] C. Năstăsescu, M. van den Bergh and F. van Oystaeyen, Separable functors applied to graded rings, *J. Algebra* **123** (1989), 397-413.
[28] M. D. Rafael, Separable functors revisited, Comm. in Algebra 18 (1990), 1445-1459.

[29] A. del Rio, Categorical methods in graded ring theory, Publicacions Math. 72 (1990), 489-531.

[30] D. Radford, Minimal quasi-triangular Hopf algebras, J. Algebra 157 (1993), 285-315.

[31] D. Radford, The order of the antipode in a finite-dimensional Hopf algebra is finite, Amer. J. Math. 98 (1976), 333-355.

[32] B. Pareigis, When Hopf algebras are Frobenius algebras, J. Algebra 18 (1971), 588-596.

[33] H.-J. Schneider, Principal homogeneous spaces for arbitrary Hopf algebras, Israel J. Math. 72 (1990), 167-195.

[34] D. Ţeanu, F. Van Oystaeyen, The Wedderburn-Malcev Theorem for comodule algebras, Comm. Algebra, to appear.

[35] M. E. Sweedler, Hopf algebras, Benjamin, New York, 1969.

[36] M. Takeuchi, Finite-dimensional representation of the quantum Lorentz group, Comm. Math. Phys. 144 (1992), 557-580.

[37] F. Van Oystaeyen, Y. Xu and Y. Zhang, Induction and Coinduction for Hopf extension, Sci. in China Ser. B 39 (1996), 246-263.

[38] J.H.C. Whitehead, Combinatorial homotopy, II, Bull. Amer. Math. Soc. 55 (1949), 453-496.