Abstract.
In the first part, we introduce the notion of fractional statistics in the sense of Haldane. We illustrate it on simple models related to anyon physics and to integrable models solvable by the Bethe ansatz.
In the second part, we describe the properties of the long-range interacting spin chains. We describe its infinite dimensional symmetry, and we explain how the fractional statistics of its elementary excitations is an echo of this symmetry.
In the third part, we review recent results on the Yangian representation theory which emerged from the study of the integrable long-range interacting models.

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1 Haldane’s fractional statistics.

1.1 Definition.

Haldane [1] has recently introduced a notion of fractional statistics which is independent of the dimension of space. This notion is not based on the monodromy properties of the $N$-particle wave functions, but on the way the number of available single-particle states varies when particles are added into the system. More precisely, consider a system with a total number of particles $N = \sum_j N_j$, with $N_j$ the number of particles of the species $j$. Consider now adding a particle of the species $i$ into the system without changing its size and the boundary conditions. Keeping fixed the positions of the $N$ particles of the original system, the wave function of the new $(N + 1)$-body system can be expanded in a basis of wave functions for the added particle. We denote by $D_i$ the dimension of this basis. The important point is that this dimension may depend on the numbers $N_j$ of particles in the original system. Assuming that this dependence is linear, Haldane defines the “statistical interaction” through the relation [1]:

$$\frac{\partial D_i}{\partial N_j} = -g_{ij}.$$  

(1)

Clearly, for bosons the numbers of available single-particle states are independent of the numbers $N_j$, and $g_{ij} = 0$. For fermions, the numbers of available single-particle states decrease by one for each particle added, and $g_{ij} = \delta_{ij}$.

One of the ideas underlying the introduction of the generalized Pauli principle (1) is the fact that bosons and fermions can be considered on an equal footing as far as state counting is concerned. Indeed, for bosons or fermions, the number of states of $N$ identical particles distributed among $G$ accessible orbitals can be written in a unified way as:

$$W_{b,f} = \frac{(D_{b,f} + N - 1)!}{N! (D_{b,f} - 1)!}$$

with $D_b(N) = G$ for bosons, and $D_f(N) = G - N + 1$ for fermions. The dimensions $D_{b,f}$ are the numbers of accessible states for the $N^\text{th}$ particle to be added. As it should be, $D_f$ decreases by one unit each time a fermion is added. This is generalized to fractional statistics by assuming that the total number of states with $\{N_j\}$ particles is:

$$W = \prod_i \frac{[D_i(\{N_j\}) + N_i - 1]!}{N_i! [D_i(\{N_j\}) - 1]!},$$

(2)

where $D_i(\{N_j\})$ is obtained by integrating (1):

$$D_i(\{N_j\}) + \sum_j g_{ij} N_j = G^0_i,$$

(3)

with $G^0_i \equiv D_i(\{0\})$ a constant, which is interpreted as the number of available single-particle states when no particle is present in the system. Namely, $G^0_i$ are the bare numbers of single-particle states.
1.2 Anyon-inspired examples.

The first example \[1\] is a very naive example. Let us model anyons as charged particles carrying a magnetic flux. Consider now a collection of \( \{N_i\} \) anyons with charges \( q_i \) and flux \( \phi_i \) on a disc through which goes a total magnetic flux \( \phi_B \). If we assume that the charged particles do not interact, the number \( D_i \) of accessible states for a particle of charge \( q_i \) in the disc is:

\[
D_i \simeq q_i \frac{\phi_B}{\phi_0}
\]

where \( \phi_0 = h/e \) is the flux quantum. Since the anyons carry magnetic flux, each time we introduce an anyon in the system we increase the total magnetic flux by an amount equal to the anyon flux. Therefore, the total magnetic flux is linear in the anyon numbers:

\[
\phi_B \simeq \phi_0 B + \sum_j \phi_j N_j.
\]

As a consequence, the numbers of accessible states depend on the number of anyons present in the system, and we have:

\[
g_{ij} = -\frac{\partial D_i}{\partial N_j} = -\frac{q_i \phi_j}{\phi_0}
\]

The quantity \( \theta_{ij} = \pi (g_{ij} + g_{ji}) \) coincides with the Bohm-Aharonov phase obtained by moving an anyon of charge \( q_i \) and flux \( \phi_i \) around another anyon of charge \( q_j \) and flux \( \phi_j \).

The second example is based on a study of the elementary excitations of the fractional quantum Hall effect \[2\]. So, we consider electrons moving on a plane in which there is a uniform transverse magnetic field \( B \). We denote by \( z = x + iy \) the complex coordinate in this plane, and by \( l_B = \sqrt{\hbar / e B} \) the magnetic length. If the electrons are not interacting, the energy spectrum is described by the Landau levels. In an appropriate gauge, a basis of wave functions in the first Landau level is given by:

\[
\psi_j(z, \bar{z}) = z_j \exp(-|z|^2/2l_B^2).
\]

These functions are peaked around the circle \( |z|^2 = j l_B^2 \). To confine the electrons inside a disc of radius \( R \), we impose that \( j l_B^2 \leq \pi R^2 \). This is equivalent to imposing that \( j \leq N_\phi \) where \( N_\phi \) is the number of flux quanta going through the disc, i.e. \( \phi_B = \pi R^2 B = N_\phi \phi_0 \). If there are \( N_e \) electrons in the system, the filling factor \( \nu \) is defined as:

\[
\nu = \frac{N_e}{N_\phi} = \frac{\# of electrons}{\# of flux quanta} \tag{4}
\]

If the electrons are subject to the coulomb interaction, the degeneracy between the Landau states is removed. The energy spectrum was numerically studied in ref.\[4\]. It can be described as follows. Suppose that the filling factor \( \nu \) is slightly less than the fraction \( 1/m \), namely:

\[
N_\phi = m(N_e - 1) + n \tag{5}
\]

for some integer \( n \). Then, there will be a gap in the energy spectrum in the first Landau level. The gap is of order \( e^2/l_B \). There are a large number of states above the gap but only few below the gap. These low energy states are the accessible states for \( n \) excitations carrying each a unit flux quantum. These excitations are called quasi-holes. The number of accessible states for the quasi-hole excitations can be understood using Laughlin ansatz for the quasi-hole wave functions \[3\], which for a filling factor just below \( 1/m \) as in eq.(5) are defined by:

\[
\Psi(z) = e^{-\sum_j |z|^2/2l_B^2} \prod_{i<j} (z_i - z_j)^m P(z) \tag{6}
\]
where $z_j$ are the coordinates of the $N_e$ electrons, and $P(z)$ is a polynomial, symmetric in these variables. These wave functions are ansatz for the states which are below the gap. So different choices of polynomial corresponds to different accessible states for the quasi-hole excitations. These polynomials are constrained by the fact that the electrons are confined in the disc; i.e. the electron angular momenta should be less than $N\phi$. Therefore,

$$\text{degree}(P) + m(N_e - 1) \leq N\phi$$

or equivalently, the degree of $P$ is less than the number of quasi-holes. Hence, the number of accessible states to the quasi-holes is the number of symmetric polynomials in $N_e$ variables and of degree less than $n$ in each of the variables. This number is $\frac{(N_e+n)!}{N_e!n!}$. It agrees with the numerical simulations $^{[4]}$. Comparing with eq.(2), we see that, at fixed number of flux quanta $N\phi$, the effective dimension of the quasi-hole Hilbert space is:

$$D_{qh}(n) = 2 + \frac{1}{m}(N\phi - n)\quad (7)$$

It depends linearly in the quasi-hole number. Eq.(7) gives the quasi-hole statistical interaction:

$$g_{qh} = -\frac{\partial D_{qh}}{\partial n} = \frac{1}{m}\quad (8)$$

These examples illustrate how low energy collective excitations may possess a statistics with a fractional character.

### 1.3 The Bethe ansatz and fractional statistics.

We now give another example based on integrable models $^{[5]}$. We will describe how the Bethe ansatz equations can be reinterpreted in such way that they code a statistical interaction between particles, assuming that particles of different momenta belong to different species. Although it is useful in revealing the fractional statistics of the elementary excitations of the exactly solvable models, this remark does not provide a new way of solving the Bethe ansatz equations.

In the Bethe ansatz approach to integrable models, the information is encoded in the two-body $S$-matrix. We denote it by $S(k)$ with $k$ the relative momentum of the scattering particles; we have $S(k) = -\exp(-i\theta(k))$, where $\theta(k)$ is the phase shift and it is odd in $k$. The eigenstates of the $N$-body hamiltonian (with periodic boundary condition) are labeled by $N$ (pseudo-)momenta $\{k_r\}$ ($r = 1, \cdots, N$), which are solutions of the Bethe ansatz equations:

$$e^{ik_r L} = \prod_{s \neq r} S(k_s - k_r) \quad \text{for all } r,\quad (9)$$

with $L$ the length of the system. The energy $E$ of the eigenstate $|k_1, \cdots, k_N\rangle$ is $E = \sum_r \epsilon^0(k_r)$, where $\epsilon^0(k)$ is some universal function, e.g. $\epsilon^0(k) = k^2$.

The information about the statistical interactions is encoded in the Bethe ansatz equations (9), if it is rewritten in appropriate form. Indeed, taking as usual the logarithm of eq.(9), the Bethe ansatz equations become:

$$L k_r = \sum_{s \neq r} \theta(k_r - k_s) + 2\pi I_r \quad (10)$$
with $\theta(k)$ the phase shift. Here $\{I_r\}$ ($r = 1, \cdots, N$) is a set of integers or half integers, depending on $N$ being odd or even, which one may choose to serve as the quantum numbers labeling the eigenstates, instead of the momenta $\{k_r\}$. For simplicity, let us consider the cases when these $I_r$’s have been chosen to be all different, (this is the usual situation). The ground state corresponds to an equidistribution of the integral quantum numbers $\{I_r\}$ in an interval centered around the origin: $I_{r+1} - I_r = 1$. The excited states correspond to particle/hole excitations in the integral lattice for the $\{I_r\}$:

$$I_{r+1} - I_r = 1 + M^h_r,$$

where $M^h_r$ is the number of holes, i.e. unoccupied integer numbers, between $I_{r+1}$ and $I_r$. The description of the states by the quantum numbers $\{I_r\}$ is a fermionic description since there cannot be two integers taking the same value. In this description, the statistics is simple, but the energy is a complicated function of the $\{I_r\}$.

We now change to the momentum description in the thermodynamical limit, $N \to \infty$ at fixed density $D = N/L$. It is convenient to introduce the variable $x = r/N$ which varies from zero to one. The pseudo-momenta $k_r$, the integers $I_r$ and the numbers of holes $M_r$ are all functions of $x$. As usual, we define the density $\rho(k)$ of particles of pseudo-momentum $k$ and the density of holes $\rho_h(k)$ by:

$$\rho(k) = \frac{1}{L(k_{r+1} - k_r)} = D \left( \frac{dx}{dk} \right),$$

$$\rho_h(k) = \frac{I_{r+1} - I_r - 1}{L(k_{r+1} - k_r)} = \rho(k) M(k).$$

The density of particles is normalized by $\int dk \rho(k) = D$. The energy is then given by:

$$\frac{E}{L} = \int_{-\infty}^{+\infty} dk \rho(k) \epsilon^0(k).$$

Thus there is no interaction energy between particles of different pseudo-momenta. Contrary to the fermion description, in the momentum description the dynamics looks simple but as we will see, the statistics is non-trivial.

The statistical interaction is hidden in the Bethe ansatz equations. Taking the difference of eqs. (10) for $r$ and $(r + 1)$ in the limit $N \to \infty$ gives (13):

$$\rho_h(k) + \rho(k) + \int_{-\infty}^{+\infty} \frac{dk'}{2\pi} \phi(k, k') \rho(k') = \rho^0(k) = \frac{1}{2\pi}$$

with $\phi(k, k')$ the derivative of the phase shift: $\phi(k, k') = \theta'(k - k')$. Eq. (13) is the well known thermodynamical limit of the Bethe ansatz equations. A close comparison with the state-counting equation (3) shows that they are identical provided that the following identification are made:

$$G_i^0/L \longleftrightarrow \rho^0(k) \equiv \frac{1}{2\pi},$$

$$N_i/L \longleftrightarrow \rho(k),$$

$$D_i(\{N_j\})/L \longleftrightarrow \rho_h(k).$$

with the discrete sum replaced by the integral over momentum. This last identification is quite natural: the hole density $\rho_h(k)$ clearly represents the density of available states for
an additional particle to be added. Furthermore, from eq.(13) we derive the formula of the statistical interaction in the momentum description:

\[ g(k, k') = \delta(k - k') + \frac{1}{2\pi} \phi(k, k') . \]  

(17)

This shows that the dynamical interaction, which is summarized in the two-body phase shift, is transmuted into a statistical interaction. Similarly as the boson-fermion equivalence in one-dimension, Eq.(17) provides an illustration of a phenomenon familiar in one-dimension: the translation of a dynamical equation into a statistical equation.

A simple example is provided by the Calogero-Sutherland model [7]. This is a model of particles interacting through a $1/r^2$ potential. It is integrable, and its two-body $S$-matrix is 

\[ S(k) = -\exp[-i\pi(\lambda - 1) \text{sign}(k)], \]

with \( \lambda \) the coupling constant. Therefore, \( \phi(k, k') = 2\pi(\lambda - 1)\delta(k - k') \), and the statistical interaction is:

\[ g(k, k') = \lambda \delta(k - k') . \]  

(18)

The Bethe ansatz equation then reads:

\[ \frac{1}{2\pi} = \rho_h(k) + \lambda \rho(k) . \]  

(19)

The bare energy is \( \epsilon^0(k) = k^2 \). The coupling constant \( \lambda \) governs the statistical interaction: if the density of particles of momentum \( k \) increases by a unit, then the holes density decreases by \( \lambda \). Eq.(18) shows that the statistical interaction is purely between particles with identical momenta. In this respect, the Calogero-Sutherland system appears clearly as an ideal gas of particles with a fractional statistics. This property of the Calogero-Sutherland model is also apparent in its recently computed correlation functions [8]. Finally, from eq.(19) we see that the duality \( \lambda \leftrightarrow 1/\lambda \), which exchanges the system with coupling \( \lambda \) and \( 1/\lambda \), corresponds to exchanging particles and holes.

1.4 Thermodynamics.

Knowing how to enumerate states, it is then possible to study the thermodynamics. In the thermodynamic limit, the numbers of particles \( \{N_j\} \), as well as the bare numbers of available states \( \{G_j^0\} \), become infinite. But the occupation numbers \( n_i = (N_i/G_i^0) \) remain finite. The entropy is \( S = k_B \log W \) with \( k_B \) the Boltzman constant. A notion of generalized ideal gas was introduced in ref.[10]. By definition, a system is called a generalized ideal gas if (a) its total energy with \( \{N_j\} \) particles is simply given by

\[ E = \sum_j N_j \epsilon_j^0 \]  

(20)

with constant \( \epsilon_j^0 \), and if (b) its states are counted according to eq.(2).

For such gases, the thermodynamic potential \( \Omega \equiv -PV \) at equilibrium can be evaluated by minimizing \( \Omega = E - TS - \sum_j N_j \mu_j \) with respect to the variation of the densities \( n_i \). Here \( T \) is the temperature and \( \mu_i \) the chemical potential for the species \( i \). The resulting thermodynamics was described in ref.[10]:

\[ \Omega = -k_B T \sum_i G_i^0 \log \left( \frac{1 + w_i}{w_i} \right) , \]  

(21)
where the functions \( w_i \) are determined by the equations:

\[
\log(1 + w_i) + \sum_j g_{ji} \log \left( \frac{w_j}{1 + w_j} \right) = \frac{\epsilon_i^0 - \mu_i}{k_B T}. \tag{22}
\]

The quantities \( w_i \) possess a clear physical meaning: they are equal to the mean value of accessible states per particles of species \( i \), \( w_i = D_i(\{N_j\})/N_i \).

These relations completely specify the thermodynamics of the generalized ideal gas. The other thermodynamical quantities can be derived from the relation:

\[
d\Omega = -SdT - \sum_i N_i d\mu_i - PdV.
\]

In particular, the occupation numbers \( n_i \) are obtained from

\[
n_i G_0^i = -\frac{\partial\Omega}{\partial\mu_i}.
\]

It gives:

\[
w_i g^g(1 + w_i)^{1-g} = \exp\left(\frac{(\epsilon_i^0 - \mu)}{k_B T}\right), \tag{23}
\]

For \( g = 0 \) (or \( g = 1 \)), we recover the bosonic (or fermionic) occupation numbers. Eq.(23) possesses a \( g \leftrightarrow 1/g \) duality: \( w_i(T; g) = w_i(-T/g; 1/g) \). This is the analogue of the particle/hole duality of the Calogero models which we mentioned in the previous section.

For \( g \neq 0,1 \), a special example with only one energy level, is that of anyons in the lowest Landau level. In ref.[10], the anyon thermodynamical potential was computed in the strong magnetic field limit using a diagrammatic expansion; in ref.[10], it was derived from a state counting approach similar to those described in section (1.2).

The Bethe ansatz solvable models also provide examples of generalized ideal gas, since in the momentum description the energy is given by eq.(12), (i.e. there is no interaction between the particles of different momenta), and the entropy is \( S = k_B \log W \) with \( W \) given by eq.(3) with the correspondence (15,16). The thermodynamics of such gas is called the thermodynamic Bethe ansatz (TBA). It was developed by Yang and Yang in ref.[1]. The equations governing the TBA are eqs.(21,22) with the correspondence (14,15,16).

2 Long-Range Interacting Models.

We now present another model whose elementary excitations obey a fractional statistics. It is the XXX spin chain with long range interaction introduced by Haldane and Shastry [12], see also [13]. This is a variant of the spin half Heisenberg chain, with exchange inversely proportional to the square distance between the spins. It possesses the remarkable properties that its spectrum is highly degenerate and additive, and that the elementary excitations are spin half objects obeying a half-fractional statistics intermediate between bosons and fermions.

There is a large family of integrable long range interacting spin chains which are defined as follows. We consider a spin chain with \( N \) sites, labeled by integers \( i,j, \cdots \) ranging from 1 to \( N \). On each sites there is a spin variable \( \sigma_i \) which takes two values: \( \sigma_i = \pm \). The hamiltonians, which are all \( su(2) \) invariant, are of the following form:

\[
H = \sum_{i \neq j} h_{ij} (P_{ij} - 1) \tag{24}
\]
where $P_{ij}$ is the operator which exchanges the spins at the sites $i$ and $j$. For translation invariance $h_{ij} = h(i - j)$. Demanding the integrability of the model selects the functions $h$. The possible choices are:

$$h(x) = \begin{cases} \frac{\gamma^2}{(\sinh \gamma x)^2}, & \text{hyperbolic model (}\gamma\text{ real)} \\ \frac{(\pi/N)^2}{(\sin \frac{\pi}{N})^2}, & \text{trigonometric model} \\ \mathcal{P}(x), & \text{elliptic model.} \end{cases}$$

where $\mathcal{P}(x)$ is the Weierstrass function. When $\gamma \to \infty$, the hyperbolic model reduces to the Heisenberg spin chain: $h_{ij} = \delta_{i,j+1} + \delta_{j,i+1}$, and for $\gamma \to 0$, the interaction becomes the $1/x^2$ exchange. The hyperbolic model has not been completely solved for general $\gamma$, although a partial list of eigenstates is known. The elliptic model is even more intriguing since it interpolates between the Heisenberg spin chain of finite length and the trigonometric model.

The Haldane-Shastry spin chain is the trigonometric model. In the thermodynamical limit, $N \to \infty$, it reduces to the $1/x^2$ exchange model, but it also possesses remarkable properties at finite $N$. Notably, its hamiltonian commutes with an infinite dimensional algebra whose two first generators are:

$$Q_0 = \sum_i \vec{S}_i$$

$$Q_1 = \sum_{i \neq j} \cotg \left( \frac{\pi(i - j)}{N} \right) \vec{S}_i \times \vec{S}_j$$

with $\vec{S}_i$ the spin operators acting on the site $i$. The first generators are the usual su(2) generators. Together with the second ones, they form a representation of the su(2) Yangian, (which is a deformation of the su(2) current algebra, see section 3 for an introduction to the Yangians). This infinite dimensional symmetry is at the origin of the large degeneracy of the spectrum. The fact that the hamiltonian is Yangian invariant at finite $N$ is particular to the Haldane-Shastry spin chain; in the Heisenberg spin chain, the Yangian symmetry only appears in the thermodynamical limit.

In order to grasp the rules describing the spectrum, we first construct few eigenstates. Clearly, the ferromagnetic vacuum $|\Omega\rangle = |+ + \cdots +\rangle$ is an eigenstate: its energy is zero. The eigenstates in the one-magnon sector are the plane waves $|k\rangle = \sum_n \exp(i2\pi k n/N) \sigma_n^- |\Omega\rangle$, with pseudo-momentum $k$, $1 \leq k \leq (N - 1)$: the one-magnon energy is $\epsilon(k) = \left( \frac{\pi}{N} \right)^2 k(k - N)$. In the two-magnon sectors, i.e. for states of the form $|\psi\rangle = \sum_{n,m} \psi_{n,m} \sigma_n^- \sigma_m^- |\Omega\rangle$, the eigenstates which are not degenerate with the zero or one-magnon eigenstates are labeled by two pseudo-momenta $k_1, k_2$, with $1 \leq k_1, k_2 \leq (N - 1)$. They are given by:

$$|\psi[^{k_1,k_2}_{n,m}]\rangle = (k_1 - k_2) \left( \omega^n k_1 + m k_2 + \omega^m k_1 + n k_2 \right) - \frac{\omega^n + \omega^m}{\omega^n - \omega^m} \left( \omega^n k_1 + m k_2 - \omega^n k_1 + n k_2 \right)$$

with $\omega = \exp(i2\pi/N)$. Note that these wave functions vanish if $k_1 = k_2$ but also if $|k_1 - k_2| = 1$. The energy of $|\psi[^{k_1,k_2}_{n,m}]\rangle$ is $E = \epsilon(k_1) + \epsilon(k_2)$.

From the two-magnon computation we learn two properties of the spectrum: (i) it is additive, e.g. the two-magnon energy is the sum of the one-magnon energies, but (ii) the
pseudo-momenta satisfy a selection rule: they are neither equal nor they differ by a unit. These rules are the general rules, and the full spectrum can be described as follows. To each eigenstate multiplet is associated a set of pseudo-momenta \{k_p\} which are non-consecutive integers ranging from 1 to \((N-1)\). The energy of an eigenstate \(|\{k_p\}\rangle\) with pseudo-momenta \{k_p\} is:

\[
H|\{k_p\}\rangle = \left(\sum_p \epsilon(k_p)\right)|\{k_p\}\rangle \quad \text{with} \quad \epsilon(k) = \left(\frac{\pi}{N}\right)^2 k(k - N)
\]

(27)

Furthermore, the degeneracy of the multiplet with pseudo-momenta \{k_p\} is described by its su(2) representation content as follows. Encode the pseudo-momenta in a sequence of \((N-1)\) labels 0 or 1 in which the 1’s indicate the positions of the pseudo-momenta; add two 0’s at both extremities of the sequence which now has length \((N+1)\). Since the pseudo-momenta are neither equal nor consecutive, two labels 1 cannot be adjacent. The sequence corresponding to the ferromagnetic vacuum is a line of 0, those of the one-magnon states have \(N\) label 0 and only one label 1, and so on. A sequence can be decomposed into the product of elementary motifs, which are series of \((Q+1)\) consecutives 0’s. The multiplicity of the spectrum is recovered if to each elementary motif of length \((Q+1)\) we associate a spin \(Q/2\) representation of su(2). The representation content of the full sequence is then given by the tensor product of its elementary motifs.

The magnons are the excitations over the ferromagnetic vacuum; the excitations over the antiferromagnetic vacuum are conveniently described in terms of spinons. For \(N\) even, the antiferromagnetic vacuum corresponds to the alternating sequence of symbols 010101⋯010. The excitations are obtained by flipping and moving the symbols 0 and 1. Let us give the sequences corresponding to the first few excitations over the antiferromagnetic vacuum, (for concreteness we choose \(N = 10\)):

\[
\begin{align*}
0 & 1 0 1 0 1 0 1 0 1 0 , \quad \text{antiferromagnetic vacuum (o)} \\
0 & 1 0 1 1 0,0,0 1 0 , \quad \text{a two-spinon excitations (2a)} \\
0 & 1 0,0,0 1 0 1 0 1 0 1 , \quad \text{a two-spinon excitations (2b), etc...}
\end{align*}
\]

We have inserted a \(x\) between any two consecutive labels 0. These crosses represent the spinon excitations, their number is the spinon number. Note that there is no one-spinon excitation for \(N\) even. By convention, we will say that consecutive crosses not separated by any label 1 correspond to spinons in the same orbital, while crosses separated by labels 1 correspond to spinons in different orbitals. From the rules described above, it follows that the degeneracy of the excitations (2a) and (2b) are different: it is three in the case (2a) and four in the case (2b). These degeneracy are recovered by giving a su(2) spin half to the spinons and by assuming that spinons in the same orbital are in a fully symmetric states. Hence, in the case (2a), there are two spinons in the same orbital and therefore they form a spin one representation of su(2), and in the case (2b), the two spinons are in two different orbitals and therefore they form a su(2) representation isomorphic to the tensor product of two spin half representations of su(2). The fact that the spinons are spin half excitations can also be seen by looking at the excitations of a spin chain of length \(N\) with \(N\) odd.

This description of the states generalizes to the full spectrum. We can classify the sequences by their number \(M\) of pseudo-momenta. The spinon number \(N_{sp}\) of a sequence is
then defined by $M = \frac{N - N_{sp}}{2}$. Since $M$ is an integer, $(N - N_{sp})$ is always even: this means that the spinons are always created by pairs. A sequence of pseudo-momenta $\{k_p; p = 1, \ldots, M\}$, in the $N_{sp}$ spinon sector, can be decomposed into $(M + 1)$ elementary motifs where, as before an elementary motif is a series of consecutive 0. We call the elementary motifs the accessible orbitals to the spinons. At fixed $N_{sp}$, there are $N_{orb} = \left(1 + \frac{N - N_{sp}}{2}\right)$ orbitals. Hence, a sequence of pseudo-momenta $\{k_p\}$ corresponds to the filling of the $N_{orb}$ orbitals with respective spinon occupation numbers $n_p = (k_{p+1} - k_p - 2)$, with $k_0 = -1$ and $k_{M+1} = N + 1$ by convention. Since an elementary motif of length $(Q + 1)$ corresponds to a spin $Q/2$ representation of $su(2)$, the full degeneracy of the sequences is then recovered by assuming that the spinons are spin half objects which behave as bosons in each orbitals.

The spinons are not bosons but “semions”: they obey a half fractional statistics. This follows from the fact that the number of available orbitals varies with the total occupation number [1]. Indeed, at spinon number $N_{sp}$, the number of orbitals is $N_{orb} = \left(1 + \frac{N - N_{sp}}{2}\right)$. Therefore, we have the statistical interaction:

$$g_{sp} = -\frac{\partial N_{orb}}{\partial N_{sp}} = 1/2$$

The fractional statistics of the spinons is also apparent in the spin-spin correlation function [17]. In the following section, we will describe how the fractional statistics of the spinons is encoded in the Yangian representation theory.

The spinon description of spectrum is very similar to the description of the excitations of the Heisenberg spin chain given by Faddeev and Takhtajan [18]. Note that the model is gapless. Its low energy properties belong to the same universality class as the Heisenberg model. The low energy, low temperature, behavior is described by the level one $su(2)$ WZW conformal field theory. The spinon formulation of the Haldane-Shastry spin chain provides a new quasi-particle description of the states in the WZW model [19].

3 Algebraic Solution of the Long-Range Interacting Models.

In this section we review few of the new results on integrable models and on the Yangian representation theory which emerged from the study of the long-range interacting models. But we first need to recall standard result concerning the algebraic Bethe ansatz, cf. e.g. [20].

3.1 Algebraic Bethe ansatz and Yangians.

We introduce the basic notion of the algebraic Bethe ansatz, using the quantum Heisenberg chain as an example. We consider a chain of length $N$: on each site there is a spin variable $\sigma_j$. We denote by $S_j^{ab}$, $a, b = 1, 2$, the spin operators satisfying the $su(2)$ commutation relations:

$$\left[ S_j^{ab}, S_k^{cd} \right] = \delta_{jk} \left( \delta^{cb} S_j^{ad} - \delta^{ad} S_j^{cb} \right)$$

The Heisenberg hamiltonian is:

$$H = \sum_{k=1}^{N} \sum_{ab} S_k^{ab} S_{k+1}^{ba} = \sum_{k=1}^{N} (P_{k,k+1} - 1)$$
Here, we have assumed periodic boundary conditions. As is well known, in order to preserve the integrability the spin operators $S^{ab}_{k}$ should act on the spin half representation of $\text{su}(2)$. So, the spin variables take only two values, $\sigma_{j} = \pm$, and the operator $S^{ab}_{j}$ which acts only the $j^{th}$ spin is represented by the canonical matrix $|a\rangle\langle b|$.

The algebraic Bethe ansatz goes in few steps.

1. The first step consists in constructing the local monodromy matrices $T_{j}(u)$. These matrices are $2 \times 2$ matrices whose elements $T^{ab}_{j}(u)$ are operators. The matrices $T_{j}(u)$ are defined by:

$$T^{ab}_{j}(u) = u\delta^{ab} + \lambda S^{ab}_{j}$$

(30)

where $u$ is a complex number, called the spectral parameter, and $\lambda$ a coupling constant. Note that the matrix $T_{j}(u)$ only acts on the $j^{th}$ spin. The important point is that we can compute the commutation relations between its matrix elements. These relations can be gathered into the famous relations of the algebraic Bethe ansatz, see e.g. [20]:

$$R(u - v)(T(u) \otimes 1)(1 \otimes T(v)) = (1 \otimes T(v))(T(u) \otimes 1)R(u - v)$$

(31)

where $R(u)$ is Yang’s solution of the Yang-Baxter equation, $R(u) = u - \lambda P$, with $P$ the exchange operator $P(x \otimes y) = y \otimes x$.

2. The second step consists in constructing the complete monodromy matrix, which we denote by $T(u)$. It is obtained by taking the ordered product of the local monodromy matrices. Namely,

$$T^{ab}(u) = \sum_{a_{2} \cdots a_{N}} T^{a_{2}a_{3}}_{1}(u) T^{a_{2}a_{3}}_{2}(u) \cdots T^{a_{N}b}_{N}(u)$$

(32)

It admits an $(\frac{1}{u})$-expansion:

$$u^{-N} T^{ab}(u) = \delta^{ab} + \frac{\lambda}{u} \left( \sum_{k} S^{ab}_{k} \right) + \frac{\lambda^{2}}{u^{2}} \left( \sum_{j<k} \sum_{d} S^{ad}_{j} S^{db}_{k} \right) + \cdots$$

The crucial point is the fact that the complete monodromy matrix (32) satisfy the relations (31) if the local monodromy matrices do. These relations are equivalent to the following quadratic commutation relations:

$$(u - v) \left[ T^{ab}(u), \ T^{cd}(v) \right] = \lambda \left( T^{cb}(u)T^{ad}(v) - T^{cb}(v)T^{ad}(u) \right)$$

(33)

An important consequence of the relations (31) is that the transfer matrix $T(u)$, which is the trace of the monodromy matrix, $T(u) = tr(T(u)) = T^{11}(u) + T^{22}(u)$, is a generating function of commuting hamiltonians:

$$[ T(u) , \ T(v) ] = 0$$

The Heisenberg hamiltonian is recovered by expanding the logarithm of the trace to first order: $H \propto \partial_{u} \log T(u) \big|_{u=0}$.

Another generating function of commuting quantities is given by the quantum determinant $\text{det}_{q}T(u)$. It is defined by:

$$\text{det}_{q}T(u) = T^{22}(u - \lambda)T^{11}(u) - T^{21}(u - \lambda)T^{12}(u)$$

(34)
It commutes with all the matrix elements of the monodromy matrix: \[ \det_q T(u), T^{ab}(v) = 0. \]

The quadratic algebra (33) is called a su(2) Yangian [21]. More precisely, consider a T-matrix satisfying the commutation relations (31) or (33), and normalized to have a quantum determinant equal to one: \( \det_q T(u) = 1 \). Assume that the T-matrix possesses a \( \frac{1}{u} \)-expansion as follows:

\[
T^{ab}(\lambda) = \delta^{ab} + \lambda \sum_{n=0}^{\infty} u^{-n-1} t^{ab}_{(n)}
\]  

(35)

Then, the su(2) Yangian is the associative algebra generated by the elements \( t^{ab}_{(n)} \). For these elements, the relations (33) are equivalent to:

\[
\begin{align*}
\left[ t^{ab}_{(0)}, t^{cd}_{(m)} \right] &= \delta^{cb} t^{ad}_{(m)} - \delta^{ad} t^{cb}_{(m)} \\
\left[ t^{ab}_{(n+1)}, t^{cd}_{(m)} \right] - \left[ t^{ab}_{(n)}, t^{cd}_{(m+1)} \right] &= \lambda \left( t^{cb}_{(m)} t^{ad}_{(n)} - t^{ad}_{(n)} t^{cb}_{(m)} \right)
\end{align*}
\]

(36)

Note that with the quantum determinant constraint, the \( \frac{1}{u} \)-expansion of the monodromy matrix can be reconstructed from its two first components \( t^{ab}_{(0)} \) and \( t^{ab}_{(1)} \). The relations (36) clearly shows the Yangians are deformation of loop algebras.

The next step consists in diagonalizing the transfer matrix. The algebraic Bethe ansatz provides a way to perform this diagonalization inside a finite dimensional irreducible representation of the su(2) Yangian. Similarly as for the unitary representations of su(2), any finite dimensional irreducible Yangian representation is specified by an highest weight vector \( |\Omega\rangle \). It is characterized by the following equations:

\[
T(u)|\Omega\rangle = \left( \begin{array}{cc} f_1(u) & 0 \\ \ast & f_2(u) \end{array} \right) |\Omega\rangle
\]

(37)

where \( f_1(u) \) and \( f_2(u) \) are C-number functions, not operators. The product of these functions is related to the quantum determinant by: \( \det_q T(u) = f_2(u - \lambda) f_1(u) \). Due to the fact that the quantum determinant commutes with the T-matrix, only the ratio \( f_1(u)/f_2(u) \) encodes the data of the representation. Moreover, the Yangian representation with highest weight vector \( |\Omega\rangle \) is finite dimensional if and only if this ratio satisfies [21]:

\[
\frac{f_1(u)}{f_2(u)} = \frac{P(u + \lambda)}{P(u)}
\]

(38)

for some polynomial \( P(u) \). These polynomials are called Drinfel’d polynomials. The condition (38) is the analogue of the fact that finite dimensional su(2) representations correspond to half integer spins.

All the states in an irreducible Yangian representation are obtained by iterative actions of \( T^{21}(u) \) on \( |\Omega\rangle \):

\[
|\Psi\rangle = T^{21}(u_1) T^{21}(u_2) \cdots T^{21}(u_M) |\Omega\rangle
\]

(39)

The Bethe states, which are eigenstates of the transfer matrix, are of this form, but for particular values of the parameters \( u_p \). The relations determining these \( u_p \)'s are called the Bethe ansatz equations. They can be summarized as follows. Let us define a polynomial \( Q(u) \) of degree \( M \) whose roots are the \( u_p \)'s:

\[
Q(u) = \prod_{p=1}^{M} (u - u_p)
\]

(40)

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The state (39) is then an eigenstate of the transfer matrix $\mathcal{T}(u)$ if the roots $u_p$ of $Q(u)$ are such that this polynomial is solution of the following difference equation:

$$t(u)Q(u) = f_1(u)Q(u - \lambda) + f_2(u)Q(u + \lambda)$$

(41)

where $t(u)$, a polynomial of degree $N$, is the eigenvalue of the transfer matrix on the Bethe state (39). Notice that eq. (41) at the same time gives the equations determining the Bethe roots $u_p$ and the eigenvalue $t(u)$. Eq. (41) was introduced by Baxter in its solution of the 8-vertex model [22].

Following an idea due to Sklyanin [23], the Bethe eigenstates can then be rewritten in terms of the polynomial $Q(u)$. Since the operator $T^{21}(u)$ is a polynomial of degree $(N - 1)$, let us assume that we can factorized it as follows,

$$T^{21}(u) = \lambda S^- \cdot \prod_{k=1}^{N-1} (u - x_k)$$

(42)

where the $x_k$ are operators and $S^- = \sum_j S_{j1}^2$. It follows from the relations (33) that the $x_k$ are commuting operators. The Bethe eigenstates (39) are then given by

$$|\Psi\rangle = (S^-)^M Q(x_1) Q(x_2) \cdots Q(x_{N-1}) |\Omega\rangle$$

(43)

The eqs. (41,43) reflect the separation of the variables, since the eigenstates are determined from the solutions of one equation for a function of one variable only.

Finalizing the solution of the models consists in analyzing the Bethe ansatz equations and their solutions. This can analytically be done only in the thermodynamical limit, along the lines outlined in section 1.3.

### 3.2 Quantization of the spectral parameter.

The long-range interacting models cannot be solved using the algebraic Bethe ansatz. This follows from the fact the hamiltonian commutes with the $T$-matrix, and therefore non-degenerate eigenstates cannot be obtained by iterative action of the lowering operators $T^{21}(u)$. Nevertheless, the tools of the algebraic Bethe ansatz are useful for constructing integrable long range interacting models and for deciphering the symmetries of these models.

To illustrate this fact, we now consider su(2) generalizations of the Calogero-Sutherland models. These models describe $M$ particles interacting by long range forces. Their positions are parameterized by complex numbers $z_i$, $i = 1, \cdots, M$, and each particle carries a spin $\sigma = \pm$. The Hamiltonian is:

$$H_D = \sum_{j=1}^M (z_j \partial z_j)^2 - \sum_{i \neq j} \lambda (P_{ij} + \lambda) \frac{z_i z_j}{(z_i - z_j)^2}$$

(44)

where $\lambda$ is a coupling constant and $P_{ij}$ exchanges the spins of the particles $i$ and $j$. Notice we recover the Haldane-Shastry spin chain in the static limit $\lambda = \infty$.

The construction of these models relies on the definition a monodromy matrix in which the spectral parameter has been quantized. More precisely, let us consider the monodromy matrix (32) but in which the spectral parameters have been shifted to $(u - \hat{D}_i)$, where the $\hat{D}_i$
are operators, commuting among themselves and with the spin operators. More precisely, we define a $\hat{T}$-matrix by [24]:

$$\hat{T}^{ab}(u) = \sum_{a_2 \cdots a_N} \hat{T}_{a_1}^{a_2}(u) \hat{T}_{a_2}^{a_3}(u) \cdots \hat{T}_{a_N}^{a_N}(u)$$  \hspace{1cm} (45)

with

$$T_i^{ab}(u) = \frac{(u - \hat{D}_i) \delta^{ab} + \lambda S_i^{ab}}{u - \hat{D}_i}$$  \hspace{1cm} (46)

The operators $\hat{D}_i$ we consider are defined as follows [24]:

$$\hat{D}_i = z_i \partial_{z_i} + \lambda \sum_{j > i} \theta_{ij} K_{ij} - \lambda \sum_{j < i} \theta_{ji} K_{ij}$$  \hspace{1cm} (47)

where $\theta_{ij} = \frac{z_i - z_j}{z_i - z_j}$ and $K_{ij}$ the operators which exchange the particles at positions $z_i$ and $z_j$: $K_{ij}z_j = z_i K_{ij}$. They obey the defining relations of a degenerate affine Hecke algebra:

$$[\hat{D}_i, \hat{D}_j] = 0$$
$$[K_{i,i+1}, \hat{D}_k] = 0 \text{ if } k \neq i, i + 1$$

$$K_{i,i+1}\hat{D}_i - \hat{D}_{i+1}K_{i,i+1} = -\lambda$$  \hspace{1cm} (48)

In the mathematics literature, the role of the affine Hecke algebra in this context was revealed by Cherednik [25]. In the physics literature, operators similar but different to the $\hat{D}_i$ were introduced by Polykronakos [26]. Notice that these relations imply that:

$$[K_{ij}, \hat{A}_M(u)] = 0, \quad \text{with} \quad \hat{A}_M(u) = \prod_{i=1}^M (u - \hat{D}_i)$$  \hspace{1cm} (49)

I.e. $\hat{A}_M(u)$ is symmetric by permutation of the particles. This property follows from

$$[K_{i,i+1}, (u - \hat{D}_i)(u - \hat{D}_{i+1})] = 0,$$

which is valid for all $i$.

Since the operators $\hat{D}_i$ commute, the $\hat{T}$-matrix (45) satisfies the RTT relation (31). However, the positions and the spin variables are totally uncoupled since the operators $\hat{D}_i$ commute with the spin operators. In order to couple them, we define a projection $\pi$ which consists in replacing the permutation $K_{ij}$ by the permutation $P_{ij}$ after it has been moved to the right of an expression. One can view this projection as the result of acting on wave functions totally symmetric under simultaneous permutations of the positions and of the spins. In more mathematical words, this procedure consists in quotienting the algebra generated by the permutations $K_{ij}$ and $P_{ij}$ by the left ideal generated by $(K_{ij} - P_{ij})$. We use it to eliminate the permutations of the particles by replacing them with those of the spins.

The transfer matrix $T(u)$ defined by

$$T(u) = \pi(\hat{T}(u))$$  \hspace{1cm} (50)

will then satisfy the Yang-Baxter equation if we can replace the projection of the product $(1 \otimes \hat{T}(u))(\hat{T}(u) \otimes 1)$ by the product of the projections. Since, $\hat{A}_M(u)$ is symmetric under permutation, it is equivalent to check this property for $\hat{T}'(u) = \hat{A}_M(u)\hat{T}(u)$. For this to be
true, $\hat{T}(u)$ applied on a totally symmetric wave function must still be a totally symmetric wave function. Equivalently, we must have:

$$\pi \left( K_{ij} \hat{T}(u) \right) = P_{ij} \pi \left( \hat{T}(u) \right)$$

(51)

Since the permutation groups are generated by the permutations $K_{i,i+1}$ and $P_{i,i+1}$, eq. (51) is equivalent to: $\pi \left( K_{i,i+1}\hat{T}(u)\hat{T}_{i+1}(u) \right) = P_{i,i+1}\pi \left( \hat{T}_{i}(u)\hat{T}_{i+1}(u) \right)$, with $\hat{T}_{i}(u)$ defined in (46).

This is guaranteed if the commutation relations of the degenerate Hecke algebra (48) are satisfied. Thus, the relations (48) are the necessary relations for this $T$-matrix to satisfy the RTT-relation.

An alternative presentation of this $T$-matrix was obtained in ref. [24]:

$$T_{ab}(u) = \delta_{ab} + \lambda \sum_{i,j=1}^{M} S_{ab}^{ij} \left( \frac{1}{u - L} \right)_{ij}$$

(52)

where $L$ is the matrix defined by: $L_{ij} = \delta_{ij} z_{i} \partial_{z_{j}} + (1 - \delta_{ij}) \lambda \theta_{ij} P_{ij}$, with $\theta_{ij} = z_{i}/(z_{i} - z_{j})$. In eq. (52), the projection $\pi$ has been explicitly done.

The immediate consequences of this construction are the following. Since the $T$-matrix (50) satisfies the relation (31) it defines a representation of the $su(2)$ Yangian. As explained in the previous section, the relation (31) implies that $T(u) = tr(T(u))$ is a generating function of commuting hamiltonian. However, $T(u)$ is not Yangian invariant since it does not commute with $T$ itself. A clever choice consists in choosing the quantum determinant $\det_{q}T(u)$ as the generating function of commuting hamiltonians. It is the projection of the quantum determinant of $\hat{T}(u)$:

$$\det_{q}T(u) = \pi \left( \frac{\hat{\Delta}_{M}(u + \lambda)}{\Delta_{M}(u)} \right)$$

(53)

where $\hat{\Delta}_{M}(u)$ is defined in eq. (49). The hamiltonian (44) is the $u^{-2}$-term in (53). It is therefore Yangian invariant. This invariance has recently been imbeded in a bigger algebra which is a deformation a $W_{\infty}$ algebra [27]. The quantum determinant (53) has been diagonalized in ref. [24] by directly diagonalizing the operators $\hat{D}_{i}$.

### 3.3 Application to the Haldane-Shastry spin chain.

We now explain how the previous construction can be used to derive the fractional selection rules satisfied by the eigenstates of the Haldane-Shastry spin chain.

As mentioned in section 2, the Haldane-Shastry spin chain is Yangian invariant. Therefore, there exists a $T$-matrix commuting with the hamiltonian (24) and satisfying the relations (31). It was constructed in [24]. It is the limit $u, \lambda \to \infty$ with $x = u/\lambda$ fixed, of the $T$-matrix (52). Its expression is:

$$T_{ab}(x) = \delta_{ab} + \sum_{i,j=1}^{N} S_{ab}^{ij} \left( \frac{1}{x - L} \right)_{ij}$$

(54)

with $L_{ij} = (1 - \delta_{ij}) \theta_{ij} P_{ij}$, $\theta_{ij} = z_{i}/z_{ij}$ with $z_{ij} = z_{i} - z_{j}$, and $S_{ab}^{ij}$ is the canonical matrix $|a\rangle\langle b|$ acting on the $i^{th}$ spin only. For any values of the complex numbers $z_{j}$, the transfer matrix (54) form a representation of the exchange algebra (49) with $u$ changed into $x$ and $\lambda.$
normalized to one. The trigonometric spin chain corresponds to \( z_j = \omega^j \) with \( \omega \) a primitive \( N^{th} \) root of the unity. For these values of \( z_j \), the transfer matrix \( T(x) \) commutes with the hamiltonian \( (24) \).

In the representation \( (54) \), the quantum determinant is a pure number for any values of the \( z_j \)'s given by:

\[
det_q T(x) = 1 + \sum_{i,j=1}^{N} \left( \frac{1}{x - \theta_{ij}} \right) \frac{\Delta_N(x + 1)}{\Delta_N(x)} \tag{55}
\]

with \( \Delta_N(x) \) the characteristic polynomial of the \( N \times N \) matrix \( \Theta \) with entries \( \theta_{ij} \): \( \Delta_N(x) = \text{det}(x - \Theta) \). For the Haldane-Shastry spin chain \( z_j = \omega^j \) and we have:

\[
\Delta_N(x) = \prod_{j=1}^{N} \left( x + \frac{N + 1}{2} - j \right) \tag{56}
\]

Since the monodromy matrix \( (54) \) commutes with the hamiltonian, the long-range interacting spin chain cannot be solved using the algebraic Bethe ansatz. A way to solve it consists first in decomposing the spin chain Hilbert space into irreducible sub-representation of the Yangian, and then in computing the energy in each of these irreducible blocks. For the values \( z_j = \omega^j \), the representation \( (54) \) is completely reducible. Each irreducible sub-representation possesses a unique highest weight vector \( | \Lambda \rangle \) which is annihilated by \( T_{12}(x) \) and which is an eigenvector of the diagonal components of the transfer matrix, as in eq.(37).

In ref.\[24\], it was shown that the corresponding eigenvalues of \( T_{11}(x) \) and \( T_{22}(x) \) can be expressed in terms of two polynomials \( P(x) \) and \( Q(x) \):

\[
T(x)| \Lambda \rangle = \frac{Q(x + 1)}{Q(x)} \begin{pmatrix} P(x + 1) \\ P(x) \\ 0 \\ * \\ 1 \end{pmatrix} | \Lambda \rangle \tag{57}
\]

These polynomials characterize the irreducible sub-representations. The polynomials \( Q(x) \) and \( P(x) \) are not independent, since the quantum determinant \( (54) \) take the same value in any of the irreducible blocks. They should satisfy:

\[
\Delta_N(x) = P(x) Q(x) Q(x - 1). \tag{58}
\]

Therefore, the roots of \( P(x) \) and \( Q(x) \) are among those of \( \Delta_N(x) \). This implies that \( Q(x) \) factorized as:

\[
Q(x) = \prod_{p=1}^{M} (x + \frac{N + 1}{2} - k_p) \tag{59}
\]

where the \( \{ k_p \} \) are integers between 1 and \( (N - 1) \). The equation \( (58) \) then admits solutions if and only if the roots of \( Q(x) \) are not adjacent, or equivalently, if and only if the integers \( \{ k_p \} \) are neither equal nor adjacent. These integers will be identified with the rapidities labeling the eigenmultiplets of the spin chain.

This provides a purely algebraic way to recover the rapidity selection rule. It also shows that the fractional statistics of the spinon excitations is an echo of the Yangian symmetry.

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