Spin-1/2 sub-dynamics nested in the quantum dynamics of two coupled qutrits

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Abstract

In this paper, we investigate the quantum dynamics of two spin-1 systems, \( \hat{S}_1 \) and \( \hat{S}_2 \), adopting a generalised \((\hat{S}_1 + \hat{S}_2)^2\)-nonconserving Heisenberg model. We show that due to its symmetry property, the nine-dimensional dynamics of the two qutrits exactly decouples into the direct sum of two sub-dynamics living in two orthogonal four- and five-dimensional subspaces. Such a reduction is further strengthened by our central result, consisting of the fact that in the four-dimensional dynamically invariant subspace, the two qutrit quantum dynamics, with no approximations, is equivalent to that of two noninteracting spin-1/2s. The interpretative advantages stemming from such a remarkable and nonintuitive nesting are systematically exploited, and various intriguing features that consequently emerge in the dynamics of the two qutrits are deeply scrutinised. The possibility of exploiting the dynamical reduction brought to light in this paper for exactly treating the time-dependent versions of our Hamiltonian model is briefly discussed as well.

Keywords: two coupled qutrit Hamiltonian models, symmetry-based emergence of qubit subdynamics, quantum entanglement, quantum mechanics

(Some figures may appear in colour only in the online journal)

1. Introduction

Interacting spin systems with \( s > 1/2 \) reveal a rich variety of phenomena in condensed matter and atomic physics. For example, spin models with a higher spin length may exhibit novel topological phases described by a hidden-order parameter [1]. Moreover, various strongly
interacting spin-boson systems can be mapped onto coupled spin models [2–4]. Apart from the methods used to solve various spin-1/2 systems analytically, in general, models with \( s > 1/2 \) are highly complex and do not permit analytical treatment.

In this paper, we consider a system of two interacting spin-1 systems denoted by \( \hat{S}_1 \) and \( \hat{S}_2 \), respectively living in the Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), in a physical model described by the time-independent Hamiltonian

\[
\hat{H} = \mu (g_1 \hat{B}_1 \hat{S}_1^z + g_2 \hat{B}_2 \hat{S}_2^z) + J_0 \hat{S}_1 \cdot \hat{S}_2 + \hat{D}_{12} \cdot \hat{S}_1 \cdot \hat{S}_2.
\]

The first two terms in equation (1) describe the interaction of the two spins with two (generally different) parallel local magnetic fields oriented along the \( \hat{z} \)-axis, \( \hat{B}_1 \) and \( \hat{B}_2 \), with the assumption of the scalar \( g \)-factors, \( g_1 \) and \( g_2 \). The third term represents the Heisenberg isotropic exchange interaction of coupling strength \( J_0 \), while the last term, through the second-order traceless Cartesian tensor \( \hat{D}_{12} \), accounts for symmetric spin–spin anisotropic couplings stemming from the dipole–dipole (d–d) interaction and anisotropic exchange interaction.

Part of the motivation for the present work stems from the growing interest in qutrits, which are three-state quantum systems. Qutrits, and qudits in general, offer numerous advantages over qubits beyond the obvious exponential increase in their Hilbert space. For example, qutrits allow us to construct new types of quantum protocols [5, 6] and entanglement [7], Bell inequalities resistant to noise [8], larger violations of nonlocality [9], more secure quantum communication [10, 11], and the optimisation of the Hilbert space dimensionality versus control complexity [12], amongst other things. To this end, efficient recipes for the manipulation of qutrits [13, 14] and qudits [15] have been proposed.

Although we only consider two interacting spin-1 systems, our model covers a broad range of physical situations. For example, in solid-state physics the coupling between two molecules, which in their ground state possess a total angular momentum (effective spin) \( S = 1 \), is described using the Hamiltonian model (1), with the proviso that the spin–orbit effect can be neglected [16]. An optical lattice of two wells, each containing a single atom of spin-1, provides another possible physical scenario wherein manipulation of the atom–atom coupling constants is within experimental reach [17]. In addition, the interaction between nanomagnets with a total spin of 1, which is of great interest in quantum computing, is described by the Hamiltonian model (1) [18]. Recently, it was shown that the interaction between two separated nitrogen-vacancy centres in a diamond can be described by a Heisenberg spin-1 model [19]. Moreover, spin-1 models can be realised in a linear ion crystal by using atomic species with three metastable levels driven by laser fields [20, 21].

In this paper, we investigate the quantum dynamics generated by the \( \hat{S}_2 \)-nonconserving Hamiltonian model (1) where \( \hat{S}_2 = (\hat{S}_1 + \hat{S}_2)^2 \). Our main result is that the overall nine-dimensional dynamics may be investigated in two-, four- and five-dimensional subspaces of well-defined symmetry. In the 4D-subspace, the Hamiltonian \( \hat{H} \) may be mapped into the Hamiltonian of two noninteracting spin-1/2 systems. The consequences of this remarkable reduction are deeply scrutinised, and in particular, the time evolution of the entanglement between the two qutrits is investigated by evaluating the negativity of the compound system.

The paper is organised as follows: in sections 2 and 3 we present the spin-1 model and discuss its symmetry properties. In section 4 we study the odd-parity 4D-subspace dynamics of the model and show that it is equivalent to a single spin-3/2 system. Based on this, in section 5 we provide an analysis of the entanglement negativity of the states, which belong to the 4D-subspace. In section 6 we discuss the properties of the even-parity 5D-subspace, which
is equivalent to a single spin-2 system. Finally, in section 7 conclusive remarks are given together with a possible application of our treatment to Hamiltonian models where the two qutrits are subjected to time-dependent magnetic fields.

2. The model

Let us suppose that our system possesses $C_2$-symmetry with respect to the $\hat{z}$ direction. In this case, the $D_{12}$ matrix takes the form

$$D_{12} = \begin{pmatrix} d_{xx} & d_{xy} & 0 \\ d_{yx} & d_{yy} & 0 \\ 0 & 0 & d_{zz} \end{pmatrix}$$

(2)

and the Hamiltonian (1) may be written as

$$\hat{H} = \hbar \omega_1 \hat{S}_1^x + \hbar \omega_2 \hat{S}_2^x + \gamma_1 \hat{S}_1^x \hat{S}_2^y + \gamma_2 \hat{S}_1^y \hat{S}_2^y + \gamma_3 \hat{S}_1^z \hat{S}_2^z + \gamma_4 \hat{S}_1^y \hat{S}_2^x + \gamma_5 \hat{S}_1^x \hat{S}_2^y + \gamma_6 \hat{S}_1^y \hat{S}_2^x + \gamma_7 \hat{S}_1^z \hat{S}_2^z,$$

(3)

where the Pauli operators $\hat{\Sigma}_i^k$ ($i = 1, 2; k = x, y, z$) for a spin-1 system are related to the spin-1 operator components as

$$\hat{S}_i^x = \frac{\hbar}{\sqrt{2}} \hat{\Sigma}_i^x, \quad \hat{S}_i^y = \frac{\hbar}{\sqrt{2}} \hat{\Sigma}_i^y, \quad \hat{S}_i^z = \hbar \hat{\Sigma}_i^z.$$

(4)

The seven real parameters appearing in equation (3) are given by

$$\omega_1 = \mu g_1 B_1, \quad \omega_2 = \mu g_2 B_2^c,$$

$$\gamma_1 = \frac{\hbar^2}{2} (J_0 + d_{xx}), \quad \gamma_2 = \frac{\hbar^2}{2} (J_0 + d_{yy}), \quad \gamma_3 = \hbar^2 (J_0 + d_{zz}),$$

$$\gamma_4 = \frac{\hbar^2}{2} d_{xy}, \quad \gamma_5 = \frac{\hbar^2}{2} d_{yx},$$

(5)

In this paper, we wish to keep the considerations as general as possible, without the restrictions of a specific physical situation. Hence, hereafter we do not attribute any specific symmetry constraints to the real parameters appearing in the Hamiltonian model (3). In this manner, our model includes several of those from the literature as special cases. These include the XXX ($\gamma_1 = \gamma_2 = \gamma_3$), XXZ ($\gamma_1 = \gamma_5$) and XYZ models for two qutrits subjected to an inhomogeneous magnetic field, generalised with the inclusion of the Dzyaloshinskii–Moriya (DM) interaction ($\gamma_1x = -\gamma_5$) [22]. In addition, from our Hamiltonian model one may easily recover a lot of other models, e.g. the XX and XY models ($\gamma_1 = 0$) with (or without) the contribution derived by the DM interaction and the presence (or not) of a homogeneous or inhomogeneous magnetic field, recently taken as a starting point for investigating the appearance of thermal entanglement in the system of two interacting qutrits [23–25].

3. Canonical transformation of $\hat{H}$ based on symmetry

The following symmetry transformation of $\hat{H}$

3.1. The canonical transformation

$$\hat{S}_i' = U \hat{S}_i U^{-1},$$

(6)

where $U$ is a unitary matrix, is such that

$$\hat{S}_i' = \begin{pmatrix} S_{1x}' \\ S_{1y}' \\ S_{1z}' \end{pmatrix} = \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} S_{1x} \\ S_{1y} \\ S_{1z} \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} S_{1x} \\ S_{1y} \\ S_{1z} \end{pmatrix},$$

(7)

where $c_1 = \cos \theta$, $s_1 = \sin \theta$, $\tan \theta = \frac{d_{xy}}{d_{xx} - d_{yy}}$.

In this manner, we wish to keep the considerations as general as possible, without the restrictions of a specific physical situation. Hence, hereafter we do not attribute any specific symmetry constraints to the real parameters appearing in the Hamiltonian model (3). In this manner, our model includes several of those from the literature as special cases. These include the XXX ($\gamma_1 = \gamma_2 = \gamma_3$), XXZ ($\gamma_1 = \gamma_5$) and XYZ models for two qutrits subjected to an inhomogeneous magnetic field, generalised with the inclusion of the Dzyaloshinskii–Moriya (DM) interaction ($\gamma_1x = -\gamma_5$) [22]. In addition, from our Hamiltonian model one may easily recover a lot of other models, e.g. the XX and XY models ($\gamma_1 = 0$) with (or without) the contribution derived by the DM interaction and the presence (or not) of a homogeneous or inhomogeneous magnetic field, recently taken as a starting point for investigating the appearance of thermal entanglement in the system of two interacting qutrits [23–25].
\begin{align*}
\hat{\Sigma}_{1z}^x &= -\hat{\Sigma}_{1z}^z, \quad \hat{\Sigma}_{1z}^y = -\hat{\Sigma}_{1z}^y, \quad \hat{\Sigma}_{1z}^z = \hat{\Sigma}_{1z}^z, \\
\hat{\Sigma}_{2z}^x &= -\hat{\Sigma}_{2z}^z, \quad \hat{\Sigma}_{2z}^y = -\hat{\Sigma}_{2z}^y, \quad \hat{\Sigma}_{2z}^z = \hat{\Sigma}_{2z}^z.
\end{align*}
\tag{6}
\]

is canonical, such that \( \hat{H} \rightarrow \hat{H} \), which implies the existence of a unitary time-independent operator accomplishing the transformation given by equation (6) which, by construction, is a constant of motion. Because the transformation (6) is nothing but a rotation of \( \pi \) around the \( \hat{z} \)-axis of each spin, we can write the unitary operator accomplishing this transformation as
\[
\hat{K} = e^{i\pi \hat{\Sigma}_{1z}^z} \otimes e^{i\pi \hat{\Sigma}_{2z}^z} = e^{i\pi \hat{\Sigma}_{1z}^z} \otimes e^{i\pi \hat{\Sigma}_{2z}^z} = 1 - 2((\hat{\Sigma}_{1z}^z)^2 + (\hat{\Sigma}_{2z}^z)^2) + 4(\hat{\Sigma}_{1z}^z)^2(\hat{\Sigma}_{2z}^z)^2.
\tag{7}
\]

The matrix representation of the operator \( \hat{K} \) in the standard ordered basis
\[
\{ |11\rangle, |10\rangle, |1 - 1\rangle, |01\rangle, |00\rangle, |0 - 1\rangle, | -11\rangle, | -10\rangle, | -1 - 1\rangle \}
\tag{8}
\]
is diagonal,
\[
\hat{K} = \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\tag{9}
\]

Equation (9) suggests the possibility of expressing \( \hat{K} \) as
\[
\hat{K} = \cos(\pi \hat{\Sigma}_{\text{tot}}^z),
\tag{10}
\]
with \( \hat{\Sigma}_{\text{tot}}^z = \hat{\Sigma}_{1z}^z + \hat{\Sigma}_{2z}^z \) being the total spin of the composed system along the \( z \)-direction. Equation (10) shows that the constant of motion \( \hat{K} \) is indeed a parity operator with respect to the collective Pauli spin variable \( \hat{\Sigma}_{\text{tot}}^z \), since in correspondence to its integer eigenvalues \( M = 2, 1, 0, -1, -2 \), \( \hat{K} \) has eigenvalues \( +1 \) and \( -1 \) depending on the parity of \( M \).

The existence of this constant of motion subdivides the 9D Hilbert space of the system into two dynamically invariant and orthogonal subspaces corresponding to the two eigenvalues \( +1 \) and \( -1 \) of \( \hat{K} \). The subspace relative to \( K = 1 \) (\( K = -1 \)), and then to even (odd) values of \( M \), will hereafter be referred to as even- (odd-) parity subspace. It can be easily seen that the unitary and hermitian operator
transforms the operator \(\hat{K}\) as follows

\[ \hat{k} = \hat{U}^{\dagger} \hat{K} \hat{U} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \] (12)

As a consequence, by transforming \(\hat{H}\) into \(\hat{H} = \hat{U}^{\dagger} \hat{H} \hat{U}\), we obtain a new Hamiltonian \(\hat{H}\) whose matrix form consists of two blocks, one of dimension four, related to the eigenvalue \(-1\) of the new constant of motion \(\hat{K}\), and the other of dimension five related to the eigenvalue \(+1\) of \(\hat{k}\), representing the two orthogonal sub-dynamics. The new Hamiltonian \(\hat{H}\) can be written as

\[ \hat{H} = \hat{P}_1 \hat{H} \hat{P}_1 + \hat{P}_2 \hat{H} \hat{P}_2, \] (13)

where we introduced the hermitian operator \(\hat{P}_1 (\hat{P}_2)\) projecting a generic state of the total Hilbert space \(\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2\) into the \(\hat{K}\)-invariant subspace \(\mathcal{H}_1 (\mathcal{H}_2)\) relative to its eigenvalue \(-1\) \((+1)\) such that \(\hat{P}_1 \hat{H} \hat{P}_1 (\hat{P}_2 \hat{H} \hat{P}_2)\) consists of the upper (lower) block of \(\hat{H}\), or better in a matrix with the same dimension (nine) of \(\hat{H}\), but with nonvanishing entries only in the upper (lower) four- (five-) dimensional block.

It is worth noticing that the arguments leading to the possibility of representing the Hamiltonian in accordance with equation (13) hold their validity even for a more general Hamiltonian model \(\hat{H}_{\text{gen}}\) obtainable from \(\hat{H}\) adding terms commuting with \(\hat{k}\), e.g. \((\hat{\Sigma}_4)^2\), \(\hat{\Sigma}_4^x \hat{\Sigma}_4^y \hat{\Sigma}_4^z\), and \(\hat{\Sigma}_4^x \hat{\Sigma}_4^y \hat{\Sigma}_4^z \hat{\Sigma}_4^z\).

\[ \hat{H}_{\text{gen}} = \hat{H} + \text{terms commuting with } \hat{k}. \] (14)

However, we confine ourselves to the Hamiltonian model (3) since it is comparatively more accessible in the laboratory, and in addition, as we will show in the following sections, it generates interesting quantum dynamical behaviour.

Setting
\[
\begin{align*}
\Omega_+ &= \omega_1 + \omega_2, \\
\Omega_- &= \omega_1 - \omega_2, \\
\gamma_1 &= \gamma_1 - i(\gamma_{xy} + \gamma_{yx}), \\
\gamma_2 &= \gamma_1 + i(\gamma_{xy} - \gamma_{yx}),
\end{align*}
\]

(15)

the 4 × 4 block reads

\[
\hat{H}_L \equiv \begin{pmatrix}
\hbar\omega_1 & \gamma_2 & \gamma_1 & 0 \\
\gamma_2^* & \hbar\omega_2 & 0 & \gamma_1 \\
\gamma_1^* & 0 & -\hbar\omega_2 & \gamma_2 \\
0 & \gamma_1^* & \gamma_2^* & -\hbar\omega_1
\end{pmatrix}
\]

(16)

The four states of the original basis are

\[
|e_1\rangle = |10\rangle, |e_2\rangle = |01\rangle, |e_3\rangle = |0 - 1\rangle, |e_4\rangle = |-10\rangle.
\]

(17)

The lower block of \(\hat{H}\) is represented by the 5 × 5 matrix

\[
\hat{H}_L \equiv \begin{pmatrix}
\hbar\Omega_+ + \gamma_1 & 0 & \gamma_1 & 0 & 0 \\
0 & \hbar\Omega_- - \gamma_1 & \gamma_2 & 0 & 0 \\
\gamma_1^* & \gamma_2^* & 0 & \gamma_1 & \gamma_2 \\
0 & 0 & \gamma_1^* & -\hbar\Omega_- - \gamma_1 & 0 \\
0 & 0 & \gamma_2^* & 0 & -\hbar\Omega_+ + \gamma_1
\end{pmatrix}
\]

(18)

where the five states of the original basis are

\[
|e_5\rangle = |11\rangle, |e_6\rangle = |1 - 1\rangle, |e_7\rangle = |00\rangle, \\
|e_8\rangle = |-11\rangle, |e_9\rangle = |-1 - 1\rangle.
\]

(19)

Equation (13) implies that the quantum dynamics of two qutrits interacting according to the model of equation (3), factorises into an effective spin-3/2 system and an effective spin-2 system.

We note that the mathematical steps leading from equation (3) to equation (13) reproduce analogous results, even if we use, \textit{mutatis mutandis}, the same Hamiltonian model where qudits systematically substitute the appearing qutrits. Of course, the dimensions of the dynamically invariant subspaces existing in the case of the qudits strictly depends on the dimension of their Hilbert space.

In the next section, we will show that in the case of qutrits a further aspect of such reducibility of the quantum dynamics of the system emerges, with physically transparent and far-reaching consequences.

4. Four-dimensional sub-dynamics

The eigenvectors of the Hamiltonian \(\hat{H}_L\) (16) may be exactly derived by solving the fourth degree relative secular equation, obtaining \(|\psi\rangle = \sum_{k=1}^{4} \alpha_{k} |e_k\rangle\), where the coefficients \(\alpha_{k}\) are given in appendix A in equations (A.1)–(A.4). The corresponding eigenvalues are

\[
\mathcal{E}_1 = E_1 + E_2, \quad \mathcal{E}_2 = E_1 - E_2, \quad \mathcal{E}_3 = -\mathcal{E}_2, \quad \mathcal{E}_4 = -\mathcal{E}_6.
\]

(20)
where
\[
E_1 = \sqrt{\frac{(\hbar \Omega \gamma)^2}{4} + |\gamma|^2},
\]
\[
E_2 = \sqrt{\frac{(\hbar \Omega \gamma)^2}{4} + |\gamma|^2}.
\]

The four eigenvalues of \(\tilde{\hat{H}}_{-}\), in view of equation (20), may be obtained by summing elements of the two pairs \([E_1, -E_1]\) and \([E_2, -E_2]\) in all possible ways. This circumstance hints that the quantum dynamics of the two qutrits restricted to the four-dimensional Hilbert subspace generated by \(|e_k\rangle\) with \(k = 1, 2, 3, 4\), is traceable to that of two effective noninteracting spin-1/2 systems, respectively described by two bi-dimensional traceless Hamiltonians \(\hat{H}_1\) and \(\hat{H}_2\), with the eigenvalues \(\pm E_1\) and \(\pm E_2\).

To verify this intuition, we search for a mapping between the two qutrit original basis states in (17) and the two spin-1/2 basis, that is \(\{|++\rangle, |+\rangle, |-+\rangle, |--\rangle\}\), in accordance with which the generic eigenstate \(|\psi\rangle\) of \(\tilde{\hat{H}}_{-}\) may be represented as a tensorial product between an eigenstate of \(\hat{H}_1\) and an eigenstate of \(\hat{H}_2\). Such a mapping consists simply of
\[
|10\rangle \leftrightarrow |++\rangle,
|01\rangle \leftrightarrow |+-\rangle,
|0 - 1\rangle \leftrightarrow |-+\rangle,
|-10\rangle \leftrightarrow |--\rangle,
\]
where we define the effective spin-1/2 states as \(\sigma_i^\pm = \pm i \sigma_i\) with \(i = 1, 2\). Indeed, it is straightforward to show that the sub-dynamics of the two spin-1 systems, which interact according to (3), and are related to the \(\vec{K}\)-invariant subspace of dimension four, characterised by the eigenvalue \(\tilde{\gamma} = -\tilde{K}\), may be reinterpreted as the dynamics of two decoupled effective spin-1/2 systems. Indeed, we can write \(\tilde{\hat{H}}_{-}\) as
\[
\tilde{\hat{H}}_{-} = \hat{H}_1 \otimes \hat{1}_2 + \hat{1}_1 \otimes \hat{H}_2,
\]
where we define
\[
\hat{H}_1 = \frac{\hbar (\omega_1 + \omega_2)}{2} \sigma_1^z + (\gamma_x - \gamma_y) \sigma_1^y + (\gamma_y + \gamma_x) \sigma_1^x,
\]
\[
\hat{H}_2 = \frac{\hbar (\omega_1 - \omega_2)}{2} \sigma_2^z + (\gamma_x + \gamma_y) \sigma_2^y - (\gamma_y - \gamma_x) \sigma_2^x.
\]

The physical interpretation of this sub-dynamics in terms of two spin-1/2 systems is clear and direct: \(\hat{H}_1\) (\(\hat{H}_2\)) describes a fictitious spin-1/2 system immersed in an effective magnetic field \(B_1^{\text{eff}}\) (\(B_2^{\text{eff}}\)) expressible as
\[
B_1^{\text{eff}} = \left(\gamma_x - \gamma_y, (\gamma_y + \gamma_x), \frac{\hbar \mu}{2} (g_1 B_1^i + g_2 B_2^i)\right),
\]
\[
B_2^{\text{eff}} = \left(\gamma_x + \gamma_y, (\gamma_y - \gamma_x), \frac{\hbar \mu}{2} (g_1 B_1^i - g_2 B_2^i)\right).
\]
such that we have $\hat{H}_- = \sum_{i=1}^2 \hat{\sigma}_i \cdot \mathbf{B}^{\text{eff}}$. 

Since $\hat{H}_-$ of equation (23) describes two decoupled spin-1/2 systems, the eigenvectors of $\hat{H}_-$ may be written in the following factorised form

$$
\hat{H}_- \rightarrow \begin{cases}
|\psi_1\rangle \otimes |\psi_{21}\rangle \rightarrow |\psi_1\rangle, \\
|\psi_1\rangle \otimes |\psi_{22}\rangle \rightarrow |\psi_2\rangle, \\
|\psi_{12}\rangle \otimes |\psi_{21}\rangle \rightarrow |\psi_3\rangle, \\
|\psi_{12}\rangle \otimes |\psi_{22}\rangle \rightarrow |\psi_4\rangle,
\end{cases}
$$

(26)

where \{ |\psi_{11}\rangle, |\psi_{12}\rangle \} (\{ |\psi_{21}\rangle, |\psi_{22}\rangle \}) are the eigenvectors of $\hat{H}_1$ ($\hat{H}_2$) given explicitly in appendix A. The corresponding eigenenergies for each state are given by equation (20).

We emphasise that two qutrit systems may be prepared in a state whose evolution is dominated by one admissible Bohr frequency, which is only exactly mappable in the time evolution of a single spin-1/2 system subjected to an appropriate magnetic field (see equation (25)). In other words, the quantum dynamics of two qutrits generated by the Hamiltonian (3) possesses the symmetry properties which lead to such peculiar dynamical behaviour. Finally, we note that the unitary operator $\hat{U}$ which transforms $\hat{H}$ into the direct sum of $\hat{H}_-$ and $\hat{H}_+$ is independent of time might provide significant advantages, in view of [31], even when $\hat{H}$ is time-dependent—at least in its 4D dynamically invariant subspace—having demonstrated that the quantum dynamics induced by $\hat{H}_-$ may be traced back to that of two effective spin-1/2 systems.

4.1. States with a specific structure invariant in time

The odd-parity subspace of both $\hat{H}$ and $\hat{H}_\text{gen}$ of equation (14) is spanned by the eigenvectors \{ |10\rangle, |01\rangle, |0 - 1\rangle, | - 10\rangle \} of $\hat{\Sigma}_z^{\text{tot}}$. Inside such a subspace, the two spin-1 systems may be prepared in the following normalised generic superposition of two (generic as well) eigenstates of $\hat{\Sigma}_z^{\text{tot}}$ of eigenvalue $M = \pm 1$, which peculiarly share the same pair of amplitudes in the respective two sub-bases \{ |10\rangle, |01\rangle \} and \{ |0 - 1\rangle, | - 10\rangle \}, namely

$$
|\Psi\rangle = a|c|10\rangle + d|01\rangle + b|c|0 - 1\rangle + d| - 10\rangle,
$$

(27)

where the four complex amplitudes $ac, ad, bc, bd$ fulfil the normalisation condition

$$
(|a|^2 + |b|^2)(|c|^2 + |d|^2) = 1.
$$

(28)

Equations (27) and (28) individuate a proper subclass of states (initial conditions) sharing a characterising expansion structure in the standard basis of the odd-parity subspace.

If the quantum dynamics of the two coupled spin-1 systems is governed by a Hamiltonian model $\hat{H}_\text{gen}$, the initial state given by equation (27) would evolve assuming the form

$$
|\Psi(t)\rangle = a(t)c(t)|10\rangle + d(t)|01\rangle + b(t)c'(t)|0 - 1\rangle + d'(t)| - 10\rangle,
$$

(29)

with $a(0) = a$, $b(0) = b$, $c(0) = c'$, $d(0) = d$ and where, in general, $c(t) \neq c'(t)$ and $d(t) \neq d'(t)$. Stated another way, it is legitimate to claim that under $\hat{H}_\text{gen}$ the initial state of the two spin-1 systems $|\Psi(0)\rangle$ evolves without preserving its initial structure. This result is not surprising in view of the fact that the four eigenvectors of $\hat{\Sigma}_z^{\text{tot}}$ involved in the initial state $|\Psi(0)\rangle$ are not eigenstates of $\hat{H}_\text{gen}$. On this basis, it appears rather unexpected that
the time evolution of $|\Psi(0)\rangle$ determined by the Hamiltonian model (3) adopted in this paper imposes the special condition $c(t) = c'(t)$ and $d(t) = d'(t)$ at any time instant $t$, namely

$$|\Psi(t)\rangle = a(t)[c(t)|10\rangle + d(t)|01\rangle]$$

$$+ b(t)[c(t)|01\rangle - 1 + d(t)|-10\rangle].$$

This is indeed an interesting result, meaning that there is a sort of time invariance of the structure imposed on the initial state given by equation (27). In the appendix B, we report the explicit expressions of the coefficients $a(t)$, $b(t)$, $c(t)$ and $d(t)$.

The properties possessed by our Hamiltonian model and brought to light in the introduction of this section provide the basis for an easy interpretation of the result of equation (30), making transparent how the existence of effective spin-1/2 sub-dynamics is reflected in the quantum dynamics of two spin-1 systems. Using the mapping expressed by equation (22), the initial state $|\Psi(0)\rangle$ is indeed immediately seen to correspond to the following factorised initial state of the two fictitious spin-1/2 systems

$$|\Psi = |\ +| \ - \rangle \otimes |\ +| \ - \rangle$$

(31)

In view of equation (23), it is then easy to deduce that this state evolves into the state

$$|\Psi = |\ +| \ + \rangle \otimes |\ +| \ - \rangle$$

(32)

clearly keeping its initial factorisation at any time instant $t$.

Looking at equation (28), one might wonder whether the state of each effective spin-1/2 system should be subjected to its own normalisation condition, that is, whether one should require $|a|^2 + |b|^2 = 1$ together with $|c|^2 + |d|^2 = 1$. The answer is negative since the only probabilities we are indeed interested in, which are of experimental meaning for our system, are the joint probabilities relative to the two spin-1/2 systems. This consideration of course complies with equation (28).

In order to better appreciate and strengthen the interplay between the quantum dynamics of the two spin-1 systems and that of the two spin-1/2 systems in the odd-parity subspace of the total Hilbert space $H$, we now evaluate and discuss the time dependence of the mean value of some exemplary and transparent physical observables of the two spin-1 systems on the state $|\Psi(t)\rangle$ given in equation (30).

It is possible to demonstrate that at any time instant $t$

$$\langle \Psi(t)|\hat{S}_z^1 + \hat{S}_z^2|\Psi(t)\rangle = \hbar \frac{|a(t)|^2 - |b(t)|^2}{|a(t)|^2 + |b(t)|^2},$$

(33a)

$$\langle \Psi(t)|\hat{S}_z^1 - \hat{S}_z^2|\Psi(t)\rangle = \hbar \frac{|c(t)|^2 - |d(t)|^2}{|c(t)|^2 + |d(t)|^2},$$

(33b)

where $a(t)$, $b(t)$, $c(t)$, $d(t)$ are given in appendix B. The right-hand-side expressions of these two equations suggest an interpretation in terms of the mean values of the appropriate ‘physical observables’ related to the two spin-1/2 systems on the state $|\Psi(t)\rangle$ given in (32). It is indeed easy to persuade oneself that

$$\frac{|a(t)|^2 - |b(t)|^2}{|a(t)|^2 + |b(t)|^2} = \langle \Psi(t)|\hat{S}_z^1|\Psi(t)\rangle,$$

(34a)
\[ \frac{|c(t)|^2 - |d(t)|^2}{|c(t)|^2 + |d(t)|^2} = (\bar{\Psi}(t)|\hat{\Sigma}_2|\Psi(t)). \quad (34b) \]

Equation (34a) clearly discloses that the temporal behaviour of the total magnetisation of the compound spin-1 systems is entirely traceable back to the time dependence of the \( z \)-component of the first fictitious spin-1/2 systems. This asymmetry may be fully understood with the help of equation (30) observing that \( a(t) \) and \( b(t) \) are proportional to the time-dependent amplitudes measuring the maximum and the minimum admissible value of \( \hat{S}^z \equiv \hat{S}^z_1 + \hat{S}^z_2 \), respectively, provided one takes equation (28) appropriately into account. Equation (34b) transparently relates the time dependence of \( \hat{S}^z_1 - \hat{S}^z_2 \) to the time evolution of the mean value of the \( z \)-component of the second fictitious spin-1/2 system. It is possible to capture the origin of such a result changing the role of \( a(t) \) and \( b(t) \) with that of \( c(t) \) and \( d(t) \), respectively, which amounts to rewriting equation (30) as follows:

\[
|\bar{\Psi}(t)| = c(t)[a(t)|10\rangle + b(t)|0\rangle|0 - 1\rangle] + d(t)[a(t)|01\rangle + b(t)|10\rangle|0\rangle]. \quad (35)
\]

Applying to this equation the same arguments used for interpreting equation (34a), we can easily appreciate the interplay between the mean value of \( \hat{S}^z_1 - \hat{S}^z_2 \) and that of \( \sigma_z \) at any time instant.

It is worth noting that when the amplitudes \( a \) and \( b \) (\( c \) and \( d \)), appearing in equation (31), are fixed in such a way that \( |\bar{\Psi}(0)\rangle \) is an eigenstate of \( \hat{H}_1 \otimes \mathbb{I}_2 \) (\( \mathbb{I}_1 \otimes \hat{H}_2 \)), then the mean value of \( \hat{S}^z_1(t) + \hat{S}^z_2(t) \) \( (\hat{S}^z_1(t) - \hat{S}^z_2(t)) \) does not evolve in time, even if the mean values of \( \hat{S}^z_1(t) \) and \( \hat{S}^z_2(t) \) do (unless this special choice is simultaneously made for both amplitude pairs \( (a, b) \) and \( (c, d) \)). In addition, we emphasise that once more, as a consequence of the sub-dynamics exhibited by \( \hat{H} \) in the odd-parity 4D subspace, the mean value of the magnetisation of the compound system exhibits sinusoidal oscillations at the frequency \( \frac{2\pi}{T} \) where \( T \) is given by equation (21a). This behaviour is directly related to equations (33a) and (34a). In view of equations (27), (28) and (31), this occurs for any initial state \( |\Psi(0)\rangle \) of the two spin-1 systems, such that \( |\bar{\Psi}(0)\rangle \) is not an eigenstate of \( \hat{H}_1 \otimes \mathbb{I}_2 \).

In order to illustrate the time dependence of the magnetisation, let us assume for simplicity that the initial state is given by equation (27) characterised by equal amplitudes, namely

\[ a(0) = b(0) = c(0) = d(0) = \frac{1}{\sqrt{2}}. \quad (36) \]

Under such a condition it is easy to get

\[ \langle \hat{S}^z(t) \rangle = \mathcal{A}(\rho_t) \cos \left( \frac{1 + \rho_t^2}{\rho_t} - \phi(\rho_t) \right) + C(\rho_t), \quad (37) \]

with
where $\epsilon_j$ and $\gamma_j$ ($j = 1, 2$) are defined in equations (A.2) and (15), respectively.

We note that in view of equations (37) and (38), and also taking into account the $\rho_1$ dependence of $\epsilon_j$, upon increasing the external parameter $\Omega = \omega_1 + \omega_2$, which amounts to appropriately acting upon the magnitudes of the two magnetic fields $B_{z1}$ and $B_{z2}$, the amplitude of the magnetisation oscillations increases from 0 to $\gamma_1$ when $\rho_1$ goes from 0 to 1, then decreases and asymptotically vanishes for large $\rho_1$. At the same time, the frequency of these oscillations goes down to its minimum value of 2 (the adimensional frequency with respect to the adimensional $\tau$) and then increases asymptotically as $\rho_1$. It is worth noticing that both the amplitude and the frequency of the magnetisation are invariant under the change of $\rho_1$ with $1/\rho_1$.

Looking at $\phi(\rho_1)$ we may instead easily deduce that under the same change of $\rho_1$, this phase constant undergoes a change of $\pi$.

In figure 1 these features characterising the time behaviour of the magnetisation are plotted against the dimensionless time $\tau$ for the following exemplary values of $\rho_1$, while keeping $\gamma_1$ invariant:

$$\rho_1 = 10, 1, 0.1; \quad \text{Re}[\gamma_1] = \frac{3}{5} |\gamma_1|.$$  

(39)
The features exhibited by the time evolution of $\langle \hat{S}^z \rangle$ may be physically understood by tracing back to the coincidence existing between such time behaviour and that of the mean value of the Pauli matrix $\hat{\sigma}^z_1$ relative to the first of the two fictitious spin-1/2s, as clearly expressed by equation (34a). It is indeed possible to demonstrate that by changing the magnetic field $\hat{B}_1^{\text{eff}} \equiv (\hat{B}_{1x}^{\text{eff}}, \hat{B}_{1y}^{\text{eff}}, \hat{B}_{1z}^{\text{eff}})$ to the related magnetic field

$$\tilde{\hat{B}}_1 \equiv \left( \sqrt{\frac{B_{1x}^{\text{eff}} - |\hat{B}_1^{\text{eff}}|}{2}}, \sqrt{\frac{B_{1z}^{\text{eff}} - |\hat{B}_1^{\text{eff}}|}{2}}, \sqrt{(B_{1x}^{\text{eff}})^2 + (B_{1y}^{\text{eff}})^2 - |\hat{B}_1^{\text{eff}}|} \right),$$

(40)

realises the change from $\rho$ into $\rho_1^*$ in physical terms. At the same time, it is possible to show that the first fictitious spin-1/2 driven by $\hat{B}_1^{\text{eff}}$ exhibits a sinusoidal time evolution for the mean value of $\hat{\sigma}^z_1$, coincident with what the first spin-1/2 would have under $\hat{B}_1^{\text{eff}}$, except for the emergence of a difference of phase of $\pi$. Figure 2 instead reports the time behaviour of $\langle \hat{S}_1^z(t) \rangle$ and $\langle \hat{S}_2^z(t) \rangle$, assuming that the two spin-1 systems are initially prepared in the following state

$$\epsilon_1(|10\rangle + |01\rangle) + \gamma_1(|0 - 1\rangle + |1 - 0\rangle) \sqrt{2(\epsilon_1^2 + |\gamma_1|^2)},$$

(41)

that is, in terms of the two spin-1/2s

$$|\psi_{11}\rangle \otimes \frac{|+\rangle_2 + |-\rangle_2}{\sqrt{2}}.$$

(42)

The quantity $\epsilon_1$ and the eigenstate $|\psi_{11}\rangle$ of $\hat{H}_1$ are defined in appendix B.

Figure 2 displays the time independence of the magnetisation in the evolution of the system from its initial state (41) together with the time dependence of $\langle \hat{S}^z_1(t) \rangle$ and $\langle \hat{S}^z_2(t) \rangle$, which manifests clearly that the initial state is not a stationary state of $\hat{H}$. The time invariance of $\langle \hat{S}^z_1(t) \rangle$ is certainly traceable back to the stationarity of the first fictitious spin-1/2 system.
5. The negativity of the two qutrits in their four-dimensional subspace

In this section, we wish to study the negativity, introduced by Vidal and Werner in [26], possessed by the two qutrit system in a generic pure state, belonging to the four-dimensional dynamically invariant subspace \( \mathcal{H} \), as well as investigate the emergence of a peculiar behaviour in the time evolution of such a parameter, which is adoptable to measure the entanglement established between the two spins. The negativity of a bi-partite system constituting two qudits, whose individual Hilbert spaces are of the dimension \( d_1 \) and \( d_2 \) respectively, may be defined, for both pure and mixed states [26, 27], as

\[
\mathcal{N}_\rho = \frac{\|\hat{\rho}_T\| - 1}{d - 1},
\]

where \( d = \min\{d_1, d_2\} \) and \( \hat{\rho}_T \) is the partial transpose of the matrix \( \rho \) representing the state of the total system \( (A + B) \) with respect to the subsystem \( B \). The symbol \( \|\| \) is the trace norm, which for a hermitian matrix, is nothing but the sum of the absolute values of its eigenvalues. As a consequence, the negativity of the state \( \rho \) is simply the sum of the absolute values of the negative eigenvalues of \( \hat{\rho}_T \), which is hermitian, such that \( \text{Tr}(\hat{\rho}_T) = 1 \). The value of \( \mathcal{N}_\rho \) ranges from 0 to 1 [30] and is independent of the factorised orthonormal basis which we choose to represent the matrix \( \hat{\rho} \). In addition, \( \mathcal{N}_\rho \) is independent of the subsystem with respect to which we choose to calculate the partial transpose, given the properties \( \rho \rho = \rho_T \rho_T \) and \( \|X\| = \|X_T\| \) for any operator \( X \).

A generic pure state \( \hat{\rho} = |\Psi\rangle \langle \Psi| \) belonging to \( \mathcal{H} \) may be expanded as \( |\Psi\rangle = \sum_{k=1}^{d_1} c_k |\psi_k\rangle \) \( (\sum_{k=1}^{d_1} |c_k|^2 = 1) \) in view of equation (17). The corresponding eigenvalues of \( \hat{\rho}_T \) are

\[
\Upsilon_1 = 1 - x, \quad \Upsilon_2 = x, \quad \Upsilon_3 = \sqrt{x(1-x)}, \quad \Upsilon_4 = -\Upsilon_3,
\]

with \( x = |c_1|^2 + |c_4|^2 \). Therefore, the negativity of a generic pure state can be written as

\[
\mathcal{N}_\rho = \sqrt{x(1-x)},
\]

which is well defined (since \( x \in [0, 1] \)) and reaches its maximum value \( \mathcal{N}_\rho^{\text{max}} = 1/2 \) at \( x = 1/2 \). Thus, in the four-dimensional dynamically invariant subspace of \( \hat{H} \), the negativity exhibited by the two coupled qutrits in a pure state reaches 1/2 as an upper limit. Consequently, the negativity of the two qutrits, since a generic mixed state \( \hat{\rho} = \sum_{n=1}^{d} p_n |\psi_n\rangle \langle \psi_n| \) with \( |\psi_n\rangle \) in \( \mathcal{H} \) and \( \sum_{n=1}^{d} p_n = 1, p_n \geq 0 \), possesses the same upper bound 1/2, since [26]

\[
\mathcal{N}_\rho \left( \sum_{\nu} |\psi_\nu\rangle \langle \psi_\nu| \right) \leq \sum_{\nu} p_{\nu} \mathcal{N}_\rho (|\psi_\nu\rangle \langle \psi_\nu|) \leq \frac{1}{2}.
\]

The existence of a such an upper limit is directly traceable to the easily demonstrable circumstance in which every pure state in \( \mathcal{H} \) possesses a Schmidt decomposition, with at most two nonvanishing Schmidt coefficients, namely \( k_1 \) and \( k_2 \), expressible as

\[
k_1 = \sqrt{|c_1|^2 + |c_4|^2}, \quad k_2 = \sqrt{|c_2|^2 + |c_3|^2}, \quad k_3 = 0.
\]

When \( k_1 k_2 > 0 \) the concurrence \( (C(|\psi\rangle)) \) of the two qutrits obtained by Cereceda [28], on the lines of the \( I \)-concurrence first introduced in [29], reaches its maximum value of 3/2. Since in such a case [30]

\[
C(|\psi\rangle) = \sqrt{3. \mathcal{N}(|\psi\rangle)},
\]

\[C(|\psi\rangle) = \sqrt{3. \mathcal{N}(|\psi\rangle)},\]
an upper bound for the negativity equal to 1/2 emerges in accordance with our previous conclusion. Thus, no pure state in $\mathcal{H}$ exhibits maximum entanglement ($C(|\psi\rangle) = 1$).

It is possible to show that the generic normalised entangled state of the two qutrits in $\mathcal{H}$, which saturates the negativity at the value $N = \frac{1}{2}$, up to a global phase factor, may be parametrically represented as

$$
|\Psi\rangle = \frac{1}{\sqrt{2}}\left\{ (\cos(\theta)|1\rangle + e^{i\phi}\sin(\theta)|-1\rangle)_{1} \otimes |0\rangle_{2} + e^{i\phi}|0\rangle_{1} \otimes (\cos(\theta')|1\rangle + e^{i\phi'}\sin(\theta')|1\rangle)_{2}\right\},
$$

(49)

where $\theta, \theta', \phi, \phi'$ and $\Phi$ freely run in $[0, 2\pi]$.

We stress that the existence of the upper bound 1/2 for the negativity of a generic pure state belonging to $\mathcal{H}$ cannot be extended to $\mathcal{H'}$. It is easy to persuade oneself of this proposition, considering the pure normalised state

$$
\frac{1}{\sqrt{3}}(|11\rangle + |00\rangle + |1-1\rangle) \equiv \tilde{k}_{0}|11\rangle + \tilde{k}_{0}|00\rangle + \tilde{k}_{-1}|1-1\rangle - 1 - 1).
$$

(50)

Since it is in the Schmidt form, we can write its negativity as follows [28, 30]:

$$
\mathcal{N}_{\tilde{\rho}} = \tilde{k}_{0} + \tilde{k}_{0} + \tilde{k}_{0} = 1,
$$

(51)

which means maximum entanglement.

We point out the possibility that in the parameter space of the Hamiltonian model given by equation (3) there may exist specific examples, built by diminishing the number of independent parameters appearing in $\hat{H}$, with the eigenvectors in $\mathcal{H}$ belonging to the class of states expressed by equation (49). We do not investigate this possibility in its generality, confining ourselves to a noticeable example whose four eigenvectors in $\mathcal{H}$ are all in states of maximum (1/2) negativity. To this end, we claim that when $\omega_{1} = \omega_{2}$ (homogeneous magnetic field) and $\gamma_{0} = \gamma_{2}$ (when the $C_{2}$-symmetric tensor $D_{12}$ is symmetric), the four eigenvectors in $\mathcal{H}$ of the corresponding Hamiltonian model exhibit $\mathcal{N}_{\tilde{\rho}} = \frac{1}{2}$, meaning that each of them may be written in the form (49) where $\theta = \theta', \phi$ and $\Phi$ are appropriate expressions of the parameters in $\hat{H}$.

5.1. The pure states of $\mathcal{H}$ which saturate $N_{\tilde{\rho}}$ and evolve with one Bohr frequency only

In this section, we are going to investigate the negativity of a special class of pure states of $\mathcal{H}$ by exploiting the advantages stemming from the possibility of describing the four-dimensional dynamics of the two spin-1 systems in terms of two decoupled spin-1/2s. We concentrate on the set of pure states of $\mathcal{H}$ whose evolution is dominated by one admissible Bohr frequency only. This set consists of any pure state expressible as a linear superposition of just two eigenstates of $\hat{H}$ in equation (16). Exploiting the mapping postulated by equation (22) we are allowed to use the language of the two fictitious spin-1/2s to fully characterise this set of pure states of the two spin-1s. To this end, it is useful to consider two classes of states. The first one simply encompasses nonstationary, normalised and factorised states of the two spin-1/2s, wherein one and only one of them is stationary, that is

$$
|\psi_{1}\rangle \otimes (\zeta_{1}^{+})_{2} + \zeta_{1}^{-}(-_{2}),
$$

$$
|\psi_{2}\rangle \otimes (\zeta_{1}^{+})_{2} + \zeta_{1}^{-}(-_{2}),
$$

$$
(\zeta_{1}^{+})_{1} + \zeta_{1}^{-}(-_{1}) \otimes |\psi_{2}\rangle,
$$

(52)

$$
(\zeta_{1}^{+})_{1} + \zeta_{1}^{-}(-_{1}) \otimes |\psi_{2}\rangle,
$$
where \( \xi \) and \( \zeta \) are complex coefficients fulfilling the normalisation condition for the two spin-1/2 states. Besides the states described by equation (52) there exist nonfactorisable, normalised states of the two fictitious spin-1/2s, also generating the states of two spin-1s, whose evolution is once more only dominated by an admissible Bohr frequency. This second class may be represented as follows:

\[
\psi_{11}\psi_{21} + \psi_{12}\psi_{22},
\]

\[
\psi_{11}\psi_{22} + \psi_{12}\psi_{21},
\]

(53)

with \( a \) and \( b \) satisfying the normalisation condition for the compound system. It is easy to convince oneself that with the help of equations (22), (52) and (53), one can generate all and only the six linear combinations of pairs of stationary states of the two spin-1s, evolving with one Bohr frequency out of a set of only four admissible characteristic Bohr frequencies obtained in accordance with equation (20). Thus, the language of the two spin-1/2s, whose dynamics is governed by the Hamiltonian \( \tilde{\mathcal{H}} \) given by equation (23), provides a simple way for individuating all the states of interest in this section, with the foreseeable advantage of reducing the quantum dynamics of the two qutrits to that of one or two qubits.

We begin analysing the negativity of the states described by equation (52). With the help of equation (22) it is possible to demonstrate that the following condition

\[
|\zeta|^2 = |\xi|^2 = \frac{1}{2},
\]

(54)

characterises the subset of this first class having \( \mathcal{N}_b = 1/2 \). The coefficients \( \gamma_1 \) and \( \gamma_2 \) are defined in equation (15), while \( \epsilon_j \) may be expressed in terms of the coupling constants and the frequencies appearing in the Hamiltonian model (3) as follows

\[
\epsilon_j = \frac{\hbar(\omega_1 - (-1)^j\omega_2)}{2} + E_j.
\]

(55)

The subset fulfilling condition (54) may be represented by the following four ( \( j = 1, 2 \) ) parametric (\( \Phi \in [0, 2\pi] \)) states of the two spin-1 systems

\[
|\Psi_{1i}\rangle = \frac{\epsilon_j((0) + e^{i\Phi}(1)) + \gamma((0 - 1) + e^{i\Phi}(1))}{\sqrt{2(\epsilon_j^2 + |\gamma|^2)}}.
\]

(56a)
Figure 4. The plot of $x(t) + 1/2$ (blue dotted line) and $\langle \Psi(t) | \Psi(t) \rangle$ (green dashed-dotted line) where $|\Psi(t)\rangle$ is the initial condition corresponding to $|\Psi_0\rangle$ in (56a), $\Phi = 0$ and $\langle \Psi(t) | \Psi(t) \rangle$ is the related evolved state. The plot shows that at the time instants $\tau_2 = k\pi (k = 0, 1, 2, \ldots)$ the two spin-1 systems come back to their initial condition.

Figure 5. The plot of $x(t) + 1/2$ (blue dotted line), $\langle \Psi(t) | \Psi(t) \rangle$ (green dashed-dotted line) and $\langle \Phi | \Psi(t) | \Phi \rangle$ (red dashed line) where $|\Psi(t)\rangle$ is the initial condition corresponding to $|\Psi_1\rangle$ in (56a) with $\Phi = 0$, $|\Psi(t)\rangle$ is the related evolved state and $|\Psi(t) | \Phi \rangle$ is the state corresponding to $|\Psi(t)\rangle$ in (56a) with $\Phi = 0.28$. The plot shows that the second point in which $x(t) + 1/2 = 0$ corresponds to the state $|\Psi(t)\rangle$.

$$
|\Psi_{12}\rangle = \frac{\gamma_3(|10\rangle + e^{i\phi}|01\rangle) - c_\epsilon(|01\rangle + e^{i\phi}|10\rangle)}{\sqrt{2(\epsilon^2 + |\gamma|^2)}}.
$$

(56b)
We emphasise that the existence of a maximum for the negativity of a pure state belonging to $\mathcal{H}$ is a property of such a subspace, meaning that a pure state which cannot be expanded in the basis given in equation (17) might possess negativity higher than 1/2, which is what happens, for example, to the state in equation (50), orthogonal to $\mathcal{H}$. The time evolution of the negativity of the two spin-1 systems prepared in the state $|\Psi_{11}\rangle$ with $\Phi=0$ is reported in figure 3, and exhibits a double periodic return to the condition of maximum negativity.

To interpret this behaviour, in figure 4 we plot $x(t) + 1/2$ simultaneously with $|\Psi_{11}\rangle$, providing evidence of the periodic return of the system to its initial condition, as well as the periodic involvement of another state of maximum negativity.

It is possible to show that under the condition represented in figure 3, the state of $\mathcal{M}_{\rho} = 1/2$, which is different from the initial one periodically reached by the system, is of the form (56a) with $\rho \simeq 0.28$, as we can see in figure 5.

Besides reaching the maximum admissible value of negativity, the states belonging to the first class parametrically expressed by equation (52) cannot go below a minimum nonvanishing value of the same negativity. This can easily be calculated as

$$N_{\rho min} = \frac{\epsilon_j |\gamma_j|}{\epsilon_j + |\gamma_j|^2} = \frac{\rho_j}{1 + \rho_j^2},$$

with $\rho_j = \frac{\gamma_j}{\epsilon_j} (j = 1, 2)$.

This minimum value is assumed in correspondence with the following eight states of the two spin-1/2s.

$|\Psi_{21}\rangle = \frac{\gamma^2_1 |0\rangle + e^{i\phi} |0 - 1\rangle + \gamma^2_2 |01\rangle + e^{i\phi} |00\rangle}{\sqrt{2(\gamma^2_1 + \gamma^2_2)}},$

$$|\Psi_{22}\rangle = \frac{\gamma^2_1 |0\rangle + e^{i\phi} |0 - 1\rangle - \gamma^2_2 |01\rangle + e^{i\phi} |00\rangle}{\sqrt{2(\gamma^2_1 + \gamma^2_2)}}. \quad (56c)$$

$|\Psi_{22}\rangle = \frac{\gamma^2_1 |0\rangle + e^{i\phi} |0 - 1\rangle - \gamma^2_2 |01\rangle + e^{i\phi} |00\rangle}{\sqrt{2(\gamma^2_1 + \gamma^2_2)}}. \quad (56d)$
\[ |\psi_1\rangle \otimes |+\rangle, \quad |\psi_2\rangle \otimes |-\rangle, \quad (58a) \]
\[ |+\rangle_1 \otimes |\psi_2\rangle, \quad |-\rangle_1 \otimes |\psi_2\rangle, \quad (58b) \]

with \( j = 1, 2 \).

Generating these states of minimum negativity from the set described by equation (52), the condition on \( \zeta \) and \( \xi \) may easily be expressed in the form \( \zeta \xi = 0 \), provided \( |\zeta|^2 + |\xi|^2 = 1 \).

Summing up, the negativity of a generic pure state belonging to the class given by equation (52) possesses an upper bound of less than 1 (precisely 1/2) and a lower bound, which is strictly positive, given by equation (57).

The amplitude of the negativity oscillations exhibited by \( |\Psi_1\rangle \) is generally \( \Phi \)-dependent. One might then wonder whether there exist values of \( \Phi \) for which such an amplitude reaches its maximum possible value: \( \frac{1}{2} - \frac{\sqrt{\gamma}}{\sqrt{|c|^2 + |d|^2}} \) or in other terms, whether the evolved state coincides, up to a global phase factor, with the state \( |\psi_{11} \rangle \otimes |\sigma\rangle \) with \( |\sigma\rangle = |+\rangle_2 \) or \( |\sigma\rangle = |-\rangle_2 \), in accordance with equation (58). We find that the positive or negative answer to such a question depends on the appropriate choice of the region of the parameter space related to the Hamiltonian model depending on the initial state.

To appreciate this point, we start from the particular couple of states
\[ |\psi_{11}\rangle \otimes |\pm\rangle_2, \quad (59) \]
with minimum negativity and sharing the stationary state of the first fictitious spin 1/2.

We now seek the times at which the negativity restores its initial value as well as the times at which it reaches its maximum value of 1/2. Since we are considering pure states evolving with one Bohr frequency only, the function \( x(t) = |c(t)|^2 + |d(t)|^2 \) is periodic; we are then sure that the equation \( x(t) = x(0) \) admits infinitely many solutions. This conclusion is valid in every point of the parameter space of the Hamiltonian model.

Concerning whether the system reaches states of maximum negativity we solve the equation \( x(t) = 1/2 \), which assumes the same simple form for both states (59)
\[ \sin^2(\rho_2) = \frac{(1 + \rho_2^2)}{8\rho_2^2} \equiv |s|^2, \quad (60) \]
with \( \rho_2 = E / 8 \) and \( \rho_2 = \frac{\zeta}{|\zeta|^2} \). We note that in view of equation (55), the adimensional parameter \( \rho_2 \) is strictly positive, and that equation (60) is solvable only in the \( \rho_2 \)-interval \([ \sqrt{2} - 1, \sqrt{2} + 1 ] \). Thus, only when the two spin-1s are in a well-confined region of the parameter space of the Hamiltonian (3) does the time evolution of the states (59) exhibit negativity oscillations of period \( \pi \), from their common \( \rho_2 \)-dependent minimum value to their maximum value (1/2)—as displayed in figures 6–8—where the negativity of both states \( |\psi_{11}\rangle \otimes |\pm\rangle \) is reported for exemplary different values of \( \rho_2 \). In these plots we fix \( \rho_1 = 1 + \sqrt{2} \) determining the same minimum value of the negativity.

The time interval between the two time instants at which \( x(t) = 1/2 \) in any complete oscillation is \( \pi - 2 \arccos(|s|) \), whereas the relative midpoints are \( \pi/2 + k\pi \), with \( k = 0, 1, 2, \ldots \). It is worth noticing the disappearance of the secondary minima when \( \rho_2 \) assumes its highest
possible value of $1 + \sqrt{2}$. Moreover, we observe from figures 6 and 7 that the closer $\rho_2$ is to 1, the deeper the secondary minima at the midpoints ($\pi/2 + k\pi$, $k = 0, 1, 2, \ldots$) are. Figure 8 shows that when $\rho_2 = 1$, the two spin-1 systems recover their initial negativity with periodicity $\pi/2$. It is possible to show that with periodicity $\pi$, the two spin-1 systems come back to their initial state too, and that under the special condition $\rho_2 = 1$, the state of minimum negativity reached at $\pi/2$ is $|\psi_{11}\rangle \otimes |\pm\rangle$ (with $j = 1, 2$) of the two fictitious spin-1/2s; the parameter space region is identified by $\rho_1 = 1 + \sqrt{2}$ and $\rho_2 = 1$.

It is of relevance to observe that the negativity of states (59) for any fixed value of $\rho_1$ is invariant under the change of $\rho_2$ with $1/\rho_2$ and of $\rho_2$ with $1/\rho_2$. As a consequence, plots six coincide with the one referring to $\rho_1 = \sqrt{2} - 1$ and $\rho_2 = 1/2$. This characteristic invariance of the negativity time evolution is a consequence of the way that $\hat{H}$ changes under the canonical transformation $\Sigma_i^+ \rightarrow \Sigma_i^+ \Sigma_i^0 \rightarrow -\Sigma_i^0 \Sigma_i^+ \rightarrow -\Sigma_i^+$, describing a rotation of $\pi$ around the $x$-axis.

Had we started from a pair of states of minimum negativity $|\psi_{11}\rangle \otimes |\pm\rangle$, we should have derived exactly the same equation (60), as well as figures coincident with figures 6–8.

It is worth noticing that analogous considerations may be developed for the other four states given in equation (58b) with comparable conclusions concerning the time behaviour of the relative negativity. This time, however, $\rho_2$ must be replaced by $\rho_1 = |\alpha|/|\beta|$ and the relevant adimensional time becomes $\tau = \frac{\Delta t}{\pi}$. 

\[\text{Figure 7. The time evolution of } \mathcal{K} \text{ when the two spin-1 systems are initially prepared in the states corresponding to the states } |\psi_{1j}\rangle \otimes |\pm\rangle \text{ (} j = 1, 2 \text{) of the two fictitious spin-1/2s; the parameter space region is identified by } \rho_1 = 1 + \sqrt{2} \text{ and } \rho_2 = 1.\]

\[\text{Figure 8. The time evolution of } \mathcal{K} \text{ when the two spin-1 systems are initially prepared in the states corresponding to the states } |\psi_{1j}\rangle \otimes |\pm\rangle \text{ (} j = 1, 2 \text{) of the two fictitious spin-1/2s; the parameter space region is identified by } \rho_1 = 1 + \sqrt{2} \text{ and } \rho_2 = 1.\]
Figure 9. The time evolution of $\hat{N}_\rho$ when the two spin-1 systems are initially prepared in the states corresponding to the states $|\pm \rangle \otimes |\psi_j \rangle$ ($j = 1, 2$) of the two fictitious spin 1/2s; the parameter space region is identified by $\rho_1 = 1 + \sqrt{2}$ and $\rho_2 \approx 1.7$.

Figure 10. The time evolution of $\hat{N}_\rho$ when the two spin-1 systems are initially prepared in the states corresponding to the states $|\pm \rangle \otimes |\psi_j \rangle$ ($j = 1, 2$) of the two fictitious spin 1/2s; the parameter space region is identified by $\rho_1 = 1 + \sqrt{2}$ and $\rho_2 \approx 1.3$.

Figure 11. The time evolution of $\hat{N}_\rho$ when the two spin-1 systems are initially prepared in the states corresponding to the states $|\pm \rangle \otimes |\psi_j \rangle$ ($j = 1, 2$) of the two fictitious spin 1/2s; the parameter space region is identified by $\rho_1 = 1 + \sqrt{2}$ and $\rho_2 = 1$. 
The time instants at which $N/\rho = \rho_2^{1/2}$ satisfy an equation like (60), mutatis mutandis, and thus the admissible domain for $\rho_2$ coincides with that of $\rho_1$.

We stress that even if the domain of variability of $\rho_1$ and $\rho_2$ is the same, each of them indeed singles out a proper region in the parameter space of $\tilde{H}$. It turns out that these two regions overlap the original Hamiltonian models, whose parameters satisfy the domain conditions for both $\rho_1$ and $\rho_2$. In such a common region of the parameter space, it then happens that all eight states of minimum negativity evolve—as given by equations (58)—exploring the full domain of the negativity compatible with $\tilde{H}$.

In figures 9–11, we plot the time evolution of $\mathcal{N}_\rho(t)$ when the system is initially prepared in the states $|\pm\rangle \otimes |\psi_2\rangle$ ($j = 1, 2$) assuming $\rho_1 = 1 + \sqrt{2}$ and the same three values for $\rho_2$ as given before. We notice that in a different way from figures 6–11, the three cases display a $\rho_2$-dependent level of the minimum negativity, and the disappearance of all the secondary minima since $\rho_1 = 1 + \sqrt{2}$.

Figure 11 shows the remarkable invariance of negativity with time when $\rho_2 = 1$. This fact may be immediately understood observing that in view of equation (57), $\mathcal{N}^{\min}_\rho = \mathcal{N}^{\max}_\rho = 1/2$ for all four states $|\pm\rangle \otimes |\psi_2\rangle$ ($j = 1, 2$). We emphasise the time invariance of negativity against the nonstationarity of these initial states.

6. Five-dimensional dynamics

As shown in section 3, the quantum dynamics of the two coupled spin-1 systems is reducible to two quantum sub-dynamics, with the first one described by equation (16) and second one by equation (18). This lucky mathematical occurrence, which leads us to trace back the quantum dynamics of the two spin-1 systems to that of two noninteracting spin-1/2 systems in the four-dimensional invariant subspace, cannot emerge in the other invariant subspace, essentially because dimension five is a prime number. Then, in the spirit of the previous section, the only observation we may make is that in such five-dimensional subspace, the quantum dynamics of the two spin-1 systems may be mapped into that of a spin-2. Unfortunately the effective (through the appropriate mapping) representation of $\tilde{H}_5$ in terms of spin-2 operators is very involved, appears to be strongly nonlinear, and is practically impossible to relate to a convincing physical scenario. This is why we do not proceed further in this direction, confining ourselves to the consideration of particular conditions which easily provide the possibility of extracting the useful properties possessed by our model.

The first aspect relating to the model that deserves attention is that by comparing the reduced matrices given by equations (16) and (18), it is possible to note that the parameter $\gamma_z$ influences the sub-dynamics in the five-dimensional dynamically invariant subspace of $\tilde{H}$ only. As a consequence, we may choose specific values of this parameter without modifying the dynamical properties of the system in the four-dimensional dynamically invariant subspace. It is possible to convince oneself that for $\gamma_z = 0$, the eigensolutions of $\tilde{H}_+ \psi$ may be found exactly, and are given in appendix C.

Furthermore, it is possible to verify that if we assume

$$
\begin{align*}
\gamma_x &= \gamma_y = \gamma \\
\gamma_{xy} &= -\gamma_{yx} = D_z
\end{align*}
$$

(61)
the five-dimensional block is reduced to two one-dimensional blocks and a three-dimensional one, as can be appreciated from what follows:

\[
\begin{pmatrix}
\Omega_+ + \gamma_c & 0 & 0 & 0 & 0 \\
0 & \Omega_- - \gamma_c & 2(\gamma + iD_c) & 0 & 0 \\
0 & 2(\gamma - iD_c) & 0 & 2(\gamma + iD_c) & 0 \\
0 & 0 & 2(\gamma - iD_c) & -\Omega_- - \gamma_c & 0 \\
0 & 0 & 0 & 0 & -\Omega_+ + \gamma_c
\end{pmatrix}
\]

(62)

The previous specific conditions (61) clearly have an interesting physical meaning: the first condition imposes an isotropic XY-exchange interaction, while the second one takes into account the antisymmetric exchange, or the Dzyaloshinskii–Moriya interaction \(D \cdot (\hat{S}_1 \times \hat{S}_2)\) with \(D \equiv (0, 0, D_z)\). This model is well known in the literature and has been studied in connection with the properties of thermal entanglement [22].

It is interesting to point out, moreover, that in this instance, the three-dimensional block may be described in terms of the single spin-1 Pauli operator defined in equation (4), and the relative Hamiltonian precisely reads

\[
\tilde{H}_3 = 2\gamma \hat{S}_x^x - 2D_z \hat{S}_y^y + \Omega_- \hat{S}_z^z - \gamma_c (\hat{S}_z^z)^2.
\]

(63)

We immediately see that putting \(\gamma_c = 0\) we have the SU(2) three-dimensional fictitious sub-dynamics of a single spin-1, subjected to the effective external magnetic field \(B_1 \equiv (2\gamma, 2D_z, \Omega_-)\), so that we may write \(\tilde{H}_3 = \sum B_i \cdot \hat{S}_i\). This observation is particularly significant in light of the interplay between the new results obtained for the SU(2) bi-dimensional time-dependent dynamics [31, 32], with the results reported in [33]. In this way, under the conditions (61) and \(\gamma_c = 0\), we may also study analytically and know exactly the five-dimensional sub-dynamics of the two spin-1 systems in a time-dependent scenario—more precisely when the two magnetic fields are time dependent—so that we have \(\omega_1(t)\) and \(\omega_2(t)\). This target is, however, out of the scope of this paper.

Finally, it is worth pointing out that contrary to the conditions on \(\gamma_c\), the conditions (61) modify the dynamics in the four-dimensional subspace, too. In this instance, indeed, we obtain a four-dimensional sub-dynamics of the two spin-1s which is well described in terms of the two decoupled fictitious spin-1/2s, in which the first spin is only subjected to a magnetic field in the \(z\)-direction, while the second spin is immersed in a magnetic field, with a direction that depends on the coupling parameters of the model. This can be appreciated and easily verified by equations (24), providing the conditions (61). Therefore, under the conditions (61), both the sub-dynamics are exactly treatable—or in other words, the full model may be exactly solved.

7. Conclusive remarks and outlooks

In this paper, we have examined the quantum dynamics of two spin-1s, coupled in accordance with the generalised \(\hat{S}_z^2\)-nonconserving Heisenberg Hamiltonian model \(\hat{H}\) given in equation (3). Such a model encompasses a large variety of physical situations, and its investigation aims to bring to light the existence of static and dynamical properties which will be useful for interpreting the physical behaviour of a pair of two coupled three-state quantum systems, regardless of the specific scenario under scrutiny.
Our first central result is that at each point of the wide parameter space $\hat{H}$, there exist two dynamically invariant subspaces, with four and five dimensions, which are namely the only two $(\pm 1)$-eigenspaces of the parity, and the constant of the motion operator $\hat{K}$ defined in equation (10). This subdivision of the total nine-dimensional Hilbert space of the system stems from the invariance of $\hat{H}$ under a rotation of $\pi$ around the $z$-axis. This reveals that the quantum dynamics of our system breaks into that of two fictitious systems—namely a spin-$3/2$ and a spin-$2$—whose quantum evolutions are generated by effective Hamiltonians acting upon their own Hilbert spaces, and which are isomorphic to the two dynamically invariant subspaces of $\hat{H}$ with parity $-1$ and $+1$ respectively.

A careful analysis of the dynamical properties of our system in its four-dimensional invariant subspace leads to our second nonintuitive and original result. This consists of the fact that the quantum evolution in such a subspace is exactly traceable to that of two noninteracting fictitious spin-$1/2$s, subjected to their own, also fictitious, magnetic fields. Thanks to such a factorisation property, one may thus expect to be in the condition to interpret and foresee the time evolution of the two qutrits, which are prepared in a generic state, and live in the four-dimensional dynamically invariant subspace of $\hat{H}$, thereby exploiting the underlying and simple evaluation of the quantum evolutions of two decoupled fictitious spin-$1/2$s.

This is the reason why most of the paper is dedicated to the investigations of properties exhibited by the system in the invariant odd parity subspace, confining the considerations on the properties of the system evolving in the invariant even-parity subspace to section 6.

We demonstrate that the interplay between the quantum dynamics of the two spin-$1$s and that of the compound system of the two fictitious spin-$1/2$s in the four-dimensional subspace $\mathcal{H}$, under scrutiny, amounts to the emergence of some dynamical constraints on the time evolution of some classes of states and mean values of observables relative to the two qutrit system. Indeed, we show the existence of a class of states of the two qutrits, whose given expansion structure in the standard basis of the odd parity invariant subspace keeps such an initial structure invariant in time. Moreover, we find that the time evolution of the total magnetisation of the two spin-$1$s is expressible in a remarkably simple way in terms of the time-dependent amplitudes of the evolved state with the assigned initial structure. All these results may be easily interpreted, adopting the point of view of the two noninteracting spin-$1/2$s nested in the quantum dynamics of the system restricted in $\mathcal{H}$. Their time evolution offers the key which will lead to a full understanding of the physical origin of the peculiar properties exhibited by our system when it evolves from any state belonging to the class described by equation (27).

To investigate whether and how the underlying sub-dynamics of the two noninteracting spin-$1/2$s impose constraints on the time evolution of the entanglement, which are established between the two qutrits when they are initially prepared in a pure state of $\mathcal{H}$, we have studied the negativity, bringing to light the existence of an upper limit equal to $1/2$. The origin of such a bound is strictly related to the specific four factorised (standard basis) states of the nine-dimensional total Hilbert space of the two spin-$1$s, generating the four-dimensional subspace $\mathcal{H}$. In turn, this fact brings to light the role of the $C_2$-symmetry possessed by $\hat{H}$, responsible for the existence of the two specific dynamically invariant subspaces.

It is of relevance to emphasise that our model could even be extended to include nonlinear terms without breaking such $C_2$-symmetry of the Hamiltonian, thus originating once again in the subdivision of the total Hilbert space found with our model. The circumstance that the restriction of $\hat{H}$ into $\mathcal{H}$ may be mapped into a model of two noninteracting spin-$1/2$s is instead directly linked to the specific model adopted in this paper, meaning that in the presence of the appropriate nonlinear terms, added to our model, the two spin-$1/2$s would interact.
Once more, we may also take advantage of such a mapping from a dynamical point of view, foreseeing the existence of special classes of nonstationary states of the two qutrits which manifest peculiar entanglement properties, both static and dynamical. We are thus led to the four classes of states of $\mathcal{H}$, given in equation (52), evolving with an admissible Bohr frequency, and also showing that each class only possesses a lower nonvanishing bound of negativity besides the common upper bound. The paper reports an analysis revealing the region of the parameter space of $\hat{H}$, wherein a state of minimum negativity evolves which periodically oscillates between this initial value and $1/2$. In particular, we succeed in constructing nonstationary states of $\hat{H}$ whose negativity keeps the value $1/2$ at any time instant.

The advantage of reducing the interacting spin-1 particles to effective spin-1/2 dynamics rests in the possibility of using the vast knowledge of spin-1/2 dynamics in order to describe the more complex spin-1 dynamics. We can indeed simulate the spin-1 dynamics by using spin-1/2 particles in an experiment, as long as we bear in mind the limitations of the initial model, i.e. we cannot simulate the most general spin-1 model, only the special case that allows for this reduction.

In section 6 we have shown that under specific relations among the parameters appearing in $\hat{H}$, the underlying dynamics of the effective spin-2 system can be solved exactly.

We note that when the spin-1 particles are prepared in a superposition of states belonging to different subspaces, the present method allows one to find the evolution in a straightforward manner, because the different components of this superposition would evolve separately. The evolution of the component belonging to the dynamically invariant four-dimensional subspace can be found by using the knowledge of the evolution of the spin-1/2 particles. The evolution of the other component may be calculated when $\gamma_z = 0$, for which the paper reports the exact eigensolutions. We stress that making such a specific choice for $\gamma_z$ has no influence on the dynamical behaviour of the component in $\mathcal{H}$.

Recently, a systematic approach has been reported for generating the exactly solvable quantum dynamics of a single spin-1/2, subjected to a time-dependent magnetic field. The present work then suggests that the appearance of a nested spin-1/2 sub-dynamics in $\mathcal{H}$, which is treatable in the time-dependent case, may lead to the construction of exactly solvable time-dependent scenarios, wherein the two qutrits are subjected to two generally different nonconstant magnetic fields.

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Appendix A. Eigenvectors of the four-dimensional dynamics

The eigenvectors of the four-dimensional block, in terms of the two spin-1 standard basis states are

\[ |\psi_1\rangle = \frac{1}{N}[\epsilon_1|e_2\rangle|10\rangle + \epsilon_1\gamma_2^*|01\rangle + \gamma_1^*|2\rangle|0 - 1\rangle + \gamma_1^*\gamma_2^*| - 10\rangle] \]

(A.1a)

\[ |\psi_2\rangle = \frac{1}{N}[\epsilon_1|e_2\rangle|10\rangle - \epsilon_1|e_2|01\rangle - \gamma_1\gamma_2|0 - 1\rangle - \gamma_1^*| - 10\rangle] \]

(A.1b)

\[ |\psi_3\rangle = \frac{1}{N}[\gamma_1^*|e_2\rangle|10\rangle + \gamma_1^*\gamma_2^*|01\rangle - \epsilon_1|e_2|0 - 1\rangle - \epsilon_1\gamma_1^*| - 10\rangle] \]

(A.1c)

\[ |\psi_4\rangle = \frac{1}{N}[\gamma_1\gamma_2|10\rangle - \gamma_1\gamma_2|01\rangle - \epsilon_1|e_2|0 - 1\rangle + \epsilon_1|e_2| - 10\rangle] \]

(A.1d)

where the quantity \( \epsilon_j \) (\( j = 1, 2 \)) is related to the parameters appearing in the Hamiltonian model (3) as follows:

\[ \epsilon_j = \frac{\hbar(\omega_1 - (-1)^j\omega_2)}{2} + E_j \]

(A.2)

and \( N = N_1N_2 \) is the normalisation factor with

\[ N_1 = \sqrt{\epsilon_1^2 + |\gamma_1|^2} \quad N_2 = \sqrt{\epsilon_2^2 + |\gamma_2|^2} \]

(A.3)

and finally \( \gamma_1 \) and \( \gamma_2 \) are given in equation (15).

In view of the reinterpretation of the four-dimensional sub-dynamics in terms of two decoupled spin-1/2s—as explicitly made in equation (23)—the four eigenvectors of the dynamics under scrutiny may be written as factorised states of the two fictitious spin 1/2s—as given in equation (26)—where the eigenstates of the single spin-1/2 of \( \hat{H}_1 \) (\( |\psi_{11}\rangle \) and \( |\psi_{12}\rangle \)) and \( \hat{H}_2 \) (\( |\psi_{21}\rangle \) and \( |\psi_{22}\rangle \)) are

\[ |\psi_{11}\rangle = \frac{\epsilon_1|+\rangle_1 + \gamma_1^*|\rangle_1}{N_1} \]

(A.4a)

\[ |\psi_{12}\rangle = \frac{\gamma_1|+\rangle_1 - \epsilon_1|\rangle_1}{N_1} \]

(A.4b)

\[ |\psi_{21}\rangle = \frac{\epsilon_2|+\rangle_2 + \gamma_2^*|\rangle_2}{N_2} \]

(A.4c)

\[ |\psi_{22}\rangle = \frac{\gamma_2|+\rangle_2 - \epsilon_2|\rangle_2}{N_2} \]

(A.4d)

Appendix B. The time-dependent coefficients of the specific structure
nonchanging in time

The time-dependent coefficients of the state $|\Psi(t)\rangle$ in (30) take the form

$$
\begin{align*}
 a(t) &= ak_1(t) + bk_1^r(t) \\
 b(t) &= ak_1(t) + bk_1^l(t) \\
 c(t) &= ck_2(t) + dk_2^r(t) \\
 d(t) &= ck_2(t) + dk_2^l(t)
\end{align*}
$$

(B.1)

where

$$
\begin{align*}
 k_j^r(t) &= \frac{\epsilon_j^2 e^{-E_j t} + |\gamma_j|^2 e^{+E_j t}}{\epsilon_j^2 + |\gamma_j|^2} \\
 k_j^l(t) &= -2i\frac{\epsilon_j^2 \gamma_j}{\epsilon_j^2 + |\gamma_j|^2} \sin\left(\frac{E_j}{\hbar} t\right) \\
 k_j^r(t) &= -2i\frac{\epsilon_j^2 \gamma_j}{\epsilon_j^2 + |\gamma_j|^2} \sin\left(\frac{E_j}{\hbar} t\right) \\
 k_j^l(t) &= \frac{|\gamma_j|^2 e^{-2E_j t} + \epsilon_j^2 e^{2E_j t}}{\epsilon_j^2 + |\gamma_j|^2}.
\end{align*}
$$

(B.2a)

(B.2b)

(B.2c)

(B.2d)

Appendix C. The eigenvectors and eigenvalues of the five-dimensional dynamics

For $\gamma = 0$, the secular equation relative to $\hat{H}_\gamma$ becomes a bi-quadratic equation, and so it can be solved exactly. In this instance, the eigenvectors read $(ab \leftrightarrow \pm)$

$$
\begin{align*}
|\psi_{2\gamma}\rangle &= -\gamma|11\rangle - \frac{\Omega_+}{\Omega_-} \gamma_2 |1 - 1\rangle + \hbar \Omega_+ |00\rangle \\
&\quad + \frac{\Omega_+}{\Omega_-} \gamma_2^* |1 - 11\rangle + \gamma_2^* |1 - 1 - 1\rangle \\
|\psi_{6\gamma}\rangle &= \frac{1}{\gamma_1} \left[ (\gamma_\chi - \gamma_\gamma)^2 + (\gamma_\gamma + \gamma_\gamma)^2 \mp \frac{E_0 (\pm \hbar \Omega_+ - \pm \hbar \Omega_- E_0)}{\Omega_+ - \Omega_-} \right] |11\rangle \\
&\quad + \gamma_2 \gamma_2^* \left[ \frac{h \Omega_+ - E_0}{h \Omega_+ - E_0} \right] |1 - 1\rangle + \left( h \Omega_+ \pm E_2 \right) |00\rangle \\
&\quad + \gamma_2 \left( \frac{\pm h \Omega_+ - E_0}{\pm h \Omega_+ - E_0} \right) |1 - 11\rangle + \gamma_2^* |1 - 1 - 1\rangle \\
|\psi_{6\gamma}\rangle &= \left( \frac{\mp \hbar \Omega_+ - E_0}{\pm \hbar \Omega_+ - E_0} \right) |1 - 1\rangle + \left( h \Omega_+ \pm E_2 \right) |00\rangle \\
&\quad + \gamma_2 \left( \frac{\mp h \Omega_+ - E_0}{\mp h \Omega_+ - E_0} \right) |1 - 11\rangle + \gamma_2^* |1 - 1 - 1\rangle \\
\end{align*}
$$

(C.1a)

(C.1b)
\[ |\psi_{S/0}\rangle = -\frac{1}{\gamma_1} \left[ (\gamma_6 - \gamma) + (\gamma_{xy} + \gamma_{y})^2 \mp E_6 (h \Omega_+ \pm E_6) \right. \]

\[ + \frac{\gamma_2 \gamma_2^* (\mp h \Omega_+ - E_6)}{h \Omega_+ - E_6} + \frac{\gamma_2 \gamma_2^* (\pm h \Omega_+ - E_6)}{-h \Omega_+ - E_6} \right] (11) \]

\[ + \frac{\gamma_2 (\mp h \Omega_+ - E_6)}{\pm h \Omega_+ - E_6} (1 - 1) + (h \Omega_+ \pm E_4) |00\rangle \]

\[ + \frac{\gamma_2 (\mp h \Omega_+ - E_6)}{\pm h \Omega_+ - E_6} (-11) + \gamma |1 - 1 - 1\rangle \]  \quad (C.1c)

with the relative eigenvalues

\[ E_5 = 0 \]  \quad (C.2a)

\[ E_6 = -\left\{ \hbar^2 (\omega_1^2 + \omega_2^2) + 2 \left[ \gamma_2^2 + \gamma_2^* \right] - \left( \hbar^2 \omega_1^2 \right) + 4 \hbar^2 \omega_2 (\gamma_6 + \gamma_{xy} + \gamma_{y}) + \left( \gamma_2^2 + \gamma_2^* \right) \right\}^{1/2} \]  \quad (C.2b)

\[ E_7 = -E_6 \]  \quad (C.2c)

\[ E_8 = -\left\{ \hbar^2 (\omega_1^2 + \omega_2^2) + 2 \left[ \gamma_2^2 + \gamma_2^* \right] + \left( \hbar^2 \omega_2 \right) + 4 \hbar^2 \omega_2 (\gamma_6 + \gamma_{xy} + \gamma_{y}) + \left( \gamma_2^2 + \gamma_2^* \right) \right\}^{1/2} \]  \quad (C.2d)

\[ E_9 = -E_8 \]  \quad (C.2e)

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