HYBRID SUBCONVEXITY AND THE PARTITION FUNCTION

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Abstract. We give an upper bound for the error term in the Hardy-Ramanujan-Rademacher formula for the partition function. The main input is a new hybrid subconvexity bound for the central value \( L(1/2, f \times (q \cdot s)) \) in the \( q \) and spectral parameter aspects, where \( f \) is a Hecke-Maass cusp form for \( \Gamma_0(N) \) and \( q \) is a fundamental discriminant.

1. Introduction

Of the many fruits of the famous collaboration of Hardy and Ramanujan, the invention of the circle method is among the most lasting and influential. Their first application of the circle method was the discovery in 1918 of an asymptotic formula for the partition function \( p(n) \) which is precise enough to compute the exact value of \( p(n) \) when \( n \) is sufficiently large. The Hardy-Ramanujan formula is, in the notation of [14],

\[
p(n) = \sum_{q=1}^{\nu} A_q(n) \phi_q(n) + O(\alpha n^{-1/4}), \quad \nu = \lfloor \alpha \sqrt{n} \rfloor,
\]

where \( \alpha \) is any fixed positive real number, \( A_q(n) \) is a finite sum of \( 24q \)-th roots of unity, and

\[
\phi_q(n) = \sqrt{\left( \exp(C \lambda_n/q) \right) / \lambda_n}, \quad C = \pi \sqrt{2/3}, \quad \lambda_n = \sqrt{n - 1/24}.
\]

Since \( p(n) \) is an integer and the error term tends to zero as \( n \to \infty \), the exact value of \( p(n) \) is the nearest integer to the sum in (1.1) for all sufficiently large \( n \). However, the series obtained by replacing \( \nu \) by \( \infty \) in (1.1) diverges because \( \phi_q(n) \gg n^{1/2} \) and for each \( n \) there are infinitely many \( q \) for which \( A_q(n) \gg \sqrt{q} \) (see [18] for the latter fact).

In 1936, by carefully refining the contour used in the Hardy-Ramanujan circle method, Rademacher [20] showed that a slight modification of the sum in (1.1) leads to an absolutely convergent infinite series whose value equals \( p(n) \). In fact, the only modification required is replacing the exponential function in the definition of \( \phi_q(n) \) by the hyperbolic sine function, that is,

\[
p(n) = \sum_{q=1}^{\infty} A_q(n) \tilde{\phi}_q(n), \quad \tilde{\phi}_q(n) = \frac{\sqrt{q}}{\pi \sqrt{2} \sqrt{n}} \frac{d}{dn} \left( \sinh(C \lambda_n/q) \right),
\]

Suppose that we truncate Rademacher’s series at \( q = \nu \), as in (1.1), and write

\[
p(n) = \sum_{q=1}^{\nu} A_q(n) \tilde{\phi}_q(n) + R(n, \nu).
\]

How fast does \( R(n, \nu) \) decay as \( n \to \infty \)? Since \( \tilde{\phi}_q(n) \) is exponentially large when \( q \ll \sqrt{n} \), it makes sense to set \( \nu = \lfloor \alpha \sqrt{n} \rfloor \) for some \( \alpha > 0 \), as in (1.1). Rademacher showed in [20] that \( R(n, \alpha \sqrt{n}) \ll n^{-1/4} \), which matches the error term in (1.1). It is apparent that any improvement to this bound requires a careful study of the \( A_q(n) \), which are generalized Kloosterman sums given by

\[
A_q(n) = \sqrt{-i} \sum_{d \mod q \atop (d, q) = 1} \tilde{\nu}_q \left( \left( \begin{array}{cc} a & \ast \\ q & d \end{array} \right) \right) e \left( \frac{a + d}{24q} - \frac{nd}{q} \right),
\]
where \( ad \equiv 1 \pmod q \), \( e(x) = e^{2\pi i x} \), and \( \nu_q : \text{SL}_2(\mathbb{Z}) \to \mathbb{C} \) is the multiplier system for the Dedekind eta function given by
\[
\nu_q \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \frac{\eta\left( \frac{az + b}{cz + d} \right)}{\sqrt{cz + d} \, \eta(z)}, \quad \eta(z) = e\left( \frac{1}{24} z \right) \prod_{k=1}^{\infty} (1 - e(kz)).
\]

A priori, \(|A_q(n)| \leq q\), which is enough to show that the series \( \sum q^{1/2} \) converges absolutely since we have \( \phi_q(n) \ll q^{5/2} \) for \( q \gg \sqrt{n} \). In 1937, Lehmer [18] showed that
\[
|A_q(n)| < 2^{\omega_\alpha(q)} \sqrt{q},
\]
where \( \omega_\alpha(q) \) is the number of distinct odd primes dividing \( q \). He concluded [19] that
\[
R(n, \alpha \sqrt{n}) \ll_\alpha n^{-1/2} \log n.
\]

This upper bound is essentially best possible if one estimates the sum of the absolute values of the terms in \( R(n, \nu) \). Any improvement must take into account the sign changes of the \( A_q(n) \). Lehmer’s record held until 2010, when Folsom and Masri [13] broke the \(-1/2\) exponent barrier when \( 24n - 1 \) is squarefree. This exponent was improved in 2015 by the first author and Ahlgren [3], then again in 2018 by Ahlgren and Dunn [4], who proved that for all \( n \) and for all \( \epsilon > 0 \) we have
\[
R(n, \alpha \sqrt{n}) \ll_{\alpha, \epsilon} n^{-1/2 - 1/147 + \epsilon}.
\]

The main result of this paper is an improvement on this upper bound.

**Theorem 1.1.** Write \( 24n - 1 = tu^2 \), where \( t \) is squarefree. Then
\[
R(n, \alpha \sqrt{n}) \ll_{\alpha, \epsilon} t^{-1/36} \omega^{-1/6} n^{-1/2 + \epsilon}.
\]

The upper bound in Theorem 1.1 follows from an estimate (see (2.10) below) for the weighted sums of Kloosterman sums
\[
\sum_{q \leq x} A_q(n) q^{-1}.
\]

Lehmer’s inequality (1.3) yields the trivial upper bound \( \sqrt{x} \log x \) for the sum (1.4). We will give an improved bound in the critical range \( x \asymp \sqrt{n} \). In Section 2 we relate, via a Kuznetsov trace formula, the sum (1.4) to a sum of coefficients of Maass cusp forms \( u_j \) of half-integral weight whose transformation law involves the eta multiplier \( \nu_j \). This closely follows the arguments in [3] and [4]. We depart from the arguments in [3] and [4] at Proposition 2.2 below, which is a Waldspurger-type formula relating the cusp form coefficient \( \mu_j(d) \) to the central value \( L\left( \frac{1}{2}, \varphi \times \chi_d \right) \), where \( \varphi \) is the Shimura lift of \( u_j \) associated to the discriminant \( d \) (see Section 3 for notation and details). Then in Section 4 using a version of Motohashi’s formula proved by the second author in [21], we prove the hybrid subconvexity bound
\[
L\left( \frac{1}{2}, \varphi \times \chi_d \right) \ll \epsilon \left( |\epsilon||d|^N \right)^{\frac{1}{2}} \left( \log(\text{lcm}(|d|, N)) \right)^{\frac{1}{2}},
\]
where \( N \) is the level of \( \varphi \) and \( r \) is the spectral parameter of \( \varphi \). This generalizes a result of Young [24] to arbitrary level \( N \). (Note that (1.5) is not subconvex in the \( N \) aspect, but it is in the \( d \) and \( r \) aspects.) This leads to our bound (2.10) from which Theorem 1.1 follows quickly.

### 2. Sums of Kloosterman sums

In this section we outline the proof of Theorem 1.1 postponing many of the details to later sections. In what follows, we will use the same setup and notation as [3], with the notable exception that we are normalizing the Maass form coefficients differently (see (2.11) below); for more details, consult that paper, especially Section 2. Through the rest of the paper, we will encounter the Bessel functions \( I_{\nu} \), \( J_{\nu} \), and \( K_{\nu} \), and the Whittaker functions \( M_{\mu, \nu} \) and \( W_{\mu, \nu} \). Definitions and properties of these functions can be found in Sections 10 and 13 of [10].
Let $S_k(N, \nu, r)$ denote the vector space of Maass cusp forms of weight $k$ on $\Gamma_0(N)$ with multiplier system $\nu$ and spectral parameter $r$. Each function $u \in S_k(N, \nu, r)$ satisfies

$$u \left( \frac{az + b}{cz + d} \right) = \nu(\gamma) \left( \frac{cz + d}{|cz + d|} \right)^k u(z) \quad \text{for all } \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(N)$$

and

$$-\Delta_k u := (-y^2 \partial_x^2 + \partial_y^2) + iky \partial_x u = \left( \frac{4}{y^4} + 2y^2 \right) u$$

and vanishes at the cusps of $\Gamma_0(N)$. We will be primarily interested in the space $S_{1/2}(1, \nu, r)$. It is shown in Section 5 of [3] that $S_{1/2}(1, \nu, r) = \{0\}$ unless $r = i/4$ or $r > 0$. Let $S_{1/2}^r(1, \nu, r)$ denote the span of the union of all of the spaces $S_{1/2}(1, \nu, r)$ for $r > 0$ and let $\{u_j\}_{j \geq 1}$ denote an orthonormal basis for $S_{1/2}^r(1, \nu, r)$ with $r_j$ the spectral parameter of $u_j$. For convenience in working with the Hecke operators, we will normalize the Fourier coefficients of $u_j$ by

$$u_j(z) = \sum_{n \equiv 1 \pmod{24}} \mu_j(n) W_{\frac{1}{2} \nu n(n), r_j} \left( \frac{z}{6} n | y \right) e \left( \frac{1}{24} n x \right). \tag{2.1}$$

These coefficients are related to the $\rho_j(n)$ in [3] by $\rho_j(n) = \mu_j(24n - 23)$.

Theorem 4.1 of [3] is a Kuznetsov trace formula for the sums $A_c(n)$ since $A_c(n) = \sqrt{-1} S(1, 1 - n, c, \chi)$ in the notation of that paper. To state the formula, we first let $\phi : [0, \infty) \to \mathbb{C}$ be a four times continuously differentiable function satisfying

$$\phi(0) = \phi'(0) = 0, \quad \phi^{(j)}(x) \ll x^{-2-\varepsilon} \quad (j = 0, \ldots, 4) \quad \text{as } x \to \infty,$$

and let $\tilde{\phi}$ denote the integral transform

$$\tilde{\phi}(r) = \cosh(\pi r) \int_0^\infty K_{2ir}(u) \phi(u) u^{-1} du.$$  

Then for $n \geq 1$ we have

$$\sum_{c=1}^\infty \frac{A_c(n)}{c} \phi \left( \frac{\pi \sqrt{24n - 1}}{6c} \right) = \frac{1}{8} \sqrt{24n - 1} \sum_{r_j \geq 0} \frac{\mu_j(1) \mu_j(1 - 24n)}{\cosh \pi r_j} \tilde{\phi}(r_j).$$

We choose $\phi = \phi_{a, x, T}$ as in Section 6 of [3], where $a = \frac{n}{8} \sqrt{24n - 1}$ and $T = x^{1/3}$. Breaking the spectral sum into three ranges as in the proof of [3, Proposition 9.2], we find that

$$\sum_{x \leq n \leq 2x} \frac{A_c(n)}{c} \ll x^{4/6} \log x + M_1 + M_2 + \sum_{\ell = 0}^\infty M_3 \left( \frac{\ell x}{x} \right),$$

where

$$M_1 = \sqrt{24n - 1} \sum_{0 < r_j < a/8x} \frac{\mu_j(1) \mu_j(1 - 24n)}{\cosh(\pi r_j)} r_j^{-3/2} e^{-r_j/2},$$

$$M_2 = x \sum_{a/8x \leq r_j < a/x} \frac{\mu_j(1) \mu_j(1 - 24n)}{\cosh(\pi r_j)},$$

$$M_3(A) = \sqrt{24n - 1} A^{3/2} \min \left( 1, \frac{x^{1/3}}{A} \right) \sum_{A \leq r_j < 2A} \frac{|\mu_j(1) \mu_j(1 - 24n)|}{\cosh(\pi r_j)}.$$

After applying the Cauchy-Schwarz inequality to each sum above, we see that we need an upper bound for the sums

$$\sum_{r_j \leq x} \frac{|\mu_j(1)|^2}{\cosh(\pi r_j)} \quad \text{and} \quad (24n - 1) \sum_{r_j \leq x} \frac{|\mu_j(1 - 24n)|^2}{\cosh(\pi r_j)} \tag{2.2}$$
for all \( n \geq 1 \). One estimate is provided by Theorem 4.1 of [7]: for all \( m \neq 0 \) we have
\[
|m| \sum_{r_j \leq x} \left| \mu_j(24m - 23) \right|^2 \lesssim x^{-\text{sgn}(m)/2} \left( x^2 + |m|^{1/2} \right) (|m|x)^{\varepsilon}.
\]

This result uses (1.3) and the fact that the negatively indexed Fourier coefficients in weight 1/2 are positively indexed Fourier coefficients in weight \(-1/2\) (see (2.17) of [3]). When \( x \) is large compared with \( |m| \), say \( x \gg |m|^{1/4} \) the estimate (2.3) is essentially sharp, so it suffices as an estimate for the first sum in (2.2). We require an estimate in the complementary range \( x \ll |m|^{1/4} \) for the second sum. It will depend on the factorization of \( 1 - 24n \); for the remainder of the section, write
\[
1 - 24n = dw^2, \quad \text{where} \quad d \equiv 1 \pmod{24} \quad \text{is a negative fundamental discriminant.}
\]

There are Hecke operators \( T_p^\alpha \) on the spaces \( S_{1/2}(1, \nu_\eta, r) \) for primes \( p \geq 5 \) (see Section 2.6 of [3]); they act on Fourier expansions via
\[
T_p^\alpha u_j(z) = \sum_{n=1}^{\infty} \left( p\mu_j(p^2n) + p^{-1/2} \left( \frac{12n}{p} \right) \mu_j(n) + p^{-1}\mu_j(n/p^2) \right) W_{\frac{1}{2}\text{sgn}(n), ir_j} \left( \frac{x}{n} |n/y| e(\frac{1}{24}nx) \right),
\]

where \( \mu_j(n/p^2) = 0 \) if \( p^2 \nmid n \). We can choose our basis \( \{ u_j \} \) such that each \( u_j \) is a an eigenform of \( T_p^\alpha \) for all \( p \geq 5 \), i.e.
\[
T_p^\alpha u_j = \lambda_j(p) u_j.
\]

If \( \mu_j(d) = 0 \) then (2.4) implies that \( \mu_j(dw^2) = 0 \) for all \( w \). Since \( u_j \neq 0 \) there exists a squarefree \( d \) such that \( \mu_j(d) \neq 0 \); in what follows we will assume that \( d \) is chosen with this property.

Our next aim is to construct the Shimura lift \( S_d(u_j) \) associated to the fundamental discriminant \( d \); for \( d > 0 \) this is done in Section 5 of [3], but here we need it for \( d < 0 \). Define numbers \( a(n) \) by the formal identity
\[
\mu_j(d) \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = L(s + \frac{1}{2}, \chi_d) \sum_{m=1}^{\infty} \left( \frac{12}{m} \right) \frac{\mu_j(dm^2)}{m^{s-1}},
\]

where \( \chi_d = (\frac{d}{\cdot}) \), and let
\[
v(z) = S_d(u_j)(z) = 2\sqrt{y} \sum_{n \neq 0} a(|n|) K_{2ir_j}(2\pi |n|y) e(nx).
\]

Then \( v \) satisfies \( T_p v = \left( \frac{12}{p} \right) \chi_0(p)v \) for all \( p \geq 5 \), where \( T_p \) is the usual Hecke operator in weight 0 that acts on Fourier expansions as
\[
(T_p v)(z) = 2y^{1/2} \sum_{n \neq 0} (a(|n|) + \chi_0(p)p^{-1}a(n/p)) K_{2ir_j}(2\pi |n|y) e(nx)
\]

(see [8] Corollary 5.2 for a proof of this; note that the assumption \( d > 0 \) is not used in the proof). Here \( \chi_0 \) is the trivial character modulo 6. Thus \( a(p) = \left( \frac{12}{p} \right) \chi_0(p) \). We claim that \( v \) is also an eigenform of \( T_2 \) and \( T_3 \). Let \( p \in \{ 2, 3 \} \). Then by (2.5) we have \( a(p^e) = p^{-\ell/2} \). If \( n = p^e n' \) with \( p \nmid n' \) then (2.5) yields
\[
a(pn) = \chi_0(p)\left( \frac{12}{m} \right) \frac{m^{\ell+1}}{\sqrt{m}} \mu_j(dm^2)
= p^{-(\ell+1)/2}a(n') = p^{-1/2}a(p^e)a(n') = p^{-1/2}a(n).
\]

Thus \( T_p v = p^{-1/2}v \) for \( p \in \{ 2, 3 \} \). It remains to show that \( v \) is actually a Maass cusp form of weight 0.

**Proposition 2.1.** Let \( d \equiv 1 \pmod{24} \) be negative and squarefree. Then \( S_d(u_j) \in S_0(6, \chi_0, 2r_j) \). Furthermore, \( S_d(u_j) \) has eigenvalue \(-1\) under the Atkin-Lehner involutions \( W_2 \) and \( W_3 \).
We will prove Proposition 2.1 and define \( W_2 \) and \( W_3 \) in the next section.

The Shimura lift allows us to estimate \( \mu_j(dw^2) \) in terms of \( \mu_j(d) \). Suppose that \( (w, 6) = 1 \). By (2.6) we have

\[
w\mu_j(dw^2) = \left( \frac{12}{w} \right) \mu_j(d) \sum_{\ell | w} \mu(\ell) \chi_d(\ell) \ell^{-1/2} a(n) \ll |\mu_j(d)| w^\varepsilon \max_n |a(n)|.
\]

A result of Kim and Sarnak [17] gives \( |a(n)| \ll |n|^{\theta + \varepsilon} \) where \( \theta = \frac{7}{64} \). It follows that for \( n \geq 1 \) we have

\[
(2.7) \quad (24n - 1) \sum_{r_j \leq x} |\mu_j(1 - 24n)|^2 \cosh(\pi r_j) \ll |dw|^{2\theta + \varepsilon} \sum_{r_j \leq x} |\mu_j(d)|^2 \cosh(\pi r_j),
\]

where \( 1 - 24n = dw^2 \).

For each \( j \), choose an orthogonal basis \( B_j \) of the subspace of \( S_0(6, \chi_0, 2r_j) \) of forms with eigenvalue \(-1\) under \( W_2 \) and \( W_3 \), where each element of \( B_j \) is an eigenform for all \( T_p \). We arithmetically normalize each \( \varphi \in B_j \) so that \( a(1) = 1 \). Then

\[
\sum_{r_j \leq x} |\mu_j(d)|^2 \cosh(\pi r_j) \leq \sum_{r_j \leq x} \frac{1}{\cosh(\pi r_j)} \sum_{\varphi \in B_j} \sum_{\mu_j(d) = \varphi} |\mu_j(d)|^2.
\]

In the next section we will prove the following formula.

**Proposition 2.2.** Suppose that \( d \equiv 1 \pmod{24} \) is negative. Then for each \( \varphi \in B_j \) we have

\[
(2.8) \quad \frac{\pi}{3} |d| \sum_{S_\mu(u_j) = \varphi} |\mu_j(d)|^2 = \langle \varphi, \varphi \rangle^{-1} \left| \Gamma\left( \frac{3}{4} + ir_j \right) \right|^2 L\left( \frac{1}{2}, \varphi \times \chi_d \right),
\]

where \( L(s, \varphi \times \chi_d) \) is the analytic continuation of the \( L \)-function

\[
L(s, \varphi \times \chi_d) = \sum_{n=1}^{\infty} \frac{a(n) \chi_d(n)}{n^s}.
\]

Corollary 0.3 of [17] gives the upper bound \( \langle \varphi, \varphi \rangle^{-1} \ll (1 + r_j)^{\varepsilon} e^{2\pi r_j} \). So by Stirling’s formula we have

\[
\frac{|d|}{\cosh(\pi r_j)} \sum_{S_\mu(u_j) = \varphi} |\mu_j(d)|^2 \ll (1 + r_j)^{\frac{3}{2} + \varepsilon} L\left( \frac{1}{2}, \varphi \times \chi_d \right).
\]

Thus

\[
|d| \sum_{r_j \leq x} |\mu_j(d)|^2 \cosh(\pi r_j) \ll \sum_{r_j \leq x} (1 + r_j)^{\frac{3}{2} + \varepsilon} L\left( \frac{1}{2}, \varphi \times \chi_d \right).
\]

In Section 4 we prove a hybrid subconvexity bound, uniform in both \( d \) and \( r_j \), for the central \( L \)-values appearing above. Theorem 4.1 together with H"{o}lder's inequality, shows that

\[
(2.9) \quad |d| \sum_{0 < r_j < x} |\mu_j(d)|^2 \cosh(\pi r_j) \ll |d|^{\frac{3}{2} + \varepsilon} (|d| x^2)^\varepsilon.
\]

We return to the sums \( M_\ell \) for \( \ell = 1, 2, 3 \) above. By (2.8), (2.7), and the Cauchy-Schwarz inequality, we find that

\[
\sqrt{24n - 1} \sum_{0 < r_j < x} |\mu_j(1)\mu_j(1 - 24n)| \cosh(\pi r_j) \ll x^{3/4} \left( x^{5/4} + \min \left( |dw|^2 x^{-1/4}, x^{5/4} |d|^{1/6} u^\theta \right) \right) (nx)^\varepsilon.
\]

For \( M_1 \) and \( M_2 \) we use that \( \min(a, b) \leq b \) to see that

\[
M_1 \ll \sqrt{24n - 1} \sum_{T=0}^{\infty} e^{-T/2} \sum_{T < r_j < T+1} |\mu_j(1)\mu_j(1 - n)| \cosh(\pi r_j) \ll |d|^{1/6 + \varepsilon} w^{\theta + \varepsilon},
\]

\[\text{An analogous formula (which can be proved in a similar way) holds when } d \text{ is positive, but we will not need it here.}\]
and
\[ M_2 \ll x^{-1}|d|^{2/3}w^{1+\theta}(nx)^\varepsilon. \]

For \( M_3(A) \) we use that \( \min(a, b) \leq a^{2/5}b^{3/5} \) to get
\[ M_3(A) \ll \min(1, A^{-1}x^{1/3}) \left( A^{1/2} + |d|^{1/5}w^{(1+3\theta)/5} \right) (nA)^\varepsilon, \]
from which it follows that
\[ \sum_{\ell=0}^\infty M_3 \left( \frac{\ell q}{x} \right) \ll \left( x^{1/6} + |d|^{1/5}w^{(1+3\theta)/5} \right) (nx)^\varepsilon. \]

Thus we have
\[ \sum_{x \leq c \leq 2x} \frac{A_c(n)}{c} \ll \left( x^{1/6} + |d|^{1/5}w^{(1+3\theta)/5} + x^{-1}|d|^{2/3}w^{1+\theta} \right) (nx)^\varepsilon. \]

Applying Lehmer’s bound [13] in the range \( c \leq |d|^a w^\beta \) we find that
\[ \sum_{c \leq x} \frac{A_c(n)}{c} \ll \sum_{c \leq |d|^a w^\beta} \frac{A_c(n)}{c} + \sum_{c > |d|^a w^\beta} \frac{A_c(n)}{c} \ll \left( |d|^{a/2}w^{\beta/2} + x^{1/6} + |d|^{1/5}w^{(1+3\theta)/5} + |d|^{2/3-a}w^{1-\beta+\theta} \right) (nx)^\varepsilon. \]

We choose \( a = 4/9 \) and \( \beta = 2/3 \) to balance the first and third terms and this yields
\[ (2.10) \quad \sum_{c \leq x} \frac{A_c(n)}{c} \ll \left( x^{1/6} + |d|^{2/9}w^{1/3} \right) (nx)^\varepsilon. \]

Arguing as in Section 10 of [3] we conclude that
\[ R(n, \alpha \sqrt{n}) \ll \alpha |d|^{-19/36+\varepsilon}w^{-7/6+\varepsilon}. \]

This completes the proof of Theorem [11].

3. The Shimura Lift and the Waldspurger Formula

In this section we prove Propositions [2, 2] and [22]. We will follow the arguments given in Sections 8–10 of [11] with modifications following the ideas in [3, 6].

We first need Poincaré series of weight 0 for \( \Gamma_6 = \Gamma_0(6)/\{\pm I\} \). Let \( \Gamma_\infty = \{ \pm \left( \begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix} \right) \; : \; n \in \mathbb{Z} \} \). For any \( m \in \mathbb{Z} \) and for \( \text{Re}(s) > 1 \) define
\[ F_m(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_6} f_m(\gamma z, s), \]
where
\[ f_m(z, s) = \begin{cases} y^s & \text{if } m = 0, \\ \frac{\Gamma(s)}{2\pi \sqrt{|m|} \Gamma(2s)} M_{0, s-\frac{1}{2}}(4\pi |m| y e(m x)) & \text{if } m \neq 0. \end{cases} \]

The function \( F_0(z, s) \) is the nonholomorphic Eisenstein series for \( \Gamma_6 \) (see [16, Chapter 15]). For \( m \neq 0 \) we have the following analogue of Proposition 3 of [11]. It can be proved similarly, with only very minor modifications.

**Proposition 3.1.** For \( r \geq 0 \) let \( B_r \) denote a Hecke-Maass orthogonal basis of the finite-dimensional vector space \( S_0(6, \chi_0, r) \). For \( \varphi \in B_r \), write
\[ \varphi(z) = 2\sqrt{7} \sum_{m \neq 0} a_\varphi(m) K_{ir}(2\pi |m| y e(mx)). \]
Then \( F_m(z, s) \) has a meromorphic continuation to \( \text{Re}(s) > 0 \) and

\[
\text{Res}_{s = \frac{1}{2} + ir} (2s - 1) F_m(z, s) = 2 \sum_{\varphi \in B_r} (\varphi, \varphi)^{-1} a_{\varphi}(m) \varphi(z).
\]

For each \( \ell \mid 6 \) let \( W_{\ell} \) denote any matrix with determinant \( \ell \) of the form \( W_{\ell} = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \) with \( a, b, c, d \in \mathbb{Z} \). Then the map \( \varphi(z) \mapsto \varphi(W_{\ell}z) \) is an Atkin-Lehner involution on \( S_6(\chi, \gamma) \) which does not depend on the choice of \( a, b, c, d \). It is convenient to choose

\[
W_1 = \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right), \quad W_2 = \left( \begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix} \right), \quad W_3 = \left( \begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix} \right), \quad W_6 = \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right).
\]

Since the Atkin-Lehner involutions commute with the action of the Hecke operators, we can choose the orthogonal basis \( B_r \) in Proposition 3.1 to have the additional property that each \( \varphi \) satisfies

\[
\varphi(W_{\ell}z) = \alpha(\ell) \varphi(z), \quad \text{where} \quad \alpha(\ell) = \pm 1.
\]

Additionally, since \( \varphi(W_{\ell}W_{\ell'}z) = \varphi(W_{\ell'}z) \), where \( \ell'' = \ell'/(\ell, \ell')^2 \), we see that the only valid sign patterns for \( (\alpha(1), \alpha(2), \alpha(3), \alpha(6)) \) are \((+, +, +, +), (+, -, -, +), \) and \((+, -, -, -)\). It follows that

\[
(3.1) \quad \text{Res}_{s = \frac{1}{2} + ir} (2s - 1) \sum_{\ell \mid 6} \mu(\ell) F_m(W_{\ell}z, s) = 8 \sum_{\varphi \in B_r^*} (\varphi, \varphi)^{-1} a_{\varphi}(m) \varphi(z),
\]

where \( \mu \) is the Möbius function and \( B_r^* \) is the subset of \( B_r \) containing only the \( \varphi \) for which \( \alpha(\ell) = \mu(\ell) \).

To simplify the notation, let

\[
F_m^*(z, s) = \frac{1}{4} \sum_{\ell \mid 6} \mu(\ell) F_m(W_{\ell}z, s).
\]

A straightforward modification of the proof of Proposition 5 of [2] shows that the Fourier expansion of \( F_m^*(z, s) \) is of the form

\[
(3.2) \quad F_m^*(z, s) = f_m(z, s) + c_m(s) y^{1-s} + \sqrt{y} \sum_{n \neq 0} c_m(n, s) K_{s - \frac{1}{2}}(2\pi |n| y) e(n z),
\]

where \( c_m(n, s) \in \mathbb{C} \) and

\[
c_m(s) = \begin{cases} 
\frac{\sqrt{\pi} \Gamma(s - \frac{1}{2})}{4 \Gamma(s)} \sum_{\ell \mid 6} \frac{\mu(\ell)}{\ell^s} \sum_{0 < \ell \equiv 0(6)} \frac{\varphi(c)}{\ell^{2s}} & \text{if} \ m = 0, \\
\frac{\pi^{s-\frac{1}{2}}}{4(s-\frac{1}{2}) \Gamma(s)} \sum_{\ell \mid 6} \frac{\mu(\ell)}{\ell^s} \sum_{0 < \ell \equiv 0(6)} \frac{R_c(m)}{\ell^{2s}} & \text{if} \ m \neq 0,
\end{cases}
\]

where \( \varphi(c) \) is the Euler totient function, and \( R_c(m) \) is the Ramanujan sum with modulus \( c \). By a standard calculation, \( c_m(s) \) can be written in terms of \( \zeta(2s) \) and \( \zeta(2s - 1) \). The exact evaluations are not important for us here, only that \( c_m(s)/c_0(s) \) is meromorphic in \( \text{Re}(s) > 0 \) with no pole at \( s = \frac{1}{2} + ir \) if \( r \neq 0 \).

Let \( D \equiv 1 \pmod{24} \) be a positive integer and let \( \mathcal{Q}_D \) denote the set of (indefinite) integral binary quadratic forms \( Q(x, y) = [a, b, c] (x, y) = ax^2 + bxy + cy^2 \) with discriminant \( D = b^2 - 4ac \) and with \( 6 \mid a \).

Let \( \Gamma_6^* \) denote the group generated by \( \Gamma_6 \) and \( \{ W_{\ell} : \ell \mid 6 \} \). A matrix \( \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_6^* \) acts on \( Q \in \mathcal{Q}_D \) via

\[
(\gamma Q)(x, y) = \frac{1}{\det \gamma} Q(Dx - By, -Cx + Ay).
\]

The set \( \Gamma_6 \backslash \mathcal{Q}_D \) contains finitely many equivalence classes and forms a group under Gauss composition. For \( r \in \{1, 5, 7, 11\} \), let \( \mathcal{Q}_D^{(r)} \) be the subset of \( \mathcal{Q}_D \) comprising those \( [a, b, c] \) for which \( b \equiv r \pmod{12} \). In Section 3 of [3] it is shown that

\[
\mathcal{Q}_D = \bigcup_{\ell \mid 6} W_{\ell} \mathcal{Q}_D^{(1)}.
\]

When \( D \) is a square we have the following explicit description of \( \Gamma_6 \backslash \mathcal{Q}_D \), which is a straightforward generalization of Lemma 3 of [3].
Lemma 3.2. Let \( \omega \in \{-1, 1\} \). If \( D = d^2 \) then
\[
\{[0, \omega|d|, c] : 0 \leq c < |d|\}
\]
is a complete set of representatives for \( \Gamma_6 \backslash Q_D^{(r)} \), where \( r \equiv \omega|d| \pmod{12} \).

For \( Q \in Q_D \) let \( S_Q \) denote the geodesic in \( \mathcal{H} \) connecting the (real) roots of \( Q(z, 1) = 0 \). Explicitly, \( S_Q \) is the set of points satisfying
\[
a|z|^2 + b \text{Re} z + c = 0.
\]
When \( a \neq 0 \), \( S_Q \) is a semicircle in \( \mathcal{H} \) which we orient clockwise if \( a > 0 \), counterclockwise if \( a < 0 \). When \( a = 0 \), \( S_Q \) is the vertical line \( \text{Re}(z) = -c/b \) which we orient downward. If \( D \) is not a square then the group \( \Gamma_Q \subseteq \Gamma_6 \) of automorphs of \( Q \) is infinite cyclic, and if \( D \) is a square then \( \Gamma_Q \) is trivial. In either case let \( C_Q = \Gamma_Q \backslash S_Q \). Then \( C_Q \) is a closed geodesic (of finite length) on \( \Gamma_6 \backslash \mathcal{H} \) if \( D \) is not a square, and it is an infinite geodesic otherwise.

Following Section 9 of [11], we would like to integrate the function \( \partial_z F^*_m(z, s) \) over \( C_Q \); when \( D \) is not a square there is no issue, but when \( D \) is a square the integral does not converge. In that case we will integrate the function
\[
F^*_m(Q, z, s) = \sum_{\ell \mid 6} \mu(\ell) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_6} f_m(\gamma W_iz, s),
\]
where \( a_1, a_2 \in \mathbb{P}^1(Q) \) are the endpoints of \( C_Q \). The function \( F^*_m(Q, z, s) \) is studied in Section 3 of [6]. We claim that the endpoints \( a_1 \) and \( a_2 \) are related by
\[
a_2 = W_6 \gamma a_1
\]
for some \( \gamma \in \Gamma_6 \). This is straightforward to see for the set of forms in Lemma 3.2 and thus holds for all forms in \( Q_D \). Since \( F^*_m(Q, z, s) \) is invariant under \( z \mapsto W_6z \), the Fourier expansion of \( F^*_m(Q, z, s) \) is the same at \( a_1 \) and \( a_2 \). Thus, by [32], the Fourier expansion of \( F^*_m(Q, z, s) \) at the cusp \( a_j \) is of the form
\[
c_{m,j}(s)y^{-s} + \sqrt{y} \sum_{n \neq 0} c_{m,j}(n, s)K_{s-\frac{1}{2}}(2\pi|n|y)\ell(nx),
\]
where \( c_{m,1}(s) = c_{m,2}(s) \) and \( c_{m,1}(n, s) = c_{m,2}(n, s) \).

For the rest of this section, let \( D = dd' \) be a factorization of \( D \) into negative discriminants \( d, d' \equiv 1 \pmod{24} \) where \( d \) is squarefree. The generalized genus character associated to this factorization is
\[
\chi_d(Q) = \begin{cases} (d \overline{m}) & \text{if } \gcd(a, b, c, d) = 1 \text{ and } m = Q(x, y) \text{ for some } x, y \in \mathbb{Z}, \\ 0 & \text{if } \gcd(a, b, c, d) > 1. \end{cases}
\]
The following proposition evaluates the cycle integrals in terms of the Kloosterman sums
\[
S(p_0, q_0, c, \nu_0) = \sum_{(a \ b \ c \ d) \in \Gamma_\infty \backslash \Gamma_6/\Gamma_\infty} \tilde{\nu}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) e\left( \frac{pa + qd}{24c} \right),
\]
where \( p, q \equiv 1 \pmod{24} \) and \( p_0 = \frac{p + 23}{24} \), \( q_0 = \frac{q + 23}{24} \). For uniform notation, when \( D \) is not a square we define \( F^*_m(Q, z, s) = F^*_m(z, s) \).

\[\text{We are following the convention in [13].}\]
Proposition 3.3. Let $D$ be a positive integer with $D \equiv 1 \pmod{24}$ and let $D = dd'$ be a factorization of $D$ into negative integers such that $d \equiv 1 \pmod{24}$ is squarefree. For $m \geq 0$ and $\Re(s) > 1$ we have

\begin{equation}
(3.3) \sum_{Q \in \Gamma_0 \backslash \mathcal{Q}_D^{(1)}} \chi_d(Q) \int_{C_Q} i \partial_z F_{m,Q}^*(z,s) dz
= \begin{cases} 
\left( \frac{\Gamma(\frac{s+1}{2})}{\Gamma(\frac{s}{2})} \right) \sqrt{\frac{n}{|m|}} \left( \frac{d}{n} \right) \chi_d(Q) \left( \frac{m^2}{n^2} d, d', s \right) & \text{if } m \neq 0, \\
0 & \text{if } m = 0,
\end{cases}
\end{equation}

where

\[ R(p,q,s) = 2\sqrt{\pi} \sqrt{-1} \sum_{c > 0} \frac{S(p_0,q_0,c,v_q)}{c} J_{s-\frac{1}{2}} \left( \frac{\pi \sqrt{|pq|}}{6c} \right), \quad p_0 = p + 23 \frac{1}{24}. \]

Proof. We closely follow the proof of Proposition 5.1 of [6]. In the notation of [6] we have

Let $D = ab,c$ and write

\[ F_{m,Q}^*(z,s) = \Gamma(s) \frac{P_{m,m,Q}(z,s)}{2s+3 |m| \Gamma(\frac{s+1}{2})^2}. \]

Let $T_m(d,d')$ denote the left-hand side of (3.3) and write

\[ 2i \partial_z F_{m,Q}^*(z,s) = \sum_{\ell \in \mathbb{Z}} \mu(\ell) \sum_{\gamma \in \Gamma_0 \backslash \Gamma_6} f_{2,m}(\gamma W_l z, s) \frac{d(\gamma W_l z)}{dz}, \]

where $f_{2,m}(z,s) = \phi_{2,m}(y,s)e(mx)$ and (using (9.2) of [11])

\[ \phi_{2,m}(y,s) = \begin{cases} 
y^{-s-1} & \text{if } m = 0, \\
s^{s-1} (2\pi y)^{-1} \frac{\Gamma(s)}{\Gamma(2s)} M_{1,s-\frac{1}{2}}(4\pi my) & \text{if } m > 0.
\end{cases} \]

To handle the cases when $D$ is a square and nonsquare together, we have written $\sum'$ to indicate that the terms with $\gamma W_l a_j = \infty$ should be excluded when $D$ is a square. Then

\[ T_m(d,d') = \frac{1}{2} \sum_{Q \in \Gamma_0 \backslash \mathcal{Q}_D^{(1)}} \chi_d(Q) \sum_{\ell \in \mathbb{Z}} \mu(\ell) \sum_{\gamma \in \Gamma_0 \backslash \Gamma_6} \int_{C_Q} f_{2,m}(\gamma W_l z, s) d(\gamma W_l z). \]

For each $Q \in \mathcal{Q}_D$ let $\Gamma_Q$ denote the stabilizer of $Q$ (note that when $D$ is a square, $\Gamma_Q$ is trivial). Then

\[ \sum' \int_{C_Q} f_{2,m}(\gamma W_l z, s) d(\gamma W_l z) = \sum' \int_{S_Q} f_{2,m}(\gamma W_l z, s) d(\gamma W_l z). \]

Making the change of variable $\gamma W_l z \mapsto z$ we have

\[ T_m(d,d') = \frac{1}{2} \sum_{Q \in \Gamma_0 \backslash \mathcal{Q}_D^{(1)}} \chi_d(Q) \sum_{\ell \in \mathbb{Z}} \mu(\ell) \sum_{\gamma \in \Gamma_0 \backslash \Gamma_6} \int_{S_Q} f_{2,m}(z,s) dz. \]

As explained in [6], the map $(\gamma, \ell, Q) \mapsto \gamma W_l Q$ is a bijection from $\Gamma_\infty \backslash \Gamma_6 / \Gamma_Q \times \{1,2,3,6\} \times \Gamma_6 \backslash \mathcal{Q}_D^{(1)}$ to $\Gamma_\infty \backslash \mathcal{Q}_D$, and $\chi_d(\gamma W_l Q) = \chi_d(Q)$. Furthermore, if $[a,b,c] = W_l Q$ for some $Q \in \mathcal{Q}_D^{(1)}$ then $\mu(\ell) = \left( \frac{12}{b} \right)$. When $D$ is not a square, each quadratic form $[a,b,c]$ in the set $\Gamma_\infty \backslash \mathcal{Q}_D$ has $a \neq 0$. Now suppose that $D$ is a square and let $Q \in \mathcal{Q}_D$. If $a_0$ is one of the roots of $Q(x,y) = 0$ then $\gamma W_l a_0 Q = a_0 W_l Q$. Thus the bijection $(\gamma, \ell, Q) \mapsto \gamma W_l Q$ translates the condition $\gamma W_l a_j \neq 0$ to the condition $a_j \neq \infty$. The quadratic forms $[a,b,c] \in \Gamma_\infty \backslash \mathcal{Q}_D$ with one of $a_j = \infty$ are precisely those with $a = 0$. Thus, for all $D$, we have

\[ T_m(d,d') = \frac{1}{2} \sum_{Q \in \Gamma_\infty \backslash \mathcal{Q}_D} \left( \frac{12}{b} \right) \chi_d(Q) \int_{S_Q} f_{2,m}(z,s) dz. \]
Each $S_Q$ with $Q = [a, b, c]$ and $a > 0$ can be parametrized by
$$z = \text{Re} z_Q - e^{-i\theta} \text{Im} z_Q, \quad 0 \leq \theta \leq \pi,$$
where
$$z_Q = -\frac{b}{2a} + i \frac{\sqrt{D}}{2a}$$
is the apex of the geodesic. Thus
$$\int_{S_Q} e(mx) \phi_{2,m}(y, s) dz = e \left(-\frac{mb}{2a}\right) H_m \left(\frac{\sqrt{D}}{2a}\right),$$
where
$$H_m(t) = i t \int_0^\pi e(-mt \cos \theta) \phi_{2,m}(t \sin \theta, s) e^{-i\theta} \, d\theta.$$ 
It follows that
\begin{equation}
T_m(d, d') = \sum_{e=1}^{\infty} H_m \left(\frac{\sqrt{D}}{12c}\right) \sum_{b(12c) \atop b^2 \equiv D(24c)} \left(\frac{12}{b}\right) \chi_d \left(\left[6c, b, \frac{b^2-D}{24c}\right]\right) e \left(-\frac{mb}{12c}\right).
\end{equation}

We claim that the inner sum in (3.4) equals zero if $m = 0$. Indeed, let $r$ be an integer satisfying $(\frac{12}{r}) = -1$ and $r^2 \equiv 1 \pmod{24(c, d)}$, and replace $b$ by $rb$ in the sum. By [6, Lemma 3.1, P4] the $\chi_d$ factor is invariant under this change of variable. It follows that the sum equals zero.

If $m \neq 0$ then by Lemma 7 of [11] we have
$$H_m(t) = \frac{2 \sqrt{\pi} \Gamma(\frac{m+1}{2}) t^{1/2}}{\Gamma(\frac{m}{2})} J_{m - \frac{1}{2}}(2\pi|m|t).$$
We finish the proof by applying Proposition 4.2 of [6], which states that
\begin{equation}
\sum_{b(12c) \atop b^2 \equiv D(24c)} \left(\frac{12}{b}\right) \chi_d \left(\left[6c, b, \frac{b^2-D}{24c}\right]\right) e \left(-\frac{mb}{12c}\right) = 4 \sqrt{-3i} \frac{1}{S(p_0, q_0, c, \nu)} \sum_{n|m, c} \left(\frac{12}{m/n}\right) \left(\frac{d}{n}\right) \sqrt{n} S \left(\frac{m^2}{24n^2} d + \frac{23}{24} d' + \frac{23}{24} c, \nu, \nu\right).
\end{equation}
(note that $\sqrt{-i} S(p_0, q_0, c, \nu) = K(p_0 - 1, q_0 - 1; c)$ in the notation of [6]). In that paper it is assumed that $(m, 6) = 1$, but that assumption is only used in one step of the proof, namely the key identity just above (4.13) in [6]. For $(m, 6) > 1$ that identity reads
\begin{equation}
\sum_{j(2u) \atop \epsilon(u)} e \left(-\frac{d(3j^2 + j)/2 + 3}{u}\right) e \left(\frac{m(6j + 1)}{12u}\right) + e \left(\frac{-m(6j + 1)}{12u}\right) = 0.
\end{equation}
By splitting the sum along $j = 2\ell$ and $j = 2\ell + 1$ we see that (3.5) is equivalent to the identity
$$\sum_{h \in \{\pm 1, \pm 7\}} \left(\frac{12}{h}\right) \sum_{\ell(c) \atop \ell(c)} e \left(-\frac{6d\ell^2 + (dh + m)\ell}{c}\right) = 0,$$
which is Lemma 5.5 of [6]. This completes the proof. \qed
We are ready to prove Proposition \[2.1\] Let \(C(s)\) be a function which is analytic in \(\Re(s) > 1\). Then by Proposition \[3.3\] we have

\[
\sum_{Q \in \Gamma \setminus Q^{(1)}} \chi_d(Q) \int_{C_Q} i\partial_z (F_{m,Q}^*(z,s) - C(s)F_{0,Q}^*(z,s)) \, dz = \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{1}{2}\right) D^+ \sum_{n|m} \left(\frac{12}{m/n}\right) \left(\frac{d}{n}\right) n^{-\frac{s}{2}} R\left(m^2 n^2 d, d', s\right).
\]

We would like to extend this identity to a neighborhood of \(\Re(s) = \frac{1}{2}\), so that we can apply Proposition \[3.4\] Note that Corollary 3.6 of \[12\] shows that \(R(p,q,s)\) has a meromorphic continuation to \(\mathbb{C}\) with poles on the line \(\Re(s) = \frac{1}{2}\). If \(D\) is not a square then each \(C_Q\) is compact so \[3.6\] automatically holds in the region \(\Re(s) > \frac{1}{2} - \varepsilon\). Suppose that \(D\) is a square. Then the Fourier expansion of \(F_{m,Q}^*(z,s) - C(s)F_{0,Q}^*(z,s)\) at either of the endpoints of \(C_Q\) is

\[
\pm (c_m(s) - C(s)c_0(s)) y^{1-s} + G(z,s),
\]

where \(G(z,s)\) decays exponentially as \(y \to \infty\). Choosing \(C(s) = c_m(s)/c_0(s)\), we find that the integrals in \[3.6\] are convergent for \(\Re(s) > \frac{1}{2} - \varepsilon\). Furthermore, we have

\[
\lim_{s \to \frac{1}{2} \pm i r} \Res_{s=\frac{1}{2} \pm i r} (F_{m,Q}^*(z,s) - C(s)F_{0,Q}^*(z,s)) = \lim_{s \to \frac{1}{2} \pm i r} \Res_{s=\frac{1}{2} \pm i r} F_{m,Q}^*(z,s)
\]

because \(C(s)F_{0,Q}^*(z,s)\) is analytic at \(s = \frac{1}{2} \pm i r\), \(r \neq 0\), and because \(F_{m,Q}^*(z,s)\) differs from \(F_{0,Q}^*(z,s)\) by an analytic function.

Corollary 3.6 of \[12\] shows that the poles of \((2s-1)R(p,q,s)\) are simple and lie at the points \(\frac{1}{2} \pm r_j\), where \(r_j\) is the spectral parameter of \(u_j\) as in Section \[2\]. From Proposition \[3.3\] \[3.4\] and \[3.7\] we have

\[
2 \sum_{\varphi \in \mathcal{B}} \langle \varphi, \varphi \rangle^{-1} a_{\varphi}(m) \sum_{Q \in \Gamma \setminus Q^{(1)}} \chi_d(Q) \int_{C_Q} i\partial_z \varphi(z) \, dz = D^+ \sum_{n|m} \left(\frac{12}{m/n}\right) \left(\frac{d}{n}\right) n^{-\frac{s}{2}} \Res_{s=\frac{1}{2} \pm i r} (2s-1) \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{s}{2}\right) R\left(m^2 n^2 d, d', s\right).
\]

To compute the residue on the right-hand side of \[3.8\], we will follow the argument given in Section 8 of \[11\]. We need Poincaré series for weight 1/2 with multiplier system \(\nu_\eta\) on \(\Gamma_1 = \text{PSL}_2(\mathbb{Z})\). These appear as Fourier coefficients of the resolvent kernel \(G_{s,\frac{1}{2}}(z,z')\) from Theorem 3.1 of \[12\]. For \(m \equiv 1 \pmod{24}\) and \(\Re(s) > 1\) define

\[
F_{s,\frac{1}{2}}(z,s) = \frac{6\Gamma(s - \frac{1}{4}\text{sgn}(m))}{\pi|m|\Gamma(2s)} \sum_{\gamma \in \Gamma \cap \Gamma_1} \nu^{-1}_\eta(\gamma) \left(\frac{cz + d}{|cz + d|}\right)^{-\frac{1}{2}} M_{\text{sgn},s-\frac{1}{2}}^{1/2} \left(\frac{\pi}{2}|m|\Im\gamma z\right) e\left(\frac{1}{2}m\Re\gamma z\right).
\]

This is related to the Poincaré series \(P_m(z,s)\) from Proposition 8 of \[1\] by

\[
F_{s,\frac{1}{2}}(z,s) = \left(\frac{6}{\pi|m|}\right)^{3/4} y^{1/4} \Gamma(s - \frac{1}{4}\text{sgn}(m)) \Gamma(2s) P_m(z,s).
\]

Thus, by that proposition, we have

\[
F_{1/2}(s) = \frac{6\Gamma(s - \frac{1}{4}\text{sgn}(m))}{\pi|m|\Gamma(2s)} M_{\text{sgn},s-\frac{1}{2}}^{1/2} \frac{1}{2}|m|y e\left(\frac{1}{24}mx\right)
\]

\[
+ \frac{6}{\sqrt{\pi}} \sum_{n \equiv 1(24)} |mn|^{-1/2} \frac{1}{\Gamma(s - \frac{1}{2}\text{sgn}(m))} R(m, n, 2s - \frac{1}{2}) W_{\text{sgn}(n),s-\frac{1}{2}}^{1/2} \frac{1}{2}|n|y e\left(\frac{1}{24}nx\right).
\]
Thus by (3.8) we have
\[
S = \frac{6}{\sqrt{\pi} |mn|^{\frac{1}{2}}} \Res_{s=\frac{1}{2}+\frac{1}{2\pi}i}(2s-1)F_{\frac{1}{2},\frac{1}{2}}(z, s) = \sum_{r_j=r/2} \bar{\mu}_j(m)u_j(z),
\]
where \(u_j, \mu_j,\) and \(r_j\) are defined in Section 2. By equating Fourier coefficients, it follows that
\[
\frac{6}{\sqrt{\pi} |mn|^{\frac{1}{2}}} \Res_{s=\frac{1}{2}+\frac{1}{2\pi}i}(2s-1)\Gamma(s-\frac{1}{4}\text{sgn}(m)) \Gamma(s+\frac{1}{4}\text{sgn}(n)) R(m, n, 2s-\frac{1}{2}) = \sum_{r_j=r/2} \bar{\mu}_j(m)\mu_j(n).
\]
Thus by (3.5) we have
\[
\sum_{\varphi \in B^*} a_{\varphi}(m) \sum_{Q \in (\Gamma \setminus Q_D^{(1)} \cap \mathcal{R}^n)} \chi_d(Q) \int_{C_Q} i\partial_z \varphi(z_Q) \, dz = \frac{\sqrt{\pi}}{3} D^{\frac{d}{2}} \sum_{r_j=r/2} \mu_j(d') \sum_{n|m} m \left( \frac{12}{m/n} \right) \left( \frac{d}{n} \right) n^{-3/2} \bar{\mu}_j \left( \frac{n^2}{m} \right).
\]
By (2.3) the inner sum above equals \(\bar{\mu}_j(d)\bar{a}(m),\) where \(a(m)\) is the \(m\)-th coefficient of \(S_d(u_j).\) Thus
\[(3.9) \quad \frac{\sqrt{\pi}}{3} D^{\frac{d}{2}} \sum_{r_j=r/2} \mu_j(d') \bar{\mu}_j(d)S_d(u_j) = \sum_{\varphi \in B^*} (\varphi, \varphi)^{-1} \sum_{Q \in (\Gamma \setminus Q_D^{(1)} \cap \mathcal{R}^n)} \chi_d(Q) \int_{C_Q} i\partial_z \varphi(z) \, dz,
\]
where \(S_d(u_j)\) is defined in 2.6. This identity is valid for all negative \(d, d' \equiv 1 \pmod{24}\) such that \(d\) is squarefree. Thus, by the same argument given in Section 8 of [9] we find that \(S_d(u_j) \in \mathcal{S}_0(6, \chi_0, r)\) and that \(S_d(u_j)\) has eigenvalue \(-1\) under both \(W_2\) and \(W_3.\) This completes the proof of Proposition 2.1.

We now prove Proposition 2.2. We rewrite the left-hand side of (3.9) as
\[
\frac{\sqrt{\pi}}{3} D^{\frac{d}{2}} \sum_{\varphi \in B^*} \varphi \sum_{\varphi \in B^*} \mu_j(d') \bar{\mu}_j(d).
\]
Then the linear independence of the \(\varphi\) yields the formula
\[(3.10) \quad \frac{\sqrt{\pi}}{3} D^{\frac{d}{2}} \sum_{\varphi \in B^*} \mu_j(d') \bar{\mu}_j(d) = (\varphi, \varphi)^{-1} \sum_{Q \in (\Gamma \setminus Q_D^{(1)} \cap \mathcal{R}^n)} \chi_d(Q) \int_{C_Q} i\partial_z \varphi(z) \, dz.
\]
Suppose that \(d' = d.\) Then by Lemma 3.2 the quadratic forms \([0, d, c] \text{ with } 0 \leq c < |d|\) form a complete set of representatives for \(\Gamma \setminus Q_D^{(1)}\). Furthermore,
\[
\chi_d([0, d, c]) = \left( \frac{d}{c} \right).
\]
Following the normalization (2.4), we write
\[
\varphi(z) = 2\sqrt{\frac{y}{3}} \sum_{n \neq 0} a(n)K_{ir}(2\pi|n|y)e(nx).
\]
A computation involving [10, §10.29, (10.30.2), and (10.40.2)] shows that
\[
\partial_z [\sqrt{y}K_{ir}(2\pi|n|y)e(nx)] = \pi \sin \sqrt{y}K_{ir}(2\pi|n|y)e(nx) + g(n, y)e(nx)
\]
for some function \( g(n, y) \) which satisfies \( g(-n, y) = g(n, y) \) and \( g(n, y) \ll |n|^{1/2} e^{-2\pi |n|y} \) as \( |n|y \to \infty \) and \( g(n, y) \ll_n y^{-1/2} \) as \( y \to 0 \). So if \( \text{Re}(s) > 1 \) we have

(3.11) \[
\sum_{Q \in \Gamma \setminus \mathcal{Q}_D} \chi_d(Q) \int_{C_d} i\partial_z \varphi(z) y^s \, dz
\]

\[
= -2\pi i \sum_{n \neq 0} na(n) G(n,d) \int_0^\infty y^{s+\frac{1}{2}} K_{ir}(2\pi |n|y) \, dy - 2 \sum_{n \neq 0} a(n) G(n,d) \int_0^\infty y^s g(n,y) \, dy,
\]

where \( G(n,d) \) is the Gauss sum

\[
G(n,d) = \sum_{c \mod |d|} \left( \frac{d}{c} \right) e \left( \frac{-n c}{d} \right) = i \left( \frac{d}{n} \right) \sqrt{|d|}.
\]

Since \( a(-n)G(-n,d)g(-n,y) = -a(n)G(n,d)g(n,y) \), the second sum on the right-hand side of (3.11) vanishes. By [11, (10.43.19)] we have

\[
\int_0^\infty y^{s+\frac{1}{2}} K_{ir}(2\pi |n|y) \, dy = \frac{1}{4} (\pi |n|)^{-s-\frac{1}{2}} \Gamma \left( \frac{s}{2} + \frac{ir}{2} + \frac{3}{4} \right) \Gamma \left( \frac{s}{2} - \frac{ir}{2} + \frac{3}{4} \right).
\]

It follows that

\[
\sum_{Q \in \Gamma \setminus \mathcal{Q}_D} \chi_d(Q) \int_{C_d} i\partial_z \varphi(z) y^s \, dz = \pi^{-s-\frac{1}{2}} \sqrt{|d|} \Gamma \left( \frac{s}{2} + \frac{ir}{2} + \frac{3}{4} \right) \Gamma \left( \frac{s}{2} - \frac{ir}{2} + \frac{3}{4} \right) L(s + \frac{1}{2}, \varphi \times \chi_d).
\]

Setting \( s = 0 \) and using (3.10), we obtain (2.8).

4. Cubic Moment and Subconvexity

Let \( N \geq 1 \) be a fixed integer. Denote by \( \mathcal{S}_0(N) \) the space of weight 0 Maass cusp forms of level \( \Gamma_0(N) \) with a basis \( \mathcal{B}_0(N) \) of eigenforms for Hecke operators. For a Maass form \( \varphi \in \mathcal{S}_0(N) \), denote by \( r = r(\varphi) \) the spectral parameter. The purpose of this section is to establish the following bound.

**Theorem 4.1.** Let \( T \gg 1 \), and \( q \) be a fundamental discriminant. Let \( \chi_q \) be the corresponding quadratic Dirichlet character of modulus \( |q| \). Then we have for any \( \varepsilon > 0 \)

\[
\sum_{\varphi \in \mathcal{B}_0(N)} \frac{L \left( \frac{1}{2}, \varphi \times \chi_q \right)^3}{L(1, \varphi, Ad)} \ll \varepsilon \left( T^{|q|N} \right)^{\varepsilon T \text{lcm}(|q|,N)}.\]

**Corollary 4.2.** For the above \( \varphi \in \mathcal{B}_0(N) \), we have the bound

\[
L \left( \frac{1}{2}, \varphi \times \chi_q \right) \ll \left( |r| \sqrt{|q|N} \right)^\varepsilon \left( |r| \text{lcm}(|q|,N) \right)^{\varepsilon T}.
\]

Our main tool is the version of Motohashi’s formula in [21, 22], which we recall as follows. For a Hecke eigenform \( \varphi \), we do not distinguish it from the irreducible representation \( \pi \) generated by it. Hence \( \varphi_v \) will mean \( \pi_v \) if \( \pi \simeq \otimes_v \pi_v \). Let \( S \) be a finite set of primes \( p \). Let \( \Psi_v \in \mathcal{S}(M_2(Q_v)) \) be Schwartz functions at \( v \in \{ \infty \} \cup S \). For any Hecke-Maass form \( \varphi \) which is unramified at all \( p \notin S \), we introduce for each \( v \in S \cup \{ \infty \} \)

\[
M_{3,v}(\Psi_v \mid \varphi_v) = \sum_{e_1, e_2 \in B(\varphi_v)} \text{Z}_v \left( \frac{1}{2}, \Psi_v, \beta(e_2, e_1^\vee) \right) \text{Z} \left( \frac{1}{2}, W_{e_1} \right) \text{Z} \left( \frac{1}{2}, W_{e_2} \right),
\]

where

- \( B(\varphi_v) \) is an orthogonal basis of \( \varphi_v \);
- for \( e \in B(\varphi_v) \), \( e^\vee \) is the dual vector in the dual basis, where the implicit inner product is defined in the Kirillov model;
- \( W_\ast \) is the Kirillov function of \( * \) with respect to the standard additive character \( \psi_\ast \) à la Tate;
- \( \beta(e_2, e_1^\vee) \) is the matrix coefficient related to \( e_2 \) and \( e_1^\vee \);
\[
\bullet \quad Z_v(s, \Psi_v, \beta) \text{ is the Godement-Jacquet zeta integral }
\]
\[
Z_v(s, \Psi_v, \beta) = \int_{\text{GL}_2(F_v)} \Psi_v(g)\beta(g)|\det g|_v^{s+\frac{1}{2}} dg
\]
and \(Z_v(s, W)\) is the standard local zeta integral
\[
Z_v(s, W) = \int_{F_v} W(t)|t|_v^{s-\frac{1}{2}} dt.
\]
With these local terms, we define
\[
M_3(\Psi | \varphi) := \frac{3}{\pi} \frac{L(\frac{1}{2}, \varphi)^3}{L(1, \varphi, Ad)} \cdot M_{3, \infty}(\Psi_\infty | \varphi_\infty) \prod_{p \in S} M_{3,p}(\Psi_p | \varphi_p) L(1, \varphi_p \times \bar{\varphi}_p).
\]
There is a counterpart for the Eisenstein series \(M_3(\Psi | \chi, s)\). Basically its corresponding local terms \(M_{3,v}(\Psi_v | \chi_v, s)\) are the same as \(M_{3,v}(\Psi_v | \varphi_v)\) if \(\varphi_v \simeq \pi(\chi_v|\cdot|_v, \chi_v^{-1}|\cdot|_v^{-1})\), except that we change the inner product structure to be defined in the induced model. Namely,
\[
M_3(\Psi | \chi, s) = \frac{L(\frac{1}{2} + s, \chi)^3 L(\frac{1}{2} - s, \chi^{-1})^3}{L(1 + 2s, \chi^2)L(1 - 2s, \chi^{-2})} \cdot M_3(\Psi_\infty | \chi_\infty, s)
\]
\[
\times \prod_{p \in S} M_{3,p}(\Psi_p | \chi_p, s) \frac{L_p(1 + 2s, \chi_p^2)L_p(1 - 2s, \chi_p^{-2})}{L_p(\frac{1}{2} + s, \chi_p^2)L_p(\frac{1}{2} - s, \chi_p^{-2})^3}.
\]
For any Dirichlet character \(\chi\) which is unramified at all \(p \notin S\), we introduce for each \(v \in S \cup \{\infty\}\)
\[
M_{4,v}(\Psi_v | \chi_v) = \zeta_v(1)^4 \int_{F_v} \chi_v \left( \frac{x_1 x_4}{x_2 x_3} \right)^\frac{1}{2} \prod_{i=1}^4 dx_i
\]
and the corresponding global distribution
\[
M_4(\Psi | \chi) := L(\frac{1}{2}, \chi)^2 L(\frac{1}{2}, \chi^{-1})^2 \cdot M_{4,\infty}(\Psi_\infty | \chi_\infty, s) \prod_{p \in S} M_{4,p}(\Psi_p | \chi_p) \frac{1}{L(\frac{1}{2}, \chi_p)^2 L(\frac{1}{2}, \chi_p^{-1})^2}.
\]
Write
\[
M_3(\Psi) = \sum_\varphi M_3(\Psi | \varphi) + \sum_\chi \int_{-\infty}^\infty M_3(\Psi | \chi, ir) \frac{dr}{4\pi},
\]
\[
M_4(\Psi) = \sum_\chi \int_{-\infty}^\infty M_4(\Psi | \chi, |\cdot|_L) \frac{dr}{2\pi}.
\]
Then the formula is
\begin{equation}
M_3(\Psi) + DS(\Psi) = M_4(\Psi) + DG(\Psi),
\end{equation}
where the degenerate terms \(DS(\cdot)\) and \(DG(\cdot)\) are given by
\begin{equation}
DG(\Psi) = \text{Res}_{s=\frac{3}{2}} M_4(\Psi | \cdot |_{L}),
\end{equation}
\begin{equation}
DS(\Psi) = \text{Res}_{s=\frac{1}{2}} M_3(\Psi | \cdot |_{L}).
\end{equation}
In order to apply the formula (4.1) to our problem, let \(S\) be the set of primes \(p | qN\). Let \(\chi_0\) be the Hecke character corresponding to \(\chi_q\). We first specify \(\Psi_p\) for \(p \in S\) and give the relevant estimations of \(M_{3,p}\) and \(M_{4,p}\). The choice of \(\Psi_p\) is a simple variant of those made in [8, §1.4]. For integers \(n \geq 0\), let
\[
K_0[p^n] := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}_p) \mid c = p^n \mathbb{Z}_p \right\}.
\]
For each \(p \in S\), write \(n_p \geq 1\) such that \(p^{n_p} \parallel \text{lcm}(q, N)\), and introduce
\[
\phi_0(g) := 1_{K_0[p^{n_p}]}(g) \chi_0(\det g).
\]
Let integer $t_p \geq 0$ be such that $p^{t_p} \parallel q \ (l_p \leq 1 \text{ if } p \neq 2)$. Then we take
\[
\Psi_p = \begin{cases} 
L_{n(p^{-t_p})} R_{n(p^{-t_p})} \phi_0 & \text{if } p \mid q, \\
\phi_0 & \text{if } p \nmid q,
\end{cases}
\]
where $L_{g_1}, R_{g_2}, \Psi(x) := \Psi(g_1^{-1}xg_2)$. This choice differs from [3, §1.4 & §4] in that $l_p$ may be strictly smaller than $n_p$. We first treat the weight functions $M_{3,p}(\cdot)$. The argument is almost the same as [3, Lemma 4.1], hence we give a proof with minimal amount of details.

**Lemma 4.3.** (1) The local weight $M_{3,p}(\Psi_p \mid \varphi_p) \neq 0$ only if $p^{n_p+1}$ does not divide the level of the form $\varphi \otimes \chi_0$. In this case, we have uniformly in $p$
\[
M_{3,p}(\Psi_p \mid \varphi_p) \gg p^{-n_p-l_p},
\]
where the implicit constant depends only on a constant towards the Ramanujan-Petersson conjecture.

(2) If $p \mid q$ but $p \nmid N$, then we have $M_{3,p}(\Psi_p \mid 1, s) = 0$.

(3) If $p \mid N$, then we have $M_{3,p}(\Psi_p \mid 1, s) \neq 0$ only if $n_p \geq 2l_p$. Under this condition, we have
\[
\left| \frac{\partial^n}{\partial s^n} M_{3,p}(\Psi_p \mid 1, s) \zeta_p(1+2s) \right| \ll_n p^{1-n_p-l_p} \log^n p.
\]

**Proof.** (1) Let $\pi$ be the global representation of $GL_2(\mathbb{A})$ corresponding to the newform $\varphi$. Let $(\pi_p, V_p)$ be the local component of $\pi$ at $p$. As in the proof of [3, Lemma 4.1], we have $\pi_p(\phi_0) \neq 0$ only if $V_p$ contains a non zero vector $e$ satisfying
\[
\pi_p \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) e = \chi_0(p)(ad)e, \quad \forall \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \in K_0[p^{n_p}].
\]

This is equivalent to $e(\pi_p \otimes \chi_0) \leq n_p$. Let $\mathcal{B}_0$ be an orthogonal basis of the subspace of $V_p \otimes \chi_0$ of vectors satisfying
\[
(\pi_p \otimes \chi_0) \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) . e = e, \quad \forall \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \in K_0[p^{n_p}],
\]
which contains the newvector $e_0$. This basis determines a dual basis $\mathcal{B}_0^\vee$ in the dual representation $(\pi_p^\vee \otimes \chi_0, V_p^\vee \otimes \chi_0)$, such that $e_0^\vee$ is the dual element of $e$ so that
\[
(e, e) = 1, \quad (e_0^\vee, e) = 0, \forall e \neq e_0 \in \mathcal{B}_0.
\]

For $e$ or $e_0$, let $W_e$ or $W_{e_0}$ be the corresponding Kirillov function of $e$ or $e_0^\vee$ with respect to the standard additive character $\psi_p$ resp. $\psi_p^{-1}$. The above local pairing is defined in the Kirillov model. Then we have
\[
M_{3,p}(\Psi_p \mid \varphi_p) = \text{Vol}(K_0[p^{n_p}]) \sum_{e \in \mathcal{B}_0} Z \left( \frac{1}{2}, \chi_0, n(p^{-l_p})W_e \right) Z \left( \frac{1}{2}, \chi_0, n(p^{-l_p})W_{e'} \right),
\]
where $Z(\cdot)$ is the local Rankin-Selberg zeta function for $GL_2 \times GL_1$. Each summand on the right hand side is non-negative. We drop all but the term corresponding to $e = e_0$. Writing $W_0 = W_{e_0}$, such that $\text{supp}(W_0) \subset \mathbb{Z}_p$ and $W_0(1) = 1$. If $l_p > 0$, then we have by [22, Proposition 4.6]
\[
\left| Z \left( \frac{1}{2}, \chi_0, n(p^{-l_p})W_0 \right) \right| = p^{-l_p/2}(1 - p^{-1})^{-1}, \quad \left| W_0 \right|^2 \ll_\theta 1
\]
where $\theta$ is any constant towards the Ramanujan-Petersson conjecture. While if $l_p = 0$, then we have
\[
\left| Z \left( \frac{1}{2}, \chi_0, W_0 \right) \right| = |L \left( \frac{1}{2}, \pi_p \otimes \chi_0 \right) | \gg_\theta 1.
\]

The desired bound follows readily.

(2) This follows from $e(\pi(\vert q \mid_p^s \mid \Gamma_p^\pi)^{-1} {\chi_0}) = 2e(\chi_0) > 0$.

(3) For simplicity, we only treat the case $l \geq 1$. For $\pi_p = \pi(\vert q \mid_p^s \mid \Gamma_p^\pi), \Gamma_p^\pi$ be a lattice that can be chosen as $\{e_k : 0 \leq k \leq n_p\}$, where $e_k$ can be written as linear combination of $a(p^{-l})$ translations of $e_0$ as in [22, Lemma 2.18 (i)]. Let $W_k$ resp. $W_k^\vee$ be the Kirillov function of $e_k$ (resp. $e_k^\vee$) with respect to $\psi_p$ (resp. $\psi_p^{-1}$). We deduce
\[
W_0(1)W_0^\vee(1) = (1 - p^{-1-2s})(1 - p^{-1+2s}), \quad W_1(1)W_1^\vee(1) = (p^s + p^{-s})^2,
\]
The dual weight $M_{4, p}(\Psi_p | \chi_p)$ is given by the formula
\[
\frac{M_{4, p}(\Psi_p | \chi_p)}{\zeta_p(1)^4} = \int_{\mathbb{Q}_p^4} \phi_0 \left( \begin{array}{cccc}
 x_1 & x_2 & 1 & y \\
 x_3 & x_4 & 1 & 1 
\end{array} \right) \chi_p \left( \begin{array}{cccc}
 (x_1 + p^{-l} x_3)x_4 - p^{-l} x_3x_1 & (x_2 + p^{-l} x_3)x_1 - p^{-l} x_3x_2 \\
 (x_2 - p^{-l} x_1)x_4 - p^{-l} x_3x_2 & (x_1 - p^{-l} x_3)x_1 - p^{-l} x_3x_2 
\end{array} \right) 
\] \times \left| (x_1 + p^{-l} x_3)(x_4 - p^{-l} x_3)(x_2 - p^{-l} x_1)x_4 - p^{-l} x_3x_1)^{1/2} \right|
\] (4.4)

In the rest of this paragraph, we omit the subscript $p$ for simplicity of notation. Considering the change of variables
\[
x_1 \mapsto x_3(1 + \delta), \quad x_1 \mapsto x_1 - p^{-l} \delta x_3, \quad x_2 \mapsto x_2 - p^{-2l} \delta x_3, \quad x_4 \mapsto x_4 + p^{-l} \delta x_3
\]
for any $\delta \in p^l \mathbb{Z}_p$, we get $M_4(\Psi | \chi) = \chi(1 + \delta)^{-1} M_4(\Psi | \chi)$. Thus
\[
M_4(\Psi | \chi) \neq 0 \quad \Rightarrow \quad m := c(\chi) \leq l.
\]
Note that $\phi_0$ is given in the coordinates of the Bruhat decomposition as
\[
\phi_0 \left( \begin{array}{cccc}
 z & x & 1 & y \\
 z & x & 1 & 1 
\end{array} \right) = \mathbb{I}_{(2^n z, z)}(z, u) \mathbb{I}_{(2^n z, y)}(u^{-n}x, y) \chi_0(u)^{-1}.
\]
Making the change of variables
\[
\left( \begin{array}{cccc}
 x_1 & x_2 & 1 & y \\
 x_3 & x_4 & 1 & 1 
\end{array} \right) = \left( \begin{array}{cccc}
 z & x & 1 & y \\
 z & x & 1 & 1 
\end{array} \right) = \left( \begin{array}{cccc}
 z & u & z & y \\
 z & u & z & 1 + xy 
\end{array} \right)
\]
whose Jacobian is equal to $|z|^3 |u|$, then $x \mapsto p^n x$, we get
\[
\frac{M_4(\Psi | \chi)}{\zeta_p(1)^4} = p^{-n} \int_{\mathbb{Z}_p^2} \int_{\mathbb{Z}_p^2} \chi \left( \frac{(1 + p^{-l} x)(1 - p^{-l} x)(u - p^{-l} y)}{p^{-n} x(1 - (1 + p^{-l} x)(u - p^{-l} y))} \right) \chi_0(u)^{-1} 
\] \times \left| z |dz|dudx dy \right|
\] (1) If $n = l \geq 1$, then the above integral is exactly the one studied in [3, §4], and we get uniformly
\[
|M_4(\Psi | \chi)| \ll p^{-2l}.
\]
(2) If $n > l \geq 1$, then by the change of variable $u \mapsto u + p^{-l} y$ we can further simplify
\[
\frac{M_4(\Psi | \chi)}{\zeta_p(1)^4} = p^{-n} \int_{\mathbb{Z}_p^2} \int_{\mathbb{Z}_p^2} \chi \left( \frac{(1 + p^{-l} x)(1 - p^{-l} x)(u - p^{-l} y)}{p^{-n} x(1 - (1 + p^{-l} x)(u - p^{-l} y))} \right) \chi_0(u)^{-1} \frac{dudx}{|p^{-l} x(1 - (1 + p^{-l} x)u)|^{1/2}}
\]
Applying the consecutive changes of variables $u \mapsto u^{-1}$ and $u \mapsto 1 + p^{-l} x + y$ gives
\[
\frac{M_4(\Psi | \chi)}{\zeta_p(1)^4} = p^{-n} \int_{\mathbb{Z}_p^2} \int_{\mathbb{Z}_p^2} \chi \left( \frac{(1 + p^{-l} x)(1 + y)}{p^{-l} x} \right) \chi_0(1 + p^{-l} x + y) \frac{dxdy}{|p^{-l} x(y)|^{1/2}}
\] = \frac{p^{-l}}{\zeta_p(1)} \sum_{k \geq 0, l \geq 0} \int_{p^{k+2l} x \times p^{k+2l} y} \mathbb{I}_{(2^n z, z)}(1 + y) \chi \left( \frac{(1 + x)(1 + y)}{xy} \right) \chi_0(1 + x + y) \frac{dxdy}{|xy|^{1/2}}
\] = \frac{p^{-l}}{\zeta_p(1)} \sum_{k \geq 0, l \geq 0} M_{4, l}^{k,l}(\chi_0 | \chi),
where the integrals $M_{4}^{k,l}$ are the same as those defined in \[8\ (4.4)]. In other words, the above integral is simply a partial one of the integral studied in the previous case, whose proof actually goes by bounding each summand $M_{4}^{k,l}$. We conclude the bound (4.4) in this case, too.

(3) If $l = 0$ (and $n \geq 1$), we take the definition of the dual weight formula (4.4) to see

\[
\frac{M_{4}(\Psi | \chi)}{\zeta_{p}(1)^{4}} = \int_{Q_{p}^{\infty}} \|K_{0}[p^{n}]\|^{2} x_{1} x_{2} x_{3} x_{4} \chi \frac{x_{1} x_{4}}{x_{2} x_{3}} \prod_{i} dx_{i} \left| \frac{x_{1} x_{2} x_{3} x_{4}}{x_{1} x_{2} x_{3} x_{4}} \right|^{2}
\]

which is non-vanishing only if $c(\chi) = 0$, in which case

\[
M_{4}(\Psi | \chi) = \int_{\mathbb{Z}_{p}} \chi(x_{2})^{-1} |x_{2}|^{\frac{3}{2}} d^{\infty} x_{2} \int_{p^{n}Z_{p}} \chi(x_{3})^{-1} |x_{3}|^{\frac{3}{2}} d^{\infty} x_{3} \approx p^{-\frac{3}{4}}.
\]

**Lemma 4.4.** (1) The local weight $M_{4,p}(\Psi_{p} | \chi_{p})$ is non vanishing only if $c(\chi_{p}) \leq 1$. Under this condition, we have uniformly for unitary $\chi_{p}$

\[
|M_{4,p}(\Psi_{p} | \chi_{p})| \ll \begin{cases} p^{-2i_{p}} & \text{if } i_{p} > 0 \\ p^{-\frac{t}{2}} & \text{if } i_{p} = 0.\end{cases}
\]

(2) For any integer $n \geq 0$, we have

\[
\left| \frac{\partial^{n}}{\partial s^{n}} M_{4,p}(\Psi_{p} | \chi_{p}) \right|_{s = \pm \frac{1}{2}} \ll_{n} p^{-l_{p}} \log p.
\]

**Proof.** (1) This is just a summary of the above discussion.

(2) We simply replace $\chi$ by $|\cdot|_{p}^{3}$ in the above discussion to find precise formula of the term on the left hand side. The details can be found in \[8\ §5].

At the infinite place, if $\varphi_{\infty}$ has spectral parameter $r$, then we can choose for $T \geq 1$ and $\Delta = T^{n}$

\[
M_{3,\infty}(\Psi_{\infty} | \varphi_{\infty}) = \sqrt{\pi} \cosh(\pi r) 2\Delta \left\{ \exp \left[ \frac{-(r-T)^{2}}{2\Delta^{2}} - \frac{\pi r}{2} \right] + \exp \left[ \frac{-(r+T)^{2}}{2\Delta^{2}} + \frac{\pi r}{2} \right] \right\}^{2}.
\]

This is a positive weight function, which approximates the characteristic function of the interval $[T - \Delta, T + \Delta]$. The dual weight $M_{3,\infty}(\Psi_{\infty} | \chi_{\infty})$ is studied in \[8\ §3.3], which is bounded as

\[
|M_{4,\infty}(\Psi_{\infty} | \chi_{\infty})| \ll \begin{cases} 1 & \text{for all } \chi_{\infty}, \text{ for any } A \text{ and } \chi_{\infty}(t) = t^{A} \text{ with } t > 0, x \geq (1 + |T|) \log(1 + |T|).\end{cases}
\]

**Lemma 4.5.** The above local weight $M_{3,\infty}(\Psi_{\infty} | \varphi_{\infty})$ satisfies:

(1) It is non negative for all unitary $\varphi_{\infty}$.

(2) For $\varphi_{\infty}$ with spectral parameter $r$ such that $T - 1 \leq |r| \leq T + 1$, we have $M_{3,\infty}(\Psi_{\infty} | \varphi_{\infty}) \gg_{\epsilon} T^{-\epsilon}$.

The dual weight satisfies

\[
|M_{4,\infty}(\Psi_{\infty} | \chi_{\infty})| \ll \begin{cases} 1 & \text{for all unitary } \chi_{\infty}, \text{ for any } A > 1, \chi_{\infty}(t) = t^{A} \text{ with } t > 0, x \geq |T| \log^{2}|T|.\end{cases}
\]

**Lemma 4.6.** The corresponding weight $M_{3,\infty}(\Psi_{\infty} | \mathbb{1}, s)$ vanishes at $s = 1/2$ to order one, and satisfies for any integer $n \geq 0$ and constant $A > 1$

\[
\left| \frac{\partial^{n}}{\partial s^{n}} M_{3,\infty}(\Psi_{\infty} | \mathbb{1}, s) \right|_{s = \pm \frac{1}{2}} \ll_{n,A} T^{-A}.
\]

The corresponding dual weight $M_{4,\infty}(\Psi_{\infty} | \chi_{\infty}, |\cdot|^{3})$ has a singularity at $s = \pm 1/2$ of order $\leq 2$, and satisfies for any integer $n \geq 0$

\[
\left| \frac{\partial^{n}}{\partial s^{n}} \left( s \mp \frac{1}{2} \right)^{2} M_{4,\infty}(\Psi_{\infty} | \mathbb{1}, s) \right|_{s = \pm \frac{1}{2}} \ll_{\epsilon,n} T^{1 + \epsilon}.
\]
For a proof of the above two lemmas, see [8, §5].

Inserting Lemma 4.3 (2), Lemma 4.4 (2), Lemma 4.6 into the formulas (4.2) and (4.3), we deduce

$$|DG(\Psi)| + |DS(\Psi)| \ll_{\varepsilon} T^{1+\varepsilon}q^{-1+\varepsilon}.$$  

Introduce the decomposition

$$N = N_0N_1,$$

so that $p \mid \gcd(q, N) \iff p \mid q_0 \iff p \mid N_0$, and $\gcd(N_0, N_1) = 1$. Inserting all the above local bounds in the Motohashi’s formula (4.1), we get

$$\sum_{\varphi \in \mathcal{B}_{\ell}(N)} r(\varphi) \ll_{\varepsilon} (T|q|N)^{1/2}\log T \sum_{\chi \leq T \log^2 T} |L \left( \frac{1}{2} + ir, \chi \right)|^4,$$

where the sum over $\chi$ are those Dirichlet characters of conductor dividing $q$. A spectral large sieve inequality shows that the fourth moment is bounded as

$$\sum_{\chi} \int_{|r| \leq T \log^2 T} |L \left( \frac{1}{2} + ir, \chi \right)|^4 \ll_{\varepsilon} (T|q|N)^{1+\varepsilon}.$$  

We deduce the desired bound in Theorem 4.1.

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