An equivalent electrical circuit for the Hindmarsh-Rose model

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Summary
Hardware realizations of neural networks can pave the way towards a new generation of processors due to the biological role model being superior in terms of speed and energy-efficiency compared to today’s processors. This can be achieved by deriving novel design principles for circuits being obtainable when replicating and investigating real biological neural networks in depth. This has, for example, been done by utilizing the Hindmarsh-Rose model, offering a rich repertoire of neuronal firing patterns. Our aim is to synthesize a theoretical, equivalent electrical circuit of the Hindmarsh-Rose model being well interpretable in terms of biology, since this supports the derivation of design principles from biology and can serve as a basis for a systematic circuit simplification. We do this by starting from a linearized model because this allows for a systematic approach and then first derive a linear and afterwards a nonlinear equivalent electrical circuit. The resulting circuit has a structure similar to conductance-based models, where a deployed negative impedance converter can be seen as a deeper modeling of ion pump activity. Simulation results of the proposed circuits show the functionality of the equivalent circuits.

KEYWORDS
circuit synthesis, Hindmarsh-Rose model, neural networks, neuromorphic engineering

1 | INTRODUCTION

It is commonly known that real biological nervous systems are far superior than today’s computers when it comes down to energy efficiency. Hence, even though the simulation of neuronal networks in software has attracted a lot of interest since it enables machine learning applications and artificial intelligence such as pattern recognition,1,2 great potential especially lies in the hardware realization of such networks. For this purpose, it is useful to investigate and model real biological neural networks in hardware, as a further understanding of its mechanisms can result in novel design principles for electrical circuits. The modeling depth depends on whether abstract technical applications or more bio-inspired applications are desired, which is why various neuron models are deployed. Here, a widely used model is the leaky integrate-and-fire model.3,4 In its basic form, this is a rather simple model, yet its structure is biologically motivated as it is based on modeling ion channel conductances and is hence in general well interpretable. It is often realized by...
Two biologically more accurate and also well interpretable models are the Hodgkin-Huxley\cite{7,9} and the Morris-Lecar model\cite{10,11} which have later been proposed to be memristive circuits.\cite{12,13} More abstract models are, for example, the Izhikevich\cite{14} and the Hindmarsh-Rose model\cite{15} which can mimic versatile neuronal behaviors but are less close to biology and thus less interpretable. In contrast to the original Hodgkin-Huxley and Morris-Lecar model, the more abstract models can, for example, also exhibit a bursting behavior being observed from certain neuron types.\cite{14} The Izhikevich model is especially popular in the context of field programmable gate array (FPGA) realizations,\cite{16–18} while the Hindmarsh-Rose model is either realized by integrator circuits\cite{19–21} or digitally by FPGAs.\cite{22}

In this work, we focus on the Hindmarsh-Rose model since it has already been used to replicate the locomotory circuit of \textit{C. elegans}\cite{23} and is hence a potential candidate to investigate the underlying mechanisms of this neural network. Our aim is to synthesize an equivalent electrical circuit of the Hindmarsh-Rose model starting from the governing differential equations. A circuit synthesis enabling a practical realization can in general be carried out as explained in Itoh.\cite{24} In contrast to this, we intend to find a theoretical, equivalent electrical circuit based on basic circuit elements such as resistors, capacitors, and inductors. These equivalent circuits allow for a model analysis from a circuit point of view and can hence support the design process; see, e.g., Pang et al\cite{25} and Miano et al.\cite{26} In the context of the Hindmarsh-Rose model, an equivalent circuit offers several advantages. First, the resulting circuit and the circuit elements are directly based on the underlying differential equations, allowing for an analysis of the Hindmarsh-Rose model from a circuit theoretical point of view. In particular, the role and importance of the individual circuit components for the overall functionality can be investigated by observing power flows as well as stored energy. This can serve as a basis for a model reduction and can hence lead to a simpler circuit realization while maintaining the major functionality. Second, this approach enables the derivation of a biologically well-interpretable circuit model. This supports a better interpretation of observed behavior from real biological neural networks, which is important for deriving corresponding design principles for circuits.

The authors view this work as important for a deeper understanding of the Hindmarsh-Rose model from a circuit theoretic point of view. The main contributions are as follows: first, we present a systematic circuit synthesis of the Hindmarsh-Rose model. Second, we propose an equivalent circuit for the original three-dimensional as well as for a memristive Hindmarsh-Rose model. Third, the proposed equivalent circuits allow for an analysis of the model’s behaviors with respect to circuit theory.

In order to synthesize an equivalent circuit, we make use of a systematic approach for a circuit synthesis depicted in Figure 1 and based on Ochs.\cite{27,28} In particular, we first linearize the Hindmarsh-Rose model in Section 2 in order to obtain a linearized state-space model. The latter serves as the basis for a systematic circuit synthesis\cite{27,28} discussed in Section 3, leading to a linear electrical circuit. Due to the Hartman-Grobman theorem\cite{29} stating that the local behavior of a nonlinear system near a hyperbolic equilibrium point can be characterized by the behavior of its linearization near the equilibrium point, the linear circuit can be assumed to have the same circuit structure as the nonlinear circuit. Hence, the linear circuit can be extended to a nonlinear circuit being an equivalent circuit to the original nonlinear Hindmarsh-Rose model, which is presented in Section 4. Furthermore, we briefly investigate a memristive Hindmarsh-Rose model.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{systematic_approach.png}
\caption{A systematic approach for synthesizing an equivalent electrical circuit of the Hindmarsh-Rose model}
\end{figure}
Rose model in Section 5 potentially offering a circuit structure more close to conductance-based models such as the Hodgkin-Huxley model. Simulation results in order to verify the equivalent circuits are presented in Section 6, and a conclusion is given in Section 7.

2 | LINEARIZATION OF THE HINDMARSH-ROSE MODEL

2.1 | Hindmarsh-Rose model

The (three-dimensional) Hindmarsh-Rose model is governed by the following set of equations:

\[
\begin{align*}
\dot{x}_1 &= -ax_1^3 + bx_1^2 + x_2 - x_3 + I, \\
\dot{x}_2 &= c - dx_2^2 - x_2, \\
\dot{x}_3 &= \varepsilon [s(x_1 - \phi_r) - x_3],
\end{align*}
\]

where \(x_1\) is the membrane potential; \(x_2\) is the recovery variable or spiking variable and denotes an ionic current; \(x_3\) is the bursting variable and denotes an ionic current as well; \(I\) denotes an externally applied current; \(a, b, c, d,\) and \(s\) are some positive constants; \(\varepsilon\) is a small positive constant; and \(\phi_r\) is the resting potential.

2.2 | Linearization

In the following, we derive the linearized state-space model of the Hindmarsh-Rose model, because it serves as a basis for the desired circuit synthesis. In particular, the state-space representation allows for a closed circuit representation of the set of linearized differential equations. For this purpose, we first consider the equilibrium points, which can be (implicitly) calculated to

\[
\begin{align*}
x_{2,\infty} &= c - dx_{1,\infty}^2, \\
x_{3,\infty} &= s(x_{1,\infty} - \phi_r), \\
0 &= x_{1,\infty}^3 + \frac{d-b}{a} x_{1,\infty} + \frac{s}{a} x_{1,\infty} - c + I + s\phi_r.
\end{align*}
\]

Based on the equilibrium points, the linearized set of differential equations can be expressed by

\[
\dot{x} = A(x_{1,\infty}) (x - x_{\infty}) + bI,
\]

\[
A(x_1) = \begin{bmatrix} -3ax_1^2 + 2bx_1 & 1 & -1 \\ -2dx_1 & -1 & 0 \\ \varepsilon s & 0 & -\varepsilon \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad x_{\infty} = \begin{bmatrix} x_{1,\infty} \\ x_{2,\infty} \\ x_{3,\infty} \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},
\]

where \(A(x_{1,\infty})\) denotes the Jacobian matrix evaluated at the equilibrium points.

3 | CIRCUIT SYNTHESIS OF A LINEAR ELECTRICAL CIRCUIT

Let us now synthesize an equivalent circuit of the linearized state-space model of the Hindmarsh-Rose model. This is motivated by the assumption that the resulting circuit is structurally identical to the nonlinear equivalent circuit. The synthesis can be carried out by substituting the state-space variables and matrices by physically meaningful variables and matrices, with which an abstract circuit representation is obtained. The associated, abstract circuit elements are then investigated in detail to synthesize the individual circuit elements. In this concern, we define the states as inductor current, i.e., \(i = x\) and consequently \(i_{\infty} = x_{\infty}\). In a similar way, the state derivatives are defined as inductor voltage, such that the set of differential equations can be expressed by the mesh rule.
\[
\begin{align*}
\mathbf{u} &= -R_\infty \mathbf{i} + \mathbf{e}_\infty, \\
\mathbf{u} &= L_\infty \frac{\mathrm{d}\mathbf{i}}{\mathrm{d}t}, \\
L_\infty &= L_0 \\
R_\infty &= \frac{R_0}{L_0} L_\infty A(i_{1,\infty}), \\
e_\infty &= R_\infty i_{\infty} + \frac{R_0}{L_0} L_\infty b_1,
\end{align*}
\]

where \( \mathbf{u} \) denotes the inductor voltages, \( L_\infty \) denotes an inductance matrix with a normalization inductance \( L_0 \), \( R_\infty \) denotes a resistance matrix with a normalization resistance \( R_0 \), and \( e_\infty \) denotes voltage sources. Note that the \( \infty \)-subscript indicates the dependency on the equilibrium points.

In general, the above equations show a third-order system described by a vector valued series interconnection of inductor and resistive voltage source. It can be deduced from the inductance matrix being of diagonal structure that the resulting circuit consists of three series interconnections each being made of an inductance and a resistive voltage source. The coupling between those series interconnections, however, is not clear yet, since the implementation of the resistance matrix is more complicated. In order to synthesize this resistance matrix, we decompose it into its symmetric and skew-symmetric part, because the symmetric part and skew-symmetric part can in general be realized by ideal transformers with resistors respective gyrators.\(^{28}\) This decomposition yields

\[
R_\infty = R_{s,\infty} + R_G + R_G^T, \quad \text{with } R_{s,\infty} = R_0 \\
\begin{bmatrix}
3ai_{1,\infty}^2 - 2bi_{1,\infty} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \quad \text{and } R_G = R_0 \\
\begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
-1 & 0 & 0
\end{bmatrix},
\]

where \( R_{s,\infty} \) and \( R_G + R_G^T \) are the symmetric part and the skew-symmetric part, respectively, with the gyration resistance matrix \( R_G \). Since the symmetric part is a diagonal matrix, the ideal transformers have a turns ratio of 1 and can hence be neglected. As a result, the symmetric part only consists of resistors whose values are given by the diagonal elements. The skew-symmetric part, on the other hand, results in two gyrators with gyration resistance \( R_0 \). Due to the symmetric part, the resistive voltage sources have a resistance equal to the diagonal elements of \( R_{s,\infty} \). Moreover, it follows from the skew-symmetric part that the first series interconnection is interconnected to the second and third series interconnection by a gyrator with gyration resistance \( R_0 \), respectively. The resulting equivalent circuit is depicted in Figure 2.

![Figure 2](image-url)
### 3.1 Simplification of the synthesized circuit

The circuit synthesized so far is indeed equivalent to the linearized Hindmarsh-Rose model. Compared to the equivalent circuits of the Hodgkin-Huxley or the Morris-Lecar model, it is, however, less accessible and interpretable due to the gyrators. Therefore, we aim to manually simplify the circuit in order to achieve a structure similar to the above mentioned models. This can be done by combining the gyrators into the middle series interconnection, due to which the latter becomes a parallel interconnection of a capacitor and a resistive current source. Furthermore, this leads to a cascade of the two gyrators which is similar to an ideal transformer with turns ratio 1. Since this can be neglected, the resulting circuit consists of two series interconnections parallel interconnected to a capacitor and a resistive current source. This is depicted in Figure 3A, where the corresponding circuit elements are defined by

\[
G_{1,\infty} = \frac{3au_{1,\infty}^2 - 2bu_{1,\infty}}{R_0 U_0^2}, \quad L_{2,\infty} = L_2 \frac{U_0}{2u_{1,\infty}}, \quad R_{2,\infty} = R_2 \frac{U_0}{2u_{1,\infty}}, \\
J_{1,\infty} = I + G_{1,\infty}u_{1,\infty} - l_{2,\infty} + i_{3,\infty}, \quad e_{2,\infty} = R_{2,\infty}l_{2,\infty} + u_{1,\infty}, \quad e_{3,\infty} = R_3i_{3,\infty} - u_{1,\infty}, \\
C_1 = \frac{L_0}{R_0^2}, \quad L_2 = \frac{L_0}{d}, \quad R_2 = \frac{R_0}{d}, \quad L_3 = \frac{L_0}{es^2}, \quad R_3 = \frac{R_0}{s},
\]

where \(U_0\) is a normalization voltage. The circuit structure of this model is similar to conductance-based models such as the Hodgkin-Huxley or the Morris-Lecar model, because the membrane voltage is modeled by a capacitor, the input signal is given by a current source, and ionic currents due to ion channels are modeled by additional parallel interconnections. The memory aspects of these ion channels are modeled by additional inductors, which is in contrast to the previously mentioned models, where memristors account for the memory.

### 4 NONLINEAR ELECTRICAL CIRCUIT

We now aim to synthesize an equivalent electrical circuit of the original, nonlinear Hindmarsh-Rose model. For this purpose, we derive the equations for the nonlinear circuit elements, because we assume that the nonlinear circuit is structurally identical to the linear one. Due to this, we model the nonlinear circuit as shown in Figure 3B, which is similar to the circuit shown in Figure 3A with the only difference that the linear circuit elements depending on the equilibrium have been replaced by nonlinear circuit elements denoted by a hat. Note that the latter does not hold true for \(e_3\) and \(j_1\) since they remain linear, as can be seen in the following.

We derive the definitions for the circuit elements by considering the mesh and node rules of the nonlinear circuit and by comparing them to the set of nonlinear differential equations. In particular, the mesh and node rules read

![Figure 3](image-url)

**Figure 3** Equivalent circuits of the Hindmarsh-Rose model: (A) The linear Hindmarsh-Rose circuit being a simplified version of the circuit depicted in Figure 2 with its circuit elements defined in equation (6) and (B) the nonlinear Hindmarsh-Rose circuit with its circuit elements defined in equation (9)
We compare these equations to the original set of differential equations by reformulating Equation 1, such that

\[
\begin{align*}
\dot{x}_1 &= -[ax_1^2 - bx_1]x_1 + x_2 - x_3 + I, \\
\frac{1}{dx_1} \dot{x}_2 &= -x_1 - \frac{1}{dx_1}x_2 + \frac{c}{dx_1}, \\
\frac{1}{es} \dot{x}_3 &= x_1 - \frac{1}{s}x_3 - \phi_r.
\end{align*}
\] (8)

These equations resemble the structure of the above mesh and node rules, which can be seen when associating \(x_1\) with \(u_1\), \(x_2\) with \(i_2\) and \(x_3\) with \(i_3\). It then follows that the circuit elements previously depending on the equilibrium points are now defined as

\[
\begin{align*}
\hat{G}_1 &= au_1^2 - bU_0u_1, \\
\hat{L}_2 &= L_2 \frac{U_0}{u_1}, \\
\hat{R}_2 &= R_2 \frac{U_0}{u_1}, \\
\hat{j}_1 &= I, \\
\hat{\dot{e}}_2 &= \frac{cU_0^2}{du_1}, \\
e_3 &= -\phi_r.
\end{align*}
\] (9)

while the other linear circuit elements remain the same as in the linear equivalent circuit.

### 4.1 Positive capacitor voltage: \(u_1 > 0\) V

Considering the definitions for the nonlinear circuit elements, it can be seen that the nonlinear conductance is not always physically meaningful, because it becomes negative for small but positive voltages \(u_1\), indicating an active behavior. In order to modify the circuit structure such that only positive and hence passive conductances occur for \(u_1 > 0\) while simultaneously remaining at an equivalent circuit, we rewrite the definition for \(\hat{G}_1\) such that

\[
\hat{G}_1 = \hat{G}_{1a} - \hat{G}_{1b}, \\
\hat{G}_{1a} = \frac{au_1^2}{R_0U_0^2}, \\
\hat{G}_{1b} = \frac{b}{R_0} \frac{|u_1| \sigma(u_1/U_0)}{U_0},
\] (10)

where at first, \(\sigma(\cdot)\) is the sign-function. Considering the case that \(u_1 > 0\), \(\hat{G}_{1a}\) and \(\hat{G}_{1b}\) are both passive conductances and the negative sign in front of \(\hat{G}_{1b}\) can be achieved by connecting \(\hat{G}_{1b}\) to a negative impedance converter (NIC) yielding \(i_{G_{1b}} = -i_{G_{1a}}\); see Figure 4A.

A possible realization of the NIC based on Belevitch\(^{31}\) is shown in Figure 5, which requires a negative resistor as the only active component.

### 4.2 Negative capacitor voltage: \(u_1 < 0\) V

While the conductance \(\hat{G}_1\) remains passive for negative voltages, i.e., \(u_1 < 0\), the circuit elements \(\hat{L}_2\) and \(\hat{R}_2\) become negative in this case. Hence, they are only physically meaningful for \(u_1 > 0\). In a similar way as before, we modify our circuit for the case that \(u_1 < 0\) to ensure that \(\hat{L}_2\) and \(\hat{R}_2\) remain positive. For this purpose, we rewrite the definitions for the circuit elements to
\[ L_2 = L_2 f(u_1), \quad R_2 = R_2 f(u_1), \quad f(u) = U_0 \frac{\sigma(u)}{|u|} \]

and deploy an NIC to which the left series interconnection of the equivalent electrical circuit is interconnected; see Figure 4B. Here, it holds that \( i_2 = -i'_2 \).

### 4.3 Almost vanishing capacitor voltage: \( u_1 \approx 0 \) V

Observing the definitions for the inductance \( L_2 \) and the resistance \( R_2 \), it can be seen that they tend towards infinite values when \( u_1 \to 0 \). We aim to mitigate this case by limiting the values for \( L_2 \) and \( R_2 \) as this is physically more meaningful. This can be achieved by modifying the function \( f(u) \), such that

\[ L_2 = L_2 f(u_1), \quad R_2 = R_2 f(u_1), \quad f(u) = U_0 \frac{\sigma(u)}{\epsilon + |u|} \]

where \( \epsilon \) is a small, positive constant ensuring that the denominator does not become zero. The modified function is exemplary shown in Figure 6, where the sign-function has been replaced by the tanh-function to account for a continuous switching behavior. Note that even though \( u_1 = 0 \) V now theoretically leads to circuit element values being zero, this case can be mitigated when assuming a fast transition from \( u_1 \approx 0^+ \) to \( u_1 \approx 0^- \). This is supported by the fact that \( u_1 = 0 \) is not an equilibrium point for typical parameter sets of the Hindmarsh-Rose model given in Table A1.
4.4 Equivalent circuit for positive and negative $u_1$

As has been previously discussed, the two cases for $u_1$ pose two different circuit realizations when aiming for meaningful circuit parameters. Our aim is now to combine these two cases into one circuit realization which remains an equivalent electrical circuit to the nonlinear Hindmarsh-Rose model, such that only positive inductances and resistances are required. These positive circuit elements are related to the previous defined circuit elements by

$$\tilde{L}_2 = \frac{L_2}{\sigma\left(\frac{u_1}{U_0}\right)}, \quad \tilde{R}_2 = \frac{R_2}{\sigma\left(\frac{u_1}{U_0}\right)}, \quad \tilde{G}_{1b} = \frac{G_{1b}}{\sigma\left(\frac{u_1}{U_0}\right)}.$$  \hspace{1cm} (13)

Making use of the positive circuit elements, the combination of the two cases $u_1 > 0$ and $u_1 < 0$ can be achieved by utilizing switches enabling a switching between an NIC and a short circuit, as depicted in Figure 4C. These switches act based on $\sigma\left(\frac{u_1}{U_0}\right)$, and due to this, it holds that $i_2' = \sigma\left(\frac{u_1}{U_0}\right)i_2$ and $i_2'_{G_{1b}} = -\sigma\left(\frac{u_1}{U_0}\right)i_2_{G_{1b}}$.

It should be mentioned for the approach with an NIC and switches that this introduces an additional resistance, which is why the circuit is no longer equivalent to the nonlinear Hindmarsh-Rose model. This additional resistance can be seen when evaluating the corresponding mesh rule

$$\tilde{L}_2 \frac{di_2'}{dt} = -u_1 + \dot{e}_2 - \tilde{R}_2i_2', \hspace{1cm} (14a)$$

$$\tilde{L}_2 \frac{d}{dt} i_2 \sigma\left(\frac{u_1}{U_0}\right) = -u_1 + \dot{e}_2 - \tilde{R}_2 \sigma\left(\frac{u_1}{U_0}\right)i_2, \hspace{1cm} (14b)$$

$$\tilde{L}_2 \frac{di_2}{dt} = -u_1 + \dot{e}_2 - i_2 \left[ \tilde{R}_2 + \tilde{L}_2 \frac{d\sigma\left(\frac{u_1}{U_0}\right)}{dt} \right]. \hspace{1cm} (14c)$$

In order to maintain an equivalent electrical circuit, we eliminate this additional resistance by modifying the definition for the resistance $\tilde{R}_2$ such that

$$\tilde{R}_2 = R_2f(u_1) - \tilde{L}_2 \frac{d\sigma\left(\frac{u_1}{U_0}\right)}{dt}.$$  \hspace{1cm} (15)

At first glance, the modified circuit structure now consisting of an NIC differs from conductance-based models being especially close to biology since the NIC is an active component. This can, however, be biologically motivated. In particular, the generation of action potentials is not only based on passive ion transport but also requires energy for active transport due to the activity of, for example, ion pumps regenerating the equilibrium potential.
based models, this is only superficially taken into account by a linear resistive voltage source modeling leakage current. Hence, the NIC present in our circuit model can be seen as a deeper modeling of ion pump activity.

The proposed theoretical equivalent electrical circuit shows that, in general, only one active component, namely, the negative resistor of the NIC, is required for an implementation of the Hindmarsh-Rose model. In principle, the presented circuit can also be used to derive a practical circuit implementation. In this context, the voltage-controlled switches can be implemented by for example transistors, where each switch is realized by two transistors whose control voltage $u_1$ and $-u_1$ is applied by a (inverting) buffer amplifier, respectively. The NIC can be realized by making use of an operational amplifier or by current conveyors, and the nonlinear resistors $\tilde{G}_{1a}$, $\tilde{G}_{1b}$ and $\tilde{R}_2$ can be realized by utilizing adders and multipliers. On the other hand, $L_2$ can be implemented by interconnecting an appropriate nonlinear resistor to a mutator. Since the realization of the nonlinear circuit elements is in general rather complex, it can be, however, more convenient to utilize the circuit synthesis approach of Itoh when aiming for a practical circuit realization. This approach is especially focused on an implementable circuit synthesis, where one first synthesizes an electrical circuit based on the individual differential equation and then simplifies the resulting circuit.

5 | MEMRISTIVE HINDMARCH-ROSE MODEL

In addition to the original Hindmarsh-Rose model, variations of this model have been reported as well. In this section, we briefly investigate a memristive Hindmarsh-Rose model presented in Bao et al, since the use of memristors suggests an equivalent electrical circuit closer to the ones for the Hodgkin-Huxley or Morris-Lecar model. The memristive Hindmarsh-Rose model yields

$$
\begin{align*}
\dot{x}_1 &= -ax_1^3 + bx_1^2 + x_2 + kx_1x_3 + I, \\
\dot{x}_2 &= c - dx_1^2 - x_2, \\
\dot{x}_3 &= \frac{1}{\alpha}x_1,
\end{align*}
$$

(16)

where $kx_1x_3$ is the externally applied electromagnetic induction of the newly introduced memristor, $k$ is the corresponding strength of the electromagnetic induction assumed to be positive, and $\alpha$ is a scaling factor accounting for different time scales. Compared to the original model described in Equation 1, here, the first and third differential equations are modified. Based on the insights from before, this indicates that the series interconnection with the constant inductor $L_3$ is modified, while the rest of the circuit remains unchanged. In particular, the third differential equation is the state equation of an ideal flux controlled memristor described by

$$
\begin{align*}
i_3 &= W_3(\phi)u_1, \\
W_3(\phi) &= -k\phi, \\
\dot{\phi} &= \frac{1}{\alpha}u_1,
\end{align*}
$$

(17)

where $\phi$ is the flux and denotes the third state variable. Due to this modeling, $kx_1x_3$ describes the current due to the memristor, due to which a parallel interconnection of the memristor and the capacitor can be deduced. The complete electrical circuit for the memristive Hindmarsh-Rose model is shown in Figure 7, where we again make use of an NIC with switches. Compared to the circuit for the original model, the series interconnection with the constant inductor has been replaced by a memristor. This memristor can again be seen as a model for an ionic current, and hence, the resulting overall circuit is indeed closer to the conductance-based approach as seen from the Hodgkin-Huxley and Morris-Lecar model. In this context, the absence of a voltage source indicates that the type of ions modeled by the memristor has an equilibrium potential of 0 V.

6 | SIMULATION RESULTS

Let us now simulate the equivalent circuits of the original and the memristive Hindmarsh-Rose model to verify the functionality of the proposed circuits. Here, we consider the circuits without NIC and switches as reference, since they are directly equivalent to the set of differential equations. For this purpose, we make use of the parameter sets shown in Table A1. These parameter sets are a scaled version of the parameters reported in Barrio et al. The scaling allows
changing the magnitudes of time, voltage, and current originally given by s, V, and A, respectively, to ns, mV, and mA, respectively, which are more reasonable magnitudes both in terms of biology and integrated circuits. As normalization parameters, we set \( L_0 = 1 \text{ nH}, \) \( R_0 = 1 \Omega, \) and \( U_0 = 1 \text{ V}, \) and the small positive constant ensuring the denominator of \( \frac{\sigma}{L^2} \) and \( \frac{\sigma}{R^2} \) does not become zero is chosen to be \( \epsilon = 10^{-6}. \) Moreover, we deploy \( \tanh(\frac{u}{U_0}) \) as a replacement for the sign function \( \sigma(\frac{u}{U_0}) \) to account for a continuous switching, enabling a better numerical stability. Here, \( U_s = 10\mu V \) denotes the slope of the tanh-function. Simulation results for the original nonlinear equivalent circuits without NIC and switches are obtained through LTspice. Results for the equivalent circuits with NIC and switches are obtained by utilizing an ODE solver of Matlab, since LTspice failed due to numerical issues.

**6.1 | Original Hindmarsh-Rose model**

The Hindmarsh-Rose model’s behavior is based on the interplay of a fast and a slow subsystem. Concerning our proposed theoretical circuit, the fast subsystem is described by the capacitor voltage \( u_1 \) and the inductor current \( i_1, \) while the slow subsystem is governed by the dynamics of the inductor current \( i_3. \) The qualitative behavior of the Hindmarsh-Rose model has been extensively studied in, e.g., Barrio et al., Shilnikov and Kolomiets, and Innocenti et al, and in general gives rise to a rich repertoire of neuronal firing patterns such as spiking and bursting. These studies also apply to our proposed circuit since it is equivalent to the Hindmarsh-Rose model as shown in the previous sections.

In order to verify the functionality of our circuit, let us first consider the spiking behavior which is characterized by a stable limit cycle when observing the phase space. The corresponding simulation results are shown in top row of Figure 8, where the gray curves indicate the reference simulation results.

As can be seen from Figure 8A, the state trajectory indeed reaches a stable limit cycle after four cycles. This can also be seen from Figure 8B, where the capacitor voltage representing the membrane potential of the neuron is depicted and where a consistent firing pattern is achieved after a few spikes. The time it takes to reach the limit cycle represents a firing frequency adaption mechanism and is an important feature in certain neuron types. The spikes themselves consist of a depolarization and a repolarization phase denoting the rising and decreasing of the membrane potential, respectively. A hyperpolarization phase indicating the dropping of the membrane potential below its resting potential, which is typical for some neurons, is not present. Note that both the state trajectory and the capacitor voltage coincide very well with the reference simulation results except for a minimal temporal delay.

Let us also take a look at the power flow at the right side of the NIC presented in Figure 8C, which can be interpreted as the generated energy required for ion pump activity. Typically, the ion pump activity is assumed to be approximately constant during the spiking activity and, as mentioned earlier, primarily responsible for regenerating the resting potential. As can be seen from the simulation results, however, most energy is generated during repolarization and only a smaller amount during the depolarization phase. This indicates that in this model ion pumps might not only be involved in the regeneration of the resting potential, but also in the arising of action potentials.

Let us now consider the bursting behavior as the second important firing pattern, where we in particular focus on a square-wave bursting. This behavior is characterized by a spiking period with increasing interspike intervals.
during depolarization and a quiescent period during hyperpolarization.\textsuperscript{36} Here, the number of spikes during a spiking period coincides with the amount of complete revolutions during one whole cycle observable in the phase space.\textsuperscript{36} The simulation results for this behavior are depicted in the bottom row of Figure 8, where the state trajectory and the capacitor voltage are shown in Figure 8D,E, respectively. It can be seen that the capacitor voltage indeed displays a firing pattern consisting of plateaus with multiple spikes before a hyperpolarization phase occurs, where the number of spikes matches the amount of complete revolutions during one cycle present in the state trajectory. As before, the state trajectory and the capacitor voltage coincide very well with the reference simulation marked by gray color except for a minimal temporal delay. Overall, this verifies the equivalency of the circuit depicted in Figure 5.

Let us again also take a look at the power flow of the NIC depicted in Figure 8F. Here, most energy is generated during the hyperpolarization phase while a much smaller amount is generated during the repolarization phase of the spikes. Considering the NIC again as an indicator of ion pump activity, this implies that ion pumps are mostly active during the hyperpolarization phase and to a lesser extent during the repolarization phase.

### 6.2 Memristive Hindmarsh-Rose model

In the following, we observe the behavior of the equivalent circuit for the memristive Hindmarsh-Rose model, for which purpose we make use of the parameters defined in the top of Table A1. Moreover, we set $k = 0.9 \cdot 10^3 \frac{A}{V^2}$ and $\alpha = 10^{-9}$. Our aim is to verify the functionality by simulating the spiking behavior. As can be seen from Figure 9A, the state trajectory again shows a stable limit cycle, which is supported by the capacitor voltage depicted in Figure 9B. In contrast to the original Hindmarsh-Rose model, however, no spike frequency adaption is observable. As in the case of the original Hindmarsh-Rose model, the results for the state trajectory and capacitor voltage coincide excellently with the reference simulation results highlighted in gray, verifying the equivalency of the circuit depicted in Figure 7.
Moreover, considering the power flow of the NIC shown in Figure 9C, this highlights that most energy is generated during the repolarization and only a smaller amount during the depolarization. Interpreting the NIC again as an indicator for the ion pump activity, this implies that they are mostly active during the repolarization, which coincides with the previous results.

7 CONCLUSION AND FUTURE RESEARCH

In this work, we have considered the three-dimensional Hindmarsh-Rose model as well as a memristive variant of this model with the aim to synthesize a theoretical equivalent electrical circuit. For this purpose, we have linearized the Hindmarsh-Rose model enabling a systematic approach for the circuit synthesis. In particular, we have derived a linear circuit based on the corresponding state-space model. This circuit has then been extended to a nonlinear circuit by making use of the structural equality of the linear and nonlinear circuit. The resulting circuit is in general well-interpretable since it is structurally similar to conductance-based models such as the Hodgkin-Huxley or the Morris-Lecar. In particular, it consists of a capacitance accounting for the membrane potential, additional parallel interconnections modeling ion currents and an NIC which can be interpreted as a deeper modeling of ion pump activity. This good interpretability enables a circuit-based investigation of the Hindmarsh-Rose model. In particular, it can be examined in future research how the individual circuit elements contribute to the overall biological functionality, allowing for a circuit simplification while maintaining the major functionality. Due to this, the proposed equivalent circuits can serve as a starting point towards energy-efficient hardware realizations of real biological neural networks based on the Hindmarsh-Rose model. Simulation results have verified that the presented circuits are indeed equivalent, while the power flow of the NIC has furthermore suggested that in this model, ion pump activity is predominantly present during the repolarization or hyperpolarization phase.

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CONFLICT OF INTEREST

The authors declare no potential conflict of interest.

DATA AVAILABILITY STATEMENT

Research data are not shared.

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APPENDIX A

TABLE A1 Scaled parameter set for a spiking and bursting behavior of the Hindmarsh-Rose model based on Barrio et al.16

| **Spiking parameters** |  |
|-------------------------|-------------------|
| $a = 10^6$              | $b = 3 \cdot 10^3$ |
| $c = 10^{-3}$           | $d = 5 \cdot 10^3$ |
| $s = 4$                 | $\epsilon = 0.05$ |
| $\phi = -1.6$ mV       | $I = 4$ mA         |

| **Bursting parameters** |  |
|-------------------------|-------------------|
| $a = 10^6$              | $b = 2.225 \cdot 10^3$ |
| $c = 10^{-3}$           | $d = 5 \cdot 10^3$ |
| $s = 4$                 | $\epsilon = 0.05$ |
| $\phi = -1.6$ mV       | $I = 4.2$ mA       |