LOCAL RIGIDITY OF CERTAIN ACTIONS OF SOLVABLE GROUPS ON THE BOUNDARIES OF RANK-ONE SYMMETRIC SPACES

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ABSTRACT. Let $G$ be the group of orientation-preserving isometries of a rank-one symmetric space $X$ of non-compact type. We study local rigidity of certain actions of a solvable subgroup $\Gamma \subset G$ on the boundary of $X$, which is diffeomorphic to a sphere. When $X$ is a quaternionic hyperbolic space or the Cayley hyperplane, the action we constructed is locally rigid.

1. INTRODUCTION

One of the most active areas of the study of rigidity of group actions is around the Zimmer program, in which many remarkable properties of actions of a lattice $\Gamma$ of a higher rank Lie group have been discovered. See [5] for the recent development. As pointed out in [5], in the study of actions of a lattice of a higher rank Lie group, the study of actions of a higher rank abelian group $\Gamma = \mathbb{Z}^n$, $n \geq 2$, of certain hyperbolicity plays an important role. On the other hand, Burslem and Wilkinson showed that there exists a solvable group $\Gamma$ which does not contain a higher rank abelian group such that an action on the circle $S^1$ is locally rigid [3]. In this paper, we consider locally rigid actions of solvable groups which does not contain a hyperbolic action of a higher rank abelian group.

As a higher dimensional analogue of the result of Burslem and Wilkinson, Asaoka constructed an action of a solvable group on $S^n$, $n \geq 2$ [1]. Asaoka showed that, while the action is not locally rigid, it is locally rigid in a weaker sense. One of the most important example of such a weak form of local rigidity is [6]. In [2], Asaoka studied local rigidity of an action of the same group on the torus $T^n$, which can also be viewed as a higher dimensional version of the result of Burslem and Wilkinson. In [8], the author studied local rigidity of certain action of a solvable group on the sphere. In [13], Wilkinson and Xue studied rigidity of an action of a solvable group on the torus.

In this paper, we consider a generalization of the results of [1] and [8] which can be formulated as follows. Let $X$ be a rank-one symmetric space of non-compact type, $G$ the group of orientation-preserving isometries of $X$, and $G = \text{KAN}$ an Iwasawa decomposition.
**Definition 1.1.** A subgroup $\Gamma$ of $AN \subset G$ is called a standard subgroup of $G = \text{KAN}$ if $\Gamma$ is generated by a lattice $\Lambda$ of $N$ and a nontrivial element $a \in \Lambda$ such that $a\Lambda a^{-1} \subset \Lambda$.

Let $M \subset K$ be the centralizer of $A$ in $K$ so that $P = MAN$ is a minimal parabolic subgroup of $G$. Then the homogeneous space $G/P$ is diffeomorphic to a sphere. The action of $G$ on $G/P$ by the left translation will be denoted by $l : G \to \text{Diff}(G/P)$. The following theorem, which can be referred to as $C^2$-local rigidity of $l|_{\Gamma}$ up to embedding of $\Gamma$ into $G$, is the main theorem of this paper.

**Theorem 1.2.** Let $G$ be the group of orientation-preserving isometries of a rank-one symmetric space of non-compact type, $\Gamma$ a standard subgroup of $G$, and $l|_{\Gamma}$ the action of $\Gamma$ on $G/P$ by left translations. Assume $G \neq \text{PSL}(2, \mathbb{R})$. If $\rho$ is a $C^\infty$ action of $\Gamma$ on $G/P$ sufficiently $C^2$-close to $l|_{\Gamma}$, then there is an embedding $\iota$ of $\Gamma$ into $G$ as a standard subgroup and a $C^\infty$ diffeomorphism $h$ of $G/P$ such that

$$\rho(g) = h \circ l(\iota(g)) \circ h^{-1}$$

for all $g \in \Gamma$.

While we excluded the case $G = \text{PSL}(2, \mathbb{R})$ for a technical reason, the claim also holds. In this case, $\Gamma$ can be presented as

$$\langle a, b \mid aba^{-1} = b^k \rangle$$

for some integer $k \geq 2$ and $G/P$ is diffeomorphic to a circle $S^1$. The action $l|_{\Gamma}$ admits a common fixed point and the action on the complement, which is diffeomorphic to $\mathbb{R}$, is given by

$$a \cdot x = kx, \quad b \cdot x = x + 1 \quad (x \in \mathbb{R}).$$

The local rigidity of the action follows from the result of Burslem and Wilkinson mentioned above. It is not difficult to check that the case $G = \text{SO}_0(n+1, 1)$, $n \geq 2$ is exactly the above result of Asaoka. The case $G = \text{SU}(n+1, 1)$, $n \geq 2$ for $C^3$-small perturbation is the above result of the author.

When $G = \text{Sp}(n+1, 1)$, $n \geq 2$ or $F_4^{20}$, we can show the inclusion $\Gamma \hookrightarrow G$ is locally rigid. So we obtain local rigidity in the strict sense:

**Corollary 1.3.** For $G = \text{Sp}(n+1, 1)$ ($n \geq 2$) and $F_4^{20}$, the action $l|_{\Gamma}$ of a standard subgroup $\Gamma$ of $G$ on $G/P$ is $C^2$-locally rigid; a $C^\infty$ action sufficiently $C^2$-close to $l|_{\Gamma}$ is $C^\infty$-conjugate to $l|_{\Gamma}$.

It should be pointed out that the action $l|_{\Gamma}$ is not locally rigid in the remaining cases. When $G = \text{SO}_0(n+1, 1)$, $n \geq 2$, the classification up to conjugacy of the actions of standard subgroups by left translations is given in [1]. In particular, the action $l|_{\Gamma}$ is not locally rigid. When $G = \text{SU}(n+1, 1)$, $n \geq 2$, we can also show that $l|_{\Gamma}$ is not locally rigid. See Proposition 8.4.

The proof of the main theorem can be described as follows. Set $o = eP \in G/P$. The point $o$ is the common fixed point of the action $l|_{\Gamma}$. Using Stowe’s theorem [11], we see that an action close to $l|_{\Gamma}$ also admits a common fixed point close to $o$. Moreover, by an argument similar to that of [1], a conjugacy defined around the common fixed points extends to a diffeomorphism of the whole $G/P$. 

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This content is a part of a journal article, indicating that it is intended for a scholarly context. The definitions, theorems, and corollaries are presented with mathematical rigor, typical of research in dynamical systems or related fields. The content is dense and technical, requiring a background in advanced mathematics to fully comprehend.
which means that perturbation of the homomorphism of a (non-unitary) representation theory as well as an explicit classification of rank 1 such that the Taylor expansion is determined by the derivatives of order at most r at the fixed point, where r depends on the “resonance” of the eigenvalues of the first-order derivative of the diffeomorphism at the fixed point. The last step is the classification of elements in the group $J^r(\mathbb{R}^n, 0)$ of r-jets at 0 $\in \mathbb{R}^n$ of diffeomorphisms around 0 $\in \mathbb{R}^n$.

In our case, the problem can be reduced to local rigidity of a homomorphism of $\Gamma$ into $J^3(G/P, o)$. Computing the induced homomorphism of $\Gamma$ into $J^3(G/P, o)$, and using a theorem of Malcev, we will show that the problem can be reduced to local rigidity of a homomorphism of the closure $\langle a \rangle N$ of $\Gamma$ in $G$ into $J^3(G/P, o)$. Then the problem reduces to the computation of the cohomology $H^1(n, j^3(G/P, o))^{(a)} = H^1(n, j^3(G/P, o))^a$, where $n, j^3(G/P, o)$, and $A$ denote the Lie algebras of $N$, $j^3(G/P, o)$, and $A$, respectively. The computation of such a cohomology is, as we can see in [1] and [8], one of the most difficult part of the proof. In this paper, we will compute the cohomology using some tools from (non-unitary) representation theory as well as an explicit classification of rank one simple Lie algebras. As a result, we obtain an isomorphism

$$H^1(n, j^3(G/P, o))^a = H^1(n, g)^a,$$

which means that perturbation of the homomorphism of $\langle a \rangle N$ into $J^3(G/P, o)$ is locally rigid up to embedding of $\langle a \rangle N$ into $G$. Moreover, it is not difficult to check $H^1(n, g)^a = 0$ if and only if $G = Sp(n + 1, 1)$ ($n \geq 2$) or $F_{4-20}$, in which case our result is in fact local rigidity in the strict sense.

In Section 2, we collect facts which will be used later. In particular, in Subsection 2.4 we establish a fundamental property of the action $l|_\Gamma$ of a standard subgroup $\Gamma$ on $G/P$. Section 3 is devoted to the computation of the cohomology of $n$ mentioned above. In Section 4, we compute certain cohomology of a standard subgroup $\Gamma$, vanishing of which is the assumption of the above theorem of Stowe. In Section 5, we study local rigidity of a homomorphism of a standard subgroup $\Gamma$ into the group $J^3(G/P, o)$ of 3-jets. In Section 6, we consider local rigidity of a homomorphism of a standard subgroup $\Gamma$ into the group $\mathcal{F}(G/P, o)$ of Taylor expansions of the diffeomorphisms in $\mathcal{G}(G/P, o)$, called the group of formal transformations. In Section 7, we study local rigidity of a homomorphism of a standard subgroup $\Gamma$ into the group $\mathcal{G}(G/P, o)$ of germs of diffeomorphism defined around $o \in G/P$ fixing $o \in G/P$. In Section 8, we prove the main theorem.
2. Preliminaries

2.1. **Representation of a semisimple Lie algebra.** The goal of this subsection is to introduce Theorem 2.1 and Theorem 2.2. See [12] for the detail. Theorem 2.2 is the formula for cohomology of the nilradical \( n \) of a parabolic subalgebra \( p \) of a complex semisimple Lie algebra \( g \) with the coefficient in a finite-dimensional \( g \)-module, while in Section 3 we have to compute cohomology of \( n \) with the coefficient in an infinite-dimensional \( g \)-module. The proof of Theorem 2.2 due to Casselman and Osborne [4] contains a study of an infinite-dimensional \( g \)-module. Theorem 2.1 is a consequence of the result of Casselman and Osborne, the formulation of which is due to Vogan. We will use Theorem 2.1 for our computation in Section 3.

To state Theorem 2.1, we review representation theory of semisimple Lie algebras. A standard reference is [7]. Let \( g \) be a complex semisimple Lie algebra, \( h \subset g \) a Cartan subalgebra, and \( \Delta(g, h) \subset h^* \) the system of roots. Fix a positivity on \( h^* \). Let \( \Delta^+(g, h) \subset \Delta(g, h) \) be the system of positive roots. Then the subalgebra

\[
b = h \oplus \bigoplus_{\alpha \in \Delta^+(g, h)} g_{\alpha}
\]

is called the corresponding *Borel subalgebra*. A subalgebra \( p \) of \( g \) containing \( b \) is called a *parabolic subalgebra*. A parabolic subalgebra \( p \) admits the decomposition

\[
p = l \oplus n
\]

such that

\[
n = \bigoplus_{\alpha \in \Delta(n, h)} g_{\alpha}
\]

is a nilpotent subalgebra with \( \Delta(n, h) \) contained in \( \Delta^+(g, h) \), and

\[
l = h \oplus \bigoplus_{\alpha \in \Delta(l, h)} g_{\alpha}
\]

is reductive with

\[
\Delta(l, h) = \Delta(g, h) \sim (\Delta(n, h) \cup -\Delta(n, h)),
\]

where \(-\Delta(n, h) = \{-\alpha \mid \alpha \in \Delta(n, h)\} \). The subalgebra

\[
n_- = \bigoplus_{\alpha \in -\Delta(n, h)} g_{\alpha}
\]

is called the *opposite* of \( n \) and we obtain the decomposition

\[
g = n_- \oplus l \oplus n
\]

of \( g \) as a vector space.

Let \( g \) be a complex semisimple Lie algebra, \( U(g) \) the universal enveloping algebra of \( g \), and \( Z(g) \) the center of \( U(g) \). There is an isomorphism called the *Harish-Chandra isomorphism* of \( Z(g) \) onto the algebra \( U(h)^W \) of the Weyl group \( W = W(g, h) \) invariant elements of \( U(h) \) which can be constructed as follows.
Let \( g = g_- \oplus \mathfrak{h} \oplus g_+ \) be the decomposition corresponding to the Borel subalgebra \( \mathfrak{b} = \mathfrak{h} \oplus g_+ \), where \( g_\pm = \bigoplus_{\alpha \in \pm \Delta^+(g, \mathfrak{h})} \mathfrak{g}_\alpha \). By the Poincaré–Birkhoff–Witt theorem, \( U(g) = U(\mathfrak{h}) \oplus (g_- U(g) + U(g) g_+) \).

Let \( p : U(g) \to U(\mathfrak{h}) \) be the projection onto the first component. Define the shift map \( \sigma_{\delta}(g) : U(\mathfrak{h}) \to U(\mathfrak{h}) \) by the extension of \( \mathfrak{h} \to U(\mathfrak{h}), X \to X - \delta(g) X \) as a homomorphism of algebra, where
\[
\delta(g) = \frac{1}{2} \sum_{\alpha \in \Delta^-(\mathfrak{g}, \mathfrak{h})} \alpha
\]
is the lowest form of \( g \). Then the composition \( \gamma = \sigma_{\delta}(g) \circ p : U(g) \to U(\mathfrak{h}) \) is the Harish-Chandra map. It is known that the Harish-Chandra map induces an isomorphism between \( Z(g) \) and \( U(\mathfrak{h})^W \) called the Harish-Chandra isomorphism and that the Harish-Chandra isomorphism does not depend on the choice of a positivity on \( \mathfrak{h}^* \).

Let \( g \) be a complex semisimple Lie algebra and \( Z(g) \) the center of the universal enveloping algebra \( U(g) \). A \( g \)-module is naturally a module over \( U(g) \). A representation of \( g \) is said to admit an infinitesimal character if each element of \( Z(g) \) acts by multiplication by a scalar. In this case, the action of \( Z(g) \) is described by a homomorphism of \( Z(g) \) into \( \mathbb{C} \). Via the Harish-Chandra isomorphism, we obtain a homomorphism of \( U(\mathfrak{h})^W \) into \( \mathbb{C} \). Since \( \mathfrak{h} \) is abelian, \( U(\mathfrak{h}) \) can be considered as the algebra of polynomial functions on \( \mathfrak{h}^* \). It is not difficult to see that a homomorphism of \( U(\mathfrak{h})^W \) into \( \mathbb{C} \) is the evaluation map \( \text{ev}_\lambda \) at a point \( \lambda \in \mathfrak{h}^* \) and \( \text{ev}_\lambda = \text{ev}_\mu \) if and only if \( \lambda \) and \( \mu \) have the same \( W \)-orbit. Such a \( \lambda \) is called an infinitesimal character of the representation.

A typical example of a representation with an infinitesimal character is an irreducible finite-dimensional representation. A weight vector in a \( g \)-module is highest (resp. lowest) if it is annihilated by \( g_+ \) (resp. \( g_- \)). Let \( F^g_\lambda \) be the irreducible finite-dimensional representation of \( g \) with a highest weight vector of weight \( \lambda \). By the construction of the Harish-Chandra isomorphism, we see that it has an infinitesimal character \( \lambda + \delta(g) \).

Let \( g \) be a complex semisimple Lie algebra with the decomposition \( g = n_- \oplus \mathfrak{l} \oplus n \) corresponding to a parabolic subalgebra. A \( g \)-module \( V \) is \( l \)-finite if \( V \) admits a decomposition into the sum of (possibly infinitely many) finite-dimensional representations of \( \mathfrak{l} \).

**Theorem 2.1** ([12, Corollary 3.1.6]). Let \( g \) be a complex semisimple Lie algebra with the decomposition \( g = n_- \oplus \mathfrak{l} \oplus n \) and \( V \) be a representation of \( g \) which admits an infinitesimal character \( \lambda \). Assume \( V \) is \( l \)-finite. Then \( H^\mathfrak{l}(n, V) \) also admits a decomposition into the sum of finite-dimensional representations of \( \mathfrak{l} \). Moreover, a weight \( \mu \in \mathfrak{h}^* \) appears as an \( l \)-highest weight of \( H^\mathfrak{l}(n, V) \) only if \( \mu + \delta(g) \) and \( \lambda \) have the same \( W \)-orbit.

When \( V \) is finite dimensional, using this theorem, one can completely determine \( H^\mathfrak{l}(n, V) \). For \( w \in W \), the smallest number \( n \) such that \( w \) is a product of \( n \) reflections in simple roots is called the length of \( w \), which will be denoted
by $n = \text{length}(w)$. Let $\langle \cdot, \cdot \rangle$ be the bilinear form on $\mathfrak{h}^*$ induced by the restriction of the Killing form on $\mathfrak{g}$ to $\mathfrak{h}$. A weight $\lambda \in \mathfrak{h}^*$ is $\mathfrak{g}$-dominant if $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h})$, and $\mathfrak{l}$-dominant if $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in \Delta^+(\mathfrak{l}, \mathfrak{h}) = \Delta^+(\mathfrak{g}, \mathfrak{h}) \cap \Delta(\mathfrak{l}, \mathfrak{h})$.

Define the subset $W^l$ of $W$ by

$$W^l = \{ w \in W \mid \lambda, \mathfrak{g}$-dominant $\Rightarrow w(\lambda), \mathfrak{l}$-dominant $\}. $$

**Theorem 2.2** (Kostant, [12, Theorem 3.2.3]). Let $\mathfrak{g}$ be a complex semisimple Lie algebra with the decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{l} \oplus \mathfrak{n}$ and $\mathfrak{F}^\mathfrak{g}_\lambda$ be the irreducible finite-dimensional representation of $\mathfrak{g}$ with the highest weight $\lambda$. Then

$$H^r(\mathfrak{n}, \mathfrak{F}^\mathfrak{g}_\lambda) = \bigoplus_{\mu} \mathfrak{F}^\mathfrak{g}_{\mu}$$

as an $\mathfrak{l}$-module, where the sum is taken over $\mu = w(\lambda + \delta(\mathfrak{g})) - \delta(\mathfrak{g})$ for $w \in W^l$, $r = \text{length}(w)$.

### 2.2. Classification of the simple Lie algebras of real rank one

Let $\mathfrak{g}$ be a real simple Lie algebra and $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be an Iwasawa decomposition. The dimension of $\mathfrak{a}$ is called the real rank of $\mathfrak{g}$. Assume the real rank of $\mathfrak{g}$ is one. Then there is a restricted-root decomposition

$$\mathfrak{g} = \bigoplus_{i=-2}^{2} \mathfrak{g}_i$$

such that

$$\mathfrak{n} = \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

where $\mathfrak{g}_i = \{ X \in \mathfrak{g} \mid [H, X] = i\alpha(H)X \text{ for all } H \in \mathfrak{a} \text{ for some } \alpha \in \mathfrak{a}^* \}$. Then the subalgebra

$$\mathfrak{p} = \bigoplus_{i=0}^{2} \mathfrak{g}_i$$

is a minimal parabolic subalgebra of $\mathfrak{g}$. Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathfrak{g}$. There exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_{\mathbb{C}}$ such that $\mathfrak{h} \subset (\mathfrak{g}_0)_{\mathbb{C}}$. We fix a positivity on $\mathfrak{h}^*$ such that $\Delta(\mathfrak{n}_{\mathbb{C}}, \mathfrak{h}) \subset \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$. Then $\mathfrak{p}_{\mathbb{C}} = (\mathfrak{g}_0)_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}$ is the decomposition of the parabolic subalgebra $\mathfrak{p}_{\mathbb{C}}$ as in Subsection 2.1.

By the classification of real simple Lie algebras, $\mathfrak{g}_{\mathbb{C}}$ is isomorphic to $\mathfrak{so}(n, \mathbb{C})$, $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{sp}(2n, \mathbb{C})$, or $\mathfrak{f}_4$. In each case, the system $\Delta(\mathfrak{g}, \mathfrak{h})$ of roots can be expressed as follows. Let $\langle \cdot, \cdot \rangle$ be the bilinear form on $\mathfrak{h}^*$ induced by the restriction of the Killing form on $\mathfrak{g}$ to $\mathfrak{h}$.

When $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(2n+1, \mathbb{C})$, $n \geq 1$, there is a complex basis $e_1, \ldots, e_n$ of the dual $\mathfrak{h}^*$ of $\mathfrak{h}$ with $\langle e_i, e_j \rangle = 0$ for $i \neq j$ and $|e_i|^2 = |e_j|^2$ such that

$$\Delta(\mathfrak{n}_{\mathbb{C}}, \mathfrak{h}) = \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq n \} \cup \{ \pm e_i \mid 1 \leq i \leq n \},$$

$$\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}) = \{ e_1 \pm e_j \mid 2 \leq j \leq n \} \cup \{ e_1 \},$$

$$\Delta((\mathfrak{g}_0)_{\mathbb{C}}, \mathfrak{h}) = \{ \pm e_i \pm e_j \mid 2 \leq i < j \leq n \} \cup \{ \pm e_i \mid 2 \leq i \leq n \}.$$
When $g_C = \mathfrak{so}(2n, \mathbb{C})$, $n \geq 1$, there is a basis $e_1, \ldots, e_n$ of $\mathfrak{h}^*$ with $\langle e_i, e_j \rangle = 0$ for $i \neq j$ and $|e_i|^2 = |e_j|^2$ such that
\[
\Delta(g_C, h) = \{\pm e_i \mp e_j \mid 1 \leq i < j \leq n\},
\Delta(n_C, h) = \{e_1 \pm e_j \mid 2 \leq j \leq n\},
\Delta((g_0)_C, h) = \{\pm e_i \pm e_j \mid 2 \leq i < j \leq n\}.
\]

When $g_C = \mathfrak{sl}(n, \mathbb{C})$, $n \geq 2$, there is an $n$-dimensional vector space with a bilinear form with a basis $e_1, \ldots, e_n$ satisfying $\langle e_i, e_j \rangle = 0$ for $i \neq j$ and $|e_i|^2 = |e_j|^2$ such that there is an identification of $\mathfrak{h}^*$ with the subspace $\{\sum_i c_i e_i \mid \sum c_i = 0\}$ under which
\[
\Delta(g_C, h) = \{\pm (e_i - e_j) \mid 1 \leq i < j \leq n\},
\Delta(n_C, h) = \{e_1 - e_j \mid 1 < j \leq n\} \cup \{e_i - e_n \mid 1 \leq i < n\},
\Delta((g_0)_C, h) = \{\pm (e_i - e_j) \mid 2 \leq i < j \leq n-1\}.
\]

When $g_C = \mathfrak{sp}(2n, \mathbb{C})$ with $n \geq 3$,¹ there is a basis $e_1, \ldots, e_n$ of $\mathfrak{h}^*$ with $\langle e_i, e_j \rangle = 0$ for $i \neq j$ and $|e_i|^2 = |e_j|^2$ such that
\[
\Delta(g_C, h) = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{\pm 2e_i \mid 1 \leq i \leq n\},
\Delta(n_C, h) = \{e_1 \pm e_j \mid 3 \leq j \leq n, i = 1, 2\} \cup \{2e_1, e_1 + e_2, 2e_2\},
\Delta((g_0)_C, h) = \{\pm e_i \pm e_j \mid 3 \leq i < j \leq n\} \cup \{\pm 2e_i \mid 3 \leq i \leq n\} \cup \{\pm (e_1 - e_2)\}.
\]

When $g_C = \mathfrak{f}_4$, there is a basis $e_1, \ldots, e_4$ of $\mathfrak{h}^*$ with $\langle e_i, e_j \rangle = 0$ for $i \neq j$ and $|e_i|^2 = |e_j|^2$ such that
\[
\Delta(g_C, h) = \{\pm e_1 \pm e_2 \pm e_3 \pm e_4 \} \cup \{\pm 2e_i \pm 2e_j \mid 1 \leq i < j \leq 4\} \cup \{\pm 2e_i \mid 1 \leq i \leq 4\},
\Delta(n_C, h) = \{e_1 \pm e_2 \pm e_3 \pm e_4 \} \cup \{2e_1 \pm 2e_j \mid 2 \leq j \leq 4\} \cup \{2e_1\},
\Delta((g_0)_C, h) = \{\pm 2e_i \pm 2e_j \mid 2 \leq i < j \leq 4\} \cup \{\pm 2e_i \mid 2 \leq i \leq 4\}.
\]

2.3. Vector fields on a vector space. Let $V$ be a finite-dimensional vector space over $\mathbb{R}$. At each point $x$ of $V$, there is a natural identification of the tangent space $T_x V$ with $V$ itself. Let $S(V) = \bigoplus_{r \geq 0} S^r(V)$ be the space of symmetric tensor products of $V$. Consider the space $S(V^*) \otimes V$ of $V$-valued polynomial functions on $V$, where $V^*$ is the dual of $V$. For each $f \in S(V^*) \otimes V$, we define the vector field $X_f$ on $V$ by
\[
X_f(x) = -f(x) \in V = T_x V
\]
for $x \in V$. Such a vector field will be called a polynomial vector field on $V$. A polynomial vector field corresponding to a constant function $v \in V \subset S(V^*) \otimes V$ will be called a constant vector field. Observe that a smooth vector field $X$ on $V$ is polynomial if and only if there exist $r \geq 0$ such that $\text{ad}(X_1) \ldots \text{ad}(X_r)X = 0$ for any constant vector fields $X_1, \ldots, X_r$ on $V$. The Lie algebra of polynomial vector fields will be denoted by Poly($V$). We identify Poly($V$) with $S(V^*) \otimes V$ by the above equation. Under this identification,
\[
[S^p(V^*) \otimes V, S^q(V^*) \otimes V] \subset S^{p+q-1}(V^*) \otimes V
\]

¹$\mathfrak{sp}(4, \mathbb{C})$ is isomorphic to $\mathfrak{so}(5, \mathbb{C})$.  

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for $p, q \geq 0$. So $\text{Poly}(V) = \bigoplus_{r \geq 0} S^r(V^*) \otimes V$ is naturally a graded Lie algebra. It is not difficult to check that if $V$ is a representation of a group $G$, then this identification is an isomorphism between $G$-modules.

The grading on $\text{Poly}(V)$ is convenient to describe the structure of the group of jets. For $r \geq 1$, let $j^r(V, 0)$ be the group of $r$-jets at $0 \in V$ of the diffeomorphism defined around $0$ and fixing $0$. Then its Lie algebra $j^r(V, 0)$ is naturally a quotient of the Lie algebra $\text{Poly}(V, 0) = \bigoplus_{r \geq 1} S^r(V^*) \otimes V$ of polynomial vector fields vanishing at $0 \in V$. In fact,

$$j^r(V, 0) = \text{Poly}(V, 0) / \bigoplus_{q \geq r + 1} S^q(V^*) \otimes V.$$ 

Thus $j^r(V, 0)$ can be identified with $\bigoplus_{1 \leq q \leq r} S^q(V^*) \otimes V$ as a linear space. When $V$ is a representation of a group $G$, this identification is an isomorphism between $G$-modules.

2.4. The standard actions on the boundaries of rank-one symmetric spaces.

2.4.1. The boundaries of rank-one symmetric spaces. Let $X$ be a rank-one symmetric space of non-compact type and $G$ the group of orientation-preserving isometries of $X$, Then $X = G/K$, where $K$ is a maximal compact subgroup of $G$. Fix an Iwasawa decomposition $G = KAN$. Let $M = \{k \in K \mid ak = ka \text{ for all } a \in A\}$ be the centralizer of $A$ in $K$ so that $P = MAN \subset G$ is a minimal parabolic subgroup of $G$. Then the corresponding compact manifold $G/P$, which will be called the boundary is diffeomorphic to a sphere. In fact, since $G$ is of real rank one, its Weyl group $W(G, A)$ consists of exactly two elements. Thus the Bruhat decomposition assures that the left action of $P$ on $G/P$ has exactly two orbits: One is $\{eP\}$ and the other is $PgP$ for some $g \in G$ such that $gNg^{-1} = N_-$, where $N_-$ is the opposite of $N$. Now

$$PgP = NgP = g(g^{-1}Ng)P = gN_-P.$$ 

Since the product map $N_- \times P \to G$ is a diffeomorphism onto its image, the $N_-$-orbits of $P$ in $G/P$ is diffeomorphic to $N_-$. Thus $G/P$ as a manifold is a disjoint union of a point and a Euclidean space, which must be a sphere.

2.4.2. Local structure of the left action on the boundary. To study local structure of the left action of $G$ on $G/P$ around the point $o = eP$, we use the homomorphism $l_* : g = \text{Lie}(G) \to \mathcal{X}(G/P)$ defined by

$$l_*(X)gP = \frac{d}{dt} \bigg|_{t=0} \exp(-tX)gP$$

for $gP \in G/P$. This homomorphism can be rephrased as follows. There is a natural anti-isomorphism of $g$ onto the algebra of the right-invariant vector fields on $G$. This induces an anti-homomorphism of $g$ into the space of smooth vector fields $\mathcal{X}(G/P)$ on $G/P$. Multiplying by $-1$, we obtain the homomorphism $l_* : g \to \mathcal{X}(G/P)$.

Using the embedding $i : N_- \to G/P$ defined by $i(g) = gP$ and the diffeomorphism $\exp : n_- = \text{Lie}(N_-) \to N_-$, we obtain a homomorphism $\lambda = \exp^* \circ i^* \circ l_*$ of
g into $X(n_-)$. In the local coordinate system $\exp: n_- \to G/P$ around $o \in G/P$, the homomorphism $l_*: g \to X(G/P)$ can be described in terms of notions introduced in Subsection 2.3.

**Proposition 2.3.** Let $\lambda: g \to X(n_-)$ be the homomorphism defined as above and $\text{Poly}(n_-) \subset X(n_-)$ be the subalgebra of polynomial vector fields on $n_-$. 

(i) $\lambda(n_-) \subset \text{Poly}(n_-)$.

(ii) Let $E \in g$ be the vector characterized by $[E, X] = rX$ for all $X \in g_r$. Then $\lambda(E) \in \text{Poly}(n_-)$ is the linear vector field corresponding to $\text{ad}(E) \mid n_- = -2\text{Id}_{g_r} - \text{Id}_{g_r} \in \text{gl}(n_-)$.

(iii) $\lambda(g) \subset \text{Poly}(n_-)$.

**Proof.** (i) By the construction, for $X \in n_-$, $\lambda(X)$ is the pull-back of a right-invariant vector field on $N_-$ by $\exp: n_- \to N_-$. More explicitly, the tangent vector at $Y \in n_-$ of $\lambda(X)$ is given by the differential at $t = 0$ of the curve $\gamma(t)$ on $n_-$ satisfying

$$\exp(\gamma(t)) = \exp(-tX)\exp(Y).$$

Since $N_-$ is nilpotent, the Baker–Campbell–Hausdorff formula assures that $\gamma(t)$ is a polynomial function of $tX$ and $Y$. Thus the tangent vector at $Y$ is a polynomial function of $Y$.

(ii) Observe that $i: N_- \to G/P$ is $A$-equivariant where the domain is equipped with the action by conjugation and the range with the left action. In particular, $A$ acts on $N_-$ by automorphism of Lie group. By construction of $\lambda: g \to X(n_-)$, the claim follows immediately.

(iii) By (ii), we can show that for any integer $m$, the subspace

$$\{X \in X(n_-) | [\lambda(E), X] = mX\}$$

is contained in $\text{Poly}(n_-)$. See the proof of in [8, Proposition 7.12]. Since $g = \bigoplus g_r$, we see that $\lambda(g) \subset \text{Poly}(n_-)$.

We will use the following lemma which follows immediately from the construction of the map $\lambda$.

**Lemma 2.4.** Let $\mathfrak{z} = \mathfrak{z}_{X(n_-)}(\lambda(n_-))$ be the centralizer of $\lambda(n_-)$ in $X(n_-)$.

(i) $\mathfrak{z}$ is isomorphic to $n_-$ as a Lie algebra.

(ii) $\mathfrak{z}$ is isomorphic to $n_-$ as a representation of $MA$.

(iii) $\mathfrak{z} \subset \text{Poly}(n_-)$.

**Proof.** Since $i^* \circ l_*(n_-) \subset X(N_-)$ is the space of right-invariant vector fields on $N_-$, its centralizer is the space of left-invariant vector fields. So $\mathfrak{z}$ is isomorphic to $n_-$ as a Lie algebra. Moreover, they are isomorphic as representations of $MA$. In fact, they are isomorphic to the isotropic representation at $e \in N_-$. In particular, by the same argument for (iii) of Proposition 2.3, we see that $\mathfrak{z} \subset \text{Poly}(n_-)$.
2.4.3. The standard subgroup. Let $G$ be the group of orientation-preserving isometries of a rank-one symmetric space of non-compact type and $G = KAN$ an Iwasawa decomposition. We define a subgroup $\Gamma$ of $G$ to be a standard subgroup if it is generated by a non-trivial element $a \in A$ and a lattice $\Lambda \subset N$ of $N$ satisfying $a\Lambda a^{-1} \subset \Lambda$. We will give an explicit presentation of $\Gamma$.

Recall that $n = g_1 \oplus g_2$ for the restricted-root decomposition $g = \bigoplus_r g_r$. In particular, $N$ is at most 2-step nilpotent. Thus a lattice $\Lambda$ of $N$ has a set of generators $b_1, \ldots, b_{m_1}, c_1, \ldots, c_{m_2}$ such that

$$[b_i, c_j] = e,$$

$$[b_i, b_j] \in \langle c_1, \ldots, c_{m_2} \rangle,$$

$$[c_i, c_j] = e,$$

where $m_i = \dim g_i$. For $a \in A$, $\text{Ad}(a)$ on $g$ is of the form $\text{Ad}(a)|_{g_t} = e^{rt}\text{Id}_{g_t}$, for some $t \in \mathbb{R}$. Thus, if a nontrivial $a \in A$ satisfies $a\Lambda a^{-1} \subset \Lambda$, then there exists an integer $k \geq 2$ such that $\text{Ad}(a)|_{g_t} = k^t\text{Id}_{g_t}$, so that $ab_i a^{-1} = b_i^k$, $ac_i a^{-1} = c_i^k$.

3. Cohomology of Lie algebra

Let $g = \mathfrak{t} \oplus a \oplus n$ be an Iwasawa decomposition of a real simple Lie algebra of real rank one. In this section, we compute certain cohomology of $n$. Cohomology of $n$ with the coefficient in a finite-dimensional $g$-module can be computed by using Theorem 2.2. We compute $H^1(n, g)^a$ in Subsection 3.1. As we will see in Section 8, vanishing of $H^1(n, g)^a$ is equivalent to local rigidity of our action. The goal of Subsection 3.2 is Corollary 3.9 and Corollary 3.10, which will be used in Section 5. To compute such cohomology of an infinite-dimensional $g$-module, we use Theorem 2.1.

3.1. Cohomology of finite-dimensional modules. By using Theorem 2.2 and the classification in Subsection 2.2, we obtain the following. To reduce the notation, $g_C, n_C, (g_0)_C$, and $a_C$ in Subsection 2.2 will be denoted by $g$, $n$, $l$, and $a$, respectively.

**Lemma 3.1.** Let $g$ be the complexification of a real simple Lie algebra of real rank one. Using the notation for $\Delta(g, h)$ as in Subsection 2.2,

$$H^1(n, g)^a = \begin{cases} 
F^l_{2e_2} \oplus F^l_{-2e_2} & \text{if } g = \mathfrak{so}(4, \mathbb{C}) \\
F^l_{2e_2} & \text{if } g = \mathfrak{so}(m, \mathbb{C}), m \geq 5 \\
F^l_{-e_1+2e_2-e_n} \oplus F^l_{e_1-2e_{n-1}+e_n} & \text{if } g = \mathfrak{sl}(n, \mathbb{C}), n \geq 3 \\
0 & \text{if } g = \mathfrak{sp}(2n, \mathbb{C}), n \geq 3, \text{ or } g = \mathfrak{j}_4.
\end{cases}$$

as an $l$-module, where $F^l_\lambda$ denotes the finite-dimensional irreducible $l$-module with the highest weight $\lambda$.

**Proof.** By Theorem 2.2,

$$H^1(n, F^0_\lambda) = \bigoplus_{\mu} F^l_\mu$$
as an \( l \)-module, where the sum is taken over \( \mu = r_\alpha (\lambda + \delta (g)) - \delta (g) \) for the reflection \( r_\alpha \in W^l \) in a simple root \( \alpha \). We will determine the \( \alpha \)-invariant summands.

When \( g = so(4, C) \), \( g = F^0_{e_1 + e_2} \oplus F^0_{e_1 - e_2} \) as a \( g \)-module. As \( \alpha^* \) is spanned by \( e_1 \), a weight vector is \( \alpha \)-invariant if and only if its coefficient of \( e_1 \) of the weight is 0. Since the simple roots are \( e_1 \pm e_2 \), \( \delta (g) = 2e_1 \) and \( l = h \), it is not difficult to see that

\[
H^1 (n, F^0_{e_1 \pm e_2})^\alpha = \bigoplus_{\mu} F^l_{\pm 2e_2}.
\]

When \( g = so(2n, C) \), \( n \geq 3 \), \( g = F^0_{e_1 + e_2} \) as a \( g \)-module. Using the facts that \( \alpha^* \) is spanned by \( e_1 \), the simple roots are

\[
e_i - e_{i + 1}, (i = 1, \ldots, n - 1), e_{n - 1} + e_n,
\]

and

\[
\delta (g) = \sum_{i=1}^{n} (n - i) e_i,
\]

it is easy to see that a summand \( F^l_\mu, \mu = r_\alpha (e_1 + e_2 + \delta (g)) - \delta (g) \) is \( \alpha \)-invariant only if \( \alpha = e_1 - e_2 \). Observe that a weight is \( g \)-dominant if and only if its coefficients of \( e_1, \ldots, e_n \) are positive and that a weight is \( l \)-dominant if and only if its coefficients of \( e_2, \ldots, e_n \) are positive. It follows that \( r_{e_1 - e_2} \in W^l \). Since

\[
r_{e_1 - e_2} (e_1 + e_2 + \delta (g)) - \delta (g) = 2e_2,
\]

the claim follows.

When \( g = so(2n + 1, C) \), \( n \geq 2 \), \( g = F^0_{e_1 + e_2} \) as a \( g \)-module. Since \( \alpha^* \) is spanned by \( e_1 \), the simple roots are

\[
e_i - e_{i + 1} (i = 1, \ldots, n - 1), e_n,
\]

and

\[
\delta (g) = \sum_{i=1}^{n} \frac{2n + 1 - 2i}{2} e_i,
\]

the claim follows from the same argument as the case \( g = so(2n, C) \).

When \( g = sl(n, C) \), \( n \geq 3 \), \( g = F^0_{e_1 - e_n} \) as a \( g \)-module. As \( \alpha^* \) is spanned by \( e_1 - e_n \), a weight vector is \( \alpha \)-invariant if and only if the coefficients of \( e_1 \) and \( e_n \) of the weight is the same. Since the simple roots are

\[
e_i - e_{i + 1} (i = 1, \ldots, n - 1)
\]

and

\[
\delta (g) = \sum_{i=1}^{n} \frac{n + 1 - 2i}{2} e_i,
\]

it is easy to see that a summand \( F^l_\mu, \mu = r_\alpha (e_1 - e_n + \delta (g)) - \delta (g) \) is \( \alpha \)-invariant only if \( \alpha = e_1 - e_2, e_{n - 1} - e_n \). Observe that a weight is \( g \)-dominant if and only if its coefficient of \( e_i \) is larger that of \( e_{i + 1} \) for \( i = 1, \ldots, n - 1 \) and that a weight is \( l \)-dominant if and only if its coefficient of \( e_i \) is larger that of \( e_{i + 1} \) for \( i = 2, \ldots, n - 2 \). Then it is not difficult to see that \( r_{e_1 - e_2}, r_{e_1 - e_n} \in W^l \). Since

\[
r_{e_1 - e_2} (e_1 - e_n + \delta (g)) - \delta (g) = -e_1 + 2e_2 - e_n
\]
and 
\[ r_{e_{n-1} - e_n} (e_1 - e_n + \delta(g)) - \delta(g) = e_1 - 2e_{n-1} + e_n, \]
the claim follows.

When \( g = \text{sp}(2n, \mathbb{C}) \), \( n \geq 3 \), \( g = \mathfrak{g}^{0}_{2\mathfrak{f}_1} \) as a \( g \)-module. As \( \alpha^* \) is spanned by \( e_1 + e_2 \), a weight vector is \( \alpha \)-invariant if and only if the sum of the coefficients of \( e_1 \) and \( e_2 \) of the weight is 0. Since the simple roots are 
\[ e_1 - e_{i+1} (i = 1, \ldots, n-1), \quad 2e_n, \]
and 
\[ \delta(g) = \sum_{i=1}^{n} (n+1-i) e_i, \]
it is easy to see that a summand \( F^l_{\mu} \), \( \mu = r_{\alpha}(e_1 + e_2 + \delta(g)) - \delta(g) \) is \( \alpha \)-invariant only if \( \alpha = e_2 - e_3 \). Since 
\[ r_{e_2 - e_3} (e_1 + e_2 + \delta(g)) - \delta(g) = 2e_1 - e_2 + e_3, \]
there is no \( \alpha \)-invariant summands and the claim follows.

When \( g = \mathfrak{f}_4 \), \( g = \mathfrak{g}^{0}_{2\mathfrak{f}_1,2\mathfrak{e}_3} \) as a \( g \)-module. As \( \alpha^* \) is spanned by \( e_1 \), a weight vector is \( \alpha \)-invariant if and only if the coefficient of \( e_1 \) of the weight is 0. Since the simple roots are 
\[ e_1 - e_2 - e_3 - e_4, 2e_2 - 2e_3, 2e_3 - 2e_4, 2e_4, \]
and 
\[ \delta(g) = 11e_1 + 5e_2 + 3e_3 + e_4, \]
it is easy to see that a summand \( F^l_{\mu} \), \( \mu = r_{\alpha}(2e_1 + 2e_2 + \delta(g)) - \delta(g) \) is \( \alpha \)-invariant only if \( \alpha = e_1 - e_2 - e_3 - e_4 \). Since 
\[ r_{e_2 - e_3 - e_4} (2e_1 + 2e_2 + \delta(g)) - \delta(g) = -e_1 - e_2 + e_3 + e_4, \]
there is no \( \alpha \)-invariant summands and the claim follows. \( \square \)

3.2. Cohomology of infinite-dimensional modules. Let \( \mathfrak{g} \) be the complexification of a real simple Lie algebra of real rank one. We will use the same notation for \( n, l \), and \( \alpha \) as in Subsection 3.1. A weight \( \lambda \in \mathfrak{h}^* \) is orthogonal to \( \alpha^* \) if \( \langle \lambda, \mu \rangle = 0 \) for all \( \mu \in \alpha^* \), where \( \langle \cdot, \cdot \rangle \) is the bilinear form on \( \mathfrak{h}^* \) induced by the restriction of the Killing form on \( g \) to \( h \), and is \( l \)-dominant if \( \langle \lambda, \mu \rangle > 0 \) for all \( \mu \in \Delta^+(l, h) \).

**Lemma 3.2.** Let \( \lambda \in \mathfrak{h}^* \) be the weight of an \( l \)-lowest weight vector in \( \mathfrak{n}_- \) which is not the weight of a \( g \)-lowest weight vector in \( g \). Then for \( w \in W(g, h) \), 
\[ \mu = w(\lambda - \delta(g)) - \delta(g) \]
is not orthogonal to \( \alpha^* \) if \( \mu \) is \( l \)-dominant.

**Proof.** When \( g = \text{so}(n, \mathbb{C}) \), \( n \geq 3 \), an \( l \)-lowest weight vector in \( \mathfrak{n}_- \) is a \( g \)-lowest weight vector in \( \mathfrak{g} \), so there is no \( \lambda \) satisfying the assumption. The remaining cases are \( g = \text{sl}(n, \mathbb{C}) \) \( (n \geq 3) \), \( \text{sp}(2n, \mathbb{C}) \) \( (n \geq 3) \), and \( \mathfrak{f}_4 \).
When \( g = \mathfrak{sl}(n, \mathbb{C}) \) \((n \geq 3)\), \( \lambda = -e_1 + e_{n-1} \) or \(-e_2 + e_n\). We assume for simplicity \( \lambda = -e_1 + e_{n-1} \). Then

\[
\lambda - \delta(g) = -\frac{n+1}{2} e_1 - \frac{n-3}{2} e_2 - \cdots + \frac{n-1}{2} e_{n-1} + \frac{n-1}{2} e_n.
\]

Since \( \mathfrak{a}^* \) is spanned by \( e_1 - e_2 \), a weight \( \mu \) is orthogonal to \( \mathfrak{a}^* \) if and only if the coefficients of \( e_1 \) and \( e_2 \) are the same. On the other hand, the difference between coefficients of \( e_1 \) and \( e_2 \) in \( \delta(g) \) is \( n - 1 \). It follows that the coefficients of \( e_1 \) and \( e_2 \) in \( \mu = w(\lambda - \delta(g)) - \delta(g) \) coincides for some \( w \in W(\mathfrak{g}, \mathfrak{h}) \) only if the set of coefficients in \( \lambda - \delta(g) \) contains two elements which differ by \( n - 1 \). We see that there is no such two elements. Thus \( \mu \) is not orthogonal to \( \mathfrak{a} \). The case \( \lambda = -e_2 + e_n \) follows by the same argument.

When \( \mathfrak{sp}(2n, \mathbb{C}) \) \((n \geq 3)\), \( \lambda = -e_1 - e_3 \). So

\[
\lambda - \delta(g) = -(n+1)e_1 - (n-1)e_2 - (n-1)e_3 - (n-3)e_4 \cdots - e_n.
\]

Since \( \mathfrak{a}^* \) is spanned by \( e_1 + e_2 \), a weight \( \mu \) is orthogonal to \( \mathfrak{a}^* \) if and only if the sum of the coefficients of \( e_1 \) and \( e_2 \) is 0. Moreover, if such a weight \( \mu \) is \( \mathfrak{l} \)-dominant, the coefficient of \( e_1 \) is non-negative.\(^2\) Since the coefficient of \( e_1 \) in \( \delta(g) \) is \( n \), if the coefficient of \( e_1 \) in \( \mu = w(\lambda - \delta(g)) - \delta(g) \) is non-negative, it must be 1. But as the coefficient of \( e_2 \) in \( \delta(g) \) is \( n - 1 \), the coefficient of \( e_2 \) in \( \mu \) can not be \(-1 \). Thus \( \mu \) is not orthogonal to \( \mathfrak{a}^* \).

When \( g = \mathfrak{f}_4 \), \( \lambda = -e_1 - e_2 - e_3 - e_4 \). So

\[
\lambda - \delta(g) = -12e_1 - 6e_2 - 4e_3 - 2e_4.
\]

Since \( \mathfrak{a}^* \) is spanned by \( e_1 \), a weight \( \mu \) is orthogonal to \( \mathfrak{a}^* \) if and only if the coefficient of \( e_1 \) is 0. Since the action of \( W(\mathfrak{g}, \mathfrak{h}) \) preserves the bilinear form on \( \mathfrak{h}^* \), the set \( \{ \pm e_1 \pm e_2 \pm e_3 \pm e_4 \} \cup \{ \pm 2e_i \mid 1 \leq i \leq 4 \} \) is invariant. Observe that for each \( \alpha \) in this set, \((\lambda - \delta(g), \alpha)\) is an integer multiple of \( 4\lvert e_i \rvert^2 \). Thus this is also true for \( w(\lambda - \delta(g)) \). In particular, the coefficient of \( e_1 \) in \( w(\lambda - \delta(g)) \) is even. Thus \( w(\lambda - \delta(g)) - \delta(g) \) is not orthogonal to \( \mathfrak{a}^* \).

**Corollary 3.3.** Let \( \lambda \in \mathfrak{h}^* \) be the weight of an \( \mathfrak{l} \)-lowest weight vector in \( \mathfrak{n}_- \), which is not the weight of a \( \mathfrak{g} \)-lowest weight vector in \( \mathfrak{g} \). If \( V \) is an \( \mathfrak{l} \)-finite \( \mathfrak{g} \)-module with an infinitesimal character \( \lambda - \delta(g) \), then \( H^* (\mathfrak{n}, V)^\mathfrak{g} = 0 \).

**Proof.** By Theorem 2.2, the weight \( \mu \) of an \( \mathfrak{l} \)-highest weight vector in \( H^* (\mathfrak{n}, V)^\mathfrak{g} \) is of the form \( \mu = w(\lambda - \delta(g)) - \delta(g) \) for some \( w \in W(\mathfrak{g}, \mathfrak{h}) \). Observe that a weight vector is \( \mathfrak{a} \)-invariant if and only if its weight is orthogonal to \( \mathfrak{a}^* \) and that the weight of an \( \mathfrak{l} \)-highest weight vector in a finite-dimensional \( \mathfrak{l} \)-module is \( \mathfrak{l} \)-dominant. Thus the claim is immediate from Lemma 3.2.

**Lemma 3.4.** Let \( \lambda \in \mathfrak{h}^* \) be the weight of a \( \mathfrak{g} \)-lowest weight vector in \( \mathfrak{g} \). Assume a weight \( \mu \in \mathfrak{h}^* \) is \( \mathfrak{l} \)-dominant, orthogonal to \( \mathfrak{a}^* \), and

\[
\mu = w(\lambda - \delta(g)) - \delta(g)
\]

\(^2\)The assumption of \( \mathfrak{l} \)-dominance is used only here.
for some \( w \in W(g, h) \). Then

\[
\mu = \begin{cases} 
2e_2 & \text{if } g = \mathfrak{so}(n, \mathbb{C}), n \geq 4, \lambda = -e_1 - e_2 \\
-2e_2 & \text{if } g = \mathfrak{so}(4, \mathbb{C}), \lambda = -e_1 + e_2 \\
e_1 + 2e_2 - e_n, e_1 - 2e_{n-1} + e_n & \text{if } g = \mathfrak{sl}(n, \mathbb{C}), n \geq 3 \\
2e_1 - 2e_2 + e_3 + e_4 & \text{if } g = \mathfrak{sp}(2n, \mathbb{C}), n \geq 3 \\
4e_2 + e_3 + e_4 & \text{if } g = f_4.
\end{cases}
\]

**Proof.** When \( g = \mathfrak{so}(4, \mathbb{C}), \lambda = -e_1 + e_2 \). Let us first consider the case \( \lambda = -e_1 - e_2 \). Then \( \lambda - \delta(g) = -2e_1 - e_2 \). So \( \mu = w(\lambda - \delta(g)) - \delta(g) \) is orthogonal to \( \mathfrak{a} \) only if \( w(\lambda - \delta(g)) = e_1 + 2e_2 \) and \( \mu = 2e_2 \). When \( \lambda = -e_1 - e_2 \), the claim follows from the above argument with \( e_2 \) replaced by \(-e_2\).

When \( g = \mathfrak{so}(2n+1, \mathbb{C}) \) \( (n \geq 2) \), \( \lambda = -e_1 - e_2 \). So

\[
\lambda - \delta(g) = 2n + 1 \frac{e_1}{2} - 2n - 1 \frac{e_2}{2} + \frac{e_2}{2} - e_3 - \cdots - e_n.
\]

Since the coefficient of \( e_1 \) in \( \delta(g) \) is \( \frac{2n+1}{2} \), \( \mu \) is orthogonal to \( \mathfrak{a}^* \) if and only if \( w(e_2) = -e_1 \). It is not difficult to check that \( \mu \) is \( l \)-dominant only if

\[
w(\lambda - \delta(g)) = 2n - 1 \frac{e_1}{2} + \frac{e_2}{2} + \frac{e_2}{2} + \frac{e_3}{2} + \cdots + \frac{e_n}{2}.
\]

and \( \mu = 2e_2 \).

The case \( g = \mathfrak{so}(2n, \mathbb{C}) \) \( (n \geq 3) \) is similar to the above. In this case, \( \lambda = -e_1 - e_2 \). Comparing the coefficients of \( \lambda - \delta(g) \) and \( \delta(g) \), we can show that \( \mu \) is orthogonal to \( \mathfrak{a} \) if and only if \( w(e_2) = -e_1 \). By the \( l \)-dominance, \( \mu = 2e_2 \).

When \( g = \mathfrak{sl}(n, \mathbb{C}) \) \( (n \geq 3) \), \( \lambda = -e_1 + e_n \) and

\[
\lambda - \delta(g) = -n + 1 \frac{e_1}{2} - \frac{e_2}{2} - \cdots - \frac{e_{n-1}}{2} - \frac{e_n}{2}.
\]

As the difference between coefficients of \( e_1 \) and \( e_n \) in \( \delta(g) \) is \( -1 \), we see that \( \mu \) is orthogonal to \( \mathfrak{a}^* \) if and only if \( w(e_1) = e_n, w(e_{n-1}) = e_1 \) or \( w(e_2) = e_n, w(e_n) = e_1 \). When \( w(e_1) = e_n \) and \( w(e_{n-1}) = e_1 \), \( \mu \) is \( l \)-dominant only if

\[
w(\lambda - \delta(g)) = \frac{n-3}{2} e_1 + \frac{n+1}{2} e_2 + \frac{n-5}{2} e_3 + \cdots + \frac{n-3}{2} e_{n-1} + \frac{n+1}{2} e_n
\]

and \( \mu = -e_1 + 2e_2 - e_n \). When \( w(e_2) = e_n, w(e_n) = e_1 \), \( \mu \) is \( l \)-dominant only if

\[
w(\lambda - \delta(g)) = \frac{n+1}{2} e_1 + \frac{n-3}{2} e_2 + \cdots + \frac{n-5}{2} e_{n-2} + \frac{n-3}{2} e_{n-1} + \frac{n-3}{2} e_n
\]

and \( \mu = e_1 - 2e_{n-1} + e_n \).

When \( g = \mathfrak{sp}(2n, \mathbb{C}) \) \( (n \geq 3) \), \( \lambda = -2e_1 \) and

\[
\lambda - \delta(g) = -(n+2) e_1 - (n-1) e_2 - \cdots - e_n.
\]

As the sum of the coefficients of \( e_1 \) and \( e_2 \) in \( \delta(g) \) is \( 2n - 1 \), we see that \( \mu \) is orthogonal to \( \mathfrak{a}^* \) if and only if \( w \) maps \( \{e_1, e_4\} \) onto \( \{-e_1, -e_2\} \). When \( \mu \) is \( l \)-dominant, the coefficient of \( e_1 \) in \( \mu \) is non-negative. Thus \( w(e_1) = -e_1 \) and \( w(e_4) = -e_2 \). Then \( \mu \) is \( l \)-dominant only if

\[
w(\lambda - \delta(g)) = (n+2) e_1 + (n-3) e_2 + (n-1) e_3 + (n-2) e_4 + (n-4) e_5 + \cdots + e_n.
\]
and $\mu = 2e_1 - 2e_2 + e_3 + e_4$.

When $g = f_4$, $\lambda = -2e_1 - 2e_2$ and

$$\lambda - \delta(g) = -13e_1 - 7e_2 - 3e_3 - e_4.$$  

Assume $\mu$ is orthogonal to $a^*$. Then the coefficient of $e_1$ in $w(\lambda - \delta(g))$ is 11. Let $\{c_1, c_2, c_3, c_4\}$ be the set of coefficients of $w(\lambda - \delta(g))$ with $|c_i| \geq |c_{i+1}|$. Since $W(g,h)$ preserves the bilinear form on $h^*$, $13^2 + 7^2 + 3^1 + 1^2 = \sum_i c_i^2$. It follows that $c_1 = 11$. Using the fact that $\{\pm e_1 \pm e_2 \pm e_3 \pm e_4\} \cup \{\pm 2e_i \mid 1 \leq i \leq 4\}$ is $W(g,h)$-invariant and that $\langle \lambda - \delta(g), \alpha \rangle$ is an integer multiple of $2|e_i|^2$ for each element $\alpha$ of this set, we see that the coefficients $c_i$ of $w(\lambda - \delta(g))$ are integers. Since $\{\pm 2e_i \pm 2e_j \mid 1 \leq i < j \leq 4\}$ is $W(g,h)$-invariant and the maximal value of $\langle \lambda - \delta(g), \alpha \rangle$ for $\alpha \in \{\pm 2e_i \pm 2e_j \mid 1 \leq i < j \leq 4\}$ is $2(13 + 7)$, we see $|c_1| + |c_2| = 13 + 7$. Thus $|c_2| = 9$. By the equation $13^2 + 7^2 + 3^1 + 1^2 = \sum_i c_i^2$, we obtain $|c_3| = 4$ and $|c_4| = 2$. Now it is easy to check that $\mu$ is $l$-dominant only if

$$w(\lambda - \delta(g)) = 11e_1 + 9e_2 + 4e_3 + 2e_4$$

and $\mu = 4e_2 + e_3 + e_4$. \hfill $\square$

Thus when $V$ is a $g$-module with the same infinitesimal character as that of $g$, unlike Corollary 3.3, $H^n(V) = 0$ does not necessarily vanish. In fact, when $V = g$, as we have seen in Subsection 3.1, $H^1(n,V) = 0$ if $g = so(n,C)$ or $sl(n,C)$.

We define a $g$-module $V$ to be a-bounded below if for all positive weights $\mu \in a^*$, the set $\{\langle \mu, \lambda \rangle \mid \lambda \in \text{wt}(V)\} \subset \mathbb{R}$ is bounded below, where $\text{wt}(V)$ denotes the set of weights in $V$.

**Corollary 3.5.** Let $\lambda \in h^*$ be the weight of a $g$-lowest weight vector in $g$. If $V$ is an $l$-finite $g$-module with an infinitesimal character $\lambda - \delta(g)$ which is a-bounded below, then $H^0(n,V) = 0$.

**Proof.** By Theorem 2.1, the weight $\mu$ of an $l$-highest weight vector in $H^0(n,V)$ must be $\mu$ as in Lemma 3.4. On the other hand, an $l$-highest weight vector in $H^0(n,V)$ is a $g$-highest weight vector in $V$. Since $V$ is a-bounded below, a $g$-highest weight vector in $V$ must be a highest weight vector of a finite-dimensional $g$-submodule. Thus its weight $\mu$ must be $g$-dominant. But weights $\mu$ in Lemma 3.4 are not $g$-dominant. Thus $H^0(n,V) = 0$. \hfill $\square$

We define a weight $\lambda \in h^*$ to be $a^*$-nonnegative if $\langle \mu, \lambda \rangle \geq 0$ for all positive weights $\mu \in a^*$.

**Proposition 3.6.** Assume $g = so(n,C)$, $n \geq 4$ or $sl(n,C)$, $n \geq 3$. Let $\lambda \in h^*$ be the weight of a $g$-lowest weight vector in $g$, and $V$ an $l$-finite $g$-module with an infinitesimal character $\lambda - \delta(g)$. Assume the weights of $V$ are $a^*$-nonnegative. Then $H^1(n,V) = 0$.

Assume $H^1(n,V) = 0$. We will again use the explicit description of the root system $\Delta(g,h)$ as in Subsection 2.2 to obtain a contradiction.

**Proof in the case** $g = so(4,C)$. The weight $\lambda$ of a $g$-lowest weight vector in $g$ is $\lambda = -e_1 \pm e_2$. When $\lambda = -e_1 - e_2$, by Lemma 3.4, $H^1(n,V)$ has an $l$-highest weight.
vector of weight $2e_2$. Let $f : n \to V$ be a non-zero cocycle of weight $2e_2$. Then $f(g_{e_1+e_2})$ or $f(g_{e_1-e_2})$ is non-zero. So $V$ contains a weight vector of weight $e_1 + 3e_2$ or $e_1 + e_2$.

We will show $V$ does not contain a weight vector of weight $e_1 + 3e_2$ or $e_1 + e_2$. Since $V$ is an $l$-finite $g$-module with $a^*$-nonnegative weights, a vector in $V$ generates a $g_-$-submodule which contains a $g$-lowest weight. On the other hand, the weight of a $g$-lowest weight vector in $V$ is of the form $w(\lambda - \delta(g)) + \delta(g)$. The weights appear in the $g_-$-submodule of $V$ generated by a weight vector of weight $e_1 + 3e_2$ are $e_1 + 3e_2, 4e_2, 2e_2$, while that of weight $e_1 + e_2$ are $e_1 + e_2, 4e_2, 2e_2$.

It is easy to see that none of them are of the form $w(\lambda - \delta(g)) + \delta(g)$. When $\lambda = -e_1 + e_2$, the claim follows from the above argument with $e_2$ replaced by $-e_2$. □

**Proof in the case $g = \mathfrak{so}(m, C), m \geq 5$.** By Lemma 3.4, $H^1(n, V)$ has an $l$-highest weight vector of weight $\mu = 2e_2$. Let $f : n \to V$ be a non-trivial $l$-highest cocycle of weight $\mu$. Since $g_{e_1-e_2}$ generates $n$ as an $l$-module, an $l$-highest cocycle $f$ is determined by $f|_{g_{e_1-e_2}}$.

Let $g' = h \oplus \bigoplus_{a \in \Delta(g', h)} g_a$ be the subalgebra of $g$ where

$$\Delta(g', h) = \{ \pm e_1 \pm e_2 \}.$$

Set $n' = n \cap g'$, and $l' = l \cap g'$. Now we will show that the restriction of $f$ to $n'$ gives a non-zero $l'$-highest weight vector in $H^1(n', V')$ of weight $\mu$, where $V'$ denotes the $g'$-subalgebra of $V$ generated by $f(g_{e_1-e_2})$. It suffices to show that $f|_{n'}$ is not a boundary. If $f|_{n'}$ is a boundary, there exists $\nu \in V$ with $f|_{n'} = d\nu$. Then $\nu$ is a weight vector of weight $\mu$. Such a weight vector is $l'$-highest: As $\mu$ is orthogonal to $a$, the $l$-submodule generated by $\nu$ admits an $l$-highest weight vector of weight orthogonal to $a$. By Lemma 3.4, the weight of the $l$-highest weight vector is $\mu$. Thus $\nu$ is $l$-highest. Since $\nu$ is $l$-highest, $f - d\nu$ is an $l$-highest cocycle. By $(f - d\nu)(g_{e_1-e_2}) = 0$, we obtain $f - d\nu = 0$. So $f$ is a boundary, which contradicts to the assumption.

Replacing $f$ if necessary, we may assume $V'$ admits a $g'$-infinitesimal character. By Theorem 2.1, the infinitesimal character must be $\mu + \delta(g') = e_1 + 2e_2$. By the same argument as in the proof of the case $g = \mathfrak{so}(4, C)$, $V'$ does not contain a weight vector of weight $e_1 + e_2$. This contradicts to the fact that $f(g_{e_1-e_2})$ is of weight $e_1 - e_2 + \mu = e_1 + e_2$. □

**Proof in the case $g = \mathfrak{so}(n, C), n \geq 3$.** By Lemma 3.4, $H^1(n, V)$ has an $l$-highest weight vector of weight $\mu = -e_1 + 2e_2 - e_n, e_1 - 2e_{n-1} + e_n$. Let us first consider the case $\mu = -e_1 + 2e_2 - e_n$. Let $f : n \to V$ be a non-trivial $l$-highest cocycle of weight $\mu$.

We will show $f = 0$ if $f|_{g_{e_1-e_2}} = 0$. Assume $f(g_{e_1-e_2}) = 0$. By $[g_{e_1-e_2}, g_{e_1-e_n}] = 0$ and the cocycle equation, $f(g_{e_1-e_n})$ is annihilated by $g_{e_1-e_2}$. If $f(g_{e_1-e_n}) \neq 0$, this is a weight vector of weight $\mu + e_1 - e_n = 2e_2 - 2e_n$. Set $s_{e_1-e_2} = h \oplus g_{e_1-e_2} \oplus g_{-e_1+e_2}$.

As the weights of $V$ are $a$-nonnegative, the $s_{e_1-e_2}$-module generated by $f(g_{e_1-e_n})$ is finite dimensional with a $s_{e_1-e_2}$-highest weight vector in $f(g_{e_1-e_n})$. But its weight $2e_2 - 2e_n$ is not $s_{e_1-e_2}$-dominant, which is a contradiction. Thus we
have \( f(g_{e_{n-1}}) = 0 \). Then by \([g_{e_{1}} - e_{2}, g_{e_{n-1}} - e_{n}] \subset g_{e_{1}} - e_{2}\) and the cocycle equation, \( f(g_{e_{n-1}} - e_{n}) \) is annihilated by \( g_{e_{1}} - e_{2} \). Since the weight \( \mu + e_{n-1} - e_{n} = -e_{1} + 2e_{2} + e_{n-1} - 2e_{n} \) is not \( s_{e_{1}} - e_{2} \)-dominant, by the same argument as above, we obtain \( f(g_{e_{n-1}} - e_{n}) = 0 \). As \( g_{e_{1}} - e_{2}, g_{e_{n-1}} - e_{n} \), and \( g_{e_{1}} - e_{n} \) generate \( n \) as an \( l_{+} \)-module, we see \( f = 0 \).

Let \( g' = \mathfrak{h} = \bigoplus_{\alpha \in \Delta(g', \mathfrak{h})} \mathfrak{g}_{\alpha} \) be the subalgebra of \( \mathfrak{g} \), where
\[
\Delta(g', \mathfrak{h}) = \{ \pm (e_{i} - e_{j}) | 1 \leq i < j \leq n - 1 \},
\]
and set \( n' = n \cap g' \), and \( l' = l \cap g' = l \). By the same argument as in the case of \( g = \mathfrak{so}(m, \mathbb{C}) \), we obtain an \( l' \)-highest weight vector in \( H^{1}(n', V') \) of weight \( \mu \), where \( V' \) denotes the \( g' \)-subalgebra of \( V \) generated by \( f(g_{e_{1}} - e_{2}) \).

Replacing \( f \) if necessary, we may assume \( V' \) admits a \( g' \)-infinitesimal character. Observe that
\[
\delta(g') = \frac{n - 2}{2} e_{1} + \frac{n - 4}{2} e_{2} + \cdots + \frac{n - 4}{2} e_{n-2} - \frac{n - 2}{2} e_{n-1}.
\]
By Theorem 2.1, the infinitesimal character must be
\[
\mu + \delta(g') = \frac{n - 4}{2} e_{1} + \frac{n - 2}{2} e_{2} + \cdots - \frac{n - 2}{2} e_{n-1} - e_{n}.
\]
Thus the weight of a \( g' \)-lowest weight vector is of the form \( \mu + \delta(g') + \delta(g') \) for some \( \nu \in W(g', \mathfrak{h}) \). We see that the weight of this form appears in the \( g' \)-submodule generated by \( f(g_{e_{1}} - e_{2}) \) only if \( \nu = \mu + \delta(g') + \delta(g') = e_{n-1} - e_{n} \). But \( V' \) does not contain a \( g' \)-lowest weight vector of weight \( e_{n-1} - e_{n} \). In fact, if there is such a weight vector, \( V \) also has a \( g' \)-lowest weight vector of weight \( e_{n-1} - e_{n} \). Considering the \( g \)-infinitesimal character of \( V \), we see that the weight vector is not \( g \)-lowest. So it is not annihilated by \( g_{e_{n-1}} + e_{n} \). Then applying \( g_{e_{n-1}} + e_{n} \), we obtain a \( g' \)-lowest weight vector in \( V \) of weight \( e_{n-1} - e_{n} - e_{n-1} + e_{n} = 0 \). Considering the \( g \)-infinitesimal character of \( V \) again, this is a contradiction.

**Proposition 3.7.** Assume \( g = \mathfrak{sp}(2n, \mathbb{C}), n \geq 3 \) or \( \mathfrak{l} \). Let \( \lambda \in \mathfrak{h}^{*} \) be the weight of a \( g \)-lowest weight vector in \( \mathfrak{g} \), and \( V \) an \( l \)-finite \( g \)-module with an infinitesimal character \( \lambda - \delta(g) \) which is a-bounded below. Then \( H^{1}(n, V) = 0 \).

**Proof.** in the case \( g = \mathfrak{sp}(2n, \mathbb{C}), n \geq 3 \). By Lemma 3.4, \( H^{1}(n, V) \) has an \( l \)-highest weight vector of weight \( \mu = 2e_{1} - 2e_{2} + e_{3} + e_{4} \). Let \( f : n \rightarrow V \) be a non-trivial \( l \)-highest cocycle of weight \( \mu \). Since \( n \) is generated by the \( l_{+} \)-submodule generated by \( g_{e_{2}} - e_{3} \) as Lie algebra, \( f = 0 \) if \( f|_{g_{e_{2}} - e_{3}} = 0 \).

Let \( g' = \mathfrak{h} = \bigoplus_{\alpha \in \Delta(g', \mathfrak{h})} \mathfrak{g}_{\alpha} \) be the subalgebra of \( \mathfrak{g} \) where
\[
\Delta(g', \mathfrak{h}) = \{ \pm e_{2} \pm e_{3}, \pm 2e_{2}, \pm 2e_{3} \}
\]
and set \( n' = n \cap g' \), and \( l' = l \cap g' \). By the same argument as in the case of \( g = \mathfrak{so}(m, \mathbb{C}) \), we obtain an \( l' \)-highest weight vector in \( H^{1}(n', V') \) of weight \( \mu \), where \( V' \) denotes the \( g' \)-subalgebra of \( V \) generated by \( f(g_{e_{2}} - e_{3}) \).

Replacing \( f \) if necessary, we may assume \( V' \) admits a \( g' \)-infinitesimal character. By Theorem 2.1, the infinitesimal character must be
\[
\mu + \delta(g') = (2e_{1} - 2e_{2} + e_{3} + e_{4}) + (2e_{2} + e_{3}) = 2e_{1} + 2e_{3} + e_{4}.
\]
Thus the weight of a $g'$-lowest weight vector is of the form $w(\mu + \delta(g')) + \delta(g')$ for some $w \in W(g', h)$. Observe that the coefficient of $e_2$ in $w(\mu + \delta(g')) + \delta(g')$ is non-negative. On the other hand, the weights in the $g'$-module generated by $f(g_{e_2-e_3})$, the weight of which is $e_2 - e_3 + \mu = 2e_1 - e_2 + e_4$, have negative coefficients for $e_2$. This is a contradiction. 

**Proof in the case $g = f_4$.** By Lemma 3.4, $H^1(n, V)$ has an $l$-highest weight vector of weight $\mu = 4e_2 + e_3 + e_4$. Since $n$ is generated by the $l_+$-submodule generated by $g_{e_1-e_2-e_3-e_4}$ as Lie algebra, $f = 0$ if $f|_{g_{e_1-e_2-e_3-e_4}} = 0$.

Let $g' = h \oplus \bigoplus_{\alpha \in \Delta(g', h)} g_{\alpha}$ be the subalgebra of $g$, where

$$\Delta(g', h) = \{ \pm (e_1 - e_2 - e_3) \pm e_4 \} \cup \{ \pm 2e_4 \},$$

and set $n' = n \cap g'$ and $l' = l \cap g'$. Observe that $g'$ is a reductive Lie algebra with its semisimple part isomorphic to $sl(3, \mathbb{C})$. By the same argument as in the case of $g = so(m, \mathbb{C})$, we obtain an $l'$-highest weight vector in $H^1(n', V')$ of weight $\mu$, where $V'$ denotes the $g'$-subalgebra of $V$ generated by $f(g_{e_1-e_2-e_3-e_4})$.

Replacing $f$ if necessary, we may assume $V'$ admits a $g'$-infinitesimal character. By Theorem 2.1, the infinitesimal character must be

$$\mu + \delta(g') = (4e_2 + e_3 + e_4) + (e_1 - e_2 - e_3 + e_4) = e_1 + 3e_2 + 2e_4.$$ 

Thus the weight $\nu$ of a $g'$-lowest weight vector satisfies $|\mu + \delta(g')| = |\nu - \delta(g')|$. On the other hand, the weights in the $g'$-module generated by $f(g_{e_1-e_2-e_3-e_4})$, the weight of which is $e_1 - e_2 - e_3 - e_4 + \mu = e_1 + 3e_2$, are of the form $(1 - k)e_1 + (3 + k)e_2 + ke_3 + le_4$ for a non-negative integer $k$ and an integer $l$. So

$$\nu - \delta(g') = -ke_1 + (k + 4)e_2 + (k + 1)e_3 + le_4$$

for a non-negative integer $k$ and an integer $l$. Thus $|\mu + \delta(g')| < |\nu - \delta(g')|$, which is a contradiction. 

Let $G$ be the group of orientation-preserving isometries of a rank-one symmetric space of non-compact type, $G = KAN$ an Iwasawa decomposition, and $M$ the centralizer of $A$ in $K$. Then $P = MAN$ is a minimal parabolic subgroup. Recall that we defined the homomorphism $\lambda : g \to Pol(n_-)$ in Subsection 2.4. Consider the representation of $g$ on $Pol(n_-)$ via $\lambda : g \to Pol(n_-)$. Let $l \subset g$ be the subalgebra corresponding to $MA \subset G$.

**Lemma 3.8.** The $g_{\mathbb{C}}$-module $V = Pol(n_-)_\mathbb{C}$ admits a decomposition $V = \bigoplus V_{\alpha_i}$ into a finite sum of $g_{\mathbb{C}}$-submodules, where the sum is taken over the set $\{ \alpha_i \}$ of weights of $l_{\mathbb{C}}$-lowest weight vectors in $(n_-)_\mathbb{C}$ and $V_{\alpha_i}$ is a $g_{\mathbb{C}}$-submodule with an infinitesimal character $\alpha_i - \delta(g_{\mathbb{C}})$.

**Proof.** It suffices to show that the weight of a weight vector in $V = Pol(n_-)_\mathbb{C}$ annihilated by the sum $(g_{\mathbb{C}})_-$ of negative root spaces is $\alpha_i$. Since $(g_{\mathbb{C}})_- = (n_-)_\mathbb{C} \oplus (l_{\mathbb{C}})_- \cap V(g_{\mathbb{C}})_- = V(n_-)_\mathbb{C} \cap V(l_{\mathbb{C}})_-$. By Lemma 2.4, the centralizer of $\lambda(n_-)$ in $Pol(n_-)$ is an $l$-submodule which is isomorphic to $n_-$. Thus $V(n_-)_\mathbb{C}$ is isomorphic to $(n_-)_\mathbb{C}$ as an $l_{\mathbb{C}}$-module. Thus the weight of an $l_{\mathbb{C}}$-lowest weight vector is $\alpha_i$. 

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As we mentioned in Subsection 2.3, since $n_-$ is an $l$-module,
\[ \text{Poly}(n_-) = S(n^*_-) \otimes n_- \]
as an $l$-module. Thus Poly$(n_-)_{\mathbb{C}}$ is $l_{\mathbb{C}}$-finite as a $g_{\mathbb{C}}$-module.

**Corollary 3.9.** $H^0(\mathfrak{n}, \text{Poly}(n_-))^a = 0$.

**Proof.** We will show the complexification $H^0(\mathfrak{n}_C, \text{Poly}(n_-)_C)^a_{\mathbb{C}}$ is vanished. By Lemma 3.8, it suffices to show $H^0(\mathfrak{n}_C, V_{a_i})^a_{\mathbb{C}} = 0$ for all $a_i$. Since $V_{a_i}$ admits an infinitesimal character $\alpha_i - \delta(g)$, this is immediate from Corollary 3.3 and Corollary 3.5.

**Corollary 3.10.** The map
\[ H^1(\mathfrak{n}, g)^a \to H^1(\mathfrak{n}, \text{Poly}(n_-))^a \]
induced by $\lambda : g \to \text{Poly}(n_-)$ is an isomorphism.

**Proof.** We will show the complexification $H^1(\mathfrak{n}_C, g_{\mathbb{C}})^a_{\mathbb{C}} \to H^1(\mathfrak{n}_C, \text{Poly}(n_-)_C)^a_{\mathbb{C}}$ is an isomorphism. Since $\lambda : g \to \text{Poly}(n_-)$ is injective, we have a short exact sequence
\[ 0 \to g \to \text{Poly}(n_-) \to \text{Poly}(n_-)/\lambda(g) \to 0. \]

Thus it suffices to show that $H^1(\mathfrak{n}_C, \text{Poly}(n_-)_C)/\lambda(g)_C)^a_{\mathbb{C}} = 0$ for $i = 0$ and $i = 1$. By Lemma 3.8, $V = \text{Poly}(n_-)_C$ admits a decomposition $V = \bigoplus V_{a_i}$ into $g_{\mathbb{C}}$-submodules $V_{a_i}$ with an infinitesimal character $\alpha_i - \delta(g_C)$. So its quotient $V' = \text{Poly}(n_-)_C/\lambda(g)_C$ also admits a decomposition $V' = \bigoplus V'_{a_i}$, where $V'_{a_i}$ has an infinitesimal character $\alpha_i - \delta(g_C)$. Let $a_0$ be the weight of a $g_{\mathbb{C}}$-lowest weight vector in $g_{\mathbb{C}}$. By Corollary 3.3, $H^*(\mathfrak{n}_C, V'_{a_i})^a_{\mathbb{C}} = 0$ for $a_i \neq a_0$. Moreover, since $\text{Poly}(n_-) = S(n^*_-) \otimes n_- as an $l$-module, $\text{Poly}(n_-)$ is $a$-bounded below. By Corollary 3.5, $H^0(\mathfrak{n}_C, V'_{a_0})^a_{\mathbb{C}} = 0$. Thus it remains to show $H^1(\mathfrak{n}_C, V'_{a_0})^a_{\mathbb{C}} = 0$.

When $g_{\mathbb{C}} = \mathfrak{sp}(2n, \mathbb{C})$, $n \geq 3$ or $\mathfrak{f}_4$, by Proposition 3.7, $H^1(\mathfrak{n}_C, V'_{a_0})^a_{\mathbb{C}} = 0$. So we may assume $g_{\mathbb{C}} = \mathfrak{so}(n, \mathbb{C}) (n \geq 4)$ or $\mathfrak{sl}(n, \mathbb{C}) (n \geq 3)$. By Proposition 3.6, it suffices to show that the weights of $V'_{a_0}$ are $a_{\mathbb{C}}^*$-nonnegative.

Under the isomorphism $\text{Poly}(n_-) = S(n^*_-) \otimes n_-$ as $l$-modules, the subspace spanned by weight vectors in $\text{Poly}(n_-)$ of $a^*$-negative weights is
\[ g_{-2} \oplus g_{-1} \oplus (g^*_1 \otimes g_{-2}) \subset S(n^*_-) \otimes n_. \]

On the other hand, the weights in $\lambda(n_-)$ are $a^*$-negative and by Lemma 2.4, the weights in $Z(\lambda(n_-)) = Z_{\text{Poly}(n_-)}(\lambda(n_-))$ are also $a^*$-negative.

When $g_{\mathbb{C}} = \mathfrak{so}(n, \mathbb{C})$, we may assume $g_{-2} = 0$. Then $\lambda(n_-) = Z(\lambda(n_-))$ is of dimension equal to $-1$. Thus $\text{Poly}(n_-)/\lambda(g)$ does not have $a^*$-negative weights.

When $\mathfrak{sl}(n, \mathbb{C})$, $n \geq 3$, $g_{-2}$ is of dimension one. Since $Z_{\mathfrak{sl}}(n_-) = g_{-2}$, $\lambda(n_-)$ and $Z(\lambda(n_-))$ span a subspace of dimension $2 \dim g_{-1} + \dim g_{-2}$, which is equal to the dimension of $g_{-2} \oplus g_{-1} \oplus (g^*_1 \otimes g_{-2})$. Thus weight vectors in $\text{Poly}(n_-)$ of $a^*$-negative weights are contained in $\lambda(n_-) + Z(\lambda(n_-))$. So $V'_{a_0} = V_{a_0}/\lambda(g_{\mathbb{C}})$ does not have $a_{\mathbb{C}}^*$-negative weights. \qed
Using Corollary 3.9, we can show the following lemma which will be used in the proof of Proposition 8.4.

**Lemma 3.11.** Assume \( g = \text{su}(n, 1), n \geq 3 \). Let \( N_{\text{Poly}(n, n)}(\lambda(n)) \) be the normalizer of \( \lambda(n) \) in \( \text{Poly}(n, n) \) and \( Z_{\text{Poly}(n, n)}(\lambda(n)) = \text{Poly}(n, n)^a \) the centralizer of \( \lambda(n) \) in \( \text{Poly}(n, n) \). Then \( N_{\text{Poly}(n, n)}(\lambda(n)) \cap Z_{\text{Poly}(n, n)}(\lambda(n)) = \lambda(g)^a \).

**Proof.** Set \( q = N_{\text{Poly}(n, n)}(\lambda(n)) \cap Z_{\text{Poly}(n, n)}(\lambda(n)) \). Since \( \lambda(g)^a \subset q \), it remains to show that \( q \subset \lambda(g)^a \). We will first show \( q \subset N_{\text{Poly}(n, n)}(\lambda(g)^a) \). Fix \( X \in q \). Since \([\lambda(g^{-2}), \text{Poly}(n, n)^a] \subset \lambda(g^{-2})\), we have \( X \in N_{\text{Poly}(n, n)}(\lambda(g^{-2})) \). Fix \( Y \in g^{-2} \sim \{0\} \). Then \([Y, g_1] = g^{-1}\). Applying \( \text{ad}(\lambda(Y))^2 \) to \([X, \lambda(g_1)] \subset \lambda(g_1)\), we see that \( X \in N_{\text{Poly}(n, n)}(\lambda(g_1)) \).

Identifying \( g \) with its image \( \lambda(g) \) by \( \lambda \), we see that for \( X \in g \), \( \text{ad}(X)|_{\lambda(n)} \) defines an \( \alpha \)-invariant cocycle on \( n \) with its value in \( g \). By Corollary 3.9, the map \( q \to Z^1(n, g)^a \) is injective, where \( Z^1(n, g)^a \) denotes the space of \( \alpha \)-invariant cocycles. This induces the injective map \( \lambda(g)^a \to H^1(n, g)^a \). We will show the complexification \( q_C/\lambda((g_C)^{\text{ac}}) \) is trivial. If \( q_C/\lambda((g_C)^{\text{ac}}) \neq 0 \), by Lemma 3.1, the weight of an \( \text{ac} \)-highest weight vector is \(-e_1 + 2e_2 - e_n \) or \(-e_1 - 2e_n - 1 + e_2\). Assume the weight is \(-e_1 + 2e_2 - e_n\). Then there is a weight vector \( X \in q_C \) of weight \(-e_1 + 2e_2 - e_n\). Since \( \text{ad}(X)|_{\lambda(n)} \neq 0 \), we see \([X, \lambda(g_{e_1}e_2)] = \lambda(g_{e_2}e_1)\). Applying \( \text{ad}(\lambda(g_{e_1}e_2)) \) to this equation and using \([X, \lambda(g_{-e_1}e_2)] \in \lambda((n_{-e_1})_{\text{ac}})\), we obtain \( X \in g \) which is a contradiction. The argument for \( e_1 - 2e_{n-1} + e_n \) is the same. We proved \( q_C/\lambda((g_C)^{\text{ac}}) = 0 \).

### 4. Cohomology of the Standard Subgroup

Let \( G \) be a group of orientation-preserving isometries of a rank-one symmetric space of non-compact type, and \( \Gamma \) its standard subgroup. The goal of this section is to prove 4.2. Recall that \( \Gamma \) has a finite generating set \( a, b_1, \ldots, b_{m_1}, c_1, \ldots, c_{m_2} \) as in Subsection 2.4.3.

**Lemma 4.1.** Let \( V \) be a vector space, and \( \rho_s : \Gamma \to GL(V) \) the representation defined by \( \rho_s(\lambda) = (\text{id}_V) \) and \( \rho_s(a) = s\text{id}_V \) for a constant \( s > 0 \). Then

(i) \( H^0(\Gamma, V) = 0 \) if \( s \neq 1 \);
(ii) \( H^1(\Gamma, V) = 0 \) if \( s \neq 1, k, k^2 \).

**Proof.**

(i) Since \( H^0(\Gamma, V) \) can be identified with the space \( V^\Gamma \) of \( \Gamma \)-fixed vectors in \( V \), the claim is immediate.

(ii) Let \( \tilde{\rho}_s(g) = v - \rho(g)v \) \( (g \in \Gamma) \) be the coboundary given by \( v \in V \). Given a cocycle \( \alpha : \Gamma \to V \), since \( s \neq 1 \), there is a unique \( v \in V \) satisfying \( \alpha(a) = \beta_v(a) \). So we may assume \( \alpha(a) = 0 \). For any \( b_i \in \{b_1, \ldots, b_{m_1}\} \), as \( \alpha \) is a cocycle,

\[
\alpha(ab_i) = s\alpha(b_i) + \alpha(a).
\]

On the other hand, using the relation \( ab_i = b_i^ka \),

\[
\alpha(ab_i) = \alpha(b_i^ka) = \alpha(a) + \alpha(b_i^k) = \alpha(a) + k\alpha(b_i).
\]

Thus

\[
(s - k)\alpha(b_i) = 0.
\]
Since $s \neq k$, we see $\alpha(b_{ij}) = 0$. Similarly, for any $c_i \in \{c_1, \ldots, c_m\}$, using the relation $ac_i = c_i \alpha(a)$ and the assumption $s \neq k^2$, we obtain $\alpha(c_i) = 0$. Thus the claim follows.

Since $\Gamma \subset P \subset G$, the subalgebra $p \subset g$ is invariant under the adjoint representation of $\Gamma$ on $g$. The induced representation of $\Gamma$ on $g/p$ will also be called the adjoint representation.

**Proposition 4.2.** Consider the adjoint representation of a standard subgroup $\Gamma$ on $g/p$. Then $H^1(\Gamma, g/p) = 0$.

**Proof.** Recall that $g$ is graded $g = \bigoplus g_i$ so that $p = \bigoplus g_i$. Set $V = g/p$, and $W = (\bigoplus g_i)/p$. Then $V/W = g/(\bigoplus g_i)$. To prove $H^1(\Gamma, V) = 0$, it suffices to show $H^1(\Gamma, W) = 0$ and $H^1(\Gamma, V/W) = 0$.

Since $[g_i, g_j] \subset g_{i+j}$, the adjoint representation of $n = g_1 \oplus g_2$ on $V/W$ and $V$ are trivial. Thus the representations of $\Lambda \subset N$ on $W$ and $V/W$ are also trivial. Since $a$ acts on $W$ by $k^{-1}\text{Id}_W$ and on $V/W$ by $k^{-2}\text{Id}_{V/W}$, by Lemma 4.1, $H^1(\Gamma, W) = H^1(\Gamma, V/W) = 0$.

5. Local Rigidity of the Homomorphism into the Group of Jets

In this section, using the results obtained in Section 3, we will show Proposition 5.1 which claims local rigidity in a weak sense of the homomorphism of the standard subgroup into the group of jets. Let $J^r(G/P, o)$, $r \geq 0$ be the group of $r$-jets at $o \in G/P$. The $C^s$-topology ($s \geq 0$) on $\text{Diff}(G/P)$ induces a topology on $J^r(G/P, o)$ which will also be called the $C^s$-topology. When $r \leq s$, the topology is the same as that as a Lie group, while when $s < r$, the topology is not Hausdorff. The statement of the following proposition is obviously weaker than that of our main theorem.

**Proposition 5.1.** Let $G$ be the group of orientation-preserving isometries of a rank-one symmetric space of non-compact type, $\Gamma$ a standard subgroup of $G$, and $l : P \to J^3(G/P, o)$ the homomorphism into the group of 3-jets at $o \in G/P$ induced by the action of $P$ on $G/P$ by left translations. If $\rho : \Gamma \to J^3(G/P, o)$ is a homomorphism $C^2$-close to $l|_{\Gamma}$, then there is an embedding $i$ of $\Gamma$ into $G$ as a standard subgroup and $h \in J^3(G/P, o)$ such that

$$\rho(g) = h \circ l(i(g)) \circ h^{-1}.$$  

for all $g \in \Gamma$.

Using the local coordinate system $i \circ \exp : n_- \to G/P$ around $o \in G/P$ introduced in Subsection 2.4.2, the group $J^3(G/P, o)$ can be identified with $J^3(n_-, 0)$. The induced homomorphism will also be denoted by $l : P \to J^3(n_-, 0)$.

Assume the standard subgroup $\Gamma$ is generated by $a \in A$ and a lattice $\Lambda \subset N$. Let

$$Z = Z_{J^3(n_-, 0)}(l(a))$$

be the centralizer of $l(a)$ in $J^3(n_-, 0)$. Recall that $\text{Ad}(a)|_{\theta_1} = k^2\text{Id}_{\theta_1}$ for an integer $k \geq 2$. By Proposition 2.3 (ii), we see that the action of $a$ around $0 \in n_-$ is the...
linear transformation corresponding to $k^{-2}Id_{g_{-2}} \oplus k^{-1}Id_{g_{-1}} \in \text{GL}(n_{-})$. By Sternberg's normalization [10], we see that an element $C^{2}$-close to $l(a)$ is conjugate to an element in $Z$. So to prove Proposition 5.1, we may assume $\rho(a) \in Z$.

Let $\pi: J^{3}(n_{-},0) \to J^{1}(n_{-},0) = \text{GL}(n_{-})$ be the natural projection. Let us first consider the homomorphism $\pi \circ l: P \to \text{GL}(n_{-})$. Since $\pi \circ l(a) = \begin{pmatrix} k^{-1}Id_{g_{-1}} & 0 \\ 0 & k^{-2}Id_{g_{-2}} \end{pmatrix} \in \text{GL}(n_{-}) = \text{GL}(g_{-1} \oplus g_{-2})$

and $\text{Ad}(a)|_{n} = kId_{g_{0}} \oplus k^{2}Id_{g_{2}}$, it is easy to see that $\pi \circ l(N) \subset \begin{pmatrix} Id_{g_{-1}} & * \\ 0 & Id_{g_{-2}} \end{pmatrix}$. Let $H \subset \text{GL}(n_{-})$ be the subgroup defined by

$$H = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

**Lemma 5.2.** If $f: \Gamma \to \text{GL}(n_{-})$ is a homomorphism close to $\pi \circ l|_{\Gamma}$, then $f$ is conjugate to a homomorphism $f'$ such that $f'(<\Gamma>) \subset H$.

This lemma can be shown easily by using the following theorem of Stowe:

**Theorem 5.3 ([11]).** Let $\Gamma$ be a finitely generated group, and $\rho$ a smooth action of $\Gamma$ on a manifold $M$ with a common fixed point $x_{0} \in M$. Then an action $C^{2}$-close to $\rho$ admits a common fixed point $x$ close to $x_{0}$ if the first cohomology $H^{1}(\Gamma,T_{x_{0}},M)$ with respect to the isotropic representation of $\rho$ at $x_{0}$ is vanished.

**Proof of Lemma 5.2.** As $\text{GL}(n_{-})$ acts on $M = \text{GL}(n_{-})/H$ by left translations, a homomorphism of $\Gamma$ into $\text{GL}(n_{-})$ induces an action of $\Gamma$ on $M$. Let $\sigma$ be the action induced by $\pi \circ l|_{\Gamma}$. Since $\pi \circ l(<\Gamma>) \subset H$, $\sigma$ has the common fixed point $x_{0} = eH \in M$. Given a homomorphism $f: \Gamma \to \text{GL}(n_{-})$ close to $\pi \circ l|_{\Gamma}$, the induced action of $\Gamma$ on $M$ is close to $\sigma$.

By the above observation on $\pi \circ l: P \to \text{GL}(n_{-})$ , we see that, in the isotropic representation of $\sigma$ at $x_{0}$, $\sigma$ acts on $T_{x_{0}} = g(n_{-})/h$ by multiplication by $k^{-1}$ and $\Lambda$ acts trivially. Thus by Lemma 4.1, $H^{1}(\Gamma,T_{x_{0}},M) = 0$. By Theorem 5.3, the action of $\Gamma$ on $M$ induced by $f$ also admits a common fixed point. Replacing $f$ with its conjugate, we may assume $f$ fixes $x_{0}$. This is equivalent to $f(<\Gamma>) \subset H$. \[\Box\]

By Lemma 5.2, to prove Proposition 5.1 we may assume $\pi \circ \rho(<\Gamma>) \subset H$. Then $\pi \circ \rho: \Gamma \to \text{GL}(n_{-})$ induces homomorphisms of $\Gamma$ into $\text{GL}(g_{-1})$ and $\text{GL}(g_{-2})$. By [1, Lemma 2.2], we see that

$$\pi \circ \rho(\Lambda) \subset \begin{pmatrix} Id_{g_{-1}} & * \\ 0 & Id_{g_{-2}} \end{pmatrix}.$$

The following lemma shows that the image $\rho(\Lambda)$ is contained in a connected simply-connected nilpotent Lie group.

**Lemma 5.4.** Let $L \subset \text{GL}(n,\mathbb{R}) = J^{1}(\mathbb{R}^{n},0)$ be the group of upper triangular matrices with diagonal entries 1 and $\pi_{1}^{T}: J^{r}(\mathbb{R}^{n},0) \to J^{1}(\mathbb{R}^{n},0)$ be the natural projection. Then $(\pi_{1}^{T})^{-1}(L) \subset J^{r}(\mathbb{R}^{n},0)$ is a connected simply-connected nilpotent Lie group.
Proof: We will first show that \( \hat{L} := (\pi^1)^{-1}(L) \) is contractible. Consider the one-parameter subgroup of linear transformations of \( \mathbb{R}^n \) given by the diagonal matrices

\[
\left\{ \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\
0 & e^{\lambda_2 t} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & e^{\lambda_n t} \end{pmatrix} \right| t \in \mathbb{R},
\]

where we fixed constants \( 1 < \lambda_1 < \cdots < \lambda_n < 2 \). The action of this group on \( \text{GL}(n, \mathbb{R}) \) preserves the subgroup \( L \). So this group acts on \( \hat{L} = (\pi^1)^{-1}(L) \). We will show that this action defines a flow on \( \hat{L} \) such that all points \( g \in \hat{L} \) converges to \( e \in \hat{L} \) as \( t \to \infty \). If such a flow exists, then \( \hat{L} \) is contractible. Observe that an element in \( J'(\mathbb{R}^n, 0) \) can be represented by a unique polynomial transformation of \( \mathbb{R}^n = \{ (x_1, \ldots, x_n) \} \) of degree at most \( r \). Using this representation, we can describe the flow on \( \hat{L} \) explicitly: The coefficient of \( x_{j_1} \cdots x_{j_m} \) in the \( i \)-th component is multiplied by \( \exp((\lambda_{j_1} - \lambda_{j_2} - \cdots - \lambda_{j_m}) t) \). By the definition of \( \hat{L} \) and the condition \( 1 < \lambda_1 < \cdots < \lambda_n < 2 \), we see that \( \lambda_{j_1} - \lambda_{j_2} - \cdots - \lambda_{j_m} < 0 \) except \( \lambda_1 - \lambda_1 = 0 \). It follows that any points in \( \hat{L} \) converge to \( e \in L \) as \( t \to \infty \).

To prove \( \hat{L} \) is nilpotent, we consider the adjoint representation of the above one-parameter subgroup on the Lie algebra \( \hat{l} = \text{Lie}(\hat{L}) \). By the above description of the action on the Lie group \( \hat{L} \), it follows that the action on \( \hat{l} \) is diagonal and that the eigenvalues are negative for \( t > 0 \). A Lie algebra which admits such a representation is nilpotent. We proved \( \hat{l} \) is nilpotent.

Since \( \Lambda \) is a lattice of \( N \), we can use the following theorem of Malcev:

**Theorem 5.5 ([9]).** Let \( N \) and \( V \) be connected simply-connected nilpotent Lie groups, and \( H \) a uniform subgroup of \( N \). Then any continuous homomorphism \( f: H \to V \) can be extended uniquely to a continuous homomorphism \( \hat{f}: N \to V \).

It follows that a homomorphism \( \rho: \Gamma \to j^{3}(n_{-}, 0) \) satisfying

\[
\pi \circ \rho(\Lambda) \subset \left\{ \begin{pmatrix} \text{Id}_{g-1} & * \\
0 & \text{Id}_{g-2} \end{pmatrix} \right\}
\]

extends uniquely to a continuous homomorphism \( \hat{\rho}: \langle a \rangle N \to j^{3}(n_{-}, 0) \), where \( \langle a \rangle N \) is the closure of \( \Gamma \) in \( AN \subset G = KAN \). In fact, by Theorem 5.5, the restriction \( \rho|_{\Lambda}: \Lambda \to j^{3}(n_{-}, 0) \) can be extended to a continuous homomorphism \( \overline{\rho}|_{\Lambda}: N \to j^{3}(n_{-}, 0) \). Moreover, using the uniqueness of the extension, we see that \( \overline{\rho}|_{\Lambda} \) is a continuous extension of \( \rho|_{\Lambda} \), where \( \tilde{\Lambda} = \bigcup_{n \in \mathbb{Z}} a^{n}\Lambda a^{-n} \). Define a map \( \hat{\rho}: \langle a \rangle N \to j^{3}(n_{-}, 0) \) by \( \hat{\rho}(a^{n}g) = \rho(a^{n})\overline{\rho}(g) \) for \( n \in \mathbb{Z}, g \in \tilde{\Lambda} \). Since \( \hat{\rho} \) is a continuous map which is an extension of \( \rho, \hat{\rho} \) is also a group homomorphism.

Let \( j^{3}(n_{-}, 0) \) be the Lie algebra of \( j^{3}(n_{-}, 0) \). Since \( \text{Ad}(a) \) on \( n_{-} \) is diagonal with eigenvalues \( k^{-1} \) and \( k^{-2} \), \( \text{Ad}(a) \) on

\[
j^{3}(n_{-}, 0) = \bigoplus_{1 \leq r \leq 3} (S^{r}(n_{-}^{+}) \otimes n_{-})
\]

is diagonal with eigenvalues \( k^{i}, -1 \leq i \leq 5 \). Let \( j^{3}(n_{-}, 0)_{i} \) be the eigenspace corresponding to \( k^{i} \).
**Lemma 5.6.** Let \( \tilde{\rho} : (a)N \to J^3(n_-,0) \) be a continuous homomorphism such that \( \rho(a) \in Z \) is sufficiently close to \( l(a) \) and \( \tilde{\rho}_* : n \to J^3(n_-,0) \) its differentiation at \( e \in (a)N \). Then

\[
\tilde{\rho}_*(g_i) \subset J^3(n_-,0)_i.
\]

In other words, \( \tilde{\rho}_* \) is \( \alpha \)-invariant.

**Proof.** For any \( h \in Z \), \( \text{Ad}(h) \) preserves the decomposition \( J^3(n_-,0) = \bigoplus_i J^3(n_-,0)_i \). Since \( \tilde{\rho} : (a)N \to J^3(n_-) \) is a group homomorphism,

\[
\text{Ad}(\tilde{\rho}(a)) \circ \tilde{\rho}_* = \tilde{\rho}_* \circ \text{Ad}(a).
\]

In particular, \( \tilde{\rho}_*(g_i) \) is contained in the eigenspace of \( \text{Ad}(\tilde{\rho}(a)) \) for eigenvalue \( k^i \). As \( \tilde{\rho}(a) \in Z \) is close to \( l(a) \), we see that \( \tilde{\rho}_*(g_i) \subset J^3(n_-,0)_i \).

Let \( \rho : \Gamma \to J^3(n_-,0) \) be a homomorphism \( C^2 \)-close to \( l|_\Gamma \) such that \( \rho(a) \in Z \) and \( \tilde{\rho} : (a)N \to J^3(n_-,0) \) its continuous extension. Since \( \rho|_\Lambda \) is \( C^2 \)-close to \( l|_\Lambda \), we see that

\[
\pi \circ \tilde{\rho}_* : n \to J^2(n_-,0)
\]

is close to \( \pi \circ l_*|_n : n \to J^2(n_-,0) \), where \( \pi : J^3(n_-,0) \to J^2(n_-,0) \) is the natural projection and \( l_* : p \to J^3(n_-,0) \) is the differentiation of \( l \). By Lemma 5.6, \( \tilde{\rho} \) is \( \alpha \)-invariant. While \( \rho \) is only \( C^2 \)-close (not \( C^3 \)-close) to \( l \), using the \( \alpha \)-invariance, we can show that \( \tilde{\rho}_* : n \to J^3(n_-,0)_i \) is close to \( l_*|_n : n \to J^3(n_-,0)_i \).

**Lemma 5.7.** Let \( f : n \to J^3(n_-,0) \) be an \( \alpha \)-invariant homomorphism of Lie algebras such that \( \pi \circ f : n \to J^2(n_-,0) \) is close to \( \pi \circ l_*|_n \). Then \( f \) is close to \( l_*|_n \).

**Proof.** Set \( j_{i,j} = J^3(n_-,0)_i \cap S^{j+1}(\mathfrak{s}_n^n) \otimes \mathfrak{n}_- \) so that \( [j_{i,j}, j_{i',j'}] \subset j_{i+j',j+j'} \). Then

\[
\begin{align*}
J^3(n_-,0)_1 &= j_{1,0} \oplus j_{1,1} \oplus j_{1,2}, \\
J^3(n_-,0)_2 &= j_{2,1} \oplus j_{2,2}.
\end{align*}
\]

By assumption, for \( X \in \mathfrak{g}_i \) \((i = 1,2)\), \( f(X) \in J^3(n_-,0)_i \) and its \( j_{i,j} \)-component \( f(X)_{i,j} \) is close to the \( j_{i,j} \)-component \( l(X)_{i,j} \) of \( l(X) \) for \( j \leq 1 \).

We will first show that \( f|_{\mathfrak{g}_2} \) is close to \( l_*|_{\mathfrak{g}_2} \). Fix \( X \in \mathfrak{g}_2 \). It suffices to show that \( f(X)_{2,2} \) is close to \( l_*|_{\mathfrak{g}_2} \). Since \( [\mathfrak{g}_1, \mathfrak{g}_2] = 0 \), \( [f(Y), f(X)] = 0 \) for all \( Y \in \mathfrak{g}_1 \). The \( j_{3,2} \)-component of this equation is

\[
[f(Y)_{1,0}, f(X)_{2,2}] + [f(Y)_{1,1}, f(X)_{2,1}] = 0.
\]

Since \( f(X)_{i,j} (j \leq 1) \) is close to \( l_*|_{\mathfrak{g}_2} \), we see that

\[
[l_*|_{\mathfrak{g}_1}, f(X)_{2,2} - l_*|_{\mathfrak{g}_2}]
\]

is close to 0. Using the fact that \( [\mathfrak{g}_{-2}, \mathfrak{g}_1] = \mathfrak{g}_{-1} \), we see that

\[
l_*|_{\mathfrak{g}_1}, \mathfrak{g}_{-2} = \mathfrak{g}_{-1}
\]

under the identification \( j_{1,0} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \). Moreover, under the identification

\[
S^3(\mathfrak{s}_{-1}^n) \otimes \mathfrak{n}_- = \bigoplus_{0 \leq m \leq 3, i=1,2} S^m(\mathfrak{s}_{-1}^n) \otimes S^{3-m}(\mathfrak{s}_{-2}^n) \otimes \mathfrak{g}_{-i},
\]

the subspace \( j_{2,2} \) can be identified with \( (S^3(\mathfrak{s}_{-1}^n) \otimes \mathfrak{g}_{-1}) \oplus (S^2(\mathfrak{s}_{-1}^n) \otimes \mathfrak{g}_{-2}^n) \otimes \mathfrak{g}_{-2} \). By these observations, we see that \( f(X)_{2,2} - l_*|_{\mathfrak{g}_2} \) is close to 0.
It remains to show that $f|_{g_1}$ is close to $l_*$ on $g_1$. Fix $X \in g_1$. It suffices to show that $f(X)_{1,2}$ is close to $l_*(X)_{1,2}$. Since $[g_1,g_1] \subset g_2$ and $f|_{g_2}$ is close to $l_*|_{g_2}$, we see that
\[
[l_*(X)_{1,0}, f(Y)_{1,2} - l_*(Y)_{1,2}] + [l_*(Y)_{1,0}, f(X)_{1,2} - l_*(X)_{1,2}]
\]
is close to 0 for all $Y \in g_1$. Using the fact that $[X,g_1] = g_{-1}$ for any $X \in g_{-2} \sim \{0\}$ and identifying $i_{1,2}$ with $S^3(g_{-1}) \otimes g_{-2}$, it is not difficult to check that $f(X)_{1,2} - l_*(X)_{1,2}$ is close to 0.

The next step of the proof is to show that there is $h \in Z$ such that $h\tilde{\rho}(N)h^{-1} = l(N)$. In other words,
\[
\text{Ad}(h) \circ \tilde{\rho}_*(n) = l_*(n).
\]
As $\tilde{\rho}_*: n \to j^3(n_{-},0)$ is close to $l_*|_{n}$, the existence of such $h \in Z$ is an immediate consequence of the following.

**Lemma 5.8.** The map $H^1(n, p)^a \to H^1(n, j^3(n_{-},0))^a$ induced by $l_* : p \to j^3(n_{-},0)$ is an isomorphism.

**Proof.** Recall that we obtained the isomorphism
\[
H^1(n, g)^a \to H^1(n, \text{Poly}(n_{-}))^a
\]
in Corollary 3.10.

For an $AN$-module $V$, $H^0(n, V)^a \neq 0$ only if $V^a \neq 0$. In general, since the Lie algebra $n$ is generated by $g_1$ on which $\text{Ad}(a)$ acts by multiplication by $k$, $H^1(n, V)^a \neq 0$ only if $\text{Ad}(a)$ on $V$ has an eigenvalue $k^i$.

We see that $p$ is an $n$-submodule of $g$ and that $\text{Ad}(a)$ acts diagonally on $g/p$ with eigenvalues $k^{-1}$, $k^{-2}$. Thus $H^1(n, p)^a$ is isomorphic to $H^1(n, g)^a$.

Recall that $j^3(n_{-},0) = \text{Poly}(n_{-})/\bigoplus_{r \geq 4}(S^r(n_{+}) \otimes n_{-})$ under the identification $\text{Poly}(n_{-}) = \bigoplus_{r \geq 0}(S^r(n_{+}) \otimes n_{-})$, where $\text{Poly}(n_{-},0) = \bigoplus_{r \geq 1}(S^r(n_{+}) \otimes n_{-})$ is the polynomial vector fields vanishing at 0 in $V$. Since $\text{Ad}(a)$ acts diagonally on $\text{Poly}(n_{-})/\text{Poly}(n_{-},0)$ with eigenvalues $k^{-1}$, $k^{-2}$, we see that $H^1(n, \text{Poly}(n_{-}))^a$ is isomorphic to $H^1(n, \text{Poly}(n_{-},0))^a$. Moreover, as $\text{Ad}(a)$ acts diagonally on $\bigoplus_{r \geq 4}(S^r(n_{+}) \otimes n_{-})$ with eigenvalues $k^i$ ($i \geq 2$), we see that $H^1(n, \text{Poly}(n_{-},0))^a$ is isomorphic to $H^1(n, j^3(n_{-},0))^a$.

Replacing $\rho$ with its conjugate, we may further assume that $\tilde{\rho}(N) = l(N)$. It remains to show that a continuous homomorphism $\tilde{\rho} : \langle a \rangle N \to j^3(n_{-},0)$ close to $l|_{\langle a \rangle N}$ with $\tilde{\rho}(a) \in Z$ and $\tilde{\rho}(N) = l(N)$ satisfies $\tilde{\rho}(a) = l(a)$. In fact, if this claim holds, the image of the homomorphism $\rho : \Gamma \to j^3(n_{-},0)$ is contained in $l(\langle a \rangle N)$. As $l : P \to j^3(n_{-},0)$ is an isomorphism onto its image, Proposition 5.1 follows immediately.

Let $\tilde{\rho} : \langle a \rangle N \to j^3(n_{-},0)$ be a continuous homomorphism close to $l|_{\langle a \rangle N}$ with $\tilde{\rho}(a) \in Z$ and $\tilde{\rho}(N) = l(N)$. We will show the element $z_0 = \tilde{\rho}(a)l(a)^{-1} \in Z$ close to the identity $e \in Z$ is in fact exactly $e$. By the above assumption and the equation $\text{Ad}(\tilde{\rho}(a)) \circ \tilde{\rho}_* = \rho_* \circ \text{Ad}(a)$, we see that $\text{Ad}(z_0)$ fixes each element of $l_*(n)$. Thus to show $z_0 = e$, it suffices to show that $H^0(n, z) = 0$, where $z = \text{Lie}(Z)$. Since $H^0(n, z) = H^0(n, j^3(G/P,0)^a) = H^0(n, j^3(G/P,0))^a$, by the same argument as the
proof of Lemma 5.8, we can show \( H^0(n, 3) = H^0(n, \text{Poly}(n_-))a \). By Corollary 3.9, \( H^0(n, 3) = 0 \). We finished the proof of Proposition 5.1.

When \( G = \text{Sp}(n + 1, 1) \), \( n \geq 2 \) or \( F_{4}^{-20} \), using \( H^1(n, g) = 0 \), we can show local rigidity in the strict sense.

**Corollary 5.9.** When \( G = \text{Sp}(n + 1, 1) \), \( n \geq 2 \) or \( F_{4}^{-20} \), the homomorphism \( l|_{\Gamma} : \Gamma \to j^3(G/P, o) \) is \( C^2 \)-locally rigid.

**Proof.** By Lemma 3.1 and Lemma 5.8, we see \( H^1(n, 3(n-, 0))a = 0 \). Thus for a continuous homomorphism \( \tilde{\rho} : (a) N \to j^3(n-, 0) \) close to \( l|(a) N \) with \( \tilde{\rho}(a) \in Z \), there is \( h \in Z \) such that \( h \tilde{\rho}(g)h^{-1} = l(g) \) for \( g \in N \). By the same argument as the proof of Proposition 5.1, the claim follows.

---

6. Local Rigidity of the Homomorphism into the Group of Formal Transformations

Let \( \mathcal{F}(M, p) \) be the set of equivalence classes of diffeomorphisms defined around a point \( p \) of a manifold \( M \) and fixing \( p \in M \) where two diffeomorphisms are equivalent if and only if their Taylor expansions at \( p \in M \) are the same. As \( \mathcal{F}(M, p) \) has a natural group structure as a quotient of the group \( \mathcal{G}(M, p) \) of germs at \( p \) of diffeomorphisms, we call \( \mathcal{F}(M, p) \) the group of formal transformations at \( p \in M \). The goal of this section is to show the following weak local rigidity of a homomorphism into the group of formal transformations.

**Proposition 6.1.** Let \( G \) be the group of orientation-preserving isometries of a rank-one symmetric space of non-compact type, \( \Gamma \) a standard subgroup of \( G \), and \( l : P \to \mathcal{F}(G/P, o) \) the homomorphism into the group of formal transformations at \( o \in G/P \) induced by the action of \( P \) on \( G/P \) by left translations. If \( \rho : \Gamma \to \mathcal{F}(G/P, o) \) is a homomorphism \( C^2 \)-close to \( l|_{\Gamma} \), then there is an embedding \( \iota \) of \( \Gamma \) into \( G \) as a standard subgroup and \( h \in \mathcal{F}(G/P, o) \) such that

\[
\rho(g) = h \circ l(\iota(g)) \circ h^{-1}
\]

for all \( g \in \Gamma \).

As we proved Proposition 5.1, the rigidity of a homomorphism into the group of jets, Proposition 6.1 is an immediate consequence of the following proposition. Moreover, when \( G = \text{Sp}(n + 1, 1) \), \( n \geq 2 \) or \( F_{4}^{-20} \), by Corollary 5.9, we obtain local rigidity of \( l|_{\Gamma} : \Gamma \to j^3(G/P, o) \). Let \( \pi^r : \mathcal{F}(M, p) \to j^r(M, p), r \geq 0 \), be the natural projection from the group of formal transformations onto the group of \( r \)-jets of diffeomorphisms.

**Proposition 6.2.** Let \( \Gamma \subset P \subset G \) be a standard subgroup and \( l : P \to \mathcal{F}(G/P, o) \) be the homomorphism induced by the left action of \( P \) on \( G/P \). If \( \rho : \Gamma \to \mathcal{F}(G/P, o) \) is a group homomorphism such that \( \pi^3 \circ \rho = \pi^3 \circ l|_{\Gamma} : \Gamma \to j^3(G/P, o) \), then \( \rho, l|_{\Gamma} : \Gamma \to \mathcal{F}(G/P, o) \) are conjugate.

**Proof.** Since \( \pi^3 \circ \rho(a) = \pi^3 \circ l(a) \), by Sternberg’s normalization, \( \rho(a) \) and \( l(a) \) are conjugate. So we may assume \( \rho(a) = l(a) \). Under this assumption, we will
show $\rho = l_{|\Gamma}$. By induction, it suffices to show that for $r \geq 3$, a group homomorphism $\rho : \Gamma \to J^{r+1}(G/P, o)$ such that $\pi_{r+1}^{l_\rho} \circ \rho = \pi^{l_\rho} \circ l_{|\Gamma}$ and $\rho(a) = \pi^{l_\rho} \circ l(a)$ satisfies $\rho = \pi^{r+1} \circ l_{|\Gamma}$, where $\pi_{r+1}^{l_\rho} : J^{r}(G/P, o) \to J^{r}(G/P, o)$ $(s > r)$ denotes the natural projection from the group of $s$-jets to the group of $r$-jets. Using Theorem 5.5 and Lemma 5.4, by the same argument as before, we see that $\rho$ extends to a continuous homomorphism $\hat{\rho} : (\langle a \rangle N) \to J^{r+1}(G/P, o)$.

To prove the proposition, it suffices to show that the differential $\hat{\rho}_* : n \to J^{r+1}(G/P, o)$ of $\hat{\rho}$ at $e \in \langle a \rangle N$ is equal to $(\pi^{r+1} \circ l)_* |_{\langle n \rangle}$, where $(\pi^{r+1} \circ l)_* : p \to J^{r+1}(G/P, o)$ is the differential of $\pi^{r+1} \circ l : P \to J^{r+1}(G/P, o)$. To show the difference $\hat{\rho}_* - (\pi^{r+1} \circ l)_* : n \to J^{r+1}(G/P, o)$ is vanished, we consider the adjoint representation of $a \in \Gamma$ as in Section 5. Recall that $\text{Ad}(a)|_{\mathfrak{g}_1} = k^1 \text{Id}_{\mathfrak{g}_1}$, and that $\text{Ad}(\pi^{r+1} \circ l(a))$ on $J^{r+1}(G/P, o)$ can be identified the representation of $\text{Ad}(a)$ on $\oplus_{1 \leq q \leq r} S^q(\mathfrak{n}_*) \otimes \mathfrak{n}_\ast$. Let $(\pi^{r+1}_\ast)_* \hat{\rho}$ be the differential at $e$ of the projection $\pi^{r+1}_\ast : J^{r+1}(G/P, o) \to J^r(G/P, o)$. As the projections onto $r$-jets of $\hat{\rho}$ and $\pi^{r+1} \circ l$ coincide, the image of the difference $\hat{\rho}_* - (\pi^{r+1} \circ l)_* |_{\langle n \rangle}$ is contained in the kernel of $((\pi^{r+1}_\ast))_*$. On the other hand, for any $X \in \mathfrak{g}_1$,

\[ k(\hat{\rho}_* - (\pi^{r+1} \circ l)_*)(X) = \hat{\rho}_* - (\pi^{r+1} \circ l)_\ast(\text{Ad}(a)X) = \text{Ad}(\rho(a)) \circ \hat{\rho}_*(X) - \text{Ad}(\pi^{r+1} \circ l(a)) \circ (\pi^{r+1} \circ l)_\ast(X) = \text{Ad}(\pi^{r+1} \circ l(a)) \circ (\hat{\rho}_* - (\pi^{r+1} \circ l)_\ast)(X). \]

Observe that $\text{Ad}(\pi^{r+1} \circ l(a))$ on the kernel of $(\pi^{r+1}_\ast)_*$ can be identified with $\text{Ad}(a)$ on $\mathcal{G}(\mathfrak{n}_\ast) \otimes \mathfrak{n}_\ast$, whose eigenvalues are $k^i$, $r - 1 \leq i \leq 2r + 1$. Since $r \geq 3$, it follows that $(\hat{\rho}_* - (\pi^{r+1} \circ l)_\ast)(X) = 0$. As the Lie algebra $\mathfrak{n}$ is generated by $\mathfrak{g}_1$, we obtain $
abla = (\pi^{r+1} \circ l)_\ast |_{\langle n \rangle}$.

### 7. Local Rigidity of Local Actions

Let $\mathcal{G}(G/P, o)$ be the group of germs at $o = eP \in G/P$ of diffeomorphisms in $\text{Diff}(G/P, o)$. As $P$ is the stabilizer at $o$ of the action of $G$ on $G/P$ by left translations, we obtain a group homomorphism of $P$ into $\mathcal{G}(G/P, o)$, which will also be denoted by $l : P \to \mathcal{G}(G/P, o)$. The goal of this section is to show weak local rigidity of local actions of standard subgroups:

**Proposition 7.1.** Let $G$ be the group of orientation-preserving isometries of a rank-one symmetric space of non-compact type, $\Gamma$ a standard subgroup of $G$, and $l : P \to \mathcal{G}(G/P, o)$ the homomorphism into the group of germs at $o \in G/P$ of diffeomorphisms around $o \in G/P$ induced by the action of $P$ on $G/P$ by left translations. If $\rho : \Gamma \to \mathcal{G}(G/P, o)$ is a homomorphism $C^2$-close to $l_{|\Gamma}$, then there is an embedding $i$ of $\Gamma$ into $G$ and $h \in \mathcal{G}(G/P, o)$ such that
\[ \rho(g) = h \circ l(i(g)) \circ h^{-1} \]
for all $g \in \Gamma$.

By Proposition 6.1, to prove Proposition 7.1 it suffices to show the following proposition which claims that a local action close to $l_{|\Gamma}$ is determined by its Taylor expansions at $o \in G/P$. Moreover, when $G = \text{Sp}(n + 1, 1)$, $n \geq 2$ or $F_4^{20}$, as
we mentioned in Section 6, the homomorphism of $\Gamma$ into $\mathcal{F}(G/P, o)$ is locally rigid in the strict sense. Thus using the following proposition, we see that the homomorphism of $\Gamma$ into $\mathcal{G}(G/P, o)$ is also locally rigid in the strict sense.

**Proposition 7.2.** Let $\Gamma$ be a standard subgroup of $P$, $\rho : \Gamma \rightarrow \mathcal{G}(G/P, o)$ a homomorphism, and $\pi : \mathcal{G}(G/P, o) \rightarrow \mathcal{F}(G/P, o)$ the natural projection. If $\pi \circ \rho = \pi \circ l_{|\Gamma} : \Gamma \rightarrow \mathcal{F}(G/P, o)$, then $\rho, l_{|\Gamma} \in \text{Hom}(\Gamma, \mathcal{G}(G/P, o))$ are conjugate.

The remaining of this section is devoted to the proof of Proposition 7.2. Since $\pi \circ \rho(a) = \pi \circ l(a) \in \mathcal{F}(G/P, o)$, by Sternberg’s normalization [10], we may assume $l(a) = \rho(a)$. Set $\tilde{\Lambda} = \Gamma \cap N = \bigcup_{i \in \mathbb{Z}} a^i \Lambda a^{-i}$. We will show $l_{|\tilde{\Lambda}} = \rho_{|\tilde{\Lambda}}$.

As $o \in G/P$ is the common fixed point of $l(P)$, $l$ induces an action of $P$ on the complement $X = (G/P) \sim \{o\}$. Using the simply transitivity of the left action of $N$ on $X$, we can define an $n = \text{Lie}(N)$-valued $l(N)$-invariant 1-form $\omega$ on $X$ as follows. Let $x \in X$ be the fixed point of the action of $a$ on $X$. Then under the identification $N \rightarrow X$, $g \rightarrow gx$ of $N$ with $X$, we define the $n$-valued 1-form $\omega$ on $X$ to be the pull-back of the Maurer–Cartan form on $N$. Observe that a diffeomorphism $f$ of $X$ is contained in $l(N)$ if and only if $f^* \omega = \omega$. Moreover, for $g_1, g_2 \in N$, the Taylor expansions at $o \in G/P$ of $l(g_1)$ and $l(g_2)$ coincide if and only if $g_1 = g_2$. So $l_{|\tilde{\Lambda}} = \rho_{|\tilde{\Lambda}}$ if and only if $\rho(\tilde{\Lambda})$ preserves $\omega$.

While the 1-from $\omega \in \Omega^1(X; n)$ cannot be extended smoothly on $G/P$, by Proposition 2.3 (iii), it is rational around $o \in G/P$ in the local coordinate $\exp : n_- \rightarrow G/P$ around $o \in G/P$. In particular, for any diffeomorphisms $f, g$ defined around $o \in G/P$ fixing $o$, if the Taylor expansions at $o \in G/P$ of $f$ and $g$ are the same, then $f^* \omega - g^* \omega$ is a smooth 1-form around $o \in G/P$. Thus for each $g \in \tilde{\Lambda}$,

$$
\Phi(g) = \rho(g)^* \omega - l(g)^* \omega
$$

is a germ of a smooth 1-form defined around $o \in G/P$. Since $\omega$ is $l(N)$-invariant, $\Phi(g) = \rho(g)^* \omega - \omega$. So $\omega$ is $\rho(\tilde{\Lambda})$-invariant if and only if $\Phi(\tilde{\Lambda}) = 0$. Observe that for $g, h \in \tilde{\Lambda}$,

$$
\Phi(gh) = \rho(h)^* (\rho(g)^* \omega - \omega) + \rho(h)^* \omega - \omega = \rho(h)^* \Phi(g) + \Phi(h).
$$

Thus $\Phi : \tilde{\Lambda} \rightarrow \Omega^1(G/P, o; n)$ is a cocycle, where $\Omega^1(G/P, o; n)$ denotes the space of germs at $o \in G/P$ of $n$-valued 1-forms defined around $o \in G/P$. Moreover, for $g \in \tilde{\Lambda}$,

$$
\Phi(aga^{-1}) = l(a^{-1})^* \rho(g)^* l(a)^* \omega = l(a^{-1})^* (\rho(g)^* \omega - \omega = l(a^{-1})^* (\text{Ad}(a) \circ \rho(g)^* \omega - \omega = l(a^{-1})^* (\text{Ad}(a) \circ \rho(g)^* \omega - \text{Ad}(a) \circ \omega) = l(a^{-1})^* \text{Ad}(a) \circ \Phi(g).
$$

Now Proposition 7.2 is a consequence of the following lemma.

**Lemma 7.3.** A $(a)$-equivariant cocycle $\Phi : \tilde{\Lambda} \rightarrow \Omega^1(G/P, o; n)$ is vanished.\(^3\)

\(^3\) Since $\Omega^1(G/P, o; n)(a) = 0$, this is equivalent to $H^1(\tilde{\Lambda}, \Omega^1(G/P, o; n))(a) = 0$. 

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Proof. Let $\Phi: \tilde{\Lambda} \to \Omega^1(G/P, o; n)$ be a $\langle a \rangle$-equivariant cocycle. Recall that the lattice $\Lambda$ has a set of generators $b_1, \ldots, b_{n_1}, c_1, \ldots, c_{n_2}$ such that $[b_i, c_j] = e, [b_i, b_j] \in \langle c_1, \ldots, c_{n_2} \rangle, [c_i, c_j] = e, ab_i a^{-1} = b_j^k, ac_i a^{-1} = c_i^k$. Thus, to prove $\Phi = 0$, by the $\langle a \rangle$-equivariance, it suffices to show that $\Phi(g) = 0$ for all $g \in \Lambda$ with $ag a^{-1} = g^k$. Fix $g \in \tilde{\Lambda}$ with $ag a^{-1} = g^k$. Then

$$l(a^{-1})^* \text{Ad}(a) \circ \Phi(g) = \sum_{j=0}^{k-1} \rho(g^j)^* \Phi(g).$$

Recall that we have the local coordinate system $\exp : \tilde{n}_- \to G/P$ around $o \in G/P$ in which the differential $T_o l(a)$ at $o$ of $l(a)$ is of the form $k^{-1} \text{id}_{g_{-1}} \oplus k^{-2} \text{id}_{g_{-2}}$ and that of $l(g^j a)$ is of the form

$$\left( \begin{array}{cc} k^{-1} \text{id}_{g_{-1}} & u_j \\ 0 & k^{-2} \text{id}_{g_{-2}} \end{array} \right)$$

for some $u_j : g_{-2} \to g_{-1}$. As we fixed a local coordinate system around $o \in G/P$, an $n$-valued 1-form around $o \in G/P$ can be considered as an $n^* \otimes n$-valued smooth function around $o \in G/P$. So to prove $\Phi(g) = 0$, it suffices to show that an $n^* \otimes n$-valued smooth function $F$ defined around $o$ satisfying

$$\text{Ad}(a)F(x) = \sum_{j=0}^{k-1} F(\rho(g^j a)(x)) T_x \rho(g^j a)$$

is vanished around $o \in G/P$. Fix a norm on $n_-$. There is a neighborhood $U$ of $o \in G/P$ such that

- $F$ is defined on $U$,
- $\rho(g^j a)x \in U$ for $j = 0, \ldots, k-1$ and $x \in U$, and
- $\|T_x \rho(g^j a)\| < k^{-1} + \epsilon$ for $j = 0, \ldots, k-1$ and $x \in U$,

where $\|A\| = \sup_{v \in n_-} \|Av\|/\|v\|$ denotes the operator norm. Moreover, fixing a norm on $n$, since $\text{Ad}(a)$ on $n$ is diagonal with eigenvalues $k$ and $k^2$, $\|\text{Ad}(a)^{-1}\| = k^{-1}$ with respect to the induced norm on $\text{gl}(n)$. Moreover, we obtain the induced norm on $n^* \otimes n$. Then

$$\sup_{x \in U} \|F(x)\| = \sup_{x \in U} \left\| \sum_{j=0}^{k-1} \text{Ad}(a)^{-1} F(\rho(b^j a)(x)) T_x \rho(b^j a) \right\|$$

$$\leq \sum_{j=0}^{k-1} \left\| \text{Ad}(a)^{-1} \right\| \sup_{x \in U} \|F(\rho(b^j a)(x))\| \sup_{x \in U} \left\| T_x \rho(b^j a) \right\|$$

$$\leq \sum_{j=0}^{k-1} k^{-1} \sup_{x \in U} \|F(x)\| (k^{-1} + \epsilon)$$

$$= (k^{-1} + \epsilon) \sup_{x \in U} \|F(x)\|.$$

It follows that $\sup_{x \in U} \|F(x)\| = 0$. Thus $\Phi(g) = 0$. $\square$
8. Local rigidity of group actions

Let $G$ be a group of orientation-preserving isometries of a rank-one symmetric space of non-compact type with an Iwasawa decomposition $G = KAN$, $P$ a minimal parabolic subgroup of $G$ containing $AN$, $l : G \to \text{Diff}(G/P)$ the action by the left multiplication, $\Gamma = \langle a, \Lambda \rangle$ the standard subgroup generated by $a \in A$ and a lattice $\Lambda \subset N$ with $a\Lambda a^{-1} \subset \Lambda$.

Let $\rho$ be an action of $\Gamma$ on $G/P$ sufficiently close to $l|_\Gamma$. Since $\Gamma \subset P$, the original action $l|_{\Gamma}$ admits a common fixed point $o = P \in G/P$. We will use Theorem 5.3 to show that $\rho$ also admits a common fixed point close to $o$. It suffices to show that the first cohomology with respect to the isotropic representation $dl|_{\Gamma} : \Gamma \to \text{GL}(T_o(G/P))$ is vanished. Under the natural identification of $T_o(G/P)$ with $g/p$, the isotropic representation at $o \in G/P$ of the left action is identified with the adjoint representation of $\Gamma$ on $g/p$. By Proposition 4.2, the cohomology is vanished. Thus $\rho$ admits a common fixed point close to $o$.

Conjugating $\rho$ by a diffeomorphism of $G/P$ which maps the common fixed point of $\rho$ to $o$, we may assume that $\rho$ has a common fixed point $o$. By Proposition 7.1, we may assume that for each $g \in \Gamma$, the germs at $o \in G/P$ of $\rho(g)$ and $l(g)$ are the same. To prove Theorem 1.2, it remains to show the following proposition. Moreover, when $G = \text{Sp}(n + 1, 1)$, $n \geq 2$ or $F_4^{\text{-20}}$, Corollary 1.3 also follows from this proposition.

**Proposition 8.1.** Assume $G \neq \text{PSL}(2, \mathbb{R})$. Let $\rho : \Gamma \to \text{Diff}(G/P,o)$ be an action of $\Gamma$ on $G/P$ with a common fixed point $o$ whose germs at $o$ coincides with that of $l|_{\Gamma} : \Gamma \to \text{Diff}(G/P,o)$. Then $\rho, l|_{\Gamma} \in \text{Hom}(\Gamma, \text{Diff}(G/P,o))$ are conjugate.

The outline of the proof can be described as follows. The left action of $N$ on $G/P$ has a unique common fixed point $o$, while its action on the complement $(G/P) \sim \{o\}$ is simply transitive. Thus if we fix a point $x \neq o \in G/P$, we obtain a natural identification of $(G/P) \sim \{o\}$ with $N$. Then the conjugacy around $o \in G/P$ can be considered as a $\Gamma$-equivariant function “at infinity” of $N$. Lemma 8.3 implies that such a function can be extended $\Lambda$-equivariantly. As $\Lambda$ is a normal subgroup of $\Gamma$, we can deduce the $\Gamma$-equivariance.

Let us begin with an easy lemma that gives a sufficient condition for the existence of an equivariant extension of a function.

**Lemma 8.2.** Let $\Lambda$ be a group with a generating set $S$, and $X, Y$ manifolds on which $\Lambda$ acts smoothly. Assume there is an open subset $U$ of $X$ such that

1. $X = \bigcup_{g \in \Lambda} gU$, and
2. for $g \in \Lambda$, $gU \cap U \neq \emptyset$ only if $g \in S$.

If $f$ is a smooth map from $X$ into $Y$ such that $sf(x) = f(sx)$ for $x \in U$ and $s \in S$, then there is a unique $\Lambda$-equivariant smooth map $\tilde{f}$ from $X$ into $Y$ such that $\tilde{f}|_U = f$ on $U$.

**Proof.** By the assumption (i) on $U$, for any $x \in X$, there is $g \in \Lambda$ such that $gx \in U$. Thus it suffices to show that for any $x \in X$, $\tilde{f}(x) = g^{-1}f(gx)$ does not depend on the choice of $g \in \Lambda$ with $gx \in U$. If $g_1x, g_2x \in U$, by the assumption (ii), there is
Thus the claim follows.

A finitely generated group \( \Lambda \) is said to have exactly one end if the Cayley graph \( \Delta = \text{Cay}(\Lambda, S) \) of \( \Lambda \) with respect to a finite generating set \( S \) has the following property: For any finite subgraph \( F \subset \Delta \), there is a finite subgraph \( F' \) containing \( F \) such that the complement \( \Delta \sim F' \) is connected. It is known that this condition does not depend on the choice of a finite generating set.

**Lemma 8.3.** Let \( \Lambda, S, X, Y, \) and \( U \) as in Lemma 8.2. Assume further that \( S \) is a finite set, \( \Lambda \) has exactly one end, and the center of \( \Lambda \) is infinite. Let \( f \) be a smooth map from \( X \) into \( Y \) such that for any \( g \in \Lambda \), there is a compact subset \( K_g \) of \( X \) such that \( gf(x) = f(gx) \) for \( x \in X \sim K_g \). Then there is a \( \Lambda \)-equivariant smooth map \( \tilde{f} \) from \( X \) into \( Y \) and a compact subset \( \tilde{K} \) of \( X \) such that \( \tilde{f} = f \) on \( X \sim \tilde{K} \).

**Proof.** Set \( K = \bigcup_{s \in S} K_s \) so that \( sf(x) = f(sx) \) for all \( s \in S \) and \( x \in X \sim K \). By the assumption (ii) on \( U \), there are at most finitely many elements \( g \in \Lambda \) such that \( SgU \cap K \neq \emptyset \). As \( \Lambda \) has one end, there is a finite subset \( F \) of \( \Lambda \) such that \( SgU \cap K = \emptyset \) for \( g \in \Lambda \sim F \) and that the complement of \( \text{Cay}(\Lambda, S) \) for \( F \) is connected. As the center of \( \Lambda \) is infinite, we may choose an element \( c \in \Lambda \) in the center so that \( c \notin F \). As \( c \) commutes with any elements in \( \Lambda \), \( cU \) also satisfies the assumptions (i) and (ii). By Lemma 8.2, there is a unique \( \Lambda \)-equivariant smooth extension \( \tilde{f} \) of \( f|_{cU} \) to \( X \). Observe that for any \( g \in \Lambda \) with \( \tilde{f} = f \) on \( gU \), if \( sg \in \Lambda \sim F \), \( s \in S \), then \( \tilde{f} = f \) on \( sgU \). Since \( \tilde{f} = f \) on \( cU \), using the above observation \( m \) times, we see that \( \tilde{f} \) coincides with \( f \) on \( gU \). We proved \( \tilde{f} = f \) on \( \bigcup_{g \in \Lambda \sim F} gU \).

We claim that \( \tilde{f}(x) \neq f(x) \) only if \( x \in \bigcup_{g \in \Lambda} gK \). Assume there exists \( x \in X \) such that \( \tilde{f}(x) \neq f(x) \) and \( \Lambda x \cap K = \emptyset \). By the assumption (ii), there are at most finitely many \( g \in \Lambda \) such that \( gFU \cap FU \neq \emptyset \). So there exist \( g \in \Lambda \) and \( s \in S \) such that \( \tilde{f}(gx) \neq f(gx) \) and \( \tilde{f}(sgx) = f(sgx) \). Since \( \tilde{f} \) is \( \Lambda \)-equivariant, this contradicts to the condition \( sf(x) = f(sx) \) for \( s \in S \) and \( x \in X \sim K \). Thus the claim follows. By the assumption (ii), we see that there are at most finitely many \( g \in \Lambda \) such that \( gK \setminus FU \neq \emptyset \). Let \( \tilde{K} \) be the union of \( gK \) over such \( g \in \Lambda \). Then \( \tilde{f} = f \) on \( X \sim \tilde{K} \).

**Proof of Proposition 8.1.** As we assume \( G \neq \text{PSL}(2, \mathbb{R}) \), the Lie group \( N \) is diffeomorphic to \( \mathbb{R}^n \) for some \( n \geq 2 \). So its lattice \( \Lambda \) has exactly one end. Moreover, the center of \( \Lambda \) is infinite. As the left action of \( N \) on \( X = (G/P) \sim \{o\} \) is simply transitive, the action of \( \Lambda \) on \( X \) is properly discontinuous and cocompact. So there are a finite generating set \( S \) of \( \Lambda \) and an open subset \( U \) of \( X \) satisfying the assumptions of Lemma 8.3. Since the germs at \( o \) of \( l|_{\Gamma} \) and \( \rho \) are the same, for

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4 In our application below, we can choose \( U \) so that \( \tilde{U} \) is compact. In this case, the conclusion of Lemma 8.3 follows immediately from this statement.
each \( g \in \Lambda \), there is a compact subset \( K_g \) of \( X \) such that \( l(g) = \rho(g) \) on \( X \setminus K_g \). Applying Lemma 8.3, we obtain a \( \Lambda \)-equivariant smooth map \( \tilde{f} : X \to X \) which is identity outside of a compact subset, where the domain is equipped with the \( \Lambda \)-action induced by \( l \) and the range with the action induced by \( \rho \). It remains to show that the extension \( h : G/P \to G/P \) of \( \tilde{f} \) by \( h(o) = o \) is a conjugacy between \( l|_\Gamma \) and \( \rho \).

By the \( \Lambda \)-equivariance, \( h \) is a covering map over \( G/P \), which is diffeomorphic to the sphere \( S^n \), \( n \geq 2 \). So \( h \) is a diffeomorphism of \( G/P \). We will show the \( (a) \)-equivariance of \( h \). For any \( x \in X \), we can choose \( g \in \Lambda \) so that \( l^g(x) = l(g)(x) \) is sufficiently close to \( o \). So we may choose \( g \in \Lambda \) satisfying \( h \circ l^g(x) = \rho^a \circ h \circ l^g(x) \).

By the definition of \( \Gamma \), \( a g^{-1} a^{-1} \) is an element of \( \Lambda \). Using the \( \Lambda \)-equivariance of \( h \),

\[
h \circ l^g(x) = h \circ l^g a^{-1} a^{-1} \circ h \circ l^g(x) = \rho^a \circ h \circ l^g(x) = \rho^a \circ h(x),
\]

which shows the \( (a) \)-equivariance of \( h \). So \( h \) is \( \Gamma \)-equivariant and thus a conjugacy between \( l|_\Gamma \) and \( \rho \).

Finally, we will show that the action \( l|_\Gamma \) of \( \Gamma \) on \( G/P \) is not locally rigid if \( G = SU(n + 1, 1) \), \( n \geq 2 \).

**Proposition 8.4.** When \( G = SU(n + 1, 1) \), \( n \geq 2 \), the action \( l|_\Gamma \) of a standard subgroup \( \Gamma \) of \( G \) on \( G/P \) is not \( C^2 \)-locally rigid.

**Proof.** Let \( l : P \to J^3(G/P, o) \) be the homomorphism induced by the action of \( P \) on \( G/P \) by the left translation. We will show that there is an automorphism \( \phi \) of the group \( AN \) close to the identity such that

- \( \phi|_A = id_A \), \( \phi(N) = N \), and
- the homomorphisms \( l \circ \phi \), \( l|_{AN} \) of \( AN \) into \( J^3(G/P, o) \) are not conjugate.

Let \( G_1 \) be the group of automorphisms of \( AN \) that fix \( A \) and preserve \( N \) and \( G_2 \) the subgroup of \( J^3(G/P, o) \) consisting of elements commuting with \( l(A) \) and normalizing \( l(N) \). It suffices to show that the dimension of \( G_1 \) is larger than that of \( G_2 \). It is easy to see that the Lie algebra of \( G_1 \) can be identified with the space \( \text{Der}(n)^a \) of \( a \)-equivariant derivations of \( n \). On the other hand, the Lie algebra of \( G_2 \) can be identified with the subalgebra of \( J^3(G/P, o) \) consisting of elements centralizing \( l_*(a) \) and normalizing \( l_*(n) \). By Lemma 3.11, this subalgebra is equal to \( l_*(g^a) \). Since the codimension of \( l_*(g^a) \subset \text{Der}(n)^a \) is equal to the dimension of \( H^1(n,g)^a \neq 0 \), the claim follows.

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