Scaling of Fock-space propagator and multifractality across the many-body localization transition

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We implement a recursive Green function method to extract the Fock space (FS) propagator and associated self-energy across the many-body localization (MBL) transition, for one-dimensional interacting fermions in a random onsite potential. We show that the typical value of the imaginary part of the local FS self-energy, $\Delta_t$, related to the decay rate of an initially localized state, acts as a probabilistic order parameter for the thermal to MBL phase transition; and can be used to characterize critical properties of the transition as well as the multifractal nature of MBL states as a function of disorder strength $W$. In particular, we show that a fractal dimension $D_s$ extracted from $\Delta_t$ jumps discontinuously across the transition, from $D_s < 1$ in the MBL phase to $D_s = 1$ in the thermal phase. Moreover, $\Delta_t$ follows an asymmetrical finite-size scaling form across the thermal-MBL transition, where a non-ergodic volume in the thermal phase diverges with a Kosterlitz-Thouless like essential singularity at the critical point $W_c$, and controls the continuous vanishing of $\Delta_t$ as $W_c$ is approached. In contrast, a correlation length ($\xi$) extracted from $\Delta_t$ exhibits a power-law divergence on approaching $W_c$ from the MBL phase.

I. INTRODUCTION

A many-body localized (MBL) phase is a fascinating non-equilibrium state of matter which can originate in isolated quantum systems in the presence of disorder and interactions [1–4]. In the MBL phase, high-energy eigenstates violate the Eigenstate Thermalization Hypothesis (ETH) [5–7], and local memory can be retained up to arbitrarily long times under time evolution. After the early landmark papers [8, 9], a mathematical proof [10] and numerous numerical studies [11–16] have provided strong evidence in favour of the existence of the MBL phase in one dimension. However, the universal properties of the transition from the thermal or ETH phase to the MBL phase remain under active debate [17–23]. This is mainly due to the challenge to “first principles” numerical verification of theoretical scenarios, posed by the exponentially growing Fock-space dimension ($N_F$) with system size $L$ [11, 13, 17, 24, 25]. To complement calculations on microscopic models, several phenomenological renormalisation group-based approaches have been employed [18, 26–28], which suggest a Kosterlitz-Thouless (KT)-like scenario for the MBL transition.

A further complementary approach to MBL is to consider it as an effective ‘Fock-space (FS) localization’ problem of a fictitious particle on the complex, correlated FS graph (or ‘lattice’) [16, 29–31]. This has led to crucial insights such as the role of strong FS correlations in stabilising the MBL phase [30–33], multifractality of eigenstates therein [17, 34–37] and, importantly, a numerical scaling theory of the MBL transition in terms of FS inverse participation ratios (IPR) of eigenstates, which is consistent with a KT-like scenario [35, 36]. Here we ask the question: can a scaling theory of the MBL transition be formulated in terms of FS propagators?

In this work we answer this question in the affirmative, by studying the propagator or Green function on the FS lattice for a fermionic chain with $L \leq 22$. In particular, we extract the local Feenberg self-energy [30, 38, 39] from the diagonal elements of the FS propagator. As known since Anderson’s seminal paper [40], the typical value of the imaginary part of the Feenberg self-energy, $\Delta_t$, acts as a probabilistic order parameter in the thermodynamic limit for Anderson transitions [40–43]. This quantity has recently been employed to construct a self-consistent mean-field theory of the MBL transition on the Fock space [30, 31]. However, numerically exact evaluation of the FS self-energy for the microscopic models of MBL, and analysis of its behaviour across the MBL transition, is scarce. Here we fill this void and show that the MBL transition from ergodic extended states in the thermal phase to multifractal states in the MBL phase [17, 34–37, 44–46], is manifest in an anomalous scaling of $\Delta_t$ with the Fock-space dimension. This, to-
gether with a scaling theory of the MBL transition based on the order parameter \( \Delta_t \), that is consistent with the KT-like scenario, constitutes the central result of the work.

The FS propagator contains information about the ergodic/non-ergodic nature of the phase, as different eigenstates contribute to it, based on their energy and amplitudes over FS lattice sites. From a technical point of view, computation of the propagator in principle requires all eigenstates. This restricts the system sizes accessible to numerical exact diagonalization (ED), which for \( L > 18 \) can access only a limited number of eigenstates [47]. To access larger sizes, comparable to those accessible via parallelised shift-invert method [47] or POLFED [48], but at significantly cheaper computational cost, we compute the FS propagators using a standard recursive Green function method [49–51], but adapted to the FS graph. Moreover, for the reasons discussed below, we study a scaled version of the self-energy, viz. \( \Delta | \sqrt{L} \) and refer to it as \( \Delta_t \), throughout the rest of the paper for notational convenience. Employing the recursive method, we obtain the following main results:

1. The typical value \( \Delta_t \) of the self-energy is finite in the thermal phase. It vanishes \( \propto N_F^{-1(1-D_s)} \) in the MBL phase and at the critical point, where \( D_s < 1 \) is a fractal dimension reflecting the multifractal nature of the states. \( D_s \) changes discontinuously across the MBL transition, from \( D_s < 1 \) to \( D_s = 1 \) throughout the thermal phase.

2. The finite-size scaling of \( \Delta_t \) as a function of disorder strength \( (W) \) is consistent with an asymmetric finite-size scaling form [35, 36, 52, 53] across the MBL transition. Scaling on the thermal side is controlled by a non-ergodic volume scale \( \Lambda \), which diverges with an essential singularity \( \Lambda \sim \exp (b/\sqrt{W}) \), \( |\delta W = (W_c - W) | \sim O(1) \) at a critical disorder \( (W_c) \), redolent of a KT-like transition. Scaling on the MBL side by contrast is controlled by a FS correlation length \( (\xi) \), which exhibits a power-law divergence on approaching criticality. Moreover, the scaling theory implies that in the thermodynamic limit \( \Delta_t \) vanishes continuously on approaching the transition from the thermal side as \( \sim \exp (-b/\sqrt{W}) \) with \( b \sim O(1) \).

As already mentioned, multifractal characterization of MBL states [17, 34] and numerical scaling theory consistent with a KT-like MBL transition have been obtained via study of eigenstate IPRs [35, 36]. However our work reveals for the first time the multifractality and KT-type critical scaling in terms of a FS order parameter for the thermal-MBL transition. Additionally, the FS order parameter, being associated with an inverse decay time of localized initial states, provides a truly dynamical characterization of MBL transition unlike the static properties studied based on eigenstates in the previous studies [17, 34–36].

II. MODEL

We study the following standard model [11–13, 15–17, 54] of MBL for a fermionic chain with an i.i.d. random onsite potential \( \epsilon_i \in [-W, W] \) of strength \( W \) on \( i = 1, \ldots, L \) sites and nearest-neighbour repulsion \( (V) \),

\[
\mathcal{H} = t \sum_{i=1}^{L-1} (c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) + \sum_{i=1}^{L} \epsilon_i \hat{n}_i + V \sum_{i=1}^{L-1} \hat{n}_i \hat{n}_{i+1}.
\]

(1)

Here \( c_i^\dagger (c_i) \) is the fermion creation (annihilation) operator for site \( i \), with number operator \( \hat{n}_i = c_i^\dagger c_i \). We choose \( t = 0.5 \) and \( V = 1 \) to be consistent with earlier studies; and study the model in the half-filled sector at ‘infinite temperature’, which corresponds to the middle of the many-body energy spectrum. In this case, the model shows a thermal to MBL transition at a critical disorder \( W_c \approx 3.7 - 4.2 \) [17]. In this work, we take the critical disorder \( W_c \) as 3.75. Variation of \( W_c \) between \( 3.5 - 4 \) leads to comparably good scaling collapse in our finite-size scaling analysis.

To describe the many-body system in Fock space, we employ the occupation-number basis \( \{|I\} \) of particles on the real-space sites, \( |I\rangle = |n_1^{(I)}, n_2^{(I)}, \ldots, n_L^{(I)}\rangle \) with \( n_i^{(I)} \in 0 \text{ or } 1 \). In this basis, the Hamiltonian Eq. (1) takes the form of a tight-binding model [30, 33, 55]

\[
\mathcal{H} = \sum_{I,J} T_{IJ} |I\rangle \langle J| + \sum_{I} E_{I} |I\rangle \langle I|,
\]

(2)

but on the FS lattice [Fig. 1(a)]. Here, the FS "hopping" \( T_{IJ} = t \) when \( |I\rangle \) and \( |J\rangle \) are connected by a single nearest-neighbor hop in real-space, and \( T_{IJ} = 0 \) otherwise. The onsite "disorder" potential at FS site \( I \) is \( E_{I} = \sum_{i} \epsilon_{i} n_{i}^{(I)} + V \sum_{i} n_{i}^{(I)} n_{i+1}^{(I)} \), and is a combination of the real-space disorder potential and the nearest-neighbor interaction.

The disorder-averaged many-body density of states for the model is a Gaussian as a function of energy \( E \), with the mean \( \propto L \) and variance \( \rho_{E}^2 = (t^2 + W^2/6 + V^2/8) L/2 \) [55]. As a result, to treat different system sizes on the same footing and for the theory to have a well-defined thermodynamic limit [31], we scale the parameters and work with a rescaled \( \tilde{\mathcal{H}} = \mathcal{H}/\sqrt{L} \) [30, 31, 55]. This leads directly to the scaled version of the self-energy mentioned above. We also set the mean many-body energy to zero, by transforming \( \tilde{\mathcal{H}} \) to \( \tilde{\mathcal{H}} = 1 \text{ Tr } \tilde{\mathcal{H}}/N_F \) for each disorder realization, since the middle of the spectrum fluctuates with disorder realization for a finite system.

III. RECURSIVE GREEN’S FUNCTION METHOD

Elements of the retarded FS propagator \( G(E) = [(E + i\eta)1 - \tilde{\mathcal{H}}]^{-1} \) are computed by implementing the standard
recursive method \([49–51, 56, 57]\) at the middle (\(D_{\text{regulator}}\)) \(\eta\) of the energy spectrum. We choose the broadening (or phase. At \(W\) by the dashed black line (linear fit) for one value of \(t\) \(\langle\cdot\rangle\) in Appendix A.

It is this locality that allows for an efficient implementation of the method for the FS lattice, as detailed in Appendix A. We calculate the diagonal elements \(G_{II}(E) = \langle I|G(E)|I\rangle\), where \(I\) is the middle slice \(M\) [Fig. 1(a)]. We extract one of the important characterizations of the localization properties in the FS, the imaginary part \(\Delta_I\) of the Feenberg self-energy \(\Sigma_I(E) = X_I(E) - i\Delta_I(E) \equiv G^{I\dagger}_{II}(E) - (E + i\eta - \xi_I)\) at a FS site \(I\). As mentioned in the introduction, the analogous quantity on the real-space lattice was quintessential in the development of the concept of localization \([40]\) for non-interacting systems. We consider the distributions of \(\Delta_I\), in particular its disorder-averaged typical value, denoted by \(\Delta_c\). The latter is the geometric mean calculated by averaging over different disorder realizations and different Fock space sites using \(\ln \Delta_I = \langle \ln \Delta_I \rangle_{I,\{\epsilon_i\}}\), where \(\langle \cdot \rangle_{I,\{\epsilon_i\}}\) denotes averaging over \(I \in M\) and disorder realizations \(\{\epsilon_i\}\). Depending on \(L\), \(-150\) to \(10,000\) disorder realizations are employed to generate the distribution of \(\Delta_I\) (Appendix C). We have checked that our results converge with number of disorder samples for all the system sizes, \(L = 12 – 22\), that we study, and the details are given in Appendix C.

**IV. MULTIFRACTALITY AND SCALING THEORY FROM THE IMAGINARY PART OF FEENBERG SELF-ENERGY**

The imaginary part \(\Delta_I(E)\) of the self-energy determines the energy-resolved decay time \([\Delta_I(E)]^{-1}\) of a localized initial state \(|I\rangle\) or, alternatively, the life time of an excitation with energy \(E\) created at the FS lattice site \(I\) \([40]\). As such, \(\Delta_I(E)\) directly encodes information about localization or lack thereof. In the thermal phase, the typical value \(\Delta_t \sim O(1)\), as the initial state decays in a finite time, whereas in the MBL phase, \(\Delta_t \to 0\) in the thermodynamic limit \(N_F \to \infty\).

Numerical results for \(\Delta_t\) as a function of \(N_F\) are shown in Fig. 1(b). \(\Delta_t\) indeed decreases as a function of \(W\) from an \(O(1)\) value in the thermal phase to a value which in the MBL phase vanishes exponentially rapidly with \(L\). Deep in the thermal phase at weak disorder, \(\Delta_t\) is independent of \(N_F\). By contrast, in the MBL phase \((W > W_c)\), \(\Delta_t\) decays as a power-law, \(\propto N_F^{-(1-D_s)}\) with \(0 < D_s < 1\). As discussed later, \(D_s\) is a spectral fractal dimension which characterizes the multifractality of the MBL states. It depends on \(W\) as shown in Fig. 1(c), where \(D_s\) has been obtained from the finite-size scaling analysis discussed below. Note that at intermediate disorder in the ETH phase, \(\Delta_t\) initially decays with \(N_F\) for small \(L\), before showing an increasing trend, presumably towards its finite asymptotic value in the thermodynamic limit [Fig. 1(b)]. This indicates that the systems are in the critical regime at intermediate disorder.

The bare data itself conforms to the expectation that in the thermodynamic limit \(\Delta_t\) vanishes in the MBL phase and approaches a finite \(O(1)\) value in the thermal phase. But to analyze compellingly the critical properties of \(\Delta_t\), we perform a scaling collapse of the data using the following finite-size scaling ansatz \([35, 36, 52]\),

\[
\frac{\ln \Delta_I}{\Delta_c} = \begin{cases} 
F_{\text{vol}} \left( \frac{N_F}{\Lambda} \right) & : W < W_c \\
F_{\text{lin}} \left( \frac{\ln N_F}{\xi} \right) & : W > W_c,
\end{cases}
\]

where \(\Delta_c = \Delta_I(W=W_c) \sim N_F^{-(1-D_s)}\). The scaling ansatz states that in the ETH phase, \(\Delta_t\) follows a ‘volumic’ scaling form where the finite-size scaling is controlled by a Fock-space volume scale, \(\Lambda\) \([35, 36, 52]\). In the MBL phase, on the other hand, the scaling form is ‘linear’ with the scaling controlled by a Fock-space lengthscale, \(\xi\) \([35, 36, 52]\). Taking \(W_c = 3.75\), the good collapse of the data shown in Fig. 2 suggests that the above scaling forms are appropriate. As shown in Appendix B, varying \(W_c\) between \(-3.5–4\) leads to comparably good scaling.
FIG. 2. Finite-size scaling collapse of \(\ln(\Delta_t/\Delta_c)\) using respectively the volumetric and linear scaling [Eq. (3)] in the thermal (upper panel) and MBL (lower panel) phase, with \(W_c = 3.75\). Upper panel: The asymptotic scaling of \(\ln(\Delta_t/\Delta_c) \sim 0.5\ln(N_F/S)\) deep in the thermal phase is determined by the exponent \((1 - D_c) \simeq 0.5\) at \(W = W_c\), where \(\Delta_c \sim N_F^{-(1-D_c)}\). Inset shows the KT-like essential singularity of the non-ergodic volume, \(\Lambda \sim \exp(b/\sqrt{|\delta W|})\) with \(\delta W = (W_c - W)\), near \(W_c\). Lower panel: Deep in the MBL phase, asymptotic scaling gives \(\ln(\Delta_t/\Delta_c) \propto (\ln N_F/\xi)\). The inset shows the divergence of the correlation length, \(\xi \sim |\delta W|^{-\beta}\) with \(\beta \simeq 0.7\), on approaching the transition from the MBL phase. System sizes from \(L = 12 - 20\) were used to obtain the scaling.

The ‘non-ergodic’ volume \(\Delta(W)\) [52] extracted from the scaling collapse of \(\ln(\Delta_t/\Delta_c)\) in the thermal phase \((W < W_c)\) is shown in the upper panel of Fig. 2 (inset). \(\Delta\) diverges at the critical point with a KT-like essential singularity, \(\Lambda \sim \exp(b/\sqrt{|\delta W|})\) with \(\alpha \simeq 0.5\), where \(\delta W = (W_c - W)\) and \(b \sim O(1)\). This kind of KT-type singularity has been predicted by a phenomenological RG theory [18, 26–28], albeit based on a real-space picture. Throughout the ETH phase, since the eigenstates are understood to be ergodic [35], we expect \(\Delta_c \sim O(1)\) asymptotically in the limit \(N_F \gg 1\). This implies for the volumetric scaling function in Eq. (3) that \(\mathcal{F}_{\text{vol}}(x) \sim (1-D_c)\ln x\) for \(x \gg 1\). As evinced in the upper panel of Fig. 2, this is indeed the asymptotic form of the scaling function, \(\mathcal{F}_{\text{vol}}(x) \sim 0.49\ln x\) (see Appendix B for details) from which we estimate \(D_c \simeq 0.51\). This is an excellent agreement with \(D_c \simeq 0.50\) obtained by fitting the raw data of \(\Delta_c\) against \(N_F\) for \(W_c = 3.75\) [Fig. 1(b)]. A significant conclusion from this is that \(D_c = 1\) throughout the ETH phase and jumps discontinuously to a value \(D_s < 1\) at the MBL transition, whereafter it decreases smoothly with increasing \(W\) [Fig. 1(c)]. This is in complete consonance with the discontinuity across the MBL transition of the fractal dimension characterising the Fock-space IPRs of the eigenstates [35, 36]. Furthermore, the form of the divergence of \(\Lambda\) and the asymptotic form of \(\mathcal{F}_{\text{vol}}\) implies that in the thermodynamic limit, \(\Delta_c\) vanishes continuously as the transition is approached from the ETH side as \(\Delta_c \sim \Lambda^{-(1-D_c)} \sim \exp[-(1 - D_c)/|\delta W|\alpha]\).

In the MBL phase for \(W > W_c\), the scaling collapse yields a diverging correlation length, \(\xi \sim (W - W_c)^{-\beta}\) with \(\beta \simeq 0.7\), as shown in the lower panel of Fig. 2 (inset). For \(x = (\ln N_F/\xi) \gg 1\), the asymptotic scaling is \(\mathcal{F}_{\text{lin}}(x) \sim -(1 - D_c)x\) (Appendix B), as indeed seen in Fig. 2 lower panel. From the form of \(\mathcal{F}_{\text{lin}}(x)\), \(\xi = (1 - D_c)(D_c - D_s)\), which implies a divergent \(\xi\) as \(D_s \to D_c\), i.e. \(W \to W_c\) (Appendix B). Note that \(D_s < 1\) implies that the MBL critical point is actually a part of the MBL phase itself. This is consistent with the understanding that the entire MBL phase is critical in the sense that it is multifractal, and the MBL transition can be understood as the terminal end point of the line of fixed points.

Having established the critical scaling of the FS order parameter \(\Delta_t\) and the volume (length) scale in the thermal (MBL) phase, we now discuss how the system-size scaling \(\Delta_t \sim N_F^{-(1-D_c)}\) in Fig. 1(b), (c) for \(W > W_c\) is related to multifractality in the MBL phase. MBL eigenstates are known to be multifractal in nature [17, 34–37], i.e. non-ergodic but extended over \(\sim N_F^\alpha (0 < D < 1)\) FS sites. Similar to the case of the local density of states discussed in Ref. [58], the multifractality can be deduced from the dependence of the typical value and the distribution of \(\Delta_t\), on the broadening parameter \(\eta\) for large but finite \(N_F\). In this limit, \(\Delta_t\) saturates as a function of \(\eta\) below an energy scale \(\eta_c \sim N_F^{-z(1)} \ll \eta\), whereas \(\Delta_t \sim \eta^{\theta}\) for \(\eta \gg \eta_c\). Here the exponent \(\theta > 0\) and the spectral fractal dimension \(D_s = 1 - z\theta\) lies between 0 and 1 [58]. Physically, \(\eta_c\) signifies the presence of an energy scale much larger than the mean level spacing \(N_F^{-(1-D_c)}\) for multifractal states for any finite system. In our calculations, \(\eta \sim N_F^{-(1-D_c)} \ll \eta\), thus we expect \(\Delta_t \sim N_F^{-\theta}\). This is indeed the behaviour for \(\Delta_t(N_F)\) that we find in Fig. 1(b) over the entire MBL range \((W > W_c)\). It is also completely consistent with the asymptotic form of the scaling function [Fig. 2 (lower panel)] and the \(D_s\) extracted from it [Fig. 1(b)]. The spectral dimension \(D_s\) can be shown to be the same as the fractal dimension \(D\) extracted from the eigenstates under very general considerations [58]. We note that the behaviour \(\Delta_t \sim N_F^{-(1-D_s)}\) is different from a self-consistent theory [30, 31] of MBL, where the thermodynamic limit of \(N_F \to \infty\) is taken first before taking the \(\eta \to 0\) limit. As a result, one obtains \(\Delta_t \sim \eta\) and such a theory does not in effect distinguish between Anderson localized and multifractal states.

Further insight into the multifractal behaviour may be gained by analysing the tail of the probability distribution function \(P(\Delta)\) of \(\Delta_t\) over disorder realizations. For finite \(\eta\) and/or \(N_F\), one expects [58] the distribution to be cut off at a maximum value \(\Delta = \Delta_{\text{max}}\), where
possible connections between the Fock-space propagators and critical properties, e.g. how rare thermal regions in real space [36] contribute. It would also be interesting to explore further the connection [36] between real-space and Fock-space critical fractal dimension.

In the MBL phase, the scaling of $\Delta_{\text{max}}$ with the Fock space dimension $N_F$ is shown to follow a power-law $\Delta_{\text{max}} \propto N_F^z$ for disorder strength $\eta$. The exponent $z$, obtained from $\Delta_{\text{max}} \sim \eta^{-1}$ in (a), is shown to be $< 1$ in the MBL phase, consistent with the multifractal scaling of the self-energies. The solid line is merely a guide to the eye.

$\Delta_{\text{max}} \sim 1/\eta_c \sim N_F^z$ for $\eta < \eta_c$. As shown in Fig. 3(a), we indeed find that $\Delta_{\text{max}}$ directly extracted from the numerical data follows a power-law $\sim N_F^z$. The exponent $z$ extracted from $\Delta_{\text{max}}$ is shown in Fig. 3(b), and indeed satisfies $0 < z < 1$. This further confirms the multifractal scaling of the self-energy in the MBL phase.

V. CONCLUSION

Using a recursive Green function method, we have in summary obtained the critical scaling across the MBL transition by calculating the self-energy associated with the local Fock-space propagator, the typical value of the imaginary part of which, $\Delta_{\text{max}}$, acts as an order parameter for the MBL transition. The finite-size scaling of $\Delta_{\text{max}}$ implies the existence of a non-ergodic volume with a KT-like essential singularity, which directly controls the continuous vanishing of $\Delta_{\text{max}}$ on approach to the transition from the ergodic side, and the multifractal nature of the MBL phase was demonstrated via determination of the spectral fractal dimension $D_s$, which was shown to change discontinuously across the transition. While our focus here has been on the local FSB propagator, the recursive Green function method also gives access to the non-local propagator, a question of immediate future interest which can potentially provide insights into the critical scaling of a FS localization length, and also enable study of the inhomogeneous nature of MBL eigenstates on the FS lattice [36]. It would also be interesting to explore further the connection [36] between real-space and Fock-space critical properties, e.g. how rare thermal regions in real space [59–61] affect the FS self-energy. In the same vein, possible connections between the Fock-space propagators and one-particle density matrix and real-space propagators [62–67] also remains a question for future work.

Appendix A: Recursive Green function Method

In the recursive method, we calculate different elements of the Green function by starting from one of the two minimally connected Fock-space sites $|111\ldots000\rangle$ and $|000\ldots111\rangle$ and by adding the next slice connected by hopping to the earlier slice at each step. Here we start with the top site $|111\ldots000\rangle$ as shown in Fig. 1(a). At each iteration step, the method inverts a matrix containing the Hamiltonian elements for the added slice and is, therefore, of size $N_{\ell'} \times N_{\ell}$, where $N_{\ell'}$ is the size of the added slice. Hence, this method avoids the inversion of the full Hamiltonian in Eq. (2). The maximum size of the matrix $\langle N_M \times N_M \rangle$ that is inverted during the calculation is determined by the number of sites $N_M$ in the middle slice, which is the largest slice of the Fock space [Fig. 1(a)]. For $L$ between 10 and 22, $N_M$ is between $\sim 1-2$ orders of magnitude lower than the full Fock-space dimension $N_F = (L^3_0)$. It is this substantial reduction that provides the advantage to achieve the system size $L = 22$ even with serial computation. The method can be applied to even larger systems like $L = 24$ with parallelization.

At the $\ell$-th step of recursion, let $H^{(\ell)}$ be the Hamiltonian corresponding to the part of the Fock space lattice containing all slices up to the $\ell$-th slice. The corresponding Green function for these $\ell$ slices is given by $G^{(\ell)} = \left[ E^+ \mathbb{1} - H^{(\ell)} \right]^{-1}$, where $E^+ = E + i\eta$. Using Eq. (2) and the local structure of the Fock space lattice, the Hamiltonian corresponding to all slices up to the $(\ell + 1)$-th slice can be written as:

$$H^{(\ell+1)} = \left( H^{(\ell)} T_{rl} H^{(0)(\ell+1)} \right),$$

$$H^{(0)(\ell+1)} = \sum_{I,\ell} \mathcal{E}_{I,\ell+1} |I,\ell + 1\rangle \langle I,\ell + 1|,$$

$$T_{rl} = \sum_{I \in \ell} \sum_{J \in (\ell+1)} T_{I,J} |I,\ell + 1\rangle \langle I,\ell|,$$

where $H^{(0)(\ell+1)}$ contains the part of the Hamiltonian corresponding to the $(\ell + 1)$-th slice disconnected from the rest of the Fock lattice, $\mathcal{E}_{I,\ell+1}$ contains the hopping elements for hopping from the $\ell$-th slice to the $(\ell + 1)$-th slice and $T_{rl} = T_{rl}^\dagger$. |I,\ell⟩, representing the state |I⟩, explicitly mentions the slice index $\ell$ which |I⟩ belongs to. Similarly, the Green function at the $(\ell + 1)$-th step can be
written as

\[
G^{(\ell+1)} = \begin{pmatrix} G_r^{(\ell+1)} & G_r^{(\ell+1)} \\ G_l^{(\ell+1)} & G_l^{(\ell+1)} \end{pmatrix}.
\]

Here the subscript \( \ell \) denotes the matrix elements of the left part (first to the \( \ell \)-th slice), and \( r \) denotes the elements of the \((\ell+1)\)-th slice, as described in Fig. A1. Then, at the \((\ell+1)\)-th step of recursion, the Green function \( G^{(\ell+1)} \) and the Hamiltonian \( H^{(\ell+1)} \) have the following matrix equation.

\[
\left[ E^+ \mathbb{1} - H^{(\ell+1)} \right] G^{(\ell+1)} = \mathbb{1}
\]

\[
\begin{pmatrix} (G^{(\ell)})^{-1} & T_r, \\ T_l, & (G_0^{(\ell+1)})^{-1} \end{pmatrix} \begin{pmatrix} G_r^{(\ell+1)} & G_r^{(\ell+1)} \\ G_l^{(\ell+1)} & G_l^{(\ell+1)} \end{pmatrix} = \mathbb{1}
\]

Here \((G^{(\ell)})^{-1} = (E^+ \mathbb{1} - H^{(\ell)}) \) and \((G_0^{(\ell+1)})^{-1} \) is given by \([E^+ \mathbb{1} - H^{(0)(\ell+1)}]\).

From the above equation, we can obtain the following equation.

\[
G^{(\ell+1)} = G^{(\ell)} + G^{(\ell)} T_r G_r^{(\ell+1)},
\]

\[
\langle s | G^{(\ell+1)} | s' \rangle = \langle s | G^{(\ell)} | s' \rangle + \langle s | G^{(\ell)} (T_\ell | \ell \rangle \langle \ell + 1 | G^{(\ell+1)} | s' \rangle \rangle; \quad s, s' \leq \ell.
\]

Note that \( \langle s | A | s' \rangle = A(s, s') \) denotes an element of a rectangular matrix \( A \), as each of the slices, indexed by \( s \) and \( s' \), contains different numbers of lattice points in general. For the same reason, \( T_\ell \) is also a rectangular matrix containing hopping matrix elements between \( \ell \)-th and \((\ell+1)\)-th slices. Therefore,

\[
G^{(\ell+1)}(s, s') = G^{(\ell)}(s, s') + G^{(\ell)}(s, \ell) T_\ell G^{(\ell+1)}(\ell + 1, s'), \quad s, s' \leq \ell.
\]

Similarly, starting from Eq. (A1),

\[
G^{(\ell+1)}(\ell + 1, s) = G^{(\ell+1)}(\ell + 1, \ell + 1) T_\ell G^{(\ell)}(\ell, s), \quad s \leq \ell,
\]

(A2a)

\[
G^{(\ell+1)}(s, \ell + 1) = G^{(\ell)}(s, \ell) T_\ell G^{(\ell+1)}(\ell + 1, \ell + 1), \quad s \leq \ell.
\]

(A2b)

Therefore, for \( s, s' \leq \ell \),

\[
G^{(\ell+1)}(s, s') = G^{(\ell)}(s, s') + G^{(\ell)}(s, \ell) T_\ell G^{(\ell+1)}(\ell + 1, \ell + 1) T_\ell G^{(\ell)}(\ell, s'), \quad s, s' \leq \ell.
\]

(A2c)

and on-slice elements

\[
G^{(\ell+1)}(\ell + 1, \ell + 1) = \left[ (G_0^{(\ell+1)})^{-1} - T_\ell G^{(\ell)}(\ell, \ell) T_\ell \right]^{-1}.
\]

(A2d)

Using Eqs. (A2), we can calculate the desired elements of the Green function, avoiding the inversion of a matrix having the dimension of the full Fock space.

**Appendix B: Scaling of \( \Delta_t \)**

**Asymptotic behaviour of the scaling functions:**

In the ergodic phase, the scaling ansatz Eq. (3) reads

\[
\ln \Delta_x = \ln \Delta_x + \mathcal{F}_{\text{vol}}(N_F/\Lambda)
\]

with the critical \( \Delta_c = C_c N_F^{-(1-D_c)} \) and \( C_c \) a constant; equivalently,

\[
\mathcal{F}_{\text{vol}} \left( \frac{N_F}{\Lambda} \right) = \ln \Delta_x - \ln C_c + (1 - D_c) \ln \Lambda + (1 - D_c) \ln N_F.
\]

(B1)

Deep in an ergodic phase, both \( \Lambda \) and \( \Delta_c \) are \( \mathcal{O}(1) \). From Eq. (B1), for \( x = N_F/\Lambda \gg 1 \) the asymptotic large-\( x \) behaviour of \( \mathcal{F}_{\text{vol}}(x) \) is then

\[
\mathcal{F}_{\text{vol}}(x) \sim (1 - D_c) \ln x
\]

(B2)

up to \( \mathcal{O}(1) \) corrections. Its slope gives the fractal exponent \( (1 - D_c) \) for \( W = W_c \), which is also found to be consistent with the numerical asymptotic fit in Fig. 1(b).

On the other hand, in the MBL phase \( W > W_c \), \( \Delta_t = C_m N_F^{-(1-D_c)} \) (Fig. 1(b)), where \( C_m \) and \( D_c \) both depend on \( W \). The scaling ansatz Eq. (3) then gives

\[
\mathcal{F}_{\text{lin}} \left( \frac{\ln N_F}{\xi} \right) = \ln C_m \frac{N_F}{\Lambda} - (1 - D_c) \ln N_F - \frac{C_c}{\xi},
\]

(B3)

with \( \xi \) defined as

\[
\xi = \frac{(1 - D_c)}{D_c - D_s}.
\]

(B4)

As \( D_s \to D_c^+ \), i.e. as \( W \to W_c^+ \), we see that \( \xi \) diverges. Deep in the MBL phase, where \( x = (\ln N_F)/\xi \gg 1 \), the asymptotic scaling is clearly

\[
\mathcal{F}_{\text{lin}}(x) \sim -(1 - D_c)x.
\]

(B5)

Finally, returning to the ergodic phase, Eq. (B1) with \( x = N_F/\Lambda \) can be cast as

\[
\Delta_t = \Lambda^{-(1-D_c)} \exp \left[ \mathcal{F}_{\text{vol}}(x) - (1 - D_c) \ln x + \ln C_c \right]
\]

(B6)

and is determined by \( W \), and hence for any finite \( \Lambda \) no matter how large, the thermodynamic limit corresponds to \( x \to \infty \). From Eq. (B2) the argument of the exponential in Eq. (B6) is \( \mathcal{O}(1) \), so in the thermodynamic limit the order parameter \( \Delta_t \) vanishes continuously as

\[
\Delta_t \sim \Lambda^{-(1-D_c)}
\]

(B7)

and is controlled by \( \Lambda \).

**Dependence of scaling results on \( W_c \):** For the finite-size scaling shown in Fig. 2, \( W_c = 3.75 \) was employed. Fig. A3 gives corresponding results for the scaling collapse on choosing \( W_c = 3.5 \) and 4. As seen, the quality of scaling is quite robust to the variation in \( W_c \).
Appendix C: Convergence of the data with the number of FS sites and disorder realizations

We first verify that the statistics of the imaginary part of the self-energy ($\Delta_I$) is not affected by considering only the middle slice of the Fock space to generate the distribution. For both the ergodic and MBL phases, Fig. A2 gives a representative example of the fact that the distributions of $\Delta_I$ ($P(\Delta_I)$ or $P(\Delta_I/\eta)$) are barely affected whether one considers $I \in M$ or $I \in \text{FS}$. 

In Table I, we show the number of disorder realizations considered for different sizes to generate the distributions of $\Delta_I$. The x-axis represents the fraction of the largest number of disorder samples considered for the largest number of disorder realizations considered (Table I). We have also verified that the distributions do not change further with an increasing number of disorder realizations. Fig. A5 shows the convergence of $\ln \Delta_I$ for different system sizes at three different disorder strengths, $W = 1.0$ (in the thermal phase), $4.0$ (near the critical point), and $6.0$ (in the MBL phase). The x-axis represents the fraction of the maximum number of disorder samples considered for the calculation of $\ln \Delta_I$ as shown in Table I. For example, for $L = 18$ the disorder sample size is changed from 400 to 1000 in steps of 100.

However, the quantity $\Delta_I$ is not expected to be self-averaging, i.e., there can be sample-to-sample fluctuations even in the thermodynamic limit. This kind of phenomenon is well known for conductance and its universal sample-to-sample fluctuations in studies [68, 69] of single-particle Anderson localization. Still, a scaling theory for the Anderson localization transition can be constructed using the typical or average value of the conductance [69]. In a similar spirit, we consider the typical or average value of the corresponding number of disorder realizations employed to generate the distributions of $\Delta_I$ at each $W$ point are shown in the first two columns. The third column estimates the product of number of disorder realizations and the number of FS sites in the middle slice.

![Figure A1](image1.png)

FIG. A1. (a) The figure on the top left corner is similar to Fig. 1(a), but rotated by 90°. The larger figure shows a part of it, up to $\ell$th slice, where the grey shaded area shows each slice. Orange circles and solid blue lines represent the FS sites and hopping, respectively. (b) As we keep on adding the next slices, the Green function of the left part gets updated via the iterative Eqs. (A2).

![Figure A2](image2.png)

FIG. A2. The probability distributions of the imaginary part $\Delta_I$ of the self-energy, considering both $I \in M$ and $I \in \text{FS}$. These match well with each other, in both the thermal phase (shown in (a)) and MBL phase ((b)). Data shown in the figure are for system size $L = 14$.

| $L$ | # disorder realizations | # of total data points |
|-----|------------------------|------------------------|
| 12  | $10^4$                 | $6 \times 10^4$        |
| 14  | $5 \times 10^3$        | $8 \times 10^5$        |
| 16  | $2 \times 10^3$        | $11 \times 10^5$       |
| 18  | $10^3$                 | $17 \times 10^5$       |
| 20  | 500                    | $27 \times 10^5$       |
| 22  | 130                    | $23 \times 10^5$       |

TABLE I. $L$ and corresponding number of disorder realizations employed to generate the distributions of $\Delta_I$ at each $W$ point are shown in the first two columns. The third column estimates the product of number of disorder realizations and the number of FS sites in the middle slice.

In Fig. A4, we show the relative change in the same quantity with respect to the value of $\Delta_I$ for the largest sample size for all system sizes. The relative change is always less than 1% of the values plotted in Fig. 1(b), even for smaller sample sizes for all system sizes up to $L = 22$. However, the quantity $\Delta_I$ is not expected to be self-averaging, i.e., there can be sample-to-sample fluctuations even in the thermodynamic limit. This kind of phenomenon is well known for conductance and its universal sample-to-sample fluctuations in studies [68, 69] of single-particle Anderson localization. Still, a scaling theory for the Anderson localization transition can be constructed using the typical or average value of the conductance [69]. In a similar spirit, we consider the typical or average value of the corresponding number of disorder realizations employed to generate the distributions of $\Delta_I$ at each $W$ point are shown in the first two columns. The third column estimates the product of number of disorder realizations and the number of FS sites in the middle slice.

![Figure A5](image3.png)
FIG. A3. Finite-size scaling collapse of \( \ln(\Delta_t/\Delta_c) \) in direct parallel to that of Fig. 2 for \( W_c = 3.75 \), but here shown for \( W_c = 3.5 \) (left panel) and \( W_c = 4.0 \) (right panel).

FIG. A4. Relative percentage change in \( \ln\Delta_t \) as the sample size is changed from 0.4 times the maximum sample size to the maximum sample size. The relative change is defined as \( \left| \frac{\ln\Delta_t - \ln\Delta_t^{(1)}}{\ln\Delta_t^{(1)}} \right| \), where \( \ln\Delta_t^{(1)} \) is calculated considering all the available samples. This shows the value of \( \ln\Delta_t \) is always within 1% of the values plotted in Fig. 1(b) even for smaller sample sizes, and for all system sizes up to \( L = 22 \).

We show here only the convergence for \( W = 4.0 \), although the convergence holds for all the disorder strengths. We would also like to emphasize that the typical value \( \Delta_t \) for the local FS self-energy \( \Delta_I \) is obtained by averaging over both disorder realizations and FS sites \( I \) in the middle slice of the FS lattice [Fig. 1(a)]. Since the number of sites in the middle slice increases exponentially with system size, the effective number of samples, albeit not independent, over which statistics is accumulated, is \( \sim 10^6 \) for \( L \geq 14 \), see Table I. This leads to \( \Delta_I \) and \( P(\ln\Delta_I) \) converging quite rapidly for larger system sizes like \( L = 22 \) even when the number of independent disorder realizations is smaller for larger \( L \), as we demonstrate here.

[1] R. Nandkishore and D. A. Huse, Many-body localization and thermalization in quantum statistical mechanics, Annual Review of Condensed Matter Physics 6, 15 (2015).
FIG. A5. Convergence of $\ln \Delta_t$ for different number of disorder samples i.e. realizations. The maximum number of disorder samples used in these calculations are shown in Table I. For each system size the number of samples is varied from 0.4 times the maximum sample size to the maximum sample size. The convergence is tested for three disorder values $W = 1.0$ (in the thermal phase), 4.0 (near the transition), and 6.0 (in the MBL phase). Values of $L$ corresponding to the colors are shown in the legend of (a).

FIG. A6. Convergence of the distribution of $\ln \Delta_t$ for different sample sizes at disorder strength $W = 4.0$ near the transition. Similar to Fig. A4, the sample size is changed, as shown in the legend of (a), from 0.5 times the maximum sample size to the maximum sample size for all system sizes. For all sample sizes and system sizes considered, the distribution is seen to be the same.
[2] D. A. Abanin and Z. Papić, Recent progress in many-body localization, *Annalen der Physik* 529, 1700169 (2017).

[3] F. Alet and N. Laflorencie, Many-body localization: An introduction and selected topics, *Comptes Rendus Physique* 19, 498 (2018).

[4] D. A. Abanin, E. Altman, I. Bloch, and M. Serbyn, Colloquium: Many-body localization, thermalization, and entanglement, *Rev. Mod. Phys.* 91, 021001 (2019).

[5] J. M. Deutsch, Quantum statistical mechanics in a closed system, *Phys. Rev. A* 43, 2046 (1991).

[6] M. Srednicki, Chaos and quantum thermalization, *Phys. Rev. E* 50, 888 (1994).

[7] M. Srednicki, The approach to thermal equilibrium in quantized chaotic systems, *Journal of Physics A: Mathematical and General* 32, 1163 (1999).

[8] D. Basko, I. Aleiner, and B. Altshuler, Metal–insulator transition in a weakly interacting many-electron system with localized single-particle states, *Annals of Physics* 321, 1126 (2006).

[9] I. V. Gornyi, A. D. Mirlin, and D. G. Polyakov, Interacting electrons in disordered wires: Anderson localization and low-t transport, *Phys. Rev. Lett.* 95, 206603 (2005).

[10] J. Z. Imbrie, On many-body localization for quantum spin chains, *Journal of Statistical Physics* 163, 998 (2016).

[11] V. Ograneyyan and D. A. Huse, Localization of interacting fermions at high temperature, *Phys. Rev. B* 75, 155111 (2007).

[12] M. Zaïdarić, T. Prosen, and P. Prelovšek, Many-body localization in the Heisenberg XXZ magnet in a random field, *Phys. Rev. B* 77, 064426 (2008).

[13] A. Pal and D. A. Huse, Many-body localization phase transition, *Phys. Rev. B* 82, 174411 (2010).

[14] J. H. Bardarson, F. Pollmann, and J. E. Moore, Unbounded growth of entanglement in models of many-body localization, *Phys. Rev. Lett.* 109, 017202 (2012).

[15] J. A. Kjäll, J. H. Bardarson, and F. Pollmann, Many-body localization in a disordered quantum Ising chain, *Phys. Rev. Lett.* 113, 107204 (2014).

[16] M. Serbyn, Z. Papić, and D. A. Abanin, Criterion for many-body localization-delocalization phase transition, *Phys. Rev. X* 5, 041047 (2015).

[17] D. J. Luitz, N. Laflorencie, and F. Alet, Many-body localization edge in the random-field heisenberg chain, *Phys. Rev. B* 91, 081103 (2015).

[18] P. T. Dumitrescu, A. Goremykina, S. A. Parameswaran, M. Serbyn, and R. Vasseur, Kosterlitz-thouless scaling at many-body localization phase transitions, *Phys. Rev. B* 99, 094205 (2019).

[19] M. Kiefer-Emmanouilidis, R. Unanyan, M. Fleischhauer, and J. Sirker, Evidence for unbounded growth of the number entropy in many-body localized phases, *Phys. Rev. Lett.* 124, 243601 (2020).

[20] J. Šuntajs, J. Bonča, T. c. v. Prosen, and L. Vidmar, Quantum chaos challenges many-body localization, *Phys. Rev. E* 102, 062144 (2020).

[21] R. K. Panda, A. Scardicchio, M. Schulz, S. R. Taylor, and M. Žnidarič, Can we study the many-body localisation transition?, *EPL (Europhysics Letters)* 128, 67003 (2020).

[22] D. Abanin, J. Bardarson, G. De Tomasi, S. Gopalakrishnan, V. Khemani, S. Parameswaran, F. Pollmann, A. Potter, M. Serbyn, and R. Vasseur, Distinguishing localization from chaos: Challenges in finite-size systems, *Annals of Physics* 427, 168415 (2021).

[23] P. Sierant, D. Delande, and J. Zakrzewski, Thouless time analysis of anderson and many-body localization transitions, *Phys. Rev. Lett.* 124, 186601 (2020).

[24] R. Modak and S. Mukerjee, Many-body localization in the presence of a single-particle mobility edge, *Phys. Rev. Lett.* 115, 230401 (2015).

[25] V. Khemani, D. N. Sheng, and D. A. Huse, Two universality classes for the many-body localization transition, *Phys. Rev. Lett.* 119, 075702 (2017).

[26] A. Goremykina, R. Vasseur, and M. Serbyn, Analytically solvable renormalization group for the many-body localization transition, *Phys. Rev. Lett.* 122, 040601 (2019).

[27] A. Morningstar and D. A. Huse, Renormalization-group study of the many-body localization transition in one dimension, *Phys. Rev. B* 99, 224205 (2019).

[28] A. Morningstar, D. A. Huse, and J. Z. Imbrie, Many-body localization near the critical point, *Phys. Rev. B* 102, 125134 (2020).

[29] B. L. Altshuler, Y. Gefen, A. Kamenev, and L. S. Levitov, Quasiparticle lifetime in a finite system: A nonperturbative approach, *Phys. Rev. Lett.* 78, 2803 (1997).

[30] D. E. Logan and S. Welsh, Many-body localization in Fock space: A local perspective, *Phys. Rev. B* 99, 045131 (2019).

[31] S. Roy and D. E. Logan, Fock-space correlations and the origins of many-body localization, *Phys. Rev. B* 101, 134202 (2020).

[32] A. Altland and T. Micklitz, Field theory approach to many-body localization, *Phys. Rev. Lett.* 118, 127202 (2017).

[33] S. Ghosh, A. Acharya, S. Sahu, and S. Mukerjee, Many-body localization due to correlated disorder in Fock space, *Phys. Rev. B* 99, 165131 (2019).

[34] A. De Luca and A. Scardicchio, Ergodicity breaking in a model showing many-body localization, *EPL (Europhysics Letters)* 101, 37003 (2013).

[35] N. Macé, F. Alet, and N. Laflorencie, Multifractal scalings across the many-body localization transition, *Phys. Rev. Lett.* 123, 180601 (2019).

[36] S. Roy and D. E. Logan, Fock-space anatomy of eigenstates across the many-body localization transition, *Phys. Rev. B* 104, 174201 (2021).

[37] G. De Tomasi, I. M. Khaymovich, F. Pollmann, and S. Warzel, Rare thermal bubbles at the many-body localization transition from the fock space point of view, *Phys. Rev. B* 104, 024202 (2021).

[38] E. Feenberg, A note on perturbation theory, *Phys. Rev.* 74, 206 (1948).

[39] E. N. Economou, *Green’s Functions in Quantum Physics* (Springer, Berlin, 2006).

[40] P. W. Anderson, Absence of diffusion in certain random lattices, *Phys. Rev.* 109, 1492 (1958).

[41] R. Abou-Chacra, D. J. Thouless, and P. W. Anderson, A selfconsistent theory of localization, *Journal of Physics C Solid State Physics* 6, 1734 (1973).

[42] E. N. Economou and M. H. Cohen, Existence of mobility edges in anderson’s model for random lattices, *Phys. Rev. B* 5, 2931 (1972).

[43] D. C. Licciardello and E. N. Economou, Study of localization in anderson’s model for random lattices, *Phys. Rev. B* 11, 3697 (1975).

[44] C. L. Bertrand and A. M. García-García, Anomalous...
thouless energy and critical statistics on the metallic side of the many-body localization transition, Phys. Rev. B 94, 144201 (2016).

[45] E. J. Torres-Herrera and L. F. Santos, Extended nonergodic states in disordered many-body quantum systems, Annalen der Physik 529, 1600284 (2017), arXiv:1610.02035 [cond-mat.dis-nn].

[46] M. Serbyn, Z. Papic, and D. A. Abanin, Thouless energy and multifractality across the many-body localization transition, Phys. Rev. B 96, 104201 (2017).

[47] F. Pietracaprina, N. Macé, D. J. Luitz, and F. Alet, Shift-invert diagonalization of large many-body localizing spin chains, SciPost Phys. 5, 45 (2018).

[48] P. Sierant, M. Lewenstein, and J. Zakrzewski, Polynomially filtered exact diagonalization approach to many-body localization, Phys. Rev. Lett. 125, 156601 (2020).

[49] P. A. Lee and D. S. Fisher, Anderson localization in two dimensions, Phys. Rev. Lett. 47, 882 (1981).

[50] A. MacKinnon, The conductivity of the one-dimensional disordered Anderson model: a new numerical method, Journal of Physics C: Solid State Physics 13, L1031 (1980).

[51] A. MacKinnon and B. Kramer, The scaling theory of electrons in disordered solids: Additional numerical results, Zeitschrift für Physik B Condensed Matter 53, 1 (1983).

[52] I. García-Mata, O. Giraud, B. Georgeot, J. Martin, R. Dubertrand, and G. Lemarié, Scaling theory of the anderson transition in random graphs: Ergodicity and universality, Phys. Rev. Lett. 118, 166601 (2017).

[53] N. Laflolencie, G. Lemarié, and N. Macé, Chain breaking and kosterlitz-thouless scaling at the many-body localization transition in the random-field heisenberg spin chain, Phys. Rev. Research 2, 042033 (2020).

[54] B. Bauer and C. Nayak, Area laws in a many-body localized state and its implications for topological order, Journal of Statistical Mechanics: Theory and Experiment 2013, P09005 (2013).

[55] S. Welsh and D. E. Logan, Simple probability distributions on a fock-space lattice, Journal of Physics: Condensed Matter 30, 405601 (2018).

[56] J. A. Verges, Computational implementation of the Kubo formula for the static conductance: application to two-dimensional quantum dots, Computer Physics Communications 118, 71 (1999).

[57] Prabhakar and A. Mukherjee, Fock Space Recurrent Green’s functions for bound complexes in partially filled bands of interacting systems, arXiv e-prints, arXiv:2111.10781 (2021), arXiv:2111.10781 [cond-mat.str-el].

[58] B. L. Altshuler, L. B. Ioffe, and V. E. Kravtsov, Multifractal states in self-consistent theory of localization: analytical solution (2016), arXiv:1610.00758 [cond-mat.dis-nn].

[59] W. De Roeck and F. m. c. Huveneers, Stability and instability towards delocalization in many-body localization systems, Phys. Rev. B 95, 155129 (2017).

[60] D. J. Luitz, F. m. c. Huveneers, and W. De Roeck, How a small quantum bath can thermalize long localized chains, Phys. Rev. Lett. 119, 150602 (2017).

[61] I.-D. Potirniche, S. Banerjee, and E. Altman, Exploration of the stability of many-body localization in $d > 1$, Phys. Rev. B 99, 205149 (2019).

[62] S. Bera, H. Schomerus, F. Heidrich-Meisner, and J. H. Bardarson, Many-body localization characterized from a one-particle perspective, Phys. Rev. Lett. 115, 046603 (2015).

[63] T. L. M. Lezama, S. Bera, H. Schomerus, F. Heidrich-Meisner, and J. H. Bardarson, One-particle density matrix occupation spectrum of many-body localized states after a global quench, Phys. Rev. B 96, 060202 (2017).

[64] M. Hopjan and F. Heidrich-Meisner, Many-body localization from a one-particle perspective in the disordered one-dimensional bose-hubbard model, Phys. Rev. A 101, 063617 (2020).

[65] M. Hopjan, F. Heidrich-Meisner, and V. Alba, Scaling properties of a spatial one-particle density-matrix entropy in many-body localized systems, Phys. Rev. B 104, 035129 (2021).

[66] T. Orito and K.-I. Imura, Multifractality and fock-space localization in many-body localized states: One-particle density matrix perspective, Phys. Rev. B 103, 214206 (2021).

[67] A. Jana, V. R. Chandra, and A. Garg, Local density of states and scattering rates across the many-body localization transition, Phys. Rev. B 104, L140201 (2021).

[68] P. A. Lee and A. D. Stone, Universal conductance fluctuations in metals, Phys. Rev. Lett. 55, 1622 (1985).

[69] K. Slevin, P. Markoš, and T. Ohtsuki, Reconciling conductance fluctuations and the scaling theory of localization, Phys. Rev. Lett. 86, 3594 (2001).