Full-rank Valuations and Toric Initial Ideals

Lara Bossinger*  

Abstract  

Let \( V(I) \) be a polarized projective variety or a subvariety of a product of projective spaces and let \( A \) be its (multi-)homogeneous coordinate ring. To a full-rank valuation \( \nu \) on \( A \) we associate a weight vector \( w_\nu \). Our main result is that the value semi-group of \( \nu \) is generated by the images of the generators of \( A \) if and only if the initial ideal of \( I \) with respect to \( w_\nu \) is prime. As application we prove a conjecture by [BLMM17] connecting the Minkowski property of string polytopes to the tropical flag variety. For Rietsch-Williams’ valuation for Grassmannians we identify a class of plabic graphs with non-integral associated Newton–Okounkov polytope (for \( \text{Gr}_k(\mathbb{C}^n) \) with \( n \geq 6 \) and \( k \geq 3 \)).

1 Introduction

In the context of toric degenerations\(^1\) of projective varieties the study of full-rank valuations on homogeneous coordinate rings (see Definition 1) is very popular. This goes back to a result of Anderson [And13]: if the semi-group, which is the image of the valuation (called value semi-group) is finitely generated, the valuation defines a toric degeneration. Therefore, a hard and central question is whether a given valuation has finitely generated value semi-group or not.

For example, the valuations from birational sequences in [FFL17] are of full rank, and constructed to define toric degenerations of flag and spherical varieties. But for a general valuation arising in this setting it remains unknown if its value semi-group is finitely generated. In the recent paper [Bos21] a new class of valuations from birational sequences for Grassmannians is constructed. The author applies results of this paper to identify those giving toric degenerations.

To gain more control over the valuation it is moreover desirable to identify algebra generators of \( A \) whose valuation images generate the value semi-group. Such generators are called a Khovanskii basis, introduced by Kaveh-Manon in [KM19]. They construct full rank valuations with finite Khovanskii bases from maximal prime cones of the tropicalization of a polarized projective variety. In this paper, we complement their work by taking the opposite approach: starting from a full rank valuation we give bases from maximal prime cones of the tropicalization of a polarized projective variety. In this paper, we prove a conjecture by [AKT20] connecting the Minkowski property of string polytopes to the tropical flag variety. For Rietsch-Williams’ valuation for Grassmannians we identify a class of plabic graphs with non-integral associated Newton–Okounkov polytope (for \( \text{Gr}_k(\mathbb{C}^n) \) with \( n \geq 6 \) and \( k \geq 3 \)).

Throughout the paper for \( n \in \mathbb{Z}_{\geq 0} \) let \( [n] \) denote \( \{1, \ldots, n\} \). Let \( X \) be a subvariety of the product of projective spaces \( \mathbb{P}^{k_1-1} \times \cdots \times \mathbb{P}^{k_s-1} \). In particular, if \( s = 1 \) then \( X \) is a polarized projective variety. Its (multi-)homogeneous coordinate ring \( A \) is given by \( \mathbb{C}[x_{ij} | i \in [s], j \in [k_i]]/I \). Here \( I \) is a prime ideal in \( S := \mathbb{C}[x_{ij} | i \in [s], j \in [k_i]] \), the total coordinate ring of \( \mathbb{P}^{k_1-1} \times \cdots \times \mathbb{P}^{k_s-1} \). Further, \( I \) is homogeneous with respect to the \( \mathbb{Z}_{\geq 0} \)-grading on \( S \). By \( \bar{x}_{ij} \in A \) we denote the cosets of variables \( x_{ij} \). Let \( d \) be the Krull-dimension of \( A \).

A valuation \( \nu : A \setminus \{0\} \to \mathbb{Z}^d \) has full rank if its image (the value semi-group \( S(A, \nu) \)) spans a sublattice of full rank in \( \mathbb{Z}^d \). It is homogeneous if it respects the grading on \( A \). From now on we only consider full-rank valuations. In Definition 6 we define the weighting matrix of \( \nu \) as \( M_\nu := (\nu(\bar{x}_{ij}))_{ij} \in \mathbb{Z}^{d \times (k_1 + \cdots + k_s)} \).

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\(^1\)A toric degeneration of a projective variety \( X \) is a flat morphism \( \pi : X \to \mathbb{A}^m \) with generic fiber \( \pi^{-1}(t) \) for \( t \neq 0 \) isomorphic to \( X \) and \( \pi^{-1}(0) \) a projective toric variety.
1. Introduction

By means of higher Gröbner theory (see for example, [KM19, §8]), we consider the initial ideal \( \text{in}_{M_0}(I) \subset S \) of \( I \) with respect to \( M_0 \) (see Definition 4). Our main result is the following theorem. It is formulated in greater detail in Theorem 1 below.

**Theorem.** Let \( \nu : A \setminus \{0\} \to \mathbb{Z}^d \) be a full-rank valuation with full rank weight matrix \( M_0 \in \mathbb{Z}^{d \times (k_1 + \cdots + k_s)} \) for the presentation \( S/I \) of \( A \). Then

\[
S(A, \nu) \text{ is generated by } \{ \nu(\bar{x}_{ij}) \}_{i \in [s], j \in [k_i]} \text{ if and only if } \text{in}_{M_0}(I) \text{ is prime.}
\]

The theorem has some very interesting implications in view of toric degenerations and Newton–Okounkov bodies. Consider a valuation of form \( \nu : A \setminus \{0\} \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}^{d-s} \) with \( \nu(f) = (\deg f, \cdot) \) for all \( f \in A \). Without loss of generality by [IW18, Remark 2.6] we may assume that any full-rank homogeneous valuation is of this form. The Newton–Okounkov cone \( C(A, \nu) \subset \mathbb{R}^s \times \mathbb{R}^{d-s} \) is the cone over its image. The Newton–Okounkov body \( \Delta(A, \nu) \) is then the intersection of \( C(A, \nu) \) with the subspace \( \{(1, \ldots, 1)\} \times \mathbb{R}^{d-s} \), for details consider Definition 3. Newton–Okounkov bodies are in general pretty wild objects, they are convex bodies but need neither be polyhedral nor finite. However, if \( S(A, \nu) \) is finitely generated they are rational polytopes (this case for \( s = 1 \) is treated by Anderson).

Recall, that if \( S(A, \nu) \) is generated by \( \{ \nu(\bar{x}_{ij}) \}_{i \in [s], j \in [k_i]} \) (i.e. \( \{ \bar{x}_{ij} \}_{i \in [s], j \in [k_i]} \) is a Khovanskii basis for \( (A, \nu) \)) this has a number of useful consequences. For example, in this case the associated graded algebra of \( \nu \) can be presented as \( S/\text{in}_{M_0}(I) \). The following corollary is another such consequence and crucial for our application to flag varieties:

**Corollary.** Let \( \nu : A \setminus \{0\} \to \mathbb{Z}^d \) be a full-rank valuation with \( M_0 \in \mathbb{Z}^{d \times (k_1 + \cdots + k_s)} \) the weighting matrix of \( \nu \) for the presentation \( S/I \) of \( A \). Assume additionally \( \text{in}_{M_0}(I) \) is prime, hence \( S(A, \nu) \) is generated by \( \{ \nu(\bar{x}_{ij}) \}_{i \in [s], j \in [k_i]} \). Then the Newton–Okounkov polytope is the Minkowski sum:\n
\[
\Delta(A, \nu) = \text{conv}(\nu(\bar{x}_{i1}))_{i \in [k_1]} + \cdots + \text{conv}(\nu(\bar{x}_{ij}))_{j \in [k_i]}.
\]

We continue our study by considering monomial maps as appear for example in [MS18]. Let \( \phi_\nu : S \to \mathbb{C}[y_1, \ldots, y_d] \) be the homomorphism defined by sending a generator \( x_{ij} \) to the monomial in \( y_k \)'s with exponent vector \( \nu(\bar{x}_{ij}) \). Its kernel \( \ker(\phi_\nu) \subset S \) is a toric ideal, see (3.5).

Further analyzing our weighting matrices, we associate to each a weight vector. Let \( w_\nu \in \mathbb{Z}^{k_1 + \cdots + k_s} \) be a weight vector associated to \( M_0 \) satisfying \( \text{in}_{M_0}(I) = \text{in}_{w_\nu}(I) \) (see Lemma 3). The following lemma reveals the relation between \( w_\nu \) and the toric ideal \( \ker(\phi_\nu) \), see also Lemma 2.

**Lemma.** For every full-rank valuation \( \nu : A \setminus \{0\} \to \mathbb{Z}^d \) we have \( \text{in}_{w_\nu}(I) \subset \ker(\phi_\nu) \). In particular, \( \text{in}_{w_\nu}(I) \) is monomial-free and \( w_\nu \) is contained in the tropicalization of \( I \) (in the sense of [MS15], see (3.2)).

We apply our results to two classes of valuations.

**Grassmannians and valuations from plabic graphs.** We consider a class of valuations on the homogeneous coordinate rings of Grassmannians defined in [RW19]. In the context of cluster algebras and cluster duality for Grassmannians, they associate a full-rank valuation \( \nu_G \) to every plabic graph \( G \) with certain properties [Pos06].

For \( k < n \) denote by \( \binom{n}{k} \) the set of \( k \)-element subsets of \( [n] \). Consider the Grassmannian with its Plücker embedding \( \text{Gr}_k(C^n) \hookrightarrow \mathbb{P}^{\binom{n}{k}-1} \). We obtain its homogeneous coordinate ring \( \mathbb{C}[p_{J}] \) \( J \in \binom{n}{k} \). In particular, \( I_{k,n} \) is the Plücker ideal defining the Grassmannian. The elements \( \bar{p}_J \in A_{k,n} \) are Plücker coordinates. Applying our main theorem we identify a class of plabic graphs for \( \text{Gr}_k(C^n) \) with \( k \geq 3 \) and \( n \geq 6 \) for which the Newton–Okounkov body \( \Delta(I_{k,n}, \nu_G) \) is non-integer (see Theorem 3).

Related to [RW19], in [BFF⁺18] the authors associate plabic weight vectors \( \nu_G \) for \( \mathbb{C}[p_{J}] \) to the same plabic graphs \( G \) (see Definition 15). They study these weight vector for \( \text{Gr}_2(C^n) \) and \( \text{Gr}_3(C^6) \).

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2For two polytopes \( A, B \subset \mathbb{R}^d \) their Minkowski sum is defined as \( A + B := \{ a + b \mid a \in A, b \in B \} \subset \mathbb{R}^d \).
and show that, in these cases, they lie in the tropical Grassmannian [SS04], i.e. the tropicalization of $I_{k,n}$. With our methods, we show that for $Gr_2(\mathbb{C}^n)$ their weight yields the same toric degeneration as Rietsch–Williams’ Newton–Okounkov polytope (see Proposition 3 for details):

**Proposition.** For every plabic weight vector $w_G$ for $Gr_2(\mathbb{C}^n)$ we have $\text{in}_{M_{t_F}}(I_{2,n}) = \text{in}_{w_G}(I_{2,n})$. In particular, for the associated graded of $v_G$ we have $\text{gr}_{w_G}(A_{2,n}) = \mathbb{C}[p_{ij}]_{ij} / \text{in}_{w_G}(I_{2,n})$.

For more general Grassmannians Theorem 3 below shows that this is not always the case.

**Flag varieties and string valuations.** We consider string valuations [Kav15, FFL17] on the homogeneous coordinate ring of the full flag variety $\mathcal{F}_n$.

They were defined to realize string parametrizations [Lit98, BZ01] of Lusztig’s dual canonical basis in terms of Newton–Okounkov cones and polytopes.

Consider the algebraic group $SL_n$ and its Lie algebra $\mathfrak{s}l_n$ over $\mathbb{C}$. Fix a Cartan decomposition and take $\Lambda = \mathbb{Z}^{n-1}$ to be the weight lattice. It has a basis of fundamental weights $\omega_1, \ldots, \omega_{n-1}$ and every dominant integral weight, i.e. $\lambda \in \mathbb{Z}_{\geq 0}^{n-1}$ yields an irreducible highest weight representation $V(\lambda)$ of $\mathfrak{s}l_n$. The Weyl group of $\mathfrak{s}l_n$ is the symmetric group $S_n$. By $w_0 \in S_n$ we denote its longest element.

For every reduced expression $w_0$ of $w_0 \in S_n$ and every dominant integral weight $\lambda \in \mathbb{Z}_{\geq 0}^{n-1}$, there exists a string polytope $Q_{\omega_0}(\lambda) \subset \mathbb{R}^{\frac{n(n-1)}{2}}$. Its lattice points parametrize a basis for $V(\lambda)$. The string polytope for the weight $\rho = \omega_1 + \cdots + \omega_{n-1}$ is the Newton–Okounkov polytope for the string valuation $w_{\omega_0}$ on the homogeneous coordinate ring of $\mathcal{F}_n$.

We embed $\mathcal{F}_n$ into a product of projective spaces as follows: first, consider the embedding into the product of Grassmannians $Gr_1(\mathbb{C}^n) \times \cdots \times Gr_{n-1}(\mathbb{C}^n)$. Then concatenate with the Plücker embeddings $Gr_k(\mathbb{C}^n) \hookrightarrow \mathbb{P}^{k(n-k)}$ for every $1 \leq k \leq n-1$. This yields the (multi-)homogeneous coordinate ring $A_n$ of $\mathcal{F}_n$ as $\mathbb{C}[p_{ij}]_{J \subset [n]} / I_n$. Our main result applied to string valuations yields the following, for more details see Theorem 2.

**Theorem.** Let $w_0$ be a reduced expression of $w_0 \in S_n$ and consider the string valuation $v_{\omega_0}$. If $\text{in}_{M_{\omega_0}}(I_n)$ is prime, then

$$Q_{\omega_0}(\rho) = \text{conv}(v_{\omega_0}(\bar{p}_j))_{J \in ([n]_1)} + \cdots + \text{conv}(v_{\omega_0}(\bar{p}_j))_{J \in ([n]_1)}.$$

A central question concerning string polytopes is the following: fix a reduced decomposition $w_0$ and let $\lambda = a_1 \omega_1 + \cdots + a_{n-1} \omega_{n-1}$ with $a_i \in \mathbb{Z}_{\geq 0}$. Is the string polytope $Q_{\omega_0}(\lambda)$ equal to the Minkowski sum $a_1Q_{\omega_0}(\omega_1) + \cdots + a_{n-1}Q_{\omega_0}(\omega_{n-1})$ of fundamental string polytopes?

If equality holds for all $\lambda$, we say $w_0$ has the Minkowski property. In [BLMM17] the authors define weight vectors $w_{\omega_0}$ for every $w_0$. They conjecture a relation between the Minkowski property of $w_0$ and the weight vector $w_{\omega_0}$ lying in a maximal prime cone of the tropical flag variety, i.e. the tropicalization of $I_n$. A corollary of our main theorem proves an even stronger version of their conjecture. It can be summarized as follows (for details see Corollary 4):

**Corollary.** Let $w_0$ be a reduced expression of $w_0 \in S_n$ and consider the weight vector $w_{\omega_0}$. Then $\text{in}_{M_{\omega_0}}(I_n)$ is prime if and only if $w_{\omega_0}$ has the Minkowski property.

The paper is structured as follows. We recall preliminaries on valuations, toric degenerations and Newton–Okounkov bodies in §2. We then turn to quasi-valuations and weighting matrices in §3 and prove our main result Theorem 1. We make the connection to weight vectors and tropicalization and prove the above mentioned result Lemma 2. In §4 we apply our results to string valuations for flag varieties and in §5 to valuations from plabic graphs for Grassmannians. The Appendix contains background information for §5.

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2 Notation

We recall basic notions on valuations and Newton–Okounkov polytopes as presented in [KK12]. Let $A$ be the homogeneous coordinate ring of a projective variety or more generally the (multi-)homogeneous coordinate ring of a subvariety of a product of projective spaces, which can be described as below.

The total coordinate ring of $\mathbb{P}^{k_1-1} \times \cdots \times \mathbb{P}^{k_s-1}$ for $k_1, \ldots, k_s \geq 1$ is $S := \mathbb{C}[x_{ij} | i \in [s], j \in [k_i]]$. It is graded by $\mathbb{Z}_{\geq 0}$ as follows. Let $\{\epsilon_i\}_{i \in [s]}$ denote the standard basis of $\mathbb{Z}^s$. The degree of coordinates is given by $\deg x_{ij} := \epsilon_i \in \mathbb{Z}^s$ (see e.g. [CLS11, Example 5.2.2]) for all $i \in [s], j \in [k_i]$. For $u \in \mathbb{Z}^{k_1+\cdots+k_s}$, let $x^u$ denote the monomial $\prod_{i \in [s], j \in [k_i]} x_{ij}^{u_{ij}} \in S$. We fix the lexicographic order on $\mathbb{Z}^s$ and consider $f = \sum a_u x^u \in S$. Then

$$\deg f := \max_{\text{lex}} \{\deg x^u \mid a_u \neq 0\}.$$ 

Let $X$ be subvariety of $\mathbb{P}^{k_1-1} \times \cdots \times \mathbb{P}^{k_s-1}$ and $A$ its homogeneous coordinate ring of Krull-dimension $d$. Then $A = S/I$ for some prime ideal $I \subset S$ that is homogeneous with respect to the $\mathbb{Z}_{\geq 0}$-grading. The $\mathbb{Z}_{\geq 0}$-grading on $S$ induces a $\mathbb{Z}_{\geq 0}$-grading on $A$, which we denote $A = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} A_m$. We call a grading of this form a positive (multi-)grading.

To define a valuation on $A$ we fix a linear order $\prec$ on the additive abelian group $\mathbb{Z}^d$.

**Definition 1.** A map $\nu : A \setminus \{0\} \to (\mathbb{Z}^d, \prec)$ is a valuation, if it satisfies for $f, g \in A \setminus \{0\}, c \in \mathbb{C}^*$

(i) $\nu(f + g) \geq \min\{\nu(f), \nu(g)\}$,
(ii) $\nu(cf) = \nu(f)$, and
(iii) $\nu(fg) = \nu(f) + \nu(g)$, where $f \in A \setminus \{0\}$.

If we replace (ii) by $\nu(fg) \geq \nu(f) + \nu(g)$ then $\nu$ is called a quasi-valuation (also called loose valuation in [Tei03]). Let $\nu : A \setminus \{0\} \to (\mathbb{Z}^d, \prec)$ be a valuation. The image $\{\nu(f) \mid f \in A \setminus \{0\}\} \subset \mathbb{Z}^d$ forms an additive semi-group. We denote it by $S(A, \nu)$ and refer to it as the value semi-group. The rank of the valuation is the rank of the sublattice generated by $S(A, \nu)$ in $\mathbb{Z}^d$. We are interested in valuations of full rank, i.e. $\text{rank}(\nu) = d$.

One naturally defines a $\mathbb{Z}^d$-filtration on $A$ by $F_{\nu,a} := \{f \in A \setminus \{0\} \mid \nu(f) \geq a\} \cup \{0\}$ (and similarly $F_{\nu,a}$). The associated graded algebra of the filtration $\{F_{\nu,a}\}_{a \in \mathbb{Z}^d}$ is

$$\text{gr}_{\nu}(A) := \bigoplus_{a \in \mathbb{Z}^d} F_{\nu,a}/F_{\nu,a-}. \tag{2.1}$$

If the filtered components $F_{\nu,a}/F_{\nu,a-d}$ are at most one-dimensional for all $a \in \mathbb{Z}^d$, we say $\nu$ has one-dimensional leaves. If $\nu$ has full rank by [KM19, Theorem 2.3] it also has one-dimensional leaves. Moreover, in this case there exists an isomorphism $\mathbb{C}[S(A, \nu)] \cong \text{gr}_{\nu}(A)$ by [BG09, Remark 4.13]. To define a $\mathbb{Z}_{\geq 0}$-filtration on $A$ induced by $\nu$ we make use of the following standard trick that can be found in [Bay82, Proposition 1.8] or [Cal02, Lemma 3.2].

**Lemma 1.** Let $F$ be a finite subset of $\mathbb{Z}^d$. Then there exists a linear form $e : \mathbb{Z}^d \to \mathbb{Z}_{\geq 0}$ such that for all $m, n \in F$ we have $m \prec n \Rightarrow e(m) > e(n)$ (note the switch!).

Assume $S(A, \nu)$ is finitely generated, more precisely assume it is generated by $\{\nu(x_{ij})\}_{i \in [s], j \in [k_i]}$. In this case $\{x_{ij}\}_{i \in [s], j \in [k_i]}$ is called a Khovanskii basis for $(A, \nu)$. Now choose a linear form as in the

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4 We assume the image of $\nu$ lies in $\mathbb{Z}^d$ for simplicity. Without any more effort, we could replace it by $\mathbb{Q}^d$. The same is true for all (quasi)-valuations considered in the rest of the paper.

5 This term was introduced in [KM19] generalizing the notion of SAGB basis.
3. Quasi-valuations with weighting matrices

lemma for $F = \{v(\bar{x}_i)\}_{i \in [s], j \in [k]} \subset \mathbb{Z}^d$. We construct a $\mathbb{Z}_{\geq 0}$-filtration on $A$ by $F_{\leq m} := \{f \in A \setminus \{0\} \mid e(v(f)) \leq m\} \cup \{0\}$ for $m \in \mathbb{Z}_{\geq 0}$. Define similarly $F_{< m}$. The associated graded algebra satisfies

$$\text{gr}_v(A) \cong \bigoplus_{m \geq 0} F_{\leq m}/F_{< m}. \quad (2.2)$$

For $f \in A \setminus \{0\}$ denote by $\overline{f}$ its image in the quotient $F_{\leq e(v(f))}/F_{< e(v(f))}$, hence $\overline{f} \in \text{gr}_v(A)$. We obtain a family of $\mathbb{C}$-algebras containing $A$ and $\text{gr}_v(A)$ as fibers (see e.g. [And13, Proposition 5.1]) that can be defined as follows:

**Definition 2.** The Rees algebra associated with the valuation $v$ and the filtration $\{F_{\leq m}\}_m$ is the flat $\mathbb{C}[t]$-subalgebra of $A[t]$ defined as

$$R_{v, e} := \bigoplus_{m \geq 0} (F_{\leq m})t^m. \quad (2.3)$$

It has the properties that $R_{v, e}/tR_{v, e} \cong \text{gr}_v(A)$ and $R_{v, e}/(1-t)R_{v, e} \cong A$. In particular, if $A$ is $\mathbb{Z}_{\geq 0}$-graded, it defines a flat family over $\mathbb{A}^1$ (the coordinate on $\mathbb{A}^1$ given by $t$). The generic fiber is isomorphic to $\text{Proj}(A)$ and the special fiber is the toric variety $\text{Proj}(\text{gr}_v(A))$.

It is desirable to have a valuation that encodes the grading of $A$: a valuation $v$ on $A = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} A_m$ is called *homogeneous*, if for $f \in A$ with homogeneous decomposition $f = \sum_{m \in \mathbb{Z}_{\geq 0}} a_m f_m$ we have

$$v(f) = \min \{v(f_m) \mid a_m \neq 0\}.$$  

The valuation is called *fully homogeneous*, if $v(f) = (\deg f, \cdot) \in \mathbb{Z}^* \times \mathbb{Z}^{d-s}$ for all $f \in A$. Given a full rank homogeneous valuations by [IW18, Remark 2.6] without loss of generality one may assume it is fully homogeneous.

Introduced by Lazarsfeld-Mustăţă [LM09] and Kaveh-Khovanskii [KK12] we recall the definition of Newton–Okounkov body.

**Definition 3.** Let $v : A \setminus \{0\} \to (\mathbb{Z}^* \times \mathbb{Z}^{d-s}, \prec)$ be a fully homogeneous valuation. The *Newton–Okounkov body* of $(A, v)$ is now defined as

$$\Delta(A, v) := \text{cone}(S(A, v) \cup \{0\}) \subset \mathbb{R}^d. \quad (2.4)$$

The Newton–Okounkov body of $(A, v)$ is now defined as

$$\Delta(A, v) := \text{cone}(S(A, v) \cup \{0\}) \cap (\{1, \ldots, 1\}) \times \mathbb{R}^{d-s}. \quad (2.5)$$

For $\mathbb{Z}_{\geq 0}$-graded $A$, Anderson showed in [And13] that if $\text{gr}_v(A)$ is finitely generated, $\Delta(A, v)$ is a rational polytope. Moreover, it is the polytope associated to the normalization of the toric variety $\text{Proj}(\text{gr}_v(A))$.

Dealing with polytopes throughout the paper we need the notion of *Minkowski sum*. For two polytopes $A, B \subset \mathbb{R}^d$ it is defined as

$$A + B := \{a + b \mid a \in A, b \in B\} \subset \mathbb{R}^d. \quad (2.6)$$

Consider polytopes of form $P_v(\lambda) := C(A, v) \cap (\lambda \times \mathbb{R}^{d-s})$ for $\lambda \in \mathbb{R}_{\geq 0}$. It is a central question whether $P_v(\varepsilon_1) + \cdots + P_v(\varepsilon_s) = \Delta(A, v)$. Our main result (Theorem 1 below) treats this question.

3 Quasi-valuations with weighting matrices

We briefly recall some background on higher-dimensional Gröbner theory and quasi-valuations with weighting matrices as in [KM19, §3.1&8.1]. For classical results in Gröbner theory we rely on [Eis95,
§15]. Then we define for a given valuation an associated quasi-valuation with weighting matrix. The central result of this paper is Theorem 1. It is proved in full generality below, and applied to specific cases of valuations in §4 and §5.

As above let $A$ be a finitely generated algebra and domain of Krull-dimension $d$. We assume $A$ has a presentation of form $\pi: S \rightarrow A$, such that $A = S/\ker(\pi)$. Here $S$ denotes the total coordinates ring of $\mathbb{P}^{k_1-1} \times \cdots \times \mathbb{P}^{k_s-1}$, with $\mathbb{Z}^s_{>0}$-grading as defined above. Let $I := \ker(\pi)$ be homogeneous with respect to the $\mathbb{Z}^s_{>0}$-grading. To simplify notation, let $\pi(x_{ij}) := \bar{x}_{ij}$ for $i \in [s], j \in [k_i]$ and $n := k_1 + \cdots + k_s$. For simplicity we always consider (quasi-)valuations with image in $\mathbb{Z}^d$, but without much more effort all results extend to the case of $\mathbb{Q}^d$.

**Definition 4.** Let $f = \sum a_u x^u \in S$ with $u \in \mathbb{Z}^n$, where $x^u = x_1^{u_1} \cdots x_n^{u_n}$. For $M \in \mathbb{Z}^{d \times n}$ and a linear order $\prec$ on $\mathbb{Z}^d$ we define the *initial form* of $f$ with respect to $M$ as

$$\text{in}_M(f) := \sum_{Mm = \min_\prec \{Mu|a_u \neq 0\}} a_m x^m. \quad (3.1)$$

We extend this definition to ideals $I \subset S$ by defining the *initial ideal* of $I$ with respect to $M$ as $\text{in}_M(I) := \{\text{in}_M(f) \mid f \in I\} \subset S$.

Note that, by definition, if the ideal $I$ is homogeneous with respect to the $\mathbb{Z}^s_{>0}$-grading, then so is every initial ideal of $I$. Using the *fundamental theorem of tropical geometry* [MS15, Theorem 3.2.3] we recall the related notion of *tropicalization* of an ideal $I \subset S$:

$$\text{trop}(I) = \{w \in \mathbb{R}^n \mid \text{in}_w(I) \text{ is monomial-free}\}. \quad (3.2)$$

By the *Structure Theorem* [MS15, Theorem 3.3.5] we can choose a fan structure on trop$(I)$ in such a way that it becomes a subfan of the Gröbner fan of $I$: $w$ and $v$ lie in the relative interior of a cone $C \subset \text{trop}(I)$ (denoted $w, v \in C^0$) if $\text{in}_w(I) = \text{in}_v(I)$. For this reason we adopt the notation $\text{in}_C(I)$, which is defined as $\text{in}_w(I)$ for arbitrary $w \in C^0$. Further, we call a cone $C \subset \text{trop}(I)$ *prime*, if $\text{in}_C(I)$ is a prime ideal.

Coming back to the more general case of weighting matrices, we say that $M \in \mathbb{Z}^{d \times n}$ lies in the *Gröbner region* GR$(I)$ of an ideal $I \subset S$, if there exists a monomial order $\preceq$ on $\mathbb{S}$ such that

$$\text{in}_\preceq(\text{in}_M(I)) = \text{in}_\preceq(I).$$

Such a monomial order is called *compatible with $M$*. If the ideal $I \subset S$ is (multi-)homogeneous with respect to the $\mathbb{Z}^s_{>0}$-grading, then by [KM19, Lemma 8.7] we have $\mathbb{Q}^{d \times n} \subset \text{GR}(I)$. To a given matrix $M \in \mathbb{Z}^{d \times n}$ one associates a quasi-valuation as follows. As above, fix a linear order $\prec$ on $\mathbb{Z}^d$.

**Definition 5.** Let $\tilde{f} = \sum a_u x^u \in S$ and define $\tilde{v}_M : S \setminus \{0\} \rightarrow (\mathbb{Z}^d, \prec)$ by $\tilde{v}_M(\tilde{f}) := \min_\prec \{Mu \mid a_u \neq 0\}$. By [KM19, Lemma 3.2], there exists a quasi-valuation $v_M : A \setminus \{0\} \rightarrow (\mathbb{Z}^d, \prec)$ given for $f \in A$ by

$$v_M(f) := \max_\prec \{\tilde{v}_M(\tilde{f}) \mid \tilde{f} \in S, \pi(\tilde{f}) = f\}.$$

It is called the *quasi-valuation with weighting matrix* $M$.

We denote the associated graded algebra (defined analogously as in (2.1) for valuations) of the quasi-valuation $v_M$ by $\text{gr}_M(A)$. It has the property

$$\text{gr}_M(A) \cong S/\text{in}_M(I). \quad (3.3)$$

In particular, $\text{gr}_M(A)$ inherits the $\mathbb{Z}^s_{>0}$-grading of $S$, as $I$ is homogeneous with respect to this grading.

From the definition, it is usually hard to explicitly compute the values of a quasi-valuation $v_M$. The following proposition makes it more computable, given that $M$ lies in the Gröbner region of

---

*The initial form of an element $f = \sum a_u x^u \in S$ with respect to a monomial order $\prec$ is the monomial $a_m x^m$ of $f$ which satisfies $a_m \neq 0$ and $x^m$ is minimal with respect to $\prec$ among all monomials of $f$ with non-zero coefficient.*
I. Recall, that if \( C_\prec \subset \text{GR}^d(I) \) is a maximal cone with associated monomial ideal \( \text{in}_\prec(I) \), then \( B := \{ \bar{x}^\alpha \mid x^\alpha \notin \text{in}_\prec(I) \} \) is a vector space basis for \( A \), called \textit{standard monomial basis.} The monomials \( x^\alpha \notin \text{in}_\prec(I) \) are called \textit{standard monomials.} In general, a vector space basis \( B \subset A \) is called adapted to a valuation \( \mathfrak{v} : A \setminus \{0\} \to (\mathbb{Z}^d, \prec) \), if \( F_{\mathfrak{v} \circ \prec} \cap B \) is a vector space basis for \( F_{\mathfrak{v} \circ \prec} \) for every \( \alpha \in \mathbb{Z}^d \).

**Proposition.** ([KM19, Proposition 3.3]) Let \( M \in \text{GR}^d(I) \) and \( B \subset A \) be a standard monomial basis for the monomial order \( \prec \) on \( S \) compatible with \( M \). Then \( B \) is adapted to \( \mathfrak{v}_M \). Moreover, for every element \( f \in A \) written as \( f = \sum \bar{x}^\alpha a_\alpha \) with \( \bar{x}^\alpha \in B \) and \( a_\alpha \in \mathbb{C} \) we have

\[
\mathfrak{v}_M(f) = \min_{\prec} \{ M\alpha \mid a_\alpha \neq 0 \}. \tag{3.4}
\]

**Remark 1.** The proposition implies that \( \mathfrak{v}_M \) is homogeneous: for \( \alpha \in \mathbb{Z}_{\geq 0}^{k_1 + \cdots + k_s} \), define \( m_\alpha := (\sum_{j=1}^{k_1} \alpha_{ij}, \ldots, \sum_{j=1}^{k_s} \alpha_{sj}) \in \mathbb{Z}_{\geq 0}^{d} \). Then \( \bar{x}^\alpha \in A_{m_\alpha} \) for all \( \bar{x}^\alpha \in B \). For \( f \in A \) the unique expression \( f = \sum \bar{x}^\alpha a_\alpha \) is also its homogeneous decomposition and by the proposition

\[
\mathfrak{v}_M(f) = \min_{\prec} \{ \mathfrak{v}_M(\bar{x}^\alpha) \mid a_\alpha \neq 0 \}.
\]

From our point of view, quasi-valuations with weighting matrices are not the primary object of interest. In most cases we are given a valuation \( \mathfrak{v} : A \setminus \{0\} \to (\mathbb{Z}^d, \prec) \) whose properties we would like to know. In particular, we are interested in the generators of the value semi-group and if there are only finitely many. The next definition establishes a connection between a given valuation and weighting matrices. It allows us to apply techniques from Kaveh-Manon for quasi-valuations with weighting matrices to other valuations of our interest.

**Definition 6.** Given a valuation \( \mathfrak{v} : A \setminus \{0\} \to (\mathbb{Z}^d, \prec) \). We define the \textit{weighting matrix of} \( \mathfrak{v} \) associated with the presentation \( S/I \) of \( A \) by

\[
M_{\mathfrak{v}} := (\mathfrak{v}(\bar{x}_{ij}))_{i \in [s], j \in [k_i]} \in \mathbb{Z}^{d \times n}.
\]

That is, the columns of \( M_{\mathfrak{v}} \) are given by the images \( \mathfrak{v}(\bar{x}_{ij}) \) for \( i \in [s], j \in [k_i] \).

Note we slightly abuse notation and write \( M_{\mathfrak{v}} \) instead of \( M_{\mathfrak{v}}(S/I) \) as we fix the presentation of \( A \) from the start. The following corollary is obtained by an argument very similar to the proof of [KM19, Proposition 4.2]. We therefore leave its proof to the reader.

**Corollary 1.** Let \( M \in \mathbb{Z}^{d \times n} \) be of full rank with \( d \) the Krull-dimension of \( A \). If \( \text{in}_M(I) \) is prime, then \( \mathfrak{v}_M \) is a valuation whose value semi-group \( S(A, \mathfrak{v}_M) \) is generated by \( \{ \mathfrak{v}_M(\bar{x}_{ij}) \}_{i \in [s], j \in [k_i]} \). In particular, the associated Newton–Okounkov body is given by

\[
\Delta(A, \mathfrak{v}_M) = \sum_{i=1}^s \text{conv}(\mathfrak{v}_M(\bar{x}_{ij})) \mid j \in [k_i]).
\]

To a full rank valuation \( \mathfrak{v} : A \setminus \{0\} \to (\mathbb{Z}^d, \prec) \) we associate a homomorphism of polynomial rings, called \textit{monomial map of} \( \mathfrak{v} \) (see e.g. [MS18]):

\[
\phi_\mathfrak{v} : S \to \mathbb{C}[y_1, \ldots, y_d] \quad \text{by} \quad \phi_\mathfrak{v}(x_{ij}) := y^{\mathfrak{v}(\bar{x}_{ij})}.
\]

The image \( \text{im}(\phi_\mathfrak{v}) \) is naturally isomorphic to the semi-group algebra \( \mathbb{C}[S(\mathfrak{v}(\bar{x}_{ij})_{ij})] \), where \( S(\mathfrak{v}(\bar{x}_{ij})_{ij}) \) is the semi-group generated by \( \{ \mathfrak{v}(\bar{x}_{ij}) \mid i \in [s], j \in [k_i] \} \subset \mathbb{Z}^d \). We have

\[
S/\ker(\phi_\mathfrak{v}) \cong \mathbb{C}[S(\mathfrak{v}(\bar{x}_{ij})_{ij})] \subset \mathbb{C}[S(A, \mathfrak{v})] \cong \text{gr}_\mathfrak{v}(A). \tag{3.5}
\]

Therefore, \( \ker(\phi_\mathfrak{v}) \) is a binomial prime ideal. The height of \( \ker(\phi_\mathfrak{v}) \) is \( n - \text{rank}(S(\mathfrak{v}(\bar{x}_{ij})_{ij})) = n - \text{rank}(M_{\mathfrak{v}}) \).

**Lemma 2.** For every full rank valuation \( \mathfrak{v} : A \setminus \{0\} \to (\mathbb{Z}^d, \prec) \) we have \( \text{in}_{M_{\mathfrak{v}}}(I) \subset \ker(\phi_\mathfrak{v}) \).
3. Quasi-valuations with weighting matrices

Proof. Let \( f \in I \) and consider \( \text{in}_{M_\nu}(f) = \sum_{i=1}^s c_i x^{u_i} \) with \( c_i \in \mathbb{C} \) and \( u_i \in \mathbb{Z}_{\geq 0}^n \) for all \( i \). So \( M_{u_i} = M_{u_j} : = a \), i.e. \( v(x^{u_i}) = v(x^{u_j}) \) and \( x^{u_i}, x^{u_j} \in F_{F_{\geq 0}} \) for all \( 1 \leq i, j \leq s \). As \( v \) has one-dimensional leaves, we have \( \sum_{i=1}^s c_i x^{u_i} \in F_{F_{\geq 0}} \), which implies it is zero in \( \text{gr}_v(A) \). Hence, by (3.5) we have \( \text{in}_{M_\nu}(f) \in \ker(\phi_\nu) \). \( \square \)

Before stating the main theorem relating a given valuation \( v \) with the (quasi-)valuation with weighting matrix \( M_\nu \) we prove the following proposition that is used in the proof.

**Proposition 1.** Assume \( I \) is homogeneous with respect to the \( \mathbb{Z}_{\geq 0}^n \)-grading and Krull dimension of \( A = S/I \) is \( d \). Let \( v : A \setminus \{0\} \to (\mathbb{Z}^d, \prec) \) be a full rank valuation with \( M_\nu \in \mathbb{Z}^{d \times n} \) the weighting matrix of \( v \). Then

\[
\nu = \nu_{M_\nu} \iff \text{in}_{M_\nu}(I) \text{ is prime and rank}(M_\nu) = d.
\]

**Proof.** “\( \Rightarrow \)” Assume \( v = \nu_{M_\nu} \), then \( \mathbb{C}[S(A, v)] \cong S/\text{in}_{M_\nu}(I) \) by (3.3). In particular, \( \text{in}_{M_\nu}(I) \) is binomial and prime. Further, \( \text{rank}(M_\nu) = \text{rank}(S(A, v)) = \text{Krull dimension of } A \).

“\( \Leftarrow \)” Assume \( \text{in}_{M_\nu}(I) \) is prime and \( M_\nu \) has rank equal to the Krull dimension of \( A \). By [KM19, Theorem 2.17] we obtain \( v = \nu_{M_\nu} \) if and only if \( \text{in}_{M_\nu}(I) = \ker(\phi_\nu) \). By Lemma 2 we have \( \text{in}_{M_\nu}(I) \subseteq \ker(\phi_v) \). As \( \text{rank}(M_\nu) = \text{rank}(S(A, v)) \) both ideals have the same height and are therefore equal. \( \square \)

**Theorem 1.** Assume \( I \) is homogeneous with respect to the \( \mathbb{Z}_{\geq 0}^n \)-grading and Krull dimension of \( A = S/I \) is \( d \). Let \( v : A \setminus \{0\} \to (\mathbb{Z}^d, \prec) \) be a full rank valuation with associated weighting matrix \( M_\nu \in \mathbb{Z}^{d \times n} \). Then,

\[
S(A, v) \text{ is generated by } \{v(\bar{x}_{ij})\}_{i \in [s], j \in [k_i]} \iff \text{in}_{M_\nu}(I) \text{ is prime and rank}(M_\nu) = d.
\]

Both directions of the proof rely on the equality \( v = \nu_{M_\nu} \). The equality is false in general, a counterexample is given in §4 Example 3. For “\( \Leftarrow \)” the equality follows from Proposition 1 while for “\( \Rightarrow \)” it follows from a direct computation.

**Proof.** As \( I \) is homogeneous with respect to a positive grading, \( M_\nu \) lies in the Gröbner region of \( I \). Let \( B \subset A \) be the standard monomial basis adapted to \( v_{M_\nu} \).

“\( \Leftarrow \)” Assume \( \text{in}_{M_\nu}(I) \) is prime and \( M_\nu \) has full rank. Then we obtain by Proposition 1 \( S(A, v) = S(A, v_{M_\nu}) \cong S/\text{in}_{M_\nu}(I) \). Hence, by Corollary 1 \( S(A, v) \) is generated by \( \{v(\bar{x}_{ij})\}_{ij} \).

“\( \Rightarrow \)” Assume \( S(A, v) \) is generated by \( \{v(\bar{x}_{ij})\}_{i \in [s], j \in [k_i]} \). We have \( v_{M_\nu}(\bar{x}^\alpha) = M_\nu \alpha = v(\bar{x}^\alpha) \) for \( \bar{x}^\alpha \in B \) by (3.4). We need to prove \( v = v_{M_\nu} \) then \( \text{in}_{M_\nu}(I) \) is prime and \( \text{rank}(M_\nu) = \text{Krull dimension of } A \) by Proposition 1. This follows from:

**Claim:** \( v_{M_\nu}(\bar{x}^\alpha) \neq v_{M_\nu}(\bar{x}^\beta) \) for \( \bar{x}^\alpha, \bar{x}^\beta \in B \) with \( \alpha \neq \beta \).

**Proof of claim:** Assume there exist \( \bar{x}^\alpha, \bar{x}^\beta \in B \) with \( \alpha \neq \beta \) and \( v_{M_\nu}(\bar{x}^\alpha) = v_{M_\nu}(\bar{x}^\beta) \). Then

\[
v(\bar{x}^\alpha) = M_\nu \alpha = v_{M_\nu}(\bar{x}^\alpha) = v_{M_\nu}(\bar{x}^\beta) = M_\nu \beta = v(\bar{x}^\beta).
\]

This implies that \( v(\bar{x}^\alpha + \bar{x}^\beta) > v(\bar{x}^\alpha) = v(\bar{x}^\beta) \). In particular, \( v(\bar{x}^\alpha + \bar{x}^\beta) \) does not lie in the semi-group span \( \{v(\bar{x}_{ij}) \mid i \in [s], j \in [k_i]\} = S(A, v) \), a contradiction by assumption. \( \square \)

Knowing the generators of \( S(A, v) \) explicitly has a number of crucial consequences for applications. Given the theorem, they now follow directly from \( \text{in}_{M_\nu}(I) \) being prime:

**Corollary 2.** Assume \( I \) is homogeneous with respect to the \( \mathbb{Z}_{\geq 0}^n \)-grading and generated by elements \( f \in I \) with \( \deg f > \epsilon_i \) for all \( i \in [s] \). Let \( v : A \setminus \{0\} \to (\mathbb{Z}^d, \prec) \) be a full rank valuation with \( M_\nu \in \mathbb{Z}^{d \times n} \) of full rank. Assume additionally that \( \text{in}_{M_\nu}(I) \) is prime. Then \( v \) is homogeneous and

(i) \( \text{gr}_v(A) \cong S/\text{in}_{M_\nu}(I) \),

(ii) \( \{\bar{x}_{ij}\}_{i \in [s], j \in [k_i]} \) is a Khovanskii basis for \( (A, v) \), and

(iii) \( \Delta(A, v) = \sum_{i=1}^s \text{conv}(v(\bar{x}_{ij}) \mid j \in [k_i]) \).
3.1 From weighting matrix to weight vector and tropicalization

In this section we summarize how to pass from a weighting matrix to a weight vector which is desirable for applications. We assume the ideal $\mathcal{I} \subset S$ is homogeneous with respect to the $\mathbb{Z}^d$-grading on $S$ and that $\mathcal{A} = S/I$ has Krull dimension $d$. We consider weighting matrices that lie in the Gröbner region of $\mathcal{I}$ and for simplicity we assume they have integer entries.

**Definition 7.** Let $M \in \mathbb{Z}^{d \times n}$ and denote by $M_1, \ldots, M_n$ its columns. A linear map $e : \mathbb{Z}^d \to \mathbb{Z}$ is called an order preserving projection with respect to $M$ and $\mathcal{I}$, if for $eM := (e(M_1), \ldots, e(M_n))$ we have $\text{in}_M(I) = \text{in}_{eM}(I)$.

The following lemma a reformulation of classical results in Gröbner theory (see e.g. [Bay82, Cal02, KM19]). We include it for completeness in view of applications.

**Lemma 3.** For every $M \in \mathbb{Z}^{d \times n}$ the order preserving projections with respect to $M$ and $\mathcal{I}$ are organized in a polyhedral cone $\sigma_M^\circ(I) \subset \mathbb{R}^d$. Moreover, there exist $w \in \mathbb{R}^n$ with $\text{in}_w(I) = \text{in}_M(I)$.

**Proof.** Let $\{R_1, \ldots, R_s\}$ be a reduced Gröbner basis for $\text{in}_M(I)$. Assume the initial form of $R_i$ is of weight $Mm_i$ for $l \in [s]$. So we have $R_i = \text{in}_M(R_i) + \sum_{j=1}^{q_i} u_j R_j^{l_j}$ with $Mm_i \prec Mu_j^{l_j}$ for all $j \in [q_i]$. We define the (open) polyhedral cone of order preserving projections

$$
\sigma^\circ_M := \{ e \in \mathbb{R}^d \mid (e, M(m_l - u_j^{l_j})) < 0 \text{ for all } l \in [s], j \in [q_i] \}.
$$

Applying the linear map defined by $M$ we obtain a set $E^\circ_M := \{ eM \mid e \in \sigma^\circ_M \}$ and

$$
E_M \subset \{ w \in \mathbb{R}^n \mid \langle w, m_l - u_j^{l_j} \rangle < 0 \text{ for all } l \in [s], j \in [q_i] \} := C^\circ_M.
$$

By Lemma 1, $\sigma^\circ_M$ is nonempty, hence $C^\circ_M$ is nonempty and by definition an open cone in the Gröbner fan of $\mathcal{I}$.

**Corollary 3.** Consider $\nu : A \setminus \{0\} \to (\mathbb{Z}^d, \prec)$ a full rank valuation with associated weighting matrix $M_\nu \in \mathbb{Z}^{d \times n}$. Then $\text{in}_{M_\nu}(I)$ is monomial-free and $C_{M_\nu}$ is a cone in trop($\mathcal{I}$).

**Proof.** By Lemma 2 we have $\text{in}_{M_\nu}(I)$ is contained in a binomial prime ideal.

4 Application: (tropical) flag varieties and string cones

In this subsection we focus on a particular valuation on the Cox ring of the full flag variety. The valuation was defined in the context of representation theory by [FFL17] (see also [Kav15]). It is of particular interest, as it realizes Littelmann’s string polytopes [Lit98, BZ01] as Newton–Okounkov polytopes. Applying Theorem 1 to this valuation, we solve a conjecture by [BLMM17] relating the Minkowski property of string cones (see Definition 8) to the tropical flag variety.

Consider $\mathcal{F}_n$, the full flag variety of flags of vector subspaces of $\mathbb{C}^n$. Its dimension as a projective variety is $N := \binom{n(n-1)}{2}$. We embed it into the product of Grassmannians $\text{Gr}_1(\mathbb{C}^n) \times \cdots \times \text{Gr}_{n-1}(\mathbb{C}^n)$. Further embedding each Grassmannian via its Plücker embedding into projective space we obtain $\mathcal{F}_n \hookrightarrow \mathbb{P}^{(1)} \times \cdots \times \mathbb{P}^{(n-1)}$. Set $K := (n) + \cdots + (n-1)$. In this way, we get the (multi-)homogeneous coordinate ring $A_n$ as a quotient of the total coordinate ring of the product of projective spaces $S := \mathbb{C}[p_J \mid 0 \neq J \subseteq [n]]$. Namely $A_n \cong S/I_n$. In this case $S$ is multigraded by $\mathbb{Z}_{\geq 0}^n \succ : \deg(p_J) := \epsilon_k$ for $|J| = k$ and $\{e_k \}_{k=1}^{n-1}$ standard basis of $\mathbb{Z}^{n-1}$. The variables $p_J$ are called Plücker variables and their cosets $\bar{p}_J \in A_n$ are Plücker coordinates. The ideal $I_n \subset S$ is generated by the quadratic Plücker relations, see [MS05, Theorem 14.6]. It is homogeneous with respect to the $\mathbb{Z}_{\geq 0}^{n-1}$-grading induced from $S$. The tropical flag variety, denoted trop($\mathcal{F}_n$) (as defined in [BLMM17]) is the tropicalization of the ideal $I_n \subset S$. 
Realizing $\mathcal{F}_n = SL_n/B$, where $B \subset SL_n$ are upper triangular matrices, one can make use of the representation theory of $SL_n$ to define valuations on $A_n$. We summarize the necessary representation-theoretic background below.

Let $S_n$ denote the symmetric group and $w_0 \in S_n$ its longest element. By $w_0$, we denote a reduced expression $s_{i_1} \ldots s_{i_N}$ of $w_0$ in terms of simple transpositions $s_i := (i, i+1)$. Let $\mathfrak{s}l_n$ be the Lie algebra of $SL_n$ and fix a Cartan decomposition $\mathfrak{s}l_n = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ into lower, diagonal, and upper triangular traceless matrices. Let $R = \{\varepsilon_i - \varepsilon_j\}_{i < j}$ be the root system of type $A_{n-1}$, where $\{\varepsilon_i\}_{i=1}^n$ is the standard basis of $\mathbb{R}^n$. Every $\varepsilon_i - \varepsilon_{i+1}$ defines an element $f_i \in \mathfrak{n}^-$. 

Denote the weight lattice by $\Lambda \cong \mathbb{Z}^{n-1}$. It is spanned by the fundamental weights $\omega_1, \ldots, \omega_{n-1}$ and we set $\Lambda^+ := \mathbb{Z}_{\geq 0}^{n-1}$ with respect to the basis of fundamental weights. For every $\lambda \in \Lambda^+$ we have an irreducible highest weight representation $V(\lambda)$ of $\mathfrak{sl}_n$. It is cyclically generated by a highest weight vector $v_\lambda$ (unique up to scaling) over the universal enveloping algebra $U(\mathfrak{n}^-)$. In particular, for every reduced expression $w_0 = s_{i_1} \ldots s_{i_N}$ the set $S_{w_0} := \{f_{i_1}^{m_1} \cdots f_{i_N}^{m_N}(v_\lambda) \in V(\lambda) \mid m_i \geq 0\}$ is a spanning set for the vector space $V(\lambda)$.

**Example 1.** Given a fundamental weight $\omega_k$, we have $V(\omega_k) = \Lambda^k \mathbb{C}^n$. We can choose the highest weight vector as $v_{\omega_k} := e_1 \wedge \cdots \wedge e_k$, where $\{e_i\}_{i=1}^n$ is a basis of $\mathbb{C}^n$. In this context Plücker coordinates $\rho_{(j_1, \ldots, j_k)} \in A_n$ are identified with dual basis elements $(\varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_k})^* \in V(\omega_k)^*$. 

Littelmann [Lit98] introduced in the context of quantum groups and crystal bases the so called (weighted) string cones and string polytopes $Q_{w_0}(\lambda)$. The motivation is to find monomial bases for the irreducible representations $V(\lambda)$. Littelmann identifies a linearly independent subset of the spanning set $S_{w_0}$ by introducing the notion of adapted string (see [Lit98, p. 4]) referring to a tuple $(a_1, \ldots, a_N) \in \mathbb{Z}_{\geq 0}^n$. His basis for $V(\lambda)$ consists of those elements $f_{i_1}^{m_1} \cdots f_{i_N}^{m_N}(v_\lambda)$ for which $(a_1, \ldots, a_N)$ is adapted.

For a fixed reduced expression $w_0$ of $w_0 \in S_n$ and $\lambda \in \Lambda^+$ he gives a recursive definition of the string polytope $Q_{w_0}(\lambda) \subset \mathbb{R}^N$ ([Lit98, p. 5], see also [BZ01]). The lattice points $Q_{w_0}(\lambda) \cap \mathbb{Z}^N$ are the adapted strings for $w_0$ and $\lambda$. The string cone $Q_{w_0} \subset \mathbb{R}^N$ is the convex hull of all $Q_{w_0}(\lambda)$ for $\lambda \in \Lambda^+$. The weighted string cone is defined as 

$$Q_{w_0} := \text{conv}\left( \bigcup_{\lambda \in \Lambda^+} \{\lambda\} \times Q_{w_0}(\lambda) \right) \subset \mathbb{R}^{n-1+N}.$$ 

By definition, one obtains the string polytope from the weighted string cone by intersecting it with the hyperplanes $\{\lambda\} \times \mathbb{R}^N$. In this context, a central question is the following:

**Definition 8.** A reduced expression $w_0$ of $w_0 \in S_n$ has the (strong) Minkowski property, if $Q_{w_0}(\lambda) = a_1Q_{w_0}(\omega_1) + \cdots + a_{n-1}Q_{w_0}(\omega_{n-1})$ holds for all $\lambda = 1 + \sum_{k=1}^{n-1} a_k \omega_k \in \Lambda^+$. 

If $Q_{w_0}(\rho) = Q_{w_0}(\omega_1) + \cdots + Q_{w_0}(\omega_{n-1})$ for $\rho := \omega_1 + \cdots + \omega_{n-1}$, we say $w_0$ satisfies MP.

String polytopes can be realized as Newton–Okounkov polytopes, as was done by [Kav15, FFL17]. The corresponding valuations are defined on the coordinate ring of $G/U := \text{Spec}(\mathbb{C}[G/U])$, where $U \subset B$ are upper triangular matrices with 1’s on the diagonal. By [VP89] $\mathbb{C}[G/U]$ is isomorphic to the Cox ring $\text{Cox}(\mathcal{F}_n)$ of the flag variety. As a basis for the Picard group we fix the homogeneous line bundle associated to the fundamental weights $L_{\omega_1}, \ldots, L_{\omega_{n-1}}$ (see e.g. [Bri04, p.15]). This way we obtain

$$\text{Cox}(\mathcal{F}_n) = \bigoplus_{\lambda \in \Lambda^+} H^0(\mathcal{F}_n, L_{\lambda}) \cong \bigoplus_{\lambda \in \Lambda^+} V(\lambda)^*,$$

where the isomorphism is due to the Borel-Weil theorem. For every $w_0$ there exists a fully homogeneous valuation $\nu_{w_0} : \text{Cox}(\mathcal{F}_n) \setminus \{0\} \to \Lambda^+ \times \mathbb{Z}^N$ called string valuation. By [FFL17, §11] it has the properties

$$C(\text{Cox}(\mathcal{F}_n), \nu_{w_0}) = \mathcal{Q}_{w_0} \quad \text{and} \quad P_{\nu_{w_0}}(\lambda) = Q_{w_0}(\lambda) \quad \text{for all } \lambda \in \Lambda^+. \quad (4.1)$$

---

7The element $f_i \in \mathfrak{n}^-$ is the elementary matrix with only non-zero entry 1 in the $(i, i+1)$-position.
As described above, we fixed $\mathcal{F}_{n} \hookrightarrow \mathbb{P}(1)^{-1} \times \cdots \times \mathbb{P}(n^{-1})^{-1}$ and want to consider the string valuations on the homogeneous coordinate ring $A_{n}$. The following proposition is therefore very useful.

As a consequence of the standard monomial theory for flag varieties it follows from [LB09, Theorem 12.8.3]. We give the proof for completion. To simplify notation let $Z := \mathbb{P}(1)^{-1} \times \cdots \times \mathbb{P}(n^{-1})^{-1}$. Recall that Schubert varieties are of form $X(w) := BwB/B \subset SL_{n}/B$ for $w \in S_{n}$ and that for $w = w_{0} s_{k}$ they are divisors in $\mathcal{F}_{n}$.

Proposition 2. We have $A_{n} \cong \text{Cox}(\mathcal{F}_{n})$.

Proof. Let $L_{k}$ be the ample generator of $\text{Pic}(\mathbb{P}(1)^{-1})$. By [LB09, Theorem 11.2.1] it pulls back to the line bundle $L_{\omega_{k}}$ of the Schubert divisor $X(w_{0} s_{k}) \subset \mathcal{F}_{n} \to Z$ and $\text{Pic}(\mathcal{F}_{n})$ is generated by $\{L_{\omega_{k}}\}_{k=1}^{n}$. For every $\lambda = \sum_{k=1}^{n-1} a_{k} \omega_{k} \in \mathbb{Z}_{\geq 0}^{n-1}$ by [LB09, Theorem 12.8.3] we have a surjection

$$H^{0}(Z, L_{\omega_{1}}^{a_{1}} \otimes \cdots \otimes L_{\omega_{n-1}}^{a_{n-1}}) \to H^{0}(\mathcal{F}_{n}, L_{\lambda})$$

Note that $S = \bigoplus_{a \in \mathbb{Z}_{\geq 0}^{n-1}} H^{0}(Z, L_{\omega_{1}}^{a_{1}} \otimes \cdots \otimes L_{\omega_{n-1}}^{a_{n-1}})$, hence we have a surjection $\pi : S \to \text{Cox}(\mathcal{F}_{n})$. Its kernel is the multi-homogeneous ideal generated by elements $f \in H^{0}(Z, L_{\omega_{1}}^{a_{1}} \otimes \cdots \otimes L_{\omega_{n-1}}^{a_{n-1}})$ vanishing on $\mathcal{F}_{n}$ for varying $(a_{1}, \ldots, a_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$. In particular, $\ker(\pi) = I_{n}$ and $\text{Cox}(\mathcal{F}_{n}) \cong S/I_{n} = A_{n}$.\]

From now on we consider for a fixed reduced expression $w_{0} = s_{i_{1}} \cdots s_{i_{n}}$ the valuation $v_{\omega_{0}}$ on $A_{n}$. We explain how to compute $v_{\omega_{0}}$ on Plücker coordinates, following [FFL17]. We use the total order $<$ on $\mathbb{Z}^{N}$ defined by $m < n$ if and only if, $\sum_{i=1}^{N} m_{i} < \sum_{i=1}^{N} n_{i}$ or $\sum_{i=1}^{N} m_{i} = \sum_{i=1}^{N} n_{i}$ and $m > \text{lex} n$. Now $v_{\omega_{0}}$ can be computed explicitly\footnote{The action of $n^{-}$ on $\bigwedge^{k} \mathbb{C}^{n}$ necessary to make explicit computations is given by $f_{1,\ldots,j}(e_{i_{1}} \wedge \cdots \wedge e_{i_{k}}) = \sum_{s} f_{s} = f_{1,\ldots,j}(e_{i_{1}} \wedge \cdots \wedge e_{i_{k}}) \wedge e_{i_{s}}$, where $f_{1,\ldots,j} e_{i_{s}} = \delta_{i_{s},i_{1}} e_{i_{s+1}}$ and $f_{s} = f_{1,\ldots,j}(e_{i_{1}} \wedge \cdots \wedge e_{i_{k}})$} on Plücker coordinates by [FFL17, Proposition 2]:

$$v_{\omega_{0}}(\bar{p}_{j_{1},\ldots,j_{k}}) = \min \{ m \in \mathbb{Z}^{N} \mid f^{m}(e_{1} \wedge \cdots \wedge e_{k}) = e_{j_{1}} \wedge \cdots \wedge e_{j_{k}} \},$$

for $\{j_{1}, \ldots, j_{k} \} \subset \{1, \ldots, N\}$, where $f^{m} = f_{1}^{m_{1}} \cdots f_{N}^{m_{N}} \in U(n^{-})$.

We want to apply the results from §3 to the given valuations of form $v_{\omega_{0}} : A_{n} \setminus \{0\} \to (\mathbb{Z}^{N}, <)$. Let therefore $M_{\omega_{0}} := (v_{\omega_{0}}(\bar{p}_{j}))_{0 \neq J \subseteq \{1, \ldots, N\} \in \mathbb{Z}^{N \times K}}$ be the matrix whose columns are given by the images of Plücker coordinates under $v_{\omega_{0}}$. Consider as in [BLMM17] the linear form $e : \mathbb{Z}^{N} \to \mathbb{Z}$ given by

$$-e(m) := 2^{-1} m_{1} + 2^{-2} m_{2} + \ldots + 2^{-N-1} m_{N-1} + m_{N}.$$  

Definition 9. For a fixed reduced expression $w_{0}$ the weight of the Plücker variable $p_{j}$ is $e(v_{\omega_{0}}(\bar{p}_{j}))$. We define the weight vector $w_{\omega_{0}} := (e(v_{\omega_{0}}(\bar{p}_{j})))_{0 \neq J \subseteq \{1, \ldots, N\}} \in \mathbb{R}^{K}$.

The weight vectors $w_{\omega_{0}}$ are computed in [BLMM17] for $n = 4$ and $n = 5$. The authors verify that they lie in the tropical flag variety and conjecture this is true in general.

Lemma 4. For every reduced expression $w_{0}$ we have $\text{in}_{M_{\omega_{0}}}(I_{n}) = \text{in}_{w_{\omega_{0}}}(I_{n})$. In particular, $w_{\omega_{0}} \in \text{trop}(\mathcal{F}_{n})$.

Proof. The linear map $e : \mathbb{Z}^{N} \to \mathbb{Z}$ is constructed using the recursive recipe given in [Cal02, Proof of Lemma 3.2]. Caldero uses it to define the filtration $\{F_{m}\}_{m \in \mathbb{Z}}$ on $A_{n}$. In Proposition 3.1 and Corollary 3.2 of [Cal02] he shows that this $\mathbb{Z}$-filtration and the $\mathbb{Z}^{N+(n-1)}$-filtration $\{F_{\omega_{0}}\}$ have the same associated graded algebra, namely $\mathbb{C}[S(A_{n}, v_{\omega_{0}})]$. To be precise, given a relation in $A_{n}$ coming from a lifting of a minimal relation in $S(A_{n}, v_{\omega_{0}})$ Caldero shows that in the associated graded algebras with respect to $\{F_{m}\}_{m \in \mathbb{Z}}$ and $\{F_{\omega_{0}}\}$ the same initial relation holds. The generators of $I_{n}$ are naturally liftings of minimal relations in $S(A_{n}, v_{\omega_{0}})$ as they are of minimal total degree $2$. So in particular, $\text{in}_{M_{\omega_{0}}}(I_{n}) = \text{in}_{w_{\omega_{0}}}(I_{n})$. The rest follows from Corollary 4.
Table 1: Isomorphism classes of string polytopes (up to unimodular equivalence) for \( n = 4 \) and \( \rho \) depending on \( \tilde{w}_0 \), the property MP, the weight vectors \( w_{\tilde{w}_0} \) as in Definition 9, and primeness of the initial ideals \( \text{in}_{w_{\tilde{w}_0}}(I_4) \). For details see [BLMM17, §5]

| String 1: \( s_1s_2s_1s_3s_2s_1 \) | MP | weight vector \( -w_{\tilde{w}_0} \) | \( \text{in}_{w_{\tilde{w}_0}}(I_4) \) prime |
|-----------------|-----|---------------------|------------------|
| yes | (0, 32, 24, 7, 0, 16, 6, 48, 38, 30, 0, 4, 20, 52) | yes |
| yes | (0, 16, 48, 7, 0, 32, 6, 24, 22, 54, 0, 4, 36, 28) | yes |
| yes | (0, 4, 36, 28, 0, 32, 24, 6, 22, 54, 0, 16, 48, 7) | yes |
| yes | (0, 4, 20, 52, 0, 16, 48, 6, 38, 30, 0, 32, 24, 7) | yes |

| String 2: \( s_1s_2s_1s_2s_1s_2 \) | yes | (0, 32, 18, 14, 0, 16, 12, 48, 44, 27, 0, 8, 24, 56) | yes |
| String 3: \( s_2s_1s_2s_1s_2s_1s_2 \) | yes | (0, 8, 24, 56, 0, 16, 48, 12, 44, 27, 0, 32, 18, 14) | yes |
| String 4: \( s_1s_2s_1s_2s_1s_2 \) no | (0, 16, 12, 44, 0, 8, 40, 24, 24, 56, 15, 0, 32, 10, 26) | no |

Example 2. Consider the reduced expression \( \tilde{w}_0 = s_1s_2s_1s_3s_2s_1 \) for \( w_0 = s_1 \). We compute \( v_{\tilde{w}_0}(\tilde{p}_{i1}) = (0, 1, 0, 0, 0, 0) \) and the weight of \( \tilde{p}_{i1} = c(0, 1, 0, 0, 0, 0) = -1(1 \cdot 2^1) = -16 \). Similarly, we obtain weights for all Plücker variables and (ordered lexicographically by their indexing sets)

\[-w_{\tilde{w}_0} = (0, 32, 24, 7, 0, 16, 6, 48, 38, 30, 0, 4, 20, 52).\]

Table 1 contains all weight vectors (up to sign) for \( F \) constructed this way.

Lemma 5. For every reduced decomposition \( w_n \) of \( w_0 \in S_n \) we have \( \text{rank}(M_{w_0}) = N + n - 1 \), i.e \( M_{\tilde{w}_0} \) is of full rank.

Proof. We first restrict our attention to a submatrix of \( M_{\tilde{w}_0} \) having only those columns corresponding to \( v_{\tilde{w}_0}(\tilde{p}_I) \) for \( I \in \binom{[n]}{k} \), denote it by \( M_{\tilde{w}_0}|_{I=1} \), for any \( k \in [n-1] \). This matrix corresponds to a full-rank valuation on the homogeneous coordinate ring of the Grassmannian under the Plücker embedding. So, \( \text{rank}(M_{\tilde{w}_0}|_{I=1}) = k(n - k) + 1 \). In particular, we deduce for \( n \geq 4 \)

\[ \text{rank}(M_{\tilde{w}_0}) = \min \left\{ \sum_{k=1}^{n-1} (k(n - k) + 1), N + n - 1 \right\} = N + n - 1. \]

Direct computations reveals the statement is also true for \( n \leq 3 \). \( \square \)

Based on computational evidence for \( n \leq 5 \) the authors of [BLMM17] state the following conjecture relating the Minkowski property with the tropical flag variety.

Conjecture 1. For \( n \geq 3 \) and \( \tilde{w}_0 \) a reduced expression of \( w_0 \in S_n \) we have \( w_{\tilde{w}_0} \in \text{trop}(F_n) \). Moreover, if \( w_{\tilde{w}_0} \) lies inside the relative interior of a maximal prime cone of \( \text{trop}(F_n) \), then \( \tilde{w}_0 \) satisfies MP.

We already proved the first part of the conjecture in Lemma 4. The second part is a consequence of Theorem 1 as we explain below.

Theorem 2. Let \( \tilde{w}_0 \) be a reduced expression of \( w_0 \in S_n \) and consider \( w_{\tilde{w}_0} \in \mathbb{R}^K \). If \( \text{in}_{w_{\tilde{w}_0}}(I_n) \) is prime, then \( S(A_n, w_{\tilde{w}_0}) \) is generated by \( \{v_{\tilde{w}_0}(\tilde{p}_J) \mid J \neq 0 \} \), i.e. the Plücker coordinates form a Khovanskii basis for \( (A_n, v_{\tilde{w}_0}) \). Moreover, we have

\[ Q_{\tilde{w}_0}(\rho) = \text{conv}(v_{\tilde{w}_0}(\tilde{p}_J))_{J \in \binom{[n]}{1^\rho}} + \cdots + \text{conv}(v_{\tilde{w}_0}(\tilde{p}_J))_{J \in \binom{[n]}{1^\rho}}. \]
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Proof. We have $M_{\omega_0}$ is of full rank and $\text{in}_{\omega_0}(I_n) = \text{in}_{M_{\omega_0}}(I_n)$. So, $\text{in}_{M_{\omega_0}}(I_n)$ is prime by assumption. We apply Theorem 1 and deduce that $S(A_n, v_{\omega_0})$ is generated by $\{v_{\omega_0}(p_J) \mid 0 \neq J \subseteq [n]\}$. Then

$$\Delta(A_n, v_{\omega_0}) = \text{conv}(v_{\omega_0}(p_J))_{J \in ([n])} + \cdots + \text{conv}(v_{\omega_0}(p_J))_{J \in ([n]) \setminus [J]}.$$ 

By [BFL17, §11] $Q_{\omega_0}(\rho) = \Delta(A_n, v_{\omega_0})$, so the claim follows.

As a corollary of the above theorem, we prove an even stronger version of the [BLMM17]-conjecture.

**Corollary 4.** Let $w_0$ be a reduced expression of $w \in S_n$ and consider $w_{\omega_0} \in \mathbb{R}^K$. Then

$$\text{in}_{\omega_0}(I_n) \text{ is prime } \iff w_0 \text{ has the (strong) Minkowski property.}$$

**Proof.** "$\Rightarrow$" By Theorem 2, $S(A_n, v_{\omega_0})$ is generated by $\{v_{\omega_0}(p_J) \mid J \subseteq [n]\}$, which are ray generators of $C(A_n, v_{\omega_0})$. Further, the generators are of form $v_{\omega_0}(p_J) = (\omega_k, \cdot)$ for all $k \in [n-1]$ and $|J| = k$. In particular, if $\lambda = \sum_{k=1}^{n-1} a_k \omega_k \in \Lambda^+$ we obtain by (4.1)

$$Q_{\omega_0}(\lambda) = \sum_{k=1}^{n-1} a_k Q_{\omega_0}(\omega_k).$$

"$\Leftarrow$" Assume $w_0$ has the strong Minkowski property. Then, by the reverse argument from above, $S(A_n, v_{\omega_0})$ is generated by $\{v_{\omega_0}(p_J) \mid J \subseteq [n]\}$. Applying Theorem 1, which we can do by the proof of Theorem 2, it follows that $\text{in}_{\omega_0}(I_n)$ is prime.

**Example 3.** Corollary 4 implies what we have seen from computations already, namely that $w_0 = s_1 s_3 s_2 s_3 s_1 s_2 \in \text{String 4}$ does not satisfy MP. The reason is that the element

$$v_{\omega_0}(p_2 p_{134} + p_{1} p_{234}) \geq \min \{v_{\omega_0}(p_2 p_{134}), v_{\omega_0}(p_{1} p_{234})\}$$

is missing as a generator for $S(A_4, v_{\omega_0})$. As $v_{\omega_0}(p_2 p_{134}) = v_{\omega_0}(p_{1} p_{234}) = (1, 0, 1, 0, 0)$, $\text{we deduce } v_{\omega_0}(p_2 p_{134} + p_{1} p_{234}) \succ (1, 0, 1, 1, 0)$. Hence, this element can not be obtained from the images of Plücker coordinates under $v_{\omega_0}$ and therefore $v_{\omega_0}(p_2 p_{134} + p_{1} p_{234})$ has to be added as a generator for $S(A_4, v_{\omega_0})$. In this example, $v$ and $v_{\omega_0}$ are in fact different:

$$v_{\omega_0}(p_2 p_{134} + p_{1} p_{234}) = (1, 0, 1, 0, 0), \text{ while } v(p_2 p_{134} + p_{1} p_{234}) \succ (1, 0, 1, 1, 0) =: a.$$ 

In particular, $v_{\omega_0}$ does not have 1-dimensional leaves as the quotient $F_{v_{\omega_0} \geq a} / F_{v_{\omega_0} \succ a}$ is two-dimensional. For $v$ this is not the case, as $p_2 p_{134} = p_{1} p_{234}$ in $\text{gr}_v(A_4)$.

5 Application: Rietsch-Williams valuation from plabic graphs

In this section we apply Theorem 1 from §3 to the valuation $v_G$ defined by Rietsch-Williams for Grassmannians using the cluster structure and Postnikov’s plabic graphs in [RW19]. We identify a class of plabic graphs with non-integral associated Newton–Okounkov polytope. The same combinatorial objects are used in [BFF+18] to define weight vectors. For $Gr_2(\mathbb{C}^n)$ we show that these weight vectors yield the same toric degeneration of $Gr_2(\mathbb{C}^n)$ as the corresponding Newton–Okounkov polytope.

Consider the Grassmannian with its Plücker embedding $Gr_k(\mathbb{C}^n) \hookrightarrow \mathbb{P}^{\binom{n}{k}}$. In this setting, its homogeneous coordinate ring $A_{k,n}$ is a quotient of the polynomial ring in Plücker variables $p_J$ with $J \subseteq [n]$ of cardinality $k$, denoted $J \in \binom{[n]}{k}$. We quotient by the Plücker ideal $I_{k,n} \subset \mathbb{C}[p_J | J \in \binom{[n]}{k}])$, a homogenous prime ideal generated by all Plücker relations. More precisely, for $K \in \binom{[n]}{k-1}$ and $L \in \binom{[n]}{k}$ let $\text{sgn}(j; K, L) := (-1)^{\# \{i \in L | i < j\}} + \# \{k \in K | k > j\}$. Then the associated Plücker relation (see e.g. [MS15, §4.3]) is of form

$$R_{K,L} := \sum_{j \in L} \text{sgn}(j; K, L)p_{K \cup \{j\} \cup \{k \in K | k > j\}}.$$
Then $A_{k,n} = \mathbb{C}[p_{j,n}]/I_{k,n}$. By [Sco06], $A_{k,n}$ has the structure of a cluster algebra [FZ02]: clusters are sets of algebraically independent algebra generators of $A_{k,n}$ over an ambient field of rational functions. Together with certain combinatorial data (e.g. a quiver) a cluster forms a seed. They are related by mutation, a procedure creating new seeds from a given one, that recovers the whole algebra after possibly infinitely many recursions. In particular, certain subsets of Plücker coordinates $\bar{p}_j \in A_{k,n}$ are clusters. These special clusters are encoded by combinatorial objects called plabic graphs $^9$ [Pos06]. We recall plabic graphs in the Appendix A.1 below.

For every plabic graph $\mathcal{G}$ (or more generally every seed) for $\Gr_k(\mathbb{C}^n)$ in [RW19] they define a valuation $v_\mathcal{G} : A_{k,n} \setminus \{0\} \to \mathbb{Z}^d$ where $d := k(n-k) = \dim \Gr_k(\mathbb{C}^n)$. The images of Plücker coordinates $v_\mathcal{G}(\bar{p}_j)$ can be computed using the combinatorics of the plabic graph $\mathcal{G}$. Please consider Appendix A.2 for the precise definition of the valuation and how to compute it. In what follows we use the terminology summarized there without further explanation.

Figure 1: The plabic graph $\mathcal{G}^{\text{rec}}$ of type $\pi_{3,5}$ with perfect orientation and source set $\{1,2\}$. Faces are labelled by Young tableaux as described in §A.1.

Let $\mathcal{G}$ be a reduced plabic graph for $\Gr_k(\mathbb{C}^n)$ with perfect orientation chosen such that $[k]$ is the source set, see e.g. Figure 1. Consider the weighting matrix $M_\mathcal{G} := M_{v_\mathcal{G}}$ of $v_\mathcal{G}$ as defined in Appendix A.2. The columns of $M_\mathcal{G}$ are $v_\mathcal{G}(\bar{p}_j)$ for $J \in \left( \begin{array}{c} n \cr k \end{array} \right)$ and the rows $M_1, \ldots, M_{d+1}$ are indexed by the faces of the plabic graph $\mathcal{G}$. Denote the boundary faces of $\mathcal{G}$ by $F_1, \ldots, F_n$, where $F_i$ is adjacent to the boundary vertices $i$ and $i+1$. Hence, $F_n = F_\mathcal{G}$ is the face that does not contribute to the image of $v_\mathcal{G}$ as $M_n = (0, \ldots, 0)$ and we omit this row of $M$. Order the rows of $M_\mathcal{G}$ such that $M_i$ is the row corresponding to the face $F_i$ in $\mathcal{G}$. In the following lemma we compute the columns of $M$ corresponding to boundary faces of $\mathcal{G}$ explicitly.

**Lemma 6.** Let $r \in [n-1]$ and $J = \{j_1, \ldots, j_s\} \in \left( \begin{array}{c} n \cr k \end{array} \right)$ with $j_1 < \ldots < j_s \leq k < j_{s+1} < \cdots < j_k$. Set $[k] \setminus \{j_1, \ldots, j_s\} = \{i_1, \ldots, i_{k-s}\}$ with $i_1 < \cdots < i_{k-s}$. Then the $J^\text{th}$ entry of the column $M_r$ is

$$(M_r)_J = \# \{ l \mid r \in [i_l, j_{k-l+1} - 1] \},$$

where $[i_l, j_{k-l+1} - 1]$ is the cyclic interval in $\mathbb{Z}/n\mathbb{Z}$.

**Proof.** Let $\mathbf{f} = \{\mathbf{p}_{j_1}, \ldots, \mathbf{p}_{j_k}\}$ be a flow (see Definition 14) from $[k]$ to $J$, where $\mathbf{p}_{j_i}$ denotes the path with sink $j_i$. The paths $\mathbf{p}_{j_r}$ for $r \leq s$ are “lazy paths”, starting and ending at $j_r$ without moving. Let $[k] \setminus \{j_1, \ldots, j_s\} = \{i_1, \ldots, i_{k-s}\}$ with $i_1 < \cdots < i_{k-s}$. Hence, for $k-l+1 > s$ we have non-trivial paths $\mathbf{p}_{j_{k-l+1}}$ with source $i_l$ and sink $j_{k-l+1}$. To its left are all boundary faces $F_r$ with $r$ in the cyclic interval $[i_l, j_{k-l+1} - 1]$. \qed

Recall, the tropicalization of an ideal defined in (3.2). The tropical Grassmannain [SS04] refers to the tropicalization of the Plücker ideal $I_{k,n} \subset \mathbb{C}[p_{j,n}]$, we denote it by $\text{trop}(\Gr_k(\mathbb{C}^n)) \subset \mathbb{R}^{\binom{n}{k}}$. It contains an $n$-dimensional linear subspace $L_{I_{k,n}} := \{ w \in \mathbb{R}^{\binom{n}{k}} \mid \text{in}_w(I_{k,n}) = I_{k,n} \}$, called lineality space, see [SS04, page 393]. In the following corollary we treat columns of $M$ as weight vectors for Plücker variables.

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$^9$To be precise, we consider only reduced plabic graphs of trip permutation $\pi_{n-k,n}$, for details see Appendix A.1.
5. Application: Rietsch-Williams valuation from plabic graphs

Corollary 5. For all \( r \in [n-1] \) we have \( M_r \in L_{k,n} \).

Proof. Consider a Plücker relation \( R_{K,L} \) with \( K \in \binom{n}{k-1} \) and \( L \in \binom{n}{k+1} \) of form (5.1). Every term in \( R_{K,L} \) equals \( \pm p_{J}p_{J'} \) for some \( J, J' \in \binom{n}{k} \). We can rewrite the formula for \( (M_r)_J \) as follows

\[
(M_r)_J = \begin{cases} 
|J \setminus [r-1]|, & \text{if } r < k, \\
|J \cap [r+1, n]|, & \text{if } r \geq k.
\end{cases}
\]

Let \( J = \{j_1, \ldots, j_k\}, J' = \{j'_1, \ldots, j'_k\} \) and define the sequence \( S := (j_1, \ldots, j_k, j'_1, \ldots, j'_k) \). For any set \( N \) we denote by \( S \cap N \) the sequence obtained from \( S \) when deleting all entries that are not elements of \( N \). Similarly, let \( S \setminus N \) be the sequence obtained from \( S \) when deleting all entries that do belong to \( N \). Further, for any sequence \( S' \) let \( |S'| \) denote its length. Then

\[
(M_r)_J + (M_r)_{J'} = \begin{cases} 
|S \setminus [r-1]|, & \text{if } r < k, \\
|S \cap [r+1, n]|, & \text{if } r \geq k.
\end{cases}
\]

Note that the right hand side only depends on the entries of \( S \) regardless of the ordering. Further, for all pairs \( J, J' \) corresponding to monomials in \( R_{K,L} \) the sequence \( S \) is the same up to reordering. Hence, \( \text{im}_{M_r}(R_{K,L}) = R_{K,L} \) for all \( r \in [n-1] \).

Recall the plabic weight vector \( \mathbf{w}_G \) from Definition 15. The following proposition establishes the connection to what we have seen in §3. In terms of the weighting matrix \( M_G \), we observe

\[
\mathbf{w}_G = \sum_{e_j \text{ interior face of } M_j} \mathbf{e}_j
\]

where the sum contains exactly those \( M_j \) corresponding to interior faces of \( G \).

| \( p_{12} \) | \( e_{35} \) | \( e_{25} \) | \( e_{45} \) | \( e_{15} \) | \( e_{12} \) | \( e_{23} \) | \( e_{34} \) | \( \mathbf{g}_{G^{\text{rec}}} \) | \( f_{35} \) | \( f_{25} \) | \( f_{45} \) | \( f_{15} \) | \( f_{12} \) | \( f_{23} \) | \( f_{34} \) |
|-------|--------|--------|--------|--------|--------|--------|--------|----------------|--------|--------|--------|--------|--------|--------|--------|
| \( p_{13} \) | 0 0 0 0 0 0 0 0 | \( p_{13} \) | 0 -1 0 1 0 1 0 |
| \( p_{14} \) | 0 0 0 0 1 1 0 0 | \( p_{14} \) | -1 0 0 1 0 0 1 |
| \( p_{15} \) | 0 0 0 0 1 1 1 1 | \( p_{15} \) | 0 0 0 1 0 0 0 |
| \( p_{23} \) | 0 0 0 1 1 0 0 | \( p_{23} \) | 0 0 0 0 0 1 0 |
| \( p_{24} \) | 0 0 0 1 1 1 0 | \( p_{24} \) | -1 1 0 0 0 0 1 |
| \( p_{25} \) | 0 0 0 1 1 1 1 | \( p_{25} \) | 0 1 0 0 0 0 0 |
| \( p_{34} \) | 0 1 0 1 2 1 0 | \( p_{34} \) | 0 0 0 0 0 0 1 |
| \( p_{35} \) | 0 1 0 1 2 1 1 | \( p_{35} \) | 1 0 0 0 0 0 0 |
| \( p_{45} \) | 1 1 0 1 2 2 1 | \( p_{45} \) | 0 0 1 0 0 0 0 |

Table 2: The images of Plücker coordinates under the valuation \( \mathbf{v}_{G^{\text{rec}}} \) for \( G^{\text{rec}} \) as in Figure 1 on the left and under the valuation \( \mathbf{g}_{G^{\text{rec}}} \) on the right. See §A.2&A.3 for details.

Example 4. Consider the plabic graph \( G = G^{\text{rec}} \) with perfect orientation from Figure 1 and source set [2]. We compute \( \deg_G(p_J) \) and \( \mathbf{v}_G(p_J) \) for all \( J \in \binom{[5]}{2} \). Order the faces of \( G \) by

\[
F_{35} = \varnothing, \quad F_{25} = \varnothing, \quad F_{45} = \varnothing, \quad F_{15} = \varnothing, \quad F_{12} = \varnothing, \quad F_{23} = \varnothing, \quad \text{and} \quad F_{34} = \varnothing.
\]

For example, consider \( J = \{2, 4\} \). There are two flows, \( f_1 \) and \( f_2 \) from [2] to \( J = \{2, 4\} \). Both consist of only one path from 1 to 4. One of them, say \( f_1 \), has faces labelled by \( \varnothing \) to its left while the other \( f_2 \) has faces \( \varnothing \) to its left. Then with respect to the above order of coordinates (corresponding to faces of \( G \)) on \( \mathbb{Z}^7 \) we have

\[
\text{wt}(f_1) = (0, 0, 0, 0, 1, 1, 1, 0) \quad \text{and} \quad \text{wt}(f_2) = (0, 1, 0, 0, 1, 1, 1, 0).
\]

As \( \deg_G(f_1) = 0 \) and \( \deg_G(f_2) = 1 \), we have \( \mathbf{v}_G(p_{24}) = (1, 0, 1, 1, 0, 0) \) and \( \deg_G(p_{24}) = 0 \). All other \( \mathbf{v}_G(p_J) \) and \( \deg_G(p_J) \) can be recovered from Table 2.
5. Application: Rietsch-Williams valuation from plabic graphs

Proposition 3. For every plabic graph $\mathcal{G}$ for $\text{Gr}_2(\mathbb{C}^n)$ we have $\text{in}_{M_\mathcal{G}}(I_{2,n}) = \text{in}_{w_\mathcal{G}}(I_{2,n})$.

In general, for $k \geq 3$ and $n \geq 6$ the statement of the proposition is false, see Example 5 and Theorem 3 below.

Proof. Fix the plabic graph $\mathcal{G} = \mathcal{G}^{\text{rec}}$. We show that for every Plücker relation $R_{ijkl} := p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk}$ with $1 \leq i < j < k < l \leq n$ we have $\text{in}_{M_\mathcal{G}}(R_{ijkl}) = \text{in}_{w_\mathcal{G}}(R_{ijkl})$. As the $R_{ijkl}$ form a Gröbner basis for both initial ideals it follows in $M_\mathcal{G}(I_{2,n}) = \text{in}_{w_\mathcal{G}}(I_{2,n})$. For arbitrary plabic graphs the claim follows as both initial ideals change in the same way under the mutation move (M1). To see this compare [RW19, §13] with [BFF18, §6] (or for more details [Bos18, §3.3.4]). So without loss of generality we can focus on $\mathcal{G}^{\text{rec}}$.

Consider $\mathcal{G}^{\text{rec}}$ as in Figure 2 and order the faces by $E_3, \ldots, E_{n-1}, F_1, \ldots, F_n$. By Corollary 5 we may focus only on the columns of $M_{\mathcal{G}^{\text{rec}}}$ corresponding to the interior faces $E_3, \ldots, E_{n-1}$. We compute the leading monomials of the flow polynomials and the entries of $w_{\mathcal{G}^{\text{rec}}}$:

\[
\begin{align*}
\nu_{\mathcal{G}^{\text{rec}}}(\widehat{p}_{1i}) &= (0, \ldots, 0, 1, \ldots, 0), & w_{\mathcal{G}^{\text{rec}}}(\widehat{p}_{1i}) &= 0, \\
\nu_{\mathcal{G}^{\text{rec}}}(\widehat{p}_{2i}) &= (0, \ldots, 0, 1, 1, \ldots, 0), & w_{\mathcal{G}^{\text{rec}}}(\widehat{p}_{2i}) &= 0, \\
\nu_{\mathcal{G}^{\text{rec}}}(\widehat{p}_{ij}) &= (1, \ldots, 1, 0, 0, \ldots, 0), & w_{\mathcal{G}^{\text{rec}}}(\widehat{p}_{ij}) &= i - 2,
\end{align*}
\]

where $i < j$ and the 1’s in $\nu_{\mathcal{G}^{\text{rec}}}(\widehat{p}_{ij})$ correspond to the faces $E_3, \ldots, E_i$. In particular, for $R_{ijkl}$ with $1 \leq i < j < k < l \leq n$ we deduce $\text{in}_{M_{\mathcal{G}^{\text{rec}}}}(R_{ijkl}) = -p_{ik}p_{jl} + p_{il}p_{jk} = \text{in}_{w_{\mathcal{G}^{\text{rec}}}}(R_{ijkl})$. □

Example 5. For $\text{Gr}_3(\mathbb{C}^6)$ we consider the plabic graph $\mathcal{G}$ on the left in Figure 3 and compute the images of Plücker coordinates under $\nu_{\mathcal{G}} : A_{3,6} \setminus \{0\} \to \mathbb{Z}^9$ (see Table 3 below). With respect to the order on variables as indicated in Table 3 we fix the lexicographic order on $\mathbb{Z}^9$.

Take the following four-term Plücker relation written as a sum of two binomials:

\[(p_{123}p_{456} - p_{124}p_{356}) + (p_{125}p_{346} - p_{126}p_{345}) =: f_1 + f_2\]
We compute the flow polynomials for \( \bar{f}_1 \) and \( \bar{f}_2 \) and obtain: \( \psi_G(\bar{f}_1) = \psi_G(\bar{f}_2) = (1, 2, 3, 2, 1, 1, 3, 1) \). Comparing with the valuations of Plücker coordinates we deduce \( \psi_G(\bar{f}_1) \) does not lie in the semigroup-span of \( \{ \psi_G(p_{ijk}) \mid 1 \leq i < j < k \leq 6 \} \). Hence, by Theorem 1 the associated initial ideal in \( M_G(I_{3,6}) \) is not prime. Further, computing initial forms of \( f_1 + f_2 \) we see \( \psi_{w_2}(f_1 + f_2) = f_1 + f_2 \), but \( \psi_{M_G}(f_1 + f_2) = f_1 \). So, \( \psi_{M_G}(I_{3,6}) \neq \psi_{w_2}(I_{3,6}) \). For the plabic graph on the right side of Figure 3 a similar argument works considering the Plücker relation \( (p_{123}p_{456} - p_{145}p_{236}) + (p_{146}p_{235} - p_{156}p_{234}) \).

**Definition 10.** We say a plabic graph \( G \) is *hexagonal*, if locally around six consecutive boundary vertices it is equivalent up to moves (M2) and (M3) to the arrangement depicted on the left in Figure 4.

**Remark 2.** We call such plabic graphs hexagonal in alignment with \( [\text{BCMN19}] \) where a criterion for non-prime cones in trop(Gr\(_3(C^n)\)) is given in terms of *hexagonal* tropical line arrangement. We believe that a combinatorial algorithm transforming hexagonal plabic graphs for Gr\(_3(C^n)\) into hexagonal tropical line arrangements should exist.

![Figure 4: On the left: the local structure of a hexagonal plabic graph. On the right: a possible labelling with perfect orientation of a hexagonal plabic graph.](image)

**Theorem 3.** Let \( G \) be a hexagonal plabic graph. Then in \( M_G(I_{k,n}) \) is not prime. Moreover, in \( M_G(I_{k,n}) \neq \psi_{w_G}(I_{k,n}) \) and \( \Delta(A_{k,n}, \psi_G) \) is not integral.

**Proof.** We will show that the value semigroup \( S(A_{k,n}, \psi_G) \) associated with the valuation of a plabic graph as on the right side of Figure 4 is not generated by \( \{ \psi_G(p_{ij}) \mid J \in \binom{[n]}{k} \} \) and how this is enough to deduce the claims of the Theorem. Assuming this statement is true by the isomorphism of \( [\text{BCMN19}] \) (see §A.3) the same is true for the valuation \( g_G : A_{k,n} \setminus \{ 0 \} \to \mathbb{Z}^d \) by sending every Plücker coordinate to its \( g \)-vector (see §A.3). While \( \psi_G \) depends on the perfect orientation given to \( G \), the valuation \( g_G \) purely depends on the combinatorial type of \( G \) (on the associated quiver to be precise). If \( G' \) is a hexagonal plabic graph obtained from \( G \) by permuting the labelling of the boundary vertices this induces a natural isomorphism between \( S(A_{k,n}, g_{G'}) \cong S(A_{k,n}, g_G) \). Combining with \( [\text{BCMN19}] \) we obtain

\[
S(A_{k,n}, \psi_{G''}) \cong S(A_{k,n}, g_{G'}) \cong S(A_{k,n}, g_G) \cong S(A_{k,n}, \psi_G).
\]

So without loss of generality that \( G \) is perfectly oriented and of form as on the right in Figure 4. Consider the Plücker relation

\[
f_1 + f_2 := (p_{[k]}p_{[k-3] \cup \{k+1,k+2,k+3\}} - p_{[k-1]}p_{[k+1]}p_{[k-3] \cup \{k+2,k+3\}}) + (p_{[k-1] \cup \{k+2\}}p_{[k-3] \cup \{k+1,k+3\}} - p_{[k-1]}p_{[k+1]}p_{[k-3] \cup \{k+1,k+2\}}).
\]

**Claim:** The image of \( \bar{f}_1 \) under \( \psi_G \) does not lie in the semigroup span of \( \{ \psi_G(p_{ij}) \mid J \in \binom{[n]}{k} \} \).
We compute (the lowest degree terms of) the flow polynomials

\[
\begin{align*}
\bar{p}[k] &= 1, \\
\bar{p}[k-1] \cup \{k+1\} &= x_F(1 + x_C), \\
\bar{p}[k-1] \cup \{k+2\} &= x_C x_F x_{F_{k+1}}, \\
\bar{p}[k-1] \cup \{k+3\} &= x_C x_F x_{F_{k+1}} x_{F_{k+2}}, \\
\bar{p}[k-3] \cup \{k+1,k+2\} &= x_A x_B x_C^2 x_{F_{k-2}}^2 x_{F_{k-1}}^2 x_F x_{F_{k+1}}, \\
\bar{p}[k-3] \cup \{k+1,k+3\} &= x_A x_B x_C^2 x_{F_{k-2}}^2 x_{F_{k-1}}^2 x_F x_{F_{k+1}} x_{F_{k+2}} (1 + x_D + \text{h.o.t.}), \\
\bar{p}[k-3] \cup \{k+2,k+3\} &= x_A x_B x_C^2 x_D x_{F_{k-2}} x_{F_{k-1}} x_{F_{k+1}} x_{F_{k+2}} x_{F_{k+3}} (1 + \text{h.o.t.}), \\
\bar{p}[k-3] \cup \{k+1,k+2,k+3\} &= x_A x_B x_C^2 x_D x_{F_{k-2}} x_{F_{k-1}} x_{F_{k+1}} x_{F_{k+2}} x_{F_{k+3}}.
\end{align*}
\]

Ordering the variables corresponding to faces of \( G \) by \( A, B, C, D, F_{k-2}, F_{k-1}, F_k, F_{k+1}, F_{k+2} \) followed by the ones not displayed in Figure 4 we obtain

\[
\begin{align*}
\vartheta_G (\bar{p}[k] \bar{p}[k-3] \cup \{k+1,k+2,k+3\}) &= (1, 1, 1, 2, 1, 2, 3, 2, 1, 0, \ldots, 0), \\
\vartheta_G (\bar{p}[k-1] \cup \{k+1\} \bar{p}[k-3] \cup \{k+2,k+3\}) &= (1, 1, 2, 1, 1, 2, 3, 2, 1, 0, \ldots, 0), \\
\vartheta_G (\bar{p}[k-1] \cup \{k+2\} \bar{p}[k-3] \cup \{k+1,k+3\}) &= (1, 1, 3, 0, 1, 2, 3, 2, 1, 0, \ldots, 0), \\
\vartheta_G (\bar{p}[k-1] \cup \{k+3\} \bar{p}[k-3] \cup \{k+1,k+2\}) &= (1, 1, 3, 1, 1, 2, 3, 2, 1, 0, \ldots, 0), \\
\vartheta_G (\bar{f}_1) &= \vartheta_G (\bar{f}_2) = (1, 1, 3, 1, 1, 2, 3, 2, 1, 0, \ldots, 0).
\end{align*}
\]

If \( \vartheta_G (\bar{f}_1) \) would lie in the semigroup span of the values of Plücker coordinates, by Corollary 5 it has to be of the form

\[
\vartheta_G (\bar{f}_1) = \vartheta_G (\bar{p}_1) + \vartheta_G (\bar{p}_2),
\]

for \( I_1, I_2 \subseteq \binom{[n]}{2} \) such that \( I_1 \cap I_2 = [k-3] \) and \( I_1 \cup I_2 = [k+3] \). The values of such Plücker coordinates resemble the ones in the case of \( \text{Gr}_3(\mathbb{C}^6) \) from Table 3. This can be seen by making the following identifications: identify faces of the plabic graph \( G \) and the one on the left in Figure 3:

\[
\begin{align*}
\varnothing &\leftrightarrow A, \hspace{1cm} \varnothing &\leftrightarrow B, \hspace{1cm} \varnothing &\leftrightarrow C, \hspace{1cm} \varnothing &\leftrightarrow D, \hspace{1cm} \varnothing &\leftrightarrow F_{k-2}, \ldots, \varnothing &\leftrightarrow F_{k+2},
\end{align*}
\]

and identify Plücker coordinates \( \bar{p}_{i,j,l} \leftrightarrow \bar{p}_{[k-3] \cup \{i+k-3,j+k-3,l+k-3\}} \) for \( 1 \leq i < j < k \leq 6 \). As in Example 5 we see that the value \((1,1,3,1,1,2,3,2,1,\ldots)\) does not arise as a sum of any two corresponding values in Table 3, so the claim follows.

Lemma 7 and Theorem [BCM19] stated in \( \text{§A.3} \) imply that \( M_G \) is of full rank. Then by Theorem 1 it follows that \( \text{in}_{M_G} (I_{k,n}) \) is not prime. Further, we have \( \text{in}_{M_G} (f_1 + f_2) = f_1 \) and \( \text{in}_{M_G} (f_1 + f_2) = f_1 + f_2 \). Hence, \( \text{in}_{M_G} (I_{k,n}) \neq \text{in}_{M_G} (I_{k,n}) \). Lastly, the ray spanned by \( \vartheta_G (\bar{f}_1) \) gives a vertex of \( \Delta(A_{k,n}, \vartheta_G) \) that is of form

\[
\left( \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, 0, \ldots, 0 \right),
\]

which makes \( \Delta(A_{k,n}, \vartheta_G) \) non-integral.

\[
A \hspace{1cm} \text{Appendix for \( \text{§5} \)}
\]

A.1 Plabic graphs

We review the definition of plabic graphs due to Postnikov [Pos06]. This section is closely oriented towards [RW19] and [BFF+18].
Table 3: The images of Plücker coordinates under the valuation $v_G$ for the plabic graph $G$ as on the left in Figure 3.

Definition 11. A plabic graph $G$ is a planar bicolored graph embedded in a disk (up to homotopy). It has $n$ boundary vertices numbered $1, \ldots, n$ in a clockwise order. Boundary vertices lie on the boundary of the disk and are not coloured. Additionally, there are internal vertices coloured black or white. Each boundary vertex is adjacent to a single internal vertex.

For our purposes we assume that plabic graphs are connected and that every leaf of a plabic graph is a boundary vertex. We first recall the four local moves on plabic graphs.

(M1) If a plabic graph contains a square of four internal vertices with alternating colours, each of which is trivalent, then the colours can be swapped. So every black vertex in the square becomes white and every white vertex becomes black (see Figure 5).

(M2) If two internal vertices of the same colour are connected by an edge, the edge can be contracted and the two vertices can be merged. Conversely, any internal black or white vertex can be split into two adjacent vertices of the same colour (see Figure 5).

(M3) If a plabic graph contains an internal vertex of degree 2, it can be removed. Equivalently, an internal black or white vertex can be inserted in the middle of any edge (see Figure 6).

(R) If two internal vertices of opposite colour are connected by two parallel edges, they can be reduced to only one edge. This can not be done conversely (see Figure 6).

The equivalence class of a plabic graph $G$ is defined as the set of all plabic graphs that can be obtained from $G$ by applying (M1)-(M3). If in the equivalence class there is no graph to which (R) can be applied, we say $G$ is reduced. From now on we only consider reduced plabic graphs.
Definition 12. Let $G$ be a reduced plabic graph with boundary vertices $v_1, \ldots, v_n$ labelled in a clockwise order. We define the trip permutation $\pi_G$ as follows. We start at a boundary vertex $v_i$ and form a path along the edges of $G$ by turning maximally right at an internal black vertex and maximally left at an internal white vertex. We end up at a boundary vertex $v_{\pi(i)}$ and define $\pi_G = [\pi(1), \ldots, \pi(n)] \in S_n$.

It is a fact that plabic graphs in one equivalence class have the same trip permutation. Further, it was proven by Postnikov in [Pos06, Theorem 13.4] that plabic graphs with the same trip permutation are connected by moves (M1)-(M3) and are therefore equivalent. Let $\pi_{n-k,n} = (k+1, k+2, \ldots, n, 1, 2, \ldots, k)$. From now on we focus on plabic graphs $G$ with trip permutation $\pi_G = \pi_{n-k,n}$. Each path $v_i$ to $v_{\pi_{n-k,n}(i)}$ defined above, divides the disk into two regions. We label every face in the region to the left of the path by $i$. After repeating this for every $1 \leq i \leq n$, all faces have a labelling by an $k$-element subset of $[n]$. Every such $k$-element subset defines a Young diagram that fits into an $(n-k) \times k$-rectangle of boxes. We denote by $\mathcal{P}_G$ the set of all such subsets (resp. their associated Young diagrams) for a fixed plabic graph $G$. The cardinality of $\mathcal{P}_G$ is $d + 1$.

A face of a plabic graph is called internal, if it does not intersect with the boundary of the disk. Other faces are called boundary faces. Following [RW19] we define an orientation on a plabic graph. This is the first step in establishing the flow model introduced by Postnikov, which we use to define plabic degrees on the Plücker coordinates.

Definition 13. An orientation $O$ of a plabic graph $G$ is called perfect, if every internal white vertex has exactly one incoming arrow and every internal black vertex has exactly one outgoing arrow. The set of boundary vertices that are sources is called the source set and is denoted by $I_O$.

Postnikov showed in [Pos06] that every reduced plabic graph with trip permutation $\pi_{n-k,n}$ has a perfect orientation with source set of order $k$. See Figure 1 for a plabic graph with trip permutation $\pi_{3,5}$.

Index the standard basis of $Z^{\mathcal{P}_G} = Z^{d+1}$ by the faces of the plabic graph $G$, where $d = k(n-k)$. Given a perfect orientation $O$ on $G$, every directed path $p$ from a boundary vertex in the source set to a boundary vertex that is a sink, divides the disk in two parts. The weight $\text{wt}(p) \in Z_{\geq 0}^{\mathcal{P}_G}$ has entry 1 in the position corresponding to a face $F$ of $G$, if $F$ is to the left of $p$ with respect to the orientation. The degree $\text{deg}_G(p) \in Z_{\geq 0}$ is defined the number of internal faces to the left of the path. The boundary face between the boundary vertices $k$ and $k+1$ never lies to the left of any path and therefore is also referred to as $F_G$.

Definition 14. For a set of boundary vertices $J$ with $|J| = |I_O|$, we define a $J$-flow as a collection of self-avoiding, vertex disjoint directed paths with sources $I_O - (J \cap I_O)$ and sinks $J - (J \cap I_O)$. Let $I_O - (J \cap I_O) = \{j_1, \ldots, j_r\}$ and $f = \{p_{j_1}, \ldots, p_{j_r}\}$ be a flow where each path $p_{j_i}$ has sink $j_i$. Then the weight of the flow is $\text{wt}(f) := \text{wt}(p_{j_1}) + \cdots + \text{wt}(p_{j_r})$. Similarly, we define the degree of the flow as $\text{deg}_G(f) = \text{deg}_G(p_{j_1}) + \cdots + \text{deg}_G(p_{j_r})$. By $\mathcal{F}_J$ we denote the set of all $J$-flows in $G$ with respect to $O$.

A.2 Valuation and plabic degree

In [RW19] Rietsch-Williams use the cluster structure on $\text{Gr}_k(\mathbb{C}^n)$ (due to Scott, see [Sco06]) to define a valuation on $\mathbb{C}(\text{Gr}_k(\mathbb{C}^n)) \setminus \{0\}$ for every seed. In fact, a plabic graph $G$ defines a seed in the corresponding cluster algebra. A combinatorial algorithm associates a quiver with $G$ (see e.g. [RW19, Definition 3.8]). The corresponding cluster is a set of Plücker coordinates $\hat{p}_J$, where $J$ is a face label in $G$ as described above.

Let $A_{k,n} := \text{Gr}(\mathbb{C}^n)$. We recall the definition of the valuation $v_G$ from [RW19, §8]. By [Pos06, §6] for every plabic graph $G$ there exists a map $\Phi_G : (\mathbb{C}^*)^{\mathcal{P}_G} \to \text{Gr}_k(\mathbb{C}^n)$ sending $(x_\mu)_{\mu \in \mathcal{P}_G}$ to $A \in \mathbb{C}^{n \times k}$, where

$$A_{ij} = (-1)^{|i'|} \sum_{\text{path } p : i \to j} x^{\text{wt}(p)} \in \mathbb{C}[x_\mu]_{\mu \in \mathcal{P}_G}.$$
Here $x^{\text{wt}(p)}$ denotes the monomial with exponent vector \( \text{wt}(p) \in \{0,1\}^{P_\mathbb{G}} \). Fix an order on the coordinates \( \{x_\mu\}_{\mu \in P_\mathbb{G}} \) and let \( \prec \) be the total order on \( \mathbb{Z}^{P_\mathbb{G}} \) be the corresponding lexicographic order (see [RW19, Definition 8.1]). The pullback of \( \Phi_\mathbb{G}^* \) satisfies \( \Phi_\mathbb{G}^*(\bar{p}_J) \in \mathbb{C}[x_\mu^{\pm 1}|\mu \in P_\mathbb{G}] \) for \( J \in \binom{n}{k} \). Moreover, every polynomial in Plücker coordinates has a unique Laurent polynomial expression in \( \{x_\mu\}_{\mu \in P_\mathbb{G}} \). Suppose \( f \in A_{k,n} \) has expression \( \sum_{i=1}^n a_i x^{m_i} \) for \( a_i \in \mathbb{C} \) and \( m_i \in \mathbb{Z}^{P_\mathbb{G}} \). Then the lowest term valuation \( v_\mathbb{G} : \mathbb{C}(\text{Gr}_k(\mathbb{C}^n)) \setminus \{0\} \to (\mathbb{Z}^{P_\mathbb{G}}, \prec) \) is defined as \[
 v_\mathbb{G}(f) := \min_{\text{lex}}\{m_i \mid a_i \neq 0\}. \]

For a rational function \( h = \frac{f}{g} \) it is defined as \( v_\mathbb{G}(h) := v_\mathbb{G}(f) - v_\mathbb{G}(g) \). For Plücker coordinates the images of \( v_\mathbb{G} \) can be computed explicitly using the combinatorics of the plabic graph. For \( J \in \binom{n}{k} \) let \( f_J \in F_J \) be the flow with \( \deg_\mathbb{G}(f_J) = \min_{v_\mathbb{G}}\{\deg_\mathbb{G}(f) \mid f \in F_J\} \). Then on a Plücker coordinate \( \bar{p}_J \in A_{k,n} \) the valuation \( v_\mathbb{G} \) is given by \[
 v_\mathbb{G} (\bar{p}_J) = \text{wt}(f_J) \in \mathbb{Z}^{P_\mathbb{G}}. \tag{A.1} \]

In fact, \( P_\mathbb{G} \) contains one label \( \mu \) whose corresponding face never contributes to the weight of any flow. With our convention it is the boundary face between the vertices \( k \) and \( k+1 \), call it \( F_\mathbb{G} \). Therefore, we can omit the corresponding variable and have \( v_\mathbb{G} : A_{k,n} \setminus \{0\} \to \mathbb{Z}^{P_\mathbb{G}} - \mathbb{G} \cong \mathbb{Z}^d \). In particular, this implies that the rank of \( v_\mathbb{G} \) is \( d = \dim(\text{Gr}_k(\mathbb{C}^n)) \) which is one less than the Krull-dimension of \( A_{k,n} \).

In order to have a valuation that fits into our framework, we slightly modify \( v_\mathbb{G} \) and define \[
 \hat{v}_\mathbb{G} : A_{k,n} \setminus \{0\} \to \mathbb{Z}^{d+1} \text{ given by } \hat{v}_\mathbb{G}(f) = (\deg f, v_\mathbb{G}(f)). \tag{A.2} \]

Note that \( \hat{v}_\mathbb{G} \) is a full-rank valuation on \( A_{k,n} \). Further, \( M_\mathbb{G} \in \mathbb{Z}^{(d+1) \times (\binom{n}{k})} \) differs from \( M_\mathbb{G} \in \mathbb{Z}^{d \times (\binom{n}{k})} \) by the row \( (1, \ldots, 1) \in \mathbb{Z}^{(\binom{n}{k})} \). As \( I_{k,n} \) is homogeneous, seen as a weight vector, we have \( (1, \ldots, 1) \in L_{I_{k,n}} \).

In particular, \[
 \text{in}_{M_\mathbb{G}}(I_{k,n}) = \text{in}_{M_\mathbb{G}}(I_{k,n}). \tag{A.3} \]

In [BFF+18] they define closely related to the valuation the following notion of degree for Plücker variables in \( \mathbb{C}[p_J]_J \) and associate a weight vector in \( \mathbb{Z}^{(\binom{n}{k})} \). For an example, consider Example 4 above.

**Definition 15.** For \( J \in \binom{n}{k} \) and a plabic graph \( \mathcal{G} \), the plabic degree of the Plücker variable \( p_J \) is defined as \[
 \deg_{\mathcal{G}}(p_J) := \min\{\deg_\mathbb{G}(f) \mid f \in F_J\} \in \mathbb{Z}_{\geq 0}. \]

It gives rise to the plabic weight vector \( \mathbf{w}_\mathcal{G} \in \mathbb{Z}^{(\binom{n}{k})} \) defined by \( (\mathbf{w}_\mathcal{G})_J := \deg_{\mathcal{G}}(p_J) \).

By [PSW09, Lemma 3.2] and its proof, the plabic degree is independent of the choice of the perfect orientation. We therefore fix the perfect orientation by choosing the source set \( I_\mathcal{O} = [k] \). The following proposition guarantees that the degree (and the valuation) are well-defined. It is a reformulation of the original statement adapted to our notion degree.

**Proposition.** ([RW19, Corollary 12.4]) There is a unique \( J \)-flow in \( \mathcal{G} \) with respect to \( \mathcal{O} \) with degree equal to \( \deg_\mathcal{G}(p_J) \).

### A.3 [GHKK18]’s g-vector valuation vs. [RW19]’s valuation

Gross, Hacking, Keel and Kontsevich construct vector space bases for cluster algebras in [GHKK18]. Among other powerful applications their so-called theta basis can be used to construct toric degenerations for partially compactified cluster varieties.

For the Grassmannian their toric degeneration can be formulated in terms of valuations and Newton-Okounkov bodies. They rely on Fomin and Zelevinsky’s principal coefficients (introduced in [FZ07])
and the associated multiweights for cluster monomials called $g$-vectors. Generalized $g$-vectors are introduced in [GHKK18, Definition 5.10] for elements of the theta basis. For any seed $s$ the assignment of its $g$-vector to a theta basis element extends to a full-rank valuation $g_s : A_{k,n} \setminus \{0\} \to \mathbb{Z}^d$. For us the following straightforward lemma is important.

**Lemma 7.** For a plabic graph $G$ consider the set of Plücker coordinated $\bar{p}_J$, where $J \in \mathcal{P}_G - \emptyset$. Then the matrix with rows $g_G(\bar{p}_J)$ is (up to permutation of the rows) the identity matrix.

The following result is a direct consequence of the main theorem in [BCMN19]:

**Theorem (BCMN19).** For every plabic graph $G$ there exists a linear map $\bar{p}_G^* : \mathbb{R}^d \to \mathbb{R}^d$ inducing a unimodular equivalence between $\Delta(A_{k,n}, v_G) \cong \Delta(A_{k,n}, g_G)$.

**Example 6.** We consider the plabic graph $G_{rec}$ for $Gr_2(\mathbb{C}^5)$ as in Figure 1. For a lattice $N \cong \mathbb{Z}^7$ we fix an ordered basis $\{e_{35}, e_{25}, e_{45}, e_{12}, e_{23}, e_{34}\}$ corresponding to the faces of $G_{rec}$. Let $M := N^*$ be the dual lattice with dual basis (in order) $\{f_{35}, f_{25}, f_{45}, f_{12}, f_{23}, f_{34}\}$. Then $\bar{p}_{G_{rec}} : N \otimes \mathbb{Z} \mathbb{R} \to M \otimes \mathbb{Z} \mathbb{R}$ with respect to our chosen bases is given by the matrix

\[
\begin{pmatrix}
0 & 1 & -1 & 0 & 0 & -1 & 1 \\
1 & 0 & 1 & -1 & 0 & 0 & -1 \\
1 & 0 & 1 & 0 & -1 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 1 & -1 \\
0 & -1 & 1 & 0 & -1 & 1 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & -1 & 1
\end{pmatrix}
\]

The Newton–Okounkov cone $C(A_{2,5}, v_{G_{rec}})$ is cut out by the tropicalized Marsh-Rietsch superpotential $W = W_1 + \cdots + W_5$ (defined in [MR13]). Expressed in the Plücker coordinates corresponding to $G_{rec}$ it is of form

\[
W_1 = \frac{\bar{p}_{34}}{\bar{p}_{35}} + \frac{\bar{p}_{23}\bar{p}_{45}}{\bar{p}_{35}\bar{p}_{25}} + \frac{\bar{p}_{12}\bar{p}_{45}}{\bar{p}_{15}\bar{p}_{25}}, W_2 = \frac{\bar{p}_{25}}{\bar{p}_{12}}, W_3 = \frac{\bar{p}_{15}}{\bar{p}_{25}} + \frac{\bar{p}_{12}\bar{p}_{35}}{\bar{p}_{15}\bar{p}_{25}}, W_4 = \frac{\bar{p}_{25}}{\bar{p}_{35}} + \frac{\bar{p}_{23}\bar{p}_{45}}{\bar{p}_{34}\bar{p}_{35}}, W_5 = \frac{\bar{p}_{25}}{\bar{p}_{45}}.
\]

The convention to make sense of $\bar{p}^*$ as written above is $\bar{p}_{ij} := z^{f_{ij}}$. Note that the image of $v_{G_{rec}}$ lies inside a hyperplane defined by $e_{45} = 0$. By [RW19] the Newton–Okounkov polytope associated to $v_{G_{rec}} : A_{2,5} \setminus \{0\} \to N$ is

\[
\Delta(A_{2,5}, v_{G_{rec}}) = \{W_1^\text{trop} \geq 0\} \cap \{W_2^\text{trop} \geq -1\} \cap \{W_3^\text{trop} \geq 0\} \cap \{W_4^\text{trop} \geq 0\} \cap \{W_5^\text{trop} \geq 0\}.
\]

For example, $W_5^\text{trop} = f_{35} - f_{45}$ is the normal vector for the inequality $e_{35} - e_{45} \geq 0$. The lattice points of $\Delta(A_{2,5}, v_{G_{rec}})$ can be found on the left in Table 2. The tropicalized Gross-Hacking-Keel-Kontsevich potential $W_{GHKK} = \vartheta_1 + \cdots + \vartheta_5$ cuts out the Newton–Okounkov cone $C(A_{2,5}, g_{G_{rec}})$. Expressed in our seed we have

\[
\vartheta_1 = z^{-e_{15}}(1 + z^{-e_{25}}(1 + z^{-e_{35}})), \vartheta_2 = z^{-e_{12}}, \vartheta_3 = z^{-e_{23}}(1 + z^{-e_{25}}), \vartheta_4 = z^{-e_{34}}(1 + z^{-e_{25}}), \vartheta_5 = z^{-e_{45}}.
\]

The image of $\Delta(A_{2,5}, v_{G_{rec}})$ under $\bar{p}_{G_{rec}}^*$ is

\[
P = \{\vartheta_1^\text{trop} \geq 0\} \cap \{\vartheta_2^\text{trop} \geq -1\} \cap \{\vartheta_3^\text{trop} \geq 0\} \cap \{\vartheta_4^\text{trop} \geq 0\} \cap \{\vartheta_5^\text{trop} \geq 0\}.
\]

It is the Newton–Okounkov polytope for a linearly equivalent divisor. The Newton–Okounkov polytope $\Delta(A_{2,5}, g_{G_{rec}})$ associated to $g_{G_{rec}} : A_{2,5} \setminus \{0\} \to M$ equals $\{W_{GHKK} \leq 0\} \cap \{\sum f_{ij} = 1\}$. It is unimodularly equivalent to $P$. The polytope $\Delta(A_{2,5}, g_{G_{rec}})$ is sent to $P$ by the a linear map $g_{G_{rec}} : M \otimes \mathbb{Z} \mathbb{R} \to M \otimes \mathbb{Z} \mathbb{R}$ defined by

\[
\begin{pmatrix}
1 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 \\
-2 & -2 & -2 & -1 & 0 & -1 & -1 \\
-1 & 0 & 1 & 0 & 0 & 1 & 0 \\
-1 & 0 & 1 & -1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0
\end{pmatrix}.
\]
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Instituto de Matemáticas UNAM Unidad Oaxaca, Antonio de León 2, altos, Col. Centro, Oaxaca de Juárez, CP. 68000, Oaxaca, México

E-mail address: lara@im.unam.mx