Abstract. Let $x$ denote a diffusion process defined on a closed compact manifold. In an earlier article, the author introduced a new approach to constructing admissible vector fields on the associated space of paths, under the assumption of ellipticity of $x$. In this article, this method is extended to yield similar results for degenerate diffusion processes. In particular, these results apply to non-elliptic diffusions satisfying Hörmander’s condition.
1 Introduction

Let $X_1, \ldots, X_n$ and $V$ denote smooth vector fields on a closed compact manifold $M$. We fix a point $o \in M$ and a positive time $T$ and consider the Stratonovich stochastic differential equation (SDE)

$$dx_t = \sum_{i=1}^{n} X_i(x_t) \circ dw_i + V(x_t) dt, \quad t \in [0, T]$$

(1.1)

where $w = (w_1, \ldots, w_n)$ is a Wiener process in $\mathbb{R}^n$. Assume that the vector $V(x)$ lies within the span of $X_1(x), \ldots, X_n(x)$, for all $x \in M$. The solution process $x$ is a random variable taking values in the space of paths

$$C_o(M) = \{ \sigma : [0, T] \mapsto M/\sigma(0) = o \},$$

an infinite-dimensional manifold with tangent bundle consisting of fibers

$$T_oC_o(M) = \{ r : [0, T] \mapsto TM/ r_0 = 0, r_t \in T_{\sigma(t)}M \forall t \in [0, T] \}.$$

The law $\gamma$ of $x$, as a measure on $C_o(M)$, can then be considered as a generalized version of Wiener measure on $C_0(\mathbb{R}^n)$. A major goal in stochastic analysis is to extend the rich body of results that have been developed for the Wiener measure to this non-linear setting.

The Cameron-Martin space, i.e. the space of paths $\{ \sigma : [0, T] \mapsto \mathbb{R}^n, \sigma_0 = 0 \}$ with finite energy

$$\int_0^T ||\dot{\sigma}_t||^2 dt$$

provides a geometrical framework for the Wiener measure and plays a central role in its analysis. Therefore, in addressing the problem raised above, it is natural to seek an analogue of the Cameron-Martin space for the measure $\gamma$. A reasonable candidate for such an analogue is the set of vector fields on the space $C_o(M)$ that admit an “integration by parts” formula of the type described in the following

**Definition 1.1** A vector field $\eta$ on $C_o(M)$ is admissible (with respect to $\gamma$) if there exists an $L^1$ function $\text{Div}(\eta)$ such that the relation\(^2\)

$$\int_{C_o(M)} \eta(\Phi)d\gamma = \int_{C_o(M)} \Phi \text{Div}(\eta)d\gamma$$

(1.2)

holds for a dense class of smooth functions $\Phi$ on $C_o(M)$.

\(^2\)The integrals in (1.2) will usually be written as expectations in the sequel.
The construction of admissible vector fields is an important problem that has been studied by many authors in the last three decades. A breakthrough in the problem was achieved by Driver [6] in 1992, following important partial results by Bismut [5]. Driver proved that parallel translation along $x$ of Cameron-Martin paths in $T_o M$ produces admissible vector fields on $C^0(M)$. A fundamental innovation in [6] is the use of the rotation-invariance of the Wiener process. This property also plays a crucial role in the present work.

The work of Bismut and Driver stimulated a great deal of activity in this area and the problem is still being widely studied (cf., e.g. Driver [7], Hsu [9] and [10], Enchev & Stroock [8], Elworthy, Le Jan & Li [7]). Much of this work has dealt with the elliptic case, where the vector fields $X_1, \ldots, X_n$ in (1.1) are assumed to span $TM$ at all points of $M$. In [1], the author introduced a new approach to the problem of constructing admissible vector fields on path space, again in the elliptic setting. The purpose of the present article, the third in a series of papers on this theme (cf. [1] and [2]), is to extend this approach to the case of degenerate diffusions.

The central object of study in the author’s approach is the Itô map $g : w \mapsto x$ defined by equation (1.1). This is used to lift the problem from the manifold $M$ to $\mathbb{R}^n$, where classical integration by parts theorems can be applied. “Lifting” is defined as follows.

**Definition 1.2** A process $r$ taking values in $\mathbb{R}^n$ is said to be a lift of $\eta$ to $C^0(\mathbb{R}^n)$ (via the Itô map) if the following diagram commutes:

$$
\begin{array}{ccc}
TC^0(\mathbb{R}^n) & \xrightarrow{dg} & TC^0(M) \\
\uparrow r & & \uparrow \eta \\
C^0(\mathbb{R}^n) & \xrightarrow{g} & C^0(M)
\end{array}
$$

The idea in [1] is to simultaneously construct a vector field $\eta$ on $C^0(M)$ and an admissible lift $r$ of $\eta$ to $C^0(\mathbb{R}^n)$. In particular, this requires that $r$ take the form

$$
r_t = \int_0^t A(s)dw_s + \int_0^t B(s)ds
$$

where $A$ and $B$ are continuous adapted processes taking values in $so(n)$ (the space of skew-symmetric $n \times n$ matrices) and $\mathbb{R}^n$ respectively. Processes of this form thus comprise the tangent bundle $TC^0(\mathbb{R}^n)$ in the above diagram.

For test functions $\Phi$ on $C^0(M)$, one then has

$$
E[(\eta \Phi)(x)] = E[r(\Phi \circ g)(w)]
$$

This method had previously been employed by Malliavin in his probabilistic approach to the hypoellipticity problem [12].

Since $g$ is non-differentiable in the classical sense the derivative $dg$ must be interpreted in the extended sense of Malliavin. As this type of regularity is now generally well-understood by stochastic analysts, this point will not be emphasized in the paper (cf. e.g. the monographs [3], [13], [14], [15] for an introduction to the Malliavin calculus).

For test functions we use the set of smooth cylindrical functions on $C^0(M)$.
\[\begin{align*}
= & \mathbb{E}[\Phi \circ g(w) \text{Div}(r)] \\
= & \mathbb{E}[\Phi(x) \mathbb{E}[\text{Div}(r)/x]].
\end{align*}\]
where \(\text{Div}\) denotes the divergence operator in the classical Wiener space. Thus \(\eta\) is admissible with divergence
\[\text{Div}(\eta)(x) = \mathbb{E}[\text{Div}(r)/x].\]

An important consequence of the ellipticity assumption is the fact that every non-anticipating vector field on \(C_0(M)\) can be written in the form
\[\eta_t = \sum_{i=1}^{n} h_i(t) X_i(x_t)\]
where \(h_i, i = 1, \ldots, n\) are real-valued process, adapted to the filtration of \(x\). In the highly non-generic situation where the vector fields \(\{X_i\}\) commute, \(x_t\) becomes a function of \(w_t\) and the problem trivializes. The argument in [1] sets up a duality between the processes \(h\) and \(r\), the lift of \(\eta\), in which (in the non-commuting case) the commutators \([X_i, X_j]\) play an explicit role.

The point of departure for the present work is the a priori selection of an additional collection of vector fields \(\{V_I : I \in \mathcal{I}\}\) on \(M\) such that
\[\{V_I(x) : I \in \mathcal{I}\} \text{ span } T_x M, \ \forall x \in M. \quad (1.3)\]
Thus in the elliptic case \(\{V_I\}\) can be taken to be the set \(\{X_1, \ldots, X_n\}\), whereas in the hypoelliptic case (where the diffusion process (1.1) is degenerate but Hörmander’s condition holds), one can choose \(\{V_I\} = \text{Lie}(X_1, \ldots, X_n)\), the Lie algebra generated by the vector fields \(X_1, \ldots, X_n\). We construct admissible vector fields on \(C_0(M)\) in the form
\[\eta_t = \sum_{I \in \mathcal{I}} h_I(t) V_I(x_t).\]
Somewhat surprisingly, it proves to be possible to trade ellipticity in \(\{X_1, \ldots, X_n\}\) for ellipticity in \(\{V_I\}\). This enables us to establish our results under very general hypotheses.

The layout of the paper is as follows. Section 2 contains background material. The results here are well-known, for the most part. Theorem 2.1 asserts that Riemann integrals of continuous adapted paths have divergence given by an Itô integral, while Theorem 2.2 states that Itô integrals with continuous adapted skew-symmetric integrands are divergence free. The former result follows easily from the Girsanov theorem, the latter from the infinitesimal rotation-invariance of the Wiener measure. Theorem 2.5 gives a relationship between a vector field \(\eta\) along the path \(x\) and the lift of \(\eta\) to the Wiener space. This relationship, expressed in terms of the derivative of the stochastic flow of the SDE (1.1) and the inverse flow, plays a key role in Section 3.

Section 3 contains the main results of the paper. Theorem 3.1 gives the construction of a class of vector fields on \(C_0(M)\) as functions of \(x\), under hypotheses that allow the
SDE (1.1) to be degenerate. The proof of Theorem 3.1 follows the above outline, and is an extension of the argument in [1]. An essential step in the proof is the decomposition of non-tensorial terms in the lift obtained from Theorem 2.5, into tensorial plus skew-symmetric parts.

Theorem 3.2 is a variation on Theorem 3.1 that exhibits a vector field on \( C_o(M) \) with given divergence. In particular, we obtain a class of vector fields with divergence expressed in terms of Ricci curvature. The interest of this result lies in the fact that formulae of this type arise in the work of other authors, e.g Driver [6] and Elworthy, Le Jan & Li [8], where they are obtained using different methods. In Example 3.3, Theorem 3.2 is applied to obtain vector fields with divergence having no extraneous dependence on the Wiener path \( w \). This property is important in applications of the theorem that require a degree of regularity of the divergence such as the study of quasi-invariance (this point is discussed in the remark directly preceding Example 3.3). Theorem 3.4 is an intrinsic formulation of Theorem 3.1 that does not depend on the choice of a basis \( \{V_i\} \). The proof of this result requires the introduction of a tensor that enables us to express the Levi-Civita connection on \( M \) in terms of a connection intrinsic to the diffusion process (1.1). In Theorem 3.7, we apply our theory to gradient systems. As a consequence (Corollary 3.8), we obtain Driver’s result cited above.

In Section 4, we consider the special case where the vector fields \( X_1, \ldots, X_n \) are linearly independent. In this case, the problem under consideration simplifies considerably and our argument simplifies accordingly. We conclude with an example where the SDE (1.1) takes values in the Heisenberg group \( G \). In this case we obtain explicit formulae for a class of admissible vector fields \( \eta \) on \( C_o(G) \).

### 2 Background material

#### 2.A Divergence theorems for Wiener space

We present two such results. These concern the transformation of the Wiener measure under Euclidean motions (the first under translations, the second under rotations).

Let \( \Omega \) denote the measure space for the Wiener process, equipped with the filtration

\[
\mathcal{F}_t = \sigma\{w_s \mid s \leq t\}.
\]

**Theorem 2.1** Let \( h : \Omega \times [0, T] \mapsto \mathbb{R}^n \) be a continuous adapted path. Then the process \( \int_0^T h_s ds \) is admissible (with respect to the Wiener measure) and

\[
\text{Div} \left[ \int_0^T h_s ds \right] = \int_0^T h_s \cdot dw_s
\]

where \(-\cdot\) on the right of the equation denotes the Euclidean inner product.

**Proof.** The result follows easily from the Girsanov theorem, which implies that for \( \Phi \in C_0^\infty (C_0(\mathbb{R}^n)) \) and \( \epsilon \in \mathbb{R} \),

\[
E \left[ \Phi(w + \epsilon \int_0^T h_s ds) \right] = E[\Phi(w)G_\epsilon(w)]
\]

(2.1)
where
\[
G_\epsilon(w) = \epsilon \int_0^T h_s \cdot dw_s - \frac{\epsilon^2}{2} \int_0^T \|h_s\|^2 ds.
\]
Differentiating each side of (2.1) wrt \( \epsilon \) and setting \( \epsilon = 0 \) gives the theorem.

**Theorem 2.2** Let \( A : \Omega \times [0,T] \to \text{so}(n) \) be a continuous adapted process. Then \( \int_0^T Adw \) is admissible and
\[
\text{Div}\left[ \int_0^T Adw \right] = 0.
\]

**Proof.** Define a process \( \theta^\epsilon_t = \exp \epsilon (A_t) \) where \( \exp \) denotes matrix exponentiation. Then \( \theta^\epsilon_t \) is an adapted \( O(n) \)-valued matrix process with \( \theta^0_t = I \). It follows from the infinitesimal rotation-invariance of the Wiener measure that the law of the process \( w^\epsilon = \int_0^T \theta^\epsilon_t dw_t \) is invariant under \( \epsilon \). Hence for \( \Phi \in C_0^\infty(C^0(R^n)) \), we have
\[
E[\Phi(w^\epsilon)] = E[\Phi(w)].
\]
As before, differentiating in \( \epsilon \) and setting \( \epsilon = 0 \) gives the result.

2. B Geometric preliminaries

In this section we introduce some geometric machinery that will be needed in Section 3. We adopt the summation convention throughout the paper: whenever an index in a product (or a bilinear form) is repeated, it will be assumed to be summed on.

First, let \( [g_{jk}] \) be the Riemannian metric defined on \( M \) by
\[
g^{jk} = a^j_I a^k_I
\]
where
\[
V_I = a^I_I \frac{\partial}{\partial x_j}, \quad I \in \mathcal{I}
\]
is the expression of the vector fields in local coordinates (note that the matrix \( [g^{jk}] \) is non-degenerate by the spanning condition (1.3)).

Denote the corresponding inner product by \((.,.)\). It is easy to see that
\[
V = (V, V_I) V_I, \quad \forall V \in TM.
\]
Let \( \tilde{\nabla} \) denote the Levi-Civita covariant derivative corresponding to this metric.

The following constructions were introduced by Elworthy, Le Jan and Li (cf. [8]). Assume the set of vectors \( \{X_1(x), \ldots, X_n(x)\} \) span a subspace \( E_x \) of \( T_xM \) of constant dimension as \( x \) varies in \( M \) and define \( E \) to be the subbundle of \( TM \)
\[
E = \bigcup_{x \in M} E_x.
\]
Then $E$ becomes a Riemannian bundle under the inner product induced on $E$ by the linear maps
\[ X(x) : (h_1, \ldots, h_n) \in \mathbb{R}^n \mapsto h_i X_i(x) \] (2.3)
from the Euclidean space $\mathbb{R}^n$.

There is a metric connection $\nabla$ on $E$ compatible with the metric $\langle \cdot, \cdot \rangle$. This connection (termed the Le Jan-Watanabe connection in [8]), is defined by
\[
\nabla_V Z = X(x) d_V (X^* Z), \quad Z \in \Gamma(E), V \in T_x M,
\]
where $d$ represents the derivative of the function
\[ x \in M \mapsto X(x)^* Z(x) \in \mathbb{R}^n. \]

The corresponding Riemann curvature tensor is defined by
\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,
\]
and the Ricci tensor by
\[
\text{Ric}(X) = R(X, e_i) e_i
\]
where $\{e_i\}$ is an orthonormal basis of $E_x$.

**Lemma 2.3**

(i) $\langle Y, X_i \rangle = Y, \quad \forall Y \in E.$

(ii) $\text{Ric}(Y) = R(Y, X_i) X_i, \quad \forall Y \in TM.$

**Proof.** See [3. Sec. 2]

2.C Flow-related theorems

**Lemma 2.4** Let $g_t : M \mapsto M$ denote the stochastic flow $x_0 \mapsto x_t$ defined by the SDE (1.1). Define $Y_t : T_{x_0} M \mapsto T_{x_t} M$ and $Z_t : T_{x_t} M \mapsto T_{x_0} M$ by $Y_t \equiv dg_t$ and $Z_t \equiv Y_t^{-1}$. Let $B$ denote a vector field on $M$ and $d$ the stochastic time differential. Then
\[
d[Z_t B(x_t)] = Z_t \left( [X_i, B](x_t) \circ dw_i + [V, B](x_t) dt \right).
\]

**Proof.**

Let $D_t$ denote the stochastic covariant differential along the path $x_t$, with respect to the Levi-Civita $\nabla$ connection defined above. Then differentiating with respect to the initial point $o$ in (1.10) gives
\[
D_t Y = \nabla_{Y_t} X_t \circ dw_t + \nabla_{Y_t} V dt.
\]

We then have
\[
D_t Z = D_t (Y_t^{-1})
\]

\[\text{From this point on, we assume that all vector fields appearing in the equations are evaluated at } x_t.\]
= \overline{-Z_t D_t Y Z_t}
= \overline{-Z_t \left( \overline{\nabla}_{\overline{t} \overline{d}_t} X_i \circ dw_i + \overline{\nabla}_{\overline{t} \overline{d}_t} V dt \right)}

where \( Id_t \) denotes the identity map on \( T_{x_t} M \). Thus

\[
d(Z_t B) = D_t Z B + Z_t \overline{\nabla}_{dx_t} B
\]

\[
= \overline{-Z_t \left( \overline{\nabla}_B X_i \circ dw_i + \overline{\nabla}_B V dt \right) + Z_t \left( \overline{\nabla}_X B \circ dw_i + \overline{\nabla}_V B dt \right)}
\]

d[Z_t B(x_t)] = Z_t (\overline{[X, B]}(x_t) \circ dw_i + \overline{[V, B]}(x_t) dt),
\]
as required.

**Theorem 2.5** Let \( r : \Omega \times [0, T] \mapsto \mathbb{R}^n \) be an Itô process. Then the path \( \eta \equiv dg(w)r \) is given by

\[
\eta_t = Y_t \int_0^t Z_s X_i(x_s) \circ dr_i \quad (2.4)
\]

**Proof.** Note that \( \eta \) is a vector field along the path \( x \). Let \( U_s : T_o \mapsto T_{x_s} M \) denote stochastic parallel translation along \( x \).

Differentiating in (1.1) with respect to \( w \) gives the following covariant equation for \( \eta \)

\[
D_t \eta = \overline{\nabla}_{\eta} X_i(x_t) \circ dw_i + X_i(x_t) \circ dr_i + \overline{\nabla}_\eta V(x_t) dt \quad (2.5)
\]

\[
\eta_0 = 0.
\]

We write (2.5) as

\[
d(U_t^{-1} \eta) = U_t^{-1} \overline{\nabla}_{\eta} X_i(x_t) \circ dw_i + U_t^{-1} X_i(x_t) \circ dr_i + U_t^{-1} \overline{\nabla}_{\eta} V(x_t) dt.
\]

Denoting the path \( t \mapsto U_t^{-1} \eta \) by \( y \), we note that the equation for \( y \) has the form

\[
dy = M_j(t) y_t \circ dw_i + M_0(t) y_t + U_t^{-1} X_i(x_t) \circ dr_i \quad (2.6)
\]

where \( M_j(t), j = 1, \ldots, n \) are linear operators on \( T_o M \).

On the other hand, differentiation in (1.1) with respect to the initial point \( o \) gives the following equation for \( \overline{Y}_t \equiv U_t^{-1} Y_t \)

\[
d\overline{Y} = M_i(t) \overline{Y}_t \circ dw_i + M_0(t) \overline{Y}_t dt \quad (2.7)
\]

\[
\overline{Y}_0 = I.
\]

Equation (2.6) can be solved in terms of \( \overline{Y} \) using an operator version of the familiar “integrating factor” method used to solve first order linear ODE’s. Noting, then, that \( \overline{Y}^{-1} \) is an integrating factor for (2.6) and using this to solve for \( y \) gives

\[
y_t = \overline{Y}_t \int_0^t \overline{Y}_s^{-1} U_s^{-1} X_i(x_s) \circ dr_i.
\]
Divergence theorems in path space III

Writing (2.8) in terms of $\eta$ and $Y$, we obtain (2.4).

Remarks
1. Theorem 2.5 gives an alternative proof of the “lifting” equation (3.2) in [1].
2. Suppose $\eta$ in (2.4) has the form $\eta_t = X_i(x_t)h_i(t)$ for an $\mathbb{R}^n$-valued process $h = (h_1, \ldots, h_n)$. Then, writing

$$X = [X_1 \ldots X_n]$$

and solving for $dr$ in (2.4), we have

$$Z_tX(x_t) \circ dr = d\left[Z_tX(x_t)h_t\right].$$

This equation suggests that $r$ can be considered as a type of “covariant derivative” of $h$ along $x$, where the operator $Z_tX(x_t)$ plays the role of backward parallel translation.

3 Divergence theorems for degenerate diffusions

3.A A divergence theorem

Let $X$ be as defined in (2.3). Then the SDE (1.1) may be written

$$dx = X(x_t) \circ d\tilde{w}$$

where

$$d\tilde{w} = dw + X(x_t)^*V(x_t)dt$$

and the adjoint map is defined using the metric $<\ldots, \ldots>$ on $E$ (so $X(x)^*$ is a left inverse for $X(x)$). By the Girsanov theorem, the law $\tilde{\nu}$ of $\tilde{w}$ is equivalent to the law $\nu$ of $w$, with Radon-Nikodym derivative $\frac{d\tilde{\nu}}{d\nu}$ given by

$$G(w) = \exp\left( \int_0^T X(x_t)^*V(x_t) \cdot dw - \frac{1}{2} \int_0^T ||X(x_t)^*V(x_t)||^2 dt \right).$$

Suppose that $r$ is an admissible lift for the vector field $\eta$ under the map $\tilde{g} : \tilde{w} \mapsto x$. Then

$$E[\eta\phi(x)] = E\left[G(w) \cdot r(\Phi \circ \tilde{g})(w)\right]$$

$$= E\left[\Phi \circ \tilde{g}(w)Div(G \cdot r)\right].$$

$$E\left[\Phi \circ \tilde{g}(w)\{G \cdot Div(r) - r(G)\}\right].$$

Thus $\eta$ is admissible.

In view of this discussion, there is no loss in generality in assuming $V = 0$ and we shall assume in the sequel that this is the case.\footnote{It is clear that our argument will work for non-zero drift, however this reduction to the case $V = 0$ simplifies the later calculations.}
We introduce the following tensors \( \{ S_I \} \) and \( \{ T_I \} \) associated to the vector fields \( \{ V_I \} \)
\[
S_I(X) = \nabla_{V_I}X + [X, V_I], \quad X \in E,
\]
and
\[
T_I(X) = S_I(X) - <\nabla_{V_I}X_i, X > X_i, \quad X \in E.
\]

**Theorem 3.1** Let \( r = (r_1, \ldots, r_n) \) be a path in the Cameron-Martin space of \( \mathbb{R}^n \) and define \( \{ h_I : I \in \mathcal{I} \} \) by the linear stochastic system
\[
dh_I = (X_i, V_I) \dot{r}_i dt - (T_J(\circ dx), V_I) h_J
\]
\[
h_I(0) = 0.
\]
Then the vector field \( \eta_t \equiv h_I(x_t) h_I(t), \quad t \in [0, T] \) is admissible on \( C^0(M) \).

**Proof.**

We first note that Theorem 2.5 implies that \( r \) is lift of \( \eta \) if \( r \) satisfies
\[
X_i dr_i = Y_i d[Z_t \eta_t]
\]
Substituting \( \eta_t = h_I(x_t) h_I(t) \) into (3.2) and using Lemma 2.4, we have
\[
X_i dr_i = V_I \circ dh_I + [X_J, V_I] h_J \circ dw_j
\]
Writing the Lie bracket term involving \( X_j \) in terms of the connection \( \nabla \) and using Lemma 2.3 (i) gives
\[
[X_J, V_I] = S_I(X_J) - \nabla_{V_I}X_J
\]
Denote
\[
G^{IJ}_I = <\nabla_{V_I}X_i, X_J> - <\nabla_{V_I}X_J, X_i>
\]
Then we have
\[
[X_J, V_I] = G^{IJ}_I X_i + T_I(X_J)
\]
Substituting this into (3.3) gives
\[
X_i dr_i = V_I \circ dh_I + G^{IJ}_I h_I \circ dw_j + T_I(\circ dx) h_I.
\]
We note that, more generally, a semimartingale path \( \hat{r} \) is a lift of \( h_I V_I \) if equation (3.5) holds with the left hand side replaced by the Stratonovich differential \( X_i \circ \dot{r}_i \).

Suppose now the coefficient functions \( \{ h_I \} \) satisfy the system
\[
X_i dr_i = V_I \circ dh_I + T_I(\circ dx) h_I,
\]
\[
h_I(0) = 0.
\]
Then
\[ X_i [dr_i + G^{ij}_I \circ h_I \circ dw_j] = V_I \circ dh_I + G^{ij}_I X_i h_I \circ dw_j + T_I (X_j) h_I \circ dw_j \]
So if we define
\[ \tilde{r}_i = r_i + \int_0^\cdot G^{ij}_I h_I \circ dw_j. \] (3.7)
then (3.3) holds with \( r \) replaced by \( \tilde{r} \). It follows that \( \tilde{r} \) is a lift of \( \eta \), where
\[ \eta_t = h_I (t) V_I (x_t). \] (3.8)
Furthermore, the skew-symmetry of the functions \( G^{ij}_I \) in the upper indices and Theorem 2.2 imply that the Stratonovich integral in (3.7) can be written as a Riemann integral plus a divergence-free Itô integral. It follows from Theorems 2.1 and 2.2 that \( \tilde{r} \) is admissible. Note also that by (2.2), the processes \( h_I \) defined by (3.1) satisfy equation (3.3).

We have thus shown that \( \tilde{r} \) is an admissible lift to the Wiener space of the vector field \( \eta \) in (3.8). In view of Definition 1.2, we have for any test function \( \Phi \) on \( C_c (M) \)
\[ E[(\eta \Phi)(x)] = E[r(\Phi \circ g)(w)] \]
\[ = E[\Phi \circ g(w) \text{Div}(r)] \]
\[ = E[\Phi(x)E[\text{Div}(r)/x]]. \]
Thus \( \eta \) is admissible and
\[ \text{Div}(\eta)(x) = E[\text{Div}(r)/x]. \]

3.B Computation of the divergence
In order to compute the divergence of the vector field \( \eta \) in Theorem 3.1, it is necessary to convert the Stratonovich integral in (3.7) into Itô form. The relation between the Stratonovich and Itô differentials is formally
\[ G^{ij}_I h_I \circ dw_j = G^{ij}_I h_I dw_j + \frac{1}{2} d(G^{ij}_I h_I)dw_j. \] (3.9)
Write
\[ \alpha^{kij}_I = <\nabla_{X_k} \nabla_{V_I} X_i, X_j > + <\nabla_{V_I} X_i, \nabla_{X_k} X_j > \]
\[ - <\nabla_{X_k} \nabla_{V_I} X_j, X_i > - <\nabla_{V_I} X_j, \nabla_{X_k} X_i > \] (3.10)
and
\[ \beta^k_I = -(T_I (X_k), V_I) h_I. \] (3.11)
Then by (3.1) and (3.4)
\[ dG^{ij}_I = \alpha^{kij}_I dw_k + \{\ldots\} d \]
and
\[ dh_I = \beta^k_I dw_k + \{\ldots\} dt. \]
Substituting these into (3.9) and using the Itô rules
\[dw_i dw_j = \delta_{ij} dt, \quad dw_i dt = 0\]
we see that the Ito-Stratonovich correction term in (3.9) is
\[\frac{1}{2} (\alpha_I^{ik} h_I + G_I^{ik} \beta_I^f) dt. \quad (3.12)\]
Thus (3.7) becomes
\[\tilde{r}_i = r_i + \int_0^T G_I^{ij} h_I dw_j + \frac{1}{2} \int_0^T (\alpha_I^{ik} h_I + G_I^{ik} \beta_I^f) dt.\]
As remarked in the proof of Theorem 3.1, the Itô integral has divergence zero and using Theorem 2.1 we obtain
\[\text{Div}(\tilde{r}) = \int_0^T \left( \tilde{r}_i + \frac{1}{2} (\alpha_I^{ik} h_I + G_I^{ik} \beta_I^f) \right) dw_i.\]
Hence
\[\text{Div}(\eta) = E \left[ \int_0^T \left( \tilde{r}_i + \frac{1}{2} (\alpha_I^{ik} h_I + G_I^{ik} \beta_I^f) \right) dw_i \right]. \quad (3.13)\]
where the \(\alpha\)’s and \(\beta\)’s are given in (3.10) and (3.11).

By adjusting the right hand side in equation (3.1) by the addition of a suitably chosen drift term, the above argument can easily be modified to give

**Theorem 3.2** Let \(\gamma : \Omega \times [0,T] \to \mathbb{R}^n\) be a \(C^1\) adapted process and define \(\{h_I\}\) by
\[h_I(0) = 0 \quad \text{and} \quad dh_I = \left( (d\gamma_i - \frac{1}{2} \alpha_I^{ik} \beta_I^f dt)X_i + (T_J(dx) - \frac{1}{2} \alpha_J^{ik} X_i dt) h_J, V_I \right).\]
Then the vector field \(\eta_I = h_I V_I\) is admissible and for every test function \(\Phi\) on \(C_0(M)\), we have
\[E[(Z\Phi)(x)] = E[\Phi(x) \int_0^T \gamma_i dw_i].\]
The proof of Theorem 3.2 is an easy modification of the argument above, where we replace \(r\) by the path
\[\tilde{r}_i = \gamma_i - \frac{1}{2} \int_0^T (\alpha_I^{ik} h_I + G_I^{ik} \beta_I^f) dt.\]
The essential point is that the correction term (3.12) in the computation of the divergence does not explicitly involve the path \(r\).

**Corollary to Theorem 3.2** Given any path \(r\) in the Cameron-Martin space of \(\mathbb{R}^n\), we can construct an admissible vector field \(\eta\) on \(C_0(M)\) such that
\[E[(\eta\Phi)(x)] = E[\Phi(x) \int_0^T \left( \tilde{r}_i + \frac{1}{2} < \text{Ric}(\eta), X_i > (x_t) \right) dw_i]. \quad (3.15)\]
Remarks
1. Formula (3.12) is similar to those appearing in the work of Driver [6], [7] and Elworthy, Le Jan & Li [8].

2. The appearance of the conditional expectation in (3.14) and (3.15) entails a loss of information concerning the regularity of the function $\text{Div}(\eta)$. This point is crucial in certain applications of the results presented here. For example, the regularity of $\text{Div}(\eta)$ plays a major role in recent work of the author [4] in which the admissibility of $\eta$ is used, in the elliptic setting, to establish quasi-invariance of the law of $x$ under the flow generated by $\eta$ on $C_0(M)$.

With this in mind, we note that by choosing the process $\gamma$ in (3.14) appropriately, we can eliminate the extraneous dependence of the integral on $w$ and thus circumvent this problem. The next example illustrates this point.

Example 3.3 Suppose $B$ is a smooth vector field on $M$, $\rho$ is a deterministic $C^1$ real-valued function, and define

$$\gamma_i(t) = \int_0^t \rho_t(B, X_i)(x_i) dt$$

so

$$\int_0^T \gamma_i dw_i = \int_0^T \rho_t(B, X_i)dw_i.$$

Using the Levi-Civita connection $\tilde{\nabla}$ to write this in Stratonovich form we have

$$\int_0^T \rho_t(B, X_i)dw_i =$$

$$\int_0^T \rho_t(B, X_i) \circ dw_i - \frac{1}{2} \int_0^T \rho_t\left( (\tilde{\nabla} X_i B, X_i) + (B, \tilde{\nabla} X_i X_i) \right) dt =$$

$$\int_0^T \rho_t(B, \circ dx) - \frac{1}{2} \int_0^T \rho_t\left( (\tilde{\nabla} X_i B, X_i) + (B, \tilde{\nabla} X_i X_i) \right) dt \quad (3.16)$$

Since (3.16) is measurable with respect to $x$, (3.14) becomes

$$\text{Div}(\eta) = \int_0^T \rho_t(B, \circ dx) - \frac{1}{2} \int_0^T \rho_t\left( (\tilde{\nabla} X_i B, X_i) + (B, \tilde{\nabla} X_i X_i) \right) dt.$$

In particular, $\text{Div}(\eta)$ is an explicit function of the path $x$.

3.C A basis-free formulation of the argument
Assume now that $M$ is a Riemannian manifold. In this case we can formulate the preceding argument intrinsically, i.e. in a way that does not depend on the choice of a basis $\{V_i\}$.
Let \( \tilde{\nabla} \) denote the Levi-Civita covariant derivative with respect to the Riemannian metric on \( M \) and \( \tilde{D} \) the corresponding covariant stochastic differential. As before, \( \langle \cdot, \cdot \rangle \) and \( \nabla \) will denote the inner product and the connection on the subbundle \( E \) introduced in Section 2.B.

We define
\[
T(X, Y) = \tilde{\nabla}_Y X - \nabla_Y X, \quad Y \in TM, X \in E, \tag{3.17}
\]
noting that \( T \) is tensorial in both arguments.

Let \( r : [0, T] \times \Omega \to \mathbb{R}^n \) be an Itô semimartingale
\[
dr_k(t) = b^{kj}(t)dw_j + c^k(t)dt
\]
where \( b^{kj} \) and \( c^k \) are adapted continuous processes. Then differentiation in equation (1.1) gives the following covariant equation for the path \( \eta \equiv dg(w) \)
\[
\tilde{D}_t \eta = \tilde{\nabla}_\eta X_i \circ dw_i + X_i \circ dr_i
\]
where
\[
G^{ij}_\eta = \langle \nabla X_k (\nabla \eta X_k), X_j \rangle - \langle \nabla X_j (\nabla \eta X_k), X_k \rangle.
\]
Thus
\[
\tilde{D}_t \eta = \langle \nabla X_j, X_i \rangle X_j \circ dw_i + T(X_i, \eta) \circ dw_i + X_i (\circ dr_i + G^{ij}_\eta \circ dw_j).
\] (3.18)

We now have

**Theorem 3.4** Let \( r \) be any Cameron-Martin path in \( \mathbb{R}^n \) and define a vector field \( \eta \) along \( x \) by the covariant SDE
\[
\tilde{D}_t \eta = \left[ \langle \nabla \eta X_j, \cdot \rangle X_j + T(\cdot, \eta) \right](\circ dx) + X_i \dot{r}_i dt \tag{3.19}
\]
\( \eta(0) = 0. \)

Then \( \eta \) is admissible and for any test function \( \Phi \) on \( C_c(M) \),
\[
E[(\eta \Phi)(x)] = E\left[ \Phi(x) \int_0^T (\dot{r}_i - \frac{1}{2} \alpha_i) dw_i \right], \tag{3.20}
\]
where
\[
\alpha_i(t) = \left[ \langle \nabla X_k (\nabla \eta X_k), X_i \rangle + \langle \nabla \eta X_k, \nabla X_k X_i \rangle \right. \\
- \left. \langle \nabla X_k (\nabla \eta X_k), X_k \rangle - \langle \nabla \eta X_i, \nabla X_k X_k \rangle \right](x_t).
\]
Proof. We note that equation (3.18) implies \( \tilde{r} \) is a lift of \( \eta \), where
\[
\tilde{r}_i = r_i - \int_0^1 G^i_j \circ dw_j.
\] (3.21)

Since the functions \( G^i_j \) are skew-symmetric in the indices \( j \) and \( i \), Theorems 2.1 and 2.2 imply that \( \tilde{r} \) is an admissible vector field on the Wiener space. As before, for any test function \( \Phi \) on \( C_0(M) \), we have
\[
E[\mathcal{D}\Phi(x)\eta] = E[\Phi(x)\text{Div}(\tilde{r})].
\]
and it follows that \( \eta \) is admissible as claimed.

As in Section 2.B, the divergence \( \text{Div}(\tilde{r}) \) is computed by converting the Stratonovich integrals in (3.21) into Itô form and applying Theorem 2.1. This yields (3.20) and so completes the proof.

Remark 3.5
It is clear that the argument used to prove Theorem 3.4 is valid in more generality, with the deterministic Cameron-Martin path \( r \) replaced by any \( (x\text{-measurable}) \) random path of the form
\[
r = \int_0^1 A(s)dw_s + \int_0^1 B(s)ds.
\] (3.22)
where \( A : \Omega \times [0,T] \to so(n) \) and \( B : \Omega \times [0,T] \to \mathbb{R}^n \) are continuous adapted processes. In view of Theorems 2.1 and 2.2, it is natural to consider the Wiener space \( C_0(\mathbb{R}^n) \) as a manifold with tangent bundle \( \bigcup_w T_wC_0(\mathbb{R}^n) \), where each fiber \( T_wC_0(\mathbb{R}^n) \) consists of paths of the form (3.22).

For each such path \( r = r(x) \), equation (3.19) produces a vector field \( \eta \) on \( C_0(M) \) that is then lifted to a vector field \( \tilde{r} \) on \( C_0(\mathbb{R}^n) \) by equation (3.21). We summarize these constructions as follows.

Define
\[
H(r) = (r, \eta), \quad r \in TC_0(\mathbb{R}^n)
\]
and let
\[
\pi : TC_0(\mathbb{R}^n) \to C_0(\mathbb{R}^n)
\]
denote the bundle projection.

Then the chain of maps in Theorem 3.4 and its proof is illustrated by the commutative diagram
3.D Gradient systems

Suppose $M$ is an isometrically embedded submanifold of a Euclidean space $\mathbb{R}^N$. Define $X_i = P e_i$, $1 \leq i \leq N$ where $e_1, \ldots, e_N$ is the standard orthonormal basis of $\mathbb{R}^N$ and $P(x)$ is orthogonal projection onto the tangent space $T_x M$. Then the diffusion process $x$ in equation (1.1) is a Brownian motion in $M$.

In this case the connection $\nabla$ coincides with the Levi-Civita connection on $M$ (cf. [8]), hence the tensor $T$ defined in (3.17) is zero. Equation (3.19) thus becomes

$$\tilde{D}_t \eta = \langle \nabla_d X_j, \circ dx \rangle X_j + X_i \dot{r}_i dt \quad (3.23)$$

A further reduction results from

Lemma 3.6 For all $V \in TM$ and $W \in \mathbb{R}^N$

$$\langle \nabla_V X_j, W \rangle X_j = 0.$$

Proof. Using the classical representation of the Levi-Civita connection and denoting the Frechet derivative by $d$, we have

$$\langle \nabla_V X_j, W \rangle X_j = P e_j \langle P dP(V) e_j, W \rangle = P e_j \langle e_j, dP(V) P W \rangle = P dP(V) P W.$$

Differentiating the relation $P^2 = P$ gives

$$dP(V) P + P dP(V) = dP(V).$$

Thus

$$dP(V) P = dP(V) - P dP(V) = Q dP(V)$$

where $Q = I - P$. Then

$$P dP(V) P = P Q dP(V) = 0$$

and the result follows.

In view of Lemma 3.6, equation (3.22) reduces to

$$\tilde{D}_t \eta = X_i \dot{r}_i.$$

Hence

$$\eta_t = U_t \int_0^t U_s^{-1} X_i \dot{r}_i. \quad (3.24)$$

where $U$ denotes parallel translation along $x$. This yields

Theorem 3.7 If $r$ is any (random, $x$-adapted) path such that $\dot{r} \in L^2[0, T]$ then the vector field $\eta$ defined by (3.24) is admissible.

By Nash’s embedding theorem, every finite-dimensional Riemannian manifold can be realized this way.
In particular, let \( h \) be any path in the Cameron-Martin space of \( T_o(M) \) and define
\[
r_i = \int_0^t < U_t \dot{h}_t, X_i > \, dt, \quad i = 1, \ldots, N.
\]
Then the integral in (3.24) becomes \( h_t \) and we obtain the following result of Driver (cf. [6])

**Corollary 3.8** For every path \( h \) in the Cameron-Martin space of \( T_o(M) \), the vector field \( \eta_t \equiv U_t h_t \) is admissible.

Finally, we note that every adapted vector field on \( C_o(M) \) with an admissible lift to the Wiener space is obtained from Theorem 3.4. Denote the process \( \eta \) in Theorem 3.4 by \( \eta' \). Then we have

**Proposition 3.9** Suppose \( \eta \) is an adapted vector field on \( C_o(M) \) such that
\[
\eta = dg(w)r
\]
for some \( r \in T C_0(\mathbb{R}^n) \). Then there exists \( \tilde{r} \in T C_0(\mathbb{R}^n) \) such that \( \eta = \eta' \).  

Proof. This follows immediately from equations (3.18) and (3.19). We define \( \tilde{r} \) by
\[
\tilde{r}_i = r_i + \int_0^t \nabla^j \eta \circ dw_j, \quad i = 1, \ldots, n.
\]

4 Linearly independent diffusion coefficients

In this section we consider the special case where the vectors \( \{X_1(x), \ldots, X_n(x)\} \) are linearly independent at every point \( x \in M \). As we shall see, this implies that the Wiener path \( w \) is a function of the solution \( x \) of the SDE (1.1) i.e.
\[
w = \Theta(x)
\]
where \( \Theta \) is a measurable function on \( C_o(M) \). In this case the following simplified version of the method used in Section 3 produces admissible vector fields on \( C_o(M) \).

Choose \( r \) to be any process of the form
\[
r_t = \int_0^t A(s)dw_s + \int_0^t B(t)dt, \quad t \in [0, T]
\]
(4.1)
where \( A \) and \( B \) are continuous adapted processes with values in \( so(n) \) and \( \mathbb{R}^n \) and define \( \eta \) by (2.4), i.e.
\[
\eta_t = Y_t \int_0^t Z_s X_i(x_s) \circ dr_s,
\]
\footnote{In the elliptic case there is a topological obstruction to this condition, i.e. if \( M \) has non-zero Euler characteristic then it is impossible. However, the condition is reasonable in the non-elliptic case.}
By Theorems 2.1, 2.2 and 2.5, \( r \) is an admissible lift of \( \eta \), hence \( \eta(w) = \eta(\Theta(x)) \) is an admissible vector field on \( C_o(M) \).

We now study how the formulae in Section 3 reduce in the linearly independent case. As before, define \( X(x) : \mathbb{R}^n \rightarrow T_xM \) by

\[
X(x)(h_1, \ldots, h_n) = X_i(x)h_i.
\]

We will need the following result.

**Lemma 4.1** The vectors \( X_1(x), \ldots, X_n(x) \) are linearly independent if and only if

\[
X(x)^*X(x) = I_{\mathbb{R}^n}.
\]

Since Lemma 4.1 is elementary, the proof will be omitted.

Assume now that \( \{X_1, \ldots, X_n\} \) are linearly independent. Then Lemma 4.1 enables us to solve the SDE (1.1) for \( w \) in terms of \( x \) and obtain

\[
dw = X(x_t)^* \circ dx,
\]

thus \( w = \Theta(x) \), as claimed above. We also have

**Corollary to Lemma 4.1** For \( a_i \in C^\infty(M), i = 1, \ldots, n \) and \( V \in TM \)

\[
\nabla_V(a_iX_i) = V(a_i)X_i.
\]

In particular

\[
\nabla_VX_i = 0, \quad i = 1, \ldots, n.
\]

The corollary implies that the functions \( G^{ij}_I \) in (3.4) are all zero. Furthermore, the tensors \( T_I \) in Section 3 take the form

\[
T_I(aX_i) = a[X_i, V_I], \quad i = 1, \ldots, n
\]

for \( a \in C^\infty(M) \). Theorem 3.1 then becomes

**Theorem 4.2** Suppose the process \( r \) is defined as in (4.1) and the functions \( h_I \) are chosen to satisfy

\[
\begin{align*}
\langle dh_I \rangle &= (X_i, V_I) \circ dr_i - \langle [X_i, V_J], V_I \rangle h_J \circ dw_i \\
h_I(0) &= 0.
\end{align*}
\]

Then the vector field \( \eta = h_IV_I \) is admissible and

\[
\text{Div}(\eta) = \int_0^T B_i(t)dw_i.
\]
Example 4.3

Let \( M \) be the Heisenberg group, i.e. the Lie group \( \mathbb{R}^3 \) with group multiplication
\[
(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = \left( a_1 + b_1, a_2 + b_2, a_3 + b_3 + \frac{1}{2}(a_1 b_2 - b_1 a_2) \right).
\]

Let
\[
X_1 = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z},
X_2 = \frac{\partial}{\partial y} - \frac{x}{2} \frac{\partial}{\partial z},
\]
and define \( V_1 = X_1, V_2 = X_2, \) and
\[
V_3 = [V_1, V_2] = \frac{\partial}{\partial z}.
\]

Then
\[
[X_1, V_2] = V_3
\]
\[
[X_2, V_1] = -V_3
\]
\[
[X_i, V_j] = 0, \ i + j \neq 3.
\]

Thus equation (4,2), which we write in the form
\[
V_I \circ dh_I = X_i \circ dr_i - [X_i, V_I] h_I \circ dw_i
\]
becomes
\[
V_1 \circ dh_1 + V_2 \circ dh_2 + V_3 \circ dh_3 = X_1 \circ dr_1 + X_2 \circ dr_2 + V_3 (h_1 \circ dw_2 - h_2 \circ dw_1).
\] (4.3)

Since the vectors \( \{V_1, V_2, V_3\} \) are linearly independent equation (4.3) has a unique solution, given by
\[
h_1 = r_1
\]
\[
h_2 = r_2
\]
\[
h_3 = \int_0 r_1 \circ dw_2 - r_2 \circ dw_1.
\] (4.4)

As point of interest, we note that if \((w_1, w_2)\) is substituted for \((r_1, r_2)\) then the integral in (4.4) becomes the Levy area (this is not, however, an admissible choice of \( r \)).
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