SOME CONVOLUTION INEQUALITIES IN REALIZED HOMOGENEOUS BESOV AND TRIEBEL–LIZORKIN SPACES

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Abstract. Using the realizations, we study some convolution inequalities in the realized homogeneous Besov spaces \( \dot{B}_{p,q}^{s}(\mathbb{R}^{n}) \) and the realized homogeneous Triebel–Lizorkin spaces \( \dot{F}_{p,q}^{s}(\mathbb{R}^{n}) \). We also deduce for the homogeneous Sobolev spaces \( \dot{W}_{p}^{m}(\mathbb{R}^{n}) \) in certain sense.

1. Introduction

We study some properties of the convolution in homogeneous Besov spaces \( \dot{B}_{p,q}^{s}(\mathbb{R}^{n}) \) and homogeneous Triebel–Lizorkin spaces \( \dot{F}_{p,q}^{s}(\mathbb{R}^{n}) \). This type of properties has been studied by Peetre [13] Chapter 8 considering \( \dot{B}_{p,q}^{s}(\mathbb{R}^{n}) \), see also Bourdaud [3]. As these spaces are defined modulo polynomials, since \( \|f\|_{\dot{B}_{p,q}^{s}} = \|f\|_{\dot{F}_{p,q}^{s}} = 0 \) if and only if, \( f \) is a polynomial on \( \mathbb{R}^{n} \), then in our investigation, we will consider realized homogeneous Besov spaces \( \dot{\tilde{B}}_{p,q}^{s}(\mathbb{R}^{n}) \) and realized homogeneous Triebel–Lizorkin spaces \( \dot{\tilde{F}}_{p,q}^{s}(\mathbb{R}^{n}) \), which are defined in the tempered distributions space \( \mathcal{S}'(\mathbb{R}^{n}) \). We will employ the notation \( \dot{\tilde{A}}_{p,q}^{s}(\mathbb{R}^{n}) \) for either \( \dot{B}_{p,q}^{s}(\mathbb{R}^{n}) \) or \( \dot{F}_{p,q}^{s}(\mathbb{R}^{n}) \), the notation \( \dot{\tilde{A}}_{p,q}^{s}(\mathbb{R}^{n}) \) for either \( \dot{\tilde{B}}_{p,q}^{s}(\mathbb{R}^{n}) \) or \( \dot{\tilde{F}}_{p,q}^{s}(\mathbb{R}^{n}) \) and their initials \( \dot{B} \) and \( \dot{F} \), respectively. Also, we will omit the symbol \( \mathbb{R}^{n} \) in notations since all function spaces which occur in this work are defined on \( \mathbb{R}^{n} \). We will also use the following two notations:

- If \( f \in \mathcal{S}' \), \([f]_{P}\) denotes the equivalence class of \( f \) modulo all polynomials on \( \mathbb{R}^{n} \).
- \( \mathcal{E}' \) is the set of distributions with compact support in \( \mathbb{R}^{n} \).

So in the convolution, we essentially prove an estimate in \( \dot{\tilde{A}}_{p,q}^{s}(\mathbb{R}^{n}) \) (see below, Theorem 2.2 and Remark 2.2 in which we explain why we work with the realized spaces) using the convergence in \( \mathcal{S}'_{c} \) (the space of tempered distributions modulo polynomials \( P_{\nu} \)).

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of degree < ν), where for any 4-tuples (n, s, p, q) and throughout this paper the number ν ∈ ℤ₀ is defined by
\[ ν := ([s − n/p] + 1)_+ \text{ if } s − n/p \notin ℤ₀ \text{ or } q > 1 \text{ in } B\text{-case } (p > 1 \text{ in } F\text{-case}) \]
\[ ν := s − n/p \text{ if } s − n/p \in ℤ₀ \text{ and } 0 < q \leq 1 \text{ in } B\text{-case } (0 < p \leq 1 \text{ in } F\text{-case}), \]
(see \[ \| \cdot \| \]), with \([t]\) denotes the greatest integer less than or equal to \(t ∈ ℜ\).

**Notation and plan of the paper.** As usual, \(Γ\) denotes the set of natural numbers, \(Γ₀ = ℤ₀ \cup \{0\}\) the integers and \(Γ\) the real numbers. For \(a ∈ Γ\) we put \(a± := \max(0, a)\). The symbol \(→\) indicates a continuous embedding. For \(0 < p \leq ∞\) we denote by \(∥ ∥_p\) the quasi-norm of \(L_p\). We will use the parameters \(s, p, q\) as \(s, p, q \in Γ\) or \(p < ∞\) in the \(F\)-case along the paper unless otherwise stated. For a function \(θ\) defined on \(Γ^n\), we set \(θₙ := λ^{-n}θ(λ^{-1}(·))\) for all \(λ > 0\) and \(θ(x) := θ(−x)\). The standard norms in the Schwartz space \(S\) are defined by
\[ ζₙ(f) := \sup_{|α| ≤ m} \sup_{x ∈ Γ^n} (1 + |x|)^{|α|} |f^{(α)}(x)|, \quad (m ∈ ℤ₀). \]
For \(f ∈ L₁\),
\[ Ff(x) = ˆf(ξ) := \int_{Γ^n} e^{-ix·ξ} f(x) dx \]
is the Fourier transform and \(F^{-1}f(x) := (2π)^{-n} ˆf(−x)\) is the inverse Fourier transform. The operators \(F\) and \(F^{-1}\) are extended to the whole \(S’\) in the usual way.

For \(k ∈ ℤ₀∪{∞}\), \(P_k\) denotes the set of all polynomials on \(Γ^n\) of degree < \(k\) (in particular \(P₀ = \{0\}, P₁ = \{c\},…, P∞ the set of all polynomials). \(S_k\) will be used for the set of all \(ϕ ∈ S\) such that \(〈u, ϕ〉 = 0 \quad (∀u ∈ P_k\) its topological dual is \(S'_k\). The mapping which takes any \([f]_P\) to the restriction of \(f\) to \(S_k\) is an isomorphism from \(S'/P_k\) onto \(S'_k\).

The constants \(c, c₁, …\) are strictly positive, depend only on the fixed parameters \(n, s, p, q, …\), their values may change from line to line.

This work is organized as follows. In Section \[2\] we state the main results. In Section \[3\] we collect some needed tools. Section \[4\] is devoted to the proofs. In the last section, we give applications and an extension to Sobolev homogeneous spaces.

**2. Statement of the main results**

The Littlewood–Paley decomposition plays a major role here, then once and for all, we fix two functions \(ρ\) and \(γ\), where \(ρ\) is a positive \(C∞\) and radial such that \(0 ≤ ρ ≤ 1\), \(ρ(ξ) = 1\) for \(|ξ| ≤ 1\) and \(ρ(ξ) = 0\) for \(|ξ| ≥ 3/2\), and \(γ(ξ) := ρ(ξ) − ρ(2ξ)\) which is supported by \(1/2 ≤ |ξ| ≤ 3/2\). Then we define the operators \(Q_j\) and \(S_j\) (\(∀j ∈ ℤ\)) by \(Q_j f := γ(2^{-j}(·)) f\) and \(S_j f := ρ(2^{-j}(·)) f\). We also fix a positive and radial function \(γ\) \(∈ D(Γ^n \setminus \{0\})\) such that \(γγ = γ\). We associate \(Q_j\) (\(∀j ∈ ℤ\)) defined by \(Q_j f := γ(2^{-j}(·)) f\). Now, for brevity we set \(ω := ω(p, q)\) such that
\[ 1/ω = 1/p − 1/q \quad \text{if } p ≤ \min(1, q), \quad ω = q’ \text{ if } p > 1 \quad \text{or } q ≤ p ≤ 1, \]
where, here and throughout the paper, \(q’ := q/(q − 1)\) if \(q > 1\) and \(q’ := ∞\) if \(0 < q ≤ 1\). So we have our first result:
THEOREM 2.1. Let $\omega$ be given as in (2.1). We put $r := \min(1, p)$ and $\mu := -s + (n/p - n)_+$. Let $f \in \dot{A}^s_{p, q}$ and $\theta \in \dot{A}^\mu_{r, \omega}$. Then $\sum_{j \in \mathbb{Z}} \dot{Q}_j \theta \ast Q_j f$ converges in $S'_f$, to an element denoted by $\theta \ast f$, such that

\[
\|\theta \ast f\|_p \leq c \|\theta\|_{A^\mu_{r, \omega}} \|f\|_{A^s_{p, q}}
\]

holds, where the constant $c > 0$ is independent of $f$ and $\theta$ (with $c = 1$ if $p \geq 1$).

Remark 2.1. By taking $\lambda$ instead of $\theta$ in the above theorem (recall $\lambda := \lambda^{-n} \theta(\lambda^{-1}(\cdot))$), we obtain a generalization of [13, Theorem 1, p. 156] given in B-case for $p \geq 1$ and $q = \infty$, see Proposition 2.1 below. Note that owing to (2.1), $\omega \geq r$, then $\dot{B}^\mu_{r, r} \hookrightarrow \dot{A}^\mu_{r, \omega}$, in particular $\dot{B}^\mu_{1, 1} \hookrightarrow \dot{A}^\mu_{1, q'}$ for $p \geq 1$ which covers the result given in the previous reference.

Secondly and similarly to (2.2), we wish to give an inequality for the usual convolution. Since in (2.2) taking $\theta \ast f$ instead of $\theta \ast f$ is not true in general (see Subsection 5.1 below), we then pass to $\dot{\Lambda}^s_{p, q}$, where the distributions vanishing at infinity play an important role.

**Definition 2.1.** We say that a distribution $f \in S'$ vanishes at infinity if $\lim_{ \lambda \to 0} f(\lambda^{-1}(\cdot)) = 0$ in $S'$. The set of all such distributions is denoted by $\dot{C}_0$.

Examples of such distributions are:

(i) $f \in \dot{C}_0$ if $f \in L_p (1 \leq p < \infty)$; (ii) $\partial_j f \in \dot{C}_0$ if either $f \in L_\infty$ or $f \in \dot{C}_0$.

Using the notion of the realization, see e.g., [2], we now recall the definition of $\dot{A}^s_{p, q}$ according to [4] or [11]:

The space $\dot{A}^s_{p, q}$ is the set of $f \in S'$ such that $[f]_F \in \dot{A}^s_{p, q}$ and $F(\alpha) \in \dot{C}_0$ ($\forall \alpha$), and one of the following three conditions:

1. There is no supplementary condition if either $s < n/p$, or $s = n/p$ and $0 < q \leq 1$ in $B$-case ($0 < p \leq 1$ in $F$-case); here $\nu = 0$.
2. $f$ is of class $C^{\nu-1}$ and $f^{(\beta)}(0) = 0$ for $|\beta| \leq \nu - 1$, if either $s - n/p \in \mathbb{R}^+ \setminus \mathbb{N}_0$, or $s - n/p \in \mathbb{N}$ and $0 < q \leq 1$ in $B$-case ($0 < p \leq 1$ in $F$-case); here either $\nu = [s - n/p] + 1$ or $\nu = s - n/p$, respectively; here $\nu \geq 1$.
3. $f$ is of class $C^{\nu-1}$ with $f^{(\beta)}(0) = \sum_{j \geq 0} (Q_j f)^{(\beta)}(0)$, $|\beta| \leq \nu - 1$, if $s - n/p \in \mathbb{N}_0$ and $q > 1$ in $B$-case ($p > 1$ in $F$-case); here $\nu = s - n/p + 1 \geq 1$.

Endowed with $\|f\|_{\dot{A}^s_{p, q}} := \|[f]_F\|_{\dot{A}^s_{p, q}}$, it is a quasi-Banach space. Then we have the following statement:

**Theorem 2.2.** Let $r$, $\mu$ and $\omega$ be real numbers given as in Theorem 2.1. Then there exists a constant $c > 0$ (with $c = 1$ if $p \geq 1$) such that

\[
\|\theta \ast f\|_p \leq c \|\theta\|_{\dot{A}^\mu_{r, \omega}} \|[f]_F\|_{\dot{A}^s_{p, q}}
\]

holds, for all $f \in \dot{A}^s_{p, q}$ and all $\theta$ satisfying $[\theta]_p \in \dot{A}^\mu_{r, \omega}$ and either $\theta \in S$ or $\theta \in S'$. 

Remark 2.2. The condition on $f$ guarantees a “good” representative. Indeed, if we replace the assumption $f \in \dot{A}^s_{p, q}$ by only $[f]_F \in \dot{A}^s_{p, q}$, it is possible to fall on a wrong choice of representative which yields a contradiction. For instance,
assume that (2.3) is valid in that case. Let $f$ be a nonzero polynomial on $\mathbb{R}^n$, then 
\[ \|f\|_{p,\alpha} = 0. \]
We take $\theta := \delta$ (Dirac distribution at the origin), it is not difficult to get $|\theta|p \in \dot{B}^0_{p,\infty} (0 < p \leq \infty)$, see e.g., the beginning of Subsection 5.1 then
- if $0 < q \leq p \leq 1$, then $\|\theta|p\|_{\dot{B}^0_{p,q}(\mathbb{R}^n)} = 0$,
- if $1 < p \leq \infty$ and $0 < q \leq 1$, then $\|\theta|p\|_{\dot{B}^0_{p,\infty}} \|f\|_{p,q} = 0$,
however $\theta * f = f$, thus it is impossible to satisfy (2.3) since its left-hand side is $\infty$.

**Remark 2.3.** If $\theta \in \mathcal{S}_\infty$, then Theorem 2.2 holds with only $|f|p \in \dot{A}^*_{p,q}$. Indeed, by Lemma 5.1 (see below) $\theta * f \in \mathcal{S}'$, and if $[f_1]P = [f_2]P = f$, then $f_1 - f_2 = \mathcal{P} \in \mathcal{P}_\infty$ and $\mathcal{P} * \theta = 0$. Recall that $\mathcal{F}(x^\alpha * \theta) = c \theta^{(\alpha)} = 0$, since $\theta^{(\beta)}(0) = 0$ for all $\alpha, \beta \in \mathbb{N}_0$.

In connection with the assertion in [13] pp. 156–159 given for the homogeneous Besov spaces, we have:

**Proposition 2.1.** Let $r, \mu$ and $\omega$ be real numbers given as in Theorem 2.1
(i) There exists a constant $c > 0$ (with $c = 1$ if $p \geq 1$) such that
\[ \lambda^{-r} \left\| \sum_{j \in \mathbb{Z}} \hat{Q}_j \theta_\lambda * Q_j f \right\|_p \leq c \|\theta\|_{\dot{A}^*_\omega} \|f\|_{\dot{A}^*_\omega}, \quad (\forall \lambda > 0) \]
holds, for all $f \in \dot{A}^*_p$ and all $\theta \in \dot{A}^*_\omega$,
(ii) There exists a constant $c > 0$ (with $c = 1$ if $p \geq 1$) such that
\[ \lambda^{-\mu} \left\| \theta_\lambda * f \right\|_p \leq c \|\theta\|_{\dot{A}^*_\omega} \|f\|_{\dot{A}^*_\omega}, \quad (\forall \lambda > 0) \]
holds, for all $f \in \dot{A}^*_p$ and all $\theta$ satisfying $|\theta|p \in \dot{A}^*_\omega$ and either $\theta \in \mathcal{S}$ or $\theta \in \mathcal{E}'$.

In (2.4), (2.5), the particular case $\lambda := 2^k (k \in \mathbb{Z})$ yields an interesting situation when the $\ell_q(\mathbb{Z})$ quasi-norm is taken of these affirmations, according to [13] Remark 1, p. 159) at least for the $B$-case, see also [3] in case $p \geq 1$; $(\ell_q(\mathbb{Z})$ is the set of all sequences $(a_k)_{k \in \mathbb{Z}} \subset \mathbb{C}$ such that $\|(a_k)\|_{\ell_q} := (\sum_{k \in \mathbb{Z}} |a_k|^q)^{1/q} < \infty$). Namely we have the following two results, where we start with the $B$-spaces.

**Theorem 2.3.** Let $r := \min(1, p), t := \min(1, p, q)$ and $\mu := -s + (n/p - n)_+$. Let $f \in \dot{B}^0_{p,q}$ and $\theta \in \dot{B}^\mu_{r,t}$. Then $\sum_{j \in \mathbb{Z}} \hat{Q}_j \theta_{2^k} * Q_j f$ converges in $\mathcal{S}'$ to an element denoted by $\theta_{2^k} \otimes f$ for all $k \in \mathbb{Z}$. Moreover, there exists a constant $c > 0$ (with $c = 1$ if $p \geq 1$) such that
\[ \left( \sum_{k \in \mathbb{Z}} 2^{ks} \|\theta_{2^k} \otimes f\|_p^q \right)^{1/q} \leq c \|\theta\|_{\dot{B}^\mu_{r,t}} \|f\|_{\dot{B}^0_{p,q}} \]
holds, for all such $f$ and $\theta$.

(ii) Let $f \in \dot{B}^0_{p,q}$. Let $\theta$ be such that $|\theta|p \in \dot{B}^\mu_{r,t}$ and either $\theta \in \mathcal{S}$ or $\theta \in \mathcal{E}'$. Then
\[ \left( \sum_{k \in \mathbb{Z}} 2^{ks} \|\theta_{2^k} \otimes f\|_p^q \right)^{1/q} \leq c \|\theta\|_{\dot{B}^\mu_{r,t}} \|f\|_{\dot{B}^0_{p,q}} \]
holds. The positive constant $c$ does not depend on $f$ and $\theta$; (if $p \geq 1$ then $c = 1$).
For the case of the $F$-spaces, we use poised homogeneous spaces of Besov $B_{p,q}^{s,a}$, introduced in e.g., [12], see Section 3 below.

**Theorem 2.4.** Let $a > n/\min(p,q)$ and $t := \min(1,q)$.

(i) Let $f \in \hat{F}_{p,q}^{s,a}$ and $\theta \in B_{1,t}^{-s,a}$. Then $\sum_{j \in \mathbb{Z}} \hat{Q}_j \theta_{2^{-k}} * Q_j f$ converges in $S'_p$ to an element denoted by $\theta_{2^{-k}} \circledast f$ for all $k \in \mathbb{Z}$. Moreover, there exists a constant $c > 0$ such that

$$
\left\| \left( \sum_{k \in \mathbb{Z}} 2^{ksq} |\theta_{2^{-k}} \circledast f|^q \right)^{1/q} \right\|_p \leq c \|\theta\|_{B_{1,t}^{-s,a}} \|f\|_{F_{p,q}}
$$

holds, for all such $f$ and $\theta$.

(ii) Let $f \in \hat{F}_{p,q}^{s,a}$. Let $\theta$ be such that $[\theta]_p \in B_{1,t}^{-s,a}$ and either $\theta \in S'_\alpha$ or $\theta \in \mathcal{E}'$. Then

$$
\left\| \left( \sum_{k \in \mathbb{Z}} 2^{ksq} |\theta_{2^{-k}} * f|^q \right)^{1/q} \right\|_p \leq c \|\theta\| p \|f\|_{F_{p,q}}
$$

holds. The positive constant $c$ does not depend on $f$ and $\theta$.

3. Preliminaries

This section contains preparations, definitions and a characterization for realized homogeneous spaces. We first recall that the operators $Q_j$ and $S_j$ take values in the space of analytical functions of exponential type, see Paley–Wiener theorem, e.g., [15] Theorem 29.2, p. 311] or [16] Remark 2.3.1/2, p. 45]. They are defined on $S'_\infty$ and $S'_s$, respectively, since $Q_j f(x) = 0$ if and only if, $f \in \mathcal{P}_\infty$. For brevity, we make use of the following convention:

If $f \in S'_\infty$ we define $Q_j f := Q_j f_1$ for all $f_1$ such that $[f_1]_p = f$.

We will exploit the following assertions:

**Lemma 3.1.** Let $k \in \mathbb{N}_0 \cup \{\infty\}$. If $f \in S'_k$ ($S_k$, respectively) and $\theta \in S_k$ ($\mathcal{E}'$, respectively), then $\theta * f \in S'_s$ ($S_s$, respectively).

**Proof.** Assume that $f \in S'_k$ and $\theta \in S_k$. For all $\varphi \in \mathcal{D}$ we obtain $\langle \theta * f, \varphi \rangle = \langle f, \hat{\theta} \ast \hat{\varphi} \rangle$. Clearly, $\hat{\theta} \ast \hat{\varphi} \in S_k$ since $F(\hat{\theta} \ast \hat{\varphi}) = (2\pi)^n (F^{-1} \hat{\theta})(\hat{\varphi})$ and $(F^{-1} \hat{\theta})(\alpha) = 0$ for all $|\alpha| < k$. The assertion follows since $\mathcal{D}$ is dense in $\mathcal{S}$.

Now suppose that $f \in S_k$ and $\theta \in \mathcal{E}'$. It suffices to observe that $F(\theta * f) = \hat{\theta} \hat{f}$ and both $\hat{f}(\alpha)(0) = 0$ if $|\alpha| < k$ and $\theta$ is a function of class $C^\infty$ with $|\theta(\xi)| \leq c_{\beta,m} (1 + |\xi|)^m$ for all $m \in \mathbb{N}_0$ and all $\beta \in \mathbb{N}_0^n$, see e.g., [9] Theorem 1.7.5, p. 20].

**Lemma 3.2.** Let $N \in \mathbb{N}_0$. There exist constants $c_1, c_2 > 0$ and a number $m \in \mathbb{N}_0$, such that for all $\varphi \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$ and all $\psi \in \mathcal{D}$, (we put $\tilde{\psi} := \varphi(2^{-j}(\cdot))$ and $\tilde{\psi}_j := \psi(2^{-j}(\cdot))$, it holds:

(i) $\|\varphi_j * f\|_p \leq c_1 2^{-jN} \zeta_m(f)$ for all $f \in S$ and all $j \in \mathbb{N}_0$.

(ii) $\|\varphi_j * f\|_p + \|\psi_j * f\|_p \leq c_2 2^{jN} \zeta_m(f)$ for all $f \in S_N$ and all $j \in \mathbb{Z} \cap \mathbb{N}$. 

Proof. For (i), it suffices in \( \varphi_j \ast f(x) = \int_{\mathbb{R}} f(x-y) \varphi_j(y) dy \) to apply the \( N \)-th degree Taylor formula with integral remainder of \( y \to f(x-y) \) at \( x \). However for (ii), we change the roles between \( f \) and \( \varphi_j \) as \( \int_{\mathbb{R}} f(y) \varphi_j(x-y) dy \), then again the \( N \)-th degree Taylor formula of \( y \to \varphi_j(x-y) \) at \( x \). Similarly for \( \psi_j f \). See \[11\] for more details.

This lemma yields the convergence of the Littlewood–Paley decomposition in the following sense: we have \( f = \sum_{j \in \mathbb{Z}} Q_j f \) in either \( S_{\infty} \) or \( S'_{\infty} \), and \( f = S_k f + \sum_{j > k} Q_j f \) in either \( S \) or \( S' \). On the other hand, we will use make use of the following classical inequalities (see e.g., \[10\] Remark 1, p. 18, Remark 2, p. 28):

**Proposition 3.1.**

(i) If \( 0 < p \leq q < \infty \) and \( R > 0 \), then it holds \( \| f \|_q \leq c R^{n/p - n/q} \| f \|_p \) for all \( f \in L_p \), satisfying supp \( \hat{f} \subseteq \{ \xi \in \mathbb{R}^n : |\xi| \leq R \} \).

(ii) If \( 0 < p \leq 1 \) and \( R > 0 \), then \( \| f \ast g \|_p \leq c R^{n/p - n/q} \| f \|_p \| g \|_p \) for all \( f, g \in S' \) satisfying that supp \( \hat{f} \) and supp \( \hat{g} \) are subsets of \( \{ \xi \in \mathbb{R}^n : |\xi| \leq R \} \). If \( p = 1 \) then \( c = 1 \), that is Young’s inequality.

For more details about \( \dot{B}_{p,q}^s \) and \( \dot{F}_{p,q}^s \), we refer to e.g., \[6,10,13\], however we recall definitions and some properties.

**Definition 3.1.** The homogeneous Besov space \( \dot{B}_{p,q}^s \) and Triebel–Lizorkin space \( \dot{F}_{p,q}^s \) are the sets of \( f \in \mathcal{S}_{\infty} \) such that \( \| f \|_{\dot{B}_{p,q}^s} := \left( \sum_{j \in \mathbb{Z}} 2^{jsq} \| Q_j f \|_p^q \right)^{1/q} < \infty \) and \( \| f \|_{\dot{F}_{p,q}^s} := \left( \sum_{j \in \mathbb{Z}} 2^{jsq} |Q_j f|^q \right)^{1/q} < \infty \), respectively.

\( \dot{A}_{p,q}^s \) are quasi-Banach spaces for the above quasi-seminorms, and do not depend on functions \( \rho \) and \( \gamma \). We have in particular:

- \( \mathcal{S}_{\infty} \hookrightarrow \dot{A}_{p,q}^s \hookrightarrow \mathcal{S}'_{\infty} \),
- the homogeneity property \( \| f \|_{\dot{A}_{p,q}^s} \equiv \lambda^{n/p - s} \| f(\lambda \cdot) \|_{\dot{A}_{p,q}^s} \) (\( \forall \lambda > 0, \forall f \in \dot{A}_{p,q}^s \)) (see \[5\] Proposition 5) and \[6\] Proposition 8),
- an equivalent quasi-seminorm \( \| f \|_{\dot{A}_{p,q}^s} \equiv \sum_{m=0}^\infty \| f^{(m)} \|_{\dot{A}_{p,q}^{-m}} \) (\( \forall m \in \mathbb{N}, \forall f \in \dot{A}_{p,q}^s \)) (see \[5\] Proposition 5) and \[6\] Proposition 8),
- and some embeddings \( \dot{A}_{p,q}^{s_1} \hookrightarrow \dot{A}_{p,q}^{s_2} \) if \( q_1 < q_2 \), \( \dot{B}_{p,q}^{s_1} \hookrightarrow \dot{B}_{p,q}^{s_2} \) if \( s_1 > s_2 \), \( \dot{F}_{p,q}^{s_1} \hookrightarrow \dot{F}_{p,q}^{s_2} \) if \( s_1 > s_2 \).

**Definition 3.2.** Let \( a > 0 \). The poised homogeneous space of Besov \( \dot{B}_{p,q}^{s,a} \) is the set of \( f \in \mathcal{S}'_{\infty} \) such that \( \| f \|_{\dot{B}_{p,q}^{s,a}} := \left( \sum_{j \in \mathbb{Z}} 2^{jsq} \| (1 + 2^j |\cdot|)^a Q_j f \|_p^q \right)^{1/q} < \infty \).

The most properties of \( \dot{B}_{p,q}^s \) are hold for the \( \dot{B}_{p,q}^{s,a} \)-case, e.g., \( \dot{B}_{p,q}^{s,0} = \dot{B}_{p,q}^s \), \( \dot{B}_{p,q}^{s,a} \hookrightarrow \dot{B}_{p,q}^{s_2,a} \) if \( q \leq q_1 \), \( \dot{B}_{p,q}^{s,a} \hookrightarrow \dot{B}_{p,q}^{s_2,a} \) if \( s - n/p = s_2 - n/p_2 \) and \( p < p_1, \ldots \).

In connection with the number \( \nu \) (see Section \[14\]) we have the following characterization, see \[4\] Proposition 4.6:
Proposition 3.2. For all $f \in \mathcal{A}_{p,q}'$, the series $\sum_{j \in \mathbb{Z}} Q_j f$ converges in $\mathcal{S}'_\nu$ to an element denoted by $\sigma_\nu(f)$ which satisfies $f = |\sigma_\nu(f)|_p$ in $\mathcal{S}'_\infty$ and $\partial^\alpha \sigma_\nu(f) \in \mathcal{C}_0$ for all $|\alpha| = \nu$.

By this proposition we have a continuous linear mapping $f \mapsto \sigma_\nu(f)$ from $\mathcal{A}_{p,q}'$ to $\mathcal{S}'_\nu$. In this context we recall the notion of realization.

Definition 3.3. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. Let $E$ be a vector subspace of $\mathcal{S}'_\infty$. A realization of $E$ in $\mathcal{S}'_k$ is a continuous linear mapping $\sigma : E \to \mathcal{S}'_k$ such that $[\sigma(f)]_p = f$ $(\forall f \in E)$. The image set $\sigma(E)$ is called the realized space of $E$.

4. Proofs

We begin by the following assertion, an estimate of Nikol’skij-type representation method.

Proposition 4.1. Let $r$, $\mu$ and $\omega$ be real numbers given as in Theorem 2.1.

Let $a_1, a_2, b_1, b_2$ be positive numbers such that $0 < a_1 < b_1$ and $0 < a_2 < b_2$. Let $(u_j)_{j \in \mathbb{Z}}$ and $(v_j)_{j \in \mathbb{Z}}$ be sequences in $\mathcal{S}'$, such that

- $\hat{a}_j$ and $\hat{v}_j$ have compact supports in the annulus $a_1 2^j \leq |\xi| \leq b_1 2^j$ and $a_2 2^j \leq |\xi| \leq b_2 2^j$, respectively,
- $A := (\sum_{j \in \mathbb{Z}} 2^{j\nu q} |u_j|_\nu^q)^{1/q} < \infty$ and $B := (\sum_{j \in \mathbb{Z}} 2^{j\mu \omega} |v_j|_\nu^\omega)^{1/\omega} < \infty$.

(i) Then $\sum_{j \in \mathbb{Z}} u_j * v_j$ converges in $\mathcal{S}'_\nu$ to an element denoted by $u \odot v$, satisfying

\begin{equation}
\| u \odot v \|_p \leq cAB,
\end{equation}

where the positive constant $c$ depends only on the parameters $n, s, p, q, a_1, a_2, b_1$ and $b_2$, with $c = 1$ if $p \geq 1$.

(ii) If $0 < p < \infty$, then we can replace $A$ and $B$ by $A_1 := \| (\sum_{j \in \mathbb{Z}} 2^{j\nu q} |u_j|_\nu^q)^{1/q} \|_p$ and $B_1 := \| (\sum_{j \in \mathbb{Z}} 2^{j\mu \omega} |v_j|_\nu^{\omega})^{1/\omega} \|_r$, with $A_1 < \infty$ and $B_1 < \infty$, respectively, in the preceding statement.

Proof. Step 1: proof of (i). Substep 1.1: convergence in $\mathcal{S}'_\nu$. We first note that if $b_1 < a_2$ or $b_2 < a_1$ then $u_j * v_j = 0$, so these cases are excluded. We introduce a radial and positive function $\eta \in D(\mathbb{R}^n \setminus \{0\})$ supported in $\min(a_1/2, a_2/2) \leq |\xi| \leq \max(2b_1, 2b_2)$, and $\eta(\xi) = 1$ for $\min(a_1, a_2) \leq |\xi| \leq \max(b_1, b_2)$. We define the operators $\eta_j := \eta(2^{-j}D)$ $(\forall j \in \mathbb{Z})$, i.e., $\eta_j f := \eta(2^{-j}(-)) f$, then we have $u_j * v_j = \eta_j(u_j * v_j)$. Taking $\eta_j$’s properties into account, we can write $\langle u_j * v_j, \varphi \rangle = \langle u_j * v_j, \eta_j \varphi \rangle$ for all $\varphi \in \mathcal{S}'$, and prove

\begin{equation}
\sum_{j \in \mathbb{Z}} \| \langle u_j * v_j, \eta_j \varphi \rangle \| < \infty.
\end{equation}

The case $1 < p \leq \infty$: We begin by $|\langle u_j * v_j, \eta_j \varphi \rangle| \leq \| u_j * v_j \|_p \| \eta_j \varphi \|_{p'}$. By Lemma 3.2 $\| \eta_j \varphi \|_{p'} \leq c_{j,m}(\varphi) \min(2^{-jN}, 2^j\nu)$. Then using Young’s inequality, we get
(4.3) \[ \sum_{j \in \mathbb{Z}} |\langle u_j * v_j, \eta_j \varphi \rangle| \leq c_1 \zeta_m(\varphi) \sum_{j \in \mathbb{Z}} \min(2^{-jN}, 2^{j\nu}) (2^{js}\|u_j\|_p)(2^{-sj}\|v_j\|_1) \]
\[ \leq c_2 AB. \]

The case \( 0 < p < 1 \): We estimate the first term in (4.2) as \( |\langle u_j * v_j, \eta_j \varphi \rangle| \leq \|u_j * v_j\|_1 \|\eta_j \varphi\|_\infty \). Then, \( \|\eta_j \varphi\|_\infty \leq \zeta_m(\varphi) \min(2^{-jN}, 2^{j\nu}) \) (see Lemma 3.2). Also, by definition of \( \nu \) we have \( \nu + n/p - n > 0 \). Then choosing an integer \( N > n/p - n \) and using both Bernstein and convolution inequalities, we obtain
\[ \sum_{j \in \mathbb{Z}} |\langle u_j * v_j, \eta_j \varphi \rangle| \leq c_1 \zeta_m(\varphi) \sum_{j \in \mathbb{Z}} 2^{j\nu.setp}\|u_j\|_p \min(2^{-jN}, 2^{j\nu}) (2^{js}\|u_j\|_p)(2^{-sj}\|v_j\|_1) \]
\[ \leq c_2 AB \sum_{j \in \mathbb{Z}} 2^{j\nu.setp}\|u_j\|_p \min(2^{-jN}, 2^{j\nu}) \leq c_4 AB. \]

**Substep 1.2: proof of (4.1).** First, assume that \( 1 \leq p \leq \infty \). By inequalities of Young and Hölder with exponents \( q \) and \( q' \) for \( q > 1 \), we have
\[ \sum_{j \in \mathbb{Z}} \|u_j * v_j\|_p \leq \sum_{j \in \mathbb{Z}} (2^{js}\|u_j\|_p)(2^{-j\nu}\|v_j\|_1) \leq AB. \]

For \( 0 < q < 1 \), we use the following elementary inequality
\[ \left( \sum a_j \right)^d \leq \sum a_j^d, \quad \forall a_j \geq 0, \quad 0 < d \leq 1, \]
in the second term of (4.3) with \( d := q \), then
\[ \sum_{j \in \mathbb{Z}} (2^{js}\|u_j\|_p)(2^{-j\nu}\|v_j\|_1) \leq \left( \sum_{j \in \mathbb{Z}} (2^{js}\|u_j\|_p)^d \right)^{1/q} \sup_{k \in \mathbb{Z}} (2^{-k\nu}\|v_k\|_1) \leq AB. \]

Clearly that (4.5) and (4.6) describe the behaviour of constant \( c \) in the right-hand side of (4.1), that is \( c = 1 \).

Second, let \( 0 < p < 1 \). By using (4.6) with \( d := p \) and convolution inequality, we get
\[ \sum_{j \in \mathbb{Z}} (2^{js}\|u_j\|_p)(2^{-j\nu}\|v_j\|_1) \leq \left( \sum_{j \in \mathbb{Z}} (2^{js}\|u_j\|_p)^p \right)^{1/p} \leq \left( \sum_{j \in \mathbb{Z}} (2^{js}\|u_j\|_p)^p (2^{j(n/p-n-s)}\|v_j\|_p)^p \right)^{1/p}. \]

- If \( p < q \), we apply Hölder inequality with exponents \( q/p \) and \( \omega/p \), the last term in (4.8) is bounded by \( cAB \).

- If \( q \leq p < 1 \), we use (4.6) with \( d := q/p \), and the last term in (4.8) is bounded by
\[ c_1 \left( \sum_{j \in \mathbb{Z}} (2^{js}\|u_j\|_p)^d (2^{j(n/p-n-s)}\|v_j\|_p)^d \right)^{1/q} \leq c_2 AB. \]

**Step 2: proof of (ii).** **Substep 2.1: convergence in \( S'_\nu \).** We use the notations of Substep 1.1. From (4.3), if \( 1 < p < \infty \), as \( (2^{js}\|u_j\|_p)(2^{-sj}\|v_j\|_1) \leq A_1 B_1 \)
Since \( \theta \in \mathbb{Z} \), then (4.2) holds. Similar if \( 0 < p \leq 1 \), i.e., as in (4.1) since it holds \((2^{js} \|u_j\|_p)(2^{j(n/p-n-s)} \|v_j\|_p) \leq A_1 \) \( \forall j \in \mathbb{Z} \).

Substep 2.2: proof of (4.1) with \( A_1 \) and \( B_1 \). Assume that \( 1 \leq p < \infty \). By H"{o}lder inequality with exponents \( q \) and \( q' \) for \( q' > 1 \), and by Minkowski inequality with respect to \( L_p \), we have

\[
(4.9) \quad \left\| \sum_{j \in \mathbb{Z}} |u_j * v_j| \right\|_p \leq \left\| \int_{\mathbb{R}^n} \left( \sum_{j \in \mathbb{Z}} (2^{js}|u_j(\cdot - y)|)(2^{j} |v_j(y)|) \right) dy \right\|_p \\
\leq \int_{\mathbb{R}^n} \left\| \left( \sum_{j \in \mathbb{Z}} 2^{jsq}|u_j(\cdot - y)|^q \right)^{1/q} \right\|_p \left( \sum_{j \in \mathbb{Z}} 2^{-jsq'} |v_j(y)|^{q'} \right)^{1/q'} dy = A_1 B_1.
\]

For \( 0 < q \leq 1 \), we use (4.6) with \( d := q \) in the first line of (4.9), then

\[
(4.10) \quad \left\| \sum_{j \in \mathbb{Z}} |u_j * v_j| \right\|_p \leq \left\| \int_{\mathbb{R}^n} \left( \sum_{j \in \mathbb{Z}} 2^{jsq}|u_j(\cdot - y)|^q \right)^{1/q} \sup_{k \in \mathbb{Z}} 2^{-ks}|v_k(y)| dy \right\|_p \\
\leq \int_{\mathbb{R}^n} \left\| \left( \sum_{j \in \mathbb{Z}} 2^{jsq}|u_j(\cdot - y)|^q \right)^{1/q} \sup_{k \in \mathbb{Z}} 2^{-ks}|v_k(y)| dy = A_1 B_1.
\]

As above in (4.5) and (4.7), inequalities (4.9) – (4.10) show that the constant \( c \) in the right-hand side of (4.1) is equal to 1.

Now, the case \( 0 < p < 1 \). We first suppose \( p < q \) (here \( q \in [0, \infty) \)). From (4.5) and by H"{o}lder inequality with exponents \( q/p \) and \( \omega/p \), we get

\[
(4.11) \quad \left( \sum_{j \in \mathbb{Z}} \|u_j * v_j\|_p^p \right)^{1/p} \leq c \left( \sum_{j \in \mathbb{Z}} (2^{js} \|u_j\|_p)^q \right)^{1/q} \left( \sum_{j \in \mathbb{Z}} (2^{j(n/p-n-s)} \|v_j\|_p)^\omega \right)^{1/\omega}.
\]

Since \( p \leq \omega \), by Minkowski inequality, the right-hand side of (4.11) is bounded by

\[
c_1 \left( \sum_{j \in \mathbb{Z}} (2^{js} \|u_j\|_p)^q \right)^{1/q} \left( \sum_{j \in \mathbb{Z}} (2^{j(n/p-n-s)} \|v_j\|_p)^\omega \right)^{1/\omega} \leq c_2 A_1 B_1.
\]

Second, suppose that \( q \leq p < 1 \). Again, from (4.5) and since \( \ell_q(\mathbb{Z}) \hookrightarrow \ell_p(\mathbb{Z}) \), we have

\[
\left( \sum_{j \in \mathbb{Z}} \|u_j * v_j\|_p^p \right)^{1/p} \leq c_1 \left( \sum_{j \in \mathbb{Z}} (2^{js} \|u_j\|_p)^p \right)^{1/p} \sup_{k \in \mathbb{Z}} 2^{k(n/p-n-s)} \|v_k\|_p \\
\leq c_1 B_1 \left( \sum_{j \in \mathbb{Z}} (2^{js} \|u_j\|_p)^p \right)^{1/p} \leq c_2 B_1 \left( \sum_{j \in \mathbb{Z}} (2^{js} \|u_j\|_p)^q \right)^{1/q} \leq c_2 A_1 B_1.
\]

Hence the proof is complete. \( \square \)

**Proof of Theorem 2.1.** It suffices to apply Proposition 4.1 with both \( u_j := Q_j f \) and \( v_j := \hat{Q}_j \theta \).

**Proof of Theorem 2.2.** Let \( f \in \dot{A}_{p,q}^s \) and \( \theta \in \mathcal{E}'(\mathcal{S}) \), respectively, be such that \( \| \theta \|_p \in \dot{A}_{p,\omega}^s \).
we have θ ∗ f in S′, implies θ ∗ f ∈ S′ (see [9], p. 21) and [15], Theorem 3.2, p. 317). Also, for all ϕ ∈ S′, we have θ ∗ ϕ ∈ S′, and indeed, in case θ ∈ S′ this follows by Lemma 3.1 (in case θ ∈ S it suffices to apply the Fourier’s properties. Then, by Proposition 3.2, for all ϕ ∈ S′, it holds

\[(4.12) \quad (θ ∗ f, ϕ) = (f, θ ∗ ϕ) = \sum_{j \in \mathbb{Z}} (Q_j f, θ ∗ ϕ) = \sum_{j \in \mathbb{Z}} (Q_j f, Q_j θ, ϕ).
\]

Hence θ ∗ f = θ ⊗ f in S′, and Proposition 3.1 with u_j := Q_j f and v_j := Q_j θ gives

\[\|θ ⊗ f\|_p \leq c\|θ\|_p \|f\|_p \|θ\|_p^* \quad (\forall k \in \mathbb{N}_0).
\]

Step 2. We set g_k := θ ∗ f − θ ∗ (S_{−k} f) for k ∈ \mathbb{N}_0. Then the sequence \( (g_k)_{k \in \mathbb{N}_0} \) has the following properties:

\( g_k \) tends to θ ∗ f pointwise; indeed, assume that p ≥ 1, then by Young and Bernstein inequalities, we have \( \|Q_j f ∗ Q_j θ\|_\infty \leq \|Q_j f\|_p \|Q_j θ\|_p \leq c2^{jn/p} \|Q_j f\|_p \|Q_j θ\|_1 \), then

\[(4.14) \quad |g_k(x) − θ ∗ f(x)| \leq c_1 \sum_{j \leq −k} (2^j \|Q_j f\|_p)(2^{−s})|\tilde{Q}_j θ|_1 2^{jn/p}
\]

then we use the embeddings \( A^s_{p,q} \hookrightarrow B^s_{p,∞} \) and \( \tilde{A}^s_{p,q} \hookrightarrow \tilde{B}^s_{1,∞} \), on the one hand. On the other, suppose 0 < p < 1, and again by Young and Bernstein inequalities, we have \( \|Q_j f ∗ Q_j θ\|_\infty \leq \|Q_j f\|_\infty \|Q_j θ\|_1 \leq c2^{n(2/p−1)} \|Q_j f\|_p \|Q_j θ\|_p \), then

\[(4.15) \quad |g_k(x) − θ ∗ f(x)| \leq c_1 \sum_{j \leq −k} (2^j \|Q_j f\|_p)(2^{n(1/p−s)}j)|\tilde{Q}_j θ|_p 2^{jn/p}
\]

and use both \( A^s_{p,q} \hookrightarrow \tilde{B}^s_{p,∞} \) and \( \tilde{A}^s_{p,q} \hookrightarrow \tilde{B}^s_{1,∞} \). Now, it suffices to take k → +∞ in, both, (4.14) and (4.15), to obtain the desired result.

Finally, by writing (4.13) as \( \int_{\mathbb{R}^n} |g_k(x)|^p dx \leq c\|θ\|_p \|f\|_p \|θ\|_p^* \), if p < ∞, and applying Fatou’s lemma to the sequence \((|g_k|^p|_k\in\mathbb{N}_0 \) recall that \( |g_k|^p \) tends to \( |θ ∗ f|^p \) also pointwise), inequality (2.3) follows. However, if p = ∞, we take an arbitrary \( ε > 0 \), then there exists a number \( k_0 \in \mathbb{N}_0 \) such that

\[|θ ∗ f(x)| \leq |g_k(x) − θ ∗ f(x)| + \|g_k\|_\infty \leq ε + \|g_k\|_\infty \quad (\forall k \geq k_0, \forall x \in \mathbb{R}^n);
\]

but \( \|g_k\|_\infty \leq c\|θ\|_p \|f\|_p \|θ\|_p^* \), for all \( k \in \mathbb{N}_0 \) (see again (4.13)). By arbitrariness of ε, we deduce estimate (2.3). The proof is complete. \[\square\]
Proof of Proposition 2.1. It suffices to apply the homogeneity argument (see Section 9) and Theorems 2.1 and 2.2.

Proof of Theorem 2.3. The convergence of \( \sum_{j \in \mathbb{Z}} Q_j \theta_{2^{-k}} * Q_j f \) in \( S_p' \) can be done as in the proof of Proposition 4.1/Substep 1.1. The details will be omitted.

As above, its limit will be denoted by \( \theta_{2^{-k}} \circ f \).

Step 1: proof of (i). Substep 1.1: the case \( p \geq 1 \). Applying Young's inequality, we obtain \( \| \theta_{2^{-k}} \circ f \|_p \) is bounded by \( \sum_{j \in \mathbb{Z}} \| Q_j f \|_p \| Q_j \theta_{2^{-k}} \|_1 \). By the identity

\[
Q_j \theta_{2^{-k}} = 2^{kn} Q_{j-k} \theta(2^k (-))\text{,}
\]
we have \( 2^{k} \| \theta_{2^{-k}} \circ f \|_p \) is bounded by \( \sum_{j \in \mathbb{Z}} 2^{k} 2^{-k} \| Q_{j-k} f \|_p \| Q_{j-k} \theta \|_1 \); we set \( l := j - k \), then

\[
(4.16) \quad \left( \sum_{k \in \mathbb{Z}} 2^{k}\| \theta_{2^{-k}} \circ f \|_p^q \right)^{1/q} \leq \left( \sum_{k \in \mathbb{Z}} 2^{l(q)} 2^{-l} \| Q_{l+k} f \|_p \| Q_{l+k} \theta \|_1 \right)^{1/q}.
\]

- If \( q > 1 \), then by Minkowski inequality, we estimate (4.16) by

\[
\sum_{k \in \mathbb{Z}} 2^{k}\| \theta_{2^{-k}} \circ f \|_p^q \leq \left( \sum_{j \in \mathbb{Z}} 2^{j} 2^{-j} \| Q_{j-k} f \|_p \| Q_{j-k} \theta \|_1 \right) \leq \| f \|_{B_{p, q}} ^q \| \theta \| \| B_{1, q}^1 \|.
\]

- If \( 0 < q \leq 1 \), then by using (4.6) we again estimate (4.16) by

\[
\left( \sum_{j \in \mathbb{Z}} 2^{-j} \| Q_{j} f \|_p \right)^{1/q} \quad \leq \quad \| f \|_{B_{p, q}} \| \theta \| \| B_{1, q}^1 \|.
\]

Observe that \( c = 1 \) in the right-hand side of above inequalities.

Substep 1.2: the case \( 0 < p < 1 \). By using (4.16), the convolution in \( L_p \) and (4.10), we have \( 2^{k} \| \theta_{2^{-k}} \circ f \|_p \leq c \sum_{j \in \mathbb{Z}} 2^{j} 2^{(n/p-n-q)} \| Q_{j} f \|_p \| Q_{j-k} \theta \|_p ^{1/p} \) is bounded by

\[
(4.18) \quad c \left( \sum_{j \in \mathbb{Z}} 2^{j} 2^{(n/p-n-q)} \| Q_{j} f \|_p \right)^{1/p} \quad \leq \quad \| f \|_{B_{p, q}} \| \theta \| \| B_{p, q}^1 \|.
\]

Now, we estimate \( \left( \sum_{k \in \mathbb{Z}} 2^{k}\| \theta_{2^{-k}} \circ f \|_p^q \right)^{1/q} \) as the following (we separate the estimate with respect to \( q \) into two cases):

- If \( p \leq q \) (here \( q \in ]0, \infty[ \)), by both (4.18) and Minkowski inequality, we have the bound

\[
c_1 \left( \sum_{k \in \mathbb{Z}} 2^{(n/p-n-q)} \| Q_{l+k} f \|_p ^{p/q} \right)^{1/p} \quad \leq \quad c_2 \| f \|_{B_{p, q}} \| \theta \| \| B_{p, q}^1 \|.
\]

- If \( q < p \), by using again both (4.15) and (4.16) with \( d := q/p \), we have the bound

\[
c_1 \left( \sum_{k \in \mathbb{Z}} 2^{(n/p-n-q)} \| Q_{l+k} f \|_p ^{p/q} \right)^{1/p} \quad \leq \quad c_2 \| f \|_{B_{p, q}} \| \theta \| \| B_{p, q}^1 \|.
\]

Therefore, the desired estimates hold.
Step 2: proof of (ii). This can be done as in Step 2 of the proof of Theorem 2.2. We briefly outline it. We fix $k \in \mathbb{Z}$ and introduce the sequence $(g_{l,k})_{l \in \mathbb{N}_0}$ defined by $g_{l,k} := \theta_{2^{-k} \cdot f - \theta_{2^{-k}}} \ast (S_{-l}f)$ for $l \in \mathbb{N}_0$, which satisfies (as above) both

$$\left(\sum_{k \in \mathbb{Z}} 2^{kq}\|g_{l,k}\|_{q}^{p}\right)^{1/q} \leq c\|\mathcal{H}_{p_{r},}\|_{p_{r}}\|f\|_{p_{r}}\|_{p_{r}} \quad (\forall l \in \mathbb{N}_0),$$

and $g_{l,k}$ tends to $\theta_{2^{-k} \cdot f}$ pointwise as $l \to +\infty$ for all $k \in \mathbb{Z}$. Hence it suffices to apply twice the Fatou lemma in the last inequality. □

Proof of Theorem 2.4. Using the Peetre-maximal function

$$Q^{\alpha}_{r,a}f(x) := \sup_{y \in \mathbb{R}^n} (1 + |2^j y|)^{-\alpha}|Q_j f(x-y)| \quad (x \in \mathbb{R}^n, j \in \mathbb{Z}, a > 0),$$

we have at our disposal the following characterization of $F$-spaces (see e.g., [8, p. 45]):

Proposition 4.2. Let $a > n/\min(p,q)$. Then the expression $\|f\|_{F_{p,a}^{q}} := \|\sum_{k \in \mathbb{Z}} 2^{kq}|Q^{\alpha}_{r,a}f|^q\|^{1/q}_{p}$ is an equivalent quasi-seminorm in $F_{p,a}^{q}$.

For the convergence of $\sum_{j \in \mathbb{Z}} \tilde{Q}_j \theta_{2^{-k} \cdot f}$ in $S_{\nu}^{\alpha}$ to $\theta_{2^{-k} \cdot f}$, the same technique used in the proof of Proposition 4.1 will be applied here, but some changes are needed; so we use the same notations. For all $\varphi \in S_{\nu}^{\alpha}$, we have $\|\tilde{Q}_j \theta_{2^{-k} \cdot f}, \eta_j\varphi\|$ is bounded by $c \min(2^{-jN}, 2^{j \nu})\|\tilde{Q}_j \theta_{2^{-k} \cdot f}, \eta_j\varphi\|$ (for all $j \in \mathbb{Z}$), where $\eta_j$ is defined such that $\eta_j(\tilde{Q}_j \theta_{2^{-k} \cdot f}, \varphi) = \tilde{Q}_j \theta_{2^{-k} \cdot f}$ (for all $j \in \mathbb{Z}$). Now, by (4.16) and for a real $a > n/\min(p,q)$, it suffices to observe that $\|\tilde{Q}_j \theta_{2^{-k} \cdot f}\|_{p}$ is bounded by

$$\left\| \int_{\mathbb{R}^n} \tilde{Q}_{j-\kappa} \theta(y) Q_j f(\cdot - 2^{-k} y) dy \right\|_p \leq \|Q^{\alpha}_{r,a}f\|_p \int_{\mathbb{R}^n} |\tilde{Q}_{j-\kappa} \theta(y)|(1 + 2^{j-\kappa} |y|)^{a} dy \leq c 2^{-k} \| f \|_{F_{p,a}^{q}} \| \tilde{\theta}_{r,a} \|_p.$$

Hence the convergence of the series $\sum_{j \in \mathbb{Z}} \tilde{Q}_j \theta_{2^{-k} \cdot f}$ for all $k \in \mathbb{Z}$.

Step 1: proof of (i). We estimate $\left(\sum_{k \in \mathbb{Z}} 2^{kq}\|Q_j f(\cdot - 2^{-k} y)\|_{q}^{1/q}\right)^{1/q}$ as the following: if $q > 1$, using the identity (4.16) and Minkowski inequality (twice), then it is bounded by

$$\int_{\mathbb{R}^n} \left(\sum_{k \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} 2^{kq}|Q_j f(\cdot - 2^{-k} y)\|_{q}^{1/q}\right)\right)^{1/q} dy \leq \int_{\mathbb{R}^n} \sum_{l \in \mathbb{Z}} 2^{-ls}\|\tilde{Q}_l \theta(y)\|_{q} \left(\sum_{k \in \mathbb{Z}} 2^{kq}\|Q_{k+l} f(\cdot - 2^{-k} y)\|_{q}^{1/q}\right)^{1/q} dy \leq \left(\sum_{m \in \mathbb{Z}} 2^{m}\|Q^{\alpha}_{r,a}f\|_{q}^{1/q}\right)^{1/q} \sum_{l \in \mathbb{Z}} 2^{-ls}\|\|_{1};$$

if $0 < q \leq 1$, by (4.16) and (4.6) with $d := q$, then as above the desired term is bounded by

$$\left(\sum_{m \in \mathbb{Z}} 2^{m}\|Q^{\alpha}_{r,a}f\|_{q}^{1/q}\right)^{1/q} \sum_{l \in \mathbb{Z}} 2^{-ls}\|\|_{1};$$
\[
\left( \sum_{k \in \mathbb{Z}} \left\{ \int_{\mathbb{R}^n} 2^{ls} |Q_j f (\cdot - 2^{-k} y) \tilde{Q}_{j-k} \theta(y)|dy \right\}^q \right)^{1/q} \leq \left( \sum_{j \in \mathbb{Z}} 2^{jqs} |Q_j^{\alpha,q} f|^q \sum_{l \in \mathbb{Z}} \|2^{-ls}((1 + |\cdot|)^{\alpha} \tilde{Q}_{l} \theta_{1})\|^q \right)^{1/q}.
\]

Then we calculate the \( L_q \) quasi-norm of \((\sum_{k \in \mathbb{Z}} 2^{ksq} \sum_{j \in \mathbb{Z}} \tilde{Q}_{j} \theta_{2-k} \ast Q_j f)^{1/q}\) and the desired estimate is obtained.

**Step 2: proof of (ii).** Similar to Step 2/proof of Theorem 2.2.

5. **Concluding Remarks**

5.1. **Applications.** 1. For any function \( f \), we define the differences \( \Delta^m_n f := \sum_{j=0}^{m} \left( \int f (\cdot + jh) \right) \delta_{j} \ast f \), where \( \delta_{j} \) is the Dirac distribution at the point \( x := -jh \). As \( Q_k \delta_{-jh} := 2^{-kn} F^{-\gamma} (2^k (\cdot + jh)) \), then \( \| \delta_{-jh} \|_p \in B_{n,q}^{\gamma} \) (n \( < \infty \)). We now see when \( \| \delta_{-jh} \|_p \in B_{r,\omega}^{\gamma}: \)

\[ \begin{align*}
\forall p \geq 1; \quad & \text{if } r := 1 \text{ and } \mu := -s, \quad \text{we have } -s = n - n, \quad \omega = \infty \text{ (i.e., } 0 < q \leq 1). \\
\forall 0 < p < 1; \quad & \text{if } r = p \text{ and } \mu := -s, \quad n/p - n, \quad \omega = \infty \text{ (i.e., } q \leq p < 1). 
\end{align*} \]

Consequently, \( \forall q \in [0,1], \forall p \geq q \) and \( \forall m \in \mathbb{Z} \), by Theorem 2.2 it holds

\[
\| \Delta^m_n f \|_p \leq c \| f \|_{p,q} (\forall f \in B_{p,q}^{0}, \forall h \in \mathbb{R}^n).
\]

This estimate fails with only the assumption \( [f]_p \in B_{p,q}^{0} \). Indeed, let \( f(x) := x^n \), then \( \| f \|_{p,q} = 0 \), while \( \Delta^m_n f(x) = m! h^n \) (\( \forall x, h \in \mathbb{R}^n \)), implies \( \| \Delta^m_n f \|_p = \infty \).

2. Let \( \varphi \) be a \( C^\infty \) function on \( \mathbb{R} \) such that \( \varphi(t) = 1 \) for \( t \leq e^{-3} \) and \( \varphi(t) = 0 \) for \( t \geq e^{-2} \). For \( \alpha > -n \) and \( \beta \geq 0 \), we set \( \theta_{\alpha,\beta}(x) := |x|^\alpha (-\log|x|)^{-\beta} \varphi(|x|), \ x \in \mathbb{R}^n \). This type of functions have been studied in e.g., [7, p. 82]. We have \( \theta_{\alpha,\beta} \in \mathcal{E}' \), indeed, let \( \varphi \in C^\infty \), then using polar coordinates and as \( \sup_{|r| \leq e^{-2}} |\varphi(x)| \leq c < \infty \), we find

\[
\int_{S^{n-1}} \int_0^{e^{-2}} r^{n+\alpha-1}(r e^{-\beta} |\varphi(r)| |\varphi(ry)|) r dr dy \leq c 2^{-\beta} e^{-2(\alpha+n)} \| \varphi \|_\infty.
\]

To continue, we need to introduce inhomogeneous Besov \( B_{p,q}^s \) and Triebel–Lizorkin \( F_{p,q}^s \) spaces \( p < \infty \) in \( F_{p,q}^s \) case. We denote by \( A_{p,q}^s \) for either \( B_{p,q}^s \) or \( F_{p,q}^s \), and use the abbreviations \( B, F \) to indicate them.

**Definition 5.1.** The spaces \( B_{p,q}^s \) and \( F_{p,q}^s \) are the sets of \( f \in S' \) such that

\[
\| f \|_{B_{p,q}^s} := \| S_0 f \|_p + \left( \sum_{j \geq 1} 2^{jsq} |Q_j f|_p^q \right)^{1/q} < \infty,
\]

\[
\| f \|_{F_{p,q}^s} := \| S_0 f \|_p + \left( \sum_{j \geq 1} 2^{jsq} |Q_j f|_p^q \right)^{1/q} < \infty,
\]

respectively.
PROPOSITION 5.1. (See e.g., [17] p. 98) Let $s$ be such that $s > (n/p - n)_+$. Then $\|f\|_p + \|f\|^{p^*}_{A^{p,q}_s}$ is an equivalent quasi-norm in $A^{p,q}_s$.

Assume that $\beta > 0$. By [14] Lemmas 1-2, pp. 44-47 we have, e.g., $\theta_{\alpha,\beta} \in A^{\alpha+n/r}_{r,\omega}$ for $\alpha \neq 0$, $\alpha + n/r > (n/r - n)_+$ and $\beta \omega > 1$ ($\beta > 1$ in $F$-case); also for $\alpha = 0$ and $(\beta + 1) \omega > 1$ ($\beta > 1$ in $F$-case). But in that case, by Proposition 5.1 $A^{\alpha+n/r}_{r,\omega} \to A^{\alpha+n/r}_{r,\omega}$; (at now $r \in [0, \infty]$, $r \neq \infty$ in $F$-case, and $\omega \in [0, \infty]$).

Now, for all $s < 0$, $\alpha := -s - n$, $r := \min(1, p)$ and $\omega$ as in (2.1), it holds $\|\theta_{-s-\eta,\beta} * f\|_p \leq c\|f\|_{\hat{A}^{\mu+q}_s}$ (for all $f \in \hat{A}^{\mu+q}_s$), where $\beta \omega > 1$ ($\beta > 1$ in $F$-case) for $\alpha \neq 0$, and $(\beta + 1) \omega > 1$ ($\beta > 1$ in $F$-case) for $0 < \beta < 1$ and $\alpha = 0$.

REMARK 5.1. We note that other cases on the parameters (e.g., the case $\theta_{\alpha,\beta} \in A^{\mu}_{r,\omega}$ with $\mu > \alpha + n/r$) can be obtained from the properties of $\theta_{\alpha,\beta}$, see [14], etc.

3. Let $\mathcal{X}$ be the characteristic function of the unit cube $[-1, 1]^n$ in $\mathbb{R}^n$. Clearly that $\hat{\mathcal{X}}(\xi) = i^{-n} \prod_{j=1}^n (e^{i\xi_j} - e^{-i\xi_j}) / \xi_j$. Using the development of $\prod_{j=1}^n (e^{i\xi_j} - e^{-i\xi_j})$ (see [1] §8.1/(13), p. 98), and we define a function $\psi \in S_\infty$ by $\psi(\xi) := \gamma(\xi) / (\xi_1 \cdots \xi_n)$, we find

$$Q_k \mathcal{X}(x) = 2^{-kn} (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \xi} \left\{ e^{i((\xi_1 + \cdots + \xi_n) \cdots \xi_n)} - \sum_{j=1}^n e^{i((\xi_1 + \cdots + \xi_n) \cdots \xi_j)} \right\} \psi(2^{-k}\xi) \, d\xi,$$

which implies

$$i^n Q_k \mathcal{X}(x) = \psi(2^k(x_1 + 1, \ldots, x_n + 1)) - \sum_{j=1}^n \psi(2^k(x_1 + 1, \ldots, x_j - 1, \ldots, x_n + 1))$$

$$+ (-1)^2 \sum_{j=1}^{n-1} \sum_{j_1 + j_2 = 2}^n \psi(2^k(x_1 + 1, \ldots, x_j, \ldots, x_j - 1, \ldots, x_n + 1))$$

$$+ (-1)^3 \sum_{j=1}^{n-2} \sum_{j_1 + j_2 + j_3 = 3}^n \psi(2^k(x_1 + 1, \ldots, x_j, x_j - 1, \ldots, x_n + 1))$$

$$+ \cdots + (-1)^{n} \psi(2^k(x_1 + 1, \ldots, x_n - 1)) \quad \text{(there are $2^n$ terms)}.$$

Consequently, we have $\|Q_k \mathcal{X}\|_u \leq 2^{n/\alpha} 2^{-kn/a} \|\psi\|_u$ for all $u \in [0, \infty]$, $a := \min(1, u)$, and all $k \in \mathbb{Z}$. On the other hand, as $\mathcal{X} \in L_1 \cap L_u$, then $S_0 \mathcal{X} = \mathcal{X} - \sum_{k \geq 1} Q_k \mathcal{X}$ implies

$$\|S_0 \mathcal{X}\|_u = \|\mathcal{X}\|_u + \left( \sum_{k \geq 1} \|Q_k \mathcal{X}\|_u \right)^{1/\alpha} \leq \|\mathcal{X}\|_u + c_1 \left( \sum_{k \geq 1} 2^{-kn/a} \right)^{1/\alpha} \leq c_2.$$
All these facts give \( X \in B^t_{u,v} \) with, either \( t < n/u \), or \( t = n/u \) and \( v = \infty \).

We now turn to the application of the above results, looking for \( [X]\) for \(:= -s + 1 - s \). By Proposition 5.3, we have \( [X]\) belongs to \( B^{-s}_{1,q} \) for \( -n < s < 0 \) and \( 0 < q \leq \infty \), belongs to \( B_{1,\infty}^{n/p} \) with \( 0 < q \leq 1 \).

- For \( p > 1 \): here \( \mu := -s \) and \( r := 1 \). By Proposition 5.3, we have \( [X]\) belongs to \( \dot{B}^s_{1,q} \) for \( -n < s < 0 \) and \( 0 < q \leq \min(1,q) \), belongs to \( \dot{B}_{1,\infty}^{n/p} \) with \( 0 < q \leq p \leq 1 \).

We conclude that for, either \( -n < s < 0 \) and \( 0 < q \leq \infty \), or \( s = -n \) and \( 0 < q \leq 1 \), it holds \( \|X*f\|_p \leq c\|f\|_p \|\theta\|_p \) where the constant \( c \) depends only on \( n,s,p,q \).

### 5.2. An extension to homogeneous Sobolev spaces

The homogeneous Sobolev spaces \( \dot{W}^m_p (1 \leq p \leq \infty , m \in \mathbb{N}_0) \) is the set of distributions \( f \) such that \( f^{(\alpha)} \in L_p \) for all \( \alpha = m \) and endowed with the seminorm \( \|f\|_{\dot{W}^m_p} := \sum_{|\alpha| = m} \|f^{(\alpha)}\|_p \). The quotient \( \dot{W}^m_p / \mathcal{P}_m \) is a Banach space in \( S'_m \) for this norm.

**Theorem 5.1.** Let \( 1 \leq p < \infty \) and \( m \in \mathbb{N}_0 \). There exists a constant \( c > 0 \) such that \( \|\theta*F\|_{\dot{W}^m_p} \leq c\|\theta\|_p \|F\|_{\dot{B}^m_p} \) for all \( F \in \dot{W}^m_p \) and all \( \theta \) satisfying \( \|\theta\|_p \in \dot{B}^m_{1,1} \) and either \( \theta \in S \) or \( \theta \in E' \).

**Proof.** We note that \( \theta * f \) is well defined, see Lemma 2.1 or [9, p. 21] or [15, p. 317].

First, we have \( \dot{W}^m_p \hookrightarrow \dot{B}^m_{p,\infty} \). Indeed, since \( \|Q_j g\|_p \leq c\|g\|_p \) \( \forall g \in L_p \), then \( L_p \hookrightarrow \dot{B}^m_{p,\infty} \); now \( \|f\|_{\dot{B}^m_{p,\infty}} = \sum_{|\alpha| = m} \|f^{(\alpha)}\|_{\dot{B}^m_{p,\infty}} \) yields the desired embedding. Let now \( f \in \dot{W}^m_p \). We have \( f^{(\alpha)} \in \hat{C}_0 \) \( \forall |\alpha| = m \), see example (i) just after Definition 2.1. Consequently \( f^{(\alpha)} \in \dot{B}^m_{p,\infty} \) \( \forall |\alpha| = m \), and by Theorem 2.2 it holds \( \|\theta*f^{(\alpha)}\|_p \leq c\|\theta\|_p \|f^{(\alpha)}\|_{\dot{B}^m_{p,\infty}} \). Since \( \theta*f^{(\alpha)} = (\theta*f)^{(\alpha)} \), then the desired estimate follows.

**Remark 5.2.** Similar to Theorem 5.1's proof, \( \|\theta*f\|_{\dot{W}^m_{p+k}} \leq c\|\theta\|_p \|f\|_{\dot{B}^m_{p+k}} \) \( (k = 1, 2, \ldots) \) for all \( f \in \dot{W}^m_p \) and all \( \theta \) satisfying \( \|\theta\|_p \in \dot{B}^m_{1,1} \) and either \( \theta \in S \) or \( \theta \in E' \). On the other hand, Subsection 5.1 can be adapted according to Theorems 2.3 and 5.1.

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