Abstract. Motivated by Homotopy Type Theory, we introduce type-theoretic algebraic weak factorisation systems and show how they give rise to models of Martin-Löf type theory. This is done by showing that the comprehension category associated to a type-theoretic algebraic weak factorisation system satisfies the assumptions necessary to apply the Bénabou-Giraud coherence theorem. We then provide methods for constructing examples of type-theoretic algebraic weak factorisation systems including some based on normal fibrations.

Introduction

Algebraic weak factorisation systems (awfs’s for short), introduced in [14] and studied extensively in [5, 6, 12, 23], are a succinct categorification of weak factorisation systems (wfs’s for short), in which lifting properties of morphisms are replaced by lifting structures satisfying a naturality condition. The aim of this paper is to introduce and study type-theoretic awfs’s, a special kind of awfs’s motivated by homotopy-theoretic models of Martin-Löf’s type theory [1]. Our main results show that type-theoretic awfs allow us to define models of Martin-Löf’s type theory and describe general methods to obtain examples of type-theoretic awfs. As an application, we obtain a uniform account of the groupoid model [17] and of models based on simplicial and cubical sets [7, 11]. In these models, dependent types are interpreted as morphisms having an algebra structure for the monad of the type-theoretic awfs.

Our construction of models of type theory from type-theoretic awfs is obtained in two steps. The first step is to define a non-split comprehension category from a type-theoretic awfs (Proposition 2.2) and show that this comprehension category is equipped with choices of $\Sigma$-types, $\Pi$-types and $\text{Id}$-types (Theorem 2.11) that are pseudo-stable, in the sense of [19], i.e. commuting with pullback up to canonical isomorphism. The second step is to apply a coherence theorem and turn this non-split comprehension category into a split one equipped with strictly stable choices $\Sigma$-types, $\Pi$-types and $\text{Id}$-types (Theorem 1.5), as required to have a genuine model of Martin-Löf’s type theory. The coherence theorem used in the second step, due to Bénabou and Giraud [13], is based on the right 2-adjoint of the inclusion of split Grothendieck fibrations into Grothendieck fibrations (see [27] and [22] for an illuminating 2-categorical perspective). This method was already used by Hofmann [15] in order to remedy the coherence issues affecting the interpretation of Martin-Löf type theory in locally cartesian closed categories [26], thus accounting for $\Sigma$-types, $\Pi$-types and extensional $\text{Id}$-types, and then extended for intensional $\text{Id}$-types by Warren [28].

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The key difference between type-theoretic awfs’s and ordinary wfs’s is that the former allow us to satisfy the pseudo-stability conditions for \( \text{Id} \)-types necessary to apply the Bénabou-Giraud coherence theorem, while the latter do not, even in concrete examples such as simplicial sets \([19, 28]\). For this reason, when considering wfs’s for the semantics of Martin-Löf type theory as in \([1, 20]\), it is necessary to consider another method for splitting Grothendieck fibrations, based on the left 2-adjoint of the inclusion of split Grothendieck fibrations into Grothendieck fibrations \([19]\). The right 2-adjoint has the advantage of being easier to define and work with and of possessing the correct universal property to interpret the syntax of type theory in the usual functorial style \([9]\).

Building on ideas in \([3]\), we will show that the presence of extra algebraic structure in a type-theoretic awfs makes it possible to construct pseudo-stable \( \text{Id} \)-types in the associated comprehension category. Furthermore, developing further the theory of uniform fibrations of \([11]\), we can show that examples of type-theoretic awfs are abundant. The first example we provide of a type-theoretic awfs is in the category of groupoids \( \text{Gpd} \).

The first main result of this paper is Theorem 4.7, which isolates sufficient structure on a category in order to produce a type-theoretic awfs of uniform fibrations. We call a category equipped with such structure a type-theoretic suitable topos (Definition 4.6); we show that, in particular, any Grothendieck topos equipped with an interval object with connections is an example of a type-theoretic suitable topos. The main technical machinery used in the proof of Theorem 4.7 is Proposition 4.5 where we show that given a type-theoretic suitable topos, the resulting awfs of uniform fibrations can be equipped with a stable functorial choice of path objects, which is the structure necessary to produce pseudo-stable identity types. This result fills the gap between the theory developed in \([11]\) and its intended application to the construction of models of Martin-Löf type-theory.

We further advance the theory of \([11]\) by introducing a stronger notion of algebraic fibration which we call a normal uniform fibration. A normal uniform fibration consists of a uniform fibration that enjoys an extra property: the canonical lifts preserve degeneracies. These can be seen as a generalization of normal cloven isofibrations in groupoids, as explained in Remark 5.5. We show that the arguments in \([11]\) can be modified so as to accommodate this new normality property. With this, we are able to show in Theorem 5.2 that any suitable topos admits an awfs of normal uniform fibrations.

Our second main result is Theorem 6.12 which shows that one of the requirements for a type-theoretic suitable topos \( \mathcal{E} \) can be avoided by making use of a normal uniform fibration. The requirement on \( \mathcal{E} \) is that for any object \( X \in \mathcal{E} \), the reflexivity map \( r_X : X \to X^I \) – that maps a point of \( X \) to the constant, or degenerate, path on it – is a member of a distinguished class of monomorphisms \( \mathcal{M} \) whose members are to be thought as generating cofibrations. This assumption clearly holds if we consider \( \mathcal{M} \) to be the class of all monomorphisms. However, if \( \mathcal{M} \) is a presheaf topos, this condition fails if we restrict our attention to the class of decidable monomorphisms (i.e. those whose image is level-wise constructively decidable). As noted in \([21]\), it is important to consider decidable monomorphisms when trying to model univalent universes. This issue is also
relevant to the question of whether the path types and the identity types coincide in the cubical type theory of [7].

Outline of the paper. Section 1 contains a brief review of the interpretation of type dependency using comprehension categories as well as for the coherence theorem for the right adjoint splitting construction. In Section 2 we introduce the notion of type-theoretic algebraic weak factorisation system, and prove that the induced comprehension category supports pseudo-stable $\Pi$, $\Sigma$ and $\text{Id}$ types. We then move on to the construction of examples of type-theoretic awfs. In Section 3 we revisit the Hofmann-Streicher groupoid model. In Section 4 we show how to construct a type-theoretic awfs out of a type-theoretic suitable topos by applying the theory of uniform fibrations of [11]. In the last part of the paper we work with the new notion of normal uniform fibrations; in Section 5 we introduce the awfs of normal uniform fibrations and in Section 6 we show that it is type-theoretic.

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1. Preliminaries

We recall from [18] that a comprehension category over a category $\mathcal{C}$ consists of a strictly commutative diagram of categories:

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\chi} & \mathcal{C} \\
\downarrow^{\rho} & & \downarrow^{\text{cod}} \\
\mathcal{C} & &
\end{array}
$$

such that $\rho : \mathcal{E} \to \mathcal{C}$ is a Grothendieck fibration, $\text{cod} : \mathcal{C} \to \mathcal{C}$ is the codomain functor and $\chi : \mathcal{E} \to \mathcal{C}$ maps Cartesian arrows in $\mathcal{E}$ to pullback squares in $\mathcal{C}$. We will usually refer to a comprehension category by the tuple $(\mathcal{C}, \rho, \chi)$ consisting of the base category, the fibration and the comprehension functor; alternatively we may denote the comprehension category by the pair $(\rho, \chi)$ if the category $\mathcal{C}$ is easily inferable from the context. A cleavage for $(\mathcal{C}, \rho, \chi)$ consists of a choice of lifts for the fibration $\rho$, i.e. for each $u : \Delta \to \Gamma$ in $\mathcal{C}$ and $A$ over $\Gamma$, a Cartesian morphism $u^* : A[u] \to A$ over $u$. We will refer to $A[u]$ as the reindexing of $A$ along $u$. A cleavage is split if it preserves the identities and composition of $\mathcal{C}$. A split comprehension category is a comprehension category equipped with a split cleavage.

A split comprehension category $(\mathcal{C}, \rho, \chi)$ provides a natural setting to interpret the basic judgements and the structural rules of a dependent type theory. Briefly, the interpretation goes as follows. A dependent type $\Gamma \vdash A$ type is interpreted as an object $A$ in the fibre of $\rho$ over an object $\Gamma$. Substitution is interpreted with the use of the split cleavage. Context extension is modelled via the comprehension functor $\chi$: for an
object $A$ in the fibre over $\Gamma$ we obtain a morphism $\chi_A : \Gamma.A \to \Gamma$ whose domain $\Gamma.A$ is the interpretation of the context extension. The operation of weakening is modelled by reindexing an object $A$ along a projection $\chi_B : \Gamma.B \to \Gamma$, to avoid overloading the notation, we will denote by $\chi_{B.A} : \Gamma.B.A \to \Gamma.B$ the comprehension of $A[\chi_B]$.

In order to model additional logical structure (i.e. dependent sums, products and identity types) we will require a split comprehension category to be equipped with additional choices of the desired logical structure. These choices must be made in such a way that they cohere strictly with respect to the canonical split cleavage. If a comprehension category lacks a cleavage we can require a choice to cohere with respect to all Cartesian arrows of the comprehension category in a suitably functorial way; we refer to this as a pseudo-stability condition. We will provide a description of strict and pseudo stability for the case of intensional identity types, leaving to the reader the cases of dependent sums and products, which are analogous.

**Definition 1.1.** A choice of $\text{Id}$-types on a comprehension category $(C, \rho, \chi)$ consists of an operation that assigns to each object $A$ in the fibre over some $\Gamma \in C$, a tuple $(\text{Id}_A, r_A, j_A)$ where:

1. $\text{Id}_A$ is an object in the fibre over $\Gamma.A.A$.
2. $r_A$ is a section of $\text{Id}_A$ over the diagonal morphism $\delta_A$, i.e. a factorisation of $\delta_A$ as shown:

$$
\begin{array}{c}
\Gamma.A.A.
\end{array}
\begin{array}{c}
\text{Id}_A
\end{array}
\begin{array}{c}
\chi_{\text{Id}_A}
\end{array}
\begin{array}{c}
\Gamma.A.
\end{array}
\begin{array}{c}
\delta_A
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
\Gamma.A.A.
\end{array}
\begin{array}{c}
r_A
\end{array}
$$

3. $j_A$ is an operation that takes a pair $(C, t)$ consisting of an object $C$ in the slice over $\Gamma.A.A.\text{Id}_A$ and a section $t$ of $C$ over $r_A$, as in the following solid arrowed diagram:

$$
\begin{array}{c}
\Gamma.A
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
\Gamma.A.A.\text{Id}_A.C
\end{array}
\begin{array}{c}
\chi_C
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
\Gamma.A.A.\text{Id}_A
\end{array}
\begin{array}{c}
r_A
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
\Gamma.A.A.\text{Id}_A
\end{array}
\begin{array}{c}
j_A(C, t)
\end{array}
$$

to a section $j_A(C, t)$ of $C$ (shown as the dotted arrow in the diagram) making both triangles commute.

We will refer to a choice of $\text{Id}$-types by $(\text{Id}, r, j)$.

We observe that a choice of $\text{Id}$-types is a direct translation to the setting of a comprehension category of the formation, introduction, elimination and computation rules for intensional identity types in the syntax.

**Definition 1.2.** A choice of $\text{Id}$-types $(\text{Id}, r, j)$ on a split comprehension category $(C, \rho, \chi)$ is said to be strictly stable if for every morphism $\sigma : \Delta \to \Gamma$ in the base category, and for every object $A$ in the fibre over $\Gamma$, the following condition is satisfied:

$$(\text{Id}_A[\sigma], r_A[\sigma], j_A[\sigma]) = (\text{Id}_{A[\sigma]}, r_{A[\sigma]}, j_{A[\sigma]})$$
Remark 1.3. The notation \((\text{id}_A[\sigma], r_A[\sigma], j_A[\sigma])\) is technically incorrect, since \(\text{id}_A\) has context \(\Gamma.A.A\) and \(\sigma\) has codomain \(\Gamma\). We will allow ourselves this kind of notational abuse, trusting the reader to infer the precise meaning from the context.

It is straightforward to verify that the definition of strict-stability for \(\text{id}\)-types incarnates the syntactic admissible rule establishing the stability of identity types under substitution.

Definition 1.4. A choice of \(\text{id}\)-types \((\text{id}, r, j)\) in a comprehension category is said to be **pseudo-stable** if for any Cartesian arrow \(f: B \rightarrow A\) over a morphism \(\sigma: \Delta \rightarrow \Gamma\) in the base, there is a Cartesian arrow \(\text{id}_f: \text{id}_B \rightarrow \text{id}_A\) over the canonical morphism \(\delta_f: \Delta.B.B \rightarrow \Gamma.A.A\) and such that, the assignment \(f \mapsto \text{id}_f\) is functorial, i.e. \(\text{id}_\text{id}_A = \text{id}_\text{id}_A\) and \(\text{id}_{f eq} = \text{id}_f \circ \text{id}_g\). Moreover, \(\text{id}_f\) coheres appropriately with the additional structure \(r\) and \(j\).

Notice that the definition of pseudo-stability (as opposed to that of strict-stability) makes no reference to an explicit cleavage, and thus can be defined in a general comprehension category and in particular in non-split comprehension categories. This is of interest as non-split comprehension categories are easier to come by in nature. Unfortunately, it is only split comprehension categories equipped with strictly-stable choices of \(\text{id}\)-types that provide a sound interpretation of dependent type theory equipped with intentional identity types. We can remedy this by applying a well-known construction by Giraud and Bénabou [13] that replaces a comprehension category \((C, \rho, \chi)\) with an equivalent split one \((C, \rho^R, \chi^R)\), universally as a right adjoint functor. For this reason we call this construction the **right adjoint splitting**.

The following important coherence result establishes the connection between pseudo-stability in a comprehension category and strict-stability on its right adjoint splitting. The proof for dependent sums and products can be found in [15, Theorem 2] and although the result was stated in the context of locally cartesian closed categories (lccc) and categories with attributes (CW\(\mathcal{A}\)), it can be adapted to the language of comprehension categories. The coherence theorem for the case of identity types is proved in [28, Theorem 2.48].

Theorem 1.5 (Coherence Theorem). Let \((C, \rho, \chi)\) be a comprehension category equipped with pseudo-stable choices of \(\Sigma, \Pi\) and \(\text{id}\) types. Then the right adjoint splitting \((C, \rho^R, \chi^R)\) is equipped with strictly-stable choices of \(\Sigma, \Pi\) and \(\text{id}\) and the counit \(\varepsilon_\rho : (C, \rho^R, \chi^R) \rightarrow (C, \rho, \chi)\) is an equivalence of comprehension categories that preserves each choice of logical structure strictly. \(\square\)

Remark 1.6. The coherence theorem for comprehension categories can be understood in two complementary manners. The most immediate one is that we can think of the right adjoint splitting construction as a mechanism to build models of dependent type theory with dependent sums, products and identity types in split comprehension categories form the weaker notion of pseudo-stability in non-split comprehension categories. Alternatively, the theorem is telling us that we can work inside a non-split comprehension category as if we were working in a model of type theory with dependent sums, products and identity types, provided that the comprehension category is equipped with pseudo-stable choices of the corresponding logical structure.
We proceed to recall some of the basic theory on algebraic weak factorisation systems and on orthogonal categories of arrows. A **functorial factorisation** \((Q, L, R)\) on a category \(\mathcal{C}\) consists of an operation that assigns to each arrow \(f : X \to Y\) a factorisation \(X \xrightarrow{Lf} Qf \xrightarrow{Rf} Y\) functorial on \(f\) in the obvious way. There are induced endofunctors \(L, R : \mathcal{C} \to \mathcal{C}^\to\) which are canonically copointed and pointed respectively; that is, there is a counit map \(\epsilon : L \to 1\) and a unit map \(\eta : 1 \to R\).

We denote the category of \((R, \eta)\)-algebras as \(R\text{-Map}\) and the category of \((L, \epsilon)\)-coalgebras by \(L\text{-Map}\). We refer to the objects of \(R\text{-Map}\) by \(R\)-maps and to the objects of \(L\text{-Map}\) as \(L\)-maps. Notice that there are faithful (but not full) forgetful functors down to the arrow category:

\[
L\text{-Map} \to \mathcal{C}^\to \quad \text{and} \quad R\text{-Map} \to \mathcal{C}^\to
\]

**Remark 1.7.** We can recognise \(R\)-maps as pairs \((f, s)\) where \(f : X \to Y\) is a morphism and \(s : Qf \to X\) is a filler for the square defining the counit \(\epsilon_f\). Dually \(L\)-maps correspond to pairs \((g, \lambda)\) where \(g : A \to B\) is a morphism and \(\lambda : B \to Qg\) is a filler for the square defining the unit \(\eta_f\).

Let \((g, \lambda) : A \to B\) be an \(L\)-map, \((f, s) : X \to Y\) an \(R\)-map and \((h, k) : g \to f\) a morphism in the arrow category between them. Out of this data, we can construct a canonical filler for the square \((h, k)\); this is given by \(\lambda := s \circ Q(h, k) \circ \lambda : B \to X\) where \(Q(h, k) : Qg \to Qf\) is the map obtained from the functorial factorisation applied to \((h, k)\). These canonical fillers satisfy naturality conditions with respect to morphism of \(L\)-maps and \(R\)-maps.

**Definition 1.8.** An **algebraic weak factorisation system** or awfs on a category \(\mathcal{C}\) consists of the following data:

1. a functorial factorisation \((Q, L, R)\) on \(\mathcal{C}\).
2. an extension of the pointed endofunctor \((R, \eta)\) to a monad \((R, \eta, \mu)\).
3. an extension of the copointed endofunctor \((L, \epsilon)\) to a comonad \((L, \epsilon, \delta)\).
4. there is a canonical map \(\Delta : LR \to RL\) defined using the monad and comonad structure. We require this map to be a distributive law.

We will refer to the awfs just as \((L, R)\).

**Remark 1.9.** Item 4 of Definition 1.8 can be considered as a technical requirement and can be safely ignored for our proposes.

Notice that we can now talk about algebras for the monad \((R, \eta, \mu)\). We denote \(R\text{-Alg}\) the category of such algebras. Dually, we denote by \(L\text{-Coalg}\) the category of coalgebras of \((L, \epsilon, \delta)\). We observe that there are full and faithful functors \(R\text{-Alg} \hookrightarrow R\text{-Map}\) and \(L\text{-Coalg} \hookrightarrow L\text{-Map}\). We refer to the objects of \(R\text{-Alg}\) and \(L\text{-Coalg}\) respectively as \(R\)-algebras and \(L\)-coalgebras.

We have that the category \(R\text{-Alg}\) (and also \(R\text{-Map}\)) is closed under ‘vertical’ composition: that is if \((f, s) : X \to Y\) and \((f', s') : Y \to Z\) are \(R\)-algebras then there is a canonical \(R\)-algebra structure \(s' \cdot s\) on the composite \(f' \cdot f\). In fact, finding such a vertical composition operation provides a complete characterisations of the awfs [2 Theorem 4.15].

Let us consider a \(R\)-algebra \((f, s)\) and a pullback square \((h, k) : f' \to f\) then, there exists a unique \(R\)-algebra structure \(s'\) on \(f'\) making \((h, k)\) a morphism of \(R\)-algebras. The same result holds for \(R\)-maps.
We recall some notions regarding categories of arrows and of orthogonality in the algebraic setting. By a category of arrows over $C$ we mean a functor $u: J \to C$ where $J$ is a (possibly small) category. A right $J$-map consists of a pair $(f, \theta)$ where $f: X \to Y$ is an arrow of $C$ and $\theta$ is a right lifting operation against $J$: that is, $\theta$ assigns to each commutative square of the form $(l, m): u_i \to f$, with $i \in J$, a filler $\theta(i)$. These fillers, in addition, are compatible with the arrows in $J$ in the obvious way.

Given a pair of right $J$-maps $(f, \theta)$ and $(f', \theta')$, a right $J$-map morphism consists of a square $(\alpha, \beta): f \to f'$ such that for every $i \in J$, the triangle created by the corresponding choices of diagonal fillers commute.

Let us consider a category of arrow $u: J \to C$, and from this we can define a new category $J^{\square}$ consisting of right $J$-maps $(f, \theta)$ together with the corresponding morphisms; moreover there is a functor $u^{\square}: J^{\square} \to C$ forgetting the lifting structure. It can be shown that this operation defines a contravariant functor denoted by $(-)^{\square}$.

In a completely analogous manner, we can define the concepts of left $J$-map and left $J$-map morphism, the dual functor $/\square(-)$. It turns out that this forms an adjunction, which generalises the classical Galois connection between orthogonal classes of maps:

$$
\text{CAT}/C \to \downarrow \downarrow \uparrow \uparrow (\text{CAT}/C)^{\text{op}}
$$

we refer to this as the orthogonality adjunction.

In the following proposition, we exhibit the relation between awfs and orthogonal categories of arrows. A proof can be found in [5, Lemma 1].

**Proposition 1.10.** Let $(L, R)$ be an awfs on $C$. Then, there are lifting functors over $C$ as shown in the following commutative diagram:

$$
\text{R-Alg} \xrightarrow{\text{lift}} (L-\text{Coalg})^{\square} \\
\text{R-Map} \xrightarrow{\text{lift}} (L-\text{Map})^{\square}
$$

All functors are full and faithful and only the diagonal one is an equivalence. There is a functor $(L-\text{Map})^{\square} \to \text{R-Map}$ but it is not, in general, an equivalence.

We say that an AWFS $(L, R)$ is algebraically-free on a category of arrows $J$ if there is a functor $\eta: J \to L-\text{Coalg}$ over $C$, such that the composition

$$
\text{R-Alg} \xrightarrow{\text{lift}} (L-\text{Coalg})^{\square} \xrightarrow{\eta^{\square}} (J)^{\square}
$$

is an isomorphism of categories. This is a categorification of the well-known notion of cofibrant generation for normal weak factorisation systems. Arguably the most important result regarding algebraically-free awfs is the algebraic version of Quillen’s small object argument due to Garner [12, Theorem 4.4].

The following result regarding algebraically-free awfs is implicit in the literature (cf. [11, Theorem 6.9] for example).
Proposition 1.11. If \((L, R)\) is algebraically-free on some category of arrows \(J\), then there are maps back-and-forth over the identity on \(C\) as shown:

\[ \text{R-Map} \xleftarrow{\sim} \text{R-Alg} \]

This proposition shows that when working with an algebraically-free awfs \((L, R)\) (as will be the case in Section \(\S\)), any construction made using R-maps can be functorially transported to a construction using R-algebras, and vice-versa.

2. Type Theoretic Algebraic Weak Factorisation Systems

In this section we introduce the notion of a type-theoretic algebraic weak factorisation system. We show how a type-theoretic awfs induces a comprehension category structure equipped with pseudo-stable choices of the relevant logical structure of a Martin-Löf type system. We begin by making the connection between comprehension categories and awfs.

Lemma 2.1. Let \((L, R)\) be an awfs over \(C\). The functor \(\text{R-Map} \to C\) mapping an R-map \((f, s)\) to \(\text{cod}(f)\) is a Grothendieck fibration. Moreover, the Cartesian arrows are the morphisms of R-maps whose underlying square is a pullback square.

Proposition 2.2. For a given awfs \((L, R)\) on a category \(C\), the following commutative diagram is a comprehension category:

\[
\begin{array}{ccc}
\text{R-Map} & \xrightarrow{U} & C \\
\downarrow & & \downarrow \text{cod} \\
\circ & & \circ
\end{array}
\]

where the horizontal functor is the forgetful one. We will call this the comprehension category induced by \((L, R)\).

We have stated Lemma 2.1 and Proposition 2.2 with respect to the category \(\text{R-Map}\) however, the same results hold for the category \(\text{R-Alg}\).

We proceed to investigate when the comprehension category induced by an awfs is equipped with additional logical structure.

Proposition 2.3. Let \((L, R)\) be an awfs on \(C\). Then the comprehension category induced by \((L, R)\) is equipped with a pseudo-stable choice of \(\Sigma\)-types.

Proof. Recall that for a given morphism \(f : X \to \Gamma\), the pullback functor along \(f\) has a left adjoint \(\Sigma_f : C/X \to C/\Gamma\), which is given explicitly by composition. Let us assume that \((f, s) : X \to \Gamma\) is an R-map, then the fact that R-maps can be vertically composed in a coherent way implies that this adjunction can be lifted to the category \(\text{R-Map}\) as follows:

\[
\begin{array}{ccc}
\text{R-Map}/\Gamma & \xleftarrow{\sim} & \text{R-Map}/X \\
\downarrow & & \downarrow \text{cod} \\
\circ & & \circ
\end{array}
\]

where the slice category \(\text{R-Map}/\Gamma\) (and analogously \(\text{R-Map}/X\)) has as objects R-maps of the form \(f : X \to \Gamma\) and as arrows morphisms of R-maps over the identity on \(\Gamma\).
This implies that an analogous argument to that of [15, Theorem 2] for the construction of $\Sigma$-types in CwA’s can be carried out in the setting of comprehension categories in order to construct a pseudo-stable choice of $\Sigma$-types. □

The case of dependent products is more complicated. We will require the following property on an awfs.

**Definition 2.4.** An awfs $(L, R)$ on a category $C$ satisfies the **exponentiability** property if the following condition is satisfied: for any two maps $g : Z \to Y$ and $f : Y \to X$ in the image of $R-\text{Map} \to C^{\to}$, the categorical exponential $\Pi_f g \in C/X$ exists. This map enjoys the following universal property: for any $(h : W \to X) \in C/X$, there is a bijective correspondence, natural on $h$, between maps $h \to \Pi_f g$ in $C/X$ and maps $f^*h \to g$ in $C/Y$.

Any locally cartesian closed category will satisfy the exponentiability property with respect to any given awfs $(L, R)$.

We need a way to coherently lift an exponential morphism $\Pi_f g \in C/X$ to the slice $R-\text{Map}/X$. For this reason we introduce the following fundamental concept.

**Definition 2.5.** Let $(L, R)$ be an awfs on a category $C$. A **functorial Frobenius structure** is given by a lift of the pullback functor as shown:

\[
\begin{array}{ccc}
R-\text{Map} \times_{C} L-\text{Map} & \xrightarrow{PB} & L-\text{Map} \\
\downarrow & & \downarrow \\
C^{\to} \times_{C} C^{\to} & \xrightarrow{PB} & C^{\to}
\end{array}
\]

where $PB(f, g)$ denotes the pullback of $g$ along $f$.

With this definitions in place, we can now state and prove the following proposition.

**Proposition 2.6.** Consider an awfs $(L, R)$ on $C$ satisfying the exponentiability property and equipped with a functorial Frobenius structure. Then the comprehension category induced by $(L, R)$ has a pseudo-stable choice of $\Pi$-types.

**Proof.** Consider a $R$-map $(f, s) : X \to \Gamma$. By [11, Proposition 6.5 and Proposition 6.7] we obtain a lift of the push-forward functor along $f$ as shown:

\[\Pi_f : (L-\text{Map})^2/X \to (L-\text{Map})^2/\Gamma\]

We can apply the back-and-forth functors $R-\text{Map} \leftrightarrow L-\text{Map}^2$ of Proposition [11.10] to obtain the following lift:

\[\Pi_f : R-\text{Map}/X \to R-\text{Map}/\Gamma\]

It also follows from [11, Proposition 6.7] that for any $R$-maps $g, f$ and for any Cartesian square $(h, k) : g \to f$ in $R-\text{Map}$, the Beck-Chevalley isomorphism $BC : k^*\Pi_f \to \Pi_g h^*$ lifts to an isomorphism of $R$-maps.

These two facts are enough to reproduce the arguments of [15, Theorem 2] in order to obtain a psuedo-stable choice of $\Pi$-types for the comprehension category induced by $(L, R)$. □
Remark 2.7. As shown in the foregoing proof, a functorial Frobenius structure guarantees the existence of lifts $\Pi_f : R\text{-Map}/A \to R\text{-Map}/\Gamma$ of the push-forward functor for each R-map $(f, s)$. However, it does not guarantee that the universal property of the categorical exponential $\Pi_f g$ also lifts to $R\text{-Map}$. Fortunately, this is not necessary for the construction of pseudo-stable choices of $\Pi$-types since we only need the universal property to hold at the level of $\mathcal{C}$.

In order to obtain pseudo-stable $\Sigma$ and $\Pi$ types, the only part we had to ‘categorify’ to the setting of $R\text{-Map}$ was the relevant formation rule. Indeed, once we had done this, the rest of the argument goes through just as it did in the non-algebraic case, i.e. as in the original reference [15]. The case for intensional identity types is more complicated. Here the extra algebraic structure is essential, it will allow us to keep track of the necessary information needed to coherently produce the ‘elimination terms’ (i.e. the fillers $j$ of Item 3 from Definition 1.1) for the choice of $\text{Id}$-types.

Let us recall the following notion. A functorial factorisation of the diagonal is a functor $P : \mathcal{C} \to \mathcal{C} \to \mathcal{C}$ that acts on a map $f : X \to Y$ as shown:

\[
P : f \mapsto (X \xrightarrow{r_f} PX \xrightarrow{\rho_f} X \times_Y X)
\]

such that the composition $\rho_f \cdot r_f$ equals the diagonal morphism $\delta_f : X \to X \times_Y X$. We say that a functorial factorisation of the diagonal is stable if whenever we have a Cartesian square $(h, k) : f' \to f$ in $\mathcal{C}$, the resulting square $\rho_{[h,k]} : \rho_{f'} \to \rho_f$ is also Cartesian. We denote a (stable) functorial factorisation of the diagonal by $P = (r, \rho)$, where $r, \rho : \mathcal{C} \to \mathcal{C}$ denote the induced functors from the two legs of the factorisation respectively. The following notion was first described in [3, Definition 3.3.3].

Definition 2.8. Let $(L, R)$ be an awfs on $\mathcal{C}$. A stable functorial choice of path objects (or sfpo for conciseness) consists of a lift of a stable functorial factorisation of the diagonal $P$ as shown in the following diagram:

\[
\begin{array}{ccc}
R\text{-Map} & \xrightarrow{P} & L\text{-Map} \times_{\mathcal{C}} R\text{-Map} \\
\downarrow & & \downarrow \\
\mathcal{C} & \xrightarrow{P} & \mathcal{C} \times_{\mathcal{C}} \mathcal{C}
\end{array}
\]

Proposition 2.9. Let $(L, R)$ be an awfs equipped with a sfpo of the form $P = (r, \rho)$. Then $(L, R)$ is equipped with the structure of a pseudo-stable choice of $\text{Id}$-types

Proof. We need to construct a choice $(\text{Id}, r, j)$ of $\text{Id}$-types (Definition 1.1). The choices for $\text{Id}$ and $r$ are canonically given by the stable functorial choice of path objects. These satisfy the coherence properties of Definition 1.4.

Since the maps $r_f$ are equipped with an $L$-map structure, we have lifts against $R$-maps. Using this, we obtain a choice of canonical elimination terms (i.e. $j$-terms).

We are left to verify that this choice is coherent. For this, it is sufficient to show that given a Cartesian morphism of $R$-maps $(h, k) : f' \to f$, a $R$-map $q : \mathcal{C} \to PX$, and a commutative diagram as in the right of the following figure; the diagram in the left...
where \( q^* : C^* \to PX \) is defined as the pullback of \( q \) along \( P(h,k) \). The arrows denoted by \( j \) are the canonical choices of lifts. The arrow \( d^* \) is the pullback of the map \( d \) along \( P(h,k) \), i.e. it is defined to be the unique arrow \( d^* : X' \to C^* \) such that:

\[
q^* \circ d^* = r f' \quad \text{and} \quad P(h,k)^* \circ d^* = d \circ h.
\]

(1)

We split the problem into two. First consider the following diagram equipped with the corresponding canonical lifts:

\[
\begin{array}{ccc}
X' & \xrightarrow{d^*} & C^* \\
\downarrow{r f'} & & \downarrow{q} \\
PX' & \xrightarrow{j(d^*)} & PX \\
\end{array}
\begin{array}{ccc}
P(h,k)^* & \xrightarrow{P(h,k)} & C \\
\downarrow{j(d)} & & \downarrow{q} \\
PX' & \xrightarrow{j(d)} & PX \\
\end{array}
\]

note that since the Cartesian square \( q^* \to q \) is a morphism of \( R \)-maps, we obtain that \( j = P(h,k)^* \circ j(d^*) \). Now consider the following lifting problem

\[
\begin{array}{ccc}
X' & \xrightarrow{h} & X \\
\downarrow{r f'} & & \downarrow{q} \\
PX' & \xrightarrow{j(d)} & PX \\
\end{array}
\begin{array}{ccc}
C^* & \xrightarrow{P(h,k)^*} & C \\
\end{array}
\]

once more, since the square \( r f' \to r f \) is morphism of \( L \)-maps, we obtain that \( j' = j(d) \circ P(h,k) \). Finally we notice that Eq. (1) tells us that the outer squares of the two previous diagrams are equal, implying that they have the same lift \( j = j' \); thus \( P(h,k)^* \circ j(d^*) = j(d) \circ P(h,k) \) as needed.

\[\square\]

We have so far described what additional structure a awfs must posses in order to obtain pseudo-stability of the relevant logical structure in the comprehension category induced by it. This motivates the following definition.

**Definition 2.10.** A **type-theoretic awfs** consists of the following data:

1. a category \( C \) equipped with an awfs \((L,R)\) satisfying the exponentiability property.
2. a functorial Frobenius structure on \((L,R)\).
3. a stable functorial choice of path objects on \((L,R)\).

The following theorem follows immediately.

**Theorem 2.11.** Let \((L,R)\) be an awfs on \( C \) with the structure of a type-theoretic awfs. Then the comprehension category induced by \((L,R)\) is equipped with pseudo-stable choices of \( \Sigma, \Pi \) and \( \text{Id} \)-types.
Proof. Apply Proposition 2.3, Proposition 2.6 and Proposition 2.9.

The problem now is to obtain examples of type-theoretic algebraic weak factorisation systems. It is to this that we turn our attention next.

3. Type Theoretic AWFS on Groupoids

The aim of this section is to provide a first example of a type-theoretic awfs by revisiting the original Hofmann-Streicher model [17] on the category of groupoids using the theory we have exposed so far. Explicitly, we construct an algebraic weak factorisation system \((C_f, F_f)\) on the category \(\mathbf{Gpd}\) of groupoids and functors.

Consider \(f : X \to Y\) a functor between groupoids. The comma category of \(f\), denoted by \(\downarrow f\), has as objects tuples \((a, b, p)\) with \(a \in X\), \(b \in Y\) and \(p : b \to fa\). We have that \(\downarrow f\) is again a groupoid, and moreover the construction is functorial: \(\downarrow (-) : \mathbf{Gpd}^{\to} \to \mathbf{Gpd}\).

This forms the middle part of a functorial factorisation assigning to each \(f : X \to Y\):

\[X \xrightarrow{C_f} \downarrow f \xrightarrow{F_f} Y\]

where \(C_f(a) = (a, fa, \text{id}_{fa})\) and \(F_f(a, b, p) = b\).

**Proposition 3.1.** The functorial factorisation \((\downarrow (-), C_f, F_f)\) is an algebraic weak factorisation system on \(\mathbf{Gpd}\). The \(C_f\)-maps are characterised as strong deformation retractions while the \(F_f\)-maps are normal isofibrations.

Proof. We start by examining the structures of the \(C_f\)-maps and the \(F_f\)-maps. We know that an \(F_f\)-map structure on a map \(f : X \to Y\) corresponds to a lift \(s\) as shown on the diagram on the left of the following figure:

\[\xymatrix{ X \ar[r]^{f} & Y \ar[dl]_{\downarrow f} }\]

\[\xymatrix{ X \ar[r]^{C_f} & \downarrow f }\]

\[\xymatrix{ A \ar[r]^{g} & B \ar[dl]_{\downarrow g} }\]

A closer analysis will show that \(s\) equips \(f : X \to Y\) with the structure of a normal isofibration. An \(L\)-map structure on a map \(g : A \to B\), is given by a lift \(\lambda\) as shown on the diagram on the right of the previous figure. The structure obtained from such a lift \(\lambda\) can be decomposed as \(\lambda(b) = (\lambda_1(b), \lambda_2(b))\) where \(\lambda_1 : B \to A\) corresponds to a retraction of \(g\) and \(\lambda_2 : \text{id}_B \to g \circ \lambda_1\) corresponds to a natural transformation constant on the image of \(f\). This information corresponds to the structure of a strong deformation retraction.

We proceed to the construction of the corresponding structures of a comonad and a monad for \(C_f\) and \(F_f\) respectively. We provide a brief description and leave the details to the reader. The comultiplication \(\delta_f : \downarrow f \to \downarrow C_f f\) for \(C_f\) is defined as follows:

\[\delta_f : (a, b, p) \mapsto (a, (a, b, p), (1_a, p) : (a, b, p) \to (a, fa, 1_{fa}))\]

Similarly we have that the endofunctor \(F_f\) has a multiplication \(\mu_f : \downarrow F_f \to \downarrow f\) given by:

\[\mu_f : ((a, b, p), \tilde{b}, \tilde{p} : \tilde{b} \to b) \mapsto (a, \tilde{b}, p \circ \tilde{p})\]
Remark 3.2. Notice that the definition of the multiplication \( \mu : \downarrow \mathbf{F} f \rightarrow \downarrow f \) for \( \mathbf{F} \), uses the fact that paths (i.e. morphisms in groupoids) can be composed. Moreover, the fact that this composition is strictly associative and unital is crucial in proving the monad axioms.

Remark 3.3. A close analysis of the category of \( \mathbf{F} \)-algebras and \( \mathbf{F} \)-maps, reveals that these are precisely the categories of split isofibrations and of normal isofibrations respectively. This implies that the category \( \mathbf{Gpd} \) satisfies the exponentiability condition (see Definition 2.4) with respect to the awfs \( (\mathcal{C}_t, \mathbf{F}) \) since, although \( \mathbf{Gpd} \) is not locally cartesian closed, it is well known that isofibrations can be exponentiated [8].

We now proceed to show that the awfs \( (\mathcal{C}_t, \mathbf{F}) \) in \( \mathbf{Gpd} \) has a functorial Frobenius structure.

**Proposition 3.4.** The awfs \( (\mathcal{C}_t, \mathbf{F}) \) is equipped with a functorial Frobenius condition.

**Proof.** We need to show that pulling back a \( \mathcal{C}_t \)-map along an \( \mathbf{F} \)-map is uniformly a \( \mathcal{C}_t \)-map. Consider \((g, \lambda) : A \rightarrow Y \) a \( \mathcal{C}_t \)-map and \((f, s) : X \rightarrow Y \) an \( \mathbf{F} \)-map. Let \( g' : A \times_Y X \rightarrow X \) be the pullback of \( g \) along \( f \). We define a \( \mathcal{C}_t \)-map structure \( \lambda' \) on \( g' \) which, by Proposition 3.1, corresponds to a strong deformation retraction \((g', \lambda'_1, \lambda'_2)\). Making use that \( f \) correspond to a normal isofibration, we can find for each point \( x \in X \), a point \( x' \in X \) and a lift \( \lambda'_2(x) \) of \( \lambda_2(fx) \) as shown:

\[
\begin{array}{ccc}
X & \xrightarrow{\lambda'_2(x)} & X' \\
\downarrow fx & & \downarrow g fx \\
\downarrow & & \\
\end{array}
\]

We define \( \lambda'_1(x) = (\lambda_1(fx), x') \), the homotopy \( \lambda'_2 : 1 \rightarrow g' \circ \lambda'_1 \) is defined using the top arrow in the previous diagram.

Finally, we turn our attention to identity types. We start by noticing that the category \( \mathbf{Gpd} \) has a stable and functorial factorisation of the diagonal given on a map \( f : X \rightarrow Y \) by:

\[
\begin{array}{ccc}
X & \xrightarrow{r_f} & Pf \\
& \xrightarrow{\rho_f} & X \times_Y X \\
\end{array}
\]

where the objects of \( Pf \) are tuples \((a, a', p)\) such that \( p : a \rightarrow a' \) is a morphism in \( X \) over the identity, i.e. \( fa = fa' \) and \( fp = \text{id}_{fa} \). The map \( r_f \) is given by \( a \mapsto (a, a, \text{id}_a) \) and the map \( \rho_f \) is given by \( (a, b, p) \mapsto (a, b) \).

**Proposition 3.5.** The awfs \( (\mathcal{C}_t, \mathbf{F}) \) is equipped with a stable and functorial choice of path objects.

**Proof.** Consider an \( \mathbf{F} \)-map \((f, s) : X \rightarrow Y \), we need to uniformly provide a \( \mathcal{C}_t \)-map structure to \( r_f \) and an \( \mathbf{F} \)-map structure to \( \rho_f \).

Let us define \( \lambda_1 := t_f : P_w f \rightarrow X \) the canonical target map. We define the natural transformation \( \lambda_2 : \text{id}_{P_w f} \rightarrow r_f \circ t_f \) by \( \lambda_2(a, a', p) := (p, \text{id}_{a'}) : (a, a', p) \rightarrow (a', a', \text{id}_{a'}) \). This corresponds to a strong deformation retraction structure on \( r_f \).
An \( F \)-map structure on \( \rho_f \) corresponds to a normal isofibration. Consider a morphism \( (\alpha, \beta) : (b, b') \to (a, a') \) in \( X \times Y \) and an object \( (a, a', p) \in Pf \) over \( (a, a') \). We find the lift \( (\alpha, \beta) : (b, b', q) \to (a, a', p) \) by setting \( q := \beta \circ p \circ \alpha^{-1} : b \to b' \).

The following theorem follows.

**Theorem 3.6.** The category \( \text{Gpd} \) is equipped with the structure of a type-theoretic awfs.

*Proof.* Apply Proposition 3.1, Proposition 3.4 and Proposition 3.5.

Applying Theorem 2.11 we obtain a model of dependent type theory with \( \Pi, \Sigma \) and \( \text{Id} \)-types. This is essentially the same model as the Hofmann-Streicher one.

4. Type Theoretic AWFS from Uniform Fibrations

In this section we will investigate how to obtain type-theoretic awfs in the setting of uniform fibrations of \cite{11}. As we will see, this will provide a major source of examples of categories equipped with type-theoretic awfs, including some important ones in the categories of simplicial and cubical sets. For the convenience of the reader, we will include a brief description of the construction of the awfs of uniform fibrations in the context of a Grothendieck topos.

We begin by recalling the pushout-product construction. This is of outmost importance in both, the construction and the manipulation of the awfs of uniform fibrations. Let us consider a Grothendieck topos \( \mathcal{E} \), the pushout-product bifunctor:

\[
- \hat{\times} - : \mathcal{E} \rightarrow \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E}
\]

is defined on a pair of arrows \( f : X \to Y \) and \( g : A \to B \) as the universal dotted arrow shown in the following pushout diagram:

\[
\begin{array}{ccc}
X \times A & \xrightarrow{f \times A} & Y \times A \\
\downarrow & & \downarrow \\
X \times B & \xrightarrow{Y \times g} & Y \times B
\end{array}
\]

This construction enjoys several useful properties. We refer the reader to \cite{24} for further details.

The following notion plays a fundamental role in the theory of uniform fibrations. An *interval object* in \( \mathcal{E} \) consists of an object \( I \) together with two morphisms \( \delta^0, \delta^1 : I \to I \) respectively called the left and right *endpoint inclusions*; these morphisms are required to be disjoint, i.e., the pullback of one along the other matches the initial object. We require the following additional structure. The *connection operations* on \( I \) are given by...
\( c^k : I \times I \to I \) for \( k \in \{0, 1\} \), making the following diagrams commute:

\[
\begin{array}{ccc}
I & \overset{\delta^k \times 1}{\longrightarrow} & I \times I \\
\downarrow{\epsilon} && \downarrow{c^k} \\
I & \underset{\delta^k}{\longleftarrow} & I
\end{array}
\quad \begin{array}{ccc}
I & \overset{\delta^{1-k} \times 1}{\longrightarrow} & I \times I \\
\downarrow{\delta^k} && \downarrow{c^k} \\
I & \underset{\delta^k}{\longleftarrow} & I
\end{array}
\]

Connections correspond to special type of degeneracy maps that can be pictured as the two possible deformations of the square \( I \times I \) into its diagonal fixing, respectively, each endpoint.

With this in place, we proceed to describe the construction of uniform fibrations. Our starting point is the following definition.

**Definition 4.1.** A suitable topos consists of a tuple \((E, I, M)\) where \( E \) is a Grothendieck topos equipped with an interval object \( I \) with connections and a class \( M \) of arrows in \( E \) satisfying the following conditions:

- (M1) the objects of \( M \) are monomorphisms
- (M2) the initial map \( \emptyset \to X \) is in \( M \) for every \( X \in E \).
- (M3) the objects of \( M \) are closed under pullback along any arrow in \( E \).
- (M4) the elements of \( M \) are closed under pushout-product with the endpoint inclusions, i.e. for each \( j \in M \), we have that \( \delta^k \hat{\times} j \in M \).

The objects of \( M \) are called generating monomorphisms.

**Remark 4.2.** The objects of \( M \) are thought of as generating cofibrations in analogy to the results of [25].

Given a suitable topos \((E, I, M)\) let us denote by \( M_{\hat{\times}} \) the category that has as objects maps of the form \( \delta^k \hat{\times} j \) with \( j \in M \) and \( k \in \{0, 1\} \) and whose morphisms are given by squares of the form \( \delta^k \hat{\times} \sigma : (\delta^k \hat{\times} j') \to (\delta^k \hat{\times} j) \) induced by functoriality of the pushout-product applied to Cartesian squares \( \sigma : j' \to j \) between generating monomorphisms. We consider \( M_{\hat{\times}} \) to be a category of arrows by taking the inclusion into \( E \rightarrow \).

**Construction 4.3.** Let us consider a suitable topos \((E, I, M)\). The category of arrows of uniform fibrations denoted by:

\[
\text{UniFib} \to E^{\rightarrow}
\]

is defined to be the right orthogonal category of arrow to \( M_{\hat{\times}} \); that is, we have that \( \text{UniFib} := M_{\hat{\times}}^{\rightarrow} \).

Using Garner’s small object argument, it is possible to construct an awfs \((C_1, F)\) on \( E \) algebraically-free on \( M_{\hat{\times}} \); i.e. such that the category of \( F \)-algebras coincides with that of uniform fibrations. We will call \((C_1, F)\) the awfs of uniform fibrations on \( E \).

**Remark 4.4.** The construction of \((C_1, F)\) requires some technicalities since we cannot apply Garner’s small object argument directly to \( M_{\hat{\times}} \) because of set-theoretical constraints. Instead, we consider \( I \) the subclass of \( M \) consisting of those monomorphisms whose codomain lie in a fixed, dense subcategory of \( M \). We construct the awfs \((C_1, F)\) by applying Garner’s small object argument to \( I_{\hat{\times}} \) instead. By [11, Theorem 9.1] we
obtain that $\mathcal{I}_\Delta^\otimes = \mathcal{M}_\Delta^\otimes = \text{UniFib}$. We will refer to $\mathcal{I}_\Delta$ as the category of generating trivial cofibrations.

We will make use of an auxiliary awfs which we describe. Consider $\mathcal{M}$ as a category of arrows on $E$ by taking the morphisms to be Cartesian squares. It is possible to construct a second awfs $(\mathcal{C}, \mathcal{F}_1)$ algebraically-free on $\mathcal{M}$, this awfs $(\mathcal{C}, \mathcal{F}_1)$ satisfy the properties of a suitable awfs (see [11, Definition 7.1]) as shown in [11, Theorem 9.1]. We call $(\mathcal{C}, \mathcal{F}_1)$ the awfs of trivial uniform fibrations.

We proceed to show that, under some extra hypothesis, the awfs $(\mathcal{C}, \mathcal{F}_1)$ of uniform fibrations on a suitable topos is type-theoretic. One of the main results of [11] is [11, Theorem 8.8] which states that the awfs of uniform fibrations on a suitable topos has a functorial Frobenious structure. This theorem implies that we only need to construct a stable functorial choice of path objects on $(\mathcal{C}_1, \mathcal{F})$. For this, we require the following construction. Given a topos $E$ equipped with an interval object $I$, there is a natural way to construct a stable and functorial factorisation of the diagonal: for a morphism $f : B \to A$, consider $B \xrightarrow{r_f} Pf \xrightarrow{\rho_f} B \times_A B$ where $Pf$ and the morphism $r_f : B \to Pf$ are given as in the pullback square in the following diagram:

we use the abbreviation of $(-)^I$ for the exponential object $\text{hom}(I, -)$ and denote by $!: I \to \bot$ the terminal map. The second leg of the factorisation $\rho_f : Pf \to B \times_A B$ is given by the universal property of $B \times_A B$ applied to the canonical source and target maps $s_f, t_f : Pf \to B$. We denote the factorisation by $\mathcal{P}_f$ indicating that it was constructed in this way from the interval $I$.

Let us provide an alternative construction of this factorisation which makes evident some intermediate steps and uses the adjunction $\begin{array}{c} - \otimes f I \vdash \text{hom}(f, -) \end{array}$ given by the pushout-product and pullback-exponential. Denote by $i : \delta I \to I$ the boundary inclusion of the interval object. The following diagram expands the previous one, i.e. the exterior part
is exactly the one of Figure (2).

The intermediate arrows $\lambda_f$, $\alpha_f$ and $\beta_f$ are given intuitively as follows. $\lambda_f$ maps a pair of points in $B \times_A B$ to the same pair of points but now in $B^0$. $\alpha_f$ maps a similar pair of points $(b_1, b_2)$ to the reflexivity (constant) path on $f(b_1) = f(b_2)$. $\beta_f$ also maps a point $b$ to the reflexivity path on $f(b)$.

With this in place and with the help of the machinery of uniform fibrations, we are now able to state and prove the following proposition.

**Proposition 4.5.** Consider a suitable topos $(E, I, M)$ and let $(C_t, F)$ be a corresponding awfs of uniform fibrations on $E$. Suppose that the following additional hypothesis hold:

(M5) Maps in $M$ are closed under pushout-product against the boundary inclusion $i : \partial I \rightarrow I$, i.e. for any $j \in M$, we have that $i \times j \in M$.

(M6) For any $f : B \rightarrow A$ in $E$, the first leg map $r_f : B \rightarrow Pf$ from the factorisation of the diagonal $P_I = \langle r, \rho \rangle$ induced from the interval object, lifts to a stable functorial choice of path objects for $(C_t, F)$:

$$\text{F-Map} \rightarrow C_t\text{-Map} \times_E \text{F-Map}$$

Proof. We will divide the proof into two parts. For this, recall that the factorisation of the diagonal $P_I = \langle r, \rho \rangle$ is divided into two functors $r, \rho : E^\rightarrow \rightarrow E^\rightarrow$.

Claim 1. The functor $\rho : E^\rightarrow \rightarrow E^\rightarrow$ lifts to a functor $\rho : \text{F-Map} \rightarrow \text{F-Map}$.

Proof of Claim. Since the awfs of trivial uniform fibrations $(C, F_t)$ is suitable (see [11, Definition 7.1]), we have that the functor $\delta^k \sim -$ lifts to the category $\text{C-Map}$ and by [11, Lemma 8.4] we have that $\delta^k \sim -$ also factors though the category $S$ of homotopy equivalences (see [11, Definition 4.1 and Lemma 8.1]). Combining this two facts, we obtain a lift of $\delta^k \sim -$ as shown:

$$\text{C-Map} \rightarrow \text{C-Map} \times_{E^\rightarrow} S$$
By [11, Proposition 8.5], we have a functor $C\text{-Map} \times_{E} S \to C_t\text{-Map}$ over $E^\to$, composing with the one above, we obtain a lift of $\delta^k\hat{x}^\rightarrow$ as shown:

$$\begin{array}{c}
\xymatrix{C\text{-Map} \ar[r]^-{\delta^k\hat{x}^\rightarrow} & C_t\text{-Map}}
\end{array}$$

Using the good behaviour of the orthogonality functors with respect to the pushout-product construction (see [11, Proposition 5.9]) together with the hypothesis (M5), we observe that the functor $i\hat{x}^\rightarrow$ lifts to the category $C\text{-Map}$ as shown:

$$\begin{array}{c}
\xymatrix{C\text{-Map} \ar[r]^-{i\hat{x}^\rightarrow} & C\text{-Map}}
\end{array}$$

Applying these two lifts together with the fact that $(C,F_t)$ is algebraically-free on the category of arrows $M \to E^\rightarrow$ as witnessed by the functor $\eta : M \to C\text{-Coalg}$ we obtain the following diagram:

$$\begin{array}{c}
\xymatrix{M \ar[r]^-{\hat{\eta}} & C\text{-Map} \ar[r]^-{i\hat{x}^\rightarrow} & C\text{-Map} \ar[r]^-{\delta^k\hat{x}^\rightarrow} & C_t\text{-Map}}
\end{array}$$

where $\hat{\eta}$ denotes the functor $\eta$ composed with the forgetful functor from $C$-algebras to $C$-maps.

By symmetry of the pushout-product functor, we obtain a natural isomorphism $i\hat{x}^\rightarrow \delta^k\hat{x}^\rightarrow \cong \delta^k\hat{x}i\hat{x}^\rightarrow$. We can transfer the algebraic structure along this natural isomorphism in order to obtain the following lift:

$$\begin{array}{c}
\xymatrix{M \ar[r] & C_t\text{-Map} \ar[r]^-{\delta^k\hat{x}^\rightarrow} & C_t\text{-Map} \ar[r]^-{i\hat{x}^\rightarrow} & E^\rightarrow}
\end{array}$$

Taking the coproduct of these lifts for $k = 0, 1$ we obtain a lift of $i\hat{x}^\rightarrow$ as shown:

$$\begin{array}{c}
\xymatrix{M_{\hat{x}} \ar[r]^-{i\hat{x}^\rightarrow} & C_t\text{-Map}}
\end{array}$$

Using that $C_t\text{-Map} \cong F\text{-Alg}$ by Proposition [11,10] and that $(C_t,F)$ is algebraically-free on $M_{\hat{x}}$ we can apply [11, Proposition 5.9] to the previous lift in order to obtain the following:

$$\begin{array}{c}
\xymatrix{F\text{-Alg} \ar[r]^-{\hat{\text{hom}}(1,-)} & (M_{\hat{x}})^{\square} \ar[r]^-{\cong} & F\text{-Alg}}
\end{array}$$

Observing the top pullback square of Figure (3) we notice that the morphism $\rho_f : Pf \to B \times_{A} B$ is obtained in the following two steps:

$$f \mapsto \hat{\text{hom}}(i,f) \mapsto \langle \alpha_f, \lambda_f \rangle^* \hat{\text{hom}}(i,f) = \rho_f$$
i.e. by first applying $\text{hom}(i, -)$ and then pulling back along $\langle \alpha_f, \lambda_f \rangle$. Thus since we have lifts of $\text{hom}(i, -)$ and of the pullback functor to the category of $F$-algebras, we obtain a lift of $\rho$ as shown:

$$
\begin{array}{ccc}
F\text{-Alg} & \xrightarrow{\text{hom}(i, -)} & F\text{-Alg} \\
\rho \searrow & & \nearrow \text{PB}(-, (\alpha, \lambda)) \\
& F\text{-Alg} &
\end{array}
$$

Since we are working in an algebraically-free awfs, we have lifts back-and-forth between $R\text{-Alg}$ and $R\text{-Map}$ over $\mathbb{E}^\to$ (Proposition 1.11), and thus we can transfer the lift of $\rho$ from the category of $R$-algebras to that of $R$-maps. □

**Claim 2.** The functor $r : \mathbb{E}^\to \to \mathbb{E}^\to$ lifts to a functor $r : F\text{-Map} \to C_t\text{-Map}$.

**Proof of Claim 2.** We first show that $r : \mathbb{E}^\to \to \mathbb{E}^\to$ lifts to a functor $r : F\text{-Map} \to S$ where $S$ is the category of strong homotopy equivalence (see [11, Definition 4.1 and Lemma 8.1]).

For this we make use of the fact that the target map functor (that takes a map $f : B \to A$ to a map $t_f : Pf \to B$) lifts to a functor from $F\text{-Map}$ to $F_t\text{-Map}$. Using that we have a lift $\delta^1 \times - : C\text{-Map} \to C_t\text{-Map}$ as shown in the proof of Claim 1, we can transpose using [11, Proposition 5.9] to obtain a lift of $\text{hom}(\delta^1, -)$:

$$
\begin{array}{ccc}
F\text{-Alg} & \xrightarrow{\text{hom}(\delta^1, -)} & F_t\text{-Alg} \\
\nearrow & & \searrow \\
& F_t\text{-Alg} &
\end{array}
$$

Looking at Figure (3) we notice that $t_f : Pf \to B$ is obtained by applying $\text{hom}(\delta^1, -)$ to $f$ and then pulling back along $\langle \beta_f, \text{id}_B \rangle$, thus the functor mapping $f \mapsto t_f$ lifts as shown:

$$
\begin{array}{ccc}
F\text{-Alg} & \xrightarrow{\text{hom}(\delta^1, -)} & F_t\text{-Alg} \\
\nearrow & & \searrow \text{PB}(-, (\beta, \text{id})) \\
& F_t\text{-Alg} &
\end{array}
$$

(4)

since both awfs in question are algebraically-free we can apply Proposition 1.11 to obtain the desired lift.

Let us return to the task of finding a lift of the functor $r : \mathbb{E}^\to \to \mathbb{E}^\to$ to a functor $r : F\text{-Map} \to S$. For this, we show that for each uniform fibration $(f, s) : B \to A$ the target map $t_f : Pf \to B$ is a strong homotopy retraction of $r_f : B \to Pf$.

Looking again at Figure (3) it is clear that $t_f \circ r_f = \text{id}_B$. Thus we are left with the task of constructing an homotopy $H : r_f \circ t_f \sim \text{id}_{ Pf }$, for this consider the following commutative diagram:

$$
\begin{array}{ccc}
Pf & \xrightarrow{(r_f \circ t_f, \text{id}_{ Pf })} & Pf^{\partial I} \\
\text{B} \circ t_f \downarrow & & \downarrow t_f^{\partial I} \\
\text{B}^{\partial I} & \xrightarrow{t_f^{\partial I}} & \text{B}^{\partial I}
\end{array}
$$
where the top horizontal arrow is given by the universal property of the product $\text{Pf}^\Omega \cong \text{Pf} \times \text{Pf}$. This gives us (by universal property) an arrow into the pullback:

$$\tilde{H} : \text{Pf} \to B^I \times_{B^\Omega} \text{Pf}^\Omega.$$  

We already have a lift of the target map $t_{(-)} : \text{F-Map} \to \text{F}_{t^*}\text{Map}$. Combining this with the fact that $\hat{\text{hom}}(i, -)$ lifts to $\text{F}_{t^*}\text{Map}$ (by similar arguments to those used in the proof of Claim [1]), we find that $\hat{\text{hom}}(i, t_{(-)})$ lifts to a functor:

$$\begin{array}{c}
\text{F-Map} \\
\downarrow \hat{\text{hom}}(i, t_{(-)})
\end{array} \quad \begin{array}{c}
\text{F}_{t^*}\text{Map}
\end{array}$$

let’s apply this to $f$ to obtain a uniform trivial fibration $\hat{\text{hom}}(i, t_f)$.

By item (M2) of Definition [11] we have that for every object $X \in \mathcal{E}$, the map $\emptyset \to X$ is in $\mathcal{M}$. Using this, we obtain a morphism $H$ as the canonical filler in the following diagram:

$$\begin{array}{c}
\emptyset \\
\downarrow \downarrow \\
\text{Pf} \\
\downarrow \downarrow \\
B^I \times \text{Pf}^\Omega \\
\uparrow \uparrow \\
\text{Pf} \\
\downarrow \downarrow \\
B^I \times_{B^\Omega} \text{Pf}^\Omega
\end{array}$$

It is straightforward to verify that this $H$ is actually an homotopy from $\tau_f \circ t_f$ to $\text{id}_{\text{Pf}}$. This shows that $t_f$ is a strong deformation retract of $\tau_f$, but every strong deformation retraction is in particular a strong homotopy equivalence.

So far we have given the action on objects of the desired lift $\tau : \text{F-Map} \to \mathcal{S}$. We have to show that this construction is functorial on $f$. For this, consider a morphism of $\text{F-Map}$ $(h, k) : f' \to f$. Using the fact that the factorisation of the diagonal is functorial, we obtain the following diagram:

$$\begin{array}{c}
B' \\
\downarrow \tau_{f'} \\
\text{Pf}' \\
\downarrow t_{f'} \\
B
\end{array} \quad \begin{array}{c}
\text{Pf} \\
\downarrow \tau_f \\
\text{Pf} \\
\downarrow t_f \\
B
\end{array} \quad \begin{array}{c}
B'
\end{array}$$

Notice that the bottom square is a morphism of $\text{F}_{t^*}\text{Map}$ since it is the result of applying the lift of $t_{(-)}$ of Figure [1] to the square $(h, k)$.

Let us prove that $(h, P(h, k)) : \tau_{f'} \to \tau_f$ is a morphism of strong homotopy equivalences. Looking at the definition of a morphism of homotopy equivalences (in the paragraph before [11] Lemma 8.1), we observe that the only thing we need to show is
that the following diagram commutes:

\[
\begin{array}{ccc}
Pf' & \overset{P(h,k)}{\rightarrow} & Pf \\
\downarrow^{H'} & & \downarrow^{H} \\
Pf' \overset{P(h,k)}{\rightarrow} Pf
\end{array}
\]

where the left and right horizontal arrows are the homotopies witnessing that \(r_f\) and \(r_I\) respectively are strong deformation retracts. For this, we make use of the naturality of the filling operations. Consider the following two diagrams:

\[
\begin{array}{ccc}
\emptyset & \overset{\emptyset}{\rightarrow} & Pf \\
\downarrow^{Pf'} & \downarrow^{f^I \times_{B^01} Pf^I} & \downarrow^{h^I \circ (i,t_f)} \\
\emptyset & \overset{\emptyset}{\rightarrow} & Pf
\end{array}
\]

The left square of the top diagram is a morphism in \(M\) since it is trivially Cartesian. The right square of the bottom diagram is a morphism of \(F_t\)-maps since it is the result of applying the lift \(h^I \circ (i,t_f) : F-\text{Map} \rightarrow F_t-\text{Map}\) to the square \((h,k)\) which is, by hypothesis, a morphism of \(F\)-maps. We obtain that the corresponding lifts cohere.

Since the construction of the maps \(H\) and \(H'\) is functorial (given by a universal property), we have that the following diagram commutes:

\[
\begin{array}{ccc}
Pf' & \overset{P(h,k)}{\rightarrow} & Pf \\
\downarrow^{H'} & \downarrow^{H} & \downarrow^{H'} \\
B^I \overset{B^I \times_{B^01} Pf'^I}{\rightarrow} Pf^I \overset{h^I \times_{B^01} P(h,k)^I}{\rightarrow} B^I \times_{B^01} Pf^I
\end{array}
\]

this means that that the composition of the bottom horizontal arrows in the previous two lifting diagrams coincide, this makes the lift \(L\) in both diagrams the same morphism, and thus we obtain that:

\[
H \circ P(h,k) = L = P(h,k)^I \circ H'
\]

as required. This concludes the construction of the lift \(r : F-\text{Map} \rightarrow S\).

We will now argue that we also have a lift of the functor \(r : E \rightarrow E\) to a functor \(r : E_{\rightarrow} \rightarrow C-\text{Map}\). This follows form two easy observations. First, since the factorisation of the diagonal is stable, we can conclude that \(r\) preserves Cartesian squares; and thus by item (M6) in the hypothesis of the theorem, we have that \(r\) lifts to \(M\) seen as a
category of arrows. This is shown in the following diagram:

\[ \mathcal{E} \rightarrow \mathcal{M}. \]

Secondly, consider the unit \( \eta_{\mathcal{M}} : \mathcal{M} \rightarrow \square(\mathcal{M}^{\square}) \) of the orthogonality adjunction \( \square(-) \vdash (-)^{\square} \) and notice that since \((C,F_1)\) is algebraically-free on \( \mathcal{M} \), we obtain a morphism in the slice over \( \mathcal{E} \rightarrow \mathcal{M} \):

\[ \mathcal{M} \xrightarrow{\eta_{\mathcal{M}}} \text{C-Map}. \]

We can compose these last two lifts to obtain \( r : \mathcal{E} \rightarrow \text{C-Map} \).

Finally, we can combine the lifts \( r : F\text{-Map} \rightarrow S \) and \( r : \mathcal{E} \rightarrow \text{C-Map} \) and apply \[ \text{Proposition 8.5} \] in order to obtain the desired lift of \( r \):

\[ \begin{array}{ccc}
F\text{-Map} & \xrightarrow{r} & \text{C-Map} \\
\downarrow & & \downarrow \\
\text{S} & \rightarrow & \text{C}_t\text{-Map}
\end{array} \]

The proof of the proposition now follows immediately by Claim 1 and Claim 2.

We summarise the discussion so far. In order to do this, it is appropriate to introduce the following definition.

**Definition 4.6.** A *type-theoretic suitable topos* consists of a suitable topos \((\mathcal{E}, I, \mathcal{M})\) (see Definition 4.1) which moreover satisfy the conditions (M5) and (M6) in the hypothesis of Proposition 4.5.

The notion of a type-theoretic suitable topos enable us to concisely state the following main result.

**Theorem 4.7.** Let \((\mathcal{E}, I, \mathcal{M})\) be a type-theoretic suitable topos, and let \((C_1,F)\) be the awfs of uniform fibrations on \( \mathcal{E} \). Then \((C_1,F)\) is equipped with the structure of a type-theoretic awfs.

**Proof.** The result follows immediately by applying \[ \text{Theorem 8.8} \] and Theorem 4.7.

We proceed to look at some examples of type-theoretic suitable topos. Our main examples follow from the following easy result.

**Proposition 4.8.** Consider \( \mathcal{E} \) be a Grothendieck topos equipped with an interval object \( I \) with connections. Let \( \mathcal{M}_{\text{all}} \) be the class that consists of all monomorphisms of \( \mathcal{E} \). Then the tuple \((\mathcal{E}, I, \mathcal{M}_{\text{all}})\) is a type-theoretic suitable topos.

**Proof.** We need to verify conditions (M1)-(M6) from the definition of type-theoretic suitable topos. By elementary properties of monomorphisms, it is clear that (M1)-(M3) hold. Condition (M4) follows because in the topos \( \mathcal{E} \) the pushout-product construction \( \delta^k \times j \) for a given monomorphism \( j : A \rightarrow B \), computes the join (or union) of the subobjects \( \delta^k \times B : B \rightarrow I \times B \) and \( I \times j : I \times A \rightarrow I \times B \), which is again a subobject of \( I \times B \) and in particular a monomorphism. The same arguments applies for condition (M5). Finally condition (M6) follows since for any map \( f : X \rightarrow Y \), the morphism \( t_f : X \rightarrow Pf \) is the section of the target map \( t_f : Pf \rightarrow X \) and in particular, it is a monomorphism.
Example 4.9. We can instantiate Proposition 4.8 on the presheaf toposes of simplicial sets $sSet$ and of cubical sets $cSet$ equipped with the obvious choices of interval objects given by the representable 1-simplex and 1-cube respectively. We thus obtain type-theoretic awfs on $sSet$ and $cSet$. Using [11, Theorem 9.9] we observe that in the case of $sSet$, the underlying morphism of a uniform fibration corresponds with the classical notion of a Kan fibration.

Although the proof of Theorem 4.7 uses only constructive arguments, it has been pointed out, for example in [21], that in order to construct a univalent universe à la Hofmann-Streicher [16] in a constructive setting, it is necessary to restrict the category $M_{all}$ of generating monomorphisms to that of decidable ones; i.e. those monomorphism $i : A \to B$ in $sSet$ or $cSet$ (or more generally in any presheaf category) that have level-wise decidable image.

Remark 4.10. It turns out that the arguments in this section will not apply if we take $M_{dec}$ as the category of generating monomorphisms, where $M_{dec}$ is the subclass of $M_{all}$ of decidable monomorphisms (for either $sSet$ or $cSet$). The issue lies on verifying condition (M6), i.e. that the first leg of the factorisation of the diagonal $r_f : X \to Pf$ for any morphisms $f : X \to Y$ lies in the class $M_{dec}$. Intuitively, the morphism $r_f$ maps an object of $x$ of $X$ to the degenerate path on $x$, this morphism is not decidable because, in general, it is not possible to decide degeneracies [4]. However, see [10].

5. Normal Uniform Fibrations

In this section, we develop the notion of a normal uniform fibration in the context of a suitable topos $(E, I, M)$ (Definition 4.1). Recall from Remark 4.4 (and from [11, Definition 7.3]) that the category of arrows of uniform fibrations was constructed by right orthogonality form that of generating trivial cofibrations $I_{=} \to E$. We define a new category of arrows:

$$I^n_{=} \to E^{-1}$$

such that a right $I^n_{=}$-map will consist of a uniform fibration with an extra ‘normality’ property.

The idea is that $I^n_{=} \to E^{-1}$ will be obtained form $I_{=} \to E^{-1}$ by adding, for each generating monomorphism $i : A \to B$ and for $k \in \{0, 1\}$, the coherence square on the left of the following diagram:

$$\begin{array}{c}
B +_A (I \times A) \xrightarrow{sq_k(i)} B \\
\delta^k \xi \downarrow \quad \downarrow \epsilon \times B \\
I \times B \quad \quad B
\end{array}$$

and

$$\begin{array}{c}
A \\
\delta^k \xi \downarrow \quad \downarrow \epsilon \times A \\
I \times A \\
\epsilon \times B \\
B +_A (I \times A) \xrightarrow{sq_k(i)} B
\end{array}$$

where the map $sq_k(i) : B +_A (I \times A) \to B$ is the universal map out of the pushout as described on the right of the previous diagram. The arrows $\epsilon \times B$ and $\epsilon \times A$ are the
projections from the second component of the product. We refer to the square on the left of \( \square \) as the \( k \)-squash square of \( i : A \to B \) and we will denote it by

\[
squash_k(i) : \delta^k \hat{\times} i \to \text{id}_B.
\]

The name follows the intuition of squashing the mapping cylinder in the direction of the interval (i.e. the filling direction). The following technical result about squash squares will be needed in what follows.

**Lemma 5.1.** Let \( k \in \{0, 1\} \) and consider monomorphisms \( i : A \to B \) and \( j : C \to D \). Then applying the pushout-product functor \( (j \hat{\times} -) : E^\to \to E^\to \) to the \( k \)-squash square of \( i : A \to B \), produces the \( k \)-squash square of \( j \hat{\times} i \); that is:

\[
j \hat{\times} (\text{squash}_k(i)) \cong \text{squash}_k(j \hat{\times} i) : \delta^k \hat{\times} (j \hat{\times} i) \to \text{id}_{D \times B}
\]

**Proof.** If we apply \( (j \hat{\times} -) : E^\to \to E^\to \) to the \( k \)-squash square of \( i : A \to B \), using that the pushout-product is symmetric and associative, we will get the following square:

\[
dom(\delta^k \hat{\times} (j \hat{\times} i)) \xrightarrow{\Theta} D \times B
\]

\[
\delta^k \hat{\times} (j \hat{\times} i)
\]

\[
I \times (D \times B) \xrightarrow{\epsilon \times (D \times B)} D \times B
\]

where we only need to verify that the top horizontal arrow \( \Theta \) is the squash morphism, that is, we need to verify that \( \Theta = \text{sq}_k(j \hat{\times} i) : \text{dom}(\delta^k \hat{\times} (j \hat{\times} i)) \to D \times B \), but this follows since the diagram commutes. \( \square \)

We now proceed to construct the arrow category \( \mathcal{I}_\times^u \to E^\to \) that will generate the category of normal uniform fibrations. We do this as follows. First let us denote by \( \ulcorner \) the ‘walking arrow’, that is the poset with two objects \( 0 < 1 \) considered as a category, this has the structure of an interval object in \( \text{Cat} \) and we denote the inclusions by:

\[
\ast \xrightarrow{0} \ulcorner \xrightarrow{1} \ulcorner
\]

We define \( \mathcal{I}_\times^u := \ulcorner \times \mathcal{I}_\times \), where \( \mathcal{I}_\times \) is the generating category of uniform fibrations. The functor down to \( E^\to \) is determined by the following two properties.

1. The following diagram commutes:

\[
\mathcal{I}_\times \xrightarrow{\rho_0} \mathcal{I}_\times^u \xrightarrow{\rho_1} \mathcal{I}_\times
\]

where the map \( \epsilon_{\text{cod}} : \mathcal{I}_\times \to E^\to \) sends an object \( i \in \mathcal{I}_\times \) to the identity arrow on the codomain of \( i \) (recall that \( \mathcal{I}_\times = \mathcal{I} + \mathcal{I} \)).

2. For \( k \in \{0, 1\} \) and for each \( i : A \to \Delta^n \) in \( \mathcal{I} \), the functor \( u_{\Delta^n}^i \) takes the arrow in \( \Delta \times \mathcal{I}_\times \) of the form \( \ulcorner \times i : \{0\} \times i \to \{1\} \times i \), to the \( k \)-squash square of \( i \); i.e. \( \text{squash}_k(i) : \delta^k \hat{\times} i \to \text{id}_{\Delta^n} \).
in other words, \( T^n \rightarrow E \rightarrow \) is a natural transformation: \( u \rightarrow \epsilon_{\text{cod}} : T^n \rightarrow E \rightarrow \) whose components are the k-squash squares.

We define \( \text{NrmUniFib} \rightarrow E \rightarrow \) to be the category of arrows of right \( T^n \rightarrow \) maps in \( E \), and we call its objects normal uniform fibrations. Using Garner’s small object argument \([12, \text{Theorem 4.4}]\), we obtain the following result.

**Theorem 5.2.** There is an algebraically-free awfs on the category of arrows \( T^n \rightarrow E \rightarrow \), denoted by \( (\text{NC}_t, \text{NF}) \), whose category of NF-algebras is that of normal uniform fibrations. □

Let us observe that the forgetful functor into \( E \rightarrow \) factors through the category of uniform fibrations, i.e. we have a commutative diagram:

\[
\begin{array}{ccc}
\text{NrmUniFib} & \xrightarrow{U} & \text{UniFib} \\
\downarrow & & \downarrow \\
E \rightarrow & & E \rightarrow 
\end{array}
\]

Moreover, we can prove the following lemma.

**Lemma 5.3.** The forgetful functor \( U : \text{NrmUniFib} \rightarrow \text{UniFib} \) is fully-faithful.

**Proof.** This follows intuitively by noticing that the structure of a normal uniform fibration does not add any new lifting problems to that of a uniform fibration; this is because the only new vertical arrows we are adding are identities and every morphism have a unique lift against them. Concretely, if \( (f, \phi) \in \text{NrmUniFib} \) and if \( (f, \theta) \in \text{UniFib} \), then both lifting structures \( \phi \) and \( \theta \) produce lifts against the exactly the same squares, the difference is that \( \phi \) may have additional coherence properties. □

In the following proposition we characterise those uniform fibration structures that are normal. We use the following terminology: we say that a morphism \( \theta : I \times B \rightarrow X \) is degenerate in the lifting direction if it factors through the projection \( \rho_1 : I \times B \rightarrow B \) via some arrow \( \theta^* : B \rightarrow X \); we call \( \theta^* \) the lifting degeneracy of \( b \).

**Proposition 5.4.** Let \( (f, \theta) \in \text{UniFib} \) then the following are equivalent:

1. \( (f, \theta) \) is a normal uniform fibration.
2. For any generating monomorphism \( i : A \rightarrow A \) in \( I \) (i.e. with \( A \in \mathbb{A} \)) and for any square:

\[
\begin{array}{ccc}
A +_A (I \times A) & \xrightarrow{a} & X \\
\delta^k \times i & \downarrow & \text{id}_X \\
I \times A & \xrightarrow{\theta_i(a,b)} & Y \\
\end{array}
\]

if the square factors through the squash square of \( i \) as \( \delta^k \times i \rightarrow \text{id}_A \rightarrow (a^*, b^*) \rightarrow f \), then the lift \( \theta_i(a, b) \) is degenerate in the lifting direction with \( a^* \) as lifting degeneracy.
(3) For any generating monomorphism \( i : A \rightarrow B \) in \( \mathcal{M} \) and for any square:

\[
\begin{array}{ccc}
B +_A (I \times A) & \xrightarrow{a} & X \\
\downarrow \delta^k \circ i & & \downarrow f \\
I \times B & \xrightarrow{\theta_i(a,b)} & Y
\end{array}
\]

if the square factors through the squash square of \( i \) as

\[
\begin{array}{ccc}
\delta^k \circ i & \xrightarrow{\text{squash}_1(a^*,b^*)} & f \\
\downarrow & & \downarrow \\
\text{id}_B & \xrightarrow{(a^*,b^*)} & f
\end{array}
\]

then the lift \( \theta_i(a,b) \) is degenerate in the lifting direction with \( a \) as lifting degeneracy.

**Proof.** Let us first assume that \( (f, \theta) \) is a normal uniform fibration. It is easy to see that item (2) holds, for this consider the diagram:

\[
\begin{array}{ccc}
\mathcal{A} +_A (I \times A) & \xrightarrow{\text{squash}_1(a^*,b^*)} & X \\
\downarrow \delta^k \circ i & & \downarrow f \\
I \times A & \xrightarrow{\theta_x(i)} & Y
\end{array}
\]

it is clear that the lifts cohere because the left square is by definition a morphism in (the image of) \( \mathcal{T}^n_{\mathcal{A}} \rightarrow \mathcal{E} \).

It is also easy to see that (2) implies (1), this follows since the uniform fibration structure \( \theta \) already provides lifts against all lifting problems coming from \( u_{\otimes} : \mathcal{T}_{\mathcal{A}} \rightarrow \mathcal{E}^{\rightarrow} \). So we only need to verify that it coheres with all the squares coming from \( u_{\otimes} \): \( \mathcal{T}_{\mathcal{A}} \rightarrow \mathcal{E}^{\rightarrow} \). So we only need to verify that it coheres with the squash squares, but these squares are precisely those as in the hypothesis of item (2).

It is clear that (3) implies (2). For the converse let us first observe, using that colimits in \( \mathcal{E} \) are universal, that any monomorphism \( i : A \rightarrow B \), is the colimit over the generalised elements with domain on the dense subcategory \( \mathcal{A} \); that is

\[
i \cong \colim_{\mathcal{A} \in \mathcal{A}} A^x(i)
\]

where for each \( x : \mathcal{A} \rightarrow \mathcal{B} \) we denote by \( x^*(i) \) the pullback of \( i \) along \( x \). Now, since \( \delta^k \circ - : \mathcal{E}^{\rightarrow} \rightarrow \mathcal{E}^{\rightarrow} \) is cocontinuous, we have that:

\[
\colim_{\mathcal{A} \in \mathcal{A}} (\delta^k \circ (x^*(i))) \cong i\colim_{\mathcal{A} \in \mathcal{A}} (\delta^k \circ (x^*(i))) \cong \delta^k \circ i
\]

Now let us suppose that (2) holds, and we have a diagram as in item (3). Then for each generalised element \( x : \mathcal{A} \rightarrow \mathcal{B} \) with \( \mathcal{A} \in \mathcal{A} \), we have a square:

\[
\begin{array}{ccc}
\mathcal{A} +_{\mathcal{A}} (I \times \mathcal{A}) & \xrightarrow{\gamma_x} & B +_A (I \times A) \\
\downarrow \delta^k \circ x^*(i) & & \downarrow \delta^k \circ i \\
I \times \mathcal{A} & \xrightarrow{\theta_{\gamma_x(i)}} & I \times B \\
\downarrow \iota & & \downarrow b \\
I \times X & \xrightarrow{\theta_i(a,b)} & Y
\end{array}
\]

where the left square is the colimit inclusion corresponding to \( x : \mathcal{A} \rightarrow \mathcal{B} \). The commutation of the respective triangle is obtained by the universal property of the colimit.
Finally, if the square on the right factors through a squash square
\[ \delta^k \circ \text{squash} \xrightarrow{i} \text{id}_B \xrightarrow{(a^*, b^*)} f \]
then (by naturality) the outer square also factor through a squash square and thus the lift \( \theta_{x^*} \) is degenerate with \( a^* \circ x \) as lifting degeneracy. This implies by the uniqueness of the universal property, that also \( \theta_i \) is degenerate with \( a^* \) as lifting degeneracy. \( \square \)

**Remark 5.5.** To guide our intuition towards normal uniform fibrations, we can compare the notions of normality for cloven isofibrations in groupoids and for uniform fibrations in simplicial sets. For this, we consider the awfs of (normal) uniform fibrations on simplicial sets constructed from the suitable topos structure consisting of the 1-simplex as the interval object and the class \( M_{\text{all}} \) of all monomorphisms as the class of generating monomorphisms.

It is not hard to show that the following are pullback squares:

\[
\begin{array}{ccc}
\text{NrmFib} & \xrightarrow{N} & \text{NrmUniFib} \\
\downarrow & & \downarrow \\
\text{ClFib} & \xrightarrow{N} & \text{UniFib} \\
\downarrow & & \downarrow \\
\text{Grd} & \xrightarrow{N} & \text{sSet} \\
\end{array}
\]

Here the categories ClFib and NrmFib are those of cloven isofibrations and normal cloven isofibrations in groupoids while the horizontal arrows are given by the nerve functor and its respective lifts.

This shows how the notion of uniform fibration (respectively normal uniform fibration) is a generalisation to higher dimensions of the notion of cloven isofibration (respectively normal cloven isofibrations).

The category of arrows of *normal trivial cofibrations* is defined to be the category of NC\(_T\)-maps with respect to the awfs of normal uniform fibrations Theorem 5.2. Alternatively, it is the left orthogonal category of arrows of NrmUniFib. We will denote it by NrmTrivCof.

We would like a way to characterise normal trivial cofibrations; however a complete characterisation remains elusive. The next best thing we can have is a general method for constructing normal trivial cofibrations from a structure that is easier to handle, and this we can do. For this we recall the following notion.

Consider a suitable topos \( (E, I, M) \). We define a **strong deformation retraction** to be a tuple \( (g, r, h) \) such that \( (g, r, h, \epsilon) \) is a strong homotopy equivalence where epsilon denotes the constant homotopy (see [11, Definition 4.1]). We have that strong deformation retractions and morphisms of such form a category of arrows, which we denote \( SDR \to E^{-}\).

It is easy to verify that there is a functor over \( E^{-}\) form the category of strong deformation retracts to that of strong homotopy equivalences:

\[
\begin{array}{c}
SDR \\
\xrightarrow{S}
\end{array}
\]
the action on objects is given by \((g, r, h) \mapsto (g, r, h, e)\).

In the next proposition, we observe that every strong deformation retract has (uniformly) the structure of a normal trivial cofibration. Normality is an essential ingredient in the proof, in particular, a similar result would not hold for uniform fibrations.

**Proposition 5.6.** There is a functor from the category strong deformation retracts \(\text{SDR}\) to that of normal trivial cofibrations \(\text{NrmTrivCof}\) as shown in the following diagram:

\[
\begin{array}{ccc}
\text{SDR} & \xrightarrow{\Psi} & \text{NrmTrivCof} \\
\end{array}
\]

*Proof.* Let \((g, r, h) \in \text{SDR}\) which we assume to be 0-oriented (the other case being analogous). We have to define \(\Psi(g, r, h) := (g, \Psi h)\) with \(\Psi h\) a left \(\text{NrmUniFib}\)-map structure for \(g\). To do this, let’s consider a normal uniform fibration \((f, \phi)\) and a square \((a, b) : g \to f\) for which we will construct a lift \(\Psi h_f : B \to X\) as shown:

\[
\begin{array}{ccc}
A & \xrightarrow{a} & X \\
\downarrow{g} & & \downarrow{f} \\
B & \xrightarrow{b} & Y \\
\end{array}
\]

We first consider the lift \(H : I \times B \to X\), in the following square (which commutes because the deformation retraction is 0-oriented), produced by the normal uniform fibration structure of \(f\):

\[
\begin{array}{ccc}
B & \xrightarrow{f} & A & \xrightarrow{a} & X \\
\downarrow{\delta^0 \times B} & & \downarrow{\psi h_f} & & \downarrow{f} \\
I \times B & \xrightarrow{h} & B & \xrightarrow{b} & Y \\
\end{array}
\]

and we define \(\psi h_f := H \cdot (\delta^1 \times B)\), that is, the lift \(\psi h_f\) is defined to be \(H\) on restricted to the top of the cylinder \(I \times B\).

The verification that \(\psi h_f \cdot g = a\) is straightforward, while the verification that \(f \circ \psi h_f = b\) requires the use of the extra property of normality for \(f\). \(\square\)

### 6. Type-Theoretic AWFS from Normal Uniform Fibrations

In order to equip the awfs \((\text{NC}_1, \text{NF})\) of normal uniform fibrations with the structure of a type-theoretic awfs we require a functorial Frobenius structure and a stable functorial choice of path objects. In this section, we show how to do construct these.

We focus first with the construction of a stable functorial choice of path objects. We work in the context of a suitable topos \((\mathcal{E}, \mathcal{I}, \mathcal{M})\) that in addition satisfies hypothesis \((M5)\) from Proposition 4.5.

Recall form the discussion preceding Proposition 4.5 that a suitable topos has a canonical stable and functorial factorisation of the diagonal, called \(\mathcal{P}_1\), which is constructed via exponentiation by the interval. Our objective is to lift this factorisation to a stable functorial choice of path objects. That is, we need to exhibit a lift of \(\mathcal{P}_1\) as shown in the following diagram:

\[
\begin{array}{ccc}
\text{NrmUniFib} & \xrightarrow{\mathcal{P}_1} & \text{NrmTrivCof} \times \text{NrmUniFib} \\
\end{array}
\]
Notice that we can split the problem in two. If we denote by \( r, \rho : E \to E \) the two legs of the sfpo (i.e. by composing \( P_1 \) with the two projections from the pullback), then it is sufficient to show that there are lifts of these functors as in the following diagram.

\[
\begin{array}{ccc}
\text{NrmUniFib} & \xrightarrow{r} & \text{NrmTrivCof} \\
\text{NrmUniFib} & \xrightarrow{\rho} & \text{NrmUniFib}
\end{array}
\]

In order to prove this, we make use of the following lemmas whose proofs are found as part of the proof of Proposition 4.5.

**Lemma 6.1.** There is a lift of the functor \( r : E \to E \) to the category of strong deformation retracts as shown:

\[
\begin{array}{ccc}
E & \xrightarrow{r} & E \\
\downarrow & & \downarrow \\
E & \xrightarrow{r} & E
\end{array}
\]

\[\square\]

**Lemma 6.2.** There is a lift of the functor \( \rho : E \to E \) to the category of uniform fibrations as shown:

\[
\begin{array}{ccc}
\text{UniFib} & \xrightarrow{\rho} & \text{UniFib} \\
\downarrow & & \downarrow \\
E & \xrightarrow{\rho} & E
\end{array}
\]

\[\square\]

Making use of Lemma 6.1 and Proposition 5.6, it is easy to see that we obtain the desired lift of \( r : E \to E \).

**Lemma 6.3.** There is a lift the functor \( r : E \to E \) to the category of uniform trivial cofibrations as shown:

\[
\begin{array}{ccc}
\text{NrmUniFib} & \xrightarrow{r} & \text{NrmTrivCof} \\
\downarrow & & \downarrow \\
E & \xrightarrow{r} & E
\end{array}
\]

**Proof.** We construct the desired lift as the following composite:

\[
\begin{array}{ccc}
\text{NrmUniFib} & \xrightarrow{r} & \text{SDR} & \xrightarrow{\Psi} & \text{NrmTrivCof} \\
\downarrow & & \downarrow & & \downarrow \\
E & \xrightarrow{r} & E & \xrightarrow{\Psi} & E
\end{array}
\]

where the lift in the leftmost square is the forgetful functor, that on the middle square is that form Lemma 6.1 and the lift in the rightmost square is the one form Proposition 5.6.

\[\square\]

**Remark 6.4.** The proof of Claim 2 (Proposition 4.5), which shows that the functor \( r : E \to E \) lifts to the category of left maps of the awfs of uniform fibrations, relied crucially on the hypothesis (M6). This says that the image of \( r \) lands on the class \( \mathcal{M} \) of generating monomorphisms of the suitable topos. As noted in Remark 4.10 hypothesis (M6) does not hold if we consider \( \mathcal{M}_{\text{dec}} \), the class of decidable monos in the context.
of a presheaf topos. However notice that the proof of Lemma 6.3 does not require hypothesis (M6). In other words, the extra ‘normality’ condition on the category of uniform fibrations allows us to get rid of this requirement.

The construction of the lift for the other functor \( \rho : E^{-} \rightarrow E^{-} \) is not quite as direct; we will need to briefly recall the construction of the uniform fibration structure produced by Lemma 6.2.

Let us consider a map \( f : X \rightarrow Y \) in \( E \); recall (from the discussion before Proposition 4.5) that the second leg of the factorisation of the diagonal, \( \rho_f : Pf \rightarrow X \times Y \) is alternatively obtained as a pullback of the map \( \text{h} \text{om}(i,f) \) where \( i : \partial I \rightarrow I \) stands for the inclusion of the boundary of the interval.

Let us assume for now that \( (f, \theta) \) is a uniform fibration. We know that right orthogonal categories of arrows are closed under pullbacks, thus to give a uniform fibration structure to \( \rho_f \) it is sufficient to give one to \( \text{h} \text{om}(i,f) \).

Now, in order to construct a uniform fibration structure for \( \text{h} \text{om}(i,f) \), let us consider a lifting problem with respect to a morphism of the generating category of arrows \( I \times B \) of uniform fibrations; i.e. a square of the form \( \delta^k \hat{\chi} : \delta^k \times j \rightarrow \text{h} \text{om}(i,f) \) where \( j : A \rightarrow B \) is in \( I \), for which we will show how to construct a lift. This is shown in the left side of the following diagram.

Transposing along the adjunction \( (i \times -) \vdash \text{h} \text{om}(i,-) \) we obtain a square as on the right of the previous diagram. We use that the pushout-product construction is symmetric and associative, and in particular we obtain that \( i \times (\delta^k \hat{\chi} j) \) is a generating monomorphism, thus we find a lift for the square on the right of the previous diagram, denoted by \( \rho_\theta \).

With this in place we can now state and prove the following lemma. We will make use of the explicit construction of the uniform fibration structure \( \rho \theta \) described above.

**Lemma 6.5.** There is a lift of the functor from Lemma 6.2 as shown:

\[
\begin{array}{c}
N\text{rmUniFib} \xrightarrow{0} N\text{rmUniFib}
\end{array}
\]

**Proof.** Since the forgetful functor \( N\text{rmUniFib} \rightarrow \text{UniFib} \) is fully faithful (Lemma 5.3), and using that right orthogonal categories are closed under pullbacks; it is sufficient to prove that given \( (f, \psi) \) a normal uniform fibration, the uniform fibration structure \( \rho \psi \) of \( \text{h} \text{om}(i,f) \), described in the foregoing discussion, is also normal.
Using Proposition [5.4] we need to show that for any generating monomorphism \( j : A \rightarrow B \) the lifts in the diagram on the left of the following figure cohere:

\[
\begin{array}{ccc}
B +_A (I \times A) \xrightarrow{\text{squash}^i} & B \xrightarrow{U^*} & X^I \\
\downarrow \delta^k \hat{\gamma} j & \downarrow \text{dom}(\delta^k \hat{\gamma} (i \hat{\gamma} j)) & \downarrow \text{hom}(i,f) \\
I \times B & \xrightarrow{\text{hôm}(i,\cdot)} & I \times (I \times B) \xrightarrow{\text{squash}^i (i \hat{\gamma} j)}
\end{array}
\]

\[
\begin{array}{ccc}
dom(\delta^k \hat{\gamma} (i \hat{\gamma} j)) \xrightarrow{\text{dom}(\delta^k \hat{\gamma} (i \hat{\gamma} j))} & I \times B \xrightarrow{U^*} & X \\
\downarrow \delta^k \hat{\gamma} (i \hat{\gamma} j) & \downarrow \text{dom}(\delta^k \hat{\gamma} (i \hat{\gamma} j)) & \downarrow \text{dom}(\delta^k \hat{\gamma} (i \hat{\gamma} j)) \\
I \times (I \times B) \xrightarrow{\text{hôm}(i,\cdot)} & I \times B \xrightarrow{U^*} & X
\end{array}
\]

by transposing the whole diagram along \((i \hat{\gamma} \cdot) \vdash \text{hôm}(i,\cdot)\), and using the symmetry and associativity of the pushout-product, we obtain the lifting problem as on the right of the previous diagram, for which we need to show that the lifts cohere. The lift \( \rho \theta_j \) on the left (on either diagram) is, by construction, the lift obtained from the uniform fibration structure \( \rho \theta \) on \( \text{hôm}(i,f) \). The result follows by applying Lemma [5.1]. \( \square \)

We synthesise these results in the following proposition.

**Proposition 6.6.** Consider a suitable topos \((E, I, M)\) satisfying condition \((M5)\). Then the stable functorial factorisation of the diagonal \( P_I \) lifts to a stable functorial choice of path objects for the awfs of normal uniform fibrations; as shown in the following diagram:

\[
\begin{array}{ccc}
\text{NrmUniFib} \xrightarrow{P_I} \text{NrmTrivCof} \times_C \text{NrmUniFib} \\
\downarrow \downarrow \\
E^\rightarrow \xrightarrow{P_I} E^\rightarrow \times_E E^\rightarrow
\end{array}
\]

**Proof.** This follows by applying Lemma [6.3] to lift the functor \( r : E^\rightarrow \rightarrow E^\rightarrow \) and by applying Lemma [6.2] and Lemma [6.5] to lift the functor \( \rho : E^\rightarrow \rightarrow E^\rightarrow \). \( \square \)

We turn our attention to the proof that the category of arrows of normal uniform fibrations has a functorial Frobenius structure. The structure is given by adapting the functorial Frobenius structure on uniform fibrations constructed in [11, Theorem 8.8]. Throughout this section, we will work on an arbitrary suitable topos \((E, I, M)\).

We first need a couple of lemmas.

**Lemma 6.7.** Let \( i : A \rightarrow B \) be a monomorphism, and let \( f : X \rightarrow B \) be any map. Then the following holds:

1. There is an isomorphism
   \[
   \delta^k \hat{\gamma} (f^* i) \cong (I \times f)^* (\delta^k \hat{\gamma} i)
   \]

2. Pulling back the k-squash square of \( i \) along the square \((I \times f, f)\) produces the k-squash square of \( f^* i \); concretely, for \( k \in \{0, 1\} \), there is an isomorphism:
   \[
   \text{squash}_k (f^* i) \cong (I \times f, f)^* (\text{squash}_k (i))
   \]
Proof. To show item (1), let us first consider the following cube:

\[
\begin{array}{c}
\text{\(f^* A\)}} \\
\downarrow \quad \downarrow \quad \downarrow f^* i \\
\delta_k \times f^* A \\
\downarrow \quad \downarrow \quad \downarrow f \\
X \\
\downarrow \quad \downarrow \quad \downarrow \pi \\
\text{\(I \times (f^* A)\)}} \\
\downarrow \quad \downarrow \quad \downarrow 1 \times f^* i \\
\text{\(I \times X\)}} \\
\downarrow \quad \downarrow \quad \downarrow \iota \\
\text{\(I \times B\)}}
\end{array}
\]

here, the square on the top is the pullback of \(i\) along \(f\). It is straightforward to verify that all squares pointing from left to right are Cartesian, and notice that the squares on the left and right are the outer squares used for defining the pushout-products \(\delta^k \hat{\times} (f^* i)\) and \(\delta^k \hat{\times} i\) respectively. All of this implies that there is a comparison map \(\delta^k \hat{\times} (f^* i) \rightarrow (I \times f)^*(\delta^k \hat{\times} i)\), which is an isomorphism because colimits in \(E\) are universal. Item (2) follows directly form item (1). \(\square\)

The next lemma is a technical result about the squares \(\theta^k \hat{\times} i : i \rightarrow \delta^k \hat{\times} i\) from [\ref{bib:types} Lemma 4.3].

Lemma 6.8. For any morphism \(i : A \rightarrow B\) the square \(\theta^k \hat{\times} i : i \rightarrow \delta^k \hat{\times} i\) depicted bellow:

\[
\begin{array}{c}
A \\
\downarrow i \\
B \\
\delta^{1-k} \times B
\end{array}
\]

\[
\begin{array}{c}
B +_A (I \times A) \\
\downarrow \text{\(\delta^k \hat{\times} i\)} \\
I \times B
\end{array}
\]

is Cartesian.

Proof. The proof uses once again the fact that colimits in \(E\) are universal. Let us compute the pullback of \(\delta^k \hat{\times} i\) along \(\delta^{1-k} \times B\). By universality of colimits, this is the same as pulling back the diagram defining \(B +_A (I \times A)\) and then calculating the colimit.

We can observe in the following picture, the result of first pulling back the defining diagram of \(B +_A (I \times A)\) which appears as the upper span of the right-most square on
Let us notice that the pullback of $\delta^k \times B$ (respectively $\delta^k \times A$) along $\delta^{1-k} \times B$ (respectively $\delta^{1-k} \times A$) is empty since the interval has disjoint endpoints. We conclude that the colimit of the upper span of the left-most square on the cube must be equal to $A$ and moreover, the universal arrow down to $B$ has to be $i : A \to B$. □

Consider a generating monomorphism $i : A \to B$ and a uniform fibration $f : X \to B$, then there are two possible trivial uniform cofibration structures on the map $\delta^k \hat{\times} (f^* i)$: the first one is the canonical one, i.e. the one given by the fact that $f^* i$ is also a generating monomorphism. The second one is the one provided by the functorial Frobenius structure on uniform fibrations using the isomorphism $\delta^k \hat{\times} (f^* i) \cong (I \times f)^* (\delta^k \hat{\times} i)$ of Lemma 6.7.

These two are actually the same structure as we show in the following lemma.

Lemma 6.9. Consider $i : A \to B$ be a generating monomorphism and $f : X \to B$ a uniform fibration. Then the two possible trivial uniform cofibration structures on $\delta^k \hat{\times} (f^* i)$ coincide.

Proof. Let us denote by $\lambda^1$ and $\lambda^2$, respectively, the canonical trivial uniform cofibration structure on $\delta^k \hat{\times} (f^* i)$ and the one obtained by applying the functorial Frobenius structure.

In order to prove they are the same, let us consider $g : Z \to Y$ a uniform fibration and a square $(a, b) : \delta^k \hat{\times} (f^* i) \to g$. Without loss of generality, let us denote by $\lambda^1$ and $\lambda^2$ the two fillers of this square given by the uniform trivial cofibration structure with the same name.

We have to show that $\lambda^1 = \lambda^2$. If we go over the proof of ??, just before concluding, we made use of a retract diagram in order to transfer the structure of a trivial cofibration to the desired morphism (since trivial cofibrations are closed under retracts). In our situation, this retract diagram is given by the two left-most squares shown bellow:

\[
\begin{array}{ccc}
\delta^k \hat{\times} (f^* i) & \xrightarrow{\delta^k \hat{\times} \delta^k \hat{\times} (f^* i)} & \delta^k \hat{\times} (f^* i) \\
\theta^k \hat{\times} \delta^k \hat{\times} (f^* i) & \xrightarrow{\delta^k \hat{\times} \delta^k \hat{\times} (f^* i)} & \delta^k \hat{\times} (f^* i) \\
t & \xrightarrow{f} & f \\
b & \xrightarrow{a} & Z \\
\end{array}
\]

where the left-most square is $\theta^k \hat{\times} \delta^k \hat{\times} (f^* i)$. Now, the square $\delta^k \hat{\times} \delta^k \hat{\times} (f^* i) \to f$ has a lift which we denote by $\lambda$, notice that by definition, the lift $\lambda^2$ is equal to $\lambda \cdot t$ where $t$ is the
horizontal arrow on the lower left part of the diagram. Moreover, we have that the lift of the outer square is \( \lambda^1 \). Thus if we want to show that \( \lambda^1 = \lambda^2 \) it is sufficient to show that the square \( \theta_k \hat{\otimes} \delta_k \hat{\otimes} (f^*i) \) is a morphism of trivial uniform cofibrations.

To show this, we use that the pushout-product is symmetric and associative, and thus \( \theta_k \hat{\otimes} \delta_k \hat{\otimes} (f^*i) \cong \delta_k \hat{\otimes} \theta_k \hat{\otimes} (f^*i) \). From this, we see that the square is a morphism of trivial uniform cofibrations provided the square \( \theta_k \hat{\otimes} (f^*i) \) is a morphism of generating monomorphisms, i.e. if it is Cartesian. But this is precisely the statement of Lemma 6.8.

We now have enough tools to show that the functorial Frobenius structure on uniform fibrartion given by [11, Theorem 8.8] can be extended to a functorial Frobenius structure on normal uniform fibrations. We start with the following proposition.

**Proposition 6.10.** There is a lift of the pullback functor as shown:

\[
\begin{array}{ccc}
\mathcal{I}_X^n \times_{\text{UniFib}} & \xrightarrow{\text{PB}} & \text{NrmTrivCof} \\
\downarrow & & \downarrow \\
\mathcal{E}^{-} \times_{\mathcal{E}} & \xrightarrow{\text{PB}} & \mathcal{E}^{-}
\end{array}
\]

**Proof.** Object-wise, this follows directly from [11, Theorem 8.8]. To see this, we notice that there are no more objects in \( \mathcal{I}_X^n \) than in \( \mathcal{I}_X \) thus we can apply the functorial Frobenius structure for uniform fibrations. Then we use the functor \( \text{TrivCof} \rightarrow \text{NrmTrivCof} \), obtain by functoriality of the left orthogonal functor \( \Box (-) \) applied to the forgetful functor \( \text{NrmUniFib} \rightarrow \text{UniFib} \).

For the morphism case, we first notice that the only morphisms in \( \mathcal{I}_X^n \) that we need to consider are the squash squares. Thus let us consider a cospan of squares as in the following diagram:

\[
\begin{array}{ccc}
\text{sq}_k(i) & \xrightarrow{\delta_k \hat{\otimes} i} & B \\
X' & \xrightarrow{f'} & I \times B \\
\downarrow m & & \downarrow e \times B \\
X & \xrightarrow{f} & B
\end{array}
\]

such that the vertical square is the squash square of a generating monomorphism \( i : A \rightarrow B \) and the horizontal square is a morphism of uniform fibrations \( (m, e \times B) : f' \rightarrow f \).

We need to verify that pulling back the squash square along the morphism of uniform fibrations is a morphism of normal trivial cofibrations.
The first thing we do is to split this cospan of squares into two, by factoring through the pullback square of \( f \) along \( \epsilon \times B \). That is we obtain the following diagrams:

\[
\begin{array}{ccc}
I \times X & \xrightarrow{I \times f} & I \times B \\
\epsilon \times X & \searrow & \epsilon \times B \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & B
\end{array}
\quad \begin{array}{ccc}
X' & \xrightarrow{f'} & I \times B \\
m^* & \searrow & \delta^k \hat{\times} \hat{\times} \\
\downarrow & & \downarrow \\
I \times X & \xrightarrow{I \times f} & I \times B
\end{array}
\]

where the dotted arrow \( m^* : X' \rightarrow (I \times X) \) is obtain by universal property. Notice that composing the two cospans of squares along their common face, produces the original one. Notice also that the two horizontal squares are morphisms of uniform fibrations.

Let us focus first on the cospan of the right. The identity morphism \( \text{id} : (\delta^k \hat{\times} \hat{\times}) \rightarrow (\delta^k \hat{\times} \hat{\times}) \) is a morphism of trivial uniform cofibrations, thus if we pull-back this along the morphism of uniform fibrations \( (f', I \times f) : m^* \rightarrow \text{id}_{\delta^k \hat{\times} \hat{\times}} \) we obtain a morphism of trivial uniform cofibrations by [11, Theorem 8.8] to which we can apply the functor \( \text{TrivCof} \rightarrow \text{NrmTrivCof} \) to obtain a morphism of normal trivial cofibrations.

With this we have reduced the situation to the cospan of squares on the left of the previous diagram. Using item (2) of Lemma 6.7 we see that the pullback of the squash square of \( i : A \rightarrow B \) along the square \( (I \times f, f) : \epsilon \times X \rightarrow \epsilon \times B \) is the squash square of \( f^* i : f^* A \rightarrow X \). This square is a morphism in \( I \times X \) provided that the canonical trivial normal cofibration structure of \( \delta^k \hat{\times} (f^* i) \) is the same as that obtained from the functorial Frobenius structure; but this follows from Lemma 6.9. \( \Box \)

We are now ready to state and prove the following proposition.

**Proposition 6.11.** Let \((\mathcal{E}, I, M)\) be a suitable topos. Then the awfs \((\text{NCt}, \text{NF})\) of normal uniform fibrations has a functorial Frobenius structure.

**Proof.** Using the lift of Proposition 6.10 and the forgetful functor \( \text{NrmUniFib} \rightarrow \text{UniFib} \), we find a lift of the pullback functor as one shown:

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\text{PB}} & \text{NrmTrivCof} \\
\downarrow & & \downarrow \\
\mathcal{E} \rightarrow & \rightarrow & \mathcal{E} \rightarrow
\end{array}
\]

The fact that we can extend this structure from \( \mathcal{E} \) to the whole category \( \text{NrmTrivCof} \) follows from [11, Proposition 6.8]. \( \Box \)

The following main theorem follows immediately from the results obtained so far.
**Theorem 6.12.** Consider a suitable topos $\mathcal{E}$ satisfying condition $(M5)$. Then the awfs $(\text{NC}, \text{NF})$ of normal uniform fibrations has the structure of a type-theoretic awfs.

*Proof.* Follows from Proposition 6.6 and Proposition 6.11.

---

**7. Conclusions**

We have shown that by making use of algebraic techniques it is possible to obtain sufficient structure to model a version of Martin-Löf’s dependent type theory, equipped with dependent sums, products and intensional identity types. In order to do so, we introduced the notion of a type-theoretic algebraic weak factorisation system. There are two main reasons for the interest in this notion, as opposed to its non-algebraic counterpart. First, the condition of pseudo-stability for intensional identity types can be obtained from the more natural notion of a path-objects. Second, by making use of the theory of uniform fibrations of [11], we have shown that type-theoretic awfs are abundant.

Moreover, by slightly adapting the methodology in [11], we are able to produce a type-theoretic awf of normal uniform fibrations. This allows us to circumvent one of the requirements that the interval path-object factorisation need to satisfy in order to produce a stable functorial choice of path objects. With this we are able to carry out some arguments in a constructive meta-theory instead of a classical one.

In a nutshell, we have shown that most of the type-theoretic properties that are present in the non-algebraic approaches to the categorical semantics of type theory, have a direct categorification in the language of awfs. We expect that this approach can be extended to accommodate additional kinds of logical structure such as $W$-types and universes. The payoff of working with the additional algebraic structure is that we are able to apply the right adjoint splitting to obtain models, which in some other approaches to the semantics of dependent type theory has been abandoned in favor of other methods (such as the left adjoint splitting) due to the difficulties imposed by the pseudo-stability conditions [19].

Future work includes adapting the definition of type-theoretic awfs in order to include the relevant structure needed to produce models with additional logical structure. Of particular interest is the case of universes; we would like to identify sufficient additional structure that a type-theoretic awfs should posses in order to model these. Afterwards, we could ask if it is possible to adapt the methodology of uniform fibrations to produce such structure. The models based on uniform fibrations in cubical sets would provide useful guidance to develop this theory. Additionally we could investigate under which circumstances the universes produced in this manner are univalent.

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