THE ORTHOGONAL COMPLEMENT RELATIVE TO THE FUNCTOR EXTENSION OF THE CLASS OF ALL GORENSTEIN PROJECTIVE MODULES

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Abstract

In this paper, we study the pair \((GP(R), GP(R)^\perp)\) where \(GP(R)\) is the class of all Gorenstein projective modules. We prove that it is a complete hereditary cotorsion theory, provided \(l.Ggldim(R) < \infty\). We discuss also, when every Gorenstein projective module is Gorenstein flat.

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1 Introduction

Throughout the paper, all rings are associative with identity, and an \(R\)-module will mean left \(R\)-module unless explicitly stated otherwise.

Let \(R\) be a ring, and let \(M\) be an \(R\)-module. As usual, we use \(pd_R(M)\), \(id_R(M)\), and \(fd_R(M)\) to denote, respectively, the classical projective dimension, injective dimension, and flat dimension of \(M\). We denote by \(M^+ = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})\) the character module of \(M\).

For a two-sided Noetherian ring \(R\), Auslander and Bridger [1] introduced the \(G\)-dimension, \(Gdim_R(M)\), for every finitely generated \(R\)-module \(M\). They showed that \(Gdim_R(M) \leq pd_R(M)\) for all finitely generated \(R\)-modules \(M\), and equality holds if \(pd_R(M)\) is finite.

Several decades later, Enochs and Jenda [5, 6] introduced the notion of Gorenstein projective dimension (\(G\)-projective dimension for short), as an extension of \(G\)-dimension to modules that are not necessarily finitely generated, and the Gorenstein injective dimension (\(G\)-injective dimension for short) as a dual notion of Gorenstein projective dimension. Then, to complete the analogy with the classical homological dimension, Enochs, Jenda, and Torrecillas [8] introduced the Gorenstein flat dimension. Some references are [2, 3, 4, 5, 6, 8, 12].

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Recall that an $R$-module $M$ is called \textit{Gorenstein projective}, if there exists an exact sequence of projective $R$-modules:

$$P : \cdots \longrightarrow P_{-1} \longrightarrow P_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \cdots$$

such that $M \cong \text{Im}(P_0 \to P_1)$ and such that the functor $\text{Hom}_R(\_, Q)$ leaves $P$ exact whenever $Q$ is a projective $R$-module. The complex $P$ is called a \textit{complete projective resolution}.

The \textit{Gorenstein injective} $R$-modules is defined dually.

An $R$-module $M$ is called \textit{Gorenstein flat}, if there exists an exact sequence of flat $R$-modules:

$$F : \cdots \longrightarrow F_{-2} \longrightarrow F_{-1} \longrightarrow F_0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \cdots$$

such that $M \cong \text{Im}(F_0 \to F_1)$ and such that the functor $I \otimes_R \_ - \text{leaves} F$ exact whenever $I$ is a right injective $R$-module. The complex $F$ is called a complete flat resolution.

The Gorenstein projective, injective, and flat dimensions are defined in terms of resolutions and denoted by $\text{Gpd}_R(\_)$, $\text{Gid}_R(\_)$, and $\text{Gfd}_R(\_)$, respectively (see [3, 7, 12]).

\textbf{Notation.} By $\mathcal{P}(R)$ and $\mathcal{I}(R)$ we denote the classes of all projective and injective $R$-modules, respectively, and by $\mathcal{P}(R)$ and $\mathcal{I}(R)$ we denote the classes of all modules with finite projective dimensions and injective dimensions, respectively. Furthermore, we let $\mathcal{G}\mathcal{P}(R)$ and $\mathcal{G}\mathcal{I}(R)$ denote the classes of all Gorenstein projective and injective $R$-modules, respectively.

In [2], the authors proved the equality

$$\sup\{\text{Gpd}_R(M) \mid M \text{ is a (left) } R\text{-module}\} = \sup\{\text{Gid}_R(M) \mid M \text{ is a (left) } R\text{-module}\}.$$  

They called the common value of the above quantities the \textit{left Gorenstein global dimension} of $R$ and denoted it by $l.\text{Ggldim}(R)$. Similarly, they set

$$l.w\text{Ggldim}(R) = \sup\{\text{Gfd}_R(M) \mid M \text{ is a (left) } R\text{-module}\}$$

which they called the \textit{left weak Gorenstein global dimension} of $R$.

Given a class $\mathcal{X}$ of $R$-modules we set:

$$\mathcal{X}^\perp = \ker\text{Ext}_R^1(\mathcal{X}, \_), = \{M \mid \text{Ext}_R^1(X, M) = 0 \text{ for all } X \in \mathcal{X}\}.$$  

$$\mathcal{X} = \ker\text{Ext}_R^1(\_, \mathcal{X}) = \{M \mid \text{Ext}_R^1(M, X) = 0 \text{ for all } X \in \mathcal{X}\}.$$  

The class $\mathcal{X}^\perp$ (resp., $\perp \mathcal{X}$) is usually called the \textit{right (resp., left) orthogonal complement relative to the functor $\text{Ext}_R^1(\_, \_)$ of the class $\mathcal{X}$}.

\textbf{Definition 1.1 (Precovers and Preenvelopes).} Let $\mathcal{X}$ be any class of $R$-modules and let $M$ be an $R$-module.

\begin{itemize}
  
  \item An $\mathcal{X}$-\textit{precover} of $M$ is an $R$-homomorphism $\phi : X \to M$ where $X \in \mathcal{X}$ and such that the sequence

  $$\text{Hom}_R(X', X) \xrightarrow{\text{Hom}_R(X', \phi)} \text{Hom}_R(X', M) \longrightarrow 0$$

  is exact for every $X' \in \mathcal{X}$. An $\mathcal{X}$-precover is called \textit{special}, if $\phi$ is surjective and $\ker(\phi) \in \mathcal{X}^\perp$.

\end{itemize}
An $\mathcal{X}$-preenvelope of $M$ is an $R$-homomorphism $\varphi : M \to X$ where $X \in \mathcal{X}$ and such that the sequence,

$$\text{Hom}_R(X,X') \xrightarrow{\text{Hom}_R(\varphi,X')} \text{Hom}_R(M,X') \longrightarrow 0$$

is exact for every $X' \in \mathcal{X}$. An $\mathcal{X}$-preenvelope is called special, if $\varphi$ is injective and $\text{coker}(\varphi) \in \perp \mathcal{X}$.

For more details about precovers (and preenvelopes), the reader may consult [7, Chapters 5 and 6].

**Definition 1.2** ([12], Resolving classes 1.1). For any class $\mathcal{X}$ of $R$-modules.

- We call $\mathcal{X}$ projectively resolving, if $\mathcal{P}(R) \subseteq \mathcal{X}$ and for every short exact sequence $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ with $X'' \in \mathcal{X}$ the conditions $X' \in \mathcal{X}$ and $X \in \mathcal{X}$ are equivalent.

- We call $\mathcal{X}$ injectively resolving, if $\mathcal{I}(R) \subseteq \mathcal{X}$ and for every short exact sequence $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ with $X' \in \mathcal{X}$ the conditions $X'' \in \mathcal{X}$ and $X \in \mathcal{X}$ are equivalent.

A pair $(\mathcal{X}, \mathcal{Y})$ of classes of $R$-modules is called a cotorsion theory [7], if $\mathcal{X} \perp \mathcal{Y}$ and $\perp \mathcal{Y} = \mathcal{X}$. In this case, we call $\mathcal{X} \cap \mathcal{Y}$ the kernel of $(\mathcal{X}, \mathcal{Y})$. Note that each element $K$ of the kernel is a splitter in the sense of [11], i.e., $\text{Ext}^1_R(K, K) = 0$. If $\mathcal{C}$ is any class of modules, then $(\perp \mathcal{C}, (\perp \mathcal{C}) \perp)$ is easily seen be a cotorsion theory, called a cotorsion theory generated by $\mathcal{C}$ (see [13, Definition 1.10]). A cotorsion theory $(\mathcal{X}, \mathcal{Y})$ is called complete [13], if every $R$-module has a special $\mathcal{Y}$-preenvelope (or equivalently every $R$-module has a special $\mathcal{X}$-precover; see [13, Lemma 1.13]). A cotorsion theory $(\mathcal{X}, \mathcal{Y})$ is said to be hereditary [10], if whenever $0 \longrightarrow L' \longrightarrow L \longrightarrow L'' \longrightarrow 0$ is exact with $L, L'' \in \mathcal{X}$, then $L'$ is also in $\mathcal{X}$, or equivalently, if $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is exact $M', M \in \mathcal{Y}$, then $M''$ is also in $\mathcal{Y}$.

The aim of this paper is the study of the pair $(\mathcal{G}P(R), \mathcal{G}P(R) \perp)$.

**Note:** Below, we have only proved the results concerning the Gorenstein projective modules. The proofs of the Gorenstein injective ones are dual, and we can find a dual of the results using in the proofs in [12].

## 2 Lemmas

In this section, we recall some fundamental results about Gorenstein projective modules and dimensions. These results are extracted from the work of Holm in [12].

The first lemma shows that the class of Gorenstein projective modules is projectively resolving:

**Lemma 2.1** ([12], Theorems 2.5). Let $R$ be a ring. Then, The class $\mathcal{G}P(R)$ is projectively resolving. Moreover, it is closed under direct sums and direct summands.

The next lemma study the $\mathcal{G}P(R)$-precovers of $R$-modules with finite Gorenstein projective dimension.
Lemma 2.2 ([12], Theorems 2.10). Let $M$ be an $R$-module with finite Gorenstein projective dimension $n$. Then $M$ admits a surjective $\mathcal{GP}(R)$-precover, $\varphi : G \to M$, where $K = \ker \varphi$ satisfies $\text{pd}_R K = n - 1$ (if $n = 0$, this should be interpreted as $K = 0$).

In the following lemma, Holm gave a functorial description of the finite Gorenstein projective dimension of modules.

Lemma 2.3 ([12], Theorem 2.20). Let $M$ be an $R$-module with finite Gorenstein projective dimension, and let $n$ be an integer. Then the following conditions are equivalent:

1. $\text{Gpd}_R(M) \leq n$.
2. $\text{Ext}_R^i(M, L) = 0$ for all $i > n$, and all $R$-modules $L$ with finite $\text{pd}_R(L)$.
3. $\text{Ext}_R^i(M, Q) = 0$ for all $i > n$, and all projective $R$-modules $Q$.
4. For every exact sequence of $R$-module $0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$ where $G_0, \ldots, G_{n-1}$ are Gorenstein projectives, $K_n$ is also Gorenstein projective.

Recall that the finitistic projective dimension of $R$ is defined as:

$$\text{FPD}(R) = \sup \{ \text{pd}_R(M) \mid M \text{ is an } R\text{-module with } \text{pd}_R(M) < \infty \}$$

Lemma 2.4 ([12], Theorems 2.28). For any ring $R$ there is an equality

$$\text{FPD}(R) = \sup \{ \text{Gpd}_R(M) \mid M \text{ is an } R\text{-module with finite Gorenstein projective dimension} \}.$$

3 Main results

We begin with the following theorem:

Theorem 3.1. For any ring $R$, the following holds:

1. $\text{Ext}_R^i(G, M) = 0$ for all $i > 0$, all $G \in \mathcal{GP}(R)$, and all $M \in \mathcal{GP}(R)^\perp$.
2. $\text{Ext}_R^i(M, G) = 0$ for all $i > 0$, all $G \in \mathcal{GI}(R)$, and all $M \in \mathcal{GI}(R)$.
3. $\mathcal{GP}(R)^\perp$ and $\mathcal{GI}(R)$ are projectively resolving.
4. $\mathcal{GP}(R)^\perp$ and $\mathcal{GI}(R)$ are injectively resolving.

Proof. (1) Let $M$ and $G$ be an arbitrary elements of $\mathcal{GP}(R)^\perp$ and $\mathcal{GP}(R)$, respectively, and let $n > 1$ be an integer. Pick an exact sequence $0 \to G' \to P_1 \to \cdots \to P_n \to G \to 0$ where all $P_i$ are projectives. By the projectively resolving of $\mathcal{GP}(R)$ (Lemma 2.1), $G'$ is clearly Gorenstein projective. Consequently, we have $\text{Ext}_R^n(G, M) = \text{Ext}_R^n(G', M) = 0$, as desired.

(2) By a dual argument to (1).

(3) We claim that $\mathcal{GP}(R)^\perp$ is projectively resolving. Using the long exact sequence in homology, we conclude that $\mathcal{GP}(R)^\perp$ is closed by extension, i.e., if $0 \to M \to M' \to M'' \to 0$ is an exact sequence where $M$ and $M''$ are in $\mathcal{GP}(R)^\perp$, then so is $M'$. In addition, from the
definition of Gorenstein projective modules, it is clear that $\mathcal{P}(R) \subseteq \mathcal{GP}(R)$. Now, consider a short exact sequence $0 \to M \to M' \to M'' \to 0$ where $M'$ and $M''$ are in $\mathcal{GP}(R)^\perp$. For an arbitrary Gorenstein projective $R$-module $G$, consider a short exact sequence $0 \to G \to P \to G' \to 0$ where $P$ is projective and $G'$ is Gorenstein projective (such a sequence exists by the definition of Gorenstein projective modules). From the long exact sequence of homology, we have:

$$\cdots \to \text{Ext}^1_R(G', M'') \to \text{Ext}^2_R(G', M) \to \text{Ext}^2_R(G', M') \to \cdots$$

Then, $\text{Ext}^2_R(G', M) = 0$ since $\text{Ext}^2_R(G', M') = 0$ (from (1) above). Accordingly, $\text{Ext}^1_R(G, M) = \text{Ext}^2_R(G', M) = 0$, as desired.

(4) We claim that $\mathcal{GP}(R)^\perp$ is injectively resolving. Clearly, $I(R) \subseteq \mathcal{GP}(R)^\perp$, and $\mathcal{GP}(R)^\perp$ is closed by extension. Now, consider a short exact sequence $0 \to M \to M' \to M'' \to 0$ where $M$ and $M'$ belongs to $\mathcal{GP}(R)^\perp$. Using the long exact sequence of homology, for all Gorenstein projective module $G$, we have

$$\cdots \to \text{Ext}^1_R(G, M') \to \text{Ext}^1_R(G, M'') \to \text{Ext}^2_R(G, M) \to \cdots$$

Thus, from (1), $\text{Ext}^1_R(G, M'') = 0$. Hence, $M'' \in \mathcal{GP}(R)^\perp$. Consequently, $\mathcal{GP}(R)^\perp$ is injectively resolving.

From the above theorem, we conclude the following two corollary.

**Corollary 3.2.** For any ring $R$,

1. $\mathcal{P}(R) = \mathcal{GP}(R) \cap \mathcal{GP}(R)^\perp$.

2. $I(R) = \mathcal{GI}(R) \cap I(GI(R))$.

**Proof.**

(1) Let $M \in \mathcal{GP}(R) \cap \mathcal{GP}(R)^\perp$ and consider a short exact sequence $0 \to M' \to P \to M \to 0$ where $P$ is projective. Since $\mathcal{GP}(R)^\perp$ is projectively resolving (by Theorem 3.1), $M' \in \mathcal{GP}(R)^\perp$. Then, $\text{Ext}^1_R(M, M') = 0$. Therefore, this short exact sequence splits. Consequently, $M$ is a direct summand of $P$, and then projective.

(2) By a dual proof to (1).

**Corollary 3.3.**

1. [12, Proposition 2.27] Every Gorenstein projective (resp., injective) module with finite projective (resp., injective) dimension is projective (resp., injective).

2. Every Gorenstein projective (resp., injective) module with finite injective (resp., projective) dimension is projective (resp., injective).

**Proof.**

(1) If $M$ is a Gorenstein projective module with finite projective dimension, then $M \in \mathcal{GP}(R) \cap \mathcal{GP}(R)^\perp$ (by Lemma 2.3). Consequently, $M$ is projective (by Corollary 3.2). The injective case is dual.

(2) Note that every module $I$ with $\text{id}_R(I) := n < \infty$ belongs to $\mathcal{GP}(R)^\perp$. Indeed, by the definition of the Gorenstein projective modules, for each Gorenstein projective module $G$ we can find an exact sequence $0 \to G \to P_{n-1} \to \ldots \to P_0 \to G' \to 0$ where all $P_i$ are projective and $G'$ is Gorenstein projective. Thus, we have $\text{Ext}^1_R(G, I) = \text{Ext}^{n+1}_R(G', I) = 0$. 


Now, if $M$ is a Gorenstein projective module with finite injective dimension, then $M \in GP(R) \cap GP(R)^\perp$. Accordingly, by Corollary 3.2, $M$ is projective.

Dually, we can prove that every module with finite projective dimension is an element of $^\perp GI(R)$. Consequently, by Corollary 3.2, every Gorenstein injective module with finite projective dimension is injective.

The main result of this paper is the following theorem:

**Theorem 3.4.** If $l.Ggldim(R) < \infty$, then $(GP(R), GP(R)^\perp)$ and $(^\perp GI(R), GI(R))$ are complete, hereditary cotorsion theories.

**Proof.** (1) To show that $(GP(R), GP(R)^\perp)$ is a cotorsion theory, we have to prove that $^\perp (GP(R)^\perp) = GP(R)$. Let $M$ be an element of $^\perp (GP(R)^\perp)$. Since $\text{Gpd}_R(M) < \infty$ and from Lemmas 2.2 and 2.3, $M$ admits a surjective $GP(R)$-precover $\phi: G \twoheadrightarrow M$ where $K = \ker(\phi) \subseteq GP(R)^\perp$. Then, $G$ is a special $GP(R)$-precover of $M$, and the short exact sequence $0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0$ splits since $\text{Ext}^1_R(M, K) = 0$. Thus, $M$ is a direct summand of $G$. Hence, $M$ is Gorenstein projective (by Lemma 2.1). Consequently, $^\perp (GP(R)^\perp) \subseteq GP(R)$, while the other inclusion is clear. Therefore, $(GP(R), GP(R)^\perp)$ is a cotorsion theory, and every $R$-module has a special $GP(R)$-precover. This implies that $(GP(R), GP(R)^\perp)$ is complete. Moreover, since $GP(R)$ is projectively resolving and $GP(R)^\perp$ is injectively resolving, this cotorsion theory is hereditary.

(2) To prove the dual Gorenstein injective result, we use the dual result of Lemmas 2.1 and 2.2.

**Proposition 3.5.** If $l.Ggldim(R) < \infty$, then $GP(R)^\perp = \overline{P(R)} = \overline{I(R)} = ^\perp GI(R)$.

**Proof.** Clearly, by Lemma 2.3, $\overline{P(R)} \subseteq GP(R)^\perp$. Now, let $M \in GP(R)^\perp$ and $N$ be an arbitrary $R$-module, and set $n := l.Ggldim(R)$. We have, $\text{Gpd}_R(N) \leq n$. Then, by Lemma 2.3, we can find an exact sequence

$$0 \rightarrow G \rightarrow P_n \rightarrow ... \rightarrow P_1 \rightarrow N \rightarrow 0$$

where all $P_i$ are projective and $G$ is Gorenstein projective. Thus, by Theorem 3.1, for all $j > 0$, $\text{Ext}^j_{P(R)}(N, M) = \text{Ext}^j_{P(R)}(G, M) = 0$. Consequently, $\text{id}_{P(R)}(M) \leq n$. Using [2, Corollary 2.7], $\overline{P(R)} = \overline{I(R)}$ since $l.Ggldim(R) < \infty$. Then, $M \in \overline{P(R)}$. Accordingly, $GP(R)^\perp = \overline{P(R)}$. Similarly, we prove that $^\perp GI(R) = \overline{I(R)}$. This finishes the proof.

**Proposition 3.6.** If $GP(R) = ^\perp (\overline{P(R)})$ and $GP(R)^\perp = \overline{P(R)}$, then $\text{FPD}(R) = l.Ggldim(R)$.

**Proof.** From [13, Theorem 2.2], every $R$-module admits a special $GP(R)^\perp$-preenvelope. On the other hand, by hypothesis, $(GP(R), GP(R)^\perp)$ is the cotorsion theory generated by $\overline{P(R)}$. Then, $(GP(R), \overline{P(R)})$ is a complete cotorsion theory. Therefore, every $R$-module $M$ has a special $GP(R)$-precover.

The inequality $\text{FPD}(R) \leq l.Ggldim(R)$ follows from Lemma 2.4. Now, suppose that $\text{FPD}(R) \leq n$ and let $M$ be an arbitrary $R$-module. We claim that $l.Ggldim(R) < \infty$. From the first part of this proof, $M$ admits a special $GP(R)$-precover. Then, there is an exact sequence $0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0$ where $G$ is Gorenstein projective and $K \subseteq GP(R)^\perp = \overline{P(R)}$. Thus, $\text{pd}_R(K) \leq n$, and so $\text{Gpd}_R(M) \leq n + 1$. Hence, $l.Ggldim(R) \leq n + 1 < \infty$. Consequently, by Lemma 2.4, $l.Ggldim(R) = \sup \{\text{Gpd}_R(M) \mid \text{Gpd}_R(M) < \infty\} = \text{FPD}(R)$.

From the above propositions, we conclude the following characterization of the left Gorenstein global dimension of a ring $R$, provided $\text{FPD}(R) < \infty$. 

Corollary 3.7. If FPD$(R) < \infty$, then the following are equivalent:

1. $\sup \text{Ggldim}(R) < \infty$.
2. $\mathcal{G}P(R) = \perp \mathcal{P}(R)$ and $\mathcal{G}P(R) \perp = \overline{\mathcal{P}(R)}$.

Proof. (1) $\Rightarrow$ (2) The first equality follows from Lemma 2.3, whereas the second follows from Proposition 3.5.

(2) $\Rightarrow$ (1) Follows from Proposition 3.6.

Now, we discuss the rings over which “every Gorenstein projective module is Gorenstein flat”.

Proposition 3.8. For any ring $R$, the following are equivalent:

1. Every Gorenstein projective module is Gorenstein flat.
2. $I^+ \in \mathcal{G}P(R)^\perp$ for every right injective $R$-module $I$.
3. $(F^+)^+ \in \mathcal{G}P(R)^\perp$ for every flat $R$-module $F$.

Proof. (1) $\Rightarrow$ (2) Let $I$ be a right injective $R$-module. Since every Gorenstein projective $R$-module is Gorenstein flat, and by the definition of the Gorenstein flat modules, we have $\text{Tor}_R^1(I, G) = 0$ for all $G \in \mathcal{G}P(R)$. By adjointness, we have $\text{Ext}_R^1(G, I^+) = (\text{Tor}_R^1(I, G))^+ = 0$. Consequently, $I^+ \in \mathcal{G}P(R)^\perp$.

(2) $\Rightarrow$ (1) Consider a complete projective resolution

$$P: \quad \cdots \to P_{-2} \xrightarrow{f_2} P_{-1} \xrightarrow{f_1} P_0 \xrightarrow{f_0} P_1 \xrightarrow{f_1} P_2 \to \cdots$$

We decompose it into a short exact sequences $0 \to G_i \to P_i \to G'_i \to 0$ where $G_i = \ker(f_i)$ and $G'_i = \text{Im}(f_i)$. From [12, Observation 2.2], $G_i$ and $G'_i$ are Gorenstein projectives. Now, let $I$ be a right injective $R$-module. By hypothesis, we have $(\text{Tor}_R^1(I, G'_i))^+ = \text{Ext}_R^1(G'_i, I^+) = 0$. Then, $\text{Tor}_R^1(I, G'_i) = 0$. Therefore, $0 \to I \otimes_R G_i \to I \otimes_R P_i \to I \otimes_R G'_i \to 0$ is exact. Thus, $I \otimes_R -$ keeps the exactness of $P$. Then, $P$ is a complete flat resolution. Consequently, every Gorenstein projective module is Gorenstein flat.

(2) $\Rightarrow$ (3) Let $F$ be a flat $R$-module. Then, $F^+$ is a right injective $R$-module. Consequently, $(F^+)^+ \in \mathcal{G}P(R)^\perp$.

(3) $\Rightarrow$ (2) Let $I$ be a right injective $R$-module. There exists a flat $R$-module $F$ such that $F \to I^+ \to 0$ is exact. Then, $0 \to (I^+)^+ \to F^+$ is exact. However, $0 \to I \to (I^+)^+$ is exact (by [9, Proposition 3.52]). Thus, $0 \to I \to F^+$ is exact, and then $I$ is a direct summand of $F^+$. Hence, $I^+$ is a direct summand of $(F^+)^+$. On the other hand, it is easy to see that $\mathcal{G}P(R)^\perp$ is closed under direct summands. Consequently, $I^+ \in \mathcal{G}P(R)^\perp$, as desired.

Proposition 3.9. For any ring $R$, $\sup \{\text{Gfd}_R(M) \mid M \text{ is Gorenstein projective} \} = 0$ or $\infty$. 

**Proof.** Note that if \( \text{Gfd}_R(M) \leq n \), then we have \( \text{Tor}_i^R(I, M) = 0 \) for all \( i > n \). Indeed, the case \( n = 0 \) follows directly from the definition of the Gorenstein flats modules, whereas the case \( n > 0 \) is deduced from the first case by an \( n \)-step projective resolution of \( M \). Suppose that \( \sup \{ \text{Gfd}_R(M) \mid M \text{ is Gorenstein projective} \} = n < \infty \). Then, \( \text{Ext}^{n+1}_R(G, I^+) = (\text{Tor}^{n+1}_R(I, G))^+ = 0 \) for every right injective module \( I \) and every Gorenstein projective module \( G \). However, for every Gorenstein projective module \( G \) we can find an exact sequence \( 0 \to G \to P_{n-1} \to \ldots \to P_0 \to G' \to 0 \) where all \( P_i \) are projective and \( G' \) is Gorenstein projective. Thus, \( \text{Ext}^1_R(G, I^+) = \text{Ext}^{n+1}_R(G', I^+) = 0 \). So, \( I^+ \in \mathcal{GP}(R)^\perp \) for every right injective module \( I \). Then, by Proposition 3.8, every Gorenstein projective module is Gorenstein flat. Consequently, \( \sup \{ \text{Gfd}_R(M) \mid M \text{ is Gorenstein projective} \} = 0 \), as desired.

A direct consequence of the above proposition is the following corollary:

**Corollary 3.10.** If \( l.w.Ggldim(R) < \infty \), then every Gorenstein projective \( R \)-module is Gorenstein flat.

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