Einstein’s equations from Einstein’s inertial motion and Newton’s law for relative acceleration

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We show that Einstein’s equation for $R^{\mu 0}(P)$ for nonrelativistic matter-sources in $P$ and for arbitrarily strong gravitational fields, is identical with Newton’s equation for the relative radial acceleration of neighboring freefalling test-particles, spherically averaged.— With Einstein’s concept of inertial motion (≡ freefalling-nomoving), inertial worldlines (≡ geodesics) in Newtonian experiments can intersect repeatedly. This is evidence for the space-time curvature encoded in $R^{\mu 0}$.— These two laws of Newton and Einstein are explicitly identical, if one uses (1) our adapted space-time slicing (generated by the radial 4-geodesics of the primary observer with worldline through $P$ and $\hat{u}_{\text{obs}}(P) = \hat{e}_0(P)$, (2) our adapted Local Ortho-Normal Bases, LONBs (radially parallel with the primary observer’s LONB), and (3) Riemann normal 3-coordinates (centered at the primary observer). Hats on indices denote LONB components.— Our result: Full general relativity follows from Newton’s law of relative acceleration by using Lorentz covariance and energy-momentum conservation combined with the Bianchi identity.— The gravitational field equation of Newton-Gauss and Einstein’s field equation for $R^{\mu 0}(P)$ are both linear in gravitational fields, if the primary observer (≡ worldline through $P$) is inertial.

Einstein’s principle of equivalence between fictitious forces and gravitational forces is formulated as a precise equivalence theorem with explicit equations of motion from general relativity. With this equivalence theorem, the gravitational field equation of 19th-century Newton-Gauss physics and Einstein’s field equation for $R^{\mu 0}(P)$ are both bilinear in the gravitational forces for non-inertial primary observers.

$R^{\mu 0} = -\text{div} \hat{E}_\mu$ and $R^{i 0} = -(\text{curl} \hat{B}_i/2)$ hold exactly in general relativity for inertial primary observers, if one uses our space-time slicing and our LONB’s. The gravitoelectric $\hat{E}_\mu$ and the gravitomagnetic $\hat{B}_i$ fields are defined and measured exactly with nonrelativistic test-particles via $(d/dt)(p_i)$ and $(d/dt)(S_i)$ in direct correspondence with the electromagnetic $(\hat{E}, \hat{B})$ fields. The $(\hat{E}_\mu, \hat{B}_i)$ fields are identical with the Ricci connection for any observer’s LONBs and for displacements along any observer’s worldline, $(\omega_{ik})_0$.

In the explicit equations of particle-motion of general relativity (using our adapted space-time slicing and our adapted LONBs), there are precisely two gravitational forces equivalent to the two fictitious forces on the worldline of the observer: the force from $\hat{E}_\mu$, equivalent to the fictitious force measured by an accelerated observer, and the force from $\hat{B}_i$, equivalent to the fictitious Coriolis force for a rotating observer.

The exact Ricci curvature component $R^{i 0}$ can be measured with non-relativistic test particles. $(R^{i 0}, R^{i 0})_0$ are linear in the gravitational fields $(\hat{E}_\mu, \hat{B}_i)$ for inertial primary observers with worldlines through $P$. For non-inertial primary observers, $(R^{i 0}, R^{i 0})_0$ are bilinear in the gravitational fields $(\hat{E}_\mu, \hat{B}_i)$, which are the only gravitational fields in $(R^{i 0}, R^{i 0})_0$.

Einstein’s $R^{i 0}$ equation for non-relativistic matter and for inertial primary observers gives the Gauss law, $\text{div} \hat{E}_\mu = -4\pi G_N \rho_{\text{mass}}$, and Einstein’s $R^{0 0}$ equation gives the gravito-magnetic Ampère law, $\text{curl} \hat{B}_i = -16\pi G_N J_{\text{mass}}$.— For relativistic matter and inertial primary observers, Einstein’s $R^{i 0}$ equation gives $\text{div} \hat{E}_\mu = -4\pi G_N (\hat{\rho}_s + 3\hat{\rho})$, where $\hat{\rho}_s$ is the energy density, and $(3\hat{\rho})$ is the trace of the momentum-flow 3-tensor, both in the frame with $\hat{u}_{\text{obs}} = \hat{e}_0$. Einstein’s $R^{i 0}$ equation for an inertial observer gives the relativistic gravito-magnetic Ampère law, $\text{curl} \hat{B}_i = -16\pi G_N J_{L}$.

The remaining six Ricci components, $R^{ij}$, involve the curvature of space-space plaquettes, which are unmeasurable with non-relativistic particles in quasi-local experiments.

With our primary-observer-adapted spacetime splitting and LONBs, the equations of motion of general relativity for particles without nongravitational forces and for a noninertial primary observer are (1) form-identical with the 19th-century equations of Newtonian mechanics for nonrelativistic particles, (2) form-identical with the equations of motion for special relativity with the obvious replacements $(\hat{E}, \hat{B}) \Rightarrow (\hat{E}_\mu, \hat{B}_i)$ and $q \Rightarrow e \equiv \text{total energy of particle}$.— We formulate the precise theorem of equivalence of fictitious and gravitational forces in the equations of motion.

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I. GRAVITO-ELECTRIC AND GRAVITO-MAGNETIC FIELDS $\vec{E}_g$, $\vec{B}_g$

Our exact and general operational definition in arbitrary spacetimes of the gravitoelectric field $\vec{E}_g$ measured by any observer is probably new.

In contrast to the literature, we use no perturbation theory on a background geometry, no “weak gravitational fields”, no “Newtonian limit”.

For an observer with his Local Ortho-Normal Bases (LONBs) on his worldline our exact operational definition of the gravitoelectric field $\vec{E}_g$ is by measuring the acceleration of quasistatic freefalling test-particles in analogy to the operational definition of the ordinary electric field. We must replace the particle’s charge $q$ by its rest mass $m$. This gives the operational definition of the gravitoelectric field,

$$m^{-1} \frac{d}{dt} p_i \equiv E_i^{(g)}$$

\[ \iff \vec{a}_{\text{eff}} = \vec{E}^{(g)} = \vec{g}, \]  

(1)

for freefalling, quasistatic test particles.

The local time-interval $dt$ is measured on the observer’s wristwatch. The measured 3-momentum is $p_i$, and the measured acceleration of the quasistatic particle relative to the observer is $\vec{a}_g = \vec{g}$ ≡ gravitational acceleration.

We use the method of Élie Cartan, who gives vectors $\vec{V}$ (and tensors) at any spacetime point $P$ by their components $V^\alpha$ in the chosen Local Ortho-Normal Basis. — The LONB-components $V^\alpha$ are directly measurable. This is in stark contrast to coordinate-basis components $V^\mu$, which are not measurable before one has obtained $g_{\alpha\beta}$ by solving Einstein’s equations for the specific problem at hand. — Cartan’s method uses coordinates only in the mapping from an event $P$ to the event-coordinates, $P \Rightarrow x^\mu_P$.

It is important to distinguish LONB-components, denoted by hats, from coordinate-basis components, denoted without hats in the notation of Misner, Thorne, and Wheeler [1]. We denote spacetime LONB-indices by $(\hat{a}, \hat{b}, ...)$, 3-space LONB-indices by $(\hat{i}, \hat{j}, ...)$, and coordinate-indices by Greek letters.

LONBs off the observer’s worldline are not needed in Eq. (1), because a particle released from rest will still be on the observer’s worldline after an infinitesimal time $\delta t$, since $\delta s \propto (\delta t)^2 \Rightarrow 0$, while $\delta v \propto \delta t \neq 0$.

Arbitrarily strong gravitoelectric fields $\vec{E}_g$ of general relativity can be measured exactly with freefalling test-particles which are quasistatic relative to the observer in Galilei-type experiments, Eq. (1). — But this same measured $\vec{E}_g$ is exactly valid for relativistic test-particles in the equations of motion of general relativity, Eqs. (25), and in the field equations.

For the gravitomagnetic field $\vec{B}_g$, we give the exact operational definition in arbitray spacetimes, which is analogous to the modern definition of the ordinary magnetic field: $\vec{B}_g$ is defined via the gravitational torque causing the spin-precession of quasistatic freefalling test particles with spin $\vec{S}$ and free of nongravitational torques (or by quasistatic gyroscopes). In the gyro-magnetic ratio $q/(2m)$ of electromagnetism, the particle charge $q$ must be replaced by the rest-mass $m$, therefore the gyro-gravitomagnetic ratio is $(1/2)$,

$$\frac{d}{dt} S_i \equiv \{ \vec{S} \times (\vec{B}_g/2) \};$$

(2)

$$\Omega^i_\text{gyro} = - (\vec{B}_g/2), \quad \text{quasistatic gyroscope, (3)}$$

1/2 = gyro-gravitomagnetic ratio.

The gravitomagnetic field $\vec{B}_g$ can also be measured by the deflection of freefalling test particles, nonrelativistic relative to the observer,

$$\frac{d}{dt} \vec{v} = m [ \vec{E}_g + (\vec{v} \times \vec{B}_g) ].$$

(4)

The second term, $\vec{F}_g = m(\vec{v} \times \vec{B}_g)$ is the Coriolis force of Newtonian physics, which arises in rotating reference frames. It has the same form as the Lorentz force of electromagnetism with the charge $q$ replaced by the rest mass $m$ for a nonrelativistic particle. — Our definition of $\vec{B}_g$ agrees with Thorne et al [2].

The gravitomagnetic field $\vec{B}_g$ of general relativity has been measured by Foucault with gyroscopes precessing relative to his LONBs, $\vec{e}_\text{free} = \vec{e}_\text{obs}$ and $\vec{e}_i = (\text{East, North, vertical})$. Recently, $\vec{B}_g$ has been measured on Gravity Probe B by gyroscope precession relative to LONBs given by the line of sight to quasars resp. distant stars without measurable proper motion.

In 1893, Oliver Heaviside, in his paper A Gravitational and Electromagnetic Analogy [3], gave the same operational definition of the gravitoelectric field $\vec{E}_g$, and he postulated the gravitomagnetic field $\vec{B}_g$ in analogy to Ampère-Maxwell.

For an inertial observer (freefalling-nonrotating) with worldline through $P$, $(\vec{E}_g, \vec{B}_g)_P$ are zero.

Arbitrarily strong gravitomagnetic field $\vec{B}_g$ of general relativity can be measured exactly with the precession of quasistatic gyroscopes Eq. (2), or with the Coriolis-deflection of nonrelativistic test particles, Eq. (4). — But this same measured $\vec{B}_g$ is exactly valid for relativistic test-particles in the equations of motion, Eqs. (25), and in the field equations.
A. Ricci’s LONB-connection

Relative to an airplane on the shortest path (geodesic) from Zurich to Chicago, the Local Ortho-Normal Bases (LONBs), chosen to be in the directions “East” and “North”, rotate relative to the geodesic (relative to parallel transport) with a rotation angle $\delta \alpha$ per path length $\delta s$, i.e. with the rotation rate $\omega = (\text{d} \alpha / \text{d}s)$.

For an infinitesimal displacement $\delta \hat{\vec{D}}$ along a geodesic in any direction, the infinitesimal rotation angle $\delta \alpha$ of LONBs is given by a linear map encoded by the Ricci rotation coefficients $\omega_i$:

$$\delta \alpha = \omega_i \delta \hat{\vec{D}}_i.$$  

The Ricci rotation coefficients are also called connection coefficients, because they connect the LONBs at infinitesimally neighboring points by a rotation relative to the infinitesimal geodesic between these points (i.e. relative to parallel transport).

For non-geodesic displacement curves, the tangents for infinitesimal displacements are infinitesimal geodesics. For the connection coefficients, it is irrelevant, whether the displacement curve is geodesic or non-geodesic, only the tangent vectors matter, either the coordinate basisvector, $\partial_t = \hat{e}_t$, or the LONB-vector, $\partial \hat{e}_k = \hat{e}_k$.

For Ricci connections (LONB rotations), it is irrelevant, whether we are in curved space or e.g. in the Euclidean plane with polar coordinates $(r, \phi)$ and LONBs $(\hat{e}_r, \hat{e}_\phi)$, where for a displacement vector $\hat{e}_k \gamma$ the LONB-rotation angle is $(\omega_i \hat{e}_k)_\gamma = 1/r$.

The Ricci rotation coefficients $\omega_i$ are directly measurable because of their LONB-displacement-index. But for computations of curvature in Sect. \cite{LVB} a line-integral along a displacement curve $\mathcal{C}$ calls for infinitesimal displacement vectors with contravariant components $\delta \hat{e}_k^\gamma$, the infinitesimal difference of coordinates $\delta \vec{x}_\gamma$. This calls for a linear map from infinitesimal coordinate-displacement vectors $\delta \hat{e}_k^\gamma$ (input) to the corresponding measured LONB-rotation angles $\delta \alpha$ (output),

$$\delta \alpha = \omega_i \delta \hat{e}_i^\gamma.$$  

This linear map is the LONB-connection 1 - form $\omega_i$, given explicitly by covariant components $\omega_i = (\equiv 1$-form components). In this paper, Greek indices always refer to a coordinate basis. Using 1-form components makes the line-integral free of metric factors $g_{\mu \nu}$,

$$\alpha(\mathcal{C}) = \int_{\mathcal{C}} \omega_i \, \text{d}x^\gamma.$$  

A line-integral with $\text{d}x^\gamma$ cries out to have some 1-form $\sigma_\gamma$ as an integrand. Vice versa, the connection 1-form $\omega_i$ cries out to be contracted with a contravariant displacement vector to give a rotation angle, $\delta \alpha = \omega_i \delta x_i^\gamma$. This is index-matching.

The rotation of the chosen LONBs $(\hat{e}_x, \hat{e}_y)$ relative to the geodesic from $P$ to $Q$ (i.e. relative to parallel transport) is given by the rotation matrix,

$$\begin{pmatrix} \hat{e}_x^Q \\ \hat{e}_y^Q \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \hat{e}_x^P \\ \hat{e}_y^P \end{pmatrix}.$$  

For infinitesimal displacements, the infinitesimal rotation matrix is,

$$\begin{pmatrix} \hat{e}_x^Q \\ \hat{e}_y^Q \end{pmatrix} = \begin{pmatrix} 1 + \alpha (0 & 1) \\ -\alpha (1 & 0) \end{pmatrix} \begin{pmatrix} \hat{e}_x^P \\ \hat{e}_y^P \end{pmatrix}.$$  

The infinitesimal LONB-rotation matrix $\delta R_{ij}^\gamma$ is given by the linear map from the infinitesimal coordinate-displacement vector $D_i^\gamma$:

$$\delta R_{ij}^\gamma = (\omega_{ij})_\gamma \delta D_i^\gamma,$$  

$$\omega_{ij}^\gamma = -\omega_{ji}^\gamma = \omega_{ij}^\gamma = \text{rotation angle in } [\hat{1}, \hat{2}] \text{ plane}.$$  

The coefficients $(\omega_{ij})_\gamma$ are the connection 1-form components.

The rotational change of a LONB vector relative to parallel transport under a displacement in the coordinate $x^\gamma$ is called the covariant derivative of $\hat{e}_i$ with respect to the coordinate $x^\gamma$ and denoted with the symbol $\nabla_\gamma$,

$$\nabla_\gamma \hat{e}_i^\gamma = \omega_i^\gamma.$$  

Combining Eqs. (5) and (6) gives,

$$\nabla_\gamma \hat{e}_i^\gamma = (\omega_i^\gamma)_\gamma.$$  

In 3-space, for an infinitesimal displacement from $P$ to $Q$, the LONBs $\hat{e}_i$ rotate relative to parallel transport, which can be given by (1) the tangent vector to the geodesic curve and (2) the spin-axes of two transported gyroscopes. The infinitesimal rotations are given by the antisymmetric matrix with components $(\omega_{ij}^\gamma)_\gamma$, $(\omega_{ij}^\gamma)_\gamma$, $(\omega_{ij}^\gamma)_\gamma$.

In curved (1+1)-spacetime, the Lorentz transformation of the chosen LONBs relative to a given displacement geodesic is a Lorentz boost $L_{\hat{b}}^\alpha$:

$$\begin{pmatrix} \hat{e}_i^\gamma \\ \hat{e}_j^\gamma \end{pmatrix}^Q = \begin{pmatrix} \cosh \chi & \sinh \chi \\ \sinh \chi & \cosh \chi \end{pmatrix} \begin{pmatrix} \hat{e}_i^P \\ \hat{e}_j^P \end{pmatrix}.$$  

rapidity $\equiv \chi$ is additive, $\tanh \chi \equiv v/c$.

For infinitesimal displacements, the infinitesimal Lorentz boost $L_{\hat{b}}^\alpha$, is,

$$\begin{pmatrix} \hat{e}_i^\gamma \\ \hat{e}_j^\gamma \end{pmatrix}^Q = \begin{pmatrix} 1 + \chi (0 & 1) \\ -\chi (1 & 0) \end{pmatrix} \begin{pmatrix} \hat{e}_i^P \\ \hat{e}_j^P \end{pmatrix}.$$  

In space-time, we denote vectors by a bar, $\bar{V}$, in 3-space, we denote vectors by an arrow, $\vec{V}$.

In curved (3+1)-spacetime, and with two lower indices, $\omega_{\hat{a}\hat{b}}$ is antisymmetric for Lorentz boosts and for rotations,

$$\delta L_{\hat{a}\hat{b}} = (\omega_{\hat{a}\hat{b}})_\gamma \delta D_i^\gamma,$$  

$$\omega_{i\hat{b}} = -\omega_{\hat{b}i} = X_{i\hat{b}}.$$  

For a displacement in observer-time, the exact Ricci connection coefficients $(\omega_{\hat{a}\hat{b}})_\hat{i}$ of general relativity can be measured in quasistatic experiments. But these Ricci connection coefficients predict the motion of relativistic particles with the equations of motion, Eqs. (25, 30).
B. \((\vec{E}_g, \vec{B}_g)\) identical with Ricci connection in time \(\langle \omega_{ab} \rangle_0\)

Our gravitoelectric field \(\vec{E}_g\) is identically equal to the negative of the Ricci Lorentz-boost coefficients for a displacement in time,

\[
E_i^{(g)} = -\langle \omega_i \rangle_0. \tag{9}
\]

The proof is immediate, see the next equation: from the point of view of the observer with his LONBs along his worldline, the gravitational acceleration \(a_i = a_i^{(\text{ff particle})}\) of freefalling quasistatic test-particles (starting on the observer’s worldline) is by definition identical to the exact gravitoelectric field \(E_i\) of general relativity from Eq. (1). — But from the point of view of freefalling test-particles, the acceleration of the quasistatic observer with his LONBs is by definition identical to the exact Ricci LONB-Lorentz-boost coefficients \(\langle \omega_{i0} \rangle_0\),

\[
E_i^{(g)} \equiv ([a_i]^{(\text{ff particle})}_{\text{relat.to.obs.}})_{\text{quasistatic}} = - ([a_i]^{(\text{obs.})}_{\text{relat.to.gyro}})_{\text{quasistatic}} \equiv -\langle \omega_{i0} \rangle_0. \tag{10}
\]

The exact Ricci connection coefficients \(\langle \omega_{i0} \rangle_0\) of general relativity in arbitrarily strong gravitational fields can be directly measured in quasistatic experiments by the acceleration of freefalling test particles relative to the LONBs of the observer as Galilei did.

For the gravitomagnetic 3-vector field \(\vec{B}_i^{(g)}\), we introduce its Hodge-dual antisymmetric 2-tensor \(B_{ij}^{(g)}\),

\[
B_{ij}^{(g)} \equiv \varepsilon_{ijk}B_{k}^{(g)}, \tag{11}
\]

where \(\varepsilon_{ijk}\) is the Levi-Civita tensor, totally antisymmetric, whose primary definition is given in a LONB,

\[
\varepsilon_{123} \equiv +1 \text{ for LONB with positive orientation,} \tag{12}
\]
\[
B_{12} = B_{3} \text{ and cyclic permutations.}
\]

The Levi-Civita tensor in a coordinate-basis in 3 dimensions, \(\varepsilon_{\alpha\beta\gamma}\), cannot be known before Einstein’s equations have been solved for the specific problem at hand.

Recall from Eq. (3) that the gyro-precession relative to the observer equals \((-\vec{B}_g/2)\), hence

\[
(-B_{ij}^{(g)}/2) \equiv [(\Omega_{ij})^{(\text{gyro})}_{\text{relat.to.obs.}}]_{\text{quasistatic}} = -[(\Omega_{ij})^{(\text{obs.})}_{\text{relat.to.gyro}}]_{\text{quasistatic}} \equiv -\langle \omega_{i0} \rangle_0. \tag{13}
\]

The exact Ricci connection coefficients \(\langle \omega_{i0} \rangle_0\) of general relativity can be directly measured in quasistatic experiments by the precession of gyroscopes relative to the LONBs of the observer as Foucault did.

All Ricci connection coefficients for a displacement in time, \(\langle \omega_{ab} \rangle_0\), which have all indices in LONBs, can be directly and exactly measured in arbitrarily strong gravitational fields of general relativity using freefalling test particles and gyroscopes which are quasistatic relative to the observer with \(\vec{u}_{\text{obs}} = \vec{e}_0\) in Galilei-type and Foucault-type experiments. Therefore, it is superfluous to use relativistic test-particles to measure the Ricci connection coefficients for a displacement in time.

Cartan’s LONB-connection 1-form \(\langle \omega_{ab} \rangle_0\), with its displacement index in the coordinate basis, are measurable as soon as a coordinatization \(P \Rightarrow x^\mu_P\) is chosen. No metric coefficients \(g_{\alpha\beta}\) are needed.

In striking contrast, the connection coefficients for coordinate bases, the Christoffel symbols, \((\Gamma^\alpha_{\beta\gamma})_\gamma \equiv \Gamma^\alpha_{\beta\gamma}\), have no direct physical-geometric meaning, they are not measurable until the metric fields \(g_{\mu\nu}(x)\) have been obtained by either solving Einstein’s equations or going out and measuring distances, time-intervals, angles, and Lorentz-boost-angles in a fine-mesh coordinate grid. — We write Christoffel connection-1-form coefficients with a bracket: \((\Gamma^\alpha_{\beta\gamma})_\gamma \equiv \Gamma^\alpha_{\beta\gamma}\). Inside the bracket are the coordinate-basis transformation-indices \((\alpha, \beta)\), outside the bracket is the coordinate-displacement index \(\gamma\).

Conclusion:

- Ricci connection coefficients \(\langle \omega_{ab} \rangle_0\) are directly measurable for given LONBs,
- Cartan connection coefficients \(\langle \omega_{ab} \rangle_0\) are measurable for given LONBs after a coordinatization-mapping \(P \Rightarrow x^\mu_P\) is chosen, but no metric is needed,
- Christoffel connection coefficients \((\Gamma^\alpha_{\beta\gamma})_\delta \equiv \Gamma^\alpha_{\beta\gamma}\) cannot be known until one has solved Einstein’s equations for the problem at hand in order to obtain the metric \(g_{\alpha\beta}\) for the chosen coordinate system.

The connection-coefficients for local ortho-normal bases (Ricci and Cartan) are more efficient than Christoffel’s connection-1-form coefficients: In two spatial dimensions, Ricci and Cartan connections each need only one rotation angle for a given displacement. In contrast, the Christoffel-connection 1-form \((\Gamma^\alpha_{\beta\gamma})_\gamma\) for a given displacement needs four numbers for the coordinate-basis transformation, two for stretching/compressing basis vectors, one for skewing them, and one for rotating them.
C. Equations of motion with general LONBs and coordinates

We obtain the equations of motion in terms of Ricci coefficients from (1) the equations of motion in terms of \((E^i, B^k)\) for quasistatic freefalling-nonrotating test-particles, Sect. II (2) the identity of \((E^{(s)}_i, B^{(s)}_j)\) with the Ricci connection for a displacement in time, \((\omega_{\alpha\beta})_0\), Sect. IIA

\[
m^{-1} \frac{d}{dt} p_i + (\omega^{(\alpha)}_i)_0 = 0, \quad i = 1, 2, 3,
\]

\[
\frac{d}{dt} S_i + (\omega^{(\alpha)}_i)_0 S^\alpha = 0.
\]

At the level of first derivatives of vectors, there cannot be curvature effects, we are at the level of special relativity, and the Lorentz-covariant extension of Eqs. (14) is trivial: we must include spatial displacements in \((dx^{\mu}/dt)\) and spatial components of the vector \(p^\alpha\). This Lorentz-covariant extension of Eq. (14) gives the relativistic geodesic equation and Fermi spin transport equation,

\[
\frac{d}{dt} p^\alpha + (\omega^\alpha_\mu)_\mu p^\beta \frac{dx^\mu}{dt} = 0,
\]

\[
\frac{d}{dt} S^\alpha + (\omega^\alpha_\beta)_\beta S^\beta \frac{dx^\mu}{dt} = 0.
\]

The covariant derivative \(\nabla_\mu\) of vector- and tensor-fields can be defined by

\[
(\nabla_\mu \hat{V})_\alpha \equiv \partial_\mu (\hat{V}_\alpha)
\]

for LONBs parallel in direction \(\partial_\mu\).

Hence,

freefalling-nonrotating particles:

\[
\frac{Dp}{D\tau} = 0, \quad \frac{DS}{D\tau} = 0.
\]

For arbitrary LONBs \(\bar{e}_\alpha\), the covariant derivative of vector fields follows directly from the definition of the Ricci connection coefficients in Sect. IIA

\[
(\nabla_\gamma \hat{V})_\alpha = \partial_\gamma \hat{V}_\alpha + (\omega^\alpha_\beta)_\gamma \hat{V}^\beta,
\]

\[
\nabla_\gamma \hat{e}_\beta = \hat{e}_\alpha (\omega^\alpha_\beta)_\gamma
\]

The fundamental observational rocks for general relativity are \((E^i, B^k)\), measured by the acceleration resp. precession of freefalling-nonrotating test-particles resp. gyroscopes which are quasistatic relative to the observer, as in the experiments of Galilei resp. Foucault. Apart from a minus sign and a factor 2, these measured fields are identical with Ricci’s LONB-connection for a displacement along the observer’s worldline.

II. NEWTON-INERTIAL MOTION VERSUS GENERAL-RELATIVITY-INERTIAL

This paper is based on two pillars:

1. Einstein’s revolutionary concept of inertial motion as freefalling-nonrotating, which replaces the first law of Newton, his law of inertia,

2. The Newtonian law on relative acceleration of neighboring freefalling nonrelativistic test-particles, spherically averaged.

We never assume a “Newtonian limit of general relativity”, and we never assume “weak gravity”.

From the above two pillars, it immediately follows that space-time is curved, i.e. the curvature of \([t, x^r]\)-plaquettes does not vanish, a straightforward, but revolutionary insight demonstrated in Sec. IIC

Incorporating the local Minkowski metric of special relativity leads directly to Riemannian geometry.

From these inputs alone,

We derive the exact curvature component \(R^{\beta\alpha}(P)\) of general relativity from Newtonian relative acceleration experiments of freefalling test-particles,

\[
[R^{\beta\alpha}]_{GR \text{ exact}} = -\frac{\partial}{\partial r} \left[ a_{\text{relative}} > \text{quasistatic avg.}_{\text{spherical av}} \right] r = 0.
\]

where one takes the spherical average of the relative radial acceleration of freefalling quasistatic test-particles starting at radial distance \(r\).

Einstein’s Ricci-0-0 curvature of general relativity is exactly determined by an experiment with quasistatic test-particles.

A. Newton’s method to find inertial motion

Newton’s first law, the law of inertia, states that force-free particles move in straight lines and at constant velocities. The same law of inertia holds in special relativity.

Newton’s law of inertia is valid only relative to inertial reference frames. Inertial frames are defined as those frames in which the law of inertia is valid. Inertial frames can be constructed operationally either using three force-free particles moving in non-coplanar directions, or using one force-free particle plus two gyroscopes with non-aligned spins axes.

1. There is a grave problem with this standard formulation of Newton’s first law: force-free particles do not exist e.g. for solar-system dynamics, gravitational forces are always present, therefore the operational construction of an inertial frame is a very difficult task.
In a few cases, gravity is irrelevant: (1) for particle physics, gravity is negligible, since the relative magnitude of the gravitostatic compared to electrostatic force between two protons is extremely small, $10^{-36}$, (2) for the motion of polished balls on a polished horizontal table, gravity cannot act, it is orthogonal to the table.

Newton’s solution for this problem: He wrote in the *Principia* [4] that for finding “true motion” equivalent to “absolute motion” (i.e. motion relative to inertial frames), distinguished from “relative motion”, and in particular true (absolute) acceleration (i.e. acceleration relative to inertial frames), distinguished from relative acceleration, one must first subtract the gravitational effects of all celestial bodies. In the *Scholium* on space and time, at the end of the initial “Definitions”, Newton writes on p. 412 of [4]: “True motion is neither generated nor changed except by forces impressed upon the moving body.” On p. 414:

- “It is certainly very difficult to find out the true motions of individual bodies. ... Nevertheless, the case is not hopeless. For it is possible to draw evidence ... from the forces that are the causes ... of the true motions.”

In order to find inertial frames, Newton invokes the forces from his second law. Newton implies that without his second law, his first law would be “hopeless”. — On p. 415, at the very end of this Scholium, Newton concludes: “In what follows, a fuller explanation will be given of how to determine true motion from their causes ... For this was the purpose for which I composed the following treatise.”

Essentially by this method, the local inertial center-of-mass frame for the solar system is determined today, the “International Celestial Reference Frame” of 1998.

Note the dramatic difference between the two aspects of inertial frames: (1) It is easy to establish the non-rotating frame by using two gyroscopes (Foucault) or, much less precisely, with the rotating-bucket experiment of Newton. (2) It is very work-intensive to establish a frame without linear acceleration relative to an inertial frame in the mechanics of the solar system.

It is remarkable that most textbooks on classical mechanics pretend that Newtonian inertial frames can be constructed operationally using force-free particles, although force-free particles do not exist for the mechanics of the solar system.

B. Poincaré’s man under permanent cloud cover

An entirely different point was discussed by Henri Poincaré in “Science and Hypothesis” in 1904 [5]: “Suppose a man were transported to a planet, the sky of which was constantly covered with a thick curtain of clouds, so that he could never see the other stars. On that planet he would live as if it were isolated in space. But he would notice that it rotates, either by measuring the planet’s ellipsoidal shape (... which could be done by purely geodesic means), or by repeating the experiment of Foucault’s pendulum. The absolute rotation of this planet might be clearly shown in this way.”

On such a planet with permanent dense cloud cover, it is impossible to use Newton’s method “to draw evidence ... from the forces that are the causes ... of the true motions,” since the Sun is invisible. On such a planet, it is impossible to decide, whether the planet is on an orbit around the Sun, i.e. in non-inertial motion in the sense of Newton, or moving in a straight line with constant velocity, i.e. in inertial motion in the sense of Newton. If the planet was sufficiently small, tidal effects from the Sun would be undetectable. It follows that

- Newton’s first law is operationally empty for a person on a very small planet with permanent cloud cover: One pillar of Newtonian physics is destroyed.

C. Einstein’s revolutionary concept of inertial motion

But one cannot remove one pillar from a theory without putting in its place a new pillar:

- The new pillar is Einstein’s revolutionary re-definition of inertial motion: Inertial motion and the local inertial frame in general relativity are operationally defined by a freefalling particle together with the spin axes (not parallel) of two comoving gyroscopes, i.e. freefalling-nonrotating motion.

Freefalling particles determine and define the straightest timelike worldlines, geodesics. They have maximal proper time between any two nearby points on the worldline, $\int d\tau = \text{maximum}$.

Spacetime curvature, i.e. curvature of space-time plaquettes $[t, x]$ etc. in distinction to space-space plaquettes $[x, y]$ etc., follows directly from *Newtonian experiments* on relative acceleration of neighbouring freefalling test particles with Einstein’s concept of inertial motion and geodesics: In a Gedanken-experiment, consider a vertical well drilled down through an ideal ellipsoidal Earth from the North Pole through the center of the Earth to the South Pole. Drop a pebble and afterwards another pebble. The pebbles will fall freely (geodesic worldlines) from the North Pole to the center of the Earth, rise to the surface of the Earth at the South Pole and fall back again. The two geodesics will cross again and again, direct evidence of space-time curvature from *Newtonian experiments*. In Sect. IV

- we compute the space-time curvature component $R^0_i = (\mathcal{R}^0_i)_{\text{0i}}$ of general relativity for $[0i]$ space-time plaquettes exactly from *Newtonian experiments* on relative acceleration of neighbouring freefalling test particles, quasistatic relative to $\dot{e}_0$, spherically averaged, using the revolutionary concept of inertial motion as freefalling-nonrotating.
III. EQUVALENCE OF FICTITIOUS AND GRAVITATIONAL FORCES IN EQUATIONS OF MOTION

We do not trace the history of the equivalence principle in Einstein’s writings beginning in 1907.

Starting with Einstein’s theory of relativity of 1915, we prove the two equivalence theorems with their exact explicit equations of motion,

1. for a non-inertial observer the equivalence and exact equality of gravitational forces and fictitious forces in the equations of motion, Sect. III D

2. for an inertial observer the exact explicit vanishing of \((d/dt)(p_i)\) and \((d/dt)(S_i)\) for inertial particles in Sect. III C

Our explicit equalities presented hold, if and only if one uses our adapted spacetime slicing and our adapted LONBs presented in Sect. III B

The reader might want to first read our results in Sects. III D and III C afterwards our method in Sect. III E and finally our discussion of fundamentals in Sect. III A

A. Fundamentals

Fictitious forces, e.g. centrifugal forces seen by a rotating observer in his rotating reference frame, have been important since Huygens, Newton, Leibniz, and Hooke. Fictitious forces are also called inertial forces. — Because the centrifugal force vanishes on the worldline of a rotating observer, this force will not play any role for the equivalence between fictitious and gravitational forces in the equations of motion.

Equivalence principle for \(\vec{E}_g\): For an observer accelerated with \(\vec{a}_{\text{obs}}\) relative to freefall, who is inside a windowless elevator or windowless space-ship or in fog, and measures the acceleration \(\vec{a}_{\text{freefall}}\) of freefalling test-particles at his position, it is impossible to distinguish between (1) particle-acceleration caused by the fictitious force due to the observer’s own acceleration relative to freefall, and (2) particle-acceleration caused by gravitational forces due to the attraction by matter-sources (Earth, Sun, Moon, etc). Therefore, this fictitious force and this gravitational force are equivalent in the equations of motion.

Equivalence principle for \(\vec{B}_g\): For an observer rotating with angular velocity \(\Omega_{\text{obs}}\) relative to comoving gyro-spin axes at his position, who is in a windowless elevator or spaceship or in fog, it is impossible to distinguish (1) a fictitious Coriolis force \(\vec{F} = 2m \left[\vec{\Omega} \times \vec{v}\right]\) due to his own rotation, from (2) a gravito-magnetic Lorentz-force \(\vec{F} = m \left[\vec{v} \times \vec{B}_g\right]\) from the gravitational field \(\vec{B}_g\) generated by mass-currents. The gravitomagnetic field \(\vec{B}_g\) has been postulated by Heaviside in 1893 in analogy to the magnetic field in electromagnetism, and for general relativity, the gravito-magnetic field is defined in Eqs. (2-4). Both forces are written here for nonrelativistic particles. For an observer in a windowless spaceship, a fictitious Coriolis force and a gravito-magnetic Lorentz-force are fundamentally indistinguishable, hence equivalent.

For an observer in a windowless spaceship, the mass-sources of gravitational fields (Sun, etc) are unknown. Therefore, for such an observer the gravitational field equations are useless, in 19th-century physics Gauss’s law for gravity, \(\text{div}\ \vec{E}_g = -4\pi G \rho_{\text{mass}}\).

Newton has emphasized that two reference systems rotating relative to each other are equivalent, unless one considers the forces, Sec. II A Einstein’s principle of equivalence is no longer valid, as soon as (1) the source-masses of gravity are seen (observer not “under a permanent cloud cover”) and (2) the gravitational forces are explicitly known by solving the gravitational field equation of Gauss-Newton or Einstein. Conclusion: the Principle of Equivalence is not valid at the level of the gravitational field equations.

- The equivalence principle states that fictitious forces are equivalent to gravitational forces in the equations of motion. The equivalence principle holds for first time-derivatives of particle momenta and spins on the worldline of the primary observer.

- But the equivalence cannot hold for relative acceleration (tidal acceleration) and for relative precession of two particles at different positions, i.e. the equivalence cannot hold for second derivatives of particle momenta and spins, one derivative in time and one derivative in space (to compare particles at different locations): The equivalence cannot hold for the curvature tensors. — For an accelerated observer in Minkowski spacetime, there are no tidal forces, no relative acceleration of freefalling particles, and no spacetime-curvature.

- The equivalence cannot hold, if in the equations of motion at \(P\), the particle is initially, at \(t_P\), off the primary observer’s world-line, because the tidal forces would be relevant.

- The equivalence cannot hold in the gravitational field equations of Einstein or Gauss, \(\text{div}\vec{E}_g = -4\pi G \rho_{\text{mass}}\), with their matter-sources. — For an accelerated observer in Minkowski space, there are no matter sources.

We focus on one primary observer. In the equations of motion, e.g. in \([(d/dt)(p_i)](t_1) = \lim_{\delta t \rightarrow 0}[p_i(t_1 + \delta t) - p_i(t_1)]/\delta t\), spacetime curvature is absent, if and only if particles start at \(t_1\) on the worldline of the primary observer.

Many auxiliary local observers are needed to measure the momenta \(\vec{p}\) and spins \(\vec{S}\) of test-particles at \(t = t_1 + \delta t\), when they are no longer on the worldline of the primary observer. We shall see that our auxiliary observers...
cannot be inertial, even if the primary observer is inertial (unless spacetime curvature vanishes). Every auxiliary local observer, with his worldline through \( Q \) and \( \bar{u}_{\text{obs}}(Q) = \bar{e}_\gamma(Q) \) and with his local spatial axes \( \bar{e}_\gamma \), is in a one-to-one relationship with the Ricci connection coefficients \( (\omega^\alpha_\beta)_0 \) at \( Q \).

An observer can be non-inertial in exactly two fundamental ways:

1. observer’s acceleration relative to freefall,
2. observer’s rotation relative to spin axes of comoving gyroscopes.

Correspondingly, there are exactly two fictitious forces in the equations of motion on the worldline of a non-inertial observer, equivalent to two gravitational forces:

1. the fictitious force measured by an observer accelerated relative to freefall, which is equivalent to Newton’s force in a gravito-electric field, \( \bar{g} \equiv \bar{E}_R \).
2. the fictitious Coriolis force measured by an observer rotating relative to gyro axes, which is equivalent to the gravito-magnetic Lorentz force due to \( \bar{B}_g \) postulated by Heaviside in 1893.

The remaining two of the four fictitious forces cannot contribute in the equations of motion on the worldline of a primary observer, because these forces vanish on his worldline. Therefore these two fictitious forces cannot occur in the equivalence theorem,

1. the fictitious centrifugal force,
   \[
   \vec{F}/m = [\vec{\Omega}_{\text{obs}} \times (\vec{r} \times \vec{\Omega}_{\text{obs}})],
   \]
2. the fictitious force due to an observer’s non-uniform rotation,
   \[
   \vec{F}/m = [\vec{r} \times (d\vec{\Omega}_{\text{obs}}/dt)],
   \]
   both vanish on the worldline of the primary rotating observer, \( r = 0 \).

To put this subsection “Fundamentals” in perspective, we print the criticism of the equivalence principle by J.L. Synge [7]: “Does the principle of equivalence mean that the effects of the gravitational field are indistinguishable from the effects of the observer’s acceleration? If so, it is false. ... Either there is a gravitational field or there is none, according to the Riemann tensor. ... This has nothing to do with the observer’s worldline. ... The Principle of equivalence ... should now be buried.”

Synge mixes up two entirely different levels, first time-derivatives of momenta versus second derivatives of momenta around a plaquette:

1. Equations of motion, acceleration with first time-derivatives of momenta for particles starting at one point on the observer’s worldline. — The equivalence principle holds for the equations of motion, where curvature is invisible. The equivalence principle makes no statements about relative acceleration and curvature.

2. Equations of relative acceleration of neighbouring freefalling test-particles starting at two different positions with second derivatives of momenta around a plaquette with one time-derivative and one spatial derivative to obtain relative acceleration, hence space-time curvature.

B. The primary-observer-adapted space-time slicing and LONB-field

For the theorem of equivalence between fictitious forces and gravitational forces, the equations of motion \( \nabla_\bar{p} \bar{p} = 0 \) and \( \nabla_\bar{p} \bar{S} = 0 \) for inertial particles are not instructive by themselves, because they do not make explicit the fundamental distinction between inertial and non-inertial observers.

The more explicit equations of motion for inertial particles, the geodesic equation, \( (d/dt) p_i + (\omega^a_i)_\mu p^\mu (dx^\mu/dt) = 0 \), and the Fermi spin-transport equation, \( (d/dt) S_i + (\omega^a_i)_\mu S^\mu (dx^\mu/dt) = 0 \), are not instructive by themselves, because they contain, depending on the formalism used, \( 6 \times 4 = 24 \) Ricci LONB-connection coefficients \( (\omega^a_i)_\gamma \), resp. Cartan LONB-connection coefficients \( (\omega^a_i)_\gamma \), since they are antisymmetric in the lower-index pair \([a\bar{b}]\), hence 6 pairs, and they have 4 displacement indices. — There are \( 4 \times 10 = 40 \) Christoffel connection coefficients \( (\Gamma^\alpha_i)_\gamma \), hence 6 pairs, and there are 4 values for \( \alpha \). — These 40 resp. 24 connection coefficients are highly uninformative.

For the exact equations of motion demonstrating explicitly the equivalence theorems for both non-inertial for inertial observers, we need a primary-observer-adapted spacetime splitting and LONB-field:

- our primary-observer-adapted space-time splitting uses fixed-time slices which are generated by radial 4-geodesics starting Lorentz-orthogonal to the worldline of the primary observer,
- the time-coordinate \( t \) all over the fixed-time slice \( \Sigma_t \) is defined as the time measured on the wristwatch of the primary observer,
- our Local Ortho-Normal Bases, LONBs, are radially parallel, i.e. parallel along radial 4-geodesics from the primary observer’s LONB,

LONBs radially parallel to primary observer’s LONB,
• 3-coordinates on the slices $\Sigma_t$ are not needed in the equations of motion.

In the equations of motion, for $(d/dt)(p_i)$, and for the equivalence theorems, we work with LONBs exclusively, hence with Ricci connection coefficients $(\omega^a_{\hat{b} \hat{c}})_{\hat{d}}$. We do so, because the LONB-components $p^i$ can be measured directly, while the coordinate-basis components $p_i$ cannot be measured without having first solved Einstein’s equations to obtain the metric in the coordinate basis, $g_{\mu \nu}$.

Our slicing and LONB-field are needed for the primary observer’s direct determination of momentum components $p^i(t_1 + \delta t)$ and spin components $S^i(t_1 + \delta t)$ off his worldline: The primary observer measuring $\bar{p}(t)$ of a particle off his worldline, needs a parallel transport of the particle’s $\vec{p}(t)$ on the radial geodesic on $\Sigma_t$ to his own LONB at the time $t$. In this parallel transport, the components $p^i$ stay unchanged, because we have chosen the LONBs radially parallel. This makes the explicit equations simple.

Our crucial result: with our LONBs radially parallel,

- the connection coefficients for spatial displacements vanish on the entire worldline of the primary observer,

\[
[(\omega^a_{\hat{b} \hat{c}})_i]_{(r \rightarrow 0, \ t)} = 0, \quad (19)
\]

\[
\hat{i} = (1, 2, 3), \quad \hat{a}, \hat{b} = (0, 1, 2, 3).
\quad (20)
\]

- For a non-inertial observer, the connection coefficients for displacements in time are non-zero,

\[
(\omega^i_{\hat{b} \hat{c}})_0 = E_i^0(x),
\]

\[
(\omega^i_{\hat{b} \hat{c}})_0 = B_{ij}^0(x).
\quad (21)
\]

- For an inertial primary observer, all connection coefficients vanish on the primary observer’s entire worldline,

\[
[(\omega^a_{\hat{b} \hat{c}})_{\hat{d}}]_{(r \rightarrow 0, \ t)} = 0, \quad \hat{a}, \hat{b}, \hat{d} = (0, 1, 2, 3).
\quad (22)
\]

A non-inertial and an inertial observer have necessarily different worldlines (displacement curves), but only the tangent vector in $P$ matters in the connection-1-form coefficients $(\omega^a_{\hat{b} \hat{c}})\hat{c}$. The tangent-vector in $P$ is the same, if the two observers are instantaneously comoving in $P$.

Our radially parallel spatial LONB-field $\vec{e}_i$ in curved spacetime is consistent with the 3-globally parallel basis vector-field $\partial_i$ in Newton’s Euclidean 3-space with Cartesian coordinates.

In the equations of motion, evaluated on the worldline of the observer, it is irrelevant that our LONB’s $(\vec{e}_\theta, \vec{e}_\phi, \vec{e}_0, \vec{e}_t)$ are not parallel in the $(\theta, \phi)$ directions.

Our radially parallel $\vec{e}_i$ field in curved spacetime is unique in being consistent with Newtonian observers at a given time being at relative rest, hence basis vectors $\partial_i$ spatially parallel in the language of affine geometry of Felix Klein of 1872.

Our extended fields of LONBs define an extended reference frame for an inertial or for a non-inertial primary observer along their entire worldlines. — Spatially, this holds outwards until radial 4-geodesics cross, e.g. from multiply-imaged quasars.

\[ d/dt \left( p_i \right), \quad d/dt \left( S_i \right) = 0, \quad (23) \]

where $t$ is the time measured on the worldline of the primary observer.

For an inertial observer and with our adapted slicing and adapted LONBs, all Ricci connection coefficients $(\omega^a_{\hat{b} \hat{c}})\hat{c}$ vanish. For an inertial observer with a general non-adapted slicing and non-adapted LONBs, there are $6 \times 4 = 24$ non-vanishing Ricci connection coefficients $(\omega^a_{\hat{b} \hat{c}})\hat{c}$, which would be needed in the geodesic equation and the Fermi spin-transport equation with non-adapted LONBs and non-adapted spacetime-slicing.

For an inertial observer with our adapted slicing and LONBs, the equivalence theorem of general relativity states that all gravitational fields vanish which is equivalent with the vanishing of all $24$ Ricci connection coefficients $(\omega^a_{\hat{b} \hat{c}})\hat{c}$.

D. Equivalence theorem for noninertial observers

Our three fundamental results for the equivalence theorem in the equations of motion of general relativity for non-inertial observers are given first, the derivations follow afterwards:

- Our results are explicitly valid, if and only if one uses our primary-observer-adapted slicing of spacetime, generated by radial 4-geodesics starting Lorentz-orthogonal to the primary observer’s worldline, and our primary-observer-adapted LONBs, which are radially parallel to the LONB of the primary observer as described in Sec. III B.

- First result: For any test-particle and an observer who is non-relativistic relative to this particle, our exact equations of motion in arbitrarily strong gravitational fields of general relativity are identical with
the 19th-century Newton-Heaviside-Maxwell equations of motion,
\[ \frac{d}{dt}(p_i) = m [\vec{E}_g + \vec{v} \times \vec{B}_g]_i, \]
gravity in general relativity,
\[ + q [\vec{E} + \vec{v} \times \vec{B}]_i, \]
el.mag. in general relativity.
Every term in this exact equation of general relativity is identical to the terms known in the 19th century. This conceptual framework is different in general relativity, but the equations are the same.

(1) Newton's gravitational force, the gravito-electric force due to \( \vec{E}_g = \hat{g} \) in Heaviside's notation of 1893, defined for general relativity in Eq. (1).

(2) Heaviside's gravito-magnetic force due to \( \vec{B}_g \), postulated 1893 \[2\], defined for general relativity in Eq. (2), and form-identical with the Lorentz magnetic force due to \( \vec{B} \) of Ampère and Maxwell.

(3) Maxwell's electromagnetic forces due to \( (\vec{E}, \vec{B}) \).

Our exact equation of motion of general relativity in our primary-observer-adapted spacetime slicing and LONBs, Eqs. (24) and (25), has only two gravitational fields, \( \vec{E}_g \) and \( \vec{B}_g \), versus 40 Christoffel symbols, utterly non-instructive, for general coordinates (both in general relativity and in Newtonian physics with its universal time).

- Second result: For relativistic test-particles, the exact equations of motion of general relativity in arbitrarily strong gravitational fields and electromagnetic fields are,
\[ \frac{d}{dt}(p_i) = \varepsilon [\vec{E}_g + \vec{v} \times \vec{B}_g]_i \]
gravity in GR
\[ + q [\vec{E} + \vec{v} \times \vec{B}]_i, \]
el.mag. in general relativity
\[ \frac{d}{dt} \varepsilon = \varepsilon \vec{v} \cdot \vec{E}_g \]
gravity in general relativity
\[ + q \vec{v} \cdot \vec{E}, \]
el.mag. in general relativity.

The energy and 3-momentum of the test-particle,
\[ \varepsilon = \text{total particle-energy} = \gamma m, \quad \gamma = (1 - v^2/c^2)^{-1/2}, \]
energy of photon \( \varepsilon = h \nu \gamma \),
\[ \vec{p} = \text{3-momentum} = \gamma m \vec{v}, \quad \text{for photon} \; \vec{p} = h \nu \varepsilon \vec{v}. \]

With our primary-observer-adapted space-time slicing and LONBs,
1. the terms due to electromagnetic fields in general relativity are explicitly identical with the equations in special relativity,
2. the terms due to gravitational fields in general relativity are form-identical with the terms for electromagnetic fields in Minkowski spacetime of special relativity, except for the replacements,
\[ \vec{E}, \vec{B} \Rightarrow (\vec{E}_g, \vec{B}_g) \]
\[ q \Rightarrow \varepsilon. \]

It is remarkable that these simple exact equations of motion for general relativity with arbitrarily strong gravitational fields are missing in textbooks.

Eqs. (24) and (25) are exact in curved spacetime. But equations of motion have only first derivatives of particle momenta, therefore they are independent of curvature, which needs second derivatives of vectors around a space-time plaquette.

- Third result: Theorem of equivalence between fictitious forces and gravitational forces in the exact equations of motion of general relativity for relativistic particles in arbitrarily strong gravitational fields:

(1) the equivalence between the fictitious force measured by an accelerated observer and a gravito-electric field \( \vec{E}_g \) generated by sources of matter,

(2) the equivalence between the fictitious Coriolis force measured by a rotating observer and a gravito-magnetic Lorentz force from a gravitomagnetic field \( \vec{B}_g \) generated by matter-currents.

Our fundamental definitions of \( \vec{E}_g \) and \( \vec{B}_g \) in Eqs. (1) and (3), inserted in the exact equation of motion of general relativity, Eq. (25), for particles free of non-gravitational forces give,
\[ \frac{d}{dt}(p_i) = \varepsilon [\vec{E}_g + \vec{v} \times \vec{B}_g]_i. \]

These equations hold for relativistic particles in general relativity. — The first of these two equations agrees with Landau and Lifshitz, Classical Mechanics \[6\], Eq. (39.7) for the nonrelativistic case.

The exact equivalence between gravitational forces and fictitious forces in the equation of motion for general relativity and for relativistic particles is evident by comparing the two right-hand-sides in Eq. (26).

\[ \text{equivalence of fictitious and gravitational forces:} \]
\[ \vec{a}_{\text{obs}} = -\vec{E}_g, \]
\[ \vec{\Omega}_{\text{obs}} = \frac{1}{2}\vec{B}_g. \]

It is a crucial fact that the exact \( \vec{E}_g = -\vec{a}_{\text{obs}} \), and \( \vec{B}_g/2 = \vec{\Omega}_{\text{obs}} \), can be measured with freefalling test-particles and gyroscopes which are quasistatic relative to the observer, Eqs. (4) and (5), but they predict the motion of relativistic particles in Eq. (26).

Arbitrarily strong gravito-electric and gravito-magnetic fields can be measured by quasi-static methods in the instantaneous comoving inertial frame, since only infinitesimally small velocities and gyro-precession angles arise after an infinitesimally short time.

Our results in this subsection look very familiar. But our results are entirely new:
(1) Our exact results for general relativity with arbitrarily strong gravitational fields are new. No “Newtonian approximation”, no weak field limit, no perturbation theory.

(2) The exact definition of $E_g$ in arbitrarily strong gravitational fields is new, Sec. III.

(3) The identity of $(E_g, B_g)$ with the Ricci connection for a displacement in time, $(\omega_{i0})_0$, is new, Sec. IB.

(4) Our exact equations of motion in arbitrarily strong gravitational fields, Eqs. (24, 26), are new.

(5) Our equations Eqs. (24, 26) for the equivalence of fictitious and gravitational forces in the equations of motion are new.

It is remarkable that most textbooks on general relativity do not discuss the equivalence of fictitious forces and gravitational forces in the equations of motion. The few textbooks, which do discuss this equivalence, are not specific about which fictitious force is equivalent to which gravitational force. But most important: these textbooks do not give equations. Words without equations is unusual for an important topic in theoretical physics.

It is also remarkable that in textbooks there is no discussion of measurements by rotating observers, no equation demonstrating the equivalence of the fictitious Coriolis force with the gravitomagnetic Lorentz force, no discussion of the fundamental impossibility to distinguish these two forces in the equations of motion.

The old paradigm for fundamentals in general relativity has been that one should work with general coordinates. Only for applications to special situations, Schwarzschild, Kerr, Friedmann-Robertson-Walker, should adapted coordinates be used.

The new paradigm for fundamentals in general relativity is that one must choose adapted spacetime slicing and adapted LONBs to exhibit (1) the identity of 19th-century Newtonian equations of motion and the equations of motion of general relativity for nonrelativistic test-particles, (2) the identity of the equations of motion of special relativity in an electromagnetic field and the equations of motion of general relativity with the obvious replacements $(E, B) \Rightarrow (E_g, B_g)$ and $q \Rightarrow \varepsilon$.

The derivation of our equations of motion in gravitational plus electromagnetic fields, Eqs. (26), follows from (1) the geodesic equation plus the electromagnetic term and (2) our primary-observer-adapted space-time slicing and LONBs,

\[ \frac{d}{dt} p_i = - (\omega_{i0})_0 p^a \left( \frac{dx^a}{dt} \right) + q F_{ia} u^a = - (\omega_{i0})_0 p^a + q F_{ia} u^a. \]  \hspace{1cm} (28)

With our primary-observer-adapted slicing and LONBs, radially parallel LONBs, of Sec. XIIIIB, the connection coefficients nonzero only for displacement in time along worldline of primary observer,

\[ \frac{d}{dt} p_i = \pm (\omega_{i0})_0 p^a, \]  \hspace{1cm} (29)

\[ (\omega_{i0})_0 = - E_i^{(g)}, \]

\[ (\omega_{ij})_0 = - \frac{1}{2} B_{ij}^{(g)}, \]

Using the 4-momentum components $p^0 = \varepsilon$, and $p^i = \varepsilon v^i$, we obtain Eqs. (25).

To derive the equations of motion for a non-inertial observer, Eqs. (25), the Local Inertial Coordinate System and the associated Local Inertial Frame used in all textbooks are useless.

The time-evolution equation for spin, the Fermi spin transport equation, Eqs. (15) simplify in our primary-observer-adapted LONBs and slicing. Since the spin 4-vector in the rest frame of a particle has by definition no time-component, we have $p^0 S_0 = 0$, and we can eliminate $S_0$

\[ p^0 S_0 = 0 \quad \Rightarrow \quad S^0 = \vec{v} \cdot \vec{S}, \]

\[ \frac{d}{dt} (S_j) = (\vec{S} \times \vec{B}_g/2)_j + (\vec{v} \cdot \vec{S}) E_i^{(g)}, \]  \hspace{1cm} (30)

where $\vec{v} \equiv (d\vec{x}/dt)$ is the 3-velocity of the particle.

For a comoving gyroscope, the second term in Eq. (30) vanishes, and this equation for relativistic gyroscopes becomes identical to the equation for nonrelativistic gyroscopes, Eq. (2). For a non-comoving gyroscope, the second term is the gravitational analogue of the Thomas precession (1927) in classical electrodynamics [8].

E. Connection-1-forms versus curvature tensor

Textbooks give an asymmetric status to connections, e.g. Ricci’s $(\omega^a_b)_c$, versus curvature tensors, e.g. Riemann’s $(R^a_b)_{cd}$.

For dynamics, the equations of motion and the gravitational field equations are two equally important legs. In Wheeler’s words: “matter tells spacetime how to curve, curved spacetime tells matter how to move.”

In the equations of motion, the measured forces are encoded in the Ricci connection coefficients $(\omega^a_b)_0$ with our adapted slicing and LONBs. These observables are $E_g$ and $B_g$, directly measured, the observational rock, on which the equations of motion are built. — The equations of motion do not involve the Riemann tensor.

Observers are in a one-to-one relation with Ricci’s connection for a time-like displacement, $(\omega^a_b)_0$. Hence, observers and their worldlines are as fundamental as Ricci’s connection for a time-like displacement.
IV. SPACETIME CURVATURE FROM NEWTONIAN EXPERIMENTS

In Sect. II C we have considered the Newtonian experiment of dropping pebbles down a well drilled all the way through the Earth. We have explained, how this Newtonian experiment together with Einstein’s revolutionary concept of inertial motion and geodesics gives direct evidence of space-time curvature.

We now show by explicit computation, how the space-time curvature-coefficient $R^{00}$ of general relativity is completely determined by Newtonian experiments alone.

In general relativity, (3+1)-spacetime is not embedded in some higher dimensional spacetime, and there is no observational evidence for such an embedding. Therefore, curvature of (3+1)-spacetime always means intrinsic curvature. Generally, intrinsic curvature is determined by measurements intrinsic to the space considered. The intrinsic curvature of a surface of an apple at a point $P$ can be determined by measuring angles and lengths on the surface of the apple near $P$.

A. Curvature from deficit tangent-rotation angle around closed curves

The sum of inner angles in a triangle is always 1800 in the Euclidean plane.

For $N$-polygons as boundaries of simply connected areas in the Euclidean plane, the sum of inner angles is not useful, it is $(N-2)\pi$ and goes to infinity for $N \to \infty$. It is much more useful to consider the sum of tangent-rotation angles $\alpha_i$. For border-curves $C$ of simply connected areas $A$, the integral of tangent-rotation angles is always a full rotation, 2$\pi$, in the Euclidean plane,

$$\sum_i \alpha_i^{(\text{tangents})} = \oint_C \left( \frac{d\alpha}{ds} \right)^{\text{(tangents)}} ds = 2\pi.$$

In a non-Euclidean 2-space, e.g. on the surface of an apple, $N$-polygons and curves $C$ which are the boundary of a simply connected area $A$, $C = \partial A$, have a sum, resp. integral, of tangent-rotation angles different from a full rotation of $2\pi$,

$$\sum_i \alpha_i^{(\text{tangents})} = \oint_C \left( \frac{d\alpha}{ds} \right)^{\text{(tangents)}} ds \equiv 2\pi - \delta_C,$$

$$\delta_C \equiv \text{deficit rotation angle. } (31)$$

As an example, we compute the deficit angle $\delta$ for a geodesic triangle on the surface of a spherical Earth:

1. start at the North-pole and go South along the zero-degree-meridian to the equator,
2. go East along the equator to the 90$^\circ$ meridian,
3. follow the 90$^\circ$ meridian back to the North-pole.

This is a geodesic triangle with three tangent-rotation angles of $\pi/2$, the total tangent-rotation angle is $3\pi/2$, hence the deficit angle is $\delta = \pi/2$.

The intrinsic Gauss curvature $R_G$ at any point $P$ of any 2-dimensional space can be operationally defined by the deficit angle $\delta$ divided by the measured area $A$ inside the curve $C$ for the length of the curve around $P$ going to zero,

$$R_G = \lim_{C \to 0} \left( A^{-1} \delta_C \right). \quad (32)$$

For our chosen geodesic triangle on the surface of a spherical Earth, the area of $A$ can be measured entirely on the surface of the Earth, an intrinsic measurement on the surface. The result is $A = r^2_{\text{Earth}} \frac{\pi}{2}$. The deficit angle is $\delta = \pi/2$, hence the Gauss intrinsic curvature is $R_G = \frac{\pi}{2 r^2_{\text{Earth}}}$.

The primary definition of intrinsic curvature in elementary geometry is given by Eqs. (31) and (32) in terms of the deficit rotation angle for geodesic triangles and polygons.

B. Curvature from LONB-rotation angle around closed curves

Cartan used a different method to compute the same deficit angle $\delta_C$ around a curve $C$.

Cartan’s method for computing curvature is missing in many textbooks on general relativity, and it is not taught in most graduate programs in gravitation and cosmology. Therefore, we give a short introduction to Cartan’s method.

Instead of using as a path an arbitrary closed curve $C = \partial A$, and instead of considering the rotation of the tangent vectors along this curve $C$, Cartan used (1) the rotation of the chosen LONB-field, already crucial for Secs. II and III (2) the closed path along a coordinate plaquette $[\hat{e}_\alpha, \hat{e}_\beta] \equiv [\partial_\alpha, \partial_\beta]$ in the chosen coordinate system. Therefore, Cartan’s method needs the LONB-rotation for a displacement in a coordinate, $(\omega_{ij})_\mu = \hat{e}_i \Rightarrow \hat{e}_j$. Cartan needs a coordinatization, i.e. a mapping $P \Rightarrow x^\mu$, but the metric $g_{\mu\nu}$ is not needed to compute the Riemann tensor with Cartan’s method. The metric $g_{\mu\nu}$ cannot be known until Einstein’s equations are solved for the specific problem.

In curved 2-space, LONB-rotations are given by an angle, no need for a rotation matrix, $(\omega_{ij})_\mu \Rightarrow \omega_\mu$.

In the Euclidean plane, the total LONB-rotation angle along any closed curve $C = \partial A$ is always zero,

$$\oint_C \left( \frac{d\alpha}{dx} \right)^{\text{(LONBs)}} \equiv \omega_\mu : \oint_C \omega_\mu dx^\mu = 0.$$

For a Cartesian LONB-field in Euclidean 2-space, the LONB-rotation angle is identically zero. For the LONB-field aligned with polar coordinates in the Euclidean 2-space, (1) the LONB rotation angle is nonzero for a displacement in the direction $\partial_\phi$, (2) the LONB-field is singular at the origin, hence it is not admitted, if the origin is inside $A$. 

In curved 2-space, the total rotation angle of LONBs relative to infinitesimal geodesic pieces along the positively oriented boundary \( C = \partial A \) gives the deficit angle,

\[
\int_C \omega_\mu \, dx^\mu \equiv - \delta C \equiv - \text{deficit angle.} \quad (33)
\]

The line-integral calls for the 1-form \( \bar{\omega} \) with covariant components \( \omega_\mu \). Using 1-form components \( \omega_\mu \) causes the absence of metric factors \( g_{\mu\nu} \) in the line-integral (??) over curved space.

For two areas touching along a piece of common boundary, the total deficit angle is additive, because the LONB-rotation angles cancel along the common boundary due to the opposite direction of the boundary curves. The areas are also additive. Therefore, the total deficit angle divided by the total area, \( (\delta C / A) \) is the same for any (simply connected) area on a perfect sphere. — For the surface of an apple, an infinitesimal curve \( C \) around any point \( P \) gives the local measure of intrinsic curvature, the Gauss curvature at \( P \) of Eq. (32),

\[
R_G = \lim_{C \to 0} \left( A^{-1} \delta C \right).
\]

As an example of Cartan’s method of LONB-rotation angles around a closed curve, we compute the intrinsic curvature of the surface of a spherical Earth We choose the LONB field \( (\bar{e}_{\text{East}}, \bar{e}_{\text{North}}) \). For the closed boundary-path \( \partial A \), we must avoid the singular points of this LONB field, North pole and South pole:

1. go South along \( 0^0 \) meridian, start an infinitesimal \( \delta \theta \) away from North pole, end at equator: LONB rotation angle = 0,
2. go East along equator to \( 90^0 \) meridian: LONB rotation angle = 0,
3. go North along \( 90^0 \) meridian to \( \delta \theta \) before North pole: LONB rotation angle = 0,
4. go West along \( \delta \theta = \) infinitesimal constant, from \( 90^0 \) meridian to \( 0^0 \) meridian:

The total LONB-rotation angle is negative, \( \alpha = - \pi / 2 \), the deficit angle is positive, \( \delta = \pi / 2 \), measured area \( A = (\pi / 2) r_{\text{Earth}}^2 \), intrinsic curvature \( R_G = r_{\text{Earth}}^{-2} \).

C. Cartan’s curvature equation in two dimensions

We convert the line-integral for the deficit angle \( \delta \) of Eq. (33) using Stoke’s theorem,

\[
\delta C = - \int_{C=\partial A} \omega_\mu \, dx^\mu = - \int_A (\text{curl} \bar{\omega}) \cdot dA. \quad (34)
\]

In two dimensions, curl \( \bar{\omega} \) and the Gauss curvature \( R_G \) are both scalars under rotation and space reflections (even under parity). The rotation deficit angle \( \delta \) and the oriented area \( A \) are both pseudoscalars.

Cartan curvature eq. in 2 dimensions

\[
R_G = - \text{curl} \bar{\omega}, \quad (35)
\]

We now need Cartan’s tools: connection 1-form, exterior derivative, and Cartan’s curvature 2-form.

In curvilinear coordinates (in curved space or flat space), the antisymmetric covariant derivative of a vector field \( \bar{V} \) in covariant components is equal to the ordinary antisymmetric derivative,

\[
[\nabla_\nu V_\mu - \nabla_\mu V_\nu ] = [ \partial_\nu V_\mu - \partial_\mu V_\nu ].
\]

This holds, because the Christoffel connection coefficients \( (\Gamma^\alpha_{\beta\lambda})_\delta \) are symmetric in the last two indices.

We shall see that covariant derivatives play no role in Cartan’s method of computing curvature.

In the calculus of forms (here for Riemannian geometry, which has a metric), a vector, if and only if it is given by its covariant components is called a 1-form and denoted by a tilde (instead of an arrow or a bar), \( \bar{\sigma} \) with 1-form components \( \sigma_\mu \). The index structure of differential forms is trivial, therefore one often drops indices,

\[
1\text{-form } \bar{\sigma} \leftrightarrow 1\text{-form components } \sigma_\mu.
\]

The antisymmetric derivative of a 1-form \( \bar{\sigma} \) is called exterior derivative, denoted by the symbol \( d \) in \( d \bar{\sigma} \),

\[
d \bar{\sigma} \leftrightarrow (d \bar{\sigma})_{\mu\nu} = \partial_\mu \sigma_\nu - \partial_\nu \sigma_\mu.
\]

The exterior derivative of a 1-form \( \bar{\sigma} \) produces \( d \bar{\sigma} \), a covariant antisymmetric 2-tensor, which by definition is a 2-form. — General 2-forms are defined as antisymmetric covariant 2-tensors.

In 3-space, the Hodge-dual star-operation of \( d \bar{\sigma} \) is formed with the Levi-Civita tensor, which is defined by \( (1) \) totally antisymmetric tensor, \( (2) \) \( \varepsilon_{123} \equiv +1 \) in a LONB with positive orientation, as discussed in Eq. (12). Applying the Hodge dual-star-operation to the exterior derivative of a 1-form in 3-space, \( * d \bar{\sigma} \), gives the curl of the vector field \( \bar{\sigma} \),

\[
(* d \bar{\sigma})_\lambda = (\text{curl} \bar{\sigma})_\lambda.
\]

Hence, Cartan’s curl equation for curvature in two spatial dimensions, Eq. (33), can be re-written in the calculus of forms,

Cartan’s curvature scalar in 2 dim.:

\[
R = * d \bar{\omega} = \frac{1}{2} \varepsilon^{\mu\nu} [ \partial_\mu \omega_\nu - \partial_\nu \omega_\mu ]. \quad (36)
\]

Cartan’s equation for the curvature 2-form components in 2 dimensions,

\[
R_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu . \quad (37)
\]

The geometric meaning of the two antisymmetric covariant indices in \( (d \omega)_{\mu\nu} \):

1. find the infinitesimal Lorentz transformation of LONBs under a first displacement along an observer’s worldline, \( \bar{e}_\nu \equiv \partial_\nu \), to obtain \( \omega_\nu \).
(2) compare this quantity on a neighbouring worldline separated by a second displacement \( \partial_\mu \) to obtain \( \partial_\mu \omega_\nu \),

(3) take the difference with the same in opposite order \( \mu \leftrightarrow \nu \) to obtain \( (\partial_\mu \omega_\nu - \partial_\nu \omega_\mu) \), which means going around the closed displacement curve \( C = \partial A \), the coordinate plaquette \( [\partial_\mu, \partial_\nu] \).

The displacement indices must necessarily be an antisymmetric covariant index-pair. Only in this case is the displacement plaquette \( \text{closed} \). Example on Earth: if you go East by 10\(^\circ\), go North by 10\(^\circ\), go West by 10\(^\circ\), go South by 10\(^\circ\), you have a \( \text{closed plaquette} \). If instead, generated by \( \partial_\mu \), you go East 100 km, North 100 km, go West 100 km, go South 100 km, you have no \( \text{closed path} \). A coordinate-displacement plaquette, formed by \( [\partial_\mu, \partial_\nu] \), is \( \text{closed} \), as it must be for Eq. (33). But a plaquette formed by \( [\partial_\mu, \partial_\nu] \) is not \( \text{closed} \). This is the reason, why the displacement-plaquette indices must be coordinate indices.

The curvature 2-form \( R_{\mu\nu} \) in 2 dimensions, Eq. (37), written suppressing the displacement index-pair, i.e. without the plaquette index-pair \([\mu\nu]\), is denoted by script-\( R \),

\[
\text{curvature from deficit rotation angle in 2-dim, without LONB-rotation indices:}
\]

\[
R = d\tilde{\omega},
\]

which is Cartan’s curvature equation in 2 dimensions.

Up to now in Sect. XV a LONB-rotation in 2 dimensions has been given by one number, the rotation angle. We now start using again the \textit{matrix notation} for infinitesimal LONB-rotations and Lorentz-boosts \( \omega_\hat{a}_b \) for a given displacement in the coordinate basis \( \partial_\nu \equiv \bar{e}_\nu \), which gives \textit{Cartan’s connection 1-form} \( (\omega_\hat{a}_b)_\nu \). Taking the exterior derivative with \( \partial_\mu \) as in Eq. (37), but now in the notation of an infinitesimal LONB-transformation-matrix, gives,

\[
\text{Cartan’s curvature 2-form} = (R^\hat{a}_{\hat{b}})_{\mu\nu}.
\]

\[
\text{LONB transformation indices} = [\hat{a}, \hat{b}],
\]

\[
\text{plaquette for displacements in coord.} = [\mu, \nu].
\]

This completes our derivation of Cartan’s curvature equation in (1+1)-spacetime and in 2-space, which started with the total rotation angle of LONBs around a closed curve, Sec. XV B

\[
(R^\hat{a}_{\hat{b}})_{\mu\nu} = (d\omega^\hat{a}_{\hat{b}})_{\mu\nu},
\]

\[
R^\hat{a}_{\hat{b}} = d\omega^\hat{a}_{\hat{b}}.
\]

\[\text{D. Exact Ricci curvature } R^{\hat{a}\hat{b}} \text{ from relative acceleration of quasistatic particles in (1+1)-spacetime} \]

We prove the crucial new theorem stated in the title of this section, valid in \textit{arbitrarily strong gravitational fields}. As a preparation, we start with (1+1)-dimensional spacetime in this subsection.

We compute the curvature in (1+1)-spacetime with our spacetime slicing \( \Sigma_t \) and our radially parallel LONBs. We also need a spatial coordinate \( P \Rightarrow x_P \). We choose the Riemann normal 1-coordinate \( x \) with \( g_{xx} \equiv 1 \), i.e. the coordinate \( x_P \) is equal to the measured distance from the observer to \( P \) on \( \Sigma_t \).

\[\bullet \text{ For curvature computations in this first paper, we consider an inertial primary observer. This is sufficient to derive Einstein’s } R^{\hat{a}\hat{b}} \text{ equation from Newtonian experiments.} \]

In a second paper, spacetime curvature will be computed for a non-inertial observer.

We choose the spacetime coordinate-plaquette \( [\partial_\mu, \partial_\nu] \) formed on one side by the worldline of the primary observer at \( x = 0 \).

\[\text{(1) For a displacement along a worldline of an inertial primary observer at } r = 0,\]

\[\text{along inertial worldline: } [(\omega_{\hat{a}\hat{b}})]_{x=0} = 0.\]

\[\text{(2) With LONBs spatially parallel, the LONB Lorentz-boost angles vanish in the } x \text{ direction, } t = \text{ fixed,} \]

\[\text{along spatial geodesics: } (\omega_{\hat{a}\hat{b}})_x = 0.\]

\[\text{(3) For an auxiliary observer at } x = \text{ fixed and infinitesimal, the LONB-Lorentz-boost angle for a displacement along his worldline, i.e. the relativistic connection coefficients } (\omega_{\hat{a}\hat{b}}), \text{ are determined exactly by experiments with quasistatic inertial test particles initially at rest relative to the auxiliary observer, as in Eqs. (19),} \]

\[\text{along worldline } \delta x \text{ constant:}
\]

\[
(\omega_{\hat{a}\hat{b}})_t = -E_x(g),
\]

\[
= -(dv/dt)_{\text{quasistatic inertial particle}} = -a_x.
\]

With our choice of the spatial coordinate, the coordinate distance \( x \) is the measured geodesic distance. Around the infinitesimal \([t, x]\)-plaquette of the observer, the coordinate time \( t \) is the time measured by the observer, to first order in \( x \). The measured velocity of quasistatic particles is \( v = dx/dt \).

The relative (tidal) acceleration of two neighbouring inertial quasistatic test particles, one constantly at rest relative to the primary inertial observer, the other one starting at rest relative to the primary and to the auxiliary observers,

\[
\partial_x[(\omega_{\hat{a}\hat{b}})_t] = -\text{div} \bar{E}_g = -\partial_x[(dv/dt)_{\text{freefalling}}].
\]
where $\partial_x F^a_{\mu} = \text{div} \vec{E}_a^\mu$, because the LONBs are radially parallel at all times.

The space-time curvature is obtained by considering the \textit{deficit Lorentz-boost angle} $\delta$ of LONBs around our infinitesimal $[t, x]$-plaquette,

$$\delta = - \oint_{[t, x]} (\omega_{\hat{z} \hat{r}})_\mu dx^\mu, \quad \text{(42)}$$

where the displacements in $x$ do not give a contribution, since the LONBs are radially parallel, and the displacement along the primary observer’s worldline at $x = 0$ does not give a contribution, since the primary observer is inertial, hence the LONBs along his worldline are self-parallel.

The deficit Lorentz-boost angle $\delta$ per measured plaquette area is per definition Cartan’s $(1+1)$ Riemann curvature $(\mathcal{R}^i_{\hat{z} \hat{r}})_t$. With coordinate indices equal to LONB indices at $\vec{R}_0$, $\vec{e}_a = \vec{e}_a$, the contraction of the two spatial indices gives the Ricci component $\mathcal{R}^i_{\hat{z} \hat{r}}$.

For Cartan’s curvature equation in three and more dimensions, we omit all equations which are not needed for Cartan’s wedge $\sigma$.

\section*{E. Cartan’s curvature equation in more than two dimensions}

For Cartan’s curvature equation in three and more dimensions, we need the \textit{antisymmetric product} of two 1-forms $\hat{\sigma}$ and $\rho$, called the \textit{exterior product} and denoted by a \textit{wedge}, $\hat{\sigma} \wedge \rho$,

\begin{equation}
\text{exterior product: } [\hat{\sigma} \wedge \rho]_{\mu \nu} = \sigma_\mu \rho_\nu - \sigma_\nu \rho_\mu. \quad \text{(44)}
\end{equation}

The \textit{Hodge-star-dual} of the exterior product of two 1-forms in three spatial dimensions, $\star ([\hat{\sigma} \wedge \rho])_i$, is equal to the vector product $[\hat{\sigma} \times \rho]_i$,

\begin{equation}
[\star ([\hat{\sigma} \wedge \rho])_i] = \varepsilon_{ijk} \sigma^j \rho^k = [\hat{\sigma} \times \rho]_i. \quad \text{(45)}
\end{equation}

We omit all equations which are not needed for Cartan’s curvature equation.

In dimensions $n$ higher than 2, a connection \textit{Lorentz transformation} of LONBs is given by a $(n \times n)$-matrix. For a \textit{coordinate-displacement} $\mu$, it is given by Cartan’s \textit{connection-1-form} $(\omega^a_{\hat{z} \hat{r}})_\mu$.

The curvature, the \textit{deficit Lorentz transformation} under a displacement around the coordinate plaquette $[\mu, \nu]$, divided by the plaquette area, is given by,

\begin{equation}
(R^a_{\hat{z} \hat{r}})_{\mu \nu} : \quad \hat{a}, \hat{b} = \text{LONB indices of deficit Lorentz-transf.,} \quad \mu, \nu = \text{coord. plaquette displacement-indices.} \quad \text{(46)}
\end{equation}

Dropping the displacement indices around the plaquette gives the \textit{curvature-2-form} $\mathcal{R}^a_{\hat{z} \hat{r}}$.

In more than two dimensions, Cartan’s curvature formula has an \textit{extra term}, which \textit{vanishes} for our primary-inertial-observer adapted LONBs. — We report Cartan’s curvature equation from [1],

\begin{equation}
\mathcal{R}^a_{\hat{z} \hat{r}} = d \omega^a_{\hat{b} \hat{r}} + \omega^a_{\hat{c} \hat{r}} \wedge \omega^c_{\hat{b} \hat{r}}. \quad \text{(47)}
\end{equation}

The \textit{second term vanishes} on the worldline of a \textit{inertial primary observer} with his \textit{radially parallel LONBs}, because all connection coefficients vanish on his worldline.

\section*{F. Exact Ricci curvature $R^{00}$ from relative acceleration of quasistatic particles in $(3+1)$ space-time}

\textit{Along} the worldline of a primary observer, his LONBs and our coordinate bases are identical, $\vec{e}_a = \vec{e}_a$.

\textit{Away} from the worldline of a primary inertial observer, $\vec{e}_a = \vec{e}_a$ continues to hold to \textit{first order in $r$} for our choice of slicing $\Sigma_t$, LONBs, and 3-coordinates.

In curved $(3+1)$ space-time, a primary inertial observer measures the \textit{spherical average} $<...>$ of the \textit{radial accelerations} $a_{\text{radial}} = (d\nu_r / dt)$ of freefalling test-particles initially at measured radial distance $r$. The test-particles at $r$ can be chosen \textit{quasi-static} relative to the primary observer at $r = 0$, and they will remain quasi-static within an infinitesimal time-interval. The \textit{spherical average} is equal to the average over the \textit{three principal directions} $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$. Hence, the generalization of Eq. (43) to $(3+1)$ spacetime is,

\begin{equation}
(R^{00})_{\text{exact}} = (-\partial_t < a_{\text{radial}} >)_{\text{spherical}}. \quad \text{(48)}
\end{equation}

Both equations are wrong for:

- a non-inertial primary observer,
- secondary observers (to obtain the relative acceleration) whose LONBs are not radially parallel with the LONBs of the primary observer.

\textbf{Conclusion:}

- For a \textit{freefalling-nonrotating primary observer} with $\vec{u}_{\text{obs}} = \vec{e}_0$, the \textit{exact} Ricci component $R^{00}$ of general relativity for arbitrarily \textit{strong gravity} fields is \textit{unambiguously determined}. 

---

\textit{Note:} Some symbols and expressions might not be fully rendered due to limitations in text formatting. The above content represents a natural text reading of the provided document page.
by measuring the relative acceleration of neighbouring freefalling test-particles which are quasi-static relative to the observer.

\( R_{00} \) is linear in the gravitational fields.

Observers which are not freefalling-nonrotating will be treated in a separate paper. For a uniformly rotating-freefalling observer, there is only one fictitious force, the centrifugal force, causing the radial acceleration of quasi-static test-particles,

\[
\vec{a}_{\text{centrifugal}} = \left[ \vec{\Omega}_{\text{obs}} \times (\vec{r} \times \hat{x}) \right] = \frac{1}{4} (\vec{B}_g \times [\vec{r} \times \hat{B}_g]). \tag{49}
\]

The radial derivative of the centrifugal acceleration at \( r = 0 \) gives an additional term for \( R_{00}^{\text{extra}} \),

\[
R_{00}^{\text{extra}} = -\partial_r < \hat{a}_{\text{radial}} >_{\text{extra}} < \hat{a}_{\text{spherical}} >_{\text{quasistatic}} = -\text{const.} \vec{B}_g^2. \tag{50}
\]

G. Einstein’s \( R_{00} \) equation for nonrelativistic matter-sources identical with Newton’s equation of relative acceleration

The exact equality between the space-time curvature \( R_{00} \) and the relative acceleration of freefalling particles, spherically averaged, for a freefalling-nonrotating observer with worldline through \( P \) and \( \vec{u}_{\text{obs}}(P) = \vec{v}_0(P) \), Eq. (48), gives the proof that,

- In arbitrarily strong gravitational fields, Newton’s law for acceleration of freefalling particles at \( \delta r \), measured relative to a freefalling-nonrotating observer and spherically averaged,

\[
< a_{\text{radial}} >_{\text{inertial obs.}} = -G_N M \delta r (\delta r)^{-2}, \tag{51}
\]

is explicitly identical

with Einstein’s \( R_{00} \) equation for source-matter non-relativistic relative to this inertial observer,

\[
R_{00} = 4\pi G_N \rho_{\text{mass}}. \tag{52}
\]

Only after choosing an inertial primary observer and LONBs radially parallel, is \( \text{div} \vec{E}_g \) uniquely defined for a given physical situation. With these two choices and only with these choices, we obtain,

\[
R_{00} = -(\text{div} \vec{E}_g)_{\text{LONBs rad. parallel}} = 4\pi G_N \rho_{\text{mass}}. \tag{53}
\]

The explicit field equation for non-inertial observers is bilinear in the gravitational fields \((\vec{E}_g, \vec{B}_g)\). These explicit field equations will be given in a second paper.

For \( R_{00} \), a tensor component, it is irrelevant, whether the primary observer is inertial or non-inertial, and it is irrelevant, whether we choose LONBs away from the primary worldline radially parallel or not (which dictates the choice for auxiliary observers). But \( \text{div} \vec{E}_g \) involves Ricci connections, all depends on inertial versus non-inertial primary observer (field equation linear in \( \vec{E}_g \) versus bilinear) and it depends on LONBs radially parallel or not (two fields, \( \vec{E}_g \) and \( \vec{B}_g \), versus more than a hundred connection coefficients).

It has often been emphasized that a fundamental difference between general relativity and Newtonian physics is the non-linearity of Einstein’s equations versus the linearity of the Newton-Gauss field equation \( \text{div} \vec{E}_g = -4\pi G_N \rho_{\text{mass}} \). Nothing could be farther from the truth:

- Einstein’s \( R_{00}(P) \) equation with nonrelativistic source-matter and the gravitational field equation of 19th-century Newton-Gauss physics are explicitly identical.

Both field equations are:

1. linear for a freefalling-nonrotating observer with worldline through \( P \), with \( \vec{u}_{\text{obs}} = \vec{v}_0 \), and with our radially parallel LONBs,

2. nonlinear for non-freefalling and/or rotating observers with worldlines through \( P \), because fictitious forces are equivalent to gravitational forces, Sec. III, and e.g. the centrifugal term is quadratic in \( \vec{B}_g \), Eq. (50).

H. From nonrelativistic source-matter to the full Einstein equations

Using,

1. Lorentz covariance,

2. the contracted second Bianchi identity to satisfy the covariant conservation of the energy-momentum-stress tensor,

one directly obtains Einstein’s equations, as explained in all textbooks,

\[
G^{\hat{a}\hat{b}} = 8\pi G_N T^{\hat{a}\hat{b}}, \tag{54}
\]

\[
G^{\hat{a}\hat{b}} = R^{\hat{a}\hat{b}} - \frac{1}{2} \eta^{\hat{a}\hat{b}} R. \tag{55}
\]

This completes our derivation of Einstein’s equations of general relativity from Newtonian experiments on relative acceleration.
I. Comparison with “heuristic derivations” of Einstein’s equations

There are two important differences between our paper and the literature:

The first crucial difference is that our paper gives a rigorous derivation of Einstein’s $R^{00}$ equation from nothing more than Newtonian physics and Einstein’s concept of inertial motion, while the literature only gives “heuristic derivations” of Einstein’s equations.

Struikmann [9], for his Sec. 3.2.1 gives the title: Heuristic “Derivation” of the Field Equations (“... “ marks by Struikmann).

Wald [10], in contrast to a rigorous derivation, writes in Section 4.3, “a clue is provided”, “the correspondence suggests the field equation”, but Wald gives no equation for the correspondence.

Weinberg [11] in Sec. 7.1 takes the “weak static limit”, makes a “guess”, and argues with “number of derivatives”.

Misner, Thorne, and Wheeler [12] give “Six Routes to Einstein’s field equations” in Box 17.2: (1) to model geometrodynamics after electrodynamics, (2) to take the variational principle with only a scalar linear in second derivatives of the metric and no higher derivatives, (3) “again electromagnetism as a model”, (4) superspace, (5) field equation for spin 2, (6) Sakharov’s view of gravitation as an elasticity of space.

In contrast, this paper gives a rigorous derivation of Einstein’s equations from Newtonian physics, not heuristic arguments.

The second crucial difference between the literature and our paper: For the $R^{00}$ field equation at a given point $P$, one can distinguish the matter-source “non-relativistic versus relativistic”. But the term “Newtonian limit” is meaningless for the $R^{00}$ field equation, because this Einstein equation is identical to the 19th-century Newtonian field equation: both equations are identical and linear for an inertial observer with worldline through $P$, both equations are identical and non-linear for non-inertial observers.

J. Outlook

In this paper, the field equation for $R^{00}(P)$ has been treated only for an inert primary observer with worldline through $P$ and with $\ddot{u}_{\text{obs}}(P) = c_0(P)$. This field equation of Einstein is linear in the gravitational field $E_g$, and it is explicitly identical with the Newton-Gauss field equation, $\nabla E_g = -4\pi G_N \rho_{\text{mass}}$.

A first extension: The field equation for non-inertial observers will be given in a companion paper. Again, the 19th-century Newtonian equation and Einstein’s $R^{00}$ equation for non-relativistic source-matter are exactly and explicitly identical for arbitrarily strong gravitational fields. But for non-inertial observers, both the field equation of 19th-century Newtonian physics and Einstein’s $R^{00}$ equation are bilinear in the gravitational fields.

A second extension: One asks the question, can this linearity for inertial observers be extended to an equation and its solution all over the universe? — In the paper [13], we have derived and solved exactly the linear field equation for Einstein’s angular-momentum constraint at any point $P_0$ from the angular-momentum input of an arbitrary distribution of matter on the past light-cone of $P_0$. The output is $\vec{B}_g$ at $P_0$, given as a simple explicit linear retarded integral over the matter-angular-momentum input on the past light-cone of $P_0$ back to the big bang.

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