A Spacetime Geometry picture of Forest Fire Spreading and of Quantum Navigation

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Abstract

The problem of finding null geodesics in a stationary Lorentzian spacetime is known to be equivalent to finding the geodesics of a Randers-Finsler structure. This latter problem is equivalent to finding the motion of charged particles moving on a Riemannian manifold in a background magnetic field or equivalently, by a generalization of Fermat’s principle, to Zermelo’s problem of extremizing travel time of an aeroplane in the presence of a wind. In this paper this triad of equivalences is extended to include recent model of the spread of a forest fire which uses form of Huyghen’s principle. The construction may also be used to solve a problem in quantum control theory in which one seeks a control Hamiltonian taking an initial state of a quantum mechanical system with its own Hamiltonian to a desired final state in least time. The associated stationary spacetime may be thought of as defined on an extended quantum phase space (Souriau’s evolution space), the space of quantum states being complex projective space equipped with its Fubini-Study Kähler metric. It is possible that this spacetime viewpoint may provide insights relevant for our understanding of quantum gravity.
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1 Introduction

Over the last few years it has emerged [1, 2, 3, 4] that there is a close connection between three superficially unrelated problems.

- The motion of light rays in a stationary spacetime.
- A particular kind of Finsler geometry due to Randers [5].
- A variational problem generalising the notion of a geodesic of a Riemannian metric due to Zermelo [6].

The interest of these connections goes beyond General Relativity: there are applications to a number of other areas of physics. For instance to the propagation of sound in a wind [7, 8], the behaviour of quasi-particles in graphene sheets [9], problems in quantum control theory [10, 11, 12, 13, 14, 15, 16] and most recently the spread of wild fires [17]. This latest, rather surprising, application makes use of Huyghen’s principle and this motivates the present paper the aim of which is to both explore in more detail this aspect from a spacetime point of view but also to expand upon from the same standpoint the results of [15, 16]. The common idea will be the systematic exploitation of the Hamilton-Jacobi and Eikonal equations.

2 Randers-Finsler metrics, Zermelo’s problem and null geodsics

In the interest of making this paper self-contained we start with the a brief review of the three legs of the correspondence described in [2]. It will also serve to establish our notation and conventions.

2.1 Finsler geometry

In Riemannian geometry the function on the tangent bundle given by

$$F = \sqrt{a_{ij}(x)v^i v^j}$$  \hspace{1cm} (2.1)

where $a_{ij}$ is the Riemannian metric, is homogeneous degree one in the fibre coordinate $v^i$ and serves to define a norm \^1 on the tangent space at each point. Moreover one may define geodesics as critical points of the length functional on curves $x^i(\lambda)$ with tangent vectors $v^i = \frac{dx^i}{d\lambda}$ given by

$$S[x^i(\lambda)] = \int F d\lambda.$$

\(1\)sometimes, confusingly for a physicist, called a Minkowski norm
Note that the functional $S[x^i(\lambda)]$ is invariant under reparametrisations $\lambda \to g(\lambda)$. In Finsler geometry one replaces the right hand side of (2.1) by any any homogeneous function of degree one of the tangent vector provided it defines a norm. The co-dimensions one convex surface in the tangent space given by
\[
F = 1 ,
\]
is called the indicatrix. For the Riemannian case it is an ellipsoid and is centro-symmetric. In general the indicatrix of a Finsler structure is neither an ellipsoid nor centro-symmetric.

The special case we shall be most interested in is that of Randers for which
\[
F = \alpha + \beta , \quad \alpha = \sqrt{a_{ij}v^iv^j} , \quad \beta = b_iv^i .
\]
The indicatrix is an ellipsoid but displaced from the origin by $b_i$. Thus we require that its length with respect to the Riemannian metric $a_{ij}$, i.e. $\sqrt{a^{ij}a_{ij}}$ be less than unity. Randers-Finsler geodesics satisfy
\[
\frac{d^2x^i}{ds^2} + \left\{ _{j \ k}^{i} \right\} \frac{dx^j}{ds} \frac{dx^k}{ds} = a^{ij}F_{jk} \frac{dx^k}{ds} ,
\]
where $ds = \alpha d\lambda$ is arc length with respect to the Riemannian metric $a_{ij}$, $\left\{ _{j \ k}^{i} \right\}$ are the Christoffel symbols of the Riemannian metric $a_{ab}$ and $F_{ij} = \partial_i b_j - \partial_j b_i$. If $F_{ij}$ vanishes then Randers-Finsler geodesics are just ordinary geodesics of the Riemannian metric $a_{ij}$. If $F_{ij} \neq 0$ then they are what are sometimes called magnetic geodesics, i.e. the paths of charged particles moving in a magnetic field $F_{ij}$. Note that while two Randers-Finsler structures with one-forms $b_i$ and $\tilde{b}_i$ for which
\[
b_i - \tilde{b}_i = \partial_i h ,
\]
for some function $h$, have the same Randers-Finsler geodesics they are nevertheless not equivalent as Randers-Finsler structures.

2.1.1 The Funk-Randers metric

The fore-going remarks may be illustrated by the following celebrated example

Given a convex domain in Euclidean space Funk introduced a distance function $d(x, y)$ which is asymmetric $d(x, y) \neq d(y, x)$ which generalises a standard construction of a symmetric distance function by Cayley in the case of the unit ball. In general the symmetrisation $\frac{1}{2}(d(x, y) + d(y, x))$ of the Fink distance is called the Hilbert distance. In what follows we shall specialise to the case of the unit ball. In that case the Funk distance function arises from a Randers Finsler metric for which [18, 19]
\[
F = \sqrt{(1 - x^i x^i)v^iv^j + (v^j x^j)^2 + (x^k v^k)} , \quad (2.7)
\]
Evidently since
\[ \partial_i b_j - \partial_j b_i = 0, \]  
the Finsler geodesics are unaffected by the one-form \( b_a \) so the Finsler geodesics coincide with the geodesics of the symmetrized Finsler function, the metric of which is given by
\[ a_{ij} dx^i dx^j = \frac{dx^i dx^j (1 - x^k x^k) + (dx^j x^j)^2}{(1 - x^i x^i)^2}. \]  
(2.9)

In polar coordinates \( x^i = r(\cos \phi, \sin \phi) \)
\[ ds^2 = \frac{dr^2}{(1 - r^2)^2} + \frac{r^2}{1 - r^2} d\phi^2 \]  
(2.10)
\[ = \frac{1}{4} \left\{ d\psi^2 + \sinh^2 \psi d\phi^2 \right\}, \]  
(2.11)
where \( r = \tanh \frac{\psi}{2} \). This is a metric of constant curvature \(-\frac{1}{4}\) and the coordinates \( x^a \) are Beltrami coordinates in which the geodesics are straight lines (see e.g. [20]). One therefore has
\[ b_i dx^i = \frac{1}{2} d \ln \cosh \frac{\chi}{2} = d \frac{1}{\sqrt{1 - r^2}}. \]  
(2.12)

The full Funk distance function may be expressed as
\[ d(x, y) = \frac{1}{2} d_{\mathbb{H}^2} + \frac{1}{\sqrt{1 - x^i x^i}} - \frac{1}{\sqrt{1 - y^j y^j}}, \]  
(2.13)
where \( d_{\mathbb{H}^2} \) is the standard Hyperbolic distance function on the unit disc and may be expressed in Beltrami-coordinates as
\[ d_{\mathbb{H}^2} = \cosh^{-1} \left( \frac{1 - x^i y^i}{\sqrt{1 - x^j x^j} \sqrt{1 - y^k y^k}} \right). \]  
(2.14)

### 2.2 The Zermelo Problem

An example of Zermelo’s problem is faced by the pilot of an aircraft flying in the presence of a wind of speed \( W^i \) relative to the ground, seeks to minimize journey time, given that the aircraft flies at constant speed relative to the air. If the journey is of any distance, for example a transatlantic flight, the pilot must take into account the curvature of the earth and so in the absence of wind the optical route would be a great circle, i.e a geodesic of a Riemannian manifold with metric \( h_{ij} \). The pair \( \{ h_{ij}, W^i \} \) are referred to as the Zermelo data.

According to [1] this problem is equivalent to finding the shortest length of a curve in a manifold equipped with a Randers-Finsler metric with Randers data \( \{ a_{ij}, b_i \} \). The equivalence and the relationship between the two sets of data is conveniently expressed in terms of a third problem, finding the motion of a light ray moving in a time-independent Lorentzian spacetime.
2.3 Stationary Metrics

A general stationary Lorentzian metric may be expressed as

\[ g_{\mu\nu}dx^\mu dx^\nu = -V^2(dt + \omega_i dx^i)^2 + \gamma_{ij} dx^i dx^j, \]  

(2.15)

where \( V, \omega_i, \gamma_{ij} \) are independent of the time coordinate \( t \). In general the time coordinate \( t \) is not unique since one may always make the replacement

\[ t \to t - f(x^i), \quad \omega_i \to \omega_i + \frac{\partial f}{\partial x^i}. \]  

(2.16)

Locally at least one may regard the spacetime as an \( \mathbb{R} \) bundle over the spatial manifold, i.e. the space of orbits of the Killing vector field \( \frac{\partial}{\partial t} \), and \( \omega_i \) are the components of the horizontal connection, known in this context as the Sagnac connection \[21\]. If the curvature \( F_{ij} = \partial_i \omega_j = \partial_j \omega_i \) vanishes one may, at least locally, choose \( f \) to make \( \omega_i \) vanish. Such spacetimes are said to be static and admit an extra time-reversal symmetry \( t \to -t \).

Evidently we have

\[ g_{ij} = \gamma_{ij} - V^2 \omega_i \omega_j. \]  

(2.17)

Fermat’s principle arises from the Randers structure given by

\[ a_{ij} = V^{-2}\gamma_{ij}, \quad b_i = -\omega_i. \]  

(2.18)

(2.19)

This is equivalent to a Zermelo structure of the form

\[ h_{ij} = \frac{1}{1 + V^2 g^{lm} \omega_l \omega_m} g_{ij}, \]  

(2.20)

\[ W^i = V^2 g^{ij} \omega_j. \]  

(2.21)

We have

\[ \gamma_{ij} = g_{ij} + V^2 \omega_i \omega_j, \]  

(2.22)

\[ \gamma^{ij} = g^{ij} - \frac{V^2 \omega^i \omega^j}{1 + V^2 g^{lm} \omega_l \omega_m}, \]  

(2.23)

with

\[ \omega^i = g^{ij} \omega_j, \quad W^i = \frac{V^2 \gamma^{ij} \omega_j}{1 - V^2 g^{lm} \omega_l \omega_m}, \]  

(2.24)

\[ 1 - V^2 \gamma^{ij} \omega_i \omega_j = \frac{1}{1 + V^2 g^{ij} \omega_i \omega_j}. \]  

(2.25)

Note that

\[ |W^i|^2 = h_{ij} W^i W^j = a_{ij} b^i b^j = |b|^2. \]  

(2.26)
In terms of the Zermelo data the spacetime metric (2.15) is

\[ ds^2 = \frac{V^2}{1 - h_{ij} W^i W^j} \left[ -dt^2 + h_{ij} (dx^i - W^i dt)(dx^j - W^j dt) \right]. \]  \hspace{1cm} (2.27)

Note that both the Randers-Finsler structure and the Zermelo structures do not depend upon the conformal equivalence class of the metric (2.15). In other words they are unchanged by the replacements (known as Weyl rescaling)

\[ V^2 \rightarrow \Omega^2 V^2, \quad \gamma_{ij} \rightarrow \Omega^2 \gamma_{ij}, \quad \omega_i \rightarrow \omega_i, \]  \hspace{1cm} (2.28)

where \( \Omega \) is an arbitrary non-vanishing function of both the space and time coordinates.

Note also that under (2.28) we gave

\[ g_{ij} \rightarrow \Omega^2 g_{ij}, \quad g^{ij} \rightarrow \Omega^{-2} g^{ij}. \]  \hspace{1cm} (2.29)

Clearly therefore, we may pick \( \Omega \) to be given by

\[ \Omega^2 = 1 - h_{ij} W^i W^j, \]  \hspace{1cm} (2.30)

in which case (2.15) or equivalently (2.27) may be replaced by the metric

\[ -dt^2 + h_{ij} (dx^i - W^i dt)(dx^j - W^j dt). \]  \hspace{1cm} (2.31)

### 2.4 Wave Fronts

If one thinks of light propagation and adopts a covariant spacetime perspective, the associated wave fronts are null hypersurfaces of spacetime \( S(x^\mu) = \text{constant} \) satisfy

\[ g^{\mu\nu} \partial_\mu S \partial_\nu S = 0, \quad \frac{dx^\mu}{d\lambda} = g^{\mu\nu} \partial_\nu S. \]  \hspace{1cm} (2.32)

and the associated light rays \( x^\mu = x^\mu(\lambda) \) are their null generators satisfying

\[ g^{\mu\nu} \partial_\mu S \partial_\nu = 0, \quad \frac{dx^\mu}{d\lambda} = g^{\mu\nu} \partial_\nu S. \]  \hspace{1cm} (2.33)

Four statements follow directly from (2.32, 2.33).

- The rays lie in the wave fronts

\[ \frac{dS}{d\lambda} = \frac{dx^\mu}{d\lambda} \partial_\mu S = 0. \]  \hspace{1cm} (2.34)

- The light rays have null tangent vectors:

\[ g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0. \]  \hspace{1cm} (2.35)
• The light rays are geodesic
\[ \frac{dx^\nu}{d\lambda} \nabla_\nu \frac{dx^\mu}{d\lambda} d_\lambda = 0, \] (2.36)
where \( \nabla_\mu \) is the covariant derivative with respect to the Levi-Civita associate to the metric \( g_{\mu\nu} \).

• The parameter \( \lambda \) is an affine parameter
\[ \frac{dx^\nu}{d\lambda} \nabla_\nu \frac{dx^\mu}{d\lambda} = 0. \] (2.37)

Only the last statement is not invariant under conformal rescaling of the spacetime metric
\[ g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}. \] (2.38)

### 2.4.1 Randers viewpoint

In the Randers framework equation (2.32) becomes
\[ a^{ij} (\partial_i S + b_i \partial_r S) (\partial_j S + b_j \partial_r S) = \left( \partial_t S \right)^2. \] (2.39)

Separating off the time-dependence
\[ S = -t + W(x^i) \] (2.40)
yields
\[ a^{ij} (\partial_i W - b_i) (\partial_j W - b_j) = 1, \] (2.41)
\[ \frac{dx^i}{d\lambda} = a^{ij} (\partial_j W - b_j), \] (2.42)
\[ \frac{dt}{d\lambda} = (1 - a^{ij} b_i b_j) + a^{ij} b_i \partial_j W. \] (2.43)

### 2.4.2 Zermelo viewpoint

In the Zermelo framework the Hamilton-Jacobi equation becomes
\[ h^{ij} \partial_i S \partial_j S - (\partial_t S + W^i \partial_i S)^2 = 0, \] (2.44)
which may be rewritten, taking the positive square root, as
\[ \partial_t S = \sqrt{(h^{ij} \partial_i S \partial_j S) - W^i \partial_i S} = G(x, p_i) = \sqrt{h^{ij} p_i p_j - W^i p_i}, \] (2.45)
where \( p_i = \partial_i S \). Now \( G \) coincides with the expression given in equation (14) of [2] for the moment map generating the Zermelo flow on the cotangent space of the spatial manifold.
In fact equation (3.92) is in striking agreement with the Hamilton-Jacobi equation obtained in section §2.2 of [22] by entirely different arguments.

Further progress typically depends upon being able to continue to separate variables either because further ignorable coordinates arise from Killing vector fields generating isometries of the metric \( h_{ab} \) or because of the existence of higher rank Killing tensor fields. Note that by Froebenius’s theorem, we may only introduce as many ignorable coordinates as there are mutually commuting Killing vector fields.

2.5 Winds which are Killing vector fields

Consider the metric

\[- dt^2 + h_{ij}(dx^i - W^i dt)(dx^j - W^j dt) = g_{\mu\nu} dx^\mu dx^n u \]  

(2.46)

Using Hamilton-Jacobi theory one has that null geodesics are given by

\[ \frac{dx^\mu}{d\lambda} = g^{\mu\nu} \partial_\nu S, \quad g^{\mu\nu} \partial_\mu S \partial_\nu S = 0, \]  

(2.47)

where \( \lambda \) is an affine parameter and \( i = 1, 2, \ldots m \). Since

\[ g_{\mu\nu} = \begin{pmatrix} -1 + W^2 & -W_i \\ -W_i & h_{ij} \end{pmatrix} \]  

(2.48)

\[ g^{\mu\nu} = \begin{pmatrix} -1 & -W_k \\ -W_j & h^{jk} - W^j W^k \end{pmatrix} \]  

(2.49)

with \( W^2 = h_{ij} W^i W^j \), and \( W_i = h_{ij} W^j \) etc. In the present case the second equation become

\[ (\partial_t S + W^i \partial_i S)^2 = h^{ij} \partial_i S \partial_j S \]  

(2.50)

If \( S = -t + F(x^i) \) we have

\[ (E - W^i \partial_i F)^2 = h^{ij} \partial_i F \partial_j F \]  

(2.51)

and

\[ \frac{dt}{d\lambda} = E - W^k \partial_k F \]  

(2.52)

\[ \frac{dx^i}{d\lambda} = EW^i + (h^{ik} - W^i W^k) \partial_k F. \]  

(2.53)

Now suppose that \( W^i \) is a Killing vector of \( h_{ij} \). We may find adapted coordinates \( \phi, x^A, A = 2, 3, \ldots, m \) such that \( W^i = \Omega \delta^i_\phi \) where \( \Omega \) is a constant. The metric \( h_{ij} \) thus takes the form

\[ h_{ij} dx^i dx^j = h_{AB} dx^A dx^B + 2h_{A\phi} dx^A d\phi + h_{\phi\phi} d\phi^2. \]  

(2.54)
That is, $h_{ij}$ is independent of $\phi$ and $\phi$ is thus and “ignorable” or “cyclic” coordinate.

We may set $S = -Et + J\phi + W(x^A)$, with $J$ a constant and find

$$ (E - \Omega J)^2 = h^{ij} \partial_i F \partial_j F. \quad (2.55) $$

Thus

$$ \frac{1}{|E - \Omega J|} F, \quad (2.56) $$

satisfies the Hamilton-Jacobi equation of the metric $h_{ij}$ and hence

$$ T^i = \frac{1}{|E - \Omega J|} h^{ij} \partial_j F \quad (2.57) $$

is a unit tangent vector to a geodesic. We have

$$ \frac{dt}{d\lambda} = |E - \Omega J|, \quad (2.58) $$

and

$$ \frac{dx^i}{d\lambda} = (E - \Omega J)W^i + h^{ik} \partial_k F. \quad (2.59) $$

That is

$$ \frac{1}{|E - \Omega J|} \frac{dx^i}{d\lambda} = W^i + T^i, \quad (2.60) $$

whence

$$ \frac{dx^i}{dt} = W^i + T^i, \quad (2.61) $$

where $T^i$ is a unit tangent vector to a geodesic of the metric $h_{ij}$.

3 Forest Fires

In the remarkable recent paper [17] it has been pointed that the equations of Richards for large scale wildfire spreads (see e.g. [22]) may be cast in terms of a Finsler function $F(x, y, u, v)$ giving the distance between successive firelines as the fire advances according to Huyghen’s principle. Here $(x, y) = x^i$ are spatial coordinates and $(u, v) = v^i$ velocities with $i = 1, 2$ and $F(x^i, v^i)$ is homogeneous degree one in $v^i : F(x^i, \lambda v^i) = \lambda F(x^i, v^i)$. A particular example, known as the hemi-spherical elliptic model, discussed in [17] is of Randers type. It belongs to a slightly wider class of three one-parameter families of two dimensional Randers-Finsler metrics discussed in [23]. There is also some overlap with some of the the spacetimes described in [2]. They are described in detail in our notation and conventions in the following sub-section.
3.1 The models

The Randers-Finsler metrics described in [23] share the feature that they are given in iso-thermal coordinates, and that in these coordinates the Finsler geodesics are circles.

- **I**: $S^2$

\[
a_{ij}dx^idx^j = 4 \frac{dx^2 + dy^2}{(1 + x^2 + y^2)^2}, \quad b_i dx^i = 2\lambda \frac{ydx - xdy}{(1 + x^2 + y^2)}.
\]  

If \((x, y) = \tan \frac{\theta}{2} (\cos \phi, \sin \phi)^T\),

\[
a_{ij}dx^idx^j = d\theta^2 + \sin^2 \theta d\phi^2, \quad b_i dx^i = -2\lambda \sin^2 \frac{\theta}{2} d\phi.
\]  

- **II**: $E^2$

\[
a_{ij}dx^idx^j = dx^2 + dy^2, \quad b_i dx^i = \frac{1}{2} \lambda (ydx - xdy).
\]  

If \((x, y) = \rho (\cos \phi, \sin \phi)^T\),

\[
a_{ij}dx^idx^j = d\rho^2 + \rho^2 d\phi^2, \quad b_i dx^i = \frac{1}{2} \lambda \rho^2 d\phi.
\]  

- **III**: $H^2$

\[
a_{ij}dx^idx^j = 4 \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}, \quad b_i dx^i = 2\lambda \frac{ydx - xdy}{(1 - x^2 - y^2)}.
\]  

If \((x, y) = \operatorname{th} \frac{\psi}{2} (\cos \phi, \sin \phi)^T\),

\[
a_{ij}dx^idx^j = d\psi^2 + \sinh^2 \psi d\phi^2, \quad b_i dx^i = 2\lambda \sinh^2 \frac{\psi}{2} d\phi.
\]  

It is case III with $\lambda = 1$ which is discussed in [17].

Note that if $\lambda = 1$, then the 2-form $db$ coincides with the area element on $S^2$, $E^2$ or $H^2$ respectively. Thus the rays coincide with magnetic geodesics, that is the trajectories of charged particles moving in a uniform magnetic field of strength $\lambda$ on $S^2$, $E^2$ or $H^2$ respectively. In particular, for $E^2$ we have the Larmor problem, and the case of $S^2$ is closely connected with the motion of a particle in the neighbourhood of a magnetic monopole.

Further geometric insight in case I may be afforded by recalling that if $\tau, \theta, \phi$ are Euler angles for $SU(2)$ then a left-invariant basis for the its Lie algebra $\mathfrak{su}(2)$, is given by

\[
\sigma^1 = \sin \tau d\theta - \sin \theta \cos \tau d\phi
\]  

\[
\sigma^2 = \cos \tau d\theta + \sin \theta \cos \tau d\phi
\]  

\[
\sigma^3 = d\tau + \cos \theta d\phi.
\]
The spacetime metric in case $I$ is

$$ds_I^2 = -(dt - 2\lambda \sin^2 \frac{\theta}{2} d\phi)^2 + d\theta^2 + \sin^2 \theta d\phi^2.$$  (3.74)

let $\tau = \frac{t}{\lambda} - \phi$, then one finds that

$$ds_I^2 = -\lambda^2 (\sigma^3)^2 + (\sigma^1)^2 + (\sigma^2)^2$$  (3.75)

which is a left-invariant Lorentzian metric on $SU(2)$. It’s best known appearance in general relativity is in the Taub-NUT metric. Since the coordinate $\tau$ is necessarily periodic, the spacetime admits closed timelike curves, CTC’s. That is, the spacetime is a model of a time-machine.

Case $II$ may be regarded as a Wigner-Inonu contraction because

$$ds_{II} = -(dt + \frac{1}{2} \lambda (ydx - xdy))^2 + dx^2 + dy^2$$  (3.76)

is a left-invariant metric on the Heisenberg group, sometimes call Nill, for which $dx, dy, dt - \frac{1}{2} \lambda (ydx - xdy)$ provide a left-invariant basis of one-forms.

3.2 The hemi-spherical elliptic model

These appear to be so-called because the fire is envisaged as propagating on a hemi-spherical shaped piece of terrain with unit radius and geometric height function $z = \sqrt{1 - x^2 - y^2}$. From the Randers point of view, it is perhaps puzzling that the metric $a_{ij}$ is not the metric induced on the hemi-sphere from the flat metric on Euclidean space $\mathbb{E}^3 = (x, y, z)$ which would have constant positive curvature $1$, but a metric of constant curvature $-\frac{1}{4}$ which in fact is induced on the top sheet of the hyperboloid $t = \sqrt{1 + x^2 + y^2}$ from the flat Lorentzian metric of three dimensional Minkowski spacetime $\mathbb{E}^{2,1} = (t, x, y)$.

The puzzle is resolved when one evaluates the Zermelo metric $h_{ij}$. This is given by [17]

$$h_{ij}dx^i dx^j = \frac{(1 - y^2)dx^2 + 2xydxdy + (1 - x^2)dy^2}{1 - x^2 - y^2}.$$  (3.77)

The wind is given

$$W^i \frac{\partial}{\partial x^i} = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}.$$  (3.78)

In case $III$ the spacetime metric is

$$ds_{III}^2 = -(dt - 2\lambda \sinh^2 \frac{\psi}{2})^2 + d\psi^2 + \sinh^2 \psi d\phi^2,$$  (3.79)

If $\lambda = \sqrt{2}$ and one replaces $\frac{\psi}{2}$ by $r$ one obtains the celebrated metric of Godel as given in §3 of [24].
More generally if we define
\begin{align*}
\sigma^1 &= \sin \tau d\psi - \sin \psi \cos \tau d\phi \\
\sigma^2 &= \cos \tau d\psi + \sin \psi \cos \tau d\phi \\
\sigma^3 &= d\tau + 2 \sinh^2 \frac{\psi}{2} d\phi ,
\end{align*}
(3.80)
we obtain a left-invariant basis of one forms for \( \mathfrak{sl}(2, \mathbb{R}) \) and the spacetime metric may be expressed as
\begin{align*}
ds_{III}^2 &= -\lambda^2 (\sigma^3)^2 + (\sigma^1)^2 + (\sigma^2)^2 \\
&= -\lambda^2 (\tau + \cosh \psi d\phi)^2 + d\psi^2 + \sinh^2 \psi d\phi^2 ,
\end{align*}
(3.81)
which is a family of left-invariant Lorentzian metrics on \( SL(2, \mathbb{R}) \). If \( \lambda^2 = 1 \), we have, up to a factor of 4, the bi-invariant metric on \( SL(2, \mathbb{R}) \). This coincides with three-dimensional anti-de-Sitter spacetime \( AdS_3 \) which has the topology \( S^2 \times \mathbb{R} \) and closed timelike curves. However if one passes to the universal covering space \( \tilde{SL}(2, \mathbb{R}) \) the CTC’s are eliminated. By contrast, if \( \lambda^2 > 1 \) the metrics admit closed timelike curves, even if one declines to identify the coordinate \( \tau \) since \( g_{\phi\phi} \) is negative for values of \( \psi \) for which \( \sinh^2 \frac{\psi}{2} > \frac{1}{\lambda^2} \).

To see the concrete relation to more conventional representations of anti-de-Sitter spacetime define
\begin{align*}
Z^1 &= \sqrt{1 + r^2} e^{iT} , \\
Z^2 &= re^{i\Phi} ,
\end{align*}
(3.85)
which embeds \( AdS_3 \equiv SL(2, \mathbb{R}) \) isometrically into \( \mathbb{H}^2 \equiv \{ \mathbb{C}^2, |dZ^2|^2 - |dZ^1|^2 \} \) as the quadric
\[ |Z^1|^2 - |Z^2|^2 = 1 . \]
(3.86)
The induced metric is
\begin{equation}
ds^2 = \frac{1}{4} \left\{ -(1 + r^2) dT^2 + \frac{dr^2}{1 + r^2} + r^2 d\phi^2 \right\} .
\end{equation}
(3.87)
To recover (3.84) multiply the rhs of (3.87) with \( \lambda = 1 \) by 4 and set
\begin{align*}
r &= \sinh \frac{\psi}{2} , \\
T &= \frac{1}{2}(\tau - \phi) , \\
\Phi &= \frac{1}{2}(\tau + \phi) , \\
\phi - \tau &= t .
\end{align*}
(3.88)
To recover (3.68) define the Lorentzian analogue of stereo-graphic coordinate \( z = x + iy \) by
\begin{equation}
z = \frac{Z^2}{Z^1} .
\end{equation}
(3.89)
The analogy works as follows. One identifies the hyperbolic plane with the upper-sheet of the two sheeted hyperboloid :
\[ t^2 - x^2 - y^2 = 1 \]
(3.90)
in three-dimensional Minkowski spacetime \( \mathbb{E}^{2,1} \equiv \{ \mathbb{R}^3, -dt^2 + dx^2 + dy^2 \} \). One now maps every point on the upper sheet to the intersection on the spacelike plane \( t = -1 \) with the straight line joining \( (t, x, y), t \geq 1 \) with \( (-1, 0, 0) \). The reader unfamiliar with this standard construction may find more details in the appendix of [20].
3.3 Wave Fronts and Hamilton-Jacobi equation

Adopting the Zermelo picture, the Hamilton-Jacobi equation becomes

\[ h^{ij} \partial_i S \partial_j S - (\partial_t S + W^i \partial_i S)^2 = 0, \]  

(3.91)

which may be rewritten, taking the positive square root, as

\[ \partial_t S = \sqrt{h^{ij} \partial_i S \partial_j S - W^i \partial_i S} = G(x, p_i) = \sqrt{h^{ij} p_i p_j - W^i p_i} \]  

(3.92)

where \( p_i = \partial_i S \). Now \( G \) coincides with the expression given in equation (14) of [2] for the moment map generating the Zermelo flow on the cotangent space of the spatial manifold. In fact equation (3.92) is in striking agreement with the Hamilton-Jacobi equation obtained in section \( \S 2.2 \) of [22] by entirely different arguments.

4 Quantum Navigation

There has been interest recently in the quantum control problem of finding a time independent Hamiltonian \( \hat{H}_c \) such that a state \( |\psi_{\text{initial}}\rangle \) of a system with Hamiltonian \( \hat{H}_0 \) is driven to a state \( |\psi_{\text{final}}\rangle \) in the shortest possible time subject to a bound on \( \text{Tr} \hat{H}_c^2 \) \( [10, 11, 12, 13, 14, 15, 16] \). The Hamiltonian \( \hat{H}_0 \) generates the natural motion or drift of the system in the absence of external intervention by an experimenter.

As we shall show in this section, the problem is equivalent to a special case of Zermelo’s problem and hence to finding a Randers-Finsler geodesic. By the theory expounded earlier this is equivalent to finding a null geodesic in a stationary spacetime. In order to understand this connection it is necessary to recast the standard well known formalism of quantum mechanics in which states correspond to vectors in a Hilbert space to a more geometrical formulation in which physically distinct states correspond to points in a particular type of Riemannian manifold \( [25, 26, 27, 28] \). Since this particular formalism is less familiar than it perhaps deserves and we will begin with a short summary.

4.1 The space of physically distinct quantum states

It was one of Dirac’s greatest achievements to have recognised, shortly after Heisenberg’s discovery of his eponymous uncertainty relations, the profound connection between quantum mechanics and Hamiltonian Mechanics. However not even he realised that one may regard quantum mechanics as a special case of Hamiltonian mechanics \( ^2 \). For that to be possible, it is necessary

\(^2\)However I owe to Dorje Brody the observation that this idea seems to be presaged in [29]
to pass from the traditional formulation of Hamiltonian mechanics, which passes by a Legendre transformation from Lagrange’s equations

\[ p_a = \frac{\partial L}{\partial \dot{q}^a}, \quad \dot{p}_a = \frac{\partial L}{\partial q^a} \]  

(4.93)

defined on the 2n dimensional tangent bundle \( TQ \) of an \( n \)-dimensional configuration space \( Q \) with local coordinates \( q^a, v^a = \dot{q}^a \) to Hamilton’s equations

\[ \dot{q}_a = \frac{\partial H}{\partial p^a}, \quad \dot{p}_a = -\frac{\partial H}{\partial q^a} \]  

(4.94)

defined on the cotangent bundle \( T^*Q \) with local coordinates \( q^a, p_a \) to a more general formulation in which Hamilton’s equations are seen as defining a flow on a 2n-dimensional symplectic manifold \( P \) with local coordinates \( x_i, i = 1, 2, \ldots 2n \) equipped with a closed 2-form \( \omega_{ij} = -\omega_{ji} \) of maximal rank i.e. \( \det \omega_{ij} \neq 0 \), and given by

\[ \dot{x}^i = \omega^{ij} \frac{\partial H}{\partial x^j}, \]  

(4.95)

where \( \omega^{ij} \) are the components of the inverse of \( \omega \). We recover the traditional viewpoint if \( P = T^*Q, x^i = q^i, p_{a+n} \) and \( \omega = d\theta \), where \( \theta = p_a dq^a \) is the canonical one-form.

Dirac did recognise that the space of physically distinct quantum states is the set of rays, i.e. vectors \( |\Psi\rangle \in \mathbb{C}^{n+1} \) with basis \( \{ |n\rangle, |n+1\rangle \} \), in a Hilbert space \( \mathcal{H} \), which here we take to be of finite dimension \( n + 1 \), subject to the equivalence

\[ |\Psi\rangle \equiv \lambda |\Psi\rangle \]  

(4.96)

What Dirac and others of his generation did not seem to have realised\(^3\) is that Schrödinger’s equation

\[ \frac{i}{\hbar} \frac{d|\Psi\rangle}{dt} = \hat{H}|\Psi\rangle \]  

(4.97)

becomes, when projected onto the space of physically distinct quantum states \( P \), a Hamiltonian flow with respect to a symplectic form \( \omega \) and Hamiltonian function.

\[ H = \langle \Psi | \hat{H} | \Psi \rangle. \]  

(4.98)

In fact \( P, \omega \) in this case is complex projective space \( \mathbb{CP}^n \equiv U(n+1)/U(n) \times U(1) \), which is not only a symplectic manifold, but an Einstein-Kähler manifold.

To say that a 2n-dimensional symplectic manifold is Kähler is to say that it admits a Riemannian, i.e. positive definite, metric \( a_{ij} \) such that if \( \nabla_i \) is the associated covariant derivative operator, then the symplectic form is covariantly constant:

\[ \nabla_i \omega_{jk} = 0, \]  

(4.99)

\(^3\)but see previous footnote
and that
\[ g_{ij} \omega^{im} \omega^jn = g^{mn}. \] (4.100)
From this we deduce that \( I^i_j = g^{ik} \omega_{kj} \) is covariantly constant and satisfies
\[ I^i_k I^k_j = -\delta^i_j. \] (4.101)
It follows that \( I^i_j \) is an integrable complex complex structure for the manifold \( P \), in other words we may consistently introduce charts of \( n \) complex coordinates with overlap functions which are locally holomorphic to cover our manifold. To express things more briefly a Kähler manifold is one whose holonomy group is contained within \( U(n) \subset SO(2n) \). Finally, to say that a Riemannian manifold is Einstein is to say that the the Ricci tensor satisfies
\[ R_{ij} = \Lambda g_{ij}, \] (4.102)
for some constant \( \Lambda \).

The standard flat Kähler structure on \( \mathbb{C}^{n+1} \) is given by
\[ ds^2 = |dZ^k|^2, \quad \omega = \frac{1}{i} dZ^k \wedge d\bar{Z}^k = dp_k \wedge dq^k \] (4.103)
where \( Z^k = \frac{1}{\sqrt{2}}(q^k + ip^k) \). The space of unormalised states is thus a flat Kähler manifold with
\[ ds^2 = \langle d\Psi | d\Psi \rangle. \] (4.104)
where
\[ |\Psi \rangle = Z^k |k\rangle \] (4.105)
and \( \{|k\} \) is an orthonormal basis.

One may check that Schrödinger’s equation may be written as
\[ i \frac{dZ^k}{dt} = \frac{\partial H(Z, \bar{Z})}{\partial \bar{Z}^k} \] (4.106)
or
\[ \frac{dq^k}{dt} = \frac{\partial H(p, q)}{\partial p_k}, \quad \frac{dp_k}{dt} = -\frac{\partial H(p, q)}{\partial q^k}. \] (4.107)
The general Kähler manifold admits local complex coordinates \( Z^k \) and a so-called Kähler potential \( K(Z, \bar{Z}) \) such that
\[ ds^2 = \frac{\partial^2 K}{\partial Z^m \partial \bar{Z}^n} dZ^m \otimes_d d\bar{Z}^n, \quad \omega = \frac{1}{i} \frac{\partial^2 K}{\partial Z^m \partial \bar{Z}^n} dZ^m \wedge d\bar{Z}^n. \] (4.108)
For the flat Kähler structure on \( \mathbb{C}^{n+1} \) we have
\[ K = Z^k \bar{Z}^k = \langle \Psi | \Psi \rangle. \] (4.109)
Because a Kähler manifold is a symplectic manifold, it also admits local Darboux coordinates such that
\[ \omega = dp_k \wedge dq^k. \] (4.110)

In the case of the flat coordinates used above, they are both complex and Darboux. For a general Kähler manifold this need not be the case.

We now give a construction of the Fubini-Study metric on \( \mathbb{CP}^n \) the space of physically distinct quantum states. We first normalise our state vectors
\[ \langle \Psi | \Psi \rangle = 1. \] (4.111)

The space of normalised states may be identified with the \( 2n+1 \) sphere \( S^{2n+1} = U(n+1)/U(n) \), with its standard unit round metric
\[ ds_{2n+1}^2 = \langle d\Psi | d\Psi \rangle. \] (4.112)

Moreover \( U(n) \subset SO(2n) \) acts transitively and isometrically. We now eliminate the remaining phase freedom in (4.111). Thus we restrict \( \lambda \) to be an element of \( U(1) \),
\[ \lambda = e^{i\psi}, \] (4.113)
and identify points on \( S^{2n+1} \) under the action of this \( U(1) \). Thus \( \mathbb{CP}^n \equiv S^{2n+1}/U(1) \). This construction of \( S^{2n+1} \) as a \( U(1) \) bundle over \( \mathbb{CP}^n \) is known to topologists as the Hopf fibration. The obits of \( U(1) \) in \( S^{2n+1} \) are called the Hopf fibres.

To obtain the \( SU(n)/\mathbb{Z}_n \) invariant Fubini-Study metric \( g_{ab}(x) \) on \( \mathbb{CP}^n \) we project the round metric (4.112) orthogonally to the Hopf fibres. Thus
\[ ds_{2n+1}^2 = \frac{1}{4}(d\psi + A_a dx^a)^2 + g_{ij}(x^c)dx^idx^j \] (4.114)
where \( i = 1, 2, \ldots 2n \) and find that (up to factor?) that locally the Kähler form is given by
\[ \omega = dA_i dx^i, \] (4.115)
and if
\[ g_{ij}dx^idx^j = \langle d\Psi | d\Psi \rangle - |\langle \Psi | d\Psi \rangle|^2. \] (4.116)

For more details of this general formalism and how it relates to the language of q-bits and their entanglement the reader may consult \[31\].

4.1.1 The Bloch Sphere

If \( n = 2 \) the space of physically distinguishable states is known as the Bloch sphere \[30\]. Geometrically this is the same as a sphere of radius \( \frac{1}{2} \). Thus If \( Z^1, Z^2 \) are affine coordinates
for \( \mathbb{C}^2 \) then \( S^2 \equiv \mathbb{C} \mathbb{P}^1 = \{(Z^1, Z^2) | (Z^1, Z^2) \equiv \lambda(Z^1, Z^2), \lambda \in \mathbb{C}^* \} \), and recalling that \( S^3 \equiv SU(2) \) we may parameterize then in terms of a radius \( r = \sqrt{|Z^1|^2 + |Z^2|^2} \) and Euler angles \( \{(\psi, \theta, \phi) | 0 \leq \psi < 4\pi, 0 \leq \theta < \pi, 0 \leq \phi < 2\pi \} \) as

\[
Z^1 = re^{i(\psi + \phi) \cos \theta / 2} \tag{4.117}
\]

\[
Z^2 = re^{i(\psi - \phi) \sin \theta / 2} \tag{4.118}
\]

The Hopf fibration \( S^3 \to \mathbb{C} \mathbb{P}^1 \) is given by \( \{(\psi, \theta, \phi) \to (\theta, \phi) \) and the Hopf fibres are given by \( r = 1, \theta, \phi = \text{constant} \) The metric on \( \mathbb{C}^2 \equiv \mathbb{E}^4 \) is

\[
d s^2 = |dZ^1|^2 + |dZ^2|^2 = dr^2 + \frac{r^2}{4} \left( (d\psi + \cos \theta d\phi)^2 + d\theta^2 + \sin^2 \theta d\phi^2 \right) \tag{4.119}
\]

The inhomogeneous coordinate on \( \mathbb{C} \mathbb{P}^1 \) is

\[
\zeta = \frac{Z^1}{Z^2} = e^{i\phi} \cot \frac{\theta}{2} \tag{4.120}
\]

which geographers call stereographic coordinates in terms of which the Fubini-Study metric on \( \mathbb{C} \mathbb{P}^1 \) is

\[
d s^2 = \frac{|d\zeta|^2}{(1 + |\zeta|^2)^2} = \frac{1}{4} (d\theta^2 + \sin^2 \theta d\phi^2) \tag{4.121}
\]

Evidently stereographic coordinates do not provide a Darboux chart since the the area element is

\[
\frac{i}{2} \frac{d\zeta \wedge d\bar{\zeta}}{(1 + |\zeta|^2)^2}. \tag{4.122}
\]

However if we define

\[
a = \frac{\zeta}{\sqrt{1 + |\zeta|^2}}, \quad \iff \quad \zeta = \frac{a}{\sqrt{1 - |a|^2}}, \tag{4.123}
\]

we find

\[
\frac{i}{2} \frac{d\zeta \wedge d\bar{\zeta}}{(1 + |\zeta|^2)^2} = \frac{i}{2} da \wedge d\bar{a}. \tag{4.124}
\]

Thus \( a = p + iq = e^{i\phi} \cos(\theta / 2) \) is a Darboux chart for \( \mathbb{C} \mathbb{P}^1 \). and the total area of \( \mathbb{C} \mathbb{P}^1 \) with respect to its Fubini-Study metric is

\[
\int_{|a|^2 \leq 1} dpdq = \pi = \frac{1}{4} 4\pi \tag{4.125}
\]

as expected. Geographers call the map from \( S^2 \) to the unit disc \textit{Lambert’s Polar Azimuthal Equal Area Projection} Note that while this projection is a symplecto-morphism or canonical transformation, i.e. preserves the symplectic form, and hence the area element, but it does not preserve the complex structure, in other words \( a \) is not a locally holomorphic function of \( \zeta \).
The previous theory generalises in an almost trivial way to an $n+1$ state system. One introduces $n$ inhomogeneous coordinates $\zeta^a, a = 1, 2, \ldots n$ for $\mathbb{C}P^n$. The Kähler form is

$$\omega = \frac{i}{2} \partial \zeta^a \partial \bar{\zeta}^a K d\zeta^a \wedge d\bar{\zeta}^a, \quad K = \ln(1 + \bar{\zeta}^a \zeta^a),$$ (4.126)

and Darboux coordinates are

$$a^a = \frac{\zeta^a}{\sqrt{1 + |\zeta|^2}}, \quad \zeta^a = \frac{a^a}{\sqrt{1 - |a|^2}},$$ (4.127)

with $|z|^2 = z^a \bar{z}^a$ etc and the Einstein summation convention is adopted. If $a^a = p_a + \sqrt{-1}q^{n+a}$ we have, in this Darboux chart

$$\omega = dp_a \wedge dq^a.$$ (4.128)

Thus a random quantum state, i.e the perfectly ignorant density matrix, consists of quantum states uniformly distributed inside the unit ball $(p_a)^2 + (q^a)^2 \leq 1$ and Schrödinger's equation is quite literally Hamilton's original equation in these coordinates. It is of interest to write out the general Hamiltonian function in terms of $p_a$ and $q^a$, not least, since the system is clearly integrable: the states evolve under a one parameter subgroup of $SU(n+1)$.

In more detail we have

$$|\Psi\rangle = \frac{1}{\sqrt{1 + |\zeta|^2}} (\zeta^a |a\rangle + |n+1\rangle) = a^a |a\rangle + \sqrt{1 - |a|^2} |n+1\rangle$$ (4.129)

$$H = \langle \Psi | \hat{H} | \Psi \rangle = \bar{a}^a H_{ab} a^b + (1 - |a|^2) H_{(n+1)(n+1)} + \sqrt{1 - |a|^2} (\bar{a}^a H_{a(n+1)} + H_{(n+1)a} a^a)$$ (4.130)

### 4.2 Extended Phase Space

In both classical mechanics and quantum mechanics the time coordinate $t$ and the phase space coordinates $x^a$ are on a very different footing. Firstly one is allowed to use arbitrary coordinates on the symplectic manifold $\{ P, \omega \}$, just as one is allowed to use arbitrary coordinates on a Riemannian or pseudo-Riemannian manifold $\{ M, g \}$. Moreover by analogy to isometries, which are diffeomorphisms $f : M \to M$ under which the metric is invariant

$$f_* g = g,$$ (4.131)

where $f_*$ denotes the pull-back map, one defines symplectomorphisms or canonical transformations as coordinate transformations $f : P \to P$ which leave invariant the symplectic form $\omega$,

$$f_* \omega = \omega.$$ (4.132)
By contrast the time coordinate is regarded as “absolute” and one typically does not consider coordinate transformations mixing $t$ and $x^a$.

This gives rise to no significant difficulties if the Hamiltonian $H$ is independent of time and Hamilton’s equations are an autonomous set of first order ordinary differential equations on $P$ but it becomes inconvenient if one is considering, as one does in control theory, time dependent Hamiltonians, when Hamiltonians are non-autonomous, or when one wishes to make use of symmetries of Hamilton’s equations, such as Galilei or Lorentz transformations which mix $x^a$ and $t$.

To this end, one may pass to a $2n+1$ dimensional extended phase space, sometime called Evolution Space \([32]\), $V = P \times \mathbb{R}$ with coordinates $X^\alpha = t, x^a$, equipped with a closed so-called pre-symplectic 2-form

$$\Omega = \omega - dH \wedge dt.$$  \hfill (4.133)

The 2-form $\Omega$ is pre-symplectic rather than because it has a kernel $V^\mu$,

$$\Omega_{\alpha\beta} V^\beta = 0$$ \hfill (4.134)

where $V^\beta$ is the tangent vector to the lift to $E$ of the solutions $t = t(\lambda), \ x^a = x^a(\lambda)$ of Hamilton’s equations, that is

$$V^0 = \frac{dt}{d\lambda}, \quad V^a = \frac{dx^a}{d\lambda}.$$ \hfill (4.135)

In the special case that the symplectic manifold is a co-tangent bundle of some configuration space $P = T^*Q$, then

$$\Omega = d\Theta, \quad \Theta = \theta - Hdt = p_i dq^i - Hdt$$ \hfill (4.136)

and $\{E, \Theta\}$ is a a special case of a contact manifold with globally defined contact 1-form $\Theta$, which by definition has exterior derivative $d\Theta$ of maximum rank. Diffeomorphisms $f : E \to E$ such that

$$f_* \Theta = \Theta$$ \hfill (4.137)

are called “contact transformations” or “contacto-morphisms”.

Thus if $Q = \mathbb{R}^3$ and $H = \frac{1}{2m} p^2$ or $H = \sqrt{p^2c^2 + m^2c^4}$, one may check that Galilei or Lorentz transformations are contacto-morphisms respectively.

### 4.3 Extended Quantum Statespace as a Stationary Spacetime

For our present purposes we may choose $h_{ij}$ to be the Fubini-Study metric on $\mathbb{CP}^n$ and $W^i$ the vector field generating the drift. In other words, $W^i$ is a Killing wind generating the one parameter subgroup of $SU(n+1)$ with moment map

$$\langle \Psi | \hat{H}_0 | \Psi \rangle.$$ \hfill (4.138)
Thus we see that the time-independent quantum control problem is solved by null geodesics moving of the stationary spacetime on the extended quantum phase space $\mathbb{C}P^n \times \mathbb{R}$ equipped with the Lorentzian metric (2.31).

### 4.3.1 Example: a spin $\frac{1}{2}$ system on the Bloch Sphere

The spin is caused to precess with the Larmor frequency $\omega_L$ by an external magnetic field. Thus

$$h_{ij}dx^i dx^j = \frac{1}{4}\left\{d\theta^2 + \sin^2 \theta d\phi^2\right\}, \quad W^i = \omega_L \delta^i_\phi. \quad (4.139)$$

and the stationary metric (2.27) becomes

$$ds^2 = -dt^2 + \frac{1}{4}\left\{d\theta^2 + \sin^2 \theta (d\phi - \omega_L dt)^2\right\}. \quad (4.140)$$

Since $\phi$ is ignorable, we set $F = h\phi + \Theta(\theta)$ and find

$$\left(\frac{d\Theta}{d\theta}\right)^2 + \frac{h^2}{\sin^2 \theta} = 4(\mathcal{E} - h\omega_L)^2. \quad (4.141)$$

We then have

$$\frac{d\phi}{d\lambda} = \frac{4h}{\sin^2 \theta}, \quad (4.142)$$

$$\frac{d\theta}{d\lambda} = \pm \sqrt{4(\mathcal{E} - h\omega_L)^2 - \frac{h^2}{\sin^2 \theta}}, \quad (4.143)$$

$$\frac{dt}{d\lambda} = (\mathcal{E} - h\omega_L). \quad (4.144)$$

It is clear that the problem is equivalent to the usual problem but with the replacement

$$\phi \to \tilde{\phi} = \phi - \omega_L t, \quad (4.145)$$

which corresponds to the device of passing to the interaction picture. adopted in [15].

### 4.3.2 Example a spin 1 system on $\mathbb{C}P^2$

$\mathbb{C}P^2$ first came to the attention of physicists as a “gravitational instanton” [33]. Later it was recognised as the state space of a spin one system [34].

If one uses a coordinate system adapted to the action of $U(2)$ subgroup of the $SU(3)/\mathbb{Z}_3$ subgroup of the isometry group of $\mathbb{C}P^2$, the Fubini-Study metric is [35]

$$h_{ij}dx^i dx^j = \frac{dt^2}{(1 + r^2)^2} + \frac{r^2}{4(1 + r^2)^2}(d\psi + \cos \theta d\phi)^2 + \frac{r^2}{4(1 + r^2)}(d\theta^2 + \sin^2 \theta d\phi^2). \quad (4.146)$$
The coordinates \( \psi, \theta, \phi \) are Euler angles on \( SU(2) \) and \( 0 \leq r \leq \infty \). Introducing \( u = 1/r \) we can add a \( \mathbb{CP}^1 \) at infinity to obtain a two-chart atlas covering the entire, compact manifold. Alternatively, if \( r = \tan \chi, \ 0 \leq \chi \leq \pi/2 \), we have

\[
|f_i| = \frac{1}{r} \left\{ \sin^2 \theta \cos^2 \chi (d\psi + \cos \theta d\phi)^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2) \right\},
\]

If \( |\Psi\rangle = Z' |\Psi_I\rangle \), where \( |\Psi_I\rangle \) are an orthonormal basis of states, \( \langle \Psi_i | \Psi_j \rangle = \delta_{ij} \), and if \( Z^3 \neq 0 \), we may parametrise the space of distinguishable quantum states as

\[
\begin{align*}
\frac{Z^1}{Z^3} &= \tan \chi \cos(\frac{\theta}{2}) e^{i\frac{\psi}{2}} \\
\frac{Z^2}{Z^3} &= \tan \chi \sin(\frac{\theta}{2}) e^{i\frac{\phi}{2}}.
\end{align*}
\]

There are two commuting Killing vectors \( \partial_\psi \) and \( \partial_\phi \) which suggest that we choose for the wind

\[
W^i \partial_i = \omega_\psi p_\psi + \omega_\phi p_\phi.
\]

Note that \( SU(3) \) is of rank 2 and therefore this may be the most general choice.

We assume that

\[
S = -\mathcal{E} t + h_\psi \psi + h_\phi \phi + W.
\]

and find that

\[
4 \left( \frac{\partial W}{\partial \chi} \right)^2 + \frac{4}{\sin^2 \chi} \left\{ \left( \frac{\partial W}{\partial \theta} \right)^2 + \frac{1}{\sin^2 \theta} (h_\phi - \cos \theta h_\psi)^2 \right\} + \frac{4}{\sin^2 \chi \cos^2 \chi} h_\psi^2 = (\mathcal{E} - \omega_\psi h_\psi - \omega_\phi h_\phi)^2.
\]

This clearly separates. We set \( W = X(\chi) + \Theta(\theta) \) and require

\[
\left( \frac{d\Theta}{d\theta} \right)^2 + \frac{1}{\sin^2 \theta} (h_\phi - \cos \theta h_\psi)^2 = h^2
\]

and

\[
\left( \frac{dX}{d\chi} \right)^2 + \frac{4h^2}{\sin^2 \chi} + \frac{4h_\psi^2}{\sin^2 \chi \cos^2 \chi} = (\mathcal{E} - \omega_\psi h_\psi - \omega_\phi h_\phi)^2.
\]

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