Log-Optimal Portfolio Selection Using the Blackwell Approachability Theorem

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Abstract. We present a method for constructing the log-optimal portfolio using the well-calibrated forecasts of market values. Dawid’s notion of calibration and the Blackwell approachability theorem are used for computing well-calibrated forecasts. We select a portfolio using this “artificial” probability distribution of market values. Our portfolio performs asymptotically at least as well as any stationary portfolio that redistribute the investment at each round using a continuous function of side information. Unlike in classical mathematical finance theory, no stochastic assumptions are made about market values.

1 Introduction

The model of stock market considered in this paper is the one studied, among others, by Breiman [5], Algoet and Cover [2], and Cover [8].

Consider an investor who can access $k$ financial instruments (asset, bond, cash, return of a game, etc.), and who can rebalance his wealth in each round according to a portfolio vector $b = (b(1), \ldots, b(k))$.

The evolution of the market in time is represented by a sequence of market vectors $x_1, x_2, \ldots$. A market vector $x = (x(1), \ldots, x(d))$ is a vector of $k$ nonnegative numbers representing price relatives for a given trading period. That is, the $j$th component $x(j)$ of $x$ expresses the ratio of the closing and opening prices of asset $j$. In other words, $x(j)$ is the factor by which capital invested in the $j$th asset grows during the trading period. We suppose that these components are bounded $x(j) \in [\lambda_1, \lambda_2]$ for all $1 \leq j \leq k$, where $0 < \lambda_1 < \lambda_2 < \infty$.

We assume that the assets are arbitrarily divisible, and they are available for buying or for selling in unbounded quantities at the current price at any given trading period; there are no transaction costs.

The investor is allowed to diversify his capital at the beginning of each trading period according to a portfolio vector $b = (b(1), \ldots, b(d))$. The $j$th component $b(j)$ of the vector $b$ denotes the proportion of the investors capital invested in asset $j$. We assume that $b(j) \geq 0$ for all $j$ and $\sum_{j=1}^{k} b(j) = 1$. This means that the investment strategy is self-financing and consumption of capital is excluded. The nonnegativity of the components of $b$ means that short selling and buying stocks on margin are not permitted. Let $\Gamma$ denotes the simplex of portfolio vectors $b$. 
Let \( S_0 \) denote the investors initial capital. Then, at the end of the trading period, the investors wealth becomes

\[
S_1 = S_0(b \cdot x) = S_0 \sum_{j=1}^{k} b(j)x(j),
\]

where " \( \cdot \) " denotes the inner product.

We consider the market process in the game-theoretic framework as a game between two players: Market and Investor. Shortly, the market process is described by the protocol on Fig 1.

Following Cover and Ordentlich [9], at the beginning of time period \( t \) Investor and Market can observe side information \( z_t \) which is an element of some compact topological space \( C \). At any round \( t \) Investor announces a portfolio vector \( b_t \) and after that Market announces a market vector \( x_t \). Investor updates his wealth \( S_t = S_{t-1} \cdot (b_t, x_t) \). Investor and Market can use all information known before their moves. In particular, at the round \( t \) both players can use the history \( \sigma_t = (z_1, x_1, \ldots, z_{t-1}, x_{t-1}, z_t) \) of the market-process.

\[
\text{FOR } t = 1, 2 \ldots \\
\text{Market announces a signal } z_t. \\
\text{Investor announces a portfolio } b_t. \\
\text{Market announces a market vector } x_t. \\
\text{Investor updates his wealth } S_t = S_{t-1} \cdot (b_t, x_t) \\
\text{ENDFOR}
\]

**Fig. 1.** Protocol of portfolio game

An investment strategy is a function \( b_t = b(\sigma_t) \) from the set of all histories to the simplex of all portfolios (a more precise definition see in Section 2). This strategy is also called constantly rebalanced portfolio.

Starting with an initial wealth \( S_0 \), after \( T \) trading periods, the investment strategy \( b(\cdot) \) achieves the wealth

\[
S_T = S_0 \prod_{t=1}^{T} (b(\sigma_t) \cdot x_t).
\]

For simplicity, in what follows we assume that \( S_0 = 1 \).

In modeling the behavior of the evolution of the market, two main approaches have been considered in the theory of sequential investment. In probabilistic approach, we assume that market vectors \( x_t \) are realization of a sequence of random process \( X_t \), where \( t = 1, 2, \ldots \), and describe a statistical model.

If the market process \( X_t \) is memoryless, i.e., it is a sequence of independent and identically distributed (i.i.d.) random market vectors then it was shown by
Morvai [13] that the portfolio
\[ b^* = \arg\max_b \mathbb{E}(\log(b \cdot X_1)) \]
is asymptotically optimal in the following sense: for any portfolio vector \( b \) with finite \( \mathbb{E}((\log(b \cdot X_1))^2) \)
\[ \lim\inf_{T \to \infty} \frac{1}{T} \log \frac{S_T^*}{S_T} \geq 0 \]
almost surely, where \( S_T^* = \prod_{t=1}^{T} (b^* \cdot X_t) \) and \( S_T = \prod_{t=1}^{T} (b \cdot X_t) \).

But i.i.d. model is insufficient if the market vectors of different trading periods have a statistical dependence, which seems to be the case in real-world markets. In general case, we consider an arbitrary random process \( X_1, X_2, \ldots \) generating market vectors \( x_1, x_2, \ldots \).

Algoet [1] and Algoet and Cover [2] constructed so-called log-optimum portfolio. Let \( X_1, X_2, \ldots \) be an arbitrary stationary and ergodic process. Denote \( X_{t-1} = X_1, \ldots, X_{t-1} \). A strategy \( b^*(\cdot) \) is called log-optimal portfolio strategy if
\[ \mathbb{E}(\log(b^*(X_{t-1}) \cdot X_t)) = \max_{b(\cdot)} \mathbb{E}(\log(b(X_{t-1}) \cdot X_t)). \] (1)

Let \( S_T^* \) denotes the capital achieved by a log-optimum portfolio strategy \( b^*(\cdot) \) after \( T \) trading periods. Then for any stationary and ergodic process \( X_t \) and for any other investment strategy \( b(\cdot) \)
\[ \lim\inf_{T \to \infty} \frac{1}{T} \log \frac{S_T^*}{S_T} \geq 0 \] (2)
almost surely. Such a strategy is called universal with respect to the process \( X_1, X_2, \ldots \).

Moreover, Algoet [1] and Györfi and Schäfer [10] shown that there exists a strategy uniformly universal with respect to the class of all stationary and ergodic processes. This means that a strategy \( b^*(\cdot) \) exists such that for any stationary and ergodic process \( X_1, X_2, \ldots \) asymptotic inequality (2) holds almost surely. Györfi and Schäfer called this scheme a histogram-based investment strategy. Györfi et al. [11] extended this result to the kernel-based case.

We have to emphasize the basic condition of the probabilistic model: the behavior of the market is not affected by the actions of the investor using the strategy under investigation.

Another “worst-case” approach allows the market sequence \( x_1, x_2, \ldots \) to take completely arbitrary values, and no stochastic model is imposed on the mechanism generating the price relatives. This approach was pioneered by Cover [8]. Cover showed that there exists an investment strategy \( b^*_T \) (so-called universal portfolio) that perform almost as well as the best portfolio in the sense that for any market sequence \( x_1, x_2, \ldots \)
\[ S_T^* \geq c T^{-\frac{1}{2}} S_T(b) \]

1 In what follows \( \log \) denotes logarithm on the base 2
2 That is true when \( X_t \) are uniformly bounded.
for all $T$, where $c$ is a positive constant and $S_T^\ast = \prod_{t=1}^T (b^\ast \cdot x_t)$ is the wealth achieved by the universal portfolio strategy and $S_T(b) = \prod_{t=1}^T (b \cdot x_t)$ is the wealth achieved by arbitrary constant portfolio $b$. The universal portfolio is defined as the mixture

$$b^\ast_T = \frac{\int b \prod_{t=1}^{T-1} (b \cdot x_t) P(db)}{\int \prod_{t=1}^{T-1} (b \cdot x_t) P(db)},$$

where $P$ is the $1/2$-Dirichlet distribution on $\Gamma$. Further development of this approach see in Cover and Ordentlich [9], Vovk and Watkins [14], Blum and Kalai [4], and so on. In this approach the achieved wealth is compared with that of the best in a class of reference strategies. The class of reference strategies considered by Cover [8] is that the class of all constant portfolios defined by all vectors $b \in \Gamma$. Cover and Ordentlich [9] extended this method for a case where side information in form of states from a finite set can be used by a reference strategy.

Using prediction with expert advice approach, Vovk [15] constructed a prediction strategy that perform almost as well as an arbitrary prediction strategy presented by a stationary continuous function of the side information. This result can be applied for construction of the portfolio log-optimal with respect to the reference class consisting of all arbitrary continuous portfolios $b(\sigma_t)$.

The advantage of this worst-case approach is that it avoids imposing statistical models on the stock market and the results hold for all possible sequences $x_1, x_2, \ldots$. In this sense this approach is extremely robust.

In Section 2, we follow the combined worst-case and stochastic approach. We construct “an artificial probability distribution” for market values. This distribution is defined by well-calibrated forecasts, these forecasts are constructed using game-theoretic Blackwell approachability theorem. No stochastic assumptions are made about the market values for constructing such forecasts. We construct the log-optimal portfolio by scheme (1), where the mathematical expectation $E$ is over probability distribution defined by well-calibrated forecasts. Our log-optimal portfolio performs asymptotically at least as well as any stationary portfolio that redistribute the investment at each round using a continuous function of the side information.

## 2 Main result

**Blackwell approachability theorem.** We will define our randomized strategy for universal portfolio selection using the Blackwell approachability theorem.

Recall some standard notions of the theory of games. Consider a game between two players with finite sets of their moves (pure strategies) $I = \{s_1, \ldots, s_N\}$ and $J = \{a_1, \ldots, a_M\}$. A mixed strategy of the first player is a probability distribution on $I$ presented by a vector $P = (p(1), \ldots, p(N))$, where $p(1) + \ldots + p(N) = 1$ and $p(i) \geq 0$ for all $i$. Denote by $P(I)$ the set of all mixed strategies of the first player. Analogously, let $P(J)$ be the set of all mixed strategies of the second player.
Consider an infinitely repeatable game, where at each round $t$ the first player announces a mixed strategy $P_t \in P(I)$ and the second player announces a pure strategy $j_t \in J$. They can announce their moves independently or, in the adversarial setting, where the second player announces an element $j_t$ after the first player announces the element $P_t$. In the adversarial setting, the second player can use $P_t$ in his strategy for choosing $j_t$.

A vector-valued payoff function $f(s_i, a_j) \in \mathbb{R}^d$ be given, where $d$ is an arbitrary, $s_i \in I$, and $a_j \in J$. As usual, $f(P, a_j) = \sum_{i=1}^{N} f(s_i, a_j)p(i)$ and $f(P, Q) = \sum_{i=1}^{N} \sum_{j=1}^{M} f(s_i, a_j)p(i)q(j)$, where $P = (p(1), \ldots, p(N))$ and $Q = (q(1), \ldots, q(M))$ are elements of $P(I)$ and $P(J)$ respectively.

By randomized online strategy of the first player we mean an infinite sequence $P_1, P_2, \ldots$ of his mixed strategies.

We consider the $l_1$ norm $\|x\|_1 = \|x\|_1 = \sum_{i=1}^{d} |x_i|$, where $x \in \mathbb{R}^d$. For any subset $S \subseteq \mathbb{R}^d$ and any vector $x \in \mathbb{R}^d$, the distance from $x$ to $S$ is defined as $\text{dist}(x, S) = \inf_{y \in S} \|x - y\|$.

Following Blackwell a set $S \subseteq \mathbb{R}^d$ is called approachable if a randomized online strategy $P_1, P_2, \ldots$ of the first player exists such that for any sequence $j_1, j_2, \ldots$ moves of the second player

$$\lim_{T \to \infty} \text{dist}\left(\frac{1}{T} \sum_{t=1}^{T} f(i_t, j_t), S\right) = 0$$

holds for $P$-almost all sequences $i_1, i_2, \ldots$ of the first player moves, where $P = \prod P_i$ is the overall probability distribution on trajectories $i_1, i_2, \ldots$ of the first player moves generated by its mixed strategies $P_1, P_2, \ldots$.

Blackwell [3] proposed a generalization of the minimax theorem for the case of the vector-payoff functions. In particular, he proved the following theorem.

**Theorem 1.** A closed convex subset $S \subseteq \mathbb{R}^d$ is approachable by the first player if and only if for every mixed strategy $Q \in P(J)$ of the second player a mixed strategy $P \in P(I)$ of the first player exists such that $f(P, Q) \in S$.

We apply this theorem for the log-optimal portfolio construction.

**Optimal portfolio construction.** The game of redistribution of the finance between Investor and Market is played according to the protocol (of perfect information, in the sense that either player can see the other players moves made so far) presented on Fig 1.

After Market first move the game proceeds in rounds numbered by the positive integers $t$. At the beginning of each round $t = 1, 2, \ldots$ Investor is given

\[\begin{align*}
&\text{Consider an infinitely repeatable game, where at each round } t \text{ the first player announces a mixed strategy } P_t \in P(I) \text{ and the second player announces a pure strategy } j_t \in J. \text{ They can announce their moves independently or, in the adversarial setting, where the second player announces an element } j_t \text{ after the first player announces the element } P_t. \text{ In the adversarial setting, the second player can use } P_t \text{ in his strategy for choosing } j_t. \\
&\text{A vector-valued payoff function } f(s_i, a_j) \in \mathbb{R}^d \text{ be given, where } d \text{ is an arbitrary, } s_i \in I, \text{ and } a_j \in J. \text{ As usual, } f(P, a_j) = \sum_{i=1}^{N} f(s_i, a_j)p(i) \text{ and } f(P, Q) = \sum_{i=1}^{N} \sum_{j=1}^{M} f(s_i, a_j)p(i)q(j), \text{ where } P = (p(1), \ldots, p(N)) \text{ and } Q = (q(1), \ldots, q(M)) \text{ are elements of } P(I) \text{ and } P(J) \text{ respectively.} \\
&\text{By randomized online strategy of the first player we mean an infinite sequence } P_1, P_2, \ldots \text{ of his mixed strategies.} \\
&\text{We consider the } l_1 \text{ norm } \|x\|_1 = \|x\|_1 = \sum_{i=1}^{d} |x_i|, \text{ where } x \in \mathbb{R}^d. \text{ For any subset } S \subseteq \mathbb{R}^d \text{ and any vector } x \in \mathbb{R}^d, \text{ the distance from } x \text{ to } S \text{ is defined as } \text{dist}(x, S) = \inf_{y \in S} \|x - y\|. \\
&\text{Following Blackwell a set } S \subseteq \mathbb{R}^d \text{ is called approachable if a randomized online strategy } P_1, P_2, \ldots \text{ of the first player exists such that for any sequence } j_1, j_2, \ldots \text{ moves of the second player}\n\end{align*}\]
some signal \( z_t \) relevant to predicting the portfolio \( b_t \). The signal is taken from the signal space \( C \). Investor then announces his prediction \( b_t \), taken from the simplex \( \Gamma \) of all portfolios. The quality of the portfolio in is measured by the wealth obtained using this portfolio.

Recall that for any market vector \( x = (x(1), \ldots, x(k)) \), \( x(i) \in [\lambda_1, \lambda_2] \), where \( \lambda_1 \) and \( \lambda_2 \) are real numbers such that \( 0 < \lambda_1 < \lambda_2 \).

Let \( \epsilon > 0 \). Define an \( \epsilon \)-net (\( \epsilon \)-grid) \( A_\epsilon = \{a_1, \ldots, a_M\} \) in the set of all market vectors \( [\lambda_1, \lambda_2]^k \) such that for any market vector \( a \in [\lambda_1, \lambda_2]^k \) an element \( a_i \in A_\epsilon \) exists satisfying \( \|a - a_i\| < \epsilon \).

For any vector \( a \in A_\epsilon \), let \( \delta[a] = (0, \ldots, 1, \ldots, 0) \) be a probability distribution concentrated on element \( a \) of the set \( A_\epsilon \). In this vector of dimension \( M \), the \( i \)th coordinate is 1, all other coordinates are 0.

Let \( P(A_\epsilon) \) be the set of all probability distributions on the set \( A_\epsilon \) and \( \mathcal{P}_\epsilon = \{s_1, \ldots, s_N\} \) be an \( \epsilon \)-net in \( P(A_\epsilon) \). Note that any \( P \in P(A_\epsilon) \) is an \( M \)-dimensional vector. For any \( P \in P(A_\epsilon) \) an \( s_i \in \mathcal{P}_\epsilon \) exists such that \( \|P - s_i\| < \epsilon \).

We use signals, or side information: an element \( z_t \) of some set \( C \) is announced by \( Market \) at the beginning of any round \( t \). We suppose that the set \( C \) is a compact topological space.

Following Vovk [15], we consider the space of infinite histories \( \mathcal{C} = (C \times A)^\infty \) consisting of bi-infinite sequences \( \ldots, z_{-1}, x_{-1}, z_0, x_0, z_1, x_1, \ldots, z_t, x_t, \ldots \) and equipped with the product topology. The space \( \mathcal{C} \) is compact in this topology.

An investment strategy is a function \( b : (C \times A)^\infty \times C \times \mathcal{N} \rightarrow \Gamma \), where \( \mathcal{N} \) is the set of all integers. It maps each history \( \sigma_t = (\ldots, z_{t-1}, x_{t-1}, z_t) \) and the current time \( t \) to the chosen portfolio \( b(\sigma_t, t) \). In this paper we will only be interested in continuous prediction strategies (according to the traditional point of view going back to Brouwer, only continuous prediction strategies can be computable). An especially natural class of strategies is formed by the stationary prediction strategies \( b : (C \times A)^\infty \times C \rightarrow \Gamma \), which do not depend on time explicitly; since the origin of time is usually chosen arbitrarily, this appears a reasonable restriction. A stationary strategy \( b \) maps each history \( \sigma_t = (\ldots, z_{t-1}, x_{t-1}, z_t) \) to the chosen portfolio \( b(\sigma_t) \).

Now we apply Theorem 1. Consider a two-players infinitely repeated game, where the first player moves are elements of the \( N \)-element set \( I = \mathcal{P}_\epsilon \) and the second player moves are from the \( M \)-element set \( J = A_\epsilon \). At any round \( t \) the first player outputs a forecast \( p_t \in \mathcal{P}_\epsilon \) and the second player outputs an outcome \( x_t \in A_\epsilon \).

The values of payoff function \( f \) are vectors of dimension \( MN \):

\[
f(s_t, a) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ s_t - \delta[a] \\ \vdots \\ 0 \end{pmatrix},
\]
where \( s_i \) is the \( i \)th vector of the set \( P_\varepsilon = \{ s_1, \ldots, s_N \} \) and \( a \) is an arbitrary vector of the set \( A_\varepsilon = \{ a_1, \ldots, a_M \} \), \( 0 \) is the \( M \)-dimensional zero vector, and \( s_i - \delta[a] \) is the difference of two \( M \)-dimensional column vectors, which is the \( i \)th component of the complex vector \( f(s_i, a) \).

We suppose that some (trivial) initial history \( \ldots, z_{-1}, x_{-1}, z_0, x_0 \) be given as the starting point of the game.

We now define a convex set in the space \( R^{MN} \). We consider vectors in \( R^{MN} \) as complex vectors of dimension \( N \) with the vector components from \( R^M \): \( \bar{X} = (\bar{x}_1, \ldots, \bar{x}_N)' \), where \( \bar{x}_i \in R^M \) for \( 1 \leq i \leq N \).

We define the closed convex set of such complex vectors:

\[
U = \left\{ \bar{X} : \sum_{k=1}^{N} \| \bar{x}_k \| \leq \epsilon \right\}.
\]

By definition a randomized strategy of the first player is a sequence \( P_1, P_2, \ldots \) of probability distributions on the set \( I = P_\varepsilon \). Let \( x_1, x_2, \ldots \) be a sequence of moves of the second player.

A set \( \mathcal{C} \) is approachable if a randomized strategy \( P_1, P_2, \ldots \) of the first player exists such that \( \lim_{T \to \infty} \text{dist}(f(p_t, x_t), \mathcal{C}) = 0 \) almost surely regardless the second player moves \( x_1, x_2, \ldots \), where the trajectory \( p_1, p_2, \ldots \) is distributed according to the probability distribution \( P = \prod P_t \) and \( p_t \in P_\varepsilon \) for all \( t \).

Recall that the set of all bi-infinite sequences \( \mathcal{C} \) is the compact topological space. Let \( C_\varepsilon = \{ c_1, \ldots, c_K \} \) is a finite net in \( \mathcal{C} \) with the properties defined in the proof of Theorem 2 (see below).

By Theorem 1 the closed convex set \( S \) is approachable if and only if for each \( Q \in P(A_\varepsilon) \) an \( P \in P(\mathcal{P}_\varepsilon) \) exists such that \( f(P, Q) \in U \). Indeed, we can take \( P = \delta[s_k] \), where \( s_k \in \mathcal{P}_\varepsilon \) such that \( ||s_k - Q|| < \varepsilon \), where \( \delta[s_k] \) is the probability distribution on \( \mathcal{P}_\varepsilon \) concentrated on \( s_k \). In this case \( f(P, Q) = f(s_k, Q) \).

Then a randomized strategy \( P_1, P_2, \ldots \) of the first player exists, where \( P_i \) are probability distributions on the set \( \mathcal{P}_\varepsilon \) such that for any \( 1 \leq j \leq K \) regardless of that sequence \( x_1, x_2, \ldots \) was announced by the second player the sequence of the vector-valued payoffs

\[
\tilde{m}(p_t, x_t) = \frac{1}{T} \sum_{t=1}^{T} f(p_t, x_t) = \left( \frac{1}{T} \sum_{t=1}^{T} I_{\{p_t=s_1, \sigma_t=c_1\}}(s_1 - \delta[x_t]) \right) \ldots \left( \frac{1}{T} \sum_{t=1}^{T} I_{\{p_t=s_N, \sigma_t=c_1\}}(s_N - \delta[x_t]) \right).
\]

\( P \)-almost surely approaches to the set \( \mathcal{C} \), where \( P = \prod P_t \) is the overall probability distribution generated by the randomized strategy and the trajectory \( p_1, p_2, \ldots \) is distributed by the measure \( P \).

Let us turn to the portfolio game defined by the protocol presented on Fig 1. In this game, let \( x_1, x_2, \ldots \) be a sequence of market values and \( \sigma_1, \sigma_2, \ldots \) be a sequence of all histories announced in the process of the game.
In order to apply the Blackwell approachability theorem, we make this game finite. Assume that \( x_t \in A_t \) and \( \sigma_t \in C_t \) for all \( t \).

Let \( p_1, p_2, \ldots \) be a sequence of the well-calibrated forecasts governed by the probabilistic strategy \( P_1, P_2, \ldots \).

Let \( N_T(s, i, j) = |\{t : p_t = s, 0 \leq t \leq T, x_t = a_i, \sigma_t = c_j\}| \) and \( M_T(s, j) = |\{t : p_t = s, 0 \leq t \leq T, \sigma_t = c_j\}| \), where \( 1 \leq i \leq M \), \( 1 \leq j \leq K \), and let \( s = (s(1), \ldots, s(M)) \) be an arbitrary element of the net \( \mathcal{P}_e \).

Approachability of the set \( S \) implies that there exists a randomized strategy \( P_1, P_2, \ldots \) such that for any sequence \( \ldots, z_{-1}, z_0, z_0, z_1, \ldots, z_t, x_t, \ldots \) for any \( 1 \leq j \leq K \)

\[
\limsup_{T \to \infty} \frac{1}{T} \sum_{s \in \mathcal{P}_e} \sum_{s_1 = s}^{1 \leq i \leq M} |N_T(s, i, j) - M_T(s, j)s(i)| \leq \epsilon
\]  

(3)

for almost all sequences \( p_1, p_2, \ldots \) distributed according to the overall probability distribution generated by \( P_1, P_2, \ldots \). We call forecasts \( p_1, p_2, \ldots \) satisfying (3) \( \epsilon \)-calibrated. If (3) holds for each \( \epsilon > 0 \) then these forecasts are called well-calibrated.

Let \( X \) be a random variable distributed according to the probability distribution \( P\{X = a_i\} = s(i) \) for every \( 1 \leq i \leq M \). For any probability distribution \( s \in \mathcal{P}_e \) define the optimal portfolio

\[
b^* = \arg\max_b E_{X \sim s}(\log(b \cdot X)).
\]

(4)

We can rewrite (4) also as \( b^* = \arg\max_b \sum_{i=1}^{M} (\log(b \cdot a_i))s(i) \). Using a randomized forecast \( p_t \in \mathcal{P}_e \) distributed according with respect to \( P_t \) existing by the Blackwell approachability theorem, we can define at any round \( t \) of the game presented on Fig 1 the random portfolio

\[
b^*_t = \arg\max_b E_{X \sim p_t}(\log(b \cdot X)),
\]

(5)

where \( p_t \in \mathcal{P}_e \) is the random forecast announced at round \( t \).

The following theorem asserts that portfolio (5) is almost surely log-optimal with respect to the class of all portfolios presented by continuous functions from the history.

**Theorem 2.** The randomized portfolio strategy \( b^*_t \) defined by (5) is almost surely log-optimal for the class of all continuous portfolio strategies \( b(\sigma_t) \) using in the game presented on Fig 1:

\[
\liminf_{T \to \infty} \frac{1}{T} \log \frac{S_T^*}{S_T} \geq 0
\]

(6)

for almost all trajectories \( p_1, p_2, \ldots \), where \( S_T^* = \prod_{t=1}^{T} (b^*_t \cdot x_t) \) is the wealth achieved by of the universal portfolio strategy, \( S_T = \prod_{t=1}^{T} (b(\sigma_t) \cdot x_t) \) is the wealth achieved by an arbitrary portfolio \( b(\sigma_t) \), and \( \sigma_t \) is the history at any round \( t \).\(^4\)

\(^4\) \( \sigma_t = \ldots, z_{-1}, z_{-1}, z_0, x_0, z_1, x_1, \ldots, x_{t-1}, z_t \)
Proof. The complete proof is based on the construction which is repeated for an infinite series of nets \( C_{\epsilon_n}, A_{\epsilon_n}, P_{\epsilon_n}, n = 1, 2, \ldots \), approximating the sets \( C, A, \) and \( P(A) \) with increasing degree of accuracy \( \epsilon_n \to 0 \) as \( n \to \infty \).

For simplicity of presentation, we give the proof only for one of such nets. Given \( \epsilon > 0 \), we consider such an approximation up to \( \epsilon > 0 \) and corresponding nets \( C_{\epsilon}, A_{\epsilon}, \) and \( P_{\epsilon} \) and prove optimality of the universal portfolio up to \( O(\epsilon) \).

Replace the sums with market values and histories on their approximations from the corresponding \( \epsilon \)-nets. Let us estimate the loss of accuracy as a result of such replacements.

Notice that for any \( b \in C, x \in A \) and \( a \in A_{\epsilon} \) such that \( \|x - a\| < \epsilon \) we have
\[
|b \cdot x - (b \cdot a)| = |(b \cdot (x - a))| \leq \|b\|\|x - a\| \leq \|x - a\| < \epsilon.
\]
Then \( \ln(b \cdot x) - \ln(b \cdot a) = \ln\left(1 + \frac{b \cdot (x - a)}{b \cdot a}\right) \leq \frac{|b \cdot (x - a)|}{b \cdot a} \leq \epsilon/\lambda_1, \) where \( \ln \) is the natural logarithm.

Let \( b(\cdot) \) be an arbitrary continuous stationary portfolio strategy. Given \( \epsilon > 0 \) consider a sufficiently accurate approximating net \( C_{\epsilon} = \{c_1, \ldots, c_K\} \) in the set \( C \) of all histories satisfying the following property: for each \( \sigma \in C \) an \( c_i \in C_{\epsilon} \) exists such that \( \|b(\sigma) - b(c_i)\| < \epsilon. \)

So, we can change all sums on \( O(\epsilon) \) as a result of these replacements of market values and histories. Assuming that market values and histories are now elements of the corresponding finite nets and by continuity of the function \( b(\cdot) \), we obtain the estimate for the wealth of arbitrary portfolio \( b(\cdot) \) up to \( O(\epsilon) \):
\[
\frac{1}{T} \log S_T = \frac{1}{T} \sum_{t=1}^{T} \log(b(z_t) \cdot x_t) = \frac{1}{T} \sum_{j=1}^{K} \sum_{s \in P_j} \sum_{p_i = s}^{M} N_T(s, i, j) \log(b(c_j) \cdot a_i) + O(\epsilon), \quad (7)
\]
where \( N_T(s, i, j) = \{|t : p_t = s, 0 \leq t \leq T, x_t = a_i, z_t = c_j\}|. \)

Let \( s = (s(1), \ldots, s(M)) \). By (3), for any \( 1 \leq j \leq K \) almost surely
\[
\frac{1}{T} \sum_{s \in P_j} \sum_{p_i = s}^{M} \sum_{1 \leq i \leq M} |N_T(s, i, j) - M_T(s, j)|s(i) \leq \epsilon + o(1)
\]
as \( T \to \infty. \)

Then for any \( 1 \leq j \leq K \), it holds almost surely
\[
\frac{1}{T} \sum_{s \in P_j} \sum_{p_i = s}^{M} N_T(s, i, j) \log(b(c_j) \cdot a_i) =
\]
\footnote{Since the complete construction is based on the sequence of \( \epsilon_k \)-nets, where \( \epsilon_k \to 0 \) as \( k \to \infty \), for each continuous function \( b(\cdot) \) an \( \epsilon_k \)-net exists such that this property holds.}
\[
\frac{1}{T} \sum_{s \in \mathcal{P}} \sum_{p_t = s} M(s, j) \sum_{i=1}^{M} \log(b(c_j) \cdot a_i)s(i) + O(\epsilon) + o(1) = \\
\frac{1}{T} \sum_{s \in \mathcal{P}} M(s, j) \sum_{p_t = s} E_{X \sim s}(\log(b(c_j) \cdot X)) + O(\epsilon) + o(1) \leq \\
\frac{1}{T} \sum_{s \in \mathcal{P}} M(s, j) \sum_{p_t = s} \sum_{i=1}^{M} \log(b_i^* \cdot a_i)s(i) + O(\epsilon) + o(1) = \\
\frac{1}{T} \sum_{s \in \mathcal{P}} \sum_{p_t = s} \sum_{i=1}^{M} N_T(s, i, j) \log(b_i^* \cdot a_i) + O(\epsilon) + o(1) = \\
\frac{1}{T} \sum_{t=1}^{T} \sum_{s \in \mathcal{P}} \sum_{p_t = s} \frac{1}{T} \sum_{i=1}^{M} N_T(s, i, j) \log(b_i^* \cdot a_i) + O(\epsilon) + o(1)
\]

as \( T \to \infty \).

Summing (8) by \( j \), we obtain

\[
\sum_{j=1}^{K} \sum_{s \in \mathcal{P}} \frac{1}{T} \sum_{i=1}^{M} N_T(s, i, j) \log(b_i^* \cdot a_i) + O(\epsilon) + o(1) = \\
\frac{1}{T} \sum_{t=1}^{T} \log(b_i^* \cdot x_t) + O(\epsilon) + o(1) = \frac{1}{T} \log S_t^* + O(K\epsilon) + o(1)
\]

almost surely, where \( S_t^* \) is the wealth achieved by the optimal portfolio.

Relations (7), (8), and (9) imply that almost surely \( \liminf_{T \to \infty} \frac{1}{T} \log S_T^* \geq -c\epsilon \), where \( c \) is a positive constant.

### 3 Some remarks on rate of convergence

In this section following Mannor and Stoltz [12] we discuss rate of convergence in the optimality condition (6). This rate is defined by the rate of convergence of a calibrated forecaster in the Blackwell approachability theorem (3) and on the infinite series of \( \epsilon \)-nets \( C_{\epsilon} \), \( A_{\epsilon} \), and \( \mathcal{P}_{\epsilon} \). We assume also that all portfolios functions \( b(\cdot) \) are Lipschitz continuous.

The proof of the approachability theorem gives rise to an implicit strategy, as indicated in Blackwell [3]. We start from a variant the Blackwell theorem for \( \ell^2 \) norm \( \| \cdot \|_2 \). Denote \( d_C(x) \) the projection of \( x \) in \( \ell^2 \)-norm onto \( C \). According to the proof of this theorem (see Blackwell [3] or Cesa-Bianchi and Lugosi [7], Section 7), at each round \( t > 2 \) and with the notations above, the forecaster should pick his action \( p_t \) at random according to a distribution \( P_t \) on the set \( \mathcal{P}_\epsilon \) such that \( (\bar{m}(P_t, a_j) - d_C(\bar{m}(P_t, a_j))) \cdot (f(P_t, a_j) - \bar{m}(P_t, a_j)) \leq 0 \) for all \( a_j \in A_{\epsilon} \).

Proof of the Blackwell theorem from [3] and convergence theorem for Hilbert space-valued martingales of Chen and White [6] provide uniform convergence...
rates of sequence of empirical payoff vectors \( \vec{m}(p_t, a_t) \) to the target set \( C \): there exists an absolute constant \( c \) such that for any \( \delta > 0 \) for all strategies of Market and for all \( T \), with probability \( 1 - \delta \),

\[
\| \vec{m}(p_t, x_t) - d_C(\vec{m}(p_t, x_t)) \|_2 \leq c \sqrt{\frac{\log \frac{1}{\delta}}{T}},
\]

where \( c \) is a positive constant, \( \| \cdot \|_2 \) is the Euclidian norm in \( R^{NM} \), and \( x_t \in A_\epsilon \) for all \( t \).

The suitable choice of \( \epsilon \) is when \( \epsilon \sim T^{-\frac{1}{k+1}} \). Further, taking into account the precision \( \mu \) of approximating the sets \( A, P(A) \) and \( C \) in (8) and (9) and assuming that \( b(\cdot) \) is Lipshiz continuous, we should choose the suitable \( \mu \) to minimize the sum \( \mu + T(1/\mu)^{k+1} \), where \( M = O((1/\mu)^{M-1}) \). Combining series of nets like in V’yugin [16], we obtain rate of convergence \( O\left(\frac{1}{(\log T)^{\frac{1}{k+1}-\nu}}\right) \) in (6), where \( \nu \) is an arbitrary small positive real number, and \( k \) is the number of assets. More precise, for any \( \delta > 0 \), with probability \( 1 - \delta \),

\[
\frac{1}{T} \log S_T \geq \frac{1}{T} \log S_T - c(\log(T/\log \frac{1}{\delta}))^{-\frac{1}{k+1}+\nu}
\]

for all \( T \), where \( c \) is a constant and \( S_T = \prod_{t=1}^{T} (b(\sigma_t) \cdot x_t) \) is the wealth achieved by an arbitrary Lipshiz continuous portfolio \( b(\cdot) \).

4 Conclusion

In this paper we study the method for constructing the log-optimal portfolio in a game-theoretic framework. No stochastic assumptions are made about market values. Instead, we define “an artificial probability distribution” for market values using the method of calibration. Using this distribution, we construct the log-optimal portfolio by the standard scheme (1), where the mathematical expectation \( E \) is over probability distribution defined by well-calibrated forecasts. Our log-optimal portfolio performs asymptotically at least as well as any stationary portfolio that redistribute the investment at each round using a continuous portfolio.
function of the side information. This performance is almost surely, where the corresponding probability distribution is an internal distribution of the probabilistic algorithm computing well-calibrated forecasts on the base of the Blackwell approachability theorem.

The drawback of this approach is the very poor bounds of the rate convergence. Note that no rate of convergence exists for portfolio strategies universal with respect to the class of all stationary and ergodic processes.

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