ON MULTIPLE LEBESGUE FUNCTIONS

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Abstract. We introduce a notion being a \( k \)-fold Lebesgue function for measure preserving transformations, where any 2-fold Lebesgue function is just ordinary Lebesgue. We discuss how this new metrical isomorphisms invariant of dynamical systems is related to others classical notions in ergodic theory, mostly focusing on its spectral aspects. In particular, for transformations with sufficiently many multiple Lebesgue functions we treat the corresponding multiple analogs of very well-known problems of Banach and Rokhlin.

1. Introduction

By a transformation \( T \) we mean an invertible measure-preserving map defined on a non-atomic standard Borel probability space \((X, \mathcal{F}, \mu)\). The spectral properties of \( T \) are those of the induced (Koopman’s) unitary operator on \( L^2(\mu) \) defined by

\[
\hat{T} : L^2(\mu) \to L^2(\mu); \quad \hat{T} f(x) = f(Tx).
\]

In the sequel, we keep the notation \( L^2_0(\mu) \) for the main \( \hat{T} \)-invariant subspace of centered functions \( f \) (i.e. \( \mu(f) = 0 \)) in \( L^2(\mu) \) and \( C(f_1, f_2, \ldots) \), called the component, for the minimal closed \( \hat{T} \)-invariant subspace in \( L^2(\mu) \), containing functions \( f_1, f_2, \ldots \). Usually, in ergodic theory, we talk about reduced spectral invariants of \( \hat{T} \) that are restricted to \( L^2_0(\mu) \) because of the trivial rest. For example, the oldest unsolved problem, associated with Banach, asks whether there exists a transformation with simple Lebesgue spectrum (i.e. whether there exists a Lebesgue function \( f \) such that \( C(f) = L^2_0(\mu) \)). Let us remind that by the Bochner theorem we can find to each \( \phi \in L^2(\mu) \) a unique (spectral) measure \( \sigma_\phi \) on the unique circle of the complex plain \( \mathbb{T} = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \) with the following moments.

\[
\sigma_\phi(n) = \int_{\mathbb{T}} \lambda^n d\sigma_\phi(\lambda) = \int_X \hat{T}^n \phi \cdot \overline{\phi} d\mu, \quad n \in \mathbb{Z}.
\]

And, as above, in the classical setting, \( f \) to be a Lebesgue function means just \( \sigma_f \) to be the Lebesgue measure or, equivalently,

\[
\int_X \hat{T}^n f \cdot \overline{f} d\mu = 0
\]

if \( n \neq 0 \), and \( \int_X |f|^2 d\mu = 2\pi \).

It is well known that any Koopman operator is just a unitary operator that is additionally multiplicative on the set of bounded functions. Besides, needless to say that it is still unknown which unitary operator can be realized as Koopman’s one. There were invented many other invariants of transformations, but the classification

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problem is unsolved even in certain subclasses as, for example, the sets of rank one or mixing transformations. It all can be considered as an easy motivation in study of other invariants.

One of the ways to get involved some piece of the multiplicative structure is to care about higher order moments. It is not a new isomorphisms invariant, but still not in active use, because, probably, of harder calculation.

One more our motivation is to look a bit differently at the class of transformations with Lebesgue spectrum keeping in mind the nice Host result (see [9]) that if there is a counterexample to Rokhlin’s problem on multiple mixing, then it must have at least an absolutely continuous component in its spectrum.

To understand the motivation better, let us rewrite classical notions mixing and multiple mixing according to the functional point of view.

1.1. Mixing case. From now on we keep for mixing the following version of the definition. We say that a transformation \( T \) is mixing iff for any \( f, g \in L^2(\mu) \)

\[
\int_X \hat{T}^n f \cdot \overline{g} d\mu \to \int_X f d\mu \cdot \int_X \overline{g} d\mu \text{ as } n \to \infty,
\]

or, in an equivalent form, for any \( f, g \in L^2(\mu) \)

\[
\int_X \hat{T}^n f \cdot \overline{g} d\mu \to 0 \text{ as } n \to \infty.
\]

Following, for example, [3], and ..., we say that \( T \) is mixing in some component \( B = C(f_1, f_2, \ldots) \) (and \( B \) is said to be a mixing component ) iff for any \( f, g \in B \) the assumption (1) is true.

For every transformation \( T \) there exist many decompositions of the space \( L^2(\mu) \) into the orthogonal sum of at most countably many components of the form \( C(f_i) \) \((i = 1, 2, \ldots)\) that are called cyclic. Obviously, \( T \) is mixing if all the components \( C(f_i) \) are mixing ones. This implies that to check being mixing for \( T \) is just to check that the moments \( \sigma_{f_i}(n) \) vanish for any \( i \).

The notion being \( k \)-fold mixing \((k \geq 2)\) for a transformation \( T \) can be defined as follows. For any collection of \( k \) bounded functions \( f_1, f_2, \ldots, f_k \) in \( L^2(\mu) \) and for any sequence of \( k - 1 \) integers \( n_1(m), n_2(m), \ldots, n_{k-1}(m) \) \((m = 1, 2, \ldots)\), if \( n_i(m) - n_j(m) \to \infty \) and \( n_i(m) \to +\infty \) for any \( i \neq j \), then

\[
\int_X \hat{T}^{n_i(m)} f_1 \cdot \hat{T}^{n_2(m)} f_2 \cdots \hat{T}^{n_{k-1}(m)} f_{k-1} \cdot \overline{T}_k d\mu \to
\]

\[
\int_X f_1 d\mu \cdot \int_X f_2 d\mu \cdots \int_X f_{k-1} d\mu \cdot \int_X \overline{T}_k d\mu \text{ as } m \to +\infty.
\]

Obviously, \( k \)-fold mixing implies mixing of all smaller orders, and the famous unsolved Rokhlin problem on multiple mixing asks whether there exists a (2-fold) mixing transformation which is not mixing of all orders (i.e. not multiple mixing).

**Definition 1.1.** Following [14], we say that a transformation \( T \) is \( k \)-fold mixing \((k \geq 2)\) functionally in some component \( C(g_1, g_2, \ldots) \) if there exists a dense subset \( A \) of \( C(g_1, g_2, \ldots) \) such that (2) is satisfied for any collection of \( k \) functions \( f_1, f_2, \ldots, f_k \) in \( A \).

Trivially, all the above definitions are equivalent for \( k = 2 \). Nevertheless, for higher orders it is a tricky question, because there is not much known here. To avoid such unpleasant things as, say, treating of questions how many bounded functions
some component may have and what is the set of functions in a fixed component such that integrals in (2) exist for sufficiently large $m$, we slightly modify the set $A$.

**Definition 1.2.** We say that a transformation $T$ is $k$-fold mixing ($k \geq 2$) in some component $C(g_1, g_2, \ldots)$ if it is $k$-fold mixing functionally for the subset $A$, where $A$ contains any bounded function in $C(g_1, g_2, \ldots)$.

One can easily construct transformations (see Example 2.2 below) with some orthogonal decomposition of $L_0^2(\mu)$ into $3$-fold mixing cyclic components $C(f_i)$, where every $C(f_i)$ is not mixing. Thus the transformations are not $3$-fold mixing.

Besides, in spite of the case $k = 2$, $3$-fold mixing even in $L_0^2(\mu)$ does not imply $3$-fold mixing in the whole $L^2(\mu)$ (see Example 2.3). Let us also mention that there exist transformations with a mixing component that is not $3$-fold mixing and contains a dense set of bounded functions (see [14]). Adding all we see that there is not much of the expected similarities for higher orders of mixing if we do not add some regularities conditions between components and the constant. One of the ways to do it is the following.

Fix a transformation $T$, and $k \geq 2$. Take some orthogonal decomposition

$$L_0^2(\mu) = \bigoplus_{i \geq 1} C(f_i), \text{ then } L_0^2(\mu) = \bigoplus_{i \geq 0} C(f_i),$$

where $f_0$ is a non-zero constant function. Let us say that this decomposition is **sufficiently ($k$-)good** if

$$\forall i \int_X |f_i|^{2k-2} d\mu < \infty.$$

For any sufficiently good decomposition, it can easily be checked that the transformation $T$ is $k$-fold mixing if and only if for any collection of indexes $i_1, i_2, \ldots, i_k$ ($i_1^2 + i_2^2 + \ldots + i_k^2 \neq 0$) and for any sequence of $k-1$ integers $n_1(m), n_2(m), \ldots, n_{k-1}(m)$ ($m = 1, 2, \ldots$), if $n_i(m) - n_j(m) \to \infty$ and $n_i(m) \to \infty$ as $m \to +\infty$ for any $i \neq j$, then

$$\int_X \hat{T}^{n_1(m)} f_{i_1} \cdot \hat{T}^{n_2(m)} f_{i_2} \cdots \hat{T}^{n_{k-1}(m)} f_{i_{k-1}} \cdot f_{i_k} d\mu \to 0 \text{ as } m \to +\infty. \quad (3)$$

Similarly, take a transformation $T$ and its cyclic component $C(f) \subseteq L_0^2(\mu)$. Suppose $\int_X |f|^{2k-2} d\mu < \infty$ and $C(f)$ contains a dense subset of bounded functions. Then, $T$ is $k$-fold mixing in the component $C(f)$ if and only if (3) are all true, where we replaced every $f_{i_j}$ ($j = 1, 2, \ldots, k$) by $f$.

By the spectral theorem, for any transformation with simple spectrum, we can write $L_0^2(\mu) = C(f)$ for some $f$. Applying the Alexeyev theorem (see [5]), $f$ can be taken bounded. In the general case, we stress that there is still no examples of (even not necessarily $k$-fold mixing) transformations without sufficiently good decompositions of $L_0^2(\mu)$.

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1 In fact, in [14], there was no taking into account that integrals in (2) may not exist for some functions. Therefore, we really need some $A$. 

1.2. Lebesgue case.

**Definition 1.3.** We say that a non-zero function \( f \in L^2(\mu) \) is \( k \)-fold Lebesgue \((k \geq 2)\) if for any collection \((n_1, n_2, \ldots, n_{k-1})\) of pairwise different non-zero integers and for any \( m \leq k \)

(i): \( \hat{T}^{n_1}f \cdot \hat{T}^{n_2}f \cdot \cdots \cdot \hat{T}^{n_{m-1}}f \in L^2(\mu) \),

(ii): \( \int_X \hat{T}^{n_1}f \cdot \hat{T}^{n_2}f \cdot \cdots \cdot \hat{T}^{n_{k-1}}f \cdot f d\mu = 0 \).

Obviously, for \( k = 2 \) the only difference is we admit to get any positive \( L^2 \)-norm for Lebesgue functions. We remark that in the theory of real-valued random processes the above assumption (i) is somehow related to the most common finiteness restriction of the \( k^{th} \) moment (i.e. \( \int_X |f|^k d\mu < \infty \)). Thus developing of the corresponding theory is a bit of different and independent interest. However, in our opinion, (i) is most adapted to the purposes of this paper.

We begin with the following simple observation.

**Proposition 1.4.** Let \( T \) be any transformation and \( f \) be any \( k \)-fold Lebesgue function for \( T \). The following are true:

(i): The function \( f \) is \( k \)-fold Lebesgue as well.

(ii): If \( T \) is ergodic, then \( f \in L^2_0(\mu) \).

(iii): If \( f \in L^2_0(\mu) \), then \( T \) is \( k \)-fold mixing functionally in \( C(f) \).

In the sequel, let us restrict ourselves to the first non-trivial case \( k = 3 \) mostly. We do hope all we deal below with can be naturally extended to the general case.

Now we introduce a notion of some \( k \)-fold analog of transformations with Lebesgue spectrum.

According to the mixing case discussion above, it looks natural to leave the following naive definition.

Fix some collection \( \{f_i\}_{i \in I}, (I \subseteq \mathbb{N}) \) such that \( L^2_0(\mu) = C(\{f_i\}_{i \in I}) \) for a transformation \( T \), and every \( f_i \) is a 3-fold Lebesgue function. Denote \( B = \{\hat{T}^n f_i : i \in I, n \in \mathbb{Z}\} \cup \{f_0\} \), where \( f_0 \) is a fixed non-zero constant function, and any element \( \hat{T}^{n_1} f_{i_1} \) is different to \( \hat{T}^{n_2} f_{i_2} \) iff \( n_1 \neq n_2 \) or \( i_1 \neq i_2 \) (they can be equal as elements of \( L^2_0(\mu) \)). One can say that the transformation \( T \) has 3-fold Lebesgue functions generated spectrum if for any collection \( \varphi_1, \varphi_2, \varphi_3 \) of pairwise different elements of \( B \)

\[
\int_X \varphi_1 \cdot \varphi_2 \cdot \varphi_3 d\mu = 0.
\]

At the first side this definition looks attractive, since, for example, 3-fold Lebesgue functions generated spectrum implies 2-fold Lebesgue functions generated spectrum. The first reason why this definition is not so good is the fact that for \( k = 2 \) it gives the homogeneous Lebesgue spectrum only (i.e. \( L^2_0(\mu) \) is the orthogonal sum of cyclic Lebesgue components). Besides, unitary operators with Lebesgue spectrum (i.e. with Lebesgue measure of maximal spectral type) can easily have non-homogeneous Lebesgue spectrum. For Koopmans operators there is still no

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2 By spectral arguments we can prove that for any transformation holding at least one Lebesgue function there is a lot of real-valued different ones, but we do not believe that the analogous statement is true even for 3-fold Lebesgue functions.
The main reason why this definition is not so good is the following theorem.

**Theorem 1.5.** There is no such transformations.

This implies that we really need to weaken the above conditions. In our opinion, the weakest setting comes if we treat Lebesgue components separately.

**Definition 1.6.** We say that a transformation \( T \) has \( k \)-fold Lebesgue functions generated spectrum if for some collection of \( k \)-fold Lebesgue functions \( f_1, f_2, \ldots \)

\[ L^2_0(\mu) = C(f_1, f_2, \ldots) \]

First note that, clearly, any 2-fold Lebesgue functions generated spectrum is just ordinary Lebesgue spectrum.

Now note that, if there exist transformations with just one 3-fold Lebesgue function generated spectrum, then we can omit our above discussion concerning relations between components for such transformations. If so, it then can be considered as a solution of a \( k \)-fold analog of the Banach problem in positive.

The main result is the following theorem.

**Theorem 1.7.** There is no transformations with one 3-fold Lebesgue function generated spectrum.

Further note that, in spite of the case \( k = 2 \), Examples 2.2 and 2.3 suggest that transformations with 3-fold Lebesgue functions form a bigger class. Indeed, recall one says that a typical transformation has some property (or the property is said to be typical) if the set of elements satisfying the property contains a dense \( G_\delta \) subset in the Polish group of all transformations equipped with weak (coarse) topology. We offer the following simple theorem.

**Theorem 1.8.** For a typical transformation of the space \((X, F, \mu)\), there exist \( k \)-fold Lebesgue functions \((k = 3, 4, \ldots)\). Moreover, these functions can be chosen as finitely many-valued ones.

Does there exist a transformation with no multiple Lebesgue functions?

We do not know.

Finally note that other certain questions of some fundamental origin come almost automatically as well. To be precise, denote by \( L(k) \) \((L^*(k))\) the set of all transformations with \( k \)-fold Lebesgue functions generated spectrum (with at least one \( k \)-fold Lebesgue function)

What are relations between \( L(k) \) or \( L^*(k) \) for different \( k \)?

In particular,

Is it true that \( L(2) \subset \cap_k L(k) \)?

For bravery we can call these questions as Lebesgue analogs of Rokhlin’s problem. Working on this, we prove that \( L(3) \) is not included in \( L(4) \) (see Example 2.4).

2. **Key examples and proofs**

**Example 2.1.** Bernoulli shifts.
Let $T$ be a Bernoulli shift. Take its standard realization on the space 
$$(X, \mathcal{F}, \mu), \ X = \times_{i \in \mathbb{Z}} X_i, \ X_i = \mathbb{N}$$
for any $i$. Fix some orthogonal basis, say $\{\varphi_i\}_i$, in the subspace of $L^2(X, \mu)$-functions depending on the 0-coordinate only. Clearly, we can assume that every $\varphi_i$ is a bounded function and a constant function is one of the basis elements.

The reader will easily check that every non-constant function of the form
$$\hat{T}^i_1\varphi_{j_1} \cdot \hat{T}^i_2\varphi_{j_2} \cdots \hat{T}^i_m\varphi_{j_m} \ (-\infty < i_1 < i_2 < \ldots < i_m < +\infty)$$
is $k$-fold Lebesgue for any $k$, and, consequently, every Bernoulli shift has $k$-fold Lebesgue functions generated spectrum for any $k$.

It also should be noted that spectrum of any Bernoulli shift can not be generated by finitely many $k$-fold Lebesgue functions, since its multiplicity function is unbounded.

**Example 2.2.** Ergodic transformations with discrete spectrum.

Simple exotic examples of transformations with Lebesgue and mixing components appear if we do not require restrictions for $k = 2$. Indeed, take any ergodic transformation $T$ with discrete spectrum. Then we get a certain orthogonal decomposition
$$L^2_\mu = \bigoplus_{i \geq 1} C(f_i),$$
where $\{f_1, f_2, \ldots\}$ is the set of all but the constant eigenfunctions of $\hat{T}$. Obviously, every $f_i$ is a 3-fold Lebesgue function and every $C(f_i)$ is a 3-fold mixing component.

To get the same for any $k \geq 3$, it is enough to restrict ourselves to transformations with no roots of the unity among eigenvalues of all their eigenfunctions. For example, every ordinary ergodic shift on $\mathbb{R}/\mathbb{Z}$ is as it is required.

**Example 2.3.** of a 3-fold mixing in $L^2_\mu$ transformation that is not (3-fold) mixing.

Let $T$ be any 3-fold mixing transformation on the space $(X, \mathcal{F}, \mu)$. Consider a $\mathbb{Z}/2\mathbb{Z}$-extension, say $S$, of $T$ defined by
$$S(x, y) = (Tx, y + 1), \ (x, y) \in Y = X \times \mathbb{Z}/2\mathbb{Z};$$
here $S$ preserves the product measure, say $\nu$, of $\mu$ and the Haar measure on $\mathbb{Z}/2\mathbb{Z}$. Clearly, $S$ is not mixing, because there is a non-constant eigenfunction.

We claim that $S$ is 3-fold mixing in $L^2_\nu$. Indeed, first note that every function $\varphi \in L^2_\nu$ is the orthogonal sum $\varphi_+ + \varphi_-$ according to the decomposition $L^2_\nu(Y, \nu) = H^+_0 \oplus H^-_0$ into $\hat{S}$-invariant components
$$H^+_0 = \{f \in L^2_\nu : f(x, y + 1) \equiv \pm f(x, y)\}.$$ 
Clearly, $\hat{S} \varphi_\pm = (\pm 1)\hat{T}^* \varphi_\pm$; here the transformation $T^*$ is defined by $T^*(x, y) = (Tx, y)$. Take then any triple $f, g, h \in L^2_\nu$ of bounded functions. By the simple calculation, for any $n_1(m), n_2(m)$
$$(n_1(m) - n_2(m) \to \infty \text{ and } n_i(m) \to \infty \text{ i = 1, 2 as } m \to +\infty )$$
we get
$$\int_Y \hat{S}^{n_1(m)} f \cdot \hat{S}^{n_2(m)} g \cdot T_+ d\nu \to 0 \text{ as } m \to +\infty,$$
since each of 8 summands $\int_Y \hat{S}^{n_1(m)} f_\pm \cdot \hat{S}^{n_2(m)} g_\pm \cdot T_\pm d\nu$ is zero or tends to zero as $m \to +\infty$. 
**Example 2.4.** of an ergodic transformation whose 3-fold Lebesgue functions generated spectrum can not be 4-fold Lebesgue functions generated.

Suppose \( \alpha \) is an irrational; then by \( \Lambda \) we denote a countable subgroup of \( T = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \) generated by \(-1\) and \( \exp(2\pi i \alpha) \). It is well known that there exists a unique (up to an isomorphism) ergodic transformation, say \( T \), with discrete spectrum such that all its eigenvalues are \( \Lambda(T) = \Lambda \). Arguing as in Example 2.2, we see that all but \( f_1 \) eigenfunctions \( f_\lambda \) of \( T \) are 3-fold Lebesgue, where \( f_\lambda \) are eigenfunctions of \( \hat{T} \) with eigenvalues \( \lambda \in \Lambda \). Thus \( T \) has 3-fold Lebesgue functions generated spectrum.

Let us prove that the spectrum of \( T \) can not be 4-fold Lebesgue functions generated. To get this, it is sufficient to show that every 4-fold Lebesgue function is orthogonal to \( f_{-1} \).

Assume the converse. Then \(-1 \in \text{supp} f_L \), where \( f_L = \sum_\lambda c_\lambda f_\lambda \), and \( \text{supp} f_L = \{ \lambda \in \Lambda : c_\lambda \neq 0 \} \). By the spectral theorem, for any \( \varphi, f \in L^2(\mu) \)

\[ \varphi \in C(f) \iff \text{supp} \varphi \subseteq \text{supp} f, \quad \text{and} \quad C(\varphi) \perp C(f) \iff \text{supp} \varphi \cap \text{supp} f = \emptyset. \]

Therefore \( f_{-1} \in C(f_L) \).

Besides, it is well known that every ergodic transformation \( S \) with discrete spectrum is rigid (i.e. \( \hat{S}^{k_i} \rightarrow E \) for some \( k_i \rightarrow \infty \) in the weak (strong) operator topology, where \( E \) is the neutral element).

It implies that for any pair of mutually different non-zero integers \( (n_2, n_3) \)

\[ \forall n \int_X \hat{T}^{n_2} f_L \cdot \hat{T}^{n_3} f_L \cdot f_{-1} d\mu = 0. \]

Thus

\[ C(f_L) \perp C(\hat{T}^{n_2} f_L \cdot \hat{T}^{n_3} f_L). \]

It gives

\[ \forall n_2 \neq n_3 \neq 0 \neq n_2 \int_X f_{-1} \cdot \hat{T}^{n_2} f_L \cdot \hat{T}^{n_3} f_L \cdot f_L d\mu = 0. \]

Applying rigidity again, we get

\[ C(f_L) \perp C(\hat{T}_{-1}^{n_3} f_L \cdot f_L); \]

here \( \hat{T}_{-1}^{n_3} f_L \cdot f_L \in L^2(\mu) \), since |\( f_{-1} | \) is a constant function. Then

\[ \forall n_3 \neq 0 \int_X f_{-1} \cdot f_{-1} \cdot \hat{T}^{n_3} f_L \cdot f_L d\mu = 0, \]

and, consequently,

\[ \int_X f_{-1} \cdot f_{-1} \cdot \hat{T}^L d\mu = 0, \quad \int_X f_{-1} \cdot \hat{T}^L d\mu = 0. \]

This contradiction concludes the proof.

*Proof. of Proposition 1.4.* It is an easy application of von Neumann ergodic theorem for unitary operators and left to the reader. \( \square \)
Proof. of Theorem 1.8. It is an easy application of the fact that a typical transformation $T$ is isomorphic to a $G$-extension for any finite abelian group $G$ (see [2]). Namely, take a realization of $T$ as a $\mathbb{Z}/m\mathbb{Z}$-extension for an $m$. Hence there is the standard decomposition of $L^2(Y, \nu)$ into the orthogonal sum of $T$-invariant components

$$H_\chi = \{ f \in L^2(Y, \nu) : f(x, y) = \chi(y) \cdot \varphi(x), \varphi \in L^2(X, \mu) \},$$

where $\chi(y)$ are characters of $\mathbb{Z}/m\mathbb{Z}$, $Y = X \times \mathbb{Z}/m\mathbb{Z}$. Every non-zero element of $H_\chi$ ($\chi \neq 1$) is a $k$-fold Lebesgue function for an appropriate $m$. $\square$

Proof. of Theorems 1.5 and 1.7. It is an easy application of the following technical theorem

**Theorem 2.5.** Let $f$ be a 3-fold Lebesgue function for a transformation $T$; then there exists $n \neq 0$ such that $\hat{T}^n f \cdot f \notin C(f) \oplus C(1)$. $\square$

The proof of Theorem 2.5 will be published elsewhere.

3. Closing remarks and questions

Looking at transformations treated above that are mostly of discrete or Lebesgue maximal spectral types, one can think that dealing with any $k$-fold Lebesgue function $f$ is just a matter of some power of $f$ and convolutions for spectral types. However, applying group extensions, one can easily construct (see constructions in [4], [7], [10]) transformations with $\sigma_f \perp \sigma_2 \perp \sigma_1 \perp \sigma_f$. It also should be noted that for a typical transformation any measure of the maximal spectral type is disjoint with its convolution. This means that $\sigma_g \perp \sigma_f \perp \sigma_f$ for every pair $f, g \in L^2_0(\mu)$.

Transformations and, a bit more generally, group actions (i.e. group representations by transformations) are main objects to study in modern ergodic theory. Investigations describing roughly what is changed if we go to actions of larger groups or back are one of the steadily well-developing aspects of ergodic theory. The reader can easily produce the notion being a $(k$-fold) Lebesgue function, as in the case $k = 2$, to actions of any countable groups. Traditional ($k$-fold) problems can be stated for general group actions as well.

Suppose a transformation $T$ belongs to $L(k)$; then by $m_k(T)$ we denote the minimal cardinality of collections of $k$-fold Lebesgue functions that generate the spectrum of $T$.

**What can we say about possible values of $m_k(T)$?**

Let us mention that, for example, $m_3(T)$ is infinite for every ergodic transformation $T$ with discrete spectrum. Related question is based on an easy inequality

$$m_2(T) \leq m_k(T),$$

if they both exist.

**Can we say the same for other pairs $n \leq k$?**

It should be also noted that all above questions are most important in the main subclass of weakly mixing transformations. It would be interesting to see any new effects that restriction may imply.

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