LOGARITHMIC CONNECTIONS ON PRINCIPAL BUNDLES OVER A RIEMANN SURFACE

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ABSTRACT. Let $E_G$ be a holomorphic principal $G$–bundle on a compact connected Riemann surface $X$, where $G$ is a connected reductive complex affine algebraic group. Fix a finite subset $D \subset X$, and for each $x \in D$ fix $w_x \in \text{ad}(E_G)_x$. Let $T$ be a maximal torus in the group of all holomorphic automorphisms of $E_G$. We give a necessary and sufficient condition for the existence of a $T$–invariant logarithmic connection on $E_G$ singular over $D$ such that the residue over each $x \in D$ is $w_x$. We also give a necessary and sufficient condition for the existence of a logarithmic connection on $E_G$ singular over $D$ such that the residue over each $x \in D$ is $w_x$, under the assumption that each $w_x$ is $T$–rigid.

1. Introduction

Let $X$ be a compact connected Riemann surface. Given a holomorphic vector bundle $E$ on $X$, a theorem of Weil and Atiyah says that $E$ admits a holomorphic connection if and only if the degree of every indecomposable component of $E$ is zero (see [We], [At]). Now let $G$ be a connected complex affine algebraic group and $E_G$ a holomorphic principal $G$–bundle on $X$. Then $E_G$ admits a holomorphic connection if and only if for every holomorphic reduction of structure group $E_H \subset E_G$, where $H$ is a Levi factor of some parabolic subgroup of $G$, and for every holomorphic character $\chi$ of $H$, the degree of the associated line bundle

$$E_H(\chi) = E_H \times^\chi \mathbb{C} \longrightarrow X$$

(1.1)

is zero [AB]. Our aim here is to investigate the logarithmic connections on $E_G$ with fixed residues, where $(G, E_G)$ is as above. More precisely, fix a finite subset $D \subset X$ and also fix

$$w_x \in \text{ad}(E_G)_x$$

for each $x \in D$, where $\text{ad}(E_G)$ is the adjoint vector bundle for $E_G$. We investigate the existence of logarithmic connections on $E_G$ singular over $D$ such that residue is $w_x$ for every $x \in D$.

Let $\text{Aut}(E_G)$ denote the group of all holomorphic automorphisms of $E_G$; it is a complex affine algebraic group. Fix a maximal torus

$$T \subset \text{Aut}(E_G).$$

This choice produces a Levi factor $H$ of a parabolic subgroup of $G$ as well as a holomorphic reduction of structure group $E_H \subset E_G$ to $H$ [BBN]. This pair $(H, E_H)$ is determined by $T$ uniquely up to a holomorphic automorphism of $E_G$. 

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The group $\text{Aut}(E_G)$ acts on the vector bundle $\text{ad}(E_G)$. An element of $\text{ad}(E_G)$ will be called $T$–rigid if it is fixed by the action of $T$; some examples are given in Section 4.2.

We prove the following (see Theorem 5.3):

**Theorem 1.1.** The following two are equivalent:

1. There is a $T$–invariant logarithmic connection on $E_G$ singular over $D$ with residue $w_x$ at every $x \in D$.
2. The element $w_x$ is $T$–rigid for each $x \in D$, and

$$\text{degree}(E_H(\chi)) + \sum_{x \in D} d\chi(w_x) = 0$$

for every holomorphic character $\chi$ of $H$, where $E_H(\chi)$ is the line bundle in (1.1), and $d\chi: \text{Lie}(H) \rightarrow \mathbb{C}$ is the homomorphism of Lie algebras corresponding to $\chi$.

The Lie algebra $\mathbb{C}$ being abelian the homomorphism $d\chi$ factors through the conjugacy classes in $\text{Lie}(H)$, so it can be evaluated on the elements of $\text{ad}(E_H)$.

We also prove the following (see Theorem 5.1):

**Theorem 1.2.** Assume that each $w_x$ is $T$–rigid. Then there is a logarithmic connection on $E_G$ singular over $D$, with residue $w_x$ at every $x \in D$, if and only if

$$\text{degree}(E_H(\chi)) + \sum_{x \in D} d\chi(w_x) = 0,$$

where $E_H(\chi)$ and $d\chi(w_x)$ are as in Theorem 1.1.

2. Logarithmic connections and residue

2.1. Preliminaries. Let $G$ be a connected reductive affine algebraic group defined over $\mathbb{C}$. A Zariski closed connected subgroup $P \subset G$ is called a parabolic subgroup if $G/P$ is a projective variety [Bo, 11.2], [Hu]. The unipotent radical of a parabolic subgroup $P \subset G$ will be denoted by $R_u(P)$. The quotient group $P/R_u(P)$ is called the Levi quotient of $P$. A Levi factor of $P$ is a Zariski closed connected subgroup $L \subset P$ such that the composition $L \hookrightarrow P \twoheadrightarrow P/R_u(P)$ is an isomorphism [Hu, p. 184]. We note that $P$ admits Levi factors, and any two Levi factors of $P$ are conjugate by an element of $R_u(P)$ [Hu, § 30.2, p. 185, Theorem].

The multiplicative group $\mathbb{C} \setminus \{0\}$ will be denoted by $\mathbb{G}_m$. A torus is a product of copies of $\mathbb{G}_m$. Any two maximal tori in a complex algebraic group are conjugate [Bo, p. 158, Proposition 11.23(ii)].

By a homomorphism between algebraic groups or by a character we will always mean a holomorphic homomorphism or a holomorphic character.

2.2. Logarithmic connections. Let $X$ be a compact connected Riemann surface. Fix a finite subset

$$D := \{x_1, \ldots, x_n\} \subset X.$$

The reduced effective divisor $x_1 + \ldots + x_n$ will also be denoted by $D$. 
Let
\[ p : E_H \rightarrow X \quad (2.1) \]
be a holomorphic principal \( H \)-bundle on \( X \), where \( H \) is a connected affine algebraic group defined over \( \mathbb{C} \). The Lie algebra of \( H \) will be denoted by \( \mathfrak{h} \). Let
\[ dp : T E_H \rightarrow p^* T X \quad (2.2) \]
be the differential of the map \( p \) in (2.1), where \( T E_H \) and \( T X \) are the holomorphic tangent bundles of \( E_H \) and \( X \) respectively; note that \( dp \) is surjective. The action of \( H \) on \( E_H \) produces an action of \( H \) on \( T E_H \). This action on \( T E_H \) clearly preserves the subbundle \( \text{kernel}(dp) \). Define
\[ \text{ad}(E_H) := \text{kernel}(dp)/H \rightarrow X, \]
which is a holomorphic vector bundle on \( X \); it is called the adjoint vector bundle for \( E_H \). We note that \( \text{ad}(E_H) \) is identified with the vector bundle \( E_H \times^H \mathfrak{h} \rightarrow X \) associated to \( E_H \) for the adjoint action of \( H \) on its Lie algebra \( \mathfrak{h} \). So the fibers of \( \text{ad}(E_H) \) are Lie algebras isomorphic to \( \mathfrak{h} \). Define the Atiyah bundle for \( E_H \)
\[ \text{At}(E_H) := (T E_H)/H \rightarrow X. \]
The action of \( H \) on \( T E_H \) produces an action of \( H \) on the direct image \( p_* T E_H \). We note that
\[ \text{At}(E_H) = (p_* T E_H)^H \subset p_* T E_H \]
(see [At]). Taking quotient by \( H \), the homomorphism \( dp \) in (2.2) produces a short exact sequence
\[ 0 \rightarrow \text{ad}(E_H) \rightarrow \text{At}(E_H) \xrightarrow{d'p} T X \rightarrow 0, \quad (2.3) \]
where \( d'p \) is constructed from \( dp \); this is known as the Atiyah exact sequence for \( E_H \).

The subsheaf \( T X \otimes \mathcal{O}_X(-D) \) of \( T X \) will be denoted by \( T X(-D) \). Now define
\[ \text{At}(E_H, D) := (d'p)^{-1}(T X(-D)) \subset \text{At}(E_H), \]
where \( d'p \) is the projection in (2.3). So from (2.3) we have the exact sequence of vector bundles on \( X \)
\[ 0 \rightarrow \text{ad}(E_H) \xrightarrow{i_0} \text{At}(E_H, D) \xrightarrow{\sigma} T X(-D) \rightarrow 0, \quad (2.4) \]
where \( \sigma \) is the restriction of \( d'p \); this will be called the \textit{logarithmic Atiyah exact sequence} for \( E_H \).

A logarithmic connection on \( E_H \) singular over \( D \) is a holomorphic homomorphism
\[ \theta : T X(-D) \rightarrow \text{At}(E_H, D) \quad (2.5) \]
such that \( \sigma \circ \theta = \text{Id}_{T X(-D)} \), where \( \sigma \) is the homomorphism in (2.4). Note that giving such a homomorphism \( \theta \) is equivalent to giving a homomorphism \( \varpi : \text{At}(E_H, D) \rightarrow \text{ad}(E_H) \) such that \( \varpi \circ i_0 = \text{Id}_{\text{ad}(E_H)} \), where \( i_0 \) is the homomorphism in (2.4).
2.3. **Residue of a logarithmic connection.** Given a vector bundle \( W \) on \( X \), the fiber of \( W \) over any point \( x \in X \) will be denoted by \( W_x \). For any \( \mathcal{O}_X \)-linear homomorphism \( f : W \to V \) of holomorphic vector bundles, its restriction \( W_x \to V_x \) will be denoted by \( f(x) \).

From (2.3) and (2.4) we have the commutative diagram of homomorphisms

\[
\begin{array}{cccccc}
0 & \to & \text{ad}(E_H) & \overset{i_0}{\to} & \text{At}(E_H, D) & \overset{\sigma}{\to} & TX(-D) & \to & 0 \\
\| & & & \downarrow j & & & \downarrow \iota & & \\
0 & \to & \text{ad}(E_H) & \overset{i}{\to} & \text{At}(E_H) & \overset{d'p}{\to} & TX & \to & 0 \\
\end{array}
\]

(2.6)
on \( X \). So for any point \( x \in X \), we have

\[ d'p(x) \circ j(x) = \iota(x) \circ \sigma(x) : \text{At}(E_H, D)_x \to (TX)_x = T_xX. \]

Note that \( \iota(x) = 0 \) if \( x \in D \), therefore in that case \( d'p(x) \circ j(x) = 0 \). Consequently, for every \( x \in D \) there is a homomorphism

\[ R_x : \text{At}(E_H, D)_x \to \text{ad}(E_H)_x \]

(2.7)

uniquely defined by the identity \( i(x) \circ R_x(v) = j(x)(v) \) for all \( v \in \text{At}(E_H, D)_x \). Note that

\[ R_x \circ i_0(x) = \text{Id}_{\text{ad}(E_H)_x}, \]

where \( i_0 \) is the homomorphism in (2.6). Therefore, from (2.4) we have

\[ \text{At}(E_H, D)_x = \text{ad}(E_H)_x \oplus \text{kernel}(R_x) = \text{ad}(E_H)_x \oplus TX(-D)_x; \]

(2.8)

note that the composition \( \text{kernel}(R_x) \hookrightarrow \text{At}(E_H, D)_x \overset{\sigma(x)}{\to} TX(-D)_x \) is an isomorphism.

For any \( x \in D \), the fiber \( TX(-D)_x \) is identified with \( \mathbb{C} \) using the Poincaré adjunction formula \([GH,\ p.\ 146]\). Indeed, for any holomorphic coordinate \( z \) around \( x \) with \( z(x) = 0 \), the image of \( z \frac{\partial}{\partial z} \) in \( TX(-D)_x \) is independent of the choice of the coordinate function \( z \); the above mentioned identification between \( TX(-D)_x \) and \( \mathbb{C} \) sends this independent image to \( 1 \in \mathbb{C} \). Therefore, from (2.8) we have

\[ \text{At}(E_H, D)_x = \text{ad}(E_H)_x \oplus \mathbb{C} \]

(2.9)

for all \( x \in D \).

For a logarithmic connection \( \theta : TX(-D) \to \text{At}(E_H, D) \) as in (2.5), and any \( x \in D \), define

\[ \text{Res}(\theta, x) := R_x(\theta(1)) \in \text{ad}(E_H)_x, \]

(2.10)

where \( R_x \) is the homomorphism in (2.7); in the above definition \( 1 \) is considered as an element of \( TX(-D)_x \) using the identification of \( \mathbb{C} \) with \( TX(-D)_x \) mentioned earlier.

The element \( \text{Res}(\theta, x) \) in (2.10) is called the **residue**, at \( x \), of the logarithmic connection \( \theta \).
2.4. **Extension of structure group.** Let $M$ be a complex affine algebraic group and 
\[ \rho : H \rightarrow M \]
a homomorphism. As before, $E_H$ is a holomorphic principal $H$–bundle on $X$. Let 
\[ E_M := E_H \times^\rho M \rightarrow X \]
be the holomorphic principal $M$–bundle obtained by extending the structure group of 
$E_H$ using $\rho$. So $E_M$ is the quotient of $E_H \times M$ obtained by identifying $(y, m)$ and 
$(y h^{-1}, \rho(h) m)$, where $y, m$ and $h$ run over $E_H$, $M$ and $H$ respectively. Therefore, we
have a morphism 
\[ \hat{\rho} : E_H \rightarrow E_M, \ y \mapsto (y, e_M), \]
where $(y, e_M)$ is the equivalence class of $(y, e_M)$ with $e_M$ being the identity element of $M$.
The homomorphism of Lie algebras $d\rho : \mathfrak{h} \rightarrow \mathfrak{m} := \text{Lie}(M)$ associated to $\rho$ produces a
homomorphism of vector bundles 
\[ \alpha : \text{ad}(E_H) \rightarrow \text{ad}(E_M). \quad (2.11) \]
The maps $\hat{\rho}$ and $d\rho$ together produce a homomorphism of vector bundles
\[ \tilde{A} : \text{At}(E_H) \rightarrow \text{At}(E_M). \]
This map $\tilde{A}$ produces a homomorphism 
\[ A : \text{At}(E_H, D) \rightarrow \text{At}(E_M, D), \quad (2.12) \]
which fits in the following commutative diagram of homomorphisms 
\[ \begin{array}{ccc}
0 & \rightarrow & \text{ad}(E_H) \\
\downarrow \alpha & & \downarrow A \\
0 & \rightarrow & \text{ad}(E_M) \\
\end{array} \]
\[ \begin{array}{ccc}
\text{At}(E_H, D) & \rightarrow & \text{At}(E_M, D) \\
\downarrow \sigma & & \downarrow \text{TX}(\neg D) \\
\text{TX}(\neg D) & \rightarrow & 0 \\
\end{array} \quad (2.13) \]
where the top exact sequence is the one in (2.4) and the bottom one is the corresponding 
sequence for $E_M$.

If $\theta : \text{TX}(\neg D) \rightarrow \text{At}(E_H, D)$ is a logarithmic connection on $E_H$ as in (2.5), then 
\[ A \circ \theta : \text{TX}(\neg D) \rightarrow \text{At}(E_M, D) \quad (2.14) \]
is a logarithmic connection on $E_M$ singular over $D$. From the definition of residue in 
(2.10) it follows immediately that
\[ \alpha(\text{Res}(\theta, x)) = \text{Res}(A \circ \theta, x) \quad (2.15) \]
for all $x \in D$. This proves the following:

**Lemma 2.1.** With the above notation, if $E_H$ admits a logarithmic connection $\theta$ singular over $D$ with residue $w_x \in \text{ad}(E_H)_x$ at each $x \in D$, then $E_M$ admits a logarithmic connection $\theta' = A \circ \theta$ singular over $D$ with residue $\alpha(w_x)$ at each $x \in D$. 

**Proof.**
3. Connections with given residues

3.1. Formulation of residue condition. Fix a holomorphic principal \( H \)-bundle \( E_H \) on \( X \), and fix elements

\[
   w_x \in \text{ad}(E_H)_x
\]

for all \( x \in D \). Consider the decomposition of \( \text{At}(E_H, D)_x \) in \([2.9]\). For any \( x \in D \), let

\[
   \ell_x := \mathbb{C} \cdot (w_x, 1) \subset \text{ad}(E_H)_x \oplus \mathbb{C} = \text{At}(E_H, D)_x
\]

be the line in the fiber \( \text{At}(E_H, D)_x \). Let

\[
   \mathcal{A} \subset \text{At}(E_H, D)
\]

be the subsheaf that fits in the short exact sequence

\[
   0 \to \mathcal{A} \to \text{At}(E_H, D) \to \bigoplus_{x \in D} \text{At}(E_H, D)_x / \ell_x \to 0. \tag{3.1}
\]

Note that the composition

\[
   \text{ad}(E_H)_x \xrightarrow{i_0(x)} \text{At}(E_H, D)_x \xrightarrow{\phi_x} \text{At}(E_H, D)_x / \ell_x
\]

is injective, hence it is an isomorphism, where \( i_0 \) is the homomorphism in \([2.13]\); this composition will be denoted by \( \phi_x \). Therefore, from \([2.4]\) and \((3.1)\) we have a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & \text{ad}(E_H) \otimes \mathcal{O}_X(-D) & \to & \mathcal{A} & \xrightarrow{\sigma_1} & TX(-D) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \text{ad}(E_H) & \xrightarrow{i_0} & \text{At}(E_H, D) & \xrightarrow{\sigma} & TX(-D) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \bigoplus_{x \in D} \text{ad}(E_H)_x & \xrightarrow{\bigoplus_{x \in D} \phi_x} & \bigoplus_{x \in D} \text{At}(E_H, D)_x / \ell_x & \to & 0 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & & \tag{3.2}
\end{array}
\]

where all the rows and columns are exact; the restriction of \( \sigma \) to the subsheaf \( \mathcal{A} \) is denoted by \( \sigma_1 \).

**Lemma 3.1.** Consider the space of all logarithmic connections on \( E_H \) singular over \( D \) such that the residue over every \( x \in D \) is \( w_x \). It is in bijection with the space of all holomorphic splittings of the short exact sequence of vector bundles

\[
   0 \to \text{ad}(E_H) \otimes \mathcal{O}_X(-D) \to \mathcal{A} \xrightarrow{\sigma_1} TX(-D) \to 0
\]

on \( X \) in \((3.2)\).
Proof. Let $\theta : \text{TX}(-D) \to \text{At}(E_H, D)$ be a logarithmic connection on $E_H$ singular over $D$ such that the residue over every $x \in D$ is $w_x$. From the definition of residue and the construction of $\mathcal{A}$ it follows that

$$\theta(\text{TX}(-D)) \subseteq \mathcal{A} \subseteq \text{At}(E_H, D).$$

Therefore, $\theta$ defines a holomorphic homomorphism

$$\theta' : \text{TX}(-D) \to \mathcal{A}.$$

Evidently, we have $\sigma_1 \circ \theta' = \text{Id}_{\text{TX}(-D)}$. So $\theta'$ is a holomorphic splitting of the exact sequence in the lemma.

To prove the converse, let $\theta_1 : \text{TX}(-D) \to \mathcal{A}$ be a holomorphic homomorphism such that $\sigma_1 \circ \theta_1 = \text{Id}_{\text{TX}(-D)}$. Consider the composition $\nu \circ \theta_1 : \text{TX}(-D) \to \text{At}(E_H, D)$, where $\nu$ is the homomorphism in (3.2). This defines a logarithmic connection on $E_H$ singular over $D$, because $\sigma \circ \nu \circ \theta_1 = \text{Id}_{\text{TX}(-D)}$ by the commutativity of (3.2).

From (3.2) it follows immediately that $\nu \circ \theta_1(\text{TX}(-D)_x) = \ell_x \subseteq \text{At}(E_H, D)_x$ for every $x \in D$. Now from the definition of residue it follows that the residue of the connection $\nu \circ \theta_1$ at any $x \in D$ is $w_x$. \qed

3.2. Extension class. The short exact sequence in Lemma 3.1 determines a cohomology class

$$\beta \in H^1(X, \text{Hom}(\text{TX}(-D), \text{ad}(E_H) \otimes \mathcal{O}_X(-D))) = H^1(X, \text{ad}(E_H) \otimes K_X),$$

where $K_X = (\text{TX})^*$ is the holomorphic cotangent bundle of $X$. Therefore, $E_H$ admits a logarithmic connection singular over $D$ with residue $w_x \in \text{ad}(E_H)_x$ at each $x \in D$ if and only if the cohomology class $\beta$ in (3.3) vanishes.

Let $\rho : H \to M$ be a homomorphism of affine algebraic groups. Let $E_M := E_H \times^\rho M$ be the principal $M$–bundle over $X$ obtained by extending the structure group of $E_H$ to $M$ by $\rho$. Consider the homomorphism $\alpha$ in (2.11). It produces a homomorphism

$$\overline{\rho} : H^1(X, \text{ad}(E_H) \otimes K_X) \to H^1(X, \text{ad}(E_M) \otimes K_X).$$

From (2.13) we have a commutative diagram

$$
\begin{array}{ccc}
0 & \to & \text{ad}(E_H) \otimes \mathcal{O}_X(-D) \\
\downarrow & & \downarrow \sigma_1 \\
0 & \to & \text{ad}(E_M) \otimes \mathcal{O}_X(-D)
\end{array}
\begin{array}{ccc}
\longrightarrow & \to & \mathcal{A} \\
& \sigma_1 \rightarrow & \text{TX}(-D) \\
& \| & \\
& \| & \\
\longrightarrow & \to & 0 \\
\longrightarrow & \to & \mathcal{A}(E_M) \\
\longrightarrow & \to & \text{TX}(-D) \\
\longrightarrow & \to & 0
\end{array}
$$

where the top exact sequence is the one in Lemma 3.1 and the bottom one is the same sequence for $E_M$. From this diagram it follows that the cohomology class in $H^1(X, \text{ad}(E_M) \otimes K_X)$ for the short exact sequence in Lemma 3.1 for $E_M$ coincides with $\overline{\rho}(\beta)$, where $\beta$ is the cohomology class in (3.3).
Corollary 3.2. Assume that $E_H$ admits a holomorphic reduction of structure group $E_J \subset E_H$ to a complex algebraic subgroup $J \subset H$. The cohomology class $\beta$ in (3.3) is contained in the image of the natural homomorphism $H^1(X, \text{ad}(E_J) \otimes K_X) \hookrightarrow H^1(X, \text{ad}(E_H) \otimes K_X)$.

3.3. A necessary condition for logarithmic connections with given residue. Let $\theta$ be a logarithmic connection on $E_H$ singular over $D$. Take any character

$$\chi : H \to \mathbb{G}_m.$$ 

The group $H$ acts on $\mathbb{C}$; the action of $h \in H$ sends any $c \in \mathbb{C}$ to $\chi(h) \cdot c$. Let

$$E_H(\chi) := E_H \times^x \mathbb{C} \to X$$ 

be the holomorphic line bundle over $X$ associated to $E_H$ for this action of $H$ on $\mathbb{C}$. Since $\mathbb{G}_m$ is abelian, the adjoint vector bundle for $E_H(\chi)$ is the trivial holomorphic line bundle $\mathcal{O}_X$ over $X$. The above logarithmic connection $\theta$ induces a logarithmic connection on $E_H(\chi)$ (see (2.14)); this induced logarithmic connection on $E_H(\chi)$ will be denoted by $\theta^\chi$.

For any $x \in D$, let

$$\text{Res}(\theta^\chi, x) \in \mathbb{C}$$

be the residue of $\theta^\chi$. The residue $\text{Res}(\theta, x)$ defines a conjugacy class in the Lie algebra $\mathfrak{h}$, because any fiber of $\text{ad}(E_H)$ is identified with $\mathfrak{h}$ uniquely up to conjugation. From (2.15) it follows immediately that $\text{Res}(\theta^\chi, x)$ coincides with $d\chi(\text{Res}(\theta, x))$, where $d\chi : \mathfrak{h} \to \mathbb{C}$ is the homomorphism of Lie algebras associated to $\chi$; note that since the Lie algebra $\mathbb{C}$ is abelian, the homomorphism $d\chi$ factors through the conjugacy classes in $\mathfrak{h}$.

As $\theta^\chi$ is a logarithmic connection on the line bundle $E_H(\chi)$ with residue $d\chi(\text{Res}(\theta, x))$ at each $x \in D$, using a computation in [Oh] it follows that

$$\text{degree}(E_H(\chi)) + \sum_{x \in D} d\chi(\text{Res}(\theta, x)) = 0$$

(see [BDP] Lemma 2.3)).

Therefore, we have the following:

Lemma 3.3. Let $E_H$ be a holomorphic principal $H$–bundle on $X$. Fix

$$w_x \in \text{ad}(E_H)_x$$

for every $x \in D$. If there is a logarithmic connection on $E_H$ singular over $D$ with residue $w_x$ at every $x \in D$, then

$$\text{degree}(E_H(\chi)) + \sum_{x \in D} d\chi(w_x) = 0$$

for every character $\chi$ of $H$, where $E_H(\chi)$ is the associated holomorphic line bundle, and $d\chi$ is the homomorphism of Lie algebras corresponding to $\chi$.

Let $\rho : H \to M$ be an injective homomorphism to a connected complex algebraic group $M$ and $E_M := E_H \times^\rho M$ the holomorphic principal $M$–bundle on $X$ obtained by
extending the structure group of $E_H$ using $\rho$. As before, the Lie algebras of $H$ and $M$ will be denoted by $\mathfrak{h}$ and $\mathfrak{m}$ respectively. Using the injective homomorphism of Lie algebras

$$d\rho : \mathfrak{h} \rightarrow \mathfrak{m} \quad (3.4)$$

associated to $\rho$, we have an injective homomorphism $\alpha$ as in (2.11). For every $x \in D$, fix an element $w_x \in \text{ad}(E_H)_x$.

**Lemma 3.4.** Assume that $H$ is reductive. There is a logarithmic connection on $E_H$ singular over $D$ with residue $w_x$ at each $x \in D$ if there is a logarithmic connection on $E_M$ singular over $D$ with residue $\alpha(x)(w_x)$ at each $x \in D$, where $\alpha$ is the homomorphism in (2.11).

**Proof.** The adjoint action of $H$ on $\mathfrak{h}$ makes it an $H$–module. On the other hand, the homomorphism $\rho$ composed with the adjoint action of $M$ on $\mathfrak{m}$ produces an action of $H$ on $\mathfrak{m}$. The injective homomorphism $d\rho$ in (3.4) is a homomorphism of $H$–modules. Since $H$ is reductive, there is an $H$–submodule $V \subset \mathfrak{m}$ which is a complement of $d\rho(\mathfrak{h})$, meaning

$$\mathfrak{m} = d\rho(\mathfrak{h}) \oplus V.$$ 

Let $\eta : \mathfrak{m} \rightarrow \mathfrak{h}$ be the projection constructed from this decomposition of $\mathfrak{m}$; in particular, we have $\eta \circ d\rho = \text{Id}_\mathfrak{h}$.

Since the above $\eta$ is a homomorphism of $H$–modules, it produces a projection

$$\widehat{\eta} : \text{ad}(E_M) \rightarrow \text{ad}(E_H)$$

such that $\widehat{\eta} \circ \alpha = \text{Id}_{\text{ad}(E_H)}$, where $\alpha$ is the homomorphism in (2.11).

Now if $\theta : \text{At}(E_M, D) \rightarrow \text{ad}(E_M)$ is a logarithmic connection on $E_M$ singular over $D$ with residue $\alpha(x)(w_x)$ at each $x \in D$, consider the composition

$$\widehat{\eta} \circ \theta \circ A : \text{At}(E_H, D) \rightarrow \text{ad}(E_H),$$

where $A$ is constructed in (2.12). Evidently, it is a logarithmic connection on $E_H$ singular over $D$ with residue $w_x$ at each $x \in D$. \qed

### 4. $T$–Rigid Elements of Adjoint Bundle

**4.1. Definition.** As before, $H$ is a complex affine algebraic group and $p : E_H \rightarrow X$ a holomorphic principal $H$–bundle on $X$. An automorphism of $E_H$ is a holomorphic map $F : E_H \rightarrow E_H$ such that

- $p \circ F = p$, and
- $F(zh) = F(z)h$ for all $z \in E_H$ and $h \in H$.

Let $\text{Aut}(E_H)$ be the group of all automorphisms of $E_H$. We will show that $\text{Aut}(E_H)$ is a complex affine algebraic group.

First consider the case of $H = \text{GL}(r, \mathbb{C})$. For a holomorphic principal $\text{GL}(r, \mathbb{C})$–bundle $E_{\text{GL}}$ on $X$, let $E := E_{\text{GL}} \times^{\text{GL}(r, \mathbb{C})} \mathbb{C}^r$ be the holomorphic vector bundle of rank $r$ on $X$ associated to $E_{\text{GL}}$ for the standard action of $\text{GL}(r, \mathbb{C})$ on $\mathbb{C}^r$. Then $\text{Aut}(E_{\text{GL}})$ is identified with the group of all holomorphic automorphisms $\text{Aut}(E)$ of the vector bundle $E$ over the
identity map of $X$. Note that $\text{Aut}(E)$ is the Zariski open subset of the complex affine space $H^0(X, \text{End}(E))$ consisting of all global endomorphisms $f$ of $E$ such that $\det(f(x_0)) \neq 0$ for a fixed point $x_0 \in X$; since $x \mapsto \det(f(x))$ is a holomorphic function on $X$, it is in fact a constant function. Therefore, $\text{Aut}(E_{\text{GL}})$ is an affine algebraic variety over $\mathbb{C}$.

For a general $H$, fix an algebraic embedding $\rho : H \hookrightarrow \text{GL}(r, \mathbb{C})$ for some $r$. For a holomorphic principal $H$–bundle $E_H$ on $X$, let $E_{\text{GL}} := E_H \times^\rho \text{GL}(r, \mathbb{C})$ be the holomorphic principal $\text{GL}(r, \mathbb{C})$–bundle on $X$ obtained by extending the structure group of $E_H$ using $\rho$. The injective homomorphism $\rho$ produces an injective homomorphism $\rho' : \text{Aut}(E_H) \hookrightarrow \text{Aut}(E_{\text{GL}})$.

The image of $\rho'$ is Zariski closed in the algebraic group $\text{Aut}(E_{\text{GL}})$. Hence $\rho'$ produces the structure of a complex affine algebraic group on $\text{Aut}(E_H)$. This structure of a complex algebraic group is independent of the choices of $r$, $\rho$. Therefore, $\text{Aut}(E_H)$ is an affine algebraic group. Note that $\text{Aut}(E_H)$ need not be connected, although the automorphism group of a holomorphic vector bundle is always connected (as it is a Zariski open subset of a complex affine space).

The Lie algebra of $\text{Aut}(E_H)$ is $H^0(X, \text{ad}(E_H))$. The group $\text{Aut}(E_H)$ acts on any fiber bundle associated to $E_H$. In particular, $\text{Aut}(E_H)$ acts on the adjoint vector bundle $\text{ad}(E_H)$. This action evidently preserves the Lie algebra structure on the fibers of $\text{ad}(E_H)$.

Let $\text{Aut}(E_H)^0 \subset \text{Aut}(E_H)$ be the connected component containing the identity element. Fix a maximal torus $T \subset \text{Aut}(E_H)^0$.

An element $w \in \text{ad}(E_H)_x$, where $x \in X$, will be called $T$–rigid if the action of $T$ on $\text{ad}(E_H)_x$ fixes $w$.

Consider the adjoint action of $H$ on itself. Let

$$\text{Ad}(E_H) := E_H \times^H H \longrightarrow X$$

be the associated holomorphic fiber bundle. Since this adjoint action preserves the group structure of $H$, the fibers of $\text{Ad}(E_H)$ are complex algebraic groups isomorphic to $H$. More precisely, each fiber of $\text{Ad}(E_H)$ is identified with $H$ uniquely up to an inner automorphism of $H$. The corresponding Lie algebra bundle on $X$ is $\text{ad}(E_H)$.

The group $\text{Aut}(E_H)$ is the space of all holomorphic sections of $\text{Ad}(E_H)$. For any $x \in X$, the action of $\text{Aut}(E_H)$ on the fiber $\text{ad}(E_H)_x$ coincides with the one obtained via the composition

$$\text{Aut}(E_H) \xrightarrow{\text{ev}_x} \text{Ad}(E_H)_x \xrightarrow{\text{ad}} \text{Aut}(\text{ad}(E_H)_x),$$

where $\text{ev}_x$ is the evaluation map that sends a section of $\text{Ad}(E_H)$ to its evaluation at $x$, and $\text{ad}$ is the adjoint action of the group $\text{Ad}(E_H)_x$ on its Lie algebra $\text{ad}(E_H)_x$.

Therefore, an element $w \in \text{ad}(E_H)_x$ is $T$–rigid if and only if the adjoint action of $\text{ev}_x(T) \subset \text{Ad}(E_H)_x$ on $\text{ad}(E_H)_x$ fixes $w$. 
4.2. **Examples.** The center of $H$ will be denoted by $Z_H$. Let $E_H$ be a holomorphic principal $H$–bundle on $X$. Since $Z_H$ commutes with $H$, for any $t \in Z_H$, the map $E_H \rightarrow E_H, z \mapsto zt$ is $H$–equivariant. Therefore, we have $Z_H \subset \text{Aut}(E_H)$. The principal $H$–bundle $E_H$ is called simple if $Z_H = \text{Aut}(E_H)$. Note that if $E_H$ is simple then every element of $\text{ad}(E_H)$ is $T$–rigid, where $T$ is any maximal torus in $\text{Aut}(E_H)^0$.

Let $H$ be connected reductive, and let $E_H$ be stable. Then $Z_H$ is a finite index subgroup of $\text{Aut}(E_H)$. Therefore, any maximal torus of $\text{Aut}(E_H)^0$ is contained in $Z_H$. This implies that every element of $\text{ad}(E_H)$ is $T$–rigid, where $T$ is any maximal torus in $\text{Aut}(E_H)^0$.

Take $E_H$ to be the trivial holomorphic principal $H$–bundle $X \times H$. Then the left–translation action of $H$ identifies $H$ with $\text{Aut}(E_H)$. Also, $\text{ad}(E_H)$ is the trivial vector $X \times h$, where $h$ is the Lie algebra of $H$. Let $T$ be a maximal torus of $H = \text{Aut}(E_H)$. Then an element $v \in h = \text{ad}(E_H)x$ is $T$–rigid if and only if $v \in \text{Lie}(T)$.

5. **A criterion for logarithmic connections with given residue**

5.1. **Logarithmic connections with $T$–rigid residue.** As in Section 2.1, $G$ is a connected reductive affine algebraic group defined over $\mathbb{C}$. Let $E_G$ be a holomorphic principal $G$–bundle over $X$. Fix a maximal torus $T \subset \text{Aut}(E_G)^0$, where $\text{Aut}(E_G)^0$ as before is the connected component containing the identity element of the group of automorphisms of $E_G$.

We now recall some results from [BBN], [BP].

As in (4.1), define the adjoint bundle $\text{Ad}(E_G) = E_G \times^G G$. For any point $y \in X$, consider the evaluation homomorphism

$$\varphi_y: T \rightarrow \text{Ad}(E_G)_y, \ s \mapsto s(y).$$

Then $\varphi_y$ is injective and its image is a torus in $G$ [BBN, p. 230, Section 3]. Since $G$ is identified with $\text{Ad}(E_G)_y$ uniquely up to an inner automorphism, the image $\varphi_y(T)$ determines a conjugacy class of tori in $G$; this conjugacy class is independent of the choice of $y$ [BBN, p. 230, Section 3], [BP, p. 63, Theorem 4.1]. Fix a torus

$$T_G \subset G$$

(5.1)

in this conjugacy class of tori. The centralizer

$$H := C_G(T_G) \subset G$$

(5.2)

of $T_G$ in $G$ is a Levi factor of a parabolic subgroup of $G$ [BBN, p. 230, Section 3], [BP, p. 63, Theorem 4.1]. The principal $G$–bundle $E_G$ admits a holomorphic reduction of structure group

$$E_H \subset E_G$$

(5.3)

to the above subgroup $H$ [BBN, p. 230, Theorem 3.2], [BP, p. 63, Theorem 4.1]. Since $T_G$ is in the center of $H$, the action of $T_G$ on $E_H$ commutes with the action of $H$, so $T_G \subset \text{Aut}^0(E_H)$ (this was noted in Section 4.2). The image of $T_G$ in $\text{Aut}^0(E_H)$ coincides
with $T$. This reduction $E_H$ is minimal in the sense that there is no Levi factor $L$ of some parabolic subgroup of $G$ such that

- $L \subset H$, and
- $E_G$ admits a holomorphic reduction of structure group to $L$.

(See [BBN, p. 230, Theorem 3.2].)

The above reduction $E_H$ is unique in the following sense. Let $L$ be a Levi factor of a parabolic subgroup of $G$ and $E_L \subset E_G$ a holomorphic reduction of structure group to $L$ satisfying the condition that $E_G$ does not admit any holomorphic reduction of structure group to a Levi factor $L'$ of some parabolic subgroup of $G$ such that $L' \subset L$. Then there is an automorphism $\varphi \in \text{Aut}(E_G)^0$ such that $E_L = \varphi(E_H)$ [BP, p. 63, Theorem 4.1]. In particular, the subgroup $L \subset G$ is conjugate to $H$.

The Lie algebras of $G$ and $H$ will be denoted by $\mathfrak{g}$ and $\mathfrak{h}$ respectively. The inclusion of $\mathfrak{h}$ in $\mathfrak{g}$ and the reduction in (5.3) together produce an inclusion $\text{ad}(E_H) \hookrightarrow \text{ad}(E_G)$. This subbundle $\text{ad}(E_H)$ of $\text{ad}(E_G)$ coincides with the invariant subbundle $\text{ad}(E_G)^T$ for the action of $T$ on $\text{ad}(E_G)$ [BBN, p. 230, Theorem 3.2], [BP, p. 61, Proposition 3.3] (this action is explained in Section 4.1), in other words,

\[
\text{ad}(E_H) = \text{ad}(E_G)^T \subset \text{ad}(E_G). \tag{5.4}
\]

For every $x \in D$ fix a $T$–rigid element

\[
w_x \in \text{ad}(E_G)_x \tag{5.5}
\]

(see Section 4.1). Since each $w_x$ is $T$–rigid, from (5.4) we conclude that

\[
w_x \in \text{ad}(E_H)_x \quad \forall \ x \in D. \tag{5.6}
\]

So $w_x$ determines a conjugacy class in $\mathfrak{h}$. For any character $\chi$ of $H$, the corresponding homomorphism of Lie algebras $d\chi : \mathfrak{h} \rightarrow \mathbb{C}$ factors through the conjugacy classes in $\mathfrak{h}$, because $\mathbb{C}$ is abelian. Therefore, we have $d\chi(w_x) \in \mathbb{C}$.

**Theorem 5.1.** There is a logarithmic connection on $E_G$ singular over $D$, and with $T$–rigid residue $w_x$ at every $x \in D$ (see (5.5)), if and only if

\[
\text{degree}(E_H(\chi)) + \sum_{x \in D} d\chi(w_x) = 0 \tag{5.7}
\]

for every character $\chi$ of $H$, where $E_H(\chi)$ is the holomorphic line bundle on $X$ associated to $E_H$ for $\chi$, and $d\chi$ is the homomorphism of Lie algebras corresponding to $\chi$.

**Proof.** Assume that there is a logarithmic connection on $E_G$ singular over $D$ such that the residue at each $x \in D$ is $w_x$. Since the group $H$ is reductive, from Lemma 3.4 it follows that $E_H$ admits a logarithmic connection singular over $D$ such that the residue at each $x \in D$ is $w_x$ (see (5.6)). Now from Lemma 3.3 we know that (5.7) holds for every character $\chi$ of $G$.

To prove the converse, assume that (5.7) holds for every character $\chi$ of $H$. We will show that $E_G$ admits a logarithmic connection singular over $D$ such that the residue at each $x \in D$ is $w_x$. 

Since $E_G$ is the extension of structure group of $E_H$ using the inclusion of $H$ in $G$ (see (5.3)), a logarithmic connection on $E_H$ induces a logarithmic connection on $E_G$ (this is explained in Section (2.4)). Therefore, in view of (2.13) and (5.6), the following proposition completes the proof of the theorem.

**Proposition 5.2.** There is a logarithmic connection on $E_H$ singular over $D$ such that the residue over any $x \in D$ is $w_x \in \text{ad}(E_H)_x$.

**Proof.** The connected component of the center of $H$ containing the identity element coincides with $T_G$ in (5.1). Define the quotient groups

$$S := H/T_G, \quad Z := H/[H, H].$$

So $S$ is semisimple, and $Z$ is a torus. The projections of $H$ to $S$ and $Z$ will be denoted by $p_S$ and $p_Z$ respectively. Let $E_S$ (respectively, $E_Z$) be the principal $S$–bundle (respectively, $Z$–bundle) on $X$ obtained by extending the structure group of $E_H$ using $p_S$ (respectively, $p_Z$). Consider the homomorphism

$$\varphi : H \rightarrow S \times Z, \quad h \mapsto (p_S(h), p_Z(h)). \quad (5.8)$$

It is surjective with finite kernel, hence it induces an isomorphism of Lie algebras. Let $E_{S \times Z}$ be the principal $S \times Z$–bundle on $X$ obtained by extending the structure group of $E_H$ using $\varphi$. Note that $E_{S \times Z} \cong E_S \times_X E_Z$. Since $\varphi$ induces an isomorphism of Lie algebras, we have

$$\text{ad}(E_H) = \text{ad}(E_{S \times Z}) = \text{ad}(E_S) \oplus \text{ad}(E_Z) \quad (5.9)$$

and

$$\text{At}(E_H) = \text{At}(E_{S \times Z}), \quad \mathcal{A} = \mathcal{A}_{E_{S \times Z}},$$

where $\mathcal{A}_{E_{S \times Z}}$ is constructed as in (5.1) for $(E_{S \times Z}, \{w_x\}_{x \in D})$ (see (5.9)). Consequently, $E_H$ admits a logarithmic connection singular over $D$, with residue $w_x$ for all $x \in D$, if and only if $E_{S \times Z}$ admits a logarithmic connection singular over $D$ with residue $w_x$ for all $x \in D$.

For $x \in D$, let

$$w_x = w_x^s \oplus w_x^z, \quad w_x^s \in \text{ad}(E_S)_x, \quad w_x^z \in \text{ad}(E_Z)_x$$

be the decomposition given by (5.9). Consider the short exact sequences

$$0 \rightarrow \text{ad}(E_S) \otimes \mathcal{O}_X(-D) \rightarrow \mathcal{A}_{E_S} \xrightarrow{\sigma_{1,S}} TX(-D) \rightarrow 0 \quad (5.10)$$

and

$$0 \rightarrow \text{ad}(E_Z) \otimes \mathcal{O}_X(-D) \rightarrow \mathcal{A}_{E_Z} \xrightarrow{\sigma_{1,Z}} TX(-D) \rightarrow 0, \quad (5.11)$$

as in Lemma 3.1 for the data $(E_S, \{w_x^s\}_{x \in D})$ and $(E_Z, \{w_x^z\}_{x \in D})$ respectively. Let

$$q_S : \mathcal{A}_{E_S} \oplus \mathcal{A}_{E_Z} \rightarrow \mathcal{A}_{E_S}, \quad q_Z : \mathcal{A}_{E_S} \oplus \mathcal{A}_{E_Z} \rightarrow \mathcal{A}_{E_Z}$$

be the projections. Note that

$$\mathcal{A}_{E_S} \oplus \mathcal{A}_{E_Z} \supset \text{kernel}(\sigma_{1,S} \circ q_S - \sigma_{1,Z} \circ q_Z) = \mathcal{A}_{E_S} \times_{TX(-D)} \mathcal{A}_{E_Z} = \mathcal{A}_{E_{S \times Z}}.$$

Therefore, giving a holomorphic splitting of the exact sequence

$$0 \rightarrow \text{ad}(E_{S \times Z}) \otimes \mathcal{O}_X(-D) \rightarrow \mathcal{A}_{E_{S \times Z}} \rightarrow TX(-D) \rightarrow 0$$
Let $E$ be the element corresponding to $\beta$ of $\text{ad}(\pi)$ such that (5.10) splits holomorphically.

Consider the homomorphism of character groups $\text{Hom}(Z, \mathbb{C}_m) \rightarrow \text{Hom}(H, \mathbb{C}_m)$ given by the projection $p_z$. It is an isomorphism because being semisimple $[H, H]$ does not admit any nontrivial character. Since $E_Z = E_H \times^{p_z} Z$, for any character $\chi \in \text{Hom}(Z, \mathbb{C}_m)$, the holomorphic line bundle $E_Z(\chi) = E_Z \times^x \mathbb{C}$ is identified with the holomorphic line bundle $E_H(\chi \circ p_Z)$.

A holomorphic line bundle $L$ on $X$ admits a logarithmic connection singular over $D$ with residue $\lambda_x \in \mathbb{C}$ for every $x \in D$ if and only if

$$\text{degree}(L) + \sum_{x \in D} \lambda_x = 0$$

(see [BDP, Lemma 2.3]). Since $Z = H/[H, H] = (\mathbb{C}_m)^d$ for some $d$, it follows that $E_Z$ admits a logarithmic connection singular over $D$ with residue $w_x^z$ at each $x \in D$ if and only if for each $1 \leq i \leq d$, the line bundle $E_Z(\pi_i)$ admits a logarithmic connection singular over $D$ with residue $d\pi_i(w_x^z)$ at each $x \in D$, where $\pi_i : Z = (\mathbb{C}_m)^d \rightarrow \mathbb{C}_m$ is the projection to the $i$-th factor. From this and the given condition in (5.7) we conclude that $E_Z$ admits logarithmic connection singular over $D$ with residue $w_x^z$ for all $x \in D$.

To complete the proof of the proposition we need to show that $E_S$ admits logarithmic connection singular over $D$ such that the residues over each $x \in D$ is $w_x^s$. We will show that (5.10) splits holomorphically.

Let

$$\beta \in H^1(X, \text{Hom}(TX(-D), \text{ad}(E_S) \otimes \mathcal{O}_X(-D)))) = H^1(X, \text{ad}(E_S) \otimes K_X)$$

be the extension class for (5.10) as in (3.3). The exact sequence in (5.10) splits holomorphically if and only if

$$\beta = 0.$$ (5.13)

The Lie algebra of $S$ will be denoted by $\mathfrak{s}$. Consider $\mathfrak{s}$ as a $S$–module using the adjoint action of $S$ on $\mathfrak{s}$. Since $S$ is semisimple, the Killing form

$$\kappa : \mathfrak{s} \times \mathfrak{s} \rightarrow \mathbb{C}, \quad (v, w) \mapsto \text{trace(ad}_v \circ \text{ad}_w),$$

is nondegenerate, where $\text{ad}_u(u') := [u, u']$. Therefore, the Killing form induces an isomorphism $\mathfrak{s} \rightarrow \mathfrak{s}^*$ of $S$–modules. This isomorphism produces a holomorphic isomorphism of $\text{ad}(E_S)$ with the dual vector bundle $\text{ad}(E_S)^*$. Now Serre duality gives

$$H^1(X, \text{ad}(E_S) \otimes K_X) = H^0(X, \text{ad}(E_S)^*)^* = H^0(X, \text{ad}(E_S))^*.$$ (5.12)

Let

$$\beta' \in H^0(X, \text{ad}(E_S))^*$$

be the element corresponding to $\beta$ (defined in (5.12)) by the above isomorphism. Then

$$\beta'(\gamma) = \int_X \kappa(\widehat{\beta}, \gamma), \quad \forall \gamma \in H^0(X, \text{ad}(E_S)), \quad (5.14)$$
where $\hat{\beta}$ is an $\text{ad}(E_S)$–valued $(1, 1)$–form on $X$ which represents the cohomology class $\beta$ using the Dolbeault isomorphism.

As before, $\text{Aut}(E_H)^0 \subset \text{Aut}(E_H)$ (respectively, $\text{Aut}(E_G)^0 \subset \text{Aut}(E_G)$) is the connected component containing the identity element, and $T \subset \text{Aut}(E_G)^0$ is the fixed maximal torus. Since $T$ is abelian, from [5.4] it follows immediately that

$$T \subset \text{Aut}(E_H)^0 \subset \text{Aut}(E_G)^0.$$ 

Therefore, the maximal torus $T \subset \text{Aut}(E_G)^0$ (see [5.1]) is also a maximal torus of $\text{Aut}(E_H)^0$. Since $T_G$ is the connected component, containing the identity element, of the center of $H$, and $T$ is the image of $T_G$ in $\text{Aut}(E_H)^0$, it now follows that the maximal torus of $\text{Aut}(E_S)^0$ is trivial. Hence every holomorphic section of $\text{ad}(E_S)$ is nilpotent.

Take any nonzero element $\gamma \in H^0(X, \text{ad}(E_S))$. Following the proof of [AB, Proposition 3.9], using $\gamma$ we construct a holomorphic reduction of the structure group of $E_S$ to a parabolic subgroup of $S$ as follows. For each $x \in X$, since $\gamma(x) \in \text{ad}(E_S)_x$ is nilpotent, there is a parabolic Lie subalgebra $\mathfrak{p}_x \subset \text{ad}(E_S)_x$ canonically associated to $\gamma(x)$ [AB, p. 340, Lemma 3.7]. Exponentiating $\mathfrak{p}_x$ we get a proper parabolic subgroup $P_x \subset \text{Ad}(E_S)_x$ associated to $\gamma(x)$. Since there are only finitely many conjugacy classes of nilpotent elements of $\mathfrak{s}$, and the algebraic subvariety of $\mathfrak{s}$ defined by the nilpotent elements has a natural filtration defined using the adjoint action of $S$, there is a finite subset $C \subset X$ such that the conjugacy classes of $P_x$, $x \in X \setminus C$, coincide. Fix a parabolic subgroup $P \subset S$ in this conjugacy class.

For any $x \in X \setminus C$, consider the projection map

$$\xi_x : (E_S)_x \times S \longrightarrow \text{Ad}(E_S)_x, \quad (z, s) \longmapsto \overset{\sim}{(z, s)},$$

where $\overset{\sim}{(z, s)}$ is the equivalence class of $(z, s)$. Define

$$(E_P)_x := \{ z \in (E_S)_x | \xi_x(z, g) \in P_x, \forall g \in P \}.$$ 

For the natural action of $S$ on $(E_S)_x$, the action of $P \subset S$ preserves $(E_P)_x$. Since $P$ is a parabolic subgroup of $S$, its normalizer $N_S(P)$ is $P$ itself [Hu, p. 143, Corollary B]. So the action of $P$ on $(E_P)_x$ is transitive (and also free, since the $G$–action on $(E_S)_x$ is free). Therefore, we have a holomorphic reduction of structure group $E_P \subset E_S$ to $P \subset S$ over $X \setminus C$. This holomorphic reduction defines a holomorphic section $\eta : X \setminus C \longrightarrow E_S/P$ which is meromorphic over $X$. Since $S/P$ is a projective variety, the above section $\eta$ extends holomorphically to a section $\overset{\sim}{\eta} : X \longrightarrow E_S/P$. This defines a holomorphic reduction of structure group $E_P \subset E_S$ to $P$.

From Corollary 3.2 it follows that the cohomology class $\beta$ in (5.12) lies in the image of the natural homomorphism $H^1(X, \text{ad}(E_P) \otimes K_X) \longrightarrow H^1(X, \text{ad}(E_S) \otimes K_X)$. Therefore, $\beta$ is represented by an $\text{ad}(E_P)$–valued $(1, 1)$–form $\overset{\sim}{\beta}$ on $X$.

Now for all $x \in X \setminus C$, the element $\gamma(x) \in \text{ad}(E_S)_x$ lies in the Lie algebra $\mathfrak{u}_x$ of the unipotent radical of $\text{ad}(E_P)_x$ [Hu, p. 186, Corollary A], and $\overset{\sim}{\beta}(x) \in \mathfrak{p}_x$. Therefore, $\kappa(\overset{\sim}{\beta}(x), \gamma(x)) = 0$, since $\mathfrak{u}_x$ is the orthogonal complement $(\text{ad}(E_P)_x)^\perp$ with respect to
the Killing form on $\text{ad}(E_S)_x$. Hence from (5.14) we have $\beta'(\gamma) = 0$. This proves (5.13), and completes the proof of the proposition.

As noted before, Proposition 5.2 completes the proof of Theorem 5.1.

5.2. $T$–invariant logarithmic connections with given residue. The automorphism group $\text{Aut}(E_G)$ has a natural action on the space of all logarithmic connections on $E_G$ singular over $D$. Given a maximal torus $T \subset \text{Aut}(E_G)^0$, by a $T$–invariant logarithmic connection we mean a logarithmic connection on $E_G$ singular over $D$ which is fixed by the action of $T$.

**Theorem 5.3.** Let $E_G$ be a holomorphic principal $G$–bundle on $X$, where $G$ is reductive. Fix $w_x \in \text{ad}(E_G)_x$ for each $x \in D$. Fix a maximal torus $T \subset \text{Aut}(E_G)^0$. The following two are equivalent:

1. There is a $T$–invariant logarithmic connection on $E_G$ singular over $D$ with residue $w_x$ at every $x \in D$.
2. The element $w_x$ is $T$–rigid for each $x \in D$, and (5.7) holds for every character $\chi$ of $H$.

**Proof.** Let $\theta$ be a $T$–invariant logarithmic connection on $E_G$ singular over $D$ with residue $w_x$ at every $x \in D$. Since $\theta$ is $T$–invariant, its residues are also $T$–invariant. Hence $w_x$ is $T$–rigid for each $x \in D$. From Theorem 5.1 we know that (5.7) holds for every character $\chi$ of $H$.

Now assume that the second statement in the theorem holds. From Theorem 5.1 we know that there is a logarithmic connection on $E_G$ singular over $D$ with residue $w_x$ at every $x \in D$.

As noted in Section 4.2, for a holomorphic principal $M$–bundle $E_M$ on $X$, the center $Z_M$ of $M$ is contained in the automorphism group $\text{Aut}(E_M)$. It is straight-forward to check that the action of $Z_M \subset \text{Aut}(E_M)$ on the space of all logarithmic connections on $E_M$ is trivial.

Since $T_G$ is contained in the center of $H$ (see (5.2)), and $T$ is the image of $T_G$ in $\text{Aut}(E_H)$, every logarithmic connection on the principal $H$–bundle $E_H$ in Proposition 5.2 is $T$–invariant. Consequently, from Proposition 5.2 it follows that $E_G$ admits a $T$–invariant logarithmic connection singular over $D$ with residue $w_x$ at every $x \in D$.

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