Extensions of the Schwarzschild solution into regions of non-zero energy density and pressure

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Abstract

We present solutions of the Einstein equations that extend the static Schwarzschild solution in empty space into regions of non-zero energy density $\rho$ and radial pressure $p = w\rho$, where $w$ is a constant equation of state parameter. For simplicity we focus mainly on solutions with constant $\rho$. For $w = 0$ we find solutions both with and without a singularity at the origin. We propose that our explicit non-singular solution with $w = -1$ describes the interior of a black hole, which is a form of vacuum energy. We verify that its entropy is consistent with the Bekenstein-Hawking entropy, however one needs to assume the Hawking temperature. We further suggest that this idea can perhaps be applied to the dark energy of the observable universe, if one views the latter as arising from black holes as pockets of vacuum energy. We estimate the average density of such a dark energy to be $\bar{\rho}_{de} \approx 10^{-30} \text{g/cm}^3$. We also present solutions with non-constant $\rho \propto 1/r^2$.

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I. INTRODUCTION

The Schwarzschild solution \[1\] to Einstein’s equations is based on a zero stress-energy tensor \(T_{\mu\nu}\). It is static and spherically symmetric, and the total mass is \(M\). It was eventually understood that it has an event horizon at the Schwarzschild radius \(r_s = 2MG/c^2\), and this initiated the development of the theory of black holes. The solution has a true singularity at the origin \(r = 0\). The standard point of view is that anything that falls into the black hole beyond \(r_s\) will eventually reach the singularity, and perhaps quantum gravity effects can resolve this singularity. It is not clear that this is the only possible resolution. It would thus be desirable to discover a classical resolution of the singularity within the context of General Relativity which predicts some properties of the interior of black holes, and this is the subject of this work.

Theories for the internal structure of black holes may even be testable now that gravitational waves from black hole mergers have been detected \[2\]. These experimental results were the main motivation for our work. There may be some signature of the internal structure of a black hole in the gravitational wave signal, however we will not address this complicated aspect here.

As a simplifying approximation, we introduce a length scale \(r_0\) such that stress-energy tensor satisfies

\[
\begin{align*}
T_{\mu\nu} &= 0 \text{ for } r > r_0, \\
T_{\mu\nu} &\neq 0 \text{ for } r \leq r_0.
\end{align*}
\]

(1)

Thus, for \(r > r_0\) the solution must be equal to the Schwarzschild solution with total mass \(M\). In finding solutions, one needs to match the Schwarzschild solution at \(r = r_0\). Although this is a rather straightforward approach, it’s not clear from the beginning that interesting exact solutions exist. However, we will present many such solutions, some of which extend to the interior of a black hole.

Let us summarize our main results. Henceforth we set the speed of light \(c = 1\). Planck’s constant \(\hbar \neq 0\) will only be relevant in Section \[VI\] in connection with the Hawking temperature. In the next section we present the Einstein equations in a form that we could not find in the literature and are easier to solve. We mainly focus on solutions with constant non-zero energy density \(\rho\) in the region \(r < r_0\). In Section \[IV\] we find solutions with zero radial pressure \(p\), which is commonly attributed to matter. We present solutions with and without singularities at \(r = 0\), the latter being simpler.
In Section V we consider non-zero radial pressure \( p = w \rho \) where \( w \) is a constant, focusing on non-singular solutions, and we present such solutions for arbitrary \( w \). The case \( w = -1 \) is especially interesting since the energy density and pressures are equivalent to vacuum energy, i.e. \( T_{\mu \nu} = -\rho \, g_{\mu \nu} \). In Section VI we study the black hole limit \( r_0 = r_s \). This solution has no singularity at the origin and does not have the peculiarity that the effective speed of light vanishes inside the black hole as was the case for \( w = 0 \). We propose that this is the correct equation of state of the interior of a black hole. As a check of this idea we show that the classical entropy of the black hole is consistent with the Bekenstein-Hawking entropy if one assumes that the temperature is the Hawking temperature up to a factor of two. The detailed nature of the matter inside the black hole that leads to \( w = -1 \) is not possible to address based on this work.

Having found these new solutions to the static Einstein equations, which are on solid mathematical ground, we proceed to try to apply them to cosmology. This necessarily should be viewed as more speculative since this is no longer a static situation. Nevertheless, our attempts are are in line with standard approaches to cosmology. For instance, in the current universe, which has a large matter component (about 30%), this matter content is treated as a fluid with zero pressure, where the “particles” are actual galaxies. However the galaxies are treated as static sources in spite of the expansion of the universe, in the sense that mergers of galaxies, accretion of galaxies, etc. are ignored, and one still obtains reliable predictions. Black holes are also not static in that they inevitably grow, but for the above reasons we will treat them as static sources of gravitation as an approximation. Having stated these caveats, in Section VII we propose that the currently observed non-zero dark energy arises from the pockets of vacuum energy inside black holes. This kind of dark energy is not a conventional cosmological constant because it changes with time due to the simple fact that the total energy density of black holes changes with time. We present this idea since the predictions are very good: we roughly estimate the average energy density \( \bar{\rho}_{de} \) of such a proposed dark energy to be \( \bar{\rho}_{de} \approx 10^{-30} \, g/cm^3 \) which is surprisingly close to the measured value (“de” here refers to dark energy); however this estimate is perhaps fortuitously too good (see the discussion below.)

Finally in Section VIII we present a solution with non-constant energy density \( \rho \propto 1/r^2 \). However we do not offer a potentially physical interpretation of this solution.
II. THE EINSTEIN EQUATIONS

The general spherically symmetric and static metric is defined by the line element

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -e^{2a}dt^2 + e^{2b}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \] (2)

Further calculations are largely based on Weinberg’s comprehensive book [3]. The non-zero Christoffel symbols are

\[
\begin{align*}
\Gamma^t_{tr} &= \Gamma^t_{rt} = a', \\
\Gamma^r_{rr} &= b', \\
\Gamma^r_{tt} &= a' e^{2(a-b)} \\
\Gamma^\varphi_r &= \Gamma^\varphi_r = 1/r \\
\Gamma^\theta_r &= \Gamma^\theta_r = 1/r^2 \\
\Gamma^\varphi_\theta &= -r e^{-2b} \\
\Gamma^\theta_\varphi &= -\sin \theta \cos \theta \\
\Gamma^\varphi_\varphi &= \cot \theta \\
\end{align*}
\] (3)

where \( a' \) denotes the derivative with respect to \( r \), i.e. \( a' = da/dr \), etc. The Einstein equations are

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \] (7)

where the non-zero components of \( G_{\mu\nu} \) are

\[
\begin{align*}
G_{tt} &= e^{2(a-b)} \left( \frac{2b'}{r} - \frac{1}{r^2} \right) + \frac{e^{2a}}{r^2} \\
G_{rr} &= \frac{2a'}{r} + \frac{1}{r^2} (1 - e^{2b}) \\
G_{\theta\theta} &= \frac{1}{\sin^2 \theta} G_{\varphi\varphi} = e^{-2b} r^2 \left( a'' + a'^2 - a'b' + (a' - b')/r \right) \\
\end{align*}
\] (8)

We take the following form for the stress-energy tensor:

\[
\begin{align*}
T_{tt} &= \rho e^{2a}, \\
T_{rr} &= p e^{2b} \\
T_{\theta\theta} &= \frac{1}{\sin^2 \theta} T_{\varphi\varphi} = p_\theta r^2 \\
\end{align*}
\] (9)

where \( \rho \) is the energy density and \( p, p_\theta \) are pressures. The metric factors \( e^{2a} \) and \( e^{2b} \) are necessary in the above equation: for instance for vacuum energy \( T_{tt} = -\rho g_{tt} \). The pressures in \( T_{\theta\theta} \) and \( T_{\varphi\varphi} \) must be identical due to the first equality in (10).

It will be convenient to rescale \( \rho \) and the pressures as follows

\[
\begin{align*}
\hat{\rho} &= 8\pi G \rho, \\
\hat{p} &= 8\pi G p, \\
\bar{p} &= 8\pi G p_\theta \\
\end{align*}
\] (10)
such that \( \hat{\rho}, \hat{p} \) and \( \tilde{p} \) all have dimensions of inverse length squared. (If \( \rho \) has units of mass per volume, then reintroducing \( c, \hat{\rho} \) has units of inverse time squared.) The Einstein equations now read

\[
\frac{2b'}{r} + \frac{1}{r^2} (e^{2b} - 1) = \hat{\rho} e^{2b} 
\]

(14)

\[
\frac{2a'}{r} - \frac{1}{r^2} (e^{2b} - 1) = \hat{p} e^{2b} 
\]

(15)

\[
a'' + a'^2 - a'b' + (a' - b')/r = \tilde{p} e^{2b}
\]

(16)

These equations are generally difficult to solve due to the second order equation (16). However one can replace the latter with a first order equation by differentiating (15) and using (14). One obtains a somewhat complicated equation, however it will turn out to be very useful:

\[
\tilde{p} = (4(r^2\hat{\rho} - 1))^{-1} \left[ 2r^2\hat{p}^2 (2r^2 - 1) + 2\hat{p} (r^2\hat{\rho}(1 + r^2) + r^2 - 2) + r \left( 2\hat{p}(b' + r^2\hat{p}') - 2\hat{p}' - r\hat{\rho}^2 \right) \right] 
\]

(17)

This equation can be understood as a statement of energy-momentum conservation since Bianchi identities ensure that \( T_{\mu\nu} \) is covariantly conserved \( D_{\mu}T_{\mu\nu} = 0 \)

We will refer to \( \hat{\rho} \) as the radial pressure and \( \tilde{p} \) as the orbital pressure. One novelty of our work is that the form of the above equations (14), (15), and (17) are easier to solve analytically compared to other equivalent versions in the literature. In particular, when \( \hat{\rho} \) and \( \hat{p} \) are specified, then equations (14) and (15) already determine solutions, up to constants of integration and the boundary condition at \( r = r_0 \); then the orbital pressure \( \tilde{p} \) is determined by (17).

III. THE SCHWARZSCHILD SOLUTION

To warm up, it is instructive to reproduce the known Schwarzschild solution from the above equations. Here, we assume a central point mass \( M \) at \( r = 0 \), and for \( r > 0 \) we have \( \hat{\rho} = \hat{p} = \tilde{p} = 0 \), namely \( r_0 = 0 \). Using (14) and (15) one finds

\[
2b = -\log \left( 1 - r_s/r \right), \quad 2a = \alpha + \log \left( 1 - r_s/r \right)
\]

(18)

where \( r_s \) and \( \alpha \) are constants of integration. Imposing that \( g_{\mu\nu} \) approaches the Minkowski empty space metric as \( r \to \infty \), and using \( g_{tt} = -e^{2a} \to -1 - 2\Phi(r) \) where \( \Phi(r) = -MG/r \)
is the Newtonian potential, one finds $\alpha = 0$ and

$$e^{2a} = \left(1 - \frac{r_s}{r}\right), \quad e^{2b} = \left(1 - \frac{r_s}{r}\right)^{-1}$$  \hspace{1cm} (19)

where $r_s$ is the Schwarzschild radius

$$r_s = 2MG$$  \hspace{1cm} (20)

It is important to note that the above solution depends on determining the constant of integration and that (16) is automatically satisfied due to the equivalent equation (17).

IV. SOLUTIONS WITH CONSTANT NON-ZERO ENERGY DENSITY, AND ZERO RADIAL PRESSURE

A. General Solutions ignoring the boundary conditions.

In this section we consider solutions with constant energy density $\rho$, and with zero radial pressure, i.e. $p = 0$, which is commonly associated with cold matter. We will ignore the boundary condition (1) at $r = r_0$ for the remainder of this sub-section, but will impose it later.

Since $\hat{\rho}$ has units of inverse length squared, let us define the scale $\ell$

$$\hat{\rho} \equiv 1/\ell^2, \quad \hat{p} = 0$$  \hspace{1cm} (21)

where $\ell$ has units of length and is constant in $r$. The solution to (14) and (15) is

$$2b = -\log \left(1 - \frac{r^2}{3\ell^2} - \frac{\beta\ell}{r}\right)$$  \hspace{1cm} (22)

$$2a = \alpha - \log(r/\ell) + \sum_{x\text{-roots}} \left(\frac{\log(x/\ell - r/\ell)}{1 - x^2/\ell^2}\right)$$  \hspace{1cm} (23)

where $\alpha$ and $\beta$ are constants of integration, and $x$-roots are the three roots of the cubic algebraic equation

$$3\beta - 3(x/\ell) + (x/\ell)^3 = 0$$  \hspace{1cm} (24)

B. Solution for all $r$ with no singularity at $r = 0$ and its Black Hole limit.

Let us first impose that there is no singularity in $b(r)$ at $r = 0$. Then $\beta = 0$ and the $x$-roots are simply

$$x/\ell = 0, \quad x/\ell = \pm\sqrt{3}$$  \hspace{1cm} (25)
By redefining the constant $\alpha$ one finds the solution

$$e^{2a} = \frac{C^2}{\sqrt{1 - \frac{r^2}{3\ell^2}}}, \quad e^{2b} = \left(1 - \frac{r^2}{3\ell^2}\right)^{-1}$$

where $C$ is a real constant.

Next we require that the above solution matches the Schwarzschild solution at $r = r_0$:

$$\left(1 - \frac{r_s}{r_0}\right)^{-1} = \left(1 - \frac{r_0^2}{3\ell^2}\right)^{-1} \quad (27)$$

$$1 - \frac{r_s}{r_0} = C^2 \left(1 - \frac{r_0^2}{3\ell^2}\right)^{-1/2} \quad (28)$$

The first equation is easily interpreted since it simply implies

$$\frac{r_s}{r_0} = \frac{r_0^2}{3\ell^2}, \quad \Rightarrow \quad \frac{4}{3} \pi r_0^3 \rho = M \quad (29)$$

Multiplying the two equations, one concludes

$$C^2 = \left(1 - \frac{r_0^2}{3\ell^2}\right)^{3/2} \quad (30)$$

Requiring $C^2$ to be real implies $r_0^2/2\ell^2 \leq 1$, and together with (29)

$$r_s \leq r_0. \quad (31)$$

Finally, using (17) one finds the orbital pressure has the simple expression:

$$\tilde{p} = \frac{1}{4\ell^2} \left(\frac{3\ell^2}{r^2} - 1\right)^{-1} \quad (32)$$

Recall there is no matter for $r > r_0$ where the solution is the usual Schwarzschild solution. The equation (31) implies that the Schwarzschild radius is generally inside the region with a distribution of matter where $r < r_0$, so that our solution does not represent the interior of a black hole, except possibly for $r_s = r_0$, which will be considered below. It is important to note that our solution is not singular at $r = 0$.

**Black Hole limit?** Consider the limiting case $r_s = r_0$, which is in principle allowed by (31), where $r_0$ approaches $r_s$ from above. In this case, all the matter is inside the Schwarzschild radius, and our solution arguably extends the Schwarzschild solution into the interior of the event horizon of a black hole, which is rather intriguing. At $r = 0$ the metric is

$$ds^2 = -C^2 dt^2 + dr^2 \quad (33)$$
where $C$ can be interpreted as an effective speed of light. However (29) and (30) imply that the effective speed of light $C = 0$. This would seem to be consistent with the fact that light cannot escape the event horizon, since it is slowed down to zero speed everywhere inside, namely, $C = 0$ for all $r < r_s$. It is as if time has stopped so no longer exists. Also note that the orbital pressure $\tilde{\rho} = 0$ at $r = 0$. However it is not entirely clear that $C = 0$ is physically sensible, and we think it actually is not; we will thus not deliberate further on this limit in this paper. Fortunately in Section V we will find a more physically appealing solution with a different equation of state, namely $p = w\rho$, with $w = -1$, as for vacuum energy.

C. Solutions with $\beta \neq 0$.

Let us now turn to solutions with a singularity at the origin where $\beta \neq 0$. The solutions are more complicated, but nevertheless have analytic expressions.

Define

$$\tilde{\beta} = \frac{3\beta}{2} \left( 1 - \sqrt{1 - \frac{4}{9\beta^2}} \right)$$

(34)

The roots to (24) are then

$$x_1/\ell = e^{i\pi/3} \tilde{\beta}^{1/3} + e^{-i\pi/3} \tilde{\beta}^{-1/3}$$

(35)

$$x_2/\ell = -\tilde{\beta}^{1/3} - \tilde{\beta}^{-1/3}$$

(36)

$$x_3/\ell = -e^{2i\pi/3} \tilde{\beta}^{1/3} - e^{-2i\pi/3} \tilde{\beta}^{-1/3}$$

(37)

We express the solution in the above form in order to properly keep track of branches.

In order for the metric to be real, we require that $\beta$ is real. There are two cases to consider:

Case 1. Here $|\beta| > 2/3$ and $\tilde{\beta}$ is real. In general the $x_i$ are complex, however we only require that $a(t)$ is real, which is compatible with complex $x_i$. For instance when $0 < \tilde{\beta} < 1$, then it turns out that $x_3 = x_1^*$ and $x_2$ is real, which implies that $a(r)$ in (23) is real. On the other hand when $\tilde{\beta} < 0$, $x_2 = x_1^*$ and $x_3$ is real so that again $a(r)$ is real. Incidentally, it turns out that if $0 < \tilde{\beta} < 1$ is rational, then so is $\beta$. For instance $\tilde{\beta} = 9/10$ corresponds to $\beta = 181/270$.

Case 2. Here $|\beta| < 2/3$, and

$$\tilde{\beta} = \frac{3\beta}{2} - i\sqrt{1 - 9\beta^2/4}$$

(38)
One sees that $|\tilde{\beta}| = 1$ such that $\tilde{\beta}$ is a pure phase. Let us parameterize it as

\[ \tilde{\beta} = e^{-3\pi i \gamma} \quad \gamma \equiv \frac{1}{3\pi} \arccos(3\beta/2) \quad (39) \]

A nice feature of this case is that the $x$-roots are real:

\[ \{x_1, x_2, x_3\}/\ell = \{2\cos\left(\pi(\gamma - \frac{1}{3})\right), -2\cos(\pi\gamma), 2\cos\left(\pi(\gamma + \frac{1}{3})\right)\} \quad (40) \]

These roots satisfy

\[ x_1 + x_2 + x_3 = 0 \quad (41) \]

The case $\beta = 0$ of the last section corresponds to $\gamma = 1/6$ and $\tilde{\beta} = -i$.

Cases 1 and 2 are separated by $\beta = 2/3$, where $\gamma = 0$ and $\{x_1, x_2, x_3\}/\ell = \{1, -2, 1\}$. The roots $x/\ell = 1$ lead to singularities in the expression (23) for $a(r)$, and won’t be further considered here.

Rather than attempt to solve all such cases, for illustrative and simplifying purposes, we will limit ourselves to Case 2 where all $x$-roots are real. Returning to (23), for each $x_i$ one has $\log((x_i - r)/\ell) = \log |(x_i - r)/\ell|$ or $\log |(x_i - r)/\ell| + i\pi$ depending on the sign of $(x_i - r)$. If $r$ is such that the $i\pi$ is required, then this just leads to a constant that can be absorbed into the constant $\alpha$. Consequently, defining

\[ \nu_i = \frac{1}{\left(1 - x_i^2/\ell^2\right)} \quad (42) \]

one finds the solution

\[ e^{2b} = \left(1 - \frac{r^2}{3\ell^2} - \frac{\beta\ell}{r}\right)^{-1} \quad (43) \]

\[ e^{2a} = C^2 \left(\frac{\ell}{r}\right)^3 \prod_{i=1}^{3} \left|1 - \frac{r}{x_i}\right|^{\nu_i} \quad (44) \]

Above, $C^2$ is required to be a real constant which implicitly depends on $\beta$. Matching the above solution to the Schwarzschild solution at $r = r_0$, one obtains

\[ \frac{r_s}{r_0} = \frac{r_0^3}{3\ell^2} + \frac{\beta\ell}{r_0} \quad (45) \]

\[ C^2 = \frac{r_0}{\ell} \left(1 - \frac{r_s}{r_0}\right)^3 \prod_{i=1}^{3} \left|1 - \frac{r_0}{x_i}\right|^{-\nu_i} \quad (46) \]
V. SOLUTIONS WITH NON ZERO RADIAL PRESSURE

Here we consider
\[ \hat{\rho} = 1/\ell^2, \quad \hat{p} = w \hat{\rho} \] (47)
where \( \ell \) and \( w \) and constants. In cosmology, for matter, radiation, and vacuum energy, \( w = 0, 1/3, -1 \) respectively, and these are still interesting cases in our context.

A. General \( w \)

The solution of the last section still applies with modified exponents \( \nu_i \). Namely (43) and (44) still apply, where the \( x_i \)satisfy the same cubic equation (24), however now
\[ \nu_i = 1 + w \frac{x_i^2}{\ell^2} \] (48)
Matching to the Schwarzschild solution at \( r = r_0 \), one again obtains (45) and (46) with these modified exponents \( \nu_i \).

For physical reasons, and also for simplicity, let us consider non-singular solutions in \( e^{2b} \) where \( \beta = 0 \). Then, as before, \( \{x_i\}/\ell = \{0, \pm \sqrt{3}\} \). One sees that the \( x_i = 0 \) root just cancels the log \( r \)'s in (23), such that the \( 1/r \) factor in (44) is cancelled by the \( x_i = 0 \) root.

One obtains
\[ e^{2b} = \left(1 - \frac{r^2}{3\ell^2}\right)^{-1}, \quad e^{2a} = C^2 \left|\frac{r^2}{3\ell^2}\right|^\nu, \quad \nu = -(1 + 3w)/2 \] (49)
Matching to the Schwarzschild solution at \( r = r_0 \), one finds
\[ \frac{r_s}{r_0} = \frac{r_0^2}{3\ell^2}, \quad C^2 = \left(1 - \frac{r_s}{r_0}\right) \left|\frac{r_s}{r_0}\right|^{-\nu} \] (50)

B. The special case \( w = -1 \)

Unless \( w = -1 \), \( C^2 \) equals 0 or \( \infty \) as \( r_0 \to r_s \). When \( w = -1 \), \( \nu = 1 \), and the above solution simplifies considerably. Remarkably, from (17) one obtains the non-trivial result from this complicated expression that \( \tilde{p} = -1/\ell^2 \). Thus when \( w = -1 \), all pressures are entirely consistent with vacuum energy, i.e. \( T_{\mu\nu} = -\rho g_{\mu\nu} \):
\[ \{\hat{\rho}, \hat{p}, \tilde{p}\} = \{1, -1, -1\}/\ell^2 \] (51)
In the next section, we will apply this solution to the interior of a black hole.
VI. THE INTERIOR OF A BLACK HOLE AS VACUUM ENERGY

A. The solution inside the event horizon

Consider the non singular solution of the last section ($\beta = 0$) with $w = -1$ where $\nu = 1$. Recall that by construction for $r > r_0$ the solution is the Schwarzschild one. Let $r_0 \rightarrow r_s$, approaching the limit from above. In this limit, all the matter is inside the event horizon at $r_s$ and can be interpreted as a black hole of mass $M$. It turns out that in this limit $C^2 = 1$, which is more physically sensible than $C^2 = 0$ in Section IV where $w = 0$. In fact, unless $w = -1$, $C^2$ is equal to either 0 or $\infty$, so that $w = -1$ is the only physically sensible possibility. Inside the black hole $r < r_s$ the solution is quite simple.

$$e^{2a} = 1 - \frac{r^2}{3\ell^2}, \quad e^{2b} = \left(1 - \frac{r^2}{3\ell^2}\right)^{-1} \quad (r < r_s)$$

where $r_s/r_0 = r_0^2/3\ell^2$, which just implies $r_0 \rightarrow r_s$. The original black hole singularity at $r = 0$ of the Schwarzschild solution no longer exists. Furthermore, inside the black hole, the energy and pressures are interpreted as vacuum energy due to (51).

B. Black Hole Entropy

Basic laws of thermodynamics, with zero chemical potential, imply

$$TdS = dU + p dV$$

Here $U$ is the internal energy, so that $U = \rho V$. One has $dU = \rho dV + d\rho V$ and the $\rho dV$ term is cancelled when $p = -\rho$. Thus $dS/d\rho = V/T$, which implies $S = V(\rho - \rho_0)/T$, where $\rho V = M$ is the mass of the black hole and $\rho_0$ is a constant of integration. Now, $M/T$ is a constant, and if $M = 0$, then nothing exists, and the entropy $S$ must be zero. Thus we take the integration constant to be proportional to $\rho$.

$$S = \kappa \frac{M}{T} = \kappa \frac{r_s}{2GT}$$

for some constant $\kappa$, which we cannot predict.

Thus far, our analysis has been purely classical with $\hbar = 0$. Although we cannot justify the following based on this classical analysis, let us nevertheless identify the temperature
\[ T = T_H \] where \( T_H \) is the Hawking temperature [6]

\[ T_H = \frac{\hbar}{k_B} \frac{1}{8\pi GM} \tag{55} \]

with \( k_B \) equal to Boltzmann’s constant. Identifying the area \( A = 4\pi r_s^2 \), one finds an entropy which is proportional to the area [5, 6]. For \( \kappa = 1/2 \), which corresponds to \( \rho_0 = \rho/2 \), one finds the Bekenstain-Hawking entropy

\[ S = \frac{k_B}{4\ell_p^2} A \tag{56} \]

where \( \ell_p = \sqrt{\hbar G} \) is the Planck length. One point of view is that one can chose the constant of integration \( \rho_0 \) such that \( \kappa = 1/2 \) in order to match with the Bekenstein-Hawking entropy; however as stated, we could not present arguments to justify this based on our classical analysis. Furthermore it is not possible to predict the nature of the matter inside the black hole beyond its equation of state \( w = -1 \).

VII. COULD THE DARK ENERGY OF THE OBSERVABLE UNIVERSE ORIGINATE FROM THE VACUUM ENERGY INSIDE BLACK HOLES?

In the last section, we have proposed that the interior of a black hole consists of vacuum energy. Just as the matter content of the universe is treated as a fluid, where the “particles” are individual galaxies, black holes can be considered as pockets of vacuum energy, and on average constitute a total vacuum energy density \( \rho_{\text{de}} \). One should distinguish between the local effect of black holes, which are classically point masses beyond the Schwarzschild radius, verses their global cosmological effects. We suggest that the latter can perhaps be interpreted as dark energy. In the following, we only make rough estimates.

A. The density of the proposed dark energy for the observable universe

Let \( \rho_{\text{de}} \) denote the total vacuum energy of the universe due to black holes at the current time. Then

\[ \rho_{\text{de}} = \frac{M_{\text{bh-total}}}{V_{\text{total}}} \tag{57} \]

where \( M_{\text{bh-total}} \) is the sum total of the masses of all black holes in the universe and \( V_{\text{total}} \) is the total volume of the universe. We can give a rough estimate of \( \rho_{\text{de}} \) as follows. The
average $\bar{\rho}_{\text{de}}$ is

$$\bar{\rho}_{\text{de}} = \frac{V_{\text{bh}}}{V_{\text{total}}} N_g \overline{N}_{\text{bh}} \bar{\rho}_{\text{bh}} = \frac{N_g \overline{N}_{\text{bh}} \overline{M}_{\text{bh}}}{V_{\text{total}}}$$ (58)

where $V_{\text{bh}}$ is the average volume of a black hole, $N_g$ is the total number of galaxies, $\overline{N}_{\text{bh}}$ is the average number of black holes per galaxy, $\bar{\rho}_{\text{bh}}$ is the average density of a black hole, and $\overline{M}_{\text{bh}}$ is the average mass of a black hole. If $\rho_{\text{total}}$ is the total energy density of the universe, then $\bar{\rho}_{\text{de}}/\rho_{\text{total}} = M_{\text{bh-total}}/M_{\text{total}} < 1$ where $M_{\text{total}}$ is the total mass of the universe. It needs to be emphasized that $\bar{\rho}_{\text{de}}$ in (58) is not constant in the evolution of the universe, and is thus not an ordinary cosmological constant.

The above formula (58) for $\bar{\rho}_{\text{de}}$ is compatible with the measured dark energy density $\rho_{\text{de}} = 0.7 \times 10^{-29}$ g/cm$^3$ [7, 8], for some reasonable estimates of the parameters. For instance, let us take the known estimates $N_g = 10^{11}$, $\overline{N}_{\text{bh}} = 10^8$ as in our own galaxy, and the estimate $V_{\text{total}} = 4 \times 10^{86}$ cm$^3$. The average black hole mass $\overline{M}_{\text{bh}}$ is harder to estimate since it can range from $10 M_\odot$ to $10^9 M_\odot$. Given this situation, it makes sense to take the geometric mean of these extreme limits. Thus we take $\overline{M}_{\text{bh}} = 10^5 M_\odot$. Then (58) gives $\bar{\rho}_{\text{de}} \approx 10^{-30}$ g/cm$^3$, which is surprisingly close to the measured value. However our estimates were rough approximations, so this apparently excellent agreement is likely to be fortuitous. In particular, our estimate for $\overline{M}_{\text{bh}}$ is perhaps over estimated, unless there are many as yet undetected intermediate mass black holes. In any case, more accurate estimates of $\bar{\rho}_{\text{de}}$ along the above lines are likely to give the same value within an order of magnitude or so.

**VIII. SOME SOLUTIONS WITH NON-CONSTANT ENERGY DENSITY**

For the previous sections, we considered constant energy density $\rho$. In this section we consider the case

$$\hat{\rho} = \frac{\sigma}{r^2}, \quad \hat{p} = 0$$ (59)

where $\sigma$ is a constant. One can find an exact solution for any $\sigma$, which we do not present here. The case $\sigma = 1$ is particularly simple:

$$e^{2b} = \frac{c_1 r}{\ell}, \quad e^{2a} = \left( \frac{c_2 \ell}{r} \right) e^{c_1 r/\ell}$$ (60)
where \( c_{1,2} \) are constants of integration, and \( \ell \) is a length scale. Matching to the Schwarzschild solution, one finds

\[
c_1 = \frac{\ell}{r_0} \left( 1 - \frac{r_s}{r_0} \right)^{-1}, \quad c_2 = \left( \frac{r_0}{\ell} - \frac{r_s}{\ell} \right) e^{-c_1 r_0 / \ell}
\]

Note that this solution is unavoidably singular at \( r = 0 \), and we don’t have much more to say about it here.

**IX. CONCLUDING REMARKS**

We already summarized our results and proposals in the Introduction, so let us just remark on some open questions, of which there are many.

★ We proposed that the interior of a black hole is vacuum energy. What is interesting about this proposal is that it has nothing to do with Quantum Mechanics nor Quantum Gravity. We only introduced \( \hbar \) in order to compare with the Bekenstein-Hawking entropy. Is there indeed a purely classical resolution of the original black hole singularity as we proposed? A related question is whether the temperature \( T \) in the black hole entropy formula \((54)\) is necessarily the Hawking temperature \( T_H \) which does depend on \( \hbar \). For \( \kappa = 1/2 \), or equivalently, \( T = 2T_H \), our proposed entropy agrees with the Bekenstein-Hawking entropy, however we could not justify this based on our classical analysis. String theory models suggest that the temperature should indeed equal \( T_H \) \[10\].

★ We have proposed that the vacuum energy inside black holes could perhaps explain the current dark energy density of the observable universe, which is a late time inflation. The vacuum energy density due to black holes is not constant in time. Could the same idea be applied to the very early universe, where the origin of inflation is a single primitive black hole? Certainly the energy density in the early universe was large enough for such a black hole to momentarily exist. We have some preliminary results in this direction \[9\].

★ Is there a physical application of the singular solutions for zero radial pressure presented in Section IV in particular those for Case 1, which were not studied in much detail?

★ Are there physically sensible solutions for the interior of a black hole where the matter is deeply inside, i.e. \( r_0 < r_s \)? Based on \((65)\) this may be possible for non-zero \( \beta \) where \( r_0^2 / 3\ell^2 > 1 - \beta \ell / r_0 \).
Since the measurement of gravitational waves from black hole mergers is now possible \[2\], is it feasible to detect any potential internal structure of a black hole, as of the kind proposed in this work?

**Note added:** After completing this article we were informed of previous works on non-singular black holes with negative pressure from a different approach. In particular we wish to cite the works \[11-14\] and references therein. We thank R. Brustein for pointing this out and for discussions.
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