APPLICATIONS OF ZERO-FREE REGIONS ON AVERAGES AND
SHIFTED CONVOLUTION SUMS OF HECKE EIGENVALUES

JISEONG KIM

ABSTRACT. By assuming Vinogradov-Korobov type zero-free regions and the generalized
Ramanujan-Petersson conjecture, we establish nontrivial upper bounds for almost
all short sums of Fourier coefficients of Hecke-Maass cusp forms for $SL(n, \mathbb{Z})$. As
applications, we obtain nontrivial upper bounds for the averages of shifted sums involving
coefficients of the Hecke-Maass cusp forms for $SL(n, \mathbb{Z})$. Furthermore, we present a
conditional result regarding sign changes of these coefficients.

Keywords. Automorphic form; Hecke eigenvalue; Shifted sum; Sign changes; Zero-free
region

1. Introduction

There are many interesting results regarding the behavior of multiplicative functions that
concern mean values within short intervals and their applications. For a recent overview
of these results, we refer the reader to [15] and [3]. The utilization of Hecke relations
reveal that the normalized Fourier coefficients $\{A_F(m, 1, ..., 1)\}$ of Hecke-Maass forms for
$SL(n, \mathbb{Z})$ (see Subsection 1.2) are multiplicative, and it is conjectured that these are divisor-
bounded as indicated by the generalized Ramanujan-Petersson conjecture (see Lemma
3.1). In addition, there have been extensive research on analytic properties of automor-
phic L-functions, which is useful for studying the behavior of $A_F(m, 1, ..., 1)$. Given this
background, one would seek to apply techniques from multiplicative number theory to au-
tomorphic L-functions. In this paper, we show that simple methods outlined in [13] and
[15] can be applied to study $\{A_F(m, 1, ..., 1)\}$ under the generalized Ramanujan-Petersson
conjecture and the $SL(n, \mathbb{Z})$ Vinogradov-Korobov zero-free regions (see subsection 1.3,
(1.24) respectively). One of the main results in this paper is as follows.

Theorem 1.1. Assume the generalized Ramanujan-Petersson conjecture and the $SL(n, \mathbb{Z})$
Vinogradov-Korobov zero-free regions. Let $X$ be sufficiently large, and let $e^{(\log X)^{1+\epsilon}} \ll \epsilon h_1 \leq h_2 \ll \epsilon X^{1-\epsilon}$ where $\epsilon > 0$ is fixed small. Then

$$\frac{1}{X} \int_X^{2X} \left| \frac{1}{h_1} \sum_{m=x}^{x+h_1} A_F(m, 1, ..., 1) - \frac{1}{h_2} \sum_{m=x}^{x+h_2} A_F(m, 1, ..., 1) \right|^2 \, dx \ll F, \epsilon (\log X)^{-2/3+\epsilon}.$$  

As applications, we obtain nontrivial upper bounds for shifted convolution sums involving $\{A_F(m, 1, ..., 1)\}$ and investigate sign changes of them. Before discussing our results,
let us review some relevant studies on holomorphic Hecke cusp forms.

1.1. Holomorphic Hecke cusp forms. Let $\mathbb{H} = \{z = x + iy : x \in \mathbb{R}, y \in (0, \infty)\}, G = SL(2, \mathbb{Z})$. Define $j_\gamma(z) = (cz + d)$ where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is
called a modular form of weight $k$ if it satisfies the modularity

$$f(\gamma z) = j_\gamma(z)^k f(z)$$
for all $\gamma \in SL(2, \mathbb{Z})$. It is well-known that $f(z)$ has a Fourier expansion at the cusp $\infty$ given by

$$f(z) = \sum_{n=0}^{\infty} b_n e(nz)$$

where $e(z) = e^{2\pi iz}$, and the normalized Fourier coefficients $a(n)$ of $f(z)$ are defined by

$$a(n) := b_n n^{-\frac{k-1}{2}}.$$  

The set of all modular forms of a fixed weight of $k$ forms a vector space, denoted by $M_k$. The set of modular forms belonging to $M_k$ that have a constant term of zero is denoted by $C_k$. For the details, see [5, Chapter 14]. The $n$th Hecke operator $T_n$ on $C_k$ is defined by

$$(T_n f)(z) := \frac{1}{n} \sum_{d|n} a_k \sum_{b \pmod{d}} f \left( \frac{az + b}{d} \right).$$

It is well-known that there exists an orthonormal basis $S_k$ of $C_k$, consisting of eigenfunctions of all the Hecke operators $T_n$. These eigenfunctions are called Hecke cusp forms. If $f$ is a Hecke cusp form, its eigenvalues $\lambda_f(n)$ of the $n$th Hecke operator satisfy

$$a(n) = a(1)\lambda_f(n),$$

and the function $n \mapsto \lambda_f(n)$ is multiplicative and real valued. For any Hecke cusp form $f$, it is known that

$$\sum_{n=1}^{X} \lambda_f(n) \ll_{f, \epsilon} X^{\frac{1}{12} + \epsilon}$$

for sufficiently large $X$ (see [17]). Therefore, when $x \in [X, 2X]$, we have

$$\sum_{n=x}^{x+h} \lambda_f(n) \ll \sum_{n=1}^{x+h} \lambda_f(n) \ll \sum_{n=1}^{x+h} \lambda_f(n) \ll_{f, \epsilon} X^{1/12 + \epsilon}$$

for $1 \leq h \leq X$.

Moreover, for any small fixed $\epsilon > 0$ and $h \gg X^{\epsilon}$, we can apply Shiu’s theorem (see Lemma 2.1) to deduce that

$$\sum_{n=x}^{x+h} \lambda_f(n) \ll \sum_{n=x}^{x+h} |\lambda_f(n)| \ll h \prod_{p \in \mathcal{P}} \left( 1 + \frac{|\lambda_f(p)|-1}{p} \right)$$

where $\mathcal{P}$ is the set of all primes. According to the Sato-Tate law, we have

$$\prod_{p \in \mathcal{P}} \left( 1 + \frac{|\lambda_f(p)|-1}{p} \right) \ll (\log X)^{\frac{1}{12}}$$

for sufficiently large $X$ (see [10], [20]). Note that $\frac{8}{3\pi}-1 \sim 0.1511$...

In [8], Jutila proved that for any fixed $\epsilon > 0$ and $1 \leq l \leq X^{2/3}$,

$$\sum_{n=1}^{X} \lambda_f(n)\lambda_f(n+l) \ll_{f, \epsilon} X^{2/3 + \epsilon}.$$  

We now apply the aforementioned bounds to get the following theorem.

**Theorem 1.2.** Let $\epsilon > 0$ be small fixed. Let $h, X$ be integers such that $1 \leq h \ll X^{2/3}$. Then

$$\frac{1}{h} \left| \sum_{n=x}^{x+h} \lambda_f(n) \right| \ll_{f, \epsilon} \max \left( h^{1/2}, X^{1/4 + \epsilon} \right)$$

for almost all $x \in [X, 2X]$, indicating that the proportion of integers in $[X, 2X]$ that satisfy the given inequality approaches 1 as $X$ tends to infinity.
Proof. To apply the Chebyshev inequality, let us consider

\begin{equation}
\frac{1}{X} \sum_{x=1}^{2X} \left| \frac{1}{h} \sum_{n=x}^{x+h} \lambda_f(n) \right|^2.
\end{equation}

By squaring out the inner sums, we can represent (1.8) as

\begin{equation}
\frac{1}{X h^2} \sum_{x=1}^{2X} \sum_{h=0}^{h} \sum_{n=x}^{x+h-l} (2 - \delta_0(l)) \lambda_f(n) \lambda_f(n + l)
\end{equation}

where $\delta_0(l) = 1$ for $l = 0$ and 0 otherwise. The diagonal terms ($l = 0$) in (1.8) contribute

\begin{equation}
\frac{1}{X h^2} \sum_{x=1}^{2X} \sum_{n=x}^{x+h} \lambda_f(n)^2.
\end{equation}

By using the fact that $\sum_{x=1}^{x} \lambda_f(n)^2 \sim c_f X$ for some constant $c_f$ (see [5, (14.56)]), (1.9) is bounded by $O_f(h^{-1})$. The off-diagonal terms ($1 \leq l \leq h$) in (1.8) contribute

\begin{equation}
\frac{1}{X h^2} \sum_{l=1}^{h} \sum_{x=X}^{x+h} \lambda_f(n) \lambda_f(n + l).
\end{equation}

Therefore, by applying Deligne’s bound $\lambda_f(n) \ll n^\epsilon$, (1.10) is bounded by

\begin{equation}
\frac{1}{X h^2} \left( \sum_{l=1}^{h} (h - l) \sum_{x=X}^{x+h-l} \lambda_f(n) \lambda_f(n + l) + \sum_{l=1}^{h} (h - l)^2 h^2 X^{2/3 + \epsilon} \right).
\end{equation}

By using (1.7), we have

\begin{equation}
\sum_{l=1}^{h} (h - l) \sum_{x=X}^{x+h-l} \lambda_f(n) \lambda_f(n + l) \ll \epsilon h^2 X^{2/3 + \epsilon}.
\end{equation}

Therefore, (1.10) is bounded by

\begin{equation}
O_f, \epsilon \left( X^{-1/3 + \epsilon} + h X^{\epsilon - 1} \right).
\end{equation}

By the assumption on the range of $h$, (1.9) is bounded by

\begin{equation}
O_f, \max \left( h^{-1}, X^{-1/3 + \epsilon} \right).
\end{equation}

By applying the Chebyshev inequality, the proof is completed. \hfill \Box

Note that the upper bound in Theorem 1.2 is better than the trivial bound provided by (1.6) when $h$ exceeds a certain power of $\log X$, and is better than the trivial bound provided by (1.5) when $h \ll X^{1/2 + \epsilon}$. It’s worth emphasizing that the upper bound for the shifted sums played a pivotal role in the proof of Theorem 1.2

1.2. Hecke-Maass cusp forms for $\text{SL}(n, \mathbb{Z})$. Let $h^n = GL(n, \mathbb{R})/(O(n, \mathbb{R}) \cdot \mathbb{R}^*)$, where $\mathbb{R}^*$ is the set of all invertible elements in $\mathbb{R}$. For any element $z$ in $h^n$, by the Iwasawa decomposition (see [2, Proposition 1.2.6]), it can be represented as $z = x \times y$, where

\begin{equation}
x = \begin{bmatrix}
1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\
1 & x_{2,2} & x_{2,3} & \cdots & x_{2,n} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
1 & x_{n-1,n} & 1 \\
1 & x_{n,n}
\end{bmatrix},
y = \begin{bmatrix}
y_1 y_2 \cdots y_{n-1} \\
y_1 y_{n-2} \cdots \\
\vdots \\
y_1
\end{bmatrix}
\end{equation}
with \( x_{i,j} \in \mathbb{R} \) and \( y_i > 0 \). Let \( v = (v_1, v_2, \ldots, v_{n-1}) \in \mathbb{C}^{n-1} \) and \( m = (m_1, m_2, \ldots, m_{n-2}, m_{n-1}) \in \mathbb{Z}^{n-1} \). Let

\[
M(m_1, m_2, \ldots, m_{n-1}) := \begin{bmatrix}
m_1 m_2 \cdots m_{n-2} m_{n-1} \\
\vdots & \ddots & m_1 \\
& & & m_1
\end{bmatrix},
\]

and let \( I_v(z) \) be defined by

\[
I_v(z) := \prod_{i=1}^{n-1} \prod_{j=1}^{n-1} y_i^{b_{i,j}} v_{ij}.
\]

For any \((n-1) \times (n-1)\) matrix \( \gamma \), define

\[
\gamma^+ = \begin{bmatrix} \gamma & 1 \\ \end{bmatrix}.
\]

For any ring \( B \), let \( U_n(B) \) be the set of all upper triangular matrices of the form

\[
\begin{bmatrix}
I_{r_1} & & \\
& I_{r_2} & \\
& & \ddots & \\
& & & I_{r_k} \\
\end{bmatrix}
\]

for some natural numbers \( r_1, r_2, \ldots, r_k \) such that \( \sum_{i=1}^{k} r_i = n \), where \( I_{r_i} \) denotes the \( r_i \times r_i \) identity matrix, and \( * \) denotes any arbitrary element in \( B \). Let \( \psi_m \) be the character of \( U_n(\mathbb{R}) \) defined by \( \psi_m(u) := e^{2\pi i (m_1 u_{1,2} + m_2 u_{2,3} + \cdots + m_{n-1} u_{n-1,n})} \)

\[
u = \begin{bmatrix} 1 & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\
& 1 & u_{2,3} & \cdots & u_{2,n} \\
& & \ddots & \ddots & \ddots \\
& & & 1 & u_{n-1,n} \\
\end{bmatrix}.
\]

For \( z \in h^n \) and \( m \neq (0, 0, \ldots, 0) \), the Jacquet-Whittaker function \( W_{Jacquet}(z; v, \psi_m) \) is defined by

\[
W_{Jacquet}(z; v, \psi_m) = \int_{U_n(\mathbb{R})} I_v(w_n \cdot u \cdot z) \psi_m(u) d^* u,
\]

where

\[
w_n = \begin{bmatrix} (-1) \left( \frac{m}{2} \right) \\
& \ddots \\
& & 1 \\
& & & 1 \\
\end{bmatrix}.
\]

Let \( F(z) \) be a Hecke-Maass form for \( SL(n, \mathbb{Z}) \) given by

\[
F(z) = \sum_{\gamma \in \Gamma(n-1, \mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-1}=0}^{\infty} \frac{A_F(m_1, m_2, \ldots, m_{n-1})}{\prod_{k=1}^{n-1} |m_k|^{\frac{\kappa(n-k)}{2}}} \times W_{Jacquet}(M^\gamma z; v, \psi_{1,1,\ldots,1,m_{n-1}}),
\]

(when \( F(z) \) is a non-zero Hecke-Maass cusp form of type \( v = (v_1, v_2, \ldots, v_{n-1}) \in \mathbb{C}^{n-1} \) for \( SL(n, \mathbb{Z}) \), we have the aforementioned expansion. see [2, Theorem 9.3.11]). We have the following multiplicative relations

\[
A_F(m, 1, \ldots, 1) A_F(m_1, m_2, \ldots, m_{n-1}) = \sum_{\prod_{k=1}^{n-1} c_k = m} A_F\left( \frac{m_1 c_1}{c_1}, \frac{m_2 c_1}{c_2}, \ldots, \frac{m_{n-1} c_{n-2}}{c_{n-1}} \right).
\]
For details, we refer the reader to [2, Chapter 5, Chapter 9].

1.3. The generalized Ramanujan-Petersson conjecture. In order to prove that \( \mathcal{A}_F(m,1,...,1) \) is divisor bounded (see Lemma 3.1), it is necessary to assume the generalized Ramanujan-Petersson conjecture. \( L_F(s) \) has an Euler product

\[
L_F(s) = \prod_{p \in \mathbb{P}} \left(1 - A_F(p,1) p^{-s} + A_F(1,p,1) p^{-2s} - \cdots + (-1)^{n-1} A_F(1,...,1,p) p^{-(n-1)s} + (-1)^n p^{-ns}\right)^{-1}
\]

(1.20)

for some \( a_{F,j}(p) \in \mathbb{C} \). The generalized Ramanujan-Petersson conjecture claims

\[
|a_{F,j}(p)| = 1. \tag{1.21}
\]

To study short averages of \( \mathcal{A}_F(m,1,...,1) \), we will utilize the following result, which provides the upper bound for the long average of these values.

Lemma 1.3 ([17]). For sufficiently large \( X \),

\[
\sum_{m=1}^{X} A_F(m,1,...,1) \ll F, \epsilon X^{\frac{n^2 n!}{2} + \epsilon}. \tag{1.22}
\]

1.4. Zero-free regions and shifted convolution sum. To establish results analogous to Theorem 1.2 on \( \mathcal{A}_F(m,1,...,1) \), it suffices to obtain certain upper bounds for the following shifted sums

\[
\sum_{m=1}^{X} A_F(m,1,...,1) \overline{A_F(m+l,1,...,1)}
\]

for \( l \in [1,X] \) where \( \overline{A_F(m+l,1,...,1)} \) is the complex conjugate of \( A_F(m+l,1,...,1) \). Obtaining nontrivial upper bounds or asymptotics of the above shifted sum has many applications. As far as the author is aware, such results do not yet exist in the literature when \( n \geq 3 \). As an alternative, let us consider successful approaches in multiplicative number theory. Let \( s = \sigma + it \) where \( \sigma, t \in \mathbb{R} \). Define

\[
L_F(s) := \sum_{m=1}^{\infty} \frac{A_F(m,1,...,1)}{m^s}.
\]

The generalized Riemann hypothesis claims that there are no nontrivial zeros of \( L_F(s) \) for \( \sigma > \frac{1}{2} \). The current state of knowledge of the zero-free region is, there are no nontrivial zeros of \( L_F(s) \) where

\[
\sigma > 1 - \frac{c_F}{\log |t + 3|}
\]

where \( c_F > 0 \) is a constant depending on the conductor of \( F \). In contrast, for the Riemann zeta function \( \zeta(s) \), it is known that there are no nontrivial zeros where

\[
\sigma > 1 - \frac{c_1}{(\log |t + 3|)^{2/3} \log \log |t + 3|}
\]

and \( |t| \gg 1 \) for some constant \( c_1 > 0 \). This zero-free region, characterized by the power of the logarithm in the denominator is \( 2/3 \), is called the Vinogradov-Korobov zero-free region. Using this zero-free region, it can be shown that for any \( A > 0 \) and any small fixed \( \epsilon > 0 \), we have

\[
\sum_{p=p}^{Q} \frac{1}{p^{1+\epsilon t}} \ll_A (\log X)^{-A} \quad \text{when} \quad t > (\log X)^{A+2}. \tag{1.23}
\]
for \( e^{(\log X)^{2/3+\varepsilon}} \ll P < Q \ll X \) (see [13, Lemma 2]). By applying the above inequality, the authors of [13] show that nontrivial results concerning the Liouville function in short interval can be obtained by using methods that are relatively simpler than the methods presented in [15] (note that the methods outlined in [15] are sophisticated and comprehensive, demonstrating improved results in cases where broad zero-free regions are not available). This motivates us to prove an analog of (1.23), under the assumption of the generalized Ramanujan-Petersson conjecture (see (1.21)) and the existence of \( c_F \) such that \( L_F(s) \) satisfies the following zero-free region (we will refer to this zero-free region as the \( SL(n, \mathbb{Z}) \) Vinogradov-Korobov zero-free region):

\[
\sigma > 1 - \frac{c_F}{(\log |t + 3|)^{2/3} \log \log |t + 3|}.
\]

**Remark 1.4.** Let \( L(s, \text{sym}^2 f_j) \) be the symmetric square \( L \)-function of \( j \)th Hecke-Maass cusp forms \( f_j \) (with the Laplacian eigenvalue \( \lambda_j := \frac{1}{4} + t_j^2, t_j > 0 \)) for \( SL(2, \mathbb{Z}) \) (for details, see [5, Chapter 15]). Define

\[
J = \{ j : t_j \leq T \log^2 T, D_j(s) \text{ has a zero in } \sigma \geq 1 - \eta, |t| \leq \log^3 T \}
\]

where \( L(s, \text{sym}^2 f_j) = \zeta(2s)D_j(s) \). Luo [11] proved that

\[
|J| \ll T^{1/5}
\]

if \( \eta \) is sufficiently small. A direct application of [21, Theorem 1.2] can show that almost all \( L \)-functions associated to Hecke-Maass cusp forms for \( SL(n, \mathbb{Z}) \) (with some restrictions on the size of analytic conductor) has the zero-free region \( \sigma > 1 - \alpha, t \gg 1 \) for sufficiently small \( \alpha > 0 \).

**Lemma 1.5.** Let \( \varepsilon > 0 \) be small fixed, and let \( A \) be sufficiently large. Assume the zero-free region (1.24) and the generalized Ramanujan-Petersson conjecture. Then for \( e^{(\log X)^{2\varepsilon}} \ll t \ll X \), we have

\[
\sum_{\substack{p < P_1 \leq P \leq P_2 \leq P_1 \log \log X}} A_F(p, 1, \ldots, 1) p^{1+it} \ll_F (\log X)^{-A}
\]

for all \( e^{(\log X)^{2/3+\varepsilon}} < P_1 < Q_1 < e^{(\log X)^{1-\varepsilon}} \).

**Proof.** The proof of this basically follows from the proof of [13, Lemma 2]. Let \( \kappa = \frac{1}{\log X}, T = \frac{|t|+1}{2} \). By using Perron’s Formula, we have

\[
\sum_{\substack{p < P_1 \leq P \leq P_2 \leq P_1 \log \log X}} A_F(p, 1, \ldots, 1) p^{1+it} = \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} \log L_F(s + 1 + it) \frac{Q_1^s - P_1^s}{s} ds + O(R_F)
\]

where

\[
R_F = \max \left( \sum_{m=1}^{Q_1} \frac{|A_F(m, 1, \ldots, 1)|}{m \max (1, T \log Q_m)} \left( \frac{Q_1}{m} \right)^\kappa, \sum_{m=1}^{4P_1} \frac{|A_F(m, 1, \ldots, 1)|}{m \max (1, T \log P_m)} \left( \frac{P_1}{m} \right)^\kappa \right)
\]

(see [19, Corollary 5.3]). Note that \( O_e(P_1^{-1+\varepsilon}) \) arises from a simple application of Lemma 3.1 \( (A_F(p^k, 1, \ldots, 1) \ll d_m(p^k)) \) to

\[
\sum_{\substack{p \leq 2, p \nmid [P_1, Q_1] \cap \mathbb{P}}} \frac{|A_F(p^k, 1, \ldots, 1)|}{p^{k(1+it)}}.
\]
By applying the upper bound (3.1) \((A_F(m,1,...,1) \ll \epsilon m^t)\), it is straightforward to show that \(R_F \ll_F, \epsilon Q_l^1/T\). Let \(\sigma_0 = \frac{\epsilon e^{\gamma}}{(\log X)^{1/2}(\log \log X)^{1/2}}\). By shifting the line of integration in (1.26) to \([-\sigma_0-iT,-\sigma_0+iT]\) and changing variables, we have

\[
\sum_{p \leq P_1} \frac{A_F(p,1,...,1)}{p^{1+it}} = \frac{1}{2\pi i} \int_{-iT}^{iT} \log L_F(1-\sigma_0+it+iu) \frac{Q_1^{-\sigma_0+iu} - P_1^{-\sigma_0+iu}}{-\sigma_0+iu} du + O_{F,\epsilon} \left(\frac{Q_l^1}{T}\right) + O_F \left(P_1^{-1}\right).
\]

Note that a standard application of the Borel-Caratheodory theorem gives (see [1, Section 4]),

\[
\log L_F(\sigma_1 + it + iu) \ll_F |\log (t + u + 2)|
\]

for \(\sigma_1 \in [1 - \sigma_0, 1 + \kappa]\). Therefore, by applying (1.28) to (1.27), (1.25) is bounded by \((\log X)^{-A}\).

**Remark 1.6.** Without assuming the zero-free region, the argument presented in the proof of Lemma 1.5 gives that for \(f_j\) where \(t_j \leq T \log^2 T\) and \((\log T)^3 > X\),

\[
\sum_{p \leq P_1} \frac{1 + \lambda_{f_j}(p^2)}{p^{1+it}} \ll_{f_j} (\log X)^{-A}
\]

holds for almost all \(j \in J\). Additionally, it can be directly deduced that almost all L-functions associated to Hecke-Maass cusp forms for \(SL(n,\mathbb{Z})\) satisfy (1.25).

In [6], Jääsaari showed that under the assumption of the generalized Lindelöf hypothesis and a weak version of the Ramanujan-Petersson conjecture, when \(F\) is a nontrivial Hecke-Maass cusp form for \(SL(n,\mathbb{Z})\),

\[
\frac{1}{X} \int_X^{2X} \left| \sum_{m=x}^{x+h} A_F(m,1,...,1) \right|^2 dx \sim B_F X^{1 - \frac{1}{n}}
\]

holds for some constant \(B_F\), where \(h \in [X^{1 - \frac{1}{n} + \epsilon}, X^{1 - \epsilon}]\) for some small fixed \(\epsilon > 0\). In the following subsection, we consider the wider range \(h \in [e^{\log X^{1 - \epsilon}}, X^{1 - \epsilon}]\).

**1.5. Main results.**

**The mean-square version of Theorem 1.1.** Let \(\epsilon > 0\) be small fixed, and let \(X\) be sufficiently large. Let \(e^{(\log X)^{1 - \epsilon}} \ll \epsilon h \ll X^{1 - \epsilon}\). Assume the generalized Ramanujan-Petersson conjecture on \(F\) and the \(SL(n,\mathbb{Z})\) Vinogradov-Korobov zero-free region (1.24). Then

\[
\frac{1}{X} \int_X^{2X} \left| \sum_{m=x}^{x+h} A_F(m,1,...,1) \right|^2 dx \ll_{F,\epsilon} h^2 (\log X)^{-2/3 + \epsilon}.
\]

This theorem implies the following corollaries.

**Corollary 1.7.** Let \(\epsilon > 0\) be small fixed, and let \(X\) be sufficiently large. Let \(e^{(\log X)^{1 - \epsilon}} \ll \epsilon h \ll X^{1 - \epsilon}\). Assume the generalized Ramanujan-Petersson conjecture on \(F\) and the \(SL(n,\mathbb{Z})\) Vinogradov-Korobov zero-free region (1.24). Then

\[
\frac{1}{h} \sum_{m=x}^{x+h} A_F(m,1,...,1) \ll_{F,\epsilon} (\log X)^{1/2 + \epsilon}
\]

for almost all \(x \in [X,2X]\).
Corollary 1.8. Let $\epsilon > 0$ be small fixed, and let $X$ be sufficiently large. Let $e^{(\log X)^{1-\epsilon}} \ll \epsilon, h \ll X^{1-\epsilon}$. Assume the generalized Ramanujan-Petersson conjecture on $F$ and the zero-free region (1.24). Then
\[
\frac{1}{Xh^2} \sum_{l=0}^{h} |h-l| \left( \sum_{m \leq X} A_F(m, 1, ..., 1)\overline{A_F(m + l, 1, ..., 1)} \right) \ll_{F, \epsilon} (\log X)^{-2/3+\epsilon}. 
\]
The $SL(n, \mathbb{Z})$ Sato-Tate conjecture, coupled with the generalized Ramanujan-Petersson conjecture, implies that for sufficiently large $X$,
\[
\prod_{p \leq X} \left( 1 + \frac{|A_F(p, 1, ..., 1)| - 1}{p} \right) \asymp (\log X)^{\kappa_n - 1}
\]
where
\[
\kappa_n := \frac{1}{n!(2\pi)^{n-1}} \int_{\theta_1, \theta_2, ..., \theta_n = 0} |e^{i\theta_1} + e^{i\theta_2} + ... + e^{i\theta_n}| \prod_{1 \leq j < k \leq n} \left| e^{i\theta_j} - e^{i\theta_k} \right|^2 d\theta_1 d\theta_2 ... d\theta_n - 1
\]
(see [7, (1.16)]). For example, $\kappa_3 \approx 0.8911, \kappa_4 \approx 0.8853$, and $\kappa_5 \approx 0.8863$. Therefore, by using (2.3) and (1.31), we have
\[
\frac{1}{h} \sum_{l=1}^{h} \sum_{m \leq X} A_F(m, 1, ..., 1)\overline{A_F(m + h, 1, ..., 1)} \ll X (\log X)^{2\kappa_n - 2}
\]
under the generalized Ramanujan-Petersson conjecture. By assuming the $SL(n, \mathbb{Z})$ Vinogradov-Korobov zero-free region instead of the Sato-Tate conjecture, when $\kappa_n > 5/6$, we can obtain a superior upper bound for the shifted sums compared to (1.32).

Corollary 1.9. Let $\epsilon > 0$ be small fixed and let $X$ be sufficiently large. Let $e^{(\log X)^{1-\epsilon}} \ll \epsilon, h \ll X^{1-\epsilon}$. Assume the generalized Ramanujan-Petersson conjecture on $F$ and the zero-free region (1.24). Then
\[
\frac{1}{X(2h-1)} \sum_{l=0}^{h} \left( \sum_{m \leq X} A_F(m, 1, ..., 1)\overline{A_F(m + l, 1, ..., 1)} \right) \ll_{F, \epsilon} (\log X)^{-1/3+\epsilon}. 
\]

Proof. The left-hand side of (1.33) can be represented as
\[
\frac{1}{X} \int_X^{2X} \left( \frac{1}{2h-1} \sum_{m \geq [x-h]} A_F(m, 1, ..., 1) \right) A_F([x], 1, ..., 1) dx.
\]
Note that
\[
\int_X^{2X} |A_F([x], 1, ..., 1)|^2 dx \ll \sum_{n \leq X} |A_F(n, 1, ..., 1)|^2 \ll F X
\]
(see [2, Remark 12.1.8]). By applying Hölder’s inequality, Theorem 1.7, and (1.35), we see that
\[
\int_X^{2X} \left( \frac{1}{2h-1} \sum_{m \geq [x-h]} A_F(m, 1, ..., 1) \right) A_F([x], 1, ..., 1) dx \ll F, X (\log X)^{-1/3+\epsilon}.
\]
Remark 1.10. If the zero-free region

\[
\sigma > 1 - \frac{c_1}{(\log |t| + 3)^\theta}
\]

holds for some constants \(c_F > 0\), \(\theta < 1\), then it is easy to show that we can replace \((\log X)^{-1/3+\epsilon}\) in Theorem 1.7 with \((\log X)^{\theta-1-\epsilon}\) (the cancellation \(\log P/\log Q\) comes from the truncation in Lemma 3.3). Therefore, we only need to replace \(P = e^{(\log X)^{2/3+\epsilon}}\) with \(P = e^{(\log X)^{\theta+\epsilon}}\) in the beginning of Section 2). Since the Vinogradov-Korobov zero-free region is proved for the Riemann zeta function, one could get similar cancellations on the symmetric square L-functions. For example, without assuming the zero-free region, one may apply (1.29).

We say that a multiplicative function \(f: \mathbb{N} \to \mathbb{R}\) has \(k\) sign changes in \([1, x]\) if there exist integers \(1 \leq m_1 < m_2 < \ldots < m_{k+1} \leq x\) such that \(\prod_{i=1}^{k+1} f(m_i) \neq 0\) and \(f(m_i)f(m_{i+1}) < 0\) for all \(i \leq k\). In [14], it has been demonstrated that when \(f(n)\) is negative for some \(n\), and \(f(n)\) is non-zero for a positive proportion, then, as \(\psi(x)\) tends toward infinity, almost every interval \([x, x + \psi(x)]\) has a sign change (see [14, Corollary 4]). As an application of (1.37), let us talk about the sign changes of \(\{A_F(m, 1, \ldots, 1)\}_{m=1}^X\). Since we do not know that \(A_F(m, 1, \ldots, 1)\) is non-zero for a positive proportion, we cannot apply the aforementioned result.

By assuming the Ramanujan-Petersson conjecture, it can be proved that

\[
\sum_{m \leq x} |A_F(m, 1, \ldots, 1)| \gg x (\log x)^{\frac{1}{2} - 1}.
\]

This lower bound can be derived from the work of Wirsing in [22] along with the following lower bound:

\[
\sum_{p \leq x} |A_F(p, 1, \ldots, 1)| \log p \geq \frac{1}{n} \sum_{p \leq x} |A_F(p, 1, \ldots, 1)|^2 \log p
\]

\[
= \frac{1}{n} (1 + o(1)) x
\]

(see [7, (1.15)]). When \(F\) is self-dual, \(A_F(m, 1, \ldots, 1) \in \mathbb{R}\) for all \(m \in \mathbb{N}\). Hence, for the self-dual Hecke-Maass cusp forms \(F\), if

\[
\sum_{m \leq x} A_F(m, 1, \ldots, 1) = o \left( \sum_{m \leq x} |A_F(m, 1, \ldots, 1)| \right)
\]

and

\[
\sum_{m \leq x} |A_F(m, 1, \ldots, 1)| \neq 0,
\]

then

\[
\sum_{m \leq x} (|A_F(m, 1, \ldots, 1)| \pm A_F(m, 1, \ldots, 1)) > 0,
\]

which indicates that \([x, x+h]\) has a sign change. Therefore, we will prove the following corollary in Section 3.

Corollary 1.11. Let \(\epsilon > 0\) be sufficiently small and let \(X\) be sufficiently large. Let \(F\) be a self-dual Hecke-Maass cusp form for \(SL(n, \mathbb{Z})\). Assume the generalized Ramanujan-Petersson conjecture on \(F\), the zero-free region (1.37) for some \(\theta < \frac{1}{n}\). Then the number of sign changes in \(\{A_F(m, 1, \ldots, 1)\}_{m=1}^X\) is \(\gg_{F, \epsilon} \frac{X}{(\log X)^{2/3+\epsilon}}\).

Remark 1.12. We use the generalized Ramanujan-Petersson conjecture to give the upper bounds of second moments, shifted convolution sums of Hecke eigenvalue squares over some subsets in long intervals.
Remark 1.13. Applying methods from [12] can deduce results similar to our results. The methods in [12] rely on understanding the relationship between values of multiplicative functions on $p$ and Dirichlet characters $p^it$ over primes $p$ instead of assuming the $SL(n, \mathbb{Z})$ Vinogradov-Korobov zero-free regions. The results from the above methods may hold for a wider range of $h$, but the resulting cancellations are weaker compared to our finding. Moreover, we lack a comprehensive understanding of the correlations between $A_F(p, 1, \ldots, 1)$ and $p^it$. Specifically, under the generalized Ramanujan-Petersson conjecture, the best possible upper bound for the left-hand side of (1.30) from the methods (see [12, Theorem 1.7]) is

$$(\log X)^{-\frac{0.1241(1)}{321+\frac{h}{d}}}.$$ 

Note that the above upper bound is larger than (1.32).

2. Lemmas

From now on, let $\epsilon$ be a small, fixed positive number, and the value of $\epsilon$ may differ from one occurrence to another. Let $P = e^{(\log X)^{2/3+\epsilon}}$ and $Q = e^{(\log X)^{1-\epsilon}}$. Let $S_d$ be the set of integers in $[X/d, 2X/d]$ having at least one prime factor in $[P, Q]$. We denote $S_d^\prime := \{n \in [X/d, 2X/d] : n \notin S_d\}$. For convenience we denote $S$ instead of $S_1$, $S'$ instead of $S_1'$. For simplicity, we use

$$\prod_p a_p, \sum_p a_p \text{ instead of } \prod_{p \in P} a_p, \sum_{p \in P} a_p$$

respectively, for any $a_p$. Let $d_n(m)$ be the $n$th divisor function (the coefficients of $m^{-s}$ of $\zeta(s)^n$). For convenience, let $\text{GVK}$ stand for the $SL(n, \mathbb{Z})$ Vinogradov-Korobov zero-free region (1.24) and let GRP stand for the generalized Ramanujan-Petersson conjecture. We will apply the following lemma to obtain upper bounds for mean values of multiplicative functions over intervals.

Lemma 2.1. (Variants of Shiu’s theorem) Let $\alpha, \beta \geq 1$, $\epsilon > 0$. Let $g$ be a non-negative multiplicative function such that

$$(2.1) \quad g(p^n) \leq \alpha^n \quad \text{for all } p \in P, n \in \mathbb{N},$$

$$g(n) \leq \beta n^{\frac{3}{400}} \quad \text{for all } n \in \mathbb{N}.$$ 

Then for $2 \leq X^\epsilon \leq Y \leq X, 1 \leq H \leq X$,

$$(2.2) \quad \frac{1}{Y} \sum_{n = X}^{X+Y} g(n) \ll_{\alpha, \beta, \epsilon} X \prod_{p = 1}^{X} \left(1 + \frac{g(p) - 1}{p}\right),$$

$$(2.3) \quad \sum_{l = 1}^{H} \sum_{n = X}^{X+Y} g(n)g(n + l) \ll_{\alpha, \beta, \epsilon} HY X \prod_{p = 1}^{X} \left(1 + \frac{g(p) - 1}{p}\right)^2.$$

Proof. See [16, Lemma 2.3].

By using Lemma 2.1, it is easy to check that typical numbers in $[X/d, 2X/d]$ are contained in $S_d$. By using (2.1), we have

$$(2.4) \quad \left|\{n \in [X, 2X] : n \notin S_d\}\right| \ll X \log P / \log Q^d \ll X (\log X)^{-1/3+\epsilon}/d.$$

Lemma 2.2. Let $h \geq 1$. For sufficiently large $X$,

$$(2.5) \quad \sum_{l = 1}^{h} \sum_{n = X}^{X+2X} 1_{S'}(n)1_{S'}(n + l) \ll hX (\log P / \log Q)^2$$

Proof. This arises from a direct application of (2.3) with $g(n) = 1_{S'}(n)$. □
The following lemma will be needed after Lemma 3.5.

**Lemma 2.3.** (Dirichlet mean value theorem) For any \( \{a_n\}_{n \in \mathbb{N}} \subset \mathbb{C} \),

\[
\int_0^T \left( \sum_{n=X}^{2X} \frac{a_n}{n^s} \right)^2 \, dt \ll (T + X) \sum_{n=X}^{2X} |a_n|^2.
\]

**Proof.** See [18, Theorem 9.2]. □

### 3. Proof of Theorems and Corollaries

The following lemma states that \( \mathcal{A}_F(m, 1, \ldots, 1) \) satisfies the divisor bound, under \( \text{GRP} \).

**Lemma 3.1.** Assume \( \text{GRP} \). Then for any \( k \in \mathbb{N} \) and \( p \in \mathbb{P} \),

\[
\mathcal{A}_F(p^k, 1, 1, \ldots, 1) \leq d_n(p^k).
\]

**Proof.** By the Euler product of \( L_F(s) \),

\[
\mathcal{A}_F(p^k, 1, 1, \ldots, 1) = \sum_{r_1, \ldots, r_n \geq 0} \prod_{1 \leq j \leq n} a_{F,j}(p)^{r_j}.
\]

By using \( \text{GRP} \), we have

\[
\sum_{r_1, \ldots, r_n \geq 0} \prod_{1 \leq j \leq n} a_{F,j}(p)^{r_j} \leq \sum_{r_1, \ldots, r_n \geq 0} 1 \leq d_n(p^k).
\]

Note that \( \mathcal{A}_F(p, 1, \ldots, 1) \leq d_n(p) = n \). In order to apply Lemma 2.1, we also need to show that for arbitrarily small fixed \( \epsilon > 0 \),

\[
A_F(m, 1, \ldots, 1) \ll \epsilon \cdot m^\epsilon.
\]

By the Euler product (1.20) and \( \text{GRP} \), we see that

\[
A_F(m, 1, \ldots, 1) \ll d_n(m).
\]

Given that \( d_n(m) \leq d_2(m)^n \), and utilizing \( d_2(m) \leq e^{(\log 2^{1+o(1)}) \frac{\log m}{\log \log m}} \), we can establish (3.1). By applying Lemma 2.1 along with the bound \( A_F(p, 1, \ldots, 1) \leq n \), we have

\[
\sum_{m=1}^{X} |A_F(m, 1, \ldots, 1)|^4 \ll X (\log X)^{n^4 - 1}.
\]

#### 3.1. Proof of Theorem 1.1

This subsection is devoted to proving Theorem 1.1. In order to prove Theorem 1.1, we need the following lemmas.

**Lemma 3.2.** For sufficiently large \( X \),

\[
\sum_{p=1}^{X} \frac{|A_F(p, 1, \ldots, 1)|^2}{p} = \sum_{p=1}^{X} \frac{1}{p} + O_F(1).
\]

**Proof.** The proof of this theorem is essentially in line with [5, Theorem 5.13]. Let \( \tilde{F} \) be the dual of \( F \). The Rankin-Selberg \( L \)-function is given by

\[
L_{F \times \tilde{F}}(s) = \sum_{n=1}^{\infty} \frac{|A_F(n, 1, \ldots, 1)|^2}{n^s},
\]

and it is meromorphic with a simple pole at \( s = 1 \). Furthermore, there exists a constant \( d_F > 0 \) such that

\[
L_{F \times \tilde{F}}(\sigma + it) \neq 0 \]
for $\sigma > 1 - \frac{d_F}{(\log |t| + 3)}$ with $|t| \gg 1$ (see [4]). Therefore, by using [5, Theorem 5.13], we have

$$\sum_{p=1}^{X} |A_F(p, 1, ..., 1)|^2 \Lambda(p) = x + o_F(x)$$

where $\Lambda$ is the von-Mangoldt function. Hence, applying partial summation over $p$ gives (3.3).

\[\Box\]

**Lemma 3.3.** Assume GRP and GVK. Let $(\log X)^{n^4 - 1/3 - \epsilon} \ll \epsilon h = o(X)$. Then

$$\int_X^{2X} \left| \frac{1}{h} \sum_{m=x}^{x+h} A_F(m, 1, ..., 1) \sigma(m) \right|^2 dx \ll_F \epsilon X (\log X)^{-2/3+\epsilon}.$$  

**Proof.** By using the Cauchy-Schwarz inequality, we see that

$$\int_X^{2X} \left| \frac{1}{h} \sum_{m=x}^{x+h} A_F(m, 1, ..., 1) \sigma(m) \right|^2 dx \ll \frac{1}{h^2} \int_X^{2X} \left| \sum_{m=x}^{x+h} \sigma(m) \right|^2 dx \ll \frac{1}{h^2} \left( \int_X^{2X} \left( \sum_{m=x}^{x+h} \sigma(m) \right)^2 \right)^{1/2} \left( \int_X^{2X} \left( \sum_{m=x}^{x+h} |A_F(m, 1, ..., 1) \sigma(m)|^2 \right)^2 \right)^{1/2}.$$  

By squaring out the inner sum over $m$ and using Lemma 2.2, we have

$$\int_X^{2X} \left( \sum_{m=x}^{x+h} \sigma(m) \right)^2 dx \ll h \left( \sum_{l=1}^{h} \sum_{m=X}^{X+2h} \sigma(m) \sigma(m+l) + \sum_{m=X}^{X+2h} \sigma(m) \right)$$

$$\ll h^2 X \left( \frac{\log P}{\log Q} \right)^2 + hX \frac{\log P}{\log Q}$$

$$\ll \epsilon h^2 X (\log X)^{-2/3+2\epsilon}.$$  

In addition, by squaring out the last integrand in (3.7), we see that

$$\int_X^{2X} \left( \sum_{m=x}^{x+h} |A_F(m, 1, ..., 1) \sigma(m)|^2 \right)^2 dx \ll h \sum_{m=x}^{X+2h} |A_F(m, 1, ..., 1) \sigma(m)|^4$$

$$+ h \sum_{l=1}^{h} \sum_{m=X}^{X+2h} |A_F(m, 1, ..., 1) \sigma(m)|^2 |A_F(m+l, 1, ..., 1) \sigma(m+l)|^2.$$  

By applying Lemma 2.1 and 3.2, we have

$$h \sum_{l=1}^{h} \sum_{m=X}^{X+2h} |A_F(m, 1, ..., 1) \sigma(m)|^2 |A_F(m+l, 1, ..., 1) \sigma(m+l)|^2 \ll_F \epsilon h^2 (2X+h)(\log X)^{-2/3+2\epsilon}.$$  

Hence, by using the assumption $(\log X)^{n^4 - 1/3 - \epsilon} \ll \epsilon h = o(X)$ and (3.2), we see that

$$\int_X^{2X} \left| \sum_{m=x}^{x+h} A_F(m, 1, ..., 1) \sigma(m) \right|^2 dx \ll_F \epsilon h^2 X (\log X)^{-2/3+\epsilon}.$$  

The proof is completed. \[\Box\]

**Remark 3.4.** In order to obtain results similar to (3.6), one may avoid assuming GRP by employing certain large sieve techniques when $h$ is bigger than $X^\epsilon$. However, the resulting upper bounds will be weaker than (3.6) (see [5, Theorem 7.14]).
Lemma 3.5. (Parseval’s bound) Let $X$ and $A$ be sufficiently large. Let $e^{(\log X)^{1+\epsilon}} \ll h_1 \leq h_2 = O_{\epsilon}(X^{1-\epsilon})$. Then

$$
\frac{1}{X} \int_X^{2X} \left| \frac{1}{h_1} \sum_{m=x}^{x+h_1} A_F(m, 1, \ldots, 1)S(m) - \frac{1}{h_2} \sum_{m=x}^{x+h_2} A_F(m, 1, \ldots, 1)S(m) \right|^2 \, dx
$$

$$
\ll \max_{i=1,2} \left( \int_{e^{(\log X)^{1+\epsilon}}}^{Xh_i^{-1}} \left| \sum_{m=X}^{2X} \frac{A_F(m, 1, \ldots, 1)S(m)}{m^{1+it}} \right|^2 \, dt + \max_{T > Xh_i^{-1}} \int_T^{2T} \left| \sum_{m=X}^{2X} \frac{A_F(m, 1, \ldots, 1)S(m)}{m^{1+it}} \right|^2 \, dt \right)
$$

$$
+ (\log X)^{-A}.
$$

Proof. The proof of this basically follows from [15, Lemma 14]. The 1-bounded condition in [15, Lemma 14] can be readily eliminated (as shown in [9, Lemma 2.1]).

Let $K$ be the set of integers $m \in S$ such that $p^2 + m$ for all $p \in [P, Q] \cap \mathbb{P}$, and let $K^c = \{ m \in S : m \notin K \}$. Then

$$
\sum_{m=X}^{2X} \frac{A_F(m, 1, \ldots, 1)S(m)}{m^{1+it}} = \sum_{m=X}^{2X} \frac{A_F(m, 1, \ldots, 1)S(m)}{m^{1+it}} + \sum_{m=X}^{2X} \frac{A_F(m, 1, \ldots, 1)S(m)}{m^{1+it}}.
$$

The first summand on the right-hand side of (3.11) can be represented as

$$
\sum_{p=P}^{Q} \sum_{m=1 \atop p|m}^{p^{1+it}} \frac{A_F(m, 1, \ldots, 1)}{m^{1+it}} \sum_{1 \atop p|m}^{p^{1+it}} \frac{1}{|\{ q \in [P, Q] \cap \mathbb{P} : q|m \}| + 1}
$$

(see [5, section 17.3], or [15, (9)]).

Lemma 3.6. Assume GRP. Let $X, T$ be sufficiently large. Then

$$
\int_0^T \left| \sum_{m=X}^{2X} \frac{A_F(m, 1, \ldots, 1)S(m)}{m^{1+it}} \right|^2 \, dt \ll_F \frac{(T/X + 1)(\log X)^{n-1}}{P^{1/2}}.
$$

Proof. By applying Lemma 2.3 and the Cauchy-Schwarz inequality,

$$
\int_0^T \left| \sum_{m=X}^{2X} \frac{A_F(m, 1, \ldots, 1)S(m)}{m^{1+it}} \right|^2 \, dt \ll \frac{(T + O(X))}{X^2} \sum_{m \in K^c} |A_F(m, 1, \ldots, 1)|^2 1_S(m)
$$

$$
\ll \frac{(T + O(X))}{X^2} \left( \sum_{m=X}^{2X} |A_F(m, 1, \ldots, 1)|^2 \right)^{1/2} \left( \sum_{m \in K^c} 1_S(m) \right)^{1/2}.
$$

By the definition of $K^c$, we see that

$$
\sum_{m \in K^c} 1_S(m) \ll \sum_{p \in P} \sum_{m=1 \atop p|m}^{p^{1+it}} 1 \ll X \sum_{p \in P} \frac{1}{p^{1+it}} \ll \frac{X}{P}.
$$

Therefore, by combining (3.2) and (3.14), the last term in (3.13) is bounded by

$$
\ll_F \frac{(T/X + 1)(\log X)^{n-1}}{P^{1/2}}.
$$
Lemma 3.7. Assume GRP. Let $X$ be sufficiently large, and let $2 \leq H \ll X^{1-\varepsilon}$. For $v \in I := [H \log P - 1, H \log Q]$, define

\begin{equation}
\mathcal{N}_{v,H}(s) := \sum_{p = 1}^{e^{\frac{v}{X}}} A_F(p, 1, \ldots, 1) \frac{p^s}{p^s} \tag{3.15}
\end{equation}

\begin{equation}
\mathcal{R}_{v,H}(s) := \sum_{m = X}^{2X} A_F(m, 1, \ldots, 1) \frac{1}{m^s} \left| \left\{ q \in [P, Q] : q \mid m \right\} \right| + 1.
\end{equation}

Then

\begin{equation}
\int_{e^{(\log X)^2}}^{T} \left| \sum_{m = X}^{2X} A_F(m, 1, \ldots, 1) 1_{S}(m) \right|^2 \frac{dt}{m^{1+it}} \ll_{F,e} \frac{T + X}{H X} (\log X)^{n^2 - 1}
\end{equation}

\begin{equation}
+ \frac{T}{X \log X} (\log X)^{n^2 - 1} + H \log \frac{Q}{P} \sum_{j \in I} \int_{e^{(\log X)^2}}^{T} \left| N_{j,H}(1 + it) \right|^2 \mathcal{R}_{j,H}(1 + it) \frac{dt}{m^{1+it}}.
\end{equation}

Proof. The proof of this basically follows from [14, Lemma 12]. By applying (3.11), (3.12), we see that

\begin{equation}
\sum_{m = X}^{2X} A_F(m, 1, \ldots, 1) 1_{S}(m) = \sum_{j \in I} N_{j,H}(1 + it) \mathcal{R}_{j,H}(1 + it) + \sum_{m = X}^{2X} \frac{b(m)}{m^{1+it}}
\end{equation}

\begin{equation}
+ \sum_{m = X}^{2X} \frac{c(m)}{m^{1+it}}
\end{equation}

\begin{equation}
\left| \sum_{m = X}^{2X} A_F(m, 1, \ldots, 1) 1_{S}(m) \right|^2 - \sum_{j \in I} N_{j,H}(1 + it) \mathcal{R}_{j,H}(1 + it).
\end{equation}

By using Lemma 2.1, we have

\begin{equation}
\sum_{m = X}^{2X} \frac{b(m)}{m^{1+it}} \ll_{F,e} \frac{e^{\frac{\pi}{X^2}}}{X^2} \sum_{m = X}^{2X} |A_F(m, 1, \ldots, 1)|^2
\end{equation}

\begin{equation}
\ll_{F,e} \frac{e^{\frac{\pi}{X^2}}}{X^2} \frac{X}{H} (\log X)^{n^2 - 1}.
\end{equation}

Therefore, by using Lemma 2.3, we see that

\begin{equation}
\int_{e^{(\log X)^2}}^{T} \left| \sum_{m = X}^{2X} b(m) \right|^2 \frac{dt}{m^{1+it}} \ll (T + X) \sum_{m = X}^{2X} \frac{b(m)}{m}
\end{equation}

\begin{equation}
\ll (T + X) \sum_{m = X}^{2X} \frac{b(m)}{m}
\end{equation}

\begin{equation}
\ll_{F,e} \frac{T}{X \log X} (\log X)^{n^2 - 1}.
\end{equation}
By using (3.2) and Lemma 2.3, we have
\[
\int_{e^{(\log X)^2}}^{T} \left| \sum_{\substack{Q \leq P \leq X \atop P^k \equiv m \pmod{m+1}}} \frac{c(m)}{m+1} \right|^2 dt \ll \frac{T + X}{X^2} \left( \sum_{m=1}^{2X} \left| c(m) \right|^2 \right)
\]
(3.19)
\[
\ll_{F} \frac{T + X}{X^2} \left( \frac{X}{P} \right)^{1/2} \left( \sum_{m=1}^{2X} \left| A_F(m, 1, ..., 1) \right|^4 \right)^{1/2}
\]
\[
\ll_{F} \frac{T + X (\log X)^{A}}{P^{1/2}}.
\]

Finally, by applying the Cauchy-Schwarz inequality, we see that
\[
\int_{e^{(\log X)^2}}^{T} \left| \sum_{j \in I} N_{j,H}(1 + it) R_{j,H}(1 + it) \right|^2 dt \ll (H \log Q - H \log P) \sum_{j \in I} \int_{e^{(\log X)^2}}^{T} \left| N_{j,H}(1 + it) \right|^2 |R_{j,H}(1 + it)|^2 dt.
\]
(3.20)
\]

Let us consider
\[
\sum_{j \in I} \int_{e^{(\log X)^2}}^{T} \left| N_{j,H}(1 + it) \right|^2 |R_{j,H}(1 + it)|^2 dt.
\]
(3.21)

For $e^{(\log X)^2} \ll T \ll X$, by using Lemma 1.5, we have $|N_{j,H}(1 + it)| \ll_{F,A} (\log X)^{-A}$. Therefore, (3.21) is bounded by
\[
(\log X)^{-2A} \sum_{j \in I} \int_{e^{(\log X)^2}}^{T} \left| R_{j,H}(1 + it) \right|^2 dt
\]
for $e^{(\log X)^2} \ll T \ll X$. Using Lemma 2.2, Lemma 2.3 and (3.3), we see that
\[
\int_{e^{(\log X)^2}}^{T} \left| R_{j,H}(1 + it) \right|^2 dt \ll_{F} (T + X e^{-\frac{\pi}{\Phi}})/(X e^{-\frac{\pi}{\Phi}}).
\]

Therefore, (3.21) is bounded by
\[
(3.22) \quad H \log \frac{Q}{P} (\log X)^{-2A} \left( \frac{TQ}{X} + 1 \right).
\]

3.2. Proof of Theorem 1.1.

Proof. Let us split the integrand in (1.1) by
\[
\left( \frac{1}{h_1} \sum_{m=x}^{x+h_1} A_F(m, 1, ..., 1) S(m) - \frac{1}{h_2} \sum_{m=x}^{x+h_2} A_F(m, 1, ..., 1) S(m) \right)
\]
\[
+ \left( \frac{1}{h_1} \sum_{m=x}^{x+h_1} A_F(m, 1, ..., 1) S'(m) - \frac{1}{h_2} \sum_{m=x}^{x+h_2} A_F(m, 1, ..., 1) S'(m) \right).
\]

By using Lemma 3.5, we see that
\[
(3.23) \quad \frac{1}{X} \int_{X}^{2X} \left| \frac{1}{h_1} \sum_{m=x}^{x+h_1} A_F(m, 1, ..., 1) S(m) - \frac{1}{h_2} \sum_{m=x}^{x+h_2} A_F(m, 1, ..., 1) S(m) \right|^2 dx
\]
is bounded by
By applying Lemma 3.7, (3.22), and choosing \( H = (\log X)^{\frac{3}{2}} \), we have

\[
\left( 3.24 \right) \quad \max_{T \geq Xh^{-1}} \frac{Xh^{-1}}{T} \int_{T}^{2T} \left| \sum_{m \in \mathbb{H}} A_F(m, 1, \ldots, 1) 1_{S}(m) \right|^2 \, dt
\]

for \( i = 1, 2 \), and for sufficiently large \( A \). Using a similar argument as in (3.25), we see that

\[
\left( 3.25 \right) \quad \max_{Xh^{-1} < T \leq X} \frac{Xh^{-1}}{T} \int_{T}^{2T} \left| \sum_{m \in \mathbb{H}} A_F(m, 1, \ldots, 1) 1_{S}(m) \right|^2 \, dt \ll_{F, \epsilon} \frac{(\log X)^{1/3 - \epsilon}}{h^1} \left( \frac{Q}{h_1} + 1 \right) + \frac{(\log X)^{n^2 - 1}}{H} \ll_{F, \epsilon} (\log X)^{-\frac{3}{2} + 3}
\]

for \( i = 1, 2 \). In addition, by applying (1.35), we have

\[
\max_{T \geq X} \frac{Xh^{-1}}{T} \int_{T}^{2T} \left| \sum_{m \in \mathbb{H}} A_F(m, 1, \ldots, 1) 1_{S}(m) \right|^2 \, dt \ll_{F, \epsilon} \frac{h^{-1} X}{T} \sum_{m \in \mathbb{H}} |A_F(m, 1, \ldots, 1)|^2 \ll_{F, \epsilon} h^{-1}
\]

for \( i = 1, 2 \). Therefore, (3.24) is bounded by

\[
\left( 3.26 \right) \quad (\log X)^{-\frac{3}{2}}.
\]

By using Lemma 3.3, we see that

\[
\left( 3.27 \right) \quad \frac{1}{X} \int_{X}^{2X} \left| \frac{1}{h_1} \sum_{m \in X} A_F(m, 1, \ldots, 1) 1_{S}(m) \right|^2 \, dx \ll_{F, \epsilon} (\log X)^{-2/3 + \epsilon}.
\]

Combining (3.26) and (3.27), we get (1.1).

3.3. **Proof of Theorem 1.7.**

**Proof.** By using the Chebyshev inequality and Theorem 1.1, we see that

\[
\frac{1}{h} \sum_{m = x}^{x + h} A_F(m, 1, \ldots, 1) \ll_{F, \epsilon} (\log X)^{-1/3 + \epsilon}
\]

for almost all \( x \in [X, 2X] \). By (1.22), we have

\[
\frac{1}{X^{1-\epsilon}} \sum_{m \in X} A_F(m, 1, \ldots, 1) \ll_{F, \epsilon} X^{\frac{n^2-n-1+\epsilon}{n^2-1+\epsilon}} \ll_{F, \epsilon} (\log X)^{-1/3 + \epsilon}.
\]

\( \square \)
3.4. Proof of Corollary 1.8. In this subsection, we prove Corollary 1.8 by applying Theorem 1.1.

Proof. For convenience we assume that $X \in \mathbb{N}$. Let us consider

\[
(3.28) \quad \frac{1}{X} \sum_{h=x}^{2X} \left| \frac{1}{h} \sum_{m=x}^{x+h} A_F(m, \ldots, 1) \right|^2.
\]

By squaring out the inner sums and applying the upper bound $A_F(m, \ldots, 1) \ll \epsilon m^\epsilon$, (3.28) can be represented as

\[
\frac{1}{h^2} \sum_{l=h}^{2X} |h-l| \left( \frac{1}{h} \sum_{m=x}^{x+h} A_F(m, \ldots, 1) A_F(m+l, \ldots, 1) \right) + O_F \left( \frac{1}{h} \right) + O_F(\epsilon X^{\frac{1}{2}-1}).
\]

By applying Theorem 1.1 and (1.22), we see that (3.28) is bounded by

$$
\ll_F \epsilon (\log X)^{-2/3 + \epsilon}.
$$

Since $\frac{1}{h}hX^{\frac{1}{2}-1} \ll \epsilon (\log X)^{-2/3 + \epsilon}$, the proof is completed. \qed

3.5. Proof of Corollary 1.11.

Proof. Let $h' = e^{(\log X)^{1+\epsilon}}$. By using (1.38), we have

\[
(3.29) \quad \left| \left\{ x \in \left[ \frac{X}{2}, X - h' \right] : \sum_{m=x}^{x+h'} |A_F(m, \ldots, 1)| > \frac{h'(\log X)^{\frac{1}{2}-1}}{\log \log X} \right\} \right| \gg X.
\]

By combining the above inequality with Remark 1.10, we see that (3.30)

\[
\sum_{m=x}^{x+h'} A_F(m, \ldots, 1) \ll_F \epsilon h'(\log X)^{\theta + \epsilon - 1} = o_{F, \epsilon} \left( \frac{h'(\log X)^{\frac{1}{2}-1}}{(\log \log X)^2} \right) = o_{F, \epsilon} \left( \sum_{m=x}^{x+h'} |A_F(m, \ldots, 1)| \right)
\]

for a positive proportion of $x \in \left[ \frac{X}{2}, X - e^{(\log X)^{1+\epsilon}} \right)$. Therefore, there are $x \leq m_1 < m_2 \leq x + h'$ such that $A_F(m_1, \ldots, 1) A_F(m_2, \ldots, 1) < 0$ for a positive proportion of $x \in \left[ \frac{X}{2}, X - e^{(\log X)^{1+\epsilon}} \right)$. Hence, the number of sign changes is $\gg_F \epsilon X^{\frac{1}{2}}$. \qed

4. Acknowledgements

The author would like to thank his advisor Professor Xiaqing Li for her helpful advice on zero-free regions and the generalized Ramanujan-Petersson conjecture. The author wishes to express his thanks to Professor Andrew Granville, Professor Kaisa Matomaki, and the anonymous referee for their helpful comments.

References

1. Étienne Fouvry and Satadal Ganguly. Strong orthogonality between the Möbius function, additive characters and Fourier coefficients of cusp forms. Compos. Math., 150(5):763–797, 2014.
2. Dorian Goldfeld. Automorphic forms and $L$-functions for the group $GL(n, \mathbb{R})$, volume 99 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2015. With an appendix by Kevin A. Broughan. Paperback edition of the 2006 original [MR2254662].
3. Andrew Granville, Adam J. Harper, and K. Soundararajan. A new proof of Halász’s theorem, and its consequences. Compos. Math., 155(1):126–163, 2019.
4. Peter Humphries and Farrell Brumley. Standard zero-free regions for Rankin-Selberg $L$-functions via sieve theory. Math. Z., 292(3-4):1105–1122, 2019.
5. Henryk Iwaniec and Emmanuel Kowalski. Analytic number theory, volume 53 of American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2004.
6. Jesse Jääsaari. On short sums involving Fourier coefficients of Maass forms. J. Théor. Nombres Bordeaux, 32(3):761–793, 2020.
7. Yujiao Jiang, Guangshi Lü, and Zhiwei Wang. Exponential sums with multiplicative coefficients without the Ramanujan conjecture. Math. Ann., 379(1-2):589–632, 2021.
[8] Matti Jutila. The additive divisor problem and its analogs for Fourier coefficients of cusp forms. I. Math. Z., 223(3):435–461, 1996.
[9] Jiseong Kim. On Hecke eigenvalues of cusp forms in almost all short intervals. International Journal of Number Theory, 18(04):1–14, 2021.
[10] Guangshi Liu. Sums of absolute values of cusp form coefficients and their application. J. Number Theory, 139:29–43, 2014.
[11] WZ Luo. Values of symmetric square L-functions at 1. Journal für die reine und angewandte Mathematik, 506(506):215–235, 1999.
[12] Alexander P. Mangerel. Divisor-bounded multiplicative functions in short intervals. Res. Math. Sci., 10(1):Paper No. 12, 47, 2023.
[13] Kaisa Matomäki and Maksym Radziwiłł. A note on the liouville function in short intervals. arXiv, 2015.
[14] Kaisa Matomäki and Maksym Radziwiłł. Sign changes of Hecke eigenvalues. Geom. Funct. Anal., 25(6):1937–1955, 2015.
[15] Kaisa Matomäki and Maksym Radziwiłł. Multiplicative functions in short intervals. Ann. of Math. (2), 183(3):1015–1056, 2016.
[16] Kaisa Matomäki, Maksym Radziwiłł, and Terence Tao. Correlations of the von Mangoldt and higher divisor functions II: divisor correlations in short ranges. Math. Ann., 374(1-2):793–840, 2019.
[17] Jaban Meher and M. Ram Murty. Oscillations of coefficients of Dirichlet series attached to automorphic forms. Proc. Amer. Math. Soc., 145(2):563–575, 2017.
[18] Hugh L. Montgomery. Topics in multiplicative number theory. Lecture Notes in Mathematics, Vol. 227. Springer-Verlag, Berlin-New York, 1971.
[19] Hugh L. Montgomery and Robert C. Vaughan. Multiplicative number theory. I. Classical theory, volume 97 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2007.
[20] R. W. K. Odoni. Solution of a generalised version of a problem of Rankin on sums of powers of cusp-form coefficients. Acta Arith., 104(3):201–223, 2002.
[21] Jesse Thorner and Asif Zaman. An unconditional GL(n) large sieve. Advances in Mathematics, 378:107529, 2021.
[22] E. Wirsing. Das asymptotische Verhalten von Summen über multiplikative Funktionen. II. Acta Math. Acad. Sci. Hungar., 18:411–467, 1967.

Email address: Jkim51@olemiss.edu

University of Mississippi, Department of Mathematics