HIGH-ORDER CONSERVATIVE POSITIVITY-PRESERVING
DG-INTERPOLATION FOR DEFORMING MESHES AND
APPLICATION TO MOVING MESH DG SIMULATION OF
RADIATIVE TRANSFER

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Abstract. Solution interpolation between deforming meshes is an important component for
several applications in scientific computing, including indirect arbitrary-Lagrangian-Eulerian and
re zoning moving mesh methods in numerical solution of partial differential equations. In this paper,
a high-order, conservative, and positivity-preserving interpolation scheme is developed based on the
discontinuous Galerkin solution of a linear time-dependent equation on deforming meshes. The
scheme works for bounded but otherwise arbitrary mesh deformation from the old mesh to the new
one. The cost and positivity preservation (with a linear scaling limiter) of the DG-interpolation are
investigated. Numerical examples are presented to demonstrate the properties of the interpolation
scheme. The DG-interpolation is applied to the rezoning moving mesh DG solution of the radiative
transfer equation, an integro-differential equation modeling the conservation of photons and involving
time, space, and angular variables. Numerical results obtained for examples in one and two spatial
dimensions with various settings show that the resulting rezoning moving mesh DG method maintains
the same convergence order as the standard DG method, is more efficient than the method with a
fixed uniform mesh, and is able to preserve the positivity of the radiative intensity.

Key words. DG-interpolation, conservative, positivity-preserving, moving mesh DG method,
MMPDE, radiative transfer equation

AMS subject classifications. 65M50, 65M60, 65M70, 65R05, 65.75

1. Introduction. Solution interpolation or remapping between two deforming
meshes is an important component for several applications in scientific computing,
including arbitrary-Lagrangian-Eulerian (ALE) methods in computational fluid dy-
namics [1, 2, 3, 4, 5, 8, 10, 11, 12, 15, 25, 26, 35] and rezoning moving mesh methods in
general numerical solution of partial differential equations (PDEs) [13, 28, 29, 38, 46].
If not designed properly, a scheme for the interpolation can lead to violation of con-
servation of some important physical quantities, deterioration of accuracy, and/or
introduction of spurious negative values in supposedly nonnegative variables.
Some of the earliest work on conservative interpolation between two deforming
meshes grew out of the development of ALE methods [15]. Depending on the relation
between the old (Lagrangian) and new (rezoned) meshes, we can classify mesh-to-
mesh interpolation algorithms as integral-remapping or advection-remapping ones. If
the two meshes are completely independent of one another or have the same con-
nectivity but are arbitrarily displaced with respect to each other, one needs to use
integral-remapping interpolation which involves finding the intersections of the cells

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of two meshes. In [10], a conservative interpolation method is proposed that assumes piecewise constant fields and simplifies the problem of computing the volume of intersection of old and new cells into a surface integral by invoking the divergence theorem. However, the first-order nature of the method leads to excessive diffusion. The approach is extended to second order to improve its diffusive characteristics in [11]. In [35], a second-order accurate, conservative, and sign-preserving local remapping algorithm for a positive, scalar, cell-centered function is developed based on the intersection, which can be written in flux form if two meshes have the same connectivity. Then the authors simplify it as a face-based donor-cell method, which avoids finding the cell intersections but requires the displacements to be small to maintain the positivity of the remapping variables. The main drawback of this type of method is the difficulty of evaluating the integrals for arbitrary meshes, especially in higher dimensions.

When the old and new meshes have the same connectivity, they can be viewed as a deformation of each other and advection-remapping can be used. It is shown in [12] that if the physical time step is made sufficiently short such that node trajectories are confined to the nearest neighbor cells (and thus the magnitude of the deformation is small), then the remapping can be written as a flux-form convection algorithm. An incremental remapping method based on the solution of convection equations is developed in [12]. A linearity-and-bound preserving conservative interpolation scheme is introduced in [25]. A main advantage of an advection-based scheme is that it does not require finding the intersections of old and new mesh cells. However, the connection between advection equations and conservative interpolation/remapping does not seem to be well understood as assumptions and discretization errors of using advection methods for interpolation/remapping are not easily identified.

There is a different approach of advection-remapping where the interpolation is viewed as solving a linear convection PDE over a pseudo-time interval. For example, Li et al. [28] use such an interpolation scheme in a moving mesh finite element method. A conservative interpolation scheme is proposed and used by Tang and Tang [38] for finite volume computation of hyperbolic equations. This scheme seems to work only for small mesh deformation. A divergence-free-preserving interpolation algorithm is developed in [13] for the moving mesh finite element computation of the incompressible Navier-Stokes equations. It is worth pointing out that only one pseudo-time step is used in their computation since the mesh deformation is very small. The idea of [13] is extended in [29] to develop a second-order conservative interpolation scheme for use with a moving mesh DG method. Anderson et al. [1] propose a method for remapping the state variables of single material ALE based on solving convection equations using semi-discrete DG methods and three nonlinear approaches to enforce monotonicity of the remapping variables. Its multi-material extension and combination with the Lagrangian phase can be found in [2]. For more remapping/interpolation methods, the interested reader is referred to [4, 8, 26, 46] and the references therein.

The objective of this paper is to develop an arbitrary high-order conservative interpolation scheme and present an analysis for its cost and positivity preservation, two issues that have hardly been studied for interpolation/remapping for deforming meshes. The scheme is based on solving a linear convection equation with a moving mesh DG method for spatial discretization and an explicit third-order Runge-Kutta scheme for time discretization. This DG-interpolation scheme is shown to be mass-conservative and applicable for bounded but otherwise arbitrary mesh deformation. Moreover, it is shown that the cost of the DG-interpolation is in the order of the
number of mesh vertices multiplied by the number of pseudo-time steps needed to integrate the convection equation from the pseudo-time zero corresponding to the old mesh and the pseudo-time one corresponding to the new mesh. This number of pseudo-time steps depends on the magnitude of mesh deformation relative to the size of mesh elements in general. It stays constant as the mesh is being refined when the mesh deformation is in the order of the minimum element diameter, a typical situation in the moving mesh solution of conservation laws with an explicit scheme. On the other hand, the number of pseudo-time steps increases as the mesh is being refined if the mesh deformation only stays bounded. A typical scenario of this is in the moving mesh solution of PDEs with a fixed physical time step size or with an implicit scheme. Another issue is positivity preservation. Generally speaking, the DG-interpolation alone may not preserve the positivity/nonnegativity of the function to be interpolated. We consider a limiter [33, 44, 45] that uses a linear scaling around the positive cell average while conserving the cell average and maintaining the convergence order of the DG discretization. We show analytically and verify numerically that the DG-interpolation with the limiter can preserve the positivity of the function to be interpolated.

As an application example, we study the use of the DG-interpolation scheme in the rezoning moving mesh DG solution of the radiative transfer equation (RTE). The RTE is an integro-differential equation modeling the interaction of radiation with scattering and absorbing media and having important applications in various fields in science and engineering. It involves time, space and angular variables and contains an integral term in angular directions while being hyperbolic in space. The challenges for its numerical solution include the needs to handle with its high dimensionality, the presence of the integral term, the development of discontinuities and sharp layers in its solution along spatial directions, and appearance of spurious negative values in the supposedly nonnegative radiative intensity. These challenges make adaptive high-order DG methods amenable to the numerical solution of RTE. Indeed, DG methods have been considered for RTE. For example, a quasi-Lagrangian moving mesh DG method is proposed in [42] and the preservation of nonnegativity of the radiative intensity is investigated in [32, 40, 43] for the DG solution of RTE on a fixed mesh.

We consider a rezoning moving mesh DG method (instead a quasi-Lagrangian one) for the numerical solution of RTE. It typically includes three steps, mesh redistribution/adaptation, solution interpolation from the old mesh to the new one, and solution of the physical equation on the new mesh. The method has the advantages that these steps are independent of each other and existing schemes can be used for each step. Moreover, the task seems to be simpler here than that with a quasi-Lagrangian moving mesh method that strongly couples the effects of mesh movement with the discretization of RTE. For the current situation, we deal separately with a scalar function/equation on a moving mesh for the second step (interpolation) and the discretization of RTE on a fixed mesh for the third step. In our computation, we use the positivity-preserving DG method of [32, 40, 43] for spatial variables and the discrete-ordinate method (DOM) [30] for angular variables. For adaptive mesh generation (the first step), we use a moving mesh method [18, 20, 21, 22, 23] which is known to produces a nonsingular moving mesh [19]. We use the DG-interpolation for the second step. The whole computation can be made positivity preserving when the computation at the second and third steps can be made positivity preserving.

The outline of the paper is as follows. The high-order DG-interpolation is developed and its cost, mass conservation, and positivity preservation are analyzed in §2. The moving mesh PDE (MMPDE) method is described in §3. Numerical results

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obtained for one- and two-dimensional examples are presented in §4 to demonstrate the high-order accuracy and positivity-preserving features and the cost of the DG-interpolation. A rezoning moving mesh DG method for RTE is described in §5 and numerical examples with various settings in one and two spatial dimensions are given in §6. Finally, §7 contains the conclusions.

2. High-order conservative positivity-preserving DG-interpolation. In this section we present an interpolation scheme from an old simplicial mesh to a new one with high-order accuracy, mass conservation, and positivity preservation. The scheme works in any dimension although we restrict our discussion in one and two dimensions for notational simplicity.

Let $\mathcal{D} \subset \mathbb{R}^d$ ($d = 1$ and $2$) be a polygonal bounded domain. Assume that we are given nonsingular simplicial meshes $\mathcal{T}_h^{old}$ and $\mathcal{T}_h^{new}$ on $\mathcal{D}$ that have the same number of elements and vertices and the same connectivity. They differ only in the location of vertices and can be considered a deformation of each other. They can also be regarded as a moving mesh at different time instants. In this work, we use the MMPDE moving mesh method (see §3) to generate such a mesh.

The interpolation problem between $\mathcal{T}_h^{old}$ and $\mathcal{T}_h^{new}$ is equivalent to the numerical solution of the differential equation \cite{1, 23, 28, 29}

\begin{equation}
\frac{\partial u}{\partial \varsigma}(x, \varsigma) = 0, \quad (x, \varsigma) \in \mathcal{D} \times (0, 1]
\end{equation}

on the moving mesh $\mathcal{T}_h(\varsigma)$ obtained as a linear interpolant of $\mathcal{T}_h^{old}$ and $\mathcal{T}_h^{new}$ in the pseudo-time $\varsigma \in [0, 1]$. In particular, $\mathcal{T}_h(\varsigma)$ has the same number of elements and vertices and the same connectivity as $\mathcal{T}_h^{old}$ and $\mathcal{T}_h^{new}$ and its nodal positions and velocities are given by

\begin{align}
(2.2) & \quad x_i(\varsigma) = (1 - \varsigma)x_i^{old} + \varsigma x_i^{new}, \quad i = 1, ..., N_v \\
(2.3) & \quad \dot{x}_i = x_i^{new} - x_i^{old}, \quad i = 1, ..., N_v.
\end{align}

The initial condition is

\begin{equation}
(2.4) \quad u(x, 0) = u_0(x), \quad x \in \mathcal{D}
\end{equation}

where $u_0(x)$ is the original function defined on $\mathcal{T}_h^{old}$. We define the piecewise linear mesh velocity function as

\begin{equation}
(2.5) \quad \dot{X}(x, \varsigma) = \sum_{i=1}^{N_v} \dot{x}_i \phi_i(x, \varsigma),
\end{equation}

where $\phi_i$ is the linear basis function associated with the vertex $x_i$.

We consider the numerical solution of (2.1) using a quasi-Lagrangian moving mesh DG method \cite{34, 42}. Let $K$ be an arbitrary element of $\mathcal{T}_h(\varsigma)$. Denote the basis functions of degree up to $r \geq 1$ on $K$ by $\phi_{K,j}^{[j]}$, $j = 1, ..., n_b = (r + d)!/(d!r!)$. Notice that $n_b = r + 1$ for $d = 1$ and $n_b = (r + 1)(r + 2)/2$ for $d = 2$. The DG finite element space is defined as

\begin{equation}
(2.6) \quad V_h^r(\varsigma) = \{ v \in L^2(\mathcal{D}) : v|_K \in P^r(K), \forall K \in \mathcal{T}_h(\varsigma) \},
\end{equation}

where $P^r(K)$ stands for the space of polynomials of degree at most $r$ on $K$. Then any DG approximation polynomial $u_h \in V_h^r(\varsigma)$ can be expressed as

\begin{equation}
(2.7) \quad u_h(x, \varsigma) = \sum_{j=1}^{n_b} u_j^{[j]} K(\varsigma) \phi_{K,j}^{[j]}(x, \varsigma), \quad x \in K, \quad K \in \mathcal{T}_h(\varsigma)
\end{equation}
where $u_h^j$, $j = 1, ..., n_b$, are the degrees of freedom. Without causing confusion, hereafter we will suppress the subscript “$h$” in $u_h$, i.e., we will write $u_h$ as $u$. We note that the basis functions depend on $\zeta$ due to the movement of the vertices. From the fact that $K$ is a simplex, it is not difficult to show that

$$\frac{\partial \phi_h^j}{\partial \zeta}(x, \zeta) = -\nabla \phi_h^j(x, \zeta) \cdot \dot{X}(x, \zeta), \quad \text{a.e. in } D.$$  \hspace{1cm} (2.8)

For the weak formulation of (2.1), multiplying it by a test function $v \in V_h^r(\zeta)$ and integrating the resulting equation over $K$, we obtain

$$\int_K \frac{\partial u}{\partial \zeta} v dx = 0.$$  \hspace{1cm} (2.9)

On the other hand, from the Reynolds transport theorem we have

$$\frac{d}{ds} \int_K u v dx = \int_K \frac{\partial}{\partial \zeta} (u v) dx + \int_{\partial K} u v \cdot n_K ds$$

$$= \int_K \left( \frac{\partial u}{\partial \zeta} + u \frac{\partial v}{\partial \zeta} \right) dx + \int_{\partial K} u v \cdot n_K ds,$$

where $n_K$ is the outward unit normal to the boundary $\partial K$. Using (2.8) (with $\phi_h^j$ being replaced by $v$) and (2.9) in the above equation, we get

$$\frac{d}{ds} \int_K u v dx + \int_{\partial K} v \left( -u \dot{X} \cdot n_K \right) ds + \int_K (u \dot{X}) \cdot \nabla v dx = 0.$$  \hspace{1cm} (2.10)

The boundary integral term is replaced by a numerical flux in the DG approximation. Thus, the semi-discrete moving mesh DG solution for (2.1) is to seek $u \in V_h^r(\zeta)$, $0 < \zeta \leq 1$ such that

$$\frac{d}{ds} \int_K u v dx + \sum_{e \in \partial K} \int_e v F_e(u_h^\text{in}, u_h^\text{out}) ds + \int_K (u \dot{X}) \cdot \nabla v dx = 0, \quad \forall v \in V_h^r(\zeta)$$  \hspace{1cm} (2.11)

where $F_e(u_h^\text{in}, u_h^\text{out}) \approx -u \dot{X} \cdot n_K$ is a numerical flux defined on $e \in \partial K$, $u_h^\text{in}$ denotes the value of $u$ on $K$, and $u_h^\text{out}$ is the value of $u$ on the element (denoted by $K'$) sharing the common edge $e$ with $K$. We use the local Lax-Friedrichs numerical flux, viz.,

$$F_e(u_h^\text{in}, u_h^\text{out}) = \frac{1}{2} \left( -u_h^\text{in} \dot{X}^e - u_h^\text{out} \dot{X}^e \right) \cdot n_K^e - \alpha_e (u_h^\text{out}^e - u_h^\text{in}^e), \quad \forall e \in \partial K$$  \hspace{1cm} (2.12)

where $\dot{X}^e$ denotes the restriction of $\dot{X}$ on $e$ and

$$\alpha_e = \max \left( |\dot{X}^e \cdot n_K^e|, |\dot{X}^e \cdot n_K^e| \right).$$  \hspace{1cm} (2.13)

Note that the numerical flux is zero on the boundary due to the fact that the boundary does not move, i.e., $\dot{X}^e \cdot n_K^e = 0$ if $e \in \partial D$. For any interior edge $e$, it is not difficult to see that $\alpha_e = |\dot{X}^e \cdot n_K^e|$ and

$$F_e(u_h^\text{in}, u_h^\text{out}) = -\frac{|\dot{X}^e \cdot n_K^e|}{2} u_h^\text{in} + \frac{|\dot{X}^e \cdot n_K^e|}{2} u_h^\text{out} - \frac{|\dot{X}^e \cdot n_K^e|}{2} |\dot{X}^e \cdot n_K^e| u_h^\text{in}.$$
which is actually an upwind flux. It is known that this numerical flux satisfies several properties including consistency, monotonicity, Lipschitz continuity, and conservative-
ness. The last one reads as

\begin{equation}
F_e(u^{\text{in}}_K, u^{\text{out}}_K) + F_e(u^{\text{in}}_{K^r}, u^{\text{out}}_{K^r}) = 0.
\end{equation}

In our computation, the second and third terms in the left of (2.11) are computed using Gaussian quadrature rules.

The third-order explicit total variation diminishing (TVD) Runge-Kutta scheme is used to discretize (2.11) in time. To describe the scheme, we rewrite (2.11) into

\begin{equation}
\frac{d}{ds} \int_K u v \, dx = -A(u, v)|_K,
\end{equation}

\begin{equation}
A(u, v)|_K = \sum_{e \in \partial K} \int_e v F_e(u^{\text{in}}_K, u^{\text{out}}_K) \, ds + \int_K (u \hat{X}) \cdot \nabla v \, dx.
\end{equation}

Let the time instants be

\begin{equation}
0 = \varsigma^0 < \varsigma^1 < \cdots < \varsigma^N < \varsigma^{N+1} = 1, \quad \text{and} \quad \Delta \varsigma^\nu = \varsigma^{\nu+1} - \varsigma^\nu.
\end{equation}

The third-order explicit TVD Runge-Kutta scheme for (2.15) reads as

\begin{equation}
\begin{cases}
\int_{K^{\nu}(1)} u^{(1)} v^{\nu(1)} \, dx = \int_{K^{\nu}} u^{\nu} v^{\nu} \, dx - \Delta \varsigma^\nu A(u^{\nu}, v^{\nu})|_{K^{\nu}}, \\
\int_{K^{\nu}(2)} u^{(2)} v^{\nu(2)} \, dx = \frac{1}{3} \int_{K^{\nu}} u^{\nu} v^{\nu} \, dx \\
\quad \quad \quad + \frac{1}{6} \left( \int_{K^{\nu}(1)} u^{(1)} v^{\nu(1)} \, dx - \Delta \varsigma^\nu A(u^{(1)}, v^{\nu(1)})|_{K^{\nu}(1)} \right), \\
\int_{K^{\nu+1}} u^{\nu+1} v^{\nu+1} \, dx = \frac{1}{3} \int_{K^{\nu}} u^{\nu} v^{\nu} \, dx \\
\quad \quad \quad + \frac{1}{2} \left( \int_{K^{\nu}(2)} u^{(2)} v^{\nu(2)} \, dx - \Delta \varsigma^\nu A(u^{(2)}, v^{\nu(2)})|_{K^{\nu}(2)} \right),
\end{cases}
\end{equation}

where \(u^{(1)}, v^{\nu(1)}, K^{\nu(1)}\) are stage values at \(\varsigma = \varsigma^{\nu+1}, u^{(2)}, v^{\nu(2)}, K^{\nu(2)}\) are the values at \(\varsigma = \varsigma^{\nu+\frac{1}{2}}\), and \(u^{\nu+1}, v^{\nu+1}, K^{\nu+1}\) are at \(\varsigma = \varsigma^{\nu+1}\). It is emphasized that the coordinates of the vertices and the volume of \(K\) need to be updated at these stages.

Especially, as will be seen in §2.1, a special update scheme for the element volume may be needed for the scheme to satisfy the so-called geometric conservation law [37, 39]. It is also worth pointing out that the test functions at \(\varsigma^{(1)}\), \(K^{\nu(1)}\), \(K^{\nu(1)}\) are stage values at \(\varsigma = \varsigma^{\nu+1}, u^{(2)}, v^{\nu(2)}, K^{\nu(2)}\) are the values at \(\varsigma = \varsigma^{\nu+\frac{1}{2}}\), and \(u^{\nu+1}, v^{\nu+1}, K^{\nu+1}\) are at \(\varsigma = \varsigma^{\nu+1}\). It is emphasized that the coordinates of the vertices and the volume of \(K\) need to be updated at these stages.

The time step size \(\Delta \varsigma\) is chosen to ensure the stability of the scheme [9], i.e.,

\begin{equation}
\Delta \varsigma = \frac{C_{\text{cfl}}}{\max_{e,K} |X^e \cdot n_K^e|} \cdot \min(h_{\text{old}, e}^{\text{min}}, h_{\text{new}, e}^{\text{min}}),
\end{equation}

where \(C_{\text{cfl}}\) is a constant typically chosen to be less than \(1/(2r + 1)\), \(h_{\text{old}, e}^{\text{min}}\) and \(h_{\text{new}, e}^{\text{min}}\) are the minimum element diameters for \(T_{\text{old}, e}\) and \(T_{\text{new}, e}\), respectively, and the maximum is taken over all edges and elements.

From the theory of DG and TVD Runge-Kutta scheme (e.g., see [47]), we can expect that the above described DG-interpolation scheme is \((r+1)\)th order in space.
and third order in time for problems with smooth solutions, viz., $O(\Delta t^3) + O(h^{r+1})$,
where $h$ denotes the maximum element diameter. Particularly, the scheme is second order for $r = 1$ and third order for $r = 2$. For $r > 2$, we may need to use a Runge-Kutta method of higher order or use smaller time step size than (2.19) to balance the spatial and temporal error terms.

It is emphasized that the above described scheme does not require any prior conditions on the meshes $T_h^{old}$ and $T_h^{new}$. Particularly, it works when the mesh has large deformation although more time steps may be needed. The cost of the scheme is discussed in §2.3.

2.1. The geometric conservation law (GCL). GCL stands for geometric identities that hold in continuous form. They may no longer hold in a discrete setting especially in the computation with moving meshes [37, 39]. A simple verification for satisfying GCL is to use uniform flow reproduction, i.e., to check if the underlying scheme produces a uniform flow if the initial flow is uniform. Theoretical and numerical analysis (e.g., see [6, 14]) shows that satisfying GCL is neither a necessary nor a sufficient condition for the stability of a scheme but often helps improve the accuracy and stability of the computation. We study (2.17) here for the satisfaction of GCL.

Taking $u \equiv 1$ in (2.16) and using $F_{\nu}(1,1) = -X^\nu \cdot n_K^\nu$ and the divergence theorem, we have

$$A(1,v)|_K = \sum_{e \in \partial K} \int e v(-X_\nu \cdot n_K^\nu)ds + \int_\nu X \cdot \nabla vd\xi$$

$$= -\int_\nu v \nabla \cdot \dot{X} d\xi = -\nabla \cdot \dot{X}|_K \int_\nu v d\xi = -\frac{|K|}{|K|} \nabla \cdot \dot{X}|_K \int_\nu \dot{v} d\xi.$$  

(2.20)

Combining this equation with (2.15) and taking $u \equiv 1$ and $v = 1$, we get

$$(2.21) \frac{d}{dt}|_K = |K| \nabla \cdot \dot{X}|_K,$$

which is the GCL governing the evolution of the volume of element $K$. On the other hand, taking $u_\nu = 1$, $u^{(1)} = 1$, $u^{(2)} = 1$, $u^{r+1} = 1$, and $\dot{v} = 1$ in (2.17), we obtain

$$\begin{cases} |K^{\nu,(1)}| = |K^{\nu}| + \Delta \zeta^{\nu}|K^{\nu}| \nabla \cdot \dot{X}|_K^{\nu}, \\ |K^{\nu,(2)}| = \frac{3}{4}|K^{\nu}| + \frac{1}{2}\left((|K^{\nu,(1)}| + \Delta \zeta^{\nu}|K^{\nu,(1)}| \nabla \cdot \dot{X}|_K^{\nu,(1)})ight), \\ |K^{\nu+1}| = \frac{3}{4}|K^{\nu}| + \frac{1}{2}\left((|K^{\nu,(2)}| + \Delta \zeta^{\nu}|K^{\nu,(2)}| \nabla \cdot \dot{X}|_K^{\nu,(2)})ight), \end{cases}$$

(2.22)

which can be used to update the volume of $K$ at the three Runge-Kutta stages. The above equation can also be obtained by applying the third-order Runge-Kutta scheme directly to (2.21).

**Lemma 2.1.** The fully-discrete moving mesh DG scheme (2.17) reproduces the uniform flow, i.e., $u_\nu \equiv 1$ implies $u^{r+1} \equiv 1$, if the element volume is updated according to (2.22).

This lemma can be proved by taking $u_\nu \equiv 1$ in (2.17) and using (2.20) and (2.22).

As mentioned above, the volume of $K$ at different Runge-Kutta stages can be obtained using (2.22). It can also be calculated directly using the coordinates of the vertices. Interestingly, it can be verified that these two approaches are the same in one dimension but different in two and higher dimensions. In the latter case, (2.22) needs to be used for uniform flow reproduction and thus GCL satisfaction.

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2.2. Mass conservation. In this subsection we show that the DG-interpolation scheme (2.11) and (2.17) conserves the mass.

Lemma 2.2. The semi-discrete moving mesh DG scheme (2.11) conserves the mass, \( \int_D u \, dx \).

This lemma can be proved by taking \( v = 1 \) in (2.11), summing the resulting equation over all elements, re-arranging the terms according to interior and boundary edges, and using (2.14) and the fact that the numerical flux vanishes on the boundary.

Lemma 2.3. The fully discrete moving mesh DG scheme (2.17) conserves the mass, i.e.,

\[
\sum_{K^\nu+1} \int_{K^\nu+1} u^{\nu+1} \, dx = \sum_{K^\nu} \int_{K^\nu} u^\nu \, dx, \quad \nu = 0, 1, \ldots
\]

This lemma can be proved similarly as for Lemma 2.2.

Remark 2.4. Similarly, we can prove that the first-order forward Euler scheme and the second-order explicit TVD Runge-Kutta scheme also conserve the mass when applied to (2.11).

2.3. Cost of the DG-interpolation. We now investigate the cost of the DG-interpolation scheme (2.17). We start with noticing that the cost of each time step of the scheme is \( O(N_v) \) and the total cost is \( O(N_v N_\varsigma) \), where \( N_v \) is the number of the mesh vertices and \( N_\varsigma \) is the number of time steps to reach \( \varsigma = 1 \). Note that this total cost is the cost for each interpolation of the function from the old mesh to the new one. The key to the estimation of this cost is to estimate \( N_\varsigma \).

To this end, we recall that the CFL stability condition (2.19). Since \( \dot{X} \) is piecewise linear, from (2.3) we have

\[
\max_{e,K} |\dot{X} \cdot n_e^K| \sim \max_i |x_i^{old} - x_i^{new}|.
\]

Then, (2.19) becomes

\[
(2.24) \quad \Delta \varsigma = \frac{C_{cfl}}{\max_i |x_i^{old} - x_i^{new}|} \cdot \min(h_{min}^{old}, h_{min}^{new}).
\]

This indicates that \( \Delta \varsigma \) and thus \( N_\varsigma \) depend on the magnitude of mesh deformation relative to the size of mesh elements. In the following we consider two special cases.

Case 1. In the first case we consider the situation where

\[
(2.25) \quad \max_i |x_i^{old} - x_i^{new}| = O(\min(h_{min}^{old}, h_{min}^{new})).
\]

Then, (2.24) implies that the DG-interpolation only takes a constant number of time steps to reach \( \varsigma = 1 \) and its total cost is \( O(N_v) \).

An extreme situation for (2.25) is that the mesh is fixed. Then we have \( \max_i |x_i^{old} - x_i^{new}| = 0 \) and the upper bound of (2.24) becomes infinity, which means just one step is needed for the DG-interpolation.

In the context of the moving mesh solution of first-order hyperbolic equations, \( T_h^{old} \) and \( T_h^{new} \) correspond to meshes at consecutive time steps, i.e., \( T_h^{old} = T_n^m \) and \( T_h^{new} = T_n^{m+1} \), where \( n \) stands for the index for the physical time step, and the time step size used for integrating the physical equations is typically chosen as

\[
(2.26) \quad \Delta t = O\left( \min(h_{min}^{old}, h_{min}^{new}) \right)
\]
to ensure stability. If the mesh velocities are bounded, i.e.,
\[
(2.27) \quad \max_i \left| \frac{x^n_i - x^{n+1}_i}{\Delta t} \right| = O(1) \quad \text{or} \quad \max_i |x^n_i - x^{n+1}_i| = O(\Delta t),
\]
then (2.26) implies (2.25). As a consequence, we can expect that the cost for each DG-interpolation in the moving mesh solution of hyperbolic equations is \(O(N_v)\).

**Case 2.** In this case we consider the situation with
\[
(2.28) \quad \max_i |x^{old}_i - x^{new}_i| = O(1).
\]
Then (2.24) means that the number of the time steps needed is
\[
(2.29) \quad N_\varsigma = O\left(\frac{1}{\min(h^{old}_{\min}, h^{new}_{\min})}\right),
\]
which is \(O(N^{1/4})\) at the minimum (where \(N\) is the number of elements). It clearly indicates that \(N_\varsigma\) increases as the mesh is being refined.

A typical scenario for this case is when the physical PDE is integrated with an implicit scheme and the physical time step size \(\Delta t\) is taken independent of the mesh size (in contrast to (2.26)). Then we have \(\max_i |x^{old}_i - x^{new}_i| = O(\Delta t)\) and
\[
(2.30) \quad N_\varsigma = O\left(\frac{\Delta t}{\min(h^{old}_{\min}, h^{new}_{\min})}\right),
\]
which increases as the mesh is being refined.

**Remark 2.5.** The condition (2.25) has been used in [41] to restrict the mesh movement in the moving mesh WENO solution of conservation laws.

**Remark 2.6.** It is interesting to mention that the interpolation schemes in [13, 38] for the rezoning moving mesh methods and in [12, 25, 35] for ALE methods can be viewed as the one-step implementation of some explicit schemes for integrating (2.1) on a moving mesh. These schemes have been observed [12, 13, 25, 35, 38] to work only for small mesh deformation. This may be explained using (2.24) and (2.25), i.e., (2.25) (which implies small mesh deformation) needs to be held if we want the right-hand side of (2.24) to be constant. The analysis in this subsection also shows that multiple steps are needed if large mesh deformation is allowed.

### 2.4. Preservation of positivity.

It should be pointed out that the above-described DG-interpolation scheme (2.17) cannot preserve the positivity of the solution in general. Nevertheless, it can be shown that cell average \(\bar{u}_{K^{v+1}} \geq 0\) for all \(K^{v+1}\) if cell average \(\bar{u}_{K^v} \geq 0\) and DG polynomial \(u_{K^v}\) is nonnegative at some special quadrature points [45] (Gauss-Lobatto quadrature points in one dimension) for all \(K^v\). The proof is not given here since it is similar to those in [33, 44, 45] for constructing high-order positivity-preserving (PP) DG schemes on fixed meshes for scalar conservation laws. Once cell averages are shown to be nonnegative, a linear scaling limiter (PP limiter) can be used to modify the DG polynomial such that it is nonnegative at the special quadrature points; see [33, 44, 45] or §5.1. It is known that this PP limiter maintains the convergence order of the original DG schemes.

### 3. The MMPDE moving mesh method.

In this section we describe the generation of the new mesh \(T_{h}^{\text{new}}\) from the old one \(T_{h}^{\text{old}}\) using the MMPDE moving
mesh method [21, 22, 23]. We use here a new implementation of the method proposed in [18]. Adaptive meshes generated using this method are used in §4 for the numerical examination of the DG-interpolation scheme and in §6 for the numerical solution of the radiative transfer equation.

To describe the MMPDE method, we introduce a computational mesh $T_c = \{\xi_1, ..., \xi_{N_c}\}$ which serves as an intermediate variable, and an almost uniform reference computational mesh $\hat{T}_c = \{\xi_1, ..., \xi_{N_c}\}$ which keeps fixed in the computation. A key idea of the MMPDE moving mesh method is to view any nonuniform mesh as a uniform one in some metric $\mathbb{M}$ [17, 23]. $\mathbb{M} = \mathbb{M}(x)$ is a symmetric and uniformly positive definite matrix-valued function defined on $\mathcal{D}$. It provides the information needed for determining the size, shape, and orientation of the mesh elements throughout the domain. Various metric tensors have been proposed; e.g., see [23, 24]. We use here a metric tensor based on the Hessian of the computed solution. To be specific, we consider a physical variable $u$ and denote its finite element approximation by $u_h$. Let $H_K$ be a recovered Hessian of $u_h$ on $K \in T_h$ for a mesh $T_h$. A number of strategies can be used for Hessian recovery for finite element approximations; e.g., see [7, 27, 48, 49]. Least square fitting [48] is used in our computation. Denoting

$$|H_K| = Q \text{diag}(|\lambda_1|, ..., |\lambda_d|)Q^T,$$

where $Q \text{diag}(|\lambda_1|, ..., |\lambda_d|)Q^T$ is the eigen-decomposition of $H_K$, the metric tensor is then defined as

$$M_K = \det \left( \mathbb{I} + |H_K| \right)^{-\frac{1}{2d}} \left( \mathbb{I} + |H_K| \right), \quad \forall K \in T_h$$  

(3.1)

where $\mathbb{I}$ is the identity matrix and $\det(\cdot)$ is the determinant of a matrix. The metric tensor (3.1) is known [24] to be optimal for the $L^2$-norm of linear interpolation error. For situations with several physical variables, we first compute the metric tensor for each of the variables and then obtain the final metric tensor by matrix intersection.

When $T_h$ is uniform in the metric $\mathbb{M}$ in reference to the computational mesh $T_c$, it is known [17, 23] that it satisfies the equidistribution and alignment conditions,

$$|K| \sqrt{\frac{\det(M_K)}{|D_c|}} = \frac{\sigma_h |K_c|}{|D_c|}, \quad \forall K \in T_h$$  

(3.2)

$$\frac{1}{d} \text{tr}(F_K'^{-1}M_K^{-1}(F_K')^{-T}) = \det(F_K'^{-1}M_K^{-1}(F_K')^{-T})^{\frac{1}{2}}, \quad \forall K \in T_h$$  

(3.3)

where $F'_K$ is the Jacobian matrix of the affine mapping: $F_K : K_c \in T_c \rightarrow K \in T_h$, $M_K$ is the average of $\mathbb{M}$ over $K$, $\text{tr}(\cdot)$ denotes the trace of a matrix, and

$$|D_c| = \sum_{K_c \in T_c} |K_c|, \quad \sigma_h = \sum_{K \in T_h} |K|\det(M_K)^{\frac{1}{2}}.$$

The condition (3.2) determines the size of elements through the metric tensor $M$. On the other hand, (3.3), derived from requiring $K$ (measured in the metric $M_K$) to be similar to $K_c$ (measured in the Euclidean metric), determines the shape and orientation of $K$ through $M$ and shape of $K_c$. An energy function associated with these conditions is given by

$$\mathcal{I}_h(T_h, T_c) = \frac{1}{3} \sum_{K \in T_h} |K|\det(M_K)^{\frac{1}{2}} \left( \text{tr}(F_K'^{-1}M_K^{-1}(F_K')^{-T}) \right)^{\frac{3d}{2}}$$  

(3.4)

$$+ \frac{1}{3} d^{\frac{3d}{2}} \sum_{K \in T_h} |K|\det(M_K)^{\frac{1}{2}} \left( \det(F_K')\det(M_K)^{\frac{1}{2}} \right)^{-\frac{3}{2}},$$
which is a Riemann sum of a continuous functional developed in [16] based on mesh equidistribution and alignment.

Note that $I_h(T_h, T_c)$ is a function of the vertices $\xi_i, i = 1, ..., N_v$, of $T_c$ and the vertices $x_i, i = 1, ..., N_v$, of $T_h$. Here, we adopt an indirect approach with which we take $T_h$ as $T_{h, old}$, minimize $I_h(T_{h, old}, T_c)$ with respect to $T_c$, and obtain the new physical mesh through the relation between $T_{h, old}$ and newly obtained $T_c$. The mesh equation is defined as the gradient of the energy function (the MMPDE approach), i.e.,

$$\frac{d\xi_i}{dc} = -\frac{\det(M(x_i))^{\frac{1}{2}}}{\tau} \left( \frac{\partial I_h(T_{h, old}, T_c)}{\partial \xi_i} \right)^T, \quad i = 1, ..., N_v$$

where $\partial I_h/\partial \xi_i$ is considered as a row vector and $\tau > 0$ is a parameter used to adjust the response time of mesh movement to the changes in $M$.

Let $J = (F_K)^{-1} = E_K, E_K^{-1}$ with $E_K = [x^K_1 - x^K_0, ..., x^K_N - x^K_0]$ and $E_{K, i} = [\xi^K_i - \xi^K_0, ..., \xi^K_N - \xi^K_0]$, and define the function $G$ associated with the energy (3.4) as

$$G(J, \det(J)) = \frac{1}{3} \det(M_{K, 1})^{\frac{1}{2}} (\text{tr}(J M_{K, 1}^{-1} J^T))^{\frac{3}{2}} + \frac{d}{3} \det(M_{K, 1})^{\frac{1}{2}} \left( \frac{\det(J)}{\det(M_{K, 1})^{\frac{1}{2}}} \right)^{\frac{3}{2}}.$$  

Using the notion of scalar-by-matrix differentiation, the derivatives of $G$ with respect to $J$ and $\det(J)$ can be found [18] as

$$\frac{\partial G}{\partial J} = \frac{d}{2} \det(M_{K, 1})^{\frac{1}{2}} (\text{tr}(J M_{K, 1}^{-1} J^T))^{\frac{3}{2}} - \frac{d}{2} \det(M_{K, 1})^{-\frac{1}{2}} \det(J)^{\frac{1}{2}},$$

$$\frac{\partial G}{\partial \det(J)} = \frac{1}{2} \frac{d}{d} \det(M_{K, 1})^{-\frac{1}{2}} \det(J)^{\frac{1}{2}}.$$  

With these formulas, we can rewrite (3.5) as (cf. [18])

$$\frac{d\xi_i}{dc} = \frac{\det(M(x_i))^{\frac{1}{2}}}{\tau} \sum_{K \in \omega_i} |K| \nu_{i_K}, \quad i = 1, ..., N_v$$

where $\omega_i$ is the element patch associated with the vertex $x_i$, $i_K$ is the local index of $x_i$ on $K$, and $\nu_{i_K}^K$ is the local velocity contributed by the element $K$ to the vertex $i_K$.

The local velocities $\nu_{i_K}^K$, $i_K = 1, ..., d$, are given by

$$\begin{bmatrix}
\nu_{i_K}^K \\
\nu_{i_K}^K \\
\vdots \\
\nu_{i_K}^K
\end{bmatrix} = -E_K^{-1} \frac{\partial G}{\partial J} \frac{\det(E_{K, i})}{\det(E_K)} E_{K, i}^{-1}, \quad \nu_0^K = - \sum_{i_K = 1}^d \nu_{i_K}^K.$$

Note that the velocities for the boundary nodes need to be modified properly. For example, the velocities for the corner vertices should be set to be zero. For other boundary vertices, the velocities should be modified such that they only slide along the boundary and do not move out of the domain.

Starting with the reference computational mesh $\hat{T}_c$ as the initial mesh, the mesh equation (3.9) is integrated over a physical time step for the case with numerical solution of RTE (cf. §6) or from $\zeta = 0$ to $\zeta = 1$ for DG-interpolation testing (cf. §4).

The obtained new mesh is denoted by $T_{c, new}$. Note that $T_{h, old}$ is kept fixed during the integration and forms a correspondence with $T_{c, new}$, i.e., $T_{h, old} = \Phi_h(T_{c, new})$. Then the new physical mesh $T_{h, new}$ is defined as $T_{h, new} = \Phi_h(T_c)$, which can be computed using linear interpolation.

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4. Numerical results for DG-interpolation. In this section we present numerical results in one and two dimensions to demonstrate the accuracy and positivity preservation property of the DG-interpolation scheme with PP limiter. The CFL number for pseudo-time stepping taken to be $1/4$ for $P^1$-DG and $1/6$ for $P^2$-DG in one dimension, and $1/4$ for $P^1$-DG and $1/9$ for $P^2$-DG in two dimensions. In the computation, $\mathcal{T}_h^{old}$ is taken as a uniform mesh and $\mathcal{T}_h^{new}$ is obtained using the MMPDE moving mesh method described in the previous section. More specifically, the metric tensor is first computed on the current mesh for a given function (with its Hessian recovered based on nodal values via quadratic least squares fitting) and then a new physical mesh is obtained through integrating the mesh equation (3.9) from $\varsigma = 0$ to $\varsigma = 1$ with $\tau = 0.01$ and linear interpolation. This procedure is iterated five times.

No restriction is imposed on the mesh deformation.

Example 4.1. In this test, we choose the function as

$$u(x) = \cos^2(\pi x) + 10^{-14}, \quad x \in (0, 1).$$

Fig. 1 shows the meshes and numerical solutions obtained by $P^2$-DG interpolation with or without PP limiter from the old mesh to the new one. It demonstrates that the PP limiter is able to maintain the positivity of the solution. The convergence history is plotted in Fig. 2(a,b), which show that the PP DG-interpolation has the expected convergence order in both $L^1$ and $L^\infty$ norm. Fig. 2(c) shows the number ($N_\varsigma$) of time steps used to reach $\varsigma = 1$ as $N$ increases for the PP DG-interpolation. One can see that the curves for $P^1$-DG and $P^2$-DG are almost parallel to $N_\varsigma = N \max_i |x_i^{old} - x_i^{new}|$, which is consistent with the analysis for Case 2 in $\S 2.3$ (cf. (2.29)). For this example, $\max_i |x_i^{old} - x_i^{new}|$ stays almost constant (about 0.023 for large $N$) as the mesh is being refined.

![Fig. 1. Example 4.1. The meshes ($N = 40$) and numerical solutions obtained by $P^2$-DG interpolation with or without PP limiter.](image)
Example 4.1. (a) and (b): The convergence history. (c): The number of time steps used to reach $\varsigma = 1$ is plotted against $N$ for the PP DG-interpolation. The “analytical” stands for the curve $N_{\varsigma} = N \max_i |x_i^{\text{old}} - x_i^{\text{new}}|$.

Example 4.2. We consider
\[ u(x, y) = 1 - \tanh \left( 50 \left( (x - 0.5)^2 + (y - 0.5)^2 - \frac{1}{16} \right) \right) + 10^{-14}, \quad (x, y) \in (0, 1) \times (0, 1) \]
which has a sharp jump around the circle $(x - 0.5)^2 + (y - 0.5)^2 = 1/16$.

Fig. 3 shows the meshes of $N = 1600$ and corresponding solution contours obtained by $P^2$-DG interpolation with PP limiter. The mesh elements where the solution become negative and the PP limiter has been applied are indicated by blue dots. The convergence history is plotted in Fig. 4(a,b) while the number of time steps to reach $\varsigma = 1$ as $N$ increases for PP DG-interpolation is plotted in Fig. 4(c) where the curves for $P^1$-DG and $P^2$-DG have a similar increase rate $N_{\varsigma} = \sqrt{N} \max_i |x_i^{\text{old}} - x_i^{\text{new}}|$, verifying the estimate (2.29). ($\max_i |x_i^{\text{old}} - x_i^{\text{new}}| \approx 0.007$ for large $N$ for this example.)
5. Application of DG-interpolation to moving mesh DG simulation of RTE.

In this section, as an application, we consider the use of the DG-interpolation in a rezoning moving mesh DG method for the numerical solution of RTE in one and two spatial dimensions. Our goal is to show that the method maintains high-order accuracy of DG schemes while preserving the positivity of the radiative intensity.

The rezoning moving mesh method is illustrated in Fig. 5. As one can see, it involves three independent steps, generating the new mesh, interpolating the solution from the old mesh to the new one, and solving the RTE on the new mesh. In this work, we use the MMPDE moving mesh method described in §3 to generate the new mesh, the DG-interpolation scheme of §2 to interpolate the physical variables between the old and new meshes, and a high-order PP DG scheme of [32, 40, 43] to solve the RTE on the new mesh $T_{h}^{n+1}$. Since the first two steps have been discussed in previous sections, we focus on the last step in this section. The RTE is an integro-

differential equation modeling the conservation of photons [36]. We consider a case with isotropically scattering radiative transfer. The governing equation for this case reads as

$$\frac{1}{c} \frac{\partial I}{\partial t} + \Omega \cdot \nabla I + \sigma_t I = \frac{\sigma_s}{4\pi} \int_S I(x, \tilde{\Omega}, t) d\tilde{\Omega} + q,$$

where $c$ is the speed of photons, $\mathbf{x}$ is the spatial variable, $\nabla$ is the gradient operator with respect to $\mathbf{x}$, $\Omega$ is the unit angular variable, $S$ is the unit sphere, $t$ is time, $I(\mathbf{x}, \Omega, t)$ is the radiative intensity in the direction $\Omega$, $\sigma_s \geq 0$ is the scattering coefficient of the medium, $\sigma_t$ is the extinction coefficient of the medium which includes absorption and scattering, and $q(\mathbf{x}, \Omega, t)$ is a given source term. The vector $\mathbf{x}$ is described by the Cartesian coordinates $x, y, z$ while $\Omega = (\zeta, \eta, \mu)$ is usually described by

![Diagram of rezoning moving mesh method]

Fig. 5. Illustration of the rezoning moving mesh method.

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a polar angle $\beta$ measured with respect to the $z$ axis and a corresponding azimuthal angle $\varphi$. Denoting $\mu = \cos \beta$, $\zeta = \sin \beta \cos \varphi$, $\eta = \sin \beta \sin \varphi$ then

$$dx = dx dy dz, \quad d\Omega = \sin \beta d\beta d\varphi = -d\mu d\varphi.$$ 

In this work we consider the numerical solution of (5.1) in one and two spatial dimensions.

5.1. Positivity-preserving DG scheme for RTE in one dimension. The one-dimensional form of (5.1) reads as

$$\frac{1}{c} \frac{\partial I}{\partial t} + \mu \frac{\partial I}{\partial x} + \sigma_l I = \frac{\sigma_s}{2} \int_{-1}^{1} I(x, \mu, t) d\mu + q,$$ 

where $x \in (a, b)$, $\mu \in (-1, 1)$, and $t \in (0, T]$. The initial and boundary conditions are

$$I(x, \mu, 0) = I_0(x, \mu), \quad x \in (a, b)$$

and

$$\begin{cases} I(a, \mu, t) = I_l(\mu, t), & 0 < \mu \leq 1, \ 0 < t \leq T \\ I(b, \mu, t) = I_r(\mu, t), & -1 \leq \mu < 0, \ 0 < t \leq T. \end{cases}$$

We first use the discrete ordinate method (DOM) [30] to discretize (5.2) in the angular variable. Consider a Gauss-Legendre quadrature rule with weights $w_m$ and nodes $\mu_m$, $m = 1, \ldots, N_a$. We define the discrete-ordinate approximation for RTE as

$$\frac{1}{c} \frac{\partial I_m}{\partial t} + \mu_m \frac{\partial I_m}{\partial x} + \sigma_l I_m = \frac{\sigma_s}{2} \sum_{m' = 1}^{N_a} w_{m'} I_{m'} + \tilde{q}_m, \quad m = 1, \ldots, N_a$$

where $I_m = I_m(x, t) \approx I(x, \mu_m, t)$.

For temporal discretization, if we use an explicit scheme, we will have to take a small time step as $O(1/c)$ to ensure stability. To avoid this, we use the backward Euler scheme and have

$$\frac{\sigma_l I_m^{n+1}}{c} + \mu_m \frac{\partial I_m^{n+1}}{\partial x} = \frac{\sigma_s}{2} \sum_{m' = 1}^{N_a} w_{m'} I_{m'}^{n+1} + \tilde{q}_m^{n+1}, \quad m = 1, \ldots, N_a$$

where $I_m^{n+1} \approx I_m(x, t_{n+1})$ and

$$\tilde{q}_m^{n+1} = \tilde{q}_m + \frac{1}{c \Delta t} I_m^n, \quad \Delta t = t^{n+1} - t^n.$$ 

We now consider the DG spatial discretization for (5.4) on $T_h^{n+1}$. We only consider here for the case with $\mu_m > 0$, as a similar procedure can be used for $\mu_m < 0$. Assume that the cells of $T_h^{n+1}$ can be written as

$$K_i^{n+1} = (x_{i-1/2}^{n+1}, x_i^{n+1}), \quad i = 1, \ldots, N.$$ 

Multiplying (5.4) with a test function, integrating the resulting equation over $K_i^{n+1}$, taking integration by part for the second term, and applying the upwind numerical flux at the cell boundaries, we obtain the DG formulation as: find $I_m^{n+1} \in V_h(t_{n+1})$ such that, for $\forall \phi \in P^r(K_i^{n+1})$, $i = 1, \ldots, N$,

$$\int_{K_i^{n+1}} \sigma_l I_m^{n+1} \phi dx - \mu_m \int_{K_i^{n+1}} I_m^{n+1} \phi' dx + \mu_m (I_m^{n+1} \phi)|_{x_{i+1/2}^{n+1}}$$

$$= \int_{K_i^{n+1}} \sigma_s \psi_i^{n+1} \phi dx + \int_{K_i^{n+1}} \tilde{q}_m^{n+1} \phi dx + \mu_m (I_m^{n+1} \phi)|_{x_{i-1/2}^{n+1}},$$
where \( I_{m,i}^{n+1} = I_{m,i}^{n+1}(x)_{|K_i^{n+1}} \), \( \tilde{I}_n \) is the DG-interpolant of \( I_n \) from \( T_h^n \) to \( T_h^{n+1} \), and

\[
\Psi_{i}^{n+1} = \sum_{m=1}^{N_a} w_m I_{m,i}^{n+1}, \quad \hat{q}_{m,i}^{n+1} = \frac{1}{c\Delta t} \tilde{I}_m^{n+1}.
\]

Notice that the unknown variables in different angular directions are coupled in (5.6) through \( \Psi_{i}^{n+1} \). The so-called source iteration (SI) [31] is commonly employed to solve the equations separately. Denote the \( \ell \)-th iterate of the solution by \( I_{m,i}^{n+1,(\ell)} \).

Then the scheme reads as

\[
(5.7) \quad \int_{K_i^{n+1}} I_{m,i}^{n+1,(\ell+1)}(\sigma_t \phi - \mu_m \phi')dx + \mu_m(I_{m,i}^{n+1,(\ell+1)} \phi)|_{x_{i+1/2}}^{x_{i+1}} = \int_{K_i^{n+1}} (\sigma_s \Psi_i^{n+1,(s)} + \hat{q}_{m,i}^{n+1} \phi) dx + \mu_m(I_{m,i}^{n+1,(\ell+1)} \phi)|_{x_{i+1/2}}^{x_{i+1}}, \quad \forall \phi \in P^r(K_i^{n+1})
\]

where \( \Psi_i^{n+1,(s)} = \sum_{m=1}^{N_a} w_m I_{m,i}^{n+1,(s)} \) and \( I_{m,i}^{n+1,(s)} \) is taken as \( I_{m,i}^{n+1,(\ell+1)} \) when it is available and otherwise as \( I_{m,i}^{n+1,(\ell)} \). The sweeping direction in space is indicated in Fig. 6. The iteration is stopped when the maximum norm of the difference between any two consecutive iterates is smaller than \( 10^{-12} \). The radiative intensity is positive in

\[
\mu_m > 0:
\]

\[
\mu_m < 0:
\]

**Fig. 6.** Mesh sweeping directions for \( \mu_m > 0 \) (top) and \( \mu_m < 0 \) (bottom).

physics. However, a numerical approximation may contain negative values especially for high-order methods. The appearance of spurious negative values could lead to instability in the computation and slow iterative convergence. Thus, it is important to develop schemes that preserve the positivity of the radiative intensity. To this end, we mention that it has been proved in [32] any \( P^r \)-DG scheme (including the one described above) produces the positive cell averages for the one-dimensional RTE on fixed meshes provided that both the inflow boundary condition from the upstream cell (including the physical boundary condition for the first cell) and the source term are positive and the initial condition is nonnegative. As a consequence, the scaling PP limiter [33, 44] can be used to modify the DG schemes to preserve the positivity of the radiative intensity. In the following we briefly describe the procedure. The reader is referred to [32, 40] for the detail.

Denote the set of the \( r + 1 \) Gauss-Lobatto points on \( K_i^{n+1} \) as

\[
G_i^{n+1} = \{ x_{i-1/2}^{n+1}, x_{i+1/2}^{n+1}, \ldots, x_{i-r}^{n+1}, x_{i,r}^{n+1} = x_{i+1/2}^{n+1} \},
\]

and let \( I_{m,i}^{n+1,(\ell+1)} \) be the cell average of \( I_{m,i}^{n+1,(\ell+1)} \) on \( K_i^{n+1} \). Then the scaling limiter is defined as

\[
(5.9) \quad \hat{I}_{m,i}^{n+1,(\ell+1)} = \theta_i^{n+1}(I_{m,i}^{n+1,(\ell+1)} - I_{m,i}^{n+1,(\ell+1)}) + I_{m,i}^{n+1,(\ell+1)},
\]

where

\[
\theta_i^{n+1} = \min \left\{ \left| \frac{\hat{I}_{m,i}^{n+1,(\ell+1)}}{I_{m,i}^{n+1,(\ell+1)} - \hat{I}_{m,i}^{n+1,(\ell+1)}} \right|, 1 \right\}, \quad \bar{x}_i^{n+1} = \min_{\hat{x} \in G_i^{n+1}} \left\{ I_{m,i}^{n+1,(\ell+1)}(\hat{x}), 0 \right\}.
\]
The modified polynomial $\tilde{I}^{n+1,(\ell+1)}_{m,i}$ is positive at all of the Gauss-Lobatto points and satisfies

$$\sum_{i=1}^{N} |K^{n+1}_{i}| \tilde{I}^{n+1,(\ell+1)}_{m,i} = \sum_{i=1}^{N} |K^{n+1}_{i}| I^{n+1,(\ell+1)}_{m,i},$$

$$|\tilde{I}^{n+1,(\ell+1)}_{m,i} - I^{n+1,(\ell+1)}_{m,i}| = O(h^{r+1}).$$

For the above discussion and the positivity property of the DG-interpolation with PP limiter, we can claim the positivity preservation for the rezoning moving mesh DG method for the one-dimensional RTE (5.2). More specifically, assume both the inflow physical boundary and the source term are positive. Then, $I^{n+1}_{m,i}(\hat{x}_{i,\gamma}) \geq 0 \forall \hat{x}_{i,\gamma} \in G_{i}^{n}$ and $\tilde{I}^{n}_{m,i} \geq 0$ for all $i$ and $m$ imply that $I^{n+1}_{m,i}(\hat{x}_{i,\gamma}) \geq 0 \forall \hat{x}_{i,\gamma} \in G_{i}^{n+1}$ and $\tilde{I}^{n+1}_{m,i} \geq 0$ for all $i$ and $m$. Moreover, the scheme has the same convergence order as the DG scheme without using the PP limiter.

### 5.2. Positivity-preserving DG scheme for RTE on triangular meshes.

The two-dimensional form of (5.1) reads as

$$\frac{1}{c} \frac{\partial I}{\partial t} + \Omega \cdot \nabla I + \sigma_{I} I = \frac{\sigma_{s}}{4\pi} \int_{S} I(x, y, \Omega, t) d\Omega + q,$$

where $(x, y) \in \mathcal{D}$, $t \in (0, T]$, $\Omega = (\zeta, \eta)$, and

$$\zeta = \sqrt{1 - \mu^{2} \cos \varphi} \in (-1, 1), \ \eta = \sqrt{1 - \mu^{2} \sin \varphi} \in (-1, 1), \ \mu \in (-1, 1), \ \varphi \in (0, 2\pi).$$

The initial and inflow boundary conditions are

$$I(x, y, \Omega, 0) = I_{0}(x, y, \Omega), \quad (x, y) \in \mathcal{D}, \quad \Omega \in S$$

$$I(x, y, \Omega, t) = I_{b}(x, y, \Omega, t), \quad (x, y) \in \partial \mathcal{D}_{in}, \quad \Omega \in S, \quad t \in (0, T].$$

Here, $I_{0}(x, y, \Omega)$ and $I_{b}(x, y, \Omega, t)$ are given functions, $\partial \mathcal{D}_{in} = \{(x, y) \in \partial \mathcal{D} \mid n(x, y) \cdot \Omega < 0\}$, and $n(x, y)$ is the unit outward normal vector of the boundary. It is worth pointing out that no boundary condition is needed in $\Omega$-direction.

Once again, we use the DOM for the discretization in $\Omega$. Specifically, a Legendre-Chebyshev quadrature rule with weights $w_{m}$’s and nodes $\Omega_{m} = (\zeta_{m}, \eta_{m})$’s, $m = 1, \ldots, N_{a}$ is used to approximate the integral in (5.11). The meanings of $N_{l}$ and $N_{c}$ are given below. The nodes $\Omega_{m} = (\zeta_{m}, \eta_{m})$’s are given by

$$\zeta_{m} = \sqrt{1 - \mu^{2} \cos \varphi_{j}}, \quad \eta_{m} = \sqrt{1 - \mu^{2} \sin \varphi_{j}}, \quad m = (i - 1)N_{c} + j,$$

where $\mu_{i}, \ i = 1, \ldots, N_{l}$ denote the roots of the Legendre polynomial of degree $N_{l}$ and $\varphi_{j} = (2j - 1)\pi/N_{c}$, $j = 1, \ldots, N_{c}$ are the nodes based on a Chebyshev polynomial. Once the discrete angles are defined, the DOM approximation in $\zeta, \eta$, the DG discretization in $(x, y)$, and the PP limiter for (5.11) are similar to those in one dimension. To save space, we omit the detail here. The interested reader is referred to [42, 43]. We remark that the PP limiter uses a set of special quadrature points on triangle $K$ [45]. The limiter guarantees the nonnegativity of the approximate radiative intensity $\tilde{I}^{n+1,(\ell+1)}_{m,k}$ at the quadrature points while maintaining the mass conservation and high-order accuracy if the cell averages are nonnegative. Ling et al. [32] give a counterexample showing that $P^{r}$- or $Q^{r}$-DG schemes on rectangular meshes can result in negative cell averages for the two-dimensional RTE even if both
the inflow boundary value and the source term are positive and the initial condition
is nonnegative. On the other hand, we have not observed in our limited numerical
experience that \( P^r \)-DG schemes lead to negative cell averages on triangular meshes
(cf. §6) and thus we use the scaling PP limiter in our computation. It is interesting
to point out that the rotational PP limiter on triangular meshes [43] can be used for
situations with negative cell averages. Since this limiter is non-conservative, we will
not discuss it further in this work.

To conclude this section, we emphasize that, since the DG-interpolation with
PP limiter is positivity-preserving, our rezoning moving mesh DG method with DG-
interpolation is positivity-preserving as long as the physical PDE solver on a fixed
mesh is positivity-preserving.

6. Numerical results for RTE. In this section we present numerical results
obtained for the one- and two-dimensional versions of RTE using the rezoning moving
mesh DG method with and without the positivity-preserving (PP) limiter as described
in the previous section. For comparison purpose, we consider three variants of the
DG method.

- The fixed mesh (FM) DG method with PP limiter: The PP limiter is applied
to the DG solution of RTE;
- The moving mesh (MM) DG method with PP limiter: The PP limiter is
applied to both the DG solution of RTE and the DG-interpolation;
- The MM DG method without PP limiter: The PP limiter is applied to neither
the DG solution of RTE nor the DG-interpolation.

The numerical results are presented to demonstrate the performance of the DG-
interpolation scheme in the adaptive moving mesh solution of RTE. They also show
that the proposed moving mesh DG method with PP limiter can maintain high-order
accuracy of the DG method, preserve the positivity of radiative intensity, and be able
to adapt the mesh to the dynamic structures in the solution.

Unless otherwise stated, we use the Gauss-Legendre \( P_8 \) and Legendre-Chebyshev
\( P_8-T_8 \) rules to discretize angular variables for one- and two-dimensional problems,
respectively, and take the final time \( T = 0.1 \) and the time step size \( \Delta t = 2 \times 10^{-4} \).
For mesh movement, we take \( \tau = 0.01 \). The photon speed is \( c = 3 \times 10^8 \). For
the cases with exact solutions, the error in the computed solution is measured in the
(global) \( L^1 \) and \( L^\infty \) norm, i.e., \( \int_0^T \| e_h(\cdot, t) \|_{L^1} \, dt, \int_0^T \| e_h(\cdot, t) \|_{L^\infty} \, dt \).

Example 6.1. (A discontinuous example of 1D RTE for the absorbing-scattering
model.) In this example we take the scattering coefficient \( \sigma_s = 1 \), the extinction
coefficient and source term as

\[
\sigma_t = \begin{cases} 
1, & \text{for } 0 \leq x < 0.2 \\
900, & \text{for } 0.2 \leq x < 0.6 \\
90, & \text{for } 0.6 \leq x \leq 1
\end{cases}
\]

and

\[
q(x, \mu, t) = \begin{cases} 
100e^{-t}, & \text{for } 0 \leq x < 0.2 \\
1, & \text{for } 0.2 \leq x < 0.6 \\
1000e^{3t}, & \text{for } 0.6 \leq x \leq 1.
\end{cases}
\]

The initial condition is \( I(x, \mu, 0) = 15x \) and the boundary condition is

\[
\begin{align*}
I(0, \mu, t) &= 0, & \text{for } 0 < \mu \leq 1, & 0 < t \leq T \\
I(1, \mu, t) &= 15 + 2t, & \text{for } -1 \leq \mu < 0, & 0 < t \leq T.
\end{align*}
\]

The solution of this problem has two sharp layers. Since its analytical form is
unavailable, for comparison purpose we take the numerical solution obtained by the
\( P^2 \)-DG method with PP limiter and a fixed mesh of \( N = 10000 \) as the reference

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solution. The solutions at the final time in the directions \( \mu = 0.5255 \) and 0.9603 obtained by the moving mesh \( P^1 \)-DG and \( P^2 \)-DG methods (\( N = 40 \)) with and without PP limiter are shown in Fig. 7. We can see that the computed radiative intensity can have negative values for both \( P^1 \)-DG and \( P^2 \)-DG and for fixed and moving meshes while those using the PP limiter can stay nonnegative.

The mesh trajectories for the moving mesh \( P^2 \)-DG method with PP limiter are shown in Fig. 10(a) which demonstrates the ability of the method to concentrate mesh points in the regions of sharp layers.

The solution in the direction \( \mu = -0.1834 \) and 0.1834 obtained by the moving mesh \( P^2 \)-DG method (\( N = 80 \)) with PP limiter is compared with the \( P^2 \)-DG method with PP limiter and the fixed mesh of \( N = 80 \) and 1280 in Fig. 8 and 9, respectively. The results show that the moving mesh solution (\( N = 80 \)) is more accurate than those with fixed meshes of \( N = 80 \) and 1280. The figures also show that our moving mesh method with PP limiter has the ability to preserve the radiative intensity positivity.

To show the cost of the DG-interpolation in the moving mesh DG method with PP limiter, we plot the average number of time steps \( N_\tau \) in Fig. 10(b,c). One can see that \( N_\tau \) is increasing as the mesh is being refined when a fixed time step size \( \Delta t = 1/5000 = 2 \times 10^{-4} \) is used. On the other hand, \( N_\tau \) stays almost constant when the time step size is chosen as \( \Delta t = 0.5 \min(h_{\text{old}}^\|, h_{\text{new}}^\|) \) and 0.1 \( \min(h_{\text{old}}^\|, h_{\text{new}}^\|) \) and is larger for the former than the latter. These are consistent with the analysis in \S2.3.

![Graphs showing the computed radiative intensity for different DG methods and mesh conditions.](image)

**Fig. 7. Example 6.1.** The computed radiative intensity at the final time in the directions \( \mu = 0.5255 \) and 0.9603 are obtained by the moving mesh \( P^1 \)-DG (Top) and \( P^2 \)-DG (Bottom) methods with and without PP limiter and with \( N = 40 \). The dots represent the radiative intensity at the first Gauss-Lobatto points on each cell.

**Example 6.2.** (An accuracy test of 2D RTE for the absorbing-scattering model.)

In this example, we take \( \sigma_t = 22000 \), \( \sigma_s = 1 \). The source term and initial and boundary conditions are chosen such that the exact solution is given by

\[
I(x, y, \zeta, \eta, t) = e^t \left( (\zeta^2 + \eta^2) \cos^4 \left( \frac{\pi}{2}(x + y) \right) + 10^{-14} \right).
\]
Fig. 8. Example 6.1. The computed radiative intensity at the final time in the direction $\mu = -0.1834$ obtained by the moving mesh $P^2$-DG method ($N = 80$) with PP limiter is compared with those obtained by the fixed mesh $P^2$-DG method with PP limiter of $N = 80$ and 1280. The dots represent the radiative intensity at the mid-points on each cell.

For this problem, the computed radiative intensity can have negative values for both the $P^1$-DG and $P^2$-DG methods. The error and convergence order for $P^2$-DG methods is shown in Table 1. (The results for $P^1$-DG method are omitted here to save space. They are similar to those for $P^2$-DG.) We can see that the third-order convergence for $P^2$-DG is achieved for fixed and moving meshes and with or without PP limiter. The average number of time steps $N_\tau$ used in the DG-interpolation for the moving mesh DG method is small (almost one) for relatively coarse meshes and then increases as the mesh is being refined. This is because the mesh deformation over a time step (with a fixed time step size) is small compared to the minimum element diameter for small $N$ and then becomes larger for large $N$. This observation is consistent with that for the previous one-dimensional example and the analysis in §2.3.

Example 6.3. (A discontinuous example of 2D RTE for the transparent model.)

In this test, we take $\sigma_t = 0$, $\sigma_s = 0$, $q = 0$, $\zeta = 0.3$, and $\eta = 0.5$. The computational domain is $(0, 1) \times (0, 1)$. The initial and boundary conditions are

$$I(x, y, \zeta, \eta, 0) = \begin{cases} \varepsilon, & \text{for } y < \frac{y}{\zeta} x \\ \cos^6 \left( \frac{\pi}{2} y \right), & \text{otherwise} \end{cases}$$

$$I(0, y, \zeta, \eta, t) = \cos^6 \left( \frac{\pi}{2} y \right) \cos^{10}(t),$$

$$I(x, 0, \zeta, \eta, t) = \varepsilon,$$

where $\varepsilon = 10^{-14}$. The exact solution of this example is

$$I(x, y, \zeta, \eta, t) = \begin{cases} \varepsilon, & \text{for } y < \frac{y}{\zeta} x \\ \cos^6 \left( \frac{\pi}{2} (y - \frac{y}{\zeta} x) \right) \cos^{10}(t - \frac{y}{\zeta} x), & \text{otherwise} \end{cases}$$

which is discontinuous along $y = \frac{y}{\zeta} x$. 

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Fig. 9. Example 6.1. The computed radiative intensity at the final time in the direction \( \mu = 0.1834 \) obtained by the moving mesh \( P^2\)-DG method (\( N = 80 \)) with PP limiter is compared with those obtained by the fixed mesh \( P^2\)-DG method with PP limiter of \( N = 80 \) and 1280. The dots represent the radiative intensity at the mid-points on each cell.

Fig. 10. Example 6.1. (a): The mesh trajectories are obtained by the moving mesh \( P^2\)-DG method with PP limiter and \( N = 80 \). (b) and (c): The average number of time steps used in the DG-interpolation for the moving mesh DG method with PP limiter.

The radiative intensity obtained by the moving mesh \( P^2\)-DG method with PP limiter and \( N = 1600 \) are plotted in Fig. 11(a) and the radiative intensity cut along the line \( y = 0.495 \) obtained with and without PP limiter is shown in Fig. 11(b). The cells where the PP limiter has been applied are marked with white dots. The computed radiative intensity can have negative values for this example for the DG schemes without PP limiter.

The contours of the radiative intensity obtained by the \( P^2\)-DG method with PP limiter on a moving mesh of \( N = 1600 \) and fixed meshes of \( N = 1600 \) and 57600 are shown in Fig. 12(a,b,c). The corresponding cut along the line \( y = 0.495 \) is plotted in Fig. 12(d,e). The results show that the moving mesh solution (\( N = 1600 \)) is more accurate than that with the fixed mesh of \( N = 1600 \) and is comparable with that with the fixed mesh of \( N = 57600 \). The figures also show that our moving mesh \( P^2\)-DG method with PP limiter produces the positive radiative intensity.
Table 1
Example 6.2. Error and convergence order for three \( P^2 \)-DG methods.

| \( N \)  | \( L^1 \)-error | order | \( L^\infty \)-error | order | limiter(%) | \( N_c \) |
|------|----------------|-------|---------------------|-------|-----------|-------|
| Fixed mesh \( P^2 \)-DG method with PP limiter |
| 1600 | 7.400E-07      |       | 1.031E-05          |       | 5.00      | -     |
| 6400 | 9.242E-08      | 3.00  | 1.295E-06          | 2.993 | 2.50      | -     |
| 25600| 1.152E-08      |       | 1.639E-07          | 2.982 | 1.25      | -     |
| 57600| 3.407E-09      |       | 4.909E-08          | 2.973 | 0.83      | -     |
| Moving mesh \( P^2 \)-DG method with PP limiter |
| 1600 | 8.163E-07      |       | 2.561E-05          |       | 5.00      | 1.02  |
| 6400 | 1.008E-07      | 3.017 | 3.711E-06          | 2.787 | 2.50      | 1.07  |
| 25600| 1.177E-08      |       | 3.538E-07          | 3.391 | 1.25      | 1.17  |
| 57600| 3.441E-09      |       | 9.399E-08          | 3.269 | 0.83      | 1.67  |
| Moving mesh \( P^2 \)-DG method without PP limiter |
| 1600 | 8.117E-07      |       | 2.561E-05          |       | -         | 1.02  |
| 6400 | 1.007E-07      | 3.011 | 3.711E-06          | 2.787 | -         | 1.07  |
| 25600| 1.175E-08      |       | 3.538E-07          | 3.391 | -         | 1.17  |
| 57600| 3.440E-09      |       | 9.398E-08          | 3.269 | -         | 1.67  |

The error and convergence history in the \( L^1 \) norm are shown in Fig. 13(a) for the fixed and moving mesh DG methods with PP limiter. One can see that both fixed and moving meshes lead to almost the same convergence order. It is worth pointing out that we cannot expect the fixed/moving mesh DG can achieve the optimal order for this problem since the solution is discontinuous. The actual order of \( P^1 \)-DG is about 0.5th and 1st for \( P^2 \)-DG. Moreover, the figures show that a moving mesh produces a more accurate solution than a fixed mesh of the same number of elements for this example.

To show the efficiency of the moving mesh DG method with PP limiter, we plot the average number of time steps used in the DG-interpolation in Fig. 13(b), which indicates that \( N_c \) increases as the mesh is being refined. We also plot the \( L^1 \) norm of the error against the CPU time in Fig. 13(c). It shows that the moving mesh DG method is more efficient than the fixed mesh method and \( P^2 \)-DG is more efficient than \( P^1 \)-DG.

It is interesting to mention that a quasi-Lagrangian moving mesh DG method has been developed in [42] for RTE. Compared to the method in the current work, it does not require interpolation of the physical variables between old and new meshes although extra work is needed to compute a convection term in the DG formulation of RTE that is caused by mesh movement. It is unclear to us yet how to preserve the radiative intensity in the quasi-Lagrangian method, which is an interesting future research topic. To obtain a rough comparison, we plot in Fig. 14 the \( L^1 \) norm of the error against CPU time for both the quasi-Lagrangian and rezoning moving mesh DG methods (without PP limiter). We can see that both methods have comparable efficiency while the rezoning method is slightly more efficient when the mesh is not very fine. As the mesh is being refined, the DG-interpolation will need more steps and become more expensive, and then the quasi-Lagrangian method becomes more efficient. It should also be pointed that this comparison is done with a fixed time step.
size. The situation may be different when a variable time step size is used.

Fig. 11. Example 6.3. (a): The radiative intensity contours are obtained by the moving mesh \( P^2 \)-DG method with PP limiter (\( N = 1600 \)). The white dots represent the cells where the PP limiter has been applied. (b): The radiative intensity cut along the line \( y = 0.495 \) is obtained by the moving mesh \( P^2 \)-DG method with and without PP limiter (\( N = 1600 \)).

Fig. 12. Example 6.3. The radiative intensity contours (and mesh) at \( t = 0.1 \) is obtained by \( P^2 \)-DG method with PP limiter. (d) and (e): The radiative intensity cut along the line \( y = 0.495 \).

7. Conclusions. In the previous sections we have presented a high-order DG-interpolation scheme for deforming unstructured meshes based on the pseudo-time-dependent linear equation (2.1). Such a scheme can be used for indirect arbitrary-Lagrangian-Eulerian and rezoning moving mesh methods in numerical solution of partial differential equations. We have shown that the scheme is conservative. It is also positivity-preserving when a linear scaling limiter is used. The scheme places no restrictions on the deformation of the old mesh to the new one. The cost of the scheme has been investigated. The total cost of each use of the DG-interpolation is \( \mathcal{O}(N_v N_\varsigma) \), where \( N_v \) is the number of mesh vertices and \( N_\varsigma \) is the number of time steps used to integrate (2.1) from \( \varsigma = 0 \) to \( \varsigma = 1 \). It is shown that \( N_\varsigma \) depends on the magnitude of mesh deformation relative to the size of mesh elements. It stays constant as the
As an application example, we have considered the use of the DG-interpolation scheme in the rezoning moving mesh DG solution of RTE. RTE has been discretized in our computation in angular directions using the discrete ordinate method, in space using the DG method, and in time using the backward Euler scheme. At each time step, the new mesh is generated using the MMPDE moving mesh method and then the radiative intensity is interpolated from the old mesh to the new one using the DG-interpolation scheme. Numerical results obtained for examples in one and two spatial dimensions with various settings have demonstrated that the resulting rezoning moving mesh DG method is 2nd-order with $P_1$-DG and 3rd-order with $P_2$-DG, more efficient than the method with a fixed mesh, and able to preserve the positivity of the radiative intensity when the PP limiter is used. It is also shown that the scheme is comparable in efficiency for not very fine meshes with a quasi-Lagrangian moving mesh DG method developed in [42] for RTE when a fixed time step size is used. It is still unclear if the latter can be made to preserve the positivity of the radiative intensity.
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