Error Bounds for Numerical Integration of Oscillatory Bessel Transforms with Algebraic or Logarithmic Singularities

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Abstract In this paper, we present and analyze the Clenshaw-Curtis-Filon methods for computing two classes of oscillatory Bessel transforms with algebraic or logarithmic singularities. More importantly, for these quadrature rules we derive new computational sharp error bounds by rigorous proof. These new error bounds share the advantageous property that some error bounds are optimal on $\omega$ for fixed $N$, while other error bounds are optimal on $N$ for fixed $\omega$. Furthermore, we prove from the presented error bounds in inverse powers of $\omega$ that the accuracy improves greatly, for fixed $N$, as $\omega$ increases.

Keywords: oscillatory, Bessel transforms singularities, Clenshaw-Curtis-Filon methods, quadrature rules error bounds.

1 Introduction

Highly oscillatory Bessel transforms arise widely in mathematical and numerical modeling of oscillatory phenomena in many areas of sciences and engineering such as astronomy, electromag-
netics, acoustics, scattering problems, physical optics, electrodynamics, and applied mathematics [2, 3, 10, 16]. In this paper, we focus on new computational sharp error bounds of the quadrature rules for singular oscillatory Bessel transforms of the forms

\begin{align}
I_1[f] &= \int_0^b x^\alpha f(x)J_m(\omega x)dx, \\
I_2[f] &= \int_0^b x^\alpha \ln(x)f(x)J_m(\omega x)dx,
\end{align}

where \( f(x) \) is suitably smooth in \([0, b]\), \( J_m(z) \) is the Bessel function \( \text{[1]} \) of the first kind and of order \( m \) with \( \text{Re}(m) > -1 \), \( \omega \) is a large parameter, \( b \) are real and finite, and \( \alpha > -1 \). In particular, it should be noticed that transforms (1.1) and (1.2) are integrals with algebraic and logarithmic singularities, respectively.

In most of the cases, such integrals cannot be calculated analytically and one has to resort to numerical methods [9]. The numerical evaluation can be difficult when the parameter \( \omega \) is large, because in that case the integrand is highly oscillatory. The singularities of algebraic or logarithmic type and possible high oscillations of the integrands in (1.1) and (1.2) make the above integrals very difficult to approximate accurately using standard methods, e.g., Gaussian quadrature rules. It is well known [9] that a prohibitively large number of quadrature points is needed if one uses a classic rule such as Gaussian quadratures, or any quadrature method based on (piecewise) polynomial interpolation of the integrands.

In the last few decades, much progress has been made in developing numerical schemes for generalized Bessel transform \( \int_a^b f(x)J_m(\omega g(x))dx \) without singularity. For example, the modified Clenshaw-Curtis method [28] was introduced for efficiently computing \( \int_0^1 f(x)J_m(\omega x)dx \) for \( m \) being an integer in 1983; the Levin method [21], Levin-type method [27], and generalized quadrature rules [11, 32] were also available for approximating \( \int_a^b f(x)J_m(\omega x)dx \) for \( \text{Re}(m) > -1 \). However, the Levin method, Levin-type methods and generalized quadrature rules cannot be used if \( 0 \notin [a, b] \). In addition, based on a diffeomorphism transformation, the reference [33] extended the Filon-type method to the efficient computation of the integrals \( \int_0^1 f(x)J_m(\omega g(x))dx \) with the exotic oscillator \( g(x) \) satisfying that for \( r \geq 0 \), and \( g(0) = g'(0) = \cdots = g^{(r)}(0) = 0, g^{(r+1)}(0) \neq 0, g'(x) \neq 0 \) for \( x \in (0, 1] \), where \( \text{Re}(m) > 1/(r + 1) \). In many situations the accuracy of the Filon-type method proposed in [33] is significantly higher than that of other methods. As a matter of fact, it requires the solution of a linear system that becomes more ill-conditioned as the number of interpolation nodes increases, and one has to adopt higher-order digit arithmetic to get the required accuracy. Furthermore, to avoid the Runge phenomenon, the Clenshaw-Curtis-Filon-type method [34] based on Clenshaw-Curtis points is designed for computing Bessel transform.
\[ \int_a^b f(x) J_m(\omega x) dx \] without singularity. Here, it should be also mentioned that the homotopy perturbation method in [4, 5] was presented to compute \[ \int_a^b f(x) J_m(\omega x) dx \]. Recently, Chen [6, 7] also proposed two different complex integration methods for approximating \[ \int_a^b f(x) J_m(\omega x) dx \] if \( 0 \not\in [a, b] \). To the best of our knowledge, so far little research has been done on the numerical computation of the integrals (1.1) and (1.2) with an algebraic or logarithmic singularity.

Consequently, our aim is to demonstrate high efficiency of the proposed quadrature rules for such integrals (1.1) and (1.2) by constructing error bounds. In the next section, we propose the Clenshaw-Curtis-Filon methods for computing the integrals (1.1) and (1.2). Here, the required modified moments can be efficiently calculated by a recurrence relation. Section 3 sets up new and computational sharp error bounds of these quadrature rules by theory analysis. In Section 4, we design a higher order method and derive its error estimate in inverse power of \( \omega \). From these new error bounds, it can be seen that for fixed \( \omega \), the error bounds are optimal on \( N \), while for fixed \( N \) the error bounds are optimal on \( \omega \). Moreover, for fixed \( N \), the larger the values of \( \omega \), the higher the accuracy.

## 2 Clenshaw-Curtis-Filon methods for computing (1.1) and (1.2)

Chebyshev interpolation has precisely the same effect as taking partial sum of an approximation Chebyshev series expansion [24]. Suppose that \( f(x) \) is absolutely continuous on \([0, b]\). Let \( P_N f(x) \) denote an interpolant of \( f(x) \) of degree \( N \) in the Clenshaw-Curtis points

\[
x_k = \frac{b - a}{2} + \frac{b + a}{2} \cos \left( \frac{k\pi}{N} \right), \quad k = 0, 1, \ldots, N.
\]  

(2.3)

Then, the polynomial \( P_N f(x) \) can be expressed by (see [24, Eq. 6.27, 6.28])

\[
P_N f(x) = \sum_{j=0}^{N} a_j T_j^{**}(x), \quad \text{where} \quad a_j = \frac{2}{N} \sum_{k=0}^{N} f(x_k) T_j^{**}(x_k),
\]  

(2.4)

where the double primes indicate that the first and last terms of the sum are to be halved, \( T_j^{**}(x) \) denotes the shifted Chebyshev polynomial of the first kind of degree \( j \) on \([0, b]\). The coefficients \( a_j \) can be computed efficiently by FFT [8, 30].

The Clenshaw-Curtis-Filon (CCF) methods for (1.1) and (1.2) are defined, respectively, as follows,

\[
I_{1}^{CCF}[f] = \int_{0}^{b} x^\alpha P_N f(x) J_m(\omega x) dx = b^{\alpha+1} \sum_{j=0}^{N} a_j M_j,
\]  

(2.5)
and

\[ P_{CCF}^{[2]}[f] = \int_0^b x^\alpha \ln(x) P_N f(x) J_m(\omega x) dx \]

\[ = b^{\alpha+1} \sum_{j=0}^N a_j [\ln(b) M_j + \tilde{M}_j], \quad (2.6) \]

where, for \( r = b\omega, \)

\[ M_j = \int_0^1 x^\alpha T_j^*(x) J_m(r x) dx, \quad (2.7) \]

\[ \tilde{M}_j = \int_0^1 x^\alpha \ln(x) T_j^*(x) J_m(r x) dx, \quad (2.8) \]

are called the modified moments, where \( T_j^*(x) \) denotes the shifted Chebyshev polynomial on \([0, 1]\), and which can be computed efficiently, as described below.

**Fast computations of the modified moments:**

The homogeneous recurrence relation of the modified moments \( M_j \), was provided by Piessens [20][29], as follows:

\[
\frac{r^2}{16} M_{j+4} + [(j + 3)(j + 3 + 2\alpha) + \alpha^2 - m^2 - \frac{r^2}{4}] M_{j+2} \\
+ [4(m^2 - \alpha^2) - 2(j + 2)(2\alpha - 1)] M_{j+1} \\
- [2(j^2 - 4) + 6(m^2 - \alpha^2) - 2(2\alpha - 1) - \frac{3r^2}{8}] M_j \\
+ [4(m^2 - \alpha^2) - 2(j - 2)(2\alpha - 1)] M_{j-1} \\
+ [(j - 3)(j - 3 - 2\alpha) + \alpha^2 - m^2 - \frac{r^2}{4}] M_{j-2} + \frac{r^2}{16} M_{j-4} = 0, \quad (2.9) 
\]

It is worth to notice that

\[ \frac{\partial}{\partial \alpha} M_j = \tilde{M}_j. \]

Therefore, by differentiating the above recurrence relation (2.9) with respect to \( \alpha \), we find \( \tilde{M}_j \) satisfying the following recurrence relation:

\[
\frac{r^2}{16} \tilde{M}_{j+4} + [(j + 3)(j + 3 + 2\alpha) + \alpha^2 - m^2 - \frac{r^2}{4}] \tilde{M}_{j+2} \\
+ [4(m^2 - \alpha^2) - 2(j + 2)(2\alpha - 1)] \tilde{M}_{j+1} \\
- [2(j^2 - 4) + 6(m^2 - \alpha^2) - 2(2\alpha - 1) - \frac{3r^2}{8}] \tilde{M}_j \\
+ [4(m^2 - \alpha^2) - 2(j - 2)(2\alpha - 1)] \tilde{M}_{j-1} \\
+ [(j - 3)(j - 3 - 2\alpha) + \alpha^2 - m^2 - \frac{r^2}{4}] \tilde{M}_{j-2} + \frac{r^2}{16} \tilde{M}_{j-4} \\
= -2(\alpha + j + 3) M_{j+2} + 4(2\alpha + j + 2) M_{j+1} + 4(3\alpha + 1) M_j \\
+ 4(2\alpha - j + 2) M_{j-1} + 2(j - \alpha - 3) M_{j-2}. \quad (2.10) 
\]
Because of the symmetry of the recurrence relation of the Chebyshev polynomials $T_j(x)$, it is convenient to get $T_{-j}(x) = T_j(x)$, $j = 1, 2, \ldots$, and, consequently $T^*_j(x) = T^*_j(x)$, $M_{-j} = M_j$ and $\tilde{M}_{-j} = \tilde{M}_j$. It can be verified easily that both (2.9) and (2.10) are valid, not only for $j \geq 5$, but for all integers of $j$. Unfortunately, for (2.9) and (2.10) both the forward recursion and the backward recursion are asymptotically unstable [20, 29]. Nevertheless, in practical applications the instability is less pronounced if $\omega \geq 2j$. Practical experiments demonstrate that $M_j$ and $\tilde{M}_j$ can be computed accurately using the forward recursion as long as $\omega \geq 2j$. But for $\omega < 2j$ the loss of significant figures increases and recursion in the forward direction is no longer applicable. In this case Lozier’s algorithm [22] or Oliver’s algorithm [25] has to be used. This means that both (2.9) and (2.10) have to be solved as a boundary value problem with six initial values and two end values. The solution of this boundary value problem requires the solution of a linear system of equations having a band structure. The end value can be estimated by the asymptotic expansions in [20] or can be set equal to zero. The Lozier’s algorithm incorporates a numerical test for determining the optimum location of the endpoint, when the end value is set to be zero. The advantage is that a user-required accuracy is automatically obtained, without computation of the asymptotic expansion. For details one can refer to [20, 22, 25, 29]. To start the recurrence relation with $k = 0, 1, 2, 3, \ldots$, we only need $M_0, M_1, M_2,$ and $M_3$. By plugging the shifted Chebyshev polynomials $T_0^*(x) = 1$, $T_1^*(x) = 2x - 1$, $T_2^*(x) = 8x^2 - 8x + 1$ and $T_3^*(x) = 32x^3 - 48x^2 + 18x - 1$ on $[0, 1]$ into (2.7), we obtain

$$
M_0 = G(\omega, m, \alpha),
$$

$$
M_1 = 2G(\omega, m, \alpha + 1) - M_0,
$$

$$
M_2 = 8G(\omega, m, \alpha + 2) - 4M_1 - 3M_0,
$$

$$
M_3 = 32G(\omega, m, \alpha + 3) - 6M_2 - 15M_1 - 10M_0.
$$

Then, it is apparent from the above equalities that

$$
\tilde{M}_0 = \frac{\partial}{\partial \alpha} G(\omega, m, \alpha),
$$

$$
\tilde{M}_1 = 2\frac{\partial}{\partial \alpha} G(\omega, m, \alpha + 1) - \tilde{M}_0,
$$

$$
\tilde{M}_2 = 8\frac{\partial}{\partial \alpha} G(\omega, m, \alpha + 2) - 4\tilde{M}_1 - 3\tilde{M}_0,
$$

$$
\tilde{M}_3 = 32\frac{\partial}{\partial \alpha} G(\omega, m, \alpha + 3) - 6\tilde{M}_2 - 15\tilde{M}_1 - 10\tilde{M}_0,
$$

where, from [1, p.480], [15, p.676] and [23, p.44], we find several moments formulae as follows,
for $\Re(m + \alpha) > -1$,

$$
G(\omega, m, \alpha) = \int_0^1 x^\alpha J_m(rx)dx
= \frac{2^\omega \Gamma\left(\frac{m+\alpha+1}{2}\right)}{\Gamma\left(\frac{m-\alpha+1}{2}\right)} + \frac{1}{r^\alpha}[(\alpha + m - 1)J_m(r)S_{a-1,m-1}(r) - J_{m-1}(r)S_{a,m}(r)],
$$

(2.15)

$$
G(\omega, m, \alpha) = \frac{r^m}{2^m(\alpha + m + 1)\Gamma(m + 1)} {\Gamma(m + \frac{\alpha + 1}{2})\over \Gamma(m + \frac{\alpha - 1}{2})} F_2(\alpha + m + 1; \frac{\alpha + m + 3}{2}, m + 1; -\frac{r^2}{4}),
$$

(2.16)

$$
G(\omega, m, \alpha) = \frac{\Gamma(m+\frac{\alpha+1}{2})}{\pi^\alpha r^\alpha} \sum_{j=0}^{\infty} (m + 2j + 1)\Gamma(m+\frac{\alpha+1}{2}+j)\frac{J_m(r) - \cos(\pi(\mu - \nu)/2)Y_r(z)}{\Gamma(\frac{\nu+1}{2} - j)},
$$

(2.17)

where $S_{\mu,\nu}(z), \Gamma(z), _1 F_2(\mu; \nu, \lambda; z)$ denote a Lommel function of the second kind, the gamma function, a class of generalized hypergeometric function, respectively. Moreover, _1 F_2(\mu; \nu, \lambda; z) converges for all $|z|$. From [32, p.346], $S_{\mu,\nu}(z)$ can be expressed in terms of _1 F_2(\mu; \nu, \lambda; z), namely,

$$
S_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu + \nu + 1)(\mu + \nu + 1)} _1 F_2\left(1; \frac{\mu - \nu + 3}{2}, \frac{\mu + \nu + 3}{2}; -\frac{z^2}{4}\right)
- \frac{2\nu^{-1}\Gamma(m+\frac{\alpha+1}{2})}{\pi^\alpha} (\nu(z) - \cos(\pi(\mu - \nu)/2)Y_r(z)),
$$

(2.18)

where $Y_r(z)$ is a Bessel function of the second kind of order $\nu$. The right-hand sides of (2.11-2.14) involve the derivatives of the generalized hypergeometric function with respect to the parameter $\alpha$, which have been shown in [19]. The required derivatives of the gamma function are also described in terms of the Psi (Digamma) function $\psi_0(z)$, such as

$$
\Gamma'(z) = \Gamma(z)\psi_0(z).
$$

(2.19)

The efficient implementation of the modified moments is based on the fast computation of the Lommel functions $S_{\mu,\nu}(z)$ and the hypergeometric function _1 F_2(\mu; \nu, \lambda; z). Excellent references in this area are [14] [31]. Obviously, when programming the proposed algorithm in a language like Matlab, we can calculate the values of $\Gamma(z), J_\mu(z)$ and _1 F_2(\mu; \nu, \lambda; z) by invoking the built-in functions ‘ gamma(z),’ ‘ besselj(m, z),’ and calling mfun(‘ hypergeom’, [\mu], [\nu, \lambda], z) from Maple, respectively.

The computation of $S_{\mu,\nu}(z)$:

(1) For large $|z|$ and $|\arg z| < \pi$, we can calculate efficiently $S_{\mu,\nu}(z)$ by truncating the following asymptotic expansion (see [31] pp. 351-352)) in inverse powers of $z$:

$$
S_{\mu,\nu}(z) = z^{\mu-1} \left\{ 1 - \frac{(\mu - 1)^2 - \nu^2}{z^2} + \frac{(\mu - 1)^2 - \nu^2]}{z^4} \right. \left[ \frac{(\mu - 3)^2 - \nu^2]}{z^4} \right. \left. + \ldots \right.

+ (-1)^p \left[ \frac{(\mu - 1)^2 - \nu^2]}{z^{2p}} \right. \left[ (\mu - 2p + 1)^2 - \nu^2] \right. \left. \frac{1}{z^{2p}} \right] + O(z^{\mu-2p-2}),
$$

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(2) For small $|z|$, we prefer to compute $S_{\mu,\nu}(z)$ using (2.18).

So, when $r = b\omega$ is large, such as $r \geq 50$, we prefer to compute the moments using (2.15). When $r = b\omega$ is small, for example $r < 50$, the moments (2.17) are available. This may be due to the property that $J_m(r)$ is a fast decreasing function of $m$ when $m > r$. Practical experiments also demonstrate that $J_m(r)$ can decrease to zero quite rapidly when $m$ is a little larger than $r$. Fortunately, the moments (2.16) is available for all $r$.

### 3 Error bounds of the CCF methods (2.5) and (2.6)

To obtain results that are absolutely reliable for numerical computations, it is necessary to construct a upper bound for the corresponding error. In the following, we will consider new and computational error bounds. These new error bounds share that for fixed $N$, the error bounds are optimal on $\omega$, while for fixed $\omega$ the error bounds are optimal on $N$.

In the following, in order to derive these new error bounds in inverse powers of $\omega$, we first give Lemmas 3.1 and 3.2.

**Lemma 3.1** For every $t \in [0, b]$ ($b > 0$) and $\alpha > -1$, it is true that, for $\omega \geq 1$,

\[
\int_0^t x^\alpha J_m(\omega x)dx = \begin{cases} 
O\left(\frac{1}{\omega^{\alpha+1}}\right), & \text{if } -1 < \alpha < 0, \\
O\left(\frac{1}{\omega}\right), & \text{if } \alpha \geq 0,
\end{cases}
\]  

(3.20)

\[
\int_0^t x^\alpha \ln(x)J_m(\omega x)dx = \begin{cases} 
O\left(\frac{1+\ln(\omega)}{\omega^{\alpha+1}}\right), & \text{if } -1 < \alpha \leq 0, \\
O\left(\frac{1}{\omega}\right), & \text{if } \alpha > 0.
\end{cases}
\]  

(3.21)

**Proof:** We divide our proof in three steps.

(1) For $-1 < \alpha < 0$, setting $y = \omega x$ yields that

\[
\int_0^t x^\alpha J_m(\omega x)dx = \frac{1}{\omega^\alpha + 1} \int_0^{\omega t} y^\alpha J_m(y)dy,
\]

\[
\int_0^t x^\alpha \ln(x)J_m(\omega x)dx = \frac{1}{\omega^{\alpha+1}} \left[ \int_0^{\omega t} y^\alpha \ln(y)J_m(y)dy - \ln(\omega) \int_0^{\omega t} y^\alpha J_m(y)dy \right].
\]

Obviously, whether the integral upper limit $\omega t$ in the right-side of the above two formulae is finite or not, by convergence tests for improper integrals (*Cauchy’s test* or *Dirichelet’s test*), we know that the resulting defect or infinite integrals are convergent. It leads to the first identities in (3.20) and (3.21).

(2) For $\alpha = 0$, combining $\int_0^\omega J_m(t)dt = 1$ [11 p.486] and the moments formula (2.15), we have

\[
\int_0^t J_m(\omega x)dx = \frac{1}{\omega} \int_0^{\omega t} J_m(y)dy = O\left(\frac{1}{\omega}\right).
\]
If $0 < \omega t \leq 1$, from [11, p.362] and [26], we have

$$|J_m(x)| \leq 1, \ m \geq 0, \ x \in \mathbb{R}. \quad (3.22)$$

So, the first identity in (3.21) follows that

$$\left| \int_0^\omega \ln(x) J_m(\omega x) dx \right| = \frac{1}{\omega} \left| \int_0^{\omega a} \ln\left(\frac{y}{\omega}\right) J_m(y) dy \right|$$

$$\leq \frac{1}{\omega} \int_0^{\omega a} |\ln(y) - \ln(\omega)| |J_m(y)| dy$$

$$\leq \frac{1}{\omega} \int_0^{\omega a} ( - \ln(y) + \ln(\omega)) dy$$

$$= \frac{1 + \ln(\omega)}{\omega}. \quad (3.23)$$

If $\omega t > 1$, from the proof of (3.23), we then obtain

$$\left| \int_0^\omega \ln(x) J_m(\omega x) dx \right| = \frac{1}{\omega} \left| \int_0^{\omega a} \ln\left(\frac{y}{\omega}\right) J_m(y) dy \right|$$

$$\leq \frac{1}{\omega} \int_0^{\omega a} (\ln(y) - \ln(\omega)) J_m(y) dy + \frac{1}{\omega} \int_1^{\omega a} (\ln(y) - \ln(\omega)) J_m(y) dy$$

$$\leq \frac{1 + \ln(\omega)}{\omega} + \frac{1}{\omega} \int_1^{\omega a} (\ln(y) - \ln(\omega)) J_m(y) dy. \quad (3.24)$$

Using the mean value theorem for integrals, we have

$$\int_1^{\omega a} (\ln(y) - \ln(\omega)) J_m(y) dy = -\ln(\omega) \int_1^{\omega a} J_m(y) dy + \ln(\omega) \int_1^{\omega a} J_m(y) dy, \text{ for } 1 \leq \xi \leq \omega t.$$

Then, it follows that

$$\left| \int_1^{\omega a} (\ln(y) - \ln(\omega)) J_m(y) dy \right| \leq \ln(\omega) \left| \int_1^{\omega a} J_m(y) dy \right| + |\ln(\omega)| \left| \int_1^{\omega a} J_m(y) dy \right|$$

$$= O(1 + \ln(\omega)), \quad (3.25)$$

which is due to the fact that both $\int_1^{\omega a} J_m(y) dy$ and $\int_1^{\omega a} J_m(y) dy$ converge by referring to the identity $\int_0^\omega J_m(t) dt = 1$ and the moments formula (2.15). Thus, combining (3.24) and (3.25) yields the first identity in (3.21).

(3) For $\alpha > 0$, by integrating by parts and noting the differential relation [1,pp.361,439]

$$d[x^{m+1} J_{m+1}(\omega x)] = \omega x^{m+1} J_m(\omega x) dx, \quad (3.26)$$

and together with the first identities in (3.20) and (3.21), we find

$$\int_0^\omega x^\alpha J_m(\omega x) dx = \frac{1}{\omega} \int_0^\omega x^{\alpha - m-1} d[x^{m+1} J_{m+1}(\omega x)]$$

$$= \frac{1}{\omega} \left[ x^\alpha J_{m+1}(\omega x) \right]_0^\omega - (\alpha - m - 1) \int_0^\omega x^{\alpha - 1} J_{m+1}(\omega x) dx$$

$$= O\left( \frac{1}{\omega} \right).$$

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Similarly, we obtain
\[ \int_0^t x^\alpha \ln(x) J_m(\omega x) dx = O\left(\frac{1}{\omega}\right) . \]
This completes the proof.

From Lemma 3.1, we prove Lemma 3.2.

**Lemma 3.2** For \( f \in C[0, b], \alpha > -1 \) and \( \omega \geq 1 \), it is true that,
\[
\begin{align*}
\left| \int_0^b t^\alpha f(t) J_m(\omega t) dt \right| & \leq C_1(\omega)(|f(b)| + \int_0^b |f'(t)| dt), \quad (3.27) \\
\left| \int_0^b t^\alpha \ln(t) f(t) J_m(\omega t) dt \right| & \leq C_2(\omega)(|f(b)| + \int_0^b |f'(t)| dt), \quad (3.28)
\end{align*}
\]
where
\[
C_1(\omega) = \begin{cases} 
\frac{c_1}{\omega^{\alpha+1}}, & \text{if } -1 < \alpha < 0, \\
\frac{c_2}{\omega}, & \text{if } \alpha \geq 0,
\end{cases}
\]
\[
C_2(\omega) = \begin{cases} 
\frac{c_3(1+\ln(\omega))}{\omega^{\alpha+1}}, & \text{if } -1 < \alpha < 0, \\
\frac{c_4}{\omega}, & \text{if } \alpha \geq 0,
\end{cases}
\]
and \( c_k(k = 1, 2, 3, 4) \) are four constants independent of \( \omega \) and \( f \).

**Proof:** Setting \( F(t) = \int_0^t x^\alpha J_m(\omega t) dt, \) \( t \in [0, b] \), together with Lemma 3.1, we then have
\[
C_1(\omega) = ||F(t)||_\infty = \begin{cases} 
\frac{c_1}{\omega^{\alpha+1}}, & \text{if } -1 < \alpha < 0, \\
\frac{c_2}{\omega}, & \text{if } \alpha \geq 0,
\end{cases}
\]
These together implies that, by integrating by parts,
\[
\left| \int_0^b t^\alpha f(t) J_m(\omega t) dt \right| = \left| \int_0^b f(t) dF(t) \right| = \left| f(t)F(t) \right|_0^b - \int_0^b F(t)f'(t) dt \leq |f(b)||F(b)| + \int_0^b |F(t)||f'(t)| dt \leq |f(b)||F(t)||_\infty + \int_0^b |f'(t)| dt||F(t)||_\infty = C_1(\omega)(|f(b)| + \int_0^b |f'(t)| dt).
\]
It is now obvious that the assertion \((3.27)\) holds. The proof of \((3.28)\) can be completed by the method analogous to that used above.

Meanwhile, it should also be noted that the following Lemma 3.3 also plays an important role in the construction of error bounds.
Lemma 3.3 (see [34]) Let \( n \) be a nonnegative integer. If \( f \) is analytic with \( |f(z)| \leq M \) in the region \( \mathcal{E}_\rho \), bounded by the ellipse with foci \( \pm 1 \) and major and minor semi-axes whose lengths sum to \( \rho > 1 \), then for \( x \in [-1, 1] \),

\[
||f^{(n)}(x) - P_N^{(n)}f(x)||_\infty \leq \frac{2M(N+1)^2}{(\rho^N - \rho^{-N})(2n-1)!} \sum_{j=0}^n \left( \frac{2\rho}{(\rho - 1)^2} \right)^{n+1-j},
\]

where \( (2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1) \) and \( (-1)!! = 1 \).

Based on the above Lemmas 3.1-3.3, we derive error bounds in inverse powers of \( \omega \) in the following Theorems 3.1-3.2. For \( f \in C^2[0, b] \), the error bound of the CCF methods (2.5) is shown as follows.

**Theorem 3.1** Assume that \( f \in C^2[0, b] \). Then the absolute error of the CCF methods (2.5) for \( \omega \geq 1, \alpha > -1 \) and every fixed \( N \), satisfies

\[
|I_1[f] - I_1^{CCF}[f]| \leq \min \left\{ \frac{\rho^{n+1}}{n+1} ||f(x) - P_Nf(x)||_\infty, \right.
\]

\[
bC_1(\omega)||f'(x) - P'_Nf(x)||_\infty, \]

\[
\left. \frac{C_1(\omega)}{\alpha} ||f''(b) - P''_Nf(b)|| + b(1 + \frac{1}{2} \alpha - m - 1)||f'''(x) - P''_Nf(x)||_\infty \right\}.
\]

**Proof:** From the definition of \( P_Nf(x) \), it is obvious that

\[
f(0) - P_Nf(0) = f(b) - P_Nf(b) = 0.
\]

In the following the proof will be split into three parts.

(1) For the first inequality in (3.30), it follows at once from (3.22) that

\[
|I_1[f] - I_1^{CCF}[f]| = \left| \int_0^b x^n(f(x) - P_Nf(x))J_m(\omega x)dx \right|
\]

\[
\leq \int_0^b |x^n(f(x) - P_Nf(x))J_m(\omega x)|dx
\]

\[
\leq \int_0^b x^n dx ||f(x) - P_Nf(x)||_\infty
\]

\[
= \frac{b^{n+1}}{n+1} ||f(x) - P_Nf(x)||_\infty.
\]

(2) By using Lemma 3.2 and the identities (3.31), the second inequality in (3.30) follows that

\[
|I_1[f] - I_1^{CCF}[f]| = \left| \int_0^b x^n(f(x) - P_Nf(x))J_m(\omega x)dx \right|
\]

\[
\leq C_1(\omega)||f(b) - P_Nf(b)|| + \int_0^b |f'(x) - P'_Nf(x)|dx
\]

\[
\leq bC_1(\omega)||f'(x) - P'_Nf(x)||_\infty.
\]
(3) Since \( f(x) - P_N f(x) \) and \( f'(x) - P'_N f(x) \) can be expanded in terms of Maclaurin expansions, as follows,

\[
f(x) - P_N f(x) = f(0) - P_N f(0) + (f'(0) - P'_N f(0))x + \frac{f''(\eta_1) - P''_N f(\eta_1)}{2} x^2
\]

\[
f'(x) - P'_N f(x) = f'(0) - P'_N f(0) + (f''(\eta_2) - P''_N f(\eta_2))x, \quad 0 < \eta_1 < x,
\]

then we have

\[
\left\| \frac{f(x) - P_N f(x)}{x} \right\| = \left\| \frac{x(f'(x) - P'_N f(x)) - (f(x) - P_N f(x))}{x^2} \right\|
\]

\[
= \left\| \frac{(f''(\eta_2) - P''_N f(\eta_2)) - \frac{1}{2} (f''(\eta_1) - P''_N f(\eta_1))}{x^2} \right\|
\]

\[
\leq \left\| f''(\eta_2) - P''_N f(\eta_2) \right\| + \frac{1}{2} \left\| f''(\eta_1) - P''_N f(\eta_1) \right\|
\]

\[
\leq \frac{3}{2} \| f''(x) - P''_N f(x) \|_{\infty}.
\]

By integrating by parts and using Lemma 3.2, together with (3.26), (3.31), (3.32), and due to the limit

\[
\lim_{x \to 0^+} \frac{f(x) - P_N f(x)}{x} = f'(0) - P'_N f(0),
\]

we then obtain

\[
|I_1[f] - I_1^{CCF}[f]| = \left| \int_0^b x^{m+1} \frac{f(x) - P_N f(x)}{x} J_m(\omega x) dx \right|
\]

\[
= \frac{1}{\omega} \left| \int_0^b x^{m+1} \frac{f(x) - P_N f(x)}{x} d[x^{m+1} J_m(\omega x)] \right|
\]

\[
= \frac{1}{\omega} \left| \int_0^b x^{m+1} \frac{f(x) - P_N f(x)}{x} J_m(\omega x) \right|_0^b
\]

\[
- \frac{1}{\omega} \int_0^b x^{m+1} J_m(\omega x) dx \left[ x^{m+1} \frac{f(x) - P_N f(x)}{x} \right]_0^b
\]

\[
\leq \frac{1}{\omega} \left| \int_0^b x^\omega f(x) - P_N f(x) J_m(\omega x) dx \right|
\]

\[
+ \frac{1}{\omega} \left| \int_0^b x f'(x) - P'_N f(x) J_m(\omega x) dx \right|
\]

\[
\leq \frac{1}{\omega} \left[ C_1(\omega) \left( \frac{f(b) - P_N f(b)}{b} \right) + \int_0^b \left| \frac{f(x) - P_N f(x)}{x} \right| dx \right]
\]

\[
+ \frac{C_1(\omega)}{\omega} \left| f'(b) - P'_N f(b) \right| + \int_0^b \left| f'(x) - P'_N f(x) \right| dx
\]

\[
\leq \frac{C_1(\omega)}{\omega} \left[ |f'(b) - P'_N f(b)| + b(1 + \frac{3}{2} |\alpha - m - 1|) \| f''(x) - P''_N f(x) \|_{\infty} \right]. (3.33)
\]

We have thus proved the theorem.

For \( f \in C^3[0, b] \), the error bound of the CCF methods (2.6) is presented as follows.
Theorem 3.2 Assume that $f \in C^3[0, b]$. Then the absolute error of the CCF methods (7.6) for $\omega \geq 1, \alpha > -1$ and every fixed $N$, satisfies

$$
|l_2[f] - l_2^{CCF}[f]| \leq \min \left\{ \begin{array}{ll}
\frac{b^{\alpha+1}(1-(\alpha+1)\ln(b))}{\alpha+1} \|f(x) - P_N f(x)\|_\infty, & \text{if } 0 < b \leq 1, \\
\frac{2+b^{\alpha+1}(\alpha+1)\ln(b)+1}{\alpha+1} \|f(x) - P_N f(x)\|_\infty, & \text{if } b > 1,
\end{array} \right.
$$

(3.34)

**Proof:** (1) For the first inequalities in (3.34), it follows that

$$
|l_2[f] - l_2^{CCF}[f]| = \left| \int_0^b x^\alpha \ln(x)(f(x) - P_N f(x)) J_m(\omega x) \, dx \right|
$$

$$
\leq \int_0^b |x^\alpha \ln(x)(f(x) - P_N f(x)) J_m(\omega x)| \, dx
$$

$$
\leq \int_0^b |x^\alpha \ln(x)| \|f(x) - P_N f(x)\|_\infty \, dx
$$

$$
= \left\{ \begin{array}{ll}
\frac{b^{\alpha+1}(1-(\alpha+1)\ln(b))}{\alpha+1} \|f(x) - P_N f(x)\|_\infty, & \text{if } 0 < b \leq 1, \\
\frac{2+b^{\alpha+1}(\alpha+1)\ln(b)+1}{\alpha+1} \|f(x) - P_N f(x)\|_\infty, & \text{if } b > 1.
\end{array} \right.
$$

The second inequality in (3.34) can be proved by the same method as employed in the proof of the second inequality in (3.30).

(3) Since $f(x) - P_N f(x)$, $f'(x) - P_N' f(x)$ and $f''(x) - P_N'' f(x)$ can be expanded in terms of Maclaurin expansions, as follows,

$$
f(x) - P_N f(x) = f(0) - P_N f(0) + (f'(0) - P_N' f(0)) x + \frac{f''(0) - P_N'' f(0)}{2} x^2 + \frac{f'''(\xi_1) - P_N''' f(\xi_1)}{6} x^3
$$

$$
= (f'(0) - P_N' f(0)) x + \frac{f''(0) - P_N'' f(0)}{2} x^2 + \frac{f'''(\xi_1) - P_N''' f(\xi_1)}{6} x^3, \quad 0 < \xi_1 < x,
$$

$$
f'(x) - P_N' f(x) = f'(0) - P_N' f(0) + (f''(0) - P_N'' f(0)) x + \frac{f'''(\xi_2) - P_N''' f(\xi_2)}{2} x^2, \quad 0 < \xi_2 < x,
$$

$$
f''(x) - P_N'' f(x) = f''(0) - P_N'' f(0) + (f'''(\xi_3) - P_N''' f(\xi_3)) x, \quad 0 < \xi_3 < x,
$$

we then have

$$
\left| \frac{(f(x) - P_N f(x))''}{x} \right| = \left| \frac{x^2(f'''(x) - P_N''' f(x)) - 2x(f''(x) - P_N'' f(x)) + 2(f(x) - P_N f(x))}{x^3} \right|
$$

$$
= \left| (f'''(\xi_3) - P_N''' f(\xi_3)) - (f'''(\xi_2) - P_N''' f(\xi_2)) - \frac{1}{3} (f'''(\xi_1) - P_N''' f(\xi_1)) \right|
$$

$$
\leq \frac{\alpha^2}{3} \|f'''(x) - P_N''' f(x)\|_\infty, \quad \text{(3.35)}
$$

By integrating by parts and using Lemma 3.2, together with (3.26), (3.31), (3.35), and noting that

$$
\lim_{x \to 0^+} x^{\alpha+1} \ln(x) \frac{f(x) - P_N f(x)}{x} = 0,
$$

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using the same argument as in the proof of (3.33), we can easily carry out the proof of the third inequality in (3.34).

The proof of the theorem is now complete.

Remark 1 By transferring the integral interval \([0, b]\) into \([-1, 1]\), these estimates \(\|f(x) - P_N f(x)\|_\infty, \|f'(x) - P'_N f(x)\|_\infty, \|f''(x) - P''_N f(x)\|_\infty\) in Theorems 3.1-3.2, have been given by Lemma 3.3.

As shown in the above Theorems 3.1-3.2, for fixed \(N\), we give these error bounds in inverse powers of \(\omega\). In the sequel, for fixed \(\omega\), we consider error bounds in inverse powers of \(N\). Here, to derive these error bounds, we first establish the following Lemma 3.4.

Lemma 3.4 For every \(j \geq 1\) and fixed \(\omega\), it is true that

\[
\int_0^b x^\alpha T_j^*(x) J_m(\omega x) dx = \begin{cases} 
O\left(\frac{1}{j^{\alpha+\epsilon}}\right), & \text{if } -1 < \alpha < -\frac{1}{2}, \\
O\left(\frac{1}{j^{\alpha+\epsilon/2}}\right), & \text{if } \alpha \geq -\frac{1}{2},
\end{cases} \tag{3.36}
\]

\[
\int_0^b x^\alpha \ln(x) T_j^*(x) J_m(\omega x) dx = \begin{cases} 
O\left(\frac{\ln(j)}{j^{\alpha+\epsilon}}\right), & \text{if } -1 < \alpha < -\frac{1}{2}, \\
O\left(\frac{1}{j^{\alpha+\epsilon}}\right), & \text{if } \alpha = -\frac{1}{2}, \\
O\left(\frac{1}{j^{\alpha+\epsilon/2}}\right), & \text{if } \alpha > -\frac{1}{2},
\end{cases} \tag{3.37}
\]

Proof: For \(-1 \leq t \leq 1\), three transformations \(x = \frac{b}{2} + \frac{b}{2} t, t = \cos \theta\) and \(\theta = \pi - 2\varphi\), yields that

\[
\int_0^b x^\alpha T_j^*(x) J_m(\omega x) dx = (-1)^j 2^\alpha b^\alpha+1 \int_0^\pi \cos(2 j \varphi) \sin^{2\alpha+1}(\varphi) \cos(\varphi) J_m(b \omega \sin^2(\varphi)) d\varphi. \tag{3.38}
\]

(1) In the case of \(\alpha \geq -\frac{1}{2}\): Based on differential relations \(\cos(2 j \varphi) d\varphi = \frac{1}{2j} d \sin(2 j \varphi)\) and \(\sin(2 j \varphi) d\varphi = -\frac{1}{2j} d \cos(2 j \varphi)\), we can derive the second identity in (3.36) by integrating (3.38) by parts twice.

(2) In the case of \(-1 < \alpha < -\frac{1}{2}\): Setting \(\varphi = \frac{\theta}{2j}\), we have

\[
\int_0^b x^\alpha T_j^*(x) J_m(\omega x) dx = (-1)^j 2^\alpha b^\alpha+1 \frac{1}{j^{2\alpha+2}} \int_0^{j\pi} u^{2\alpha+1} \cos(u) \left(\frac{\sin(\frac{u}{2j})}{\frac{u}{2j}}\right)^{2\alpha+1} \cos\left(\frac{u}{2j}\right) J_m(b \omega \sin^2\left(\frac{u}{2j}\right)) du. \tag{3.39}
\]

Now that the right-side improper integral in (3.39) is convergent, it is evident to see that the first identity in (3.36) holds.
Similarly, according to the fact that
\[
\int_0^b x^\alpha \ln(x) T_j^{+\alpha}(x) J_m(\omega x)dx = (-1)^j 4b^{\alpha+1} \int_0^\frac{\pi}{2} \cos(2j\varphi) \sin^{2\alpha+1}(\varphi) \ln(\sin(\varphi)) \cos(\varphi) J_m(b\omega \sin^2(\varphi))d\varphi
\]
\[+[\ln(b) + (2^\alpha - 1) \ln 2] \int_0^b x^\alpha T_j^{+\alpha}(x) J_m(\omega x)dx,
\]
and the logarithmic relation
\[
\ln\left(\sin\left(\frac{\alpha}{2j}\right)\right) = \ln\left(\frac{\sin\left(\frac{\pi}{2j}\right)}{\pi}\right) + \ln(2j) - \ln(u),
\]
together with the assertion (3.36), by the same procedure in the proof of (3.36), we then obtain the desired result (3.37). We have thus proved the lemma.

**Remark 2** Using the asymptotic theory of Fourier integrals (see Erdélyi [12, 13], Piessens [20, 29]) established this asymptotic expansion for \( j \to \infty \):
\[
\int_0^1 x^\alpha T_j(x) J_m(\omega x)dx = 2^{-\alpha-1} \int_0^1 (1 + x)^\alpha T_j(x) J_m(\omega(x + 1)/2)dx
\]
\[\sim -2^{-\alpha-2} J_j(\omega) j^{-2} + (-1)^j / 2^{-3m-3\alpha-2} \frac{\omega^m}{\Gamma(m+1)} \cos((\alpha + 1)\pi) \Gamma(2\alpha + 2) j^{-2\alpha-2m-2}.
\]
(3.40)

Then, by differentiating (3.40) with respect to \( \alpha \) and using (2.19), we have
\[
\int_0^1 x^\alpha \ln(x) T_j(x) J_m(\omega x)dx
\]
\[\sim 2^{-\alpha-2} \ln(2) J_j(\omega) j^{-2} + (-1)^j / 13 \ln(2) 2^{-3m-3\alpha-2} \frac{\omega^m}{\Gamma(m+1)} \cos((\alpha + 1)\pi) \Gamma(2\alpha + 2) j^{-2\alpha-2m-2}
\]
\[+ (-1)^j / 2^{-3m-3\alpha-2} \frac{\omega^m}{\Gamma(m+1)} \sin((\alpha + 1)\pi) \Gamma(2\alpha + 2) j^{-2\alpha-2m-2}
\]
\[+ (-1)^j / 2^{-3m-3\alpha-1} \frac{\omega^m}{\Gamma(m+1)} \cos((\alpha + 1)\pi) \Gamma(2\alpha + 2) j^{-2\alpha-2m-2}
\]
\[+ (-1)^j / 2^{-3m-3\alpha-1} \frac{\omega^m}{\Gamma(m+1)} \cos((\alpha + 1)\pi) \Gamma(2\alpha + 2) j^{-2\alpha-2m-2} \ln(j).
\]
(3.41)

However, for each fixed \( j \), the asymptotic expansions (3.40) and (3.41) are divergent for \( \omega \). Moreover, the estimates in (3.36) and (3.37) can not be directly derived from these asymptotic formulas (3.40) and (3.41) particularly for \( \alpha > 0 \).

With the aid of the above Lemma 3.4, we derive error bounds in inverse powers of \( N \) as follows.
**Theorem 3.3** Assume that $f(x)$ has an absolutely continuous $(k - 1)$st derivative $f^{(k-1)}(x)$ on $[0, b]$ and a $k$th derivative $f^{(k)}(x)$ of bounded variation $V_k$ for some $k \geq 1$. Then, for every fixed $\omega$, and $N \geq k$, the absolute errors of the CCF methods (2.5) and (2.6) satisfy

$$ |I_1[f] - I_1^{CCF}[f]| = \begin{cases} O\left(\frac{1}{N^{k+1}}\right), & \text{if } -1 < \alpha < -\frac{1}{2}, \\ O\left(\frac{1}{N^{\alpha+1}}\right), & \text{if } \alpha \geq -\frac{1}{2}. \end{cases} (3.42) $$

$$ |I_2[f] - I_2^{CCF}[f]| = \begin{cases} O\left(\frac{\ln(N)}{N^{k+1}}\right), & \text{if } -1 < \alpha < -\frac{1}{2}, \\ O\left(\frac{\ln(N)}{N^{\alpha+1}}\right), & \text{if } \alpha = -\frac{1}{2}, \\ O\left(\frac{1}{N^{\alpha+1}}\right), & \text{if } \alpha > -\frac{1}{2}. \end{cases} (3.43) $$

**Proof:** Recalling that $T_{n}^{**}(x) = T_{n}(\frac{2x}{b} - 1)$ and (2.3), we have

$$ T_{2pN\pm j}^{**}(x_k) = T_{2pN\pm j}\left(\frac{2x_k}{b} - 1\right) = T_{2pN\pm j}\left(\cos\left(\frac{k\pi}{N}\right)\right) = \cos\left(\frac{jk\pi}{N}\right) = T_{j}\left(\cos\left(\frac{k\pi}{N}\right)\right) = T_{j}\left(\frac{2x_k}{b} - 1\right) = T_{j}^{**}(x_k), \quad (3.44) $$

for each $j, k = 0, 1, \ldots, N$ and $p = 1, 2, \ldots$.

From (2.4) and (3.44), together with the discrete orthogonality of Chebyshev polynomials (see [24, Section 4.6]), we obtain

$$ P_N T_{pN\pm j}^{**}(x) = \begin{cases} T_{N-j}^{**}(x), & \text{if } p \text{ is odd,} \\ T_{j}^{**}(x), & \text{if } p \text{ is even.} \end{cases} \quad (3.45) $$

Then, we can see directly from (2.5), (2.6) and (3.45) that

$$ I_1^{CCF}[T_{pN\pm j}^{**}(x)] = \begin{cases} I_1[T_{N-j}^{**}(x)], & \text{if } p \text{ is odd,} \\ I_1[T_{j}^{**}(x)], & \text{if } p \text{ is even.} \end{cases} \quad (3.46) $$

$$ I_2^{CCF}[T_{pN\pm j}^{**}(x)] = \begin{cases} I_2[T_{N-j}^{**}(x)], & \text{if } p \text{ is odd,} \\ I_2[T_{j}^{**}(x)], & \text{if } p \text{ is even.} \end{cases} \quad (3.47) $$

Therefore, we can deduce from Lemma 3.4, (3.46) and (3.47), that the sums of aliasing errors
for the CCF methods (2.5) and (2.6) can be estimated for $p$ being a positive integer by

$$
\sum_{j=0}^{N} I_{1}^{CCF}[T_{pN+j}^{{*}}(x)] = \begin{cases} 
\sum_{j=0}^{N} I_{1}[T_{N-j}^{*}(x)] = \sum_{j=0}^{N} \int_{0}^{b} x^\alpha T_{N-j}^{*}(x)J_m(\omega x)dx, & \text{if } p \text{ is odd}, \\
\sum_{j=0}^{N} I_{1}[T_{N-j}^{*}(x)] = \sum_{j=0}^{N} \int_{0}^{b} x^\alpha T_{N-j}^{*}(x)J_m(\omega x)dx, & \text{if } p \text{ is even},
\end{cases}
$$

and

$$
\sum_{j=0}^{N} I_{2}^{CCF}[T_{pN+j}^{{*}}(x)] = \begin{cases} 
\sum_{j=0}^{N} I_{2}[T_{N-j}^{*}(x)] = \sum_{j=0}^{N} \int_{0}^{b} x^\alpha \ln(x)T_{N-j}^{*}(x)J_m(\omega x)dx, & \text{if } p \text{ is odd}, \\
\sum_{j=0}^{N} I_{2}[T_{N-j}^{*}(x)] = \sum_{j=0}^{N} \int_{0}^{b} x^\alpha \ln(x)T_{N-j}^{*}(x)J_m(\omega x)dx, & \text{if } p \text{ is even},
\end{cases}
$$

Moreover, we can find that Theorem 4.2 in [30] implies the estimate

$$|\alpha_j| = O\left(\frac{1}{x^ {k+1}}\right).$$

Combining (3.48), (3.49) and (3.50), and by a similar way as shown in the proof of Theorem 5.1 in [30], we can deduce the desired results (3.42) and (3.43).

4 Extension to a higher order method and error estimate

The choice of the extreme points as interpolation points for highly oscillatory integrals is not only a technical necessity but also can improve the accuracy significantly [27, 32, 34]. Moreover, only by adding derivative information of $f(x)$ at the endpoints 0 and $b$ can the asymptotic order of the method be improved [17, 18, 27, 32, 34]. By the above-mentioned particular observations, the higher order CCF methods for (1.1) and (1.2) can be defined, respectively, by

$$
I_{1}^{HCCF}[f] = \int_{0}^{b} x^\alpha P_{N+2s}f(x)J_m(\omega x)dx \\
= b^ {\alpha+1} \sum_{j=0}^{N+2s} b_j M_j,
$$

$$
I_{2}^{HCCF}[f] = \int_{0}^{b} x^\alpha \ln(x)P_{N+2s}f(x)J_m(\omega x)dx \\
= b^ {\alpha+1} \sum_{j=0}^{N+2s} b_j [\ln(b)M_j + \tilde{M}_j],
$$

where $P_{N+2s}f(x)$ is the special Hermite interpolation polynomial [17, 34] of $f(x)$ at the Clenshaw-Curtis points $c_n = \frac{n}{N} + \frac{b}{2} \cos\left(\frac{\pi n}{N}\right)$ on $[0, b]$ satisfying

$$
P_{N+2s}^{(k)}f(0) = f^{(k)}(0), \quad P_{N+2s}f(c_n) = f(c_n), \quad P_{N+2s}^{(k)}f(b) = f^{(k)}(b),$$

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for \( n = 1, 2, \ldots, N - 1 \) and \( k = 0, 1, \ldots, s \); it can be evaluated in \( O(N \log N) \) operations \([34]\). Moreover, the polynomial \( P_{N+2s} f(x) \) can be expressed by

\[
P_{N+2s} f(x) = \sum_{j=0}^{N+2s} b_j T_j^*(x), \tag{4.54}
\]

where the coefficients \( b_j \) can be computed efficiently by an algorithm \([34]\).

Here, we establish error bounds in inverse powers of \( N \) as follows.

**Theorem 4.4** Assume that \( f \in C^{s+2}[0, b] \). Then the absolute errors of the higher order CCF methods \((4.51)\) and \((4.52)\) for \( \alpha > -1 \) and every fixed \( N \), satisfy

\[
|I_1[f] - I_1^{HCCF}[f]| = O\left(\frac{C_1(\omega)}{\omega^{s+1}}\right), \tag{4.55}
\]

\[
|I_2[f] - I_2^{HCCF}[f]| = O\left(\frac{C_2(\omega)}{\omega^{s+1}}\right). \tag{4.56}
\]

**Proof:** Based on the differential relation \((3.26)\), we have

\[
I_1[f] - I_1^{HCCF}[f] = \int_0^b x^\alpha (f(x) - P_{N+2s} f(x)) J_m(\omega x) dx
\]

\[
= \frac{1}{\omega} \int_0^b \frac{f(x) - P_{N+2s} f(x)}{x^{s+1}} d\left(\frac{x^{m+1} J_m(\omega x)}{x^{s+1}}\right), \tag{4.57}
\]

\[
I_2[f] - I_2^{HCCF}[f] = \int_0^b x^{\alpha+1} \ln(x) (f(x) - P_{N+2s} f(x)) J_m(\omega x) dx
\]

\[
= \frac{1}{\omega} \int_0^b \frac{f(x) - P_{N+2s} f(x)}{x^{s+1}} d\left(\frac{x^{m+1} J_m(\omega x)}{x^{s+1}}\right). \tag{4.58}
\]

By using \((4.53)\) and resorting to integrating \((4.57)\) and \((4.58)\) by parts \( s + 1 \)-time, respectively, together with Lemma 3.2, we establish the desired results as in the proof of the third inequalities in Theorems 3.1-3.2.

**5 Conclusion**

This paper presents a series of new sharp error bounds of the Clenshaw-Curtis-Filon methods for two classes of oscillatory Bessel transforms with algebraic or logarithmic singularities. From the above error bounds, it is worth noting that the required accuracy can be obtained by using derivatives of \( f(x) \) at the end-points or adding the number of the interior node points. Moreover, the Clenshaw-Curtis-Filon methods possesses the advantageous property that for fixed \( N \) the accuracy increases when oscillation becomes faster, which can be also obtained directly from these error bounds in inverse powers of \( \omega \).
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