A NEWCOMER’S GUIDE TO ZETA FUNCTIONS OF
GROUPS AND RINGS

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Abstract. These notes grew out of lectures given at the LMS-EPSRC
Short Course on Asymptotic Methods in Infinite Group Theory, University of Oxford, 9-14 September 2007, organised by Dan Segal.

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Date: June 9, 2009.

Key words and phrases. Subgroup growth, representation growth, nilpotent groups, $p$-adic integration, Kirillov theory, Igusa’s local zeta function, local functional equations.
1. **Introduction**

1.1. **Zeta functions of nilpotent groups.** A finitely generated group $G$ has only finitely many subgroups of each finite index. The zeta function of such a group is the Dirichlet generating function encoding these numbers. If, for $m \in \mathbb{N}$, there are $a_m = a_m(G)$ subgroups of index $m$ in $G$, the zeta function of $G$ is defined as

$$
\zeta_G(s) := \sum_{m=1}^{\infty} a_m m^{-s} = \sum_{H \leq_f G} |G : H|^{-s}.
$$

Here $s$ is a complex variable. Zeta functions were introduced into infinite group theory as tools to study groups of polynomial subgroup growth. In fact, it is well-known that a series like (1.1) converges on the complex right-half plane $\{s \in \mathbb{C} | \Re(s) > \alpha\}$ if and only if the numbers

$$
s_m := s_m(G) := \sum_{i \leq m} a_i
$$

grow at most polynomially of degree $\alpha$, i.e. $s_m = O(1 + m^\alpha)$. We therefore call

$$
\alpha_G := \inf\{\alpha | \exists c > 0 \forall m : \sum_{i \leq m} a_i < c(1 + m^\alpha)\}
$$

the abscissa of convergence of $\zeta_G(s)$.

We call groups with this property groups with polynomial subgroup growth (PSG). The subgroup growth of any group $G$ is the same as the subgroup growth of $G/R(G)$, where $R(G) := \bigcap_{N \triangleleft_f G} N$, the finite residual of $G$, is the intersection of the group’s normal subgroups of finite index. In studying subgroup growth, we may thus assume without loss of generality that the group $G$ is residually finite, i.e. that its finite residual is trivial. Finitely generated, residually finite groups of polynomial subgroup growth have been characterised as the virtually soluble groups of finite rank ([28]). This class of groups includes the class of finitely generated, torsion-free nilpotent (or $T$-)groups. It was this class of PSG-groups for which zeta functions were first introduced as a means to study asymptotic and arithmetic aspects of subgroup growth (cf. [16]).

Let $G$ be a $T$-group. It is not difficult to see that, owing to the nilpotency of $G$, the zeta function $\zeta_G(s)$ has an Euler factorisation

$$
\zeta_G(s) = \prod_{p \text{ prime}} \zeta_{G,p}(s)
$$

into local (or Euler) factors $\zeta_{G,p}(s) := \sum_{i=0}^{\infty} a_p p^{-is}$, indexed by the primes $p$, enumerating subgroups of $p$-power index. This generalises the familiar Euler product decomposition satisfied by the Riemann zeta function

$$
\zeta(s) := \sum_{m=1}^{\infty} m^{-s} = \prod_{p \text{ prime}} \zeta_p(s),
$$

---

1 We have chosen this definition of $\alpha_G$ to ensure that $\alpha_G = -\infty$ if $G$ is finite.
where $\zeta_p(s) := \frac{1}{1-p^{-s}}$. While (1.4) reflects the Fundamental Theorem of Arithmetic that every positive integer can be written as the product of prime powers in an essentially unique way, the identity (1.3) reflects the fact that every finite nilpotent group is the direct product of its Sylow $p$-subgroups. In fact, $\zeta(s)$ is the zeta function of the infinite cyclic group, making (1.4) a special case of (1.3). Indeed, it is well known that, for all $n \in \mathbb{N}$, there is a unique subgroup of index $n$ in $\mathbb{Z}$, namely $n\mathbb{Z}$. It is instructive to see how this generalises to abelian groups of higher rank.

**Example 1.1.** For $n \in \mathbb{N}$, let $\mathbb{Z}^n$ be the free abelian group of rank $n$. Then

$$(1.5) \quad \zeta_{\mathbb{Z}^n}(s) = \zeta(s)\zeta(s-1)\cdots \zeta(s-(n-1)).$$

The monograph [30] contains no fewer than five proofs of this beautiful formula. We will add another, new one, in Section 2.5. We observe that this formula allows us to give precise asymptotic information about the numbers $s_m(\mathbb{Z}^n)$ of subgroups of index at most $m$ in $\mathbb{Z}^n$. Indeed, one can deduce from (1.5) that

$$s_m(\mathbb{Z}^n) \sim n^{-1}\zeta(n)\zeta(n-1)\cdots \zeta(2)m^n \quad \text{as } m \to \infty.$$  

For example, using the identity $\zeta(2) = \pi^2/6$, we see that

$$s_m(\mathbb{Z}^2) \sim (\pi^2/12)m^2 \quad \text{as } m \to \infty.$$  

**1.2. Zeta functions of rings.** We will see in Section 1.3 below that the study of zeta functions of nilpotent groups may – at least to a certain extent – be reduced to the study of zeta functions of suitable rings. By a ring we mean an additive group of finite rank, carrying a bi-additive product, not necessarily commutative or associative. Given a ring $L$, its (subring) zeta function is defined as the Dirichlet generating series

$$\zeta_L(s) = \sum_{m=1}^{\infty} b_m m^{-s} = \sum_{H \leq L} |L : H|^{-s},$$

where, for $m \in \mathbb{N}$, $b_m = b_m(L)$ denotes the number of subrings of index $m$ in $L$ and $s$ is again a complex variable. By properties of the underlying additive group of $L$ alone (essentially the Chinese Remainder Theorem), this zeta function also satisfies an Euler product decomposition

$$(1.6) \quad \zeta_L(s) = \prod_{p \text{ prime}} \zeta_{L,p}(s)$$

into Euler factors $\zeta_{L,p}(s) := \sum_{i=0}^{\infty} b_p p^{-is}$, enumerating subrings of finite $p$-power index. It is worth pointing out that, for each prime $p$, the Euler factor $\zeta_{L,p}(s)$ is the zeta function of the $\mathbb{Z}_p$-algebra $L_p := L \otimes \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the ring of $p$-adic integers.

In the study of nilpotent groups, nilpotent Lie rings play an important role (which motivated our choice of notation ‘$L$’ for a general ring). A Lie
ring is a finitely generated abelian group with a bi-additive product \([,]\) (called ‘Lie-bracket’) satisfying the Jacobi-identity
\[
\forall x, y, z \in L : \ [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0
\]
and, for all \(x \in L\), \([x, x] = 0\). The lower central series of \(L\) is defined inductively via \(\gamma_1(L) := L, \gamma_i(L) := [\gamma_{i-1}(L), L]\) for \(i \geq 2\). We say that a Lie ring \(L\) is nilpotent of class \(c\) if \(\gamma_{c+1}(L) = \{0\}\) but \(\gamma_c(L) \neq \{0\}\). For example, a Lie ring is nilpotent of class 1 if and only if it is abelian, and nilpotent of class 2 if and only if the derived ring is central, i.e. if \(L' := [L, L] \leq Z(L)\).

**Example 1.2.** Let \(sl_2(\mathbb{Z})\) be the Lie ring of traceless integral \(2 \times 2\)-matrices with Lie bracket \([x, y] := xy - yx\). It has a \(\mathbb{Z}\)-basis consisting of the matrices
\[
e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
which satisfy the relations \([h, e] = 2e, [h, f] = -2f, [e, f] = h\). A non-trivial computation shows that, for odd prime \(p\),
\[
\zeta_{sl_2(\mathbb{Z}), p}(s) = \zeta_{sl_2(\mathbb{Z}_p)}(s) = \zeta_p(s)\zeta_p(s - 1)\zeta_p(2s - 1)\zeta_p(2s - 2)\zeta_p(3s - 1)^{-1}
\]
whereas, for \(p = 2\),
\[
\zeta_{sl_2(\mathbb{Z}), 2}(s) = \zeta_{sl_2(\mathbb{Z}_2)}(s) = \zeta_2(s)\zeta_2(s - 1)\zeta_2(2s - 1)\zeta_2(2s - 2)(1 + 6 \cdot 2^{-2s} - 8 \cdot 2^{-3s}).
\]
This was first proved in [14]. We will sketch an alternative proof in Section 2.6.

**1.3. Linearisation.** Whilst it is possible to analyse the Euler factors of zeta functions of nilpotent groups directly ([16, Section 2]), it is often useful to exploit the fact that the study of subgroup growth of nilpotent groups can be linearised, i.e. reduced to the study of subring growth of suitable (nilpotent Lie) rings associated with these groups. Let \(G\) be a \(T\)-group. The Malcev correspondence assigns to \(G\) a \(\mathbb{Q}\)-Lie algebra \(L = L(G)\) which contains a Lie subring \(L = L(G)\). The dimension of \(L\) as a \(\mathbb{Q}\)-vector space (and thus the torsion-free rank of \(L\) as a \(\mathbb{Z}\)-module) coincides with the *Hirsch length* \(h(G)\) of \(G\), the number of infinite cyclic factors in a polycyclic series for \(G\). It can also be shown that \(L\) is nilpotent of class \(c\), where \(c\) is the nilpotency class of \(G\). It has the property that, for almost all (i.e. all but finitely many) primes \(p\),
\[
(1.7) \quad \zeta_{G, p}(s) = \zeta_{L, p}(s)
\]
(see [16, Section 4] for details).

The exclusion of a finite number of primes is a recurrent phenomenon in the theory of zeta functions of nilpotent groups and of rings.

If \(G\) is nilpotent of class 1, i.e. abelian, there is of course nothing to do: we choose \((L, +) = (G, \cdot)\), with trivial ring structure. If \(G\) is nilpotent of class 2, i.e. if \(G' \leq Z(G)\), we may choose
\[
(1.8) \quad L := Z(G) \oplus G/Z(G),
\]
with Lie bracket induced from taking commutators in the groups. It satisfies the identities (1.7) for all primes, i.e. \( \zeta_G(s) = \zeta_L(s) \) in this case.

We illustrate the passage from nilpotent groups to nilpotent Lie rings with an important and prototypical example and some of its generalisations.

**Example 1.3.** The group

\[
G := \begin{pmatrix}
1 & Z & Z \\
0 & 1 & Z \\
0 & 0 & 1
\end{pmatrix}
\]

is called the discrete Heisenberg group of \( 3 \times 3 \)-upper-unitriangular matrices over the integers. It can easily be seen to be nilpotent of class 2 and of Hirsch length 3. In fact, its centre \( Z(G) \) coincides with the derived group \( G' \), which is the infinite cyclic subgroup generated by the matrix

\[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

It is not hard to see that the Lie ring \( L \) constructed in (1.8) has a presentation

\[
L = \langle x, y, z \mid [x, y] = z, [x, z] = [y, z] = e \rangle.
\]

It can be shown that

\[
\zeta_G(s) = \zeta_L(s) = \zeta(s)\zeta(s-1)\zeta(2s-3)\zeta(2s-2)\zeta(3s-3)^{-1}.
\]

This was first proved in [16]. We will prove this in Proposition 2.11 and sketch another proof in Section 2.6.

**Example 1.4.** The Heisenberg group has many aspects that may be generalised. For instance, it is the free nilpotent group of nilpotency class 2 on two generators. In general, given integers \( c, d \geq 2 \), the free nilpotent group \( F_{c,d} \) on \( d \) generators and nilpotency class \( c \) may be defined as the quotient

\[
F_{c,d} := F_d/\gamma_{c+1}(F_d)
\]

of the free group \( F_d \) on \( d \) letters by the \( c + 1 \)-th term of its lower central series. The groups \( F_{2,d} \), for example, have a presentation

\[
F_{2,d} = \langle x_1, \ldots, x_d, y_11, y_12, \ldots, y_{d-1}d \mid [x_i, x_j] = y_{ij}, \text{ all other } [,] \text{ trivial} \rangle.
\]

The associated Lie rings \( L_{2,d} \) have identical presentations.

Computing explicit formulae for zeta functions of groups is in general very difficult, even if the groups have quite a transparent structure. For the zeta functions \( \zeta_{F_{c,d}}(s) \), explicit formulae are only known for the cases \( (1, d) \) (cf. Example 1.1) and \( (c, d) \in \{(2,2),(2,3),(3,2)\} \). For example (cf. [15, 2.7.1]),

\[
\zeta_{F_{2,3}}(s) = \zeta_{Z^3}(s)\zeta(2s-4)\zeta(2s-5)\zeta(2s-6)\zeta(3s-6)\zeta(3s-7)\zeta(3s-8)\zeta(4s-8)^{-1}\prod_{p\text{ prime}} W_{2,3}(p,p^{-s}),
\]

where \( W_{2,3}(p,p^{-s}) \) is the p-adic zeta function.
where
\[
W_{2,3}(X,Y) = 1 + X^3Y^2 + X^4Y^2 + X^5Y^2 - X^4Y^3 - X^5Y^3 \\
- X^6Y^3 - X^7Y^4 - X^9Y^4 - X^{10}Y^5 - X^{11}Y^5 \\
- X^{12}Y^5 + X^{11}Y^6 + X^{12}Y^6 + X^{13}Y^6 + X^{16}Y^8.
\]

We have seen that the theory of zeta functions of nilpotent groups can, to a great extent, be reduced to the study of the zeta functions of nilpotent Lie rings. It is worth recalling, however, that the theory of zeta functions of rings we are about to present applies to much more general rings.

1.4. **Layout of the paper.** In Section 2 we study local and global aspects of subring zeta functions of rings, reviewing some of the methods available to study these functions. By the linearisation results outlined in the introduction, these yield, as corollaries, theorems about (generic local) zeta functions of $T$-groups. We put particular emphasis on connections with the theory of linear homogeneous diophantine equations and on local functional equations.

Some of the manifold generalisations and variations of the concept of the zeta function of a group or ring are reviewed in Section 3. We concentrate on ideal (or normal) zeta functions of rings (or nilpotent groups, respectively) and representation zeta functions of nilpotent, arithmetic and $p$-adic analytic groups.

In Section 4 we present a collection of what we believe are major open questions and conjectures in the area.

We use the following notation.

- $\mathbb{N}$: the set $\{1, 2, \ldots\}$ of natural numbers
- $I = \{i_1, \ldots, i_l\} <$ the finite set of natural numbers $i_1 < \cdots < i_l$
- $I_0$: the set $I \cup \{0\}$ for $I \subseteq \mathbb{N}$
- $[k]$: the set $\{1, \ldots, k\}$, $k \in \mathbb{N}$
- $S_n$: the symmetric group on $n$ letters
- $M^t$: the transpose of a matrix $M$
- $v_p$: the $p$-adic valuation ($p$ a prime)
- $\mathbb{Z}_p$: the ring of $p$-adic integers
- $\mathbb{Q}_p$: the field of $p$-adic numbers
- $|x|_p$: the $p$-adic absolute value of a $p$-adic number, defined by $|x|_p := p^{-v_p(x)}$
- $\|S\|_p$: the $p$-adic absolute value of a set $S$ of $p$-adic numbers, defined by $\|S\|_p := \max\{|s|_p| s \in S\}$
- $[\Lambda]$: the homothety class $\mathbb{Q}_p^n\Lambda$ of a (full) lattice $\Lambda$ in $\mathbb{Q}_p^n$
- $\delta_P$: the ‘Kronecker delta’ which is equal to 1 if the property $P$ holds and equal to 0 otherwise

Given a set $f$ of polynomials and a polynomial $g$, we write $gf$ for $\{gf| f \in f\}$, and $(f)$ for the ideal generated by $f$. 
2. Local and global zeta functions of groups and rings

Let \( L \) be a ring. Given equation (1.6), the problem of studying the zeta function \( \zeta_L(s) \) is reduced to the problem of understanding the Euler factor \( \zeta_{L,p}(s) \), \( p \) prime, and the analytic properties of their Euler product. The following are natural questions:

1. What do the local factors \( \zeta_{L,p}(s) \) have in common? What is their structure?
2. How do the local factors vary with the prime \( p \)?

In the following subsections we will explore some of the existing methods to analyse local zeta functions of rings, and will address both of these questions.

2.1. Rationality and variation with the prime. In all the examples we have seen, the local factors all shared a number of features. In particular, they were all rational functions in the parameter \( p-s \). This is no coincidence:

**Theorem 2.1.** [16, Theorem 3.5] For all primes \( p \), the local zeta function \( \zeta_{L,p}(s) \) is a rational function in \( p-s \), i.e. there is a rational function \( W_p(Y) = \frac{P(Y)}{Q(Y)} \in \mathbb{Q}(Y) \) such that

\[
W_p(p-s) = \zeta_{L,p}(s).
\]

The proof of this theorem uses deep results from the theory of \( p \)-adic integration, which we survey to some degree below.

Theorem 2.1 asserts that the sequence \((b_p(L))\) of the numbers of subrings of \( L \) of index \( p^i \) satisfies a strong regularity property: it is easy to see that a generating function of the form \( \sum_{i=0}^{\infty} b_p \cdot t^i \) is rational in the variable \( t \) if and only if there is a finite linear recurrence relation on the coefficients \( b_p \), the length of which is determined by the degree of the denominator (cf. [34, Theorem 4.1.1]). In other words, the numbers of finite index subalgebras of \( L_p \) are already determined by the numbers of subalgebras in some finite quotient of \( L_p \).

\textit{A priori}, Theorem 2.1 does not give us any information on the shape of the rational functions \( W_p \). In particular, it does not tell us how the lengths of these recurrence relations depend on the prime, or when they set in. In the examples above we observe that the denominators are all of the form

\[
\prod_{i \in I} (1 - p^{a_i} \cdot b_i Y) \text{ for suitable non-negative integers } a_i, b_i.
\]

This, too, is a general phenomenon.

**Theorem 2.2.** [5] For each \( n \in \mathbb{N} \) there exists a finite index set \( I_n \), and finitely many pairs \((a_i, b_i)_{i \in I_n}\) of natural numbers such that, if \( L \) is a ring of additive rank \( n \), for all primes \( p \) the denominator polynomial \( Q_p(Y) \in \mathbb{Q}[Y] \) in Theorem 2.1 can be taken to divide \( \prod_{i \in I_n} (1 - p^{a_i} \cdot b_i Y) \).

Theorem 2.2 implies that the degrees in \( p^{-s} \) of the denominator polynomials \( Q_p(Y) \) are bounded when \( L \) ranges over all rings of a given rank \( n \). In particular, there is a uniform upper bound on the lengths of the recurrence relations satisfied by the sequences \((b_p(L))_i\) for fixed \( L \) as \( p \) ranges over
the primes. It also shows that the coefficients of $Q_p(Y)$ are polynomials in $p$, so that the denominators of the Euler factors are really polynomials in $p$ and $p^{-s}$. The proof of Theorem 2.2 relies on non-constructive methods from model theory. No procedure is known to describe explicitly (even just a reasonably small superset of) the factors of the denominator of the local zeta functions of a given ring.

The numerators of the Euler factors have, in general, a far more complicated and interesting structure. In all of the examples we have encountered so far, the coefficients of the polynomials $P_p(Y), too, were – at least for almost all primes $p$ – polynomials in $p$. It was known already to the authors of [16] that this is not a general feature. Their paper contains examples of zeta functions of nilpotent groups whose local factor at the prime $p$ depends on how the rational prime $p$ behaves in a number field. The right framework to explain this phenomenon, however, was not discovered until much later.

**Theorem 2.3.** [11, Theorem 1.3] Let $L$ be a ring. There are smooth algebraic varieties $V_t$, $t \in [m]$, defined over $Q$, and rational functions $W_t(X,Y) \in Q(X,Y)$ such that, for almost all primes $p$,

$$
(2.1) \quad \zeta_{L,p}(s) = \sum_{t=1}^{m} c_t(p) W_t(p, p^{-s}),
$$

where $c_t(p)$ denotes the number of $\mathbb{F}_p$-rational points\(^2\) of $V_t$, the reduction modulo $p$ of $V_t$.

We will remark on the proof of this theorem at the end of Section 2.4.

In general, the numbers of $\mathbb{F}_p$-rational points of the reduction modulo $p$ of varieties defined over $\mathbb{Q}$ will not be polynomials in $p$, as the following example shows.

**Example 2.4.** Let $E$ be the elliptic curve defined by the equation $y^2 = x^3 - x$. For a prime $p$ we denote by $c(p)$ the number of $\mathbb{F}_p$-rational points of $V_t$, the reduction modulo $p$ of $E$, i.e.

$$
c(p) := |\{(x,y) \in \mathbb{F}_p^2| y^2 = x^3 - x\}|.
$$

It is known ([21, §18.4]\(^3\)) that, if $p \equiv 3 \mod (4)$, then $c(p) = p$. If, however, $p \equiv 1 \mod (4)$, then $c(p) = p - (\pi + \pi)$, where $\pi$ is the complex number satisfying $p = \pi \bar{\pi}$ and $\pi \equiv 1 \mod (2 + 2i)$.

It is not clear a priori that varieties with such ‘wild’ arithmetical behaviour can occur in the description of zeta functions of rings given in (2.1).

\(^2\)The formulation given here follows from the original formulation in [11] by the inclusion-exclusion principle.

\(^3\)The discrepancy with the formula given in [21, §18.4, Theorem 5] comes from the fact that there $c(p)$ refers to the number of *projective* points of $E$, which includes also a point at infinity (cf. Example 2.14). This should also have been taken into account in [13, Example 1].
In [7, 8] du Sautoy gave an example of a class-2-nilpotent Lie ring (or, equivalently, class-2-nilpotent group) whose local zeta functions involve the cardinalities \( c(p) \) associated with the elliptic curve in Example 2.4. In particular, he proved that the zeta function of this Lie ring is not ‘finitely uniform’. We say that \( \zeta_L(s) \) is \textit{finitely uniform} if there are finitely many rational functions \( W_i(X, Y) \in \mathbb{Q}(X, Y), \; i \in I \), a finite index set, such that for every prime \( p \) there exists an \( i = i(p) \) such that \( \zeta_{L,p}(s) = W_i(p, p^{-s}) \). We say that \( \zeta_L(s) \) is \textit{uniform} if it is finitely uniform for \(|I| = 1\) and \textit{almost uniform} if there exists a rational function \( W(X, Y) \) such that \( \zeta_{L,p}(s) = W(p, p^{-s}) \) for almost all \( p \).

We will revisit du Sautoy’s example in Section 3, where we will look at the \textit{ideal} zeta function of this particular Lie ring, counting only ideals of finite index. For this variant, we will be able to give an explicit formula for the local zeta functions, illustrating Theorem 2.23 (or rather its analogue for ideal zeta functions of rings) in this particular case. It seems worth pointing out, however, that so far all the zeta functions \( \zeta_L(s) \) of rings \( L \) for which explicit formulae are known are finitely uniform.

For future reference we study in some detail an important sample family of varieties with very ‘uniform’ reduction behaviour modulo \( p \). They play a key role in explicit formulae for zeta functions of rings.

### 2.2. Flag varieties and Coxeter groups.

Let \( V \) denote an \( n \)-dimensional vector space over a field \( k \). For each \( i \in [n-1] \), the set \( G_{n,i}(k) \) of subspaces of \( V \) of dimension \( i \) can be given the structure of a smooth projective variety over \( k \), called the \( i \)-th Grassmannian of \( V \). Given a prime power \( q \) we obtain \( G_{n,i}(\mathbb{F}_q) \). We define, for \( 1 \leq i < n \), the polynomial

\[
\binom{n}{i}_X := \prod_{j=0}^{i-1} \frac{(X^{n-j} - 1)}{(X^{i-j} - 1)} \in \mathbb{Z}[X].
\]

It is not hard to prove that \(|G_{n,i}(\mathbb{F}_q)| = \binom{n}{i}_q \in \mathbb{Z}[q] \). For example, the cardinality \(|\mathbb{P}^{n-1}(\mathbb{F}_q)|\) of the \( n-1 \)-dimensional projective space of lines in \( \mathbb{F}_q^n \) is given by \( \binom{n}{1}_q = (q^n - 1)/(q - 1) = 1 + q + \cdots + q^{n-1} \).

More generally, let \( I = \{i_1, \ldots, i_l\} <_r \) be a subset of \([n-1] \). A \textit{flag} of \textit{type} \textit{I} in \( V \) is a sequence \((V_i)_{i \in I} \) of subspaces of \( V \) satisfying

\[
\{0\} \subsetneq V_{i_1} \subsetneq V_{i_2} \subsetneq \cdots \subsetneq V_{i_l} \subsetneq V
\]

and, for all \( i \in I \), \( \dim(V_i) = i \). A flag is called \textit{complete} if it is of type \( I = [n-1] \). The set of flags of type \( I \) can be given the structure of a smooth projective variety over \( k \). If \( k = \mathbb{F}_q \), we obtain the variety of flags of type \( I \) in \( \mathbb{F}_q^n \). We define the polynomial

\[
\binom{n}{I}_X := \binom{n}{i_1}_X \binom{i_1}{i_2}_X \cdots \binom{i_{l-1}}{i_l}_X \in \mathbb{Z}[X].
\]

The numbers \( \binom{n}{I}_q \) are called \textit{q-binomial coefficients} or \textit{Gaussian polynomials}. One easily proves inductively that the number of flags of type \( I \) in \( \mathbb{F}_q^n \) is given
Definition 2.5. Let $S_n$ be the symmetric group of $n$ letters with standard (Coxeter) generators $s_1, \ldots, s_{n-1}$ (in cycle notation these are the transpositions $s_i = (i \ i+1)$). Let $w \in S_n$. The (Coxeter) length $\ell(w)$ is the length of a shortest word in the generators $s_i$ representing $w$. The (left) descent type $D_L(w)$ is the set $\{i \in [n-1] \mid w(i+1) < w(i)\}$.

It can be shown that

$$D_L(w) = \{i \in [n-1] \mid \ell(s_i w) < \ell(w)\}. \quad (2.3)$$

Proposition 2.6. Let $q$ be a prime power. For all $I \subseteq [n-1]$

$$\binom{n}{I}_{q} = \sum_{w \in S_n, D_L(w) \subseteq I} q^{\ell(w)}.$$ 

Proof. We first prove the proposition for $I = [n-1]$. In this case, $\binom{n}{[n-1]}_q$ gives the number of complete flags $(V_i)_{i \in [n-1]}$ in the finite vector space $F^n_q$. These may be also viewed as the cosets $GL_n(F_q)/B(F_q)$, where $B$ denotes the Borel subgroup of upper-triangular matrices in $GL_n$. It is well-known that the algebraic group $GL_n$ satisfies a Bruhat decomposition

$$GL_n = \bigcup_{w \in S_n} BwB$$

(where we identify permutations in $S_n$ with permutation matrices in $GL_n$, acting from the left on unit column vectors, say). Therefore

$$GL_n(F_q)/B(F_q) = \bigcup_{w \in S_n} B(F_q)wB(F_q)/B(F_q).$$

The disjoint pieces $\Omega_w(F_q) := B(F_q)wB(F_q)/B(F_q)$, $w \in S_n$, are called Schubert cells. It can be shown that each Schubert cell $\Omega_w(F_q)$ is an affine space over $F_q$ of dimension given by the length $\ell(w)$. Indeed, a complete set of representatives of $B(F_q)wB(F_q)/B(F_q)$, of size $q^{\ell(w)}$, is obtained in the following way: start with the permutation matrix corresponding to $w$. Substitute an arbitrary entry in $F_q$ for each of the zeros of this matrix which is not positioned anywhere below or to the right of a 1. We conclude that

$$\binom{n}{[n-1]}_q = \left|GL_n(F_q)/B(F_q)\right| = \left|\bigcup_{w \in S_n} B(F_q)wB(F_q)/B(F_q)\right|$$

$$= \sum_{w \in S_n} |\Omega_w(F_q)| = \sum_{w \in S_n} q^{\dim(\Omega_w)} = \sum_{w \in S_n} q^{\ell(w)}$$

This proves the proposition in the special case $I = [n-1]$. 

by the polynomial $\binom{n}{I}_{q} \in \mathbb{Z}[q]$. For example, the number of complete flags in $F^3_q$ is given by $(1 + q + q^2)(1 + q) = 1 + 2q + 2q^2 + q^3$.

For further applications we shall need an expression for the polynomials $\binom{n}{I}_X$ in terms of Coxeter group theoretic notions.
Example 2.7. Let $n = 5$. The Schubert cell $\Omega_w$ indexed by the element $w = (1532) \in S_5$ may be identified with the set of matrices of the form

\[
\begin{pmatrix}
* & * & 1 & 0 & 0 \\
* & * & 0 & * & 1 \\
* & 1 & 0 & 0 & 0 \\
* & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

where $*$ may take any value in $\mathbb{F}_q$. Note that there are 7 $*$'s, reflecting the fact that $\ell(w) = 7$. Indeed, a shortest word representing $w$ is $s_2 s_3 s_1 s_4 s_3 s_2 s_1$.

The descent type of $w$ is $D_L(w) = \{2, 4\}$.

In the general case, given $I = \{i_1, \ldots, i_l\} \subseteq [n - 1]$, $\binom{n}{I}_q$ is the number of flags $(V_i)_{i \in I}$, $\dim(V_i) = i$, in $\mathbb{F}_q^n$. These are in 1–1-correspondence with cosets $\text{GL}_n(\mathbb{F}_q)/B_I(\mathbb{F}_q)$, where $B_I(\mathbb{F}_q)$ is the parabolic subgroup consisting of matrices of the form

\[
\begin{pmatrix}
\gamma_{i_1} & * & * & * \\
0 & \gamma_{i_2 - i_1} & * & * \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \gamma_{n-i_l}
\end{pmatrix},
\]

where $\gamma_i \in \text{GL}_i(\mathbb{F}_q)$.

Among the Schubert cells $\Omega_w(\mathbb{F}_q)$ which are being identified by passing to cosets of $B_I(\mathbb{F}_q)$ there is a unique one with minimal dimension. It is not hard to see that these cells are exactly the cells indexed by elements $w \in S_n$ with $D_L(w) \subseteq I$, and that they constitute a set of representatives for the cosets $\text{GL}_n(\mathbb{F}_q)/B_I(\mathbb{F}_q)$. We obtain

\[
\binom{n}{I}_q = |\text{GL}_n(\mathbb{F}_q)/B_I(\mathbb{F}_q)| = \sum_{w \in S_n, D_L(w) \subseteq I} |\Omega_w(\mathbb{F}_q)| = \sum_{w \in S_n, D_L(w) \subseteq I} q^{\ell(w)}.
\]

This proves Proposition 2.6 in general.

2.3. Counting with $p$-adic integrals. The idea to employ tools from the theory of $p$-adic integration to count subgroups and subrings is as old as the subject. It was first put to work in [16], and was further developed in [11] and [40]. All of these $p$-adic integrals are in some sense generalisations of Igusa’s local zeta function, which we describe first. This will allow us to give a first proof of formula (1.5) for the zeta functions of abelian groups. We will also show how a formulation in terms of $p$-adic integrals enables us to express the local zeta functions of the Heisenberg Lie ring (cf. Example 1.3)
in terms of the generating function associated with a polyhedral cone (or, equivalently, a system of linear homogeneous diophantine equations), which we may evaluate to confirm formula (1.9). We will study these in some detail in Section 2.4.

The $p$-adic integrals we consider are all variants of Igusa’s local zeta function. Given a polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$, Igusa’s local zeta function associated with $f$ is the $p$-adic integral

$$Z_f(s) := \int_{\mathbb{Z}_p^n} |f(x)|_p^s d\mu^{(n)}.$$ 

Here, $\mathbb{Z}_p$ are the $p$-adic integers, $\mu^{(n)}$ is the (additive) Haar measure on $\mathbb{Z}_p^n$ (normalised such that $\mu^{(n)}(\mathbb{Z}_p^n) = 1$), $| \cdot |_p$ denotes the $p$-adic norm (defined by $|a|_p := p^{-v_p(a)}$, where $v_p(a) = r$ if $a = p^r b$ with $p \nmid b$), and $s$ is a complex variable. (For a reminder about the Haar measure on $\mathbb{Z}_p^n$, see [9, Section 1.6].)

Igusa’s local zeta function associated with the polynomial $f$ is a good tool to understand the sequence $(N_m)$, where $N_m$ denotes the number of solutions of the congruence $f(x) \equiv 0 \bmod (p^m)$. These numbers may be encoded in a Poincaré series

$$P_f(t) := \sum_{m=0}^{\infty} p^{-nm} N_m t^m.$$ 

This Poincaré series is related to the $p$-adic integral via the formula

$$(2.4) \quad P_f(p^{-s}) = \frac{1 - p^{-s} Z_f(s)}{1 - p^{-s}}.$$ 

Indeed, $p^{-nm} N_m$ is the measure of the set $\{ x \in \mathbb{Z}_p^n \mid v_p(f(x)) \geq m \}$ and thus

$$\mu^{(n)}(\{ x \in \mathbb{Z}_p^n \mid v_p(f(x)) = m \}) = p^{-nm} N_m - p^{-n(m+1)} N_{m+1}.$$ 

Thus

$$Z_f(s) = \sum_{m=0}^{\infty} \mu^{(n)}(\{ x \in \mathbb{Z}_p^n \mid v_p(f(x)) = m \}) p^{-sm}$$

$$= \sum_{m=0}^{\infty} \left( p^{-nm} N_m - p^{-n(m+1)} N_{m+1} \right) p^{-sm}$$

$$= P_f(p^{-s}) - p^s (P_f(p^{-s}) - 1)$$

$$= (1 - p^s) P_f(p^{-s}) + p^s,$$

which is equivalent to (2.4). As an example of the above formula, consider the integral

$$Z(s) := \int_{\mathbb{Z}_p} |x|_p^s d\mu^{(1)}.$$
Observing that the associated Poincaré series equals
\[ P(p^{-s}) = \sum_{m=0}^{\infty} (p^{-1-s})^m = \zeta_p(s + 1) = \frac{1}{1 - p^{-1-s}} \]
we deduce that
\[
Z(s) = \frac{1 - p^{-1}}{1 - p^{-1-s}} = (1 - p^{-1})\zeta_p(s + 1).
\]

We now explore how $p$-adic integrals may be used to count subgroups, by giving a first proof of formula (1.5). Recall that we may consider $\mathbb{Z}^n$ as a ring with trivial multiplication, so counting subgroups and counting subrings is the same thing in this case. It suffices to prove that, for each prime $p$,
\[
\zeta_{\mathbb{Z}^n,p}(s) = \zeta_{\mathbb{Z}^n}(s) = \zeta_p(s)\zeta_p(s - 1)\cdots\zeta_p(s - (n - 1)).
\]

The first equation is clear. For the second equation, assume that $\mathbb{Z}^n = \mathbb{Z}_p e_1 \oplus \cdots \oplus \mathbb{Z}_p e_n$ as $\mathbb{Z}_p$-module, and set $\Gamma := \text{GL}_n(\mathbb{Z}_p)$. Then subgroups of $\mathbb{Z}_p^n$ of finite index may be identified with right $\Gamma$-cosets of $n \times n$-matrices over $\mathbb{Z}_p$ with non-zero determinant. Indeed, every such subgroup may be generated by $n$ generators, whose coordinates with respect to the chosen basis may be encoded in the rows of an $n \times n$-matrix over $\mathbb{Z}_p$. Two such matrices $M_1$ and $M_2$ correspond to the same subgroup if and only if there is an element $\gamma \in \Gamma$ such that $M_1 = \gamma M_2$. In fact, one sees easily that these matrices may be chosen to lie in the set $\text{Tr}(n, \mathbb{Z}_p)$ of upper-triangular matrices over $\mathbb{Z}_p$, so that subgroups $H$ correspond to cosets $U M$, where $M \in \text{Tr}(n, \mathbb{Z}_p)$ and $U := \Gamma \cap \text{Tr}(n, \mathbb{Z}_p)$.

Now choose, for each $H \leq \mathbb{Z}^n$, a representative $M_H$ in $U M =: \mathcal{M}(H)$, the $U$-coset in $\text{Tr}(n, \mathbb{Z}_p)$ corresponding to $H$. Notice that
\[
|\mathbb{Z}_p^n : H| = |\det(M_H)|_p^{n-1},
\]
and that
\[
\mu(\mathcal{M}(H)) = (1 - p^{-1})^n \prod_{i=1}^{n} |(M_H)_{ii}|_p^{i-1}
\]
where $\mu$ denotes the additive Haar measure on $\text{Tr}(n, \mathbb{Z}_p) \cong \mathbb{Z}_p^{(n+1)/2}$, normalised so that $\mu(\text{Tr}(n, \mathbb{Z}_p)) = 1$. We thus obtain a partition
\[
\text{Tr}(n, \mathbb{Z}_p) = \bigcup_{H \leq \mathbb{Z}^n} \mathcal{M}(H) \cup \text{Tr}_0(n, \mathbb{Z}_p),
\]
where $\text{Tr}^0(n, \mathbb{Z}_p)$ denotes the set of $n \times n$-upper-triangular matrices over $\mathbb{Z}_p$ with zero determinant, of Haar-measure zero) and compute

$$\sum_{H \leq \mathbb{Z}_p^n} |H|^{-s} = \sum_H |\det(M_H)|_p^s$$

$$= \sum_H \mu(\mathcal{M}(H))^{-1} \mu(\mathcal{M}(H)) \prod_{i=1}^n |(M_H)_{ii}|_p^s$$

$$= \sum_H (1 - p^{-1})^{-n} \prod_{i=1}^n |(M_H)_{ii}|_p^{-i} \int_{\mathcal{M}(H)} \prod_{i=1}^n |(M_H)_{ii}|_p^s d\mu$$

$$= (1 - p^{-1})^{-n} \int_{\text{Tr}(n, \mathbb{Z}_p)} \prod_{i=1}^n |M_{ii}|_p^{s-i} d\mu$$

$$= (1 - p^{-1})^{-n} \int_{\mathbb{Z}_p^n} \prod_{i=1}^n |x_i|_p^{s-i} d\mu^{(n)}$$

$$= (1 - p^{-1})^{-n} \prod_{i=1}^n \int_{\mathbb{Z}_p} |x_i|_p^{s-i} d\mu^{(1)}$$

$$= \prod_{i=1}^n \zeta_p(s - (i - 1)), \quad (2.5)$$

which proves (2.6).

Note that we managed to compute each of the local factors of $\zeta_{\mathbb{Z}_p^n}(s)$ by expressing it as an integral over the affine space $\text{Tr}(n, \mathbb{Z}_p)$ of upper-triangular matrices. The integrand in this integral is a simple function of the diagonal entries of the matrices. How does this approach vary if we consider rings with nontrivial multiplication? For arbitrary rings $L$, our above analysis carries through up to (and including) equation (2.8). In general, however, not every coset $\mathcal{U}M$ will correspond to a subring of $L$. We therefore need to describe conditions for such a coset to define a subring.

Let us return to Example 1.3 of the discrete Heisenberg group. Its associated Lie ring $L$ has a $\mathbb{Z}$-basis $(x, y, z)$, where $[x, y] = z$ is the only nontrivial relation. To compute its local zeta function at the prime $p$, we need to count subalgebras in the $\mathbb{Z}_p$-algebra $L_p := \mathbb{Z}_p \otimes L$. The rows of a matrix $M = (M_{ij}) \in \text{Tr}(3, \mathbb{Z}_p)$ encode the generators of a full additive sublattice of $\mathbb{Z}_p^3$. To determine whether such a matrix gives rise to a subalgebra we need to check whether this sublattice is closed under taking Lie brackets of its generators. In this case it is easy to see that the only condition we need to check is

$$[M_{11}x + M_{12}y + M_{13}z, M_{22}y + M_{23}z] \in \langle M_{33}z \rangle_{\mathbb{Z}_p}.$$  

Using the commutator relation $[x, y] = z$ and the bilinearity of the Lie bracket $[,]$, we see that this condition is equivalent to

$$M_{33} | M_{11}M_{22}. \quad (2.10)$$
Note that this divisibility condition is equivalent to the inequality of \( p \)-adic valuations
\[
v_p(M_{33}) \leq v_p(M_{11}) + v_p(M_{22}).
\]
We thus obtain
\[
\sum_{H \leq L_p} |L_p : H|^{-s} = (1 - p^{-1})^{-3} \int_{\{ M_{11} \leq 3 \}} \prod_{i=1}^{3} |M_{ii}|^{-s-i} d\mu(6)
\]
\[
= (1 - p^{-1})^{-3} \int_{\{ x \in \mathbb{Z}_p^3 | x_3|x_1x_2 \}} |x_1|^{-s-1} |x_2|^{-s-2} |x_3|^{-s-3} d\mu(3)
\]
\[
= \sum_{\{ m \in \mathbb{N}_0^3 | m_3 \leq m_1 + m_2 \}} (p^{-s})^{m_1} (p^{1-s})^{m_2} (p^{2-s})^{m_3}.
\]
It is not hard to compute this sum explicitly (see Proposition 2.11 below).

It is useful, however, to observe that it may be interpreted as a generating function associated with a system of linear homogeneous diophantine equations.

### 2.4. Linear homogeneous diophantine equations.

Let \( \Phi \) be an \( r \times m \) matrix over \( \mathbb{Z} \) (without loss of generality of rank \( r \)), and consider the system of linear equations
\[
\Phi \alpha = 0,
\]
where \( \alpha^t = (\alpha_1, \ldots, \alpha_m) \) and \( \mathbf{0} \in \mathbb{N}_0^r \). The set of non-negative integral solutions of (2.11) form a commutative monoid \( \mathcal{E} := \{ \alpha \in \mathbb{N}_0^m | \Phi \alpha = 0 \} \) with identity under addition. One approach to study this monoid is to investigate the generating function
\[
E(X) := E_\Phi(X) := \sum_{\alpha \in \mathcal{E}} X^\alpha,
\]
where \( X^\alpha = X_1^{\alpha_1} \ldots X_m^{\alpha_m} \) is a monomial in variables \( X_1, \ldots, X_m \).

The generating functions \( E_\Phi(X) \) have been intensely studied by Stanley and others ([34, Chapter 4.6], [33, Chapter I]). It can be proved, for example, that \( E_\Phi(X) \) is always a rational function in the variables \( X_1, \ldots, X_m \), with denominator of the form \( \prod_{\beta \in CF(E)} (1 - X^\beta) \), where \( \beta \) ranges over the finite set \( CF(E) \) of completely fundamental solutions to \( \Phi \). (A solution \( \beta \) to (2.11) is called fundamental if, whenever \( \beta = \gamma + \delta \) for \( \gamma, \delta \in \mathcal{E} \), \( \gamma = \delta \) or \( \delta = \beta \). A solution \( \beta \) to (2.11) is called completely fundamental if, whenever \( n \beta = \gamma + \delta \) for \( \gamma, \delta \in \mathcal{E} \), then \( \gamma = n_1 \beta \) for some \( 0 \leq n_1 \leq n \).)

We will also consider the closely related generating function
\[
\overline{E}(X) := \overline{E}_\Phi(X) := \sum_{\alpha \in \overline{\mathcal{E}}} X^\alpha,
\]
where \( \overline{\mathcal{E}} := \{ \alpha \in \mathbb{N}_0^m | \Phi \alpha = 0 \} \), the semigroup of the positive integral solutions of (2.11). \( \overline{E}(X) \) is also a rational function in the variables \( X_1, \ldots, X_m \). The following result of Stanley will be of great importance in applications to
zeta functions of rings. We denote by $1/X$ the vector of inverted variables $(1/X_1, \ldots, 1/X_m)$.

**Theorem 2.8.** [34, Theorem 4.6.14] Assume that $\mathcal{E} \neq \emptyset$ and set $d := \dim(\mathcal{E})$, where $\mathcal{E}$ is the cone of non-negative real solutions to (2.11). Then

$$E(X) = (-1)^d E(1/X).$$

**Example 2.9.** If $r = 0$ we obtain $E = \mathbb{N}_0^m$, with completely fundamental solutions $\{(1,0,\ldots,0),\ldots,(0,0,\ldots,0,1)\}$, yielding

$$E(X) = \sum_{\alpha \in \mathbb{N}_0^m} X_\alpha = \prod_{i=1}^m \frac{1}{1-X_i} \quad \text{and} \quad E(X) = \sum_{\alpha \in \mathbb{N}^m} X_\alpha = \prod_{i=1}^m \frac{X_i}{1-X_i}.$$ 

**Example 2.10.** Consider the matrix $\Phi = (1,1,-1,-1)$. It can be shown that the (completely) fundamental solutions of the equation $\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 = 0$ are $(1,0,1,0)$, $(1,0,0,1)$, $(0,1,1,0)$ and $(0,1,0,1)$. Note that there is one non-trivial relation between these solutions:

$$(1,0,1,0) + (0,1,0,1) = (1,0,0,1) + (0,1,1,0) = (1,1,1,1)) = (1,1,1,1).$$

This can be used to show (see [33, I.11] for details) that

$$E_\Phi(X_1, X_2, X_3, X_4) = \frac{1 - X_1X_2X_3X_4}{(1 - X_1X_3)(1 - X_1X_4)(1 - X_2X_3)(1 - X_2X_4)}.$$ 

Note that we obtain nothing more complicated if we allow inequalities rather than equalities in (2.11). Indeed, an inequality may always be expressed in terms of an equality by introducing a slack variable. The generating functions enumerating integral points in rational polyhedral cones (intersections of finitely many rational half-spaces) may therefore be expressed in terms of generating functions associated with linear homogeneous diophantine equations. For example, $m_3 \leq m_1 + m_2$ if and only if there exists $m_4 \in \mathbb{N}_0$ such that $m_3 + m_4 = m_1 + m_2$ or, equivalently, $m_1 + m_2 - m_3 - m_4 = 0$. We obtain the generating function enumerating non-negative solutions of the inequality by taking the generating function associated with the equality by setting the variable corresponding to the slack variable to 1. From Example 2.10 we get, for instance,

$$\sum_{\{m \in \mathbb{N}_0^3 : m_3 \leq m_1 + m_2\}} X_1^{m_1} X_2^{m_2} X_3^{m_3} = E_\Phi(X_1, X_2, X_3, 1) = \frac{1 - X_1X_2X_3}{(1 - X_1X_3)(1 - X_1)(1 - X_2X_3)(1 - X_2)}.$$ 

In Section 2.3 we showed how the local zeta functions of the Heisenberg Lie rings can be expressed in terms of the rational function given in (2.14). We summarise this result in
Proposition 2.11. [16, Proposition 8.1] Let $L$ be the Heisenberg Lie ring (cf. Example 1.3). Then, for all primes $p$, the local zeta function of $L$ equals

\[
\zeta_{L,p}(s) = E_{\Phi}(p^{-s}, p^{1-s}, p^{2-s}, 1) = \frac{1 - p^{3-3s}}{(1 - p^{-s})(1 - p^{1-s})(1 - p^{2-2s})(1 - p^{3-2s})} = \zeta_p(s)\zeta_p(s-1)\zeta_p(2s-2)\zeta_p(2s-3)\zeta_p(3s-3)^{-1}.
\]

We have thus expressed the local factors of the zeta function of the Heisenberg Lie ring in terms of the generating function associated with a linear homogeneous equation (or, equivalently, a rational polyhedral cone). The feasibility of this approach was a direct consequence of the divisibility condition (2.10). In general, things are not that simple, as the following example shows.

Example 2.12. Let us reconsider the Lie ring $\mathfrak{sl}_2(\mathbb{Z})$ from Example 1.2. Fix a prime $p$. It is not hard (and a recommended exercise; cf. [14]) to show that the coset $U\mathcal{M}$ of a matrix $M \in \text{Tr}(3, \mathbb{Z}_p)$ encodes the coordinates of generators of a subring of $\mathfrak{sl}_2(\mathbb{Z}_p)$ if and only if

\[
v_p(M_{22}) \leq v_p(4M_{12}M_{23}), \quad v_p(M_{22}) \leq v_p(4M_{12}M_{33}) \quad \text{and} \quad v_p(M_{22}M_{33}) \leq v_p(M_{11}M_{22}^2 + 4M_{22}M_{13}M_{23} - 4M_{12}M_{23}^2).
\]

(2.15)

In general, the condition for a coset to define a subalgebra may be described by a finite number of inequalities in the $p$-adic values of polynomials in the matrix entries. If these polynomials are monomials (as is the case for the Heisenberg Lie ring; cf. (2.10)), the computation of the local zeta function reduces to the computation of the generating function of a rational polyhedral cone (or system of linear homogeneous diophantine equations). In general, a resolution of singularities – a tool from algebraic geometry – may be used to remedy the situation. It allows for a partition of the domain of integration into pieces on which the integral may be expressed in terms of generating functions of polyhedral cones. The pieces are indexed by the $\mathbb{F}_p$-points of certain algebraic varieties defined over $\mathbb{F}_p$. These kinds of $p$-adic integrals, called cone integrals, were introduced in [11]. A comprehensive introduction to cone integrals may be found in [13, Sections 4 and 5].

The description of local zeta functions of groups and rings in terms of cone integrals has far reaching applications for the analysis of analytic properties of global zeta functions (cf. Section 2.7).

2.5. Local functional equations. The zeta functions of the rings we have presented so far as examples all share a remarkable property: their local factors generically exhibit a palindromic symmetry on inversion of the prime $p$. More precisely, almost all of the Euler factors satisfy a local functional equation of the form

\[
\zeta_{L,p}(s)|_{p \rightarrow p^{-1}} = (-1)^a p^{b-cs} \zeta_{L,p}(s),
\]

(2.16)
where \(a, b, c\) are integers which are independent of the prime \(p\). In the present section we explain and give an outline of the proof of the following theorem.

**Theorem 2.13.** [40, Theorem A] Let \(L\) be a ring of additive rank \(n\). There are smooth projective varieties \(V_t\), defined over \(\mathbb{Q}\), and rational functions \(W_t(X,Y) \in \mathbb{Q}(X,Y), t \in \mathbb{M}\), such that, for almost all primes \(p\), the following hold:

1. \[
\zeta_{L,p}(s) = \sum_{t=1}^{m} b_t(p) W_t(p,p^{-s}),
\]

   where \(b_t(p)\) denotes the number of \(\mathbb{F}_p\)-rational points of \(\overline{V_t}\), the reduction modulo \(p\) of \(V_t\).

2. Setting \(b_t(p^{-1}) := p^{-\dim(V_t)} b_t(p)\) the following functional equation holds:

\[
\zeta_{L,p}(s) \big|_{p \to p^{-1}} = (-1)^n p^{(n^2 - n)s} \zeta_{L,p}(s).
\]

Note that the advance of Theorem 2.13 over Theorem 2.3 consists in the assertion (2). The notation ‘\(p \to p^{-1}\)’ needs some justification. If \(b_t(p)\) is a polynomial in \(p\), \(b_t(p^{-1})\) is with the rational number obtained by evaluating this polynomial at \(p^{-1}\). This follows from the fact that the varieties \(\overline{V_t}\) are smooth and projective. In general, the above definition is motivated by properties of the numbers of \(\mathbb{F}_p\)-rational points of such varieties, which follow from the Weil conjectures. More precisely, let \(V\) be a smooth projective variety defined over the finite field \(\mathbb{F}_p\). By deep properties of the Hasse-Weil zeta function associated with \(V\), there are complex numbers \(\alpha_{rj}, 0 \leq r \leq 2 \dim(V), 1 \leq j \leq t_r\) for suitable non-negative integers \(t_r\), such that the number \(b_V(p)\) of \(\mathbb{F}_p\)-rational points of \(V\) can be written as

\[
b_V(p) = \sum_{r=0}^{2 \dim(V)} (-1)^r \sum_{j=1}^{t_r} \alpha_{rj}.
\]

(Note that the numbers \(t_r\) may well be zero; cf. the examples given in Section 2.2.) Furthermore, for each \(r \in [2 \dim(V)]_0\) the multisets

\[\{\alpha_{rj} \mid j \in [t_{2 \dim(V) - r}]\}\]

and

\[\left\{\frac{p^{\dim(V)}}{\alpha_{rj}} \mid j \in [t_r]\right\}\]

coincide. Thus,

\[
b_V(p^{-1}) := p^{-\dim(V)} b_V(p) = \sum_{r=0}^{2 \dim(V)} (-1)^r \sum_{j=1}^{t_r} \alpha_{rj}^{-1}
\]

may be interpreted as the expression we obtain by inverting the terms \(\alpha_{rj}\) in (2.19) (even if they are not, in general, powers of the prime \(p\)).

Before we give an outline of the proof of Theorem 2.13, let us revisit Example 2.4.
Example 2.14. Let $E$ denote the elliptic curve defined by the equation $y^2 = x^3 - x$. For a prime $p$, denote this time by $b(p)$ the number of projective points of $E$ over $\mathbb{F}_p$, i.e.

$$b(p) := |\{(x : y : z) \in \mathbb{P}^2(\mathbb{F}_p) \mid y^2z = x^3 - xz^2\}|.$$  

Clearly $b(p) = c(p) + 1$, where $c(p)$ was defined in Example 2.4: we simply add the point $(0 : 1 : 0)$ ‘at infinity’. The results quoted there imply that

$$b(p) = \begin{cases} 1 + p & \text{if } p \equiv 3 \mod (4) \text{ and} \\ 1 - (\pi + \overline{\pi}) + p & \text{otherwise,} \end{cases}$$

where $\pi \overline{\pi} = p$. Note that this last equation implies that $\pi^{-1} = \overline{\pi}/p$ and $\overline{\pi}^{-1} = \pi/p$, so that

$$1 - (\pi^{-1} + \overline{\pi}^{-1}) + p^{-1} = p^{-1}(p - (\pi + \overline{\pi}) + 1) = p^{-1}b(p) = b(p)|_{p\rightarrow p^{-1}},$$

by definition of the latter.

Outline of proof of Theorem 2.13: (For details see [40, Sections 2 and 3].) The proof falls into two parts. The first is of a combinatorial and Coxeter group theoretic nature. It consists in proving the following general result about generating functions.

Proposition 2.15. Let $n \in \mathbb{N}$ and, let $(W_I(p^{-s}))_{I \subseteq [n-1]}$ be a family of functions in $p^{-s}$ with the property that

$$(2.20) \quad \forall I \subseteq [n-1] : W_I(p^{-s})|_{p\rightarrow p^{-1}} = (-1)^{|I|} \sum_{J \subseteq I} W_J(p^{-s}).$$

Then the function

$$(2.21) \quad W(p^{-s}) := \sum_{I \subseteq [n-1]} \binom{n}{I}_{p^{-1}} W_I(p^{-s})$$

(with the polynomials $(\binom{n}{I})_{X}$ defined as in (2.2)) satisfies

$$(2.22) \quad W(p^{-s})|_{p\rightarrow p^{-1}} = (-1)^{n-1}p^{\binom{n}{2}}W(p^{-s}).$$

Remark 2.16. We do not need to specify the operation $p \rightarrow p^{-1}$ in (2.20) at this stage; the left hand sides of these equations could be defined in terms of the right hand sides. We do not assume the functions $W_I(p^{-s})$ to be rational in $p^{-s}$. In practice, we will apply Proposition 2.15 to families of rational functions $W_I(p^{-s})$ which are themselves of the form (2.17), and we define $p \rightarrow p^{-1}$ as in Theorem 2.13. What is understood, however, is that the inversion of the prime extends linearly to $W(p^{-s})$, and that, of course, $(\binom{n}{I})_{p^{-1}} |_{p\rightarrow p^{-1}} = (\binom{n}{I})_{p}.$

Proof of Proposition 2.15. We utilise the Coxeter group theoretic description of the numbers $(\binom{n}{I})_{p}$ given in Proposition 2.6. It is a well-known fact (\cite[Page 612.0x792.0} 19}...
Section 1.8) that there is a unique longest element \( w_0 \in S_n \), namely the inversion, such that, for all \( w \in S_n \),

\[
\ell(w) + \ell(ww_0) = \ell(w_0) = \binom{n}{2}
\]

and

\[
D_L(ww_0) = D_L(w^c).
\]

Here, given \( I \subseteq [n-1] \) we write \( I^c \) for \([n-1] \setminus I\). We also need the following Lemma.

**Lemma 2.17.** [39, Lemma 7] Under the hypotheses of Proposition 2.15, for all \( I \subseteq [n-1] \),

\[
\sum_{I \subseteq J} W_J(p^{-s})|_{p \to p^{-1}} = (-1)^{n-1} \sum_{F \subseteq J} W_J(p^{-s}).
\]

**Proof.** We have

\[
\sum_{I \subseteq J} W_J(p^{-s})|_{p \to p^{-1}} = \sum_{I \subseteq J} (-1)^{|I|} \sum_{S \subseteq J} W_S(p^{-s}) = \sum_{R \subseteq [n-1]} c_R W_R(p^{-s}),
\]

say, where

\[
c_R = \sum_{R \cup J \subseteq J} (-1)^{|J|} = (-1)^{|R \cup J|} \sum_{S \subseteq (R \cup J)^c} (-1)^{|S|}
\]

\[
= (-1)^{|R \cup J|} (1 - 1)^{|R \cup J|^c} = \begin{cases} (-1)^{n-1} & \text{if } R \supseteq I^c, \\ 0 & \text{otherwise.} \end{cases}
\]

This proves Lemma 2.17. \( \square \)

We compute

\[
W(p^{-s})|_{p \to p^{-1}} = \sum_{I \subseteq [n-1]} \binom{n}{I} p^I W_I(p^{-s})|_{p \to p^{-1}} \tag{2.21}
\]

\[
= \sum_{I \subseteq [n-1]} \left( \sum_{w \in S_n, D_L(w) \subseteq I} p^{\ell(w)} \right) W_I(p^{-s})|_{p \to p^{-1}} \quad \text{Prop. 2.6}
\]

\[
= \sum_{w \in S_n} p^{\binom{n}{2} - \ell(ww_0)} \sum_{D_L(ww_0) \subseteq I} W_I(p^{-s})|_{p \to p^{-1}} \tag{2.23}
\]

\[
= (-1)^{n-1} p^{\binom{n}{2}} \sum_{w \in S_n} p^{-\ell(ww_0)} \sum_{D_L(ww_0) \subseteq I} W_I(p^{-s}) \quad \text{Lemma 2.17, (2.24)}
\]

\[
= (-1)^{n-1} p^{\binom{n}{2}} \sum_{I \subseteq [n-1]} \left( \sum_{w \in S_n, D_L(ww_0) \subseteq I} p^{-\ell(ww_0)} \right) W_I(p^{-s})
\]

This proves Proposition 2.15. \( \square \)
We now proceed to the second part of the proof of Theorem 2.13. It consists in proving that the local zeta function \( \zeta_{L,p}(s) \) of a ring \( L \) of additive rank \( n \) may be written as

\[
(1 - p^{-ns})^{-1} W(p^{-s}),
\]

where \( W(p^{-s}) \) is of the form (2.21) for suitable (rational) functions \( W_I(p^{-s}) \), satisfying the hypotheses (2.20) of Proposition 2.15. This will require both algebro-geometric and combinatorial methods (which are similar to but markedly different from the ones used to study cone integrals). It may be instructive to see this done in a familiar special case first.

Example 2.18. We will see below in Example 2.19 that the local zeta functions of the abelian group \( \mathbb{Z}^n \) may be written as

\[
\zeta_{\mathbb{Z}^n,p}(s) = \frac{1}{1 - X_n} \sum_{I \subseteq [n-1]} \binom{n}{I} p^{-1} \prod_{i \in I} X_i \frac{1}{1 - X_i},
\]

where, for \( i \in [n] \), \( X_i := p^{i(n-i)-is} \). One checks immediately that the functions

\[
W_I(p^{-s}) := \prod_{i \in I} \frac{X_i}{1 - X_i}
\]

satisfy (2.20). Indeed, the operation \( p \to p^{-1} \) simply amounts to an inversion of the ‘variables’ \( X_i \), as they are monomials in \( p \) and \( p^{-s} \), and

\[
\frac{X^{-1}}{1 - X^{-1}} = - \left( 1 + \frac{X}{1 - X} \right).
\]

More conceptually, the validity of the equations (2.20) may be regarded as a consequence of Theorem 2.8 in the special case studied in Example 2.9, as we may view \( W_I(p^{-s}) \) as obtained from the rational generating function in variables \( X_i \), counting positive integral solutions of an (empty) set of linear homogeneous diophantine equations in \( |I| \) variables, where the variables \( X_i \) are substituted by certain monomials in \( p \) and \( p^{-s} \). A variation of this basic idea will be crucial for the proof of Theorem 2.13.

Given a prime \( p \), our task is to enumerate full additive sublattices of the \( n \)-dimensional \( \mathbb{Z}_p \)-algebra \( L_p \subset \mathbb{Q}_p \otimes L_p \) which are subalgebras, i.e. which are closed under multiplication. It is easy to see that, given such a lattice \( \Lambda \), there is a unique lattice \( \Lambda_0 \) in the homothety class \([\Lambda] := \mathbb{Q}_p^*\Lambda \) of \( \Lambda \) such that the subalgebras contained in \([\Lambda] \) are exactly the multiples \( p^m\Lambda_0 \), \( m \in \mathbb{N}_0 \). Indeed, given any sublattice \( \Lambda \) of \( L_p \), and \( e \in \mathbb{Z} \), clearly \( (p^e\Lambda)^2 \subset p^e\Lambda \) if and only if \( p^e\Lambda^2 \subset \Lambda \). Let \( e_0 := \min\{e \in \mathbb{Z} | p^e\Lambda \subset L_p \text{ and } p^e\Lambda^2 \subset \Lambda \} \), and set \( \Lambda_0 := p^{e_0}\Lambda \). Evidently, \( \Lambda_0 \) only depends on the homothety class of \( \Lambda \). We thus have

\[
\zeta_{L_p}(s) = (1 - p^{-ns})^{-1} \sum_{[\Lambda]} |L_p : \Lambda_0|^{-s}.
\]
We set
\[ W(p^{-s}) := \sum_{[\Lambda]} |L_p : \Lambda_0|^{-s}. \]

It remains to show that \( W(p^{-s}) \) is of the form (2.21), with rational functions \( W_I(p^{-s}) \) to which Proposition 2.15 is applicable. We will achieve this by first partitioning the set of homothety classes of lattices into finitely many parts, indexed by the subsets \( I \) of \([n - 1]\), reflecting (aspects of) their elementary divisor types. On each of these parts, we will describe the indices \( |L_p : \Lambda_0| \) in terms of algebraic congruences, and then encode the numbers of solutions to these congruences in terms of a suitable \( p \)-adic integral \( W_I(p^{-s}) \) so that the family \( (W_I(p^{-s}))_{I \subseteq [n-1]} \) satisfies the ‘inversion properties’ (2.20). The proof of the latter will require sophisticated methods from algebraic geometry, which we can only sketch here.

The reader will note the analogy with the proof of equation (2.4), which also proceeded by expressing the numbers of certain congruences in terms of the Haar measure of suitable sets.

We recall from Section 2.3 that lattices in \( L_p \) are in 1−1-correspondence with cosets \( \Gamma M \), where \( \Gamma = \text{GL}_n(\mathbb{Z}_p) \) and \( M \in \text{Tr}(n, \mathbb{Z}_p) \), where the rows of \( M \) encode coordinates of generators of \( \Lambda \) with respect to a fixed basis \( (l_1, \ldots, l_n) \) for \( L_p \) as \( \mathbb{Z}_p \)-module. For \( r \in [n] \), let \( C_r \) denote the matrix of the linear map given by right-multiplication with the basis element \( l_r \) with respect to this basis. It is then not hard to show (cf. the proof of [11, Theorem 5.5]) that the lattice corresponding to the coset \( \Gamma M \) is a subalgebra if and only if
\[ \forall i, j \in [n]: M_i \sum_{r \in [n]} C_r m_{jr} \in (M_k | k \in [n])_{\mathbb{Z}_p}, \]

where \( M_i \) denotes the \( i \)-th row of \( M \). This condition is easy to check if \( M \) may be chosen to be diagonal; in this case, condition (2.26) is satisfied if, for all \( k \in [n] \), the \( k \)-th entries of all the vectors on the left hand side are divisible by \( M_{kk} \), the \( k \)-th diagonal entry of \( M \). In general, however, the coset \( \Gamma M \) will not contain a diagonal element. One way around this is to choose a different basis for \( L_p \). Indeed, by the Elementary Divisor Theorem, the coset \( \Gamma M \) does contain an element of the form \( D\alpha^{-1} \), where \( \alpha \in \Gamma \) and
\[ D = D(I, r_0) = p^{\sum_{i} r_i} \text{diag}(p^{\sum_{i} r_i}, \ldots, p^{\sum_{i} r_i}, 1, \ldots, 1) \]

for a set \( I = \{i_1, \ldots, i_t\} \subseteq [n - 1] \) and a vector \( (r_0, r_{i_1}, \ldots, r_{i_t}) =: r_0 \in \mathbb{N}_0 \times \mathbb{N}^t \) (both depending only on \( \Gamma M \)). Setting \( r := (r_{i_1}, \ldots, r_{i_t}) \), we say that the homothety class \( [\Lambda] \) of \( \Lambda \) is of type \((I, r)\) (or sometimes, by abuse of notation, of type \( I \)) and write \( \nu([\Lambda]) = (I, r) \) (or \( \nu([\Lambda]) = I \), respectively).
The matrix $\alpha$ is only unique up to right-multiplication by an element of

$$\Gamma_{I,r} := \left\{ \begin{pmatrix} \gamma_{i_1} & * & \cdots & * & * \\
p^{r_1} * & \gamma_{i_2-i_1} & * & \cdots & * \\
p^{r_1+r_2} * & p^{r_2} * & * & \cdots & * \\
& \cdots & \cdots & \cdots & \cdots \\
& \cdots & \cdots & \cdots & \cdots \\
p^{r_1+\cdots+r_{i-1}} * & p^{r_{i-1}+\cdots+r_{i-1}} * & \cdots & p^{r_{n-1}} * & \gamma_{n-i} * \end{pmatrix} \right\},$$

where $\gamma_i \in \Gamma_i := \text{GL}_i(\mathbb{Z}_p)$, and $*$ stands for an arbitrary matrix with entries in $\mathbb{Z}_p$ of the respective size. As an immediate and useful corollary, we deduce a formula for the number of lattices of given type $(I,r)$:

$$(2.27) \quad \left| \{ [\Lambda] \mid \nu([\Lambda]) = (I,r) \} \right| = |\Gamma : \Gamma_{I,r}| = \mu(\Gamma)/\mu(\Gamma_{I,r}) = \binom{n}{I} p^{n-1} \prod_{i \in I} r_i (n-i).$$

Here, $\mu$ denotes the Haar measure on the group $\Gamma$ normalised so that $\mu(\Gamma) = (1-p^{-1}) \cdots (1-p^{-n})$. It is a crucial observation that it coincides with the restriction of the additive Haar measure on $\text{Mat}_n(\mathbb{Z}_p) \cong \mathbb{Z}_p^n$, normalised so that $\mu(\text{Mat}_n(\mathbb{Z}_p)) = 1$.

We consider the $n \times n$-matrix of $\mathbb{Z}$-linear forms

$$\mathcal{R}(y) = (L_{ij}(y)) \in \text{Mat}_n(\mathbb{Z}[y]),$$

where $L_{ij}(y) := \sum_{k \in [n]} \lambda^k_{ij} y_k$, encoding the structure constants $\lambda^k_{ij}$ of $L$ with respect to the chosen basis, that is $l_i l_j = \sum_{k \in [n]} \lambda^k_{ij} l_k$. Right-multiplication by $\alpha$ now yields that the subalgebra condition (2.26) is equivalent to

$$(2.28) \quad \forall i \in [n] : D \mathcal{R}_{(i)}(\alpha) D \equiv 0 \mod (D_{ii}),$$

where $\mathcal{R}_{(i)}(\alpha) := \alpha^{-1} \mathcal{R}(\alpha[i]) (\alpha^{-1})^t$, as a quick calculation shows. (Here we write $\alpha[i]$ for the $i$-th column of the matrix $\alpha$.) Considering these matrix congruences modulo a common modulus, this is equivalent to

$$(2.29) \quad \forall i, r, s \in [n] : \mathcal{R}_{(i)}(\alpha)_{rs} p^{r_0 + \sum_{s \leq i \in I} r_i + \sum_{r \leq i \in I} r_i + \sum_{i > j \in I} r_i} \equiv 0 \mod (p^{\sum_{s \in I} r_s})$$

which may in turn be reformulated as

$$(2.30) \quad r_0 \geq \sum_{i \in I} r_i - \min \left\{ \sum_{i \in I} r_i, \sum_{l \leq i \in I} r_i + \sum_{r \leq i \in I} r_i + \sum_{i > j \in I} r_i + v_{irs}(\alpha) \mid (i, r, s) \in [n]^3 \right\},$$

where $v_{irs}(\alpha) := \min \{ v_p (\mathcal{R}_{(i)}(\alpha)_{rs}) \mid i \leq l, \rho \geq r, \sigma \geq s \}$.

We observe that this description of the quantity $m([\Lambda])$ is in terms which are linear in the $(r_i)_{i \in I}$ and terms $v_{irs}(\alpha)$, which only depend on $\alpha$. Moreover, by construction the $v_{irs}(\alpha)$ only depend on the coset $\alpha B$, where $B \subset
GL_n(\mathbb{Z}_p) is the Borel subgroup of upper-triangular matrices. (This is the purpose of using inequalities rather than equalities in their definition.) As in the proof of the identity (2.4), this allows us to express the numbers of lattice classes \([\Lambda]\) of given type \(\nu([\Lambda])\) and invariant \(m([\Lambda])\) in terms of the Haar measure of the set on which the integrand of a certain \(p\)-adic integral is constant. More precisely, we set, for \((i, r, s) \in [n]^3\),

\[
f_{irs}(y) := \{(R_i(y))_{i,s} \mid t \leq i, \rho \geq r, \sigma \geq s\}
\]

and

\[
Z_I((s_i)_{i \in I}, s_n) := \int_{p\mathbb{Z}_p^I \times \Gamma} \prod_{i \in I} |x_i|_{sp}^{|s_i|} \left( \prod_{i \in I} x_i \left| \prod_{i \leq r+s+\delta_i} \right. \right) f_{irs}(y) \left| \left. \prod_{I \in \mathcal{S}} d\chi_I dy. \right.
\]

(Here, we extended the \(p\)-adic absolute value to a set \(\mathcal{S}\) of \(p\)-adic numbers by setting \(\parallel S \parallel_p := \min \{ |s|_p \mid s \in \mathcal{S} \}\). We denoted by \(d\chi_I = dx_1 \cdots dx_i\) the Haar measure on \(p\mathbb{Z}_p^I\).) This \(p\)-adic integral has been expressly set up so that, for each \(I \subseteq [n-1]\),

\[
\sum_{\nu([\Lambda]) = I} |L_p : \Lambda_0|^{-s} = \binom{n}{I}^{-1} W_I(p^{-s}),
\]

say, where

\[
W_I(p^{-s}) := \frac{Z_I((s_i + n - i + s_{\nu([\Lambda])})_{i \in I}, s_{n})}{(1 - p^{-1})^I \mu(\Gamma)},
\]

so that

\[
W(p^{-s}) = \sum_{I \subseteq [n-1]} \binom{n}{I}^{-1} W_I(p^{-s}).
\]

We would like to establish that the functions \(W_I(p^{-s})\) satisfy the inversion property (2.20). Let us first confirm this in the abelian case.

**Example 2.19** (abelian groups revisited). If \(L_p\) is abelian, i.e. if the multiplication on \(L_p\) is trivial, all the sets of polynomials \(f_{irs}\) are equal to \(\{0\}\), so (2.31) takes the form

\[
Z_I((s_i)_{i \in I}, s) = \int_{(p\mathbb{Z}_p)^I \times \Gamma} \prod_{i \in I} |x_i|_{sp}^{s_i+s_n} d\chi_I dy
= \mu(\Gamma) \prod_{i \in I} \int_{p\mathbb{Z}_p} |x_i|_{sp}^{s_i+s_n} dx_i = \mu(\Gamma)(1 - p^{-1})^I \prod_{i \in I} \frac{p^{-1-s_i-s_n}}{1 - p^{-1-s_i-s_n}}
\]

and thus (cf. (2.25))

\[
W_I(p^{-s}) = \prod_{i \in I} \frac{p^{I(n-i)-s_i}}{1 - p^{I(n-i)-s_i}}.
\]
(Of course we could have deduced this immediately from (2.27), avoiding any reference to \( p \)-adic integrals.)

We note that in the formula (2.31), the variables \( x \) enter monomially. If the same were true for the variables \( y \), the inversion properties (2.20) would follow from the following proposition, generalising a result of Stanley:

**Proposition 2.20.** [40, Proposition 2.1] Let \( s, t \in \mathbb{N}_0 \) and, for \( \sigma \in [s], \tau \in [t] \), let \( L_{\sigma \tau}(n) \) be \( \mathbb{Z} \)-linear forms in the variables \( n_1, \ldots, n_r \). Let \( X_1, \ldots, X_r, Y_1, \ldots, Y_s \) be independent variables and set

\[
Z^\circ(X, Y) := \sum_{n \in \mathbb{N}^r} \prod_{\rho \in [r]} X_\rho^{n_\rho} \prod_{\sigma \in [s]} \min_{r \in [\sigma]} \{ L_{\sigma \tau}(n) \},
\]

\[
Z(X, Y) := \sum_{n \in \mathbb{N}^r} \prod_{\rho \in [r]} X_\rho^{n_\rho} \prod_{\sigma \in [s]} \min_{r \in [\sigma]} \{ L_{\sigma \tau}(n) \}.
\]

Then

\[
Z^\circ(X^{-1}, Y^{-1}) = (-1)^r Z(X, Y).
\]

For \( t \leq 1 \) this follows immediately from Theorem 2.8, as \( Z^\circ(X, Y) \) (and \( Z(X, Y) \)) may be interpreted in terms of the generating functions \( E(X) \) (and \( E(X) \), respectively) associated with the empty set of equations in \( r \) variables. The general case follows from an adaptation of the proof of [34, Proposition 4.16.14].

In general, the inversion properties (2.20) can be proved by making the integral (2.31) ‘locally monomial’ in the variables \( y \). This is achieved by applying a ‘principalisation of ideals’, a tool from algebraic geometry. More precisely, we apply the following deep result to the ideal \( \prod_{(i, r, s) \in [n]^3} (f_{irs}(y)) \), defining a subvariety of the homogeneous space \( X = \text{GL}_n/B \).

**Theorem 2.21.** [41, Theorem 1.0.1] Let \( \mathcal{I} \) be a sheaf of ideals on a smooth algebraic variety \( X \). There exists a principalisation \( (Y, h) \) of \( \mathcal{I} \), that is, a sequence

\[
X = X_0 \leftarrow h_1 \ X_1 \leftarrow \cdots \leftarrow h_i \ X_i \leftarrow \cdots \leftarrow h_r \ X_r = Y
\]

of blow-ups \( h_i : X_i \to X_{i-1} \) of smooth centres \( C_{i-1} \subset X_{i-1} \) such that

a) The exceptional divisor \( E_i \) of the induced morphism \( h' = h_r \circ \cdots \circ h_2 : X_i \to X \) has only simple normal crossings and \( C_i \) has simple normal crossings with \( E_i \).

b) Setting \( h := h_r \circ \cdots \circ h_1 \), the total transform \( h^*(\mathcal{I}) \) is the ideal of a simple normal crossing divisor \( \tilde{E} \). If the subscheme determined by \( \mathcal{I} \) has no components of codimension one, then \( \tilde{E} \) is an \( \mathbb{N} \)-linear combination of the irreducible components of the divisor \( E_r \).

The existence of a principalisation lies as deep as Hironaka’s celebrated resolution of singularities in characteristic zero [17]. See [41] for details.
2.6. A class of examples: 3-dimensional $p$-adic Lie algebras. Constructing an explicit principalisation for a given family of ideals $(f_{ir}(y))_{ir}$ is in general very difficult. In the special case that $L_p$ is an anti-symmetric (not necessarily nilpotent or Lie) $\mathbb{Z}_p$-algebra of dimension 3, however, the approach of Theorem 2.13 leads to an explicit, unified expression for the zeta function of $L_p$.

**Theorem 2.22.** [24, Theorem 1] Let $L$ be a 3-dimensional $\mathbb{Z}_p$-Lie algebra. Then there is a ternary quadratic form $f(x) \in \mathbb{Z}_p[x_1, x_2, x_3]$, unique up to equivalence, such that, for $i \geq 0$,

$$\zeta^p_{L}(s) = \zeta^{3}_{\mathbb{Z}_p}(s) - Z_f(s - 2) \zeta_p(2s - 2) \zeta_p(s - 2) p^{(2-s)(i+1)} (1 - p^{-1})^{-1},$$

where $Z_f(s)$ is Igusa’s local zeta function associated with $f$.

The form $f(x)$ in Theorem 2.22 may be defined explicitly in terms of the structure constants of $L_p$ with respect to a chosen basis; different bases give rise to equivalent forms (see [24] for details).

This result yields, in particular, a uniform expression for the zeta functions of all 3-dimensional $\mathbb{Z}_p$-Lie algebras we have seen so far (and others, e.g. [23]). For example, the forms $f(x)$ for the abelian algebra $\mathbb{Z}_p^3$, the Heisenberg Lie algebra and the ‘simple’ Lie algebra $\mathfrak{sl}_2(\mathbb{Z}_p)$ are $0$, $x_3^2$ and $x_3^2 - 4x_1x_2$, respectively.

Using the setup of Section 2.5, the key to proving Theorem 2.22 is the observation that only the functions $W_r(p^{-s})$ with $1 \in I$ differ from the ‘abelian’ functions (2.25). Indeed, if $1 \notin I$, the conditions (2.29) hold for all $r_0 \in \mathbb{N}_0$. If $1 \in I$ then they hold if and only if

$$p^{r_0}(\mathcal{R}_1(\alpha))_{23} \equiv 0 \mod (p^{r_1})$$

and a quick calculation shows that, for $\alpha = (\alpha_{ij}) \in \Gamma_3$,

$$\det(\alpha)(\mathcal{R}_1(\alpha))_{23} = L_{23}(\alpha[1])\alpha_{11} - L_{13}(\alpha[1])\alpha_{21} + L_{12}(\alpha[1])\alpha_{31}.$$

Setting

$$f(x) := L_{23}(x)x_1 - L_{13}(x)x_2 + L_{12}(x)x_3$$

we see that (2.33) holds if and only if

$$r_0 \geq r_1 - v_p(f(\alpha[1])).$$

The computation of the integral (2.31) is thus no harder than the computation of the Igusa zeta function associated with the quadratic polynomial $f(x)$.

We note that Theorem 2.22 also yields a complete description of the possible poles of zeta functions of 3-dimensional $\mathbb{Z}_p$-Lie algebras, as the poles of Igusa’s local zeta function of quadratic forms are well understood (cf. [24, Corollary 1.2]). In higher dimensions, such a description is entirely elusive. Also, Theorem 2.22 shows explicitly the relationship between $\zeta_L(s)$ and $\zeta_{pL}(s)$ if $L$ is of dimension 3. No such formula is known in higher dimensions.
2.7. **Global zeta functions of groups and rings.** Let $G$ be a group with polynomial subgroup growth. As noted in the introduction, the degree of polynomial subgroup growth of $G$ is encoded in an analytic invariant of the group’s zeta function, namely its abscissa of convergence $\alpha$ (cf. (1.2)). The following is a deep result.

**Theorem 2.23.** [11, Theorem 1.1] Let $G$ be a $T$-group.

1. The abscissa of convergence $\alpha$ of its zeta function $\zeta_G(s)$ is a rational number, and $\zeta_G(s)$ can be meromorphically continued to $\Re(s) > \alpha - \delta$ for some $\delta > 0$. The continued function is holomorphic on the line $\Re(s) = \alpha$ except for a pole at $s = \alpha$.

2. Let $b + 1$ denote the multiplicity of the pole of $\zeta_G(s)$ at $s = \alpha$. There exists a real number $c \in \mathbb{R}$ such that

$$\sum_{i \leq m} a_i \sim c \cdot m^\alpha (\log m)^b \quad \text{as } m \to \infty.$$ 

The proof of Theorem 2.23 given in [11] proceeds via an analysis of the (local) 'cone integrals' mentioned above.

Whilst it is a remarkable fact that global zeta functions of nilpotent groups always allow for some analytic continuation beyond their abscissa of convergence, it is not the case that they may all be continued to the whole complex plane, as is the case for abelian groups or the Heisenberg group. In fact, numerous groups have been found for which there are natural boundaries for analytic continuation (cf. [15, Chapter 7]). Surprisingly little is known about the abscissa of convergence $\alpha$ and the pole order $b + 1$ in general.

### 3. Variations on a theme

The theme of counting subobjects of finite index in a nilpotent group or a ring may be varied in several interesting ways.

3.1. **Normal subgroups and ideals.** One of the forerunners of the very concept of the zeta function of a group is the Dedekind zeta function of a number field, one of the most classical objects in algebraic number theory. Given a number field $k$, with ring of integers $\mathcal{O}$, the Dedekind zeta function of $k$ is defined as the Dirichlet series

$$\zeta_k(s) := \sum_{a \in \mathcal{O} : a | O} |O : a|^{-s},$$

where the sum ranges over the ideals of finite index in $\mathcal{O}$. Owing to our understanding of the ideal structure in the Dedekind ring $\mathcal{O}$ we have a very good control of arithmetic and analytic properties of this important function. In particular, we know that it allows for an analytic continuation to the whole complex plane and has a simple pole at $s = 1$. Its residue at this pole encodes important arithmetic information about the number field $k$, given by the class number formula.
Given a general ring $L$, its ideal zeta function is defined as the Dirichlet series
\[ \zeta^\triangleright_L(s) := \sum_{m=1}^{\infty} b^\triangleright_m m^{-s} = \sum_{H \triangleleft L} |L : H|^{-s}, \]
where $b^\triangleright_m = b^\triangleright_m(L)$ is the number of ideals in $L$ of index $m$. Similarly, the normal zeta function of a nilpotent group $G$ is defined as
\[ \zeta^\triangleright_G(s) := \sum_{m=1}^{\infty} a^\triangleright_m m^{-s} = \sum_{H \triangleleft G} |G : H|^{-s}, \]
where $a^\triangleright_m = a^\triangleright_m(G)$ denotes the number of normal subgroups of $G$ of index $m$.

Both the ideal zeta function of a ring $L$ and the normal zeta function of a nilpotent group $G$ satisfy an Euler product decomposition
\[ \zeta^\triangleright_L(s) = \prod_{p \text{ prime}} \zeta^\triangleright_{L,p}(s), \quad \zeta^\triangleright_G(s) = \prod_{p \text{ prime}} \zeta^\triangleright_{G,p}(s) \]
into local factors enumerating subobjects of $p$-power index. Fortunately, also the study of normal subgroup growth can be linearised using the Lie ring introduced in Section 1.3. By [16, Section 4] we have, for almost all primes $p$,
\[ \zeta^\triangleright_{G,p}(s) = \zeta^\triangleright_{L(G),p}(s) \]
where $L(G)$ is the nilpotent Lie ring associated with $G$ (cf. Section 1.3).

**Example 3.1.** Let $G$ be the discrete Heisenberg group from Example 1.3. It can be shown that
\[ \zeta^\triangleright_G(s) = \zeta^\triangleright_L(s) = \zeta(s)\zeta(s-1)\zeta(3s-2). \]
Note again that the equation (3.1) holds for all primes $p$.

In many ways, the theory of ideal zeta functions of nilpotent groups is similar to the theory of their (subgroup) zeta functions. In particular, their local factors are also rational in $p^{-s}$, and analogues of Theorems 2.2, 2.3 and 2.23 hold. The first explicitly computed example of a non-uniform zeta function is the normal zeta function of a class-2-nilpotent group.

**Example 3.2.** In [7] du Sautoy showed that both the subgroup and the normal subgroup zeta function of the following class-2-nilpotent group are not finitely uniform. He defined
\[ G := \langle x_1, \ldots, x_6, y_1, y_2, y_3 | \forall i, j : [x_i, x_j] = R(y)_{ij}, \text{ all other } [,] \text{ trivial}, \rangle \]
where
\[ R(y) = \begin{pmatrix} 0 & y_3 & y_1 & y_2 \\ -R(y)^t & 0 & y_1 & y_3 \\ y_2 & 0 & y_1 & y_3 \\ y_1 & y_3 & y_2 & 0 \end{pmatrix}. \]
Notice that the polynomial $\det(R(y)) = y_1y_2^2 - y_1^3 - y_2^2y_3$ defines the projective elliptic curve $E$ considered in Example 2.14. It can be shown (cf. [36, p. 1031]) that, for $p \neq 2$,

$$\zeta_{G,p}(s) = \zeta_{Z_6}(s)(W_1(p, p^{-s}) + b(p)W_2(p, p^{-s})),$$

where $b(p)$ is the number defined in Example 2.14 and

$$W_1(X, Y) = \frac{1 + X^6Y^7 + X^7Y^7 + X^{12}Y^8 + X^{13}Y^8 + X^{19}Y^{15}}{(1 - X^{18}Y^9)(1 - X^{14}Y^8)(1 - X^8Y^7)}$$

$$W_2(X, Y) = \frac{(1 - Y^2)X^6Y^5(1 + X^{13}Y^8)}{(1 - X^{18}Y^9)(1 - X^{14}Y^8)(1 - X^8Y^7)(1 - X^7Y^5)}.$$

Using the identity $b(p)|_{p \rightarrow p-1} = p^{-1}b(p)$ established earlier, the functional equation

$$\zeta_{G,p}(s)|_{p \rightarrow p-1} = -p^{36-15s}\zeta_{G,p}(s)$$

follows immediately. The local subgroup zeta functions $\zeta_{G,p}(s)$ have not been calculated explicitly.

The methods used to perform the calculations in Example 3.2 rely on the fact that the (square root of the) determinant of the matrix of relations $R(y)$ defines a smooth hypersurface in the projective space over the centre of the group. Together with the algebro-geometric fact that every smooth plane curve defined over $\mathbb{Q}$ may be defined by the determinant of a suitable matrix of linear forms, one can, in this way, force any such curve to take on the role played by the elliptic curve in Example 3.2 in the normal zeta function of a class-2-nilpotent group. We refer to [37] for details.

Equation (3.2) is a special case of an analogue of Theorem 2.13 for normal zeta functions of class-2-nilpotent Lie rings ([40, Theorem C]). To what extent this symmetry phenomenon extends to normal zeta functions of other (Lie) rings is largely mysterious. Examples due to Woodward ([15]) show that this may or may not hold in Lie rings of higher nilpotency classes, and in certain soluble Lie rings.

### 3.2. Representations

Another variant of the theme of counting subgroups in a group consists in enumerating the group’s finite-dimensional irreducible complex representations. Again, the concept of a zeta function is helpful to study these if the group has – at least up to some natural equivalence relation – only finitely many irreducible complex representations of each finite dimension, and if these numbers grow at most polynomially. We call an (abstract or profinite) group $G$ rigid if, for every $n \in \mathbb{N}$, the number $r_n(G)$ of (continuous, if $G$ is profinite,) irreducible complex representations of $G$ of dimension $n$ is finite. We say that a rigid group $G$ has polynomial representation growth (PRG) if, for each $m \in \mathbb{N}$, the number of representations of $G$ of dimension at most $m$ is bounded above by a polynomial in $m$,.
i.e. $\sum_{i \leq m} r_i(G) = O(1 + m^\alpha)$ for some $\alpha \in \mathbb{R}$. As in the case of counting subgroups, we define a Dirichlet generating function

$$\zeta_{irr}^T(s) := \sum_{n=1}^{\infty} r_n(G)n^{-s} = \sum_{\rho}(\dim(\rho))^{-s},$$

where $\rho$ ranges over the finite-dimensional irreducible complex representations of $G$, called the representation zeta function of $G$. It defines a convergent function on the complex half-plane determined by the infimum of these $\alpha$.

No general characterisation of rigid or PRG groups is known. In the current section we will concentrate on results regarding three classes of groups: finitely generated torsion-free nilpotent (or $T$-)groups, arithmetic groups and compact $p$-adic analytic groups.

As we will see in Section 3.2.1, $T$-groups are ‘rigid up to twisting with one-dimensional representations’. The growth of the numbers of the ensuing equivalence classes, called ‘twist-isoclasses’, is polynomial, and the associated representation zeta functions satisfy Euler product decompositions, indexed by the primes, analogous to the context of counting subgroups.

The Kirillov orbit method offers a suitable ‘linearisation’ of the problem of counting twist-isoclasses of representations of $p$-power dimension, and we may once again use our arsenal of tools from $p$-adic integration to study the Euler factors (at least for almost all primes).

It is known that arithmetic groups are PRG if and only if they satisfy the Congruence Subgroup Property (CSP). In Section 3.2.2 we review results that show that the representation zeta functions of these groups, too, satisfy an Euler product decomposition, indexed by all places of the underlying number field (including the archimedean ones). The non-archimedean factors are zeta functions associated with compact $p$-adic analytic groups.

As we shall see, these are also rational functions, albeit not solely in the parameter $p^{-s}$.

3.2.1. $T$-groups. A $T$-group has infinitely many one-dimensional irreducible representations: it has infinite abelianisation, and the group of one-dimensional representations of $\mathbb{Z}^n$, i.e. of homomorphisms of $\mathbb{Z}^n$ to $\mathbb{C}^*$, is isomorphic to $(\mathbb{C}^*)^n$. Tensoring with one-dimensional representations will thus give us an infinitude of $m$-dimensional representations for every $m$ for which such representations exist. Fortunately, this is all that needs fixing. More precisely, given a $T$-group $G$, we denote by $R_n(G)$ the set of $n$-dimensional irreducible complex representations of $G$. Given $\sigma_1, \sigma_2 \in R_n(G)$, we say that $\sigma_1$ and $\sigma_2$ are twist–equivalent if there exists a one-dimensional representation $\chi \in R_1(G)$ such that $\sigma_1 = \chi \otimes \sigma_2$. The classes of this equivalence relation are called twist-isoclasses. The set $R_n(G)$ has the structure of a quasi–affine complex algebraic variety whose geometry was analysed by Lubotzky and Magid. They proved in [27, Theorem 6.6] that, for every $m \in \mathbb{N}$, there is a finite quotient $G(m)$ of $G$ such that every $m$-dimensional irreducible
representation of $G$ is twist-equivalent to one that factors through $G(m)$. In particular, the number $c_m = c_m(G)$ of twist-classes of irreducible $m$-dimensional representations is finite. The representation zeta function of $G$ is defined (cf. [19]) by

$$\zeta_{\text{irr}}^G(s) := \sum_{m=1}^{\infty} c_m m^{-s}. \tag{3.3}$$

Furthermore, the function $m \mapsto c_m$ is multiplicative. Indeed, this follows from Lubotzky and Magid’s result together with the group-theoretic fact that the finite nilpotent groups $G(m)$ are the direct products of their Sylow $p$-subgroups and the representation-theoretic fact ([3, (10.33)]) that the irreducible representations of direct products of finite groups are exactly the tensor products of irreducible representations of their factors. Thus

$$\zeta_{\text{irr}}^G(s) = \prod_{p \text{ prime}} \zeta_{\text{irr}}^{G,p}(s), \quad \text{where} \quad \zeta_{\text{irr}}^{G,p}(s) := \sum_{i=0}^{\infty} c_{p^i} p^{-is}.$$

As in the case of saturable pro-$p$-groups (see Section 10.2 in Klopsch’s lecture notes), there is a close connection between representations of $T$-groups and co-adjoint orbits. This generalisation of Kirillov’s orbit method to the discrete setting of $T$-groups is due to Howe. In [18] he shows that (twist-classes of) irreducible representations in a $T$-group $G$ are parametrised by co-adjoint orbits of certain (additive) characters on the associated Lie ring $L(G)$. More precisely, we write $\hat{L}$ for the group $\text{Hom}(L, \mathbb{C}^*)$, and $\text{Ad}^*$ for the co-adjoint action of $G$ on $\hat{L}$ Denote by $L'$ the Lie subring of $L$ corresponding to the group’s derived group $G'$. We say that a character $\psi \in \hat{L}$ is rational on $L'$ if its restriction to $L'$ is a torsion element, i.e. if $\psi(nL') \equiv 1$ for some $n \in \mathbb{N}$. The smallest such $n$ is called the period of $\psi$. Howe’s principal result now states that a character’s co-adjoint orbit is finite if and only if the character is rational on $L'$, and that finite $\text{Ad}^*$-orbits in $\hat{L}$, $\Omega$ say, of characters of odd period, are in $1-1$-correspondence with (twist-classes of) finite-dimensional representations $U_\Omega$ of $G$ of dimension $|\Omega|^{1/2}$ (see [40, Section 3.4] for details).

To effectively enumerate twist-classes of finite-dimensional representations of $G$ we thus have to deal with two problems: given a character $\psi \in \hat{L}$ of finite period, we firstly need to determine the size of its co-adjoint orbit. Secondly, to control over-counting, we have to determine the size of the co-adjoint orbit of the restriction of $\psi$ to $L'$. From now on, we will restrict ourselves to the case that the nilpotence class of $G$ is 2. In this case, the latter task is trivial, as the co-adjoint action on the restriction of characters to $L'$, which is central, is trivial.

As in the case of saturable pro-$p$ groups, we associate with a character $\psi \in \hat{L}$ the bi-additive antisymmetric map

$$b_\psi : L \times L \to \mathbb{C}^*, \quad (x, y) \mapsto \psi([x, y]).$$
Note that $b_\psi$ only depends on the restriction of $\psi$ to $L'$. We define

$$\text{Rad}_\psi := \text{Rad}(b_\psi) = \{x \in L | \forall y \in L : b_\psi(x, y) = 1\}.$$  

One can show that, if $\psi$ is rational on $L'$ (so its co-adjoint orbit is finite by Howe’s result) and $|L : \text{Rad}|$ is coprime to finitely many ‘bad primes’, depending only on $G$, then $\text{Rad}_\psi$ is the Lie ring corresponding to the stabiliser subgroup $\text{Stab}_G(\psi)$ of $\psi$ under the co-adjoint action. Then, by the Orbit Stabiliser Theorem, the index $|L : \text{Rad}_\psi|$ equals the size of the co-adjoint orbit of $\psi$. The Kirillov correspondence now implies that the representation associated with the orbit of $\psi$ has degree $|L : \text{Rad}_\psi|^{-1/2}$.

Recall that, for a class-2-nilpotent group, finite co-adjoint orbits are parameterized by rational characters on $L'$ of finite period. For a prime $p$ and $N \in \mathbb{N}_0$, we write $\Psi_N$ for the set of $\psi \in \hat{L}'$ of period $p^N$. By Howe’s results we have

**Theorem 3.3.** [40, Corollary 3.1] Let $G$ be a class-2-nilpotent $T$-group. Then, for almost all primes $p$,

$$\zeta_{\text{irr}}^{\text{ct}}_{G, p}(s) = \sum_{N \in \mathbb{N}_0, \psi \in \Psi_N} |L : \text{Rad}_\psi|^{-s/2}. \quad (3.4)$$

Assume that $\text{rk}(G/G') = d$ and $\text{rk}(G') = d'$, say, and let $p$ be a prime for which (3.4) holds. To compute the right hand side of this equation effectively, we identify $\Psi_N$ with $W_{p,N} := (\mathbb{Z}/(p^N)^d \setminus p(\mathbb{Z}/(p^N))^{d'}$ as additive groups, and let $\mathcal{R}(y) \in \text{Mat}(d, \mathbb{Z}[y_1, \ldots, y_{d'}])$ be the matrix of linear forms encoding the commutator structure of $G$, i.e. $\mathcal{R}(y)_{ij} = \sum_{k=1}^{d'} \lambda_{ij}^k y_k$ if $G$ is generated by $e_1, \ldots, e_d$ subject to the relations $[e_i, e_j] = \sum_{k=1}^{d'} \lambda_{ij}^k f_k$, say, where $G/G' = \langle e_1 G', \ldots, e_d G' \rangle$ and $G' = \langle f_1, \ldots, f_{d'} \rangle$.

A simple computation shows that if $\psi \in \Psi_N$ corresponds to $\ell \in W_{p,N}$, then the index of $\text{Rad}_\psi$ in $L$ equals the index of the system of linear congruences

$$\mathcal{R}(\ell) x \equiv 0 \mod (p^N) \quad (3.5)$$

where $x \in \mathbb{Z}_p^d$, say. This index can be easily computed from the elementary divisors of the matrix $\mathcal{R}(\ell)$. Recall that $\mathcal{R}(\ell)$ is said to have elementary divisor type $m = (m_1, \ldots, m_d) \in [N]_0^d$ – written $\nu(\mathcal{R}(\ell)) = m$ – if there are matrices $\beta, \gamma \in \text{GL}_d(\mathbb{Z}/p^N)$ such that

$$\beta \mathcal{R}(\ell) \gamma \equiv \begin{pmatrix} p^{m_1} & \cdots \\ \vdots \\ p^{m_d} \end{pmatrix}$$

and $m_1 \leq \cdots \leq m_d$. Given $N \in \mathbb{N}_0$ and $m \in \mathbb{N}_0^d$ we set

$$\mathcal{N}_{N, m} := \{\ell \in W_{p,N} | \nu(\mathcal{R}(\ell)) = m\}.$$
It is now easy to see that
\[ (3.6) \quad \zeta_{G,p}^{irr}(s) = \sum_{N \in \mathbb{N}_0, m \in \mathbb{N}_0^2} \mathcal{N}_{N,m} \mathcal{P}^{N \mathcal{P} + (m_1 + m_2)s/2}. \]

This ‘Poincaré series’ may, in analogy to equation (2.4), be expressed in terms of a $p$-adic integral. The integrand of this (in general quite complicated) integral is defined in terms of the minors of the matrix $\mathcal{R}(y)$. This approach yields immediately the rationality of (almost all of) the local representation zeta functions of $T$-groups, which was first established in [19] by model-theoretic means (and for all primes $p$). The general case (of $T$-groups of arbitrary nilpotency class) is complicated by having to account for overcounting when we run over the characters of $L'$. This can also be formulated in terms of elementary divisors of matrices of forms. See [40, Section 2.2] for details. We illustrate the computations outlined above with a familiar example.

**Example 3.4.** Let $G$ be the discrete Heisenberg group from Example 1.3. Here $d = 2$ and $d' = 1$. For all primes $p$ and $N \in \mathbb{N}_0$ we have $W_{p,N} = (\mathbb{Z}/(p^N))^\times$. The commutator matrix $\mathcal{R}(y)$ is given by

\[ \mathcal{R}(y) = \begin{pmatrix} 1 & y \\ -y & 0 \end{pmatrix} \]

and therefore

\[ \mathcal{N}_{N,m} = \begin{cases} 1 & \text{if } N = 0, \\ (1 - p^{-1})p^N & \text{if } N \in \mathbb{N} \text{ and } m_1 = m_2 = 0, \\ 0 & \text{otherwise}. \end{cases} \]

Thus, for all primes $p$,

\[
\zeta_{G,p}^{irr}(s) = \sum_{N \in \mathbb{N}_0, m \in \mathbb{N}_0^2} \mathcal{N}_{N,m} \mathcal{P}^{N \mathcal{P} + (m_1 + m_2)s/2}
= 1 + \sum_{N \in \mathbb{N}} (1 - p^{-1})p^{(1-s)N}
= (1 - p^{-s})/(1 - p^{1-s}),
\]

or, equivalently,

\[
\zeta_{G}^{irr}(s) = \sum_{m=1}^{\infty} \phi(m)m^{-s} = \zeta(s - 1)\zeta(s)^{-1},
\]

where $\phi$ denotes the Euler totient function. This was first proved in [31, Theorem 5], by entirely different means.

Notice that the local factors of the representation zeta function of the Heisenberg group all satisfy the functional equation

\[ \zeta_{G,p}^{irr}(s)|_{p \to p^{-1}} = p \zeta_{G,p}^{irr}(s). \]

This generalises in the following way:
Theorem 3.5. [40, Theorem D] Let $G$ be a $T$-group with derived group $G'$ of Hirsch length $d'$. Then, for almost all primes $p$,
\[
\zeta_{G,p}^{\text{irr}}(s)\big|_{p\to p-1} = p^{d'}\zeta_{G,p}^{\text{irr}}(s).
\]

3.2.2. Arithmetic groups. Let $k$ be a number field with ring of integers $\mathcal{O}$ and let $G = G(\mathcal{O}_S)$ be an arithmetic lattice in a semisimple, simply connected and connected $k$-defined algebraic group $G$ or, for short, an arithmetic group. Recall that $G$ is said to have the Congruence Subgroup Property (CSP) if every finite index subgroup of $G$ is a congruence subgroup. (See Section 3 of Nikolov’s notes for definitions of these terms.) Recall further that, if $G$ is rigid, the representation zeta function of $G$,
\[
\zeta_{G}^{\text{irr}}(s) = \sum_{n=1}^{\infty} r_n(G)n^{-s},
\]
has finite abscissa of convergence if and only if $G$ has polynomial representation growth (PRG).

Theorem 3.6. [29, Theorems 1.2 and 1.3] Let $G$ be an arithmetic group. Then $G$ has PRG if and only it has the CSP.

Assume from now on that $G$ is an arithmetic group with the CSP.

Proposition 3.7. [26, Proposition 4.6] There is a subgroup $G_0$ of $G$ of finite index in $G$ such that
\[
\zeta_{G_0}^{\text{irr}}(s) = \zeta_{G(\mathbb{C})}^{\text{irr}}(s)|_{S_{\infty}} \prod_{v \not\in S} \zeta_{L_v}^{\text{irr}}(s),
\]
where $S_{\infty}$ denotes the set of archimedean valuations of $k$, $L_v$ is an open subgroup of $G(\mathcal{O}_v)$ and $\zeta_{G(\mathbb{C})}^{\text{irr}}(s)$ (resp. $\zeta_{L_v}^{\text{irr}}(s)$) enumerates irreducible rational (resp. continuous) representations of $G(\mathbb{C})$ (resp. $L_v$).

The fact that we need to pass to a finite index subgroup in Proposition 3.7 is insubstantial if we are mainly interested in the representation zeta function’s abscissa of convergence. Indeed, we have the following

Lemma 3.8. ([26, Corollary 4.5]) If $G_0$ is a finite index subgroup of the rigid PRG group $G$, then the abscissae of convergence of the zeta functions $\zeta_{G}^{\text{irr}}(s)$ and $\zeta_{G_0}^{\text{irr}}(s)$ coincide.

Example 3.9. Let $G = SL_n(\mathbb{Z})$. It is well-known that $SL_n(\mathbb{Z})$ satisfies the CSP if and only if $n \geq 3$. In this case, Proposition 3.7 yields that
\[
\zeta_{\text{SL}_n(\mathbb{Z})}^{\text{irr}}(s) = \zeta_{\text{SL}_n(\mathbb{C})}^{\text{irr}}(s) \prod_{p \text{ prime}} \zeta_{\text{SL}_n(\mathbb{Z}_p)}^{\text{irr}}(s).
\]

Already at first glance the Euler product (3.7) differs from the Euler factorisations we have encountered before by the presence of a factor ‘at infinity’. The Euler factor $\zeta_{G(\mathbb{C})}^{\text{irr}}(s)$ is, however, comparatively well understood. In particular, we know its abscissa of convergence in certain cases.
Theorem 3.10. [26, Theorem 5.1] If $G(\mathbb{C})$ is defined as above then the abscissa of convergence of $\zeta_{G(\mathbb{C})}^{\text{irr}}(s)$ is equal to $\rho/\kappa$, where $\rho = \text{rk}(G)$ and $\kappa = |\Phi^+|$ is the number of positive roots.

The proof of Theorem 3.10 is based the fact that the rational representations of these groups are combinatorially parametrised by their highest weights; see [26, Section 5] for details.

Example 3.11. The group $\text{SL}_2(\mathbb{C})$ has a unique irreducible representation of each finite dimension. Thus

$$\zeta_{\text{SL}_2(\mathbb{C})}^{\text{irr}} = \sum_{m=1}^{\infty} m^{-s} = \zeta(s).$$

Indeed, the abscissa of convergence of the Riemann zeta function is $1 = 1/1 = \rho/\kappa$.

Theorem 3.12. [1, Theorem 1.2] Let $G$ be an arithmetic group which satisfies the CSP. Then the abscissa of convergence of $\zeta_G^{\text{irr}}(s)$ is a rational number.

The proof of this deep result uses sophisticated tools from algebraic geometry, model theory and the representation theory of finite groups of Lie type. We only remark that whilst its conclusion is analogous to one of the conclusions of Theorem 2.23, its proof requires substantially different methods.

3.2.3. Compact $p$-adic analytic groups. The groups $L_v$ in Proposition 3.7 are compact $p$-adic analytic groups. Let, more generally, $G$ be a finitely generated profinite group. It is well-known (cf. Section 10.1 in Klopsch’s lecture notes) that the number $r_n(G)$ of isomorphism classes of continuous irreducible $n$-dimensional complex representations of $G$ is finite if and only if $G$ is $\text{FAb}$, i.e. if and only if every open subgroup of $G$ has finite abelianisation.

Theorem 3.13. [22, Theorem 1.1] Let $G$ be a compact $\text{FAb}$ $p$-adic analytic group with $p > 2$. Then there are natural numbers $n_1, \ldots, n_k$ and functions $f_1(p^{-s}), \ldots, f_k(p^{-s})$, rational in $p^{-s}$, such that

$$\zeta_G^{\text{irr}}(s) = \sum_{i \in [k]} n_i^{-s} f_i(p^{-s}).$$

This deep result takes a more complicated form than the rationality results for Euler factors we have met before. It should not surprise us, however, that the representation zeta function of a $p$-adic analytic group is not, in general, a rational function just in $p^{-s}$: whereas the continuous representations of a pro-$p$ group clearly all have dimension a power of $p$ (as they factor over finite index normal subgroups of the group), a $p$-adic analytic group is only $\text{virtually}$ pro-$p$, i.e. it has a pro-$p$ subgroup of finite index. The natural numbers $n_1, \ldots, n_k$ in Theorem 3.13 can be interpreted as the dimensions of the representations of the quotient of $G$ by a normal, finite index pro-$p$ subgroup.
As the work on representation zeta functions for $T$-groups sketched in Section 3.2.1, the proof of Theorem 3.13 is based on a Kirillov orbit method for compact $p$-adic analytic groups.

Explicit examples of representation zeta functions of compact $p$-adic groups are thin on the ground. In [22], Jaikin gives the example of $\zeta_{\text{SL}_2(\mathbb{Z}_p)}(s)$ for odd $p$. (Note, however, that the Euler product over the local factors (including $p = 2$ and ‘infinity’) only counts ‘congruence representations’ of $\text{SL}_2$, as $\text{SL}_2$ does not satisfy the CSP.) In [25], formulae are developed for the representation zeta functions of the principal congruence subgroups $\text{SL}_3^k(\mathbb{Z}_p)$ for all primes $p$ and $k \in \mathbb{N}$ ($k \geq 2$ if $p = 2$), and the abscissa of convergence of $\zeta_{\text{SL}_3(\mathbb{Z}_p)}(s)$ is determined. A result on functional equations of representation zeta functions of pro-$p$ groups in globally defined families can also be found in this paper (cf. Theorem 10.3 in Klopsch’s lecture notes).

3.3. Further variations.

3.3.1. Nilpotent groups. Besides the zeta functions counting all subgroups, normal subgroups and representations of a $T$-group $G$, people have studied the zeta functions enumerating subgroups of $G$ which are isomorphic to $G$ ([16]), the ‘pro-isomorphic’ zeta functions enumerating subgroups whose profinite completion is isomorphic to the profinite completion of $G$ ([12, 2], and the zeta functions enumerating subgroups up to conjugacy ([40, Section 3.2]). The last two satisfy Euler product decompositions into Euler factors which are rational in $p^{-s}$.

3.3.2. Compact $p$-adic analytic groups. Let $G$ be a compact $p$-adic analytic group. Recall that such a group is virtually pro-$p$. In [4] du Sautoy proved that the ‘local’ zeta function

$$\zeta_{G,p}(s) = \sum_{n=0}^{\infty} a_{p^n}(G)p^{-ns}$$

of $G$ is rational in $p^{-s}$. He also proved that the ‘global’ zeta function $\zeta_G(s)$ counting all finite-index subgroups is rational in $p^{-s}, n_1^{-s}, \ldots, n_k^{-s}$ for natural numbers $n_1, \ldots, n_k$ (analogous to Theorem 3.13), and established similar results for zeta functions counting normal subgroups, $r$-generator subgroups and subgroups up to conjugacy in compact $p$-adic analytic groups. We refer to [30, Chapter 16] for details. In [10] du Sautoy showed the rationality of certain generating functions enumerating the class numbers of (i.e. the total numbers of conjugacy classes in) families of finite groups associated with compact $p$-adic analytic groups.

3.3.3. Finite $p$-groups. The methods used to study the subgroup growth of nilpotent or $p$-adic analytic groups have found applications in the enumeration of finite $p$-groups. Given a prime $p$ and natural numbers $c$ and $d$, let $f(n,p,c,d)$ denote the number of (isomorphism classes of) $d$-generator
p-groups of order $p^n$ and nilpotency class at most $c$. We define the Dirichlet generating function
\[
\zeta_{c,d,p}(s) := \sum_{n=0} f(n, p, c, d) p^{-ns}.
\]

In [6] du Sautoy proved that these generating series are rational in the parameter $p^{-s}$ (cf. [30, Section 16.4] for an exposition). It follows easily from the structure theorem for finite abelian $p$-groups that
\[
(3.8) \quad \zeta_{1,d,p}(s) = \zeta_p(s) \zeta_p(2s) \cdots \zeta_p(ds).
\]
In [35] it was proved that, for all primes $p$,
\[
(4.1) \quad \zeta_{2,2,p}(s) = \zeta_p(s) \zeta_p(2s) \zeta_p(3s)^2 \zeta_p(4s).
\]
No other explicit formulae of this kind are known.

4. Open problems and conjectures

4.1. Subring and subgroup zeta functions.

Conjecture 4.1. [16, p. 188] Let $F_{c,d}$ denote the free class-$c$-nilpotent group on $d$ generators. Then $\zeta_{F_{c,d},p}(s)$ and $\zeta_{\triangleright F_{c,d},p}(s)$ are almost uniform, i.e. there are rational functions $W_{c,d}(X,Y), W_{\triangleright c,d}(X,Y) \in \mathbb{Q}(X,Y)$ such that, for almost all primes $p$,
\[
\zeta_{F_{c,d},p}(s) = W_{c,d}(p,p^{-s})
\]
\[
\zeta_{\triangleright F_{c,d},p}(s) = W_{\triangleright c,d}(p,p^{-s}).
\]

Conjecture 4.2. Let $L$ be a class-$c$-nilpotent Lie ring of rank $n$ with upper central series $(Z_i(L))_i, i = 0, \ldots, c$. Set $n_i := \text{rk}(L/Z_i(L))$ (so $n_0 = n = \text{rk}(L)$). Then, for almost all primes $p$,
\[
\deg_{p^{-s}}(\zeta_{L,p}^2(s)) = -\sum_{i=0}^{c} n_i
\]
\[
\lim_{s \to -\infty} (p^{-s})^{\sum_{i=1}^{c} n_i} \zeta_{L,p}^2(s) = (-1)^n p^{n_2}. 
\]

Note that, for the primes $p$ for which $\zeta_{L,p}(s)$ satisfies a functional equation of the form
\[
(4.3) \quad \zeta_{L,p}^2(s)|_{p^{-s} = 1} = (-1)^n p^{n_2} \zeta_{L,p}^2(s),
\]
these are simple corollaries of (4.3). In particular, Conjecture 4.2 holds if $c \leq 2$ (cf. [40, Theorem C]). For higher classes, however, it is known that the equation (4.3) does not hold in general. All known examples (cf., e.g., [15]) nevertheless satisfy equations (4.1) and (4.2).

Problem 4.3. Characterise nilpotent Lie rings for which the functional equation (4.3) holds for almost all primes $p$.

A ‘conjectural’ characterisation has been given in [15, Chapter 4].
Conjecture 4.4. Let $L$ be a class-2-nilpotent Lie ring with rk$(L/L') = d$, rk$(Z(L)) = m$ and rk$(L/Z(L)) = r$. Let $\alpha^\circ$ denote the abscissa of convergence of $\zeta^\circ_L(s)$. Then

$$\alpha^\circ = \max_{k \in [m]} \left\{ d, \frac{k(m + d - k) + 1}{r + k} \right\}.$$  

That $\alpha^\circ$ is greater or equal to the right hand side was proved in [32]. Equality has been proved, in particular, for the free class-2-nilpotent groups $F_{2,d}$ in [38]. More generally, we ask

Problem 4.5. Given a ring $L$, determine the abscissae of convergence of its subring and ideal zeta functions, respectively.

Problem 4.6. Given a ring $L$, determine (a small superset of) the natural numbers $a_i, b_i$ occurring in the denominators of its local (ideal) zeta functions (cf. Theorem 2.2).

It follows from [11] that the abscissa of convergence of a ring’s global zeta function is a simple function of these integers. Problem 4.6 is thus strictly harder than Problem 4.5. Even (partial) answers for specific families of Lie rings as nilpotent or soluble Lie rings or ‘simple’ Lie rings like $sl_n(\mathbb{Z})$ would be very interesting.

4.2. Representation zeta functions.

Problem 4.7. ([26, Problem 4.2]) Characterise rigid groups, and groups of polynomial representation growth (PRG).

Problem 4.8. Let $G$ be a $\mathcal{T}$-group with representation zeta function $\zeta^\text{irr}_G(s)$. Is the abscissa of convergence of $\zeta^\text{irr}_G(s)$ a rational number? Does $\zeta^\text{irr}_G(s)$ admit analytic continuation beyond its abscissa of convergence? Interpret the abscissa of convergence and the poles of the Euler factors of $\zeta^\text{irr}_G(s)$ in terms of the structure of $G$.

A positive answer to this problem would imply asymptotic statements about the numbers of twist-isoclasses of representations of $\mathcal{T}$-groups, analogous to Part B of Theorem 2.23.

Problem 4.9. Let $SL^k_n(\mathbb{Z}_p) := \ker(SL_n(\mathbb{Z}_p) \to SL_n(\mathbb{Z}/(p^k\mathbb{Z})))$ denote the $k$-th congruence subgroup of $SL_n(\mathbb{Z}_p)$. How do the functions $\zeta^\text{irr}_{SL^k_n(\mathbb{Z}_p)}(s)$ vary with the prime $p$? What are the abscissae of convergence of the zeta functions $\zeta^\text{irr}_{SL^k_n(\mathbb{Z}_p)}(s)$? What about other ‘classical’ $p$-adic analytic groups?

Problem 4.10. Let $G$ be an arithmetic group satisfying the CSP. Does its representation zeta function $\zeta^\text{irr}_G(s)$ admit analytic continuation beyond its rational (Theorem 3.12) abscissa of convergence?

Again, a positive answer would give us control over the asymptotic of the numbers $r_n(G)$ as $n$ tends to infinity.
5. Exercises

Exercise 1. Let $q$ be a prime power, $n \in \mathbb{N}$, and $I \subseteq [n-1]$. Show that the number of flags of type $I$ in $\mathbb{P}_p^n$ is equal to $\binom{n}{I}_q$.

Exercise 2. Prove equation (2.14) directly.

Exercise 3. (cf. Example 3.1) Let $p$ be a prime, and $L_p = \mathbb{Z}_p \otimes L$, where $L$ is the Heisenberg Lie ring. In the setup of Section 2.5, show that a coset $\Gamma M$ corresponds to an ideal if and only if $M_{33} \mid M_{11}$ and $M_{33} \mid M_{22}$. Deduce that, for all primes,

$$\zeta_{L_p}^\gamma(s) = \sum_{H \triangleleft L_p} |L_p : H|^{-s} = \frac{1}{(1 - p^{-s})(1 - p^{-1-s})(1 - p^{-2-3s})}.$$ 

Exercise 4 ($\star$). Let $L$ be a ring of additive rank $n$. Using the setup and notation of Section 2.3, show that a matrix $M = (M_{ij}) \in \text{Tr}_3(\mathbb{Z}_p)$ encodes the generators of an ideal if and only if

$$\forall i \in [n] : D\alpha_i^{-1} R(\alpha[i]) \equiv 0 \mod D_{ii},$$

(This is the ‘ideal’-analogue of equation (2.28).)

Exercise 5 ($\star$). For $n \in \mathbb{N}$, let $L(n) = \mathbb{Z}^n$, considered as a ring with component-wise multiplication. Show that, for all primes $p$,

$$\zeta_{L(2),p}^\gamma(s) = \frac{(1 + p^{-s})^2}{(1 - p^{-s})(1 - p^{-1-3s})}.$$ 

Show that, for all $n \in \mathbb{N}$ and all primes $p$,

$$\zeta_{L(n),p}(s) = \zeta_p(s)^n,$$

where $\zeta_p(s) = (1 - p^{-s})^{-1}$.

Exercise 6 ($\star$). Let $G$ be the group defined in Example 3.2. Show that, for $p \neq 2$,

$$\zeta_{G,p}^\text{irr}(s) = W_1(p,p^{-s}) + b(p)W_2(p,p^{-s}),$$

where

$$W_1(X_1, X_2) = \frac{1 - X_2^3}{1 - X_1^3X_2^2}, \quad W_2(X_1, X_2) = \frac{(X_1 - 1)(X_2 - 1)X_2^2}{(1 - X_1^3)(1 - X_1^3X_2^2)}$$

and $b(p)$ is defined as in Example 2.14. Deduce the assertion of Theorem 3.5 in these cases.

Exercise 7. Establish formula (3.8).

Acknowledgements. I am indebted to Mark Berman, Benjamin Klopsch and Alexander Stasinski, whose careful comments greatly improved these notes.
References

1. N. Avni, "Arithmetic groups have rational representation growth," arXiv:0803.1331, 2008.
2. M. Berman, "Uniformity and functional equations for zeta functions of $\mathbb{Q}$-split algebraic groups," preprint, 2007.
3. C. W. Curtis and I. Reiner, Methods of representation theory, with applications to finite groups and orders, vol. 1, John Wiley & Sons, 1981.
4. M. P. F. du Sautoy, Finitely generated groups, p-adic analytic groups and Poincaré series, Ann. of Math. (2) 137 (1993), no. 3, 639–670.
5. , Zeta functions of groups and rings: uniformity, Israel J. Math. 86 (1994), 1–23.
6. , Counting p-groups and nilpotent groups, Publ. Math. I.H.E.S. 92 (2000), 63–112.
7. , A nilpotent group and its elliptic curve: non-uniformity of local zeta functions of groups, Israel J. Math. 126 (2001), 269–288.
8. , Counting subgroups in nilpotent groups and points on elliptic curves, J. Reine Angew. Math. 549 (2002), 1–21.
9. , Zeta functions of groups: The quest for order versus the flight from ennui, Groups St. Andrews 2001 in Oxford, London Math. Soc. Lecture Note Ser., 304, Cambridge University Press, 2003, pp. 150–189.
10. M. P. F. du Sautoy, Counting conjugacy classes, Bull. London Math. Soc. 37 (2005), no. 1, 37–44.
11. M. P. F. du Sautoy and F. J. Grunewald, Analytic properties of zeta functions and subgroup growth, Ann. of Math. 152 (2000), 793–833.
12. M. P. F. du Sautoy and A. Lubotzky, Functional equations and uniformity for local zeta functions of nilpotent groups, Amer. J. Math. 118 (1996), no. 1, 39–90.
13. M. P. F. du Sautoy and D. Segal, Zeta functions of groups, New horizons in pro-p groups, Progr. Math., Birkhauser, Boston MA, 2000, pp. 249–286.
14. M. P. F. du Sautoy and Gareth Taylor, The zeta function of $\text{SL}_2$ and resolution of singularities, Math. Proc. Cambridge Philos. Soc. 132 (2002), no. 1, 57–73.
15. M. P. F. du Sautoy and L. Woodward, Zeta functions of groups and rings, Lecture Notes in Mathematics 1925, Springer Verlag, 2008.
16. F. J. Grunewald, D. Segal, and G. C. Smith, Subgroups of finite index in nilpotent groups, Invent. Math. 93 (1988), 185–223.
17. H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II, Ann. of Math. (2) 79 (1964), 109–203; ibid. (2) 79 (1964), 205–326.
18. R. E. Howe, On representations of discrete, finitely generated, torsion-free, nilpotent groups, Pacific J. Math. 73 (1977), no. 2, 281–305.
19. E. Hrushovski and B. Martin, Zeta functions from definable equivalence relations, math.LO/0701011 on arxiv.org, 2007.
20. J. E. Humphreys, Reflection groups and Coxeter groups, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990.
21. K. Ireland and M. Rosen, A classical introduction to modern number theory, GTM 84, Springer, 1982.
22. A. Jaikin-Zapirain, Zeta function of representations of compact $p$-adic analytic groups, J. Amer. Math. Soc. 19 (2006), no. 19, 91–118.
23. B. Klopsch, Zeta functions related to the pro-p group $SL_1(\Delta_p)$, Math. Proc. Cambridge Philos. Soc. 135 (2003), 45–57.
24. B. Klopsch and C. Voll, Zeta functions of 3-dimensional p-adic Lie algebras, arXiv:0710.1970v1, to appear in Math. Z., 2007.
25. , Representation zeta functions of compact p-adic Lie groups, in preparation, 2008.
26. M. Larsen and A. Lubotzky, *Representation growth of linear groups*, J. Eur. Math. Soc. (JEMS) **10** (2008), no. 2, 351–390.
27. A. Lubotzky and A. R. Magid, *Varieties of representations of finitely generated groups*, Mem. Amer. Math. Soc. **58** (1985), no. 336, xi+117 pp.
28. A. Lubotzky, A. Mann, and D. Segal, *Finitely generated groups of polynomial subgroup growth*, Israel J. Math. **82** (1993), no. 1-3, 363–371.
29. A. Lubotzky and B. Martin, *Polynomial representation growth and the congruence subgroup growth*, Israel J. Math. **144** (2004), 293–316.
30. A. Lubotzky and D. Segal, *Subgroup growth*, Birkhäuser Verlag, 2003.
31. C. Nunley and A. R. Magid, *Simple representations of the integral Heisenberg group*, Contemp. Math. **82** (1989), 89–96.
32. P. M. Paajanen, *On the degree of polynomial subgroup growth in class-2-nilpotent groups*, Israel J. Math. **157** (2007), 323–332.
33. R. P. Stanley, *Combinatorics and commutative algebra*, Birkhäuser, 1996, second edition.
34. ______, *Enumerative combinatorics*, Cambridge Studies in Advanced Mathematics, 49, vol. 1, Cambridge University Press, 1997.
35. C. Voll, *Zeta functions of groups and enumeration in Bruhat-Tits buildings*, Ph.D. thesis, University of Cambridge, 2002.
36. ______, *Zeta functions of groups and enumeration in Bruhat-Tits buildings*, Amer. J. Math. **126** (2004), 1005–1032.
37. ______, *Functional equations for local normal zeta functions of nilpotent groups*, Geom. Func. Anal. (GAFA) **15** (2005), 274–295, with an appendix by A. Beauville.
38. ______, *Normal subgroup growth in free class-2-nilpotent groups*, Math. Ann. **332** (2005), 67–79.
39. ______, *Counting subgroups in a family of nilpotent semidirect products*, Bull. London Math. Soc. **38** (2006), 743–752.
40. ______, *Functional equations for zeta functions of groups and rings*, arXiv:math/0612511, to appear in Ann. of Math., 2006.
41. J. Wlodarczyk, *Simple Hironaka resolution in characteristic zero*, J. Amer. Math. Soc. **18** (2005), no. 4, 779–822.

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