Anatomy of the magnetic catalysis by renormalization-group method

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Abstract

We first examine the scaling argument for a renormalization-group (RG) analysis applied to a system subject to the dimensional reduction in strong magnetic fields, and discuss the fact that a four-Fermi operator of the low-energy excitations is marginal irrespective of the strength of the coupling constant in underlying theories. We then construct a scale-dependent effective four-Fermi interaction as a result of screened photon exchanges at weak coupling, and establish the RG method appropriately including the screening effect, in which the RG evolution from ultraviolet to infrared scales is separated into two stages by the screening-mass scale. Based on a precise agreement between the dynamical mass gaps obtained from the solutions of the RG and Schwinger-Dyson equations, we discuss an equivalence between these two approaches. Focusing on QED and Nambu–Jona-Lasinio model, we clarify how the properties of the interactions manifest themselves in the mass gap, and point out an importance of respecting the intrinsic energy-scale dependences in underlying theories for the determination of the mass gap. These studies are expected to be useful for a diagnosis of the magnetic catalysis in QCD.

1. Introduction

Strong magnetic fields confine charged fermions in the lowest Landau levels (LLLS), and they enjoy the properties of the \((1+1)\)-dimensional chiral fermions with the dispersion relation \([-eB = (0, 0, eB), \ eB > 0]:\]

\[
\epsilon_{\text{LLL}} \pm p_z. \tag{1}
\]

Intuitively, this is a consequence of the formation of the small cyclotron orbit with the radius \(\sim 1/[eB]^{1/2}\) and the residual free motion along the field. It turned out that this dimensional reduction gives rise to rich physics phenomena. Especially, the magnetic catalysis of the chiral symmetry breaking and the chiral magnetic effect have been addressed by many authors (see, e.g., Refs. [1, 2] and Refs. [3, 4, 5, 6] for reviews).

The clear statement on the physical mechanism of the magnetic catalysis was due to Gusynin, Miransky, and Shovkovy in terms of a simple four-Fermi interaction \([7].\) By solving the gap equation of the NJL model, they found a mass gap

\[
m_{\text{dyn}} = \sqrt{eB} \exp\left(-\frac{\pi}{\rho_{\text{LLL}} G_{\text{NJL}}}ight), \tag{2}
\]

where \(\rho_{\text{LLL}}\) and \(G_{\text{NJL}}\) are the density of states in the LLL and a dimensionful coupling constant of the four-Fermi interaction, respectively. Their core observation is seen in the similarity between the mass gap and the energy gap of superconductivity which is given by \(\Delta \sim \omega_D \exp[-c'/(\rho_F G')]\) with \(\omega_D\) and \(\rho_F\) being the Debye frequency and the density of states near the Fermi surface, respectively. Also, \(G'\) and \(c'\) are a coupling constant and a positive number, respectively. In fact, this similarity is originated from the dimensional reduction in the low-energy domains of the both theories, i.e., in the LLL and in the vicinity of the Fermi surface.

We can clearly see the consequence of the dimensional reduction by focusing on QED in the weak coupling regime. From the rainbow approximation of the Schwinger-Dyson (SD) equation, the mass gap was obtained as

\[
m_{\text{dyn}} \simeq \sqrt{eB} \exp\left(-\frac{\pi}{2\alpha}\right), \tag{3}
\]

with an unscreened photon propagator in the early studies \([8, 9, 10],\) and also

\[
m_{\text{dyn}} \simeq \sqrt{2eB} \alpha^{1/3} \exp\left(-\frac{\pi}{\alpha \log(C/\pi)}\right), \tag{4}
\]

with a screened photon propagator \([11, 12].\) Here, \(\alpha = e^2/4\pi\) and \(C\) is a certain constant of order one. The constant was analytically obtained as \(C = 1\) when the momentum dependence of \(m_{\text{dyn}}\) is neglected. The authors of Refs. \([11, 12]\) observed that the gap equation always has a nontrivial solution irrespective of the size of the coupling constant, indicating that the strong magnetic fields cause the dynamical symmetry breaking without support of any other nonperturbative dynamics. This reminds us of the well-known fact that any weak attractive interaction causes superconductivity.

Our main assertion in this Letter is that all these aspects of the magnetic catalysis can be understood with
the Wilsonian renormalization group (RG) analysis. We will show that the emergence of the dynamical mass gap is informed from the RG flow for the effective four-Fermi operator that goes into the Landau pole. Bearing it in mind that four-Fermi operators are irrelevant in ordinary (3+1)-dimensional systems, we will clearly see from the RG point of view that the magnetic catalysis of the dynamical symmetry breaking is intimately related to the dimensional reduction. Our approach shares the philosophy with the analysis of (color) superconductivity by the RG method [13, 14, 15, 16, 17, 18, 19, 20].

Our ultimate goal is to consistently understand the enhancement of the chiral symmetry breaking at zero or low temperature, and the inverse magnetic catalysis near the chiral phase transition temperature in QCD. There has been a discrepancy between the estimates of the chiral condensate from the lattice QCD simulation and typical model calculations [21]. For the magnetic catalysis and the inverse catalysis to be compatible with each other, it appeared to be important to explain a mechanism which makes the dynamical mass gap stay as small as the QCD scale $\Lambda_{\text{QCD}}$ even in a strong magnetic field $eB \gg \Lambda_{\text{QCD}}$ [22] (see also Refs. [23, 24, 25]). The method of renormalization group is a potentially useful tool to obtain a clear insight on this issue on the basis of an argument of the hierarchy which we will elaborate in the present paper.

However, to the best of our knowledge, even the correct form of the mass gap in weak-coupling gauge theories has not been obtained by the RG analyses in the presence of the screening effect. Therefore, before discussing the strong-coupling regime in QCD, one should understand how the screening effects are reflected in the parametric form of the mass gap in a clear way. Moreover, it is a generic issue to establish a systematic way of including the screening effects in the RG analyses, which will be important in a variety of systems. Note, for example, that there was an issue of the color magnetic screening in the RG analysis on the color superconductivity [17].

We will show that all of the results in Eqs. (2), (3), and (4) from the SD equations are precisely obtained from the solutions of the RG equations. Furthermore, we will clarify the origins of the overall factor of $\sqrt{eB}$ and the exponents in the language of the RG method. We will find that the properties of the interactions in the model/theory are directly reflected in the parametric dependences of the dynamical mass on the coupling constant and the magnitude of $eB$. Ultimately, these studies will be useful for a diagnosis of the magnetic catalysis in QCD. We will come back to this point with a brief comment on the perspective in the last section.

More specifically, we will closely look into the screening effect on the photon propagator. It would be instructive to mention a successful application of the RG method to color superconductivity in dense quark matter, where an appropriate treatment of the dynamical screening effect on the magnetic gluons was important for obtaining the correct magnitude of the gap [17, 20].

We should also mention that the RG analysis of the magnetic catalysis at weak coupling was performed in Refs. [26, 27]. Also, the magnetic catalysis in QCD was investigated on the basis of both the SD and RG equations in Ref. [28]. However, roles of the screening effect arising from the quark loop in the magnetic field have not been identified thus far, and we are not aware of the RG analysis in the literature of which the result agrees with that from the SD equation (4).

As we will discuss later in more detail, the screening effect should be appropriately incorporated in the derivation of the RG equation, since the screening mass sets an intrinsic energy scale of the underlying theory in between the ultraviolet and infrared regimes. The essential technique was recently developed for the analysis of the RG flow in “magnetically induced QCD Kondo effect” [29]. In the present Letter, we will show that the same technique successfully works for the analysis of the magnetic catalysis at weak coupling.

The structure of this Letter is the following. We first show the connection between the magnetic catalysis and the dimensional reduction which can be understood from a simple discussion of the scaling dimensions. Next, we construct an effective four-Fermi interaction from the underlying weak-coupling theory, i.e., QED, and appropriately include the energy-scale dependence of the tree-level interaction. Based on these discussions, we derive the RG equations and obtain the dynamical mass gap from their solutions. We confirm that the energy-scale dependence of the interaction is necessary for obtaining the correct form of the gap, which was however missing in the previous analyses. Finally, we discuss the correspondences between the RG and SD analyses, and the crucial roles of the photon/gluon propagators in the magnetic catalysis. The derivation of the RG equation is briefly summarized in an appendix.

2. Infrared scaling dimensions

We begin with looking into an analogy between the systems in the strong magnetic field and at high density. In the presence of a large Fermi sphere, the low-energy excitations near the Fermi surface show the dimensional reduction: The two-dimensional phase space tangential to the large Fermi sphere is degenerated, and the energy dispersion depends only on the momentum normal to the sphere. Then, the dimensional reduction enhances the infrared (IR) dynamics, leading to the instabilities near the Fermi surface. Based on the analogy with this mechanism, Gusynin et al. clearly pointed out that the chiral symmetry breaking occurs in the strong magnetic field no matter how weak the coupling is [7].

One can see possible emergence of the IR instability from a simple argument of the scaling dimensions. The kinetic term for the LLL reads

$$S_{\text{kin}}^{\text{skin}}\big|_{\text{LLL}} = \int dt \int dp_z \bar{\psi}_{\text{LLL}}(p_z)(i\partial_\gamma^0 - p_z \gamma^3)\psi_{\text{LLL}}(p_z),$$

(5)
where we have suppressed the label specifying the location of the cyclotron center on the transverse plane. From this kinetic term, one can find the IR scaling dimension of the LLL fermion field when the excitation energy goes down toward zero as \( \epsilon_{\text{LLL}} \to s_0' \epsilon_{\text{LLL}} (t \to s^{-1} t) \) with \( s < 1 \). Since the LLL fermion has the \((1+1)\) dimensional dispersion relation \((1)\), the longitudinal momentum \( p_z \) also scales as \( p_z \to s p_z \). On the other hand, the transverse momentum does not scale, because it serves just as the label of the degenerated states and does not appear in the dispersion relation \((1)\). Therefore, when the kinetic term \((5)\) is invariant under the scale transformation \((1)\), the four-Fermi interaction model which do not respect the scale dependences in the underlying theory.

3. Effective interactions from underlying theories

To build a bridge between the underlying gauge theories and the scaling-dimension argument for the four-Fermi operator, we first construct an effective four-Fermi interaction based on the underlying theory. It is very crucial to correctly take into account the energy-scale dependence of the interactions in gauge theories which are mediated by gauge bosons. Elaborating this point, we will obtain the correct form of the dynamical mass, and will show that the reliable effective theory gives a different RG evolution from that in the scale-independent four-Fermi interaction model which do not respect the scale dependences in the underlying theory.

In QED as the underlying theory, we shall introduce the photon propagator in the strong magnetic field that has a screening mass \([11, 12, 30, 31, 32, 33]\). The explicit form in a non-covariant gauge reads \([11, 12]\)

\[
iD_{\mu\nu}(q) = g_{\mu\nu} \frac{q^2 - m^2}{q^2} + \frac{q_{\perp}^\mu q_{\perp}^\nu}{q^2} + \frac{q_{\parallel}^\mu q_{\parallel}^\nu + q_{\perp}^\mu q_{\perp}^\nu}{(q^2)^2}, \tag{7}
\]

where \( g_{\mu\nu} = \text{diag}(1, 0, 0, -1) \), \( g_{\perp}^{\mu\nu} = \text{diag}(0, -1, -1, 0) \), and \( q_{\parallel,\perp} = g_{\parallel,\perp} q_0 \). The screening mass \( m^2_\perp = 2\alpha eB/\pi \) comes from the one-loophoton self-energy composed of the LLL fermion lines in the vanishing frequency limit.\(^1\) The other terms are coupled to neither the LLL fermion loop nor the scattering LLL fermions because the LLL fermion current, \( j_{\text{LLL}}^\mu = \bar{\psi}_{\text{LLL}} \gamma^\mu \psi_{\text{LLL}} \), is longitudinal to the magnetic field. Thus, those terms are completely irrelevant in the present discussion.

Now, we note that the dispersion relation of the LLL fermion is given by the \((1+1)\)-dimensional form \((1)\), while the photons live in the ordinary four dimensions. Therefore, we define the effective coupling constant \( G \) by integrating out the transverse momentum in the photon propagator as \([22, 25, 29]\)

\[
G(q_0^2) \equiv \int \frac{d^2 q_\perp}{(2\pi)^2} \frac{(-ie)^2}{q_\perp^2 - q_\parallel^2 - m^2_\perp} e^{-\frac{q_\perp^2}{2\Lambda^2}}, \tag{8}
\]

where the exponential factor comes from the transverse part of the fermion wave functions. This effective coupling allows us to identify the dimensionally reduced effective interaction discussed in Eq. \((6)\) which however possesses an appropriate energy dependence as we will see below.

Importantly, we have an intrinsic energy scale given by the screening mass \( \Lambda_{\text{sc}} \equiv m_\gamma \) which cuts off the IR region of the transverse momentum integral. Therefore,

\[\text{Figure 2: Scattering diagrams contributing to the RG flow in the magnetic catalysis. Yellow blobs capture the the running effective coupling constants which are defined in terms of the exchanged gauge-boson propagators in the weak-coupling theory.}\]
the result of the integral depends on if the scale of interest \( \Lambda \) is larger than \( \Lambda_{\text{sc}} \) [Region (I): \( \Lambda > \Lambda_{\text{sc}} \)] or is in the deeper IR region \( \Lambda < \Lambda_{\text{sc}} \) [Region (II): \( \Lambda < \Lambda_{\text{sc}} \)]. As summarized in Fig. 1, one needs to examine the RG evolution in these regions separately. On the other hand, the upper boundary of the integral is, in the both cases, given by \( \Lambda_{\text{UV}} \equiv \sqrt{2eB} \) which appears in the exponential factor of Eq. (8) and also corresponds to the energy scale where the higher Landau levels start to contribute. Based on this scale-dependent four-Fermi interaction, we investigate the RG evolution in the next section.

4. RG analysis at weak coupling

We derive the RG equation for the coupling constant of the effective four-Fermi interaction in terms of the Wilsonian renormalization group. More specifically, we compute the scattering amplitudes for the fermion and antifermion pair that forms the chiral condensate (see Fig. 2), and integrate out the excited states. As we have already learned that the four-Fermi operator is marginal in the LLL, we anticipate that the loop integral in the scattering amplitude generates a logarithm, and renormalizing the effective coupling constant with this logarithm will drive the system toward a strong coupling regime.

As shown in Fig. 2, the leading-order (tree-level) scattering amplitude is given by the one-photon exchange diagram:

\[
\mathcal{M}_0 = G(q_{\parallel}^2) .
\]  

The relevant scattering channels for the formation of the chiral condensate are those between the pairs carrying the opposite chiralities, so that the two relevant spinor structures are given by \([\bar{u}_{\parallel R/L}(p_{\parallel}^{(3)}) \gamma_{\parallel\mu} u_{\parallel R/L}(p_{\parallel}^{(1)})] [\bar{v}_{\perp L/R}(p_{\perp}^{(4)}) \gamma_{\perp\nu} v_{\perp L/R}(p_{\perp}^{(2)})]\) with the spinors of the fermion \( u \) and the antifermion \( v \) in the LLL. The scattering amplitudes are the same for the both channels, and below these trivial spinor structures will be suppressed for notational simplicity.

Because the coupling constant \( G(q_{\parallel}^2) \) has the scale dependence discussed in the previous section, the tree-level amplitude also contributes to the RG evolution. This situation is somewhat, though not exactly, similar to that in the color superconductivity in dense quark matter where the scale dependence of the dynamical screening mass in the tree-level diagram modifies the exponent in the critical temperature [17].

In the present case, when the energy and momentum scales of the fermion are of the order of \( \Lambda \), the momentum transfer is \(-\Lambda^2 \lesssim q_{\parallel}^2 \lesssim 0\), where we are only interested in the space-like region contributing to the fermion scatterings. Now, when integrating out the fermionic degrees of freedom in the shell \((\Lambda - \delta \Lambda) \sim \Lambda\), we obtain the increment of \( \mathcal{M}_0(\Lambda) \) as

\[
\mathcal{M}_0(\Lambda - \delta \Lambda) - \mathcal{M}_0(\Lambda) = G(-{(\Lambda - \delta \Lambda)^2}) - G(-\Lambda^2) .
\]  

In Region II, the dependence on \( \Lambda \) goes away, because the screening scale \( \Lambda_{\text{sc}} \) dominates over \( \Lambda \). However, in

Figure 3: Diagram for the scattering between the particle and antiparticle.

Region I, the screening mass is negligible, so that there is the dependence on \( \Lambda \). Performing the transverse-momentum integral in Eq. (8), we have

\[
G(-{(\Lambda - \delta \Lambda)^2}) - G(-\Lambda^2) \sim \alpha \int_{(\Lambda - \delta \Lambda)^2}^{\Lambda^2} \frac{dq_{\perp}^2}{q_{\perp}^4} .
\]  

Therefore, the result at the tree level is summarized as

\[
\mathcal{M}_0(\Lambda - \delta \Lambda) - \mathcal{M}_0(\Lambda) \sim \begin{cases} 0 & \text{Region I} \vspace{0.1cm} \\ 2 \alpha \log \left( \frac{\Lambda}{\Lambda_{\text{sc}}} \right) & \text{Region II} \end{cases} .
\]  

This intrinsic scale dependence of the effective coupling constant partly drives the RG evolution, which should be taken into account in the RG equation below.

Next, let us proceed to the one-loop amplitude shown in Figs. 2 and 3. It is written down as

\[
-iM_1 = (-iG)^2 \int \frac{d^2k_{\parallel}}{(2\pi)^2} \left[ \bar{u}(p_{\parallel}^{(3)}) \gamma_{\parallel\mu} S(k_{\parallel}) \gamma_{\parallel\nu} u(p_{\parallel}^{(1)}) \right] \left[ \bar{v}(p_{\perp}^{(4)}) \gamma_{\perp\nu} \gamma_{\perp\mu} S(k_{\perp}) - P_{\parallel} \gamma_{\parallel\nu} \bar{v}(p_{\perp}^{(2)}) \right] ,
\]  

where \( P_{\parallel} = p_{\parallel}^{(1)} + p_{\parallel}^{(2)} \). The LLL propagator in the Ritus basis is given by

\[
S(k) = \frac{i\Gamma_{\parallel}}{k_{\parallel}^2 + i\epsilon} \mathcal{P}_+ ,
\]  

with the spin projection operator \( \mathcal{P}_{\pm} = (1 \pm i\gamma^1\gamma^5)/2 \) in the direction of the magnetic field.

We are left with the two-dimensional loop integral in Eq. (13), which is a natural consequence of the dimensional reduction. After performing the elementary \( k^3 \)-integral as explained in Appendix A, one finds the origin of the magnetic catalysis:

\[
\mathcal{M}_1 = G^2 \int \frac{dk_{\parallel}}{2\pi} \frac{1}{|k_{\parallel}|} .
\]  

The same result is obtained for the both scattering channels, so that we have again suppressed the trivial spinor structures which are common to the tree-level amplitude. The remaining one-dimensional integral has the anticipated logarithmic IR divergence. When the scale goes down from \( \Lambda \) to \( \Lambda - \delta \Lambda \), the increment of the one-loop amplitude is found to be

\[
\mathcal{M}_1(\Lambda - \delta \Lambda) - \mathcal{M}_1(\Lambda) = \frac{G^2(\Lambda)}{\pi} \log \frac{\Lambda}{\Lambda - \delta \Lambda} .
\]  

The logarithm is absorbed by the renormalization of the effective coupling constant \( G \), which leads to the RG evolution.
From the scattering amplitudes in Eqs. (12) and (16), the RG equation is obtained as
\[
\frac{d}{d\Lambda} G(\Lambda) = -2\alpha - \frac{1}{\pi} G^2(\Lambda) \quad \text{Region I,} \tag{17a}
\]
\[
\frac{d}{d\Lambda} G(\Lambda) = -\frac{1}{\pi} G^2(\Lambda) \quad \text{Region II.} \tag{17b}
\]

The first term in Eq. (17a) comes from the tree-level amplitude according to the intrinsic scale dependence of the coupling constant shown in Eq. (12). This term is absent in Eq. (17b), because the soft momentum transfer is cut off by the screening mass. The clear distinction between these two regions will result in an important consequence, as first discussed in the analysis of magnetically induced QCD Kondo effect [29]. The other terms in these equations, which come from the logarithmic contribution in Eq. (16), have minus signs, so that these terms drive the RG flow toward the Landau pole when the interaction is attraction ($G > 0$) as explicitly seen below.

When the scale is reduced from $\Lambda_0$ to $\Lambda$ ($\Lambda > \Lambda_{sc}$) where the initial scale $\Lambda_0$ is of the order of $\sqrt{eB}$, we use the RG equation (17a) in Region I. With the initial coupling $G(\Lambda_0) = \alpha \log(2eB/\Lambda_0^2) \ll 1$ from Eq. (8), we find the solution as
\[
G(\Lambda) \simeq \sqrt{2\alpha} \pi \tan\left(-\sqrt{\frac{\alpha}{\pi}} \log \frac{\Lambda^2}{2eB}\right). \tag{18}
\]
Here, the $\Lambda_0$ dependence appears in the higher-order terms of $\alpha$, and can be neglected. Then, we find the running coupling constant at the lower boundary of Region I as
\[
G(\Lambda_{sc}) \simeq \alpha \log \frac{2eB}{m_f^2} + \frac{\alpha^2}{6\pi} \left(\log \frac{2eB}{m_f^2}\right)^3, \tag{19}
\]
where we performed an expansion with respect to $\alpha \log(2eB/m_f^2) = \alpha \log(\pi/\alpha) \ll 1$. The leading term of order $\alpha$ corresponds to the tree-level coupling constant (8), while the subsequent term explains the growth of the coupling constant driven by the quantum effect in the scale region $\Lambda_{sc} < \Lambda < \Lambda_0$.

When the scale $\Lambda$ enters Region II, we solve the RG equation (17b) with the initial condition at $\Lambda_0 = \Lambda_{sc}$ which was obtained from the RG evolution in Region I in Eq. (19). In this way, the evolutions in the two regions are smoothly connected. We find the solution in Region II as
\[
G(\Lambda) = \frac{G(\Lambda_{sc})}{1 + \pi^{-1}G(\Lambda_{sc}) \log(\Lambda/\Lambda_{sc})}. \tag{20}
\]
Clearly, this solution has a Landau pole at
\[
\Lambda_{IR} = \Lambda_{sc} e^{-\pi/G(\Lambda_{sc})}, \tag{21}
\]
which indicates the emergence of the dynamical IR scale.

The presence of the strong four-Fermi interaction induces a minimum of the effective potential at a nonzero value of the chiral condensate. The associated dynamical mass is of the order of the emergent scale $\Lambda_{IR}$, while the order of the chiral condensate is given in combination with the transverse degeneracy factor $\sim eB\Lambda_{IR}$. Intuitively, the condensate is squeezed along the magnetic field within the size of the cyclotron motion $\sim 1/\sqrt{eB}$, and has a number of copies distributed with the density $eB/(2\pi)$ in the transverse plane [22] (see also a discussion in the last section). Therefore, the scale of the dynamical mass gap is explicitly
\[
m_{dyn} \simeq m_f \exp\left\{-\frac{\pi}{\alpha \log(\pi/\alpha)} + \log\left(\frac{\pi}{\alpha}\right)^{1/6}\right\}
\]
\[
= \sqrt{2eB} \alpha^{1/3} \exp\left\{-\frac{\pi}{\alpha \log(\pi/\alpha)}\right\}, \tag{22}
\]
where the first and second terms in the exponential correspond to those in Eq. (19).

Notice that, no matter how small the coupling constant in the underlying theory is, the solution in Eq. (20) has the Landau pole as long as the interaction is attractive. This is consistent with the aforementioned observation made in the study of the SD equations [11, 12]. By the use of RG analyses, this fact is even more clearly seen in the RG flow informed by the beta function. In short, when the magnitude of a magnetic field increases, the effect of the magnetic field shifts the UV fixed point that determines the critical coupling strength for the chiral symmetry breaking. Eventually, the fixed point merges with the one at the origin, and the beta function is completely pushed out from the positive region, leaving a vanishing critical coupling strength and the beta function entirely in the negative region (see Refs. [34, 35] for pedagogical discussions). This occurs only for the dimensional reason, as also discussed in a little bit different way in Sec. 2 and implied by the integral in Eq. (15), and thus is a generic consequence of the dimensional reduction. We can immediately read off the beta function from Eq. (17a) and (17b) as
\[
\beta_{1}(\Lambda) = -2\alpha - \frac{\pi}{\alpha} G^2(\Lambda), \tag{23a}
\]
\[
\beta_{11}(\Lambda) = -\frac{1}{\pi} G^2(\Lambda), \tag{23b}
\]
for Region I and II, respectively. Clearly, the beta function is negative for any value of the coupling strength $G(\Lambda)$, indicating that the broken phase is favored at zero temperature regardless of the value of the coupling strength $\alpha$.

We remark on the contributions of the higher Landau levels. As is clear from Eq. (22), the location of the Landau pole is exponentially smaller than the Landau level spacing $\sim \sqrt{eB}$. Therefore, the higher Landau levels are decoupled from the low-energy dynamics of the LLL and should not have any significant impact on the dynamical mass, consistently to the previous observations [36]. Also, for sufficiently strong magnetic fields such that $eB \gg m_f^2$ with $m_f$ being the intrinsic fermion mass, the dynamical mass, or $\Lambda_{IR}$, emerges much earlier than the $m_f$ when the energy scale goes down. Therefore, the $m_f$ also should not have any significant impact.
5. Discussions and concluding remarks

Based on the RG analysis in the previous sections, we discuss an equivalence between the analyses by the RG and SD equations, and digest our RG analysis to identify the origins of the parametric forms of the dynamical mass gap (22) in the screened QED as well as those in the NJL model and the unscreened QED [cf., Eqs. (2) and (3)].

First, we would like to highlight the fact that our result (22) has the same parametric form as that of the mass gap (4) obtained from the SD equation. This is not an accidental coincidence. The RG equations (17a) and (17b) correspond to the resummation of the multiple ladder diagrams for the multiple photon exchanges. On the other hand, the analysis by the SD equation is based on the rainbow approximation, which holds only the planar diagrams of the fermion self-energy. Cutting the intermediate fermion propagator in the SD equation provides the same ladder diagrams which we have included in the RG approach. Thus, solving the SD equation in the rainbow approximation is essentially equivalent to solving the RG equation in the leading order. Note also that the numerical constant $C$ in the exponent takes one in the both results. This agreement originates from a correspondence between the regularizations involved in the analyses. The result from the SD equation is obtained assuming that the fermion self-energy is a constant without a momentum dependence, that is, $m_{dy}/$, which regularizes an IR divergence of the SD equation in the $(1+1)$ dimensions. This regularization corresponds to the sharp cutoff scheme in our RG analysis. The value of $C$ could depend on such a regularization as implied by a slightly different number $C = 1.82$ from the numerical result [11, 12].

Now, we can identify the origins of the overall factors of $\sqrt{\alpha eB}$, $\alpha^{1/3}$, and the exponent in Eq. (4) in the language of the RG method. We should first note that the dimensionful quantities enter in the RG evolutions (17a) and (17b) only through the energy scales $\Lambda_{UV}$ and $\Lambda_{IR}$ that specify the hierarchy and the initial conditions. Therefore, the emergent IR scale has to appear in dimensionless combinations with those scales.

We clearly see in Eqs. (20) and (21) that the initial energy scale $\Lambda_0 = \Lambda_{IR}$ in Region II results in the factor of $\sqrt{\alpha eB}$ in the final result (22), and also that the initial condition $G(\Lambda_{IR})$ in the exponent. Therefore, the dependence of the mass gap on the magnetic field comes from that of the screening mass $\sim \sqrt{\alpha eB}$. Also, we could obtain the logarithmic exponent in Eq. (4) from the initial condition (19) resulting from the evolution in Region I: The separation of the RG evolutions in the two regions naturally led us to take the initial condition $G(\Lambda_{IR})$ at the intermediate scale $\Lambda_{IR}$ instead of at the UV scale $\sim \sqrt{\alpha eB}$. Note that this exponent is independent of the magnetic field. This is because the only two available scales are both proportional to $eB$.

The origin of the factor of $\alpha^{1/3}$ is more subtle. In the passing from Eq. (21) to Eq. (22), we notice that a part of this factor comes from the initial energy scale $\Lambda_{UV}$ in Region II and the other part comes from the initial condition $G(\Lambda_{IR})$ in the exponent. The latter contribution was correctly obtained as the result of the RG evolution in Region I, as discussed below Eq. (19). In a word, without the energy-scale dependence of the tree-level contribution in Eq. (17a), one cannot reproduce this factor.

The above observations clearly indicate the importance of respecting the interaction properties in the underlying theory. Therefore, it is useful to compare with the results in the NJL model (2) and the unscreened QED (3), and identify the origins of the differences in the RG analyses.

**QED with unscreened photons.**— Without the screening effect, the relevant RG equation is only Eq. (17a) in the entire scale regions. The solution is given in Eq. (18).

As the scale $\Lambda$ decreases, this solution hits the Landau pole. Therefore, the dynamical IR scale is obtained from the following equation:

$$-\frac{\alpha}{2\pi} \log \frac{\Lambda_{IR}^2}{2\pi eB} \sim \frac{\pi}{2},$$

(24)

which yields the mass gap

$$m_{dyn} \simeq \sqrt{2eB} \exp \left(-\frac{\pi}{2\alpha} \right).$$

(25)

This result agrees with that from the SD equation (3) up to an order-one constant factor. In unscreen QED, there is only one scale, that is, the initial scale $\Lambda_{UV}$ for the entire RG evolution. This explains the overall factor of $\sqrt{eB}$ and the independence of the exponent from $eB$.

More in detail, as we have already discussed, the screening properties are reflected in the exponent. Without the screening effect, the exponential suppression is parametrically weaker than that in the screened QED (22). The overall factor of $\sqrt{\alpha eB}$ comes from the initial scale of the RG equation, of which the value is, however, different from that in screened QED: While it was $\Lambda_{IR} \sim \sqrt{\alpha eB}$, it is now $\Lambda_{UV} \sim \sqrt{\alpha eB}$. This also means the absence of the overall power factor of $\alpha$ in unscreened QED.

**NJL model.**— The coupling constant $G_{NJL}$ in the NJL model does not have any energy-scale dependence in its tree-level Lagrangian. Therefore, the RG equation for the NJL model is formally the same as Eq. (17b) for Region II, but with an initial scale at the UV region $\Lambda_0 \sim \sqrt{\alpha eB}$. Since there is no photon propagator in Eq. (8), the effective coupling is, as the result of the Gaussian integral, given by $\rho_{LLL} G_{NJL}$ where $\rho_{LLL} = eB/(2\pi)$ is the density of states in the LLL. Plugging the initial scale and initial condition into Eq. (21), one can precisely reproduce the mass gap (2).

Although this result is similar to that in QED, the magnetic field dependence is different. As discussed just above, the exponent in QED is independent of the magnetic field, because of the absence of the dimensionful quantities other than $\Lambda_{UV}$ and $\Lambda_{IR}$ which are both proportional to $eB$. On the other hand, in NJL mode, the
dimensional coupling constant $G_{\text{NJL}}$ appears in combination with the factor of $eB$ in the density of states $\rho_{\text{LL}}$. Therefore, the mass gap in the NJL model increases much faster than that in QED with an increasing $eB$ due to the diminishing exponential suppression. When discussing QCD, this behavior may not be regarded physical because of involved intrinsic energy-scale dependences in the intermediate- to low-energy QCD, which implies one of limitations of the NJL model.

It will be interesting to investigate an extension to asymptotic-free theories which offer an additional intrinsic IR scale, i.e., the QCD scale $\Lambda_{\text{QCD}}$. Indeed, there has been an important issue that, when $eB \gtrsim \Lambda_{\text{QCD}}$, typical effective models of QCD fail to explain the linear dependence of the chiral condensate on $eB$, i.e., $\langle \bar{q}q \rangle \sim eB$ which was observed in the lattice QCD simulations [21] (see also Ref. [37] for a brief summary). It was pointed out that the dynamical mass should stay at $m_{\text{dyn}} \sim \Lambda_{\text{QCD}}$ with increasing $B$, because the dimensional reduction leads to a factorization which is roughly $\langle \bar{q}q \rangle \sim eB m_{\text{dyn}}$ [22] (see also Refs. [23, 24, 25]). Also, to explain the inverse magnetic catalysis [38, 21], this saturating behavior of $m_{\text{dyn}}$ is thought to be important for the thermal excitations not to acquire too huge a mass gap to restore the chiral symmetry [22, 25]. Thus, those issues at zero and finite temperatures appear to problem of explaining the saturation of the mass gap with an increasing $B$. In Refs. [22, 25], the IR-dominant interactions are shown to be important for reproducing the saturation on the basis of the SD equation. In the above RG analyses, it is clear that the mass gap has to be $m_{\text{dyn}} \sim \sqrt{eB}$ in any theory/model containing only $\Lambda_{\text{UV}}$ and $\Lambda_{\text{IR}}$. For example, NJL model may not work as an effective model of QCD in the study of the magnetic catalysis for this reason. It is then interesting to see how the intrinsic scale $\Lambda_{\text{QCD}}$ modifies the chart of the hierarchy in Fig 1 and manifests itself in the mass gap in the language of the RG and the functional RG [39, 35, 34, 40, 41, 42, 28, 43].

In summary, we have closely looked into the magnetic catalysis phenomena by means of the RG method with a special care of the scale separation in the RG evolution. Especially, we elaborated the treatment of the screening effect on the photon propagator, and showed its crucial role in the determination of the dynamical mass gap. Our result on the mass gap agrees with that from the SD equation [11, 12].

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Appendix A. Logarithm from the one-loop scattering amplitude

Inserting the LLL propagator (14) into the one-loop scattering amplitude (13), we have

$$M_1 = i(-iG)^2 \int \frac{d^2k_\parallel}{(2\pi)^2} \frac{i[\bar{u}(p^{(3)}_\parallel)\gamma\mu(k_\parallel)p_\parallel] + \gamma^\nu u(p^{(1)}_\parallel)}{k_\parallel + i\epsilon} \times \frac{i[v(p^{(4)}_\parallel)\gamma\mu(k_\parallel - P_\parallel)p_\parallel] + \gamma^\nu \bar{v}(p^{(2)}_\parallel)}{(k_\parallel - P_\parallel)^2 + i\epsilon}.$$  (A.1)

To proceed, we first perform the contour integral for the $k^0$, which is given by the contributions from the residues of four poles at $k^0 = \pm k_z - i \text{sgn}(k^0)e$ and $k^0 = \pm k_z + P^0 \mp P_z - i \text{sgn}(k^0)e$. After inserting these poles into the numerator, one finds that the chirality projection operator $P_\pm = (1 \pm i\gamma^5)/2$ naturally arises through an identity

$$\langle \gamma^0 \pm \gamma^3 \rangle P_\pm = \langle \gamma^0 \pm \gamma^3 \rangle Q_\pm .$$  (A.2)

To pick up the contributions to the scatterings between the particle and antiparticle pairs carrying opposite chiralities, one can use useful formulas

$$\gamma^{\mu\nu}_{\parallel} \gamma^{\alpha\beta}_{\parallel} = g^{\mu\alpha} \gamma^{\nu\beta} - g^{\mu\beta} \gamma^{\nu\alpha} + g^{\nu\alpha} \gamma^{\mu\beta} + g^{\nu\beta} \gamma^{\mu\alpha},$$  (A.3)

and

$$\langle \gamma^0 \mp \gamma^3 \rangle Q_\pm = \langle \gamma^0 \mp \gamma^3 \rangle P_\mp .$$  (A.4)

where the LLL spinors are orthogonal to $P_\mp$, i.e., $P_\mp u = P_\mp v = 0$.

After performing straightforward algebraic calculation, the scattering amplitude is obtained as

$$M_1 = G^2[\gamma^{\mu\nu}_{\parallel}R][\gamma^{\alpha\beta}_{\parallel}L] \int \frac{dk_z}{2\pi} \left[ \frac{\theta(k_z - P_z) - \theta(-k_z)}{k_z - (P^0 + P_z)/2} \right] + G^2[\gamma^{\mu\nu}_{\parallel}L][\gamma^{\alpha\beta}_{\parallel}R] \int \frac{dk_z}{2\pi} \left[ \frac{\theta(k_z) - \theta(-k_z + P_z)}{k_z + (P^0 - P_z)/2} \right],$$  (A.5)

where the shorthand notations $[\gamma^{\mu\nu}_{\parallel}R/L][\gamma^{\alpha\beta}_{\parallel}L/R]$ denote $[\bar{u}/R/L(p^{(3)}_\parallel)\gamma^{\mu}_{\parallel}u_{R/L}(p^{(1)}_\parallel)] [v^{\alpha}_{R/L}(p^{(4)}_\parallel)\gamma^{\nu}_{L/R}v_{R/L}(p^{(2)}_\parallel)]$, respectively. The LLL spinors with the definite chiralities are $u_{R/L} = Q_\pm u$ and $v_{R/L} = Q_\pm v$. Since the energies and momenta of the scattering particles are less than $\Lambda$, we have $P^0, P_z \lesssim \Lambda$, and thus the logarithmic contribution arises in the form of the integral (15) for both the scattering channels up to a numerical factor.

7
