A NEW OBSTRUCTION FOR RIBBON-MOVES OF 2-KNOTS: 2-KNOTS FIBRED BY THE PUNCTURED 3-TORI AND 2-KNOTS BOUNDED BY HOMOLOGY SPHERES

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Abstract. This paper gives a new obstruction for ribbon-move equivalence of 2-knots.

Let \( K \) and \( K' \) be 2-knots. Let \( K \) and \( K' \) are ribbon-move equivalent. One corollary to our main theorem is as follows. A 2-dimensional fibered knot whose fiber is the punctured 3-dimensional torus is not ribbon-move equivalent to any 2-dimensional knot whose Seifert hypersurface is a punctured homology sphere.

1. Introduction

In this paper we give a new obstruction for ribbon move-equivalence of 2-knots. The author’s papers [16, 17] gave some obstructions. This paper gives new results (§3).

One of the new results is as follows. This theorem is deduced from our main theorem (Theorem 3.2).

**Theorem 1.1.** (same as Theorem 3.3) Let \( K \) be a 2-dimensional fibered knot whose fiber is the punctured 3-dimensional torus. Let \( P \) be a 2-dimensional knot whose Seifert hypersurface is a punctured homology sphere. Then \( K \) is not ribbon-move equivalent to \( P \).

**Note.** Let \( M \) be a closed \( n \)-manifold. A punctured (manifold) \( M \) is a manifold with boundary, \( M - ( \text{an open } n\text{-ball embedded trivially in } M) \).

This paper is based on the author’s preprint [15]. We work in the smooth category.

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2. Review of Ribbon-moves of 2-knots

In this section we review the definition of ribbon-moves.

An (oriented) 2-(dimensional) knot is a smooth oriented submanifold $K$ of $S^4$ which is diffeomorphic to the 2-sphere. We say that 2-knots $K_1$ and $K_2$ are equivalent if there exists an orientation preserving diffeomorphism $f : S^4 \to S^4$ such that $f(K_1) = K_2$ and that $f|_{K_1} : K_1 \to K_2$ is an orientation preserving diffeomorphism. Let $id : S^4 \to S^4$ be the identity. We say that 2-knots $K_1$ and $K_2$ are identical if $id(K_1) = K_2$ and $id|_{K_1} : K_1 \to K_2$ is an orientation preserving diffeomorphism.

If $W$ is a subset of a manifold $Z$, then let $\overline{W}$ mean the closure of $W$ in $Z$. In this paper we often omit to explain what $Z$ is if it is easy to understand what $Z$ is.

**Definition 2.1.** Let $K_1$ and $K_2$ be 2-knots in $S^4$. We say that $K_2$ is obtained from $K_1$ by one ribbon-move if there is a 4-ball $B$ embedded trivially in $S^4$ with the following properties.

1. $K_1$ coincides with $K_2$ in $S^4 - B$. This identity map from $\overline{K_1 - B}$ to $\overline{K_2 - B}$ is orientation preserving.

2. $B \cap K_1$ is drawn as in Figure 2.1.1. $B \cap K_2$ is drawn as in Figure 2.1.2.
Figure 2.1.1
We regard $B$ as $(\text{a close 2-disc}) \times \{s | 0 \leq s \leq 1\} \times \{-1 \leq t \leq 1\}$. We let $B_t = (\text{a close 2-disc}) \times \{s | 0 \leq s \leq 1\} \times \{t\}$. Then $B = \bigcup B_t$, where $\{t | -1 \leq t \leq 1\}$. In Figure 2.1.1 and 2.1.2, we draw $B_{-0.5}, B_0, B_{0.5} \subset B$. We draw $K_1$ and $K_2$ by the bold line. The fine line denotes $\partial B_t$.

$B \cap K_1$ (resp. $B \cap K_2$) is diffeomorphic to $D^2 \Pi (S^1 \times \{s | 0 \leq s \leq 1\})$, where $\Pi$ denotes the disjoint union.

$B \cap K_1$ has the following properties: $B_t \cap K_1$ is empty for $-1 \leq t < 0$ and $0.5 < t \leq 1$. $B_0 \cap K_1$ is diffeomorphic to $D^2 \Pi (S^1 \times \{0 \leq s \leq 0.3\}) \Pi (S^1 \times \{0.7 \leq s \leq 1\})$. $B_{0.5} \cap K_1$ is diffeomorphic to $(S^1 \times \{0.3 \leq s \leq 0.7\})$. $B_t \cap K_1$ is diffeomorphic to $S^1 \Pi S^1$ for $0 < t < 0.5$. (Here we draw $S^1 \times \{0 \leq s \leq 1\}$ to have the corner in $B_0$ and in $B_{0.5}$. Strictly to say, $B \cap K_1$ in $B$ is a smooth embedding which is obtained by making the corner smooth naturally.)
$B \cap K_2$ has the following properties: $B \cap K_2$ is empty for $-1 \leq t < -0.5$ and $0 < t \leq 1$. $B_0 \cap K_2$ is diffeomorphic to $D^2$ II $(S^1 \times \{0 \leq s \leq 0.3\}) \cup (S^1 \times \{0.7 \leq s \leq 1\})$. $B_{-0.5} \cap K_2$ is diffeomorphic to $(S^1 \times \{0.3 \leq s \leq 0.7\})$. $B_t \cap K_2$ is diffeomorphic to $S^1 \cup S^1$ for $-0.5 < t < 0$. (Here we draw $D^2 \cup (S^1 \times \{0 \leq s \leq 0.3\}) \cup (S^1 \times \{0.7 \leq s \leq 1\})$ to have the corner in $B_0$ and in $B_{-0.5}$. Strictly to say, $B \cap K_1$ in $B$ is a smooth embedding which is obtained by making the corner smooth naturally.)

In Figure 2.1.1 (resp. 2.1.2) there are an oriented cylinder $S^1 \times \{0 \leq s \leq 1\}$ and an oriented disc $D^2$ as we stated above. We do not make any assumption about the orientation of the cylinder and the disc. The orientation of $B \cap K_1$ (resp. $B \cap K_2$) coincides with that of the cylinder and that of the disc.

Suppose that $K_2$ is obtained from $K_1$ by one ribbon-move and that $K'_2$ is equivalent to $K_2$. Then we also say that $K'_2$ is obtained from $K_1$ by one ribbon-move. If $K_1$ is obtained from $K_2$ by one ribbon-move, then we also say that $K_2$ is obtained from $K_1$ by one ribbon-move.

**Definition 2.2.** Two 2-knots $K_1$ and $K_2$ are said to be ribbon-move equivalent if there are 2-knots $K_1 = \hat{K}_1, \hat{K}_2, ..., \hat{K}_{r-1}, \hat{K}_r = K_2$ ($r \in \mathbb{N}, p \geq 2$) such that $\hat{K}_i$ is obtained from $\hat{K}_{i-1}$ ($1 < i \leq r$) by one ribbon-move.

**Problem 2.3.** Let $K_1$ and $K_2$ be 2-knots. Find a necessary (resp. sufficient, necessary and sufficient) condition that $K_1$ and $K_2$ are ribbon-move equivalent.

In [16] the author proved the following.

**Theorem 2.4.** ([16]) (1) If 2-knots $K$ and $K'$ are ribbon-move equivalent, then

$$\mu(K) = \mu(K'),$$

where $\mu(\cdot)$ denotes the $\mu$-invariant of 2-knots.

(2) Let $K_1$ and $K_2$ be 2-knots in $S^3$. Suppose that $K_1$ are ribbon-move equivalent to $K_2$. Let $W_i$ be an arbitrary Seifert hypersurface for $K_i$. Then the torsion part of $\{H_1(W_1) \oplus H_1(W_2)\}$ is congruent to $G \oplus G$ for a finite abelian group $G$.

(3) Not all 2-knots are ribbon-move equivalent to the trivial 2-knot.

(4) The converse of (1) is not true. The converse of (2) is not true.

**Note.** See [16] for the $\mu$-invariant of 2-knots.

Furthermore, in [17] the author proved the following Theorem 2.7.
Definition 2.5. Let $K$ be a 2-knot $\subset S^4$. Let $N(K)$ be the tubular neighborhood of $K$ in $S^4$. Let $\alpha : \pi_1(S^4 - N(K)) \to H_1(S^4 - N(K); \mathbb{Z})$ be the abelianization. Note that any 1-cycle is oriented naturally by using the orientation of $K$ and that of $S^4$. We define the canonical isomorphism $\beta : H_1(S^4 - N(K); \mathbb{Z}) \to \mathbb{Z}$ by using this orientation. Let $\tilde{X}_K^\infty$ be the covering space associated with $\beta \circ \alpha : \pi_1(S^4 - N(K)) \to \mathbb{Z}$. We call $\tilde{X}_K^\infty$ the canonical infinite cyclic covering space of the complement $S^4 - N(K)$ of $K$. We also call $\tilde{X}_K^\infty$ the canonical infinite cyclic covering space for $K$. See [6, 11, 28] for canonical infinite cyclic covering spaces for details.

Note 2.6. In this paper, if we regard a tubular neighborhood as a fiber bundle naturally, it is the close disc (not the open disc) that the fiber of the fiber bundle is. That is, we have the following. Let $A$ be a $a$-submanifold in a $b$-manifold $B$ ($a, b \in \mathbb{N} \cup \{0\}$). Then the tubular neighborhood of $A$ in $B$ is a fiber bundle over $A$ whose fiber is the $(b - a)$-dimensional close disc.

Theorem 2.7. ([17]) Let $K$ and $K'$ be 2-knots. Suppose that $K$ and $K'$ are ribbon-move equivalent. Then there is an isomorphism

$$c : \text{Tor}H_1(\tilde{X}_K^\infty; \mathbb{Z}) \to \text{Tor}H_1(\tilde{X}_{K'}^\infty; \mathbb{Z}),$$

where the homomorphism $c$ is not only one as $\mathbb{Z}$-modules but also one as $\mathbb{Z}[t, t^{-1}]$-modules, with the following properties.

1. Let $x, y \in \text{Tor}H_1(\tilde{X}_K^\infty; \mathbb{Z})$. Then we have

$$\text{lk}(x, y) = \text{lk}(c(x), c(y)),$$

where $\text{lk}( )$ denotes the Farber-Levine pairing. That is, the Farber-Levine pairing on $\text{Tor}H_1(\tilde{X}_K^\infty; \mathbb{Z})$ is equivalent to that on $\text{Tor}H_1(\tilde{X}_{K'}^\infty; \mathbb{Z})$.

2. Let $\alpha : H_1(\tilde{X}_K^\infty; \mathbb{Z}) \to \mathbb{Z}_p$ be a homomorphism. Note that there is a homomorphism $\alpha' : H_1(\tilde{X}_{K'}^\infty; \mathbb{Z}) \to \mathbb{Z}_p$ such that $\alpha|_{\text{Tor}} = (\alpha'|_{\text{Tor}}) \circ c$. Then we have $\tilde{\eta}(K, \alpha) = \tilde{\eta}(K', \alpha') \in \mathbb{Q}/\mathbb{Z}$. That is, the set of the values of the $\mathbb{Q}/\mathbb{Z}$-valued $\tilde{\eta}$ invariants for $K$ is equivalent to that for $K'$.

Note. See [17] for the $\tilde{\eta}$-invariants of 2-knots, the Farber-Levine pairing, and the Alexander module.
3. Main results

Definition 3.1. Let $K$ be a 2-knot $\subset S^4$. Let $\widetilde{X}_K^\infty$ be the canonical infinite cyclic covering space for $K$ (see Definition 2.5). Let $\mathcal{M} = \{M_1, ..., M_m\} (m \in \mathbb{N})$ be a set such that $M_i$ is an open oriented 3-submanifold $\subset \widetilde{X}_K^\infty$ and that $[M_i] \in H^\infty_3(\widetilde{X}_K^\infty; \mathbb{R})$. Here, $H^\infty(X; \mathbb{R})$ denotes the $\infty$-chain homology group with the $\mathbb{R}$ coefficient.

The set $\mathcal{M} = \{M_1, ..., M_m\}$ is called an $o$-set for $K$ if $\mathcal{M}$ satisfies the following conditions.

(1) $M_i$ intersects $M_j$ transversely ($i \neq j$).
(2) $M_i \cap M_j$ intersects $M_k$ transversely ($i \neq j, j \neq k$, and $k \neq i$).
(3) $M_i \cap M_j \cap M_k$ is an oriented submanifold $S \coprod R$, where $\coprod$ denotes the disjoint union, with the following conditions.

(i) $S$ is a disjoint union of circles. We do not assume the number of circles. The number of circles may be \begin{align*}
\text{zero} & \quad \text{finite and nonzero} \\
\text{infinite.} & \quad \text{infinite.}
\end{align*}
(ii) $R$ is the empty set $\phi$ or is diffeomorphic to a single line $\mathbb{R}$ with the following property $(\ast)$.

$(\ast)$ Let $V$ be any Seifert hypersurface for $K$. Let $\pi : \widetilde{X}_K^\infty \to S^4 - N(K)$ be the projection map. Then $\pi^{-1}(V) = \coprod_{i=\infty}^\infty V_i$ and each $V_i$ is diffeomorphic to $V$. Let this $\mathbb{R}$ satisfy the following.

(a) This $\mathbb{R}$ and $V_i$ intersect transversely.
(b) The algebraic number of the points, (this $\mathbb{R}) \cap V_i$, is one for each $i$.
(c) The geometric number of the points, (this $\mathbb{R}) \cap V_i$, is finite for each $i$.

Note. In this paper, the fact that $\mathbb{R}$ is a submanifold $\subset \widetilde{X}_K^\infty$ means the following: for each $p \in \mathbb{R}$, there is an open set $U \subset \widetilde{X}_K^\infty$ such that $p \in U$ and that $\mathbb{R} \cap (U - \mathbb{R}) = \phi$.

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1 Let $X$ be a topological space. The infinity chain homology group $H^\infty_i(X; \mathbb{Z})$ is defined by using the infinity chain group $C^\infty_i = \{\Sigma \nu \sigma | \sigma \text{ is a } i\text{-simplex. } \nu \in \mathbb{Z}\}$. The number of nonzero $\nu$ may be infinite. 

Recall that the homology group $H^\infty_1(X; \mathbb{Z})$ is defined by using the chain group $C_i = \{\Sigma \nu \sigma | \sigma \text{ is a } i\text{-simplex. } \nu \in \mathbb{Z}\}$. The number of nonzero $\nu_i$ is finite. 

The infinity chain homology group $H^\infty_i(X; \mathbb{R})$ is defined by using the infinity chain group $C^\infty_i \oplus \mathbb{R}$.

Recall that the homology group $H^\infty_1(X; \mathbb{R})$ is defined by using the infinity chain group $C_i \oplus \mathbb{R}$. 

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Let $\mathcal{M} = \{M_1, \ldots, M_m\}$ be an $o$-set for $K$. Take three elements $M_i, M_j, M_k$. Then $M_i \cap M_j \cap M_k$ is $S \Pi R$ as above. Define the $o$-invariant

$$o(M_i, M_j, M_k) = \begin{cases}
0 & \text{if } R \text{ is the empty set } \phi \\
1 & \text{if } R \cong \mathbb{R}.
\end{cases}$$

Our main theorem is the following.

**Theorem 3.2.** Let $K$ and $K'$ be 2-knots $\subset S^4$. Suppose that $K$ is ribbon-move equivalent to $K'$. Let $\mathcal{M} = \{M_1, \ldots, M_m\} (m \in \mathbb{N})$ be an $o$-set for $K$. Then there is an $o$-set $\mathcal{M}' = \{M'_1, \ldots, M'_m\}$ for $K'$ such that $o(M_i, M_j, M_k) = o(M'_i, M'_j, M'_k)$ for any $i, j, k$.

The results in [16, 17] (Theorem 2.4, 2.7 in this paper) does not imply Theorem 3.2 (see Note 3.6). However, this paper prove Theorem 3.2. Hence Theorem 3.2 is new.

Theorem 3.2 implies Theorem 3.3.

**Theorem 3.3.** (same as Theorem 1.1) Let $K$ be a 2-dimensional fibered knot whose fiber is the punctured 3-dimensional torus. Let $P$ be a 2-dimensional knot whose Seifert hypersurface is a punctured homology sphere. Then $K$ is not ribbon-move equivalent to $P$.

**Note 3.4.** There is a 2-dimensional fibered knot whose fiber is the punctured torus (see [12 3 8]). There is a 2-dimensional knot $Z$ whose Seifert hypersurface is the punctured Poincaré sphere (see [34]). By [16], $\mu(Z) \neq 0$. Take the connected sum $Z \# Z$. By [16], $\mu(Z \# Z) = 0$. Hence there is a 2-knot $J$ whose Seifert hypersurface is a punctured homology 3-sphere such that $\mu(J) \begin{cases}
= 0 \\
\neq 0.
\end{cases}$

**Proof of Theorem 3.3.** By the definition of $K$, $\widetilde{X}_K^\infty \cong T^3 \times \mathbb{R}$. Let $T^3 = S_1^1 \times S_2^1 \times S_3^1$. Let $M_1 = p_1 \times S_2^1 \times S_3^1 \times \mathbb{R}$, $M_2 = S_1^1 \times p_2 \times S_3^1 \times \mathbb{R}$, and $M_3 = S_1^1 \times S_2^1 \times p_3 \times \mathbb{R}$, where $p_i$ is a point $\in S_1^1$. Then we have the following.

1. $\{M_1, M_2, M_3\}$ is an $o$-set.
2. $M_1 \cap M_2 \cap M_3 = \mathbb{R}$.
3. $o(M_1, M_2, M_3) = 1$.

We prove by contradiction. We suppose that $K$ is ribbon-move equivalent to $P$. By Theorem 3.2, there is an $o$-set $\{M'_1, M'_2, M'_3\}$ for the 2-knot $P$ such that $o(M'_1, M'_2, M'_3) = 1$. Hence $M'_1 \cap M'_2 \cap M'_3$ represents a nonzero element $\in H_1^\infty(\widetilde{X}_P^\infty; \mathbb{R})$. Hence $[M'_i] (i = 1, 2, 3)$ represents a...
nonzero element $\in H_3^\infty(\widetilde{X}_P; \mathbb{R})$. Hence $H_3^\infty(\widetilde{X}_P; \mathbb{R})$ is not congruent to 0. However, since a punctured homology sphere is a Seifert hypersurface for the 2-knot $P$, $H_3^\infty(\widetilde{X}_P; \mathbb{R}) \cong 0$. We arrived at a contradiction. Hence the initial condition is false. That is, $K$ is not ribbon-move equivalent to $P$.

Note that, only in the $\mu(K) \neq \mu(P)$ case, Theorem 2.4 implies that $K$ is not ribbon equivalent to $P$.

Note 3.5. In the similar manner in the above proof, we have the following. Suppose that $\{M_1, M_2, M_3\}$ is an o-set for the trivial 2-knot. Then $o(M_1, M_2, M_3) = 0$.

Note 3.6. Theorem 2.4 2.7 cannot prove the $\mu(K) = \mu(P)$ case of Theorem 3.3. However, Theorem 3.2 (and Theorem 3.3) can.

We have the following Theorem 3.7. Compare Theorem 3.7 to Note 3.5. Hence the converse of Theorem 3.2 is not true.

Theorem 3.7. There is a 2-knot $K$ with the following properties.
(1) For any o-set $\{M_1, M_2, M_3\}$ for $K$, $o(M_1, M_2, M_3) = 0$.
(2) $K$ is not ribbon-move equivalent to the trivial knot.

Proof. The 2-knot $Z$ in Note 3.4 is an example because of Theorem 2.4.(1).

Compare the following theorem to the above Theorem 3.2 (resp. 3.3).

Theorem 3.8. There is a nontrivial 2-knot $K$ with the following properties.
(1) $K$ is ribbon-move equivalent to the trivial 2-knot.
(2) There are a 3-dimensional open oriented submanifold $M$ such that $[M] \in H_3^\infty(\widetilde{X}_K; \mathbb{R})$ and a 2-dimensional open oriented submanifold $N$ such that $[N] \in H_2^\infty(\widetilde{X}_K; \mathbb{R})$ with the following properties.
(i) $M \cap N \cong \mathbb{R}$.
(ii) This $\mathbb{R}$ satisfies the $(\ast)$ in (3)(ii) in Definition 3.1.

Proof of Theorem 3.8. An example is the spun knot of the trefoil knot.

4. (1,2)-pass-moves and ribbon-move surgeries of $S^4$
In order to prove our main theorem (Theorem 3.2), we use the (1,2)-pass-moves for 2-knots. [16] defined the (1,2)-pass-moves for 2-knots.
**Definition 4.1.** Let $K_1$ and $K_2$ be 2-links in $S^4$. We say that $K_2$ is obtained from $K_1$ by one $(1,2)$-pass-move if there is a 4-ball $B$ embedded in $S^4$ with the following properties.

We draw $B$ as in Definition 1.1.

1. $K_1$ coincides with $K_2$ in $\overline{S^4 - B}$. This identity map from $\overline{K_1 - B}$ to $\overline{K_2 - B}$ is orientation preserving. Note that this condition on $K_i$ implies that $K_1$ coincides with $K_2$ in $\overline{S^4 - B}$.

2. $B \cap K_1$ is drawn as in Figure 4.1.1. $B \cap K_2$ is drawn as in Figure 4.1.2.

We suppose that each vector $\overrightarrow{x}$, $\overrightarrow{y}$ in Figure 4.1.1 (resp. 4.1.2) is a tangent vector of each disc at a point. (Note we use $\overrightarrow{x}$ (resp. $\overrightarrow{y}$) for different vectors.) The orientation of each disc in Figure 4.1.1 (resp. Figure 4.1.2) is determined by the each set $\{\overrightarrow{x}, \overrightarrow{y}\}$. We do not make any assumption about the orientations of the annuli in the Figure 4.1.1 (resp. Figure 4.1.2). The orientation of $B \cap K_1$ coincides with that of the disjoint union of the two discs and the annuli.
Figure 4.1.1
Suppose that $K_2$ is obtained from $K_1$ by one (1,2)-pass-move and that $K_2'$ is equivalent to $K_2$. Then we also say that $K_2'$ is obtained from $K_1$ by one (1,2)-pass-move.

If $K_1$ is obtained from $K_2$ by one (1,2)-pass-move, then we also say that $K_2$ is obtained from $K_1$ by one (1,2)-pass-move.

2-links $K_1$ and $K_2$ are said to be (1,2)-pass-move equivalent if there are 2-links $K_1 = \hat{K}_1, \hat{K}_2, ..., \hat{K}_{p-1}, \hat{K}_p = L_2$ ($p \in \mathbb{N}, p \geq 2$) such that $\hat{K}_i$ is obtained from $\hat{K}_{i-1}$ ($1 < i \leq p$) by one (1,2)-pass-move.

**Note.** In [25] the author defined $(p, q)$-pass-moves for $p, q \in \mathbb{N}$. The (1,2)-pass move here is the $(p, q)$-pass-move in the $p = 1$ and $q = 2$ case there.
Before [25], the author defined other local moves in [19]. The local moves in [19] are the \((p, p)-\text{pass-moves}\) in [25].

[16] proved:

**Theorem 4.2.** ([16]) Let \(K\) and \(K'\) be 2-knots. The following conditions (1) and (2) are equivalent.

1. \(K\) is \((1,2)\)-pass-move equivalent to \(K'\).
2. \(K\) is ribbon-move equivalent to \(K'\).

Furthermore, if \(K\) is obtained from \(K'\) by one ribbon-move, then \(K\) is obtained from \(K'\) by one \((1,2)\)-pass-move.

Let \(K_\prec\) and \(K_\succ\) be 2-knots. Suppose that \(K_\prec\) is ribbon-move equivalent to \(K_\succ\). By Theorem 4.2, \(K_\prec\) is \((1,2)\)-pass-move equivalent to \(K_\succ\). Therefore, in order to prove our main result (Theorem 3.2) in §3, it suffices to prove the case when \(K'\) is obtained from \(K\) by one \((1,2)\)-pass-move in a 4-ball \(B\) embedded in \(S^4\).

Next we state a relation among surgeries, \((1,2)\)-pass-moves and ribbon-moves.

**Definition 4.3.** Let \(M\) be an \(m\)-manifold or an \(m\)-manifold with boundary. Make a product manifold \(M \times [0, 1]\). Identify \(M \times \{0\}\) with \(M \times \{1\}\). Attach handles \(h^p\) are attached to \(M \times [0, 1]\), where \(h^p \cap (M \times [0, 1]) \subset M \times \{1\}\). Let \(M' = (\partial(h^p \cup (M \times [0, 1]))) - M \times \{0\} - (\partial M) \times [0, 1]\), where \((\partial M) \times [0, 1] = \emptyset\) if \(\partial M = \emptyset\).

If we do the above procedure, we say that \(M'\) is obtained from \(M\) by the surgery by using the above handles. (In other words, if we do the above procedure, we sometimes do not explain that we use \(M \times [0, 1]\).)

See [31] for surgeries.

**Theorem 4.4.** Let \(K\) be a 2-knot \(\subset S^4\). Take a 4-ball \(B \subset S^4\). Let \(K \cap B\) be as in Figure 4.1.1. Then there is a submanifold \(P \cap Q \subset B\) with the following properties.

1. \(P \cong S^1\).
2. \(Q \cong S^2\).
3. \(P \cap Q = \emptyset\). The linking number of \(P\) and \(Q\) is one if we give an orientation to \(P \cap Q\).
4. Regard \(\begin{array}{c} B \times \{1\} \\ S^4 \times \{1\} \end{array}\) as \(\begin{array}{c} B \times [0, 1] \\ S^4 \times [0, 1] \end{array}\).

Attach a 5-dimensional 2-handle along \(P\) with the trivial framing and a 5-dimensional 3-handle along \(Q\) with the trivial framing. Then
the new 4-manifold made from \( \left\{ \frac{B \times \{1\}}{S^4 \times \{1\}} \right\} \) by this surgery is diffeomorphic to \( \left\{ \frac{B}{S^4} \right\} \) again.

(5) The new knot in the new \( S^4 \) obtained by these surgeries is the knot made from \( K \) by one \((1,2)\)-pass-move in \( B \). That is, \( K \cap ( \text{the new } B ) \) is as in Figure 4.1.2.

Note. By Theorem 4.2, the new knot is ribbon-move equivalent to \( K \).

Definition 4.5. The set of these surgeries in Theorem 4.4 is called the \textit{ribbon move surgery} of \( S^4 \) along \( P \amalg Q \).
Proof of Theorem 4.4. See Figure 4.5.1.

Figure 4.5.1

Note that $K \cap B$ is a disjoint union of the cylinder $A$ and a set $D$ of the two discs. Take a 3-dimensional 1-handle $h^1$ trivially embedded in $B$. The manifold $\partial h^1 - \partial B$ can be regarded as $A$. Take a 4-dimensional 2-handle $h^2$ trivially embedded in $B$. The manifold $\partial h^2 - \partial B$ can be regarded as $D$.

Embed $S^1$ in $B$ so that the linking number of $S^1$ and $h^2$ in $B$ is one, let $P$ denote this $S^1$. Embed $S^2$ in $B$ so that the linking number of $S^2$ and $h^1$ in $B$ is one, let $Q$ denote this $S^2$. Note that we can define the linking number of $\{ P \amalg h^2 \}

\{ Q \amalg h^1 \}$ in $B$ as above if we give an orientation to $\{ P \amalg h^2 \}

\{ Q \amalg h^1 \}$. Note that the attaching part of $h^*$ is fixed at $\partial B$. 

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Then we can suppose that the linking number of $P(\cong S^1)$ and $Q(\cong S^2)$ in $B$ is one if we give an orientation to $P \amalg h^2 \amalg Q \amalg h^1$ so that we do not change the linking number of $P \amalg h^2 \amalg Q \amalg h^1$ in $B$. This completes the proof.

**Note 4.6.** Take a Seifert hypersurface $V$ for $K$. We can suppose that $V \cap B = h^1 \amalg h^2$. Proof. The idea of the proof is Pontrjagin-Thom construction. See e.g. P. 49 of [10].

5. **Ribbon-move surgeries of canonical infinite cyclic covering spaces $X^\infty_K$**

Let $X_K$ denote $S^4 - N(K)$. Note that $N(K)$ is diffeomorphic to $K \times (a$ close 2-disc). Let $V$ be a Seifert hypersurface for $K$. Let the manifold $V \cap X_K$ be called $V$ again. Let $N(V)$ be the tubular neighborhood of $V$ in $X_K$. Then $N(V) \cong V \times [-1, 1]$. Let $Y_K = X_K - N(V)$.

Let $NV$ denote $N(V)$. Let $X^\infty_K$ be the canonical infinite cyclic covering space for $K$ (see Definition 2.5). We can regard $X^\infty_K = \cup_{i=-\infty}^{\infty} (NV_i \cup Y_i)$ ($i \in \mathbb{Z}$), where $\{NV_i\}$ is made from $\{NV = V \times [-1, 1]\}$ naturally. We sometimes abbreviate $\{Y_{Ki}\}$ to $\{Y_i\}$. We can regard $NV_i = V_i \times [-1, 1]$.

Suppose that $Y_j \cap NV_i \begin{cases} \neq \phi & \text{if } i = j, i = j + 1 \\ = \phi & \text{if } i \neq j, i \neq j + 1. \end{cases}$

Note we can make $X_K$ from $NV$ and $\{Y_i\}$ by using an attaching map which is different from the attaching map in making $X^\infty_K$.

Take $P \amalg Q \subset B$ as in Definition 4.5. We define $\left\{ \begin{array}{c} P_i \\ Q_i \end{array} \right\}$ so that $\left\{ \begin{array}{c} \Pi_{i=\infty}^i P_i \\ \Pi_{i=\infty}^i Q_i \end{array} \right\}$ is the lift of $\left\{ \begin{array}{c} P \\ Q \end{array} \right\}$.

We define that $\left\{ \begin{array}{c} (\Pi_{j=-\infty}^j P_j) \cap Y_i \\ (\Pi_{j=-\infty}^j Q_j) \cap Y_i \end{array} \right\}$ is $\left\{ \begin{array}{c} P_i \\ Q_i \end{array} \right\}$.

Note that we can let $P_i \cap NV_j = \phi$ and $Q_i \cap NV_j = \phi$ for each $(i, j)$.

We use the following kind of surgeries from now on.

**Definition 5.1.** Let $N$ be an $n$-submanifold or an ‘$n$-submanifold with boundary’ of an $m$-manifold $M$. Let $N \times [0, 1]$ be a submanifold $\subset M$. 

\[16\]
Identify \( N \times \{0\} \) with \( N \). Suppose that handles \( h^p \) which are embedded in \( M \) are attached to \( N \times [0, 1] \), where \( h^p \cap (N \times [0, 1]) \subset N \times \{1\} \). Suppose that \( h^p \cap (N \times [0, 1]) = \emptyset \).

Let \( N' = (\partial(h^p \cup (N \times [0, 1]))) - N \times \{0\} - (\partial N) \times [0, 1], \) where \( (\partial N) \times [0, 1] = \emptyset \) if \( \partial N = \emptyset \).

If we do the above procedure, we say that \( N' \) is obtained from \( N \) by the surgery by using the above embedded handles. (In other words, if we do the above procedure, we sometimes do not explain that we use \( N \times [0, 1] \).)

Note. We sometimes abbreviate ‘submanifold with boundary’ to submanifold.

Compare Definition 5.1 to Definition 4.3.

Claim 5.2. There are surgeries along \( P_i \coprod Q_i (\subset Y_{K_i}) \) with the following property (*).

(*) Let \( Y'_{K_i} \) denote the new manifold made from \( Y_{K_i} \). Let \( K' \) be obtained from \( K \) by the one ribbon-move in \( B \). It holds that \( Y'_{K_i} \) is \( Y_{K_i}' \). It holds that \( \Pi_{i=-\infty}^{\infty}(NV_i \cup Y'_{K_i}) = \tilde{X}_{K'}^{\infty} \).

Note. \( \{P_i, Q_i\} \) may be a nonvanishing cycle.

Definition 5.3. The set of these surgeries is called the ribbon-move surgery of the infinite cyclic covering space \( \tilde{X}_{K}^{\infty} \) along \( P_i \coprod Q_i \).

Compare Definition 5.3 to Definition 4.5.

Proof of Claim 5.2. Carry out the ribbon surgery along \( P \coprod Q (\subset Y_K \subset S^4) \). Corresponding to this ribbon-surgery, we can carry out surgeries on \( Y_{K_i} \) along \( P_i \coprod Q_i \) to satisfy the conditions in Claim 5.2.

6. Proof of Theorem 3.2 (main theorem)

Proof of Theorem 3.2. As we state right after Theorem 4.2, it suffices to prove the case where a 2-knot \( K \) is obtained from a 2-knot \( K' \) by using a single \((1,2)\)-pass-move.

Let \( M = \{\tilde{M}_1, ..., \tilde{M}_m\} (m \in \mathbb{N}) \) be an \( \omega \)-set for \( K \).

We can suppose that \( M_\omega \) and \( NV_i \cup \tilde{V}_i \) intersect transversely.

Note that \( M_\omega \) represents an element \( \in H_3^\infty(\tilde{X}_{K}^{\infty}; \mathbb{R}) \). Recall that \( V_i \) is a compact oriented manifold and that \( \partial V_i \subset \partial(\tilde{X}_{K}^{\infty}) \).
Hence \[
\begin{aligned}
\{ \widetilde{M}_s \cap NV_i, \widetilde{M}_s \cap V_i \} \text{ is a compact oriented manifold } \{ F \times [-1,1], F, \}
\end{aligned}
\]
where the following hold.

1. \( F \) is a closed oriented surface. (\( F \) may not be connected.)
2. Let \( e_t \) be the embedding map \( F \times \{t\} \hookrightarrow V_i \times \{t\} \). Then \( e_t \) and \( e'_t \)
   are the same embedding maps (\( -1 \leq t \leq 1, -1 \leq t' \leq 1 \)) if we identify \( F \times \{t\} \)
   with \( F \times \{t'\} \) and if we do \( V_i \times \{t\} \) with \( V_i \times \{t'\} \).

Let \( \widetilde{M}_s \cap NV_i = F \times [-1,1] \) be called \( M^*_s NV_i \).

Claim 6.1. We can suppose that the manifold \( F \times [-1,1] = \widetilde{M}_s \cap NV_i \)
is embedded in the manifold \( NV_i - (h^1 \times [-1,1]) \).

Proof. Take the cocore \( C \) of \( h^1 \subset V_i \). Note \( C \) is a 2-disc. We can
suppose that \( F \) and \( C \) intersect transversely, if they intersect. The
intersection \( F \cap C \) is a set of circles. These circles are the boundaries
of discs \( \hat{D} \) such that these discs \( \hat{D} \) are embedded in \( C \). (Note that the
disks \( \hat{D} \) may intersect each other but our proof may not mind that.)

Regard these 2-discs \( \hat{D} \) as the cores of 4-dimensional 2-handles. Carrying
out surgeries on \( \widetilde{M}_s \) by attaching these 2-handles \( \hat{D} \) along \( F \cap C \), let
\( \widetilde{M}_s \cap (h^1 \times [0,1]) = \phi \) and \( (\widetilde{M}_s \cap V_i) \cap h^1 = \phi \).

Note: If the disks \( \hat{D} \) intersect each other, use the isotopy of \( \hat{D} \).

Here, if necessary, we carry out these surgeries on \( \widetilde{M}_s (s = 1, ..., m) \).
These surgeries are done in the interior of the tubular neighborhood
\( N(h^1 \times [-1,1]) \) of \( h^1 \times [-1,1] \) in \( \widetilde{X}^\infty_K \). Note that
\( N(h^1 \times [-1,1]) \) is a compact set. (See Note 2.6) Suppose that
\( \alpha(M_{\alpha}, M_{\beta}, M_{\gamma}) = 1 \) for a set \( \{ \alpha, \beta, \gamma \} \). By using isotopy of \( \widetilde{M}_s \), we can
suppose that \( \widetilde{M}_\alpha \cap \widetilde{M}_\beta \cap \widetilde{M}_\gamma \) does not intersect \( N(h^1 \times [-1,1]) \) before
these surgeries.

Then we have the following. Suppose that we obtain a new intersection \( \widetilde{M}_\alpha \cap \widetilde{M}_\beta \cap \widetilde{M}_\gamma \) after these surgeries. (Note that the new intersection
consists of triple points and is an oriented 1-dimensional manifold.)
Then the new intersection is a disjoint union of circles because of the following.

1. These surgeries are done in the interior of the compact set \( N(h^1 \times [-1,1]) \).
2. The triple point set does not exist in \( N(h^1 \times [-1,1]) \) before these surgeries. (Because of the above procedure.)
Therefore, even if we change $\tilde{M}_*$ in the above procedure, the $o$-invariant $o(\tilde{M}_\alpha, \tilde{M}_\beta, \tilde{M}_\gamma)$ for an arbitrary set $\{\alpha, \beta, \gamma\}$ does not change. This completes the proof.

Suppose that $o(\tilde{M}_\alpha, \tilde{M}_\beta, \tilde{M}_\gamma) = 1$ for a set $\{\alpha, \beta, \gamma\}$. Then $\tilde{M}_\alpha \cap \tilde{M}_\beta \cap \tilde{M}_\gamma = \mathbb{R}$.

**Claim 6.2.** The geometric number of the points, $(\text{this } \mathbb{R}) \cap V_i$, is one for each $i$.

**Proof.** Recall Definition 3.1. The algebraic number of the points, $(\text{this } \mathbb{R}) \cap V_i$, is one for each $i$. The geometric number of the points, $(\text{this } \mathbb{R}) \cap V_i$, is finite for each $i$.

We can suppose that $(\text{this } \mathbb{R}) \cap P_i = \phi$ for each $i$. Because: If it is not an empty set, use the isotopy of $P_i$.

We can suppose that $(\text{this } \mathbb{R}) \cap Q_i = \phi$ for each $i$. Because: If it is not an empty set, use the isotopy of $Q_i$.

If the geometric number, $(\text{this } \mathbb{R}) \cap V_i$, is not one, carry out surgeries on $V_i$ by using 4-dimensional 1-handles with the following properties.

1. Each of the handles is embedded in $\tilde{X}_K^\infty$.
2. The core of each of the handles is a 1-dimensional ‘submanifold with boundary’ of this $\mathbb{R}$.

Note that, since the geometric number of the points, $(\text{this } \mathbb{R}) \cap V_i$, is finite, the number of these surgeries is finite.

Note that the new $V_i$ is orientable. Because: We can suppose that the two points along which each of the above 1-handles is attached have the opposite orientations.

This completes the proof.

We can suppose that $\tilde{M}_*$ and $Y_i$ intersect transversely.

Let $M_{*i} = Y_i \cap \tilde{M}_*$ ($* = 1, ..., m$). Then we have the following.

1. $\partial M_{*i} = N_{s_{i-1}} \cup N_{s_{i+1}} \subset \partial Y_i$.
2. $M_{*i} \cap NV_i = N_{s_{i-1}}$.
3. $M_{*i} \cap NV_{i+1} = N_{s_{i+1}}$.
4. For each set $\{\alpha, \beta, \gamma\} \subset \{1, ..., m\}$, $M_{\alpha i} \cap M_{\beta i} \cap M_{\gamma i}$ is a disjoint union $S \cup I$, where $S$ is a set of circles and where $I$ is the empty set or a single segment $I$ with the following properties:
   - (i) (One of $\partial I \subset NV_i$.
   - (ii) (The other of $\partial I \subset NV_{i+1}$.

Note that $\partial I$ is two points. Note that $(\tilde{M}_\alpha \cap \tilde{M}_\beta \cap \tilde{M}_\gamma) \cap V_i$ is geometrically one point. (See Claim 6.2.)

**Claim 6.3.** We can suppose that $Q_i \cap M_{*i} = \phi$.  

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Proof. See Figure 4.1. We can take $Q$ in the tubular neighborhood of $h^1$ in $S^4$. Let $C(\widetilde{X}_K^K)$ be the collar neighborhood of $X_K^K$. Take the above $N(h^1 \times [-1, 1])$, which is the tubular neighborhood of $h^1 \times [-1, 1]$ in $\widetilde{X}_K^K$.

Then it holds that we can take $Q_i$ in $U = N(h^1 \times [-1, 1]) \cup C(\widetilde{X}_K^K)$. Take the above $N(h^1 \times [-1, 1])$, which is the tubular neighborhood of $h^1 \times [-1, 1]$ in $\widetilde{X}_K^K$.

Then it holds that we can take $Q_i$ in $U = N(h^1 \times [-1, 1]) \cup C(\widetilde{X}_K^K)$. Take the above $N(h^1 \times [-1, 1])$, which is the tubular neighborhood of $h^1 \times [-1, 1]$ in $\widetilde{X}_K^K$.

Next we consider $P_i \cap M_{s_i}$. (Recall that $P_i \cong S^1$.) We can suppose that $P_i \cap M_{s_i}$ is a finite set of points. Each point is oriented by $P_i$, $M_{s_i}$, and $S^4$.

Claim. We can suppose that the orientations of these points are same.

Proof. Suppose that there are two points $(\subset P_i \cap M_{s_i})$ such that the two points sit side by side in $P_i$ and that they have different orientations.

Carry out a surgery on $M_{s_i}$ along the two points by using a 4-dimensional 1-handle with the following properties.

(1) The handle is embedded in $Y_i$.
(2) The core of the handle is a 1-dimensional ‘submanifold with boundary’ of $P_i$. (Note $P_i \cong S^1$.)

Note that the new $M_{s_i}$ is orientable. Because: The two points along which the above 1-handle is attached have the opposite orientations.

Repeat this surgery.

If necessary, we carry out these surgeries on all $M_{s_i}(i = 1, ..., m)$. These surgeries are done in the interior of the compact set $W = N(P_i) \cap \{ N(h^1 \times [-1, 1]) \cup C(\widetilde{X}_K^K) \}$, where we have the following.

(1) $N(P_i)$ is the tubular neighborhood of $P_i$ in $Y_i$.
(2) We take $N(h^1 \times [-1, 1])$ so that $P_i \cap (\text{the interior of } N(h^1 \times [-1, 1])) \neq \emptyset$ and that $P_i \cap (\text{the interior of } N(h^1 \times [-1, 1]))$ is connected.

Suppose that the triple point set $\widetilde{M}_\alpha \cap \widetilde{M}_\beta \cap \widetilde{M}_\gamma$ intersects $W$ before these surgeries. By using isotopy of $\widetilde{M}$, we can suppose that $\widetilde{M}_\alpha \cap \widetilde{M}_\beta \cap \widetilde{M}_\gamma$ does not intersect $W$. Suppose that the triple point set $\widetilde{M}_\alpha \cap \widetilde{M}_\beta \cap \widetilde{M}_\gamma$ is obtained after these surgeries. Then the new triple point set $\widetilde{M}_\alpha \cap \widetilde{M}_\beta \cap \widetilde{M}_\gamma$ is a disjoint union of circles. Because we have the following.

(1) These surgeries are done in the interior of the compact set $W$. 

\[20\]
(2) The triple point set does not exist in $W$ before these surgeries. (Because of the above procedure.)

Therefore, the $o$-invariant $o(\tilde{M}_\alpha, \tilde{M}_\beta, \tilde{M}_\gamma)$ for an arbitrary set $\{\alpha, \beta, \gamma\}$ does not change. This completes the proof.

Claim 6.4. We can suppose that there is a compact 3-dimensional ‘submanifold with boundary’ $E \subset X_K = S^4 - N(K)$ such that $E \cong \text{(the punctured } S^1 \times S^2)\text{) and that } \partial E = Q$. (Recall $Q \cong S^2$.)

Proof. We can take $Q$ in $t = 0$ in Figure 4.1.1. Take a 3-ball $G$ in $t = 0$ in Figure 4.1.1 so that $\partial G = Q$. Note that $G \cap K$ is a single circle $S^1$.

Attach a 4-dimensional 2-handle $h^2$ to $G$ along this $S^1$, where the following hold.

(1) $h^2$ is embedded in $S^4$.
(2) the core of $h^2$ is a 2-dimensional ‘submanifold with boundary’ of $K$.

After this surgery by this $h^2$, $G$ is changed into $E$ as in Claim 6.4. This completes the proof.

Note that $E \cap P$ is a single point by the construction (see Figure 4.1.1 and Figure 4.5.1). Note that $E \cap V \neq \phi$.

Let $\tilde{E}$ be the lift of $E$ associated with the projection map $\tilde{X}_K \to X_K$. Let $E_{Y_i} = \tilde{E} \cap Y_i$. Let $E_{NV_i} = \tilde{E} \cap NV_i$.

Note that $\partial E_{Y_i} - \partial Y_i = Q_i$ by the construction (see Figure 4.1.1 and Figure 4.5.1). Note that we do not suppose that the orientation of $\partial E_{Y_i}$ coincides with that of $Q_i$. Their orientations may coincide or may not. We determine the orientation of $E_{Y_i}$ (and hence that of $\partial E_{Y_i} - \partial Y_i$) after several lines from here. Take the tubular neighborhood $N(Q_i)$ of $Q_i$ in $Y_i$. Let us $\tilde{E}_{Y_i} - N(Q_i)$ call $E_{Y_i}$ again.

Claim. We have $E_{Y_i} \cap M_{s_i} = \phi$.

Proof. We can suppose that $E_{Y_i} \subset N(h^1 \times [-1, 1]) \cap C(\tilde{X}_K)$. We can suppose that $M_{s_i}$ exists outside $N(h^1 \times [-1, 1]) \cap C(\tilde{X}_K)$. Therefore $E_{Y_i} \cap M_{s_i} = \phi$.

Let $\nu$ be the number of the points $P_i \cap M_{s_i}$ ($\nu \in \{0\} \cup N$). Take $\nu$ copies of $E_{Y_i}$. Let each $E_{Y_i}$ be parallel each other. Note $\partial N(Q_i) \cong S^1 \times S^2$. We can suppose the following.

(1) The intersection (each $E_{Y_i}$) $\cap \partial N(Q_i)$ is a 2-sphere.
This 2-sphere is $p \times S^2 \subset S^1 \times S^2 \cong \partial N(Q_i)$, where $p$ is a point $\in$ (this $S^1$).

(3) Each 2-sphere is parallel to other 2-spheres in $S^1 \times S^2 \cong \partial N(Q_i)$.

We give an orientation to each $E_Y i$ so that the orientation of $P_i \cap E_Y i$ is the opposite one of that of $P_i \cap M_{si}$. Note that we do not suppose the orientation of $E_Y i$ coincides with that of $Q_i$.

Carry out surgeries on $M_{si} \coprod (\nu$ copies of $E_Y i)$ by using 4-dimensional $\nu$-1-handles with the following properties. (Note that $M_{si} \cap E_Y i = \phi$ by the above Claim.)

(1) The handles are embedded in $Y_i$.

(2) The handles are attached along two points such that one point is in $P_i \cap (\nu$ copies of $E_Y i)$ and that the other point is in $P_i \cap M_{si}$. (Note that the orientation of $M_{si}$ is compatible with that of $E_Y i$.)

(3) The core of each 1-handle is a 1-dimensional ‘submanifold with boundary’ of $P_i$. (Note $P_i \cong S^1$.)

Thus we made a new submanifold $M_{si}^\#$ from $M_{si}, \nu$ copies of $E_Y i$, and ‘the above $\nu$-1-handles’ by these surgeries. Note that these surgeries can avoid making any self-intersection of $M_{si}^\#$ for each $i$ by using the above 1-handles appropriately. Note that $\partial M_{si}^\# - \partial Y_i = \partial E_Y i - \partial Y_i$ and that $\partial M_{si}^\# - \partial Y_i$ is a set of the 2-spheres $\subset \partial N(Q_i)$.

Claim. These surgeries do not change $I \subset M_{ai} \cap M_{bi} \cap M_{si}$ for each $\{\alpha, \beta, \gamma\}$.

Proof. If necessary, we carry out these surgeries on all $M_{si}(i = 1, \ldots, m)$. These surgeries are done in the interior of the compact set $N(P_i)$, where $N(P_i)$ is the tubular neighborhood of $P_i$ in $Y_i$. Suppose that the triple point set $\tilde{M}_\alpha \cap \tilde{M}_\beta \cap \tilde{M}_\gamma$ intersects $N(P_i)$ before these surgeries. By using isotopy of $M_*$, we can suppose that $\tilde{M}_\alpha \cap \tilde{M}_\beta \cap \tilde{M}_\gamma$ does not intersect $N(P_i)$. Suppose that the triple point set $\tilde{M}_\alpha \cap \tilde{M}_\beta \cap \tilde{M}_\gamma$ is obtained after these surgeries. Then the new triple point set $\tilde{M}_\alpha \cap \tilde{M}_\beta \cap \tilde{M}_\gamma$ is a disjoint union of circles. Because we have the following.

(1) These surgeries are done in the interior of the compact set $N(P_i)$.

(2) The triple point set does not exist in $N(P_i)$ before these surgeries. (Because of the above procedure.)

This completes the proof.

Carry out the ribbon-move surgery along $P_i$ and $Q_i$. Then, by Claim 3.2 we have the following.
(1) $\widetilde{X}_K^\infty$ is changed into $\widetilde{X}_{K'}^\infty$.
(2) Let $Y_{K_i}$ be changed into $Y'_{K_i}$. Then $Y'_{K_i}$ is $Y_{K'i}$. Recall that $Y_{K_i}$ and $Y_i$ are same (it is written in §5 and before Definition 5.1).

After this ribbon-move surgery, we can carry out the following surgeries.

Take 3-dimensional 3-handles in new $Y'_{K_i} = Y_{K'i}$. Strictly to say, the 3-dimensional 3-handles are in $Y'_{K_i} - (Y'_{K_i} \cap Y_{K_i})$. Attach these 3-dimensional 3-handles to $M^*_{si}$ along all of the 2-spheres $\partial M^*_{si} - \partial Y_{K_i} = \partial E_{Y_{i}} - \partial Y_{K_i}$. Thus we obtain a new 3-manifold $M'_{si}$ with boundary.

When we change $M^*_{si}$, we do not change
$I(\subset \tilde{M}_\alpha \cap \tilde{M}_\beta \cap \tilde{M}_\gamma )$, $\partial M^*_{si}(\subset \partial Y_{K_i})$, or ($\partial E_{Y_i} \cap \partial Y_{K_i}$).

Note: In the above procedure, we may move $I$ by isotopy. However, we do not change the diffeomorphism type of $I$. Furthermore, we do not move $\partial I$.

Hence we have the following.

(1) $\partial M'_{si} = \partial M^*_{si} \amalg (\partial E_{Y_i} \cap \partial Y_{K_i})$.
(2) $\widetilde{M}^* = (\coprod_{i=\infty}^{\infty} M'_{si}) \cup (\coprod_{i=-\infty}^{\infty}\{\nu \text{ copies of } E_{NV_i}\}) \cup (\coprod_{i=-\infty}^{\infty} M^*_{NV_i})$ is an open 3-manifold without boundary. (Note that $E_{NV_i}$ is defined after a few lines from the proof of Claim 6.4. Note that $M^*_{NV_i}$ is defined before a few lines from Claim 6.1.)

$\widetilde{M}^*$ is a submanifold of $\widetilde{X}_{K'}^\infty$. $\widetilde{M}^*$ represents an element $\in H^3_3(\widetilde{X}_{K'}^\infty; \mathbb{R})$.

(3) $\{\tilde{M}'_1, ..., \tilde{M}'_m\} (m \in \mathbb{N})$ is an $o$-set for $K'$.
(4) For each set $\{\alpha, \beta, \gamma\} \subset \{1, ..., m\}$, we have
\[ o(\tilde{M}_{\alpha}', \tilde{M}_{\beta}', \tilde{M}_{\gamma}') = o(M_{\alpha}, M_{\beta}, M_{\gamma}). \]

7. Problems

Here, we submit Problem 2.3 again.

**Problem 7.1.** (essentially same as Problem 2.3) Classify 2-knots by the ribbon-move equivalence.

In particular, the following problems interest us.

**Problem 7.2.** Is there a nontrivial 2-knot $K$ with the following properties?
(1) A Seifert hypersurface of $K$ is a punctured integral homology sphere.
(2) $K$ is ribbon-move equivalent to the trivial knot.

**Problem 7.3.** Is there a nontrivial 2-knot $K$ with the following properties?
(1) A Seifert hypersurface of $K$ is a punctured integral homology sphere.
(2) $\mu(K) = 0$.
(3) $K$ is not ribbon-move equivalent to the trivial knot.

[34] proved that a Seifert hypersurface of the five twist spun knot of the trefoil knot is the punctured Poincaré homology sphere. (Note [3,4] quotes this result.) This 2-knot is called $Z$.

**Problem 7.4.** Is $\{Z \# Z \# (-Z)\}$ ribbon-move equivalent to the trivial knot? (Note that $-Z$ is the 2-knot which has the opposite orientation of $Z$.)

We introduce another local move.

**Definition 7.5.** Let $K$ be a 2-knot $\subset S^4$. Embed $S^1 \times D^3$ trivially in $S^4$, where $S^1$ is a circle and where $D^3$ is a close 3-disc. Suppose that the following hold.
(1) $K \cap (S^1 \times D^3)$ is $(S^1 \times I) \coprod (S^1 \times I)$, where $I$ is the interval.
(2) $K \cap (S^1 \times D^3)$ is

\[
\begin{array}{c}
S^1 \times D^3 \\
\end{array}
\]

where we have the following.
(i) The bold line and its interior in the above figure represent the 3-disc $D^3$.
(ii) The arrows of finite lines represent a submanifold of $K$.
(iii) $S^1 \times$ (each of the two arrows) means each of the above $(S^1 \times I) \coprod (S^1 \times I)$.

Fix this chart of $S^1 \times D^3 \subset \mathbb{R}^4$.

Let $K'$ be a 2-knot with the following properties.
(1) $K \cap (S^1 \times D^3)$ is

\[
\begin{array}{c}
S^1 \times D^3 \\
\end{array}
\]

in the above chart.
Then we say that $K'$ is obtained from $K$ by one XO-move.

If $K''$ is obtained from $K$ by a sequence of XO-moves then we say that $K''$ is XO-move equivalent to $K$.

All $n$-twist spun knots could be XO-move equivalent to the trivial knot. The obstructions for ribbon-moves in \cite{16, 17} could not be obstructions for XO-moves.

**Problem 7.6.** (1) Are all 2-knots XO-move equivalent to the trivial knot?
(2) Is the $o$-invariant an obstruction for XO-moves?

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