Estimation of errors of quadrature formula for singular integrals of Cauchy type with special forms

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Abstract

In this work, we consider the singular integrals of Cauchy type of the forms

\[ J(f, x) = \frac{\sqrt{1-x^2}}{\pi} \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2(t-x)}} \, dt, \quad -1 < x < 1. \]

and

\[ \Phi(f, z) = -\frac{\sqrt{z^2-1}}{\pi} \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2(t-z)}} \, dt, \quad z \notin [-1, 1]. \]

which are understood as Cauchy principal value integrals. Quadrature formulas (QFs) for singular integrals (SIs) (1) and (2) are of the forms

\[ J(f, x) = \sum_{k=0}^{N} A_k(x) f(t_k) + R_N(f, x), \quad -1 < x < 1. \]

and

\[ \Phi(f, z) = \sum_{k=0}^{N} B_k(z) f(t_k) + R_N^*(f, z), \quad z \notin [-1, 1]. \]

where \( z \) is complex variable with \( |Re(z)| > 1 \). With the help of linear spline interpolation, we have proved the rate of convergence of the errors of QFs (3) and (4) for different classes (i.e. \( H^\alpha([-1, 1], K), C^{m, \alpha}[-1, 1], W^r[-1, 1] \)) of density function \( f(t) \). It is shown that approximation by spline possesses more advantages than other kinds of approximation: it requires the minimum smoothness of density function \( f(x) \) to get good order of decreasing errors.

1 Introduction

The importance of singular integrals (SIs) of the form (1) and (2) and their numerical solution are given in many researchers work ([1]-[3]) and literatures cited therein. Many of them are based on the approximation of density function \( f(t) \) with Chebyshev polynomials.

Note that in (2), the function \( \sqrt{z^2-1} \) is understood as a single-valued branch in the plane of complex variable with cut along the interval \([-1,1]\) such that \( \sqrt{z^2-1} = z + O(z^{-1}) \)

1This paper is dedicated to the memory of Professor Israilov M.I. (1934-2010). It was published in collection of papers entitled "Differential Equations and Inverse Problems, Press FAN, Tashkent, 1986, pp. 236-258 (in Russian)
for the large $z > 0$. In the future, under $W = \arcsinz$ only a branch of the function for which $|\Re(W)| < \frac{\pi}{2}$ will be understood.

In this paper, we construct efficient quadrature formulas (QFs) for SIs (1) and (2) using linear spline interpolation. Obtained QFs provide uniform convergence for any singular point $x \in (-1, 1)$ and any $z \not\in [-1, 1]$.

## 2 Construction of the quadrature formula

In order to write the exact form of coefficients of the quadrature formula (3) and (4), we introduce the following notations

$$G_k(x) = \frac{|t_k \sqrt{1 - x^2} - x \sqrt{1 - t_k^2}|}{\sqrt{1 - x^2 + \sqrt{1 - t_k^2}}},$$

$$g_k = \frac{1}{\pi h_k}(\arcsin t_{k+1} - \arcsin t_k),$$

$$F_k(z) = \frac{1}{\pi h_k} \left( \frac{zt_{k+1} - 1}{z - t_{k+1}} - \arcsin \frac{zt_k - 1}{z - t_k} \right).$$

If $x \neq t_k$, then the coefficients of QFs (3) are computed by the formulas

$$A_0(x) = \frac{t_1 - x}{\pi h_0} \ln G_1(x) - \sqrt{1 - x^2}g_0,$$

$$A_k(x) = \frac{t_{k+1} - x}{\pi h_k} \ln G_{k+1}(x) + \frac{x - t_{k-1}}{\pi h_{k-1}} \ln \frac{G_k(x)}{G_{k-1}(x)}$$

$$- \sqrt{1 - x^2} (g_k - g_{k-1}), \quad k = 1, ..., N - 1,$$

$$A_N(x) = \frac{t_N - x}{\pi h_{N-1}} \ln G_{N-1}(x) + \sqrt{1 - x^2}g_{N-1}.$$  \hfill (5)

As $G_k(\pm 1) = 1$ for all $k$, then $A_k(\pm 1) = 0$ for $k = 0, ..., N$. These correspond with the fact in [6] that $J(f, x)|_{x = \pm 1} = 0$ is independent from the value of $f(\pm 1)$.

If $x$ coincides with the nodes $t_k$, $(k = 1, ..., N - 1)$, then the coefficients $A_j(t_k), j \neq k - 1, k, k + 1$ are computed by (5). If $k$ in (5) is replaced by $k - 1$ and $k + 1$ and $x = t_k$ is put, then coefficients $A_{k-1}(t_k)$ and $A_{k+1}(t_k)$ are again computed respectively by (5) and for $A_k(t_k)$ we have

$$A_k(t_k) = \frac{1}{\pi} \ln \frac{G_{k+1}(t_k)}{G_{k-1}(t_k)} - \sqrt{1 - t_k^2} (g_k - g_{k-1}), \quad k = 1, ..., N - 1.$$  \hfill (6)
coefficients of the QFs (4) have the form
\[ B_0(z) = (z - t_1)F_0(z) + \sqrt{z^2 - 1}g_0, \]
\[ B_k(z) = (z - t_{k+1})F_k(z) - (z - t_{k-1})F_{k-1}(z) + \sqrt{z^2 - 1}(g_k - g_{k-1}), \quad k = 1, \ldots, N - 1 \]
\[ B_N(z) = (t_{N-1} - z)F_{N-1}(z) - \sqrt{z^2 - 1}g_{N-1}. \] (7)

Let us derive coefficients of QFs which are given by (5) and (7). As we know the linear spline \( S_N(t) \) interpolating the given function \( f \) on the grid \( \Delta : -1 = t_0 < t_1 < \ldots < t_{N-1} < t_N = 1 \) for \( t \in [t_j, t_{j+1}] \) has the form
\[ S_N(t) = \frac{1}{h_k}[(t_{k+1} - t)f(t_k) + (t - t_k)f(t_{k+1})] \] (8)

Replacing \( f(t) \) in (1) with \( S_N(t) \) we have
\[ J(S_N, x) = \frac{\sqrt{1 - x^2}}{\pi} \sum_{k=0}^{N-1} \frac{1}{h_k} \int_{t_k}^{t_{k+1}} \frac{(t_{k+1} - t)f(t_k) + (t - t_k)f(t_{k+1})}{\sqrt{1 - t^2(t - x)}} dt. \]
\[ = \frac{\sqrt{1 - x^2}}{\pi} \left[ \frac{1}{h_0} \int_{-1}^{t_1} \frac{(t_1 - t)dt}{\sqrt{1 - t^2(t - x)}} f(t_0) + \frac{1}{h_{N-1}} \int_{t_{N-1}}^{1} \frac{(t - t_{N-1})dt}{\sqrt{1 - t^2(t - x)}} f(t_N) \right. \]
\[ + \sum_{k=1}^{N-2} \frac{1}{h_k} \left( \int_{t_k}^{t_{k+1}} \frac{(t - t_k)dt}{\sqrt{1 - t^2(t - x)}} + \int_{t_{k+1}}^{t_{k+2}} \frac{(t_{k+2} - t)dt}{\sqrt{1 - t^2(t - x)}} \right) \left( f(t_{k+1}) \right). \] (9)

Introducing notations
\[ J(k, x) = \int_{t_k}^{t_{k+1}} \frac{dt}{\sqrt{1 - t^2(t - x)}}, \quad J_1(k, x) = \int_{t_k}^{t_{k+1}} \frac{tdt}{\sqrt{1 - t^2(t - x)}}, \]
and using easy checking formulas
\[ \int \frac{xdt}{\sqrt{1 - t^2(\sqrt{1 - x^2} + \sqrt{1 - t^2})}} = ln \frac{1 + \sqrt{1 - x^2}\sqrt{1 - t^2} + xt}{\sqrt{1 - x^2} + \sqrt{1 - t^2}} + C, \]
\[ \int \frac{tdt}{\sqrt{1 - t^2(\sqrt{1 - x^2} + \sqrt{1 - t^2})}} = -ln(\sqrt{1 - x^2} + \sqrt{1 - t^2}) + C, \]
obviously we have

\[ J(k, x) = \frac{1}{1 - x^2} \ln \left| \frac{(t - x)(1 + \sqrt{1 - x^2} \sqrt{1 - t^2} + xt)}{(\sqrt{1 - x^2} + \sqrt{1 - t^2})^2} \right|_{t=t_k}^{t=t_{k+1}}, \]

writing

\[ (t - x)(1 + \sqrt{1 - x^2} \sqrt{1 - t^2} + xt) = (\sqrt{1 - x^2} + \sqrt{1 - t^2})(t\sqrt{1 - x^2} - x\sqrt{1 - t^2}), \]

we obtain

\[ J(k, x) = \frac{1}{1 - x^2} \ln \left| \frac{t\sqrt{1 - x^2} - x\sqrt{1 - t^2}}{(\sqrt{1 - x^2} + \sqrt{1 - t^2})^2} \right|_{t=t_k}^{t=t_{k+1}} = \frac{1}{1 - x^2} \ln \frac{G_{k+1}}{G_k(x)}. \tag{10} \]

Next, having the following relation

\[ J_1(k, x) = xJ(k, x) + \pi h_k g_k. \tag{11} \]

Eq. (9) can be rewritten as

\[ J(S_N, x) = \frac{\sqrt{1 - x^2}}{\pi} \left\{ \frac{1}{h_0} \left[ t_1 J(0, x) - J_1(0, x) \right] f(-1) + \frac{1}{h_{N-1}} \left[ J_1(N - 1, x) - t_{k-1} J_1(N - 1, x) \right] f(1) + \sum_{k=1}^{N-1} \left[ \frac{1}{h_k} \left( t_{k+1} J(k, x) - J_1(k - 1, x) \right) + \frac{1}{h_{k-1}} \left( J_1(k - 1, x) - t_{k-1} J(k - 1, x) \right) \right] f(t_k) \right\} \tag{12} \]

Substituting (10) and (11) into (12) and simplifying the expressions, we arrive at (5) for finding the coefficients \( A_k(x) \) of the QFs (3). Furthermore, we can derive the Eq. (6) from (12) and (10)-(11) as follows

\[ A_k(t_k) = \frac{\sqrt{1 - x^2}}{\pi} \left\{ \frac{1}{h_k} \left( t_{k+1} J(k, t_{k+1}) - J_1(k, t_k) \right) + \frac{1}{h_{k-1}} \left( J_1(k - 1, t_k) - t_{k-1} J(k - 1, t_k) \right) \right\} = \frac{\sqrt{1 - x^2}}{\pi} \left[ J(k, t_k) - J(k - 1, t_k) - \pi (g_k - g_{k-1}) \right] = \frac{1}{\pi} \ln \frac{G_{k+1}(t_k)}{G_{k-1}(t_k)} - \sqrt{1 - t_k^2} (g_k - g_{k-1}), \ k = 1, ..., N - 1. \tag{13} \]
In order to derive the coefficients $B_k$ of the QFs (4) we use the combination of the integrals

$$J^*(k, z) = \int_{t_k}^{t_{k+1}} \frac{dt}{\sqrt{1 - t^2}(t - z)}, \quad J_1^*(k, z) = \int_{t_k}^{t_{k+1}} \frac{t \, dt}{\sqrt{1 - t^2}(t - z)}. \quad (14)$$

First $J^*(k, z)$ is computed at $z = x$, where $x$ is any number such that $|x| > 1$. Then continuing analytical function $J^*(k, x)$ along the intervals $(-\infty, -1), (1, \infty)$ on the plane of complex variable $z$ with cut along the interval $[-1, 1]$, we obtain

$$J^*(k, z) = \frac{1}{\sqrt{z^2 - 1}} \arcsin \frac{zt - 1}{z - t} \bigg|_{t=t_{k+1}}^{t=t_k} = \frac{\pi h_k}{\sqrt{z^2 - 1}} F_k(z). \quad (15)$$

For $J_1^*$

$$J_1^*(k, z) = z J^*(k, z) + g_k \pi h_k. \quad (16)$$

Replacing $x$ into $z$ in (12) and using (14)-(16), we obtain the coefficients $B_k$ of the QFs (4).

3 Estimation of errors

Let us introduce the following classes of functions:

1. $H^\alpha([-1, 1], K)$ is a class function satisfying Holder condition on the interval with the index $\alpha$ and constant $K$.
2. $C_m^\alpha[-1, 1] = \left\{ f(t) : f^{(m)} \in H^\alpha([-1, 1], K_m) \right\}$
3. $CC_{\triangle}^\alpha[-1, 1] = \left\{ f(t) : f(t) \in C[-1, 1] \text{ and } f(t) \in C^\alpha[t_k, t_{k+1}] \right\}$.
4. $W^r[-1, 1] = \left\{ f(t) : f^{(r-1)}(t) \text{ is absolutely continuous and } ess \sup_{|t| \leq 1} |f^{(r)}| = M_r \right\}.$
5. $CW_{\triangle}^2[-1, 1] = \left\{ f(t) : f(t) \in C[-1, 1] \text{ and } f(t) \in W^r[t_k, t_{k+1}] \right\}$.

Everywhere we use the notation $\|f\|_C = \|f(t)\|_{C[-1, 1]}$ as a norm of the function. Note that $M_r = \|f^{(r)}\|_C$ for any $f(t) \in C^r[-1, 1]$.

Now we prove the following theorems with respect to QFs (3) and (4).

**Theorem 1** Let $f(t)$ be a function belonging to one of the classes of functions $W^1[-1, 1], CC^1_\triangle$ or $CW^2_\triangle[-1, 1]$. Then for the errors of QFs (3) the estimations

$$\|R_N(f, x)\|_C \leq L \frac{L \ln N}{N^\beta},$$

are true for all $x \in (-1, 1)$, where $L$ and $\beta$ are given in the Table 1.
Table 1: For QFs (3)

| Classes of functions | $\beta$ | $L$ |
|----------------------|--------|-----|
| $W^1[-1,1]$          | 1      | $\frac{4\gamma M_1}{\pi} \left(1 + \frac{\pi \sqrt{2}}{2\gamma \ln N}\right)$ |
| $CC_{\Delta}^{1,\alpha}[-1,1]$ | $1 + \alpha$ | $\frac{2\gamma^{1+\alpha} K_1}{\pi} \left(1 + \frac{\pi \sqrt{2}}{2\gamma \ln N}\right)$ |
| $CW^2_{\Delta}[-1,1]$ | 2      | $\frac{\gamma^2 M_2}{\pi} \left(1 + \frac{\pi \sqrt{2}}{\gamma \ln N}\right)$ |

Remark 1: In the case of uniform grids, $\gamma = 2$.

**Theorem 2** Let $f(t)$ satisfy the conditions of Theorem 1. Then the errors of QFs (4) are

$$\max_z |R^*_N(f,z)| \leq L \frac{L\ln N}{N^\beta}, \quad L^* = \sqrt{L^2 + \left(\frac{L_1^*}{\ln N}\right)^2}$$

where $L$, $L_1$ and $\beta$ are given in the Table 2.

Next theorem is again related to QFs (3) but in different classes of functions:

**Theorem 3** Let $f(t)$ be a function belonging to one of the classes of functions

$$H^\alpha([-1,1], K), \ W^1[-1,1], \ CC_{\Delta}^{1,\alpha} \ or \ CW^2_{\Delta}[-1,1].$$

Then the error terms of QFs (3) satisfy the following estimations

$$||R_N(f,x)||_C \leq L_2 \frac{\ln N}{N^\beta},$$

for all $x \in (-1,1)$, where $L_2$ and $\beta$ are given in the Table 3.

Remark 2: Note that the main terms of $L_2$ in the Theorem 3 for the last three classes of functions is twice less than the main terms of $L$ in the Theorem 1.
In estimation of the error of QFs (3) we use the idea of [4] (see also [3]) and the following Lemmas.

**Lemma 1** Let $S_N(t)$ be linear spline (8) interpolating $f(t)$ on the grid $\Delta$, and let $t \in [t_k, t_{k+1}]$. Then for the estimate of error $r_N(f, t) = S_N(t) - f(t)$, we obtain

$$||r_N(f, t)||_C = r_N^*(h_k),$$

where $r_N^*(h_k)$ is defined by the Table (4)

| Classes of functions | $L_2$ |
|----------------------|--------|
| $H^\alpha([-1, 1], K)$ | $\alpha$ | \[
\frac{2^{2-\alpha} \gamma^\alpha K}{\pi} \left(1 + \left(2 + \frac{1}{\alpha}\right) \frac{2^{2-2\alpha}}{\gamma^\alpha \ln N}\right)\]
| $W^1[-1, 1]$ | 1 | \[
\frac{2\gamma M_1}{\pi} \left(1 + \frac{12\pi}{\gamma \ln N}\right)\]
| $CC^1,\alpha[-1, 1]$ | $1 + \alpha$ | \[
\frac{\gamma^{1+\alpha} K_1}{\pi} \left(1 + \frac{12\pi}{\gamma \ln N}\right)\]
| $CW^2_\Delta[-1, 1]$ | 2 | \[
\frac{\gamma^2 M_2}{2\pi} \left(1 + \frac{\pi \sqrt{24\pi}}{\gamma \ln N}\right)\]

Lemma 1 is proved as Theorem 2.1 which is shown in [4].

**Lemma 2** Let $S_N(t)$ be linear spline defined by (8). Then

$$r_N(f, t) \in H^1([-1, 1], \tilde{K}),$$

where $\tilde{K}$ is given in Table (5).

| Classes of functions | $L_2$ |
|----------------------|--------|
| $H^\alpha([-1, 1], K)$ | $\frac{1}{2\alpha} K h_k^\alpha$ |
| $W^1[-1, 1]$ | $\frac{1}{2} h_k$ |
| $CC^1,\alpha[-1, 1]$ | $\frac{1}{4} K_1 h_k^{1+\alpha}$ |
| $CW^2_\Delta[-1, 1]$ | $\frac{1}{8} h_k^2$ |
Table 5: Error of the linear spline (8)

| Classes of functions  | $K$          |
|-----------------------|-------------|
| $W^1[-1,1]$           | $2M_1$      |
| $CC^1_{\alpha}[-1,1]$ | $K_1 h_k^\alpha$ |
| $CW^2_{\alpha}[-1,1]$ | $M_2 h_k$  |

**Proof of the Lemma 2.** Consider three cases:

(a) $t, t' \in [t_k, t_{k+1}]$,  
(b) $|t - t'| \geq h_k$  
(c) $\tau \in [t_{k-1}, t_k]$, $t' \in [t_k, t_{k+1}]$, $|t - t'| \leq h_k$

I. Let $f(t) \in W^1[-1,1]$. Then in the case (a), using representation (8) we have

$$|r_N(f; t) - r_N(f; t')| = \frac{1}{h_k} \left| (t - t')[f(t_{k+1}) - f(t_k)] - h_k[f(t) - f(t')] \right|$$

$$= \frac{1}{h_k} \left| (t - t') \int_{t_k}^{t_{k+1}} f'(s) ds - h_k \int_{t'}^{t} f'(s) ds \right| \leq 2M_1 |t - t'|.$$ 

In the case (b), in accordance with Lemma 1, we get

$$|r_N(f; t) - r_N(f; t')| \leq |r_N(f; t)| + |r_N(f; t')| \leq h_k M_1 \leq M_1 |t - t'|.$$ 

Using the case (a), in the case (c) we obtain

$$|r_N(f; t) - r_N(f; t')| \leq |r_N(f; t) - r_N(f; t_k)| + |r_N(f; t_k) - r_N(f; t')|$$

$$\leq 2M_1 |t_k - t| + 2M_1 |t - t'| = 2M_1 |t - t'|.$$ 

II. Now let $f(t) \in CC^1_{\alpha}[-1,1]$. In the case (a)

$$|r_N(f; t) - r_N(f; t')| \leq \frac{1}{h_k} \left| (t - t')[f(t_{k+1}) - f(t_k)] - h_k[f(t) - f(t')] \right|$$

$$= \frac{1}{h_k} \left| (t - t')(t_{k+1} - t_k)f'(\theta_1) - h_k(t - t')f'(\theta_2) \right|$$

$$= \left| (t - t') \right| \left| f'(\theta_1) - f'(\theta_2) \right| \leq K_1 |t - t'| |\theta_1 - \theta_2|^\alpha$$

$$\leq K_1 h_k^\alpha |t - t'|.$$ 

In the case (b), due to Lemma 1, we have

$$|r_N(f; t) - r_N(f; t')| \leq K_1 h_k^\alpha |t - t'|.$$
It is obvious in the case (c) that
\[ |r_N(f; t) - r_N(f; t')| \leq K_1 h_k |t - t'|. \]

**III.** Let \( f(t) \in CW_2^\Delta [-1, 1] \). In case (a), we have
\[ |r_N(f; t) - r_N(f; t')| = |t - t'| |f'(\theta_1) - f(\theta_2)| \]
\[ = |t - t'| \left| \int_{\theta_1}^{\theta_2} f''(s) ds \right| \leq M_2 h_k |t - t'|. \]

The cases (b) and (c) are proved in a similar way as the case (a). So that the proof of the Lemma 2 follows from the above obtained errors.

Now it is easy to prove the following lemma.

**Lemma 3** Let \( S_N(t) \) be linear spline defined by (8) and \( f(t) \in H^\alpha([-1, 1], K) \). Then
\[ r_N(f; t) \in H^\alpha([-1, 1], 2^{2-\alpha} K). \]

**Prove of the Theorem 1.** Since
\[ \int_{-1}^{1} \frac{dt}{\sqrt{1 - x^2(t - x)}} = 0, \]
the reminder term of QFs (3) can be represented as
\[ R_N(f, x) = \frac{\sqrt{1 - x^2}}{\pi} \int_{-1}^{1} \frac{r_N(f, t) - r_N(f, x)}{\sqrt{1 - x^2(t - x)}} dt. \quad (17) \]

For definiteness, let us prove the Theorem 1 in case \( 0 \leq x \leq 1 \) (the case \(-1 \leq x \leq 0 \) is considered analogically). Fixing the number \( 0 < \delta_N < \frac{1}{2} \) and dividing the integral in (17) into three parts to yield
\[ R_N(f, x) = \frac{\sqrt{1 - x^2}}{\pi} \left( \int_{-1}^{x-\delta_N} + \int_{x+\delta_N}^{x+\delta_N} + \int_{x+\delta_N}^{1} \right) \frac{r_N(f, t) - r_N(f, x)}{\sqrt{1 - x^2(t - x)}} dt \]
\[ = \frac{\sqrt{1 - x^2}}{\pi} \left( J_1 + J_2 + J_3 \right). \quad (18) \]
First assume that \( \delta_N < 1 - x \). Then due to (10), for \( J_1 \) we have

\[
|J_1| = 2||r_N(f, x)||_C \left| \int_{x - \delta_N}^{x} \frac{dt}{\sqrt{1 - x^2(t - x)}} \right|
\]

\[
= \frac{2}{\sqrt{1 - x^2}} ||r_N(f, x)||_C \ln \left| \frac{t\sqrt{1 - x^2} - x\sqrt{1 - t^2}}{\sqrt{1 - x^2} + \sqrt{1 - t^2}} \right|_{t = x - \delta_N}.
\]

It is not hard to show that

\[
\varphi_1(x, \delta_N) = \frac{t\sqrt{1 - x^2} - x\sqrt{1 - t^2}}{\sqrt{1 - x^2} + \sqrt{1 - t^2}} \bigg|_{t = x - \delta_N}
\]

\[
= x + \frac{1}{\delta_N} \left( 1 - x^2 - \sqrt{1 - x^2} \sqrt{1 - (x - \delta_N)^2} \right).
\]

This is a function of \( x \) which strictly decreases on \([0, \frac{\delta_N}{2}]\) and strictly increases on \([\frac{\delta_N}{2}, 1]\), and

\[
\varphi_1(0, \delta_N) = \frac{\delta_N}{1 + \sqrt{1 - \delta_N^2}}, \quad \varphi_1(\delta_N/2, \delta_N) = \frac{\delta_N}{2}, \quad \varphi_1(1, \delta_N) = 1.
\]

Hence,

\[
|J_1| \leq \frac{2}{\sqrt{1 - x^2}} ||r_N(f, x)||_C \ln \frac{2}{\delta_N}.
\]  \( \text{(19)} \)

For \( J_2 \), we use Lemma 2

\[
|J_2| \leq \tilde{K} \int_{x - \delta_N}^{x + \delta_N} \frac{dt}{\sqrt{1 - t^2}} = \tilde{K} [\arcsin(x + \delta_N) - \arcsin(x - \delta_N)].
\]

Let

\[
\varphi_2(x, \delta_N) = \arcsin(x + \delta_N) - \arcsin(x - \delta_N).
\]

Since \( 0 \leq x \leq 1 - \delta_N \) and by assumption \( \delta_N \leq 1 - x \), derivative of \( \varphi_2(x, \delta_N) \) is positive and

\[
\varphi_2(x, \delta_N) \leq \varphi_2(1 - \delta_N, \delta_N) = \arcsin 2\sqrt{\delta_N(1 - \delta_N)}.
\]

From this and the known inequality \( \arcsin \alpha \leq \frac{\pi}{2} \alpha, (0 \leq \alpha \leq \frac{\pi}{2}) \) it follows that

\[
\varphi_2(x, \delta_N) \leq \pi \sqrt{\delta_N}.
\]

Hence

\[
|J_2| \leq \tilde{K} \pi \sqrt{\delta_N}.
\]  \( \text{(20)} \)
For $J_3$, we have

$$|J_3| = 2||r_N(f, x)||_C \left| \int_{x+\delta_N}^1 \frac{dt}{\sqrt{1-x^2}(t-x)} \right|$$

$$= \frac{2}{\sqrt{1-x^2}}||r_N(f, x)||_C \left| -\ln \frac{t\sqrt{1-x^2} - x\sqrt{1-t^2}}{\sqrt{1-x^2} + \sqrt{1-t^2}} \right|_{t=x+\delta_N}.$$

We may show that the function

$$\varphi_3(x, \delta_N) = \frac{t\sqrt{1-x^2} - x\sqrt{1-t^2}}{\sqrt{1-x^2} + \sqrt{1-t^2}} \bigg|_{t=x+\delta_N} = -\varphi_1(x, -\delta_N),$$

strictly increases on $[0, 1-\delta_N]$ and strictly decreases from $\frac{\delta_N}{1+\sqrt{1-\delta_N}}$ to 1. So that

$$|J_3| \leq \frac{2}{\sqrt{1-x^2}}||r_N(f, x)||_C \ln \frac{2}{\delta_N}. \quad (21)$$

It follows from the errors of (19)–(21) and (24) that

$$||R_N(f, x)||_C \leq \frac{4}{\pi}||r_N(f, x)||_C \ln \frac{2}{\delta_N} + \tilde{K} \sqrt{\delta_N}. \quad (22)$$

Now consider the case $\delta_N > 1-x$. Write

$$R_N(f, x) = \frac{\sqrt{1-x^2}}{\pi} \left( \int_{x-\delta_N}^{x+\delta_N} + \int_{-1}^{x-\delta_N} \right) \frac{r_N(f, t) - r_N(f, x)}{\sqrt{1-x^2}(t-x)} dt$$

$$= \frac{\sqrt{1-x^2}}{\pi} (J_1^* + J_2^*) \quad (23)$$

Integral $J_1^*$ is estimated as $J_1$. Due to Lemma 2

$$|J_2^*| \leq \tilde{K} \int_{x-\delta_N}^{1} \frac{dt}{\sqrt{1-t^2}} = \tilde{K} \arcsin \sqrt{1-(x-\delta_N)^2}.$$ 

Since $0 < x \leq 1$, $\delta_N > 1-x$ and due to the inequality

$$1 - (x-\delta_N)^2 = (1-x+\delta_N)(1+x-\delta_N) < 4\delta_N.$$

we obtain

$$|J_2^*| \leq \tilde{K} \arcsin 2\sqrt{\delta_N} \leq \tilde{K} \pi \sqrt{\delta_N}.$$ 

Substituting the errors of $J_1^*$ and $J_2^*$ into (23), we arrive at estimation (22).

In order to determine the errors of estimation for every classes of functions in Theorem 1, we use the results of Lemma 1 and 2 and set $\delta_N = \frac{2}{N^2}$. Viz:
I. Let \( f(t) \in W^1[-1, 1] \). Then
\[
||R_N(f, x)||_C \leq \frac{2}{\pi} M_1 h \ln \frac{2}{\delta_N} + 2M_1 \sqrt{\delta_N} \\
= \frac{4M_1 \gamma}{\pi} \left( 1 + \frac{\pi \sqrt{2}}{2\gamma \ln N} \right) \ln N.
\]

II. If \( f(t) \in CC^{1, \alpha} \triangle [-1, 1] \), then
\[
||R_N(f, x)||_C \leq \frac{K_1}{\pi} h^{1+\alpha} \ln \frac{2}{\delta_N} + 2K_1 h^\alpha \sqrt{\delta_N} \\
= \frac{2K_1 \gamma^{1+\alpha}}{\pi} \left( 1 + \frac{\pi \sqrt{2}}{2\gamma \ln N} \right) \ln N \frac{1}{N^{1+\alpha}}.
\]

III. If \( f(t) \in CW^2 \triangle [-1, 1] \), then
\[
||R_N(f, x)||_C \leq \frac{M_2 \gamma^2}{2\pi N^2} \ln \frac{2}{\delta_N} + \frac{2M_2 \gamma}{N} \sqrt{\delta_N} \\
= \frac{M_2 \gamma^2}{\pi} \left( 1 + \frac{\pi \sqrt{2}}{\gamma \ln N} \right) \ln N \frac{1}{N^2}.
\]

Theorem 1 is proved.

**Proof of the Theorem 2** is carried out by the famous scheme of the formula Sokhotskii-Plemenger (see [5]), principle maximum module for analytical function and results of Theorem 1 and Lemma 1.

**Proof of the Theorem 3.** Let the remainder term of QFs (3) be divided into three parts
\[
R_N(f, x) = \frac{\sqrt{1-x^2}}{\pi} \left( \int_{-1}^{x-\delta_N} + \int_{x+\delta_N}^{x} + \int_{x}^{1} \right) \frac{r_N(f, t) - r_N(f, x)}{\sqrt{1-x^2}(t-x)} dt \\
= \frac{\sqrt{1-x^2}}{\pi} \left( \tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3 \right). \tag{24}
\]

In the proof of Theorem 1, we have already seen that the case \( \delta_N < 1-x \), is adequate for the estimations of \( \tilde{J}_1 \) and \( \tilde{J}_3 \) i.e.
\[
|\tilde{J}_1| + |\tilde{J}_3| \leq \frac{4}{\pi} \sqrt{1-x^2} ||r_N(f, x)||_C \ln \frac{2}{\delta_N}. \tag{25}
\]

For the estimation of \( \tilde{J}_2 \), we consider the function
\[
T(x, \varepsilon, \sigma) = \sqrt{1-x^2} \int_{x-\varepsilon}^{x+\varepsilon} \frac{|t-x|^\alpha-1}{\sqrt{1-t^2}} dt, \tag{26}
\]
where \(0 < \sigma \leq 1\), \(1 - x \geq \varepsilon\), \(0 < \varepsilon < \frac{1}{2}\).

It is obvious that

\[
T(x, \varepsilon, \sigma) \leq 2\varepsilon^\sigma \int_0^1 \frac{\sqrt{1-x^2y^{\sigma-1}}}{\sqrt{1-(x+\varepsilon y)^2}} dy
\]

\[
= 2\varepsilon^\sigma \left[ \int_0^{1/2} \frac{\sqrt{1-x^2y^{\sigma-1}}}{\sqrt{1-(x+\varepsilon y)^2}} dy + \int_{1/2}^1 \frac{\sqrt{1-x^2y^{\sigma-1}}}{\sqrt{1-(x+\varepsilon y)^2}} dy \right]
\]

\[
= 2\varepsilon^\sigma [T_1(x, \varepsilon, \sigma) + T_2(x, \varepsilon, \sigma)].
\] (27)

Let \(k_0 = [1/\varepsilon]\), since \(1 - x \geq \varepsilon\), then for some \(k, 1 \leq k \leq k_0\) the inequality

\[k\varepsilon \leq 1 - x < (k + 1)\varepsilon,
\]

takes place. From this and \(0 < y \leq \frac{1}{2}\) it follows that

\[
\frac{1 - x^2}{1 - (x + \varepsilon y)^2} = \frac{(1-x)(1+x)}{(1-x-\varepsilon y)(1+x+\varepsilon y)} \leq \frac{1-x}{1-x-\varepsilon/2} \leq \frac{k+1}{k-1/2} \leq 4,
\]

for all \(k \geq 1\). Hence

\[
T_1(x, \varepsilon, \sigma) = 2 \int_0^{1/2} y^{\sigma-1} dy = \frac{2^{1-\sigma}}{\sigma}.
\] (28)

Furthermore

\[
T_2(x, \varepsilon, \sigma) = \frac{1}{2^{\sigma-1}} \int_{1/2}^1 \frac{\sqrt{1-x^2}}{\sqrt{1-(x+\varepsilon y)}} dy.
\]

Let \(\varepsilon \leq 1 - x < 2\varepsilon\). Then

\[
\frac{1 - x^2}{1 - (x + \varepsilon y)^2} \leq \frac{1-x}{1-x-\varepsilon/2} \leq \frac{2}{1-y},
\]

and therefore

\[
T_2(x, \varepsilon, \sigma) = 2^{1-\sigma} \sqrt{2} \int_{1/2}^1 (1-y)^{1/2} dy = 2^{2-\sigma}.
\] (29)

If for some \(k \leq 2\) the inequality

\[k\varepsilon \leq 1 - x \leq (k + 1)\varepsilon
\]
takes place, then
\[
\frac{1 - x^2}{1 - (x + \varepsilon y)^2} \leq \frac{1 - x}{1 - x - \varepsilon y} \leq \frac{k + 1}{k - 1} \leq 3,
\]
therefore
\[
T_2(x, \varepsilon, \sigma) \leq 2^{1-\sigma} \sqrt{3} \cdot \frac{1}{2} < 2^{2-\sigma}.
\]  
(30)

From (27)-(30) it follows that
\[ T(x, \varepsilon, \sigma) \leq 2^{1-\sigma} \left( 2 + \frac{1}{\sigma} \right) \varepsilon^\sigma. \]  
(31)

Now for \( \tilde{J}_2 \) we have
\[
\tilde{J}_2 \leq \frac{K}{\pi} \sqrt{1 - x^2} \int_{x-\delta_N}^{x+\delta_N} \frac{|t - x|^{\sigma-1}}{\sqrt{1 - t^2}} = \frac{K}{\pi} T(x, \varepsilon, \sigma),
\]  
(32)

where \( \sigma = \alpha \) for the class \( H^\alpha([-1, 1], K) \) and \( \sigma = 1 \) for rest classes of functions. Assuming \( \sigma = \frac{2}{N} \) and from the Lemmas 2, 3 and inequalities (25) and (31)-(32) we get the assertion of the Theorem 3.

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