CONFORMAL NET REALIZABILITY OF TAMBARA–YAMAGAMI CATEGORIES AND GENERALIZED METAPLECTIC MODULAR CATEGORIES

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This paper is dedicated to Karl-Henning Rehren on the occasion of his 60th birthday

Abstract. We show that all isomorphism classes of even rank Tambara–Yamagami categories arise as \( \mathbb{Z}_2 \)-twisted representations of conformal nets. As a consequence, we show that their Drinfel’d centers are realized by (generalized) orbifolds of conformal nets associated with (self-dual) lattices. The quantum double subfactors of even rank Tambara–Yamagami categories are Bisch–Haagerup subfactors and we describe their (dual) principal graphs.

For every abelian group of odd order the Drinfel’d centers of the associated Tambara–Yamagami categories give a fusion ring generalizing the Verlinde ring \( \text{Spin}(2n+1) \) in the case of \( \mathbb{Z}_{2n+1} \). We classify all generalized metaplectic modular categories, i.e. unitary modular tensor category with those fusion rules and show that they are realized as \( \mathbb{Z}_2 \)-orbifolds of conformal nets associated with lattices. We further show that twisted doubles of generalized dihedral groups of abelian groups of odd order are group-theoretical generalized metaplectic modular categories and vice versa.

We give some examples of twisted orbifolds of conformal nets and show how generalized metaplectic modular categories arise by condensation of simpler ones.

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1. Introduction

Local conformal nets on the circle axiomatizing chiral conformal field theory using von Neumann algebras. The local algebras turn out to be factors, i.e. von Neumann algebras with trivial centers and conformal nets give rise to subfactors, i.e. unital inclusion of factors in many different ways. Via their representation theory, conformal nets also give rise to (braided) $C^*$-tensor categories.

In particular, in [KLM01], Kawahigashi, Longo, and Müger have introduced the notion of a completely rational conformal net and showed that the representation category of such a net is a unitary modular tensor category. A natural question is if all unitary modular tensor categories arise this way in the following sense: Given any abstract unitary modular tensor category $C$, is there always a completely rational conformal net $\mathcal{A}$ which realizes $C$, i.e. $\text{Rep}(\mathcal{A})$ is braided equivalent to $C$? Finding a solution for a family of categories goes under the name of reconstruction program. One difficulty is that such a net $\mathcal{A}$ realizing $C$ (if it exists) is far from unique. For example, every net $\mathcal{A} \otimes \mathcal{B}$ with $\mathcal{B}$ a holomorphic net, i.e. a completely rational conformal net with trivial representation category $\text{Rep}(\mathcal{B}) \cong \text{Hilb}$, also realizes $C$. Every even self-dual positive lattice $\Gamma$ gives a holomorphic net $\mathcal{A}_\Gamma$ but the classification of even self-dual positive lattices itself is a hopeless problem.

If $\mathcal{F}$ is a unitary fusion category, which means that it does not need to be braided, then its Drinfel’d center $Z(\mathcal{F})$ is a unitary modular tensor category [Müg03b]. In this case, we may ask as—a special case of the reconstruction program—if all Drinfel’d centers of fusion categories are realized by conformal nets, i.e. if for any unitary fusion category $\mathcal{F}$ there is a completely rational conformal net $\mathcal{A}$ which realizes $Z(\mathcal{F})$. A positive answer to this question would imply that all unitary fusion categories and all finite index finite depth subfactors arise from conformal nets using higher representation theory [Bis16b, Bis16a].

If $\mathcal{A}$ is a net realizing $Z(\mathcal{F})$ for some unitary fusion category $\mathcal{F}$, we call $\mathcal{A}$ a quantum double net. In this case, there is a holomorphic conformal net $\mathcal{B}$, and a proper action of the fusion ring hypergroup $K_\mathcal{F}$ on $\mathcal{B}$, such that $\mathcal{A}$ is the fixed point net $\mathcal{B}^{K_\mathcal{F}}$ [Bis17]. The unitary fusion category $\mathcal{F}$ can be reconstructed as the category of solitons coming from $\alpha$-induction applied to the inclusion $\mathcal{A} \subset \mathcal{B}$. Conversely, if a finite hypergroup $Q$ acts properly on a holomorphic net $\mathcal{B}$, then the fixed point net $\mathcal{A}^Q$ is a quantum double net with $\text{Rep}(\mathcal{A}^Q)$ braided equivalent to $Z(\mathcal{F})$ for a unitary fusion category $\mathcal{F}$ which is a categorification of $Q$, i.e. $K_\mathcal{F} \cong Q$.

The main goal of this paper is to establish quantum double nets and therefore a reconstruction for a certain family of unitary fusion categories, namely so-called Tambara–Yamagami categories of even rank.
Namely, we prove that for every unitary fusion category $\mathcal{F}$ with $\text{Irr}(\mathcal{F}) = G \cup \{\rho\}$ for some finite group $G$ of odd order and fusion rules\footnote{these fusion rules are also denoted by $G + 0$ in \cite{EG14}}

\[
[\rho][\rho] = \sum_{g \in G} [g], \qquad [g][\rho] = [\rho][g] = [\rho], \qquad [g][h] = [gh],
\]

there is a net $\mathcal{A}$ with $\text{Rep}(\mathcal{A})$ braided equivalent to $Z(\mathcal{F})$. We note that Tambara–Yamagami categories are classified \cite{TY98} and that $G$ is necessarily abelian.

It is conjectured that unitary fusion categories are coming from models in low-dimensional physics. In particular, it is believed, that one can use the data to obtain a model in statistical physics whose critical limit is a conformal field theory.

One might ask:

**Question 1.1.** Where do Tambara–Yamagami categories come from?

One of the easiest rational CFT models are in physical language sigma models or chiral Wess–Zumino–Witten models with target space a (metric) torus, i.e. euclidean space $\mathbb{R}^n$ compactified by a necessarily even positive lattice $L$. Here $L$ can be seen as the level $H^4_+ (B^n, \mathbb{Z})$ for the $n$-torus $\mathbb{T}^n$, see \cite{Hen17a}. These models can be described by a conformal net $\mathcal{A}_L$.

Indeed, the reconstruction of Tambara–Yamagami categories can be done by taking $\mathbb{Z}_2$-orbifolds of the nets $\mathcal{A}_L$ and we can give an answer to Question 1.1 for the case of even rank:

**Theorem 1.2** (Theorem 3.8). Tambara–Yamagami categories of even rank arise as the category of $\mathbb{Z}_2$-twisted representations of a net $\mathcal{A}_L$.

Using our reconstruction result, we can make several interesting observations and connections. We introduce the notion of a generalized metaplectic modular category associated with an abelian group $A$ of odd order which is a unitary modular tensor categories with fusion rules depending on $A$ which generalizes the Verlinde fusion rules of $\text{Spin}(2k+1)_2$ for the case $A = \mathbb{Z}_{2k+1}$, see Definition 4.4. We generalize the classification of metaplectic modular categories up to braided equivalence in \cite{ACRW16} to generalized metaplectic modular categories, see Theorem 4.8, 4.11 and show that generalized metaplectic modular categories are determined by its modular data, see Theorem 4.12. Again, we have a similar reconstruction result:

**Theorem 1.3** (Theorem 4.10). All odd generalized metaplectic modular categories are be realized by $\mathbb{Z}_2$-orbifolds of conformal nets associated with even lattices.

We can use the classification to show that $\text{Rep}(\mathcal{A}_{\text{Spin}(2p+1)_2})$ of the loop group net $\mathcal{A}_{\text{Spin}(2p+1)_2}$ is braided equivalent to the unitary modular tensor category $\mathcal{C}_{\text{Spin}(2p+1)_2}$ for all $p \in \mathbb{N}$ and is modular in the sense of \cite{KL05}.

We give some results on how (generalized) metaplectic modular categories can be obtained from simpler building blocks via condensation, i.e. as the category of local modules with respect to a commutative $Q$-system. We show how metaplectic modular categories can often be realized as condensation of $\text{Spin}(p)_2$, their reverses, and semion categories. This is always possible if no prime factor of $n$ is a Pythagorean prime.

In Section 5, we give several relations of generalized metaplectic modular categories and Tambara–Yamagami categories to generalized dihedral groups, i.e. groups of the form $\text{Dih}(A) := A \rtimes \mathbb{Z}_2$ for some abelian group $A$, where $\mathbb{Z}_2$ acts by $a \mapsto a^{-1}$.

We introduce a generalization of Tambara–Yamagami categories in Subsection 5.1. They are extensions of pointed unitary fusion categories of generalized dihedral groups and are Morita equivalent to generalized metaplectic modular categories.
In Subsection 5.2 we show that any twisted quantum double of a generalized dihedral group $\text{Dih}(A)$ with $|A|$ odd is a generalized metaplectic modular category. Conversely, a generalized metaplectic modular category which is the Drinfel’d center of a unitary fusion category is braided equivalent to the twisted double of a generalized dihedral group. In particular, a generalized metaplectic modular category is group theoretical if and only if it is Witt trivial, the Drinfel’d center of a unitary fusion category.

The Longo–Rehren subfactor associated with (odd) Tambara–Yamagami categories are Bisch–Haagerup subfactors $M^H \subset M \rtimes K$. We determine the associated $G$-kernel for $G = \langle H, K \rangle$ and describe the principal and dual principal graphs, see Subsection 5.3.

We show that Drinfel’d center of Tambara–Yamagami categories based on an abelian group $A$ of odd order can be realized as $\mathbb{Z}_2$-twisted orbifolds of $\text{Dih}(A)$-fixed point nets of holomorphic nets associated with even self-dual lattices, see Subsection 5.4.

As an outlook, and a posterior motivation of this work, we remark that the just mentioned result implies that the two modular data from the unitary modular tensor categories $\mathcal{C}_{\text{Spin}(2n+1)}$ and the (twisted) quantum double of $D_m$ for $m$ odd, respectively, which serve as an ingredient for a grafting in $[\text{EG}14]$ are both the modular data of generalized metaplectic modular categories. In particular, the modular data $[\text{EG}14]$ Proposition 7b) which conjectural is the modular data for $G + n$ near group categories $[\text{EG}14]$ with $n = |G|$ factorizes as $(S, T) = (S' \otimes S_{(G,q)}, T' \otimes T_{(G,q)})$, where $(S_{(G,q)}, T_{(G,q)})$ is the Weil representation associated with $G$ and a non-degenerate quadratic form $q$, or equivalently the modular data of a pointed UMTC $\mathcal{C}(G,q)$. The (interesting) factor $(S', T')$ can be thought to be build of the modular data of two generalized metaplectic modular categories associated with groups $G$ and $G'$, respectively, where $|G'| = |G| + 4$. For example, for $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ and $G' = \mathbb{Z}_{13}$ one gets the modular data $(S', T')$ of a factor of the quantum double of a $\mathbb{Z}_3 \times \mathbb{Z}_3 + 9$ near group category. The generalized metaplectic modular category associated with $G'$ is a twisted double of $S_3 \cong \text{Dih}(\mathbb{Z}_3 \times \mathbb{Z}_3)$. This modular data also corresponds to the double of the even part of the Haagerup subfactor, see also [\text{Bis}17] Remark 5.14. The argument can be generalized to other near group categories, and near group categories with a certain Lagrangian correspond to Izumi–Haagerup categories.

In Section 6 we give a relation of the constructed models to generalized orbifolds and topological defects. We show that some of the examples allow to construct twisted orbifolds by generalized dihedral groups, see Subsection 6.1. We briefly discuss the harmonic analysis for actions of the Tambara–Yamagami fusion hypergroup on the obtained models. Finally, we discuss generalized Kramers–Wannier dualities as in [\text{FFRS}04] using the work [\text{BKL}R16]. Namely, let $\mathcal{F}$ be a fusion category, then following [\text{FFRS}04] a simple object $X$ gives rise to a duality defect if every simple subobject of $X \otimes X$. Therefore the generating object of a Tambara–Yamagami category describes a generalized Kramers–Wannier duality. We explain this in the setting of local conformal nets in Subsection 6.3.

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2. Preliminaries

2.1. Unitary fusion categories and unitary modular tensor categories. A unitary fusion category $\mathcal{F}$ is a rigid $C^*$-tensor category with simple tensor unit $1$, i.e. $\text{Hom}(1, 1) \cong \mathbb{C}$, such that
the Frobenius–Perron dimension of equation is called conjugate equation \([LR97]\).

Because of its spherical structure a braided unitary fusion category is a premodular category, i.e. a unitary modular tensor category (UMTC)\([Reh90, Müg03c]\). For \(\mathcal{C}\) a braided fusion category, we denote by \(\mathcal{C}^{\text{rev}}\) the braided fusion category with the opposite braiding \(\varepsilon^{-}(\rho, \sigma) = \varepsilon(\sigma, \rho)^{*}\).

Let \(C\) be a braided unitary fusion category. For a full subcategory \(\mathcal{D}\) we define the Müger centralizer
\[
\mathcal{D}' \cap \mathcal{C} = \{\rho \in \mathcal{C} : \varepsilon(\rho, \sigma) = \varepsilon(\sigma, \rho)^{*} \text{ for all } \sigma \in \mathcal{D}\}.
\]

The category \(\mathcal{C}\) is called non-degenerately braided if the Rehren–Müger center \(\mathcal{C}' \cap \mathcal{C}\) is equivalent to the trivial (braided) fusion category, i.e. \(\mathcal{C}' \cap \mathcal{C} \cong \text{Hilb}\). A non-degenerate braided unitary fusion category \(\mathcal{C}\) is a unitary modular tensor category (UMTC)\([Reh90, Müg03c]\).

From every unitary fusion category \(\mathcal{F}\), we get the unitary modular tensor category \(Z(\mathcal{F})\), the (unitary) Drinfel’d center of \(\mathcal{F}\). A braided unitary fusion category is a unitary modular tensor category if and only if \(Z(\mathcal{C})\) is braided equivalent to \(\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}\)[Müg03b, Corollary 7.11].

We say two unitary modular tensor categories \(\mathcal{C}\) and \(\mathcal{D}\) are (unitarily) Witt equivalent if there are unitary fusion categories \(\mathcal{F}\) and \(\mathcal{G}\), such that \(\mathcal{C} \boxtimes Z(\mathcal{F})\) is braided equivalent to \(\mathcal{D} \boxtimes Z(\mathcal{G})\). Witt equivalence is an equivalence relation and the equivalence classes form an abelian group under the multiplication \([\mathcal{C}][\mathcal{D}] = [\mathcal{C} \boxtimes \mathcal{D}]\) with identity [Hilb] = \([Z(\mathcal{F})]\) and inverse \([\mathcal{C}]^{-1} = [\mathcal{C}^{\text{rev}}]\), see [DMNO13].

Let \(\mathcal{C}\) be a unitary braided fusion category of rank \(n = |\text{Irr}(\mathcal{C})|\). We define for \(\lambda, \mu \in \text{Irr}(\mathcal{C})\)
\[
Y_{\lambda\mu} = \text{tr}_{\lambda \otimes \mu} \left[ \varepsilon(\lambda, \mu)^{*} \varepsilon(\mu, \lambda) \right] = \ild{\lambda} \mu \; ,
\]
\[
\omega_{\lambda} = (d\lambda)^{-1} \cdot \text{tr}_{\lambda \otimes \lambda} \left[ \varepsilon(\lambda, \lambda) \right] \; , \; \omega_{\lambda} \cdot \text{id}_{\lambda} = \ild{\lambda} \; ,
\]
and the following \(n \times n\)-matrices
\[
S_{\lambda\mu} = (\dim N_{\mathcal{C}})^{-\frac{1}{2}} Y_{\lambda\mu} \; , \quad T_{\lambda\mu} = e^{-\pi i c_{\text{top}}/12} \delta_{\lambda\mu} \omega_{\lambda} \; ,
\]
where the topological central charge \( c_{\text{top}} \equiv c_{\text{top}} \mod 8 \) is defined by

\[
c_{\text{top}} = \frac{4 \arg(z)}{\pi} \quad \text{where} \quad z = \sum_{\rho \in \text{Irr}(\mathcal{C})} (d\rho)^2 \cdot \omega_\rho.
\]

The matrices \( S \) and \( T \) obey the relations of the partial Verlinde modular algebra: \( TSTST = S \), \( CTC = T \), and \( CSC = S \) \cite{Reh90, BEK99}, where \( C_{\rho\rho'} = \delta_{\rho,\rho'} \) is the charge conjugation matrix. The matrix \( S \) is unitary if and only if \( \mathcal{C}' \cap \mathcal{C} \) is trivial \cite{Reh90}, i.e. \( \mathcal{C} \) is a unitary modular tensor category. In this case, we have \( (ST)^3 = C = S^2 \), thus \( (S,T) \) gives a unitary representation of \( \text{SL}(2,\mathbb{Z}) \) on \( \mathcal{C}' \cap \text{Irr}(\mathcal{C}) \).

The pair \( (S,T) \) is called the modular data associated with a unitary modular tensor category \( \mathcal{C} \) and we have the Verlinde formula \cite{Reh90}:

\[
N_{i,j}^k = \sum_{\ell} \frac{S_{j,\ell} S_{i,\ell} S_{k,\ell}}{S_{0,\ell}}.
\]

We say two modular data \( (S,T) \) and \( (S',T') \) are equivalent, if there is a \( 3 \) a third root of unity \( \zeta \) and a bijection \( \sigma \) from the index set of \( (S,T) \) to the index set of \( (S',T') \) which fixes first element (corresponding to \( 1 \) of the index set, such that \( S'_{\sigma(i),\sigma(j)} = S_{i,j} \) and \( T'_{\sigma(i),\sigma(j)} = \zeta T_{i,j} \) for all possible indices. The modular data associated with \( \mathcal{C} \) up to equivalence is an invariant of the modular tensor category \( \mathcal{C} \).

Let \( \mathcal{C} \) be a unitary modular tensor category and \( \rho \in \text{Irr}(\mathcal{C}) \). The Frobenius–Schur indicator \( \nu_\rho \) is in terms of \( (S,T) \) by Bantay’s formula \cite{Ban97, GR16}:

\[
\nu_\rho = \sum_{\sigma, \tau} S_{\sigma,1} S_{\tau,1} N_{\sigma,\tau}^0 T_{\sigma,\tau}^2 \in \{0, \pm 1\}.
\]

2.2. Pointed unitary fusion categories. A unitary fusion category \( \mathcal{F} \) is called pointed if every irreducible/simple object \( \rho \in \mathcal{C} \) is invertible, i.e. has dimension \( d\rho = 1 \). Then \( G = \text{Irr}(\mathcal{C}) \) forms a finite group under the multiplication \( [\rho][\sigma] = [\rho \otimes \sigma] \) and \( \mathcal{F} \) is tensor equivalent to the category of \( G \)-graded finite-dimensional Hilbert spaces \( \text{Hilb}_G^\omega \) for some \( \omega \in Z^3(G, \mathbb{T}) \), where the associator of simple objects \( H_\rho \equiv \mathcal{C} \) is given by

\[
(H_\rho \otimes H_h) \otimes H_k \xrightarrow{\omega(g,h,k)} H_{\rho} \otimes (H_h \otimes H_k).
\]

Up to tensor equivalence this only depends on the class \( [\omega] \in H^3(G, \mathbb{T}) \), see \cite{ENO10}. A strict model for \( \text{Hilb}_G^\omega \) is \( \langle \alpha_g : g \in G \rangle \subseteq \text{End}_G(N) \) for a type III factor \( N \) and a \( G \)-kernel, i.e. map \( \alpha : G \to \text{Aut}(N) \) which is a lift of a homomorphism \( \chi : G \to \text{Out}(N) \) having obstruction \( [\omega] \), cf. e.g. \cite{Sin80, Jon80, IK02}.

Example 2.1. For a pointed unitary fusion category with two objects the Frobenius–Schur indicator of the generator is a complete invariant, namely for \( \text{Hilb}_{\mathbb{Z}_2}^{\omega,\pm} = \langle \alpha_0 = \text{id}, \alpha_1 \rangle \) with \( H^3(\mathbb{Z}_2, \mathbb{T}) = \{[\omega_+] = 0, [\omega_-] \equiv \mathbb{Z}_2 \} \) we have \( \nu_{\alpha_0} = \pm 1 \), respectively.

Let \( \mathcal{C} \) be a pointed unitary modular tensor category. Because of the braiding \( G = \text{Irr}(\mathcal{C}) \) is a finite abelian group. The category \( \mathcal{C} \) is up to braided equivalence characterized by a cohomology class \( [(\omega, c)] \in H^3_{\text{ab}}(G, \mathbb{T}) \) in the abelian Eilenberg–MacLane cohomology. It is a fact that \( [(\omega, c)] \) is determined by the quadratic form \( q(a) = c(a,a) \), which equals the twist, see e.g. \cite{EGN15}. We use the convention \( q([\rho])_{1_{R \otimes R}} = \varepsilon(\rho, \rho) \) for \( [\rho] \in \text{Irr}(\mathcal{C}) \).

A map \( b : G \times G \to \mathbb{T} \) with \( b(g,h) = b(h,g) \) and \( b(g+h,k) = b(g,k)b(h,k) \) and \( b(g, \cdot) \equiv 1 \) if and only if \( g = 0 \), is called a non-degenerate symmetric bicharacter.

Let us see \( G \) as an additive group and let \( \partial q(a, b) = q(a)q(b)q(a + b)^{-1} \). Then \( \partial q \) is a non-degenerate symmetric bicharacter. In this case, we call the quadratic form \( q \) non-degenerate.
Conversely, we call a map $q: G \rightarrow \mathbb{T}$ with $q(na) = q(a)^{n^2}$ for every $n \in \mathbb{Z}$, such that $\partial q$ is a non-degenerate symmetric bicharacter a non-degenerate quadratic form on $G$.

Therefore, every pointed unitary modular tensor category is characterized by the pair $(G, q)$ and we denote such a pointed unitary modular tensor category by $\mathcal{C}(G, q)$.

A pair $(G, q)$ of a finite abelian group $G$ and a non-degenerate quadratic form $q$ on $G$ is called a metric group. We say two metric groups $(G, q)$ and $(G', q')$ are equivalent if there is an isomorphism $\phi: (G, q) \rightarrow (G', q')$, i.e. an isomorphism $\phi: G \rightarrow G'$, such that $q = q' \circ \phi$. In this case, we write $(G, q) \sim (G', q')$. We note that $\mathcal{C}(G, q)$ and $\mathcal{C}(G', q')$ are braided equivalent if and only if $(G, q) \sim (G', q')$. For $\{(G_i, q_i) \colon i = 1, 2\}$ two metric groups we define their direct sum to be the metric group $(G_1, q_1) \oplus (G_2, q_2) = (G_1 \oplus G_2, q_1 \oplus q_2)$ with $(q_1 \oplus q_2)(g_1, g_2) = q_1(g_1)q_2(g_2)$.

**Example 2.2.** The UMTC $\mathcal{C}_{SU(n+1)}$ is pointed and braided equivalent to $\mathcal{C}(\mathbb{Z}_{n+1}, q)$ with $q(x) = \exp\left(\frac{\pi i}{n+1} x^2\right)$. In particular, for $\mathcal{C}_{SU(2)} = (\alpha) \cong \mathcal{C}(\mathbb{Z}_2, q)$ with $[\alpha^2] = [\text{id}]$ we have $\nu_{\alpha} = -1$, since $q(x) = \exp(\pi i x^2/2)$ does not come from a bicharacter and using $[\text{LN14}]$ Lemma 4.4.

The modular data of $\mathcal{C}(G, q)$ is given by

$$S_{g,h} = \frac{1}{\sqrt{|G|}}b(g,h), \quad T_{g,h} = \delta_{g,h}e^{-\pi i c_{\text{top}}/12}q(a),$$

where the topological central charge $c_{\text{top}} \equiv c_{\text{top}} \pmod{8}$ is determined via a Gauß sum:

$$e^{\pi i c_{\text{top}}/4} = \frac{1}{\sqrt{|G|}} \sum_{g \in G} q(g).$$

This modular data $(S, T) = (S_{(G,q)}, T_{(G,q)})$ is sometimes called the Weil representation associated with $(G, q)$.

### 2.3. Bicharacters and odd rank pointed unitary modular tensor categories.

Let $G$ be a finite abelian group of odd order and $\langle \cdot, \cdot \rangle: G \times G \rightarrow \mathbb{T}$ a non-degenerate symmetric bicharacter on $G$. Define $q: G \rightarrow \mathbb{T}$ by $q(g) = \langle g, g \rangle^{-1}$ and $\partial q: G \times G \rightarrow \mathbb{T}$ by $\partial q(g, h) = q(g)q(h)q(g + h)^{-1}$ as above. Then $\partial q(g, h) = (g, h)^2$ and $\partial q$ is a non-degenerate symmetric bicharacter, since $|G|$ is odd. In particular, $q$ is a non-degenerate quadratic form on $G$. Again since $|G|$ is odd $\langle \cdot, \cdot \rangle$ is determined by $q$.

Conversely, every non-degenerate quadratic form $q$ gives a non-degenerate symmetric bicharacter $\langle \cdot, \cdot \rangle$ with $q(g) = \langle g, g \rangle^{-1}$. Namely, $(g, h) := \partial g(g, h)\frac{1}{2}(\text{Exp}(G) + 1)$ where $\text{Exp}(G)$ is the exponent of $G$. Since $G$ is odd we can define $g' = \frac{1}{2}(\text{Exp}(G) + 1) \cdot g$ with $g = 2g'$ for every $g \in G$. We note that in general $q$ must take values in $\mathbb{T}_{2\text{Exp}(G)}$, where $\mathbb{T}_n = \{z \in \mathbb{T} : z^n = 1\}$, but since $|G|$ is odd we have $q(g) = q(g')^4$ and $q$ actually takes values in $\mathbb{T}_{\text{Exp}(G)}$. Then $q(g)^{-1} = \partial q(g, g') = \partial q(g, g)\frac{1}{2}(\text{Exp}(G) + 1)$ and therefore $\langle g, g \rangle = q(g)^{-1}$. Finally, $\langle \cdot, \cdot \rangle$ is a symmetric bicharacter and non-degenerate since $\partial q$ is non-degenerate.

The following lemma will be useful.

**Lemma 2.3.** Let $\mathcal{C}$ be a pointed UMTC of odd rank, i.e. $\mathcal{C}$ braided equivalent to $\mathcal{C}(G, q)$ for an odd metric group $(G, q)$. Then the corresponding $[\omega] \in H^3(G, \mathbb{T})$ is trivial. In particular, $\mathcal{C}$ is tensor equivalent to $\text{Hilb}_G$.

**Proof.** As above there is a symmetric bicharacter $\langle \cdot, \cdot \rangle$ with $q(g) = \langle g, g \rangle^{-1}$ and the cocycle must therefore be trivial by $[\text{LN14}]$ Lemma 4.4. \qed
2.4. Q-systems. Let \( F \) be a unitary fusion category. An (irreducible) Q-system \( \Theta = (\theta, w, x) \) in \( F \) is a triple \( \theta \in F \) with \( \dim \text{Hom}(1, \theta) = 1 \), and isometries \( w \in \text{Hom}(1, \theta) \) and \( x \in \text{Hom}(\theta, \theta \otimes \theta) \), such that \((x \otimes \text{id}_\theta) \circ x = (\text{id}_\theta \otimes x) \circ x \) and \( x^* \circ (w \otimes \text{id}_\theta) = x^* \circ (\text{id}_\theta \otimes w) = \delta^{-1} \text{id}_\theta \) where \( \delta = \sqrt{\text{det} w} \). In other words, \( \theta \) is an algebra object with unit \( e = \delta^{-1/2} w \) and associative multiplication \( \mu = \delta^{-1/2} x^* \). A Q-system \( \Theta \) in a braided fusion category \( C \) is called commutative if \( x = \varepsilon(\theta, \theta) x \), see [BKLR15, BKL15]. If \( \Theta \) is a Q-system in a unitary fusion category \( F \), then \( F \) and \( \Theta \) are (weakly monoidally) Morita equivalent and \( Z(F) \) is braided equivalent to \( Z(\Theta) \) [Müg03a, Müg03b].

We denote by \( F_\Theta \) the category of right \( \Theta \)-modules, see [BKLR15, BKL15]. If \( C \) is braided and \( \Theta \) is commutative, then \( C_\Theta \) has the structure of a fusion category and the category of local modules \( C^0_\Theta \) has the structure of a braided fusion category. Let us assume that \( C \) is a modular tensor category, then \( C_\Theta \) is a braided, then \( C_\Theta \) has the structure of a fusion category and the category of \( \Theta \)-modules, see [BKLR15, BKL15]. If \( C \) is braided and \( \Theta \) is commutative, then \( C_\Theta \) has the structure of a fusion category and the category of local modules \( C^0_\Theta \) has the structure of a braided fusion category. Let us assume that \( C \) is a modular tensor category, then \( C_\Theta \) is a braided, then \( C_\Theta \) has the structure of a fusion category and the category of \( \Theta \)-modules, see [BKLR15, BKL15]. If \( C \) is braided and \( \Theta \) is commutative, then \( C_\Theta \) has the structure of a fusion category and the category of local modules \( C^0_\Theta \) has the structure of a braided fusion category. Let us assume that \( C \) is a modular tensor category, then \( C_\Theta \) is a braided, then \( C_\Theta \) has the structure of a fusion category and the category of local modules \( C^0_\Theta \) has the structure of a braided fusion category.

We say that an injective tensor functor \( \iota : D \to F \) from a braided unitary fusion category \( D \) to a unitary fusion category \( F \) is central, if there is a braided functor \( \iota : D \to Z(F) \), such that \( \iota = F \circ \iota \), i.e. the following diagram commutes

\[
\begin{array}{ccc}
Z(F) & \xrightarrow{\iota} & F \\
\downarrow & & \downarrow \ \\
D & \xrightarrow{\iota} & F
\end{array}
\]

Here \( F : Z(F) \to F \) denotes the forgetful functor.

A braided unitary fusion category is called symmetric if \( C' \cap C = C \). Let \( G \) be a finite group and consider the symmetric unitary fusion category \( \text{Rep}(G) \) of finite dimensional unitary representations of \( G \) with the usual tensor product. The regular representation defines a canonical Q-system \( \Theta_G \in \text{Rep}(G) \). If \( F \) has a Tannakian subcategory, i.e. if there is a central functor \( \iota : \text{Rep}(G) \to F \), then we denote \( F_G := F_{\iota(\Theta_G)} \), which is also called the de-equivariantization of \( F \).

If \( C \) is a unitary modular tensor category and \( \iota : \text{Rep}(G) \to C \) braided, then \( C_G \) is a G-crossed braided extension of the unitary modular tensor category \( C^0_G \) which we take for the purpose of this article as a definition, cf. M"uger’s characterization [Tur10, Appendix 5, Thm 4.1]. This characterization also says that every G-crossed braided category is of the form \( C_G \). We refer to [Müg03, 8.119, 8.123] for the definition of a G-crossed braided category. We note that in this case the inclusion has the structure of a central functor \( \iota : (C^0_G)^{\text{rev}} \to C_G \). Also the following converse is true, which is probably well-known to experts.

Proposition 2.4. Let \( D \) be a unitary modular tensor category and \( F = \bigoplus_g F_g \) a (faithfully) \( G \)-graded extension of \( F_e = D \) together with a central structure \( \iota : D^{\text{rev}} \to F \) on the canonical inclusion functor. Then \( F \) is a G-crossed braided extension of \( D \).

Proof. By [Bis17, Prop. 5.12] there is a commutative Q-system \( \Theta \) in \( C := \iota(D^{\text{rev}})^{\text{rev}} \cap Z(F) \), such that \( C_\Theta \) is equivalent to \( F \) and \( C^0_\Theta \) is braided equivalent to \( C \). Similarly to the proof of [Bis17, Prop. 5.17] it follows that \( \Theta \) is the Q-system of a group subfactor \( M_G \subset M \) and since \( \Theta \) is commutative we have a braided functor \( \iota : \text{Rep}(G) \to C \), such that \( C^0_G \) is braided equivalent to \( D \) and \( C_G \cong F \). \( \square \)

2.5. Even lattices. A (positive) even lattice is a finitely generated free abelian group \( L \) with a positive definite inner product \( \langle \cdot, \cdot \rangle : L \times L \to \mathbb{R} \), such that \( \langle x, x \rangle \in 2\mathbb{N} \) for all \( x \in L \setminus \{0\} \). Then \( V_L = L \otimes \mathbb{Z} \mathbb{R} \) is an euclidean space, with inner product \( \langle \cdot, \cdot \rangle \) defined by bilinear continuation. The number of generators, or equivalently \( \dim(V_L) \), is called the rank of \( L \). The dual lattice is \( L^* = \{ x \in V_L : \langle x, L \rangle \subset \mathbb{Z} \} \cong \text{Hom}(L, \mathbb{T}) \). Since \( L \) is even, we have \( L \subset L^* \) and \( G_L = L^*/L \) is a finite abelian group with non-degenerate quadratic form \( q_L : G_L \to \mathbb{T} \) given by \( q_L(x) = \exp(\pi i \langle x, x \rangle) \).
Thus every even lattice \( L \) gives a metric group \((G_L, q_L)\), which is called the discriminant group of \( L \). If \( H \leq G_L \) is an isotropic subgroup, i.e. \( q_L | H \equiv 1 \), we get a new even lattice \( L \oplus H = \{ x + h : x \in L, [h] \in H \} \). This gives a one-to-one correspondence between overlattices \( M \supset L \) and isotropic subgroups \( H \leq G_L \) given by \( H \mapsto M := L \oplus H \supset L \) and \( M \mapsto H := M/L \leq G_L \). A lattice \( L \) is called self-dual if \( L^* / L \) is trivial. If \( L \) and \( M \) are even lattices then \( LM := L \times M \) is an even lattice with \( G_{LM} = G_L \times G_M \) and \( q_{LM} = q_L \times q_M \). We say that an even lattice \( L \) is a mirror of \( L \) if there is an isomorphism \( \phi : (G_L, q_L) \to (G_L, q_L) \). Then we get a self-dual lattice \( \Gamma = LL \oplus \Delta^0(G_L) \) via the isotropic “diagonal” subgroup \( \Delta^0(G_L) = \{ (x, \phi(x)) \in G_L \times G_L : x \in G_L \} \leq G_L \times G_L \).

We will often refer to the \( A_n, E_{6,7,8} \) root lattices whose discriminant groups are listed in Table 1. We note that \( E_7 \) and \( E_6 \) are mirrors of \( A_1 \) and \( A_2 \), respectively. There is a unique over lattice \( A_1E_7 \oplus Z_2 \supset A_1E_7 \) and two overlattices \( A_2E_6 \oplus Z_3 \supset A_2E_6 \) which are both isomorphic to \( E_8 \).

| \( L \) | \( G_L \) | \( q_L \) |
|---|---|---|
| \( A_n \) | \( Z_{n+1} \) | \( \exp \frac{\pi i x^2}{x+1} \) |
| \( E_6 \) | \( Z_3 \) | \( \exp \frac{\pi i x^2}{3} \) |
| \( E_7 \) | \( Z_3 \) | \( \exp \frac{\pi i x^2}{2} \) |
| \( E_8 \) | \( Z_1 \) | 1 |

Table 1. Discriminant groups for \( A_n, E_{6,7,8} \) root lattices

2.6. Conformal nets. We denote by \( \mathcal{I} \) the set of proper (i.e. open, non-empty, and non-dense) intervals \( I \subset S^1 \) on the circle and by \( J' = S^1 \setminus \mathcal{I} \). Let us denote the group of orientation preserving diffeomorphisms of the circle \( S^1 \) by \( \text{Diff}_+ (S^1) \). We note that the Möbius group \( \text{Möb} \) is naturally a subgroup of \( \text{Diff}_+ (S^1) \). By a (local) conformal net \( \mathcal{A} \), we mean a local Möbius covariant net on the circle, which is diffeomorphism covariant. Although, we do not use diffeomorphism covariance, all nets we consider have this property.

More precisely, a conformal net associates with every interval \( I \in \mathcal{I} \) a von Neumann algebra \( \mathcal{A}(I) \subset \mathcal{B}(\mathcal{H}) \) on a fixed Hilbert space \( \mathcal{H} = \mathcal{H}_\mathcal{A} \), such that the following properties hold:

A. **Isotony.** \( I_1 \subset I_2 \) implies \( \mathcal{A}(I_1) \subset \mathcal{A}(I_2) \).

B. **Locality.** \( I_1 \cap I_2 = \emptyset \) implies \( [\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\} \).

C. **Möbius covariance.** There is a unitary representation \( U \) of \( \text{Möb} \) on \( \mathcal{H} \), such that \( U(g)A(I)U(g)^* = A(gI) \).

D. **Positivity of energy.** \( U \) is a positive energy representation, i.e. the generator \( L_0 \) (conformal Hamiltonian) of the rotation subgroup \( U(z \mapsto e^{i\theta}z) = e^{i\theta L_0} \) has positive spectrum.

E. **Vacuum.** There is a (up to phase) unique rotation invariant unit vector \( \Omega \in \mathcal{H} \) which is cyclic for the von Neumann algebra \( \mathcal{A} := \bigvee_{I \in \mathcal{I}} \mathcal{A}(I) \).

F. **Diffeomorphism covariance.** There is a projective unitary representation \( U \) of \( \text{Diff}_+ (S^1) \) extending the representation \( U \) of \( \text{Möb} \), such that for all \( I \in \mathcal{I} \)

\[
U(g)A(I)U(g)^* = A(gI), \quad g \in \text{Diff}_+ (S^1),
\]

\[
U(g)xU(g)^* = x, \quad x \in \mathcal{A}(I'), g \in \text{Diff}_+ (I').
\]

where \( \text{Diff}_+ (I) = \{ g \in \text{Diff}_+ (S^1) : g \mid I' = \text{id}_{I'} \} \).

Let \( I \in \mathcal{I} \). By Haag duality \( \mathcal{A}(I') = \mathcal{H} \mathcal{A}(I') \). We have \( U(g) \in \mathcal{A}(I) \) for each \( g \in \text{Diff}(I) \) and \( \text{Vir}_\mathcal{A}(I) := \{ U(g) : g \in \text{Diff}(I) \}^\vee \subset \mathcal{A}(I) \) defines a subnet of \( \mathcal{A} \), the so-called Virasoro net associated with \( \mathcal{A} \). The positive energy representation of \( \text{Diff}_+ (S^1) \) restricted to \( \mathcal{H}_{\text{Vir}_\mathcal{A}} = \bigvee_I \text{Vir}_\mathcal{A}(I)\Omega \) is an irreducible positive energy representation of \( \text{Diff}_+ (S^1) \) with an \( \text{Möb} \) invariant vector \( \Omega \) (see [Car04, CW05]). Such representations are completely classified by the central charge \( c \) and \( \text{Vir}_\mathcal{A} \cong \text{Vir}_c \) for some unique \( c > 0 \).
A conformal net $\mathcal{A}$ is called completely rational if it fulfills the split property, i.e. for $I_0, I \in \mathcal{I}$ with $\overline{I_0} \subset I$ the inclusion $\mathcal{A}(I_0) \subset \mathcal{A}(I)$ is a split inclusion, namely there exists an intermediate type I factor $M$, such that $\mathcal{A}(I_0) \subset M \subset \mathcal{A}(I)$.

**H.** is strongly additive, i.e. for $I_1, I_2 \in \mathcal{I}$ two adjacent intervals obtained by removing a single point from an interval $I \in \mathcal{I}$ the equality $\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I)$ holds.

**I.** for $I_1, I_3 \in \mathcal{I}$ two intervals with disjoint closure and $I_2, I_4 \in \mathcal{I}$ the two components of $(I_1 \cup I_3)'$, the $\mu$-index of $\mathcal{A}$

$$\mu(\mathcal{A}) := [(\mathcal{A}(I_2) \vee \mathcal{A}(I_4))' : \mathcal{A}(I_1) \vee \mathcal{A}(I_3)]$$

(which does not depend on the intervals $I_i$) is finite.

**Remark 2.5.** It was recently shown, that diffeomorphism covariance implies the split property. Further, diffeomorphism covariance, split property and finite $\mu$-index implies strong additivity. Thus finite $\mu$-index is equivalent to completely rationality if we assume diffeomorphism covariance.

A representation $\pi$ of a strongly additive net $\mathcal{A}$ is a family of (unital) representations $\pi = \{\pi_I : \mathcal{A}(I) \to B(\mathcal{H}_\pi)\}_{I \in \mathcal{I}}$ on a common Hilbert space $\mathcal{H}_\pi$ which are compatible, i.e. $\pi_J \circ \mathcal{A}(I) = \pi_I$ for $I \subset J$. Every representation $\pi$ with $\mathcal{H}_\pi$ separable—for every choice of an interval $I_0 \in \mathcal{I}$—turns out to be equivalent to a representation localized in $I_0$, i.e. $\rho$ on $\mathcal{H}$, such that $\rho_I = \text{id}_\mathcal{A}(I,J)$ for $J \cap I_0 = \emptyset$. Then Haag duality implies that $\rho_I$ is an endomorphism of $\mathcal{A}(I)$ for every $I \in \mathcal{I}$ with $I \supset I_0$, which we also denote by $\rho$. The statistical dimension of a representation $\rho$ localized in $I$ is given by the square root of the Jones index of the Jones–Wassermann subfactor $d\rho = [\mathcal{A}(I) : \rho(\mathcal{A}(I))]^{\frac{1}{2}}$. By Cor. 39 of every representation of a completely rational conformal net is a direct sum of representations with finite statistical dimension. For convenience we will restrict to representations with finite statistical dimension, thus every representation is a finite direct sum of irreducible representations and we obtain a semisimple category.

Thus we can realize the category of representations of $\mathcal{A}$ with finite statistical dimension which are localized in $I$ inside the rigid $C^*$-tensor category of endomorphisms $\text{End}_0(N)$ of the type III factor $N = \mathcal{A}(I)$ and the embedding turns out to be full and replete. We denote this category by $\text{Rep}^I(\mathcal{A})$. In particular, this gives the representations of $\mathcal{A}$ the structure of a tensor category 

It has a natural braiding, which is completely fixed by asking that if $\rho$ is localized in $I_1$ and $\sigma$ in $I_2$ where $I_1$ is left of $I_2$ inside $I$, then $\varepsilon(\rho, \sigma) = 1$ [FRS99]. Let $\mathcal{A}$ be completely rational conformal net, then by Cor. 39 of $\text{Rep}^I(\mathcal{A})$ is a UMTC and $\mu_\mathcal{A} = \text{Dim}(\text{Rep}^I(\mathcal{A}))$. A completely rational conformal net is called holomorphic, if $\text{Rep}^I(\mathcal{A})$ is trivial, i.e. equivalent to the category finite dimensional Hilbert spaces Hilb, or equivalently $\mu_\mathcal{A} = 1$.

We write $\mathcal{A} \subset \mathcal{B}$ or $\mathcal{B} \supset \mathcal{A}$ if there is a representation $\pi = \{\pi_I : \mathcal{A}(I) \to B(\mathcal{H}_\pi)\}_{I \in \mathcal{I}}$ on $\mathcal{H}_\pi$ and an isometry $V : \mathcal{H}_\mathcal{A} \to \mathcal{H}_\mathcal{B}$ with $V\mathcal{H}_\mathcal{A} = \mathcal{H}_\mathcal{B}$ and $VU_{\mathcal{A}}(g) = U_{\mathcal{B}}(g)V$. We furthermore ask that $Va = \pi_I(a)V$ for $I \in \mathcal{I}$, $a \in \mathcal{A}(I)$. Define $p$ to be the orthogonal projection onto $\mathcal{H}_{\mathcal{A}} = \pi_I(\mathcal{A}(I))\Omega$. Then $pV$ is a unitary equivalence of the nets $\mathcal{A}$ on $\mathcal{H}_\mathcal{A}$ and $\mathcal{A}_0$ defined by $\mathcal{A}_0(I) = \pi_I(\mathcal{A}(I))p$ on $\mathcal{H}_{\mathcal{A}_0}$.

The inclusion $\mathcal{A} \subset \mathcal{B}$ is called finite index if the Jones index $[\mathcal{B}(I) : \mathcal{A}(I)]$ is finite. In this case, $\mathcal{A}$ is completely rational if and only if $\mathcal{B}$ is completely rational [Lon03]. The conformal net $\mathcal{B}$ is characterized by a commutative Q-system $\Theta$ in $\text{Rep}(\mathcal{A})$ [LR95] and $\text{Rep}(\mathcal{A})$ is braided equivalent to $\text{Rep}(\mathcal{B})^0_{\Theta}$ [BKL15].

Let $\mathcal{B}$ be a completely rational conformal net, we note that two extensions $\mathcal{A} \supset \mathcal{B}$ and $\tilde{\mathcal{A}} \supset \mathcal{B}$ are isomorphic, if and only if they have equivalent Q-systems in $\text{Rep}(\mathcal{B})$, which can be taken as a definition for the purpose of this paper.
Definition 2.6 ([KL05]). A diffeomorphism covariant completely rational net \( \mathcal{A} \) with central charge \( c \) is called modular if for

\[
\chi_\rho(\tau) = \text{tr} \left( e^{2\pi i \tau (L_0^\rho - c/24)} \right)
\]

we have a representation of \( \text{SL}(2, \mathbb{Z}) \) with \( T^x = \text{diag}(e^{2\pi i (L_0^\rho - c/24)}) \) and

\[
\begin{align*}
\chi_\rho(-1/\tau) &= \sum_{\nu \in \text{Irr}(\text{Rep}(\mathcal{A}))} S^\chi_{\rho,\nu} \chi_\nu(\tau), \\
\chi_\rho(\tau + 1) &= \sum_{\nu \in \text{Irr}(\text{Rep}(\mathcal{A}))} T^\chi_{\rho,\nu} \chi_\nu(\tau),
\end{align*}
\]

such that \( (S^\chi, T^x) \) coincides with the (categorical) modular data \( (S, T) \) of \( \text{Rep}(\mathcal{A}) \), i.e. \( S = S^\chi \) and \( T = \omega_3 T^x \) for some third root of unity \( \omega_3 \).

By the spin-statistic theorem ([GL95]) the requirement on \( T \) is \( c_{\text{top}}(\text{Rep}(\mathcal{A})) \equiv c \pmod{8} \). Conformal nets with \( c < 1 \) and the nets \( \mathcal{A}_{SU(2)} \) are modular [Xu00b, Xu01], cf. also [KL05].

2.7. Orbifold theories. Fixed points of conformal nets under group actions, so-called orbifolds, were studied in [Xu00a, Müg05]. An automorphism of a conformal net \( \mathcal{A} \) is a compatible family \( \{\alpha_I \in \text{Aut}(\mathcal{A}(I))\} \) of automorphisms which preserve the vacuum, i.e. \( (\Omega, \alpha_I(a)\Omega) = (\Omega, a\Omega) \) for all \( a \in \mathcal{A}(I) \). The group of all automorphisms of \( \mathcal{A} \) is denoted by \( \text{Aut}(\mathcal{A}) \).

Let \( G \leq \text{Aut}(\mathcal{A}) \) be a finite group, then the fixed point net \( \mathcal{A}^G \subset \mathcal{A} \) given by \( \mathcal{A}^G(I) = \{a \in \mathcal{A}(I) : \alpha_I(a) = a \text{ for all } \alpha \in G\} \) is a finite index subnet with index \( [\mathcal{A} : \mathcal{A}^G] = |G| \). The Q-system \( \Theta \) in \( \text{Rep}(\mathcal{A}^G) \) giving \( \mathcal{A} \supset \mathcal{A}^G \) is the regular representation of \( G \) and \( \text{Rep}(\mathcal{A}) = \text{Rep}(\mathcal{A}^G)^0 \).

We identify \( S^1 \setminus \{-1\} \) with \( \mathbb{R} \) and fix \( I \in \mathbb{R} \). For \( \alpha \in \text{Aut}(\mathcal{A}) \), we say \( \pi \) is an \( \alpha \)-representation of \( \mathcal{A} \), if \( \pi \) is a representation of \( \mathcal{A} \) on \( \mathbb{R} \), such that \( \pi_{I_\ell} = \text{id}_{\mathcal{A}(I_{\ell-})} \) and \( \pi_{I_\ell} = \alpha_I \) for \( I_{\ell-} < I < I_{\ell+} \). We define \( G_\text{Rep}(\mathcal{A}) = G_\text{Rep}^I(\mathcal{A}) \) to be the category of representations of \( \mathcal{A} \) on \( \mathbb{R} \) which are finite direct sums of \( \alpha_g \)-representations for \( \alpha_g \in G \).

The category generated from \( \alpha^+ \)-induction of \( \text{Rep}^I(\mathcal{A}^G) \) for \( \mathcal{A}(I)^G \subset \mathcal{A}(I) \) is equivalent to \( G_\text{Rep}^I(\mathcal{A}) \) which implies that \( G_\text{Rep}(\mathcal{A}) \) is tensor equivalent to \( \text{Rep}(\mathcal{A}^G)^G \). In particular, the category \( G_\text{Rep}(\mathcal{A}) \) is a \( G \)-crossed braided extension of \( \text{Rep}^I(\mathcal{A}) \). Furthermore, \( \text{Rep}^I(\mathcal{A}^G) \) is braided equivalent to \((G_\text{Rep}^I(\mathcal{A}))^G \). We refer to [Müg05] for more details.

2.8. Generalized orbifolds. Generalized orbifolds in conformal nets were introduced by the author in [Bis17]. A (finite) hypergroup \( K \) is a finite set, which is the basis of a (finite-dimensional) \( C^* \)-algebra \( \mathbb{C} K \), such that the identity 1 \( \in K \), the set \( K \) is closed under adjoints (i.e. \( K^* = K \)), the multiplication restricts to a map \( m: K \times K \to \text{Conv}(K) = \{\sum_{k \in K} \lambda_k k : \sum_k \lambda_k = 1 \text{ and } \lambda_k \geq 0\} \), and we have the following antipode law: \( 1 < k \cdot \ell \text{ for some } k, \ell \in K \) if and only if \( k = \ell^* \). Here, for \( \ell \in K \) we write \( \ell < \sum_k \lambda_k k \in \text{Conv}(K) \) if \( \lambda_\ell > 0 \).

A finite group \( G \) is a hypergroup with the usual multiplication and \( g^* = g^{-1} \). Conversely, a hypergroup, such that the multiplication \( m: K \times K \to \text{Conv}(K) \) takes values in \( K \) is a finite group.

There is an obvious notion of a subhypergroup \( L \leq K \) and the double quotient \( K/L \) is again a hypergroup. If \( G \) is a finite group we have the Tambara–Yamagami hypergroup \( K = G \sqcup \{\rho\} \) with \( \rho^* = \rho = gp = pg \) for all \( g \in G \) and \( \rho^2 = \frac{1}{|G|} \sum g \). More generally, if \( F \) is a unitary fusion category, then we have the associated fusion hypergroup given by the renormalized basis \( K_F = \{dp^{-1}[\rho] : [\rho] \in \text{Irr}(F)\} \) of the complexified fusion ring or fusion algebra \( K_0(F) \otimes_{\mathbb{Z}} \mathbb{C} \).

Let \( \mathcal{A} \) be a conformal net. A quantum operation on \( \mathcal{A} \) is a compatible family \( \phi = \{\phi_I: \mathcal{A}(I) \to \mathcal{A}(I)\} \) of extremal normal unital completely positive maps which are vacuum preserving and have an adjoint \( \phi_I^\# \) with \( (a\Omega, \phi_I(b)\Omega) = (\phi_I^\#(a\Omega, b\Omega)) \) for all \( a, b \in \mathcal{A}(I) \). The set of all quantum operations on \( \mathcal{A} \) is denoted by \( \text{QuOp}(\mathcal{A}) \). We note that \( \text{Aut}(\mathcal{A}) \subset \text{QuOp}(\mathcal{A}) \). One of the main
result of [Bis17] can be restated as follows. There is a one-to-one correspondence between finite index subnets $B \subset A$ and finite hypergroups $Q \leq \text{QuOp}(A)$, where by $Q \leq \text{QuOp}(A)$ we mean a finite subset $Q$ which forms a hypergroup with multiplication given by composition and adjoints given by $\phi^* = \phi^\#$. The correspondence is given by $Q \mapsto A^Q \subset A$ where $A^Q$ is the fixed point net $A^Q(I) = \{a \in A(I) : \phi_1(a) = a \text{ for all } \phi \in Q\}$. If $A$ is completely rational and $B \subset A$ finite index then the unique $Q \leq \text{QuOp}(A)$ with $A^Q = B$ is isomorphic to the double quotient $K_F/\text{KRep}(A)$, where $F$ is the category generated by $\alpha^+$-induction for the inclusion $B(I) \subset A(I)$. In pure analogy with the group case we denote $Q\text{–}\text{Rep}^I(A) := F$.

Thus for any finite hypergroup $Q \leq \text{QuOp}(A)$ there is a unitary fusion category $Q\text{–}\text{Rep}^I(A)$ extending $\text{Rep}^I(A)$, such that $Q \cong K_{Q\text{–}\text{Rep}(A)}/\text{KRep}(A)$. Furthermore, $Q\text{–}\text{Rep}^I(A)$ generalizes $G\text{–}\text{Rep}^I(A)$ in the case $Q = G$ is a finite group. The inclusion $\text{Rep}(A) \to Q\text{–}\text{Rep}(A)$ has naturally the structure of a central functor $\text{Rep}(A)^\text{rev} \to Q\text{–}\text{Rep}(A)$, such that $\text{Rep}(A^Q)$ is braided equivalent to the Müger centralizer $(\text{Rep}(A)^\text{rev})' \cap Z(Q\text{–}\text{Rep}(A))$ of $\text{Rep}(A)^\text{rev}$ in the Drinfel’d center $Z(Q\text{–}\text{Rep}(A))$. In the case that $A$ is holomorphic, $Q\text{–}\text{Rep}^I(A)$ is a categorification of $Q$, i.e. $K_Q\text{–}\text{Rep}^I(A) = Q$ and $\text{Rep}(A^Q)$ is braided equivalent to $Z(Q\text{–}\text{Rep}(A))$.

3. Realization of Tambara–Yamagami categories and their centers

3.1. Changing Frobenius–Schur indicators. We show that starting with a unitary modular tensor category $\bar{C}$ with certain properties, we obtain a new (twisted) unitary modular tensor category $\hat{C}$ with the same fusion rules but different Frobenius–Schur indicators. We call the unitary modular tensor category $S = C_{\text{SU}(2)}$, the semion category.

**Proposition 3.1** (Changing the Frobenius–Schur Indicator). Let $C = C_0 \oplus C_1$ be a $\mathbb{Z}_2$-graded unitary modular tensor category and $\alpha \in C_0$ with $\langle \alpha \rangle$ braided equivalent to $\text{Rep}(\mathbb{Z}_2)$. Let $S = \langle \tau \rangle$ the semion category and let $\Theta$ be the $\mathbb{Z}_2$ $Q$-system associated with $[\theta] = [\text{id}] \oplus [\alpha \otimes \tau \otimes \bar{\tau}]$. Then

1. $\hat{C} := \langle C \boxtimes S \boxtimes S^{\text{rev}} \rangle^0_0$ has the same fusion rules as $C$, i.e. there is a map $\text{Irr}(C) \to \text{Irr}(\hat{C})$: $\rho \mapsto \hat{\rho}$ giving an isomorphism of Grothendieck rings.
2. $C \to \hat{C}$ is involutive, i.e. $\hat{C}$ is braided equivalent to $C$.
3. $\hat{C}$ is braided equivalent to the subcategory $\langle \hat{\rho}_0 := \rho_0 \boxtimes \text{id} \boxtimes \text{id}, \hat{\rho}_1 := \rho_1 \boxtimes \tau \boxtimes \text{id} : \rho_i \in C_i \rangle$ of $C \boxtimes S \boxtimes S^{\text{rev}}$. In particular, objects in $C_1$ have opposite Frobenius-Schur indicators compared to the corresponding objects in $C_1$, i.e. $\nu_{\hat{\rho}_1} = -\nu_{\rho_1}$ for all (self-dual) $\rho_1 \in C_1$.

**Proof.** We have subcategories $\hat{C} = \langle \rho_0 \boxtimes \text{id} \boxtimes \text{id}, \rho_1 \boxtimes \tau \boxtimes \text{id} : \rho_i \in C_i \rangle$ and $D = \langle \alpha \boxtimes \text{id} \boxtimes \bar{\tau}, \text{id} \boxtimes \tau \boxtimes \text{id} \rangle$ of $C \boxtimes S \boxtimes S^{\text{rev}}$. Then it follows that $D \cong S \boxtimes S^{\text{rev}}$. By Müger’s theorem [Müg03c, Theorem 4.2] we have $C \boxtimes S \boxtimes S^{\text{rev}} \cong \hat{C} \boxtimes D$. Then the canonical algebra $\Theta$ in $D$ is $[\text{id} \boxtimes \text{id} \boxtimes \text{id}] \oplus [\alpha \boxtimes \tau \boxtimes \bar{\tau}]$ and $\langle C \boxtimes S \boxtimes S^{\text{rev}} \rangle^0_0 \cong \hat{C}$. Choosing $\hat{C} \cong \langle \rho_0 \boxtimes \text{id} \boxtimes \text{id}, \rho_1 \boxtimes \text{id} \boxtimes \bar{\tau} : \rho_i \in C_i \rangle$ and $D = \langle \alpha \boxtimes \tau \boxtimes \text{id}, \text{id} \boxtimes \text{id} \boxtimes \bar{\tau} \rangle$ gives braided equivalent tensor categories. Therefore, we have $\hat{C} \cong \langle \rho_0 \boxtimes \text{id} \boxtimes \text{id}, \rho_1 \boxtimes \tau \boxtimes \bar{\tau} : \rho_i \in C_i \rangle \cong C$. \[\square\]

**Proposition 3.2.** Let $A$ be a completely rational net with $C := \text{Rep}(A)$ fulfilling the assumption of Proposition 3.1. Then there is a completely rational net $\hat{A}$ with $\text{Rep}(A) \cong \hat{C}$.

$A$ is a $\mathbb{Z}_2$-orbifold $A = A^{\mathbb{Z}_2}_L$ of the conformal net $A_L$ associated with an even lattice $L$, there is a proper $\mathbb{Z}_2$-action on $A_{LE_8}$, such that we can choose $\hat{A} = A^{\mathbb{Z}_2}_L$.

**Proof.** Let $A_{A_1E_7}$ be the conformal net associated with the lattice $A_1E_7$, then $\text{Rep}(A_{A_1E_7})$ is braided equivalent to $S \boxtimes S^{\text{rev}}$. Consider, the $\mathbb{Z}_2$-simple current extension $\hat{A} = (A \otimes A_{A_1E_7}) \times \mathbb{Z}_2$ w.r.t. $\alpha \otimes \alpha_{1,1}$, where $\alpha_{1,1}$ correspond to $\tau \boxtimes \bar{\tau}$. Then it follows directly that $\text{Rep}(A)$ is braided equivalent to $\hat{C}$.\[12\]
Let us consider \( \mathcal{A}_L^{\mathbb{Z}_2} \otimes \mathcal{A}_{A_1 E_7} \). Then we can make a \( \hat{\mathbb{Z}}_2 \times \hat{\mathbb{Z}}_2 \)-simple current extension giving \( \mathcal{A}_{LE_8} \). We get a \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) action with
\[
\mathcal{A}_{(1,0)}^{(1,0)} = \mathcal{A}_L^{\mathbb{Z}_2} \otimes \mathcal{A}_{E_8} \\
\mathcal{A}_{(0,1)}^{(1,1)} = \mathcal{A}_L \otimes \mathcal{A}_{A_1 E_7} \\
\mathcal{A}_{(1,1)}^{(1,1)} = (\mathcal{A}_L^{\mathbb{Z}_2} \otimes \mathcal{A}_{A_1 E_7}) \times_{111} \mathbb{Z}_2
\]
and we can choose \( \hat{\mathcal{A}} = \mathcal{A}_{(1,1)}^{(1,1)} \).

\[\square\]

**Example 3.3.** Let \( \mathcal{A}_{SU(2)} \) be the loop group net of \( SU(2) \) at level \( k \) [Was98, Xu00b], then it follows that \( \text{Rep}(\mathcal{A}_{SU(2)}^k) \) is braided equivalent to \( \mathcal{C}_{SU(2)}^k \) using the classification [FK93], cf. [Hen17b]. The simple objects are \( \{ \rho_0, \rho_\frac{1}{2}, \ldots, \rho_\frac{k}{2} \} \) with fusion rules
\[
[\rho_i] \times [\rho_j] = \bigoplus_{i+j+\ell \leq k, i+j - \ell \text{ even}} [\rho_\ell].
\]

The unitary modular tensor category \( \mathcal{C} = \mathcal{C}_{SU(2)}^k \) fulfills the assumption of Proposition 3.1. Since in \( \mathcal{C} \) the generating object \( \rho_{\frac{1}{2}} \) has trivial Frobenius–Schur indicator, it turns out that \( \hat{\mathcal{C}} \) is what could be called the Jones–Kaufmann (modular tensor) category cf. [Wan10].

Therefore we get a completely rational net \( \hat{\mathcal{A}}_{SU(2)}^k = (\mathcal{A}_{SU(2)}^k \otimes \mathcal{A}_{A_1 E_7}) \times \mathbb{Z}_2 \) realizing the Jones–Kaufmann category \( \hat{\mathcal{C}} \). We can replace \( \mathcal{A}_{SU(2)}^k \) by the net constructed in [Bis16b] which realizes \( (\mathcal{C}_{SU(2)^k})^{\text{rev}} \), to obtain a conformal net realizing \( \hat{\mathcal{C}}^{\text{rev}} \).

We note that \( \mathcal{C} \rightarrow \hat{\mathcal{C}} \) corresponds to the \( \mathbb{Z}_2 \)-twist of the UMTC \( \mathcal{C}_{SU(2)}^k \). Namely, Kazhdan and Wenzl showed in [KW93] that fusion categories with \( SU(N)^k \) fusion rules are \( \mathcal{C}_{SU(N)^k} \) possibly twisted by an element of \( \mathbb{Z}_N \).

### 3.2. Changing \( H^3 \) in \( G \)-crossed braided categories of orbifold nets.

We briefly generalize the result from \( \mathbb{Z}_2 \) to an arbitrary finite group \( G \). We note that the Frobenius–Schur indicator in Prop. 3.2 comes from a class in \( H^3(\mathbb{Z}_2, \mathbb{T}) \) which classifies the \( \mathbb{Z}_2 \)-extension \( \mathbb{Z}_2 \rightarrow \text{Rep}(\mathcal{A} \times_{\alpha} \mathbb{Z}_2) \) of \( \text{Rep}(\mathcal{A} \times_{\alpha} \mathbb{Z}_2) \).

In [ENO10] it is shown that \( G \)-crossed braided extensions \( \mathcal{F} \) of \( \mathcal{C} \) are parametrized by an group homomorphism \( c: G \rightarrow \text{Pic}(\mathcal{C}) \), \( M \in H^2(G, \mathbb{Z}_2) \) and \( t_\mathcal{F} \in H^3(G, \mathbb{C}^*) \), such that certain obstructions \( c_1(c) \in H^3(G, \mathbb{C}^*) \) and \( c_2(c, M) \in H^4(G, \mathbb{C}^*) \) vanish.

Let \( \mathcal{A} \) be a completely rational net \( G \leq \text{Aut}(\mathcal{A}) \), then \( G \rightarrow \text{Rep}(\mathcal{A}) \) is a \( G \)-crossed braided category which is a \( G \)-graded extension of \( \text{Rep}(\mathcal{A}) \). The classification of \( G \)-extensions involves a \( [\varphi] \in H^3(G, \mathbb{T}) \) which we can twist as follows. Assume \( \mathcal{B} \) is a holomorphic net with an action of \( G \) such that \( G \rightarrow \text{Rep}(\mathcal{B}) \cong \text{Hilb}_{\mathbb{T}}^G \) for some \( [\omega] \in H^3(G, \mathbb{T}) \). Then we can take the diagonal action of \( G \) on \( \mathcal{A} \otimes \mathcal{B} \) and get that \( G \rightarrow \text{Rep}(\mathcal{A} \otimes \mathcal{B}) \) which has the same fusion rules as \( G \rightarrow \text{Rep}(\mathcal{A}) \) but gives the class \( [\varphi + \omega] \in H^3(G, \mathbb{T}) \). Evans and Gannon announced that for every finite group \( G \) and every \( [\omega] \in H^3(G, \mathbb{T}) \) there is a conformal net \( \mathcal{A}_{G,\omega} \) with \( \text{Rep}(\mathcal{A}_{G,\omega}) \) braided equivalent to \( Z(\text{Hilb}_{\mathbb{T}}^G) \), thus from the Lagrangian Q-system coming from the induction functor \( I: \text{Hilb}_{\mathbb{T}}^G \rightarrow Z(\text{Hilb}_{\mathbb{T}}^G) \) we get a holomorphic extension \( \mathcal{B}_{G, \omega} = \mathcal{A}_{G, \omega} \times \text{Rep}(G) \) with \( G \rightarrow \text{Rep}(\mathcal{A}) \cong \text{Hilb}_{\mathbb{T}}^G \).

Thus we have proven

**Proposition 3.4.** Let \( \mathcal{C} \) be a UMTC and \( \mathcal{F} \) a \( G \)-crossed braided extension of \( \mathcal{C} \) with \( t_\mathcal{F} \in H^3(G, \mathbb{T}) \). If there is a completely rational net \( \mathcal{A} \) realizing \( \mathcal{F}^G \), then there is a completely rational net realizing \( \mathcal{F}^G \) for every \( [\varphi] \in H^3(G, \mathbb{T}) \), where \( \hat{\mathcal{F}} \) is the \( G \)-crossed braided extension of \( \mathcal{C} \) similar to \( \mathcal{F} \), but with \( t_{\hat{\mathcal{F}}} = [\varphi] \) instead of \( \varphi \).
3.3. Realization of pointed unitary modular tensor categories. Let \( \mathcal{C} \) be a pointed UMTC, then \( \mathcal{C} \) is braided equivalent to \( \mathcal{C}(G,q) \). It follows from [Deg] that there is an even positive lattice \( L = (L, \langle \cdot, \cdot \rangle) \), such that \( (G,q) \) is equivalent to the discriminant form \( (G_L, q_L) \) with \( G_L = L^*/L \) and \( q_L(\ell + \ell') = \exp(\pi i \langle \ell, \ell' \rangle) \).

Let us consider the conformal net \( A_L \) associated with \( L \), see [DX06, Bis12]. From [DX06] it follows that \( \text{Rep}(A_L) \) has \( G_L \) fusion rules, namely the sectors are \( \{ [\rho_{m+L}] : m + L \in L^*/L \} \) with fusion rules \( [\rho_{m+L}] [\rho_{n+L}] = [\rho_{m+n+L}] \). Further, it is shown that the spectrum \( \text{spec}(L_0^{p+L}) \) of the conformal Hamiltonian \( L_0^{p+L} \) in the sector \( [\rho_{m+L}] \) equals \( \{ \frac{1}{2} \langle x, x \rangle : x \in m + L \} \). Using the spin-statistics theorem [GL90] it follows:

**Proposition 3.5.** Let \( L \) be an even lattice, then \( \text{Rep}(A_L) \) is braided equivalent to \( \mathcal{C}(G_L, q_L) \).

Thus the classical result [Nik79] about lifting metric groups to even lattices, implies the following reconstruction result for conformal nets, which expect to be well-known to experts.

**Theorem 3.6.** Let \( \mathcal{C} \) be a pointed UMTC, then there is an even lattice \( L \), such that \( \text{Rep}(A_L) \) is braided equivalent to \( \mathcal{C} \).

**Remark 3.7.** The same is true for vertex operator algebras using that for \( V_L \) the vertex operator algebra associated with the lattice \( L \) the category of \( V_L \) modules is braided equivalent to \( \mathcal{C}(G_L, q_L) \) by [DL94], see also [Hoh03].

3.4. Realization of Tambara–Yamagami doubles for odd groups. In this section we will prove the following main reconstruction theorem.

**Theorem 3.8.** Let \( \mathcal{F} \) be a unitary fusion category with Tambara–Yamagami fusion rules of even rank. Then:

1. There is an even lattice \( L \), and a proper \( \mathbb{Z}_2 \)-action on \( A_L \), such that the category of \( \mathbb{Z}_2 \)-twisted representations \( \mathbb{Z}_2 \text{-Rep}(A_L) \) is tensor equivalent to \( \mathcal{F} \).
2. There is an even lattice \( M = LL \) of rank 0 (mod 8), and a proper \( \mathbb{Z}_2 \)-action on \( A_M \), such that \( \text{Rep}(A_M^{\mathbb{Z}_2}) \) is braided equivalent to \( Z(\mathcal{F}) \).
3. There is a self-dual even lattice \( \Gamma \) of rank 0 (mod 8), and a proper action of the hypergroup \( K_{\mathcal{F}} \) on \( A_{\Gamma} \), such that \( \text{Rep}(A_{\Gamma}^{K_{\mathcal{F}}}) \) is braided equivalent to \( Z(\mathcal{F}) \).

Associated with an abelian group \( G \) and a non-degenerate symmetric bicharacter \( \langle \cdot, \cdot \rangle : G \times G \to \mathbb{T} \) there are two unitary fusion category \( \mathcal{TY}(G, \langle \cdot, \cdot \rangle, \pm) \) with irreducible sectors \( \{ [\rho_g], [\rho_{\pm}] : g \in G \} \cong G \cup \{ \rho_{\pm} \} \) having the following fusion rules [Izu01]:

\[
[\rho_g][\rho_h] = [\rho_{g+h}], \quad [\rho_{\pm}][\rho_g] = [\rho_{g}][\rho_{\pm}] = [\rho_{\pm}], \quad [\rho_{\pm}][\rho_{\pm}] = \bigoplus_{g \in G} [\rho_g].
\]

Every unitary fusion category with this fusion rules is of the above form by the classification [TY98].

Let us consider the Drinfel’d center \( Z(\mathcal{TY}(G, \langle \cdot, \cdot \rangle, \pm)) \). The objects are [Izu01]:

\[
[\rho_i^g] = (\rho_g, \varepsilon_i^g), \quad g \in G, \ i = 0, 1, \\
[\rho_{\pm}^{(g,i)}] = (\rho_{\pm}, \varepsilon_{\pm}^{(g,i)}), \quad g \in G, \ i = 0, 1, \\
[\sigma_{g,h}] = (\rho_g \oplus \rho_h, \varepsilon_{g,h}), \quad g < h, \ g, h \in G.
\]
The modular data is given by [Izu01]:

\[
S = \frac{1}{2n} \begin{pmatrix}
|\rho_i| & |\rho_i^{(h,j)}| & |\sigma_{h,k}|

\sqrt{\langle g, h \rangle}^2 & \langle g, h \rangle & 2\langle g, h + k \rangle

* & (1 - 1)^i \omega_y \omega_h \sum_k (k - (g + h), k) & 0

* & 0 & 2(\langle h, h' \rangle \langle h, k' \rangle + \langle k, k' \rangle \langle h, h' \rangle)
\end{pmatrix},
\]

\[
T = \text{diag} \left( \begin{array}{ccc}
|\rho_i| & |\rho_i^{(g,a)}| & |\sigma_{h,k}|

\langle g, g \rangle & (-1)^i \omega_y \langle h, k \rangle & \langle h, k \rangle
\end{array} \right).
\]

Here \( \omega_y \) is defined as follows. Let \( a : G \to \mathbb{T} \) be a function satisfying \( a(g)a(h) = \langle g, h \rangle a(g + h) \) and \( a(g) = a(-g) \) for all \( g, h \in G \). Let \( \hat{a} \) be the finite Fourier transform of \( a \) given by:

\[
\hat{a}(g) = \frac{1}{\sqrt{|G|}} \sum_{h \in G} \langle g, h \rangle f(h)
\]

and define \( \omega_y = \sqrt{\pm \hat{a}(g)} \).

The subcategory \( \mathcal{G} := \langle \rho_0^g : g \in G \rangle \) is a pointed subcategory. If \( |G| \) is of odd order \( \mathcal{G} \) is a UMTC, namely it is braided equivalent to \( \mathcal{C}(G, \bar{q}) \), where \( \bar{q}(a) = \langle a, a \rangle \).

**Proposition 3.9.** Let \( G \) be an abelian group of odd order, then \( Z(\mathcal{Y}(G, \langle \cdot, \cdot \rangle, \pm)) \) is braided equivalent to \( \mathcal{MP}(G, \langle \cdot, \cdot \rangle, \pm) \). Here \( \bar{q}(a) = \langle a, a \rangle \) and \( \mathcal{MP}(G, \langle \cdot, \cdot \rangle, \pm) \) is a unitary modular tensor category of rank 4 + (|G| - 1)/2 with global dimension \( 4|G| \).

The inclusion functor \( \iota : \mathcal{C}(G, \bar{q}) \to \mathcal{Y}(G, \langle \cdot, \cdot \rangle, \pm) \) mapping \( [g] \) to \( \rho_0^g \) is central.

If there is a central injective functor \( \mathcal{C}(G, \bar{q}) \to \mathcal{Y}(G', \langle \cdot, \cdot \rangle, \pm) \) with \( |G'| = |G| \), then \( (G, \bar{q}) \sim (G', \bar{q}') \).

**Proof.** That the functor \( \iota \) is central can be seen from the half-braidings. We define \( \mathcal{MP}(G, \langle \cdot, \cdot \rangle, \pm) \) to be the Müger centralizer \( \iota(\mathcal{C}(G, \bar{q}))' \cap Z(\mathcal{F}) \) which is modular by [Müg03c, Theorem 4.2].

Conversely, assume we have an injective central functor \( \iota : \mathcal{C}(G, \bar{q}) \to \mathcal{Y}(G', \langle \cdot, \cdot \rangle, \pm) \). From the modular data we see that the pointed part of \( \mathcal{Y}(G', \langle \cdot, \cdot \rangle, \pm) \) has \( z_2 \times G' \) fusion rules and because \( |G'| \) is odd, the repletion of \( \iota(\mathcal{C}(G', \bar{q}')) \subset Z(\mathcal{Y}(G', \langle \cdot, \cdot \rangle, \pm)) \) is the unique pointed fusion subcategory of rank \( |G'| \). This gives a braided injective functor \( \mathcal{C}(G, \bar{q}) \to \mathcal{C}(G', \bar{q}') \). Since \( |G'| = |G| \) it is a braided equivalence and we can conclude \( (G, \bar{q}) \sim (G', \bar{q}') \). \( \square \)

**Proposition 3.10.** Let \( \mathcal{A} \) be a completely rational conformal net with a proper action of \( \mathbb{Z}_2 \). If \( \text{Rep}(\mathcal{A}) \) is braided equivalent to \( \mathcal{C}(G, \bar{q}) \) for some odd abelian group \( G \) and \( \mathcal{A} \) has a \( \mathbb{Z}_2 \)-twisted soliton of dimension \( \sqrt{|G|} \), then the category of \( \mathbb{Z}_2 \)-twisted solitons \( \mathcal{Z}_2 - \text{Rep}(\mathcal{A}) \) is equivalent to \( \mathcal{Y}(G, \langle \cdot, \cdot \rangle, \nu) \), where \( \nu \in \{ \pm \} \) and \( \langle \cdot, \cdot \rangle : G \to \mathbb{T} \) is the unique non-degenerate bicharacter determined by \( \bar{q}(a) = \langle g, g \rangle \) for all \( g \in G \).

Furthermore, there is a proper \( \mathbb{Z}_2 \)-action on \( \bar{A} = \mathcal{A} \otimes \mathcal{A}_{E_8} \), such that the category \( \mathcal{Z}_2 - \text{Rep}(\bar{A}) \) is equivalent to \( \mathcal{Z}_2 \otimes \mathcal{Y}(G, \langle \cdot, \cdot \rangle, \nu) \).

**Proof.** We have \( \text{Rep}(\mathcal{A}) \subset \mathcal{F} \) and \( \text{Dim}(\mathcal{F}) = 2|G| \), thus \( \text{Irr}(\mathcal{F}) \cong G \cup \{ \rho \} \) and since \( \mathcal{F} \) is \( \mathbb{Z}_2 \)-graded we have \( [g][\rho] = [\rho][g] \) and therefore by Frobenius reciprocity \( [\rho][\rho] = [\rho][\bar{\rho}] = \bigoplus_{g \in G} [g] \), thus \( \mathcal{F} \) has Tambara–Yamagami fusion rules.

Using [Bis17, Proposition 5.11] we have a central injective functor \( \text{Rep}(\mathcal{A})^{\text{rev}} \cong \mathcal{C}(G, \bar{q}) \to \mathcal{F} \). Thus \( \mathcal{F} \) is equivalent to \( \mathcal{Y}(G, \langle \cdot, \cdot \rangle, \pm) \) by Proposition 3.9.

The last statement follows from Proposition 3.2 by considering the \( \mathbb{Z}_2 \)-orbifold \( \mathcal{Z}_2 \subset \bar{A} = \mathcal{A} \otimes \mathcal{A}_{E_8} \) using the same argument as in Section 3.2. \( \square \)
Remark 3.11. One can also check, e.g. by inspecting the modular data or using $\alpha$-induction, that the Frobenius–Schur indicators of generating object of the representation category of the $\mathbb{Z}_2$-orbifold net coincides with the Frobenius–Schur indicators of the Tambara–Yamagami category. This is an alternative proof of the fact that $\mathcal{A}$ and $\hat{\mathcal{A}}$ give Tambara–Yamagami categories with opposite Frobenius–Schur indicator.

By Theorem 3.9 there is an even positive lattice $L$, such that $\mathcal{A}_L$ realizes $\mathcal{C}(G,q)$, i.e. $\text{Rep}(\mathcal{A}_L)$ is braided equivalent to $\mathcal{C}(G,q)$.

Proposition 3.12. Let $(G,q)$ be a metric group with $|G|$ odd and let $L$ be an even lattice, such that $\text{Rep}(\mathcal{A}_L)$ is braided equivalent to $\mathcal{C}(G,q)$. Let $\sigma$ be the reflection of $L$ and $\mathcal{A}_L^{\mathbb{Z}_2} := \mathcal{A}_L^{(\sigma)}$ the associated $\mathbb{Z}_2$-orbifold. Then there is an irreducible $\mathbb{Z}_2$-twisted soliton $\rho$ of $\mathcal{A}_L$, with dimension $\sqrt{|n|}$. In particular, the category of $\mathbb{Z}_2$-twisted solitons of $\mathcal{A}_L$ is equivalent to the Tambara–Yamagami category $\mathcal{TY}(G,\langle \cdot, \cdot \rangle, \pm)$ with bicharacter $\langle \cdot, \cdot \rangle$ determined by $q(g) = \langle g, g \rangle^{-1}$.

Proof. By Proposition 3.10 we only have to find an irreducible $\mathbb{Z}_2$-twisted soliton $\rho$ with $d\rho = \sqrt{|G|}$.

Indeed, following the notation in [DX06] consider $(L^*/L)^\sigma$ to be the cosets fixed under $\sigma$. We get $(L^*/L)^\sigma = \{x \in L^*: 2x \in L\}/L = \{L\}$ and $|L^*/L|/(L^*/L)^\sigma| = |G|$. Then it follows from [DX06] Proposition 4.25, Corollary 4.31, that there is an irreducible soliton $\beta$ with $d\beta = \sqrt{|G|}$. □

Now we are able prove Theorem 3.8.

Proof of Theorem 3.8. Let $G$ be an odd abelian group and $\langle \cdot, \cdot \rangle$ a non-degenerate symmetric bicharacter, which is determined by its quadratic form $q(g) = \langle g, g \rangle$. By Theorem 3.9 we can choose even lattices $L_+$, $L_-=L_+\mathbb{E}_8$ and $\bar{L}$, such that $\text{Rep}(\mathcal{A}_{L_+})$ realizes $\mathcal{C}(G,q)$ and $\text{Rep}(\mathcal{A}_{\bar{L}})$ realizes $\mathcal{C}(G,\bar{q})$. Using the $\mathbb{Z}_2$-action on $\mathcal{A}_{L_+}$ as in Proposition 3.12 and the associated $\mathbb{Z}_2$-action on $\mathcal{A}_{L_-} \cong \mathcal{A}_{L_+} \otimes \mathcal{A}_{\mathbb{E}_8}$ as in Proposition 3.10 we get the first statement.

The other statements follow from general theory. For example, we have $\text{Rep}(\mathcal{A}_{L_{\pm}} \otimes \mathcal{A}_{\bar{L}})$ is braided equivalent to $Z(\mathcal{T\hat{Y}}(G,\langle \cdot, \cdot \rangle, \pm \nu))$, respectively, and we can choose $M_{\pm} = L_{\pm} \mathbb{E}_8$. For the third statement the self-dual lattices are obtained by diagonally gluing $\Gamma = (L_{\pm} \mathbb{E}_8) \oplus \Delta(G)$, respectively. □

4. Realization and classification of generalized metaplectic modular categories

4.1. Generalized metaplectic modular categories from Tambara–Yamagami categories. Let $G = (G, +)$ be an abelian finite group of odd order with a symmetric non-degenerate bicharacter $\langle \cdot, \cdot \rangle$ and $q(g) = \langle g, g \rangle^{-1}$. It is convenient to choose a set $G_+$ of “positive elements” of $G$, such that $G = G_+ \sqcup \{0\} \sqcup -G_+$. We denote by $|g|$ the element $\pm g \in G_+ \sqcup \{0\}$.

As in Proposition 3.9 we denote by $\mathcal{MP}(G,\langle \cdot, \cdot \rangle, \pm)$ the unitary modular tensor category given by the Müger centralizer $\mathcal{C}(G,\bar{q})' \cap Z(\mathcal{T\hat{Y}}(G,\langle \cdot, \cdot \rangle, \pm))$. From the braided equivalence:

$$Z(\mathcal{T\hat{Y}}(G,\langle \cdot, \cdot \rangle, \pm)) \cong \mathcal{MP}(G,\langle \cdot, \cdot \rangle, \pm) \boxtimes \mathcal{C}(G,\bar{q})$$

we can read off that $\mathcal{MP}(G,\langle \cdot, \cdot \rangle, \pm)$ has rank $|G| - 1)/244$ and that $\text{Irr}(\mathcal{MP}(G,\langle \cdot, \cdot \rangle, \pm)) = \{\rho_0, \rho_{\pm}^{(0,1)}, \sigma_{h,-h}: i = 0,1; h \in G_+\}$. From the modular data (11) of $Z(\mathcal{T\hat{Y}}(G,\langle \cdot, \cdot \rangle, \pm))$ we get the
following modular data of $\mathcal{MP}(G, \langle \cdot, \cdot \rangle, \pm)$:

$$S = \frac{1}{2\sqrt{n}} \begin{pmatrix} [\rho_0^0] & [\rho_0^{0,1}] & [\sigma_{h,-h}] \\ [\rho_0^0] & 1 & 2 \\ [\sigma_{k,-k}] & * & 2(h,k)^2 \\ [\sigma_{h,-h}] & 0 & (h,k)^2 \end{pmatrix},$$

$$T = e^{-\pi i c_{\text{top}}/12} \cdot \text{diag} \left( 1, (1)^i \omega_g \right),$$

where $c_{\text{top}}$ (mod 8) is determined from $(G,q)$ by (3). Thus the fusion rules of $\mathcal{MP}(G, \langle \cdot, \cdot \rangle, \pm)$ denoting $[\text{id}] := [\rho_0^0]$ and $[\alpha] := [\rho_0^{0,1}]$ are

$$[\alpha]^2 = [\text{id}], \quad [\alpha][\rho_0^{0,1}] = [\rho_0^{0,1}], \quad [\alpha][\sigma_{h,-h}] = [\sigma_{h,-h}],$$

$$[\rho_0^{0,1}][\rho_0^{0,1}] = [\text{id}] + \sum_{-h < k} [\sigma_{h,-h}], \quad [\rho_0^{0,1}][\rho_0^{0,1}] = [\alpha] + \sum_{-h < h} [\sigma_{h,-h}],$$

$$[\sigma_{h,-h}] = [\rho_0^{0,1}] + [\rho_0^{0,1}], \quad [\sigma_{h,-h}] = [\text{id}] + \sum_{-h < h} [\sigma_{h,-h}],$$

$$[\rho_0^{0,1}][\sigma_{h,-h}] = [\rho_0^{0,1}][\sigma_{h,-h}], \quad [\sigma_{h,-h}][\sigma_{h,-h}] = [\text{id}] + [\alpha] + [\sigma_{2h,-2h}],$$

$$[\rho_0^{0,1}][\sigma_{g,-h}][\sigma_{h,-h}] = [\rho_0^{0,1}][\sigma_{g,-h}][\sigma_{h,-h}] + [\rho_0^{0,1}][\sigma_{g,-h}][\sigma_{h,-h}].$$

**Example 4.1.** For $p \in \mathbb{N}$ consider the unitary modular tensor category $C_{\text{Spin}(2p+1)} = C(B_{p,2})$ (often called $C_{\text{SO}(2p+1)}$ in the literature). Let $\rho \in C_{\text{Spin}(2p+1)}$ be one of the two irreducible generating objects with dimension $\sqrt{2p+1}$, then one can calculate from the modular data that $\nu_\rho = (-1)^{[\frac{p+1}{2}]}$. Since $C_{\text{Spin}(2p+1)}$ is braided equivalent to $C_{\text{SU}(2p+1)}$, it follows that we have a braided equivalence

$$C_{\text{Spin}(2p+1)} \cong \mathcal{MP} \left( \mathbb{Z}_{2p+1}, \langle \cdot, \cdot \rangle_{A_{2p}}, (-1)^{\frac{p+1}{2}} \right)$$

where $\langle x, y \rangle_{A_{2p}} = \exp(-\pi i xy/(p+1))$.

Namely, the modular data of this category is given by the Kac–Peterson $S, T$-matrices $[KPS\text{4}]$ with $c = 2p$ and because $C_{\text{Spin}(2p+1)}$ is a metaplectic modular category which are classified by $S, T$, see $[ACRW\text{16}]$ and below. This clarifies the relationship between doubles of Tambara–Yamagami categories and $C(B_{p,2})$ quantum group categories.

**Example 4.2.** Consider the $A_{2p}$ lattice. Then $A_{2p}^*/A_{2p} \cong \mathbb{Z}_{2p+1} := \{ -p, \ldots, +p \}$ with bicharacter and $\langle x, y \rangle_{A_{2p}} = \exp(-\pi i xy/(p+1))$ and quadratic form $q(x) = \langle x, x \rangle_{A_{2p}}^{-1}$ and $\text{Rep}(A_{2p})$ is braided equivalent to $C(\mathbb{Z}_{2p+1}, q)$. Then $A_{2p} \cong A_{\text{SU}(2p+1)}$ $[\text{Sta95, Xu09, Bis12}]$ and $A_{2p}^* \cong A_{\text{Spin}(2p+1)}$ by $[Xu00\text{a}].$

As an application we get the following.

**Proposition 4.3.** Let $p \in \mathbb{N}$, then $\text{Rep}(A_{\text{Spin}(2p+1)}^*)$ is braided equivalent to $C_{\text{Spin}(2p+1)}$. In particular, $A_{\text{Spin}(2p+1)}$ is modular (see Definition 2.7).

**Proof.** It follows directly that $\text{Rep}(A_{\text{Spin}(2p+1)}^*)$ is braided equivalent to $\mathcal{MP}(\mathbb{Z}_{2p+1}, \langle \cdot, \cdot \rangle_{A_{2p}}, \nu_\rho)$ and by (4) we have to only check that the Frobenius–Schur indicator of the generating object equals $\nu_\rho = (-1)^{[\frac{p+1}{2}]}$. But the value of $\nu_\rho$ is distinguished by the twist $\omega_\rho$ thus by the $T$-matrix. Equivalently, one can use Bantay’s formula (2). Thus it is enough to check that the net is modular.
The central charge of $\mathcal{A}_{2p}$ and thus of thus $\mathcal{A}_{\text{Spin}(2p+1)}$ is $c = 2p$. Since every irreducible representation in $\text{Rep}(\mathcal{A}_{\text{Spin}(2p+1)})$ comes from a module of the affine Lie algebra and $T_{\rho\rho} = \text{diag}(e^{2\pi i (T_0 - p/12)})$ by the Guido–Longo spin statistics theorem [GL95] the T-matrix coincides with the Kac–Peterson T-matrix. Since also the $S$-matrices coincide, we get that that $\mathcal{A}$ is modular. □

4.2. Classification of generalized metaplectic modular categories. Let $n$ be odd and $G = \mathbb{Z}_n$, then the fusion rules of $\mathcal{MP}(G, \langle \cdot, \cdot \rangle, \pm)$ are the fusion rules of $\mathcal{Spin}(n)_2$, see Example [11]. Any unitary modular tensor category with these fusion rules is called a metaplectic modular category and metaplectic modular categories up to braided equivalence have been classified in [ACR W16]. Here we allow $G$ to be an arbitrary abelian group of odd order and make the following definition.

Definition 4.4. Let $G = (G, +)$ be a finite abelian group with $|G|$ odd. A unitary modular tensor category of rank $(|G| + 7)/2$ with irreducibles $\{[1], [\alpha], [\rho], [\alpha \rho], [\sigma_g] : g \in G_+\}$ having the commutative fusion rules

$$\begin{align*}
[\alpha]^2 &= [1], \\
[\alpha][\sigma_g] &= [\sigma_g], \\
[\rho]^2 &= [1] + \sum_{g \in G_+} [\sigma_g], \\
[\alpha][\rho] &= [\alpha \rho], \\
[\sigma_g][\sigma_g] &= [1] + [\alpha] + [\sigma_{2g}], \\
[\sigma_g][\sigma_h] &= [\sigma_{g+h}] + [\sigma_{g-h}],
\end{align*}$$

for all $g, h \in G_+$ with $g \neq h$ is called a generalized metaplectic modular category (based on $G$).

Thus by Subsection [11] we get that a generalized metaplectic modular category based on $G$ is a unitary modular tensor category which has the same fusions rules as $\mathcal{MP}(G, \langle \cdot, \cdot \rangle, \pm)$. In particular, $\mathcal{MP}(G, \langle \cdot, \cdot \rangle, \nu)$ is a generalized metaplectic modular category based on $G$.

Lemma 4.5. Let $C$ be a generalized metaplectic modular category, then $\alpha$ has trivial twist $\omega_\alpha = 1$. In particular, $C$ contains a unique Tannakian subcategory $\text{Rep}(\mathbb{Z}_2)$.

This statement has been proven for $G = \mathbb{Z}_n$ in [ACR W16].

Proof. From the fusion rules it follows that $\langle \alpha \rangle$ is braided equivalent to $C(\mathbb{Z}_2, q)$ with $q(g) = \omega_\alpha^g$ where $\omega_\alpha$ is the twist of $[\alpha]$ and $\omega_\alpha^4 = 1$.

From $Y_{\alpha, \sigma} = \sum N_{\alpha, \sigma}^{\alpha, \rho} \omega_{\rho}^{\omega_{\rho}} dp$ we get $Y_{\alpha, \alpha} = \omega_\alpha^2$ and $Y_{\alpha, \sigma} = 2\omega_\alpha$ for every $\sigma = \sigma_\rho$. If we use that $Y_{\alpha, \bullet}$ are multiple of characters [Reh90] of the fusion rules, namely $d_{ij}^{-1} Y_{ij} Y_{kj} = \sum_m N_{ik}^m Y_{mj}$, we get $Y_{\sigma_\rho} = Y_{id, \alpha} + Y_{\alpha, \alpha} + Y_{\sigma, \alpha}$ or equivalently $4\omega_\alpha^4 = 1 + \omega_\alpha^2 + 2\omega_\alpha$ which has only the solution $\omega_\alpha = 1$ fulfilling $\omega_\alpha^4 = 1$. □

Lemma 4.6. Let $C$ be a generalized metaplectic modular category for some odd abelian group $G$, which has a unique Tannakian subcategory $\text{Rep}(\mathbb{Z}_2)$ by Lemma [2.3]. Then $\mathbb{Z}_2$ is a Tambara–Yamagami category based on $G$ and $\mathcal{C}_G^{\text{t}} \cong \mathcal{C}(G, q)$ for some non-degenerate quadratic form $q$ on $G$.

Proof. Let $\Theta$ be the (unique) $\mathbb{Q}$-system with $\theta = [\text{id}] \oplus [\alpha]$. Since $\alpha \otimes \sigma_\rho \cong \sigma_\rho$ we have that each free module $\sigma_\rho \otimes \theta \in \mathcal{C}_\Theta$ splits into two modules $\beta^\pm_\rho$. Let us write $G = \bigoplus_i (g_i)$ with $g_i \in G_+$ of order $n_i$. Let $F_i \subset \mathcal{C}_\Theta$ be the full subcategory $\langle \sigma_{g_i} \otimes \theta \rangle = \langle \beta^+_g \rangle$ generated by $\sigma_{g_i} \otimes \theta$. It follows from the fusion rules that $F_i$ is a pointed category of rank $1 + 2(n_i - 1)/2 = n_i$ since the rank is odd and $1 < \sigma_{g_i} \sigma_{g_i}$ it follows that $\beta^+_g$ and $\beta^-_g$ are each others inverses. Thus $F_i = \langle \beta^+_g \rangle$ and $F = \bigoplus F_i$ has $G$ fusion rules.

The free $\Theta$-module $\hat{\rho} = \rho \otimes \theta$ is irreducible and has dimension $\sqrt{|G|}$. From the fusion rules follows that $\rho \otimes \Theta \hat{\rho} \cong \bigoplus_{\alpha \in \text{Irr}(F_i)} \alpha$. Since $\mathcal{C}_0 = F$ is modular it must be braided equivalent to $\mathcal{C}(G, q)$ for a non-degenerate quadratic form $q$. □

Proposition 4.7. The generating objects $[\rho] = [\rho^+_0] = [\rho^-_0]$ and $[\alpha \rho] = [\rho^0_{\pm 1}]$ in $\mathcal{MP}(G, \langle \cdot, \cdot \rangle, \pm)$ have Frobenius-Schur indicator given by $\nu_\rho = \nu_{\alpha \rho} = \pm 1$, respectively.
Proof. The α-induction of the object ρ is the generating object in the Tambara–Yamagami category whose Frobenius–Schur indicator gives the sign ± of \( \mathcal{T} \mathcal{V}(G, \langle \cdot, \cdot \rangle, \pm) \), therefore it is enough to show that \( \nu_\rho = \nu_{\alpha_\rho^+} \). But if \( \bar{\rho} = \rho \) and \( \bar{R}_\rho = \nu_\rho R_\rho \) is a standard solution, we have that \( (R, \bar{R}) \) gives a standard solution for \( \alpha_\rho^+ \) as for example in [Reh00, Lemma 2.2]. \( \square \)

Let \((G, \langle \cdot, \cdot \rangle)\) with \( G \) a finite abelian group and \( \langle \cdot, \cdot \rangle \) a bicharacter on \( G \). We say \((G, \langle \cdot, \cdot \rangle)\) and \((G', \langle \cdot, \cdot \rangle')\) are equivalent if there is a group isomorphism \( \phi: G \rightarrow G' \), such that \( \langle \phi(g), \phi(h) \rangle' = \langle g, h \rangle \) for all \( g, h \in G \). The same way, we say two bicharacters \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle' \) if there is a group automorphism \( \phi \) of \( G \), such that \( \langle \phi(g), \phi(h) \rangle' = \langle g, h \rangle \) for all \( g, h \in G \).

We have proven that every generalized metaplectic modular category \( \mathcal{C} \) is braided equivalent to \( \mathcal{MP}(G, \langle \cdot, \cdot \rangle, \nu) \), where \( \nu \) is the Frobenius–Schur indicator of any irreducible generating object and \( [\langle G, \langle \cdot, \cdot \rangle \rangle] \) is fixed by either \( Z(\mathcal{C}_{\mathbb{Z}_2}) \) being braided equivalent to \( \mathcal{C} \boxtimes \mathcal{C}(G, \bar{q}) \) or equivalently, by \( \mathcal{C}_{\mathbb{Z}_2}^{\nu} \) being braided equivalent to \( \mathcal{C}(G, q) \), where \( \bar{q}(g) = \langle g, g \rangle \). Therefore we have proven:

**Theorem 4.8.** Let \( G \) be an abelian group of odd order. Generalized metaplectic modular categories based on \( G \) up to braided equivalence are in one-to-one correspondence with pairs \( (\langle \cdot, \cdot \rangle, \nu) \), where \([\langle \cdot, \cdot \rangle]\) is the equivalence class of a non-degenerate symmetric bicharacter on \( G \) and \( \nu \in \{\pm\} \).

The correspondence is given by associating \( \mathcal{MP}(G, \langle \cdot, \cdot \rangle, \pm) \) with \([\langle \cdot, \cdot \rangle]\), respectively.

**Remark 4.9.** Let \( \mathcal{C} \) be a generalized metaplectic modular category based on an abelian group \( G \) of odd order. Then \([\langle \cdot, \cdot \rangle]\) can be recovered from \( q(g) = \langle g, g \rangle^{-1} \) for \( q \) a quadratic form on \( G \), such that \( \mathcal{C}_{\mathbb{Z}_2}^{\nu} \) is braided equivalent to \( \mathcal{C}(G, q) \). The sign is given by the Frobenius–Schur indicator of the generating object as in Proposition 4.7.

Therefore the proof of Theorem 3.8 shows:

**Theorem 4.10.** All generalized metaplectic modular categories (see Definition 4.7) are realized by \( \mathbb{Z}_2 \)-orbifolds of conformal nets associated with lattices.

We have a decomposition into Sylow groups

\[
(G, \langle \cdot, \cdot \rangle) = \bigoplus_p (A_p, \langle \cdot, \cdot \rangle_p) \iff (G, q) = \bigoplus_p (A_p, q_p)
\]

over primes \( p > 2 \). Each \((A_p, q_p)\) can be written as a direct sum of \((\mathbb{Z}_{p^k}, q_{p^k, \pm})\) following [BJ15, Theorem 2.1], cf. [Wal63, Nik79] where the two quadratic forms are given by \( q_{p^k, \pm}(x) = \exp(2\pi i ax^2/p^k) \) for some \( a \) with Jacobi symbol \( (\frac{a}{p}) = \pm 1 \), respectively. Since with \( k \geq 1 \) the only relation [Nik79, Proposition 1.8.1 and 1.8.2] is

\[
(\mathbb{Z}_{p^k}, q_{p^k, 0}) \oplus (\mathbb{Z}_{p^k}, q_{p^k, 0}) \cong (\mathbb{Z}_{p^k}, q_{p^k, 0}) \oplus (\mathbb{Z}_{p^k}, q_{p^k, 0})
\]

it follows that on \( \mathbb{Z}_{p^k}^n \) for \( n \geq 1 \) there are two isomorphism classes of metric groups: \((\mathbb{Z}_{p^k}^n, q_{p^k, +})^{\oplus n} \) and \((\mathbb{Z}_{p^k}^n, q_{p^k, +})^{\oplus (n-1)} \oplus (\mathbb{Z}_{p^k}^n, q_{p^k, -}) \). Therefore, for fixed \( G \) with

\[
G \cong \bigoplus_{i=1}^k \mathbb{Z}_{q_i}^{n_i}
\]

where \( q_1, \ldots, q_k \) are distinct prime powers and \((n_1, \ldots, n_k) \in \mathbb{N}^k \), there are exactly \( 2^k \) isomorphism classes of metric groups or equivalently bicharacters based on \( G \). Together, we have:

**Theorem 4.11.** Let \( G \) be a finite abelian group of odd order with the decomposition into \( k \) summands as in (4.2), then there are \( 2^{k+1} \) braided equivalence classes of generalized metaplectic modular categories based on \( G \).
We remember that the modular data $(S,T)$ up to equivalence is an invariant of a unitary modular tensor category $\mathcal{C}$. It is has been believed that the modular data is a complete invariant but while writing this paper a counterexample has appeared in a preprint [MS17]. Nevertheless, for generalized metaplectic modular categories we have the following:

**Theorem 4.12.** The modular data up to equivalence is a complete invariant for generalized metaplectic modular categories, i.e. the braided isomorphism class of an odd generalized metaplectic modular category is determined by its modular data $(S,T)$.

**Proof.** Let $\mathcal{C}$ be an odd generalized metaplectic modular category with modular data $(S,T)$. The fusion rules of $\mathcal{C}$ are determined by $S$ via the Verlinde formula (1). Let $G$ be the finite abelian group given by the fusion rules of objects of dimension as in the proof of Lemma 4.6. Then objects of dimension two are naturally indexed by $G_+$ which specifies a unique non-degenerate bicharacter $\langle \cdot, \cdot \rangle$ such that $\langle \pm g, \pm g \rangle = T_{g,g}/T_{0,0}$ for all $g \in G_+$. Let $k$ be an object with dimension $\sqrt{|G|}$.

Then Bantay’s formula (2) gives the Frobenius–Schur indicator $\nu$ for this object and from the classification it follows that $\mathcal{C}$ is braided equivalent to $\mathcal{MP}(G,\langle \cdot, \cdot \rangle, \nu)$.

4.3. (Generalized) metaplectic modular categories from condensation. We remember that if $\Theta$ is a commutative Q-system in a unitary modular tensor category we get a new unitary modular tensor category $\mathcal{C}_\Theta^0$. This process is also called condensation, since it corresponds to condensation in topological phases of matter. It also correspond to local extension by the Q-system $\Theta$ if $\mathcal{C}$ is realized by a local conformal net.

We show that certain metaplectic modular categories can be obtained from the basic examples $\mathcal{C}_{\text{Spin}(2n+1)}$ its reverse and semion categories using condensation by finite abelian groups, which are also called simple current extensions. These give simple relations in the Witt group of unitary modular tensor categories.

**Proposition 4.13.** Let $(G_i)_{i=1,\ldots,n}$ be a finite family of abelian groups of odd order and $\mathcal{C}_i = \mathcal{MP}(G_i,\langle \cdot, \cdot \rangle_i, \nu_i)$. Then in

$$\mathcal{C} = \prod_{i=1}^n \mathcal{C}_i$$

we have a $\mathbb{Z}_2^n$ commutative Q-system. Let $K \subset \mathbb{Z}_2^n$ be the subgroup of even codes. Then $\mathcal{C}_K^0$ is braided equivalent to $\mathcal{MP}(G,\langle \cdot, \cdot \rangle, \nu)$, where

$$G = \bigoplus_{i=1}^n G_i, \quad \langle \cdot, \cdot \rangle = \bigoplus_{i=1}^n \langle \cdot, \cdot \rangle_i, \quad \nu = \prod_{i=1}^n \nu_i.$$ 

**Proof.** Taking a conformal net realization $\mathcal{A}^{\mathbb{Z}_2^n}_L \cong \mathcal{A}^{\mathbb{Z}_2^n}_{L_1} \otimes \cdots \otimes \mathcal{A}^{\mathbb{Z}_2^n}_{L_n}$ of $\mathcal{C}$ from [10] let us consider the intermediate net $\mathcal{A}^{\Delta(\mathbb{Z}_2^n)}_L \subset \mathcal{A}_L \cong \mathcal{A}_{L_1} \otimes \cdots \otimes \mathcal{A}_{L_n}$ with $\Delta(\mathbb{Z}_2^n) = \{(0,\ldots,0), (1,\ldots,1)\} \cong \mathbb{Z}_2$.

On the one hand, $\mathcal{A}^{\Delta(\mathbb{Z}_2^n)}_L$ is a simple current extension of $\mathcal{A}_L$ by $K$, thus $\text{Rep}(\mathcal{A}^{\Delta(\mathbb{Z}_2^n)}_L)$ is braided equivalent to $\mathcal{C}_K^0$. On the other hand, $\Delta(\mathbb{Z}_2^n) - \text{Rep}(\mathcal{A}_L) \cong TF(G,\langle \cdot, \cdot \rangle, \nu)$ and therefore $\text{Rep}(\mathcal{A}^{\Delta(\mathbb{Z}_2^n)}_L)$ is braided equivalent to $\mathcal{MP}(G,\langle \cdot, \cdot \rangle, \nu)$.

In purely categorical terms, we have that $\mathcal{C}^{\mathbb{Z}_2^n}$ is equivalent to

$$\mathcal{F} = \prod_{i=1}^n \mathcal{T}Y(G_i,\langle \cdot, \cdot \rangle_i, \nu_i)$$

and there is an obvious injective functor $\mathcal{T}Y(G,\langle \cdot, \cdot \rangle, \nu) \to \mathcal{F}$. Then similarly, one can check that there are braided equivalences $\mathcal{C}_K^0 \cong \mathcal{T}Y(G,\langle \cdot, \cdot \rangle, \nu)^{\mathbb{Z}_2^n} \cong \mathcal{MP}(G,\langle \cdot, \cdot \rangle, \nu)$.

□
Example 4.14. Let \( p \neq 2 \) be a prime number and \( n \in \mathbb{Z} \). Then \( \mathbb{Z}_{pn} \) has up to equivalence two bicharacters. In other words, there are up to equivalence two metric groups \((\mathbb{Z}_{pn}, q_\pm)\). One is \( q_+ := q_{A_{pn-1}} \) from Table I. We have to distinguish two cases:

(1) If \( p \equiv 3 \pmod{4} \), then we denote \( q_- = q_+^* \) which is inequivalent to \( q_+ \).

(2) If \( p \equiv 1 \pmod{4} \), then \( q_+ \) is equivalent to \( q_- \) and therefore there exists an inequivalent quadratic form which we denote by \( q_- \).

Namely, the quadratic form \( q_+ \) of \( \mathbb{Z}_{pn} \) can be represented as \( q_+(x) = \exp(2\pi i x^2 m/p^n) \), where \( m \) is given by \( p^n = 2m + 1 \). Therefore \( q_+ \sim q_\pm \) if and only if \( mx^2 \equiv -m \pmod{p^n} \) has a solution \( x \neq 0 \pmod{p^n} \). But \( x^2 \equiv -1 \pmod{p^n} \) has a solution if and only if \( x^2 \equiv -1 \pmod{p} \) has a solution due to Gauß. By the Law of Quadratic Reciprocity, \( x^2 \equiv -1 \pmod{p} \) has a solution if and only if \( p \equiv 1 \pmod{4} \). So we conclude that \( q_+ \sim q_\pm \) if and only if \( p \equiv 1 \pmod{4} \).

This shows that many (generalized) metaplectic modular categories arise from condensations of \( \mathcal{C}_{\text{Spin}(p^n)_2}^\pm \) and semion categories.

Proposition 4.15. We have:

(1) Let \( N = \prod_{i=1}^r p_i^{n_i} \) with \( p_i \equiv 3 \pmod{4} \) for all \( i = 1, \ldots, r \). Then all \( 2^{r+1} \) metaplectic modular categories for \( \mathbb{Z}_N \) can be obtained from condensing products of \( \mathcal{C}_{\text{Spin}(p_i^{n_i})_2}^\pm \) and \( S^\pm \).

Otherwise, at least \( 2^{r-k+1} \) of the \( 2^{r+1} \) metaplectic modular categories arise this way, where \( k = |\{i : p_i \equiv 1 \pmod{4}\}| \).

(2) Let \( G \equiv \bigoplus_{i=1}^r \mathbb{Z}_{q_i} \) be an odd abelian group with \( q_1 = p_1^{n_1}, \ldots, q_r = p_r^{n_r} \) distinct prime powers and \( (n_1, \ldots, n_r) \in \mathbb{N}^r \) and let \( k = \{|i : p_i \equiv 1 \pmod{4} \}| \). Then at least \( 2^{k-r+1} \) of the \( 2^{r+1} \) generalized metaplectic modular categories arise from condensation of \( \mathcal{C}_{\text{Spin}(p_i^{n_i})_2}^\pm \) and \( S^\pm \).

(3) All odd generalized metaplectic modular categories can be obtained from condensing products of the following list of unitary modular tensor categories:

- \( \mathcal{C}_{\text{Spin}(p^n)_2}^\pm \) with odd \( p \) prime and \( n \in \mathbb{N} \),
- metaplectic modular categories \( \mathcal{M}_{\text{Spin}}(\mathbb{Z}_{p^n}, \tilde{q}, +) \) with odd \( p \) and \( n \in \mathbb{N} \), such that \( p \equiv 1 \pmod{4} \) (here \( \tilde{q} \) is a non-degenerate quadratic form with \( \tilde{q} \not\sim q_+ = q_{A_{pn-1}} \)), and
- the two semion categories \( S^\pm \).

Example 4.16. The 8 braided equivalence classes of (generalized) metaplectic modular categories with \( |G| = 15 \) (rank 11) are given by \( \mathcal{C}_{\text{Spin}(15)_2}^\pm \), the condensation \( \mathcal{C}_{\text{Spin}(13)_2}^\pm \otimes \mathcal{C}_{\text{Spin}(3)_2}^\pm \) and the four twists \( \hat{C} = (C \otimes S^\pm \otimes S^-)^{0}_{E_2} \) of these.

Everything in this subsection can be proved using only tensor categories. We could have therefore proved the reconstruction results Theorem 4.10 and similarly Theorem 3.8 by proving the reconstruction only for cyclic groups of prime power orders.

5. Several relations to generalized dihedral groups

5.1. Generalized Tambara–Yamagami categories and generalized dihedral groups. Let \( G = (G, \cdot) \) be a abelian group of odd order seen as a multiplicative group. Let us consider the generalized dihedral group \( \text{Dih}(G) = G \times_{\alpha} \mathbb{Z}_2 \), where \( \text{Aut}(G) \ni \alpha : g \mapsto g^{-1} \), i.e. \( \text{Dih}(G) = G \sqcup G \tau \), with \( \tau^2 = e \) and \( \tau g \tau = g^{-1} \). Consider the following fusion rules of \( \text{Dih}(G) \cup \{\rho_+, \rho_-\} \):

\[
\begin{align*}
|\rho_\pm|^2 &= \sum_{g \in G} |g|, & |\rho_\pm||\rho_\mp| &= \sum_{g \in G} |g\tau|, \\
|g||\rho_\pm| &= |\rho_\pm||g| = |\rho_\pm|, & |\tau||\rho_\pm| &= |\rho_\pm| = |\rho_\mp|.
\end{align*}
\]

These fusion rules can be seen as a generalization of Tambara–Yamagami fusion rules.
Proposition 5.1. Let $A$ be a completely rational conformal net with $\text{Rep}(A)$ a generalized metaplectic modular category based on a finite abelian group $G$ of odd order and $B = A \rtimes \mathbb{Z}_2$ the $\mathbb{Z}_2$-simple current extension (e.g. for example $A = A_L^{\mathbb{Z}_2} \subset B = A_L$ from Theorem 4.10). Then the category of $B$–$B$ sectors coming from $A \subset B$ has the fusion rules (7).

Proof. Let $n = |G|$. The modular invariant [BEK99, BEK00] for the inclusion $A = A(I) \subset B = B(I)$ for the unique $\mathbb{Z}_2$-simple current extension $A \subset B = A \rtimes \mathbb{Z}_2$ can calculated to be:

$$Z = |\chi_{id} + \chi_{\alpha}|^2 + \sum_{h \in G \setminus \{0\}} |\chi_{\sigma_{n-h}}|^2 = |\chi_0 + \chi_{r}|^2 + \sum_{i=1}^{\frac{n-1}{2}} 2|\chi_i|^2.$$ 

We get $\text{tr}(Z) = n + 1$ and $\text{tr}(ZZ^*) = 2n + 2$, which gives [BEK99 Corollary 6.10] that $|\text{Irr}(B^0C_B)| = 2n + 2$. Further we have $\dim C^0_B = \dim(B^0C_B) = n$, $\dim C_{\mathbb{Z}_2} = \dim(B^0C_B^+) = 2n$ and $\dim(C_{\mathbb{Z}_2}^{\alpha}) = \dim(B^0C_B) = n$. 

We have that $D_{\pm} := B^0C_B^\pm$ are Tannaka–Yamagami categories, see Lemma 4.6. Let us denote $\text{Irr}(B^0C_B) = G \cup \{\rho_{\pm}\}$. Since $A \subset B$ is a fixed point under an outer $\mathbb{Z}_2$-action the canonical endomorphism $\gamma \in D := \mathbb{Z}_2C_{\mathbb{Z}_2}$ is of the form $[\gamma] = [id] \bigoplus [\tau]$ for some $\tau \in D$ with $[\tau]^2 = [id]$. By a simple counting argument we know that $\text{Irr}(D) = G \cup \rho_{\pm} \cup \{\alpha_k : k = 1, \ldots, n\}$, with $\alpha_k$ automorphisms. Since $\tau \not\in D_\pm$ see e.g. [Bis17, Lemma 5.8], we get that $[\tau] \in \{[\alpha_k] : k = 1, \ldots, h\}$. 

The $\mathbb{Z}_2$-grading on $D_\pm$ gives a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading on $D$. Since $D := \langle D_+, D_- \rangle = \langle \rho_+, \rho_- \rangle$ we have $[\tau] \prec [\rho_+] [\rho_-]$, which is equivalent to $[\tau][\rho_+] \prec [\rho_+]$. Therefore we get $[\tau][\rho_+] = [\rho_+]$. But this implies $[\rho_+] [\rho_-] = \sum_{g \in G} [g\tau]$. Finally, $[\tau][g][\tau] = [g^{-1}]$ follows from how $\mathbb{Z}_2$ acts on $D_0$. □

The purely categorical formulation of this proposition is:

Proposition 5.2. Let $C$ be a generalized metaplectic modular category based on $G$ with $|G|$ odd, then there is a unique Tannakian subcategory $\text{Rep}(\mathbb{Z}_2)$ and $D = \mathbb{Z}_2C_{\mathbb{Z}_2}$ has the fusion rules (7).

The following is a classical extension problem of finite groups with cocycles. We consider the extension of groups

$$1 \rightarrow A \rightarrow \text{Dih}(A) \rightarrow \text{Dih}(A)/A \cong \mathbb{Z}_2 \rightarrow 1.$$

For $[\omega] \in H^3(\text{Dih}(A), T)$, by restriction we get elements $[\omega \mid A] \in H^3(A, T)$ and $[\omega \mid (\tau)] \in H^3(\mathbb{Z}_2, T)$ for every order two element $\tau \in \text{Dih}(A)$.

Lemma 5.3. Let $A$ be an odd abelian group. Then the restriction map $H^3(\text{Dih}(A), T) \rightarrow H^3(A, T) \oplus H^3(\mathbb{Z}_2, T)$ is injective. In particular, there are two classes $[\omega_{\pm}]$ in

$$\{[\omega] \in H^3(\text{Dih}(A), T) : [\omega \mid A] \in B^3(A, T)\},$$

which are distinguished by $[\omega_{\pm} \mid (\tau)] = [\pm] \in H^3(\mathbb{Z}_2, T) = \{[\pm]\}$.

Proof. $H^3(\text{Dih}(A), T)$ can be calculated by the second page of the Lyndon–Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(\mathbb{Z}_2, H^q(A, T)) \implies H^{p+q}(\text{Dih}(A), T)$$

and since $A$ is odd one can calculate that

$$H^3(\text{Dih}(A), T) \cong E_2^{3,0} \oplus E_2^{0,3} \cong H^3(A, T)^T \oplus \mathbb{Z}_2$$

and there are no differentials in the spectral sequence since they would connect 2-groups with $p$-groups for odd $p$. Restriction to $A$ is the projection onto the first component and restriction to any subgroup of order 2 injects onto the second component. Alternative, one can show that the cohomology group in [IK02 Definition 2.3] vanishes. □
Proposition 5.4. Let \( A \) be an odd a abelian group. There are up to equivalence two pointed \( \mathbb{Z}_2 \)-extensions of \( \text{Hilb}_A \) with fusion rules given by \( \text{Dih}(A) \). They are \( \mathcal{F}_+ = \text{Hilb}_{\text{Dih}(A)} \) and \( \mathcal{F}_- = \text{Hilb}^{\omega_-}_{\text{Dih}(A)} \) which are co-commutative by \([\text{IK}02, \text{Corollary 9.9}]\), thus it is a group subfactor by an outer action of a Kac algebra by Ocneanu’s characterization, see \([\text{Izu}93, \text{Szy}94, \text{Lon}94, \text{Dav}96, \text{Sun}97, \text{Izu}98]\), and which is co-commutative by \([\text{Izu}97, \text{Izu}98]\), and which is co-commutative by \( \text{Dih}(A) \), and for each order two element \( \tau \in \text{Dih}(A) \) we have that the subcategory \( \langle \tau \rangle \) is equivalent to \( \text{Hilb}_{\mathbb{Z}_2} \) and therefore \( \nu_\tau = \pm 1 \) for every order two element \( \tau \in \mathcal{F}_\pm \).

Proof. By Lemma \([5.3]\) the only extensions are \( \text{Hilb}^{\omega_\pm}_{\text{Dih}(A)} \), and for each order two element \( \tau \) we obtain doubles of \( \text{Dih}(A) \), and thus \( \text{Hilb}_{\mathbb{Z}_2} \) is braided equivalent to \( \mathcal{C}(A,G) \) for some quadratic form \( q \) and thus tensor equivalent to \( \text{Hilb}_A \) by Lemma \([2.3]\) we get \( \nu_\tau = 0 \in \text{Hom}(G,\mathbb{T}) \). Since \( \nu_\tau \) is a Q-system we get that \( \nu_\tau = 0 \in \text{Hilb}(\mathbb{Z}_2,\mathbb{T}) \). This implies \( \nu_\tau = 0 \in \text{Hilb}(\mathbb{Z}_2,\mathbb{T}) \) by Lemma \([5.3]\) \( ∎ \)

Proposition 5.5. Let \( C \) be a generalized metaplectic modular category and \( D = \mathbb{Z}_2 \mathbb{C}_{\mathbb{Z}_2} \) as in Proposition \([5.2]\). Then the pointed subcategory \( D^\times \subset D \) is equivalent to \( \text{Hilb}_{\text{Dih}(A)} \).

Proof. We have that \( D^\times \) is equivalent to \( \text{Hilb}^\omega_{\text{Dih}(A)} \) for some \( \omega \in H^3(A,\mathbb{T}) \). Since \( D^0 = \mathbb{C}_{\mathbb{Z}_2}^0 \) is braided equivalent to \( \mathcal{C}(A,q) \) for some quadratic form \( q \) and thus tensor equivalent to \( \text{Hilb}_A \) by Lemma \([2.3]\) we get \( \nu_\tau = 0 \in \text{Hom}(G,\mathbb{T}) \). Since \( \nu_\tau \) is a Q-system we get that \( \nu_\tau = 0 \in \text{Hilb}(\mathbb{Z}_2,\mathbb{T}) \). This implies \( \nu_\tau = 0 \in \text{Hilb}(\mathbb{Z}_2,\mathbb{T}) \) by Lemma \([5.3]\) \( ∎ \)

Proposition 5.6. Let \( A \) be an abelian group of odd order. Let \( C \) be a generalized metaplectic modular category based on \( A \), i.e. \( C \cong \mathcal{M}(A,\langle \cdot,\cdot \rangle,\pm) \). Then the even part \( C_0 \) of \( C = C_0 \oplus C_1 \) is tensor equivalent to \( \text{Rep}(\text{Dih}(A)) \). As a braided fusion category \( C_0 \) is degenerate with Müger center braided equivalent to \( \text{Hilb}(\mathbb{Z}_2) \).

Proof. By Theorem \([1.10]\) there is an even lattice \( L \) with \( L^*/L \cong A \) and a \( \mathbb{Z}_2 \)-action on \( A_L \), such that \( \text{Rep}(A_L^\omega) \) is braided equivalent to \( \mathcal{M}(A,\langle \cdot,\cdot \rangle,\pm) \). Since \( \text{Rep}(A_L^\omega) \) is tensor equivalent to \( \text{Hilb}_A \) and in particular the obstruction in \( H^3(A,\mathbb{T}) \) vanishes by Lemma \([2.3]\) we can consider the crossed product extension \( A_L^\omega(I) \subset A_L(I) \subset A_L(I) \rtimes A \). From the branching rules it easily follows that \( \theta = [\text{id}] \oplus [\alpha] \oplus \bigoplus_{a \in A^\omega} 2[\sigma_a] \), thus the inclusion has depth two and thus it is a crossed product by an outer action of a Kac algebra by Ocneanu’s characterization, see \([\text{Izu}93, \text{Szy}94, \text{Lon}94, \text{Dav}96, \text{Sat}97, \text{Izu}98]\), and which is co-commutative by \([\text{IK}02, \text{Corollary 9.9}]\), thus it is a group subfactor \( N^G \subset N \) by \([\text{VK}74, \text{Izu}91, \text{BS}93]\). It is also of the form, \( M_{\mathbb{Z}_2} \subset M \rtimes A \) with \( A,\mathbb{Z}_2 = \text{Dih}(A) \) with trivial 3-cocycle by Proposition \([5.5]\). Thus we can conclude that \( G = \text{Dih}(A) \). \( ∎ \)

5.2. Doubles of generalized dihedral groups. In this section we want to clarify the relation between doubles of generalized dihedral groups and generalized metaplectic modular categories.

Let \( A \) be an abelian group of odd order. Using the Galois correspondence of Longo–Rehren subfactors \([\text{Izu}00]\), we obtain doubles of \( \text{Dih}(A) \). Therefore, let \( C \) be a generalized metaplectic modular category based on \( A \), i.e. \( C \) is braided equivalent to \( \mathcal{M}(A,\langle \cdot,\cdot \rangle,\nu) \). There is a unique Tannakian subcategory \( \text{Rep}(\mathbb{Z}_2) \subset C \). We have braided equivalences \( Z(\mathbb{Z}_2 \mathbb{C}_{\mathbb{Z}_2}^0) \cong Z(C) \cong C \boxtimes C^{\text{ev}} \). Further, \( \text{Hilb}_{\text{Dih}(A)} \subset \mathbb{Z}_2 \mathbb{C}_{\mathbb{Z}_2} \) by Proposition \([5.5]\). Using the Galois correspondence, we have a Tannakian subcategory \( \text{Rep}(\mathbb{Z}_2) \subset C \boxtimes C^{\text{ev}} \), such that \( (C \boxtimes C^{\text{ev}})^0_{\mathbb{Z}_2} \cong Z(\text{Hilb}_{\text{Dih}(A)}) \). We will show that it is possible to twist this construction and to obtain all twisted doubles of \( \text{Dih}(A) \).

Let \( A \) be an abelian group and \( \hat{A} = \text{Hom}(A,\mathbb{T}) \) the Pontryagin dual. The canonical pairing \( q_{\text{can}} : \hat{A} \oplus A \to \mathbb{T} \) given by \( q_{\text{can}}(\chi, a) = \chi(a) \) is a non-degenerate quadratic form on \( \hat{A} \oplus A \). The Drinfel’d center \( Z(\text{Hilb}_A) \) is braided equivalent to \( \mathcal{C}(\hat{A} \oplus A, q_{\text{can}}) \). Let \( (G, q) \) be a metric group, we say a subgroup \( L \subseteq G \) is Lagrangian if \( |L|^2 = |G| \) and \( q \mid L = 1 \).

Proposition 5.7. Let \( A \) be an abelian group of odd order. The following metric groups are equivalent:

1. \( (A \oplus A, q \oplus \bar{q}) \), where \( q \) is a non-degenerate bicharacter on \( A \).
2. \( (A \oplus \hat{A}, q_{\text{can}}) \) where \( q_{\text{can}} \) is the canonical pairing.
3. \( (G, q) \) a metric group admitting a Lagrangian subgroup \( L \cong \hat{A} \) and \( C(G, q)_A \cong \text{Hilb}_A \).
Proof. For each of the \((G, q)\) we have that \(C(G, q)\) is braided equivalent to \(Z(\text{Hilb}_A)\). Namely, in (1), using Lemma [2.3] we have \(Z(\text{Hilb}_A) \cong Z(C(A, q)) \cong C(A \oplus A, q \oplus q)\). In (3), we have \(C(G, q)\) that is braided equivalent to \(Z(C(G, q)_A)\), since \(\hat{A}\) gives rise to a Lagrangian algebra [DMNO13].

**Proposition 5.8.** Let \(A\) be an abelian group and \((G, q)\) be a metric group based on \(A\) as in Proposition [7.7] Then \(\mathcal{MP}(G, q, \pm)\) is braided equivalent to \(Z(\text{Hilb}_{\text{Dih}(A)}^{\omega^\pm})\), where \([\omega^\pm] = 0 \in H^3(\text{Dih}(A), \mathbb{T})\) and \([\omega^-]\) is the unique order two element in \(H^3(\text{Dih}(A), \mathbb{T})\).

Proof. Let \(C_{\pm} = \mathcal{MP}(A, \langle \cdot, \cdot \rangle, \pm)\) and we can consider the modular tensor category \(C := (C_{\mu} \boxtimes C_{\nu} \boxtimes C_{\mu \nu})^0_{Z_2}\). It follows that \(C\) is braided equivalent to \(\mathcal{MP}(A \oplus A, \langle \cdot, \cdot \rangle \oplus \langle \cdot, \cdot \rangle, \mu \nu)\). Namely, \(C_{\mu} \boxtimes C_{\nu} \boxtimes C_{\mu \nu} \cong \mathcal{F}^Z_{x \times Z_2}\) with \(\mathcal{F} = \mathcal{T} \mathcal{Y}(A, \langle \cdot, \cdot \rangle, \nu) \boxtimes \mathcal{T} \mathcal{Y}(A, \langle \cdot, \cdot \rangle, \mu)\) and \(\mathcal{F}\) contains the subcategory \(\mathcal{F}_0 \cong \mathcal{T} \mathcal{Y}(A \oplus A, \langle \cdot, \cdot \rangle \oplus \langle \cdot, \cdot \rangle, \pm)\) and we have \(\mathcal{F}^Z_0 \cong (C_{\mu} \boxtimes C_{\nu} \boxtimes C_{\mu \nu})^0_{Z_2}\). There is a Lagrangian subgroup \(L\) in the pointed modular tensor category \(D = C^0_{Z_2}\) and we have \(\mathcal{F}^Z_0 \cong \text{Hilb}_A\), which by restriction gives a Lagrangian algebra \(\Theta\) in \(C := \mathcal{MP}(A \oplus A, \langle \cdot, \cdot \rangle \oplus \langle \cdot, \cdot \rangle, \mu \nu)\). In the case, \(\mu = \nu\) we have discussed above that \(C_\Theta \cong \text{Hilb}_{\text{Dih}(A)}\). In the case, \(\mu = -\nu\), the fusion rules do not change and we have \(C_\Theta \cong \text{Hilb}_{\text{Dih}(A)}\) for some class \([\omega] \in H^3(\text{Dih}(A), \mathbb{T})\). But \(\text{Hilb}_{\text{Dih}(A)}\) is an extension of \(\text{Hilb}_A\) and therefore \(\omega \in [\omega^\pm]\). Finally, the sign is determined by the Frobenius–Schur indicator as in Proposition [4.7].

The following example shows that we can also twist by certain elements in \(H^3(A, \mathbb{T})\).

**Example 5.9.** Take \(C(\mathbb{Z}_3, q) \cong C_{\text{SU}(3)} \cong \text{Rep}(A_{\mathbb{Z}_3})\), i.e. \(q(x) = e^{\pi i x^2/3}\). Then there is a unique Lagrangian subgroup \(A \cong \mathbb{Z}_3\). It corresponds to the conformal embedding \(A_{\mathbb{Z}_3} \subset A_{\mathbb{Z}_3}\). One can check that \(C_{\text{Spin}(\mathbb{Z}_3)}\) is braided equivalent to \(Z(\text{Hilb}_{\text{Dih}(\mathbb{Z}_3)})\) for a generator \([\omega] \in H^3(\text{Dih}(\mathbb{Z}_3), \mathbb{T})\). We note that \(\text{Dih}(\mathbb{Z}_3)\) is isomorphic to the symmetric group \(S_3\).

It follows that, \(\text{Rep}(A_{\text{Spin}(\mathbb{Z}_3)})\) is braided equivalent to \(Z(\text{Hilb}_{S_3})\). We get the other twist by using that for \(L = A_2^2 \oplus \mathbb{Z}_3\) the net \(\mathcal{A}_L\) realizes \(C(\mathbb{Z}_3, q')\) with \(q'(x) = e^{4 \pi i x^2/3}\). Then \(A_2^2\) realizes \(Z(\text{Hilb}_{S_3})\) for some \(\omega'\) with \([\omega' \mid \mathbb{Z}_3] = [\omega \mid \mathbb{Z}_3]\). The trivial cocycle is realized by \(A_1^2 \oplus A_2 E_6\). The other three elements \(H^3(S_3, \mathbb{T})\) with non-trivial 2-part can be obtained by the twisting as in Proposition [3.2]. This way we can realize all twisted doubles of \(S_3\) by a \(\mathbb{Z}_2\)-orbifold \(A_1^2 \mathcal{Z}_2\) of a conformal net associated with a lattice.

We also see that the six generalized metaplectic modular categories

\[
\mathcal{MP}(\mathbb{Z}_3^2, \langle \cdot, \cdot \rangle_{\text{can}}, \pm), \quad \mathcal{MP}(\mathbb{Z}_3, \langle \cdot, \cdot \rangle_{\text{can}}, \pm)
\]

are all braided equivalent to some \(Z(\text{Hilb}_{S_3})\) with all six possible cohomology classes \([\omega] \in H^3(S_3, \mathbb{T})\) arising. In particular, this shows that the twisted doubles of \(S_3\) have two different fusion rules depending if \([\omega \mid \mathbb{Z}_3]\) is trivial or not.

A UFC \(F\) is called group-theoretical if there is a finite group \(G\) and \([\omega] \in H^3(G, \mathbb{T})\), such that \(Z(\mathcal{F})\) is braided equivalent to \(Z(\text{Hilb}_{S_3})\), or equivalently, \(F\) is (weakly monoidally) Morita equivalent to \(\text{Hilb}_{S_3}^\mathbb{C}\) [Mig03a, Mig03b]. In [GNN09, Theorem 4.6] it is shown that \(\mathcal{T} \mathcal{Y}(G, \langle \cdot, \cdot \rangle, \pm)\) is group theoretical if and only if \((G, \langle \cdot, \cdot \rangle)\) contains a Lagrangian, i.e. \(L \leq G\) with \([L]^2 = |G|\) and \(\langle \cdot, \cdot \rangle \mid L \equiv 1\). For \(G\) odd this is equivalent to \(L\) being a Lagrangian subgroup of \((G, q)\).

Assume \((G, q)\) is a metric group with a Lagrangian subgroup \(\hat{A}\). Then \(C(G, q)\) has a Tannakian subcategory \(\text{Rep}(A)\). The de-equivariantization \(C(G, q)_A\) is tensor equivalent to \(\text{Hilb}_\mathbb{A}\) for some \([\omega] \in H^3(A, \mathbb{T})\) [DGN010, Proposition 4.58] and \(C(G, q)\) is braided equivalent to \(Z(\text{Hilb}_{\mathbb{A}}^\mathbb{C})\) [DMNO13, Proposition 4.8].

**Proposition 5.10.** Let \((G, q)\) be a metric group of odd order with Lagrangian subgroup \(\hat{A}\) and \(\langle \cdot, \cdot \rangle\) the non-degenerate bi-character on \(G\), such that \(q(x) = (x, x)^{-1}\). Then \(\mathcal{MP}(G, \langle \cdot, \cdot \rangle, \pm)\) is braided equivalent to \(Z(\text{Hilb}_{\text{Dih}(A)}^{\omega^\pm})\) for some \([\omega^\pm] \in H^3(\text{Dih}(A), \mathbb{T})\).
In this case, \( C(G,q) \) is tensor equivalent to \( \text{Hilb}_A^\omega \) with \([\omega] = [\omega_\pm \upharpoonright A] \). Let \( \tau \in \text{Dih} (\hat{A}) \) with \( (\tau) \cong \mathbb{Z}_2 \) then \([\omega_\pm \upharpoonright (\tau)] \cong [\pm] \) in \( H^3 (\mathbb{Z}_2, \mathbb{T}) \), respectively, i.e. every simple self-dual element \( \tau \in \text{Hilb}_\omega^{\mathbb{Z}_2} (A) \) has Frobenius-Schur indicator \( \nu_\tau = \pm 1 \), respectively.

**Proof.** Let \( A^{\mathbb{Z}_2} \) be a realization of \( \mathcal{M} \mathcal{P} (G, \cdot , \cdot , \pm) \) from Theorem 4.10. We can consider the simple current extension \( A_L \times A \) of \( A_L \) by \( A \), in other words the Lagrangian subgroup \( A \leq G = L^*/L \) gives a self-dual lattice \( \Gamma = L \oplus A \) and \( A_L \times A \) can be identified with \( A_T \). The category of \( A \) twisted representations of \( A_T \) is equivalent to \( \text{Hilb}_A^\omega \) for some \([\omega] \in H^3 (A, \mathbb{T}) \) by the above discussion or [Müg10] 3.6 Corollary, see [Bis17] Theorem 1.7. By considering the inclusion \( A^{\mathbb{Z}_2}_L \subset A_L \subset A_T \) it follows from the branching rules that \( A^{\mathbb{Z}_2}_L \subset A_T \)

\[
[\theta] = [\text{id}] \oplus [\alpha] \oplus \bigoplus_{a \in A} 2[\sigma a]
\]

and the inclusion has depth 2, thus by [Bis17] Corollary 1.2 we have that \( A^{\mathbb{Z}_2}_L \) is an orbifold \( A_T^H \) of \( A_T \) for some group \( H \triangleright A \) of order \( 2|A| \). A similar argument as in the proof of Proposition 5.6 shows that \( H = \text{Dih} (A) \). The category of \( H \)-twisted representations of \( A_T \) is a \( \mathbb{Z}_2 \)-extension \( \text{Hilb}_H^\omega \) of \( \text{Hilb}_A^\omega \) with \( \tilde{\omega} \upharpoonright A = \omega \). As in Proposition 4.7 it follows that the \( \nu_\tau = \pm 1 \).

**Proposition 5.11.** Let \( A \) be an abelian group of odd order, \([\tilde{\omega}] \in H^3 (\text{Dih} (A), \mathbb{T}) \), and \([\omega] \in H^3 (A, \mathbb{T}) \) given by \( \omega = \tilde{\omega} \upharpoonright A \). Further, assume that \( Z (\text{Hilb}_A^\omega) \) is pointed, i.e. braided equivalent to \( \mathcal{C} \langle G_A, \omega, q_{A, \omega} \rangle \) for some metric group \((G_A, \omega, q_{A, \omega})\).

Then \( Z (\text{Hilb}_\omega^{\text{Dih} (A)}) \) is braided equivalent to \( \mathcal{M} \mathcal{P} (\mathcal{C} (G_A, \omega, q_{A, \omega}), \nu_\tau) \), where \( \nu_\tau \) is the Frobenius-Schur indicator of any order two element \( \tau \in \text{Hilb}_\omega^{\text{Dih} (A)} \).

**Proof.** We have a Tannakian subcategory \( \text{Rep} (\text{Dih} (A)) \) of \( \mathcal{C} := Z (\text{Hilb}_\omega^{\text{Dih} (A)}) \), such that \( \mathcal{C} \cong \text{Hilb}_\omega^{\text{Dih} (A)} \). From the Galois correspondence we get a Tannakian subcategory \( \text{Rep} (\mathbb{Z}_2) \) of \( \mathcal{C} \), such that \( \mathcal{C}^{\mathbb{Z}_2} \cong Z (\text{Hilb}_A^\omega) \) which by assumption is pointed. Thus \( \mathcal{C}^{\mathbb{Z}_2} \) is \( \mathbb{Z}_2 \)-crossed braided extension \( \mathcal{F} \) of \( Z (\text{Hilb}_A^\omega) \) which is equivalent to the relative center \( Z (\text{Hilb}_A^\omega) \) by [GNN09] Theorem 3.5. Let \((\rho, \varepsilon_\rho) \in Z (\text{Hilb}_A^\omega) \) irreducible, where \( \varepsilon_\rho = \{ \varepsilon_\rho (\sigma) : \rho \sigma \rightarrow \sigma \rho \}_{\sigma \in \text{Hilb}_A^\omega} \) and assume \( \nu_\rho = \rho \). Then because of the half-braiding, we have \([\rho][g] = [g][\rho] \) for all irreducible sectors \([g] \) in \( \text{Hilb}_A^\omega \). Thus \([g][\tau][g^{-1}] = \rho \trans [\tau] \rho \) and because \([A] \) is odd, we have \( \bigoplus_{g \in A} [g][\tau][g^{-1}] \subset \rho \) and \( d \rho \geq |A| \). Via a counting argument involving the global dimensions we get \( \text{Irr} (\mathcal{F}) = G \cup \{ (\rho, \varepsilon_\rho) \} \) and \( d \rho = |A| \). Thus can conclude that \( \mathcal{F} \) is tensor equivalent to \( \mathcal{I} \mathcal{Y} (G, \cdot , \cdot , \pm) \) with \( q_{A, \omega} (g) = (g, g) \).

**Remark 5.12.** We note that \( Z (\text{Hilb}_A^\omega) \) is pointed if and only if \([\omega] \in H^3 (A, \mathbb{T}) \) and \([\omega] \in H^3 (\text{Dih} (A), \mathbb{T}) \) by [MN01] Corollary 3.6, see also [Ng03] Proposition 4.1. Here \( \psi^* : H^3 (G, \mathbb{T}) \rightarrow \text{Hom} (\Lambda^3 G, \mathbb{T}) \) is given by

\[
[\psi^* ([\omega]) (x, y, z) = \prod_{\omega \in S_3} \omega (\sigma (x), \sigma (y), \sigma (z))^{\text{sign} (\sigma)}
\]

For example, let \( G = (\mathbb{Z}_n)^3 \) we have the cocycle \( \omega (x, y, z) = \zeta_n^{xyyzxz} \) and \( Z (\text{Hilb}_A^\omega) \) is not pointed [Ng03] Example 4.5.

**Lemma 5.13.** Let \( A \) be an abelian group of odd order, \( \tilde{\omega} \in H^3 (\text{Dih} (A), \mathbb{T}) \), and \( \omega = \tilde{\omega} \upharpoonright A \). Then \([\omega] \in H^3 (A, \mathbb{T}) \).

**Proof.** Let \( \varphi = \psi^* \omega \in \text{Hom} (\Lambda^3 (A), \mathbb{T}) \). For any \( c : A \times A \times A \rightarrow \mathbb{T} \) let us denote by \( c^\tau \) the map \( c^\tau : A \times A \times A \rightarrow \mathbb{T} \) with \( c^\tau (g, h, k) = c (g^{-1}, h^{-1}, k^{-1}) \).

Let \( \tau \in \text{Dih} (A) \) with \( \tau^2 = 1 \) and let

\[
\xi (g, h) = \frac{\tilde{\omega} (\tau, g, h) \tilde{\omega} (g^{-1}, h^{-1}, \tau) \tilde{\omega} (g^{-1}, \tau, h)}{\tilde{\omega} (g^{-1}, \tau, h)}
\]

25
which correspond to the associator \((g^{-1}h^{-1})\tau \rightarrow \tau(gh)\) in \(\text{Hilb}_{\text{Dih}(A)}\). Considering the associator \(\omega(g,h,k)\) of \(\tau((gh)k) \rightarrow \tau(g(hk))\) and \(\varphi^r(g,h,k)\) of \(((g^{-1}h^{-1})k^{-1})\tau \rightarrow (g^{-1}(h^{-1}k^{-1}))\tau\) in \(\text{Hilb}_{\text{Dih}(A)}\). We get (similarly to [Izu16 Lemma 2.5]) that \(\omega = \omega^r\) with \(\omega_g(h,k) = \frac{\xi(h,k)\xi(g,hk)}{\xi(g,h)\xi(h,k)}\) for the above \(\xi: A \times A \rightarrow T\). From the symmetry of \(\omega\) follows \(\varphi = \varphi^r\) and since \(\varphi \in \text{Hom}(\Lambda^3(A), T)\), we have \(\varphi^r = \varphi^{-1}\). Thus \(\varphi^2 = 1\) and since \(A\) is odd we have \(\varphi = 1\) which is equivalent to \(\omega \in H^3(A, T)\).

In other words, the restriction gives a split exact sequence

\[
\{0\} \rightarrow H^3(\mathbb{Z}_2, \mathbb{T}) \overset{\text{res}}{\longrightarrow} H^3(\text{Dih}(A), \mathbb{T}) \overset{\text{res}}{\longrightarrow} H^3(A, T) \rightarrow \{0\}
\]

where \(H^3(A, T) \oplus H^3(\mathbb{Z}_2, \mathbb{T}) \rightarrow H^3(\text{Dih}(A), \mathbb{T})\) is given by Proposition 5.11. Further, the proof of Corollary 5.13 shows that \(H^3(A, T) \cong H^3(A, T)\) if \(\omega\) is group theoretical then it admits an Lagrangian \(\tilde{A}\) and \(\text{MP}(G, \langle \cdot, \cdot \rangle, \pm)\) is the Frobenius–Schur indicator \([\tilde{\omega}] \in H^3(\mathbb{Z}_2, T)\) and \((G, \langle \cdot, \cdot \rangle, \pm)\) as above.

Corollary 5.14. All twisted doubles of generalized dihedral groups \(\text{Dih}(A)\) with \(A\) of odd order are generalizes metaplectic modular categories.

Namely, \(Z(\text{Hilb}_{\text{Dih}(A)})\) is braided equivalent to \(\mathcal{MP}(G_{A,\omega}, \langle \cdot, \cdot \rangle, \nu)\), where \(\omega = \tilde{\omega} \uparrow A\) and \(\nu\) is the Frobenius–Schur indicator \([\tilde{\omega}] \in H^3(\mathbb{Z}_2, T)\) and \((G_{A,\omega}, \langle \cdot, \cdot \rangle, \pm)\) as above.

Corollary 5.15. A Tambara–Yamagami category \(\mathcal{F}\) based on an abelian group of odd order is group theoretical if and only if \(\mathcal{F} \cong \mathcal{T}\mathcal{Y}(G_{A,\omega}, \langle \cdot, \cdot \rangle, \pm)\) for some abelian group \(A\) of odd order and some \(\omega \in H^3(A, T)\).

In this case, \(\mathcal{F}\) is Morita equivalent to \(\text{Hilb}_{\text{Dih}(A)} \otimes \tilde{\text{Hilb}}_{\tilde{A}}\) or equivalently \(\text{Hilb}_{\text{Dih}(A)} \otimes \text{Hilb}_{\tilde{A}}\), where \([\omega] \in H^3(\text{Dih}(A), T)\) is characterized by \([\omega] \uparrow A = [\omega]\) and \([\omega] \uparrow \mathbb{Z}_2 = [\pm] \in H^3(\mathbb{Z}_2, T)\), respectively.

Proof. If \(\mathcal{T}\mathcal{Y}(G, \langle \cdot, \cdot \rangle, \pm)\) is group theoretical then it admits an Lagrangian \(\tilde{A}\) and \(\mathcal{MP}(G, \langle \cdot, \cdot \rangle, \pm)\) is braided equivalent to \(Z(\text{Hilb}_{\text{Dih}(A)})\) as in Proposition 5.11. We only need to observe that we have braided equivalences:

\[
Z(\mathcal{T}\mathcal{Y}(G_{A,\omega}, \langle \cdot, \cdot \rangle, \pm)) \cong \mathcal{MP}(G_{A,\omega}, \langle \cdot, \cdot \rangle, \pm) \otimes \mathcal{C}(G_{A,\omega}, q_{A,\omega})
\]

\[
\cong Z(\text{Hilb}_{\text{Dih}(A)}) \otimes Z(\text{Hilb}_{\tilde{A}})
\]

which shows the if part and the second statement.

Corollary 5.16. All twisted doubles of generalized dihedral groups of odd abelian groups can be realized by a \(\mathbb{Z}_2\)-orbifold net \(\mathcal{A}_L^{\mathbb{Z}_2}\) of a conformal net associated with a lattice.

5.3. Quantum double subfactors of Tambara–Yamagami categories and Bisch–Haagerup subfactors.

Proposition 5.17. Let \(A\) be an abelian group of odd order. Consider \(G = \text{Dih}(A) \times A\), and let \(H = \langle \tau \rangle \cong \mathbb{Z}_2\) and \(\Delta(A)\) the diagonal embedding of \(A\) in \(A \times A \subset \text{Dih}(A) \times A\). Then \(G = \langle \Delta(A), H\rangle\). The Longo–Rehren subfactor \(S \subset T\) associated with \(\mathcal{T}\mathcal{Y}(A, \langle \cdot, \cdot \rangle, \pm)\) is a Bisch–Haagerup subfactor \(M^H \subset M \rtimes \Delta(A)\) for a \(G\)-action on \(M = S \rtimes \mathbb{Z}_2\).
Example 5.19. The principal graph associated with Proof. A Tambara–Yamagami categories associated with Corollary 5.18. Let order on $A$ and it follows that the action of $Z$.

Proof. We use the conformal net realization of the Longo–Rehren subfactor. With $M = A_L(I) \otimes A_L(I)$ we have that the Longo–Rehren subfactor is of the form:

$$M^{((\tau, 1))} \subset M \subset M \times_{(g, g^{-1})} A$$

and it follows that the action of $G \to \text{Out}(M)$ is characterized by the dual category of the inclusion $A_L^\otimes(I) \otimes A_L(I) \subset A_L(I) \otimes A_L(I) \equiv M$, where $(\tau, 1) = \tau \otimes \text{id}$ and $(g, \bar{g}) = g \otimes \bar{g}$.

We claim that with $H = \langle (\tau, 1) \rangle \cong Z_2$ and $K = \langle (g, \bar{g}) : g \in G \rangle \cong A$ we get $G = \langle K, H \rangle = \{(g, \bar{h}) : g \in \text{Dih}(A), h \in A \} \cong \text{Dih}(A) \times A$. Namely, since $(g, \bar{g})(\tau, 1)(\bar{g}^{-1}, g^{-1})(\tau, 1) = (g^2, 1)$ and $(g, \bar{g})(\tau, 1)(\bar{g}, g)(\tau, 1) = (\bar{g}^2, g^2)$ and since $|A|$ is odd we get $\{(g, \bar{h}) : g, h \in A \} \subset G$ and therefore it comes from a $G$-kernel for some $[\omega] \in H^3(G, \mathbb{T})$. To show that it comes from a $G$-action, we have to show that the obstruction $[\omega] \in H^3(G, \mathbb{T})$ vanishes. Because of the tensor product form $\omega$ is a product $H^3(\text{Dih}(A), \mathbb{T}) \oplus H^3(A, \mathbb{T})$ and therefore vanishes by Proposition 5.3. \qed

Corollary 5.18. Let $A$ be a cyclic group of odd order, then the Longo–Rehren subfactors of Tambara–Yamagami categories associated with $A$ are all conjugated.

Proof. Since $A$ is cyclic $H^2(A, \mathbb{T})$ and $H^2(\mathbb{Z}_2, \mathbb{T})$ vanish. Thus the Bisch–Haagerup subfactor associated with $A$ as described in Proposition 5.17 is unique up to conjugacy. \qed

It is straightforward to determine the (dual) principal graph for the Longo–Rehren subfactor associated with $\mathcal{Y}(A, \langle \cdot, \cdot \rangle, \pm)$. Let $n = 2k + 1 = |A|$ and let $A_+ = \{h \in A : -h < h \}$ for some order on $A$, or equivalently, let $A = \{0\} \cup A_+ \cup -A_+$. Thus we have $|A_+| = k$. The dual principal graph $\Gamma'$ is given by

- **even vertices ($M^K$ sectors):** There are $(2 + k)$ vertices which are given by $(\text{id}, g), (\alpha, g)$ and $(\sigma_{(g, -h)}, g)$, where $g \in A$ and $h \in A_+$.
- **odd vertices ($M \times H$ sectors):** There are $n$ vertices which are given by $\ell(\text{id}, g) = \ell(h, g + h)$, where $g, h \in A$.
- **edges:** $\ell(\text{id}, g)$ is connected to $(\text{id}, g), (\alpha, g)$ and $\{(\sigma_{(g-h,h-g)}, h) : h \in A, g - h \neq 0\}$ for all $g \in A$.

The principal graph $\Gamma$ is given by

- **even vertices ($M \times H$ sectors):** There are $n^2 + 1$ vertices given by $(g, h)$ and $(\rho, \rho)$, where $g, h \in A$.
- **odd vertices ($M$ sectors):** There are $n$ vertices given by $\ell(\text{id}, g) = \ell(h, g + h)$, where $g, h \in A$.
- **edges:** $\ell(\text{id}, g)$ is connected to $\{(h, h + g) : h \in A\}$ and $(\rho, \rho)$ for all $g \in A$.

**Example 5.19.** For $G = \mathbb{Z}_3$ the (dual) principal graph is given by

$$\Gamma' = \begin{array}{c}
\star \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\end{array}$$

This graph also appeared in [Bur15].

5.4. Tambara–Yamagami realizations are $\mathbb{Z}_2$-twisted and generalized orbifolds. Let $\mathcal{A}$ be a conformal net with a proper $\mathbb{Z}_2$-action. The orbifold net $\mathcal{A}^{\mathbb{Z}_2}$ has always the trivial $\mathbb{Z}_2$-simple current extension which recovers $\mathcal{A}$. We call a non-trivial $\mathbb{Z}_2$-simple current extension of $\mathcal{A}^{\mathbb{Z}_2}$ a $\mathbb{Z}_2$-twisted orbifold of $\mathcal{A}$ cf. [KL06, Section 3] for twisted orbifolds of holomorphic nets.

We remember that if $\mathcal{A}$ is a conformal net and $K$ a finite hypergroup, then there is the notion of a proper action of $K$ on $\mathcal{A}$ and the fixed-point net $A^K$ is the generalized orbifold of $\mathcal{A}$, see [Bis17].

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Proposition 5.20. Let $\mathcal{F}$ be a Tambara–Yamagami category associated with a group $A$ of odd order. Then there is a self-dual lattice $\Gamma$ and a proper $\text{Dih}(A)$ action on $\mathcal{A}_\Gamma$, such that there is a $\mathbb{Z}_2$-twisted orbifold of $\mathcal{A}_{\Gamma}\text{Dih}(A)$, which realizes $Z(\mathcal{F})$.

Namely, there is a lattice $L$, such that $\Gamma = L \oplus \hat{A}$, with an action of $\mathbb{Z}_2^2$, such that

\[
\begin{align*}
&\text{Rep}(\mathcal{A}_L^{(1,0)}) \cong Z(\mathcal{F}) \\
&\text{Rep}(\mathcal{A}_L^{(1,1)}) \cong Z(\text{Hilb}_{\text{Dih}(A)}) \\
&\text{Rep}(\mathcal{A}_L^{(0,1)}) \cong Z(\mathcal{F}^{\text{op}})
\end{align*}
\]

with $\mathcal{A}_L^{(1,1)} = \mathcal{A}_{\Gamma}\text{Dih}(A)$.

Further, $\mathcal{A}_L^{(1,0)}$ and $\mathcal{A}_L^{(0,1)}$ are generalized orbifolds of $\mathcal{A}_\Gamma$, with respect to proper actions of the hypergroup of the Tambara–Yamagami fusion rules associated with $A$. In the same way $\mathcal{A}_L^{\mathbb{Z}_2^2}$ is a generalized orbifold with respect to the hypergroup of the generalized Tambara–Yamagami fusion rules based on $\text{Dih}(A)$ in (7).

Proof. Assume $\mathcal{F} \cong \mathcal{TY}(A, \langle \cdot, \cdot \rangle, \nu)$. By Theorem 5.10 there are lattices $M$ and $\hat{M}$ and $\mathbb{Z}_2$-actions on $\mathcal{A}_M$ and $\mathcal{A}_{\hat{M}}$, such that $\mathcal{A}_M^{\mathbb{Z}_2}$ and $\mathcal{A}_{\hat{M}}^{\mathbb{Z}_2}$ realize $\mathcal{MP}(A, \langle \cdot, \cdot \rangle, \nu)$ and $\mathcal{MP}(A, \langle \cdot, \cdot \rangle^{-1}, \nu)$, respectively. Choose $L = M \oplus \hat{M}$, then as in the proof of Proposition 5.10 there is an action of $\text{Dih}(A)$ on $\mathcal{A}_\Gamma$ with

\[
\mathcal{A}_L^{(1,1)} = \mathcal{A}_{\Gamma}\text{Dih}(A) \subset \mathcal{A}_\Gamma,
\]

using the above $\mathbb{Z}_2 \times \mathbb{Z}_2$-action on $\mathcal{A}_L \cong \mathcal{A}_M \otimes \mathcal{A}_{\hat{M}}$. The rest follows immediately. \qed

Remark 5.21. There is a twisted version, where

\[
\begin{align*}
&\text{Rep}(\mathcal{A}_L^{(1,1)}) \cong Z(\text{Hilb}_{\text{Dih}(A)}^{[\omega]}) \\
&\text{Rep}(\mathcal{A}_L^{(0,1)}) \cong Z(\mathcal{F}^{\text{op}})
\end{align*}
\]

with $\mathcal{F}_- \cong \mathcal{TY}(A, \langle \cdot, \cdot \rangle, -\nu)$ for $\mathcal{F} \cong \mathcal{TY}(A, \langle \cdot, \cdot \rangle, \nu)$.

Remark 5.22. If we take $\text{Dih}(A)$ for $|A|$ odd, we can consider the element $c_\tau = \frac{1}{|A|} \sum_{g \in A} g \tau \in \mathbb{C}[\text{Dih}(A)]$. Then $c_\tau c_\tau = \frac{1}{|A|} \sum_{g \in A} g$ and $K = \{A, c_\tau\}$ represents the $A$-Tambara–Yamagami hypergroup. Let us now assume $|A|$ is odd. The action $\alpha$ of $\text{Dih}(A)$ from Proposition 5.20 gives an action of $K$ by $\phi(c_\tau) = \frac{1}{|A|} \sum_{g \in A} \alpha_g$, but this action is not proper since $\phi(c_\tau)$ fails to be extremal if $A$ is non-trivial. On the other hand, the two $\mathbb{Z}_2$-twisted orbifolds of the $\text{Dih}(A)$-orbifold $\mathcal{A}_{\Gamma}\text{Dih}(A)$ give proper actions of $K$ and therefore “generalized $K$-orbifolds”.

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It is enlightening to draw the lattice of intermediate nets for \( \mathcal{A}_L^{\mathbb{Z}_2 \times \mathbb{Z}_2} \equiv \mathcal{A}_\Gamma^{K_{\text{gen}}} \subset \mathcal{A}_\Gamma \):

\[
\begin{array}{c}
\mathcal{A}_\Gamma \\
\downarrow \\
\mathcal{A}_\Gamma^G \\
\downarrow \\
\mathcal{A}_\Gamma^{(gr)} \\
\downarrow \\
\mathcal{A}_\Gamma^{(G,r)} \\
\downarrow \\
\mathcal{A}_\Gamma^{K_+} \\
\downarrow \\
\mathcal{A}_\Gamma^{K_-} \\
\downarrow \\
\mathcal{A}_\Gamma^{\text{Dih}(A)} \\
\downarrow \\
\mathcal{A}_\Gamma^{K_{\text{gen}}} \\
\end{array}
\]

where the dotted part has to be filled by the respective intermediate groups of \( \text{Dih}(A) \). We note that for subgroups \( G \subset A \) the net \( \mathcal{A}_\Gamma^G \) is the conformal net associated the lattice \( L \oplus \hat{A}^G \), where \( \hat{A}^G = \{ \chi \in \hat{A} : \chi(g) = 1 \text{ for all } g \in G \} \). The hypergroup \( K_{\text{gen}} \) is the hypergroup associated with the generalized Tambara–Yamagami category based on \( \text{Dih}(A) \) (see Subsection 5.1) which can be seen as \( K_+ \times_A K_- \), namely a relative product over \( A \) of two Tambara–Yamagami hypergroups \( K_\pm \) based on \( A \). All solid lines are \( \mathbb{Z}_2 \)-orbifolds.

6. \textbf{Generalized orbifolds and defects}

6.1. \textbf{Holomorphic nets from twisted orbifolds.} The following is an analogue of the twisted orbifold construction in VOAs.

**Definition 6.1.** Let \( \mathcal{A} \) be a holomorphic net, and \( G \leq \text{Aut}(\mathcal{A}) \) a finite group. A holomorphic net \( \mathcal{B} \) is called a \textit{twisted} \( G \)-\textit{orbifold} of \( \mathcal{A} \) if it is a holomorphic extension \( \mathcal{B} \supset \mathcal{A}_G \).

If \( \mathcal{A} \) is holomorphic and \( G \leq \text{Aut}(\mathcal{A}) \) the category of \( G \)-twisted representations \( \mathcal{G} = \text{Rep}(\mathcal{A}) \) is tensor equivalent to \( \text{Hilb}^{\omega}_{G} \) for some \( [\omega] \in H^3(G, \mathbb{T}) \). More explicitly, by using \( \alpha^+ \)-induction applied to \( \mathcal{A}(I)^G \subset \mathcal{A}(I) \) for a fixed interval \( I \subset S^1 \setminus \{-1\} \) we get a \( G \)-kernel \( \{ [\alpha_g] : g \in G \} \subset \text{Out}(\mathcal{A}(I)) \) which can be lifted to \( \text{Aut}(G) \) if the associated obstruction in \( H^3(G, \mathbb{T}) \) vanishes. Let us assume that the obstruction vanishes, i.e. \( \omega \) is a coboundary. In this case, we say that \( G \) acts \textit{anomaly free}. Let us choose a trivialisation of \( \omega \). We can consider the crossed product \( \mathcal{A}(I) \rtimes G \), which gives rise to a relatively local extension \( \mathcal{A} \rtimes G \) of \( \mathcal{A}^G \) which is a net on the universal cover of \( S^1 \) or on the restriction \( S^1 \setminus \{-1\} \). Namely, by definition the Q-system of \( \mathcal{A}(I)^G \subset \mathcal{A}(I) \rtimes G \) is in \( \text{Rep}^I(\mathcal{A}^G) \) which characterizes a non-local extension \cite{LR95,LR04}. Let us denote by \( \mathcal{A}^{+/G} \) the intermediate net \( \mathcal{A}^G \subset \mathcal{A}^{+/G} \subset \mathcal{A} \rtimes G \) obtained by the left center construction \cite{BKLR16}. The left and right center \( \mathcal{B}^\pm \) of a net \( \mathcal{B} \) on the real line are defined by

\[
\begin{align*}
\mathcal{B}^+(a, b) & := \mathcal{B}(a, b) \cap \mathcal{B}(-\infty, a)', \\
\mathcal{B}^-(a, b) & := \mathcal{B}(a, b) \cap \mathcal{B}(b, \infty)',
\end{align*}
\]

respectively, where we consider the net in the real line picture \( \mathbb{R} \cong S^1 \setminus \{-1\} \). The nets are local and therefore extend to \( S^1 \).

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Lemma 6.2. Let $A$ be holomorphic with an anomaly free action of $G$ as above. The right center $[A \times G]^-$ of $A \times G \supset A^G$ is $A$ and the left center $A//G$ is holomorphic. The extensions $A//G \supset A^G$ and $A \supset A$ are isomorphic if and only if $G$ is trivial.

Proof. Let $\psi_g \in \{(t, t') \in I \} = \{\psi \in [A \times G](I) : \forall x = \beta_g(x)\psi \text{ for all } x \in A(I)\}$. Then we have $\psi_g \in \{(t, t') \in K \} = \{\psi \in [A \times G](I) : \forall x = \alpha_g(x)\psi \text{ for all } x \in A(I)\}$. Let $I_{r, l}$ be the left and right component of $K \cap I'$, respectively. Since $\alpha_g$ is a right soliton, we have $\psi_g = \psi_g$ for $x \in A(I_l)$ and $\psi_g = \alpha_g(x)\psi_g$ for $x \in A(I_r)$. Thus $A(I_l) = A(I_l) \cap [A \times G](I + \mathbb{R}_{\geq 0}) \subset [A \times G]^-(I_l)$. Thus $A \subset [A \times G]^-$.

We have $A//G(I_r) = A(I_r) \cap [A \times G](I - \mathbb{R}_{\geq 0}) = [A \times G]^+(I_r) \cap A(I_r)$, thus the left center does coincide with $A$ only if $G$ is trivial.

The $\mu$ index of the left and right center coincide, but the right center is $A$, which is holomorphic, thus $A//G$ is holomorphic.

We note that $A//G$ itself can still be isomorphic to $A$. For example, if we consider the $\mathbb{Z}_2$-action $A_{\mathbb{Z}_2}^G = A_{D_4}$, then it follows that $A//\mathbb{Z}_2$ is isomorphic to $A_{D_4}$. It is well-believed that $A_{D_4}$ is the only holomorphic conformal net with central charge 8. This is only a conjecture for conformal nets. If the conjecture was true it would imply that $A//G$ is isomorphic to $A_{D_4}$ for every finite group $G \leq \text{Aut}(A_{D_4})$ acting anomaly free.

Lemma 6.3. $A^G(I) \subset [A \times G](I)$ is irreducible.

For an outer action of $G$ on a factor $M$, the inclusion $M^G \subset M \times G$ is only irreducible for $G$ being trivial. But in our case, we take fixed point with the gauge action and the crossed product with solitons.

Proof. Let $\theta$ be the dual canonical endomorphism. We have to show that $\dim \text{Hom}(id, \theta) = 1$, but we have a $G$-grading and $\theta = \bigoplus_g \theta_g$, where $\theta_e$ is the dual canonical endomorphism for $A^G(I) \subset A(I)$ and $\dim \text{Hom}(id, \theta_e) = 1$.

Proposition 6.4. The extension $A \times G \supset A$ defines an $A^G$-topological defect (actually a phase boundary) between $A$ and $A//G$ and this property determines $A//G$ uniquely. Namely, $A//G$ is characterized to be the unique holomorphic extension $B \supset A^G$ giving $A \times G$ the structure of an $A^G$-topological defect between $A$ and $B$.

There is action of the hypergroup $K_{\text{Rep}(G)}$ on $A//G$, such that $(A//G)^{K_{\text{Rep}(G)}} \cong A^G$.

We note that for $G$ abelian $K_{\text{Rep}(G)}$ is a group and can be identified with the Pontryagin dual $\hat{G}$. On the other hand, if $G$ is non-abelian, we get an action of the genuine hypergroup $K_{\text{Rep}(G)}$.

In general, for $A$ a holomorphic net and $G \leq \text{Aut}(A)$ there is a unique class $[\omega] \in H^3(G, \mathbb{T})$, such that $G - \text{Rep}(A) \cong \text{Hilb}_G^\omega$. For every $H \leq G$, such that $\omega \mid H$ is a coboundary, i.e. $H$ acts anomaly free, we can form $A//H$. This choice is classified by $H^2(H, \mathbb{T})$.

On the other hand, a holomorphic twisted $G$-orbifolds of $A$ is by definition a holomorphic extension of $A^G$. These are in one-to-one correspondence with equivalence classes of Lagrangian $Q$-systems in $\text{Rep}(A^G) \cong Z(\text{Hilb}_G^\omega)$. But these are classified by the same data $[\omega]$ [27], so the above construction gives all twisted orbifolds.

Example 6.5. Let $(G, \langle \cdot, \cdot \rangle, \nu)$ be a triple consisting of an abelian group $G$ of odd order, a non-degenerate symmetric bicharacter $\langle \cdot, \cdot \rangle$, and a sign $\nu$. Let us consider the UMTCs $C_+ = MP(G, \langle \cdot, \cdot \rangle, \nu)$ and $C_- = MP(\langle \cdot, \cdot \rangle, \nu)^\text{rev}$. We find pairs of lattices $L_\pm$ and $\mathbb{Z}_2$-automorphisms $\alpha_\pm$ of $A_{L_\pm}$, such that $\text{Rep}(A_{L_\pm}^{\alpha_\pm})$ is braided equivalent to $C_\pm$, respectively. Then there is a self-dual lattice $\Gamma = (L_+ \oplus L_-) \oplus G$ and an action of $H = \hat{G}$ on $A_\Gamma$, such that $A_\Gamma^H = A_{L_+ \oplus L_-}$. This action extends to an action of $Dih(H)$ on $A_\Gamma$ by Proposition 6.20. Therefore we can form $A//\text{Dih}(H)$ or
more generally, \( A^{\phi K} \) for \( K \leq \text{Dih}(H) \) which gives many examples twisted holomorphic orbifolds of conformal nets.

**Example 6.6.** Let \( L_1 = A_2 E_8 \), \( L_2 = E_6 E_8 \) and \( \Gamma = E_8^3 \). There are \( \mathbb{Z}_2 \)-actions such that we have braided equivalences

\[
\text{Rep}(\mathcal{A}^{\mathbb{Z}_2}_{A_2 E_8}) \cong \text{MP}(\mathbb{Z}_3, (\cdot, \cdot), +), \quad \text{Rep}(\mathcal{A}^{\mathbb{Z}_2}_{E_6 E_8}) \cong \text{MP}(\mathbb{Z}_3, (\cdot, \cdot), +),
\]

where \( (x, y) = e^{2\pi i xy/3} \). We get an action of \( S_3 \cong \text{Dih}(\mathbb{Z}_3) \) on \( \mathcal{A}_\Gamma \) and can form \( \mathcal{B} := \mathcal{A}_\Gamma^{\mathbb{Z}_3} \). With the help of the computer program KAC [Sch], we determine that the weight one subspace should have dimension 456 and that \( \mathcal{B} \) should correspond to 64 in Schellekens’ list [Sch93]. We therefore conjecture that \( \mathcal{A}_\Gamma^{\mathbb{Z}_3} \) is the conformal net \( \mathcal{A}_{\text{Ni}(D_{10}E_7^2)} \) associated with the lattice \( \text{Ni}(D_{10}E_7^2) \). Here \( \text{Ni}(L) \) is the Niemeier lattice [Nie73] with root lattice \( L \).

6.2. **Generalized orbifolds and hypergroup character tables.** Let \( G \) be an abelian group of odd order which we see as a multiplicative group. Let \( K = G \cup \{ \tau \} \) the hypergroup associated with the Tambara–Yamagami fusion rules, i.e. \( \tau g = g \tau = \tau = \tau^* \) for \( g \in G \) and \( \tau^2 = \frac{1}{|G|} \sum_{g \in G} g \).

Then we have the dual hypergroup \( \hat{K} = \{ 1, \varepsilon, c_\chi : \chi \in \hat{G} \setminus \{ 1 \} \} \) with \( \varepsilon c_\chi = c_\chi \varepsilon = c_\chi, \varepsilon^2 = 1 \) with relations:

\[
c_\chi c_\xi = \begin{cases} \frac{1}{2}(1 + \varepsilon) & \chi = \chi^{-1}, \\ c_\chi \chi & \text{otherwise}, \end{cases}
\]

and the canonical pairing \( \langle \cdot, \cdot \rangle_K : \hat{K} \times K \to \mathbb{C} \). The character table is easily determined to be (\( g \in G \setminus \{ \varepsilon \}, \chi \in \hat{G} \setminus \{ 1 \} \)):

\[
\begin{array}{c|cc}
\langle \cdot, \cdot \rangle_K & 1 & g & \tau \\
\hline
1 & 1 & 1 & 1 \\
\varepsilon & 1 & 1 & -1 \\
c_\chi & 1 & \chi(g) & 0
\end{array}
\]

(8)

Namely, there is a self-dual lattice \( \Gamma = L\bar{L} \oplus \hat{G} \) given by the glueing of two lattice \( L \) and \( \bar{L} \) and a proper hypergroup action of \( K \) on \( \mathcal{A}_\Gamma \), such that \( \text{Rep}(\mathcal{A}^K_\Gamma) \cong Z(\mathcal{F}) \) for some Tamabara–Yamagami category \( \mathcal{F} \) with \( K\mathcal{F} = K \). The dual canonical endomorphism of the inclusion \( \mathcal{A}^K_\Gamma(I) \subset \mathcal{A}_\Gamma(I) \) is given by

\[
[\theta] = \bigoplus_{k \in K} [\rho_k] = [(0, 0)] \oplus [(\alpha, 0)] \oplus \bigoplus_{g \in G^\times} [\theta(g, -g), \bar{g}]
\]

with the correspondence

\[
[(0, 0)] \leftrightarrow 1 \quad [(\alpha, 0)] \leftrightarrow \varepsilon \quad [(\sigma(g, -g), \bar{g})] \leftrightarrow \chi(\cdot) = \langle g, \cdot \rangle \quad (g \in G^\times).
\]

As in [Bis17] this gives an action of \( V : K \to B(\mathcal{H}_\mathcal{A}_\Gamma) \)

\[
\mathcal{H}_\mathcal{A}_L = \bigoplus_{k \in K} \mathcal{H}_k, \quad V(k) = \bigoplus_{k \in K} (\hat{k}, k)_K.
\]

which extends to a *-representation of \( CK \). With \( M := \mathcal{A}_\Gamma(I) \supset \iota(N) := \mathcal{A}^K_\Gamma(I) \) we have \( M = \bigoplus_{k \in K} M_k \), where \( M_k = \iota(N) \psi_k \) and \( \psi_k \in \text{Hom}(\iota, \iota \rho_k) \) is a charged intertwiner. We have \( M_\bullet \Omega \subset \mathcal{H}_\bullet \) which gives a “hypergrading” \( m_k m_\ell \in \bigoplus_{n \leq k \ell} M_n \) for \( m, n \in M_\bullet \).
6.3. Defects and Kramer–Wannier duality. Let \( \mathcal{A} \) be a completely rational conformal net which we see by restriction as a net on \( \mathbb{R} \). Let \( \mathcal{B} \supset \mathcal{A} \) be a possibly non-local extension and \( \Theta \in \text{Rep}(\mathcal{A}) \) be the corresponding \( \mathbb{Q} \)-system, see [LR04].

By [BKLR16] the irreducibles of the fusion category of \( \mathcal{B} \)-\( \mathcal{B} \) sectors, or equivalently \( \mathfrak{g}_\text{Rep}(\mathcal{A}) \) describes phase boundaries (or defects) on Minkowski space between \( \mathcal{B}_2 \) and itself. Here \( \mathcal{B}_2 \supset \mathcal{A} \otimes \mathcal{A} \) is the full CFT on Minkowski space coming from the full center construction of \( \Theta \), see [BKL15]. By \( \mathcal{A} \otimes \mathcal{A} \) we denote the net on Minkowski space, which we identify with the product \( \mathbb{R} \times \mathbb{R} \) of two light rays, defined by \( (\mathcal{A} \otimes \mathcal{A})(I \times J) = \mathcal{A}(I) \otimes \mathcal{A}(J) \). A phase boundary (between \( \mathcal{B}_2 \) and \( \mathcal{B}_2 \) which is transparent for \( \mathcal{A}_2 = \mathcal{A} \otimes \mathcal{A} \) is a quadrilateral inclusion of nets \( \mathcal{A}_2(O) \subset \mathcal{B}_2^{I,J}(O) \subset \langle \mathcal{B}_2^2(O), \mathcal{B}_2^2(O) \rangle =: \mathcal{D}(O) \) on a common Hilbert space with \( \mathcal{D}(O) \) being a factor and \( \mathcal{A}_2(O) \subset \mathcal{B}_2^{I,J}(O) \) being isomorphic to \( \mathcal{A}_2(O) \subset \mathcal{B}_2(O) \), such that \( \mathcal{B}_2(O_I) \) commutes with \( \mathcal{B}_2^I(O_r) \) for \( O_I \) space-like left of \( O_r \), see [BKLR16] for details.

Invertible objects \( \{ \alpha_g \}_{g \in G} \) with \( [\alpha_g][\alpha_h] = [\alpha_{gh}] \) for a finite group \( G \) correspond to group-like defects or gauge transformations. An object \( \rho \) with fusion rules
\[
[rho][rho] = \bigoplus_{g \in G} [alpha_g]
\]
correspond to a duality defect. This can be seen as generalization of Kramer–Wannier duality of the Ising model with fusion rules \( [\sigma][\sigma] = [1] + [\varepsilon] \) and \( [\varepsilon]^2 = [1] \), see [FFRS04].

Let \( G \) be a finite abelian group of odd order. For \( \mathcal{F} = \mathcal{TV}(G, \langle \cdot, \cdot \rangle, \pm) \) Theorem 3.8 gives lattices \( L, \bar{L} \) and a \( \mathbb{Z}_2 \)-action on \( \mathcal{A}_L \). We can consider the net \( \mathcal{A}_G^2 \otimes \mathcal{A}_L \) or alternatively, the “reflected” chiral net \( \mathcal{A}_G^2 \otimes \mathcal{A}_L = \mathcal{A}_G^{K,F} \). Then there is a full CFT \( \mathcal{B}_2 \supset \mathcal{A}_2 := \mathcal{A}_G^2 \otimes \mathcal{A}_L \) corresponding to the holomorphic extension \( \mathcal{A}_G \supset \mathcal{A}_G^{K,F} \).

Proposition 6.7. The phase boundaries in the sense of [BKLR16] which are defects between \( \mathcal{B}_2 \) and itself and are invisible for \( \mathcal{A}_2 \) correspond to the elements of the Tambara–Yamagami hypergroup \( K = G \cup \{ \tau \} \).

Proof. Namely, \( \mathcal{B}_2 \supset \mathcal{A}_G^2 \otimes \mathcal{A}_L^\tau \) is the full center of the inclusion \( \mathcal{A}_L \supset \mathcal{A}_L^\tau \) and the phase boundaries for \( \mathcal{B}_2 \supset \mathcal{A}_G^2 \otimes \mathcal{A}_L^\tau \) are given by \( \mathcal{A}_G \supset \mathcal{A}_L \) sectors or equivalently irreducibles in \( \mathbb{Z}_2 \text{Rep}(\mathcal{A}_G^2) \). It can be checked that the sectors in \( \mathbb{Z}_2 \text{Rep}(\mathcal{A}_G^2)^+ \cong \mathcal{F} \) exactly correspond to phase boundaries that preserve the intermediate net \( \mathcal{A}_2 = \mathcal{A}_G^2 \otimes \mathcal{A}_L \).

The algebra \( \mathcal{B}_2(I) \) is generated by \( \mathcal{A}_2(I) \) and charged intertwiners \( \{ \psi_1, \psi_2, \psi_3 : \chi \in \hat{G} \setminus \{ 1 \} \} \) with \( \psi_i \in \text{Hom}(\iota, \iota \rho_i) \), see [BKL15]. A phase boundary (condition) gives relations between the generators \( \psi^l_\chi \) and \( \psi^r_\chi \) of \( \mathcal{B}_2^{I,J}(O) \), respectively. Using the character table \( \mathbb{S} \) we get the following relations:
- A group like defect \( g \in G \) correspond to a gauge automorphism \( \psi^l_\chi = \chi(g)\psi^r_\chi \) and \( \psi^l_{1/\varepsilon} = \psi^r_{1/\varepsilon} \).
- The duality defect \( \tau \), due to the zero entry in the character table \( \mathbb{S} \), gives independent fields \( \sigma_{\chi} = \psi^l_{\chi} \) and \( \mu_{\chi} = \psi^r_{\chi} \) while \( \varepsilon := \psi^l_{1/\varepsilon} = -\psi^r_{1/\varepsilon} \).

The “Kramers–Wannier duality” of the conformal Ising model (the unique full CFT \( \mathcal{B} \supset \text{Vir}_{1/2} \otimes \overline{\text{Vir}}_{1/2} \))
\[
(1, \sigma, \varepsilon) \leftrightarrow (1, \mu, -\varepsilon)
\]
generalizes to a duality from the defect \( \tau \) given by
\[
(1, \sigma_{\chi_2}, \ldots, \sigma_{\chi_{|G|}}, \varepsilon) \leftrightarrow (1, \mu_{\chi_2}, \ldots, \mu_{\chi_{|G|}}, -\varepsilon)
\]
for the heterotic theory \( \mathcal{B}_2 \supset \mathcal{A}_G^2 \otimes \mathcal{A}_L \) associated with \( \mathcal{F} \). We note that, because the Ising category is modular, the Ising hypergroup is self-dual and there is a correspondence between fields
and defects, namely both are indexed by $\mathbb{Z}_2 \cup \{\tau\} \cong \{1, \sigma, \varepsilon\}$. For $G \cup \{\tau\}$ with $G$ of odd order this correspondence breaks down. The defects are indexed by $G \cup \{\tau\}$ while the fields are indexed by $\{1, \sigma_1, \ldots, \sigma_{|G|}, \varepsilon\}$.

It seems interesting to study the above dualities in statistical physics models on a lattice in terms of high/low temperature duality. We believe that such a duality may arise from gauging certain spin models. The golden way would be to find corresponding lattice models whose continuum limit at criticality recovers the in this section constructed conformal field theories. Such models might be easy enough to be interesting for physical implementations and applications in topological quantum computing.

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