Performance of nonconforming spectral element method for Stokes problems

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Abstract
A nonconforming spectral element method for the Stokes problem on nonsmooth domains has been proposed in Mohapatra et al. (J Comput Appl Math 372:112696, 2020). The main focus of this article is to study the performance of this method for Stokes problems on smooth curvilinear domains and Stokes problem with mixed boundary conditions. Various test cases are considered including the generalized Stokes problem in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) to verify the exponential accuracy of the method.

Keywords Stokes equations · Velocity · Pressure · Spectral element · Nonconforming · Exponential accuracy · PCGM

Mathematics Subject Classification 65N35 · 65F10 · 35J57

1 Introduction

The stationary Stokes equations are linearized form of the stationary Navier–Stokes equations. These equations describe the flow of the incompressible fluids. Stokes problem arise in various applications in physics and engineering. The numerical solution of stationary Stokes equations has been extensively studied in the literature. Finite difference method, finite-element method, Spectral methods, discontinuous Galerkin methods, Least-squares finite-element methods, and meshless methods etc. are popular methods among the numerical techniques to solve the Stokes problem.
Standard central differences do not give stable discretizations on uniform grids to Stokes equations due to the failure of the discrete inf-sup condition (Chen 2022). The marker and cell method (MAC Scheme) is a simple and more efficient numerical scheme for solving Stokes and Navier–Stokes problems. Finite difference MAC scheme has been studied in Chen (2022), Ito and Qiao (2008). Details of the other finite difference schemes can be found in Song et al. (2020), Strikwerda (1984a), Strikwerda (1984b). The finite volume MAC scheme has been studied in Rui and Li (2017) and the references therein.

The finite-element method for Stokes problem has been widely studied. Standard Galerkin formulation is viewed as a saddle-point problem. Mixed FEM imposes restrictions such as inf-sup or Ladyzhenskaya–Babuska–Brezzi (LBB) condition while choosing approximation spaces for different unknowns because of which same order polynomials cannot be used for different unknowns. Therefore, it uses two different finite-element spaces for velocity and pressure, respectively. The analysis of mixed FEM is based on the theory of saddle point problem which has been developed in Babuska (1973), Brezzi (1974). Use of different order polynomial spaces makes computation cumbersome. In addition, choosing a suitable pair of polynomial order spaces is not easy in general. Mass conservation is another issue in the approximation of the solution of the Stokes problem.

To overcome the problem of using different finite-element spaces for velocity and pressure variables, in stabilized finite-element method, the standard bilinear form is modified such that any pair of finite-element spaces can be chosen. Stabilized finite-element method for the Stokes problem has been introduced in Brezzi and Douglas (1988), Hughes and Franca (1987), Hughes et al. (1986). Details of several other stabilized finite-element formulations can be found in Arnold et al. (1984), Barrenechea and Valentin (2002), Blasco (2007), Boffi et al. (2008), Codina (2001), Douglas and Wang (1989) and the references therein. Divergence-free finite-element methods have been studied in Blank (2014), Crouzeix and Raviart (1973), Mu and Ye (2017), Wang et al. (2009) and the references therein. Divergence-free methods eliminate pressure variable from the saddle point system and result in positive definite linear systems and also avoid the mass conservation issue.

Least-squares methods have various advantages when applied to systems of differential equations (Aziz et al. 1985; Bochev and Gunzburger 1998; Eason 1976; Gunzburger and Bochev 2009; Jiang 1998). Least-squares based methods do not require LBB condition while choosing the approximation spaces and also leads to symmetric positive definite linear systems when applied to linear problems. In the least-squares formulation Stokes problem is transformed into a first-order system. There are different first order formulations for Stokes system. For example velocity–vorticity–pressure formulation (Amara et al. 2003; Bochev and Gunzburger 2009; Bramble and Pasciak 1996; Cai et al. 1995; Chang and Yang 2002; Duan and Liang 2003; Dubois 2002; Jiang 1998), velocity–stress–pressure formulation (Bochev and Gunzburger 1995; Bramble and Pasciak 1996; Kim and Shin 2002) and acceleration–pressure formulation (Chang 1990). Mass conservation issues also arise in least-squares methods because of the inclusion of the continuity equation in minimizing functional (Deang and Gunzburger 1998). The mass conserving properties of the least-squares methods have been addressed also in Bolton and Thatcher (2005), Chang and Nelson (1997), Proot and Gerritsma (2006).

Stokes problem has also been studied using spectral/spectral element methods. The spectral method has been proposed for this problem in Schumack et al. (1991). Proot and Gerritsma (2002a, b, 2006) have proposed a least-squares spectral element scheme for Stokes equations in velocity–vorticity–pressure formulation. Numerical results conclude that the pressure variable to be one order less accurate compared to the velocity variable as pressure is not prescribed on the boundary. The least-squares spectral collocation method has been studied
in Heinrichs (2004), Kim et al. (2003). If the data in the given problem is analytic then these methods gives exponential convergence.

Nonconforming methods such as mortar finite-element methods (Belgacem 2000; Belgacem et al. 2002), discontinuous Galerkin methods (Burman and Stamm 2010; Cockburn et al. 2002; Li et al. 2020; Montlaur et al. 2008; Montlaur and Fernandez-Mendez 2014), and some others in (Apel et al. 2020; Bochev et al. 2012, 2013) provide a numerical approximation to the Stokes equations. Meshless methods always avoid the problem of mesh generation on the domains with complex geometries. Meshless methods for the Stokes problem have been studied in Ahmad et al. (2020), Desimone et al. (1998), Li (2015), Tan et al. (2013), Traska et al. (2016).

In this article, we have also considered the generalized Stokes problem. This problem occurs in the numerical treatment of the time dependent Navier–Stokes equations. Generalized Stokes equations look very similar to the Stokes equations. This problem has been studied in Barrenechea and Valentin (2002), Burman and Hansbo (2006), Butt (2018), Calgaro and Laminie (2000), Chou (1997), Codina (2001), Larin and Reusken (2007), Nafa (2009), Sarin and Samesh (1998).

In Mohapatra et al. (2020) an exponentially accurate non-conforming least-squares spectral element method for Stokes equations on non-smooth domains has been proposed. This is different from the standard least-squares FEM formulations, where the Stokes system is converted into a first order system as mentioned earlier. The minimizing functional in the least-squares formulation includes the residuals in the partial differential equations and residuals in the boundary conditions in appropriate Sobolev norms. The method is nonconforming and higher order spectral element functions have been used. Various numerical examples with singular solutions were presented to verify the exponential accuracy of the method. In Mohapatra and Ganesan (2016), authors have studied the spectral element method for Oseen equations with applications to Navier–Stokes equations. They have presented various numerical examples on square domains in \( \mathbb{R}^2 \) with Dirichlet boundary conditions.

In this article, we study the performance of this method for Stokes problems on smooth curvilinear domains with mixed boundary conditions. The normal equations in the least-squares formulation are solved using preconditioned conjugate gradient method without storing the matrix. The exponential accuracy of the method is verified through various numerical tests.

This paper is organized as follows. In Sect. 2, we have considered the generalized Stokes equations on smooth domains. The discretization of the domain and the numerical scheme are described. Numerical results are presented in Sect. 3. Finally we conclude with Sect. 4. Some necessary notations to describe the numerical formulation are given in the appendix and also the stability estimate is stated.

Here we give some notations and define required function spaces. Let \( \Omega \in \mathbb{R}^2 \), be an open bounded set with sufficiently smooth boundary \( \partial \Omega \). \( H^m(\Omega) \) denotes the Sobolev space of functions with square integrable derivatives of integer order less than or equal to \( m \) in \( \Omega \) equipped with the norm:

\[
\|u\|_{H^m(\Omega)}^2 = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^2(\Omega)}^2.
\]

Furthermore, let \( I = (-1, 1) \). Then we define fractional norms \( (0 < s < 1) \) by

\[
\|w\|_{s,I}^2 = \|w\|_{0,I}^2 + \int_I \int_I \frac{|w(\xi) - w(\xi')|^2}{|\xi - \xi'|^{1+2s}} \, d\xi \, d\xi'.
\]
where \( I \) denotes an interval contained in \( \mathbb{R} \). Moreover
\[
\|w\|_{1+s,I}^2 = \|w\|_{0,I}^2 + \left\| \frac{\partial w}{\partial \xi} \right\|_{s,I}^2 + \left\| \frac{\partial w}{\partial \eta} \right\|_{s,I}^2.
\]
We shall denote the vectors by bold letters. For example
\[
\mathbf{u} = (u_1, u_2)^T, \quad H^k(\Omega) = H^k(\Omega) \times H^k(\Omega), \quad \text{etc.}
\]
The norms are given by
\[
\|\mathbf{u}\|_{k,\Omega}^2 = \|u_1\|_{k,\Omega}^2 + \|u_2\|_{k,\Omega}^2 \quad \text{for} \quad \mathbf{u} \in H^k(\Omega),
\]
\[
\|\mathbf{u}\|_{s,I}^2 = \|u_1\|_{s,I}^2 + \|u_2\|_{s,I}^2, \quad \text{etc.}
\]

2 Discretization and numerical formulation

In this section, we describe the discretization of the domain and derive the numerical formulation. We consider the generalized Stokes problem with Dirichlet boundary condition to describe the numerical formulation.

2.1 Generalized Stokes problem

Consider the generalized Stokes equations in \( \Omega \subset \mathbb{R}^2 \), with sufficiently smooth boundary \( \partial\Omega = \Gamma \) (as shown in Fig. 1).

\[
\begin{aligned}
\alpha \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in} \quad \Omega, \\
-\nabla \cdot \mathbf{u} &= h \quad \text{in} \quad \Omega, \\
\mathbf{u} &= \mathbf{g} \quad \text{on} \quad \partial\Omega.
\end{aligned}
\]

Here, \( \mathbf{u} \) is the velocity field, \( p \) is the pressure and \( \alpha, \nu > 0 \). Assume that the positive parameters \( \alpha, \nu \) are not simultaneously zero. When \( \alpha = 0 \) it reduces to Stokes problem.

Let \( h \in L^2(\Omega) \) and such that \( \int_\Omega h \, dx = 0 \) and \( \mathbf{g} \) satisfies the compatibility condition \( \int_\Gamma \mathbf{g} \cdot \mathbf{n} = 0 \), where \( \mathbf{n} \) is the unit outward normal to \( \Gamma \). Let \( \mathbf{f} \in L^2(\Omega), \mathbf{g} \in H^1(\partial\Omega) \).

Then the generalized Stokes problem (1–3) has the solution \((\mathbf{u}, p) \in H^1(\Omega) \times L^2(\Omega)\), where \( p \) is unique up to an additive constant. \( p \) can be obtained uniquely in \( L^2(\Omega)/\mathbb{R} \) or in \( L^2_0(\Omega) = \{ u \in L^2(\Omega) | \int_\Omega u = 0 \} \) (Burman and Hansbo 2006). Furthermore, a regularity estimate also holds good (see Appendix A1).

2.2 Discretization and spectral element functions

The domain \( \Omega \) is divided into \( L \) quadrilaterals \( \Omega_1, \Omega_2, \ldots, \Omega_L \) as shown in Fig. 1 (some of them are curvilinear elements). Let \( \mathbf{x} = (x_1, x_2) \) be any point in the domain. A set of nonconforming spectral element functions are defined on these elements which are a sum of tensor products of polynomials of degree \( W \). Let \( S \) denote the master element \( S = (-1, 1)^2 \). Now, there is an analytic map \( M_l(\xi, \eta) \) from \( S \) to \( \Omega_l \) which has an analytic inverse (blending function mapping Gordan and Hall 1973):

\[
x_1 = X_1^l(\xi, \eta) \quad \text{and} \quad x_2 = X_2^l(\xi, \eta).
\]

Define the spectral element functions \( \hat{\mathbf{u}}_l \) and \( \hat{p}_l \) on \( S \) by
\[
\hat{\mathbf{u}}(\xi, \eta) = \sum_{i=0}^W \sum_{j=0}^W a_{i,j} \xi^i \eta^j, \quad \hat{p}(\xi, \eta) = \sum_{i=0}^W \sum_{j=0}^W b_{i,j} \xi^i \eta^j.
\]
Then, $\mathbf{u}_l$ and $p_l$ on $\Omega_l$ are given by

$$\mathbf{u}_l(x_1, x_2) = \hat{\mathbf{u}}(M_l^{-1})$$
and

$$p_l(x_1, x_2) = \hat{p}(M_l^{-1}).$$

Let $\Pi^{L,W} = \{ \{ \mathbf{u}_l \}_{1 \leq l \leq L}, \{ p_l \}_{1 \leq l \leq L} \}$ be the space of spectral element functions consisting of the above tensor products of polynomials of degree $W$.

### 2.3 Numerical formulation

Let $\mathcal{L}(\mathbf{u}, p)$ and $\mathcal{D}\mathbf{u}$ be the differential operators for the momentum equations and the continuity equation, respectively. Thus

$$\mathcal{L}(\mathbf{u}, p) = \alpha \mathbf{u} - \nu \nabla^2 \mathbf{u} + \nabla p,$$

and

$$\mathcal{D}(\mathbf{u}) = -\nabla \cdot \mathbf{u}.$$ 

Let $J_l(\xi, \eta)$ be the Jacobian of the mapping $M_l(\xi, \eta)$ from $S = (-1, 1)^2$ to $\Omega_l$ for $l = 1, 2, ..., L$. Now

$$\int_{\Omega_l} |\mathcal{L} \hat{\mathbf{u}}_l|^2 \, dx_1 \, dx_2 = \int_{S} |\mathcal{L} \hat{\mathbf{u}}_l|^2 J_l \, d\xi \, d\eta.$$ 

Define $\mathcal{L}_l \hat{\mathbf{u}}_l = \mathcal{L} \hat{\mathbf{u}}_l / \sqrt{J_l}$. Then

$$\int_{\Omega_l} |\mathcal{L} \mathbf{u}_l|^2 \, dx_1 \, dx_2 = \int_{S} |\mathcal{L} \hat{\mathbf{u}}_l|^2 \, d\xi \, d\eta.$$ 

Similarly, we define $\mathcal{D}_l \hat{\mathbf{u}}_l = \mathcal{D} \hat{\mathbf{u}}_l / \sqrt{J_l}$.

Let $\hat{x} = (\xi, \eta)$. Let $\mathbf{f}_l(\hat{x}) = f(M_l(\xi, \eta)), h_l(\hat{x}) = h(M_l(\xi, \eta))$, for $l = 1, 2, \ldots, L$. Define

$$\mathbf{F}_l(\hat{x}) = \mathbf{f}_l(\hat{x}) \sqrt{J_l(\hat{x})} \quad \text{and} \quad H_l(\hat{x}) = h_l(\hat{x}) \sqrt{J_l(\hat{x})}.$$
Now, consider the boundary condition \( \mathbf{u} = \mathbf{g} \) on \( \partial \Omega \). Let \( \gamma_s \subseteq \partial \Omega \cap \Omega_l \) be the image of \( \xi = 1 \) under the mapping \( M_l \) which maps \( S \) to \( \Omega_l \). Let

\[
\mathbf{g}_l = \mathbf{g}(M_l(1, \eta)).
\]

We now define the least-squares functional

\[
\mathcal{R}^{L, W}(\mathbf{u}, p) = \sum_{l=1}^{L} \left\| \mathcal{L}_l \mathbf{u}_l - \mathbf{F}_l \right\|_{0, S}^2 + \sum_{l=1}^{L} \left\| \mathcal{D}_l \mathbf{u}_l - H_l \right\|_{1, S}^2
\]

\[
+ \sum_{\gamma_s \subseteq \partial \Omega \cap \Omega_l} \left( \left\| [\mathbf{u}] \right\|_{0, \gamma_s}^2 + \sum_{k=1}^{2} \left\| [\mathbf{u}_k] \right\|_{2, \gamma_s}^2 + \left\| p \right\|_{2, \gamma_s}^2 \right) + \sum_{\gamma_s \subseteq \partial \Omega \cap \Omega_l} \left\| \mathbf{u}_l - \mathbf{g}_l \right\|_{3, \gamma_s}^2. \tag{4}
\]

We choose our approximate solution the unique \( (\mathbf{z}, q) \in \Pi^{L, W} \) which minimizes the functional \( \mathcal{R}^{L, W}(\mathbf{u}, p) \) overall \( (\mathbf{u}, p) \in \Pi^{L, W} \). Here, \( \Pi^{L, W} \) denotes the space of spectral element functions.

The details of the jump terms \( \left\| [\mathbf{u}] \right\|_{0, \gamma_s}^2 \), \( \left\| [\mathbf{u}_k] \right\|_{2, \gamma_s}^2 \) and \( \left\| p \right\|_{2, \gamma_s}^2 \) in (4) are described in the appendix A2. The functional \( \mathcal{R}^{L, W}(\mathbf{u}, p) \) is closely related to the quadratic form \( \mathcal{V}^{L, W}(\mathbf{u}, p) \) which is also defined in A2.

**Error estimate**

**Theorem 1** Let \( (\mathbf{z}, q) \) minimize \( \mathcal{R}^{L, W}(\mathbf{u}, p) \). Then, for \( W \) large enough there exists constants \( C \) and \( b \) (being independent of \( W \)) such that the estimate

\[
\sum_{l=1}^{L} \left\| \mathbf{z}_l(\hat{\mathbf{x}}) - \mathbf{u}_l(\hat{\mathbf{x}}) \right\|_{2, S}^2 + \sum_{l=1}^{L} \left\| q_l(\hat{\mathbf{x}}) - p_l(\hat{\mathbf{x}}) \right\|_{1, S}^2 \leq Ce^{-bW} \tag{5}
\]

holds true.

Proof of the this theorem easily follows from Theorem 4.2 in Mohapatra et al. (2020).

**Remark** After obtaining a nonconforming solution a set of corrections can be made such that velocity variable \( \mathbf{z} \) becomes conforming (Kishore Kumar 2014; Schwab 1998). So \( \mathbf{z} \in H^1(\Omega) \) and we have the following error estimate

\[
\left\| \mathbf{u} - \mathbf{z} \right\|_{1, \Omega} + \left\| p - q \right\|_{0, \Omega} \leq Ce^{-bW}. \tag{6}
\]

**Residue computations and preconditioner**

The solution will be obtained at Gauss–Legendre–Lobatto (GLL) quadrature points by minimizing the residue \( \mathcal{R}^{L, W}(\mathbf{u}, p) \). The normal equations obtained from the minimization will be solved using preconditioned conjugate gradient method (PCGM) efficiently. The details of the residual computations in each element and procedure of solving the normal equations are shown in detail in Dutt et al. (2007), Kishore Kumar (2014), Mohapatra et al. (2020).

We use the following quadratic form

\[
\mathcal{U}^{L, W}(\mathbf{u}, p) = \sum_{l=1}^{L} \left\| \mathbf{u}_l \right\|_{2, S}^2 + \sum_{l=1}^{L} \left\| p_l \right\|_{1, S}^2. \tag{7}
\]

as a preconditioner which is spectrally equivalent to \( \mathcal{V}^{L, W}(\mathbf{u}, p) \) (defined in appendix A2).
Fig. 2 Discretization of $[0, 1]^2$

3 Numerical results

Here, we verify the exponential convergence of the numerical scheme by considering various numerical examples. The numerical examples include the Stokes equations on curvilinear domains, Stokes problem with mixed boundary conditions, and generalized Stokes equations in $\mathbb{R}^2$ and $\mathbb{R}^3$. Spectral element functions of higher order of degree $W$ are used and uniform $W$ is used for all the elements in the discretization. Let $z$ and $q$ be the approximate solutions of the velocity $u$ and pressure $p$, respectively. $\| E_u \|_1 = \frac{\| u - z \|_1}{\| u \|_1}$ denotes the relative error in $u$ in $H^1$ norm, $\| E_p \|_0 = \frac{\| p - q \|_0}{\| p \|_0}$ denotes relative error in pressure in $L^2$ norm and $\| E_c \|_0$ denotes error in continuity equation in $L^2$ norm. ‘Iter’ denotes the total number of iterations required to reach the desired accuracy. In the case of Dirichlet boundary value problem, pressure is specified to be zero at one point of the domain to ensure the uniqueness in each problem. The numerical code was written in FORTRAN 90. Computations were performed on intel core i7-6700 CPU@3.40 GHz × 8 with 32 GB RAM.

Example 1: Generalized Stokes problem on $[0, 1]^2$

Consider the generalized Stokes equation (1–3) with $\alpha = 1$, $\nu = 1$ and $h = 0$ on $[0, 1]^2$. Chosen the data, such that

$$\begin{align*}
u_1 &= \sin \pi x_1 \sin \pi x_2, \\
u_2 &= \cos \pi x_1 \cos \pi x_2, \\
p &= 150 \left( x_1 - \frac{1}{2} \right) \left( x_2 - \frac{1}{2} \right) + c.
\end{align*}$$

The domain $[0, 1]^2$ is divided into 4 elements with equal step size $h = \frac{1}{2}$ in both $x_1$ and $x_2$ directions (see Fig. 2). The approximate solution is obtained and the relative errors $\| E_u \|_1$ and $\| E_p \|_0$ for various values of $W$ are shown in Table 1. Table 1 also shows the total number of iterations of PCGM to reach the achieved accuracy and the error in the continuity equation $\| E_c \|_0$. One can see that the errors $\| E_u \|_1$, $\| E_p \|_0$ and $\| E_c \|_0$ decays exponentially. Figure 3 shows the graph of the log of the relative errors $\| E_u \|_1$ and $\| E_p \|_0$ against $W$. The curves are almost linear. This shows the exponential decay of the errors.
Table 1 Error $\|E_u\|_1$, $\|E_p\|_0$ and $\|E_c\|_0$ for different $W$

| $W$ | $\|E_u\|_1$ | $\|E_p\|_0$ | $\|E_c\|_0$ | Iter |
|-----|-------------|-------------|-------------|------|
| 2   | 5.139862909E−01 | 1.547329906E−01 | 1.177314788E−00 | 22   |
| 3   | 9.320542876E−02 | 5.59518233E−02  | 3.465516273E−02 | 116  |
| 4   | 1.644416626E−02 | 6.011788360E−03 | 3.739501208E−03 | 227  |
| 5   | 1.611022198E−03 | 6.748736448E−04 | 3.739501208E−03 | 442  |
| 6   | 1.693357558E−04 | 6.570180017E−05 | 4.172145413E−04 | 761  |
| 7   | 2.471931274E−05 | 7.480811937E−06 | 6.553699066E−05 | 1204 |
| 8   | 3.451649675E−06 | 4.728984810E−07 | 8.701982919E−06 | 1788 |

Fig. 3 Log of relative error vs. $W$

**Example 2: Stokes problem involving Reynolds number**

Consider the following equation on $\Omega = [\frac{1}{2}, \frac{1}{2}] \times [0, 1]$

$$-\frac{1}{Re} \Delta u + \nabla p = f \text{ in } \Omega,$$
$$\nabla \cdot u = 0 \text{ in } \Omega,$$
$$u = 0 \text{ on } \Gamma.$$

Choose the data such that

$$u_1 = 1 - e^{\lambda x_1} \cos(2\pi x_2),$$
$$u_2 = \frac{\lambda}{2\pi} e^{\lambda x_1} \sin(2\pi x_2),$$
$$p = \frac{1}{2} e^{2\lambda x_1} + c.$$

Here, $\lambda = \frac{Re}{2} - \sqrt{\frac{(Re)^2}{4} + 4\pi^2}$ and $Re$ is Reynolds number (Mu and Ye 2017).
The domain is divided into 4 elements with step size \( h = \frac{1}{2} \) in both directions. We have obtained the approximate solution of the given Stokes system for different values of \( Re = 1, 10, 100, 1000 \). Table 2 shows the relative errors \( \| E_u \|_1 \) and \( \| E_p \|_0 \) for various values of \( W \) for \( Re = 1, 10 \). Table 3 shows the relative errors \( \| E_u \|_1 \) and \( \| E_p \|_0 \) for various values of \( W \) for \( Re = 100, 1000 \).

Figure 4a shows the graph of the log of the relative error \( \| E_u \|_1 \) vs. \( W \) and Fig. 4b shows the graph of log of the relative error \( \| E_p \|_0 \) vs. \( W \) for \( Re = 1, 10, 100 \) and 1000. Both the graphs shows that the error decays exponentially. One can see that the iteration count is high for \( Re = 1000 \) to achieve the relative errors of \( O(10^{-7}) \) and \( O(10^{-6}) \), but the iteration count is not high to achieve the accuracy of \( O(10^{-4}) \) or \( O(10^{-5}) \).

**Example 3: Stokes problem on annular domain**

Consider the Stokes problem (problem (1)–(3) with \( \alpha = 0, \nu = 1 \) and \( h = 0 \)) on the annular domain \( \Omega = \{(r, \theta) : 1 \leq r \leq 4 \text{ and } 0 \leq \theta \leq \frac{\pi}{2}\} \) with Dirichlet boundary condition on the boundary.
| W  | $Re = 1$  | $Re = 10$ | Iter |
|----|-----------|-----------|------|
|    |         $\|E_u\|_1$ | $\|E_u\|_1$ | $\|E_u\|_1$ | $\|E_p\|_0$ | $\|E_p\|_0$ | $\|E_p\|_0$ | Iter |
| 4  | 2.366975497E−01 | 4.34268366E−01 | 12 | 1.131661632E−01 | 1.198261163E−01 | 40 |
| 5  | 4.869971324E−02 | 1.76876076E−01 | 129 | 1.184006906E−02 | 1.227193197E−02 | 120 |
| 6  | 8.623443013E−03 | 8.20262592E−02 | 321 | 3.004801200E−03 | 1.629441123E−03 | 195 |
| 7  | 1.518313601E−03 | 1.30971235E−02 | 733 | 2.839200284E−04 | 3.512941780E−04 | 346 |
| 8  | 2.667438050E−04 | 3.12678320E−03 | 1206 | 3.181384304E−05 | 4.256880275E−05 | 561 |
| 9  | 4.391290170E−05 | 6.57675621E−04 | 1862 | 1.053751269E−06 | 9.059197820E−06 | 967 |
| 10 | 7.408947632E−06 | 1.56727777E−04 | 2539 | 1.718964312E−07 | 2.683927048E−07 | 1843 |
### Table 3 Relative errors against $W$ for $Re = 100, 1000$

| $W$ | $Re = 100$ | $Re = 1000$ |
|-----|------------|-------------|
|     | $\|E_u\|_1$ | $\|E_p\|_0$ | Iter | $\|E_u\|_1$ | $\|E_p\|_0$ | Iter |
| 4   | 4.270224166E−01 | 1.971826154E−01 | 26 | 3.7944941768E−01 | 2.480889414E−01 | 36 |
| 5   | 4.244262501E−02 | 1.618248455E−02 | 179 | 3.8779299147E−02 | 2.947725331E−02 | 654 |
| 6   | 3.958119651E−03 | 1.747850562E−03 | 325 | 2.3764211892E−03 | 5.438172800E−04 | 1703 |
| 7   | 3.201794986E−04 | 1.504988745E−04 | 514 | 1.9140466720E−04 | 5.837954856E−05 | 3190 |
| 8   | 1.26502314E−05  | 3.527247772E−05 | 748 | 1.5032229108E−05 | 5.156661742E−06 | 4569 |
| 9   | 3.187641118E−06 | 2.098243693E−06 | 936 | 3.5580252816E−06 | 1.602969000E−06 | 5721 |
| 10  | 3.037312630E−07 | 1.211241028E−07 | 1271 | 7.451543690E−07  | 6.117219799E−07  | 7160 |
Table 4 \( \|E_u\|_1, \|E_p\|_0 \) and \( \|E_c\|_0 \) for various values of \( W \)

| \( W \) | \( \|E_u\|_1 \)     | \( \|E_p\|_0 \)     | \( \|E_c\|_0 \)     | Iter |
|--------|----------------|----------------|----------------|-----|
| 2      | 3.73710182E−01 | 1.000929095E−00 | 558.6790137E−00 | 4   |
| 3      | 5.47325472E−02 | 1.019245995E−01 | 149.7030088E−00 | 91  |
| 4      | 9.27471997E−03 | 1.026087077E−02 | 17.92350320E−00 | 178 |
| 5      | 2.27183319E−03 | 4.526209660E−03 | 7.01641256E−00  | 360 |
| 6      | 4.61843438E−04 | 6.17589621E−04  | 1.262648344E−00 | 451 |
| 7      | 4.52049220E−05 | 1.124195597E−04 | 1.507379559E−01 | 935 |
| 8      | 8.47350232E−06 | 5.888507439E−06 | 2.12266602E−02  | 1252|
| 9      | 7.82707160E−07 | 1.396124323E−06 | 2.061182562E−03 | 1930|
| 10     | 1.52518872E−07 | 1.042122306E−07 | 3.131737628E−04 | 2748|

Fig. 6 Log of relative errors against \( W \)

The domain is divided into 4 curvilinear elements, as shown in Fig. 5. Blending elements have been used (Gordan and Hall 1973). The data is chosen such that

\[
\begin{align*}
    u_1 &= 20x_1x_2^3, \\
    u_2 &= 5(x_1^4 - x_2^4), \\
    p &= 60x_1^2x_2 - 20x_2^3 + c.
\end{align*}
\]

Table 4 shows the relative errors \( \|E_u\|_1, \|E_p\|_0 \) and \( \|E_c\|_0 \) for various values of \( W \). Figure 6 shows the log of the relative errors against \( W \). This shows that the error decays exponentially in \( \|E_u\|_1 \) and \( \|E_p\|_0 \) norms.

**Example 4. Stokes problem on a square domain with a circular hole**

Consider the Stokes problem on a square domain \([0, 1]^2\) with a circular hole, where the circle is centered at \((0.5, 0.5)\) with radius 0.2 (see Fig. 7). As shown in Fig. 7, we have the boundaries on the sides of the unit square and also on the circle.
The data are chosen, such that

\[ u_1 = x_1 + x_2^2 - 2x_1x_2 + x_1^3 - 3x_1x_2^2 + x_2x_1^2, \]
\[ u_2 = -x_2 - 2x_1x_2 + x_2^3 - 3x_2x_1^2 + x_2^3 - x_1x_2^2, \]
\[ p = x_1x_2 + x_1 + x_2 + x_1^3x_2^2 + c. \]

The given domain is decomposed into 4 elements, as shown in Fig. 7. Table 5 shows the relative errors \( \| \mathbf{E}_u \|_1 \) and \( \| \mathbf{E}_p \|_0 \) for various values of \( W \).

So far, we have considered the Stokes problem with Dirichlet condition on the boundary. Here, we consider Stokes problem with mixed boundary conditions. We consider the following Neumann type boundary conditions on some part of the boundary of the domain Benes and Kucera (2016), Manouzi (1990)

\[ \gamma_N(u, p) = \frac{\partial u}{\partial n} - pn = g^N \text{ or} \]
\[ \gamma_N(u, p) = \left( (\nabla u + \nabla u^T) - pl \right) n = g^N \]

where \( n = (n_1, n_2) \) is unit outward normal vector and \( I \) is \( 2 \times 2 \) identity matrix.

Details of the existence and regularity of the solution of the Stokes problem with mixed boundary conditions can be found in Benes and Kucera (2016). The numerical method proposed in this article also works for mixed boundary conditions. In this case, we add the following term to the minimizing functional \( \mathcal{R}_{L,W}(u, p) \) defined in (4)

\[ \sum_{\gamma_N \subseteq \Gamma^N \cap \Omega_l} \left\| \gamma_N(u, p) - g_{\gamma_N}^N \right\|_{\frac{1}{2}, \gamma_N}^2, \]

where \( \Gamma^N \) is the part of the boundary of the domain on which the Neumann type of boundary condition is specified.
Example 5: Stokes problem with mixed boundary conditions on a square domain

Consider the Stokes problem (Eqs. (1)–(3) with $\alpha = 0$ and $\nu = 1$) on $[0, 1]^2$ with mixed boundary conditions. Dirichlet boundary condition is considered on the sides $x = 0$, $x = 1$ and $y = 1$ and the following Neumann type boundary condition is taken on the side $y = 0$:

$$\frac{\partial u}{\partial n} - p n = g^N.$$

Chosen the data such that the exact solution of the problem is

$$u_1 = \sin \pi x_1 \sin \pi x_2,$$
$$u_2 = \cos \pi x_1 \cos \pi x_2,$$
$$p = 150 \left( x_1 - \frac{1}{2} \right) \left( x_2 - \frac{1}{2} \right).$$

The domain $[0, 1]^2$ is divided into 4 elements with equal step size $h = \frac{1}{2}$ in each direction. The relative errors $\|E_u\|_1$, $\|E_p\|_0$ and $\|E_c\|_0$ for various values of $W$ are shown in Table 6. The errors decays very faster and the decay of the error $\|E_c\|_0$ shows that the method has good mass conservative property.

Example 6: Stokes problem with mixed boundary conditions on an annular domain

Consider the Stokes problem on the annular domain which was considered in the example 3. The following Neumann type boundary condition is considered on the side $y = 0$:

$$\left( (\nabla u + \nabla u^T) - p I \right) n = g^N.$$

Dirichlet boundary conditions are considered on the other parts of boundary of the annular domain.

We have considered the same data as in example 3 and the domain is divided into 4 elements (see Fig. 5). Table 7 shows the relative errors $\|E_u\|_1$ and $\|E_p\|_0$ for various values of $W$. The error decays quickly and this shows the exponential accuracy of the numerical method. The iteration count is also less compared to the number of iterations in Example 3.
Consider the generalized Stokes problem (problem \((1–3)\)) in \(\mathbb{R}^3\) with \(\alpha = 1\) on the domain \([-1, 1]^3\) with Dirichlet boundary condition on the boundary. Let \(x = (x_1, x_2, x_3)\) be a point in the domain and \(u = (u_1, u_2, u_3)\) be the velocity vector. The force function and boundary data are chosen such that the exact solution of the given problem is given by

\[
\begin{align*}
\ u_1(x_1, x_2, x_3) &= 4x_1^2x_2x_3(1 - x_1)^2(1 - x_2)(1 - x_3)(x_3 - x_2), \\
\ u_2(x_1, x_2, x_3) &= 4x_1x_2^2x_3(1 - x_1)(1 - x_2)^2(1 - x_3)(x_1 - x_3), \\
\ u_3(x_1, x_2, x_3) &= 4x_1x_2x_3^2(1 - x_1)(1 - x_2)(1 - x_3)^2(x_2 - x_1), \\
\ p(x_1, x_2, x_3) &= -2x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3 - x_1 - x_2 - x_3.
\end{align*}
\]

Only one element is considered (i.e., \([-1, 1]^3\)) and obtained the approximate solution of the generalized Stokes problem for \(v = 1, 10\). Table 8 shows the errors \(\|E_u\|_1, \|E_p\|_0\) and \(\|E_c\|_0\) against different values of \(W\) for \(v = 1\) and Table 9 shows for \(v = 10\).

The results shows that the errors \(\|E_u\|_1, \|E_p\|_0\) and \(\|E_c\|_0\) decays exponentially. We have presented the error in the continuity equation \(\|E_c\|_0\) against \(W\) in few other examples also.
in this section. The decay of $\|E_c\|_0$ shows the mass conservation property of the method. Similar behavior has been observed in all the other examples too.

4 Conclusions and future work

In this article, we have studied the performance of the nonconforming least-squares spectral element method for Stokes problems on smooth domains. The generalized Stokes equation, Stokes problem with mixed boundary conditions, and also the Stokes problem on curvilinear domains were considered. The spectral approximation is nonconforming and same order spectral element functions are used for both velocity and pressure variables. The numerical results show that the method is exponentially accurate in both $u$ and $p$. Since the numerical method is least-squares, the obtained linear system is symmetric positive definite. In addition to these advantages, the numerical scheme has good mass conservation property. The decay of the error in continuity equation in $L^2$ norm shows that the method works very well while conserving the mass. Studying the performance of this approach for unsteady flow problems on curvilinear domains is in progress. The Stokes interface problem is under consideration for future work.

Appendix

Here, we state the regularity estimate for the generalized Stokes problem (1)–(3) and define the jumps in $u$, $p$ and $u_{x_k}$ in different Sobolev norms. Finally we state the stability estimate.

A1. Regularity estimate

The following fundamental regularity estimate is based on ADN (Agmin–Douglis–Nirenberg) theory (Agmon et al. 1964).

Let $\Omega$ be an open bounded subset of class $C^r$, $r = \max(m + 2, 2)$. For $u \in W^{1,2}(\Omega)$, $p \in L^2(\Omega)$ being solutions of the generalized Stokes equations (1)–(3) and for $f \in W^{m,2}(\Omega)$, $h \in W^{m+1,2}(\Omega)$ and $g \in W^{m+2,2}(\Gamma)$, then $u \in W^{m+2,2}(\Omega)$, $p \in W^{m+1,2}(\Omega)$ and there exists a constant $C_0(\alpha, m, \Omega)$ such that

$$
\|u\|_{W^{m+2,2}(\Omega)} + \|p\|_{W^{m+1,2}(\Omega)/\mathbb{R}} \leq C_0 \left( \|f\|_{W^{m,2}(\Omega)} + \|h\|_{W^{m+1,2}(\Omega)} + \|g\|_{W^{m+2,2}(\Gamma)} \right).
$$

A2. Stability estimate

Since the approximation is nonconforming, to enforce the continuity along the inter element boundaries we introduce the jumps in $u$, $u_{x_1}$, $u_{x_2}$ and $p$ in suitable Sobolev norms. Let the edge $\gamma_s$ be common to the adjacent elements $\Omega_l$ and $\Omega_m$. Assume that edge $\gamma_s$ is the image of $\eta = 1$ under the map $M_l$ which maps $S$ to $\Omega_l$ and also the image of $\eta = -1$ under the map $M_m$ which maps $S$ to $\Omega_m$. Then the jumps along the inter-element boundaries are defined as

$$
\left\| \left[ u \right] \right\|^2_{0,\gamma_s} = \left\| \hat{u}_m(\xi, -1) - \hat{u}_l(\xi, 1) \right\|^2_{0,\gamma_s},
$$

$$
\left\| \left[ u_{x_k} \right] \right\|^2_{1,\gamma_s} = \left\| (\hat{u}_m)_{x_k}(\xi, -1) - (\hat{u}_l)_{x_k}(\xi, 1) \right\|^2_{1,\gamma_s},
$$
\[
\|\|\|p\|\|_2^\frac{1}{2},\gamma_s = \|\hat{p}_m(\xi, -1) - \hat{p}_l(\xi, 1)\|_2^\frac{1}{2},I.
\]

Here, \(I = (-1, 1)\). The expressions on the right hand side in the above equation are given in the transformed coordinates \(\xi\) and \(\eta\).

Let us consider the boundary condition. Let \(\gamma_s \subseteq \partial \Omega \cap \Omega_l\) be the image of \(\xi = 1\) under the mapping \(M_l\) which maps \(S\) to \(\Omega_l\). Then

\[
|||\mathbf{u}_l|||_2^\frac{1}{2},\gamma_s = |||\hat{\mathbf{u}}_l(1, \eta)|||_2^\frac{1}{2},I.
\]

Let \(\mathbf{u}, p \in \Pi^{L,W}_{\Omega_1}\). We now define the quadratic form

\[
\mathcal{V}_{L,W}(\mathbf{u}, p) = \sum_{l=1}^{L} |||\mathcal{L}(\mathbf{u}_l, p_l)|||_{0,\Omega_l}^2 + \sum_{l=1}^{L} |||\mathbf{D}\mathbf{u}_l|||_{1,\Omega_l}^2 + \sum_{\gamma_s \subseteq \partial \Omega \cap \Omega_l} \left( |||\mathbf{u}|||_{0,\gamma_s}^2 + \sum_{k=1}^{2} |||\mathbf{u}_{x_k}|||_{2,\gamma_s}^2 + |||p|||_{2,\gamma_s}^2 \right) + \sum_{\gamma_s \subseteq \partial \Omega \cap \Omega_l} |||\mathbf{u}_l|||_{3,\gamma_s}^2.
\]

(9)

Then, we have the following result.

**Theorem** For \(W\) large enough, there exists a constant \(C > 0\) such that the estimate

\[
\mathcal{U}_{L,W}(\mathbf{u}, p) \leq C (\ln W)^2 \mathcal{V}_{L,W}(\mathbf{u}, p)
\]

(10)

holds. Where \(\mathcal{U}_{L,W}(\mathbf{u}, p)\) is defined in Sect. 2 (see (7)). The proof of this one is very similar to Theorem 4.1 in (Mohapatra et al. 2020).

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