Abstract

The threshold network model is a type of finite random graphs. In this paper, we introduce a generalized threshold network model. A pair of vertices with random weights is connected by an edge when real-valued functions of the pair of weights belong to given Borel sets. We extend several known limit theorems for the number of prescribed subgraphs to show that the strong law of large numbers can be uniform convergence. We also prove two limit theorems for the local and global clustering coefficients.

1 Introduction

Complex networks have been an attractive research topic for a decade. Particularly, many real-world graphs are characterized by the small diameter, high clustering (abundance of connected triangles), and fat-tail degree distributions. Degree distributions often follow the truncated power law, which is called the scale-free property of networks [1, 4, 15]. Both growing and static network models are capable of generating scale-free networks.

Here we are concerned with asymptotic properties of a class of static network models called the threshold network model, which is generated on a graph with a common distribution of vertex degrees. We associate the random variable \( X_i \) with vertex \( i \). Now we introduce Borel measurable functions \( f_c : (\mathbb{R}^d)^2 \rightarrow \mathbb{R} \) with \( f_c(x,y) = f_{c,i}(y,x) \) for all \( i \in \{1, \ldots, n\} \). Let \( \mathcal{B}(\mathbb{R}) \) be the Borel \( \sigma \)-field of \( \mathbb{R} \). For a given finite collection of Borel measurable sets \( \mathcal{C} = \{B_1, \ldots, B_l\} \) with \( B_i \subset \mathcal{B}(\mathbb{R}) \), we connect vertices \( i \) and \( j \) (\( i \neq j \)) if \( f_{c,i}(X_i, X_j) \in B_i \) for all \( i \in \{1, \ldots, l\} \). In other words, we form an edge \( (i,j) \) if \( \prod_{l'} I_{B_{l'}} \left( f_{c,i}(X_i, X_j) \right) = 1 \) for \( i \neq j \), where \( I_A(x) \) denotes the indicator function, i.e., \( I_A(x) = 1 \) for \( x \in A \) and \( I_A(x) = 0 \) otherwise. Thus we obtain a random graph \( G_C(X_1, \ldots, X_n) \). If there exist two collections of Borel sets \( \mathcal{C} = \{B_1, \ldots, B_l\} \) and \( \mathcal{C}' = \{B'_1, \ldots, B'_l\} \) with \( B_i \subset B'_i \) for all \( i \in \{1, \ldots, l\} \), then \( \mathbb{P}\left\{(i,j) \in G_C(X_1, \ldots, X_n)\right\} \leq \mathbb{P}\left\{(i,j) \in G_{C'}(X_1, \ldots, X_n)\right\} \) holds by simple coupling.

This random graph generalizes the threshold network model studied in [5, 6, 8–10, 12, 14]. By choosing \( l = 1, B_1 = (\theta, \infty) \) for some \( \theta \in \mathbb{R} \), \( f_{c,i}(x,y) = x + y \), we reproduce the model in [5, 6, 8–10]. In the context of social networks, a model with \( l = 2, B_1 = (\theta, \infty), B_2 = (-\infty, c) \) \( (\theta, c \in \mathbb{R}) \), \( f_{c,i}(x,y) = x + y \), and
$f_2^2(x,y) = |x-y|$ (or $f_2^2(x,y) = |x-y|/(x+y)$) was proposed [12]. General limit theorems are shown in [14] when $l = 1$, $B_1 = (\theta, \infty)$, and $f_2^2(x,y) = |x-y|$.

In Sec. 2, we state several general limit theorems for the number of prescribed subgraphs. By using $U$-statistics, the strong law of large numbers, the central limit theorem, and the law of the iterated logarithm are stated for global properties of the model. We also state a limit distribution for a local property. These are generalizations of Theorems 1, 4, and 5 of [9] and Theorems 1, 2, and 3 of [14]. In Sec. 3, we show that the strong law of large numbers for the number of prescribed subgraphs is uniform convergence on so-called the VC class of Borel sets, generalizing Theorem 1a of [14]. In Sec. 4, we show limit theorems for the clustering coefficient, which quantifies the abundance of connected triangles in a graph in a specific ways. Particularly, we show the strong law of large numbers for the local clustering coefficient (Theorem 2) and the global clustering coefficient (Theorem 3). Theorems 2 and 3 are main results of this paper. In Sec. 5, we present several examples of limit degree distributions.

2 General Results

In this section, we show limit theorems for the number of prescribed subgraphs. Let us begin with notations [14]. For $m \in \{2, \ldots, n\}$, we consider a graph $H = (V_H, E_H)$ on the ordered set of $m$ vertices $V_H = \{v_1, \ldots, v_m\}$ and the edge set $E_H$. For another graph $H' = (V_{H'}, E_{H'})$ on $m$ vertices, we say $H' \sim H$ if $V_{H'} = V_H$ and $E_{H'} = E_H$ for some reordering of vertices. Thus $\mathcal{A}_H = \{H' : H' \sim H\}$ is the set of all graphs isomorphic to $H$. Let us define $\mathcal{A}_m = \bigcup_{i=1}^{m} \mathcal{A}_{H_i}$, where $H_i$ is an arbitrarily chosen graph on $m$ vertices. The collection of all triangles and graphs on three vertices that consist of two connected vertices and an isolated vertex is an example of $\mathcal{A}_3$. The collection of cliques on $m$ vertices and the graphs on $m$ vertices with $m$ isolated vertices is an example of $\mathcal{A}_m$. The definition of $\mathcal{A}_m$ guarantees the symmetrical property of the kernel function $h_{\mathcal{A}_m} : (\mathbb{R}^d)^m \to \mathbb{R}$ given by

$$h_{\mathcal{A}_m}(x_1, \ldots, x_m) = I_{\mathcal{A}_m}(G_C(x_1, \ldots, x_m)),$$

(1)

where $G_C(x_1, \ldots, x_m)$ denotes a realization of the random graph $G_C(X_1, \ldots, X_m)$. Then we define

$$\tilde{U}_n(\mathcal{C}, \mathcal{A}_m) = \sum_{1 \leq i_1 < \cdots < i_m \leq n} h_{\mathcal{A}_m}(X_{i_1}, \ldots, X_{i_m}),$$

i.e., the number of subgraphs belonging to the collection $\mathcal{A}_m$ in the random graph $G_C(X_1, \ldots, X_n)$. We also define

$$U_n(\mathcal{C}, \mathcal{A}_m; i) = \sum_{1 \leq i_2 < \cdots < i_m \leq n, i_2, \ldots, i_m \neq i} h_{\mathcal{A}_m}(X_{i_1}, X_{i_2}, \ldots, X_{i_m}),$$

i.e., the number of subgraphs that include vertex $i$ and belong to $\mathcal{A}_m$ in the random graph $G_C(X_1, \ldots, X_n)$.

Note that $U_n(\mathcal{C}, \mathcal{A}_m; i), 1 \leq i \leq n$ are identical in distribution and the following relation holds:

$$\frac{\sum_{i=1}^{n} U_n(\mathcal{C}, \mathcal{A}_m; i)/\binom{n-1}{m-1}}{n} = \frac{m \tilde{U}_n(\mathcal{C}, \mathcal{A}_m)}{\binom{n}{m}}.$$

This implies that the global property $\tilde{U}_n(\mathcal{C}, \mathcal{A}_m)/\binom{n}{m}$ is the arithmetic mean of the local properties $U_n(\mathcal{C}, \mathcal{A}_m; 1)/\binom{n-1}{m-1}, \ldots, U_n(\mathcal{C}, \mathcal{A}_m; n)/\binom{n-1}{m-1}$.

We define

$$F(\mathcal{C}, \mathcal{A}_m) = E[h_{\mathcal{A}_m}(X_1, \ldots, X_m)],$$

$$\zeta(\mathcal{C}, \mathcal{A}_m) = \text{Var}(E[h_{\mathcal{A}_m}(X_1, \ldots, X_m)|X_1]),$$

and assume $\zeta(\mathcal{C}, \mathcal{A}_m) > 0$. Since $\tilde{U}_n(\mathcal{C}, \mathcal{A}_m)/\binom{n}{m}$ is a $U$-statistic [17] obtained from the symmetric kernel $h_{\mathcal{A}_m}$, the strong law of large numbers (SLLN), the central limit theorem (CLT) and the law of the iterated logarithm (LIL) are derived from general results for the $U$-statistics, namely, Theorem A (SLLN) and Theorem B (LIL) in Section 5.4, and Theorem A (CLT) in Section 5.5 of [17]:

2
**Fact 1** (SLLN for global property). As \( n \to \infty \),
\[
\frac{\hat{U}_n(C, A_m) \binom{n}{m}}{\binom{n}{m}} \to F(C, A_m), \quad \text{almost surely.}
\]

**Fact 2** (CLT for global property). As \( n \to \infty \),
\[
\sqrt{\frac{n}{m^2 \zeta(C, A_m)}} \left[ \frac{\hat{U}_n(C, A_m) \binom{n}{m}}{\binom{n}{m}} - F(C, A_m) \right] \Rightarrow Z,
\]
where \( \Rightarrow \) stands for convergence in distribution and \( Z \) is a standard normal random variable.

**Fact 3** (LIL for global property).
\[
\limsup_{n \to \infty} \sqrt{\frac{n (\log \log n)^{-1}}{2m^2 \zeta(C, A_m)}} \left| \frac{\hat{U}_n(C, A_m) \binom{n}{m}}{\binom{n}{m}} - F(C, A_m) \right| = 1, \quad \text{almost surely.}
\]

There are the direct generalization of Theorem 4 of [9] and Theorems 1, 2, and 3 of [14] to the present model.

**Remark 1.** Generally, when the \( f'_c \) is asymmetric (e.g. directed graph), the number of graphs isomorphic to a graph \( H \) on \( m \) vertices is \( \binom{n}{m} \cdot m! \). The limit theorems above are valid by replacing the normalizing factor \( \binom{n}{m} \) with \( \binom{m}{m} \cdot m! \).

By generalizing Theorems 1 and 5 of [9], we obtain the following asymptotic behavior:

**Fact 4.** As \( n \to \infty \),
\[
\frac{U_n(C, A_m; 1) \binom{n-1}{m-1}}{\binom{n-1}{m-1}} \Rightarrow U(C, A_m),
\]
where
\[
U(C, A_m) = \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} h_{A_m}(x_1, x_2, \ldots, x_m) F(dx_2) \cdots F(dx_m).
\]

**Remark 2.** Almost sure convergence theorem for \( U_n(C, A_m; 1) \binom{n-1}{m-1} \) in the sense of Theorem 2 in Sec. 4.1 can be proved by a simple modification of the proof of Theorems 1 and 5 of [9].

### 3 Uniform Property

The Vapnik-Chervonenkis approach is well known in the context of the statistical learning theory. Particularly, it is useful in showing uniform convergence for limit theorems [7, 16]. In this section, we show that SLLN for global property (Fact 1) is uniform convergence on the VC class of the Borel sets, which extends the special case treated in [14].

Let \( M \) be a set and \( \mathcal{D} \) be a family of subsets of \( M \). For \( A \subseteq M \) let \( \Delta^D(A) = 2^\#(A \cap \mathcal{D}) \), where \( \#(A \cap \mathcal{D}) \) denotes the number of sets in \( A \cap \mathcal{D} = \{ A \cap D : D \in \mathcal{D} \} \). Let \( m_D(n) = \max_{A \subseteq M} \{ \Delta^D(A) : |A| = n \} \) for \( n = 0, 1, 2, \ldots \), where \( |A| \) denotes the number of elements in \( A \), or if \( |M| < n \) let \( m_D(n) = m_D(|M|) \). We define an indicator of the family \( \mathcal{D} \):
\[
S(\mathcal{D}) = \begin{cases} 
\sup \{ n : m_D(n) = 2^n \} & \text{if } \mathcal{D} \text{ is non-empty}, \\
-1 & \text{if } \mathcal{D} \text{ is empty}.
\end{cases}
\]

The family \( \mathcal{D} \) is called a Vapnik-Chervonenkis (VC) class of sets if \( S(\mathcal{D}) < +\infty \). For example, the collection of half intervals \( \mathcal{D} = \{ (-\infty, x) : x \in \mathbb{R} \} \) is a VC class on \( \mathbb{R} \) with \( S(\mathcal{D}) = 1 \). Based on Chapter 4.5 of [7], we have
Corollary 1. For any $\mathcal{D} \subset 2^M$ and $\mathcal{D}' \subset 2^M$ (resp. $2^N$), if $S(\mathcal{D}) < \infty$ and $S(\mathcal{D}') < \infty$ then $S(\mathcal{D} \cup \mathcal{D}') < \infty$ and $S(\mathcal{D} \cap \mathcal{D}') < \infty$ (resp. $S(\mathcal{D} \times \mathcal{D}') < \infty$)

where $\mathcal{D} \cup \mathcal{D}' = \{D \cup D': D \in \mathcal{D}, D' \in \mathcal{D}'\}$, $\mathcal{D} \cap \mathcal{D}' = \{D \cap D': D \in \mathcal{D}, D' \in \mathcal{D}'\}$ and $\mathcal{D} \times \mathcal{D}' = \{D \times D': D \in \mathcal{D}, D' \in \mathcal{D}'\}$.

For a given function $h: M \to \mathbb{R}$, the subgraph of $h$ is the set

$$D_h = \{(x, t) \in M \times \mathbb{R} : 0 \leq t \leq h(x) \text{ or } h(x) \leq t \leq 0\}.$$  

A class of functions $\mathcal{H}$ is a VC-subgraph class if the collection $\mathcal{D}_\mathcal{H} = \{D_h : h \in \mathcal{H}\}$ is a VC class of sets.

For a class of real-valued measurable functions $\mathcal{H}$ on $M^m$ for a fixed integer $m$, Arcones and Giné [2] proved the following uniform SLLN for i.i.d. sequence $\{X_i\}_{i=1,2,\ldots}$ on $M$:

**Lemma 1** (Corollary 3.3 of [2]). If $\mathcal{H}$ is a measurable VC-subgraph class of functions with $\mathbb{E}[\sup_{h \in \mathcal{H}} |h(X_1, \ldots, X_m)|] < +\infty$, then

$$\sup_{h \in \mathcal{H}} \left[ \frac{1}{n} \sum_{1 \leq i_1 < \cdots < i_m \leq n} h(X_{i_1}, \ldots, X_{i_m}) - \mathbb{E}[h(X_1, \ldots, X_m)] \right] \to 0, \text{ almost surely, as } n \to \infty.$$  

In order to use Lemma 1, we rewrite the kernel function $\mathbf{1}$ and show that the family of the kernel functions indexed by the VC class of Borel sets is a VC-subgraph class. Indeed, $\mathcal{H}$ is a VC class then $\mathcal{D}_\mathcal{H}$ is a VC class of sets.

For a given collection $\mathcal{A}_m$ of graphs on $m$ vertices, we define a set $\tilde{\mathcal{A}}_m$ on $\mathbb{R}^{m(2)}$ as follows. For each graph $H \in \mathcal{A}_m$, we associate a set $\tilde{H}$ on $\mathbb{R}^{m(2)}$. When a pair of vertices in $H$ has an edge, the corresponding coordinate of $\tilde{H}$ is occupied by $B$. Otherwise, it is occupied by $B^c$.

Each coordinate corresponds to a potential edge of the graph with a lexicographic order. For example, if $m = 4$, then $\tilde{H} = B \times B^c \times B^c \times B \times B^c \times B^c$. Then we define the set $\tilde{\mathcal{A}}_m = \bigcup_{H \in \mathcal{A}_m} \tilde{H}$. Note that $G(x_1, \ldots, x_m) \in \tilde{\mathcal{A}}_m$ is equivalent to the event that the realization of the random graph with weights $x_1, \ldots, x_m$ is in $\mathcal{A}_m$. Finally, we obtain the rewritten form of the kernel function:

$$h^B_{\tilde{\mathcal{A}}_m}(x_1, \ldots, x_m) = I_{\tilde{\mathcal{A}}_m} (G(x_1, \ldots, x_m)) = I_{G^{-1}(\tilde{\mathcal{A}}_m)}(x_1, \ldots, x_m),$$  

where $G^{-1}(\tilde{\mathcal{A}}_m)$ denotes the inverse image of $\tilde{\mathcal{A}}_m$.

Let $\mathcal{D}$ be a VC class of Borel sets on $\mathbb{R}$. For a fixed collection $\mathcal{A}_m$, we consider the class of kernel functions $\mathcal{H}_\mathcal{D} = \{h^B_{\mathcal{A}_m}(x_1, \ldots, x_m) : B \in \mathcal{D}\}$. By Corollary 1 if $\mathcal{D}$ is a VC class of sets in general, then the class of indicators $\{1_B : D \in \mathcal{D}\}$ is a VC-subgraph class. From Eq. (2), if the collection $G^{-1}(\tilde{\mathcal{A}}_m) = \{G^{-1}(\tilde{\mathcal{A}}_m) : B \in \mathcal{D}\}$ is a VC class then $\mathcal{H}_\mathcal{D}$, which is a collection of indicator functions, is a VC-subgraph class on $\mathbb{R}^m$. Indeed, $G^{-1}(\tilde{\mathcal{A}}_m)$ is a VC class based on Theorem 4.2.3 of [7] and Corollary 1 above, and $\mathcal{H}_\mathcal{D}$ is a VC-subgraph class. Finally, we find the uniform version of Fact 1 by Lemma 1.

**Theorem 1.** If $\mathcal{D}$ be a VC class of Borel sets, then for a fixed $f_c$ and $\mathcal{A}_m$,

$$\sup_{B \in \mathcal{D}} \left[ \frac{1}{n} \sum_{1 \leq i_1 < \cdots < i_m \leq n} h^B_{\tilde{\mathcal{A}}_m}(X_{i_1}, \ldots, X_{i_m}) - \mathbb{E}[h^B_{\tilde{\mathcal{A}}_m}(X_1, \ldots, X_m)] \right] \to 0,$$  

almost surely, as $n \to \infty$.

Theorem 1 can be extended to general $d$ and $l$. It generalizes the uniform SLLN in Theorem 1a of [14], which deals with the collection of half intervals as a VC class of sets.
4 Clustering Coefficient

Real-world networks are often equipped with high clustering, that is, a large number of connected triangles. The clustering coefficient quantifies the density of triangles in a graph (see [1, 15] for review). In this section, we study the limit theorems for the clustering coefficient.

4.1 Local Clustering Coefficient

We assume \( d = l = 1 \); extensions of the following results to general \( d \) and \( l \) is straightforward. We consider a random graph \( G_B(X_1, \ldots, X_n) \) for a given Borel set \( B \) and \( f_c \equiv f_c^1 \). Then we define

\[
D_n(i) = \sum_{1 \leq j \leq n, j \neq i} h_D(X_i, X_j),
\]

\[
T_n(i) = \sum_{1 \leq j, k \leq n, j, k \neq i} h_T(X_i, X_j, X_k),
\]

where \( h_D(x, y) = I_B(f_c(x, y)) \) and \( h_T(x, y, z) = I_B(f_c(x, y)) \cdot I_B(f_c(y, z)) \cdot I_B(f_c(z, x)) \), i.e., \( D_n(i) \) is the degree of vertex \( i \) and \( T_n(i) \) is the number of triangles including vertex \( i \). The local clustering coefficient \( C_n(i) \) of vertex \( i \) is given by

\[
C_n(i) = \frac{T_n(i)}{\binom{D_n(i)}{2}} \cdot I_{\{D_n(i) \geq 2\}} + w \cdot I_{\{D_n(i) = 0, 1\}}
\]

for an indeterminate \( w \). The second term represents the singular part for which the local clustering coefficient is not defined in physics literature and applications. Here we retain this term to assess the contribution of vertices with degree 0 or 1. If it is necessary to restrict \( C_n(i) \in [0, 1] \), we must substitute a real value on \([0, 1]\) into \( w \). If we substitute 0 into \( w \), the contribution of these vertices to \( C_n(i) \) is ignored. If we substitute 1, this contribution is implied to be the maximum because vertices with degree more than one satisfies \( C_n(i) \leq 1 \). Now we define

\[
V_n(i) = \sum_{1 \leq j, k \leq n, j, k \neq i} h_V(X_i, X_j, X_k),
\]

where \( h_V(x, y, z) = I_B(f_c(x, y)) \cdot I_B(f_c(y, z)) \cdot I_B(f_c(z, x)) \), which represents the number of vertex pairs \((j, k)\) such that both vertex \( j \) and vertex \( k \) are connected to vertex \( i \). We note the relation: On \( \{D_n(i) \geq 2\} \) or equivalently \( \{V_n(i) \geq 1\}, \)

\[
\left( \frac{D_n(i)}{2} \right) = V_n(i),
\]

which leads to

\[
C_n(i) = \frac{T_n(i)}{V_n(i)} \cdot I_{\{V_n(i) \geq 1\}} + w \cdot I_{\{V_n(i) = 0\}}.
\]

We also define

\[
C(i) = \frac{E_T(X_i)}{E_D(X_i)^2} \cdot I_{\{E_D(X_i) > 0\}} + w \cdot I_{\{E_D(X_i) = 0\}},
\]

where

\[
E_D(X_i) = \int_R h_D(X_i, y)F(dy),
\]

\[
E_T(X_i) = \int_R \int_R h_T(X_i, y, z)F(dy)F(dz).
\]

We consider \( C_n(i; x), C(i; x), D_n(i; x), T_n(i; x), \) and \( V_n(i; x) \), which are random variables \( C_n(i), C(i), D_n(i), T_n(i) \) and \( V_n(i) \) restricted to the subspace such that \( \{X_i = x\} \). For example, \( T_n(1; x) = \sum_{2 \leq j, k \leq n} h_T(x, X_j, X_k) \). We obtain the following asymptotic results for \( C_n(i) \):

\[
\]
Theorem 2. As $n \to \infty$,
(i) For any $x \in \mathbb{R}$, $C_n(1;x) \to C(1;x)$, almost surely.
In particular,
(ii) $C_n(1) \to C(1)$, almost surely.

Proof. For an arbitrary fixed $x \in \mathbb{R}$, we first prove

$$
\mathbb{E}[h_V(x,X_2,X_3)] = 0 \iff \mathbb{P}(V_n(1;x) = 0 \text{ for all } n \geq 1) = 1.
$$

(4)

Indeed, if $\mathbb{E}[h_V(x,X_2,X_3)] = 0$ then

$$
\mathbb{E}[V_n(1;x)] = \mathbb{E}\left[ \sum_{2 \leq j < k \leq n} h_V(x,X_j,X_k) \right] = \sum_{2 \leq j < k \leq n} \mathbb{E}[h_V(x,X_j,X_k)] = 0
$$

for all $n \geq 1$. Conversely, if $\mathbb{E}[V_n(1;x)] = 0$ for all $n \geq 1$ then

$$
\mathbb{E}[h_V(x,X_2,X_3)] = \mathbb{E}[V_3(1;x)] = 0.
$$

Therefore, we obtain

$$
\mathbb{E}[h_V(x,X_2,X_3)] = 0 \iff \mathbb{E}[V_n(1;x)] = 0 \text{ for all } n \geq 1.
$$

Since $V_n(1;x)$ is nonnegative,

$$
\mathbb{E}[V_n(1;x)] = 0 \iff \mathbb{P}(V_n(1;x) = 0) = 1 \text{ for all } n \geq 1.
$$

Moreover, $\{V_n(1;x) = 0\}$ is nonincreasing with $n$, which implies

$$
\mathbb{P}(V_n(1;x) = 0) = 1 \text{ for all } n \geq 1 \iff \mathbb{P}(V_n(1;x) = 0 \text{ for all } n \geq 1) = 1.
$$

Thus we have Eq. (5). By definition, $V_n(1;x)$ is invariant under any permutation on $\{x_2,x_3,\ldots,x_n\}$, and $V_n(1;x)$ is nondecreasing, i.e.,

$$
V_n(1;x)(x_2,x_3,\ldots,x_n) \leq V_{n+1}(1;x)(x_2,x_3,\ldots,x_n,x_{n+1})
$$

(5)

for all $n \geq 1$. Therefore $\mathbb{P}(V_n(1;x) = 0 \text{ for all } n \geq 1)$ equals to zero or one by the Hewitt-Savage zero-one law (see Theorem 36.5 of [3]). So we have

$$
\mathbb{E}[h_V(x,X_2,X_3)] > 0 \iff \mathbb{P}(V_n(1;x) = 0 \text{ for all } n \geq 1) = 0 \iff \mathbb{P}(V_n(1;x) \geq 1 \text{ for some } n \geq 1) = 1.
$$

Using Eq. (6), $\{V_n(1;x) \geq 1 \text{ for some } n \geq 1\}$ is equivalent to the event

$\{\exists \mathbb{N} \geq 1 \text{ s.t. } V_n(1;x) \geq 1 \text{ for all } n \geq N\}$. Hence we obtain

$$
\mathbb{E}[h_V(x,X_2,X_3)] > 0 \iff \mathbb{P}(\exists \mathbb{N} \geq 1 \text{ s.t. } V_n(1;x) \geq 1 \text{ for all } n \geq N) = 1
$$

$$
\iff \mathbb{P}(\exists \mathbb{N} \geq 1 \text{ s.t. } C_n(1;x) = T_n(1;x)/V_n(1;x) \text{ for all } n \geq N) = 1.
$$

(6)

Since $h_T(x_2,x_3)$ and $h_V(x_2,x_3)$ are symmetric functions of $x_2$ and $x_3$, we define $U$-statistics

$$
\frac{T_n(1;x)}{\binom{n-1}{2}} = \frac{1}{\binom{n-1}{2}} \sum_{2 \leq j < k \leq n} h_T(x,X_j,X_k),
$$

$$
\frac{V_n(1;x)}{\binom{n-1}{2}} = \frac{1}{\binom{n-1}{2}} \sum_{2 \leq j < k \leq n} h_V(x,X_j,X_k).
$$

We have the following SLLN by Theorem A in Section 5.4 of [17]: As $n \to \infty$,

$$
\frac{T_n(1;x)}{\binom{n-1}{2}} \to \mathbb{E}[h_T(x,X_2,X_3)], \text{ almost surely},
$$

(7)

$$
\frac{V_n(1;x)}{\binom{n-1}{2}} \to \mathbb{E}[h_V(x,X_2,X_3)], \text{ almost surely}.
$$

(8)
Based on Eqs. (7) and (8), the corresponding clustering coefficient
\[ C_n(1; x) = \frac{T_n(1; x)}{V_n(1; x)} = \frac{T_n(1; x)/(n-1)}{V_n(1; x)/(n-1)} \]
converges to \( \mathbb{E}[h_T(x, X_2, X_3)]/\mathbb{E}[h_V(x, X_2, X_3)] \), almost surely as \( n \to \infty \). By Eq. (6), we have
\[ \mathbb{P} \left( \lim_{n \to \infty} C_n(1; x) = \mathbb{E}[h_T(x, X_2, X_3)]/\mathbb{E}[h_V(x, X_2, X_3)] \right) = 1. \] (9)

On the other hand, Eqs. (4) and (9) imply that \( \mathbb{P}(\lim_{n \to \infty} C_n(1; x) = w) = 1 \). With the relation \( \mathbb{E}[h_V(x, X_2, X_3)] = \mathbb{E}[h_D(x, X_2)]^2 \), we obtain
\[ C_n(1; x) \to C(1; x) = \frac{\mathbb{E}[h_T(x, X_2, X_3)]}{\mathbb{E}[h_D(x, X_2)]^2} \cdot I(I_{\mathbb{E}[h_D(x, X_2)]>0} + w \cdot I(\mathbb{E}[h_D(x, X_2)] = 0), \]
almost surely as \( n \to \infty \). Particularly, we have by using Fubini’s theorem,
\[ \mathbb{P} \left( \lim_{n \to \infty} C_n(1) = C(1) \right) = \int_{\mathbb{R}} \mathbb{P} \left( \lim_{n \to \infty} C_n(1; x) = C(1; x) \right) F(dx) = \int_{\mathbb{R}} 1 \cdot F(dx) = 1. \]
This completes the proof. \( \square \)

4.2 Global Clustering Coefficient

The global clustering coefficient is defined by
\[ C_n = \frac{1}{n} \sum_{i=1}^{n} C_n(i). \]
Since it is a symmetric function of \((x_1, \ldots, x_n)\), we can prove SLLN for \( C_n \) by using the ergodic theory.

Theorem 3. As \( n \to \infty \),
\[ C_n \to \mathbb{E}[C(1)], \quad \text{almost surely.} \]

Proof. For simplicity, we only deal with the case \( \mathbb{E}[C(1)] = 0 \). For general cases, we can prove the theorem by replacing \( C(1) \) by \( C(1) - \mathbb{E}[C(1)] \). Let \( \mathbf{x} = (x_1, x_2, \ldots) \) be an infinite vector and \( \mathbf{x}_k = x_k \). We define measure-preserving transformation \( T_n \) for the product measure \( \mathbb{P} \) by
\[ (T_n \mathbf{x})_k = \begin{cases} x_{k+1} & \text{if } 1 \leq k \leq n-1, \\ x_1 & \text{if } k = n, \\ x_k & \text{otherwise,} \end{cases} \]
for each \( n \geq 1 \). By denoting \( C_n(i; \mathbf{x}) = C_n(i; x_i) \), a realization of \( C_n \) is represented by
\[ C_n(\mathbf{x}) = \frac{1}{n} \sum_{i=0}^{n-1} C_n(1; T_n^i \mathbf{x}). \]
For arbitrary fixed \( \varepsilon > 0 \), we define
\[ C_n^\varepsilon(1; \mathbf{x}) = (C_n(1; \mathbf{x}) - \varepsilon) \cdot I_{A_\varepsilon}, \]
\[ S_n^\varepsilon(\mathbf{x}) = \sum_{i=0}^{n-1} C_n^\varepsilon(1; T_n^i \mathbf{x}), \]
where \( A_\varepsilon = \{ \mathbf{x} : \limsup_{n \to \infty} C_n(\mathbf{x}) > \varepsilon \} \). Using the maximal ergodic theorem (see Theorem 24.2 of [3]),
\[ \int_{M_{\infty}^\varepsilon} C_n^\varepsilon(1; \mathbf{x}) d\mathbb{P} \geq 0 \]
for every $n \geq 1$, where $M^c_n = \{x : \sup_{1 \leq j \leq n} S^c_{kj}(x) > 0\}$. On the other hand, we have

$$M^c_n \uparrow \left\{ x : \sup_{k \geq 1} S^c_{kj}(x) > 0 \right\} = \left\{ x : \sup_{k \geq 1} \frac{S^c_{kj}(x)}{k} > 0 \right\} = \left\{ x : \sup_n c_n(x) > \varepsilon \right\} \cap A_\varepsilon = A_\varepsilon,$$

as $n \to \infty$ by Eq. (10). From the dominated convergence theorem and Theorem 2, we derive

$$0 \leq \int_{M^c_n} C^c_n(1; x) d\mathbb{P} \to \int_{A_\varepsilon} [C(1; x) - \varepsilon] d\mathbb{P},$$

as $n \to \infty$.

Let $\mathcal{I}_n$ be the class of sets that are invariant under all permutations of the first $n$ coordinates and $\mathcal{I} = \bigcap_{n=1}^{\infty} \mathcal{I}_n$. It is easy to check that $A_\varepsilon \in \mathcal{I}$. Since $\mathbb{P}(A)$ equals to zero or one for any $A \in \mathcal{I}$ by the Hewitt-Savage zero-one law (see Theorem 36.5 of [3]), the conditional expectation $\mathbb{E}[C(1)|\mathcal{I}]$ equals to $\mathbb{E}[C(1)]$, almost surely. This leads to

$$0 \leq \int_{A_\varepsilon} [C(1; x) - \varepsilon] d\mathbb{P} = \int_{A_\varepsilon} C(1; x) d\mathbb{P} - \varepsilon \mathbb{P}(A_\varepsilon) = \int_{A_\varepsilon} \mathbb{E}[C(1; x)|\mathcal{I}] d\mathbb{P} - \varepsilon \mathbb{P}(A_\varepsilon),$$

by Eq. (11) and $\mathbb{E}[C(1)] = 0$. Then we have $\mathbb{P}(A_\varepsilon) = 0$ for any $\varepsilon > 0$ and therefore $\lim \sup_{n \to \infty} C_n \leq 0$, almost surely. Repeating the same argument for $-C_n$, we have $\lim \inf_{n \to \infty} C_n \geq 0$, almost surely. This completes the proof. \qed

Here we show a simple example for Theorem 3. Consider an i.i.d. sequence $X_1, \ldots, X_n$ such that $\mathbb{P}(X_i = 1) = p$ and $\mathbb{P}(X_i = 0) = 1 - p$ for all $i = 1, \ldots, n$. Let $B_i = (\theta, \infty)$ and $f_c(x, y) = x + y$. We set a threshold $\theta$ such that $0 < \theta < 1$. In this case, a pair of vertices $i$ and $j$ with $i \neq j$ is disconnected if and only if $X_i = X_j = 0$. By direct computation, we have

$$\mathbb{E}[C(1)] = p \cdot C(1; 1) + (1 - p) \cdot C(1; 0) = p \cdot \frac{p^2 + 2p(1 - p)}{1} + (1 - p) \cdot \frac{p^2}{1} = 1 - p(1 - p)^2.$$

In order to calculate $C_n$, let $S_n = \sum_{i=1}^{n} X_i$, that is, the number of vertices with $X_i = 1$. We use the symbols $x_i$, $s_n$, and $c_n$ as realization of random variables $X_i$, $S_n$ and $C_n$ respectively. If $s_n = 0$, the graph consists of $n$ isolated vertices. In this case $c_n = w$. If $s_n = 1$, the graph is the star in which only one central vertex has $n - 1$ edges and other $n - 1$ vertices are connected only to the center. So we obtain

$$c_n = \frac{1}{n} \{0 \cdot 1 + w \cdot (n - 1)\} = \left(1 - \frac{1}{n}\right) \cdot w.$$

If $2 \leq s_n \leq n - 2$, $s_n$ vertices with $x_i = 1$ have $n - 1$ edges, and the other $n - s_n$ vertices are connected only to the vertices with $x_i = 1$. So we have

$$c_n = \frac{1}{n} \left\{ \frac{(n-1)}{2} - \frac{(n-s_n)}{2} \cdot s_n + \frac{s_n}{2} \cdot (n - s_n) \right\} = 1 - \frac{(n - s_n)(n - 1 - s_n)s_n}{n(n - 1)(n - 2)}.$$

If $s_n = n - 1$ or $n$, we obtain the complete graph, and $c_n = 1$. Noting

$$1 - \frac{(n - s_n)(n - 1 - s_n)s_n}{n(n - 1)(n - 2)} = \begin{cases} 1 & \text{if } s_n = 0, n - 1, n, \\ 1 - \frac{1}{n} & \text{if } s_n = 1, \\ 1 - \frac{(n-s_n)(n-1-s_n)s_n}{n(n-1)(n-2)} & \text{otherwise}, \end{cases}$$

we have

$$C_n = \left[ 1 - \frac{(n - S_n)(n - 1 - S_n)S_n}{n(n - 1)(n - 2)} \right] \cdot \left\{ I_{\{2, \ldots, n\}}(S_n) + r \cdot I_{\{0, 1\}}(S_n) \right\}$$

$$= \left[ 1 - \frac{S_n}{n} \left( 1 - \frac{S_n}{n - 1} \right) \left( \frac{S_n}{n - 2} \right) \right] \cdot \left\{ 1 + (r - 1) \cdot I_{\{0, 1\}}(S_n) \right\}$$

$$\to 1 - (1 - p)^2 p = \mathbb{E}[C(1)], \quad \text{almost surely (}n \to \infty).$$
The last convergence comes from SLLN for the i.i.d. sequence.

One of our motivations to study limit theorems for the clustering coefficients is to make a clear distinction between the proportion of triangles in an entire graph and the clustering coefficients. By Eq. (66) of [9], the normalized number of triangles including vertex 1 converges to $E_T(x)$, almost surely for each realization $x$ of $X$, where the normalization constant is equal to $(n!)^{-1}$. In the same way, the degree of vertex 1 normalized by $n - 1$ converges to $E_D(x)$, almost surely. Thus, when $E_D(x) > 0$, the local clustering coefficient converges almost surely to $E_T(x)/E_D(x)^2$, that is, the limit of the normalized number of triangles divided by the square of the limit of the normalized degree. The mean field result corresponding to Theorem 2 is found in Eq. (3) of [18]. The denominator equals to $E_T(x)/E_D(x)^2$, almost surely as $N \to \infty$, where $k(x)$ is the degree of the vertex 1 and $N$ denotes the number of vertices. Equation (30) of [5] corresponds to the normalized number of triangles. These heuristic results are consistent with our rigorous result. Several examples for the global clustering coefficient are calculated in [10].

In practice, we may substitute 0 into $w$ and consider

$$
\tilde{C}_n = \frac{1}{n - \text{number of vertices with degree 0 or 1}} \sum_{i=1}^{n} C_n(i)
$$

instead of $C_n$. Using the same arguments of Theorems 2 and 3 it is easy to prove that

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} I_{\{j\}}(V_n(i)) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} I_{\{w\}}(C_n(i)) = \mathbb{P}(C(1) = w) = \mathbb{P}(E_D(X_1) = 0), \quad \text{almost surely.}
$$

The last equality follows from the definition of $C(1)$, i.e., Eq. (3). Noting that

$$
\tilde{C}_n = \frac{1}{1 - (1/n) \sum_{i=1}^{n} I_{\{j\}}(V_n(i))} \cdot C_n,
$$

we have the following:

**Corollary 2.** As $n \to \infty$,

$$
\tilde{C}_n \to \frac{1}{1 - \mathbb{P}(E_D(X_1) = 0)} \cdot \mathbb{E}[C(1)], \quad \text{almost surely.}
$$

## 5 Examples

In this section, we show examples of the limit degree distribution, i.e., $m = 2$ and $A_2$ is chosen as the collection of all possible edges in the limit theorem (Fact 4). We consider the case $l = 1$ with $f_c \equiv f^1_c$.

We assume that the random variable $X_1$ is absolutely continuous so that it has a probability density function $f$. Let supp $f = \{x \in \mathbb{R} : f(x) \neq 0\}$ be the support of $f$.

We first set $B_1 = (\theta, \infty)$ and $f_c(x, y) = x + y$, i.e., the threshold network model in which an edge $(i, j)$ forms if $\theta < X_i + X_j$ for a given threshold $\theta \in \mathbb{R}$ [5, 6, 9, 10]. By calculating the characteristic function of $D = U(B_1, A_2)$, namely, the density of edges connected to vertex 1, we obtain the following results:

**Lemma 2.** 1. If there exists $a \in \mathbb{R}$ such that supp $f = [a, \infty)$, then

$$
D \sim \begin{cases} 
\delta_1(dk) & \text{if } \theta \leq 2a, \\
I_{(1-F(\theta-a), 1)}(k) \cdot \frac{f(\theta-F^{-1}(1-k))}{f(F^{-1}(1-k))} \, dx & \text{if } \theta > 2a,
\end{cases}
$$
2. If there exists \( b \in \mathbb{R} \) such that \( \text{supp} \, f = (-\infty, b] \), then
\[
D \sim \begin{cases} 
(1 - F(\theta - b)) \cdot \delta_0(dk) & \text{if } \theta < 2b, \\
+I_{(0,1-F(\theta-b))}(k) \cdot \frac{f(\theta - F^{-1}(1-k))}{f(F^{-1}(1-k))} dk & \text{if } \theta \geq 2b.
\end{cases}
\]

3. If there exist \( a, b \in \mathbb{R} \) such that \( \text{supp} \, f = [a, b] \), then
\[
D \sim \begin{cases} 
\delta_1(dk) & \text{if } \theta \leq 2a, \\
I_{(1-F(\theta-a),1)}(k) \cdot \frac{f(\theta - F^{-1}(1-k))}{f(F^{-1}(1-k))} dk & \text{if } 2a < \theta < a + b, \\
(1 - F(\theta - b)) \cdot \delta_0(dk) & \text{if } \theta = a + b, \\
+I_{(0,1-F(\theta-b))}(k) \cdot \frac{f(\theta - F^{-1}(1-k))}{f(F^{-1}(1-k))} dk & \text{if } a + b < \theta < 2b, \\
\delta_0(dk) & \text{if } \theta \geq 2b.
\end{cases}
\]
Furthermore, if \( f \) is symmetric on \( \text{supp} \, f \), then
\[
D \sim I_{(0,1)}(k) dk \quad \text{if } \theta = a + b.
\]

4. If \( \text{supp} \, f = (-\infty, \infty) \), then
\[
D \sim I_{(0,1)}(k) \cdot \frac{f(\theta - F^{-1}(1-k))}{f(F^{-1}(1-k))} dk
\]
for any \( \theta \in \mathbb{R} \).

**Example 1.** (Exponential distribution) If the random variable \( X_1 \) has the probability density function
\[
f(x) = \begin{cases} 
\lambda e^{-\lambda x} & \text{if } x \geq 0, \\
0 & \text{otherwise},
\end{cases}
\]
for a given \( \lambda > 0 \), then
\[
D \sim \begin{cases} 
\delta_1(dk) & \text{if } \theta \leq 0, \\
I_{(e^{-\lambda \theta},1)}(k) \cdot \frac{e^{-\lambda \theta}}{\theta} dk + e^{-\lambda \theta} \cdot \delta_1(dk) & \text{if } \theta > 0.
\end{cases}
\]

**Example 2.** (Pareto distribution) If
\[
f(x) = \begin{cases} 
\frac{c}{x} (\frac{a}{x})^{c+1} & \text{if } x \geq a, \\
0 & \text{otherwise},
\end{cases}
\]
for given \( a, c > 0 \), then
\[
D \sim \begin{cases} 
\delta_1(dk) & \text{if } \theta \leq 2a, \\
I_{((\frac{a}{\theta})^{c+1})}(k) \cdot \left(\frac{a}{\theta} - \frac{c+1}{\theta} \right) \cdot \delta_0(dk) & \text{if } \theta > 2a.
\end{cases}
\]

The distribution of \( D \) of these two examples is proportional to \( k^{-\alpha} \). The exponent \( \alpha \) equals 2 in Example 1 and \( 1 + 1/c \) in Example 2. Because of a lower cutoff of \( f \) in both examples, the limit distributions have weights on \( \delta_1 \).
for which the limit distribution is represented by:

\[ X \in \{ 5, 6, 9, 10 \} \]

More precisely, an edge \( \langle \theta \rangle \) distributed according to the exponential distribution (Eq. (12)). This is the model proposed in [12]. Because the kernel function of this model is

\[ \text{then} \]

\[ D \sim \begin{cases} 
\delta_1(dk) & \text{if } \theta \leq 0, \\
I_{(1-\theta, 1)}(k)dk + (1-\theta) \cdot \delta_1(dk) & \text{if } 0 < \theta < 1, \\
I_{(0, 1)}(k)dk & \text{if } \theta = 1, \\
(\theta - 1) \cdot \delta_0(dk) + I_{(0, 2-\theta)}(k)dk & \text{if } 1 < \theta < 2, \\
\delta_0(dk) & \text{if } \theta \geq 2.
\end{cases} \]

In this case, the limit distribution is mixture of the uniform distribution and the delta measure.

Example 3. (Uniform distribution) If

\[ f(x) = \begin{cases} 
1 & \text{if } 0 \leq x \leq 1, \\
0 & \text{otherwise},
\end{cases} \]

then

\[ D \sim \begin{cases} 
\delta_0(dk) & \text{if } \theta_1 < \theta_2 \leq 0, \\
e^{-\lambda \theta_2} \cdot \delta_0(dk) & \text{if } 0 < \theta_1 < \theta_2, \\
+I_{(0,1-\theta)}(k) \cdot e^{-\lambda \theta_2} \frac{1}{(1-k)^2} dk & \text{if } \theta_1 = 0 < \theta_2, \\
e^{-\lambda \theta_2} \cdot \delta_0(dk) & \text{if } 0 < \theta_1 < \theta_2 \leq 0, \\
+I_{(0,1-\theta)}(k) \cdot e^{-\lambda \theta_2} \frac{1}{(1-k)^2} dk & \text{if } \theta_1 = 0 \leq \theta_2,
\end{cases} \]

Finally, we deal with an example with \( l = 2 \). For fixed \( \theta \in \mathbb{R} \) and \( c \in [0, \infty) \), we choose \( B_1 = (\theta, \infty), B_2 = (0, c) \), \( f_c^1(x, y) = x + y \), and \( f_c^2(x, y) = |x - y| \). We consider the case in which \( X_1 \) is distributed according to the exponential distribution (Eq. (12)). This is the model proposed in [12]. Because the kernel function of this model is

\[ h(x, x_2) = \begin{cases} 
I_{[-c+x, c+x]}(x_2) & \text{if } \frac{-c+c}{2} \leq x \leq \frac{\theta+c+c}{2}, \\
I_{[\theta-x, c+x]}(x_2) & \text{if } \frac{-c+c}{2} \leq x \leq \frac{\theta+c+c}{2}, \\
0 & \text{if } x \leq \frac{-c+c}{2},
\end{cases} \]

the limit distribution \( D = U(C_{\theta,c}, A_2) \) is the following:

\[ D \sim \begin{cases} 
(1 - e^{-\lambda (\theta-c)/2}) \delta_0(dk) & \text{if } c \leq \theta, \\
I_{(0,2c-\lambda \theta_2-\lambda \theta_2)}(k) \cdot g(k) \frac{1}{2 \sinh(\lambda c)} dk & \text{if } c \leq \theta, \\
I_{(0, \lambda \theta_2-\lambda \theta_2)}(k) \cdot \frac{2e^{-\lambda \theta_2}}{2 \sinh(\lambda c)} dk & \text{if } 0 \leq \theta \leq c, \\
I_{(0,1-\lambda \theta_2)}(k) \cdot \frac{1}{2 \sinh(\lambda c)} dk & \text{if } 0 \leq \theta \leq c, \\
I_{(0, \lambda \theta_2-\lambda \theta_2)}(k) \cdot \frac{2e^{-\lambda \theta_2}}{2 \sinh(\lambda c)} dk & \text{if } -c \leq \theta \leq 0, \\
I_{(0,1-\lambda \theta_2)}(k) \cdot \frac{1}{2 \sinh(\lambda c)} dk & \text{if } \theta \geq 0,
\end{cases} \]

where

\[ g(k) = \frac{4e^{-\lambda \theta}}{(k + \sqrt{k^2 + 4e^{-\lambda (\theta+c)}})^2 + 4e^{-\lambda (\theta+c)}} + \frac{1}{2 \sinh(\lambda c)} \].

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