Black Hole Superradiance in $f(R)$ Gravities

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Abstract

We study the superradiance amplification factor (SAF) for a charged massive scalar wave scattering off small and slowly rotating Kerr-Newman black holes in $f(R)$ gravity immersed in asymptotically flat and de-Sitter spacetimes. The $f(R)$-Kerr-Newman family solution induces an extra distinguishable effect on the contribution of the black hole’s electric charge to the metric and that in turn affects the SAFs and their frequency ranges. While our analysis are general, but we present the numerical results for the Starobinsky and Hu-Sawicki $f(R)$ models of gravity as our working examples. In the case of asymptotically flat spacetime, the SAFs predicted in Starobinsky $f(R)$ model are not distinguishable from those of GR while for Hu-Sawicki model the SAFs can be weaker or stronger than those of GR within the frequency parameters space. In the case of asymptotically de-Sitter spacetime, superradiance scattering does not occur in Starobinsky model while in Hu-Sawicki model the results of SAFs and their frequency regimes are different from standard ones which can have important astronomical implications.
1 Introduction

In systems with the capability to dissipate energy there is the possibility of superradiance in which radiation is enhanced. This phenomenon occurs in various branches of physics such as in quantum mechanics [1] and relativity [2], see [3] for a review. One useful method to study the superradiance is to look for the scattering of scalar fields by certain systems in which the scattered field obtains a larger energy compared to the incident field. Black holes are the favorite candidates for the superradiance to occur since the event horizon (EH) provides a dissipative mechanism [4]. Given the fact that black holes appear in different vacuum solutions of the Einstein field equations, the EH provides a natural dissipation mechanism for superradiance. In other words, in a composed system of black hole and the external field, superradiance is equivalent to the energy extraction of vacuum by the superradiant scattering. This phenomenon is more interesting in the light of the detection of the gravitational wave signal such as in “GW150914” (originated by the binary black hole merger) or from the “Event Horizon Telescope” (EHT) [8] which demonstrates the reality of black holes in nature.

Historically, the study of black hole superradiance stem from the seminal works of Zeldovich [2] and Misner [9] who predicted the possibility of amplification of some waves by Kerr black holes. Teukolsky [10] has presented the master equation for the Kerr geometry from the linearized bosonic perturbations (scalar, electromagnetic and gravitational) which turns out to be separable. Indeed, by using this master equation for each of scalar, electromagnetic and gravitational waves scattering off a Kerr black hole, Teukolsky and Press were able to show that there are some superradiant modes [11]. While there is no mathematical proof for the absence of superradiance for the fermionic fields, but Unruh [12] and Chandrasekhar [13] demonstrated this conclusion for the massless and massive Dirac fields scattered by a Kerr black hole respectively. Bekenstein [14], by discovering the relation between the superradiance and Hawking’s area theorem, was able to show that this phenomena can be understood through the classical laws of black hole mechanics.

Superradiance phenomena is not restricted to black holes arising from general relativity (GR) but it would happen in any extended theory of gravity that admits black hole solutions. To have an analytic study of superradiance amplification for Kerr-like black holes in extended theories of gravity we have to work in slow-rotation limit [15] while going beyond this limit requires numerical analysis [16, 17, 18]. One important motivation for studying superradiance in black hole solutions of modified gravity is that the geometric structure of black holes indeed contains some information about the modified theory of gravity in the strong field limit. For instance, it is shown in [15] that for the slowly-rotating black hole solutions predicted by quadratic gravity the amplification factor of superradiance decreases. This means that the superradiance phenomena is sensitive to the geometric structure of black holes so that one may use this phenomenon as a

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1 Recall that in context of curved spacetime physics there is also another energy extraction phenomenon known as the “Penrose process” [5] which commonly thought to be as particle analog of superradiance. Even though, the nature of these two energy extraction processes is generally distinct [4], however under some circumstances one can find some interesting connections between them, see [6].
tool to shed light on gravity in the strong field limit. The analysis of [19, 20] for the Kerr black holes in scalar-tensor theories show that the underlying phenomena is sensitive to the presence of matter too. An interesting issue following the superradiance phenomena in the context of alternative theories of gravity is the stability analysis of the superradiant modes [21]-[26]. Of course, in the context of standard GR numerous studies have been performed with a variety of assumptions on the background geometry as well as the field perturbations (see e.g. [27]-[43] and references therein). Note that the Kerr superradiant instability arising from the hypothetical ultralight bosons such as axions, as one of the candidates for dark matter, have interesting theoretical as well as observational implications. For the case of real bosonic fields, the cloud disperses for a long time so, depending on the boson masses, it is expected to generate the gravitational wave signals in specific range of frequencies [44]. However, for the case of complex bosonic fields, the gravitational wave emission is suppressed so at the final state of instability a composite system of Kerr black hole plus an external bosonic structure remains [45]. This phenomenon can be used to test some fundamental paradigms in theoretical physics such as the no-hair conjuncture [46]. Finally, due to the spin down instability i.e. the transfer of energy and angular momentum from Kerr black hole to the bosonic cloud, it is possible to impose some constraints on the boson fields [47].

With these discussions in mind in this work we would like to address the natural question that what is the effect of curvature corrections on the superradiance phenomenon? We would like to address the consequence of curvature corrections on the scalar wave amplification and whether or not the deviations from standard GR affect the black hole superradiance. For this purpose, we focus on a natural extension of GR, the so called $f(R)$ gravity. From theoretical standpoint, one advantage of $f(R)$ gravity compared to other theories of modified gravity is the absence of ghost instabilities [48, 49]. The cosmological and astrophysical implications of $f(R)$ gravity scenarios have been studied extensively, see for example [50, 48, 49, 51]. Due to importance and far reaching implications of $f(R)$ theories in cosmology and astrophysics, it is well motivated to study the black hole superradiance in $f(R)$ gravity. For this purpose, we study the superradiance for the most general black hole solution including the black hole rotation and charge (Kerr-Newman type solution). This type of black hole solution allows us to study the non-linear interplay between gravity and electromagnetism. Contrary to the usual belief that the real black holes in sky are mostly electrically neutral, however, some processes in both classical and relativistic frameworks indicate a small non-zero charge for the black holes [52]. Theoretically, there are some mechanisms such as the imbalance between the mass of protons and electrons within the ionized plasma around the black hole and or the twisting of magnetic field lines due to rotation which allow the black hole to be charged, see [53].

While our analysis are for general $f(R)$ theories, but we present the numerical results for the two most interesting examples of $f(R)$ gravity: the Starobinsky model [54] and the Hu-Sawicki model [55]. The Starobinsky model is the first inflationary model which is well consistent with the cosmic microwave background data such as the Planck observations [56, 57]. The Hu-Sawicki model, on the other hand, may be viewed as a counterpart to $ΛCDM$ while at the same time satisfying the standard tests of the solar system via a mechanism known as the “chameleon
screening”. In order to provide a realistic study, our discussion will cover both asymptotically flat and de-Sitter Kerr-Newman black hole spacetimes.

The rest of the paper is organized as follows. After an overview of Kerr-Newman black hole solutions in $f(R)$ theory in Sec. 2, we determine the relevant superradiance conditions for asymptotically flat and de-Sitter Kerr-Newman black hole spacetimes in Sec. 3. In Sec. 4 the analytic expressions for the superradiance amplification factor (SAF) of $f(R)$ Kerr-Newman black hole and charged massive scalar field are presented with Starobinsky and Hu-Sawicki models as our case studies. The summary and discussions are presented in Sec. 5. We work in natural units $c = \hbar = k_B = G_N = 1$.

## 2 Kerr-Newman Black Hole in $f(R)$ Gravity

We study the charged black hole solutions in $f(R)$ modified gravity which are either asymptotically flat or are in dS space, so we assume the spacetime has a constant curvature $R = R_0$. The action of the system is

$$S = \int d^4 \sqrt{|g|} \left( f(R) + 2\kappa^2 \mathcal{L}_{\text{em}} \right), \quad \kappa^2 = 8\pi,$$

in which $R$ is the Ricci scalar and $g$ denotes the determinant of the metric $g_{\mu\nu}$. There is an electric field which has filled the spacetime with the Lagrangian density $\mathcal{L}_{\text{em}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ in which $A_\mu$ is the vector potential and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor obeying the Maxwell equation $\partial_\mu (\sqrt{-g} F^{\mu\nu}) = 0$.

Varying action (1) with respect to the inverse metric, we obtain the modified Einstein equations

$$R_{\mu\nu} f'(R_0) - \frac{1}{2} g_{\mu\nu} f(R_0) = 8\pi T_{\mu\nu},$$

where $T_{\mu\nu} = F_{\mu\alpha} F^\alpha_{\nu} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$ is the stress-energy tensor of the electromagnetic field. Taking the trace of Eq. (2) in the absence of matter sources, one obtains the constant curvature scalar

$$R_0 = \frac{2 f(R_0)}{f'(R_0)} \equiv 4\Lambda_f,$$

where $\Lambda_f$ is the cosmological constant associated with the curvature constant $R_0$ so the cases $R_0 = 0$, $R_0 > 0$ and $R_0 < 0$, corresponds to the flat, de-Sitter and anti de-Sitter spacetimes, respectively. Having defined $\Lambda_f$, Eq. (2) can now be rewritten as

$$R_{\mu\nu} = \Lambda_f g_{\mu\nu} + \frac{8\pi}{f'(R_0)} T_{\mu\nu}.$$

Adopting the standard Boyer-Lindquist coordinate $(t, r, \vartheta, \phi)$, the four-dimensional axisymmetric and stationary solution in $f(R)$ gravity with a constant curvature scalar $R_0$ is given by
\[ ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta \]
\[ = -\frac{\Delta_r}{\rho^2\chi^2}(dt - a\sin^2\vartheta d\phi)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\vartheta}d\vartheta^2 + \frac{\Delta_\vartheta \sin^2\vartheta}{\rho^2\chi^2}(a\ dt - (r^2 + a^2)d\phi)^2 , \]
\[ t \in (-\infty, \infty), \quad r \in (0, \infty), \quad \vartheta \in [0, \pi], \quad \phi \in [0, 2\pi) , \]

with
\[ \rho^2 \equiv r^2 + a^2\cos^2\vartheta , \]
\[ \Delta_r \equiv (r^2 + a^2) \left(1 - \frac{R_0}{12}r^2\right) - 2Mr + \frac{q^2}{f'(R_0)} , \]
\[ \Delta_\vartheta \equiv 1 + \frac{R_0}{12}a^2\cos^2\vartheta , \]
\[ \chi \equiv 1 + \frac{R_0}{12}a^2 . \]

With the above metric, the potential vector as well as the electromagnetic field tensor required in Eq. (2) take the following forms
\[ A_\alpha = \frac{qr}{\chi \rho^2}(-1, 0, 0, a\sin^2\vartheta) , \]

and
\[ F_{\alpha\beta} = \begin{pmatrix}
0 & \frac{q(r^2-a^2\cos^2\vartheta)}{\rho^2\chi} & \frac{qra^2\sin2\vartheta}{\rho^2\chi} & 0 \\
-\frac{q(r^2-a^2\cos^2\vartheta)}{\rho^2\chi} & 0 & 0 & \frac{aq\sin^2\vartheta(r^2-a^2\cos^2\vartheta)}{\rho^2\chi} \\
-\frac{qra^2\sin2\vartheta}{\rho^2\chi} & 0 & 0 & -\frac{qra^2(r^2+a^2)\sin2\vartheta}{\rho^2\chi} \\
0 & -\frac{aq\sin^2\vartheta(r^2-a^2\cos^2\vartheta)}{\rho^2\chi} & -\frac{qra^2(r^2+a^2)\sin2\vartheta}{\rho^2\chi} & 0
\end{pmatrix} . \]

A distant observer may interpret the above solution as a Kerr-Newman family of black hole with mass \(M\), the angular momentum per unite mass \(a \equiv J/M\) and the electric charge \(q\).

Compared to the case of GR, here the contribution of the black hole's electrical charge to the metric is modified by the factor \(f'(R_0)^{-1/2}\) as seen from the definition of \(\Delta_r\). For simplicity, from now on we use the notion \(Q \equiv q\sqrt{f'(R_0)}\). However, note that the electrical charge of the black hole as measured by the distant observer is \(q\) and not \(Q\), as is evident from the vector potential and the field strength in Eqs. (8) and (9), respectively.

Defining the horizon via \(g_{rr} = \Delta_r = 0\), we obtain the following quadratic equation
\[ R_0r^4 + (R_0a^2 - 12)r^2 + 24Mr - 12(a^2 + Q) = 0 , \]

which yields four roots, \(r_1, ..., r_4\).
For the flat spacetime \((R_0 = 0)\) the above equation has two positive real roots: 
\[ r_{c,h} = M \mp \sqrt{M^2 - (a^2 + Q^2)}, \]
representing the positions of the Cauchy and the event horizons respectively. However, if \(R_0 > 0\), we have three positive roots \(r_{c,h}\) and \(r_H\) in which \(r_H\) represents the cosmological horizon. For the case of \(R_0 < 0\) there are just two positive roots (as in the case of flat spacetime).

For convenience, we define the following parameters,
\[
\Omega_h \equiv \frac{a}{r_h^2 + a^2}, \quad \Omega_H \equiv \frac{a}{r_H^2 + a^2}, \quad \Phi_h \equiv \frac{Qr_h}{(r_h^2 + a^2)}, \quad (11)
\]
which respectively represent the angular velocity \(\Omega\) on the surfaces of event horizon and cosmological horizon and the electric potential \(\Phi\) on the surface of event horizon.

3 Condition for Superradiance Modes

To study superradiance, we consider a complex scalar field \(\Psi\) with mass \(\mu_s\) which is charged under the \(U(1)\) gauge field with the electric charge coupling \(e\). The corresponding Klein-Gordon equation is
\[
(\Box - \mu_s^2)\Psi = \frac{1}{\sqrt{|g|}} D_\alpha (\sqrt{|g|} g^{\alpha\beta} D_\beta \Psi) - \mu_s^2 \Psi = 0, \quad (12)
\]
where the covariant derivative is given by \(D_\alpha = \partial_\alpha - ieA_\alpha\).

To solve the Klein-Gordon equation, we introduce the following ansatz
\[
\Psi = e^{-i\omega t + im\phi} R(r) S(\vartheta), \quad (13)
\]
with the positive oscillation frequency \(\omega > 0\) and the azimuth angular number \(m\). Inserting the above ansatz into (12) and focusing on the slowly rotating black holes, \(a\omega \ll 1\), we obtain the following separated differential equations for \(R(r)\) and \(S(\vartheta)\)
\[
\Delta_r \frac{d}{dr} \left( \Delta_r \frac{d}{dr} R(r) \right) + \left[ \chi^2 (\omega^2 + a^2 - ma - eQr)^2 - \Delta_r (\lambda + \mu_s^2 r^2) \right] R(r) = 0, \quad (14)
\]
and
\[
\sin \vartheta \frac{d}{d\vartheta} \left( \sin \vartheta \frac{d}{d\vartheta} S(\vartheta) \right) + \left( \lambda \sin^2 \vartheta - \frac{\mu_s^2 a^2}{4} \sin^2 2\vartheta - m^2 \right) S(\vartheta) = 0. \quad (15)
\]
Here \(\lambda \equiv l(l+1)\) denotes the angular separation constant with non-negative angular momentum index \(l \geq 0\). From now on, we define \(\varepsilon_f \equiv \frac{\varepsilon}{\sqrt{r'(R_0)}}\) with \(\varepsilon \equiv eQ\) representing the joint coupling of the scalar field and the black hole electric charges. Increasing the value of \(\varepsilon\), this coupling becomes large and we enter the strong coupling limit when \(\varepsilon > 1\).\(^2\)

\(^2\)This regime seems to be relevant for the expected black holes in our universe with even small charges [61].
Defining the new field variable \( u(r) \equiv \sqrt{r^2 + a^2} \, R(r) \) and going to the tortoise coordinate defined via \( dr_* = \frac{r^2 + a^2}{\Delta r} \, dr \), after some algebra Eq. (14) takes the following Schrödinger-like form

\[
\frac{d^2 u(r_*)}{dr_*^2} + V_{\text{eff}}(r) u(r_*) = 0 ,
\]

with the effective potential given by

\[
V_{\text{eff}} = \chi^2 \left[ \omega - \frac{ma}{r^2 + a^2} - \frac{\varepsilon f r}{(r^2 + a^2)} \right]^2 - \frac{\Delta r}{(r^2 + a^2)^2} \left[ \lambda + \mu_s^2 r^2 + \sqrt{r^2 + a^2} \, \frac{d}{dr} \left( \frac{r \Delta r}{(r^2 + a^2)^{3/2}} \right) \right].
\]

Now, by equating the Wronskian for regions near the event horizon \( W_h = u_h \frac{d u_h^*}{d r_*} - u_h^* \frac{d u_h}{d r_*} \) with its other counterparts at infinity and on cosmological horizon \( W_{\infty(H)} = u_{\infty(H)} \frac{d u_{\infty(H)}^*}{d r_*} - u_{\infty(H)}^* \frac{d u_{\infty(H)}}{d r_*} \), we arrive at the following conditions

\[
\mathcal{A}_T^2 - \mathcal{A}_R^2 = \frac{\omega - (m + \frac{\varepsilon f r_h}{a}) \Omega_h}{\sqrt{\omega^2 - \mu_s^2}} \mathcal{A}_T^2 , \quad \text{(flat background)}
\]

\[
\text{and}
\]

\[
\mathcal{A}_T^2 - \mathcal{A}_R^2 = \frac{\omega - (m + \frac{\varepsilon f r_h}{a}) \Omega_h}{\omega - (m + \frac{\varepsilon f r_H}{a}) \Omega_H} \mathcal{A}_T^2 , \quad \text{(dS background)}
\]

for the flat and de-Sitter spacetimes, respectively.
In order for the superradiance to take place the amplitude of the reflected wave must exceed the amplitude of the incident wave so the following frequency conditions must be met

\[
\mu_s < \omega < (m + \frac{\varepsilon f h}{a})\Omega_h, \quad \text{(flat background)} \tag{22}
\]

and

\[
(m + \frac{\varepsilon f H}{a})\Omega_H < \omega < (m + \frac{\varepsilon f h}{a})\Omega_h, \quad \text{(dS background)} \tag{23}
\]

for flat and de-Sitter backgrounds respectively.

At first glance, however, one might imagine that modifications in the frequency conditions (22) and (23) are just a renormalization of the black hole’s electric charge \(q\) or \(\varepsilon\). Although mathematically it seems to be true, but physically this is not the case. In fact, the physical electrical charge of the black hole as measured by a distant observer is still \(q\) (as we have already addressed through Eqs. (8) and (9)), meaning that the correction induced by \(f(R)\) modified gravity on the black hole’s charge are distinct from each other. So, in essence in Eqs. (22) and (23), we deal with a new distinguishable contribution which comes directly from gravitational corrections. The aforementioned equations indicates that the \(f(R)\) correction affects the superradiance conditions compared to GR (with \(f'(R_0) = 1\) and \(\varepsilon_f = \varepsilon\)). More specifically, in the presence of \(f(R)\) correction with \(f'(R_0) \neq 1\), the threshold superradiance frequency, \(\omega_t \equiv (m + \frac{\varepsilon f h}{a})\Omega_h\), is modified relative to its GR counterpart. Since the onset of superradiance instability in the composed system consisting of Kerr-Newman black hole and the massive scalar field is characterized by \(\omega = \omega_t\), the displacement in the threshold frequency can be of phenomenological importance. The threshold frequency, in essence, is a boundary with marginal stability which separates stable \((\omega > \omega_t)\) and unstable composed system under question. With these discussions in mind, in next section, we investigate the effects of curvature modifications on the range of superradiance frequency as well as the power of superradiance for both asymptotically flat and de-Sitter spacetimes.

Before proceeding, however, let us here mention an interesting point. If we take the limit \(R_0 \to 0\) in Eq. (23), its lower bound does not coincide with Eq. (22) since \(\Omega_H\) goes to zero as \(R_0 \to 0\). This mismatch was already seen in Kerr-de-Sitter black holes [62] where the authors have argued that, despite the oddity of this difference, there seems to be something else going on. Inspired from Ref. [62] one can conclude that when \(R_0 > 0\), superradiance always occurs if Eq. (23) is satisfied. However, for \(\omega < \mu_s\) the tunnelling probability (proportional to \(A_T^2\)) becomes much smaller than that for \(\mu_s < \omega < (m + \frac{\varepsilon f H}{a})\Omega_H\) so the superradiance amplification is extremely suppressed. So, as \(R_0 \to 0\), the superradiance amplification vanishes for waves in the range \((m + \frac{\varepsilon f H}{a})\Omega_H < \omega < \mu_s\), which is consistent with condition (22).

\section{Superradiance Amplification Factors}

Despite the fact that the Teukolsky’s equation (in particular the radial equation (14)) can not be solved analytically, but some approximate methods have been developed. In this section,
using the “analytical asymptotic matching” (AAM) method, proposed first by Starobinsky [63], we obtain the cross section of a scalar wave scattering off a $f(R)$-charged Kerr black hole. This enables us to detect the effect of $f(R)$ correction on the “amplification factor” $Z_{lm} \equiv \frac{A_{lm}^{g}}{A_{lm}^{t}} - 1$, a dimensionless quantity which its positive value indicates a superradiant amplification from the black hole. So far, we have imposed the slowly rotating approximation $a \omega \ll 1$. However, to employ the AAM method we have to impose the additional approximation that the Compton wavelength of the propagating particle is very large compared to the size of the black hole, i.e. $\mu_{s}r_{h} \ll 1$.

The main point in AAM method is that one can split the space outside the event horizon into two limits: region near the horizon ($r - r_{h} \ll \omega^{-1}$) known as the “near-region”, and region very far from the horizon ($r - r_{h} \gg M$) known as the “far-region”. The exact solutions derived for the above two asymptotic regions are matched in an overlapping region where $M \ll r - r_{h} \ll \omega^{-1}$. In the following, to provide an analytic expression for the cross section of a scalar wave scattering off a $f(R)$-charged Kerr black hole, we solve the radial equation (14) using the above approximations.

4.1 Asymptotically flat spacetime: $R_{0} = 0$

In this subsection we preset the superradiance analysis for a black hole located in an asymptotically flat spacetime, $R_{0} = 0$.

(i) Near-region solution: First we obtain the solution for the near-region.

Performing the change of variable $x = \frac{r - r_{h}}{r_{h} - r_{c}}$ and plugging $\Delta_{r} \frac{d}{dr} = (r_{h} - r_{c})x(x + 1)\frac{d}{dx}$ into Eq. (14), we obtain

$$x^{2}(x + 1)^{2}\frac{d^{2}R}{dx^{2}} + x(x + 1)(2x + 1)\frac{dR}{dx} + \left[\eta^{2} - L(l + 1)x(x + 1) - \mu_{s}^{2}((r_{h} - r_{c})x + r_{h})^{2}x(x + 1)\right]R = 0,$$

where we have defined $\eta \equiv \frac{r_{h}^{2} - a^{2}}{r_{h} - r_{c}}(\omega - (m + \frac{e_{r}r_{h}}{a})\Omega_{h})$.

For regions near the horizon, we can approximate $\mu_{s}^{2}((r_{h} - r_{c})x + r_{h})^{2} \approx \mu_{s}^{2}r_{h}^{2}$ and further by applying the Compton wavelength approximation, the above equation simplifies to

$$x^{2}(x + 1)^{2}\frac{d^{2}R}{dx^{2}} + x(x + 1)(2x + 1)\frac{dR}{dx} + (\eta^{2} - L(l + 1)x(x + 1))R = 0.$$

The most general solution of Eq. (25) is given in terms of ordinary hypergeometric functions $2F_{1}(a, b; c; z)$

$$R(x) = C_{1}x^{-in}(1 + x)^{-in} 2F_{1}(-l - 2in, l + 1; 1 - 2in; -x) + C_{2}x^{in}(1 + x)^{in} 2F_{1}(-l, l + 1; 1 + 2in; -x).$$

Imposing the ingoing boundary condition, and also using the following identities

$$2F_{1}(a, b; c; z) = z^{1-c}(1 - z)^{c-a-b} 2F_{1}(1 - a, 1 - b; 2 - c; z),$$
$$2F_{1}(a, b; c; z) = 2F_{1}(b, a; c; z),$$

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the solution (26) finally reads off

\[ R_{\text{near}} = C \left( \frac{x}{x+1} \right)^{i\eta} _2F_1(-l, l+1; 1 - 2i\eta; -x). \]  

(29)

To match the above solution to the solution from the far-region, we consider the behaviour of above solution at large \( x \), yielding

\[ R_{\text{near-large } x} \sim C \left( \frac{\Gamma(2l+1)\Gamma(1-2i\eta)}{\Gamma(l+1-2i\eta)\Gamma(l+1)} x^l + \frac{\Gamma(-2l-1)\Gamma(1-2i\eta)}{\Gamma(-l)\Gamma(l-2i\eta)} x^{-l-1} \right), \]  

(30)

where the approximation \((x+1)^{i\eta} \approx x^{i\eta}\) along with the following asymptotic behaviour of the hypergeometric function has been used

\[ \lim_{x \to \infty} _2F_1(a, b; c; -x) = \frac{\Gamma(b-a)\Gamma(c)}{\Gamma(c-a)\Gamma(b)} x^{-a} + \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(c-b)\Gamma(a)} x^{-b}. \]  

(31)

(ii) Far-region solution: Now we consider the solution for the far-region. In the asymptotic region, \( x \to \infty \) \((x \gg 1, r \gg r_h)\), Eq. (24) is simplified to

\[ \frac{d^2R}{dx^2} + \frac{2}{x} \frac{dR}{dx} + \left( \xi^2 - \frac{l(l+1)}{x^2} \right) R = 0, \]  

(32)

where \( \xi \equiv (r_h - r_c) \sqrt{\omega^2 - \mu_s^2} \).

The solution of the above equation is given in terms of the confluent hypergeometric function of the second kind \( U(a, b, z) \) and the generalized Laguerre polynomial \( L_n^{(a)}(z) \) as follows:

\[ R = \exp(-i\xi x) \left( A_1 \ x^l \ U(l+1, 2l+2, 2i\xi x) + A_2 \ x^l \ L_{-l-1}^{(2l+1)}(2i\xi x) \right). \]  

(33)

Using the following expression

\[ L_n^{(a)}(z) = \frac{(-1)^n}{n!} U(-n, a+1, z), \]  

(34)

and also the identity \( U(a, b, z) = z^{1-b} U(1+a-b, 2-b, z) \), the solution in Eq. (33) takes the following form

\[ R_{\text{far}} = \exp(-i\xi x) \left( A_1 \ x^l \ U(l+1, 2l+2, 2i\xi x) + A_2 \ x^{-l-1} \ U(-l, -2l, 2i\xi x) \right). \]  

(35)

To match the above far-region solution to the near-region solution, now we consider the small \( x \) limit of the above solution. Using the Taylor expansion \( \lim_{z \to 0} U(a, b, z) \approx \frac{\Gamma(1-b)}{\Gamma(1+a-b)} + \ldots \) the approximate form of the above solution at small \( x \) is given by

\[ R_{\text{far-small } x} \sim A_1 \ \frac{\Gamma(-2l-1)}{\Gamma(-l)} x^l + A_2 \ \frac{\Gamma(2l+1)}{\Gamma(l+1)} x^{-l-1}. \]  

(36)
In order to compute the scalar wave fluxes at infinity, we have to connect the coefficients $A_1$ and $A_2$ with coefficients $A_I$ and $A_R$ in the infinity limit of the radial solution (18). To do so, we first expand the far region solution (35) at infinity as

$$A_1 = C \frac{\Gamma(-l)\Gamma(2l + 1)\Gamma(1 - 2i\eta)}{\Gamma(l + 1 - 2i\eta)\Gamma(l + 1)\Gamma(-2l - 1)},$$

$$A_2 = C \frac{\Gamma(l + 1)\Gamma(-2l - 1)\Gamma(1 - 2i\eta)}{\Gamma(-l)\Gamma(2l + 1)\Gamma(-l - 2i\eta)}.$$  

Equating Eqs. (30) and (36) we obtain

$$(39)$$

In order to compute the scalar wave fluxes at infinity, we have to match the coefficients $A_1$ and $A_2$.

Now by applying the approximations $\frac{1}{x} \sim \frac{\xi}{\sqrt{\omega^2 - \mu_s^2 r}}$, $\exp(\pm i\xi x) \sim \exp(\pm i\sqrt{\omega^2 - \mu_s^2 r})$ and then matching the above solution with the radial solution

$$R_\infty(r) \sim A_I \frac{\exp(-i\sqrt{\omega^2 - \mu_s^2 r^*})}{r} + A_R \frac{\exp(i\sqrt{\omega^2 - \mu_s^2 r^*})}{r}, \quad (r \to \infty),$$

we obtain

$$(40)$$

Finally, by substituting the relevant expressions for $A_1$ and $A_2$, we obtain

$$(41)$$

and

$$(42)$$

$$(43)$$

$$A_I = \frac{C}{\sqrt{\omega^2 - \mu_s^2}} \left[ \frac{(2i)^{-l-1}\xi^{-l}\Gamma(-l)\Gamma(2l + 1)\Gamma(2l + 2)(\Gamma(1 - 2i\eta))^2}{\Gamma(-2l - 1)(\Gamma(l + 1 - 2i\eta)\Gamma(l + 1))^2} + \frac{(2i)^{-l-1}\xi^{l+1}\Gamma(-2l)\Gamma(l + 1)\Gamma(-2l - 1)(\Gamma(1 - 2i\eta))^2}{\Gamma(2l + 1)(\Gamma(-l)\Gamma(-l - 2i\eta))^2} \right].$$
Model I: \( f = R + \alpha R^2, \ \alpha > 0 \) \[54\] 
\( f''(0) = 2\alpha \)

Model II: \( f^{n=1} = R - \frac{\gamma^2 c_1 (R/\gamma^2)^m}{1+c_2 (R/\gamma^2)^m} \) \[55\] 
\( 1 - c_1 \frac{2c_1c_2}{\gamma^2} \)

Table 1: Two viable cosmological \( f(R) \) models in asymptotically flat spacetime.

As a result, the amplification factor can then be computed via

\[
Z_{lm} = \frac{|A_R|^2}{|A_I|^2} - 1.
\]

(45)

Figure 1: Percentage amplification factor \( Z_{lm} \) (Eq. (45) \( \times \) 100) in terms of the frequency \( \omega \) for a charged massive scalar waves with mass \( \mu_s = 0.1 \) and modes: \( l = 0 = m \) (top raw), \( l = 1 = m \) (bottom raw) scattering off a \( f(R) \)-Kerr-Newman black hole with electric charge \( q = 0.1M \). The electric coupling \( \varepsilon \) is 0.5 (left panel) and 1 (right panel).

4.1.1 Analysis with Viable \( f(R) \) Models

Now using the above expressions we are able to look for the effects of \( f(R) \) modified gravity on the superradiant amplification. To do so, we study two viable cosmological \( f(R) \) models, as listed in Table 1 as our case studies.
In order to prevent the ghost and tachyonic instabilities the following conditions are required to be satisfied for viable $f(R)$ theories [64]

$$f'(R_0) = \frac{df}{dR}\bigg|_{R_0} > 0, \quad f''(R_0) = \frac{d^2f}{dR^2}\bigg|_{R_0} \geq 0,$$

(46)

The $f(R)$ models listed in Table 1 satisfy the conditions of obtaining the Kerr-Newman black holes in asymptotically flat spacetime, i.e. $f(0) = 0$ and $f'(0) \neq 0$. Even though model I explicitly satisfies both of the above stability conditions, but the satisfaction of these conditions for model II depends on the parameters $c_{1,2}$. In this model, to satisfy both stability conditions (46), either of the following three combinations of parameters should be satisfied

$c_{1,2}$: $(0 < c_1 < 1, \ c_2 \geq 0); \ (c_1 < 0, \ c_2 \leq 0); \ (c_1 = 0 = c_2)$.

To have a view of the effects of $f(R)$ modification on the amplification factor, we have plotted the behaviour of $Z_{lm}$ given in Eq. (45) in Figs. 1 and 2. Generally speaking, we see the amplification factors of ground state ($l = 0 = m$) and the first excited state ($l = 1 = m$) are affected differently. Fig. 1 clearly shows that in some frequencies, $Z_{00}$ grows as $f'(0)$ changes from 0.5 to 2 while for $Z_{11}$ this trend is reversed. Furthermore, the ground state mode has a bigger superradiance parameter space as $f'(0)$ moves from 0.5 to 2 while it becomes smaller for the excited mode.

While rotation ($a \neq 0$) has a negligible effect on the ground state mode but for the mode $Z_{11}$ it has significant effects on the power as well as the frequency range of superradiance. As another interesting point, the behaviour of these modes are sensitive to the values of the electrical charge coupling $\varepsilon$. For the ground stat when approaching $\varepsilon = 1$ the amplitude and the superradiance frequency range become smaller as $f'(0)$ moves from 0.5 to 2, while this trend is reversed for the excited mode. In large coupling limit there appears some resonance peaks for the mode $Z_{11}$ in some given frequencies $\omega_{\text{res}}$ as depicted in Fig. 2. The resonance frequencies $\omega_{\text{res}}$ as well as the amplitude of these peaks grows as $f'(0)$ increases. However, we have found that the large coupling regime does not support any solution for the case of the ground state mode $Z_{00}$. Note that such large amplifications may have interesting astrophysical applications.
Table 2: Two viable cosmological $f(R)$ models in asymptotically de-Sitter spacetime.

| Model I: $f(R) = R + \alpha R^2$, $\alpha > 0$ | Ref | $f'(R_0 = \frac{12}{L^2})$ | $f''(R_0 = \frac{12}{L^2})$ |
|--------------------------------------------|-----|------------------|------------------|
| $f = R + \alpha R^2$, $\alpha > 0$    | [54] | $1 + \frac{24\alpha}{L^2}$ | $2\alpha$ |
| Model II: $f^{n=1} = R - \gamma^2 \frac{c_1 (R/\gamma^2)^n}{1+c_2 (R/\gamma^2)^n}$ | [55] | $\frac{(1-c_1)\gamma L^4+24c_2\gamma^2 L^2+144c_2^2}{(12c_2+\gamma^2 L^2)^2}$ | $\frac{2c_1 c_2 \gamma^4 L^6}{(12c_2+\gamma^2 L^2)^3}$ |

Table 2: Two viable cosmological $f(R)$ models in asymptotically de-Sitter spacetime.

The shape of the resonance peaks is similar to Breit-Wigner (BW) form [65] since their heights and widths are respectively finite and very narrow with no infinities and zeros, as in Dirac delta function. Such peaks have been interpreted as the very long lived quasi-normal modes (corresponding to quasi-bound states) with $\omega_I \ll \omega_R$ so that $\omega_R \sim \omega_{\text{res}}$, see the discussions in [19]. Historically, the existence of such a long-lived quasi-normal modes can be traced to the work of Detweiler [66]. However, for some more recent works on these weakly damped quasi-normal modes see [67, 68, 69]. As a consequence, the BW-shaped resonances in Fig. 2 address the existence of stable quasi-bound modes in asymptotically flat $f(R)$-Kerr-Newman black holes, specifically in large coupling regime 3.

It should be noted that model I and the case $c_1 = 0$ in model II are similar to the case of GR with $f'(0) = 1$. However, the cases $0 < c_1 < 1$, $c_2 \geq 0$ and $c_1 < 0$, $c_2 \leq 0$ in model II can have smaller or bigger values than $f'(0) = 1$. As a result, from the two models listed in Table 1, only in model II with the aforementioned cases for $c_{1,2}$, the superradiance amplification as well as its frequency ranges are distinguishable from those of GR. However, in the coming subsection we show that the model I becomes distinguishable from GR if we place the black hole in an asymptotically de-Sitter spacetime.

4.2 Asymptotically de-Sitter spacetime: $R_0 > 0$

Given the fact that the de-Sitter spacetime is bounded (observer can not see beyond the cosmological horizon $r_H$) so now the near and far regions approximations are altered to $r_H - r_h \ll \omega^{-1}$ and $r_H - r_h \gg M$ respectively. To maintain the validity of these approximations, we have to assume that the cosmological horizon is also much larger than the event horizon, $r_H \gg r_h$. This means that for the spacetime near the vicinity of the charged Kerr black the large limit of the near-region solution in Eq. (30) is applicable here. So, in the following, we focus on the solution in the far regions. Inspired by the far region approximation $r_H - r_h \gg M$, the black hole’s mass, electric charge and angular momentum parameters do not play important roles and can be ignored for a distant observer near to the cosmological horizon ($M \sim 0$, $Q \sim 0$, $a \sim 0$). Namely, $\Delta_r \approx r^2 (1 - \frac{r_h^2}{L^2})$ where $L$ is the radius of the de-Sitter spacetime defined as $L \equiv \sqrt{\frac{12}{R_0}}$.

Now, the radial wave equation (14) represents the propagation of a charged massive scalar

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3It should be mentioned that the boundary condition for the quasi-normal modes differs from superradiance and here we only provide an analogy for the resonance peaks compared to the quasi-normal modes.
Figure 3: Percentage amplification factor $Z_{lm}$ (Eq. (57) $\times 100$) vs. the frequency $\omega$ for $\mu_s = 0.1$ and modes: $l = 0 = m$ (top row) and $l = 1 = m$ (bottom row) scattering off a GR-Kerr-Newman-dS black hole (left panels) and $f(R)$-Kerr-Newman-dS black hole in model I (right panels) with the electrical charge $q = 0.1M$. For the de-Sitter length we set value $L = 500$.

Figure 4: Same as Fig. 3 but for $f(R)$ model II. The general behaviour of plots do not change for any value of $c_2 > 0$ and $\gamma > 0$ or $\gamma < 0$ (here we set $\gamma = 20$). The curves with $c_1 = 0 = c_2$ have the same predictions as in GR.
field (with frequency $\omega$ and angular momentum $l$) in a pure dS spacetime,

$$\Delta_r \frac{d^2 R}{dr^2} + \frac{d\Delta_r}{dr} \frac{dR}{dr} + \left[ \frac{\omega^2 r^4}{\Delta_r} - \left( l(l + 1) + \mu^2 r^2 \right) \right] R = 0. \quad (47)$$

In terms of the new coordinates $y \equiv 1 - \frac{r^2}{L^2}$ and the new field variable

$$R \equiv y^{i\omega/2} (1 - y)^{l/2} G(y), \quad (48)$$

the differential equation (47) is cast into

$$y(1 - y) \frac{d^2 G}{dy^2} + \left( (1 + iL\omega) - (l + \frac{5}{2} + iL\omega)y \right) \frac{dG}{dy} - \frac{1}{4} \left( (l + iL\omega)(l + 3 + iL\omega) + L^2 \mu^2 \right) G = 0. \quad (49)$$

Defining the following parameters

$$\beta_1 = 1 + iL\omega, \quad \beta_{2,3} = \frac{3 \pm \sqrt{9 - 4L^2 \mu^2} + 2(l + iL\omega)}{4}, \quad (50)$$

the radial equation takes the well known Gaussian hypergeometric differential equation,

$$y(y - 1) \frac{d^2 G}{dy^2} + \left( (\beta_2 + \beta_3 + 1)y - \beta_1 \right) \frac{dG}{dy} + \beta_2 \beta_3 G = 0. \quad (51)$$

For non-integer $\beta_1$, the general solution of the differential equation can be written as [70]

$$R = (1 - y)^{l/2} \left[ D_1 y^{i\omega/2} \, _2F_1(\beta_2, \beta_3; \beta_1; y) + D_2 y^{-i\omega/2} \, _2F_1(\beta_2 - \beta_1 + 1, \beta_3 - \beta_1 + 1, 2 - \beta_1; y) \right], \quad (52)$$

with constant coefficients $D_{1,2}$.

Considering the small $r$ limit, i.e. $y \to 1$, the solution is

$$R = (B_1 D_1 + C_1 D_2) r^l + (B_2 D_1 + C_2 D_2) r^{-l-1}, \quad (53)$$

with

$$B_1 = -\frac{\pi \Gamma(\beta_1)}{L^l \sin \left( \frac{(\beta_2 + \beta_3 - \beta_1)\pi}{\pi} \right) \Gamma(1 + \beta_2 + \beta_3 - \beta_1) \Gamma(\beta_1 - \beta_2) \Gamma(1 - \beta_3)},$$

$$C_1 = -\frac{\pi \Gamma(2 - \beta_1)}{L^l \sin \left( \frac{(\beta_2 + \beta_3 - \beta_1)\pi}{\pi} \right) \Gamma(1 + \beta_2 + \beta_3 - \beta_1) \Gamma(1 - \beta_2) \Gamma(1 - \beta_3)},$$

$$B_2 = \frac{\pi \Gamma(\beta_1) L^{l+1}}{\sin \left( \frac{(\beta_2 + \beta_3 - \beta_1)\pi}{\pi} \right) \Gamma(1 + \beta_2 - \beta_1) \Gamma(1 + \beta_3 - \beta_1) \Gamma(1 + \beta_1 - \beta_2 - \beta_3)},$$

$$C_2 = \frac{\pi \Gamma(2 - \beta_1) L^{l+1}}{\sin \left( \frac{(\beta_2 + \beta_3 - \beta_1)\pi}{\pi} \right) \Gamma(1 + \beta_2 - \beta_1) \Gamma(1 + \beta_3 - \beta_1) \Gamma(1 + \beta_1 - \beta_2 - \beta_3)}. \quad (54)$$
Now by matching the solution (53) with solution (30), we can solve for the coefficients $D_{1,2}$ as

$$D_1 = \frac{C_2\alpha_1 - C_1\alpha_2}{B_1C_2 - B_2C_1}, \quad D_2 = \frac{B_2\alpha_1 - B_1\alpha_2}{B_2C_1 - B_1C_2}, \quad (55)$$

where $\alpha_{1,2}$ are obtained from (30) as follows

$$\alpha_1 = C(r_h - r_c)^{-l}\frac{\Gamma(2l + 1)\Gamma(1 - 2i\eta)}{\Gamma(l + 1 - 2i\eta)\Gamma(l + 1)},$$

$$\alpha_2 = C(r_h - r_c)^{l+1}\frac{\Gamma(-2l - 1)\Gamma(1 - 2i\eta)}{\Gamma(-l)\Gamma(-l - 2i\eta)}. \quad (56)$$

As the last step, we have to expand the equation (52) around the cosmological horizon $r_H$, i.e. $y \to 0$ and subsequently compare it with the solution (19) on $r_H$. As a result, the amplification factor is given by

$$Z_{lm} = \frac{|D_1|^2}{|D_2|^2} - 1. \quad (57)$$

The amplification factors $Z_{00}$ and $Z_{11}$ for the $f(R)$ model I are displayed in Fig. 3. Note that the $f(R)$ model I (the Starobinsky model) is well consistent with cosmological observations for early universe cosmology and inflation with the parameter $\alpha \sim 10^{10}$ [71]. Interestingly, as displayed in Fig. 3, we find that for the mentioned value of $\alpha$ there is no superradiance since both $Z_{00}$ and $Z_{11}$ are negative in the limit of our interest where $M \omega \ll 1$. As a result, we conclude that for the Kerr-Newman black hole in a de-Sitter background the Starobinsky model with the required value of parameter $\alpha$ does not supports the superradiance phenomena.

Our analysis show that for the $f(R)$ model II superradiance can be either enhanced or reduced. Note that in the case of $c_1 = 0 = c_2$ the results of this model coincides with those of GR, independent of the value of $\gamma$. Clearly one can see from Table 2 that for values $0 < c_1 \leq 1$, $c_2 > 0$ with any arbitrary value for $\gamma$ the required conditions $f''(R_0) > 0$ and $f'''(R_0) > 0$ are always satisfied. As revealed in Fig. 4, the behaviour of SAFs for $Z_{00}$ and $Z_{11}$ are different from what is shown for the model I in Fig. 3. For the ground state mode $Z_{00}$ we see that the superradiance is weakened but the frequency parameter space becomes wider compared to the case of GR. Also, similar to GR, one can see some resonance peaks at the start and at the end of the superradiance parameter space. However, for the mode $Z_{11}$, the resonance peak appears only in the large coupling regime. The frequency of this resonance peak is slightly smaller than in the case of GR while its amplitude is equal to the case of GR. For the rest the amplitude and the parameter space of superradiance are respectively stronger and wider than in the case of GR.

### 5 Summary and Conclusion

The details of the black hole superradiance amplification depend both on the geometry of black hole and the wave dynamics in the modified theories of gravity. Therefore, it is an interesting question to study the superradiance phenomenon in modified theories of gravity. In this work,
we have studied this question for a charged massive scalar wave scattering off small and slow rotating $f(R)$-Kerr-Newman black holes in asymptotically flat and de-Sitter spacetimes respectively. While our analysis were general, but as case studies we have presented our results for two $f(R)$ models, the Starobinsky model [54] and the Hu-Sawicki model [55]. The main feature that distinguishes this black hole solution from its standard counterpart is that here the contribution of the black hole’s charge to the metric carries an additional effect given by the factor $\frac{1}{\sqrt{f'(R_0)}}$.

We have argued that this extra effect is not degenerate with the black hole’s electric charge so it leaves distinguishable imprints in $f(R)$ models of gravity.

We have found that the induced curvature correction affects the underlying phenomenon so black hole superradiance scattering may provide a platform to distinguish GR from $f(R)$ theories. Below we summarize our results for the cases of asymptotically flat and dS spacetimes separately.

- **Asymptotically flat spacetime:**

  In the case of asymptotically flat spacetime we have found that only in $f(R)$ model II the superradiance amplification is distinguishable from those of GR with $f'(0) = 1$. In the weak coupling limit $\varepsilon \leq 1$ our analysis explicitly show that for the ground state mode we have a bigger superradiance parameter space with stronger amplitude as $f'(0)$ moves from 1 towards larger values. However, for the excited modes the frequency range becomes smaller and the amplitude becomes weaker. In the large coupling regime $\varepsilon \gg 1$, we have observed some Breit-Wigner shaped resonances describing the quasi-bound states with their peak frequencies increasing as $f'(0)$ changes from 1 towards larger values. Generally, though the SAFs in the Starobinsky $f(R)$ model I are not distinguishable from those of GR, but depending on model parameters, SAFs in model II could be weaker or stronger than in GR.

- **Asymptotically dS spacetime:**

  In the case of asymptotically dS spacetime the predictions of superradiance scattering in both $f(R)$ models are different from the predictions of the asymptotically flat spacetime. We have shown that the Starobinsky model with the required values of the free parameter $\alpha$ to be consistent with the inflationary predictions does not admit superradiance. However, in the Hu-Sawicki model the amplitudes as well as the frequency ranges are different from those of GR. In this model, we also have seen some resonance peaks in SAFs corresponding to the quasi-bound states. These peaks appear at different frequencies than in GR which may have interesting astrophysical implications.

At the end, it is necessary to point out two issues. First, the difference in the range of superradiance frequency between $f(R)$ gravity and GR can have phenomenological importance. Indeed, there are some ranges of frequency in which GR does not support superradiance while superradiance occurs in $f(R)$ gravity. This can happen reversely as well, i.e. there are frequency ranges in which $f(R)$ does not support superradiance while it occurs in GR. As a result the shift
in superradiance regime relative to GR may be an important observational tool to distinguish between GR and \( f(R) \) theories. Second, we comment that since \( f(R) \) theories are equivalent to the generalized Brans-Dicke gravity the results obtained here can be viewed as spacial cases of the general scalar-tensor theories.

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