Ruin probabilities for two collaborating insurance companies

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Abstract

In this note we find a formula for the supremum distribution of spectrally positive or negative Lévy processes with a broken linear drift. This gives formulas for ruin probabilities in the case when two insurance companies (or two branches of the same company) divide between them both claims and premia in some specified proportions. As an example we consider gamma Lévy process, $\alpha$-stable Lévy process and Brownian motion. Moreover we obtain identities for Laplace transform of the distribution for the supremum of Lévy processes with randomly broken drift and on random intervals.

Keywords: Lévy process, distribution of supremum of a stochastic process, ruin probability, gamma Lévy process, $\alpha$-stable Lévy process
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1 Introduction

In this paper we study the supremum distribution of a spectrally positive or negative Lévy process with a piecewise linear drift. We find exact formulas for the distribution of supremum which are expressed by one-dimensional densities of a given Lévy process. The results can be applied to find ruin probabilities in the case when two insurance companies (or two branches of the same company) divide between them both claims and premia in some specified proportions. Moreover the formulas can be used for a two-node tandem queue (see Lieshout and Mandjes [9]). In Avram et al. [2] the related problem is investigated if the accumulated claim amount is modeled by a Lévy process that admits negative exponential moments. They find exact formulas for ruin probabilities expressed by ordinary ruin probabilities when the accumulated claim amount process is spectrally negative or a compound Poisson process with exponential claims. Additionally they find asymptotic behavior of ruin probabilities under the Cramér assumption. In Foss et al. [7] the same problem is investigated as in Avram et al. [2] but the subexponential claims are admitted and an asymptotic behavior of ruin probabilities on finite and infinite time horizon is found. In the models analyzed in this contribution we assume that the accumulated claim amount process is a spectrally positive or a spectrally negative Lévy process with one-dimensional density functions. We find exact formulas for ruin probabilities expressed by one-dimensional densities of an underlying Lévy process. The main difference of our models and models of Avram et al. [2] is that we admit heavy tailed claims and we provide explicit formulas of ruin probabilities both on finite and infinite time horizon unlike Avram et al. [2] where it is done only on infinite time horizon.

The layout of the rest of the article is the following. In this section we recall the formulas which will be used in the main results. The next section contains the main results that is the distribution of supremum of a Lévy process with a broken drift and examples. In the section we outline how to apply the main results to ruin probabilities for two collaborating insurance companies. The last section deals with the identities for Laplace transform of the distribution for supremum of Lévy processes with randomly broken drift and on random intervals.
In Michna et al. [12] a joint distribution of the random variable $Y(T)$ and $\inf_{t<T} Y(t)$ was found where $Y$ is a spectrally negative Lévy process (we will consider real stochastic processes with time defined on the non-negative half real line).

**Theorem 1.** If $Y$ is a spectrally negative Lévy process and the one-dimensional distributions of $Y$ are absolutely continuous then

$$
\mathbb{P}(\inf_{t<T} Y(t) < -u, Y(T) + u \in dz) = dz \int_0^T \frac{z}{T-s} p(z, T-s) p(-u, s) ds,
$$

where $T, u > 0$, $z \geq 0$ and $p(x,s)$ is a density function of $Y(s)$ for $s > 0$.

**Remark 1.** We do not expose a drift of the process $Y$ but it can be incorporated in the process $Y$.

If $X$ is a spectrally positive Lévy process then $X = -Y$ and we get the following corollary.

**Corollary 1.** If $X$ is a spectrally positive Lévy process and the one-dimensional distributions of $X$ are absolutely continuous then

$$
\mathbb{P}(\sup_{t<T} X(t) \leq u, X(T) \in dz) = dz \left[ f(z, T) - \int_0^T \frac{u-z}{T-s} f(z-u, T-s) f(u, s) ds \right],
$$

where $T, u > 0$, $z \in (-\infty, u]$ and $f(x,s)$ is a density function of $X(s)$ for $s > 0$.

Integrating the last formula with respect to $z$ we get the following theorem (see Michna et al. [12] and Michna [11]).

**Theorem 2.** If the one-dimensional distributions of $X$ are absolutely continuous then

$$
\mathbb{P}(\sup_{t<T} X(t) > u) = \mathbb{P}(X(T) > u) + \int_0^T \mathbb{E}(X(T-s))^+ \frac{f(u, s)}{T-s} ds,
$$

where $x^- = -\min\{x, 0\}$ and $f(u,s)$ is a density function of $X(s)$ for $s > 0$. 

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Let us now find the joint distribution of supremum and the value of the process for any spectrally negative Lévy process. It will easily follow from Corollary 1 and the duality lemma.

**Corollary 2.** If $Y$ is a spectrally negative Lévy process and the one-dimensional distributions of $Y$ are absolutely continuous then

$$
P(\sup_{t<T} Y(t) \leq u, Y(T) \in dz) = \left[ p(z,T) - u \int_0^T \frac{p(u, T - s)}{T - s} p(z - u, s) ds \right] dz,
$$

where $T, u > 0$, $z \in (-\infty, u]$ and $p(x, s)$ is a density function of $Y(s)$ for $s > 0$.

**Proof.** By the duality lemma (see e.g. Bertoin [3]) we have that $X((t - s) - T) - X(T) \overset{d}{=} Y(t)$ in the sense of finite dimensional distributions for $t \leq T$ ($X(t -)$ means the left-hand side limit at $t$). Thus we get

$$
P(\sup_{t<T} X(t) \leq u, X(T) \in dz) =
\begin{align*}
&\P(\sup_{t<T} X((T - t) -) \leq u, X(T) \in dz) \\
&= \P(\sup_{t<T} (X((T - t) -) - X(T)) \leq u - z, X(T) \in dz) \\
&= \P(\sup_{t<T} Y(t) \leq u - z, -Y(T) \in dz).
\end{align*}
$$

Substituting $u' = u - z$ and $z' = -z$ and using Corollary 1 we obtain the formula. 

The following theorem can be found in a more general form in Takács [17].

**Theorem 3.** If $Y$ is a spectrally negative Lévy process and the one-dimensional distributions of $Y$ are absolutely continuous then

$$
P(\sup_{t<T} Y(t) > u) = u \int_0^T \frac{p(u, s)}{s} ds,
$$

where $p(u, s)$ is the density function of $Y(s)$.

**Proof.** It follows directly from Kendall’s identity (see Kendall [8] or e.g. Sato [15], Th. 46.3). 

□
2 Main results and examples

In this section we analyze the distribution of supremum for both $X(t) - c(t)$ and $Y(t) - c(t)$ where $X$ is a spectrally positive Lévy process and $Y$ is a spectrally negative Lévy process and

$$c(t) = \begin{cases} c_1 t & \text{if } t \in [0, T] \\ c_2(t - T) + c_1 T & \text{if } t \in (T, \infty) \end{cases},$$

where $c_1, c_2 \geq 0$. Since we now expose the drift of the process we will assume that densities of $X(s)$ and $Y(s)$ are $f(x, s)$ and $p(x, s)$, respectively (unlike the previous section where a drift was incorporated in the processes).

**Theorem 4.** If $S > T$ ($S$ is finite or $S = \infty$) and $X(t)$ is absolutely continuous with density $f(x, t)$ then

$$\mathbb{P}(\sup_{t<S} (X(t) - c(t)) > u) = A + B :=$$

$$\mathbb{P}(\sup_{t<T} (X(t) - c_1 t) > u)$$

$$+ \mathbb{P}(\sup_{t<T} (X(t) - c_1 t) \leq u, \sup_{0<t<S-T} (X(t + T) - X(T) - c_2 t) > u - X(T) + c_1 T),$$

where

$$A = \mathbb{P}(X(T) - c_1 T > u) + \int_0^T \frac{\mathbb{E}(X(T - s) - c_1 (T - s))}{T - s} f(u + c_1 s, s) \, ds$$

and

$$B = \int_0^\infty \mathbb{P}(\sup_{t<S-T} X(t) - c_2 t > z) f(-z + u + c_1 T, T) \, dz$$

$$- \int_0^\infty z \mathbb{P}(\sup_{t<S-T} X(t) - c_2 t > z) \, dz$$

$$\cdot \int_0^T \frac{f(u + c_1 s, s)}{T - s} f(-z + c_1 (T - s), T - s) \, ds \cdot$$
Proof. The decomposition $A + B$ we get as follows

\[ P(\sup_{t<S} (X(t) - c(t)) > u) = A + B := \]

\[ P(\sup_{t<T} (X(t) - c_1 t) > u) \]

\[ + P((\sup_{t<T} (X(t) - c_1 t) \leq u, \sup_{T<t<S} (X(t) - c_2 (t - T) - c_1 T) > u)) \]

\[ = P(\sup_{t<T} (X(t) - c_1 t) > u) \]

\[ + P(\sup_{t<T} (X(t) - c_1 t) \leq u, \sup_{0<t<S-T} (X(t + T) - X(T) - c_2 t) > u - X(T) + c_1 T) . \]

The formula for $A$ we directly get from Theorem 2. Let $F(dx, dz)$ be the joint distribution of $(\sup_{t<T} (X(t) - c_1 t), X(T) - c_1 T)$. Then the formula for $B$ follows from the strong Markov property and Corollary 1 that is

\[ B = \int_0^u \int_{-\infty}^u P(\sup_{t<S-T} (X(t) - c_2 t) > u - z) F(dx, dz) \]

\[ = \int_{-\infty}^u P(\sup_{t<S-T} (X(t) - c_2 t) > u - z) f(z + c_1 T, T) dz \]

\[- \int_{-\infty}^u P(\sup_{t<S-T} (X(t) - c_2 t) > u - z) dz \]

\[ \cdot \int_0^T \frac{u - z}{T - s} f(z - u + c_1(T - s), T - s) f(u + c_1 s, s) ds \]

and substituting $z' = u - z$ we obtain the final formula. \[ \square \]

Similarly we get a formula for spectrally negative Lévy processes.

**Theorem 5.** If $S > T$ ( $S$ is finite or $S = \infty$ ) and $Y(t)$ is absolutely continuous with density $p(x, t)$ then $P(\sup_{t<S} (Y(t) - c(t)) > u) = A + B$ where

\[ A = P(\sup_{t<T} (X(t) - c_1 t) > u) = u \int_0^T \frac{p(u + c_1 s, s)}{s} ds \]
\[ B = \int_0^\infty \mathbb{P}(\sup_{t < S - T} (Y(t) - c_2 t) > z)p(-z + u + c_1 T, T)dz \\
- u \int_0^\infty \mathbb{P}(\sup_{t < S - T} (Y(t) - c_2 t) > z)dz \\
\cdot \int_0^T \frac{p(-z + c_1 s, s)}{T - s}p(u + c_1 (T - s), T - s)ds. \]

Proof. Using Corollary 2 and Th. 3 we proceed the same way as in the proof of Th. 4. \[\square\]

The application of Th. 4 leads to the following example with Brownian motion (see Mandjes [10] and Lieshout and Mandjes [9] or Avram et al. [2]).

**Example 1.** If \( W \) is the standard Brownian motion then

\[ \mathbb{P}(\sup_{t < \infty} (W(t) - c(t)) > u) = \]

\[ \Phi(-uT^{-1/2} - c_1 \sqrt{T}) + e^{-2c_1 u} \Phi(-uT^{-1/2} + c_1 \sqrt{T}) + e^{-2c_2(u + c_1 T - c_2 T)} \Phi(uT^{-1/2} + (c_1 - 2c_2)\sqrt{T}) - e^{2(c_2 - c_1)u + 2c_2 c_2 T - 2c_1 c_2 T} \Phi(-uT^{-1/2} + (c_1 - 2c_2)\sqrt{T}). \]

Indeed using Theorem 7 and

\[ \mathbb{P}(\sup_{t < T} (W(t) - ct) > u) = \Phi(-uT^{-1/2} - c \sqrt{T}) + e^{-2cu} \Phi(-uT^{-1/2} + c \sqrt{T}) \]

and

\[ \mathbb{P}(\sup_{t < \infty} (W(t) - ct) > u) = e^{-2cu} \]

for \( c \geq 0 \) (see e.g. Asmussen and Albrecher [1]) we get

\[ \mathbb{P}(\sup_{t < \infty} (W(t) - c(t)) > u) = A + B, \quad (3) \]

where

\[ A = A(c_1, T, u) := \Phi(-uT^{-1/2} - c_1 \sqrt{T}) + e^{-2uc_1} \Phi(-uT^{-1/2} + c_1 \sqrt{T}) \quad (4) \]
and

\[ B = e^{-2c_2(u+c_1T-c_2T)}\Phi(uT^{-1/2}+(c_1-2c_2)\sqrt{T}) \]
\[ - \frac{e^{-c_1u-c_2^2T/2}}{2\pi} \int_0^\infty z e^{(c_1-2c_2)z} \, dz \int_0^T (T-s)^{-3/2} s^{-1/2} e^{-\frac{z^2}{2(T-s)}} \frac{dz}{2s} \, ds. \]

Let us take \( c = c_1 = c_2 \geq 0 \) in eq. (3). Then \( A + B = e^{-2uc_2} \) and the second summand of \( A \) and the first one of \( B \) sum up to \( e^{-2uc_2} \) thus we get

\[ \frac{e^{-cu-c_2^2T/2}}{2\pi} \int_0^\infty z e^{-cz} \, dz \int_0^T (T-s)^{-3/2} s^{-1/2} e^{-\frac{z^2}{2(T-s)}} \frac{dz}{2s} \, ds = \Phi(-uT^{-1/2}-c\sqrt{T}). \]

Thus using the last identity for \( c = 2c_2 - c_1 \geq 0 \) we get the second term of \( B \).

Similarly let us take \( c = c_1 \) and \( c_2 = 0 \) in eq. (3). Then \( A + B = 1 \) and the first summand of \( A \) and the first one of \( B \) sum up to \( 1 \) thus we get

\[ \frac{e^{cu-c^2T/2}}{2\pi} \int_0^\infty z e^{-cz} \, dz \int_0^T (T-s)^{-3/2} s^{-1/2} e^{-\frac{z^2}{2(T-s)}} \frac{dz}{2s} \, ds = \Phi(-uT^{-1/2}+c\sqrt{T}). \]

Thus using the last identity for \( c = c_1 - 2c_2 > 0 \) we get the second term of \( B \).

Example 2. Let \( 0 < T < S < \infty \) and \( W \) be the standard Brownian motion then

\[
\mathbb{P}(\sup_{t<T}(W(t) - c(t)) > u) = \]
\[
A(c_1, T, u) + \frac{1}{\sqrt{2\pi T}} \int_0^\infty A(c_2, S - T, z) e^{-\frac{(u+c_1T-z)^2}{2T}} \, dz \]
\[
- \frac{e^{-uc_1-c_2^2T/2}}{2\pi} \int_0^\infty z e^{c_1z} A(c_2, S - T, z) \, dz \int_0^T s^{-1/2} (T-s)^{-3/2} e^{-\frac{z^2}{2(T-s)}} \frac{dz}{2s} \, ds,
\]

where \( A(c_1, T, u) \) is defined in eq. (7).

Example 3. Let \( X(t) \) be gamma Lévy process with the density

\[
f(x, t) = \frac{\delta^t}{\Gamma(t)} x^{t-1} e^{-\delta x} I_{\{x>0\}},
\]

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where $\delta > 0$ and $c(t)$ be defined in eq. (2). Using Th. 4 we give the explicit formulas of $\mathbb{P}(\sup_{t,S}(X(t) - c(t)) > u)$ for both $T < S < \infty$ and $S = \infty$, respectively. For $T < S < \infty$, we have that

$$A = \frac{\delta^T}{\Gamma(T)} \int_0^\infty x^{T-1} e^{-\delta x} \, dx$$
$$+ \delta^T e^{-\delta u} \int_0^T (u + c_1 s)^{s-1} e^{-c_1 \delta s} \int_0^{c_1(T-s)} (c_1(T-s) - x) x^{T-s-1} e^{-\delta x} \, dx$$
$$=: A(c_1, T, u)$$

and

$$B = \frac{\delta^S e^{-\delta(u+c_1 T)}}{\Gamma(T) \Gamma(S - T)} \int_0^{u+c_1 T} (u + c_1 T - z)^{T-1} e^{\delta z} \int_z^{c_2(S-T)} x^{S-T-1} e^{-\delta x} \, dx$$
$$+ \frac{\delta^S e^{-\delta(u+c_1 T)}}{\Gamma(T)} \int_0^{u+c_1 T} (u + c_1 T - z)^{T-1} \int_0^{S-T} (z + c_2(s-T))^{s-1} e^{-c_2 \delta s} ds$$
$$\cdot \int_0^{c_2(S-T-s)} (c_2(S-T-s) - x)x^{S-T-s-1} e^{-\delta x} \, dx$$
$$- \delta^T e^{-\delta(u+c_1 T)} \int_0^{c_1 T} z e^{\delta z} A(c_2, S - T, z) \, dz$$
$$\cdot \int_0^{c_1(T-z)} (u + c_1 s)^{s-1} (c_1(T-s) - z)^{T-s-1} \int_0^{\Gamma(s) \Gamma(T-s+1)} ds.$$

For $S = \infty$, we additionally assume that $c_2 \delta > 1$. In this case, since $X(t)$ has finite variation, in view of Th. 4 in [12] we have

$$\mathbb{P}(\sup_{t<\infty}(X(t) - c_2 t) > z) = \frac{c_2 \delta - 1}{\delta} e^{-\delta z} \int_0^\infty \delta^s (z + c_2 s)^{s-1} e^{-c_2 s} ds, \quad z > 0.$$

Let us notice that $A$ is the same as in the case $T < S < \infty$ and using the
above expression we get

\[ B = \frac{(c_2 \delta - 1) \delta T - 1 e^{-\delta(u + c_1 T)}}{\Gamma(T)} \int_{0}^{u + c_1 T} (u + c_1 T - s)^{-1} ds \\
\cdot \frac{\delta^s}{\Gamma(s)} (z + c_2 s)^{-1} e^{-\delta c_2 s} d\delta \\
- (c_2 \delta - 1) \delta T - 1 e^{-\delta(u + c_1 T)} \int_{c_1 T}^{c_1} z d\delta \\
\cdot \frac{T - c_1}{\Gamma(s)} (u + c_1 s)^{-1} (c_1 (T - s) - z)^{T - s - 1} ds \int_{0}^{T} \delta^t (z + c_2 t)^{-1} e^{-\delta c_2 t} dt. \]

Example 4. Let \( Z(s) \) be an \( \alpha \)-stable Lévy process totally skewed to the right (that is with \( \beta = 1 \) see e.g. Janicki and Weron [6] or Samorodnitsky and Taqqu [14]) with \( 1 < \alpha < 2 \) and expectation zero then its density function is the following

\[ f(x, s) = \frac{1}{\pi s^{1/\alpha}} \int_{0}^{\infty} e^{-t} \cos \left( ts^{-1/\alpha} x - t^{\alpha} \tan \frac{\pi \alpha}{2} \right) dt \]

(see e.g. Nolan [13]). Then (see Michna et al. [12]) for \( c > 0 \)

\[ A(c, \infty, u) := \mathbb{P}(\sup_{t < \infty} (Z(t) - ct) > u) = \]

\[ \frac{c}{\pi} \int_{0}^{\infty} s^{-1/\alpha} ds \int_{0}^{\infty} e^{-t} \cos \left( ts^{-1/\alpha} (u + cs) - t^{\alpha} \tan \frac{\pi \alpha}{2} \right) dt \]

and for any \( c \) and \( T > 0 \)

\[ A(c, T, u) := \mathbb{P}(\sup_{t < T} (Z(t) - ct) > u) = \]

\[ \frac{1}{\pi T^{1/\alpha}} \int_{0}^{T} dx \int_{0}^{\infty} e^{-t} \cos \left( t^{1/\alpha} (x + cT) - t^{\alpha} \tan \frac{\pi \alpha}{2} \right) dt \\
+ \frac{1}{\pi} \int_{0}^{T} \frac{\mathbb{E}(Z(T-s) - c(T-s))^{-}}{(T-s)^{1/\alpha}} ds \\
\cdot \int_{0}^{\infty} e^{-t} \cos \left( ts^{-1/\alpha} (u + cs) - t^{\alpha} \tan \frac{\pi \alpha}{2} \right) dt \]

where

\[ \mathbb{E}(Z(s) - cs)^{-} = -\frac{1}{\pi s^{1/\alpha}} \int_{-\infty}^{\infty} x dx \int_{0}^{\infty} e^{-t} \cos \left( ts^{-1/\alpha} (x + cs) - t^{\alpha} \tan \frac{\pi \alpha}{2} \right) dt. \]
Thus using Th. [4] for $S > T > 0$ (allowing also $S = \infty$ and putting $\infty - T = \infty$) we get

$$\mathbb{P}(\sup_{t < S}(Z(t) - c(t)) > u) = A + B = A + B_1 - B_2,$$

where $A = A(c_1, T, u)$ (see eq. (5)) and

$$B_1 = \frac{1}{\pi T^{1/\alpha}} \int_{0}^{\infty} A(c_2, S - T, z) \, dz \cdot \int_{0}^{\infty} e^{-t^\alpha} \cos \left( tT^{-1/\alpha}(-z + u + c_1 T) - t^\alpha \tan \frac{\pi \alpha}{2} \right) \, dt$$

and

$$B_2 = \frac{1}{\pi^2} \int_{0}^{\infty} z A(c_2, S - T, z) \, dz \int_{0}^{\infty} \frac{ds}{(T - s)^{1/\alpha} + 1} \int_{0}^{T} e^{-t^\alpha} \cos \left( ts^{-1/\alpha}(u + c_1 s) - t^\alpha \tan \frac{\pi \alpha}{2} \right) \, dt \cdot \int_{0}^{\infty} e^{-w^\alpha} \cos \left( w(T - s)^{-1/\alpha}(-z + c_1 (T - s)) - w^\alpha \tan \frac{\pi \alpha}{2} \right) \, dw.$$

3 Two collaborating insurance companies

Let us consider two insurance companies which split the amount they pay out of each claim in proportions $\delta_1 > 0$ and $\delta_2 > 0$ where $\delta_1 + \delta_2 = 1$, and receive premiums at rates $p_1 > 0$ and $p_2 > 0$, respectively (see Avram et al. [2]). Then the corresponding risk processes are

$$R_i(t) = x_i + p_i t - \delta_i X(t),$$

where $i = 1, 2$, $x_i > 0$ and $X(t)$ is an accumulated claim amount up to time $t$. One can be interested in the instant when at least one insurance company is ruined

$$\tau_{or}(x_1, x_2) = \inf\{t > 0 : R_1(t) < 0 \text{ or } R_2(t) < 0\}$$
and in the instant when both insurance companies are simultaneously ruined
\[ \tau_{\text{sim}}(x_1, x_2) = \inf \{ t > 0 : R_1(t) < 0 \text{ and } R_2(t) < 0 \} . \]

Let the ultimate ruin probabilities be
\[ \psi_{\text{or}}(x_1, x_2) = \mathbb{P}(\tau_{\text{or}}(x_1, x_2) < \infty), \quad \psi_{\text{sim}}(x_1, x_2) = \mathbb{P}(\tau_{\text{sim}}(x_1, x_2) < \infty) \]
and
\[ \psi_1(x_1) = \mathbb{P}(\tau_1(x_1) < \infty), \quad \psi_2(x_2) = \mathbb{P}(\tau_2(x_2) < \infty), \]
where \( \tau_i(x_i) = \inf \{ t > 0 : R_i(t) < 0 \} \) for \( i = 1, 2 \). One can also be interested in the following ruin probability
\[ \psi_{\text{and}}(x_1, x_2) = \mathbb{P}(\tau_1(x_1) < \infty \text{ and } \tau_2(x_2) < \infty) \]
and the following relation is easily to notice
\[ \psi_{\text{and}}(x_1, x_2) = \psi_1(x_1) + \psi_2(x_2) - \psi_{\text{or}}(x_1, x_2) . \]

Let us put \( u_i = x_i/\delta_i \) and \( c_i = p_i/\delta_i \) where \( i = 1, 2 \). Then we get
\[ \tau_{\text{or}}(x_1, x_2) = \inf \{ t > 0 : X(t) > u_1 + c_1 t \text{ or } X(t) > u_2 + c_2 t \} \]
and
\[ \tau_{\text{sim}}(x_1, x_2) = \inf \{ t > 0 : X(t) > u_1 + c_1 t \text{ and } X(t) > u_2 + c_2 t \} . \]

If the lines \( y = u_1 + c_1 t \) and \( y = u_2 + c_2 t \) do not cross each other in the first quadrant then the ruin probabilities \( \psi_{\text{or}}(x_1, x_2) \) and \( \psi_{\text{sim}}(x_1, x_2) \) reduce to ordinary ruin probabilities of a risk process with a linear drift. If they cross each other in the first quadrant and e.g. \( u_1 < u_2 \ (c_1 > c_2) \) then
\[ \psi_{\text{or}}(x_1, x_2) = \mathbb{P}(\sup_{t < \infty} (X(t) - c(t)) > u_1) , \quad (6) \]
where \( c(t) \) is defined in eq. (2) with \( T = \frac{(u_2 - u_1)}{(c_1 - c_2)} \) (we take \( c(t) = \min(u_1 + c_1 t, u_2 + c_2 t) - u_1 \).

Similarly, if the lines have a common point in the first quadrant and e.g. \( u_2 < u_1 \ (c_2 > c_1) \) then
\[ \psi_{\text{sim}}(x_1, x_2) = \mathbb{P}(\sup_{t < \infty} (X(t) - c(t)) > u_1) , \quad (7) \]
where \( c(t) \) is defined in eq. (2) with \( T = \frac{(u_2 - u_1)}{(c_1 - c_2)} \) (we take \( c(t) = \max(u_1 + c_1 t, u_2 + c_2 t) - u_1 \)).
Example 5. Let $X(t)$ be the standard Brownian motion. Then using eq. (6) and Example 1 we get for $u_1 < u_2$ and $c_1 > c_2$:

$$
\psi_{or}(x_1, x_2) = 
\Phi(a(-u_1, -c_1)) + e^{-2c_1u_1}\Phi(a(-u_1, c_1))
+ e^{-2c_2u_2}\Phi(a(u_1, c_1 - 2c_2)) - e^{-2(c_1 - 2c_2)u_1 - 2c_2u_2}\Phi(a(-u_1, c_1 - 2c_2)),
$$

where $a(u, c) = uT^{-1/2} + c\sqrt{T}$, $T = (u_2 - u_1)/(c_1 - c_2)$, $u_i = x_i/\delta_i$ and $c_i = p_i/\delta_i$ for $i = 1, 2$. This formula recovers the result of Avram et al. [2] Eq. (67). The same way we obtain the formula for $\psi_{sim}(x_1, x_2)$.

In a similar we can consider ruin probabilities on a finite time horizon.

4 Randomly broken drift and random interval

In the fluctuation theory there are many interesting identities for Lévy processes and an exponentially distributed time e.g. the distribution of supremum on an exponentially distributed time interval (see e.g. Bertoin [3] Sec. VI. 2. and Sec. VII). Thus let us consider a spectrally positive Lévy process $X$ with a randomly broken drift that is let us assume that $T$ (see eq. (2)) is an exponentially distributed random variable with mean $1/\lambda$ independent of the process $X$. Moreover let us investigate two cases $S = \infty$ (see Th. 4) and $S - T = V$ is a positive random variable independent of the process $X$ and the random variable $T$. We put

$$
\varphi_i(\gamma) = \ln \mathbb{E} \exp(-\gamma(X(1) - c_i)), \ i = 1, 2
$$

where $\gamma \geq 0$ and $\frac{d}{d}\varphi_i(\lambda), i = 1, 2$ the inverse function of $\varphi_i$.

Theorem 6. If $X$ is a spectrally positive Lévy process and $T$ is an exponential random variable with mean $1/\lambda > 0$ independent of $X$ then for any
\[ \gamma > \varphi_1(\lambda) \]

\[ \mathbb{E} e^{-\gamma \sup_{t<T+V}(X(t) - c(t))} = \]

\[ \mathbb{E} e^{-\gamma \sup_{t<T}(X(t) - c_1 t)} + \frac{\gamma \lambda}{\varphi_1(\gamma) - \lambda} \left[ \frac{1 - \mathbb{E} e^{-\gamma \sup_{t<V}(X(t) - c_2 t)}}{\gamma} - \frac{1 - \mathbb{E} e^{-\varphi_1(\lambda) \sup_{t<V}(X(t) - c_2 t)}}{\varphi_1(\lambda)} \right] \tag{8} \]

where \( V \) is a positive random variable independent of \( X \) and \( T \).

Proof. Observe that for \( \gamma > 0 \)

\[ \mathbb{E} e^{-\gamma \sup_{t<T+V}(X(t) - c(t))} = 1 - \gamma \int_0^\infty e^{-\gamma u} \mathbb{P} \left( \sup_{t<T+V} X(t) - c(t) > u \right) du \]

and

\[ \mathbb{P} \left( \sup_{t<T+V} (X(t) - c(t)) > u \right) = \mathbb{P} \left( \sup_{t<T} (X(t) - c_1 t) > u \right) + \mathbb{P} \left( \sup_{t<T} (X(t) - c_1 t) \leq u, \sup_{t<V} (X(t + T) - X(T) - c_2 t) > u - X(T) + c_1 T \right). \]

Thus we have that

\[ \int_0^\infty e^{-\gamma u} \mathbb{P} \left( \sup_{t<T+V} X(t) - c(t) > u \right) du = I_1 + I_2, \tag{9} \]

where

\[ I_1 := \int_0^\infty e^{-\gamma u} \mathbb{P} \left( \sup_{t<T} X(t) - c_1 t > u \right) du = \frac{1 - \mathbb{E} e^{-\gamma \sup_{t<T}(X(t) - c_1 t)}}{\gamma} \]

and

\[ I_2 := \int_0^\infty e^{-\gamma u} \mathbb{P} \left( \sup_{t<T} (X(t) - c_1 t) \leq u, \sup_{t<V} (X(t + T) - X(T) - c_2 t) > u - X(T) + c_1 T \right) du. \]
By the fact that $T$ is exponentially distributed and independent of $X$ and $V$ we have

$$I_2 = \lambda \int_0^\infty e^{-\lambda s} ds \int_0^\infty e^{-\gamma u} du \cdot \mathbb{P} \left( \sup_{t<s} (X(t) - c_1 t) \leq u, \sup_{t<V} (X(t+s) - X(s) - c_2 t) > u - X(s) + c_1 s \right) du.$$ 

Moreover, by the independence of $X(t+s) - X(s) - c_2 t, t \geq 0$ and $X(s) - c_1 s$ and the fact that

$$\mathbb{P} \left( \sup_{t<s} (X(t) - c_1 t) \leq u, u - X(s) + c_1 s \leq z \right) = 0, \quad z < 0$$

we have

$$I_2 = \lambda \int_0^\infty e^{-\lambda s} ds \int_0^\infty e^{-\gamma u} du \int_0^\infty \mathbb{P} \left( \sup_{t<V} (X(t) - c_2 t) > z \right) \cdot \mathbb{P} \left( \sup_{t<s} (X(t) - c_1 t) \leq u, u - X(s) + c_1 s \in dz \right) du \int_0^\infty \mathbb{P} \left( \sup_{t<V} (X(t) - c_2 t) > z \right) \cdot \mathbb{P} \left( \inf_{t<s} (u - X(t) + c_1 t) > 0, u - X(s) + c_1 s \in dz \right).$$

Due to Suprun [16] (see also Bertoin [4] Lemma 1) we have that

$$\int_0^\infty e^{-\lambda s} \mathbb{P} \left( \inf_{t<s} (u - X(t) + c_1 t) > 0, u - X(s) + c_1 s \in dz \right) ds = \left[ e^{-\varphi_1(\lambda)z} W^{(\lambda)}(u) - \mathbb{I}(u \geq z) W^{(\lambda)}(u - z) \right] dz,$$

where $\mathbb{I}(\cdot)$ is the indicator function and $W^{(\lambda)} : [0, \infty) \to [0, \infty)$ is a continuous and increasing function such that

$$\int_0^\infty e^{-\gamma x} W^{(\lambda)}(x) dx = \frac{1}{\varphi_1(\gamma) - \lambda}, \quad \gamma > \frac{1}{\varphi_1(\lambda)}.$$
Consequently, for $\gamma > \frac{1}{\varphi_1'(\lambda)}$

$$I_2 =$$

$$\lambda \int_0^\infty \int_0^\infty e^{-\gamma u} \mathbb{P}\left(\sup_{t < V} (X(t) - c_2 t) > z\right)$$

$$\cdot \left[ e^{-\varphi_1'(\lambda)z} W(\lambda)(u) - \mathbb{I}(u \geq z) W(\lambda)(u - z) \right] dz \, du$$

$$= \lambda \int_0^\infty e^{-\varphi_1'(\lambda)z} \mathbb{P}\left(\sup_{t < V} (X(t) - c_2 t) > z\right) dz \int_0^\infty e^{-\gamma u} W(\lambda)(u) du$$

$$- \lambda \int_0^\infty \mathbb{P}\left(\sup_{t < V} (X(t) - c_2 t) > z\right) dz \int_0^\infty \mathbb{I}(u \geq z) e^{-\gamma u} W(\lambda)(u - z) du$$

$$= \frac{\lambda}{\varphi_1(\gamma) - \lambda} \left[ \int_0^\infty e^{-\varphi_1'(\lambda)z} \mathbb{P}\left(\sup_{t < V} (X(t) - c_2 t) > z\right) dz 

- \int_0^\infty e^{-\gamma z} \mathbb{P}\left(\sup_{t < V} (X(t) - c_2 t) > z\right) dz \right]$$

$$= \frac{\lambda}{\varphi_1(\gamma) - \lambda} \left[ 1 - \frac{\mathbb{E}e^{-\varphi_1'(\lambda)\sup_{t < V}(X(t) - c_2 t)}}{\varphi_1(\lambda)} \right] - \frac{1 - \mathbb{E}e^{-\gamma \sup_{t < V}(X(t) - c_2 t)}}{\gamma}$$.

\[\square\]

**Corollary 3.** Under the assumption of Theorem 6, if $V = \infty$, then

$$\mathbb{E}e^{-\gamma \sup_{t < \infty}(X(t) - c(t))} = \frac{\gamma \lambda \varphi_2'(0)[\varphi_2(\gamma) - \varphi_2(\varphi_1'(\lambda))]}{\varphi_2(\gamma)(\varphi_1(\gamma) - \lambda)\varphi_2(\varphi_1'(\lambda))}.$$ (10)

*If $V$ is an exponentially distributed random variable with mean $1/\theta > 0$ independent of $X$ and $T$ then*

$$\mathbb{E}e^{-\gamma \sup_{t < T+V}(X(t) - c(t))} = \gamma \lambda \theta \frac{\varphi_2(\theta) - \varphi_1'(\lambda)}{\varphi_1(\lambda) \varphi_2'(\theta) [\varphi_1(\gamma) - \lambda]}.$$ (11)

**Proof.** Case $V = \infty$. It is well-known that

$$\mathbb{E}e^{-\gamma \sup_{t < T}(X(t) - c_1 t)} = \frac{\lambda}{\lambda - \varphi_1(\gamma)} \left( 1 - \frac{\gamma}{\varphi_1'(\lambda)} \right),$$ (12)

where $\gamma > 0$ (see e.g. Bertoin [3] eq. (3) p. 192 or Th. 4.1 in Dębicki and Mandjes [5]). Moreover, by Th. 3.2 in Dębicki and Mandjes [5] (or going
with $\lambda$ to 0 in the previous identity), it follows that

$$\mathbb{E} \exp \left( -\gamma \sup_{t<\infty} (X(t) - c_2 t) \right) = \frac{\gamma \varphi'_2(0)}{\varphi_2(\gamma)}.$$

Consequently, by (8) for $\gamma > 0$

$$\mathbb{E} e^{-\gamma \sup_{t<\infty} (X(t) - c(t))} = \frac{\lambda}{\lambda - \varphi_1(\gamma)} \left[ 1 - \frac{\gamma}{\varphi_1(\lambda)} \right] + \frac{\gamma \lambda}{\varphi_1(\gamma) - \lambda} \left[ 1 - \frac{\gamma}{\varphi_2(\gamma)} \frac{\varphi_2(\theta)}{\varphi_2(\varphi_2(\theta))} \right]$$

$$= \frac{\gamma \lambda \varphi'_2(0)[\varphi_2(\gamma) - \varphi_2(\varphi_1(\lambda))]}{\varphi_2(\gamma)(\varphi_1(\gamma) - \lambda)\varphi_2(\varphi_1(\lambda))}.$$

Case $V$ exponentially distributed. Using (12), for $\gamma \geq 0$ we have that

$$\mathbb{E} \exp \left( -\gamma \sup_{t<V} (X(t) - c_2 t) \right) = \frac{\theta}{\theta - \varphi_2(\gamma)} \left( 1 - \frac{\gamma}{\varphi_2(\theta)} \right).$$

Recalling (12), for $\gamma > 0$ it follows that

$$\mathbb{E} e^{-\gamma \sup_{t<T+V} X(t) - c(t)} = \frac{\lambda}{\lambda - \varphi_1(\gamma)} \left[ 1 - \frac{\gamma}{\varphi_1(\lambda)} \right]$$

$$+ \frac{\gamma \lambda}{\varphi_1(\gamma) - \lambda} \left[ 1 - \frac{\theta}{\theta - \varphi_2(\gamma)} \frac{1 - \gamma}{\varphi_2(\theta)} - \frac{\theta}{\theta - \varphi_2(\varphi_2(\theta))} \frac{1 - \varphi_1(\lambda)}{\varphi_1(\lambda)} \right]$$

$$= \gamma \lambda \theta \frac{\varphi_2(\theta) - \varphi_1(\lambda)}{\varphi_1(\lambda) \varphi_2(\theta)[\varphi_1(\gamma) - \lambda]}.$$

□

**Corollary 4.** Let $W$ be the standard Brownian motion. Then

$$\varphi_1(\gamma) = \frac{1}{2} \gamma^2 + c_1 \gamma, \quad \varphi_2(\gamma) = \frac{1}{2} \gamma^2 + c_2 \gamma,$$

$$\varphi_1(\lambda) = \sqrt{c_1^2 + 2\lambda - c_1}, \quad \varphi_2(\lambda) = \sqrt{c_2^2 + 2\lambda - c_2}.$$
Consequently, for $\gamma > \sqrt{c_1^2 + 2\lambda - c_1}$

$$
\mathbb{E}e^{-\gamma \sup_{t<\infty} (W(t) - c(t))} = 
\frac{\gamma \lambda c_2 (\frac{1}{2} \gamma^2 + c_2 \gamma - c_1^2 - \lambda - (c_2 - c_1) \sqrt{c_1^2 + 2\lambda + c_1 c_2})}{(\frac{1}{2} \gamma^2 + c_1 \gamma - \lambda)(\frac{1}{2} \gamma^2 + c_2 \gamma)(c_1^2 + \lambda + (c_2 - c_1) \sqrt{c_1^2 + 2\lambda - c_1 c_2})}
$$

and

$$
\mathbb{E}e^{-\gamma \sup_{t<T+V} (W(t) - c(t))} = 
\frac{\gamma \lambda \theta^{\theta - c_1^2 - \lambda - (c_2 - c_1)} \sqrt{c_1^2 + 2\lambda + c_1 c_2}}{(c_1^2 + 2\lambda - c_1)(\sqrt{c_1^2 + 2\theta - c_2})(\frac{1}{2} \gamma^2 + c_1 \gamma - \lambda)}
$$

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