1 Introduction

The goal of this note is to prove the following statement.

**Theorem 1** There exists a $C^\infty$ foliation with 3 singular points on the two-dimensional torus such that any lifting of a leaf of this foliation on the universal covering of the torus is a dense subset of the covering.

This foliation is nonorientable and it has a transversal measure. The measure can be given locally out of the singularities by a closed 1-form.

**Half translation surfaces.** Let $M$ be a two-dimensional surface with a flat metric which has only conical singularities. The surface is called a half translation if the holonomy along any path on the surface is either $Id$ or $-Id$. It is clear that the angles around the conical points should be an integer multiple of $\pi$. The surface is called a translation if the holonomy is always $Id$.

A half translation structure gives rise to a family of foliations on the surface parametrized by the circle $S^1$. To construct a foliation from this family fix a tangent vector on the surface and move it to all the points of the surface by means of parallel translation. Since the holonomy is $\pm Id$ we obtain a foliation.

A half translation surface can have only a finite group of automorphisms (with the only one exception of a flat torus). But if we forget about the metric and remember only the underlying affine structure on the surface we can have an interesting group of automorphisms preserving affine structure.

The idea of the proof of the theorem 1 is the following. We construct a half translation torus with 3 singular points such that the underlying affine
structure has a big group of automorphisms. In particular there is an au-
omorphism of the torus which acts trivially in the first homology of the torus
but is a locally hyperbolic map (in affine coordinates on the torus). The
expanding foliation of this automorphism will be the foliation mentioned in
the theorem 1.

Remarks. There is a classical result of A. Weil [We] about oriented fol-
iation with finite number of singularities on $T^2$. The deviation of such a
foliation is always bounded i.e. any lifting of any leaf of the foliation to the
universal covering of the torus is contained in a finite neighborhood of a strait
line. First example of a nonoriented foliation with unbounded deviation was
constructed by A.D. Anosov [An1], [An2]. In [EMZ] you can find a good
introduction to translation surfaces.

2 Construction of the foliation

Construction of the half translation structure on the torus. The half
translation structure on the torus will be obtained from a usual flat torus by
the operation of folding.

Definition of folding. Let $[AB]$ be a geodesic segment on a flat surface
M. Cut M along $[AB]$ and denote by $C_+$ and $C_-$ two centers of different
shores of the cut. Glue together segment $[AC_+]$ with $[C_+B]$ and $[AC_-]$ with
$[C_-B]$. On the resulting surface we obtain two new singular points with
‘angles’ $\pi$ (these points appear from $C_+$ and $C_-$) and a singular point with
‘angle’ $4\pi$.

\[
\begin{array}{c}
A & \bullet & \bullet & C & \bullet & B \\
\end{array}
\]

\[
\begin{array}{c}
A & \bullet & \bullet & C_+ & \bullet & \bullet & C_- & \bullet & \bullet & B \\
\end{array}
\]

The torus we are looking for is the flat torus $\mathbb{R}^2/(3\mathbb{Z} \oplus \mathbb{Z})$ folded along
the segment $[(-1,0); (1,0)]$. We denote this torus by $T$. The structure of the
folded torus is given on the fig.2. The arrows connect pieces of the border of
the rectangle that should be glued.
Affine automorphisms of $T$. Decomposition on cylinders. Consider the horizontal foliation (i.e. the foliation whose leaves are parallel to the $x$-axis) on the torus $T$ and remove the leaf passing through singular points. It cuts $T$ into the cylinder with height 1 and length 3. The horizontal foliation fills the cylinder by closed leaves of length 3. There is an obvious automorphism $\gamma_h$ of $T$ preserving the horizontal foliation. It fixes the singular leaf and it makes a twist of the cylinder. This map can be given by the formula $\gamma_h(x, y) = (x + 3y (\text{mod } 3), y)$.

Now consider the vertical foliation. This foliation has three connected singular leaves. They split $T$ into two cylinders $1 \times 1$ and $2 \times 1$ (here and later the cylinder $m \times n$ is the cylinder with the height $m$ and the length $n$). There is an automorphism $\gamma_v$ of $T$ which makes the full twist of the cylinder $1 \times 1$ and it makes the half twist of the cylinder $2 \times 1$. The automorphism $\gamma_v$ fixes the singular point having the conical angle $4\pi$ and permutes two singular points having the conical angle $\pi$.

There is a natural map from the group of affine automorphisms of $T$ to $\text{PSL}(2, \mathbb{Z})$. It is given by the action of the automorphisms on the tangent space of $T$. The images of $\gamma_h$ and $\gamma_v$ are given by

$$
\begin{pmatrix}
1 & 3 \\
0 & 1
\end{pmatrix}, \begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix}
$$

These matrixes are defined only up to a sign,

The automorphism we are looking for is $(\gamma_v \gamma_h^{-1} \gamma_v)^4$.

Construction of the foliation. We denote by $\mathcal{F}_1$ the expanding foliation of the map $\gamma_v \gamma_h^{-1} \gamma_v$ and by $\mathcal{F}_2$ the contracting foliation. These foliations are parallel to the directions $(\sqrt{3}, 1)$ and $(\sqrt{3}, -1)$.

Lemma 2 Any leaf of the foliation $\mathcal{F}_1$ is a dense subsets of the torus.
The proof of this lemma is standard. It follows from the absence of separatrices joining singular points of the foliation.

3 Proof of the theorem

Action on homology. We note by $e_1$ the class in the homology group of a nonsingular leaf of the horizontal foliation and we note by $e_2$ the class of homology of a leaf of the vertical cylinder $1 \times 1$. It is clear that $e_1, e_2$ form a basis for $H_1(T, \mathbb{Z})$.

Lemma 3 The action of the induced homomorphism $\gamma_v\gamma_h^{-1}\gamma_v$ on the first homology of $T$ with respect to the basis $(e_1, e_2)$ is given by the matrix

$\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}$

Proof. Since $\gamma_h$ makes the single twist of the horizontal cylinder $1 \times 3$ and $\gamma_v$ makes the single twist of the vertical cylinder $1 \times 1$ the action of $\gamma_h$ and $\gamma_v$ in $H_1(T, \mathbb{Z})$ is given by the matrices

$\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}$

$\begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix}$

It proves the lemma. □

Lemma 3 implies that the induced action of $(\gamma_v\gamma_h^{-1}\gamma_v)^4$ on $H_1(T, \mathbb{Z})$ is trivial.

Action on the universal covering of torus $T$. The universal covering of $T$ can be viewed as the usual plane $\mathbb{R}^2$ folded along the segments $[(n, 3k - 1), (n, 3k + 1)]$, $n, k \in \mathbb{Z}$. After folding the origin $(0, 0)$ of the plane gives rise to two singular points of the angle $\pi$. We denote these points by $(0, 0_+)$ and $(0, 0_-)$. The action of $\gamma_v\gamma_h^{-1}\gamma_v$ on $T$ can be lifted to the action on the folded plane in such a way that point $(0, 0_+)$ is fixed. All singular points of type $(m, 3n_+)$ can be naturally identified with $H_1(T, \mathbb{Z})$. So these points are invariant under the fourth power of the automorphism $\gamma_v\gamma_h^{-1}\gamma_v$ (lemma 3).

Denote by $\tilde{F}_1$ and $\tilde{F}_2$ the liftings of the foliations $F_1$ and $F_2$ on the universal covering of the torus $T$. Denote by $\tilde{f}_1$ the leaf of $\tilde{F}_1$ starting at the point $(0, 0_+)$ and by $\tilde{f}_2$ the leaf of $\tilde{F}_2$ starting in $(3, 0_+)$

Lemma 4 The leaf $\tilde{f}_1$ of the foliation $\tilde{F}_1$ comes arbitrary close to the point $(3, 0_+)$. 

Proof. The point \(\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)\) is an intersection of the leaves \(\tilde{f}_1\) and \(\tilde{f}_2\). The transformation \((\gamma_v\gamma_h^{-1}\gamma_v)^4\) preserves points \((0,0_+\) and \((3,0_+)\) so it preserves leaves \(\tilde{f}_1\) and \(\tilde{f}_2\). It means that the point \((\gamma_v\gamma_h^{-1}\gamma_v)^4\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)\) is also a point of the intersection of \(\tilde{f}_1\) and \(\tilde{f}_2\). Now use the fact that the action of \((\gamma_v\gamma_h^{-1}\gamma_v)^4\) contracts \(\tilde{f}_2\). Applying the map \((\gamma_v\gamma_h^{-1}\gamma_v)^4\) inductively we construct a sequence of intersections of \(\tilde{f}_1\) and \(\tilde{f}_2\) which tends to the point \((3,0_+)\). □

Now we give a proof of the theorem 1 for the foliation \(\tilde{F}_1\) and its leaf \(\tilde{f}_1\). Since \(\gamma_v\gamma_h^{-1}\gamma_v\) performs a cyclic permutation of points \((\pm 3,0_+\), \((0,\pm 1_+)\) (lemma 1) the leaf \(\tilde{f}_1\) approaches arbitrary close to all these points. It means that the closure of \(\tilde{f}_1\) in \(\mathbb{R}^2\) contains the leaves of \(\tilde{F}_1\) passing through fore points \((\pm 3,0_+\), \((0,\pm 1_+)\). Since \(\tilde{F}_1\) is periodic in \(\mathbb{R}^2\) the same argument tells that the leaf \(\tilde{f}_1\) approaches arbitrary close to every singular point of the type \((3n,k_+)\). So its closure in the plane contains the closure of the union of all leaves of the foliation \(\tilde{F}_1\) passing through singular points of type \((3n,k_+)\). The last closure is the whole plane (lemma 2). □

**Final remarks.** The leaf \(\tilde{f}_1\) constructed in the theorem 1 is everywhere dense in the plane. One can see that this leaf propagates in the plane with a logarithmic speed.

Here is an optical interpretation of the constructed foliation. Consider the two dimensional plane \(\mathbb{R}^2\) with the integer lattice \(\mathbb{Z}^2\). Put in any vertex of the lattice a 90° rotation invariant cross in such a way that all the picture is \(\mathbb{Z}^2\) invariant (fig.3). Send a light in the direction parallel to the bissectrix of crosses and consider it reflections from edges of crosses. The constructed ray corresponds to a leaf of some foliation on the two-dimensional torus with 3 singular points.
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