QUANTUM TORUS ALGEBRAS AND B(C) TYPE TODA SYSTEMS

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Abstract. In this paper, we construct a new even constrained B(C) type Toda hierarchy and derive its B(C) type Block type additional symmetry. Also we generalize the B(C) type Toda hierarchy to the N-component B(C) type Toda hierarchy which is proved to have symmetries of a coupled $\bigotimes^N QT_+$ algebra (N-folds direct product of the positive half of the quantum torus algebra $QT$).

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1. Introduction

The Toda lattice hierarchy as a completely integrable system has many important applications in mathematics and physics including the representation theory of Lie algebras and random matrix models [1-3]. The Toda system has many kinds of reductions or extensions, for example the B and C type Toda hierarchies [2,4], extended Toda hierarchy (ETH) [5], bigraded Toda hierarchy (BTH) [6-11] and so on. There are some other generalizations called multi-component Toda systems [2,12] which are...
useful in the fields of multiple orthogonal polynomials and non-intersecting Brownian motions.

The multicomponent 2D Toda hierarchy was considered from the point of view of the Gauss-Borel factorization problem, the theory of multiple matrix orthogonal polynomials, non-intersecting Brownian motions and matrix Riemann-Hilbert problem [12]–[15]. In fact the multicomponent 2D Toda hierarchy in [13] is a periodic reduction of the bi-infinite matrix-formed two dimensional Toda hierarchy. In [16], we generalize the multicomponent Toda hierarchy to an extended multicomponent Toda hierarchy including extended logarithmic flow equations. Later by a commutative algebraic reduction on the extended multicomponent Toda hierarchy, we get an extended \( Z_N \)-Toda hierarchy [17] which might be useful in Gromov-Witten theory.

This paper is organized in the following way. In Section 2, we recall some basic knowledge about the B(C) type Toda hierarchy. We construct a new even constrained B(C) type Toda hierarchy and derive its Block type additional symmetry in Section 3. Next, in Section 4 we generalize the B(C) type Toda hierarchy to a new \( N \)-component B(C) type Toda hierarchy. In the last section, we construct the symmetry of the \( N \)-component B(C) type Toda hierarchy which constitutes a coupled \( \otimes^N QT_+ \) algebra (\( N \)-folds direct product of the positive half of the quantum torus algebra \( QT \)).

2. The B(C) Type Toda Hierarchy

In this section, some basic facts about the B(C) type Toda hierarchy are reviewed. One can refer to [2][4] for more details about the B(C) type Toda hierarchy (or BTH(CTH)).

Then the BTH hierarchy is defined in the Lax forms as

\[
\begin{align*}
\partial_{x_{2n+1}} L_1 &= \left[-(L_1^{2n+1})_-, L_1 \right], \\
\partial_{y_{2n+1}} L_1 &= \left[-(L_2^{2n+1})_-, L_1 \right], \\
\partial_{x_{2n+1}} L_2 &= \left[(L_1^{2n+1})_+, L_2 \right], \\
\partial_{y_{2n+1}} L_2 &= \left[(L_2^{2n+1})_+, L_2 \right],
\end{align*}
\]

(1)

where the Lax operator \( L_i \) is given by a pair of infinite matrices

\[
L_1 = \sum_{-\infty < i < 1} \text{diag}[a_i^{(1)}(s)]\Lambda^i, \quad L_2 = \sum_{-1 < i < \infty} \text{diag}[a_i^{(2)}(s)]\Lambda^i,
\]

(3)

with \( \Lambda = (\delta_{j-i})_{i,j\in\mathbb{Z}} \), and \( a_i^{(k)}(s) \) and \( a_i^{(1)}(s) \) depending on \( x = (x_1, x_3, x_5, \cdots) \) and \( y = (y_1, y_3, y_5, \cdots) \), such that

\[
a_1^{(1)}(s) = 1 \quad \text{and} \quad a_1^{(2)}(s) \neq 0 \quad \forall s
\]
and satisfies the BTH(CTH) constraint \[2\]

\[ L_i^T = -JL_iJ^{-1}, \quad L_i^T = -KL_iK^{-1}, \]  

(4)

where \( J = ((-1)^{i+j}δ_{i+j,0})_{i,j \in \mathbb{Z}} \), \( K = ΛJ \) and \( T \) refers to the matrix transpose. The BTH constraint is explicitly showed as

\[ a_i^{(k)}(s) = (-1)^{i+1}a_i^{(k)}(-s - i), \quad k = 1, 2. \]  

(5)

The CTH constraint means

\[ a_i^{(k)}(s) = (-1)^{i+1}a_i^{(k)}(-s - i - 1). \]  

(6)

The Lax equation for the BTH(CTH) can be expressed as a system of equations of the Zakharov-Shabat type:

\[ \partial_{x_{2n+1}}(L_1^{2n+1}+) - \partial_{x_{2n+1}}(L_1^{2n+1}+) + [(L_1^{2n+1}+), (L_1^{2n+1}+)] = 0, \]  

(7)

\[ \partial_{y_{2n+1}}(L_2^{2n+1}+) - \partial_{y_{2n+1}}(L_2^{2n+1}+) - [(L_2^{2n+1}+), (L_2^{2n+1}+)] = 0, \]  

(8)

\[ \partial_{y_{2n+1}}(L_1^{2n+1}+) + \partial_{x_{2n+1}}(L_1^{2n+1}+) - [(L_1^{2n+1}+), (L_1^{2n+1}+)] = 0, \]  

(9)

\[ -\partial_{y_{2n+1}}(L_2^{2n+1}+) - \partial_{x_{2n+1}}(L_2^{2n+1}+) - [(L_2^{2n+1}+), (L_2^{2n+1}+)] = 0. \]  

(10)

When \( m = n = 0 \), one can get the B type Toda equation

\[ \partial_{x_1}a_1^{(2)}(1) = a_1^{(2)}(1)a_0^{(1)}(1), \quad \partial_{x_1}a_{-1}^{(2)}(s) = a_{-1}^{(2)}(s)(a_0^{(1)}(s) - a_0^{(1)}(s - 1)) \quad (s \geq 2), \]  

(11)

\[ \partial_{y_1}a_0^{(2)}(s) = a_{-1}^{(2)}(s) - a_{-1}^{(2)}(s + 1) \quad (s \geq 1), \]  

by considering the corresponding constraint. Also one can get the C type Toda equation

\[ \partial_{x_1}a_1^{(2)}(0) = 2a_1^{(2)}(0)a_0^{(1)}(0), \quad \partial_{x_1}a_{-1}^{(2)}(s) = a_{-1}^{(2)}(s)(a_0^{(1)}(s) - a_0^{(1)}(s - 1)) \quad (s \geq 1), \]  

(12)

\[ \partial_{y_1}a_1^{(1)}(s) = a_{-1}^{(1)}(s) - a_{-1}^{(1)}(s + 1) \quad (s \geq 0), \]

The Lax operator of the BTH(CTH) \[37\] has the representation

\[ L_1 = W_1A^{-1}W_1^{-1} = S_1A S_1^{-1}, \]  

(13)

\[ L_2 = W_2A^{-1}W_2^{-1} = S_2A^{-1} S_2^{-1}, \]  

(14)

where

\[ S_1(x, y) = \sum_{i \geq 0} \text{diag}[c_i(s; x, y)]Λ^{-i}, \quad S_2(x, y) = \sum_{i \geq 0} \text{diag}[c_i'(s; x, y)]Λ^i \]  

(15)

and

\[ W_1(x, y) = S_1(x, y)ε(x, Λ), \quad W_2(x, y) = S_2(x, y)ε(x, Λ^{-1}) \]  

(16)
with \( c_0(s; x, y) = 1 \) and \( c'_0(s; x, y) \neq 0 \) for any \( s \), and \( \xi(x, \Lambda^\pm) = \sum_{n \geq 0} x_{2n+1} \Lambda^{\pm 2n+1} \).

For the B type Toda hierarchy, under an appropriate choice \((W_1, W_2)\) satisfies
\[
J^{-1}W_i^T J = W_i^{-1}, \quad i = 1, 2.
\]

(17)

For the C type Toda hierarchy, under an appropriate choice \((W_1, W_2)\) satisfies
\[
K^{-1}W_i^T K = W_i^{-1}, \quad i = 1, 2.
\]

(18)

The wave operators evolve as
\[
\partial_{x_{2n+1}} S_1 = -(L_1^{2n+1})^- S_1, \quad \partial_{y_{2n+1}} S_1 = -(L_2^{2n+1})^- S_1, \quad \partial_{x_{2n+1}} W_1 = (L_1^{2n+1})^+ W_1, \quad \partial_{y_{2n+1}} W_1 = -(L_2^{2n+1})^- W_2,
\]

(19)

(20)

\[
\partial_{x_{2n+1}} S_2 = (L_1^{2n+1})^+ S_2, \quad \partial_{y_{2n+1}} S_2 = (L_2^{2n+1})^+ S_2, \quad \partial_{x_{2n+1}} W_2 = (L_1^{2n+1})^+ W_2, \quad \partial_{y_{2n+1}} W_2 = -(L_2^{2n+1})^- W_2.
\]

(21)

(22)

At last, we end this section with the introduction of the additional symmetries of the BTH(CTH). The Orlov-Shulman operator \([4]\) is defined as
\[
M_1 = W_1 \varepsilon W_1^{-1}, \quad M_2 = W_2 \varepsilon^* W_2^{-1},
\]

(23)

where
\[
\varepsilon = \text{diag}[s] \Lambda^{-1}, \quad \varepsilon^* = -J \varepsilon J^{-1},
\]

satisfying
\[
[L_i, M_i] = 1, \quad \partial_{x_{2n+1}} M_i = [(L_1^{2n+1})^+, M_i], \quad \partial_{y_{2n+1}} M_i = [-(L_2^{2n+1})^-, M_i].
\]

(24)

To construct the Block symmetry of the BTH, the following lemma should be introduced.

**Lemma 1.** The following identities hold true
\[
\Lambda^{-1} \varepsilon \Lambda = J^{-1} \varepsilon^T J, \quad \Lambda \varepsilon^* \Lambda^{-1} = J^{-1} \varepsilon^* T J,
\]

(25)

\[
\varepsilon = K \varepsilon^T K^{-1}, \quad \varepsilon^* = K \varepsilon^* T K^{-1}.
\]

(26)

For the BTH, using the above lemma, one can derive
\[
M_i^T = JL_i^{-1} M_i J_i L_i J^{-1}.
\]

(27)

For the CTH, using the above lemma, one can derive
\[
M_i^T = KM_i K^{-1}.
\]

(28)
The additional symmetry \[4\] of the BTH can be defined by introducing the additional independent variables \(x_{m,l}\) and \(y_{m,l}\),

\[
\partial_{x_{m,l}} W_1 = -A_{ml}(M_1, L_1)_{\cdot} W_1, \quad \partial_{y_{m,l}} W_1 = -A_{ml}(M_2, L_2)_{\cdot} W_1, \tag{29}
\]

\[
\partial_{x_{m,l}} W_2 = A_{ml}(M_1, L_1)_{\cdot} W_2, \quad \partial_{y_{m,l}} W_2 = A_{ml}(M_2, L_2)_{\cdot} W_2, \tag{30}
\]

where

\[
A_{ml}(M_i, L_i) = M_i^m L_i^l - (-1)^l L_i^{l-1} M_i^m L_i. \tag{31}
\]

For the case of the CTH, the operator \(A_{ml}\) will become

\[
A_{ml}(M_i, L_i) = M_i^m L_i^l - (-1)^l L_i^l M_i^m. \tag{32}
\]

These additional flows form a coupled \(W_\infty\) Lie algebra \([4]\).

3. The even constrained BTH(CTH)

In this section, for a new constrained BTH(CTH), the Lax operator \(L\) is given by an infinite matrices \(L\) as

\[
L = L_1^{2N} = L_2^{2M} = \sum_{-2M < i \leq 2N} \text{diag}[a_i(s)] \Lambda^i, \tag{33}
\]

with \(a_{2N}(s) = 1\), and for the BTH, it satisfies the B type constraint

\[
L^T = JLJ^{-1}, \tag{34}
\]

and for the CTH, it satisfies the C type constraint

\[
L^T = KKL^{-1}. \tag{35}
\]

Then the constrained BTH(CTH) hierarchy is defined in the Lax forms as

\[
\partial_{x_{2n+1}} L = -(L_{\frac{2n+1}{2M}}), L \quad \text{and} \quad \partial_{y_{2n+1}} L = -(L_{\frac{2n+1}{2M}}), L, \tag{36}
\]

\[
\partial_{x_{2n+1}} L = [(L_{\frac{2n+1}{2M}})_+, L] \quad \text{and} \quad \partial_{y_{2n+1}} L = [(L_{\frac{2n+1}{2M}})_+, L], \quad n = 0, 1, 2, \ldots \tag{37}
\]

The Lax operator of the constrained BTH(CTH) \([37]\) has the representation

\[
L = W_1 \Lambda^{2N} W_1^{-1} = W_2 \Lambda^{-2M} W_2^{-1}, \tag{38}
\]

where

\[
S_1(x, y) = \sum_{i \geq 0} \text{diag}[c_i(s; x, y)] \Lambda^{-i}, \quad S_2(x, y) = \sum_{i \geq 0} \text{diag}[c_i'(s; x, y)] \Lambda^i \tag{39}
\]

and

\[
W_1(x, y) = S_1(x, y)e^{\xi(x, \Lambda)}, \quad W_2(x, y) = S_2(x, y)e^{\xi(y, \Lambda^{-1})} \tag{40}
\]
with \( c_0(s; x, y) = 1 \) and \( c'_0(s; x, y) \neq 0 \) for any \( s \), and \( \xi(x, \Lambda^{\pm1}) = \sum_{n \geq 0} x^{2n+1} \Lambda^{\pm2n+1} \).

Under an appropriate choice \((W_1, W_2)\) of the constrained BTH(CTH) satisfies

\[
J^{-1}W_i^T J = W_i^{-1}, \quad (K^{-1}W_i^T K = W_i^{-1}), \quad i = 1, 2. \tag{41}
\]

The wave operators evolve according to

\[
\begin{align*}
\partial_{x^{2n+1}} S_1 &= -(L^{2n+1})_- S_1, \quad \partial_{y^{2n+1}} S_1 = -(L^{2n+1})_- S_1, \tag{42} \\
\partial_{x^{2n+1}} W_1 &= (L^{2n+1})_+ W_1, \quad \partial_{y^{2n+1}} W_1 = -(L^{2n+1})_- W_1, \tag{43} \\
\partial_{x^{2n+1}} S_2 &= (L^{2n+1})_+ S_2, \quad \partial_{y^{2n+1}} S_2 = (L^{2n+1})_+ S_2, \tag{44} \\
\partial_{x^{2n+1}} W_2 &= (L^{2n+1})_+ W_2, \quad \partial_{y^{2n+1}} W_2 = -(L^{2n+1})_- W_2. \tag{45}
\end{align*}
\]

The Orlov-Shulman operator \( \tilde{M}_i \) will be defined as

\[
\tilde{M}_1 = W_1 \varepsilon_{2N} W_1^{-1}, \quad \tilde{M}_2 = W_2 \varepsilon_{-2M} W_2^{-1}, \tag{46}
\]

where

\[
\varepsilon_{2N} = \frac{1}{2N} \text{diag}[s] \Lambda^{-2N}, \quad \varepsilon_{-2M} = -\frac{1}{2M} \varepsilon^T \Lambda^{2M},
\]

satisfying

\[
[L, M_i] = 1, \quad \partial_{x^{2n+1}} \tilde{M}_i = [(L^{2n+1})_+, \tilde{M}_i], \partial_{y^{2n+1}} \tilde{M}_i = [-(L^{2n+1})_-, \tilde{M}_i]. \tag{47}
\]

**Lemma 2.** The difference of two Orlov-Schulman operators \( \tilde{M}_i \) for constrained BTH hierarchy has following B type property:

\[
L^T (\tilde{M}_1 - \tilde{M}_2)^T = J(L\tilde{M}_1 - L\tilde{M}_2)J^{-1}, \tag{48}
\]

and for constrained CTH hierarchy has following C type property:

\[
L^T (\tilde{M}_1 - \tilde{M}_2)^T = K(L\tilde{M}_1 - L\tilde{M}_2)K^{-1}. \tag{49}
\]

**Proof.** It is easy to find the two Orlov-Schulman operators can be expressed as

\[
\tilde{M}_1 = \frac{M_1 L_1^{2N}}{2N}, \quad \tilde{M}_2 = -\frac{M_2 L_2^{2M}}{2M}. \tag{50}
\]

Putting equation (50) into \((\tilde{M}_1 - \tilde{M}_2)^T\) can lead to

\[
\begin{align*}
(\tilde{M}_1 - \tilde{M}_2)^T &= \frac{JL_1^{2N} M_1 L_1 J^{-1}}{2N} + \frac{JL_2^{2M} M_2 L_2 J^{-1}}{2M} \tag{51} \\
&= \frac{JL_1^{2N} M_1 L_1 J^{-1}}{2N} + \frac{JL_2^{2M} M_2 L_2 J^{-1}}{2M} \tag{52} \\
&= \frac{J(M_1 L_1^{2N} - 2NL_1^{2N}) J^{-1}}{2N} + \frac{J(M_2 L_2^{2M} + 2ML_2^{2N}) J^{-1}}{2M}. \tag{53}
\end{align*}
\]
which can further lead to eq. (48). For the CTH, one can do the similar calculation as

\[(\bar{M}_1 - \bar{M}_2)^T = \frac{KL_1^{-2N}M_1K^{-1}}{2N} + \frac{KL_2^{-2M}M_2K^{-1}}{2M}\] (54)

\[= \frac{KL_1^{-2N}M_1K^{-1}}{2N} + \frac{KL_2^{-2M}M_2K^{-1}}{2M}\] (55)

\[= \frac{K(M_1L_1^{-2N} - 2NL_1^{-2N})K^{-1}}{2N} + \frac{K(M_2L_2^{-2M} + 2ML_2^{-2})K^{-1}}{2M}\] (56)

which can further lead to eq. (49)

In above calculation, the commutativity between \(L\) and \(\bar{M}_1 - \bar{M}_2\) is already used. Till now, the proof is finished.

For the constrained BTH(CTH), we need the following operator

\[B_{m,l} = (\bar{M}_1 - \bar{M}_2)^mL^l, \ m \in \mathbb{Z}_{odd}^+, l \in \mathbb{Z}_+^+\] (57)

One can easily check that for the BTH

\[B_{m,l}^T = JB_{m,l}J^{-1}, \ m \in \mathbb{Z}_{odd}^+,\] (58)

and for the CTH

\[B_{m,l}^T = KB_{m,l}K^{-1}, \ m \in \mathbb{Z}_{odd}^+.\] (59)

That means it is reasonable to define additional flows of the constrained BTH(CTH) as

\[\frac{\partial L}{\partial c_{m,l}} = \left[-(B_{m,l})_+, L\right], \ m \in \mathbb{Z}_{odd}^+, l \in \mathbb{Z}_+^+.\] (60)

**Proposition 3.** For the BTH(CTH), the flows (60) can commute with original flows of the BTH(CTH), namely,

\[\left[\frac{\partial}{\partial c_{m,l}}, \frac{\partial}{\partial x_k}\right] = 0, \quad \left[\frac{\partial}{\partial c_{m,l}}, \frac{\partial}{\partial y_k}\right] = 0, \quad l \in \mathbb{Z}_+, m, k \in \mathbb{Z}_{odd}^+,\]

which hold in the sense of acting on \(W_i\) or \(L\).

**Theorem 4.** The flows in eq. (60) about additional symmetries of constrained BTH(CTH) compose following Block type Lie algebra

\[\partial_{c_{m,l}}, \partial_{c_{s,k}} = (km - sl)\partial_{c_{m+s-1,k+l-1}}, \ m, s \in \mathbb{Z}_{odd}^+, k, l \in \mathbb{Z}_+,\] (61)

which holds in the sense of acting on \(W_i\) or \(L\).
4. Multicomponent B(C) type Toda hierarchy

In this section we will introduce the multicomponent B type Toda hierarchy (MBTH) and multicomponent C type Toda hierarchy (MCTH). In the following, we denote $E_{\mathbb{Z} \times \mathbb{Z}}$ as the bi-infinite identity matrix and $E_{N \times N}$ as the $N \times N$ identity matrix. We also denote $E_{kk}$ as a $N \times N$ matrix which is 1 at the position of the $k$-th row and $k$-th column and 0 for other elements. The Lax operators $L_1, L_2$ of the MBTH(MCTH) are given by a pair of infinite matrices

$$L_1 = \sum_{-\infty < i \leq 1} \text{diag}[b_i(s)]\bar{\Lambda}^i, \quad L_2 = \sum_{-1 \leq i < \infty} \text{diag}[c_i(s)]\bar{\Lambda}^i,$$

(62)

where $b_i(s), c_i(s)$ are matrices of size $N \times N$ and $\bar{\Lambda}^i = \Lambda^i \otimes E_{N \times N}$ and they satisfy the B type(C Type) constraint \[2\]

$$L_i^T = -J L_i J^{-1} (L_i^T = -K L_i K^{-1}),$$

(63)

where $J = ((-1)^i\delta_{i+j,0})_{i,j \in \mathbb{Z}} \otimes E_{N \times N}$, $K = \Lambda J \otimes E_{N \times N}$. Here the product $\otimes$ is the Kronecker product between a matrix of size $\mathbb{Z} \times \mathbb{Z}$ and a matrix of size $N \times N$. Let us first introduce some convenient notations as $\bar{E}_{kk} = E_{\mathbb{Z} \times \mathbb{Z}} \otimes E_{kk}$.

The Lax operators of the MBTH(MCTH) (37) can have the following dressing structure

$$L_1 = W_1\bar{\Lambda} W_1^{-1} = S_1\bar{\Lambda} S_1^{-1},$$

(64)

$$L_2 = W_2\bar{\Lambda} W_2^{-1} = S_2\bar{\Lambda}^{-1} S_2^{-1},$$

(65)

where

$$W_1(x, y) = S_1(x, y)(e^{\xi(x, \Lambda)} \otimes E_{N \times N}), \quad W_2(x, y) = S_2(x, y)(e^{\xi(y, \Lambda^{-1})} \otimes E_{N \times N}).$$

(66)

Now we define matrix operators $C_{kk}, \bar{C}_{kk}, B_{jk}, \bar{B}_{jk}$ as follows

$$C_{kk} := W_1\bar{E}_{kk} W_1^{-1}, \quad \bar{C}_{kk} := W_2\bar{E}_{kk} W_2^{-1},$$

$$B_{jk} := W_1\bar{E}_{kk} \bar{\Lambda} W_1^{-1}, \quad \bar{B}_{jk} := W_2\bar{E}_{kk} \bar{\Lambda}^{-j} W_2^{-1}.$$  

(67)

Now we give the definition of the multicomponent B(C) type Toda hierarchy (MBTH).

**Definition 1.** The multicomponent B(C) type Toda hierarchy is a hierarchy in which the dressing operators $S_1, S_2$ satisfy following Sato equations

$$\partial_{t_{jk}} S_1 = -(B_{jk}) S_1, \quad \partial_{t_{jk}} S_2 = (B_{jk}) S_2,$$

$$\partial_{t_{jk}} S_1 = -(\bar{B}_{jk}) S_1, \quad \partial_{t_{jk}} S_2 = (\bar{B}_{jk}) S_2.$$  

(68)

Then one can easily get the following proposition about $W_1, W_2$.  


For the multicomponent CTH, we define the operator $B$ firstly we define the operator $B$

To construct the additional quantum torus symmetry of the multicomponent BTH, the Orlov-Shulman
operator of the MBTH(MCTH) will be defined as

Proposition 6. The Lax equations of the MBTH(MCTH) are as follows

$$\frac{\partial t_{jk}}{t_{jk}} \mathcal{L}_j = [(B_{jk}^+) \mathcal{L}_j, \quad \frac{\partial t_{jk}}{t_{jk}} C_{ss} = [(B_{jk}^+) C_{ss}, \quad \frac{\partial t_{jk}}{t_{jk}} \bar{C}_{ss} = [(B_{jk}^+) \bar{C}_{ss}], \quad \frac{\partial t_{jk}}{t_{jk}} \mathcal{L}_j = [(B_{jk}^+) \mathcal{L}_j, \quad \frac{\partial t_{jk}}{t_{jk}} C_{ss} = [(B_{jk}^+) C_{ss}, \quad \frac{\partial t_{jk}}{t_{jk}} \bar{C}_{ss} = [(B_{jk}^+) \bar{C}_{ss}] \tag{72}$$

$$\frac{\partial t_{jk}}{t_{jk}} \mathcal{L}_j = [(\bar{B}_{jk}) \mathcal{L}_j, \quad \frac{\partial t_{jk}}{t_{jk}} C_{ss} = [(\bar{B}_{jk}) C_{ss}, \quad \frac{\partial t_{jk}}{t_{jk}} \bar{C}_{ss} = [(\bar{B}_{jk}) \bar{C}_{ss}] \tag{73}$$

5. Symmetries of MBTH(MCTH)

To introduce the additional symmetries of the MBTH(MCTH). The Orlov-Shulman
operator of the MBTH(MCTH) will be defined as

$$\mathcal{M}_1 = \mathcal{W}_1 (\varepsilon \otimes E_{N \times N}) \mathcal{W}_1^{-1}, \quad \mathcal{M}_2 = \mathcal{W}_2 (\varepsilon^* \otimes E_{N \times N}) \mathcal{W}_2^{-1}, \quad R_{ij} = \mathcal{W}_i (E \otimes E_{jj}) \mathcal{W}_i^{-1} \tag{74}$$

To construct the additional quantum torus symmetry of the multicomponent BTH, firstly we define the operator $B_{mnj}^{(i)}$ as

$$B_{mnj}^{(i)} = \mathcal{M}_i^m \mathcal{L}_i^n R_{ij} - (-1)^n R_{ij} \mathcal{L}_i^n \mathcal{M}_i^m \mathcal{L}_i \tag{76}$$

For the multicomponent CTH, we define the operator $B_{mnj}^{(i)}$ as

$$B_{mnj}^{(i)} = \mathcal{M}_i^m \mathcal{L}_i^n R_{ij} - (-1)^n R_{ij} \mathcal{L}_i^n \mathcal{M}_i^m \tag{77}$$

For any matrix operator $B_{mnj}^{(i)}$ in (77), one has

$$\frac{\partial B_{mnj}^{(i)}}{\partial t_{kj}} = [(\mathcal{L}_1^k R_{1j}^{(i)}, B_{mnj}^{(i)}], \quad k \in \mathbb{Z}_+^{odd} \tag{78}$$

$$\frac{\partial B_{mnj}^{(i)}}{\partial t_{kj}} = [(\mathcal{L}_2^k R_{2j}^{(i)}, B_{mnj}^{(i)}], \quad k \in \mathbb{Z}_+^{odd} \tag{79}$$

Then we can derive the following lemma.

Lemma 7. The following identities hold true

$$\bar{\Lambda}^{-1} (\varepsilon \otimes E_{N \times N}) \bar{\Lambda} = \mathcal{J}^{-1} (\varepsilon^T \otimes E_{N \times N}) \mathcal{J}, \quad \bar{\Lambda} (\varepsilon^* \otimes E_{N \times N}) \bar{\Lambda}^{-1} = \mathcal{J}^{-1} (\varepsilon^T \otimes E_{N \times N}) \mathcal{J} \tag{80}$$

$$\varepsilon (\varepsilon \otimes E_{N \times N}) = \mathcal{K} (\varepsilon^T \otimes E_{N \times N}) \mathcal{K}^{-1}, \quad \varepsilon^* \otimes E_{N \times N} = \mathcal{K} (\varepsilon^T \otimes E_{N \times N}) \mathcal{K}^{-1} \tag{81}$$
Then for the MBTH, by (41) and (80), we can derive

\[ \mathcal{M}_1^T = (W_1(\varepsilon \otimes E_{N \times N})W_1^{-1})^T = (W_1^{-1})^T(\varepsilon^T \otimes E_{N \times N})W_1^T \]

\[ = \mathcal{J}W_1^{-1}(\varepsilon^T \otimes E_{N \times N})\mathcal{J}W_1^{-1}\mathcal{J}^{-1} \]

\[ = \mathcal{J}W_1\tilde{\Lambda}^{-1}(\varepsilon \otimes E_{N \times N})\tilde{\Lambda}W_1^{-1}\mathcal{J}^{-1} \]

\[ = \mathcal{J}\mathcal{L}_i^{-1}\mathcal{M}_1\mathcal{L}_1\mathcal{J}^{-1}, \]  \hfill (82)

Using the second equation in eq. (80), we can also derive

\[ \mathcal{M}_2^T = \mathcal{J}\mathcal{L}_2^{-1}\mathcal{M}_2\mathcal{L}_2\mathcal{J}^{-1}. \]  \hfill (83)

Similarly, for the CTH, we can derive

\[ \mathcal{M}_i^T = \mathcal{K}\mathcal{M}_i\mathcal{K}^{-1}. \]  \hfill (84)

Because of the constraints (63) on the Lax operators for the MBTH(MCTH), we can have the following proposition.

**Proposition 8.** For the MBTH, it is sufficient to ask for

\[ B_{mnj}^{(i)T} = -\mathcal{J}B_{mnj}^{(i)T}\mathcal{J}^{-1}, \]  \hfill (85)

**Proof.** From (63) and (82), we have

\[ (\mathcal{M}_i^m\mathcal{L}_i^n R_{ij})^T = R_{ij}^T(\mathcal{L}_i^n)^T(\mathcal{M}_i^m)^T = (-1)^l\mathcal{J}\mathcal{L}_i^{-1}\mathcal{J}\mathcal{L}_i^{-1}\mathcal{M}_i^m\mathcal{L}_i\mathcal{J}^{-1} \]

\[ = \mathcal{J}(-1)^nR_{ij}\mathcal{L}_i^{-1}\mathcal{M}_i^m\mathcal{L}_i\mathcal{J}^{-1}. \]

Since \( J^T = J^{-1} = J \). Therefore \( B_{mnj}^{(i)} \) will satisfy the B type condition. \( \square \)

Similarly, the following proposition can also be got.

**Proposition 9.** For the MCTH, the following C type condition must hold true

\[ B_{mnj}^{(i)T} = \mathcal{K}B_{mnj}^{(i)}\mathcal{K}^{-1}, \]  \hfill (86)

Now for the MBTH we will denote the matrix operator \( D_{mnj} \) as

\[ D_{mnj} := \varepsilon^m\mathcal{M}_i q^n\mathcal{L}_i R_{ij} - \mathcal{L}_i^{-1}R_{ij}q^{-n}\mathcal{L}_i \varepsilon^m\mathcal{M}_i\mathcal{L}_i, \]  \hfill (87)
which further leads to
\[
D_{imnj} = \sum_{p,s=0}^{\infty} \frac{m^p(n \log q)^s(M^p_iL^s_jR_{ij} - (-1)^sR_{ij}L^{s-1}_iM^p_iL_i)}{pl!s!} = \sum_{p,s=0}^{\infty} \frac{m^p(n \log q)^sB^{(i)}_{psj}}{pl!s!}.
\] (88)

Then the following calculation will lead to the B(C) type anti-symmetry property of
\[D_{imnj}\] as
\[
D^T_{imnj} = -J(D_{imnj})J^{-1} = -J(D_{imnj}J^{-1}).
\] (90)

Now for the MCTH we will denote the matrix operator \(D_{mnj}\) as
\[
D_{imnj} := e^{mM_iq^nL_iR_{ij} - R_{ij}q^{-n}L_i}e^{mM_i}.
\] (89)

Therefore we get the following important B(C) type condition which the matrix operator \(D_{imnj}\) satisfies
\[
D^T_{imnj} = -J(D_{imnj})J^{-1}(D_{imnj} = -KD_{imnj}K^{-1}).
\] (90)

Then basing on a quantum parameter \(q\), the additional flows for the time variable \(t_{ij}^{m,n}t^{sij}_{m,n}\) are defined as follows
\[
\frac{\partial S_1}{\partial t_{ij}^{m,n}} = -(B^{(i)}_{mnj})_+S_1, \quad \frac{\partial S_1}{\partial t^{sij}_{m,n}} = -(D_{imnj})_-S_1, \quad \frac{\partial S_2}{\partial t_{ij}^{m,n}} = (B^{(i)}_{mnj})_-S_2, \quad \frac{\partial S_2}{\partial t^{sij}_{m,n}} = (D_{imnj})_+S_2,
\] (91, 92)
or equivalently rewritten as
\[
\frac{\partial L_1}{\partial t_{ij}^{m,n}} = -[(B^{(i)}_{mnj})_-, L_1], \quad \frac{\partial M_1}{\partial t^{sij}_{m,n}} = -[(D_{imnj})_-, M_1],
\] \[
\frac{\partial L_2}{\partial t_{ij}^{m,n}} = [(B^{(i)}_{mnj})_+, L_2], \quad \frac{\partial M_2}{\partial t^{sij}_{m,n}} = [(D_{imnj})_+, M_2].
\] (93, 94)

Generally, one can also derive
\[
\partial_{t_{l,k}}(D_{imnj}) = [-(D_{lkp})_-, D_{imnj}],
\] (95)
\[ \partial_{t^{ip}_{i,k}}(D_{2mnj}) = [(D_{ilkp})_{+}, D_{2mnj}]. \]  

(96)

This further leads to the commutativity of the additional flow \( \frac{\partial}{\partial t^{ij}_{m,n}} \) with the flow \( \partial_{t_{jn}} \), \( \partial_{\bar{t}_{jn}} \) in the following theorem.

**Theorem 10.** The additional flows of \( \partial_{t^{ij}_{i,k}} \) are symmetries of the multicomponent BTH(CTH), i.e. they commute with all \( \partial_{t_{jn}} \), \( \partial_{\bar{t}_{jn}} \) flows of the multicomponent BTH(CTH).

Comparing with the additional symmetry of the single-component BTH(CTH), the additional flows \( \partial_{t^{ij}_{i,k}} \) of the multicomponent BTH(CTH) form the following \( N \)-folds direct product of the \( W_{\infty} \) algebra as following

\[ [\partial_{t^{ir}_{ps}_{a,b}}, \partial_{t^{ic}_{a,b}}] \mathcal{L}_k = \delta_{ij} \delta_{rc} \sum_{\alpha,\beta} C_{\alpha \beta}^{(ps)(ab)} \partial_{t^{ic}_{a,b}} \mathcal{L}_k, \quad i, j, k = 1, 2; 1 \leq r, c \leq N. \]

Now it is time to identify the algebraic structure of the additional \( t^{ij}_{i,k} \) flows of the multicomponent BTH(CTH).

**Theorem 11.** The additional flows \( \partial_{t^{ij}_{i,k}} \) of the multicomponent BTH(CTH) form the coupled \( \bigotimes^N QT_+ \) algebra (\( N \)-folds direct product of the positive half of the quantum torus algebra \( QT_+ \)), i.e.,

\[ [\partial_{t^{ir}_{n,m}_{a,b}}, \partial_{t^{id}_{a,b}}] = \delta_{cd} \delta_{rj} (q^{ml} - q^{nk}) \partial_{t^{ir}_{n+l,m+k}}, \quad n, m, l, k \geq 0; \quad 1 \leq r, j \leq N; \quad c = d = 1, 2. \]  

(97)

**Proof.** One can also prove this theorem as following by rewriting the quantum torus flow in terms of a combination of \( \partial_{t^{ij}_{m,n}} \) flows

\[ [\partial_{t^{ir}_{n,m}_{a,b}}, \partial_{t^{id}_{a,b}}] \mathcal{L}_i \]

\[ = \left[ \sum_{p,s=0}^{\infty} \frac{n^p (m \log q)^s}{p! s!} \partial_{t^{ir}_{ps}_{a,b}} \sum_{a,b=0}^{\infty} \frac{l^a (k \log q)^b}{a! b!} \partial_{t^{id}_{a,b}} \right] \mathcal{L}_i \]

\[ = \sum_{p,s=0}^{\infty} \sum_{a,b=0}^{\infty} \frac{n^p (m \log q)^s}{p! s!} \frac{l^a (k \log q)^b}{a! b!} \left[ [\partial_{t^{ir}_{ps}_{a,b}}, \partial_{t^{id}_{a,b}}] \mathcal{L}_i \right] \]

\[ = \sum_{p,s=0}^{\infty} \sum_{a,b=0}^{\infty} \frac{n^p (m \log q)^s}{p! s!} \frac{l^a (k \log q)^b}{a! b!} \sum_{\alpha,\beta} C_{\alpha \beta}^{(ps)(ab)} \delta_{rj} \partial_{t^{ir}_{n,m+k}} \mathcal{L}_i \]

\[ = (q^{ml} - q^{nk}) \sum_{\alpha,\beta=0}^{\infty} \frac{(n + l)^{\alpha} ((m + k) \log q)^{\beta}}{\alpha! \beta!} \delta_{cd} \partial_{t^{ir}_{n,m+k}} \partial_{t^{id}_{a,b}} \mathcal{L}_i. \]

\[ \square \]
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References

[1] M. Toda, Wave propagation in anharmonic lattices, J. Phys. Soc. Jpn. 23(1967) 501-506.
[2] K. Ueno, K. Takasaki, Toda lattice hierarchy, In “Group representations and systems of differential equations” (Tokyo, 1982), 1-95, Adv. Stud. Pure Math. 4, North-Holland, Amsterdam, 1984.
[3] R. Dijkgraaf, E. Witten, Mean field theory, topological field theory, and multimatrix models, Nucl. Phys. B 342(1990), 486-522.
[4] J. P. Cheng, K. L. Tian and J. S. He. The additional symmetries for the BTL and CTL hierarchies. J. Math. Phys. 51 (2011) 053515.
[5] G. Carlet, B. Dubrovin, Y. Zhang, The Extended Toda Hierarchy, Mosc. Math. J. 4(2004), 313-332.
[6] G. Carlet, The extended bigraded Toda hierarchy, Journal of Physics A: Mathematical and Theoretical 39(2006), 9411-9435.
[7] C. Z. Li, J. S. He, K. Wu, Y. Cheng, Tau function and Hirota bilinear equations for the extended bigraded Toda Hierarchy, J. Math. Phys. 51(2010), 043514.
[8] G. Carlet, J. van de Leur, Hirota equations for the extended bigraded Toda hierarchy and the total descendent potential of $\mathbb{P}^1$ orbifolds, Journal of Physics A: Mathematical and Theoretical, 46(2013), 405205-405220.
[9] C. Z. Li, J. S. He, Y. C. Su, Block type symmetry of bigraded Toda hierarchy, J. Math. Phys. 53(2012), 013517.
[10] C. Z. Li, Solutions of bigraded Toda hierarchy, Journal of Physics A: Mathematical and Theoretical, 44, 255201(2011), arXiv:1011.4684.
[11] C. Z. Li, J. S. He, Dispersionless bigraded Toda hierarchy and its additional symmetry, Reviews in Mathematical Physics 24(2012), 1230003.
[12] M. Mañas, L. Martínez Alonso, The multicomponent 2D Toda hierarchy: dispersionless limit, Inverse Problems, 25(2009), 11.
[13] M. Mañas, L. Martínez Alonso, and C. Álvarez Fernández, The multicomponent 2D Toda hierarchy: discrete flows and string equations, Inverse Problems, 25(2009), 065007.
[14] C. Álvarez Fernández, U. Fidalgo Prieto, M. Mañas, The multicomponent 2D Toda hierarchy: generalized matrix orthogonal polynomials, multiple orthogonal polynomials and Riemann–Hilbert problems, Inverse Problems, 26(2010), 055009.
[15] C. Álvarez Fernández, U. Fidalgo Prieto, and M. Mañas, Multiple orthogonal polynomials of mixed type: Gauss-Borel factorization and the multi-component 2D Toda hierarchy, Advances in Mathematics, 227(2011), 1451-1525.
[16] C. Z. Li, J. S. He, On the extended multi-component Toda hierarchy, Math. Phys., Analysis and Geometry 17(2014), 377-407.
[17] C. Z. Li, J. S. He, The extended $\mathbb{Z}_N$-Toda hierarchy, Theor. Math. Phys. 185(2015), 1614-1635.