Existence of density for solutions of mixed stochastic equations

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Abstract. We consider a mixed stochastic differential equation $dX_t = a(t, X_t) \, dt + b(t, X_t) \, dW_t + c(t, X_t) \, dB^H_t$ driven by independent multidimensional Wiener process and fractional Brownian motion. Under Hörmander type conditions we show that the distribution of $X_t$ possesses a density with respect to the Lebesgue measure.

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1. Introduction

In this paper we study a so-called mixed stochastic differential equation (SDE) in $\mathbb{R}^d$

$$X_t = X_0 + \int_0^t a(s, X_s) \, ds + \int_0^t b(s, X_s) \, dW_s + \int_0^t c(s, X_s) \, dB^H_s \quad (1.1)$$
driven by a multidimensional standard Wiener process and a multidimensional fractional Brownian motion (fBm) with Hurst parameter $H \in (1/2, 1)$ (see next section for precise definitions). Recently such equations gained a lot of attention thanks to their modeling features. There is already a large literature devoted to them; the few papers we cite here give an extensive overview of existing results. The unique solvability result in the form suitable for our needs is obtained in the paper [9]; although the result is formulated there for equations with delay, it is a fortiori valid for usual equations. The paper [8] contains useful estimates of the solution and results on its integrability. Finally, we mention the paper [7], where the Malliavin differentiability of the solution is obtained.

The main aim of this article is to provide conditions under which the solution to (1.1) has a density with respect to the Lebesgue measure. For Itô SDEs, such issues were addressed by many authors, see [4] and references therein. Existence and regularity of density for SDEs driven by fBm we proved in [1, 6, 5] under Hörmander type conditions. The recent paper [2] contains a generalization of these
results to equations driven by Gaussian rough paths, in particular, it allows to deduce the existence of a smooth density of the solution to (1.1) with Stratonovich integral with respect to the Wiener process. However, the machinery used in that article is quite sophisticated, and here we use a more direct approach.

The paper is organised as follows. In Section 2 we introduce our notation, describe the main object and briefly discuss Malliavin calculus of variations for fractional Brownian motion. In Section 3, we prove that the distribution of the solution $X_t, t > 0$ possesses density w.r.t. Lebesgue measure under a simplified version of the Hörmander condition. Section 4 contains the result on existence and smoothness of the density under a strong version of the Hörmander condition. The Appendix contains some technical lemmas and the Norris lemma for a mixed SDE.

2. Preliminaries

2.1. Definitions and notation

Throughout the paper, $|\cdot|$ will denote the absolute value of a number, the Euclidean norm of a vector, and the operator norm of a matrix. $\langle \cdot, \cdot \rangle$ stays for the usual scalar product in the Euclidean space. We will use the symbol $C$ to denote a generic constant, whose value is not important and may change from one line to another. We will write a subscript if a constant is relevant or if its value depends on some parameters.

For a matrix $A = (a_{i,j})$ of arbitrary size, we denote by $a_i$ its $i$-th row and by $a_{\cdot,j}$ its $j$-th column.

The classes of continuous and $\theta$-Hölder continuous functions on $[a, b]$ will be denoted respectively by $C[a, b]$ and $C^\theta[a, b]$. For a function $f : [a, b] \to \mathbb{R}$ denote by $\|f\|_{\infty, [a, b]}$ its supremum norm and by

$$\|f\|_{\theta, [a, b]} = \sup_{a \leq s < t \leq b} \frac{|f(t) - f(s)|}{|t - s|^\theta}$$

its $\theta$-Hölder seminorm. If there is no ambiguity, we will use the notation $\|f\|_{\infty}$ and $\|f\|_{\theta}$.

Finally, for a function $h \in C(\mathbb{R}^d)$ denote by $\partial_x h = (\frac{\partial}{\partial x_1} h_1, \ldots, \frac{\partial}{\partial x_d} h_d)$ its gradient and by $\partial^2_{xx} h = (\frac{\partial^2}{\partial x_i \partial x_j} h)_{i,j=1,...,d}$ its second derivative matrix.

2.2. Main equation and assumptions

For a fixed time horizon $T > 0$, let $\{\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P}\}$ be a standard stochastic basis. Equation (1.1) is driven by two independent sources of randomness: an $m$-dimensional $\mathbb{F}$-Wiener process $\{W_t = (W^1_t, \ldots, W^m_t), t \in [0,T]\}$ and an $l$-dimensional fBm $\{B^H_t = (B^{H,1}_t, \ldots, B^{H,l}_t), t \geq [0,T]\}$ with Hurst index $H \in (1/2, 1)$, i.e. a centered Gaussian process having the covariance

$$\mathbb{E} \left[ B^{H,i}_t B^{H,j}_s \right] = \frac{\delta_{i,j}}{2} (t^{2H} + s^{2H} - |t-s|^{2H}).$$
It is well known that the fBm $B^H$ has a modification with $\gamma$-Hölder continuous path for any $\gamma < H$, in the following we will assume that the process itself is Hölder continuous.

Equation (1.1) is understood as a system of SDEs on $[0, T]$

$$X_t^i = X_0^i + \int_0^t a_i(s, X_s) \, ds + \sum_{j=1}^m \int_0^t b_{i,j}(s, X_s) \, dW_s^j + \sum_{k=1}^l \int_0^t c_{i,k}(s, X_s) \, dB_s^{H,k},$$

$i = 1, \ldots, d$, with a non-random initial condition $X_0 \in \mathbb{R}^d$. In this equation, the integral w.r.t. $W$ is understood in a usual Itô sense, the one w.r.t. $B^H$ is understood in a pathwise sense, as Young integral. More information on its definition and properties can be found in [3].

The coefficients $a_i, b_{i,j}, c_{i,k} : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $i = 1, \ldots, d$, $j = 1, \ldots, m$, $k = 1, \ldots, l$ are assumed to satisfy the following conditions:

A1 for all $t \in [0, T]$ \quad $a(t, \cdot), b(t, \cdot) \in C^1(\mathbb{R}^d)$, $c(t, \cdot) \in C^2(\mathbb{R}^d)$;
A2 for all $t \in [0, T], x \in \mathbb{R}^d$

$$|a(t,x)| + |b(t,x)| + |c(t,x)| \leq C(1 + |x|);$$
A3 for all $t \in [0, T], x \in \mathbb{R}^d$ \quad $|\partial_x c(t,x)| \leq C$;
A4 there exists $\beta > 0$ such that for all $t, s \in [0, T], x \in \mathbb{R}^d$

$$|c(t,x) - c(s,x)| \leq C|t-s|^\beta(1 + |x|), \quad |\partial_x c(t,x) - \partial_x c(s,x)| \leq C|t-s|^\beta.$$

The continuous differentiability implies that $a, b, \partial_x c$ are locally Lipschitz continuous. Therefore, by [9 Theorem 4.1], equation (1.1) has a unique solution which is Hölder continuous of any order $\theta \in (0, 1/2)$.

2.3. Ad hoc Malliavin calculus

Here we summarize some facts from the Malliavin calculus of variations, see [4] for a deeper exposition. Denote by $S[0, T]$ the set the of step functions of the form $f(t) = \sum_{k=1}^l c_k 1_{(a_k, b_k)}(t)$ defined on $[0, T]$. Let $L_H^2[0, T]$ denote the separable Hilbert space obtained by completing $S[0, T]$ w.r.t. the scalar product

$$\langle f, g \rangle_{L_H^2[0, T]} = \int_0^T \int_0^T f(t)g(s)\phi(t,s) \, dt \, ds,$$

where $\phi(t,s) = H(2H - 1)|t-s|^{2H-2}$.

Consider the product space

$$\mathcal{S} = (L_H^2[0, T])^l \times (L^2[0, T])^m.$$

It is also a separable Hilbert space with a scalar product

$$\langle f, g \rangle_{\mathcal{S}} = \sum_{i=1}^l \langle f_i, g_i \rangle_{L_H^2[0, T]} + \sum_{i=l+1}^{l+m} \langle f_i, g_i \rangle_{L^2[0, T]}.$$

The map

$$\mathcal{J} : (1_{[0,t_1]}, \ldots, 1_{[0,t_l]}, 1_{[0,s_1]}, \ldots, 1_{[0,s_m]}) \mapsto (B_{t_1}^{H,1}, \ldots, B_{t_l}^{H,1}, W_{s_1}^1, \ldots, W_{s_m}^m)$$

can be extended by linearity to $S[0, T]^{l+m}$. It appears that for $f, g \in S[0, T]^{l+m}$

$$\mathbb{E} [\langle \mathcal{J}(f), \mathcal{J}(g) \rangle] = \langle f, g \rangle_{\mathcal{S}},$$

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so \( \mathcal{I} \) can be extended to an isometry between \( \mathcal{H} \) and a subspace of \( L^2(\Omega; \mathbb{R}^{m+l}) \).

For \( \xi = F(\mathcal{I}(f_1), \ldots, \mathcal{I}(f_n)) \), where \( f_1, \ldots, f_n \in \mathcal{H} \) and \( f_i = (f_{i,1}, \ldots, f_{i,m+1}) \), \( i = 1, \ldots, n \), \( F : \mathbb{R}^{n(m+l)} \to \mathbb{R} \) is a continuously differentiable finitely supported function, define the Malliavin derivative \( D\xi \) as an element of \( \mathcal{H} \), whose \( j \)-th coordinate equal to

\[
\sum_{i=1}^{n} \partial_{(i-1)(l+m)+j} F(\mathcal{I}(f_1), \ldots, \mathcal{I}(f_n)) f_{i,j}, j = 1, \ldots, l + m.
\]

Denote for \( p \geq 1 \) by \( \mathbb{D}^{1,p} \) the closure of the space of smooth cylindrical random variables with respect to the norm

\[
\|\xi\|_{\mathbb{D}^{1,p}} = \mathbb{E} [ |\xi|^p + \|D\xi\|_{\mathcal{H}} ]^{1/p}.
\]

\( \mathbb{D} \) is closable in this space and its closure will be denoted likewise. Finally, the Malliavin derivative is a (possibly, generalized) function from \([0, T]\) to \( \mathbb{R}^{l+m} \), so we can introduce the notation

\[
D\xi = \left\{ D_t\xi = (D_{t}^{H,1}\xi, \ldots, D_{t}^{H,l}\xi, D_{t}^{W,1}\xi, \ldots, D_{t}^{W,m}\xi), t \in [0, T] \right\}.
\]

We say that \( \xi \in \mathbb{D}^{1,p}_{loc} \) if exists a sequence \( \{\xi_n(\omega), \Omega_n\}_{n \geq 1} \) such that \( \Omega_n \subset \Omega_{n+1} \) for \( n \geq 1 \), \( P(\Omega \setminus \bigcup_{n \geq 1} \Omega_n) = 0 \); \( \xi_n \in \mathbb{D}^{1,p} \) and \( \xi|_{\Omega_n} = \xi_n|_{\Omega_n} \) for all \( n \geq 1 \).

For the reader convenience we state here the theorem concerning the Mallivian differentiability of the solution to (1.1) in the case of SDE with non-homogeneous coefficients. The proof is similar to that of [7, Theorem 2]

**Theorem 2.1.** Suppose that coefficients \( a, b, c \) of (1.1) satisfy the assumptions

B1 for all \( t \in [0, T] \) \( a(t, \cdot), b(t, \cdot) \in C^1(\mathbb{R}^d) \), \( c(t, \cdot) \in C^2(\mathbb{R}^d) \);

B2 \( a, b, \partial_{x}a, \partial_{x}b, \partial_{x}c, \partial_{x}^2c \) are bounded;

B3 there exists \( \beta > 0 \) such that for all \( t, s \in [0, T], x \in \mathbb{R}^d \)

\[
|c(t,x) - c(s,x)| \leq C|t-s|^\beta (1 + |x|), \quad |\partial_{x}c(t,x) - \partial_{x}c(s,x)| \leq C|t-s|^\beta.
\]

Then \( X_t \in \bigcap_{p \geq 1} \mathbb{D}^{1,p} \).

### 3. Existence of density under simplified Hörmander condition

In this section we prove that a solution to (1.1) possesses density of a distribution under a quite strong condition, which we call a simplified Hörmander condition. More precisely, we will assume in this section that the coefficients of (1.1) satisfy

\[
\text{span}\{c, \kappa_1(0, X_0), b, \kappa_2(0, X_0) \mid 1 \leq k \leq l, 1 \leq j \leq m \} = \mathbb{R}^d. \tag{3.1}
\]

The first step to establish the existence of density is to show the (local) Malliavin differentiability of the solution to (1.1).

**Theorem 3.1.** If the coefficients of (1.1) satisfy the assumptions A1–A4, then \( X_t \in \bigcap_{p \geq 1} \mathbb{D}^{1,p}_{loc} \).

**Proof.** Define \( \Omega_n = \{ \omega : \|X(\omega)\|_{\infty, [0, t]} < n \}, n \geq 1 \). Obviously, \( \Omega_n \subset \Omega_{n+1}, n \geq 1 \) and, since \( \|X(\omega)\|_{\infty, [0, t]} < \infty \) a.s., \( \bigcup_{n \geq 1} \Omega_n = \Omega \). Consider a smooth function \( \psi = \psi(x), x \in \mathbb{R} \) such that
Define the matrix-valued process absolutely continuous with respect to the Lebesgue measure in

Proof. of the Mallivain covariation matrix 2.1.2]) and thanks to the previous theorem, it is enough to verify the non-degeneracy and the simplified Hörmander condition

\[ J_{t} \]

where

\[ J(t) = \sum_{k=1}^{m} \int_{0}^{t} \frac{\partial b_{i,k}(s,X_s)}{\partial x_r} J_{s,0}(r,j) \, ds + \sum_{q=1}^{l} \int_{0}^{t} \frac{\partial c_{i,q}(s,X_s)}{\partial x_r} J_{s,0}(r,j) \, dB_{s,q}^{H}. \]

Since the functions \(a_n,b_n,c_n\) satisfy assumptions B1–B3, in view of Theorem 2.1, \(X_i \in \mathbb{D}^{1,p}\). It is not hard to see that \(X_i(\omega) = X_t(\omega)\) for \(\omega \in \Omega_n\), which concludes the proof. \(\square\)

Now we are to prove the main result of this section.

**Theorem 3.2.** Suppose that the coefficients of (1.1) satisfy assumptions A1–A4 and the simplified Hörmander condition (3.1). Then for all \(t > 0\) the law of \(X_t\) is absolutely continuous with respect to the Lebesgue measure in \(\mathbb{R}^d\).

**Proof.** By the classical condition for existence of density (see e.g. [4, Theorem 2.1.2]) and thanks to the previous theorem, it is enough to verify the non-degeneracy of the Mallivain covariation matrix \(M(t) = (M_{i,j}(t))_{i,j=1,...,d}\) with \(M_{i,j}(t) = \langle DX_i^t, DX_j^t \rangle_{\mathcal{F}}\). Define the matrix-valued process \(J_{t,0} = (J_{t,0}(i,j))_{i,j=1,...,d}\) as the solution to

\[ J_{t,0}(i,j) = \delta_{i,j} + \sum_{r=1}^{d} \left[ \int_{0}^{t} \frac{\partial a_{i,j}(s,X_s)}{\partial x_r} J_{s,0}(r,j) \, ds \right. \]

\[ + \sum_{k=1}^{m} \int_{0}^{t} \frac{\partial b_{i,k}(s,X_s)}{\partial x_r} J_{s,0}(r,j) \, dB_{s,k}^{H} + \sum_{q=1}^{l} \int_{0}^{t} \frac{\partial c_{i,q}(s,X_s)}{\partial x_r} J_{s,0}(r,j) \, dB_{s,q}^{H}. \]

where \(\delta_{i,j} = 1_{i=j}\) is the Kronecker delta. The system above is linear, hence, possesses a unique solution. In view of Lemma [A.1], \(J_{t,0}\) is non-degenerate; denoting \(J_{t,s} = J_{t,0}^{-1} J_{s,0}^{1}\) and applying Lemma [A.2] one can write

\[ M(t) = \sum_{k=1}^{m} \int_{0}^{t} (J_{t,s}^{-1} b_{i,k}(s,X_s)) (J_{t,s}^{-1} b_{i,k}(s,X_s))' \, ds \]

\[ + \sum_{q=1}^{l} \int_{0}^{t} \int_{0}^{t} \phi(s,u) (J_{t,s}^{-1} c_{i,q}(s,X_s)) (J_{t,u}^{-1} c_{i,q}(u,X_u))' \, ds \, du = J_{t,0} C_{t} J_{t,0}^{1}, \]

where

\[ C_{t} = \sum_{k=1}^{m} \int_{0}^{t} (J_{s,0}^{-1} b_{i,k}(s,X_s)) (J_{s,0}^{-1} b_{i,k}(s,X_s))' \, ds \]

\[ + \sum_{q=1}^{l} \int_{0}^{t} \int_{0}^{t} \phi(s,u) (J_{s,0}^{-1} c_{i,q}(s,X_s)) (J_{u,0}^{-1} c_{i,q}(u,X_u))' \, ds \, du. \]

Again, due to the invertibility of \(J_{t,0}\), \(M_t\) is invertible if and only if so is \(C_t\). Assuming the contrary, there exists a non-zero vector \(v \in \mathbb{R}^d\) such that \(\nu' C_{t} v = 0\). Write

\[ \nu' C_{t} v = \sum_{k=1}^{m} \| \langle J_{t,0} b_{i,k}(\cdot,X_s), v \rangle \|_{L^2[0,t]}^2 + \sum_{q=1}^{l} \| \langle J_{t,0} c_{i,q}(\cdot,X_s), v \rangle \|_{L^2[0,t]}^2. \]
Since the functions
\[ s \mapsto \langle J_{s,t}^{-1} b, V(s, X_s), \rangle, \quad k = 1, \ldots, m, \]
\[ s \mapsto \langle J_{s,t}^{-1} c, q(s, X_s), \rangle, \quad q = 1, \ldots, l \]
are continuous, they must be equal zero for all \( s \in [0, t] \). For \( s = 0 \) we get
\[
\sum_{i=1}^{d} b_{i,k}(0, X_0) v_i = 0, k = 1, \ldots, m;
\]
\[
\sum_{i=1}^{d} c_{i,q}(0, X_0) v_i = 0, q = 1, \ldots, l.
\]
This, however, contradicts the assumption \( (3.1) \). Consequently, \( M_t \) is invertible, as required.

\[ \square \]

4. Existence of density under strong Hörmander condition

In this section we consider a homogeneous version of \( (1.1) \):

\[ X_t = X_0 + \int_0^t a(X_s) \, ds + \int_0^t b(X_s) \, dW_s + \int_0^t c(X_s) \, dB^H_s. \]  \hspace{1cm} (4.1)

In this section we assume that Hurst index \( H \in (1/2, 2/3) \), and some \( \theta \in ((H - 1/2)/(3 - 4H), 1/2) \) is fixed. The role of the restriction \( \theta > (H - 1/2)/(3 - 4H) \) will become clear in the proof of the Norris lemma for \( (4.1) \) (Lemma A.5). Now we just remark that the expression \( (H - 1/2)/(3 - 4H) \) is increasing for \( H \in (1/2, 3/4) \) and is equal to 1/2 for \( H = 2/3 \), so the upper bound \( H < 2/3 \) arises naturally.

We impose the following condition on the coefficients of \( (4.1) \):

\[ C1 \; a, b, c \in C^\infty_p(\mathbb{R}^d) \] with all derivatives bounded.

Under this assumption the solution is infinitely differentiable in the Malliavin sense: \( X_t \in \bigcap_{k,p=1}^{\infty} \mathbb{D}^{k,p} = \mathbb{D}^\infty \), which can be shown similarly to its differentiability under B1–B3.

The aim of this section is to investigate the existence of a density and properties of this density of a distribution of \( X_t \) under the strong Hörmander condition, which reads as follows.

Set \( V_0 = a, V_j(\cdot) = b_{j, \cdot} \) for \( j = 1, \ldots, m \) and \( V_{j+m}(\cdot) = c_{j, \cdot} \) for \( j = 1, \ldots, l \).

Using the Lie bracket \([\cdot, \cdot]\), define the set
\[ Y_k = \{ [V_{i_1}, \ldots, \{ V_{i_{k-1}}, V_{i_k} \}, \ldots], (i_1, \ldots, i_k) \in \{ 1, \ldots, d \}^k \}. \]

It is said that the vector field \( Y_0 = \{ V_j \}_{j=1, \ldots, m+l} \) satisfies the Hörmander condition at the point \( X_0 \), if for some positive integer \( n_0 \) one has
\[ \text{span} \left\{ V(X_0), V \in \bigcup_{k=1}^{n_0} Y_k \right\} = \mathbb{R}^d. \] \hspace{1cm} (4.2)

The main result of this section is the following theorem.
Theorem 4.1. Assume that coefficients of (4.1) satisfy assumption C1 and the Hörmander condition (4.2). Then the law of $X_t$ for all $t > 0$ possesses a smooth density with respect to the Lebesgue measure in $\mathbb{R}^d$.

Proof. Using the usual condition for existence of a smooth density (see e.g. [4 Theorem 2.1.4]) and taking into account that all moments of the Jacobian $J_{1,s}$ are finite, it is enough to show that the matrix inverse to the reduced Malliavin covariance matrix of $X_t$ possesses all moments.

Recall from Theorem 3.2 that the reduced Malliavin covariance matrix of the solution to (4.1) can be written as

$$C(t) = \sum_{k=1}^{m} \int_0^t \left( J_{s,0}^{-1} b_{s,k}(X_s) (J_{s,0}^{-1} b_{s,k}(X_s))' \right) \, ds$$

To simplify the notation, we assume from now that $t = 1$. We are to prove that $E[|\det C_t|^{-p}] < \infty$ for all $p \geq 1$. Due to [4 Lemma 2.3.1] it suffices to prove that the entries of $C_t$ possess all moments and for any $p \geq 2$ there exists $C_p$ such that for all $\varepsilon > 0$ it holds

$$\sup_{\|v\|=1} P \left\{ \langle v, C_1 v \rangle \leq \varepsilon \right\} \leq C_p \varepsilon^p.$$

Write

$$\langle v, C_1 v \rangle = \sum_{k=1}^{m} \left\| \left( J_{s,0}^{-1} b_{s,k}(X_s), v \right) \right\|_{L^2[0,1]}^2 + \sum_{q=1}^{l} \left\| \left( J_{s,0}^{-1} c_{s,q}(X_s), v \right) \right\|_{L^2_H[0,1]}^2.$$

It is well known that $\|f\|_{L^2_H[0,1]} \leq \|f\|_{L^2[0,1]}$. Therefore,

$$\langle v, C_1 v \rangle \geq C \sum_{k=1}^{m} \|G_k\|_{L^2_H[0,1]}, \text{ where } G_k = \left( J_{s,0}^{-1} V_k(X_s), v \right).$$

Applying [1 Lemma 4.4] we get that

$$\langle v, C_1 v \rangle \geq C \sum_{k=1}^{m} \|G_k\|_{L^2_H[0,1]}^{2(3+1/\theta)} \|G_k\|_{\theta}^{2(2+1/\theta)}$$

for $\theta > H - 1/2$. Thus,

$$P \left\{ \langle v, C_1 v \rangle \leq \varepsilon \right\} \leq P \left\{ C \sum_{k=1}^{m} \|G_k\|_{L^2_H[0,1]}^{2(3+1/\theta)} \|G_k\|_{\theta}^{2(2+1/\theta)} \leq \varepsilon \right\}.$$

From [1 Lemma 4.5] and Theorem A.5 we obtain the following estimate

$$P \left\{ C \sum_{k=1}^{m} \|G_k\|_{L^2_H[0,1]}^{2(3+1/\theta)} \|G_k\|_{\theta}^{2(2+1/\theta)} \leq \varepsilon \right\} \leq C \varepsilon^p + \min_{k=1,\ldots,m+l} P \left\{ \|\langle v, J_{0} V_k(X_s) \rangle \|_{\infty} \leq \varepsilon^\alpha \right\}.$$
Now let $V$ be a bounded vector field with bounded derivatives of all order. The chain rule implies

$$J_{t,0}^{-1}V(X_t) = V(X_0) + \int_0^t J_{s,0}^{-1}([V_0, V] + \frac{1}{2} \sum_{k=1}^{m+l} [V_k, [V, V]])(X_s) \, ds$$

$$+ \sum_{k=1}^m \int_0^t J_{s,0}^{-1}[V_k, V](X_s) \, dW_s + \sum_{k=m+1}^{l+m} \int_0^t J_{s,0}^{-1}[V_k, V](X_s) \, dB_{s}^{H}.$$ 

Thus, applying Theorem A.5 once more, we obtain

$$P \left\{ \| \langle v, J_{0}V(X_{t}) \rangle \|_{\infty} < \varepsilon \right\} \leq C \varepsilon^{p} + \frac{1}{m} P \left\{ \left\| \langle v, J_{0}^{-1}[V_{k}, V](X_{t}) \rangle \right\|_{\infty} \leq \varepsilon^{q} \right\}.$$ 

Let $n_0$ be the integer from the Hörmander condition. Iterating our consideration above, we obtain

$$P \left\{ \langle v, C_{1}v \rangle \leq \varepsilon \right\} \leq C \varepsilon^{p} + \min_{V \in \bigcup_{k=1}^{n_0} Y_k} P \left\{ \left\| \langle v, J_{0}^{-1}V(X_{t}) \rangle \right\|_{\infty} \leq \varepsilon^{q} \right\}$$

for all $\varepsilon$ small enough. Since $\{V(x_0), V \in \bigcup_{k=1}^{n_0} Y_k \}$ spans $\mathbb{R}^d$, there exists $v$ such that $\langle v, V(x_0) \rangle \neq 0$. Hence, there exists $\varepsilon_0(p)$ such that for all $\varepsilon < \varepsilon_0(p)$ the second term vanishes. As a result,

$$P \left\{ \langle v, C_{1}v \rangle \leq \varepsilon \right\} \leq C_p \varepsilon^{p}$$

for all $\varepsilon \leq \varepsilon_0(p)$, as required. \hfill \Box

Appendix A. Technical lemmas

The following two lemmas concern the Jacobian of the flow generated by the solution $X$ to equation (1.1). These are quite standard facts, so we just sketch the proofs.

**Lemma A.1.** Under assumptions A1–A4 the matrix valued process $J_{t,0} = (J_{t,0}(i, j))_{i,j=1,...,d}$ given by (3.2) has an inverse $Z_{t,0} = (Z_{t,0}(i, j))_{i,j=1,...,d}$ for all $t > 0$. Moreover, $\{Z_{t,0}, t \geq 0\}$ satisfies the following system of equations

$$Z_{t,0}(i, j) = \delta(i, j) - \sum_{r=1}^{d} \left[ \int_0^t \frac{\partial a_{r}}{\partial x_{j}}(s, X_{s})Z_{s,0}(r, j) \, ds\right.$$

$$- \sum_{k=1}^{m} \int_0^t \frac{\partial b_{r,k}}{\partial x_{j}}(s, X_{s})Z_{s,0}(i, r) \, dW_{s}^{k} - \sum_{q=1}^{l} \int_0^t \frac{\partial c_{r,q}}{\partial x_{j}}(s, X_{s})Z_{s,0}(i, r) \, dB_{s}^{H,q} \left. + \frac{1}{2} \sum_{u=1}^{d} \sum_{v=1}^{d} \int_0^t \frac{\partial b_{r,u}}{\partial x_{u}}(s, X_{s}) \frac{\partial b_{r,v}}{\partial x_{v}}(s, X_{s})Z_{s,0}(i, r) \, ds \right].$$

**Proof.** The equation (A.1) is linear, thus possesses a unique solution $Z_{t,0}$. So we need to verify that $Z_{t,0}J_{t,0} = J_{t,0}Z_{t,0} = I_{d},$ the identity matrix. The equality clearly holds for $t = 0$. To show it for $t > 0$, it is enough to show that the differentials of $Z_{t,0}J_{t,0}$ and of $J_{t,0}Z_{t,0}$ vanish. But this can be routinely checked using the Itô formula. \hfill \Box

Denote for $t \geq s$ $J_{t,s} = J_{t,0}^{-1}$. 

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Lemma A.2. Under assumptions A1–A4, the Malliavin derivatives of the solution to (1.1) are given by

$$D^{W,k}_s X_t = J_{t,s} b_{.,k}(s,X_s) \mathbf{1}_{s \leq t}, \ k = 1, \ldots, m, \quad (A.2)$$

$$D^{H,q}_s X_t = J_{t,s} c_{.,q}(s,X_s) \mathbf{1}_{s \leq t}, \ q = 1, \ldots, l. \quad (A.3)$$

Proof. The argument is exactly the same for both equations, so we prove only (A.2). Evidently, $D^{W,k}_s X_t = 0$ for $s > t$, so suppose that $s \leq t$. Due to the closedness of the derivative, we can freely differentiate (1.1) as if the integrals were finite sums, in particular, using the chain rule, we can write for $i = 1, \ldots, d$

$$D^{W,k}_s \int_0^t a_i(u,X_u) \, du = \int_0^t D^{W,k}_s a_i(u,X_u) \, du = \sum_{r=1}^d \int_s^t \frac{\partial}{\partial x_r} a_i(u,X_u) D^{W,k}_s X^r_u \, du$$

and similarly

$$D^{W,k}_s \int_0^t c_{i,q}(u,X_u) \, dB^H_{u,q} = \sum_{r=1}^d \int_s^t \frac{\partial}{\partial x_r} c_{i,q}(u,X_u) D^{W,k}_s X^r_u \, dB^H_{u,q}, \ q = 1, \ldots, l,$$

$$D^{W,k}_s \int_0^t b_{i,j}(u,X_u) \, dW^j_u = \sum_{r=1}^d \int_s^t \frac{\partial}{\partial x_r} b_{i,j}(u,X_u) D^{W,k}_s X^r_u \, dW^j_u, \ j = 1, \ldots, m, \ j \neq k.$$

To differentiate the integral w.r.t. $W^k$, approximate it by an integral sum and note that we will have an extra term corresponding to the derivative of the increment of $W^k$ on the interval containing $s$. Passing to the limit, we get

$$D^{W,k}_s \int_0^t b_{i,k}(u,X_u) \, dW^k_u = b_{i,k}(s,X_s) + \sum_{r=1}^d \int_s^t \frac{\partial}{\partial x_r} b_{i,k}(u,X_u) D^{W,k}_s X^r_u \, dW^k_u.$$

Therefore, we have for $s \leq t$ the following linear equation on $D^{W,k}_s X_u$:

$$D^{W,k}_s X_t = b_{.,k}(s,X_s) + \sum_{r=1}^d \left[ \int_s^t \frac{\partial}{\partial x_r} a(u,X_u) D^{W,k}_s X^r_u \, du + \sum_{q=1}^l \int_s^t \frac{\partial}{\partial x_r} c_{.,q}(u,X_u) D^{W,k}_s X^r_u \, dB^H_{u,q} + \sum_{j=1}^m \int_s^t \frac{\partial}{\partial x_r} b_{.,j}(u,X_u) D^{W,k}_s X^r_u \, dW^j_u \right].$$

On the other hand, from (3.2) we can write

$$J_{t,0} = J_{s,0} + \sum_{r=1}^d \left[ \int_s^t \frac{\partial a_i}{\partial x_r}(u,X_u) J_{u,0} \, du + \sum_{k=1}^m \int_s^t \frac{\partial b_{i,k}}{\partial x_r}(s,X_s) J_{u,0} \, dW^k_u + \sum_{q=1}^l \int_s^t \frac{\partial c_{i,q}}{\partial x_r}(s,X_s) J_{u,0} \, dB^H_{u,q} \right],$$

which, upon multiplying by $J_{s,0}^{-1} b_{.,k}(s,X_s)$ on the right leads to the same equation on $J_{u,0} b_{.,k}(s,X_s)$ as that on $D^{W,k}_s X^r_u$. Hence, by uniqueness, we get the desired result.

Further we establish a simple estimate on the Itô integral of a Hölder continuous integrand.
Lemma A.3. Let \( \{ f(t), t \in [0, T] \} \) be an \( \mathcal{F} \)-adapted stochastic process such that \( \mathbb{E} \left[ \| f \|_{p}^{p} \right] < \infty \) for all \( p \geq 1 \), and \( 0 < \delta < \Delta \leq T \). Then for all \( s, t, u \in [0, T] \) such that \( u < s < t, t - s < \delta, t - u \leq \Delta \) it holds

\[
\left| \int_{s}^{t} (f(v) - f(u)) \, dW_{v} \right| \leq \Delta^{\theta} \delta^{1/2} \xi_{\Delta, \delta},
\]

where \( \mathbb{E} \left[ \xi_{\Delta, \delta}^{p} \right] < C_{p} \mathbb{E} \left[ \| f \|_{\Theta}^{p} \right] \) for all \( p \geq 1 \).

**Proof.** It suffices to establish the required result for \( p \) large enough, then one can get deduce it for all \( p \geq 1 \) with the help of Jensen’s inequality.

By the Garsia–Rodemich–Rumsey inequality, we get

\[
\left| \int_{s}^{t} (f(v) - f(u)) \, dW_{v} \right| \leq C |t - s|^{1/4} \left( \int_{s}^{t} \int_{s}^{t} \frac{|f_{x}^{y} (f(v) - f(u)) \, dW_{v}|^{8}}{|x - y|^{4}} \, dx \, dy \right)^{1/8}
\]

\[
\leq C \Delta^{\theta} \delta^{1/2} \xi_{\Delta, \delta},
\]

where

\[
\xi_{\Delta, \delta} = \Delta^{-\theta} \delta^{-1/4} \left( \int_{s}^{t} \int_{s}^{t} \frac{|f_{x}^{y} (f(v) - f(u)) \, dW_{v}|^{8}}{|x - y|^{4}} \, dx \, dy \right)^{1/8}.
\]

For \( p > 8 \) the Hölder inequality entails that

\[
\mathbb{E} \left[ \xi_{\Delta, \delta}^{p} \right] \leq \Delta^{-\theta p} \delta^{-p/4} (t - s)^{2(p/8 - 1)} \int_{s}^{t} \int_{s}^{t} \mathbb{E} \left[ \frac{|f_{x}^{y} (f(v) - f(u)) \, dW_{v}|^{p}}{|x - y|^{p/2}} \right] \, dx \, dy
\]

\[
\leq C_{p} \Delta^{-\theta p} \delta^{-2} \int_{s}^{t} \int_{s}^{t} \mathbb{E} \left[ \frac{(f_{x}^{y} |f(v) - f(u)|^{2} \, dv)^{p/2}}{|x - y|^{p/2}} \right] \, dx \, dy
\]

\[
\leq C_{p} \Delta^{-\theta p} \delta^{-2} \mathbb{E} \left[ \| f \|_{\Theta}^{p} \right] \Delta^{	heta} \delta^{2} = C_{p} \mathbb{E} \left[ \| f \|_{\Theta}^{p} \right].
\]

Hence, we arrive at the desired statement. \( \square \)

We also need the result concerning the pathwise regularity property of \( X \). It establishes certain exponential integrability of the Hölder seminorm of \( X \), so it is an interesting result on its own.

Theorem A.4. Let \( \{ X_{t}, t \in [0, T] \} \) be the solution to (4.1). Assume that \( a, b, c \) satisfy the assumption C1. Then \( X \in C^{\theta} [0, T] \) for \( \theta \in (0, 1/2) \) and \( \mathbb{E} \left[ \exp \left\{ K \| X \|_{\Theta}^{q} \right\} \right] < \infty \) for all \( q \in (0, q^{*}), K > 0 \), where

\[
q^{*} = \frac{4H}{2(H + \theta) + 1} \wedge \frac{2H + 1}{4H}.
\]

In particular, \( \mathbb{E} \left[ \| X \|_{\Theta}^{p} \right] < \infty \) for all \( p > 0 \).

**Proof.** Define for \( \varepsilon \in (0, T] \)

\[
\| X \|_{\Theta, \varepsilon} = \sup_{0 \leq t - \varepsilon \leq s < t \leq T} \frac{|X_{t} - X_{s}|}{(t - s)^{\theta}}.
\]
Evidently, \( \|X\|_\theta \leq \|X\|_{\theta,e} + 2\varepsilon^{-\theta} \|X\|_\infty \). It follows from [8] equation (4) that
\[
\|X\|_{\theta,e} \leq C_1 \left( \|r^b\|_{\theta} + \Lambda_\mu \left( 1 + \|X\|_\infty e^{\mu - \theta} \right) \right),
\]
for any \( \varepsilon \in (0,C_2\Lambda_\mu^{-1/\mu}] \) where \( C_1,C_2 \) are some positive constants, \( \mu \in (1/2,H) \), \( \Lambda_\mu = \|B_H\|_\mu + 1 \), \( r^b = \int_0^t b(X_s) \, dW_s \). Therefore, setting \( \varepsilon = C_2\Lambda_\mu^{-1/\mu} \), we obtain
\[
\|X\|_{\theta} \leq C_1 \left( \|r^b\|_{\theta} + \Lambda_\mu + 2 \|X\|_\infty \Lambda^{\theta/\mu} \right) \leq C \left( \|r^b\|_{\theta} + \Lambda_\mu + \|X\|_\infty' + \Lambda q'\theta/\mu \right),
\]
where \( p' > 1 \), and \( q' = p'/ (p' - 1) \) is the exponent conjugate to \( p' \). Therefore,
\[
\|X\|_\theta' \leq C \left( \|r^b\|_{\theta}' + \Lambda_\mu' + \|X\|_\infty' q' + \Lambda q'\theta/\mu \right).
\]
Evidently, \( q^* < 1 \), so it follows from [7] Lemma 1 that \( \mathbb{E} \left[ \exp \left\{ K \|r^b\|_{\theta}' \right\} \right] < \infty \) for all \( K > 0 \). Further, \( \Lambda_\mu \) is an almost surely finite supremum of a centered Gaussian family, so by Fernique’s theorem \( \mathbb{E} \left[ \exp \left\{ K\Lambda_\mu^z \right\} \right] < \infty \) for any \( K > 0 \), \( z \in (0,2) \).

Finally, by [8] Corollary 4, \( \mathbb{E} \left[ \exp \left\{ K \|X\|_\infty^z \right\} \right] < \infty \) for all \( K > 0 \), \( z < 4H/(2H + 1) \).

Now if \( p' > 1 \) is close to \( 4Hq^{-1}(2H + 1)^{-1} \) (thanks to the bound on \( q \) such choice is possible) and \( \mu \) is close to \( H \), then \( q' \) is close to \( 4H/(4H - q(2H + 1)) \), and \( qq'\theta/\mu \) is close to \( 4q\theta/(4H - q(2H + 1)) \), which is less than 2. Indeed, the last statement is equivalent to \( q(2\theta + 2H + 1) < 4H \), which is true thanks to the restriction on \( q \).

Thus, we get the desired integrability. \( \square \)

The following result is a Norris type lemma for mixed SDEs. It is a crucial result to prove existence of density under the Hörmander condition. Loosely speaking, this statement says that if \( \Delta \) is close to \( 1 \), then \( \varepsilon \) is close to \( 1 \).

**Theorem A.5.** Assume that \( H \in (1/2,2/3) \), \( \theta \in (\theta_*,1/2) \), where
\[
\theta_* = \frac{H - \frac{1}{2}}{3 - 4H}
\]
and that \( a,b,c \) in (A.4) are \( \mathcal{F} \)-adapted processes satisfying \( \mathbb{E} \left[ \|a\|_\infty^p + \|b\|_\theta^p + \|c\|_\theta^p \right] < \infty \) for all \( p \geq 1 \). Then exists \( q > 0 \) such that for all \( p \geq 1 \), \( \varepsilon > 0 \)
\[
P \left\{ \|Y\|_\infty < \varepsilon \text{ and } \|b\|_\infty + \|c\|_\infty > \varepsilon^q \right\} \leq C_p e^{p\varepsilon^q}.
\]

**Proof.** Here we imitate the proof of in [1] Proposition 3.4. For notational simplicity, we assume that \( T = 1 \). For some positive integers \( M \) and \( r \) denote \( \Delta = 1/M \), \( \delta = \Delta/r \) and define the following uniform partitions of \([0,1]\): \( T_N = N\delta \), \( N = 0, \ldots, M \); \( t_n = \ldots \).
Further, fix some $\tilde{H} \in (1/2,H)$ and write for $N = 0, \ldots, M - 1$, $n = Nr, \ldots, (N+1)r - 1$ (so that $t_n \in [T_n, T_{n+1})$, $i = 1, \ldots, d$

$$\langle c_i(T_N), B^H_{n+1} - B^H_n \rangle + \langle b_i(T_N), W_{n+1} - W_n \rangle \leq |Y^i_{n+1} - Y^i_n| + \delta \|a\|_\infty$$

$$= \sum_{i=1}^{l} \langle c_i(T_N), B^H_{n+1} - B^H_n \rangle + \sum_{i=1}^{l} \langle b_i(T_N), dW^i_s \rangle \bigg|_{t_n}^{t_{n+1}}$$

$$\leq 2 \|Y\|_\infty + \delta \|a\|_\infty + C\Delta^\theta \delta^\tilde{H} \|c_i\|_{0} \|B^H\|_{\tilde{H}} + \Delta^\theta \delta^{1/2} \xi_{n+1} \xi_{n} =: S,$$

where in the last step we have used the Young–Love inequality (see e.g. [5, Proposition 1]) and Lemma A.3.

For processes $\xi, \zeta$ denote

$$V_N(\xi, \zeta) = \sum_{n=0}^{N-1} (\xi_{n+1} - \xi_{n}) (\zeta_{n+1} - \zeta_{n});$$

we remind that the summation is in fact over $t_n \in [T_n, T_{n+1})$. Squaring the both sides of (A.5), summing over $n = Nr, \ldots, (N+1)r - 1$ and then taking the square root we get

$$\left( \sum_{u,v=1}^{m} b_{i,u}(T_N) b_{i,v}(T_N) V_N(W^u, W^v) + \sum_{u,v=1}^{l} c_{i,u}(T_N) c_{i,v}(T_N) V_N(B^H,u, B^H,v) \right)^{1/2} \leq C\Delta^{1/2} \delta^{1/2} S,$$

Therefore,

$$\sum_{u=1}^{m} |b_{i,u}(T_N)| V_N(W^u, W^u)^{1/2} + \sum_{v=1}^{l} |c_{i,v}(T_N)| V_N(B^H,v, B^H,v) \leq C \left( \sum_{1 \leq u < v \leq m} |b_{i,u}(T_N)|^{1/2} |b_{i,v}(T_N)|^{1/2} V_N(W^u, W^v)^{1/2} \right.$$  

$$+ \sum_{1 \leq u \leq v \leq l} |c_{i,u}(T_N)|^{1/2} |c_{i,v}(T_N)|^{1/2} V_N(B^H,u, B^H,v)^{1/2}$$  

$$+ \sum_{u=1}^{m} \sum_{v=1}^{l} |b_{i,u}(T_N)|^{1/2} |c_{i,v}(T_N)|^{1/2} V_N(W^u, B^H,v)^{1/2} + \Delta^{1/2} \delta^{1/2} S \right).$$

Further, for arbitrary $f \in C^0[0,1]$,

$$\left| \Delta \sum_{N=0}^{M-1} |f(T_N) - f(T_{N+1})| \right| \leq \|f\|_{0} \Delta^{1/2},$$
which yields

\[
\sum_{u=1}^{m} \| b_{i,u} \|_{L^1[0,1]} + \sum_{v=1}^{l} \| c_{i,v} \|_{L^1[0,1]} \leq \sum_{u=1}^{m} \left( \Delta^\theta \| b_{i,u} \|_\theta + \Delta \sum_{N=0}^{M-1} \| b_{i,u}(T_N) \| \right) + \sum_{v=1}^{l} \left( \Delta^\theta \| c_{i,v} \|_\theta + \Delta \sum_{N=0}^{M-1} \| c_{i,v}(T_N) \| \right)
\]

\[
\leq \sum_{u=1}^{m} \left( \Delta^\theta \| b \|_\theta + \Delta^{1/2} \| b \|_\infty \sum_{N=0}^{M-1} \| \Delta^{1/2} - V_N(W^u, W^v)^{1/2} \| \right) + \Delta^{1/2} \Delta^{1/2-H} \sum_{N=0}^{M-1} \| b_{i,u}(T_N) \| V_N(W^u, W^v)^{1/2}
\]

\[
+ \sum_{v=1}^{l} \left( \Delta^\theta \| c \|_\theta + \Delta^{1/2} \Delta^{1/2-H} \| c \|_\infty \sum_{N=0}^{M-1} \| \Delta^{1/2} \Delta^{H-1/2} - V_N(B^{H,u}, B^{H,v})^{1/2} \| \right) + \Delta^{1/2} \Delta^{1/2-H} \sum_{N=0}^{M-1} \| c_{i,v}(T_N) \| V_N(B^{H,u}, B^{H,v})^{1/2}
\]

Therefore, using (A.9), we arrive at

\[
\| b \|_{L^1[0,1]} + \| c \|_{L^1[0,1]} \leq C \left( \Delta^\theta \left( \| b \|_\theta + \| c \|_\theta \right) + \Delta^{1/2} \Delta^{1/2-H} \| b \|_\infty \sum_{N=0}^{M-1} \sum_{u,v=1}^{m} \| \Delta^{1/2} \Delta^{H-1/2} \| \right) V_N(W^u, W^v)^{1/2} + \Delta^{1/2} \Delta^{1/2-H} \sum_{N=0}^{M-1} \sum_{u,v=1}^{m} \| \Delta^{1/2} \Delta^{H-1/2} \| \right)
\]

\[
+ \Delta^\theta \| c \|_\theta + \Delta^{1/4} \Delta^{3/4-H} \| b \|_\infty R^W + \Delta^{1/4} \Delta^{(1-H)/2} \| c \|_\infty + \| b \|_\infty R^{W,B} + \delta^{-H} S
\]

where

\[
R^W = \Delta^{3/4} \Delta^{1/4} \sum_{N=0}^{M-1} \sum_{u,v=1}^{m} \| \Delta^{1/2} \Delta^{H-1/2} \| \right) V_N(W^u, W^v)^{1/2},
\]

\[
R^B = \Delta^{H-3/2} \Delta^{1/2} \sum_{N=0}^{M-1} \sum_{u,v=1}^{l} \| \Delta^{1/2} \Delta^{H-1/2} \| \right) V_N(B^{H,u}, B^{H,v})^{1/2},
\]

\[
R^{W,B} = \Delta^{3/4} \Delta^{H-2/2} \sum_{N=0}^{M-1} \sum_{u,v=1}^{m} \| \Delta^{1/2} \Delta^{H-1/2} \| \right) V_N(W^u, B^{H,v})^{1/2}.
\]

(A.10)
Further we use the following interpolation inequality, valid for any \( f \in C^\theta[0,1] \) and \( \gamma < 1 \):
\[
\|f\|_\infty \leq C\left( \gamma \|f\|_\theta + \gamma^{-1/\theta} \|f\|_{L^1[0,1]} \right).
\]
for any \( \gamma \leq 1 \). Thus,
\[
\|b\|_\infty + \|c\|_\infty \leq C\left( \|b\|_\theta + \|c\|_\theta \right) \gamma + C\gamma^{-1/\theta} \left( \|b\|_\theta + \|c\|_\theta \right) \Delta^\theta
\]
\[
+ \left( \|b\|_\infty + \|c\|_\infty \right) \left[ \Delta^{-1/4} \delta^{3/4-H} R^W + \Delta^{H-1} \delta^{1-H} R^B + \Delta^{-1/4} \delta^{(1-H)/2} R^{W,B} \right]
\]
\[
+ \delta^{-H} \|Y\|_\infty + \delta^{1-H} \|a\|_\infty + \Delta^\theta \delta^{H-\theta} \|c\|_\theta \|B^H\|_B + \Delta^\theta \delta^{1/2-H} \xi_{\Delta,\delta}\right).
\]
(A.11)
Now we want to put
\[\Delta^\beta \sim \varepsilon^\beta, \delta \sim \varepsilon^\alpha, \gamma \sim \varepsilon^\eta, \alpha > \beta > 0, \eta > 0,\]
so that in the right-hand side of (A.11), the exponents of \( \varepsilon \) are positive for all terms except \( \|Y\|_\infty \). Since \((H-1/2)/\theta \leq (3 - 4H) < 1\), it is possible to take \( \beta/\alpha \in ((H-1/2)/\theta, (3 - 4H)) \) so that both \( \theta \beta + (1/2 - H) \alpha \) and \( -\beta/4 + (3/4 - H) \alpha \) are positive. Also \((H-1)\beta + (1 - H)\alpha = (1 - H)(\alpha - \beta) > 0, -\beta/4 + (1 - H)\alpha/2 > -\beta/4 + (3/4 - H) > 0, \theta \beta + (H - H) \alpha > \theta \beta + (1/2 - H) \alpha > 0 \). Therefore, by choosing \( \eta \) small enough we can make all needed exponents positive.
Thus, for some \( \kappa > 0 \) and \( C_1 > 0 \) we have
\[
\|b\|_\infty + \|c\|_\infty \leq C_1 \|Y\|_\infty \varepsilon^{-\lambda} + C_1 \varepsilon^\lambda \left( \|b\|_\infty + \|c\|_\infty \right) \left[ R^W + R^B + R^{W,B} \right]
\]
\[
+ \|b\|_\theta + \|c\|_\theta + \|a\|_\infty + \|c\|_\theta \|B^H\|_B + \xi_{\Delta,\delta},
\]
where \( \lambda = H \alpha + \eta/\theta \). Consequently, for \( \varepsilon \) small enough
\[
P \left\{ \|b\|_\infty + \|c\|_\infty > \varepsilon^{\kappa/2} \text{ and } \|Y\|_\infty < \varepsilon^{\lambda+\kappa} \right\}
\]
\[
\leq P \left\{ R^W \geq \varepsilon^{-\kappa/3} \right\} + P \left\{ R^B \geq \varepsilon^{-\kappa/3} \right\} + P \left\{ R^{W,B} \geq \varepsilon^{-\kappa/3} \right\}
\]
\[
+ P \left\{ \|b\|_\theta + \|c\|_\theta + \|c\|_\theta \|B^H\|_B + \xi_{\Delta,\delta} \geq \varepsilon^{-\kappa/3} \right\}.
\]
Now the statement follows by applying Lemmas A.3 and A.6 and the Chebyshev inequality.

**Lemma A.6.** Let \( R^W, R^B \) and \( R^{W,B} \) be given by (A.10) and (A.6). Then we have for any \( h > 1 \) the following concentration inequalities
\[
P \left\{ R^W \geq h \right\} \leq \frac{C}{\Delta} \exp\left( -Ch^2 \right), \quad \text{(A.12)}
\]
\[
P \left\{ R^B \geq h \right\} \leq \frac{C}{\Delta} \exp\left( -Ch^2 \right), \quad \text{(A.13)}
\]
\[
P \left\{ R^{W,B} \geq h \right\} \leq \frac{C}{\Delta} \exp\left( -Ch^2 \right). \quad \text{(A.14)}
\]

**Proof.** By [1] Lemma 3.1 we have for \( h > 0 \)
\[
P \left\{ \left| \Delta^{1/2} - V_N(W^u, W^u)^{1/2} \right| \Delta^{-1/4} \delta^{-1/4} \geq h \right\} \leq C \exp\left( -Ch^2 \right). \quad \text{(A.15)}
\]
Further, let $u \neq v$. Since $W^u$ and $W^v$ are independent, and $W^v$ has independent increments, then conditional on $W^v$, $V_N(W^v, W^u) \Delta^{-1/2} \delta^{-1/2}$ has a centered Gaussian distribution with the variance $V_N(W^v, W^v) \Delta^{-1}$. Therefore,

$$P \left\{ |V_N(W^u, W^v)|^{1/2} \Delta^{-1/4} \delta^{-1/4} \geq h \right\}$$

$$= E \left[ P \left\{ |V_N(W^u, W^v)|^{1/2} \Delta^{-1/2} \delta^{-1/2} \geq h^2 \right\} \mid W^v \right] \leq C E \left[ \exp \left\{ -\frac{h^4 \Delta}{4V_N(W^v, W^v)} \right\} \right]$$

$$\leq C \exp \left\{ -\frac{h^4 \Delta}{4(\Delta^{1/2} + v^2)} \right\} + P \left\{ |\Delta^{1/2} - V_N(W^v, W^v)|^{1/2} \geq v \right\}$$

$$\leq C \exp \left\{ -\frac{h^4 \Delta}{8(\Delta + v^2)} \right\} + C \exp \left\{ -C \frac{v^2}{\Delta^{1/2} \delta^{1/2}} \right\},$$

where we have used (A.15). Setting $v^2 = h^2 \Delta$ and recalling that $\Delta \geq \delta$ we get

$$P \left\{ |V_N(W^u, W^v)|^{1/2} \Delta^{-1/4} \delta^{-1/4} \geq h \right\} \leq C \exp \left\{ -C h^2 \right\}.$$

Combining this with (A.15), we get

$$P \left\{ R^W \geq h \right\} \leq \sum_{N=0}^{M-1} \sum_{u,v=1}^m P \left\{ \Delta^{-1/4} \delta^{-1/4} |\Delta^{1/2} - V_N(W^u, W^v)|^{1/2} \geq hm^2 \right\}$$

$$\leq C \delta \exp \left\{ -C h^2 \right\}.$$

Using the inequalities from [1, Lemma 3.2] and repeating the last step, we get (A.13).

The estimate (A.14) is proved similarly to (A.12), so we omit some details. Write

$$P \left\{ |V_N(W^u, B^{H,v})|^{1/2} \Delta^{-1/4} \delta^{-H/2} \geq h \right\}$$

$$= E \left[ P \left\{ |V_N(W^u, B^{H,v})|^{1/2} \Delta^{-1/2} \delta^{-H} \geq h^2 \right\} \mid B^{H,v} \right] \leq C E \left[ \exp \left\{ -\frac{h^4 \Delta \delta^{2H-1}}{4V_N(B^{H,v}, B^{H,v})} \right\} \right]$$

$$\leq C \exp \left\{ -\frac{h^4 \Delta \delta^{2H-1}}{4(\Delta^{1/2} \delta^{H-1/2} + v^2)} \right\} + P \left\{ |\Delta^{1/2} \delta^{H-1/2} - V_N(B^{H,v}, B^{H,v})|^{1/2} \geq v \right\}$$

$$\leq C \exp \left\{ -\frac{h^4 \Delta \delta^{2H-1}}{8(\Delta \delta^{H-1} + v^2)} \right\} + C \exp \left\{ -C \frac{v^2}{\Delta \delta^{2H-1} \delta} \right\}.$$

Setting $v^2 = h^2 \Delta \delta^{2H-1}$ and taking into account that $\Delta \geq \delta$, we arrive at

$$P \left\{ |V_N(W^u, B^{H,v})|^{1/2} \Delta^{-1/4} \delta^{-H/2} \geq h \right\} \leq C \exp \left\{ -C h^2 \right\}.$$

From here (A.14) is deduced similarly to (A.12). \qed

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