Hedging The Risk In The Continuous Time Option Pricing Model With Stochastic Stock Volatility

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Abstract

In this work, I address the issue of forming riskless hedge in the continuous time option pricing model with stochastic stock volatility. I show that it is essential to verify whether the replicating portfolio is self-financing, in order for the theory to be self-consistent. The replicating methods in existing finance literature are shown to violate the self-financing constraint when the underlying asset has stochastic volatility. Correct self-financing hedge is formed in this article.

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It has been indicated by empirical observations that stock price volatility is stochastic. Considerable amount of analytical and numerical work has been devoted to pricing derivatives when the underlying stock has stochastic volatility [1,2]. For example, there were early works done by Merton [1], and that by Garman etc [2]. In current existing finance literature, various authors constructed riskless hedging portfolios in different ways [3,10]. With the principle of no-arbitrage, the riskless hedging portfolio should match the return of a riskless loan. This will result in the partial differential equation satisfied by the option prices. In the situation where the underlying asset has stochastic volatility, investors’ preferences, such as the risk premium on the stock, will get involved explicitly.

In this work, I would like to address the issue of hedging the risk for the above system. In the continuous time models, it is necessary to make sure that the replicating portfolio is self-financing as it was assumed to be. This verification is essential for the theory to be self-consistent. In the following, I show that in spite of the final correctness of the derived PDE for option prices, the hedging strategies in some current finance literature [3,10] turned out to violate the self-financing condition that was assumed to hold. Arguments and corrections are given in this work when the underlying asset has stochastic volatility.

Below, we follow the reference [3] and its notations. Let us first define a probability space $(\Omega, Q, F)$. Consider the stock price obeying the stochastic process

$$dP = \alpha Pdt + \sigma PdZ_1,$$

where the volatility $\sigma$ is described by another mean-reverting process

$$d\sigma = \beta(\bar{\sigma} - \sigma)dt + \gamma dZ_2.$$

Here, both $Z_1$ and $Z_2$ are one dimensional Brownian motions. The co-quadratic process of $Z_1$ and $Z_2$ is assumed to be $[Z_1, Z_2] = t\delta$. For the process $\sigma$ described above, there is non-zero chance for the volatility to be negative. However, the following argument of ours will remain unchanged when other positive processes for the volatility are used, such as for the process $dln\sigma = \beta(\ln\sigma - lns)dt + \gamma dZ_2$. 

2
To hedge away the risk, a portfolio was constructed to have two call options and one stock [3]. The two options have different maturity dates $T_1$ and $T_2$. Denote $\tau_1 = T_1 - t$ and $\tau_2 = T_2 - t$. The option price was assumed to be $H(P, \sigma, \tau)$, a function of stock price $P$, the stock volatility $\sigma$, and $\tau = T - t$. In the following, we may use short notations $H(\tau_1) = H(P, \sigma, \tau_1), H(\tau_2) = H(P, \sigma, \tau_2)$. According to the reference [3], a trading strategy was proposed to be

$$\phi = (\phi_1, \phi_2, \phi_3) = (1, \omega_2, \omega_3)$$

(0.3)

where one has

$$\begin{align*}
\phi_1 &= 1, \\
\phi_2 &= \omega_2 = -\frac{H_2(P, \sigma, \tau_1)}{H_2(P, \sigma, \tau_2)}, \\
\phi_3 &= \omega_3 = -H_1(P, \sigma, \tau_1) + \frac{H_2(P, \sigma, \tau_1)H_1(P, \sigma, \tau_2)}{H_2(P, \sigma, \tau_2)}.
\end{align*}$$

(0.4)

Here one uses the conventions $H_2(P, \sigma, \tau) = \partial H(P, \sigma, \tau)/\partial \sigma, H_1(P, \sigma, \tau) = \partial H(P, \sigma, \tau)/\partial P, H_3(P, \sigma, \tau) = \partial H(P, \sigma, \tau)/\partial \tau = -\partial H(P, \sigma, \tau)/\partial t$. The wealth process of this portfolio is thus given by $V(\phi) = \phi_1 H(\tau_1) + \phi_2 H(\tau) + \phi_3 P$. Assuming this portfolio is self-financing, the change of the portfolio value will therefore take the following form:

$$dV = \phi_1 dH(\tau_1) + \phi_2 dH(\tau_2) + \phi_3 dP,$$

(0.5)

which can be seen being riskless after substituting the trading strategy explicitly. This change of the wealth process should match the return of a riskless loan, leading to a partial differential equation satisfied by the option price. Combining this with some general equilibrium consideration, one therefore obtains the option valuation PDE when the underlying stock volatility is stochastic [3].

However, we would like to note that it is essential to fully justify the above assumption that the corresponding portfolio is indeed self-financing. In spite of the fact that the final PDE based on this portfolio is correct accidentally, it is essential to check explicitly $dV = \phi_1 dH(\tau_1) + \phi_2 dH(\tau_2) + \phi_3 dP$ holds within the continuous-time framework for the theory to
be self-consistent. It is shown below that this particular trading strategy defined by Eq.(1.4) is not self-financing.

Suppose that we have found the price of European call option from Eq.(4) of the reference [3], with appropriate boundary condition \( H(P, \sigma, \tau) = \max(0, P(T) - K) \) at time \( t = T \). Let us substitute this back into the trading strategy \( \phi \) defined before. The wealth is \( V = V(\phi) = \phi_1 H(\tau_1) + \phi_2 H(\tau_2) + \phi_3 P \). Therefore, from simple rule of stochastic calculus, we should have

\[
dV = dH(\tau_1) + \omega_2 dH(\tau_2) + \omega_3 dP + (H(\tau_2) d\omega_2 + d[\omega_2, H(\tau_2)] + P d\omega_3 + d[P, \omega_3]). \tag{0.6}
\]

Denote \( dW = H(\tau_2) d\omega_2 + d[\omega_2, H(\tau_2)] + P d\omega_3 + d[P, \omega_3] \). If we find that \( dW = 0 \), then, the original trading strategy is self-financing. Otherwise, the proposed trading strategy is not self-financing, and there is self-inconsistence in the theory. We can compute \( dW \) explicitly, although the computation is a bit tedious. In general, \( dW \) will take the form

\[
dW = f^{(0)}(P, \sigma, \tau_1, \tau_2) dt + f^{(1)}(P, \sigma, \tau_1, \tau_2) dZ_1 + f^{(2)}(P, \sigma, \tau_1, \tau_2) dZ_2, \tag{0.7}
\]

where the functions \( f^{(0)}, f^{(1)}, f^{(2)} \) can be found, given the European option price. One can easily see that \( d[\omega_2, H(\tau_2)] = K_1(P, \sigma, \tau_1, \tau_2) dt \) and \( d[P, \omega_3] = K_2(P, \sigma, \tau_1, \tau_2) dt \), where both \( K_1 \) and \( K_2 \) are some functions of \( P, \sigma, \tau_1, \) and \( \tau_2 \). Hence, \( f^{(1)} \) and \( f^{(2)} \) solely come from the contributions of expanding \( H_2(\tau_2) d\omega_2 \) and \( P d\omega_3 \). Expanding these two terms will also contribute to \( f^{(0)} \). After some tedious calculation, we obtain \( f^{(1)} dZ_1 + f^{(2)} dZ_2 \) explicitly. The first coefficient is found to be:

\[
f^{(1)} = (H(\tau_2) - PH_1(\tau_2))(-\frac{H_{21}(\tau_1)}{H_2(\tau_2)} + \frac{H_2(\tau_1)}{H_2^2(\tau_2)} H_{21}(\tau_2)) \sigma P +
- PH_{11}(\tau_1) \sigma P + P \frac{H_2(\tau_1)}{H_2(\tau_2)} H_{11}(\tau_2) \sigma P. \tag{0.8}
\]

We have also found the second coefficient:

\[
f^{(2)} = (H(\tau_2) - PH_1(\tau_2))(-\frac{H_{22}(\tau_1)}{H_2(\tau_2)} + \frac{H_2(\tau_1)}{H_2^2(\tau_2)} H_{22}(\tau_2)) \gamma +
- PH_{12}(\tau_1) \gamma + P \frac{H_2(\tau_1)}{H_2(\tau_2)} H_{12}(\tau_2) \gamma. \tag{0.9}
\]
When $\tau_1 = \tau_2$, we see that $f^{(1)} = f^{(2)} = 0$. However, for general $\tau_1 \neq \tau_2$, it is true that

$$f^{(1)} \neq 0, f^{(2)} \neq 0,$$

indicating that in general $dW$ is non-zero (almost surely) in the probability space $(\Omega, Q, F)$.

One observes that the hedging portfolio constructed by reference [3] is not self-financing in the continuous time framework.

In order to remedy the situation, we may construct a trading strategy in different way:

$$\phi = (\phi_1, \phi_2, \phi_3) = \left( \frac{Vx_1}{H(\tau_1)}, \frac{Vx_2}{H(\tau_2)}, \frac{Vx_3}{P} \right)$$

(0.11)

where $x_1 + x_2 + x_3 = 1$ is assumed, and $V = V(P, \sigma, \tau_1, \tau_2)$ is to be determined. We choose the x’s to be

$$\frac{x_1}{H(\tau_1)} = a, \frac{x_2}{H(\tau_2)} = a\omega_2, \frac{x_3}{P} = \omega_3a,$$

(0.12)

with $a = 1/[H(\tau_1) + \omega_2H(\tau_2) + P\omega_3]$, and with $\omega_2, \omega_3$ defined as before. The process $V(P, \sigma, \tau_1, \tau_2)$ is defined by the differential equation

$$\frac{dV}{V} = (\phi_1H(\tau_1) + \phi_2H(\tau_2) + \phi_3P)^{-1} \times \left[ -H_3(\tau_1) + \frac{1}{2}H_{11}(\tau_1)\sigma^2P^2 + H_{12}(\tau_1)\delta\gamma\sigma P + \frac{1}{2}H_{22}(\tau_1)\gamma^2 - \frac{H_2(\tau_1)}{H_2(\tau_2)}[-H_3(\tau_2) + \frac{1}{2}H_{11}(\tau_2)\sigma^2P^2 + H_{12}(\tau_2)\delta\gamma\sigma P + \frac{1}{2}H_{22}(\tau_2)\gamma^2] \right]dt.$$

(0.13)

Suppose that we have obtained European call option price $H(P, \sigma, \tau)$ from Eq.(4) of the reference [3] with proper boundary condition. Substitute $H$ into the above equation and it will determine the finite-variation process $V$. With this $V$ substituted into the trading strategy as defined above, one can easily check that the self-financing constraint is satisfied. Therefore, there will be no inconsistency in the derivation of PDE for option pricing.

In one recent interesting article, the path-integral technique was used to do perturbation for the option pricing model when the underlying asset has stochastic volatility [10]. However, we would like to note that for constant volatility, the usual way in section one of the paper [10] to construct the riskless portfolio of one long option $f$ and $\partial f/\partial S$ share short stock is not self-financing trading strategy, as shown by Bergman, and by Musiela &
Apart from this, when the volatility is random, it is known in finance community that in general a riskless hedge can not be formed from only one option and the stock. The riskless hedge constructed with one option and one stock in the paper [10] is always zero strategy. First, one can check that Eq.(8e) in the reference [10] would give $\theta_1 = 0$, because $\partial f/\partial S \neq 0$, and $\partial f/\partial V \neq 0$ (in his notations). Therefore, the same equation again will give $\theta_2 = 0$. Hence the trading strategy is always zero. We wish to note that as Wiggins and other people did, one could only form a portfolio of one stock and one option, such that the portfolio has zero co-quadratic variation with the stock [6]. It is impossible to form a riskless hedge with only one option and one stock in this case in general.

In summary, we have addressed the issue of forming a riskless hedge in continuous time option pricing model when the underlying asset has stochastic volatility. It is essential to the self-consistency of theory to explicitly justify whether the hedge is indeed self-financing as one assumed. Some confusion in existing finance literature on this aspect is cleared.

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REFERENCES

[1] R. C. Merton, “The Theory of Rational Option Pricing”, The Bell Journal of Economics and Management Science, 4, 141-183 (1973).

[2] M. Garman, “A General Theory of Asset Valuation under Diffusion State Processes”, Working Paper No. 50, UC Berkeley (1976).

[3] L. Scott, “Option Pricing when the Variance Changes Randomly: Theory, Estimation, and an Application”, Journal of Financial and Quantitative Analysis, Vol. 22, No.4, 419 (1987).

[4] J. C. Hull and A. White, “The Pricing of Options on Asset with Stochastic Volatilities”, Journal of Finance, 2, 281 (1987).

[5] J. C. Hull and A. White, “Analysis of the Bias in Option Pricing caused by Stochastic Volatility”, Advances in Futures and Option Research, Vol.3, p29 (1988).

[6] J. B. Wiggins, “Option Value under Stochastic Volatility: Theory and Empirical Estimates”, Journal of Financial Economics, 19, 351 (1987).

[7] P. P. Boyle and D. Emanuel, “Mean-dependent Options”, working paper, Accounting group, University of Waterloo, (1985).

[8] See the book by M. Musiela and M. Rutkowski, “Martingale methods in financial modelling”, published by Springer (1997), and references therein.

[9] S. Heston, “A Closed Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options”, The Review of Financial Studies 6, 327 (1993).

[10] B. E. Baaquie, “A Path-Integral Approach to Option Pricing with Stochastic Volatility: Some Exact Results”, J. Phys. I (France), Vol.7, No. 12, 1733-1753 (1997); cond-mat/9708178
[11] E. Stein and J. Stein, “Stock Price Distribution with Stochastic Volatility: An Analytic Approach”, 4, 727 (1991).

[12] C. Ball and A. Roma, “Stochastic Volatility Option Pricing”, Journal of Financial and Quantitative Analysis 29, 589 (1994)

[13] C. Bergman, UC Berkeley Ph.D. thesis (1985).