Decorated Geometric Crystals and Polyhedral Realizations of type $D_n$

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Abstract. This is a continuation of [15, 16]. We shall show that for type $D_n$ the realization of crystal bases obtained from the decorated geometric crystals in [2] coincides with the polyhedral realizations of crystal bases.

1. Introduction

In [14, 17] we introduced the polyhedral realizations of crystal bases and therein we described their explicit forms for type $A_n$, $A_n^{(1)}$ and arbitrary rank 2 Kac-Moody algebras. In [6], Hoshino described them explicitly for all simple Lie algebras. It is a kind of realizations of crystal bases that is presented as a convex polyhedral domain in an infinite/finite $\mathbb{Z}$-lattice defined by some system of linear inequalities.

In [2], Berenstein and Kazhdan introduced decorated geometric crystals for reductive algebraic groups. Geometric crystals are a kind of geometric analogue to the Kashiwara’s crystal bases ([1]). Let $I$ be a finite index set. Associated with a Cartan matrix $A = (a_{i,j})_{i,j \in I}$, define the decorated geometric crystal $\mathcal{X} = (\chi, f)$, which is a pair of geometric crystal $\chi = (X, \{e_i\}, \{\gamma_i\}, \{\epsilon_i\})$ and a certain special rational function $f$ with the condition (3.1).

If we apply the procedure called “ultra-discretization” (UD) to “positive geometric crystals”, then we obtain certain free-crystals for the Langlands dual type ([11, 13]). As for a positive decorated geometric crystal $(\chi, f, T', \theta)$ applying UD to the function $f$ and considering the convex polyhedral domain defined by the inequality $UD(f) \geq 0$, we get the crystal with the property “normal” ([9]) and furthermore as a connected component with the highest weight $\lambda$, we obtain the Langlands dual Kashiwara’s crystal $B(\lambda)$.

This result makes us recall the polyhedral realization of crystal bases as introduced above since it has similar way to get the crystal $B(\lambda)$ from certain free-crystals, defined by the system of linear inequalities. Thus, one of the main aims of this article is to show that the crystals obtained by UD from positive decorated geometric crystals and the polyhedral realizations of crystals coincide with each other for type $D_n$, which is a continuation of [15] for type $A_n$.

In [16], the explicit feature of the decoration function $f_B$ on some special geometric crystal $TB_{w_0}$ for all classical Lie algebras. In [15] for type $A_n$ the coincidence of polyhedral realizations of crystals and the realizations of crystals by the decoration functions are presented. Here, in this article we shall...
do these for type $D_n$ by using the results in [6]. For type $B_n$ and $C_n$ we shall discuss the relations of two realizations in the forthcoming paper.

The organization of the article is as follows: in Sect.2, we review the theory of crystals and their polyhedral realizations and give the explicit feature of polyhedral realization of type $D_n$. In Sect.3, first we introduce the theory of decorated geometric crystals following [2] and their positive structures and ultra-discretization. Next, we define the decoration function by using elementary characters and certain special positive decorated geometric crystal on $\mathbb{B}_{w_0} = TB_{w_0}$. Finally, the ultra-discretization of $TB_{w_0}$ is described explicitly. We describe the function $f_B$ exactly for type $D_n$ in Sect.4 following [16]. In Sect.5, for type $D_n$ we shall see the coincidence of the polyhedral realization $\Sigma_{\mu_0}[\lambda]$ and the ultra-discretization $B_{f_B,\Theta_{W_0}}(\lambda)$.

2. Crystals and polyhedral realizations

2.1. Notations. Let $g$ be a semi-simple Lie algebra over $\mathbb{Q}$ with a Cartan subalgebra $t$, a weight lattice $P \subset t^*$, the set of simple roots $\{\alpha_i : i \in I\} \subset t^*$, and the set of coweights $\{\beta_i : i \in I\} \subset t$, where $I$ is a finite index set. Let $(h, \lambda) = \lambda(h)$ be the pairing between $t$ and $t^*$, and $(\alpha, \beta)$ be an inner product on $t^*$ such that $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{\geq 0}$ and $(h_i, \lambda) = \sum_{i \in I} (\alpha_i, \alpha_i) \lambda_i$ for $\lambda \in t^*$ and $A := ((h_i, \alpha_j))_{i,j}$ is the associated Cartan matrix. Let $P^* = \{h \in t : \langle h, P \rangle \subset \mathbb{Z}\}$ and $P_+ := \{\lambda \in P : \langle h_i, \lambda \rangle \in \mathbb{Z}_{\geq 0}\}$. We call an element in $P_+$ a dominant integral weight. The quantum algebra $U_q(g)$ is an associative $\mathbb{Q}(q)$-algebra generated by the $e_i, f_i (i \in I)$, and $q^h (h \in P^*)$ satisfying the usual relations. The algebra $U_q^-(g)$ is the subalgebra of $U_q(g)$ generated by the $f_i (i \in I)$.

For the irreducible highest weight module of $U_q(g)$ with the highest weight $\lambda \in P_+$, we denote it by $V(\lambda)$ and its crystal base we denote $(L(\lambda), B(\lambda))$. Similarly, for the crystal base of the algebra $U_q^-(g)$ we denote $(L(\infty), B(\infty))$ (see [7, 8]). Let $\pi_\lambda : U_q^-(g) \rightarrow V(\lambda) \cong U_q^-(g)/\sum_i U_q^-(g)f_i^{1+|h_i, \lambda|}$ be the canonical projection and $\tilde{\pi}_\lambda : L(\infty)/qL(\infty) \rightarrow L(\lambda)/qL(\lambda)$ be the induced map from $\pi_\lambda$. Here note that $\tilde{\pi}_\lambda(B(\infty)) = B(\lambda) \cup \{0\}$.

Let crystal be a combinatorial object defined in [9], see also [14, 15, 17]. In fact, $B(\infty)$ and $B(\lambda)$ are the typical examples of crystals.

Let $B_1$ and $B_2$ be crystals. A strict morphism of crystals $\psi : B_1 \rightarrow B_2$ is a map $\psi : B_1 \cup \{0\} \rightarrow B_2 \cup \{0\}$ satisfying the following conditions: $\psi(0) = 0$, $wt(\psi(b)) = wt(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, $\varphi_i(\psi(b)) = \varphi_i(b)$ if $b \in B_1$ and $\psi(b) \in B_2$, and the map $\psi : B_1 \cup \{0\} \rightarrow B_2 \cup \{0\}$ commutes with all $\tilde{e}_i$ and $\tilde{f}_i$. An injective strict morphism is called an embedding of crystals.

Crystals have very nice properties for tensor operations. Indeed, if $(L_i, B_i)$ is a crystal base of $U_q(g)$-module $M_i (i = 1, 2)$, $(L_1 \otimes A L_2, B_1 \otimes B_2)$ is a crystal base of $M_1 \otimes_{\mathbb{Q}(q)} M_2$ ([8]).

2.2. Polyhedral Realization of Crystals. Let us recall the results in [14, 17].

Consider the infinite $Z$-lattice

$$Z^\infty := \{(\cdots, x_k, \cdots, x_2, x_1) : x_k \in \mathbb{Z} \text{ and } x_k = 0 \text{ for } k \gg 0\}.$$  

We fix an infinite sequence of indices $i = \cdots, i_k, \cdots, i_2, i_1$ from $I$ such that

$$i_k \neq i_{k+1} \text{ and } \sharp\{k : i_k = i\} = \infty \text{ for any } i \in I.$$  

Given $i$, we can define a crystal structure on $Z^\infty$ and denote it by $Z_i^\infty$ ([17, 2.4]).

Proposition 2.1 ([9]. See also [17]). There is a unique strict embedding of crystals (called Kashiwara embedding)

$$\Psi_i : B(\infty) \hookrightarrow Z_i^\infty \subset Z^\infty,$$

such that $\Psi_i(u_\infty) = (\cdots, 0, \cdots, 0, 0)$, where $u_\infty \in B(\infty)$ is the vector corresponding to $1 \in U_q^-(g)$. 

In the rest of this section, suppose $\lambda \in P_+$. Let $R_\lambda := \{r_\lambda\}$ be the crystal with the single element $r_\lambda$ satisfying the condition $\text{wt}(r_\lambda) = \lambda, \varepsilon_i(r_\lambda) = -\langle h_i, \lambda \rangle$ and $\varphi_i(r_\lambda) = 0$. Here we define the map
\begin{equation}
\Phi_\lambda : (B(\infty) \otimes R_\lambda) \sqcup \{0\} \rightarrow B(\lambda) \sqcup \{0\},
\end{equation}
by $\Phi_\lambda(0) = 0$ and $\Phi_\lambda(b \otimes r_\lambda) = \hat{\pi}_\lambda(b)$ for $b \in B(\infty)$. We set
\begin{equation*}
\tilde{B}(\lambda) := \{b \otimes r_\lambda \in B(\infty) \otimes R_\lambda \mid \Phi_\lambda(b \otimes r_\lambda) \neq 0\}.
\end{equation*}

**Theorem 2.2** ([14]).

\begin{enumerate}[(i)]
\item The map $\Phi_\lambda$ becomes a surjective strict morphism of crystals $B(\infty) \otimes R_\lambda \rightarrow B(\lambda)$.
\item $\tilde{B}(\lambda)$ is a subcrystal of $B(\infty) \otimes R_\lambda$, and $\Phi_\lambda$ induces the isomorphism of crystals $\tilde{B}(\lambda) \cong B(\lambda)$.
\end{enumerate}

By Theorem 2.2 we have the strict embedding of crystals $\Omega_\lambda : B(\lambda)(\cong \tilde{B}(\lambda)) \hookrightarrow B(\infty) \otimes R_\lambda$. Combining $\Omega_\lambda$ and the Kashiwara embedding $\Psi_\iota$, we obtain the following:

**Theorem 2.3** ([14]). There exists the unique strict embedding of crystals
\begin{equation}
\Psi_\iota^{(\lambda)} : B(\lambda) \xrightarrow{\Omega_\lambda} B(\infty) \otimes R_\lambda \xrightarrow{\Psi_\iota \otimes \text{id}} \mathbb{Z}^\infty_\iota[\lambda],
\end{equation}
such that $\Psi_\iota^{(\lambda)}(u_\lambda) = (\cdots, 0, 0, 0) \otimes r_\lambda$.

Consider the infinite dimensional vector space
\begin{equation*}
\mathbb{Q}^\infty := \{x = (\cdots, x_k, \cdots, x_2, x_1) : x_k \in \mathbb{Q} \text{ and } x_k = 0 \text{ for } k \gg 0\},
\end{equation*}
and its dual space $(\mathbb{Q}^\infty)^* := \text{Hom}(\mathbb{Q}^\infty, \mathbb{Q})$. We will write a linear form $\varphi \in (\mathbb{Q}^\infty)^*$ as $\varphi(x) = \sum_{k \geq 1} \varphi_k x_k$ ($\varphi_j \in \mathbb{Q}$) for $x \in \mathbb{Q}^\infty$.

Let $S_k = S_{k,\iota}$ on $(\mathbb{Q}^\infty)^*$ be the piecewise-linear operator as in [17] and set
\begin{equation}
\Xi_\iota := \{S_{ji} \cdots S_{ji} x_{j_0} \mid l \geq 0, j_0, \cdots, j_l \geq 1\},
\end{equation}
\begin{equation}
\Sigma_\iota := \{x \in \mathbb{Z}^\infty \subset \mathbb{Q}^\infty \mid \varphi(x) \geq 0 \text{ for any } \varphi \in \Xi_\iota\}.
\end{equation}

For a fixed $\iota = (i_k)$ and a positive integer $k$, let $k^{(-)}$ be the maximum number $m$ such that $m < k$ and $i_k = i_m$ if it exists, and 0 otherwise. We impose on $\iota$ the following positivity assumption:
\begin{equation}
\text{if } k^{(-)} = 0 \text{ then } \varphi_k \geq 0 \text{ for any } \varphi(x) = \sum_k \varphi_k x_k \in \Xi_\iota.
\end{equation}

**Theorem 2.4** ([17]). Let $\iota$ be a sequence of indices satisfying (2.22) and (2.23). Then we have $\text{Im}(\Psi_\iota)(\cong B(\infty)) = \Sigma_\iota$.

For every $k \geq 1$, let $\hat{S}_k = \hat{S}_{k,\iota}$ be the piece-wise linear operator for a linear function $\varphi(x) = c + \sum_{k \geq 1} \varphi_k x_k$ ($c, \varphi_k \in \mathbb{Q}$) on $\mathbb{Q}^\infty$ as in [14]. For the fixed sequence $\iota = (i_k)$, in case $k^{(-)} = 0$ for $k \geq 1$, there exists unique $i \in I$ such that $i_k = i$. We denote such $k$ by $i^{(\iota)}$, namely, $i^{(\iota)}$ is the first number $k$ such that $i_k = i$. Here for $\lambda \in P_+$ and $i \in I$ we set $\lambda^{(\iota)}(x) := -\beta_1^{(\iota)}(x) = \langle h_i, \lambda \rangle - \sum_{1 \leq j \leq i^{(\iota)}} \langle h_i, \alpha_{j}\rangle x_j - x_{i^{(\iota)}}$.

For $\iota$ and a dominant integral weight $\lambda$, let $\Xi_\iota[\lambda]$ be the set of all linear functions generated by $\hat{S}_k = \hat{S}_{k,\iota}$ from the functions $x_j$ ($j \geq 1$) and $\lambda^{(\iota)}(i \in I)$, namely,
\begin{equation}
\Xi_\iota[\lambda] := \{\hat{S}_{ji} \cdots \hat{S}_{ji} x_{j_0} : l \geq 0, j_0, \cdots, j_l \geq 1\}
\end{equation}
\begin{equation*}
\cup \{\hat{S}_{ji} \cdots \hat{S}_{ji} \lambda^{(\iota)}(x) : k \geq 0, i \in I, j_1, \cdots, j_k \geq 1\}.
\end{equation*}
Now we set
\begin{equation}
\Sigma_\iota[\lambda] := \{x \in \mathbb{Z}^\infty[\lambda] \subset \mathbb{Q}^\infty : \varphi(x) \geq 0 \text{ for any } \varphi \in \Xi_\iota[\lambda]\}.
\end{equation}

For a sequence $\iota$ and a dominant integral weight $\lambda$, a pair $(\iota, \lambda)$ is called *ample* if $\Sigma_\iota[\lambda] \ni \overline{0} = (\cdots, 0, 0)$. 
2.3. $D_n$-case. Let us identify the index set $I$ with $[1, n] := \{1, 2, \ldots, n\}$ in the standard way; thus, the Cartan matrix $(a_{i,j} = \langle h_i, \alpha_j \rangle)_{1 \leq i,j \leq n}$ of type $D_n$ is given by

$$a_{i,i} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i - j| = 1, (i, j) \neq (n - 1, n), (n, n - 1) \text{ or } (i, j) = (n - 2, n), (n, n - 2), \\ 0 & \text{otherwise}. \end{cases}$$

(2.11)

Following the result in [6], we shall describe the polyhedral realization of type $D_n$.

As we have seen in [12], in semi-simple setting, to describe $B(\lambda)$ we only need the finite rank $\mathbb{Z}$-lattice such as $\mathbb{Z}^N$ where $N$ is the length of the longest element in the corresponding Weyl group, that is, for a reduced longest word $\iota = (i_N, \cdots, i_2, i_1)$ the crystal $B(\lambda)$ can be embedded in $\mathbb{Z}^N \otimes R_\lambda$.

Thus, for type $D_n$ we will take the sequence $\iota_0 = (n \cdots 2 1)^{n-1}$ ($N = n(n - 1)$). Associating with this sequence, we take a doubly-indexed variables $(x^{(j)}_{j})_{(j,i) \in [1, n-1] \times [1, n]}$. Here we define the following set of linear forms:

$$\Xi_k := \{x_{i+1}^{(k)} - x_i^{(k+1)} | i = 1, \ldots, n-2\} \cup \{x_n^{(k)} - x_n^{(k+1)}, x_{n-1}^{(k)} - x_{n-2}^{(k+1)} - x_{n-1}^{(k+1)} - x_n^{(k+1)}\}$$

(2.12)

$$\cup \{x_n^{(k)} - x_n^{(k+1)}, x_{n-1}^{(k)} - x_{n-2}^{(k+1)}\} \cup \{x_r^{(j)} - x_{r+1}^{(j)} | j = k + 2, \ldots, n - 1\} \quad (k = 1, \ldots, n - 2),$$

(2.13)

$$\Xi_{n-1} := \{x_n^{(n-1)}\}, \quad \Xi_n := \{x_n^{(n-1)}\},$$

(2.14)

where we understand $x^{(j)}_{j} = 0$ unless $(j, i) \in [1, n - 1] \times [1, n]$.

To define the sets of linear forms $\Xi'_{n-1}$ and $\Xi'_n$, we prepare certain combinatorial objects “admissible patterns” (6):

$$M_n := \{\mu = (\mu_1, \mu_2, \cdots, \mu_l) | 0 < \mu_1 < \mu_2 < \cdots < \mu_l < n\},$$

where we call an element $\mu$ in $M_n$ an admissible pattern.

**Example 2.5.** For $n = 4$, we have $M_4 = \{(3, 2, 1), (3, 2), (3, 1), (3), (2, 1), (2), (1)\}$.

The following is trivial.

**Lemma 2.6.** The number of elements in $M_n$ is $2^{n-1} - 1$.

Using these admissible patterns, we shall define the linear forms: For an admissible pattern $\mu = (\mu_1, \cdots, \mu_l) \in M_n$, define

$$\varphi_{\mu}(x) := \begin{cases} \sum_{k=1}^{l} (X_{n-\mu_k}^{(\mu_1 + k - 1)} - X_{\mu_k}^{(\mu_1 + k - 1)}) & \text{if } \mu_1 = 1, \\ \sum_{k=1}^{l} (X_{n-\mu_k}^{(\mu_1 + k - 1)} - X_{\mu_k}^{(\mu_1 + k - 1)}) + X_n^{(l)} & \text{if } \mu_1 \geq 2, \end{cases}$$

(2.15)

$$\varphi'_{\mu}(x) := \begin{cases} \sum_{k=1}^{l} (X_{n-\mu_k}^{(\mu_1 + k - 1)} - X_{\mu_k}^{(\mu_1 + k - 1)}) & \text{if } \mu_1 = 1, \\ \sum_{k=1}^{l} (X_{n-\mu_k}^{(\mu_1 + k - 1)} - X_{\mu_k}^{(\mu_1 + k - 1)}) + X_n^{(l)} & \text{if } \mu_1 \geq 2, \end{cases}$$

(2.16)

$$\Xi'_{n-1} := \{\varphi_{\mu}(x) | \mu \in M_n\}, \quad \Xi'_n := \{\varphi'_{\mu}(x) | \mu \in M_n\},$$

(2.17)
Let \( \phi \in \mathfrak{g} \) be the simple Lie algebra associated with \( G \), where \( \mathfrak{g} \) is a maximal torus of \( \mathfrak{g} \). An element \( \phi \) satisfies \( \iota(\phi) = \iota(h) \phi \). Let \( G \) be the associated root data satisfying \( \iota(\phi) = \{0\} \), and \( \mathfrak{g} = \mathfrak{g}(A) = \{t, e_i, f_i (i \in I)\} \) be the simple Lie algebra associated with \( A \) over \( \mathbb{C} \) and \( \Delta = \Delta_+ \cup \Delta_- \) be the root system associated with \( \mathfrak{g} \), where \( \Delta_\pm \) is the set of positive/negative roots.

Define the simple reflections \( s_i \in \text{Aut}(t) (i \in I) \) by \( s_i(h) := h - \alpha_i(h) h_i \), which generate the Weyl group \( W \). Let \( G \) be the simply connected simple algebraic group over \( \mathbb{C} \) whose Lie algebra is \( \mathfrak{g} = \mathfrak{n}_+ \oplus t \oplus \mathfrak{n}_- \). Let \( U_\alpha := \exp \mathfrak{g}_\alpha \ (\alpha \in \Delta) \) be the one-parameter subgroup of \( G \). The group \( G \) (resp. \( U_{\pm} \)) is generated by \( \{U_\alpha | \alpha \in \Delta_\pm \} \) (resp. \( \{U_\alpha | \alpha \in \Delta_{\pm} \} \)). Here \( U_{\pm} \) is a unipotent radical of \( G \) and \( \text{Lie}(U_{\pm}) = \mathfrak{n}_\pm \). For any \( i \in I \), there exists a unique group homomorphism \( \phi_i : SL_2(\mathbb{C}) \to G \) such that \( \phi_i \left( \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right) = \exp(t e_i) \), \( \phi_i \left( \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \right) = \exp(t f_i) \) \((t \in \mathbb{C}) \).

Set \( \alpha_i^+(c) := \phi_i \left( \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \right) \), \( x_i(t) := \exp(te_i) \), \( y_i(t) := \exp(tf_i) \), \( G_i := \phi_i(SL_2(\mathbb{C})) \), \( T_i := \alpha_i^+ (\mathbb{C}^\times) \) and \( N_i := N_{G_i}(T_i) \). Let \( T \) be a maximal torus of \( G \) which has \( P \) as its weight lattice and \( \text{Lie}(T) = t \). Let \( B_{\pm}(\subset T) \) be the Borel subgroup of \( G \). We have the isomorphism \( \phi : W \to N/T \) defined by \( \phi(s_i) = N_i T/T \). An element \( \overline{s}_i := x_i(-1)y_i(1)x_i(-1) \) is in \( N_{G_i}(T_i) \), which is a representative of \( s_i \in W = N_G(T)/T \).

DEFINITION 3.1. Let \( X \) be an affine algebraic variety over \( \mathbb{C} \), \( \gamma_i, \varepsilon_i, f (i \in I) \) rational functions on \( X \), and \( e_i : \mathbb{C}^\times \times X \to X \) a unital rational \( \mathbb{C}^\times \)-action. A 5-tuple \( \chi = (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I}, f) \) is a \( G \) (or \( g \))-decorated geometric crystal if

(i) \( \{(1) \times X \} \cap \text{dom}(e_i) \) is open dense in \( \{1\} \times X \) for any \( i \in I \), where \( \text{dom}(e_i) \) is the domain of definition of \( e_i : \mathbb{C}^\times \times X \to X \).
(ii) The rational functions \( \{\gamma_i\}_{i \in I} \) satisfy \( \gamma_j(e_i^*(x)) = e^{\alpha_i \gamma_j}(x) \) for any \( i, j \in I \).
(iii) The function \( f \) satisfies

\[
(3.1) \quad f(e_i^*(x)) = f(x) + (c-1)\varphi_i(x) + (c^{-1} - 1)\varepsilon_i(x),
\]

for any \( i \in I \) and \( x \in X \), where \( \varphi_i := \varepsilon_i \cdot \gamma_i \).
(iv) \( e_i \) and \( e_j \) satisfy the following relations:
\[
\begin{align*}
\epsilon_i e_j &= e_j e_i, \\
\epsilon_i e_j &= e_j e_i, \\
\epsilon_i e_j &= e_j e_i, \\
\epsilon_i e_j &= e_j e_i, \\
\epsilon_i e_j &= e_j e_i.
\end{align*}
\]
if \( a_{ij} = a_{ji} = 0 \),
if \( a_{ij} = a_{ji} = -1 \),
if \( a_{ij} = 1, a_{ji} = -1 \),
if \( a_{ij} = -2, a_{ji} = -1 \),
if \( a_{ij} = -3, a_{ji} = -1 \).

(v) The rational functions \( \{\epsilon_i\}_{i \in I} \) satisfy \( \epsilon_i(e_i(x)) = e^{-1} \epsilon_i(x) \) and \( \epsilon_i(e_j(x)) = \epsilon_i(x) \) if \( a_{i,j} = a_{j,i} = 0 \).

We call the function \( f \) in (iii) the \textit{decoration} of \( \chi \) and the relations in (iv) are called \textit{Verma relations}.

If \( \chi = (X, \{e_i\}, \{\gamma_i\}, \{\epsilon_i\}) \) satisfies the conditions (i), (ii), (iv) and (v), we call \( \chi \) a geometric crystal.

Remark. The definitions of \( \epsilon_i \) and \( \varphi_i \) are different from the ones in e.g., \cite{2} since we adopt the definitions following \cite{10} \cite{11}. Indeed, if we flip \( \epsilon_i \rightarrow \epsilon_i^{-1} \) and \( \varphi_i \rightarrow \varphi_i^{-1} \), they coincide with ours.

Let \( T_+ \) be the category whose objects are algebraic tori over \( \mathbb{C} \) and whose morphisms are positive rational maps. Then, let \( \mathcal{U}D \) be the functor:
\[
\begin{align*}
\mathcal{U}D : & \quad T_+ \rightarrow \mathcal{G}et \\
& \quad g : T \rightarrow T' \rightarrow g' : X_*(T) \rightarrow X_*(T'),
\end{align*}
\]
which is given as in \cite{10} \cite{11} \cite{13} \cite{15}.

For a split algebraic torus \( T \) over \( \mathbb{C} \), let us denote its lattice of (multiplicative) characters (resp. co-characters) by \( X^+(T) \) (resp. \( X_+(T) \)). By the usual way, we identify \( X^+(T) \) (resp. \( X_+(T) \)) with the weight lattice \( P \) (resp. the dual weight lattice \( P^* \)).

Let \( \theta : T' \rightarrow T \) be a positive structure on a decorated geometric crystal \( \chi = (X, \{e_i\}_{i \in I}, \{\omega_i\}_{i \in I}, \{\epsilon_i\}_{i \in I}, f) \).

Applying the functor \( \mathcal{U}D \) to positive rational morphisms \( e_{i,\theta} : \mathbb{C}^* \times T' \rightarrow T' \) and \( \gamma_i \circ \theta, \epsilon_i \circ \theta, f \circ \theta : T' \rightarrow \mathbb{C} \) (the notations are as above), we obtain:
\[
\begin{align*}
\tilde{e}_i &:= \mathcal{U}D(e_{i,\theta}) : \mathbb{Z} \times X_*(T') \rightarrow X_*(T') \\
\tilde{w}_i &:= \mathcal{U}D(\gamma_i \circ \theta) : X_*(T') \rightarrow \mathbb{Z}, \\
\tilde{\epsilon}_i &:= \mathcal{U}D(\epsilon_i \circ \theta) : X_*(T') \rightarrow \mathbb{Z}, \\
\tilde{f} &:= \mathcal{U}D(f \circ \theta) : X_*(T') \rightarrow \mathbb{Z}.
\end{align*}
\]

Now, for given positive structure \( \theta : T' \rightarrow T \) on a geometric crystal \( \chi = (X, \{e_i\}_{i \in I}, \{\omega_i\}_{i \in I}, \{\epsilon_i\}_{i \in I}) \), we associate the quadruple \( (X_*(T'), \tilde{e}_i, \tilde{w}_i, \tilde{\epsilon}_i) \) with a free crystal structure (see \cite{1} 2.2) and denote it by \( \mathcal{U}D_{\theta,T}(\chi) \).

We have the following theorem:

\textbf{Theorem 3.2 (\cite{1} 2.13).} For any geometric crystal \( \chi = (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\epsilon_i\}_{i \in I}) \) and positive structure \( \theta : T' \rightarrow T \), the associated crystal \( \mathcal{U}D_{\theta,T}(\chi) = (X_*(T'), \tilde{e}_i, \tilde{w}_i, \tilde{\epsilon}_i) \) is a Langlands dual Kashiwara’s crystal.

Remark. The definition of \( \tilde{\epsilon}_i \) is different from the one in \cite{2} 6.1.] since our definition of \( \epsilon_i \) corresponds to \( \epsilon_i^{-1} \) in \cite{2}.

Now, for a positive decorated geometric crystal \( \mathcal{X} = ((X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\epsilon_i\}_{i \in I}, f), \theta, T') \), set
\[
\tilde{B}_f := \{ \tilde{x} \in X_*(T') | \tilde{f}(\tilde{x}) \geq 0 \},
\]
where \( X_*(T') \) is identified with \( \mathbb{Z}^{\dim(T')} \). Define
\[
\tilde{B}_{f,\theta} := \{ \tilde{B}_f, \tilde{e}_i | \tilde{B}_f, \tilde{w}_i, \tilde{\epsilon}_i | \tilde{B}_f \}_{i \in I}.
\]

\textbf{Proposition 3.3 (\cite{2}).} For a positive decorated geometric crystal \( \mathcal{X} = ((X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\epsilon_i\}_{i \in I}, f), \theta, T') \), the quadruple \( \tilde{B}_{f,\theta} \) in \( \tilde{B} \) is a normal crystal.
3.2. Characters. Let $\hat{U} := \text{Hom}(U, \mathbb{C})$ be the set of additive characters of $U$. The elementary character $\chi_i \in \hat{U}$ and the standard regular character $\chi^{\text{st}} \in \hat{U}$ are defined by

$$\chi_i(x_i(c)) = \delta_{i,j} \cdot c \quad (c \in \mathbb{C}, \; i \in I), \quad \chi^{\text{st}} = \sum_{i \in I} \chi_i.$$ 

Let us define an anti-automorphism $\eta : G \to G$ by

$$\eta(x_i(c)) = x_i(c), \quad \eta(y_i(c)) = y_i(c), \quad \eta(t) = t^{-1} \quad (c \in \mathbb{C}, \; t \in T),$$

which is called the positive inverse.

The rational function $f_B$ on $G$ is defined by

$$f_B(g) = \chi^{\text{st}}(\pi^+(w_0^{-1} g)) + \chi^{\text{st}}(\pi^+(w_0^{-1} \eta(g))),$$

for $g \in B w_0 B$, where $\pi^+: B^- U \to U$ is the projection by $\pi^+(bu) = u$.

3.3. Decorated geometric crystal on $\mathbb{B}_w$. For a Weyl group element $w \in W$, define $B^-_w$ by $B^-_w := B^- \cap U\pi U$ and set $\mathbb{B}_w := TB^-_w$. Let $\gamma_i : \mathbb{B}_w \to \mathbb{C}$ be the rational function defined by

$$\gamma_i : \mathbb{B}_w \to B^- \xrightarrow{\sim} T \times U^- \xrightarrow{\text{proj}} T \xrightarrow{\alpha^i} \mathbb{C}.$$ 

For any $i \in I$, there exists the natural projection $pr_i : B^- \to B^- \cap \phi(SL_2)$. Hence, for any $x \in \mathbb{B}_w$ there exists unique $v = \left(\begin{smallmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{smallmatrix}\right) \in SL_2$ such that $pr_i(x) = \phi_i(v)$. Using this fact, we define the rational function $\varepsilon_i$ on $\mathbb{B}_w$ by

$$\varepsilon_i(x) = \frac{b_{22}}{b_{21}} \quad (x \in \mathbb{B}_w).$$

The rational $\mathbb{C}^\times$-action $e_i$ on $\mathbb{B}_w$ is defined by

$$e_i(c) := x_i((c - 1) \varphi_i(x)) \cdot x_i \left( (c^{-1} - 1) \varepsilon_i(x) \right) \quad (c \in \mathbb{C}^\times, \; x \in \mathbb{B}_w),$$

if $\varepsilon_i(x)$ is well-defined. That is, $b_{21} \neq 0$, and $e_i(x) = x$ if $b_{21} = 0$.

**Remark.** The definition of $\varepsilon_i$ is different from the one in [2]. Indeed, if we take $\varepsilon_i(x) = b_{21}/b_{22}$, then it coincides with the one in [2].

**Proposition 3.4 ([2]).** For any $w \in W$, the $5$-tuple $\chi := (\mathbb{B}_w, \{e_i\}_i, \{\gamma_i\}_i, \{\varepsilon_i\}_i, f_B)$ is a decorated geometric crystal, where $f_B$ is in (3.4), $\gamma_i$ is in (3.5), $\varepsilon_i$ is in (3.6) and $e_i$ is in (3.7).

Taking the longest Weyl group element $w_0 \in W$, let $I_0 = i_1 \ldots i_N$ be one of its reduced expressions and define the positive structure on $B^-_w$ as $\Theta^\circ_{I_0} : (\mathbb{C}^\times)^N \to B^-_{w_0}$ by

$$\Theta^\circ_{I_0}(c_1, \ldots, c_N) := y_i(c_1) \cdots y_{i_N}(c_N),$$

where $y_i(c) = y_i(c) \tilde{\alpha}(c^{-1})$, which is different from $Y_i(c)$ in [13, 14, 19, 11]. Indeed, $Y_i(c) = y_i(c^{-1})$.

We also define the positive structure on $\mathbb{B}_{w_0} = T \Theta^\circ_{I_0} : T \times (\mathbb{C}^\times)^N \to \mathbb{B}_{w_0}$ by $T \Theta^\circ_{I_0}(t, c_1, \ldots, c_N) = t \Theta^\circ_{I_0}(c_1, \ldots, c_N)$. The explicit geometric crystal structure on $\mathbb{B}_{w_0} = TB^-_{w_0}$ is described in [14, 16].

3.4. Ultra-Discretization of $\mathbb{B}_{w_0} = TB^-_{w_0}$. Applying the ultra-discretization functor to $\mathbb{B}_{w_0}$, we obtain the free crystal $\mathcal{U}D(\mathbb{B}_{w_0}) = X_*(T) \times \mathbb{Z}^N$, where $N$ is the length of the longest element $w_0$. Then define the map $\hat{h} : \mathcal{U}D(\mathbb{B}_{w_0}) = X_*(T) \times \mathbb{Z}^N \to X_*(T)(= P^*)$ as the projection to the left component and set $B_{w_0}(\lambda^\vee) := \hat{h}^{-1}(\lambda^\vee)$ and $B_{f_B, \Theta^\circ_{I_0}}(\lambda^\vee) := B_{w_0}(\lambda^\vee) \cap B_{f_B, \Theta^\circ_{I_0}}$ for $\lambda^\vee \in X_*(T) = P^*$. Let $P^+_I := \{h \in P^* | \lambda_i(h) \geq 0 \text{ for any } i \in I\}$ and for $\lambda^\vee = \sum_\iota \lambda_i \Lambda_i \in P^+_I$, we define $\lambda = \sum_\iota \lambda_i \Lambda_i \in P_+$. Then, we have

**Theorem 3.5 ([2]).** The set $B_{f_B, \Theta^\circ_{I_0}}(\lambda^\vee)$ is non-empty if $\lambda^\vee \in P^+_I$ and in that case, $B_{f_B, \Theta^\circ_{I_0}}(\lambda^\vee)$ is isomorphic to $B(\lambda)^L$, which is the Langlands dual crystal associated with $g^L$. 

Theorem 3.6 ([15]). Let $\lambda^\vee \in P^*_+$. The explicit crystal structure of $B_{f_B, \Theta_{i_0}}(\lambda^\vee)$ is as follows: For $x = (x_1, \ldots, x_N) \in B_{f_B, \Theta_{i_0}}(\lambda^\vee) \subset \mathbb{Z}^N$, we have

$$
\tilde{e}_i(x) = \begin{cases} (x_1', \ldots, x_N') & \text{if } UD(f_B)(x_1', \ldots, x_N') \geq 0, \\ 0 & \text{otherwise}, \end{cases}
$$

(3.8)

where

$$
x_j' = x_j + \min_{1 \leq m < j, i_m = i} \left( n + \sum_{k=1}^m a_{i_k, i} x_k, \frac{1}{2} \sum_{k=1}^m a_{i_k, i} x_k \right),
$$

(3.9)

$$
\tilde{\varepsilon}_i(x) = \max_{1 \leq m \leq N, i_m = i} \left( x_m + \sum_{k=m+1}^N a_{i_k, i} x_k, \right),
$$

(3.10)

where $x = (x_1, \ldots, x_N)$ belongs to $B_{f_B, \Theta_{i_0}}(\lambda^\vee)$ if and only if $UD(f_B)(x) \geq 0$.

It follows immediately from [9A):

**Lemma 3.7.** Set $X_m := \sum_{k=1}^m a_{i_k, i} x_k$, $\lambda^{(i)} := \min\{X_m | 1 \leq m \leq N, i_m = i\}$ (i.e., $\lambda^{(i)}$ is as follows:

(3.11) $\tilde{e}_i(x) = \begin{cases} (x_1, \ldots, x_m - 1, \ldots, x_N) & \text{if } UD(f_B)(x_1, \ldots, x_m - 1, \ldots, x_N) \geq 0, \\ 0 & \text{otherwise}, \end{cases}$

(3.12) $\tilde{\varepsilon}_i(x) = \begin{cases} (x_1, \ldots, x_m + 1, \ldots, x_N) & \text{if } UD(f_B)(x_1, \ldots, x_m + 1, \ldots, x_N) \geq 0, \\ 0 & \text{otherwise}. \end{cases}$

Finally, due to the results in Sect.2 and in this section, we obtain the following theorem

**Theorem 3.8 ([15]).** If we have $B_{f_B, \Theta_{i_0}}(\lambda^\vee) = \Sigma_{i_0-1}[\lambda]^L$ as a subset of $\mathbb{Z}^N$, then they are isomorphic to each other as crystals, where $L$ means the Langlands dual crystal, that is, it is defined by the transposed Cartan matrix and $i_0^{-1}$ refers to $i_0$ in opposite order.

4. Explicit form of the decoration $f_B$ of type $D_n$

4.1. Generalized Minors and the function $f_B$. For this subsection, see [3, 4, 5]. Let $G$ be a simply connected simple algebraic group over $\mathbb{C}$ and $T \subset G$ a maximal torus. Let $X^*(T) := \text{Hom}(T, \mathbb{C}^*)$ and $X_*(T) := \text{Hom}(\mathbb{C}^*, T)$ be the lattice of characters and co-characters respectively. We identify $P$ (resp. $P^*$) with $X^*(T)$ (resp. $X_*(T)$) as above.

**Definition 4.1.** For $\mu \in P_+$, the principal minor $\Delta_\mu : G \to \mathbb{C}$ is defined as

$$
\Delta_\mu(u^\pm v^+) := \mu(t) \quad (u^\pm \in U^\pm, \ t \in T).
$$

Let $\gamma, \delta \in P$ be extremal weights such that $\gamma = u\mu$ and $\delta = v\mu$ for some $u, v \in W$. Then the generalized minor $\Delta_{\gamma, \delta}$ is defined by

$$
\Delta_{\gamma, \delta}(g) := \Delta_\mu((u^{-1}g)^\vee) \quad (g \in G),
$$

which is a regular function on $G$. 
Proposition 4.2 [2]. The function $f_B$ in (4.4) is described as:

$$f_B(g) = \sum_i \frac{\Delta_{w_0, \Lambda_i, \Lambda_i}(g) + \Delta_{w_0, \Lambda_i, \Lambda_i}(g)}{\Delta_{w_0, \Lambda_i, \Lambda_i}(g)}.$$  

Let $i = i_1 \cdots i_N$ be a reduced word for the longest Weyl group element $w_0$. For $t\Theta_i^{-1}(c) \in B_{w_0} = T \cdot B_{w_0}$, we get the following formula.

$$f_B(t\Theta_i^{-1}(c)) = \sum_i \Delta_{w_0, \Lambda_i, \Lambda_i}(\Theta_i^{-1}(c)) + \alpha_i(t)\Delta_{w_0, \Lambda_i, \Lambda_i}(\Theta_i^{-1}(c)).$$

4.2. Explicit form of $f_B(t\Theta_i^{-1}(c))$ of type $D_n$. The results in this subsection are given in [16]. Fix the cyclic reduced longest word $i_0 = (12 \cdots n - 1)n^{-1}$.

Theorem 4.3 [16]. For $k \in \{1, 2, \cdots, n\}$ and $c = (c_j^{(i)}) = (c_1^{(1)}, c_2^{(1)}, \cdots, c_{n-1}^{(1)}, c_n^{(1)}) \in (\mathbb{C}^*)^{(n-1)}$, we have

$$\Delta_{w_0, \Lambda_k, \Lambda_k}(\Theta_i^{-1}(c)) = \frac{i_{1(k)}^{(k)}}{c_1^{(k)}} + \frac{i_{2(k)}^{(k)}}{c_{n-2}^{(k)}} + \cdots + \frac{i_{n-1(n)}^{(k)}}{c_{n-1}^{(k)}},$$

$$\Delta_{w_0, \Lambda_{n-1, n-1}}(\Theta_i^{-1}(c)) = c_{n-1}^{(n-1)},$$

$$\Delta_{w_0, \Lambda_n, \Lambda_n}(\Theta_i^{-1}(c)) = c_n^{(n-1)}.$$  

Theorem 4.4 [16]. Let $k$ be in $\{1, 2, \cdots, n-2\}$. Then we have

$$\Delta_{w_0, \Lambda_k, \Lambda_k}(\Theta_i^{-1}(c)) = 1 + \sum_{j=1}^{k-1} \frac{c_j^{(j)}}{c_{k-j+1}}.$$  

The cases $k = n - 1, n$ will be presented in [13].

4.3. $\Delta_{w_0, \Lambda_{n-1, n-1}}(\Theta_i^{-1}(c))$ and $\Delta_{w_0, \Lambda_{n-1, n}}(\Theta_i^{-1}(c))$. To state the results for $\Delta_{w_0, \Lambda_{n-1, n-1}}(\Theta_i^{-1}(c))$ and $\Delta_{w_0, \Lambda_{n-1, n}}(\Theta_i^{-1}(c))$, we need to prepare the set of triangles $\Delta'_n$ for type $D_n$:

$$\Delta'_n := \{(j_l^{(l)} | 1 \leq k \leq l < n) | 1 \leq j_k^{(l)} \leq j_{k+1}^{(l)} < j_{k+1}^{(l+1)} \leq n \ (1 \leq k \leq l < n - 1)\}.$$  

We visualize a triangle $(j_k^{(l)})$ in $\Delta'_n$ as follows:

$$\begin{align*}
(j_k^{(l)}) = & \begin{pmatrix}
j_1^{(1)}

j_2^{(2)} & j_1^{(2)}

j_3^{(3)} & j_2^{(3)} & j_1^{(3)}

\vdots \vdotspresent \vdotspresent \vdotspresent

j_{n-1}^{(n-1)} & \cdots & j_{2}^{(n-1)} & j_1^{(n-1)}
\end{pmatrix}
\end{align*}$$

Lemma 4.5. For any $k \in \{1, 2, \cdots, n - 1\}$ there exists a unique $j$ $(1 \leq j \leq k)$ such that the $k$th row of a triangle $(j_k^{(l)})$ in $\Delta'_n$ is in the following form:

$$\begin{align*}
(j_k^{(l)}) &= (k+1, k-1, \cdots, j, j+1, j-1, j-2, \cdots, 2, 1),
\end{align*}$$

that is, we have $j_{m}^{(k)} = m$ for $m < j$ and $j_{m}^{(k)} = m + 1$ for $m \geq j$.

For a triangle $\delta = (j_k^{(l)}) \in \Delta'_n$, we list $j$'s as in the lemma: $s(\delta) = (s_1, s_2, \cdots, s_{n-1})$, which we call the label of a triangle $\delta$.

Lemma 4.6. $| \Delta'_n | = 2^{n-1}$.  

LEMMA 4.7. For any \( \delta \in \Delta'_n \) let \( s(\delta) := (s_1, s_2, \cdots, s_{n-1}) \) be its label. Then

(i) The label \( s(\delta) \) satisfies \( 1 \leq s_k \leq k + 1 \) and \( s_{k+1} = s_k \) or \( s_k + 1 \) for \( k = 1, \cdots, n - 1 \).

(ii) Each \( k \)-th row of a triangle \( \delta \) is in one of the following I, II, III, IV:

\begin{align*}
\text{I.} & \quad s_{k+1} = s_k + 1 \text{ and } s_k = s_{k-1} - 1. \\
\text{II.} & \quad s_{k+1} = s_k \text{ and } s_k = s_{k-1}.
\end{align*}

\begin{align*}
\text{III.} & \quad s_{k+1} = s_k + 1 \text{ and } s_k = s_{k-1} + 1. \\
\text{IV.} & \quad s_{k+1} = s_k \text{ and } s_k = s_{k-1} + 1.
\end{align*}

Here we suppose that \( s_0 = 1 \) and \( s_n = s_{n-2} + 1 \), which means that the 1st row must be in I, II, III or IV and the \( n-1 \)-th row is in I or IV.

Now, we associate a Laurent monomial \( m(\delta) \) in variables \((c_i^{(j)})_{(j,i)\in[1,n-1]\times[1,n]}\) with a triangle \( \delta = (j_k^{(l)}) \) by the following way.

(i) Let \( s = (s_1, \cdots, s_{n-1}) \) be the label of \( \delta \in \Delta'_n \).

(ii) Suppose \( i \)-th row is in the form I. If \( 1 \leq i \leq n - 2 \), then associate \( c_i^{(s_i)} \). For \( i = n - 1 \),

(a) if \( n + s_{n-1} \) is even, then associate \( c_{n-1}^{(s_{n-1})} \).

(b) if \( n + s_{n-1} \) is odd, then associate \( c_{n-1}^{(s_{n-1})} \).

(iii) Suppose \( i \)-th row is in the form IV. If \( 1 \leq i \leq n - 2 \), then associate \( c_i^{-1} \). For \( i = n - 1 \),

(a) if \( n + s_{n-1} \) is even, then associate \( c_{n-1}^{-1} \).

(b) if \( n + s_{n-1} \) is odd, then associate \( c_{n-1}^{-1} \).

(iv) If \( i \)-th row is in the form II or III, then associate 1.

(v) Take the product of all monomials as above for \( 1 \leq i < n \), then we obtain the monomial \( m(\delta) \) associated with \( \delta \).

Here we define the involutions \( \xi \) and \( \dashv \) on Laurent monomials in \((c_i^{(j)})_{(j,i)\in[1,n-1]\times[1,n]}\).

\begin{align}
\xi : c_i^{(k)} \mapsto c_i^{(k)}, \quad c_i^{(k)} \mapsto c_i^{(k)}, \quad c_i^{(k)} \mapsto c_i^{(k)} \quad (j \neq n - 1, n), \\
\dashv : c_i^{(j)} \mapsto c_i^{(n-j)}.
\end{align}

Let us denote the special triangle such that \( j_k^{(l)} = k + 1 \) (resp. \( j_k^{(l)} = k \)) for any \( k, l \) by \( \delta_k \) (resp. \( \delta_k \)). Indeed, we have

\begin{align}
m(\delta_k) = \begin{cases} 
\check{c}_i^{(1)} & \text{if } n \text{ is odd}, \\
\check{c}_i^{(1)} & \text{if } n \text{ is even},
\end{cases} \quad m(\delta_l) = c_i^{(n)}^{-1}.
\end{align}

Now, we present \( \Delta_{s_0\cdots s_{n-1}, s_{n-1}}(\Theta_{4n}^{-}(c)) \) and \( \Delta_{s_0\cdots s_{n-1}, s_{n-1}}(\Theta_{4n}^{-}(c)) \) for type \( D_n \):

THEOREM 4.8. For type \( D_n \), we have the explicit forms:

\begin{align}
\Delta_{s_0\cdots s_{n-1}, s_{n-1}}(\Theta_{4n}^{-}(c)) &= \sum_{\delta \in \Delta'_n \setminus \{\delta_l\}} m(\delta), \\
\Delta_{s_0\cdots s_{n-1}, s_{n-1}}(\Theta_{4n}^{-}(c)) &= \sum_{\delta \in \Delta'_n \setminus \{\delta_l\}} \xi \circ m(\delta).
\end{align}
EXAMPLE 4.9. The set of triangles $\Delta'_n$ is as follows:

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 \\
21 & 21 & 21 & 31 & 31 & 31 & 31 & 21 \\
321 & 321 & 421 & 421 & 421 & 421 & 421 & 421 \\
4321 & 5321 & 5321 & 5321 & 5421 & 5421 & 5421 & 5421 \\
1 & 2 & 2 & 1 & 2 & 2 & 2 & 2 \\
31 & 31 & 32 & 31 & 31 & 32 & 32 & 32 \\
431 & 431 & 431 & 431 & 431 & 431 & 432 & 432 \\
5421 & 5421 & 5421 & 5431 & 5431 & 5431 & 5431 & 5432 \\
\end{array}
\]

and their labels $s(\delta)$ are

\[
(2, 3, 4, 5), \quad (2, 3, 4, 4), \quad (2, 3, 3, 4), \quad (2, 2, 3, 4), \quad (1, 2, 3, 4), \quad (1, 2, 3, 3), \quad (2, 2, 3, 3), \quad (2, 3, 3, 3),
\]
\[
(2, 2, 2, 3), \quad (1, 2, 2, 3), \quad (1, 1, 2, 3), \quad (2, 2, 2, 2), \quad (1, 2, 2, 2), \quad (1, 1, 2, 2), \quad (1, 1, 1, 2), \quad (1, 1, 1, 1).
\]

Then, we have the corresponding monomials $m(\delta)$:

\[
\begin{aligned}
c_5^{(0)} & , & c_4^{(1)} & , & c_2^{(2)} & , & c_1^{(3)} & , & c_4^{(1)} & , & c_1^{(4)} & , & c_4^{(2)} & , & c_2^{(3)} \\
& , & c_2^{(3)} & , & c_2^{(2)} & , & c_3^{(3)} & , & c_2^{(4)} & , & c_3^{(3)} & , & c_2^{(5)} & , & c_3^{(4)} \\
& , & c_2^{(3)} & , & c_2^{(2)} & , & c_3^{(3)} & , & c_2^{(4)} & , & c_3^{(3)} & , & c_2^{(5)} & , & c_3^{(4)} \\
& , & c_2^{(3)} & , & c_2^{(2)} & , & c_3^{(3)} & , & c_2^{(4)} & , & c_3^{(3)} & , & c_2^{(5)} & , & c_3^{(4)} \\
& & & & & & & & & & & & & & & 1 \\
\end{aligned}
\]

Thus, we have

\[
\Delta_{\omega_0s_n, \Lambda_n, \Lambda_n}(\Theta_{\ell_0})(c) = c_3^{(1)} c_5^{(1)} + c_2^{(2)} c_4^{(1)} + c_1^{(3)} c_4^{(1)} + c_4^{(2)} + c_2^{(3)} c_4^{(2)} + c_2^{(3)} c_4^{(2)} + c_2^{(3)} c_4^{(2)} + c_2^{(3)} c_4^{(2)}
\]

5. Ultra-Discretization and Polyhedral Realizations of type $D_n$

5.1. Ultra-discretization of $UD(f_B)(x)$. Since in this section we treat the type $D_n$, we identify $P$ with $P^*$ by $\lambda \leftrightarrow \lambda^\vee$. Let us describe the explicit form of $B_{f_B, \Theta_{\ell_0}}(\lambda)$ for type $D_n$, applying the result in Theorem 3.6 and show the coincidence of the crystals $B_{f_B, \Theta_{\ell_0}}(\lambda)$ and $\Sigma_{\ell_0}[\lambda]$ in Sect.2 using Theorem 4.8.

For type $D_n$ let $\ell_0$ be as in 3.2. We shall see the explicit form of $UD(f_B)(x)$. Indeed, due to 4.2, it is sufficient to know the forms of $\Delta_{\omega_0s_n, \Lambda_n, \Lambda_n}(\Theta_{\ell_0})(c)$ and $\Delta_{\omega_0s_n, \Lambda_n, \Lambda_n}(\Theta_{\ell_0})(c)$, which are given in Theorem 4.3 and Theorem 4.3. Thus, we have

\[
UD(f_B)(t, x) = \min_{1 \leq j \leq n} (UD(\Delta_{\omega_0s_n, \Lambda_n, \Lambda_n}(\Theta_{\ell_0}))(x), UD(\alpha_j(t)) + UD(\Delta_{\omega_0s_n, \Lambda_n, \Lambda_n}(\Theta_{\ell_0}))(x))
\]

and

\[
UD(\Delta_{\omega_0s_n, \Lambda_n, \Lambda_n}(\Theta_{\ell_0}))(x)
\]

\[
= \min \{x_{i}^{(k)} - x_{i-1}^{(k+1)} | i = 1, \cdots, n - 2 \} \cup \{x_{n-1}^{(k)} + x_{n}^{(k+1)} - x_{n-2}^{(k+1)}, x_{n-1}^{(k+1)} - x_{n}^{(k+1)}
\]

\[
\cup \{x_{n-k}^{(k)} - x_{n-k+1}^{(k+1)} | k \geq 2 \} \cup \{x_{n-k}^{(j)} - x_{n-k-j}^{(j)} | k + 2 \leq j \leq n \} \} = \min \Xi_k \quad (1 \leq k \leq n - 2),
\]

\[
UD(\Delta_{\omega_0s_n, \Lambda_n, \Lambda_n}(\Theta_{\ell_0})) = x_{n-1}^{(n-1)} \in \Xi_{n-1}, \quad UD(\Delta_{\omega_0s_n, \Lambda_n, \Lambda_n}(\Theta_{\ell_0}))(x) = x_{n}^{(n-1)} \in \Xi_{n}.
\]

\[
UD(\Delta_{\omega_0s_n, \Lambda_n, \Lambda_n}(\Theta_{\ell_0}))(x) = \min \{x_{k-j}^{(j)} - x_{k-j+1}^{(j)} | j = 1, 2, \cdots, k \} = \min \Xi_k \quad (1 \leq k \leq n - 2).
\]
where \( x^{(j)}_i = UD(a_i^{(j)}) \) and we understand \( x^{(m)}_0 = 0 \). Here \( UD(\Delta_{w_0 s_h, \Lambda_k}(\Theta_k))((x)) \) \((k = n - 1, n)\) is given in the next subsection.

5.2. Explicit correspondence for \( k = n - 1, n \). Now, let us see \( UD(\Delta_{w_0 s_h, \Lambda_k}(\Theta_k))((x)) \) = \( \min \mathbb{Z}_k \) for \( k = n - 1, n \). For the purpose, we define the map from the set of labels of triangles. Let \( M_n \) be the set of admissible patterns as before and \( S_n \) be the set of labels of \( \Delta_n \) given in the next subsection.

\[
S_n := \{ s = (s_1, \ldots, s_{n-1}) | 1 \leq s_1 \leq \cdots \leq s_{n-1} < n, |s_k - s_{k-1}| \leq 1 \ (k = 1, 2, \cdots, n - 1) \}.
\]

where we set \( s_0 = 1 \) and \( s_n = s_{n-2} + 1 \). Set \( K_n := (n, n - 1, \cdots, 3, 2) \in M_n \) and for \( m \in [1, n - 1] \)

\[
U_m := (1, 1, \ldots, 1, 1).
\]

For \( a = (a_1, a_2, \cdots, a_l) \in \mathbb{Z}_l^l \), set

\[
a^{-1} := (a_l, \cdots, a_2, a_1).
\]

Now, let us define the map \( F : M_n \to S_n \) by

\[
F(\mu) = F(\mu_1, \cdots, \mu_l) := (K_n - U_{\mu_1} - \cdots - U_{\mu_l})^{-1}.
\]

We shall see the well-definedness and bijectivity of this map.

**Example 5.1.** For \( n = 5 \), we have

\[
F(4, 3, 1) = ((5, 4, 3, 2) - (1, 1, 1, 1) - (1, 1, 1) - (1))^{-1} = (2, 2, 1, 1)^{-1} = (1, 1, 2, 2),
\]

\[
F(3, 2) = ((5, 4, 3, 2) - (1, 1, 1) - (1, 1))^{-1} = (3, 2, 2, 2)^{-1} = (2, 2, 2, 3).
\]

**Lemma 5.2.** The map \( F : M_n \to S_n \) is a bijective map.

**Proof.** Note that \( \sharp M_n = \sharp S_n = 2^{n-1} - 1 \). The injectivity of \( F \) is trivial if it is well-defined. Thus, its bijectivity is evident by these facts. So, let us show the well-definedness of the map \( F \), that is, \( F(\mu) \) is in \( S_n \) for any \( \mu \in M_n \). Let us show this for \( \mu = (m_1, \cdots, m_l) \) by the induction on \( l \). For \( l = 1 \), set \( \mu = (m) \in M_n \) \((1 \leq m \leq n - 1)\). Then we have

\[
F(\mu) = (2, 3, \cdots, n - m, n - m, n - m + 1, \cdots, n - 1),
\]

which turns out to be an element in \( S_n \). For \( \mu' = (\mu_2, \cdots, \mu_l) \) with \( 1 \leq \mu_1 < \cdots < \mu_2 < n - 1 \), assume that \( F(\mu') := (a_1, \cdots, a_{n-1}) \) is an element in \( S_n \). These \( a_1, \cdots, a_{n-1} \) satisfy that

\[
a_j = j + 1 \text{ for } j = 1, 2, \cdots, n - \mu_2 \text{ and } a_{n-\mu_2+1} = n - \mu_2 + 1 = a_{n-\mu_2}.
\]

Let \( \mu_1 \) be a positive integer with \( \mu_2 < \mu_1 < n \) and \( \mu := (\mu_1, \mu_2, \cdots, \mu_l) \), which is in \( M_n \). By the definition of \( F \), we obtain

\[
F(\mu) = (a_1, \cdots, a_{n-\mu_1}, a_{n-\mu_1+1} - 1, \cdots, a_{n-1} - 1).
\]

Since \( (a_1, a_2, \cdots, a_{n-1}) \) is in \( S_n \), then we easily find that \( F(\mu) \) is also in \( S_n \). As mentioned above, the bijectivity is evident from the injectivity of \( F \) and the fact that \( \sharp M_n = \sharp S_n \) by Lemma 2.6 and Lemma 4.6.

Let us see the explicit correspondence of the set of linear forms \( \Xi_k \) and \( \Xi_k' \) to the ultra-discretizations of \( \Delta_{w_0 s_h, \Lambda_k} \) and \( \Delta_{w_0 s_h, \Lambda_k'} \).
Lemma 5.3. For $\mu \in M_n$, let $\varphi'_\mu(x)$ (resp. $\varphi_\mu(x)$) be the linear forms as in (2.16) (resp. (2.15)), and for $s \in S_n$ let us denote the corresponding triangle by $\delta_s$ and let $m(\delta_s)$ be the corresponding monomial as in Sect.4. Then we have for any $\mu \in M_n$

(5.3) \[ UD(m(\delta_F(\mu)))(x) = \varphi'_\mu(x), \]
(5.4) \[ UD(\xi \circ m(\delta_{F(\mu)})) = \varphi_\mu(x). \]

Proof. Let us show (5.3). We shall see the conditions that each variable $x^{(j)}_i$ appears in the left-hand side and the right-hand side of (5.3). Note that each coefficient of variable $x^{(j)}_i$ in both sides of (5.3) can be $\pm 1$ or 0. First, let us see the right-hand side. In the case $i \neq n-1,n$, due to the explicit description in (2.16) for $\mu = (\mu_1, \ldots, \mu_l) \in M_n$ we can see that $x^{(j)}_i$ appears in $\varphi'_\mu(x)$ iff there exists $k$ such that $j = \mu_k + k - 1, i = n - \mu_k - 1$ and $\mu_k + k - 1 \neq \mu_{k-1} + k - 2$ or $n - \mu_k - 1 \neq n - \mu_{k-1}$, which is equivalent to the condition $\mu_k < \mu_{k-1} - 1$. Similarly, $-x^{(j)}_i$ appears in $\varphi'_\mu(x)$ iff there exists $k$ such that $j = \mu_k + k - 1, i = n - \mu_k$ and $\mu_{k+1} < \mu_k - 1$. Considering the cases $i = n-1,n$ similarly, we obtain the following results for a admissible pattern $\mu = (\mu_1, \ldots, \mu_l)$:

Lemma 5.4. (i) $x^{(j)}_i (i \neq n-1,n)$ appears in $\varphi'_\mu(x)$ iff there exists $k$ such that $j = \mu_k + k - 1, i = n - \mu_k - 1$ and $\mu_k < \mu_{k-1} - 1$.
(ii) $-x^{(j)}_i (i \neq n-1,n)$ appears in $\varphi'_\mu(x)$ iff there exists $k$ such that $j = \mu_k + k - 1, i = n - \mu_k$ and $\mu_{k+1} < \mu_k - 1$.
(iii) $x^{(j)}_n$ appears in $\varphi'_\mu(x)$ iff $\mu_l \geq 2, l = j$ and $j$ is odd.
(iv) $x^{(j)}_n$ appears in $\varphi'_\mu(x)$ iff $\mu_l \geq 2, l = j$ and $j$ is even.
(v) $-x^{(j)}_n$ appears in $\varphi'_\mu(x)$ iff $\mu_l = 1, l = 1$ and $j$ is even.
(vi) $-x^{(j)}_n$ appears in $\varphi'_\mu(x)$ iff $\mu_l = 1, l = 0$ and $j$ is odd.

Next, let us see the conditions that $c^{(j)}_i$ appears in $m(\delta_s)$. By the recipe in Sect.4 to obtain $m(\delta_s)$, we have the following results for a label $s = (s_1, \ldots, s_{n-1}) \in S_n$:

Lemma 5.5. (i) $c^{(j)}_i (i \neq n-1,n)$ appears in $m(\delta_s)$ iff $j = n - s_i, s_i = s_{i-1} + 1$ and $s_{i+1} = s_i$.
(ii) $c^{(j)}_{n-1} (i \neq n-1,n)$ appears in $m(\delta_s)$ iff $j = n - s_i, s_i = s_{i-1} + 1$ and $s_{i+1} = s_i + 1$.
(iii) $c^{(j)}_{n-1}$ appears in $m(\delta_s)$ iff $j = n - s_{n-1}, n + s_{n-1}$ is odd and $s_{n-1} = s_{n-2} + 1, s_n = s_{n-1}$.
(iv) $c^{(j)}_{n-1}$ appears in $m(\delta_s)$ iff $j = n - s_{n-1}, n + s_{n-1}$ is even and $s_{n-1} = s_{n-2} + 1, s_n = s_{n-1}$.
(v) $c^{(j)}_{n-1}$ appears in $m(\delta_s)$ iff $j = n - s_{n-1}, n + s_{n-1}$ is even and $s_{n-1} = s_{n-2} + 1, s_n = s_{n-1}$.
(vi) $c^{(j)}_{n-1}$ appears in $m(\delta_s)$ iff $j = n - s_{n-1}, n + s_{n-1}$ is odd and $s_{n-1} = s_{n-2} + 1, s_n = s_{n-1} + 1$.

Here note that we assumed that $s_n = s_{n-2} + 1$ beforehand. We find that each condition in both lemmas are equivalent through the map $F$. Let us see that the conditions in Lemma 5.4 (i) and Lemma 5.5 (i) are equivalent via the map $F$. Assume that $j = \mu_k + k - 1, i = n - \mu_k - 1$ and $\mu_k < \mu_k - 1$. Since $i = n - \mu_k - 1$, for $K^{-1} = (k_1, \ldots, k_{n-1})$ we have $k_i = n - \mu_k$. And then for $s = (s_j) = F(\mu)$ we have $s_i = k_i - (k - 1) = n - \mu_k - k + 1$, which means $n - s_i = \mu_k - k + 1 = j$. It follows from $\mu_k < \mu_k - 1$ that $s_i = s_{i-1} + 1$ and $s_{i+1} = s_i$. The inverse implication can be shown similarly. The equivalence for the cases (ii)-(vi) are also shown similarly. Thus, we obtain (5.3).

Thus, by Lemma 5.3 we have

Proposition 5.6. For $k = n - 1, n$ we have the following:

(5.5) \[ UD(\Delta_{\Theta_{m,K}^{(k)}(\mu)}(x)) = \min \Xi_k. \]
Thus, if we identify $UD(\alpha_k(t))$ with $\lambda_k$ ($k = 1, 2, \cdots, n$), then the condition $UD(f_B)(\lambda, x) \geq 0$ in Theorem 3.6 is equivalent to the condition in (2.18) in Theorem 2.7.

Now, by Theorem 3.8 we obtain the following theorem:

**Theorem 5.7.** In $D_n$ case, for any dominant integral weight $\lambda$, there exists an isomorphism of crystals $B_{f_B, \Theta^{-i_0}}[\lambda] \cong \sum_{i_0}[\lambda]$ where $\sum_{i_0}[\lambda]$ is as in Theorem 2.7 and $i_0 = i_0^{-1}$.

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