Stability analysis of axisymmetric supersonic wakes using various basic states

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Abstract. Two-dimensional stability analyses are conducted of turbulent axisymmetric supersonic wakes at \( M = 2.46 \). The aim is to investigate which azimuthal modes are dominant and how the stability behaviour is influenced by the choice of basic state. To that end, axisymmetric (two dimensional) and three dimensional direct numerical simulations (DNS) with either laminar or turbulent inflow conditions were conducted of supersonic wakes to provide the respective basic states for the stability analysis. The global stability analyses were then performed by computing the temporal pulse response using forced Navier-Stokes simulations for each basic state. Using the time- and azimuthally-averaged data from the 3D DNS with turbulent inflow as basic state, an absolute instability of the axisymmetric mode at a Reynolds number, based on wake-generating body diameter and freestream velocity, \( Re_D = 100,000 \) was found. This is in contrast to results obtained earlier using an axisymmetric flow solution as the basic state. The linear stability analysis for all basic states is presented showing the temporal growth of various azimuthal modes and the respective radial mode shapes. The results are also contrasted to those obtained from nonlinear DNS.

1. Introduction

The interest in supersonic axisymmetric wakes, or base flows, has originally been due to gaining a better understanding of the dynamics of supersonic turbulent flows, and to devise methods for drag reduction, such as boat-tailing, base-bleed, etc. Later, base flows were frequently chosen as a challenging test case for numerical simulations, in particular for RANS or hybrid RANS/LES models. This is mainly due to the availability of reliable data from carefully conducted axisymmetric base flow experiments (see, e.g., Herrin & Dutton, 1989), and the fact that a complex flow is generated by a relatively simple geometry, facilitating grid generation. Furthermore, the failure of early RANS simulations to capture some of the characteristic properties of the flow, e.g. a flat base pressure distribution (Sahu et al., 1985), has motivated studies employing various turbulence models.

Nevertheless, the focus of most numerical investigations has mostly been on matching steady mean flow data from the experiments, rather than attempting to identify the most relevant instability mechanisms and understand the effect of large-scale vortical structures resulting from these instabilities on the mean flow. A reason for this might be that well resolved eddy-resolving simulations of a high Reynolds number base flow are computationally very expensive and that the resulting flow is strongly non-parallel and thus makes a meaningful linear stability analysis
more challenging. In classical linear stability theory (LST) the development of small amplitude perturbations is considered in a parallel flow, i.e. a flow containing only one inhomogeneous spatial direction. Mathematically, the problem can be formulated as an eigenvalue problem (EVP) that requires the solution of the Orr-Sommerfeld equation. In an extensive review on different classes of instabilities for spatially growing flows, Huerre & Monkewitz (1990) introduced the concept of global instability in the context of slow variations of the basic flow in the streamwise direction. If, however, the basic state is truly non-parallel, another approach is required. A multidimensional eigenvalue problem can be solved with the underlying basic state being a fully two-dimensional, steady or time-periodic non-parallel flow as presented in Theofilis (2003). However, the resulting multidimensional EVP becomes computationally expensive to solve. Alternatively, the linearised Navier–Stokes equations can be numerically integrated to conduct linear stability investigations using a two-dimensional basic state.

Sandberg & Fasel (2006) conducted complementary DNS and linear stability calculations of base flows for Reynolds numbers up to 100,000 to identify the fundamental hydrodynamic mechanisms leading to the generation of large structures. However, in all cases a laminar approach boundary layer was specified and the basic flow chosen for all linear stability computations was obtained from precursor axisymmetric (two dimensional) DNS. This paper aims at investigating the effect of using various basic states for the stability analysis. In order to obtain different basic states, axisymmetric and three dimensional direct numerical simulations (DNS) with either laminar or turbulent inflow conditions were conducted of supersonic wakes.

2. Numerical Setup

2.1. Direct Numerical Simulations

The compressible Navier–Stokes equations for conservative variables are solved in cylindrical coordinates using a newly developed finite-difference code. For the DNS presented here, a 4th-order central difference scheme with Carpenter boundary stencils is applied for the spatial discretization in the radial and streamwise directions. A spectral method is used in the azimuthal direction, enabling an axis treatment that exploits parity conditions of individual Fourier modes (Sandberg, 2011). Time marching is achieved by an ultra low-storage 4th-order Runge–Kutta scheme (Kennedy et al., 2000). The stability of the code is enhanced by a skew-symmetric splitting of the nonlinear terms (Kennedy & Gruber, 2008). In addition, an 11 point wave-number optimized filter (Bogey et al., 2009) is used after each full Runge–Kutta cycle with a weighting of 0.5 to remove grid-to-grid-point oscillations that might occur using a central scheme to capture shock waves. For the simulation requiring a turbulent inflow, a mean turbulent profile from a precursor simulation is specified at the inlet, superposed with turbulent fluctuations calculated using a digital filter technique (Touber & Sandham, 2009).

The code has been validated by carrying out simulations of oblique instability waves in supersonic boundary layers, matching linear stability results, and by running low Mach number wakes reproducing incompressible reference data. Also, turbulent pipe flow simulations have been conducted showing good agreement with recently published results (Wu & Moin, 2008).

The computational domain comprises three blocks (see figure 1), which can be classified as subdomains, containing: approach flow (block 1) and near and developing wake downstream of the body (blocks 2, 3). The size of each block, along with the corresponding number of grid points and number of subdomains in the streamwise (z) and radial (r) directions, is given in table 1 for each case. In the azimuthal direction 128 Fourier modes were employed (corresponding to 258 collocation points in physical space), resulting in a total of $78 \times 10^6$ and $177 \times 10^6$ grid points for the laminar and turbulent approach flow cases, respectively.
Table 1. Size, number of grid points and the number of processors for each block in the computational domain (see figure 1). All dimensions are normalized with the radius $R$ of the axisymmetric wake-producing body.

| Inflow condition | Block No. | Block 1 | Block 2 | Block 3 | total |
|------------------|-----------|---------|---------|---------|-------|
| Laminar          | $L_z \times L_r$ | 4.0 $\times$ 9.0 | 14.0 $\times$ 8.0 | 14.0 $\times$ 1.0 | N/A   |
| $N_z \times N_r$ | 165 $\times$ 72 | 1353 $\times$ 72 | 1353 $\times$ 144 | 0.304 $\times$ 10^6 |
| $Np_z \times Np_r$ | 5 $\times$ 4 | 41 $\times$ 4 | 41 $\times$ 8 | 512 |
| Turbulent        | $L_z \times L_r$ | 4.0 $\times$ 9.0 | 14.0 $\times$ 8.0 | 14.0 $\times$ 1.0 | N/A   |
| $N_z \times N_r$ | 288 $\times$ 168 | 1632 $\times$ 168 | 1632 $\times$ 224 | 0.688 $\times$ 10^6 |
| $Np_z \times Np_r$ | 12 $\times$ 12 | 68 $\times$ 12 | 68 $\times$ 16 | 2048 |

Figure 1. Sketch of computational domain.

2.2. Linear Stability Analysis

The multidimensional stability analysis was conducted using forced Navier–Stokes simulations. At the start of each simulation the right-hand-side of the Navier–Stokes equations, containing all spatial derivatives of the basic state, is computed and stored. The simulation is then progressed, subtracting the stored forcing term at each Runge–Kutta substep.

The result is that assuming there is no change or perturbation to the flow field, the initial condition can be maintained as a reference state, upon which the behaviour of small perturbations can be investigated. This method has been successfully used for stability investigations of airfoil flows in Jones et al. (2010). In the current case, an initial low-amplitude perturbation is added to the simulation and the pulse-response in each case is evaluated. For the stability analyses, the same numerical procedure and the same axial and streamwise grids were used as for the DNS. For all basic states considered, in the azimuthal direction, eight Fourier modes were considered.

3. Results

All simulations presented here were conducted at $M = 2.46$ and a Reynolds number, based on diameter of wake-generating body and freestream velocity, $Re_D = 1 \times 10^5$.

3.1. Direct Numerical Simulations

Instantaneous contours of the streamwise density gradient, as shown in figure 2, illustrate the considerable difference between the cases with a laminar or turbulent inflow condition. In case of a laminar approach flow the initial shear layer is laminar and the onset of transition to turbulence does not occur until $z \approx 2.5$. In contrast, using a turbulent approach flow a fully
turbulent initial shear layer can be observed. The increased mixing of the fully turbulent shear layers results in a larger turning angle of the flow at separation, leading to a significantly shorter recirculation region. As shown in Sandberg (2010), the DNS using a turbulent inflow condition shows surprising resemblance with the experimental data despite the Reynolds number being smaller than in the experiments by a factor of roughly 30. This indicates that the DNS data presented here, used as basic state for the stability analysis, is representative of the higher Reynolds number flow experimentally studied.

3.2. Stability Analysis

The multi-dimensional stability analysis via forced direct numerical simulations was performed for four different basic states. The first basic state was obtained from a converged two dimensional (axisymmetric) simulation with a laminar inflow condition for the approach boundary layer, as already considered in Sandberg & Fasel (2006). A basic state was also obtained from a converged axisymmetric simulation using a turbulent mean flow profile as inflow condition for the approach boundary layer. It was shown in Sandberg (2010) that the two axisymmetric solutions are qualitatively very similar. Further, basic states were obtained by averaging data from 3D DNS using either laminar or turbulent inflow conditions in the azimuthal
direction and in time.

Figure 3 shows the temporal response of the zeroth azimuthal (axisymmetric) mode with respect to an initial pulse imposed on the various basic states with amplitude $10^{-8}$. For the basic states obtained from axisymmetric simulations the axisymmetric mode appears stable at this Reynolds and Mach number combination, as also observed in Sandberg & Fasel (2006). The difference in inflow condition in the axisymmetric cases does not alter the temporal development of the axisymmetric mode significantly, i.e. both cases exhibit the same decay rate with only a small difference in the amplitude. This was expected in light of the highly similar basic states in these two cases.

When using basic states obtained from full 3D DNS, the axisymmetric mode is found to grow over time, with the case using a turbulent inflow boundary condition exhibiting a considerably higher growth rate than the case with a laminar inflow condition. Thus, it can be concluded that the basic states obtained from 3D DNS are subject to an absolute instability of the axisymmetric mode at these flow conditions.

In figure 4 the pulse response (initial pulse with amplitude $10^{-8}$) of higher azimuthal Fourier modes ($m \geq 1$) is shown for forced DNS using basic states obtained from either axisymmetric or 3D DNS with laminar inflow conditions. In both cases, all higher modes show temporal growth, i.e. the basic states are absolutely unstable with respect to helical modes. However, the choice of basic state plays a significant role in what growth rates are obtained and which mode is the linearly most amplified. Performing the stability analysis on the basic state from an axisymmetric simulation results in significantly higher growth rates than when using the time- and azimuthally-averaged 3D DNS data as basic state. This is somewhat surprising given that the axisymmetric mode was unstable in the latter case while stable with respect to the basic state from an axisymmetric simulation. In addition, in figure 4 (a) the most amplified modes are $m = 4, 5, 6$ while in subfigure (b) modes $m = 1, 2, 3$ exhibit the largest growth. The overall lower growth rates and lower azimuthal mode numbers being linearly most unstable are indicative of a smaller Reynolds number (Sandberg & Fasel, 2006). This suggests that the basic state obtained from the 3D DNS is representative of a lower Reynolds number wake flow.

In figure 5 the pulse response of Fourier modes $m \geq 1$ are shown for forced DNS using basic states obtained from either axisymmetric or 3D DNS with turbulent inflow conditions. As for the cases with laminar inflow conditions, for either basic state all higher modes show temporal
growth, i.e. the basic states are absolutely unstable with respect to helical modes. Figure 5 (a) shows a similar picture as 4 (a), i.e. the azimuthal modes \( m = 4, 5, 6 \) are linearly most amplified and have similar growth rates. However, in contrast to the laminar inflow condition cases, when using a basic state derived from a 3D DNS the behavior of the azimuthal modes \( m \geq 1 \) remains similar to the case using a basic state from an axisymmetric simulation, i.e. \( m = 4, 5, 6 \) are the most amplified modes and the growth rates are comparable.

In figures 6 and 7 the radial amplitude distributions of selected azimuthal modes of the conservative variable \( \rho u \) are shown at a location within the recirculation region or in the developing wake, respectively, using the basic states from 3D DNS. The main interest lies on the shape of the radial mode distributions and therefore several modes were scaled by arbitrary factors for better visibility. The amplitudes of individual Fourier modes are not considered crucial, as all modes are decoupled in small-amplitude calculations and, therefore, do not interact with each other. Comparing the radial mode shapes in the recirculating region of the wake.

**Figure 5.** Temporal development of azimuthal Fourier modes of density obtained from forced DNS using base state from axisymmetric (2D) solution (a) and using basic states obtained from time-averaged 3D DNS (b) with turbulent inflow; \( Re_D = 1 \times 10^5 \), \( M = 2.46 \).

**Figure 6.** Radial amplitude distributions of azimuthal Fourier modes of \( \rho u \) in recirculation region obtained from forced DNS using basic states obtained from 3D DNS with laminar inflow (a) and turbulent inflow (b); \( Re_D = 1 \times 10^5 \), \( M = 2.46 \).
obtained from stability analysis using the basic states with either laminar or turbulent inflow conditions (figure 6) shows qualitatively similar behavior, except for $m = 1$ which exhibits a larger number of local maxima and minima for the laminar inflow case. Due to the larger turning angle of the flow in the case of a turbulent inflow, the peaks of the radial amplitude distributions for the higher modes are shifted towards the axis at the same streamwise position. The radial profile of the axisymmetric mode $m = 1$ possess its maximum in the vicinity of the shear layer ($r \approx 1$) in the laminar inflow case, suggesting that in this case the convectively unstable shear-layer mode competes with the global mode, which in turn dominates for all higher modes indicated by their maxima being present for $0.35 \leq r \leq 0.8$ (Sandberg & Fasel, 2006).

At a streamwise position within the developing wake (figure 7) a similar trend is observed, i.e. $m = 1$ is the azimuthal mode that shows the greatest difference between the two basic states used. Most mode-shapes only exhibited one maximum, contracted to a smaller radial extent due to the lesser circumference of the wake at the location further downstream. In the absence of a recirculation region, only the shear-layer mode is present at the location within the trailing wake.

In Sandberg & Fasel (2006) it was suggested that the growth of individual azimuthal modes without changing their shape indicates the presence of global modes. Comparing results from linearized Navier–Stokes simulations with temporal simulations the existence of global modes were verified. Here, the evolution of the two-dimensional modes is investigated in the case of using a basic state obtained from the 3D DNS with turbulent inflow condition. Representative

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{Radial amplitude distributions of azimuthal Fourier modes of $\rho u$ in developing wake obtained from forced DNS using basic states obtained from 3D DNS with laminar inflow (a) and turbulent inflow (b); $Re_D = 1 \times 10^5$, $M = 2.46$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8.png}
\caption{Two dimensional mode-shape of Fourier mode $m = 4$ of $\rho u$ for two instances obtained from forced DNS calculations using basic state obtained from time-averaged 3D DNS with turbulent inflow; $Re_D = 1 \times 10^5$, $M = 2.46$.}
\end{figure}
Figure 9. Time-averaged radial amplitude distributions of azimuthal Fourier modes of $\rho u$ in recirculation region (left) and in developing wake (right); from 3D DNS with laminar inflow (lines) and turbulent inflow (lines with open circles); $z = 2.5$ and $z = 7$ for laminar inflow case, and $z = 1$ and $z = 6$ for laminar inflow case; $Re_D = 1 \times 10^5$, $M = 2.46$.

for other unstable modes, the fourth azimuthal Fourier mode is shown in figure 8 for two time-instants. Even though the amplitude increases one order of magnitude from $t = 85$ to $t = 100$, the shape of the mode changes only marginally, in particular close to the base. This suggests that global modes are also present using basic states obtained by time- and azimuthally averaging 3D DNS data.

Finally, the radial profiles obtained directly from temporally averaging the individual azimuthal Fourier modes of the 3D DNS with either laminar or turbulent inflow condition are shown in figure 9. For both inflow options, the amplitude of the second mode $m = 1$ is by far the largest of all shown within the recirculation region and the radial mode-shapes are similar for both cases. However, the radial distribution differs considerably from stability results, suggesting that in the DNS mode $m = 1$ might be generated mainly through non-linear interaction of other modes. It can also be observed that $m = 1$ is much smaller in amplitude in the developing wake for the turbulent inflow case, indicating that the turbulent approach flow reduces flapping of the wake. For modes with $m \geq 2$ the radial distributions found in the developing wake are similar for both inflow conditions and resemble in shape those obtained from stability analysis.
4. Conclusions

Direct numerical simulations were conducted of supersonic axisymmetric wakes at $M = 2.46$ and a Reynolds number $Re_D = 100,000$ with laminar and turbulent inflow conditions for the approach boundary layer. The DNS data were averaged in time and in the azimuthal direction in order to provide basic states for multi-dimensional stability analysis which was conducted using a forced DNS approach. Furthermore, basic states were obtained from axisymmetric (2D) DNS with laminar or turbulent mean inflow conditions.

The multi-dimensional stability analyses were conducted for all basic states by computing the temporal pulse response of azimuthal Fourier modes with respect to each basic state. Using the time- and azimuthally-averaged data from the 3D DNS with turbulent inflow as basic state, an absolute instability of the axisymmetric mode was found. This is in contrast to results obtained using an axisymmetric flow solution as the basic state, where an absolute stability could not be found at this Reynolds number. When considering the cases with laminar inflow, the helical modes $m = 4, 5, 6$ are the most linearly amplified modes using the basic state from an axisymmetric simulation while the stability analysis using the basic state based on time- and azimuthally-averaged DNS results in modes $m = 1, 2, 3$ being linearly most amplified with considerably lower growth rates. In contrast, the turbulent inflow cases result in similar behavior of the temporal development of azimuthal modes with $m \geq 1$, regardless of whether the basic state was obtained from 2D or 3D DNS.

As the forced DNS evolve in time it was found that the shape of the mode changes only marginally, in particular close to the base, suggesting that global modes are present using basic states obtained by time and azimuthally averaging 3D DNS data. The radial mode shapes obtained from time-averaged DNS data differ from the mode shapes from forced DNS mainly for mode $m = 1$, suggesting that $m = 1$ might be generated primarily through non-linear interaction of other modes.

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