UNIVERSALITY IN UNITARY RANDOM MATRIX ENSEMBLES WHEN THE SOFT EDGE MEETS THE HARD EDGE

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Dedicated to Percy Deift on the occasion of his sixtieth birthday

Abstract. Unitary random matrix ensembles

\[ Z_{n,N}^{-1}(\det M)^\alpha \exp(-N \text{Tr} V(M)) \, dM \]

defined on positive definite matrices \( M \), where \( \alpha > -1 \) and \( V \) is real analytic, have a hard edge at 0. The equilibrium measure associated with \( V \) typically vanishes like a square root at soft edges of the spectrum. For the case that the equilibrium measure vanishes like a square root at 0, we determine the scaling limits of the eigenvalue correlation kernel near 0 in the limit when \( n, N \to \infty \) such that \( n/N - 1 = \mathcal{O}(n^{-2/3}) \). For each value of \( \alpha > -1 \) we find a one-parameter family of limiting kernels that we describe in terms of the Hastings-McLeod solution of the Painlevé II equation with parameter \( \alpha + 1/2 \).

1. Introduction and statement of results

We deal in this paper with the following question.

Question: Given a unitary random matrix ensemble

\[ \frac{1}{Z_{n,N}}(\det M)^\alpha \exp(-N \text{Tr} V(M)) \, dM, \quad \alpha > -1, \quad (1.1) \]

defined on positive definite Hermitian matrices \( M \) of size \( n \times n \), where the real analytic potential \( V \) is such that the equilibrium measure associated with \( V \) has a density on \([0, \infty)\) that vanishes like a square root at 0. What are the scaling limits around 0 of the eigenvalue correlation kernel as \( n, N \to \infty \), \( n/N \to 1 \)?

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One expects a universality type result for this situation where the limiting kernels do not depend on the exact form of $V$. This is in line with known universality results in the usual unitary random matrix ensembles of the form

$$
\frac{1}{Z_{n,N}} |\det M|^{2\alpha} \exp(-N \text{Tr} W(M)) dM, \quad \alpha > -1/2
$$

(1.2)
defined on all Hermitian matrices $M$ of size $n \times n$, [2, 3, 6, 7, 12, 18]. In (1.1) and (1.2) and throughout the paper we assume that the confining potentials $V$ and $W$ are real analytic and satisfy

$$
\lim_{x \to +\infty} \frac{V(x)}{\log(x^2 + 1)} = +\infty, \quad \text{and} \quad \lim_{x \to \pm \infty} \frac{W(x)}{\log(x^2 + 1)} = +\infty.
$$

(1.3)
The random matrix ensemble (1.1) is restricted to positive definite matrices, thereby creating a hard edge at 0. This has an influence on the local eigenvalue behavior near 0 since eigenvalues are always positive.

It is well-known, see e.g. [9, 19], that the eigenvalue correlation kernel for the ensemble (1.1) has the form

$$
K_{n,N}(x,y) = x^{\frac{\alpha}{2}} y^{\frac{\alpha}{2}} e^{-\frac{\alpha}{2} V(x)} e^{-\frac{\alpha}{2} V(y)} \sum_{j=0}^{n-1} p_j^{(N)}(x) p_j^{(N)}(y),
$$

(1.4)
where $p_j^{(N)}$ is the $j$-th degree orthonormal polynomial with respect to the weight $x^\alpha e^{-\alpha V(x)}$ on $[0, \infty)$. The limiting mean eigenvalue density

$$
\psi_V(x) = \lim_{n,N \to \infty} \frac{1}{n} K_{n,N}(x,x)
$$

is the minimizer of the weighted energy

$$
I_V(\psi) = \iint \log \frac{1}{|x - y|} \psi(x) \psi(y) \, dx \, dy + \int V(x) \psi(x) \, dx
$$

(1.5)
taken over probability density functions $\psi$ supported on $[0, \infty)$, see [9, 20]. It does not depend on $\alpha$. As $V$ is real analytic, the support of $\psi_V$ consists of a finite union of intervals [11].

We can then distinguish two situations. The first is that 0 does not belong to the support $S_V$ of $\psi_V$. Then the hard edge at 0 does not lead to a new phenomenon. Generically one finds that the eigenvalue correlation kernel (1.4) has the sine kernel

$$
\frac{\sin \pi(x - y)}{\pi(x - y)}
$$

(1.6)
Figure 1. The density $\psi_{V_c}$ for $V_c(x) = \frac{1}{2c}(x - 2)^2$ with $c$ equal to 0.7 (left figure), 1 (middle figure) and 1.2 (right figure).

as scaling limit in the bulk and the Airy kernel
\[
\frac{\text{Ai}(x) \text{Ai}'(y) - \text{Ai}'(x) \text{Ai}(y)}{x - y}
\]
as scaling limit at the (soft) edge points of $S_{V}$. These are also the generic scaling limits for the correlation kernel associated with the matrix ensemble (1.2), see [2, 9, 10, 12, 21].

The second situation is that 0 belongs to $S_{V}$. In that case, one typically finds that $\psi_{V}$ is unbounded at 0 with a square root type singularity at 0. Then instead of the Airy kernel we are led to a Bessel kernel
\[
\frac{J_{\alpha}(\sqrt{x})\sqrt{y}J'_{\alpha}(\sqrt{y}) - J_{\alpha}(\sqrt{y})\sqrt{x}J'_{\alpha}(\sqrt{x})}{2(x - y)}
\]
that describes the local eigenvalue correlations near 0, see [15, 22, 23].

Changing parameters in $V$ we may create a transition from the Airy kernel to the Bessel kernel. This happens for example if
\[
V(x) = V_c(x) = \frac{1}{2c}(x - 2)^2.
\]

When considered on $\mathbb{R}$ this is a shifted and scaled GUE ensemble. For $c < 1$ we have
\[
\psi_{V_c}(x) = \frac{1}{2\pi c} \sqrt{4c - (x - 2)^2}, \quad \text{for } x \in [2 - 2\sqrt{c}, 2 + 2\sqrt{c}].
\]
These are all semi-circle laws which tend as $c \to 1^-$ to the semi-circle law on $[0, 4]$. Then the left soft edge meets with the hard edge at 0. For $c > 1$ we have
\[
\psi_{V_c}(x) = \frac{1}{2\pi c\sqrt{x}}(x + a)\sqrt{b - x}, \quad \text{for } x \in [0, b],
\]
with
\[
a = \frac{4}{3} + \frac{2}{3}\sqrt{1 + 3c} \quad \text{and} \quad b = \frac{4}{3} + \frac{4}{3}\sqrt{1 + 3c},
\]
which has a square-root singularity at 0. See Figure 1 for the density \( \psi_V \) for the values \( c = 0.7, c = 1 \) and \( c = 1.2 \).

Thus our Question asks about the transitional case. What happens when the soft edge meets the hard edge? Note that this transition from soft to hard edge is different from the one considered in [4] where the authors start from a hard edge situation and let the parameter \( \alpha \) increase to infinity. Other types of singular edge behaviors in unitary random matrix ensembles have been studied in [16], where a soft edge coincides with a spectral singularity, and in [8], where the limiting mean eigenvalue density vanishes faster than in the regular case.

Remark 1.1. Our Question is related to the calculation of the so-called Janossy densities for unitary random matrix ensembles

\[
\frac{1}{Z_{n,N}} \exp(-N \text{Tr} V(M)) dM
\]

in an asymptotic limit. Indeed, it was shown in [5] that the Janossy densities for an interval \( I \subset \mathbb{R} \) are expressed in terms of the orthogonal polynomials for the weight \( e^{-NV(x)} \) restricted to the complement of \( I \). If \( \psi_V \) vanishes like a square root at a left edge point \( a \), and if \( I = (-\infty, a] \), then after linear scaling we can reduce this to the case \( a = 0 \), and then the Janossy densities are expressed in terms of the orthogonal polynomial kernel (1.4) with \( \alpha = 0 \).

Our first result is that there is indeed a one-parameter family of limiting kernels that depends on \( \alpha \).

**Theorem 1.2.** For every \( \alpha > -1 \), there is a one-parameter family of kernels \( K^{\text{soft/hard}}_{\alpha}(x, y; s) \) depending on \( s \in \mathbb{R} \), such that the following holds. Let \( V \) be real analytic on \([0, \infty)\) such that \( \psi_V \) vanishes like a square root at 0 and such that there are no other singular points in the external field \( V \). Then there are positive constants \( c = c_{1,V} \) and \( c_{2,V} \) such that

\[
\lim_{n,N \to \infty} \frac{1}{(cn)^{2/3}} K_{n,N} \left( \frac{x}{(cn)^{2/3}}, \frac{y}{(cn)^{2/3}} \right) = K^{\text{soft/hard}}_{\alpha}(x, y; s) \quad (1.9)
\]

whenever \( n, N \to \infty \) such that

\[
\lim_{n \to \infty} n^{2/3} \left( \frac{n}{N} - 1 \right) = L \in \mathbb{R} \quad (1.10)
\]

and \( s = c_{2,V} L \). The limit (1.9) holds uniformly for \( x \) and \( y \) in compact subsets of \((0, \infty)\).

See equation (3.3) below for explicit formulas for the constants \( c_{1,V} \) and \( c_{2,V} \). We also refer to [12] and Section 3 below for the notion and the classification of singular points in an external field.
In contrast to other works on universality, see e.g. [7, 12], we do not have to go through an explicit Riemann-Hilbert steepest descent analysis in order to prove Theorem 1.2. Instead we prove Theorem 1.2 by relating the random matrix ensemble (1.1) to the random matrix ensemble (1.2) defined on all $n \times n$ Hermitian matrices where

$$W(x) = \frac{1}{2}V(x^2) \quad \text{for } x \in \mathbb{R}$$

and with $\alpha$ replaced by $\alpha \pm 1/2$. The equilibrium measure $\psi_W$ for (1.2) vanishes quadratically at 0. The universal limiting kernels for this situation were found in [7]. They are constructed out of $\psi$-functions associated with the Hastings-McLeod solution of the Painlevé II equation. Using this connection we find the following representation for the limiting kernels $K_{\alpha}^{\text{soft/hard}}(x, y; s)$.

**Theorem 1.3.** Let $\alpha > -1$. Then the kernels $K_{\alpha}^{\text{soft/hard}}(x, y; s)$ have the form

$$K_{\alpha}^{\text{soft/hard}}(x, y; s) = \frac{f_{\alpha}(x; s)g_{\alpha}(y; s) - f_{\alpha}(y; s)g_{\alpha}(x; s)}{\pi(x - y)}$$

(1.11)

where $f_{\alpha}$ and $g_{\alpha}$ are solutions of the system of differential equations

$$\frac{d}{dx} \begin{pmatrix} f_{\alpha}(x; s) \\ g_{\alpha}(x; s) \end{pmatrix} = \begin{pmatrix} 2q + \alpha/(2x) & 2 + (q^2 + r + \frac{s}{2})/x \\ -2x - q^2 + r - \frac{s}{2} & -2q - \alpha/(2x) \end{pmatrix} \begin{pmatrix} f_{\alpha}(x; s) \\ g_{\alpha}(x; s) \end{pmatrix}$$

(1.12)

where $q = q(s)$ is the Hastings-McLeod solution of the Painlevé II equation with parameter $\alpha + 1/2$

$$q'' = sq + 2q^3 - \alpha - \frac{1}{2}$$

(1.13)

and $r = r(s) = q'(s)$. The solutions of (1.12) are characterized by the asymptotic behavior

$$f_{\alpha}(x; s) = x^{-1/4}\cos \left( \frac{4}{3}x^{3/2} + sx^{1/2} - \frac{1}{2}\pi(\alpha + \frac{1}{2}) \right) + O(x^{-3/4}),$$

(1.14)

$$g_{\alpha}(x; s) = -x^{1/4}\sin \left( \frac{4}{3}x^{3/2} + sx^{1/2} - \frac{1}{2}\pi(\alpha + \frac{1}{2}) \right) + O(x^{-1/4}),$$

(1.15)

as $x \to +\infty$.

**Remark 1.4.** The Hastings-McLeod solution of (1.13) with $\alpha = -1/2$ was introduced in [15]. It is characterized by the asymptotic behavior $q(s) \sim \text{Ai}(s)$ as $s \to +\infty$ where $\text{Ai}$ is the usual Airy function. For
general $\alpha > -1$, $\alpha \neq -1/2$, the Hastings-McLeod solution of (1.13) is characterized by the asymptotic behavior $q(s) \sim (\alpha + 1/2)/s$ as $s \to +\infty$ and $q(s) \sim \sqrt{-s}/2$ as $s \to -\infty$, see [13, Chapter 11.7]. It was shown in [7] that the Hastings-McLeod solution has no poles on the real line and so the system (1.12) is well-defined for every $s \in \mathbb{R}$.

Note that $p_1 = q^2 + r + \frac{\alpha}{2}$ satisfies
\[ p'' = 2p^2 - sp + \frac{(p')^2 - \alpha^2}{2p} \]
and $p_2 = q^2 - r + \frac{\alpha}{2}$ satisfies
\[ p'' = 2p^2 - sp + \frac{(p')^2 - (\alpha + 1)^2}{2p} \]
which are versions of the Painlevé XXXIV equation, see [13].

The functions $f_\alpha$ and $g_\alpha$ also satisfy the following linear differential equation with respect to $s$
\[ \frac{\partial}{\partial s} \begin{pmatrix} f_\alpha(x; s) \\ g_\alpha(x; s) \end{pmatrix} = \begin{pmatrix} q & 1 \\ -x & -q \end{pmatrix} \begin{pmatrix} f_\alpha(x; s) \\ g_\alpha(x; s) \end{pmatrix} \tag{1.16} \]
which together with (1.12) constitutes the Lax pair for Painlevé XXXIV, see [13, page 175]. We will not use (1.16) in what follows. In [16], a limiting kernel is obtained which is built out of solutions to the Lax pair for Painlevé XXXIV as well. However, the relevant solution to Painlevé XXXIV in this paper is different from ours, and not related to the Hastings-McLeod solution to Painlevé II.

**Remark 1.5.** The system (1.12) can also be considered for complex values of $x$. The functions $f_\alpha$ and $g_\alpha$ have asymptotics in the complex $x$-plane given by
\[ f_\alpha(x; s) = \frac{1}{2x^{1/4}} e^{\frac{\pi i}{2}(\alpha + \frac{1}{2})} e^{-i(\frac{1}{2}x^{3/2} + sx^{1/2})} (1 + O(x^{-1/2})) \tag{1.17} \]
\[ g_\alpha(x; s) = \frac{x^{1/4}}{2i} e^{\frac{\pi i}{2}(\alpha + \frac{1}{2})} e^{-i(\frac{1}{2}x^{3/2} + sx^{1/2})} (1 + O(x^{-1/2})) \tag{1.18} \]
uniformly as $x \to \infty$ in the sector $\varepsilon < \arg x < 2\pi - \varepsilon$ for any $\varepsilon > 0$. The positive real line $\arg x = 0$ is an anti-Stokes line for the system (1.12), where the asymptotic behaviors (1.17) and (1.18) are not valid. As usual one has a two-term asymptotic approximation along the anti-Stokes line which leads to (1.14) and (1.15).

**Remark 1.6.** Besides the eigenvalue correlation kernel, one might be interested in the limiting distribution of the (re-scaled) smallest eigenvalue of the ensemble (1.1). These distributions are given by Fredholm
determinants of the limiting kernels (1.11). For general $\alpha$ we do not know if these kernels allow Painlevé expressions but for $\alpha = 0$ we can relate them to the well-known Tracy-Widom distribution [21]

$$F_{TW}(x) = \exp \left( - \int_x^\infty (y-x)^2 q(y) \, dy \right) \quad (1.19)$$

where $q(y)$ is the Hastings-McLeod solution of Painlevé II.

Indeed, since $V$ is real analytic on $[0, \infty)$ we can also consider $V$ on $(-\delta, \infty)$ for some $\delta > 0$. For small enough $\delta$ the equilibrium measure associated with $V$ on $(-\delta, \infty)$ will not depend on $\delta$ and it will vanish like a square root at 0. Then for the ensemble (1.1) with $\alpha = 0$ and defined on Hermitian matrices with eigenvalues $\geq -\delta$ we have that the distribution of the smallest eigenvalue tends to the Tracy-Widom distribution (1.19) as $n \to \infty$. Here we assume for convenience that $N = n$; otherwise under the assumption (1.10) we find a shifted version of (1.19). Moreover, denoting the smallest eigenvalue by $\lambda_1$ we have for $x > 0$,

$$\mathbb{P}(\lambda_1 > x) = \tilde{\mathbb{P}}(\lambda_1 > x \mid \lambda_1 > 0) = \frac{\tilde{\mathbb{P}}(\lambda_1 > x)}{\mathbb{P}(\lambda_1 > 0)}. \quad (1.20)$$

Here $\mathbb{P}$ denotes the probability in the ensemble (1.1) on positive-definite matrices, while $\tilde{\mathbb{P}}$ is the probability in the ensemble (1.1) on Hermitian matrices with eigenvalues $\geq -\delta$. Hence for an appropriate constant $c > 0$ we have if $\alpha = 0$ and $N = n$,

$$\lim_{n \to \infty} \mathbb{P}(cn^{2/3} \lambda_1 > x) = \frac{F_{TW}(-x)}{F_{TW}(0)}. \quad (1.21)$$

For $\alpha \neq 0$, the right-hand side of (1.21) will be different, but the scaling with $n^{2/3}$ in the left-hand side will be the same.

2. **A Quadratic Transformation**

In this section we relate the correlation kernels (1.4) for the random matrix ensemble (1.1) to the eigenvalue correlation kernel for the random matrix ensemble (1.2) with

$$W(x) = \frac{1}{2} V(x^2), \quad x \in \mathbb{R} \quad (2.1)$$
and parameter $\alpha \pm 1/2$ instead of $\alpha$. We use $K_{n,N}^{(2\alpha \pm 1)}(x,y)$ to denote the correlation kernel. Thus

\[
K_{n,N}^{(2\alpha \pm 1)}(x,y) = |x|^\alpha |y|^\alpha e^{-\frac{N}{4} W(x)} e^{-\frac{N}{4} W(y)} \sum_{j=0}^{n-1} P_j^{(N)}(x) P_j^{(N)}(y),
\]

(2.2)

where $P_j^{(N)}$ is the $j$-th degree orthonormal polynomial with respect to the weight $|x|^{2\alpha \pm 1} e^{-NV(x)}$ on $\mathbb{R}$.

To distinguish this kernel from the kernel (1.4) that is related to $V$ and parameter $\alpha$, we write in this section

\[
K_{n,N}(x,y) = K_{n,N}^{(\alpha,V)}(x,y).
\]

The following result is not new, see e.g. [1, Appendix B] and [14, Exercises 4.4]. For the convenience of the reader we have included a proof.

**Proposition 2.1.** Let $V$ be defined on $[0, \infty)$ and let $W$ be defined by (2.1). Assume that (1.3) is satisfied. Then for $\alpha > -1$, we have

\[
K_{n,N}^{(\alpha,V)}(x,y) = \frac{1}{2} (xy)^{-\frac{\alpha}{4}} \left( K_{2n,2N}^{(2\alpha+1,W)}(\sqrt{x}, \sqrt{y}) + K_{2n,2N}^{(2\alpha+1,W)}(\sqrt{x}, -\sqrt{y}) \right),
\]

(2.3)

while for $\alpha > 0$, we have in addition

\[
K_{n,N}^{(\alpha,V)}(x,y) = \frac{1}{2} (xy)^{-\frac{\alpha}{4}} \left( K_{2n,2N}^{(2\alpha-1,W)}(\sqrt{x}, \sqrt{y}) - K_{2n,2N}^{(2\alpha-1,W)}(\sqrt{x}, -\sqrt{y}) \right).
\]

(2.4)

**Proof.** In the proof we use the abbreviations

\[
K_{n,N} = K_{n,N}^{(\alpha,V)}, \quad K_{n,N}^+ = K_{n,N}^{(2\alpha+1,W)}, \quad K_{n,N}^- = K_{n,N}^{(2\alpha-1,W)}.
\]

We also write $p_n^{(N)}$ for the orthonormal polynomial of degree $n$ with respect to the weight $w(x) = x^\alpha e^{-NV(x)}$ on $[0, \infty)$, and $P_n^{(N)}$ and $Q_n^{(N)}$ for the orthonormal polynomials with respect to the weights $w_1(x) = |x|^{2\alpha+1} e^{-NW(x)}$ and $w_2(x) = |x|^{2\alpha-1} e^{-NW(x)}$, respectively. Note that $p_n^{(N)}$ and $P_n^{(N)}$ are well-defined for $\alpha > -1$, while $Q_n^{(N)}$ is only defined for $\alpha > 0$.

Since $w_1$ is an even weight on $\mathbb{R}$, the even degree orthogonal polynomials are even and the odd degree orthogonal polynomials are odd. Therefore there exists a polynomial $\tilde{p}_n^{(N)}$ of degree $n$ such that

\[
P_{2n}^{(2N)}(x) = \tilde{p}_n^{(N)}(x^2)
\]

(2.5)
For \( k < n \), we then have
\[
0 = \int_{-\infty}^{\infty} P_{2n}^{(2N)}(x) x^{2k} |x|^{2\alpha + 1} e^{-2NW(x)} \, dx \\
= 2 \int_{0}^{\infty} \tilde{p}_n^{(N)}(x^2) x^{2k} x^{2\alpha + 1} e^{-NV(x^2)} \, dx \\
= \int_{0}^{\infty} \tilde{p}_n^{(N)}(s) s^{k} s^{\alpha} e^{-NV(s)} \, ds, \tag{2.6}
\]
where we put \( s = x^2 \). A similar calculation shows that
\[
1 = \int_{-\infty}^{\infty} \left( P_{2n}^{(2N)}(x) \right)^2 |x|^{2\alpha + 1} e^{-2NW(x)} \, dx \\
= \int_{0}^{\infty} \left( \tilde{p}_n^{(N)}(s) \right)^2 s^{\alpha} e^{-NV(s)} \, ds, \tag{2.7}
\]
Therefore \( \tilde{p}_n^{(N)} \) has the orthogonality conditions that characterize the orthonormal polynomial \( p_n^{(N)} \) and it follows that
\[
P_{2n}^{(2N)}(x) = p_n^{(N)}(x^2). \tag{2.8}
\]
If \( \alpha > 0 \), we find in a similar way that
\[
Q_{2n+1}^{(2N)}(x) = xp_n^{(N)}(x^2). \tag{2.9}
\]
From (2.2), we now get for \( x, y \geq 0 \),
\[
K_{2n+2N}(x, \sqrt{y}) + K_{2n,2N}(\sqrt{x}, -\sqrt{y}) = (xy)^{\alpha + 1} e^{-N(V(x)+V(y))} \\
\times \left( \sum_{j=0}^{n-1} P_j^{(2N)}(\sqrt{x}) P_j^{(2N)}(\sqrt{y}) + \sum_{j=0}^{n-1} P_j^{(2N)}(\sqrt{x}) P_j^{(2N)}(-\sqrt{y}) \right). \tag{2.10}
\]
Since \( P_j^{(2N)} \) is even for even \( j \) and odd for odd \( j \), we see that the odd \( j \)-terms in the two sums of (2.10) cancel out, while the even \( j \)-terms are equal. We then obtain by (2.8) and (1.4) that
\[
K_{2n,2N}(\sqrt{x}, \sqrt{y}) = 2(xy)^{\alpha + 1} e^{-N(V(x)+V(y))} \sum_{j=0}^{n-1} P_{2j}^{(2N)}(\sqrt{x}) P_{2j}^{(2N)}(\sqrt{y}) \\
= 2(xy)^{\alpha + 1} e^{-N(V(x)+V(y))} \sum_{j=0}^{n-1} p_j^{(N)}(x) p_j^{(N)}(y) \\
= 2(xy)^{\alpha + 1} e^{-N(V(x)+V(y))} K_{n,N}(x, y).
\]
This proves (2.3).

If $\alpha > 0$ and if we use (2.9) we obtain (2.4) in a similar way. □

We will also connect the density $\psi_V$ of the equilibrium measure for $V$ with the density $\psi_W$ of the equilibrium measure for $W$. The equilibrium density $\psi_V$ minimizes the energy (1.5). It is characterized as the unique probability density function on $[0, \infty)$ with the property that for some constant $\ell_V$,

\[
2 \int \log |x - y| \psi_V(y) \, dy - V(x) + \ell_V = 0 \quad \text{for } x \in \text{supp } \psi_V,
\]

(2.11)

\[
2 \int \log |x - y| \psi_V(y) \, dy - V(x) + \ell_V \leq 0 \quad \text{for } x \in [0, \infty).
\]

(2.12)

The density $\psi_W$ is the limiting mean eigenvalue density for the matrix ensemble (1.2) as $n, N \to \infty$ in such a way that $n/N \to 1$. Then $\psi_W$ minimizes the weighted energy

\[
I_W(\psi) = \iint \log \frac{1}{|x - y|} \psi(x)\psi(y) \, dx \, dy + \int W(x) \psi(x) \, dx
\]

(2.13)

among all probability density functions $\psi$ on $\mathbb{R}$. The density $\psi_W$ is uniquely characterized by the following variational conditions for some constant $\ell_W$:

\[
2 \int \log |x - y| \psi_W(y) \, dy - W(x) + \ell_W = 0 \quad \text{for } x \in \text{supp } \psi_W,
\]

(2.14)

\[
2 \int \log |x - y| \psi_W(y) \, dy - W(x) + \ell_W \leq 0 \quad \text{for } x \in \mathbb{R}.
\]

(2.15)

Lemma 2.2. We have that

\[
\psi_W(x) = |x|\psi_V(x^2) \quad \text{for } x \in \mathbb{R}.
\]

Proof. We first show that $|x|\psi_V(x^2)$ is indeed a probability density. This follows from the positivity of $\psi_V$ and the fact that

\[
\int_{-\infty}^{\infty} |x|\psi_V(x^2) \, dx = 2 \int_{0}^{\infty} x\psi_V(x^2) \, dx = \int_{0}^{\infty} \psi_V(s) \, ds = 1.
\]

To prove part (i) of the lemma, it is now sufficient to show that $|x|\psi_V(x^2)$ satisfies the variational conditions (2.14) and (2.15). We
have
\[ \int_{-\infty}^{\infty} \log |x - y| \log |y\psi(y^2)| dy \]
\[ = \int_{-\infty}^{\infty} \log |x - y| y\psi(y^2) dy + \int_{0}^{\infty} \log |x + y| y\psi(y^2) dy \]
\[ = \int_{0}^{\infty} \log |x^2 - y^2| y\psi(y^2) dy \]
\[ = \frac{1}{2} \int_{0}^{\infty} \log |x^2 - s| \psi(s) ds. \]

This relation implies (2.14) and (2.15) by virtue of (2.11) and (2.12). Since these variational conditions uniquely characterize \( \psi_W \), the lemma follows. \( \square \)

In exactly the same way we find the following relation between the equilibrium densities of the sets \( S_V \) and \( S_W \). Thus \( \omega_V \) is the unique probability density on \( S_V \) that minimizes the (unweighted) energy
\[ \int \int \log \frac{1}{|x - y|} \omega(x) \omega(y) dx dy. \]

**Lemma 2.3.** Let \( \omega_V \) be the equilibrium density of the set \( S_V \) and let \( \omega_W \) be the equilibrium density of \( S_W = \{x \mid x^2 \in S_V \} \). Then
\[ \omega_W(x) = |x|\omega_V(x^2) \quad \text{for } x \in S_W. \]  (2.16)

**3. Proof of Theorem 1.2**

The proof of Theorem 1.2 is based on the following result for the matrix ensemble (1.2) for the case that \( \psi_W \) vanishes quadratically at 0. There is a one-parameter family of limiting kernels that are connected with the Hastings-McLeod solution of the Painlevé II equation \( q'' = sq + 2q^3 - \alpha \). We denote these kernels by \( K_{\alpha, \Pi}^{\text{crit}}(u, v; s) \) and we refer to (4.4) below for the precise form of \( K_{\alpha, \Pi}^{\text{crit}}(u, v; s) \). The superscript \( \Pi \) refers to the classification of singular points for the matrix ensemble (1.2) see [12, 17] according to which there are three cases

**Singular case I:** Equality holds in (2.15) for some \( x \in \mathbb{R} \setminus \text{supp } \psi_W \). Any such point \( x \) is then a singular exterior point in the external field \( W \).

**Singular case II:** \( \psi_W \) vanishes at an interior point of its support. Such a point is a singular interior point in the external field \( W \).
Singular case III: \( \psi_W \) vanishes at an edge point to higher order than a square root. Such a point is a singular edge point in the external field \( W \).

The following result was established in [7].

**Theorem 3.1.** ([7, Theorem 1.2]) Let \( W \) be real analytic on \( \mathbb{R} \) such that (1.3) holds. Suppose that \( \psi_W(0) = \psi'_W(0) = 0 \) and \( \psi''_W(0) > 0 \), and assume that there no other singular points besides 0. Define constants

\[
c_1,W = \frac{\pi}{8} \psi''_W(0) > 0 \quad \text{and} \quad c_2,W = \frac{\pi}{c_1,W} \omega_W(0) > 0
\]

where \( \omega_W \) is the density of the equilibrium measure of the support of \( \psi_W \). Let \( n, N \to \infty \) such that \( \lim_{n,N \to \infty} n^{2/3}(n/N - 1) = L \in \mathbb{R} \) exists and put \( c = c_1,W \) and \( s = c_2,W L \). Then

\[
\lim_{n,N \to \infty} \frac{1}{(cn)^{1/3}} K^{(\alpha, W)}_{n,N} \left( \frac{x}{(cn)^{1/3}}, \frac{y}{(cn)^{1/3}} \right) = K^{\text{crit}, II}_{\alpha}(x, y; s)
\]

uniformly for \( x, y \) in compact subsets of \( \mathbb{R} \setminus \{0\} \).

Now we are ready for the proof of Theorem 1.2 with constants

\[
c_1,V = \frac{\pi}{2} \lim_{x \to 0^+} \left[ x^{-1/2} \psi_V(x) \right] \quad \text{and} \quad c_2,V = \frac{2\pi}{c_1,V} \lim_{x \to 0^+} \left[ \sqrt{x} \omega_V(x) \right],
\]

where \( \omega_V \) is the density of the equilibrium measure of the support \( S_V \) of \( \psi_V \).

**Proof of Theorem 1.2.** We define

\[
W(x) = \frac{1}{2} V(x^2), \quad x \in \mathbb{R}.
\]

Then by Lemma 2.2 we have that \( \psi_W(x) = |x| \psi_V(x^2) \), which implies by the assumptions on \( \psi_V \) near 0, that

\[
\psi_W(0) = \psi'_W(0) = 0, \quad \psi''_W(0) = 2 \lim_{x \to 0^+} \left[ x^{-1/2} \psi_V(x) \right] > 0.
\]

It follows from Lemma 2.2 that there are no other singular points besides 0 in the external field \( W \) (in fact, the absence of type I singular points follows from the proof of the lemma). The conditions of Theorem 3.1 are therefore satisfied. From (3.3), (3.1) and Lemmas 2.2 and 2.3 it follows that

\[
c_{1,V} = 2c_{1,W}, \quad \text{and} \quad c_{2,V} = 2^{2/3} c_{2,W}.
\]
Then if \( n, N \to \infty \) such that (1.10) and if \( c = c_{1, V} \) and \( s = c_{2, V} L \), we get from from Theorem 3.1 that

\[
\lim_{n, N \to \infty} \frac{1}{(cn)^{1/3}} K_{2n, 2N}^{(2\alpha+1, W)} \left( \frac{x}{(cn)^{1/3}}, \frac{y}{(cn)^{1/3}} \right) = K_{\alpha+1/2}^{\text{crit}, II} (x, y; s),
\]

(3.5)

and if \( \alpha > 0 \) we also have

\[
\lim_{n, N \to \infty} \frac{1}{(cn)^{1/3}} K_{2n, 2N}^{(2\alpha-1, W)} \left( \frac{x}{(cn)^{1/3}}, \frac{y}{(cn)^{1/3}} \right) = K_{\alpha-1/2}^{\text{crit}, II} (x, y; s),
\]

(3.6)

Then straightforward calculations based on (2.3) and (3.5) show that the limit (1.9) exists with

\[
K_{\alpha}^{\text{soft}/\text{hard}} (x, y; s) = \frac{1}{2} (xy)^{-1/4} \left( K_{\alpha+1/2}^{\text{crit}, II} (\sqrt{x}, \sqrt{y}; s) + K_{\alpha+1/2}^{\text{crit}, II} (\sqrt{-x}, -\sqrt{y}; s) \right),
\]

(3.7)

while if \( \alpha > 0 \), then from (2.4) and (3.6) we also get

\[
K_{\alpha}^{\text{soft}/\text{hard}} (x, y; s) = \frac{1}{2} (xy)^{-1/4} \left( K_{\alpha-1/2}^{\text{crit}, II} (\sqrt{x}, \sqrt{y}; s) - K_{\alpha-1/2}^{\text{crit}, II} (\sqrt{-x}, -\sqrt{y}; s) \right).
\]

(3.8)

\[\square\]

4. Proof of Theorem 1.3

To prove Theorem 1.3, we recall the definition of the kernel \( K_{\alpha}^{\text{crit}, II} \) from [7]. It uses the Hastings-McLeod solution \( q = q_\alpha \) of the Painlevé II equation

\[
q'' = sq + 2q^3 - \alpha.
\]

(4.1)

Given \( \alpha > -1/2 \) and \( s \in \mathbb{R} \), we abbreviate \( q = q_\alpha (s) \) and \( r = q'_\alpha (s) \), and we let \( \left( \Phi_{\alpha, 1} (z; s) \right) \) be the unique solution of the equation

\[
\frac{d}{dz} \left( \Phi_{\alpha, 1} \right) = \begin{pmatrix} -4iz^2 - i(s + 2q^2) & 4zq + 2ir + \alpha/z \\ 4zq - 2ir + \alpha/z & 4iz^2 + i(s + 2q^2) \end{pmatrix} \begin{pmatrix} \Phi_{\alpha, 1} \\ \Phi_{\alpha, 2} \end{pmatrix},
\]

(4.2)

with asymptotics

\[
e^{i(\frac{3}{4}z^3 + sz)} \begin{pmatrix} \Phi_{\alpha, 1} (z; s) \\ \Phi_{\alpha, 2} (z; s) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O(z^{-1}),
\]

(4.3)

uniformly as \( z \to \infty \) in the sector \( \epsilon < \arg z < \pi - \epsilon \) for any \( \epsilon > 0 \). Since \( q_\alpha \) has no real poles for \( \alpha > -1/2 \), \( \Phi_{\alpha, 1} \) and \( \Phi_{\alpha, 2} \) are well-defined for real \( s \). The functions \( \Phi_{\alpha, 1} \) and \( \Phi_{\alpha, 2} \) extend to analytic functions on \( \mathbb{C} \setminus (-i\infty, 0] \), with branch points in 0. We denote these extensions also
by $\Phi_{\alpha,1}$ and $\Phi_{\alpha,2}$. In the kernel $K_{\alpha}^{\text{crit,II}}$ we need their values on the real line. Indeed, we have for real $x$, $y$, and $s$,

$$
K_{\alpha}^{\text{crit,II}}(x,y;s) = -e^{\frac{1}{2} \pi i \alpha \text{sgn}(x) + \text{sgn}(y)} \frac{\Phi_{\alpha,1}(x;s) \Phi_{\alpha,2}(y;s) - \Phi_{\alpha,1}(y;s) \Phi_{\alpha,2}(x;s)}{2\pi i (x-y)}. \tag{4.4}
$$

We first rewrite $K_{\alpha}^{\text{crit,II}}$ in a ‘real’ form, using the functions (see also [3]),

$$
\begin{pmatrix} F_{\alpha,1}(x;s) \\ F_{\alpha,2}(x;s) \end{pmatrix} = e^{\frac{1}{2} \pi i \alpha \text{sgn}(x)} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} \Phi_{\alpha,1}(x;s) \\ \Phi_{\alpha,2}(x;s) \end{pmatrix}, \quad x \in \mathbb{R} \setminus \{0\}. \tag{4.5}
$$

Then we get from (4.4) that

$$
K_{\alpha}^{\text{crit,II}}(x,y;s) = \frac{F_{\alpha,1}(x;s) F_{\alpha,2}(y;s) - F_{\alpha,1}(y;s) F_{\alpha,2}(x;s)}{\pi (x-y)} \tag{4.6}
$$

and from (4.2) we get the differential equation

$$
\frac{d}{dx} \begin{pmatrix} F_{\alpha,1} \\ F_{\alpha,2} \end{pmatrix} = \begin{pmatrix} 4xq + \frac{\alpha}{x} & 4x^2 + s + 2q^2 + 2r \\ -4x^2 - s - 2q^2 + 2r & -4xq - \frac{\alpha}{x} \end{pmatrix} \begin{pmatrix} F_{\alpha,1} \\ F_{\alpha,2} \end{pmatrix}. \tag{4.7}
$$

The following properties are crucial.

**Lemma 4.1.** Both $F_{\alpha,1}(x;s)$ and $F_{\alpha,2}(x;s)$ are real for real $x$ and $s$. In addition, $F_{\alpha,1}$ is an even function and $F_{\alpha,2}$ is an odd function, and they have the following asymptotic behaviors

$$
F_{\alpha,1}(x;s) = \cos \left( \frac{4}{3} x^3 + sx - \frac{1}{2} \pi \alpha \right) + O(x^{-1}), \tag{4.8}
$$

$$
F_{\alpha,2}(x;s) = -\sin \left( \frac{4}{3} x^3 + sx - \frac{1}{2} \pi \alpha \right) + O(x^{-1}), \tag{4.9}
$$

as $x \to +\infty$.

**Proof.** The proof is based on the symmetries of the Riemann-Hilbert problem that characterizes $\begin{pmatrix} \Phi_{\alpha,1} \\ \Phi_{\alpha,2} \end{pmatrix}$.

Let $\Gamma = \bigcup_j \Gamma_j$ be the contour consisting of four straight rays oriented to infinity, where

$$
\Gamma_1 : \text{arg } z = \frac{\pi}{6}, \quad \Gamma_2 : \text{arg } z = \frac{5\pi}{6}, \quad \Gamma_3 : \text{arg } z = -\frac{5\pi}{6}, \quad \Gamma_4 : \text{arg } z = -\frac{\pi}{6}
$$

see Figure 2. The contour $\Gamma$ divides the complex plane into four sectors $S_j$, $j = 1, 2, 3, 4$ as shown in Figure 2.

For $\alpha > -1/2$ and $s \in \mathbb{R}$ we look for $\Psi_{\alpha}(z;s)$ that satisfies the following Riemann-Hilbert problem.
Figure 2. The contour $\Gamma$ consisting of four straight rays oriented to infinity.

(a) $\Psi_\alpha : \mathbb{C} \setminus \Gamma \to \mathbb{C}^{2 \times 2}$ is analytic in $\mathbb{C} \setminus \Gamma$.
(b) $\Psi_\alpha$ satisfies the following jump relations on $\Gamma \setminus \{0\}$,

\[
\begin{align*}
\Psi_{\alpha,+}(z; s) &= \Psi_{\alpha,-}(z; s) \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 0 \end{pmatrix}, \quad \text{for } z \in \Gamma_1, \quad (4.10) \\
\Psi_{\alpha,+}(z; s) &= \Psi_{\alpha,-}(z; s) \begin{pmatrix} 1 & 0 \\ -e^{\pi i \alpha} & 0 \end{pmatrix}, \quad \text{for } z \in \Gamma_2, \quad (4.11) \\
\Psi_{\alpha,+}(z; s) &= \Psi_{\alpha,-}(z; s) \begin{pmatrix} 1 & e^{-\pi i \alpha} \\ 0 & 1 \end{pmatrix}, \quad \text{for } z \in \Gamma_3, \quad (4.12) \\
\Psi_{\alpha,+}(z; s) &= \Psi_{\alpha,-}(z; s) \begin{pmatrix} 1 & -e^{\pi i \alpha} \\ 0 & 1 \end{pmatrix}, \quad \text{for } z \in \Gamma_4. \quad (4.13)
\end{align*}
\]

(c) $\Psi_\alpha$ has the following behavior at infinity,

\[
\Psi_\alpha(z; s) = (I + \mathcal{O}(1/z))e^{-i(\frac{\alpha}{2}z^2 + sz)}\sigma_3, \quad \text{as } z \to \infty. \quad (4.14)
\]

Here $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ denotes the third Pauli matrix.
(d) $\Psi_\alpha$ has the following behavior near the origin. If $\alpha < 0$,

\[
\Psi_\alpha(z; s) = \mathcal{O} \begin{pmatrix} |z|^{\alpha} & |z|^{\alpha} \\ |z|^{\alpha} & |z|^{\alpha} \end{pmatrix}, \quad \text{as } z \to 0. \quad (4.15)
\]
and if $\alpha \geq 0$,

$$
\Psi_\alpha(z; s) = \begin{cases}
\mathcal{O} \left( \begin{array}{cc}
|z|^{-\alpha} & |z|^{-\alpha} \\
|z|^{-\alpha} & |z|^{-\alpha}
\end{array} \right), & \text{as } z \to 0, \ z \in S_1 \cup S_3, \\
\mathcal{O} \left( \begin{array}{cc}
|z|^\alpha & |z|^{-\alpha} \\
|z|^{-\alpha} & |z|^{-\alpha}
\end{array} \right), & \text{as } z \to 0, \ z \in S_2, \\
\mathcal{O} \left( \begin{array}{cc}
|z|^{-\alpha} & |z|^\alpha \\
|z|^{-\alpha} & |z|^{-\alpha}
\end{array} \right), & \text{as } z \to 0, \ z \in S_4.
\end{cases}
$$

(4.16)

It was shown in [7] that the Riemann-Hilbert problem for $\Psi_\alpha$ has a unique solution and that

$$
\begin{pmatrix}
\Psi_\alpha(z; s) \\
\Phi_{\alpha,1}(z; s) \\
\Phi_{\alpha,2}(z; s)
\end{pmatrix} = \begin{cases}
\Psi_\alpha(z; s) \begin{pmatrix} 1 \\ e^{-\pi i \alpha} \end{pmatrix}, & \text{for } z \in S_1, \\
\Psi_\alpha(z; s) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & \text{for } z \in S_2, \\
\Psi_\alpha(z; s) \begin{pmatrix} 1 \\ e^{\pi i \alpha} \end{pmatrix}, & \text{for } z \in S_3, \\
\Psi_\alpha(z; s) \begin{pmatrix} 0 \\ e^{-\pi i \alpha} \end{pmatrix}, & \text{for } z \in S_4, \ \text{Re} z > 0, \\
\Psi_\alpha(z; s) \begin{pmatrix} 0 \\ e^{\pi i \alpha} \end{pmatrix}, & \text{for } z \in S_4, \ \text{Re} z < 0.
\end{cases}
$$

(4.17)

It is easy to check that $\sigma_1 \Psi_\alpha(z; s) \sigma_1$ and $\sigma_1 \Psi_\alpha(-z; s) \sigma_1$ with $\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ also satisfy the conditions of the Riemann-Hilbert problem for $\Psi_\alpha$. Therefore by the uniqueness of the solution we have

$$
\Psi_\alpha(z; s) = \sigma_1 \Psi_\alpha(z; s) \sigma_1 = \sigma_1 \Psi_\alpha(-z; s) \sigma_1.
$$

(4.18)

From the first equality in (4.18) it follows (as noted in Remark 2 of [7]) that

$$
e^{\frac{1}{2} \pi i \alpha \text{sgn}(x)} \Phi_{\alpha,2}(x; s) = e^{\frac{1}{2} \pi i \alpha \text{sgn}(x)} \Phi_{\alpha,1}(x; s), \quad \text{for } x \in \mathbb{R} \setminus \{0\} \text{ and } s \in \mathbb{R},
$$

(4.19)

which shows that $F_{\alpha,1}(x; s)$ and $F_{\alpha,2}(x; s)$ are real, since we obtain from

(4.5) and (4.19) that

$$
F_{\alpha,1}(x; s) = \text{Re} \left[ e^{\frac{1}{2} \pi i \alpha \text{sgn}(x)} \Phi_{\alpha,1}(x; s) \right], \\
F_{\alpha,2}(x; s) = \text{Im} \left[ e^{\frac{1}{2} \pi i \alpha \text{sgn}(x)} \Phi_{\alpha,1}(x; s) \right].
$$

(4.20)
From (4.14), (4.17), and (4.20) we obtain the asymptotic behaviors (4.8) and (4.9).

The symmetry \( \Psi_\alpha(z; s) = \sigma_1 \Psi_\alpha(-z; s) \sigma_1 \) yields by (4.17) that
\[
e^{\frac{i}{2} \pi \alpha \text{sgn}(x)} \Phi_{\alpha, 2}(x; s) = e^{\frac{i}{2} \pi \alpha \text{sgn}(-x)} \Phi_{\alpha, 1}(-x; s), \quad \text{for } x \in \mathbb{R} \setminus \{0\},
\]
which in view of (4.19) and (4.20) shows that \( F_{\alpha, 1} \) is even and \( F_{\alpha, 2} \) is odd.

Now we can give the proof of Theorem 1.3.

**Proof of Theorem 1.3** From (3.7) and (4.6), we get
\[
K^{\text{soft/hard}}_{\alpha}(x, y; s) = \frac{1}{\pi (xy)^{1/4}} \left( \frac{F_{\alpha + \frac{1}{2}, 1}(\sqrt{x}) F_{\alpha + \frac{1}{2}, 2}(\sqrt{y}) - F_{\alpha + \frac{1}{2}, 1}(\sqrt{y}) F_{\alpha + \frac{1}{2}, 2}(\sqrt{x})}{\pi (\sqrt{x} - \sqrt{y})} + \frac{F_{\alpha + \frac{1}{2}, 1}(\sqrt{x}) F_{\alpha + \frac{1}{2}, 2}(\sqrt{y}) - F_{\alpha + \frac{1}{2}, 1}(\sqrt{y}) F_{\alpha + \frac{1}{2}, 2}(\sqrt{x})}{\pi (\sqrt{x} + \sqrt{y})} \right).
\]
(4.21)

Putting the fractions onto a common denominator, and using the even/odd properties of Lemma 4.1, we find
\[
K^{\text{soft/hard}}_{\alpha}(x, y; s) = \frac{1}{\pi (xy)^{1/4}} \left( \frac{F_{\alpha + \frac{1}{2}, 1}(\sqrt{x}) \sqrt{y} F_{\alpha + \frac{1}{2}, 2}(\sqrt{y}) - F_{\alpha + \frac{1}{2}, 1}(\sqrt{y}) \sqrt{x} F_{\alpha + \frac{1}{2}, 2}(\sqrt{x})}{x - y} \right).
\]
(4.22)

which leads to (1.11) with
\[
f_{\alpha}(x; s) = x^{-1/4} F_{\alpha + \frac{1}{2}, 1}(\sqrt{x}; s),
\]
(4.23)
and
\[
g_{\alpha}(x; s) = x^{1/4} F_{\alpha + \frac{1}{2}, 2}(\sqrt{x}; s).
\]
(4.24)

The system of differential equations (1.12) follows from (4.7), (4.23), and (4.24) and the asymptotic behavior (1.14)-(1.15) follows from (4.8)-(4.9) and (4.23)-(4.24).

This completes the proof of Theorem 1.3.

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