ON CONVEX HYPERSURFACES IN SPACE FORMS
AND EIGENVALUE ESTIMATES FOR DIFFERENTIAL
FORMS

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Abstract. We apply Reilly’s formula and Hsiung-Minkowski formulas to obtain some geometric and analytic results. In the first part, we obtain some sharp integral inequalities on a convex hypersurface in space forms and show their rigidities. In the second part, we give some sharp lower bounds of the first eigenvalue for the Hodge Laplacian acting on differential forms on a hypersurface in a Riemannian manifold. We also give some sharp estimates for the first nonzero Steklov eigenvalue for differential forms.

1. Introduction

Integral formulas have always been an important tool for studying various analytical and geometric problems on Riemannian manifolds. The Reilly’s formula [16] and the Hsiung-Minkowski formulas [10] are typical examples that have yielded many classical results. Often, these formulas produce some integral identities or inequalities where the vanishing of the integrand produces useful geometric consequences. Despite the simplicity of such an idea, this method achieves many results. Let us for example mention the work of Ros [18] who has combined the Reilly and Minkowski formulas to show that a compact embedded hypersurface in \( \mathbb{R}^n \) with one of the higher order mean curvatures being constant is a sphere. On the analytic side, it is well-known that the first non-zero eigenvalue of the Laplacian with respect to various boundary conditions gives very important analytic information of a Riemannian manifold. Integral formulas is one of the main tools in obtaining various eigenvalue estimates. For example, Choi and Wang [3] used the Reilly’s formula to prove that for a compact orientable minimal hypersurface
Σ of a compact orientable \((N^n, g)\) with Ricci curvature bounded from above by \(k > 0\), the first nonzero eigenvalue of \(\Sigma\) satisfies \(\lambda_1(\Sigma) \geq k/2\), this is closely related to a conjecture of Yau [26].

In this paper, we apply some integral formulas to obtain some geometric and analytic results. In the first part, using conformal vector fields on space forms and applying Reilly’s and Hsiung-Minkowski formulas, we obtain some integral inequalities on a convex hypersurface in space forms, and discuss its sharpness. They can be regarded as some rigidity results in space forms. Here is an example of our results, in which we show a characterization of a convex hypersurface in space forms, which is a particular case of Theorem 3.1 and Theorem 3.2:

**Theorem 1.1.** Let \(\Sigma\) be a compact convex hypersurface of \(\mathbb{R}^n\) which encloses \(\Omega \ni O\). Suppose \(\Sigma\) is circumscribed by the geodesic sphere \(S_{r_1}\) centered at \(O\). Then

\[
n\text{Vol}(\Omega) \leq r_1 \int_{\Sigma} \sigma_0 \leq r_1^2 \int_{\Sigma} \sigma_1 \leq \cdots \leq r_1^n \int_{\Sigma} \sigma_{n-1}
\]

where \(\sigma_k\) is the normalized \(k\)-th mean curvature of \(\Sigma\) and \(\sigma_0 = 1\). The equality holds if and only if \(\Sigma\) is \(S_{r_1}\). There are analogous results for \(\mathbb{H}^n\) and \(\mathbb{S}^n\). For example, if \(\Sigma\) is a convex hypersurface contained in the open hemisphere centered at \(O\) and \(r_1 < \pi/2\), then

\[
\int_{\Omega} ((n + 1) \cos^2 r - 1) \leq \sin^2 r_1 \int_{\Sigma} \sigma_1 \leq \frac{\sin^3 r_1}{\cos r_1} \int_{\Sigma} \sigma_2 \\
\leq \cdots \leq \frac{\sin^n r_1}{\cos^{n-2} r_1} \int_{\Sigma} \sigma_{n-1}.
\]

Here \(r\) is the distance from \(O\). The equality holds if and only if \(\Sigma\) is \(S_{r_1}\).

This generalizes Garay’s result [7] that if a convex hypersurface \(\Sigma\) in \(\mathbb{R}^n\) has mean curvature \(H = (n - 1)\sigma_1 \leq \frac{n(n-1)\text{Vol}(\Omega)}{r^2\text{Area}(\Sigma)}\), then \(\Sigma\) is the sphere of radius \(r\).

In the second part of this paper, we will apply Reilly’s formula to obtain some lower bounds for the first nonzero eigenvalue of the Hodge Laplacian acting on differential forms on the boundary \(\Sigma\) of a Riemannian manifold \((N, g)\) whose Ricci curvature is bounded from below. Our main results, Theorem 4.2 and 4.4, are natural extensions of the results of Escobar [4], Xia [24], Wang-Xia [22] and Raulot-Savo [13]. For example, in Theorem 4.4 we extend the results of Xia [24] and Raulot-Savo [13]:
**Theorem 1.2.** Let \((N^n, g)\) be a compact orientable Riemannian manifold with boundary \(\Sigma\). Suppose the Bochner curvature \(W^r\) or \(W^{n-r}\) on \(N\) is bounded from below by \(k \geq 0\). Assume that the lowest \(q\)-curvature \(s_q\) of \(\Sigma\) is nonnegative, where \(q = \min\{r, n-r\}\). Then for \(1 \leq r \leq n-1\), we have

\[
2\lambda'_{1,r} = 2\lambda''_{1,r-1} \geq k + s_r s_{n-r} + \sqrt{(s_r s_{n-r})^2 + 2s_r s_{n-r}k},
\]

where \(\lambda'_{1,r}\) (resp. \(\lambda''_{1,r}\)) is the first nonzero eigenvalue of the Hodge Laplacian on the exact (resp. co-exact) \(r\)-forms on \(\Sigma\). The equality can hold only when \(k = 0\), with the \(r\)-curvatures and the \((n-r)\)-curvatures being positive constants. If, furthermore, \((N, g)\) has non-negative Ricci curvature, then the equality holds if and only if \((N, g)\) is isometric to a Euclidean ball. The condition on Ricci curvature can be removed if \(r = 1\) or \(n - 1\).

The notions \(W^r\) and \(s_r\) will be explained in Section 4. Let us just mention that when \(r = 1\), \(s_1\) is the minimum eigenvalue of the second fundamental form of \(\Sigma\) and \(s_{n-1}\) is the minimum of its mean curvature, \(W^1\) is just the Ricci curvature and \(\lambda_1 = \lambda''_{1,0}\) is the first nonzero eigenvalue of the Laplacian on functions on \(\Sigma\).

We will also give a sharp lower bound of \(\lambda'_{1,r}\) in terms of the first nonzero Steklov eigenvalues for differential forms, as well as some lower and upper bounds for the Steklov eigenvalues in terms of \(\lambda'_{1,r}\) (Theorem 4.4). It is also interesting to see that when \(n = 2\), a simple extension of the result of Hang-Wang \([8]\) gives an improvement of Choi-Wang’s result mentioned above. Indeed, we can prove that \(\lambda_1(\Sigma) \geq k\) and the estimate is sharp (Theorem 4.5). It may have some independent interest.

This paper is organized as follows. In Section 2, we set up the notations and introduce the Reilly and Hsiung-Minkowski formulas. In Section 3, we establish some integral inequalities of convex hypersurfaces in space forms. In Section 4, we prove the various estimates for the Hodge Laplacian eigenvalues and also Steklov eigenvalues for differential forms on a manifold with boundary.

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2. Reilly’s formula and Hsiung-Minkowski formulas

Let us first set up the notations. Throughout this paper, \((N, g)\) will denote an \(n\)-dimensional connected oriented Riemannian manifold \((n \geq 2)\) (with or without boundary). If \(N\) is without boundary then we will always denote a closed hypersurface on \(N\) by \(\Sigma\), otherwise we assume \(N\) is compact has a smooth compact boundary \(\partial N = \Sigma\). We will denote the Levi-Civita connection on \(N\) and \(\Sigma\) by \(\nabla\) and \(\nabla\) respectively. The Laplacian on \(N\) and \(\Sigma\) will be denoted by \(\Delta\) and \(\Delta\) respectively and let \(\text{Rc}\) be the Ricci curvature of \(N\). We define \(\nu\) to be the unit outward normal on \(\Sigma\) w.r.t. \(N\) and \(A(X, Y) = \langle \nabla_X \nu, Y \rangle\) and \(H = \text{tr} A\) to be the second fundamental form and the mean curvature of \(\Sigma\) respectively.

Reilly’s formula states that:

**Theorem 2.1.**\(^{[16]}\) Let \(f\) be a smooth function on \(N\), \(z = f|_\Sigma\) and \(u = \frac{\partial f}{\partial \nu}\). Then

\[
\int_N (\Delta f)^2 - |\nabla^2 f|^2 = \int_N \text{Rc}(\nabla f, \nabla f) + \int_\Sigma 2u\Delta z + Hu^2 + A(\nabla z, \nabla z).
\]

(2.1)

To state the Hsiung-Minkowski formulas, we define

**Definition 2.1.** A vector field \(X\) on \((N^n, g)\) is conformal if there is a smooth function \(\alpha\) on \(N\) such that

\[
\mathcal{L}_X g = 2\alpha g.
\]

As \(\mathcal{L}_X g(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X)\), it is easy to see that \(\alpha = \frac{1}{n} \text{div} X\) if \(X\) is conformal.

Next, we define the \(r\)-th mean curvature of \(\Sigma\) in \(N\) as follows. Suppose \(k_1, \ldots, k_{n-1}\) are the principal curvatures of \(\Sigma\) (eigenvalues of the second fundamental form \(A\)), we define \(H_r\) by the identity

\[
\prod_{i=1}^{n-1} (1 + k_it) = H_0 + H_1t + H_2t^2 + \cdots + H_{n-1}t^{n-1}.
\]

We define the normalized \(r\)-th mean curvatures to be \(\sigma_r = \frac{H_r}{\binom{n-1}{r}}\).

We are now ready to state the Hsiung-Minkowski formulas:

**Theorem 2.2.**\(^{[10]} [17]\) Suppose \(N^n\) is a space form. Let \(X\) be a conformal vector field on \(N\) and \(\nu\) be an unit normal vector field of \(\Sigma\). Then

\[
\int_\Sigma (\text{div} X) \sigma_{r-1} = n \int_\Sigma \sigma_r \langle X, \nu \rangle
\]
for $1 \leq r \leq n - 1$. For $r = 1, 2$, this holds with no assumption on $N$ being a space form.

**Proof.** Let us illustrate the proof for $r = 1$ for example. Let $W$ be the vector field on $\Sigma$ defined by $\sum_{i=1}^{n-1} \langle X, e_i \rangle e_i$, where $\{e_i\}_{i=1}^{n-1}$ is a local orthonormal field on $\Sigma$ with $\nabla_{e_i} e_j(p) = 0$, then

$$\text{div}W(p) = \sum_{i=1}^{n-1} \nabla_{e_i} \langle X, e_i \rangle = \sum_{i=1}^{n-1} \langle \nabla_{e_i} X, e_i \rangle + \langle X, \nabla_{e_i} e_i \rangle$$

$$= \sum_{i=1}^{n-1} \frac{1}{2} \mathcal{L}_X g(e_i, e_i) + \langle X, \nu \rangle \langle \nu, \nabla_{e_i} e_i \rangle$$

$$= \frac{(n-1)}{n} \text{div}X - \langle X, \nu \rangle H.$$

The result follows by applying divergence theorem. \hfill \Box

## 3. Some integral inequalities in space forms

In this section, we first study the conformal vector fields on space forms. We then apply Reilly’s and Hsiung-Minkowski formulas to deduce some sharp integral inequalities for convex hypersurfaces in space forms.

By rescaling we can assume the curvature of a space form is $K = 0, \pm 1$. We will denote the space forms

$$(\mathbb{R}^n, dr^2 + r^2 d\theta^2), \quad (\mathbb{H}^n, dr^2 + \sinh r^2 d\theta^2) \quad \text{and} \quad (\mathbb{S}^n, dr^2 + \sin r^2 d\theta^2)$$

by $N_0$, $N_{-1}$ and $N_1$ respectively, i.e. $N_K$ is a space form of curvature $K$. Here $r$ denotes the distance from a fixed point $O$ in $N_K$ and $d\theta^2$ is the metric of the standard sphere $\mathbb{S}^{n-1}$. We define $X = \nabla f$, where

$$f = f_K = \begin{cases} 
\frac{1}{2} r^2 & \text{on } N_0 \\
\cosh r & \text{on } N_{-1} \\
-\cos r & \text{on } N_1 
\end{cases} \quad \text{(3.1)}$$

will give a conformal vector field on $N_K$. In fact, if $X = \nabla f$ then

$$\mathcal{L}_X g = 2\nabla^2 f.$$

For $f_K$ such defined, direct computation gives

$$\nabla f = f' \partial_r \quad \text{and} \quad \nabla^2 f = f'' g \quad \text{(3.2)}$$
and so $X$ is conformal. For later use, we define the functions $f'_K = S_K = C_K = C$. Equivalently, $S$ and $C$ are the unique solution to
\begin{align*}
C''_K + KC_K &= 0, \quad C_K(0) = 1, \quad C'_K(0) = 0, \\
S''_K + KS_K &= 0, \quad S_K(0) = 0, \quad S'_K(0) = 1. \tag{3.3}
\end{align*}

From (3.2), we have

**Lemma 3.1.** On $N_K$, if $X$ is chosen by (3.1), then
\[ 
\nabla V X = C_K V \quad \text{and} \quad \text{div} X = \Delta f = nC_K.
\]

Thus the Hsiung-Minkowski formulas become
\[ 
\int_{\Sigma} C_K \sigma_{r-1} = \int_{\Sigma} \sigma_r \langle X, \nu \rangle
\] (3.4)
for $1 \leq r \leq n - 1$.

Using (3.2), by direct calculations, we have

**Lemma 3.2.** On $N_K$, for $f$ chosen by (3.1), we have
\[ 
(\Delta f)^2 - |\nabla f|^2 - Rc(\nabla f, \nabla f) = (n - 1)(nC^2 - KS^2).
\]

**Lemma 3.3.** Suppose $\Omega$ is a region in $N_K$ which has smooth boundary $\partial \Omega$, if $f$ is chosen as in (3.1), then
\[ 
2u \Delta z + Hu^2 = (n - 1) \langle \nabla f^2, \nu \rangle - H \langle X, \nu \rangle^2,
\]
where $u = \frac{\partial f}{\partial \nu}$ and $z = f |_{\Sigma}$.

**Proof.** We have $u = \langle \nabla f, \nu \rangle = \langle X, \nu \rangle$. Let $\{e_i\}_{i=1}^{n-1}$ be an orthonormal basis of $T_p \Sigma$ such that $\nabla e_i e_j (p) = 0$. Then
\[ 
\Delta z(p) = \sum_{i=1}^{n-1} e_i e_i f = \sum_{i=1}^{n-1} e_i \langle X, e_i \rangle = \sum_{i=1}^{n-1} \langle \nabla e_i X, e_i \rangle + \langle X, \nabla e_i e_i \rangle
\]
\[ 
= \sum_{i=1}^{n-1} f'' + \langle X, \nu \rangle \langle \nu, \nabla e_i e_i \rangle
\]
\[ 
= (n - 1) f'' - Hu.
\]
The result follows from (3.2). \qed

**Lemma 3.4.** With the notations in Theorem 2.1, suppose $\Omega$ is a region in $N_K$ which has smooth boundary $\partial \Omega$, if $X$ is chosen as in (3.1), then
\[ 
\int_{\Sigma} H \langle X, \nu \rangle^2 = (n - 1) \int_{\Omega} ((n + 1)C^2 - 1) + \int_{\Sigma} A(\nabla z, \nabla z).
\]
Proof. Recall $S = f'$ and $S'' = -KS$. By the formula $\Delta(h(r)) = h'' + (\Delta r)h' = h'' + (n - 1)\frac{S}{h}h'$, we compute

$$\frac{1}{2}\Delta(f^2) = nC^2 - KS^2.$$ 

So by divergence theorem,

$$(n - 1)\int_{\Sigma} \langle \nabla(S^2), \nu \rangle = (n - 1)\int_{\Omega} \Delta(S^2) = 2(n - 1)\int_{\Omega} (nC^2 - KS^2).$$ 

Therefore by (2.1) (Reilly’s formula), Lemma 3.2 and Lemma 3.3,

$$\int_{\Sigma} H \langle X, \nu \rangle^2 = (n - 1)\int_{\Omega} (nC^2 - KS^2) + \int_{\Sigma} A(\nabla z, \nabla z)$$
$$= (n - 1)\int_{\Omega} ((n + 1)C^2 - 1) + \int_{\Sigma} A(\nabla z, \nabla z).$$ 

□

We now state our first main result:

**Theorem 3.1.** Suppose $\Sigma$ is a compact convex hypersurface (i.e. the second fundamental form $A \geq 0$) in $N_K$. Suppose $O$ is in the interior $\Omega$ of $\Sigma$ in $N_K$ and $B_{r_0} \subset \Omega \subset B_{r_1}$, where $B_r$ is the closed geodesic ball of radius $r$ centered at $O$. Then

1. If $N_K = \mathbb{R}^n$, we have $n\text{Vol}(\Omega) \leq r_1 \text{Area}(\Sigma)$.

2. If $N_K = \mathbb{H}^n$, we have

$$\int_{\Omega} ((n + 1) \cosh^2 r - 1) \leq \sinh(r_1) \cosh(r_1) \text{Area}(\Sigma).$$

3. If $N_K = \mathbb{S}^n$, and furthermore, $\Sigma$ is contained in the closed hemisphere centered around $O$, we have

$$\int_{\Omega} ((n + 1) \cos^2 r - 1) \leq \sin(r_1) \cos(r_0) \text{Area}(\Sigma).$$

In all cases, the equality holds if and only if $r_0 = r_1$ and $\Sigma$ is the geodesic sphere of radius $r_1$ centered at $O$.

Proof. By [19], $\Sigma$ bounds a geodesically convex region, in particular, $\langle X, \nu \rangle = \langle S_K \partial_r, \nu \rangle \geq 0$. By (3.3),

$$\int_{\Sigma} H \langle X, \nu \rangle^2 \leq \int_{\Sigma} H \langle X, \nu \rangle S_K(r) \leq S_K(r_1) \int_{\Sigma} H \langle X, \nu \rangle$$
$$= (n - 1)S_K(r_1) \int_{\Sigma} C_K. \quad (3.5)$$
We have
\[
\int_{\Sigma} C_K \begin{cases} 
= \text{Area}(\Sigma), & K = 0 \\
\leq \cosh r_1 \text{Area}(\Sigma), & K = -1 \\
\leq \cos r_0 \text{Area}(\Sigma), & K = 1
\end{cases}
\] (3.6)

On the other hand, by Lemma 3.4,
\[
\int_{\Sigma} H \langle X, \nu \rangle^2 \geq (n - 1) \int_{\Omega} ((n + 1)C_K^2 - 1).
\]

Thus the inequalities in all three cases hold.

Suppose the equality holds, since \( \int_{\Sigma} H \langle X, \nu \rangle = (n - 1) \int_{\Sigma} C_K \geq 0 \), from (3.5), we have \( S_K(r) \equiv S_K(r_1) \) if \( \int_{\Sigma} C_K = 0 \), we have \( K = 1 \) and \( r \equiv \pi/2 \). From this we can easily deduce that \( \Sigma \) is the equator, i.e. the geodesic sphere of radius \( \pi/2 \).

\[ \square \]

**Remark 1.** We remark that when \( N_K = S^n \), there is no loss of generality to assume that \( \Sigma \) is contained in a closed hemisphere, since by the result of [1], \( \Sigma \) is either totally geodesic (i.e. a great hypersphere) or is contained in an open hemisphere.

Using similar techniques, we have

**Theorem 3.2.** Let \( \Sigma \) be a compact convex hypersurface of \( N_K \) which encloses \( \Omega \). If \( K = 1 \), we assume that \( \Sigma \) is contained in the open hemisphere centered at \( O \). Suppose \( B_{r_0} \subset \overline{\Omega} \subset B_{r_1} \) \( (r_1 < \pi/2 \) if \( K = 1 \), where \( B_r \) is the closed geodesic ball of radius \( r \) centered at \( O \). Then

(1) When \( N = \mathbb{R}^n \), then
\[
n \text{Vol}(\Omega) \leq r_1^2 \int_{\Sigma} \sigma_1 \leq r_1^3 \int_{\Sigma} \sigma_2 \leq \cdots \leq r_1^n \int_{\Sigma} \sigma_{n-1}.
\]

(2) When \( N = \mathbb{H}^n \), then
\[
nV_{-1}(\Omega) \leq \sinh^2 r_1 \int_{\Sigma} \sigma_1 \leq \frac{\sinh^3 r_1}{\cosh r_0} \int_{\Sigma} \sigma_2 \leq \frac{\sinh^4 r_1}{\cosh^2 r_0} \int_{\Sigma} \sigma_3 \leq \cdots \leq \frac{\sinh^n r_1}{\cosh^{n-2} r_0} \int_{\Sigma} \sigma_{n-1}.
\]

(3) When \( N = S^n \), then
\[
nV_1(\Omega) \leq \sin^2 r_1 \int_{\Sigma} \sigma_1 \leq \frac{\sin^3 r_1}{\cos r_1} \int_{\Sigma} \sigma_2 \leq \frac{\sin^4 r_1}{\cos^2 r_1} \int_{\Sigma} \sigma_3 \leq \cdots \leq \frac{\sin^n r_1}{\cos^{n-2} r_1} \int_{\Sigma} \sigma_{n-1}.
\]
Here $V_K(\Omega) = \frac{1}{n} \int_{\Omega} ((n + 1)C^2_K - 1)$. Any equality of the above inequalities holds if and only if $\Sigma$ is the geodesic sphere $S_{r_1}$ centered at $O$.

**Proof.** Recall $H = (n - 1)\sigma_1$. Lemma 3.4 implies

$$\int_{\Sigma} \sigma_1 \langle X, \nu \rangle^2 = nV_K(\Omega) + \frac{1}{n - 1} \int_{\Sigma} A(\nabla x, \nabla x) \geq nV_K(\Omega).$$

On the other hand, since $\langle X, \nu \rangle = S_K(r) \langle \partial_r, \nu \rangle$, we have $\langle X, \nu \rangle^2 \leq S_K(r_1)^2$. So

$$\int_{\Sigma} \sigma_1 \langle X, \nu \rangle^2 \leq S_K(r_1)^2 \int_{\Sigma} \sigma_1.$$ 

We will prove the remaining cases by induction, and let us do it only on $S^n$, since the other two cases are similar. The induction step is done by observing

$$\int_{\Sigma} \sigma_k \leq \frac{1}{\cos r_1} \int_{\Sigma} \sigma_k \cos r = \frac{1}{\cos r_1} \int_{\Sigma} \sigma_{k+1} \langle X, \nu \rangle \leq \frac{\sin r_1}{\cos r_1} \int_{\Sigma} \sigma_{k+1},$$

which we have used (3.4).

Suppose any of the equalities holds, then $\langle X, \nu \rangle^2 = S_K(r)^2 \langle \partial_r, \nu \rangle^2 \equiv S_K(r_1)^2$ and hence $r \equiv r_1$, i.e. it is the sphere centered at $O$ with radius $r_1$.

**Remark 2.** Recall that $\sigma_1 = H/(n - 1)$. When $N = \mathbb{R}^n$, by Theorem 3.2 we have $\max_{\Sigma} H \geq \frac{n(n-1)V_K(\Omega)}{S_K(r)^2 \text{Area}(\Sigma)} = \frac{n(n-1)V_K(\Omega)}{r^2 \text{Area}(\Sigma)}$. This recovers a result due to O.Garay [7] that if $\Sigma \subset \mathbb{R}^n$ is convex and

$$H \leq \frac{n(n-1)\text{Vol}(\Omega)}{r^2 \text{Area}(\Sigma)},$$

then $\Sigma$ is the sphere of radius $r$. The condition can also be replaced by

$$\sigma_k \leq \frac{n\text{Vol}(\Omega)}{r^{k+1} \text{Area}(\Sigma)}.$$ 

To state the next result, we need to have a notion of the center of mass of $\Sigma$ in $N_K$. Let $O$ be a fixed point on $N_K$. We use the following models for $N_K$:

$$N_K = \{ x \in \mathbb{R}^{n+1} : (x^0, x^1, \cdots, x^n) = (C_K(r), S_K(r)\theta), \, r \geq 0, \theta \in S^{n-1} \}$$

with metric induced from

$$\sum_{i=1}^{n} (dx^i)^2 + K(dx^0)^2.$$
We can assume that \( O = (C_K(0), 0) = (1, 0, \cdots, 0) \). We say that \( \Sigma \) has the center of mass at \( O \) if
\[
\int_{\Sigma} (x^1, \cdots, x^n) = 0.
\] (3.7)

With this understood, we have

**Lemma 3.5.** Suppose \( \Sigma \) is a compact hypersurface in \( N_K \) whose center of mass is \( O \), then its first non-zero eigenvalue of \( \Delta \) satisfies
\[
\lambda_1(\Sigma) \leq \frac{\int_{\Sigma}((n - 1) - KS^2_K|\partial^T_r|^2)}{\int_{\Sigma}S^2_K},
\]
where \( \partial^T_r \) is the tangential component of \( \partial_r \) onto \( T\Sigma \).

**Proof.** By min-max principle [26], we have \( \lambda_1 \int_{\Sigma} f^2 \leq \int_{\Sigma} |\nabla f|^2 \) if \( \int_{\Sigma} f = 0 \). In particular,
\[
\lambda_1 \sum_{i=1}^{n} \int_{\Sigma} (x^i)^2 = \lambda_1 \sum_{i=1}^{n} \int_{\Sigma} S^2_K \leq \sum_{i=1}^{n} \int_{\Sigma} |\nabla x^i|^2 = \int_{\Sigma} ((n - 1) - K|\nabla x^0|^2) = \int_{\Sigma} ((n - 1) - K|\nabla C_K|^2) = \int_{\Sigma} ((n - 1) - KS^2_K|\partial^T_r|^2).
\]

**Theorem 3.3.** For a compact convex hypersurface \( \Sigma \) of \( N_K \), assume that its center of mass is \( O \) if \( K \neq 0 \) and it is inside the open hemisphere centered at \( O \) if \( K = 1 \), then
\[
\lambda_1(\Sigma) \leq \frac{A_K(\Sigma) \max_{\Sigma} H}{nV_K(\Omega)},
\] (3.8)
where \( A_K(\Sigma) = \frac{1}{n-1} \int_{\Sigma}((n - 1) - KS^2_K|\partial^T_r|^2) \) and \( V_K(\Omega) = \frac{1}{n} \int_{\Omega}((n + 1)C^2_K - 1) \). The equality holds if and only if \( \Sigma \) is a geodesic sphere.

**Proof.** Note that \( A_0 \) and \( V_0 \) are independent of the center of mass, so we can assume that in all cases, its center of mass is \( O \). By Lemma 3.4
\[
\int_{\Sigma} HS^2_K \geq \int_{\Sigma} H\langle X, \nu \rangle^2 = n(n-1)V_K(\Omega) + \int_{\Sigma} A(\nabla z, \nabla z) \geq n(n-1)V_K(\Omega).
\]
On the other hand, by Lemma 3.5 \( \int_{\Sigma} S^2_K \leq \frac{n-1}{\lambda_1(\Sigma)} A_K(\Sigma) \). Therefore
\[
\int_{\Sigma} HS^2_K \leq \frac{n-1}{\lambda_1(\Sigma)} A_K(\Sigma) \max_{\Sigma} H.
\]
The inequality (3.8) follows. If the equality holds, we can proceed as in the proof of Theorem 3.1 to show that \( \Sigma \) is a geodesic sphere. \( \square \)

**Remark 3.** In the case where \( N_K = \mathbb{R}^n \), Theorem 3.3 implies that if \( \Sigma \) is convex and \( H \leq \frac{\lambda_1 n \text{Vol}(\Omega)}{\text{Area}(\Sigma)} \), then \( \Sigma \) is a sphere. This is also a result of O. Garay [7].

4. **Applications to eigenvalue estimates**

In this section, we will prove several lower bounds of the first nonzero eigenvalue of the Hodge Laplacian on differential forms on the boundary \( \Sigma \) of a Riemannian manifold \((N, g)\). These results are the natural generalizations of some results in [4], [8], [11], [13], [22] and [24].

First we set up some notations. Fix \( x \in \Sigma \) and let \( k_1(x), \ldots, k_{n-1}(x) \) be the principal curvatures of \( \Sigma \) at \( x \) w.r.t. the outward unit normal \( \nu \). We define the \( r \)-curvatures (not to be confused with the \( r \)-th mean curvature) to be all the possible sums \( k_{i_1}(x) + \cdots + k_{i_r}(x) \) where \( i_1 < \cdots < i_r \). We can assume \( k_1(x) \leq \cdots \leq k_{n-1}(x) \), then we define the lowest \( r \)-curvature to be

\[
s_r(x) = k_1(x) + \cdots + k_r(x).
\]

We also define

\[
s_r(\Sigma) = \min_{x \in \Sigma} s_r(x).
\]

Note that the second fundamental form is bounded from below by \( s_1 \) and \( s_{n-1}(x) = H \) is the mean curvature. It is easy to see that if \( l \leq m \), \( \frac{s_l}{l} \leq \frac{s_m}{m} \), and that \( s_l \geq 0 \) implies \( s_m \geq 0 \).

We denote by \( \bar{d} \) and \( \bar{\delta} \) to be the exterior derivative and its adjoint w.r.t. the \( L^2 \) inner product on \((N, g)\) respectively. The Hodge Laplacian \( \bar{\Delta} \) of a \( p \)-form on \((N, g)\) is defined by

\[
\bar{\Delta} \alpha = -(\bar{d} \bar{\delta} + \bar{\delta} \bar{d}) \alpha
\]

for \( \alpha \in \Omega^p(N) \). Our sign is chosen such that \( \bar{\Delta} \) is the second derivative for functions on \( N = \mathbb{R} \). Recall the Bochner formula (see e.g. [12] p.218 Theorem 50):

\[
-\bar{\Delta} \alpha = \bar{\nabla}^* \bar{\nabla} \alpha + W^r(\alpha)
\]

where \( W^r \) is a self-adjoint endomorphism on \( \Omega^r(N) \), which is determined by the Riemann curvature tensor on \((N, g)\). This term is sometimes called the Bochner curvature term. When \( r = 1 \), \( W^1 = Rc \) and by [3], \( W^r \geq r(n-r) \gamma \) where \( \gamma \) is the lowest eigenvalue of the curvature.
operator on $(N, g)$. However, $W^r \geq 0$ is usually much weaker than the curvature operator being nonnegative.

We define the shape operator $S = \nabla \nu$ on $T\Sigma$ and define $S^r : \Omega^r(\Sigma) \to \Omega^r(\Sigma)$ by

$$S^r \alpha(X_1, \cdots, X_r) = \sum_{j=1}^r \alpha(X_1, \cdots, S(X_j), \cdots, X_r).$$

We also define $S^0$ to be zero. For example, if $\alpha$ is a 1-form, then $S_1^1 \alpha(X) = \alpha(S(X))$. Observe that $S^{n-1} \alpha = H \alpha$ and that the eigenvalues of $S^r$ are exactly the $r$-curvatures of $\Sigma$, therefore

$$\langle S^r \alpha, \alpha \rangle \geq s_r(\Sigma) |\alpha|^2.$$

We define $\lambda'_{k,r}$ (respectively $\lambda''_{k,r}$) to be the $k-th$ nonzero eigenvalue for the exact (respectively co-exact) $r$-forms on $\Sigma$. By Hodge decomposition theorem and Hodge duality (e.g. [23]), we have

$$\begin{cases}
\lambda_{1,r}(\Sigma) = \min \{ \lambda'_{1,r}(\Sigma), \lambda''_{1,r}(\Sigma) \}, \\
\lambda''_{1,r}(\Sigma) = \lambda'_{1,r+1}(\Sigma), \\
\lambda''_{1,r}(\Sigma) = \lambda'_{1,n-1-r}(\Sigma).
\end{cases}$$

From this we see that to determine $\lambda_{1,r}$, it suffices to determine $\lambda'_{1,r}$ for $1 \leq r \leq \lfloor \frac{n}{2} \rfloor$.

The following formula is the generalization of Reilly’s formula to differential forms.

**Theorem 4.1.** ([13] Theorem 3) Let $\alpha \in \Omega^r(N)$, $r \geq 1$, then

$$\int_N |d\alpha|^2 + |\delta\alpha|^2 - |\nabla\alpha|^2 = \int_N W^r(\alpha, \alpha) - 2 \int_{\Sigma} \langle \nu \alpha, \delta i^* \alpha \rangle + \int_{\Sigma} B(\alpha, \alpha)$$

where the boundary term is given by

$$B(\alpha, \alpha) = \langle S^r(i^* \alpha), i^* \alpha \rangle + \langle S^{n-r}(i^* \bar{\alpha}), i^* \bar{\alpha} \rangle.$$

Here $i : \Sigma \to N$ is the inclusion and $\bar{\alpha} : \Omega^r(N) \to \Omega^{n-r}(N)$ is the Hodge star operator on $N$. We will also denote by $d$ and $\delta$ the exterior derivative and its adjoint on $\Sigma$ respectively.

The classical Reilly’s formula (Theorem 2.1) can be recovered by setting $\alpha = df \in \Omega^1(N)$.

We now state our first main result in this section.

**Theorem 4.2.** Let $(N^n, g)$ be a compact orientable Riemannian manifold with boundary $\Sigma$. Suppose $W^r$ or $W^{n-r}$ on $N$ is bounded from
below by \( k \geq 0 \). Assume that \( s_q \geq 0 \) where \( q = \min\{r, n-r\} \). Then for \( 1 \leq r \leq n-1 \), we have

\[
2\lambda'_{1,r} = 2\lambda''_{1,r-1} \geq k + s_r s_{n-r} + \sqrt{(s_r s_{n-r})^2 + 2 s_r s_{n-r} k}.
\]  

(4.1)

If the equality holds, then \( k = 0 \), the \( r \)-curvatures constantly equal \( s_r > 0 \) and the \((n-r)\)-curvatures constantly equal \( s_{n-r} > 0 \). If, furthermore, \((N, g)\) has non-negative Ricci curvature, then the equality holds if and only if \((N, g)\) is isometric to a Euclidean ball. The condition on Ricci curvature can be removed if \( r = 1 \) or \( n-1 \).

**Proof.** Note that by Hodge decomposition theorem and Hodge duality, \( \lambda'_{1,r} = \lambda''_{1,r-1} = \lambda'_{1,n-r} \) and by (4.5) below, both \( W_r \) and \( W_{n-r} \) are bounded from below by \( k \).

Let \( \phi \) be an co-exact \((r-1)\)-eigenform on \( \Sigma \) with eigenvalue \( \lambda = \lambda''_{1,r-1} = \lambda'_{1,r} \), i.e. \( \Delta \phi = -\delta d \phi = -\lambda \phi \). Then \( \omega = d \phi \) is an exact \( r \)-eigenform with eigenvalue \( \lambda \). By Theorem 2 of [2] (p.148), there exists an \((r-1)\)-form \( \phi \) on \( N \) such that \( \delta d \phi = 0 \) and \( i^\ast \phi = \phi \) on \( \Sigma \). Let \( \omega = d \phi \). Then

\[
\begin{cases}
\overline{d \omega} = \overline{\delta \omega} = 0 & \text{on } N \\
i^\ast \omega = \omega & \text{on } \Sigma.
\end{cases}
\]

Using Reilly’s formula on \( \omega = d \phi \),

\[
0 \geq \int_N -|\nabla \omega|^2
= \int_N W^r (d \phi, d \phi) + \int_\Sigma -2 \langle \iota_\nu \omega, \delta \omega \rangle + \langle S^r (i^\ast \omega), i^\ast \omega \rangle + \langle S^{n-r} (i^\ast \pi \omega), i^\ast \pi \omega \rangle
\geq k \int_N |d \phi|^2 + \int_\Sigma -2\lambda \langle \iota_\nu \omega, \phi \rangle + s_r |i^\ast \omega|^2 + s_{n-r} |i^\ast \pi \omega|^2
= k \int_N \langle \overline{\phi}, \overline{\delta d \phi} \rangle + s_r \int_\Sigma |d \phi|^2 + s_{n-r} \int_\Sigma |i^\ast \pi \omega|^2.
\]

(4.2)

The condition \( s_q \geq 0 \) implies \( s_{n-r} \geq 0 \). From the above, as \( \int_\Sigma \langle \phi, \iota_\nu \omega \rangle = \int_N |\omega|^2 > 0 \), this shows that \( 2\lambda \geq k \), which proves (4.1) in the case where \( s_{n-r} = 0 \). So in the following we can assume \( s_{n-r} > 0 \).
As \( \|i^\ast \varpi\|^2 = |\iota_v \varpi|^2 \) and \( \int_\Sigma |d\phi|^2 = \int_\Sigma \langle \phi, \delta d\phi \rangle = \lambda \int_\Sigma |\phi|^2 \), the inequality (4.2) becomes

\[
0 \geq \int_\Sigma - (2\lambda - k) \langle \phi, \iota_v \varpi \rangle + s_r \lambda |\phi|^2 + s_{n-r} |\iota_v \varpi|^2 \\
= \int_\Sigma s_{n-r} \left| \iota_v \varpi - \frac{\lambda - k/2}{s_{n-r}} \phi \right|^2 + \left( \frac{s_r}{s_{n-r}} \right) \left( \frac{\lambda - k/2}{s_{n-r}} \right)^2 |\phi|^2 \\
\geq \int_\Sigma \left( \frac{s_r}{s_{n-r}} \lambda - \frac{(\lambda - k/2)^2}{s_{n-r}} \right) |\phi|^2.
\]

As \( \phi \) is not identically zero, we conclude that

\[
(\lambda - k/2)^2 \geq s_r s_{n-r} \lambda = 2c\lambda
\]

where \( 2c = s_r s_{n-r} \). This implies either

\[
\lambda - \frac{k}{2} \leq c - \sqrt{c^2 + ck} \quad \text{or} \quad \lambda - \frac{k}{2} \geq c + \sqrt{c^2 + ck}.
\]

In view of (4.2), we conclude that the second case holds. i.e.

\[
2\lambda \geq k + s_r s_{n-r} + \sqrt{(s_r s_{n-r})^2 + 2s_r s_{n-r} k}.
\]

Suppose the equality holds, then from (4.2), \( \nabla \varpi = 0 \). As \( \varpi \) is parallel, \( |\varpi|^2 \) is constant, and as \( i^\ast \varpi = \omega \), this constant is nonzero, which we can assume to be 1. The curvature term \( W^r \) is given by (see e.g. [12] p.218 Theorem 50):

\[
W^r(\varpi) = \frac{1}{2} \sum_{i,j=1}^n \theta^i \cdot \theta^j \cdot \overrightarrow{R}(e_i, e_j) \varpi
\]

where \( \{e_j\}_{j=1}^n \) is a local orthonormal frame on \( N \), \( \{\theta^j\}_{j=1}^n \) is its dual frame and \( \overrightarrow{R} \) is the curvature operator on \( (N, g) \). Here \( \theta^i \cdot \alpha = \theta^i \wedge \alpha - \iota_{e_i} \alpha \) is the Clifford multiplication on a differential form \( \alpha \). Since \( 0 = \nabla \varpi \), we have \( \nabla^2 \varpi = 0 \) and so \( \overrightarrow{R}(e_i, e_j) \varpi = 0 \). Therefore from (4.2)

\[
0 = \frac{1}{2} \sum_{i,j=1}^n \theta^i \cdot \theta^j \cdot \overrightarrow{R}(e_i, e_j) \varpi, \varpi = \langle W^r(\varpi), \varpi \rangle = k|\varpi|^2 = k.
\]

So we now have \( \lambda = s_r s_{n-r} > 0 \). Therefore from (4.3),

\[
\iota_v \varpi = \frac{\lambda}{s_{n-r}} \phi = s_r \phi.
\]

From this and (4.2), (4.3), we see that \( S^r \equiv s_r \) and \( S^{n-r} \equiv s_{n-r} \), i.e. the \( r \)-curvatures and the \( (n-r) \)-curvatures are constants.
Now we suppose, furthermore, that $Rc \geq 0$. As $|\omega|^2 = 1$,
\[
\text{Area}(\Sigma) = \int_{\Sigma} |\omega|^2 = \int_{\Sigma} (|d\phi|^2 + |\iota_\nu \omega|^2) = \int_{\Sigma} \lambda_1 |\phi|^2 + |\iota_\nu \omega|^2 = \left(\frac{s_r + s_{n-r}}{s_r}\right) \int_{\Sigma} |\iota_\nu \omega|^2.
\]

On the other hand, by Stokes theorem,
\[
\text{Vol}(N) = \int_N |d\phi|^2 = \int_N \langle \phi, \iota^* d\phi \rangle + \int_{\Sigma} \langle \iota^* \phi, \iota_\nu d\phi \rangle = \int_{\Sigma} \langle \phi, \iota_\nu \omega \rangle = \frac{1}{s_r} \int_{\Sigma} |\iota_\nu \omega|^2.
\]
From these we have
\[
\frac{\text{Area}(\Sigma)}{\text{Vol}(N)} = s_r + s_{n-r}.
\]
Recall that we have $\frac{s_l}{m} \leq \frac{s_m}{m}$ for $l \leq m$, so $s_r + s_{n-r} \leq \frac{r}{n-1}s_{n-1} + \frac{n-r}{n-1}s_{n-1} = \frac{n}{n-1}s_{n-1}$. Thus
\[
\frac{\text{Area}(\Sigma)}{\text{Vol}(N)} \leq \frac{n}{n-1}s_{n-1}.
\]

By \cite{18} Theorem 1, as $Rc \geq 0$, we conclude that $(N, g)$ is isometric to a Euclidean ball.

Using $\nabla_X(\overline{\alpha}) = \overline{\nabla_X \alpha}$ and $\theta^j \cdot \overline{\alpha} = \overline{\theta^j \cdot \alpha}$, we have, by (1.4),
\[
\langle W^r(\overline{\omega}), \overline{\omega} \rangle = \langle W^{n-r}(\overline{\omega}), \overline{\omega} \rangle.
\]
As $W^1 = Rc$ and $k = 0$, so the condition $Rc \geq 0$ is redundant for $r = 1$ or $n - 1$. Finally, it is well-known that (see e.g. \cite{6})
\[
\lambda^{1,r}_{1}(S^{n-1}) = r(n-r).
\]
From this it is easy to see that the equality holds on any Euclidean ball, with $k = 0$.

**Remark 4.** Theorem 4.2 is an extension of Theorem 1 in \cite{24}, which corresponds to our result when $k = 0$ and $r = 1$.

To state our next result, we need to define the Steklov eigenvalues, as follows. Let $\alpha \in \Omega^r(\Sigma)$, $r = 0, \cdots, n-1$. Then there exists a unique $r$-form $\overline{\alpha} \in \Omega^r(N)$ such that (see e.g. \cite{20} Theorem 3.4.6)
\[
\begin{cases}
\Delta \overline{\alpha} = 0 & \text{on } (N, g), \\
\iota^* \overline{\alpha} = \alpha, \ i_\nu \overline{\alpha} = 0 & \text{on } \Sigma.
\end{cases}
\]
We define the Steklov operator $T^r : \Omega^r(\Sigma) \to \Omega^r(\Sigma)$ by

$$
T^r \alpha = \iota_\nu d\alpha.
$$

By [14] Theorem 11, $T^r$ is an elliptic nonnegative self-adjoint pseudo-differential operator of order one. Thus the eigenvalue problem

$$
T^r \alpha = p\alpha
$$

has a discrete spectrum

$$
0 \leq p_{1,r}(N) \leq p_{2,r}(N) \leq \cdots.
$$

We will write $p_{k,r}$ for $p_{k,r}(N)$. Here we use the convention in [14] that $p_{1,r}$ is the smallest nonnegative eigenvalue of $T^r$. Thus in the classical case where $r = 0$, i.e. for $f \in C^\infty(\Sigma)$, $\overline{f}$ being the unique harmonic extension of $f$ to $N$ and

$$
Tf = T^0 f = \frac{\partial \overline{f}}{\partial \nu},
$$

the first nonnegative eigenvalue of $T$ is zero, corresponding to the constant functions on $\Sigma$. So in our convention, $p_{1,0} = 0$ and $p_{2,0}$ is the smallest positive eigenvalue, usually called the first Steklov eigenvalue of $N$. We will simply denote $p_{2,0}$ by $p_2$.

We remark that the first eigenvalue of $T^r$ satisfies the min-max principle ([14] Theorem 11):

$$
p_{1,r}(N) = \inf \left\{ \frac{\int_N |\overline{\delta \phi}|^2 + |\overline{\delta \phi}|^2}{\int_\Sigma |\phi|^2} : 0 \neq \overline{\phi} \in \Omega^r(N), \iota_\nu \overline{\phi} = 0 \right\}. \quad (4.7)
$$

When $r = 0$, we also have the following min-max principle for the smallest nonzero Steklov eigenvalue (see for example [9] p.113):

$$
p_2(N) = p_{2,0}(N) = \inf \left\{ \frac{\int_N |\nabla \phi|^2}{\int_\Sigma |\phi|^2_{\Sigma}} : 0 \neq \overline{\phi} \in C^\infty(N), \int_\Sigma \overline{\phi} = 0 \right\}. \quad (4.8)
$$

Let us record a Reilly type inequality here, which generalizes [11] Theorem 3.1. Recall that $C_K, S_K$ are defined by (3.3) and $\Sigma$ is said to have the center of mass at $O$ if (3.7) is satisfied.

**Theorem 4.3.** Let $\Sigma$ be a closed hypersurface in the $n$-dimensional space form $(N_K, g)$ and $\Omega$ be the region bounded by $\Sigma$. Suppose that the center of mass of $\Sigma$ is $O$ if $K \neq 0$. Then for $k = 1, \cdots, n - 1$,

$$
p_2(\Omega) \left( \int_\Sigma C_K \sigma_{k-1} \right)^2 \leq \int_\Omega (n - KS_K^2) \int_\Sigma \sigma_k^2.
$$
The equality holds for some \( k \) if and only if either \( \sigma_k \) is identically zero or \( \Omega \) is a geodesic ball around \( O \).

**Proof.** For all \( K \) we can assume that the center of mass is \( O \), i.e. \( \int_{\Sigma} x^i = 0 \) for \( i = 1, \ldots, n \). By (1.8),

\[
p_2 \sum_{i=1}^{n} \int_{\Sigma} (x^i)^2 \leq \sum_{i=1}^{n} \int_{\Omega} |\nabla x^i|^2.
\]

As

\[
n = \sum_{i=1}^{n} |\nabla x^i|^2 + K |\nabla x^0|^2 = \sum_{i=1}^{n} |\nabla x^i|^2 + KS_K^2,
\]

we have

\[
p_2 \int_{\Sigma} S_K^2 = p_2 \sum_{i=1}^{n} \int_{\Sigma} (x^i)^2 \leq \int_{\Omega} (n - KS_K^2).
\]

Multiply the above by \( \int_{\Sigma} \sigma_k^2 \), we have, by Cauchy-Schwarz inequality,

\[
p_2 \left( \int_{\Sigma} \langle X, \sigma_k \nu \rangle \right)^2 \leq p_2 \int_{\Sigma} S_K^2 \int_{\Sigma} \sigma_k^2 \leq \int_{\Omega} (n - KS_K^2) \int \sigma_k^2.
\]

The inequality follows by applying the Hsiung-Minkowski formula (Equation (3.4)).

Suppose the equality holds and \( \sigma_k \) is not identically zero, then \( x^i \) are Steklov eigenfunctions associated to \( p_2 \), thus by Lemma 3.1, \( CV = \nabla_{\nu} X = p_2 X \). But then for any tangential vector \( V \) of \( \Sigma \), we have \( V(S_K^2) = V(|X|^2) = 2\langle \nabla_V X, X \rangle = 2(CV, \nabla_{\nu} X) = 0 \). Thus \( S_K(r) \) is constant and \( \Omega \) is a geodesic ball of radius \( r \), where \( \frac{S_K(r)}{C_K(r)} = \frac{1}{p_2} \).

**Remark 5.** By the Hodge-deRham theorem for manifolds with boundary (20 Theorem 2.6.1), any cohomology class of the deRham cohomology space (with real coefficients) \( H^{\prime}_{dR}(N, \overline{d}) \) is uniquely represented by \( \overline{\phi} \in \Omega^{\prime}(N) \) such that

\[
\begin{align*}
  d \overline{\phi} &= \delta \overline{\phi} = 0 \quad \text{on } N, \\
  i_{\nu} \overline{\phi} &= 0 \quad \text{on } \Sigma.
\end{align*}
\]

We will denote the space of all such \( \overline{\phi} \) by \( \mathcal{H}^{\prime}(N) \). So from (1.7), we see that \( p_{1,r} \) is positive if and only if \( \mathcal{H}^{\prime}(N) = 0 \). Therefore we are interested in \( p_{1,r} \) only when \( \mathcal{H}^{\prime}(N) = 0 \).

By Hodge duality, the relative deRham cohomology space (cf. 20 p.103) \( H_{dR}^{\prime}(N, \overline{d}) \) is isomorphic to the vector space

\[
\mathcal{H}^{\prime}_R(N) = \{ \overline{\phi} \in \Omega^{\prime}(N) : d \overline{\phi} = \delta \overline{\phi} = 0 \quad \text{on } N, i_{\nu} \overline{\phi} = 0 \quad \text{on } \Sigma \},
\]
called the space of Dirichlet harmonic fields.

**Theorem 4.4.** Let \((N^n, g)\) be a compact orientable Riemannian manifold with boundary \(\Sigma\). Let \(r = 1, \ldots, n - 1\). We assume \(p_{1,r-1}\) is non-trivial if \(r > 1\) (corresponding to \(H^{r-1}(N) = 0\)). Suppose the Bochner curvature \(W_r\) of \(N\) is bounded from below by \(k\), the \(r\)-curvatures of \(\Sigma\) are bounded from below by \(l\) and \(s_{n-r} \geq 0\). Let \(\lambda = \lambda'_{1,r}(\Sigma) = \lambda''_{1,r-1}(\Sigma)\) and let \(p\) to be \(p_{1,r-1}\) if \(r > 1\) and \(p_2 = p_{2,0}\) if \(r = 1\). Then

1. We have the following upper bound for \(p\):
   \[
s_{n-r}p \leq \lambda - \frac{k}{2} + ((\lambda - \frac{k}{2})^2 - s_{n-r}l\lambda)^{\frac{1}{2}}. \tag{4.9}
   \]
2. Assume \(l \leq 0\), then we have the following lower bounds for \(p\) and \(\lambda\):
   \[
s_{n-r}p \geq \lambda - \frac{k}{2} - ((\lambda - \frac{k}{2})^2 - s_{n-r}l\lambda)^{\frac{1}{2}}. \tag{4.10}
   \]
   \[
\lambda \geq \frac{s_{n-r}p^2 + kp}{2p - l}. \tag{4.11}
   \]
3. Assume \(k \geq 0\) and \(l \geq 0\). We have either
   \[
\lambda \geq s_{n-r}p + \frac{k}{2} \tag{4.12}
   \]
   or
   \[
\lambda \geq \frac{s_{n-r}p^2 + kp}{2p - l}, \tag{4.13}
   \]
   provided that it is well-defined. (If \(\lambda \leq s_{n-r}p + \frac{k}{2}\) and \(s_{n-r} > 0\), we will show that \(2p - l > 0\), see also Remark\[\PageIndex{6}\]).
4. Assume \(s_r \geq 0\), and \(H^r_R(N) = 0\). Then
   \[
2\lambda \geq k + s_{r,p_{1,n-1-r}} + s_{n-r}p. \tag{4.14}
   \]
   If \(r = 1\), the condition \(H^1_R(N) = 0\) can be replaced by \(s_{n-1} > 0\) and \(k \geq 0\).
5. The inequalities (4.10) and (4.11) are actually strict (if \(l \leq 0\)).
   Any of the equality cases in (4.9), (4.13) or (4.14) can hold only when \(k = 0\), with the \(r\)-curvatures and \((n - r)\)-curvatures both being positive constants.

Suppose \((N, g)\) has non-negative Ricci curvature. Then the equality in (4.9) or (4.13) holds if and only if \(r \geq \frac{n}{2} + 1\) or \(r = 1\), and \((N, g)\) is isometric to a Euclidean ball. The condition on Ricci curvature can be removed if \(r = 1\). The equality case in (4.14) can hold if and only if \(r = 1\), \(n \geq 4\) and \((N, g)\) is a Euclidean ball.
Proof. Let \( \phi \) be an co-exact \((r - 1)\)-eigenform on \( \Sigma \) with eigenvalue \( \lambda = \lambda_{1,r-1}' = \lambda_{1,r}' \), i.e. \( \Delta \phi = -\delta d \phi = -\lambda \phi \). Then \( \omega = d \phi \) is an exact \( r \)-eigenform on \( \Sigma \) and by [20] Lemma 3.4.7, there exists an \((r - 1)\)-form \( \bar{\phi} \) on \( N \) such that

\[
\begin{align*}
-\Delta \bar{\phi} &= (\overline{d \delta + \delta d}) \bar{\phi} = 0 \quad \text{on } N, \\
i^* \bar{\phi} &= \phi, \quad i^* \delta \bar{\phi} = 0 \quad \text{on } \Sigma.
\end{align*}
\]

By Stokes theorem,

\[
\int_N |\overline{d \delta \phi}|^2 = \int_N \langle \delta \phi, \delta \delta \delta \phi \rangle + \int_\Sigma \langle i^* \bar{\delta} \bar{\phi}, \iota_\nu d \delta \bar{\phi} \rangle = \int_N \langle \delta \bar{\phi}, -\delta \delta d \bar{\phi} \rangle = 0.
\]

So we have \( \overline{d \delta \phi} = \delta d \phi \). Let \( \bar{\omega} = \overline{d \phi} \), then \( \bar{\omega} \) is a harmonic field, i.e. \( d \bar{\omega} = \delta \bar{\omega} = 0 \).

By applying Reilly’s formula (Theorem 4.1) on \( \bar{\omega} = \overline{d \phi} \), and following exactly the same steps in the proof of Theorem 4.2,

\[
0 \geq \int_N -|\nabla \bar{\omega}|^2 \geq -(2 \lambda - k) \int_\Sigma \langle \phi, \iota_\nu \bar{\omega} \rangle + l \lambda \int_\Sigma |\phi|^2 + s_{n-r} \int_\Sigma |\iota_\nu \bar{\omega}|^2.
\]

(4.15)

We now prove (1) and (2) together. Let us first assume \( l \geq 0 \). As \( \omega \neq 0 \), \( \int_\Sigma \langle \phi, \iota_\nu \omega \rangle = \int_N |\omega|^2 > 0 \), thus by (4.15), we have

\[
2 \lambda - k \geq 0.
\]

(4.16)

The inequality (4.9) (and also (4.10)) is trivial if \( s_{n-r} = 0 \), so we assume \( s_{n-r} > 0 \). Let \( k = 2a \), \( U = (\int_\Sigma |\iota_\nu \bar{\omega}|^2)^{\frac{1}{2}} \) and \( Z = (\int_\Sigma |\phi|^2)^{\frac{1}{2}} > 0 \). So by Cauchy Schwarz inequality,

\[
s_{n-r} U^2 + l \lambda Z^2 \leq 2(\lambda - a) \int_\Sigma \langle \phi, \iota_\nu \bar{\omega} \rangle \leq 2(\lambda - a) U Z.
\]

(4.17)

By completing the square,

\[
s_{n-r} \frac{U}{Z} \leq \lambda - a + ((\lambda - a)^2 - s_{n-r} l \lambda_1)^{\frac{1}{2}}.
\]

(4.18)

Let us for the time being assume \( r > 1 \). We claim that

\[
p_{1,r-1} \leq \frac{\int_\Sigma \langle \phi, \iota_\nu \bar{\omega} \rangle}{\int_\Sigma |\phi|^2}.
\]

(4.19)

By the Friedrichs decomposition for harmonic fields, as \( \bar{\omega} \) is exact, there is a unique co-exact \((r - 1)\)-form \( \tilde{\phi} \) on \( N \) such that (see [20] Theorem 2.4.8 and its proof):

\[
\overline{d \tilde{\phi}} = \bar{\omega} \quad \text{on } N, \quad \iota_\nu \tilde{\phi} = 0 \quad \text{on } \Sigma.
\]
Let $\phi' = i^* \tilde{\phi}$, then as $\tilde{\phi} = 0$,
\[
\int_{\Sigma} \langle \phi', \iota_{\nu} \omega \rangle = \int_{N} \langle \nabla \tilde{\phi}, \nabla \omega \rangle - \langle \tilde{\phi}, \delta \omega \rangle = \int_{N} |\nabla \tilde{\phi}|^2 = \int_{N} |\tilde{\phi}|^2 + |\tilde{\phi}|^2.
\]
Thus by (4.17),
\[
p_{1,r-1} \leq \frac{\int_{\Sigma} \langle \phi', \iota_{\nu} \omega \rangle}{\int_{\Sigma} |\phi'|^2}. \tag{4.20}
\]
On the other hand, we have
\[
\int_{\Sigma} \langle \phi', \iota_{\nu} \omega \rangle = \int_{N} |\nabla \tilde{\phi}|^2 = \int_{N} |\tilde{\phi}|^2 = \int_{\Sigma} \langle \phi, \iota_{\nu} \omega \rangle. \tag{4.21}
\]
As $d\tilde{\phi} = d\phi$, we also have $d\phi' = d\phi$, so
\[
\lambda \int_{\Sigma} |\phi|^2 = \int_{\Sigma} |d\phi|^2 = \int_{\Sigma} \langle d\phi', d\phi \rangle = \int_{\Sigma} \langle \phi', \delta d\phi \rangle = \lambda \int_{\Sigma} \langle \phi', \phi \rangle.
\]
We conclude that $\phi' - \phi \perp \phi$ and thus $\int_{\Sigma} |\phi'|^2 \geq \int_{\Sigma} |\phi|^2$. Combining this with (4.21), (4.20), we can get (4.19). By Cauchy Schwarz inequality,
\[
p_{1,r-1} \leq \frac{\int_{\Sigma} \langle \phi', \iota_{\nu} \omega \rangle}{\int_{\Sigma} |\phi'|^2} \leq \frac{\int_{\Sigma} \langle \iota_{\nu} \omega \rangle^2}{\int_{\Sigma} \langle \phi, \iota_{\nu} \omega \rangle}, \tag{4.22}
\]
which implies
\[
p_{1,r-1}^2 \leq \frac{U^2}{Z^2}.
\]
Putting this into (4.18), we obtain (4.9)
\[
s_{n-r}p_{1,r-1} \leq \lambda - a + ((\lambda - a)^2 - s_{n-r}l\lambda)^{1/2}. \tag{4.23}
\]
We now claim that this is also true for $l \leq 0$. Actually, in this case, by (4.15) and (4.22),
\[
2\lambda - k \geq s_{n-r} \frac{\int_{\Sigma} \langle \iota_{\nu} \omega \rangle^2}{\int_{\Sigma} \langle \phi, \iota_{\nu} \omega \rangle} + l\lambda \frac{\int_{\Sigma} |\phi|^2}{\int_{\Sigma} \langle \phi, \iota_{\nu} \omega \rangle} \geq s_{n-r}p_{1,r-1} + \frac{l\lambda}{p_{1,r-1}}.
\]
Rearranging, we have
\[
s_{n-r}p_{1,r-1}^2 + kp_{1,r-1} \leq (2p_{1,r-1} - l)\lambda \tag{4.24}
\]
which implies (4.9) and (4.10) (regardless of whether $s_{n-r} = 0$). Also, (4.11) follows immediately from (4.24).

We have completed the proofs of (1) and (2) except for the case where $r = 1$. For $r = 1$, the proofs proceed in the same way except we have to replace (4.22) by
\[
p_2 = p_{2,0}(N) \leq \frac{\int_{\Sigma} \langle i^* \tilde{\phi}, \iota_{\nu} \omega \rangle}{\int_{\Sigma} |\phi|^2} \leq \frac{\int_{\Sigma} \langle \iota_{\nu} \omega \rangle^2}{\int_{\Sigma} \langle i^* \tilde{\phi}, \iota_{\nu} \omega \rangle}. \tag{4.25}
\]
This is true due to the min-max principle for \( p_2 \) (Equation (4.8)), together with the fact that \( \int_{\Sigma} i^* \phi, w_\nu d\phi = \int_N (|\nabla \phi|^2 + \phi \Delta \phi) = \int_N |\nabla \phi|^2 \) and \( \int_{\Sigma} \phi = -\frac{1}{\lambda} \int_{\Sigma} \Delta \phi = 0 \).

We now prove (3). If \( s_{n-r} = 0 \), then (4.12) becomes \( \lambda \geq \frac{k}{2} \) which is true in view of (4.16). We can now assume \( s_{n-r} > 0 \). Suppose \( \lambda - \frac{k}{2} \leq s_{n-r}p \), then by (4.9), we have

\[
0 \leq s_{n-r}p - (\lambda - \frac{k}{2}) \leq ((\lambda - \frac{k}{2})^2 - s_{n-r}l \lambda)^{\frac{1}{2}}.
\]

Squaring this inequality gives \( s_{n-r}p^2 + kp \leq (2p - l)\lambda \). From this we see that \( p > \frac{k}{2} \) and (4.12) follows.

For (4), we can put \( l = s_r \) in (4.2) and using (4.22) or (4.25) to obtain

\[
2\lambda - k \geq s_{n-r}p + s_r \frac{\int_{\Sigma} |i^* \omega|^2}{\int_{\Sigma} (l_\nu \omega, \phi)} = s_{n-r}p + s_r \frac{\int_{\Sigma} |i^* \omega|^2}{\int_{\Sigma} |\omega|^2}. \quad (4.26)
\]

As \( \omega \) is co-closed and \( H^r_R(N) \cong H^r_{dR}(N, \mathbb{R}) = 0 \), it is also co-exact. So by (14) Proposition 14, \( \frac{\int_{\Sigma} |i^* \omega|^2}{\int_N |\omega|^2} \geq p_{1,n-r} \), and (4.14) follows. If \( r = 1, k \geq 0 \) and \( s_{n-1} > 0 \), then by (20) (Theorem 2.6.4, Corollary 2.6.2 and Theorem 2.6.1), \( H^1_R(N) = 0 \), thus this later condition can be dropped.

We now prove (5). Suppose the equality sign in any of the inequalities (4.9), (4.10), (4.11), (4.13) and (4.14) holds, then by (4.2), \( \nabla \omega = 0 \). We can then argue as in the proof of Theorem 4.2 that \( k = 0 \).

If any inequality sign of the inequalities (4.9), (4.10), (4.11) or (4.13) becomes an equality sign, then one of the inequalities in (4.9) or (4.10) is an equality. Assume one of these holds. The inequalities (4.22) (or (4.25)) and (4.2) then become equations. So we have the \( r \)-curvatures are constantly equal to \( s_r = l, t_\nu \omega = p \phi \) and the \( (n-r) \)-curvatures are equal to the constant \( s_{n-r} \). In particular, \( S^{n-r} = s_{n-r} \).

We now show that \( \lambda = s_{n-r} s_r \). To do this we make use of the following formulas:

\[
\begin{align*}
\delta i^* \alpha = i^* \overline{\delta \alpha} + t_\nu \nabla_\nu \alpha - S^{r-1} (t_\nu \alpha) + H t_\nu \alpha & \quad \text{for} \ \alpha \in \Omega^r(N), \\
* S^r(\alpha) + S^{n-1-r}(\alpha) = H * \alpha & \quad \text{for} \ \alpha \in \Omega^r(\Sigma), \\
* * \alpha = (-1)^{(n-1-r)} \alpha & \quad \text{for} \ \alpha \in \Omega^r(\Sigma), \\
\overline{\delta} \alpha = - \sum_{j=1}^n t e_j \nabla e_j \alpha & \quad \text{for} \ \alpha \in \Omega^r(N).
\end{align*}
\]

(4.27)

Here \( * : \Omega^r(\Sigma) \rightarrow \Omega^{n-1-r}(\Sigma) \) is the Hodge star operator on \( \Sigma \) and \( \{e_j\}_{j=1}^n \) is a local orthonormal frame on \( N \). The last two formulas are
standard and are included here just for convenience (e.g. [20]). For the first two formulas, see [13] Section 2 and 6. Using (4.27), we compute

$$\delta d^* \phi = \delta d^* \tilde{\phi} = \delta d^* \tilde{\omega}$$

$$= i^* \delta \omega + t_v \nabla \omega - S^{r-1} (t_v \omega) + H t_v \omega$$

$$= i^* \left( \sum_{j=1}^n t_{e_j} \nabla \omega \right) - S^{r-1} (t_v \omega) + \left( (-1)^{n(r-1)} * S^{n-r} (t_v \omega) + S^{r-1} (t_v \omega) \right)$$

$$= (-1)^{n(r-1)} * S^{n-r} (t_v \omega)$$

$$= (-1)^{n(r-1)} s_{n-r} * t_v \omega$$

$$= s_{n-r} t_v \omega.$$  

This implies

$$-\lambda \phi + s_{n-r} p \phi = -(d \delta + d \delta) i^* \phi + s_{n-r} t_v \omega = -\delta d i^* \tilde{\phi} + s_{n-r} t_v \omega = 0.$$  

As $s_{n-r} p = \lambda \pm (\lambda^2 - s_{n-r} s_r) \frac{\lambda}{2}$, we conclude that $-\lambda + \lambda \pm (\lambda^2 - s_{n-r} s_r) \frac{\lambda}{2} = 0$, or

$$\lambda = s_{n-r} s_r = s_{n-r} p > 0.$$  

This shows that $s_r = l > 0$ which contradicts the assumption of (2), thus the inequalities (4.10) and (4.11) must be strict.

We can now proceed in exactly the same way as the proof of Theorem 4.2 to show that $N$ must be a Euclidean ball if $Rc \geq 0$, which we can w.l.o.g. assume to be the standard unit ball $B^n$. But then by [15] Corollary 4,

$$p_{1,r-1}(B^n) = \begin{cases} r & \text{if } r \geq \frac{n}{2} + 1, \\ \frac{n+2}{n} (r - 1) & \text{if } 2 \leq r \leq \frac{n}{2} + 1. \end{cases} \quad (4.28)$$

As $s_m(S^{n-1}) = m$ and by (4.6), we conclude that if $r > 1$, the equality in (4.9) or (4.13) holds if and only if $r \geq \frac{n}{2} + 1$. For $r = 1$, it is well-known that $p_{2,0}(B^n) = 1$, from this we can also conclude that the equality in (4.9) or (4.13) holds if and only if $N$ is a Euclidean ball.

Suppose the equality in (4.14) holds, then by (4.22) or (4.25), $u_v \omega = p \phi$ and by the same reason as above, the $r$-curvatures are constantly equal to $s_r$, the $(n-r)$-curvatures are constantly $s_{n-r}$, and $\lambda = s_{n-r} p$. In particular, $s_r \neq 0$ in view of (4.14), so from (4.26), we have

$$p_{1,n-1-r} = \frac{\int_\Sigma |i^* \omega|^2}{\int_N |\omega|^2} = \frac{\lambda \int_\Sigma |\phi|^2}{\int_\Sigma (t_v \omega, \phi)} = \frac{\lambda}{p}.$$  

In view of (4.14), we deduce that $p = s_r$. We can then proceed as before to conclude that if $Rc \geq 0$, then $(N, g)$ is a Euclidean ball. But then by
(4.6) and (4.28), the equality cannot be attained on a Euclidean ball if \( r > 1 \). If \( r = 1 \), then from (4.28) we see that the equality is attained if and only if \( n \geq 4 \), on a Euclidean ball.

\[ \square \]

**Remark 6.**

1. Escobar ([5] Theorem 8) showed that if \( k \geq 0 \), then \( p_{2,0} > \frac{s_1}{2} \), so (4.13) is well-defined. Also, (4.14) is a generalization [4, Theorem 9] and [14, Theorem 8, Theorem 9].

2. Theorem 4.4 (1) is an extension of [22] Theorem 1.1, in which they provided an upper bound for \( p_2 \) which corresponds to our result when \( k = 0 \) and \( r = 1 \).

3. We suspect that (4.13) holds whenever \( k \geq 0 \) and in this case we have \( 2p > s, r \geq l \), but we are unable to show it for the time being.

In [25], Yang and Yau proved that for a compact Riemann surface \( \Sigma \) of genus \( g \), for any metric on \( \Sigma \), \( \lambda_1(\Sigma) \text{Area}(\Sigma) \leq 8\pi(1 + g) \). Combining this result with Theorem 4.2 and 4.4, we have several corollaries. Let us only state the following:

**Corollary 4.1.** If \( N = S^3 \), then under the assumptions of Theorem 4.2 we have
\[
(2 + s_{n-1}s_1 + \sqrt{(s_{n-1}s_1)^2 + 4s_{n-1}s_1})\text{Area}(\Sigma) < 16\pi(g + 1).
\]

**Example 4.1.** Although the estimate of Theorem 4.2 is not sharp when \( k \neq 0 \), \( r = 1 \), by examining the case where \( \Sigma \) is a geodesic circle of radius \( \rho \) in a hemisphere \( (\lambda_1 = \lambda''_{1,0}(\Sigma) = 1/\sin^2(\rho)) \), it is found that the error is within \( 1/2 \). Indeed, in this case, \( k = 1 \), \( s_1 = s_{n-1} = \cot \rho \), we have
\[
\lambda_1 - \frac{1}{2}(k + s_1^2 + \sqrt{s_1^4 + 2s_1^2k}) = \frac{1}{2}\csc^2 \rho - \frac{1}{2}\sqrt{\csc^4 \rho - 1} \leq \frac{1}{2}.
\]

The error tends to zero as \( \rho \to 0 \).

On the other hand, by [5] Example 5, the first nonzero Steklov eigenvalue of the geodesic ball of radius \( \rho \) in \( S^2 \) is computed to be \( \cot \rho + \tan \frac{\rho}{2} \).

By direct computations, it is found that the error in Theorem 4.4 (2) is
\[
\lambda_1 - \frac{s_{n-1}s_1^2 + kp_2}{2p_2} = \frac{\tan^2(\rho/2)}{2 - \cos \rho} \text{which is (very) slightly better than that of Theorem 4.2.}
\]

The following result is another immediate consequence of Theorem 4.2 which can be regarded as the analogue of Theorem 2 of Hang-Wang [8] (see also [24] Corollary 1).

**Corollary 4.2.** Let \( (N^n, g) \) be a compact orientable Riemannian manifold with boundary \( \Sigma \). Suppose the Ricci curvature of \( N \) is nonnegative, \( s_r(\Sigma)s_{n-r}(\Sigma) \geq r(n-r) = \lambda''_{1,r-1}(S^{n-1}) \geq \lambda''_{1,r-1}(\Sigma) \) for some \( r = 1, \ldots, n-1 \), and \( W^r \) is nonnegative, then \( (N, g) \) is isometric to the unit ball in \( \mathbb{R}^n \).
Theorem 4.2 gives a quick proof of the following result, which is the $K \geq 0$ analogue of Theorem 4.5:

**Corollary 4.3.** Suppose $(N^2, g)$ be a compact surface with (not necessarily connected) boundary $\gamma$ with the Gaussian curvature $K \geq 0$. If the geodesic curvature $k_g$ of $\gamma$ satisfies $k_g \geq l > 0$, then its length $L(\gamma) \leq \frac{2\pi}{l}$. The equality holds if and only if $(N, g)$ is isometric to the Euclidean disk of radius $1/l$.

**Proof.** By Gauss-Bonnet theorem, $2\pi \chi(N) = \int_N K + \int_{\partial N} k_g > 0$, thus $\gamma$ has only one component. By Theorem 4.2, $\lambda_1(\gamma) \geq l^2$. The equality holds if and only if $N$ is a Euclidean disk of radius $1/l$. As $\lambda_1(\gamma) = \left(\frac{2\pi}{L(\gamma)}\right)^2$, the result follows. $\square$

In [3], Choi and Wang proved that if $(N^n, g)$ is a compact orientable manifold whose Ricci curvature is bounded from below by $k > 0$ and $\Sigma$ is an embedded orientable minimal hypersurface in $N$, then $\lambda_1(\Sigma) \geq \frac{k}{2}$. Since their proof are essentially the same as that of Theorem 4.2, their result can be improved slightly to $\lambda_1(\Sigma) > \frac{k}{2}$. This is related to Yau’s conjecture [26]. It is easy to see that the coordinate functions are eigenfunctions of a minimal hypersurface of $S^n$ (whose Ricci curvature is $n-1$) with eigenvalue $n-1$. Yau conjectured that the first eigenvalue is actually $n-1$. Escobar also have a similar conjecture in [4].

In the two-dimensional case, an embedded minimal submanifold is reduced to a simple closed geodesic, the result of Choi-Wang can be improved to $\lambda_1 \geq k$, by a result of Toponogov [21] on the length of a closed geodesic. More generally, we have the following result which is a simple extension of the result in [8], which may have some independent interest:

**Theorem 4.5.** Let $(N^2, g)$ be a closed surface with Gaussian curvature $K \geq 1$. Let $\gamma$ be a simple closed curve in $N$ which separates $N$ into $N_1$, $N_2$. Suppose its geodesic curvature w.r.t. the outward normal of $N_1$ satisfies $k_g \geq l \geq 0$. Then its length $L(\gamma) \leq \frac{2\pi}{\sqrt{1+l^2}}$, which is equivalent to $\lambda_1(\gamma) \geq 1 + l^2$ (as $\lambda_1(\gamma) = \left(\frac{2\pi}{L(\gamma)}\right)^2$), and also $\text{Area}(N_2) \leq \text{Area}(B_r)$, where $B_r$ is the disk of radius $r = \cot^{-1}(-l)$ in the standard sphere $S^2$. If $L(\gamma) = \frac{2\pi}{\sqrt{1+l^2}}$ then $N_1$ is isometric to $B_{\pi-r}$. If, moreover, $\text{Area}(N_2) = \text{Area}(B_r)$, then $(N, g)$ is $S^2$. The condition for the area can be dropped if $l = 0$.

**Proof.** By [8] Theorem 3, we have $L(\gamma) \leq \frac{2\pi}{\sqrt{1+l^2}}$. The equality holds if and only if $(N_1, g)$ is isometric to the disk $B_{r'} \subset S^2$, $r' = \cot^{-1}(l)$.
Therefore if \( L(\gamma) = 2\pi \), \((N_1, g)\) is isometric to the standard hemisphere. But then \( k_g = 0 \), thus we can apply the same argument to \( N_2 \) to deduce that \((N, g)\) is \( S^2 \). In general, if \( L(\gamma) = \frac{2\pi}{\sqrt{1+l_2^2}} \), then by Gauss-Bonnet theorem, as \( N \) is a topological sphere,

\[
\text{Area}(B_r) + \text{Area}(B_{r'}) = 4\pi = \int_{N_2} K + \int_{N_1} K = \int_{N_2} K + \text{Area}(B_{r'}) \geq \text{Area}(N_2) + \text{Area}(B_{r'}). 
\]

So if \( \text{Area}(N_2) = \text{Area}(B_r) \), then \( K = 1 \) on \( N \) and so \((N, g)\) is \( S^2 \). □

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