The Gromov–Hausdorff Distance between Vertex Sets of Regular Polygons Inscribed in a Single Circle

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Abstract—We calculate the Gromov–Hausdorff distance between vertex sets of regular polygons endowed with the round metric. We give a full answer for the case of $n$- and $m$-gons with $m$ divisible by $n$. We also calculate all distances to 2-gons and 3-gons.

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In this work we study the space of all metric compacts considered up to isometry that is endowed with the Gromov–Hausdorff metric. Note that the accurate values of the Gromov–Hausdorff distance between the specific metric spaces are known just for a small number of cases. For instance, in work [1] Grigor’ev et al. computed the Gromov–Hausdorff distance for 1-spaces, the metric spaces with one nonzero distance. In work [2], Tuzhilin computed the Gromov–Hausdorff distance between a segment and a circle, and in work [3] Lim et al. computed the Gromov–Hausdorff distance between the spheres of different dimensions, between the vertices of regular polygons and a circle, and between different regular polygons inscribed in a single circle. Lim et al. [3] posed the problem of computing the last distance and provided an example of solution for an $m$-gon and $(m + 1)$-gon. We make some further advances in solving this problem, namely, completely study the case of $m$- and $n$-gons under the condition that $m$ is divisible by $n$ and compute all the distances to 2-gons and 3-gons.

1. PRELIMINARY INFORMATION

Let $X$ be an arbitrary metric space. The distance between points $x, y \in X$ is denoted by $d(x, y)$ or $|xy|$. For nonempty $A, B \subset X$ we define

$$d_H(A, B) = \max \{ \sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b) \}.$$ 

Definition. The value $d_H(A, B)$ is called the Hausdorff distance between $A$ and $B$. Suppose that $X$ and $Y$ are metric spaces. The triad $(X', Y', Z')$ consisting of the metric space $Z'$ and two its subsets $X'$ and $Y'$ isometric to, respectively, $X$ and $Y$, is called the realization of the pair $(X, Y)$. We put $d_{GH}(X, Y) = \inf \{ r : \exists(X', Y', Z'), d_H(X', Y') \leq r \}.$

Definition. The value $d_{GH}(X, Y)$ is called the Gromov–Hausdorff distance between the metric spaces $X$ and $Y$.

Definition. For the sets $X, Y$ the correspondence between $X$ and $Y$ denotes $R \subset X \times Y$ such that for any $x \in X$ there exists a $y \in Y$ for which $(x, y) \in R$ and, conversely, for any $y \in Y$ there exists an $x \in X$ for which $(x, y) \in R$. If $X, Y$ are metric spaces, then we define the distortion of the correspondence $R$ as follows: $\text{dis}R = \sup \{ ||x_1x_2| - |y_1y_2|| : (x_1, y_1), (x_2, y_2) \in R \}$. We denote the set of all correspondences between $X$ and $Y$ by $\mathcal{R}(X, Y)$.

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**Theorem 1** [4]. For arbitrary metric spaces \( X \) and \( Y \) it is true that
\[
d_{GH}(X, Y) = \frac{1}{2} \inf \{ \text{dis}R : R \in \mathcal{R}(X, Y) \}.
\]

Lim et al. [3] considered the following metric transform. For a metric space \((X, d_X)\), let us consider a pseudometric space \((X, u_X)\), in which the mapping \(u_X : X \times X \to \mathbb{R}_+\) is defined as follows:
\[
u_X(x, y) = \inf\{ \max_{0 \leq i \leq n-1} d_X(x_i, x_{i+1}) : x_0 = x, \ldots, x_n = y \}.
\]
We define the metric space \(U(X)\) as the factor space \((X, u_X)\) by the following equivalence relation:
\[
x \sim y \iff u_X(x, y) = 0.
\]

**Theorem 2** [3]. For any metric spaces \( X \) and \( Y \) the inequality holds:
\[
d_{GH}(X, Y) \geq d_{GH}(U(X), U(Y)).
\]

**Definition.** A metric space \( X \) is referred to as the simplex if all its nonzero distances are the same. A simplex of cardinality \( m \) in which all nonzero distances are \( \lambda > 0 \) is denoted by \( \lambda \Delta_m \).

Suppose that \( X \) is an arbitrary metric space and \( m \) is a cardinal number not exceeding the cardinality of \( X \). By \( D_m(X) \) we denote the family of all possible divisions of the space \( X \) into \( m \) nonempty subsets. Suppose that \( A, B \subset X \) are arbitrary nonempty subsets of \( X \), then we assign
\[
|AB| = \inf \{ |ab| : a \in A, b \in B \}.
\]

For the partitioning \( D = \{X_i\}_{i \in I} \in D_m(X) \) we define the following values:
\[
diam D = \sup_{i \in I} \text{diam} X_i, \quad \alpha(D) = \inf \{|X_iX_j| : i \neq j \}.
\]

**Theorem 3** [1]. Let \( X \) be an arbitrary bounded metric space and \( m = \# \Delta \leq \# X \). Then
\[
2d_{GH}(\lambda \Delta, X) = \inf_{D \in D_m(X)} \max \{ \text{diam} D, \lambda - \alpha(D), \text{diam} X - \lambda \}.
\]

## 2. MAIN RESULTS

For any natural \( n \geq 2 \) by \( P_n \) we denote the vertex set of a regular \( n \)-gon inscribed in the unit circle \( S^1 \) endowed by the intrinsic metric. Note that \( P_2 \) is a pair of diametrically opposite points. We endow the sets \( P_n \) by the metric induced by the circle. For \( m, n \geq 2 \) we put \( p_{m,n} = d_{GH}(P_m, P_n) \).

**Proposition 1** [3]. For any \( m \geq 2 \) it is true that
\[
d_{GH}(S^1, P_m) = \frac{\pi}{m}, \quad d_{GH}(P_m, P_{m+1}) = \frac{\pi}{m+1}.
\]

Let us proceed to the main results of this work.

**Lemma 1.** Let \( n \geq 2, p \in \mathbb{N} \). Then for any \( i, j = 1, 2, \ldots, n \) and \( k, l = 0, 1, \ldots, p - 1 \) it is true that
\[
\left| \min (|i - j|, n - |i - j|) - \min \left( \left| i - j + \frac{k - l}{p} \right|, n - \left| i - j + \frac{k - l}{p} \right| \right) \right| \leq \frac{|k - l|}{p}.
\]

**Proof.** We assign \( S = \min (|i - j|, n - |i - j|) - \min \left( \left| i - j + \frac{k - l}{p} \right|, n - \left| i - j + \frac{k - l}{p} \right| \right) \).

Note that, because \( \frac{|k - l|}{p} < 1 \),
\[
\left| i - j + \frac{k - l}{p} \right| = \begin{cases} i - j + \frac{k - l}{p}, & \text{if } i - j > 0; \\ \frac{|k - l|}{p}, & \text{if } i - j = 0; \\ j - i + \frac{l - k}{p}, & \text{if } i - j < 0. \end{cases}
\]
If \( i - j = 0 \), then
\[
S = \left| \min (0, n) - \min \left( \frac{|k - l|}{p}, n - \frac{|k - l|}{p} \right) \right| = \frac{|k - l|}{p}.
\]
Without loss in generality, we will think that \( i - j > 0 \). Consider several cases separately.

1. Assume that \( i - j < \frac{n}{2} \). If \( i - j + \frac{k - l}{p} > \frac{n}{2} \), then \( i - j = \frac{n - 1}{2} \) and \( \frac{k - l}{p} > \frac{1}{2} \). Hence,
\[
S = \left| \frac{n - 1}{2} - \left( n - \left( \frac{n - 1}{2} + \frac{k - l}{p} \right) \right) \right| = \frac{|k - l|}{p} - 1 < \frac{|k - l|}{p},
\]
where the last inequality holds due to \( \frac{k - l}{p} > \frac{1}{2} \). If \( i - j + \frac{k - l}{p} \leq \frac{n}{2} \), then
\[
S = \left| i - j - \left( i - j + \frac{k - l}{p} \right) \right| = \frac{|k - l|}{p}.
\]

2. Assume that \( i - j = \frac{n}{2} \). Then
\[
S = \left| \frac{n}{2} - \min \left( \frac{n}{2} + \frac{k - l}{p}, \frac{n}{2} + \frac{l - k}{p} \right) \right| = \frac{|k - l|}{p}.
\]

3. Assume that \( i - j > \frac{n}{2} \). If \( i - j + \frac{k - l}{p} < \frac{n}{2} \), then \( i - j = \frac{n + 1}{2} \) and \( \frac{l - k}{p} > \frac{1}{2} \). Hence,
\[
S = \left| \frac{n - 1}{2} - \left( \frac{n + 1}{2} + \frac{k - l}{p} \right) \right| = \frac{|l - k|}{p} - 1 < \frac{|k - l|}{p},
\]
where the last inequality holds due to \( \frac{l - k}{p} > \frac{1}{2} \). If \( i - j + \frac{k - l}{p} \geq \frac{n}{2} \), then
\[
S = \left| n - i + j - \left( n - i + j - \frac{k - l}{p} \right) \right| = \frac{|k - l|}{p}.
\]

The lemma is proved.

**Theorem 4.** Let \( 2 \leq n \leq m \) and let \( m \) be divisible by \( n \), then
\[
p_{n,m} = \frac{\pi}{n} - \frac{\pi}{m}.
\]

**Proof.** Suppose that \( u_1, \ldots, u_n \) are vertices of \( P_n \) and \( v_1, \ldots, v_m \) are vertices of \( P_m \). We prove that \( p_{n,m} \geq \frac{\pi}{n} - \frac{\pi}{m} \). By Theorem 2
\[
p_{n,m} \geq d_{GH} (\textbf{U}(P_m), \textbf{U}(P_n)).
\]
Note that \( \textbf{U}(P_m), \textbf{U}(P_n) \) are simplices of cardinality \( m \) and \( n \) with pairwise nonzero distances \( \frac{2\pi}{m} \) and \( \frac{2\pi}{n} \), respectively. Then, by Theorem 3
\[
p_{n,m} \geq d_{GH} (\textbf{U}(P_m), \textbf{U}(P_n)) = \frac{1}{2} \max \left\{ \frac{2\pi}{m} - \frac{2\pi}{n}, -\frac{2\pi}{m} \right\} \geq \frac{\pi}{n} - \frac{\pi}{m}.
\]

Now, let us prove the upper bound. Suppose that \( m = pn \), where \( p \in \mathbb{N} \). Then \( \frac{\pi}{n} - \frac{\pi}{m} = \frac{(p - 1)\pi}{pn} \). We construct a correspondence \( R \in \mathcal{R}(P_n, P_m) \) such that \( \text{dis} R \leq \frac{2(p - 1)\pi}{pn} \):
\[
R = \bigcup_{i=1}^{n} \{(u_i, v_{pi-k}) : k = 0, 1, \ldots, p - 1\}.
\]
Then, by Lemma 1 for any \(i, j = 1, 2, \ldots, n\) and \(k, l = 0, 1, \ldots, p - 1\) it is true that

\[
|d(u_i, u_j) - d(v_{pi-l}, v_{pj-k})| = \left| \frac{2\pi}{n} \min (|i - j|, n - |i - j|) - \frac{2\pi}{pm} \min (|pi - pj + k - l|, pm - |pi - pj + k - l|) \right|
\]

\[
\leq \frac{2\pi|k - l|}{pm} \leq \frac{2(p - 1)\pi}{pm}.
\]

Thus, \(\text{dis}R \leq \frac{2(p - 1)\pi}{pm}\), and this completes the proof.

**Theorem 5.** Let \(m \geq 2\), then

\[
p_{2,m} = \begin{cases} 
\frac{\pi}{2} - \frac{\pi}{2m}, & \text{if } \ m \ \text{is odd;} \\
\frac{\pi}{2} - \frac{\pi}{m}, & \text{if } \ m \ \text{is even.}
\end{cases}
\]

**Proof.** The case of even \(m\) directly follows from Theorem 4. Now, suppose that \(m\) is an odd number. Suppose that \(u_1\) and \(u_2\) are vertices of \(P_2\) and \(v_1, \ldots, v_m\) are vertices of \(P_m\). Note that \(P_2\) is the simplex of cardinality 2 with a side of length \(\pi\), that is, \(P_2 = \pi \Delta_2\). Then, by Theorem 3

\[
d_{GH}(P_2, P_m) = \frac{1}{2} \inf_{D \in \mathcal{D}_2} \max \{ \text{diam}D, \pi - \alpha(D), \text{diam}P_m - \pi \},
\]

where \(\mathcal{D}_2\) is the set of divisions of \(P_m\) into two nonempty subsets. In our case \(\alpha(D) = \frac{2\pi}{m}\) and \(\text{diam}P_m = \pi - \frac{\pi}{2m}\). Therefore,

\[
d_{GH}(P_2, P_m) = \frac{1}{2} \inf_{D \in \mathcal{D}_2} \max \{ \text{diam}D, \pi - \frac{\pi}{2m} \}.
\]

Let us show that for any \(D \in \mathcal{D}_2\) the following inequality holds:

\[
\text{diam}D \geq \pi - \frac{\pi}{2m}.
\]

Assume that for the partitioning \(D = \{X_1, X_2\}\) the converse is true, that is, \(d = \text{diam}D < \pi - \frac{\pi}{2m}\). Without loss in generality, we can think that \(\text{diam}X_1 = d\). Then, there exist vertices \(v_i, v_j \in P_m\), such that \(v_i, v_j \in X_1\) and \(d(v_i, v_j) = d\). Consider the vertices \(v_k, v_l \in P_m\), neighboring to \(v_i\) and \(v_j\), respectively, and not lying inside the smaller circular arc \(S^1\) connecting \(v_i\) to \(v_j\). The vertex \(v_k\) cannot belong to the partitioning set \(X_1\), because, otherwise,

\[
d = \text{diam}X_1 \geq d(v_k, v_j) = d(v_i, v_j) + \frac{2\pi}{m} = d + \frac{2\pi}{m} > d.
\]

Similarly, \(v_l \in X_2\). Then

\[
d = \text{diam}D \geq \text{diam}X_2 \geq d(v_k, v_l) \geq d(v_i, v_j) + \frac{2\pi}{m} = d + \frac{2\pi}{m} > d.
\]

Thus, the theorem is proved.

**Theorem 6.** Let \(m \geq 3\) and let \(r\) be a remainder in division of \(m\) by 3, then

\[
p_{3,m} = \begin{cases} 
\frac{\pi}{3} - \frac{\pi}{m}, & \text{if } \ r = 0; \\
\frac{\pi}{3} - \frac{r\pi}{3m}, & \text{if } \ r \neq 0.
\end{cases}
\]
Proof. The case $r = 0$ immediately follows from Theorem 4. Now, let $r > 0$. Note that $P_3$ is a simplex of cardinality 3 with a side of length $\frac{2\pi}{3}$, that is, $P_3 = \frac{2\pi}{3} \Delta_3$. Then, by Theorem 3
\[ d_{GH}(P_3, P_m) = \frac{1}{2} \inf_{D \in \mathcal{D}_3} \max \{ \text{diam} D, \frac{2\pi}{3} - \alpha(D), \text{diam} P_m - \frac{2\pi}{3} \}, \]
where $\mathcal{D}_3$ is the set of divisions of $P_m$ into three nonempty subsets. In our case $\alpha(D) = \frac{2\pi}{m}$ and $\text{diam} P_m \leq \pi$. From Proposition 1 it follows that $p_{3,4} = \frac{\pi}{4}$. Now, we assume that $m \geq 5$. Hence,
\[ d_{GH}(P_3, P_m) = \frac{1}{2} \inf_{D \in \mathcal{D}_3} \max \{ \text{diam} D, \frac{2\pi}{3} - \frac{2\pi}{m} \}. \]

We assign $q = \left[ \frac{m}{3} \right]$. Suppose that $r = 1$. Consider the following partitioning $D = \{ X_1, X_2, X_3 \}$:
\[ X_1 = \{ v_1, v_2, \ldots, v_q \}, \quad X_2 = \{ v_{q+1}, v_{q+2}, \ldots, v_{2q} \}, \quad X_3 = \{ v_{2q+1}, v_{2q+2}, \ldots, v_m \}. \]
Then $\text{diam} D = \left( \frac{m-1}{3} \right) \frac{2\pi}{m} = \frac{2\pi}{3} - \frac{2\pi}{3m}$. We show that for any $D \in \mathcal{D}_3$ the inequality holds:
\[ \text{diam} D \geq \frac{2\pi}{3} - \frac{2\pi}{3m}. \]
Assume that for partitioning $D = \{ X_1, X_2, X_3 \}$ the converse is true, that is, $d = \text{diam} D < \frac{2\pi}{3} - \frac{2\pi}{3m}$.
Without loss in generality, we assume that the set $X_1$ contains more than one point. Then there exist vertices $v_i, v_j \in P_m$ such that $v_i, v_j \in X_1$ and $d(v_i, v_j) = \text{diam} X_1$. We show that any vertex $v_k \in X_1$ must lie inside the smaller circular arc $S^1$ connecting $v_i, v_j$. Assume that $v_k \in X_1$ lies outside this arc. Then the circle is divided into three arcs: $v_iv_j, v_jv_k, v_kv_i$. The length of each of them must be no larger than $\text{diam} X_1 \leq d$. Then
\[ 2\pi = d(v_i, v_j) + d(v_j, v_k) + d(v_k, v_i) \leq 3d < 2\pi - \frac{2\pi}{m} < 2\pi. \]
Thus, each of the sets $X_1, X_2$, and $X_3$ is a set of consecutive vertices $P_m$ of diameter not large than $\frac{2\pi}{3} - \frac{2\pi}{3m}$. Then
\[ 2\pi - \frac{6\pi}{m} = \text{diam} X_1 + \text{diam} X_2 + \text{diam} X_3 \leq 2\pi - \frac{6\pi}{m} - \frac{2\pi}{m} < 2\pi - \frac{6\pi}{m}. \]
This means that $d_{GH}(P_3, P_m) = \frac{\pi}{3} - \frac{\pi}{3m}$. Suppose that $r = 2$. Consider the following partitioning $D = \{ X_1, X_2, X_3 \}$:
\[ X_1 = \{ v_1, v_2, \ldots, v_q \}, \quad X_2 = \{ v_{q+1}, v_{q+2}, \ldots, v_{2q} \}, \quad X_3 = \{ v_{2q+1}, v_{2q+2}, \ldots, v_m \}. \]
Then
\[ \text{diam} D = \left( \frac{m-2}{3} \right) \frac{2\pi}{m} = \frac{2\pi}{3} - \frac{4\pi}{3m}. \]
Repeating the arguments for the case $r = 1$, we obtain the fact that for any $D \in \mathcal{D}_3$ the inequality holds:
\[ \text{diam} D \geq \frac{2\pi}{3} - \frac{4\pi}{3m}. \]
Hence, $d_{GH}(P_3, P_m) = \frac{\pi}{3} - \frac{2\pi}{3m}$. The theorem is proved.

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CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

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