Mass and charge fluctuations and black hole entropy

Ashok Chatterjee* and Parthasarathi Majumdar†
Theory Group, Saha Institute of Nuclear Physics, Kolkata 700 064, India.

The effects of thermal fluctuations of the mass (horizon area) and electric charge, on the entropy of non-rotating charged macroscopic black holes, are analyzed using a grand canonical ensemble. Restricting to Gaussian fluctuations around equilibrium, and assuming a power law type of relation between the black hole mass, charge and horizon area, characterized by two real positive indices, the grand canonical entropy is shown to acquire a logarithmic correction with a positive coefficient proportional to the sum of the indices. However, the root mean squared fluctuations of mass and charge relative to the mean values of these quantities turn out to be independent of the details of the assumed mass-area relation. We also comment on possible cancellation between log (area) corrections arising due to fixed area quantum spacetime fluctuations and that due to thermal fluctuations of the area and other quantities.

I. INTRODUCTION

The notion of entropy of a spacetime such as a black hole presumes the existence of microstates whose degeneracy $g$ is measured by that entropy. It is conceptually difficult to think of microstates of a classical stationary spacetime. Yet even vacuum solutions of Einstein’s equation like the Schwarzschild and the Kerr black holes are supposed to possess an entropy equal to a quarter of their horizon area (in Planckian units) [1], [2]. The only way in which microstates might arise in these cases is within a quantum mechanical description of such spacetimes themselves (and not merely of the matter fields propagating in them) which allow for dynamical fluctuations. In other words, the entropy of black holes must be taken to originate from quantum degeneracy of spacetime geometry, and consequently, can only be properly analyzed within a quantum theory of gravitation [1].

Of all attempts to construct a ‘quantum general relativity’, one that perhaps comes closest to dealing directly with fluctuations of spacetime geometry (within a canonical framework) is Loop Quantum Gravity (LQG) [3]. Yet even in this approach, the only situation that can be reliably analyzed is one in which the black hole is isolated, i.e., has a horizon with a fixed (macroscopic) area $A \gg 1$ (in units of Planck area). From a statistical thermodynamics standpoint, this corresponds to employing a microcanonical ensemble to compute the entropy [4]. Microstates leading to the microcanonical entropy $\hat{S}_{MC} \equiv \log g$ are identified with boundary states of a three dimensional Chern-Simons theory ‘living’ on the horizon [4]. The local Lorentz invariance is gauge-fixed to a local SU(2) on an appropriate spatial slice. Thus one may take this SU(2) to be the gauge group of the Chern Simons theory. However, boundary conditions associated with the ‘isolatedness’ of the horizon have been argued [4] to further gauge-fix the SU(2) to a residual U(1). In either case, the degeneracy of the boundary states of the Chern Simons theory yields (for macroscopically large horizon areas) a term logarithmic in the area together with an infinite series in inverse powers of the area [5], [6], over and above the Bekenstein-Hawking entropy $S_{BH} = A/4$ (in Planckian units),

$$\hat{S}_{MC} = S_{BH} - \frac{3}{2} \log S_{BH} + \text{const.} + O(S_{BH}^{-1}) \quad \text{for SU(2)}$$

$$= S_{BH} - \frac{1}{2} \log S_{BH} + \text{const.} + O(S_{BH}^{-1}) \quad \text{for U(1)}$$

(1)

The negative sign of the dominant logarithmic correction in both cases is due to the fact that invariance of the horizon states under the gauge group effectively reduces the degeneracy as compared to the leading order term (area law) where the invariance is not taken into account. Notice that each term of the infinite series in eq. (1) is finite and unambiguously calculable.

The problem with an isolated black hole is that it neither radiates nor absorbs anything. This is unsatisfactory from a physical standpoint since classically a black hole absorbs everything, and with quantum matter around, it radiates all kinds of quanta in a Planckian distribution [2] at the Hawking temperature given by the surface gravity on the horizon. Thus, the appropriate equilibrium ensemble to probe black hole thermodynamics ought to be the canonical

---

*email: ashok@theory.saha.ernet.in
†email: partha@theory.saha.ernet.in
ill-founded assumptions often lead to claims regarding corrections to the area law \cite{12} which we find spurious.

The situation is more favourable for asymptotically anti-de Sitter black holes which, for a certain range of parameters, can indeed be in stable thermal equilibrium \cite{7}. For such spacetimes the area (energy) fluctuations within a canonical ensemble lead, within a saddle-point approximation, to an additional correction term in the canonical entropy, over and above the infinite series of corrections to the area law found for the microcanonical entropy \cite{8},

\[ \delta_{th} S_C = \frac{1}{2} \log S_{BH} - \frac{1}{2} \log(n - 1) + \text{const}, \tag{2} \]

where \( n \) is the index appearing in the assumed power law relation between area of the horizon and energy. This additional correction has a positive sign, as expected from the fact that thermal fluctuations tend to disorder the system and therefore always increase the canonical entropy. The area-dependent part is universal in that it is independent of the index characterizing the assumed mass-area power-law relation. We may remark that eq. (2) is valid in the limit of large horizon areas and for fixed (i.e. thermally non-fluctuating) charge or angular momentum of the black hole. For asymptotically flat black holes, \( n = 1/2 \), implying that the entropy as well as the free energy acquires an imaginary part. This indicates a breakdown of the saddle-point approximation used to compute the canonical entropy from the canonical partition function. It is not unlikely that this breakdown is indicative of the thermal instability ensuing in the canonical ensemble for such spacetimes, as mentioned earlier. On the other hand, for all anti-de Sitter black holes, \( n = 3/2 \), and therefore the calculation above is reliable.

One immediate fallout of this additional correction is that it begins to compete with the microcanonical corrections to the area law. The net canonical entropy now takes the form

\[ S_C = \tilde{S}_{MC} + \delta_{th} S_C \]
\[ = S_{BH} - \log S_{BH} - \frac{1}{2} \log(n - 1) + \text{const.} + \ldots \ldots \tag{3} \]

for the \( SU(2) \) case, while for the \( U(1) \) case, the two leading logarithmic corrections to the area law simply cancel each other out \cite{9}. At this point the cancellation is not much more than a point of curiosity, since there is still an infinite series of other (power-law) corrections hidden in the \ldots in eq. (3). However, if such cancellations are proven to work for all orders in powers of inverse area, one may ponder about symmetries, non-renormalization theorems etc.

Recall that so far the discussion of the canonical entropy has assumed fixed charges/angular momenta of the black holes. In this paper we consider the effect of including thermal fluctuations of the electric charge for non-rotating spherically symmetric black hole spacetimes. In other words, we consider the generalization of the previous canonical ensemble formulation to a grand canonical ensemble framework where the electrostatic potential of the black hole plays the role of a chemical potential. Thus, the heat bath which the black hole interacts with is now electrically charged. The use of a saddle-point approximation in this case involves important technical changes which are incorporated in our treatment. The object here is to determine whether the universality found in the canonical treatment survives the generalization. Once again, we make no more assumption than is absolutely essential. In particular, we adhere to the area spectrum strictly as derived in the LQG formulation \cite{3}, specialized to the case of large macroscopic areas where the spectrum becomes equally spaced.\footnote{Only in one other approach, involving a proposal for coherent states \cite{10} within LQG, is there a proper \textit{derivation} of an equally-spaced area spectrum \cite{11} which does not involve ad hoc assumptions regarding the black hole mass spectrum. Such ill-founded assumptions often lead to claims regarding corrections to the area law \cite{12} which we find spurious.}

In this context, we may note that two recent papers \cite{13}, \cite{14} have explored the effect of thermal energy and charge fluctuations for a grand canonical ensemble of AdS Reissner-Nordstrom black holes. While our result for the leading thermal fluctuation correction to the grand canonical entropy for this particular black hole matches with the results quoted in \cite{13}, the implied assumption in those papers regarding the independence of energy and charge fluctuations, as also about their relative size, are not tenable even for large black holes, as we shall show in what follows. Consequently, the conjectured complete cancellation (including angular momentum fluctuations) between the microcanonical and thermal logarithmic corrections to the area law, as proposed in \cite{14}, appears to be without basis.

The paper is organized as follows: in section 2 we present a grand canonical ensemble formulation of small fluctuations for general equilibrium statistical mechanical systems. This is then used in section 3 to compute the leading
logarithmic corrections to the grand canonical entropy, via a Poisson resummation procedure followed by a saddle point approximation to perform the integrals in the grand canonical partition function. In section 4. we study the relative sizes of fluctuations in more detail to show that they indeed ensure that the approximation of small Gaussian fluctuations is valid for large area black holes, when the charge is also allowed to grow with area in a certain way. However that is shown not to be true for black holes of vanishingly small charge considered in [13]. We discuss our results and conclude in section 5.

II. MASS AND CHARGE FLUCTUATIONS IN THE GRAND CANONICAL ENSEMBLE

For a thermodynamic system with Hamiltonian \( \hat{H} \) and charge operator \( \hat{Q} \) in equilibrium with a heat and charge reservoir at temperature \( T \) and electrostatic potential \( \Phi \), the grand partition function is

\[
Z_G = \text{Tr} \exp \left\{-\beta (\hat{H} - \Phi \hat{Q})\right\} \tag{4}
\]

When the spectra of \( \hat{H} \) and \( \hat{Q} \) are both continuous with eigenvalues \( E \) and \( Q \) respectively, the partition sum in eq. (4) can be conveniently rewritten as an integral

\[
Z_G = \int dE \ dQ \ \rho(E,Q) \ \exp \left\{-\beta (E - \Phi Q)\right\}, \tag{5}
\]

where the density of states \( \rho(E,Q) \) may be taken to be a smooth function and \( \beta \equiv T^{-1} \).

Defining the microcanonical entropy \( S_{MC}(E,Q) \equiv \log \rho(E,Q) \), we get

\[
Z_G = \int dE \ dQ \ \exp \left\{S_{MC} - \beta (E - \Phi Q)\right\}, \tag{6}
\]

Assuming that the integral is dominated by the saddle point of the exponent at \( E = M, \ Q = Q_0 \), we expand the exponential in the integrand up to terms quadratic in the fluctuations \( \delta E \equiv E - M, \ \delta Q \equiv Q - Q_0 \)

\[
S_{MC}(E,Q) - \beta(E - \Phi Q) = S_{MC}(M,Q_0) - \beta(M - \Phi Q_0)
+ \frac{1}{2} \left[ (S_{MC,EE})_{M,Q_0} \delta E^2 + (S_{MC,QQ})_{M,Q_0} \delta Q^2 + 2 (S_{MC,EQ})_{M,Q_0} \delta E \delta Q \right], \tag{7}
\]

where, \( S_{MC,EE} = \partial^2 S_{MC}/\partial E^2, \) etc. In writing eq. (7) above we have made use of the saddle point conditions

\[
(S_{MC,E})_{M,Q_0} = \beta = T^{-1} \tag{8}
\]

\[
(S_{MC,Q})_{M,Q_0} = -\beta \ \Phi = -\Phi/T \tag{8}
\]

which are the usual relations for the temperature and potential in the microcanonical ensemble.

The presence of cross terms involving \( \delta E \) and \( \delta Q \) in the exponent in eq.(6) (using (7)) implies that the energy and charge fluctuations cannot, in general, be treated independently. Evaluating the Gaussian integrals over the fluctuations yields,

\[
Z_G = \frac{2\pi}{\sqrt{\det \Omega}} \ \exp \left\{S_{MC}(M,Q_0) - \beta M + \beta \Phi Q_0\right\}, \tag{9}
\]

where, the Hessian matrix

\[
\Omega = \begin{pmatrix}
S_{MC,EE} & S_{MC,EQ} \\
S_{MC,EQ} & S_{MC,QQ}
\end{pmatrix}_{M,Q_0} \tag{10}
\]

is assumed to be negative definite to ensure stability under small fluctuations. The necessary and sufficient conditions for this are

\[\text{Observe that} \ S_{MC} \text{ defined as above is precisely what we called } \tilde{S}_{MC} \text{ in [8] where we defined } S_{MC} \equiv \log g(E) \text{ with } g(E) \text{ being the degeneracy of states with energy } E. \text{ This distinction between the two definitions of microcanonical entropy is relevant for logarithmic corrections to the area law [8].}\]
\[ T \Omega = S_{MC,EE}|_{M,Q_0} + S_{MC,QQ}|_{M,Q_0} < 0 \]
\[ \det \Omega = \left[ S_{MC,EE} S_{MC,QQ} - S_{MC,EE}^2 \right]|_{M,Q_0} > 0 \]  
(11)

which necessarily imply
\[ S_{MC,EE}|_{M,Q_0} < 0, \text{ and } S_{MC,QQ}|_{M,Q_0} < 0. \]  
(12)

Note that while these conditions together imply the first of the necessary and sufficient conditions (11) for stability of \( \Omega \), they are not sufficient to guarantee the second one.

Using the microcanonical relations for temperature and potential, we may express the necessary conditions for stability in terms of the heat capacity \( C_Q \equiv \left( \frac{\partial E}{\partial T} \right)_Q \) and the capacitance \( C \equiv \left( \frac{\partial Q}{\partial \Phi} \right)_E \) in the following way
\[ C_Q > 0 \text{ and } C \Phi < T \left( \frac{\partial Q}{\partial T} \right)_E. \]  
(13)

The more stringent necessary and sufficient conditions can also be similarly expressed in terms of \( C_Q \) and \( C \).

The grand partition function, evaluated in the saddle point approximation, can now be substituted in the standard thermodynamic relation in the presence of a chemical (electrostatic) potential
\[ S_G = \beta M - \beta Q \Phi + \log Z_G, \]  
(14)

so as to yield the grand canonical entropy
\[ S_G = S_{MC} - \frac{1}{2} \log \det \Omega + \text{const.} \]  
(15)

It is pertinent at this point to probe the validity of the saddle point approximation because it is used throughout this paper. One way to do this is to use this approximation to compute the mean-squared fluctuations of the energy and charge,
\[ \Delta E^2 \equiv \langle \delta E^2 \rangle = Z^{-1}_G \int dE \, dQ \, (E - M)^2 e^{S_{MC} - \beta E + \beta \Phi Q} \]
\[ \Delta Q^2 \equiv \langle \delta Q^2 \rangle = Z^{-1}_G \int dE \, dQ \, (Q - Q_0)^2 e^{S_{MC} - \beta E + \beta \Phi Q} \]  
(16)

and to ensure that these have certain desirable properties, e.g., that they are both positive and relatively small. Computing appropriate Gaussian integrals, it is easy to show that
\[ \Delta E^2 = - (\Omega^{-1})_{11} = - \frac{S_{MC,QQ}(M,Q_0)}{\det \Omega} \]
\[ \Delta Q^2 = - (\Omega^{-1})_{22} = - \frac{S_{MC,EE}(M,Q_0)}{\det \Omega}. \]
\[ \langle \delta E \, \delta Q \rangle = - (\Omega^{-1})_{12} = \frac{S_{MC,EE}(M,Q_0)}{\det \Omega}. \]  
(17)

Using the inequalities (11), (12) both mean-squared fluctuations are clearly seen to be positive, as they ought to be. The cross-correlation term, which is generally nonzero, explicitly demonstrates why it is erroneous to treat these fluctuations as independent, as already remarked.

### III. Fluctuation Contribution to the Entropy of Large Black Holes

The canonical black hole entropy has been argued [9] to be determined by the ADM Hamiltonian which describes quantum fluctuations on the horizon treated as an inner boundary of spacetime, essentially because the bulk Hamiltonian obeys the quantum Hamiltonian constraint. Unfortunately, the spectrum of this boundary Hamiltonian is yet
to be determined in LQG. Thus, as in [8], we assume that energy spectrum is a function of the discrete area spectrum (well-known in LQG [3]) and a discrete charge spectrum. The charge spectrum is of course equally spaced in general; for large macroscopic black holes the area spectrum is equally spaced as well.

In a basis in which both the area and charge operators are simultaneously diagonal, the grand canonical partition function can be expressed as

$$Z_G = \sum_{m,n} g(m,n) \exp -\beta [E(A_m,Q_n) - \Phi Q_n] ,$$  \hspace{1cm} (18)

where, \(g(m,n)\) is the degeneracy corresponding to the area eigenvalue \(A_m\) and charge eigenvalue \(Q_n\). Using a generalization of the Poisson resummation formula

$$\sum_{m,n} f(m,n) = \sum_k \int dx \ dy \ \exp \{-i(kx + ly)\} f(x,y)$$  \hspace{1cm} (19)

and assuming that the partition sum is dominated by the large eigenvalues \(A_m\), \(Q_n\), it can be expressed as a double integral

$$Z_G = \int dx \ dy \ \exp -\beta \{E(A(x),Q(y)) - \Phi Q(y)\} g(A(x),Q(y)) .$$  \hspace{1cm} (20)

Note that the transition from the discrete sum to the integral for \(Z_G\) requires only that the dominant eigenvalues are large compared to the fundamental units of discreteness which for the area is the Planck area and for the charge is the electronic charge. These conditions are of course fulfilled for all astrophysical black holes.

Changing variables in eq. (20) form \(x, y\) to \(E, Q\)

$$Z_G = \int dE \ dQ \ \mathcal{J}(E,Q) \ g(E,Q) \ \exp \{-\beta(E - \Phi Q)\}$$

$$= \int dE \ dQ \ \rho(E,Q) \ \exp \{-\beta(E - \Phi Q)\} ,$$  \hspace{1cm} (21)

where, the Jacobian \(\mathcal{J} = |E_x|^{-1} |Q_y|^{-1}\), and \(\rho = \mathcal{J}(E,Q) \ g(E,Q)\) is the density of states. Employing the saddle point approximation and using eq. (14) one obtains

$$S_G(M,Q_0) = \tilde{S}_{MC}(M,Q_0) - \frac{1}{2} \log \Delta + const.$$  \hspace{1cm} (22)

where, using \(S_{MC} = \log \rho = \tilde{S}_{MC} + \log \mathcal{J}\) we have defined

$$\Delta \equiv \det \Omega \ \mathcal{J}^2 = \det \Omega \ (E_x)^2 \ (Q_y)^2|_{M,Q_0}$$

$$= \det \Omega \ (A_E)^2 \ (A_x)^2 \ (Q_y)^2|_{M,Q_0} ,$$  \hspace{1cm} (23)

and \(\Omega\) is defined in eq. (10).

Since the microcanonical entropy is known to be only a function of the horizon area even for charged non-rotating black holes [4,5], one can express \(\det \Omega\) as

$$\det \Omega = (S_{MC,A})^2 \ [(A_{EE}) (A_{QQ}) - (A_{EQ})^2]_{M,Q_0}$$

$$+ S_{MC,AA} S_{MC,A} [(A_E)^2 A_{QQ} + (A_Q)^2 A_{EE} - 2 A_E A_Q A_{EQ}]_{M,Q_0}$$  \hspace{1cm} (24)

Now, to obtain the leading thermal fluctuation correction to the area law (over and above corrections occurring in \(S_{MC}\)) it is adequate to retain only the area term in \(S_{MC}\) in eq. (24), yielding

$$\det \Omega = \frac{1}{16} \ [A_{EE} A_{QQ} - (A_{EQ})^2]_{M,Q_0} .$$  \hspace{1cm} (25)

With both the area and charge spectrum being equally spaced (i.e., linear in the quantum numbers \(x\) and \(y\)), we therefore obtain the leading thermal fluctuation correction

$$\Delta = \frac{C}{16} \ [A_{EE} A_{QQ} - (A_{EQ})^2] \ (A_E)^2|_{M,Q_0}$$  \hspace{1cm} (26)
where $C$ is a constant related to the area and charge spectra.

To proceed further, it is necessary to specify the functional dependence of the area of the horizon $A$ on the energy $E$ and the charge $Q$. As mentioned earlier, this is not yet known within the LQG framework. We adopt here the alternative of postulating an asymptotic power law relationship between these quantities for large $A$

$$E = a \ A^\alpha + b \ Q^2 \ A^{-\beta}, \quad (27)$$

where, $a$, $b$ are real positive dimensional coefficients dependent on the Newton constant and the cosmological constant, and $\alpha$, $\beta$ are dimensionless real positive exponents. An equivalent way of rewriting eq. (27) is

$$E = a \ A^\alpha \left[ 1 + b' \ Q^2 \ A^{-\beta'} \right], \quad (28)$$

where $b'$ is similar to $b$ above and likewise for $\beta'$. Both forms of power-law relation subsume all four dimensional charged non-rotating black holes. Of these, only those spacetimes which are asymptotically anti-de Sitter, i.e., have constant negative curvature at infinity, can be considered to be in stable thermal equilibrium, so long as their (outer) horizon radius exceeds $(-\Lambda)^{-1/2}$ where $\Lambda$ is the negative cosmological constant.\(^4\)

We may remark that there is no essential loss of generality in choosing the functional relation in eqs (27, 28) to be quadratic in the charge; in fact our analysis goes through for any positive power of $Q$.

Observe that the microcanonical ensemble definition of temperature is $T \equiv (S_{MC,E})^{-1}$. On physical grounds one expects that $T \geq 0$ which implies a limit on the charge $Q^2 \leq (a\alpha/b\beta) \ A^{\alpha+\beta}$ using eq. (27). This inequality is saturated by the extremal charge $Q^2_{ext} = (a\alpha/b\beta) A^{\alpha+\beta}$ for which $T = 0$. We also define, for future use, the ratio $q \equiv Q_0/Q_{ext}$ which must satisfy $-1 \leq q \leq 1$.

The computation of the derivatives in eq. (26) is now straightforward,

$$A_{,E}|_{M,Q_0} = \frac{1}{a} \ W(q, \alpha, \beta) \ A^{1-\alpha}, \quad A_{,EE}|_{M,Q_0} = - \frac{1}{a^2} \ X(q, \alpha, \beta) \ A^{1-2\alpha}$$

$$A_{,QQ}|_{M,Q_0} = - \frac{b}{a} \ Y(q, \alpha, \beta) \ A^{1-\alpha-\beta}$$

$$A_{,EQ}|_{M,Q_0} = \left( \frac{b}{a^3} \right)^{1/2} \ q \ Z(q, \alpha, \beta) \ A^{1-\frac{2\alpha}{2} - \frac{3\beta}{2}}. \quad (29)$$

The dimensionless functions $W$, $X$, $Y$ and $Z$ are explicitly calculable and are regular and non-vanishing for $0 \leq |q| < 1$; however they have poles at $q = \pm 1$ with $W \sim (1 - q^2)^{-1}$ and $X$, $Y$, $Z \sim (1 - q^2)^{-3}$ as $|q| \to 1$.

In terms of these functions, the leading correction is given by

$$\Delta = \frac{C}{16} \ b \ \frac{X \ Y - q^2 \ Z^2}{W^2} \ A^{-(\alpha+\beta)}, \quad (30)$$

so that the grand canonical entropy

$$S_G = \tilde{S}_{MC} + \frac{1}{2} \ (\alpha + \beta) \ \log A + O(A^0) \ \text{for} \ A >> (-\Lambda)^{-1} \ \text{and} \ 0 \leq |q| < 1. \quad (31)$$

Note that the leading logarithmic correction is independent of $q$. However, as we approach extremality $|q| \to 1$, $\Delta$ diverges as $(1 - q^2)^{-4}$, the $O(A^0)$ terms become large and the approximation of large horizon areas becomes invalid. This happens because as we approach extremality, $T \to 0$ and all fluctuations basically freeze out. Furthermore, as we shall argue below, the result also cannot be trusted for $q \to 0$, although for somewhat different reasons.

**IV. RELATIVE FLUCTUATIONS FOR LARGE BLACK HOLES**

Using the area law, we find, for adS black holes with $A >> \ell^2$,\(^4\)

\(^4\)In fact we only consider very large black holes with $A \gg (-\Lambda)^{-1/2}$ for which (27) is assumed to be valid.
\[ \Delta E^2 = -\frac{1}{4} \left( \frac{A_{QQ}}{\det \Omega} \right)_{M,Q_0} \]
\[ \Delta Q^2 = -\frac{1}{4} \left( \frac{A_{EE}}{\det \Omega} \right)_{M,Q_0} , \]  

(32)

where, eq. (25) is to be used for \( \det \Omega \). Substituting the derivatives in eq. (29), we obtain the rms fluctuations in mass and charge

\[ \Delta E = 2a \left( \frac{Y}{XY - q^2 Z^2} \right)^{1/2} A^\alpha - \frac{1}{2} \]
\[ \Delta Q = 2 \sqrt{\frac{a}{b}} \left( \frac{X}{XY - q^2 Z^2} \right)^{1/2} A^{(\alpha + \beta - 1)/2} . \]

(33)

On the other hand, we have for the mean (equilibrium) values

\[ M = a A^\alpha \left( 1 + \frac{\alpha}{\beta} q^2 \right) \]

(34)

and

\[ Q_0 = q \left( \frac{a \alpha}{b \beta} \right)^{1/2} A^{(\alpha + \beta)/2} . \]

(35)

Hence the relative fluctuations are

\[ \frac{\Delta E}{M} = \left( 1 + \frac{\alpha}{\beta} q^2 \right)^{-1} \left( \frac{Y}{XY - q^2 Z^2} \right)^{1/2} S_{BH}^{-1/2} \]
\[ \frac{\Delta Q}{Q_0} = \frac{1}{q} \left[ \frac{\beta X}{\alpha (XY - q^2 Z^2)} \right]^{1/2} S_{BH}^{-1/2} . \]

(36)

Thus, although the thermal fluctuation correction to the entropy has a degree of non-universality, the relative fluctuations nevertheless turn out fall off with \( A \) for large \( A \) in a universal manner, scaling like \( A^{-1/2} \) independent of the details of the energy-area relation. This universal fall-off shows that the small fluctuation approximation employed in our analysis is valid for large area black holes as long as \( q \) is not small. This requires the black hole charge to grow as a moderate proper fraction of the extremal charge which increases with area.

However, for \( |q| \to 0 \), while the relative rms energy fluctuation is bounded, the relative charge fluctuation blows up as \( q^{-1} \), thereby invalidating the small fluctuation (Gaussian) approximation for charge. This happens not only for the \( Q_0 \sim 0 \) case considered in ref. [13] (where it also follows from the relation for the rms fluctuation in the charge quantum number, but appears to have been overlooked) but for any value of \( Q_0 \) which does not scale as \( Q_{ext}(A) \) for large \( A \).

V. DISCUSSION

The primary result of this paper is the leading thermal fluctuation correction \( \frac{1}{2}(\alpha + \beta) \log A \) in the grand canonical entropy (31). It is obvious that when charge fluctuations are taken into account in addition to energy fluctuations of a black hole, the logarithmic correction due to such fluctuations depends on the exponents appearing in the relation linking horizon area, mass and charge. This is quite unlike the case studied in Ref. [8] where the charge fluctuations were ignored; the energy fluctuations in that case produced a correction \( \frac{1}{4} \log A \) independent of the assumed area-energy relation. In this sense, charge fluctuations endow the thermal fluctuation correction with a certain degree of ‘non-universality’ absent in the earlier case. However, what is perhaps noteworthy is that the coefficient appearing in the leading thermal correction to the area law is proportional to the sum of the exponents appearing in the area-charge-energy relation (27), and not related to any one of the exponents alone. Another way of saying the same thing is that, if we use eq. (28) to express the area-charge-energy relation, the leading correction is in fact independent of \( \alpha \) and simply depends on \( \beta' \), which of course is the same as \( \alpha + \beta \) . In (28), the contribution of the charge is expressed as an additional factor multiplying the energy-area relation used in the canonical ensemble case [8]. The fact that
the leading thermal correction is proportional only to the exponent appearing in this additional factor reaffirms our earlier result of universality in absence of charge fluctuations. To this extent then, one could say that the leading thermal correction with charge fluctuations is at least ‘partly universal’.

For the adS Reissner Nordstrom black hole, the area-charge-energy relation is given by

$$E(A, Q) = \frac{1}{2\ell^2} \left( \frac{A}{4\pi} \right)^{3/2} \left[ 1 + \frac{4\pi\ell^2}{A} + \frac{(4\pi\ell^2Q^2)}{A^2} \right],$$

where, $\ell \equiv (-\Lambda/3)^{-1/2}$. For $A \gg \ell^2$ and moderate values of $|q|$ in the range $0 < |q| < 1$, we have here $\alpha = 3/2$, $\beta' = 2$, in the notation used in (28, implying that the leading thermal correction is $\log A$. This result agrees with that found in Ref. [13] for this black hole in the limit $Q_0 \to 0$ (or equivalently $q \to 0$). This limit is however pathological since, as argued above, the mean-squared charge fluctuation relative to its mean (equilibrium) value diverges in this limit, thereby invalidating the Gaussian approximation which has been used to compute the thermal fluctuation correction. We have further demonstrated above that it is not at all necessary to go to this small charge limit in order to compute the leading thermal fluctuation correction. We may reiterate that our result eq. (31) actually encompasses a larger class of adS black holes which may include non-rotating adS dilatonic black holes as well [16].

It follows that the mechanism proposed in Ref. [14] to argue a possible complete cancellation of logarithmic corrections in the grand canonical computation of the entropy stands on loose ground. In contrast, such a cancellation has already been shown earlier [9] between the logarithmic contribution in the $U(1)$ calculation of the microcanonical entropy [6] of non-rotating black holes and the universal canonical correction found by us earlier [8], both of which appear to be robust. Generalization of these considerations to black holes with rotation stands prominently on our agenda for the near future.

[1] J. D. Bekenstein, Phys. Rev. D7 (1973) 2333.
[2] S. W. Hawking, Comm. Math. Phys. 43 (1975) 221.
[3] For a recent pedagogical review, see A. Ashtekar and J. Lewandowski, Class. Quant. Grav. 21 (2004) R53.
[4] A. Ashtekar, J. Baez and K. Krasnov, Adv. Theor. Math. Phys. 4 (2000) 1 and references therein.
[5] R. Kaul and P. Majumdar, Phys. Rev. Lett. 84 (2000) 5255; Phys. Lett. B439 (1998) 267.
[6] S. Das, R. Kaul and P. Majumdar, Phys. Rev. D63 (2001) 044019.
[7] S. W. Hawking and D. N. Page, Commun. Math. Phys. 87 (1983) 577.
[8] S. Das, R. Kaul and P. Majumdar, Phys. Rev. D63 (2001) 044019.
[9] A. Chatterjee and P. Majumdar, Black hole entropy : quantum vs thermal fluctuations, ArXiv:gr-qc/0303030; see also, S. Das, P. Majumdar and R. Bhaduri, Class. Quant. Grav. 19 (2002) 2355.
[10] T. Thiemann and O. Winkler, Class. Quant. Grav. 18 (2001) 2561.
[11] A. Dasgupta, Counting the apparent horizon, ArXiv:hep-th/0310069.
[12] S. Hod, Higher Order Corrections to the area and entropy, ArXiv:hep-th/0405235.
[13] G. Gour and A.J.M. Medved, Class. Quant. Grav. 20 (2003) 3307.
[14] A.J.M. Medved, A comment on black hole entropy, ArXiv:gr-qc/0406044.
[15] L. D. Landau and E. M. Lifschitz, Statistical Mechanics, Reading, Massachusetts, USA (1974).
[16] S. J. Poletti, J. Twamley and D. L. Wiltshire, Phys. Rev. D51 (1995) 5720.