Energy transfer and dissipation in forced isotropic turbulence

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A model for the Reynolds number dependence of the dimensionless dissipation rate $C_\varepsilon$ is derived from the dimensionless Kármán-Howarth equation, resulting in $C_\varepsilon = C_{\varepsilon,\infty} + C/R_L$, where $R_L$ is the integral scale Reynolds number. The coefficients $C$ and $C_{\varepsilon,\infty}$ arise from asymptotic expansions of the dimensionless second- and third-order structure functions. The model equation is fitted to data from direct numerical simulations (DNS) of forced isotropic turbulence for integral scale Reynolds numbers up to $R_L = 5875$ ($R_\lambda = 435$), which results in an asymptote for $C_\varepsilon$ in the infinite Reynolds number limit $C_{\varepsilon,\infty} = 0.47 \pm 0.01$. Since the coefficients in the model equation are scale-dependent while the dimensionless dissipation rate is not, the scale dependences of the coefficients were modeled by an ad hoc profile function such that they cancel out, leaving the model equation scale-independent, as it must be. The profile function was compared to DNS data with very good agreement, provided we restrict the comparison to scales small enough to be well resolved in our simulations.

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I. INTRODUCTION

In recent years there has been much interest in the fundamentals of turbulent dissipation. This interest has centered on the approximate expression for the dissipation rate $\varepsilon$ which was given by Taylor in 1935\(^1\) as

$$\varepsilon = C_\varepsilon U^3/L,$$

(1)

where $U$ is the root-mean-square velocity and $L$ is the integral scale. Many workers in the field refer to equation (1) as the *Taylor dissipation surrogate*. However, others rearrange it to define the coefficient $C_\varepsilon$ as the nondimensional dissipation rate, thus:

$$C_\varepsilon = \frac{\varepsilon}{U^3/L}.$$  

(2)

In 1953 Batchelor\(^2\)\(^3\) presented evidence to suggest that the coefficient $C_\varepsilon$ tended to a constant value with increasing Reynolds number. In 1984 Sreenivasan\(^4\) showed that in grid turbulence $C_\varepsilon$ became constant for Taylor-Reynolds numbers greater than about 50. Later still, in 1998, he presented a survey of investigations of both forced and decaying turbulence\(^5\), using direct numerical simulation (DNS), which established the now characteristic curve of $C_\varepsilon$ plotted against the Taylor-Reynolds number $R_\lambda$ (e.g. see Fig. 1).

We begin with a short review of the relevant literature. Unless otherwise stated, the quoted DNSs used the pseudospectral method, and we will report on results for forced isotropic turbulence only. Some of the below mentioned numerical results are shown in Fig. 1 alongside our data.

Jiménez *et al.*\(^6\) attained Taylor-scale Reynolds numbers up to $R_\lambda = 170$, with the highest $R_\lambda$ simulation run up to $0.3\tau$, where $\tau$ denotes the large eddy turnover time. In view of the short runtime the simulation might still be transient. Aliasing errors were reduced by a combination of random grid shifts and truncation. The system was forced by using negative viscosity for wavenumbers $k \leq 2.5$ maintaining $k_{max}\eta$ and hence $\varepsilon$ constant. The authors reported an asymptotic value for the dimensionless dissipation rate $C_{\varepsilon,\infty} \simeq 0.7$. The statistics were calculated from five to ten realizations for a short runtime, that is given the sample rate and runtime the realizations were strongly correlated. Regarding resolution requirements, the authors point out that $k_{max}\eta = 1$ is the absolute minimum while $k_{max}\eta = 2$ is desirable.

In the work of Wang *et al.*\(^7\) the forcing was implemented by maintaining the kinetic
energy in the two lowest wavenumber shells constant with an energy spectrum following $k^{-5/3}$. The measured asymptote $C_{\varepsilon,\infty}$ lay in the region $0.42 \leq C_{\varepsilon,\infty} \leq 0.49$. Using the same method without dealiasing, Cao et al.\textsuperscript{8} focussed mainly on the structure of pressure, but data is provided in their Table 1 from which $C_{\varepsilon}$ can be calculated. The initial condition was similar to our DNS as $E(k,0) \sim k^4 \exp(k/k_0)^2$ with $k_0 \simeq 5$ and the system evolved for ten large eddy turnover times before measurements were taken.

Yeung and Zhou\textsuperscript{9} present time-averaged results from simulations using a partially dealiased code with stochastic forcing, covering a $R_\lambda$-range of $38 \leq R_\lambda \leq 240$ for about four large eddy turnover times. The resolution was relatively high as all runs satisfy $k_{\text{max}}\eta \geq 1.5$.

A partially dealiased code with stochastic forcing was also used by Donzis et al.\textsuperscript{10} who simulated flows with Taylor-scale Reynolds number up to $R_\lambda = 390$. The data points for $C_{\varepsilon}$ at different $R_\lambda$ were fitted to the expression $C_{\varepsilon} = A(1 + \sqrt{1 + (B/R_\lambda)^2})$ with $A \simeq 0.2$ and $B \simeq 92$, leading to an asymptote $C_{\varepsilon,\infty} \simeq 0.4$.

The investigation by Bos et al.\textsuperscript{11} presents results from DNS, LES and EDQNM calculations for Reynolds numbers up to $R_\lambda = 100$ for DNS and $R_\lambda = 2000$ for EDQNM. The authors tested different initial conditions such as Gaussian-shaped initial energy spectra and the von Kármán-spectrum and found no dependence on the choice of initial spectrum once the system has reached a stationary state. However, the transient to a steady state was found to be shorter for a von Kármán-spectrum than for Gaussian-shaped initial spectra. They measured $C_{\varepsilon,\infty} \simeq 0.5$ for the asymptote of the dimensionless dissipation rate.

Variations of the initial conditions were also studied by Goto and Vassilicos\textsuperscript{12}, mainly altering the low wavenumber behaviour and the peak wavenumber of the initial spectra. The results for $C_{\varepsilon}$ show a dependence on the different low wavenumber forms of the initial spectra, in contrast the location of the peak of the initial spectrum had no significant influence on $C_{\varepsilon}$. What is interpreted as a dependence on the form of the initial spectra could actually be due to differences in the forcing method. The system is kept statistically stationary by fixing the magnitude of the velocity field modes for wavenumbers smaller than the peak wavenumber of the initial spectra, which in some cases leads to a very large forcing range. The low wavenumber form of the initial spectrum is thus maintained during the evolution of the velocity field, such that it is no longer purely a feature of the initial condition but rather a permanent feature imposed by the forcing scheme. The observed dependence of $C_{\varepsilon}$ on the choice of initial spectrum could therefore be due to differences in the forcing spectrum.
Kaneda et al.\textsuperscript{13} conducted the largest DNS of forced isotropic turbulence so far on grids of up to $4096^3$ collocation points reaching $R_\lambda = 1201$ in single precision and $R_\lambda = 732$ in double precision, both at minimum resolution of $k_{\text{max}}\eta = 1$. The system was maintained statistically stationary by using negative viscosity for wavenumbers $k \leq 2.5$ in order to keep the total energy constant. Data has been collected from single realizations only, resulting in an asymptotic value for $C_\varepsilon$ in the range $0.4 \leq C_{\varepsilon,\infty} \leq 0.5$. The largest $R_\lambda$ simulation was only conducted for a short time, thus this run might still be transient.

The most recent high resolution DNS results for the dimensionless dissipation rate were presented by Yeung et al.\textsuperscript{14}. Four simulations spanning a Taylor-scale Reynolds number range of $140 \leq R_\lambda \leq 1000$ on $2048^3$ and $4096^3$ collocation points were conducted, at resolutions between $11.2 \leq k_{\text{max}}\eta \leq 1.3$, resulting in $0.449 \leq C_\varepsilon \leq 0.47$. Due to the computational cost incurred by simulations of this size, the runtime in steady state was relatively short, the simulation corresponding to $R_\lambda = 1000$ was stopped after $3.59\tau$. During steady state 20 snapshots were taken to populate the ensemble, so samples were taken every $0.18\tau$. Thus the ensemble consisted of realizations that are statistically correlated. The authors point out that longer runtime would be preferable, but argue that since intense fluctuations in $\varepsilon$ are relatively short lived, ensemble averaging over snapshots close in time will still improve statistics. We are unsure how the authors determine that the system reached a steady state. Our largest simulation ran for about $15\tau$ (in what we thought was steady state) to make sure that it really had reached steady state, as the total energy fluctuates on relatively long timescales. Complete dealiasing was implemented and the system was forced at large scales.

In contrast to the various pseudospectral DNSs of incompressible turbulent flows cited here, Pearson et al.\textsuperscript{15} used a sixth-order finite difference scheme with large-scale $\delta(t)$-correlated forcing for DNS of slightly compressible flows, leading to $C_\varepsilon \simeq 0.5$.

Having summarized numerical results on the topic we now briefly turn to experimental results. Pearson et al.\textsuperscript{16} measured $C_\varepsilon \simeq 0.5$ for a number of shear flows. Different flow types were investigated by Burattini et al.\textsuperscript{17}, and Mazellier et al.\textsuperscript{18} studied turbulence in a wind tunnel generated from a variety of different grid geometries including fractal grids. In the fractal case they found a significantly lower asymptote for $C_\varepsilon$, namely $C_{\varepsilon,\infty} \simeq 0.065$, which suggests that $C_{\varepsilon,\infty}$ may not be universal constant.

In summary, the asymptotic value $C_{\varepsilon,\infty} \simeq 0.5$ is a well-established numerical result which
shows good agreement with experimental work, except for fractal-generated grid turbulence. In his 1968 lecture notes, Saffman made two comments about the expression that we have given here as equation (1). These were: “This result is fundamental to an understanding of turbulence and yet still lacks theoretical support” and “the possibility that \( A \) (i.e. our \( C_\varepsilon \)) depends weakly on the Reynolds number can by no means be completely discounted”. More than forty years on, the question implicit in his second comment has been comprehensively answered by the survey papers of Sreenivasan, along with a great deal of subsequent work by others, some of which we have cited here. However, while some theoretical work has indicated an inverse proportionality between \( C_\varepsilon \) and Reynolds number, this has been limited to low (i.e. non-turbulent) Reynolds numbers or based on a mean-field approximation or restricted to providing an upper-bound. Hence his first comment is still valid today; and this lack of theoretical support remains an impediment to the development of turbulence phenomenology and hence turbulence theory.

As we have seen before, an approach based on the dimensionless dissipation \( C_\varepsilon \), the ratio of the dissipation to the surrogate expression \( U^3/L \), can be a helpful way of looking at things. In the present paper, we examine the behaviour of \( C_\varepsilon \) with increasing Reynolds number by means of a simple model based on the Kármán-Howarth equation and supported by direct numerical simulation (DNS). We find that this description captures the observed dependence of \( C_\varepsilon \), thus providing a direct theoretical route from the Navier-Stokes equation to dissipation rate scaling. We begin with a description of our DNS, before presenting a theoretical analysis followed by numerical results.

II. NUMERICAL METHOD

We used a pseudospectral DNS, with full dealiasing implemented by truncation of the velocity field according to the two-thirds rule. Time advancement for the viscous term was performed exactly using an integrating factor, while the non-linear term was stepped forward in time using Heun’s method, which is a second-order predictor-corrector routine. Each simulation was started from a Gaussian-distributed random field with a specified energy spectrum, which followed \( k^4 \) for the low-\( k \) modes. Measurements were taken after the simulations had reached a stationary state. The system was forced by negative damping,
with the Fourier transform of the force $f$ given by

$$f(k, t) = (\varepsilon_W/2E_f)u(k, t) \quad \text{for} \quad 0 < |k| < k_f;$$

$$= 0 \quad \text{otherwise}, \quad (3)$$

where $u(k, t)$ is the instantaneous velocity field (in wavenumber space). The highest forced wavenumber, $k_f$, was chosen to be $k_f = 2.5k_{\text{min}}$, where $k_{\text{min}} = 2\pi/L_{\text{box}} = 1$ is the lowest resolved wavenumber. As $E_f$ was the total energy contained in the forcing band, this ensured that the energy injection rate was $\varepsilon_W = \text{constant}$. It is worth noting that any method of energy injection employed in the numerical simulation of isotropic turbulence is not experimentally realizable. The present method of negative damping has also been used in other investigations\textsuperscript{6,13,25,26}, albeit not necessarily such that $\varepsilon_W$ is maintained constant (although note the theoretical analysis of this type of forcing by Doering and Petrov\textsuperscript{27}). Also, note that the correlation between the force and the velocity is restricted to the very lowest wavenumbers.

For each Reynolds number studied, we used the same initial spectrum and input rate $\varepsilon_W$. The only initial condition changed was the value assigned to the (kinematic) viscosity. Once the initial transient had passed, the velocity field was sampled every half a large-eddy turnover time, $\tau = L/U$, where $L$ denotes the average integral scale and $U$ the rms velocity. The ensemble populated with these sampled realizations was used, in conjunction with the usual shell averaging, to calculate statistics. Simulations were run using lattices of size $128^3$, $256^3$, $512^3$, $1024^3$ and $2048^3$, with corresponding Reynolds numbers ranging from $R_\lambda = 41.8$ up to $434.8$. The smallest wavenumber was $k_{\text{min}} = 2\pi/L_{\text{box}} = 1$ in all simulations, while the maximum wavenumber satisfied $k_{\text{max}}\eta \geq 1.30$ for all runs except one which satisfied $k_{\text{max}}\eta \geq 1.01$, where $\eta$ is the Kolmogorov dissipation length scale. The integral scale, $L$, was found to lie between $0.23L_{\text{box}}$ and $0.17L_{\text{box}}$. It can be seen in Figure 2 of McComb \textit{et al.}\textsuperscript{28} that a small-scale resolution of $k_{\text{max}}\eta > 1.6$ is desirable in order to capture the relevant dissipative physics. Evidently, this would restrict the attainable Reynolds number of the simulated flow, and the reference suggests that $k_{\text{max}}\eta \geq 1.3$ would still be acceptable (containing $\sim 99.5\%$ of dissipative dynamics\textsuperscript{29}). In contrast, at $k_{\text{max}}\eta \simeq 1$ a non-negligible part of dissipation is not taken into account. Most high resolution DNSs of isotropic turbulence try to attain Reynolds numbers as high possible and thus opt for minimal resolution requirements. As this paper focuses on dissipative physics, the simulations have been conducted following a
TABLE I. A summary of the main parameters for our numerical simulations. The values quoted for the dissipation rate \( \varepsilon \) and its standard deviation \( \sigma \), the rms velocity \( U \) and the integral scale \( L \), are ensemble- and shell-averaged mean values.

| \( R_L \) | \( R_\lambda \) | \( \nu_0 \) | \( N \) | \( \varepsilon \) | \( \sigma \) | \( U \) | \( L/L_{\text{box}} \) | \( k_{\text{max}} \eta \) |
|-----------|-----------|-----------|-----|--------|--------|-----|----------------|------|
| 81.5      | 41.8      | 0.01      | 512 | 0.097  | 0.010  | 0.581| 0.22           | 9.57 |
| 83.7      | 42.5      | 0.01      | 128 | 0.094  | 0.015  | 0.581| 0.23           | 2.34 |
| 88.2      | 44.0      | 0.009     | 128 | 0.096  | 0.009  | 0.587| 0.22           | 2.15 |
| 101.4     | 48.0      | 0.008     | 128 | 0.096  | 0.013  | 0.586| 0.22           | 1.96 |
| 105.7     | 49.6      | 0.007     | 128 | 0.098  | 0.011  | 0.579| 0.20           | 1.77 |
| 146.5     | 60.8      | 0.005     | 512 | 0.098  | 0.009  | 0.589| 0.20           | 5.68 |
| 158.6     | 64.2      | 0.005     | 128 | 0.099  | 0.011  | 0.607| 0.21           | 1.37 |
| 287.8     | 89.4      | 0.0025    | 512 | 0.101  | 0.006  | 0.605| 0.19           | 3.35 |
| 360.1     | 101.3     | 0.002     | 256 | 0.099  | 0.009  | 0.607| 0.19           | 1.41 |
| 432.6     | 113.3     | 0.0018    | 256 | 0.100  | 0.008  | 0.626| 0.20           | 1.31 |
| 785.2     | 153.4     | 0.001     | 512 | 0.098  | 0.011  | 0.626| 0.20           | 1.70 |
| 1026.3    | 176.9     | 0.00072   | 512 | 0.102  | 0.009  | 0.626| 0.19           | 1.31 |
| 1529.0    | 217.0     | 0.0005    | 1024| 0.100  | 0.008  | 0.63 | 0.19           | 2.02 |
| 2414.6    | 276.2     | 0.0003    | 1024| 0.100  | 0.009  | 0.626| 0.18           | 1.38 |
| 3535.0    | 335.2     | 0.0002    | 1024| 0.102  | 0.008  | 0.626| 0.18           | 1.01 |
| 5875.5    | 435.2     | 0.00011   | 2048| 0.102  | 0.010  | 0.614| 0.17           | 1.30 |

more conservative approach, where the emphasis has been put on higher resolution, thus necessarily compromising to some extent on Reynolds number. Large-scale resolution has only relatively recently received attention in the literature. As mentioned above, the largest scales of the flow are smaller than a quarter of the simulation box size. Details of the individual runs are summarized in Table I.

Our simulations have been well validated by means of extensive and detailed comparison with the results of other investigations\textsuperscript{29}. These include the Taylor-Green vortex\textsuperscript{30,31}; measurements of the isotropy, Kolmogorov constant and velocity-derivative skewness; advection of a passive scalar; and a direct comparison with the freely-available pseudospectral code.
FIG. 1. Variation of the dimensionless dissipation coefficient $C_\varepsilon$ with Taylor-Reynolds number $R_\lambda$ from our direct numerical simulations. Other investigations of forced turbulence are presented for comparison. The black line is a fit of the expression $A(1 + \sqrt{1 + (B/R_\lambda)^2})$ to our data only.

It can be seen from Fig. 1 that our results reproduce the characteristic behaviour for the plot of $C_\varepsilon$ against $R_\lambda$, and agree closely with other representative results in the literature\textsuperscript{7,8,10,13,33}. We note that the data presented for comparison was obtained using negative damping (with variable $\varepsilon_W$)\textsuperscript{13}, stochastic noise\textsuperscript{10,33}, or maintaining a $k^{-5/3}$ energy spectrum within the forced shells\textsuperscript{7,8}. These methods for energy injection have been discussed in\textsuperscript{11}.

III. A DIMENSIONLESS KÁRMÁN-HOWARTH EQUATION FOR FORCED TURBULENCE

The use of stirring forces with the energy equation in spectral space (i.e. with the Lin equation) is well established,

$$\frac{\partial E(k, t)}{\partial t} = T(k, t) - 2\nu_0 k^2 + W(k, t),$$

(4)
where \( \nu_0 \) is the kinematic viscosity, \( E(k, t) \) and \( T(k, t) \) are the energy and transfer spectra, respectively, and \( W(k, t) = 4\pi k^2 \langle \mathbf{u}(-k, t) \cdot \mathbf{f}(k, t) \rangle \) is the work spectrum of the stirring force, \( \mathbf{f}(k, t) \). (See, for example, Ref. 34 and Ref. 35. For conciseness we will not denote the time dependence from now on.) But this is not the case with the Kármán-Howarth equation (KHE), which is its real-space equivalent. Accordingly, we obtain the equivalent KHE by Fourier transformation of the Lin equation (with forcing) as

\[
-\frac{3}{2} \frac{\partial U^2}{\partial t} + \frac{3}{4} \frac{\partial S_2(r)}{\partial t} = -\frac{1}{4r^2} \frac{\partial}{\partial r} \left( r^4 S_3(r) \right) + \frac{3\nu_0}{2r^4} \frac{\partial}{\partial r} \left( r^4 \frac{\partial S_2(r)}{\partial r} \right) - I(r) ,
\]

where the longitudinal structure functions are defined as

\[
S_n(r) = \langle \left( [\mathbf{u}(\mathbf{r}) - \mathbf{u}(\mathbf{0})] \cdot \hat{r} \right)^n \rangle ,
\]

and the input \( I(r) \) is given in terms of \( W(k) \), the work spectrum of the stirring forces, by

\[
I(r) = 3 \int_0^\infty dk \ W(k) \left[ \frac{\sin kr - kr \cos kr}{(kr)^3} \right].
\]

Here \( I(r) \) is interpreted as the total energy injected into all scales \( r \). Note that we may make the connection between \( W(k) \) and the injection rate for the numerical simulations by

\[
I(0) = \int_0^\infty dk W(k) = \varepsilon_W ,
\]

where the energy injection rate \( \varepsilon_W \) is defined in (3).

If we were to apply (5) to freely-decaying turbulence, we would set the input term \( I(r) \) equal to zero, to give

\[
-\frac{3}{2} \frac{\partial U^2}{\partial t} = -\frac{3}{4} \frac{\partial S_2}{\partial t} - \frac{1}{4r^2} \frac{\partial}{\partial r} \left( r^4 S_3 \right) + \frac{3\nu_0}{2r^4} \frac{\partial}{\partial r} \left( r^4 \frac{\partial S_2}{\partial r} \right) .
\]

Of course, for the case of free decay, we may also set \( (3/2)\partial U^2/\partial t = -\varepsilon \), after which we obtain the form of the KHE which is familiar in the literature (e.g. see Ref. 35 or Ref. 36). However, it can lead to problems if this substitution is retained (as is sometimes the case) for forced turbulence, for which it is not valid.

If, on the other hand, we are considering forced turbulence which has reached a stationary state, then we may set \( \partial U^2/\partial t = \partial S_2/\partial t = 0 \), whereupon (5) reduces to the appropriate KHE for forced turbulence,

\[
I(r) = -\frac{1}{4r^2} \frac{\partial}{\partial r} \left( r^4 S_3(r) \right) + \frac{3\nu_0}{2r^4} \frac{\partial}{\partial r} \left( r^4 \frac{\partial S_2(r)}{\partial r} \right) .
\]
As an aside, we note that this form for the forced KHE has several important differences from other approaches which have appeared in the literature\textsuperscript{33,37}. Previous approaches incorrectly retained the dissipation rate in the equation and essentially introduced an approximate ad hoc ‘correction’ in order to take account of the forcing. This is, for example, presented for the third-order structure function as

\[ S_3(r) = -\frac{4\varepsilon r}{5} + Z(r) + 6\nu_0 \frac{\partial S_2}{\partial r}, \quad (11) \]

where \( Z(r) \) is the ad hoc correction\textsuperscript{33}. In contrast, we note that the origin of \( \varepsilon \) in the KHE was \( \partial U^2/\partial t \), which is zero for a stationary system, and instead show how its role is now played by the energy input function, \( I(r) \). Thus, in our approach, instead of equation (11), we have

\[ S_3(r) = -\frac{4}{r^4} \int_0^r dy \, y^4 I(y) + 6\nu_0 \frac{\partial S_2}{\partial r}, \quad (12) \]

where \( I(r) \) is calculated directly from the work spectrum, and is not approximated. Taking the limit \( r \to 0 \) in equation (7) and invoking stationarity, for small scales we measure \( I(r) = \varepsilon W = \varepsilon \), and so recover the Kolmogorov form of the KHE\textsuperscript{38}.

Returning to our form of the forced KHE, equation (10), we now introduce the dimensionless structure functions \( h_n(\rho) \) which are given by

\[ S_n(r) = U^n h_n(\rho), \quad (13) \]

where \( \rho = r/L \). Substitution of these into (10) leads to

\[ I(\rho) = -\frac{1}{4\rho^4} \frac{\partial}{\partial \rho} \left( \rho^4 h_3(\rho) \right) \frac{U^3}{L} + \nu_0 U^2 \frac{3}{2\rho^4} \frac{\partial}{\partial \rho} \left( \rho^4 \frac{\partial h_2(\rho)}{\partial \rho} \right). \quad (14) \]

Then, with some rearrangement, the forced KHE (10) takes the dimensionless form

\[ I(\rho) \frac{L}{U^3} = -\frac{1}{4\rho^4} \frac{\partial}{\partial \rho} \left( \rho^4 h_3(\rho) \right) + \frac{1}{R_L} \frac{3}{2\rho^4} \frac{\partial}{\partial \rho} \left( \rho^4 \frac{\partial h_2(\rho)}{\partial \rho} \right), \quad (15) \]

with \( R_L = UL/\nu_0 \) the Reynolds number based on the integral scale. For conciseness we introduce coefficients \( A_3 \) and \( A_2 \),

\[ A_3(\rho|R_L) = -\frac{1}{4\rho^4} \frac{\partial}{\partial \rho} \left( \rho^4 h_3(\rho) \right), \quad (16) \]

and

\[ A_2(\rho|R_L) = \frac{3}{2\rho^4} \frac{\partial}{\partial \rho} \left( \rho^4 \frac{\partial h_2(\rho)}{\partial \rho} \right), \quad (17) \]
where the notation $A_i(\rho|R_L), \ (i = 1, 2)$ indicates that both coefficients could have a secondary dependence on $R_L$, as we will discuss in Section III B. The dimensionless KHE (15) expressed in terms of $A_2$ and $A_3$ becomes

$$I(\rho) \frac{L}{U^3} = A_3(\rho|R_L) + \frac{A_2(\rho|R_L)}{R_L}.$$  \hspace{1cm} (18)

This simple scaling analysis has extracted the integral scale as the relevant lengthscale, and $R_L$ as the appropriate Reynolds number, for studying the behaviour of $C_\varepsilon$. This was noted by Batchelor\textsuperscript{2,3}, despite which it has become common practice to study $C_\varepsilon = C_\varepsilon(R_\lambda)$, as demonstrated by Fig. 1.

The input may be expressed in terms of an amplitude $\varepsilon_W$ and a dimensionless shape function $\phi(\rho)$,

$$I(\rho) = \varepsilon_W \phi(\rho),$$  \hspace{1cm} (19)

where $\phi(\rho)$ contains all of the scale-dependent information and, as required by equation (8), satisfies $\phi(0) = 1$.

A. The limit of $\delta(k)$-forcing

Figure 2 illustrates the shape of $\phi(\rho)$ and shows the effect of varying the forcing band defined in equation (3), using data from our $R_\lambda = 276$ run. As we reduce the width of the forcing band, we approach the limit of $\delta$-function forcing in wavenumber space, corresponding to $\phi(\rho) = 1 \ \forall \ \rho$. Although this limit cannot be studied using DNS, for theoretical convenience we can consider the limit analytically (or, alternatively, restrict our attention to scales for which $\phi(\rho) \approx 1$) before addressing the complication added by general forcing.

Now let us consider the dimensionless KHE for the case of $\delta(k)$-forcing, where $I(\rho) = \varepsilon_W = \varepsilon$. Equation (18) becomes

$$\frac{\varepsilon_W L}{U^3} = A_3(\rho|R_L) + \frac{A_2(\rho|R_L)}{R_L},$$  \hspace{1cm} (20)

from which, since $\varepsilon = \varepsilon_W$ from stationarity, and using equation (2), we have

$$C_\varepsilon = A_3(\rho|R_L) + \frac{A_2(\rho|R_L)}{R_L}.$$  \hspace{1cm} (21)

From the well-known phenomenology associated with Kolmogorov’s inertial-range theories\textsuperscript{38}, as the Reynolds number tends to infinity, we know that we must have $A_2/R_L \to 0$ and $A_3 \to C_{\varepsilon,\infty} \equiv \text{constant}$.
Increasing $k_f$

**FIG. 2.** The dimensionless input shape function $\phi(\rho)$, as defined by equation (19). The effect of varying the forcing band in wavenumber space, $0 < k < k_f$, is illustrated, showing the limit of $\delta(k)$-forcing. Presented for $R_\lambda = 276$ data.

Equation (21) can also be rewritten as

$$\varepsilon = A_3(\rho|R_L) \frac{U^3}{L} + A_2(\rho|R_L) \frac{\nu_0 U^2}{L^2} . \tag{22}$$

The first term on the RHS is essentially the Taylor surrogate, while the second term is a viscous correction. It has been shown$^{22}$ that, for the case of decaying turbulence, the surrogate $U^3/L$ represents the maximum inertial transfer, $\varepsilon_T$, more accurately than the dissipation rate. The same is shown later for forced turbulence in Fig. 4, since the input rate (hence $\varepsilon$) is kept constant. Thus, the forced KHE is expressing the equivalence of the rates at which energy is transferred and dissipated (or injected) as $\nu_0 \to 0$. At finite viscosity, there is a contribution to the dissipation rate which has not passed through the cascade. In terms of our rearranged model equation (22),

$$\varepsilon = C_{\varepsilon,\infty} \frac{U^3}{L} + \nu_0 \frac{A_2(\rho|R_L)U^2}{L^2} \to \varepsilon_T \quad \text{as} \quad \nu_0 \to 0 , \tag{23}$$

where, from equation (16), the asymptotic value denoted by $C_{\varepsilon,\infty}$ is given by the expression

$$C_{\varepsilon,\infty} = \lim_{\nu_0 \to 0} A_3(\rho|R_L) = - \lim_{\nu_0 \to 0} \frac{1}{4\rho^4} \frac{\partial}{\partial \rho} \left( \rho^4 h_3(\rho) \right) . \tag{24}$$
B. Asymptotic expansion of the structure functions in inverse powers of $R_L$

In the limit of $\delta(k)$-forcing or for scales smaller than the energy injection scale we have $I(\rho) = \varepsilon_W = \varepsilon$, hence the dimensionless KHE (15) reads

$$C_\varepsilon = -\frac{1}{4\rho^4} \frac{\partial}{\partial \rho} \left( \rho^4 h_3(\rho) \right) + \frac{1}{R_L} \frac{3}{2\rho^4} \frac{\partial}{\partial \rho} \left( \rho^4 \frac{\partial h_2(\rho)}{\partial \rho} \right).$$  \hspace{1cm} (25)

Following Lundgren\textsuperscript{39}, we note that for large $R_L$ the term with the highest derivative is multiplied by the small parameter $R_L^{-1}$, hence we are faced with singular perturbation problem. Therefore we consider outer asymptotic expansions of the structure functions in negative powers of $R_L$, a technique applied to singular perturbation problems (see e.g.\textsuperscript{40}, Chapter X). We study here only the outer expansions as we have rescaled the KHE with respect to the integral scale $L$.

The outer expansions of the dimensionless structure functions in powers of $R_L^{-1}$ read

$$h_2(\rho) = h_2^{(0)}(\rho) + \frac{1}{R_L} h_2^{(1)}(\rho) + O \left( \frac{1}{R_L^2} \right),$$  \hspace{1cm} (26)

and

$$h_3(\rho) = h_3^{(0)}(\rho) + \frac{1}{R_L} h_3^{(1)}(\rho) + O \left( \frac{1}{R_L^2} \right).$$  \hspace{1cm} (27)

Substituting the expansions (26) and (27) into (25) we obtain up to first order in $R_L^{-1}$

$$C_\varepsilon = -\frac{1}{4\rho^4} \frac{\partial}{\partial \rho} \left( \rho^4 h_3^{(0)}(\rho) \right) + \frac{1}{R_L} \left[ \frac{3}{2\rho^4} \frac{\partial}{\partial \rho} \left( \rho^4 \frac{\partial h_2^{(0)}(\rho)}{\partial \rho} \right) - \frac{1}{4\rho^4} \frac{\partial}{\partial \rho} \left( \rho^4 \frac{\partial h_3^{(1)}(\rho)}{\partial \rho} \right) \right] + O \left( \frac{1}{R_L^2} \right),$$  \hspace{1cm} (28)

where the terms $h_2^{(0)}$, $h_3^{(0)}$ and $h_3^{(1)}$ do not depend on $R_L$. We can write this in terms of the constant $C_{\varepsilon,\infty}$ and a new constant $C$ (with respect to $R_L$)

$$C_{\varepsilon,\infty} = -\frac{1}{4\rho^4} \frac{\partial}{\partial \rho} \left( \rho^4 h_3^{(0)}(\rho) \right),$$  \hspace{1cm} (29)

and

$$C = \frac{3}{2\rho^4} \frac{\partial}{\partial \rho} \left( \rho^4 \frac{\partial h_2^{(0)}(\rho)}{\partial \rho} \right) - \frac{1}{4\rho^4} \frac{\partial}{\partial \rho} \left( \rho^4 \frac{\partial h_3^{(1)}(\rho)}{\partial \rho} \right),$$  \hspace{1cm} (30)

where both coefficients are a priori scale-dependent (i.e. dependent on a lengthscale) while $C_\varepsilon$ is not, hence the scale dependences must cancel out. In fact, since $C_{\varepsilon,\infty}$ is a constant with respect to $\rho$ by the four-fifths law, the scale dependence between the two terms on the RHS of (30) must cancel out. This leads us to a model equation in the limit of $\delta(k)$-forcing

$$C_\varepsilon = C_{\varepsilon,\infty} + \frac{C}{R_L},$$  \hspace{1cm} (31)

where $C_{\varepsilon,\infty}$ and $C$ are constants with respect to $R_L$ and $\rho$. 

13
1. Physical interpretation of the constants $C_{\varepsilon,\infty}$ and $C$

In order to discuss the origins of the two constants $C_{\varepsilon,\infty}$ and $C$, we first identify what $A_2$ and $A_3$ represent. Consider the energy balance given in equation (10),

$$I(\rho) = \Pi(\rho) + \varepsilon(\rho),$$  \hspace{1cm} (32)

where $I(\rho)$ was interpreted (below equation (7)) as the total rate of energy injection into all scales $> \rho$; $\Pi(\rho)$ is the rate of energy transfer to all scales $< \rho$; and $\varepsilon(\rho)$ is the rate of dissipation by all scales $> \rho$. Taking $\varepsilon(\rho)$ to the other side, we can express this relationship in words as

$$\text{(energy available to transfer to scales } < \rho) = \text{(energy input into scales } > \rho) - \text{(energy dissipated by scales } > \rho).$$

This energy balance illustrated in Fig. 5.

By multiplying equation (10) through by $L/U^3$, we arrived at equation (18), from which we deduce that $A_3(\rho|R_L)$ is the dimensionless energy transfer rate to scales $< \rho$, and $A_2(\rho|R_L)/R_L$ measures the dissipation by scales $> \rho$. Using the outer expansion (27), we can write

$$A_3(\rho|R_L) = A_3^{(0)}(\rho) + \frac{A_3^{(1)}(\rho)}{R_L} + O\left(\frac{1}{R_L^2}\right),$$  \hspace{1cm} (33)

where $A_3^{(0)}(\rho) = C_{\varepsilon,\infty}$ is the energy transfer rate in the infinite Reynolds number limit. At finite $R_L$, the observed transfer rate will be less than this, since some energy is dissipated even at large scales. The finite Reynolds number correction $A_3^{(1)}(\rho)/R_L < 0$ is therefore a viscous correction, and represents the energy that cannot be transferred because it has already been dissipated at larger scales.

The outer expansion of $A_2$ has a leading-order contribution $A_2^{(0)}(\rho)$, which gives a measure of the strength of viscous effects in the infinite Reynolds number limit. Note that the infinite Reynolds number limit does not correspond to the Euler equation (that is to inviscid evolution), as the symmetry-breaking viscous term is still present\textsuperscript{35}. At finite Reynolds number, however, this is again an overestimation: to have $A_3^{(0)}(\rho) = C_{\varepsilon,\infty} > A_3(\rho)$ available to transfer to scales $< \rho$, there has to have been less dissipation by scales $> \rho$. Instead, the combination $A_2^{(0)}(\rho) + A_3^{(1)}(\rho) < A_2^{(0)}(\rho)$ gives a corrected measure of the viscous effects by scales $> \rho$. This is illustrated in Fig. 3. Contributions to transfer and dissipation are
FIG. 3. The transfer and dissipation to order $1/R_L$ through the scale $\rho$ with (a) the measured rates, and (b) the terms separated into transfer and viscous contributions.

shown to order $1/R_L$, with part (a) showing the actual rates, and part (b) showing the $R_L$-independent transfer rate along with the compensated dissipation. The sum of the two remains constant; just the partition into transfer and dissipation is modified.

For the case of $\delta(k)$-forcing, the input $I(\rho) = \varepsilon_W$ and we find the model equation (31). The coefficient

$$C = A_2^{(0)}(\rho) + A_3^{(1)}(\rho)$$

in the model equation (31) is therefore a measure of the viscous action, reduced to compensate for the difference between the ‘actual’ transfer rate and $C_{\varepsilon,\infty}$.

IV. MODELLING THE SCALE DEPENDENCE OF COEFFICIENTS $A_2$ AND $A_3$ WITH AN AD HOC PROFILE FUNCTION

The model equation (31) has been derived in the limit of $\delta(k)$-forcing, that is constant forcing at all scales in real space, which is unattainable in DNS. In order to apply the model (31) to analyze DNS results, we return now to a general forcing represented by the input term $I(\rho)$, where we will find it convenient to introduce a profile function $H(\rho)$ as follows.

To begin, we use $\varepsilon_W = \varepsilon$ to rewrite equation (19) as $I(\rho)L/U^3 = C_{\varepsilon}\phi(\rho)$, such that equation (18) becomes

$$C_{\varepsilon} = \frac{A_3(\rho|R_L)}{\phi(\rho)} + \frac{A_2(\rho|R_L)}{R_L\phi(\rho)}.$$

However, the fact that the left hand side of (35) is constant with respect to the dimensionless
scale variable $\rho = r/L$ means that the separate dependences on $\rho$ on the right hand side must cancel. In order to separate out the scale-dependent effects, we seek semi-empirical decompositions for $A_3(\rho)$ and $A_2(\rho)$ which satisfy the following conditions:

1. The $\rho$-dependence of the terms on the RHS of (35) must cancel, since the LHS is a constant;

2. As $\rho \to 0$, we have $A_3(\rho)/\phi(\rho) \to 0$ and so $A_2(\rho)/R_L\phi(\rho) \to C_\varepsilon(R_L)$ (it is entirely viscous);

3. As $\rho \to \infty$, we have $A_2(\rho) \to 0$ and $A_3(\rho)/\phi(\rho) \to C_\varepsilon(R_L)$ (it is entirely inertial);

4. As $R_L \to \infty$: $C_\varepsilon(R_L) \to C_\varepsilon,\infty = \text{constant}$.

It is easily verified that these constraints are satisfied by the following expressions,

\[
\frac{A_3(\rho|R_L)}{\phi(\rho)} = C_\varepsilon [1 - H(\rho)] ;
\]

and

\[
\frac{A_2(\rho|R_L)}{R_L\phi(\rho)} = C_\varepsilon H(\rho) ,
\]

where we have introduced an *ad hoc* profile function $H(\rho)$, which in general must satisfy the conditions

\[
\lim_{\rho \to 0} H(\rho) = 1 \quad \text{and} \quad \lim_{\rho \to \infty} H(\rho) = 0. \tag{38}
\]

The behaviour of the profile function at small and intermediate scales is also constrained by our knowledge of the structure functions. At small scales, the structure functions behave as $S_n \sim r^n$, which implies that $H(\rho) \simeq 1 - a\rho^2$ for some scale-independent coefficient $a$. For large enough Reynolds numbers, in the inertial range of scales, $S_2 \sim r^\gamma$ which leads to $H(\rho) \sim \rho^{\gamma-2}$, with $\gamma(R_L) \to 2/3$ as $R_L$ is increased. Based on these additional constraints, we have chosen a suitable profile function to represent the scale dependence to be

\[
H(\rho) = \left[1 + \frac{a\rho^2}{1 + b\rho^\gamma}\right]^{-1} ,
\]

where $a$, $b$ and $\gamma$ are Reynolds number dependent and obtained by fitting to numerical results. We note that the actual values of these fit parameters do not affect our model (31) since the scale dependence cancels out.
FIG. 4. Variation with Taylor-Reynolds number of the dissipation rate $\varepsilon$, maximum inertial transfer rate $\varepsilon_T$ and Taylor surrogate $U^3/L$, all scaled on the injection rate $\varepsilon_W$.

V. NUMERICAL RESULTS

In Fig. 4 we show separately the behaviour of the dissipation rate $\varepsilon$, the maximum inertial flux $\varepsilon_T$ and the Taylor surrogate $U^3/L$, where each of these quantities was scaled on the constant injection rate $\varepsilon_W$. This is the basis of our first observation. We see that the decrease of $C_\varepsilon$, with increasing Reynolds number, is caused by the increasing value of the surrogate in the denominator, rather than by decay of the dissipation rate in the numerator, as this remains fixed at $\varepsilon = \varepsilon_W$. This is the exact opposite of the case for freely decaying turbulence, where the actual dissipation rate decreases with increasing Reynolds number, while the surrogate remains fairly constant\textsuperscript{22}. The figure also shows how both $\varepsilon_T/\varepsilon_W$ and $U^3/(L\varepsilon_W)$ increase at low $R_\lambda$, while $\varepsilon/\varepsilon_W$ is constant (as required by the energy balance in forced isotropic turbulence). Therefore $U^3/L$ represents $\varepsilon_T$ better than $\varepsilon$. Furthermore, we observe that $\varepsilon/\varepsilon_T = \varepsilon_W/\varepsilon_T \to 1$ from above as the Reynolds number is increased, corresponding to the onset of an inertial range\textsuperscript{34}.

Figure 5 shows the balance of energy represented by the dimensionless equation given as (18). For small scales ($\rho < \lambda/L$ for the case $R_\lambda = 276$ shown) the input term satisfies
FIG. 5. Dimensionless energy balance in the Kármán-Howarth equation, as expressed by equation (18). $R_A = 276$. The Taylor microscale is labelled for comparison.

$I(r) \approx \varepsilon_W = \varepsilon$, as expected since such scales are not directly influenced by the forcing. We note that the second- and third-order structure functions may be obtained from the energy and transfer spectra, respectively, using

$$S_2(r) = 4 \int dk \, E(k) \, a(kr)$$

and

$$S_3(r) = 12 \int dk \, \frac{T(k) \, \partial a(kr)}{k^2 \, \partial r},$$

where the function $a(x)$ is

$$a(x) = \frac{1}{3} - \frac{\sin x - x \cos x}{x^3},$$

with the derivatives of $a(kr)$ calculated analytically. This procedure was introduced by Qian\textsuperscript{41,42} and more recently used by Tchoufag \textit{et al}\textsuperscript{43}: the underlying transforms may be found in the book by Monin and Yaglom\textsuperscript{36}, see their equations (12.75) and (12.141"'). From these expressions, the non-linear and viscous terms $A_3$ and $A_2$ given by equations (16) and (17), are calculated using

$$A_3(\rho) = -\frac{3L}{U^3} \int_0^\infty dk \, T(k) \left[ \sin kL\rho - kL\rho \cos kL\rho \right] \frac{1}{(kL\rho)^3},$$

(43)
and
\[ A_2(\rho) = \frac{6\nu_0 L}{U^3} \int_0^\infty dk \kappa^2 E(k) \left[ \frac{\sin kL\rho - kL\rho \cos kL\rho}{(kL\rho)^3} \right]. \] (44)

In order to test our model for the dimensionless dissipation rate, we fitted an expression of the form (31), but with an arbitrary power-law dependence \( R_{\rho L}^p \), to data obtained with the present DNS, and it was found to agree very well, as shown in Fig. 6(a). The exponent was measured to be \( p = -1.00 \pm 0.02 \) and so supports the model equation, with the constants given by \( C_{\epsilon,\infty} = 0.47 \pm 0.01 \) and \( C = 18.9 \pm 1.3 \).

A more graphic demonstration of this fact is given in Fig. 6(b). The standard procedure of using a log-log plot to identify power-law behaviour is unavailable in this case, due to the constant asymptote. For this reason, we subtracted the estimated asymptotic value, and plotted \( C_\epsilon - C_{\epsilon,\infty} \) against \( R_L \) on logarithmic scales. This allowed us to identify power-law behaviour consistent with \( R_{\rho L}^{-1} \). We also tested the effect of varying our estimate of the value of the asymptote \( C_{\epsilon,\infty} \). It can be seen that the results were insensitive to this at the lower Reynolds numbers, where the \( R_{\rho L}^{-1} \)-dependence is being tested. At higher \( R_L \), the viscous contribution represented by \( C/R_L \) becomes negligible and instead we become strongly dependent on the actual value of \( C_{\epsilon,\infty} \).

This model may be compared to other work in the literature. Sreenivasan\(^4\) compared experimental results for free decay to the expression for very low Reynolds numbers,
\[ C_\epsilon = \frac{15}{R_{\lambda}} \sqrt{\frac{\pi}{2}}. \] (45)

This used the isotropic relation \( \varepsilon = 15\nu_0 U^2/\lambda^2 \) (where \( \lambda \) is the Taylor microscale) and the approximation \( L/\lambda \simeq (\pi/2)^{1/2} \) \(^2\) \(^3\). Note that, while \( 15\sqrt{\pi/2} = 18.8 \), compared to \( C = 18.9 \pm 1.3 \) found in the present analysis, this expression involves \( R_{\lambda} \) rather than \( R_L \).

Later, Lohse\(^20\) used ‘variable range mean-field theory’ to find an expression for the dimensionless dissipation coefficient by matching small \( r \) and inertial range forms for the second-order structure function, and obtained
\[ C_\epsilon = C_{\epsilon,\infty} \sqrt{1 + \frac{5b^3}{4R_{\lambda}^2}}, \] (46)

where \( b = S_2(r)/(\varepsilon r)^{2/3} \) such that \( C_{\epsilon,\infty} = (h_2(1)/b)^{3/2} \). At low Reynolds numbers, the author reported \( C_\epsilon = 18/R_{\lambda} \). The asymptotic value was calculated by Pearson, Krogstad and van der Water\(^16\), who used \( h_2(1) \simeq 1.25 \) and \( b \simeq 2.05 \), to be \( C_{\epsilon,\infty} \simeq 0.48 \), which agrees with our result \( C_{\epsilon,\infty} = 0.47 \pm 0.01 \) within one standard error.
In an alternative approach, Doering and Foias used the longest lengthscale affected by forcing, $l$, to derive upper and lower bounds on $C_\varepsilon$,

$$\frac{4\pi^2}{\alpha^2 Re} \leq C_\varepsilon \leq \left( \frac{a}{Re} + b \right)$$

for constants $a, b$, where $Re = U l / \nu_0$ and $\alpha = L_{\text{box}} / l$. While the upper bound resembles the present model, it is important to note that where these authors have obtained an inequality, we have an equality. Furthermore, the inequality (47) does not predict a Reynolds number dependence. The constant $b$ is undetermined and hence any model $b' + a'R_L^\alpha$ for $\alpha < 0$ could satisfy the upper bound (47) as long as $b' \leq b^{44}$. By substitution of the relation $R_L = C_\varepsilon R_\lambda^2 / 15$ into (47) and solving for $C_\varepsilon$ a $R_\lambda$-form for the upper bound $A(1 + \sqrt{1 + (B/R_\lambda)^2})$ can be derived. This was fitted to data by Donzis, Sreenivasan and Yeung, with $A \simeq 0.2$ and $B \simeq 92$ giving reasonable agreement, such that $C_{\varepsilon,\infty} \simeq 0.4$.

Later still, Bos, Shao and Bertoglio employed the idea of a finite cascade time to relate the expressions for $C_\varepsilon$ in forced and decaying turbulence. Using a model spectrum, they then derived a form for $C_\varepsilon$ and found the asymptotic value $C_{\varepsilon,\infty} = 0.53$ with the Kolmogorov constant $C_K = 1.5$. Note that when we used their formula, with the value $C_K = 1.625$ instead (which is probably more representative), this led to $C_{\varepsilon,\infty} = 0.47$, as found in the present work. With a simplified model spectrum, the authors then showed how their expression reduced to $C_\varepsilon = 19/R_L$ for low Reynolds numbers (when $E(k) \sim k^4$ at low $k$) in agreement with $C = 18.9 \pm 1.3$ found here (within one standard error).

The expression for $A_2(\rho)/R_L \phi(\rho)$ given by (37) was fitted to the present DNS data to find $a$, $b$ and $\gamma$, as defined in (39). This also fixed the form for $A_3(\rho)/\phi(\rho)$, as given by (36). The fit was performed up to the integral scale, $\rho = 1$, as shown in Fig. 7(a) by the vertical dash-dot line, above which the simulations become less well resolved. Clearly, agreement is good for $\rho < 1$. Figure 7(b) then uses the measured function $\phi(\rho)$ to plot the equivalent fit to DNS data for $A_3$ and $A_2/R_L$. The scale dependence of $A_2$ and $A_3$ is, therefore, well modeled by our choice of profile function, $H(\rho)$. As a consequence, the scale dependence in equation (35) cancels out in such a way that $C_\varepsilon(R_L)$ can still be modeled using equation (31), despite finite forcing introducing the scale-dependent function $\phi(\rho)$ to the input term. Note that this procedure also shows that $C$ is not scale-dependent. The scale dependences of $A_3(\rho)$ and $A_2(\rho)$, which we have shown to cancel out, are both contained in $C$ in the limit of $\delta(k)$ forcing. Results for $R_\lambda = 177$ and $R_\lambda = 335$ can be found in Chapter 6.3 of Ref. 29.
VI. CONCLUSIONS

We have presented a new form of the KHE for forced turbulence which differs from the one commonly found in the literature. In deriving this equation from the Lin equation, we have obtained a scale-dependent energy input term (7). Our new form of the general KHE, equation (5), correctly reduces to the well-known form for decaying turbulence.

By scaling the forced KHE into a dimensionless form (18), we see that an appropriate Reynolds number for studying the variation of the dimensionless dissipation, $C_\varepsilon$, is that corresponding to the integral scale, $R_L$. In the limit of $\delta(k)$-forcing, or for scales well below the influence of any forcing, the dimensionless equation has been used to derive the simple model (31) for the balance of inertial and viscous contributions to the dimensionless dissipation rate by asymptotic expansions of the dimensionless structure functions in inverse powers of $R_L$. The new model has been fitted to the present DNS data with very good agreement. It also shows that the behaviour of the dimensionless dissipation rate, as found experimentally, is entirely in accord with the Kolmogorov (K41) picture of turbulence and, in particular, with Kolmogorov’s derivation of his four-fifths law^38, the one universally accepted result in turbulence.

The individual scale dependences of the inertial and viscous terms, $A_3$ and $A_2$, have been modeled by a single profile function $H(\rho)$ for general forcing in such a way that they cancel out. This profile function has subsequently been fitted to DNS data in order to verify numerically that the scale dependences of $A_3$ and $A_2$ do indeed cancel out. This underpins the applicability of the model given by (31) to general, finite forcing.

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FIG. 6. (a) The expression given in equation (31) fitted to present DNS data resulting in $C_{\varepsilon,\infty} = 0.47$ and $C = 18.9$. (b) Log-log plot of the present DNS results for $C_{\varepsilon}$ against Reynolds number, once the estimate of the asymptote is subtracted. The solid line represents a slope of $-1.00$. The effect of varying our estimate of the asymptote $C_{\varepsilon,\infty}$ is shown by the three different symbols.
FIG. 7. (a) The fit for $A_2(\rho)/R_L \phi(\rho)$, as given by equation (37), to present DNS data. This also determines $A_3(\rho)/\phi(\rho)$; see equation (36). The fit was performed for the region $\rho \leq 1$. (b) The equivalent fit for $A_3(\rho)$ and $A_2(\rho)/R_L$ using the measured function $\phi(\rho)$, showing that the scale dependences of $A_2$ and $A_3$ cancel out independently of the additional scale dependence introduced by finite forcing. In both parts, $C_\varepsilon$ is indicated by the horizontal dotted line.
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