Construction of positivity preserving numerical schemes for multidimensional stochastic differential equations

Nikolaos Halidias
Department of Mathematics
University of the Aegean
Karlovassi 83200 Samos, Greece
email: nikoshalidias@hotmail.com

October 10, 2013

Abstract

In this note we work on the construction of positive preserving numerical schemes for systems of stochastic differential equations. We use the semi discrete idea that we have proposed before proposing now a numerical scheme that preserves positivity on multidimensional stochastic differential equations converging strongly in the mean square sense to the true solution.

Keywords: Explicit numerical scheme, multidimensional super linear stochastic differential equations, positivity preserving.

AMS subject classification: 60H10, 60H35.

1 Introduction

Throughout, let \( T > 0 \) and \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P}) \) be a complete probability space, meaning that the filtration \( \{\mathcal{F}_t\}_{0 \leq t \leq T} \) satisfies the usual conditions, i.e. is right continuous and \( \mathcal{F}_0 \) includes all \( \mathbb{P} \)-null sets, and let an \( m \)-dimensional Wiener process \( W(t) \) defined on this space.

Let the following multidimensional stochastic differential equation,

\[
x(t) = x(0) + \int_0^t a(x(s))ds + \sum_{j=1}^m \int_0^t b_j(x(s))dW_j(s),
\]

where \( a(\cdot) : \mathbb{R}^d \to \mathbb{R}^d \), \( b_j(\cdot) : \mathbb{R}^d \to \mathbb{R}^d \), \( j = 1, \ldots, m \), \( x(0) = x \) and \( x : \Omega \to \mathbb{R}^d \) is \( \mathcal{F}_0 \) measurable. That is, \( a(x) = (a_1(x), \ldots, a_d(x)) \), \( b_j(x) = (b_{1j}, \ldots, b_{dj}) \) where \( j = 1, \ldots, m \).

**Assumption A** Suppose that there exists some functions \( f_i(x, y) : \mathbb{R}^{2d} \to \mathbb{R} \) and \( g_{ij}(x, y) : \mathbb{R}^{2d} \to \mathbb{R} \) such that \( f_i(x, x) = a_i(x) \) and \( g_{ij}(x, x) = b_{ij}(x) \) with \( x = (x_1, \ldots, x_d) \), \( y = (y_1, \ldots, y_d) \), \( i = 1, \ldots, d \) and \( j = 1, \ldots, m \). Let \( \mathbb{E}\|x\|_p^2 < A \) for some \( p > 2 \). Suppose further that \( f, g \) satisfy the following condition

\[
|f_i(x_1, y_1) - f_i(x_2, y_2)| + |g_{ij}(x_1, y_1) - g_{ij}(x_2, y_2)| \leq C_R (\|x_1 - x_2\|_2 + \|y_1 - y_2\|_2),
\]
for any \( R > 0, i = 1, \ldots, d \) and \( x_1, x_2, y_1, y_2 \) such that \( \|x_1\|_2 \vee \|x_2\|_2 \vee \|y_1\|_2 \vee \|y_2\|_2 \leq R \), where the constant \( C_R \) depends on \( R \) and \( x \vee y \) denotes the maximum of \( x, y \). Here

\[
\|x\|_p = \sqrt[p]{\sum_{i=1}^{d} x_i^p}
\]

where \( x = (x_1, \ldots, x_d) \).

Let the equidistant partition \( 0 = t_0 < t_1 < \ldots < t_N = T \) and \( \Delta = T/N \). We propose the following numerical scheme,

\[
y_i(t) = x_i + \int_{t_{i-1}}^{t} f_i(y(s), y(\hat{s})) ds + \sum_{j=1}^{m} \int_{t_{i-1}}^{t} g_{ij}(y(s), y(\hat{s})) dW_j(s).
\]

(2)

for \( i = 1, \ldots, d \). Note that \( y(s) = (y_1(s), \ldots, y_d(s)) \) and \( y(\hat{s}) = (y_1(\hat{s}), \ldots, y_d(\hat{s})) \), with \( y_i(\hat{s}) = y_i(t_k) \) when \( s \in [t_k, t_{k+1}] \) and \( y_i(0) = x_i \). This is again, in general, a system of stochastic differential equations and we suppose that has a unique strong solution. However, in practice, we will choose \( f, g \) in a way that the resulting system has less than \( d \) dependent equations and/or having known explicit solution. An interesting choice that we will see in our example is that the resulting system is not in fact a system of SDEs but \( d \) independent equations. Each of these equations is linear with known explicit solution and we solve it independently of the others.

Below we state a convergence, in the mean square sense, theorem of \( y_t \) to the true solution as \( \Delta \downarrow 0 \). The proof of this theorem is exactly the same as in [1], changing the absolute values by the Euclidean norm in \( \mathbb{R}^d \).

**Theorem 1** Suppose Assumption A holds and (2) has a unique strong solution. Let also

\[
E(\sup_{0 \leq t \leq T} \|x_t\|_2^p) \vee E(\sup_{0 \leq t \leq T} \|y_t\|_2^p) < A,
\]

for some \( p > 2 \) and \( A > 0 \). Then the semi-discrete numerical scheme (2) converges to the true solution of (1) in the mean square sense, that is

\[
\lim_{\Delta \to 0} E \sup_{0 \leq t \leq T} \|y_t - x_t\|_2^2 = 0.
\]

(3)

**Proof.** We use the same arguments as in [1], [2]. Set \( \rho_R = \inf \{ t \in [0, T] : \|x(t)\|_2 \geq R \} \) and \( \tau_R = \inf \{ t \in [0, T] : \|y(t)\|_2 \geq R \} \). Let \( \theta_R = \min \{ \tau_R, \rho_R \} \). Using exactly the same arguments as in [1] we obtain,

\[
E \left( \sup_{0 \leq t \leq T} \|y(t) - x(t)\|_2^2 \right) \leq E \left( \sup_{0 \leq t \leq T} \|y(t \wedge \theta_R) - x(t \wedge \theta_R)\|_2^2 \right) + \frac{2^{p+1} \delta A}{p} + \frac{(p-2)2A}{p\delta^{p-2}R^p}.
\]

First, let us estimate the quantity \( E \|y(t \wedge \theta) - y(\hat{t} \wedge \theta)\|_2^2 \), beginning with,

\[
|y_i(t \wedge \theta) - y_i(\hat{t} \wedge \theta)|^2 \leq \int_{t \wedge \theta}^{\hat{t} \wedge \theta} f_i(y_s, y_s) ds + \sum_{j=1}^{m} \int_{t \wedge \theta}^{\hat{t} \wedge \theta} g_{ij}(y_s, y_s) dW_j(s) \leq
\]

\[
C \left( \left( \int_{t \wedge \theta}^{\hat{t} \wedge \theta} f_i(y_s, y_s) ds \right)^2 + \sum_{j=1}^{m} \left| \int_{t \wedge \theta}^{\hat{t} \wedge \theta} g_{ij}(y_s, y_s) dW_j(s) \right|^2 \right).
\]
Taking expectations, using Ito’s isometry and the fact that $|f_i(y_s, y_s)|, |g_{ij}(y_s, y_s)| \leq C_R$ we have that,

$$\mathbb{E}|y_i(t \wedge \theta) - y_i(\tilde{t} \wedge \theta)|^2 \leq C_R \Delta,$$

and from this it follows that

$$\mathbb{E}\|y(t \wedge \theta) - y(\tilde{t} \wedge \theta)\|_2^2 \leq C_R \Delta,$$

Next we work on the quantity $\mathbb{E}\left(\sup_{0 \leq t \leq T} \|y(t \wedge \theta_R) - x(t \wedge \theta_R)\|_2^2\right)$ to get

$$|x_i(t \wedge \theta_R) - y_i(t \wedge \theta_R)|^2 =$$

$$\left|\int_0^{t \wedge \theta_R} (f_i(x_s, x_s) - f_i(y_s, y_s))ds + \sum_{j=1}^m \int_0^{t \wedge \theta_R} (g_{ij}(x_s, x_s) - g_{ij}(y_s, y_s))dW_j(s)\right|^2$$

$$\leq C \left(\int_0^{t \wedge \theta_R} |f_i(x_s, x_s) - f_i(y_s, y_s)|^2ds + \sum_{j=1}^m \int_0^{t \wedge \theta_R} |g_{ij}(x_s, x_s) - g_{ij}(y_s, y_s)|dW_j(s)\right)^2$$

$$\leq C_R \int_0^{t \wedge \theta_R} (\|x_s - y_s\|_2^2 + \|x_s - y_s\|_2^2)ds + C \sum_{j=1}^m \int_0^{t \wedge \theta_R} |g_{ij}(x_s, x_s) - g_{ij}(y_s, y_s)|dW_j(s)\right|^2$$

We can write now,

$$\sup_{0 \leq t \leq s} |x_i(t \wedge \theta_R) - y_i(t \wedge \theta_R)|^2$$

$$\leq C_R \int_0^s (\|x(r \wedge \theta_R) - y(r \wedge \theta_R)\|_2^2 + \|x(r \wedge \theta_R) - y(r \wedge \theta_R)\|_2^2) dr$$

$$+ C \sum_{j=1}^m \sup_{0 \leq t \leq s} \left|\int_0^{t \wedge \theta_R} (g_{ij}(x_s, x_s) - g_{ij}(y_s, y_s))dW_j(s)\right|^2.$$
2 Example

Consider the following multidimensional SDE.

\[ x(t) = x + \int_0^t \left( x(s) - \|x(s)\|^2 x(s) \right) ds + \int_0^t x(s)dW(s). \]

Assumption B Assume that \( \mathbb{E}|\ln x_i| + \mathbb{E}x_i^{2p} < C \) for all \( i = 1, \ldots, d \) for some \( p > 2 \) and \( C > 0 \).

Consider the following system of SDEs,

\[ y_i(t) = x_i + \int_0^t \left( y_i(s) - \|y(s)\|^2 y_i(s) \right) ds + \int_0^t y_i(s)dW(s). \]

Each of the above equation has only one unknown function and is linear with known explicit solution and thus preserves positivity.

Theorem 2 Under Assumption B we have

\[ \mathbb{E}\left( \sup_{0 \leq t \leq T} \|x_t\|_2^2 \right) < A, \quad \mathbb{E}\left( \sup_{0 \leq t \leq T} \|y_t\|_2^2 \right) < A \]

for some \( A > 0 \).

Proof. For the moment bound of the true solution we use Lemma 3.2 of [4] since the drift coefficient satisfies the monotonicity condition and the diffusion term the linear growth condition.

We will prove now that the approximate solution has bounded moments. Set the stopping time \( \theta_R = \inf \{ t \in [0, T] : y_i(t) > R \} \). Using Ito’s formula on \( y_i^q(t \wedge \theta_R) \) (with \( q = 2p \)) we obtain

\[ y_i^q(t \wedge \theta_R) = x_i^q + \int_0^{t \wedge \theta_R} y_i^q(s) \left( q - \|y(s)\|^2 + \frac{q(q-1)}{2} \right) ds + \int_0^{t \wedge \theta_R} qy_i^q(s)dW(s). \]

Taking expectations we arrive at the following inequality,

\[ \mathbb{E}y_i^q(t \wedge \theta_R) \leq \mathbb{E}x_i^q + \left( q + \frac{q(q-1)}{2} \right) \int_0^{t \wedge \theta_R} \mathbb{E}y_i^q(s)ds. \]

Using now Gronwall’s inequality we get that \( \mathbb{E}y_i^q(t \wedge \theta_R) < A \) and using Fatou’s lemma we arrive at the bound \( \mathbb{E}y_i^q(t) < A \) for \( i = 1, \ldots, d \). Therefore, we have that \( \mathbb{E}(\|y_t\|_q^2) < A \).

Using again Ito’s formula on \( y_i^p(t) \) we obtain,

\[ y_i^p(t) \leq x_i^p + \left( p + \frac{p(p-1)}{p} \right) \int_0^t y_i^p(s)ds + \int_0^t py_i^p(s)dW(s). \]

Taking the supremum over \([0, T]\) we have,

\[ \sup_{0 \leq t \leq T} y_i^p(t) \leq C \left( x_i^p + \int_0^T y_i^p(s)ds + \sup_{0 \leq t \leq T} \int_0^t y_i^p(s)dW(s) \right). \]
Theorem 3 For the true solution we have $x_i(t) > 0$ a.s. for $i = 1, \ldots, d$

**Proof.** Set the stopping time $\theta_R = \inf\{t \in [0, T] : x_i(t) < \frac{1}{R}\}$. Using Ito’s formula on $\ln x_i(t \wedge \theta_R)$ we obtain

$$
\ln x_i(t \wedge \theta_R) = \ln x_i + \int_0^{t \wedge \theta_R} \frac{1}{x_i(s)} (x_i(s) - \|x(s)\|^2 x_i(s)) - \frac{1}{2} x_i^2(s) ds + \int_0^{t \wedge \theta_R} \frac{1}{x_i(s)} x_i(s) dW(s).
$$

Taking absolute values and then expectations, using Jensen inequality and then Ito’s isometry on the diffusion term we arrive at

$$
\mathbb{E} |\ln x_i(t \wedge \theta_R)| \leq \mathbb{E} |\ln x_i| + \int_0^{t \wedge \theta_R} (1 + \mathbb{E}\|x(s)\|^2) ds + \sqrt{T} \leq C.
$$

We have used also the moment bound for the true solution (see Theorem 2). Therefore

$$
\ln R \ P(\{\theta_R \leq R\}) < C,
$$

and thus $P(\{\theta_R \leq R\}) \to 0$ as $R \to \infty$. But

$$
P(\{x_i(t) \leq 0\}) = P\left(\bigcap_{R=1}^{\infty} \{x_i(t) < \frac{1}{R}\}\right) = \lim_{R \to \infty} P\left(\{x_i(t) < \frac{1}{R}\}\right) \leq \lim_{R \to \infty} P(\{\theta_R < t\}) = 0.
$$

In this example we have $a(x) = x - \|x\|^2 x$ and $b(x) = x$ with $x = (x_1, \ldots, x_d)$, that is $m = 1$. We choose $f_i(x, y) = x_i - \|y\|^2 x_i$ and $g_i(x, y) = x_i$ where $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$. It is easy to see that $f_i(x, x) = a(x)$ and $g_i(x, x) = b(x)$. Moreover, $f_i, g_i$ satisfies Assumption A. Therefore, our proposed numerical scheme preserves positivity and converges strongly in the mean square sense to the true solution.

**Conclusion** Concerning super linear SDEs it is well known that the usual Euler scheme diverges and the tamed-Euler scheme [5] does not preserve positivity. For scalar SDEs there are some numerical schemes that preserves positivity (see for example [1], [2], [3], [6] and the references therein) but for the multidimensional case it is not clear how can be extended. So we extend here our semi discrete method ([4]), that we have proposed before for scalar SDEs, to the multidimensional case. There is also the possibility in some multidimensional SDEs to combine the semi discrete method with another method designed for scalar SDEs in order to construct positivity preserving numerical schemes. Let us note that we can use the semi discrete method also in the case when the diffusion term is super linear. Our goal in the future is to apply the semi discrete method to more complicated systems of SDEs.
References

[1] Halidias, N. (2013). A novel approach to construct numerical methods for stochastic differential equations. *Numer Algor.*, to appear.

[2] Higham, D.J., Mao, X., Szpruch, L. (2012). Convergence, Non-negativity and Stability of a New Milstein Scheme with Applications to Finance. *preprint*.

[3] Liu, W., Mao X. *Strong convergence of the stopped Euler-Maruyama method for nonlinear stochastic differential equations*, Applied Mathematics and Computation 223 (2013), 389-400.

[4] D. Higham, X. Mao, A. Stuart, *Strong convergence of Euler-type methods for nonlinear stochastic differential equations*, SIAM J. Numer. Anal, vol. 40, (2002), pp. 1041-1063.

[5] M. Hutzenthaler, A. Jentzen, P. E. Kloeden, *Strong convergence of an explicit numerical method for SDEs with nonglobally Lipschitz continuous coefficients*, Ann. App. Probab. Volume 22, Number 4 (2012), 1611-1641.

[6] Neuenkirch, A., Szpruch, L. (2012). First order strong approximations of scalar SDEs with values in a domain. *preprint*.