On Quantum Special Kähler Geometry

Stefano Bellucci♣, Alessio Marrani▼ and Raju Roychowdhury♠

♣ INFN - Laboratori Nazionali di Frascati,
Via Enrico Fermi 40,00044 Frascati, Italy
bellucci@lnf.infn.it

▼ Stanford Institute for Theoretical Physics
Department of Physics, 382 Via Pueblo Mall, Varian Lab,
Stanford University, Stanford, CA 94305-4060, USA
marrani@lnf.infn.it

♠ Dipartimento di Scienze Fisiche, Federico II University,
Complesso Universitario di Monte S. Angelo,
Via Cintia, Ed. 6, I-80126 Napoli, Italy
raju@na.infn.it

Abstract

We compute the effective black hole potential $V_{BH}$ of the most general $\mathcal{N} = 2,d = 4$ (local) special Kähler geometry with quantum perturbative corrections, consistent with axion-shift Peccei-Quinn symmetry and with cubic leading order behavior.

We determine the charge configurations supporting axion-free attractors, and explain the differences among various configurations in relations to the presence of “flat” directions of $V_{BH}$ at its critical points.

Furthermore, we elucidate the role of the sectional curvature at the non-supersymmetric critical points of $V_{BH}$, and compute the Riemann tensor (and related quantities), as well as the so-called $E$-tensor. The latter expresses the non-symmetricity of the considered quantum perturbative special Kähler geometry.
1 Introduction

The Attractor Mechanism was discovered in the mid 90’s [1]-[5] in the context of dynamics of scalar fields coupled to BPS (Bogomol’ny-Prasad-Sommerfeld [6]) black holes (BHs). In recent years, a number of studies (see e.g. [7]-[11] for recent reviews, and lists of Refs., see also [12]) have been devoted to the investigation of the properties of extremal BH attractors. This renewed interest can be essentially be traced back to the (re)discovery of new classes configurations of scalar fields at the BH horizon, which do not saturate the BPS bound. When embedded into a supergravity theory, such non-BPS configurations break all supersymmetries at the BH event horizon.

The geometry of the scalar manifold determines the various classes of BPS and non-BPS attractors. The richest case study is provided by the theory in which the Attractor Mechanism was originally discovered, namely by $\mathcal{N} = 2$, $d = 4$ ungauged supergravity coupled to $n_V$ Abelian vector multiplets. In such a theory, the scalar fields coordinatize a Kähler manifold of (local) special type (see e.g. [13], [14], and [15], and Refs. therein), determined by an holomorphic prepotential function $\mathcal{F}$. In general, (local, as understood throughout unless otherwise noted) special Kähler (SK) geometry admits three classes of extremal BH attractors (see e.g. [16] for an analysis in symmetric SK geometry):

- $\frac{1}{2}$-BPS (preserving four supersymmetries out of the eight pertaining to asymptotical $\mathcal{N} = 2$, $d = 4$ superPoincaré algebra);
• non-BPS with non-vanishing $\mathcal{N} = 2$ central charge function $Z$ (shortly named non-BPS $Z \neq 0$);
• non-BPS with vanishing $Z$ (shortly named non-BPS $Z = 0$).

1.1 Quantum Corrections to Prepotential

Dealing with the stringy origin of $\mathcal{N} = 2$, $d = 4$ supergravity, the classical prepotential $F$ receives quantum (perturbative and non-perturbative) corrections, of polynomials or non-polynomial (usually polylogarithmic) nature, which in some cases can spoil the holomorphicity of $F$ itself (see e.g. [17]-[30]).

A typical (and simple) example is provided by the large volume limit of $CY_3$-compactifications of Type IIA superstrings, which determines a SK geometry with purely cubic $F$ at the classical level. Thus, the sub-leading nature of the quantum corrections constrains the most general polynomial correction to $F$ to be at most of degree two in the moduli, with a priori complex coefficients. Moreover, some symmetries can further constrain the structure of such sub-leading polynomial quantum corrections to $F$. As shown in [31], the only polynomial quantum perturbative correction to classical cubic $F$ which is consistent with the perturbative (continuous) axion-shift symmetry [32] is the constant purely imaginary term ($i = 1, ..., n_V$ throughout):

$$F_{\text{class}} = \frac{1}{3!} d_{ijk} z^i z^j z^k \longrightarrow F_{\text{quant-pert.}} = \frac{1}{3!} d_{ijk} t^i t^j t^k + i\xi, \quad \xi \in \mathbb{R},$$ (1.1)

where $d_{ijk}$ is the real, constant, completely symmetric tensor defining the cubic geometry (which is then usually named $d$-SK geometry [33][34]). All other polynomial perturbative corrections (quadratic, linear and real constant terms in the scalar fields $z^i$’s) can be proved not to affect the classical $d$-SK geometry, also because the Kähler potential is insensitive to their presence [31].

The explicit form of the quantum corrections to $F$ depends on the superstring theory under consideration, and non-trivial relations among the various corrections arise due to the (perturbative and non-perturbative) dualities relating the various superstring theories.

For instance, in a certain class of compactifications of the heterotic $E_8 \times E_8$ superstring over $K_3 \times T^2$, the whole quantum-corrected $F$ reads (see e.g. [18], and Refs. therein)

$$F^{\text{het}} = stu - s \sum_{a=4}^{n_V} (\tilde{t}^a)^2 + h_1(t, u, \tilde{t}) + f_{\text{non-pert.}}(e^{-2\pi s}, t, u, \tilde{t}),$$

$$z^1 \equiv s, \quad z^2 \equiv t, \quad z^3 \equiv u, \quad z^a \equiv \tilde{t}^a.$$ (1.2)

This compactifications exhibits the peculiar feature that the axio-dilaton $s$ belongs to a vector multiplet, and this determines the presence of ($T$-symmetric) quantum perturbative string-loop corrections and non-perturbative corrections, as well. The tree-level, classical term

$$F^{\text{het}}_{\text{class}} = stu - s \sum_{a=4}^{n_V} (\tilde{t}^a)^2$$ (1.3)

is the prepotential of the so-called generic Jordan sequence

$$\frac{SU(1,1)}{U(1)} \times \frac{SO(2, n_V - 1)}{SO(2) \times SO(n_V - 1)}$$ (1.4)
of homogeneous symmetric SK manifolds (see e.g. [31] and [16, 11], and Refs. therein). Notice that $F_{\text{class}}$ given by Eq. (1.3) exhibits its maximal (non-compact) symmetry, namely $SO(1, n_V - 2)$, pertaining to its $d = 5$ uplift. Non-renormalization theorems state that all quantum perturbative string-loop corrections are encoded in the 1-loop contribution $h_1$, made out of a constant term, a purely cubic polynomial term and a polylogarithmic part (see e.g. [18] and Refs. therein). Finally, $f_{\text{non-pert.}}$ encodes the non-perturbative corrections, exponentially suppressed in the limit $s \to \infty$ (see e.g. [35, 18, 21, 30]).

As mentioned above, superstring dualities play a key role in relating the quantum corrected $F$’s in various theories. In the considered framework, the Type IIA/heterotic duality allows for an identifications of the relevant scalar fields (moduli of the geometry of the internal manifold in stringy language) such that the heterotic prepotential (1.2) becomes structurally identical to the one determined by Type IIA compactifications over Calabi-Yau threefolds ($CY_3$). Within this latter framework, the $F$ governing the resulting low-energy $N = 2$, $d = 4$ supergravity is of purely classical origin. Indeed, there are only Kähler structure moduli, and the axio-dilaton $s$ belongs to an hypermultiplet; this leads to no string-loop corrections, and all corrections to the large volume limit cubic prepotential come from the world-sheet sigma-model [36]. In particular, as shown in [37, 35], there are no 1-, 2- and 3-loop contributions. It is here worth pointing out that the non-perturbative, world-sheet instanton corrections (which we will disregard in the treatment below) spoil the continuous nature of the axion-shift symmetry, by making it discrete [32].

Thus, the relevant part of the prepotential $F$ in Type IIA compactifications reads ($n_V = h_{1,1}$)

$$F_{\text{IIA}} = \frac{1}{3!} C_{ijk} t^i t^j t^k + W_0 t^i - i \chi \frac{\zeta(3)}{16\pi^3}. \quad (1.5)$$

The $C_{ijk}$ are the real classical intersection numbers, determining the classical $d$-SK geometry in the large volume limit. On the other hand, the quantum perturbative contributions from 2-dimensional CFT on the world-sheet are encoded only in a linear and in a constant term:

- the linear term is determined by
  $$W_0 t^i = \frac{1}{4!} c_2 : J_i = \frac{1}{4!} \int_{CY_3} c_2 \wedge J_i, \quad (1.6)$$
  which are the real expansion coefficients of the second Chern class $c_2$ of $CY_3$ with respect to the basis $J_i$ of the cohomology group $H^4 (CY_3, \mathbb{R})$, dual to the basis of the $(1, 1)$-forms $J_i$ of the cohomology $H^2 (CY_3, \mathbb{R})$. The linear term $W_0 t^i$ has been shown to be reabsorbed by a suitable symplectic transformation of the period vector; thus, in the dual heterotic picture it has just the effect of a constant shift in $\text{Im} s$ [39]; see e.g. also discussion in [18].

- The constant term $-i \frac{\chi \zeta(3)}{16\pi^3} (\equiv i \xi$ in Eq. (1.1)) in (1.5) is the only relevant one, as proved in general in [31]. It is determined by the Riemann zeta-function $\zeta$, by the Euler character $\chi$ of $CY_3$, and it has a 4-loop origin in the non-linear sigma-model [36, 37, 35]. It is worth noticing that $\chi = 0$ for self-mirror $CY_3$’s, such that all have $\xi = 0$. Furthermore, some

\[ |\chi| \lesssim 10^3 \Leftrightarrow \frac{\chi \zeta(3)}{16\pi^3} \sim \mathcal{O}(1). \]

This motivates the statement that attractor solutions with $\xi = 0$ can be (in certain BH charge configurations) a good approximation for the solutions computed with $\xi \neq 0$ (see e.g. the remark after Eq. (3.42) of [18]).
arguments lead to argue that (at least) for some particular self-mirror models (such as the so-called FHSV one \cite{40} and the octonionic magic \cite{41}), non-perturbative, world-sheet instanton corrections vanish, as well (see e.g. discussion in Sects. 12 and 13 of \cite{42}, and Refs. therein). As a consequence, such models, up to suitable symplectic transformations of the period vector, would have their classical cubic prepotential unaffected by any perturbative and non-perturbative correction.

It should be here pointed out that CY3-compactifications of Type IIB do not admit a large volume limit; moreover, in Type IIB the Attractor Eqs. only depend on the complex structure moduli (which in supergravity description are the scalars of the $\mathcal{N} = 2$ vector multiplets). The solutions to $\mathcal{N} = 2$, $d = 4$ Attractor Eqs. for the resulting SK geometries were studied (in proximity of the Landau-Ginzburg point) in \cite{43} for the particular class of Fermat CY3’s with $n_V = 1$, and in \cite{44} for a particular CY3 with $n_V = 2$.

In \cite{45}, extending the BPS analysis of \cite{18}, the $\mathcal{N} = 2$, $d = 4$ Attractor Eqs. were studied in the simplest case of perturbative quantum corrected $d$-SK geometry, namely in the SK geometry with $n_V = 1$ scalar fields, described (in a special coordinates) by the holomorphic Kähler gauge-invariant prepotential $\mathcal{F}$,

$$\mathcal{F} = t^3 + i\xi, \quad (1.7)$$

which, up to overall rescaling, is nothing but $\mathcal{F}_{\text{quant-pert.}}$ of Eq. (1.1) for $n_V = 1$. Despite the (apparently) minor correction to the classical prepotential, in \cite{45} new phenomena, absent in the classical limit $\xi = 0$, were observed:

- The “separation” of attractors, namely the existence of multiple stable solutions to the Attractor Eqs. (for a given BH charge configuration). This can be ultimately related to the existence of basins of attractions \cite{46,47,48} (i.e. asymptotical boundary conditions) in the radial evolution dynamics of scalar fields in the extremal BH background.

- The “transmutation” of attractors, namely the change in the supersymmetry preserving features of stable critical points of $V_{BH}$, depending on the value of the quantum parameter $\xi$, suitably “renormalized” in terms of the relevant BH charges. For example, by varying such a “renormalized” quantum parameter, a $\frac{1}{2}$-BPS attractor becomes non-BPS (and vice versa). This can ultimately be related to the lack of an orbit structure in the space of BH charges; this is no surprise, by noticing that the SK geometry determined by $\mathcal{F}$ given by Eq. (1.7) is generally not symmetric nor homogeneous.

1.2 Critical “Flat” Directions of Black Hole Potential

Let us now shortly recall the fundamentals of the Attractor Mechanism. In the critical implementation given in \cite{5}, the Attractor Mechanism related extremal BH attractors to stable critical points $z_H^i$ of a suitably defined BH effective potential $V_{BH}$:

$$z_H (Q) : \left. \frac{\partial V_{BH} (z, Q)}{\partial z^i} \right|_{z=z_H (Q)} = 0 \quad (1.8)$$

where $Q$ denotes the $Sp(2n_V + 2, \mathbb{R})$-vector of magnetic and electric BH charges (see Eq. (2.9) below). The $n_V$ complex Eqs. (1.8) are usually called Attractor Equations. Then, a critical

\[2\] In \cite{45} $\xi$ was named $\lambda$.

\[3\] The subscript “$H$” denotes the value at the event horizon of the considered extremal BH.
point $z_H(Q)$ is an attractor in strict sense iff the (Hermitian) $2n_V \times 2n_V$ Hessian matrix $\mathcal{H}^{V_{BH}}$ of $V_{BH}$ evaluated at $z_H(Q)$ is positive definite:

$$\mathcal{H}^{V_{BH}} \bigg|_{z=z_H(Q)} \geq 0,$$

(1.9)

with “$\geq 0$” here expressing the non-negativity of the $2n_V$ eigenvalues.

As shown in [50, 51], in $N = 2$, $d = 4$ ungauged supergravities with homogeneous (not necessarily symmetric) SK manifolds (as well as in $N > 2$-extended ungauged $d = 4$ supergravities, which we however do not consider here) the critical matrix $\mathcal{H}^{V_{BH}} \big|_{\partial V_{BH}=0}$ has the following general signature: all strictly positive eigenvalues, up to some eventual vanishing eigenvalues (massless Hessian modes), which have been proved to be “flat” directions of $V_{BH}$ itself.

Thus, one can claim that in all homogeneous SK geometries the critical points of $V_{BH}$ satisfying the “non-degeneracy” condition

$$V_{BH} |_{\partial V_{BH}=0} \neq 0$$

(1.10)

are all stable, up to some eventual “flat” directions. Such directions of the SK scalar manifold $\mathcal{M}_{SK}$ coordinatize the so-called moduli space $\mathfrak{M} \subseteq \mathcal{M}$ of the considered (class of) solution(s) to Eqs. (1.8). In other words, such “flat” directions span a subset of the scalar fields which is not stabilized by the Attractor Eqs. (1.8) at the BH event horizon in terms of the BH charges $Q$. It is worth pointing out that, somewhat surprisingly, the existence of “flat” directions at the critical points of $V_{BH}$ does not plague the thermodynamical macroscopic description of extremal BHs with inconsistencies. Indeed, at the considered class of critical points, $V_{BH}$ does not actually turn out to depend on the unstabilized scalars; therefore, through the relation [5]

$$S_{BH}(Q) = \pi V_{BH} |_{\partial V_{BH}=0},$$

(1.11)

the BH entropy $S_{BH}$ can be consistently defined. Notice that the condition (1.10) implies the (classical) Attractor Mechanism\footnote{Attractor Mechanism can be consistently implemented at the quantum level, at least in some frameworks, for instance within the so-called entropy function formalism (see e.g. [8] and Refs. therein, see also [12]). See also [52] (and Refs. therein) for recent developments concerning Attractor Mechanism and higher derivatives corrections to Einstein (super)gravity theories.} to work only for the so-called “large” BHS, i.e. for those BHs with non-vanishing classical entropy.

As known since [5], “flat” directions cannot arise at $\frac{1}{2}$BPS critical points of $V_{BH}$. This is no more true for the remaining two classes of non-supersymmetric critical points, namely for non-BPS $Z \neq 0$ and non-BPS $Z = 0$ ones [50, 51]. Tables 2 and 3 of [51] respectively list the moduli spaces of non-BPS $Z \neq 0$ and non-BPS $Z = 0$ attractors for symmetric SK geometries, whose classification is known after [53] (see also [34] and [31], as well as Refs. therein). Let us mention that non-BPS $Z \neq 0$ moduli spaces are nothing but the symmetric real special scalar manifolds of the corresponding $N = 2, d = 5$ supergravity.

1.3 Quantum Removal/Survival of Critical “Flat” Directions

It should be pointed out clearly that the issue of the “flat” directions of $V_{BH}$ at its critical points, reported in Subsect. 1.2 hold only at the classical, Einstein supergravity level. It is conceivable that such “flat” directions are removed by quantum (perturbative and/or non-perturbative) corrections. Consequently, at the quantum (perturbative and/or non-perturbative) regime, no moduli spaces for attractor solutions might exist at all (and also the actual attractive nature of
the critical points of $V_{BH}$ might be destroyed). However, this might not be the case for $N = 8$, or for some particular charge configurations in $N < 8$ supergravities (see below).

By relating the issues reported in Subsects. 1.1 and 1.2, one might thus ask about the fate of classical “flat” directions of $V_{BH}$ at its (non-BPS) critical points, in presence of quantum (perturbative and/or non-perturbative) corrections to the prepotential $F$ of SK geometry.

This issue, crucial in order to understand the features of the Attractor Mechanism in the quantum regime (and thus its consistent embedding in the high-energy theories whose supergravity is an effective low-energy limit, namely superstrings and $M$-theory), was started to be investigated in [54], and it is the object of the investigation carried out in the present paper.

Let us start by recalling the simplest symmetric $d$-SK geometries, and their eventual non-BPS “flat” directions. For our purpose, it will suffice to consider only the so-called $t^3$ and $st^2$ models:

- The $t^3$ model is based on the rank-1 symmetric SK manifold

$$\frac{SU(1,1)}{U(1)},$$

endowed with prepotential ($z^1 \equiv t$, $I m t < 0$)

$$F = t^3,$$

which is the classical limit $\xi \to 0$ of Eq. (1.7). As yielded by the analysis of [53], it is an isolated case in the classification of symmetric SK geometries (see also [31]). Furthermore, such a model can also be conceived as the “$s = t = u$ degeneration” of the so-called $stu$ model [53]-[62], or equivalently as the “$s = t$ degeneration” of the so-called $st^2$ model (see below). Beside the $1/2$-BPS attractors, the $t^3$ model (whose $d = 5$ uplift is pure $N = 2$, $d = 5$ supergravity) admits only non-BPS $Z \neq 0$ critical points of $V_{BH}$ with no “flat” directions (and thus no associated moduli space) [16, 51].

- The $st^2$ model is based on the rank-2 symmetric SK manifold

$$\left(\frac{SU(1,1)}{U(1)}\right)^2,$$

endowed with prepotential ($z^1 \equiv s$, $z^2 \equiv t$, $I m s < 0$, $I m t < 0$)

$$F = st^2.$$

It has one non-BPS $Z \neq 0$ “flat” direction, spanning the moduli space $SO(1,1)$ (namely, the scalar manifold of the $st^2$ model in $d = 5$), but no non-BPS $Z = 0$ “flat” directions at all. Such a model is the smallest (i.e. the fewest-moduli) symmetric model exhibiting a non-BPS $Z \neq 0$ “flat” direction. Remarkably, the $st^2$ model constitutes the unique example of homogeneous $d$-SK geometry with $n_V = 2$ scalar fields [34, 33]. Furthermore, as evident from the structure of the cubic norm (see e.g. the discussion in [34], as well as Eq. (3.2.3) and Sect. 5 of [63]), the $st^2$ model is the unique $n_V = 2$ SK geometry to be uplifted to anomaly-free pure $(1,0)$, $d = 6$ supergravity (at least in presence of neutral matter).

As mentioned at the end of Subsect. 1.1, the non-homogeneous model “$t^3 + i\xi$” (with prepotential given by Eq. (1.7)) was studied in [45]. The “$t^3 + i\xi$” model can be conceived as the
prototype of quantum perturbative corrected SK geometry, because it is the $n_V = 1$ SK geometry with the most general quantum perturbative correction consistent with the (continuous, perturbative) axion-shift symmetry [31]. However, since the $t^3$ model has no non-BPS “flat” directions at all, the study performed in [45] is not relevant for the aforementioned issue of the fate of the moduli spaces of classical attractors in the quantum regime.

From the above analysis, the $st^2$ model is the simplest example of SK geometry in which the study of the fate of classical non-BPS $Z \neq 0$ moduli space can be investigated in quantum perturbative regime, namely considering the “$st^2 + i \xi$” model, whose prepotential in special coordinates reads

$$F = st^2 + i \xi. \quad (1.16)$$

Notice that Eq. (1.16) is the unique homogeneous $n_V = 2$ determination of Eq. (1.1). Such a study was performed in [54], within the (supergravity analogues of the) so-called magnetic ($D_0 - D_4$), electric ($D_2 - D_6$) and $D_0 - D_6$ BH charge configurations. As somewhat intuitively expected, in the magnetic and electric configurations the classical non-BPS $Z \neq 0$ moduli space $SO(1,1)$ was shown not to survive after the introduction of the quantum parameter $\xi \neq 0$. Interestingly, the investigation of [54] showed the that the quantum removal of classical “flat” directions occurs more often towards repeller directions (thus destabilizing the whole critical solution, and destroying the attractor in strict sense), rather than towards attractive directions.

Surprisingly, the study of [54] also revealed that the $D_0 - D_6$ configuration exhibits a qualitatively different phenomenon, namely that the non-BPS $Z \neq 0$ classical “flat” direction survives the considered quantum perturbative corrections effectively encoded in the “$+ i \xi$” term in Eq. (1.16), despite acquiring a non-vanishing axionic part.

**Aim and Plan of Paper**

This unexpected fact was not completely understood in [54], and it is the starting point of the present investigation, which aims at thoroughly investigating, within the effective BH potential formalism, the $d$-SK geometries with the most general quantum perturbative correction consistent with continuous Peccei-Quinn axion-shift symmetry, namely the SK geometries with prepotential (in special coordinates) given by Eq. (1.1). As already found in the simple cases investigated in [45] ($n_V = 1$) and [54] ($n_V = 2$), the Attractor Eqs. (especially the non-supersymmetric ones) cannot be solved analytically for a generic BH charge configuration, because they turn out to be algebraic Eqs. of high ($> 4$) order. However, by explicitly computing $V_{BH}$ for the prepotential (1.1), we will explain the peculiarity of the $D_0 - D_6$ configuration as being, in presence of $\xi \neq 0$, somewhat the “minimal” configuration which does not support axion-free attractor solutions. In light of new results concerning the relation between the so-called sectional curvature of matter charges at the (non-supersymmetric) critical points of $V_{BH}$ and the BH entropy $S_{BH}$, we will then compute the relevant tensors characterizing the quantum SK geometry (1.1), namely the Riemann tensor and related contractions, and the $E$-tensor.

The plan of the paper is as follows.

In Sect. 2 we explicitly compute the effective BH potential $V_{BH}$ for the most general quantum perturbatively corrected SK geometry consistent with continuous axion-shift symmetry, namely the one with prepotential (1.1), in full generality, i.e. for an arbitrary number $n_V$ of vector multiplets and for a generic configuration $Q$ of BH charges. We then determine the axion-free-supporting Bh charge configurations, commenting on the role of $D_0 - D_6$, and (partially) explaining the findings of [54].

Sect. 3 is devoted to the computation of the $\mathcal{N} = 2$ central charge $Z$ and the related matter charges $D_i Z$ in the considered framework. Such a computations allows one to draw some general
statements on the $\frac{1}{2}$-BPS solutions, connecting to the few results already known from literature \[18\].

In Sect. 4 the role of the so-called $E$-tensor in SK geometry (and in the Attractor Mechanism within) is recalled, and its explicit computations for the geometry \[1.1\] is presented. By performing the classical limit $\xi \to 0$, the $E$-tensor for a generic $d$-SK geometry is explicitly obtained. The factorizability of some functional dependences for the classical $E$-tensor is explicitly found, highlighting the possibility to uplift the theory to $d = 5$. The same does not happen when $\xi \neq 0$, thus confirming the well known fact that only $d$-SK geometry admits an uplift to $d = 5$ (see e.g. \[64\] and Refs. therein).

Then, in Sect. 5 a number of original results are derived, pointing out the role of the so-called sectional curvature of matter charges $R$ in the theory of non-supersymmetric attractors. Indeed, $R$ vanishes at $\frac{1}{2}$-BPS attractors, but it is proportional to the critical value of $V_{BH}$ and thus, through Eq. \[1.11\], to the BH entropy $S_{BH}$. In particular, in symmetric SK geometries it has the same sign of the quartic invariant $I_4$ at non-BPS $Z \neq 0$ critical points, whereas it is opposite to (the double of) $I_4$ at non-BPS $Z = 0$ critical points, thus being strictly negative in both cases.

Since $R$ is nothing but the contraction of the Riemann tensor with the matter charges vectors (i.e. with covariant derivatives of $Z$ itself), interesting role of $R$ at non-supersymmetric critical points of $V_{BH}$ elucidated in Sect. 6 calls for an explicit computation of the Riemann tensor itself. This is carried out in Sect. 6 where also the Ricci tensor and the Ricci scalar curvature are determined. We proceed by exploiting two different approaches, one merely based on Kähler geometry (Subsect. 6.1) and the second one (Subsect. 6.2) based instead on the fundamental constraints of SK geometry (see Eq. \[4.5\] below). We explicitly show the equivalence of these two approaches, by shortly commenting on the results of \[53\] and on the eventual (unlikely) Einstein nature of the SK geometries \[1.1\].

Finally, Sect. 7 makes a brief comment and outlook, and lists some of the various open issues, originated or highlighted by the present investigation, which we leave for future study.

Three Appendices conclude the paper. They respectively contain computational details concerning $V_{BH}$ (App. A), the $E$-tensor (App. B), and the Riemann tensor (App. C).

## 2 Effective Black Hole Potential

As recalled in previous Section and as firstly found in \[31\], the most general holomorphic prepotential with leading cubic behavior consistent with (perturbative, continuous) Peccei-Quinn axion-shift symmetry \[32\], and which affects the Kähler potential $K$ of SK geometry, reads

$$F(X;\xi) = \frac{1}{3!} d_{ijk} X^i X^j X^k X^0 + i\xi (X^0)^2,$$

which is nothing but Eq. \[1.1\] before projectivizing, and before switching to special coordinates and suitably fixing the Kähler gauge (see below). Let us recall once again that $i = 1, ..., n_V$ throughout ($n_V$ denoting the number of Abelian vector multiplets coupled to the $\mathcal{N} = 2$, $d = 4$ supergravity one), and $\xi \in \mathbb{R}$.

Aim of the present Section is to compute the effective BH potential $V_{BH}$ for the SK geometry determined by the holomorphic prepotential \[2.1\]. Below we will present only the main formulae, addressing the reader to Appendix A for the further details of the calculations.

A general formula determining the kinetic vector matrix $\mathcal{N}_{\Lambda \Sigma}$ reads (see e.g. \[64\]) (A =
\[ \mathcal{N}_{\Lambda \Sigma} = \mathcal{F}_{\Lambda \Sigma} + 2i \frac{Im(\mathcal{F}_{\Lambda \Omega}) Im(\mathcal{F}_{\Theta \Delta}) X^{\Omega} X^{\Delta}}{Im(\mathcal{F}_{\Theta \Xi}) X^{\Theta} X^{\Xi}}. \quad (2.2) \]

After projectivizing, it is convenient to switch to the so-called special coordinates (see e.g. [14] and Refs. therein), defined by \((a = 1, \ldots, n_V)\)

\[ e^a_i (z) \equiv \frac{\partial (X^a)}{\partial z^i} \equiv \delta^a_i, \quad (2.3) \]

where \((x^i, \lambda^i \in \mathbb{R})\)

\[ z^i \equiv x^i - i\lambda^i \quad (2.4) \]

are the \(n_V\) complex scalar fields, and further suitably fix the Kähler gauge as

\[ X^0 \equiv 1. \quad (2.5) \]

Within such a framework, one can thus write:

\[ Im[\mathcal{F}_{\Lambda \Sigma}(z; \xi)] = \begin{pmatrix} \frac{1}{2}d_{ijk}Im(z^i z^j z^k) + 2\xi & -\frac{1}{2}d_{jkl}Im(z^k z^l) \\ -\frac{1}{2}d_{jkl}Im(z^k z^l) & d_{ijkl}Im(z^k z^l) \end{pmatrix}, \quad (2.6) \]

and the block components of \(\mathcal{N}_{\Lambda \Sigma}\) are computed in Appendix A, the final results being given by Eqs. (A.17), (A.19) and (A.20).

In order to compute the \(V_{BH}\) governing the Attractor Mechanism [1, 2, 3, 4, 5], it is worth recalling that in \(\mathcal{N} = 2, d = 4\) ungauged Maxwell-Einstein supergravity the following expression holds [3, 4, 14]:

\[ V_{BH} = |Z|^2 + g^{ij}(D_j Z) \overline{D_j Z}, \quad (2.7) \]

where \(Z\) is the \(\mathcal{N} = 2\) central charge function. On the other hand, an equivalent (and independent on the number of supercharge generators) expression of \(V_{BH}\) reads [5]

\[ V_{BH} = -\frac{1}{2} Q^T \mathcal{M}(\mathcal{N}) Q. \quad (2.8) \]

\(Q\) is the \((Sp(2n_V + 2, \mathbb{R}))\)-vector of magnetic and electric charges, which in the special coordinate basis of \(\mathcal{N} = 2\) theory reads as follows:

\[ Q = \begin{pmatrix} p^0 \\ p^i \\ q_0 \\ q_i \end{pmatrix}. \quad (2.9) \]

The \((2n_V + 2) \times (2n_V + 2)\) real symmetric symplectic matrix \(\mathcal{M}(\mathcal{N})\) is defined as [14, 3, 4]

\[ \mathcal{M}(\mathcal{N}) = \mathcal{M}(Re(\mathcal{N}), Im(\mathcal{N})) \equiv \begin{pmatrix} Im(\mathcal{N}) + Re(\mathcal{N})(Im(\mathcal{N}))^{-1} Re(\mathcal{N}) & -Re(\mathcal{N})(Im(\mathcal{N}))^{-1} \\ -(Im(\mathcal{N}))^{-1} Re(\mathcal{N}) & (Im(\mathcal{N}))^{-1} \end{pmatrix}. \quad (2.10) \]
Thus, in order to compute $V_{BH}$ for the $\mathcal{N} = 2, d = 4$ specified by the (perturbative) quantum corrected holomorphic prepotential (2.1), one has to compute the inverse of matrix $\text{Im} \mathcal{N}_{\Lambda \Sigma}$.

It is also convenient to further simplify the notation, by recalling the definitions used in [64], and suitably changing them (taking into account the presence of effective quantum parameter $\xi$):

\begin{align*}
d_{ij} &\equiv d_{ijk}\lambda^k; \\
d_i &\equiv d_{ijk}\lambda^j\lambda^k; \\
\nu &\equiv \frac{1}{3!}d_{ijk}\lambda^i\lambda^j\lambda^k; \\
\tilde{\nu} &\equiv \nu + \frac{1}{4}\xi; \\
h_{ij} &\equiv d_{ijk}x^k; \\
h_i &\equiv d_{ijk}x^jx^k; \\
h &\equiv d_{ijk}x^i x^j x^k, \\
\end{align*}

(2.11-2.17)

thus e.g. yielding

\[ h_{ij}\lambda^i \lambda^j = d_{i} x^i. \]  

(2.18)

By further introducing “rescaled dilatons” [64]

\[ \hat{\lambda}^i \equiv \frac{\lambda^i}{\nu^{1/3}} \Rightarrow \frac{1}{3!}d_{ijk}\hat{\lambda}^i\hat{\lambda}^j\hat{\lambda}^k = 1, \]  

(2.19)

one can then define the following quantities:

\begin{align*}
\hat{d}_{ij} &\equiv d_{ijk}\hat{\lambda}^k = \nu^{-1/3}d_{ij}; \\
\hat{d}_i &\equiv d_{ijk}\hat{\lambda}^j\hat{\lambda}^k = \nu^{-2/3}d_i. \\
\end{align*}

(2.20-2.21)

Let us also recall Eq. (31) of [45], giving the expression of covariant metric tensor $g_{ij}$ for the prepotential (2.1) within the assumptions (2.3)-(2.5):

\begin{equation}
g_{ij} = g_{ij} = -\frac{1}{4(\nu - \frac{1}{2} \xi)} \left[d_{ij} - \frac{d_i d_j}{4(\nu - \frac{1}{2} \xi)}\right] = -\nu^{1/3}\frac{1}{4(\nu - \frac{1}{2} \xi)} \left[\hat{d}_{ij} - \frac{\nu \hat{d}_i \hat{d}_j}{4(\nu - \frac{1}{2} \xi)}\right].
\end{equation}

(2.22)

The corresponding inverse metric ($g^{ij}g_{jk} = \delta^i_k$) is computed as

\begin{equation}
g^{ij} = g^{ij} = -4(\nu - \frac{1}{2} \xi) \left[d^{ij} - \frac{\lambda^i \lambda^j}{2(\nu + \xi)}\right] = -4(\nu - \frac{1}{2} \xi) \left[\nu^{-1/3}\hat{d}^{ij} - \frac{\nu^{2/3}\hat{\lambda}^i \hat{\lambda}^j}{2(\nu + \xi)}\right]
\end{equation}

(2.23)

where

\[ d^{ij}d_{jk} = \delta^i_k \Rightarrow \hat{d}^{ij}\hat{d}_{jk} = \delta^i_k. \]  

(2.24)

\[ \text{Notice that in [64] a different notation was used, namely:} \]

\begin{align*}
\kappa_{ij} &\equiv d_{ijk}\lambda^k; \\
\kappa_i &\equiv d_{ijk}\lambda^j\lambda^k; \\
\kappa &\equiv d_{ijk}\lambda^i\lambda^j\lambda^k = 6\nu; \\
\kappa^{ij}\kappa_{ij} &\equiv \delta^i_i.
\end{align*}

(2.25)
The limit $\xi \to 0$ consistently yields the analogue results for $d$-SKG, given by Eqs. (2.4) and (2.6) of [64]:

$$\lim_{\xi \to 0} g_{ij} = -\frac{1}{4} \nu^{-2/3} \left( \hat{d}_{ij} - \frac{\hat{d}_i \hat{d}_j}{4} \right) \equiv \tilde{g}_{ij}; \quad (2.25)$$

$$\lim_{\xi \to 0} g^{ij} = 2 \nu^{2/3} \left( \hat{\lambda}^i \hat{\lambda}^j - 2 \hat{d}^{ij} \right) \equiv \check{g}^{ij}, \quad \check{g}^{ij} \check{g}_{jk} \equiv \delta^i_k, \quad (2.26)$$

where $\tilde{g}_{ij}$ and $\check{g}^{ij}$ denote the covariant and contravariant \textit{classical} ($\xi \to 0$) metric tensor.

After long but straightforward computations (detailed in Appendix A), the following explicit
expression of $V_{BH}$ is achieved:

$$V_{BH} \left( x^i, \tilde{\lambda}^i, \nu; Q, \xi \right) = \frac{1}{h_4} \left( 1 - \left( \frac{\xi}{3} \right)^2 \frac{\xi}{\nu} \right)^{-1}.$$

By using the results (A.25) and (A.26) of Appendix A, it is easy to check that in the classical limit $\xi \to 0$ Eq. (2.27) yields the effective BH potential $\tilde{V}_{BH} \left( x^i, \tilde{\lambda}^i, \nu; Q, 0 \right)$ for a generic $d$-SKG,
given by Eq. (2.13) of [64], which we report here for ease of comparison:

\[
2 \lim_{\xi \to 0} V_{BH} = 2 \tilde{V}_{BH} = \\
\left[ \frac{\nu}{2} (1 + 4 \tilde{g}) + \frac{h^2}{36 \nu} + \frac{3}{4 \nu} \tilde{g}^{ij} h_{ij} \right] (p^0)^2 + \\
+ \left( \frac{4 \nu \tilde{g}_{ij}}{4 \nu} (h_i h_j + \tilde{g}^{mn} h_{im} h_{nj}) \right) p^i p^j + \\
+ \frac{1}{\nu} \left( q^2_0 + 2 x^i q_0 q_i + \left( x^i x^j + \frac{1}{4} \tilde{g}^{ij} \right) q_0 q_j \right) + \\
+ 2 \left[ \nu \tilde{g}_i - \frac{h}{12 \nu} h_i - \frac{1}{8 \nu} \tilde{g}^{jm} h_{mj} \right] p^0 p^j + \\
- \frac{1}{3 \nu} \left[ -h p^0 q_0 + 3 q_0 p^i h_i - \left( h x^i + \frac{3}{4} \tilde{g}^{ij} h_j \right) p^0 q_i + 3 \left( h_j x^i + \frac{1}{2} \tilde{g}^{jm} h_{mj} \right) q_i p^j \right].
\]

(2.28)

\( \tilde{g}_{ij} \) and \( \tilde{g}^{ij} \) have been respectively defined in (2.25) and (2.26) of Appendix A, with contractions consistently defined as

\[
\tilde{g}_i \equiv \tilde{g}_{ij} x^j = \lim_{\xi \to 0} g_{ij} x^j; \quad \tilde{g} \equiv \tilde{g}_{ij} x^i x^j = \lim_{\xi \to 0} g_{ij} x^i x^j.
\]

(2.29)

\( (2.30) \)

2.1 Axion-Free-Supporting Configurations

Let us now consider the terms of \( V_{BH} \) given by Eq. (2.27) which are linear in the axions \( x^i \)'s; they reads as follows:

\[
V_{BH}\text{linear in } \{x^i\} = \frac{1}{27} \left( 1 - \left( \frac{4}{3} \right)^2 \frac{\xi^2}{27} \right)^{-1}.
\]

\[
\left[ \begin{array}{c}
- \left( \frac{4}{3} \right)^2 \frac{\xi^2}{27} \frac{\nu^{1/3}}{D} (d_j x^l) \hat{d}_i + \\
2 \left[ + \frac{\nu^2}{D} \left( 1 - \left( \frac{4}{3} \right)^2 \frac{\xi^2}{27} \right) 12 A_i + \\
+ \frac{3}{2} \left( 1 - \left( \frac{4}{3} \right)^2 \frac{\xi^2}{27} \right) \xi^{2/3} \frac{D}{\nu} A^{kl} h_{ik} \hat{d}_l \right] \\
- \frac{3}{2} \frac{\nu^{2/3}}{D} d_i x^j q_0 p^j + \\
+ 2 \left[ \frac{3}{2} \nu^{2/3} \frac{D}{\nu} d_j x^i + \frac{1}{12} \left( 1 - \left( \frac{4}{3} \right)^2 \frac{\xi^2}{27} \right) A^{ik} h_{jk} \right] q_i p^j + \\
+ 2 x^i q_0 q_i
\end{array} \right]
\]

(2.31)

As a consequence, for a \( d \)-SKG corrected by \( \xi \neq 0 \) (with prepotential given by Eq. (2.1)), only two axion-free-supporting BH charge configuration exists, namely the electric \( (D2 - D6) \) and
magnetic \((D0 - D4)\) ones:

\[
\text{electric} : \quad Q_{el} = \begin{pmatrix} p_0 \\ 0 \\ 0 \\ q_i \end{pmatrix}; \quad (2.32)
\]

\[
\text{magnetic} : \quad Q_{magn} = \begin{pmatrix} 0 \\ p^i \\ q_0 \\ 0 \end{pmatrix}. \quad (2.33)
\]

For such BH charge configurations \(x^i = 0 \forall i\) is a(n at least) particular solution of the axionic Attractor Eqs.:

\[
\frac{\partial V_{BH}}{\partial x^i} \bigg|_{Q=Q_{el}} = 0 \iff x^i = 0 \forall i; \quad (2.34)
\]

\[
\frac{\partial V_{BH}}{\partial x^i} \bigg|_{Q=Q_{magn}} = 0 \iff x^i = 0 \forall i. \quad (2.35)
\]

This fact is a major difference with respect to the classical limit \(\xi \to 0\), in which the linear term in \(x^i\)'s proportional to \(q_0 p^0\) (see Eq. (2.31)) vanishes. Indeed, it consistently holds that

\[
2 \lim_{\xi \to 0} V_{BH, \text{linear in } x^i} = 2 \tilde{V}_{BH, \text{linear in } x^i} = \frac{1}{\nu} 2x^i q_0 q_i + 2 \nu \tilde{g}_{ij} p^0 p^j - \frac{1}{2 \nu} \tilde{g}^{ik} h_{kj} q_i q_j. \quad (2.36)
\]

This implies that also the Kaluza-Klein \((D0 - D6)\) BH charge configuration

\[
KK : \quad Q_{KK} = \begin{pmatrix} p_0 \\ 0 \\ q_0 \\ 0 \end{pmatrix} \quad (2.37)
\]

supports axion-free (at least particular) attractor solutions \([64]\). Thus, besides the classical limits of Eqs. (2.34) and (2.35), namely:

\[
\frac{\partial \tilde{V}_{BH}}{\partial x^i} \bigg|_{Q=Q_{el}} = 0 \iff x^i = 0 \forall i; \quad (2.38)
\]

\[
\frac{\partial \tilde{V}_{BH}}{\partial x^i} \bigg|_{Q=Q_{magn}} = 0 \iff x^i = 0 \forall i, \quad (2.39)
\]

for \(\xi = 0\) it also holds that

\[
\frac{\partial \tilde{V}_{BH}}{\partial x^i} \bigg|_{Q=Q_{KK}} = 0 \iff x^i = 0 \forall i. \quad (2.40)
\]

The non-axion-free-supporting nature of the \(D0 - D6\) BH charge configuration in perturbatively quantum corrected \(d\)-SKG (determined by the holomorphic prepotential (2.1)) is consistent with, and sheds new light on, the results of \([54]\).

Such a paper (developing the analysis of \([43]\)) addressed the issue of the fate of the unique non-BPS \(Z \neq 0\) flat direction in the \(N = 2, d = 4\) ungauged Maxwell-Einstein supergravity described by Eq. (2.1) with \(n_V = 2\) (i.e. the so-called “\(st^2 + i\xi\) model”). By analyzing the (supergravity analogues of the) \(D0 - D4, D2 - D6\) and \(D0 - D6\) charge configurations, the following results were obtained:
• In $D0-D4$ and $D2-D6$ charge configurations the classical solutions ($\xi = 0$) were found to lift at the quantum level ($\xi \neq 0$). Remarkably, it was found that the quantum lift occurs more often towards repeller directions (thus destabilizing the whole critical solution, and destroying the attractor in strict sense), rather than towards attractor directions.

• The $D0-D6$ charge configuration yielded a somewhat surprising result: the classical solution gets modified at the quantum level, acquiring a non-vanishing axionic part. However, despite being no more purely imaginary, such a quantum non-BPS $Z \neq 0$ solution still exhibits a flat direction. The origin of such a deep difference among electric/magnetic and $D0-D6$ configurations was unclear in [54], but it is clarified (and further generalized to an arbitrary number $nV$ of Abelian vector multiplets) from the results of the analysis performed above: due to the very structure of $V_{BH}$ (see Eqs. (2.27 and (2.31)) for $\xi \neq 0$, the electric/magnetic still support axion-free solutions, whereas the $D0-D6$ configuration do not.

On the other hand, the persistence of the flat direction also in presence of quantum generated axions is still not completely understood, and we left the study of such issues for future work.

3 Central Charge and Matter Charges

As given by Eq. (2.7), the effective BH potential $V_{BH}$ enjoys a rewriting in terms of the $N = 2$, $d = 4$ central charge $Z$ and of its covariant derivatives $D_iZ$ (usually named matter charges), which is therefore worth computing.

In order to do this, let us recall that under the assumptions (2.3)-(2.5) the holomorphic prepotential (2.1) reduces to

$$\mathcal{F}(z; \xi) \equiv \frac{1}{3!} d_{ijk} z^i z^j z^k + i\xi. \quad (3.1)$$

Furthermore, the Kähler potential reads ($F_i \equiv \partial \mathcal{F} / \partial z^i$; see e.g. [14, 65])

$$K = -\log \left\{ i \left[ 2(\mathcal{F} - \mathcal{F}) + (\mathcal{F}^* - z^i)(\mathcal{F} + \mathcal{F}^*) \right] \right\} =
-\log \left[ -\frac{i}{3!} d_{ijk}(z^i - z^j)(z^j - z^k)(z^k - z^i) - 4\xi \right] =
-\log \left( 8 \left( \nu - \frac{\xi}{2} \right) \right), \quad (3.2)$$

where definitions (2.14) and (2.13) were used. Eq. (3.2) thus implies

$$\exp (-K) = 8 \left( \nu - \frac{\xi}{2} \right) \Leftrightarrow \exp (K/2) = \frac{1}{2\sqrt{2\nu - \xi}}, \quad (3.3)$$

with the global condition of consistency (relevant also for previous treatment, see for instance Eq. (2.22))

$$2\nu - \xi > 0. \quad (3.4)$$

Therefore, by recalling its very definition (see e.g. [5] and Refs. therein)

$$Z \equiv e^{K/2}(X^A q_A - F_{A\dot{A}}) \equiv e^{K/2}W, \quad (3.5)$$
where $W$ is the holomorphic superpotential, and under the assumptions \(2.3\) - \(2.5\), the \(N = 2, d = 4\) central charge function for the holomorphic prepotential \((2.1)\) can be computed to be:

$$
\begin{align*}
Z(x^j, \bar{\lambda}^j, \nu; Q, \xi) &= \frac{1}{2\sqrt{2\nu - \xi}} W \left(x^j, \bar{\lambda}^j, \nu; Q, \xi\right) = \\
&= \frac{1}{2\sqrt{2\nu - \xi}} \left[ q_0 + q_i x^i - \frac{p_0}{\nu} \nu^{2/3} \hat{d}_i x^i + \frac{p_0}{\nu} h - \frac{p}{\nu} h_i + \nu^{2/3} \frac{p_0}{\nu} \hat{d}_i + \\
&+ i\nu^{1/3} \left(-q_i \bar{\lambda}^i - \frac{p_0}{\nu} \hat{d}_{ij} x^i x^j + p \nu^{2/3} - 2 \frac{\xi}{\nu^{1/3}} p^0 + p^i \hat{d}_i x^j\right) \right];
\end{align*}
$$

\(D_i Z \left(x^j, \bar{\lambda}^j, \nu; Q, \xi\right) = \frac{1}{2\sqrt{2\nu - \xi}} D_i W \left(x^j, \bar{\lambda}^j, \nu; Q, \xi\right) = \\
\frac{1}{2\sqrt{2\nu - \xi}} \left\{ \begin{array}{l}
q_i + \frac{p_0}{\nu} h_i - \frac{p_0}{\nu} \nu^{2/3} \hat{d}_i + \\
- p \nu h_i + i\nu^{1/3} \left(-p \nu x^j + p^j\right) \hat{d}_i + \\
- i \frac{\nu^{2/3}}{(2\nu - \xi)} \hat{d}_i \end{array} \right\}.
$$

(3.6)

\(D_i Z \equiv \partial_i Z + \frac{1}{2} (\partial_i K) Z; \quad D_i W \equiv \partial_i W + (\partial_i K) W. \quad (3.7)\)

Clearly, due to the different Kähler weights of \(Z\) and \(W\) (respectively \((1, -1)\) and \((2, 0)\)), the covariant differential operator acting on them has different definitions, namely:

$$
\begin{align*}
D_i Z &= \partial_i Z + \frac{1}{2} (\partial_i K) Z; \\
D_i W &= \partial_i W + (\partial_i K) W.
\end{align*}
$$

(3.8)

(3.9)

Notice that in the limit \(\xi \to 0\) Eqs. \((3.8)\) and \((3.9)\) exactly matches with known results for \(d\)-SK geometries, given by Eq.(4.9) and (4.10) of \(64\). It is worth remarking that Eq. \((3.6)\) yields that the holomorphic superpotential \(W\) gets modified, with respect to its classical (\(\xi \to 0\)) counterpart, only by a \textit{global} shift of its imaginary part:

$$
\begin{align*}
W \left(x^j, \bar{\lambda}^j, \nu; Q, \xi\right) &= W \left(x^j, \bar{\lambda}^j, \nu; Q, 0\right) - 2\xi p^0.
\end{align*}
$$

(3.10)

In particular, for \textit{axion-free} critical solutions (supported for \(\xi \neq 0\) only by the charge configurations \((2,3)\) and \((2,3)\)) it holds that the superpotential \(W\) \textit{(on-shell} for axions \(x^i\)'s\) is purely imaginary and real, respectively:

$$
\begin{align*}
W \left(x^j = 0, \bar{\lambda}^j, \nu; Q_{axi}, \xi\right) &= i\nu^{1/3} \left(-q_i \bar{\lambda}^i + p \nu^{2/3} - 2 \frac{\xi}{\nu^{1/3}} p^0\right); \\
W \left(x^j = 0, \bar{\lambda}^j, \nu; Q_{magn}, \xi\right) &= q_0 + \nu^{2/3} p^i \hat{d}_i.
\end{align*}
$$

(3.11)

(3.12)

Concerning supersymmetric critical points of \(V_{BH}\), the \(\left(\frac{1}{2}\right)\)BPS conditions

$$
D_i W = 0 \quad \forall i = 1, ..., n_V
$$

(3.13)
for \textit{axion-free} critical solutions within the charge configurations (2.32) and (2.33) respectively read (∀i = 1, ..., nV):

\[
D_i W \left( x^j = 0, \hat{\lambda}^j, \nu; Q_{el}, \xi \right) = 0 \iff q_i - \frac{p^0_i}{4} \nu^{2/3} \hat{d}_i + \frac{1}{2} \nu^{1/3} \left( -q_j \hat{\lambda}^j + p^0 j \nu^{2/3} - 2 \frac{\xi}{\nu^{1/3}} \nu^0 \right) = 0; \quad (3.14)
\]

\[
D_i W \left( x^j = 0, \hat{\lambda}^j, \nu; Q_{magn}, \xi \right) = 0 \iff p^j \hat{d}_{ij} - \frac{1}{2} \nu^{1/3} \left( q_0 + p^0 j \nu^{2/3} \hat{d}_j \right) = 0, \quad (3.15)
\]

reducing to nV (ξ-parametrized) real algebraic Eqs. in nV real unknowns \{\hat{\lambda}^i, \nu\}.

In [18] the \textit{axion-free} $\frac{1}{2}$-BPS critical points of $V_{BH}$ determined by the holomorphic prepotential (3.1) were determined by introducing the Kähler gauge-invariant sections

\[
Y \equiv Z \begin{pmatrix} L^0 \\ L^i \\ M_0 \\ M_i \end{pmatrix} = \exp(K) \begin{pmatrix} X^0 \\ X^i \\ F_0 \\ F_i \end{pmatrix} \quad (3.16)
\]

and evaluating the identities of the SK geometries (see e.g. [14, 65] and Refs. therein) along the BPS conditions (3.13), thus obtaining (Ξ ∈ R)

\[
\begin{align*}
Y^0 &= \frac{1}{3} (\Xi + ip^0) ; \\
Y^i &= ip^i \frac{(\Xi + ip^0)}{\Xi}.
\end{align*} \quad (3.17)
\]

In the case of Ξ ≠ 0, the ξ-dependent value of $V_{BH}$ at its $\frac{1}{2}$-BPS \textit{axion-free} critical points can be computed to be

\[
V_{BH,BPS,axion-free} = -2 \left[ \Xi + \frac{(p^0)^2}{\Xi} \right] \left( q_0 - 2\xi \Xi + \frac{\xi^2}{2} \Xi \right), \quad (3.18)
\]

where Ξ satisfies the ξ-parametrized Eq. (see Eq. (3.34) of [18]):

\[
3p^0 q_0 + p^j q_i = 6\xi \Xi p^0, \quad (3.19)
\]

along with the condition [18]

\[
(q_0 - 2\Xi) d_{ijk} p^i p^j p^k > 0. \quad (3.20)
\]

On the other hand, the $\frac{1}{2}$-BPS \textit{axion-free} solutions with Ξ = 0 are necessarily supported only by the electric configuration (2.32), and the dependence on ξ drops out: the resolution of the Attractor Eqs. in terms of the sections $Y^\Lambda$’s and the determination of the critical value of $V_{BH}$ go as for a generic $d$-SK geometry [57].

Concerning \textit{non-axion-free} supersymmetric (if any) and non-supersymmetric (either \textit{axion-free} or \textit{non-axion-free}) critical points of $V_{BH}$, the case study becomes much more complicated.

As yielded by the analysis of $t^3 + i\xi$ model (nV = 1) [45] and of $st^2 + i\xi$ model (nV = 2 particular case) [54], in general the corresponding Attractor Eqs. are higher-order algebraic Eqs. which cannot be solved analytically, but only investigated numerically. Furthermore,

\*Eq. (3.13) fixes a typo in Eq. (3.35) of [18]. For the configuration D0 – D6 (which however, as explicitly shown above, is \textit{not} axion-free-supporting for ξ ≠ 0) this was noticed in [45].
interesting phenomena occur, such as: the “separation” of attractors (related to presence of basins of attraction / area codes in the dynamical system describing the radial evolution of the scalar fields in the BH space-time background) [45]; the “transmutation” of the supersymmetry-preserving properties of the attractors [45]; and the “lifting” (with or without removal) of the “flat” directions of the critical potential, which exist in the classical (ξ = 0) regime, at least for symmetric d-SK geometries [54].

Despite the lack of analytical expressions of non-supersymmetric (non-BPS Z ≠ 0 and/or non-BPS Z = 0) critical points of VBH for ξ ≠ 0, many issues are still to be carefully investigated (we list some of them in the concluding Sect. 7).

The intricacy of the SK geometry described by the holomorphic prepotential (2.1) (or, equivalently by Eq. (3.1)) calls for a deeper analysis of the fundamental quantities characterizing such a geometry, and also for a deeper understanding of the conditions determining the (various classes of) critical points of VBH itself. The study of these issues, needed for a deeper investigation of the dynamics of the Attractor Mechanism in the generally non-homogeneous geometries under consideration, will be the object of Sects. 4 and 6.

4 E-tensor

The first quantity we want to determine is the so-called E-tensor. This rank-5 tensor was firstly introduced in [34] (see also the treatment of [53]), and it expresses the deviation of the considered geometry from being symmetric. Its definition reads (see e.g. [9] for a recent treatment, and Refs. therein):

\[ E_{mijkl} = \frac{1}{3} D_m D_i C_{jkl}. \]  

(4.1)

This definition can be elaborated further, by recalling the properties of the so-called C-tensor Cijk. This is a rank-3 tensor with Kähler weights (2, -2), defined as (see e.g. [14, 66, 67]):

\[ C_{ijk} \equiv \langle D_i D_j V, D_k V \rangle = e^K (\partial_i N_{\Lambda\Sigma}) D_j X^\Lambda D_k X^\Sigma = e^K (\partial_i X^\Lambda) (\partial_j X^\Sigma) (\partial_k X^\Xi) \partial_\Xi \partial_\Sigma F_{\Lambda}. \]

\[ \equiv e^K W_{ijk}, \quad \overline{D_l} W_{ijk} = 0, \]  

where the second line holds only in special coordinates. Cijk is completely symmetric and covariantly holomorphic:

\[ C_{ijk} = C_{(ijk)}; \]  

(4.3)

\[ \overline{D_l} C_{jkl} = 0. \]  

(4.4)

Furthermore, it enters the fundamental constraints on the Riemann tensor Rijkl of SK geometry (see e.g. [14, 66, 67], [14] and Refs. therein; see also e.g. [68] and [9] for more recent reviews):

\[ R_{ijkl} = -g_{ij} g_{kl} - g_{il} g_{kj} + C_{ikm} \overline{C}_{jlm} g^{mn}. \]  

(4.5)

The Bianchi identities for Rijkl (see e.g. [66]) and constraints (1.3) yield the following result:

\[ D_l [C_{jkl}] = 0, \]  

(4.6)

\footnote{Notice that the third of Eqs. (1.2) correctly defines the Riemann tensor Rijkl, and it is actual the opposite of the one which may be found in a large part of existing literature. Indeed, such a formulation yields negative values of the constant scalar curvature homogeneous symmetric non-compact SK manifolds, as given by the treatment of [69].}
where (round) square brackets denote (symmetrization) anti-symmetrization with respect to enclosed indices throughout. Due to its holomorphic Kähler weight, the covariant derivative of $C_{ijk}$ reads:

$$D_i C_{jkl} = D_i (C_{jkl}) = \partial_i C_{jkl} + (\partial_i K) C_{jkl} + \Gamma_{ij}^m C_{mkl} + \Gamma_{ik}^m C_{mj} + \Gamma_{il}^m C_{mjk},$$  \hspace{1cm} (4.7)

where the Christoffel connection $\Gamma$ is defined as

$$\Gamma_{ij}^m \equiv -g^{mn} \partial_i g_{nj}.$$

By using Eqs. (4.2)-(4.8), $E_{mijkl}$ defined by (4.1) can thus be further elaborated as follows:

$$E_{mijkl} = \frac{1}{3} D_m (D_i C_{jkl}) = C_{p}^{(kl} Z^i_j Z^m_l Z^{p} Z^{m} = \frac{4}{3} g_{(ii|kk)} C_{ijkl} = 2 g_{(i|kk)} C_{ijkl} = E_{mijkl},$$  \hspace{1cm} (4.9)

It thus holds that $E_{mijkl} = 0$ globally in (homogeneous) symmetric SK manifolds, defined by the covariant constancy of $R_{ijkl}$ itself:

$$D_m R_{ijkl} = 0.$$

Eq. (4.10), through the covariant holomorphicity of $C_{ijk}$ and the constraints (4.5), yields the global covariant constancy of $C_{ijkl}$ itself, and thus the global vanishing of $E_{mijkl}$:

$$D_i C_{jkl} = D_i (C_{jkl}) = 0 \Rightarrow E_{mijkl} = 0,$$

which in turn, through Eq. (4.9), implies

$$C_{p}^{(kl|} g^{i} C_{ijkl} = \frac{4}{3} g_{(ii|kk)} C_{ijkl} \iff g^{i} R_{(i|kk)} C_{ijkl} = \frac{2}{3} g_{(i|kk)} C_{ijkl}.$$  \hspace{1cm} (4.12)

It is worth noticing that, while (4.10) defines the symmetry of a Kähler manifold, Eq. (4.11) (or equivalently Eq. (4.12)) is a necessary (but not necessarily sufficient) condition of symmetry.

Recently, in [7] the $E$-tensor was used in the expression of the value of $V_{BH}$ at its non-BPS $Z \neq 0$ critical points (see also the treatment in [70], and Refs. therein):

$$V_{BH,nBPS,Z \neq 0} = \left[ 4 |Z|^2 + \Delta \right]_{nBPS,Z \neq 0},$$  \hspace{1cm} (4.13)

where $(Z^7 \equiv g^{ij} D_j Z)$

$$\Delta \equiv -\frac{3}{4} E_{mijkl} Z^7 Z^i Z^j Z^k Z^l Z^m = \frac{3}{4} E_{mijkl} Z^7 Z^i Z^j Z^k Z^l Z^m,$$

such that (see e.g. [7 9]).

$$\left[ \begin{array}{c} g^{77} (D_i Z) D_7 Z \\ |Z|^2 \end{array} \right]_{nBPS,Z \neq 0} = 3;$$

$$\Delta_{nBPS,Z \neq 0} = 0 \iff \left( E_{mijkl} Z^7 Z^i Z^j Z^k Z^l Z^m \right)_{nBPS,Z \neq 0} = 0,$$  \hspace{1cm} (4.16)
where in the last step the non-degeneracy of the cubic norm $C_{ijk} \mathbf{Z}^j \mathbf{Z}^k$ (at least at non-BPS $Z \neq 0$ critical points of $V_{BH}$) was used. Therefore, Eq. (4.11) (or equivalently Eq. (4.12)) is a sufficient (but not necessary) condition for the so-called “rule of three” (4.13) to hold at non-BPS $Z \neq 0$ critical points of $V_{BH}$.

These results (and further relations with the sectional curvature treated further below; see Eqs. (5.21)-(5.25) as well as the treatment given in [9]) call for an explicit determination of the $E$-tenso in the SK geometries described by the holomorphic prepotential (3.1), and through the limit $\xi \to 0$, in a generic $d$-SK geometry.

Thus, by a long but straightforward algebra (detailed in Appendix B), the covariant derivative of the $C$-tensor can be written as follows:

$$D_i C_{jkl} = \frac{i}{2\nu} \left( \frac{\nu - \xi}{\nu} \right)^2 \left[ -\frac{(\nu - \xi)}{\nu^2 \xi^2} \nu^{2/3} (d_{ij} d_{kl} + d_{ik} d_{jl} + d_{il} d_{jk}) + \right. $$ $$\left. -2 (\nu - \xi) \nu^{-1/3} (d_{ij} d_{mkl} + d_{ik} d_{mjl} + d_{il} d_{mjk}) \hat{d}^{mn} + \right. $$ $$\left. + \nu^{2/3} (d_{ij} d_{kl} + d_{ik} d_{kl} + d_{il} d_{ij} + d_{il} d_{ij}) \right]. \quad (4.17)$$

Notice that for $\xi \neq 0$ there is no way to make $D_i C_{jkl} = 0$ globally. This confirms the result of [53] that, with the exception of the sequence of the minimal coupling sequence $\mathbb{C}P^n$, all homogeneous symmetric (non-compact) SK are given by $d$-geometries (namely, by the $\xi \to 0$ limit of prepotential (3.1)). Thus, the SK geometries described by the holomorphic prepotential (3.1) are not symmetric, nor homogeneous (at least of the $d$-type studied and classified in [53], and Refs. therein).

Through definition (4.11) and Eq. (4.17), the $E$-tensor can then be explicitly computed:

$$\hat{E}_{\bar{m}ijkl} = \frac{1}{3} \hat{E}_{\bar{m}} D_i C_{jkl} = \frac{1}{3} \left[ \hat{E}_{\bar{m}} D_i C_{jkl} - (\hat{E}_{\bar{m}} K) D_i C_{jkl} \right] = $$

$$= \frac{1}{12 \cdot 2^4 \left( \frac{\nu - \xi}{\nu} \right)^2} \left[ (2\nu - 7\xi) \frac{\nu^{4/3}}{4(\nu - \xi)} \left( \tilde{d}_{ij} \tilde{d}_{kl} + \tilde{d}_{ik} \tilde{d}_{jl} + \tilde{d}_{il} \tilde{d}_{jk} \right) \tilde{d}_m + \right.$$ $$\left. - \frac{\nu^{4/3}}{2(\nu - \xi)} \left( \tilde{d}_{ij} \tilde{d}_{kl} + \tilde{d}_{ik} \tilde{d}_{jl} + \tilde{d}_{il} \tilde{d}_{jk} \right) \tilde{d}_m + \right.$$ $$\left. - \frac{(\nu - \xi)}{\nu^2 \xi^2} \nu^{1/3} \left( d_{ijm} \tilde{d}_{kl} + d_{ikm} \tilde{d}_{jl} + d_{ilm} \tilde{d}_{jk} + d_{ijm} \tilde{d}_{kl} + d_{ilm} \tilde{d}_{jk} \right) \right] + $$ $$\left. + 2\nu^{1/3} \left( \tilde{d}_{im} \tilde{d}_{jkl} + \tilde{d}_{jm} \tilde{d}_{ikl} + \tilde{d}_{km} \tilde{d}_{ijl} + \tilde{d}_{il} \tilde{d}_{mjk} \right) + \right.$$ $$\left. -2 (\nu - \xi) \nu^{1/3} \left( d_{ijm} \tilde{d}_{kl} + d_{ikm} \tilde{d}_{jl} + d_{ilm} \tilde{d}_{jk} \right) \right] \quad (4.18)$$

where it is easy to show that

$$\frac{\partial d^{mn}}{\partial \lambda^n} = -d_{ijm} d^{np} d^n = -\nu^{-2/3} d_{ijm} \tilde{d}^p \tilde{d}^m. \quad (4.19)$$

By standard symmetrization procedures and using Eq. (4.19), Eq. (4.18) can be further
elaborated as follows:

\[
\mathcal{E}_{\mu\nu\rho\sigma} = -\frac{1}{3 \cdot 2^7} \left( \frac{\nu - \frac{5}{2}}{\nu + \frac{5}{2}} \right)^3 \left[ 4 \hat{a}_{ij}(d_{jkl}) - 3 \frac{(\nu - \frac{5}{2})}{\nu + \frac{5}{2}} \hat{a}_{ij}\hat{a}_{kl} \right] \nu^{4/3} \hat{a}_m + \\
+12 \frac{(\nu - \frac{5}{2})^2}{\nu + \frac{5}{2}} \nu^{1/3} d_{m(1)} \hat{a}_{kl} - 2^4 \left( \nu - \frac{5}{2} \right) \nu^{1/3} \hat{a}_m(d_{jkl}) + \\
-12 \left( \nu - \frac{5}{2} \right)^2 \nu^{-2/3} d_{m(1)} d_{n(1)} \hat{a}_{kl} \hat{a}_{mn} + \\
+\frac{3}{2^7} \nu^{5/3} \hat{a}_m \hat{a}_{ij}\hat{a}_{kl} 
\] .

(4.20)

It is here worth remarking that the observation made above that for \( \xi \neq 0 \) it is not possible to make \( D_\xi C_{jkl} = 0 \) globally does not imply that \( \mathcal{E}_{\alpha\beta\gamma\delta} = 0 \), and/or \( \mathcal{E}_{\alpha\beta\gamma\delta} = 0 \), locally, namely on a (set of) point(s), eventually at non-BPS Z \( \neq 0 \) critical points of \( V_{BH} \). Thus, the interesting question arises (which we leave for future investigation) whether for some value(s) of \( \xi \) itself the “rule of three” (4.15) still holds at non-BPS Z \( \neq 0 \) critical points of \( V_{BH} \) in SK geometries determined by the prepotential (3.1).

Let us here recall that, as explicitly found in [71], at least in some homogeneous non-symmetric d-SK geometries, the “rule of three” (4.15) still holds, despite the fact that \( \mathcal{E}_{\alpha\beta\gamma\delta} \) does not vanish globally.

Before concluding this Section, let us notice that in the limit \( \xi \rightarrow 0 \) the result (4.20) yields the expression of the \( E \)-tensor for a generic d-SK geometry, namely:

\[
\mathcal{E}_{\alpha\beta\gamma\delta,\xi=0} = -\frac{1}{3 \cdot 2^7} \nu^{5/3} \nu^{-5/3} \left[ 4 \hat{a}_{ij}(d_{jkl}) - 3 \hat{a}_{ij}\hat{a}_{kl} \right] \hat{a}_m + \\
+12 d_{m(1)} \hat{a}_{kl} - 16 \hat{a}_m(d_{jkl}) + \\
-12 d_{m(1)} d_{n(1)} \hat{a}_{kl} \hat{a}_{mn} 
\] .

(4.21)

It is worth noticing that Eq. (4.21) yields that the tensor

\[
\hat{\mathcal{E}}_{\alpha\beta\gamma\delta,\xi=0} \equiv \nu^{5/3} \mathcal{E}_{\alpha\beta\gamma\delta,\xi=0} 
\]

is independent on \( \nu \), but it rather depends only on the “rescaled dilatons” \( \hat{\lambda}^i \)’s (recall definitions (2.19)-(2.21)):

\[
\frac{\partial \hat{\mathcal{E}}_{\alpha\beta\gamma\delta,\xi=0}}{\partial \nu} = 0. 
\]

(4.22)

By looking at Eq. (4.20), it is easy to realize that the same does not happen for \( \xi \neq 0 \): the non-vanishing of the quantum parameter \( \xi \) does not allow for an overall factorization of the dependence of \( \mathcal{E}_{\alpha\beta\gamma\delta} \) on \( \nu \) and/or other (shifted and/or rescaled) variables. In other words, \( \xi \) entangles the dependence of \( \mathcal{E}_{\alpha\beta\gamma\delta} \) on \( \nu \) with the dependence on \( \hat{\lambda}^i \)’s, and thus the “\( \xi \neq 0 \) analogue” of \( \mathcal{E}_{\alpha\beta\gamma\delta,\xi=0} \) (defined in (4.22)) cannot be introduced. This fact is related to the impossibility to uplift the quantum perturbatively corrected SK geometry described by the prepotential (3.1) to \( d = 5 \) space-time dimensions. Indeed, as it is well known, in general only d-SK geometries can be uplifted to \( d = 5 \) (see e.g. [64] and Refs. therein).

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5 Sectional Curvature at Critical Points

In the present Section we reconsider the non-supersymmetric criticality conditions for the effective BH potential $V_{BH}$ of an $\mathcal{N} = 2, d = 4$ Maxwell-Einstein supergravity coupled to a generic number $n_V$ of Abelian vector supermultiplets. We will find that in both classes ($Z \neq 0$ and $Z = 0$) of its non-BPS critical points, the critical value of $V_{BH}$ (and thus, through the Bekenstein-Hawking entropy-area formula, the classical BH entropy) is proportional to the local value of the so-called sectional curvature of matter charges.

Within the present study, this general result then motivates the explicit computation (carried out in the next Section in two different, but equivalent, approaches) of the Riemann tensor, Ricci tensor and Ricci scalar curvature for the SK geometries determined by the prepotential (3.1), as well for generic $d$-SK geometry, obtained as the classical limit $\xi \to 0$ of these former ones. This latter calculation extends to the inclusion of the most general axion-shift-symmetric quantum perturbative correction (see discussion in Introduction) the results on the curvature of non-compact SK manifolds, found long time ago in [53].

Along the lines of the elaborations of [9] (see also [70]), we will now determine a “non-BPS $Z = 0$ analogue” of the “rule of three” (4.15). Such a “non-BPS $Z = 0$ analogue” is an hitherto unaddressed issue in literature (for instance, not considered in the fairly general treatment of [58], nor in [9]). In order to derive such a result, let us contract the constraints (4.5) by $Z^i Z^j Z^k Z^l$, obtaining

$$\mathcal{R}(Z) \equiv R_{ijkl} Z^i Z^j Z^k Z^l. \quad (5.1)$$

Therefore, by recalling the non-BPS $Z = 0$ criticality conditions for $V_{BH}$:

$$C_{ijk} Z^k = 0, \quad (5.2)$$

as well as the definition of sectional curvature \[^8\](of the matter charges) (see e.g. [72] for a recent use; notice the different definition used here, consistent with the one adopted in [9]: see Eq. (3.1.2.11) therein)

$$\mathcal{R}(Z) \equiv R_{ijkl} Z^i Z^j Z^k Z^l, \quad (5.3)$$

it follows that at (“large”) non-BPS $Z = 0$ critical points of $V_{BH}$ it holds that:

$$\left( Z_i Z^i \right)^2_{n_BPS, Z = 0} = \left[ g^{ij} \partial_i Z^i \partial_j Z^j \right]^2_{n_BPS, Z = 0} = -\frac{1}{2} \mathcal{R}(Z)_{n_BPS, Z = 0} > 0. \quad (5.4)$$

The result (5.4) holds for all $\mathcal{N} = 2, d = 4$ ungauged Maxwell-Einstein supergravities, not only for the ones with symmetric scalar manifolds, and it implies that the sectional curvature of the matter charges $\mathcal{R}(Z)$ to be strictly negative at non-BPS $Z = 0$ critical points of $V_{BH}$.

Moreover, for symmetric (and actually also for homogeneous non-symmetric...) SK manifolds, recalling that along the non-BPS $Z = 0$-supporting charge orbits the quartic invariant $\mathcal{I}_4$ is positive, it further holds that (see e.g. [16] and [9])

$$\left[ g^{ij} \partial_i Z^i \partial_j Z^j \right]_{n_BPS, Z = 0} = \sqrt{-\frac{1}{2} \mathcal{R}(Z)_{n_BPS, Z = 0}} = \sqrt{\mathcal{I}_4}. \quad (5.5)$$

\[^8\]Notice that in general the Riemann tensor $R_{ijkl}$, the Ricci tensor $\mathcal{R}_i$, the Ricci scalar curvature $\mathcal{R}$ and the sectional curvature $\mathcal{R}$ itself are all real quantities.
thus yielding the relation
\[ R(Z)_{\mid_{n_{BP} S, Z=0}} = -2I_4 < 0. \]  
(5.6)

Eq. (5.6) is to be contrasted with the analogue result obtained in [9] for ("large") non-BPS \( Z \neq 0 \) critical points of \( V_{BH} \) in symmetric SK geometries (see Eq. (3.1.1.2.23), as well as Eq. (3.1.1.2.20), therein):
\[ R(Z)_{\mid_{n_{BP} S, Z\neq0}} = -6 |Z|^4 \frac{3}{8} I_4 < 0. \]  
(5.7)

Thus, at least in symmetric SK geometries, at various classes of "large" critical points of \( V_{BH} \) the sectional curvature of the matter charges \( R(Z) \) takes the following values:
\[ R(Z) = \begin{cases} \frac{1}{2} - BPS : 0; \\ n_{BP} S, Z \neq 0 : \frac{3}{8} I_4 < 0; \\ n_{BP} S, Z = 0 : -2I_4 < 0. \end{cases} \]  
(5.8)

Correspondingly, through the celebrated Bekenstein-Hawking entropy-area formula [73] and its implementation through the Attractor Mechanism [5]
\[ S_{BH} = \pi \frac{A_H}{4} = \pi V_{BH} \mid_{\partial V_{BH} = 0}, \]  
(5.9)
at ("large") non-BPS critical points of \( V_{BH} \) in (at least symmetric) \( N = 2, d = 4 \) ungauged Maxwell-Einstein supergravities, the value of the classical BH entropy is proportional to the local value of \( R(Z) \) itself:
\[ S_{BH} = \pi \left( \frac{A_H}{4} \right) = \pi \left( \frac{1}{2} \right) \sqrt{2} \left| \frac{3}{2} I_4 \right| \left| R(Z) \right|; \]  
(5.10)

Eqs. (5.8) (and consequently Eqs. (5.10)) hold on-shell, i.e. at the various classes of critical points of \( V_{BH} \). Actually, they can be "unified" into an off-shell (i.e. global) relation, involving \( R(Z) \) along with the true-vector (vanishing on-shell) \( \partial_i V_{BH} \). In order to determine such a relation, let us evaluate the definition of sectional curvature of matter charges (5.3) along the constraints (4.5), thus obtaining:
\[ R(Z) = -2 \left( Z_i \overline{Z}^i \right)^2 + g^{nm} C_{ikn} C_{jm} \overline{Z}^i \overline{Z}^j Z_k Z_l. \]  
(5.11)

Now, by differentiating Eq. (2.7) and using the defining relations of SK geometry (see e.g. 65 and Refs. therein), one can then write [5]
\[ D_i V_{BH} = \partial_i V_{BH} = 2 \overline{Z} Z_i + i C_{ijk} \overline{Z}^j Z^k \Leftrightarrow C_{ijk} \overline{Z}^j Z^k = -i \left( \partial_i V_{BH} - 2 \overline{Z} Z_i \right). \]  
(5.12)

By using Eq. (5.12), Eq. (5.11) can thus be recast in the following way:
\[ R(Z) = -2 \left( Z_i \overline{Z}^i \right)^2 + g^{kl} \left( \partial_k V_{BH} - 2 \overline{Z} Z_k \right) \left( \partial_l V_{BH} - 2 \overline{Z} Z_l \right) = 2 \overline{Z} \overline{Z}^{\prime} \left( 2 |Z|^2 - Z \overline{Z} \right) + g^{kl} \left[ (\partial_k V_{BH}) \overline{Z} \overline{V}_{BH} - 2 Z (\partial_k V_{BH}) \overline{Z} \overline{Z} - 2 Z \left( \overline{Z} \overline{V}_{BH} \right) Z_l \right]. \]  
(5.13)
Eq. (5.13) is nothing but an equivalent rewriting of the sectional curvature of after charges in SK geometry, given by Eq. (5.11). By consistently using the criticality conditions of $V_{BH}$ defining the various classes of ("large") critical points of $V_{BH}$ itself (namely: 1/2-BPS - see Eq. (3.13) -, non-BPS $Z = 0$ - see Eq. (5.2) -, and non-BPS $Z \neq 0$ - see Eq. (5.16) below), the three on-shell relations (5.8) are obtained.

Aside, let us also notice that the constraints (4.5) clearly yield a constrained expression for the Ricci tensor (and for the Ricci scalar curvature) of a SK manifold, in which the partial (and complete) contractions of the $C$-tensor with its complex conjugate play a key role. Namely, Eq. (4.5) respectively imply:

$$R^j_i \equiv g^{k}{}_{l} R^i_{lkj} = -(n_V + 1) g^{j}_{i} + g^{j}{}_{k} g^{m}{}_{n} C_{imk} C_{jnl};$$  \hspace{1cm} (5.14)  

$$R \equiv g^{ij} R_{ij} = g^{ij} R_{ij} = -(n_V + 1) n_V + g^{ij} g^{kl} C_{imk} C_{jnl}. \hspace{1cm} (5.15)$$

From the discussion at the end of Subsect. 6.1, it will be clear that the first terms in the right-hand sides of Eqs. (5.14) and (5.15) are the contribution of the "quadratic sector" of the SK geometry (in which $C_{ijk} = 0$, as a consequence of its very definition (4.2); notice that the contributions of such a "quadratic sector" are missing in rigid SK geometry, see e.g. [15] and [69]).

A further elaboration for ("large") non-BPS $Z \neq 0$ critical points of $V_{BH}$ can be performed by plugging the non-BPS $Z \neq 0$ criticality condition of $V_{BH}$ (see e.g. [9])

$$D_i \log Z = -\frac{i}{2} \frac{1}{|Z|^2} C_{ij} Z^i Z^j; \hspace{1cm} (5.16)$$

into Eq. (6.1), thus getting

$$R(Z)_{nBPS,Z\neq0} = \left[2 Z_i Z^i \left(2|Z|^2 - Z_i Z^i\right)\right]_{nBPS,Z\neq0}; \hspace{1cm} (5.17)$$

$$\Updownarrow$$

$$\left(Z_i Z^i\right)^2 - 2g^{ij} Z_i Z^i |Z|^2 + \frac{1}{2} R(Z) = 0; \hspace{1cm} (5.18)$$

$$\Updownarrow$$

$$\left(Z_i Z^i\right)^{\pm} = |Z|^2 \pm \sqrt{|Z|^4 - \frac{1}{2} R(Z)}; \hspace{1cm} (5.19)$$

$$\Updownarrow$$

$$|Z|^2 = \frac{1}{4} \frac{R(Z)}{|Z_i Z^i|} + \frac{1}{2} Z_i Z^i; \hspace{1cm} (5.20)$$

where the subscript "nBPS, $Z \neq 0" has been suppressed for simplicity's sake. Notice that, also when $R(Z) > 0$ satisfying $|Z|^4 - \frac{1}{2} R(Z) \geq 0$, only one brach of $Z_i Z^i$ should be consistent with the fact that $Z_i Z^i > 0$.

Result (5.17), holding for all $N = 2$, $d = 4$ ungauged Maxwell-Einstein supergravities, not only for the ones with symmetric scalar manifolds, consistently reduces to Eq. (5.7) when the "rule of three" (4.15) holds, as it is the case for symmetric SK manifolds (see discussion above).

By recalling Eqs. (4.13) and (4.14), Eq. (5.20) thus implies that

$$R(Z)_{non-BPS,Z\neq0} = -\left[2 \left(3 + \frac{\Delta}{|Z|^2}\right) \left(1 + \frac{\Delta}{|Z|^2}\right) |Z|^4\right]_{non-BPS,Z\neq0}; \hspace{1cm} (5.21)$$
or, more explicitly (evaluation at “large” non-BPS $Z \neq 0$ critical points of $V_{BH}$ understood)

$$R(Z) = -\frac{9}{8} \frac{|Z|^4}{(N_3(Z))^2} \left[ 4N_3(Z)|Z|^2 - E(Z, \overline{Z}) \right] \left[ \frac{4}{3} N_3(Z)|Z|^2 - E(Z, \overline{Z}) \right]. \quad (5.22)$$

where

$$N_3(Z) \equiv C_{ijk}Z^iZ^jZ^k; \quad (5.23)$$

$$E(Z, \overline{Z}) \equiv E_{ijklm}Z^iZ^jZ^kZ^lZ^m. \quad (5.24)$$

Results (5.21) and (5.22) relate $R(Z), N_3(Z)$ and $E(Z, \overline{Z})$ at “large” non-BPS $Z \neq 0$ critical points of $V_{BH}$ in generic $\mathcal{N} = 2, d = 4$ ungauged Maxwell-Einstein supergravities, and they consistently reduce to Eq. (5.7) (at least) for symmetric SK manifolds. They are consistent with the treatment performed in [7, 9, 70], see e.g. Eq. (3.1.2.17) of [70], here reported for ease of comparison (evaluation at non-BPS $Z \neq 0$ critical points of $V_{BH}$ understood):

$$\frac{3}{4} \frac{1}{|Z|^2} \frac{E(Z, \overline{Z})}{N_3(Z)} - 1 = \frac{R(Z)}{2 |Z|^2 Z\overline{Z}} = 2 \frac{R(Z)}{C_{ijkm}Z^iZ^jZ^kZ^lZ^m}. \quad (5.25)$$

Furthermore, through the definition (4.14), Eqs. (5.21) and (5.22) are implied also by Eq. (5.20). Notice that, while $R(Z)$ is a real quantity, $E_{ijklm}, E(Z, \overline{Z})$ and $\Delta$ are generally complex. But, (at least) at non-BPS $Z \neq 0$ critical points, $\Delta$, or equivalently the ratio $\frac{E(Z, \overline{Z})}{N_3(Z)}$, becomes real (consistent with Eq. (4.14); see also Eq. (276) of [7] and Eq. (5.17) of [70]).

6 Riemann Tensor

The new results obtained in previous Section call for an explicit computation of the Riemann tensor, Ricci tensor and Ricci scalar curvature for the SK geometries determined by the prepotential (3.1), as well for generic $d$-SK geometry, obtained as the classical limit $\xi \to 0$ of these former ones. We will do this in the present Section, carrying out the calculation in two different, but (proved to be) equivalent, ways.

6.1 First Approach

The first approach conceives SK geometry as a particular Kähler geometry, and therefore one starts with nothing but the standard formula of Riemann tensor:

$$R_{ijkl} = g^{im} \left( \partial_j \overline{\partial_l} \partial_m K \right) \partial_l \overline{\partial_k} \partial_m K - \overline{\partial_l} \partial_k \partial_m K. \quad (6.1)$$

After long but straightforward algebra (detailed in Appendix C), the Riemann tensor of the
SK geometry determined by the prepotential (3.1) is computed as

\[
R_{ijkl} = R_{ijkl} = -\frac{\nu^{2/3}}{32 \left( \nu - \frac{\xi}{2} \right)^2} \cdot \left\{ -\frac{\nu - \frac{\xi}{2}}{\nu + \xi} \hat{d}_{ik} \hat{d}_{jl} + 2 \hat{d}_{ij} \hat{d}_{kl} + 2 \hat{d}_{il} \hat{d}_{jk} + \frac{\nu^2}{4(\nu - \frac{\xi}{2})} \hat{d}_{ij} \hat{d}_{kl} \hat{d}_{il} + \frac{\nu}{2(\nu - \frac{\xi}{2})} \left( \hat{d}_{ij} \hat{d}_{kl} \hat{d}_{il} + \hat{d}_{ik} \hat{d}_{jl} \hat{d}_{il} + \hat{d}_{il} \hat{d}_{jk} \hat{d}_{il} + \hat{d}_{kl} \hat{d}_{ij} \hat{d}_{il} + \hat{d}_{il} \hat{d}_{jk} \hat{d}_{ik} + \hat{d}_{kl} \hat{d}_{ij} \hat{d}_{ik} + \hat{d}_{kl} \hat{d}_{ij} \hat{d}_{jk} \hat{d}_{il} \hat{d}_{kl} \hat{d}_{il} \hat{d}_{jl} \hat{d}_{kl} \hat{d}_{ij} \hat{d}_{kl} \hat{d}_{ij} \hat{d}_{kl} \hat{d}_{il} \hat{d}_{jm} \hat{d}_{mn} \right) \right\}. \quad (6.2)
\]

In the classical limit (\(\xi \to 0\)), the expression of the Riemann tensor in a generic \(d\)-SK geometry is easily obtained:

\[
R_{ijkl,\xi=0} = R_{ijkl,\xi=0} = -\frac{1}{32} \nu^{-4/3} \cdot \left\{ -\hat{d}_{ik} \hat{d}_{jl} + 2 \hat{d}_{ij} \hat{d}_{kl} + 2 \hat{d}_{il} \hat{d}_{jk} + \frac{\nu^2}{4} \hat{d}_{ij} \hat{d}_{kl} \hat{d}_{il} + \frac{\nu}{2} \left( \hat{d}_{ij} \hat{d}_{kl} \hat{d}_{il} + \hat{d}_{ik} \hat{d}_{jl} \hat{d}_{il} + \hat{d}_{il} \hat{d}_{jk} \hat{d}_{il} + \hat{d}_{kl} \hat{d}_{ij} \hat{d}_{il} + \hat{d}_{il} \hat{d}_{jk} \hat{d}_{ik} + \hat{d}_{kl} \hat{d}_{ij} \hat{d}_{ik} + \hat{d}_{kl} \hat{d}_{ij} \hat{d}_{jk} \hat{d}_{il} \hat{d}_{kl} \hat{d}_{ij} \hat{d}_{kl} \hat{d}_{il} \hat{d}_{jm} \hat{d}_{mn} \right) \right\}. \quad (6.3)
\]

Notice that both Eqs. (6.2) and (6.3) have all the symmetry properties suitable to the Riemann tensor.

Consequently, the Ricci tensor and Ricci curvature scalar can respectively be computed as follows (recall \(n_V\) denotes the number of Abelian vector multiplets coupled to gravity multiplet,
or equivalently the complex dimension of the considered SK manifold:

\[ R_{\bar{\gamma}\bar{\delta}} \equiv g^{\bar{\gamma}\bar{\delta}} R_{\bar{\alpha}\bar{\beta}} = -\frac{1}{16} \frac{\nu^{4/3}}{(\nu - \xi)^2} \left[ \frac{\nu \nu^3 + \frac{5}{2} (\nu + 2) \nu^2 \xi - \frac{3}{2} \nu \xi^2 - \frac{3}{4} (4 \nu + 3) \xi^3}{(\nu + \xi)^2 (\nu - \xi)^3} \right] \nu^{4/3} R_{i\bar{j}} + \right.

\[ + \frac{1}{16} \left[ 4 \nu \nu^2 + 2 (\nu + 3) \nu \xi - (2 n \nu + 3) \xi^2 \right] \nu^{1/3} R_{i\bar{j}} + \]

\[ + \frac{1}{4} \nu^{-2/3} d_{ikn} d_{jlm} \hat{d}^{kl} \hat{d}^{mn} \]

\[ = R_{i\bar{j}} \]  

\[ (6.4) \]

\[ R \equiv g^{\gamma\delta} R_{\gamma\delta} = -n \nu (n \nu + 1) - \frac{9}{2} \left( \frac{\nu - \xi}{\nu + \xi} \right) \nu + \frac{3}{2} n \nu - \frac{\nu - \xi}{\nu + \xi} d_{ikn} d_{jlm} \hat{d}^{kl} \hat{d}^{mn} \hat{d}^{il}. \]  

\[ (6.5) \]

Thence, in the classical limit (\( \xi \to 0 \)), the expression of the Ricci tensor and Ricci scalar curvature in a generic d-SK geometry is easily obtained, respectively:

\[ R_{\bar{\gamma}\bar{\delta}, \xi=0} \equiv g^{\bar{\gamma}\bar{\delta}} R_{\bar{\alpha}\bar{\beta}, \xi=0} = -\frac{1}{16} \nu^{-2/3} \left( n \nu \hat{d}_{\bar{i}\bar{j}} - 4 n \nu \hat{d}_{\bar{i}\bar{j}} - 4 d_{ikn} d_{jlm} \hat{d}^{kl} \hat{d}^{mn} \hat{d}^{il} \right) = R_{i\bar{j}, \xi=0}; \]  

\[ (6.6) \]

\[ R_{\xi=0} \equiv g^{ij} R_{ij, \xi=0} = -n \nu (n \nu + 1) + \frac{3}{2} n \nu - d_{ikn} d_{jlm} \hat{d}^{kl} \hat{d}^{mn} \hat{d}^{il} = \]

\[ = -n \nu^2 + \frac{n \nu}{2} - d_{ikn} d_{jlm} \hat{d}^{kl} \hat{d}^{mn} \hat{d}^{il}. \]  

\[ (6.7) \]

Let us notice that both Eqs. (6.4) and (6.6) have the symmetry properties suitable for Ricci tensor.

As pointed out at the end of Sect. 4 the symmetricity conditions (4.10) cannot be satisfied for prepotential (3.1) with \( \xi \neq 0 \). As it is well known, all symmetric spaces are Einstein spaces (see e.g. [74], and [75] for a comprehensive list of Refs.), i.e. with a Ricci tensor satisfying

\[ \exists \Lambda \in \mathbb{R} : R_{\gamma\delta} = \Lambda g_{\gamma\delta} \Rightarrow R = n \nu \Lambda, \]  

\[ (6.8) \]

and then with a constant Ricci scalar curvature, whose sign is the one of the real constant \( \Lambda \) itself. However, the opposite does not generally hold true: not all Einstein spaces are symmetric. Thus, it is reasonable to ask whether the considered quantum SK geometries determined by prepotential (3.1) can be Einstein. By recalling Eq. (2.22) and using Eq. (6.4), the condition
for such geometries to be Einstein can be written as follows:

\[
\frac{1}{4} \left( \nu + \xi \right) \left( \frac{\nu - \xi}{\nu + \xi} \right)^2 - \frac{4\nu(\nu + 2(n + 1)}{\nu + \xi} - \frac{4(n + 1)}{\nu + \xi} \left( \nu + \xi \right) \left( \frac{\nu - \xi}{\nu + \xi} \right)^2 \hat{d}_{ij} + \right]
\]

and it seems to us that such an Eq. does not admit solutions for any value of the real constants \( \xi \) and \( \Lambda \).

The situation is pretty different for the classical limit \( (\xi \to 0) \), determining the so-called \( d \)-SK geometries (described by prepotential (3.1) with \( \xi = 0 \)). For such geometries, by recalling Eq. (2.25) and using Eq. (6.6), the condition to be Einstein reads

\[
\frac{1}{4} \left( nV \hat{d}_{ij} - 4d_{ikn}d_{jlm} \right) \Rightarrow \hat{d}_{ij} = \frac{1}{4} \left( \nu - \frac{\xi}{\nu + \xi} \right) \hat{d}_{ij},
\]

As found in [53] (see also [33]), a (proper) subset of solutions to Eq. (6.10) is given by the symmetric \( d \)-SK geometries, satisfying the conditions of symmetricity (4.10)-(4.12). (At least) in such geometries, the \( d \)-tensor satisfies the following relation ([53, 10, 64]; see also the treatment given in [19], and Refs. therein):

\[
d_{\rho(kl)d_{ij}n}a^{pr}a^{ns}d_{rsq} = \frac{4}{3} \delta_{m}^{n}(kd_{ij}),
\]

which is a consequence of Eq. (4.12), and in fact can be further elaborated by using the second relation (involving the Riemann tensor) in Eq. (4.12) itself. In Eq. (6.11) \( a^{ij} \) is a sort of “rescaled” metric tensor, defined as (recall Eq. (2.26); see e.g. [64] for further elucidation of \( d = 5 \) origin of such a quantity):

\[
a^{ij} = \frac{1}{4} \nu^{-2/3} \hat{g}^{ij} = \frac{1}{2} \left( \hat{\lambda}^{i} \hat{\lambda}^{j} - 2\hat{d}^{ij} \right).
\]

Let us also notice that, from Eq. (6.6) the constancy of the Ricci scalar curvature is necessary but not sufficient condition for Einstein, and in turn for symmetric, spaces. In other words, it holds:

\[
\text{symmetric} \iff Einstein \iff constant R.
\]

Eq. (6.6) yields the condition \( (\Omega \in \mathbb{R}) \)

\[
- nV (nV + 1) - \frac{9}{2} \left( \nu - \frac{\xi}{\nu + \xi} \right) \nu + \frac{3}{2} \left( \nu - \frac{\xi}{\nu + \xi} \right) \nu + \xi - \frac{\nu - \frac{\xi}{\nu + \xi}}{\nu + \xi} d_{ikn}d_{jlm} \hat{d}^{ik} \hat{d}^{lm} = \Omega,
\]

and it seems that it is not possible to have \( R \) constant for SK geometries determined by (3.1) with \( \xi \neq 0 \). On the other hand, Eq. (6.7) yields the condition

\[
R_{\xi=0} = -nV + \frac{nV}{2} - d_{ikn}d_{jlm} \hat{d}^{ik} \hat{d}^{lm} = \Omega.
\]

As pointed out above, a (proper) set of solutions to condition (6.15) is given by the symmetric \( d \)-SK geometries. As for all Einstein spaces, for symmetric \( d \)-SK spaces it holds that

\[
\Omega = \Lambda nV.
\]
The results of [53] yields $\Lambda = -\frac{2}{3}n_V$ for the four irreducible symmetric $d$-SK geometries (which are nothing but the “magic” ones) and $\Lambda = -\frac{(n^2 - 2n_V + 3)}{n_V}$ for the reducible sequence $SU(1,1) \times SO(2, n_V - 1)$ (and $\Lambda = -(n_V + 1)$ for the minimal coupling $\mathbb{C}P^{n_V}$ sequence, whose prepotential is however quadratic).

Analogously to the comment made at the and of Sect. 4, it is here worth noticing that Eqs. (6.3), (6.6) and (6.7) respectively yield that the quantities

$$R_{i \bar{j} k, \xi = 0} = \nu^{1/3} R_{i \bar{j} k, \xi = 0};$$

$$R_{i \bar{j}, \xi = 0} = \nu^{2/3} R_{i \bar{j}, \xi = 0};$$

$$R_{\xi} = 0;$$

are independent on $\nu$, but they rather depend only on the “rescaled dilatons” $\hat{\lambda}$’s (recall definitions (2.19)-(2.21)):

$$\frac{\partial R_{i \bar{j} k, \xi = 0}}{\partial \nu} = 0;$$

$$\frac{\partial R_{i \bar{j}, \xi = 0}}{\partial \nu} = 0;$$

$$\frac{\partial R_{\xi}}{\partial \nu} = 0.$$

By looking at Eqs. (6.2), (6.4) and (6.5), it is easy to realize that the same does not happen for $\xi \neq 0$: the non-vanishing of the quantum parameter $\xi$ does not allow for an overall factorization of the dependence of $R_{i \bar{j} k}$, $R_{i \bar{j}, \xi = 0}$ and $R$ on $\nu$ and/or other (shifted and/or rescaled) variables. In other words, $\xi$ entangles the dependence of $R_{i \bar{j} k}$, $R_{i \bar{j}, \xi = 0}$ and $R$ on $\nu$ with the dependence on $\hat{\lambda}$’s, and thus the “$\xi \neq 0$ analogues” of $R_{i \bar{j} k, \xi = 0}$ and $R_{i \bar{j}, \xi = 0}$ (respectively defined in (6.17) and (6.18)) cannot be introduced. As already pointed out at the end of Sect. 4 this fact is related to the impossibility to uplift the quantum perturbatively corrected SK geometry described by the prepotential (3.1) to $d = 5$ space-time dimensions. Indeed, as it is well known, in general only $d$-SK geometries can be uplifted to $d = 5$ (see e.g. [64] and Refs. therein).

6.2 Second Approach

The second approach is actually the one considered in [53]: the constraints (4.5), characterizing, among others, a Kähler geometry to be special, are exploited in order to compute the Riemann tensor itself, yielding the same results given by Eq. (6.2) and (6.3), respectively for the prepotential (3.1) and its classical limit $\xi \to 0$ ($d$-SK geometry). The same can explicitly be proved to hold for the Ricci tensor (6.4) and the Ricci scalar (6.5), and for their respective classical limits (6.6) and (6.7).

Thus, the approaches respectively based on (6.1) and (6.5) have been proved to be equivalent, by explicitly computing the expressions of the Riemann tensor $R_{i \bar{j} k}$, of Ricci tensor $R_{i \bar{j}}$ and of Ricci scalar curvature $R$ of a SK geometry of arbitrary complex dimension $n_V$ and determined by the holomorphic prepotential (3.1) (also considering the corresponding limit of $d$-SK geometry, obtained by letting the quantum parameter $\xi \to 0$). As previously mentioned, by including in the prepotential the most general quantum perturbative correction consistent with the Peccei-Quinn axion-shift symmetry (32) (see discussion in the Introduction), the results and considerations of Sect. 6 are an extension of the findings of [53] to the quantum perturbative regime.
7 Conclusion

It is clear that the present investigation (completing, extending and generalizing the work of [45] and [54]) does not conclude the study of quantum (perturbative) SK geometries. Only some venues have been considered in the vast realm of quantum geometries of the moduli spaces of superstring theories. Many issues still deserve a deeper understanding and call for a thorough analysis, and we leave them for further future study. Below, we list only some of the most appealing ones (to us).

1. It would be interesting to determine the extent of validity of the so-called "rule of three" (4.15), which is nothing but the sum rule determining the value of $V_{BH}$ at its non-BPS $Z \neq 0$ critical points. While its "non-BPS $Z = 0$ analogue" (5.4) has general validity, (4.15) does not hold in general. Firstly noticed in [58], the "rule of three" (4.15) has been proved to hold in symmetric SK geometries [16], in (at least some of the) homogeneous non-symmetric $d$-SK geometries [71] (and in $\mathcal{N} > 2$-extended supergravities admitting non-supersymmetric attractors with non-vanishing central charge matrix [76, 7]). The most general results for $d$-SK geometries currently available are given in [58], but they are depending on the particular considered BH charge configurations; thus, it would be nice to see whether the "rule of three" (4.15) still holds in a generic BH charge configuration. On the other hand, since the condition (4.16) of validity of the "rule of three" does not imply symmetricity (nor homogeneity), it would be nice to see if and how the "rule of three" works in the quantum corrected SK geometries (3.1).

2. In the present paper we explained the peculiarity of the $D0-D6$ configuration in presence of the most general axion-shift-symmetric quantum perturbative parameter $\xi$. The $D0-D6$ configuration turns out to be the somewhat "minimal" configuration which does not support axion-free critical points of $V_{BH}$. But we did not yet completely explain the results of the investigation of [54]. Namely, we did not explain why the classical non-BPS $Z \neq 0$ "flat" direction of $V_{BH}$ of the $sl^2$ model gets non-renormalized (despite acquiring a non-vanishing axion) when switching $\xi$ on. We leave the investigation of this issue (within $d$-SKG geometries of arbitrary complex dimension $n_V$) for future study.

3. An issue concerning both $d$-SK geometries and their quantum corrected counterparts (3.1) is the generality of the axion-free solutions (if any) to the Attractor Eqs.. As found in [64], the axion-free-supporting BH charge configurations in $d$-SK geometries are the electric ($D2-D6$), magnetic ($D0-D4$) and $D0-D6$ ones, whereas in the present work we obtained that for SK geometries determined by the prepotential (3.1) only electric and magnetic configurations support purely imaginary critical points of $V_{BH}$. It would be interesting to analyze the degree of generality of axion-free solutions (in a model-independent fashion, if possible) in these frameworks.

4. Concerning $d$-SKG geometries, the expression of the $\frac{1}{2}$-BPS attractors is known in the most explicit form possible [57], and (going beyond symmetric cases) there are various explicit (but charge-dependent) results for non-BPS $Z \neq 0$ critical points of $V_{BH}$ (see e.g. [58]). On the other hand, there are currently no general results on the explicit form of non-BPS $Z = 0$ critical points of $V_{BH}$ within the same SK geometry. Thus, it would be interesting to determine such expression and use it to elaborate the "non-BPS $Z = 0$ analogue" (5.4) (obtained in the present paper) of the "rule of three" (4.15).

5. Still very little is known on the explicit expression of the critical points of the quantum perturbatively corrected BH potential $V_{BH}$ given by Eq. (2.27). The complete analysis of
\( \frac{1}{2} \)-BPS critical points (beyond the axion-free results of [18]; see the end of Sect. 3) should be based on the implementation of \( \frac{1}{2} \)-BPS conditions (3.13) through the formula (3.7). More interestingly, the non-BPS \((Z \neq 0 \text{ and } Z = 0)\) critical points of (2.27) still need to be completely determined and studied.

6. The phenomena of “splitting” of attractors [45], “transmutation” of attractors [45], and “lifting” of moduli spaces of attractors [54], even if explicitly found by studying models with only one or two complex scalar field(s), are likely to characterize the quantum perturbatively corrected SK geometry (3.1) for an arbitrary complex dimension \( n_V \). Thus, it would be worth studying more in depth such phenomena, eventually relating them with the presence of particular symmetry groups acting in transitive or non-transitive way on the (generally non-homogeneous) scalar manifold.

7. By extending the results obtained in [54] (at least in the magnetic and electric configurations) to the presence of more than one “flat” direction, and including the effects of non-perturbative corrections (see e.g. [35] [18] [21] [30]), one would lead to conjecture that only a (very) few classical attractors do remain attractors in strict sense at the quantum level. Consequently, at the quantum (perturbative and non-perturbative) level the set of actual extremal BH attractors should be strongly constrained and reduced. As already noticed in the Conclusion of [54] itself, in \( \mathcal{N} = 8, d = 4 \) supergravity the (“large”) \( \frac{1}{8} \)-BPS and non-BPS BHs critical points of \( V_{BH,\mathcal{N}=8} \) exhibit 40 and 42 “flat” directions, respectively [77] [50]. Within the possibility of \( \mathcal{N} = 8 \) supergravity to be a finite theory of quantum gravity (see e.g. [78] and [79], and Refs. therein), it would be interesting to understand whether these “flat” directions may be removed at all by perturbative and/or non-perturbative quantum effects.

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A Details of Computation of $V_{BH}$

The symplectic-covariant holomorphic sections determined by the prepotential (2.1) read

$$F_\Lambda(X;\xi) = \frac{\partial F(X;\xi)}{\partial X^\Lambda}; \quad \begin{cases} \Lambda = 0 & : F_0(X;\xi) = -\frac{1}{2}d_{ijk} \frac{X^i X^j X^k}{(X^0)^2} + 2i\xi X^0; \\ \Lambda = i & : F_i(X) = \frac{1}{2}d_{ijk} \frac{X^i X^k}{X^0}. \end{cases}$$

(A.1)

thus yielding the following check of homogeneity of degree 2 of $F(X;\xi)$ in $X^\Lambda$’s:

$$F_\Lambda(X;\xi) X^\Lambda = F_0(X;\xi) X^0 + F_i(X) X^i = 2F(X;\xi).$$

(A.2)

Let us now move to evaluate the various components of the symmetric matrix:

$$F_{\Lambda\Sigma}(X;\xi) = \frac{\partial^2 F(X;\xi)}{\partial X^\Lambda \partial X^\Sigma} = \frac{\partial^2 F(X;\xi)}{\partial X^\Lambda \partial X^\Sigma} = \frac{\partial^2 F(X;\xi)}{\partial X^\Lambda \partial X^\Sigma}.$$ (A.3)

They read:

$$(\Lambda, \Sigma) = (0, 0) : F_{00}(X;\xi) = \frac{1}{3}d_{ijk} \frac{X^i X^j X^k}{(X^0)^3} + 2i\xi;$$  

(A.4)

$$(\Lambda, \Sigma) = (0, i) : F_{0i}(X) = \frac{1}{2}d_{ijk} \frac{X^i X^k}{(X^0)^2};$$  

(A.5)

$$(\Lambda, \Sigma) = (i, j) : F_{ij}(X) = d_{ijk} \frac{X^k}{X^0},$$  

(A.6)

or in matrix form:

$$F_{\Lambda\Sigma} = \begin{pmatrix} F_{00}(X;\xi) & F_{0j}(X) \\ F_{i0}(X) & F_{ij}(X) \end{pmatrix} = \begin{pmatrix} \frac{1}{3}d_{ijk} \frac{X^i X^j X^k}{(X^0)^3} + 2i\xi & -\frac{1}{2}d_{ijk} \frac{X^i X^k}{X^0} \\ -\frac{1}{2}d_{i0k} \frac{X^k}{X^0} & d_{ijk} \frac{X^k}{X^0} \end{pmatrix}. $$ (A.7)

Notice that only $F_{00}$ changes by additive constant “$2i\xi$” with respect to the classical case of $d$-SK geometry ($\xi = 0$).

In order to explicitly compute the various terms of Eq. (2.2), let us start observing that Eq. (A.1) yields

$$Im[F_{\Lambda\Sigma}(X;\xi)] = \begin{pmatrix} \frac{1}{3}d_{ijk} Im\left[\frac{X^i X^j X^k}{(X^0)^3}\right] + 2\xi & \frac{1}{2}d_{jk} Im\left[\frac{X^k X^j}{(X^0)^2}\right] \\ -\frac{1}{2}d_{i0k} Im\left[\frac{X^k}{(X^0)^2}\right] & d_{ijk} Im\left[\frac{X^k}{X^0}\right] \end{pmatrix}. $$ (A.8)

Through Eqs. (A.7) and (A.8), one can then compute:

$$Im[F_{\Lambda\Sigma}(X;\xi)] X^\Lambda X^\Sigma = \left(\frac{1}{3}d_{ijk} Im\left[\frac{X^i X^j X^k}{(X^0)^3}\right] + 2\xi\right) (X^0)^2 +$$

$$-d_{i0k} Im\left[\frac{X^k}{(X^0)^2}\right] X^0 X^i + d_{ijk} Im\left[\frac{X^k}{X^0}\right] X^i X^j;$$

(A.9)

$$Im[F_{\Lambda\Omega}(X^0 X^\Omega)] = (Im F_{\Lambda\Omega})(Im F_{\Sigma\Omega})(X^0)^2 + (Im F_{\Lambda\Omega})(Im F_{\Sigma\Omega}) X^0 X^j +$$

$$+ (Im F_{\Lambda\Omega})(Im F_{\Sigma\Omega}) X^0 X^i + (Im F_{\Lambda\Omega})(Im F_{\Sigma\Omega}) X^i X^j.$$ (A.10)
Within the assumptions (2.3)-(2.5), one can thus write:

$$\mathcal{F}_{\Lambda \Sigma}(z; \xi) = \left( \frac{1}{3} d_{ijk} z^i z^j z^k + 2i \xi - \frac{i}{3} d_{ijl} z^i z^l \right), \tag{A.11}$$

yielding Eq. (2.6). Through the definition (2.4), one can further elaborate as follows:

$$d_{ijk} \text{Im} \left( z^k \right) = -d_{ijk} \lambda^k; \tag{A.12}$$

$$d_{ikl} \text{Im} \left( z^k z^l \right) = -2d_{ikl} x^k \lambda^l; \tag{A.13}$$

$$d_{ijk} \text{Im} \left( z^i z^j z^k \right) = -d_{ijk} \left( 3x^i x^j \lambda^k - \lambda^i \lambda^j \lambda^k \right). \tag{A.14}$$

Thus, the denominator of the second term in Eq. (2.2) explicitly reads:

$$\text{Im} \left[ \mathcal{F}_{\Lambda \Sigma}(X; \xi) \right] X^\Lambda X^\Sigma = \frac{4}{3} d_{ijk} \lambda^i \lambda^j \lambda^k + 2\xi. \tag{A.15}$$

On the other hand, the $(\Lambda, \Sigma = 0, 0)$-component of the numerator of the second term in Eq. (2.2) reads

$$\text{Im} (\mathcal{F}_{00}) \text{Im} (\mathcal{F}_{0\Delta}) X^\Omega X^\Delta = \left( \frac{1}{3} d_{ijk} \lambda^i \lambda^j \lambda^k - id_{ijk} x^i \lambda^j \lambda^k \right)^2 +$$

$$+ 4\xi^2 + \frac{4}{3} \xi d_{ijk} \lambda^i \lambda^j \lambda^k - 4i \xi d_{ijk} x^i \lambda^j \lambda^k. \tag{A.16}$$

Thus, through Eqs. (2.2), (2.6) and (A.12)-(A.14), the following result is achieved:

$$N_{00} = \mathcal{F}_{00} + 2i \left( \text{Im} \mathcal{F}_{00} \right) \text{Im} \left[ \mathcal{F}_{0\Omega}(X; \xi) \right] X^\Omega X^\Delta =$$

$$= \frac{1}{3} d_{ijk} x^i x^j x^k + 9\xi \frac{d_{ij} x^i \lambda^j \lambda^k}{(2d_{pqr} \lambda^p \lambda^q \lambda^r + 3\xi)} +$$

$$+ i \left\{ d_{ijk} x^i x^j \lambda^k - \frac{1}{6} d_{ijk} \lambda^i \lambda^j \lambda^k - \frac{1}{4} \xi + 3 \left[ \frac{4}{3} \xi^2 - \left( d_{ijk} x^i \lambda^j \lambda^k \right)^2 \right] \right\}. \tag{A.17}$$

It is worth noticing that in the classical limit $\xi \to 0$ one reobtains the known result for $N_{00}$ in d-SKG [64] (see also App. A of [80]).

Similarly, the $(\Lambda, \Sigma = 0, i)$-component of the numerator of the second term in Eq. (2.2) can be computed to be:

$$\left( \text{Im} \mathcal{F}_{00} \right) \left( \text{Im} \mathcal{F}_{i\Delta} \right) X^\Omega X^\Delta = d_{ijk} \lambda^j \lambda^k \left( d_{pqr} x^p \lambda^q \lambda^r + \frac{i}{3} d_{pqr} \lambda^p \lambda^q \lambda^r + 2i \xi \right). \tag{A.18}$$

Thus, through Eqs. (2.2), (2.6) and (A.12)-(A.14), one obtains:

$$N_{0i} = \mathcal{F}_{0i} + 2i \left( \text{Im} \mathcal{F}_{00} \right) \text{Im} \left[ \mathcal{F}_{i\Omega}(X; \xi) \right] X^\Omega X^\Delta =$$

$$= -\frac{1}{2} d_{ijk} x^j x^k - \frac{3}{2} \left( \frac{\xi}{d_{lmn} \lambda^m \lambda^n + \xi} - d_{ijk} \lambda^j \lambda^k +$$

$$+ i \left[ -d_{ijk} x^j \lambda^k + \frac{d_{pqr} x^p \lambda^q \lambda^r}{(2d_{lmn} \lambda^m \lambda^n + \xi)} d_{ijk} \lambda^j \lambda^k \right] \right), \tag{A.19}$$
which in the classical limit $\xi \to 0$ is consistent with the known result for $\mathcal{N}_0$ in $d$-SKG [64] (see also App. A of [80]).

The expression of $\mathcal{N}_{ij}$ is (almost) the same of the classical ($\lambda \to 0$) case [64] (see also App. A of [80]), namely:

$$
\mathcal{N}_{ij} = \bar{\mathcal{F}}_{ij} + 2i \left( \frac{Im \mathcal{F}_{i\alpha}(\lambda)}{Im [\mathcal{F}_{\alpha\xi}(X;\lambda)]} \right) \bar{X}^\alpha X^\xi \bar{X}^\omega X^\omega = \\
= d_{ijk} x^k + i \left[ d_{ijk} \lambda^k - \frac{d_{ikm}d_{ijn} \lambda^m \lambda^j \lambda^n}{(\frac{1}{7}d_{pqr} \lambda^p \lambda^q \lambda^r + \xi)} \right].
$$

In order to compute $V_{BH}$ by using Eq. (2.8) and subsequent definitions, it is convenient to introduce the symmetric matrix

$$
\mathcal{A}_{ij} \equiv \frac{1}{12} \frac{\nu^{1/3}}{\tilde{\nu}} \left( \tilde{d}_{ij} - \frac{\nu \tilde{d}_i \tilde{d}_j}{4} \right),
$$

whose inverse reads

$$
\mathcal{A}^{ij} = 12 \frac{\tilde{\nu}}{\nu^{1/3}} \left[ \tilde{d}^{ij} - \frac{\nu^{2/3} \tilde{\lambda}_i \tilde{\lambda}_j}{2 (\nu - \frac{\xi}{2})} \right], \quad \mathcal{A}^{ij} \mathcal{A}_{jk} \equiv \delta^i_k,
$$

along with the related contractions

$$
\mathcal{A}_i \equiv \mathcal{A}_{ij} x^j; \quad \mathcal{A} \equiv \mathcal{A}_{ij} x^i x^j.
$$

Notice that there is no simple relations between the symmetric matrices $g_{ij}$ and $\mathcal{A}_{ij}$, respectively given by Eqs. (2.22) and (A.21). Generally, they are proportional only in the classical limit:

$$
\lim_{\xi \to 0} \mathcal{A}_{ij} = \frac{1}{12} \nu^{-2/3} \left( \tilde{d}_{ij} - \frac{\tilde{d}_i \tilde{d}_j}{4} \right) = - \frac{1}{3} \tilde{g}_{ij};
$$

$$
\lim_{\xi \to 0} \mathcal{A}^{ij} = 12 \nu^{2/3} \left( \tilde{d}^{ij} - \frac{\tilde{\lambda}_i \tilde{\lambda}_j}{2} \right) = - 3 \tilde{g}^{ij}.
$$
Finally, one can then compute the following expressions:

\[ \text{Re} \, N_{00} = \frac{1}{3} h + \frac{3}{4} \xi \frac{\nu^{2/3}}{\nu} \hat{d}_i x^i; \quad (A.27) \]

\[ \text{Re} \, N_{0i} = -\frac{1}{2} h_i - \frac{3}{8} \xi \frac{\nu^{2/3}}{\nu} \hat{d}_i = -\frac{1}{2} \frac{\partial \text{Re} \, N_{00}}{\partial x^i}; \quad (A.28) \]

\[ \text{Re} \, N_{ij} = \frac{1}{2} \frac{\partial^2 \text{Re} \, N_{00}}{\partial x^i \partial x^j}; \quad (A.29) \]

\[ \text{Im} \, N_{00} = -\hat{\nu} \left( 1 - 12 A \right) + \frac{9}{16} \hat{\xi}^2; \quad (A.30) \]

\[ \text{Im} \, N_{0i} = -12 \hat{\nu} A_i = -\frac{1}{2} \frac{\partial \text{Im} \, N_{00}}{\partial x^i}; \quad (A.31) \]

\[ \text{Im} \, N_{ij} = 12 \hat{\nu} A_{ij} = \frac{1}{2} \frac{\partial \text{Im} \, N_{00}}{\partial x^i \partial x^j}; \quad (A.32) \]

\[ (\text{Im} \, N)^{-1 \, 00} = -\frac{1}{\hat{\nu}} \left( 1 - \frac{9}{16} \hat{\xi}^2 \right)^{-1}; \quad (A.33) \]

\[ (\text{Im} \, N)^{-1 \, 0i} = -\frac{1}{\hat{\nu}} \left( 1 - \frac{9}{16} \hat{\xi}^2 \right)^{-1} x^i; \quad (A.34) \]

\[ (\text{Im} \, N)^{-1 \, ij} = -\frac{1}{\hat{\nu}} \left( 1 - \frac{9}{16} \hat{\xi}^2 \right)^{-1} \begin{bmatrix} x^i x^j - \frac{1}{12} \left( 1 - \frac{9}{16} \hat{\xi}^2 \right) A_{ij} \end{bmatrix}. \quad (A.35) \]

The equivalent matrix expressions of Eqs. (A.27)-(A.35) respectively read

\[ \text{Re} \, N_{\Lambda \Sigma} = \begin{pmatrix} \frac{1}{3} h + \frac{3}{4} \xi \frac{\nu^{2/3}}{\nu} \hat{d}_i x^i & -\frac{1}{2} h_j - \frac{3}{8} \xi \frac{\nu^{2/3}}{\nu} \hat{d}_j \\ -\frac{1}{2} h_i - \frac{3}{8} \xi \frac{\nu^{2/3}}{\nu} \hat{d}_i & h_{ij} \end{pmatrix}; \quad (A.36) \]

\[ \text{Im} \, N_{\Lambda \Sigma} = -\hat{\nu} \begin{pmatrix} 1 - 12 A - \frac{9}{16} \hat{\xi}^2 & 12 A_j \\ 12 A_i & -12 A_{ij} \end{pmatrix}; \quad (A.37) \]

\[ (\text{Im} \, N_{\Lambda \Sigma})^{-1} = -\frac{1}{\hat{\nu}} \left( 1 - \frac{9}{16} \hat{\xi}^2 \right)^{-1} \begin{pmatrix} 1 & x^j \\ x^i & x^i x^j - \frac{1}{12} \left( 1 - \frac{9}{16} \hat{\xi}^2 \right) A_{ij} \end{pmatrix}. \quad (A.38) \]

In the classical limit (\( \xi \to 0 \)) all above expressions yield the known results for d-SKG [64] (see also App. A of [80]).

All above results then yield to the explicit expression of \( V_{BH} \) given by Eq. (2.27).

**B** **Details of Computation of E-Tensor**

In order to compute the \( E \)-tensor for the prepotential (3.1), let us start by splitting the differential operator \( \hat{\partial} \) according to Eq. (2.4):

\[ \hat{\partial}_i \equiv \frac{\partial}{\partial z^i} = \frac{1}{2} \left( \frac{\partial}{\partial x^i} + i \frac{\partial}{\partial \lambda^i} \right). \quad (B.1) \]
This, by its very definition (4.2), the $C$-tensor can be computed to be (see also Eq. (3.2))

$$C_{ijk} = \exp(K) d_{ijk} = \frac{3}{4} \frac{d_{ijk}}{(d_{lmn})^{1/3} \lambda^m \lambda^n - 3\xi} = \frac{1}{8} \frac{d_{ijk}}{\nu - \frac{\xi}{2}},$$

(B.2)

thus implying that the $C$-tensor simply gets multiplicatively renormalized with respect to its classical ($\xi \to 0$) expression:

$$C_{ijk} = \frac{\nu}{\left(\nu - \frac{\xi}{2}\right)} C_{ijk,\xi=0}. \tag{B.3}$$

Thence, by observing that

$$\partial_i d_{jk} = \frac{i}{2} \frac{\partial}{\partial \lambda^i} d_{jk} = \frac{i}{2} d_{ijk}; \tag{B.4}$$

$$\partial_i d_j = \frac{i}{2} \frac{\partial}{\partial \lambda^i} d_j = 2d_{ij} = 2\nu^{1/3} \tilde{d}_{ij}; \tag{B.5}$$

$$\partial_i \nu = \frac{i}{2} \frac{\partial}{\partial \lambda^i} \nu = \frac{i}{4} d_i = \frac{i}{4} \nu^{2/3} \tilde{d}_i, \tag{B.6}$$

one can compute that

$$\partial_i C_{jkl} = -\frac{i}{2^6} \frac{\nu}{\left(\nu - \frac{\xi}{2}\right)^2} \tilde{d}_i \tilde{d}_{jkl}, \tag{B.7}$$

and (consistent with Eq. (3.2)):

$$\partial_i K = -\frac{i}{4} \frac{\nu^{2/3}}{\left(\nu - \frac{\xi}{2}\right)} \tilde{d}_i. \tag{B.8}$$

Using Eqs. (2.22), (2.23) and (4.8), the connection $\Gamma$ is computed as follows:

$$-\frac{i}{3} \Gamma_{ij}^m = -\frac{\nu^{2/3}}{6(\nu + \xi)} \tilde{d}_i \tilde{d}_j \tilde{d}_m - \frac{5}{24} \frac{\nu^{8/3}}{(\nu + \xi)\left(\nu - \frac{\xi}{2}\right)^2} \tilde{d}_i \tilde{d}_j \tilde{d}_m +$$

$$+ \frac{\nu^{2/3}}{6} \left(\nu - \frac{\xi}{2}\right) \tilde{d}_i \delta_j^m + \frac{\nu^{2/3}}{6} \left(\nu - \frac{\xi}{2}\right) \tilde{d}_j \delta_i^m - \frac{1}{3} d_{ij} \tilde{d}^m_\nu \nu^{-1/3}$$

$$= \frac{2}{3} \Gamma_{(ij)}^m, \tag{B.9}$$

and therefore:

$$-\frac{i}{3} \left(\nu - \frac{\xi}{2}\right) \Gamma_{ij}^m C_{mkl} = -\frac{\nu^{2/3}}{6(\nu + \xi)} \tilde{d}_i \tilde{d}_j \tilde{d}_k \tilde{d}_l - \frac{5}{24} \frac{\nu^{5/3}}{(\nu + \xi)\left(\nu - \frac{\xi}{2}\right)^2} \tilde{d}_i \tilde{d}_j \tilde{d}_k \tilde{d}_l +$$

$$+ \frac{\nu^{2/3}}{6} \left(\nu - \frac{\xi}{2}\right) \tilde{d}_i \tilde{d}_j \tilde{d}_k \tilde{d}_l + \frac{\nu^{2/3}}{6} \left(\nu - \frac{\xi}{2}\right) \tilde{d}_j \tilde{d}_k \tilde{d}_l - \frac{1}{3} \nu^{-1/3} d_{ij} d_{mn} d_{mkl} \tilde{d}^m_\nu.$$  

(B.10)

Through some straightforward elaborations, all this leads to Eq. (4.17), and then to Eqs. (4.18) and (4.20).
C Details of Computation of Riemann Tensor

In order to compute the Riemann tensor \( R_{ijkl} \) given by Eq. (6.1) (i.e. working in the approach considered in Subsect. 6.1), one needs to recall Eqs. (3.2), (2.22), (2.23), (B.1) and (B.4)-(B.6), which leads to the following results:

\[
\overline{\partial}_i \overline{\partial}_j \partial_m K = \frac{i}{2} \frac{\partial}{\partial \lambda} g_{mij} = \left[ \partial_{mij} - \frac{\nu}{2} \left( \partial_{mij} \hat{a}_i + \partial_{mij} \hat{a}_j + \partial_{mij} \tilde{a}_m \right) + \frac{\nu^2}{4} \left( \partial_{mij} \hat{a}_i + \partial_{mij} \tilde{a}_m \right) \right];
\]

(C.1)

\[
g^{\mu\nu} \left( \overline{\partial}_\mu \overline{\partial}_\nu g_{mij} \right) \partial_k \overline{\partial}_\mu \overline{\partial}_\nu K = g^{\mu\nu} \left( \overline{\partial}_\mu g_{mij} \right) \partial_k \overline{\partial}_\mu \overline{\partial}_\nu K = \left[ \partial_{mij} - \frac{\nu}{2} \left( \partial_{mij} \hat{a}_i + \partial_{mij} \tilde{a}_m \right) + \frac{\nu^2}{4} \left( \partial_{mij} \hat{a}_i + \partial_{mij} \tilde{a}_m \right) \right];
\]

(C.2)

All above results are the basic ingredients, through some simple algebra, of the result (6.2).

References

[1] S. Ferrara, R. Kallosh and A. Strominger, \( N = 2 \) Extremal Black Holes, Phys. Rev. D52, 5412 (1995), hep-th/9508072.
[2] A. Strominger, *Macroscopic Entropy of $\mathcal{N} = 2$ Extremal Black Holes*, Phys. Lett. **B383**, 39 (1996), hep-th/9602111.

[3] S. Ferrara and R. Kallosh, *Supersymmetry and Attractors*, Phys. Rev. **D54**, 1514 (1996), hep-th/9602136.

[4] S. Ferrara and R. Kallosh, *Universality of Supersymmetric Attractors*, Phys. Rev. **D54**, 1525 (1996), hep-th/9603090.

[5] S. Ferrara, G. W. Gibbons and R. Kallosh, *Black Holes and Critical Points in Moduli Space*, Nucl. Phys. **B500**, 75 (1997), hep-th/9702103.

[6] G. W. Gibbons and C. M. Hull, *A Bogomol’ny Bound for General Relativity and Solitons in $\mathcal{N} = 2$ Supergravity*, Phys. Lett. **B109**, 190 (1982).

[7] L. Andrianopoli, R. D’Auria, S. Ferrara and M. Trigiante, *Extremal black holes in supergravity*, Lect. Notes Phys. **737**, 661 (2008), hep-th/0611345.

[8] A. Sen, *Black Hole Entropy Function, Attractors and Precision Counting of Microstates*, arXiv:0708.1270.

[9] S. Bellucci, S. Ferrara, R. Kallosh and A. Marrani, *Extremal Black Hole and Flux Vacua Attractors*, Lect. Notes Phys. **755**, 115 (2008), arXiv:0711.4547 [hep-th].

[10] S. Ferrara, K. Hayakawa and A. Marrani, *Lectures on Attractors and Black Holes*, Fortsch. Phys. **56**, 993 (2008), arXiv:0805.2498 [hep-th].

[11] S. Bellucci, S. Ferrara, M. Günaydin and A. Marrani, *SAM Lectures on Extremal Black Holes in $d = 4$ Extended Supergravity*, arXiv:0905.3739 [hep-th].

[12] A. Sen, *Black hole entropy function and the attractor mechanism in higher derivative gravity*, JHEP **0509**, 038 (2005), hep-th/0506177.

[13] A. Strominger, *Special Geometry*, Commun. Math. Phys. **133**, 163 (1990).

[14] A. Ceresole, R. D’Auria and S. Ferrara, *The Symplectic Structure of $\mathcal{N} = 2$ Supergravity and Its Central Extension*, Nucl. Phys. Proc. Suppl. **46** (1996), hep-th/9509106.

[15] D. S. Freed, *Special Kähler manifolds*, Commun. Math. Phys. **203**, 31 (1999), hep-th/9712042.

[16] S. Bellucci, S. Ferrara, M. Günaydin and A. Marrani, *Charge orbits of symmetric special geometries and attractors*, Int. J. Mod. Phys. **A21**, 5043 (2006), hep-th/0606209.

[17] G. L. Cardoso, D. Lust and T. Mohaupt, *Modular symmetries of $\mathcal{N} = 2$ black holes*, Phys. Lett. **B388**, 266 (1996), hep-th/9608099.

[18] K. Behrndt, G. L. Cardoso, B. de Wit, R. Kallosh, D. Lust and T. Mohaupt, *Classical and quantum $\mathcal{N} = 2$ supersymmetric black holes*, Nucl. Phys. **B488**, 236 (1997), hep-th/9610105.

[19] K. Behrndt, *Quantum corrections for $D = 4$ black holes and $D = 5$ strings*, Phys. Lett. **B396**, 77 (1997), hep-th/9610232.
[20] K. Behrndt and T. Mohaupt, *Entropy of $\mathcal{N}=2$ black holes and their M-brane description*, Phys. Rev. D56, 2206 (1997), hep-th/9611140.

[21] K. Behrndt, G. L. Cardoso, I. Gaida, *Quantum $\mathcal{N}=2$ supersymmetric black holes in the $S-T$ model*, Nucl. Phys. B506, 267 (1997), hep-th/9704095.

[22] J. M. Maldacena, A. Strominger and E. Witten, *Black hole entropy in M-theory*, JHEP 1997, 002 (1997), hep-th/9711053.

[23] G. L. Cardoso, B. de Wit and T. Mohaupt, *Corrections to macroscopic supersymmetric black hole entropy*, Phys. Lett. B451, 309 (1999), hep-th/9812082.

[24] G. L. Cardoso, B. de Wit and T. Mohaupt, *Deviations from the area law for supersymmetric black holes*, Fortsch. Phys. 48, 49 (2000), hep-th/9904005.

[25] G. L. Cardoso, B. de Wit and T. Mohaupt, *Macroscopic entropy formulae and non-holomorphic corrections for supersymmetric black holes*, Nucl. Phys. B567, 87 (2000), hep-th/9906094.

[26] G. L. Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Stationary BPS solutions in $\mathcal{N}=2$ supergravity with $R^2$ interactions*, JHEP 0012, 019 (2000), hep-th/0009234.

[27] G. L. Cardoso, D. Lüst and J. Perz, *Entropy maximization in the presence of higher-curvature interactions*, JHEP 0605, 028 (2006), hep-th/0603211.

[28] G. L. Cardoso, V. Grass, D. Lüst and J. Perz, *Extremal non-BPS Black Holes and Entropy Extremization*, JHEP 0609, 078 (2006), hep-th/0607202.

[29] G.L. Cardoso, B. de Wit and S. Mahapatra, *Black hole entropy functions and attractor equations*, JHEP 0703, 085 (2007), hep-th/0612225.

[30] G.L. Cardoso, B. de Wit and S. Mahapatra, *Subleading and non-holomorphic corrections to $\mathcal{N}=2$ BPS black hole entropy*, HEP 0902, 006 (2009), arxiv:0808.2627 [hep-th].

[31] S. Cecotti, S. Ferrara and L. Girardello, *Geometry of Type II Superstrings and the Moduli of Superconformal Field Theories*, Int. J. Mod. Phys. A4, 2475 (1989).

[32] R. D. Peccei and H. R. Quinn, *Constraints imposed by CP conservation in the presence of instantons*, Phys. Rev. D16, 1791 (1977). R. D. Peccei and H. R. Quinn, *CP conservation in the presence of instantons*, Phys. Rev. Lett. 38, 1440 (1977). R. D. Peccei and H. R. Quinn, *Some aspects of instantons*, Nuovo Cim. A41, 309 (1977).

[33] B. de Wit and A. Van Proeyen, *Special geometry, cubic polynomials and homogeneous quaternionic spaces*, Commun. Math. Phys. 149, 307 (1992), hep-th/9112027.

[34] B. de Wit, F. Vanderseypen and A. Van Proeyen, *Symmetry structure of special geometries*, Nucl. Phys. B400, 463 (1993), hep-th/9210088.

[35] P. Candelas, X. C. De La Ossa, P. S. Green and L. Parkes, *A Pair of Calabi-Yau Manifolds as an Exactly Soluble Superconformal Theory*, Nucl. Phys. B359, 21 (1991). P. Candelas, X. C. De La Ossa, P. S. Green and L. Parkes, *An Exactly Soluble Superconformal Theory from a Mirror Pair of Calabi-Yau Manifolds*, Phys. Lett. B258, 118 (1991).
[36] L. Alvarez-Gaume, D. Z. Freedman, *Geometrical Structure and Ultraviolet Finiteness in the Supersymmetric Sigma Model*, Commun. Math. Phys. **80**, 443 (1981).

[37] M. T. Grisaru, A. van de Ven and D. Zanon, *Four Loop Beta Function for the \(N=1\) and \(N=2\) Supersymmetric Nonlinear Sigma Model in Two Dimensions*, Phys. Lett. **B173**, 423 (1986). M. T. Grisaru, A. van de Ven and D. Zanon, *Two Dimensional Supersymmetric Sigma Models on Ricci Flat Kähler Manifolds are not Finite*, Nucl. Phys. **B277**, 388 (1986). M. T. Grisaru, A. van de Ven and D. Zanon, *Four Loop Divergences for the \(N=1\) Supersymmetric Nonlinear Sigma Model in Two Dimensions*, Nucl. Phys. **B277**, 409 (1986).

[38] S. Hosono, A. Klemm, S. Theisen and Shing-Tung Yau, *Mirror symmetry, mirror map and applications to Calabi-Yau hypersurfaces*, Commun. Math. Phys. **167**, 301 (1995), hep-th/9308122.

[39] E. Witten, *Dyons of charge \(e \theta/2 \pi\)*, Phys. Lett. **B86**, 283 (1979).

[40] S. Ferrara, J. A. Harvey, A. Strominger and C. Vafa, *Second quantized mirror symmetry*, Phys. Lett. **B361**, 59 (1995), hep-th/9505162. P. S. Aspinwall, *An \(N=2\) dual pair and a phase transition*, Nucl. Phys. **B460**, 57 (1996), hep-th/9510142. D. R. Morrison and C. Vafa, *Compactifications of \(F\) theory on Calabi-Yau threefolds. 1*, Nucl. Phys. **B473**, 74 (1996), hep-th/9602114. D. R. Morrison and C. Vafa, *Compactifications of \(F\) theory on Calabi-Yau threefolds. 2*, Nucl. Phys. **B476**, 437 (1996), hep-th/9603161. J. A. Harvey and G. W. Moore, *Exact gravitational threshold correction in the FHSV model*, Phys. Rev. **D57**, 2329 (1998), hep-th/9611176. A. Klemm and M. Marino, *Counting BPS states on the Enriques Calabi-Yau*, Commun. Math. Phys. **280**, 27 (2008), hep-th/0512227. J. R. David, *On the dyon partition function in \(N=2\) theories*, JHEP **0802**, 025 (2008), arXiv:0711.1971.

[41] M. Bianchi and S. Ferrara, *Enriques and Octonionic Magic Supergravity Models*, JHEP **0802**, 054 (2008), arXiv:0712.2976 [hep-th].

[42] M. Günyaydin, *Lectures on Spectrum Generating Symmetries and \(U\)-duality in Supergravity, Extremal Black Holes, Quantum Attractors and Harmonic Superspace*, arXiv:0908.0374 [hep-th].

[43] S. Bellucci, S. Ferrara, A. Marrani and A. Yeranyan, *Mirror Fermat Calabi-Yau Threefolds and Landau-Ginzburg Black Hole Attractors*, Riv. Nuovo Cim. **029**, 1 (2006), hep-th/0608091.

[44] P. Kaura and A. Misra, *On the Existence of Non-Supersymmetric Black Hole Attractors for Two-Parameter Calabi-Yau’s and Attractor Equations*, Fortsch. Phys. **54**, 1109 (2006), hep-th/0607132.

[45] S. Bellucci, S. Ferrara, A. Marrani and A. Shcherbakov, *Splitting of Attractors in 1-modulus Quantum Corrected Special Geometry*, JHEP **0802**, 088 (2008), arXiv:0710.3559 [hep-th].

[46] A. Chou, R. Kallosh, J. Rahmfeld, Soo-Jong Rey, M. Shmakova and Wing Kai Wong, *Critical points and phase transitions in 5-d compactifications of M-theory*, Nucl. Phys. **B508**, 147 (1997), hep-th/9704142.
[47] R. Kallosh, A. D. Linde and M. Shmakova, *Supersymmetric multiple basin attractors*, JHEP **9911**, 010 (1999), [hep-th/9910021](http://arxiv.org/abs/hep-th/9910021).

[48] M. Wijnholt and S. Zhukov, *On the Uniqueness of Black Hole Attractors*, [hep-th/9912002](http://arxiv.org/abs/hep-th/9912002).

[49] G.W. Moore, *Attractors and Arithmetic*, [hep-th/9807056](http://arxiv.org/abs/hep-th/9807056). G.W. Moore, *Arithmetic and Attractors*, [hep-th/9807087](http://arxiv.org/abs/hep-th/9807087). G.W. Moore, *Les Houches Lectures on Strings and Arithmetic*, [hep-th/0401049](http://arxiv.org/abs/hep-th/0401049).

[50] S. Ferrara and A. Marrani, *N= 8 non-BPS Attractors, Fized Scalars and Magic Supergravities*, Nucl. Phys. **B788**, 63 (2008), [arXiv:0705.3866](http://arxiv.org/abs/0705.3866).

[51] S. Ferrara and A. Marrani, *On the Moduli Space of non-BPS Attractors for N= 2 Symmetric Manifolds*, Phys. Lett. **B652**, 111 (2007), [arXiv:0706.1667](http://arxiv.org/abs/0706.1667).

[52] S. Bellucci, S. Ferrara, A. Shcherbakov and A. Yeranyan, *Black hole entropy, flat directions and higher derivatives*, [arXiv:0906.4910 [hep-th]](http://arxiv.org/abs/0906.4910).

[53] E. Cremmer and A. Van Proeyen, *Classification Of Kahler Manifolds In N= 2 Vector Multiplet Supergravity Couplings*, Class. Quant. Grav. **2**, 445 (1985).

[54] S. Bellucci, S. Ferrara, A. Marrani and A. Shcherbakov, *Quantum Lift of Non-BPS Flat Directions*, Phys. Lett. **B672**, 77 (2009), [arXiv:0811.3494 [hep-th]](http://arxiv.org/abs/0811.3494).

[55] M. J. Duff, J. T. Liu and J. Rahmfeld, *Four-dimensional string/string/string triality*, Nucl. Phys. **B459**, 125 (1996), [hep-th/9508094](http://arxiv.org/abs/hep-th/9508094).

[56] K. Behrndt, R. Kallosh, J. Rahmfeld, M. Shmakova and W. K. Wong, *STU Black Holes and String Triality*, Phys. Rev. **D54**, 6293 (1996), [hep-th/9608059](http://arxiv.org/abs/hep-th/9608059).

[57] M. Shmakova, *Calabi-Yau black holes*, Phys. Rev. **D56**, 540 (1997), [hep-th/9612076](http://arxiv.org/abs/hep-th/9612076).

[58] P. K. Tripathy and S. P. Trivedi, *Non-supersymmetric attractors in string theory*, JHEP **0603**, 022 (2006), [hep-th/0511117](http://arxiv.org/abs/hep-th/0511117).

[59] K. Saraikin and C. Vafa, *Non-supersymmetric Black Holes and Topological Strings*, Class. Quant. Grav. **25**, 095007 (2008), [hep-th/0703214](http://arxiv.org/abs/hep-th/0703214).

[60] S. Nampuri, P. K. Tripathy and S. P. Trivedi, *On The Stability of Non-Supersymmetric Attractors in String Theory*, JHEP **0708**, 054 (2007), [arXiv:0705.4554](http://arxiv.org/abs/0705.4554).

[61] S. Bellucci, A. Marrani, E. Orazi and A. Shcherbakov, *Attractors with Vanishing Central Charge*, Phys. Lett. **B655**, 185 (2007), [arXiv:0707.2730](http://arxiv.org/abs/0707.2730).

[62] S. Bellucci, S. Ferrara, A. Marrani and A. Yeranyan, *stu Black Holes Unveiled*, Entropy Vol. **10**(4), 507 (2008), [arXiv:0807.3503 [hep-th]](http://arxiv.org/abs/0807.3503).

[63] L. Andrianopoli, S. Ferrara, A. Marrani and M. Trigiante, *Non-BPS Attractors in 5d and 6d Extended Supergravity*, Nucl. Phys. **B795**, 428 (2008), [arXiv:0709.3488](http://arxiv.org/abs/0709.3488).

[64] A. Ceresole, S. Ferrara and A. Marrani, *4d/5d Correspondence for the Black Hole Potential and its Critical Points*, Class. Quant. Grav. **24**, 5651 (2007), [arXiv:0707.0964 [hep-th]](http://arxiv.org/abs/0707.0964).
[65] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D’Auria, S. Ferrara, P. Fré and T. Magri, $\mathcal{N}=2$ supergravity and $\mathcal{N}=2$ super Yang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map, J. Geom. Phys. 23, 111 (1997), hep-th/9605032.

[66] L. Castellani, R. D’Auria and S. Ferrara, Special Geometry without Special Coordinates, Class. Quant. Grav. 7, 1767 (1990). L. Castellani, R. D’Auria and S. Ferrara, Special Kähler Geometry: an Intrinsic Formulation from $\mathcal{N}=2$ Space-Time Supersymmetry, Phys. Lett. B241, 57 (1990).

[67] R. D’Auria, S. Ferrara and P. Fré, Special and Quaternionic Isometries: General Couplings in $\mathcal{N}=2$ Supergravity and the Scalar Potential, Nucl. Phys. B359, 705 (1991).

[68] S. Bellucci, S. Ferrara and A. Marrani, Attractors in Black, Fortsch. Phys. 56, 761 (2008), arXiv:0805.1310 [hep-th].

[69] Z. Lu, A Note on Special Kähler Manifolds, Math. Ann. 313, 711 (1999), math/0505577.

[70] B. L. Cerchiai, S. Ferrara, A. Marrani, and B. Zumino, Duality, Entropy and ADM Mass in Supergravity, Phys. Rev. D79, 125010 (2009), arXiv:0902.3973 [hep-th].

[71] S. Ferrara and R. Kallosh, On $\mathcal{N}=8$ Attractors, Phys. Rev. D73, 125005 (2006), hep-th/0603247.

[72] L. Andrianopoli, R. D’Auria and S. Ferrara, $U$ invariants, black hole entropy and fixed scalars, Phys. Lett. B403, 12 (1997), hep-th/9703156.

[73] Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson, D. A. Kosower and R. Roiban, Three-Loop Finiteness of $\mathcal{N}=8$ Supergravity, Phys. Rev. Lett. 98, 161303 (2007), hep-th/0702112.

[74] R. Kallosh, On UV Finiteness of the Four Loop $\mathcal{N}=8$ Supergravity, JHEP 0909, 116 (2009), ArXiv:0906.3495 [hep-th].
[80] G. L. Cardoso, J. M. Oberreuter and J. Perz, *Entropy function for rotating extremal black holes in very special geometry*, JHEP 0705, 025 (2007), hep-th/0701176.