On the Approximation Lower Bound for Neural Nets with Random Weights

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Abstract

A random net is a shallow neural network where the hidden layer is frozen with random assignment and the output layer is trained by convex optimization. Using random weights for a hidden layer is an effective method to avoid the inevitable non-convexity in standard gradient descent learning. It has recently been adopted in the study of deep learning theory. Here, we investigate the expressive power of random nets. We show that, despite the well-known fact that a shallow neural network is a universal approximator, a random net cannot achieve zero approximation error even for smooth functions. In particular, we prove that for a class of smooth functions, if the proposal distribution is compactly supported, then a lower bound is positive. Based on the ridgelet analysis and harmonic analysis for neural networks, the proof uses the Plancherel theorem and an estimate for the truncated tail of the parameter distribution. We corroborate our theoretical results with various simulation studies, and generally two main take-home messages are offered: (i) Not any distribution for selecting random weights is feasible to build a universal approximator; (ii) A suitable assignment of random weights exists but to some degree is associated with the complexity of the target function.

1 INTRODUCTION

In recent years, random methods for training neural networks have attracted much attention due to their high potential in training neural networks faster, especially for large-scale training datasets and real-time processing demands [1], [2]. In this paper, we consider the random training method which trains a neural network $g_p$ with two steps:

Step I: Randomly initialize $(a_j, b_j)$ according to a given probability distribution $Q(a, b)$, and freeze them; and

Step II: Statistically estimate $c_j$ according to a given dataset $D_n = \{(x_i, y_i)\}_{i=1}^n$.

Here we focus on a shallow neural network $g_p(x) = \sum_{j=1}^{p} c_j \sigma(a_j \cdot x - b_j)$ of input $x \in \mathbb{R}^m$ with parameters $(a_j, b_j, c_j) \in \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}$ for each $j \in [p]$ and activation function $\sigma$. The setting covers a wide range of activation functions, including radial basis functions (RBFs), the hyperbolic tangent (Tanh), and the rectified linear unit (ReLU).

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The random vector functional-link (RVFL) networks \[3\], random feature expansion \[4\]–\[6\], random weight network \[7\], stochastic configuration networks \[8\] and some versions of over-parametrized networks \[9\]–\[12\] all use random training methods. A kernel function defined by the inner product of feature maps: 
\[ k(x, x') = \int_{\mathbb{R}^m} \sigma(a \cdot x - b) \sigma(a \cdot x' - b) dQ(a, b) \]
is a special case of random training methods because we can regard this as a sum of infinitely many random samples \( (a, b) \sim Q \) \[13\]–\[15\]. On the other hand, Bayesian neural networks \[16\] are not strictly a random training method as the distribution \( Q \) is a “prior” that contains the information of the dataset \( D_n \); nor is the lazy learning \[17\] as the hidden parameters are not strictly frozen.

The random training method has the remarkable trick of “convexification”. It frees us from inevitable non-convexity in the standard gradient descent training. The non-convexity is caused by hidden parameters \((a_j, b_j)\) (and not from output parameters \(c_j\)). In the random training setting, we do not optimize the parameters in Step I, but only for the output parameters \(c_j\) in Step II. This “randomization” trick is beneficial both for theory and applications, which has recently been adopted not only in practical algorithms but also in the theoretical study of deep learning \[9\], \[17\]–\[20\]. However, the expressive power of the random neural networks are less discussed. In this work, we show the expressivity limitation of random nets. Our main result is the following new approximation lower bound for shallow neural networks trained by random methods.

**Main Theorem.** Let \( f \in L^1(\mathbb{R}^m) \cap H^s(\mathbb{R}^m) \) for \( 0 \leq s \leq \infty \). Consider approximating \( f \) with a random net \( g_p = \sum_{j=1}^p c_j \sigma(a_j \cdot x - b_j) \). Draw i.i.d. samples \((a_j, b_j) \sim U([-\lambda, \lambda]^m \times [-\kappa/2, \kappa/2])\) by Step I; and determine \( c_j \) by least squares by Step II. Then, (1) the approximation error cannot attain zero, and (2) it is lower bounded as

\[
\| f - g_p \|_{L^2(\mathbb{R}^m)}^2 \geq \begin{cases} 
C_1 - C_2 \kappa \lambda^m, & 0 < \kappa \leq \vartheta \\
C_3 + C_4 \kappa \lambda^{-2s}, & \vartheta \leq \kappa \leq \infty 
\end{cases}
\]  

(1)

for some constants \( C_1, C_2, C_3, C_4 \), and \( \vartheta > 0 \) that depend on norms of \( f \), dimension \( m \) and smoothness \( s \) (but not on \( \lambda \)). Here, the right-hand side is continuous at \( \lambda = \vartheta \).

In other words, if the domain of hidden parameters is bandlimited as \( |a| \leq \lambda \) and \( |b| \leq \kappa/2 \), then the approximation error cannot be zero. The bound changes the rate at \( \lambda = \vartheta \), and it depends on the smoothness \( s \) in \( \lambda \geq \vartheta \), indicating that the tail of the parameter distribution gets lighter when the approximant \( f \) is smoother.

The key instruments to derive the lower bound are the integral representation and ridgelet analysis \[21\]–\[24\], which are developed as the harmonic analysis for neural networks. In ridgelet analysis, the hidden parameter \((a, b)\) plays a similar role as the frequency parameter, say \( \omega \), in the Fourier series expansion \( \sum_{j=-\infty}^{\infty} c_j \exp(ix \cdot \omega_j) \). Then, we can show that if the “frequency” \((a, b)\) is bandlimited, then the expressive power of the network \( g_p \) is limited.

**Notation.** \( S(\mathbb{R}) \) denotes the space of rapidly decreasing smooth functions, or the Schwartz test functions, on \( \mathbb{R} \); and \( S'(\mathbb{R}) \) denotes the space of tempered distributions, or the topological dual space of \( S(\mathbb{R}) \). \( H^s(\mathbb{R}^m) \) denotes the \( L^2 \)-Sobolev space on \( \mathbb{R}^m \) with order \( s \). For any function \( f : \mathbb{R}^m \to \mathbb{R} \), \( \hat{f}(\xi) := \int_{\mathbb{R}^m} f(x) \exp(-i\xi \cdot x) dx \) denotes the Fourier transform of \( f \). For any complex number \( z, \bar{z} \), we denote the complex conjugate of \( z \). For any subset \( A \subset X \) of an entire set \( X \), \( A^c := X \setminus A \) denotes the complement of \( A \). It should be clarified that, in this paper, we mean by the terms ‘random neural networks’, ‘random nets’, or ‘neural nets with random weights’ the same thing.
2 Ridgelet Analysis for Random Nets

In this section, we present some fundamentals of the integral representation theory and ridgelet analysis, then provide several important propositions that will be used in proving our main results.

2.1 Integral Representation of Neural Nets

The integral representation is a handy tool for the analysis of neural networks with a variable number of hidden units. Let \( V \subset \mathbb{R}^m \times \mathbb{R} \) be a space of hidden parameters \((a, b)\), and \( \mathcal{M}(V) \) be the space of signed Borel measures on \( V \). We call an element \( \mu \in \mathcal{M}(V) \) a parameter distribution.

Let \( \sigma : \mathbb{R} \rightarrow \mathbb{R} \) be an activation function. We always assume that the activation function \( \sigma \) satisfies the admissibility condition, which is defined later, a sufficient condition for the neural network to have the universal approximation property (see Proposition 1). Many typical activation functions can satisfy the condition, for example, such as radial basis functions (RBFs), the hyperbolic tangent (Tanh), and the rectified linear unit (ReLU).

The integral representation of a neural network is defined as an integral transform of the parameter distribution \( \mu \in \mathcal{M} \):

\[
S[\mu](x) := \int_{\mathbb{R}^m \times \mathbb{R}} \sigma(a \cdot x - b) \, d\mu(a, b), \quad x \in \mathbb{R}^m.
\]  

As the integral suggests, this is formally an infinite version of the ordinary finite neural network \( g_p = \sum_{j=1}^{p} c_j \sigma(a_j \cdot x - b_j) \). Whereas the finite net \( g_p \) is a weighted sum of finite hidden units \( \sigma(a_j \cdot x - b_j) \) and weight \( c_j \) with indices \( j \in [p] \), the infinite net \( S[\mu] \) is a weighted integral of infinite hidden units \( \sigma(a \cdot x - b) \) and weight function \( \mu(a, b) \) with “indices” \((a, b) \in V\).

As remarks, we can reproduce a finite net as \( g_p = S[\mu_p] \) by letting \( \mu_p = \sum_{j=1}^{p} c_j \delta(a_j, b_j) \) with Dirac measures \( \delta(a, b) \), because we assume that a parameter distribution \( \mu \) is a Borel measure, which includes both continuous densities and singular masses. In other words, the integral representation is not a counterpart of the finite models, but it is an extension of the finite models. Second, the map \( S \) is linear in \( \mu \). Since the non-linear parameters \((a_j, b_j)\) are “integrated out” in the integral representation (like “marginalize out” in the Bayesian literature), we are now free from the non-linearity of neural networks. Finally, in order to cover a wide range of activation functions, we will use a slightly extended definition of \( S[\mu] \) in the proof sections. Nevertheless, we will not use this extended version in the main sections for the sake of simplicity.

2.2 Ridgelet Transform

The ridgelet transform \( R \) is, in a nutshell, a pseudo-inverse operator to the integral representation operator \( S \). Given a function \( f \in L^2(\mathbb{R}^m) \), consider finding an unknown parameter distribution \( \mu \in \mathcal{M} \) that satisfies an integral equation \( S[\mu] = f \). As we would describe later, the solution to this integral equation is not unique, and the ridgelet transform provides a particular solution to this equation.

Given an activation function \( \sigma : \mathbb{R} \rightarrow \mathbb{C} \), we say a function \( \rho : \mathbb{R} \rightarrow \mathbb{C} \) is admissible when it satisfies the admissibility condition \( (2\pi)^{m-1} \int_{\mathbb{R}} \hat{\sigma}(\omega) \overline{\hat{\rho}(\omega)} |\omega|^{-m} \, d\omega = 1 \). In other words, this condition just requires that the \( |\omega|^{-m} \)-weighted inner product of \( \sigma \) and \( \rho \) to be finite (not zero nor infinite). Therefore, this is not a strong condition and we can find, in general, an infinite number of different \( \rho \). For example, if \( \sigma \) is Gaussian, then its Fourier transform \( \hat{\sigma} \) is again Gaussian, and we can find a “family of” particular solutions: \( \hat{\rho}(\omega) = C |\omega|^m \hat{\sigma}(\omega) \) for any
Schwartz function $\phi \in S(\mathbb{R})$ (as long as the integral is finite and not zero) with an appropriate normalizing constant $C$. We refer to [24] § 6.2 for more examples.

For every $f \in L^p(\mathbb{R}^m)(p = 1, 2)$, the ridgelet transform of $f$ with respect to $\rho \in S(\mathbb{R})$ is given by

$$R[f](a,b) := \int_{\mathbb{R}^m} f(x)\rho(a \cdot x - b)dx, \quad (a, b) \in \mathbb{R}^m \times \mathbb{R}$$

We provide two important propositions as basic preparation for the main theoretical analysis performed in Section 3. It should be noted, however, that two more theorems that extend these propositions are provided with detailed proofs in the supplementary material, by which the case of some non-integrable activation functions are covered.

**Proposition 1** (Reconstruction formula). Let $f \in L^p(\mathbb{R}^m)(p = 1, 2)$. Provided that $\rho \in S(\mathbb{R})$ is admissible with an activation function $\sigma \in S'(\mathbb{R})$, then we have

$$S[R[f]](x) = \int_{\mathbb{R}^m \times \mathbb{R}} R[f](a,b)\sigma(a \cdot x - b) dab = f(x), \quad x \in \mathbb{R}^m.$$  

We have two interpretations for Proposition 1. First, recall that $S[\mu]$ represents a neural network. Then, the reconstruction formula implies the universal approximation property, because a neural network $S[\mu]$ can express any function $f$ by letting $\mu = R[f]$. Second, recall the Fourier inversion formula: $F^{-1}[\hat{f}](x) = (2\pi)^{-m} \int_{\mathbb{R}^m} \hat{f}(\xi)\exp(i\xi \cdot x)d\xi = f(x)$. Then, we can find the clear correspondence that $S$ to $F^{-1}$, $R[f]$ to $f$, and $\sigma(a \cdot x - b)$ to $\exp(i\xi \cdot x)$. However, we should also remark the difference that by the non-uniqueness of admissible functions $\rho$, the ridgelet transform $R[f]$ is not unique either. This means that $R$ is not the strict inverse operator to $S$, but only a pseudo-inverse operator to $S$.

**Proposition 2** (Plancherel theorem). Let $f \in L^2(\mathbb{R}^m)$. Provided that $\rho$ is self-admissible, namely, it is admissible with itself ($\rho = \sigma$). Then, $\|R[f]\|_{L^2(\mathbb{R}^m \times \mathbb{R})} = \|f\|_{L^2(\mathbb{R}^m)}$.

The Plancherel theorem plays a key role in our main results. As to be displayed in the next subsection, a ridgelet spectrum $R[f]$ has a long tail. If the spectrum $R[f]$ is truncated, the Plancherel theorem implies that we cannot reconstruct $f$ without loss.

### 3 Main Results

For the sake of readability, we write $\| \cdot \| = \cdot \|L^2(\mathbb{R}^m)\|$. Given a function $f \in L^2(\mathbb{R}^m)$, we reformulate the random training method at the beginning of the introduction.

**Step I’:** Draw $\{ (a_j, b_j) \}_{j=1}^p \sim U(V)$, and let $\mathcal{M}(p) := \{ \sum_{j=1}^p c_j \delta(a_j, b_j) \mid c_j \in \mathbb{R} \}$.

**Step II’:** Let $\mu_p := \arg \min_{\mu \in \mathcal{M}(p)} \| f - S[\mu] \|^2$, and let $g_p := S[\mu_p]$.

The main goal of this section is to lower bound the approximation error $\| f - g_p \|$. Unlike the Fourier or Taylor series expansions, the rate of approximation lower bound for a finite $p$ is unknown, and it is known as a (complicated) open question (see [25] for more details). To

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1. We remark that some non-integrable activation functions such as the Tanh and ReLU cannot meet self-admissible in the present setup. In the proof sections elaborated in the supplementary material, we extend the setup to include these activation functions. However, for the sake of simplicity, we do not use the extended setup in the main sections.
Theorem 3. Let $\| f - g_p \|^2 = \min_{\mu \in \mathcal{M}(p)} \| f - S[\mu] \|^2 \geq \inf_{\mu \in \mathcal{M}(V)} \| f - S[\mu] \|^2$.

In Theorem 3, we show that the minimizer can be represented by the ridgelet spectrum.

Theorem 4 (Decay properties of ridgelet spectrum). Let $f \in L^1 \cap L^2(\mathbb{R}^m)$, and assume that the minimizer $\mu_V := \arg \min_{\mu \in \mathcal{M}(V)} \| f - S[\mu] \|$ exists. Then, the minimizer $\mu_V$ admits an expression $\mu_V = R[f]_V + \ker S$, where $R[f]_V$ denotes the restriction of the ridgelet spectrum $R[f]$ to $V$.

Finally, by integrating the decay property, we obtain a lower estimate of the tail bound $\| f \|_{V^c}$, as follows.

Theorem 5 (Main theorem). On $f$ and $\rho$, we pose the same assumptions in Theorem 4. Let $C_0 := \| f \|_{L^1(\mathbb{R}^m)}^2/\| \rho \|_{L^\infty(\mathbb{R})}$, $C_\infty := \| f \|_{H^s(\mathbb{R}^m)}^2/\| \rho \|_{L^\infty(\mathbb{R})}$, and $\vartheta := (mV_0 C_0/2)^{1/(2s+m)}$, where $V_0 := \pi^{m/2}/\Gamma(m/2 + 1)$ is the volume of the $m$-unit ball. For simplicity, we let $V = \{(a, b) \in \mathbb{R}^m \times \mathbb{R} \mid |a| \leq \lambda, |b| \leq \kappa/2\}$. For a random net $g_p = S[\mu_p]$ that is trained by Steps I and II, we have the following lower bounds:

$\| f - g_p \|^2_{L^2(\mathbb{R}^m)} \geq \| R[f] \|^2_{L^2(V^c)} (\geq 0)$

$\geq \| f \|^2_{L^2(\mathbb{R}^m)} - \left(\frac{C_0^2 V \kappa \lambda^m}{2} \right) 0 \leq \lambda \leq \vartheta$

where the bound is continuous at $\lambda = \vartheta$.

We provide the proofs of all above results in the supplementary material.

Auxiliary Interpretation of Our Theorem. To give an intuition of our main theorem, we visualize a parameter distribution and a ridgelet spectrum in Figure 1. Both figures are obtained from the dataset $D_n = \{(x_i, y_i)\}_{i=1}^n$ that is generated by function $y_i = f(x_i) = \sin 2\pi x_i$. Figure 1(a) shows the distribution of parameters $(a_j, b_j, c_j)$, which are obtained from many neural networks trained on common dataset $D_n$ by gradient descent (GD); and Figure 1(b) shows the ridgelet spectrum $R[f](a, b)$ approximated by numerical integration evaluated at each point $(a, b)$. Despite
the fact that two figures are obtained from different procedures, gradient descent and numerical integration, they have an apparent intriguing resemblance. The shapes of distributions are 10-point star shaped. In other words, the trained parameters \((a_j, b_j, c_j)\) concentrate on high intensity areas in the ridgelet spectrum. Theorem 3 provides an extended version of the theoretical derivation and explanation of this phenomenon. Based on the visualization results, one can naturally conjecture that if the parameter space is bandlimited, that is, the spectrum is truncated to a compact domain such as \(|a| \leq \lambda\) and \(|b| \leq \kappa\), the neural network loses the universal approximation property. In other words, there exists a class of functions that a bandlimited network cannot reconstruct. Overall, that is the primary idea behind our main theorem, i.e., we quantify and prove this conjecture by carefully estimating the “tail” of the spectrum.

Remark 1. There is no randomness anywhere in the main theorem. We do not need randomness condition because the key equations hold with probability 1 (besides that the data \(f\) is fixed). According to Steps I’ and II’, the LHS of \((5)\) is a random variable. However, the RHS is not a random variable but a constant because it is by definition smaller than any loss-value \(J(\mu) := \|f - S[\mu]\|^2\) for \(\mu \in \mathcal{M}(V)\). \((6)\) (= RHS of \((5)\)) indicates that the lower bound \(J := \inf_{\mu} J(\mu)\) is strictly positive (w.p.1) when and only when the spectrum \(R[f]\) is supported outside the parameter domain \(V\). Thus, if the initializer \(Q(a, b)\) (in Step I) is supported in a compact set \(V\) and \(R[f]\) has support outside \(V\), then inevitably (= w.p.1) \(J > 0\). In the main theorem, we assumed \(Q\) to be the uniform distribution \(U\), which is compactly supported. On the other hand, we may consider extensions to a fully supported distribution such as the normal distribution \(N\). For this case, in contrast, we need some high probability condition that initial parameters \(\{(a_j, b_j)\}_{j=1}^{p}\) concentrate in a certain compact domain \(V\).

Remark 2. ReLU nets can achieve zero-training-error under the norm constraint on parameters (see the reasoning below). Is this contradict to the main theorem? Here is the reasoning:

(a) Fix a target function \(f(x)\) to be approximated. (b) Fix a target accuracy \(\epsilon\). (c) We know that, if there is no constraint on the size of weights and biases that there exists \(p > 0\) so that a ReLU network of width \(p\) can approximate \(f\) to accuracy \(\epsilon\) (in practically any norm you want). (d) Now fix a neuron in this network, consider its incoming weights and biases \((a, b)\). Let \(A\) be the ceiling of \(|a|\) and \(B\) be the ceiling of \(|b|\). Write \(C = \max\{A, B\}\). (e) Let’s create \(C\) “scaled copies” of the neuron we’re considering. By this, we mean the weights/biases into each neuron are precisely \(a/A\) and \(b/B\),
while the outgoing weights copied from the original neuron. (f) Note that by adding together the outputs of these $C$ scaled neurons we precisely reproduce the output of the original neuron we had. (g) Doing this for every neuron, we’ve produced a one layer ReLU network whose weights and biases all lie in a given compact set (here the unit cube) which nonetheless gives an $\epsilon$-approximation to $f$.

—The reasoning is valid only when the units number $p$ is finite. (In practice, a finite $p$ model also suffers from a large norm.) For a homogeneous activation function $\sigma(z) = z^{\kappa}$ ($k \in \mathbb{N}$), which includes ReLU as $k = 1$, the reconstruction formula can be rewritten in the polar coordinates (see Proposition 3):

$$f(x) = c_p \int_{\mathbb{S}^{m-1} \times \mathbb{R}} (-\Delta)^{(m+k)/2} f(tu)(u \cdot x - t)^k \, dt,$$

with some constant $c_p$, where $(u,t) \in \mathbb{S}^{m-1} \times \mathbb{R} \cong \mathbb{R}^m$ denotes the polar coordinates, $-\Delta$ denotes the Laplacian w.r.t. spatial variable $x$. The rescaling (e–g) corresponds to the cancellation:

$$\int \mu(u, t)(u \cdot x - t)^k \, dt = \int C^k \mu(u, t)(u/C \cdot x - t/C)^k \, dt.$$  

Thus, if the ‘parameter distribution’ $(-\Delta)^{(m+k)/2} f(tu)$ is fully supported in $t$, then we cannot ‘rescale’ the domain of $t$ into compact. More practically, this means that the $\epsilon$-approximation in (c) will cost a large coefficient $c_j$.

**Remark 3.** The lower bound has nothing to do with the choice of $\rho$.—The first equality in (6) indicates that $\|f - S[\mu_V]\|^2$ is the lower bound, which follows from Theorem 3 and thus it is independent of the ridgelet transform, and the second equality in (6) indicates that the lower bound is given by a ridgelet spectrum, which holds only when $\rho = \sigma$ by the Plancherel (Prop.2). Therefore, the $\rho$ is replaced by the activation function $\sigma$.

**Remark 4.** Is the lower bound always positive?—We provided two lower bounds: Equations 8 (or 9) and 11 (or 10). While 8 is always non-negative and tight, 9 can be negative if, for example, we increase $\kappa$. We introduced these two because 8 is qualitative, while 9 is quantitative. 8 provides a geometric (qualitative) interpretation of the lower bound as the distance between $f$ and $S[\mu_V]$, the orthogonal projection of $f$ onto the model space $\{S[\mu] \mid \mu \in \mathcal{M}(V)\}$, and thus it is inevitably tight. 9 can be negative because we approximated $\mu_V = R[f]\vert_V$, the parameter distribution of projected $f$, from above by the decay property. At the expense of tightness, 9 provides a quantitative formula w.r.t. the smoothness $s$, dimension $m$, and norms of $f$, and thus we can instantly estimate the bound without directly computing 8.

## 4 Related Work

For a whole picture, we should recall the pioneering work by Barron [21], where a lower bound on the best approximation error for linear combinations of any fixed basis functions. Barron’s theoretical results, also known as the Kolmogorov width, indicate that “fixed basis function expansion must have a worst-case performance that is much worse than that which is proven to be achievable by certain adaptable basis function methods (such as neural nets).” It is logical to accept that the concept of fixed basis function expansion is equivalent to the meaning of neural nets with random weights. Their lower bound (if we state it informally) is estimated as $\kappa C m^{-1} p^{-1/m}$, where $\kappa$ is a universal constant not smaller than $1/(8\pi e^{\pi-1})$, $m$ denotes the input dimension, $p$ stands for the number of hidden neurons. In theory, what is missing in Barron’s bound is the distribution information of the fixed bases. In fact, for fixed $C$ and a given approximation error tolerance, the estimate $\kappa C m^{-1} p^{-1/m}$ goes to 0 as $m$ or $p$ tends to infinity; in this case, the lower bound is of impractical use to show the less effectiveness of fixed basis function approximation. In contrast, our theoretical result is independent of both $m$ and $p$. The difference lies in the assumptions on approximator and approximant. The Kolmogorov width is about the teacher-student setting
where neural nets approximate neural nets so that zero approximation error can be achieved in principle by using an infinite number of neurons, whereas our result is about where neural nets approximate functions in an abstract function space such as Sobolev class, and thus zero approximation error cannot be achieved, even if we use an infinite number of bases.

Recently, random nets have been a mathematical model for understanding the inductive bias of over-parametrized neural networks. Here, over-parametrization refers to a state of the network in which the number of parameters is vast compared to the size of the dataset. People consider over-parametrization as a strong candidate for unknown necessary conditions of the generalization of deep learning; inductive bias refers to the implicit assumptions that the learning model makes on the learning data. In deep learning, the lazy learning, the assumption that the solution is located not too far from the initial parameters, is considered as a strong candidate for the inductive bias. Random nets are often employed in the theoretical analysis as an extreme model for lazy learning because they do not move at all the hidden neurons from the initialization state. In the over-parametrized settings, people have a lot of new phenomena, such as double descent [9] and neural tangent kernel [17]. In the theoretical studies of lazy learning, including random nets, sometimes the ‘global convergence’ is claimed. However, if the model does not have universal approximation property (UAP), then the global convergence is impossible. We remark that for global convergence to be established, the UAP is always assumed either implicitly or explicitly. For example, Chizat et al. [26] consider the problem when a finite-dimensional (not random but lazy) model approximates an arbitrary function, and proved the global convergence (Theorem 2.4). However, this does not speak about the expressive power because Theorem 2.4 assumes that $Dh(w_0)$ to be surjective, which is equivalent to assume that loss can be zero. More interestingly, Yehudai and Shamir [12] and Ghorbani et al. [11] proved (under very limited settings) that the expressive power of random nets is low, while Malach et al. [19] proved a stronger lottery ticket hypothesis, which essentially claims that the expressive power is exceptionally high. These seemingly contradictory claims are, of course, consistent. Yehudai and Shamir [12] consider the problem that a finite-dimensional random net (FRN) approximates a single ReLU neuron and provided an approximation lower bound w.h.p. for finite parameter numbers $p$ to conclude low expressive power. Ghorbani et al. [11] consider the problem that an FRN approximates a quadratic function and showed asymptotic approximation error does not tend to zero (Theorem 1). Namely, these two studies focus on specific examples that FRNs cannot easily approximate. On the other hand, Malach et al. [19] consider the so-called student-teacher problem that a student FRN approximates teacher FRN, and proved that if both a student and a teacher share the common norm constraint, then the student can $\epsilon$-approximate the teacher w.h.p., which does not contradict to the previous two (and our) claims because this study focus on specific examples that FRNs can easily approximate. Compared to these results, we consider the problem that a potentially infinite-dimensional RN (with any activation) approximates an $L^2$-Sobolev function $f \in H^s(\mathbb{R}^m)$ and provided lower bounds. Thus, our results cover a wider range of functions (grouped by smoothness $s$) than previous studies.

Yarotsky [27] consider the problem that a deep ReLU net (not random, without any norm constraints on the weights) approximates an $L^\infty$-Sobolev function $f \in W^{s,\infty}([0, 1]^m)$, and proved (in Theorem 4) that if a ReLU net $\epsilon$-approximates $f$ s.t. $\|f\| \leq 1$ (i.e., in a unit ball), then the ReLU net must have at least $p_0 = c\epsilon^{-m/2s}$ units. The proof is based on the covering number arguments. Clearly, this is a different lower bound from ours (but, of course, they are not formally contradict to each other). While $H^s = W^{s,2}$ is Hilbert, but $W^{s,\infty}$ is (not Hilbert but) Banach. Therefore, we cannot directly apply our arguments to Yarotsky’s problem, and vice versa.
The ridgelet transform has been independently discovered by Murata [23], Candès [22] and Rubin [28] during 1996–1998 as a ‘harmonic analysis of neural networks’. This is a path-breaking study (not only in neural network study but) in the sparse coding theory (see overviews by Donoho [29] and by Starck, Murtagh and Fadili [30]). The ridgelet transform has been extended to Schwartz distributions by Kostadinova et al. [31], and to non-integrable activation functions such as ReLU by Sonoda and Murata [24]. The integral representation of neural network had been developed before ridgelet transform. (Recall that ridgelet transform \( R \) is a pseudo-inverse operator of integral representation operator \( S \). Thus, we can analyze \( S \) without knowing \( R \).) For example, Irie and Miyake [32], Funahashi [33] and Barron [21] used Fourier transform as an integral representation to prove the UAP. Carroll and Dickinson [34] and Ito [35] used Radon transform. In particular, the so-called Barron class (proposed in [21]) characterizes the functions that neural networks can effectively approximate. The effectiveness here is quantified as Barron’s bound, a dimension-free approximation upper bound (see overview by Kainen, Kůrková, and Sanguineti [25]). The original Barron’s theory excludes ReLU, and the upper bound is in general not tight. Thus, many authors [36]–[44] have improved and developed Barron-like theories for ReLU nets. This is notable that Ongie et al. [43] and Pahri and Nowak [44] have employed the Radon transform and developed a similar calculus with ridgelet analysis. The novelty of this study in the integral representation literature is that, besides the lower bounds, we have established the Plancherel theorem and a new reconstruction formula for \( k \)-homogeneous activation functions such as ReLU.

5 Experiments

In this section, we conduct some simulation studies to verify our theoretical results. Two toy examples for 1D function regression are used in our experiments. Consistent with our theoretical analysis, the numerical simulations aim at showing how \( \lambda \), which is used for the random assignment of input weights (and biases), would affect the expressive power of the random net. For this purpose, we present an intuitive illustration of the infeasibility of individual trivial settings of \( \lambda \). Then, we would discuss statistically the potential relationship between \( \lambda \) and the critical parameter that can show the complexity of the target function. We utilize the following 1D target function in the following Simulation 1 and Simulation 2.

\[
 f(x; \sigma) = 0.2 \exp \left( -\frac{(x - 0.4)^2}{\sigma^2} \right) + 0.5 \exp \left( -\frac{(x - 0.6)^2}{\sigma^2} \right),
\]

where \( x \in [0,1] \), \( \sigma_1 > 0 \) is a scalar index that can determine the complexity of \( f \), as mentioned in our theoretical analysis. In Simulations 1 and 2, we use the sigmoid activation function.

Simulation 1. We set \( \sigma_1 = 0.05 \) and sample 1000 instances \( \{x_i, f(x_i)\}_{i=1}^{1000} \) based on a equally spaced points on \([0,1]\), then randomly and uniformly select 500 training sample and 500 test samples. We test the performance of two random networks with \( \lambda = 1 \) and \( \lambda = 20 \). For each case, we train the network with a different number of hidden nodes \( L \), which helps with excluding the influence of \( L \) to our analysis. In Figure 2, we show the training and test approximation results for four different random networks, including (a) and (b) for the network build with \( \lambda = 1, L = 100 \), (c) and (d) for the network build with \( \lambda = 1, L = 500 \), (e) and (f) for the network build with \( \lambda = 1, L = 10000 \), (g) and (h) for the network build with \( \lambda = 20, L = 200 \), respectively. We observe that the random network with \( \lambda = 1 \) cannot approximate for this simple function approximation problem, even when the number of hidden nodes is sufficiently large. In contrast, the network with \( \lambda = 20 \) and trained with \( L = 200 \) demonstrates excellent learning and generalization performance.

Other larger values of \( \lambda \), such as \( \lambda = 50, 100, 150, 200 \) as we tested, have the same excellent performance on this regression task. It implies that the
choice of $\lambda$ has a strong impact on the random network’s expressive power, which is consistent with our theoretical results.

**Simulation 2.** Following the intuitively investigation of the role of $\lambda$ in the expressive power of random networks in **Simulation 1**, in this part, we present more statistical results for approximation with various pairs of $(\lambda, \sigma)$ so that we can summarize a general pattern empirically. Specifically, we create different forms of target function $f(x; \sigma)$ by choosing $\sigma$ as one of $\{0.01, 0.05, 0.1, 0.5\}$, and for each regression task we build random nets with $\lambda$ from $\{0.1, 0.5, 1, 5, 10, 50, 100, 200\}$, and choose a sufficiently large $L$ here $L = 10000$ in each case so that we can observe the trend as $L \rightarrow +\infty$. Similar as Simulation 1, we sample 1000 instances $\{x_i, f(x_i)\}_{i=1}^{1000}$ which are equally spaced points on $[0,1]$, then randomly and uniformly select 500 training samples and 500 test samples. For each pair of $(\lambda, \sigma)$, we run independently 50 trials and calculate the relative training error $E_k := \|\bar{f} - \bar{y}\|_2/\|\bar{f}\|_2$ for each trial, where $k = 1, 2, \ldots, 50$, $\bar{f} = (f(x_1), f(x_2), \ldots, f(x_{500}))$ represents the vector of training targets, $\bar{y} = (y_1, y_2, \ldots, y_{500})$ stands for the output vector of the random network. As a matter of fact, we only need to study the training performance to see whether a given $\lambda$ is suitable for approximating the target function produced by a given $\sigma$.

![Fig. 2. Performance of random nets with $\lambda = 1$ and $\lambda = 10$ in training and test. (a-b) $\lambda = 1, L = 100$. (c-d) $\lambda = 1, L = 500$. (e-f) $\lambda = 1, L = 10000$. (g-h) $\lambda = 20, L = 200$.](image)

Table I summarizes the averaged relative training error $E := \sum_{k=1}^{50} E_k/50$. Note that we do not provide their standard deviations here because, compared with the average values, standard deviation values may not affect the conclusion that we are aiming to verify, as we will detail later. Table I shows how the choice of $(\sigma, \lambda)$ affects the approximation ability of random networks. From the colored cells of the table, which values are tiny (magnitude between $e^{-8}$ and $e^{-6}$), we can observe that, for target function with a smaller $\sigma$ value, the larger $\lambda$ for a random net we would need to ensure random networks have a precise approximation to a target function. From the above simulations, we can see the effectiveness of the approximation by random networks is constrained to both of the network parameter distribution and the class of target functions. For a given learning task, there exists an appropriate range/distribution $D^*$ (not unique), but **NOT ANY** range/distribution, such that neural networks with random
weights assigned from $D^*$ can be a universal approximator (if the number of hidden nodes is sufficiently large). Second, the $D^*$ (for example, $[-\lambda^*,\lambda^*]$) is highly dependent upon the complexity of the target function. One needs an adequate amount of samples from the target function to provide some prior knowledge or empirical studies to discover $D^*$.

Simulation 3. To further reveal the infeasibility of the trivial range $[-1,1]$ for certain function approximation problems, we conduct similar simulations on a new target function $g(x)$, denoted as

$$
g(x) = 0.5 \cos(22\pi x^2) + 0.5 x^2, \quad x \in [0, 1].$$

Mathematically, $g(x)$ is composed of two parts: $0.5 \cos(22\pi x^2)$ and $0.5 x^2$, which represent two completely different ‘modes’ at distinct frequencies. We carry on the same sampling as Simulations 1 and 2 to generate 500 training and test points on $[0,1]$. Here, we only consider the training performance of random nets with various choices of $\lambda$. We report the comparison results for $\lambda = 1$ and $\lambda = 100$ in Figure 3. We observe that the random nets with $\lambda = 1$ are not universal approximators, although the number of hidden nodes is sufficiently large ($L = 10,000$). The network with $\lambda = 1$ can only fit the ‘mean’ curve of the original signal and fail to approximate the high-frequency ‘mode’ $0.5 \cos(22\pi x^2)$. On the other hand, for the second ‘mode’ $0.5 x^2$, the random nets $\lambda = 1$ have great approximation performance. As we observe the derivative $|g'(x)| \leq 25$ in Figure 3(c), we conjecture that in general, the ‘appropriate’ range of $\lambda$ is related to the magnitude of $|g'(x)|$, rather than independent of the target function class and training samples. Moreover, a multi-scale strategy that selects random parameters from various scopes can be beneficial, especially when the target function is complicated and composed of multiple ‘modes.’ In Figure 3(d), we find another interesting result that the training output of the network with 300 hidden neurons and weights (and biases) randomly chosen from $[-100,100]$ is not significantly affected if we remove 85 hidden neurons with weights (and biases) located in the ‘narrow’ range $[-30,30]$. It means, these hidden weights (and biases) as randomly assigned from $[-30,30]$, not to mention the ones from $[-1,1]$, have a little contribution to the approximation universality in learning.

Discussion: Our experiments support our theoretical results, which send two critical messages. (1) For a learning task, simply taking a fixed scope $[-\lambda,\lambda]$ would not make random neural nets a universal approximator, if $\lambda$ is not set properly. (2) For a Gaussian-type target function $f(x;\sigma) = \exp(-|x|^2/\sigma^2)$ ($\sigma > 0$), which is a Sobolev function and thus meets the condition of our main theorem, a large value of $\lambda$ is usually needed if $\sigma$ is small. Generally, the empirical findings can offer useful guidelines for the algorithm development of building random neural

| $\lambda$ | $\sigma = 0.01$ | $\sigma = 0.05$ | $\sigma = 0.1$ | $\sigma = 0.5$ |
|---|---|---|---|---|
| $\lambda = 0.1$ | 0.9904 | 0.9669 | 0.9419 | 0.9026 |
| $\lambda = 0.5$ | 0.9299 | 0.6627 | 0.2179 | 1.0606e-04 |
| $\lambda = 1$ | 0.9188 | 0.6546 | 0.2089 | 1.1781e-05 |
| $\lambda = 5$ | 0.8574 | 0.1253 | 0.0016 | 5.6666e-08 |
| $\lambda = 10$ | 0.5714 | 0.0064 | 5.692e-08 | 4.5881e-08 |
| $\lambda = 50$ | 0.0131 | 4.4905e-08 | 4.6897e-08 | 4.5834e-08 |
| $\lambda = 100$ | 1.1171e-07 | 1.3937e-07 | 1.0784e-07 | 1.1284e-07 |

TABLE 1
Summary of mean relative training error for various choices of ($\lambda, \sigma$).

Discussion: Our experiments support our theoretical results, which send two critical messages. (1) For a learning task, simply taking a fixed scope $[-\lambda,\lambda]$ would not make random neural nets a universal approximator, if $\lambda$ is not set properly. (2) For a Gaussian-type target function $f(x;\sigma) = \exp(-|x|^2/\sigma^2)$ ($\sigma > 0$), which is a Sobolev function and thus meets the condition of our main theorem, a large value of $\lambda$ is usually needed if $\sigma$ is small. Generally, the empirical findings can offer useful guidelines for the algorithm development of building random neural
Fig. 3. Performance for training results of $g(x)$: (a) $\lambda = 1, L = 10000$, (b) $\lambda = 100, L = 300$. (c) Derivative function $g'(x)$. (d) $N_1$: Training approximation of $g(x)$ with hidden weights (and biases) randomly assigned from $[-100, 100]$, $N_2$: Training approximation of $g(x)$ with hidden weights (and biases) randomly assigned from $[-100, 100]/[-30, 30]$, and their numerical difference $N_1 - N_2$.

nets. As a practical advice, users who use random nets for data modelling tasks should keep in mind that the setting of $\lambda$ strongly affects the model’s performance. It is better to perform simulations in a trial-and-error manner to (roughly) estimate the appropriate $\lambda$. That seems easy to implement, but indeed relies heavily on human intervention and is not an end-to-end algorithm.

6 Conclusion

In this paper, we study the lower bound of shallow neural nets’ approximation error with random weights. In particular, we investigate the impact and limitation of randomness on the network’s expressivity power. Our theoretical results show that the lower bound of training error of a random network may not be zero if one does not correctly choose the range/distribution (determined by a scalar $\lambda$) of the hidden parameters in advance. Our results hold when under the condition of bandwidth limitation, which is a stochastic limitation with finite variation and is also valid when the proposal distribution is fully supported, such as normal distribution. This finding contradicts the common saying that a shallow, random neural network is always a universal approximator regardless of the choice of hidden weights.

Our study can be generally viewed as the very first step in a road that could take us to understand far better the feasibility and effectiveness of neural nets with random weights. More in-depth analysis, either for deep neural nets or further improvements with much tighter bounds, is highly expected. More broadly, a complete theory for explaining when and why neural nets with random weights are workable or not is meaningful for this research topic. However, technically, it cannot limit itself to a simple shallow architecture case, as studied in this paper. Since we only focus on the theoretical analysis, what is missing in our work is the development of advanced algorithms to build random neural nets with guaranteed universal approximation power.
APPENDIX A
FURTHER BACKGROUND

The initial motivation of this work comes from the comments posted by Yann LeCun, see https://www.facebook.com/yann.lecun/posts/10152872571572143, in which some truth background behind randomness in neural nets was briefly revisited. We not only agree with Yann’s comments after we conduct a comprehensive literature review for this line of research but also technically question the feasibility and effectiveness of the mentioned “ELM”, since many researchers find empirically that the so-called “ELM” model often has unstable or poor results in some cases. Overall, that motivates us to investigate two pressing, however, puzzling questions: (1) Can we guarantee that random neural nets model with hidden parameters chosen from a fixed range, for example, a trivial case by letting $\lambda = 1$, is a universal approximator? (2) Given a target function $f$ with specified complexity, what is the relationship between an appropriate setting of $\lambda$ (that can lead to a universal approximator in the sense of probability) and the smoothness of $f$? Although we raise these questions, our intention is not to make any judgement on the reasonableness and correctness of the name “ELM”. Instead, we present our current study along the right track of neural nets with random weights (or random neural nets, random nets), with particular concerns on the theoretical aspects.

The appearance of randomness in neural networks can trace back to the original Rosenblatt’s perceptron [45], where the first layer is randomly connected and later Minsky and Papert’s Gamba perceptron [46] whose first layer is a bunch of linear threshold units. In the ’90s, researchers made random training methods/models reification, for example, single hidden layer feedforward networks (SLFN) with random weights [47] and random vector functional-link (RVFL) networks [48]. Algorithmically, they performed the two steps (mentioned at the beginning of the introduction section) to build the randomized learner model. However, the approximation errors of the resulted model are bounded in the statistical sense [3], implying that preferable approximation performance is not guaranteed for every random assignment of the hidden parameters if the re-given probability distribution $Q(a; b)$ is not appropriately chosen [49], [50]. In contrast, the stochastic configuration networks [8] can ensure universal approximators by enforcing certain constraints on the random assignment of the hidden parameters, rather than using the purely random way as the “good” probability distribution $Q^*(a; b)$ is unknown and data-dependent. Sonoda et al. [51] proposed the sampling regression learning method by introducing a nonparametric probability distribution of the hidden parameters of SLFNs, and fitting the output parameters via ordinary linear regression. The framework of a broad learning system [52] performs in the manner of a flat network, in which the original inputs are transferred and placed as the “mapped features” in feature nodes and the structure is expanded in a wide sense in the “enhancement nodes.”

Although we only pay attention to shallow NNRWs with Step I and Step II (mentioned in the introduction), some other techniques/models using randomness are still worth mentioning here, aiming to present the engaging readers with a big picture of the line of research. For instance, the use of randomness in deep neural nets is also concerned in terms of different viewpoints. Mongia et al. [53] showed that simple single-layer CNNs with random filters could serve as the basis for excellent texture synthesis models. Saxe et al. [7] observed that the results of the learner based on random weights are comparable to that after regular pre-training and fine-tuning processes. Giryes et al. [54] showed that under certain conditions, DNNs with random Gaussian weights could perform a stable embedding of the original data, permitting a stable recovery of the data from the features represented by the network. Reservoir computing, a new paradigm to use recurrent neural networks with fixed and randomly generated weights, has also been widely adopted in-stream data modeling tasks [55], [56]. Kernel approximation
with random features [4–6] can also be viewed as a random training method as its primary philosophy is mapping the input data to randomized low-dimensional feature space and then applying existing fast linear methods. See the recent survey paper [57]. On the other hand, random projections are well established and commonly used for dimensionality reduction [58]. Here, one utilizes a random matrix to project input patterns from a high-dimensional space to a lower-dimensional representation such that distances between these patterns are preserved with high accuracy, as stated in Johnson-Lindenstrauss lemma [59].

**APPENDIX B**

**GENERALIZED INTEGRAL REPRESENTATION**

In order to admit non-integrable activation functions such as Tanh and Relu in the main results, we define an extended version of integral representation and prove a few new results.

**Function spaces.** Let $\mathcal{F}$ and $\mathcal{G}$ be the Hilbert spaces with inner product

$$
\langle f, g \rangle_{\mathcal{F}} := \int_{\mathbb{R}^m} f(x)\overline{g(x)}\,dx,
$$

$$
\langle \mu, \nu \rangle_{\mathcal{G}} := \int_{\mathbb{R}^m \times \mathbb{R}} \mu(a, b)\nu(a, b)\,dadb.
$$

Let $\mathcal{P}_k = \text{Span}_\mathbb{R}\{b^l \mid l \in [k]\}$ be the space of all polynomial functions in $b$ of degree at most $k \geq 0$. For any measurable functions $\mu$ and $\nu$ on $\mathbb{R}^m \times \mathbb{R}$, we say that $\mu$ and $\nu$ are equivalent in $\mathcal{G}_k := \mathcal{G}/\mathcal{P}_{k-1}$ if there exists a polynomial function $\delta \in \mathcal{P}_{k-1}$ such that $\mu - \nu - \delta \in \mathcal{G}$. The quotient space $\mathcal{G}$ is a Hilbert space with inner product

$$
\langle \mu, \nu \rangle_{\mathcal{G}} := \int_{\mathbb{R}^m \times \mathbb{R}} \partial^k_b \mu(a, b)\partial^k_b \nu(a, b)\,dadb,
$$

because the null space of $\partial^k_b$ is at most $(k-1)$-th degree polynomials: $\ker \partial^k_b = \mathcal{P}_{k-1}$.

**Integral representation** $S$. We identify $\mu \in \mathcal{G}$ as the parameter distribution, and define the integral representation as

$$
S[\mu](x) := \int_{\mathbb{R}^m \times \mathbb{R}} \partial^k_b \mu(a, b)\sigma(a \cdot x - b)\,dadb, \quad \mu \in \mathcal{G}, x \in \mathbb{R}^m.
$$

This is an extended version of the parameter distributions because the distribution is now an equivalent class $[\mu](\text{mod } \mathcal{P}_{k-1})$ rather than a single function $\mu$. However, as we will see, this is a natural setup for non-integral activation functions such as Tanh and Relu. (This depends on the choice of domain, i.e. $\mathbb{R}^m \times \mathbb{R}$ for the present setting, and the harmonic functions on there.)

**Adjoint operator** $S^\ast$. For $S : \mathcal{G} \to \mathcal{F}$, we have

$$
S^\ast[f](a, b) = \int_{\mathbb{R}^m} f(x)\overline{\sigma(a \cdot x - b)}\,dx,
$$

because for any $f \in \mathcal{F}$ and $\mu \in \mathcal{G}$,

$$
\langle f, S[\mu] \rangle_{\mathcal{F}} = \int_{\mathbb{R}^m} f(x)\left[\int_{\mathbb{R}^m \times \mathbb{R}} \partial^k_b \mu(a, b)\partial^k_b \sigma(a \cdot x - b)\,dadb\right]^\ast \,dx
$$

$$
= \int_{\mathbb{R}^m \times \mathbb{R}} \partial^k_b \left[\int_{\mathbb{R}^m} f(x)\overline{\sigma(a \cdot x - b)}\,dx\right]\overline{\partial^k_b \mu(a, b)}\,dadb
$$

$$
= \langle S^\ast[f], \mu \rangle_{\mathcal{G}},
$$

as long as one of the integrals exists.
**Ridgelet transform** \( R \). For a function \( \rho \in L^2(\mathbb{R}) \), we define the ridgelet transform of \( f \in L^2(\mathbb{R}^m) \) by

\[
R[f](a,b) := \int_{\mathbb{R}^m} f(x)\rho(a \cdot x - b)dx, \quad (a, b) \in \mathbb{R}^m \times \mathbb{R}.
\]

In the proof section, we use the Fourier form

\[
R[f](a,b) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\omega)\overline{\hat{\rho}(\omega)e^{i\omega b}}d\omega.
\]

**Proof:** Since \( \rho(a \cdot x - b) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\rho}(\omega)e^{i\omega(a \cdot x - b)}d\omega \),

\[
R[f](a,b) = \frac{1}{2\pi} \int_{\mathbb{R}^m} f(x) \int_{\mathbb{R}} \hat{\rho}(\omega)e^{-i\omega(a \cdot x - b)}d\omega dx
= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\omega)\overline{\hat{\rho}(\omega)e^{i\omega b}}d\omega.
\]

\[\square\]

**Reconstruction formula.** We say a function \( \rho \in L^2(\mathbb{R}) \) is admissible if it satisfies

\[
(2\pi)^{m-1} \int_{\mathbb{R}} |\hat{\rho}(\omega)|^2/|\omega|^m d\omega = 1.
\]

For example, take a Gaussian \( \rho_0(z; \tau) = \exp(-z^2/2\tau^2) \) and let \( \rho(z; \tau) := C\rho_0^{(n)}(z; \tau) \) for some constant \( C \) and \( n \in \mathbb{N} \) s.t. \( 2n - m > 0 \). Then, \( \rho \) satisfies the admissibility condition because

\[
(2\pi)^{m-1} \int_{\mathbb{R}} |\hat{\rho}(\omega)|^2/|\omega|^m d\omega = (2\pi)^{m-1} C \int_{\mathbb{R}} |\omega|^{2n-m}|\hat{\rho}_0(\omega)|^2d\omega < \infty \text{ and we can set } C \text{ for the integral to equal 1.}
\]

Obviously, non-integrable activation functions \( \sigma \) such as Tanh and Relu cannot be admissible. In the following, we show that the reconstruction is still possible when \( \sigma \) is an “anti-derivative” of some admissible function \( \rho \). For example, Relu \( \sigma \) is an anti-derivative of \( \rho \) (in the limit), because \( \sigma^{(2+n)}(z) = \lim_{\tau \to 0} \rho(z; \tau) \).

**Theorem 6.** For the activation function \( \sigma \), assume that there exists an admissible function \( \rho \in L^2(\mathbb{R}) \) and \( k \in \mathbb{N} \) such that \( \sigma^{(k)} = \rho \). Then, \( S[S^*[f]] = f \).

**Proof:**

\[
S[S^*[f]](x) = \int_{\mathbb{R}^m \times \mathbb{R}} \partial_b^k \left[ \int_{\mathbb{R}^m} f(y)\overline{\sigma(a \cdot y - b)}dy \right] \sigma(a \cdot x - b)dadb
= \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} f(y)(\hat{\rho} \ast \rho)(a \cdot (x - y))dy da
= \frac{1}{2\pi} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} f(y) \int_{\mathbb{R}} |\hat{\rho}(\omega)|^2 e^{i\omega a \cdot (x - y)}d\omega dy da
= \frac{1}{2\pi} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} f(y) \int_{\mathbb{R}} |\rho(\omega)|^2 |\omega|^m d\omega e^{ia \cdot (x - y)}dy da
= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \hat{f}(a)e^{ia \cdot x} da = f(x).
\]

\[\square\]

**Plancherel theorem.** The following isometries play an important role because we can identify \( |R[f](a, b)|^2 \) as a density function of the parameter distribution.
Theorem 7. \( \|S^*[f]\|_{2}^{2} = \|R[f]\|_{2}^{2} = \|f\|_{2}^{2}. \)

Proof: By the definition of the adjoint operator and the reconstruction formula, we have \( \|S^*[f]\|_{2}^{2} = \langle S^*[f], S^*[f] \rangle_{\mathfrak{H}} = \langle f, S^*[f] \rangle_{\mathfrak{H}} = \|f\|_{2}^{2}. \) On the other hand, observe that \( \langle S^*[f], S^*[g] \rangle_{\mathfrak{H}} = \langle R[f], R[g] \rangle_{\mathfrak{H}} \) for any \( f, g \in \mathcal{F}. \) Then, \( \|S^*[f]\|_{2}^{2} = \langle S^*[f], S^*[f] \rangle_{\mathfrak{H}} = \langle R[f], R[f] \rangle_{\mathfrak{H}} = \|R[f]\|_{2}^{2}. \)

APPENDIX C

PROOFS

C.1 Theorem 3

Proof: By the minimality, \( \mu_{V} \) satisfies the Pythagorean theorem

\[
\|S[\mu_{V}]\|^{2} + \|f - S[\mu_{V}]\|^{2} = \|f\|^{2}. \tag{29}
\]

On the other hand, \( \mu'_{V} = R[f]\rvert_{V} \) satisfies

\[
\|S[\mu'_{V}]\|^{2} + \|f - S[\mu'_{V}]\|^{2} = \|f\|^{2}, \tag{30}
\]

because, by the Plancherel theorem of ridgelet transform,

\[
\|S[\mu'_{V}]\|^{2} + \|f - S[\mu'_{V}]\|^{2} = \|S[\mu']_{V}\|^{2} + \|S[R[f] - \mu']_{V}\|^{2} = \int_{V} |R[f](a, b)|^{2} \, da \, db + \int_{\mathbb{R}^{m+1} \setminus V} |R[f](a, b)|^{2} \, da \, db \tag{31}
\]

\[
= \|R[f]\|_{L^{2}(\mathbb{R}^{m+1})}^{2} = \|f\|^{2}. \tag{34}
\]

Thus, by the uniqueness of the minimizers in the Hilbert space \( L^{2}(\mathbb{R}^{m}) \), we have \( S[\mu'_{V}] = S[\mu_{V}] \), which concludes that \( \mu_{V} = R[f]\rvert_{V} + \ker S \).

C.2 Theorem 4

In the following, we write \( \langle x \rangle := (1 + |x|^2)^{1/2} \) for short. We employ \( \|f\|_{H^{s}}^{2} := \int_{\mathbb{R}^{m}} \hat{f}(\xi)^{2}(1 + |\xi|^{2}) s \) for the \( L^{2}\)-Sobolev norm of \( f \in L^{2}(\mathbb{R}^{m}) \) with \( s \in \mathbb{R} \). Hence, the \( L^{2}\)-Sobolev space \( H^{s}(\mathbb{R}^{m}) \) is a (closed) subspace of \( L^{2}\)-functions \( f \in L^{2}(\mathbb{R}^{m}) \) that have finite Sobolev norm: \( \|f\|_{H^{s}} < \infty \).

\textbf{Boundedness.} For any \( f \in L^{1}(\mathbb{R}^{m}) \),

\[
|R[f](a, b)| \leq \int_{\mathbb{R}^{m}} |f(x)||\rho(a \cdot x - b)| \, dx \leq \|f\|_{L^{1}(\mathbb{R}^{m})} \|\rho(a \cdot x - b)\|_{L^{\infty}} = \|f\|_{L^{1}(\mathbb{R}^{m})} \|\rho\|_{L^{\infty}(\mathbb{R})}. \tag{35}
\]

\textbf{Decay property.} For any \( f \in H^{s}(\mathbb{R}^{m})(0 \leq s < \infty) \),

\[
(2\pi)^{2} \|R[f](ru, b)\|^{2} \leq \left[ \int_{\mathbb{R}} |\hat{f}(\omega u)| |\hat{\rho}(\omega/r)| \, d\omega \right]^{2} \leq \left[ \int_{\mathbb{R}} \left( |\omega u|^{s} |\hat{f}(\omega u)| |\omega|^{m-1} \hat{\rho}(\omega/r) \right) \, d\omega \right]^{2} \tag{36}
\]

\[
\leq 2 \int_{0}^{\infty} |\omega u|^{2s} |\hat{f}(\omega u)|^{2} \omega^{m-1} \, d\omega \int_{\mathbb{R}} |\omega|^{-2s/(m-1)} |\hat{\rho}(\omega/r)| / r^{2} \, d\omega \tag{37}
\]

\[
= (2\pi C_{\rho, s})^{2} |u|^{-2s-m} \int_{0}^{\infty} (|\omega u|)^{2s} |\hat{f}(\omega u)|^{2} \omega^{m-1} \, d\omega. \tag{38}
\]
Here, we assume that the following constant exists and it is finite:

\[ C_{p,s}^2 := \frac{1}{2(2\pi)^2} \int_{\mathbb{R}} \frac{|\tilde{\rho}(\omega)|^2}{|\omega|^{2s+m-1}} d\omega < \infty. \]  \hfill (41)

In particular,

\[ \int_{\mathbb{S}^{m-1}} \int_{\varepsilon}^{\delta} |R[f](ru,b)|^2 r^{m-1} dr du \leq C_{p,s}^2 \|f\|_{H^s}^2 \frac{1}{2s} \left( \frac{1}{\varepsilon^{2s}} - \frac{1}{\delta^{2s}} \right). \]  \hfill (42)

### C.3 Theorem 5

**Proof:** By Theorem 3, the approximation error \( \inf_{\mu \in \mathcal{M}(p)} \|f - S[\mu]\|_2^2 \) is lower bounded by the tail bound \( \|R[f]\|_{V^r}^2 = \|f\|^2 - \|R[f]\|_V^2 \); and by Theorem 4, we have an upper bound of the parameter “density” \( |R[f](ru,b)|^2 \lesssim \min\{1, r^{-2s-m}\} \). Here, by integrating the upper bound over the (bandlimited) domain \( V \), we quantify the tail bound \( \|R[f]\|_{V^r}^2 \) with respect to the smoothness \( s \).

By averaging the direction \( u \in \mathbb{S}^{m-1} \),

\[ \int_{\mathbb{S}^{m-1}} |R[f](ru,b)|^2 du \leq \min\{C_0^2 \Omega_{m-1}, C_2^2 r^{-2s-m}\}. \]  \hfill (43)

Here, \( \Omega_{m-1} = 2\pi^{m-1} / \Gamma(m/2) \) is the surface area of \( \mathbb{S}^{m-1} \). Therefore, the rate in \( r \) (in the average) changes at \( r = \vartheta \) s.t. \( C_0^2 \Omega_{m-1} = C_2^2 \vartheta^{-2s-m} \).

First, suppose that \( \lambda \leq \vartheta \), then, \( |R[f](a,b)| \leq C_0 \) for every \( (a,b) \in V \). Therefore,

\[ \|R[f]\|_{V^r}^2 = \int_{\mathbb{S}^{m-1}} \int_0^\lambda \int_{-\kappa/2}^{\kappa/2} |R[f](ru,b)|^2 r^{m-1} dr dudb \leq C_0^2 V_m \kappa \lambda^m =: I_0(\lambda). \]  \hfill (44)

Here, \( V_m = \pi^{m/2} / \Gamma(m/2+1) \) is the volume of \( m \)-unit ball, and we use the relation \( \Omega_{m-1}/m = V_m \).

Next, suppose that \( \lambda \geq \vartheta \), then, by \( \{42\} \),

\[ \|R[f]\|_{V^r}^2 = I_0(\vartheta) + \int_{\mathbb{S}^{m-1}} \int_{\vartheta}^\lambda \int_{-\kappa/2}^{\kappa/2} |R[f](ru,b)|^2 r^{m-1} dr dudb \leq C_0^2 V_m \kappa \vartheta^m + \frac{C_2^2 \kappa}{2s} \left( \frac{1}{\lambda^{2s}} - \frac{1}{\vartheta^{2s}} \right) \]  \hfill (47)

\[ = \frac{C_2^2 \kappa}{2s} \left\{ \lambda^{-2s} + \frac{2s - m}{m} \vartheta^{-2s} \right\}. \]  \hfill (48)

To conclude, we have the following approximation lower bound:

\[ \inf_{\mu \in \mathcal{M}(p)} \|f - S[\mu]\|_{L^2(\mathbb{R}^m)}^2 \geq \|R[f]\|_{V^r}^2 \geq 0 \]  \hfill (49)

\[ \geq \|f\|_{L^2(\mathbb{R}^m)}^2 - \|R[f]\|_{V^r}^2 \geq \|f\|_{L^2(\mathbb{R}^m)}^2 - \left\{ \frac{C_0^2 V_m \kappa \lambda^m}{2s} \left\{ \lambda^{-2s} + \frac{2s - m}{m} \vartheta^{-2s} \right\} \right\} \geq 0 \leq \lambda \leq \vartheta \]  \hfill (50)

\[ \|f\|_{L^2(\mathbb{R}^m)}^2 - \left\{ \frac{C_0^2 V_m \kappa \lambda^m}{2s} \left\{ \lambda^{-2s} + \frac{2s - m}{m} \vartheta^{-2s} \right\} \right\} \geq 0 \leq \lambda \leq \vartheta, \]  \hfill (51)

where the bound is continuous at \( \lambda = \vartheta \).
C.4 Reconstruction Formula for $k$-Homogeneous Activation Functions in the Polar Coordinates

A $k$-homogeneous activation function $\sigma(z) := z^k$ for $z \in \mathbb{R}$ with $k \in \mathbb{N}$ satisfies the $k$-homogeneity:

$$\sigma(az) = a^k \sigma(z), \quad \forall a > 0. \quad (52)$$

Hence, the Euclidean coordinates $(a, b) \in \mathbb{R}^m \times \mathbb{R}$ for $k$-homogeneous nets has an obvious redundancy:

$$\sigma(a \cdot x - b) = r^k \sigma(u \cdot x - t), \quad \text{if} \ (a, b) = (ru, rt) \quad (53)$$

where $r \in \mathbb{R}_+$ denotes the norm $\|a\|$ of $a, u \in S^{m-1}$ the direction of $a$, and $t \in \mathbb{R}$ the $b$ rescaled by $r$. To eliminate the redundancy in $r$, we rewrite the reconstruction formula in the polar coordinates $(u, t)$ as below.

**Proposition 8** (Reconstruction formula for $k$-Homogeneous Activation Functions.).

$$f(x) = c \rho \int_{S^{m-1} \times \mathbb{R}} (-\triangle)^{(m+k)/2} f(tu)(u \cdot x - t)_+^k \, du \, dt. \quad (54)$$

**Proof:** First, we rewrite the integral representation in the polar coordinates by changing the variables: $(a, b) = (ru, rt)$ with $(r, u, t) \in \mathbb{R}_+ \times S^{m-1} \times \mathbb{R}$. Suppose $\sigma^{(s)} = \rho$ for an admissible function $\rho$ such that $\rho^{(s)}$ is even.

$$S[\mu](x) = \int_{\mathbb{R}^m \times \mathbb{R}} \partial^{2s}_b \mu(a, b)(a \cdot x - b)_+^k \, da \, db \quad (55)$$

$$= \int_{\mathbb{R}_+ \times S^{m-1} \times \mathbb{R}} (\partial^{2s}_b \mu)(ru, rt)(u \cdot x - t)_+^k r^{m+k} \, dr \, du \, dt \quad (56)$$

$$= \int_{S^{m-1} \times \mathbb{R}} \left[ \int_0^\infty (\partial^{2s}_b \mu)(ru, rt)r^{m+k} \, dr \right] (u \cdot x - t)_+^k \, du \, dt \quad (57)$$

Therefore, we define the ridgelet spectrum $\hat{R}[f]$ in the polar coordinates by letting $\mu = R[f]$ (w.r.t. $\rho$) inside $\cdots$. Using the Fourier form and $\sigma^{(2s)} = \rho^{(s)}$, we have $(\partial^{2s}_b \hat{R}[f])(a, b) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\omega a)\rho^{(s)}(\omega)e^{i\omega t}d\omega$. Then, $\hat{R}[f]$ is further calculated as below.

$$\hat{R}[f](u, t) := \int_0^\infty (\partial^{2s}_b R[f])(ru, rt)r^{m+k} \, dr \quad (58)$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}_+ \times \mathbb{R}} \hat{f}(\omega ru)\rho^{(s)}(\omega)r^{m+k}e^{i\omega t}d\omega \, dr \quad (59)$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}_+ \times \mathbb{R}} \hat{f}(\omega u)\rho^{(s)}(\omega)|\omega|^{m+k}/r^{m+k+1}e^{i\omega t}d\omega \, dr \quad (60)$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} (-\triangle)^{(m+k)/2}f(\omega u)e^{i\omega t}d\omega \int_0^\infty \frac{\rho^{(s)}(r)r^{-(m+k+1)}}{\rho^{(s)}(r)r^{-(m+k+1)}}dr \quad (61)$$

$$= c_p(-\triangle)^{(m+k)/2}f(tu), \quad (62)$$

with constant $c_p := \int_0^\infty \rho^{(s)}(r)r^{-(m+k+1)}dr$. In the third equation, we changed the variable: $(\omega', r') \leftrightarrow (\omega r, |\omega|)$ with $d\omega dr = d\omega' dr'/r'$. We note that if $m+k$ is odd, then the fractional Laplacian $(-\triangle)^{(m+k)/2}$ is understood as the Fourier multiplier: $(-\triangle)^s f := (2\pi)^{-m} \int_{\mathbb{R}^m} |\xi|^{2s} \hat{f}(\xi)d\xi$. 


By the admissibility assumption on \( \rho \), we have the reconstruction formula \( S[R[f]] = f \) for ordinary \( R \). Then, automatically we have the reconstruction formula for \( \tilde{R} \):

\[
f(x) = \int_{\mathbb{S}^{m-1} \times \mathbb{R}} \tilde{R}[f](u, t)(u \cdot x - t)^k \, du \, dt = c_\sigma \int_{\mathbb{S}^{m-1} \times \mathbb{R}} (-\Delta)^{(m+k)/2} f(tu)(u \cdot x - t)^k \, du \, dt,
\]

(63) this yields the assertion. (Furthermore, we can verify this formula directly as below.)

\[
c_\sigma \int_{\mathbb{S}^{m-1} \times \mathbb{R}} (-\Delta)^{(m+k)/2} f(tu)(u \cdot x - t)^k \, du \, dt
\]

(64)

\[
= \frac{c_\sigma}{2\pi} \int_{\mathbb{S}^{m-1} \times \mathbb{R}} |\omega u|^{m+k} \hat{f}(\omega) \tilde{\sigma}(\omega) e^{i\omega u \cdot x} \, d\omega
\]

(65)

\[
= \frac{c_\sigma}{2\pi} \int_{\mathbb{S}^{m-1} \times \mathbb{R}} \hat{f}(\omega u) e^{i\omega u \cdot x} |\omega|^{m-1} \, d\omega
\]

(66)

\[
= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \hat{f}(\xi) e^{i\xi \cdot x} \, d\xi
\]

(67)

\[
= f(x).
\]

(68)

Here, we used \( \tilde{\sigma}^{(k+1)}(\omega) = 1 \) since \( \sigma(z) = z^k \hat{\sigma}^{(k+1)} = \delta \), then changed the variable from polar coordinates \( (u, |\omega|) \) to Euclidean coordinates \( \xi = |\omega|u \), and applied the Fourier inversion formula.

\[\Box\]

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