Symmetry Breaking in Stock Demand

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Scale-free distributions and correlation functions found in financial data are reminiscent of the scale invariance of physical observables in the vicinity of a critical point. Here, we present empirical evidence for a transition phenomenon, accompanied by a symmetry breaking, in the investors’ demand for stocks. We study the volume imbalance \( \Omega \) — difference between the number of shares traded in buyer-initiated and seller-initiated trades in a time interval \( \Delta t \) — conditioned on \( \Sigma \) which is defined as the local first moment of \( \Omega \) in \( \Delta t \). We find that the conditional distribution \( P(\Omega|\Sigma) \) undergoes a qualitative change in behavior as \( \Sigma \) increases beyond a critical threshold \( \Sigma_c \). For \( \Sigma < \Sigma_c \), \( P(\Omega|\Sigma) \) displays a maximum at \( \Omega = 0 \), i.e., trades in \( \Delta t \) are equally likely to be buyer initiated or seller initiated. For \( \Sigma > \Sigma_c \), \( \Omega = 0 \) becomes a local minimum and two new maxima \( \Omega_+ \) and \( \Omega_- \) appear at non-zero values of \( \Omega \), i.e., trades in \( \Delta t \) are either predominantly buyer initiated or predominantly seller initiated. We interpret these results using a Langevin equation with multiplicative noise.

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Here, we present evidence for an analogous transition phenomenon in a financial context [1,2]. Specifically, we study the statistical properties of investors’ demand for stocks — quantified as the imbalance in the number of shares transacted by buyers and sellers over a time interval \( \Delta t \). We analyze the probability distribution of demand, conditioned on its local “noise” intensity (a variance-like parameter \( \Sigma \) defined below). We find that for intensities smaller than a critical value \( \Sigma_c \), the most probable value of demand is approximately zero — neither buying nor selling behavior dominates. For intensities larger than that critical value \( \Sigma_c \), two most probable values emerge that are symmetric around zero demand, corresponding to two distinct “phases” — excess demand and excess supply [3]. Under such conditions, the market behavior is either mainly-buying or mainly-selling, spending almost equal amount of time in each state. In other words, exchanging every “buy” with a “sell” gives the same state below the critical noise intensity, whereas above this threshold, the symmetry of this exchange is broken.

In classic critical phenomena, the qualitative change in behavior accompanying a phase transition can be formalized in terms of the extrema of a phenomenological potential, or equivalently in terms of the extrema of the corresponding probability distributions [8,9]. We first follow the latter approach and study the behavior of the probability distribution of demand. Using transactions and quotes data for the 116 most-actively traded stocks [10], we quantify the demand for a stock by calculating the “volume imbalance” over a time interval \( \Delta t \), defined to be the difference between \( Q_B \), the number of shares traded in buyer-initiated trades, and \( Q_S \), the number of shares traded in seller-initiated trades in \( \Delta t \).

\[
\Omega(t) = Q_B - Q_S = \sum_{i=1}^{N} q_i a_i .
\]

Here, the indicator \( a_i = 1 \) for buyer-initiated trades (buy trades) and \( a_i = -1 \) for seller-initiated trades (sell trades) [17], \( q_i \) is the number of shares traded in transaction \( i \), and \( N \equiv N_{\Delta t} \) denotes the number of trades in \( \Delta t \).

Our analysis of the (unconditional) probability distribution \( P(\Omega) \) for each stock shows a single peak around \( \Omega = 0 \). Since previous work shows that the distribution of \( q_i \) has divergent variance [10,20], we quantify the noise intensity by computing the ‘local deviation’ [21], defined as the centered first moment,

\[
\Sigma(t) = \langle |q_i a_i - \langle q_i a_i \rangle| \rangle ,
\]

where \( \langle \ldots \rangle \) denotes ‘local’ expectation values computed from all trades in the time interval \( \Delta t \). Next, we examine the behavior of the conditional distribution \( P(\Omega|\Sigma) \) of \( \Omega \) for a given value of the local deviation \( \Sigma \) for \( \Delta t = 15 \) min, [Fig. 1(a)]. For small \( \Sigma \), we find that \( P(\Omega|\Sigma) \) is single peaked displaying a maximum at \( \Omega = 0 \). When \( \Sigma \) exceeds a critical threshold \( \Sigma_c \), the behavior of \( P(\Omega|\Sigma) \) undergoes a qualitative change, and is double peaked with two new maxima appearing at non-zero values, \( \Omega_+ \) and \( \Omega_- \), symmetric around zero. Figure 1(a) also shows that the separation between the two maxima increases with \( \Sigma \).
This qualitative change in the behavior of \( P(\Omega) \) implies that for \( \Sigma < \Sigma_c \), the most-probable value of demand is approximately zero, and possesses the symmetry that leaves the most probable value invariant under the operation \( \Omega \to -\Omega \), or at the microscopic (trade) level, under the operation \( B \to S \) of changing every buyer-initiated trade \( B \), to a seller-initiated trade \( S \). For \( \Sigma > \Sigma_c \), the two most probable values \( \Omega \pm \) are non-zero, and the \( \Omega \to -\Omega \) (\( B \to S \)) symmetry is broken. In other words, while for \( \Sigma < \Sigma_c \) buy and sell trades are equally probable in each time interval (zero demand), for \( \Sigma > \Sigma_c \), trades in each time interval are either mostly buy trades (excess demand) or mostly sell trades (excess supply) giving rise to non-zero values of \( \Omega \pm \). Identical results can be obtained by conditioning \( P(\Omega) \) on the total trade volume in \( \Delta t \), \( Q(t) \equiv Q_B + Q_S \).

Our findings are analogous to phase transition phenomena in physical systems, where the behavior of the system undergoes a qualitative change at a critical threshold of a control parameter \( T \). In such systems, the change in behavior can be quantified by an order parameter \( \Psi \) which is identically zero for values of \( T \) below (or above as the case may be) a certain critical value \( T_c \), and becomes non-zero as \( T \) crosses \( T_c \). In our problem, the “order parameter” \( \Psi \), can be identified by the location of the maxima \( \Omega \pm \) of \( P(\Omega) \). Figure 1(c) shows the change in \( \Psi \) as a function of \( \Sigma \) as described by
\[
\Psi(\Sigma) = \begin{cases} 
0 & \text{if } \Sigma < \Sigma_c \\
\Sigma - \Sigma_c & \text{if } \Sigma > \Sigma_c
\end{cases},
\]
with \( \beta \approx 1 \).

In the mean-field theory of critical phenomena (Landau-Ginzburg theory), the qualitative change in behavior of the system is attributed to the changes in symmetry of the underlying potential \[10\]. In the following, we pursue an analogous approach to understand our empirical results. Since the transition behavior that we find occurs with change in noise intensity, we follow an approach similar to those used to understand non-equilibrium phase transition phenomena \[23\]. We start with expressing the dynamics of \( \Omega \) through a deterministic differential equation,
\[
d\Omega = h_\lambda(\Omega) \, dt, \tag{4a}
\]
where \( \lambda \) is a parameter quantifying the coupling of the system to its environment. Letting \( \lambda \) fluctuate randomly with noise intensity \( \sigma \), Eq. (4a) becomes, in general, a stochastic differential equation with multiplicative noise. Since the form of \( h_\lambda(\Omega) \) is not known, and since it is not a priori clear if \( \lambda \) is an observable, we describe the dynamics of \( \Omega \) through \[23\]
\[
d\Omega = u(\Omega) dt + \sigma v(\Omega) dW_t, \tag{4b}
\]
where \( u \equiv u(\Omega) \) is the drift term, \( v \equiv v(\Omega) \) reflects the effects of multiplicative noise, \( dW_t \) is the standard Wiener differential satisfying \( \langle dW_t \, dW_{t'} \rangle = \delta(t - t') dt \), and the parameter \( \sigma \) quantifies the intensity of the noise term \[20\]. The functions \( u \) and \( v \) may be estimated from the data as a “local” mean and standard deviation of \( d\Omega \) \[28\]. The Fokker-Planck equation corresponding to Eq. (4b) which describes the evolution of the transition probability density \( \Pi \equiv P(\Omega, t | \Omega(t = 0)) \) is
\[
\frac{\partial \Pi}{\partial t} = - \frac{\partial}{\partial \Omega}[u \Pi] + \frac{\sigma^2}{2} \frac{\partial^2}{\partial \Omega^2} \left[ v^2 \Pi \right]. \tag{5}
\]

The stationary probability density \( P(\Omega) \) from Eq. (6) is
\[
P(\Omega) = \frac{A}{v^2} \exp \left( \frac{2}{\sigma^2} \int^\Omega \frac{u}{v^2} \, dx \right), \tag{6}
\]
where \( A \) is the normalization constant, and \( P \) is assumed to be normalizable. Equation (6) can be rewritten as
\[
P(\Omega) = A \exp \left( - \frac{V(\Omega)}{\sigma^2} \right), \tag{7}
\]
where the function \( V(\Omega) \) takes the meaning of a effective “stochastic” potential given by \[23\]
\[
V(\Omega) \equiv \left[ \int^\Omega \frac{u}{v^2} \, dx - \sigma^2 \ln v \right]. \tag{8}
\]

Thus, the extrema of the probability density can be translated into the extrema of the underlying stochastic potential.

As the noise intensity of the environment (\( \sigma \)) changes, the potential could change shape, acquire new minima, and consequently the system could undergo drastic changes in behavior \[29\]. Can the transition behavior we find empirically for \( P(\Omega) \) be understood in this framework? To address this question, we must examine the shape of the potential \( V(\Omega) \) for different values of \( \Sigma \), which monotonically depends on \( \sigma \).

In order to study \( V(\Omega) \) for different \( \Sigma \) empirically, we first extract the functions \( u(\Omega) \) and \( v(\Omega) \) and analyze their behavior for different values of \( \Sigma \). From Eq. (4b), \( u \) is the drift term, which for a given \( \Sigma \), can be estimated by computing the equal-time expectation value of the change \( \Delta \Omega \equiv \Omega(t + \Delta t) - \Omega(t) \) for a given \( \Omega \),
\[
u \approx \langle \Delta \Omega \rangle_{(\Omega, \Sigma)}. \tag{9}
\]
Similarly, the product \( \sigma v \) can be estimated from the “local” deviation
\[
\sigma v \approx \langle |\Delta \Omega - \langle \Delta \Omega \rangle_{(\Omega, \Sigma)}| \rangle_{(\Omega, \Sigma)} \tag{10}
\]

Figures (a) and (b) show \( u(\Omega) \) and \( v(\Omega) \) for three different values of \( \Sigma \). Clearly, the functional form \( u(\Omega) \) does not vary with \( \Sigma \) and is consistent with a linear behavior for all \( \Sigma \). Figure (b) shows that, for small \( \Sigma \), \( \sigma v(\Omega) \) is approximately flat, whereas for large \( \Sigma \), \( \sigma v \) acquires
a marked ‘peak’ around $\Omega = 0$. Except for the smallest $\Sigma$, the functional forms $\sigma v(\Omega)$ for different $\Sigma$ seem to be consistent within a multiplicative factor (related to $\sigma$) \cite{31}.

Next, we shall analyze the extrema of the stochastic potential $V(\Omega)$ for different values of $\Sigma$. From Eqs. (6) and (3), the extrema of $P(\Omega)$ correspond to the roots of the function

$$F(\Omega) \equiv \frac{dV(\Omega)}{d\Omega} = u - \sigma^2 v \frac{dv}{d\Omega}.$$ \hspace{1cm} (11)

Figure 3(a) shows $F(\Omega)$ for three different $\Sigma$. For $\Sigma < \Sigma_c$, the function $F(\Omega)$ displays only one root at $\Omega = 0$. Near $\Sigma = \Sigma_c$, $F(\Omega)$ displays an inflexion point at $\Omega = 0$. For $\Sigma > \Sigma_c$, the potential appears almost flat: the existing minimum begins to a (unstable) local maximum and displays an inflexion point at $\Omega = 0$. For $\Sigma > \Sigma_c$, the potential displays two clear minima at $\Omega = 0$, which is consistent with one maximum for $P(\Omega)$ which we find. For $\Sigma \approx \Sigma_c$, the potential appears almost flat: the existing minimum begins to change into a (unstable) local maximum and displays an inflexion point at $\Omega = 0$. For $\Sigma > \Sigma_c$, the potential displays two clear minima at $\Omega = 0$, and a local maximum at $\Omega = 0$, consistent with the bimodal nature of the distribution $P(\Omega)$ for $\Sigma > \Sigma_c$.

In summary, we investigate the dynamics of the demand $\Omega$ by examining the distribution of volume imbalance $\Omega$ for changing market conditions quantified by the local deviation $\Sigma$. We find that the distribution $P(\Omega)$ is single peaked for $\Sigma$ smaller than a critical threshold $\Sigma_c$. For $\Sigma > \Sigma_c$, the distribution $P(\Omega)$ changes to a double-peaked distribution. The analog of the order parameter $\Psi$ which describes the above qualitative change in behavior is zero for $\Sigma < \Sigma_c$ and behaves as $\sim (\Sigma - \Sigma_c)^{\beta}$ for $\Sigma > \Sigma_c$, where $\beta \approx 1$. We have also seen that the dynamics of demand can be understood in terms of a ‘stochastic’ potential which changes its behavior with $\Sigma$. As $\Sigma$ crosses the critical threshold $\Sigma_c$, the system undergoes a transition from a ‘disordered’ state with most probable demand equal to zero to an ‘ordered’ state with two phases, excess demand and excess supply.

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1. H. Takayasu, ed., *Empirical Science of Financial Fluctuations*. Springer, Berlin, 2002.
2. H. Takayasu and M. Takayasu *Physica A* 269, 24 (1999).
3. J.-P. Bouchaud, *Quantitative Finance* 1 105 (2001).
4. Y. C. Zhang, “Towards a Theory of Marginally Efficient Markets,” *Physica A* 269 30 (1999)
5. R. N. Mantegna and H. E. Stanley, *An Introduction to Econophysics: Correlations and Complexity in Finance* (Cambridge University Press, Cambridge, 2000).
6. J. P. Bouchaud and M. Potters, *Theory of Financial Risk* (Cambridge University Press, Cambridge, 2000).
7. J. D. Farmer, *Computing in Science $\&$ Engineering* 1, 26–39 (1999).
8. B. Halperin and P. Hohenberg *Rev. Mod. Phys* 49, 435 (1977).
9. H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford University Press, Oxford, 1971).
10. For records of trade times, price, and number of shares traded, and the bid-ask quotes, we analyze the data for the 116 most frequently traded US stocks from the *Trades and Quotes database* (New York Stock Exchange, 1994-95) for the 2 yr period 1994-95.
11. J. Campbell, A. W. Lo, and A. MacKinlay, *The Econometrics of Financial Markets* (Princeton University Press, 1997).
12. C. M. Lee and M. J. Ready *Journal of Finance* 46, 733 (1991).
13. Following the procedure of C. M. Lee and M. J. Ready \cite{12}, we use the prevailing quote at least 5 s prior to the trade.
14. V. Plerou et al., “Quantifying Stock Price Response to Demand Fluctuations,” e-print: cond-mat/0106657.
15. J. Hasbrouck *J. Fin. Econ.* 22 (1988) 229.
16. J. D. Farmer, “Market Force, Ecology and Evolution,” e-print: arXiv.org/9812005.
17. We distinguish buyer-initiated and seller-initiated trades defined by which of the two participants in the trade, the buyer or the seller, is more eager to trade. When such a distinction does not exist, we label the trade as indeterminate. We identify buyer and seller initiated trades using the bid and ask quotes $S_B(t)$ and $S_A(t)$ at which a market maker is willing to buy or sell respectively \cite{10}. Using the mid-value $S_M(t) = (S_A(t) + S_B(t))/2$ of the prevailing quote \cite{13,14,12}, we label a trade buyer initiated if $S(t) > S_M(t)$, and seller initiated if $S(t) < S_M(t)$. For trades occurring exactly at $S_M(t)$, we use the sign of the change in price from the previous trade to determine whether the trade is buyer or seller initiated, while if the previous trade is at the current trade price, the trade is labelled indeterminate \cite{12}. In our case, an average of $\approx 17\%$ of the trades remain indeterminate.
18. We disregard the first trade of each day as it is most often a block trade.
19. P. Gopikrishnan *et al*., *Phys. Rev. E* 62, 4493 (2000).
20. In Ref. \cite{14} it is shown that $q_t$ have a power-law distribution with exponent $\zeta \approx 3/2$ which corresponds to a divergent second moment.
21. We note that the results presented below are robust to using the ‘local deviation’ of $\Omega = (\Sigma N)$ instead of $\Sigma$.
22. Interestingly, using the variance instead of the local deviation, one obtains the mean-field theory value for the exponent, $\beta = 1/2$.
23. W. Horsthemke and R. Lefever, *Noise Induced Phase Transitions*, Springer Verlag, Berlin, 1984.
24. H. Haken, *Rev. Mod. Phys.* 47, 67 (1975).
25. H. Haken, *Synergetics: An Introduction: Nonequilibrium
phase transitions and self-organization in physics, chemistry, and biology (Springer Verlag, Berlin, 1978).

[26] In the literature, the diffusion term in Eq. (11) is attributed to fluctuations of a parameter \( \lambda \) coupled to the environment, and \( \sigma \) would then correspond to the intensity of environmental fluctuations. It is yet unclear what interpretation to ascribe to \( \lambda \) (qualitatively some measure of news coupled with price changes), and the question of whether it is an observable remains open. Therefore, we estimate the drift and diffusion term and examine their behavior as \( \Sigma \) varies.

[27] Since the number of trades \( N \) is distributed as a power-law [32], or equivalently the time interval between trades is distributed as a power-law (with an exponential truncation), the stochastic differential equations should be more precisely fractional [8,33].

[28] Our goal is to understand the change in behavior of \( P_\Omega(\Omega) \) with \( \sigma \). Our results hold under the assumption that \( \sigma \) changes much slower than the scale at which \( \Omega \) changes, so that a stationary \( P(\Omega) \) is reached.

[29] For a related discussion in the context of price fluctuations and market crashes see J.-P. Bouchaud and R. Cont, Eur. Phys. J. B 6, 543 (1998).

[30] In our problem, \( \sigma \) is not known; however, \( \Sigma \) is known, and it monotonically depends on \( \sigma \). Thus, instead of studying \( V(\Omega) \) for different \( \sigma \), we examine \( V(\Omega) \) for different \( \Sigma \). We note that although \( \Sigma \) depends monotonically on \( \sigma \), the converse need not be true. However, it is convenient to assume that changes of \( \Sigma \) largely reflect changes in \( \sigma \).

[31] Although, we consider only multiplicative noise in our description, in general there could be additive noise present as well.

[32] V. Plerou et al, Phys. Rev. E 62, R3023 (2000).

[33] F. Mainardi, M. Raberto, R. Gorenflo, and E. Scalas, Physica A 287, 468 (2000).

[34] T. E. Holy, Phys. Rev. Lett. 79, 3545 (1997).

FIG. 1. (a) Conditional density \( P(\Omega|\Sigma) \) for varying \( \Sigma \). The distribution changes from a single-peaked distribution for small \( \Sigma \) (solid line), to a double-peaked distribution for large \( \Sigma \) (dashed line). (b) Same as (a) in semi-logarithmic scale. (c) Order parameter, \( \Psi \) (position of the maxima of the distribution \( P(\Omega|\Sigma) \)), as a function of \( \Sigma \). For small \( \Sigma \), \( P(\Omega|\Sigma) \) displays a single maximum whereas for large \( \Sigma \) two maxima appear. Both \( \Sigma \) and \( \Omega \) have been normalized to zero mean and unit first moment. For a more accurate estimation of the location of the extrema, all densities were computed using the density estimator of [34].

FIG. 2. (a) The drift part \( u(\Omega) \) and (b) the variance part \( \sigma v(\Omega) \) of Eq (1), estimated from the data as the local mean and local deviation of \( \Delta \Omega \). The curves in (b) have been shifted vertically for clarity.
FIG. 3. Stochastic “force” $F(\Omega)$ for three different values of $\Sigma$ calculated from Eq. (11). The derivative $\sigma v / d\Omega$ is calculated by first fitting the function $\sigma v$ by a third order polynomial. For $\Sigma < \Sigma _c$, the function $F(\Omega)$ displays only one zero at $\Omega = 0$. For $\Sigma \approx \Sigma _c$, the existing root starts to branch into three roots, and for $\Sigma > \Sigma _c$, we find three roots: $\Omega = 0$ and $\Omega = \Omega _\pm$.

FIG. 4. Stochastic potential $V(\Omega)$ for $\Sigma < \Sigma _c$ shows one minimum at $\Omega = 0$, whereas for $\Sigma > \Sigma _c$ the potential has a maximum at $\Omega = 0$ and two new minima $\Omega _\pm$ appear. We compute $V(\Omega)$ by integrating $F(\Omega)$ [Fig. 3]. The curves have been shifted vertically for clarity.