Approximating the partition function of planar two-state spin systems*

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Abstract

We consider the problem of approximating the partition function of the hard-core model on planar graphs of degree at most 4. We show that when the activity $\lambda$ is sufficiently large, there is no fully polynomial randomised approximation scheme for evaluating the partition function unless $\text{NP} = \text{RP}$. The result extends to a nearby region of the parameter space in a more general two-state spin system with three parameters. We also give a polynomial-time randomised approximation scheme for the logarithm of the partition function.

1 Introduction

A spin system is a model of particle interaction on a graph. Every vertex of the graph is assigned a state, called a spin. A configuration assigns a spin to every vertex, and the weight of the configuration is determined by interactions of neighbouring spins.

In this paper, we consider the following two-spin model, which applies to spin systems on a graph $G = (V,E)$. The model has three parameters, $\beta$, $\gamma$ and $\lambda$. It is easiest to view these as non-negative rationals for now — we will be slightly more general later. A configuration $\sigma : V(G) \to \{0,1\}$ is an assignment of the two spins “0” and “1” to the vertices in $V$. The configuration $\sigma$ has a weight $w_G(\sigma)$, which depends upon $\beta$, $\gamma$ and $\lambda$. Let $b(\sigma)$ denote the number of edges $(u,v)$ of $G$ with $\sigma(u) = \sigma(v) = 0$, let $c(\sigma)$ be the number of edges $(u,v)$ of $G$ with $\sigma(u) = \sigma(v) = 1$ and let $\ell(\sigma)$ be the number of vertices $u$ of $G$ with $\sigma(u) = 1$. Then $w_G(\sigma) = \beta^{b(\sigma)} \gamma^{c(\sigma)} \lambda^{\ell(\sigma)}$. The partition function of the model is given by

$$Z_{\beta,\gamma,\lambda}(G) = \sum_{\sigma : V(G) \to \{0,1\}} w_G(\sigma).$$

Two important special cases are

- the case $\beta = 1$, $\gamma = 0$, which is the hard-core model, and
- the case $\beta = \gamma$, which is the Ising model.

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The hard-core model \cite{3} is a model of a gas in which vertices are either occupied by a particle (in which case they have spin 1) or unoccupied (in which case they have spin 0). The particles cannot overlap and adjacent vertices are close together, hence \( \gamma = 0 \). The Ising model is a model of ferromagnetism. In this paper we study the hard-core model and a region of nearby two-state spin systems.

1.1 Previous work

Evaluating \( Z_{\beta,\gamma,\lambda}(G) \) is a trivial computational problem if \( \beta \gamma = 1 \), because the partition function factors. In other cases, the complexity of evaluation has been studied in detail. When \( \lambda = 1 \), the problem of computing the partition function on planar \( \Delta \)-regular graphs is called Pl-Hol(\( a, b \)) in \cite{7}, where \( a \) corresponds to \( \beta \) and \( b \) corresponds to \( \gamma \). There is a dichotomy \cite{7} Theorem 1: for non-negative \( a, b \), the problem Pl-Hol(\( a, b \)) can be computed in polynomial time in the trivial cases \( ab = 1 \) and \( a = b = 0 \), and in the case of the Ising model with no external field, \( a = b \). In all other cases the problem is \#P-hard.

A standard transformation extends this dichotomy to arbitrary \( \lambda > 0 \). Consider a configuration \( \sigma: V(G) \to \{0, 1\} \) of a planar \( \Delta \)-regular graph \( G \). Counting the number of edges adjacent to a “1” spin in two ways, we have \( \Delta \ell(\sigma) = 2(\sigma) + (|E(G)| - b(\sigma) - c(\sigma)) \). Therefore,

\[
Z_{\beta,\gamma,\lambda}(G) = \lambda^{\ell(\sigma)/\Delta} Z_{\beta \lambda^{-1/\Delta}, \gamma \lambda^{1/\Delta}}(G),
\]

which is as hard to compute as Pl-Hol(\( \beta \lambda^{-1/\Delta}, \gamma \lambda^{1/\Delta} \)). Suppose \( \beta \) and \( \gamma \) are not both 0. Unless \( \lambda = 1 \), we have either \( \beta \lambda^{-1/3} \neq \gamma \lambda^{1/3} \) or \( \beta \lambda^{-1/4} \neq \gamma \lambda^{1/4} \). If \( \beta \gamma \neq 1 \) then in either case, we can conclude from above that evaluating \( Z_{\beta,\gamma,\lambda}(G) \) is \#P-hard when the input \( G \) is restricted to be a planar graph of degree at most 4.

Since the complexity of exactly evaluating the partition function is intractable, much effort has focussed on the difficulty of approximately evaluating the partition function for a given set of parameters \( \beta, \gamma \) and \( \lambda \).

The complexity of approximating the partition function of the hard-core model and the Ising model in general (not necessarily planar) graphs is well-understood. The Gibbs measure is the distribution on configurations \( \sigma: V(G) \to \{0, 1\} \) in which the probability of configuration \( \sigma \) is proportional to \( w_G(\sigma) \). This notion of Gibbs measure extends to certain infinite graphs, for example infinite regular trees, where it may or may not be unique. For the hard-core model, there is a critical point \( \lambda_c(\Delta) = (\Delta - 1)\Delta^{-1}/(\Delta - 2)\Delta \) such that the infinite \( \Delta \)-regular tree has a unique Gibbs measure if and only if \( \lambda \leq \lambda_c \). An important result of Weitz \cite{26} showed that, in every graph with maximum degree at most \( \Delta \), the correlations between spins in the hard-core model decay rapidly with distance as long as \( \lambda \leq \lambda_c \). As a result, he gives \cite{26} Corollary 2.8] a fully-polynomial (deterministic) approximation scheme (FPTAS) for evaluating the hard-core partition function on graphs of degree at most \( \Delta \) for any \( \lambda < \lambda_c \). By contrast, Sly and Sun \cite{24} Theorem 1] (see also the earlier hardness results of Sly \cite{25} and Galanis et al. \cite{10}) show that, unless \( \text{NP} = \text{RP} \), there is no fully-polynomial randomised approximation scheme (FPRAS) on \( \Delta \)-regular graphs (for \( \Delta \geq 3 \)) for any \( \lambda > \lambda_c(\Delta) \). Thus, the difficulty of approximation is resolved, apart from at the boundary \( \lambda = \lambda_c(\Delta) \).

We say that the two-spin model is ferromagnetic if \( \beta \gamma > 1 \) and antiferromagnetic if \( \beta \gamma < 1 \). For the antiferromagnetic Ising model, Sinclair et al. \cite{23} Corollary 1.2] show that there is a FPTAS for evaluating the Ising partition function on graphs of degree at most \( \Delta \) for any choice of parameters \( \beta \) and \( \lambda \) which is in the interior of the uniqueness region of the
\(\Delta\)-ary tree. By contrast, Sly and Sun [24, Theorem 2] show that, unless \(NP = RP\), there is no FPRAS on \(\Delta\)-regular graphs (for \(\Delta \geq 3\)) if \(\beta\) and \(\lambda\) are outside the uniqueness region. (So, once again, the situation is fully resolved, apart from the boundary.) The result of Sinclair et al. [23, Corollary 1.3] extends to general anti-ferromagnetic two-state spin systems in regular graphs, and also in a somewhat wider class of graphs.

For general anti-ferromagnetic two-state spin systems, the best positive result that is known is due to Li, Lu, and Yin [19]. They use a stronger notion of correlation decay than Weitz, which enables them to obtain a PTAS, even for graphs with unbounded degree. They show [19, Theorem 2] that for any finite \(\Delta \geq 3\), or for \(\Delta = \infty\), there is an FPTAS for the partition function of the two-state spin system on graphs of maximum degree at most \(\Delta\) if the parameters of the system are antiferromagnetic, and for every \(d \leq \Delta\), they lie in the interior of the uniqueness region of the infinite \(d\)-regular tree. By contrast [19, Theorem 3], the results of Sly and Sun imply that, for any finite \(\Delta \geq 3\), or for \(\Delta = \infty\), unless \(NP = RP\), there is no FPRAS for the partition function of the two-state spin system on graphs of maximum degree at most \(\Delta\) if the parameters of the system are antiferromagnetic, and for some \(d \leq \Delta\), they lie outside the interior of the uniqueness region of the infinite \(d\)-regular tree. Thus, the approximation complexity is resolved in the antiferromagnetic case, apart from at the boundaries of the uniqueness regions. Note that the result of Sun and Sly was independently discovered by Galanis, Štefankovič and Vigoda [11] for the case \(\lambda = 1\).

The situation is not completely resolved in the ferromagnetic case. Building on Jerrum and Sinclair’s FPRAS for the ferromagnetic Ising model [10], Goldberg, Jerrum and Paterson [15] gave an FPRAS for the ferromagnetic two-spin model which applies if \(\beta \geq \gamma\) and \(\lambda \leq \sqrt{\beta/\gamma}\) (or, equivalently, if \(\beta \leq \gamma\) and \(\lambda \geq \sqrt{\beta/\gamma}\)). The approximation applies without these constraints on the parameters if the input is a regular graph.

For the hard-core model, an important issue which arises in statistical physics is approximating the partition function for planar graphs, including regular lattices. While (as far as we know) there were no hardness results for this problem (until this paper) the complexity of particular algorithms have been studied. For example, Randall [21] showed that a particular MCMC algorithm provides a bad approximation on subsets of \(\mathbb{Z}^2\), because Glauber dynamics mixes slowly when \(\lambda \geq 8.066\). (By contrast, results of Restrepo et al. [22] showed that the mixing time is \(O(n \log n)\) when \(\lambda < 2.3883\), and that Weitz’s algorithm [26] gives a (deterministic) fully-polynomial-approximation scheme in this case.) Recently tree decompositions of planar graphs have been used to give FPTASes for certain partition functions on planar graphs — see [27].

### 1.2 Our contribution

Our objective is to determine whether approximating the partition function of the hard-core model is computationally intractable on planar graphs for sufficiently large \(\lambda\). It turns out that this is so. Our main result (see Theorem [1]) is that, for a wide range of two-spin parameters, there is no FPRAS, even for planar graphs with degree at most 4. The applicable range of parameters includes the hard-core model with \(\lambda \geq 312\). Thus, we show that approximation is difficult for this problem (see Corollary [2]).

An interesting difference between the general case and the planar case is that, in general, it is difficult to approximate the logarithm of the partition function, a quantity which has physical significance and is called the mean free energy. Sly and Sun (see the proofs of Theorems 1 and 2 in [24]) showed that there is a fixed \(c > 1\) such that no algorithm can
approximate $Z_{\beta,\gamma,\lambda}(G)$ within a factor $e^{V(G)}$ unless NP=RP. By contrast, we show (see Theorem 3) that, in the planar case, there is a polynomial-time approximation scheme for $\log Z_{\beta,\gamma,\lambda}(G)$ (which implies an approximation within a factor $e^{V(G)}$ since $Z_{\beta,\gamma,\lambda}(G)$ is at most $C^{V(G)}$ for a quantity $C$ which depends only on $\beta$, $\gamma$ and $\lambda$).

At a high level, our hardness result is a reduction from the optimisation problem of computing a maximum independent set in a cubic planar graph $G$ to the problem of estimating the partition function of a much larger graph, which is constructed from $G$. Each vertex of $G$ is represented by a gadget which is a “wrapped” rectangular lattice $C_\nu$ (see Figure 1). Similar to previous results of Goldberg and Jerrum [13], and Sly [25], and Sly and Sun [24], we exploit the phase transition of the gadget to enable a reduction from a hard optimisation problem. The optimisation problem from which we start (computing a maximum independent set in a cubic planar graph) plays a similar role to that of the maximum cut problem in the reduction of Sly and Sun [24]. However, there is a key difference. Since, as we discuss below, it turns out that the logarithm of our partition function is efficiently approximable, it is therefore necessary that the optimisation problem from which we start is also easy to approximate (otherwise, we would get a contradiction). This means that our reduction has to be more carefully tuned — the approximation of the partition function has to allow us to exactly solve the optimisation problem.

A key technical challenge in the proof is to characterise the Gibbs distribution of the two-spin model on the lattice gadget. We show that the spins of the vertices do exhibit long-range correlation. In fact, the gadget is almost always in one of two phases. Each of these phases are equally likely. Also, conditioned on the phase, the spins of certain vertices along the boundary of the gadget are nearly independent, and their distribution can be determined. Thus, although there is long-range correlation between spins, all of the correlation is captured by the phase. Conditioned on the phase, the spins are not very correlated. The analysis of the Gibbs distribution of the gadget uses contour arguments adapted from Dobrushin [8] and Borgs et al. [5]. Randall’s slow-mixing result is also based on contour arguments.

In statistical physics it is sometimes useful to approximate the logarithm of the partition function, even when the partition function itself cannot be approximated (for example, in the situation of Theorem 1). Bandypadhyay and Gamarnik [2] have shown how to estimate the logarithm of the partition function of the hard-core model when $\lambda$ is small and the graph is regular, with large girth. They show that, in this case, the approximate value does not depend on the graph, given its degree and size! We give (Theorem 3) an approximation scheme for the logarithm of the partition function which applies to all planar graphs, for sufficiently large $\lambda$. The algorithm is based on the decomposition technique that Baker [1] used to give approximation schemes for optimisation problems on planar graphs. There is a parameter $k$ which is governed by desired approximation quality. The graph $G$ is decomposed into pieces which are $k$-outerplanar, and therefore have bounded tree-width. The partition functions of these pieces can be calculated directly using an algorithm of Yin and Zhang [27]. These are combined to give the estimate.

2 Preliminaries and statement of results

In our main result, we will assume that the parameters $\beta$, $\gamma$ and $\lambda$ satisfy the following conditions.

$$\lambda \geq 1, \quad \beta \geq 1 > \gamma \geq 0, \quad \beta \gamma < 1, \quad \beta \lambda^{-1/4} \leq 0.238, \quad \text{and} \quad \gamma \lambda^{3/8} \leq 0.238. \quad (1)$$
Note that these conditions are satisfied by the hard-core model when \( \lambda \geq 312 \) (by setting \( \beta = 1 \) and \( \gamma = 0 \)).

The notion of a fully polynomial randomised approximation scheme (FPRAS) is defined in Section 3. Following [13, 14], we say that a real number \( z \) is efficiently approximable if there is an FPRAS for the problem of computing \( z \). For fixed efficiently approximable reals \( \beta, \gamma \) and \( \lambda \) satisfying (1), we consider the problem of (approximately) computing \( Z_{\beta,\gamma,\lambda}(G) \), given an input graph \( G \). In order to make our (negative) result as strong as possible, we restrict the input \( G \) to have degree at most 4 as well as being planar. Thus, we study the following computational problem.

\textbf{Name} DegreeFourPlanarTwoSpin(\( \beta, \gamma, \lambda \)).

\textbf{Instance} A planar graph \( G \) with maximum degree at most 4.

\textbf{Output} The value \( Z_{\beta,\gamma,\lambda}(G) \).

Our main result is the following.

\textbf{Theorem 1.} Suppose that \( \beta, \gamma \) and \( \lambda \) are efficiently approximable reals satisfying (1). There is no FPRAS for DegreeFourPlanarTwoSpin(\( \beta, \gamma, \lambda \)) unless \( \text{NP} \subseteq \text{BPP} \).

The inclusion \( \text{NP} \subseteq \text{BPP} \) would imply \( \text{NP} = \text{RP} \) [13, Theorem 2]. So, for any fixed \( \beta, \gamma \) and \( \lambda \) satisfying (1), there is no FPRAS for DegreeFourPlanarTwoSpin(\( \beta, \gamma, \lambda \)) unless \( \text{NP} = \text{RP} \).

Of course, our result has an immediate consequence for the problem of approximating the partition function in the hard-core model. Thus, Theorem 1 implies Corollary 2 for the following computational problem.

\textbf{Name} DegreeFourPlanarHardCore(\( \lambda \)).

\textbf{Instance} A planar graph \( G \) with maximum degree at most 4.

\textbf{Output} The value \( Z_{1,0,\lambda}(G) \).

\textbf{Corollary 2.} Suppose that \( \lambda \geq 312 \) is an efficiently approximable real. There is no FPRAS for DegreeFourPlanarHardCore(\( \lambda \)) unless \( \text{NP} \subseteq \text{BPP} \).

Despite Theorem 1 we show that the logarithm of the partition function can be approximated. In particular, we study the following computational problem, where, for concreteness, we use the natural logarithm (to the base \( e \)).

\textbf{Name} PlanarLogTwoSpin(\( \beta, \gamma, \lambda \)).

\textbf{Instance} A planar graph \( G \).

\textbf{Output} The value \( \log(Z_{\beta,\gamma,\lambda}(G)) \).

Our result is that there is a polynomial-time randomised approximation scheme (PRAS) for PlanarLogTwoSpin(\( \beta, \gamma, \lambda \)). A polynomial-time randomised approximation scheme is a more liberal notion of approximation than the notion of an FPRAS. See Section 3 for a definition.
Theorem 3. Suppose that $\beta$, $\gamma$ and $\lambda$ are efficiently approximable reals satisfying $\beta \geq 1 > \gamma \geq 0$ and $\lambda \geq 1$. There is a PRAS for $\text{PLANARLogTwoSpin}(\beta, \gamma, \lambda)$.

The randomness used by the algorithm promised by Theorem 3 is only needed to approximate the parameters $\beta$, $\gamma$ and $\lambda$. If these are deterministically approximable, then the approximation is deterministic.

We will need some notation to refer to the Gibbs distribution of the two-spin model on a graph $G$, which is the distribution in which the probability of each configuration is proportional to its weight. We will use $\sigma_G$ to denote a random configuration drawn from this distribution. Thus, for any configuration $\sigma : V(G) \to \{0,1\}$,

$$\Pr(\sigma_G = \sigma) = \frac{w_G(\sigma)}{Z_{\beta,\gamma,\lambda}(G)}.$$  

(In general, as here, we use boldface for the random variable and normal type for the values that it takes on.) Finally, given a subset $S$ of $V(G)$ and a configuration $\sigma : V(G) \to \{0,1\}$, let $\sigma(S) : S \to \{0,1\}$ denote the configuration induced by $\sigma$ on $S$.

3 Polynomial Randomised Approximation Schemes

Most of this section is taken from [14] and can be skipped by readers who are already familiar with randomised approximation schemes.

A randomised approximation scheme is an algorithm for approximately computing the value of a function $f : \Sigma^* \to \mathbb{R}$. (Here, $\Sigma$ is a finite alphabet, and inputs to $f$ are represented as strings over this alphabet.) The approximation scheme has a parameter $\varepsilon > 0$ which specifies the error tolerance. A randomised approximation scheme for $f$ is a randomised algorithm that takes as input an instance $x \in \Sigma^*$ (e.g., for the problem $\text{DEGREEFourPlanarTwoSpin}(\beta, \gamma, \lambda)$, the input would be an encoding of a planar graph $G$) and a rational error tolerance $\varepsilon \in (0,1)$, and outputs a rational number $z$ (a random variable of the “coin tosses” made by the algorithm) such that, for every instance $x$,

$$\Pr[e^{-\varepsilon}f(x) \leq z \leq e^{\varepsilon}f(x)] \geq \frac{3}{4}. \quad (2)$$

The randomised approximation scheme is said to be a polynomial randomised approximation scheme or PRAS if, for each $\varepsilon$, its running time is bounded by a polynomial in $|x|$. It is said to be a fully polynomial randomised approximation scheme, or FPRAS, if its running time is bounded by a polynomial in $|x|$ and $\varepsilon^{-1}$.

Note that the quantity $\frac{3}{4}$ in Equation (2) could be changed to any value in the open interval $(\frac{1}{2}, 1)$ without changing the set of problems that have randomised approximation schemes [17, Lemma 6.1]. In fact, in the proof of Theorem [14] we will assume that our FPRASes have failure probability at most 1/15.

The notion of an FPRAS is a particularly robust notion of approximability for partition functions. For such approximations, the existence of a polynomial-time algorithm that achieves a constant-factor approximation actually implies the existence of an FPRAS. The same argument that we gave to illustrate this point for the Potts model [14] also applies to the setting of this paper. For any graph $G$, denote by $k \cdot G$ the graph composed of $k$ disjoint copies of $G$. Then $Z_{\beta,\gamma,\lambda}(k \cdot G) = Z_{\beta,\gamma,\lambda}(G)^k$. So, setting $k = O(\varepsilon^{-1})$, a constant factor approximation to $Z_{\beta,\gamma,\lambda}(k \cdot G)$ will yield (by taking the $k$th root) an FPRAS for
DegreeFourPlanarTwoSpin(\(\beta, \gamma, \lambda\)). Clearly, an approximation within a polynomial factor would also suffice. Note that the same argument does not necessarily apply to log-partition functions.

4 The Gadget

We will assume throughout this section that \(\beta, \gamma\) and \(\lambda\) satisfy (1), so we do not keep repeating this condition in the statement of our lemmas.

The gadget \(C_\nu\) has vertex set \(V(C_\nu) = \mathbb{Z}/2\nu\mathbb{Z} \times \{0, \ldots, \nu\}\). Vertices \((x, y)\) and \((x', y')\) are adjacent in \(C_\nu\) if

- \(y = y'\) and \(x = x' \pm 1\) (where of course, the arithmetic is modulo \(2\nu\) since \(x\) and \(x'\) are in \(\mathbb{Z}/2\nu\mathbb{Z}\)), or
- \(x = x'\) and \(y = y' \pm 1\).

Let \(E(C_\nu)\) denote the set of edges of \(C_\nu\). See the leftmost picture in Figure 1.

![Figure 1: \(C_5\), and the vertex subsets \(B_{1,2}\), and \(B_{0,5}\).](image)

4.1 Goalposts and keyholes

Given a vertex \((x, 0)\) \(\in V(C_\nu)\) and a value \(m \in \{0, \ldots, \nu\}\) let \(B_{x,m}\) be the set containing the vertices on the rectangular (goalpost-shaped) path at distance \(m\) around the terminal. In particular, let

\[
B_{x,m} = \bigcup_{0 \leq j \leq m} \{(x - m, j), (x - j, m), (x + j, m), (x + m, j)\}.
\]

Again, the arithmetic is done modulo \(2\nu\) since \(x \in \mathbb{Z}/2\nu\mathbb{Z}\). See the middle picture in Figure 1.

When \(m = \nu\), the vertices in \(\{(x - m, j) \mid 0 \leq j \leq m\}\) coincide with the vertices in \(\{(x + m, j) \mid 0 \leq j \leq m\}\) so \(B_{x,m}\) becomes the “keyhole” which is depicted in the rightmost picture of Figure 1 (for \(x = 0\)).

We shall often be working with configurations on gadgets. For convenience the notation \(\sigma_{C_\nu}\) will be contracted to \(\sigma_\nu\), and no confusion should result.
4.2 Parity-0 ones and parity-1 ones

We say that a vertex \((x, y) \in V(C_\nu)\) has parity 1 if \(x + y\) is odd, and that it has parity 0 otherwise. Suppose that \(S\) is a subset of \(V(C_\nu)\) and that \(s \in \{0, 1\}\). We say that \(\sigma(S)\) has parity-\(s\) ones if \(\{(x, y) \in S \mid \sigma(x, y) = 1\}\) is exactly the set of parity-\(s\) vertices in \(S\).

4.3 Idealised probabilities

Define

\[
p^- = \limsup_{\nu \to \infty} \Pr(\sigma_\nu(0, 0) = 1 \mid \sigma_\nu(B_{0,\nu})\ \text{has parity-0 ones}),\]

and

\[
p^\neq = \limsup_{\nu \to \infty} \Pr(\sigma_\nu(1, 0) = 1 \mid \sigma_\nu(B_{1,\nu})\ \text{has parity-0 ones}).\]

The notation \(p^-\) is meant to connote that we are looking at the probability of a 1 at a vertex of parity \(s\), conditioned on certain parity-\(t\) ones, where \(s = t\); for \(p^\neq\) we are interested in \(s \neq t\). As we shall see later, it will turn out that \(p^- > p^\neq\). This is a non-trivial fact about the spin system: if there were no long-range correlations, we would have \(p^- = p^\neq\). The following straightforward lemma is also useful.

**Lemma 4.** \(p^\neq > 0\) and \(p^- < 1\).

**Proof.** Suppose \(\nu \geq 2\). Consider vertex \((1, 0)\) of \(C_\nu\). Let \(S = \{(1, 0), (2, 0), (0, 0), (1, 1)\}\) be the set containing \((1, 0)\) and its immediate neighbours. Let \(S' = \{(-1, 0), (0, 1), (1, 2), (2, 1), (3, 0)\}\) be the set containing the neighbours of \(S\). Given any \(\sigma : S' \to \{0, 1\}\),

\[
\Pr(\sigma_\nu(S)\ \text{has parity-1 ones} \mid \sigma_\nu(S') = \sigma) \geq \frac{\lambda}{16} < 1.
\]

Now let \(S'' = \{(-1, 0), (0, 1), (1, 0)\}\) be the neighbours of \((0, 0)\). Given any \(\sigma : S'' \to \{0, 1\}\),

\[
\Pr(\sigma_\nu(0, 0) = 1 \mid \sigma_\nu(S'') = \sigma) \leq \frac{\lambda}{1 + \lambda} < 1.
\]

The events that \(\sigma_\nu(B_{0,\nu})\) has parity-0 ones and that \(\sigma_\nu(B_{1,\nu})\) has parity-0 ones have low probability, so it may seem strange to condition on these events, but the purpose of this conditioning is to identify two phases of the idealised gadget. We will refer to certain vertices \((x, 0)\) of \(C_\nu\) as “terminals”, and it will turn out to be the case that the spins of these terminals are nearly independent of each other in the distribution of \(\sigma_\nu\).

We will study the distribution that \(\sigma_\nu\) induces on the terminals by considering an idealised distribution with two phases. In each of these two gadget phases, the spins of the terminals will be chosen independently. Some terminals will be assigned spin 1 with probability \(p^-\) and others will be assigned spin 1 with probability \(p^\neq\). This will be explained further in the next section.

4.4 Terminals

Fix positive integers \(d\) and \(k\). Let \(\nu = 2dk\) and let \(C_{k,d}\) denote the gadget \(C_\nu\). We will work with \(C_{k,d}\) for the rest of the paper. We will use both notations, \(C_\nu\) and \(C_{k,d}\), depending on
whether we want to emphasize the role of \( \nu \) or the role of \( k \) and \( d \). Similarly, the alternative notations, \( \sigma_{k,d} \) and \( \sigma_s \) will be used as convenient.

Some of the vertices around the boundary of \( C_{k,d} \) (2\( k \) of them) are designated as “terminals”. The set of “parity-1 terminals” is

\[
T^1_{k,d} = \{(4jd + 1, 0) \mid 0 \leq j \leq k - 1\}.
\]

The set of “parity-0 terminals” is

\[
T^0_{k,d} = \{(4jd + 2d, 0) \mid 0 \leq j \leq k - 1\}.
\]

Let \( T_{k,d} = T^1_{k,d} \cup T^0_{k,d} \) denote the set of terminals.

For parity \( s \in \{0, 1\} \), let \( \mu^s_{k,d} \) be the distribution on configurations \( \sigma : T_{k,d} \to \{0, 1\} \) in which the spin of each terminal is chosen independently as follows: For each parity-\( s \) terminal \((x, 0)\), set \( \sigma(x, 0) = 1 \) with probability \( p^s \) (and set \( \sigma(x, 0) = 0 \) otherwise). For each terminal \((x, 0)\) with parity \( 1 \oplus s \), set \( \sigma(x, 0) = 1 \) with probability \( p^\overline{s} \) (and set \( \sigma(x, 0) = 0 \) otherwise).

Informally, the distribution \( \mu^s_{k,d} \) will be relevant when an idealised gadget is in a phase which prefers 1-spins at parity-\( s \) terminals. In this distribution, the probability that a terminal is given spin 1 is higher if the terminal has parity \( s \) than if it has parity \( 1 \oplus s \).

Let \( \mu_{k,d} \) be the distribution on configurations \( \sigma : T_{k,d} \to \{0, 1\} \) given by \( \mu_{k,d}(\sigma) = (\mu^0_{k,d}(\sigma) + \mu^1_{k,d}(\sigma))/2 \). We will show that, provided that \( d \) is sufficiently large, the distribution of \( \sigma_{k,d}(T_{k,d}) \) is close to \( \mu_{k,d} \).

Thus, the gadget can be thought of informally as having two phases, phases 0 and 1. We will show that the gadget almost always occupies one of these two phases, and they occur with equal probability. In phase 0, the distribution of \( \sigma_{k,d}(T_{k,d}) \) is close to \( \mu^0_{k,d} \). In phase 1, the distribution of \( \sigma_{k,d}(T_{k,d}) \) is close to \( \mu^1_{k,d} \).

**Proposition 5.** There is a \( c > 1 \) such that, if \( d \) is a sufficiently large multiple of 16, \( k \) is an integer greater than or equal to 1 and \( \tau \) is a configuration \( \tau : T_{k,d} \to \{0, 1\} \), then

\[
|\Pr(\sigma_{k,d}(T_{k,d}) = \tau) - \mu_{k,d}(\tau)| \leq c^{-d} k^2.
\]

Proposition 5 is established at the end of this section. We will use contour arguments adapted from Dobrushin [8] and Borgs et al. [5]. The outline of the argument is as follows. We first define “contours” in Section 4.5. We show, in Section 4.6, that long contours are unlikely. In Section 4.7, we show that, in the absence of long contours, the spins of terminals are nearly independent. With high probability, the gadget has a phase \( s \) and there is a boundary around each terminal, whose spins are consistent with \( s \). Conditioned on \( s \), the distribution of the spins of the terminals is close to \( \mu^s_{k,d} \).

### 4.5 The Dual Gadget, trails, and contours

The dual gadget \( C'^\nu \) has vertex set \( V(C'^\nu) = \{x + \frac{1}{2} \mid x \in \mathbb{Z}/2\nu\mathbb{Z}\} \times \{y + \frac{1}{2} \mid y \in \{-1, \ldots, \nu\}\} \). Vertices \((x, y)\) and \((x', y')\) are adjacent in \( C'^\nu \) if

- \( y = y' \) and \( y \notin \{-\frac{1}{2}, \nu + \frac{1}{2}\} \), and \( x = x' \pm 1 \) (where of course, the arithmetic is modulo \( 2\nu \)),

- or

- \( x = x' \) and \( y = y' \pm 1 \).
Figure 2: Part of $C_3$ and $C_3^*$; solid lines are edges of $C_3$, dashed lines are edges of $C_3^*$. The red thickened lines are a dual pair of edges.

$E(C_3^*)$ denotes the edge set of $C_3^*$. This is illustrated in Figure 2.

There is a bijection called “duality” between edges of $C_\nu$ and edges of $C_\nu^*$. In particular, the dual of edge $e = ((x, y), (x + 1, y))$ of $C_\nu$ is $e^* = ((x + \frac{1}{2}, y - \frac{1}{2}), (x + \frac{1}{2}, y + \frac{1}{2}))$ and the dual of $e^*$ is $e$. Similarly, the dual of edge $f = ((x, y), (x, y + 1))$ of $C_\nu$ is $f^* = ((x - \frac{1}{2}, y + \frac{1}{2}), (x + \frac{1}{2}, y + \frac{1}{2}))$ and the dual of $f^*$ is $f$. We use the * operation to move between an edge and its dual, so every edge $e$ satisfies $(e^*)^* = e$.

A trail in $C_\nu^*$ is a sequence $g = v_1, \ldots, v_j$ of vertices in $V(C_\nu^*)$ such that each pair $(v_i, v_{i+1})$ is an edge of $C_\nu^*$, and no edge is used twice. A contour is a trail $g = v_1, \ldots, v_j$ in $C_\nu^*$ satisfying one of the following:

- $v_1 = v_j$, or
- The $y$-coordinate of $v_1$ and the $y$-coordinate of $v_j$ are both in $\{-\frac{1}{2}, \nu + \frac{1}{2}\}$.

The length of $g$ is $j - 1$. We say that $g$ is a cross contour if the $y$-coordinate of $v_1$ is $-\frac{1}{2}$ and the $y$-coordinate of $v_j$ is $\nu + \frac{1}{2}$ (or vice-versa). A cross contour goes from one boundary of the gadget to the other. We say that every other contour is a simple contour.

Given $\sigma : V(C_\nu) \to \{0, 1\}$, let $\sigma^*$ be the set of edges of $C_\nu^*$ which are dual to monochromatic edges. In particular, $\sigma^* = \{(u, v)^* \in E(C_\nu^*) \mid \sigma(u) = \sigma(v)\}$.

**Definition 6.** A contour of $\sigma$ is a contour $g = v_1, \ldots, v_j$ satisfying the following two properties.
• The edges of $g$ are monochromatic: That is, for all $1 \leq i < j$, $(v_i, v_{i+1}) \in \sigma^*$.

• The contour $g$ always turns at degree-4 vertices: That is, for all $1 < i < j$, if four edges of $\sigma^*$ meet at vertex $v_i$, then $v_{i-1}$ and $v_{i+1}$ differ in both the $x$ component and the $y$ component. Similarly, if four edges of $\sigma^*$ meet at $v_1 = v_j$ then $v_2$ and $v_{j-1}$ differ in both the $x$ component and the $y$ component.

Note that contours of $\sigma$ cannot cross each other, though two contours can share a vertex without crossing. Also, two contours can have a common portion (before turning off in two different directions). Finally, every edge of $\sigma^*$ is contained in at least one contour of $\sigma$.

Let $\sigma : V(C_\nu) \to \{0, 1\}$ be a configuration, and let $g$ be a contour of $\sigma$. We say that a vertex $u \in V(C_\nu)$ is adjacent to $g$ if there is an edge $(u, v) \in E(C_\nu)$ such that $e^* \in g$. The set of vertices adjacent to $g$ can be written as the union of two sets, $L(g)$ and $R(g)$, where $L(g)$ is the set of vertices which are on the left (relative to the direction of travel) when we follow the trail $g$ from $v_1$ to $v_j$, and $R(g)$ is the set of vertices which are on the right (relative to the direction of travel). See Figure 3.

A key property of contours is the following.

**Lemma 7.** Let $\sigma : V(C_\nu) \to \{0, 1\}$ be a configuration, and let $g$ be a contour of $\sigma$. Then for some $s \in \{0, 1\}$, $\sigma(L(g))$ has parity-$s$ ones and $\sigma(R(g))$ has parity-$(1 \oplus s)$ ones.

**Proof.** Pick $s \in \{0, 1\}$ such that the vertex on the left as we go from $v_1$ to $v_2$ has parity-$s$ ones. By induction on $i$, we will show that for each $i$ the vertex on the left as we go from $v_i$ to $v_{i+1}$ has parity-$s$ ones. Suppose without loss of generality that the edge $(v_{i-1}, v_i)$ increases the $x$-component (the other three cases are similar). So $v_{i-1} = (x - \frac{1}{2}, y + \frac{1}{2})$ and $v_i = (x + \frac{1}{2}, y + \frac{1}{2})$. Since $g$ is a contour of $\sigma$, $\sigma(x, y) = \sigma(x, y + 1) = s \oplus x \oplus y$. There are three cases.

1. $v_{i+1} = (x + \frac{1}{2}, y + \frac{3}{2})$. In this case the vertex $(x, y + 1)$ is still on the left as we go from $v_i$ to $v_{i+1}$.

2. $v_{i+1} = (x + \frac{1}{2}, y - \frac{1}{2})$. In this case the vertex $(x + 1, y)$ is on the left as we go from $v_i$ to $v_{i+1}$, but since $g$ is a contour of $\sigma$ we have $\sigma(x + 1, y) = \sigma(x, y) = \sigma(x, y + 1)$. So $(x + 1, y)$ has parity-$s$ ones.

![Figure 3: Left and right vertices of a contour of $\sigma$. The shaded squares represent vertices of $C_\nu$ with parity-1 ones and the unshaded squares represent vertices of $C_\nu$ with parity-0 ones.](image.png)
Lemma 8. Fix $\sigma$ side edge $C$ of side edges $R$. Rearranging gives $\ell$. Consider the connected components of the graph edges in $C$. Let $\nu > 2$. A side vertex of $C_\nu$ is a vertex $(x, y) \in V(C_\nu)$ with $y = 0$ or $y = \nu$. A side edge is an edge in $E(C_\nu)$ between two side vertices.

**Lemma 8.** Fix $\nu > 2$ and a configuration $\sigma : V(C_\nu) \to \{0, 1\}$. Let $b(\sigma)$ be the number of side edges $(u, v)$ of $C_\nu$ with $\sigma(u) = \sigma(v) = 0$ and let $c'(\sigma)$ be the number of side edges $(u, v)$ of $C_\nu$ with $\sigma(u) = \sigma(v) = 1$. Then

$$\ell(\sigma) = \frac{1}{4}(c(\sigma) - b(\sigma)) + \frac{1}{8}(c'(\sigma) - b'(\sigma)) + \nu(\nu + 1).$$

**Proof.** Let $\ell'(\sigma)$ be the number of side vertices $u$ with $\sigma(u) = 1$. Let $E'$ be the set of all side edges in $C_\nu$. By double-counting pairs $(u, (u, v))$ with $\sigma(u) = 1$ and $(u, v) \in E(C_\nu)$,

$$|E(C_\nu)| - b(\sigma) - c(\sigma)) + 2c(\sigma) = 4\ell(\sigma) - \ell'(\sigma).$$

By double-counting pairs $(u, (u, v))$ with $\sigma(u) = 1$ and $(u, v) \in E'$, we have

$$|E'| - b'(\sigma) - c'(\sigma)) + 2c'(\sigma) = 2\ell'(\sigma).$$

Rearranging gives $\ell(\sigma) = \frac{1}{4}(c(\sigma) - b(\sigma)) + \frac{1}{8}(c'(\sigma) - b'(\sigma)) + \frac{1}{4}|E(C_\nu)| + \frac{1}{8}|E'|$. Consider the configuration with alternating 0s and 1s given by $\sigma(x, y) = x \oplus y$. For this configuration $\sigma$, we have $b(\sigma) = c(\sigma) = b'(\sigma) = c'(\sigma) = 0$ and $\ell(\sigma) = \nu(\nu + 1)$, so the constant term $\frac{1}{4}|E(C_\nu)| + \frac{1}{8}|E'|$ is $\nu(\nu + 1)$. □

4.6 Long contours are unlikely

**Lemma 9.** There is a $c > 1$ such that, for all sufficiently large $h$, all $\nu > 2$, and all $U \subseteq V(C_\nu^*)$,

$$\Pr(\sigma_\nu \text{ has a simple contour of length at least } h \text{ starting in } U) \leq |U| c^{-h}.$$

**Proof.** Suppose that $g$ is a simple length-$r$ contour of a configuration $\sigma : V(C_\nu) \to \{0, 1\}$. Consider the connected components of the graph $(V(C_\nu), E(C_\nu) \setminus \{e^* \mid e \in g\})$. We say

Figure 4: Cases 1, 2, and 3. Black squares have the same spin as $\sigma(x, y)$; white squares have the opposite spin, and hatched squares can be either.

3. $v_{i+1} = (x + \frac{3}{2}, y + \frac{1}{2})$. In this case $(x + 1, y + 1)$ is on the left, and $(x + 1, y)$ is on the right. Since $g$ is a contour of $\sigma$, $\sigma(x + 1, y) = \sigma(x + 1, y + 1)$, and we know $\sigma(x, y) = \sigma(x, y + 1)$. Since the contour did not turn, the vertex $(x + \frac{3}{2}, y + \frac{1}{2})$ cannot have degree 4 in $\sigma^*$, so $\sigma(x + 1, y + 1) = s \oplus x \oplus y \oplus 1$, so $(x + 1, y + 1)$ has parity-s ones. □
that a component is “left” if it contains at least one vertex in \(L(g)\) (but no vertices in \(R(g)\)). We say that it is “right” if it contains at least one vertex in \(R(g)\) (but no vertices in \(L(g)\)). Every component is either left or right. Let \(S\) be the set of vertices in left components. Let \(S = V(C_v) \setminus S\). Let \(S' = \{(x, y) \in S \mid (x - 1, y) \in S\}\), where, as usual, the arithmetic on \(x\) is done modulo \(2\nu\).

Suppose that \(\sigma(R(g))\) has parity-\(s\) ones. By Lemma 7, this is true for some \(s \in \{0, 1\}\). Define a configuration \(\sigma^g : V(C_v) \to \{0, 1\}\) as follows: \(\sigma^g(S) = \sigma(S), \sigma^g(S')\) has parity-\(s\) ones, and, for every \((x, y) \in S \setminus S', \sigma^g(x, y) = \sigma(x - 1, y)\).

Note that \(\sigma \mapsto \sigma^g\) is a map from the set of configurations \(\sigma\) with \(g\) as a contour to the set of all configurations; further, it does not lose information, and hence is injective. Note also that \((\sigma^g)^*\) is the same as \(\sigma^s\), but with \(g\) removed and with the contours in \(S\) shifted by one. By Lemma 8,

\[
w_{CV}(\sigma) = (\beta \lambda^{-1/4})^{b(g)}(\gamma \lambda^{1/4})^{c(g)}(\lambda^{c'(g) - b'(g)})/8w_{CV}(\sigma^g)
\]

where \(b(g), c(g), b'(g), c'(g)\) are the contributions to \(b(\sigma), c(\sigma), b'(\sigma), c'(\sigma)\) coming from edges whose duals are in \(g\). As the map \(\sigma \mapsto \sigma^g\) is injective,

\[
Pr(g \subseteq \sigma^*_{\nu}) \leq (\beta \lambda^{-1/4})^{b(g)}(\gamma \lambda^{1/4})^{c(g)}(\lambda^{c'(g) - b'(g)})/8
\]

\[
\leq (\beta \lambda^{-1/4})^{b(g)}(\gamma \lambda^{3/8})^{c(g)},
\]

where we have used the facts that \(\lambda \geq 1\) and \(c'(g) \leq c(g)\). There are at most \(|U| 3^r\) relevant contours of length \(r\) in total (\(|U|\) choices of starting point, and at most three different directions at each step), so

\[
Pr(\sigma_{\nu} has a simple contour of length at least \(h\) starting in \(U\)) \leq |U| \sum_{r \geq h} 3^r \max(\beta \lambda^{-1/4}, \gamma \lambda^{3/8})^r
\]

\[
\leq |U| \frac{(3 \times 0.238)^h}{1 - 3 \times 0.238}.
\]

There is a \(c > 1\) such that \(\frac{(3 \times 0.238)^h}{1 - 3 \times 0.238} < c^{-h}\) for all sufficiently large \(h\).

**Lemma 10.** Let \(i \in \{0, 1\}\). For every \(\nu > 2\), and every simple contour \(g\) of length \(r\),

\[
Pr(g is a contour of \sigma_{\nu} \mid \sigma_{\nu}(B_{i,\nu}) has parity-0 ones) \leq \max(\beta \lambda^{-1/4}, \gamma \lambda^{3/8})^r. \tag{3}
\]

Furthermore, there is a \(c > 1\) such that, for all sufficiently large \(h\), all \(\nu > 2\), and all \(U \subseteq V(C_v)\), the conditional probability that \(\sigma_{\nu}\) has a simple contour of length at least \(h\) which contains an edge whose dual connects two vertices in \(U\), conditioned on the fact that \(\sigma_{\nu}(B_{i,\nu})\) has parity-0 ones, is at most \(|U| c^{-h}\).

**Proof.** The proof of (3) is similar to the first half of the proof of Lemma 9 except that we have to take care to choose \(S\) to be on the correct side of the contour \(g\). Previously, it did not matter whether we formed \(S\) from the left or right components, and we arbitrarily chose the former. Now we choose \(S\) (either taking all the left or all the right components) in such a way that \(S \cap B_{i,\nu} = \emptyset\). This is possible because all the vertices in \(B_{i,\nu}\) are in a single connected
component (the contour \( g \) does not cross any edges whose endpoints lie in \( B_{i,\nu} \)). Now define \( \sigma^g \) as in the proof of Lemma 9 and continue as before. This establishes (3).

For all \( 1 \leq s \leq r \), and all \( u \in U \), there are at most \( 3^s \times 4 \) contours \( v_1 \ldots v_r \) for which \( u \) is on the left as we go from \( v_{s-1} \) to \( v_s \); a choice of initial direction and direction at each step determines the contour. Summing over \( s \) and \( u \), this implies that there are at most \( 4|U|r3^r \) length-\( r \) contours with an edge whose dual connects vertices of \( U \). By (3),

\[
\Pr \left( \sigma_\nu \text{ has a simple contour of length at least } h \text{ which contains an edge whose dual connects two vertices in } U \left| \sigma_\nu(B_{i,\nu}) \text{ has parity-0 ones} \right. \right) \\
\leq 4|U| \sum_{r \geq h} r3^r \max(\beta \lambda^{-1/4}, \gamma \lambda^{3/8})^r \\
= 4|U|(3 \times 0.238)^h \sum_{t \geq 0} (t + h)(3 \times 0.238)^t \\
= 4|U|(3 \times 0.238)^h \frac{3 \times 0.238 + h(1 - 3 \times 0.238)}{(1 - 3 \times 0.238)^2}
\]

There is a \( c > 1 \) such that \( 4(3 \times 0.238)^h \frac{3 \times 0.238 + h(1 - 3 \times 0.238)}{(1 - 3 \times 0.238)^2} < c^{-h} \) for all sufficiently large \( h \).

By the upper boundary of \( C_\nu \) we mean the set of all vertices of the form \((x, \nu)\) for some \( x \).

**Lemma 11.** Let \( i \in \{0, 1\} \). There is a \( c > 1 \) such that, for all sufficiently large \( h \) and all \( \nu > h \), the probability that \( \sigma_\nu \) has a simple contour that separates the set \( \{-h+i, \ldots, h+i\} \times \{0, \ldots, h\} \) from the upper boundary of \( C_\nu \), conditioned on the fact that \( \sigma_\nu(B_{i,\nu}) \) has parity-0 ones, is at most \( c^{-h} \).

**Proof.** Note that the separating contour cannot wrap around, owing to the boundary condition that \( \sigma_\nu(B_{i,\nu}) \) has parity-0 ones. If the separating contour has length \( r+2 \) then its right-endpoint is in the set \( \{(h+i+x-\frac{1}{2}, -\frac{1}{2}) \mid 1 \leq x \leq r\} \). There is a unique choice for the edge incident to each endpoint. Thus, there are at most \( r3^r \) possible contours. By Lemma 11 the probability that \( \sigma_\nu \) has such a simple contour, conditioned on the fact that \( \sigma_\nu(B_{i,\nu}) \) has parity-0 ones, is at most

\[
\sum_{r=h-2}^{\infty} r3^r \max(\beta \lambda^{-1/4}, \gamma \lambda^{3/8})^{r+2}.
\]

Thus, the probability is at most

\[
\max(\beta \lambda^{-1/4}, \gamma \lambda^{3/8})^2 \sum_{r=h-2}^{\infty} r3^r \max(\beta \lambda^{-1/4}, \gamma \lambda^{3/8})^r,
\]

which, as in the proof of Lemma 11, is exponentially small in \( h \).

**Lemma 12.** There is a \( c > 1 \) such that, for all sufficiently large \( \nu \),

\[
\Pr(\sigma_\nu \text{ has a cross contour}) \leq c^{-\nu}.
\]

**Proof.** Let \( g \) be a cross contour. There must be at least one other cross contour \( g' \). For otherwise there would be a path \( p \) in \( C_\nu \) from \( L(g) \) to \( R(g) \) such that \( \sigma_\nu(V(p)) \) has parity-0 ones or parity-1 ones, which would violate parity. Orient \( g \) and \( g' \) in opposite senses (one
away from \( y = -\frac{1}{2} \) and one towards). Consider the connected components of the graph 
\((V(C_ν), E(C_ν) \setminus \{e^* \mid e \in g \cup g'\})\), and let \( S \) be the union of all connected components that are left of either \( g \) or \( g' \). Now proceed as in the proof of Lemma 9, using the fact that a cross contour has length at least \( ν \) and the set of possible starting points has size \( 2ν \).

\[ \text{Lemma 13.} \text{ Let } i \in \{0, 1\}. \text{ Fix } ν \geq 1. \text{ Conditioned on } σ_ν(B_{i,ν}) \text{ having parity-}s \text{ ones (for any } s \in \{0,1\}, σ_ν \text{ has no cross contour.} \]

\[ \text{Proof.} \text{ A cross contour would have to cross a side edge in } B_{i,ν}, \text{ which is impossible.} \]

\[ \text{Lemma 14.} p^- > p^p. \]

\[ \text{Proof.} \text{ Fix } ν > 2. \text{ Suppose that } σ_ν(B_{0,ν}) \text{ has parity-0 ones. If } σ_ν(0,0) = 0 \text{ then there is a simple contour of } σ_ν \text{ that separates } (0, ν) \text{ from } (0,0). \text{ (Note that, by Lemma 13, cross contours cannot separate these two vertices.) If the separating contour has length } r + 2 \text{ then its right-endpoint is in the range } (\frac{1}{2}, -\frac{1}{2}), \ldots, (r - \frac{1}{2}, -\frac{1}{2}). \text{ There is a unique choice for the edge incident to each endpoint. Thus, there are at most } r3^r \text{ possible contours. By Lemma 10 the probability that } σ_ν \text{ has such a simple contour, conditioned on the fact that } σ_ν(B_{0,ν}) \text{ has parity-0 ones, is at most} \]

\[ \sum_{r=1}^{∞} r3^r \max(βλ^{-1/4}, γλ^{3/8})^{r+2}. \]

Thus,

\[ \Pr(σ_ν(0,0) = 0 \mid σ_ν(B_{0,ν}) \text{ has parity-0 ones}) ≤ \max(βλ^{-1/4}, γλ^{3/8})^2 \sum_{r=1}^{∞} r3^r \max(βλ^{-1/4}, γλ^{3/8})^{r} \]

\[ ≤ (0.238)^2 \frac{3 \times 0.238}{(1 - 3 \times 0.238)^2} \]

\[ < 1/2. \]

Thus, \( p^- > \frac{1}{2} \).

Similarly, suppose that \( σ_ν(B_{1,ν}) \) has parity-0 ones. If there is no simple contour of \( σ_ν \) that separates \((1, ν)\) from \((1, 0)\), then \( σ_ν(1,0) = 0 \). We already saw that the probability that no such contour exists is greater than \( \frac{1}{2} \). Thus,

\[ \Pr(σ_ν(1,0) = 0 \mid σ_ν(B_{1,ν}) \text{ has parity-0 ones}) > 1/2. \]

So \( p^p < \frac{1}{2} \). Putting the two inequalities together, we have \( p^p < \frac{1}{2} < p^- \).

\[ \text{4.7 In the absence of long contours, the spins of the terminals are nearly independent} \]

The \( *\)-distance between two points \((x, y)\) and \((x', y')\) in \( V(C_ν) \) is \( \max(|x - x'|, |y - y'|) \) where \( |x - x'| \) is the minimum non-negative integer such that \( x = x' + |x - x'| \) modulo \( 2ν \) or \( x' = x + |x - x'| \) modulo \( 2ν \). The \( *\)-distance between two points \((x, y)\) and \((x', y')\) in \( V(C_ν^*) \) is defined similarly. Let \( U_{x,h} \) be the set of vertices of \( C_ν \) whose \( *\)-distance from \((x, 0)\) is at most \( h \). Vertices in \( V(C_ν) \) (or \( V(C_ν^*) \)) are \( *\)-adjacent if the \( *\)-distance between them is 1. A \( *\)-path on \( V(C_ν) \) is a sequence \( v_1, \ldots, v_h \) of vertices in \( V(C_ν) \) such that, for each \( j \in \{1, \ldots, h - 1\} \), the vertices \( v_j \) and \( v_{j+1} \) are \( *\)-adjacent. A \( *\)-path on \( V(C_ν^*) \) is defined similarly.
Definition 15. Suppose \( \nu \) and \( h \) are positive integers. An \( h \)-boundary of a vertex \((x, 0)\) is a set of vertices \( B \subseteq V(C_\nu) \) such that the following are true.

(i) \((x, 0)\) is not connected to \((x, \nu)\) in the graph \( C_\nu \setminus B \), and

(ii) \( B \) does not intersect \( U_{x,h/4} \), and

(iii) \( B \) is a subset of \( U_{x,h/2} \).

(iv) The subgraph of \( C_\nu \) induced by \( B \) is connected.

See Figure 5.

![Figure 5: Example \( h \)-boundary of vertex \((16, 0)\) with \( \nu = 20 \) and \( h = 5 \).]

Here is the relevant fact about \( h \)-boundaries. If \( B \) is an \( h \)-boundary of a vertex \((x, 0)\) and \( \sigma(B) \) has parity-\( s \) ones then there is no contour of \( \sigma \) which contains an edge whose dual connects two vertices in \( B \).

If \( B \) is an \( h \)-boundary of vertex \((x, 0)\) and \( B' \) is an \( h' \)-boundary of \((x, 0)\), then we say that \( B \) is inside of \( B' \) if every path in \( C_\nu \) from \( B' \) to \((x, 0)\) passes through \( B \). Suppose that \( B \) and \( B' \) are \( h \)-boundaries of \((x, 0)\) and, for some \( s \in \{0, 1\} \), \( \sigma(B) \) has parity-\( s \) ones and \( \sigma(B') \) has parity-\((1 \oplus s)\) ones. Then \( B \cap B' = \emptyset \), so exactly one of the following occurs.

- \( B \) is inside of \( B' \), or
- \( B' \) is inside of \( B \).
Definition 16. Suppose that $s \in \{0, 1\}$ and that $k$ and $d$ are positive integers. Let $\nu = 2kd$. Suppose that $\sigma$ is a configuration $\sigma : V(C_\nu) \rightarrow \{0, 1\}$. We say that $\sigma$ has phase $s$ if the following holds for every terminal $(x, 0)$.

- $(x, 0)$ has a $d$-boundary $B$ for which $\sigma(B)$ has parity-$s$ ones.
- For every $d$-boundary $B'$ of $(x, 0)$ for which $\sigma(B')$ has parity-$(1 \oplus s)$ ones, $B'$ is inside of $B$.

Note that a configuration $\sigma$ can have exactly one phase (phase 0 or phase 1) or it can have no phase. Suppose that $\nu \geq 1$ and that the configuration $\sigma : V(C_\nu) \rightarrow \{0, 1\}$ has phase $s$. Say that a $d$-boundary $B$ of a vertex $(x, 0)$ is consistent if $\sigma(B)$ has parity-$s$ ones. From the consistent $d$-boundaries, we want to select a canonical one, that is in some precise sense “outermost” and also “minimal”. For each terminal $(x, 0)$, let $\hat{B}_x(\sigma)$ be the union of all $d$-boundaries of $(x, 0)$ which are consistent. Observe that $B = \hat{B}_x(\sigma)$ satisfies the first three bullet points in Definition 15, but not the final one, as the subgraph $C_\nu[B]$ of $C_\nu$ induced by $B$ may not be connected. Suppose that $C_\nu[B]$ has $j$ connected components. Partition $\hat{B}_x(\sigma) = \hat{B}_x^1(\sigma) \cup \cdots \cup \hat{B}_x^j(\sigma)$ so that $C_\nu[\hat{B}_x^1(\sigma)], \ldots, C_\nu[\hat{B}_x^j(\sigma)]$ is an enumeration of these $j$ connected components. Each set $\hat{B}_x^j(\sigma)$ is itself a $d$-boundary. To see this, consider any vertex $v \in \hat{B}_x^j(\sigma)$. From the construction of $\hat{B}_x^j(\sigma)$, this vertex is contained in some $d$-boundary, which in turn is contained in $\hat{B}_x(\sigma)$. So $\hat{B}_x^j(\sigma)$ satisfies the first three bullet points in Definition 15 in addition to inducing a connected graph.

From the first bullet point of Definition 15 it follows that $\hat{B}_x^1(\sigma), \ldots, \hat{B}_x^j(\sigma)$ are nested; suppose the numbering indicates the level of nesting, with $\hat{B}_x^1(\sigma)$ being the outermost. We now want to identify a minimal $m$-boundary within $\hat{B}_x^1(\sigma)$. Let

$$\text{Ext}\hat{B}_x^1(\sigma) = \{v \in V(C_\nu) \mid \text{there is a $*$-path from $v$ to $(x, \nu)$ in $V(C_\nu) \setminus \hat{B}_x^1(\sigma)$}\}.$$  

denote the set of vertices lying in the “exterior” of $\hat{B}_x^1(\sigma)$. Finally define

$$B_x(\sigma) = \{v \in \hat{B}_x^1(\sigma) \mid v \text{ is $*$-adjacent to some vertex in Ext}\hat{B}_x^1(\sigma)\}.$$ 

Note that $B_x(\sigma)$ is a $d$-boundary of $(x, 0)$ which is consistent. (Informally, $B_x(\sigma)$ is the outermost such $d$-boundary.) To see this, observe that any path in the graph $C_\nu$ from $(x, 0)$ to $(x, \nu)$ has a last vertex in the set $\hat{B}_x^1(\sigma)$, and this vertex must be in $B_x(\sigma)$; this deals with the first bullet point in Definition 15. Consider the trail in the dual graph $C_\nu^*$ separating $B_x(\sigma)$ and $\text{Ext}\hat{B}_x^1(\sigma)$; the trail in the primal graph that shadows it at $*$-distance $\frac{1}{2}$ inside takes in all the vertices of $B_x(\sigma)$ and establishes connectivity of $C_\nu[B_x(\sigma)]$. This deals with the final bullet mark, and the remaining two are immediate.

The $d$-boundary $B_x(\sigma)$ is our desired canonical $d$-boundary, and it has the following important property. If $\sigma'$ is a configuration that agrees with $\sigma$ on $B_x(\sigma) \cup \text{Ext}\hat{B}_x^1(\sigma)$, then $B_x(\sigma') = B_x(\sigma)$. The reason is as follows. The set $B = B_x(\sigma)$ is a $d$-boundary of $(x, 0)$ which is consistent with respect to $\sigma'$, i.e., $\sigma'(B)$ has parity-$s$ ones. It therefore gets incorporated into $\hat{B}_x(\sigma') \supseteq B_x(\sigma)$ and hence into $\hat{B}_x^1(\sigma') \supseteq B_x(\sigma)$. So $\text{Ext}\hat{B}_x^1(\sigma') = \text{Ext}\hat{B}_x^1(\sigma)$ and $B_x(\sigma') = B_x(\sigma)$. This fact becomes significant when we come to consider events supported on spins in the interior $U = V(C_\nu) \setminus (B \cup \text{Ext} B)$ of some $d$-boundary $B$. Specifically, conditioning on the event $B_x(\sigma_U) = B$ (and on the phase $s$ of $\sigma_U$) is equivalent to selecting $\sigma_U(U)$ according to the Gibbs distribution, with the boundary condition “$\sigma_U(B)$ has parity-$s$ ones”. We refer to this property as canonicity of $B_x(\sigma)$. 

17
Before proceeding we need some definitions and observations concerning connected subgraphs $K$ of the graph $(V(C_\nu),\sigma^*)$. The *-diameter* of $K$ is the maximum, over pairs of vertices in $K$, of the *-distance between those vertices. We say that $K$ reaches the lower boundary of $C_\nu$ if it contains a vertex of the form $(x,-\frac{1}{2})$ for some $x$. We say that it reaches the upper boundary of $C_\nu$ if it contains a vertex of the form $(x,\nu + \frac{1}{2})$. We say that it is a cross subgraph if it reaches both boundaries. We say that $K$ wraps around if it contains the image of some path from $(0,y)$ to $(2\nu,y)$ in $\mathbb{Z} \times \{0,\ldots,\nu\}$, under the quotient map to $V(C_\nu)$. We say that $K$ is local if it is not a cross subgraph and does not wrap around. We say that $K$ intersects $U_{x,h}$ if some edge $e^*$ in $K$ is dual to an edge $e \in E(C_\nu)$ with both endpoints in $U_{x,h}$. A contour is a connected subgraph of $(V(C_\nu),\sigma^*)$ so all of the above definitions apply to contours. We refer to the connected components of $(V(C_\nu),\sigma^*)$ as $\sigma^*$-components.

**Lemma 17.** If $\sigma$ has only local contours then $\sigma^*$ has only local components.

**Proof.** Suppose to the contrary that $\sigma^*$ has non-local component $K$. Consider first the case of a cross component.

So suppose $K$ reaches both the upper and lower boundaries of $C_\nu$, and that $K$ has $2j > 0$ degree-1 vertices. (All vertices other than the degree-1 vertices have even degree, and the number of odd-degree vertices in a graph is even.) By adapting the standard algorithm for finding an Eulerian trail in a (connected) Eulerian graph, we may decompose $K$ into $j$ contours beginning and ending at degree-1 vertices. The method is as follows. Starting at a degree-1 vertex, trace out a trail in $K$ subject only to the rule that we must turn through a right-angle at any degree-4 vertex. This trail can only end at another degree-1 vertex. The trail so formed is a contour; remove the trail from $K$ and repeat $j - 1$ more times to obtain $j$ contours in total. If any edges of $K$ remain, start at any remaining vertex and trace out a closed trail that returns to the start vertex. Again, the rule is always to turn through a right-angle at any degree 4 vertex. Repeat until there are no edges remaining in $K$. We are left with $j$ non-closed contours and an unspecified number of closed ones. Whenever a non-closed contour meets a closed one, we may splice the latter into the former, reducing the number of closed contours by one. Repeating as necessary, we obtain the sought-for decomposition of $K$ into $j$ contours beginning and ending at degree-1 vertices.

If one of these $j$ contours joins the upper and lower boundaries of $C_\nu$ we are done, as we have already found a cross contour and obtained a contradiction. Otherwise, there must be at least one vertex at which a lower-to-lower contour touches a upper-to-upper contour. Simply reroute the trails at this vertex to obtain two cross contours.

Now consider the case where $K$ wraps around. We may assume that $K$ does not reach one of the boundaries of $C_\nu$, say the upper one. Trace a closed trail along the upper boundary of $K$: this trail is a contour that wraps around, providing a contradiction. $\square$

Since non-local contours are unlikely, Lemma 17 allows us to concentrate on local $\sigma^*$-components. A $\sigma^*$-component $K$ that is local has a well defined inside and outside, and a boundary that is a valid contour. (More precisely, there is a canonical contour that has exactly the same edges as the boundary of $K$.) If $K$ reaches neither the upper nor lower boundary of $C_\nu$, then we may trace clockwise around $K$, always taking the leftmost option, until we return to our starting point. This procedure yields a simple contour; we refer to vertices of $V(C_\nu)$ that lie within this contour as forming the interior of $K$, denoted Int $K$.

If $K$ reaches the lower boundary but not the top (or vice versa), then a slightly modified construction can be used. First lift $K$ to a grid: for sufficiently large $N$, there is a connected
subset $\hat{K}$ of $\{1, \ldots, N\} \times \{0, \ldots, \nu\}$ which maps bijectively to $K$ under the quotient map to $C_\nu$. Note that lifting can only increase the diameter of $\hat{K}$ relative to $K$. We now have a natural ordering of the degree-1 vertices of $K$, namely by increasing $x$-coordinate. Start at the least degree-1 vertex in this ordering and trace the boundary of $\hat{K}$ in a clockwise-leftmost fashion until the greatest degree-1 vertex is reached. This procedure yields a simple contour which partitions the vertices of $V(C_\nu)$ into an inside and an outside (containing the point $(0, \nu)$); again we refer to the former as the interior of $K$.

**Lemma 18.** Consider $\sigma : V(C_\nu) \to \{0, 1\}$. Let $h \leq \nu$ be an integer multiple of 8. Suppose

- $\sigma$ contains only local contours.
- $\sigma$ contains no contour of length at least $h/8$ that intersects $U_{x,h/2}$.
- $\sigma$ contains no simple contour separating $U_{x,h/4}$ from the upper boundary of $C_\nu$.

Then every $\sigma^*$-component that intersects $U_{x,h/2}$ has $\ast$-diameter at most $h/8$.

**Proof.** Suppose $K$ is a $\sigma^*$-component intersecting $U_{x,h/2}$. By Lemma 17, $K$ is local. The interior of $K$ contains some vertex in $U_{x,h/2}$. By assumption, the contour defined by the boundary of $K$ does not separate $U_{x,h/4}$ from the upper boundary of $C_\nu$, so it does not separate $U_{x,h/2}$ from the upper boundary of $C_\nu$. The only remaining possibility is that this contour intersects $U_{x,h/2}$, and hence has length at most $h/8$. It follows that $K$ has $\ast$-diameter at most $h/8$. 

**Lemma 19.** Suppose that $h$ is a sufficiently large multiple of 8, and $\nu \geq h$. Consider a configuration $\sigma : V(C_\nu) \to \{0, 1\}$ and a terminal $(x, 0)$. Suppose that the following are true.

- $\sigma$ has no cross contours.
- $\sigma$ has no simple contour of length at least $h/8$ that intersects $U_{x,h/2}$.
- $\sigma$ has no simple contour separating $U_{x,h/4}$ from the upper boundary of $C_\nu$.

Then, for some $s \in \{0, 1\}$,

1. $(x, 0)$ has an $h$-boundary $B$ for which $\sigma(B)$ has parity-$s$ ones.
2. There is no $h$-boundary $B'$ of $(x, 0)$ for which $\sigma(B')$ has parity-$(1 \oplus s)$ ones.
3. For any terminal $(x', 0)$ that has an $h$-boundary $B'$ for which $\sigma(B')$ has parity-$s'$ ones, if $\sigma$ has no contour separating $U_{x,h/4}$ from $U_{x',h/4}$ then $s' = s$.
4. If $\sigma(B_{0,\nu})$ has parity-$s'$ ones then $s' = s$.

**Proof.** There are no contours that wrap around, since any such contour would either intersect $U_{x,h/2}$, or would separate $U_{x,h/4}$ from the upper boundary of $C_\nu$. Thus, by Lemma 17, all $\sigma^*$-components are local. Let $S$ be the set of all vertices in $V(C_\nu)$ that are not in the interior of some $\sigma^*$-component. That is

$$S = V(C_\nu) \setminus \bigcup \{ \text{Int } K \mid K \text{ is a } \sigma^*-\text{component} \}.$$
Note that \( \sigma(S) \) has parity-\( s \) ones, for some \( s \in \{0, 1\} \). Define \( \overline{S} = V(C_\nu) \setminus S \). Note that \( \sigma(\overline{S}) \) in general has mixed parity; the salient feature is that \( \sigma(S) \) has consistent parity.

We work first towards conclusion (1) of the lemma. By Lemma 13 every \( \sigma^* \)-component that intersects \( U_{x,h/2} \) has *-diameter at most \( h/8 \). Now (informally) we will construct the required \( h \)-boundary by tracing round the inside of \( B_{x,h/2} \), making a detour towards \( (x,0) \) around any \( \sigma^* \)-components that stand in the way. (Recall that \( B_{x,h/2} = U_{x,h/2} \setminus U_{x,h/2-1} \) is the “goalpost” at distance \( h/2 \) from \( (x,0) \).) This strategy ensures we remain in the set \( S \) and, since all the \( \sigma^* \)-components are small, our detours will not be too great.

More formally, let \( W \) be the union of the set \( V(C_\nu) \setminus U_{x,h/2} \) together with any sets of the form \( \text{Int} K \) that intersect \( B_{x,h/2} \). That is,

\[
W = (V(C_\nu) \setminus U_{x,h/2}) \cup \{ \text{Int} K \mid K \text{ is a } \sigma^*\text{-component and } \text{Int}(K) \cap B_{x,h/2} \neq \emptyset \}.
\]

The set

\[
\partial W = \{ v \in V(C_\nu) \mid v \notin W \text{ and } v \text{ is } *\text{-adjacent to some vertex in } W \}
\]

is almost the \( h \)-boundary \( B \) that we seek. Observe that any set of the form \( \text{Int} K \) is contained in a maximal set of the form \( \text{Int} K' \), and the *-neighbours of \( \text{Int} K' \) are all in \( S \). Thus \( \partial W \) is a subset of \( S \), and necessarily has parity-\( s \) ones.

And as we shall see presently, \( \partial W \) satisfies the first three conditions of an \( h \)-boundary \( B \) — (i) every path from \( (x,0) \) to \( (x,\nu) \) intersects \( B \), (ii) \( B \cap U_{x,h/4} = \emptyset \), (iii) \( B \subseteq U_{x,h/2} \) — but not necessarily the final one, namely: (iv) the induced graph \( C_\nu[B] \) is connected. (There may be islands of vertices of \( \partial W \) lying outside the \( h \)-boundary we are trying to home in on.) However, we can ensure (iv) by defining \( B \) to be the subset of vertices in \( \partial W \) that can be reached from \( (x,0) \) by a path in \( C_\nu \) whose vertices all lie in \( V(C_\nu) \setminus W \).

For (i), observe that any path from \( (x,0) \) to \( (x,\nu) \) has a first vertex \( w \) in \( W \). The vertex immediately preceding \( w \) is not in \( W \) but is adjacent to a vertex in \( W \), and hence in \( B \). (ii) follows from the fact that every vertex in \( W \cap U_{x,h/2} \) is within *-distance \( h/8 \) of a vertex in \( B_{x,h/2} \). (iii) is immediate from the construction. To see (iv), denote by \( W^\circ \) the set of all vertices in \( V(C_\nu) \) that can be reached from \( (x,0) \) by a path whose vertices all lie in \( V(C_\nu) \setminus W \). Note that \( C_\nu[W^\circ] \) is connected and that \( B \subseteq W^\circ \). Let \( \varrho^* \) be the set of edges in \( C_\nu^* \) separating \( W^\circ \) and \( W \); thus, \( e \in \varrho^* \) iff \( e \) has one endpoint in \( W \) and the other in \( W^\circ \). Since \( C_\nu[W^\circ] \) is connected, the edges in \( \varrho^* \) form a trail in \( C_\nu^* \) starting and ending at vertices with \( \nu \)-coordinate \( -\frac{1}{2} \). Following this trail anticlockwise, vertices in \( W^\circ \) lie to the left and those in \( W \) to the right. In fact, the vertices immediately to the left of the \( \varrho^* \)-trail (i.e., those at *-distance \( \frac{1}{2} \) from it) are precisely the vertices forming \( B \): they are all *-adjacent to some vertex in \( W \), and no other vertices in \( W^\circ \) have this property. Thus, any two vertices in \( B \) are connected by a path, which is obtained by shadowing \( \varrho^* \) at *-distance \( \frac{1}{2} \).

For conclusion (2) of the lemma, observe that there cannot be an \( h \)-boundary \( B' \) of \( (x,0) \) such that \( \sigma(B') \) has parity-\( (s+1) \) ones, as such a \( B' \) would have to exist entirely within \( \overline{S} \), and all *-connected components of \( \overline{S} \) are small (*-diameter at most \( h/8 \)).

As for conclusion (3), it is impossible for \( s' \neq s \). Consider the connected component of \( C_\nu[S] \) containing \( B \). If \( s' \neq s \) then the boundary of this component contains a contour separating \( B \) and \( B' \), and hence \( U_{x,h/4} \) and \( U_{x',h/4} \). If \( \sigma(B_{0,\nu}) \) has parity-\( s \) ones for \( s' \neq s \) then the boundary of the connected component of \( C_\nu[S] \) containing \( B \) is a simple contour separating \( B \) from the upper boundary of \( C_\nu \) hence, separating \( U_{x,h/4} \) from the upper boundary of \( C_\nu \), establishing (4). \( \square \)
Corollary 20. Suppose that \( k \geq 1 \) that \( d \) is a sufficiently large multiple of 8 and that \( \nu = 2dk \). Suppose \( \sigma : V(C_\nu) \to \{0,1\} \) has no contours of length at least \( d/8 \). Either \( \sigma \) has phase 0 or \( \sigma \) has phase 1.

Proof. Since there are no contours of length at least \( d/8 \) of any kind, the premises of Lemma 19 are all satisfied for every terminal \( (x,0) \) and every other terminal \( (x',0) \).

The following monotonicity property is useful for comparing different contours and boundary conditions.

Lemma 21. Let \( x \in \mathbb{Z}/2\nu\mathbb{Z} \), let \( B \) be an \( h \)-boundary of \( (x,0) \) for some \( h \), and let \( B' \) be an \( h' \)-boundary of \( (x,0) \) for some \( h' \) such that \( B \) is inside of \( B' \). Then

\[
\Pr(\sigma_{\nu} \text{ has parity-0 ones at } (x,0) \mid \sigma_{\nu}(B) \text{ has parity-0 ones}) \\
\geq \Pr(\sigma_{\nu} \text{ has parity-0 ones at } (x,0) \mid \sigma_{\nu}(B') \text{ has parity-0 ones}).
\]

Proof. For each \( S \subseteq V(C_\nu) \) let \( \sigma^S : C_\nu \to \{0,1\} \) denote the configuration which has parity-0 ones exactly on \( S \). So \( \sigma^S(x,y) = 1 \) if and only if one of these two conditions hold: \( (x,y) \in S \) and \( x + y \) is even, or \( (x,y) \notin S \) and \( x + y \) is odd.

For all \( X, Y \subseteq V(C_\nu) \) we have

\[
w_{C_\nu}(\sigma^X)w_{C_\nu}(\sigma^Y) = w_{C_\nu}(\sigma^{X \cap Y})w_{C_\nu}(\sigma^{X \cup Y})(\beta \gamma)^k
\]

where \( k \) is the number of edges \( uv \in E(G) \) such that \( \{(\sigma^X(u),\sigma^X(v)),(\sigma^Y(u),\sigma^Y(v))\} = \{(0,0),(1,1)\} \) (so either \( u \in X \) and \( v \notin X \) and \( u \notin Y \) and \( v \in Y \), or \( v \in X \) and \( u \notin X \) and \( v \notin Y \) and \( u \in Y \)). In particular,

\[
w_{C_\nu}(\sigma^X)w_{C_\nu}(\sigma^Y) \leq w_{C_\nu}(\sigma^{X \cap Y})w_{C_\nu}(\sigma^{X \cup Y}).
\]

Let

\[
\mathcal{X} = \{S \mid \{(x,0)\} \cup B' \subseteq S \subseteq V(C_\nu)\}
\]

\[
\mathcal{Y} = \{S \mid B \cup B' \subseteq S \subseteq V(C_\nu)\}
\]

By the FKG inequality 9 we have

\[
\left( \sum_{S \in \mathcal{X}} w_{C_\nu}(\sigma^S) \right) \left( \sum_{S \in \mathcal{Y}} w_{C_\nu}(\sigma^S) \right) \leq \left( \sum_{S \in \mathcal{X} \cap \mathcal{Y}} w_{C_\nu}(\sigma^S) \right) \left( \sum_{S \in \mathcal{X} \cup \mathcal{Y}} w_{C_\nu}(\sigma^S) \right)
\]

where \( \mathcal{X} \cap \mathcal{Y} \) is the family of sets \( X \subseteq V(C_\nu) \) such that \( \{(x,0)\} \cup B' \cap (B \cup B') = B' \subseteq X \), and \( \mathcal{X} \cup \mathcal{Y} \) is the family of sets \( X \subseteq V(C_\nu) \) such that \( \{(x,0)\} \cup B \cup B' \subseteq X \). Finally,

\[
\Pr(\sigma_{\nu} \text{ has parity-0 ones at } (x,0) \mid \sigma_{\nu}(B') \text{ has parity-0 ones}) = \frac{\sum_{S \in \mathcal{X}} w_{C_\nu}(\sigma^S)}{\sum_{S \in \mathcal{X} \cup \mathcal{Y}} w_{C_\nu}(\sigma^S)}
\]

\[
\Pr(\sigma_{\nu} \text{ has parity-0 ones at } (x,0) \mid \sigma_{\nu}(B) \text{ has parity-0 ones}) = \frac{\sum_{S \in \mathcal{X} \cap \mathcal{Y}} w_{C_\nu}(\sigma^S)}{\sum_{S \in \mathcal{Y}} w_{C_\nu}(\sigma^S)}.
\]

Lemma 22. There is a \( c > 1 \) such that the following is true for any \( k \geq 1 \), any \( s \in \{0,1\} \), any sufficiently large \( d \) which is a multiple of 16, and any assignment \( \{B_x\} \) of \( d \)-boundaries for each terminal \( (x,0) \):
For every parity-\(s\) terminal \((x, 0)\),
\[
|\Pr(\sigma_{k,d}(x, 0) = 1 \mid \sigma_{k,d} \text{ has phase } s \text{ and } B_x(\sigma_{k,d}) = B_x) - p^\pm| \leq c^{-d}.
\]

For every parity-(1\(\oplus s\)) terminal \((x, 0)\),
\[
|\Pr(\sigma_{k,d}(x, 0) = 1 \mid \sigma_{k,d} \text{ has phase } s \text{ and } B_x(\sigma_{k,d}) = B_x) - p^\pm| \leq c^{-d}.
\]

**Proof.** By symmetry (rotating the gadget so that parity-0 vertices become parity-1 vertices and vice-versa), it suffices to prove the inequalities for \(s = 0\). For any \(m \geq 1\), define
\[
p^\pm(m) = \Pr(\sigma_m(0, 0) = 1 \mid \sigma_m(B_{0,m}) \text{ has parity-0 ones}),\text{ and} \quad p^\pm(m) = \Pr(\sigma_m(1, 0) = 1 \mid \sigma_m(B_{1,m}) \text{ has parity-0 ones}).
\]

Now note that for any \(\nu \geq m\), \(p^\nu(m)\) and \(p^\nu(m)\) (as defined above) are the same as the equivalent expressions in the gadget \(C_\nu\). In particular,
\[
p^\nu(m) = \Pr(\sigma_\nu(0, 0) = 1 \mid \sigma_\nu(B_{0,m}) \text{ has parity-0 ones}),\text{ and} \quad p^\nu(m) = \Pr(\sigma_\nu(1, 0) = 1 \mid \sigma_\nu(B_{1,m}) \text{ has parity-0 ones}).
\]

Thus, by fixing large \(\nu\) and increasing \(m\), Lemma 22 implies that \(p^\nu(m)\) is weakly decreasing in \(m\) and that \(p^\nu(m)\) is weakly increasing. Thus, \(p^\nu = \lim_{m \to \infty} p^\nu(m)\) and \(p^\nu = \lim_{m \to \infty} p^\nu(m)\). Also, for a parity-0 terminal \((x, 0)\), the target probability
\[
\Pr(\sigma_{k,d}(x, 0) = 1 \mid \sigma_{k,d} \text{ has phase } 0 \text{ and } B_x(\sigma_{k,d}) = B_x)
\]
is between \(p^\nu(d/2)\) and \(p^\nu(d/4)\). Similarly, for a parity-1 terminal \((x, 0)\), the target probability
\[
\Pr(\sigma_{k,d}(x, 0) = 1 \mid \sigma_{k,d} \text{ has phase } 0 \text{ and } B_x(\sigma_{k,d}) = B_x)
\]
is between \(p^\nu(d/4)\) and \(p^\nu(d/2)\). (Here we use crucially the canonicity property of \(B_x(\cdot)\); refer to the discussion following Definition 16.) Thus it suffices to show
\[
p^\nu(d/4) \leq p^\nu + c^{-d} \quad \text{and} \quad p^\nu(d/4) \geq p^\nu - c^{-d},
\]

First we take a qualitative step. Pick \(w \geq 8d\) sufficiently large that \(p^\nu(w) \leq p^\nu + |U_{0,d/4}||c'|^{-d/10}\) and \(p^\nu(w) \geq p^\nu - |U_{1,d/4}||c'|^{-d/16}\), where \(c'\) is the maximum of the constants given in Lemma 10 and Lemma 11. This can be done since \(d\) is sufficiently large and \(p^\nu = \lim_{m \to \infty} p^\nu(m)\), and \(p^\nu = \lim_{m \to \infty} p^\nu(m)\) though \(w\) may be quite a lot larger than \(d\).

For \(i \in \{0, 1\}\), let \(F_i\) be the event that there is a \(d/2\)-boundary \(B\) of vertex \((i, 0)\) in gadget \(C_w\) such that \(\sigma_w(B)\) has parity-0 ones. Recall from the definition that a \(d/2\)-boundary of \((i, 0)\) is a subset of \(U_{i,d/4}\). Let \(E_i\) be the event that \(\sigma_w(B_{i,w})\) has parity-0 ones.

For each \(i \in \{0, 1\}\), applying Lemma 13 and Lemma 10 with \(h = d/16\) and \(U = U_{i,d/4}\), and Lemma 11 with \(h = d/8\), we find that, the conditional probability that the following hold, conditioned on \(E_i\), is at least \(1 - 2|U_{i,d/4}||c'|^{-d/16}\).

\[
\bullet \ \sigma_w\ \text{has no cross contour}.
\]
• \(\sigma_w\) has no simple contour of length at least \(d/16\) which contains an edge between two vertices in \(U_{i,d/4}\).

• \(\sigma_w\) has no simple contour that separates \(U_{i,d/8}\) from the upper boundary of \(C_\nu\).

Now, applying Lemma 19 with \(h = d/2\) and \(\nu = w\) and \(x = i\), if all of these hold and event \(E_i\) occurs then event \(F_i\) occurs. Thus,

\[
\Pr(F_i \mid E_i) \geq 1 - 2\vert U_{i,d/4}\vert (\rho')^{-d/16}.
\]

But by Lemma 21 we have

\[
\Pr(\sigma_w(0,0) = 1 \mid F_0 \land E_0) \geq p^\leq (d/4), \text{ and }
\Pr(\sigma_w(1,0) = 1 \mid F_1 \land E_1) \leq p^\geq (d/4).
\]

So

\[
p^\leq (d/4) \leq \frac{\Pr(\sigma_w(0,0) = 1 \land F_0 \mid E_0)}{\Pr(F_0 \mid E_0)} \leq \frac{\Pr(\sigma_w(0,0) = 1 \mid E_0)}{\Pr(F_0 \mid E_0)} = \frac{p^\leq (w)}{\Pr(F_0 \mid E_0)}
\]

\[
\leq \frac{p^\leq (w)}{1 - 2\vert U_{0,d/4}\vert (\rho')^{-d/16}} \leq p^\leq (w) + 4\vert U_{0,d/4}\vert (\rho')^{-d/16},
\]

since \(2\vert U_{0,d/4}\vert (\rho')^{-d/16} \leq 1/2\).

A similar inequality holds for \(p^\geq (d/4)\):

\[
p^\geq (d/4) \geq \frac{\Pr(\sigma_w(1,0) = 1 \land F_1 \mid E_1)}{\Pr(F_1 \mid E_1)}
\]

\[
\geq \frac{\Pr(\sigma_w(1,0) = 1 \mid E_1)}{\Pr(F_1 \mid E_1)} - \Pr(\neg F_1 \mid E_1)
\]

\[
\geq p^\geq (w) - 2\vert U_{1,d/4}\vert (\rho')^{-d/16}.
\]

Thus,

\[
p^\leq (d/4) \leq p^\leq (w) + 4\vert U_{0,d/4}\vert (\rho')^{-d/16} \leq p^\leq + 5\vert U_{0,d/4}\vert (\rho')^{-d/16}, \text{ and }
\]

\[
p^\geq (d/4) \geq p^\geq (w) - 2\vert U_{1,d/4}\vert (\rho')^{-d/16} \geq p^\geq - 3\vert U_{1,d/4}\vert (\rho')^{-d/16}.
\]

The result follows by noting that \(\vert U_{i,d/4}\) is \(O(d^2)\) and picking \(c = (\rho')^{1/17}\), say. \(\square\)

We now prove the main proposition.

**Proposition 5.** There is a \(c > 1\) such that, if \(d\) is a sufficiently large multiple of 16, \(k\) is an integer greater than or equal to 1 and \(\tau\) is a configuration \(\tau : T_{k,d} \rightarrow \{0,1\}\), then

\[
\vert \Pr(\sigma_{k,d}(T_{k,d}) = \tau) - \mu_{k,d}(\tau)\vert \leq c^{-d}k^2.
\]

**Proof.** Fix \(k \geq 1\), \(d\) a sufficiently large multiple of 16, and \(\tau : T_{k,d} \rightarrow \{0,1\}\). Let \(\rho'\) be the minimum value of the constant \(c\) from the Lemmas 9, 12, and 22.

The probability \(\Pr(\sigma_{k,d}(T_{k,d}) = \tau)\) is the sum of the following probabilities (conditioned on disjoint events)

• \(\Pr(\sigma_{k,d}(T_{k,d}) = \tau \mid \sigma_{k,d} \text{ does not have a phase}) \Pr(\sigma_{k,d} \text{ does not have a phase}).\)
• (summed over all assignments $B_x$ of $d$-boundaries for each terminal $(x,0)$)

$$\Pr(\sigma_{k,d}(T_{k,d}) = \tau \mid \sigma_{k,d} \text{ has phase } 0 \text{ and for all terminals } (x,0), B_x(\sigma_{k,d}) = B_x) \times \Pr(\sigma_{k,d} \text{ has phase } 0 \text{ and for all terminals } (x,0), B_x(\sigma_{k,d}) = B_x)$$

• (summed over all assignments $B_x$ of $d$-boundaries for each terminal $(x,0)$)

$$\Pr(\sigma_{k,d}(T_{k,d}) = \tau \mid \sigma_{k,d} \text{ has phase } 1 \text{ and for all terminals } (x,0), B_x(\sigma_{k,d}) = B_x) \times \Pr(\sigma_{k,d} \text{ has phase } 1 \text{ and for all terminals } (x,0), B_x(\sigma_{k,d}) = B_x)$$

By Lemmas 9 and 12 and Corollary 20, the probability of the first of these is at most $2|V(C_{k,d}^*)|(c')^{-d/8}$. (We will use this below.)

Now consider an assignment $B_x$ of $d$-boundaries for each terminal $(x,0)$. For any two terminals $(x',0)$ and $(x'',0)$, the random variables $\sigma_{k,d}(x',0)$ are $\sigma_{k,d}(x'',0)$ are independent, conditioned on the fact that $\sigma_{k,d}$ has a given phase, and for all terminals $(x,0), B_x(\sigma_{k,d}) = B_x$. Also, by Lemma 22, for all $s \in \{0,1\}$,

- For every parity-$s$ terminal $(x',0)$,

$$\left| \Pr(\sigma_{k,d}(x',0) = 1 \mid \sigma_{k,d} \text{ has phase } s \text{ and for all terminals } (x,0), B_x(\sigma_{k,d}) = B_x) - p^s \right| \leq (c')^{-d}.$$

- For every parity-$(1 \oplus s)$ terminal $(x',0)$,

$$\left| \Pr(\sigma_{k,d}(x',0) = 1 \mid \sigma_{k,d} \text{ has phase } s \text{ and for all terminals } (x,0), B_x(\sigma_{k,d}) = B_x) - p^s \right| \leq (c')^{-d}.$$

Now, for any probabilities $a_1, b_1, \ldots, a_k, b_k$, we have

$$\left| \prod_{i=1}^{k} a_i - \prod_{i=1}^{k} b_i \right| = \left| \sum_{j=1}^{k} a_{i+1} \cdots a_{j-1}(a_j - b_j)b_{j+1} \cdots b_k \right| \leq \sum_{i=1}^{k} |a_i - b_i|,$$

so if we fix a given phase, and $\tau$ assigns spin 1 to $k'$ terminals whose parity agrees with that phase, and spin 1 to $k''$ terminals whose parity disagrees with that phase then, letting

$$\hat{p} = (p^-)^{k'} (1-p^-)^{k-k'} (p^\#)^{k'} (1-p^\#)^{k-k''},$$

we have

$$\hat{p} - 2k(c')^{-d} \leq \Pr(\sigma_{k,d}(T_{k,d}) = \tau \mid \sigma_{k,d} \text{ has the given phase and for all terminals } (x,0), B_x(\sigma_{k,d}) = B_x) \leq \hat{p} + 2k(c')^{-d}.$$
5 Proof of Theorem 1

5.1 Efficiently approximable reals

Lemma 23. Suppose that \( \beta, \gamma \) and \( \lambda \) are efficiently approximable reals satisfying (1). Then \( p^= \) and \( p^\neq \) are efficiently approximable reals.

Proof. Recall that \( p^\neq > 0 \) (Lemma 4). Let \( q \) be a multiple of 16 greater than \((2 + \log_2(1/p^\neq))/\log(c)\) where \( c \) is the constant given by Lemma 22. Consider the following algorithm.

- Input an error parameter \( 0 < \varepsilon < 1/2 \).
- Set \( m = q[\log(\varepsilon^{-1})] \).
- Compute rational approximations \( \hat{\beta}, \hat{\gamma}, \hat{\lambda} \) satisfying
  \[
  \beta e^{-\varepsilon/8|E(C_m)|} \leq \hat{\beta} \leq \beta e^{\varepsilon/8|E(C_m)|},
  \gamma e^{-\varepsilon/8|E(C_m)|} \leq \hat{\gamma} \leq \gamma e^{\varepsilon/8|E(C_m)|},
  \lambda e^{-\varepsilon/8|V(C_m)|} \leq \hat{\lambda} \leq \lambda e^{\varepsilon/8|V(C_m)|}.
  \]
- Using the algorithm of [27, Theorem 2.8], compute
  \[
  Z = \sum_{\sigma} \hat{\beta}^b(\sigma) \hat{\gamma}^c(\sigma) \hat{\lambda}^\ell(\sigma)
  \]
  \[
  Z' = \sum_{\sigma : \sigma(0,0)=1} \hat{\beta}^b(\sigma) \hat{\gamma}^c(\sigma) \hat{\lambda}^\ell(\sigma)
  \]
  \[
  Z'' = \sum_{\sigma : \sigma(1,0)=1} \hat{\beta}^b(\sigma) \hat{\gamma}^c(\sigma) \hat{\lambda}^\ell(\sigma),
  \]
  where the sums range over configurations \( \sigma \) of \( C_m \) such that \( \sigma(B_{0,m}) \) has parity-0 ones.
- Output \( Z'/Z \) as the approximation to \( p^= \), and \( Z''/Z \) as the approximation to \( p^\neq \).

For the computation of \( Z, Z' \), and \( Z'' \) we use the fact that the grid graph \( C_m \setminus B_{0,m} \) has treewidth \( m \) [4, Corollary 89]. We also use the fact that its tree decomposition is easy to compute. So this algorithm runs in time bounded by a polynomial in \( 1/\varepsilon \). We will show that the algorithm is an FPRAS for \( p^= \) and \( p^\neq \). Define

\[
W = \sum_{\sigma} \beta^b(\sigma) \gamma^c(\sigma) \lambda^\ell(\sigma)
\]
\[
W' = \sum_{\sigma : \sigma(0,0)=1} \beta^b(\sigma) \gamma^c(\sigma) \lambda^\ell(\sigma)
\]
\[
W'' = \sum_{\sigma : \sigma(1,0)=1} \beta^b(\sigma) \gamma^c(\sigma) \lambda^\ell(\sigma),
\]
where the sums range over configurations \( \sigma \) of \( C_m \) such that \( \sigma(B_{0,m}) \) has parity-0 ones.

For any \( \sigma \) we have
\[
\beta^b(\sigma) \gamma^c(\sigma) \lambda^\ell(\sigma) e^{-\varepsilon/4} \leq \beta^b(\sigma) \gamma^c(\sigma) \lambda^\ell(\sigma) \leq \beta^b(\sigma) \gamma^c(\sigma) \lambda^\ell(\sigma) e^{\varepsilon/4}
\]
This implies \( e^{-\varepsilon/4}W \leq Z \leq e^{\varepsilon/4}W \) and similarly for \( Z' \) and \( Z'' \), and therefore \( e^{-\varepsilon/2}W'/W \leq Z'/Z \leq e^{\varepsilon/2}W'/W \) and \( e^{-\varepsilon/2}W''/W \leq Z''/Z \leq e^{\varepsilon/2}W''/W \). We will show
\[
\begin{align*}
p^\pm & \leq W'/W \leq p^\varepsilon/2 \quad (4) \\
e^{-\varepsilon/2}p^\mp & \leq W''/W \leq p^\pm \quad (5)
\end{align*}
\]

\( W'/W \) and \( W''/W \) are just the probabilities that an even or odd terminal gets assigned 1 in a random configuration of \( C_m \), conditioned on a certain \( 2m \)-boundary. By Lemma 21 we have \( p^\pm \leq W'/W \) and \( W''/W \leq p^\pm \) for any \( m \), establishing the first inequality in (4) and the second inequality in (5).

By Lemma 22 there exists \( c > 1 \) such that
\[
W'/W \leq p^\pm + c^{-q\log(\varepsilon^{-1})} = p^\pm(1 + \varepsilon q\log(c)/p^\pm).
\]
Since
\[
\varepsilon(q\log(c)-1) \leq (1/2)(q\log(c)-1) \leq p^\pm/2,
\]
which is less than \( p^\pm \) by Lemma 14, we have
\[
W'/W \leq p^\pm(1 + \varepsilon) \leq e^\varepsilon.
\]
This establishes (4). Similarly, by Lemma 22
\[
W''/W \geq p^\pm - c^{-q\log(\varepsilon^{-1})} \\
\geq p^\pm(1 - \varepsilon q\log(c)/p^\pm) \\
\geq p^\pm(1 - \varepsilon/2) \\
\geq p^\pm e^{-\varepsilon}
\]
This establishes (5).

Lemma 23 gives us a way to obtain multiplicative approximations \( \hat{p}^\pm \) and \( \hat{p}^\mp \) of the real numbers \( p^\pm \) and \( p^\mp \). When we use these approximations, we will need to know that \( 1 - \hat{p}^\pm \) and \( 1 - \hat{p}^\mp \) are also good multiplicative approximations to \( 1 - p^\pm \) and \( 1 - p^\mp \), respectively. As we show below, this follows from the fact that \( p^\pm \) and \( p^\mp \) are in \( (0, 1) \) (which follows from Lemma 4 and Lemma 14). The following lemma gives us what we need. The reason for introducing the rational \( \bar{p}' \) in the statement of the lemma is that, since it is rational, it can be hard-wired into any algorithms (whereas a real number can’t be).

**Lemma 24.** Suppose that \( p \in (0, 1) \) is an efficiently approximable real number. Let \( \bar{p}' \) be a positive rational with \( p < \bar{p}' < 1 \). For any \( \delta \in (0, 1) \), and any real number \( \bar{p} \) satisfying \( e^{-\delta(1-p')/2\bar{p}} \leq p \leq e^{\delta(1-p')/2\bar{p}} \), we have \( e^{-\delta}(1 - p) \leq 1 - \bar{p} \leq e^\delta(1 - p) \).

**Proof.** Let \( \delta' = \delta(1 - \bar{p'})/2 \). Since \( \hat{p} \geq e^{-\delta'}p \geq p(1 - \delta') \geq p - \delta' \) and similarly \( p \geq \hat{p} - \delta' \), we have
\[
(1 - p) \left( 1 - \frac{\delta(1 - \bar{p}')}{2(1 - p)} \right) = (1 - p) - \delta' \leq 1 - \hat{p} \leq (1 - p) + \delta' = (1 - p) \left( 1 + \frac{\delta(1 - \bar{p}')}{2(1 - p)} \right).
\]
Thus,
\[
(1 - p)(1 - \delta'/2) \leq 1 - \hat{p} \leq (1 - p)(1 + \delta/2),
\]
which suffices. \( \square \)
The following problem is NP-complete [12].

**Name** PlanarCubicIS.

**Instance** A planar cubic graph $G$ and a positive integer $h$.

**Output** “Yes”, if $G$ contains an independent set of size $h$, and “No”, otherwise.

Suppose that $\beta$, $\gamma$ and $\lambda$ are efficiently approximable reals satisfying (11). We will give a randomised polynomial-time algorithm for PlanarCubicIS, using as an oracle, an FPRAS for DegreeFourPlanarTwoSpin$(\beta, \gamma, \lambda)$. The oracle will be used to approximate $Z_{1,\tilde{\gamma},\hat{\lambda}}(G)$, for some suitably-defined parameters $\tilde{\gamma}$ and $\hat{\lambda}$, where $\tilde{\gamma}$ is exponentially small in $|V(G)|$ and $\hat{\lambda}$ is exponentially large. From this, it will be easy to determine whether $G$ has an independent set of size $h$.

**Lemma 25.** Suppose that $\beta$, $\gamma$ and $\lambda$ are efficiently approximable reals satisfying (11). There is a polynomial-time randomised algorithm that, given a planar cubic graph $G$ with $|V(G)|$ sufficiently large, outputs planar graphs $J$ and $J'$ with maximum degree at most 4 and randomised approximation schemes for positive reals $K$, $\tilde{\gamma}$ and $\hat{\lambda}$. The running time of each of these approximation schemes is bounded from above by a polynomial in $|V(G)|$ and the desired accuracy parameter $\varepsilon$. With probability at least 14/15, the parameters satisfy $\hat{\lambda} \geq 4|V(G)|$ and $\tilde{\gamma} \leq \hat{\lambda}^{-|V(G)|}$ and

$$e^{-1/4}Z_{1,\tilde{\gamma},\hat{\lambda}}(G) \leq K \frac{Z_{\beta,\gamma,\lambda}(J')}{Z_{\beta,\gamma,\lambda}(J)} \leq e^{1/4}Z_{1,\tilde{\gamma},\hat{\lambda}}(G).$$  \hfill (6)

**Proof.** Let $G = (V, E)$ be a planar cubic graph and let $n$ denote $|V|$.

The algorithm for constructing $J$ and $J'$ uses a quantity $\delta \in (0, 1)$. It will be important for the proof that $\delta$ is sufficiently small. Rather than giving a technical definition here, we introduce upper bounds on $\delta$ in natural places throughout the proof. The reader can verify that each of these upper bounds is at least the inverse of a polynomial in $n$ (so the algorithm runs in polynomial time).

The first step is to use the given FPRASes for $\beta$, $\gamma$ and $\lambda$, and the FPRASes for $p^\geq$ and $p^\neq$ from Lemma 23 to compute values $\hat{\beta}$, $\tilde{\gamma}$, $\hat{\lambda}$, $\hat{p}^\geq$ and $\hat{p}^\neq$ satisfying

$$e^{-\delta/3} \beta \leq \hat{\beta} \leq e^{\delta/3} \beta,$$

$$e^{-\delta/3} \gamma \leq \tilde{\gamma} \leq e^{\delta/3} \gamma,$$

$$e^{-\delta/3} \lambda \leq \hat{\lambda} \leq e^{\delta/3} \lambda,$$

$$e^{-\delta/3} p^\geq \leq \hat{p}^\geq \leq e^{\delta/3} p^\geq,$$

$$e^{-\delta/3} p^\neq \leq \hat{p}^\neq \leq e^{\delta/3} p^\neq,$$

$$e^{-\delta/3} (1 - p^\geq) \leq 1 - \hat{p}^\geq \leq e^{\delta/3} (1 - p^\geq),$$

$$e^{-\delta/3} (1 - p^\neq) \leq 1 - \hat{p}^\neq \leq e^{\delta/3} (1 - p^\neq).$$

The first five lines in (7) follow directly from the definition of FPRAS in Section 3. The next two lines follow from Lemma 24, using the fact that $p^\geq$ and $p^\neq$ are in $(0, 1)$, as argued just before Lemma 24. Since $\beta \geq 1$ by (11), we can ensure that $\hat{\beta} \geq 1$ by taking $\hat{\beta}$ to be the
Let \( \mu \) correspond to \( J \). The construction of \( J \) is straightforward. Essentially, \( J \) consists of \(|V| \) copies of \( C_{k,d} \), with one copy for every vertex in \( V \). Thus, the vertex set \( V(J) \) is the set of ordered pairs \( V(J) = V \times V(C_{k,d}) \) and the edge set \( E(J) \) is given by \( E(J) = V \times E(C_{k,d}) \). We will use \( C[u] \) to denote the gadget corresponding to vertex \( u \in V \). Formally, \( C[u] \) is the graph with vertex set \( \{u\} \times V(C_{k,d}) \) and edge set \( \{u\} \times E(C_{k,d}) \). To simplify the notation, for \( u \in V \) and \( 0 \leq j \leq k-1 \), let \( T[u,j] \) denote the \( j \)th parity-1 terminal of \( C[u] \). Formally, this is the vertex \((u, (4j + 1, 0))\) of \( J \). Similarly, let \( T[u,j] \) denote the \( j \)th parity-0 terminal of \( C[u] \). Formally, this is the vertex \((u, (4j + 2, 0))\) of \( J \). Let \( T[u] \) be the set of terminals of \( C[u] \). Let \( \mu^0, \mu^1, \mu_u \) be the distributions on configurations \( \sigma : T[u] \to \{0, 1\} \) corresponding to the distributions \( \mu^0_{k,d}, \mu^1_{k,d} \) and \( \mu \) defined in Section 4.4.

To simplify the description of \( J' \), consider a planar embedding of \( G \) in which each vertex \( u \) of \( G \) is associated with three “endpoints” \( u_0, u_1 \) and \( u_2 \), which are arranged together in clockwise order in the plane. The edge set \( E \) can then be viewed as a matching \( M \) on the points \( \bigcup_{u \in V} \{u_0, u_1, u_2\} \) such that

- \((u, v) \in E \) if and only if there are exactly two points \( u_i \) and \( v_j \) such that \((u_i, v_j) \in M \),
- No two edges of \( M \) cross.

The vertex set \( V(J') \) consists of \( V(J) \), together with a set of \( n k_1 \) new vertices, called “bristles”. Formally, \( V(J') = V(J) \cup \{(u, j) \mid u \in V, 0 \leq j \leq k_1 - 1\} \). Finally, the edge set of \( J' \) consists of...
$E(J)$, together with new edges connecting the bristles to the parity-1 terminals of the gadgets, and new edges matching the parity-0 terminals of the gadgets (guided by the matching $\mathcal{M}$). The edges connecting the bristles to parity-1 terminals of the gadgets are those in the set

$$E_B = \{((u, j), T^1[u, j]), u \in V, 0 \leq j \leq k_1 - 1\}.$$ 

It is more complicated to describe the edges matching the parity-0 terminals of the gadgets. The idea (see Figure 7) that if $u_a$ is matched to $v_b$ in $\mathcal{M}$ (where $a \in \{0, 1, 2\}$ and $b \in \{0, 1, 2\}$)

$$T^0[u, 3] \quad \bullet \quad \cdots \quad \bullet \quad T^0[v, 0]$$

$$T^0[u, 2] \quad \bullet \quad \cdots \quad \bullet \quad T^0[v, 1]$$

$$T^0[u, 1] \quad \bullet \quad \cdots \quad \bullet \quad T^0[v, 2]$$

$$T^0[u, 0] \quad \bullet \quad \cdots \quad \bullet \quad T^0[v, 3]$$

Figure 7: Terminals of $u$ are matched to terminals in $v$, reversing the order.

then the parity-0 terminals $T^0[u, ak_2], \ldots, T^0[u, ak_2 + k_2 - 1]$ get matched to the parity-0 terminals $T^0[u, bk_2], \ldots, T^0[u, bk_2 + k_2 - 1]$. However, there is a further complication: To ensure that $J'$ is planar we must ensure that one of these sequences of terminals is matched in clockwise order, and the other in anti-clockwise order. Thus, let

$$E_M = \{(T^0[u, ak_2 + j], T^0[v, bk_2 + k_2 - 1 - j]) \mid u < v, (u_a, v_b) \in \mathcal{M}, 0 \leq j \leq k_2 - 1\}.$$ 

Then $E(J') = E(J) \cup E_B \cup E_M$. Note that both $J$ and $J'$ are planar as required.

We next show how to define the positive integers $k_1$, $k_2$ and $d$. Define

$$P = \begin{pmatrix} 1 - p^- & p^- \\ 1 - p^\# & p^\# \end{pmatrix}, \quad M = P \begin{pmatrix} \beta & 1 \\ 1 & \gamma \end{pmatrix} P^t, \quad W = P \begin{pmatrix} \beta & 1 \\ 1 & \gamma \end{pmatrix} \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$$

where $P^t$ denotes the transpose of the matrix $P$. Also, define

$$\tilde{P} = \begin{pmatrix} 1 - p^- & p^- \\ 1 - p^\# & p^\# \end{pmatrix}, \quad \tilde{M} = \tilde{P} \begin{pmatrix} \beta & 1 \\ 1 & \gamma \end{pmatrix} \tilde{P}^t, \quad \tilde{W} = \tilde{P} \begin{pmatrix} \beta & 1 \\ 1 & \gamma \end{pmatrix} \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$$

Note that if (7) holds then, for any $s \in \{0, 1\}$ and $s' \in \{0, 1\}$,

$e^{-\delta} P_{s,s'} \leq \tilde{P}_{s,s'} \leq e^\delta P_{s,s'},$

$e^{-\delta} M_{s,s'} \leq \tilde{M}_{s,s'} \leq e^\delta M_{s,s'},$

$e^{-\delta} W_s \leq \tilde{W}_s \leq e^\delta W_s.$

(8)
The matrix $M$ has the following informal interpretation. Suppose that two parity-0 terminals $t$ and $t'$ are adjacent in $J'$. and that $\sigma : V(J') \rightarrow \{0, 1\}$ is a configuration. If these two terminals have spins $\sigma(t)$ and $\sigma(t')$, respectively, then the edge between them contributes a factor $\left(\frac{\beta 1}{1 + \gamma}\right)_{\sigma(t), \sigma(t')}$ to $w_{J'}(\sigma)$. We will show below that, if $t$ is a terminal of $C[u]$ and the spins of $C[u]$ are chosen from the idealised distribution $\mu^0$, then the probability that the spin of terminal $t$ is $j$ is $P_{s,j}$. Thus, informally, $M_{s,s'}$ captures the expected contribution of this connection (in the idealised distribution), where $s'$ represents the phase of the gadget of terminal $t'$.

The informal interpretation of $W$ is that, given any configuration $\sigma : V(J) \rightarrow \{0, 1\}$, a parity-1 terminal $t$ which is connected to a bristle $b$ will contribute a factor $\left[\left(\frac{\beta 1}{1 + \gamma}\right)(\frac{1}{\lambda})\right]_{\sigma(t)}$ to the sum $\sum_{\sigma'} w_{J'}(\sigma')$, where the sum is over all configurations $\sigma' : V(J') \rightarrow \{0, 1\}$ which agree with $\sigma$ except possibly at the bristle $b$. This informal description is just to provide intuition — the technical details are given below. The main idea is that, if the spins of the terminals of the gadgets are chosen from the “idealised” distribution then, if the gadget of $t$ has phase $s$, then the terminal $t$ will contribute a factor of $W_{1 | \emptyset}$ to the expected contribution from this bristle.

We now introduce some calculation which will be needed to describe the algorithm’s computation of $k_1$, $k_2$, and $d$ and also to give the definitions of the real numbers $\tilde{\gamma}$ and $\tilde{\lambda}$. The first step is deriving some tedious but necessary bounds on the various quantities defined above. In particular, we will define positive rational numbers $\Delta^-$ and $\Delta^+$ and a rational number $\xi \in (0,1)$ (independent of $\delta$ and $n$, but depending on $\beta$, $\gamma$, and $\lambda$). These will be hard-wired into the algorithm. We will prove that, provided that $\delta$ is sufficiently small, each $\hat{M}_{s,s'}$ and $\hat{W}_s$ satisfies $\Delta^- \leq \hat{M}_{s,s'} \leq \Delta^+$ and $\Delta^- \leq \hat{W}_s \leq \Delta^+$. Also, each of $\hat{p}^\setminus - \hat{p}^\upsetminus$, $1 - \hat{\gamma}$, $\hat{p}^\setminus$ and $\hat{\lambda}$ is at least $\xi$. We will also prove that $\hat{p}^\setminus \leq 1$ (and, from (7), we have $\hat{\beta} \geq 1$.) Finally, we prove $\hat{\beta} \gamma \leq 1 - \xi$. Here are the details (which the reader may skip).

- By Lemmas 4 and 14, we can define positive rational numbers $p^\setminus$ and $p^\upsetminus$ such that every element $P_{s,s'}$ of matrix $P$ satisfies $p^\setminus \leq P_{s,s'} \leq p^\upsetminus$. Then

  $$(p^\setminus)^2 (2 + \beta + \gamma) \leq M_{s,s'} \leq (p^\upsetminus)^2 (2 + \beta + \gamma),$$

  so, since $\delta < 1$,

  $$e^{-1} (p^\setminus)^2 (2 + \beta + \gamma) \leq \hat{M}_{s,s'} \leq e(p^\upsetminus)^2 (2 + \beta + \gamma),$$

  so to get the required bounds, we take any $\Delta^- < e^{-1} (p^\setminus)^2 (2 + \beta + \gamma)$ and any $\Delta^+ > e(p^\upsetminus)^2 (2 + \beta + \gamma)$. The bounds on $\hat{W}_s$ are similar.

- To ensure $\hat{p}^\setminus - \hat{p}^\upsetminus \geq \xi$, choose rational numbers $\rho_1$, $\rho_2$, $\rho_3$, and $\rho_4$ such that $p^\setminus < \rho_1 < \rho_2 < \rho_3 < \rho_4 < p^\upsetminus$. These exist by Lemma 14, which guarantees that $p^\setminus < p^\setminus$. Then, if $\delta \leq \rho_4 - \rho_3$, Equation (7) guarantees $\hat{p}^\setminus \geq p^\setminus e^{-\delta} \geq p^\setminus (1 - \delta) \geq p^\setminus - \delta \geq p^\setminus - (\rho_4 - \rho_3) \geq \rho_3$. (Note that the calculation used $\rho = 1$.) Similarly, if $\delta \leq (\rho_2 - \rho_1)/2$, Equation (7) guarantees $\hat{p}^\upsetminus \leq e^\delta p^\upsetminus \leq p^\upsetminus (1 + 2\delta) \leq p^\upsetminus + 2\delta \leq p^\upsetminus + (\rho_2 - \rho_1) \leq \rho_2$. (Again, we used $\rho = 1$.) It suffices to take any $\xi < \rho_3 - \rho_2$.

- We can similarly establish $1 - \hat{\gamma} \geq \xi$ by considering a sequence of rational numbers between $\gamma$ and 1 (using the fact that $\gamma < 1$ by (11)) and we can establish $\hat{p}^\setminus \geq \xi$ by
considering a sequence of rational numbers between 0 and \( p^\varepsilon \) (using the fact that \( p^\varepsilon > 0 \) by Lemma [1]).

- Then, by (1), \( \lambda > 0 \), so taking any \( \xi < \lambda \), we can choose a rational number \( \xi' \) with \( 0 < \xi < \xi' < \lambda \). Then choosing \( \delta \leq \log(\xi'/\xi) \) ensures \( e^{-\delta} \xi' \geq \xi \) so \( \hat{\lambda} \geq e^{-\delta} \lambda \geq e^{-\delta} \xi' \geq \xi \).

- Note that the second bullet point already establishes \( \hat{p}^\varepsilon \leq 1 \).

- Finally, (1) guarantees \( \beta \gamma < 1 \), so choose rationals \( \beta' \geq \beta \) and \( \gamma' \geq \gamma \) with \( \beta' \gamma' < 1 \). Choose \( \xi \) sufficiently small that \( \beta' \gamma' < e^{-3\xi} \). Then choose \( \delta \leq \xi/2 \) to ensure

\[
\beta' \gamma \leq e^{2\delta} \beta \gamma \leq e^{\xi} \beta' \gamma' \leq e^{-2\xi} \leq 1 - \xi.
\]

We can make the following conclusions.

\[
\begin{align*}
\tilde{M}_{1,1} - \tilde{M}_{0,1} &= (p^\varepsilon - p^\varepsilon)(\beta - 1)(1 - p^\varepsilon) + (1 - \gamma(\beta - 1)) \geq \xi^3 \\
\tilde{M}_{0,1} - \tilde{M}_{0,0} &= (p^\varepsilon - p^\varepsilon)(\beta - 1)(1 - p^\varepsilon) + (1 - \gamma(\beta - 1)) \geq \xi^3 \\
\tilde{W}_1 - \tilde{W}_0 &= (p^\varepsilon - p^\varepsilon)(\beta - 1) + (1 - \gamma(\beta - 1)) \lambda \geq \xi^3.
\end{align*}
\]

We can now define \( k_2 \). Since

\[
\frac{\tilde{M}_{0,0}\tilde{M}_{1,1} - \tilde{M}_{0,1}^2}{\tilde{M}_{0,1}} = \frac{\tilde{M}_{0,0}\tilde{M}_{1,1} - \tilde{M}_{0,1}^2}{\tilde{M}_{0,1}^2} \leq \frac{\tilde{M}_{0,0} - \xi^3}{\tilde{M}_{0,1}} = 1 - \frac{\xi^3}{\tilde{M}_{0,1}} \leq 1 - \frac{\xi^3}{(\Delta^+)^2} \leq e^{-\xi^3/(\Delta^+)^2}.
\]

Then let

\[
k_2 = \left\lceil \frac{(n^2 + n)2\log(5)(\Delta^+)^2}{\xi^3} \right\rceil.
\]

Then, if we ensure that \( \delta < (\xi^3/(\Delta^+)^2)/8 \), we have

\[
\left(e^{4\delta} \frac{\tilde{M}_{0,0}\tilde{M}_{1,1}}{\tilde{M}_{0,1}^2}\right)^{k_2} \leq e^{-k_2 \xi^3/(2(\Delta^+)^2)} \leq 5^{-n^2-n}.
\]

Then define

\[
\tilde{\gamma} = \left(\frac{M_{0,0}M_{1,1}}{M_{0,1}^2}\right)^{k_2}.
\]

By (8),

\[
\tilde{\gamma} \leq e^{4\delta k_2} \left(\frac{\tilde{M}_{0,0}\tilde{M}_{1,1}}{\tilde{M}_{0,1}^2}\right)^{k_2} \leq 5^{-n^2-n}.
\]

Also, there is a randomised approximation scheme for \( \tilde{\gamma} \) whose running time is at most a polynomial in \( n \) and in the desired accuracy parameter \( \varepsilon \).
Next, we will define $k_1$. Recall that $\hat{W}_1 > \hat{W}_0$ and $\hat{M}_{1,1} > \hat{M}_{0,1}$. If $n$ is sufficiently large with respect to $\log(\Delta^+/\Delta^-) \geq \log(\hat{W}_1/\hat{W}_0)$ then there is a positive integer $k_1$ (which the algorithm can compute) satisfying

$$\frac{3k_2 \log(\hat{M}_{1,1}/\hat{M}_{0,1})}{\log(\hat{W}_1/\hat{W}_0)} + \frac{\log(4.1)n}{\log(\hat{W}_1/\hat{W}_0)} \leq k_1 \leq \frac{3k_2 \log(\hat{M}_{1,1}/\hat{M}_{0,1})}{\log(\hat{W}_1/\hat{W}_0)} + \frac{\log(4.9)n}{\log(\hat{W}_1/\hat{W}_0)}.$$

Note that $k_1 = O(n^2)$. Also,

$$(4.1)^n \leq \left(\frac{\hat{W}_1}{\hat{W}_0}\right)^{k_1} \left(\frac{\hat{M}_{0,1}}{\hat{M}_{1,1}}\right)^{3k_2}$$

and

$$\left(\frac{\hat{W}_1}{\hat{W}_0}\right)^{k_1} \left(\frac{\hat{M}_{0,1}}{\hat{M}_{1,1}}\right)^{3k_2} \leq (4.9)^n.$$

Now if we ensure $\delta \leq n^{-2.5}$ then, for sufficiently large $n$, $\delta \leq n \log(4.1)/(2k_1 + 6k_2)$ and $\delta \leq n \log(5/4.9)/(2k_1 + 6k_2)$, so

$$4^n \leq \left(e^{-2\delta} \hat{W}_1 \hat{W}_0\right)^{k_1} \left(e^{-2\delta} \hat{M}_{0,1} \hat{M}_{1,1}\right)^{3k_2}$$

and

$$\left(e^{-2\delta} \hat{W}_1 \hat{W}_0\right)^{k_1} \left(e^{2\delta} \hat{M}_{0,1} \hat{M}_{1,1}\right)^{3k_2} \leq 5^n.$$

Note that

$$5^n \leq \frac{1}{\tilde{\gamma}^{1/n}}.$$

Then define

$$\tilde{\lambda} = \left(\frac{W_1}{W_0}\right)^{k_1} \left(\frac{M_{0,1}}{M_{1,1}}\right)^{3k_2}.$$

Note that there is a randomised approximation scheme for $\tilde{\lambda}$ whose running time is bounded from above by a polynomial in $n$ and the desired accuracy parameter $\varepsilon$. Also, $\tilde{\lambda} \leq \frac{1}{\tilde{\gamma}^{1/n}}$ so $\tilde{\gamma} \leq \tilde{\lambda}^{-n}$ and $\tilde{\lambda} \geq 4^n$, as required.

Now let $k = \max(k_1, 3k_2)$. Finally, the gadget will use a parameter $d$. By Proposition 5 there is a $c > 1$ (not depending on $k$) such that, for all sufficiently large $d$ which are multiples of 16, and all configurations $\tau : T_{k,d} \rightarrow \{0, 1\}$,

$$|\Pr(\sigma_{k,d}(T_{k,d}) = \tau) - \mu_{k,d}(\tau)| \leq c^{-d}k^2.$$

The algorithm will choose $d$ to be a multiple of 16 such that $d = O(n^3)$ and

$$c^{-d}k^2 < \frac{(e^{-\delta}W_0)^{k_1 n} (e^{-\delta}M_{1,1})^{k_2 |E|}}{(e^{\delta} (\beta + \tilde{\lambda}))^{k_1 n} (e^{\delta} \beta)^{k_2 |E|} 2^{2kn} 2^n n}.$$
This can be done, since \(|E| = O(n)|. We will use below the fact that

\[
\max_{\tau : T_{k,d} \to \{0,1\}} |\Pr(\sigma_{k,d}(T_{k,d}) = \tau) - \mu_{k,d}(\tau)| < \frac{W_0^{k_1 n} M_1^{k_2 |E|}}{(\beta + \lambda)^{k_1 n} \beta^{k_2 |E|} 2^{kn} 2^{n n}},
\]

which follows from Equations (7) and (8).

Let \(K\) be the positive real given by

\[
K = \frac{2^n}{W_0^{k_1 n} M_1^{k_2 |E|}}.
\]

Note that there is a randomised approximation scheme for \(K\) whose running time is at most a polynomial in \(n\) and in the desired accuracy parameter, \(\epsilon\).

All that remains is to establish (3), which we do in the remainder of the proof. Let \(T = \bigcup_{u \in V} T[u]\) be the set of terminals in \(V(J)\). For every configuration \(\tau : T \to \{0,1\}\), let \(\text{wt}(\tau) = \sum_{\sigma \in V(J) \to \{0,1\}, \sigma(T) = \tau} wJ(\sigma).\) The quantity \(\text{wt}(\tau)\) is the contribution to \(Z_{\beta,\gamma,\lambda}(J)\) from configurations \(\sigma\) with \(\sigma(T) = \tau\). Similarly, let \(\text{wt}'(\tau)\) be the contribution to \(Z_{\beta,\gamma,\lambda}(J')\) from these configurations. Since \(V(J') = V(J)\), we have \(\text{wt}'(\tau) = \sum_{\sigma \in V(J) \to \{0,1\}, \sigma(T) = \tau} wJ'\). Let \(F(\tau)\) denote \(\text{wt}'(\tau)/\text{wt}(\tau)\). Then

\[
\frac{Z_{\beta,\gamma,\lambda}(J')}{Z_{\beta,\gamma,\lambda}(J)} = \frac{\sum_{\tau : T \to \{0,1\}} \text{wt}'(\tau)}{\sum_{\tau : T \to \{0,1\}} \text{wt}(\tau) F(\tau)} \frac{\sum_{\tau : T \to \{0,1\}} \text{wt}(\tau) F(\tau)}{Z_{\beta,\gamma,\lambda}(J)} = \mathbb{E}[F(\sigma_j(T))].
\]

We can write \(F(\tau)\) in terms of \(\beta, \gamma,\) and \(\lambda\):

\[
F(\tau) = \prod_{u \in V} \prod_{j=0}^{k_1 - 1} \left(\left(\frac{\beta}{1} \frac{1}{\gamma} \frac{1}{\lambda}\right) \tau(T[u,j])\right) \prod_{(u,v) \in M, u < v} \prod_{j=0}^{k_2 - 1} \left(\frac{\beta}{1} \frac{1}{\gamma} \tau(T[v,u,k_2+j])\right).
\]

We now define an “idealised” distribution on configurations assigning spins to the terminals. First, let \(\tilde{\sigma}\) be a random variable which is drawn uniformly from \(V \to \{0,1\}\). Each realisation \(\tilde{\sigma}\) of \(\tilde{\sigma}\) can be thought of as specifying, for every vertex \(u \in V\), a phase \(\tilde{\sigma}(u) \in \{0,1\}\) for the gadget \(C[u]\). Conditioned on the realisation \(\tilde{\sigma} = \tilde{\sigma}\), the random variable \(\hat{\sigma} : T \to \{0,1\}\) is distributed as follows: for each \(u \in V\), \(\hat{\sigma}(T[u])\) is chosen independently from the distribution \(\mu_{\tilde{\sigma}(u)}\). Note that the (unconditioned) random variable \(\hat{\sigma}\) has the property that \(\hat{\sigma}(T[u])\) is distributed as \(\mu_u\), independently of all \(\hat{\sigma}(T[u'])\) for \(u' \neq u\).

We wish to estimate \(\mathbb{E}[F(\sigma_j(T))]\), but the distribution of \(\sigma_j(T)\) is somewhat complicated. Instead, we will first estimate \(\mathbb{E}[F(\hat{\sigma})]\), and we will later use Proposition [5] to show that these two quantities are close. From the definition of \(\hat{\sigma}\), we have

\[
\mathbb{E}[F(\hat{\sigma})] = \frac{1}{2^n} \sum_{\tilde{\sigma} : V \to \{0,1\}} \mathbb{E}[F(\hat{\sigma}) | \hat{\sigma} = \tilde{\sigma}].
\]

Now we will argue that if \(\tilde{\sigma}(u) = s\) then, for every parity-0 terminal \(t\) of \(C[u]\), it is the case that \(\Pr(\hat{\sigma}(t) = j \mid \hat{\sigma} = \tilde{\sigma}) = P_{s,j}\). (To see this, consider the possible cases. If \(s = 0\) then, from the definition of \(\mu_{k,d}^0\), the probability that \(\hat{\sigma}(t) = 1\) is \(p^=\), which is \(P_{0,1}\), but the probability that \(\hat{\sigma}(t) = 0\) is \(1 - p^=\) = \(P_{0,0}\). The situation is similar if \(s = 1\).) On the other
hand, similar reasoning shows that, for every parity-1 terminal $t$ of $C[u]$, it is the case that $\Pr(\tilde{\sigma}(t) = j \mid \tilde{\sigma} = \tilde{\sigma}) = P_{1 \oplus s, j}$. Thus, we have

$$\mathbb{E}[F(\tilde{\sigma})] = \frac{1}{2^n} \sum_{\tilde{\sigma} : V \to \{0,1\}} \sum_{\tau : T \to \{0,1\}} F(\tau) \prod_{u \in V} \prod_{j=1}^{k_1} P_{\tilde{\sigma}(u), \tau(T^0[u,j])} P_{1 \oplus \tilde{\sigma}(u), \tau(T^1[u,j])}.$$ 

Plugging in $\prod$, the contribution of each $\tilde{\sigma}$ to the right-hand-side of the above equality is $2^{-n}$ multiplied by the product of the following terms:

$$\prod_{u \in V} \prod_{j=0}^{k_1-1} \sum_{s \in \{0,1\}} P_{1 \oplus \tilde{\sigma}(u), s} \left( \begin{pmatrix} \beta & 1 \\ 1 & \gamma \end{pmatrix} \right)_{s},$$

$$\prod_{u \in V} \prod_{j=1}^{k_1} \sum_{s \in \{0,1\}} P_{1 \oplus \tilde{\sigma}(u), s},$$

$$\prod_{(u,v) \in E} \prod_{j=0}^{k_2-1} \sum_{s,s' \in \{0,1\}} P_{\tilde{\sigma}(u), s} P_{\tilde{\sigma}(u), s'} \left( \begin{pmatrix} \beta & 1 \\ 1 & \gamma \end{pmatrix} \right)_{s,s'},$$

$$\prod_{u \in V} \prod_{j=3k_2}^{k_1} \sum_{s \in \{0,1\}} P_{\tilde{\sigma}(u), s}.$$

The second and fourth of these terms are equal to 1, and the first and third simplify using the matrices that we defined earlier, so we get

$$\mathbb{E}[F(\tilde{\sigma})] = \frac{1}{2^n} \sum_{\tilde{\sigma} : V \to \{0,1\}} \prod_{u \in V} W_{1 \oplus \tilde{\sigma}(u)}^{k_1} \prod_{(u,v) \in E} M_{\tilde{\sigma}(u),\tilde{\sigma}(v)}^{k_2}.$$ 

Then, plugging in our notation from earlier, we have

$$\mathbb{E}[F(\tilde{\sigma})] = \frac{1}{2^n} \sum_{\tilde{\sigma} : V \to \{0,1\}} W_1^{k_1(n-\ell(\tilde{\sigma}))} W_0^{k_1(\ell(\tilde{\sigma}))} M_{0,0}^{k_2(b(\tilde{\sigma}))} M_{0,1}^{k_2(c(\tilde{\sigma}))} M_{1,1}^{k_2(\gamma(\tilde{\sigma}))}$$

$$= \frac{W_0^{k_1 n} M_{0,1}^{k_2 |E|}}{2^n} \sum_{\tilde{\sigma} : V \to \{0,1\}} \left( \frac{M_{0,0}}{M_{0,1}} \right)^{k_2 b(\tilde{\sigma})} \left( \frac{M_{1,1}}{M_{0,1}} \right)^{k_2 c(\tilde{\sigma})} \left( \frac{W_1}{W_0} \right)^{k_1(n-\ell(\tilde{\sigma}))}.$$ 

Replacing $\tilde{\sigma}(u)$ with $1 \oplus \tilde{\sigma}(u)$, we get

$$\mathbb{E}[F(\tilde{\sigma})] = \frac{W_0^{k_1 n} M_{0,1}^{k_2 |E|}}{2^n} \sum_{\tilde{\sigma} : V \to \{0,1\}} \left( \frac{M_{1,1}}{M_{0,1}} \right)^{k_2 b(\tilde{\sigma})} \left( \frac{M_{0,0}}{M_{0,1}} \right)^{k_2 c(\tilde{\sigma})} \left( \frac{W_1}{W_0} \right)^{k_1 \ell(\tilde{\sigma})}.$$ 

Since $G$ is cubic, we can count the pairs $(v, (u,v)) \in V \times E$ with $\tilde{\sigma}(v) = 1$ in two ways to get $3\ell(\tilde{\sigma}) = 2c(\tilde{\sigma}) + (|E| - b(\tilde{\sigma}) - c(\tilde{\sigma}))$. So $b(\tilde{\sigma}) = c(\tilde{\sigma}) + |E| - 3\ell(\tilde{\sigma})$, which implies that

$$\mathbb{E}[F(\tilde{\sigma})] = \frac{W_0^{k_1 n} M_{0,1}^{k_2 |E|}}{2^n} \left( \frac{M_{1,1}}{M_{0,1}} \right)^{k_2 |E|} \sum_{\tilde{\sigma} : V \to \{0,1\}} \left[ \left( \frac{M_{0,0} M_{1,1}}{M_{0,1}^2} \right)^{k_2} \right]^{c(\tilde{\sigma})} \left( \frac{W_1}{W_0} \right)^{k_1 \ell(\tilde{\sigma})} \left( \frac{M_{0,1}}{M_{1,1}} \right)^{3k_2 \ell(\tilde{\sigma})}$$

$$= K^{-1} Z_{(1, \tilde{\gamma}, \tilde{\lambda})}(G).$$
Applying (9), proof of Proposition 5, we have
\[ F(\psi) \leq \text{restriction} \]

To emphasise that summation over \( \tau : T \to \{0,1\} \) can be broken into summation over each restriction \( \tau(T[u]) \), we will write the summation index as \( \forall u, \tau(T[u]) : T[u] \to \{0,1\} \). By (11), \( F(\tau) \leq (\beta + \lambda)^{k_1 n} \beta^2 |E| \), so we can write
\[
\psi \leq (\beta + \lambda)^{k_1 n} \beta^2 |E| \sum_{\forall u, \tau(T[u]) : T[u] \to \{0,1\}} \left| \prod_{u \in V} \Pr(\sigma_j(T[u]) = \tau(T[u])) - \prod_{u \in V} \Pr(\sigma(T[u]) = \tau(T[u])) \right|.
\]

Using the inequality \( \sum_{u} a_u - \prod_{u} b_u \leq \sum |a_u - b_u| \) valid for values \( 0 \leq a_u, b_u \leq 1 \), as in the proof of Proposition 5, we have
\[
\psi \leq (\beta + \lambda)^{k_1 n} \beta^2 |E| 2^{k_1 n} \sum_{\forall u, \tau(T[u]) : T[u] \to \{0,1\}} \max_{\tau: T_k,d \to \{0,1\}} \left| \Pr(\sigma_k,d = \tau) - \mu_k,d(\tau) \right|.
\]

Applying (9),
\[
\psi \leq (\beta + \lambda)^{k_1 n} \beta^2 |E| 2^{k_1 n} \frac{W^{k_1 n} M^{k_2 |E|}}{(\beta + \lambda)^{k_1 n} \beta^2 |E| 2^{k_1 n} n)} = \frac{W^{k_1 n} M^{k_2 |E|}}{2^n} = K^{-1}.
\]

So to establish (13), we note that
\[
1 - \frac{1}{Z_{1,\hat{\gamma},\hat{\lambda}}(G)} = 1 - \frac{1}{K \mathbb{E}[F(\sigma)]} \leq \frac{\mathbb{E}[F(\sigma_j(T))] - \mathbb{E}[F(\sigma)]}{\mathbb{E}[F(\sigma)]} \leq 1 + \frac{1}{K \mathbb{E}[F(\sigma)]} = 1 + \frac{1}{Z_{1,\hat{\gamma},\hat{\lambda}}(G)}.
\]

The result follows from the extremely crude bound \( Z_{1,\hat{\gamma},\hat{\lambda}}(G) \geq 8 \).

We can now prove our main theorem.

**Theorem 1** Let \( \beta, \gamma \) and \( \lambda \) be efficiently-approximable reals satisfying (11). There is no FPRAS for DegreeFourPlanarTwoSpin(\( \beta, \gamma, \lambda \)) unless \( \text{NP} \subseteq \text{BPP} \).

**Proof.** We will give a randomised algorithm for PlanarCubicIS, using an FPRAS for DegreeFourPlanarTwoSpin(\( \beta, \gamma, \lambda \)) as an oracle (and also using the given FPRASes for \( \beta, \gamma \) and \( \lambda \)).

After receiving an instance \( G \) and \( h \), our algorithm uses Lemma 25 which provides planar graphs \( J \) and \( J' \) with maximum degree at most 4 and also some approximation schemes for the reals \( K, \hat{\gamma} \) and \( \hat{\lambda} \). With probability at least \( 1 - 1/15 \), these satisfy \( \hat{\lambda} \geq 4^{|V(G)|} \) and \( \hat{\gamma} \leq \hat{\lambda}^{-|V(G)|} < 1 \) and Equation (13). The algorithm then makes four calls to approximation schemes, suitably powered so that each call fails with probability at most \( 1/15 \). Thus, with
probability at least $2/3$, Equation (13) is satisfied and all calls to the approximation schemes succeed. In that case, we will show how to determine (from the outputs of the approximation schemes) whether or not $G$ has an independent set of size $h$.

Let $n = |V(G)|$. By Equation (10), we have

$$e^{-1/4}K \frac{Z_{\beta,\gamma,\lambda}(J')}{{Z_{\beta,\gamma,\lambda}(J)}} \leq Z_{1,\hat{\gamma},\hat{\lambda}}(G) \leq e^{1/4}K \frac{Z_{\beta,\gamma,\lambda}(J')}{{Z_{\beta,\gamma,\lambda}(J)}}.$$

Using the given approximation schemes for $Z_{\beta,\gamma,\lambda}(J')$, $Z_{\beta,\gamma,\lambda}(J)$ and $K$, each with accuracy parameter $1/12$ and failure probability $1/15$, we can compute a value $\hat{Z}$ which, with probability at least $1 - 3/15$, satisfies

$$e^{-1/4} \hat{Z} \leq K \frac{Z_{\beta,\gamma,\lambda}(J')}{{Z_{\beta,\gamma,\lambda}(J)}} \leq e^{1/4} \hat{Z},$$

so

$$e^{-1/2} \hat{Z} \leq Z_{1,\hat{\gamma},\hat{\lambda}}(G) \leq e^{1/2} \hat{Z}. \quad (14)$$

Using the given approximation scheme for $\hat{\lambda}$ with accuracy parameter $1/2h$ and failure probability $1/15$, we can compute a value $\hat{\lambda}$ which, with probability at least $1 - 1/15$, satisfies

$$e^{-1/2h} \hat{\lambda} \leq \hat{\lambda} \leq e^{1/2h} \hat{\lambda}$$

so

$$e^{-1/2} \hat{\lambda}^h \leq \hat{\lambda}^h \leq e^{1/2} \hat{\lambda}^h. \quad (15)$$

Suppose that all four calls to the approximation schemes succeed so that (14) and (15) hold. Recall that

$$Z_{1,\hat{\gamma},\hat{\lambda}}(G) = \sum_{\sigma:V(G) \to \{0,1\}} \gamma^{c(\sigma)} \hat{\lambda}^{\ell(\sigma)}.$$

If $G$ has an independent set of size $h$ then $Z_{1,\hat{\gamma},\hat{\lambda}}(G) \geq \hat{\lambda}^h$ so, plugging in (14) and (15),

$$\hat{Z} \geq e^{-1/2} Z_{1,\hat{\gamma},\hat{\lambda}}(G) \geq e^{-1/2h} \hat{\lambda}^h \geq e^{-1} \hat{\lambda}^h.$$

Also, if $G$ has no independent set of size $h$, then $Z_{1,\hat{\gamma},\hat{\lambda}}(G) \leq 2^n \max(\hat{\lambda}^{h-1}, \hat{\lambda}^n) \leq 2^n \hat{\lambda}^{h-1}$. So, plugging in (14) and (15) and our lower bound for $\hat{\lambda}$,

$$\hat{Z} \leq e^{1/2} Z_{1,\hat{\gamma},\hat{\lambda}}(G) \leq e^{1/2} 2^n \hat{\lambda}^{h-1} \leq e^{1/2} \frac{2^n \hat{\lambda}^h}{4^n} = e^{1/2} e^{-n} \hat{\lambda}^h \leq e^{2-n} \hat{\lambda}^h.$$

As long as $n \geq 3$, $e^{-1} > e^{2-n}$, so it is possible to determine from $\hat{Z}$ and $\hat{\lambda}$ whether or not $G$ has an independent set of size $h$.

\section{Approximating the log-partition function}

We start with a preliminary lemma, which will help us to show that our approximation is sufficiently accurate.

\begin{lemma}
Suppose that $\beta$, $\gamma$ and $\lambda$ are real numbers satisfying $\beta \geq 1 > \gamma \geq 0$ and $\lambda \geq 1$. Then, for every planar graph $G$, $Z_{\beta,\gamma,\lambda}(G) \geq (1+\lambda)|V(G)|/4$.
\end{lemma}

\begin{proof}
Let $I$ be the largest colour class in a proper 4-colouring of $G$. Then $I$ is an independent set of $G$ of size at least $|V(G)|/4$. For every configuration $\sigma:V(G) \to \{0,1\}$ which assigns spin 0 to every vertex in $V(G) \setminus I$, $w_G(\sigma) \geq \lambda^{\ell(\sigma)}$. Thus $Z_{\beta,\gamma,\lambda}(G) \geq (1+\lambda)^{|I|}$.
\end{proof}
Our approximation algorithm is inspired by Baker’s approximation schemes for optimisation problems on planar graphs [1]. For a good explanation of her technique (which we use in our exposition here), see Borradaile’s notes [6]. We will use the following notation (from [6]) to decompose a planar graph $G = (V, E)$ which is embedded in the plane. We first define the \textit{level} of each vertex. Vertices on the boundary of the embedding have level 0. Then, for $i \in \{0, \ldots, n\}$, the vertices with level $i$ are those that are on the boundary on the graph formed from $G$ by deleting all vertices whose level is less than $i$. For a fixed parameter $k$, and for every $i \in \{0, \ldots, k-1\}$, let $V_i = \{v \in V \mid \text{The level of vertex } v \text{ is equal to } i \mod k\}$. Let $G_i$ be the graph $G - V_i$. By construction, $G_i$ is $(k-1)$-\textit{outerplanar}. Also, Bodlaender [4] had shown that every $k$-outerplanar graph has treewidth at most $3k - 1$. Also, this tree decomposition is easy to compute. Using a data structure of Lipton and Tarjan [20], Baker shows that the levels of vertices can be computed in $O(|V|)$ time.

We can now prove Theorem 3.

\textbf{Theorem 3.} Suppose that $\beta$, $\gamma$, and $\lambda$ are efficiently approximable reals satisfying $\beta \geq 1 > \gamma \geq 0$ and $\lambda \geq 1$. There is a PRAS for \textit{PlanarLogTwoSpin}$(\beta, \gamma, \lambda)$.

\textit{Proof.} Consider input $G = (V, E)$ with at least 3 vertices and an accuracy parameter $\varepsilon \in (0, 1)$. Let $n = |V|$ and $m = |E| \leq 3n$. Let $\beta^+, \beta^-, \lambda^+, \lambda^-, \gamma^+$ and $\gamma^-$ be rational numbers (built into the algorithm) such that $\beta^+ \geq \beta \geq \beta^- \geq 1$, $1 > \gamma^+ \geq \gamma \geq \gamma^- \geq 0$, and $\lambda^+ \geq \lambda \geq \lambda^ - \geq 1$. Let $k$ be any integer satisfying

$$k \geq \frac{32 \log (2 \lambda^+) + 96 \log (\beta^+)}{\varepsilon \log (1 + \lambda^-)}.$$ 

Then let

$$\delta = \frac{2n \log (2 \lambda^+)}{k(n + m)}.$$

Using the given FPRASes for $\beta$, $\gamma$ and $\lambda$, compute $\hat{\beta}$, $\hat{\gamma}$ and $\hat{\lambda}$ satisfying $e^{-\delta} \beta \leq \hat{\beta} \leq e^\delta \beta$, $e^{-\delta} \lambda \leq \hat{\lambda} \leq e^\delta \lambda$ and $e^{-\delta} \gamma \leq \hat{\gamma} \leq e^\delta \gamma$. As in the proof of Lemma 25, adjust the output of the FPRASes to ensure $\beta^+ \geq \hat{\beta} \geq \beta^-$, $\gamma^+ \geq \hat{\gamma} \geq \gamma^-$, and $\lambda^+ \geq \hat{\lambda} \geq \lambda^-$. The first step is to compute a value $\hat{Z}$ satisfying

$$\hat{Z} \leq Z_{\hat{\beta}, \hat{\gamma}, \hat{\lambda}}(G) \leq (2 \lambda^+)^{2n/k} (\beta^+)^{12n/k} \hat{Z}. \tag{16}$$

This step is accomplished as follows.

1. Using Baker’s algorithm, construct the graphs $G_i$ for $i \in \{0, \ldots, k-1\}$. Each of these has treewidth at most $3(k-1) - 1$.

2. Choose $i \in \{0, \ldots, k-1\}$ as follows. Let $I = \{i \mid |V_i| \leq 2n/k\}$. Note that $|I| \geq k/2$. Now consider the $2m$ endpoints of edges in $E$. Choose $i \in I$ so that $V_i$ contains at most $(2m)/|I|$ of these. Note that $|V_i| \leq 2n/k$ and the number of edges with endpoints in $V_i$ is at most $4m/k \leq 12n/k$.

3. Use the algorithm of Yin and Zhang [27, Theorem 2.8] to compute $\hat{Z} = Z_{\hat{\beta}, \hat{\gamma}, \hat{\lambda}}(G_i)$. The running time of Yin and Zhang’s algorithm is at most the product of a polynomial in $n$ and an exponential function in the treewidth of $G_i$. In order to apply the algorithm, we first express the partition function $Z_{\hat{\beta}, \hat{\gamma}, \hat{\lambda}}(G_i)$ as the solution to a Holant problem $\text{Holant}(G, \mathcal{F})$ with regular symmetric $\mathcal{F}$. See [27] for definitions and details.
4. Equation (16) now follows by noting that

\[ Z_{\beta,\gamma,\lambda}(G) = \sum_{\tau \in V(G_i) \rightarrow \{0,1\}} w_{G_i}(\tau) \sum_{\tau' : V_i \rightarrow \{0,1\}} \sum_{\ell(\tau')} \beta(\tau,\tau') \gamma(\tau,\tau'), \]

where \( \ell(\tau') \) is the number of vertices \( u \in V_i \) with \( \tau'(u) = 1 \) and \( b(\tau,\tau') \) is the sum of the number of edges \( (u,v) \) with \( u \in V(G_i) \) and \( v \in V_i \) and \( \tau(u) = \tau'(v) = 0 \) and the number of edges \( (u,v) \) with \( u \in V_i \) and \( v \in V_i \) and \( \tau(u) = \tau'(v) = 0 \) and \( c(\tau,\tau') \) is defined similarly (with spin 1). Then \( \sum_{\tau' : V_i \rightarrow \{0,1\}} \beta(\tau,\tau') \gamma(\tau,\tau') \) is at least 1 (since \( \tau' \) can assign spin 0 to every vertex in \( V_i \)) and it is at most \( 2^{2n/k} \). To finish, note that

\[ e^{-\delta(n+m)} Z_{\beta,\gamma,\lambda}(G) \leq Z_{\beta,\gamma,\lambda}(G) \leq e^{\delta(n+m)} Z_{\beta,\gamma,\lambda}(G), \]

so since

\[ \delta(n+m) \leq \frac{2n}{k} \log(2\lambda^+) \]

and (from Lemma 26)

\[ \log Z_{\beta,\gamma,\lambda}(G) \geq (n/4)(1 + \lambda^-), \]

and

\[ (\epsilon/2)(n/4) \log(1 + \lambda^-) \geq \frac{4n}{k} \log(2\lambda^+) + \frac{12n}{k} \log(\beta^+), \]

\[ e^{-\epsilon} \log(Z_{\beta,\gamma,\lambda}(G)) \leq \log(Z_{\beta,\gamma,\lambda}(G))(1 - \epsilon/2) \]

\[ \leq \log Z_{\beta,\gamma,\lambda}(G) - \delta(n+m) - \frac{2n}{k} \log(2\lambda^+) - \frac{12n}{k} \log(\beta^+) \]

\[ \leq \log Z_{\beta,\gamma,\lambda}(G) - \frac{2n}{k} \log(2\lambda^+) - \frac{12n}{k} \log(\beta^+) \]

\[ \leq \log \hat{Z}. \]

Similarly,

\[ \log \hat{Z} \leq \log Z_{\beta,\gamma,\lambda}(G) \]

\[ \leq \delta(n+m) + \log Z_{\beta,\gamma,\lambda}(G) \]

\[ \leq \frac{2n}{k} \log(2\lambda^+) + \log Z_{\beta,\gamma,\lambda}(G) \]

\[ \leq (1 + \epsilon/2) \log Z_{\beta,\gamma,\lambda}(G) \]

\[ \leq e^\epsilon \log Z_{\beta,\gamma,\lambda}(G). \]

\[ \square \]

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