COMPACTNESS OF FIXED POINT MAPS AND THE BALL-MARSDEN-SLEMROD CONJECTURE*

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Abstract. Given a parameter dependent fixed point equation \( x = F(x, u) \), we derive an abstract compactness principle for the fixed point map \( u \mapsto x^*(u) \) under the assumptions that (i) the fixed point equation can be solved by the contraction principle and (ii) the map \( u \mapsto F(x, u) \) is compact for fixed \( x \).

This result is applied to infinite-dimensional, semi-linear control systems and their reachable sets. More precisely, we extend a non-controllability result of Ball, Marsden, and Slemrod [2] to semi-linear systems. First we consider \( L^p \)-controls, \( p > 1 \). Subsequently we analyze the case \( p = 1 \).

Key words. Compact operators, compactness of fixed point maps, reachable/attainable sets, controllability of bilinear/semi-linear systems, Ball-Marsden-Slemrod conjecture

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1. Introduction. This paper was inspired by a recent result of Boussaïd, Caponigro, and Chambrion [6] (see also [5, 7]) who addressed an “old” conjecture of Ball, Marsden, and Slemrod [2] about the non-controllability of bilinear control systems with infinite-dimensional state space. For similar results concerning the linear case we refer to e.g., [21].

To be more precise, let us consider a semi-linear control system on a (possibly complex) Banach space \( X \) given by

\[
(1.1) \quad \dot{x}(t) = Ax(t) + u(t)f(x(t)), \quad x(0) = \xi_0 \in X
\]

where \( A : D(A) \to X \) is the (possibly unbounded) infinitesimal generator of a \( C^0 \)-semigroup of bounded linear operators while \( f : X \to X \) is supposed to be locally Lipschitz and linearly bounded. Moreover, the control \( t \mapsto u(t) \) is assumed to be in some appropriate \( L^p \)-space with \( p \geq 1 \). Obviously, (1.1) boils down to a bilinear control system if \( f : X \to X \) is linear.

We call \( x : [0, T] \to X \) a classical solution\(^3\) of (1.1) if \( x \) is continuously differentiable and satisfies (1.1) for all \( t \in [0, T] \) and Carathéodory solution\(^4\) if \( x \) is continuous, almost everywhere differentiable on \([0, T]\) with \( L^1 \)-derivative and satisfies (1.1) for almost all \( t \in [0, T] \) whereas a mild solution of (1.1) has to fulfill the integral equation

\[
(1.2) \quad x(t) = e^{tA}\xi_0 + \int_0^t e^{(t-s)A}u(s)f(x(s))\,ds
\]

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\(^1\)Actually, in [2] the authors assume \( f \) to be \( C^1 \) but their existence result (cf. Thm. 2.5) works for locally Lipschitz maps as well.
\(^2\)As usual, the notation \( p \geq 1 \) includes the case \( p = \infty \).
\(^3\)In particular, if \( A \) generates an analytic semigroup, the concept of a “classical solution” is often weakened in the sense that \( x : [0, T] \to X \) is required to be continuous on the closed interval \([0, T]\) but continuously differentiable only on the half-open interval \((0, T]\). Here, for simplicity, we work with the stronger notion defined above.
\(^4\)In the literature Carathéodory solutions are also called strong solutions, cf. [17, Sec. 4.2]. Moreover, absolute continuity of \( x : [0, T] \to X \) is in general not sufficient to guarantee differentiability almost everywhere. The implication “absolute continuity \( \implies \) differentiability almost everywhere with \( L^1 \)-derivative” holds if and only if \( X \) has the Radon-Nikodým property which is the case for all reflexive and, in particular, all finite dimensional Banach spaces, cf.[1, Thm. 1.2.6 and Cor.1.2.7].
for all $t \in [0,T]$. Here, $(e^{tA})_{t \geq 0}$ denotes the strongly continuous one-parameter semigroup of linear operators generated by $A$. It is well known that (1.2) has a unique mild solution, denoted by $x(\cdot, \xi_0, u)$, once $\xi_0 \in X$ and $u \in L^1([0,T], \mathbb{R})$ are fixed, cf. [2] or [17, Sec. 6.1]. For finite-dimensional $X$, the concepts of mild and Carathéodory solutions coincide, cf. [11, Chap. 2] or [20, App. C], as can be seen by the fundamental theorem of calculus [18, Thm. 7.11 and Thm. 7.20] and the fact that in finite dimensions every strongly continuous one-parameter semigroup is actually norm-continuous and thus given by the exponential series $e^{At} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$. For infinite-dimensional $X$, however, only the implications

$$\text{classical } \Rightarrow \text{ Carathéodory } \Rightarrow \text{ mild}$$

remain true, cf. [17, Sec. 6.1]. Further references including those to relevant counter-examples are given in the bibliographical notes of [17].

In the sequel, we completely focus on mild solutions. The reachable set $R^p(\xi_0)$ of (1.1) for the initial value $\xi_0 \in X$ and $L^p$ controls is then defined as:

$$R^p(\xi_0) := \bigcup_{T \geq 0} R^p_{\leq T}(\xi_0),$$

where

$$R^p_{\leq T}(\xi_0) := \{ x(t, \xi_0, u) : t \in [0,T], u \in L^p([0,T], \mathbb{R}) \}.$$

denotes the reachable set up to time $T$ and $x(\cdot, \xi_0, u)$ the corresponding unique mild solution. Finally, (1.1) is called controllable under $L^p$ controls if $R^p(\xi_0) = X$ for all $\xi_0 \in X$ or, equivalently, if for any pair $\xi_0, \eta_0 \in X$ there exist $T \geq 0$ and $u \in L^p([0,T], \mathbb{R})$ such that $x(T, \xi_0, u) = \eta_0$.

Ball, Marsden, and Slemrod [2] showed that a bilinear system\footnote{Except for minor modifications, the proof in [2] should work for Lipschitz continuous $f$ as well.} (i.e. a system of the above form with $f$ being linear and bounded) is never controllable via $L^p$-controls if $X$ is infinite-dimensional and $p > 1$, cf. [2, Thm. 3.6]. Actually, they proved that for $p > 1$ the reachable set $R^p(\xi_0)$ of any $\xi_0 \in X$ is a countable union of (relatively) compact sets and has therefore no interior points in $X$ if $X$ is an infinite dimensional Banach space. Moreover, they conjectured that this also holds for $p = 1$, cf. [2, Rem. 3.8]. Recently, Lampart [14] derived an even more striking non-controllability result for the Schrödinger equation on the unbounded domain $\mathbb{R}^n$ – but again under the assumption $p > 1$. So the original Ball, Marsden, and Slemrod conjecture remained open for almost 40 years till Boussaïd et al. [5, 6, 7] were able to answer it for the bilinear case in the affirmative. Thereafter, Chambrion and Thomann [9, 10] derived first results for abstract semi-linear systems with applications to non-linear wave and Schrödinger equations. In these two series of papers, the authors basically exploit – besides Baire’s category theorem – two\footnote{In [5, 7] the Hilbert space case allows a different treatment via Radon-measure-valued (impulsive) controls; in [10] the authors take advantage of additional smoothing properties of the linear part of the equation.} facts:

- The standard fixed point iteration\footnote{the Dyson series} for solving (1.2) yields a uniform estimate
of the following form
\[ \|x_k(t, \xi_0, u) - x_{k-1}(t, \xi_0, u)\| \leq \frac{M^{k+1}e^{(k+1)\omega t}\|f\|^k\|u\|_1}{k!}\|\xi_0\|, \]
where \( x_k(\cdot, \xi_0, u) \) denotes \( k \)-th iteration and \( M > 0, \omega > 0 \) are suitable constants, cf. e.g. [6, Prop. 11 and Prop. 15].

• The “approximate” reachable sets given as the sets of all states which can be reached up to time \( T \) from \( \xi_0 \) by the \( k \)-th iterates \( x_k(\cdot, \xi_0, u) \) are relatively compact, cf. [6, Lemma 12 and proof of Thm. 2].

Analyzing these ideas naturally raises the question whether this result can be derived from a more general principle which guarantees that relative compactness of the “approximate” reachable sets passes over to their limit, i.e. to the reachable set. More precisely, we are interested in the following problem:

*Given a parameter-dependent fixed point equation
\[ x = F(x, u) \tag{1.3} \]
such that (1.3) has a unique solution \( x^*(u) \) for all \( u \). What can be said about the compactness of the map \( u \mapsto x^*(u) \) under the assumption that \( u \mapsto F(x, u) \) is compact for fixed \( x \) ?*

In Section 2, we prove an abstract compactness principle for parameter-dependent fixed point maps, cf. Theorem 2.4, under the additional assumption that (1.3) can be solved via the contraction principle (Banach’s fixed point theorem). Moreover, we derive some corollaries of Theorem 2.4 which turn out to be quite useful for applications to ODEs and PDEs. In Subsection 3.1, we use our previous findings to show that the fixed point map \( u \mapsto x(\cdot, \xi_0, u) \) of (1.2) is compact for \( p > 1 \). This immediately allows us to generalize the non-controllability result of Ball, Marsden, and Slemrod to semi-linear systems. Unfortunately, for \( p = 1 \) the fixed point map \( u \mapsto x(\cdot, \xi_0, u) \) fails to be compact. Nevertheless, in Subsection 3.2 a “minor” detour guided by Corollary 2.13 allows us to prove relative compactness of the reachable sets
\[ \mathcal{R}^{1,r}_{\leq T}(\xi_0) := \{x(t, \xi_0, u) : t \in [0, T], \|u\|_1 \leq r\}. \]
For this reason, we can finally extend the non-controllability statement by Ball, Marsden, and Slemrod to semi-linear systems and all \( p \geq 1 \). Moreover, we expect that the derived compactness principle will reveal further non-controllability results for infinite systems whenever a contraction argument can be applied to “construct” solutions.

Concluding, we should notice that there are of course results on “exact” (local) controllability for infinite dimensional systems. Basically they fall into two categories: Either (local) controllability is obtained in a space of higher regularity which is often compactly embedded in the original one (cf. [3] and the references therein) or they fit not into the above setting like boundary control problems (cf. [15, 23]).

2. Compactness Principle for Fixed Point Maps.

**Notation and Preliminaries.** Let \( X \) be an arbitrary metric space. Its metric is usually denoted by \( d \); its open and closed balls of radius \( r \geq 0 \) and center \( x \) are referred to as \( B_r(x) \) and \( K_r(x) \), respectively. Another metric \( d' \) on \( X \) is called *strongly*
equivalent to $d$ if there exist constants $M > 0$ and $M' > 0$ such that the estimates
\begin{equation}
 d(x, y) \leq M'd(x, y) \quad \text{and} \quad d'(x, y) \leq Md(x, y)
\end{equation}
are satisfied for all $x, y \in X$.

Whenever in the following products of metric (topological) spaces occur, they will be equipped with the product topology. Obviously, if these products are finite the product topology coincides with the topology induced by the metric
\begin{equation}
 d_{\infty}(x, y) := \max_{i = 1, \ldots, n} d_i(x_i, y_i)
\end{equation}
or, alternatively, by
\begin{equation}
 d_1(x, y) := \sum_{i = 1}^n d_i(x_i, y_i) .
\end{equation}

**Remark 2.1.** Note that the above concept of equivalence of metrics is rather “strong” in the sense that two metrics $d$ and $d'$ can generate the same topology without being strongly equivalent. However, it guarantees that the corresponding uniform structures coincide and thus completeness is preserved when passing from one metric to another strongly equivalent one. Moreover, if $d$ and $d'$ are induced by norms then the above concept boils down to the standard notion of equivalence of norms.

Next, let $X, Z$ and $P$ be metric spaces and let $F : X \times P \to Z$, $G : X \to Z$ and $H : P \to Z$ be arbitrary maps. By the usual abuse of notation\footnote{Of course; if $X$ and $P$ are not disjoint then $F_x$ is not well-defined for $x \in X \cap P$. Nevertheless, for simplicity we stick to this common notation and write $F(\cdot, x)$ or $F(x, \cdot)$ instead of $F_x$ whenever confusion can occur.}, let $F_x$ and $F_u$ with $x \in X$ and $u \in P$ denote the partial maps $F_x : P \to Z$, $u \mapsto F_x(u) := F(x, u)$ and $F_u : X \to Z$, $x \mapsto F_u(x) := F(x, u)$, respectively. Then $G : X \to Z$ is termed
- **(globally) Lipschitz** if there exists a constant $L \geq 0$ such that
\begin{equation}
 d(G(x), G(y)) \leq Ld(x, y)
\end{equation}
holds for all $x, y \in X$.
- **locally Lipschitz** if for every $x_0 \in X$ there exists neighborhood $U_0 \subset X$ of $x_0 \in X$ and a constant $L \geq 0$ such that (2.3) holds for all $x, y \in U_0$.
- **Lipschitz on bounded sets** if for every bounded subset $B \subset X$ there exists a constant $L \geq 0$ such that (2.3) holds for all $x, y \in B$.

**Remark 2.2.** While an arbitrary map $G : X \to Z$ between metric spaces which is Lipschitz on bounded sets is also locally Lipschitz, the converse is in general false. However, if $X$ has the Heine-Borel property (i.e. if “closed and bounded = compact”) then both notions are equivalent.
Moreover, the map $F : X \times P \to Z$ is said to be

- **Lipschitz in $x$** if for every $u \in P$ there exists a constant $L \geq 0$ such that
  \[
  d(F_u(x), F_u(y)) \leq Ld(x, y)
  \]
  (2.4)
  holds for all $x, y \in X$.

- **Lipschitz in $x$ uniformly in $u$** if there exists a constant $L \geq 0$ such that (2.4) holds for all $x, y \in X$ and all $u \in P$.

- **Lipschitz in $x$ uniformly on bounded sets of $P$** if for every bounded set $B \subset P$ there exists a constant $L \geq 0$ such that (2.4) holds for all $x, y \in X$ and all $u \in B$.

- **Lipschitz in $x$ locally uniformly in $u$** if for every $u_0 \in P$ there exists a neighborhood $U_0$ of $u_0 \in P$ and a constant $L \geq 0$ such that (2.4) holds for all $x, y \in X$ and all $u \in U_0$.

If in the above definitions the term “Lipschitz” is replaced by *locally Lipschitz* then for each $x_0 \in X$ there exists a neighborhood $U_0 \subset X$ of $x_0 \in X$ such that the above conditions are satisfied on $U_0$ instead of $X$. If “Lipschitz” is replaced by *contractive* then the constant $L \geq 0$ can be chosen to be less than 1. For instance, $F$ is called *contractive in $x$ uniformly on bounded sets of $P$* if for every bounded set $B \subset P$ there exists a constant $0 \leq C < 1$ such that

\[
  d(F_u(x), F_u(y)) \leq Cd(x, y)
  \]

is satisfied for all $x, y \in X$ and all $u \in B$.

For $Z = X$, we say that $G : X \to X$ is an *eventual contraction* if there exists a natural number $N \in \mathbb{N}$ such that $G^N : X \to X$ is a contraction. Consequently, $F : X \times P \to X$ is termed *eventual contraction in $x$ uniformly on bounded sets of $P$* if for every bounded set $B \subset P$ there exists a natural number $N \in \mathbb{N}$ and a constant $0 \leq C < 1$ such that

\[
  d(F_u^N(x), F_u^N(y)) \leq Cd(x, y)
  \]

holds for all $x, y \in X$ and all $u \in B$. Note that the constant $C$ and the power $N \in \mathbb{N}$ may depend on $B \subset P$.

Finally, $H : P \to Z$ is called *compact* if $H$ maps bounded sets of $P$ to relatively compact sets of $Z$. Recall that in a complete metric space $Z$ a set $S \subset Z$ is *relatively compact* (i.e. has compact closure) if and only if it features the $\varepsilon$-*net property*. This means that for all $\varepsilon > 0$ there exists a finite $\varepsilon$-*net*, i.e. there exists a natural number $N \in \mathbb{N}$ and finitely many $z_i \in S$, $i = 1, \ldots, N$ such that $S$ is covered by the union of the balls $B_\varepsilon(z_i)$, $i = 1, \ldots, N$. The $\varepsilon$-net property is also called *total boundedness* or *precompactness*, cf. [16, Prelim.] and [19, Thm. A.4]. Whenever the involved metric spaces are complete we will use these terms interchangeably. Of course, in any other case one has to distinguish clearly between relative compactness and total boundedness (precompactness).

**Warning.** Here and henceforth, boundedness of sets is always meant in the metric sense, i.e. $S \subset P$ is bounded if there exists $r \geq 0$ such that $d(x, y) \leq r$ for all $x, y \in S$. 

**Remark 2.3.** (a) Compact linear maps between normed vector spaces are always bounded and therefore continuous. However, non-linear maps which are compact are not necessarily continuous. Moreover, for linear maps between normed vector spaces the following properties are equivalent: (i) bounded sets are mapped to relatively compact ones. (ii) there exists a bounded neighborhood of the origin whose image is relatively compact. This equivalence clearly fails for non-linear maps.

(b) In metric spaces which carry an additional linear structure one has to be careful because: (i) boundedness in the metric sense is in general not equivalent to boundedness with respect to the underlying vector space topology. (ii) An equivalence similar to that mentioned in part (a) does not hold any longer as bounded neighborhoods of the origin may not exist. This discrepancy led even in the linear case to different notions of compact maps in Fréchet spaces, cf. [4].

After these preliminaries, we can state the main result of this section.

**Theorem 2.4 (Compactness Principle).** Let $X$ and $P$ be complete metric spaces and let $F : X \times P \to X$ satisfy the following conditions:

(a) $F : X \times P \to X$ is a contraction in $x$ uniformly on bounded sets of $P$.

(b) $F_x : P \to X$ is continuous for all $x \in X$.

(c) $F_x : P \to X$ is compact for all $x \in X$.

Then the well-defined fixed point map $\Phi : P \to X$ which assigns to each $u \in P$ the unique fixed point of $F_u(\cdot)$, i.e. $\Phi(u) = F(\Phi(u), u)$, is continuous and compact. If additionally

(d) $F : X \times P \to X$ is locally Lipschitz in $u$ locally uniformly in $x$ then $\Phi : P \to X$ is also locally Lipschitz.

Before losing ourselves and the reader in the $\varepsilon$-details of the proof of Theorem 2.4 it is worth to sketch its road map: Lemma 2.5 and the (Lipschitz) continuity of the fixed point map are standard results, cf. [22, Prop. 1.2]. In contrast, Lemma 2.6 and Corollary 2.7 comprise the essential ingredients for proving compactness of the fixed point map as they guarantee that in each iteration of the map $F_u$ the set of iterates $\{F^n_u(x_0) \mid u \in U\}$ is relatively compact if $U \subset P$ is bounded, where $x_0$ is an arbitrary, but fixed initial point. Thus, in the final proof of Theorem 2.4 we only have to “close the gap” between $\{F^n_u(x_0) \mid u \in U\}$ and $\{F^\infty_u(x_0) \mid u \in U\}$ which is obtained via the uniform estimate (2.6).

**Lemma 2.5.** Let $F : X \times P \to X$ be locally Lipschitz in $x$ locally uniformly in $u$.

(a) If $F$ satisfies property (b) of Theorem 2.4 then it is continuous on $X \times P$.

(b) If $F$ satisfies property (d) of Theorem 2.4 then it is locally Lipschitz on $X \times P$.

The straightforward proof is left to the reader.

**Lemma 2.6.** Let $X$ and $P$ be complete metric spaces. Moreover, let $K \subset X$ be relatively compact and $B \subset P$ be bounded. If $F : X \times P \to X$ is Lipschitz in $x$ uniformly on bounded sets of $P$ and satisfies property (c) of Theorem 2.4 then $F(K \times B)$ is also relatively compact.

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9In a topological vector space $T$ a subset $B \subset T$ is called (topologically) bounded if for every neighborhood $U_0$ of the origin there exits a scalar $r > 0$ such that $B \subset rU_0$ holds. If the topology arises from a single norm then both concepts of boundedness coincide. If not – for instance in “proper” Fréchet spaces – the two notions differ significantly.
Proof. Let $B \subset P$ be bounded and $\varepsilon > 0$. Then, by assumption, there exists a constant $L > 0$ such that

\begin{equation}
(2.5) \quad d(F_u(x), F_u(y)) \leq Ld(x, y)
\end{equation}

holds for all $x, y \in X$ and all $u \in B$. Moreover, as $K$ is relatively compact we can choose a finite $\frac{L}{2\varepsilon}$-net $x_i, i = 1, \ldots, N$ of $K$. Then due to condition (c) the images $F(x_i, B), i = 1, \ldots, N$ and thus their union are relatively compact. Hence one can find a finite $\frac{2}{\varepsilon}$-net $y_j, j = 1, \ldots, M$ of $\bigcup_{i=1}^N F(x_i, B)$. Now for arbitrary $x \in K$ and $u \in B$ one can choose $x_i$ and $y_j$ such that $x \in B_{\varepsilon/2L}(x_i)$ and $F(x_i, u) \in B_{\varepsilon/2}(y_j)$. Consequently, (2.5) implies

$$d(F(x, u), y_j) \leq d(F(x, u), F(x_i, u)) + d(F(x_i, u), y_j) < Ld(x, x_i) + \frac{\varepsilon}{2} < \varepsilon$$

i.e. $y_1, \ldots, y_M$ yields a finite $\varepsilon$-net of $F(K \times B)$. Hence $F(K \times B)$ is totally bounded and thus relatively compact.

**Corollary 2.7.** Let $X$ and $P$ be complete metric spaces and let $B \subset P$ be bounded. Moreover, for $x_0 \in X$ define

$$W_0(x_0) := \{x_0\} \quad \text{and} \quad W_{n+1}(x_0) := F(W_n \times B).$$

If $F : X \times P \to X$ is Lipschitz in $x$ uniformly on bounded sets of $P$ and satisfies property (c) of Theorem 2.4, then $W_n(x_0)$ is relatively compact for all $n \in \mathbb{N}_0$.

**Proof.** Follows immediately from Lemma 2.6.

**Proof of Theorem 2.4.** It is well known, cf. e.g. [22, Prop. 1.2], that under conditions (a) and (b) the fixed point map $u \mapsto \Phi(u)$ is well-defined (due to the contraction principle) and continuous. For the sake of completeness we present the simple arguments in the following:

Let $u \in P$ be given and let $B := B_1(u)$ be a bounded neighborhood of $u$. Then by assumption there exists a uniform contraction rate $0 \leq C < 1$ for all $v \in B$. Hence one has

$$d(\Phi(v), \Phi(u)) = d(F(\Phi(v), v), F(\Phi(u), u))$$

$$\leq d(F(\Phi(v), v), F(\Phi(u), v)) + d(F(\Phi(u), v), F(\Phi(u), u))$$

$$\leq Cd(\Phi(v), \Phi(u)) + d(F(\Phi(u), v), F(\Phi(u), u))$$

and therefore

$$d(\Phi(v), \Phi(u)) \leq \frac{1}{1-C}d(F(\Phi(u), v), F(\Phi(u), u))$$

for all $v \in B$. Hence the continuity of the map $F_{\Phi(u)}$ shows that $\Phi$ is continuous at $u \in P$. If $F$ satisfies additionally condition (d) one finds a constant $L \geq 0$ and neighborhoods $U \subset X$ of $\Phi(u)$ and $V \subset P$ of $u \in P$ such that

$$d(F(x, v), F(x, w)) \leq Ld(v, w)$$

for all $x \in U$ and $v, w \in V$. Then the continuity of $\Phi$ at $u \in P$ implies that $W := V \cap \Phi^{-1}(U) \cap B$ is again a neighborhood of $u \in P$. Thus we conclude

$$d(\Phi(v), \Phi(w)) \leq \frac{1}{1-C}d(F(\Phi(v), v), F(\Phi(v), w)) \leq \frac{L}{1-C}d(v, w)$$
for all $v, w \in W$, i.e. $\Phi$ is locally Lipschitz.

Finally, let us show that $\Phi$ is also compact. To this end, let $B \subset P$ be bounded and $\varepsilon > 0$. From the standard proof of the contraction principle, cf. e.g. [22, Chap. 1], we know that $\Phi(u)$ is given by

$$\Phi(u) = \lim_{n \to \infty} F_u^n(x_0),$$

where $x_0 \in X$ can be an arbitrary initial point. Moreover, again from the contraction principle, one has the uniform estimate

$$d(F_u^n(x_0), F_u^m(x_0)) \leq \sum_{k=m}^{n-1} C^k d(F_u(x_0), x_0) \leq \frac{C^m}{1-C} d(F_u(x_0), x_0)$$

(2.6)

for all $u \in B$ and all $n > m$. Because $F(x_0, B) \subset X$ is relatively compact by assumption, there exists a constant $K \geq 0$ such that $d(F_u(x_0), x_0) \leq K$ for all $u \in B$. This implies that there exists $N \in \mathbb{N}$ such that

$$d(\Phi(u), F_u^n(x_0)) \leq \frac{KC^n}{1-C} < \frac{\varepsilon}{2}$$

for all $u \in B$ and all $n \geq N$. Furthermore, by Corollary 2.7 we conclude that \{\$F_u^n(x_0) : u \in B\}$ is relatively compact. Hence there exists a finite $\varepsilon$-net $y_1, \ldots, y_M$ of \{\$F_u^n(x_0) : u \in B\}$ and thus for all $u \in B$ we can find $y_j$ such that $d(F_u^n(x_0), y_j) < \frac{\varepsilon}{2}$. This yields the estimate

$$d(\Phi(u), y_j) \leq d(\Phi(u), F_u^n(x_0)) + d(F_u^n(x_0), y_j) < \varepsilon$$

i.e. $y_1, \ldots, y_M$ yields a finite $\varepsilon$-net of $\Phi(B)$. This completes the proof. 

**Remark 2.8.** Actually, there is a beautiful alternative to prove the above theorem, cf. [12]: Fix a bounded subset $B \subset P$ and consider the function space $C_{\text{tb}}(B, X) \subset C(B, X)$ of all continuous maps with totally bounded image. Then, under the conditions of Theorem 2.4, one can show that the map $T : \Psi \to T(\Psi)$ with $T(\Psi)(u) := F(\Psi(u), u)$ defines a contraction from $C_{\text{tb}}(B, X)$ into itself. Since $C_{\text{tb}}(B, X)$ is closed the map $T$ has a unique fixed point $\Psi_* \in C_{\text{tb}}(B, X)$ which which satisfies $F(\Psi_*(u), u) = \Psi_*(u)$ for all $u \in B$. Hence $\Psi_*$ coincides with the restriction $\Phi|_B$ of the fixed point map. Thus the image of $\Phi$ restricted to $B$ is totally bounded (\$\subset$ relatively compact).

The following two corollaries of Theorem 2.4 are particularly useful for applications in the area of ODEs and PDEs where the corresponding integral operators sometimes reveal their contractive behavior only after several iterations or, alternatively, after passing to a strongly equivalent metric, cf. Subsection 3.2 and 3.1, respectively.

**Corollary 2.9.** Let $X$ and $P$ be complete metric spaces and let $F : X \times P \to X$ satisfy the following conditions:

(a) $F : X \times P \to X$ is Lipschitz in $x$ uniformly on bounded sets of $P$.

(b) $F : X \times P \to X$ is an eventual contraction in $x$ uniformly on bounded sets of $P$.

(c) $F_x : P \to X$ is continuous for all $x \in X$. 

(d) \( F_x : P \to X \) is compact for all \( x \in X \).

Then the well-defined fixed point map \( \Phi : P \to X \) which assigns to each \( u \in P \) the unique fixed point of \( F_u(\cdot) \), i.e. \( \Phi(u) = F(\Phi(u), u) \), is continuous and compact. If additionally

(e) \( F : X \times P \to X \) is locally Lipschitz in \( u \) locally uniformly in \( x \)

then \( \Phi : P \to X \) is also locally Lipschitz in \( u \).

**Corollary 2.10.** Let \( X \) and \( P \) be complete metric spaces and let \( F : X \times P \to X \) satisfy the following conditions:

(a) For every bounded set \( B \subset P \) there exists a strongly equivalent metric \( d_B \) on \( X \) and a constant \( 0 \leq C < 1 \) such that

\[
d_B(F_u(x), F_u(y)) \leq Cd(x, y)
\]

for all \( x, y \in X \) and all \( u \in B \).

(b) \( F_x : P \to X \) is continuous for all \( x \in X \).

(c) \( F_x : P \to X \) is compact for all \( x \in X \).

Then the well-defined fixed point map \( \Phi : P \to X \) which assigns to each \( u \in P \) the unique fixed point of \( F_u(\cdot) \), i.e. \( \Phi(u) = F(\Phi(u), u) \), is continuous and compact. If additionally

(d) \( F : X \times P \to X \) is locally Lipschitz in \( u \) locally uniformly in \( x \)

then \( \Phi : P \to X \) is also locally Lipschitz in \( u \).

The following lemma serves to reduce the proof of Corollary 2.9 to that of Corollary 2.10. An almost identical construction can be found in [8], Chap. 3. Nevertheless, for the sake of self-containedness, we present the short proof.

**Lemma 2.11.** Let \( X \) be a metric space and let \( F : X \to X \) be given. Then the following two assertions are equivalent:

(a) \( F \) is Lipschitz and an eventual contraction.

(b) There exists a strongly equivalent metric \( d' \) on \( X \) such that \( F \) is a contraction with respect to \( d' \).

**Proof.** (a) \( \Rightarrow \) (b): Assume that \( F \) is an eventual contraction, i.e. there exists \( N \in \mathbb{N} \) and \( 0 \leq C < 1 \) such that

\[
d(F^n(x), F^n(y)) \leq Cd(x, y)
\]

for all \( x, y \in X \). This allows us to define \( d' \) as follows

\[
d'(x, y) := \sup_{n \in \mathbb{N}_0} \frac{d(F^n(x), F^n(y))}{C^n} < \infty.
\]

Obviously, \( d' \) yields a metric which satisfies the estimate \( d(x, y) \leq d'(x, y) \) and

\[
d'(F(x), F(y)) = \sup_{n \in \mathbb{N}_0} \frac{d(F^{n+1}(x), F^{n+1}(y))}{C^n} = \sup_{n \in \mathbb{N}_0} \frac{d(F^{n+1}(x), F^{n+1}(y))}{C^{n+1}} \leq C \frac{d'(x, y)}{C}.
\]
neighborhood of that $\Phi(2.10)$ with respect to the metric $d$.

Next, let us prove (local Lipschitz) continuity of $u(b)$.

\[ d'(x, y) = \sup_{n \in \mathbb{N}_0} \frac{d(F^n(x), F^n(y))}{C^n} = \max_{0 \leq n \leq N-1} \frac{d(F^n(x), F^n(y))}{C^n} \]

and so Lipschitz continuity of $F$ implies $d'(x, y) \leq M d(x, y)$ with

\[ M := \max_{0 \leq n \leq N-1} \frac{L^n}{C^n}, \]

where $L \geq 0$ is any Lipschitz constant for $F$.

(b) $\iff$ (a): Assume that there exists an strongly equivalent metric $d'$ such that $F$ is a contraction with contraction rate $0 \leq C' < 1$ with respect of $d'$. Hence

\[ d(F(x), F(y)) \leq M' d'(F(x), F(y)) \leq M' C' d'(x, y) \leq M M' C' d(x, y), \]

where $M$ and $M'$ are constants which satisfy (2.1). This shows that $F$ is Lipschitz continuous with $L := M M' C'$. Moreover, choosing $N \in \mathbb{N}$ such that the following estimate holds

\[ C := M M'(C')^N < 1 \]

one obtains

\[ d(F^N(x), F^N(y)) \leq M' d'(F^N(x), F^N(y)) \leq M'(C')^N d'(x, y) \leq M M'(C')^N d(x, y) = C d(x, y), \]

for all $x, y \in X$, i.e. $F$ is an eventual contraction.

Proof of Corollary 2.10. First, we briefly demonstrate that the fixed point map $\Phi : P \rightarrow X$ is well-defined. To this end, choose an arbitrary $u \in P$. Since $\{u\}$ is obviously a bounded subset of $P$ there, by assumption, exists a strongly equivalent metric $d_u$ such that $F_u : X \rightarrow X$ is a contraction. Thus the contraction principle implies that $F_u$ has a unique fixed point $\Phi(u) \in X$. Note that the strong equivalence of $d$ and $d_u$ guarantees that $X$ is complete with respect $d_u$.

Next, let us prove (local Lipschitz) continuity of $u \mapsto \Phi(u)$. Therefore, let $u \in P$ and choose a bounded and closed\(^{10}\) neighborhood of $u \in P$, e.g. $B := K_1(u)$. Since (local Lipschitz) continuity is preserved when passing to a strongly equivalent metric it suffices to show the desired result for the metric $d_B$. Now, the restriction $F|_{X \times B}$ satisfies readily the assumptions of Theorem 2.4 (with respect to the metric $d_B$). Thus we can conclude that $\Phi$ is continuous at $u \in P$, and—under assumption (d)—that $\Phi$ is even locally Lipschitz.

Finally, let $B \subset P$ be an arbitrary bounded subset of $P$. We have to show that $\Phi(B)$ is relatively compact in $X$. Due to assumption (b) we can choose a strongly equivalent metric $d_B$ on $X$. Again it suffices to prove that $\Phi(B)$ is relatively compact with respect to $d_B$. Hence, by the same argument as above, we may consider the restriction $F|_{X \times B}$ instead of $F$, and infer from Theorem 2.4 that $\Phi(B)$ is relatively compact.

\(^{10}\)Later $B$ will play the role of $P$ (cf. Theorem 2.4) and therefore we have to guarantee that $B$ constitutes a complete metric space.
Remark 2.12. One might conjecture that assumption (a) in Corollary 2.9 can be dropped as assumption (b) already guarantees a unique fixed point of $F_u(\cdot)$. But that does not work in general, because (b) does not ensure continuity of the fixed point map as the following example shows: Let $X = P = \mathbb{R}$ and define $F : X \times P \to X$ by

$$F(x, u) := \begin{cases} u & \text{for } x \in \mathbb{Q}, \\ -u & \text{for } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then $F_x = F(x, \cdot)$ is continuous and $F$ is an eventual contraction in $x$ uniformly in $u$ as one has

$$F^2_u(x) = F(F(x, u), u) = \begin{cases} u & \text{for } u \in \mathbb{Q} \\ -u & \text{for } u \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Hence

$$\Phi(u) = \begin{cases} u & \text{for } u \in \mathbb{Q}, \\ -u & \text{for } u \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

is obviously not continuous.

The final result of this section handles a scenario in which one is interested in the compactness of some composition $G \circ \Phi : P \to Z$ instead of the compactness of the fixed point map $\Phi : P \to X$ itself. The Ball-Marsden-Slemrod conjecture for $p = 1$ falls in this regime and will be tackled with the help of the following corollary in Subsection 3.2.

**Corollary 2.13.** Let $X$, $Z$ and $P$ be complete metric spaces and let $F : X \times P \to X$ and $G : X \to Z$ satisfy the following conditions:

(a) $F : X \times P \to X$ is a contraction in $x$ uniformly on bounded sets of $P$.

(b) $F_x : P \to X$ is continuous for all $x \in X$.

(c) $G : X \to Z$ is Lipschitz continuous.

(d) For $S \subset X$ and $B \subset P$ the following implication holds:

$$G(S) \text{ relatively compact and } B \text{ bounded } \implies G(F(S \times B)) \text{ relatively compact}$$

Then the well-defined fixed point map $\Phi : P \to X$ which assigns to each $u \in P$ the unique fixed point of $F_u(\cdot)$, i.e. $\Phi(u) = F(\Phi(u), u)$, and the composition $G \circ \Phi$ are continuous. Moreover, $G \circ \Phi$ is compact.

**Proof.** Obviously it suffices to focus on the compactness of $\Phi$ as continuity follows directly from Theorem 2.4 and thus continuity of $G \circ \Phi$ is evident.

To this end we can proceed as in the proof of Theorem 2.4. Let $B \subset P$ be bounded and $\varepsilon > 0$. We can choose $N \in \mathbb{N}$ such that

$$d(\Phi(u), F^n_u(x_0)) \leq \frac{KC^n}{1-C} \leq \frac{\varepsilon}{2L}$$

for all $u \in B$ and all $n \geq N$, where $L > 0$ denotes a Lipschitz constant of $G$. Moreover, assumption (d) implies that all elements of the recursively defined sequence

$$W_0(x_0) := \{x_0\} \quad \text{and} \quad W_{n+1}(x_0) := F(W_n \times B).$$
become relatively compact when applying \( G \). Therefore, the set \( \{G(F_u^N(x_0)) : u \in B\} \subset G(W_N(x_0)) \) is also relatively compact and thus admits a finite \( \frac{1}{2} \)-net \( z_1, \ldots, z_M \). Hence, for all \( u \in B \) we can find \( z_i \) such that \( d(G(F_u^N(x_0)), z_i) < \frac{\varepsilon}{2} \) and consequently one has
\[
d(G(\Phi(u)), z_i) \leq d(G(\Phi(u)), G(F_u^N(x_0))) + d(G(F_u^N(x_0)), z_i) < Ld(\Phi(u), F_u^N(x_0)) + \frac{\varepsilon}{2} < \varepsilon,
\]
i.e. \( G(\Phi(B)) \) is relatively compact as \( z_1, \ldots, z_M \) constitutes a finite \( \varepsilon \)-net. \( \square \)

Obviously, one can combine Corollary 2.9 or 2.10 with Corollary 2.13; yet it seems to be superfluous to list these straightforward modifications here.

3. Applications and the Ball-Marsden-Slemrod Conjecture. Next, we want to apply the previous findings (i) to provide a simplified approach to the non-controllability result by Ball, Marsden, and Slemrod for \( p > 1 \) and (ii) to derive a proof of the Ball-Marsden-Slemrod conjecture for \( p = 1 \), cf. Subsection 3.1 and Subsection 3.2, respectively. In doing so, we generalize earlier results in [2] and [5, 6, 7] from bilinear systems to semi-linear ones by allowing non-linear vector fields \( f \), which are Lipschitz on bounded sets and linearly bounded. A first result in the same direction can be found in [9].

Before going into the details, a few clues are advisable to “maneuver” the reader through the following two subsections: While the approach of Subsection 3.1 is completely based on Corollary 2.10, the situation in Subsection 3.2 is bit more subtle. For proving the existence of a unique fixed point map we will apply Corollary 3.8 which is revealed in the non-compactness of the fixed point map \( \Phi : u \mapsto x(\cdot, \xi_0, u) \) for \( p = 1 \), cf. Appendix A. Here, Corollary 2.13 comes into play and allows to prove relative compactness of \( R^3(\xi_0) \) directly, i.e. without Lemma 3.4, see also Remark 3.12.

Throughout Section 3, we assume that the infinitesimal generator \( A : D(A) \to X \) is of class \((M, \mu)\) with \( M \geq 1, \mu \geq 0 \), i.e. for all \( t \geq 0 \) one has the estimate
\[
\|e^{tA}\| \leq Me^{\mu t}.
\]
Note that every infinitesimal generator of a \( C^0 \)-semigroup is of some class \((M', \mu')\) with \( M' \geq 1, \mu' \geq 0 \) [17, Chap. 1, Thm. 2.2]. Hence requiring \( A \) to be of class \((M, \mu)\) serves only to fix the constants \( M \geq 1 \) and \( \mu \geq 0 \) but does not pose an actual restriction.

Moreover, we say that a vector field \( f : \mathbb{R}^+ \times X \to X \) satisfies Assumptions (A1) and (A2), respectively, if the following conditions are fulfilled.

Assumption A1: \( f : \mathbb{R}^+ \times X \to X \) is Lipschitz in \( \xi \) on bounded sets with \( L^\infty \)-Lipschitz rate, i.e. for every bounded set \( C \subset X \) there exists \( L \in L^\infty_{loc}(\mathbb{R}^+_0; \mathbb{R}^+_{0}) \) such

\( \text{In principle, by Lemma 2.11 both approaches are equivalent. Yet, the equivalent norm constructed in Lemma 2.11 cannot be expressed explicitly as an } \omega \text{-norm. In that sense the second approach is stronger than the first one (see also Remark 3.14).} \)
that

\[ \| f(t, \xi) - f(t, \eta) \| \leq L(t) \| \xi - \eta \| \]

for all \( \xi, \eta \in C \) and all \( t \in \mathbb{R}^+ \).

**Assumption A2:** \( f : \mathbb{R}^+_0 \times X \to X \) is linearly bounded, i.e. there exist \( \alpha, \beta \in L^\infty(\mathbb{R}^+_0, \mathbb{R}^+_0) \) such that

\[ \| f(t, \xi) \| \leq \alpha(t) \| \xi \| + \beta(t) \]

for all \( \xi \in X \) and all \( t \in \mathbb{R}^+_0 \).

Finally, for \( T \geq 0 \) and continuous \( f_i : \mathbb{R}^+_0 \times X \to X, i = 1, \ldots, m \) we define the integral operator

\[ F : C([0, T], X) \times L^p([0, T], \mathbb{R}^m) \to C([0, T], X) \]

\[ F(x, u)(t) := e^{tA}\xi_0 + \sum_{i=1}^{m} \int_0^t e^{(t-s)A}u_i(s)f_i(s, x(s)) \, ds. \]

**3.1. The case \( p > 1 \).**

**Theorem 3.1.** Let \( X \) be a Banach space, let \( A \) be the infinitesimal generator of a \( C^0 \)-semigroup \( (e^{tA})_{t \geq 0} \) of bounded operators on \( X \), and let \( p > 1 \). Moreover, let \( f_i : \mathbb{R}^+_0 \times X \to X, i = 1, \ldots, m \) be continuous vector fields which satisfy Assumptions (A1) and (A2). Then for all \( T \geq 0 \), all \( \xi_0 \in X \), and all \( u \in L^p([0, T], \mathbb{R}^m) \) the equation

\[ \dot{x}(t) = Ax(t) + \sum_{i=1}^{m} u_i(s)f_i(t, x(t)), \quad x(0) = \xi_0 \]

has a unique mild solution and the solution operator

\[ \Phi : L^p([0, T], \mathbb{R}^m) \to C([0, T], X) \]

which assigns to each control \( u \in L^p([0, T], \mathbb{R}^m) \) the unique mild solution of (3.2) is compact and locally Lipschitz continuous.

**Remark 3.2.** \( (a) \) The assumption \( L_i \in L^\infty_{loc}(\mathbb{R}^+_0, \mathbb{R}^+_0) \) in Theorem 3.1, cf. Assumption (A1), can be readily improved by \( L_i \in L^{q'}_{loc}(\mathbb{R}^+_0, \mathbb{R}) \) with \( q' > q \), where \( q \) denotes the conjugate exponent to \( p \), i.e. \( \frac{1}{p} + \frac{1}{q} = 1 \). The case \( q' = q \) is more delicate, because here the proof of Lemma 3.9 below will break down. Actually, the case \( q' = q \) resembles case \( p = 1 \) of Subsection 3.2 and therefore the technique used in Lemma 3.13 can be successfully employed to prove at least existence and uniqueness of solutions. A similar comment applies to \( \alpha_i, \beta_i \in L^\infty_{loc}(\mathbb{R}^+_0, \mathbb{R}^+_0) \), cf. Assumption (A2).

\( (b) \) Moreover, we expect that the continuity assumption on \( f_i \) can be weakened (with respect to \( t \)) in the sense of Carathéodory’s theorem, cf. e.g. [11, 20].

\( (c) \) Conditions which guarantee that mild solutions are actually classical or Carathéodory solutions can be found in [17, Sec. 6.1].
(d) The existence and uniqueness result of Ball, Marsden, and Slemrod given in [2, Thm. 2.5] is stronger than that in Theorem 3.1 because there Lipschitz continuity of the non-linear part is only required locally and not on bounded sets. However, their non-controllability result [2, Thm. 3.6] is considerably weaker than Corollary 3.3 below as it is restricted to bilinear systems, i.e. to bounded linear vector fields $f_i$. Related results also dealing with non-linear $f_i$ can be found in [9, 10].

As an immediate consequence of Theorem 3.1 and Lemma 3.4 below we obtain the following generalization of Ball-Marsden-Slemrod’s non-controllability statement.

**Corollary 3.3.** Let the notation and assumptions be as in Theorem 3.1. Moreover, assume that all $f_i$, $i = 1, \ldots, m$ are autonomous. Then for all $\xi_0 \in X$ and all $p > 1$ the reachable set $R^p(\xi_0)$ of

\[
\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} u_i(t) f_i(x(t)), \quad x(0) = \xi_0, \quad u_i \in L^p_{\mathrm{loc}}([0, \infty), \mathbb{R})
\]

can be written as a countable union of relatively compact sets and has therefore no interior points if $X$ is infinite-dimensional.

Here we assumed that the vector fields $f_i$ are autonomous since this is the standard setting in control theory. Of course, the result holds for time-dependent vector fields as well—but then $R^p(\xi_0)$ depends in general additionally on the initial time of (3.3) and would not be called reachable set of $\xi_0$.

**Proof of Corollary 3.3.** Theorem 3.1 and Lemma 3.4 below immediately imply that the reachable set up to time $T$ (under bounded controls $\|u\|_p \leq r$), i.e.

\[
R^p_{\leq T}(\xi_0) := \{x(t) : t \in [0, T], \|u\|_p \leq r\},
\]

is relatively compact. Moreover, the identity

\[
R^p(\xi_0) = \bigcup_{T \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} R^p_{\leq T}(\xi_0)
\]

shows that the reachable set $R^p(\xi_0)$ is a countable union of relatively compact sets. Finally, a straightforward application of Baire’s category theorem combined with fact that relatively compact sets in an infinite-dimensional Banach space are nowhere dense shows that $R^p(\xi_0)$ has no interior points.

**Lemma 3.4.** Let $S \subset C([0, T], X)$ be relatively compact. Then the evaluation set $\operatorname{Ev}(S) := \{x(t) : x \in S, t \in [0, T]\} \subset X$ is relatively compact, as well.

**Proof.** Let $\varepsilon > 0$. Since $S \subset C([0, T], X)$ is relatively compact by assumption, we can choose a finite $\frac{\varepsilon}{2}$-net $x_1, \ldots, x_N$ of $S$. Moreover, compactness of $[0, T]$ implies that the images of $x_i$, $i = 1, \ldots, N$ are also compact. Hence we can cover all $x_i([0, T])$ and their union by a finite $\frac{\varepsilon}{2}$-net $\xi_1, \ldots, \xi_M \in X$. Now, for $x \in S$ and $t \in [0, T]$ we can choose $x_i$ and $\xi_j$ such that

\[
\|x(t) - \xi_j\| \leq \|x(t) - x_i(t)\| + \|x_i(t) - \xi_j\| < \|x - x_i\|_\infty + \frac{\varepsilon}{2} < \varepsilon.
\]

This shows that $\xi_j$, $j = 1, \ldots, M$ yields a finite $\varepsilon$-net of $\operatorname{Ev}(S)$ and thus $\operatorname{Ev}(S)$ is relatively compact.
Remark 3.5. A minor modification of Lemma 3.7 in [2] to non-linear maps satisfying Assumption (A1) allows to show that the integral operator $F$ given by (IO) is weakly continuous in $u$ and thus the solution operator $u \rightarrow \Phi(u)$ will be weakly continuous, too. Hence for $p > 1$ the sets $R_{\leq T}^{p,r}(\xi_0)$ are even compact and not only relatively compact.

In order to prove Theorem 3.1, we will show that the integral operator $F$ given by (IO) satisfies the conditions of Theorem 2.4 or, more precisely, of Corollary 2.10. To this we define on $C([0,T], X)$ the weighted $\omega$-norms

$$\|x\|_\omega := \max_{t \in [0,T]} e^{-\omega t}\|x(t)\|.$$ \hfill (3.4)

Obviously, $\| \cdot \|_0$ coincides with the standard maximum norm on $C([0,T], X)$ and all $\omega$-norms are equivalent on $C([0,T], X)$. To avoid confusion, whenever necessary, we will write $C([0,T], X)_\omega$ to indicate that a particular statement about $C([0,T], X)$ holds (only) with respect to a particular $\omega$-norm.

**Note:** For simplicity, we will prove the following auxiliary results for the case $m = 1$. This is justified because the case $m > 1$ can be handled completely analogously.

**Lemma 3.6.** Let $f_i, i = 1, \ldots, m$ be continuous and $p \geq 1$. Then the operator $F : C([0,T], X) \times L^p([0,T], \mathbb{R}^m) \rightarrow C([0,T], X)$ as given in (IO) is globally Lipschitz in $u$ locally uniformly in $x$.

**Proof.** Global Lipschitz continuity of $F_x$ readily follows from the estimate

$$\|F_x(v) - F_x(u)\|_\infty = \max_{t \in [0,T]} \left| \int_0^t e^{(t-s)}A(v(s) - u(s))(f(s, x(s))\, ds \right|$$

$$\leq \max_{t \in [0,T]} \int_0^t Me^{\mu(t-s)}|v(s) - u(s)|\|f(s, x(s))\|\, ds$$

$$\leq KME^{\mu T} \int_0^T |v(s) - u(s)|\, ds$$

\hfill (3.4)

where $K \geq 0$ is defined by $K := \max_{t \in [0,T]} \|f(t, x(t))\|$ and $T^{1/q}$ results from Hölder’s inequality (with $1/p + 1/q = 1$).

To see that the above estimate can be made locally uniformly in $x$ one can proceed as follows: Due to continuity of $f$ there exist $\delta_t > 0$ and $\varepsilon_t > 0$ such that

$$\|f(s, \eta) - f(t, x(t))\| \leq 1 \quad \text{for all } s \in (t - \delta_t, t + \delta_t) \text{ and all } \eta \in B_{\varepsilon_t}(x(t)).$$

Now the collection of all open balls $B_{\varepsilon_t/2}(x(t))$, $t \in [0,T]$ yields an open cover of the compact set $x([0,T])$ and consequently there exists a finite subcover

$$B_{\varepsilon_{t_1}/2}(x(t_1)), \ldots, B_{\varepsilon_{t_N}/2}(x(t_N)) \quad \text{with } \varepsilon_i := \varepsilon_{t_i}.$$ \hfill (**)

Set $\varepsilon := \min\{\varepsilon_1, \ldots, \varepsilon_N\}$. Then for all $t \in [0,T]$ one can choose $t_i \in [0,T]$ such that $\|x(t) - x(t_i)\| < \varepsilon_i/2$ and hence for all $y \in B_{\varepsilon/2}(x) \subset C([0,T], X)$ one has

$$\|y(t) - x(t_i)\| \leq \|y(t) - x(t)\| + \|x(t) - x(t_i)\| < \varepsilon.$$
This implies
\[ \| f(t, y(t)) \| \leq \| f(t, y(t)) - f(t, x(t)) \| + \| f(t, x(t)) \| \leq 1 + K \]
and thus
\[ \sup_{y \in B_{\epsilon/2}(x)} \max_{|t| \leq T} \| f(t, y(t)) \| \leq 1 + K. \]

It follows that estimate (3.4) is even locally uniformly in \( x \) once \( K \) is replaced by \( K + 1 \).

Remark 3.7. One can significantly simplify the second part of the above proof by assuming that all \( f_i, i = 1, \ldots, m \) satisfy an additional Lipschitz condition in \( \xi \) as in Theorem 3.1.

Lemma 3.8. Let \( f_i, i = 1, \ldots, m \) be continuous and \( p > 1 \). Then for all \( x \in C([0, T], X) \) the operator \( F_x := F(x, \cdot) : L^p([0, T], \mathbb{R}^m) \to C([0, T], X) \) is compact.

Proof. Without loss of generality let \( B := K_1(0) \) be the closed unit ball of \( L^p([0, T], \mathbb{R}) \) and let \( \varepsilon > 0 \). We have to find a finite \( \varepsilon \)-net for the image \( F_x(B) \subset C([0, T], X) \). To this end, we consider the set \( K := \{ f(t, x(t)) : t \in [0, T] \} \) which is obviously compact as \( x(\cdot) \) and \( f \) are continuous. By Lemma B.1 one can choose disjoint Borel sets \( \Delta_i \subset [0, T], i = 1, \ldots, N \) and \( S_j \subset K, j = 1, \ldots, M \) as well as \( \xi_{ij} \in X \) such that the uniform approximation
\[ \| e^{tA} \xi - \sum_{i=1}^{N} \sum_{j=1}^{M} \chi_{\Delta_i}(t) \chi_{S_j}(\xi) \xi_{ij} \| \leq \frac{\varepsilon}{2T^{1/p}} \]
holds for all \( t \in [0, T] \) and all \( \xi \in K \). It follows
\[ \max_{t \in [0, T]} \left\| \int_{0}^{t} u(s) \sum_{i=1}^{N} \sum_{j=1}^{M} \chi_{\Delta_i}(t-s) \chi_{S_j}(f(s, x(s))) \xi_{ij} \, ds \right\| \]
\[ \leq \max_{t \in [0, T]} \int_{0}^{t} \left\| u(s) \right\| e^{(t-s)A} f(s, x(s)) - \sum_{i=1}^{N} \sum_{j=1}^{M} \chi_{\Delta_i}(t-s) \chi_{S_j}(f(s, x(s))) \xi_{ij} \right\| \, ds \]
\[ \leq \frac{\varepsilon}{2T^{1/p}} \int_{0}^{T} |u(s)| \, ds \leq \frac{\varepsilon \|u\|_p}{2} \leq \frac{\varepsilon}{2}, \]
for all \( u \in K_1(0) \), where we used Hölder’s inequality in the second-to-last step. Thus it suffices to show that the set
\[ \mathcal{S} := \left\{ \int_{0}^{t} u(s) \sum_{i=1}^{N} \sum_{j=1}^{M} \chi_{\Delta_i}(t-s) \chi_{S_j}(f(s, x(s))) \xi_{ij} \, ds : u \in B \right\} \subset C([0, T], X) \]
has a finite \( \varepsilon \)-net \( y_1, \ldots, y_N \) because then the previous estimates guarantee that \( y_1, \ldots, y_N \) translated by \( y_0 : t \mapsto e^{tA} \xi_0 \) yields an \( \varepsilon \)-net of \( F_x(B) \). Note that integrability of \( s \mapsto \sum_{i=1}^{N} \sum_{j=1}^{M} \chi_{\Delta_i}(t-s) \chi_{S_j}(f(s, x(s))) \xi_{ij} \) follows readily from its measurability and essential boundedness. Since \( \mathcal{S} \) is obviously bounded and all its functions take their values in the finite-dimensional subspace \( X_{\varepsilon} := \text{span}\{ \xi_{ij} : i = 1, \ldots, N, j = 1, \ldots, M \} \) we can invoke Arzelà–Ascoli’s theorem [18, Thm. 7.25] to
prove relative compactness of $S$, that is, we have to show that $S$ is equicontinuous. To this end we consider
\[
\left\| \int_0^{t+\delta} u(s) \sum_{i=1}^N \sum_{j=1}^M \chi_{\Delta_i}(t-s)\chi_{S_j}(f(s,x(s)))\xi_{ij} \, ds \right\|
\]
\[
- \int_0^{t} u(s) \sum_{i=1}^N \sum_{j=1}^M \chi_{\Delta_i}(t-s)\chi_{S_j}(f(s,x(s)))\xi_{ij} \, ds \right\|
\]
\[
= \left\| \int_t^{t+\delta} u(s) \sum_{i=1}^N \sum_{j=1}^M \chi_{\Delta_i}(t-s)\chi_{S_j}(f(s,x(s)))\xi_{ij} \, ds \right\| \leq \max_{i=1,\ldots,N} \| \xi_{ij} \| \int_t^{t+\delta} |u(s)| \, ds
\]
where the second-to-last estimate follows again from Hölder’s inequality. This clearly shows equicontinuity of $S$ and concludes the proof. 

**Lemma 3.9.** Let $f_j$, $i = 1, \ldots, m$ be globally Lipschitz in $\xi$ with $L_\infty^{\omega}$-Lipschitz rate and let $p > 1$. Then there exists $\omega \geq 0$ such that 
\[
F : C([0,T],X) \rightarrow C([0,T],X,\omega)
\]
is a contraction in $x$ uniformly on bounded sets of $L^p([0,T],\mathbb{R})$.

**Proof.** For simplicity, let $B$ be the unit ball of $L^p([0,T],\mathbb{R})$. Then for $x, y \in C([0,T],X)$, $u \in B$, and $\omega \geq 0$ one has the estimate
\[
\left\| F(x,u) - F(y,u) \right\|_\omega = \max_{t \in [0,T]} e^{-\omega t} \left\| \int_0^t e^{(t-s)A}u(s)(f(s,x(s)) - f(s,y(s))) \, ds \right\|
\]
\[
\leq \max_{t \in [0,T]} e^{-\omega t} \int_0^t |u(s)| \left\| e^{(t-s)A} \right\| \|f(s,x(s)) - f(s,y(s))\| \, ds
\]
\[
\leq \max_{t \in [0,T]} e^{-\omega t} \int_0^t M e^{\omega(t-s)} e^{\omega s} |u(s)| \|L(s) e^{-\omega s} \| x(s) - y(s) \| \, ds
\]
\[
\leq \max_{t \in [0,T]} M \|L\|_\infty e^{(\omega-\mu)t} \int_0^t e^{(\omega-\mu)s} |u(s)| \, ds \| x(s) - y(s) \| \omega
\]
\[
\leq \max_{t \in [0,T]} M \|L\|_\infty e^{(\omega-\mu)t} \left( \int_0^t e^{q(\omega-\mu)s} \, ds \right)^{1/q} \| x(s) - y(s) \| \omega
\]
where $\|L\|_\infty$ denotes the $L^\infty$-norm of $L$ on $[0,T]$ and the second-to-last estimate exploits again Hölder’s inequality. Finally, for $\omega > \mu$ we obtain
\[
\left\| F(x,u) - F(y,u) \right\|_\omega \leq \max_{t \in [0,T]} M \|L\|_\infty e^{(\omega-\mu)t} \left( \frac{e^{q(\omega-\mu)t} - 1}{q(\omega-\mu)} \right)^{1/q} \| x(s) - y(s) \| \omega
\]
\[
\leq \frac{M \|L\|_\infty}{(q(\omega-\mu))^{1/q}} \| x(s) - y(s) \| \omega.
\]
Thus, for $\omega > 0$ sufficiently large, we can achieve $\frac{M \|L\|_\infty}{(q(\omega-\mu))^{1/q}} < 1$. This completes the proof.


The following technical result provides the justification that, under Assumptions (A1) and (A2), one can actually assume that all $f_i$ are globally Lipschitz in $\xi$ which in turn allows to apply Lemma 3.9.

**Lemma 3.10.** Let $f : \mathbb{R}^+_0 \times X \to X$ satisfy Assumptions (A1) and (A2) and let $p \geq 1$. Then for every bounded set $B \subset L^p([0,T], \mathbb{R})$ and every $\xi_0 \in X$ there exists $\hat{f} : \mathbb{R}^+_0 \times X \to X$ which is globally Lipschitz in $\xi$ with $L^\infty_{L^p}$-Lipschitz rate such that every mild solution of

$$
\dot{x}(t) = Ax(t) + u(s)f(t, x(t)) \quad x(0) = \xi_0, \quad u \in B
$$
on 0, T] is a mild solution of

$$
\dot{x}(t) = Ax(t) + u(s)\hat{f}(t, x(t)) \quad x(0) = \xi_0, \quad u \in B
$$
on 0, T] and vice versa.

**Proof.** For $n \in \mathbb{N}$ let $\rho_n : X \to [0,1]$ be a globally Lipschitz cut-off function which is equal to 1 for $\|\eta\| \leq n$ and zero for $\|\eta\| \geq n + 1$, for instance

$$
\rho_n(\eta) := \begin{cases} 
1 & \text{for } \|\eta\| \leq n, \\
0 & \text{for } \|\eta\| \geq n + 1, \\
n + 1 - \|x\| & \text{for } n \leq \|\eta\| \leq n + 1.
\end{cases}
$$

The idea is, as $t \geq 0$ is restricted to the compact interval $[0,T]$, to provide a priori estimates (via Gronwall’s lemma) for the solutions of (3.5) and (3.6) such that $\hat{f}$ can be defined as $\hat{f}(t, \eta) := \rho_n(\eta - \xi_0)f(t, \eta)$ for some appropriate $n \in \mathbb{N}$. In other words we want to choose $n \in \mathbb{N}$ such that the difference between $f$ and $\hat{f}$ occurs outside of an appropriate ball $B_n(\xi_0)$ which contains all mild solutions of the initial value problems (3.5) and (3.6).

To this end we assume that $x : [0,T] \to X$ is a mild solution of (3.5) and define $\varphi(t) := e^{-\mu t}\|x(t) - e^{tA}\xi_0\|$. Then one has

$$
\varphi(t) = e^{-\mu t}\left|\int_0^t e^{-(s-t)A}u(s)f\{s, x(s)\}\, ds\right| \leq M\int_0^t e^{-\mu s}\|u(s)\|\left(\|\alpha(s)\|\|x(s)\| + \beta(s)\right)\, ds
$$

$$
\leq M\int_0^t |u(s)|\left(\|\alpha(s)e^{-\mu s}\|\|x(s) - e^{sA}\xi_0 + e^{sA}\xi_0\| + e^{-\mu s}\|\beta(s)\|\right)\, ds
$$

$$
\leq M\int_0^t |u(s)|\left(\|\alpha(s)\|\|\varphi(s) + \alpha(s)e^{-\mu s}\|\|e^{sA}\xi_0\| + e^{-\mu s}\|\beta(s)\|\right)\, ds
$$

$$
\leq M\int_0^t \alpha(s)|u(s)|\varphi(s)\, ds + M\int_0^t |u(s)|\left(\|\alpha(s)e^{-\mu s}\|\|e^{sA}\xi_0\| + e^{-\mu s}\|\beta(s)\|\right)\, ds
$$

$$
\leq M\alpha_\infty\int_0^t |u(s)|\varphi(s)\, ds + MC_p\|u\|_p(M\|\alpha\|_\infty\|\xi_0\| + \|\beta\|_\infty),
$$

where the constant $C_p := T(p-1)/p$ results from Hölder’s inequality. A straightforward application of Gronwall’s lemma [20, App. C] yields

$$
\varphi(t) \leq MC_p\|u\|_p(M\|\alpha\|_\infty\|\xi_0\| + \|b\|_\infty)e^{M\|\alpha\|_\infty\int_0^t |u(s)|\, ds}
$$
and thus
\[
\|x(t) - \xi_0\| \leq e^{\mu t} \varphi(t) + \|e^{\lambda t} \xi_0 - \xi_0\|
\]
\[
\leq M e^{\mu t} C_p \|\varphi\|_p (M \|\alpha\|_\infty \|\xi_0\| + \|\beta\|_\infty) e^{M \|\alpha\|_\infty \int_0^t |u(s)| \, ds} + (M e^{\mu t} + 1) \|\xi_0\|
\]
\[
\leq M e^{\mu t} \left(C_p K (M \|\alpha\|_\infty \|\xi_0\| + \|\beta\|_\infty) e^{M \|\alpha\|_\infty C_p K} + 2 \|\xi_0\|\right) =: R,
\]
where the last estimate exploits the assumption that \( B \subset L^p([0, T], \mathbb{R}) \) is bounded, i.e. \( \|u\|_p \leq K \) for some constant \( K \geq 0 \). Next let us choose \( N \in \mathbb{N} \) with \( N > R \) and define \( \hat{f}(t, \eta) := \rho_N(\eta - \xi_0) f(t, \eta) \). Then, by the above estimate, \( x(t) \) is also a mild solution of (3.6). Conversely, since \( \hat{f} \) also satisfies (3.1) we obtain the same a priori estimate for any solution \( \hat{x}(t) \) of (3.6), and thus \( \hat{x}(t) \) is also a solution of (3.5).

Finally, global Lipschitz continuity of \( \hat{f} \) results from the computation
\[
\|\hat{f}(t, \eta) - \hat{f}(t, \zeta)\|
\]
\[
\leq |\rho_N(\eta - \xi_0)| \|f(t, \eta) - f(t, \zeta)\| + |\rho_N(\eta - \xi_0) - \rho_N(\zeta - \xi_0)| \|f(t, \zeta)\|
\]
\[
\leq |\rho_N(\eta - \xi_0)| L(t) \|\eta - \zeta\| + |\rho_N(\eta - \xi_0) - \rho_N(\zeta - \xi_0)| (\alpha(t) \|\zeta\| + \beta(t))
\]
\[
\leq |\rho_N(\eta - \xi_0)| L(t) \|\eta - \zeta\| + (\alpha(t) \|\zeta\| + \beta(t)) \|\eta - \zeta\|
\]
\[
\leq \begin{cases}
0 & \|\eta - \xi_0\|, \|\zeta - \xi_0\| \geq N + 1, \\
(\alpha(t) (N + 1 + \|\xi_0\|) + \beta(t)) \|\eta - \zeta\| & \|\eta - \xi_0\|, \|\zeta - \xi_0\| < N + 1, \\
(\alpha(t) (N + 1 + \|\xi_0\|) + \beta(t)) \|\eta - \zeta\| & \|\zeta - \xi_0\| < N + 1 \leq \|\eta - \xi_0\|,
\end{cases}
\]
where we used the fact that \( \rho_N \) has Lipschitz rate \( L = 1 \). Interchanging the role of \( \eta \) and \( \zeta \) yields the corresponding estimate for \( \|\eta - \xi_0\| < N + 1 \leq \|\zeta - \xi_0\| \).

*Proof of Theorem 3.1.* Let \( T \geq 0 \) and \( p > 1 \) be fixed in the following. Obviously, if \( x : [0, T] \to X \) is a mild solution of (3.2) to the control \( u \in L^p([0, T], \mathbb{R}^m) \) it is a fixed point of the integral operator (10)
\[
F : C([0, T], X) \times L^p([0, T], \mathbb{R}^m) \to C([0, T], X)
\]
given by
\[
F(x, u)(t) = e^{tA} \xi_0 + \sum_{i=1}^m \int_0^t e^{(t-s)A} u_i(s) f_i(s, x(s)) \, ds.
\]
Therefore, our aim is to apply Corollary 2.10 to \( F \). To this end, we first restrict \( F \) to \( C([0, T], X) \times K_r(0) \), where \( K_r(0) \) denotes an arbitrary closed ball in \( L^p([0, T], \mathbb{R}^m) \). Consequently, due to Lemma 3.10 we can further assume that all \( f_i : \mathbb{R}^m_+ \times X \to X \) are globally Lipschitz in \( \xi \) with \( L^\infty_\text{loc} \)-Lipschitz rate.

Hence condition (a) of Corollary 2.10 follows from Lemma 3.9; conditions (b) and (d) are implied by Lemma 3.6 and condition (c) by Lemma 3.8. Thus the fixed point map \( \Phi : u \mapsto x(t, \xi, u) \) is compact and locally Lipschitz continuous on \( K_r(0) \). Since \( r > 0 \) can be chosen arbitrarily the desired result follows.
3.2. The case $p = 1$.

Theorem 3.11. Let $X$ be a Banach space and let $A$ be the infinitesimal generator of a $C^0$-semigroup $(e^{tA})_{t \geq 0}$ of bounded operators on $X$. Moreover, let $f_i : X \to X$, $i = 1, \ldots, m$ be continuous vector fields which satisfy Assumptions (A1) and (A2). Then for all $\xi_0 \in X$ the reachable set $R^1_{C,T}(\xi_0)$ of

$$
\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} u_i(t) f_i(t, x(t)), \quad x(0) = \xi_0, \quad u_i \in L^1_{\text{loc}}([0, \infty), \mathbb{R})
$$

can be written as a countable union of relatively compact sets and therefore has no interior points if $X$ is infinite-dimensional.

Remark 3.12. Note that Lemma 3.8 and Lemma 3.9—as used in the proof of Theorem 3.11—explicitly require $p > 1$. Thus we have to fill these gaps in order to tackle the case $p = 1$: Lemma 3.9 can be readily replaced by Lemma 3.13 below. However, Lemma 3.8 is more delicate—actually, it cannot be repaired because the integral operator (IO) is in general not compact for $p = 1$. A counter-example is provided in Appendix A. However, compactness of the fixed point map $\Phi : u \mapsto x(\cdot, \xi_0, u)$ is not necessary for relative compactness of the reachable set $R^1_{C,T}(\xi_0)$. Therefore, the goal is to prove relative compactness of $R^1_{C,T}(\xi_0)$ directly via Corollary 2.13. Our strategy is closely related to the approach presented in [6].

Lemma 3.13. Let $f_i : \mathbb{R}^+ \times X \to X$, $i = 1, \ldots, m$ be globally Lipschitz in $x$ with $L^\infty_{\text{loc}}$-Lipschitz rate. The iterated integral operator $F^n_u : C([0, T], X) \to C([0, T], X)$ given by (IO) satisfies the estimate

$$(3.7) \quad \|F^n_u(x)(t) - F^n_u(y)(t)\| \leq \frac{M^ne^{\mu nT}\|L\|^n_\infty\left(\int_0^t v(s) \, ds\right)^n\|x - y\|_\infty}{n!}$$

with $\|L\| := \max_{i=1,\ldots,m} \|L_i\|_\infty$ and $v(s) := \sum_{i=1}^m |u_i(s)|$. Moreover,

$$
F : C([0, T], X) \times L^1([0, T], \mathbb{R}^m) \to C([0, T], X)
$$

is an eventual contraction in $x$ uniformly on bounded sets of $L^1([0, T], \mathbb{R}^m)$.

Proof. Let $x, y \in C([0, T], X)$, $u \in L^1([0, T], \mathbb{R})$ and suppose $m = 1$. Then we obtain

$$
\|F_u(x)(t) - F_u(y)(t)\| = \left\|\int_0^t e^{(t-s)A}u(s)(f(s, x(s)) - f(s, y(s))) \, ds\right\| \\
\leq \int_0^t Me^{\mu(t-s)}|u(s)||L(s)||x(s) - y(s)|| \, ds \\
\leq Me^{\mu T}\|L\|_\infty\int_0^t |u(s)| \, ds \|x - y\|_\infty
$$

and thus (3.7) holds for $n = 1$. Next, assume that (3.7) is satisfied for some $n \in \mathbb{N}$. 

We conclude
\[
\|F_u^{n+1}(x)(t) - F_u^{n+1}(y)(t)\| = \|F_u(F_u^n(x))(t) - F_u(F_u^n(y))(t)\|
\]
\[
= \left\| \int_0^t e^{(t-s)A} u(s) \left( f(s, F_u^n(x)(s)) - f(s, F_u^n(y)(s)) \right) \, ds \right\|
\]
\[
\leq \int_0^t M e^{\omega(t-s)} |u(s)| L(s) \left\| F^n(x, u)(s) - F^n(y, u)(s) \right\| \, ds
\]
\[
\leq \frac{M^{n+1} e^{\omega(n+1)T} \|L\|_\infty^{n+1} \|x - y\|_\infty}{n!} \int_0^t |u(s)| \left( \int_0^s |u(r)| \, dr \right)^n \, ds
\]
\[
= \frac{M^{n+1} e^{\omega(n+1)T} \|L\|_\infty^{n+1} \|x - y\|_\infty}{(n + 1)!} \left( \int_0^t |u(s)| \, ds \right)^{n+1},
\]
where the last step follows via integration-by-parts. Next, let \(m \geq 1\) and choose \(\|u\|_1 := \sum_{i=1}^m \|u_i\|_1\) as norm\(^{12}\) on \(L^1([0, T], \mathbb{R}^m)\). It follows
\[
\|F_u(x)(t) - F_u(y)(t)\| \leq M e^{\omega T} \|L\|_\infty \int_0^T \sum_{i=1}^m |u_i(s)| \, ds \|x - y\|_\infty
\]
with \(\|L\|_\infty := \max_{i=1, \ldots, m} \|L_i\|_\infty\). Setting \(v(s) := \sum_{i=1}^m |u_i(s)|\), we conclude by induction
\[
\|F_u^n(x)(t) - F_u^n(y)(t)\| \leq \frac{M^n e^{\omega n T} \|L\|_\infty^n \left( \int_0^T v(s) \, ds \right)^n}{n!} \|x - y\|_\infty
\]
Finally, taking into account \(\|v\|_1 = \|u\|_1\) we obtain
\[
\|F_u^n(x) - F_u^n(y)\|_\infty \leq \frac{M^n e^{\omega n T} \|L\|_\infty^n \|u\|_1^n}{n!} \|x - y\|_\infty
\]
and hence for bounded \(B \subset L^1([0, T], \mathbb{R}^m)\) we can choose \(n \in \mathbb{N}\) such that the estimate
\[
\frac{M^n e^{\omega n T} \|L\|_\infty^n \|u\|_1^n}{n!} < 1
\]
holds for all \(u \in B\).

**Remark 3.14.** While the above estimation certainly applies to the case \(p > 1\) as well, the \(\omega\)-norm approach of the previous subsection fails for \(p = 1\). Nevertheless, we decided to use the \(\omega\)-norm technique for \(p > 1\) in order to illustrate both methods.

Next we need an appropriate replacement of Lemma 3.8. We already know (cf. Appendix A) that Lemma 3.8 cannot be “repaired” simply by an enhanced proof technique because the integral operator \(F_x : L^1([0, T], \mathbb{R}^m) \to C([0, T], X)\) is in general not compact. Therefore, the idea is to prove relative compactness of the reachable set \(\mathcal{R}_{\leq T}^1(\xi_0)\) directly via Corollary 2.13. To this end, we define an “image map”
\[
\text{Im} : C([0, T], X) \to C(X)
\]
via
\[
\text{Im}(x) := x([0, T])
\]
\(^{12}\)Obviously, this norm on \(L^1([0, T], \mathbb{R}^m)\) is induced by the 1-norm on \(\mathbb{R}^m\). Choosing another norm on \(\mathbb{R}^n\) would simply yield an equivalent norm on \(L^1([0, T], \mathbb{R}^m)\).
where $C(X)$ denotes the set of all compact subsets of $X$ equipped with the Hausdorff metric
\[ d_H(K, K') := \max \left\{ \max_{\xi \in K} \min_{\xi' \in K'} \| \xi - \xi' \|, \max_{\xi' \in K'} \min_{\xi \in K} \| \xi - \xi' \| \right\}. \]

As $X$ is complete $(C(X), d_H)$ is complete as well, cf. [13, §33.IV, Thm.]. Moreover, one has the obvious estimate
\[ (3.8) \quad d_H(\text{Im}(x), \text{Im}(y)) \leq \| x - y \|_\infty \]
for all $x, y \in C([0, T], X)$, i.e. $\text{Im}$ is Lipschitz continuous with Lipschitz rate $L = 1$.

**Proposition 3.15.** Let $f_i : \mathbb{R}_0^+ \times X \to X, i = 1, \ldots, m$ be continuous and let $F : C([0, T], X) \times L^1([0, T], \mathbb{R}) \to C([0, T], X)$ denote the integral operator given by (IO). Moreover, let $\text{Im}$ be defined as above and set $Z := (C(X), d_H)$. Then for $S \subset C([0, T], X)$ and $B \subset L^1([0, T], \mathbb{R})$ one has the implication:
\[
\text{Im}(S) \text{ relatively compact in } Z \text{ and } B \text{ bounded} \quad \Downarrow \\
\text{Im}(F(S \times B)) \text{ is relatively compact in } Z
\]
To prove Proposition 3.15 the following auxiliary results are useful.

**Lemma 3.16 (Relative Compactness Criterion)**. Let a subset $K \subset Z$ be given. Then $K$ is relatively compact if and only if $\bigcup K \subset X$ is relatively compact.

**Proof.** $\Longrightarrow$: Let $\varepsilon > 0$ and assume that $K \subset Z$ is relatively compact with respect to $d_H$. Hence there exists a finite $\frac{\varepsilon}{2}$-net $K_1, \ldots, K_N$ of $K$. Moreover, since all $K_i, i = 1, \ldots, N$ are compact their union is compact as well and hence we can choose a finite $\frac{\varepsilon}{2}$-net $\xi_1, \ldots, \xi_M$ of it. Now we claim that $\xi_1, \ldots, \xi_M$ is an $\varepsilon$-net of $\bigcup K$: Given $\xi \in \bigcup K$. One finds $K \in K$ such that $\xi \in K$. For this $K$, there exists $K_i$ such that $d_H(K, K_i) < \frac{\varepsilon}{2}$. This implies that one has $\eta \in K_i$ with $\| \xi - \eta \| < \frac{\varepsilon}{2}$. Finally, one can choose $\xi_j$ such that $\| \eta - \xi_j \| < \frac{\varepsilon}{2}$. This yields the desired estimate
\[ \| \xi - \xi_j \| \leq \| \xi - \eta \| + \| \eta - \xi_j \| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

$\Longleftarrow$: Again, let $\varepsilon > 0$ and assume that $\bigcup K \subset X$ is relatively compact. Therefore we can choose a finite $\varepsilon$-net $\xi_1, \ldots, \xi_M$ of $\bigcup K$. Set $K_0$ be the power set of $K_0 := \{ \xi_1, \ldots, \xi_M \}$. Obviously, $K_0$ is a finite collection of compact subsets of $X$. We claim that $K_0$ is the desired $\varepsilon$-net of $K$. To see this let $K \in K$ be arbitrary and define
\[ K'_i := \{ \xi \in K_0 : K_0(\xi) \cap K \neq \emptyset \} \in K_0. \]
Now it is a standard exercise (left to the reader) to show $d_H(K, K') \leq \varepsilon$ and we are done.

**Proof of Proposition 3.15.** Let $B := K_1(0) \subset L^1([0, T], \mathbb{R})$ and $S \subset C([0, T], X)$ such that
\[ \text{Im}(S) = \{ x([0, T]) : x \in S \} \subset Z \]

---

13This result should be well known but it was difficult to locate a suitable reference. One implication ("only-if"-part) can be found in [13, §21.VIII, Thm. 2].
is relatively compact. Then, according to Lemma 3.16,

\[ \text{Ev}(S) = \{ x(t) : x \in S, t \in [0, T] \} \]

is relatively compact in \( X \) and thus continuity of \( f \) implies that the closure of \( f([0, T] \times \text{Ev}(S)) \) coincides with \( f([0, T] \times \text{Ev}(S)) \) and is therefore compact in \( X \). Thus we can apply Lemma B.1 to the continuous function \( \Gamma : [0, T] \times X \to X, (t, \xi) \mapsto e^{At} \xi \) and the compact set \( K := f([0, T] \times \text{Ev}(S)) \). This yields an approximation \( \Gamma_{\varepsilon/2} \) such that (i) holds for \( \varepsilon \). Thus for arbitrary \( x \in S \) and \( u \in B \) it follows

\[
F(x, u)(t) = e^{tA} \xi_0 + \int_0^t u(s)e^{(t-s)A}f(s, x(s)) \, ds \\
= e^{tA} \xi_0 + \int_0^t u(s)\Gamma(t-s, f(s, x(s))) \, ds \\
= e^{tA} \xi_0 + \int_0^t u(s)\Gamma_{\varepsilon/2}(t-s, f(s, x(s))) \, ds \\
+ \int_0^t u(s)\left(\Gamma(t-s, f(s, x(s))) - \Gamma_{\varepsilon/2}(t-s, f(s, x(s)))\right) \, ds.
\]

Note that integrability of \( s \mapsto \Gamma_{\varepsilon/2}(t-s, f(s, x(s))) \) follows from its measurability and essential boundedness. Since continuity of \( t \mapsto e^{tA} \xi_0 \) implies that \( \{e^{tA} \xi_0 : t \in [0, T]\} \) is compact it suffices to focus on the second and third term of (3.9). For the latter, we get

\[
\begin{align*}
&\left\| \int_0^t u(s)\Gamma(t-s, f(s, x(s))) \, ds \right\| - \int_0^t u(s)\Gamma_{\varepsilon/2}(t-s, f(s, x(s))) \, ds \\
&\leq \int_0^t |u(s)|\left\| \Gamma(t-s, f(s, x(s))) - \Gamma_{\varepsilon/2}(t-s, f(s, x(s))) \right\| \, ds \\
&\leq \frac{\varepsilon}{2} \int_0^t |u(s)| \, ds \leq \frac{\varepsilon}{2}.
\end{align*}
\]

Hence we conclude

\begin{equation}
\left\| \int_0^t u(s)\Gamma(\cdot - s, f(s, x(s))) \, ds - \int_0^t u(s)\Gamma_{\varepsilon}(\cdot - s, f(s, x(s))) \, ds \right\|_{\infty} \leq \frac{\varepsilon}{2}.
\end{equation}

For the second term we get the following representation

\[
\int_0^t u(s)\Gamma_{\varepsilon/2}(t-s, f(s, x(s))) \, ds = \sum_{i=1}^N \sum_{j=1}^M \int_0^t u(s)\chi_{\Omega_i}(t-s)\chi_{\{f(s, x(s))\}} \xi_{ij} \, ds \\
= \sum_{i=1}^N \sum_{j=1}^M \int_{I_{ij}(t)} u(s) \, ds \xi_{ij}
\]

with \( I_{ij}(t) := [0, t] \cap (t-\Delta_i) \cap (f \circ (\text{id} \times x(i))^\text{t}^{-1}(S_j)). \) This shows

\[
\int_0^t u(s)\Gamma_{\varepsilon/2}(t-s, f(s, x(s))) \, ds = \sum_{i=1}^N \sum_{j=1}^M \int_{I_{ij}(t)} u(s) \, ds \xi_{ij} \\
\subset \left\{ \sum_{i=1}^N \sum_{j=1}^M \lambda_{ij} \xi_{ij} \middle| \lambda_{ij} \in [-1, 1] \right\} =: C \subset X
\]
and hence the second term is contained in the compact convex subset $C$. Thus, one can choose a finite \( \frac{\varepsilon}{2} \)-net \( y_1, \ldots, y_L \) for \( C \) which yields—due to (3.10)—a finite \( \varepsilon \)-net of

\[
\left\{ \int_0^t u(s)e^{(t-s)A}f(s,x(s))ds \, : \, t \in [0,T], x \in S, u \in B \right\}.
\]

Since the sum of relatively compact subsets is again relatively compact we conclude that \( \text{Ev} \left( F(S \times B) \right) \) is relatively compact. A further application of Lemma 3.16 proves relative compactness of \( \text{Im} \left( F(S \times B) \right) \subset Z \).

After these preliminaries we are well-prepared for proving the main result of this subsection.

**Proof of Theorem 3.11.** As in the proof of Corollary 3.3, we have to show that for fixed \( T \geq 0 \) and \( n \in \mathbb{N} \), the reachable set up to time \( T \) under bounded \( L^1 \)-controls, i.e.

\[
R_{\leq T}^{1,n} (\xi_0) := \{ x(t, \xi_0, u) : t \in [0,T], \| u \|_1 \leq n \}
\]

is relatively compact. The rest follows again immediately from Baire’s category theorem and the identity

\[
R^1 (\xi_0) = \bigcup_{T \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} R_{\leq T}^{1,n} (\xi_0).
\]

Hence, let \( T \geq 0 \) be fixed and, for simplicity, let \( m = 1 \). Moreover, by Lemma 3.10, we can assume without loss of generality that \( f := f_1 \) is globally Lipschitz in \( x \) on bounded subsets of \( L^1 ([0,T], \mathbb{R}) \).

Next, let \( B := K_n (0) \subset L^1 ([0,T], \mathbb{R}) \) denote the closed ball around the origin with radius \( n \in \mathbb{N} \). By Lemma 2.11 and 3.13, we can choose a metric \( d' \) on \( C([0,T], X) \) which is strongly equivalent to the metric induced by the maximum-norm \( \| \cdot \|_\infty \) such that the restricted integral operator \( F : C([0,T], X) \times B \to C([0,T], X) \) constitutes a uniform contraction in \( x \). Then Proposition 3.15 and Eq. (3.8) allow us to apply Corollary 2.13 to the restriction of \( F \). Finally, Lemma 3.16 yields the desired relative compactness of \( \text{Ev} (\Phi (B)) = R_{\leq T}^{1,n} (\xi_0) \). This concludes the proof.

**Appendix A. Counter-Example.**

Consider the scalar system

\[
\dot{x} (t) = u (t), \quad x (0) = 0 \quad \text{and} \quad u \in L^1 ([0,1], \mathbb{R}).
\]

and let \( \Phi : L^1 ([0,1], \mathbb{R}) \to C([0,1], \mathbb{R}) \) denote the corresponding fixed point map, cf. Theorem 3.1. Then the image of the closed unit ball of \( L^1 ([0,1], \mathbb{R}) \) under \( \Phi \) obviously contains the sequence \( (x_n)_{n \in \mathbb{N}} \) of continuous functions given by

\[
x_n (t) = \begin{cases} 
nt & \text{for } t \in [0, \frac{1}{n}], \\
1 & \text{for } t \in [\frac{1}{n}, 1],
\end{cases}
\]

as \( x_n = \Phi (u_n) \) with \( u_n (t) := n \) for \( t \in [0, \frac{1}{n}] \) and \( u_n (t) := 0 \) else. Evidently, the sequence \( (x_n)_{n \in \mathbb{N}} \) is not equicontinuous (at \( t = 0 \)) so the image \( \Phi (K_1 (0)) \) is not
Appendix B. A Technical Lemma.

Lemma B.1. Let $T \geq 0$, $K \subset X$ be compact, and $\Gamma : [0, T] \times X \rightarrow X$ be continuous. Then for every $\varepsilon > 0$ there exists $\Gamma_\varepsilon : [0, T] \times K \rightarrow X$ such that the following conditions are satisfied:

(i) For all $\xi \in K$ and $t \in [0, T]$ one has $\|\Gamma(t, \xi) - \Gamma_\varepsilon(t, \xi)\| < \varepsilon$.

(ii) The image of $\Gamma_\varepsilon$ is finite. In particular, there exist finitely many disjoint Borel sets $\Delta_i \subset [0, T]$, $i = 1, \ldots, N$ and $S_j \subset K$, $j = 1, \ldots, M$ as well as $\xi_{ij} \in X$ such that

$$
\Gamma_\varepsilon(t, \xi) = \sum_{i=1}^{N} \sum_{j=1}^{M} \chi_{\Delta_i}(t) \chi_{S_j}(\xi) \xi_{ij}
$$

for all $\xi \in K$ and $t \in [0, T]$.

Proof. Let $K \subset X$ be compact and $\varepsilon > 0$. For convenience, choose $d_\infty$ as metric on $[0, T] \times K$, cf. (2.2). Due to the compactness of $[0, T] \times K$ the restriction $\Gamma|_{[0, T] \times K}$ is uniformly continuous meaning there exists $\delta > 0$ such that

$$
\|\Gamma(t', \xi') - \Gamma(t, \xi)\| \leq \varepsilon
$$

for $d_\infty((t', \xi'), (t, \xi)) < \delta$, i.e. for $|t' - t| < \delta$ and $\|\xi' - \xi\| < \delta$. Again the compactness of $K$ implies that there exists a finite $\delta$-net $\eta_1, \ldots, \eta_M$ of $K$. Then, choosing $N \in \mathbb{N}$ such that $T/N < \delta$ allows to define $\Gamma_\varepsilon$ as follows:

$$
\Gamma_\varepsilon(t, \xi) := \begin{cases}
\Gamma\left(\frac{t}{N}, \eta_1\right) & \text{for } t \in \left[0, \frac{T}{N}\right] \text{ and } \xi \in K \cap B_\delta(\eta_1), \\
\Gamma\left(\frac{t}{N}, \eta_1\right) & \text{for } t \in \left(\frac{T}{N}, \frac{2T}{N}\right] \text{ and } \xi \in K \cap B_\delta(\eta_1), \\
\vdots & \\
\Gamma(T, \eta_1) & \text{for } t \in \left(\frac{(N-1)T}{N}, T\right] \text{ and } \xi \in K \cap B_\delta(\eta_1), \\
\Gamma\left(\frac{T}{N}, \eta_2\right) & \text{for } t \in \left[0, \frac{T}{N}\right] \text{ and } \xi \in K \cap \left(B_\delta(\eta_2) \setminus B_\delta(\eta_1)\right), \\
\vdots & \\
\Gamma(T, \eta_M) & \text{for } t \in \left(\frac{(N-1)T}{N}, T\right] \text{ and } \xi \in K \cap \left(B_\delta(\eta_M) \setminus \bigcup_{k=1}^{M-1} B_\delta(\eta_k)\right).
\end{cases}
$$

The straightforward proof that $\Gamma_\varepsilon$ satisfies (i) is left to the reader. Finally, setting $\Delta_1 := [0, \frac{T}{N}]$, $\Delta_i := \left(\frac{(i-1)T}{N}, \frac{iT}{N}\right]$ for $i = 2, \ldots, N$, $S_j := K \cap \left(B_\delta(\eta_j) \setminus \bigcup_{k=1}^{i-1} B_\delta(\eta_k)\right)$ for $j = 1, \ldots, M$ and $\xi_{ij} := \Gamma_\varepsilon\left(\frac{t}{N}, \eta_i\right)$ yields the desired representation (B.1) of $\Gamma_\varepsilon$. \(\Box\)

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