On a concept of quantum non-Markovianity weaker than CP-indivisibility

U. Shrikant, 1,2,† R. Srikanth, 1, ‡ and Subhashish Banerjee ³

1 Poornaprajna Institute of Scientific Research, Sadashivnagar, Bangalore, India.
2 Manipal Academy of Higher Education, Manipal, India.
3 IIT Jodhpur, Rajasthan, India.

The problem of defining quantum non-Markovianity has proven elusive, with various inequivalent criteria put forth to address it. We consider the question: what is the weakest notion of quantum non-Markovianity of system dynamics that would account for any aspect of memory? That is, a process indicated to be non-Markovian according to any existing criterion is also non-Markovian according to this criterion, but the converse may not be true. Correspondingly, we identify true memorylessness with the idea of the quantum dynamics being “temporally self-similar”, whereby the form of the intermediate map that propagates the state between two arbitrary time intervals is independent of the initial time $t_0$. This is identified with the quantum dynamical semigroup (QDS), renewing advocacy for a familiar candidate. Of practical importance is that our weaker criterion is useful for indicating the non-Markovianity of CP-divisible channels that manifest an effect of counteracting decoherence. This is illustrated with the examples of power-law noise (PLN) and Ornstein-Uhlenbeck Noise (OUN). For physically well motivated channels, we propose a geometric quantification of non-Markovianity according to our definition, which is shown to be applicable to stronger manifestations of quantum non-Markovianity, demonstrated here by application to various channels, including those that are CP-indivisible, P-indivisible and eternally non-Markovian. Finally, we indicate a few open questions that come up as natural consequences of this work.

I. Introduction

Practical quantum information processing must inevitably contend with quantum noise [1–4], and in particular, non-Markovian effects in the noise [5–9], especially in the context of practical applications [10]. Classical Markovianity can be defined in terms of the divisibility of a process into intermediate transitions, or equivalently in terms of the fall in distinguishability of two states. The quantum adaptation of these ideas to define quantum non-Markovianity is not straightforward, and turns to lead to inequivalent concepts, essentially because the Kolmogorov hierarchy of classical joint probability distributions cannot be transferred to the quantum case [11–15].

The classical identification of Markovianity with divisibility leads to a hierarchy of quantum divisibility criteria based on the positivity property of the intermediate map (the propagator of the dynamics between two arbitrary times) associated with a dynamical process is CP-divisible [16, 17] or P-divisible [18], or an intermediate $k$-divisible [19, 20]. The effort to unify all such definitions into a single framework is important and remains studied by various authors; in this context, cf. [21]. It is fair to say that the question of how exactly to characterize memory effects in quantum processes remains open, cf. [22].

Earlier, quantum Markovianity was identified with the quantum dynamical semigroup (QDS) [23]. This refers to the 1-parameter semigroup of maps governing the system dynamics, generated by the time-independent linear map, namely, the Lindbladian $L$, corresponding to the time-homogeneous master equation $\dot{\rho}(t) = L[\rho(t)]$ [24, 25].

More recently, this perspective of quantum Markovianity was not favored because it would count the time-inhomogeneous generator $L(t)$, which has time-dependent but positive decoherence rates in the canonical Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) equation [23, 26, 27], also as non-Markovian, even though such processes are CP-divisible.

This traditional reason for identifying QDS with Markovianity is twofold: (a) it could be justified on grounds that QDS is a reasonable quantum analogue of the classical Chapman-Kolmogorov equation [14, 27–29]. (b) it is also favored by considerations of the system-environment interaction, such as the correlation times of the environment being very small in relation to the system’s relaxation time or allowing for the Born-Markov approximation (see Appendix A). Thus, non-Markovianity was associated with strong coupling with the bath, or non-trivial initial correlation between the system and environment [1, 3]. In [30] quantum non-Markovian behavior was studied from the perspective of linear response theory.

In the present work, we shall make a fresh case for QDS as the definition of Markovianity by bringing a new support for viewpoint (a) above. Briefly, we will show that in any deviation from QDS, the intermediate map carries a memory of the initial time; this memory effect can be shown to have an operational interpretation. We geometrically quantify this weak manifestation of non-Markovianity. For consistency, our approach is shown to be robust enough to cover also stronger versions of non-Markovianity, such as CP-indivisibility and P-indivisibility. On the other hand, the viewpoint (b) requires further investigation, mainly in light of recent works that have shown that strong forms of non-Markovianity can arise without information back-flow from the environment [31].
II. QDS and temporal self-similarity

Quite generally $\mathcal{E}(t_2, t_0) = \mathcal{E}(t_2, t_1)\mathcal{E}(t_1, t_0)$, where $\mathcal{E}(t_2, t_0)$ and $\mathcal{E}(t_1, t_0)$ are completely positive (CP). Assuming the invertibility of $\mathcal{E}(t_1, t_0)$, the intermediate map, which propagates the system from $t_1$ to $t_2$ is given by:

$$\mathcal{E}(t_2, t_1) = \mathcal{E}(t_2, t_0)\mathcal{E}(t_1, t_0)^{-1}. \quad (1)$$

Clearly, this is a function of $t_0$ in general, which is evidently a kind of memory effect. An evolution where this dependence of $\mathcal{E}(t_2, t_1)$ on $t_0$ drops out, for arbitrary intermediate intervals $(t_2, t_1)$, is said to be \textit{temporally self-similar}, in the sense that the intermediate map is oblivious of the initial time, and behaves like a full map in its own right [32].

Quite generally, $\mathcal{E}(t, t_0) = \mathcal{E}(t - t_0)$, which essentially follows from the fact that the CP map can be purified to a system-environment unitary $U(t - t_0)$ acting on a product state. Therefore, from Eq. (1), we have that $\mathcal{E}(t_2, t_1) = \mathcal{E}(f(t_2 - t_0, t_1 - t_0))$ such that $f(t_2 - t_0, t_1 - t_0)$ has no dependence of $t_0$ for arbitrary $t_2, t_1$ and $t_0$. Clearly, this holds only if $f(y, x) = y - x$ for $y \geq x$. Setting $r \equiv t_2 - t_0$ and $s \equiv t_1 - t_0$, we find that $\mathcal{E}(r + s) = \mathcal{E}(s)\mathcal{E}(r)$, which is defining composition rule for QDS. Conversely, QDS satisfies temporal self-similarity since the intermediate map under QDS is just $\mathcal{E}(t_2 - t_1)$. We thus identify temporal self-similarity with QDS. We present a simple illustration of temporal self-similarity in the Appendix (B).

If the Lindbladian $\mathcal{L}(t) = \mathcal{L}$ is a constant, then from Eq. (1), $\mathcal{E}(t_2, t_1) = e^{(t_2 - t_0)\mathcal{L}}e^{-(t_1 - t_0)\mathcal{L}} = e^{(t_2 - t_1)\mathcal{L}} = \mathcal{E}(t_2 - t_1)$, i.e., we obtain the self-similar form. But the converse is not true [15]. This happens essentially when suitable continuity and limit requirements are not met. A simple example here would be the temporally self-similar map $\mathcal{E}(t) = 1$ for $t = 0$, and $\mathcal{E}(\rho) = \sum_j \Pi_j \text{Tr}(\rho \Pi_j)$ for $t > 0$, where $\Pi_j$ is any complete set of projectors. However, in physically motivated scenarios, we can assume that the channel is continuously satisfying the limit requirement $\lim_{t \to 0^+} \mathcal{E}(t) = 1$. Given these assumptions, we can identify self-similarity at the master equation level with the time-independent Lindbladian.

It may at first seem retrograde to associate memory even with time-dependent Lindbladians with positive decoherence rates. But consider the time-nonlocal representation of the master equation of the system dynamics in terms of the linear memory kernel map $\mathcal{M}$, which makes the idea of memorylessness clearer. We have $\dot{\rho}(t) = \int_0^t ds\mathcal{M}_{t-s}\rho(s) = \int_0^t ds\mathcal{M}_{t-s}\mathcal{E}(s - t_0) \mathcal{E}^{-1}(t - t_0)\rho(t) = \int_0^t ds\mathcal{M}_{t-s} \delta(t - s)\mathcal{L}\rho(t)$. This implies that $\mathcal{M}(t - s) = \delta(t - s)\mathcal{L}$, where $\delta(t - s)$ is the Dirac delta function, meaning that the dynamics remembers only the present time and has no influence from earlier times.

In Eq. (1), it can be shown that if $\mathcal{E}(t_2, t_1)$ is NCP, then the dependence on $t_0$ does not drop out. Thus, self-similarity is a stronger concept of memorylessness than CP-divisibility.

III. Examples

We shall now present two concrete CP-divisible noisy channels, namely Ornstein-Uhlenbeck noise (OUN) and power-law noise (PLN), that help exemplify the idea of memory of initial time at the level of maps (as against master equation).

The OUN model was developed in [33] in the context of the effect of non-Markovian evolution on the dynamics of entanglement. The model used was that of Gaussian noise with a colored autocorrelation function, modeling random frequency fluctuations and has its roots in the modern development of statistical mechanics [34]. In the limit of infinite noise bandwidth, this reduced to the well-known white noise which is Markovian in nature. PLN is a non-Markovian stationary noise process. The name \textit{Power Law} points to the functional relationship between the spectral density and the frequency of the noise. It is a major source of decoherence in solid state quantum information processing devices such as superconducting qubits and has a well-defined Markovian limit [35].

The canonical Kraus representation for these channels has the form $\mathcal{E}(\rho) = \sum_j \Pi_j \rho K_j^\dagger$ with $K_1(t) = \sqrt{1 + p(t)} I$ and $K_2(t) = \sqrt{1 - p(t)} Z$, corresponding to the Choi matrix, $\chi = (\mathcal{E} \otimes I)(|00\rangle + |11\rangle)$ given by:

$$\chi(t, 0) = \begin{pmatrix} 1 & 0 & 0 & p(t) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ p(t) & 0 & 0 & 1 \end{pmatrix}, \quad (2)$$

where $I$ and $Z$ are Pauli operators, and

$$p(t) = \begin{cases} \exp\left(-\frac{G}{2}(g^{-1}[\exp(-gt) - 1]) + t\right) & \text{case of OUN}, \\ \exp\left(-\frac{G^2(t+2)}{2gt+1}\right) & \text{case of PLN}. \end{cases} \quad (3)$$

Here $G$ is the inverse of the effective relaxation time, while $g$ and $1/g$ are related to the noise band width, for the OUN and PLN noises, respectively.

The corresponding master equation in its canonical form [36] is

$$\dot{\rho} = \gamma(t)(-\rho + Z\rho Z) \quad (4)$$

where [37]

$$\gamma(t) = -\frac{\dot{p}(t)}{2p(t)}. \quad (5)$$

is the decoherence rate. It follows from Eqs. (3) and (5) that

$$\gamma(t) = \begin{cases} \frac{1}{2}G(1 - e^{-gt}) & \text{case of OUN}, \\ \frac{G^2}{2gt+1} & \text{case of PLN}, \end{cases} \quad (6)$$

showing that $\gamma(t)$ remains positive for all $t$ in both these cases. Thus, neither of them is CP-divisible [38].

According to the system-environment criterion, OUN (resp., PLN) have their Markovian limit by setting $g$ (resp., $1/g$) $\to \infty$, in Eq. (3). In this limit, we may replace Eq. (3)
by
\[ p^\ast(t) = \begin{cases} e^{-Gt/2} & \text{case of OUN,} \\ e^{-Gt} & \text{case of PLN.} \end{cases} \] (7)

For the general case, satisfying Eq. (3), \( \gamma(t) \) in Eq. (5) is time-dependent, implying that Eq. (3) corresponds to the time-inhomogeneous GKSL master equation. On the other hand, in the limit where Eq. (7) holds, the rate Eq. (5), i.e., \( -\alpha^\ast/(2p^\ast) \), becomes a constant, \( \frac{\zeta}{4} \) and \( \frac{\zeta}{2} \), respectively, in which case Eq. (4) corresponds to the strict (i.e., time-homogeneous) GKSL master equation.

The Choi matrix of the intermediate map that evolves the system from time \( t_0 \) to \( t_2 \) is given by (see Appendix C, where it is derived via the formalism of dynamical maps \( A \) and \( B \) [39]):
\[ \chi(t_2, t_1) = \begin{pmatrix} 1 & 0 & 0 & \frac{p(t_2)}{p(t_1)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{p(t_2)}{p(t_1)} & 0 & 0 & 1 \end{pmatrix}. \] (8)

From Eq. (3), we have for PLN that \( p(t_j) = \exp \left(-\frac{G[t_j - t_0]g[t_j - t_0] + 2}{2\eta(t_j - t_0)^{1+2}}\right) \). From this, we readily find that the decoherence term \( \frac{p(t_2)}{p(t_1)} \) in Eq. (8) does not simplify to a form that is independent of \( t_0 \). A similar argument holds for OUN. Thus, even though these two channels are CP-divisible, they carry a memory of the initial time \( t_0 \), which is required to construct the propagator between any two arbitrary instances.

On the other hand, for the QDS-limit rates given in Eq. (7), we find \( \frac{p(t_2)}{p(t_1)} = e^{\frac{\zeta}{4}(t_2 - t_1)} \) and \( e^{\frac{\zeta}{2}(t_2 - t_1)} \), respectively, for OUN and PLN, showing that the intermediate map is oblivious of \( t_0 \).

This brings us to the important issue of empirical or practical implication of such weaker-than-CP-indistinguishable memory. Two examples may be pointed out. The weak non-Markovian nature of the OUN model, as characterized here, was made use of in [33] to study the impact of non-Markovian behavior on the prolongation of the time to entanglement sudden death (ESD); while in [40], the analogous nature in the OUN and PLN models is shown to counteract decoherence for quantum walks. In both these examples, the weaker-than-CP-indistinguishable effect acts as a memory resource.

IV. Quantifying weak non-Markovianity

The above considerations suggest that non-Markovianity in the weaker sense may be geometrically quantified by the minimum distance of an evolution from a QDS form, which additionally satisfies the aforementioned continuity and limit assumptions. Note that these assumptions are naturally satisfied in the case of physically well motivated channels. Accordingly, as one possibility to quantify non-Markovianity in this sense, one may consider minimizing the distance \( \|\mathcal{E}(t) - e^{\mathcal{L}^\ast t}\| \) for arbitrary maps, where \( \mathcal{L}^\ast \) is a time-homogeneous Lindblad generator. However, this approach can be computationally complicated to realize, given the non-convex nature of set of CP-divisible (including QDS) maps [29].

Instead of maps, we propose here to realize this geometric measure at the level of generators. We require to define a suitable measure that quantifies \( \|\mathcal{L}(t) - \mathcal{L}^\ast\| \). To this end, we consider the infinitesimal intermediate map \( \delta\mathcal{E} \) of the given channel \( \mathcal{E} \), evolving the system of Hilbert space dimension \( d \) from time \( t \) to time \( t + dt \). We have
\[ (\delta\mathcal{E})\rho(t) = T \exp \left( \int_t^{t + dt} \mathcal{L}(s) ds \right) \rho(t) = (1 + \mathcal{L}(t) dt)\rho(t). \] (9)

By the Choi-Jamiołkowski isomorphism, for any map \( \mathcal{E} \) acting on a \( d \)-dimensional system, we can associate the unique \( d^2 \times d^2 \) Choi matrix \( \hat{\mathcal{E}}(t) \equiv d(\mathcal{E} \otimes \mathbb{I})|\Phi^+\rangle \langle \Phi^+| \), where \( |\Phi^+\rangle = d^{-1/2} \sum_j |j, j\rangle \) is the maximally entangled state. The Choi matrix of the infinitesimal intermediate map \( \mathcal{E} \) (9) is thus \( d(\hat{\mathcal{E}}^\ast|\Phi^+\rangle \langle \Phi^+| + \hat{\mathcal{L}}(t) dt) \). Here, the uniqueness means that the Choi matrix is insensitive to the unitary freedom in the representation of the generator, and in particular, is independent of whether the generator has been represented in its canonical form. Let \( \delta\mathcal{E}^\ast(t) \) be the infinitesimal intermediate map of a QDS channel, and \( \mathcal{L}^\ast \) the corresponding generator. Then the difference between the two infinitesimal intermediate maps is \( \Delta\mathcal{L} \equiv \delta\mathcal{E}(t) - \delta\mathcal{E}^\ast(t) = (\mathcal{L}(t) - \mathcal{L}^\ast)dt \), from which we have:
\[ \zeta = \min_{\mathcal{L}^\ast} \frac{1}{T} \int_0^T \|\mathcal{L}(t) - \mathcal{L}^\ast\| dt, \] (10)
where \( \|A\| = \text{Tr} \sqrt{AA^\dagger} \) is the trace norm of a matrix \( A \).

Such a measure of non-Markovianity has the following desirable properties. There is an inherent normalization, whereby \( \zeta = 0 \) iff the channel is QDS and \( \zeta > 0 \) otherwise. The measure is easily computable since the general numerical optimization problem of Eq. (10) can be implemented in \( d^2 - 1 \) dimensions efficiently. The Lindbladians can be represented in their canonical forms, and those with positive Lindblad terms will form a convex set. Furthermore, for specific examples we consider, the minimization over the Lindbladian reduces to the problem of minimizing over a single parameter. The measure respects continuity, which we illustrate with a few examples. The measure is basis-independent, and has the operational meaning of non-Markovianity as deviating from the property of self-similarity and thus requiring memory of the initial time \( t_0 \).

A similar geometric approach to quantifying non-Markovianity is considered in [41], namely minimizing the distance \( \|\mathcal{E}(t = 1) - e^{\mathcal{L}^\ast} \| \) for arbitrary maps. They consider the distance of a map (snapshot of an evolution), whereas we consider the evolution over a finite period of time. The problem with considering a snapshot is that there can be infinite number of evolutions passing through the map at that instant. For example, the map Eq. (2), with a fixed \( p(t = T) \), has an infinite number of ways to assign values to the pair \( (g, G) \).

The measure Eq. (10) is now applied to a number of well-known channels. Immediately below, we consider the CP-
divisible channels of OUN and PLN to quantify their non-Markovianity in the present approach. For dephasing channels, from Eq. (4) we find $\mathcal{L}\rho = \gamma (-\rho + Z\rho Z)$. Thus,

$$\hat{L}(t) - \hat{L}^* = (\gamma^* - \gamma)(|\Phi^+\rangle \langle \Phi^+| - |\Phi^-\rangle \langle \Phi^-|),$$

(11)

where $|\Phi^-\rangle$ is the Bell state with even parity and negative phase.

For example, for OUN, it follows from Eq. (11) that

$$\zeta = \min c \frac{1}{T} \int_0^T \left[ \frac{1}{4} G (1 - e^{-g t}) - c \right] dt$$

(12)

where $\gamma^*$ is the rate of a QDS dephasing channel. The minimization may be determined numerically and would in general depend on $T$. For OUN and PLN, it is simpler to consider an estimate of $c$ to be the family’s natural QDS limit, as may be found from Eqs. (7) and (5). The corresponding plot, given in Figure (1a), may be considered as giving an upper bound on the weak non-Markovianity.

V. Application to stronger manifestations of non-Markovianity

Although the quantity $\zeta$ has been motivated to explore the idea of memory weaker than CP-indivisibility, one would expect it to be applicable to such stronger manifestations of non-Markovianity, since in those cases, the deviation from the QDS form would be greater. For completeness, we shall consider a few representative examples, one that is P-divisible but CP-indivisible, another that is P-indivisible, involving both unital and non-unital maps.

(a) An example of a model noise which is non-Markovian in the sense of CP-indivisibility but BLP Markovian \cite{18} (i.e., P-divisible), is the interesting model called “Eternally non-Markovian” (ENM) Pauli channel, proposed in \cite{38}, with the decay rate $\gamma_3(t)$ being negative for all $t > 0$, whence the name ‘eternal’. The canonical form of master equation for the evolution of qubit under this noise is the dephasing channel $\dot{\rho} = \sum_{j=1}^3 \gamma_j |\sigma_j \rho \sigma_j^\dagger - \rho\rangle$, with $\gamma_1 = \gamma_2 = 1$ and $\gamma_3(t) = -\tanh(t)$.

The measure (10), with Eq. (11), in this case is found to be $\zeta = \min c \frac{1}{T} \int_0^T \left| - \tanh(t) - c \right| dt$, for which optimal QDS channel is clearly the dephasing channel with $\gamma_1 = \gamma_2 = 1$ and $\gamma_3 = c$, and $c = 0$. Setting $T \equiv 1$, the degree of non-Markovianity here is $\int_0^1 \tanh(t) dt = \log[\cosh(1)] \approx 0.433$.

(b) As our next example, we consider random telegraph noise (RTN), which is a very well studied pure dephasing process, known to be non-Markovian according to information back-flow and CP-divisibility criteria \cite{40, 42}. The Kraus operators characterizing this process has a functional form similar to that of PLN and OUN with the function $p(t)$ in Eq. (2) having the form $p(t) = \exp\{-gt\} \left(\cos(g\omega t) + \frac{\sin(g\omega t)}{\omega}\right)$, with $\omega = \sqrt{(2g\gamma_0)^2 - 1}$. Here $g$ is the spectral band width, which is the inverse of environmental correlation time scale $\tau$, and $\gamma_0$ defines the coupling strength between the system and the environment. The decay rate (5) is found to be:

$$\gamma(t) = \frac{2\gamma_0^2}{g \left(1 + \sqrt{\frac{4g^2}{3} - 1} \cot \left(gt\sqrt{\frac{4g^2}{3} - 1}\right)\right)},$$

(13)

which vanishes in the limit $g \gg 2\gamma_0$ and represents the QDS limit of the family. In the limit $g > 2\gamma_0$, the process is described by a master equation with time-dependent all-time-positive decay rate $\gamma(t)$, and hence CP- and P-divisible. This would correspond to Markovian behavior from the divisibility perspective, but from the present point of view would be non-Markovian due to deviation from QDS. In the limit $g < 2\gamma_0$, the process becomes non-Markovian in the sense of CP-divisibility and $\gamma(t)$ oscillates between negative and positive values, giving rise to intervals of breakdown of CP-divisibility and even P-divisibility. As $g \to 0$, the noise becomes more colored, hence non-Markovian. Setting $\gamma^* = 0$ for this noise, an upper bound for the measure of non-Markovianity Eq. (10) for channels in this family parametrized by $x \equiv \frac{1}{g}$ is depicted in the Figure (1b).
(c) The above method, considered so far for unital channels, can straightforwardly be extended to the non-unital case. As a specific example, we may consider the non-Markovian (P-indivisible) amplitude damping (AD) as an example of non-unital channel, the NM-AD channel (Appendix D). The non-Markovianity measure Eq. (10) is found to be:

$$\zeta = \min_{\gamma^*} \frac{1}{T} \int_0^T |\gamma(t) - \gamma^*(1 + \sqrt{2})| dt. \quad (14)$$

Once again, an upper bound of non-Markovianity parameter $\zeta$, can be obtained by choosing $\gamma^*$ to the QDS limit of the family of non-Markovian AD channels. As in the dephasing case, we find that the general optimization of Eq. (10) reduces to minimizing over a single parameter $\gamma^*$, which can be done numerically and is depicted in the Figure (1b).

VI. Conclusions and Discussions

Given the inherent difficulty of characterizing quantum non-Markovianity, and the various methods to address it, here we consider the question: what is, reasonably, the weakest notion of non-Markovianity, that would cover any aspect of memory? We argue that this corresponds to memory of the initial time $t_0$ encoded into the form of the map that propagates the system state between two arbitrary intermediate instances of time. This intermediate map may be positive, but a dependence on $t_0$ will betoken memory. We coin the term “temporal self-similarity” for the corresponding concept of memorylessness and identify it with the quantum dynamical semigroup (QDS). For physically well motivated channels that satisfy additional natural continuity and limit conditions, this corresponds to a master equation with a time-independent Lindbladian, or equivalently, a delta memory kernel.

A number of new directions are opened up by the present work. In [31] it was shown how maximally non-Markovian system dynamics can arise from purely classical system-environment correlations. An environment assumed stationary clearly gives rise to QDS dynamics of the system, but in the above light, one might ask what are the most general system-environment correlations leading to weak non-Markovianity. This is an important issue, since it would help reconcile the somewhat conflicting definitions of non-Markovianity that arise from considerations of system dynamics and system-environment correlations.

In [43], the authors consider a resource theory of non-Markovianity wherein the Choi matrices corresponding to small-time divisible maps constitute the free states. Our work suggests the possibility of constructing a more relaxed resource theory of small-time maps, wherein resourceful states would correspond to maps of processes deviating from QDS. Earlier, we noted the instances where the weak non-Markovian nature of the OUN model, as characterized here, helps to prolong the time to ESD [33] and also counteracts decoherence in quantum walk [40]. This should pave the way for more such examples which could be envisaged to have an impact on our understanding of the field of non-Markovian phenomena.

In [19], studying the positivity of the propagator between two arbitrary times in an extended Hilbert space (divisibility), the authors defined a non-Markovianity degree, as the analogue of Schmidt number in quantum entanglement, such that the analogue of maximally entangled states are maximally non-Markovian quantum dynamics. It would be interesting to study how channels weaker than CP-indivisible fit into this hierarchy.

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A. Quantum dynamical semigroup

Open system dynamics is generally dealt with by various approaches. Nakajima-Zwanzig projection operator technique gives rise to a master equation for the evolution of the density matrix representing open system [36]:

\[ \dot{\rho}(t) = -\frac{i}{\hbar} [H_S, \rho(t)] + \int_{t_0}^{t} M_{s,t}[\rho(s)]ds, \]  

∀s ≥ t ≥ 0, where \( \dot{\rho} = \frac{d\rho}{dt} \) and \( H_S \) is system Hamiltonian. The decoherence and/or dissipation effects due to environmental interactions are taken into account by the memory kernel \( M_{s,t} \), which is understood to take into account the memory effects in the dynamics.

Under the Born-Markov and RWA assumptions, the master equation takes the Lindblad form:

\[ \dot{\rho}(t) = -\frac{i}{\hbar} [H_S, \rho(t)] + \sum_j \gamma_j [L_j \rho(t) L_j^\dagger - \frac{1}{2} \{L_j^\dagger L_j, \rho(t)\}] \]

\[ \equiv \mathcal{L}(\rho) \]  

where \( H_S \) is time-independent system Hamiltonian, \( \gamma_j \) is time-independent decay rate, and \( L_j \) are the time-independent Lindblad operators.

From the perspective of quantum statistical mechanics (i.e. knowledge of the system-environment interaction), what was traditionally referred to as Markovian usually assumes (1) the
Born-Markov approximation

\[ \mathcal{M}_{t,s} \approx \delta(t - s)\mathcal{M}_t \]  

(A3)

and (2) Rotating Wave approximation (RWA), in which the rapidly oscillating (or fluctuating) terms of the kernel are further ignored [24, 25, 38].

A noise is called white, when there is a clean separation between the system and environment time scales and the system couples to all frequencies of the environment. This is realized in the scenario where the system-environment coupling strength is weak.

If the Fourier transform of (A3) yields a flat spectral density of the power spectrum of the environment, then such a noise is called white, wherein the system couples to all frequencies of the environment. Physically, this happens when the environmental evolution time-scale is much larger than that of the system, which occurs when one takes the weak coupling limit by making the system environment coupling strength small enough.

The maps \( \mathcal{E} \) generated by the time-independent Lindbladian obey a master equation \( \dot{\mathcal{E}} = \mathcal{L}\mathcal{E} \), whose solution \( \mathcal{E}(t,t_0) = \mathcal{E}(t_2 - t_1) = \exp\{\int(t-t_0)\mathcal{L}\} \) constitutes a family of completely positive, trace preserving (CPTP) maps for all \( t > t_0 \).

The dependence of evolution on the ‘time-difference’ \( t - t_0 \) leads to the fact that the maps obey 1-parameter time-homogeneous composition

\[ \mathcal{E}_{s+t} = \mathcal{E}_s \mathcal{E}_t \]  

(A4)

with \( t \geq s \geq t_0 \).

On the other hand, for colored noise the system preferentially couples to the environment at certain frequencies [42]. In many real life situations one must deal with noises with a colored memory kernel. In this case, the generator deviates from the Lindblad form (A2) to assume the general time-dependent form in which \( H_S = H_S(t), \gamma_j = \gamma_j(t) \) and \( L_j = L_j(t) \).

In its canonical form [38, 16], the \( \{L_j(t)\}'s \) here form a set of orthonormal trace-less operators at any given time \( t \).

If \( \gamma_j(t) \) has a non-trivial time-dependence, the maps do not constitute a QDS. However, if \( \gamma_j(t) \geq 0 \), then the map is CP-divisible [16, 38], and accordingly such processes have been termed time-dependent Markovian, rather than non-Markovian.

In this case, the maps form a two-parameter family of CPTP maps with time-inhomogeneous composition law \( \mathcal{E}(t,t_0) = \mathcal{E}(t,s)\mathcal{E}(s,t_0) \) with \( t \geq s \geq t_0 \). The map \( \mathcal{E}(t,t_0) \) obeys a master equation \( \dot{\mathcal{E}}(t,t_0) = \mathcal{L}(t)\mathcal{E}(t,t_0) \), whose solution is \( \mathcal{E}(t,t_0) = \mathcal{T}\exp\{\int_{t_0}^{t} \mathcal{L}(s)ds\} \).

For this time-dependent 2-parameter map, the integro-differential equation analog of Eq. (A1) is [29]:

\[ \dot{\mathcal{E}}(t,t_0) = \int_{t_0}^{t} \mathcal{M}_{t-s}[\mathcal{E}(s,t_0)]ds, \]  

(A5)

with \( \mathcal{E}(t_0,t_0) = 1 \), where the Hamiltonian part has been absorbed into the memory kernel without loss of generality.

Here, we have used the time-convoluted form \( \mathcal{M}_{t-s} \) of the memory kernel. It is not hard to show that the full map depends on the ‘time-difference’ \( t - t_0 \) [29], but will violate the semigroup law (A4), because the intermediate map \( \mathcal{E}(t_2,t_1) \neq \mathcal{E}(t_2 - t_1) \).

B. Illustration of Self-similarity: The case of Markovian amplitude damping (AD)

Consider the amplitude damping channel (ADC), under which a quantum state \( \rho \) evolves to state \( \rho' \) via the map \( \mathcal{E}_{AD}[\rho] \rightarrow \sum_j A_j(t)\rho A_j^\dagger(t) \), with the Kraus operators \( A_j(t) \) given by \( A_1(t) = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - \lambda(t)} \end{pmatrix} \) and \( A_2(t) = \begin{pmatrix} 0 & \sqrt{\lambda(t)} \\ 0 & 0 \end{pmatrix} \), where

\[ \lambda(t) = 1 - e^{-\gamma_0 t} \]  

(B1)

is the damping factor and \( \gamma_0 \) is the vacuum bath interaction parameter [44, 45].

Now, suppose that the system evolves starting at \( t_i \), going through \( t_1 \) to \( t_2 \). Let the damping factor associated with the full map \( \mathcal{E}_{AD}(t_2,t_1) \) be \( \lambda \), and that with the initial map \( \mathcal{E}_{AD}(t_1,t_2) \) be \( \mu \). For \( \mathcal{E}_{AD}(t_2,t_1) \) the Choi matrix [46] is found to be [17, 40]:

\[ \chi(t_2,t_1) = \begin{pmatrix} 1 & 0 & 0 & \sqrt{1 - \lambda} \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{1 - \lambda} & 0 & 0 & 1 - \lambda \end{pmatrix} \]  

(B2)

while that for the intermediate evolution map \( \mathcal{E}_{AD}(t_2,t_1) \mathcal{E}_{AD}(t_1,t_2)^{-1} \) is found to be [40]:

\[ \chi(t_2,t_1) = \begin{pmatrix} 1 & 0 & 0 & \sqrt{\frac{1 - \lambda}{1 - \mu}} \\ 0 & \frac{\mu}{\mu - 1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{\frac{1 - \lambda}{1 - \mu}} & 0 & 0 & \frac{1 - \lambda}{1 - \mu} \end{pmatrix} \]  

(B3)

Note that the matrices \( \chi(t_2,t_1) \) and \( \chi(t_2,t_1) \) are of the same form provided the functions \( 1 - \lambda \) and \( 1 - \mu \) have the exponential form \( e^{kt} \) for some \( k \). In view of Eq. (B1), this is indeed the case. We thus confirm that amplitude damping is indeed temporally self-similar.

C. Sudarshan A and B matrices for PLN-OUN

The intermediate dynamics can be studied by way of the dynamical map \( A \) introduced by Sudarshan et al [39]. The map \( A \) represents the noise superoperator as a \( d^2 \times d^2 \) matrix on a vector, obtained by vectorizing the density operator, i.e.,

\[ \rho' = A(t_1,t_0) \cdot \rho. \]  

Thus, \( \rho_{jk}' = \sum_{j,k} A_{jk}E_{j,k}\rho_{jk} \). Here \( d \) is the system’s Hilbert space dimension. Given a qubit density
operator \( \begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{bmatrix} \), the channel (2) can be represented as:

\[
\begin{pmatrix}
\rho'_0 \\
\rho'_1 \\
\rho''_0 \\
\rho''_1
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & p(t) & 0 & 0 \\
0 & 0 & p(t) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\rho_{00} \\
\rho_{01} \\
\rho_{10} \\
\rho_{11}
\end{pmatrix},
\]

(C1)

where \( p(t) \) is given by the corresponding probabilities \( p^s(t) \) in Eq. (7) if temporal self-similarity holds, and by Eq. (3) in the general case.

Unlike the Choi matrix [46], the map \( A \) can be composed directly by matrix multiplication: thus \( \rho(t_2) = A(t_2, t_1) \cdot A(t_1, t_0) \rho(t_0) \). The intermediate dynamical map \( A(t_2, t_1) \) can thus be directly computed as \( A(t_2, t_1) = A(t_2, t_0) \cdot A(t_1, t_0)^{-1} \), where the inverse is the matrix inverse and assumed to be non-singular. For Eq. (C1), one readily finds that

\[
A(t_2, t_1) =
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{p(t_2)}{p(t_1)} & 0 & 0 \\
0 & 0 & \frac{p(t_2)}{p(t_1)} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

(C2)

By re-shuffling the terms of the \( A \)-matrix according to \( B_{j'j, k'k} = A_{j'k'j, k} \) [39], from Eq. (C2), one obtains the corresponding \( B \)-matrix, which is just the Choi matrix.

### D. Non-Markovian amplitude damping

Consider a qubit interacting dissipatively with a bath of harmonic oscillators, whose spectral density is given by the Lorentzian

\[
I(\omega) = \gamma_0 g^2 (2\pi (\omega_0 + \Delta - \omega^2) + g^2)^{-1},
\]

where \( g \) is the width of the spectral density function, centered at a frequency detuned from the atomic frequency \( \omega_0 \) by an amount \( \Delta \), and the rate \( \gamma_0 \) quantifies the strength of the system-environment coupling. If we assume \( \Delta = 0 \) (no detuning) i.e., when the qubit is in resonance with the central frequency of the bath, then the GKSL-like time-dependent master equation, with the rotating wave approximation, is given by

\[
\frac{d\rho_+}{dt} = \gamma(t)\{\sigma_-\rho_+(t)\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-\rho_+(t)\}}.
\]

(D1)

where \( \gamma(t) = -2\Re \frac{\dot{G}(t)}{G(t)} \) is the time-dependent decoherence rate, and \( G \) is the decoherence function given by

\[
G(t) = e^{-\frac{gt}{2}} \left( \frac{g}{l} \sinh \left( \frac{lt}{2} \right) + \cosh \left( \frac{lt}{2} \right) \right),
\]

(D2)

with \( l = \sqrt{g^2 - 2\gamma_0 g} \) and \( \sigma_\pm \) are the standard atomic raising (lowering) operators. This is the time-dependent AD channel

and the decay rate is given by

\[
\gamma(t) = -\frac{2}{|G(t)|} \frac{d|G(t)|}{dt} = 2\Re \left( \frac{\gamma_0}{\sqrt{1 - \frac{2\gamma_0}{g^2} \coth \left( \frac{1}{2} \gamma_0 \sqrt{1 - \frac{2\gamma_0}{g^2}} \right) + 1} \right).
\]

(D3)

Now, the expression Eq. (B1) for the damping factor of the ADC is replaced by \( \lambda(t) = 1 - G(t) \) [14]. In the limit \( g < 2\gamma_0 \), the decay rate (D3) oscillates, and becomes negative for certain durations giving rise to non-Markovian evolution. In the limit \( g > 2\gamma_0 \), the dynamics is time-dependent Markovian. (The point \( g = 2\gamma_0 \), however, corresponds to a point at which the time-local master equation lacks a perturbation expansion.) One readily sees that in the limit \( g \gg 2\gamma_0 \), the decay rate \( \gamma(t) = \gamma_0 \), i.e., it becomes time-independent, corresponding to a QDS evolution, the standard AD channel.