Acoustic spacetime is a four-dimensional manifold analogue to the relativistic spacetime with the reference speed of sound replacing the speed of light. In spite of the formal similarities between the two, the analogy between acoustics in fluids and general relativity is considered to be limited by different governing equations only to wave propagation in various background flows. In this paper, the linearized Einstein field equations are used for describing both sound generation and propagation by interpreting sound waves as a weak disturbance of the background metric and treating them relativistically. The quadrupole source of jet noise is found to correspond closely to the quadrupole source of gravitational waves and Lighthill’s 8th-power law is obtained. Exact solutions for the acoustic monopole (pulsating sphere) and dipole (oscillating sphere) are also reproduced in the relativistic framework. The extended analogy between general relativity and acoustics in fluids can have far-reaching implications on theoretical and experimental studies in both fields.
I. INTRODUCTION

The analogy between acoustics and general relativity is based on the formal similarity between certain relativistic phenomena and acoustics in subsonic flows of fluids when the reference speed of light (here denoted with $c_0$) is replaced by the speed of sound. It should be distinguished from the relativistic acoustics which is relevant when the speed of acoustic waves is actually comparable to the speed of light. The first observations on the similarity between special relativity and sound propagation date back to W. Gordon\textsuperscript{1}. However, the analogy is commonly attributed to W. G. Unruh\textsuperscript{2}, who used it for studying Hawking radiation by means of much better understood acoustic effects in transonic flows. Although commonly associated with quantum phenomena, Hawking radiation is almost fully describable with classical mechanics\textsuperscript{3}.

After its establishment, the analogy was further developed and promoted by M. Visser and C. Barceló in their studies of analogue models of gravity\textsuperscript{4–6}. The authors have shown that sound propagation in moving fluids can be described with the differential geometry of a curved spacetime. Similarly as Unruh, they used the analogy to achieve new insights into the gravitational phenomena with the aid of simpler Newtonian physics. For example, gravitational ergo-surfaces are observed in the analogue models as surfaces in fluids where Mach number of the background flow equals one\textsuperscript{7}. The aim of this work is to extend the analogy beyond propagation to wave generation and show that the Einstein field equations can be used universally as governing equations for acoustics in fluids.
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In contrast to the appreciable interest in the analogy for studying gravitation, the relativistic approach has been used very sporadically for illuminating acoustic phenomena. The recent works of Gregory et al.\textsuperscript{8,9} are rare examples in this direction, although Lorentz transformations, which are typical for special relativity, were used much earlier for sound propagation in uniform mean flows\textsuperscript{10,11}. Similarly as Unruh and others, they treat acoustic problems of sound propagation in uniform and non-uniform background flows with the aid of geometric algebra of flat and curved four-dimensional spacetime.

In spite of the successful implementations, the analogy between acoustics in fluids and general relativity appears to be limited by different governing equations (the Einstein field equations and conservation laws of fluid dynamics) only to four-dimensional geometrical interpretations of sound propagation. In this work we consider the full analogy, including both sound generation and propagation, based on the hypothesis that, analogously to gravitational waves, sound waves are weak disturbances of acoustic spacetime. First we introduce the concept of acoustic spacetime as already established in literature and show how it can characterize sound propagation in background flows. Then we give the essentials of the (linearized) theory of gravitation in section III, which provides a framework for treatment of waves as perturbations of the background spacetime. The distinction is made between transverse gravitational waves and longitudinal sound waves in fluids and their representations with the second-order tensor of weak metric perturbation. The analogy based on the four-dimensional acoustic spacetime is further extended to sound generation in section IV. Three main types of the sources of sound in fluids – monopole, dipole, and quadrupole – are analysed using the relativistic approach. The 8\textsuperscript{th}-power law for the quadrupole source of
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jet noise is reproduced, as well as the exact solutions for compact pulsating and oscillating spheres. In the last section the main conclusions which follow from the extended analogy are summarized and the potentials for further investigations are discussed.

II. ACOUSTIC SPACETIME ANALOGY

In order to demonstrate the analogy in the form already present in the literature, we observe a symmetric second-order metric tensor $g$, which is in a specific frame given by its components $g_{\alpha\beta}$. By definition, pseudo-Riemannian spaces, which are considered here, are differentiable manifolds supplied at each point with a metric describing their shape. We use bold symbols for second-order tensors and arrow above a symbol (for example, $\vec{v}$) for vectors. Greek letters are used for the four-dimensional components ($\alpha, \beta, ... = 0...3$, where $0$ denotes the temporal coordinate) and Latin letters for the three spatial components only (for example, $i = 1...3$). Accordingly, $x^0 = c_0 t$, where $c_0$ is constant reference speed of sound or light and $x^1$ to $x^3$ are usual spatial coordinates, for example, $x^1 = x$, $x^2 = y$, and $x^3 = z$. For relativistic quantities we use the notation very similar to Ref. 12.

In general curved spacetime, d’Alembertian of a scalar $\phi$ equals

$$\square \phi = (g^{\alpha\beta} \phi_{,\beta},_\alpha) = \frac{1}{\sqrt{-g}} (\sqrt{-g} g^{\alpha\beta} \phi_{,\beta},_\alpha).$$

(1)

Comma denotes usual derivative with respect to the coordinate which follows it (for example, $\phi_{,\alpha} = \partial \phi / \partial x^\alpha$), while semicolon is used for covariant derivative in curved manifolds ($V^\beta,_{\alpha} = V^\beta,_{\alpha} + V^\mu \Gamma^\beta_{\mu\alpha}$, where $\Gamma^\beta_{\mu\alpha}$ are Christoffel symbols). Determinant of the matrix $(g^{\alpha\beta})$ is $g = \det(g^{\alpha\beta})$. We also use Einstein’s convention, which implies summation over each letter.
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which appears in an expression once as a subscript and once as a superscript. The two positions of the letters refer to the covariant and contravariant vector bases. It is interesting to note that the d’Alembertian in Eq. (1) is also the differential operator of the massless Klein-Gordon equation, which is the simplest Lorentz-invariant equation of motion which a scalar field can satisfy\(^\text{13}\). Therefore, it is natural that the same operator appears in the relativistic framework for sound fields in fluids, which are described with a single scalar.

Adopting the mixed signature \([- + + +]\), the simplest flat (Minkowski) spacetime is given with the metric

\[
g^{\alpha\beta} = \eta^{\alpha\beta} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}.
\]

In this particular case, \(g = -1\), the spacetime is flat, covariant derivative becomes simple derivative, and the d’Alembertian becomes the classical wave operator:

\[
\Box \phi = \eta^{\alpha\beta} \phi_{,\alpha\beta} = \phi^{,\alpha}_{\alpha} = \left(-\frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} + \nabla^2\right) \phi,
\]

since \(\phi = \partial/\partial x^0 = (1/c_0)\partial/\partial t\) and \(\nabla^2 = \partial^2/\partial(x^1)^2 + \partial^2/\partial(x^2)^2 + \partial^2/\partial(x^3)^2\). Multiplication with \(\eta^{\alpha\beta}\) raises the index \(\alpha\) (or \(\beta\)). Similarly, multiplication with \(\eta_{\alpha\beta}\) lowers \(\alpha\) or \(\beta\).
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Another example of a flat spacetime is given with the metric

\[ g^{\alpha\beta} = \begin{bmatrix} -1 & 0 & 0 & -M \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -M & 0 & 0 & 1 - M^2 \end{bmatrix}, \tag{4} \]

where \( M < 1 \) is a dimensionless constant. This time, however, the d’Alembertian has a somewhat more complicated form:

\[ \Box \phi = g^{\alpha\beta} \phi_{,\alpha\beta} = \left( -\frac{1}{c_0^2} \frac{D^2}{Dt^2} + \nabla^2 \right) \phi. \tag{5} \]

Here we recognize the operator \( D/Dt = \partial/\partial t + M c_0 \partial/\partial x^3 \) of the convected wave equation, which describes sound propagation in a uniform subsonic background flow with Mach number \( M \), which is in this case directed along the \( x^3 \)-axis. Since we can orient the axes arbitrarily, it follows that the d’Alembertian in Eq. (1) can describe sound propagation in both quiescent fluid and uniform mean flow, when used with appropriate metric tensors. The metrics such as the one in Eq. (4), which typically occur in acoustic problems, are sometimes also called acoustic metrics\(^7\).

Unruh\(^2\) and others\(^7,9\) have further shown that sound propagation in non-uniform background flows can be described with metrics of curved spacetimes. In particular, one can obtain the wave operator of the Pierce equation\(^14\), which describes sound propagation in inhomogeneous and unsteady low Mach number flows, the characteristic length and time scales of which are larger than those of the acoustic perturbations. Similar metrics are used
to describe a horizon of a black hole\textsuperscript{3} and even more general forms are discussed by Bergliaffa et al.\textsuperscript{15}.

Different metrics of acoustic spacetime are thus proven to capture the effects of background flows on sound propagation, such as convection and refraction. This has been noted by several authors who used the analogy for both acoustics and relativity. However, they restrict it to sound propagation only, due to apparently different physics (governing equations)\textsuperscript{5,9}, nature of the corresponding waves, and mechanisms of their generation. In the following, we expand the analogy to sound generation and complete it by hypothesizing that sound waves in fluids are, like gravitational waves, small disturbance (curvature) of the background acoustic spacetime. In doing so, we will apply usual linearized theory of gravitation. For simplicity, we will suppose that the background spacetime is flat Minkowski spacetime. In other words, we assume a quiescent fluid with constant density \( \rho_0 \) and speed of sound \( c_0 \) in which the sound waves propagate. After the basic description of sound waves in acoustic spacetime, we will consider the mechanism of their aeroacoustic generation within the same relativistic framework.

III. SOUND WAVES IN ACOUSTIC SPACETIME

Governing equations of general relativity are the Einstein field equations\textsuperscript{12,16}. They relate curvature of spacetime, represented by the metric tensor \( g \), with its source, the stress-energy tensor \( T \):

\[
G + \Lambda g = \frac{kG}{c_0^4} T, \tag{6}
\]
where $G$ is Einstein tensor (which depends only on the metric tensor up to its second-order derivatives) and $\Lambda$ (in $\text{m}^{-2}$), dimensionless $k$ (not to be confused with wave number), and $G$ (in $\text{m}^3/(\text{kg s}^2)$) are constants. We use bold symbols for second-order tensors. Equation (6) is written in a frame-independent form. Local conservation of energy and momentum provide the additional conditions, which expressed in terms of the components in a given frame read

$$G^{\alpha\beta;\beta} = T^{\alpha\beta;\beta} = 0. \quad (7)$$

If we are not interested in steady (or slowly varying) solutions, we can adopt usual $\Lambda = 0$. In fact, we will suppose that the only disturbance of otherwise flat spacetime is due to the waves. In an appropriate frame, the metric tensor can be written as a sum of $\eta^{\alpha\beta}$ and a weak component $h^{\alpha\beta}$:

$$g^{\alpha\beta} = \eta^{\alpha\beta} + h^{\alpha\beta}, \quad (8)$$

with $|h^{\alpha\beta}| \ll 1$.

Within the first-order (linear) approximation, it can be shown\(^\text{12}\) that there always exists $\bar{h}^{\alpha\beta}$ ($|\bar{h}^{\alpha\beta}| \ll 1$):

$$\bar{h}^{\alpha\beta} = h^{\alpha\beta} - \frac{1}{2} \eta^{\alpha\beta} h^\nu_\nu, \quad (9)$$

such that

$$\bar{h}^{\alpha\beta;\beta} = 0 \quad (10)$$

and, consequently,

$$G^{\alpha\beta} = -\frac{1}{2} \square h^{\alpha\beta}. \quad (11)$$

The term $h^\nu_\nu$ is the trace of $h^{\alpha\beta}$ and Eq. (10) is the Lorenz gauge condition. For example, if the Lorenz gauge condition is not satisfied directly by $\bar{h}^{\alpha\beta}$ in Eq. (9) in a certain frame,
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one can introduce a small change of coordinates (gauging)

\[ x^\alpha \rightarrow x^\alpha + \xi^\alpha, \quad (12) \]

which transforms the metric as

\[ h_{\alpha\beta} \rightarrow h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}, \quad (13) \]

such that

\[ \Box \xi^\alpha = \xi^{\alpha,\beta,\beta} = \left( h^{\alpha\beta} - \frac{1}{2} \eta^{\alpha\beta} h^\nu_\nu \right)_{,\beta} \neq 0 \quad (14) \]

and therefore

\[ \bar{h}^{\alpha\beta} = h^{\alpha\beta} - \frac{1}{2} \eta^{\alpha\beta} h^\nu_\nu - \xi^{\alpha,\beta} - \xi^{\beta,\alpha} + \eta^{\alpha\beta} \xi^\nu_\nu \quad (15) \]

does satisfy it. Note that

\[ \bar{h}^{\alpha}_\alpha = \eta^{\alpha\beta} \bar{h}^{\alpha\beta} = \eta^{\alpha\beta} \left( h^{\alpha\beta} - \frac{1}{2} \eta^{\alpha\beta} h^\nu_\nu \right) \]

\[ = h^{\alpha}_\alpha - \frac{1}{2} \eta^{\alpha}_\alpha h^\nu_\nu = h^{\alpha}_\alpha - 2 h^{\alpha}_\alpha = -h^{\alpha}_\alpha, \quad (16) \]

so \( h^{\alpha\beta} \) and \( \bar{h}^{\alpha\beta} \) are mutually trace reverse and Eq. (9) can be inverted to

\[ h^{\alpha\beta} = \bar{h}^{\alpha\beta} - \frac{1}{2} \eta^{\alpha\beta} \bar{h}^\nu_\nu, \quad (17) \]

which will be used later. Inserting Eq. (11) into Eq. (6) with \( \Lambda = 0 \) gives the linearized Einstein field equations:

\[ \Box h^{\alpha\beta} = -\frac{2kG}{c_0^4} T^{\alpha\beta}. \quad (18) \]

First we will exclude the source term and consider only wave propagation. Since the background spacetime is flat, Eq. (18) is classical wave equation with the operator from Eq. (3), that is

\[ \Box \bar{h}^{\alpha\beta} = \left( -\frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} + \nabla^2 \right) \bar{h}^{\alpha\beta} = 0. \quad (19) \]
The simplest solution has the form of a plane wave, the real part of

$$\bar{h}^{\alpha\beta} = A^{\alpha\beta} e^{jk_\nu x^\nu},$$  \hspace{1cm} (20)

where the components of the polarization tensor, $A^{\alpha\beta}$, are complex constants and the four-vector $k^\alpha$ is null vector in the flat Minkowski spacetime: $k_\alpha k^\alpha = \eta_{\alpha\beta} k^\beta k^\alpha = 0$. For example, if we suppose that the plane wave propagates in the direction of the $x^3$-axis, $k^\alpha = [\omega/c_0, 0, 0, \omega/c_0]$, $k_\alpha = \eta_{\alpha\beta} k^\beta = [-\omega/c_0, 0, 0, \omega/c_0]$, and we can obtain the usual exponent $-j\omega(t - z/c_0)$, after replacing $x^0$ with $c_0 t$ and $x^3$ with $z$.

The gauge condition in Eq. (10) gives additional constraint

$$k^\beta A_{\alpha\beta} = 0,$$  \hspace{1cm} (21)

which follows from the equality $\bar{h}^{\alpha\beta, \nu} = jk_\nu \bar{h}^{\alpha\beta} = jk_\nu A^{\alpha\beta} e^{j k_\mu x^\mu}$. Since it involves four equations, the condition decreases the number of the unknown components $A^{\alpha\beta}$ from 10 to 6. This number can be decreased further. We note that Eq. (14) suggests that a small change of coordinates, achieved by adding a vector $\zeta^\mu$ to $\xi^\mu$ from Eq. (12), leaves equations (8) and (10) satisfied if

$$\Box \zeta^\mu = 0.$$  \hspace{1cm} (22)

This gauge invariance allows further freedom in selecting a specific gauge within the class of Lorenz gauges and the introduction of additional constraints.

A typical treatment of plane gravitational waves takes the solution of Eq. (22) (with the lowered index),

$$\zeta_\alpha = B_\alpha e^{j k_\mu x^\mu},$$  \hspace{1cm} (23)
in which\(^{17}\)

\[
B_\alpha = \frac{1}{U^\nu k_\nu} \left( -k_\alpha U^\beta B_\beta - j U^\beta A_{\alpha \beta} + j \frac{1}{2} A^\beta \gamma U_\alpha \right) \quad (24)
\]

and

\[
U^\beta B_\beta = -\frac{j}{2 U^\nu k_\nu} \left( U^\beta U^\alpha A_{\beta \alpha} + \frac{1}{2} A_\mu \gamma \right). \quad (25)
\]

Here, \(\vec{U}\) denotes dimensionless four-velocity vector. In a particle’s momentarily comoving reference frame, it is a constant timelike unit vector (time basis vector): \(U^\alpha = \delta^\alpha_0\), where \(\delta^\alpha_\beta\) is Kronecker delta. Hence, in this frame \(\vec{U} \cdot \vec{U} = -1\). For a slowly moving (non-relativistic) particle, three-dimensional velocity satisfies \(|\vec{v}| \ll c_0\) and the spatial components of the four-velocity vector are \(v^\beta/c_0\), which justifies naming it velocity. However, it should be noted that even when the background space is not flat, we can always make a background Lorentz transformation such that \(U^\beta = \delta^\beta_0\) in the specific frame. We can also point the spatial axes in this frame such that \(k^\alpha = [\omega/c_0, 0, 0, \omega/c_0]\), so the treatment here is completely general.

After the additional gauging with \(\zeta_\alpha\), Eq. (15) gives

\[
\vec{h}^{TT \alpha \beta} = \vec{h}_{\alpha \beta} - \zeta_{\alpha, \beta} - \zeta_{\beta, \alpha} + \eta_{\alpha \beta} \zeta^\nu \gamma. \quad (26)
\]

This specifies the so-called transverse-traceless (TT) gauge in which \(A^{TT \alpha} = 0\) (zero trace) and \(A^{TT \alpha}_\gamma U^\beta = A^{TT \alpha}_\gamma \delta^\beta_0 = 0\). From the second equality it follows that \(A^{TT \alpha}_\alpha = A^{TT \alpha}_0 = 0\) and from Eq. (21) \(k_\alpha A^{TT \alpha \beta} = \omega A^{TT \alpha \beta}/c_0 = 0\) for \(k^\alpha = [\omega/c_0, 0, 0, \omega/c_0]\), which explains why the
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gauge is transverse. In the transverse-traceless gauge, the polarisation tensor equals

\[
A^{TT}_{\alpha\beta} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & A^{TT}_{11} & A^{TT}_{12} & 0 \\
0 & A^{TT}_{12} & -A^{TT}_{11} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

(27)

We can also note that gauging in equations (15) and (26) does not change the order of magnitude of the metric perturbation:

\[
|h_{\alpha\beta}| \sim |\bar{h}_{\alpha\beta}| \sim |\bar{h}^{TT}_{\alpha\beta}| \ll 1.
\]

(28)

It is the maximum order of magnitude and the solution in transverse-traceless gauge is relevant for the gravitational waves, which are transverse waves with two polarizations represented by the two components \(A^{TT}_{11}\) and \(A^{TT}_{12}\). The two components are analogous to the electric and magnetic components of transverse electromagnetic waves.

Since it suppresses all longitudinal components, transverse-traceless gauge is suitable for the transverse gravitational waves. It may also be relevant for acoustics in solids, if transverse waves dominate over the longitudinal waves. However, much more suitable for longitudinal acoustic waves in fluids is Newtonian gauge, which is used for calculations of the corrections of classical Newtonian gravitational potential. To show how it can describe sound waves in acoustic spacetime, we first note that each free particle in curved spacetime obeys the geodesic equation

\[
\frac{1}{c_0} \frac{d\vec{U}}{d\tau} = 0,
\]

(29)
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where \( \tau \) is proper time \( (c_0^2d\tau^2 = -ds^2) \), where \( ds \) is interval between two infinitesimally close events in spacetime; for example, in a flat spacetime \( ds^2 = -c_0^2dt^2 + dx^2 + dy^2 + dz^2 \). The geodesic equation says that a free particle follows its world line. In a curved spacetime it can also be written as\(^{12} \)

\[
U^\alpha;\beta U^\beta = \frac{1}{c_0} \frac{dU^\alpha}{d\tau} + \Gamma^\alpha_{\mu\nu} U^\mu U^\nu = 0. \tag{30}
\]

Christoffel symbols \( \Gamma^\alpha_{\mu\nu} \) are related to the metric by the equality

\[
\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}). \tag{31}
\]

If the particle is moving slowly (compared to \( c_0 \)) in an essentially flat spacetime, its four-acceleration due to a small metric perturbation \( h_{\alpha\beta} \) is in the first order approximation of Eq. (30)

\[
\frac{dU^\alpha}{d\tau} = -c_0 \Gamma^\alpha_{00} = -\frac{c_0}{2} \eta^{\alpha\beta} (h_{\beta0,0} + h_{0\beta,0} - h_{00,\beta}), \tag{32}
\]

with \( \bar{U} = d\vec{x}/(c_0\tau) \). The condition for a non-relativistically moving particle is satisfied by weak acoustic waves as well, since the particle velocity is much smaller than the speed of sound. For such a particle, \( \tau \approx t \) and therefore the three-dimensional acceleration equals

\[
\frac{d^2x^k}{dt^2} = -\frac{c_0^2}{2} \eta^{kl} (h_{l0,0} + h_{0l,0} - h_{00,l}). \tag{33}
\]

Finally, in the Newtonian form \( |h_{l0}| = |h_{0l}| \ll |h_{00}| \) and consequently

\[
\frac{d^2x^k}{dt^2} = \frac{c_0^2}{2} h^{00,k}. \tag{34}
\]

Motion of the particle due to the wave is expressed with a single scalar \( h_{00} \), as we expect from the compressible acoustic waves in fluids. In this way we can obtain a measurable quantity (acceleration) from the relativistic analogy.
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As a summary, the physical sound wave is naturally described by the metric component $h_{00}$ in the Newtonian gauge. Lorenz gauge condition discussed above can be used first in order to solve the linearized Einstein equations with $\bar{h}^{\alpha\beta}$, but eventually we have to switch to the Newtonian gauge (rather than transverse-traceless gauge) in order to obtain classical acoustic quantities. From Eq. (34) we can determine other relevant quantities. Since acoustic velocity and potential $\phi$ are related by the equality

$$v^k = \frac{dx^k}{dt} = \phi^k,$$

we can associate $h_{00}$ with $\phi$:

$$\frac{d\phi}{dt} = \frac{c_0^2}{2} h_{00}.$$  \hfill (36)

If the fluid is quiescent, acoustic pressure and potential are related over the equality

$$p = -\rho_0 \frac{d\phi}{dt} = -\frac{\rho_0 c_0^2}{2} h_{00}.$$  \hfill (37)

Finally, for a plane sound wave, component of the particle velocity in the direction of wave propagation (say, $x^3$-axis) equals

$$v^3 = \frac{p}{\rho_0 c_0} = -\frac{c_0}{2} h_{00}$$  \hfill (38)

and, indeed, $|v^3| \ll c_0$ for $|h_{00}| \ll 1$.

It is important to note that the value of $h_{00}$ in the Newtonian gauge does not have to be of the same order of magnitude as $\bar{h}^{TT}_{\alpha\beta}$ (or $\bar{h}_{\alpha\beta}$ or $h_{\alpha\beta}$ from which it is derived) if the transverse metric perturbations dominate over the longitudinal components. In fact, we will see that for compact sources $h_{00}$ is smaller than $\bar{h}^{TT}_{\alpha\beta}$ by factor $(\omega L/c_0)^2$, where $L$ is characteristic length scale of the source and $(\omega L/c_0) \to 0$ for a point source. In order to show this, we shall study how the component $h_{00}$ is generated by different sources of sound.
IV. GENERATION OF SOUND WAVES

In the previous section we showed how metric perturbation can describe a sound wave in acoustic spacetime by purely geometric means. In this section we inspect several mechanisms of wave generation. As in general relativity at small (non-relativistic) velocities, it is the matter satisfying the conservation laws which causes the curvature of spacetime around it.

A. Aeroacoustic sound generation

In order to study sound generation, we should refer back to the linearized Eq. (18) with the source term. The source in acoustic spacetime has to satisfy the conservation laws, Eq. (7):

\[ T^{\alpha\beta}_{\alpha\beta} = 0. \] (39)

In general relativity, stress-energy tensor of a perfect fluid equals

\[ T = (\rho c_0^2 + p) \tilde{U} \otimes \tilde{U} + p g^{-1}, \] (40)

where \( \rho \) is energy density, \( p \) is pressure, \( \tilde{U} \) is four-velocity of particles \( (\tilde{U} \otimes \tilde{U} = U^\alpha U^\beta) \), and \( g \) is metric tensor. If all particles of the fluid move with small three-dimensional velocity, \( |\vec{v}| \ll c_0 \), then the approximation \( \tilde{U} \approx (1, \vec{v}/c_0) = [1, v^1/c_0, v^2/c_0, v^3/c_0] \) holds and the
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components of the stress-energy tensor in nearly flat spacetime (Eq. (2)) are

\[
T^{\alpha \beta} =
\begin{bmatrix}
\rho c_0^2 & P c_0 v^1 & P c_0 v^2 & P c_0 v^3 \\
\rho c_0 v^1 & P v^1 v^1 + p & P v^1 v^2 & P v^1 v^3 \\
\rho c_0 v^2 & P v^1 v^2 & P v^2 v^2 + p & P v^2 v^3 \\
\rho c_0 v^3 & P v^1 v^3 & P v^2 v^3 & P v^3 v^3 + p \\
\end{bmatrix},
\]

where \( P = \rho + p/c_0^2 \).

We shall notice the similarity between the spatial part of the stress-energy tensor \((T^{j k})\) and Lighthill’s tensor\(^{18}\), which is a pure aeroacoustic source of sound in fluids in free space (without boundaries). Apart from the different value of \( c_0 \) (speed of sound instead of speed of light), the main differences are that in classical fluid dynamics \( \rho \) denotes usual density of the matter, rather than energy density, and the appearance of pressure in the sum \( P = \rho + p/c_0^2 \) in the momentum and stress terms. Usually, however, \( p/c_0^2 \ll \rho \) when \( |\vec{v}| \ll c_0 \), which is in the context of aeroacoustics satisfied in subsonic flows with low Mach number value. Pressure \( p \) should not be confused with acoustic pressure. Moreover, the mass energy usually dominates over the energy of massless particles in general relativity, as well, especially at non-relativistic speeds, which makes the analogy even deeper. The exception are highly non-relativistic fluids (for example, the early universe is considered to be a soup of massless particles). It is interesting to note that such a fluid still posses compressibility and the speed of sound can be defined\(^{19}\). The maximum speed of sound is then of the same order of magnitude as the speed of light, smaller only by the factor \( \sqrt{3} \). This justifies the relativistic treatment of acoustics even in fundamental theories.
Another important difference is that Lighthill’s tensor is derived from the conservation laws after the weak acoustic part (represented by the terms with acoustic pressure or density) is shifted to the left-hand side of the wave equation. Therefore, it alone cannot satisfy the conservation laws. In contrast to this, the full $T^{\alpha \beta}$ which we consider as the source satisfies the conservation laws in Eq. (39). It is the source of the perturbations of spacetime itself, so we do not need to split it into the source and propagation parts in terms of the dynamic quantities.

In the further treatment of aeroacoustic sound generation by the stress-energy tensor, we follow Misner et al.\textsuperscript{16} and consider a single isolated source of waves, far from which the spacetime is asymptotically flat towards the infinity, with the background metric given in Eq. (2). Small metric perturbation is defined by Eq. (8) everywhere (including the source region). Linearized Eq. (18) holds under the condition in Eq. (10) even if the source is relativistic, with high subsonic velocities in the source region\textsuperscript{16}. Furthermore, we expect that only small part of the stress-energy tensor is responsible for the radiation of waves. In order to emphasize this, we can formally split it into the dominant effective stress-energy tensor, $T^{\text{eff}}_{\alpha \beta}$, and the small component $t_{\alpha \beta}$: $T_{\alpha \beta} = T^{\text{eff}}_{\alpha \beta} + t_{\alpha \beta}$. Thus, we have

$$\Box \bar{h}_{\alpha \beta} = -\frac{2kG}{c_0^4} (T^{\text{eff}}_{\alpha \beta} + t_{\alpha \beta}).$$

Such a split may resemble the one in Lighthill’s aeroacoustic analogy, but no part of the stress energy tensor, including the radiation-related part, is shifted to the left-hand side of Eq. (42). The general solution for both ingoing ($\epsilon = -1$) and outgoing ($\epsilon = +1$) wave is

$$\bar{h}_{\alpha \beta} = \frac{kG}{2\pi c_0} \int \frac{T^{\text{eff}}_{\alpha \beta} + t_{\alpha \beta}}{R} (t - \epsilon R/c_0) d^3 \bar{y},$$

(43)
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where the integral is over the entire three-dimensional space and $R = |x^i - y^i|$. The values of $T_{\alpha \beta}^{\text{eff}}$ and $t_{\alpha \beta}$ are to be evaluated at the retarded time $t - \epsilon R/c_0$.

Next, we will assume that the source is compact, so that its characteristic length scale satisfies $L \ll c_0/\omega$, where $\omega$ is characteristic angular frequency of the oscillations. This actually comes down to the condition for non-relativistic motion, $|\vec{v}| \ll c_0$, since $|\vec{v}| \sim L \omega$. For this reason, the condition is also referred to as slow-motion condition. We will also consider far geometric field ($R \gg L$). Thus, we can approximate

$$
\bar{h}_{\alpha \beta} = \frac{kG}{2\pi r c_0^4} \int [T_{\alpha \beta}^{\text{eff}} + t_{\alpha \beta}] (t - \epsilon \tau/c_0) \, \mathrm{d}^3 \vec{y},
$$

(44)

where $r$ is radial coordinate of the spherical coordinate system with the source in its origin.

From the conservation laws, Eq. (39), one can deduce the identity$^{16}$

$$
\frac{1}{c_0^2} \frac{d^2}{dt^2} \int (T_{00}^{\text{eff}} + t_{00}) x_j x_k \, \mathrm{d}^3 \vec{x} = 2 \int (T_{jk}^{\text{eff}} + t_{jk}) \, \mathrm{d}^3 \vec{x}.
$$

(45)

It relates the spatial components of the stress-energy tensor with the (scalar) component $T_{00}$, which ultimately removes the need for a split as in Lighthill’s analogy. In fact, this is allowed by the fact that the stress-energy tensor satisfies the conservation laws, which does not hold for Lighthill’s tensor. The second-order time derivative on the left-hand side of Eq. (45) corresponds to the second-order derivatives of the source terms in Lighthill’s analogy. They naturally appear when the second-order tensor should be contracted to a scalar. The integral on the left-hand side of Eq. (45) multiplied with $1/c_0^2$ represents the second moment of the mass distribution and it is called quadrupole moment tensor of the mass distribution. It is usually denoted with $I_{jk}$, which is, thus, by definition

$$
I_{jk} = \frac{1}{c_0^2} \int (T_{00}^{\text{eff}} + t_{00}) x_j x_k \, \mathrm{d}^3 \vec{x}.
$$

(46)
The quantity which appears to be more convenient for mathematical description of wave generation is reduced quadrupole moment, defined as

\[ \mathcal{I}_{jk} = I_{jk} - \frac{1}{3} \delta_{jk} I_l^l. \] (47)

From equations (44)-(46),

\[ \bar{h}_{jk} = \frac{kG}{4\pi r c_0^4} \frac{d^2}{dt^2} I_{jk}(t - cr/c_0). \] (48)

We will now limit ourselves to the derivation of Lighthill’s scaling law for the acoustic power of the quadrupole source of jet noise. For this, it is sufficient to estimate reaction of the source to the far-field radiation, which can actually be done in the acoustic near field. Expanding \( \bar{h}_{jk} \) in powers of \( r \) for \( \omega r/c_0 \ll 1 \) (near field condition) and leaving only the terms with \( \epsilon \), which correspond to the wave radiation, gives (after replacing \( \epsilon = 1 \) for an outgoing wave)

\[ \bar{h}_{jk}^{\text{react}} = -\frac{kG}{4\pi c_0^3} \frac{d^3}{dt^3} I_{jk}(t) - \frac{kG}{24\pi c_0^5} r^2 \frac{d^5}{dt^5} I_{jk}(t) + ..., \] (49)

where ... denotes the omitted higher-order terms. Using Eq. (10), we can also find

\[ \bar{h}_{0j}^{\text{react}} = -\frac{kG}{12\pi c_0^3} x^k \frac{d^4}{dt^4} I_{jk}(t) - \frac{kG}{120\pi c_0^5} r^2 x^k \frac{d^6}{dt^6} I_{jk}(t) + ... \] (50)

and

\[ \bar{h}_{00}^{\text{react}} = -\frac{kG}{12\pi c_0^3} \frac{d^4}{dt^4} I_{jj}(t) - \frac{kG}{120\pi c_0^5} (r^2 \delta^{jk} + 2x^j x^k) \frac{d^6}{dt^6} I_{jk}(t) + ... \] (51)

These components represent reaction potentials in Lorenz gauge (reaction to the radiation).

In terms of gravitation, the omitted terms which do not contribute to the radiation are corrections of the Newtonian potential producing the effects such as perihelion shift. In the
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acoustic analogy, they describe incompressible fluctuations which do not propagate into the far field.

In order to arrive at a Newtonian form which describes the acoustic radiation, we first switch back to $h_{\alpha\beta} = \bar{h}_{\alpha\beta} - \bar{h}^{\nu} \eta_{\alpha\beta}/2$ from Eq. (17) and change the coordinates to $x^\mu + \xi^\mu$, with

$$\xi_0 = - \frac{kG}{12\pi c_0^4} \frac{d^2}{dt^2} I_{ll}(t) + \frac{kG}{48\pi c_0^6} x^j x^k \frac{d^4}{dt^4} I_{jk}(t)$$

and

$$\xi_j = - \frac{kG}{8\pi c_0^5} x^k \frac{d^3}{dt^3} I_{jk}(t) + \frac{kG}{24\pi c_0^5} x^j \frac{d^3}{dt^3} I_{ll}(t).$$

In this gauge,

$$h_{00}^{\text{react}} = - \frac{kG}{20\pi c_0^4} x^j x^k \frac{d^5}{dt^5} T_{jk}(t),$$

to the lowest order, while the components $h_{0j}^{\text{react}} \sim (\omega L/c_0) h_{00}^{\text{react}}$ are of higher order. Therefore, we have derived the metric component $h_{00}$ in the Newtonian form and it should describe the acoustic longitudinal perturbation of acoustic spacetime in the near field of the isentropic quadrupole source $\rho \vec{v} \vec{v}$ in Lighthills analogy. The geodesic equation (34) gives the acceleration of a non-relativistic particle affected by the perturbation:

$$\frac{d^2 x^l}{dt^2} = \frac{c_0^2}{2} h_{00}^{\text{react}, l} = - \frac{kG}{40\pi c_0^6} \left( x^j x^k \frac{d^5}{dt^5} T_{jk}(t) \right)^l.$$

We are now ready to derive Lighthill’s scaling law for the source power. We suppose that $T_{jk}$ scales as $\rho_0 |\vec{v}|^2$, where $\rho_0$ is density of essentially incompressible fluid. From equations (45)-(47), $|\vec{T}_{jk}| \sim \rho_0 L^5$, so the acoustic particle velocity from Eq. (55) scales as

$$|\vec{v}_{ac}| \sim \frac{kG \rho_0 L^2}{c_0^4} (\omega L)^4.$$
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The intensity (energy flux) scales as

\[ |\vec{I}| \sim \rho_0 c_0 |\vec{v}_{ac}|^2 \sim \frac{k^2 G^2 \rho_0^3 L^4}{c_0} \left( \frac{|\vec{v}|}{c_0} \right)^8, \quad (57) \]

where we also replaced \( \omega L \sim |\vec{v}| \). For \( L \sim r \) in the near field, Lighthill’s 8\textsuperscript{th}-power law gives the scaling\textsuperscript{11}: \( |\vec{I}| \sim \rho_0 c_0^3 (|\vec{v}|/c_0)^8 \). Neglecting the multiplication constant which depends on the dimensionless constant \( k \), the two solutions become equivalent if we set

\[ G = \frac{c_0^2 L}{2M} \sim \frac{c_0^2}{2 \rho_0 L^2}, \quad (58) \]

where \( M \) is total mass of the source. In this way, we can identify the length scale \( L \) as acoustic Schwarzschild radius (characteristic radius of a source of gravitational waves):

\[ L = \frac{2GM}{c_0^2}. \quad (59) \]

In cosmology, Schwarzschild radius determines the length scale of a black hole and its event horizon. It makes sense that a similar concept appears to determine the length scale of an aeroacoustic source in acoustic spacetime. This points to the similar behaviour of vorticity, as the source of aeroacoustic sound, and merging black holes, as the quadrupole sources of gravitational waves.

If \( T_{00} \sim \rho c_0^2 \) dominates aeroacoustic sound generation, as in jets with combustion or significant vapour condensation\textsuperscript{11}, the acoustic power scales as \( (\omega L)^4 \sim |\vec{v}|^4 \), which is entirely because the waves in question are longitudinal (compare with squared Eq. (76) in section IV B). In contrast to this, transverse gravitational waves are characterized with a second-order tensor, which gives the order of magnitude higher by the factor \( (\omega L/c_0)^{-4} \). The dependence of amplitude on the square of frequency appears in Lighthill’s analogy as a
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consequence of the double differentiation of the source terms. However, the representation by means of acoustic spacetime seems to be more natural, since the full stress-energy tensor is used as the source of waves, as already discussed. There is no need for splitting it into the source and propagation terms, or selecting an appropriate dynamic quantity for the analogy (acoustic pressure or density), since the waves are purely geometric perturbations of spacetime. Transition from the second-order tensor to the scalar is done by expressing the components of the obtained metric in the Newtonian gauge, rather than by double divergence of the stress-energy tensor on the right-hand side. Possible effects of the mean flow on sound propagation outside the source region, such as convection and refraction, have to be taken into account with appropriate background metric replacing the metric $\eta_{\alpha\beta}$ from Eq. (8). This was discussed in section II and in more details in\textsuperscript{8} and\textsuperscript{9}.

B. Pulsating sphere

Acoustic monopole has no counterpart in the theory of gravitation or electromagnetism and pulsating spherical objects cannot produce gravitational or electromagnetic waves. The reason is that the sound waves in fluids are compressible and described with scalar potential. Therefore, it is worthwhile to study the appearance of a monopole source in acoustic spacetime.

As an example of an ideal monopole source, we will consider radiation of a compact pulsating sphere, for which the exact solution exists. We will solve Eq. (42),

$$\Box \bar{h}_{\alpha\beta} = -\frac{2kG}{c_0^2} T_{\alpha\beta} = -\frac{2kG}{c_0^2} (T_{\alpha\beta}^{\text{eff}} + t_{\alpha\beta}),$$  \hfill (60)
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Similarly as in\textsuperscript{12}. We suppose simple oscillations of $t_{\alpha\beta}$ with frequency $\omega$ and amplitude $S_{\alpha\beta}$:

$$t_{\alpha\beta} = S_{\alpha\beta} e^{-j\omega t}. \quad (61)$$

Since the source is compact, $\omega L/c_0 \ll 1$, where $L$ is radius of the sphere. Outgoing wave solution of Eq. \((60)\) in the far field has the form of a spherical wave,

$$\tilde{h}_{\alpha\beta} = \frac{A_{\alpha\beta}}{r} e^{-j\omega(t-r/c_0)}, \quad (62)$$

with $r$ denoting distance from the source and $A_{\alpha\beta}$ complex constants. We neglected any terms of order $1/r^{-2}$ (the far-field approximation). After cancelling the time dependence on both sides of Eq. \((60)\), we obtain a form of the Helmholtz equation:

$$[(\omega/c_0)^2 + \nabla^2] \left( \frac{A_{\alpha\beta}}{r} e^{j\omega r/c_0} \right) = -\frac{2kG}{c_0^4} S_{\alpha\beta}. \quad (63)$$

Integrating the left-hand side of the equation over the source region gives the following terms:

$$\int_V \frac{\omega^2 A_{\alpha\beta}}{c_0^2} \frac{e^{j\omega r/c_0}}{r} d^3 \vec{y} = \frac{4\pi L^2 \omega^2}{3} \frac{A_{\alpha\beta}}{c_0^2} e^{j\omega L/c_0}$$

$$= \frac{4\pi}{3} \left( \frac{\omega L}{c_0} \right)^2 A_{\alpha\beta}, \quad (64)$$

where $A_{\alpha\beta}$ is taken to be constant within the compact sphere, and

$$\int_V \nabla^2 \left( \frac{A_{\alpha\beta}}{r} e^{j\omega r/c_0} \right) d^3 \vec{y}$$

$$= \oint_S \vec{n} \cdot \nabla \left( \frac{A_{\alpha\beta}}{r} e^{j\omega r/c_0} \right) d^2 \vec{y}$$

$$= 4\pi L^2 \frac{d}{dr} \left( \frac{A_{\alpha\beta}}{r} e^{j\omega r/c_0} \right)_{r=L}$$

$$= 4\pi L^2 \left( \frac{-A_{\alpha\beta}}{r^2} e^{j\omega r/c_0} + \frac{j\omega A_{\alpha\beta}}{c_0 r} e^{j\omega r/c_0} \right)_{r=L}$$

$$= -4\pi A_{\alpha\beta} + j4\pi \left( \frac{\omega L}{c_0} \right) A_{\alpha\beta}, \quad (65)$$
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with \( \vec{n} \) unit vector normal to the surface of the sphere pointing outwards. We also approximated \( e^{i\omega L/c_0} \approx 1 \).

For the compressible acoustic waves, none of the two terms on the left-hand side of Eq. (63) can be negligible compared to the other one. Consequently, their amplitude must be associated with the term in Eq. (64), which is weaker than the last two terms in Eq. (65) for compact sources, which have lower orders of \( \omega L/c_0 \). Although much stronger, the latter terms describe transverse waves only. In fact, the first of the two, which is the strongest, is associated with gravitational waves (described by a second-order tensor). Longitudinal (scalar) component of the metric perturbation is removed in the context of gravitation by utilizing the gauge invariance in Eq. (22). The weaker acoustic waves are thus left out as the higher order gauge terms. As a consequence, a source oscillating spherically symmetric, such as a pulsating sphere, generates only longitudinal waves and cannot radiate transverse gravitational waves.

The leading term from Eq. (65) and Eq. (63) give the following expression for the polarization tensor of gravitational waves:

\[
A_{\alpha\beta} = \frac{kG}{2\pi c_0^4} \int_V S_{\alpha\beta} d^3 y'.
\] (66)

After comparing the term from Eq. (64) with the leading term from Eq. (65), we can conclude that the acoustic waves are weaker by the order \( (\omega L/c_0)^2 \) (only a small fraction of \( t_{\alpha\beta} \) actually generates the longitudinal waves). For the calculation of acoustic \( A_{\alpha\beta} \), the additional factor of -1/3 should be multiplied with 2 in order to include fraction of the \( \nabla^2 \) term in Eq. (63) which is of the same order as the \( \omega^2/c_0^2 \) term and also provided by the source \( S_{\alpha\beta} \). Thus, we
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obtain the polarization tensor

$$A_{\alpha\beta} = -\frac{kG}{3\pi c_0^4} \left(\frac{\omega L}{c_0}\right)^2 \int_V S_{\alpha\beta}d^3\vec{y}$$

(67)

and from Eq. (62):

$$\bar{h}_{\alpha\beta} = -\frac{kG}{3\pi c_0^4} \left(\frac{\omega L}{c_0}\right)^2 \frac{e^{j\omega r/c_0}}{r} \int_V t_{\alpha\beta}d^3\vec{y}.$$  

(68)

This metric perturbation in Lorenz gauge captures the longitudinal sound wave in acoustic spacetime due to the compact pulsating sphere.

Physically, acoustic monopole radiation can originate from a source with unsteady mass or varying Schwarzschild radius given in Eq. (59). The latter is the case with pulsating sphere, although mass injection comes similarly down to the effect of displacement of a volume fraction of the fluid surrounding it\textsuperscript{11}. The only non-zero component $\bar{h}_{00}$ equals

$$\bar{h}_{00} = -\frac{kG}{3\pi c_0^4} \left(\frac{\omega L}{c_0}\right)^2 \frac{e^{j\omega r/c_0}}{r} \int_V t_{00}d^3\vec{y}.$$  

(69)

This solution also follows from the Schwarzschild metric, which is the only spherically symmetric and asymptotically flat (for $r \to \infty$) solution of the Einstein field equations in vacuum. For linearized Eq. (60), the solution in Lorenz gauge for large $r$ (in the far field) is\textsuperscript{16} $\bar{h}_{00} = kMG/(2\pi r c_0^2)$, $\bar{h}_{0j} = \bar{h}_{jk} = 0$. The perturbation of mass $m = M - M^{\text{eff}}$, which for a pulsating sphere replaces the mass part of the volume integral of $t_{00} = (\rho - \rho^{\text{eff}})c_0^2$ and which is of interest for acoustic radiation is scaled with the factor $-2(\omega L/c_0)^2/3$ for the same reasons which lead to Eq. (67).

We can now replace

$$\int_V t_{00}d^3\vec{y} = \rho_0 c_0^2 \frac{4}{3} (L + \bar{t}e^{-j\omega t})^3 \pi \approx 4\rho_0 c_0^2 L^2 \pi \bar{t}e^{-j\omega t},$$  

(70)
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where $\overline{l} \ll L$ is amplitude of the oscillations around $L$ and obtain

$$\bar{h}_{00} = -\frac{4kG\rho_0 L^2}{3c_0^2} \left(\frac{\omega L}{c_0}\right)^2 \frac{e^{-j\omega(t-r/c_0)}}{r}.$$  \hspace{1cm} (71)

From Eq. (58) we have

$$G = \frac{3c_0^2}{8\rho_0 L^2\pi}$$  \hspace{1cm} (72)

and therefore

$$\bar{h}_{00} = -\frac{k}{2\pi} \left(\frac{\omega L}{c_0}\right)^2 \frac{\overline{l} e^{-j\omega(t-r/c_0)}}{r}.$$  \hspace{1cm} (73)

Switching from $\bar{h}_{\alpha\beta}$ to $h_{\alpha\beta}$, Eq. (17) gives

$$h_{00} = \bar{h}_{00} - \frac{1}{2} \gamma_{00} \bar{h}_{\nu} = \bar{h}_{00} + \frac{1}{2} \gamma^\nu_{\mu} \bar{h}_{\mu\nu} = \bar{h}_{00} + \frac{1}{2} \gamma^0_{00} \bar{h}_{00} = \frac{1}{2} \bar{h}_{00}$$  \hspace{1cm} (74)

and $h_{0j} = 0$. The metric perturbation is already in the Newtonian gauge and from Eq. (34) component of the particle velocity in radial direction due to the acoustic wave equals

$$v^r_{ac} = \frac{k}{8\pi} c_0 \left(\frac{\omega L}{c_0}\right)^2 \frac{\overline{l} e^{-j\omega(t-r/c_0)}}{r}.$$  \hspace{1cm} (75)

We again neglected the component $\sim 1/r^2$. The metric does not produce any transverse waves and is indeed purely acoustic.

The classical solution for a compact $(\omega L/c_0 \ll 1)$ pulsating sphere reads

$$v^r_{ac} = -c_0 \left(\frac{\omega L}{c_0}\right)^2 \frac{\overline{l} e^{-j\omega(t-r/c_0)}}{r}.$$  \hspace{1cm} (76)

Hence, we can adopt $k = -8\pi$ to match the two results. In the context of gravitation, the dimensionless constant is $k = 8\pi$. The opposite sign followed from the different signs in equations (66) and (67). It implies that mass acts as an attracting source of gravity and (its unsteady component) as a repelling source in acoustics.
C. Oscillating sphere

In this section we consider sound radiation from a sphere with radius $L$ which oscillates with velocity $\vec{v} \sim e^{-j\omega t}$ and magnitude $|\vec{v}|$ around the origin of the coordinate system. The derivation is very similar as in section IV B, except that we replace the source integral in Eq. (70) with $(r, \phi, \text{and } z$ are now cylindrical coordinates)

$$\int_V t_{0j} d^3\vec{y} = \rho_0 c_0 \int_0^L r dr \int_0^{2\pi} d\phi \int_0^L dz = \rho_0 c_0 L^3 \pi v_j. \quad (77)$$

One half of the translating sphere is acting on the surrounding fluid with the scattering cross-sectional area $L^2 \pi$. Equation (68) then gives the only non-zero components of the metric perturbation in Lorenz gauge:

$$\bar{h}_{0j} = -\frac{kG \rho_0 L^3}{3c_0^3} \left( \frac{\omega L}{c_0} \right)^2 \frac{e^{j\omega r/c_0}}{r} v_j. \quad (78)$$

The Lorenz gauge condition is $\bar{h}_{0j,j} = 0$, which is satisfied to the lowest order of $r$ because the flow around the compact sphere is incompressible ($v^j_{\ j} = 0$). However, we are interested in the compressible longitudinal component radiated to the far field, so we introduce the change of coordinates:

$$\xi_0 = -\frac{kG \rho_0 L^3}{3c_0^3} \left( \frac{\omega L}{c_0} \right)^2 \frac{e^{j\omega r/c_0}}{r} v_j x^j \quad (79)$$

and

$$\xi_j = 0. \quad (80)$$

In this gauge, equations (13) and (17) with $\bar{h}_{\nu\nu} = 0$ give

$$h_{\alpha\beta} = \bar{h}_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}. \quad (81)$$
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and so

\[
    h_{00} = \tilde{h}_{00} - \xi_{0,0} - \xi_{0,0}
    \]

\[
    = -\frac{2}{c_0} \frac{\partial \xi_0}{\partial t} = -j \frac{k}{8\pi c_0} \left( \frac{\omega L}{c_0} \right)^3 \frac{e^{j\omega r/c_0}}{r} v_j x^j
\]

and \( h_{0j}, h_{jk} = 0 \) to the lowest order of \( \omega L/c_0 \) and \( 1/r \). We have also replaced the constant \( G \) from Eq. (58):

\[
    G = \frac{c_0^2 L/2}{2M} = \frac{3c_0^2}{16\rho_0 L^2\pi},
\]

with the Schwarzschild radius \( L/2 \), since only one half of the sphere effectively pushes the fluid.

The metric perturbation is now in the Newtonian gauge and we can use Eq. (34) to obtain at the lowest order

\[
    v_{ac}^r = \frac{k c_0}{16\omega \pi} \left( \frac{\omega L}{c_0} \right)^3 |\vec{v}| \cos(\theta) \frac{e^{-j\omega(t-r/c_0)}}{r},
\]

where we also used the equality \( (v_j x^j)^r = (|\vec{v}| \cos(\theta) e^{-j\omega t, x^r})^r = |\vec{v}| \cos(\theta) e^{-j\omega t} \) in which \( \theta \) denotes the angle between \( \vec{v} \) and the position vector \( \vec{r} \). This matches the classical solution\(^{20}\) for \( k = -8\pi \) and thus confirms the value of \( k \).

V. CONCLUSION

In the preceding sections we showed that sound waves in fluids can be treated as weak perturbations (curvatures) of acoustic spacetime which obey the governing equations of general relativity with \( c_0 \) the reference speed of sound. The quadrupole source of jet noise in fluids, which is classically described by Lighthill’s aeroacoustic analogy, corresponds to the quadrupole source of gravitational waves. In the suitable Newtonian gauge, the 8th-power law was obtained. For compact sources, this is much weaker radiation than in the
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case of transverse gravitational waves, described with a second-order tensor. In fact, the longitudinal component is then completely removed in the transverse-traceless gauge. The exact solutions were also obtained for ideal monopole (pulsating compact sphere) and dipole (oscillating compact sphere). This lead to equal value of the dimensionless constant \( k \) as in general relativity, only with opposite sign.

Although only the problems involving small perturbations of the background spacetime were treated using the linearized theory of relativity, the fact that Eq. (39) is satisfied in fluids by the conservation laws suggests that the full Einstein field equations, Eq. (6) with \( k = -8\pi \), may be applicable for general acoustic problems in subsonic flows. Nevertheless, the applicability for non-linear acoustics and high Mach number flows should be further inspected. The linearized formulation for small perturbations of otherwise flat acoustic spacetime is given in Eq. (18) under the Lorenz gauge condition in Eq. (10). Physically relevant and measurable particle motion due to an acoustic wave can be obtained from the component \( h_{00} \) of the metric perturbation in the Newtonian gauge (Newtonian form), according to the geodesic equation (34).

The proposed complete analogy between the two apparently remote physics can have far-reaching consequences. In a didactic sense, it places the scalar acoustic fields next to the vector electromagnetic and second-order tensor gravitational fields in the relativistic theories of fields, allowing equivalent treatment. More specifically, quadrupole sources of gravitational waves, such as merging stars and black holes, can be compared with acoustically compact vorticity, which constitutes the aeroacoustic quadrupole source of sound in fluids. The analogy reveals new possibilities for indirect studies of gravitational waves with the
aid of acoustic experiments and calculations. It also opens important questions for further fundamental work in both areas with regard to the nature of the two types of spacetimes, as well as possible effects of boundaries in such a framework.

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