On the effective Hamiltonian for QCD: An overview and status report

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The session on effective Hamiltonians and chiral dynamics is overviewed, combined with a review on the bound-state problem. The progress during this session allows to remove all dependence on regularization in an effective interaction, thus to renormalize a Hamiltonian for the first time, and to solve front form as if they were instant-form equations, with all the advantages implied.

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1. Introduction

This community has formed in 1991 at the first light-cone meeting in Heidelberg with the ambitious aim to solve the bound-state problem in gauge field theory particularly QCD.

How far did we get? Is it fair to say that we have not yet solved our homework problem?

However, the important contributions at this [1–13] and the last meeting [14–19] let expect a faster pace in the foreseeable future. Over 80 participants show how alive the field continues to be even after 11 years. Its richness has become apparent last week. 13 speakers are alone in this session. I attempt therefore to combine an overview on the present session on “Effective Hamiltonians and chiral dynamics” with a review on effective interactions in general. This seems to be in place in view of the progress at this meeting, particularly on renormalization [3,8] and the possibility to solve instant form rather than front form equations [6] when working on the light cone.

2. Why working on the light-cone?

The Hamiltonian approach to a field theory was a no-go-topic for over fifty years. But combined with light-cone quantization and periodic boundary conditions [20], certain advantages inherent to the light-cone Hamiltonian approach were clear right from the outset, particularly

- the simple kinematical boosts and
- the simple vacuum properties.

This continues to be so. It was equally clear that a number of extremely difficult problems were on the road, among them zero modes, gauge invariance and gauge artifacts, the field theoretical many-body problem and Fock space truncation, non-perturbative renormalization, confinement, chiral phase transitions, just to name a few. Some of them have been solved, or better understood, as reviewed in [21].

What is the homework problem? Starting from the Lagrangian density $\mathcal{L}_{\text{QCD}}$, the light-cone approach to the bound-state problem [22] aims at
solving the eigenvalue equation

\[ H_{LC} \langle \Psi \rangle = M^2 \langle \Psi \rangle . \]  

If one disregards possible zero modes and works in the light-cone gauge, the (light-cone) Hamiltonian \( H_{LC} = P^{\mu} P_{\mu} \) is a well defined Fock-space operator and given in \[21\]. Its eigenvalues are the invariant mass-squares \( M^2 \) of physical particles associated with the eigenstates \( \langle \Psi \rangle \). In general, they are superpositions of all possible Fock states with its many-particle configurations. For a meson, for example, holds

\[
\begin{align*}
|\Psi_{\text{meson}}\rangle &= \sum_{i} \Psi_{q\bar{q}}(x_i, \vec{k}_{L,i}, \lambda_i) \ |q\bar{q}\rangle \\
&+ \sum_{i} \Psi_{gg}(x_i, \vec{k}_{L,i}, \lambda_i) \ |gg\rangle \\
&+ \sum_{i} \Psi_{qg\bar{q}}(x_i, \vec{k}_{L,i}, \lambda_i) \ |qg\bar{q}\rangle \\
&+ \sum_{i} \Psi_{q\bar{q}g}(x_i, \vec{k}_{L,i}, \lambda_i) \ |q\bar{q}g\rangle \\
&+ \ldots 
\end{align*}
\]

If all wave functions \( \Psi_{n}(x_i, \vec{k}_{L,i}, \lambda_i) \) are available, one can analyze hadronic structure in terms of quarks and gluons. For example, one can calculate the space-like form factor of a hadron quite straightforwardly by a sum of overlap integrals analogous to the corresponding non-relativistic formula \[21\].

3. What are possible alternatives?

This community tries hard to have a feedback to and from other fields and activities. All these approaches have their own virtues and merits. The advantiges are usually emphasized by the proponents, and therefore I shall play the devils advocate passing them shortly review, with a sometimes over-critical attitude to make the point clear.

Phenomenological models. Practically all our knowledge on hadron structure comes from phenomenological models. The constituent quark model particularly continues to have great success. Phenomenological approaches usually do not address to the lighter mesons like the pion, but they are extremely successfull for the heavier hadrons and for baryons. A particularly beautiful example was presented by Plessas \[5\]. I do not care so much that his model has about twenty parameters, with only three of them fitted explicitly. Such work is a useful guideline to experiment. I dream of the day when front-form based work produces similar results. His work also shows the extreme difficulty to relate wave functions of constituents to actual cross sections.

Chiral perturbation theory. Leutwyler \[1\] demonstrates to which precision a well based formalism can be driven. To some extent this also holds for the similar NJL-models. I cannot quote the huge body of literature but I mention in passing that they are not renormalizable, that the relation to QCD is unclear, and that they deal mostly with the very light mesons. Heavy flavors cannot be treated, see also \[1\].

Schwinger-Dyson approaches are potentially able to cope with the bound-state problem. Roberts \[2\] emphasises the chiral aspects: Free quarks have the small current mass at large momentum, increasing to the large constituent mass at small momentum. Does this feature prevail in a bound state problem, and how?

DLCQ and LC approaches. Hiller \[16\] addresses to diagonalize by DLCQ the light-cone Hamiltonian in physical space-time \((3+1)\). His renormalization à la Pauli-Villars yields promising results, but he needs a super-computer to produce them. He works in a truncated Fock space. But Ligterink \[11\] concludes that Fock-space suppression is less dangerous than believed, and both Mangin-Brinet \[7\] and Karmanov \[10\] report good stability of bound-state calculations in truncated spaces. The separation of soft and hard aspects by Schweiger \[6\] continue to be an important aspect of light-cone quantization.

Technical problems. Basis optimalization by Sugihara \[12\] and a new algorithm by van Iersel \[13\], applied here to the Yukawa model, are very important facets. In fact the break-through in non-perturbative renormalization by Frederico \[3\] and Frewer \[8\], and a new insight into the nature of Melosh-transforms by Krassnigg \[6\] represent progress on a technical level as well.

Lattice Gauge Calculations use practically all computer power in this world to generate a potential energy between quarks, but then a non-relativistic Schrödinger equation is used to cal-
culate bound states \[17\]. This is unsatisfactory. It is generally not known that LGC’s have considerable uncertainty to extrapolate their results down to such light mesons as a pion. It is equally unknown that lattice gauge calculations get \textit{always strict and linear confinement} even for QED, where we know the ionization threshold. The ‘breaking of the string’, or in a more physical language, the ionization threshold is one of the hot topics at the lattice conferences \[22\]. Moreover, in order to get the size of the pion, thus the form factor, another generation of computers is required, as well as physicists to run them. Such considerations and the lacking perspectives on precision have motivated Wilson, among other, to quit.

The new Wilson approach to QCD is based almost entirely on the front form and renormalization group analysis \[23\]. Walhout \[9\] gives an example for that. The original hope was to assemble the operators in an effective interaction according to their relevance with respect to the renormalization group. It is not unfair to state that not much of concrete hardcore technology has thus far emerged, despite the immense efforts over the years. In developing the formalism the similarity transform of Wilson and Glazek has played a major role \[24\]. The basic idea is similar as in the preceding method Hamiltonian flow by Wegner \[25\]. But as emphasized repeatedly by the latter, the similarity transform has a serious defect \[15\]: As a built-in feature, it cannot account for the block structures in a gauge-field theoretical many-body Hamiltonian and therefore should be abandoned.

In conclusion, I believe that there is not much left than to proceed with the more conventional methods. In the sequel, I will briefly review only one of them, the method of iterated resolvents.

4. The method of iterated resolvents

Instead of diagonalizing the Hamiltonian by DLCQ, one might wish to reduce the many-body problem behind a field theory to an effective one-body problem. The derivation of the effective interaction becomes then the key issue.

Because of the inherent divergencies in a gauge field theory, the QCD-Hamiltonian in 3+1 dimensions must be regulated from the outset. One of the few practical ways is vertex regularization \[21,26\], where every Hamiltonian matrix element, particularly those of the vertex interaction (the Dirac interaction proper), is multiplied with a convergence-enforcing momentum-dependent function. It can be viewed as a form factor \[21\]. The precise form of this function is unimportant here, as long as it is a function of a cut-off scale (\(\Lambda\)).

By definition, an effective Hamiltonian acts only in the lowest sector of the theory (here: in the Fock space of one quark and one anti-quark). And, again by definition, it has the same eigenvalue spectrum as the full problem. I have derived such an effective interaction by the method of iterated resolvents \[26\], that is by systematically expressing the higher Fock-space wave functions as functionals of the lower ones. In doing so the Fock-space is not truncated and all Lagrangian symmetries are preserved. The projections of the eigenstates onto the higher Fock spaces can be retrieved systematically from the \(q\bar{q}\)-projection, with explicit formulas given in \[26\].

Let me sketch the method briefly, details may be found in \[26\]. DLCQ with its periodic boundary conditions has the advantage that the LC-
Hamiltonian is a matrix with a finite number of Fock-space sectors, which we denumerate by \( n \), with \( 1 < n \leq N \). The so called harmonic resolution \( K = LP^+/(2\pi) \) acts as a natural cut-off of the particle number. As shown in Figure 4, \( K = 3 \) allows for \( N = 8 \), and \( K = 4 \) for \( N = 13 \) Fock-space sectors, for example. The Hamiltonian matrix is sparse: Most of the matrix elements are zero, particularly if one includes only the vertex interaction \( V \). For \( n \) sectors, the eigenvalue problem in terms of block matrices reads

\[
\sum_{j=1}^{N} \langle i|H_n(\omega)|j\rangle \langle j|\Psi(\omega)\rangle = E(\omega) \langle i|\Psi(\omega)\rangle, \tag{2}
\]

for \( i = 1, 2, \ldots, n \). I can always invert the quadratic block matrix of the Hamiltonian in the last sector to define the \( n \)-space resolvent \( G_n \), that is

\[
G_n(\omega) = \frac{1}{\omega - H_n(\omega)}. \tag{3}
\]

Using \( G_n \), I can express the projection of the eigenfunction in the last sector by

\[
\langle n|\Psi(\omega)\rangle = G_n(\omega) \sum_{j=1}^{n-1} \langle n|H_n(\omega)|j\rangle \langle j|\Psi(\omega)\rangle, \tag{4}
\]

and substitute it in Eq. (2). I then get an effective Hamiltonian where the number is sectors is diminished by 1:

\[
H_{n-1}(\omega) = H_n(\omega) + H_n(\omega)G_n(\omega)H_n(\omega). \tag{5}
\]

This is a recursion relation, which can repeated until one arrives at the \( q\bar{q} \)-space. The fixed point equation \( E(\omega) = \omega \) determines all eigenvalues.

For the block matrix structure as in Figure 1, with its many zero matrices, the reduction is particularly easy and transparent. For \( K = 3 \) one has a sequence of effective interactions:

\[
H_8 = T_8, \quad H_7 = T_7 + VG_8V, \quad H_6 = T_6 + VG_7V, \quad H_5 = T_5 + VG_6V. \tag{6}
\]

The remaining ones get more complicated, \( i.e. \)

\[
H_4 = T_4 + VG_7V + VG_7VG_8V + VG_7V, \quad H_3 = T_3 + VG_6V + VG_6VG_7V + VG_6V + VG_7V, \tag{7}
\]

\[
H_2 = T_2 + VG_5V + VG_5V, \quad H_1 = T_1 + VG_4V + VG_4VG_5V + VG_4V + VG_5V.
\]

For \( K = 4 \), the effective interactions in Eq. (6) are different, see for example [26], but it is quite remarkable that they are the same in Eq. (7). In fact, the effective interactions in sectors 1-4 are independent of \( K \): The continuum limit \( K \to \infty \) is then trivial, and will be taken in the sequel.

In the continuum limit, the effective Hamiltonian in the \( q\bar{q} \)-space \( H_{1} = H_{\text{eff}} \) is thus

\[
H_{\text{eff}} = T + VG_3V + VG_3VG_2V + VG_3V, \quad = T + U_{\text{conserv}} + U_{\text{change}}. \tag{8}
\]

The effective interaction has two contributions: A flavor-conserving \( U_{\text{conserv}} \) and a flavor-changing piece \( U_{\text{change}} \). The flavor-changing interaction can not get active in flavor-off-diagonal mesons.
The dressed propagators in Eq. (8) and Figure 3 are exact. The iterated resolvents resum perturbative diagrams to all orders.

Their conversion to free propagators with effective vertices \( \alpha \to \overline{\alpha}(Q) \), represented in Figure 3 by the thin lines and the open circles, respectively, is an approximation coupled with four well specified assumptions \[26\].

The wavy line in Figure 3 should not be mistaken as a single gluon exchange. The effective gluon corresponds to a particular resummation of infinitely many gluons.

Henceforward I deal only with the flavor conserving interaction.

5. The one-body equation

The effective one-body equation for flavor off-diagonal mesons (mesons with a different flavor for quark and anti-quark) becomes thus \[19, 24\]:

\[
M^2 \psi_{\lambda_1 \lambda_2}(x, \vec{k}_\perp) = \sum_{\lambda'_1 \lambda'_2} \int dx' d^2 \vec{k}'_\perp \psi_{\lambda'_1 \lambda'_2}(x', \vec{k}'_\perp) + U_{\lambda_1 \lambda_2; \lambda'_1 \lambda'_2}(x, \vec{k}_\perp; x', \vec{k}'_\perp) \psi_{\lambda'_1 \lambda'_2}(x', \vec{k}'_\perp), \tag{9}
\]

an integral equation with the kernel

\[
U_{\lambda_1 \lambda_2; \lambda'_1 \lambda'_2}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = -\frac{4m_1 m_2}{3\pi^2} \langle \overline{\alpha}(Q) \overline{\alpha}(Q) \rangle \frac{S_{\lambda_1 \lambda_2; \lambda'_1 \lambda'_2}(x, \vec{k}_\perp; x', \vec{k}'_\perp)}{\sqrt{x(1-x)x'(1-x')}}. \tag{10}
\]

Here, \( M^2 \) is the eigenvalue of the invariant-mass squared. The associated eigenfunction \( \psi = \Psi_{\lambda_1 \lambda_2 \lambda'_1 \lambda'_2} \) is the probability amplitude \( \langle x, \vec{k}_\perp; \lambda_1, \lambda_2 | \psi \rangle \) for finding a quark with momentum fraction \( x \), transversal momentum \( \vec{k}_\perp \) and helicity \( \lambda_1 \), and correspondingly the anti-quark with \( 1-x, -\vec{k}_\perp \) and \( \lambda_2 \). The (effective) quark masses \( \overline{m}_1 \) and \( \overline{m}_2 \) and the (effective) coupling constant \( \overline{\alpha} \) are given below. The mean Feynman-momentum transfer of the quarks is denoted by \( Q^2 = Q^2(x, \vec{k}_\perp; x', \vec{k}'_\perp) \),

\[
Q^2 = \frac{1}{2} [(k_1-k'_1)^2 + (k_2-k'_2)^2], \tag{11}
\]

the spinor factor \( S = S(x, \vec{k}_\perp; x', \vec{k}'_\perp) \) by

\[
S_{\lambda_1 \lambda_2; \lambda'_1 \lambda'_2}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = \frac{\overline{\alpha}(Q)}{Q^2} \frac{\overline{\alpha}(Q)}{Q^2} \frac{S_{\lambda_1 \lambda_2; \lambda'_1 \lambda'_2}(x, \vec{k}_\perp; x', \vec{k}'_\perp)}{\sqrt{x(1-x)x'(1-x')}}. \tag{12}
\]

In the expression a second regularization parameter \( m_g \) appears which conceptually is a kinematical gluon mass. The corresponding gluon mass diagram gives

\[
\overline{m}_g^2 = m_g^2 - \frac{\alpha}{4\pi} \sum_{j=1}^{n_f} m_j^2 \ln \frac{\Lambda^2}{4m_j^2}, \tag{14}
\]

see Eq.(91) of \[23\]. The physical gluon mass must vanish due to gauge invariance, thus \( \overline{m}_g = 0 \), which expresses \( m_g \) in terms of\( m_f \).

Second, it depends on \( \Lambda \) through the effective coupling \( \overline{\alpha}(Q) \equiv \overline{\alpha}(\Lambda, Q) \). The expression in Eq.(100) of \[26\] is rewritten here conveniently in terms of an arbitrary scale \( \kappa \):

\[
\frac{1}{\overline{\alpha}(\Lambda, Q)} = \frac{1}{\alpha} - \frac{11n_c - 2n_f}{12\pi} \ln \frac{\Lambda^2}{\kappa^2} + \frac{11n_c}{12\pi} \ln \frac{\mu_g^2 + Q^2}{\kappa^2}.
\]

6. Renormalization

The effective Hamiltonian in Eq.(11) depends on a regulator scale \( \Lambda \) through three quantities.

First, it depends on \( \Lambda \) through the effective quark masses \( \overline{m}_f = \overline{m}_f(\Lambda) \) which are hidden also in the Dirac spinors. They are given in Eq.(90) of \[26\] in terms of the bare \( \alpha \) and \( m_f \):

\[
\overline{m}_f^2 = m_f^2 \left( 1 + \frac{\alpha}{2\pi} \frac{n_c^2 - 1}{2n_c} \ln \frac{\Lambda^2}{m_f^2} \right). \tag{13}
\]

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\[
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\]
with \( \mu_f = 2m_f \) and \( \mu_g = 2m_g \).

Third, the Hamiltonian depends on \( \Lambda \) through the regularization function which here is the soft cut-off

\[
\mathcal{R}(Q) \equiv \mathcal{R}(\Lambda, Q) = \frac{\Lambda^2}{\Lambda^2 + Q^2}.
\]  

(16)

The dependence on the unphysical parameter \( \Lambda \) must be removed,

\[
\frac{d}{d\Lambda} H_{LC}(\overline{m}(\Lambda), \overline{\alpha}(\Lambda), \mathcal{R}(\Lambda)) = 0,
\]

(17)

as required by renormalization theory, but how? The non-perturbative renormalization of \( H \) was stuck for many years by the fact that the vertex function \( \overline{\alpha}(\Lambda) \) and the regulator \( \mathcal{R}(\Lambda) \) are so intimately coupled in Eq.(13). It was always clear that one could add non-local counter terms \[23\], but it was utterly unclear how to construct them.

The progress comes from the recent work on the \( \uparrow \downarrow \)-model \[3,8\]: Adding to \( \mathcal{R}(\Lambda, Q) \) a counterterm \( C(\Lambda, Q) \) and requiring that the sum \( R(\Lambda, Q) = \mathcal{R}(\Lambda, Q) + C(\Lambda, Q) \) be independent of \( \Lambda \), determines \( C(\Lambda, Q) \). One remains with

\[
R(\Lambda, Q) = \mathcal{R}(\Lambda, Q) + C(\Lambda, Q) = \frac{\mu^2}{\mu^2 + Q^2}.
\]

(18)

In line with renormalization theory, one then can go to the limit \( \Lambda \to \infty \) and \( \mu \) becomes a parameter of the theory.

The cut-off dependence in \( \overline{\alpha}(\Lambda, Q) \), Eq.(13), can then be removed by replacing the bare coupling constant \( \alpha \) by the cut-off dependent running coupling constant \( \alpha_\Lambda \), i.e.

\[
\alpha_\Lambda = \frac{6\pi}{11n_c - 2n_f} \frac{1}{\ln(\Lambda/\kappa)}.
\]

(19)

The renormalized vertex function,

\[
\frac{1}{\overline{\alpha}(Q)} = \frac{11n_c}{12\pi} \ln \left( \frac{\mu^2}{\mu^2 + Q^2}/\kappa^2 \right)
\]

\[
- \frac{2}{12\pi} \sum_{f=1}^{n_f} \ln \left( \frac{\mu_f^2 + Q^2}{\mu_f^2 + Q^2}/\kappa^2 \right),
\]

(20)

does not depend explicitly on \( \Lambda \), and the scale \( \kappa \) becomes an other parameter of the theory.

In completing renormalization for the masses, Eqs.(13) and (14) are first rewritten for \( n_c = 3 \) as

\[
\overline{m}_f^2 = m_f^2 + \frac{8m_f^2}{3\pi} \left( 1 - \frac{\ln m_g/\kappa}{\ln \Lambda/\kappa} \right) \alpha \ln \frac{\Lambda}{\kappa},
\]

(21)

\[
m_g^2 = \sum_{f=1}^{n_f} \frac{m_f^2}{2\pi} \left( 1 - \frac{\ln 2m_f/\kappa}{\ln \Lambda/\kappa} \right) \alpha \ln \frac{\Lambda}{\kappa}.
\]

(22)

Inserting the running coupling constant from Eq.(19) leaves us with

\[
\overline{m}_f^2 = m_f^2 + \frac{8m_f^2}{3\pi} \frac{6\pi}{33 - 2n_f} \left( 1 - \frac{\ln m_g/\kappa}{\ln \Lambda/\kappa} \right),
\]

\[
m_g^2 = \sum_{f=1}^{n_f} \frac{m_f^2}{2\pi} \frac{6\pi}{33 - 2n_f} \left( 1 - \frac{\ln 2m_f/\kappa}{\ln \Lambda/\kappa} \right).
\]

Finally, I go to the limit \( \Lambda \to \infty \) and express the bare masses in terms of the dressed ones:

\[
m_f^2 = \frac{33 - 2n_f}{49 - 2n_f} m_f^2, \quad m_g^2 = \frac{3}{49 - 2n_f} \sum_{f=1}^{n_f} m_f^2.
\]

(23)

This completes the program of renormalization: For the first time, ever, the dependence on a cut-off \( \Lambda \) has been removed completely from a field theoretical Hamiltonian. Notice that this step rests on the contributions \[3,8\] to this meeting.

7. The locking of the coupling constant

The Lagrangian for QCD has 7 parameters: the 6 flavor quark masses \( m_f \) and the coupling constant \( \alpha \). The renormalized effective Hamiltonian has one parameter more: The 6 flavor masses \( \overline{m}_f \), and the two scales \( \kappa \) and \( \mu \). This is in full accord with renormalization theory, since whatever the model, one has a scale at which one experiments.

The renormalized vertex function of Eq.(20) deserves some further discussion. Most importantly it has a finite value at \( Q = 0 \). The coupling constant locks its-self, as one says.

One should think that \( \kappa \) is entirely fixed by the coupling constant of measured at sufficiently high \( Q \). But taking, as usual, the value \( \overline{m}(M_Z) = 0.118 \) at the Z mass \( M_Z = 91.2 \text{ GeV} \), one observes a rather dramatic dependence of \( \kappa \) on the number
The vertex function $\alpha(Q)$ versus $Q$ in GeV, with different flavor numbers $n_f = 4, 5, 6$ (top to bottom); all $m_f = 350$ MeV.

Changing $n_f$ from 4 to 6 changes $\kappa$ by a factor of 4! The dependence of $\alpha_0 \equiv \alpha(0)$ on $n_f$ is less pronounced even if one puts all flavor masses equal to $m_f = 350$ MeV, as done conveniently in Eq. (24).

The corresponding functions $\alpha(Q)$ are displayed in Figure 4. The $n_f + 1$ parameters in Eq. (20) are unpleasant to work with and it is useful to introduce the approximate expression

$$\alpha_0(Q) = \frac{1}{33 - 2n_f} \ln \left( \frac{\mu_b^2}{\kappa^2} \right)$$  \hspace{1em} (25)

The only parameter $\mu_b$ is fixed by $\alpha_0$ and given in Eq. (24) as well. As shown in Figures 4 and 5 by the dashed line, $\alpha_0(Q)$ is almost un-discernible from $\alpha(Q)$.

Using the more physical mass parameters from Eq. (25) produces

$$n_f \hspace{1em} \kappa \hspace{1em} \alpha_0 \hspace{1em} \mu_g \hspace{1em} \mu_b$$

$$4 \hspace{1em} 153.1 \hspace{1em} 0.9566 \hspace{1em} 387.7 \hspace{1em} 336.7$$

$$5 \hspace{1em} 87.84 \hspace{1em} 0.5446 \hspace{1em} 434.1 \hspace{1em} 359.6$$

$$6 \hspace{1em} 45.33 \hspace{1em} 0.3848 \hspace{1em} 488.2 \hspace{1em} 467.2$$

Latest here I have to abandon my earlier conjecture [26] that a momentum-dependent vertex function could be related to confinement in any way. In fact, the above curves have so little structure that one can replace them in a bound state calculation by the constant $\alpha_0$. Henceforward I will give up thus $\kappa$ in favor of $\alpha = \alpha_0$ and change notation from $m_f$ to $m_f$.

8. The $\uparrow\downarrow$-model as an application

In light-cone parametrization, the quarks are at relative rest when $\vec{k}_\perp = 0$ and $x = \bar{x} = m_1/(m_1 + m_2)$. For very small deviations from these equilibrium values the spinor matrix is proportional to the unit matrix, with $\lambda_1, \lambda_2$

$$\langle \lambda_1, \lambda_2 | S | \lambda'_1, \lambda'_2 \rangle \sim 4m_1m_2 \delta_{\lambda_1, \lambda'_1} \delta_{\lambda_2, \lambda'_2}.$$  \hspace{1em} (27)
For very large deviations from equilibrium, particularly for \( k_\perp^2 \gg \tilde{k}_\perp^2 \), holds
\[ Q^2 \simeq \tilde{k}_\perp^2, \quad \text{and} \quad \langle \updownarrow | S | \updownarrow \rangle \simeq 2\tilde{k}_\perp^2. \] (28)

Both extremes are combined in the \( \uparrow \downarrow \)-model [19]:
\[ \frac{S}{Q^2} = \frac{4m_1m_2}{Q^2} + 2 \Rightarrow \frac{4m_1m_2}{Q^2} + 2R(\Lambda, Q), \]
with \( R(\Lambda, Q) = \frac{\mu^2}{\mu^2 + Q^2} \). (29)

It interpolates between two extremes: For small momentum transfer, the '2' generated by the hyperfine interaction is unimportant and the dominant Coulomb aspects of the first term prevail. For large momentum transfers the Coulomb aspects are unimportant and the 2 dominates. Eq. (29) therefore is replaced by

\[ M^2\psi(x, \tilde{k}_\perp) = \left[ \frac{m_1^2 + \tilde{k}_\perp^2}{x} + \frac{m_2^2 + \tilde{k}_\perp^2}{1-x} \right] \psi(x, \tilde{k}_\perp) \]
\[ - \frac{\alpha}{3\pi^2} \int \frac{dx'x'\tilde{k}_\perp^2}{x(1-x)x'(1-x')} \psi(x', \tilde{k}_\perp') \times \left( \frac{4m_1m_2}{Q^2} + \frac{2\mu^2}{\mu^2 + Q^2} \right), \] (30)

where \( \psi(x, \tilde{k}_\perp) \equiv \langle x, \tilde{k}_\perp | \uparrow \downarrow | \psi \rangle \). With the canonical 8 parameters of the \( \uparrow \downarrow \)-model [13],

\[ \alpha, \mu, m_u, m_d, m_s, m_c, m_b, m_t \]
\[ 0.690 \quad 1.33 \]
\[ \begin{array}{c}
0.46 \quad 0.46 \quad 508 \quad 1.67 \quad 5.05 \quad 174 \quad (31)
\end{array} \]

GeV MeV MeV MeV GeV GeV GeV,

all masses of the physical mesons have been calculated according to Eq. (30). They are compiled in Table 1. The empirical masses are compiled in Table 2.

Table 1
The calculated mass eigenvalues in MeV. Those for singlet-1s states are given in the lower, those for singlet-2s states in the upper triangle.

|       | \( \pi \) | \( \bar{d} \) | \( \pi \) | \( \bar{u} \) |
|-------|-------|-------|-------|-------|
| \( u \) | 768   | 871   | 2030  | 5418  |
| \( d \) | 140   | 871   | 2030  | 5418  |
| \( s \) | 494   | 494   | 2124  | 5510  |
| \( c \) | 1865  | 1865  | 1929  | 6580  |
| \( b \) | 5278  | 5278  | 5338  | 6114  |

Table 2
Empirical masses of the flavor-off-diagonal physical mesons in MeV. Vector mesons are given in the upper, scalar mesons in the lower triangle.

|       | \( \pi \) | \( \bar{d} \) | \( \pi \) | \( \bar{u} \) |
|-------|-------|-------|-------|-------|
| \( u \) | 768   | 892   | 2007  | 5325  |
| \( d \) | 140   | 896   | 2010  | 5325  |
| \( s \) | 494   | 498   | 2110  | —     |
| \( c \) | 1865  | 1869  | 1969  | —     |
| \( b \) | 5278  | 5279  | 5375  | —     |

Table 2. The agreement between the two is amazing. To the best of my knowledge there is no other model which can describe all mesons quantitatively from the \( \pi \) up to the \( \Upsilon \) from a common point of view, which here is QCD.

The proposed pion of the \( \uparrow \downarrow \)-model is rather different from the pions in the literature. I have found no evidence that the vacuum condensates are important, but I conclude that the pion is describable by a QCD-inspired theory: The very large coupling constant in conjunction with a very strong hyperfine interaction makes it a ultra strongly bounded system of constituent quarks. More then 80 percent of the constituent quark mass is eaten up by binding effects. No other physical system has such a property.

The numerical wavefunction \( \psi(x, \tilde{k}_\perp) \) can be fitted with only one free parameter, \( i.e. \)

\[ \psi(x, \tilde{k}_\perp) = \frac{N}{\sqrt{x(1-x)}} \times \left( \frac{1 + \frac{m^2(2x-1)^2 + \tilde{k}_\perp^2}{4x(1-x)p_a^2}}{1 + \frac{m^2(2x-1)^2 + \tilde{k}_\perp^2}{4x(1-x)p_a^2}} \right)^\frac{1}{2}, \] (32)

with \( p_a = 1.338m \) [13]. The explicite form of the wavefunction can used to calculate the form factor and thus the exact root-mean-square radius \( \langle r^2 \rangle = -6 \frac{dF_2(Q^2)}{dQ^2|_{Q^2=0}} \) analytically [27]. The size of the \( q\bar{q} \) wavefunction turns out as \( \langle r^2 \rangle = (0.33 \text{ fm})^2 \), half as large as the empirical value \( \langle r^2 \rangle_{\text{exp}} = (0.67 \text{ fm})^2 \).

The parameter \( p_a = 1.338m \) in Eq. (32) plays
the role of an effective Bohr momentum of the constituents in the pion. The mean momentum of the constituents is thus 40 percent larger than their mass, which means that they move highly relativistically quite in contrast to the constituents of atoms or nuclei. No wonder that potential models thus far have failed for the pion.

This completes one of my goals: I have a pion with the correct mass, and I have an analytic expression for its light-cone wave function. Eq. (32) could be used thus as a baseline for calculating the higher Fock-space amplitudes, as explained in [26]. It could well be that the wavefunction obtained from such a simple model suffices already to be consistent with recent experiments [14].

9. Discussion: Front or instant form?

Eq. (3) is a frame-frame-independent, covariant, and fully relativistic front-form equation, with certain boosts being kinematic and trivial [21]. One pays for these advantages with the fact that the transversal components for total angular momentum \( J = \vec{L} + \vec{S} \) (the spin \( \vec{S} = \frac{1}{2}(\sigma_1 + \sigma_2) \) is not to be confused with the spinor factors \( S \) ) are complicated dynamical operators in the front form, see for example [21]. Only \( J_z \) is simple and kinematic. The eigenvalues and eigenfunctions of Eq. (3) can thus not be labeled with \( J \). Despite this, Trettmann and Pauli [3], in their numerical solution of the QED-version of Eq. (3) for different \( J_z \), have done so by using the standard (non-relativistic) spectroscopic terms \( 2S + 1L^J_z \). By inspection of the numerical results they found that the eigenvalues can be arranged in multiplets which are \((2J+1)\)-fold degenerate modulo numerical accuracy. The authors could not find a plausible answer for that in terms of the light-cone formalism.

Now, we seem understand that better. In the contribution to this session, Krassnigg [3] shows that there exists a unitary tranformation \( \Omega \) which transforms the typical combination of Lepage-Brodsky spinors in Eq. (3) to an other combination with only Bjørken-Drell spinors.

Unitary transformations do not change the eigenvalue and Eq. (3) is identically transcribed to an equation for the reduced wave function \( \varphi_{s_1 s_2}(\vec{k}) \), which reads for equal masses

\[
\left[ M^2 - 4 \left( m^2 + \vec{k}^2 \right) \right] \varphi_{s_1 s_2}(\vec{k}) = \sum_{s'_1 s'_2} \int d^4 \vec{k}' \tilde{U}_{s_1 s_2; s'_1 s'_2}(\vec{k}; \vec{k}') \varphi_{s'_1 s'_2}(\vec{k}').
\]

Since the spinors \( u(k, s) \) in the kernel

\[
\tilde{U}_{s_1 s_2; s'_1 s'_2} = -\frac{8m}{3\pi^2 Q^2} R(Q) \times [\overline{u}(k_1, s_1)\gamma^\mu u(k_2, s_2)] [\overline{u}(k_2, s_2)\gamma^\mu u(k_1, s_1)],
\]

by definition are Bjørken-Drell spinors, one can not recognize the front-form origin of this equation, see [1] for further details. They are a set of four coupled integral equations in the usual momentum space. Formally spoken, they are instant-form equations in the rest frame, and, by inspection, they are invariant under spatial rotations. Its eigenfunctions can therefore be labeled as \( \varphi_{J(L)S} \), as usual, with eigenvalues \( M^2_{J(L)S} \) being \((2J+1)\)-fold strictly degenerate multiplets.

These aspects can also be reversed. Suppose that some phenomenological model yields momentum space wavefunctions \( \varphi_{s_1 s_2}(\vec{k}) \). The transformations in [3], which lead to Eq. (33), can be inverted and used to generate light-cone wave functions \( \psi_{\lambda_1\lambda_2}(x, \vec{k}_1) \) with helicities \( \lambda_1 \) and \( \lambda_2 \). These can be used then as a reasonable approximation in existing formulas for the cross-sections.

10. Perspectives

The light-cone community has not yet solved its homework problem, but it has gone a long way: (1) The role of zero modes and vacuum structure is better understood. (2) Effective interactions can be formulated and even renormalized. (3) Bound-state wavefunctions can be calculated with a technical effort comparable to or less than in the instant form. (4) Simple models can generated which are not in conflict with experiments. (5) Last not least, much of the work can be done analytically.

Despite the limited progress one should not be discouraged from continued efforts on the homework problem. After all, the Hamiltonian approach to any field theory an in any form has
been disrupted in 1949 when Feynman’s action oriented approach did appear on the scene. Not solved are the aspects of confinement. At present one does not understand its origin. Not solved are also the aspects of the chiral phase transition. But one should emphasize that the solution to the bound-state problem takes place at temperature zero, possibly after a phase transition. Due to the fit to experiment, quark masses are finite and actually large. The present approach thus can not contribute to question like “What happens if quark masses vanish?” It starts where other approaches end.

In QED, hyperfine interactions and the Lamb shift are comparable in size. For QCD, the importance of the hyperfine interaction has been quantified by the ↑↓-model, but the Lamb shift is a completely open question.

In QED, the Lamb shift arises by a photon in flight moving relative to the hydrogen bound-state. That alone suffices to give part of the answer for QCD, by Eq.(8). Going to the diagonal representation gives

\[ U_{\text{Lamb}} = V G_3 V = \sum_n V|n\rangle \frac{1}{\omega - M^2_n} \langle n|V \quad (35) \]

The symbol \( \sum_n \) refers to a summation (or integration) over all meson states and gluon states. The invariant mass squared

\[ M^2_n = \frac{M^2_b + \vec{q}_L^2}{1 - y} + \frac{\vec{q}^2}{y} \quad (36) \]

corresponds to a free colored gluon with longitudinal momentum fraction \( y \) and transversal momentum \( \vec{q}_L \). It moves back to back to a colored meson bound-state of mass \( M_b \). We have not much of an idea on the bound-state spectrum of colored mesons, neither experimentally nor theoretically. We dont even know, whether such mesons are bound at all, but I see no immediate objection why one could not give it a try by the above methods, particularly a suitably adjusted ↑↓-model.

I conclude that the calculation of the Lamb shift in QCD is an interesting and important problem particularly for the pion. Can we challenge the lattice community to get help on that?

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