On the entropy of a family of random substitutions

Johan Nilsson

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Abstract The generalised random Fibonacci chain is a stochastic extension of the classical Fibonacci substitution and is defined as the rule that is mapping $0 \mapsto 1$ and $1 \mapsto 101^{m-i}$ with probability $p_i$, where $p_i \geq 0$ with $\sum_{i=0}^{m} p_i = 1$, and where the random rule is applied each time it acts on a $1$. We show that the topological entropy of this object is given by the growth rate of the set of inflated generalised random Fibonacci words.

Keywords Generalised substitutions · Entropy calculation

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1 Introduction

Meyer sets form an important class of point sets that can be considered as generalisations of lattices [8,9]. They have recently been studied in connection with mathematical models of quasicrystals, see [2,6,10] and references therein. One of their common features is the existence of a non-trivial point spectrum. It shows up as a relatively dense set of point measures in their diffraction measure [12], even under the constraint that one only considers point measures whose intensity is at least a given positive fraction of the the central intensity.
The majority of papers so far has concentrated on deterministic Meyer sets, such as those obtained from Pisot inflations or from the projection method. All these examples have zero entropy. On the other hand, it is well known that Meyer sets with entropy exist (such as the set $2\mathbb{Z}$ combined with an arbitrary subset of $2\mathbb{Z} + 1$: compare [3]) but relatively little is known about them. Despite having entropy, the spectral result of Strungaru [12] still applies, and the point spectrum is non-trivial.

One possibility to define special classes of Meyer sets with entropy is via random inflation rules, as proposed in [5]. Here we follow their line of approach for a family of generalised Fibonacci substitutions, parametrised by a number $m \in \mathbb{N}$. The spectral nature for $m = 1$ was studied in [5, 7]. Here we reconsider this case and its generalisation from the point of view of entropy. In fact we prove that the topological entropy can simply be calculated by a suitable (and easily accessible) subset of sub-words (or factors), thus proving an (implicit) conjecture from [5] and its generalisation to the entire family.

Let us introduce the generalised random Fibonacci chain by the generalised substitution

$$
\theta : \begin{cases} 
0 &\mapsto 1 \\
1 &\mapsto 01m \\
1^{m-1} &\text{with probability } p_1 \\
&\vdots \\
1^m0 &\text{with probability } p_m
\end{cases}
$$

where $p_i \geq 0$ and $\sum_{i=0}^{m} p_i = 1$ and where the random rule is applied each time $\theta$ acts on a 1.

In [5] Godrèche and Luck define the random Fibonacci chain by the generalised substitution given by $\theta$, from (1), in the special case $m = 1$. They introduce the random Fibonacci chain when studying quasi-crystalline structures and tilings in the plane. In their paper it is claimed without proof that the topological entropy of the random Fibonacci chain is given by the growth rate of the set of inflated random Fibonacci words. In [11] this fact was proven, and we shall here give a proof of a more general result when we consider the generalised random Fibonacci substitution given by $\theta$ from (1).

Before we can state our main theorem, we need to introduce some notation. A word $w$ over an alphabet $\Sigma$ is a finite sequence $w_1w_2\ldots w_n$ of symbols from $\Sigma$. We let here $\Sigma = \{0, 1\}$. We denote a sub-word or a factor of $w$ by the shorthand $w[a, b] = w_aw_{a+1}w_{a+2}\ldots w_{b-1}w_b$ and similarly for a set of words we let $W[a, b] = \{w[a, b] : w \in W\}$. By $|\cdot|$ we mean the length of a word and the cardinality of a set. Note that $|w[a, b]| = b-a+1$.

For two words $u = u_1u_2u_3\ldots u_n$ and $v = v_1v_2v_3\ldots v_m$ we denote by $uv$ the concatenation of the two words, that is, $uv = u_1u_2u_3\ldots u_nv_1v_2\ldots v_m$. Similarly we let for two sets of words $U$ and $V$ their product be the set $UV = \{uv : u \in U, v \in V\}$ containing all possible concatenations.

Letting $\theta$ act on the word 0 repeatedly yields an infinite sequence of words $r_n = \theta^{n-1}(0)$. We know that $r_1 = 0$ and $r_2 = 1$. But $r_3$ is one of the words $1^i01^{m-i}$ for