The Duality between IIB String Theory on PP-wave and $\mathcal{N} = 4$ SYM: a Status Report

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Abstract.
The aim of this report is to give an overview of the duality between type IIB string theory on the maximally supersymmetric PP-wave and the BMN sector of $\mathcal{N} = 4$ Super Yang-Mills theory. The general features of the string and the field theory descriptions are reviewed, but the main focus of this report is on the comparison between the two sides of the duality. In particular, it is first explained how free IIB strings emerge on the gauge theory side and then the generalizations of this relation to the full interacting theory are considered. An “historical” approach is taken and the various proposals presented in the literature are described.

1. Introduction

In [1], Berenstein, Maldacena and Nastase proposed a very concrete relation between type IIB string theory on the maximally supersymmetric PP-wave and a particular subsector of the $\mathcal{N} = 4$ Super Yang-Mills theory. This proposal immediately attracted a lot of attention and triggered a great deal of activity both in the analysis of the original physical situation and in the extension of these ideas to other interesting cases. The reason for this intense activity is twofold. On the one hand, PP-waves are interesting in themselves, since they provide a perfect arena for studying string theory in backgrounds that are very different from Minkowski flat space. For instance, PP-waves can be used to test our understanding of the string dynamics in presence of non-trivial Ramond-Ramond fields and provide tractable backgrounds that are nevertheless curved and not even asymptotically flat. On the other hand the relation proposed in [1] is the first example of a direct connection between a four dimensional gauge theory and a

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string theory that can be quantized. In particular, the relation between strings on the maximally supersymmetric PP-wave and the $\mathcal{N} = 4$ Super Yang-Mills theory appears to be a “corollary” of the usual AdS/CFT duality [2]. Thus, at the moment, the PP-wave background provides the best setup for testing in a concrete example the various ideas about the gauge/string theory duality that have been developed since ’t Hooft’s seminal paper [3].

Besides the various applications, the study of string dynamics on PP-wave has a beauty in itself because it connects many ideas coming from different areas of theoretical physics, like general relativity, string perturbation theory, gauge theories, integrable systems and others. This makes the subject quite rich and clearly it is not possible to summarize all the developments in a single paper. Fortunately various reviews are already available, each of them analyzing some particular aspect of the problem. The original proposal is summarized in [4]. Ref. [5] reviews the techniques usually employed to describe the string dynamics, while [6] focuses more on the field theory side of the correspondence. The most recent reviews are [7], where the PP–wave/SYM duality proposed in [1] is described in general, and [8], where the semiclassical interpretation of the PP-wave limit and the latest developments in this area are summarized. For a systematic introduction to the various aspects of the PP–wave/SYM duality, the reader is referred to the reviews cited above. In fact, this report does not provide a detailed description of the subject, since, for instance, the explicit derivation of many technical results will be simply omitted. On the contrary, the aim of this work is to focus on the comparison between the string and the field theory side of the PP–wave/SYM duality, and to summarize all the proposals that appeared in the literature on this issue. Particular attention is devoted to see how these proposals can be interpreted in the broader context of the string/gauge theory correspondence à la ’t Hooft. In this respect it is very important to stress the connections between the setup presented in [1] and the AdS/CFT duality involving the full $\mathcal{N} = 4$ Super Yang-Mills theory originally proposed in [2]. The main goal is clearly to use string theory on PP-waves to learn more about the general properties of the AdS/CFT duality and to derive results on quantities not protected by supersymmetry.

The structure of this report is the following. In Section 2 the basic ingredients of the PP-wave/SYM duality are introduced: the plane-wave solution itself, the dynamics of free IIB strings on this background and finally the BMN dictionary [1] between the Fock space of the string states and a subset of gauge invariant composite operators on the SYM side. This relation between the physical spectra of the two descriptions can be explained in different ways. At the planar/free level all these approaches lead to equivalent results, but they inspired different proposals when the correspondence is generalized to the full interacting theory. These developments are discussed in Section 3: in particular, on the string side, the vertices describing the 3-string interaction are discussed, while on the gauge theory side the origin of the operator mixing is explained. At the light of these results, the BMN dictionary between string states and field theory operators is re-discussed. In Section 4 the analysis is extended beyond the map between
spectra and the various proposals dealing with dynamical quantities of the interacting theory are described. Particular attention is devoted to the approaches of [9] and of [10]. In the final Section we briefly describe some possible developments in the field and summarize the open problems both at the conceptual and at the technical level.

2. The basic concepts of the duality

The starting point of the BMN proposal [1] is the AdS/CFT duality. In its strong version this duality states that the \( \mathcal{N} = 4 \ SU(N) \) Super Yang-Mills theory and type IIB string theory on \( AdS_5 \times S^5 \) with \( N \) units of five form flux are exactly equivalent. This conjecture has been extensively studied in the last six years and, even if a real proof is still far away, a large amount of evidence has emerged. There are many excellent reviews about this subject [11, 12, 13, 14, 15], so there is no need to enter into a detailed description. Only few basic facts will instead be reported here:

1) One of the most important features of the AdS/CFT correspondence is the fact that both descriptions possess the same (super)symmetry group, whose bosonic part is \( SO(4,2) \times SO(6) \). On the string theory side, this corresponds to the isometry group of the \( AdS_5 \times S^5 \) background, while in the gauge theory description it represents the conformal group times the internal \( R–\)symmetry group \( SO(6) \sim SU(4) \) which rotates the four supersymmetry charges. Notice that both these groups are exact symmetries of the full interacting quantum field theory. This has far reaching implications both for the duality and for the study of the quantum effects.

2) Type IIB string theory on the \( AdS_5 \times S^5 \) background is characterised by three independent parameters: the \( N \) units of five-form fluxes on \( S^5 \), the string coupling \( g_s \), the constant vacuum expectation value of the axion \( \chi_0 \). The common radius of \( AdS_5 \) and \( S^5 \) measured in string units is \( R^2/\alpha' = \sqrt{4\pi g_s N} \). On the other hand the parameters of the \( \mathcal{N} = 4 \) SYM theory are the rank of the gauge group \( N \), the coupling \( g_{YM} \), from which one can define the ’t Hooft coupling \( \lambda = g^2_{YM} N \), and the vacuum angle \( \theta \). The dictionary between gauge and string theory parameters identifies on the one side the five–form flux and the rank of the gauge group, and on the other the coupling constants\(^\dagger\)

\[
\tau_{YM} = \frac{\theta}{2\pi} + i\frac{4\pi}{g_{YM}} = \chi_0 + i\frac{gs}{g_s}.
\]

(1)

Notice that this implies a relation between the \( AdS_5 \times S^5 \) radius and the ’t Hooft coupling

\[
\frac{R^2}{\alpha'} = \sqrt{\lambda}.
\]

(2)

This relation has an important consequence: the regime where the supergravity description of the \( AdS_5 \times S^5 \) dynamics is reliable \( R^2 \gg \alpha' \), corresponds to a strongly coupled gauge theory \( \lambda \gg 1 \). On the other hand, when the perturbative expansion of

\(^\dagger\) The \( U(N) \) matrices are normalized as \( \text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab} \). If one chooses a different normalization, then Eqs.(1) and (2) are modified. In general one has \( g^2_{YM} = 2r^2 g_s \), where \( r \) is related to the gauge group Casimir: \( \text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab} \).
gauge theory is reliable, the radius of the corresponding bulk geometry is small and
the full string dynamics is needed in the dual description. For this reason, most of the
explicit checks of the AdS/CFT duality involved protected quantities, i.e. particular
observables that do not receive quantum corrections in $\lambda$. Clearly in this case the
supergravity computation in the bulk has to match directly the perturbative SYM result.
A notable exception to this limitation is represented by the study of the Wilson loops,
see [16] and references therein for a review.

3) Since the string and the gauge theory descriptions are supposed to be equivalent,
there must be an isomorphism between the Hilbert spaces representing the spectra of
the two theories. This isomorphism is largely unknown and actually even the spectrum
itself is not fully understood. However many properties of this mapping have been
derived in the last years. For instance we know that a special role is played by the
single trace operators since they are related to single particle (string) states on the
AdS side, unless one is dealing with very “big” operators (we shall return to this point
later in Section 3.3). Multiple particle states simply correspond in the gauge theory to
products of non–coincident single trace operators separately normal ordered, while real
multi–trace operators, i.e. operators made out of many traces with fields evaluated in
the same point and globally normal ordered, should correspond to bound states
of the elementary string excitations. Of course, the AdS/CFT dictionary should map not only
the various states of the two spectra, but also their quantum numbers. In particular,
the energy of the string states is directly related to the conformal dimension of the
Corresponding gauge theory operators [17, 18]. At the level of supergravity excitations
this prediction can be tested explicitly: from the quadratic part of the effective action
one can extract the mass of the various supergravity fields which is directly related to
the conformal dimensions in gauge theory, see for instance Eq.(5.21) of [15].

4) The correspondence between gauge and string theory goes beyond the level
of the free theory. In fact, it is natural to think that the isomorphism between the
spectra implies also the coincidence of the correlation functions between string states
and gauge theory operators. This idea has been made precise in [17, 18], where it has
been proposed that the string partition function on $AdS_5 \times S^5$, subject to particular
boundary conditions at the conformal boundary, is equal to the generator of the field
theory correlators.

2.1. The maximally supersymmetric PP-wave

The gravitational wave relevant to the following analysis is [19]

$$g_{--} = g_{+-} = -2, \quad g_{++} = -\mu^2 \sum_{I=1}^{8} x_I x^I, \quad g_{IJ} = \delta_{IJ}, \quad I, J = 1, \ldots, 8,$$

$$F_{+1234} = F_{+5678} = 2\mu, \quad \phi = \text{constant}, \quad (3)$$

where $F$ is the R–R five form and $\phi$ the dilaton field. It is not difficult to see that
this field configuration solves the equations of motion of type IIB supergravity, but
what makes this background interesting is its high degree of symmetry. It possesses 14
obvious bosonic symmetries: the shifts of the light-cone coordinates $x^\pm \rightarrow x^\pm + c$ and the separate rotations of the two groups of directions (1234) and (5678). In fact the 5-form $F$ breaks the $SO(8)$ symmetry of the metric down to $SO(4) \times SO(4)$. It is worth to remark that the PP–wave solution Eq.(3) displays a discrete symmetry $Z_2$ exchanging the two $SO(4)$ groups. The shifts in the transverse coordinates $x^I$ are broken by the $g_{++}$ element of the metric. They are substituted by $x^\pm$–dependent transformations $P^I$. Moreover, the metric in Eq.(3) in invariant under the rotations in the $(+,I)$ directions $J^+I$. Summarizing, the gravitational wave in Eq.(3) displays 30 bosonic symmetries, exactly as the $AdS_5 \times S^5$ background. Also the number of fermionic symmetries is the same, and it is given by the 32 fermionic charges $Q^+, \bar{Q}^+$ and $Q^-, \bar{Q}^-$. This makes the PP–wave background particularly interesting, since it is a new maximally supersymmetric solution with non–trivial curvature. Clearly this background is closely related to the $AdS_5 \times S^5$ geometry and in fact it can be obtained [20] as a particular limit, the Penrose limit [21, 22], from the $AdS_5 \times S^5$ solution

$$ds^2 = R^2 \left[ - \cosh^2 r \, dt^2 + dr^2 + \sinh^2 r \, d\Omega_3^2 + \cos^2 \theta \, d\psi^2 + d\theta^2 + \sin^2 \theta \, d\Omega_3^2 \right]$$

$$F_5 = \frac{1}{R} (dV_{AdS_5} + dV_{S^5}) \, , \, \phi = \text{constant} \, .$$

Let us briefly recall how the two backgrounds Eqs.(4) and (3) are connected. Starting from $AdS_5 \times S^5$, we first introduce the light-cone coordinates

$$x^+ = \frac{t + \psi}{2\mu} \, , \, x^- = \mu R^2 \frac{t - \psi}{2} \, , \, \hat{r} = R \, r \, , \, y = R \theta \, ,$$

and then take the $R \rightarrow \infty$, by keeping $x^\pm$, $\hat{r}$ and $y$ fixed. Clearly the non trivial step of the procedure is in the limit itself. When $R \rightarrow \infty$, Eqs.(5) define the null geodesic $t = \psi \sim x^+ \, , \, \theta = r = 0$, but also the metric becomes large, since there is an overall factor of $R^2$. Thanks to these two properties the limit $R \rightarrow \infty$ on the geometry is (always) well defined [21] and in our case leads to the first line of Eq.(3). These considerations can be extended to the full supergravity [22] provided that the various forms, representing the gauge potentials, scale appropriately. In our case $A_4 \sim R^4$ is exactly the right behavior to give a non–trivial and well defined R–R form also in the large $R$ limit. Moreover various features of the plane wave solutions obtained in this way can be derived on general grounds [23], by using only the properties of the Penrose limit. Here we would just like to remind that the number of (super)symmetry generators can not decrease in the limit, but the algebra can be only deformed [24]. This automatically ensures that the background Eq.(3) is maximally supersymmetric, since it is derived from $AdS_5 \times S^5$.

It is also important to clarify the physical meaning of the Penrose limit just performed. First notice that the change of coordinates Eq.(5) put at the “center” of the space a particular null geodesic. Then it is clear that the Penrose limit enlarge a small neighborhood around this geodesic to be the full space, washing away all the rest of the original background. Equivalently one can say that the Penrose limit is a truncation of the physical spectrum of the original theory, where one focuses only on the excitations that are confined to live close to the null geodesic used to define the
limit. In particular, these states describe short strings \((\frac{R^2}{\alpha'} \gg 1)\) fast rotating around the equator of the five–sphere. In fact many of the features of the string states on the PP–wave background can be derived by studying these rotating strings on \(AdS_5 \times S^5\) in the semiclassical approximation \([25]\).

2.2. The free IIB strings on PP-wave

The covariant world–sheet action describing the propagation of a free type IIB superstring on the PP–wave background has been formulated in \([26, 27]\). The remarkable feature of this action is that in the light–cone gauge it reduces to a quadratic action describing free massive bosons and fermions. In fact, by imposing the light–cone gauge conditions \(X^+ = \alpha' p^+ \tau_a \equiv \alpha \tau_a\) and \(\Gamma^+ \theta = 0\) and then rescaling \((\tau, \sigma) = |\alpha' p^+| (\tau_a, \sigma_a)\), one gets from the covariant action written in \([26, 27]\)

\[
S_\text{b} = \frac{1}{4\pi \alpha'} \int d\tau \int d\sigma \left[ (\partial_\tau X)^2 - (\partial_\sigma X)^2 - \mu^2 X^2 \right], \tag{6}
\]

\[
S_\text{f} = \frac{1}{4\pi \alpha'} \int d\tau \int d\sigma \left\{ i e(\alpha) \left[ \bar{\theta} \partial_\tau \theta + \theta \partial_\tau \bar{\theta} + \theta \partial_\sigma \theta + \bar{\theta} \partial_\sigma \theta \right] - 2 \mu \bar{\theta} \theta \right\}. \tag{7}
\]

Lorentz indices have been suppressed for sake of simplicity; the 8 bosons \(X^I\) and the 8 fermions \(\theta^a\) are always contracted with a Kronecker \(\delta\) except for the mass term in \(S_\text{f}\), where \(\Pi = \sigma_3 \otimes I_{4 \times 4}\) appears. As usual, we indicate with \(\alpha'\) the Regge slope, while \(\alpha = \alpha' p^+\) is the rescaled light-cone momentum and \(e(\alpha) = 1\) if \(\alpha > 0\) and \(e(\alpha) = -1\) if \(\alpha < 0\). Moreover, we take \(\mu > 0\). From Eqs.(6) and (7) it is straightforward to derive the mode expansions, the commutation relations and the expressions for the free symmetry generators.

The most general solution of the equations of motion derived from (6) is given by:

\[
X(\tau, \sigma) = i \sqrt{\frac{\alpha'}{2}} \left[ \frac{a_0 e^{-i \omega_0 \tau}}{\sqrt{\omega_0}} - \frac{a_0^\dagger e^{i \omega_0 \tau}}{\sqrt{\omega_0}} + \sum_{n=1}^{\infty} \frac{1}{\sqrt{\omega_n}} \left( \hat{a}_n e^{-i \omega_n \tau - n \sigma} - \hat{a}_n^\dagger e^{i \omega_n \tau + n \sigma} \right) + \sum_{n=1}^{\infty} \frac{1}{\sqrt{\omega_n}} \left( \hat{a}_n e^{-i \omega_n \tau + n \sigma} - \hat{a}_n^\dagger e^{i \omega_n \tau - n \sigma} \right) \right], \tag{8}
\]

where \(\omega_n = \sqrt{n^2 + (\alpha \mu)^2}\). The conjugate momentum is \(P = \partial S_\text{b} / \partial \dot{X} = \frac{1}{2 \pi \alpha'} \partial_\tau X\) and the light-cone Hamiltonian \(\hat{H}_\text{b}\) is

\[
\hat{H}_\text{b} = \frac{1}{\alpha} \left[ \frac{\omega_0}{2} (a_0 a_0^\dagger + a_0^\dagger a_0) + \sum_{i=1}^{2} \sum_{n=1}^{\infty} \frac{\omega_n}{2} (\hat{\alpha}_n^\dagger \hat{\alpha}_n^i + \hat{\alpha}_n \hat{\alpha}_n^i) \right]. \tag{9}
\]

The canonical commutation relations \([X(\tau, \sigma), P(\tau, \sigma')] = i \delta(\sigma - \sigma')\) are satisfied if the oscillators commute as follows:

\[
[a_0, a_0^\dagger] = 1, \quad [\hat{a}_n, \hat{a}_m^\dagger] = \delta_{nm} \delta^{ij}, \tag{10}
\]

\(\dagger\) As usual the canonical Hamiltonian is defined as the \(\partial_\tau\), while the light cone Hamiltonian is identified with the \(\partial_{x^+}\) generator. However the two are almost identical, because of the light-cone gauge condition \(X^+ = e(\alpha) \tau\). The only difference is the presence of an additional sign \(e(\alpha)\) that is included in (9).
In terms of them we can compute:
\[ a_n = \frac{1}{\sqrt{2}} (\hat{a}_n + \hat{a}_{-n}) \quad \text{and} \quad a_{-n} = \frac{i}{\sqrt{2}} (\hat{a}_n - \hat{a}_{-n}) \quad \text{for} \quad n \geq 1 . \]

In terms of them we can compute:
\[ X(\tau = 0, \sigma) = x_0 + \sqrt{2} \sum_{n=1}^{\infty} x_n \cos \frac{n\sigma}{|\alpha|} + x_{-n} \sin \frac{n\sigma}{|\alpha|} \]

where
\[ x_n = i \sqrt{\frac{\alpha'}{2\omega_n}} (a_n - a_n^\dagger) \quad n \geq 0 \quad \text{and} \quad x_{-n} = i \sqrt{\frac{\alpha'}{2\omega_n}} (a_{-n} - a_{-n}^\dagger) \quad n < 0 . \]

Concerning now the fermions, we introduce the real Grassmann variables
\[ \theta = \theta^1 + i\theta^2 \quad \text{and} \quad \bar{\theta} = \theta^1 - i\theta^2 \]

In term of these variables, the equations of motion are
\[ e(\alpha) \partial_\tau \theta^1 - \mu \Pi \theta^2 = 0 \quad \text{and} \quad e(\alpha) \partial_\tau \bar{\theta}^2 + \mu \bar{\Pi} \theta^1 = 0 . \]

Their most general solution can be written as follows:
\[
\sqrt{\frac{\alpha}{\alpha'}} \theta^1 = \frac{1}{\sqrt{2}} \left( e^{-i\omega \tau n \alpha} \theta^1_0 + e^{i\omega \tau n \alpha} \theta^1_0^\dagger \right) + \sum_{n=1}^{\infty} c_n \left[ e^{-i\omega \tau n \alpha} \theta^1_n + e^{i\omega \tau n \alpha} \theta^1_n^\dagger \right] \\
+ i \sum_{n=1}^{\infty} c_n \frac{\omega_n - n}{\alpha \mu} \left[ e^{-i\omega \tau n \alpha} \Pi \theta^2_n + e^{i\omega \tau n \alpha} \Pi (\theta^2_n)^\dagger \right],
\]

and
\[
\sqrt{\frac{\alpha}{\alpha'}} \theta^2 = \frac{e(\alpha)}{\sqrt{2}} \left( i e^{i\omega \tau n \alpha} \Pi \theta^2_0 - i e^{-i\omega \tau n \alpha} \Pi \theta^2_0^\dagger \right) + \sum_{n=1}^{\infty} c_n \left[ e^{-i\omega \tau n \alpha} \theta^2_n + e^{i\omega \tau n \alpha} (\theta^2_n)^\dagger \right] \\
- i \sum_{n=1}^{\infty} c_n \frac{\omega_n - n}{\alpha \mu} \left[ e^{-i\omega \tau n \alpha} \Pi (\theta^1_n) - e^{i\omega \tau n \alpha} \Pi (\theta^1_n)^\dagger \right],
\]

where the coefficients \( c_n \) are defined as
\[ c_n = \frac{1}{\sqrt{1 + \rho_n^2}} \quad \text{and} \quad \rho_n = \frac{\omega_n - n}{\mu \alpha} . \]
From the action in Eq. (7) we can define the conjugate momenta \( \lambda^i \equiv \frac{\delta L}{\delta \dot{\sigma}^i} = -\frac{i e(\alpha)}{4 \pi \alpha} \theta^i \) and we get the Hamiltonian

\[
H_f = \frac{1}{|\alpha|} \left\{ \sum_{n=1}^{\infty} \sum_{i=1}^{2} \frac{\omega_n}{2} \left( \theta_n^i \dot{\theta}_n^i - \dot{\theta}_n^i \theta_n^i \right) + \frac{\omega_0}{2} \left( \theta_0^0 \theta_0^0 - \theta_0^0 \theta_0^0 \right) \right\} .
\] (21)

The canonical anticommutation relations \( \{ \theta^i(\sigma, \tau), \lambda^j(\sigma', \tau') \} = -i \delta^{ij} \delta(\sigma - \sigma')/2 \) imply that \( \{ \theta^i(\sigma, \tau), \theta^j(\sigma', \tau) \} = 2\pi \alpha' e(\alpha) \delta^{ij} \delta(\sigma - \sigma') \), and are satisfied if

\[
\{ \theta^i_n, \theta^j_n \} = e(\alpha) \delta_{nm} \delta^{ij} .
\] (22)

The tower of the fermionic string excitations is built by acting on the string vacuum \( |0,p^+\rangle \) with the creation operators \( \hat{b}^+_n \equiv e(\alpha) \theta^+_n \) and \( \hat{b}^+_n \equiv e(\alpha) \theta^+_n \). As for the bosonic case, in order to describe the 3–string interaction it is useful to introduce the following combinations:

\[
\sqrt{2} b_n = \hat{b} + i \hat{b}_- , \quad \sqrt{2} b^+_n = \hat{b}^+_n - i \hat{b}^- , \quad n > 0
\] (23)

\[
\sqrt{2} b^- = \hat{b} - i \hat{b}_{-} , \quad \sqrt{2} b^+_n = \hat{b}^+_n + i \hat{b}_- , \quad n > 0
\] (24)

and

\[
b^0 = \theta_0 , \quad e(\alpha) \hat{b}^+_0 = \theta^+_0 .
\] (25)

From Eq. (22) the following anticommutation relations are easily derived:

\[
\{ b_n, b^+_m \} = \delta_{nm} , \quad \{ b_n, b_m \} = \{ b^+_n, b^+_m \} = 0 .
\] (26)

In terms of the \( b_n \)'s the Hamiltonian in Eq. (21) becomes:

\[
H_f = \frac{1}{\alpha} \sum_{n=-\infty}^{\infty} \frac{\omega_n}{2} \left( b_n^+ b_n - b_n b_n^+ \right) .
\] (27)

Notice that the zero–point energy of the fermionic oscillators cancels against the bosonic contribution, as can be seen by the comparison between Eq. (9) and Eq. (27), and the full free Hamiltonian can simply be written as follows

\[
H = \frac{1}{\alpha} \sum_{n=-\infty}^{\infty} \omega_n \left( \hat{a}_n^+ \hat{a}_n + \hat{b}_n^+ \hat{b}_n \right) .
\] (28)

2.3. The BMN dictionary

It was shown that the PP–wave background can be obtained as a Penrose limit of the \( AdS_5 \times S^5 \) geometry, and that this limit can be described as a truncation of the full string spectrum to a particular subsector. It is then natural to expect from the AdS/CFT correspondence that there is a subsector of the \( \mathcal{N} = 4 \) SYM theory which is dual to IIB string theory on the PP–wave. In order to properly define the gauge invariant operators relevant to this duality we need to recall the content of the the \( \mathcal{N} = 4 \) SYM. We adopt the formulation in terms of \( \mathcal{N} = 2 \) multiplets since this facilitates the matching with the string side. In fact, this formulation has the advantage to realize explicitly the \( R \)-symmetry subgroup \( SU(2)_V \times SU(2)_H \times U(1)_J \subset SU(4) \), where \( SU(2)_V, SU(2)_H \) are respectively the internal symmetry groups of the \( \mathcal{N} = 2 \) vector multiplet and
hypermultiplet. In this way we can naturally identify the \( SU(2)_V \times SU(2)_H \) group with one of the two sets of \( SO(4) \) rotations that preserve the solution Eq.(3). The isometries of the PP–wave background contain another \( SO(4) \) group which is identified with the Euclidean rotations of the \( \mathcal{N} = 4 \) gauge theory. Finally the \( U(1)_J \) group rotates the complex scalar field \( Z \) of the vector multiplet and corresponds, on the gravity side, to the shift of \( x^- \). Actually this identification can not hold exactly since \( x^- \in \mathbb{R} \), while \( U(1)_J \) is a compact generator. It can be valid only if we focus on the gauge theory operators that have a large \( U(1)_J \) charge and then take the limit \( J \to \infty \). In fact this truncation of the \( \mathcal{N} = 4 \) spectrum corresponds to the gauge theory analogue of the Penrose limit discussed in Section 2.1. This important observation can be made precise by rewriting, thanks to Eq.(5), the generators of the shifts of \( x^+ \) (the light–cone Hamiltonian) and of \( x^- \) (the \( p^+ \) momentum) in terms of \( \partial/\partial t \) and \( \partial/\partial \psi \). Then one can translate this result on the gauge theory side obtaining

\[
\frac{H}{\mu} = \Delta - J , \quad 2\mu p^+ = \frac{\Delta + J}{R^2} ,
\]

where, following the usual AdS/CFT intuition, the energy \( -i \partial/\partial t \) has been mapped to the conformal dimension \( \Delta \) and the angular momentum \( -i \partial/\partial \psi \) to a \( U(1)_J \) charge. Notice that the factor of 2 multiplying \( p^+ \) in the second relation is consequence of the value of \( g_{--} \). Thus the gauge invariant operators we need to consider have a parametrically large \( J \) charge and conformal dimension (\( \Delta, J \sim R^2 \)), while their difference has to be finite. Because of this, we can approximate \( \Delta + J \) with \( 2J \)

\[
2\mu p^+ = \frac{\Delta + J}{R^2} \to \frac{2J}{R^2} = \frac{2J}{\alpha' \sqrt{\lambda}} ,
\]

and find that \( p^+ \) and the \( U(1)_J \) charge are identified, as anticipated. Summarizing, the Penrose limit translates on the field theory side in the double scaling limit \([28, 9]\)

\[
\Delta \to \infty , \quad J \to \infty , \quad N \to \infty \text{ with } \frac{J}{\sqrt{N}} , \quad \Delta - J , \quad g_{YM}^2 \text{ fixed} ,
\]

which instructs us to keep on the field theory side only the gauge–invariant operators containing an infinite number of \( Z \) fields. Notice also that this limit implies that the \('t\) Hooft coupling \( \lambda \) goes to infinity, which from Eq.(2) directly corresponds to the Penrose limit \( \frac{R^2}{\alpha'} \to \infty \). Nonetheless, the presence of the large quantum number \( J \) allows for the definition of a new parameter \( \lambda' \) which is finite in the double scaling limit Eq.(31). More precisely, the string dynamics in the PP–wave background is characterized by the two independent parameters

\[
g_2 \equiv \frac{J^2}{N} = 4\pi g_s (\mu \alpha' p^+)^2 , \quad \lambda' \equiv \frac{1}{(\alpha' \mu p^+)^2} = \frac{\lambda}{\sqrt{J}} .
\]

The relation involving \( \lambda' \) follows from Eq.(30) and then combining this with \( g_{YM}^2 = 4\pi g_s \), one obtains the first relation in Eq.(32). Notice that the string expression of \( \lambda' \) does not contain any power of the string coupling constant; thus it can be seen as a classical quantity related to the world–sheet dynamics. According to the usual \('t\) Hooftian intuition the string world–sheet dynamics resums the quantum perturbative expansion of
the dual gauge theory. On the contrary, the string expression for \( g_2 \) contains \( g_s \) explicitly and thus has to be interpreted as a space–time coupling, while on the SYM side \( g_2 \) is expressed in terms of classical quantities only. Thus \( g_2 \) corresponds \cite{28, 9} to the genus expansion parameter for the operators satisfying Eq.\( (31) \): on the string side this is a quantum expansion while, on the field theory side, it classifies topology of the various diagrams. This means that the sphere approximation (tree-level) in string theory resums only the planar contributions on the SYM side, the 1–loop string diagrams correspond to the field theory contributions that are planar if drawn on a torus and so on. Thus, as in the case of the full AdS/CFT duality, also for the BMN subsector one can use classical string theory to derive field theory results valid at the quantum level, with the caveat the only planar contributions have been considered. A first simple and very interesting example of such a prediction can be derived from the first of the equations in \( (29) \) which connects the string Hamiltonian \( (28) \) to the gauge theory dilatation operator

\[
\Delta - J = \frac{H}{\mu} = \sum_{n=-\infty}^{\infty} \hat{\omega}_n \left( \hat{a}_n^\dagger \hat{a}_n + \hat{b}_n^\dagger \hat{b}_n \right), \text{ with } \hat{\omega}_n = \sqrt{1 + n^2 \lambda}.
\]  

(33)  

This can be seen as an operatorial relation: \( H \) acts on the Fock space built with the string oscillators \( \hat{a}_n^\dagger \) and \( \hat{b}_n^\dagger \) and gives the tree-level (sphere) contribution to the energy of each state; \( \Delta - J \) acts on the gauge invariant operators satisfying Eq.\( (31) \) and gives the planar contribution to their conformal dimension (minus their \( J \)–charge). One can test Eq.\( (33) \) by diagonalizing the operators on both sides of the relation and then building a dictionary between string and gauge theory states by identifying the eigenvectors. This approach is reviewed in \cite{6}, thus let us now recall a different way to derive the same result. The starting point is to use the mapping between the symmetries in the two descriptions introduced at the beginning of this section. Then, as in the AdS/CFT case, one can start build a dictionary between string and gauge theory spectra by selecting the states with the same quantum numbers. The simplest possible string state is the vacuum \( |v\rangle \), \textit{i.e.} the state which is annihilated by all destruction operators \( \hat{a}_n, \hat{b}_n \) and has minimal light–cone energy. The field theory operator related to this state is a gauge–invariant operator containing only the complex fields \( Z, : \text{Tr}[Z^J] : \), where the quantum numbers \( p^+ \) and \( J \) are connected by the Eq.\( (30) \), \( J^2 = \lambda (\mu \alpha' p^+)^2 \). The next simplest states are the half–BPS states constructed, on the string side, by acting on the vacuum with the \( \hat{a}_0, \hat{b}_0 \). These creation operators transform non-trivially under the isometry group of the background and it is not difficult to find, on the field theory side, fields that transform in the same way. These fields are the SYM analogue of the harmonic oscillators and must have \( \Delta - J = 1 \), since \([H/\mu, \hat{a}_0^\dagger] = 1 \). We can construct the field theory operators corresponding to the string states created by \( \hat{a}_0^\dagger, \hat{b}_0^\dagger \) by inserting these fields as “impurities” in the long string of \( Z \) fields representing the string vacuum. In particular, the impurities associated to the eight bosonic creation operators \( \hat{a}_0^\dagger \) are represented by the covariant derivative acting on the \( Z \) field and by the four scalars in the hypermultiplet (or by the two other complex scalar fields \( (\phi^1, \phi^2) \) in the \( N = 1 \) formulation). The impurities associated to the eight fermionic creation operators \( \hat{b}_0^\dagger \) are associated with the Weyl fermions \( \lambda^a_0 \) of the vector multiplet (for the
Figure 1. The BMN operators provides a discretized model for IIB strings.

\[ \text{Z insertion} \]

\[ \text{Impurity} \]

\[ \text{Impurity} \]

\( \Pi = 1 \) chirality) and \( \bar{\psi}_\alpha^Z \) of the hypermultiplet (for the \( \Pi = -1 \) chirality). Alternatively in the \( \mathcal{N} = 1 \) language they can be represented by the gaugino \( \lambda_\alpha \), the superpartner \( \psi^Z_\alpha \) of the \( Z \) field and the superpartners \((\bar{\psi}_1^\alpha, \bar{\psi}_2^\alpha)\) of the scalar fields \((\bar{\phi}_1, \bar{\phi}_2)\). The first two impurities correspond to the insertion of string fermionic oscillators of chirality \( \Pi = 1 \), while the last two to fermionic oscillators of the opposite chirality \( \Pi = -1 \).

With this input, one can construct gauge theory operators that transform exactly as the string state considered by inserting in the string of \( Z \) of the vacuum operator the fields corresponding to the various oscillators. In order to have a half BPS operator also on the SYM side, these insertions have to be done in all possible ways. For instance the first entries of the dictionary are

\[ |0, p^+\rangle \leftrightarrow \frac{1}{\sqrt{J N_o^J}} \text{Tr}[Z^J] = O_{\text{vac}}^J, \tag{34} \]

\[ \hat{a}^5_0 |0, p^+\rangle \leftrightarrow \frac{1}{\sqrt{N_o^{J+1}}} \text{Tr}[\phi_1^Z Z^J] = O_{J},^1 \]

\[ \hat{a}^5_0 \hat{a}^5_0 |0, p^+\rangle \leftrightarrow \frac{1}{\sqrt{N_o^{J+2}(J + 2)}} \sum_{l=0}^{J} \text{Tr}[\phi_1^Z \phi_2^Z Z^{(J-l)}] = O_{J},^{12}. \]

The normalizations have been put for later convenience: \( N_o = \Gamma(\omega - 1) \frac{\lambda}{8\omega} \) and \( \omega \) is related to the space-time dimension \( 2\omega = d \). The relation Eq.(34) is so natural that one might think that it was the only possibility. However this is not true, since there are many other field theory operators with the same quantum numbers, like \( : \text{Tr}[Z^l] \text{Tr}[Z^{J-l}] : \) or other possible multi–trace generalizations. There is a simple reason explaining why the single trace operator have a special status: as already suggested in the original paper [1] the long “string” of \( Z \)’s in the SYM operators is identified with the physical IIB string. The cyclicity of the trace makes the “string” of \( Z \)’s closed; moreover \( \alpha' p^+ \) can be also seen as parametrizing the length of the IIB string as \( J \) sets the length of the field theory operator and we have \( p^+ \sim J \). Physically one can see the SYM operators as a discretized description of IIB strings, where each \( Z \) carries one bit of the total \( p^+ \) momentum. The BMN limit \( J \to \infty \) just corresponds to the continuum limit of the discretized string model provided by the \( \mathcal{N} = 4 \) SYM.

Up to this point the dictionary is directly inherited from the AdS/CFT correspondence in the supergravity limit. The new physical input comes with the
identification of the operators corresponding to the excited string states. Now, on the string side one can construct the spectrum by acting on the vacuum with the creation operators \( \hat{a}_n^\dagger, \hat{b}_n^\dagger, n \in \mathbb{Z} \). BMN proposed that the action of these operators corresponds on the field theory side to the insertion of “impurities” in the string of \( Z \) as in Eq.(34), but weighted with a \( n \)-dependent phase. In fact, one has to insert the appropriate field with a phase proportional to the level of the string oscillator \((n)\) and to the position of the insertion \((l)\) divided by the total number of the fields in the operator. Finally one has to sum over all possible positions of the insertion. In this way, one ends up with the following dictionary for the bosonic states (for \( m \neq 0 \)):

\[
\hat{a}_m^{5\dagger} \left| 0, p^+ \right\rangle \leftrightarrow \frac{1}{J \sqrt{N_{j+1}}} \sum_{l=0}^{J} \text{Tr} \left[ Z^l \phi^1 Z^{J-l} e^{2\pi i \frac{(l+1)m}{J+2}} \right] = 0, \quad (35)
\]

\[
\hat{a}_m^{6\dagger} \hat{a}_{-m}^{5\dagger} \left| 0, p^+ \right\rangle \leftrightarrow \frac{1}{\sqrt{N_{j+2}(J+2)}} \sum_{l=0}^{J} \text{Tr} \left[ \phi^1 Z^l \phi^2 Z^{J-l} e^{2\pi i \frac{(l+1)m}{J+2}} \right]. \quad (36)
\]

In the following we will refer to this type of 2-impurity operator as \( O_{m}^{12} \). At first sight the Eq.(35) seems strange since it implies that the state obtained by acting with \( \hat{a}_m^{5\dagger} \) does not have any corresponding operator on the gauge theory side. But actually this is correct since also on the string side the state in Eq.(35) does not belong to the physical spectrum. In fact, as usual in closed string theory, the physical states have to satisfy the level matching condition \( T | s \rangle = 0 \), where

\[
T = \sum_{n=-\infty}^{\pm \infty} \frac{n}{\alpha} \left( \hat{a}_n^\dagger \hat{a}_n + \hat{b}_n^\dagger \hat{b}_n \right). \quad (37)
\]

It is interesting to notice that the dictionary Eqs.(35)-(36) nicely follows from the pictorial identification between gauge invariant operators and closed strings previously discussed. This interpretation explains also some details of the correspondence Eq.(35). In fact one may wonder why on the left hand (string) side one has to use the hatted oscillators \( \hat{a}_n^\dagger, \hat{b}_n^\dagger \) and not the oscillators \( a_n^\dagger, b_n^\dagger \) that will be used in the next Section for the derivation of the 3-string vertex. The reason is simply that we used on the field theory side the phase \( \exp \left[ 2\pi i \frac{m l + 1}{J+2} \right] \) and this phase is exactly the one appearing in the mode expansion of the bosonic string coordinates in terms of the hatted oscillators, see Eq.(8)! In fact \( J \) measures the length of the SYM operator and so is mapped into \( \alpha' p^+ = \alpha \) which measures the length of the string in the light–cone gauge. The position in the trace labeled with \( l \) becomes \( \sigma \) \((l \rightarrow \sigma / \alpha')\) and \( m \) is the usual weight for the phase of the \( m \)-th string mode. In formula this implies

\[
\phi e^{2\pi i \frac{m(l+1)}{J+2}} \leftrightarrow \hat{a}_m^{\dagger} e^{-i \frac{\alpha m}{\alpha}} , \quad \phi e^{-2\pi i \frac{m(l+1)}{J+2}} \leftrightarrow \hat{a}_{-m}^{\dagger} e^{i \frac{\alpha m}{\alpha}}. \quad (38)
\]

Notice that this simple mapping is possible because in the mode expansion Eq.(8) the term proportional to \( e^{-i \frac{\alpha m}{\alpha}} \) involves only \( \hat{a}_m^\dagger \) (i.e. \( \hat{a}_m^{\dagger} \)) and not also \( \hat{a}_m^{\dagger} \) (i.e. \( \hat{a}_{-m}^{\dagger} \)). In \( \Sigma \) The phase in Eq.(35) is normalized with a factor of \( J + 2 \) simply because this is the number of elementary fields present in the field theory operator. However, in what follows we will be interested only in the leading \( J \rightarrow \infty \) behaviour. See [29] for an analysis where the phase factors are defined so that the BMN operators form representations of the superconformal group also at finite \( J \).
We can naturally define a scalar product on the gauge theory side by introducing the of the quadratic Hamiltonian, so let us check explicitly the orthogonality condition. It is easy to see that at this level the conformal dimensions match the level) and the corresponding SYM operators should have, at planar level, the conformal dimensions dictated by Eq.(33). We first work in the free (gauge) theory approximation. It is easy to see that at this level the conformal dimensions match the $\mu \alpha \to \infty$ value of the quadratic Hamiltonian, so let us check explicitly the orthogonality condition. We can naturally define a scalar product on the gauge theory side by introducing the following map:

\[ \lim_{r \to \infty} \big\langle 0, p^+ | \hat{a}_m^6 \hat{a}_m^5 \big\rangle \leftrightarrow \lim_{r \to \infty} \tilde{O}_{n}^{J,12}(x) = \lim_{r \to \infty} (r^2)^{\Delta} O_{n}^{J,12}(x) . \] (40)

Basiclly $O_{n}^{J,12}(0)$ and $\tilde{O}_{n}^{J,12}(\infty)$ are connected by means of the conformal inversion transformation $x' = x / r^2$, with $\partial x'/\partial x = C_{\mu\lambda}(x) / r^2$, $r \equiv |x|$. This explains also the factor of $(r^2)^{\Delta}$ appearing in Eq.(40). Thus the scalar product on the SYM side is

\[ \lim_{r \to \infty} \langle \tilde{O}_{n}^{J,12}(x) O_{m}^{J,12}(0) \rangle = \frac{1}{(J + 2)N_o^{J+2}} \sum_{l=0}^{J} e^{2\pi i\nu (l+1/2)} \sum_{k=0}^{J} e^{2\pi i\eta (k+1/2)} \times \langle \text{Tr}[\hat{\phi}_1 Z^l \hat{\phi}_2 Z^{J-l}] \text{Tr}[\hat{\phi}_1 Z^k \hat{\phi}_2 Z^{J-k}] \rangle \]

\[ = \frac{1}{(J + 2)} \sum_{k,l=0}^{J} e^{2\pi i\nu (l+1/2)} e^{2\pi i\eta (k+1/2)} \delta_{k,J-l} \sim \delta_{n,m} \] (41)

The second line is obtained by contracting first $\hat{\phi}_1$ and $\phi_1$ to glue the two traces and then the $\delta$ arises from the request of contracting $\hat{\phi}_2$ and $\phi_2$ in a planar way. In the last step the limit $J \to \infty$ has been taken. For the vector and fermion impurities the orthonormality condition is more subtle and requires a suitable definition of the complex
with one and two scalar impurities respectively. These have to be divided by one extra

we use the euclidean

P

each vector or fermion impurity in order to respect the normalization conditions Eqs.(43)-(46).

conjugation on the respective operators according to the rules suggested by the radial quantization [31]. For example, for double–vector impurities we define [32, 33]

\[ \tilde{O}_{\mu \nu}^{J} \equiv \frac{N_2}{2} (r^2)^J \sum_{l=0}^{\infty} e^{2 \pi i (l+1) \frac{r}{r^2}} \text{Tr} \left[ (C_{\mu \lambda} \partial_\lambda r^2 \tilde{Z}) \tilde{Z}^l (C_{\nu \rho} \partial_\rho r^2 \tilde{Z}) \tilde{Z}^{J-l} \right]_x, \]

where \( C_{\mu \lambda}(x) = \delta_{\mu \lambda} - 2 x_\mu x_\lambda / r^2 \) is the tensor associated to the conformal inversion transformation. The following identities hold for vector insertions:

\[ \frac{N_1^2}{2} \lim_{r \to \infty} \langle 0 | (r^2)^\Delta \text{Tr} \left[ C_{\mu \lambda} \partial^\lambda (r^2 \tilde{Z}) \tilde{Z}^J \right] (x) \text{Tr} \left[ \partial'_\nu \tilde{Z} \tilde{J}^l \right] (x') | 0 \rangle = \lim_{r \to \infty} C_{\mu \lambda} \left( \delta_{\lambda \nu} - \frac{2 x_\lambda x_\nu}{r^2} \right) = \delta_{\mu \nu} \] (43)

and

\[ \lim_{r \to \infty} \langle 0 | (r^2)^\Delta \text{Tr} \left[ C^{\mu \nu} \partial_\nu (r^2 \tilde{Z}) \tilde{Z}^J \right] (x) \text{Tr} \left[ \tilde{Z}^J \right] \langle x' | 0 \rangle = \lim_{r \to \infty} \partial_\mu \left( \frac{r^2}{(x - x')^2} \right) = 0 \] (44)

From Eqs.(43)–(44) it immediately follows that the vector impurities in the BMN operators behave exactly as the scalar impurities, and thanks to the Eq.(41) they satisfy the orthonormality conditions required by the string state/operator correspondence.

For fermionic impurities we define [33]

\[ \tilde{O}_{a}^{J} \equiv \frac{N_1}{\sqrt{2}} \lim_{r \to \infty} (r^2)^\Delta \text{Tr} \left[ \tilde{\lambda} \tilde{J}^a \tilde{Z}^J \right] (x), \] (45)

with \( \tilde{\lambda} \equiv \tilde{\sigma}_a^\mu x^\mu / r, \tilde{\psi} \equiv \tilde{\sigma}_a^\mu x^\mu / r \). By using these definitions we have:

\[ \frac{N_1^2}{2} \lim_{r \to \infty} \langle 0 | (r^2)^\Delta \text{Tr} \left[ \tilde{\lambda} \tilde{J}^a \tilde{Z}^J \right] \langle x \rangle \text{Tr} \left[ \lambda \tilde{J}^a \tilde{Z}^J \right] \langle x' \rangle | 0 \rangle = \delta^a_b, \] (46)

and similarly for the \( \psi \) impurities. This implies that also the tree-level evaluation of BMN correlators with fermionic impurities can be reduced to the scalar impurity case, apart from some (important) signs due to the anticommuting nature of \( \lambda \) and \( \tilde{\psi} \). Then, from Eq.(41) it follows that also the field theory operators corresponding to fermionic string excitations are correctly identified at the tree–level in the planar limit.

Let us now consider the subleading corrections. To proceed to the one–loop check of the dictionary Eq.(35), we need the precise form of the couplings in the \( \mathcal{N} = 4 \) Lagrangian, which reads in the euclidean space:

\[ L_E = \frac{1}{g_{YM}^2} \left[ F_{\mu \nu}^a F^{\mu \nu}_a + (D_\mu \phi^J)^a(D_\mu \phi^J)^a + \psi^I_a \sigma^a(D^\mu \tilde{\psi})^a + \chi^a \sigma^a(D^\mu \tilde{\chi})^a + \sqrt{2} f^{abc} \left( \psi^I_a \phi^J_b \psi^c_c + \psi^I_a \phi^J_c \psi^c_b \right) - \sqrt{2} f^{abc} \epsilon_{IJK} \left( \psi^I_a \phi^J_b \psi^K_c + \psi^I_a \phi^J_c \psi^K_b \right) \right. \]

\[ \left. \frac{1}{2} f^{abc} f^{ade} \phi^I_a \phi^J_b \phi^I_c \phi^J_d + \frac{1}{2} f^{abc} f^{ade} \phi^I_a \phi^J_b \phi^I_c \phi^J_d \right], \] (47)

\( N_1 = 1/\sqrt{N_0^{J+1}}, N_2 = 1/\sqrt{N_0^{J+2}(J+2)} \) are the normalization factors of the BMN operators with one and two scalar impurities respectively. These have to be divided by one extra \( \sqrt{2} \) factor for each vector or fermion impurity in order to respect the normalization conditions Eqs.(43)-(46). We use the euclidean \( \sigma \)-matrices \( \sigma^m = (1, i \tau^i), \bar{\sigma}^m = (1, -i \tau^i) \), where \( \tau^i \) are the Pauli matrices, and \( \text{Tr}(\sigma^m \bar{\sigma}^n) = 2 \delta^{mn} \).
where \( \phi^I \equiv (\phi^1, \phi^2, Z) \) are the three complex scalar fields of the \( \mathcal{N} = 4 \) SYM and \( \psi^I \) their fermionic superpartners. The covariant derivative reads \( (D_\mu \phi)^a = \partial_\mu \phi^a + f^{abc} A^b_\mu \phi^c \).

We consider the 2–point function of BMN operators with two scalar impurities \( \langle \bar{O}_J^i(x) O^i_J(0) \rangle \). The one–loop contribution to this correlator is given by the self–energy of the scalar fields, the gluon exchange and the four scalar interaction vertices in the last line of Eq.(47), which is useful to rewrite as:

\[
V_F = -\frac{4}{g_{YM}^2} \text{Tr} \left( \sum_{i=1}^{2} [Z, \phi_i][\bar{Z}, \bar{\phi}_i] + [\phi_1, \phi_2][\bar{\phi}_1, \bar{\phi}_2] \right) \tag{48}
\]

\[
V_D = \frac{2}{g_{YM}^2} \text{Tr} \left( \frac{1}{2} [Z, \bar{Z}]^2 + [Z, \bar{Z}][\phi_i, \bar{\phi}_i] + \frac{1}{2} \left( \sum_{j=1}^{2} [\phi_j, \bar{\phi}_j] \right)^2 \right) \tag{49}
\]

The computation is considerably simplified if we use the non–renormalization property of the two–point functions of BPS operators [34]. This property is based on the cancellation between the contribution of the self–energy, the gluon exchange and the D–term interaction in Eq.(49). Since the BPS operators are completely symmetric and traceless in the scalar fields, the F–term interaction in Eq.(48) does not contribute. The cancellation between self–energy, gluon exchange and D–terms can be shown to hold also for the BMN operators corresponding to true string excitations, as it is not sensible to the phase factors associated to the impurities and it is valid term by term in the sum over the position of the impurities along the string of \( Z \) fields [28, 9]. The difference with respect to the BPS (supergravity) operators is in the contribution of the F–term interaction Eq.(48). In fact, the presence of the phases associated to the impurities makes these operators no longer symmetric in the position of the scalar fields. If we focus on the scalar field \( \phi_1 \), the effect of the interaction Eq.(48) on the 2–point functions is summarized by the Fig.2. Notice that, as already stressed, we have to consider only the planar contributions, where the interaction can connect only contiguous fields. In particular, the term (A) displaces the impurity \( \phi_1 \) of one step to the right, the term (B) moves it one step to the left, while the remaining two leave the impurity in the same position. The resulting combinatoric factor will be proportional to the tree–level result times the respective phases of these four contributions \( i.e. \ (e^{2\pi i n_j} + e^{-2\pi i n_j} - 2) \). This picture, where the quantum corrections have the effect to move the impurities inside the BMN operators, is consistent with the physical intuition that these operators
are identified with the closed IIB strings. In fact, since each field on \( O_n^J \) is seen as a bit of the string, the effect of the F-term is that of a (discretized) one-dimensional Laplacian \( \partial^2 / \partial^2 \sigma \). Thus, as ’t Hooft suggested, the (gauge theory) quantum corrections reconstruct the world-sheet dynamics of the dual string description and the planarity requirement implies the locality of the string action.

Concerning now the space–time dependent part, we perform the computations using dimensional regularization in \( 2\omega = 4 - 2\epsilon \) dimensions. The configuration space propagator for the scalar fields is

\[
\langle \bar{\phi}^a_I(x)\phi^b_J(0) \rangle = g^2_Y M \delta^{ab} \delta_{IJ} \Delta_\omega(x) = g^2_Y M \delta^{ab} \delta_{IJ} \frac{\Gamma(\omega - 1)}{4\pi^\omega (x^2)^{\omega-1}} ,
\]

where \( \Delta_\omega(x) \) is the Green function for the Laplacian in \( 2\omega \) dimensions. The one–loop contribution to the two–point function between two BMN operators reads

\[
\langle O_{n}^{\ast J,12}(x) O_n^{J,12}(0) \rangle_{\text{1loop}} = 2g^2_Y M \left( \frac{1}{x^{2(\omega-1)}} \right)^{J+2} \frac{I(x)}{x^{2(\omega-1)}} (e^{2\pi i \frac{x}{\omega}} + e^{-2\pi i \frac{x}{\omega}} - 2)
\]

\[
\sim - \frac{8\pi^2 n^2 g^2_Y M}{J^2} \frac{1}{x^{2(\omega-1)}} \frac{1}{x^{2(\omega-1)}} \frac{I(x)}{x^{2(\omega-1)}} ,
\]

where in the last step we have taken the limit \( J \to \infty \). The quantity \( I(x) \) in Eq.(51) is the one–loop integral:

\[
I(x) = \left[ \frac{\Gamma(\omega - 1)}{4\pi^\omega} \right]^2 (x^2)^{2(\omega-1)} \int d^2 y \frac{1}{(y^2)^{2(\omega-1)}} \frac{1}{[(x-y)^2]^{2(\omega-1)}}
\]

\[
= \frac{1}{8\pi^2} \left( \frac{1}{\epsilon} + \gamma + 1 + \log \pi + \log x^2 + O(\epsilon) \right) .
\]

The result in Eq.(51) confirms that the loop expansion for the BMN operators is actually in terms of the effective parameter \( \lambda' \). Putting together the tree–level and the one–loop result we get from Eq.(51) and Eq.(52):

\[
\langle O_{n}^{\ast J,12}(x) O_n^{J,12}(0) \rangle = \left( \frac{1}{x^2} \right)^{J+2} \left( 1 - 8\pi^2 n^2 \lambda' I(x) \right) .
\]

The requirement of the orthonormality among the BMN operators in planar perturbation theory at one–loop fixes also the finite part in the renormalization procedure. In fact, by requiring that the one–loop renormalized operators

\[
O_n^{J,12} \equiv Z(\lambda', \epsilon) O_n^{J,12}_{\text{ren}}, \quad \text{with} \quad Z(\lambda', \epsilon) = \left[ 1 + \frac{n^2 \lambda'}{2} \left( \frac{1}{\epsilon} + F \right) \right] ,
\]

are orthonormalized, we get from Eq.(53):

\[
\langle O_{n,\text{ren}}^{\ast J,12}(x) O_n^{J,12}(0) \rangle = \frac{\delta_{nm}}{(x^2)^{J+2+n^2 \lambda'}} \left[ 1 + n^2 \lambda'(F - \gamma - 1 + \log \pi) \right]
\]

\[
\equiv \frac{\delta_{nm}}{(x^2)^{J+2+n^2 \lambda'}} ,
\]

which implies \( F = \gamma + 1 + \log \pi \). Thus the duality with string theory fixes a precise renormalization scheme for the field theory calculations.

From Eq.(55) we can read the anomalous dimension, \( \gamma_n = n^2 \lambda' \), of the BMN operator \( O_n^J \), which is in agreement with the duality relation in Eq.(33). This agreement
holds also at higher orders in the quantum expansion, as it was shown in [35] by means of an explicit 2-loop computation. At first sight this success in reproducing Eq. (33) with perturbative gauge theory computations appears strange. After-all the Penrose limit translates on the SYM side into a strong coupling limit $\lambda \to \infty$ and, in this regime, the use of Feynman diagrams is not justified. In principle one should compute $\langle O_n^{J,12}(x)O_n^{J,12}(0) \rangle$ exactly in $\lambda$ and $J$ and only then could one take the double scaling limit of Eq. (31). However, the 2-loop corrections to Eq. (51) are proportional to $1/J^4$ [35] which implies that they do not contribute at first order in $\lambda'$. This is likely to be a general pattern so that, at the level of 2-point functions, the ’t Hooft expansion is automatically changed term by term into a $\lambda'$-expansion $^+$. This expectation is supported by the analysis of [37], where the the string prediction Eq. (33) has been beautifully confirmed by a pure field theory argument.

3. Further developments

3.1. The 3-string vertex

As we have seen in Section 2.2, the quantization of the IIB superstring on the PP-wave background is straightforward once the light–cone gauge condition is imposed. An alternative proposal has been put forward in [38], where the superstring on the PP-wave background is described in a covariant formalism. Even if this looks a promising step toward a covariant quantization, the presence of a non–trivial R–R field gives rise to a complicated world-sheet action, and explicit computations of amplitudes in this framework have not been done yet. Thus for the computation of the couplings among the string states we are still obliged to work in the light–cone. In this gauge, the definition of the usual vertex operators for string states with $p^+ \neq 0$ is problematic (see Chapter 7 of [39]), and a different procedure has to be applied. This has been developed in [40, 41, 42, 43] for the bosonic string in the flat space and then extended to the supersymmetric case, always in flat space, in [44, 45]. This approach consists in introducing an independent Hilbert space for each external state involved in the process, and to describe the interaction by a state $|H_n\rangle$ in the tensor product of these Hilbert spaces. The state $|H_3\rangle$ acts as a generating functional for the 3-string interactions. In fact, its scalar product with 3 physical states gives the coupling $C_{3s}$ among the strings under consideration

$$C_{3s} = \langle (|1\rangle \otimes |2\rangle \otimes |3\rangle ) |H_3\rangle .$$

(56)

Notice that this approach is not particular of the light-cone gauge. In fact, generating functionals if this type were introduced long time ago [46, 47] to describe the interactions in the dual models (covariant bosonic strings, in the modern language). Moreover this technique is not limited to the computation of the 3-point functions, either. At least in principle (and actually in an explicit way for bosonic strings) it is possible to derive $^+$ However this is certainly not the case for correlators with four or more operators [36].
the form of $|H_n\rangle$. In addition, each one of these objects has an expansion in topologies, since the various string interactions are generically corrected at quantum level. Here we will focus on the simplest case: the 3-string interaction at tree-level depicted in Fig.3 and from now on it is understood that $|H_3\rangle$ indicates only the tree-level contribution.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{three_string_interaction.png}
\caption{The light-cone picture of the three string interaction vertex.}
\end{figure}

In the light-cone gauge, the explicit form of $|H_3\rangle$ is usually determined by requiring that it satisfies all the symmetries of the background. This procedure is divided into two steps, reflecting a natural classification of the symmetries in the light–cone gauge: first one imposes the *kinematical* symmetries, i.e. the symmetries which preserve the light–cone gauge condition, and then the other *dynamical* symmetries. The outcome of the first step is a ket state $|V\rangle$, which for supersymmetric theories has to be completed by a *prefactor* $P$ such that the complete vertex:

$$
|H_3\rangle \equiv P|V\rangle
$$

satisfies all the kinematical and dynamical symmetries (see Chapter 11 of [48]). Notice that the light-cone Hamiltonian itself, being the generator of the $x^+ \rightarrow x^+ + \epsilon$ transformations, is a dynamical symmetry, since the $x^+$ shifts clearly do not preserve the gauge condition $x^+ = e(\alpha)\tau$. From Fig.3 it is easy to see a key property of the dynamical generators: they can not be expressed just in terms of the free dynamics, but get corrections also from the interaction. In fact a shift from $x^+ = -\epsilon$ to $x^+ = \epsilon$ makes us pass from a 1-string state to a 2-string state. Because of this in the light-cone gauge the 3-point interaction can be seen as non-linear correction of the free Hamiltonian and for this reason we have indicated it with $|H_3\rangle$.

In the case of the PP–wave background, the presence of the discrete symmetry $\mathbb{Z}_2$, which exchanges the two $SO(4)$’s acting on the transverse coordinates to the light–cone plane (see Section 2.1), makes the derivation of $|H_3\rangle$ more subtle [49]. In fact, in order to have a $\mathbb{Z}_2$–invariant vertex $|H_3\rangle$ there are two possible choices for the kinematical vertex $|V\rangle$ and the prefactor $P$: they can be both *odd* or both *even* under the $\mathbb{Z}_2$ transformation. Both the possibilities have been considered in the literature. The first has been studied in [50, 51, 52, 53] and then in [30, 54], where an equivalent formulation of this vertex
was given and its transformation properties under the $SO(4) \times SO(4) \times \mathbb{Z}_2$ symmetry group were clarified. The second possibility was analyzed instead in [49, 55, 33].

Here we sketch a derivation of the interaction vertex based on standard path integral techniques*. Since the tree-level string dynamics is captured by a set of harmonic oscillators, we can use the usual transition amplitude [56, 57] in the coherent state phase space with variables $(a, a^*)$:

$$U(a^*, a, t'' - t') = \prod_i \frac{da^*(t_i)da(t_i)}{2\pi i} \exp\left\{\frac{1}{2}(a^*(t'')a(t') + a^*(t')a(t'))\right\}$$

$$\times \exp\left\{i \int_{t', a}^{t'', a^*} \left(\frac{1}{2t}(\dot{a}^*(t)a(t) - a^*(t)a(t) - H(a^*(t), a(t)))\right) dt\right\}, \tag{58}$$

where $l$ enumerates the intermediate times resulting in the path integral when their number grows to infinity. The limits of the path integral indicate that the variable $a(t)$ is fixed at time $t'$ and the variable $a^*(t)$ is fixed at time $t''$. $H(a^*, a)$ is the classical Hamiltonian Eq.(9). The object integrated over $t$ is seen to be the phase space action, which vanishes here due to the equations of motion. The final form of the transition amplitude is then

$$\exp\left\{\frac{1}{2} \left(\sum_{\text{initial oscillators}} (a^*_m a_m) + \sum_{\text{final oscillators}} (a^*_m a_m)\right)\right\}. \tag{59}$$

One easily verifies that the matrix element of the evolution operator takes the same form for a fermionic oscillator with phase space variables $(b, b^*)$.

The dynamics is governed by the classical equations of motion except for the interaction point that we choose to be $x^+ = 0$. The fields on the string world sheet are described by different sets of free oscillators for $x^+ > 0$ and $x^+ < 0$. In the parametrization chosen in Fig.3 the evolution for $x^+ < 0$ is determined only in terms of the bosonic and fermionic modes $a^{(3)}_n, a^{(3)}_m, b^{(3)}_m, b^{(3)}_n$ of the third string, while for $x^+ > 0$ is written in terms of the modes $a^{(i)}_n, a^{(i)}_m, b^{(i)}_m, b^{(i)}_n$ of the other two strings $i = 1, 2$. The interesting part of the problem concerns the transition from one string at $x^+ = -\epsilon$ to two strings at $x^+ = +\epsilon$ with $\epsilon$ a small positive time. During that transition we demand the world sheet to be continuous and smooth, i.e. we demand, as usual, the continuity of the phase space trajectories. This condition yields some relations among the modes in the form of a linear system. Solving this linear system by choosing a set of independent variables and substituting back in Eq.(59) one gets the final form of the vertex. Notice that by applying this procedure one gets only the exponential part $|V|$ of the vertex Eq.(57). In the path integral formalism the prefactor $\mathcal{P}$ should come from extra contributions related to the gauge fixing of the $\kappa$–symmetry at the point on the world sheet where the 3 strings join. By ignoring this complication, we get exactly the

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*a* This approach has been worked out in collaboration with P. Di Vecchia, J.L. Petersen and M. Petrini.

** Notice that $(a, a^*)$ are in this formalism complex numbers corresponding to the eigenvalues of the string oscillators $(a, a^*)$. $a^*$ is an arbitrary complex number and not necessarily the complex conjugate of $a$. The same holds for the fermionic phase space variables $(b, b^*)$ that we will shortly introduce.
kinematical part of the vertex. To see this, let us consider first the bosonic modes: the linear system is obtained by rewriting the continuity conditions:

\[ x_{m(3)} = - \sum_{n=-\infty}^{\infty} \sum_{i=1}^{2} \frac{\alpha_i}{\alpha_3} X_{mn}^{(i)} x_{n(i)}, \quad p_{m(3)} = - \sum_{n=-\infty}^{\infty} \sum_{i=1}^{2} X_{mn}^{(i)} p_{n(i)} \]

by means of Eqs.(13) and (15). Then, by choosing the \( a^* \) modes as independent variables, one can solve for the \( a \) modes getting:

\[ a^{(r)}_n = \sum_{s=1}^{3} \sum_{m=-\infty}^{\infty} N_{nm}^{rs} a^{(s)*}_m. \]

From Eqs.(61) and (59) we then get:

\[ \exp \left[ \frac{1}{2} \sum_{r,s=1}^{3} \sum_{n,m=-\infty}^{\infty} a^{(r)}_n N_{nm}^{rs} a^{(s)*}_m \right] = \frac{1}{123} \langle \psi^{(r)*} a^{(r)}_n \mid H_3 \rangle. \]

For the definition of the \( X \) matrices in Eq.(60) and the explicit form of the Neumann matrices \( N_{nm}^{rs} \) we refer to [52]. The state \( |V \rangle_b \) in the Hilbert space corresponding to the wave–function in Eq.(62) is exactly the bosonic part of the vertex worked out in [50] by imposing the kinematical constraints as Dirac \( \delta \)–functions in the momentum space.

Concerning now the fermions, the ambiguity due to the presence of the \( \mathbb{Z}_2 \) symmetry that we mentioned before shows up in the path integral formalism as the possibility to choose different boundary conditions in the zero mode sector. In [50, 51, 52, 53], closely following the flat space analysis of [45], the fermionic zero modes are treated on different footing with respect to the others modes, as they are represented in terms of the null eigenstate \( |0\rangle \) of the fermionic coordinate zero–mode \( \theta_0 \):

\[ \theta_0 |0\rangle = 0. \]

Thus the boundary conditions in the zero–mode sector are specified in terms of the \( (\theta_0, \lambda_0) \) phase space variables, instead of the coherent state variables \( (b_0, b_0^\dagger) \). By applying the same procedure discussed before for the bosonic case and choosing \( \lambda_0 \) as the independent variables for the zero–mode sector, one ends up with the state:

\[ |V \rangle_f = \exp \left[ \frac{1}{2} \sum_{r,s=1}^{3} \sum_{m,n=1}^{\infty} b^\dagger_{-m(r)} Q_{mn}^{rs} b^\dagger_{n(s)} - \sqrt{2} \Lambda \sum_{r=1}^{3} \sum_{m=1}^{\infty} Q_{m}^r b^\dagger_{-m(r)} \right] \delta_0 |0\rangle_{123}, \]

where \( \delta_0 = \prod_{a=1}^{3} (\sum_{r=1}^{3} \lambda_0^{a(r)}) \) is the delta–function imposing the conservation of the zero–mode fermionic momentum. The \( Q \)–matrices appearing in Eq.(64) are given by:

\[ Q_{mn}^{rs} = e(\alpha_r) \sqrt{\frac{\alpha_s}{\alpha_r}} [P_{(r)} U_{(r)} C^{1/2} N_{rs}^{(r)} C^{-1/2} U_{(s)} P_{(s)}]_{mn}, \]

\[ Q_{n}^r = \frac{e(\alpha_r)}{\sqrt{\alpha_r}} (1 - 4 \mu \alpha K)^{-1} (1 + \Pi K (1 + \Pi)) [P_{(r)} C^{1/2} N_{r} C^{-1/2}]_n, \]

\[ \Lambda = \alpha_1 \lambda_0^{(2)} - \alpha_2 \lambda_0^{(1)} \quad , \quad \alpha \equiv \alpha_1 \alpha_2 \alpha_3. \]

Eq.(64) is exactly the kinematical part of the fermionic vertex written in [52], to which we refer for the definition of the quantities appearing in Eqs.(65)-(66). Notice that the
state $|0\rangle$ in Eq.(63) is not the vacuum state for the string theory on the PP–wave, as $H_{(v)}|0\rangle = 4\mu \alpha_3|0\rangle$. Nonetheless, it reduces to the flat space vacuum in the $\mu \to 0$ limit. Thus this construction ensures a smooth limit of the interaction vertex to the flat space.

An alternative proposal has been made in [49, 55, 58], where the vertex is built on the true vacuum of the theory $|v\rangle$, $b_0|v\rangle = 0$. In this case, the fermionic modes $b_0, b^\dagger_0$ are treated on the same footing as the other modes, since, contrary to the flat space case, they have a non–zero energy and they are not true zero–modes of the Hamiltonian Eq.(27). Thus the coherent state variables $(b_0, b^\dagger_0)$ are used for the zero–modes as well as for all the other modes $m \neq 0$, and the resulting state is [55, 58]:

$$|V\rangle_f^{II} = \exp\left\{ \frac{1 + \Pi}{2} \left[ \sum_{r,s=1}^{3} \sum_{m,n=1}^{\infty} b^\dagger_{-m(r)} Q^r_{mn} b_{n(s)} - \sqrt{\alpha'} \sum_{r=1}^{3} \sum_{m=1}^{\infty} Q^r_m b^\dagger_{-m(r)} \right] \\
+ \frac{1 - \Pi}{2} \left[ \sum_{r,s=1}^{3} \sum_{m,n=1}^{\infty} b^\dagger_{m(r)} Q^r_{mn} b_{-n(s)} + \frac{\alpha}{\sqrt{\alpha'}} \sum_{r=1}^{3} \sum_{m=1}^{\infty} Q^r_m b^\dagger_{m(r)} \right] \right\} \times \exp\left\{ -\sum_{i=1}^{2} \frac{\alpha_i}{|\alpha_3|} b^\dagger_{0(i)} b_{0(3)} \right\} |v\rangle_{123}, \tag{68}$$

where

$$\Theta \equiv \frac{1}{\alpha_3}(\theta_{0(1)} - \theta_{0(2)}) \tag{69}.$$  

The matrices $Q$ are diagonal in the spinor space and are defined as:

$$Q^r_{mn} \equiv e(\alpha_r) \sqrt{\frac{|\alpha_n|}{|\alpha_r|}} [U^{1/2}_{(r)} C^{1/2}_{(r)} N^r_{mn} C^{-1/2}_{(r)} U^{1/2}_{(n)}]_{mn}, \tag{70}$$

$$Q^r_m \equiv \frac{e(\alpha_r)}{\sqrt{|\alpha_r|}} [U^{1/2}_{(r)} C^{1/2}_{(r)} N^r_{m}]_m. \tag{71}$$

Notice that using the relation $D_{n(r)}^{\pm 2} = U_{n(r)} + \frac{\mu \alpha_3}{n}(1 \mp \Pi)$ one can show that the matrices $Q$ and $Q$ appearing respectively in Eq.(64) and Eq.(68) coincide for the non–zero modes with positive chirality $b^\dagger_{n\dot{a}}$, $n \neq 0$, $a = 1, \ldots, 4$.

Let us now come to the supersymmetric completion of the kinematical vertices discussed so far. The dynamical supersymmetry charges of the PP–wave background are given by $Q^-, \bar{Q}^-$ from which we define the combinations

$$Q = \frac{1}{\sqrt{2}} (Q^- + \bar{Q}^-) \quad \bar{Q} = \frac{i}{\sqrt{2}} (Q^- - \bar{Q}^-). \tag{72}$$

In the $\mu \to 0$ limit $Q$ contains just left moving oscillators, while $\bar{Q}$ depends only on the right moving ones so that $Q$ and $\bar{Q}$ are the direct generalization of the supercharges usually considered in flat–space computations. These charges satisfy the algebra

$$\{Q_{\dot{a}}, Q_b\} = 2\delta_{\dot{a}b}(H + T) \quad \{\bar{Q}_{\dot{a}}, \bar{Q}_b\} = 2\delta_{\dot{a}b}(H - T) \tag{73}$$

$$\{Q_{\dot{a}}, \bar{Q}_b\} = \mu \left[ -(\gamma_{ij}\Pi)_{\dot{a}b} J^{ij} + (\gamma_{ij}\Pi)_{\dot{a}b} J^{ij} \right] \tag{74},$$

where $T$ is the operator defined in Eq.(37), which is vanishing on the physical states. As already remarked, in order for the full supersymmetry algebra to be satisfied at
the interacting level, the kinematical vertex has to be completed with a polynomial prefactor. An analogous construction also applies for the dynamical supersymmetry generators $Q$ and $\tilde{Q}$. To our knowledge there is no way to derive the prefactor from first principles, the standard approach being to write a suitable ansatz and then check that it is invariant under all symmetries [44, 45]. To proceed further, some physical inputs are then required.

In [50, 51, 53] the continuity of the interaction vertex in the $\mu \to 0$ limit is required. The vertex is thus built starting from the kinematical part

$$|V^I\rangle = |V\rangle_b \otimes |V\rangle_f \delta \left( \sum_{i=1}^{3} \alpha_i \right)$$

(75)
given by the tensor product of the states in Eq.(62) and Eq.(64). The prefactor turns out in this case to be a slight modification of the flat space prefactor computed in [45], and the full Hamiltonian and the dynamical charges read [53]:

$$|H_3^I\rangle = \left( 1 - 4\mu \alpha K \right) \tilde{K}^I K^J - \mu \alpha \delta^{IJ} v_{IJ}(Y) |V^I\rangle,$$  

(76)

$$|Q_{3a}^I\rangle = (1 - 4\mu \alpha K)^{1/2} \tilde{s}_a^I(Y) |V^I\rangle,$$  

(77)

$$|\tilde{Q}_{3a}^I\rangle = (1 - 4\mu \alpha K)^{1/2} s_{\tilde{a}}^I(Y) |V^I\rangle.$$  

(78)

Here $K$, $\tilde{K}$ and $Y$ are respectively the bosonic and fermionic constituents of the prefactor. They are found by requiring the commutation with the kinematical constraints, such that the states in Eqs.(76),(77),(78) still satisfy them. Again, in the $\mu \to 0$ limit $K$ ($\tilde{K}$) depends only on the left (right) moving oscillators. The explicit expression of these prefactor constituents is reported in [50, 53], where one can find also the explicit form of the functions $v_{IJ}, s_a^I, \tilde{s}_{\tilde{a}}^I$ as an expansion in powers of the Grassmann variables $Y$.

In [30, 54], it was shown that a vertex completely equivalent to that in Eq.(76) can be obtained by starting from the kinematical vertex Eq.(68), and, of course, by insisting on the continuity of $\mu \to 0$ limit. This requirement forces us to assign an even $\mathbb{Z}_2$ parity to the state $|0\rangle$ [63]. In this case the vacuum has to be $\mathbb{Z}_2$-odd because $|v\rangle$ and $|0\rangle$ have opposite parity [49]. The $\mu \to 0$ smoothness is the main argument for this parity assignment since it is also used in the supergravity analysis of [59].

A different approach was suggested in [49, 55, 58]: following the gauge theory intuition, the vacuum state of the string Fock space is defined to be even under the discrete $\mathbb{Z}_2$ symmetry even if this is in contrast to what the continuity of the $\mu \to 0$ limit requires. This assignment is not only natural because of the form of $O_{\Delta}^I$ (34), but it also necessary on the SYM side. In fact the overlap between three vacuum operators is not vanishing (i.e. of the same order as all the others correlators), and this is not consistent with the selection rule implied by the odd-parity previously considered. Thus we are led to give up the continuity of the flat space limit $\mu \to 0$ for the string interaction. Also in this case it is possible to build a string vertex that is invariant under the $\mathbb{Z}_2$ transformation, thus realizing this symmetry explicitly (i.e. both the interaction and the vacuum state are $\mathbb{Z}_2$ invariant at the same time). As we will see in the next section,
this approach is consistent with the proposal of [9, 60] for the comparison with the field theory. In this case, one starts from the kinematical vertex

$$|V\rangle^{II} \equiv |V\rangle_b \otimes |V\rangle_f^I \delta \left( \sum_{r=1}^{3} \alpha_r \right)$$

(79)
given by the tensor product of the bosonic state in Eq.(62) and the fermionic one given in Eq.(68). The supersymmetric completion for this vertex turns out to be very simple [33]:

$$|H_3\rangle^{II} = \sum_{r=1}^{3} H_r |V\rangle^{II} , \quad |Q_{3\dot{a}}\rangle^{II} = \sum_{r=1}^{3} Q_{r\dot{a}} |V\rangle^{II} , \quad |\bar{Q}_{3\dot{a}}\rangle^{II} = \sum_{r=1}^{3} \bar{Q}_{r\dot{a}} |V\rangle^{II} .$$

(80)

With this ansatz the relations Eqs.(73)–(74) hold also at the interacting level as a direct consequence of the free–theory algebra. This remarkably simple form has a very similar structure to what was proposed in the string bit formalism [61]. Moreover the ansatz Eq.(80) shares some striking similarity also with the behaviour of supergravity on $AdS_5 \times S^5$. The vertex derived here is related to the cubic bulk couplings derived from the compactification of IIB theory on $AdS_5 \times S^5$ [62]. Since the PP–wave background is obtained as a Penrose limit from the $AdS_5 \times S^5$ geometry, we should be able to compare the results of $|H_3\rangle$ for supergravity states with the (leading order in $J$) results obtained in $AdS_5 \times S^5$. It is interesting to notice that the bulk vertices obtained in [63] for 3 scalars are indeed proportional to $\Delta_1 + \Delta_2 - \Delta_3$, exactly as the 3–point functions derived from $|H_3\rangle^{II}$.

Let us summarize the results reported in this subsection. We have seen that, due to the lack of a covariant quantization of the string on the PP–wave background Eq.(3), the determination of a supersymmetric 3-string interaction vertex requires some extra physical inputs. This lead to an ambiguity in the construction of the vertex, and two inequivalent solutions have been proposed. The first is obtained by requiring the continuity of the flat space limit $\mu \to 0$ and results in the vertex given by Eq.(76) [50, 51, 52, 53]. The same approach is followed in [30] which discusses the $SO(4) \times SO(4)$ formulation of the vertex. These two formulations of a string vertex smoothly connected to the flat space have been shown to be equivalent in [54]. The second solution more closely follows the behaviour of supergravity in $AdS_5 \times S^5$. The idea is that, since the PP–wave background can be seen as an approximate description of $AdS_5 \times S^5$, even for small curvatures, the 3–state interaction has to be compared with the results in $AdS_5 \times S^5$ rather than with those of flat–space.

### 3.2. The operator mixing

In the previous section we introduced the BMN operators and showed with an explicit 1-loop computation that they have, at the planar level, a well defined conformal dimension. However the conformal dimensions will in general receive contributions also from non-planar diagrams. Again, at 1-loop, one can focus just on the F-term interaction Eq.(48), since the contributions of all other vertices cancel among themselves [28, 9]. The main
difference with the diagrams previously considered is that now we have to take into account interactions between non-contiguous fields. Thus, while the space-time part of the computation is basically the same as before, in this case the contraction between the two non-contiguous fields splits the BMN operator $O^J_m$ in two traces, one of length $l$ and the other of length $J-l$, see Fig.4. From this picture, it is quite natural to expect that non-planar quantum corrections will mix single and double trace operators and that the true states with definite conformal dimension will be some linear combination of the two.

At the conceptual level, the simplest way to check this is to generalize the perturbative computation done in Section 2.3. We need to consider the single trace operators (like Eqs.(34) and (36)) together with the double-trace states that can be built by multiplying the BMN operators among themselves (for instance : $O^J_k \text{Tr}[Z^{J-J'}]$ :). Let $O_\alpha$ indicate a generic operator in this set with two impurities, then the 2-point function takes the form

$$\langle O^{*,12}_\alpha(x) O^{12}_\beta(0) \rangle = \left( \frac{1}{x^2} \right)^{J+2} \left( S_{\alpha\beta} + T_{\alpha\beta} \log |x\Lambda|^{-2} \right) \quad (81)$$

The matrices $S$ and $T$ can be computed perturbatively, as was first done in [28, 9], where the overlaps among the original BMN operators were considered. However in order to derive the eigenstates of the dilatation operator at the non-planar level, we need also the overlaps between single and multi-trace operators [36, 64]. At this point it is sufficient to diagonalize $T$ in the inner product given by $S$ and, in particular, to extract the eigenvalues of $T$. These are the conformal dimensions including the first non-planar correction. Let us briefly describe the ideas of [65] that provide a much simpler and more intuitive way to derive the same results. In the planar case, the quantum correction depicted in Fig.2 was interpreted as a discretized Laplacian, whose action on the BMN operators was basically that of multiplying the original operator by $\lambda(e^{2\pi i J} + e^{-2\pi i J} - 2)$. In a similar manner, we can interpret the non-planar contractions as an operation in the set of operators $O_\alpha$ that acts non-diagonally on the single and double-trace states. In fact, we can quantitatively summarize the process of Fig.4 by means of the “operator” $h_+$ [65]

$$h_+ O^{J,12}_m = g_2 \lambda' \sum_{k=-\infty}^{\infty} \sum_{J'=0}^{J} \frac{k \sqrt{1-r \sin^2(\pi mr)}}{r \sqrt{\pi^2 (k-mr)}} O^{J',12}_{k} O^{J-J'}_{\text{vac}} \quad (82)$$

**Figure 4.** As effect of “non-local” interaction, a single trace operator splits in two.
where $0 < r = J'/J < 1$. This can be derived by rewriting the phase $\exp[2i\pi m(l+1)/(J+2)]$ of the original BMN operator $O_{J'}^{m, 12}$ in terms of the phase factors containing $J'$ which are appropriate for the shorter operator $O_{J}^{m, 12}$. Analogously also the opposite phenomenon is possible and two traces can be glued into a single-trace operator, which is usually indicated with $h_-$ [65]. Now the basic idea is to see the action of the quantum corrections on the generalized BMN operators $O_{J}$ as an Hamiltonian in a quantum mechanical system. In particular we can treat $h_+$ and $h_-$ as perturbations with respect to the planar contribution, since they are subleading in the parameter $g_2$. By applying the usual formulae of non-degenerate perturbation theory, one can easily see that there is no “energy” shift (i.e. no modification of the conformal dimensions) at the first order in $g_2$, since $(h_+ + h_-)_mn = 0$. However, from the usual first order formula $\sum_{n \neq m} (h_+ + h_-)_mn/(E_m - E_n)|n\rangle$, we get a non-trivial redefinition of the eigenstates:

$$O_{J}^{m, 12} := O_{J}^{m, 12} - \sum_{J', k} g_2 r^{3/2} \sqrt{1-r} \sin^2(\pi mr) k \frac{\sqrt{J(2(k + mr))}}{\sqrt{(k + mr)^2}} : O_{J}^{J', 12} O_{J}^{J'-J'} : . \quad (83)$$

Notice that the factor of $\lambda'$ present in Eq.(82) and typical of any 1-loop computation is not present in Eq.(83) because $(E_m - E_n) \sim \lambda'(m^2 - n^2/r^2)$. This approach can be pushed to the next orders and the conformal dimensions at order $g_2^2$ can be derived from the usual formula, $\delta \Delta_n^{(2)} = \sum_{m \neq n} |(n|(h_+ + h_-)|m\rangle|^2/(\Delta_n^{(0)} - \Delta_m^{(0)})$, obtaining

$$\Delta_m = J + 2 + \lambda' \left\{ m^2 + \frac{g_2^2}{4\pi^2} \left( \frac{1}{12} + \frac{35}{32\pi^2 m^2} \right) \right\} . \quad (84)$$

Two remarks are in order here. First, to obtain Eq.(84) we did not sum over all the contribution $m$, but we restricted ourselves to the two-impurity operators, neglecting the impurity non-preserved amplitudes. Then a word of caution is needed on the applicability of the non-degenerate perturbation theory to the second order. Even if no problems seem to be present in the computation that leads to the energy shift Eq.(84), the formula that should give the $g_2^2$-corrected eigenstates breaks down. This is due to the existence at this order of a non-trivial overlap between single and triple-trace operators which can not be disregarded (as it has been done here in the derivation of Eq.(84)). This issue has been studied in [66], where it is also suggested that the breakdown of non-degenerate perturbation theory implies that the string states on the PP-wave are generically unstable.

### 3.3. The string/gauge theory dictionary revisited

The main question now posed by the analysis of the previous section is whether (and how) the operator mixing is relevant for the BMN dictionary between string states and gauge invariant operators. This is particularly interesting because the very same pattern is present also in the full AdS/CFT correspondence, where the role of the parameters $\lambda'$ and $g_2$ is played by $\lambda$ and $1/N$. Conceptually the most simple solution would be to relate the string states and the true eigenstates of the dilatation operator so that the r.h.s. of Eq.(36) is substituted by $O_{J}^{m, 12}$ of Eq.(83). A lot of work has been done in this
direction\cite{67,68,69,32}, but eventually it turned out that neither of the approaches to
the string interaction discussed in Section 3.1 could be easily used together with this
proposed string/gauge theory mapping. Because of this, a less stringent requirement on
the dictionary was first proposed in the context of the string bit model\cite{61} and then
applied in the field theory calculations of\cite{10,70}: in order to find the SYM operators
corresponding to the string states, it is necessary to diagonalize only
\[ S_{\alpha\beta} \]
in Eq.(81),
because this represents, on the gauge theory side, the scalar product among the states
in the string Fock space. Of course, this diagonalization can be done in many different
ways, since two bases related by an orthogonality rotation have the same scalar product;
for example, the basis of the true conformal operators satisfies this requirement, since in
this case both \[ S_{\alpha\beta} \] and \[ T_{\alpha\beta} \] are diagonal. A different choice was proposed in the string
bit model\cite{71,72} and discussed in\cite{10,70} from the gauge theory perspective. It was
proposed that, at the first order in \( g^2 \), the operators dual to the string states are
\[
\Omega_{J,12} = O_{m}^{J,12} - \sum_{J',k} g_2 r^{3/2} \sqrt{1 - \frac{2 \pi^2}{3} \sin^2(\pi mr)} \frac{1}{2 \sqrt{2} \pi^2 (k - mr)^2} O_{J',12}^{J',1} O_{vac}^{J,12} \]
+ \sum_{J'} g_2 \sin^2(\pi mr) O_{J',1}^{J,1} O_{J',2}^{J,1} \quad (85)
\]
\[
\Omega_{k,J,12} O_{vac}^{J,-J'} = O_{k,J,12}^{J,1} O_{vac}^{J,-J'} - \sum_{m} \frac{g_2 r^{3/2} \sqrt{1 - \frac{2 \pi^2}{3} \sin^2(\pi mr)}}{2 \sqrt{2} \pi^2 (k - mr)^2} O_{m}^{J,12} \]
\]
\[
\Omega_{J',1}^{J,1} O_{J,-J'}^{J,2} = O_{J',1}^{J,1} O_{J,-J'}^{J,2} + \sum_{m} \frac{g_2 \sin^2(\pi mr)}{2 \sqrt{2} \pi^2 m^2} O_{m}^{J,12} \]

It can be checked that the new set of operators is orthogonal at order \( O(g_2) \). Actually
the new basis introduced here (globally denoted with \( \hat{\Omega} \)) is related to the one built
from the BMN operators in very simple way through the matrix \( S_{\alpha\beta} \) appearing in (81):
\( \hat{\Omega} = S^{-1/2} \Omega \). Eq.(85) is just the first order expansion of this relation which automatically
ensures the orthogonality at all orders in \( g_2 \) of the \( \hat{\Omega} \)'s.

The main drawback of the choice Eq.(85) is that the \( \Omega \)'s are neither operators with
definite conformal dimension nor single trace operators. Thus the dictionary involving
the \( \Omega \)'s on the one side does not have the mathematical simplicity of the one proposed
in \[ 67,68,69,32 \] and on the other does not follow the physical intuition discussed in
Section 2.3. It is then natural to ask whether the suggestive identification between the
physical IIB strings and SYM operators could be taken to hold also for \( g_2 \neq 0 \), despite
the mixing discussed in the previous Section. This point of view has been advocated
in\cite{33} mainly for the following reason: all computations on the string side performed
so far use the oscillators \[ a_n \] and \[ b_n \] which are derived from the local (on the world-
sheet) actions Eqs.(6)–(7) by solving the resulting equations of motion on the whole
complex plane. Thus the predictions we can draw on the SYM side from these string
computations do not have general validity, but are expected to hold only at the planar
level and, even more restrictive, when only the “local” interactions, like that of Fig.2, are
considered. In fact, even if the states of the string Fock space are orthogonal at the level
of the sphere, there is no compelling reason why the string 2-point functions on the torus
could not develop non-diagonal terms. Thus, according to this point of view, the string 1-loop corrections should reproduce the $g_2^2$ terms of the matrices $S$ and $T$ in Eq.(81), by producing corrections to the kinetic and mass term respectively in the effective action. Unfortunately string amplitudes in the PP-wave background at the torus level are very challenging and no explicit calculation is available in the literature. However there are interesting indications that this point of view is indeed correct, see [73], where an approach inspired by quantum mechanical perturbation theory was used to evaluate string loop amplitudes. Notice that the approach of [73] is similar in spirit to the one used in Sections 3.2 and 4.2. Moreover, following the arguments of [74], the identification between the number of traces and the number of string states should be valid as long as the SYM operators are not too “big”. A simple quantitative characterization of big operators can be derived by realizing that overlap between single and double traces is of order $\sqrt{JJ'(J - J')/N}$. If this is not negligible in the planar limit, then we are dealing with big operators. This shows that, even if the BMN operators are made of an infinite number of fields, they are never big since $\sqrt{JJ'(J - J')/N} \sim g_2/\sqrt{J} \to 0$ (this is also the reason why the corrections in Eqs.(82), (83) and (85) are all suppressed by $1/\sqrt{J}$). Thus the usual rules of AdS/CFT should apply: the single trace operators should correspond to the elementary string states while the multi-trace operators should be bound states and so they are not present in the spectrum of the free string. Moreover it was conjectured that the interaction with the multi-trace operators is not captured by the usual string interaction, but that, for this case, non-local interactions on the string world-sheet are relevant [75]. Thus also the terms of order $g_2$ in Eq.(81) can be interpreted in a more conservative way without the need of a redefinition of the BMN dictionary presented in Section 2.3. It is in fact suggestive that also on the field theory side the mixing is triggered by “non-local” contractions (see Fig.4). So it is more natural to relate these contributions to the novel interaction of [75] than to the usual string couplings of Section 3.1.

4. Comparing string and field theory results

4.1. A proposal inspired by the AdS/CFT duality

The first proposal for the comparison of the string interaction with the field theory was put forward in [9]. This proposal is motivated by the standard AdS/CFT dictionary between bulk and boundary correlation functions [17]: since the light-cone interaction vertex on the PP-wave in Eq.(56) can be understood as the generating functional of the correlation functions among string states, it is natural to put it in correspondence with the correlation functions of the dual field theory operators. A specific prescription was proposed in [9] for the leading order in $1/\mu$:

$$\frac{\langle \{1 \otimes \{2 \otimes \{3\}\} \mid H_3\rangle}{\sum \langle H_{12}^j \rangle} = C_{ijk}.$$  (86)
where $C_{ijk}$ is the coefficient appearing in the tree-level correlator among three BMN operators of R-charge $J_i$. Again this formula is very reminiscent of quantum mechanical perturbation theory and thus it was conjectured to be valid when the energy difference in the denominator is small in the $\mu \to \infty$ expansion (i.e. for impurity preserving transitions). In this case, it is easy to extract this SYM coupling by using the definition of the barred operators introduced in (40)

$$\lim_{y \to \infty} \langle \bar{O}_i(y) O_j(x) O_k(0) \rangle = C_{ijk},$$

(87)

where $C_{ijk}$ depends on the R-charges $J_i$ and on the level $m$ of the BMN operators.

From Eq.(86) it follows that the prefactor of the string interaction vertex should reduce, or at least be proportional, to the difference of the energies of the ingoing and outgoing states. However, a careful analysis [52] of the vertex proposed in [50] finally showed that this is not the case (see [51] v3). The physical reason for this failure is quite simple: if one insists in keeping a smooth flat space limit $\mu \to 0$, then the vertex should obey for $\mu = 0$ the holomorphic factorization property and should appear as a product of the left and right moving oscillators $\hat{a}^\dagger_1$ and $\hat{a}^\dagger_2$, rather than the sum which appears in the free Hamiltonian. The alternative proposal for the supersymmetric completion of the kinematical vertex Eq.(80), which has not a smooth flat space limit, fits instead very well with the proposal Eq.(86). Indeed, it is possible to show that the 3-string couplings derived from the vertex Eq.(80) are in agreement with the free field theory evaluation of the coefficients $C_{ijk}$ from the 3–point correlation functions of single trace operators [33]. The agreement holds, at first order in $1/\mu$, for arbitrary 3-point functions, regardless of the particular type of impurities, which can be scalar, vector or fermionic fields, and of the type of correlators, which can be impurity preserving or not. We focus here only on BMN operators with fermionic impurities, referring to [62, 76, 77] for the scalar and to [33] for the vector impurities. The approach here presented has been studied also from the string bit point of view in [78]. Due to the form of the prefactor in Eq.(80), we can focus only on the kinematical part of the vertex Eq.(79) and rewrite Eq.(86) as:

$$\langle i |j| k |V \rangle = C_{ijk}/C^{(0)}_{ijk},$$

(88)

where $C^{(0)}_{ijk} = \sqrt{J_1 J_2 J_3}/N$ is the combinatorial factor of the Green function among three vacuum operators Eq.(34). This factor ensures the same normalization for the two sides of Eq.(88), since the string overlap $\langle v |V \rangle$ is set equal to one.

On the field theory side, the evaluation of the 3-point function of BMN operators with fermionic impurities can be reduced, by means of the identities Eqs.(46), to the same combinatorics found in the case of scalar impurities. One has therefore, for double–impurity operators:

$$\lim_{x_3 \to \infty} \langle \bar{O}_3^{J_3}(x_3) O_2^{J_2}(x_2) O_1^{J_1}(x_1) \rangle = -\frac{J_2}{N} \sqrt{\frac{J_1 J_2}{J_3}} \frac{\sin^2(\pi y)}{\pi^2 (m - ny)^2},$$

(89)

with $J_3 = J_1 + J_2$, $y = J_2/J_3$ and $\alpha \neq \beta$. The only difference with the correlators containing scalar impurities is that one can get some extra minus signs, due to the
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The anticommuting nature of the fermions. This difference becomes important for correlators among operators containing spinors of the same flavour

$$\langle \overline{O}^{J_3}_{\alpha\alpha,n} O^{J_2}_{\alpha\alpha,m} O^{J_1}_{\alpha\alpha,\nu} \rangle = \langle \overline{O}^{J_3}_{\alpha\beta,n} O^{J_2}_{\alpha\beta,m} O^{J_1}_{\alpha\beta,\nu} \rangle - \langle \overline{O}^{J_3}_{\alpha\beta,-n} O^{J_2}_{\alpha\beta,m} O^{J_1}_{\alpha\beta,\nu} \rangle ,$$  \hspace{1cm} (90)

On the string theory side, according to the dictionary Eq.(39), one has to compute the amplitude (88) with the external states

$$A_{ab}^{\alpha} = \frac{1}{4} \left[ (Q_{nm}^{32})^2 - 2Q_{mn}^{23}Q_{nm}^{32} \right] .$$  \hspace{1cm} (91)

and to extract the first term in the $\mu \to \infty$ limit. This can be done by using Eq.(70) and the leading term [76] in the expansion of the Neumann matrices††

$$N_{nm}^{32} \sim \frac{2ny^{3/2}\sin(\pi ny)}{\pi m^2 - n^2y^2} , \hspace{1cm} \text{with } n, m > 0 \hspace{1cm} (92)$$

$$U_{(i)} \sim \frac{n}{2\mu\alpha_i} , \hspace{1cm} U_{(3)} \sim -\frac{2\mu\alpha_3}{n} . \hspace{1cm} (93)$$

By using these formulae, one can check that the string Eq.(91) and the gauge theory Eq.(89) results agree. The string amplitude related to Eq.(90) is $A_{ab}^{\alpha} = Q_{mn}^{32}Q_{nm}^{32}$. Notice that the quadratic terms have disappeared, because the fermionic nature of the oscillators implies $(\theta_n^a)^2 = 0$. Anyway also in this case, by using Eq.(92) and Eq.(93) one recovers exactly the gauge theory results Eq.(90).

So far we considered only extremal correlators, i.e. correlators where in the Born approximation there are no propagators connecting the operators $O^{J_2}$ and $O^{J_1}$. In this case the field theory computation inherits some of the properties of the 2–point functions. One may wonder whether Eq.(88) is of more general validity as it is suggested by the analysis of [60, 81]. This possibility is motivated by considering the Penrose limit of the AdS/CFT bulk–to–boundary formula [17]. Actually the simplest possible check supports this possibility. In fact the classical contribution to the non–extremal correlators vanishes in the BMN limit faster than $C_{ijk}^{(0)}$, so that the ratio Eq.(88) is zero. Correspondingly $N_{nm}^{ij}$ with $i,j = 1, 2$ tend to zero when $\mu \to \infty$ and thus all non–extremal amplitudes are vanishing at leading order in $\lambda'$ both on the string and on the field theory side.

Notice that the subleading contributions to the Neumann matrices corresponding to these correlators start with a term of the order of $\sqrt{N} = \sqrt{\lambda}/J$ [82]. It is interesting to observe that this particular $J$ dependence is already captured by the free field theory computation of the non–extremal correlators. This can be seen by considering the scaling with $N$ and $J$ of the simplest non–extremal correlator between supergravity states

$$\frac{1}{N_{o}^{J+1}} \langle \text{Tr}[\hat{Z}^J](0)\text{Tr}[\phi Z^{J_1}](x_1)\text{Tr}[\hat{\phi} Z^{J_2}](x_2) \rangle \sim \frac{\sqrt{J}}{N} .$$  \hspace{1cm} (94)

From Eq.(94) it follows that the $J$ dependence in the r.h.s. of Eq.(88) for these correlators is $1/\sqrt{J_1J_2}$, which matches with the $\alpha_i$’s dependence of the corresponding Neumann

††A detailed study of the subleading terms in the $1/\mu$ expansion can be found in [79]. See also [80], where methods of complex analysis are used to tackle a closely related problem.
matrix element in the $\mu \to \infty$ limit $N_{00}^{12} \sim 1/\mu \sqrt{\alpha_1 \alpha_2} = \sqrt{\lambda}/\sqrt{J_1 J_2}$ [79], upon using the relation Eq.(32) between $J$ and $\alpha = \alpha_1 + \alpha_2$. Moreover, the peculiar dependence on the square root of the coupling $\sqrt{\lambda}$ is reminiscent of the behaviour of other dynamical observables evaluated in the strong coupling regime of field theory by means of AdS/CFT duality, as the vacuum expectation value of the Wilson loop [16]. Notice that the string theory prediction implies a non-trivial dependence on the coupling also for the 3-point non-extremal correlators between supergravity states. This is contrary to the conjecture based on the supergravity analysis of [63] and to the field theory arguments that support it, see References in [15]. From this point of view, the most natural expectation is that the string result is recovered after a resummation [82] of the planar SYM diagrams.

One positive feature of the approach in Eq.(86) for the comparison with the field theory is that there is a clear pictorial understanding of the duality between string and gauge theory [9]. In fact in the $\mu \to \infty$ limit the kinetic term of the free string world-surface Lagrangian Eqs.(6) and (7) can be neglected with respect to the mass term. If one discretises the string in $J$ bits, these will be then represented by a bunch of independent harmonic oscillators. The combinatorics obtained by imposing the smoothness of the world-surface during the interaction is exactly reproduced by the free correlators of the gauge theory BMN operators, in which each elementary field represents a bit of the string, see Fig.5.

![String interaction in the $\mu \to \infty$ limit](image)

**Figure 5.** In the $\mu \to \infty$ limit the various “bits” of the string can not interact. Pictorially this exactly matches the field theory behaviour in the $\lambda' \to 0$ limit.

One of the open questions in this approach is to understand how to handle the space-time dependence of the field theory correlators. This problem becomes particularly important when considering the $1/\mu$ corrections. In fact, at subleading orders in $\lambda'$, Eq. (87) depends non-trivially on $x$. In order to have the well defined scaling behaviour of Eq.(87) at one loop level, it is necessary to take into account the mixing between single and double trace operators [83, 36, 64]. The puzzling point is that this mixing can modify the tree-level value of $C_{ijk}$ and spoil the agreement with the string computations described here. A closer inspection reveals that this phenomenon is just the PP-wave version of the usual peculiarity of extremal correlators, where the contribution of the
multi-trace operator is enhanced and becomes of the same order as the leading terms. In the standard AdS/CFT duality, the simplest comparison between the supergravity and field theory results is done without advocating any mixing \[84, 85\] and agreement is found. In the PP-wave case, the problem is more severe just because we deal with non-protected operators. It is interesting to notice that exactly for these correlators the requirement of planarity does not ensure the “locality” of the interaction, and diagrams connecting non contiguous impurities are as important as the others. This may be the reason for the problems in comparing the extremal correlators with the string results.

4.2. A proposal inspired by Eq.(29)

A different approach to the comparison between string interactions and gauge theory data was proposed in \[10\]. This approach was inspired by the string bit proposal of \[61\], in which it was suggested to interpret the relation between the dilatation operator and the light-cone Hamiltonian (29) \((\Delta - J = H/\mu)\) as an exact operatorial equation, including the non-linear corrections on the string theory side. This means that in the r.h.s. of Eq.(33) we should include, at first order in \(g_2\), also \(|H_3\rangle\), which is the first correction to \(H\) (see the discussion after Eq.(57)). Since these corrections are non-linear, the full Hamiltonian acts also on states that live in the \(n^{th}\) tensor product of the free string Hilbert space, see Eq.(56). Thus, also the SYM operator \(\Delta - J\) should act in a bigger space than the one considered in Section 2.3. At the light of the discussion in Section 3.2, the most natural proposal is to identify the states living in the tensor product of two or more string Hilbert spaces with multi-trace operators. This means that, at first order in \(g_2\), we should work with the sets of SYM operators \(O_\alpha\) introduced just before Eq.(81). On the one side this mapping may appear “obvious”, but on the other side we should notice that it represents a departure from the common interpretation of the AdS/CFT duality. In fact, usually multi-trace operators are mapped to bound states on the string side (see point 3 in Section 2) and not to multiparticle states. The PP-wave limit might be particular because the harmonic potential confines the excitations around \(X^I = 0\), but notice that there is still the possibility to separate the constituent particles along the \(x^+\) direction.

There are basically two different ways to check this proposal. A first possibility is to diagonalize the operators both on the string and on the gauge theory side and then compare their eigenvalues. Conceptually this is the most straightforward approach, but it is clearly quite challenging from the computational point of view. We already discussed the diagonalization procedure on the SYM side in Section 3.2, where the primary operators (83) were introduced. In fact, from Eq.(84) we can directly read the eigenvalues of \(\Delta - J\). Thus we now need to extract the \(g_2^2\) corrections to the energy of the string states. In principle this requires the computation of the string 2-point function on the torus, since we know that different powers of \(g_2\) are related to diagrams of different topology. As we already said, fullfledged string computations of this type are not available for the PP-wave background. Thus we will follow the same approach used on
the gauge theory side to derive Eq. (84): we will apply quantum mechanical perturbation theory, by interpreting the corrections to the free Hamiltonian as perturbation terms and \( g_2 \) as the small parameter governing the perturbative expansion. Since we know that the first non trivial correction to the energy is of order \( g_2^2 \), we need to include also the correction to the classical string Hamiltonian up to the same order. Thus in principle we should extend the analysis of the string interaction done in Section 3.1. However, by following [86], we can extract the relevant part of \( g_2^2 \) Hamiltonian by using the supersymmetry algebra (73) which implies that \( Q^2 \sim H \). In particular the \( g_2^2 \) correction to \( H \) relevant for the matrix element of single string state can be derived by using the \( g_2 \) correction to the supercharges presented in Section 3.1. Thus the energy shift for a string state like the one of Eq. (36) is

\[
\delta E_n^{(2)} = g_2^2 \sum_s \frac{1}{2} \frac{|\langle n | H_3 | s \rangle|^2}{E_n^{(0)} - E_s^{(0)}} + \frac{1}{8} \sum_{\dot{a},s} |\langle n | Q_{3\dot{a}} | s \rangle|^2 ,
\]

(95)

where we should sum over all states \( |s\rangle \) satisfying the level-matching condition. The first term comes from the second order perturbation theory and because of this has an overall factor of 1/2, while the second term is the first order contribution coming from the \( g_2^2 \) correction of the Hamiltonian. The factor of 1/8 is due to the trace over the spinor indices \( \text{i.e. the sum over } \dot{a} \). There are various explicit computations in the literature [86, 87, 30] that use (95) with bosonic external states in different representations of \( SO(4) \times SO(4) \). The result of these computations is that Eq. (84) is always reproduced, when the sum over \( m \) is restricted to the 2 impurity states, so that all the amplitudes entering in (95) are impurity preserving. This approximation is similar to the one discussed on the gauge theory side (see the discussion after (84)), however a physical justification of this truncation is not available.

A second possible test of this approach is to find a dictionary between string and gauge theory Hilbert spaces so that all elements of the “matricial” relation \( \Delta - J = H/\mu \) can be compared. Unfortunately the simple BMN dictionary discussed in Section 2.3 is not the right basis for the proposal here discussed and we need to generalize the BMN dictionary along the lines explained in Section 3.3. Even if it is difficult to find a clear physical principle that can substitute the idea resumed in Figure 1, it is interesting to see that the redefinition (85) yields results that are consistent with those extracted from the string vertex (76). This approach uses the matrix \( S \) appearing in (81) to define the basis of SYM operators dual to string states. This is sometimes called string basis because the quantum corrections to the dilatation operators match the results derived from the string vertex (76)

\[
S^{-1/2}T_{\alpha\beta}S^{-1/2} = \langle \alpha, \beta | H_3 \rangle^I .
\]

(96)

This relation was checked in a number of cases [70, 87, 88, 89] at the leading order in \( \lambda' \) and for various bosonic impurities. The first checks at subleading orders in the 1/\( \mu \) expansion show however a disagreement [90].
4.3. Open problems

Let us conclude by recalling the main open problems in the comparison between interacting string theory on the PP-wave (3) and the BMN subsector of $\mathcal{N} = 4$ SYM.

- We saw that the light-cone construction of the 3-string vertex is not completely unambiguous and that the $\mathbb{Z}_2$ parity can be implemented in different ways. It would be interesting to have a more thorough understanding of this freedom. This point is strictly related to the $\mu \rightarrow 0$ limit. From the point of view of the duality the smoothness of this limit appears very puzzling. In fact this would imply that the (interacting) IIB string theory in 10 dimensional flat space can be completely captured by studying the anomalous dimensions of BMN operators.

- An important point that deserves further study is the operator mixing. It is sometimes said that this is a peculiar feature of the PP-wave, where it would be impossible to separate single and multi-trace operators. However, the discussion in Section 3.2 shows that this is a general phenomenon. As is shown in [83, 91], for short operators the mixing is suppressed by a factor of $1/N$ exactly as in the PP-wave case it is suppressed by a factor of $g_2 [36, 64]$. In the light of this observation it would important to clarify the role of the mixing in the string/gauge theory dictionary, since the same problem will also appear in the full AdS/CFT correspondence.

- Let us now focus on the approach outlined in Section 4.1. Here the main open problem is to see how to extend the validity of Eq.(88) also at the subleading orders in the $1/\mu$ expansion. It is possible to study this issue in two different situations. In the case of impurity preserving amplitudes, the expansion on the string side is simple, as it involves integer powers of $\lambda'$. However, on the SYM side, the 3-point functions have a non–canonical space-time dependence [64]. Moreover, in this case a better understanding is necessary of the role played in the duality by the SYM diagrams connecting non-contiguous impurities. Another possibility is to study non-extremal correlators, where the gauge theory side of the computation is simpler, at least conceptually. In this case the string predictions are surprising. First the large $\mu$ expansion of the Neumann matrices yields non-integer powers of $\lambda'$, which probably means that the limit Eq.(31) can not be taken order by order in perturbation theory any more. Then even the simple computation in Eq.(94) presents an unexpected behaviour. It is obviously very important to analyze this amplitude more carefully.

- As for the approach outlined in Section 4.2, it is first necessary to obtain a better understanding of the impurity non-preserving amplitudes and the truncation used to derive Eq.(95). In the impurity preserving sector the proposal is in principle self-consistent for all values of $\lambda'$. However, as was already said, the first next-to-leading computation [90] shows a disagreement. It would be interesting to see whether this problem persists also at the level of eigenvalues, or it is due to the proposed dictionary (85). Finally, all computations done so far within this approach deal with bosonic impurities. As a result, only a small part of the prefactor (76) has been checked. To overcome this limitation, it is clearly important to take fermionic impurities into
account. It would be particularly interesting to see whether the non-diagonal parts of $v_{IJ}$ can be derived from field theory computations.

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