The Randomized Communication Complexity of Randomized Auctions

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Abstract

We study the communication complexity of incentive compatible auction-protocols between a monopolist seller and a single buyer with a combinatorial valuation function over $n$ items. Motivated by the fact that revenue-optimal auctions are randomized [Tha04, MV10, BCKW10, Pav11, HR15] (as well as by an open problem of Babaioff, Gonczarowski, and Nisan [BGN17]), we focus on the randomized communication complexity of this problem (in contrast to most prior work on deterministic communication).

We design simple, incentive compatible, and revenue-optimal auction-protocols whose expected communication complexity is much (in fact infinitely) more efficient than their deterministic counterparts.

We also give nearly matching lower bounds on the expected communication complexity of approximately-revenue-optimal auctions. These results follow from a simple characterization of incentive compatible auction-protocols that allows us to prove lower bounds against randomized auction-protocols. In particular, our lower bounds give the first approximation-resistant, exponential separation between communication complexity of incentivizing vs implementing a Bayesian incentive compatible social choice rule, settling an open question of Fadel and Segal [FS09].

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1 Introduction

The central goal of Algorithmic Mechanism Design is to design mechanisms that guarantee good outcomes while taking into account both (i) the selfish agents’ incentives and (ii) the ever-increasing complexity of modern applications. A fundamental question to this field is whether simultaneously satisfying both the incentive and simplicity constraints is harder than satisfying each of them separately.

In this paper we focus on one of the simplest and most studied settings in the field: a monopolist, Bayesian, revenue-maximizing seller auctioning $n$ items to a single risk-neutral buyer. An active line of work over the past two decades argues that even in this strategically-simple setting, and even for buyers with additive or unit-demand valuations, optimal mechanisms are inherently complex, e.g. they involve randomized lotteries [Tha04 MV10 BCKW10 Pav11 HR15] and are often computationally intractable [CDO+15 DDT14 CDP+14].

One particularly influential measure of complexity of mechanisms is the menu-size complexity of [HN19]: by the taxation principle, a general incentive compatible mechanism can be canonically represented as a menu, where each line or option in the menu corresponds to a (possibly randomized) allocation and a payment. The menu-size complexity of a mechanism is then the number of lines in the corresponding menu. Perhaps the single most convincing evidence for the complexity of optimal mechanisms is an example due to [DDT17], where the optimal mechanism for an additive buyer with two i.i.d. item valuations from a seemingly benign distribution (Beta(1,2)) requires an infinite and even uncountable menu-size complexity. We henceforth refer to this powerful example as the DDT example.

[DDT17] and related complexity results for optimal revenue-maximizing auctions have inspired fruitful lines of work that circumvent these barriers, e.g. by designing sub-optimal but simple mechanisms that approximate the optimal revenue (see discussion in Related work).

It is not a-priori clear, however, that the menu-size complexity by itself is an obstacle to using optimal mechanisms. For instance, the seller in the DDT example could in principle succinctly describe her mechanism as “the-optimal-auction-for-Beta(1,2) × Beta(1,2)” and even point the buyer to a explicit description in [DDT17]. However, [BGN17] recently observed that, once the mechanism is announced, the deterministic communication complexity to implement it is equal (up to rounding) to the logarithm of the menu-size complexity. In the DDT example, for the buyer to deterministically specify his favorite line in the uncountable menu, he would need to send an infinite stream of bits. [BGN17] left open the question of randomized communication complexity of optimal mechanisms. Indeed randomized communication is a natural complexity measure in this case since we already consider randomized allocations.

In this paper, inspired by [BGN17]’s open question, we formulate a notion of an incentive

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1To circumvent some worst-case pathological examples, it is common in Algorithmic Mechanism Design to restrict the buyer’s value distribution to independent (vs correlated) items, and/or restrict the combinatorial nature of buyer’s value for bundles to one of the following classes:

- additive, unit-demand $\subset$ gross-substitutes $\subset$ submodular $\subset$ XOS $\subset$ subadditive.

(See Section 2.1 for definitions.)

2Throughout the paper, we use feminine pronouns for the seller and masculine for the buyer.

3Different applications have different simplicity desiderata. (E.g. highly regulated FCC auctions vs very fast ad auctions with automated bidders vs smart contracts that require costly documentation of transaction details on a blockchain.) Ultimately, there is no universal “right” measures of complexity, and studying a variety gives us a more complete understanding.

4Technically, our definition of IC auction protocol is a special case of Bayesian incentive compatibility (BIC)-incentivizable binary dynamic mechanism (BDM) [FS09]. We discuss this connection further in Related work.
compatible (IC) auction protocol, which is a two-party (possibly randomized) interactive communication protocol between a seller and a buyer with an allocation and payment associated with every transcript of the protocol. Before presenting our results in this model, below we briefly discuss our modeling assumptions; a full definition appears in Section 2.

**Brief discussion of modeling assumptions**

Per the discussion above, we assume that the protocol and auction format are public information. The buyer privately knows his true type (or valuations of items/bundles).

We mostly focus on the total expected communication complexity of the protocol. For our protocols, we bound the interim expectation, i.e. for every buyer’s type, the communication complexity of the protocol is bounded, in expectation\(^5\) over the protocol’s randomness\(^6\). Our lower bounds hold even for ex ante expectation, i.e. even if we allowed that some buyers may know in advance that they are expected to participate in a prohibitively long protocol.

The seller in our model has no private information and is not strategic. At the end of the communication protocol she must know the allocation and payment.

We model the buyer’s strategic aspect as a complete information single-player extensive-form game with buyer’s nodes and nodes of Chance; each leaf is associated with an allocation and a payment. In practice, nodes of Chance could be implemented by a trusted seller (e.g. when the seller is an auditable firm), a trusted intermediary, a cryptographic protocol for coin tossing\(^7\) or a publicly observable, renewable\(^8\) external source of randomness.

As is common in the aforementioned literature on randomized mechanisms, we assume that the buyer is risk-neutral. In particular, we require that the protocol is interim individually rational. In direct revelation mechanisms, it is possible to transform interim to ex-post individual rationality by correlating the payment with the randomized allocation. Similarly, at the cost of a bounded increase in the communication complexity, it is possible to transform our protocols to become ex-post approximately individually rational (see Appendix C for details).

While we make little restrictions on buyer valuations, we do generally assume that the buyer’s valuation is capped at some arbitrarily large value \(U\). The complexity of our protocols\(^9\) does not depend on \(U\), e.g. \(U\) can be all the money in the universe (typically much smaller).

**Our IC auction protocols**

We design IC auction protocols that are simple, surprisingly efficient, and are exactly revenue-optimal. For instance, in Theorem 3.2 we give a revenue-optimal IC auction protocol for the DDT example.

\(^5\) In expectation vs high probability: We remark that by Markov’s inequality in expectation upper bounds on the communication complexity imply similar upper bounds w.h.p.; e.g. if the expected complexity is at most \(C\), then it is at most \(C/\alpha\) w.p. \(\geq 1 - \alpha\).

\(^6\) In fact, all our protocols happen to satisfy a slightly stronger desideratum where all the communication complexity bounds that we prove also (approximately) hold for the communication complexity of the future of any prefix of the protocol. I.e. for any setting where we bound the expected communication complexity by \(C\), it is also true that, conditioning on any history of the protocol (possibly much longer than \(C\)), the remaining expected communication complexity is \(O(C)\). This means that the buyer and seller always expect -for every run of the protocol, and at any point during the execution- that the protocol will end soon.

\(^7\) We’re mostly interested in mechanisms that are exactly revenue-optimal, while the security of cryptographic protocols always has a negligible but non-zero chance of being broken even by naive brute-force algorithms. In theory, this small chance of cheating on the coin tosses would violate the buyer’s exact incentive constraints.

\(^8\) By “renewable” we mean that at each step of the protocol the parties have access to fresh random bits not predictable in previous iterations; for example, they could look at the weather each day.

\(^9\) Except the ex-post approximately individually rational protocols in Appendix C.
where the *buyer sends less than two bits in expectation*. (In contrast, for a deterministic auction selling two items separately, merely specifying the allocation requires the buyer to send two bits!)

**Main positive result** Our main positive result is a generic transformation of an arbitrary (revenue-optimal or otherwise) IC and IR mechanism for additive, unit-demand, or general combinatorial valuations to an IC auction protocol that uses \(O(n \log(n))\), \(O(n \log(n))\), \(O(2^n)\) bits in expectation respectively. We note that our protocols work for correlated prior distributions, and even for non-monotone and negative valuations.\footnote{We assume for simplicity that all payments are non-negative.}

**Theorem** (See Theorems 3.1 and 4.1).

For any prior \(\mathcal{D}\) of Buyer’s (additive/unit-demand/combinatorial) valuations over \(n\) items bounded by maximum valuation \(U\), and any IC mechanism \(\mathcal{M}\), there is an IC auction protocol with the same expected payment and allocation, using \((O(n \log n) / O(n \log n) / O(2^n))\) bits of communication in expectation.

**Trading off revenue for even better communication efficiency** We obtain an exponentially more efficient protocol for the special case of unit-demand with *independent items*. Specifically, at the cost of an \(\varepsilon\)-fraction loss in revenue, we obtain an IC auction protocol that uses only polylog\((n)\) communication.

**Theorem** (See Theorem 5.1). Let \(\mathcal{D}\) be a distribution of independent unit-demand valuations over \(n\) items bounded by maximum valuation \(U\). Then, for any constant \(\varepsilon > 0\), there is a \((1 - \varepsilon)\)-approximately revenue-optimal IC auction protocol using polylog\((n)\) bits of communication in expectation.

Exhibiting the richness of our IC auction protocol model, this protocol is substantially different from the generic transformation in our main result, and builds on the recent *symmetric menu-size complexity* of \([\text{KMS}+19]\).

**Remark 1.1.** For simplicity of presentation we focus on the expected communication complexity. Here we briefly remark that our protocols also have desirable properties in terms of round- and random-coin-complexities. For round complexity, our protocols use \(O(\log(n))\) rounds in expectation \((O(n)\) for general combinatorial valuations). For the protocols in Theorems 3.1 and 4.1 it will be easy to see how (using trivial batching) one can further compress the number of rounds: at the cost of a constant factor increase in the communication complexity, these protocols can be compressed to \(1 + \varepsilon\) rounds in expectation. In terms of random coins, our protocols can be implemented with \(O(\log(n))\) coins in expectation \((O(n)\) for general combinatorial valuations).

**Communication complexity lower bounds**

We show that beyond the (important) special case covered by Theorem 5.1 the communication complexity of our protocols is almost the best possible, in the following strong sense:

**Theorem** (See Theorems 6.3, 7.2, and 8.1). For revenue maximization with \(n\) items, any incentive compatible auction protocol that achieves any constant factor approximation of the optimal revenue must use at least:

- \(\Omega(n)\) communication for unit-demand valuations;
• $2^{\Omega(n^{1/3})}$ communication for gross substitutes valuations;

• $2^{\Omega(n)}$ for XOS valuations.

Furthermore, any incentive compatible auction protocol obtaining more than 80% of the optimal revenue must use at least:

• $2^{\Omega(n)}$ communication for XOS valuations over independent items.

To place the result for independent items in the greater context of Algorithmic Mechanism Design, contrast it with simple-but-approximately-optimal mechanism independent subadditive valuations: [RW18] showed that a constant fraction of revenue can be guaranteed by simple mechanisms; this constant has been improved in followup works [CDW16, CM16, CZ17], but no non-trivial upper bound on the best approximation factor were known\(^\text{11}\). Assuming that efficient randomized communication is a necessary desideratum for “simple mechanism”, our result for independent items implies that the optimal approximation factor is bounded away from 1 – even for the special case of XOS valuations.

Note also that our upper and lower bounds for correlated valuations are nearly tight in the following ways:

• For unit-demand and combinatorial valuations, our upper and lower bounds nearly match (up to logarithmic factors), even though the lower bounds hold for arbitrary (constant) approximation factor vs exactly revenue-optimal in upper bounds. Furthermore the combinatorial upper bound holds for arbitrary combinatorial valuations, which are much more general than XOS valuations used in the lower bound.

• The correlation in our unit-demand lower bound is necessary by Theorem 5.1

We remark that for one interesting case an exponential gap remains:

**Open Question 1.2.** What is the randomized communication complexity of exactly revenue optimal IC auction protocols for unit demand valuations over independent items?

Our lower bound for unit-demand requires correlated items (and this is an inherent limitation of our technique). On the other hand, our protocol for unit-demand with independent items (Theorem 5.1) does not guarantee exact revenue optimality.

**Separating the complexity of implementing and incentivizing**

Our results also have implications for a question of Fadel and Segal [FS09]. They study, for any fixed social choice rule, the communication cost of selfishness, i.e. the difference in communication complexity between (i) implementing it, and (ii) implementing it in a Bayesian incentive compatible protocol. They give examples where the communication cost of selfishness is exponential, but those examples are very brittle in the sense that they rely on agents’ utilities to have unbounded (or at least exponential) precision. They ask whether the communication cost of selfishness on any (possibly contrived) social choice rule can be reduced substantially if agents’ utilities have a bounded precision [FS09 Open Question 3]. Our source of hardness is inherently different from the instances in [FS09]: we harness the combinatorial structure of the valuations rather than exploiting the long representation of high-precision numbers.

\(^{11}\)Note that this is a maximization problem, so upper bound on the approximation factor refers to an impossibility result.
In more detail, in our constructions the buyer’s utility only requires constant precision\(^{12}\) for any outcome (and the seller is not strategic, i.e. she has constant utility zero). Furthermore, for our hard instances of unit-demand valuations, we show (Remark 6.6) that the exactly revenue-optimal IC mechanism can be implemented by a randomized (non-IC) protocol using \(O(\log(n))\) communication even in the worst case, hence resolving [FS09]’s open question on the negative\(^{13}\) We remark that by [FS09, Corollary 3], this exponential separation is tight.

**Corollary 1.3** (See Remark 6.6). There exists a randomized protocol for a revenue maximization instance, in which the buyer’s valuation has constant precision, such that there is an exponential separation between the communication complexity of its approximately Bayesian IC implementation and that of its non-IC implementation.

**Remark 1.4** (Separations for deterministic vs randomized protocols).

Formally, [FS09] phrase their open question for deterministic protocols. To view Corollary 1.3 in this context, note that in our model the seller is not strategic; hence one can consider an equivalent deterministic social choice rule in a slightly different setting where the random seed (only \(O(\log(n))\) bits are necessary) to the revenue-optimal auction is replaced by a seller’s type. The requirements from the protocol in this setting is only stricter, so the communication lower bound on IC auction protocols trivially extends. On the other hand, for the non-IC auction protocol the seller can just send the buyer her type (aka the random seed).

Interestingly, this separation between the communication complexity of implementing and incentivizing optimal auctions holds in a more general sense (albeit for expected communication in randomized protocols): In Appendix A we show a non-IC auction protocol\(^{14}\) that for any buyer with unit-demand (resp. combinatorial) valuations, the exactly optimal IC mechanism can be implemented by a randomized (non-IC) protocol using \(O(\log(n))\) (resp. \(O(n)\)) communication.

**Technical highlights: infinitely more efficient auction-protocols**

Abstracting away the game theory and other detail, we explain the simple idea which is at the core of our main positive result (Theorems 3.1 and 4.1). Simplifying further, consider a randomized auction of just a single item: our goal is to compress the infinite deterministic communication complexity of a protocol where the buyer tells the seller exactly with what probability he expects to receive the item. Denote this probability of allocation by \(p\). Given \(p\), one way to allocate with probability \(p\) using unbiased coin tosses is to generate a uniformly random number \(\tau \in [0, 1]\) (whose binary representation is a uniformly random stream of bits after the decimal point), and to allocate the item iff \(p > \tau\)\(^{15}\).

The key insight: for any fixed \(p\), we don’t actually need to know \(\tau\) to infinite precision - we only need to know the prefix of \(\tau\)’s binary representation until the first bit on which it differs from \(p\). Similarly, for a fixed \(\tau\), we only need to know \(p\) to the same precision. So here is our core protocol: draw\(^{16}\) \(\tau \in [0, 1]\) uniformly at random, and ask the buyer to stream the binary representation

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\(^{12}\)We require constant precision marginal contribution per item. For unit-demand, this translates to constant precision for any outcome. For gross substitutes, etc. this translates to \(O(\log(n))\) bits to represent outcome utilities, which is still negligible.

\(^{13}\)Note that it was an open question to obtain such a separation for any social choice rule, let alone a natural and important one like revenue-maximizing auctions.

\(^{14}\)The non-IC auction protocol is closer to [FS09]’s notion of implementing (as opposed to incentivizing) a mechanism, or to [BGN17]’s definition of randomized communication complexity of auctions.

\(^{15}\)For historical context, we remark that the setup up to this point is similar to the 1-bit public-coin protocol for single-item auctions in [BGN17].

\(^{16}\)Here and in all our protocols, \(\tau\) can be drawn on the fly so the expected number of random bits is also bounded.
of \( p \) - only with enough precision to determine whether \( p > \tau \). Each time the buyer sends a bit from the binary representation of \( p \) it differs from the corresponding bit of \( \tau \) with probability \( 1/2 \); i.e. the protocol terminates with probability \( 1/2 \) after each round. Hence we reduced the infinite deterministic protocol to one where the buyer only sends 2 bits in expectation.

What happens when we bring back incentives? It’s not too hard to show that the protocol remains incentive-compatible as long as the buyer doesn’t learn anything about \( \tau \) until the end of the protocol. This is actually too good to be true, since the protocol length must depend on \( \tau \) (otherwise it would be deterministic - and hence infinite), and the buyer must know whether the protocol is continuing in order to participate. Fortunately we can argue that if the only thing the buyer learns about \( \tau \) is that the protocol is continuing, this information cannot help him cheat. Intuitively, he has already committed to the prefix of the protocol, and the extension of his strategy for the rest of the protocol is optimal conditioned on actually being asked to use it.

**Technical highlights: a characterization of randomized IC auction protocols**

It is natural to try to prove communication lower bounds of IC auction protocols via a modular approach of: (i) use Game Theory to define a restricted communication problem that we have to solve in order to obtain near-optimal revenue; and then (ii) use standard techniques from Communication Complexity (e.g. a reduction from Set Disjointness). This approach has worked successfully in other applications of communication complexity to game theory (e.g. [PSS08, Dob16b, ILM+18, GR18]). However, our non-IC auction protocol in Appendix A formally precludes such a modular approach because there is an efficient communication protocol that exactly solves the game theoretic problem we are after. (In other words, the modular approach cannot separate the communication complexity of incentivizing and implementing a social choice rule.) Instead we need to simultaneously consider the complexity and incentives constraints, in particular we need to consider the joint evolution of the buyer’s prior and incentives in an arbitrary randomized protocol.

Our main novel insight is the following simple characterization of incentive compatible communication protocols: In a general communication protocol, each buyer’s node can partition the buyer’s types in an arbitrary way. But for IC protocols, the buyer’s next bit is fully determined by his respective value for the expected allocations conditioned on sending “0” or “1”; this means that it can only partition the buyer’s types into halfspaces in valuation space (see Figure 2). Thus IC mechanisms are much less expressive.

The second part of the proof combines tools from Auction Theory and Error Correction Codes to construct, for each class of valuations, a family of priors whose (approximately) optimal mechanisms are all different. Finally, a simple counting argument shows that the total number of short IC protocols that satisfy our characterization is too small to cover all the different mechanisms.

**Related work**

For general social choice settings, [FS09] define *binary dynamic mechanism (BDM)*, which formalizes the notion of communication protocol between multiple agents with outcomes and payments associated with the protocol-tree leaves. They contrast the communication complexity of *incentivizable* vs *implementable* BDMs. Our notion of IC auction protocols is equivalent to requiring that the protocol is incentivizable.

One subtle difference between our model and [FS09] is that the latter define BDMs as deterministic, while we focus on randomized protocols. In our context we can encode the seller’s random number source as her type\(^{17}\). In this sense, our IC auction protocol is a special case of their *Bayesian*
incentive compatibility (BIC)-incentivizable BDM. But this view misses the distinction between trusting a Bayesian prior about other agents’ valuations and merely trusting the source of randomness.

Our paper resolves an open question from [FS09] of separating the communication complexity of incentivizing and implementing Bayesian incentive compatible social choice rules. Very recently, [DR20, RST+20] resolved a different open question from the same paper about separating the communication complexity of incentivizing and implementing ex-post incentive compatible social choice rules. [RST+20] also separate the communication complexity of ex-post vs dominant strategy incentive compatibility.

Our paper exhibits a strong separation between the communication complexity associated with direct revelation and general mechanisms. Related separations have been shown before by [CS04b] and [Dob16b] for social-welfare maximization with two or more strategic buyers. Specifically, [CS04b] show an exponential gap between the communication complexity of direct revelation versus interactive mechanisms. [Dob16b] shows that in several important settings, the “taxation complexity” of deterministic mechanisms is approximately equivalent to the communication complexity, but exhibits an exponential gap between the two for truthful-in-expectation mechanisms. In contrast, we consider revenue maximization with a single strategic buyer and as few as two items. Arguably, the separation for a single strategic buyer in our settings is more surprising since he communicates with a seller who doesn’t receive exogenous private information. More generally, communication complexity of (approximate) social welfare maximization in auctions with multiple strategic buyers has been extensively studied for combinatorial auctions [NS06, Seg07, DNO14, Dob16a, Dob16b, BMW16, Ass17, BMW18, EFN+19, AKSW20] and related settings [BNS07, PSS08, BBS13, BF13, DD13].

Our paper is inspired by a discussion in [BGN17] about the communication complexity of revenue-maximizing auctions. They prove that in general the deterministic communication complexity is equivalent (up to rounding) to the logarithm of the menu-size complexity. They also define a measure of randomized communication complexity of an auction, which is most closely related to [FS09]’s weaker notion of implementable protocols. They give a randomized protocol for implementing any\textsuperscript{18} incentive-compatible auction for selling a single item using 1 bit of communication and (possibly infinitely many) public random coins.

Our protocols circumvent the intractability of exactly communicating payments (to infinite precision) by replacing them with random payments while preserving expectation. Related ideas have been used before in algorithmic mechanism design, e.g. by [APTT03, BKS15].

The study of communication complexity in economics has its roots in classic works of [Bar38] and [Hay45]. Early mathematical formulations of the question were given by [Hur60, MR74, Rei84]. Outside of auctions, communication complexity has also been considered in AGT in the context of voting rules [CS05, PR06, CP11, SA12], equilibrium computation [CS04a, HM10, RW16, BRI17, GR18, GKT18, GKP19], fair division [Seg10, BNI19, PR19], interdomain routing [LSZ11], and stable matching [GNOR19].

Since the seminal [HN19], menu-size complexity has been further studied by [DDT17, BGN17, SSW18, Gon18, KMS+19]. For a buyer with additive valuations over independent items, [BGN17] prove\textsuperscript{19} an $O(n \log(n)/n)$ upper bound on the menu-size complexity of approximately-optimal mechanisms. In this special case, this translates to an upper bound of $O(n \log(n))$ on the deterministic communication complexity - slightly more efficient than our $O(n \log(n))$ upper bound on randomized \textsuperscript{18}Note that the (revenue-)optimal auction for a single item is already deterministic and uses only 1 bit of communication.

\textsuperscript{19}Theorem 1.2 of [BGN17] states a slightly weaker bound of $n \log(n)$; the stronger bound is suggested in Footnote 3 of their paper.
communication complexity\textsuperscript{20}. Our proof is arguably much simpler\textsuperscript{21}. \cite{Gon18} explores the dependence on $\varepsilon$ in the menu complexity of mechanisms with additive-$\varepsilon$-suboptimality in revenue; his main result, combined with \cite{HN19}, implies a $\Theta(\log(1/\varepsilon))$ bound on the deterministic communication complexity with two items.

For a buyer with unit-demand valuations over independent items \cite{KMS+19} define a related notion of symmetric menu-size complexity which counts the number of lines up to symmetries, and prove an $n^{\text{polylog}(n)}$ upper bound on the symmetric menu-size complexity. We use a slightly stronger notion of partition-symmetric menu-size complexity (see Definition 5.2); the bound of \cite{KMS+19} also holds for this stronger definition. We use this result for our nearly-revenue-optimal IC auction protocol. This provides further evidence that the relatively new notion of (partition)-symmetric menu complexity is a natural complexity measure for auctions.

Over the past decade, computational and menu-size complexity results of optimal auctions have motivated the design of sub-optimal but simple mechanisms that approximate the optimal revenue \cite{CHK07, HR09, CHMS10, KW12, LTY13, BILWT14, CMS15, RW18, Rub16, CM16, HN17, CZ17, HR17, CGMW18, RW18, CTT19} or require resource augmentation \cite{RTCY12, EFF+17, LP18, FFR18, BW19}. Our results suggest that, in some cases, even strong menu-size complexity lower bounds do not preclude efficient optimal mechanisms.

2 Model and definitions

Our main notion in this paper is that of IC auction protocols:

**Definition 2.1 ((IC) auction protocols).** An auction protocol consists of:

- A (possibly infinite) binary tree whose internal nodes are labeled either B (for Buyer) or C (for Chance).
- Each node of Chance has an associated probability distribution over its children.
- Each leaf node has an associated (non-negative) payment and (feasible) allocation.
- A suggested mapping from Buyer’s types to Buyer’s strategies, where a Buyer’s strategy corresponds to a choice of child for each Buyer’s node.

We say that an auction protocol is finite if it is guaranteed to terminate after a finite number of rounds with probability 1 for every Buyer’s strategy. We say that an auction protocol is individually rational (IR) if the Buyer has a strategy that guarantees expected payment 0 and empty allocation. We say that an auction protocol is IC (in-expectation) if it is finite and IR, and if the Buyer weakly prefers the suggested Buyer’s strategy corresponding to his type over any other strategy in the protocol.

\textsuperscript{20}The results are incomparable: \cite{BGN17} uses deterministic communication, whereas our protocol gives exact-revenue-optimality and allows for correlated valuations. In particular, note that in our setting $O(n \log(n))$ is tight up to $O(\log n)$ factor, whereas for approximate revenue with independent valuations, the true answer (even for deterministic communication) is conjectured to be $O(\log(n))$ \cite[Footnote 4]{BGN17}.

\textsuperscript{21}The main technical hurdle for \cite{BGN17} is a reduction to the case where the valuations are (almost) bounded by some large number $H = \text{poly}(n, \varepsilon)$ with only a negligible loss in revenue; we simply assume that the valuations are bounded by $U$, but it can be arbitrarily large. We remark that if we assume that the optimal mechanism obtains finite revenue (as is assumed in \cite{BGN17}; see Footnote 6 of their arXiv version), then it is easy to argue that for any $\varepsilon > 0$, capping the valuations by a sufficiently large $U(\varepsilon)$ preserves a $(1 - \varepsilon)$-fraction of the revenue (see Appendix D for details).
The expected communication complexity, of an auction protocol is the expected depth of the leaf reached by a worst-case Buyer’s strategy (and in expectation over nodes of Chance). Theorem 3.2 refers to the expected Buyer’s communication, which only counts the number of Buyer’s nodes on the path to the leaf.

Note that the buyer’s strategy can be assumed wlog to be deterministic.

2.1 Valuation classes

As is standard in the Algorithmic Mechanism Design literature, we consider buyers whose value for a bundle can be restricted to one of the following classes:

Definition 2.2 (Valuation classes).

A valuation function $v : 2^{[n]} \rightarrow \mathbb{R}_{\geq 0}$ may be restricted to one of the following classes:

Additive If it can be written as $v(S) = \sum_{i \in S} v_i$ for some item values $v_i$’s.

Unit-demand If it can be written as $v(S) = \max_{i \in S} v_i$ for some item values $v_i$’s.

Matroid-rank If, for some matroid $M$ and item values $v_i$, it can be written as

$$v(S) = \max_{T \text{ is independent } \in M} \sum_{i \in S \cap T} v_i.$$  

XOS 22 If item value-vectors $v_i$ of dimension $d$, it can be written as

$$v(S) = \max_{j \in \{1,\ldots,d\}} \sum_{i \in S} v_{i,j}.$$  

The aforementioned classes are related to other well-studied classes like gross-substitutes, submodular, and subadditive in the following hierarchy:

additive, unit-demand $\subset$ matroid rank $\subset$ gross substitutes $\subset$ submodular $\subset$ XOS $\subset$ subadditive.

The formal definition of gross substitutes, submodular, and subadditive is not important for our purposes; they are economically significant because they capture different natural notions of substitutability between items (see e.g. [LLN06]).

In general, we are interested in any prior distribution over valuations of any of above-mentioned types. In particular, we also consider the notion of combinatorial valuations over independent items, which has been recently used by e.g. [RW18 CDW16 CM16 CZ17].

Definition 2.3 (independent items [Sch03]). A prior distribution $\mathcal{D}$ of valuations has a latent structure of independent items if there is a latent product distribution $\mathcal{D}_1 \times \mathcal{D}_2 \cdots \times \mathcal{D}_n$ with arbitrary support such that, a sample valuation $v$ from $\mathcal{D}$ can be generated by first sampling $a_i$ from $\mathcal{D}_i$ for all $i \in [n]$, and then for every $S \in [n]$, the value of $v(S)$ is uniquely determined by $\{a_i \mid i \in S\}$.

2.2 menu-size complexity

Definition 2.4 (Menu-size complexity). By the taxation principle, any mechanism can be canonically described by the expected allocation and payment for each type. This description induces a menu, or collection of menu lines, where each menu line is the expected allocation and payment for some type. The menu-size complexity of a mechanism is the number of distinct menu lines.

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22XOS valuations are sometimes also called fractionally subadditive.
3 IC auction protocols for an additive buyer

Theorem 3.1. For any prior $D$ of Buyer’s additive valuations over $n$ items bounded by maximum valuation $U$, and any truthful mechanism $M$, there is an IC auction protocol with the same expected payment and allocation, using $O(n \log n)$ bits of communication.

Proof. First, we convert $M$ to a strategically-equivalent mechanism $M'$ where the payment is always either zero or $U$. Note that by IR, the expected payment $P$ in $M$ for every type is always at most $U$; therefore for each type we can implement expected payment $P$ by charging a payment of $U$ with probability $P/U$ (and zero otherwise). We henceforth identify each type of Buyer with the corresponding vector in $[0, 1]^{n+1}$, which describes the probability that $M'$ allocates each item to the Buyer, and the probability $(n + 1$-th coordinate) that the Buyer pays $U$. We can further identify the mechanism $M'$ with the set of allowed types/vectors in $[0, 1]^{n+1}$.

Buyer’s nodes and suggested strategy Each Buyer’s node corresponds to a choice of $n + 1$ bits. Given the Buyer’s type and mechanism $M'$, let $p_1, \ldots, p_n$ denote the probability that Buyer is allocated items $1, \ldots, n$, respectively, and let $p_{n+1} = P/U$ denote the probability that the Buyer pays $U$. The Buyer’s suggested strategy is to send, for each round $r$ and $i \in [n+1]$, the $r$-th bit in the binary representation of $p_i$.

Correcting infeasible bits We enforce that at any point in the protocol, the Buyer’s messages are consistent with some type, i.e. with the prefix of probabilities corresponding to some feasible menu line in $M'$. If only possible value for the Buyer’s next bit would possibly be consistent with the protocol’s history, the protocol continues assuming that the Buyer indeed sent this bit (formally we remove the Buyer’s node from the protocol since it is redundant).

Nodes of Chance The distribution over nodes of Chance is determined by an implicit parameter $\tau$ drawn uniformly at random from $[0, 1]$. Before each node of Chance, we will already know that $\tau$ belongs to a particular measurable subset $S \subseteq [0, 1]$. For a partition $S_L \cup S_R = S$ (to be specified below), each child of this node of Chance will correspond to $\tau$ falling in each of $S_L$ or $S_R$, which induces the probability distribution on the children. While $\tau$ plays a crucial role in defining and analyzing the protocol, we stress that it is only implicit: in the actual protocol it is drawn on the fly, with increasing precision at each node of Chance along the path of the protocol.

To define the $r$-th node of Chance along a given path, consider, for each $i \in [n+1]$, the concatenation of the $i$-th bits across the Buyer’s $r$ messages, and compare it to the first $r$ bits in the binary representation of $\tau$. If for every $i$, at least one of the bits is different, the protocol is terminated at a leaf as follows (see Payment and Allocation). Otherwise, the protocol continues in a Buyer’s node. Note that for each node of Chance, only one of its children is an internal (Buyer’s) node.

Payment and Allocation At the end of the protocol, for each $i \in [n+1]$, let $\hat{p}_i^r \in [0, 1)$ denote the number whose binary representation is the concatenation of the $i$-th bit in each of the $r$ rounds of the protocol (after correcting infeasible bits). For $i \in [n]$, the $i$-th item is allocated iff $\hat{p}_i^r > \tau$; the Buyer pays $U$ iff $\hat{p}_{n+1}^r > \tau$, and otherwise he pays zero.

23 Here we slightly abuse notation: we defined the auction protocols for binary trees, so this technically corresponds to a sub-tree of depth $n + 1$ with all Buyer’s nodes.

24 If $p_i$ has two binary representations, using either one throughout the protocol will work.
This figure depicts the first two iterations in an example protocol with one item, where the Buyer’s favorite menu line has payment probability $2/3$ (10 in binary) and item allocated with probability $1/3$ (01 in binary). Nodes marked with B (resp. C) correspond to Buyer (resp. Chance). Triangles correspond to sub-trees never visited for this particular Buyer’s valuation. In the first iteration, the Buyer sends 1,0, corresponding to the first bit in the probability of payment, allocation. Notice that 0,1 is an infeasible prefix for the Buyer since it would violate IC constraints (lower probability of payment and higher probability of allocation). At the first node of Chance, $\tau$ cannot disagree with both bits, hence the protocol proceeds to the next Buyer’s node with probability 1. In the next iteration the Buyer sends the second bit from each probability. Finally, in the second node of Chance:

- The Buyer pays $U$ and receives the item w.p. $1/4$ ($\tau < 1/4 < 1/3, 2/3$).
- The Buyer pays nothing and receives nothing w.p. $1/4$ ($\tau > 3/4 > 1/3, 2/3$).
- W.p. $1/2$ the protocol continues.
IC  The key observation for incentive compatibility is that a Buyer’s strategy is completely determined by the infinite stream of messages that it would send in the (zero-probability) event that the protocol never terminates. To see this, recall that each node of Chance has only one internal node child. Hence for any fixed Buyer’s strategy there is a unique infinite path in the tree, and every finite run of the protocol corresponds to a prefix of this path, up to some node of Chance that deviates from the path to a leaf.

Let \( \hat{p}_i \) denote the number whose binary representation is the infinite sequence of Buyer’s \( i \)-th bits in the (zero-probability) event that the protocol never terminates. Recall from the previous paragraph that a Buyer’s strategy is completely determined by the vector of \( \hat{p}_i \)'s. Note further that the \( i \)-th item is allocated at the end of the protocol iff \( \hat{p}_i > \tau \); similarly, the Buyer pays \( U \) iff \( \hat{p}_{n+1} > \tau \). Therefore, since \( \tau \) is drawn uniformly from \([0, 1] \), the probability that the Buyer is allocated item \( i \) (resp. pays \( U \)) is exactly \( \hat{p}_i \). Hence, by IC of \( M' \), the suggested strategy \( \hat{p} = p \) is optimal for the Buyer.

Communication complexity  At each round of communication, the Buyer sends \( n+1 \) bits. Also, at each round \( r \) of communication, there is probability exactly \( 1/2 \) that the \( i \)-th bit in the Buyer’s message (for each \( i \in [n+1] \)) disagrees with the \( r \)-th bit of \( \tau \). (This probability is independent across rounds, but correlated for different \( i \)'s.) After \( 2 \log(n) \) rounds, each \( i \) has probability \( 1/n^2 \) of agreeing with all of \( \tau \)'s bits. We can take a union bound over all \( i \)'s to obtain that except with probability \( 1/n \), the protocol has already terminated. In the unlikely event that the protocol continues, we can re-apply the same analysis from scratch.

Let \( r_{ub} \) denote an upper bound on the expected number of rounds in the protocol, corresponding to the worst case where the above union bound is tight. Then we have that

\[
r_{ub} \leq 2 \log(n) + r_{ub}/n.
\]

Solving the recurrence relation for \( r_{ub} \), we have that \( r_{ub} = O(\log(n)) \). Since the Buyer sends \( n+1 \) bits in each round, the total communication complexity is \( O(n \log(n)) \).

3.1 Special case: a protocol for the [DDT17] example

We can also prove a concrete (non-asymptotic) bound on the expected number of bits that the buyer sends in the DDT example. Beyond the historical importance of this specific example, our result demonstrates that our protocols are communication-efficient not only in the asymptotic sense, especially if we take advantage of the particular features of a specific distribution. We in particular highlight the fact that the Buyer in this protocol sends strictly less bits\(^{25}\) than he would with a simple deterministic auction selling each item separately.

**Theorem 3.2.** Consider the case of \( n = 2 \) items and the Buyer drawing his valuations i.i.d. from Beta(1, 2) (i.e. the distribution on \([0, 1] \) with density function \( f(x) = 2(1-x) \)). Then there is an IC auction protocol obtaining the maximum possible revenue where the Buyer sends less than two bits in expectation.

The proof is deferred to Appendix [B]\(^{25}\)\)

\(^{25}\)Here we only count communication from the Buyer and not the random coin tosses. In many scenarios random bits are cheap but informative communication is costly.
4 An extension for general valuations

The following theorem is an analogue of Theorem 3.1 for general combinatorial valuations (not necessarily subadditive or monotone). The communication complexity upper bound is parameterized by \( B \), the number of bundles ever assigned by the direct revelation mechanism. For example, for unit demand valuations, \( B \leq n + 1 \); for general valuations, \( B \leq 2^n \).

**Theorem 4.1.** Let \( \mathcal{D} \) be any prior over Buyer’s combinatorial valuations over \( n \) items bounded by maximum valuation \( U \), and any truthful mechanism \( \mathcal{M} \). Suppose that for any type and realization of randomness, \( \mathcal{M} \) only ever allocates one of \( B \) bundles. Then there is an IC auction protocol with the same expected payment and allocation using \( O(B \log(B)) \) bits of communication.

**Proof sketch.** For any type, consider a partition of \([0, 1]\) into \( B \) intervals, where the \( b \)-th interval is of length identical to the probability that \( \mathcal{M} \) allocates Bundle \( b \) to the Buyer. The rest of the proof proceeds analogously to the proof of Theorem 3.1. First, we transform \( \mathcal{M} \) into a mechanism \( \mathcal{M}' \) with payment 0 or \( U \). We henceforth identify between a type and the \( B - 1 \) probabilities that define the partition, and the probability that the Buyer pays \( U \). The nodes of Chance are parameterized by a threshold \( \tau \) drawn uniformly at random from \([0, 1]\). At each round of communication the Buyer (allegedly) sends the next bit in the binary representation of each of the \( B \) probabilities that define his type. The protocol terminates when it has received enough information to determine in which of the \( B \) intervals \( \tau \) lies and whether \( \tau \) is smaller than the probability of payment. The allocation is the bundle corresponding to this interval, and the payment is \( U \) if \( \tau \) is smaller than the probability of payment (and zero otherwise).

5 Unit-demand, independent items: trading off revenue and communication

**Theorem 5.1.** Let \( \mathcal{D} \) be a distribution of independent unit-demand valuations over \( n \) items bounded by maximum valuation \( U \). Then, for any constant \( \varepsilon > 0 \), there is a \((1 - \varepsilon)\)-approximately revenue-optimal IC auction protocol using \( \text{polylog}(n) \) bits of communication.

Our proof uses a result of \([\text{KMS}+19]\) for Partition-symmetric menus which we introduce in Section 5.1. The proof of Theorem 5.1 is given in Section 5.2.

5.1 Partition-symmetric menu-size complexity

**Symmetries**

The following is a slight strengthening of the symmetric menu-size complexity measure recently introduced by \([\text{KMS}+19]\).

**Definition 5.2** (Partition-symmetric menu-size complexity). A partition-symmetric menu line consists of a payment, (randomized) allocation, and a partitioning of items into subsets \( S_1, \ldots, S_\sigma \). We say that a direct revelation mechanism \( \mathcal{M} \) supports this partition-symmetric menu line if its menu contains a line with the same payment for any permutation of the allocation that respects the partition (i.e. permutation \( \pi \) such that \( \pi(S_i) = S_i \) for all \( i \)). The partition-symmetric menu-size complexity of \( \mathcal{M} \) is the smallest \( c \) such that \( \mathcal{M} \) can be written as the union of \( c \) partition-symmetric menu lines.
The following theorem follows from \cite{KMS+19}; the statement here is slightly stronger than the formulation of Theorem IV.5 in their paper in the sense that (i) we consider the specific symmetry group induced by a partition of the items; and (ii) we require that the allocation probabilities are rounded to a discrete set $L_\delta$. Both desiderata follow from their proof \cite{Wei20}.

**Theorem 5.3** \cite{KMS+19}. Let $\mathcal{D}$ be a distribution of independent unit-demand valuations over $n$ items. Then, for any constant $\varepsilon > 0$, there exists a unit-demand mechanism with partition-symmetric menu-size complexity at most $n^{\operatorname{polylog}(n)}$ which recovers at least $(1 - \varepsilon)$-fraction of the optimal revenue. Furthermore, for some constant $\delta > 0$ that depends on $\varepsilon$, the probabilities that the mechanism allocates each item always belong to the discrete set $L_\delta := \{1, 1 - \delta, (1 - \delta)^2, \ldots, (1 - \delta)^\frac{2}{\ln n} \} \cup \{0\}$; in particular there are only $O(\log n)$ possible probabilities.

### 5.2 Proof of Theorem 5.1

**Proof.** We begin with the partition-symmetric mechanism of \cite{KMS+19} (see Theorem 5.3). Denote its partition-symmetric menu-size complexity by $C$. In the first stage of the protocol, the Buyer chooses a partition-symmetric menu line among $C$ options, and then a subset $S_i$ is drawn by Chance from the $\sigma \leq n$ subsets in the partition. (Each subset $S_i$ is drawn with probability equal to the sum of probabilities of items in that subset.) This first stage uses $O(\log n + \log C)$ communication. We henceforth focus on implementing the mechanism restricted to $S_i$. I.e. a mechanism whose menu has a fixed payment $P$ and the set of feasible allocations is symmetric with respect to any permutation of $S_i$.

Since the set of feasible allocations is symmetric, it suffices to consider the histogram of allocation probabilities. The Buyer may assign each probability from the histogram to any item in $S_i$. Recall also that by Theorem 5.3 all the probabilities in the histogram belong wlog to a discrete set $L_\delta$ of $O(\log(n))$ feasible probabilities. In particular, the histogram can be described by $O(\log^2(n))$ bits (since the count for each probability is an integer between 0 and $|S_i| \leq n$).

The second stage of the protocol proceeds by recursively considering smaller subsets of $S_i$. The nodes of Chance are parameterized by a number $\tau$ draws uniformly at random from $[0, 1]$. At the first iteration, the Buyer’s suggested strategy is to send the histogram of probabilities for the lexicographically first half of items in $S_i$. (This is equivalent to sending the histogram for the second half of the items since the total histogram is known.) If the sum of probabilities in the first half is greater than $\tau$, the protocol recurses on the first half; otherwise it recurses on the second half. After $O(\log |S_i|)$ iterations, only one item is left. The Buyer is allocated that item and pays $P$.

**IC** We prove that the second stage of the protocol is IC and also has the same expected allocation and payment as in the original mechanism; IC of the first stage then follows from IC of the original mechanism. To show second-stage IC, let $\widehat{p}_j$ denote the probability that the Buyer assigns item $j$ in the last iteration when it is not the only remaining item. Observe that any Buyer’s strategy for the second stage is fully determined by the vector of $\widehat{p}_j$’s. By reverse induction over the iterations of the protocol, observe that the histogram of all $\widehat{p}_j$’s is exactly equal to the histogram of feasible probabilities. Finally note that at the end of the protocol, the Buyer is allocated item $j$ with probability $\widehat{p}_j$. Therefore, by IC of the original protocol, the Buyer’s suggested strategy is optimal.

---

\textsuperscript{20}We say that a mechanism is unit-demand if it never allocates more than one item to the Buyer. (This is wlog for direct revelation mechanisms with unit-demand buyers. But in general, for mechanisms where the Seller does not fully learn the Buyer’s valuation, it is not obvious how to convert a mechanism where she allocates a bundle of items to a unit-demand mechanism without increasing the partition-symmetric menu-size complexity.)
Communication complexity The first stage of the protocol requires $O(\log n + \log C)$ communication. Each iteration of the second stage requires $O(\log^2(n))$ bits to describe the histogram, and there are at most $O(\log(n))$ iterations. Hence the total communication complexity is $O(\log^3 n + \log C) = \text{polylog}(n)$.

6 Communication lower bound for unit-demand valuations

We consider revenue maximization with unit-demand valuations as an example to demonstrate our proof technique. Our framework for constructing hard instances will rely on the design properties of a set system and a vector family, which are presented in the two lemmata in the following subsection.

6.1 Combinatorial designs

Lemma 6.1. For any constant $\varepsilon, \delta > 0$, there exists a family of size-$\varepsilon n$ subsets $\mathcal{X}_{n,\varepsilon,\delta} \subset \{0,1\}^n$ such that $|\mathcal{X}_{n,\varepsilon,\delta}| = 2^{\Theta(n)}$, and the intersection between any two distinct subsets $x_1, x_2 \in \mathcal{X}_{n,\varepsilon,\delta}$ has size at most $(1 + \delta)\varepsilon^2 n$.

Proof. By Chernoff bound, the size of intersection between two random size-$\varepsilon n$ subsets $S_1, S_2$ is concentrated:

$$\Pr[|S_1 \cap S_2| - \varepsilon^2 n] \geq \delta \varepsilon^2 n] \leq e^{-\delta^2 \varepsilon^2 n/3}.$$  

We draw $2^{\theta n}$ random subsets of size $\varepsilon n$, for $\theta = \delta^2 \varepsilon^2 / 6$. Then, by a union bound over all pairs of subsets, the size of intersection between every two random subsets is concentrated with high probability. \hfill $\square$

Lemma 6.2. For any constant $\varepsilon > 0$ and large integer constant $\ell$, let $\mathcal{R}_{\ell,\varepsilon}$ be the discrete distribution supported on $\{\varepsilon^{\ell-1}, \varepsilon^{\ell-2}, \ldots, 1\}$ such that $p^{(i)} \propto \varepsilon^{\ell - i}$, where we denote $p^{(i)} := \Pr[\varepsilon^{i-1}]$ (this is approximately the “equal-revenue distribution”). Then, for any constant $\eta > 0$, there exists a family of vectors $\mathcal{C}_{N,\ell,\varepsilon,\eta} \subset \{\varepsilon^{\ell-1}, \varepsilon^{\ell-2}, \ldots, 1\}^N$ such that

- $|\mathcal{C}_{N,\ell,\varepsilon,\eta}| = 2^{\Omega(N)}$,

- and moreover, for any $m = \omega(1)$ distinct vectors in $\mathcal{C}_{N,\ell,\varepsilon,\eta}$, for all but $\eta$ fraction of $j \in [N]$, for any $i \in [\ell]$, there are $(1 \pm \eta)p^{(i)}$ fraction of these $m$ vectors whose $j$-th coordinates are $\varepsilon^{i-1}$.

Proof. We construct $\mathcal{C}_{N,\ell,\varepsilon,\eta}$ simply by independently sampling $2^N$ vectors from product distribution $\mathcal{R}_{\ell,\varepsilon}$ for arbitrarily small constant $\delta > 0$, and we show that the desired properties hold with high probability. First, the probability that two random vectors have the same value at $j$-th coordinate is $p := \sum_{i \in [\ell]} p^{(i)} \cdot p^{(i)}$ for any $j$, and therefore, the probability that the two random vectors are exactly the same is $p^N$. For $\delta < \log(1/p)/2$, by a union bound over all the pairs of random vectors, every vector is distinct with high probability. Second, for any $m$ random vectors, for any $i \in [\ell]$, $j \in [N]$, let $m_{i,j}$ be the number of vectors whose $j$-th coordinates are $\varepsilon^{i-1}$ among the $m$ random vectors, then by Chernoff bound,

$$\Pr[|m_{i,j} - p^{(i)} m| \geq \eta \cdot p^{(i)} m] \leq e^{-\eta^2 p^{(i)} m/3}.$$  

By a union bound, the probability that there exists $i \in [\ell]$ such that $m_{i,j}$ is not within $(1 \pm \eta)p^{(i)} m$ is at most $\ell \cdot e^{-\eta^2 p^{(i)} m/3}$. It follows that for any fixed $\eta$ fraction of $j \in [N]$, the probability that there exists $i \in [\ell]$ such that $m_{i,j}$ is not within $(1 \pm \eta)p^{(i)}$ for all $j$ among the $\eta$ fraction is at most $(\ell \cdot e^{-\eta^2 p^{(i)} m/3})\eta N$ (notice that $p^{(1)}$ is the smallest among all $p^{(i)}$’s). By another union bound over all
possible $\eta$ fraction of $j \in [N]$, the probability that the second property in the statement is violated for $m$ random vectors is at most
\[
\left( \frac{N}{\eta N} \right) \cdot (\ell \cdot e^{-\eta^2 \cdot p(1)m/3}) \eta N \leq (e/\eta)^{\eta N} \cdot (\ell \cdot e^{-\eta^2 \cdot p(1)m/3}) \eta N = (\ell \cdot e^{-\eta^2 \cdot p(1)m/3}) \eta N,
\]
which is $e^{-\theta m N}$ for some constant $\theta$ that does not depend on $\delta$. Since there are $(2^N_m) \leq (e \cdot 2^N/m)^m \leq e^{\delta m N}$ distinct subsets of $m$ random vectors of $C_{N, \ell, \varepsilon, \eta}$, by union bound, for $\delta < \theta$, for any fixed $m$, the second property in the statement is violated with probability at most $e^{-(\theta - \delta) m N}$. Finally, the proof finishes by taking a union bound over all $m = \omega(1)$, namely, $\sum_{m=\omega(1)} e^{-(\theta - \delta) m N} = o(1)$. \hfill \qed

### 6.2 The main lower bound result

Now we prove the following lower bound result for communication complexity of approximate revenue maximization with unit-demand valuations. Specifically, we construct a family of priors and show that most priors in the family are hard for all low-communication (almost) truthful-in-expectation randomized protocols to approximately maximize revenue.

**Theorem 6.3.** For every constant $\tau > 0$, any $\tau$-approximate (almost) truthful-in-expectation protocol for revenue maximization, where the seller has $n$ items, and the buyers have unit-demand valuations, requires $\Omega(n)$ bits of communication in expectation.

**Proof.** We first construct a family of prior distributions of the buyers’ valuations and then argue that in order to achieve any constant approximation, a protocol tree (which we will elaborate shortly) can not be shared by many prior distributions, which implies the communication complexity lower bound by a counting argument.

**Construction** For arbitrarily tiny constants $\varepsilon_1, \varepsilon_2, \delta_1, \eta > 0$ and large integer constant $\ell$ such that $\eta, \varepsilon_1(1 + \delta_1) \ll \varepsilon_2$, we take the set family $X_{n, \varepsilon_1, \delta_1}$ from Lemma 6.1 and let $N := |X_{n, \varepsilon_1, \delta_1}| = 2^{O(n)}$, and then, we take the vector family $C_{N, \ell, \varepsilon_2, \eta}$ from Lemma 6.2 with $|C_{N, \ell, \varepsilon_2, \eta}| = 2^{O(N)} = 2^{O(n)}$. We let each $x \in X_{n, \varepsilon_1, \delta_1}$ represent a subset of items. Notice that we can fix a one-to-one mapping between the coordinates of a vector in $C_{N, \ell, \varepsilon_2, \eta}$ and all the sets in $X_{n, \varepsilon_1, \delta_1}$, and therefore, for any vector $c \in C_{N, \ell, \varepsilon_2, \eta}$, $x \in X_{n, \varepsilon_1, \delta_1}$, we can denote $c(x)$ as $c$’s value at the coordinate that corresponds to $x$.

For each vector $c \in C_{N, \ell, \varepsilon_2, \eta}$, we construct a prior distribution $D_c$ of the buyers’ valuations as follows — First, for each $x \in X_{n, \varepsilon_1, \delta_1}$, we define a unit-demand valuation $v_c^x : 2^{[n]} \to \mathbb{R}_{\geq 0}$ as follows:

\[
v_c^x(S) := \begin{cases} 0 & x \cap S = \emptyset \\ c(x) & \text{otherwise.} \end{cases}
\]

Then, we let $D_c$ be the uniform distribution over $v_c^x$’s for all $x \in X_{n, \varepsilon_1, \delta_1}$. Finally, the family of prior distributions is $F = \{D_c | c \in C_{N, \ell, \varepsilon_2, \eta}\}$.

**Interpretation** The following interpretations might be helpful for reading the proof. Each $x \in X_{n, \varepsilon_1, \delta_1}$ corresponds to a set of items which are (equally) valuable to the buyer with valuation $v_c^x$. Each vector $c \in C_{N, \ell, \varepsilon_2, \eta}$ specifies for each $x \in X_{n, \varepsilon_1, \delta_1}$ how valuable such an item is to the buyer with valuation $v_c^x$. By the design property of $X_{n, \varepsilon_1, \delta_1}$, every $v_c^{x_1}, v_c^{x_2}$ with distinct $x_1, x_2$ are interested in mostly different items. By the design property of $C_{N, \ell, \varepsilon_2, \eta}$, for a large number of valuations $v_c^x$’s with distinct $c$’s but the same $x$, the values of an item in $x$ to these valuations are distributed roughly according to the “equal revenue distribution” $R_{c, \varepsilon_2}$ defined in Lemma 6.2..
An optimal truthful-in-expectation protocol for the hard instances

The first step for proving the lower bound is to show that there is a truthful-in-expectation protocol that extracts the full welfare using $O(n)$ bits of communication for the family of Bayesian instances constructed above. The protocol is as follows: the buyer sends the set $x$ that corresponds to his valuation $v_x$ to the seller, which takes $n$ bits, and then, if $x \in \mathcal{X}_{n,\varepsilon_1,\delta_1}$ (otherwise the seller stops), the seller samples an item $i$ from set $x$ uniformly at random and gives the item $i$ to the buyer and charges him $c(x)$, where $c$ corresponds to the prior $D_c$. This protocol is obviously individual rational and revenue maximizing if the buyer tells the truth. To show truthfulness in expectation, suppose the buyer's true set of interest is $x$; without loss of generality, we can assume that the buyer sends some $x' \in \mathcal{X}_{n,\varepsilon_1,\delta_1}$, because otherwise, the seller stops, and the buyer gets net utility 0, which is not better than telling the true $x$. Moreover, if the buyer sends $x' \neq x$, by the design property of $\mathcal{X}_{n,\varepsilon_1,\delta_1}$, he receives an item in $x$ with probability at most $\varepsilon_1 (1 + \delta_1)$. Hence in expectation, the net utility is at most $\varepsilon_1 (1 + \delta_1) c(x) - c(x') \leq \varepsilon_1 (1 + \delta_1) - \varepsilon_2^{\ell - 1} < 0$, where the first inequality is due to $c(x) \leq 1$ and $c(x') \geq \varepsilon_2^{\ell - 1}$, and the second is due to our choice of parameters. Thus, sending $x$ instead of $x'$ is strictly better in expectation.

Representing a protocol as a protocol tree per prior distribution

Observe that once the prior distribution is fixed, a protocol can be viewed as a protocol tree. See Figure 2 for example. Without loss of generality, the protocol tree starts with the root $B$ representing the buyer's round and then alternates between the buyer $B$ and the seller $C$ (Chance). At each round, represented by a node, the buyer or the seller can choose to send a bit 0, represented by left edge, or bit 1, represented by right edge, to the other. At a leaf, both players agree on a set of items $Y$ allocated to the buyer and a payment $q$ to the seller. The protocol is possibly randomized, and hence, at a seller's round, the seller can send bit 0 with probability $p$ and send bit 1 with probability $1 - p$, which are represented by the weights on the edges. At a buyer’s round, the buyer’s strategy depends on his valuation, but we can assume without loss of generality that the buyer always deterministically chooses a bit to send, because the buyer is strategic and hence sending the bit that has better net utility in expectation (sending the bit that maximizes the seller’s revenue if both choices are (almost) equal, and sending bit 0 if it is still a tie) is a (almost) dominant strategy for the buyer that maximizes the seller’s revenue among all (almost) dominant strategies. Therefore, the buyer’s prescribed (almost) dominant strategy can be deterministically decided by the protocol tree and his valuation.

27We assume that the seller is not strategic in the private-coin model. In the public-coin model, the seller can not be strategic, because his responses can be inferred from the public randomness and the pre-specified protocol tree, and thus, he can keep silent unless he observes that the buyer is cheating.
To make the proof easier, we show that we can without loss of generality assume some nice properties for the protocol trees, and we will only consider such protocol trees afterwards.

**Claim 6.4.** Any (almost) truthful-in-expectation protocol with $O(k)$ communication in expectation for our hard instance can be changed (with arbitrarily small loss of the approximation factor) such that

- the protocol tree has $O(k)$ depth,
- and moreover, the payment at any leaf of the protocol tree is $2^{O(k)}$.

Suppose a protocol uses $ak$ bits of communication in expectation where $a$ is a positive constant. For an arbitrarily large constant $\beta$, by Markov’s inequality, the protocol takes $\geq \beta k$ communication with probability at most $\gamma := a/\beta$. Observe that if we trim all the nodes at level $\geq \beta k$ of the protocol tree $T$, the buyer’s expected utility (before payment) is at least $1 - \gamma$ fraction of that for $T$ (for our instance, the loss is at most $\gamma$). If we further trim every node that is reached with probability $\leq 4^{-\beta k}$ for any buyer, the buyer’s expected utility loses at most another $2^{-\beta k}$, because there are at most $2^{\beta k}$ nodes left after the first trimming step. Since we introduce new leaves after trimming, we need to specify the allocation and the payment for each of them. For each new leaf, we simply let its allocation be the empty set, and we let its payment be the least possible expected payment at this node in $T$ (that is, the minimum expected payment achieved by the worst possible buyer’s responses in the subtree rooted at this node in $T$).

After the above changes, the first property obviously holds for the new protocol tree $T'$, and the second also holds, because if any leaf has payment larger than $4^{\beta k}$, then the probability of reaching that leaf (or node) in the original $T$ for any buyer is at most $4^{-\beta k}$ (otherwise the expected payment is greater than 1 for a buyer that reaches this node with probability $\geq 4^{-\beta k}$, which exceeds the largest possible buyer’s value and hence violates individual rationality), and this leaf should have been trimmed. It remains to show that the approximation factor is decreased arbitrarily little by the above changes.

To prove this, consider the buyer’s (almost) dominant strategy $s^*$ in $T$, we change the strategy in the way that the buyer makes the same response as $s^*$ at every node that will not be trimmed by the above steps and makes the worst possible responses (which minimize the expected payment) in the subtree rooted at every node that will be trimmed. This results in a new strategy $s$ that gives the buyer almost the same expected utility as $s^*$ (as we have shown, the loss is at most $\gamma + 2^{-\beta k}$). It follows that the expected payment for $s$ can only be $\gamma + 2^{-\beta k}$ (plus another negligible error if the original protocol is only almost truthful) less than that for $s^*$, since otherwise the expected net utility of $s$ is significantly better than $s^*$. Moreover, the expected payment for $s$ can not be more than that for $s^*$ by definition of $s$, and hence, the expected net utility of $s$ is same as that of $s^*$ up to negligible error. Furthermore, observe that $s$ (ignore $s$’s responses at trimmed nodes) gets the same expected utility and payment for the buyer in $T$. If $s$ is an almost dominant strategy in $T'$ (which indeed is as we will show), then we are done because we have shown the expected payment for $s$ in $T'$ (or $T$) is same (up to negligible error) as that for $s^*$ in $T$.

To see $s$ is an almost dominant strategy in $T'$, suppose for contradiction there is another strategy $s'$ with an non-negligible improvement of expected net utility over $s$ in $T'$. We extend $s'$ to a strategy for $T$ by letting it make worst possible response (which minimize the expected payment) for the nodes that will be trimmed in $T'$. Note that the extended $s'$ has the same expected net utility in $T$ as that in $T'$, which is significantly better than $s$’s expected net utility in $T'$ (and hence $s$ or $s^*$’s expected net utility in $T$). This contradicts that $s^*$ is a (almost) dominant strategy in $T$. 

One protocol tree can not be shared by many priors

Now we show the main claim that leads to the lower bound result.

**Claim 6.5.** For any constant \( \tau > 0 \) and any \( m = \omega(1) \), any single protocol tree can only achieve \( \tau \) approximation on at most \( m \) priors in \( \mathcal{F} \).

Assume for contradiction that there are \( m = w(1) \) priors \( \mathcal{D}_1, \ldots, \mathcal{D}_m \) in \( \mathcal{F} \) sharing the same protocol tree. By Lemma 6.2 for all but \( \eta \) fraction of \( x \in \mathcal{X}_n, \varepsilon_1, \delta_1 \), the empirical distribution of \( c_i(x) \)'s for \( i \in [m] \) is close to \( R_{\ell, \varepsilon_2} \) defined in Lemma 6.2, namely, the number of \( i \)'s such that \( c_i(x) = \varepsilon_2^{\ell-1} \) is \( (1 + \eta)p(t)m \), where \( p(t) \propto \varepsilon_2^{\ell-t} \). In the rest of the proof of Claim 6.5, we show that for any such \( x \), the average revenue over valuations \( v^x_{C_i} \) for all \( i \in [m] \) achieved by the protocol tree is at most \( \theta := \frac{(\varepsilon_2+1/\ell)(1+\eta)}{1-\eta} \frac{1}{\ell} \) fraction of the optimum. Notice that \( \theta \) is a constant that we can make arbitrarily small. This will finish the proof of the claim, because for at least one of \( \mathcal{D}_1, \ldots, \mathcal{D}_m \), the protocol tree achieves no more than the average of the expected revenues for \( \mathcal{D}_1, \ldots, \mathcal{D}_m \), which is at most \( \tau = \theta + \frac{\eta \varepsilon_2^{1-\ell}}{1-\eta} \) fraction of the optimal revenue (we generously assume that it achieves full revenue on the \( \eta \) fraction of \( x \in \mathcal{X}_n, \varepsilon_1, \delta_1 \) that is excluded from the above analysis, and the full revenue for any \( x \) from this \( \eta \) fraction is at most 1, which is at most \( \varepsilon_2^{1-\ell} \) times the full revenue of any \( x' \) from the other \( 1-\eta \) fraction), and \( \frac{\eta \varepsilon_2^{1-\ell}}{1-\eta} \) is arbitrarily small by our choice of parameters.

Now consider any such \( x \) that the empirical distribution of \( c_i(x) \)'s for \( i \in [m] \) is close to \( R_{\ell, \varepsilon_2} \), and let \( C_\ell \) be the set of \( c_i \)'s with \( c_i(x) = \varepsilon_2^{\ell-1} \). Without loss of generality, the buyers with valuation \( v^x_{C_{\ell t}} \) for all \( c_{\ell t} \in C_\ell \) use the same dominant strategy. Moreover, consider any \( c_{\ell t+1} \in C_{\ell t+1} \) for any \( \ell \), we denote the expected utility and payment achieved by the prescribed dominant strategy for \( v^x_{C_{\ell t}} \) by \( u_t \) and \( q_t \), respectively, and analogously, we denote \( u_{t+1} \) and \( q_{t+1} \) for \( v^x_{C_{\ell t+1}} \). If the buyer with valuation \( v^x_{C_{\ell t}} \) plays the strategy for \( v^x_{C_{\ell t+1}} \) instead, he will get expected utility \( u_{t+1}/\varepsilon_2 \) and payment \( q_{t+1} \), because by definition \( v^x_{C_{\ell t}} = v^x_{C_{\ell t+1}}/\varepsilon_2 \). By definition of (almost) dominant strategy, we have the following inequality (the inequality holds approximately when we consider almost truthful-in-expectation protocols, and the error is negligible to the later derivations)

\[
\frac{u_{t+1}}{\varepsilon_2} - q_{t+1} \leq u_t - q_t. \tag{2}
\]

Moreover, by individual rationality,

\[
q_{t+1} \leq u_{t+1}, \tag{3}
\]

and it follows that

\[
q_t \leq u_t - \frac{u_{t+1}}{\varepsilon_2} + q_{t+1} \quad \text{(Rearranging Eq. (2))}
\]

\[
\leq u_t - \frac{u_{t+1}}{\varepsilon_2} + u_{t+1} \quad \text{(By Eq. (3))}
\]

\[
= u_t - u_{t+1} \left( \frac{1}{\varepsilon_2} - 1 \right). \quad \text{(4)}
\]

Furthermore, because \( c_i(x) \)'s for \( i \in [m] \) are distributed like \( R_{\ell, \varepsilon_2} \), the sum of the revenues obtained from the \( v^x_{C_i} \)'s for all \( i \in [m] \) is at most (up to a \( (1+\eta) \) multiplicative error)

\[
\sum_{t=1}^\ell mp(t)q_t \leq \sum_{t=1}^{\ell-1} mp(t) \left( u_t - u_{t+1} \left( \frac{1}{\varepsilon_2} - 1 \right) \right) + mp(\ell)u_\ell \quad \text{(By Eq. (4) and Eq. (3))}
\]

19
\[
= mp^{(1)}u_1 + m \sum_{t=2}^\ell u_t \left( p^{(t)} - \frac{p^{(t-1)}}{\varepsilon_2} + p^{(t-1)} \right) \quad \text{(Rearranging the sum)}
\]
\[
= mp^{(1)}u_1 + m \sum_{t=2}^\ell u_t p^{(t-1)} \quad \text{(By definition of } p^{(t)})
\]
\[
= mp^{(1)}u_1 + m\varepsilon_2 \sum_{t=2}^\ell u_t p^{(t)} \leq mp^{(1)} + m\varepsilon_2 \sum_{t=2}^\ell p^{(t)} \varepsilon_2^{t-1} \quad \text{(By } u_t \leq \varepsilon_2^{t-1}),
\]
which is at most \(\varepsilon_2\) fraction of \(\sum_{t=1}^\ell mp^{(t)}\varepsilon_2^{t-1}\) plus \(mp^{(1)}\), but \(mp^{(1)}\) is only \(1/\ell\) fraction of \(\sum_{t=1}^\ell mp^{(t)}\varepsilon_2^{t-1}\) by its definition. Because \(c_i(x)\)'s for \(i \in [m]\) are distributed like \(\mathcal{R}_{\ell,\varepsilon_2}\), the optimal total revenue we can get from all the \(v_{C_i}^*\) for \(i \in [m]\) (which is equal to their total value) is at least \((1-\eta)\sum_{t=1}^\ell mp^{(t)}\varepsilon_2^{t-1}\), and hence, the average revenue achieved by the protocol tree on valuations \(v_{C_i}^*\) for \(i \in [m]\) is at most \(\frac{(\varepsilon_2+1/\ell)(1+\eta)}{1-\eta}\) fraction of the optimum.

**Finishing the proof by a counting argument**

For any constant \(\tau > 0\), suppose that the communication complexity of a \(\tau\)-approximate truthful-in-expectation protocol is \(k = o(n)\), and without loss of generality we assume that the protocol always uses up \(k\) bits. We count how many protocol trees we can have. Note that a protocol tree is determined by the \((Y, q)\) pairs on the leaves and the probabilities on the edges. Without loss of generality, we can assume that the payments and the probabilities have finite precision, namely, the probabilities are rounded to \(\{i/4^n \mid i = 0, 1, \ldots, 4^n\}\), and the payments are rounded to \(\{i/4^n \mid i = 0, 1, \ldots, 2^{O(n)}\}\). To see this, first observe that rounding can only change the payment at any leaf by at most \(1/4^n\), and similarly, it can only change the probability of reaching any leaf by \(O(1/4^n)\), and therefore, it only changes the expected utility and the expected payment for the buyer by at most \(O(2^k/4^n) = O(1/2^n)\). As we have noted along the proof, the analysis works for almost truthful-in-expectation protocols, which tolerates this extra \(O(1/2^n)\) error.

Therefore, there are at most \(2^k\) choices of \(Y\) and at most \(2^{O(n)}\) choices of \(q\), which implies at most \(2^{O(n)}\) choices of \((Y, q)\) at each leaf, and there are at most \(4^n\) choices of the probability on each edge. Since the depth of the protocol tree is no more than \(k\), there are \(2^k\) leaves and \(2^{k+1}\) edges at most. Altogether, there are at most \((2^{O(n)})^{2^k} \cdot (4^n)^{2^{k+1}} = 2^{2^k+o(n)}\) possible protocol trees. Furthermore, by Claim 6.5 these protocol trees can only beat \(\tau\)-approximation on at most \(2^{2^k+o(n)} \cdot m\) priors in total for any \(m = \omega(1)\), but there are \(2^{2^{O(n)}}\) priors in \(\mathcal{F}\). Hence, most priors in \(\mathcal{F}\) are hard for all the \(o(n)\)-communication protocols.

\[\Box\]

### 6.3 Separating the complexity of implementing and incentivizing

**Remark 6.6.** There is an \(O(\log n)\)-communication implementation of the optimal protocol for our hard instances. Combining with the lower bound, this shows an exponential separation between communication complexity of almost truthful-in-expectation implementation and that of non-truthful implementation for this protocol, even when the buyer’s valuation has constant precision.

**Proof.** A more communication-efficient non-truthful implementation is that the buyer randomly chooses an item \(i\) of interest and sends \(i\) and \(c(x)\) to the seller, and then the seller gives the item \(i\)
7 Communication lower bound for gross-substitutes valuations

In this section, we sketch how to apply our techniques to establishing sub-exponential communication complexity lower bound for gross substitutes valuations. The specific gross substitutes valuations we use in our hard instances are the matroid rank functions provided by the following lemma.

**Lemma 7.1 ([BH18 Theorem 1]).** For any \( b \geq 8 \) with \( b = 2^{o(n^{1/3})} \), there exists a family of sets \( A \subseteq \{0,1\}^n \) and a family of matroids \( \mathcal{M} = \{ M_B \mid B \subseteq A \} \) with the following properties:

- \(|A| = b\) and \(|A| = n^{1/3}\) for every \( A \in A \).
- For every \( B \subseteq A \) and every \( A \in A \), we have
  \[
  \text{rank}_{M_B}(A) = \begin{cases} 
  |A| & A \in B \\
  8 \log b & A \in A \setminus B.
  \end{cases}
  \]

**Theorem 7.2.** For every constant \( \tau > 0 \), any \( \tau \)-approximate (almost) truthful-in-expectation protocol for revenue maximization, where the seller has \( n \) items, and the buyers have gross substitutes valuations, requires \( 2^{\Omega(n^{1/3})} \) bits of communication in expectation.

**Proof.** The proof follows the same strategy as the proof of Theorem 6.3. We first construct a family of prior distributions over matroid rank valuations.

**Construction** For any \( b = 2^{o(n^{1/3})} \), we let \( A \) be the set family of size \( b \) provided by Lemma 7.1. For arbitrarily tiny constants \( \varepsilon_1, \varepsilon_2, \delta_1, \eta > 0 \) and large integer constant \( \ell \) such that \( \eta, \varepsilon_1(1+\delta_1) \ll \varepsilon_2 \), we take the set family \( X_{b, \varepsilon_1, \delta_1} \) by Lemma 6.1 and let \( N := |X_{b, \varepsilon_1, \delta_1}| = 2^{\Omega(b)} \), and then, we take the vector family \( C_{N, \ell, \varepsilon_2, \eta} \) from Lemma 6.2 with \( |C_{N, \ell, \varepsilon_2, \eta}| = 2^{\Omega(N)} = 2^{\Omega(b)} \). We let each \( x \in X_{b, \varepsilon_1, \delta_1} \) represent a sub-family \( B_x \subseteq A \) (each coordinate of \( x \) corresponds to a distinct set \( A \in A \), and this coordinate has value 1 iff \( A \in B_x \). We can fix a one-to-one mapping between the coordinates of a vector in \( C_{N, \ell, \varepsilon_2, \eta} \) and all the \( x \in X_{b, \varepsilon_1, \delta_1} \), and for any vector \( c \in C_{N, \ell, \varepsilon_2, \eta} \), \( x \in X_{b, \varepsilon_1, \delta_1} \), we denote \( c(x) \) as \( c \)'s value at the coordinate that corresponds to \( x \).

For each vector \( c \in C_{N, \ell, \varepsilon_2, \eta} \), we construct a prior distribution \( D_c \) of the buyers’ valuations as follows — First, for each \( x \in X_{b, \varepsilon_1, \delta_1} \), we define a scaled matroid rank valuation \( v^x_c : \{0,1\}^n \to \mathbb{R}_{\geq 0} \) as

\[
  v^x_c(S) = \frac{c(x)}{n^{1/3}} \cdot \text{Rank}_{M_{B_x}}(S),
\]

where \( \text{Rank}_{M_{B_x}} \) is the matroid rank function from Lemma 7.1. Then, we let \( D_c \) be the uniform distribution over \( v^x_c \)'s for all \( x \in X_{b, \varepsilon_1, \delta_1} \). Finally, the family of prior distributions is \( \mathcal{F} = \{ D_c \mid c \in C_{N, \ell, \varepsilon_2, \eta} \} \).

**Interpretation** In this instance, each \( x \in X_{b, \varepsilon_1, \delta_1} \) corresponds to a family of subsets of items that the buyer with valuation \( v^x_c \) likes the most. Each vector \( c \in C_{N, \ell, \varepsilon_2, \eta} \) specifies for each \( x \in X_{b, \varepsilon_1, \delta_1} \) how valuable such a subset of items is to the buyer with valuation \( v^x_c \). By the design property of \( X_{b, \varepsilon_1, \delta_1} \), every \( v^x_1, v^x_2 \) with distinct \( x_1, x_2 \) are interested in mostly different subsets of items. By the design property of \( C_{N, \ell, \varepsilon_2, \eta} \), for a large number of valuations \( v^x_c \)'s with distinct \( c \)'s but the same \( x \), the values of a subset of items in \( x \) to these valuations are distributed roughly according to the “equal revenue distribution” \( \mathcal{R}_{x, \varepsilon_2} \) defined in Lemma 6.2.
An optimal truthful-in-expectation protocol for the hard instances

We show that there is a truthful-in-expectation protocol that achieves optimal full revenue using $O(b)$ bits of communication for the family of Bayesian instances $\mathcal{F}$. The protocol is as follows: the buyer sends the $x$ corresponding to his valuation $v^*_x$ to the seller, which takes $b$ bits, and then, if $x \in \mathcal{X}_{b,\varepsilon_1,\delta_1}$ (otherwise the seller stops), the seller samples a set $A \in \mathcal{B}_x$ uniformly at random and gives the items in $A$ to the buyer and charges him $c(x)$. The protocol is individual rational and achieves the full revenue if the buyer tells the truth. To show it is truthful in expectation, suppose the buyer sends some other $x' \in \mathcal{X}_{b,\varepsilon_1,\delta_1}$ (otherwise, he always gets net utility 0). Then the probability that he receives a set in $\mathcal{B}_x$ is at most $\varepsilon_1(1 + \delta_1)$, and moreover, if he receives a set in $A \setminus \mathcal{B}_x$, the value he gets is $8\log b = o(1)$. Hence in expectation, the buyer’s net utility is at most $\varepsilon_1(1 + \delta_1)c(x) + o(1) - c(x') \leq \varepsilon_1(1 + \delta_1) + o(1) - \delta_2^{-1} < 0$. Therefore, sending $x$ instead of $x'$ is a better strategy for the buyer.

The rest of the proof is same as the corresponding part of the proof of Theorem 6.3, namely, we can analogously show that any constant approximate (almost) truthful-in-expectation protocol for $\mathcal{F}$ requires $2^{\Omega(b)}$ distinct protocol trees, and it follows by the same counting argument that the protocol needs $\Omega(b)$ bits communication.

Remark 7.3. For XOS valuations with $n$ items, the communication complexity lower bound for any constant approximation can be improved to $2^{\Omega(n)}$.

Proof. The proof is basically the same as that of Theorem 7.2. We point out the difference in the construction of hard instances. Given a set family $\mathcal{X}_{n,\varepsilon_0,\delta_0}$ for arbitrarily small constants $\varepsilon_0, \delta_0 > 0$ from Lemma 6.1 we let $b := |\mathcal{X}_{n,\varepsilon_0,\delta_0}| = 2^{\Omega(n)}$. Now consider a set family $\mathcal{X}_{b,\varepsilon_1,\delta_1}$ again from Lemma 6.1. For all $x \in \mathcal{X}_{b,\varepsilon_1,\delta_1}$, $x$ can represent a set family $\mathcal{B}_x$ of sets in $\mathcal{X}_{n,\varepsilon_0,\delta_0}$ (each coordinate of $x$ corresponds to a set $A \in \mathcal{X}_{n,\varepsilon_0,\delta_0}$, and this coordinate has value 1 iff $A \in \mathcal{B}_x$). Instead of the matroid rank functions $\operatorname{Rank}_{M_{\mathcal{B}_x}}$, here we use the binary XOS functions $\operatorname{BXOS}_{\mathcal{B}_x}$ in our construction, which are given by

$$\operatorname{BXOS}_{\mathcal{B}_x}(S) = \max_{A \in \mathcal{B}_x} |A \cap S|.$$  

Let $N := |\mathcal{X}_{b,\varepsilon_1,\delta_1}| = 2^{\Omega(b)}$, and we take a vector family $\mathcal{C}_{N,\ell,\varepsilon_2,n}$ from Lemma 6.2 with $|\mathcal{C}_{N,\ell,\varepsilon_2,n}| = 2^{\Omega(n)} = 2^{\Omega(b)}$. Following the notation in the proof of Theorem 7.2, we define the valuations as follows

$$v^*_x(S) = c(x) \cdot \operatorname{BXOS}_{\mathcal{B}_x}(S).$$

$\mathcal{D}_c$’s and $\mathcal{F}$ are defined as in the proof of Theorem 7.2.

8 Communication lower bound for XOS valuations with independent items

In this section, we show that beating $4/5$ approximation for XOS valuations with independent items requires exponential communication. (Note that constant-factor approximation is known (e.g., [RW18]) for more general subadditive valuations with independent items).

Theorem 8.1. For every constant $\tau > 0$, any $(\frac{4}{5} + \tau)$-approximate (almost) truthful-in-expectation protocol for revenue maximization, where the seller has $n$ items, and the buyers have XOS valuations with independent items, requires $2^{\Omega(n)}$ bits of communication in expectation.
We show that there is a truthful-in-expectation protocol that achieves optimal full revenue using \(v(S) = \max_{j \in [b]} \sum_{i \in S} a_j^{(i)}.\) In this case, \(D_1 \times D_2 \times \cdots \times D_n\) specifies a prior distribution of XOS valuations.

**Proof.** The proof follows the same strategy as the proof of the previous lower bounds. First, we construct a family of prior distributions of XOS valuations with independent items. We focus on the following special case of prior distributions of XOS valuations with independent items — Given any integer \(b\), for each item \(i \in [n]\) there is a distribution \(D_i\) over \(\mathbb{R}_{\geq 0}^b\), an XOS valuation \(v\) is generated by first sampling a vector \(a^{(i)}\) from each \(D_i\) and then defined as

\[
v(S) = \max_{j \in [b]} \sum_{i \in S} a_j^{(i)}.
\]

For each prior, a valuation is sampled according to the procedure described in the previous paragraph, from Lemma 6.1 and a vector family \(a\) be the trivial distribution with singleton support \(\{i\}\), where \(a^{(i)} \in \mathbb{R}_{\geq 0}^b\) is defined as

\[
a_j^{(i)} = \frac{x_j^{(i)}}{(2-\gamma)\varepsilon_0(n-1)}\quad\text{for all } j \in [b].
\]

Now we take another set family \(X_{n-b,\varepsilon_0,\delta_0}\) from Lemma 6.1 and a vector family \(C\) from Lemma 6.2, where \(N := \lvert X_{n,b,\varepsilon_1,\delta_1} \rvert = 2^{\Omega(n)}\) and \(M := \lvert C_{N,2,\frac{1}{2},\eta} \rvert = 2^{\Omega(N)}\). For each \(c^{(i)}\), we let \(D_n^{(i)}\) be the uniform distribution over \(\{c_{j}^{(i)}+1, y_j | j \in [N]\}\). The family of prior distributions is \(F = \{D_1 \times D_2 \times \cdots \times D_n^{(i)} | i \in [M]\}\). For each prior, a valuation is sampled according to the procedure described in the previous paragraph, and specifically, a valuation function \(v_{c^{(i)}}\), determined by \(c^{(i)}\) and \(y^{(j)}\), is given as follows

\[
v_{c^{(i)}}(S) = \max_{t \in [b]} \mathbb{I}\{n \in S\} \cdot \frac{c_j^{(i)} + 1}{2} \cdot y_j^{(j)} + \sum_{r \in S \setminus \{n\}} \frac{x_r^{(t)}}{(2-\gamma)\varepsilon_0(n-1)}.
\]

**Interpretation** In this instance, any valuation \(v_{c^{(i)}}\), when restricted to items \(\{n-1\}\), becomes a single scaled binary XOS valuation in which the clauses correspond to the scaled binary vectors \(x^{(1)}, x^{(2)}, \ldots, x^{(b)}\) (they represent pairwise nearly disjoint subsets (of items in \(\{n-1\}\)) that are equally valuable to every buyer), and each of these clause has total value \(\frac{1}{2-\gamma}\). Each binary vector \(y^{(j)}\) then decides which of these clauses \(x^{(1)}, x^{(2)}, \ldots, x^{(b)}\) interact with the item \(n\), i.e., the item \(n\) has positive contribution to the clause \(x^{(t)}\) in the valuation \(v_{c^{(i)}}\) iff \(y_t^{(j)} = 1\). (Distinct \(y^{(j)}\)’s define almost completely different interactions.) Each binary vector \(c^{(i)}\), then specifies for each \(y^{(j)}\) how large the contribution of the item \(n\) is for each clause where, according to \(y^{(j)}\), the item \(n\) has positive contribution, i.e., the item \(n\) contributes value 1 to every clause \(x^{(t)}\) it interacts with (i.e., for which \(y_t^{(j)} = 1\) in the valuation \(v_{c^{(i)}}\) if \(c_j^{(i)} = 1\) and contributes value 0 if otherwise. For a large number of valuations \(v_{c^{(i)}}\)’s with distinct \(c^{(i)}\)’s but the same \(y^{(j)}\), the contributions of the item \(n\) in these valuations (to every clause it interacts with) are distributed roughly according to the “equal revenue distribution” \(R_{2,1/2}\).

**An optimal truthful-in-expectation protocol for the hard instances**

We show that there is a truthful-in-expectation protocol that achieves optimal full revenue using \(O(b)\) bits of communication for the family of Bayesian instances \(F\). The protocol is as follows: the buyer sends the \(y^{(j)}\) corresponding to his valuation \(v_{y^{(j)}}\) to the seller, which takes \(b\) bits, and then, if
\[ y^{(j)} \in X_{y, \varepsilon_1, \delta_1} \text{ (otherwise the seller stops), the seller samples a set } A \text{ uniformly at random from the } \]
\[ \{x^{(t)} \mid t \in [b] \text{ s.t. } y^{(j)}_t = 1\} \text{ and gives the items in } A \cup \{n\} \text{ to the buyer and charges him } \]
\[ \frac{c^{(i)}_{j} + 1}{2} + \frac{1}{2 - \gamma}. \] 

It is easy to verify that the protocol is individual rational and achieves the full revenue if the buyer tells the truth. To show it is truthful in expectation, suppose the buyer sends some other \( y^{(j')} \in X_{y, \varepsilon_1, \delta_1} \) (otherwise, he always gets net utility 0). Then the probability that he receives a set in \( \{x^{(t)} \mid t \in [b] \text{ s.t. } y^{(j)}_t = 1\} \) is at most \( \varepsilon_1(1 + \delta_1) \) (in which case he gets value \( \frac{c^{(i)}_{j} + 1}{2} + \frac{1}{2 - \gamma} \)), and moreover, if he receives a set \( B \in \{x^{(t)} \mid t \in [b] \text{ s.t. } y^{(j)}_t = 0\} \), the value he gets is at most \( \frac{c^{(i)}_{j} + 1}{2} + \frac{\varepsilon_0(1 + \delta_0)}{2 - \gamma} \) (total value of \( B \cup \{n\} \) to him). His payment is always \( \frac{c^{(i)}_{j} + 1}{2} + \frac{1}{2 - \gamma} \). Hence in expectation, the buyer’s net utility is at most
\[
\frac{c^{(i)}_{j} + 1}{2} + \frac{\varepsilon_1(1 + \delta_1) + \varepsilon_0(1 + \delta_0)}{2 - \gamma} - \left( \frac{c^{(i)}_{j} + 1}{2} + \frac{1}{2 - \gamma} \right)
\]
\[
= \frac{1}{2} - \frac{1}{2 - \gamma} + \varepsilon_1(1 + \delta_1) + \varepsilon_0(1 + \delta_0), \quad (\text{By } c^{(i)}_{j}, c^{(i)}_{j'} \in \{0, 1\} \text{ and } \gamma \text{ is tiny})
\]

which is negative by our choice of \( \gamma \). Therefore, sending \( y^{(j)} \) instead of \( y^{(j')} \) is a better strategy for the buyer.

Finally, we prove the following main claim. The calculation is slightly different from that of Claim 6.3, but the idea is the same. Using this main claim, the proof can be finished by the same counting argument as in the proof of Theorem 6.3, which we omit here.

**Claim 8.2.** For any constant \( \tau > 0 \) and any \( m = \omega(1) \), any single protocol tree can only achieve \( \frac{4}{5} + \tau \) approximation on at most \( m \) priors in \( \mathcal{F} \).

Assume for contradiction that there are \( m = \omega(1) \) priors \( \mathcal{D}^{(i_1)}_n, \ldots, \mathcal{D}^{(i_m)}_n \) (ignoring the trivial \( \mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_{n-1} \text{ part} ) \text{ in } \mathcal{F} \text{ sharing the same protocol tree. By Lemma 6.2, for all but } \eta \text{ fraction of } j \in [b], \text{ the empirical distribution of } c^{(i)}_{j} \text{'s for } i \in \{i_1, \ldots, i_m\} \text{ is close to } \mathcal{R}_{2,1/2} \text{ defined in Lemma 6.2, namely, the number of } i \text{'s such that } c^{(i)}_{j} = 1 \text{ is } 1\pm \eta \cdot m. \text{ In the rest of the proof of this claim, we show that for any such } j, \text{ the average revenue over valuations } v^{y^{(j)}}_{c^{(i)}} \text{ for all } i \in \{i_1, \ldots, i_m\} \text{ achieved by the protocol tree is at most } \theta = \frac{4(1+\eta)}{4-2\gamma(1-\eta)} \cdot \frac{4}{5} \text{ fraction of the optimum, and } \theta \text{ is a constant that we can make arbitrarily close to } \frac{4}{5}. \text{ This will finish the proof of the claim, because for at least one of } \mathcal{D}^{(i_1)}_n, \ldots, \mathcal{D}^{(i_m)}_n, \text{ the protocol tree achieves no more than the average of the expected revenues for } \mathcal{D}^{(i_1)}_n, \ldots, \mathcal{D}^{(i_m)}_n, \text{ which is } \tau = \theta + \frac{n}{1-n} \cdot \frac{3}{2} \text{ fraction of the optimal revenue (we generously assume that it achieves full revenue on the } \eta \text{ fraction of } j \in [b] \text{ that is excluded in above analysis, and the full revenue for any } y^{(j)} \text{ from this } \eta \text{ fraction is at most } \frac{3}{2} \text{ times the full revenue of any } y^{(j')} \text{ from the other } 1-\eta \text{ fraction), and } \frac{n}{1-n} \cdot \frac{3}{2} \text{ can be made arbitrarily small.}

Consider any such } j \text{ that the empirical distribution of } c^{(i)}_{j} \text{'s for } i \in \{i_1, \ldots, i_m\} \text{ is close to } \mathcal{R}_{2,1/2}, \text{ and let } C_t \text{ be the set of } c^{(i)} \text{'s such that } c^{(i)}_{j} = 1 \text{ for each } t \in \{0, 1\}. \text{ Without loss of generality, the buyers with valuation } v^{y^{(j)}}_{c^{(i)}} \text{ for all } c^{(i)} \in C_t \text{ use the same dominant strategy. Moreover, consider any } c^{(i)} \in C_0 \text{ and any } c^{(i')} \in C_1, \text{ we denote the expected utility over items } [n-1], \text{ the probability of getting item } n, \text{ and the payment achieved by the prescribed dominant strategy for } v^{y^{(j)}}_{c^{(i)}} \text{ by } u_0, p_0 \text{ and } q_0, \text{ respectively, and analogously, we denote } u_1, p_1 \text{ and } q_1 \text{ for } v^{y^{(j)}}_{c^{(i')}}. \text{ Notice that if the buyer
with valuation $v^{y(j)}_{c(i)'}$ plays the strategy for $v^{g(j)}_{c(i)'}$ instead, he will get expected utility $p_1 + u_1$ (because the probability he gets item $n$ is now $p_1$, and the expected utility he gets from $[n-1]$ is now $u_1$) and payment $q_1$. By definition of (almost) dominant strategy, we have the following inequality (the inequality holds approximately when we consider almost truthful-in-expectation protocols, and the error is negligible to the later derivations)

$$p_1 + u_1 - q_1 \leq p_0 + u_0 - q_0.$$  \hspace{1cm} (6)

Moreover, by individual rationality,

$$q_1 \leq \frac{p_1}{2} + u_1,$$  \hspace{1cm} (7)

and it follows that

$$q_0 \leq p_0 + u_0 - u_1 - p_1 + q_1 \quad \text{(Rearranging Eq. (6))}$$

$$\leq p_0 + u_0 - u_1 - p_1 + \frac{p_1}{2} + u_1 \quad \text{(By Eq. (7))}$$

$$= p_0 + u_0 - \frac{p_1}{2}.$$  \hspace{1cm} (8)

Furthermore, because $v^{y(j)}_{c(i)'}$’s for $i \in \{i_1, \ldots, i_m\}$ are distributed like $R_{\ell,\varepsilon^2}$, the sum of the revenues obtained by the protocol tree from the $v^{y(j)}_{c(i)'}$’s for all $i \in \{i_1, \ldots, i_m\}$ is at most

$$\frac{(1 + \eta)m}{2} \cdot q_0 + \frac{(1 + \eta)m}{2} \cdot q_1 \leq \frac{(1 + \eta)m}{2} \cdot (p_0 + u_0 + u_1) \quad \text{(By Eq. (7), (8))}$$

$$\leq \frac{4(1 + \eta)m}{4 - 2\gamma}.$$  \hspace{1cm} (By $p_0 \leq 1$ and $u_0, u_1 \leq \frac{1}{2 - \gamma}$)

Because $v^{y(j)}_{c(i)'}$’s for $i \in \{i_1, \ldots, i_m\}$ are distributed like $R_{\ell,\varepsilon^2}$, the optimal total revenue we can get from all the $v^{y(j)}_{c(i)'}$ for $i \in \{i_1, \ldots, i_m\}$ (which is equal to their total value) is at least $\frac{(1-\eta)m}{2} \cdot \frac{3}{2} + \frac{(1-\eta)m}{2} = \frac{5(1-\eta)m}{4}$, and hence, the average revenue achieved by the protocol tree on valuations $v^{y(j)}_{c(i)'}$ for $i \in \{i_1, \ldots, i_m\}$ is at most $\frac{4(1+\eta)}{(4-2\gamma)(1-\eta)} \cdot \frac{4}{5}$ fraction of the optimum.

\[\square\]

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A more efficient non-IC protocol

In this appendix we explore the complexity of non-IC auction protocols that implement direct revelation (IC) mechanisms, i.e. whenever the Buyer follows the suggested strategy, the expected allocation and payment are still identical to those in the direct revelation mechanism. Yet due to the order in which the information is revealed to the Buyer, he has incentives to deviate from this suggested strategy. This corresponds to [FS09]'s notion of efficiently implementable mechanism (as opposed to the stronger incentivizable mechanism).

**Theorem A.1.** Let \( D \) be any prior over Buyer's combinatorial valuations over \( n \) items bounded by maximum valuation \( U \), and let \( M \) be any direct revelation mechanism. Suppose that for any type and realization of randomness, \( M \) only ever allocates one of \( B \) bundles. Then there exists a non-IC protocol implementing \( M \) using \( O(\log(B)) \) bits of communication.

**Proof.** Similarly to the proof Theorem 4.1, we consider a partition of \([0, 1]\) into \( B \) intervals, where the \( b\)-th interval is of length identical to the probability that \( M \) allocates Bundle \( b \) to the Buyer. Also, as in the proofs of Theorems 3.1 and 4.1 we transform \( M \) into a mechanism \( M' \) with payment 0 or \( U \). Notice that \( M' \) has the same expected payment and allocation as \( M \), so implementing \( M' \) is equivalent to implementing \( M \).
Like our IC auction protocols (Theorems 3.1 and 4.1), the nodes of Chance are parameterized by a uniformly random number \( \tau \in [0, 1] \). We deviate from those IC auction protocols and reveal (prefixes of) \( \tau \) to the Buyer as soon as possible. Specifically, the nodes of Chance iteratively draw the bits in the binary representation of \( \tau \). At each iteration the Buyer can terminate the protocol or ask to reveal \( \tau \) to greater precision. He terminates the protocol if he can determine both: (i) whether \( \tau \) is greater than the probability he has to pay \( U \), and (ii) which interval contains \( \tau \); in this case he simply announces his ex-post payment and allocation (1 bit for the former, \( \lfloor \log(B) \rfloor \) bits for the latter).

Note that if the Buyer follows the suggested strategy, the distributions of allocation and payment are identical to \( \mathcal{M}' \), so this protocol indeed implements \( \mathcal{M}' \).

The analysis of the communication complexity of the protocol is similar to that of Theorems 3.1 and 4.1. The Buyer must be able to determine whether \( \tau \) is greater or smaller than \( B \) numbers (the probability of payment and \( B - 1 \) interval boundaries). He can do this once the prefix of \( \tau \) he received is different than the corresponding \( B \) prefixes. Since \( \tau \) is a uniformly random number its prefix has probability 1/2 of deviating from each of the \( B \) numbers at each iteration (independently across iterations but not across numbers). After \( 2\log(B) \) iterations, the Buyer can terminate the protocol with probability \( \geq 1 - 1/B^2 \). Therefore the number of iterations is \( O(\log(B)) \) (see also Ineq. \( \text{(1)} \)); the last iteration uses \( \log(B) + O(1) \) bits, and any iteration requires before that uses two bits. So the total communication complexity is \( O(\log(B)) \).

To appreciate why the above mechanism is non-truthful, we provide a simple example:

Example A.2 (IC vs non-IC auction protocols). Consider the following mechanism \( \mathcal{M} \) for auctioning a single item: the Buyer pays 1 with probability \( p \) (otherwise zero); he receives the item with probability \( q \); and he can choose any \( p, q \in [0, 1] \) such that \( p = q^2 \). Consider a Buyer type that favors option \( q = 2/3, p = 4/9 \). In the context of our non-IC auction protocol, this induces the partition of \( [0, 1] \) into \( [0, 2/3] \cup (2/3, 1] \) (where the buyer receives the item if \( \tau \) is in the first part). If the Buyer learns that the first bit of \( \tau \) is zero (i.e. \( \tau < 1/2 \)), the Buyer would prefer to report type \( q' = 1/2, p' = 1/4 \) over his true type since he is guaranteed to receive the item anyway.

B Special case: a protocol for the [DDT17] example

In this appendix we prove the concrete (non-asymptotic) bound on the expected number of bits that the buyer sends in the DDT example.

Theorem. (Theorem 3.2 restated) Consider the case of \( n = 2 \) items and the Buyer drawing his valuations i.i.d. from Beta(1,2) (i.e. the distribution on \([0, 1]\) with density function \( f(x) = 2(1-x) \)). Then there is an IC auction protocol obtaining the maximum possible revenue where the Buyer sends less than two bits in expectation.

Proof. The optimal direct revelation mechanism for this distribution is analyzed in detail in [DDT17 Example 3]. We summarize the properties useful for our proof: The optimal mechanism partitions the Buyer’s types into four regions \( \mathcal{Z}, \mathcal{A}, \mathcal{B}, \mathcal{W} \). Buyer types in \( \mathcal{Z} \) pay zero and never receive either item. Buyer types in \( \mathcal{W} \) pay a fixed price (\( P \approx 0.5535 \)) and receive both items. Buyer types in \( \mathcal{A} \) and \( \mathcal{B} \) have a strict preference between the items (different preferences between \( \mathcal{A} \) and \( \mathcal{B} \)); they always receive their most preferred item, and receive their least preferred item with probability \( \pi \) which is always in the range \( \pi \in [1/8, 1/8 + 0.03] \). Their payment is always less than \( P \). (The exact payment and probability of allocation is slightly different for each type in \( \mathcal{A} \cup \mathcal{B} \).)
Range of $\tau$ | probability | expected communication
|-----------------|--------------|------------------|
| $\tau < 1/U$    | $1/U$        | $3 + 2\sqrt{U}$ |
| $1/U < \tau < 1/8$ | $1/8 - 1/U$  | zero            |
| $\tau \in [1/8, 1/8 + 0.03)$ | 0.03         | $3 + O(2^{-\sqrt{U}})$ |
| $\tau > 1/8 + 0.03$ | $7/8 - 0.03$ | 1               |

Table 1: Proof of Theorem 3.2 protocol for $A \cup B$ in the

A protocol for $A \cup B$ We now describe a simple IC auction protocol for the special case where the Buyer’s type is in $A \cup B$; in this mechanism the Buyer sends less than one bit in expectation. We will then show how to use this mechanism to prove the theorem for the general case. We first convert the original mechanism to one where the Buyer always pays either zero or $U$, for some arbitrarily large constant $U$; the payment of $U$ is charged with probability $q < 1/U$ (the exact probability depends on the Buyer’s type).

The nodes of Chance are parameterized by a threshold $\tau$ drawn uniformly at random from $[0, 1]$. At the end of the protocol, the Buyer should always receive the more desired item, receive the less desired item iff $\tau < \pi$, and pay $U$ iff $\tau < q$. We now consider the following protocols, depending on $\tau$ (see summary in Table 1):

- If $1/U < \tau < 1/8$ (which happens with probability arbitrarily close to $1/8$), the Buyer always receives both items and pay nothing. In this case the protocol can terminate with zero communication from the Buyer.

- If $\tau > 1/8 + 0.03$, the Buyer that will receive one item and pay nothing. The buyer sends one bit to choose which item he prefers.

- Otherwise, i.e. if $\tau \in [1/8, 1/8 + 0.03)$ or $\tau < 1/U$, the Buyer needs to send more refined information about his valuation. First, the Buyer sends his preferred item (one bit of information). Then, the Buyer enters a sub-protocol similar to the proof of Theorem 3.1. The sub-protocol is identical to both cases as the Buyer must not learn any more information about $\tau$.

  - If $\tau \in [1/8, 1/8 + 0.03)$, the Buyer’s suggested strategy is to send at each round the next bit in the binary representations of $\pi - 1/8_{0.03} \in [0, 1)$. Every $\sqrt{U}$ rounds, the Buyer should also send the next bit in the binary representation of $qU \in [0, 1)$. The protocol terminates once the Buyer sent enough information to determine whether $\pi > \tau$. Conditioning on being in this case, this happens with probability $1/2$ at each iteration, so this part of the protocol lasts $2$ rounds in expectation. In total the Buyer expects to send just over $3$ bits in this case: $1$ for choosing the item, $2$ bits in expectation from the prefix of $\pi-1/8_{0.03}$, and $O(2^{-\sqrt{U}})$ bits in expectation from the prefix of $qU$.

  - If $\tau < 1/U$, the Buyer’s suggested strategy is as above, but the protocol terminates once it learns whether $q > \tau$. In this case the buyer must waste $\sqrt{U}$ bits for every bit from the prefix of $qU$ that he actually sends. His total expected communication $1 + 2(\sqrt{U} + 1) = 3 + 2\sqrt{U}$.

The Buyer’s total expected communication is therefore at most

$$\left(\frac{1}{8} - \frac{1}{U}\right) \cdot 0 + \left(\frac{7}{8} - 0.03\right) \cdot 1 + 0.03 \cdot \left(3 + O(2^{-\sqrt{U}})\right) + \frac{1}{U} \cdot O(\sqrt{U}) = 0.935 + O(\sqrt{1/U}) < 0.94.$$
A protocol for the general case  For the general case, we add a preliminary stage of communication where the Buyer needs to communicate whether his type is in $Z$, $W$, or $A \cup B$. The encoding for each of those three options is chosen at random as follows:

- With probability 0.98, the Buyer should use 00 when his type is in $Z$, and 01 when his type is in $W$; the signal 1 is reserved for $A \cup B$, which then follows by the above 0.94-bit protocol for $A \cup B$.

- With probability 0.01 the Buyer should use 00 when his type is in $Z$, and 01 when his type is in $A \cup B$; the signal 1 is reserved for $W$. (Signal 01 is followed by the above 0.94-bit protocol.)

- With probability 0.01 the Buyer should use 00 when his type is in $A \cup B$, and 01 when his type is in $W$; the signal 1 is reserved for $Z$. (Signal 00 is followed by the above 0.94-bit protocol.)

For each of $Z, W$, the Buyer needs to send two bits with total probability 0.99, and one bit with probability 0.01. Hence his total expected communication is 1.99 bits. For $A \cup B$, the Buyer sends in expectation 1.02 bits during the preliminary stage, so his total expected communication is bounded by 1.96 bits.

C  Approximately ex-post IR protocols

In this section we show a generic transformation of IC (and interim IR) auction protocols to approximately ex-post IR auction protocols. The main idea is to allow the Buyer, before every node of Chance of the protocol, to hedge against the risk posed by the randomness of this node.

Definition C.1. We say that an auction protocol is ex-post $\varepsilon$-IR if it is IC, and whenever the Buyer follows the suggested Buyer’s strategy for his valuation type, at the end of the protocol his payment is at most $\varepsilon$ greater than his value for the allocation.

Theorem C.2. Let $\mathcal{D}$ be any prior over Buyer’s combinatorial valuations over $n$ items upper bounded by $U$. Given an IC auction protocol $\mathcal{P}$, we can transform it into an IC and ex-post $\varepsilon$-IR auction protocol $\mathcal{P}'$ with the same expected payment and allocation; for any random seed $\pi$, if the communication complexity of $\mathcal{P}$ is $C$, then the communication complexity of $\mathcal{P}'$ with the same random seed is bounded by $O(C \log(U/\varepsilon) + C^2)$.

Proof. We consider the communication protocol tree associated with $\mathcal{P}$ and iteratively transform it into $\mathcal{P}'$ starting from the root. At the beginning of the protocol, we ask the Buyer to (approximately) specify his utility (value for the allocation he will receive minus payment he will be charged) at the end of auction protocol, in expectation over protocol’s randomness and assuming he honestly follows the suggested Buyer’s strategy. Before each node of Chance, we ask the Buyer to (approximately) specify his expected utility conditioned on each outcome of the node of Chance. For now we describe and analyze the rest of the transformation as if the Buyer exactly specifies his values, and analyze the accumulated error due to finite precision later.

Let $\mathcal{U}$ denote the Buyer’s expected utility at the beginning of the protocol. Since the protocol is interim IR, $\mathcal{U} \geq 0$. At each node of Chance we constrain the probability-weighted sum of expected utilities reported by the Buyer to be exactly $\mathcal{U}$. For each outcome $x$ of the node of Chance, let $\mathcal{U}_x$ denote the expected utility reported by the Buyer conditioned on this outcome. We add $\mathcal{U}_x - \mathcal{U}$ to the payment in every leaf of the sub-tree corresponding to $x$. We now make the following observations about this transformation:
1. The transformation does not change the Buyer’s incentives conditioned on $x$ since we add the same amount to the payment in every leaf; in particular, if the sub-protocol conditioned on $x$ was IC before the transformation, it remains IC after the transformation.

2. The transformation changes the Buyer’s expected utility conditioned on $x$ from $U_x$ to $\bar{U}$ (here we use that by the previous observation, the Buyer continues to follow the suggested Buyer’s strategy).

3. In expectation over the randomness in this node of Chance, this transformation has zero net effect over the Buyer’s payment (here use the constraint that the weighted sum of $U_x$’s is $\bar{U}$). In particular, the Buyer has no incentive to misreport his utilities for the respective outcomes. (In fact, the risk-neutral Buyer is completely indifferent between any valid report of $U_x$, so even when we later constrain the communication to finite precision, he has no incentive to deviating from reporting the best approximation.) Furthermore, this guarantees that the Buyer’s incentives for arriving at this node of Chance did not change by the transformation.

Using these observations, it follows by induction that the auction protocol remains IC after the transformation, and the Buyer always has ex-post utility $\bar{U} \geq 0$.

We now revisit this transformation with restriction to finite precision. At the $i$-th node of Chance, we ask the Buyer to report $U_x$’s to within $\pm \varepsilon 2^{-i}$. Then the total error accumulated on any path of the protocol is always bounded by $\pm \varepsilon$. The communication complexity is $\sum_{i=1}^{C} \log(U/\varepsilon) + i = O(C \log(U/\varepsilon) + C^2)$.

\section{Approximately optimal revenue with finite valuations}

In this appendix, per the request of a reviewer, we outline a very short proof of the following proposition: For additive valuations over independent items, assuming a finite bound on the Buyer’s valuation has an arbitrarily small impact on revenue. Note that [BGN17] proved a stronger result with an asymptotic analysis of this finite bound; however we believe that our proof is simpler.

We prove the proposition for the case where the optimal revenue is finite; when the optimal revenue is infinite, the supremum revenue of trivial mechanisms like only auctioning the grand bundle of all items is also infinite [HN17], so the discussion of menu-size or communication complexity is less interesting.

\begin{proposition}
Let $\mathcal{D}$ be any prior over (possibly unbounded) Buyer’s additive valuations over $n$ independent items, and let $\mathcal{M}$ be any mechanism that obtains finite revenue on $\mathcal{D}$. For every $\varepsilon$ there exists $U(\varepsilon)$ such that the same mechanism ($\mathcal{M}$) obtains $(1 - \varepsilon)$-fraction of its revenue if the Buyer’s valuations are capped at $U(\varepsilon)$.

\end{proposition}

\begin{proof}
We consider the partitioning of the type-space (aka $\mathbb{R}_{\geq 0}^n$) into countable (but infinite) hyperrectangles:

$$\left(\{0\} \cup \{[2^i, 2^{i+1})\}_{i \in \mathbb{Z}}\right)^n.$$

For each hyperrectangle, we consider the contribution to $\mathcal{M}$’s revenue of types in this hyperrectangle. We arrange the hyperrectangles in decreasing order of contributions to revenue (breaking ties arbitrarily). $\mathcal{M}$’s total revenue is the (countable) sum of contributions from hyperrectangles in this sequence. Since the sum is finite, it can be approximated to within $(1 - \varepsilon)$-factor by the first $N(\varepsilon)$ terms in the sequence, for some finite $N(\varepsilon)$. Hence it suffices to take $U(\varepsilon)$ to be $n \cdot 2^{i+1}$ for the maximal $i$ used in any of the hyperrectangles in those first $N(\varepsilon)$ terms.

\end{proof}

\section*{Appendix D

Approximately optimal revenue with finite valuations}

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\end{proof}