1 Introduction

The notion of modular deformation helps to define a canonical analytic structure in the category of analytic objects. In this paper, we discuss modular deformations of complex analytic spaces. The category of germs of complex spaces is not appropriate for this purpose since the fiber of a deformation of a germ is no more a germ. Instead, we consider the category of analytic polyhedrons (see Sec.4). Let \( f : \mathcal{X} \to S \) be a deformation of an analytic space or of a polyhedron. Take a point \( \circ \) in the base \( S \) and consider the pair \((S, \circ)\) as a germ of complex space. A modular stratum is an analytic subspace \((M, \circ) \subset (S, \circ)\) such that the uniqueness property holds: any germ morphisms \( g : R \to (S, \circ) \) and \( h : R \to (M, \circ) \) induce isomorphic deformations \( f \times_S g \cong f \times_S h \) only if \( g = h \). The modular deformation is the restriction of \( f \) to \( M \), that is the deformation \( f_M \equiv f \times_S M \). Suppose that \( f : \mathcal{X} \to (S, \circ) \) is a versal deformation of the fiber \( X_\circ = f^{-1}(\circ) \) and \( M \) is the maximal modular stratum. It has the special feature: for any other versal deformation \( f' : \mathcal{X}' \to S' \) of \( X \) and an isomorphism \( i : S \to S' \) such that \( i \times f' \cong f \), the restriction of \( i \) to \( M \) is uniquely defined and \( M' \equiv i(M) \) is the maximal modular stratum in \( S' \). The isomorphism \((i \times f')_M \equiv f_M \) is also canonically defined. Moreover, the maximal modular stratum \( M \) (if it exists) possesses the semi-local property: for any point \( s \in M \) close to the marked point \( \circ \) the germ \((M, s)\) is also maximal modular for the fiber \( X_s \cong f^{-1}(s) \).

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Therefore two versal deformations \( f \) and \( f' \) can be glued together (amalgamated) along its modular strata, say \( M \) and \( M' \), provided there is given an isomorphism \( \phi : f^{-1}(s) \to f'^{-1}(s') \) for \( s \in M, s' \in M' \). This gives rise to a flat morphism \( F : X \to M \sqcup M' \) where \( M \sqcup M' \) denotes the amalgam over some open neighborhood of the points \( s \) and \( s' \). Repeating this construction, we can get a flat morphism of complex spaces \( F : X \to M \) which is a maximal modular deformation of each its fiber. This conception is close to that of fine moduli space in the sense of Mumford, [14]. For each point \( s \in M \), the automorphism group of the space \( X_s \) acts on \( M \) which may possible further amalgamation and the global amalgam may have complicated structure.

We formulate here some general results on existence and properties of maximal modular deformations in the category of complex analytic polyhedra and discuss several examples of modular deformation of isolated singularities. V.Arnold’s classification table of hypersurface singularities [3] is a rich source of such examples. For many examples of complete intersection singularities modular deformations are contained in the paper of A. Aleksandrov [1], further study see in [2].

Several complicated examples, which have been beyond the reach of ‘by hand’ calculations, were computed and studied by B.Martin [12] and by T. Hirsch and B. Martin [5]. These authors applied a specialized computer algebra program based on SINGULAR. We shall see how these new local modular families glue together and global modular families emerge. In several cases the base of a modular family can be made compact by gluing together sufficiently many local modular deformations.

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2 Rudiments of the deformation theory

Remind some basic definitions. A germ of complex analytic spaces (which will be called simply ‘germ’) is a pair \((Z, \mathcal{O}(Z))\), where \( Z \) is the germ of complex analytic set in \((\mathbb{C}^n, 0)\) for some \( n \) and \( \mathcal{O}(Z) \) is the coherent sheaf of analytic \( \mathbb{C} \)-algebras on \((\mathbb{C}^n, 0)\) such that \( \text{supp} \mathcal{O}(Z) = Z \). We use the notation \( 0 \) for the marked point in \( \mathbb{C}^n \) and in \( Z \). Any morphism of germs \((W, \mathcal{O}(W)) \to (Z, \mathcal{O}(Z))\) is a pair \((f, \phi)\) where \( f : (W, 0) \to (Z, 0) \) is a mapping of germs of analytic sets and \( \phi : f^*(\mathcal{O}(Z)) \to \mathcal{O}(W) \) is a morphism of sheaves of analytic \( \mathbb{C} \)-algebras over \( W \). Restricting \( \phi \) to the marked point yields the morphism of analytic \( \mathbb{C} \)-algebras \( \phi_0 : \mathcal{O}_0(Z) \to \mathcal{O}_0(W) \). The morphism
\((f, \phi)\) is called embedding, if \(\phi_o\) is a surjection. Vice versa, for any analytic \(\mathbb{C}\)-algebra \(A\) there exists and is uniquely defined the germ \((Z, \mathcal{O}(Z))\) such that \(A \cong \mathcal{O}_o(Z)\). In particular, the algebra corresponding to the simple point \(\circ\) is the field \(\mathbb{C}\). For an arbitrary germ \(Z\) there is the canonical embedding \(\circ \to Z\); the corresponding algebra homomorphism is the canonical morphism \(\mathcal{O}_o(Z) \to \mathbb{C}\). The kernel of this morphism is the maximal ideal in the algebra \(A = \mathcal{O}_o(Z)\); it is denoted \(\mathfrak{m}(Z)\). The germ \(Z = \circ\) is called simple point; a germ \(Z\) is called fat point, if the corresponding algebra has a finite dimension over \(\mathbb{C}\).

The fiber product operation is well defined in the category of analytic spaces; it is denoted \(f \times_X g\) or \(Y \times_X Z\) for germ morphism \(f : Y \to X, g : Z \to X\). If \(g\) is an embedding, we call the product \(f \times_X g\) restriction \(f\) to \(Z\) and denote it \(f | Z\). Let \(V\) be a \(\mathbb{C}\)-vector space of finite dimension; we denote by \(\mathbb{D}[V]\) the \(\mathbb{C}\)-algebra that is isomorphic to \(\mathbb{C} \oplus V\) as vector space with the multiplication rule \((a, u) \cdot (b, v) = (ab, av + bu)\). This is an analytic algebra of a fat point denoted \(P(V)\). All germs of this kind form a category denoted \(D\), where for any objects \(P(U), P(V)\) the set \(\text{Hom}(P(U), P(V))\) is bijective to \(\text{Hom}_{\mathbb{C}}(U, V)\). This induces a natural structure of vector space over \(\mathbb{C}\) in the set \(\text{Hom}(P(U), P(V))\); the composition of morphisms is a bilinear operation. For any germ \(Z\) there is defined the germ \(T(Z) \in D\) and the embedding \(t(Z) : T(Z) \to Z\) such that \(\mathcal{O}(T(Z)) = \mathcal{O}(Z)/\mathfrak{m}^2(Z)\). The germ \(T(Z)\) is called the tangent space of the germ \(Z\).

**Definition 1.** Let \(X\) be a complex analytic space. A **deformation** of \(X\) over a germ \((Z, \circ)\) is a pair \((f, i)\), where \(f : X \to (Z, \circ)\) is a flat morphism and \(i : X \to f^{-1}(\circ) = f \circ \circ\) is an isomorphism of complex analytic spaces. For any morphism of germs \(h : W \to Z\) and a deformation \((f, i)\) of \(X\) over \(Z\) the fiber product \(f \times_Z h : X \times_Z W \to W\) is also flat. The pair \((f \times_Z h, ji)\) is a deformation of \(X\) with base \(W\), where \(j : f^{-1}(\circ) \cong (f \times_Z h)^{-1}(\circ)\) is the canonical bijection. In the same way, one can treat deformations of \(\mathbb{Z}_{2}\)-graded, \(\mathbb{Z}\)-graded analytic spaces, deformations of germs, of fiber bundles, of coherent sheaves and the like.

A **versal** deformation of the space \(X\) is a deformation \((f, i), f : X \to S\) such that for any deformation \((g, j), g : Y \to R\) there exists a morphism of germs \(h : R \to S\) and a isomorphism \(\gamma : X \times_S R \to Y\) over \(R\) such that the
A pair \((f, i)\) is called \textit{universal} deformation, if the morphism \(h\) is unique. The above definition can be applied for the category \(D\). Suppose that a space \(X\) possesses a universal deformation \(\delta : X \to T_X\) in the category \(D\). Take an arbitrary germ \(S\) and a deformation \((f, i)\) of \(X\) over \(S\); consider the morphism \(f_T = f \times_s T_0(S)\). It is a deformation of \(X\) with the base \(T_0(S) \in D\). Due to the universality property of \(\delta\) a morphism \(D_0 f : T_0(S) \to T_X\) is defined in the category \(D\) such that \(D_0 f \times \delta \cong f_T\). It is called \textit{Kodaira-Spencer} mapping; this mapping is linear since it belongs to \(D\). Let \((f, i)\) be a versal deformation of \(X\) with a base \(S\). There exists a morphism \(t : T_X \to S\) such that \(f \times_s t \cong \delta\). Then \(D_0 f \cdot t = \text{id}\) since \(\delta\) is universal, consequently \(D_0 f\) is surjective. The pair \((f, i)\) is called \textit{miniversal} (or minimal versal), if \(D_0 f\) is a injection, hence a bijection.

**Definition 2.** \cite{16, 17} Let \(f\) be a deformation of \(X\) as above. A subgerm (stratum) \(M \subset S\) is called \textit{modular}, if for any morphisms of germs \(h : R \to M, g : R \to S\) the equation \(f \times_s h = f \times_s g\) implies that \(h = g\). A modular stratum \(M \subset S\) is called \textit{maximal}, if it contains any other modular stratum. If \(f : X \to S\) is a miniversal deformation and \(M\) is the maximal modular stratum, the restriction \(f_M\) is called maximal modular deformation of \(X\). Any universal deformation is, of course, maximal modular.

A similar conception (prorepresenting stratum) was introduced in \cite{9} for the formal deformation theory of affine schemes. See \cite{10} for further results. The notion of modular deformation was also treated in \cite{8} in a more general setting.

**Proposition 2.1** Let \(f : X \to S\) and \(f' : X' \to S'\) be miniversal deformations of a space \(X\) and \(M, M'\) are respective maximal modular strata. There exists a isomorphism \(h : M' \to M\) such that \(f'|M' \cong f|M \times h\). The morphism \(h\) is unique.

\[\begin{align*}
\begin{array}{ccc}
X & \times_S M & \to Y \\
\gamma & \times_{R^0} & \to j \\
\tau & \times & \\
X & \to & \\
\end{array}
\end{align*}\]

There exist morphisms \(g : S' \to S\) and \(g' : S \to S'\) such that \(f' \times g' \cong f, f \times g \cong f'.\) We have \(D_0 f \cdot dg = D_0 f'\) and \(D_0 f' \cdot dg' = D_0 f.\) The differentials \(dg, dg'\) are inverse one to another, since \(D_0 f, D_0 f'\) are bijections. Therefore
g is an isomorphism of germs and the mapping \( g : M' \to S \) is a modular germ. It is factorized through a uniquely defined morphism \( h : M' \to M \), since \( M \) is maximal modular.

3 Tangent cohomology and criterion of a modular stratum

Let \( g : X \) be a complex analytic space; the graded tangent sheaf \( \oplus_{q=0}^{\infty} T^q(X) \) is defined on \( X \). It is a \( \mathbb{Z}_+ \)-graded sheaf algebra \( \text{Lie} \), which means that there is defined a bracket operation that satisfies the graded commutation and Jacobi identities. Moreover, this has a natural structure of \( \mathcal{O}(X) \)-module which agrees with the Lie algebra structure in a natural way. For any \( q \geq 0 \) the sheaf \( T^q \) is a coherent \( \mathcal{O}(X) \)-sheaf. In particular, the term \( T^0(X) \) is the sheaf of tangent fields on \( X \).

The global tangent cohomology \( T^*(X) = \sum_{0}^\infty T^n(X) \) is a \( \mathbb{Z}_+ \)-graded algebra and there is a spectral sequence \( E^p \) that converges to the tangent cohomology \( T^*(X) \) with the second term \( E^{pq}_2 = H^p(X, T^q(X)) \), see [15]. The term \( T^0(X) \) is the Lie algebra of tangent fields on \( X \). If the term \( T^1(X) \) is of finite dimension, it represents the base \( T_X \) of the universal deformation \( \delta \) of \( X \) in the category \( D \) as in Sec.2. If \( g : X \to Y \) is a morphism of complex spaces, a vertical tangent field \( t \) on \( g \) is a tangent field on \( X \) with the property \( t(g^*(a)) = 0 \) for an arbitrary \( a \in \mathcal{O}(Y) \). The notation \( T^0(X/Y) \) means the space of vertical tangent fields. For any point \( y \in Y \) the restriction mapping \( T^0(X/Y) \to T^0(X_y) \) is canonically defined where \( X_y = g^{-1}(y) \) is the complex subspace of \( X \). There is a general criterion of modularity:

**Theorem 3.1** Let \( f : \mathcal{X} \to (S, \circ) \) be a deformation of a complex analytic space \( X \). Then

(i) the simple point \( \circ \in S \) is modular, if and only if the Kodaira-Spencer mapping \( D_\circ f \) is injective;
(ii) if \( D_\circ f \) is injective, then a subgerm \( M \subset S \) is modular, if for any fat point \( Z \subset M \) the restriction mapping \( T^0(\mathcal{X} \times_M Z/Z) \to T^0(X) \) is surjective; if \( D_\circ f \) is bijective, this condition is necessary as well;
(iii) \( T_\circ(M) \) coincides with the space of tangent vectors \( t \in T_\circ(S) \) that satisfy the equation \( [D_\circ f(t), v] = 0 \) for any \( v \in T^0(X) \).

A proof is similar to that of [17], Proposition 5.1.
4 Analytic polyhedrons

The existence of maximal modular deformation and some its properties were stated in [17] for an arbitrary compact complex analytic space. Here we formulate similar results for analytic polyhedra. First we recall some definitions of [18].

Definition 3. For an arbitrary integer $n$ we fix a coordinate space $\mathbb{C}^n$, i.e. a complex vector space with a marked system of linear coordinates $w_1, \ldots, w_n$. Denote by $D^n$ the closed unit polydisk in $\mathbb{C}^n$. A complex analytic $n$-polyhedron is a pair $(X, \varphi)$, where $X$ is a complex analytic subspace of a complex space $\tilde{X}$ and $\varphi : \tilde{X} \to \mathbb{C}^n$ is a holomorphic mapping (called a barrier map) such that the set $\varphi^{-1}(\bar{D}^n)$ is compact and $X = \varphi^{-1}(D^n)$. The neighborhood $\tilde{X}$ of $X$ can be contracted, i.e. $X$ is thought as the germ of a complex analytic space $\tilde{X}$ on the compact set $\tilde{X} = \varphi^{-1}(\bar{D}^n)$. The set $\partial X = \varphi^{-1}(\partial D^n)$ is the boundary of the polyhedron $X$. A morphism of polyhedra $(X, \varphi) \rightarrow (Y, \psi)$ is a pair $(f, p)$, where $f : \tilde{X} \rightarrow \tilde{Y}$ of germs of complex spaces and $p : \mathbb{C}^n \rightarrow \mathbb{C}^m$ is a coordinate projection that make the commutative diagram: $p\varphi = \psi f$.

Let $S$ be a complex analytic spaces; we call a relative analytic $n$-polyhedron (r.p.) over $S$ any pair $(X, \varphi)$, where $X$ is an open subspace a complex space $\tilde{X}$ and $\varphi : \tilde{X} \to \mathbb{C}^n \times S$ is a holomorphic mapping such that the set $\varphi^{-1}(\bar{D}^n \times S)$ is proper over $S$ and $\varphi^{-1}(D^n \times S) = X$. Let $(X, \varphi)$ be a r.p. over $S$ and $h : R \rightarrow S$ be a morphism of complex spaces. The fiber product $(X_R, \varphi_R) = (X, \varphi) \times_S R$ is a r.p. over $R$. In particular, for any point $s \in S$ the fiber of $(X, \varphi)$ is defined as the product $(X_s, \varphi_s) = (X, \varphi) \times_S s$, which is an (absolute) analytic polyhedron, called the fiber of $(X, \varphi)$.

A morphism $(X, \varphi) \rightarrow (Y, \psi)$ of relative polyhedra over $S$ is any pair $(f, p)$ of holomorphic mappings of germs $f : \tilde{X} \rightarrow \tilde{Y}$, $p : \bar{D}^n \times S \rightarrow \bar{D}^m \times S$ such that $p\varphi = \psi f$ and $p$ commutes with the projections to $S$. If $S = \circ$ this is a definition of morphism of (absolute) polyhedra.

Definition 4. Let $S$ be a germ of complex space with the marked point $\circ$, and $(X, \varphi)$ be a polyhedron; we call a deformation of this polyhedron with the base $S$ any triple $(\mathcal{X}, \Phi; \theta)$, where

(i) $(\mathcal{X}, \Phi)$ is a relative polyhedron over $S$;
(ii) $(\mathcal{X}, \Phi)$ is flat over $S$, i.e. the composition $\pi\Phi : \mathcal{X} \rightarrow \mathbb{C}^n \times S \rightarrow S$ is a flat morphism of complex analytic spaces, where $\pi$ is the projection.
(iii) $\theta : (X, \varphi) \rightarrow (\mathcal{X}, \Phi) \times_S \{\circ\}$ is an isomorphism of polyhedra. A deformation $(\mathcal{X}, \Phi; \theta)$ of a polyhedron $(X, \varphi)$ is called versal, if for any deformation
of the same polyhedron with a base $R$ there exist a morphism of germs $h : R \to S$ and a isomorphism $\alpha : (Y, \Psi) \to (X, \Phi) \times_S R$ such that $(\alpha \times \circ) \theta = \eta$. Definitions of modular stratum, maximal modular deformation of an analytic polyhedron are given in the same lines as in Definition 2.

5 Versal and modular deformations

Let $\mathbb{C}^n$ be again a coordinate space and $D^n$ be the closed unit polydisk. For an arbitrary coordinate projection $p : \mathbb{C}^n \to \mathbb{C}^m$ we have $p(D^n) = D^m$ and the sheaf $\mathcal{O}(D^n)$ is endowed with a structure of module over the sheaf algebra $\mathcal{O}(D^m)$.

**Theorem 5.1** Let $(X, \phi)$ be an analytic $n$-polyhedron such that:

(i) for any coordinate projection $p : \mathbb{C}^n \to \mathbb{C}^m$ the equations hold:

$$\text{Tor}^k_{\mathcal{O}(D^m)}(T^k_{\phi} + 1, \mathbb{C}) = 0, \ k = 0, 1, \ldots, m$$

in each point of the set $(\partial D)^m \times D^{n-m}$, where $T^* = T^*(X)$ and

(ii) $\phi$ is finite over $\partial D^n$. The polyhedron $(X, \phi)$ has a miniversal deformation $F : X \to (S, 0)$ where $S \cong g^{-1}(0)$ and $g : (T^1(X), 0) \to T^2(X)$ is a holomorphic mapping of germs such that $g(0) = 0$, $dg(0) = 0$.

**Remark.** The conditions (ii) are fulfilled for the trivial barrier $\phi = 0$, since $\mathcal{O}(D^0) = \mathbb{C}$. This implies the existence of miniversal deformation for any compact complex space. This case was studied by H.Grauert and other authors, see the survey [15]. Theorem 5.1 was proved in [18]. Note that the space $X$ that satisfies (ii) need not to be smooth at the boundary of the polyhedron. The condition (ii) for $k = 0$ means that the sheaf $T^1_{\phi}$ vanishes on the boundary of $D^n$, hence the support of this sheaf is a compact subset of the interior of $D^n$. The spectral sequence $E_2^{pq} = H^p(X, T^q) \Rightarrow T^*(X)$ yields

$$0 \to H^1(X, T^0) \to T^1(X) \to \Gamma(X, T^1) \to H^2(X, T^0) \to \ldots$$

The term $\Gamma(X, T^1)$ is of finite dimension since $T^1$ is supported by a compact set. From the Leray sequence we find the exact sequence

$$H^1(X, T^0) \to H^0(D^n, \mathcal{R}^1 \varphi (T^0)) \xrightarrow{d_3} H^2(D^n, \mathcal{R}^0 \varphi (T^0)) \oplus H^1(D^n, \mathcal{R}^0 \varphi (T^0))$$

(2)
The sheaves $\mathcal{R}^q\varphi(\mathcal{T}^0), q = 0, 1$ are coherent by Grauert’s Theorem, hence the terms $H^1$ and $H^2$ vanish in the right-hand side. Therefore we have

$$H^1(X, \mathcal{T}^0) \cong H^0(D^n, \mathcal{R}^0\varphi(\mathcal{T}^0)).$$

The sheaf $\mathcal{R}^1\varphi(\mathcal{T}^0)$ vanishes at the boundary of the polydisk since of (i). This yields that $H^1(X, \mathcal{T}^0)$ has finite dimension. Finally $\tau = \dim_\mathbb{C} T^1(X) < \infty$.

The second condition of Theorem 5.1 can be weaken as follows (ii)’ $\mathcal{R}^1\varphi(\mathcal{T}^0)|_{\partial D^n} = 0$.

**Theorem 5.2.** Let $(X, \varphi)$ be a polyhedron as in the previous theorem and $f : \mathcal{X} \to (S, \circ)$ be its miniversal deformation. Then there exists a neighborhood $S'$ of the marked point $\circ$ in $S$ and a closed subspace $M \subset S'$ such that:

(i) for any $s \in M$ the germ $(M, s)$ is maximal modular for the deformation $f$ of the fiber $X_s$.

(ii) the restriction mapping $T^0(\mathcal{X} \times_S M/M) \to T^0(X)$ is surjective and $M$ is maximal with this property;

(iii) the Zariski tangent space $T_0(M)$ is the space of all vectors $t \in T_0(S)$ satisfying the equation $[D_0 f(t), v] = 0$ for any $v \in T^0(X)$;

(iv) the support of $M$ is the set of points $s \in S'$, where the mapping $D_s f : T_s(S) \to T^1(X_s)$ is injective.

The statement (i) is essential; it means that any point of $M'$ can be taken as marked one, hence it is a globally defined complex analytic space. A proof can be done on the lines of [17] and [18]. The parts (ii) and (iv) then follow from Theorem 3.1(i). The part (iii) is a corollary of Theorem 3.1(iii), since the bracket $[Df(t), v]$ is for any $t \in T_0(S)$ the first obstruction for extension of $v$ to a tangent field on $\mathcal{X} \times_S M/M$. The statement (iv) implies that (v) if $S$ is regular, then $\text{supp} M$ is given by the equation $\dim T^1(X_s) = \dim S, s \in S'$.

**Definition 6.** Take a versal deformation $(\mathcal{X}, S)$ as in Theorem 5.1 and the maximal modular stratum $M \subset S$ as in Theorem 5.2. We call $f|M : \mathcal{X} \times_S M \to M$ maximal modular deformation. Let $\mathcal{M}$ be a complex analytic space and $Y$ be an analytic polyhedron over $\mathcal{M}$; we call a modular family the mapping $Y \to \mathcal{M}$, if it is maximal modular deformation of each fiber $Y_s, s \in \mathcal{M}$. 

6 Shrinking a polyhedron

Let \((g,q) : (Y,\psi) \to (X,\varphi)\) be a morphism of analytic polyhedra. We call it *imbedding*, if \(g\) is an imbedding, \(q\) is a coordinate projection and there exists a proper holomorphic mapping \(\rho : X \to \mathbb{C}^n\) that makes the commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow \psi & \nearrow \rho & \downarrow \varphi \\
\mathbb{C}^n & \xrightarrow{q} & \mathbb{C}^m
\end{array}
\]

In geometric terms, this means that \(\varphi_i = q_i \rho\) for \(i = 1, \ldots, m\) and

\[
Y = \{ x \in X, |\psi_{m+1}(x)| < 1, \ldots, |\psi_n(x)| < 1 \}
\]

for an appropriate numeration of coordinates in \(\mathbb{C}^n\).

**Proposition 6.1** Let \((g,q,\rho)\) be an imbedding of polyhedra such that

(i) \(\text{supp} \mathcal{T}^1(X) \subset g(Y)\) and

(ii) the mapping \(\rho : X \setminus g(Y) \to D^m \times \mathbb{C}^{n-m} \setminus D^{n-m}\)

is finite. Then we have \(T^1(Y) \cong T^1(X)\).

**Proof.** Compare cohomology of tangent sheaves in \(X\) and \(Y\). Consider the diagram

\[
\begin{array}{ccc}
H^1(Y, \mathcal{T}^0(Y)) & \cong & H^0(D^n, R^1\psi(\mathcal{T}^0(Y))) \\
\downarrow h_1 & & \downarrow i \\
H^1(X, \mathcal{T}^0(X)) & \cong & H^0(D^m, R^1\varphi(\mathcal{T}^0(X)))
\end{array}
\]

\[
\begin{array}{ccc}
& & H^0(D^n, R^0qR^1\psi(\mathcal{T}^0(Y))) \\
\downarrow \gamma & & \downarrow j \\
& & H^0(D^m, R^0qR^1\rho(\mathcal{T}^0(X)))
\end{array}
\]

The left-side isomorphisms follow from (2). The Leray spectral sequence

\[
H^r(D^m, R^s q_* (\mathcal{F})) \Rightarrow H^{r+s}(D^n, \mathcal{F}), \mathcal{F} \cong R^1\psi(\mathcal{T}^0(Y))
\]

yields the isomorphism \(i\), since the sheaf \(\mathcal{F}\) is coherent. We have \(R^r q R^0 \rho(\mathcal{T}^0(X)) = 0\) for \(r = 1, 2\) since the sheaf \(R^0 \rho(\mathcal{T}^0(X))\) is coherent in \(D^m \times \mathbb{C}^{n-m}\) and \(q\) is a coordinate projection. By applying the Leray sequence to the composition \(\varphi = q\rho\) we obtain the isomorphism in \(D^n\)

\[
R^1\varphi(\mathcal{T}^0(X)) = R^0qR^1\rho(\mathcal{T}^0(X))
\]
This yields the bijection \( j \). Further we have

\[
R^1 \psi (\mathcal{T}^0 (Y)) = R^1 \rho g (\mathcal{T}^0 (Y)) = R^1 \rho R^0 g (\mathcal{T}^0 (Y)) = R^1 \rho (\mathcal{T}^0 (X) | g (Y))
\]

since \( R^1 g = 0 \). On the other hand, the sheaf \( R^1 \rho (\mathcal{T}^0 (X)) \) vanishes in \( X \setminus g (Y) \) since \( \rho \) is finite there. Therefore the right-hand side is isomorphic to the sheaf \( R^1 \rho (\mathcal{T}^0 (X)) \). This yields the bijection \( \gamma \) and also the bijection \( h_1 \). Consider the commutative diagram

\[
\begin{array}{cccc}
\text{Ker} \; d_2 (Y) & \rightarrow & H^0 (Y, \mathcal{T}^1 (Y)) & \xrightarrow{d_2 (Y)} & H^2 (Y, \mathcal{T}^0 (Y)) \\
\downarrow k_2 & & \downarrow h_0 & & \downarrow h_2 \\
\text{Ker} \; d_2 (X) & \rightarrow & H^0 (X, \mathcal{T}^1 (X)) & \xrightarrow{d_2 (X)} & H^2 (X, \mathcal{T}^0 (X))
\end{array}
\]

The isomorphism \( h_2 \) is defined similarly to \( h_1 \) and the bijective mapping \( h_0 \) follows from (i). This diagram yields the isomorphism \( k_2 \). This together with \( h_1 \) gives

\[
T^1 (Y) \cong H^1 (Y, \mathcal{T}^0 (Y)) \oplus \text{Ker} \; d_2 (Y) \cong H^1 (X, \mathcal{T}^0 (X)) \oplus \text{Ker} \; d_2 (X) \cong T^1 (X) \, .
\]

**Theorem 6.2** Let \((Y, \psi)\), \((X, \varphi)\) be analytic polyhedra that fulfil the conditions of Theorem 5.1 and \((X, \Phi, S_X)\) and \((Y, \Psi, S_Y)\) be their versal deformations. Let \((g, q, \rho) : (Y, \psi) \rightarrow (X, \varphi)\) be an imbedding of analytic polyhedra that fulfil the conditions (i), (ii) of Proposition 6.1. Then there exists an imbedding of relative polyhedra \((G, Q, R) : (Y, \Psi, S_Y) \rightarrow (X, \Phi, S_X)\) such that \(G \times \circ = g, Q \times \circ = q, R \times \circ = \rho\) and the diagram commutes

\[
\begin{array}{cccc}
Y & \rightarrow & \mathcal{Y} & \xrightarrow{c} & \mathcal{X} & \leftarrow & X \\
\downarrow \psi & & \downarrow \Psi & & \sqrt{R} & & \downarrow \Phi & & \downarrow \varphi \\
\mathbb{C}^n \times \circ & \subset & \mathbb{C}^n \times S_Y & \xrightarrow{q} & \mathbb{C}^m \times S_X & \supset & \mathbb{C}^m \times \circ
\end{array}
\]

(3)

where \( S'_X \) and \( S'_Y \) are some neighborhoods of marked points in \( S_X \), respectively in \( S_Y \).

**Sketch of the proof.** We can write \( \rho = \varphi \times \xi \) where \( \xi : X \rightarrow \mathbb{C}^{n-m} \) is a bounded holomorphic mapping, say \( |\xi| < b \) in \( \bar{X}\) for some positive \( b \). Consider the polyhedron \((X, \tilde{\rho})\) where \( \tilde{\rho} = \varphi \times b^{-1} \xi \). It has the same boundary as \( X \) and fulfils the conditions of Theorem 5.1. Take a versal deformation \((\mathcal{X}, R, S)\) of...
this polyhedron, where $R : \mathcal{X} \to \mathbb{C}^n \times S$ is the barrier function. We can write $R = R_m \times R_{n - m} \times F$, where $R_m : \mathcal{X} \to \mathbb{C}^m$, $R_{n - m} : \mathcal{X} \to \mathbb{C}^{n - m}$, $F : \mathcal{X} \to S$, and define the mapping $\Phi = R_m \times F : \mathcal{X} \to \mathbb{C}^m \times S$. We have $R_m \times \circ = \varphi$ and the triple $(\mathcal{X}, R_m, S)$ is a deformation of the polyhedron $(X, \varphi)$. Set $\Psi = R_m \times bR_{n - m} \times F : \mathcal{X} \to \mathbb{C}^n \times S$; the polyhedron $Y = \{x \in X ; |bR_{n - m}| < 1\}$ is embedded in $\mathcal{X}$ and $Y \times \circ = Y$, $\Psi \times \circ = \psi$, that $(Y, \Psi, S)$ is a deformation of $(Y, \psi)$. The diagram (3) commutes, where $S'_X = S'_Y = S$ and $Q = q \times \text{id}_S$. Analyzing the construction of $(\mathcal{X}, R, S)$ (see [18]) we can see that $(\mathcal{X}, R_m, S)$ is isomorphic to a versal deformation $(\mathcal{X}, \Phi, S_X)$ of the polyhedron $(X, \varphi)$. Next we prove that $(Y, \Psi, S)$ is a versal deformation of $(Y, \psi)$ and Theorem follows.

**Definition.** The class of imbeddings in $(X, \varphi)$ that satisfy the conditions of Proposition 6.1 is called the germ of $(X, \varphi)$; each polyhedron $(Y, \psi)$ is a representative of this germ. By the above theorem the versal deformations of all representative are naturally isomorphic. So are the maximal modular strata $M$ and maximal modular deformations.

## 7 Amalgams and automorphisms

**Amalgam of modular deformations.** Let $f : \mathcal{X} \to M$ and $g : \mathcal{Y} \to N$ be modular families. If there are points $s \in M, t \in N$ such that the fibers $X_s$ and $Y_t$ are isomorphic, then there exist neighborhoods $U$ of $s$ and $V$ of $t$ and a uniquely defined commutative diagram

$$
\begin{array}{ccc}
    f^{-1}(U) & \overset{A}{\cong} & g^{-1}(V) \\
    \downarrow & & \downarrow \\
    U & \overset{a}{\cong} & V
\end{array}
$$

This statement follows from Theorem 6.2. The above families can be patched together along $a$ giving rise to the modular family $\mathcal{X} \sqcup_a \mathcal{Y}$ with the base $M \sqcup_A N$, which is the amalgam (coproduct) of $M$ and $N$:

$$
\begin{array}{ccc}
    f^{-1}(U) & \overset{a}{\leftarrow} & g^{-1}(V) \\
    \downarrow & & \downarrow \\
    \mathcal{X} & \to & \mathcal{X} \sqcup_a \mathcal{Y} \leftarrow \mathcal{Y} \\
    \downarrow & & \downarrow \\
    M & \overset{\alpha}{\to} & M \sqcup_A N \overset{\varphi}{\leftarrow} N
\end{array}
$$
where $\mu$ and $\nu$ are the natural morphisms. For given $f$ and $g$ several local isomorphisms $(a, A)$ may occur which can make the amalgam a complicated occasionally non-Hausdorff space. This is just the case, if the automorphism group of $f$ or of $g$ is non-trivial.

**Automorphisms.** Let $(f, i)$ be a miniversal deformation of an analytic polyhedron $X$ as above and $M$ be the maximal modular stratum in its base $S$. Let $X_K$ be the germ of $X$ on the compact set $K \doteq \text{supp } T^1 (X) \subset X$. We show that an arbitrary automorphism $a$ of $X_K$ generates an automorphism of the stratum $M$. Choose a polyhedron $X' \subset X$ that contains the germ $X_K$ such that $a$ defines a morphism $a : X' \to X$. Take a miniversal deformation $(f', i')$ of $X'$; let $S'$ be the base of $f'$. By Theorem 6.2 there exists an imbedding $j : S' \to S$ that induces an isomorphism $f \times j \sim f'$ and maps the modular stratum $M'$ to the modular stratum $M$. Consider the deformation $(f, i a)$. This is a miniversal deformation of $X'$, hence it is induced from $(f', i')$ by a germ endomorphism $\alpha : S \to S'$. It is an automorphism, since $D \alpha$ is bijective at the marked point. The restriction $\alpha| M$ is uniquely defined and the composition $\alpha_M \doteq j^{-1} \alpha| M : M \to M$ is an automorphism of the germ $M$.

**Corollary 7.1** The mapping $a \mapsto \alpha_M$ defines a homomorphism of the group $\text{Aut } (X_K)$ to the group $\text{Aut } (M)$ of automorphisms of the maximal modular germ $M$.

The kernel $\text{Aut}_0 (X_K)$ of this homomorphism is a normal subgroup and the quotient $G (X_K) = \text{Aut } (X_K) / \text{Aut}_0 (X_K)$ acts faithfully; we call it active automorphism group. Any automorphism $a$ of $X_K$ generated by a tangent field $v \in T^0 (X_K)$ acts trivially on $M$. The field $v$ generates a local holomorphic subgroup $a (\zeta)$ of automorphisms. The infinitesimal action of this group in $T_0 (S)$ is given by the commutator $t \mapsto [D_0 f (t), v]$, which is trivial in $M$ due to Theorem 6.2 (iii). Therefore $a_M = \text{id}$ that is $a \in \text{Aut}_0 (X_K)$. The active group $G(X_K)$ is discontinuous in several examples (see below).

The fibers $X_s$ are, of course, isomorphic for points $s$ in any coset of the automorphism group. The inverse statement is by no means obvious.

**Problem 1** Let $f : \mathcal{X} \to (M, o)$ be a maximal modular deformation of a space $X \cong X_o$. When does the existence of a isomorphism $X_s \cong X_t$ for some points $s, t \in M$ imply $\alpha_M (s) = t$ for an element $a \in G (X_s)$? In particular, when an isomorphism $X_s \cong X, s \in M'$ implies $s = o$?

Is it true, if the group $G(X_s)$ is finite?
Let $X \rightarrow M$ be a modular family with an irreducible base $M$ and $s \in M$. By Corollary 7.1 any element $\alpha \in \text{G} (X_s)$ of the fiber $X_s = f^{-1} (s)$ generates an automorphism $\alpha_M$ of the germ $(M, s)$. By analytic continuation, we can extend $\alpha_M$ to a uniquely defined automorphism $\alpha$ of the base $M$. Let $G$ be the automorphism group of $M$ generated by all the elements $\alpha_M$ for $\alpha \in \text{G} (X_s), s \in M$. The quotient $M/G$ can be taken as a candidate for ‘true’ moduli space.

8 Modular deformation of compact spaces

We start with very classical families that appear to be modular.

Example 1. The family $T = \{ T(\lambda) \}, \lambda \in \mathbb{C}_+$ of 1-tori parameterized by the upper half-plane $\mathbb{C}_+$; $T(\lambda)$ is the quotient of the plane $\mathbb{C}$ by the lattice generated by 1 and $\lambda$. This family is maximal modular for each point $\lambda \in \mathbb{C}_+$. The automorphism group of the family is the group $\text{Sl}(2, \mathbb{Z})$ which acts in $\mathbb{C}_+$ by fractional linear transformations. The group $\text{Aut}_0 (T(\lambda))$ is equal to the semi-direct product of the group $T(\lambda)$ generated by tangent fields on $T(\lambda)$. The group $\mathbb{Z}_2$ generated by the mapping $z \mapsto -z$ in $\mathbb{C}$ acts trivially in the base $\mathbb{C}_+$. Take the standard fundamental domain $D = \{ \lambda : |\text{Re} \lambda| < 1/2, |\lambda| > 1 \}$ and consider the restriction of the family to $D$. There are two points $\lambda_2 = i, \lambda_3 = \sqrt{-1}$ on the boundary of $D$ such that the active automorphism group $G(T(\lambda))$ has elements with non-trivial action: these are elements $a_j \in G(T(\lambda_j))$ of order 2, respectively 3. These elements are generated by rotation of the unit square by $\pi/2$ and by rotation of the rhombus by $\pi/3$, respectively. According to previous section, these groups generate transformation group $G$ of $D$. The quotient space $D/G$ is isomorphic to the complex plane $\mathbb{C}$ and the family $T$ gives rise to a family $\tilde{T}$ of tori on $\mathbb{C}$. It can be compacted by means of amalgam with the deformation $f : \mathcal{Y} \rightarrow U$ of the singular curve $Y_0$, which is the projective line with one point of transversal self-intersection. Here $U$ is the unit disk and the deformation is given in an affine chart by

$$w^2 - z(z - s)(z - 1) = 0, |s| < 1/2$$

The fibers $Y_s$ are non-isomorphic tori for $s \neq 0$, and $\dim T^1 (Y_0) = 1$, hence the family $\mathcal{Y}$ is maximal modular. It can be patched to $\tilde{T} \rightarrow \mathbb{C}$ giving rise to a modular family with base $\mathbb{C} \cup \{ s = 0 \} = \mathbb{CP}^1$. The point $s = 0$ corresponds to infinity, since a ratio of periods of the surface $Y_s$ tends to infinity as $s \rightarrow 0$.

Example 2. Generalize the above construction for curves of an arbitrary
genus $g > 1$. Consider the family of hyperelliptic Riemann surfaces $X_a$, $a \in \mathbb{C}^m$, where $X_a$ is given in the affine chart by

$$w^2 - p(z) = 0, \quad p(z) = z^m + a_1z^{m-1} + \ldots + a_m, \quad m = 2g + 2. \quad (4)$$

Suppose that $g \geq 4$, take the singular surface $X_0$ for which $a_1 = \ldots = a_m = 0$ and check that

$$\dim T^1(X_0) = 3g - 3 \quad (5)$$

We have

$$\dim T^1(X) = \dim H^0(X, T^1) + \dim H^1(X, T^0),$$

since $H^2(X, T^1) = 0$. Further,

$$\dim H^0(X, T^1) = \dim T^1_s = m - 1 = 2g + 1,$$

since the only singular point $s = (0, 0)$ is of multiplicity $m - 1$. The surface $X$ is union of two spheres that are tangent one to another at the origin. The tangent sheaf $T^0$ is generated at the origin by two fields

$$t_1 = mz \frac{\partial}{\partial z} + 2w \frac{\partial}{\partial w}, \quad t_2 = 2w \frac{\partial}{\partial z} - mz^{m-1} \frac{\partial}{\partial w}$$

over the algebra over $\mathcal{O}(X)_s$. It helps to check that $\dim H^1(X, T^0) = g - 4$, which implies (5).

Let $F_0 : Y \to S$ be a miniversal deformation of $X_0$. Prove that it is universal. The base $S$ is a piece of $\mathbb{C}^{3g-3}$ and any non-singular fiber $Y_s$ is a Riemann surface of genus $g$. Therefore, we have again $\dim T^1(Y_s) = \dim H^1(Y_s, T^0) = 3g - 3$. This implies that the dimension of $T^1(Y_s)$ is constant in the deformation $F$, since it is a upper semi-continuous function on the base. By Theorem 5.2(iv) the deformation $F$ is maximal modular and therefore universal. The family of surfaces $X_a$ is obviously a deformation of $X_0$ which yields that $\dim T^1(X_a) = 3g - 3$. Therefore any miniversal deformation $F_a$ of $X_a$ is universal too. They can be amalgamated together in a maximal modular family $F$ with a base $S$ which is an open subset in $\mathbb{C}^{3g-3}$. On the other hand, $S$ contains the base $\mathbb{C}^m$ of the family (4). The set $D \subset S$ of critical values of this deformation coincides with the variety $\Delta(p) = 0$ in $\mathbb{C}^m$, where $\Delta(p)$ is the discriminant of the polynomial $p$.

On the other hand, the Teichmüller space $T(g)$ is the base of a universal family $R(g)$, whose fibers are all non-singular Riemann surfaces of genus $g$.
with an additional structure. The automorphism group $\Gamma(g)$ of the family $R(g)$, called modular group, acts discontinuously in $T(g)$. The deformation $F$ is amalgamated with the family $R(g)$ and the base $S \setminus D$ is patched to $T(g)/\Gamma(g)$. This gives a compacting of the space $T(g)/\Gamma(g)$, whereas the discriminant set $D$ covers the boundary.

In the cases $g = 2, 3$ the deformation $F$ as above is no more modular for the surface $X_0$, since $\dim T^1(X_0) > 3g - 3$. We take the surface $X(q)$ for $q(z) = (z^2 - 1)z^{2g}$ instead. The miniversal deformation of $X(q)$ is universal, like in the case $g = 1$.

This compacting of the space $T(g)$ is, apparently, different from that of W.Baily [4].

**Example 3.** Fix an integer $n > 1$ and consider a proper holomorphic mapping of manifolds $F : T \to S$ whose fibers are complex analytic $n$-tori, see [6]. The dimension of $S$ is equal to $n^2$ and the mapping is maximal modular deformation of each fiber as in the case $n = 1$. On the other hand, the active group $G$ of $F$ is not discontinuous. Moreover, any non-empty open set $U$ contains a point $s$ such that the coset $G_s \cap U$ is infinite. The moduli space $M/G$ is not separable.

**Example 4.** Take the family $F : V \to P(m)$ of algebraic hypersurfaces of degree $m$ in $\mathbb{CP}^n$, $n \geq 3$, where $P(m)$ is the space of all homogeneous polynomials of degree $m$ and $V(f) = F^{-1}(f)$ is the hypersurface defined by the equation $f = 0$ in $\mathbb{CP}^n$. Take a polynomial $f \in P(m)$ such that the variety $V(f)$ is non-singular and choose an affine subspace $S \subset P(m)$ that contains the point $f$ and is transversal to the linear span of polynomials $z_i \partial f/\partial z_j$, $i, j = 0, ..., n$. Consider the restriction $F|S$ of this family; the Kodaira-Spencer mapping $DF|S : T_f(S) \to T^1(V(f))$ is injective. This mapping is surjective and the family $F|S$ is versal, except in the case $n = 3, m = 4$, see [6]. Moreover, this family is maximal modular, since $\dim S = (m + n)!/n!m! - (n + 1)^2$ does not depend on $f$. The automorphism group of any fiber $X$ is finite for the same cases.

In the exceptional case $n = 3, m = 4$ the miniversal deformation $F$ of any surface contains non-algebraic fibers $X_s$ that are also K3-surfaces (the canonical bundle is trivial). The base $S$ has dimension 20 and the algebraic fibers only appear for points $s$ in the dense subset $S_{\text{alg}}$ which is a countable union of 19-dimensional subspaces in the base, see more details in [6]. Therefore $\dim T^1(X_s) = 20$ on the germ $S$ and the formation $F$ is again maximal modular. On the other hand, there is no non-trivial tangent fields on a K3-surface, but its automorphism group can be infinite [13].
9 Modular deformations of singular points

The modular deformations of polyhedra with isolated singularities look similar to the above examples. Moreover, there is a relation between deformations in these two categories.

Let $D^n$ be the open unit polydisk in a coordinate space $\mathbb{C}^n$ centered at the origin. Take holomorphic functions $f_1, ..., f_k$ in $\bar{D}^n$ and consider the analytic polyhedron $X = \{ z \in D^n, f_1(z) = ... = f_k(z) = 0 \}$ endowed with the sheaf $\mathcal{O}(X) = \mathcal{O}(\bar{D}^n)/\mathcal{I}$, where $\mathcal{I}$ is the sheaf-ideal generated by these functions. The polyhedron $X$ satisfies the condition [I] if it contains only finite number of singular points.

Suppose that $k = 1$, the generating function $f$ is weighted homogeneous for certain coordinate system $z_1, ..., z_n$ in $\mathbb{C}^n$ and there is only one singular point $z = 0$ in $X$ (otherwise the dimension of the singular set of $X$ is positive). The base $M$ of the maximal modular deformation is isomorphic to the subspace of $T^1(X)$ of elements whose weight is equal to that of $f$. In particular, $M$ is a simple point for singularities of types $A, D$ and $E$ that are characterized by the inequality weight ($\tau$) < weight ($f$) for all $\tau \in T^1(X)$.

Assume now that the weights of the coordinates are equal to 1 so that $f$ is a homogeneous polynomial of degree $m > 0$. It defines a non-singular hypersurface $V(f)$ in $\mathbb{CP}^{n-1}$ and any deformation of $V(f)$ as in Example 4 generates a deformation of the analytic polyhedron $X(f) = \{ z \in D^n, f(z) = 0 \}$. We have

$$T^1(X(f)) \cong \mathbb{C}[z_1, ..., z_n]/(\partial f/\partial z_1, ..., \partial f/\partial z_n),$$

The weight grading generates a grading in the space (6).

**Proposition 9.1** Suppose that $n \geq 4$ and the case $n = m = 4$ is excluded. Then there exists a linear injective mapping

$$j : T^1(V(f)) \rightarrow T^1(X(f)),$$

whose image coincides with the subspace of weight $m$.

By diagram (9.3) of [16], p.109 we have $T^1(V(f)) = P(m)/J$, where $P(m)$ is the space of polynomials in $\mathbb{C}^n$ of degree $m$ and $J$ is the linear envelope of the polynomials $z_i \partial f/\partial z_j, i, j = 1, ..., n$. Note that the term $H^1(V(f), \theta)$ vanishes in the diagram due to Proposition 9.2. The isomorphism (6) makes the mapping (7) obvious. ▶
Corollary 9.2 For any family of homogeneous polynomials \( \{ f_s, s \in S \} \) of \( n \) variables with the only critical point \( z = 0 \) the numbers \( \dim T^1 (V (f_s)) \) and \( \dim T^1 (X (f_s)) \) stay constant. If one of the families is modular, another is also modular.

For the first statement we note that \( \dim T^1 (X (f_s)) = (m - 1)^n \). By the previous Proposition \( \dim T^1 (V (f_s)) \) is equal to the number of monomials of degree \( m \) minus \( n \), that is the dimensions depend only on \( n \) and \( m \). We have the equation for the Kodaira-Spencer mappings \( D\Phi = jDF \) where \( F \) and \( \Phi \) are the families of polyhedrons and of projective hypersurfaces, respectively. The second statement follows from injectivity of \( j \).

The modular stratum of a non weighted-homogeneous hypersurface may be singular and have imbedded primary components.

Example 5. Consider the polyhedron in \( D^3 \) defined by the equation

\[
 f_{p,q,r} (\lambda; x, y, z) \equiv x^p + y^q + z^r + \lambda xyz = 0 \tag{8}
\]

in the complex coordinates \( x, y, z \). We denote the singular germ at the origin by \( T_{p,q,r}(\lambda) \), if the equation holds \( 1/p + 1/q + 1/r = 1 \). The generating function is then weighted homogeneous and the parameter \( \lambda \) is the coordinate in a maximal modular deformation, where \( \lambda \) runs over the complex plane with few gaps, where the polyhedron contains a singular curve. This follows from Theorem 5.2(iv), since the space of elements \( t \in T^1 (X_\lambda) \) such that \( [t, v] = 0 \) for all \( v \in T^0 (X_\lambda) \) is one-dimensional. There are just three modular families of this type: \( T_{3,3,3}(\lambda) \), \( T_{4,4,2}(\lambda) \) and \( T_{6,3,2}(\lambda) \) whose Tyurina numbers \( \tau (X) = \dim T^1 (X) \) are 8, 9, 10, respectively.

In the case \( 1/p + 1/q + 1/r < 1 \) the surfaces are all isomorphic for \( \lambda \neq 0 \). We set \( \lambda = 1 \) and use the notation \( T_{p,q,r} \).

Example 6. Start with the family \( T_{3,3,3}(\lambda) \) defined for \( \lambda \in \mathbb{C} \). We have \( \tau (X) = 8 \) and the family is maximal modular for \( \lambda^3 \neq -27 \). The active group \( G = \mathbb{Z}_3 \) acts in the family by \( \lambda \mapsto \varepsilon \lambda \), where \( \varepsilon^3 = 1 \). Take the deformation \( Y \rightarrow N \) of the germ \( T_{4,3,3} \) defined in \( D^3 \times N \) by the function

\[
 F(r, s, t; x, y, z) = x^4 + y^3 + z^3 + \mu xyz + r x^3 + s_1 y + s_2 y^2 + t_1 z + t_2 z^2,
\]

where the number \( \mu \neq 0 \) is fixed. It is maximal modular for the base \( N \subset M \times \mathbb{C}^3 \) given by the ideal \( I = I_1 \cap I_0 \) where the ideal \( I_1 = (s_1, s_2, t_1, t_2) \) defines the line parameterized by the coordinate \( r \) and we have an isomorphism \( Y_{r,0,0} \simeq T_{3,3,3}(\lambda) \) for \( \lambda = r^{-1/3} \). The ideal \( I_0 \) determines a fat point at
the origin in $\mathbb{C}^5$, whose embedding dimension is equal to 5. Therefore the modular deformations $T_{3,3,3}(\lambda)$ and $Y$ are patched together to a maximal modular deformation with a compact base $M$.

**Example 7.** Consider the family $\mathcal{X}$ of curves $X_\lambda \subset D_1^2$ given by the equation 

$$(x^3 - xy^2)(ax - by) = 0, \quad \lambda = b/a \in \mathbb{CP}^1.$$ 

The family is maximal modular for all $\lambda$, except in three points $\lambda = -1, 0, 1$. These gaps can be patched by means of another modular deformations. B. Martin has considered the family $Y \rightarrow \mathbb{C}^7$ of polyhedra in $D^2 \times \mathbb{C}^7$, given by the equation

$$x^4 - x^2y^2 + s_1x + s_2y + s_3xy + s_4y^2 + s_5y^3 + s_6xy^2 + ty^4 + y^5 = 0.$$ 

He has shown that this family is maximal modular over the base $M$, whose ideal is $I(M) = J_1 \cap J_0$, where $J_1 = (s_1, ..., s_6)$ defines the line and the ideal $J_0$ has 9 generators and defines a fat point at the origin. For any point $(0, t) \in \mathbb{C}^7$, $t \neq 0$ the germ $Y_{0,t}$ is isomorphic to $X_\lambda$ for $\lambda = 2\sqrt{t}$. Therefore the family $\mathcal{Y}$ can be amalgamated with the family $\{X_\lambda\}$ patching the gap $\lambda = 0$. The gaps $\lambda = \pm 1$ are patched in a similar way. The resulting amalgam is a maximal modular family $\tilde{T}_{3,3,3}$ with the compact base $\mathbb{CP}^1$.

Note that the Milnor number of a fiber of this family is not constant; it is equal to 9, except in the points $\lambda = -1, 0, 1$, where it equals 10.

**Example 8.** Consider the family $\mathcal{Z}$ of complete intersection curves $Z_s \subset D^3 \times \mathbb{C}$ defined by two polynomials:

$$x^4 + y^4 + 2z^2 = sz - xy = 0, \quad s \in \mathbb{C}.$$ 

It is modular with $\tau(Z_s) = 9$. The curve $Z_s$ is isomorphic for $s \neq 0$ to the plane curve $x^4 + y^4 + 2s^{-2}x^2y^2 = 0$. The germ at the origin of this curve is isomorphic to the fiber $X_\lambda$ of the family in Example 7 with $\lambda = s^2$. Now the family $\mathcal{Z}$ is amalgamated with $\mathcal{X}$ mending the gap at the point $\lambda = 0$. This provides a compacting of the family $\mathcal{X}$ at this point, which is different form that of Example 7.

**Example 9.** (5) The maximal modular deformations of polyhedra $Y_{7,3,2}$, $Y_{6,4,2}$ and $Y_{6,3,3}$ can be amalgamated with the modular family $T_{6,3,2}(\lambda)$. The modular deformation of $Y_{6,4,2}$ is given by the function

$$x^6 + y^4 + z^2 + xyz + rx^4 + sx^5 + ty^3 + vz = 0.$$
in $D^3 \times \mathbb{C}^4$. It is maximal modular over the base germ $M$ defined by the ideal $I(M) = J_1 \cap J_2 \cap J_3$, where

$$J_1 = (r, s, v), \quad J_2 = (r, s^2, t^2, 20v - st), \quad J_3 = (t, 4r - s^2, v).$$

The zero set of $J_1$ is the line with the coordinate $t$; the fiber $Y(0, 0, t, 0)$, $t \neq 0$ is a surface whose germ is isomorphic to $T_{6,3,2}(\lambda)$ for $\lambda = t^{-1/3}$. The ideal $J_2$ defines the fat point at the origin and $J_3$ defines a smooth curve parameterized by $s_5$; the fiber of deformation over this curve is the union of the germ $T_{4,4,2}(\lambda)$ for some $\lambda$ and of a singular germ of multiplicity 1 at the point $(-s/2, 0, 0)$.

The maximal modular deformation $Y_{6,3,3}$ has the base, which is the union of three components. Two of them are straight lines that are symmetric under the permutation $x \leftrightarrow y$; the fibers are of type $T_{6,3,2}(\lambda)$. The third component is a curve; the general fiber of the deformation over this curve has the singular germ of type $Y_{4,3,3}$ at the origin and another singular point of multiplicity 2.

The modular deformation $Y_{7,3,2}$ is similar to the previous two cases.

**Problem 2.** These examples give rise to the general questions: let $f^*$ be a modular deformation over the punctured disk $D^*$. When there exists a modular deformation $f$ over $D$ such that $f|D^* \cong f^*$? Note that the deformation $f$ need not to be unique as we saw in Examples 8 and 9.

## 10 Concluding remarks

Let $X$ be an arbitrary hypersurface germ with isolated singularity which is not weighted homogeneous. Then the tangent space $T(M)$ to the maximal modular stratum has always positive embedding dimension. Indeed, the action of the Lie algebra $T^0(X)$ in $T^1(X)$ is nilpotent and by Engel’s theorem, there is an element $t \in T^1(X)$, $t \neq 0$ that vanishes under action of the Lie algebra. By Theorem 5.2 this yields $t \in T(M)$. This is the case for any singular germ of type $T_{p,q,r}$.

More examples can be extracted from the classification from Arnold’s list [3], where several families of singularities with constant Milnor number are listed. Such a family has also constant Tyurina number, except for a proper subvariety of the family base, where this number jumps up. Restricting the family to a certain affine subvariety $M$, yields a modular deformation. The
‘gaps’ in $M$ could be filled by means of amalgams with appropriate modular deformations. The completed variety $\tilde{M}$ is expected to be a projective algebraic. Besides, any ‘solitary’ singularity in Arnold’s list which is not included in families may have the maximal modular deformation with the base $M$ which is either a fat point or a germ of positive dimension with splitting singularity like that of $T_{6,4,2}$ and $T_{6,3,3}$ in Example 9.

A.Aleksandrov [2] has found a series of modular families of complete intersection curves in $\mathbb{C}^3$. Examples of families with constant Tyurina number $\geq 34$ were calculated in the book [10]. B.Martin [12]. T.Hirsch and B.Martin [5] are found sophisticated examples of modular deformations.

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