Housekeeping entropy in continuous stochastic dynamics with odd-parity variables

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Abstract. We investigate the decomposition of the total entropy production in continuous stochastic dynamics when there are odd-parity variables that change their signs under time reversal. The first component of the entropy production, which satisfies the fluctuation theorem, is associated with the usual excess heat that appears during transitions between stationary states. The remaining housekeeping part of the entropy production can be further split into two parts. We show that this decomposition can be achieved in infinitely many ways characterized by a single parameter $\sigma$. For an arbitrary value of $\sigma$, one of the two parts contributing to the housekeeping entropy production satisfies the fluctuation theorem. We show that for a range of $\sigma$ values this part can be associated with the breakage of the detailed balance in the steady state, and can be regarded as a continuous version of the corresponding entropy production that has been obtained previously for discrete state variables. The other part of the housekeeping entropy does not satisfy the fluctuation theorem and is related to the parity asymmetry of the stationary state distribution. We discuss our results in connection with the difference between continuous and discrete variable cases especially in the conditions for the detailed balance and the parity symmetry of the stationary state distribution.

Keywords: Brownian motion
1. Introduction

Recent advances in understanding nonequilibrium systems have been stimulated by the discovery and applications of the so-called fluctuation theorems (FTs) [1–6]. The total entropy production (EP) in the system and the heat reservoir plays a central role as a measure of the irreversibility in the nonequilibrium dynamics. The inequality for the total EP in the thermodynamic second law now becomes a corollary of the more general equality in the FTs [3–5]. In a further development, a part of the total EP associated with the excess heat is shown to satisfy a FT on its own [7]. The remaining part is the housekeeping EP that is necessary to maintain the nonequilibrium steady state. It was also shown to satisfy the separate FT [8, 9]. The excess entropy is a transient component of the total EP as it is produced during transitions between stationary states. It can therefore be interpreted as a nonadiabatic component of the total EP, while the adiabatic part corresponds to the housekeeping EP [10].

The situations become more complicated, however, when there are dynamical variables which have an odd-parity under time reversal. An example of odd-parity variables is the momentum in the underdamped description of the Brownian motion. In a series of papers [11, 12], Spinney and Ford considered the EP in both continuous and discrete stochastic systems in the presence of odd-parity variables. They found that the excess entropy part can still be singled out, which satisfies the FT. The housekeeping part is further divided into two terms $\Delta S_2$ and $\Delta S_3$. Only $\Delta S_2$ is shown to satisfy the FT. $\Delta S_3$ is associated with the asymmetry of the steady-state distribution under time reversal. It does not satisfy the FT and turns out to be transient as it vanishes when the system stays in a steady state. In a more recent study [13] on a stochastic system described by discrete variables including odd-parity ones, the housekeeping EP was found to be decomposed into $\Delta S_{DB}$ and $\Delta S_{as}$ in a manner different from that used by Spinney and Ford. A striking finding in the case of discrete states is that the detailed
balance (DB) and the parity symmetry of stationary distribution are independent conditions, which have been believed to be equivalent conditions for equilibrium. $\Delta S_{bDB}$ is a direct consequence of the breakage of the detailed balance (DB) and satisfies the FT. On the other hand, it is not obvious how the corresponding $\Delta S_2$ applied to the case of discrete states is related to the breakage of DB. $\Delta S_{as}$ is attributed to the asymmetry of the steady-state distribution as in the Spinney and Ford’s scheme, but turns out to be not transient in this case, compared with transient $\Delta S_2$. It can therefore be regarded as a more relevant component of the adiabatic EP.

It is the purpose of this paper to identify a part of the housekeeping EP as a proper measure of the DB breakage as done in the case of discrete state variables. We consider a general continuous stochastic system with odd-parity variables described by the Fokker–Planck equation including the driven Brownian motion in the underdamped limit, where the momentum serves as the odd-parity variable. As we shall see below, there is a subtle difference between the discrete-jumping process described by the master equation and the continuous-evolving process described by the Fokker–Planck equation. As a result, the direct application of the procedures used in [13] for obtaining $\Delta S_{bDB}$ to the continuous system makes the corresponding quantity ill-defined. One has to resort to a different approach to obtain a sensible result. We note that a convenient way to obtain various EPs is to use a dual or adjoint dynamics to the original dynamics [14, 15]. In this paper, we find that many versions of $\Delta S_{bDB}$ for continuous variables can be obtained via a generalized adjoint dynamics like the one used in [16] where an infinite family of the excess EPs are constructed. We will show below that, in the housekeeping EP, we can identify infinitely many expressions for the EP related to the DB breakage, $\Delta S_{\sigma DB}$, parametrized by an arbitrary number $\sigma$, all of which satisfy the FT. Spinney and Ford’s $\Delta S_2$ corresponds to a special case of $\sigma = 0$.

The DB condition arising from the micro-reversibility serves as an essential criterion for the thermodynamic equilibrium. In this regard, it is important to identify the part of the housekeeping EP that is a direct consequence of the violation of the DB. For continuous stochastic systems in the presence of odd parity variables, a part of the housekeeping EP that satisfies the FT was isolated [11, 12], but its connection to the breakage of the DB has never been made. In this paper, we make this connection and further present a family of EPs mentioned above, all of which are direct consequences of the DB breakage.

This kind of degeneracy of EPs results from the peculiar nature of the DB condition for continuous stochastic variables including odd parity ones. In section 3, A and B, we present an in-depth analysis of the DB condition for continuous stochastic variables in the presence of the odd-parity ones. A proper care must be taken since for discrete jumping processes [13], the DB condition and the parity symmetry of the stationary distributions have been shown to be two completely independent conditions. This is in contrast to what is discussed in conventional textbooks by Risken [19] and Gardiner [20], where the parity symmetry is assumed as a condition for the DB. The DB condition is defined through the transition rate in the former, which becomes equivalent to the DB through the transition probability in the latter only by assuming the parity symmetry. We therefore consider the DB condition as defined only through the transition rate, while no condition is assumed beforehand regarding the parity symmetry of the stationary distribution. For continuous stochastic variables, we will show in

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section 3 that the parity symmetry actually follows from the DB condition if the noise distribution satisfies a certain condition. This property, which is absent in discrete stochastic systems, allows us to write many equivalent expressions for the DB condition, which is in turn responsible for the existence of a family of $\Delta S_{\text{DB}}^\omega$.

In the next section, we present how various components of the EP are defined in the general continuous stochastic dynamics including the driven Brownian dynamics. The explicit expressions for the total and the excess EPs are given along with their average rates in time. Although much of the results in the section are previously known, we present them to clear up certain technical points and to set up our notations. In section 3, we present our main results. We show in detail how the housekeeping EP can be divided into two parts by applying the generalized adjoint dynamics. We study in detail how the DB and its breakage can be represented in the expressions of the EPs. In the final section, we summarize our results with discussion.

2. Entropy productions in continuous stochastic dynamics

A simple model for the continuous stochastic dynamics involving odd-parity variables is the driven Brownian dynamics for a particle of mass $m$ in $d$ dimensions, for which the position and momentum variables, $\mathbf{q} = (\mathbf{x}, \mathbf{p})$, constitute the even and odd parity variables, respectively. The equations of motion are given by

\[ \dot{\mathbf{x}} = \frac{\mathbf{p}}{m}, \]

\[ \dot{\mathbf{p}} = -\mathbf{G} \cdot \frac{\mathbf{p}}{m} + \mathbf{f}(\mathbf{q}; \lambda) + \xi(t), \tag{2} \]

where $\mathbf{G} = \{G_{ij}\}$ is a dissipation matrix with the standard notation $(\mathbf{G} \cdot \mathbf{p})_i = \sum_j G_{ij} p_j$. We consider a most general form for the force which may depend on both position and momentum as well as on some time-dependent protocol $\lambda(t)$. In the following, we will drop the expression for $\lambda$-dependence in the force for simplicity of notation. The Gaussian white noise satisfies

\[ \langle \xi_i(t) \rangle = 0, \quad \langle \xi_i(t) \xi_j(t') \rangle = 2D_{ij} \delta(t - t') \tag{3} \]

with the symmetric positive-definite diffusion matrix $\mathbf{D} = \{D_{ij}\}$. If the Einstein relation holds, $\mathbf{D} = \mathbf{G} \mathbf{T}$ with the reservoir temperature $T$. The force can be divided into the reversible and irreversible parts, $\mathbf{f} = \mathbf{f}^{\text{rev}}(\mathbf{q}) + \mathbf{f}^{\text{ir}}(\mathbf{q})$, according to its behavior under time reversal as

\[ \mathbf{f}^{\text{rev}}(\mathbf{q}) = \frac{1}{2}(\mathbf{f}(\mathbf{q}) + \mathbf{f}(\epsilon \mathbf{q})), \quad \mathbf{f}^{\text{ir}}(\mathbf{q}) = \frac{1}{2}(\mathbf{f}(\mathbf{q}) - \mathbf{f}(\epsilon \mathbf{q})), \tag{4} \]

where $\epsilon \mathbf{q} = (\mathbf{x}, -\mathbf{p})$. The reversible part transforms as $\dot{\mathbf{p}}$ under time reversal, while the irreversible one does oppositely. The Kramers equation for the probability density function (PDF) $\rho(\mathbf{q}, t)$ is given by
\[ \partial_t \rho(q, t) = -\left[ \partial_x \cdot \frac{P}{m} + \partial_p \left( -\frac{G}{m} P + f(q) - D \partial_p \right) \right] \rho(q, t). \] (5)

As we shall see later, the discussion in this paper can also be applied to a more general continuous stochastic dynamics described by the Fokker–Planck equation,

\[ \partial_t \rho(q, t) = \left[ -\partial_q A_i(q) + \partial_p \partial_p \right] \rho(q, t), \] (6)

for dynamical variables \( q = (q_1, q_2, \cdots, q_N) \), where \( \partial_i = \partial / \partial q_i \) and the summation convention is used throughout this paper. The behavior under the time reversal of \( q \) is described by the parity \( \epsilon_j = \pm 1 \) or \( \pm 1 \) for \( q \). We denote \( \epsilon q = (\epsilon_1 q_1, \epsilon_2 q_2, \cdots, \epsilon_N q_N) \). For a general diffusion matrix \( D_0(q) \), this corresponds to a set of Langevin equations with multiplicative noises. As in the Brownian dynamics, we separate the drift term into reversible and irreversible parts: \( A_i(q) = A^r_i(q) + A^\|_i(q) \), where

\[ A^r_i(q) = \frac{1}{2} \left( A_i(q) - \epsilon_i A_i(\epsilon q) \right), \] (7)

\[ A^\|_i(q) = \frac{1}{2} \left( A_i(q) + \epsilon_i A_i(\epsilon q) \right). \] (8)

Note that no summation convention is taken for \( \epsilon_i \).

Equation (6) can be written using quantum mechanical notation as \( (d/dt) |\rho(t)\rangle = H_{\text{FP}} |\rho(t)\rangle \) with \( \rho(q, t) = \langle q | \rho(t) \rangle \) and the solution is given by \( e^{H_{\text{FP}} t} |\rho(0)\rangle \). Then, the transition probability from state \( q \) to \( q' \) in infinitesimal time \( dt \) is given by

\[ \Gamma[q', t + dt, q, t] = \delta(q' - q) + (dt) \omega[q', q] \] (9)

where \( \omega[q', q] = \langle q' | H_{\text{FP}} | q \rangle \) is the transition rate matrix given as

\[ \omega[q', q] = [-\partial_i' A_i(q) + \partial_p' \partial_p D_0(q)] \delta(q' - q), \] (10)

with \( \partial_i' = \partial q_i' \). Using \( \omega\)-matrix, we can write equation (6) as \( \partial_t \rho = \int dq' \omega[q, q'] \rho(q', t) \). The stationary state \( \rho^s(q) \) satisfies \( \int dq' \omega[q, q'] \rho^s(q') = 0 \).

Given the forward path probability density \( P[q(t)] \) for a stochastic path \( q(t) \) for \( 0 \leq t \leq \tau \), the integral FT (IFT) holds for an arbitrary function \( R[q(t)] \) of the path which has the form

\[ R[q(t)] = \ln \frac{P[q(t)]}{\hat{P}[\hat{q}(t)]}, \] (11)

where \( \hat{P}[\hat{q}] \) is the path probability density for the transformed path \( \hat{q}(t) \) with a specified time dependence. The transformation \( q(t) \rightarrow \hat{q}(t) \) must have the Jacobian of unity. It is straightforward then to see that the IFT, \( \langle \exp(-R[q]) \rangle = 1 \), follows from the normalization of \( \hat{P} \) [8–10, 15].

The IFT for the total EP, \( \Delta S_{\text{tot}} \), in the system and the environment is obtained by using in equation (11) the time reversed path \( \epsilon q(\tau - t) \) for \( \hat{q}(t) \) with the time reversed protocol \( \lambda(\tau - t) \) for \( \hat{P} \). \( P \) is written as a product of the initial PDF \( \rho(q(0), 0) \) and the conditional path probability \( \Pi[q(t); \lambda(t)] \) for the system to evolve through the path \( q(t) \) starting from \( q(0) \) subject to the protocol \( \lambda(t) \). Choosing the final PDF \( \rho(q(\tau), \tau) \) of the forward path

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process as the initial PDF of the time-reversed process, we similarly write \( \hat{P} \) as a product of 
\[ \rho(q(\tau), \tau) \text{ and } \Pi^R[\epsilon q(\tau - t); \lambda(\tau - t)], \]
where the latter is the conditional path probability for the corresponding time-reversed path starting from \( \epsilon q(\tau) \) subject to the corresponding time-reversed protocol \( \lambda(\tau - t) \) indicated by the superscript R. Then we have
\[ \Delta S_{\text{tot}} = \Delta S_{\text{sys}} + \Delta S_{\text{env}}, \]
where \( \Delta S_{\text{sys}} = -\ln \rho(q(\tau), \tau) + \ln \rho(q(0), 0) \) is the system entropy change. The environmental EP \( \Delta S_{\text{env}} \) is given by the log-ratio of the conditional probabilities as
\[ \Delta S_{\text{env}} = \ln \frac{\Pi[q(t); \lambda(t)]}{\Pi^R[\epsilon q(\tau - t); \lambda(\tau - t)].} \]
It is more convenient to consider the log-ratio of the conditional probabilities during the infinitesimal time interval \( dt \). The environmental EP for the whole time interval can be obtained by integrating the following quantity over \( [0, \tau] \):
\[ dS_{\text{env}} = \ln \frac{\Gamma[q', t + dt|q, t]}{\Pi[q, t + dt|\epsilon q', t]}, \]
The conditional path probability \( \Pi \) for the whole time interval is given by the infinite product of the conditional probabilities \( \Gamma \)'s in the limit \( dt \to 0 \). The protocol is implicit in \( \omega[q', q] \). Note that the values of the protocols for the forward and the time-reversed processes are chosen to be identical in this time interval, given by \( \lambda(t) \), and the superscript R is not necessary.

For the general Fokker–Planck equation, equation (6), the conditional probability is given by the Onsager–Machlup form as [21–23]
\[ \Gamma[q', t + dt|q, t] = \frac{1}{(4\pi dt)^{N/2}|\det(D(\alpha))|^{1/2}} \times \exp \left[ -\frac{dt}{4} H^{(\alpha)}(D(\alpha))^{-1} H^{(\alpha)} - \alpha(dt)\partial_iA_i^{(\alpha)} + \alpha^2(dt)\partial_i\partial_j H^{(\alpha)}_{ij} \right], \]
where
\[ H_i = \dot{q}_i - A_i + 2\alpha \partial_j D_{ij} \]
with \( \dot{q} = (q' - q)/dt \). \( \alpha \) indicates the discretization scheme for which all the functions are evaluated at \( q^{(\alpha)} = q + \alpha(q' - q) \). \( \alpha = 0 (1/2) \) corresponds to the Itô (Stratonovich) discretization. For the Brownian dynamics, \( D_{xx} = D_{xp} = 0 \), which gives a factor of delta function enforcing the equation for \( \dot{x} \) in equation (1). Then reducing the diffusion matrix as \( D = \{D_{pp}\} \), the conditional path probability for the Brownian dynamics is given as
\[ \Gamma[q', t + dt|q, t] = \frac{\delta(x' - x - dt(p^{(\alpha)}/m))}{(4\pi dt)^{d/2}|\det(D^{(\alpha))}|^{1/2}} \times \exp \left[ -\frac{dt}{4} H^{(\alpha)} . (D^{(\alpha))}^{-1} . H^{(\alpha)} - \alpha dt \partial p . \left( -G . \frac{p^{(\alpha)}}{m} + f^{(\alpha)} \right) \right], \]
where \( H^{(\alpha)} = \dot{p} + G . p^{(\alpha)}/m - f^{(\alpha)} \) with \( f^{(\alpha)} = f(q^{(\alpha)}) \).
The environmental EP can now be calculated from equation (14). For the general Fokker–Planck case with multiplicative noises, we first note that in order to have a sensible EP for a finite time, $dS_{\text{env}}$ in equation (14) must be $O(dt)$. In particular, $|\text{det}(D(q))|$ must be equal to $|\text{det}(D(\epsilon q))|$. If there is no relation between $D(q)$ and $D(\epsilon q)$, there is no guarantee that the log-ratio of the two determinants produces a $O(dt)$-result. Therefore, we need a restriction on the time reversal property of $D_{ij}(q)$. In the following, we will restrict our discussion to the case where

$$\epsilon_i\epsilon_j D_{ij}(\epsilon q) = D_{ij}(q).$$

(18)

The same restriction and its simpler version for the diagonal $D_{ij}$ have been used in [11]. As mentioned in that reference, we also believe that all physically meaningful models are covered by this condition. In equation (14), the forward and reverse path probabilities depend on their own discretization parameters, which we call $\alpha$ and $\beta$, respectively. It has been shown [11] that, if $\alpha + \beta = 1$, i.e. if the same discretized points are used in both forward and reverse paths, the expression of $dS_{\text{env}}$ in equation (14) is independent of the discretization parameters. The result is [11]

$$dS_{\text{env}} = dt(\dot{q}_i - A_{ii}^{\text{rev}}(\bar{q}))D_{ij}^{-1}(\bar{q})(A_{ij}^{\text{rev}}(\bar{q}) - \partial_t D_{ij}(\bar{q})) - dt \partial_j A_{ij}^{\text{rev}}(\bar{q}),$$

(19)

where all the expressions turn out to be evaluated at the midpoint value $\bar{q} = (q + q')/2$, independent of $\alpha$. This particular combination of discretization parameters, $\alpha + \beta = 1$, is actually due to the multiplicative noise. For the Brownian dynamics, where the noise is additive, $\alpha$ and $\beta$ can be arbitrary and $dS_{\text{env}}$ in equation (14) is always independent of the choice of the discretization schemes [17]. For the Brownian dynamics, we have [18]

$$dS_{\text{env}} = dt(\dot{p} - f^{\text{rev}}(q)) \cdot D^{-1} \cdot \left(-\frac{G}{m} \dot{p} + f^{\text{ir}}(q)\right) - dt \partial_p f^{\text{rev}}(q).$$

(20)

The physical meaning of the above expression has been investigated and it was found that, when $f(q)$ depends on the momentum, there is an unconventional contribution to $dS_{\text{env}}$ in addition to the usual heat production into the reservoir [18].

It is more illuminating to calculate the average of the above quantities. The average of arbitrary quantities which are functions of $q$ and $q'$ at time $t$ and $t + dt$, respectively, is defined by

$$\langle B(q')C(q) \rangle = \int dq' \int dq B(q')\Gamma[q', t + dt|q, t]C(q)\rho(q).$$

(21)

In the following, the averages are expressed in terms of the currents $j(q)$ which are obtained from equation (6) as $\partial_t \rho(q, t) = -\partial_j j_j(q)$ or from equation (5) as $\partial_t \rho(q, t) = -\partial_x \cdot j_x - \partial_p \cdot j_p$. As for the force, we separate the currents into reversible and irreversible parts as $j = j^{\text{rev}} + j^{\text{ir}}$. We have explicitly

$$j_i^{\text{rev}}(q) = A_{ii}^{\text{rev}}(q)\rho(q),$$

(22)

$$j_i^{\text{ir}}(q) = (A_{ii}^{\text{ir}}(q) - \partial_j D_{ij}(q))\rho(q).$$

(23)

For the Brownian dynamics, $j^{\text{rev}} = (p/m)\rho(q)$, $j^{\text{ir}} = 0$, and
\[ \dot{j}_p^{\text{rev}} = f^{\text{rev}}(q)\rho(q), \]

\[ j_p^{ir} = \left(-G\frac{p}{m} + f^{ir}(q) - D\partial_p\right)\rho(q). \]

The average rate of the total EP can then be calculated from equations (12) and (21) as \[11, 18\]

\[ \left\langle \frac{dS_{\text{tot}}}{dt} \right\rangle = \int dq \frac{j_p^{ir}(q) D_{ij}^{-1}(q) j_j^{ir}(q)}{\rho(q)}. \] (26)

For the Brownian dynamics, we have a similar expression involving only the momentum component of the irreversible current \[j_p^{ir}\] as \[D_{ij}^{-1}\] exists only in that space. The positivity of equation (26) comes from the positive-definiteness of \[D\]. As we will discuss later, \[j_p^{ir} = 0\] if the DB condition is satisfied.

When stationary states are involved, another type of EP which satisfies the IFT can be considered. The excess EP \[\Delta S_{\text{excess}}\] arising from transitions between stationary states can be constructed by using the adjoint or dual dynamics \[11, 15\]. For a given stochastic process described by \[\omega\], an adjoint process \[\omega^*\], called *-process, can be defined as

\[ \omega^*[q', q] \equiv \omega[q, q'] \frac{\rho^s(q')}{\rho(q)}, \] (27)

where \[\rho^s(q)\] is defined at a given \[t\] as the expected stationary distribution if the protocol \[\lambda(t)\] is kept unchanged such that \[[-\partial A(q, \lambda(t)) + \partial i] \rho^s(q) = 0\]. The associated transition probability, \[\Gamma^*[q', t + dt|q, t] = \delta(q' - q) + (dt)\omega^*[q', q]\], leads to \[\Gamma[q, t + dt|q', t](\rho^s(q')/\rho^s(q))\], where the protocols for \[\rho^s(q)\] and \[\rho^s(q')\] are chosen to take the same value \[\lambda(t)\] at time \[t\].

As in the even-variable only case, the excess EP in the presence of odd-parity variables can also be obtained by using the adjoint process \[\Gamma^*\] in the following way \[11, 13\]. For an infinitesimal time interval, we define the excess EP as

\[ dS_{\text{excess}} = \ln \frac{\rho(q)\Gamma^*[q', t + dt|q', t]}{\rho^s(q')} \frac{\Gamma*[q, t + dt|q, t]}{\rho^s(q)}. \] (28)

We can write \[dS_{\text{excess}} = \ln(\rho(q)/\rho(q')) + dS_1\] where

\[ dS_1 = \ln \frac{\Gamma^*[q', t + dt|q', t]}{\Gamma*[q, t + dt|q, t]} = \ln \frac{\rho^s(q')}{\rho^s(q)}. \] (29)

which is the same as found in \[11, 13\]. If \[P^*\] is given from \[\Gamma^*\] in the denominator of equation (11), it represents a well-defined path probability since \[\int dq' \Gamma^*[q', t + dt|q, t] = 1\], which can be shown from the property that \[\int dq' \omega'[q, q']\rho^s(q') = 0\]. Therefore we can see that the excess EP for a finite time interval, \[\Delta S_{\text{excess}} = \int dS_{\text{excess}}\] satisfies the IFT.

Using equation (29), we can readily find that

\[ dS_1 = -dt \dot{\phi}\partial_\phi(q), \] (30)
where $\rho^i(q) = \exp(-\phi(q))$. For the Brownian dynamics, we have $dS_1 = -dt(\dot{q}\partial_x\phi(q) + \dot{\phi}(q))$. One can choose any distribution functions for $\rho(q)$ and $\rho(q')$ in equation (28). If the initial and final distributions are chosen as $\rho^s$, then the total excess EP is given as $\Delta S_{\text{excess}} = \int_0^T dt \dot{\lambda}(\partial\phi/\partial\lambda)$, which is exactly the familiar Hatano–Sasa expression [7].

If $\rho(q')$ is chosen as the PDF at time $t+dt$ given the PDF $\rho(q)$ at time $t$, then we have $dS_{\text{excess}} = dS_{\text{sys}} + dS_1$. In this case, the total EP is rearranged as $dS_{\text{tot}} = dS_{\text{excess}} + dS_{\text{hk}}$ from equation (12), where the remaining $dS_{\text{hk}}$, so-called the house-keeping EP, will be introduced in the next section. The average excess EP rate can also be obtained as for the environmental EP. First, we note

$$\langle \frac{dS_{\text{excess}}}{dt} \rangle = -\int dq \left( \partial_t \rho(q) \right) \ln \rho(q) = \int dq \left( \partial_t \rho(q) \right) \ln \rho(q) = \int dq \left( \partial_t \rho(q) \right) \ln \rho(q).$$

(31)

where the stationary state currents $j^s = j^{s,\text{rev}} + j^{s,\text{ir}}$ are defined similarly to equations (22) and (23). One just replaces $\rho(q)$ by $\rho^s(q)$ in the those expressions. For the Brownian dynamics, the above expression again involves only the momentum component of the currents. The first line of equation (31) explicitly exhibits a transient nature of the excess EP rate.

3. Housekeeping entropy production

We now discuss the main subject of the present paper. The remaining part of the total entropy apart from the excess EP is the housekeeping EP: $\Delta S_{\text{hk}} = \Delta S_{\text{tot}} - \Delta S_{\text{excess}}$. In the absence of odd-parity variables, the housekeeping EP satisfies the IFT. From equations (14) and (29), however, we can show that the housekeeping entropy cannot be written as the ratio of two path probabilities, and therefore does not satisfy the IFT. This is in contrast to the case where all the variables have even parity.

Although the whole housekeeping EP does not satisfy the IFT, one can identify a part of housekeeping EP that satisfies the IFT. In a discrete-jumping process, this EP denoted by $\Delta S_{\text{bDB}}$ was calculated and shown to be directly responsible for the breakage of the DB in the stationary state [13]. For the continuous stochastic dynamics, a part of the housekeeping EP denoted by $\Delta S_2$ was separated out and shown to satisfy the IFT [11]. It was obtained by adding up the infinitesimal contributions,

$$dS_2 = \ln \frac{\Gamma[q', t + dt|q, t]}{\Gamma[q, t + dt|q', t]}.$$

(32)

It is not obvious, however, how the expression in equation (32) is related to the broken DB as we shall see in more detail below. In the following, we will show that a direct attempt to construct $\Delta S_{\text{bDB}}$ for the case of continuous stochastic dynamics poses serious
problems. In the subsequent sections, we instead find a series of EPs each of which is a part of housekeeping EP and satisfies the IFT. $\Delta S_2$ found in [11] is one of them. We will show that all these EPs are associated with the DB breakage in the stationary state under a very general assumption.

In the presence of odd-parity variables, the DB condition reads

$$\omega[q', q] \rho^s(q) = \omega[q, q'] \rho^s(q').$$

(33)

Note that we do not assume the parity symmetry of the stationary distribution beforehand in contrast to conventional textbooks [19, 20]. In [20], the DB is defined via the transition probability $\Gamma$ not the transition rate $\omega$ as done here. This definition, however, already assumes the parity symmetry as a part of the DB, since the delta function part of the transition probability in equation (9) directly gives the parity symmetry. In this paper, motivated by the investigation of the DB for discrete dynamics [13], we keep the two conditions, the DB and the parity symmetry as separate independent conditions at the beginning. In section B, we show how these are related for continuous stochastic variables.

3.1. Problems with $\Delta S_{\text{DB}}$ for continuous stochastic dynamics

In order to measure the departure from the DB, we define the adjoint $\dagger$-process

$$\omega^\dagger[q', q] \equiv \omega[q, q'] \frac{\rho^s(q')}{\rho^s(q)}$$

(34)

such that the DB condition is equivalent to the condition $\omega[q', q] = \omega^\dagger[q', q]$. This is a well-defined stochastic process as one can easily see that $\int dq' \omega^\dagger[q', q] = 0$ follows from the stationarity of $\rho^s$. In terms of the transition probability, we have

$$\Gamma^\dagger[q', t + dt|q, t] = \Gamma[q, t + dt|q', t] \frac{\rho^s(q')}{\rho^s(q)} + \delta(q - q') \left(1 - \frac{\rho^s(q')}{\rho^s(q)}\right).$$

(35)

One may expect that the EP associated with the breakage of DB can then be measured by considering the path probability ratio

$$ds_{\text{DB}} \equiv \ln \frac{\Gamma[q', t + dt|q, t]}{\Gamma^\dagger[q', t + dt|q, t]}.$$  

(36)

For a discrete jumping process in the presence of odd-parity variables [13], this quantity has been calculated and the corresponding EP $\Delta S_{\text{DB}}$ during a finite time interval has been shown to satisfy the IFT. However, this procedure cannot be repeated for a continuous stochastic dynamics.

We first look at the $\dagger$-process more closely. Using the quantum mechanical description, we write a term in the right hand side of equation (34) as

$$\omega[q, q'] \rho^s(q') = \langle q'|\{ -\epsilon_j \hat{r}_j, A_i(\epsilon \hat{q}) \} + \epsilon_i \hat{r}_i \hat{r}_j D_{ij}(\epsilon \hat{q})\rangle \rho^s(\epsilon \hat{q})|q').$$

(37)

where $\hat{q}_i, \hat{r}_i$ are non-commuting operators, satisfying $[\hat{r}_j, \hat{q}_i] = \delta_{ij}$, and have the properties: $\langle q| \hat{r}_i |q'\rangle = \partial_i \delta(q - q')$ and $\langle q| \hat{q}_i |q'\rangle = q_i \delta(q - q')$. We can then move $\hat{r}_i$’s to the right inside the bracket using the commutator relation. As a result we have three terms proportional
to \( \delta(q - q') \), \( \partial \delta(q - q') \), and \( \partial \partial \delta(q - q') \), respectively. Note that the term proportional to \( \delta(q - q') \) vanishes because of the stationary condition of \( \rho^s \). From equation (34), we have

\[
\omega^\dagger[q', q] = \left[ -\epsilon_i A_i(\epsilon q)e^{\phi_s(q)} + \frac{2}{\rho_s(q)} \epsilon_i \epsilon_j (\partial_j D_j(\epsilon q) \rho_s(\epsilon q)) \right] \partial_i \partial \delta(q - q')
\]

where

\[
\phi_s(q) = \phi(q) - \phi(\epsilon q) = \ln \frac{\rho^s(\epsilon q)}{\rho^s(q)}.
\]

This dagger process can be put into the standard form like equation (10) by noting that

\[
\omega^\dagger[q', q] = [-\partial_i A_i(\epsilon q) + \partial_j D_j(\epsilon q)] \delta(q' - q)
\]

\[= [A_i(\epsilon q) \partial_i + D_j(\epsilon q) \partial_j] \delta(q' - q).
\]

By comparing the two expressions for \( \omega^\dagger \), we have

\[A_i(\epsilon q) = -\epsilon_i A_i(\epsilon q)e^{\phi_s(q)} + \frac{2 \epsilon_i \epsilon_j}{\rho^s(q)} \partial_j D_j(\epsilon q) \rho^s(\epsilon q),
\]

\[D_j(\epsilon q) = \epsilon_i \epsilon_j D_j(\epsilon q) e^{\phi_s(q)}.
\]

Using the restriction on \( D \), equation (18), we have

\[A_i(\epsilon q) = -\epsilon_i A_i(\epsilon q)e^{\phi_s(q)} + \frac{2}{\rho^s(q)} \partial_j D_j(\epsilon q) \rho^s(\epsilon q),
\]

\[D_j(\epsilon q) = D_j(\epsilon q) e^{\phi_s(q)}.
\]

Explicit expressions for the short-time transition probability \( \Gamma^\dagger[q', t + dt|q, t] \) can be obtained in equation (15) by using \( A^\dagger \) and \( D^\dagger \) in places of \( A \) and \( D \), respectively. Now when we try to calculate \( dS_{\text{BB}} \) directly from equation (36), we note that the presence of \( e^{\phi_s} \) factor in equation (44) makes the two processes \( \Gamma \) and \( \Gamma^\dagger \) with different multiplicative noises. In particular, \( \ln|\det(e^{\phi_s(q)}D(q))/\det(D(q))|^{1/2} \) resulting from equation (36) is clearly not of \( \mathcal{O}(dt) \). This in turn makes \( \Delta S_{\text{BB}} \) for a finite time interval diverge and become ill-defined.

For the Brownian dynamics, we have from equations (17), (43) and (44)

\[
\Gamma^\dagger[q', t + dt|q, t] = \frac{\delta(x' - x - dt(p^{(\alpha)}/m)e^{\phi_s})}{(4\pi e^{\phi_s} dt)^{d/2}|\det(D^{(\alpha)})|^{1/2}} \times \exp\left[ -\frac{dt}{4e^{\phi_s}} H^{(\alpha)} \cdot D^{(\alpha) - 1} \cdot H^{(\alpha)}
\right] - \alpha dt \partial_x \left( e^{\phi_s} \frac{p^{(\alpha)}}{m} \right) - \alpha dt \partial_p \left( e^{\phi_s} \left( G \cdot \frac{p^{(\alpha)}}{m} + f(q^{(\alpha)}) - 2D \cdot \partial_q \phi(q^{(\alpha)}) \right) \right)
\]

\[+ \alpha^2 dt \partial_p \cdot D \cdot \partial_q e^{\phi_s}.
\]
where

$$H^\dagger = \dot{p} - e^{\phi^\dagger} \left( G \cdot \frac{P}{m} + f(\epsilon q) - 2D \cdot \partial_\rho \phi^\dagger(\epsilon q) - 2\alpha D \cdot \partial_\rho \phi^\dagger(q) \right). \tag{46}$$

Here even if we start from an additive noise in the forward process, the dagger process becomes one with a multiplicative noise. Again, we have the same problem with the determinants. In addition, we have a delta function enforcing the equation of motion,

$$\dot{x} = \frac{P}{m} \phi^\dagger(q) \tag{47}$$

for the $\dagger$-process. This signifies the mismatch between the forward path and the one described by the $\dagger$-process.

From the above discussion, it seems impossible to construct the EP for the continuous dynamics associated with the breakage of the DB directly from the discrete stochastic dynamics. Below we introduce another stochastic process for which the path probability ratio taken with the forward process for a finite time interval is well defined. We also want this EP to represent the departure from the DB. Before we proceed, we need to investigate the DB condition in the presence of odd-parity variables more carefully.

### 3.2. The DB condition

As mentioned above, the DB condition is equivalent to setting $\omega = \omega^\dagger$. If $\rho^s$ satisfies the stationary distribution, this condition reduces to $A_i^\dagger = A_i$ and $D_{ij}^\dagger = D_{ij}$ in equations (41) and (42). We note that in a discrete jumping process, the DB and the parity symmetry of the stationary distribution are two independent conditions [13]. From the second condition equation (42), we can see that the parity symmetry, $\phi^\dagger(q) = 0$, is in general not guaranteed unless the diffusion matrix satisfies equation (18). If this relation holds, as we are assuming in this paper, then the DB is equivalent to the parity symmetry. The vanishing irreversible current in the stationary state,

$$j_i^{\text{ir}}(q) = A_i^{\text{ir}}(q) \rho^s(q) - \partial_j D_{ij}(q) \rho^s(q) = 0 \tag{48}$$

also follows from the first condition equation (43). We note that, in conventional textbooks [19], the DB condition is defined as $\omega = \omega^\dagger$ and $\rho^s(q) = \rho^s(\epsilon q)$. The condition equation (18) is then regarded as a requirement for the existence of DB.

Therefore, the DB condition is equivalent to the parity symmetry and equation (48). In the following, we will show that under a very broad assumption, these two conditions are not actually independent. Let us assume only equation (48) for all $i$. Then

$$A_i^{\text{ir}}(q) = \frac{1}{\rho^s(q)} \partial_j D_{ij}(q) \rho^s(q). \tag{49}$$

Multiplying by $\epsilon_i$ and changing $q \rightarrow \epsilon q$, we have

$$\epsilon_i A_i^{\text{ir}}(\epsilon q) = \frac{\epsilon_i \epsilon_j}{\rho^s(\epsilon q)} \partial_j D_{ij}(\epsilon q) \rho^s(\epsilon q). \tag{50}$$
But using $\epsilon_i A^\dagger_i(\epsilon \mathbf{q}) = A^\dagger_i(\mathbf{q})$ and equation (18), we have $D_\beta(\mathbf{q}) \partial_j \phi(\mathbf{q}) = D_\beta(\mathbf{q}) \partial_j \phi(\epsilon \mathbf{q})$. Thus equation (48) implies

$$D_\beta(\mathbf{q}) \partial_j \phi_\Lambda(\mathbf{q}) = 0. \quad (51)$$

Now if $D^{-1}$ exists, we have for all $j, \partial_j \phi_\Lambda(\mathbf{q}) = 0$, or $\phi_\Lambda(\mathbf{q}) = \phi_0$, a constant. But from the normalization

$$1 = \int \! \! \! d\mathbf{q} \rho^s(\mathbf{q}) = e^{-\phi_0} \int \! \! \! d\mathbf{q} \rho^s(\epsilon \mathbf{q}) = e^{-\phi_0} \int \! \! \! d\mathbf{q}' \rho^s(\mathbf{q}') = e^{-\phi_0}, \quad (52)$$

where we have used the integration variable change $\mathbf{q}' = \epsilon \mathbf{q}$ and the fact that $d\mathbf{q}' = d\mathbf{q}$. We therefore conclude that $\phi_0 = 0$ and that the two conditions are not independent, but the vanishing irreversible current in stationary state implies the parity symmetry.

For Brownian dynamics, $D^{-1}$ in the whole space does not exist since the $x$-components of the diffusion matrix vanish as

$$D_{xx} = D_{xp} = D_{px} = 0, \quad D_{pp} = D.$$ 

Therefore from equation (51) we only have $\partial_p \phi_\Lambda(x, p) = 0$ or

$$\phi(x, p) - \phi(x, -p) = \phi_\gamma(x). \quad (53)$$

Following a similar discussion to above, we have

$$\int \! \! \! d\mathbf{p} \rho^s(x, \mathbf{p}) = e^{-\phi_\gamma(x)} \int \! \! \! d\mathbf{p} \rho^s(x, -\mathbf{p}) = e^{-\phi_\gamma(x)} \int \! \! \! d\mathbf{p} \rho^s(x, \mathbf{p}). \quad (54)$$

We again conclude that $\phi_\gamma(x) = 0$ and the parity symmetry of $\rho^s$. We can easily generalize this discussion to the case of several variables where $D^{-1}$ exists at least in the subspace spanned by the odd-parity variables. We believe this is a reasonable assumption for a physical system described by a combination of even and odd parity variables. This is certainly true for the Brownian dynamics. In the following discussion, we will assume this to be true.

3.3. Generalized adjoint process

We now introduce a generalization of the $\dagger$-process for which the path probability ratio taken with the forward process for a finite time interval is well defined. It is based on the similar construction given in [16]. For given $\omega$ and an arbitrary $h(\mathbf{q})$, we define

$$\omega^{\dagger}_h[\epsilon \mathbf{q}, \mathbf{q}] = \omega[\epsilon \mathbf{q}, \epsilon \mathbf{q}'] \frac{h(\mathbf{q}')}{h(\mathbf{q})} - \delta(\mathbf{q} - \mathbf{q}') \frac{1}{h(\mathbf{q})} \int \! \! \! d\mathbf{q}'' \omega[\epsilon \mathbf{q}, \epsilon \mathbf{q}'']h(\mathbf{q}''). \quad (55)$$

The last term ensures the stochasticity of $\omega^{\dagger}_h$, i.e. $\int \! \! \! d\mathbf{q}' \omega^{\dagger}_h[\epsilon \mathbf{q}', \mathbf{q}] = 0$. If we choose $h(\mathbf{q}) = h_0(\mathbf{q}) \equiv \rho^s(\epsilon \mathbf{q})$, then the last term of equation (55) vanishes automatically and we have $\omega^{\dagger}_h[\epsilon \mathbf{q}, \mathbf{q}] = \omega^*[\epsilon \mathbf{q}', \epsilon \mathbf{q}]$ where the $*$-process is defined in equation (27). This was the choice made by Spinney and Ford [11] to construct a component of the housekeeping EP, $\Delta S_2$, that satisfies the IFT.

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Following the same procedure as in equation (38), we can rewrite equation (55) as
\[
\omega_h^\dagger[\mathbf{q}', \mathbf{q}] = \left[ -\epsilon_i A_i(\epsilon \mathbf{q}) + \frac{2}{h(q)}(\partial_j D_{\partial_j}(\mathbf{q}) h(\mathbf{q})) \right] \partial_i \delta(\mathbf{q} - \mathbf{q}') + D_{\partial_j}(\mathbf{q}) \partial_j \delta(\mathbf{q} - \mathbf{q}').
\] (56)

Note that the term proportional to \(\delta(\mathbf{q} - \mathbf{q}')\) is cancelled by the second term on the right hand side of equation (55). Writing \(\omega_h^\dagger[\mathbf{q}', \mathbf{q}] = [A_i^\dagger(\mathbf{q}) \partial_i + D_{\partial_j}(\mathbf{q}) \partial_j] \delta(\mathbf{q} - \mathbf{q}')\), we can identify
\[
A_i^\dagger(\mathbf{q}) = -\epsilon_i A_i(\epsilon \mathbf{q}) + \frac{1}{h(q)} \partial_j D_{\partial_j}(\mathbf{q}) h(\mathbf{q})),
\] (57)
\[
D_{\partial_j}(\mathbf{q}) = D_{\partial_j}(\mathbf{q}).
\] (58)

Because of equation (58), \(\ln \Gamma \Gamma^\dagger\) will now be \(O(\text{d}t)\).

We now study what the condition that \(\omega = \omega_h^\dagger\) means. The path probability ratio
\[
dS_t \equiv \ln \frac{\Gamma[\mathbf{q}', t + \text{d}t|\mathbf{q}, t]}{\Gamma_h^\dagger[\mathbf{q}', t + \text{d}t|\mathbf{q}, t]}
\] (59)
will then measure the breakage of this condition. From equation (57), the condition \(\omega = \omega_h^\dagger\) amounts to
\[
A_i^\dagger(\mathbf{q}) - \frac{1}{h(q)} \partial_j D_{\partial_j}(\mathbf{q}) h(\mathbf{q}) = 0.
\] (60)

We write this condition in terms of \(j_i^\text{s,ir}(\mathbf{q})\). From the definition of \(j_i^\text{s,ir}(\mathbf{q})\) in equation (48), we have the identities
\[
A_i^\dagger(\mathbf{q}) = \frac{j_i^\text{s,ir}(\mathbf{q})}{\rho^s(\epsilon \mathbf{q})} + \frac{1}{\rho^s(\epsilon \mathbf{q})} \partial_j D_{\partial_j}(\mathbf{q}) \rho^s(\epsilon \mathbf{q})
\] (61)
\[
= \frac{\epsilon_i j_i^\text{s,ir}(\epsilon \mathbf{q})}{\rho^s(\epsilon \mathbf{q})} + \frac{1}{\rho^s(\epsilon \mathbf{q})} \partial_j D_{\partial_j}(\mathbf{q}) \rho^s(\epsilon \mathbf{q}),
\] (62)
where in the second equality we have multiplied by \(\epsilon_i\), changed \(\mathbf{q} \rightarrow \epsilon \mathbf{q}\), and used \(\epsilon_i A_i^\dagger(\epsilon \mathbf{q}) = A_i(\epsilon \mathbf{q})\) and equation (18). If we insert this expression into equation (60), we can rewrite the condition \(\omega_h^\dagger = \omega\) as
\[
j_i^\text{s,ir}(\mathbf{q}) = \rho^s(\epsilon \mathbf{q}) D_{\partial_j}(\mathbf{q}) \left[ \frac{\partial_j h(\mathbf{q})}{h(\mathbf{q})} - \frac{\partial_j \rho^s(\epsilon \mathbf{q})}{\rho^s(\epsilon \mathbf{q})} \right],
\] (63)
or
\[
\epsilon_i j_i^\text{s,ir}(\epsilon \mathbf{q}) = \rho^s(\epsilon \mathbf{q}) D_{\partial_j}(\mathbf{q}) \left[ \frac{\partial_j h(\mathbf{q})}{h(\mathbf{q})} - \frac{\partial_j \rho^s(\epsilon \mathbf{q})}{\rho^s(\epsilon \mathbf{q})} \right].
\] (64)

We can easily see for the choices of \(h(\mathbf{q}) = h_0(\mathbf{q}) \equiv \rho^s(\epsilon \mathbf{q})\) and the one by Spinney and Ford, \(h(\mathbf{q}) = h_0(\mathbf{q}) = \rho^s(\epsilon \mathbf{q})\) that the condition is equivalent to \(j_i^\text{s,ir}(\mathbf{q}) = 0\). So in these
cases, we can say that $\Delta S_h$ measures the departure from the state of the vanishing irreversible current in the stationary state. From the discussion in the previous section, the vanishing irreversible current in the stationary state actually means the parity symmetry $\phi_A = 0$ and consequently the DB. In [11], $\Delta S_h$ is denoted by $\Delta S_2$.

The parity symmetry $\phi_A = 0$ also follows directly from the condition equation (60), since we can rewrite that equation as

$$A^i_{ir}(q) - \frac{1}{h(eq)} \partial_j D_{ij}(q) h(eq) = 0,$$

by using the same manipulations as in the steps from equation (61) to (62). Therefore, for the condition $\omega = \omega_h^\dagger$ to hold, $h$ must satisfy

$$D_{ij}(q) \left[ \frac{\partial_j h(q)}{h(q)} - \frac{\partial_j h(eq)}{h(eq)} \right] = 0.$$

For the choices of $h_0$ and $h_1$, this is just equation (51).

We can construct more general $h(q)$ such that the condition $\omega = \omega_h^\dagger$ implies the DB. Let us consider for some number $\sigma$

$$h(q) = h_\sigma(q) \equiv (\rho^\sigma(q))^{\sigma}(\rho^\sigma(eq))^{1-\sigma}.$$

Then equation (66) is rewritten as

$$(1 - 2\sigma)D_{ij}(q)\partial_j \phi_A(q) = 0.$$

For all values of $\sigma$ except for $\sigma = 1/2$, this condition again gives the parity symmetry, $\phi_A = 0$. Using equations (63) and (64), we obtain the condition $\omega = \omega_h^\dagger$ in terms of the irreversible currents as

$$j^{s,ir}_i(q) = (1 - \sigma)\rho^\sigma(q)D_{ij}(q)\partial_j \phi_A(q) = 0,$$

or

$$\epsilon_i j^{s,ir}_i(eq) = -\sigma \rho^\sigma(eq)D_{ij}(q)\partial_j \phi_A(q) = 0,$$

where we have assumed $\sigma \neq 1/2$ for the last equalities in both equations to hold.

### 3.4. Entropy productions

As we have seen in the previous section, when $h(q) = h_\sigma(q)$ for $\sigma \neq 1/2$, the corresponding EP $dS^\sigma_{DB} \equiv dS_{h_\sigma}$ defined in equation (59) can be regarded as a measure of the DB breakage. As this quantity is given by the ratio of two path probabilities, it satisfies the IFT for all $\sigma$. In order to calculate this EP, We first evaluate $\Gamma_\sigma^\dagger \equiv \Gamma_h^\dagger$ by replacing $A_i(q)$ in equation (15) by $A^{\sigma}_i(q)$ in equation (57) with $h(q) = h_\sigma(q)$ given in equation (67).

As before, both $\Gamma$ and $\Gamma^\dagger$ depend on their own discretization schemes. For general multiplicative noises, we find that, if we use the same discretization parameter for both processes, the resulting EP, $dS_\sigma$ is independent of this parameter. Since both processes are forward in time direction in this case, it actually corresponds to taking the same discretised points as in the calculation of $dS_{env}$. When the noise is additive, things are
simpler. We are free to choose the discretization parameters for both processes as the resulting EP is always independent of the parameter anyway. After some algebra, we obtain the EP as

\[
dS_{\text{BBD}}^\sigma = dt \left( q_i - A_i^\text{rev}(\bar{q}) + D_{k}(\bar{q})(\partial_k \psi_A(\bar{q})) \right) \nabla_q^{-1}(\bar{q})(A_i^\text{ir}(\bar{q}) - \partial_i D_{\beta}(\bar{q}) + D_{\beta}(\bar{q}) \partial_i \psi_{\beta}(\bar{q}))
\]

\[
- dt \left( \partial_i A_i^\text{ir}(\bar{q}) - \partial_k D_{\chi}(\bar{q}) + D_{\chi}(\bar{q}) \partial_k \psi_{\chi}(\bar{q}) \right),
\]

(71)

where we define

\[
\psi_{\beta}(\bar{q}) \equiv - \ln h_{\beta}(\bar{q}) = \phi(\epsilon \bar{q}) + \sigma \phi_A(\bar{q}).
\]

For the Brownian dynamics, we obtain

\[
dS_{\text{BBD}}^\sigma = dt \left( \bar{p} - f^\text{rev}(\bar{q}) + \partial_p \psi_{\beta}(\bar{q}) \cdot D \cdot \nabla^{-1} \cdot \left( - \frac{\bar{p}}{m} + f^\text{ir}(\bar{q}) + D \cdot \partial_p \psi_{\beta}(\bar{q}) \right) \right)

- dt \left( - \frac{\bar{p}}{m} + f^\text{ir}(\bar{q}) + D \cdot \partial_p \psi_{\beta}(\bar{q}) \right).
\]

(73)

Note that we again express the EP using the midpoint value \( \bar{q} \). When \( \sigma = 0 \), one can check that equation (71) reduces to the EP studied in [11]. (Equation (59) in that paper is the same expression as ours when \( \sigma = 0 \) and \( D_{\beta}(\bar{q}) = \delta_{\beta} D_{q}(\bar{q}). \))

The average rate of this EP can be obtained in a similar way to the case of the total EP. We obtain the following two equivalent expressions:

\[
\left\langle \frac{dS_{\text{BBD}}^\sigma}{dt} \right\rangle = \int dq \left\{ \epsilon_{ij}^\text{ir}(\epsilon \bar{q}) \rho^\text{ir}(\epsilon \bar{q}) + \sigma D_{q}(\bar{q}) \partial_q \phi_A(\bar{q}) \right\} \times D_j^{-1}(\bar{q}) \left\{ \epsilon_{ij}^\text{ir}(\epsilon \bar{q}) \rho^\text{ir}(\epsilon \bar{q}) + \sigma D_{q}(\bar{q}) \partial_q \phi_A(\bar{q}) \right\} \rho(\bar{q}),
\]

(74)

or

\[
\left\langle \frac{dS_{\text{BBD}}^\sigma}{dt} \right\rangle = \int dq \left\{ f_i^\text{ir}(\bar{q}) - (1 - \sigma) D_{q}(\bar{q}) \partial_q \phi_A(\bar{q}) \right\} \times D_j^{-1}(\bar{q}) \left\{ f_i^\text{ir}(\bar{q}) - (1 - \sigma) D_{q}(\bar{q}) \partial_q \phi_A(\bar{q}) \right\} \rho(\bar{q}).
\]

(75)

As expected, these are positive-definite quantities.

This is of course not the whole housekeeping entropy. The remaining part is denoted by \( \Delta S_{\text{as}}^\sigma = \Delta S_{\text{BB}} - \Delta S_{\text{BBD}}^\sigma \). From the relation

\[
\Delta S_{\text{env}} = \Delta S_1 + \Delta S_{\text{BBD}} + \Delta S_{\text{as}}^\sigma,
\]

(76)

and from equations (14), (29) and (59), we obtain

\[
ds_{\text{as}}^\sigma = \ln \left[ \frac{\Gamma^\star(\epsilon \bar{q}, t + dt|\bar{q}, t)}{\Gamma(\epsilon \bar{q}, t + dt|\bar{q}, t)} \right] \left[ \frac{\Gamma^\star[\bar{q}^\prime, t + dt|\bar{q}, t]}{\Gamma(\bar{q}^\prime, t + dt|\bar{q}, t)} \right].
\]

(77)

This cannot be expressed as the log-ratio of the two conditional probabilities, and therefore there is no IFT for \( \Delta S_{\text{as}}^\sigma \). Using equations (19), (30), (71) and (76), we have

\[
ds_{\text{as}}^\sigma = dt \{ \dot{q}_i \partial_i \phi_A(\bar{q}) + \sigma \partial_t \{ D_{\beta}(\bar{q}) \partial_t \phi_A(\bar{q}) \} \} - \sigma dt(\partial_t \phi_A(\bar{q})(\dot{q}_i + \epsilon_i A_i(\epsilon \bar{q}) - \partial_i D_{\beta}(\bar{q})

+ 2D_{\beta}(\bar{q}) \partial_i \phi_A(\epsilon \bar{q}) + \sigma D_{\beta}(\bar{q}) \partial_i \phi_A(\bar{q})].
\]

(78)
We can see that this expression vanishes when the stationary state distribution is symmetric, i.e. $\phi_A = 0$. Therefore, $\Delta S^\sigma_{as}$ can be regarded as the EP due to the asymmetry of the stationary state distribution. We also note that when $\sigma = 0$, we reproduce the similar one in [11] (called $dS_3$ there). The average rate of this EP can be obtained as before. We obtain

$$\left\langle \frac{dS^\sigma_{as}}{dt} \right\rangle = \int dq \left[ \phi_A(q) \partial_q \rho(q) - \sigma(\partial_q \phi_A(q)) \left\{ \frac{2c_{ji}^{s,ir}(eq)}{\rho^3(eq)} + \sigma D_j(q) \partial_j \phi_A(q) \right\} \rho(q) \right].$$

This quantity is obviously not positive definite as there is no IFT for $\Delta S^\sigma_{as}$. The first term is exactly $\langle dS_3/dt \rangle$ obtained by Spinney and Ford [11], which is a transient contribution to the EP. For any nonzero $\sigma$, however, this EP is not transient and can be regarded as a relevant part of the adiabatic housekeeping EP.

As an example, we consider the one dimensional system driven by a constant force $F$ and a momentum-dependent force $-G'p/m$. We have $\dot{x} = p/m$ and

$$\dot{p} = -\frac{G}{m}p - \frac{G'}{m}p + F + \xi(t)$$

with $\langle \xi(t)\xi(t') \rangle = 2D\delta(t - t')$ and $D = GT$. Suppose that the system has reached the stationary state described by the distribution [11]

$$\rho_s(p) = \frac{1}{\sqrt{2\pi m T_{eff}}} \exp \left\{ -\frac{(p - \langle p \rangle_s)^2}{2m T_{eff}} \right\},$$

where $T_{eff} = GT/(G + G')$ and

$$\langle p \rangle_s = \frac{mF}{G + G'}.$$ \hspace{1cm} (82)

In this case, we have

$$\phi_A(p) = -\frac{2\langle p \rangle_s p}{m T_{eff}},$$ \hspace{1cm} (83)

and therefore $D \partial_p \phi_A(p) = -2(G + G')\langle p \rangle_s/m$.

The combination in equation (74) is given by

$$\frac{f_{p,ir}(p)}{\rho^3(-p)} = \frac{(G + G')p}{m} - D \partial_p \phi(-p) = -\frac{(G + G')\langle p \rangle_s}{m}.$$ \hspace{1cm} (84)

Inserting equation (83) into equation (74), we have the EP rate in the steady state as

$$\left\langle \frac{dS^\sigma_{\text{HDB}}}{dt} \right\rangle_s = (2\sigma - 1)^2 \frac{(G + G')^2\langle p \rangle_s^2}{D m^2} = (2\sigma - 1)^2 \frac{F^2}{D},$$ \hspace{1cm} (85)

which is always positive, as expected from the IFT. Recall that we are considering the case where $\sigma \neq 1/2$. From equation (79), we have

\[ \text{doi:10.1088/1742-5468/2016/09/093205} \]
Then \( \langle dS_{\text{hk}} / dt \rangle_s = \langle d(S_{\text{bDB}}^\sigma + S_{\text{as}}^\sigma) / dt \rangle_s = F^2/D \), which is independent of \( \sigma \).

### 4. Summary and discussion

In summary, we have shown how the housekeeping EP \( \Delta S_{\text{hk}} \) for the continuous stochastic dynamics in the presence of odd-parity variables can be separated into two parts with the former satisfying the IFT. As we have seen, the presence of the odd-parity variables and the possibility of the asymmetry of the stationary state distribution makes this separation nontrivial. For the case of discrete state variables, one particular quantity \( \Delta S_{\text{bDB}}^\sigma \) stands out, which has the physical meaning of describing the departure from the DB [13]. In the continuous variable cases, however, the corresponding quantity turns out to be ill-defined. By considering a generalized adjoint process, we have shown that there is one-parameter family of the EPs for a range of values of the parameter \( \sigma \) belonging to \( \Delta S_{\text{hk}} \). The previously known EPs in [11] are included as a special case for \( \sigma = 0 \).

We have shown that the DB condition is equivalent to the nonvanishing irreversible current and the parity symmetry in the stationary state, \( j^{\text{ir}} = 0 \) and \( \phi_A = 0 \). We exploited the adjoint process in equation (55) parametrized by equation (67) to extract an EP satisfying the IFT from the log-ratio of the two path probabilities for the adjoint process and the original process, respectively. We found the two processes to be equivalent if the DB condition is satisfied. Then, the obtained EP \( \Delta S_{\text{bDB}}^\sigma \) is directly related with the breakage of the DB, characterized by both nonzero \( j^{\text{ir}} \) and the parity asymmetry \( \phi_A \neq 0 \), as seen in equation (75). The remaining part \( \Delta S_{\text{as}}^\sigma \) not satisfying the IFT is also responsible for the breakage of the DB solely due to \( \phi_A \neq 0 \), as seen in equation (79).

In this paper, we have also investigated the similarities and differences between the discrete jumping processes and the continuous stochastic processes described by the master equation and the Fokker–Planck equation, respectively. It may not be surprising that some expressions obtained in the discrete variable case cannot be directly translated into the continuous variable case. In fact, the DB condition automatically implies the parity symmetry of the stationary state distribution for the continuous variable case. When the variables are discrete, however, the two conditions are independent. We have found that this difference in the DB condition is responsible for the degeneracy in \( \Delta S_{\text{bDB}}^\sigma \) for the continuous variable case. In the discrete jumping process, the remaining part of the housekeeping EP apart from the one associated with the breakage of DB has been shown to be not transient [13]. This term measures the parity asymmetry of the stationary state distribution. We also note that the corresponding quantity in our case, \( \Delta S_{\text{as}}^\sigma \), is also not transient for nonzero values of the parameter \( \sigma \).

It is consistent with the persistency of the nonequilibrium steady state with the broken DB. Only when \( \sigma = 0 \), this contribution exists only transiently, which corresponds to the case studied by Spinney and Ford [11].

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