Massive Vector Gauge Theory and Comparison with Higgs-Connes-Lott

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Abstract

A massive vector gauge theory constructed from the matrix derivative approach of noncommutative geometry is compared with the Higgs-Connes-Lott theory. In the massive vector gauge theory, a new extra shift-like symmetry which is due to the one form constant matrix derivative allows the theory to have a mass term while keeping the gauge symmetry intact. In the Higgs-Connes-Lott theory, the transformation of scalar field makes up the deficiency of symmetry due to the mass term. Thus, when the scalar field is absent there remains no gauge symmetry just like the Proca model. In the massive vector gauge theory, the shift-like symmetry makes up the deficiency of symmetry due to the mass term even in the absence of the scalar field.

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I. Introduction

So far, the Higgs mechanism [1] is the only known viable mass generating mechanism for gauge bosons in four dimensions. In 1990, Connes and Lott [2] showed that the standard model can be obtained from the framework of noncommutative geometry. In that framework, the scalar field is treated as an element of the connection in the enlarged noncommutative space and thus the Yang-Mills-Higgs action is naturally obtained from a noncommutative version of the Yang-Mills action. And the ad hoc insertion of the negative mass squared term for spontaneous symmetry breaking is replaced by a generalization of the Dirac operator. In this noncommutative framework, symmetry breaking does not occur in a spontaneous manner, rather it is built in from the beginning by the discrete part of the generalized Dirac operator. However, both Higgs mechanism and Connes-Lott model contain one form gauge field and zero form scalar field, and their field contents and physical implications are the same. So, we will call these two Higgs-Connes-Lott theory from now on.

Recently, in Ref. [3] it was shown that one can construct a massive gauge theory without introducing a scalar field by generalizing the matrix derivative approach [4] of noncommutative geometry. In this construction, the gauge field has an extra shift-like symmetry besides the usual gauge symmetry, and that is due to the generalized matrix derivative consisting of a constant one form even matrix. In the ordinary gauge theory, if one adds a mass term to the Yang-Mills action, then gauge symmetry is broken and there remains no gauge symmetry. This is the case of the massive Proca model. However, in the massive gauge theory, the deficiency of symmetry due to the mass term is cured by the extra shift-like symmetry of the gauge field. And this extra symmetry exactly does the role of the scalar field in the Higgs-Connes-Lott theory, in which the scalar field transformation compensates the deficiency of symmetry due to the mass term and thus makes the underlying gauge symmetry maintained.

In this paper, we first briefly review the Higgs-Connes-Lott theory in Section II, then
in Section III we explain the massive vector gauge theory constructed in the framework of the matrix derivative approach of noncommutative geometry. In Section IV, we compare the symmetry properties of the Higgs-Connes-Lott theory and the massive vector gauge theory. We conclude in Section V.

II. Higgs-Connes-Lott theory

In 1995, Connes \cite{5} has modified the Connes-Lott model \cite{2} slightly based on the reality of the spectral triple \(\{\mathcal{A} \mathcal{H} \mathcal{D}\}\). Here, we shall follow this revised version, and its technical details will be referred to Ref. \cite{5}.

A spectral triple is given by an involutive algebra of operators \(\mathcal{A}\) in a Hilbert space \(\mathcal{H}\) and a selfadjoint operator \(\mathcal{D}\) in \(\mathcal{H}\). It is called even when there exists a \(\mathbb{Z}_2\) grading operator \(\Gamma\) in \(\mathcal{H}\), and otherwise it is called odd. Here, \(\Gamma\) commutes with any element in \(\mathcal{A}\), and anticommutes with \(\mathcal{D}\).

The algebra \(\mathcal{A}\) plays the role of the algebra of coordinates on the space, and it can be noncommutative, and thus the name noncommutative geometry. \(\mathcal{D}\) is usually called the generalized Dirac operator, since in the Riemannian case it just becomes the usual Dirac operator. \(\Gamma\) corresponds to the chirality operator in a \(\mathbb{Z}_2\) graded Hilbert space, and in some sense this is related to the orientation of the space that one deals with.

Another important ingredient of the Connes-Lott approach is the \(\pi\) representation of the universal differential envelop of \(\mathcal{A}\), \(\Omega^*(\mathcal{A})\), where \(\Omega^*(\mathcal{A}) = \bigoplus \Omega^k(\mathcal{A})\) such that \(\Omega^0(\mathcal{A}) = \mathcal{A}\) and \(\Omega^k(\mathcal{A}) = \{a_0\delta a_1 \cdots \delta a_k; a_0, a_1, \cdots, a_k \in \mathcal{A}\}\), the space of universal k-forms. The differential \(\delta\) satisfies \(\delta^2 = 0\), \(\delta(a_0\delta a_1 \cdots \delta a_k) = \delta a_0\delta a_1 \cdots \delta a_k \in \Omega^{k+1}(\mathcal{A})\), and the involution \(*\) is given by \((a_0\delta a_1 \cdots \delta a_k)^* = \delta a_k^* \cdots \delta a_1^* a_0^*\). Now, \(\pi\) is a map from \(\Omega^*(\mathcal{A})\) to \(\mathcal{B}(\mathcal{H})\), the space of bounded operators on \(\mathcal{H}\), given by

\[
\pi(a_0\delta a_1 \cdots \delta a_k) = \rho(a_0)[\mathcal{D}, \rho(a_1)] \cdots [\mathcal{D}, \rho(a_k)],
\]

where \(\rho\) is a faithful representation of \(\mathcal{A}\) by bounded operators on the Hilbert space \(\mathcal{H}\). Notice that, in order to respect the nilpotency of \(\delta\), \(\mathcal{D}\) should satisfy \([\mathcal{D}, [\mathcal{D}, \cdot]] = 0\) in
A tensor product of two noncommutative spaces with spectral triples, \((A_1, H_1, D_1, \Gamma_1)\) and \((A_2, H_2, D_2, \Gamma_2)\), is defined as
\[
\begin{align*}
A &= A_1 \otimes A_2, \\
H &= H_1 \otimes H_2, \\
D &= D_1 \otimes 1 + \Gamma_1 \otimes D_2, \\
\Gamma &= \Gamma_1 \otimes \Gamma_2.
\end{align*}
\]

Now, we briefly review some simple models in the Connes-Lott approach.

1. Two-point space

We take \(A = C \oplus C, \ H = C^N \oplus C^N, \) and \(D = \left( \begin{array}{cc} 0 & M \\ \overline{M} & 0 \end{array} \right), \ \Gamma = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)\)
where \(M\) is an \(N \times N\) matrix. Assume \(a = (\lambda, \lambda') \in A\), then
\[
\pi(\delta a) = [D, \rho(a)] = (\lambda - \lambda') \left( \begin{array}{cc} 0 & -M \\ M & 0 \end{array} \right).
\]

Also, from \(J = a_0 \delta a_1 = (u, u') \cdot (v' - v, v - v') = (u(v' - v), u'(v - v'))\)
and denoting it by \(J \equiv (\phi, \overline{\phi})\), we can write \(\pi(J) = \left( \begin{array}{cc} 0 & \phi M \\ \overline{\phi M} & 0 \end{array} \right)\)

The \(\pi\) representation of the curvature \(\theta\) is given by \(\pi(\theta) = \pi(\delta J + J^2) = (|\phi + 1|^2 - 1) \left( \begin{array}{cc} M \overline{M} & 0 \\ 0 & \overline{M} M \end{array} \right)\).

Thus, the Yang-Mills action is given by \(I_\nabla = \text{Tr}_\omega((\pi(\theta))^2 D_\nabla^{-n})\) where \(\text{Tr}_\omega\) is the Dixmier trace, \(n\) is the dimension of the manifold, and \(D_\nabla\) is the \(\text{“covariant derivative”}\) given by \(D_\nabla = D + \pi(J)\).

Since \(n\) is zero and the Dixmier trace becomes the usual trace in the present case, the Yang-Mills action is given by \(I_\nabla = 2(|\phi + 1|^2 - 1)^2 \text{Tr}(M \overline{M})^2\). This is just the Higgs potential with minima at \(\phi = 0, -2,\) and its type indicates explicitly broken symmetry.

2. Spinmanifold

Take \(A = C^\infty(Z) \otimes C\) where \(Z\) is a 4-dimensional spinmanifold, and let \(H = L^2(S)\)
where \(S\) is the vector bundle of spinors on \(Z\). Then, the Dirac and the chirality op-
operators become the usual ones, $\mathcal{D} = \gamma^\mu \partial_\mu$ and $\Gamma = \gamma_5$. The connection is an ordinary differential 1-form on $Z$, $\pi(\mathcal{J}) = A = A_\mu dx^\mu$. Thus, the curvature is given as the usual one $\pi(\theta) = F = dA + A^2$, and the Yang-Mills action is $I_\nabla = \int_Z \text{Tr}(F \star F)$.

3. U(1)xSU(2) model

Take $A = C^\infty(Z) \otimes \mathcal{A}_F$ where $\mathcal{A}_F = \mathbb{C} \oplus \mathcal{H}$ and $\mathcal{H} = L^2(S) \otimes \mathcal{H}_F$ where $\mathcal{H}_F = \mathcal{E} \oplus \bar{\mathcal{E}}$. Here, we will consider the lepton part only, and $\mathcal{E}$ is the finite dimensional Hilbert space whose basis is labeled by all leptons and $\bar{\mathcal{E}}$ denotes its complex conjugate Hilbert space. $\mathcal{D}$ is given by

$$\mathcal{D} = \gamma^\mu \partial_\mu \otimes I + \gamma_5 \otimes \begin{pmatrix} Y & 0 \\ 0 & \bar{Y} \end{pmatrix},$$

where $Y = \begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix}$ and the lepton mass matrix $M$ is given by $\begin{pmatrix} 0 & 1 \end{pmatrix} \otimes \text{diag}(m_e, m_\mu, m_\tau)$. The real structure $J$ is given by the charge conjugation times $J_F$ where

$$J_F(\xi, \bar{\eta}) = (\eta, \bar{\xi}) \quad \forall \xi \in \mathcal{E}, \quad \bar{\eta} \in \bar{\mathcal{E}}.$$

The algebra $\mathcal{A}$ acts on $\mathcal{E}$ and $\bar{\mathcal{E}}$ in the following manner. Let $a = (\lambda, q) \in \mathbb{C} \oplus \mathcal{H}$, then its action on $\mathcal{E}$ is given by the following. Only $q$ acts on lepton doublets, $a \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L = q \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$, and only $\lambda$ acts on lepton singlets, $ae_R = \bar{\lambda}e_R$. On $\bar{\mathcal{E}}$, only $\lambda$ acts

$$\lambda \bar{\xi} = (\bar{\lambda} \xi), \quad \forall \lambda \in \mathbb{C}, \; \xi \in \mathcal{E}, \; \bar{\xi} \in \bar{\mathcal{E}},$$

and $q$ has no action. Note that the action of $Ja^*J^{-1}$ on the lepton doublet in $\mathcal{E}$ is given by multiplication by $\lambda$, and the same action on $\bar{\mathcal{E}}$ makes $q$ act on the lepton doublet.

If we evaluate the connection and curvature and construct the Yang-Mills action following the two-point space case that we did above, we obtain the following action of the Yang-Mills-Higgs type after some calculation.

$$I_\nabla = \int d^4x \text{Tr}\left(-\frac{1}{4}(F^B_{\mu\nu})^2 - \frac{1}{4}(F^W_{\mu\nu})^2\right) \quad (3)$$
\[
\sum_{\text{leptons}} [\bar{f}_L \gamma^\mu (\partial_\mu + B_\mu + W_\mu) f_L + \bar{f}_R \gamma^\mu (\partial_\mu + B_\mu) f_R] \\
+ \sum_{f f'} [M_{ff'} \bar{f}_L \phi f' + \text{C.C.}] + [(\partial_\mu + B_\mu + W_\mu) \phi] - \alpha |\phi|^2 + \beta (|\phi|^2)
\]

Here, \(\alpha\) and \(\beta\) are positive real parameters, \(B\) and \(W\) are \(U(1)\) and \(SU(2)\) gauge fields, respectively, \(\phi\) is a complex scalar doublet, and \(f_L\) and \(f_R\) are lepton doublets and singlets, respectively.

### III. Massive vector gauge theory

In 1990, Ne’eman and Sternberg \[6\] first applied the concept of superconnection \[\] for the Higgs mechanism. They considered 0-form scalar field and 1-form gauge field as a multiplet of a superconnection and wrote down the Yang-Mills-Higgs action by inserting the negative mass squared term for the scalar field.

With the concept of matrix derivative, this was done more naturally without inserting the negative mass squared term in the noncommutative framework \[4, 8, 9\]. In this section, we first consider how one can generalize the matrix derivative from the superconnection viewpoint \[10\], then we construct the massive vector gauge theory using the generalized matrix derivative \[3\].

Consider a super (or \(Z_2\)-graded) complex vector space, \(V = V^+ \oplus V^-\). The algebra of endomorphisms of \(V\) is a superalgebra with the even or odd endomorphisms. Let \(\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-\) be a super (or \(Z_2\)-graded) vector bundle over a manifold \(M\), and \(\Omega(M) = \oplus \Omega^k(M)\) be the algebra of smooth differential forms with complex coefficients. Then, the space of \(\mathcal{E}\) valued differential forms on \(M\), \(\Omega(M, \mathcal{E})\), has a \(Z \times Z_2\) grading, and the fibers of \(\mathcal{E}\) are superspaces. Here we are mainly concerned with its total \(Z_2\) grading.

In Ref. \[10\], we showed that a generalization of the superconnection concept can yield the matrix derivative of the noncommutative geometric gauge theory \[3\]. There, the generalized superconnection is given by \[10\]

\[
\nabla = d + \omega,
\]
where $d_t = d + d_M$ is a generalization of the one form exterior derivative satisfying the derivation property, and $\omega$ is a generalized connection given by $\omega = \begin{pmatrix} \omega_0 & L_{01} \\ L_{10} & \omega_1 \end{pmatrix}$.

Here, $\omega_0$, $\omega_1$ are matrices of odd degree differential forms and $L_{01}$, $L_{10}$ are matrices of even degree differential forms. The multiplication rule is given by

$$(u \otimes a) \cdot (v \otimes b) = (-1)^{|a||v|}(uv) \otimes (ab), \quad u, v \in \Omega(M), \quad a, b \in A,$$

where $A$ is the endomorphisms of $V$. Then the tensor product of $\Omega(M)$ and $A$ belongs to the endomorphisms of $\Omega(M, E)$. In the matrix representation, $d = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}$ where $d$ inside the matrix denotes the usual 1-form exterior derivative times a unit matrix, and $d_M$ is given below. Since $d_M$ should behave as a part of the superconnection operator in a sense, we write it as a (graded) commutator operator

$$d_M = [\eta, \cdot], \quad \eta \in \Omega(M, E).$$

Then, $d_M$ should satisfy

$$d_M^2 = 0, \quad dd_M + d_M d = 0,$$

$$d_M(\alpha \beta) = (d_M \alpha) \beta + (-1)^{|\alpha|} \alpha (d_M \beta), \quad \alpha, \beta \in \Omega(M, E).$$

In Ref. [3], two simple solutions satisfying the above conditions were given by

1. $\eta = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ where $u$, $v$ are odd degree closed forms with their coefficient matrices satisfying $u^2 = v^2 \propto 1$ or $u^2 = v^2 = 0$,

2. $\eta = \begin{pmatrix} 0 & m \\ n & 0 \end{pmatrix}$ where $m$, $n$ are even degree closed forms with their coefficient matrices satisfying $mn = nm \propto 1$ or $mn = nm = 0$.

If we take the second solution with 0-form $m$, $n$, then this choice yields the so-called matrix derivative $d_M = [\eta, \cdot]$ with $\eta = \begin{pmatrix} 0 & \zeta \\ \zeta & 0 \end{pmatrix}$ where $\zeta$, $\overline{\zeta}$ are 0-form constant matrices satisfying $\zeta \overline{\zeta} = \overline{\zeta} \zeta \propto 1$.

With the use of the generalized superconnection, the curvature is now given by

$$\mathcal{F}_t = (d_t + \omega)^2 = d_0 \omega + \omega^2.$$
In this formulation, the Yang-Mills action is given by

\[ I_{YM} = \int_M \text{Tr}(F_t^* \cdot F_t) \]  \hspace{1cm} (9)

where \(*\) denotes taking dual for each entries of \( F_t \) as well as taking Hermitian conjugate.

The fermionic action is given by

\[ I_{sp} = \int_M \overline{\Psi} \gamma^\mu (dt + \omega)_\mu \Psi, \quad \Psi \in V \otimes S \]  \hspace{1cm} (10)

where \( S \) is a spinor bundle.

Now, we consider the first solution for \( d_M \) given above with \( \eta = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma' \end{pmatrix} \) where \( \sigma, \sigma' \) are constant 1-form matrices whose squares are either proportional to a unit matrix or zero. For the generalized connection \( \omega \), we set

\[ \omega = \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix} \]  \hspace{1cm} (11)

where \( A, A' \) consist of one forms only.

For a definite understanding, we consider the case where \( A, A' \) are SU(2) valued 1-form fields, and \( \sigma \) and \( \sigma' \) are proportional to a SU(2) Pauli matrix, say \( \tau_3 \):

\[
\begin{align*}
A &= \frac{i}{2} A^\mu_{\alpha} \tau_\alpha dx^\mu \equiv A_\mu dx^\mu, \\
A' &= \frac{i}{2} A'^\mu_{\alpha} \tau_\alpha dx^\mu \equiv A'_\mu dx^\mu, \\
\sigma &= \sigma' = \frac{i}{2} m \tau_3 n_\mu dx^\mu \equiv \sigma_\mu dx^\mu.
\end{align*}
\]  \hspace{1cm} (12)

Here \( \tau \)'s are Pauli matrices, \( n_\mu \) is a constant four vector, and \( m \) is a constant parameter. Throughout the paper, we use the metric \( g_{\mu \nu} = (-1, +1, +1, +1) \) and \( \epsilon_{0123} = +1 \), and the wedge product between forms is understood.

The curvature

\[ F_t = d\omega + \omega^2 \]  \hspace{1cm} (13)

is now given by

\[
F_t = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix} + \left[ \begin{pmatrix} \sigma & 0 \\ 0 & \sigma' \end{pmatrix}, \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix} \right]_+ + \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix} \cdot \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix}.
\]
The first and third terms are the usual ones and the second term is a new piece due to the matrix derivative which we calculate below. Since all the odd parts are vanishing, the upper and lower diagonal parts do not mix. Hence, we mostly consider the upper part in our calculation.

And,

\[\sigma A + A\sigma = \frac{1}{2} \left\{ [\sigma_\mu, A_\nu] - [\sigma_\nu, A_\mu] \right\} dx^\mu dx^\nu\]

\[= -\frac{1}{4} m \ n_{[\mu} \left( \begin{array}{cc} 0 & A_1 - iA_2 \\ -A_1 - iA_2 & 0 \end{array} \right)_{\nu]} dx^\mu dx^\nu\]

\[\equiv \frac{1}{2} A_{\mu\nu} dx^\mu dx^\nu. \quad (14)\]

Thus the curvature is given by

\[F_t = \left( \begin{array}{cc} F_t & 0 \\ 0 & F_t' \end{array} \right) \quad (15)\]

with

\[F_t = \frac{1}{2} (F_{\mu\nu} + A_{\mu\nu}) dx^\mu dx^\nu. \quad (16)\]

and \(F'_t\) is the same as \(F_t\) except that \(A\) is replaced by \(A'\), and \(F_{\mu\nu}\) is the usual one,

\[F_{\mu\nu} = \partial_{[\mu} A_{\nu]} + [A_\mu, A_\nu]. \quad (17)\]

Following the same calculational step as in Ref. [9], we obtain the Yang-Mills type action of massive gauge fields from Eq. (9),

\[I_{YM} = \int_M \text{Tr} (F_t^* \cdot F_t) \quad (18)\]

\[= \frac{1}{2} \int_M d^4 x \ \text{Tr} [(F_{\mu\nu} + A_{\mu\nu}) (F^{\mu\nu} + A^{\mu\nu}) + (\text{terms with } A \to A')] \]

\[= \frac{1}{2} \int_M d^4 x \ \text{Tr} [F_{\mu\nu} F^{\mu\nu} + F_{\mu\nu} A^{\mu\nu} + A_{\mu\nu} F^{\mu\nu} + A_{\mu\nu} A^{\mu\nu} + (\text{terms with } A \to A')]. \]

The fourth term provides quadratic terms homogeneous in \(A_1\) and \(A_2\):

\[\frac{1}{2} \text{Tr} A_{\mu\nu} A^{\mu\nu} = \frac{1}{2} m^2 \left[ -n_{[\mu} n_{\nu]} (A_1 A^{\mu\nu} + A_2 A^{\mu\nu}) + n_{[\mu} n_{\nu]} (A_1 A_1^{\mu\nu} + A_2 A_2^{\mu\nu}) \right]. \quad (19)\]
The second and third terms also give terms quadratic in $A$ but mixed in $A_1$ and $A_2$:

$$\frac{1}{2} \text{Tr} \left( F_{\mu\nu} A^{\mu\nu} + A_{\mu\nu} F^{\mu\nu} \right) = m \epsilon^{ab} \left( n_{\mu} A_{\alpha\nu} \partial^\mu A_b^\nu - n_{\mu} A_{\alpha\nu} \partial^\nu A_b^\mu \right) + O(A^3) \quad (20)$$

where $a, b = 1, 2$ and $\epsilon^{12} = -\epsilon^{21} = 1, \epsilon^{11} = \epsilon^{22} = 0$.

Before we perform diagonalization and obtain the propagators for these fields, we first identify the symmetry of the action. In Ref. [9], the so-called horizontality condition was used to analyze the BRST symmetry of the noncommutative geometric gauge theory. Since we use the same superconnection framework, the BRST analysis will be more convenient for finding the symmetry of the theory. In the Yang-Mills theory, the horizontality condition is given by [12, 13, 14, 15]

$$\tilde{F} \equiv \tilde{d} \tilde{A} + \tilde{A} \tilde{A} = F, \quad (21)$$

where

$$\tilde{A} = A_{\mu} dx^\mu + A_N dy^N + A_N d\bar{y}^N \equiv A + c + \bar{c},$$

$$\tilde{d} = d + s + \bar{s}, \quad d = dx^\mu \partial_\mu, \quad s = dy^N \partial_N, \quad \bar{s} = d\bar{y}^N \partial_{\bar{N}},$$

$$F = dA + AA = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu.$$

Here $y$ and $\bar{y}$ denote the coordinates in the direction of gauge orbit of the principal fiber whose structure-group is $G \otimes G$, and $c, \bar{c}$ are ghost and antighost fields. The above horizontality condition now yields the BRST and anti-BRST transformation rules for the Yang-Mills theory.

$$(dy)^1 (dy)^1 : sc = -cc,$$

$$(dy)^2 : s\bar{c} = -\bar{c}c,$$

$$(dy)^1 (d\bar{y})^1 : s\bar{c} + \bar{s}c = -[c, \bar{c}]. \quad (22)$$

In the superconnection framework, the horizontality condition is given as follows [11].

$$\tilde{\mathcal{F}}_t \equiv \tilde{d}_t \tilde{\omega} + \tilde{\omega} \cdot \tilde{\omega} = \mathcal{F}_t \quad (23)$$
where
\[
\tilde{d}_t = d_t + s + \bar{s}, \quad (24)
\]
\[
\tilde{\omega} = \omega + C + \bar{C}, \quad (25)
\]
and
\[
s = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}, \quad \bar{s} = \begin{pmatrix} \bar{s} & 0 \\ 0 & \bar{s} \end{pmatrix}, \quad C = \begin{pmatrix} c & 0 \\ 0 & c' \end{pmatrix}, \quad \bar{C} = \begin{pmatrix} \bar{c} & 0 \\ 0 & \bar{c'} \end{pmatrix}. \quad (26)
\]
The above horizontality condition yields the following BRST and anti-BRST transformation rules:

\[
(dy)^1 : s\omega = -d_tC - \omega \cdot C - C \cdot \omega, \quad (27)
\]
\[
(d\bar{y})^1 : \bar{s}\omega = -d_T\bar{C} - \omega \cdot \bar{C} - \bar{C} \cdot \omega, \quad (28)
\]
\[
(dy)^2 : sC = -C \cdot C, \quad (29)
\]
\[
(d\bar{y})^2 : \bar{s}\bar{C} = -\bar{C} \cdot \bar{C}, \quad (30)
\]
\[
(dy)^1 (d\bar{y})^1 : sC + \bar{s}\bar{C} + C \cdot \bar{C} + \bar{C} \cdot C = 0. \quad (31)
\]
Since all the odd parts vanish as before, we again consider only the upper diagonal (even) parts in our calculation. Then, the BRST and anti-BRST transformation rules for the fields appearing in the upper parts can be written as

\[
sA = -dc - [\sigma, c]_+ - [A, c]_+, \quad (32)
\]
\[
\bar{s}A = -d\bar{c} - [\sigma, \bar{c}]_+ - [A, \bar{c}]_+, \quad (33)
\]
\[
sC = -cc, \quad (34)
\]
\[
\bar{s}\bar{c} = -\bar{c}\bar{c}, \quad (35)
\]
\[
s\bar{c} + \bar{s}\bar{c} + c\bar{c} + \bar{c}c = 0, \quad (36)
\]
where
\[
c = \frac{i}{2} c_a \tau^a, \quad \bar{c} = \frac{i}{2} \bar{c}_a \tau^a, \quad a = 1, 2, 3.
\]
Now, one can check that the above BRST and anti-BRST transformations are nilpotent, 
\[
s^2 = \bar{s}^2 = 0, \quad \text{and the total curvature } F_t = dA + AA + \sigma A + A\sigma \text{ transforms as the usual}
curvature $F = dA + AA$,\n\[ sF_t = -[c, F_t]. \] (37)
Therefore, our Yang-Mills action,
\[ I_0^{YM} = \int_M \text{Tr} F^*_t F_t = \frac{1}{2} \int_M \text{d}^4x \text{Tr} [(F_{\mu\nu} + A_{\mu\nu}) (F^{\mu\nu} + A^{\mu\nu})] \] (38)
where $*$ denotes the Hodge dual, is invariant under the above given BRST(anti-BRST) transformation. Since the BRST and gauge transformations for classical fields are the same except for a switch between the classical gauge parameter and the ghost field, one can check that the action (38) is invariant under the following gauge transformation
\[ \delta A_\mu = \partial_\mu \epsilon + [A_\mu, \epsilon] + [\sigma_\mu, \epsilon] \] (39)
where $\epsilon = \frac{i}{2} \epsilon_a \tau^a \ (a = 1, 2, 3)$ is a zero form gauge parameter. In order to obtain the propagators we use the following gauge fixing term for the action (18)
\[ L_{g.f.} = \frac{1}{\xi} \text{Tr}(d_i \omega^*)^2, \] (40)
which is translated into the following condition for the action (38),
\[ L_{g.f.}^0 = \frac{1}{\xi} \text{Tr} (\partial_\mu A^\mu + \sigma_\mu A^\mu - A_\mu \sigma^\mu)^2 \] (41)
where $\sigma_\mu, A_\mu$ are given by Eq. (12). In terms of
\[ W^\mu = \frac{1}{\sqrt{2}} (A^\mu_1 \mp i A^\mu_2), \] (42)
we obtain the propagators for $W_\pm, A_3$, after some calculation:
\[ W_\pm : \triangle_{\mu, \nu}^\pm = \frac{1}{(P^\pm)^2} \left( g_{\mu\nu} + (\xi - 1) \frac{P_\mu P_\nu}{(P^\pm)^2} \right), \]
\[ A_3 : \triangle_{\mu, \nu}^3 = \frac{1}{p^4} \left( g_{\mu\nu} + (\xi - 1) \frac{p_\mu p_\nu}{p^2} \right), \] (43)
where $P_\mu = p_\mu \pm mn_\mu$. If we set $n \cdot p = n_\mu p^\mu = 0$, then the denominator of the $W$-propagator becomes $(P^\pm)^2 = p^2 + m^2 n^2$. Thus, choosing $n_\mu$ satisfying the above condition exhibits the $W$ field having a mass; $(\text{mass})^2 = m^2 n^2$. 

12
In Eq. (43), the propagators for $W_{\pm}$ look apparently different by a term involving the gauge fixing parameter $\xi$. This difference can be removed for the choice of $\xi = 1$, and for other $\xi$ values we expect that the contribution from this $\xi$ related term be cancelled by that of ghosts since the gauge fixing should not affect the physics.

It is also possible to provide mass terms for all the gauge fields by properly choosing a constant one form even matrix for the matrix derivative. For instance, if we replace $\tau_3$ appearing in $\sigma_{\mu} = \frac{i}{2} m n_{\mu} \tau_3$ with $\tau_1 + \tau_2 + \tau_3$, then all $A_1, A_2, A_3$ fields become massive. However, if we use an identity matrix for $\sigma_{\mu}$, then there will be no massive vector gauge field.

IV. Comparison of symmetries

In the Higgs-Connes-Lott theory, the gauge field transforms in the usual way

$$\delta A_{\mu} = D_{\mu} \epsilon,$$

while the scalar field transforms with an additional term

$$\delta \phi = \phi \epsilon + \epsilon \zeta$$

where $\epsilon$ is a 0-form gauge parameter and $\zeta$ is an odd part of a constant 0-form odd matrix which is the discrete part of the generalized Dirac operator. Because of this extra piece in the transformation, the odd curvature component which provides the mass term for the gauge boson transforms covariantly,

$$\delta (\partial_{\mu} \phi + A_{\mu} \phi + A_{\mu} \zeta) = \epsilon (\partial_{\mu} \phi + A_{\mu} \phi + A_{\mu} \zeta).$$

Here, the shift-like part of $\phi$-transformation related to the discrete part of the Dirac operator cancels the non-covariant transformation part of the $\zeta$ term, which provides the mass term. This makes the theory invariant under the gauge transformation even with the mass term. However, if there is no scalar field, $\phi = 0$, then the action is not invariant under $\delta A_{\mu} = D_{\mu} \epsilon$, and the action exactly resembles the Proca’s.
In the massive gauge theory, the gauge field itself has a shift-like part in its transformation due to the action of the matrix derivative

$$\delta A_\mu = D_\mu \epsilon + [\sigma_\mu, \epsilon]. \quad (47)$$

The shift-like extra piece in this case also does the role of the shift-like transformation of the scalar field in the Higgs-Connes-Lott case. And as we have seen in the previous section, the action constructed with the matrix derivative of even matrix is gauge invariant under this transformation although it includes mass terms. This is the way how the gauge theory can have a mass term for the gauge field while keeping the gauge symmetry intact even without a scalar field.

Here, we would like to note a characteristic feature of the noncommutative construction. Because of the shift-like transformation of the scalar field, one might wonder whether a simple replacement of $\phi$ with $\phi + \zeta$ yields the result of the Higgs-Connes-Lott. The answer is no. It simply shifts the vacuum by $-\zeta$, and does not change the potential shape into an inverted Mexican hat which is essential for symmetry breaking. The Connes-Lott model exactly does that job. The discrete part of the generalized Dirac operator provides the needed potential shape:

$$V(\phi) \sim [(\phi + \zeta)^2 - \zeta^2]^2.$$ 

Also, one can not have the shift-like extra transformation piece for the scalar field unless the generalized Dirac operator has a discrete piece.

In the massive vector gauge theory case, the simple replacement of $A_\mu$ with $A_\mu + \sigma_\mu$ does not do the job either. As in the Higgs-Connes-Lott case, the extra shift-like piece in the gauge field transformation, which is due to the matrix derivative of even matrix, makes this theory have the gauge symmetry even with the mass term. This mechanism exactly parallels to that of the Higgs-Connes-Lott theory, as the shift-like piece compensates the symmetry breaking piece of the transformation from the mass term.
V. Conclusion

In this paper, we construct a massive vector gauge theory which possesses both the usual gauge symmetry and a shift-like symmetry, then compare it with the Higgs-Connes-Lott theory.

In the usual construction of noncommutative geometric gauge theory, only a zero form constant odd matrix has been used for the matrix derivative or for the discrete part of the generalized Dirac operator. In the matrix derivative approach, this constant zero form odd matrix together with scalar fields appearing in the odd part of the gauge multiplet (or superconnection) give rise to the Higgs mechanism. In the Connes-Lott formalism, this constant odd matrix does the role of the generalized Dirac operator acting on a discrete space. However, by constructing the noncommutative geometric gauge theory from the superconnection viewpoint, it is also possible to use a constant one form even matrix for the matrix derivative.

The matrix derivative of constant one form even matrix provides a shift-like symmetry to the gauge field, and this shift-like symmetry transformation of gauge field does the role of the scalar field transformation in the Higgs-Connes-Lott theory, where the scalar field transformation compensates the deficiency of symmetry due to the mass term and makes the underlying gauge symmetry maintained. This way it becomes possible to construct a massive vector gauge model similar to the Proca’s while keeping its gauge symmetry intact.

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