Exact inference in structured prediction

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Abstract

Structured prediction can be thought of as a simultaneous prediction of multiple labels. This is often done by maximizing a score function on the space of labels, which decomposes as a sum of pairwise and unary potentials. The above is naturally modeled with a graph, where edges and vertices are related to pairwise and unary potentials, respectively. We consider the generative process proposed by Globerson et al. [14] and apply it to general connected graphs. We analyze the structural conditions of the graph that allow for the exact recovery of the labels. Our results show that exact recovery is possible and achievable in polynomial time for a large class of graphs. In particular, we show that graphs that are bad expanders can be exactly recovered by adding small edge perturbations coming from the Erdős-Rényi model. Finally, as a byproduct of our analysis, we provide an extension of Cheeger’s inequality.

1 Introduction

Throughout the years, structured prediction has been continuously used in multiple domains such as computer vision, natural language processing, and computational biology. Examples of structured prediction problems include dependency parsing, image segmentation, part-of-speech tagging, named entity recognition, and protein folding. In this setting, the input $X$ is some observation, e.g., social network, an image, a sentence. The output is a labeling $y$, e.g., an assignment of each individual of a social network to a cluster, or an assignment of each pixel in the image to foreground or background, or the parse tree for the sentence. A common approach to structured prediction is to exploit local features to infer the global structure. For instance, one could include a feature that encourages two individuals of a social network to be assigned to different clusters whenever there is a strong disagreement in opinions about a particular subject. Then, one can define a posterior distribution over the set of possible labelings conditioned on the input. Some classical methods for learning the parameters of the model are conditional random fields [19] and structured support vector machines [22, 24, 3]. In this work we will focus in the inference problem and assume that the model parameters have been already learned.

In the context of Markov random fields (MRFs), for an undirected graph $G = (V, E)$, one is interested in finding a solution to the following inference problem:

$$
\max_{y \in \mathcal{M}^{|V|}} \sum_{v \in V, m \in \mathcal{M}} c_v(m) \mathbb{1}[y_v = m] + \sum_{(u, v) \in E, m_1, m_2 \in \mathcal{M}} c_{u,v}(m, n) \mathbb{1}[y_u = m, y_v = n],
$$

(1)

where $\mathcal{M}$ is the set of possible labels, $c_v(m)$ is the cost of assigning label $m$ to node $v$, and $c_{u,v}(m, n)$ is the cost of assigning $m$ and $n$ to the neighbors $u, v$ respectively. Similar inference problems arise in the context of statistical physics, sociology, community detection, average case analysis, and graph partitioning. Very few cases of the general MRF inference problem are known to be exactly solvable in polynomial time. For example, Chandrasekaran et al. [9] showed that (1) can be solved exactly in polynomial time for a graph $G$ with low treewidth via the junction tree algorithm. While in the case of Ising models, Schraudolph and Kamenetsky [20] showed that the inference problem can also be solved exactly in polynomial time for planar graphs via perfect matchings. Finally, polynomial-time solvability can also stem from properties of the pairwise potential, under this view, the inference problem can be solved exactly in polynomial time via graph cuts for binary labels and sub-modular pairwise potentials [7].

Despite the intractability of maximum likelihood estimation, maximum a-posteriori estimation, and marginal inference for most models in the worst case, the inference task seems to be easier in practice than the theoretical worst case. Approximate inference algorithms can be extremely effective, often obtaining state-of-the-art results for

\footnote{In the literature, the cost functions $c_v$ and $c_{u,v}$ are also known as unary and pairwise potentials respectively.}
We call $z$.

We now present the inference task. We consider a similar problem setting to the one in [14], with the only difference

As a concrete example [13], consider the problem of trying to recover opinions of individuals in social networks.

Suppose that every individual in a social network can hold one of two opinions labeled by 1 or 0, perhaps using a classification model on their profile, but these estimates are corrupted with probability $c$ (unary observations). Foster et al. [13] generalizes the work of Globerson et al. [14], who provides results for grid lattices.

In particular, Globerson et al. [14] analyzes the hardness of approximate inference in the case where performance is biased Rademacher with parameter $p$.

Foster et al. [13] generalizes the work of Globerson et al. [14], who provides results for grid lattices, by providing results for trees and general graphs that allow tree decompositions (e.g., hypergrids and ring lattices).

Note that the above problem is challenging since there is a statistical and computational trade-off, as in several machine learning problems. The statistical part focuses on giving highly accurate labels while ignoring computational constraints. In practice this is unrealistic, one cannot afford to wait long times for each prediction, which motivated several studies on this trade-off (e.g., Chandrasekaran and Jordan [8], Bello and Honorio [5]).

However, while the statistical and computational trade-off appears in general, an interesting question is whether there are conditions for when recovery of the true labels is achievable in polynomial time. That is, conditions for when the Hamming error of the prediction is zero and can be obtained efficiently. The present work addresses this question.

In contrast to [14, 13], we study the sufficient conditions for exact recovery in polynomial time, and provide high probability results for general families of undirected connected graphs, which we consider to be a novel result to the best of our knowledge. In particular, we show that weak-expander graphs (e.g., grids) can be exactly recovered by adding small perturbations (edges coming from the Erdős-Rényi model with small probability). Also, as a byproduct of our analysis, we provide an extension of Cheeger’s inequality [10]. Finally, another work in this line was done by Chen et al. [11], where the authors consider exact recovery for edges on sparse graphs such as grids and rings. However, [11] consider the case where one has multiple i.i.d. observations of edge labels. In contrast, we focus on the case where there is a single (noisy) observation of each edge and node in the graph.

2 Notation and Problem Formulation

This section introduces the notation used throughout the paper and formally defines the problem under analysis.

Vectors and matrices are denoted by lowercase and uppercase bold faced letters respectively (e.g., $a$, $A$), while scalars are in normal font weight (e.g., $a$). Moreover, random variables are written in upright shape (e.g., $a$, $A$). For a random vector $a$, and a random matrix $A$, their entries are denoted by $a_i$ and $A_{i,j}$ respectively. Indexing starts at 1, with $A_{1,:}$ and $A_{:,i}$ indicating the $i$-th row and $i$-th column of $A$ respectively. Finally, sets and tuples are both expressed in uppercase calligraphic fonts and shall be distinguished by the context. For example, $\mathbb{R}$ will denote the set of all real numbers.

We now present the inference task. We consider a similar problem setting to the one in [14], with the only difference that we consider general undirected graphs. That is, the goal is to predict a vector of $n$ node labels $y = (y_1,\ldots,y_n)^T$, where $y_i \in \{+1,-1\}$, from a set of observations $X$ and $c$, where $X$ and $c$ correspond to corrupted measurements of edges and nodes respectively. These observations are assumed to be generated from a ground truth labeling $y^*$ by a generative process defined via an undirected connected graph $G = (V,E)$, an edge noise $p \in (0,0.5)$, and a node noise $q \in (0,0.5)$. For each edge $(u,v) \in E$, the edge observation $X_{u,v}$ is independently sampled to be $y_u y_v^* + \epsilon_{u,v}$ (good edge) with probability $1-p$, and $-y_u y_v^* + \epsilon_{u,v}$ (bad edge) with probability $p$. While for each edge $(u,v) \not\in E$, the observation $X_{u,v}$ is always 0. Similarly, for each node $u \in V$, the node observation $c_u$ is independently sampled to be $y_u^*$ (good node) with probability $1-q$, and $-y_u^* + \epsilon_u$ (bad node) with probability $q$. Thus, we have a known undirected connected graph $G$, an unknown ground truth label vector $y^* \in \{+1,-1\}^n$, and noisy observations $X \in \{-1,0,1\}^{n \times n}$ and $c \in \{-1,1\}^n$, and our goal is to predict a vector label $y \in \{-1,1\}^n$.

**Definition 1** (Biased Rademacher variable). Let $z_p \in \{+1,-1\}$ such that $P(z_p = +1) = 1-p$, and $P(z_p = -1) = p$. We call $z_p$ a biased Rademacher random variable with parameter $p$ and expected value $1-2p$.

From the definition above, we can write the edge observations as $X_{u,v} = y_u y_v^* + \epsilon_{u,v}$, where $z_{p}^{(u,v)}$ is a biased Rademacher with parameter $p$. While the node observation is $c_u = y_u^* \epsilon_u$, where $\epsilon_u$ is a biased Rademacher with parameter $q$. 2
Given the generative process, we aim to solve the following optimization problem, which is based on the maximum likelihood estimator that returns the label \( \arg \max_y P(X, y) \) (see Globerson et al. [14]):

\[
\max_y \frac{1}{2} y^\top X y + \alpha c^\top y \quad \text{subject to} \quad y_i = \pm 1,
\]

where \( \alpha = \frac{\log \frac{1-q^2}{q^2}}{\log \frac{1-q}{q}} \). In general, the above combinatorial problem is NP-hard to compute (e.g., see for results on grids [4]). Our goal is to find what structural properties of the graph \( G \) suffice to achieve, with high probability, exact recovery in polynomial time.

3 On Exact Recovery of Labels

Our approach consists of two stages, similar in spirit to [14]. We first use only the quadratic term from (2), which will give us two possible solutions, and then as a second stage, the linear term is used to decide the best between these two solutions.

3.1 First Stage

We analyze a semidefinite program (SDP) relaxation to the following combinatorial problem (3), motivated by the techniques in [2].

\[
\max_y \frac{1}{2} y^\top X y \quad \text{subject to} \quad y_i = \pm 1,
\]

We denote the degree of node \( i \) as \( \Delta_i \), and the maximum node degree as \( \Delta_{\text{max}} = \max_{i \in V} \Delta_i \). For any subset \( S \subset V \), we denote its complement by \( S^C \) such that \( S \cup S^C = V \) and \( S \cap S^C = \emptyset \). Furthermore, let \( E(S, S^C) = \{(i, j) \in E \mid i \in S, j \in S^C \text{ or } j \in S, i \in S^C\} \), i.e., \( |E(S, S^C)| \) denotes the number of edges between \( S \) and \( S^C \).

Definition 2 (Edge Expansion). For a set \( S \subset V \) with \( |S| \leq n/2 \), its edge expansion, \( \phi_S \), is defined as: \( \phi_S = |E(S, S^C)|/|S| \). Then, the edge expansion of a graph \( G = (V, E) \) is defined as: \( \phi_G = \min_{S \subset V, |S| \leq n/2} \phi_S \).

In the literature, \( \phi_G \) is also known as the Cheeger constant, due to the geometric analogue defined by Cheeger in [10].

Next, we define the Laplacian matrix of a graph and the Rayleigh quotient which are also used throughout this section.

Definition 3 (Laplacian matrix). For a graph \( G = (V, E) \) of \( n \) nodes. The Laplacian matrix \( L \) is defined as \( L = D - A \), where \( D \) is the degree matrix and \( A \) is the adjacency matrix.

Definition 4 (Rayleigh quotient). For a given symmetric matrix \( M \in \mathbb{R}^{n \times n} \) and non-zero vector \( a \in \mathbb{R}^n \), the Rayleigh quotient \( R_M(a) \), is defined as: \( R_M(a) = \frac{a^\top M a}{a^\top a} \).

We now define a signed Laplacian matrix.

Definition 5 (Signed Laplacian matrix). For a graph \( G = (V, E) \) of \( n \) nodes. A signed Laplacian matrix, \( M \), is a symmetric matrix that satisfies \( x^\top M x = \sum_{(i,j) \in E} (y_i x_i - y_j x_j)^2 \), where \( y \) is an eigenvector of \( M \) with eigenvalue 0, and \( y_i \in \{+1, -1\} \).

Note that the typical Laplacian matrix, as in Definition 3, fulfills the conditions of Definition 5 with \( y_i = +1 \) for all \( i \).

Next, we present an intermediate result for later use.

Lemma 1. Let \( G = (V, E) \) be an undirected graph of \( n \) nodes with Laplacian \( L \). Let \( M \in \mathbb{R}^{n \times n} \) be a signed Laplacian as in Definition 5 and let \( a \in \mathbb{R}^n \) be a vector such that \( \langle y, a \rangle = 0 \). Finally, let \( 1 \in \mathbb{R}^n \) be a vector of ones. Then we have that, for a given \( \delta \in \mathbb{R} \), \( R_L(a \circ y + \delta 1) \leq R_M(a) \), where the operator \( \circ \) denotes the Hadamard product.

Proof. First, note that \( L \) has a 0 eigenvalue with corresponding eigenvector \( 1 \). Also, we have that \( x^\top L x = \sum_{(i,j) \in E} (x_i - x_j)^2 \), for any vector \( x \). Then, \( (a \circ y + \delta 1)^\top L (a \circ y + \delta 1) = \sum_{(i,j) \in E} ((y_i a_i + \delta) - (y_j a_j + \delta))^2 = (y_i a_i - y_j a_j)^2 = a^\top M a \). Therefore, we have that the numerators of \( R_L(a \circ y + \delta 1) \) and \( R_M(a) \) are equal. For the denominators, one can observe that: \( (a \circ y + \delta 1)^\top (a \circ y + \delta 1) = (a \circ y)^\top (a \circ y) + 2\delta (1, a \circ y) + \delta^2 1^\top 1 = \sum_i a_i y_i a_i y_i + 2\delta (a, y) + \delta^2 n = a^\top a + \delta^2 n \geq a^\top a \), which implies that \( R_L(a \circ y + \delta 1) \leq R_M(a) \). \( \square \)

In what follows, we present our first result, which has a connection to Cheeger’s inequality [10].
That is, if we will make use of the following matrix concentration inequality for our main proof.

We now provide the SDP relaxation of problem (3). Let $Y = yy^\top$, we have that $yy^\top XY = \text{Tr}(XY) = (X, Y)$. Since our prediction is a column vector $y$, we have that $yy^\top$ is rank-1 and symmetric, which implies that $Y$ is a positive semidefinite matrix. Therefore, our relaxation to the combinatorial problem results in the following primal formulation:

$$
\max_Y \langle X, Y \rangle \text{ subject to } Y_{ii} = 1, Y \succeq 0.
$$

We will make use of the following matrix concentration inequality for our main proof.

\[\frac{\phi_{2}}{4\Delta_{\text{max}}^2} \leq \lambda_2.\]

**Remark 1.** For a given undirected graph $G$, its Laplacian matrix $L$ fulfills the conditions of Lemma 1 and Theorem 1. That is, if $M = L$ in Theorem 1 then it becomes the known Cheeger’s inequality. Therefore, our result in Theorem 1 applies for more general matrices and is of use for our next result.

We now provide the SDP relaxation of problem (5). Let $Y = yy^\top$, we have that $yy^\top XY = \text{Tr}(XY) = (X, Y)$. Since our prediction is a column vector $y$, we have that $yy^\top$ is rank-1 and symmetric, which implies that $Y$ is a positive semidefinite matrix. Therefore, our relaxation to the combinatorial problem results in the following primal formulation:

$$
\max_Y \langle X, Y \rangle \text{ subject to } Y_{ii} = 1, Y \succeq 0.
$$

\[\text{Here we dropped the constant } \epsilon_2 \text{ since it does not change the decision problem.}\]
Lemma 2 (Matrix Bernstein inequality, Theorem 1.4 in [23]). Consider a finite sequence \(\{N_k\}\) of independent, random, self-adjoint matrices with dimension \(n\). Assume that each random matrix satisfies \(\mathbb{E}[N_k] = 0\) and \(\lambda_{\text{max}}(N_k) \leq R\) almost surely. Then, for all \(t \geq 0\), 
\[
P\left(\lambda_{\text{max}}\left(\sum_k N_k\right) \geq t\right) \leq n \cdot \exp\left(\frac{-t^2/2}{\sigma^2 + RT/3}\right),
\]
where \(\sigma^2 = \|\sum_k \mathbb{E}[N_k^2]\|\).

The next theorem includes our main result and provides the conditions for exact recovery of labels with high probability.

Theorem 2. Let \(G = (V, E)\) be an undirected connected graph with Cheeger constant \(\phi_G\) and maximum node degree \(\Delta_{\text{max}}\). Then, for the combinatorial problem [3], a solution \(y \in \{y^+, y^−\}\) is achievable in polynomial time by solving the SDP based relaxation [3], with probability at least \(1 - \epsilon_1(\phi_G, \Delta_{\text{max}}, p)\), where \(p\) is the edge noise from our model, and
\[
\epsilon_1(\phi_G, \Delta_{\text{max}}, p) = 2n \cdot e^{1536\Delta_{\text{max}}^2 p(1-p)+32(1-2p)\frac{1}{\phi_G^2} \Delta_{\text{max}}^2}.
\]

Proof. Without loss of generality assume that \(y = y^+\). The first step of our proof corresponds to finding sufficient conditions for when \(Y = yy^\top\) is the unique optimal solution to SDP [3], for which we make use of the Karush-Kuhn-Tucker (KKT) optimality conditions [6]. In the following we write the dual formulation of SDP [3]:
\[
\min_V \; \text{Tr}(V) \quad \text{subject to} \quad V \succeq X, \; V \text{ is diagonal}.
\] (9)

Thus, we have that \(Y = yy^\top\) is guaranteed to be an optimal solution under the following conditions:

1. \(yy^\top\) is a feasible solution to the primal problem [3].
2. There exists a matrix \(V\) feasible for the dual formulation [9] such that \(\text{Tr}(Vyy^\top) = \text{Tr}(V)\).

The first point is trivially verified. For the second point, we assume strong duality in order to find a dual certificate. To achieve that, we make \(V_{i,i} = (XY)_{i,i}\). If \(V - X \succeq 0\) then the matrix \(V\) is a feasible solution to the dual formulation. Thus, our first condition is to have \(V - X \succeq 0\), and we conclude that \(yy^\top\) is an optimal solution to SDP [9].

For showing that \(yy^\top\) is the unique optimal solution, we further require that \(\lambda_2(V - X) > 0\). Suppose that \(\tilde{Y}\) is another optimal solution to SDP [3]. Then, from complementary slackness we have that \(\langle V - X, \tilde{Y} \rangle = 0\), and from primal feasibility \(\tilde{Y} \succeq 0\). Moreover, notice that we have \(\langle V - X, y \rangle = 0\), i.e., \(y\) is an eigenvector of \(V - X\) with eigenvalue 0. By assumption, the second smallest eigenvalue of \(V - X\) is greater than 0, therefore, \(y\) spans all of its null space. This fact combined with complementary slackness, primal and dual feasibility, entail that \(\tilde{Y}\) is a multiple of \(yy^\top\). Thus, we must have that \(\tilde{Y} = yy^\top\) because \(\tilde{Y}_{i,i} = 1\).

From the points above we arrived to the two following sufficient conditions:
\[
V - X \succeq 0 \quad \text{and} \quad \lambda_2(V - X) > 0.
\] (10)

Our next step is to show when condition (10) is fulfilled with high probability. Since we have that \(y\) is an eigenvector of \(V - X\) with eigenvalue zero, showing that \(\lambda_2(V - X) > 0\) will imply that \(V - X\) is positive semidefinite. Therefore, we focus on controlling its second smallest eigenvalue. Next, we have that:
\[
\lambda_2(V - X) > 0 \iff \lambda_2(V - X - \mathbb{E}(V - X) + \mathbb{E}(V - X)) > 0 \\
\iff \lambda_2(V - \mathbb{E}(V)) + \lambda_2(\mathbb{E}(V) - X) + \lambda_2(\mathbb{E}(V - X)) > 0 \\
\iff \lambda_1(V - \mathbb{E}(V)) + \lambda_1(\mathbb{E}(V) - X) + \lambda_2(\mathbb{E}(V - X)) > 0.
\] (11)

We now focus on condition (11) since it implies that \(\lambda_2(V - X) > 0\). For the first two summands of condition (11) we make use of Lemma 2 while for the third summand we make use of Theorem 2. From \(V_{i,i} = (XY)_{i,i}\), we have that \(V_{i,i} = y_iX_{i,i}\), thus, \(V_{i,i} = \sum_{j=1}^n y_iy_jX_{i,j} = \sum_{j=1}^n z_p^{(i,j)}(i,j) \in E\). Then, its expected value is:
\[
\mathbb{E}[V_{i,i}] = \Delta_i(1 - 2p).
\]

Note that we now write \(V\) in upright shape (i.e., \(V\)) since it contains randomness from \(X\).
Bounding the third summand of condition (11). Our goal is to find a non-zero lower bound for the second smallest eigenvalue of $\mathbb{E}[V - X]$. Notice that $\mathbb{E}[V - X] \succeq 0$ since it is a diagonally dominant matrix, and $y$ is its first eigenvector with eigenvalue $0$, i.e., $\lambda_1(\mathbb{E}[V - X]) = 0$.

Then, we write $M = \mathbb{E}[V - X]$. Now we focus on finding a lower bound for $\lambda_2(M)$. We use the fact that for any vector $a \in \mathbb{R}^n$, we have that $\mathbb{E}^a [M a] = (1 - 2p) \sum_{(i,j) \in E} (y_i a_i - y_j a_j)^2$.

We also note that $M$ has a 0 eigenvalue with eigenvector $y$. Thus, the matrix $M/(1 - 2p)$ satisfies the conditions of Theorem and we have that $\lambda_2(M/(1 - 2p)) \geq \frac{\phi_G^2}{4\Delta_{\max}}$. We conclude that,

$$\lambda_2(\mathbb{E}[V - X]) \geq (1 - 2p) \frac{\phi_G^2}{4\Delta_{\max}}.$$  

Bounding the first summand of condition (11). Let $N_p^{(i,j)} = z_p^{(i,j)} (e_i e_i^T + e_j e_j^T)$, where $e_i$ is the standard basis, i.e., the vector of all zeros except the $i$-th entry which is 1. We can now write $V = \sum_{(i,j) \in E} N_p^{(i,j)}$. Then, we have a sequence of independent random matrices $\{\mathbb{E}[N_p^{(i,j)}] - N_p^{(i,j)}\}$, where we obtain the following: $\lambda_{\max} (\mathbb{E}[N_p^{(i,j)}] - N_p^{(i,j)}) \leq 2(1 - p)$, and also $\sum_{(i,j) \in E} \mathbb{E}[(\mathbb{E}[N_p^{(i,j)}] - N_p^{(i,j)})^2] \leq 4\Delta_{\max} p(1 - p)$.

Next, we use the fact that $\lambda_{\max}(A) = -\lambda_1(-A)$ for any matrix $A$. Then, by applying Lemma 2 we obtain:

$$P\left( \lambda_1 (V - \mathbb{E}[V]) \leq \frac{(1 - 2p)\phi_G^2}{8\Delta_{\max}} \right) \leq n \cdot e^{-\frac{3(1 - 2p)^2 \phi_G^2}{1536\Delta_{\max} p(1 - p) + 32(1 - 2p)(1 - p)\phi_G^2 \Delta_{\max}}}.$$  

Bounding the second summand of condition (11). Using similar arguments to the concentration above, we now analyze $\lambda_1(\mathbb{E}[X] - X)$. Let $H^{(i,j)} = X_{i,j} (e_i e_i^T + e_j e_j^T)$. Then, we have a sequence of independent random matrices $\{H^{(i,j)} - \mathbb{E}[H^{(i,j)}]\}$ and can write $X = \sum_{(i,j) \in E} H^{(i,j)}$. Finally, we have that $\lambda_{\max}(H^{(i,j)} - \mathbb{E}[H^{(i,j)}]) \leq 2(1 - p)$, and $\mathbb{E}[(H^{(i,j)} - \mathbb{E}[H^{(i,j)}])^2] = 4p(1 - p)(e_i e_i^T + e_j e_j^T)^2$. Thus, $\|\sum_{(i,j) \in E} \mathbb{E}[(H^{(i,j)} - \mathbb{E}[H^{(i,j)}])^2]\| \leq 4\Delta_{\max} p(1 - p)$ and by applying Lemma 2 we obtain:

$$P\left( \lambda_1 (E[X] - X) \leq \frac{(1 - 2p)\phi_G^2}{8\Delta_{\max}} \right) \leq n \cdot e^{-\frac{3(1 - 2p)^2 \phi_G^2}{1536\Delta_{\max} p(1 - p) + 32(1 - 2p)(1 - p)\phi_G^2 \Delta_{\max}}}.$$  

Note that the thresholds in the concentrations above are motivated by equation (12). Finally, combining equations (12), (13), and (14), we have that:

$$P(\lambda_2(V - X) > 0) \geq 1 - 2n e^{-\frac{3(1 - 2p)^2 \phi_G^2}{1536\Delta_{\max} p(1 - p) + 32(1 - 2p)(1 - p)\phi_G^2 \Delta_{\max}}},$$

which concludes our proof. \qed

Regarding the statistical part from Theorem 2, it is natural to ask under what conditions we obtain a high probability statement. For example, one can observe that if $\phi_G / \Delta_{\max} \in \Omega(n)$ then there is an exponential decay in the probability of error. Another example would be if $\Delta_{\max} \in \mathcal{O}(\sqrt{n})$ and $\phi_G / \Delta_{\max} \in \Omega(\sqrt{n})$ then we also obtain high probability argument. Thus, we are interested in finding what classes of graphs fulfill these or other structural properties so that we obtain a high probability bound in Theorem 2. Regarding the computational complexity of exact recovery, from Theorem 2 we are solving a SDP, and any SDP can be solved in polynomial time using methods such as the interior point method.

3.2 Second Stage

After the first stage, we obtain two feasible solutions for problem (3), that is, $y \in \{y^*, -y^*\}$. To decide which solution is correct we will use the node observations $c$. Specifically, we will output the vector $y$ that maximizes the score $c^T y$. The next theorem formally states that, with high probability, $y = y^*$ maximizes the score $c^T y$ for a sufficiently large $n$.

Theorem 3. Let $y \in \{y^*, -y^*\}$. Then, with probability at least $1 - \epsilon_2(n, q)$, we have that: $c^T y^* = \max_{y \in \{y^*, -y^*\}} c^T y$, where $\epsilon_2(n, q) = e^{-\frac{\phi_G^2}{1536\Delta_{\max} p(1 - p) + 32(1 - 2p)(1 - p)\phi_G^2 \Delta_{\max}}}$.

The remaining proofs of our manuscript can be found in Appendix A.
Remark 2. From Theorems 2 and 3 we obtain that exact recovery (i.e., \( y = y^* \)) is achievable with probability at least \( 1 - \varepsilon_1(\phi_G, \Delta_{\text{max}}, p) - \varepsilon_2(n, q) \). Finally, from Theorem 3 it is clear that since the parameter \( q \in (0, 0.5) \), for a sufficiently large \( n \) we have an exponential decay of the probability of error \( \varepsilon_2 \). Thus, we focus on the conditions of the first stage and provide examples in the next section.

4 Examples of Graphs for Exact Recovery

In this section, we provide examples of classes of graphs that yield high probability in Theorem 2.

Perhaps the most important example we provide in this section is related to the smoothed analysis on connected graphs. Consider any fixed graph \( G = (V, E) \) and let \( \mathcal{R} \) be a random set of edges over the same set of vertices \( V \), where each edge \( e \in \mathcal{R} \) is independently drawn according to the Erdős-Rényi model with probability \( \varepsilon/n \) and where \( \varepsilon \) is a small (fixed) positive constant. We denote this as \( \mathcal{R} \sim \text{ER}(n, \varepsilon/n) \), then let \( \tilde{G} = (V, \mathcal{E} \cup \mathcal{R}) \) denote the random graph with the edge set \( \mathcal{R} \) added.

The model above can be considered a generalization of the classical Erdős-Rényi random graph, where one starts from an empty graph (i.e., \( G = (V, \emptyset) \)) and adds edges between all possible pairs of vertices independently with a given probability. The focus on “small” \( \varepsilon \) means that we are interested in the effect of a rather gentle random perturbation. In particular, it is known that graphs with bad expansion are not suitable for exact inference (see for instance, [1]), but certain classes such as grids or planar graphs can yield good approximation under some regimes despite being bad expanders as shown by Globerson et al. [14]. Here we consider the graph \( G \) to be a bad expander and show that with a small perturbation, exact inference is achievable.

The following result was presented by [17] in an equivalent fashion.

Lemma 3 (Theorem 2 in [17]). Let \( G = (V, E) \) be a connected graph, choose \( \mathcal{R} \sim \text{ER}(n, \varepsilon/n) \), and let \( \tilde{G} = (V, E \cup \mathcal{R}) \). Then, for every \( \varepsilon \in [1, n] \), we have that \( \phi_{\tilde{G}} \geq \frac{\varepsilon}{256 + 256 \log n} \), with probability at least \( 1 - n^{-2.2 - \frac{\log \varepsilon}{\varepsilon}} \).

The above lemma allows us to lower bound the Cheeger constant of the random graph \( \tilde{G} \) with high probability, and is of use for our first example.

Corollary 1. Let \( G = (V, E) \) be any connected graph, choose \( \mathcal{R} \sim \text{ER}(n, \log^5 n/n) \), let \( \tilde{G} = (V, E \cup \mathcal{R}) \) and let \( \Delta_{\tilde{G}} \) be the maximum node degree of \( \tilde{G} \). Then, we have that \( \phi_{\tilde{G}}/\Delta_{\tilde{G}} \in \Omega(\log^5 n) \) and \( \Delta_{\tilde{G}} \in \Omega(\log^9 n) \) with high probability. Therefore, exact recovery in polynomial time is achievable with high probability.

We emphasize the nice property of random graphs \( \tilde{G} \) shown in Corollary 1 that is, by adding a small perturbation (edges from the Erdős-Rényi model with small probability) we are able to obtain exact inference despite of \( G \) having bad properties such as being a bad expander. Our next two examples include complete graphs and \( d \)-regular expanders.

The following corollary shows that, with high probability, exact recovery of labels for complete graphs is possible in polynomial time.

Corollary 2 (Complete graphs). Let \( G = K_n \), where \( K_n \) denotes a complete graph of \( n \) nodes. Then, we have that \( \phi_{\tilde{G}}/\Delta_{\text{max}} \in \Omega(n) \). Therefore, exact recovery in polynomial time is achievable with high probability.

Another important class of graphs that admits exact recovery is the family of \( d \)-regular expanders [15], which is defined below.

Definition 6 (d-regular expander). A \( d \)-regular graph with \( n \) nodes is an expander with constant \( c > 0 \) if, for every set \( S \subset V \) with \( |S| \leq n/2 \), \( |E(S, V \setminus S)| \geq c \cdot d \cdot |S| \).

Corollary 3 (Expanders graphs). Let \( G \) be a \( d \)-regular expander with constant \( c \). Then, we have that \( \phi_{\tilde{G}}/\Delta_{\text{max}} \in \Omega(d) \). If \( d \in \Omega(\log n) \) then exact recovery in polynomial time is achievable with high probability.

5 Concluding Remarks

We considered a model where we receive a single noisy observation for each edge and each node of a graph. Our approach consisted of two stages, similar in spirit to [13]. The first stage consisted of solving solely the quadratic term of the optimization problem and was based in a SDP relaxation in order to find the structural properties of a graph that guarantee exact recovery with high probability. Given two solutions from the first stage, the second stage consisted in

\[ \alpha = \frac{1}{2}, \varepsilon = \frac{1}{256}, K = \frac{128}{\varepsilon}, C = 1, s = K \log n, \] which results with all the conditions being fulfilled in the proof of Theorem 2 in [13].
using solely the node observations and simply outputting the vector with higher score. We showed that for any graph \( G \), the term \( \phi_2/\Delta_{\text{max}} \) is related to achieve exact recovery in polynomial time. Examples include complete graphs and \( d \)-regular expanders, that are guaranteed to recover the correct labeling with high probability. While perhaps the most interesting example is related to smoothed analysis on connected graphs, where even for a graph with bad properties such as bad expansion can still be exactly recovered by adding small perturbations (edges coming from an Erdős-Rényi model with small probability).

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A Detailed Proofs

In this section, we state the proofs of Theorem 3 and Corollaries 1, 2, 3 from our manuscript.

A.1 Proof of Theorem 3

Proof. We are interested in upper bounding the probability of predicting the wrong vector \( y \), that is,

\[
P(c^\top y^* \leq -c^\top y) = P(\sum_{u \in V} \varepsilon_u^*(u) \leq 0) \leq e^{-\frac{n}{2}(1-2q)^2},
\]

where for the last equation we applied Hoeffding’s inequality.

A.2 Proof of Corollary 1

Proof. Fix \( \epsilon = \log \frac{8}{n} \). Let \( \epsilon_r(n, \epsilon) = n^{-2.2- \log^2 n} \), then from Lemma 3 we get \( \phi_G \in \Omega(\log^7 n) \) with probability at least \( 1 - \epsilon_r(n, \epsilon) \). Let \( \Delta_{\text{max}} \) be the maximum node degree of graph \( G \), then it is clear that \( \Delta_{\text{max}}^G \) is a random variable with expected value \( E[\Delta_{\text{max}}^G] \leq \Delta_{\text{max}} + \log^8 n \). By applying Markov’s inequality we obtain \( P(\Delta_{\text{max}}^G > t) \leq \frac{E[\Delta_{\text{max}}^G]}{t} \leq (\Delta_{\text{max}} + \log^8 n)/t \) for \( t > 0 \). Set \( t = \log^9 n \), then let \( \epsilon_{\Delta}(\Delta_{\text{max}}, n) = (\Delta_{\text{max}} + \log^8 n)/\log^9 n \), we have that \( \Delta_{\text{max}}^G \leq \log^9 n \) with probability at least \( 1 - \epsilon_{\Delta}(\Delta_{\text{max}}, n) \).

By using the union bound and noting that \( \epsilon_r \to 0 \) and \( \epsilon_{\Delta} \to 0 \) as \( n \to \infty \), we have that \( \phi_G^2/\Delta_{\text{max}} \in \Omega(n) \) and \( \Delta_{\text{max}}^G \in O(\log^9 n) \) with high probability. Finally, this leads to \( \epsilon_1 \to 0 \) as \( n \to \infty \), thus, exact inference is achievable in polynomial time.

A.3 Proof of Corollary 2

Proof. For any set \( S \subset V \) with \( |S| \leq n/2 \), we have that:

\[
\phi_S = \frac{|E(S, S^C)|}{|S|} = \frac{|S| \cdot |S^C|}{|S|} = |S^C| \quad \Rightarrow \quad \phi_S = \left\lfloor \frac{n}{2} \right\rfloor.
\]

Since \( G \) is a complete graph, we have that \( \Delta_{\text{max}} = n - 1 \), which yields \( \phi_G/\Delta_{\text{max}} \in \Omega(n) \). Thus, from Theorem 2 we have that \( \epsilon_1(\phi_G, \Delta_{\text{max}}, p) \to 0 \) as \( n \to \infty \).

A.4 Proof of Corollary 3

Proof. From Definition 6 we have that \( \phi_G \geq c \cdot d \). Since the graph is regular, we have that \( \Delta_{\text{max}} = d \). Therefore, \( \phi_G/\Delta_{\text{max}} \in \Omega(d) \). Finally, if \( d \in \Omega(\log n) \), then \( \epsilon_1(\phi_G, \Delta_{\text{max}}, p) \) decays in at least \( n^{-c_1} \) for some constant \( c_1 > 0 \). That is, \( \epsilon_1(\phi_G, \Delta_{\text{max}}, p) \to 0 \) as \( n \to \infty \).