Implications of JLA data for \( k \)-essence model of dark energy with given equation of state

Abhijit Bandyopadhyay\textsuperscript{a}, Anirban Chatterjee\textsuperscript{b}\textsuperscript{15}

Department of Physics, Ramakrishna Mission Vivekananda Educational and Research Institute, Belur Math, Howrah 711202, India

Received: 23 September 2019 / Accepted: 14 November 2019 / Published online: 1 February 2020

© Società Italiana di Fisica (SIF) and Springer-Verlag GmbH Germany, part of Springer Nature 2020

Abstract We investigated implications of recently released ‘Joint Light-curve Analysis’ (JLA) supernova Ia (SNe Ia) data for dark energy models with time-varying equation of state of dark energy, usually expressed as \( w(z) \) in terms of variation with corresponding redshift \( z \). From a comprehensive analysis of the JLA data, we obtain the observational constraints on the different functional forms of \( w(z) \), corresponding to different varying dark energy models often considered in the literature, viz. CPL, JBP, BA and logarithmic models. The constraints are expressed in terms of parameters (\( w_a, w_b \)) appearing in the chosen functional form for \( w(z) \), corresponding to each of the above-mentioned models. Realising dark energy with varying equation of state in terms of a homogeneous scalar field \( \phi \), with its dynamics driven by a \( k \)-essence Lagrangian \( L = V F(X) \) with a constant potential \( V \) and a dynamical term \( F(X) \) with \( X = (1/2)\nabla^\mu \phi \nabla_\mu \phi \), we reconstructed form of the function \( F(X) \). This reconstruction has been performed for different varying dark energy models at best-fit values of parameters (\( w_a, w_b \)) obtained from analysis of JLA data. In the context of \( k \)-essence model, we also investigate the variation in adiabatic sound speed squared, \( c_s^2(z) \), and obtained the domains in (\( w_a, w_b \)) parameter space corresponding to the physical bound \( c_s^2 > 0 \) implying stability of density perturbations.

1 Introduction

Measurement of redshift and luminosity distances for type Ia Supernova (SNe Ia) events [1,2] is instrumental in establishing the fact that the universe has undergone a transition from a phase of decelerated expansion to accelerated expansion during its late time phase of evolution. Other independent evidences in support of this fact come from the observations of baryon acoustic oscillation [3–7], Cosmic Microwave Background radiations [8–11] measurement of differential ages of the galaxies in GDDS, SPICES and VDSS surveys [12–15] and studies of power spectrum of matter distributions of the universe. A general label attributed to the origin of this late-time cosmic acceleration is ‘Dark Energy’ (DE). Besides, study of rotation curves of spiral galaxies [16], bullet cluster [17], gravitational lensing [18] provides indirect evidence for the existence of nonluminous matter in the present universe. Such ‘matter’,
labelled as ‘Dark Matter’ (DM), manifests its existence only through gravitational interactions. Measurements in satellite-borne experiments—WMAP [19] and Planck [20]—have established that, at the present epoch, dark energy and dark matter comprise around 96% of total energy density of the universe (~69% dark energy and ~27% dark matter). Rest ~4% is contributed by baryonic matter with negligible contribution from radiations.

There exist diverse theoretical approaches aiming construction of different models for dark energy to explain the present-day cosmic acceleration. These include the $\Lambda$–CDM model [21], which provides excellent agreement with the cosmological data. Here, ‘CDM’ refers to cold dark matter content of the universe and $\Lambda$, the cosmological constant, denotes vacuum energy density. Though this model provides a simple phenomenological solution, it is plagued with the problem of large disagreement between vacuum expectation value of energy momentum tensor and observed value of dark energy density (fine-tuning problem). This motivates the investigation of alternative models of dark energy. One of the key features of a class of such models, called varying dark energy models, is time-varying equation of state $w = p/\rho$ ($\rho$ is the energy density and $p$ the pressure of dark energy) of dark energy, which is usually expressed in terms of variation in $w$ with redshift $z$ (in $\Lambda$–CDM model $w = -1$, constant). The redshift dependence of the EOS parameter $w(z)$ in the context of the varying DE models may be constrained from the observational data. The starting point for dealing with the issue of variations in the EOS parameter $w(z)$ is to consider diverse functional forms of $w(z; w_a, w_b)$ each involving a small number of parameters (denoted in the text by symbols $w_a$, $w_b$). The observational constraints on the $z$-dependence of $w$ may then be realised in terms of constraints on the parameters $(w_a, w_b)$ for each different functional form of $w(z; w_a, w_b)$ considered. In this work, we have performed a comprehensive analysis of ‘Joint Light-curve Analysis’ (JLA) data [22–24] to obtain constraints on different functional forms of $w(z)$ often used in the literature in the context of varying dark energy models [25–36] and references there in. As benchmark, we have chosen four such models, viz. CPL [25], JBP [29,30], BA [31,32] and logarithmic model [33], and for each model, we presented the values of the parameters $(w_a, w_b)$ those fit best the observational data from JLA and also showed the regions in this parameter space at different confidence limits allowed from observational data.

Dark energy with varying equation of state may be realised theoretically in terms of dynamics of a scalar field ($\phi$). One class of such scalar field models, called ‘Quintessence’, is described in terms of standard canonical Lagrangian of the form $L = X - V(\phi)$ where $X = \frac{1}{2}\nabla_{\mu}\phi\nabla^{\mu}\phi$ is the kinetic term. There also exists alternative class of models involving Lagrangians with noncanonical kinetic terms as $L = V(\phi)F(X)$, where $F(X)$ is a function of $X$. Such models, called $k$-essence models, have interesting phenomenological consequences different from those of quintessence models. Another motivation for considering $k$-essence scalar fields is that they appear naturally in low-energy effective string theory. Such theories with noncanonical kinetic terms were first proposed by Born and Infeld to get rid of the infinite self-energy of the electrons [37–44] and were also investigated by Heisenberg in the context of cosmic ray physics [45] and meson production [46]. In this work, we consider dark energy represented in terms of a homogeneous scalar field $\phi \equiv \phi(t)$ whose dynamics is driven by a $k$-essence Lagrangian $L = VF(X)$ with a constant potential $V$. The constancy of the potential ensures existence of a scaling relation, $XF_{\phi}^2 = Ca^{-6}$ ($C$ = constant), in a Friedmann–Lemaitre–Robertson–Walker (FLRW) background space-time with scale factor $a$. We exploited the scaling relation and observational constraints on the parameters $w_a$, $w_b$, to reconstruct the forms of the function $F(X)$ for the different varying dark energy models.

In the context of $k$-essence model, we also investigate the adiabatic sound speed squared ($c_s^2$) [47]—the quantity relevant for the growth of small fluctuations in the background energy
density. Imaginary value of the sound speed \( c_s^2 < 0 \) implies instability of density perturbations. Also from causality, it requires the speed of propagation of density perturbations not to exceed the speed of light \( c_s^2 < 1 \). However, it was pointed out in [48–50] that in \( k \)-essence theories superluminal propagation of perturbations on classical backgrounds is admissible and no causal paradoxes arise. This implies the condition \( c_s^2 > 0 \) would be enough to represent the physical bound in the context of \( k \)-essence theories. For each of the varying dark energy models considered here, we find \( z \)-dependence of \( c_s^2 \) at best-fit values of parameters \( w_a, w_b \) obtained from analysis of JLA data. This has been found over the entire redshift range \( 0 < z < 1.3 \) accessible in SNe Ia observations corresponding to the JLA data. We note that at the best fit, \( c_s^2 \) is not always within its physical bound \( (c_s^2 > 0) \) for all values of \( z \) in the above-mentioned range. For each of the varying DE models, we have found the regions in \( w_a - w_b \) parameter space for which the physical bound \( c_s^2 > 0 \) is satisfied for the entire range of values of \( z \) in JLA data. The best-fit values of parameters \( (w_a, w_b) \) for each model obtained from analysis of the observational data are found to lie outside this domain corresponding to the bound \( c_s^2 > 0 \) (and also to \( 0 < c_s^2 < 1 \)), implying that observational data allow values of parameters \( (w_a, w_b) \), for which the physical bound on \( c_s^2 \) is respected, only at higher confidence limits. For example, for BA and logarithmic models the values of parameters \( (w_a, w_b) \) corresponding to \( 0 < c_s^2(z) < 1 \) for all \( z \) are allowed from observational data only beyond \( \sim 2\sigma \) confidence limits. They are allowed only at \( 3\sigma \) and beyond for CPL model and even at larger confidence limits for JBP model. For each of the models, we have found the point in \( w_a - w_b \) parameter space which belongs to the domain for which \( 0 < c_s^2 < 1 \) for all \( z \) and is maximally favoured from observational data. The form of the \( k \)-essence Lagrangian density \( F(X) \) is also reconstructed at these points.

The paper is organised as follows. In Sect. 2, we describe the methodology of analysis of the observational data and provide a brief description of the different data sets used in our analysis. In Sect. 3, we discussed different models of dark energy with varying equation of state and their realisations in terms of \( k \)-essence scalar field models. In this context, we also discussed the relevance of investigating variations in the adiabatic sound speed squared. The methodology of obtaining form of the Lagrangian density \( L = V F(X) \) for \( k \)-essence models is also described. In Sect. 4, we discussed the results on the variation in \( c_s^2 \), form of the function \( F(X) \) obtained from the analysis of the data. The conclusions are presented in Sect. 5.

### 2 Methodology of analysis of observational data

Measurement of luminosity distances and redshift of type Ia supernovae (SNe Ia) is instrumental in probing the nature of dark energy. There exist several systematic and dedicated measurements of SNe Ia events. There are different supernova surveys in different domains of redshift \( (z) \). High-redshift projects \( (z \sim 1) \) include Supernova Legacy Survey (SNLS) [51,52], the ESSENCE project [53], the Pan-STARRS survey [54–57]. The SDSS-II supernova surveys [58–62] probe the redshift regime \( 0.05 < z < 0.4 \). The surveys in the small redshift domain \( (z > 0.1) \) are the Harvard-Smithsonian Center for Astrophysics survey [63], the Carnegie Supernova Project [64–66] the Lick Observatory Supernova Search [67] and the Nearby Supernova Factory [68]. Other different compilations of SNe Ia data may also be found in [69–90] and references in [53]. Nearly one thousand of SNe Ia events were discovered in all these surveys.

The recently released “Joint Light-curve Analysis” (JLA) data [22–24] are a compilation of several low-, intermediate- and high-redshift samples including data from the full three
years of the SDSS survey, first three seasons of the five-year SNLS survey and 14 very high
redshift $0.7 < z < 1.4$ SNe Ia from space-based observations with the HST [12]. This data
set contains 740 spectroscopically confirmed SNe IA events with high-quality light curves.

In this section, we describe the methodology of analysis of JLA data to obtain bounds on
equation of state parameter $w$ of dark energy. There exist diverse statistical techniques for
analysis of JLA data. Some of these methods are discussed in detail in [33,91–94]. However,
we take the $\chi^2$ function corresponding to JLA data as [23,24]

$$\chi^2_{SN} = \sum_{i,j} (\mu_{obs}^{(i)} - \mu_{th}^{(i)}) (\Sigma^{-1})_{ij} (\mu_{obs}^{(j)} - \mu_{th}^{(j)})$$

(1)

where values of the dummy indices $i, j$ run from 1 to 740 corresponding to the 740 SNe
IA events contained in the JLA data set [23], $\mu_{th}^{(i)}$ stands for the theoretical expression for
distance modulus in a flat FRW space-time background for the $i$th entry of the JLA data set
and is given by

$$\mu_{th}^{(i)} = 5 \log \left[ \frac{d_L(z_{hel}, z_{CMB})}{\text{Mpc}} \right] + 25$$

(2)

where

$$d_L(z_{hel}, z_{CMB}) = (1 + z_{hel})r(z_{CMB}) \quad \text{with} \quad r(z) = cH_0^{-1} \int_0^z \frac{dz'}{E(z')}.$$  

(3)

$d_L$ is the luminosity distance and $r(z)$ is the comoving distance to an object corresponding
to a redshift $z$. $z_{CMB}$ and $z_{hel}$ are SNe IA redshifts in CMB rest frame and in heliocentric
frame, respectively, $c$ is the speed of light and $H_0$ is the value of Hubble parameter at the
present epoch. The function $E(z)$ in Eq. (3) is the reduced Hubble parameter given by

$$E(z) \equiv \frac{H(z)}{H_0} = \left\{ \Omega_r^{(0)}(1+z)^4 + \Omega_m^{(0)}(1+z)^3 + \Omega_{de}^{(0)} \exp \left[ 3 \int_0^z \frac{dz'}{1+z'} \frac{1+w(z')}{} \right] \right\}^{1/2}$$

(4)

where $\Omega_r^{(0)}$, $\Omega_m^{(0)}$ and $\Omega_{de}^{(0)}$ are the values of fractional energy density contributions from
radiation, matter and dark energy, respectively, at the present epoch.

$\mu_{obs}^{(i)}$ is the observed value of distance modulus at a redshift $z_i$ corresponding to $i$th entry
of the JLA data set. This is expressed through the following empirical relation as

$$\mu_{obs}^{(i)} = m_B(z_i) - M_B + \alpha X_1(z_i) - \beta C(z_i)$$

(5)

where $m_B(z_i)$ is the observed value of peak magnitude, $X_1(z_i)$ denotes time stretching of the
light curve and $C(z_i)$ is the supernova ‘colour’ at maximum brightness. $M_B$ is the absolute
magnitude which we take fixed at $M = -19$ for our work and $\alpha, \beta$ are nuisance parameters.

$\Sigma$ is the total covariant matrix given in terms of statistical and systematic uncertainties as

$$\Sigma_{ij} = \delta_{ij} \left[ (\sigma_z^2)_i + (\sigma_{int}^2)_i + (\sigma_{lensing}^2)_i + (\sigma_{mB}^2)_i + \alpha^2(\sigma_{X_1}^2)_i + \beta^2(\sigma_C^2)_i \right. \right.$$  

$$+ 2\alpha(\Sigma_{mB,x_1})_i - 2\beta(\Sigma_{mB,c})_i - 2\alpha\beta(\Sigma_{X_1,c})_i \right.$$  

$$\left. + \left[(V_0)_{ij} + \alpha^2(V_a)_{ij} + \beta^2(V_b)_{ij} + 2\alpha(V_{0a})_{ij} - 2\beta(V_{0b})_{ij} - 2\alpha\beta(V_{ab})_{ij} \right] \right.$$  

(6)

The terms in the first two lines of Eq. (6) represent the diagonal part of the covariance matrix.
These include statistical uncertainties in redshifts ($\sigma_z^2$), in SNe Ia magnitudes (owing to
intrinsic variation ($\sigma_{int}^2$) and gravitational lensing ($\sigma_{lensing}^2$), in $m_B(\sigma_{mB}^2)$, $X_1(\sigma_{X_1}^2)$ and colour
parameters $\alpha$ denote the parameters of the chosen functional dependence. On the other hand, the nuisance parameters $(\Sigma_{m,g, h_1}, \Sigma_{m,b, c}, \Sigma_{X_1, c})$ in each bin. The terms in last line of Eq. (6) involving matrices $(V_0, V_a, V_b, V_{0a}, V_{0b}, V_{ab})$ correspond to the off-diagonal part of the covariance matrix originating from statistical and systematic uncertainties. All these matrices are given by JLA group and are extensively discussed in [23, 24].

We note from Eqs. (2), (3) and (4) that evaluation of $\mu^{(i)}_{\text{th}}$ requires values of parameters $\Omega_r^{(0)}, \Omega_m^{(0)}$ and $\Omega_{de}^{(0)}$ and knowledge of functional form of equation of state (EOS) $w(z)$ of dark energy. Since in a spatially flat universe $\Omega_r^{(0)} + \Omega_m^{(0)} + \Omega_{de}^{(0)} = 1$, neglecting the value of fractional density contribution of radiation at the present epoch with respect to those from other components, we have $\Omega_{de}^{(0)} \approx 1 - \Omega_m^{(0)}$. We may also choose different functional form of dark energy EOS which we denote by a general symbol $w(z; w_a, w_b, \ldots)$, where $w_a, w_b, \ldots$ denote the parameters of the chosen functional dependence. On the other hand, the nuisance parameters $\alpha$ and $\beta$ enter in the expression for $\mu_{\text{obs}}^{(0)}$ (Eq. (5)) and the covariance matrix as well (Eq. (6)). The $\chi^2_{\text{SN}}$ function, in Eq. (1), is minimised with respect to the parameters $w_a, w_b, \Omega_m^{(0)}, \alpha, \beta$. The (best-fit) values of these parameters corresponding to the minimum value of $\chi^2$ for different chosen models of dark energy EOS are presented in Sect. 4.

Besides SNe Ia data, compilation of measurements of differential ages of the galaxies in GDDS, SPICES and VDSS surveys gives measured values of Hubble parameter at 15 different redshift values [12–15]. The $\chi^2$ function for the analysis of these observational Hubble data (OHD) may be defined as

$$
\chi^2_{\text{OHD}} = \sum_{i=1}^{15} \left[ \frac{H(w_a, w_b, \Omega_m^{(0)}; z_i) - H_{\text{obs}}(z_i)}{\Sigma_i} \right]^2
$$

where $H_{\text{obs}}(z_i)$ is the observed value of the Hubble parameter at redshift $z_i$ with 1σ uncertainty $\Sigma_i$ and $H(w_a, w_b, \Omega_m^{(0)}; z_i)$ is its theoretical value evaluated by multiplying $E(z)$ in Eq. (4) by $H_0$. Also, observation of baryon acoustic oscillations (BAO) in Sloan Digital Sky Survey (SDSS) provides measurement of correlation function of the large sample of luminous red galaxies. Using the detected acoustic peak value of a dimensionless standard ruler, $A(z_1)$ corresponding to a typical redshift $z_1 = 0.35$ may be determined. The theoretical expression for the quantity $A(z_1)$ is given by

$$
A(w_a, w_b, \Omega_m^{(0)}, z_1) = \frac{\sqrt{\Omega_m^{(0)}}}{E^{1/3}(z_1)} \left[ \frac{1}{z_1} \int_0^{z_1} \frac{dz}{E(z)} \right]^{2/3}
$$

where the parameters $w_a, w_b, \Omega_m^{(0)}$ enter in the above expression though the function $E(z)$ (Eq. (4)). The observed value of the standard ruler $A_{\text{obs}} \pm \Delta A = 0.469 \pm 0.017$, and the $\chi^2$-function for the BAO data is taken as

$$
\chi^2_{\text{BAO}} = \left[ \frac{A(w_a, w_b, \Omega_m^{(0)}; z_1) - A_{\text{obs}}}{\Delta A} \right]^2
$$

To illustrate the impact of the observational Hubble data and BAO data, we have performed a combined analysis of SNe Ia, OHD and BAO data by minimising the total $\chi^2$ function

$$
\chi^2 = \chi^2_{\text{SN}} + \chi^2_{\text{OHD}} + \chi^2_{\text{BAO}}
$$

with respect to the parameter set $(w_a, w_b, \Omega_m^{(0)}, \alpha, \beta)$. Results of the analysis are presented in Sect. 4.
3 k-Essence and varying dark energy

In this work, we investigate realisation of dark energy with varying equation of state in terms of \( k \)-essence scalar field models. We assume dark energy represented in terms of a homogeneous scalar field whose dynamics is driven by a \( k \)-essence Lagrangian with constant potential. In this context, we give below a brief outline of basic equations of \( k \)-essence model \( i.e., \)

\[
L = V(\phi) F(X) = p \tag{11}
\]

\[
\rho = V(\phi)(2XF_X - F) \tag{12}
\]

where \( L \) is the \( k \)-essence Lagrangian, \( \rho \) and \( p \), respectively, represent energy density and pressure of dark energy. \( F(X) \) is a function of \( X \), where \( X = \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi \). We consider \( k \)-essence models with constant potential \( V(\phi) = V \), which ensures the existence of scaling relation \[95,96\]

\[
XF_X^2 = Ca^{-6} \tag{14}
\]

where \( C \) is a constant. The equation of state of dark energy represented by \( k \)-essence field is given by

\[
w = \frac{p}{\rho} = \frac{F}{2XF_X - F} \tag{15}
\]

The issue of causality in the context of \( k \)-essence scalar field theories with Lorentz invariant action of the form \( S = \int d^4x \sqrt{-g} L(\phi, X) \) where \( g \) is the determinant of the FRW metric considered here and \( L \) is the Lagrangian in Eq. (11) has been discussed in detail in [50,97]. Variation in the action with respect to the scalar field gives the equation of motion of the scalar field \( \phi \) as

\[
G^{\mu\nu} \nabla_\mu \nabla_\nu \phi + 2X \frac{\partial}{\partial X} \left( \frac{\partial L}{\partial \phi} \right) - \frac{\partial L}{\partial \phi} = 0 \tag{16}
\]

where the effective metric \( G^{\mu\nu} \equiv L_X g^{\mu\nu} + L_{XX} \nabla_\mu \phi \nabla_\nu \phi \) with \( L_X \) and \( L_{XX} \) denoting \( \partial L/\partial X \) and \( \partial^2 L/\partial X^2 \) respectively, has a Lorentzian structure and describes the time evolution of the system if the following condition is satisfied \[50,98–100\]

\[
1 + 2X \frac{L_{XX}}{L_X} > 0 \tag{17}
\]

Now introducing \( c_s^2 \) as

\[
c_s^2 \equiv \left[ 1 + 2X \frac{L_{XX}}{L_X} \right]^{-1} \tag{18}
\]

it has been shown in [101], that for \( X = \frac{1}{2} \dot{\phi}^2 \), in our case \( > 0 \), the quantity \( c_s^2 \) plays the role of sound speed squared for propagation of small perturbations. However, in the context of
spatially flat FRW universe, with small perturbations, neglecting vector perturbations which decay as $a^{-2}$, the metric may be written as

$$ds^2 = (1 + 2\Phi)dt^2 - a^2(t)[(1 - 2\Phi)\delta_{ij} + h_{ij}]dx^i dx^j$$

(19)

where $\Phi$ (the gravitational Newtonian potential) is the scalar perturbation and $h_{ij}$ is traceless transverse perturbations. From the standard results of cosmological perturbation theory [101–103], it follows that perturbations in the k-essence field $\delta\phi$ which are gauge invariant are connected with the scalar metric perturbations and the dynamics of cosmological perturbation may be described by the action of the form

$$S_c = \frac{1}{2} \int d^3x d\eta \left[ \left( \frac{dv}{d\eta} \right)^2 - c_s^2 (\nabla v)^2 - m_c^2 v^2 \right]$$

(20)

where $\eta = \int dt/a(t)$ is the conformal time, $v = \sqrt{\frac{d\phi}{dX}} a(\delta\phi + \frac{1}{\sqrt{\Phi}} \frac{d\phi}{d\eta} \Phi)$, $\eta = (1/a)(da/d\eta)$, $m_c^2 = -(1/z)(d^2z/d\eta^2)$ with $z = \sqrt{\frac{d\rho}{dX} a \frac{d\phi}{d\eta}}$ and the quantity $c_s^2$ representing sound speed squared for propagation of small perturbations in Eq. (20) is given by

$$c_s^2 = \frac{d\rho/dX}{d\rho/dX}$$

(21)

A derivation for the above formula from an effective hydrodynamical description of the system is also obtained in [97].

Therefore, for the classical solutions $F(X)$ of the scaling relation Eq. (14) to be stable against small perturbations of the background energy density, the square of adiabatic sound speed should be positive ($c_s^2 > 0$). On the other hand, causality arguments require that this speed of propagation of small perturbations of the background should not exceed the speed of light, implying $c_s^2 < 1$ [104–106]. In the context of k-essence model, using Eqs. (11) and (12) in (21) we have

$$c_s^2 = \frac{F_X}{2XF_{XX} + F_X}$$

(22)

Using $1/a = 1+z$ (where $z$ is the redshift and value of scale factor $a$ at the present epoch is normalised to unity) in $H = \dot{a}/a$ we have $dt = -dz/(1+z)H$. Exploiting this result and transforming time dependences of $\rho$, $p$ and $w$ to their $z$-dependences in Eq. (13), we may also express the sound speed squared as a function of redshift $z$ as

$$c_s^2 = \frac{3w(1+w) + (1+z)dw/dz}{3(1+w)}$$

(23)

We note from Eq. (23) that the bound $c_s^2 > 0$ corresponds to

$$\text{either} \quad -(1+z)dw/dz < 3w(1+w), \; w < 0$$

$$\text{or} \quad -(1+z)dw/dz > 3w(1+w), \; w < 1$$

(24)

and the bound $c_s^2 < 1$ corresponds to

$$\text{either} \quad -(1+z)dw/dz > 3(w^2 - 1), \; w > 1$$

$$\text{or} \quad -(1+z)dw/dz < 3(w^2 - 1), \; w < 1$$

(25)

Whether the physical bound $c_s^2 > 0$ (or $0 < c_s^2 < 1$) is realised for any chosen functional form of equation of state $w(z)$ may be verified by checking the conditions given in Eq. (24)
Fig. 1 Pictorial representations of the conditions for \( c_s^2 > 0 \) and \( c_s^2 < 1 \) in the parameter space spanned by \( w \) and \(- (1 + z) \frac{dw}{dz}\) [see Eqs. (24) and (25)]. For \( w < -1 \), the region lying above the dotted line \((3w(1 + w))\) and for \( w > -1 \) region lying below the dotted line correspond to \( c_s^2 > 0 \). The shaded region corresponds to the bound \( 0 < c_s^2 < 1 \). The curves at the best-fit \((w_a, \ w_b)\) points corresponding to different parametrisations of \( w(z) \) in different varying dark energy models are also shown.

(or Eqs. (24)and (25)). These conditions are pictorially demonstrated in Fig. 1 (see [104] for details) where we have studied the effect of the conditions on the plane spanned by quantities \( w \) and \(- (1 + z) \frac{dw}{dz}\). The shaded region marked in the figure is bounded by two curves \(- (1 + z) \frac{dw}{dz} = 3w(1 + w) \) (dashed line) and \(- (1 + z) \frac{dw}{dz} = 3(w^2 - 1) \) (solid line). The curves at the best-fit \((w_a, \ w_b)\) points corresponding to different parametrisations of \( w(z) \) in different varying dark energy models are also shown.

Variation in the dark energy density \( \rho(z) \) may be expressed in terms of variation of dark energy equation of state \( w(z) \). Expression for a general functional form of \( w(z) \), in principle, involves infinite number of parameters. However, for a practical analysis, it is effective to express the functional dependence \( w(z) \) in terms a small number of parameters and consider different forms of parametrisations of \( w(z) \). We consider here four different models, often used in the literature in the context of varying dark energy, viz. CPL [25], JBP [29,30], BA [31,32] and logarithmic model [33]. Each of the models uses a specific functional form of \( w(z) \) expressed in terms of two parameters \((w_a, \ w_b)\) and is listed in Table 1 where we have also given functional form of the quantity \( Y(z) \equiv \exp \left[ 3 \int_0^z dz' \frac{1+w(z')}{1+z'} \right] \) which gives the \( z \)-dependence of the corresponding dark energy density \( \rho(z) \) (see Eq. (26)). In the plane of Fig. 1, we have also shown the curves representing the different parametrisations of \( w(z) \)
Table 1  Functional forms of equation of state \( w(z) \) of dark energy as used in different varying dark energy models. Expressions for corresponding \( z \)-dependences of dark energy density, expressed through the function \( Y(z) \) [Eq. (26)] are also given

| Model         | \( w(w_a, w_b; z) \)                                      | \( Y(z) \equiv \exp \left[ 3 \int_0^z \frac{d z'}{1 + z'} \frac{1 + w(z')}{1 + z} \right] \) |
|---------------|----------------------------------------------------------|-----------------------------------------------------------------------------------------------|
| CPL [25]      | \( w_a + w_b \frac{z}{1+z} \)                           | \( (1 + z)^3 \frac{1 + w_a + w_b}{1 + z} \exp \left( -\frac{3w_a z}{1 + z} \right) \)       |
| JBP [29,30]   | \( w_a + \frac{w_b z}{(1+z)^2} \)                       | \( (1 + z)^3 \frac{1 + w_a}{1 + z} \exp \left( \frac{3w_b z^2}{2(1+z)^2} \right) \)      |
| BA [31,32]    | \( w_a + \frac{w_b (1+z)}{1+z^2} \)                     | \( (1 + z)^3 \frac{1 + w_a}{1 + z} \left( 1 + z^2 \right)^{\frac{3w_b}{2}} \)        |
| Logarithmic [33] | \( w_a + w_b \log(1 + z) \)                          | \( (1 + z)^3 \frac{1 + w_a + w_b}{2 \log(1 + z)} \)                                      |

for the best-fit values of parameters \( w_a \) and \( w_b \) obtained from the analysis of observational data (see Sect. 4).

We finally exploit the equations of \( k \)-essence models to reconstruct the form of the function \( F(X) \). Using Eq. (13), we express energy density as a function of redshift as

\[
\rho = \rho^{(0)} Y(z), \quad \text{where} \quad Y(z) = \exp \left[ 3 \int_0^z \frac{d z'}{1 + z'} \frac{1 + w(z')}{1 + z} \right] \quad (26)
\]

where \( \rho^{(0)} \) corresponds to value of dark energy density at present epoch \( (z = 0) \). Using Eqs. (11), (12), (14) and (26), we obtain

\[
\left( \frac{4CV^2}{\rho^{(0)} \Delta^2} \right) X = \frac{Y^2(z)(1 + w(z))^2}{(1 + z)^6} \quad (27)
\]

Writing \( p = \rho w \) in Eq. (11) and then using Eq. (26), we have

\[
\left( \frac{V}{\rho^{(0)}} \right) F(X) = Y(z)w(z) \quad (28)
\]

For a given form of the equation of state \( w(z) \), the right-hand sides of Eqs. (27) and (28) may be evaluated numerically at each \( z \). Eliminating \( z \) from both the equations, one may obtain the \( X \)-dependence of the function \( F(X) \) corresponding to a given form of \( w(z) \). The dependences of \( F(X) \) on \( X \) obtained from the analysis of observational data are shown and described in Sect. 4.

4 Results of analysis of observed data

In this section, we present the results of analysis of the observational data using the methodology described in Sect. 2. We investigate implications of the observations in the context of the varying dark energy models listed in Table 1. For each of the models, we obtain the best-fit values of the parameters \( (w_a, w_b) \) along with their allowed domains at different confidence limits from the analysis. We perform analysis of SNe Ia data alone and also a combined analysis of data from SNe Ia, BAO and OHD (discussed in Sect. 2) where we freely vary the parameters \( w_a, w_b, \Omega_m^0, \alpha \) and \( \beta \) to find their best-fit values [corresponding to minimum value of \( \chi^2 \) in Eq. (10)]. The obtained best-fit values of the above parameters for different models are presented in Table 2.
Table 2  Best-fit values of parameters for different models from analysis of SNe Ia data alone and SNe + BAO + OHD. The values of $\chi^2 (\chi^2_{\text{min}})$ at the best-fit point per DOF (degrees of freedom) are also shown. The range of $z$ for which the value of $c_s^2$ evaluated at best-fit lies between 0 and 1 is also shown. Last column shows the values of $(w_a, w_b)$ corresponding to point $P_1$ (see text for details)

| Data set | Model | $w_a$ | $w_b$ | $\Omega^0_m$ | $\alpha$ | $\beta$ | $\chi^2$/DOF | Range of $z$ for which $0 < c_s^2 < 1$ | $P_1 (w_a, w_b)$ |
|----------|-------|-------|-------|-------------|-------|-------|----------------|--------------------------------|------------------|
| SNe Ia   | CPL   | $-0.63$ | $-0.93$ | $0.23$ | $0.14$ | $3.1$ | $685.42/735$ | $0.93–1.15$ | $(-0.77,1.15)$ |
|          | JBP   | $-0.59$ | $-1.16$ | $0.20$ | $0.14$ | $3.1$ | $685.39/735$ | $-$ | $(-0.49,2.22)$ |
|          | BA    | $-0.65$ | $-0.44$ | $0.22$ | $0.14$ | $3.1$ | $685.48/735$ | $0.94–1.10$ | $(-0.87,0.65)$ |
|          | Log.  | $-0.65$ | $-0.82$ | $0.24$ | $0.14$ | $3.1$ | $685.44/735$ | $0.79–1.01$ | $(-0.85,0.76)$ |
|          |       | $-0.71$ | $-1.94$ | $0.29$ | $0.14$ | $3.13$ | $708.26/751$ | $0.35–0.47$ | $-$ |
| + BAO    | JBP   | $-0.63$ | $-3.14$ | $0.28$ | $0.14$ | $3.13$ | $707.65/751$ | $0.31–0.39$ | $-$ |
| + OHD    | BA    | $-0.77$ | $-0.97$ | $0.29$ | $0.14$ | $3.13$ | $708.86/751$ | $0.45–0.53$ | $-$ |
|          | Log.  | $-0.74$ | $-1.52$ | $0.29$ | $0.14$ | $3.13$ | $708.56/751$ | $0.38–0.51$ | $-$ |

Fig. 2  Plots of $\chi^2 - \chi^2_{\text{min}}$ as a function of each individual parameter of the set $(w_a, w_b, \Omega^0_m, \alpha$ and $\beta)$. The CPL parametrisation of $w(z)$ has been used. In each of the plots, depicting variation in $\chi^2$ with one of the parameters at a time, values of the other parameters are kept fixed at their respective best-fit values as given in Table 2 (solid lines for best fits of SNe Ia data alone and dotted lines for the best fits from the combined analysis of SNe Ia, BAO and OHD). The values of $\Delta \chi^2$ viz. 1, 4 and 9, corresponding, respectively, to one parameter confidence levels of $1\sigma$, $2\sigma$ and $3\sigma$ are shown by dotted horizontal lines.
| Data set  | Model | $1\sigma$ range of $w_a$ | $1\sigma$ & $3\sigma$ range of $w_b$ | $1\sigma$ & $3\sigma$ range of $\Omega_m^0$ | $1\sigma$ & $3\sigma$ range of $\alpha$ | $1\sigma$ & $3\sigma$ range of $\beta$ |
|-----------|--------|--------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| SNe Ia    | CPL    | [0.67, -0.58]            | [-1.25, -0.60]                  | [0.20, 0.26]                    |                                 |                                 |
|           |        |                         | & [-0.76, -0.50]                | [0.15, 0.32]                    |                                 |                                 |
|           | JBP    | [-0.63, -0.55]           | [1.51, -0.77]                   | [0.17, 0.23]                    |                                 |                                 |
|           |        |                         | & [-0.70, -0.47]                | [0.12, 0.30]                    |                                 |                                 |
|           | BA     | [-0.69, -0.61]           | [0.64, -0.23]                   | [0.19, 0.25]                    | [0.13, 0.15]                    | [0.24, 0.25]                    |
|           |        |                         | & [-0.77, -0.52]                | [0.12, 0.30]                    | & [0.12, 0.16]                  | & [0.23, 0.26]                  |
|           | Log.   | [-0.69, -0.60]           | [1.12, -0.48]                   | [0.22, 0.27]                    |                                 |                                 |
|           |        |                         | & [-0.78, -0.51]                | [0.16, 0.33]                    |                                 |                                 |
| SNe Ia    | CPL    | [-0.75, -0.67]           | [-2.3, -1.63]                   | [0.27, 0.30]                    |                                 |                                 |
|           |        |                         | & [-0.84, -0.58]                | [0.24, 0.33]                    |                                 |                                 |
|           | JBP    | [-0.65, -0.58]           | [-3.39, -2.6]                   | [0.27, 0.30]                    |                                 |                                 |
|           |        |                         | & [-0.74, -0.49]                | [0.24, 0.33]                    |                                 |                                 |
|           | BA     | [-0.82, -0.74]           | [-1.24, -0.79]                  | [0.27, 0.30]                    | [0.13, 0.15]                    | [0.24, 0.25]                    |
|           |        |                         | & [-0.91, -0.65]                | [0.24, 0.33]                    | & [0.12, 0.16]                  | & [0.23, 0.26]                  |
|           | Log.   | [-0.78, -0.70]           | [-1.84, -1.22]                  | [0.27, 0.30]                    |                                 |                                 |
|           |        |                         | & [-0.87, -0.61]                | [0.24, 0.33]                    |                                 |                                 |
We also find the ranges of the individual parameters allowed at different confidence levels from the analysis of the observational data. To obtain this, we find the variation in $\chi^2$ with each of the parameters of the set $\{w_a, w_b, \Omega_m^0, \alpha, \beta\}$ at a time, keeping values of all other parameters fixed at their respective best-fit values. The confidence interval for the single parameter may then be obtained from the distribution of the function $\Delta \chi^2 \equiv \chi^2 - \chi^2_{\text{min}}$ [107]. The range of values of the parameter for which $\Delta \chi^2 \leq 1$, $\Delta \chi^2 \leq 4$ and $\Delta \chi^2 \leq 9$, respectively, correspond to 1$\sigma$ (68.3% confidence level (C.L)), 2$\sigma$ (95.4% C.L) and 3$\sigma$ (99.73% C.L) [107] allowed intervals of the parameter. We have shown in Fig. 2 the nature of dependence of $\Delta \chi^2 \equiv \chi^2 - \chi^2_{\text{min}}$ on each of the individual parameters. For demonstrative purpose, we have shown the plot for CPL model only. However, the obtained 1$\sigma$ and 3$\sigma$ ranges of the individual parameters for different models of parametrisations of $w(z)$ are given in Table 3.

![Diagram of parameter space](image)

**Fig. 3** Regions of $w_a - w_b$ parameter space allowed at 1$\sigma$, 2$\sigma$ and 3$\sigma$ confidence limits from analysis of SNe Ia data (left panel) and SNe Ia + BAO + OHD (right panel). The results for four different types of parametrisations of $w(z)$ are shown in different rows. The corresponding best-fit point and the point $P_1$ are also shown (see text for details). The region above the black solid line corresponds to $c_s^2 > 0$. The shaded region in the figure corresponds to the bound $0 < c_s^2 < 1$.

© Springer
For a comparative study of the different parametrisations of \( w(z) \), we have also displayed the joint confidence region in the parameter space of \( w_a \) and \( w_b \). To obtain this, we keep the other parameters (\( \omega_m^0, \alpha, \beta \)) at their respective best-fit values and obtain domains in \( w_a - w_b \) parameter space for which evaluated values of \( \chi^2 \) lie in the domain \( \Delta \chi^2 = \chi^2_{\text{min}} + \Delta \chi^2 \). The 1\( \sigma \), 2\( \sigma \) and 3\( \sigma \) joint confidence regions of two parameters \( (w_a \text{ and } w_b) \) correspond to \( \Delta \chi^2 < 2.30, \Delta \chi^2 < 6.17 \) and \( \Delta \chi^2 < 11.8 \) respectively [107]. The obtained joint confidence regions in \( w_a - w_b \) parameter space for the different \( w(z) \) parametrisations are shown in Fig. 3. The corresponding best-fit points are also shown in the parameter space.

In the context of \( k \)-essence model of dark energy, we investigate to what extent the condition \( 0 < c_s^2 < 1 \) is favoured from observational data for different parametrisations of the dark energy equation of state \( w(z) \). We used the different parametrisations of \( w(z) \) in Eqs. (24) and (25) to find the range of values of the parameters \( w_a \) and \( w_b \) for which the condition \( 0 < c_s^2 < 1 \) is satisfied for all \( z \) within the domain of observations. For each of the parametrisations of \( w(z) \) mentioned in Table 1, this range is shown by a shaded region in \( w_a - w_b \) plane in Fig. 3. The regions corresponding to \( c_s^2 > 0 \) only are also shown for each model. We then observe that for all the models, the shaded region corresponding to \( 0 < c_s^2 < 1 \) has no overlap with the region bounded by 1\( \sigma \) contour allowed from analysis of SNe Ia data alone. For BA and logarithmic models, the overlap is seen when one considers allowed ranges at and beyond \( \sim 2 \sigma \) confidence limits. This implies that the physical bound \( 0 < c_s^2 < 1 \) for the entire range of values of \( z \) probed by the observed data considered here is favoured from observational data (SNe Ia only) at and above \( \sim 2 \sigma \) confidence level if we

\[ (4CV^2/\rho(0)^2) X \]

![Figure 4](https://example.com/figure4.png)

**Fig. 4** Plot of \( c_s^2 \) versus \( z \) for different parametrisations of \( w(z) \) at best fit (right panel) from SNe Ia data and at \( P_1 \) (middle panel) (see text for details). Corresponding plots of \( F(X) \) versus \( X \) reconstructed at best fit and \( P_1 \) are shown in right panel.
consider parametrisations of \( w(z) \) as in BA and logarithmic models. For CPL parametrisation, the physical bound is disfavoured below \( \sim 3\sigma \), and with JBP, it is disfavoured up to even higher confidence limits. For all the different parametrisations of \( w(z) \), the physical bound is disfavoured to a larger extent from a combined analysis of SNe Ia, BAO and OHD. In the \( w_a - w_b \) parameter space shown in Fig. 3, we have also marked a point \( P_1 \) in the shaded region corresponding to \( 0 < c_s^2 < 1 \), at which the value of \( \chi^2 \) is closest to the value of \( \chi^2_{\text{min}} \) for the corresponding model. Thus, \( P_1 \) refers to the maximally favoured values of parameters \( w_a \) and \( w_b \) from observational data for which \( c_s^2 \) lies between 0 and 1 for all \( z \) values. The values of \( (w_a, w_b) \) corresponding to \( P_1 \) are shown in the last column of Table 2.

At the best-fit values of parameters \( w_a \) and \( w_b \) obtained from the combined analyses of observational data from SNe Ia, BAO and OHD for different \( (w_a, w_b) \) parametrisations, we have shown the variation in sound speed squared \( c_s^2(z) \) with redshift \( z \) in left panel of Fig. 4. We see that at the best-fit values of the parameters the calculated value of \( c_s^2 \) lies within its physical bound \( 0 < c_s^2 < 1 \) only for a very narrow range of values of \( z \). These ranges are also shown in Table 2. The variation in \( c_s^2 \) at values of \( w_a, w_b \) corresponding to the point \( P_1 \) is shown for different models in the middle panel of Fig. 4. These correspond to a monotonous variation in \( c_s^2 \) with \( z \) within its physical bound \( 0 < c_s^2 < 1 \) imposed by causality and stability.

In Sect. 3, we discussed the methodology to reconstruct the form of function \( F(X) \) for different forms of parametrisations of \( w(z) \). We have shown in right panel of Fig. 4 the obtained dependences of \( F(X) \) on \( X \), for each of the models at the corresponding best-fit values of the parameters \( (w_a, w_b) \). The figure shows that for JBP model \( F(X) \) has a monotonous dependence of \( X \), whereas for the other models (CPL, BA and Logarithmic) the function is double valued in a certain domain of \( X \).

As discussed earlier, the cosmological parameters that enter into our analysis are \( \Omega_m^0 \) and parameters \( w_a \) and \( w_b \) which parametrise the equation of state of dark energy. We have obtained best-fit values of the parameters from analysis of the SNe Ia observational data. However, for a cosmographic analysis, which is a model-independent way of processing cosmological data, the chosen parameter set is different. Basic aspects of cosmographic methodology and results of cosmographic analysis of SNe data are discussed in [108–110]. Here, we briefly discuss the cosmography constructed from DE equation of state. We also qualitatively compare results of our analysis with the features of results of cosmographic analysis in terms of cosmographic parameters.

Neglecting the contribution to the present-day energy density due to radiation \( \Omega_r^0 \) in Eq. (4), we can express the EOS parameter of dark energy as

\[
\omega(z) = -1 + \frac{1}{3} \left[ \frac{E(z)^2 - \Omega_m^0 (1 + z)^3}{E(z)^2 - \Omega_m^0 (1 + z)^3} \right] (1 + z) \tag{29}
\]

where \( ' \) in the above equation denotes derivative with respect to \( z \). We consider two cosmographic parameters: the deceleration parameter \( q(z) \) and the jerk parameters \( j(z) \), which are defined in a model-independent way as

\[
q(z) = -\frac{\dddot{a}}{a^2} \quad \text{and} \quad j(z) = \frac{\dddot{a}}{a^2} \tag{30}
\]

These parameters are relevant in describing features of expansion of the universe. \( q(z) \) is positive (negative) for a decelerating (accelerating) universe. Evolution of the jerk parameter \( j(z) \) is relevant in search for departure from \( \Lambda \)-CDM model [111]. Exploiting the \( d/dt = \)
Table 4 Expressions of cosmographic parameters $q_0$ and $j_0$ in terms of $w_a$, $w_b$, $\Omega_m^0$ ($\Omega_{de}^0 = 1 - \Omega_m^0$) for different varying dark energy models. Numerical values of $q_0$ and $j_0$ calculated using the expressions at point $P_1$ (see text for details) are also shown.

| Model      | $q_0$ ($w_a$, $\Omega_m^0$) | $j_0$ ($w_a$, $w_b$, $\Omega_m^0$) | $q_0$ at $P_1$ | $j_0$ at $P_1$ |
|------------|-----------------------------|-----------------------------------|----------------|----------------|
| CPL        | $\frac{\Omega_{de}^0[3w_b+2+9w_a(w_a+1)+2\Omega_m^0]}{2(\Omega_{de}^0+\Omega_m^0)}$ | $-0.39$ | $1.71$ |
| JBP        | $\frac{(3w_a+1)\Omega_{de}^0+\Omega_m^0}{2(\Omega_{de}^0+\Omega_m^0)}$ | $-0.08$ | $1.96$ |
| BA         | $\frac{3\Omega_{de}^0[w_b+3w_a(w_a+1)+2\Omega_m^0]}{2(\Omega_{de}^0+\Omega_m^0)}$ | $-0.52$ | $0.58$ |
| Logarithmic| $\frac{\Omega_{de}^0[3w_a^2+3w_b-3w_a-4]+2\Omega_m^0}{2(\Omega_{de}^0+\Omega_m^0)}$ | $-0.47$ | $1.36$ |

$-(1+z)H(z)d/dz$ in the above equations, we express $q(z)$ and $j(z)$ as

$$q(z) = -1 + \frac{1}{2}(1+z) \frac{[E(z)^2]'}{E(z)^2}$$  \hspace{1cm} (31)

$$j(z) = \frac{1}{2}(1+z)^2 \frac{[E(z)^2]''}{E(z)^2} - (1+z) \frac{[E(z)^2]'}{E(z)^2} + 1$$  \hspace{1cm} (32)

where ' in the above two equations denotes derivative with respect to $z$. Using Eq. (4) with $\Omega_r^0 = 0$ and $\Omega_{de}^0 = 1 - \Omega_m^0$ in Eqs. (31) and (32) and using expressions for $w(z)$ in terms of parameters $w_a$ and $w_b$ for different models considered in this work (as summarised in Table 1), we may express corresponding $z$-dependences of $q(z)$ and $j(z)$ involving parameters $w_a$, $w_b$ and $\Omega_m^0$. Putting $z = 0$, we may then obtain relationships between $w_a$, $w_b$, $\Omega_m^0$, $q_0$ and $j_0$, where $q_0$ and $j_0$ correspond to the values of deceleration parameter and jerk parameter at the present epoch. We may take these two parameters ($q_0$ and $j_0$) as cosmographic parameters relevant in this context. In the methodology of analysis described in Sect. 2, apart from the nuisance parameters $\alpha$ and $\beta$, the set ($\Omega_m^0$, $q_0$, $j_0$) instead of the set ($\Omega_m^0$, $w_a$, $w_b$) may be chosen for a cosmographic analysis. A comprehensive statistical analysis of specific DE parametrisations using SNe Ia data has been performed in [108], which directly gives cosmographic parameter values. For a qualitative comparison, we have given in Table 4, the expressions for $q_0$ and $j_0$ for different DE parametrisations in terms of $w_a$, $w_b$ and $\Omega_m^0$. As discussed in Sect. 4, for each of the varying dark energy models, we obtained a point in $w_a - w_b$ parameter space (marked by $P_1$ in Fig. 3), corresponding to the parameters satisfying the physical bound $0 < c_s^2 < 1$. Values of these best fits $P_1$ are given in the last column of Table 2. For a comparison with the results of comprehensive cosmographic analysis performed in [108], we have also given in Table 4 the numerical values of $q_0$ and $j_0$ calculated using their analytical expressions given in second and third columns of the same table, at the point $P_1$.

5 Conclusion

In this work, we have performed a comprehensive analysis of recently released ‘Joint Light-curve Analysis’ (JLA) data to investigate its implications for models of dark energy with varying equation of state parameter $w(z)$. As a benchmark, we considered four different
varying dark energy models, viz. CPL [25], JBP [29,30], BA [31,32] and Logarithmic models [33], each of which involves a specific functional form of $z$—dependence of the dark energy equation of state $w(z)$. The analytical expression for the function $w(z)$ in each case involves two parameters, denoted by $w_a$ and $w_b$. From the analysis of observational data, we have obtained best-fit values of these parameters and also their ranges allowed at $1\sigma$, $2\sigma$ and $3\sigma$ confidence level. Description of the data and methodology of analysis is discussed in detail in Sect. 2. The results of the analysis are presented in Table 2 and depicted in Fig. 3.

We make an attempt to realise the scenario of varying equation state of dark energy in terms of dynamics of a scalar field $\phi$. We assume the scalar field to be homogeneous with its dynamics driven by a $k$-essence Lagrangian $L = V F(X)$, with a constant potential $V$ and a dynamical term $F(X)$ with $X = (1/2)\nabla_{\mu}\phi\nabla_{\mu}\phi$. Consideration of constant potential ensures a scaling relation of the form $X (dF/dX)^2 = Ca^{-6}$ ($C =$ constant) in FLRW space-time background with scale factor $a$. We have exploited this to reconstruct functional form of $F(X)$ for the different varying dark energy models considered here (see Sect. 3 for details). The nature of $F(X)$ is obtained for each kind of $w(z)$ dependences corresponding to the best-fit parameters ($w_a$, $w_b$). In this context, we also obtain the dependences of $c_s^2$ on $z$. $c_s$, as mentioned earlier, is the speed with which small fluctuation in the background energy density grows. Stability of the density perturbations and causality require $c_s^2$ to lie in the domain $0 < c_s^2 < 1$. The results show that at the best fits, $c_s^2$ lies within its physical bound $0 < c_s^2 < 1$ only for a small range of values of $z$. For each of the varying DE models, we have shown the region of the $w_a - w_b$ parameter space for which $c_s^2$ lies with its physical bound for all values of $z$ accessible in SNe Ia observations. We finally find the point $P_1$ in $w_a - w_b$ parameter space for which $0 < c_s^2 < 1$ for all $z$ and which is maximally favoured from the observational data. The $z$—dependence of $c_s^2$ and form of $F(X)$ are also obtained for this point ($P_1$) for all the varying DE models. These results are discussed in detail in Sect. 4 and depicted in Fig. 4. In this section, we have also given a comparison of our results with those obtained from a comprehensive cosmographic analysis performed in [108].

Acknowledgements  We would like to thank the honourable referee for valuable suggestions. AC would like to thank University Grants Commission (UGC), India, for supporting this work by means of NET Fellowship (Ref. No. 22/06/2014 (i) EU-V and Sr. No. 2061451168).

References

1. A.G. Riess et al. Supernova Search Team, Astron. J. 116, 1009 (1998). https://doi.org/10.1086/300499 [astro-ph/9805201]
2. S. Perlmutter et al. Supernova Cosmology Project Collaboration, Astrophys. J. 517, 565 (1999). https://doi.org/10.1086/307221 [astro-ph/9812133]
3. D.J. Eisenstein et al. SDSS Collaboration, Astrophys. J. 633 (2005) 560. https://doi.org/10.1086/466512 [astro-ph/0501171]
4. Y. Zhang. arXiv:1411.5522 [astro-ph.CO]
5. E. Aubourg et al., Phys. Rev. D 92, no. 12, 123516 (2015). https://doi.org/10.1103/PhysRevD.92.123516. arXiv:1411.1074 [astro-ph.CO]
6. C. Blake, K. Glazebrook, Astrophys. J. 594, 665 (2003). https://doi.org/10.1086/376983. arXiv:astro-ph/0301632
7. H.J. Seo, D.J. Eisenstein, Astrophys. J. 598, 720 (2003). https://doi.org/10.1086/379122. arXiv:astro-ph/0307460
8. G. Hinshaw et al. WMAP Collaboration, Astrophys. J. Suppl. 180, 225 (2009) https://doi.org/10.1088/0067-0049/180/2/225. arXiv:0803.0732 [astro-ph]
9. E. Komatsu et al., Astrophys. J. Suppl. 192, 18 (2011). https://doi.org/10.1088/0067-0049/192/2/18. arXiv:1001.4538 [astro-ph]
48. E. Babichev, S. Ramazanov, A. Vikman, JCAP **1811**(11), 023 (2018). https://doi.org/10.1088/1475-7516/2018/11/023. arXiv:1807.10281 [gr-qc]

49. A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis, R. Rattazzi, JHEP **0610**, 014 (2006). https://doi.org/10.1088/1126-6708/2006/10/014. arXiv:hep-th/0602178

50. E. Babichev, V. Mukhanov, A. Vikman, JHEP **0802**, 101 (2008). https://doi.org/10.1088/1126-6708/2008/02/101. arXiv:0708.0561

51. P. Astier et al. [SNLS Collaboration], Astron. Astrophys. **447**, 31 (2006) https://doi.org/10.1051/0004-6361:20054185 arXiv:astro-ph/0510447

52. Sullivan et al., ApJ **737**, 102 (2011). https://doi.org/10.1088/0004-637X/737/2/102. arXiv:1104.1444

53. W. M. Wood-Vasey et al. [ESSENCE Collaboration], Astrophys. J. **666**, 694 (2007) https://doi.org/10.1086/518642 arXiv:astro-ph/0701041

54. Tomny et al., ApJ **750**, 99 (2012). https://doi.org/10.1088/0004-637X/750/2/99. arXiv:1203.0297

55. P. Astier et al. [SNLS Collaboration], Astron. Astrophys. **447**, 31 (2006) https://doi.org/10.1051/0004-6361:20054185 arXiv:astro-ph/0510447

56. Sullivan et al., ApJ **737**, 102 (2011). https://doi.org/10.1088/0004-637X/737/2/102. arXiv:1104.1444

57. W. Zheng, S.Y. Li, H. Li, J.Q. Xia, M. Li, T. Lu, JCAP **1408**, 030 (2014) Erratum: JCAP **1409**, E01 (2014) https://doi.org/10.1088/1475-7516/2014/09/E01, https://doi.org/10.1088/1475-7516/2014/08/030

58. J.A. Frieman et al., Astron. J. **135**, 338 (2008). https://doi.org/10.1088/0004-657X/135/1/338. arXiv:0708.2749 [astro-ph]

59. Kessler et al., ApJS **185**, 32 (2009a). https://doi.org/10.1086/595500 arXiv:0908.4274 [astro-ph]

60. Sollerman et al., ApJ **703**, 1374 (2009). https://doi.org/10.1088/0004-637X/703/2/1374. arXiv:0908.4276 [astro-ph]

61. Lampeitl et al., MNRAS **401**, 2331 (2010a). https://doi.org/10.1111/j.1365-2966.2009.15851.x arXiv:0910.2193 [astro-ph]

62. Campbel et al., ApJ **763**, 88 (2013). https://doi.org/10.1088/0004-637X/763/2/88. arXiv:1211.4480 [astro-ph]

63. Hicken et al., ApJ **700**, 331 (2009). https://doi.org/10.1088/0004-637X/700/1/331. arXiv:0901.4787 [astro-ph]

64. Contreras et al., AJ **139**, 519 (2010). https://doi.org/10.1088/0004-657X/139/2/519. arXiv:0910.3330 [astro-ph]

65. Folatelli et al., AJ **139**, 120 (2010). https://doi.org/10.1088/0004-657X/139/1/120. arXiv:0910.3317 [astro-ph]

66. Stritzinger et al., AJ **142**, 156 (2011). https://doi.org/10.1088/0004-657X/142/5/156. arXiv:1108.3108 [astro-ph]

67. M. Ganeshalingam, W. Li, A.V. Filippenko, Mon. Not. Roy. Astron. Soc. **433**, 2240 (2013). https://doi.org/10.1093/mnras/stt2312. arXiv:1303.4302 [astro-ph.CO]

68. Aldering et al., SPIE Conf. Ser., **4836**, 61

69. B. Leibundgut et al., Astrophys. J. Lett., **V 466**, 21

70. G. Goldhaber et al., ApJ, **558**, 359

71. R. Foley et al., ApJ, **626**, L11

72. J.M.Hook et al., AJ, **130**, 2788

73. S. Blondin et al., AJ, **131**, 1648

74. W.D.Li et al., Am. Inst. Phys. Conf. Ser., 103–106

75. A.V. Filippenko et al., ASP Conf. Ser. **332**

76. W. M. Wood-Vasey et al., New Astron. Rev., **48**, 637

77. J. Frieman et al., Bull. Am. Astron. Soc., **1548**

78. B. Dilday et al., Bull. Am. Astron. Soc. **1459**

79. C. Lidman et al., The Messenger **118**, 24

80. M. Hamuy et al., AJ, **112**, 2391

81. G. Miknaitis et al., PASP, **53**, 224

82. T. Matheson et al., AJ, **129**, 2352

83. W. Li et al., PASP, **113**, 1178

84. W. Li et al. PASP, **115**, 453

85. S. Jha et al., AJ, **132**, 189

86. W.J. Percival et al., MNRAS, **327**, 1297

87. G.M. Voit et al., Rev. Modern Phys., **77**, 207

88. E.V. Linder et al., Phys. Rev. Lett., **90**, 091301

89. C. Hernández-Monteagudo et al., Mon. Not. Roy. Astron. Soc. **438**(2), 1724 (2014). https://doi.org/10.1093/mnras/stt2312. arXiv:1303.4302 [astro-ph.CO]
90. J.Q. Xia, G.B. Zhao, B. Feng, H. Li, X. Zhang, Phys. Rev. D 73, 063521 (2006). https://doi.org/10.1103/PhysRevD.73.063521. arXiv:astro-ph/0511625
91. H. Li, Jun-Qing Xia, JCAP 04, 026 (2010). https://doi.org/10.1088/1475-7516/2010/04/026. arXiv:1004.2774 [astro-ph.CO]
92. J.P. Dai, Y. Yang, J.Q. Xia, Astrophys. J. 857(1), 9 (2018). https://doi.org/10.3847/1538-4357/aab49a
93. W. Zheng, H. Li, J.Q. Xia, Y.P. Wan, S.Y. Li, M. Li, Int. J. Mod. Phys. D 23, 1450051 (2014). https://doi.org/10.1142/S0218271814500515. arXiv:1403.2571 [astro-ph.CO]
94. J.Q. Xia, G.B. Zhao, H. Li, B. Feng, X. Zhang, Phys. Rev. D 74, 083521 (2006). https://doi.org/10.1103/PhysRevD.74.083521. arXiv:astro-ph/0605366
95. R.J. Scherrer, Phys. Rev. Lett. 93, 011301 (2004)
96. L.P. Chimento, Phys. Rev. D 69, 123517 (2004)
97. A. Vikman, K-essence: cosmology, causality and emergent geometry (Ludwig-Maximilians-Universitat, Munchen, 2007). Ph.D. thesis
98. Y. Aharonov, A. Komar, L. Susskind Phys Rev. 182, 1400 https://doi.org/10.1103/PhysRev.182.1400
99. C. Armendariz-Picon, E.A. Lim, JCAP 0508, 007 (2005). https://doi.org/10.1088/1475-7516/2005/08/007. arXiv:astro-ph/0505207
100. A.D. Rendall, Class. Quant. Grav. 23, 1557 (2006). https://doi.org/10.1088/0264-9381/23/5/008. arXiv:gr-qc/0511158
101. J. Garriga, V. Mukhanov, Phys. Lett. B 458, 219–225 (1999)
102. V.F. Mukhanov, H.A. Feldman, R.H. Brandenberger, Phys. Rep. 215, 203–333 (1992)
103. V. Mukhanov, Physical Foundations of Cosmology (University Press, Cambridge, 2005), p. 421
104. R. de Putter, E.V. Linder, Astropart. Phys. 28, 263 (2007). https://doi.org/10.1016/j.astropartphys.2007.05.011. arXiv:0705.0400 [astro-ph]
105. L.R. Abramo, N. Pinto-Neto, Phys. Rev. D 73, 063522 (2006). https://doi.org/10.1103/PhysRevD.73.063522. arXiv:astro-ph/0511562
106. V.H. Cárdenas, N. Cruz, S. Muñoz, J.R. Villanueva, Eur. Phys. J. C 78(7), 591 (2018). https://doi.org/10.1140/epjc/s10052-018-6066-8. arXiv:1804.02762 [gr-qc]
107. W.H. Press, S.A. Teukolsky, W.T. Vetterling, B.P. Flannery, Numerical Recipes in Fortran 77 (Cambridge University Press, Cambridge, 1997)
108. C. Escamilla-Rivera, S. Capozziello, Int. J. Mod. Phys. D 28(12), 1950154 (2019). https://doi.org/10.1142/s0218271819501542. arXiv:1905.04602 [gr-qc]
109. S. Capozziello, R. D’Agostino, O. Luongo, Int. J. Mod. Phys. D 28(10), 1930016 (2019). https://doi.org/10.1142/S0218271819300167. arXiv:1904.01427 [gr-qc]
110. M. Benetti, S. Capozziello. arXiv:1910.09975 [astro-ph.CO]
111. A Al Mamon, K. Bamba, Eur. Phys. J. C 78(10), 862 (2018). https://doi.org/10.1140/epjc/s10052-018-6355-2. arXiv:1805.02854 [gr-qc]