Detection of Particles Under Potential Barrier

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Abstract

We introduce a model detector which registers the passage of a particle through the detector location, without substantially perturbing the particle wave function. (The exact time of passage is not determined in such measurements.) We then show that our detector can operate in a classically forbidden region and register particles passing through a certain point under a potential barrier. We show that it should be possible to observe the particle’s track under the barrier.

1 Introduction

The question we would like to address in this paper is: does a particle leave a track while tunneling under a potential barrier? Semiclassical calculations of the tunneling rate are often performed using the Euclidean under-barrier trajectory of the particle, $x(\tau)$, where $\tau$ is the Euclidean (imaginary) time [1]. This trajectory does not, of course, correspond to any classical motion of the particle in the real time, but if $\tau$ is treated as a parameter, then $x(\tau)$ gives a well-defined spatial path (called the most probable escape path [2]). The question is whether this path is just a mathematical artifact, or, in some sense, the particle does follow this trajectory in the process of tunneling. For instance, if a potential barrier is set up in a cloud chamber, will the track disappear when the particle hits the barrier and reappear on the other side, or will it be continuous?

At first sight, one could think that under-barrier detection is impossible for energetic reasons. If the particle’s wave function is localized in a small region where the potential energy is $U(x)$, then, since the kinetic energy is positive, the average energy of the particle will be greater than $U(x)$. This seems to imply that any under-barrier position measurement would throw the particle above the barrier. This conclusion is indeed correct if the position is measured at a specific moment of time, which leads to wave function localization. However, when an atom in a cloud chamber is ionized by a particle, the exact time of the event is not measured. Observation of the particle track gives us approximate positions of the ionized atoms and only indicates that the particle has passed close to these positions. The particle’s wave function is not necessarily localized in such measurements.
In this paper we are going to argue that the spatial trajectory of a particle under a potential barrier is just as real as its semi-classical trajectory in the classically allowed region. The paper is organized as follows. In the next section we introduce a simple 1-dimensional model detector which can perform “non-destructive” position measurements. The detector does not substantially perturb the particle, and yet is able to register its presence in a particular region. The 3-dimensional case is considered in Section 4. The under-barrier operation of our detector in one and three dimensions is discussed, respectively, in Sections 3 and 5. The conclusions of the paper are summarized and discussed in Section 6.

2 Model detector

The simplest detector is a two-level system with states $|0\rangle$ and $|1\rangle$ which we shall read as “no particle” and “particle detected” (or “up” and “down”). In realistic detectors the two states have different energies, but this energy difference is insignificant as long as it is much smaller than the other energy scales in the problem. We shall assume for simplicity that the two states of the detector are degenerate.

The state of the system (which includes a particle and the detector) is described by a two-component wave function

$$\Psi (r, t) = \begin{bmatrix} u (r, t) \\ d (r, t) \end{bmatrix},$$

(2.1)

where $u (r, t)$ and $d (r, t)$ are the amplitudes for the particle to be at position $r$ at time $t$ with the detector “up” and “down” respectively.

The interaction between the particle and the detector should flip the state of the detector when the particle is within the sensitive region. This means that the interaction introduces non-diagonal elements in the Hamiltonian. We shall make a simple choice

$$H_{\text{int}} = V \left( r \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv \hat{V} \left( r \right),$$

(2.2)

where the function $V \left( r \right)$ rapidly decreases to zero outside the sensitivity region.

The total Hamiltonian of the system (we put $\hbar = 1$) is given by

$$H = \frac{\hat{p}^2}{2m} + U \left( r \right) + H_{\text{int}},$$

(2.3)

where $\hat{p} = -i \nabla$ is the momentum operator and $U \left( r \right)$ is the external potential describing the barrier. We note that the Hamiltonian (2.3) is formally equivalent to that of a spin-1/2 particle moving in a potential $U \left( r \right)$ and in a magnetic field $B_y = V \left( r \right) / q, B_x = B_z = 0$, where $q$ is the particle charge. A similar model was employed in [3], where the rotation of the particle spin was used to determine the time spent by the particle under the barrier. For that purpose, the magnetic field in [3] was chosen to extend over the whole under-barrier region, while here we are going to assume that the detector size is much smaller than the barrier width.

We shall first consider the operation of our detector without a barrier, taking $U \left( r \right) = 0$. To simplify the discussion here, we restrict ourselves to the 1-dimensional case. We will treat the corresponding 3-dimensional case in Sections 4 and 5.
To analyze the detection of free particles in one dimension, it is sufficient to solve the time-independent Schrödinger equation

\[
-\frac{1}{2m} \frac{d^2 \Psi}{dx^2} + \hat{V}(x) \Psi = E \Psi,
\]

(2.4)

where \( \hat{V}(x) \) is taken from (2.2). We treat the detector potential \( V(x) \) as a perturbation, and to linear order in \( V(x) \) the solution of (2.4) is

\[
\Psi(x) = \Psi^{(0)}(x) - \int G(x, x') \hat{V}(x') \Psi^{(0)}(x') dx',
\]

(2.5)

where

\[
G(x, x') = \frac{im}{p} \exp \left( ip |x - x'| \right)
\]

(2.6)

is the Green’s function for the equation (2.4), \( p \equiv \sqrt{2mE} \), and \( \Psi^{(0)}(x) \) is the unperturbed wave function describing the incident particle with detector “down”:

\[
\Psi^{(0)}(x) = e^{ipx} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

(2.7)

The asymptotic forms of \( \Psi(x) \) at large \( |x| \) are

\[
\Psi(x \to +\infty) = e^{ipx} \left( 1 - \frac{im}{p} \int \hat{V}(x') dx' \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

(2.8)

\[
\Psi(x \to -\infty) = e^{ipx} \left( 1 - e^{-2ipx} \frac{im}{p} \int \hat{V}(x') e^{2ipx'} dx' \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

(2.9)

The expressions (2.8), (2.9) directly give us the probabilities of different interaction outcomes. The term multiplied by the “up” detector state describes the wave function of the particle in case it is detected, and hence the probability of transmission with detection is

\[
w_{td} = \frac{m}{p} \int V(x) dx \left| \right|^2,
\]

(2.10)

and the probability of reflection with detection is

\[
w_{rd} = \frac{m}{p} \int V(x) e^{2ipx} dx \left| \right|^2.
\]

(2.11)

(There is no reflection without detection in the first-order approximation.) If the size \( L \) of the interaction region is large enough so that \( L \gg p^{-1} \), then the integral in (2.11) is much smaller than the corresponding integral in (2.10). This means that \( w_r \ll w_{td} \) and hence the total probability of detection is

\[
w_d = w_{td} + w_{rd} \approx w_{td} = \frac{m}{p} \int V(x) dx \left| \right|^2.
\]

(2.12)
The detection probability \((2.12)\) may be written as
\[
wd = (\bar{V} \Delta t)^2, \tag{2.13}
\]
where
\[
\bar{V} = \frac{1}{L} \int V(x) \, dx \tag{2.14}
\]
is the average interaction potential and
\[
\Delta t = \frac{mL}{p} \tag{2.15}
\]
is the time it takes the particle to traverse the detector location (the interaction region of size \(L\)). We will later compare this expression \((2.13)\) with the under-barrier case in Section 3.

The detection efficiency can be improved by increasing the number of internal states of the detector. A simple extension of our model \((2.2), (2.3)\) with a detector having \(N\) internal states is constructed by taking the interaction term \(\hat{V}(x) = V(x) \hat{M}\), where the matrix \(\hat{M}\) must have non-zero off-diagonal elements. For instance, if we choose for simplicity \(M_{ij} = 1 - \delta_{ij}\),
\[
\hat{V}(x) = V(x)
\begin{pmatrix}
0 & 1 & \cdots & 1 \\
1 & 0 & \cdots & \vdots \\
\vdots & \vdots & \ddots & 1 \\
1 & \cdots & 1 & 0
\end{pmatrix} \tag{2.16}
\]
the detection probability \((2.12)\) will be multiplied by \((N - 1)\). With a suitable choice of parameters the detection probability could be made close to 1, while the reflection probability would still be negligible. (The perturbative expressions like \((2.11), (2.12)\) are valid only when the detection probability is very small, \(wd \ll 1\); nevertheless one can show using exactly solvable potentials that it is indeed possible to make \(wd\) close to 1.)

3 Under-barrier detection in 1 dimension

In order to introduce some useful relations we shall first consider barrier penetration without a detector. The corresponding Schrödinger equation in one dimension is
\[
-\frac{1}{2m} \frac{d^2\psi}{dx^2} + U(x) \psi = E\psi. \tag{3.1}
\]
For a particle incident on the barrier in the positive \(x\) direction, the asymptotic forms of the wave function \(\psi^+(x)\) for large \(|x|\) are:
\[
\psi^+(x \to -\infty) = e^{ipx} + B e^{-ipx}, \\
\psi^+(x \to \infty) = C e^{ipx}. \tag{3.2}
\]
The tunneling probability is \(w_t = |C|^2\). For a particle incident in the negative \(x\) direction, the wave function \(\psi^-(x)\) is
\[
\psi^-(x \to -\infty) = C' e^{-ipx}, \\
\psi^-(x \to \infty) = e^{-ipx} + B' e^{ipx}. \tag{3.3}
\]
and it can be shown \[4\] that \(|B'| = |B|, \ |C'| = |C|\). We shall assume for simplicity that the barrier is symmetric, \(U(-x) = U(x)\). Then \(C' = C, B' = B\), and

\[
\psi^-(x) = \psi^+(-x).
\]  

(3.4)

An approximate form of the tunneling wave functions \(\psi^\pm(x)\) can be found in the WKB limit. Let the classical turning points of the potential be \(x = b\) and \(x = -b\), so \(U(\pm b) = E\). Then \[4\]

\[
\psi^+(x > b) \approx C \sqrt{\frac{p}{k(x)}} \exp \left[ i \int_b^x k(x') \, dx' + \frac{i\pi}{4} \right],
\]

(3.5)

\[
\psi^+(-b < x < b) \approx C \sqrt{\frac{p}{k(x)}} \exp \left[ \int_x^b k(x') \, dx' \right],
\]

(3.6)

where

\[
k(x) = \sqrt{2m|E - U(x)|}.
\]

(3.7)

A similar expression can be written for \(x < -b\). The tunneling amplitude in the WKB approximation is given by

\[
C = \exp \left[ -\int_{-b}^b k(x) \, dx \right].
\]

(3.8)

The wave function \(\psi^-(x)\) for a particle incident in the opposite direction can be found from (3.4). It is then easy to show that for \(x\) inside the barrier, \(-b < x < b\), the functions \(\psi^\pm(x)\) satisfy the relation

\[
\psi^+(x) \psi^-(x) = C \frac{p}{k(x)}, \quad -b < x < b.
\]

(3.9)

In the presence of a detector, the Schrödinger equation becomes

\[
\left( -\frac{1}{2m} \frac{d^2}{dx^2} + U(x) + \hat{V}(x) \right) \Psi = E \Psi.
\]

(3.10)

To linear order in \(V(x)\), its solution can still be written in the form (2.5), where now

\[
\Psi^{(0)}(x) = \psi^+(x) \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

(3.11)

and \(G(x, x')\) is a Green’s function of the equation (3.1):

\[
\left( -\frac{1}{2m} \frac{d^2}{dx^2} + U(x) - E \right) G(x, x') = \delta(x - x').
\]

(3.12)

The Green’s function \(G(x, x')\) is determined by eq. (3.12) only up to an arbitrary solution of the homogeneous equation (3.1). This freedom can be removed by imposing suitable boundary conditions at \(x \to \pm\infty\). The boundary conditions appropriate for a scattering problem require that \(G(x, x')\) at \(x \to \pm\infty\) contains only outgoing waves, that is only terms proportional to \(e^{ipx}\).
at \( x \to +\infty \) and to \( e^{-ipx} \) at \( x \to -\infty \). With this choice of boundary conditions, the Green’s function can be expressed in terms of the tunneling solutions \( \psi^\pm (x) \),

\[
G(x, x') = \frac{-2m}{W} \cdot \begin{cases} \psi^+(x) \psi^-(x') , & x \geq x' , \\ \psi^-(x) \psi^+(x') , & x \leq x' . \end{cases}
\] (3.13)

Here \( W \) is the Wronskian which can be evaluated using the asymptotic expressions (3.2), (3.3):

\[
W(x') = \psi^\prime_+ (x) \psi^- (x) - \psi^+ (x) \psi^\prime_- (x) = 2ipC.
\] (3.14)

Combining eqs. (2.5), (3.11), (3.13), and (3.14) we can find the asymptotic forms of the wave function of the system to the first order in \( V(x) \):

\[
\Psi (x \to +\infty) = e^{ipx} \left[ C \left( \begin{array}{c} 0 \\ 1 \end{array} \right) - \frac{im}{p} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \int V(x') \psi^+ (x') \psi^- (x') \, dx' \right],
\] (3.15)

\[
\Psi (x \to -\infty) = e^{ipx} \left( \begin{array}{c} 0 \\ 1 \end{array} \right) + e^{-ipx} \left[ B \left( \begin{array}{c} 0 \\ 1 \end{array} \right) - \frac{im}{p} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \int V(x') \psi^{+2} (x') \, dx' \right].
\] (3.16)

For each incident particle there are four possible outcomes: the particle is either transmitted or reflected and it is either detected or not. The probabilities of these events which we denote \( w_{\text{dt}}, w_{\text{dr}}, w_{\text{nt}}, \) and \( w_{\text{nr}} \), can be found directly from eqs. (3.15)–(3.16). We shall be interested in conditional probabilities of detection for transmitted and reflected particles, \( w(d|t) \) and \( w(d|r) \). For instance, \( w(d|t) \) is the probability of detection under the condition that the particle is transmitted; it is given by

\[
w(d|t) = \frac{w_{\text{dt}}}{w_t}.
\] (3.17)

where \( w_t = w_{\text{dt}} + w_{\text{nt}} \) is the total transmission probability.

We now shall explore the dependence of these probabilities on the position of the detector under the barrier. We assume that the size of the sensitive region is much smaller than the barrier width, \( L \ll b \).

From (3.16), the detection probability for a reflected particle is

\[
w(d|r) = \left| \frac{m}{p} \int V(x) \psi^{+2} (x) \, dx \right|^2,
\] (3.18)

where we have assumed that the tunneling probability \(|C|^2\) is small so that we may take \(|B|^2 \approx 1\). We see that the probability (3.18) is highest when the detector is placed near the classical turning point \( x = -b \), and that \( w(d|r) \) decreases exponentially as the detector is moved further under the barrier:

\[
w(d|r) \sim \left| \psi^+ (x_0) \right|^4,
\] (3.19)

where \( x_0 \) is the detector location.

The detection probability for transmitted particles, \( w(d|t) \), can be obtained from (3.15):

\[
w(d|t) = \frac{m}{k} \left| \int \frac{V(x)}{k(x)} \, dx \right|,
\] (3.20)
where we have used the WKB relation (3.9). As in the previous section, we can write [cf. eqs. (2.13)–(2.15)]

\[ w(d|t) \approx \left( \frac{mL}{k} \hat{V} \right)^2 = \left( \hat{V} \Delta t \right)^2, \tag{3.21} \]

where \( k(x) \) is calculated at the detector location and \( \Delta t = mL/k \) can be interpreted as the Euclidean detector traversal time, i.e. the time it takes to traverse a length \( L \) with the Euclidean momentum \( k \).

Note that unlike the reflection probability \( w(d|r) \), the detection probability (3.20) is not exponentially suppressed under the barrier and depends on the detector position only through \( k(x) \). [It should be kept in mind that the WKB-based relation (3.9) which gave rise to the simple expression (3.20) is not valid near the classical turning points \( x = \pm b \).]

An intuitive explanation of these results can be suggested. The reflection amplitude can be obtained in the path integral approach by summing over paths that originate at and return back to the negative infinity. This amplitude is dominated by the vicinity of a classical path in which the particle is reflected from the barrier at \( x = -b \). A detector placed far under the barrier is not traversed by such paths, so the detection probability is exponentially suppressed. On the other hand, the transmission amplitude is obtained by summation over paths originating at \( x = -\infty \) and ending at \( x = \infty \). All such paths traverse the location of the detector, and so the detection probability is high regardless of the detector position.

Suppose now that a large number of detectors is uniformly distributed in the classically forbidden region \( -b < x < b \) and that a particle is incident on the barrier from the negative \( x \) direction. After interaction some of the detectors will register the particle; these excited detectors form the “track” which we observe. Then if the particle is reflected, we expect that only some of the detectors near \( x = -b \) will register the particle. On the other hand, if the particle is transmitted, the track will stretch throughout the entire under-barrier region, with the highest density of detection near the turning points \( x = \pm b \) where the Euclidean momentum \( k(x) \) is small.

4 Free particle detection in 3 dimensions

We now consider detection of a free particle in 3 dimensions. The Schrödinger equation for the detector-particle system is

\[ \left( -\frac{1}{2m} \nabla^2 + \hat{V}(\mathbf{r}) \right) \Psi(\mathbf{r}) = E \Psi(\mathbf{r}). \tag{4.1} \]

We assume as before that the operator \( \hat{V}(\mathbf{r}) \) is of the form (2.2) and that the detector potential \( V(\mathbf{r}) \) rapidly decreases outside a region of size \( L \). The coordinates can always be chosen so that the detector is centered at the origin and the incoming particle moves in the positive \( x \) direction. As an example of the detector potential function we can choose

\[ V(\mathbf{r}) = V_0 \exp \left( -\frac{r^2}{L^2} \right), \tag{4.2} \]

but in most of the following discussion we will not have to assume a specific form of \( V(\mathbf{r}) \).
To linear order in $V(r)$, the solution of (4.1) is given by
\[ \Psi(r) = \Psi^{(0)}(r) - \int G(r, r') \hat{V}(r') \Psi^{(0)}(r') \, d^3r', \] (4.3)
where the unperturbed wave function $\Psi^{(0)}(r)$ is still given by (2.7),
\[ \Psi^{(0)}(r) = e^{ipx} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \] (4.4)
and the Green’s function $G(r, r')$ satisfies the equation
\[ \left( -\frac{1}{2m} \nabla^2 - E \right) G(r, r') = \delta(r - r'). \] (4.5)
The solution of (4.5) with the “outgoing wave” boundary condition at infinity is
\[ G(r, r') = \frac{1}{4\pi |r - r'|} \exp(ip |r - r'|) \] (4.6)
with $p = \sqrt{2mE}$.

At large distances from the detector ($r \gg L$), the integration over $r'$ in (4.3) is effectively performed over the range $r' \ll r$, and we can use the expansion
\[ |r - r'| = r - n \cdot r' + O(r'^2), \] (4.7)
where $n$ is a unit vector in the direction of $r$. This gives
\[ \Psi(r) \approx \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ipx} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{e^{ipr}}{4\pi r} \int e^{ip(x' - n \cdot r')} V(r') \, d^3r'. \] (4.8)

We shall assume that the detector is large compared to the particle wavelength, that is
\[ L \gg p^{-1}. \] (4.9)
Then the $r'$-integral in (4.8) is suppressed by the rapidly oscillating exponential, unless $n$ is nearly parallel to the $x$-axis, so that the angle $\theta$ between $n$ and the $x$ axis satisfies
\[ |pL \sin \theta| \lesssim 1. \] (4.10)
As a result, the particle wave function corresponding to the excited detector in (4.8) is appreciably non-zero only in a narrow cone of angular width $\theta_{\text{max}} \sim (pL)^{-1}$ as shown on Fig. 1.

The interpretation of this result is straightforward. The interaction with the detector localizes the particle within a region of size $L$, resulting in a transverse momentum uncertainty $p_\perp \sim L^{-1}$. Therefore the direction of motion after interaction may be altered by an angle
\[ \theta_{\text{max}} \sim p_\perp/p \sim (pL)^{-1}. \] (4.11)
If we distribute detectors uniformly in space, then we expect the moving particle to create a track in which (almost) every excited detector lies within a cone of angular width $\theta_{\text{max}}$ from the preceding excited detector. This can be verified directly by calculating the probability for the particle to be registered by two detectors, using the second-order perturbation expansion in $V(r)$.
5 Under-barrier detection in 3 dimensions

Turning now to the 3-dimensional barrier penetration problem, we assume the geometry illustrated in Fig. 2. The particle is described by a plane wave traveling in the positive \( x \)-direction and the barrier potential \( U (r) \) is a function only of \( x \). Hence, without the detector the problem reduces to the one-dimensional problem of Section 3. We shall assume that the detector is centered on the \( x \)-axis at \( x = x_0 \), with \( -b < x_0 < b \), and that its size is much smaller than the barrier width, \( L \ll b \).

The solution of the Schrödinger equation

\[
\left( -\frac{1}{2m} \nabla^2 + U (x) + \hat{V} (r) \right) \Psi (r) = E \Psi (r)
\] (5.1)

can still be written in the form (4.3), where now the Green’s function satisfies the equation

\[
\left( -\frac{1}{2m} \nabla^2 + U (x) - E \right) G (r, r') = \delta (r - r')
\] (5.2)

and

\[
\Psi^{(0)} = \psi^+ (x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} .
\] (5.3)

The symmetry of the problem suggests that the dependence of \( G (r, r') \) on \( y, y' \) and \( z, z' \) is reduced to the dependence on the distance in the \((y, z)\)-plane. Hence, we can represent it as a Fourier transform,

\[
G (r, r') = \int \frac{d^2 q}{(2\pi)^2} G_q (x, x') e^{iq(R-R')}.
\] (5.4)

Here, \( R \equiv (y, z) \), \( R' \equiv (y', z') \) and \( G_q (x, x') \) satisfies

\[
\left( -\frac{1}{2m} \frac{d^2}{dx^2} + U (x) - E_q \right) G_q (x, x') = \delta (x - x'),
\] (5.5)

where

\[
E_q = E - \frac{q^2}{2m}.
\] (5.6)

Equation (5.3) is identical to eq. (3.12) for the 1-dimensional problem, and we can read the solution from (3.13), (3.14):

\[
G_q (x, x') = \frac{im}{p_q C_q} \cdot \begin{cases} \psi^+_q (x) \psi^-_q (x'), & x \geq x', \\ \psi^-_q (x) \psi^+_q (x'), & x \leq x'. \end{cases}
\] (5.7)

Here, the subscripts \( q \) indicate that the corresponding quantities are evaluated for the energy \( E_q \) given by (5.6).

Suppose the detector is centered at \( x = x_0 \), \( R = 0 \). Then for \( x - x_0 \gg L \) we can use the upper line of (5.7) for the Green’s function in eq. (4.3). Using the WKB expressions (3.6) for \( \psi^+_q (x) \) and \( \psi^-_q (x) = \psi^+ (-x) \) together with eqs. (3.4), (3.8), we obtain after some algebra:

\[
\Psi (r) = \psi^+ (x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \left( \begin{array}{c} 1 \\ 0 \end{array} \right) im \int \frac{d^2 q}{(2\pi)^2} \frac{C_0 \sqrt{p_0}}{C_q \sqrt{p_q}} e^{iqR} \psi^+_q (x)
\]
\[ \cdot \int \frac{dx' V_q(x')}{\sqrt{k_q(x')} k_0(x')} \exp \left\{ - \int_{x'}^b [k_q(\xi) - k_0(\xi)] \, d\xi \right\}, \tag{5.8} \]

where

\[ V_q(x) = \int d^2R e^{-iQ R} V(x, R) \tag{5.9} \]

and a subscript "0" indicates that the corresponding quantities are taken at \( q = 0 \).

The Fourier component \( V_q(x) \) is small for \( q \gg L^{-1} \). For instance, with the detector potential \( (4.2) \),

\[ V_q(x) = \pi L^2 V_0 \exp \left( -\frac{q^2 L^2}{4} \right) \exp \left( -\frac{(x-x_0)^2}{L^2} \right). \tag{5.10} \]

We shall assume that \( L^{-1} \ll k_0(x) \) almost everywhere under the barrier, except in the vicinity of the turning points \( x = \pm b \). If we don’t place the detector near \( x = \pm b \) (which is necessary also because the WKB approximation breaks down at those points) then the integration in \( (5.8) \) is over \( x' \) which are far from \( x' = \pm b \). For these \( x' \) we can put \( k_q(x') \approx k_0(x') \) and replace \( k_q \) by \( k_0 \) everywhere in \( (5.8) \) except in the exponential, where \( k_q \) can be expanded as

\[ k_q \approx k_0 + \frac{q^2}{2k_0}. \tag{5.11} \]

The expansion \( (5.11) \) is not valid all the way up to the turning point, but one can show that \( (5.11) \) gives a good approximation for the \( \xi \)-integral in \( (5.8) \) if \( q \) is much smaller than typical values of \( k_0 \). Since \( p_q = \sqrt{2mE-q^2} \), for \( q \ll \sqrt{2mE} \) we can also set \( p_q \approx p_0 \) (although we cannot substitute \( q = 0 \) in \( C_q \) because the dependence of the latter on \( q \) is exponential). The resulting expression is

\[ \Psi (r) = \psi^+(x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} im \int \frac{d^2q}{(2\pi)^2} \frac{C_0}{C_q} e^{iq \cdot R} \psi^+_q(x) \cdot \int dx' V_q(x') \frac{d}{k_0(x')} \exp \left\{ -\frac{q^2}{2} \int_{x'}^b \frac{d\xi}{k_0(\xi)} \right\}. \tag{5.12} \]

To find the large \( x \) asymptotics of the wave function, we replace \( \psi^+_q(x) \) in \( (5.12) \) by its asymptotic form \( (3.2) \),

\[ \psi^+_q(x) = C_q e^{ip_0 x}. \tag{5.13} \]

For very large \( x \) and \( R \), the \( q \)-integration is dominated by the stationary point of the phase, \( qR + p_0 x \), which is given by

\[ Q = \frac{R}{\sqrt{R^2 + p_0^2}} = n_0 \sin \theta, \tag{5.14} \]

where \( n \) is a unit vector pointing in the direction of \( R \) and \( \theta \) is the angle between \( r \equiv (x, R) \) and the \( x \) axis, \( \tan \theta = R/x \). Using the stationary phase approximation, we find for \( x \rightarrow +\infty \):

\[ \Psi (r) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} C e^{ip x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{ip x} \cos \theta \frac{n_0}{2\pi r} \int dx' \frac{V_Q(x')}{k(x')} \exp \left[ -\frac{Q^2}{2} \int_{x'}^b \frac{d\xi}{k(\xi)} \right], \tag{5.15} \]

where the subscript "0" has been dropped.
The integral over $x'$ in (5.15) can be estimated as

$$I_Q(x_0, b) \sim \bar{V}_Q \frac{VL}{k(x_0)} \exp \left[ -\frac{Q^2}{2} \int_{x_0}^{b} \frac{d\xi}{k(\xi)} \right], \quad (5.16)$$

where $\bar{V}_Q$ is defined as in (2.14). Large values of $Q$ are suppressed in $I_Q$ both by $\bar{V}_Q$ and by the exponential factor, so we get a bell-shaped function of $Q$. As a result, the particle wave function corresponding to the excited detector state \((1 0)\) in (5.15) describes a narrow beam of angular width

$$\theta \approx \frac{Q_{\text{max}}}{p}, \quad (5.17)$$

where $Q_{\text{max}}$ is the effective cutoff introduced by $I_Q$. If the detector is close to the end of the barrier at $x = b$, so that

$$\int_{x_0}^{b} \frac{d\xi}{k(\xi)} \equiv (b - x_0) \frac{1}{k^{-1}(x_0, b)} \lesssim L^2, \quad (5.18)$$

then

$$Q_{\text{max}} \sim L^{-1}. \quad (5.19)$$

Otherwise,

$$Q_{\text{max}} \sim \left[(b - x_0) \frac{1}{k^{-1}(x_0, b)}\right]^{-\frac{1}{2}}. \quad (5.20)$$

The under-barrier wave function in the range $x_0 < x < b$ can be obtained from (5.12) by substituting the WKB expression (3.6) for $\psi^+(x)$. This gives

$$\Psi(r) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \psi^+(x) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \psi^+(x) i m \int \frac{d^2 q}{(2\pi)^2} e^{iqR} \int \frac{dx' V_q(x')}{k(x')} \exp \left[ -\frac{q^2}{2} \int_{x'}^{x} \frac{d\xi}{k(\xi)} \right]. \quad (5.21)$$

Just like in (5.16), the integral over $x'$ in (5.21) is a bell-shaped function of $q$, $I_q(x_0, x)$, but now it has an $x$-dependent width expressed in the notation of (5.18) as

$$q_{\text{max}}(x) \sim \min \left\{ L^{-1}, \left[ (x - x_0) \frac{1}{k^{-1}(x_0, x)} \right]^{-\frac{1}{2}} \right\}. \quad (5.22)$$

This width, in turn, determines the spatial width of the region in which the wave function of the particle with an excited detector is appreciably different from zero:

$$R_{\text{max}}(x) \sim q_{\text{max}}^{-1}(x) = \max \left\{ L, \left[ (x - x_0) \frac{1}{k^{-1}(x_0, x)} \right]^{\frac{1}{2}} \right\}. \quad (5.23)$$

Beyond the barrier, at $x > b$, the width $R_{\text{max}}$ is growing linearly in $x$:

$$R_{\text{max}}(x > b) = R_{\text{max}}(b) + (x - b) \theta_{\text{max}}, \quad (5.24)$$

where $\theta_{\text{max}} = q_{\text{max}}(b)/p$ as in Section 4.
Using a similar argument, one finds that the wave function of the detected particle in the range \(-b < x < x_0\) is exponentially damped compared to the unperturbed wave function \(\psi^+ (x)\):

\[
\Psi (r) = \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \psi^+ (x) - \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \psi^+ (x) \exp \left( -2 \int_0^{x_0} k (\xi) \, d\xi \right) i m \int \frac{d^2 q}{(2\pi)^2} e^{i q R} I_q (x, x_0).
\]

The region where the wave function of the particle with an excited detector is not negligibly small is outlined in Fig. 3. The wave function is exponentially decreasing with the characteristic length \(k^{-1}\) in both directions along the \(x\) axis (and with \(x\)-dependent characteristic length \(R_{\text{max}} (x)\) in all transverse directions) and has its maximum at the detector location. Note, however, that the exponential decay of the wave function in the \(x\) direction does not affect the conditional probability of detection for transmitted particles (since the unperturbed wave function without a detector decays in the same manner).

To observe the track of a tunneling particle, we distribute detectors uniformly in the under-barrier region. If the particle has tunneled out of the barrier and was registered at a position \((x, y, z)\), where \(x > b\), then the conditional probability of a detector registering the particle under the barrier at a position \((x_0, y_0, z_0)\) will be vanishing unless \((y, z)\) lies within the distance \(R_{\text{max}} (x)\) of \((y_0, z_0)\). Therefore, the detectors registering the particle under the barrier will form a line going through the barrier approximately parallel to the \(x\) axis, and the track of the particle outside the barrier will be a continuation of this line.

If \(N\) detectors register the tunneling particle, the deviation of the particle in the transverse direction will be approximately \(\sqrt{N} \cdot R_{\text{max}} \, (\Delta x)\), where \(\Delta x\) is the average distance between two excited detectors. We can compare this deviation to that obtained in the free case in Section 4. Assuming that the average Euclidean momentum \(k\) under barrier is of the same order as the particle momentum \(p\) in the free case, one can show that the under-barrier track is always narrower than the free particle track.

In actual measurements, the track width will probably be determined by the bubble size in a bubble chamber or by the photo-emulsion grain size rather than by transverse deviations from the \(x\) direction.

6 Discussion

We have considered a model detector which registers semiclassical particles traversing its location without substantially perturbing the energy of the particles. If placed sufficiently far under a potential barrier, such detectors will be sensitive to tunneling particles but will not detect reflected particles.

By distributing many detectors in a region of space, we can observe the particle’s track formed by the detectors registering the particle. As expected, the free particle track obtained in this manner is a straight line. In the presence of a potential barrier, the reflected particles create a track which does not extend much beyond the classical turning point, whereas tunneling particles will create a continuous track going through the barrier region. In the case of a plane-symmetric barrier with a particle moving orthogonally to the plane, the track is a nearly straight line. In a more general setting we expect the track to follow the Euclidean trajectory of the particle.

Having established that the under-barrier path of a particle is a (semiclassically) meaningful concept, one can now ask whether or not any meaning can be assigned to a spacetime trajectory...
in the under-barrier region. As we emphasized in the Introduction, the readings of our detectors give no information about the times when the particle passed the detector locations. One could try to determine these times by using time-dependent detectors which are turned on for a finite time interval $\Delta \tau$ and then turned off. For very small $\Delta \tau$, however, the perturbation of the particle energy incurred by the detector (of order $\Delta \tau^{-1}$) would throw the particle above the potential barrier and destroy tunneling. Therefore, we cannot time the tunneling process more precisely than up to $\Delta \tau_{\text{min}} \sim (V_0 - E)^{-1}$. This brings about the somewhat controversial issue of the tunneling time (see for instance [3] and references therein). Several definitions of the tunneling time have been suggested [3], but the most relevant for us here is the dwell time $\tau_D$, which is defined as the average time spent by the particle under the barrier. In the nonrelativistic case this time is essentially independent of the barrier width, and for a rectangular barrier is given by

$$\tau_D = \frac{1}{V_0} \sqrt{\frac{E}{V_0 - E}}. \quad (6.1)$$

It is easily verified that $\tau_D$ is always smaller than the measurement limit $\Delta \tau_{\text{min}}$, and thus non-destructive timing of the tunneling particle is impossible.

In the relativistic case, however, causality requires that a barrier of width $2b$ cannot be traversed in a time less than $2b/c$. For a sufficiently wide barrier, this time can be arbitrarily large, suggesting that the measurement limit $\Delta \tau_{\text{min}}$ can be circumvented. If relativistic under-barrier motion can indeed be timed, it would be interesting to use time-dependent detectors to analyze vacuum decay processes such as pair production in an electric field or bubble nucleation in a false vacuum. In these two cases, the evolution after tunneling is Lorentz-invariant, and appears the same to all inertial observers. It is not clear whether or not this applies to the under-barrier evolution as well. Detectors introduce a preferred reference frame, and it is quite possible that the evolution seen by an observer will depend on her velocity with respect to that frame.

Another problem that can be studied using model detectors similar to ours is the problem of under-barrier measurement in quantum cosmology. Tunneling of the whole Universe has been extensively discussed in the literature [4], and it has been conjectured that the origin of the Universe was a quantum tunneling event. The tunneling time controversy does not exist in quantum cosmology: there are no clocks external to the Universe, and the cosmological wave function does not depend on “time”. (In fact, “time” is an arbitrary coordinate label in general relativity, and physics should not depend on it.) A physically meaningful time is defined using the geometric or matter degrees of freedom (e.g., the radius of the Universe, the abundance of radioactive elements, or detectors like ours). It would be interesting to investigate how under-barrier evolution will be seen by internal observers (or how it will be registered by internal detectors).

**Acknowledgements**

We would like to thank Slava Mukhanov for stimulating our interest in this problem and for useful discussions in the course of our research. We also gratefully acknowledge discussions with Francis Low and Richard Milburn. A.V. is grateful to Alan Guth for his hospitality at MIT, where this research was completed. This work was supported in part by the National Science Foundation and by the U.S. Department of Energy.
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Figure captions

Fig. 1. Detection of a free particle. The wave function of the detected particle is non-negligible in the shaded cone.

Fig. 2. Detector placed under a potential barrier.

Fig. 3. Under-barrier detection. The wave function of the detected particle is non-negligible in the shaded region, which is bounded by $R_{\text{max}}(x)$. 
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