Uplink-Downlink Tradeoff in Secure Distributed Matrix Multiplication

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Abstract

In secure distributed matrix multiplication (SDMM) the multiplication $AB$ from two private matrices $A$ and $B$ is outsourced by a user to $N$ distributed servers. In $\ell$-SDMM, the goal is to design a joint communication-computation procedure that optimally balances conflicting communication and computation metrics without leaking any information on both $A$ and $B$ to any set of $\ell \leq N$ servers. To this end, the user applies coding with $\tilde{A}_i$ and $\tilde{B}_i$ representing encoded versions of $A$ and $B$ destined to the $i$-th server. Now, SDMM involves multiple tradeoffs. One such tradeoff is the tradeoff between uplink (UL) and downlink (DL) costs. To find a good balance between these two metrics, we propose two schemes which we term USCSA and GSCSA that are based on secure cross subspace alignment (SCSA). We show that there are various scenarios where they outperform existing SDMM schemes from the literature with respect to UL-DL efficiency. Next, we implement schemes from the literature, including USCSA and GSCSA, and test their performance on Amazon EC2. Our numerical results show that USCSA and GSCSA establish a good balance between the time spend on the communication and computation in SDMMs. This is because they combine advantages of polynomial codes, namely low time for the upload of $\left(\tilde{A}_i, \tilde{B}_i\right)_{i=1}^N$ and the computation of $O_i = \tilde{A}_i \tilde{B}_i$, with those of SCSA, being a low timing overhead for the download of $(O_i)_{i=1}^N$ and the decoding of $AB$.

Index Terms

Matrix Multiplication, Security, Interference Alignment, Distributed Computing.

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I. Introduction

Distributed matrix multiplication (DMM) is an important ingredient in many applications, including but not limited to machine learning and object recognition. Recently, coding theory has been applied to enhance the efficiency of DMM. Prominent outcomes of this research thrust are Entangled Polynomial codes [1] and PolyDot codes [2]. Although DMM can resolve computational and memory related difficulties, there are security concerns about providing information to external servers. Secure DMM (SDMM) seeks to handle these security concerns through cryptographic, coding and/or information-theoretic means. In the cryptography literature, DMM is applied amongst others in cloud computing [3], [4] and in the MapReduce framework using partially homomorphic encryption [5]. Recently, it has been shown that with proper coding, the time needed for SDMM is less than for local matrix multiplication [6]. In this work, we are interested in SDMM from a coding and information-theoretic perspective.

The main problem of SDMM is to effectively retrieve the matrix product \( \mathbf{AB} \) from \( N \) distributed servers without leaking any information on finite field left matrix \( \mathbf{A} \in \mathbb{F}^{m \times n} \) and right matrix \( \mathbf{B} \in \mathbb{F}^{n \times p} \) to any set of \( \ell \leq N \) external servers (cf. Fig 1). To this end, a user who seeks to determine \( \mathbf{AB} \), encodes matrices \( \mathbf{A} \) and \( \mathbf{B} \) individually to encoding matrices \( \{\tilde{\mathbf{A}}_i\}_{i=1}^N \) and \( \{\tilde{\mathbf{B}}_i\}_{i=1}^N \). The user then provides the \( i \)-th server with both \( \tilde{\mathbf{A}}_i \) and \( \tilde{\mathbf{B}}_i \) from which the \( i \)-th server computes \( \mathbf{O}_i = \mathbf{A}_i \mathbf{B}_i \). After the download of any \( Q \leq N \) server observations \( \mathbf{O}_j \), \( j \in Q \subseteq \{1, \ldots, N\} \), \( |Q| = Q \), the user shall be able to decode \( \mathbf{AB} \). In this context, \( Q \) is known as the recovery threshold (RT). We may measure the efficiency of SDMM through various conflicting metrics.

Two metrics related to the communication efficiency may be the upload (\( K_{UL} \)) and down-
load \( (K_{DL}) \) communication costs. They measure the ratio of total number of bits uploaded (or downloaded) vs. the number of bits attributed to uploading \((A, B)\) to \(N\) servers (or downloading \(AB\) from any \(Q\) servers). Other metrics are rather concerned with the computation complexity. Specifically, the computation complexity can be attributed to both the user and the server. The user computation complexity is comprised of encoding and decoding complexity while the server computation complexity refers to the matrix multiplication \(O_i = \tilde{A}_i\tilde{B}_i\). In case of high deviation in server computation times, minimizing the RT \(Q\) may be important in order to be robust against particularly slowly computing servers which are frequently termed as stragglers.

**TABLE I: Related work in the research area of SDMM.**

| Metric                        | Downlink cost | Recovery threshold | Tradeoff          |
|-------------------------------|---------------|--------------------|-------------------|
|                               |               |                    | Downlink costs vs.| Uplink vs.        |
|                               |               |                    | recovery threshold| downlink costs    |
| [7]–[12]                      |               | [8, 10]            | [13]              | [14, 15]          |

The most related work on SDMM based on different performance metrics is given in Table I. Initially, the research focused on minimizing the download costs for cases where either only one matrix (e.g., \(A\)) is private \([7]\) or both matrices \(A, B\) ought to be secured \([8]–[12], [16]\). For the first scenario, the capacity is fully characterized while for the latter scenario only the asymptotic capacity where the ratio of matrix dimensions satisfy \(n/min(m,p) \to \infty\) is known. The (asymptotic) capacity-achieving scheme uses a novel interference alignment scheme known as secure cross subspace alignment (SCSA) \([17]\). In this alignment scheme, undesired components are distributed over multiple subspace dimensions.\(^3\) More recently, rather than optimizing over a single metric, attempts on characterizing the tradeoff between \((a)\ K_{DL}\ and RT\ [13] \ or \ (b)\ K_{DL}\ and K_{UL}\ [14, 15] \ have been pursued. In [14, 15], a modified system model of SDMM is considered where elements of SDMM are intertwined.

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1. We use the words upload and uplink interchangeably. The same applies to download and downlink.
2. This setting is known as one-sided SDMM.
3. This is in contrast to existing subspace alignment schemes developed for wireless and cache-aided wireless networks \([18, 19]\) where one interference component is not attributed to multiple subspaces.
with private information retrieval (PIR). Specifically, a user is interested in retrieving $A B_\theta$, $\theta \in \{1, \ldots, M\}$, privately without revealing any information on $A$ and the realization of $\theta$ to the servers.

In this work, similarly to [14], [15], we analyze the tradeoff between $K_{DL}$ and $K_{UL}$ for the classical SDMM system model (instead of an intertwined one-sided SDMM-PIR model). To this end, we propose two SCSA-based, uplink cost adjustable schemes which we term USCSA and GSCSA. These two schemes allow us to flexibly balance uplink and downlink costs through respective matrix partitioning and encoding parameters. By choosing USCSA and GSCSA scheme parameters appropriately, we can benefit from advantages of both polynomial and SCSA codes. Note that these schemes resemble an SCSA-scheme developed for the problem of coded distributed batch matrix multiplication, which was proposed very recently in an independent work by Jia et al. [20]. To improve the encoding efficiency of all SDMM schemes, we suggest applying a second-order (SO) polynomial evaluation scheme that exploits the even-odd decomposition of a polynomial $u(x) = u_{\text{even}}(x^2) + xu_{\text{odd}}(x^2)$ evaluated at $x$ and $-x$. To verify our theoretical findings, we test USCSAs and GSCSAs performance against various other schemes with respect to the time needed for encoding, upload, computation, download and decoding through implementation on Amazon EC2 clusters. Our proposed schemes show a particular good balance between the time needed for upload and download. In addition, the simulations show that SO-based polynomial evaluation represents a good encoding solution, particularly for cross subspace alignment (CSA) schemes.

**Notation:** Throughout this paper, boldface lower-case and capital letters represent vectors and matrices, respectively. Specifically, for any integer $a$, we define $a_n$ to be column vector of dimension $n \times 1$ with all elements being $a$. $A = \text{diag}(x_n)$ is a diagonal $n \times n$ matrix with the $i$-th diagonal entry $A_{ii}$ corresponding to the $i$-th element $x_{ni}$ of $x_n$. We use $\circ$ and $\otimes$ to denote the Hadamard and Kronecker product, respectively. Further, for any two integers $a, b$ with $a \leq b$, we define $[a : b] \triangleq \{a, a+1, \ldots, b\}$.

**II. System Model**

We consider the problem of secure distributed matrix multiplication (SDMM). In this problem, the user has two confidential matrices $A \in \mathbb{F}^{m \times n}$ and $B \in \mathbb{F}^{n \times p}$ with elements drawn from a sufficiently large field $\mathbb{F}$. The goal of the user is to retrieve the matrix product...
AB by using N servers without revealing the identity of both A and B to the curious servers. We assume that any set \( L \subseteq [1 : N] \) of \(|L| = \ell \leq N\) collude. The system model of SDMM is shown in Fig. 1.

To ensure secure matrix multiplication, the user applies encoding functions \( f = (f_1, \ldots, f_N) \) and \( g = (g_1, \ldots, g_N) \) to encode matrices \( A \) and \( B \), respectively. Hereby, \( f_i \) and \( g_i \) denote functions that encode matrices \( A \) and \( B \) for the \( i \)-th server. \( \tilde{A}_i \) and \( \tilde{B}_i \) are the encoded versions of \( A \) and \( B \) provided to the \( i \)-th server; in other words, they are the outputs of encoding functions \( f_i \) and \( g_i \), i.e.,

\[
\tilde{A}_i = f_i(A), \quad \tilde{B}_i = g_i(B).
\]

We assume that every server is honest, thus the server observation \( O_i \) is a deterministic function of \( \tilde{A}_i \) and \( \tilde{B}_i \), i.e., \( H(O_i|\tilde{A}_i, \tilde{B}_i) = 0 \). Upon receiving all server observations \( O_{[1:N]} = (O_1, \ldots, O_N) \), the user is able to determine \( AB \) by invoking the decoding function \( d(\cdot) \), such that \( AB = d(O_{[1:N]}) \). Information-theoretically, this is equivalent to the decodability constraint

\[
H(AB|O_{[1:N]}) = 0. \tag{1}
\]

Since servers \( j \in L \) collude and the security has to be preserved, the collection of encoding matrices \( \tilde{A}_j \) and \( \tilde{B}_j \), \( \forall j \in L \), denoted by \( \tilde{A}_L \) and \( \tilde{B}_L \), do not reveal any information on private matrices \( A \) and \( B \). Thus,

\[
l(\tilde{A}_L, \tilde{B}_L; A, B) = 0, \quad \forall L \subseteq [1 : N], |L| = \ell. \tag{2}
\]

In this paper, we compare the communication cost of SDMM schemes. This cost is comprised of uplink (UL) and downlink (DL) costs. These two costs are defined as follows

\[
K_{UL} = \sum_{i=1}^{N} \frac{|\tilde{A}_i| + |\tilde{B}_i|}{n(m + p)}, \tag{3}
\]

\[
K_{DL} = \sum_{i=1}^{N} \frac{|O_i|}{mp}, \tag{4}
\]

where \( |\tilde{A}_i|, |\tilde{B}_i|, |O_i| \) denote the number of elements in \( \tilde{A}_i \), \( \tilde{B}_i \) and \( O_i \), respectively. In (3) and (4), the denominator corresponds to the number of matrix elements of \( (A, B) \) and \( AB \), respectively. In the next sections, we elaborate on achievability results on these two metrics. As far as the achievability is concerned, we review the cross subspace alignment scheme [17] applied to SDMM [10], [11] and then elaborate on our uplink cost adjustable SDMM schemes.
III. Review: SCSA in SDMM

Before discussing our scheme that balances uplink and downlink cost, we review SCSA. This scheme is parametrized by the variable \( b \in \{0, 1\} \). Thus, we write SCSA\((b)\). In the following, we describe the main ingredients of SCSA\((b)\), namely matrix partitioning, encoding at the user, matrix multiplication at the servers and the decoding at the user.

A. Matrix Partitioning

Most schemes execute horizontal and vertical matrix partitioning of both \( A \) and \( B \). To this end, we define the partitioning operator \( \text{PART}([v_A, h_A], [v_B, h_B]) \), which breaks matrix \( A \) into \( v_A h_A \) equal-size sub-matrices \( A_{ij} \in \mathbb{F}^{m/v_A \times n/h_A}, \forall i \in [1 : v_A], j \in [1 : h_A] \) and matrix \( B \) into \( v_B h_B \) sub-matrices \( B_{jk} \in \mathbb{F}^{m/v_B \times n/h_B}, \forall j \in [1 : v_B], k \in [1 : h_B] \) such that

\[
A = \begin{bmatrix}
A_{11} & A_{12} & \ldots & A_{1h_A} \\
A_{21} & A_{22} & \ldots & A_{2h_A} \\
\vdots & \vdots & \ddots & \vdots \\
A_{v_A 1} & A_{v_A 2} & \ldots & A_{v_A h_A}
\end{bmatrix}, \quad B = \begin{bmatrix}
B_{11} & B_{12} & \ldots & B_{1h_B} \\
B_{21} & B_{22} & \ldots & B_{2h_B} \\
\vdots & \vdots & \ddots & \vdots \\
B_{v_B 1} & B_{v_B 2} & \ldots & B_{v_B h_B}
\end{bmatrix}.
\]

Note that the \( \text{PART} \)-operator works under the assumption that \( m, n \) and \( p \) are multiple of \( v_A, h_A \) and \( h_B \), respectively. In SCSA\((b)\), \( b \in \{0, 1\} \), the user either applies only a partitioning

- of matrix \( A \) in the form \( \text{PART}([r, 1], [1, 1]) \) if \( b = 0 \),
- or of matrix \( B \) in the form \( \text{PART}([1, 1], [1, r]) \) if \( b = 1 \)

with \( r = N - 2 \ell \). In the sequel of this chapter, we describe latter case \((b = 1)\). For this case

\[
AB = \begin{bmatrix}
AB_1 & AB_2 & \ldots & AB_r
\end{bmatrix}.
\]

B. Encoding at the User

Based on the above partition, the user encodes matrix \( A \) and each sub-matrix \( B_j \) (destined to the \( i \)-th server) for SCSA\((1)\) individually according to:

\[
\tilde{A}^{(j)}_i = \frac{1}{j + \alpha_i} \left( A + \sum_{k=1}^{\ell} (j + \alpha_i)^k Z_{jk} \right), \quad (5)
\]

\[
\tilde{B}^{(j)}_i = B_j + \sum_{k=1}^{\ell} (j + \alpha_i)^k Z'_{jk}, \quad (6)
\]

\(^4\)Note that \( h_A = v_B \).
where \( Z_{jk}, Z'_{jk} \) represent i.i.d., uniformly distributed noise terms to ensure privacy of \( A \) and \( B \). The scalars \( \alpha_i, \forall i \in [1:N] \), are distinct elements of
\[
\mathbb{G} = \{ \alpha_i \in \mathbb{F} : j + \alpha_i \neq 0, \forall j \in [1:r]\}.
\] (7)
The user then sends the pairs
\[
(\tilde{A}_i^{(1)}, \tilde{B}_i^{(1)}), \ldots, (\tilde{A}_i^{(r)}, \tilde{B}_i^{(r)})
\]
to the \( i \)-th server. The encoding in (5) and (6) is designed such that the decoding boils down to the interpolation of the (matrix) rational function
\[
F(\alpha) = \sum_{j=1}^{r} \frac{1}{\alpha - t_j} X_j + \sum_{s=1}^{\bar{r}} \alpha^{j-1} X_{r+s},
\] (8)
satisfying the conditions
\[
F(\alpha_i) = \sum_{j=1}^{r} \tilde{A}_i^{(j)} \tilde{B}_i^{(j)}, \quad i \in [1:N]
\] (9)
for \( r + \bar{r} \leq N \). Specifically, (5) and (6) ensure that in (8)
\[
X_j = AB_j,
\] (10)
\[
X_{r+s} = \sum_{j=1}^{r} \sum_{k:k \geq s} c_{j,k,s} (AZ_{jk} + Z_{jk}B_j) + \sum_{j=1}^{r} \sum_{k,k:s} d_{j,k+k,s} Z_{jk}Z'_{jk}
\] (11)
for \( j \in [1:r] \) and \( s \in [1:\bar{r}] \). In other words, through the encoding, we separate desired matrix products \( AB_j \) from undesired matrix products (e.g., \( AZ'_{jk} \)) by assigning them, respectively, the rational or the polynomial part of \( F(\alpha) \). Interestingly, as opposed to other SDMM schemes in the literature, undesired matrix products are not solely attributed to a single power \( \alpha^{j-1} \) but rather dispersed to multiple powers.

C. Matrix Multiplication at the Servers

Upon receiving all pairs \( \{\tilde{A}_i^{(j)}, \tilde{B}_i^{(j)}\}_{j=1}^{r} \), the \( i \)-th server computes
\[
O_i = \sum_{j=1}^{r} \tilde{A}_i^{(j)} \tilde{B}_i^{(j)}
\]
\[
= \left[ \frac{1}{1+\alpha_i} \cdots \frac{1}{r+\alpha_i} \right] \otimes I_m \left[ (AB_1)^T \cdots (AB_r)^T \right]^T
\]
\[
+ \left[ \begin{array}{cccc}
1 & \alpha_i & \alpha_i^{2^\ell-1} & \cdots \\
\end{array} \right] \otimes I_m \left[ \sum_{j=1}^{r} I_{j,0}^T \sum_{j=1}^{r} I_{j,1}^T \cdots \sum_{j=1}^{r} I_{j,2^{\ell-1}}^T \right]^T,
\] (12)

\(^5\)The exact definition of the coefficients \( c_{j,k,s} \) and \( d_{j,k+k,s} \) is not of importance and is thus omitted.
where $I_{jk}$ denotes the effective interference terms. The $i$-th server output $O_i \in \mathbb{F}^{n \times p'}$ is then transferred to the user.

### D. Decoding at the User

The user receives the server responses $O_1, \ldots, O_N$. In SCSA, all undesired terms (e.g., $AZ'_{jk}$) disperse to multiple powers $\alpha_i^k$ in the range $k \in [0 : 2^\ell - 1]$. (The superposition of these terms gives $I_{jk}$.) Simultaneously, all desired terms $AB_j$ are distinguishable from each other and from the interference by their unique powers $\frac{1}{\epsilon + \alpha_i}$. In other words, the user is able to decode the desired items since it can construct a full rank decoding matrix

$$D^{\text{SCSA}} = \begin{bmatrix}
\frac{1}{1+\alpha_1} & \ldots & \frac{1}{r+\alpha_1} & 1 & \alpha_1 & \ldots & \alpha_1^{2^\ell-1} \\
\frac{1}{1+\alpha_2} & \ldots & \frac{1}{r+\alpha_2} & 1 & \alpha_2 & \ldots & \alpha_2^{2^\ell-1} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \ddots \\
\frac{1}{1+\alpha_N} & \ldots & \frac{1}{r+\alpha_N} & 1 & \alpha_N & \ldots & \alpha_N^{2^\ell-1}
\end{bmatrix}$$

from server observations $O_1, \ldots, O_N$ each of dimension $n \times p/r$ with $r = N - 2^\ell$. Thus,

$$K^{\text{SCSA}}_{DL} = \frac{N}{r} = \frac{N}{N - 2^\ell}.$$ 

Since we can reverse the partitioning in SCSA, we can show that

$$K^{\text{SCSA}(b)}_{UL} = \begin{cases} 
K^{\text{SCSA}(0)}_{UL} = \frac{N \left( \frac{m}{p} + r \right)}{1 + \frac{m}{p}} & \text{for PART } ([r, 1], [1, 1]) \\
K^{\text{SCSA}(1)}_{UL} = \frac{N \left( \frac{1}{r} + \frac{m}{p} \right)}{1 + \frac{m}{p}} & \text{for PART } ([1, 1], [1, r])
\end{cases}.$$ 

Thus, the lowest possible uplink costs is the minimum of these two cases. Thus,

$$K^{\text{SCSA}}_{UL} = \min_{b \in \{0, 1\}} K^{\text{SCSA}(b)}_{UL} = \frac{N \min \left( 1 + \frac{m}{p} r, \frac{m}{p} + r \right)}{1 + \frac{m}{p}}.$$ 

Note that the pairs $(K^{\text{SCSA}(0)}_{UL}, K^{\text{SCSA}}_{DL})$ and $(K^{\text{SCSA}(1)}_{UL}, K^{\text{SCSA}}_{DL})$ are both achievable in the UL-DL tradeoff curve.

\footnote{Note that $X_{r+k+1} = \sum_{j=1}^r I_{jk}$.}
TABLE II: Applied matrix partitioning strategies based on the proposed schemes. For all three schemes only one partitioning scenario is described in detail in the text of Section IV.

| Scheme   | Sub-Scheme | Partitioning | Comments |
|----------|------------|--------------|----------|
| SCSA     | SCSA(0)    | PART ([r, 1], [1, 1]) | r = N − 2ℓ |
|          | SCSA(1)    | PART ([1, 1], [1, r]) |          |
| USCSA(f, q, g) | USCSA(f, q, g, 0) | PART ([g, 1], [1, ℓ]) | g ∈ {f, q}, ̄g = {f, q} \ g |
|          | USCSA(f, q, g, 1) | PART ([g, 1], [1, ℓ]) |          |
| GSCSA(f, q, g) | GSCSA(f, q, g, 0) | PART ([fq, 1], [1, 1]) | g ∈ {f, q}, ̄g = {f, q} \ g |
|          | GSCSA(f, q, g, 1) | PART ([1, 1], [1, fq]) |          |

IV. UPLINK COST ADJUSTABLE SCHEMES

In SCSA(b), we can apply two partitioning/encoding scenarios – b = 0 and b = 1. Recall that SCSA(1) uses PART ([1, 1], [1, r]), i.e., the matrix A is left without partitioning while matrix B is horizontally partitioned into r sub-matrices. The user conveys to the i-th server r encoded pairs \( (\tilde{A}_i^{(j)}, \tilde{B}_i^{(j)}) \), \( j \in [1 : r] \). The transmission of multiple pairs with a low partitioning level to single servers results in an excessive use of uplink resources. To make better use of uplink resources, we propose two uplink cost adjustable SCSA schemes – uplink-adjustable SCSA (USCSA) and group-based, uplink-adjustable SCSA (GSCSA) – which guarantee better uplink performance than SCSA in exchange for an increase in the downlink costs. As opposed to the classical SCSA of section III both schemes use a more general partitioning and a modified encoding strategy. We discuss the details in the next sub-sections.

A. Matrix Partitioning

Table II specifies the options of partitioning strategies applied by the user for the different schemes. We use the flag variable \( b \in \{0, 1\} \) to differentiate between PART ([d, 1], [1, c]) \( (b = 0) \) and PART ([c, 1], [1, d]) \( (b = 1) \). In GSCSA(f, q, g, 0), we assign sub-matrices \( A_j \), \( j \in [1 : f q] \) into \( g \) groups comprised of \( g \) sub-matrices per group. Group \( k, \forall k \in [1 : ̄g] \), includes sub-matrices \( A_{[(k−1)g+1:kg]} \triangleq \{ A_{(k−1)g+1}, \ldots, A_{kg} \} \). In the sequel, we use the indexing set \( I_{k, g} \triangleq \{(k−1)g+1, \ldots, kg\} \) to refer to the \( k \)-th partitioning group comprised of \( g \) elements. Throughout the text of this section, we use \( g \in \{f, q\} \) and \( ̄g = \{f, q\} \setminus g \).
B. Encoding at the User

Now, we describe the encoding of both USCSA\((f, q, g, 0)\) and GSCSA\((f, q, v, 0)\). Under the described partitioning, the encoded matrices destined to the \(i\)-th server are

- in case of USCSA\((f, q, g, 0)\) for \(j \in [1 : g]\)
  \[
  \tilde{A}^{(j)}_i = \sum_{k=1}^{g} \frac{1}{k + (j - 1)g + \alpha_i} A_k + \sum_{p=1}^{\ell} (j + \alpha_i)^{p-1} Z_{jp}, \tag{13}
  \]
  \[
  \tilde{B}^{(j)}_i = B_j + \prod_{k=1}^{g} (k + (j - 1)g + \alpha_i) \left( \sum_{p=1}^{\ell} (j + \alpha_i)^{p-1} Z'_{jp} \right), \tag{14}
  \]
- and in case of GSCSA\((f, q, g, 0)\) for \(j \in [1 : g]\)
  \[
  \tilde{A}^{(j)}_i = \sum_{k \in I_{j,g}} \frac{1}{k + \alpha_i} A_k + \sum_{p=1}^{\ell} (j + \alpha_i)^{p-1} Z_{jp}, \tag{15}
  \]
  \[
  \tilde{B}^{(j)}_i = B_j + \prod_{k \in I_{j,g}} (k + \alpha_i) \sum_{p=1}^{\ell} (j + \alpha_i)^{p-1} Z'_{jp}, \tag{16}
  \]

where \(Z_{jp}, Z'_{jp}\) represent i.i.d. noise terms to ensure privacy. The user then sends the pairs \(\left(\tilde{A}^{(j)}_i, \tilde{B}^{(j)}_i\right)_{j=1}^{g}\) to the \(i\)-th server.

Remark 1. On the one hand, similarly to SCSA, the encoding in USCSA and GSCSA is designed to facilitate the dispersion of interference components to multiple subspaces which results in low download costs. On the other hand, simultaneously, both USCSA and GSCSA partition matrices \(A\) and \(B\) and accumulate them more efficiently than SCSA to encoding matrices \(\tilde{A}^{(j)}_i\) and \(\tilde{B}^{(j)}_i\), respectively. Overall, in comparison to SCSA, this results in lower upload costs at the expense of increased download costs.

Remark 2 (USCSA vs. GSCSA). Recall that USCSA\((f, q, g)\) partitions both \(A\) and \(B\) into \(f\) and \(q\) parts, whereas GSCSA\((f, q, g)\) partitions either \(A\) or \(B\) into \(fq\) parts. Since \(fq \geq \max(f, q)\) for \(f, g \in \mathbb{N}\), one can argue that with respect to upload costs USCSA (GSCSA) is better suited for matrices \(A \in \mathbb{R}^{m \times n}, F^{n \times p}\) where \(m \approx p\) \((m \gg p\) or \(m \ll p\)\) (cf. details in Section V).

Remark 3. The encoding strategy of matrices \(A\) and \(B\) for both USCSA and GSCSA are exchangeable. In other words, we can use the encoding strategy in (13) and (15) for matrix \(B\) and the encoding in (14) and (16) for matrix \(A\). This establishes USCSA\((f, q, g, 1)\) and GSCSA\((f, q, g, 1)\). However, for the sake of brevity, we describe the remaining steps of the scheme for Eqs. (13)-(16).
Privacy Constraint: Now consider the privacy constraint for any $\ell$ colluding servers, given by $\{i_1, \ldots, i_\ell\}$ and $\alpha_L = [\alpha_{i_1}, \ldots, \alpha_{i_\ell}]^T$ for both USCSA($f, q, g, 0$) and GSCSA($f, q, g, 0$). Further, we define $Z = \{Z_{j1}, \ldots, Z_{j\ell}\}_{j=1}^g$ and $Z' = \{Z'_{j1}, \ldots, Z'_{j\ell}\}_{j=1}^g$ for both USCSA and GSCSA.

- For USCSA($f, q, g, 0$), $\tilde{A}_L = \left[A_{i_1}^{(j)}, \ldots, A_{i_\ell}^{(j)}\right]_{j=1}^g$ and $\tilde{B}_L = \left[B_{i_1}^{(j)}, \ldots, B_{i_\ell}^{(j)}\right]_{j=1}^g$ can be represented by

$$\tilde{A}_L = I_{\tilde{g}} \otimes I_\ell \cdot \text{blkC} (\alpha_L, g, \tilde{g}) \otimes I_{\tilde{g}} \cdot \text{diagV} (\alpha_L, \tilde{g}) \otimes I_{\tilde{g}} \cdot \text{diagV} (\alpha_L, g, \tilde{g}) \otimes I_{\tilde{g}} \cdot Z,$$

$$\tilde{B}_L = I_{\tilde{g}} \otimes 1_\ell \otimes I_n \cdot \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{\tilde{g}} \end{bmatrix} \equiv \begin{bmatrix} I_{\tilde{g}} \\ I_{\tilde{g}} \end{bmatrix} \cdot \Gamma(g) \cdot \text{diagV} (\alpha_L, g, \tilde{g}) \otimes I_{\tilde{g}} \cdot Z',$$

where

$$\text{blkC} (\alpha_L, g, \tilde{g}) = \begin{bmatrix} C (\alpha_L, [1 : g]) \\ C (\alpha_L, [g + 1 : 2g]) \\ \vdots \\ C (\alpha_L, [(\tilde{g} - 2) + 1 : \tilde{g}g]) \end{bmatrix},$$

$$\text{diagV} (\alpha_L, \tilde{g}) = \begin{bmatrix} V_{\ell} (1 + \alpha_L) & 0_\ell & \ldots & 0_\ell \\ 0_\ell & V_{\ell} (2 + \alpha_L) & \ldots & 0_\ell \\ \vdots & \vdots & \ddots & \vdots \\ 0_\ell & 0_\ell & \ldots & V_{\ell} (\tilde{g} + \alpha_L) \end{bmatrix},$$

Note that these matrices differ in their dimension for USCSA and GSCSA. While for USCSA($f, q, g, 0$) $[Z] = \ell g n \times n$, $[Z'] = \ell \tilde{g} n \times \ell$, for GSCSA the dimensions are $[Z] = \ell g n \times n$, $[Z'] = \ell \tilde{g} n \times p$. 
\[ \Gamma(\tilde{g}) = \begin{bmatrix} \prod_{j=1}^{\tilde{g}} \text{diag}(j + \alpha \ell) & \ldots & 0_{\ell} \\ \vdots & \ddots & \vdots \\ 0_{\ell} & \ldots & \prod_{j=1}^{\tilde{g}} \text{diag}(j + (\tilde{g} - 1)g + \alpha \ell) \end{bmatrix}, \]

with the Cauchy and Vandermonde matrices being

\[ C(a, b) = \begin{bmatrix} \frac{1}{a_1 - b_1} & \frac{1}{a_1 - b_2} & \ldots & \frac{1}{a_1 - b_n} \\ \frac{1}{a_2 - b_1} & \frac{1}{a_2 - b_2} & \ldots & \frac{1}{a_2 - b_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_m - b_1} & \frac{1}{a_m - b_2} & \ldots & \frac{1}{a_m - b_n} \end{bmatrix} \in \mathbb{F}^{m \times n}, \]

\[ V_n(a) = \begin{bmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \ldots & a_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_m & a_m^2 & \ldots & a_m^{n-1} \end{bmatrix} \in \mathbb{F}^{m \times n} \]

for \( a = [a_1, a_2, \ldots, a_m]^T \) and \( b = [b_1, b_2, \ldots, b_n]^T \).

- Similarly, for GSCSA we can write the observations \( \tilde{A}_L = \begin{bmatrix} \tilde{A}_{i_1}^{(j)} & \ldots & \tilde{A}_{i_{\ell}}^{(j)} \end{bmatrix}_{\tilde{g}} \) and \( \tilde{B}_L = \begin{bmatrix} \tilde{B}_{i_1}^{(j)} & \ldots & \tilde{B}_{i_{\ell}}^{(j)} \end{bmatrix}_{\tilde{g}} \) as follows

\[
\begin{aligned}
\tilde{A}_L &= I_{\tilde{g}} \otimes I_{\ell} \cdot \text{diag} C(\alpha_L, g, \tilde{g}) \otimes I_{m_{f_q}} \cdot \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{m_{f_q}} \end{bmatrix} + \text{diag} V(\alpha_L, \tilde{g}) \otimes I_{m_{f_q}} \cdot Z, \\
\tilde{B}_L &= 1_{\tilde{g} \ell} \otimes B + \Gamma(\tilde{g}) \cdot \text{diag} V(\alpha_L, \tilde{g}) \otimes I_{n} \cdot Z',
\end{aligned}
\]

where

\[
\text{diag} C(\alpha_L, g, \tilde{g}) = 
\begin{bmatrix}
C(\alpha_L, [1 : g]) & 0_{\ell} & \ldots & 0_{\ell} \\
0_{\ell} & C(\alpha_L, [g + 1 : 2g]) & \ldots & 0_{\ell} \\
\vdots & \vdots & \ddots & \vdots \\
0_{\ell} & 0_{\ell} & \ldots & C(\alpha_L, [(\tilde{g} - 1)\nu + 1 : \tilde{g}g])
\end{bmatrix}
\]
From (17)-(20), it is possible to rewrite $\tilde{A}_L$ and $\tilde{B}_L$ compactly by

$$\tilde{A}_L = P \cdot (Q \otimes I \cdot A + R \otimes I \cdot Z), \quad (21)$$

$$\tilde{B}_L = S(B) + T \otimes I \cdot Z', \quad (22)$$

where

$$S(B) = \begin{cases} 
  I_q \otimes 1_\ell \otimes I_n \cdot B' & \text{for USCSA} \\
  1_q \ell \otimes B & \text{for GSCSA}
\end{cases}.$$

Since the inverses for $P$ and $R \otimes I$ in (21) and $T \otimes I$ in (22) exist, we can show that the observations of any $\ell$ colluding servers are independent of $A$ and $B$.

$$I\left(A, B; \tilde{A}_L, \tilde{B}_L\right) = I\left(A, B; (R \otimes I)^{-1} P^{-1} \tilde{A}_L, (T \otimes I)^{-1} \tilde{B}_L\right)$$

$$\overset{(21)-(22)}{=} I\left(A, B; (R \otimes I)^{-1} Q \otimes I \cdot A + Z, (T \otimes I)^{-1} S(B) + Z'\right) \leq I(A, B; Z, Z') = 0$$

Since $I\left(A, B; \tilde{A}_L, \tilde{B}_L\right) \geq 0$ in general, we infer that

$$I\left(A, B; \tilde{A}_L, \tilde{B}_L\right) = 0.$$

### C. Matrix Multiplication at the Servers

Every server $i$ multiplies its pairs and accumulates them to retrieve the server output $O_i$. Mathematically, the server output becomes for both USCSA($f, q, g, 0$) and GSCSA($f, q, v, 0$)

$$O_i = \sum_{j=1}^{\tilde{g}} \tilde{A}_i^{(j)} \tilde{B}_i^{(j)} = \sum_{j=1}^{\tilde{g}} C_{ji},$$

where e.g. for USCSA($f, q, g, 0$) we may show

$$C_{ji} = \sum_{k=1}^{\tilde{g}} \frac{1}{k + (j-1)g + \alpha_i} A_k B_j + \sum_{t=0}^{2(\ell-1)+g} \alpha_i^t I_{jk}.$$

Note that for the derivation, we use (a) the binomial expansion in the form $(j + \alpha_i)^p = \sum_{t=0}^{p} \binom{p}{t} \alpha_i^t j^{p-t}$ and (b) combine all undesired terms $A_k Z_j^p, Z_j^p B_j$ and $Z_j^p Z_j^p$ with matching exponents in $\alpha_i$ to effective interference components $I_{jt}, \forall t \in [0 : 2(\ell-1)+g]$. A similar expression on $C_{ji}$ can be derived for GSCSA but is omitted here for the sake of brevity. Overall, summing $C_{ji}$ over $j \in [\tilde{g}]$, we obtain

- for USCSA($f, q, g, 0$)

$$O_i = \sum_{j=1}^{\tilde{g}} \sum_{k=1}^{g} \frac{1}{k + (j-1)g + \alpha_i} A_k B_j + \sum_{j=1}^{\tilde{g}} \sum_{k=0}^{2(\ell-1)+g} \alpha_i^k I_{j}.$$
• and for USCSA\((f, q, g, 0)\)

\[
O_i = \sum_{j=1}^{\bar{g}} \sum_{k=1}^{g} \frac{1}{k + (j - 1)g + \alpha_i} A_{(j-1)g+k} B + \sum_{j=1}^{\bar{g}} \sum_{k=0}^{2(l-1)+g} \alpha_i^k I_{jk}.
\]

D. Decoding at the User

Similarly to SCSA, we can derive a decoding matrix for both USCSA and GSCSA by finding a linear representation of \(O_{[1:N]}\) as a function of desired sub-matrix products \(A_k B_j\) and interfering terms \(I_k' = \sum_j I_{jk}\). The general full-rank decoding matrix for both USCSA and GSCSA is given by

\[
D = \left[\begin{array}{cccc}
\frac{1}{(1+\alpha_1)} & \cdots & \frac{1}{(f_q+\alpha_1)} & 1 & \alpha_1 & \cdots & \alpha_1^{2(l-1)+g} \\
\frac{1}{(1+\alpha_2)} & \cdots & \frac{1}{(f_q+\alpha_2)} & 1 & \alpha_2 & \cdots & \alpha_2^{2(l-1)+g} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{1}{1+\alpha_N} & \cdots & \frac{1}{(f_q+\alpha_N)} & 1 & \alpha_N & \cdots & \alpha_N^{2(l-1)+g}
\end{array}\right]
\]

for \(g \in \{f, q\}\). This matrix has the dimension \(N \times Q\) with

\[
Q^{USCSA(f,q,g,b)} = Q^{GSCSA(f,q,g,b)} = f q + g + 2 \ell - 1
\]

for \(b \in \{0, 1\}\) and \(g \in \{f, q\}\). Since there are at most \(N\) server observations accessible, the user selects \(f\), \(q\), and \(g\) such that \(Q \leq N\). Overall the user recovers \(f q\) desired items from \(Q\) overall items (including \(2 \ell + g - 1, g \in \{f, q\}\), interference terms \(I_k'\)). Thus, we attain the download costs

\[
K^{USCSA(f,q,g,b)}_{DL} = \frac{f q + g + 2 \ell - 1}{f q},
\]

\[
K^{GSCSA(f,q,g,b)}_{DL} = \frac{f q + g + 2 \ell - 1}{f q}.
\]

For these downlink costs we need the following respective uplink costs.

\[
K^{USCSA(f,q,g)}_{UL} = \min_{b \in \{0, 1\}} K^{USCSA(f,q,g,b)}_{UL} = \min\left(\frac{N f}{1 + \frac{m g}{p q}}, \frac{N f}{1 + \frac{m g}{p q}}\right),
\]

\[
K^{GSCSA(f,q,g)}_{UL} = \min_{b \in \{0, 1\}} K^{GSCSA(f,q,g,b)}_{UL} = \min\left(\frac{N f}{1 + \frac{m g}{p q}}, \frac{N f}{1 + \frac{m g}{p q}}\right).
\]

Ultimately, we can flexibly balance the matrix partitioning (and ultimately the uplink costs) against the downlink costs. For the special case, when \(g = 1\) and \(\bar{g} = f q = N - 2 \ell\), USCSAs downlink and uplink costs reduce to the one of SCSA.

\(^8\)Since \(Q\) is independent of \(b \in \{0, 1\}\), in the sequel, we remove the index \(b\) wherever possible.
V. Comparison of SDMM Schemes

In the following, we first construct a lower bound on \( K_{UL} \).

*Lemma 1.* For independent and uniformly distributed matrices \( A \) and \( B \), the uplink cost \( K_{UL} \) is bounded from below by

\[
K_{UL} \geq \frac{N}{N - \ell}.
\]  

*Proof.* Details of the proof are provided in the Appendix. \( \square \)

A. USCSA vs. GSCSA

The uplink costs for both USCSA and GSCSA are \( \frac{m}{p} \) for \( m/p \in (0, 1) \) and decreasing in \( m/p \) for \( m/p \in [1, \infty) \). Thus, defining \( K_{UL}^{USCSA}(f,q,g) = N \min \left( 1, \frac{\bar{g}}{g} \right) \) and \( K_{UL}^{GSCSA}(f,q,g) = N \frac{\bar{g}}{f} \frac{1 + \frac{1}{f}}{2} \) for USCSA and GSCSA, respectively, we get for USCSA

\[
K_{UL}^{USCSA}(f,q,g) = N \min \left( 1, \frac{\bar{g}}{g} \right),
\]

and for GSCSA

\[
K_{UL}^{GSCSA}(f,q,g) = N \frac{\bar{g}}{f} \frac{1 + \frac{1}{f}}{2}.
\]

Recall that \( K_{DL}^{USCSA}(f,q,g) = K_{UL}^{GSCSA}(f,q,g) \) while \( K_{UL}^{USCSA}(f,q,g) \neq K_{UL}^{GSCSA}(f,q,g) \) in general for \( g \neq 1 \) or \( \bar{g} \neq 1 \). If \( g, \bar{g} > 1 \), we infer from (25) and (26) that for

- \( m/p \in (0, 1/\max(f,q)] \cup (\max(f,q), \infty) \): \( K_{UL}^{GSCSA}(f,q,g) \leq K_{UL}^{USCSA}(f,q,g) \) and
- \( m/p \in (1/\max(f,q), \max(f,q)] \): \( K_{UL}^{GSCSA}(f,q,g) \geq K_{UL}^{USCSA}(f,q,g) \).

In other words, for low and high ratios of \( m/p \), GSCSA\((f,q,g)\) outperforms USCSA\((f,q,g)\). Further, we may upper bound the additive gap

\[
0 \leq \frac{1}{K_{UL}^{*}} - \frac{1}{K_{UL}} \leq \frac{N - \ell}{N} - \frac{1}{K_{UL}} = 1 - \frac{\ell}{N} - \left\{ \begin{array}{ll}
\frac{2}{N \left( \frac{\bar{g}}{g} \right)} & \text{for USCSA}(f,q,g) \\
\frac{2}{N \left( \frac{1 + \frac{1}{f}}{2} \right)} & \text{for GSCSA}(f,q,g)
\end{array} \right.
\]

\[
\leq 1 - \frac{\ell}{N} - \left\{ \begin{array}{ll}
\frac{2}{N \left( \frac{\max(f,q)}{\min(f,q)} \right)} & \text{for USCSA}(f,q,g) \\
\frac{2}{N \left( \frac{\max(f,q)}{\min(f,q)} \right)} & \text{for GSCSA}(f,q,g)
\end{array} \right.
\]

\[
\leq 1 - \frac{\ell}{N} - \frac{2}{N (N - 2\ell + 1)} = 1 - \frac{\ell}{N} - \frac{2}{N^2} \cdot \frac{1}{1 - (2\ell - 1)},
\]

(28)

where (a) follows from maximizing the denominator of the last term for \( Q^{USCSA}, Q^{GSCSA} \leq N \). This suggests that the maximum additive gap \( \frac{1}{K_{UL}^{*}} - \frac{1}{K_{UL}} \) is decreasing in \( \ell/N \).
B. Uplink-Downlink-Cost-Tradeoff – Comparison with Other Schemes

We now compare the achievable reciprocal uplink and downlink cost pairs \((1/K_{UL}, 1/K_{DL})\) for various SDMM schemes. These are the aligned secret sharing scheme (A3S) \([8]\) and the gap additive secure polynomial (GASP) scheme \([9]\). Fig. 2 shows the performance of all schemes when \(N = 100, \ell = 8, g = f\) and \(m/p = 200\).

![Graph showing comparison between the achievable reciprocal uplink and downlink cost pairs for multiple schemes: (i) A3S & GASP Codes, (ii) USCSA and (iii) GSCSA. In these plots, we set \(N = 100, \ell = 8\) and \(m/p = 200\). For the upper bounds, we use \(1/K_{DL} \leq (N-\ell)/N = 0.92\) and \(1/K_{UL} \leq (N-\ell)/N = 0.92\).](image)

Recall that in this case, both \(1/K_{UL}\) and \(1/K_{DL}\) are bounded from above by \((N-\ell)/N = 0.92\) (cf. dotted line in Fig. 2). On the one hand, the lowest downlink cost of \(K_{DL} = 1/0.84\) is attained for SCSA (or USCSA and GSCSA when \(f = 1\) and \(q = N - 2\ell = 84\)) when \(1/K_{UL}^{SCSA} \leq 1/141.3 \approx 0.007\). On the other hand, the lowest uplink cost is when \(1/K_{UL} \approx 0.35\) for which \(K_{DL} \leq 42/99 \approx 0.42\). For USCSA and GSCSA, this cost pair is achievable for \(f = 42\) and \(q = 1\). For this example, we see that for almost all pairs, GSCSA outperforms all remaining schemes.

VI. DESIGN OF POLYNOMIAL EVALUATION POINTS

So far, we have focused solely on the communication aspect of CSA schemes. In this section, we propose an efficient polynomial evaluation scheme suited for the CSA encoding.

A. Polynomial Evaluation at a Single Point

Consider the evaluation of an \((n-1)\)-th degree polynomial

\[
u(x) = u_{n-1}x^{n-1} + u_{n-2}x^{n-2} + \ldots + u_1x + u_0, \quad u_{n-1} \neq 0
\]
at some point \( x \). One may decompose (29) in the form,

\[
    u(x) = u_{\text{even}}(x^2) + xu_{\text{odd}}(x^2),
\]

where

\[
    u_{\text{even}}(x) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} u_{2i} x^i,
\]

\[
    u_{\text{odd}}(x) = \sum_{i=0}^{\lfloor n/2 \rfloor - 1} u_{2i+1} x^i,
\]

are, respectively, the even and odd subsidiary polynomials. The procedure

\[
    u(x) = \cdots (u_{n-1} x + u_{n-2}) x + \cdots) x + u_0,
\]

that is starting with \( u_{n-1} \) multiplying by \( x \), adding \( u_{n-2} \), multiplying by \( x \), repeating this procedure until finally adding \( u_0 \) gives \( u(x) \). In the literature, it is known by Horner’s rule (HR) which requires \( n - 1 \) multiplications and \( n - 1 \) additions, minus one addition for each coefficient that is zero. The Horner scheme is optimal in the sense that a polynomial of degree \( n - 1 \) cannot be evaluated with fewer arithmetic operations [21]. In the sequel of this chapter, the ‘unit’ operation means a multiplication followed by an addition. Note that in general the complexity of evaluating \( p \) polynomials \( u_m(x) \) of degree \( \deg(u_m(x)) = n_i \) at a single point \( x = \bar{x} \) according to HR is \( C_{\text{EC}}^{\text{HR}}(n_1, \ldots, n_p) = \sum_{m=1}^{p} n_m \) operations.

The strategy that separately applies Horner’s rule on \( u_{\text{even}}(x^2) \) and \( xu_{\text{odd}}(x^2) \) and combines them according to (30) to \( u(x) \) is known as the second-order (SO) Horner’s rule (SO-HR) [22]. The SO-HR scheme uses \( n \) multiplications and \( n - 1 \) additions to determine \( u(x) \). This corresponds to approximately \( n \) (scalar) operations.

**B. Polynomial Evaluation at \( n \) Points**

Now consider the same polynomial \( u(x) \) that shall be evaluated at \( n \) points \( x_0, x_1, \ldots, x_{n-1} \). On closer inspection of (30) one may realize that the cost for evaluating \( u(-x) \) from \( u_{\text{even}}(x) \) and \( xu_{\text{odd}}(x) \) is only one additional multiplication. Using the SO strategy in conjuction with HR for the evaluation of \( n \) points, one needs in total \( n (n - \lfloor n/2 \rfloor) \) (instead of \( n(n - 1) \) for HR) operations.

\[\text{We count the evaluation of } x^2 \text{ as one extra multiplication.}\]
C. Second-Order Encoding of CSA schemes

For different matrix coefficients, \( j \) matrix polynomials, each evaluated at \( \alpha = \alpha_i \), i.e.,
\[
U^{(j)} (\alpha_i) = \sum_{p=0}^{d-1} U_{jp} (j + \alpha_i)^p ,
\]
\( \forall i \in [1 : N], \forall j \in [1 : \bar{j}] \) represent intermediate results of CSA encoding matrices \( \tilde{A}^{(j)}_i \) and \( \tilde{B}^{(j)}_i \). For these particular matrix polynomials, we choose \( \alpha = [\alpha_1, \ldots, \alpha_N]^T \) with \( \alpha_i \in \mathbb{G} \) such that SO-based polynomial evaluation (cf. VI-B) of \( U^{(j)} \) is applicable. Implicitly it is assumed that \( \bar{j}, N \geq 2 \). Define \( \bar{y}_D = \lfloor N/(\bar{j}+1) \rfloor \) and \( N_R = N - \bar{y}_D(\bar{j} + 1) \). For \( \alpha_i \in \mathbb{G}, \forall i \in [1 : N] \) and distinct \( (\alpha_{s(j+1)+1}) \), the relationship
\[
\begin{cases}
    j + \alpha_{(s-1)(j+1)+1} = -\left(j + \alpha_{(s-1)(j+1)+j+1}\right), & \forall j \in [1 : \bar{j}], \forall s \in [1 : \bar{y}_D] \\
    j + \alpha_{\bar{y}_D(j+1)+1} = -\left(j + \alpha_{\bar{y}_D(j+1)+j+1}\right), & \forall j \in [1 : N_R]
\end{cases}
\]
generates in total \( N_R + \bar{y}_D \bar{j} - 1 = N - \bar{y}_D - 1 \) additive inverse pairs \((x, -x)\) for which SO-HR can be used in the encoding process. In Table III, we derive the overall encoding complexity \( C_{EC}^{(SO-HR \text{ CSA})} \) of evaluating the matrix polynomial \( U^{(j)}, \forall j \in [1 : \bar{j}] \) at \( N \) points \( \alpha = \alpha_i \) chosen according to (33). \(|U|\) denotes the number of matrix elements of \( U^{(j)}, \forall j \in [1 : \bar{j}] \).

**TABLE III:** Derivation of the encoding complexity of evaluating the matrix polynomial \( U^{(j)}, \forall j \in [1 : \bar{j}] \) at \( N \) points \( \alpha = \alpha_i, \forall i \in [1 : N] \).

| Category      | #                          | Complexity [operation] | Overall [operation] |
|---------------|-----------------------------|------------------------|---------------------|
| SO-Pair       | \( N - \bar{y}_D - 1 \)    | \( (d - 1 + \lambda_+) |U| \)                   |
|               |                             | \( \bar{y}_D \bar{j} = \bar{y}_D \bar{j} - 1 \) | \( (N - \bar{y}_D - 1)(d - 1 + \lambda_+) |U| \) |
| No pair       | \( N (\bar{j} - 2) + 2 (\bar{y}_D + 1) \) | \( (d - 1) |U| \)                |
|               |                             | \( N (\bar{j} - 2) + 2 (\bar{y}_D + 1)(d - 1) |U| \) |

The final expression is as follows
\[
C_{EC}^{(SO-HR \text{ CSA})} (|U|, d, N, \bar{j}) = (N \bar{j} (d - 1) - (d - 1 - \lambda_+) (N - \bar{y}_D - 1)) |U|. \quad (34)
\]

\(^{10}\)For SCSA \( \bar{j} = r \), USCSCA and GSCSA \( \bar{j} \in \{ f, q \} \).

\(^{11}\)Note that \( N_R \leq \bar{j} \).

\(^{12}\)Typically, since \(|U| \gg 1\), the scalar multiplication of \( x^2 \) at evaluation point \( x \) is ignored in the operation count.
In this equation $\lambda_+ \in (0, 1)$ (similarly $\lambda_*$) denotes the relative time needed for a scalar addition (multiplication) in comparison to a scalar operation.

We can now derive the complexity of the CSA schemes. Consider first the encoding of SCSA(1) (cf. (5) and (6)). The encoding of $\left(\tilde{A}_j^{(j)}\right)_{i=1, j=1}^{i=N, j=r}$, on the other hand, can be realized through SO-HR (following (33)) and a subsequent multiplication with $\frac{1}{j + \alpha_i}$. Overall, this requires $C_{EC}^{(SO-HR\ CSA)}(|A|, \ell + 1, N, r) + \lambda_* r N |A|$. The encoding of $\left(\tilde{B}_j^{(j)}\right)_{i=1, j=1}^{i=N, j=r}$, on the other hand is attainable using SO-HR with $C_{EC}^{(SO-HR\ CSA)}(|B|/r, \ell + 1, N, r)$ operations. Thus, the overall encoding complexity of SCSA(1) using SO-HR becomes

$$C_{EC}^{SO-HR\ CSA(1)} = C_{EC}^{SO-HR\ CSA}(|A| + |B|/r, \ell + 1, N, r) + \lambda_* r N |A|. \tag{35}$$

Recall that in the encoding of SCSA(0), $\tilde{A}_j^{(j)}$ ($\tilde{B}_j^{(j)}$) is encoded as $\tilde{B}_j^{(j)}$ ($\tilde{A}_j^{(j)}$) for SCSA(1). This implies that the encoding complexity of SCSA(0) applying SO-HR is

$$C_{EC}^{SO-HR\ CSA(0)} = C_{EC}^{SO-HR\ CSA}(|A|/r + |B|, \ell + 1, N, r) + \lambda_* r N |B|. \tag{36}$$

Naturally, if $|A| > |B|$ ($|A| < |B|$), SCSA(0) (SCSA(1)) gives a lower encoding complexity than SCSA(1) (SCSA(0)). For given matrices $A$ and $B$, we compute the SO-HR CSA encoding complexity as

$$C_{EC}^{SO-HR\ CSA} = \min_{b \in \{0, 1\}} C_{EC}^{SO-HR\ CSA(b)}$$

$$= C_{EC}^{SO-HR\ CSA}(\max(|A|, |B|)/r + \min(|A|, |B|), \ell + 1, N, r) + \lambda_* r N \min(|A|, |B|), \tag{37}$$

whereas the encoding complexity of HR CSA is

$$C_{EC}^{HR\ CSA} = \sum_{i=1}^{N} \sum_{j=1}^{r} C_{EC}^{HR\ CSA}(\ell \max(|A|, |B|)/r + \min(|A|, |B|)) + \lambda_* r N \min(|A|, |B|)$$

$$= N r \ell \left(\max(|A|, |B|)/r + \min(|A|, |B|)\right) + \lambda_* r N \min(|A|, |B|). \tag{38}$$

Then, the theoretical gain of SO-HR CSA over HR CSA is given by

$$1 - \frac{C_{EC}^{SO-HR\ CSA}}{C_{EC}^{HR\ CSA}} = \frac{(\ell - \lambda_+)(N - \bar{y} D - 1) \left(\max(|A|, |B|)/r + \min(|A|, |B|)\right)}{C_{EC}^{HR\ CSA}}. \tag{39}$$

The complexity and the theoretical gains for SO-HR USCSA and GSCSA can be derived similarly. Details on the derivation are omitted here for the sake of brevity. The final expressions are provided in Table [X].

$^{13}\lambda_+ + \lambda_* = 1$

$^{14}$We ignore the effort of calculating $N\tilde{j}$ multiplicative inverses of $j + \alpha_i$, $j \in [1 : \tilde{j}]$, $i \in [1 : N]$, since typically $|U| \gg N\tilde{j}$. 
TABLE IV: Encoding complexity of USCSA and GSCSA

| Scheme   | Sub-Scheme | Complexity (operation) ${}^{15}$ |
|----------|------------|----------------------------------|
| USCSA    | USCSA $(f, q, g, 0)$ | $C_{SO-HR CSA}(|A| + |B|, \ell, N, \bar{g}) + N(|A| + |B|)$ |
| USCSA    | USCSA $(f, q, g, 1)$ | $C_{SO-HR CSA}(|A| + |B|, \ell, N, \bar{g}) + N(|A| + |\bar{g}| |B|)$ |
| GSCSA    | GSCSA $(f, q, g, 0)$ | $C_{SO-HR CSA}(|A| + |B|, \ell, N, \bar{g}) + N(|A| + |\bar{g}| |B|)$ |
| GSCSA    | GSCSA $(f, q, g, 1)$ | $C_{SO-HR CSA}(|A| + |B|, \ell, N, \bar{g}) + N(|A| + |\bar{g}| |B|)$ |

VII. Python Implementation

Five schemes (SCSA, USCSA, GSCSA, A3S and GASP) have been implemented using the python library MPI4py that provides the Message Passing Interface (MPI). The implementation is available on GitHub [23]. In order to create a cluster we have used StarCluster, an open source tool for cluster computing [24]. We created a cluster on Amazon EC2, consisting of $N + 1$ hosts – one user and $N$ servers, which run on the instances c3.8xlarge.

To compare the schemes, we measure the time for (a) computing the encoding matrices $(\vec{A}_i, \vec{B}_i)_{i=1}^N (T_{EC})$, (b) the upload of the encoding matrices $(T_{UL})$, (c) the server computation $(T_C)$, (d) the download of $(O_i)_{i \in Q} (T_{DL})$ and (e) the decoding of $AB (T_{DC})$. As far as the computation at the servers is concerned, we do not measure the whole time that is spent by all servers to compute $O_i$ but rather the average time $\bar{T}_C$ over the number servers. However, because all servers exhibit very similar timing overhead for the computation, the fluctuation around the mean value is very small and can therefore be neglected.

The total time, which was spent on the SDMM process is computed as the sum (cf. Fig. 3)

$$\hat{T} = T_{EC} + T_{UL} + \bar{T}_C + T_{DL} + T_{DC}. \quad (40)$$

A. Numerical Results

We test our implementation for two scenarios. In each scenario, we gradually increase the matrix dimensions of $A$ and $B$ – namely $m$, $n$ and $p$ – according to the sequence

$$c_j = 1.3c_{j-1}, \forall c \in \{m, n, p\}, \forall j \in [1 : 9]$$

$^{15}$We ignore the complexity of computing the scalars $\prod_{k=1}^g (k + (j - 1)g + \alpha_i), \forall i \in [1 : N], \forall j \in [1 : \bar{g}]$. 

and measure the times $T_{EC}$, $T_{UL}$, $T_{DL}$, $T_{DC}$ and $\hat{T}$ for $(m_k, n_k, p_k)$, $\forall k \in [0 : 9]$. We average each time measurement over 10 iterations and plot the time measurements over a single matrix dimension – namely $p$. Next, we specify the two considered scenarios.

**Scenario 1:** In this scenario, we choose $\ell = 4$, $N = 15$, $m_0 = 90$, $p_0 = 1000$ and $n_0 \in \{10, 100\}$. We call Scenario 1 with $n_0 = 10$ ($n_0 = 100$) the small-$n$ (big-$n$) scenario. All schemes apply a $\text{PART} ([r_A, 1], [1, r_B])$ matrix partitioning scheme. The schemes are parametrized as follows:

- A3S: $(r_{A}^{A3S}, r_{B}^{A3S}) = (1, 4)$,
- GASP: $(r_{A}^{GASP}, r_{B}^{GASP}) = (2, 2)$,
- SCSA: $(r_{A}^{SCSA}, r_{B}^{SCSA}) = (1, 7)$,
- USCSA: $(r_{A}^{USCSA}, r_{B}^{USCSA}) = (2, 3)$,
- GSCSA: $(r_{A}^{GSCSA}, r_{B}^{GSCSA}) = (1, 6)$

and $f = 2$ and $q = 3$ for USCSA and GSCSA. Note that in the encoding process, we apply SO-HR for A3S, SCSA, USCSA and GSCSA and in the computation process we use a hybrid Winograd-Strassen (WS) algorithm proposed in [25] for matrix multiplication in SCSA. The results for this scenario are shown in Fig. 4 ($n_0 = 10$) and in Fig. 6 ($n_0 = 100$).

**Scenario 2:** In this scenario, we set $\ell = 2$, and $N = 18$ and keep $m_0$, $p_0$ and $n_0$ as in Scenario 1. Since $\ell$ and $N$ changed (as compared to Scenario 1), we adapt the schemes
according to:

- A3S: \((r_A^{A3S}, r_B^{A3S}) = (2, 4)\),
- GASP: \((r_A^{GASP}, r_B^{GASP}) = (3, 3)\),
- SCSA: \((r_A^{SCSA}, r_B^{SCSA}) = (1, 14)\),
- USCSA: \((r_A^{USCSA}, r_B^{USCSA}) = (3, 4)\),
- GSCSA: \((r_A^{GSCSA}, r_B^{GSCSA}) = (1, 12)\)

and \(f = 3\) and \(q = 4\) for USCSA and GSCSA. We plot the results for this scenario in Fig. 3 for \(n_0 = 10\) and in Fig. 7 for \(n_0 = 100\). Now we discuss the numerical results. We discuss each time measurement individually in the following paragraphs.

1) Encoding: The encoding time is the duration needed to compute all encoding matrices \(\left(\tilde{A}_i, \tilde{B}_i\right)_{i=1}^N\). While in cross subspace alignment (CSA) schemes – SCSA, USCSA and GSCSA – the encoding matrices for server \(i\) consist of \(\bar{j}\) pairs \(\left(\tilde{A}_i^{(j)}, \tilde{B}_i^{(j)}\right)_{j=1}^{\bar{j}}\), the encoding in GASP and A3S is lighter in the sense that only a single encoding pair \(\bar{j} = 1\) is needed. Overall, we observe that GASP and A3S outperform all CSA schemes (cf. 4-7). Our results also show that SO-HR represents a viable option to reduce the encoding complexity. They are particularly relevant for CSA schemes, where the absolute time saving over HR is significant. For instance, for Scenario 1 with \(n_0 = 10\) at \(p \approx 1.6 \cdot 10^4\), SO-HR SCSA gives a relative encoding gain of 10.9% which closely matches the theoretical gain of Eq. (39) (10.8%). By comparing the performance of SO-HR for CSA schemes in Scenario 1 (\(\ell = 4\)) with Scenario 2 (\(\ell = 2\)), we could also verify that the SO-HR encoding gain decays with decreasing \(\ell\).

In the two scenarios considered in this paper, where the degree of the polynomials are relatively small, we observed that the presence of HR in conjunction with the second-order (SO) scheme is of negligible importance. However, we expect that as we increase the degree of the polynomial the influence of Horner’s rule in reducing the encoding time will become more dominant.

Alternative algorithms - amongst others the finite field fast fourier transform (FFT) [26] – have been tested in the encoding for the evaluation of matrix polynomials. However, they did not show good performance results for the two experimentation scenarios. The reasons are the following. For sparse matrix polynomials (cf. with A3S encoding) and low-degree matrix polynomials (cf. with CSA encoding), the FFT is not well suited, as the operation count, thus ultimately the running time, is dependent upon the degree \(d - 1\) of
the matrix polynomial $U^{(j)}(\alpha)$. Other issues that involve the finite field FFT are the power-of-two restriction on $d$ and the choice of a (primitive) root of unity for a given field. All these issues make the FFT not useful for the two experimentation scenarios. Alternative algorithms via division [29] have also been tested for encoding but showed inferior performance to SO-HR.

![Graphs showing time vs. $p$ for Encoding, Upload, Computation, Download, Decoding, and Total with specific labels for General and Encoding.](#)

Fig. 4: Scenario 1 – small-$n$ ($n_0 = 10$). Note that there are two legends - a general and an encoding-specific legend.
2) Upload: This time is devoted to send the encoded matrices $\tilde{A}_i$ and $\tilde{B}_i$ to servers $i \in [1 : N]$. It is easy to see that for a sufficiently large $p$, A3S outperforms all remaining schemes in all scenarios. Further, we see that in comparison to SCSA both GSCSA and USCSA reduce the upload time significantly, e.g., for $p \approx 0.8 \cdot 10^4$ the SCSA upload time is about 66% (47%) higher than of GSCSAs (USCSAs) upload time for Scenario 2 with $n_0 = 100$. The dominance of USCSA and, in particular, GSCSA over SCSA is exacerbated with increasing matrix dimension $n$ (cf. upload time of Figs. 5 and 7). Overall, the order in incurred upload time of all five schemes is in agreement with their order of upload costs.

3) Computation: This time accounts for the computation of $\tilde{A}_i$ and $\tilde{B}_i$ at each server $i$. In schemes that use polynomial codes (PC), such as A3S and GASP, each server $i$ simply
multiplies \( \tilde{A}_i \in \mathbb{F}^{m \times n} \) with \( \tilde{B}_i \in \mathbb{F}^{r \times p} \). In contrast to PC schemes, in CSA schemes such as SCSA, USCSA and GSCSA, each server \( i \) multiplies \( \tilde{A}^{(j)}_i \in \mathbb{F}^{m \times n} \) with its respective pair \( \tilde{B}^{(j)}_i \in \mathbb{F}^{r \times p} \) and accumulates all \( 16 \) matrix products according to \( \sum_{j=1}^{16} \tilde{A}^{(j)}_i \tilde{B}^{(j)}_i \). For the scenarios under study, this results in the highest computation time for SCSA and the slowest for A3S and GASP. As expected, as we increase \( n_0 \) from 10 to 100, only the absolute time for the computation increases without affecting the relative order of the schemes. Advanced matrix multiplication algorithms developed by Strassen and extended by Winograd \cite{30}, \cite{31} intertwined with regular matrix multiplication \cite{25}, show only improvements over pure regular matrix multiplication for the big-\( n \) scenario when SCSA is applied. The gain at \( p \approx 1.6 \cdot 10^4 \) for scenarios 1 and 2 are, respectively, 15.8\% and 15.6\%.

4) Download: The download time in our experiment spans the duration until the user receives all \( N \) server observations \( O_1, O_2, \ldots, O_N \in \mathbb{F}^{m \times n} \). Since \( K_{\text{DL}}^{\text{SCSA}} \leq K_{\text{DL}}^{\text{G(U)SCSA}} \leq K_{\text{DL}}^{\text{GASP}} \leq K_{\text{DL}}^{\text{A3S}} \) for a constant \( \ell \) and \( N \), we expect that for sufficiently large matrix dimensions SCSA will outperform all remaining schemes with respect to the download time. Our numerical results confirm this expectation. Further, the numerical results also verify that for all schemes the download time does not depend on the matrix dimension \( n \) (cf. download time of Figs. 4 and 6).

5) Decoding: The decoding time encompasses the time at the user (i) to construct the \( N \times N \) decoding matrix \( D \), (ii) to determine the inverse \( D^{-1} \) and (iii) to multiply the inverse \( D^{-1} \) with the set of \( N \) server observations \( O_{[1:N]} \). For the inversion we use python built-in functions instead of fast algorithms for the inversion of Vandermonde matrices \cite{27}, \cite{32}, \cite{33} in case of polynomial codes or Cauchy-Vandermonde matrices \cite{34} for CSA codes. Overall, the time for steps (i) and (ii) are (a) independent of the matrix dimensions \( m, n \) and \( p \) and (b) behave similarly for all five schemes. On the contrary, the time for step (iii) scales with \( \frac{1}{r_{A'B}} \) and \( mp \) and thus vary for different schemes. Again, the numerical results validate this behavior for large matrix dimensions. For instance, SCSA has the largest \( r_{A'B} \)-value and simultaneously has the lowest decoding time.

6) Total Time: We compute the total time according to Eq. (40) and plot the results in Figs. 4–7. While \( n_0 = 100 \) is a preferable situation for A3S and GASP (see Figs. 6 and 7) over

\footnote{In our experiments, we set \( j \) to \( j = N - 2\ell \) for SCSA and \( j = q \) for GCSA and USCSA.}

\footnote{In our experiments, the recovery threshold \( Q \) matches the number of servers \( N \).}
SCSA, SCSA tends to outperform GASP and A3S in situations where $n_0 = 10$. GSCSA and USCSA, on the other hand, possess good performance results over both scenarios irrespective of the $n_0$ value. This is mainly due to the fact that they combine advantages of GASP and A3S (i.e., upload and computation) with those of SCSA (i.e., download and decoding).

VIII. Conclusion

In this paper, we studied the secure distributed matrix multiplication problem (SDMM), where a user is interested in computing the matrix product $AB$ of two private matrices $A$ and $B$. To this end, the user outsources the multiplication job to $N$ distributed servers
without leaking any information to any set of $\ell \leq N$ colluding servers. In SDMM, the goal is to find schemes that optimally balance conflicting communication and computation metrics. One such tradeoff is the one between uplink and downlink communication efficiency. As part of this study, we first propose two uplink adjustable secure cross subspace alignment (CSA) schemes, namely USCSA and GSCSA that balance uplink and downlink communication costs. For CSA schemes, we develop a second-order (SO) encoding scheme that exploits on the symmetry property of evaluating a polynomial $u(x)$ at $x$ and $-x$. Next, we implement various SDMM schemes of the literature (including USCSA and GSCSA) in Python and compare computation and communication times using Amazon EC2 instances. Our numerical results show that USCSA and GSCSA establish a good balance between the time spend

Fig. 7: Scenario 2 – big-$n$ ($n_0 = 100$).
on the communication and computation in SDMMs. Further, we infer from our simulations that SO encoding represents a good solution to reduce the encoding complexity of CSA schemes. A future research thrust is to refine our schemes by using optimized decoding algorithms [34], [35].

**APPENDIX A**

**PROOF OF LEMMA [1]**

Secure computing of \( AB \) requires that

(i) \( H(A, B | \tilde{A}_{[1:N]}, \tilde{B}_{[1:N]}) = 0 \) (cf. decodability constraint \( (1) \)) and

(ii) \( H(A, B) = H(A, B | \tilde{A}_{L}, \tilde{B}_{L}) \) for any \( L \subseteq [N] \), \( |L| = \ell \) (cf. security constraint \( (2) \)).

Thus, we infer that

\[
H(A, B) = I(\tilde{A}_L, \tilde{B}_L; A, B | \tilde{A}_L, \tilde{B}_L)
= H(\tilde{A}_L, \tilde{B}_L | \tilde{A}_L, \tilde{B}_L) - H(\tilde{A}_L, \tilde{B}_L, A, B),
\]

where \( L^C = [1 : N] \setminus L \). Ignoring the second term of above mutual information term gives the upper bound

\[
H(A, B) \leq H(\tilde{A}_L, \tilde{B}_L | \tilde{A}_L, \tilde{B}_L).
\]  

(41)

Since there are \( \binom{N}{N-\ell} \) possible subsets \( L^C \) of non-colluding servers of size \( N - \ell \), we can sum up \( (1) \) and obtain

\[
\left( \binom{N}{N-\ell} \right) H(A, B) \leq \sum_{\substack{L^C \subseteq [1:N] \\text{ | } |L^C| = N-\ell}} H(\tilde{A}_L, \tilde{B}_L | \tilde{A}_L, \tilde{B}_L)
\]

\[
\iff H(A, B) \leq \frac{N - \ell}{N} \sum_{\substack{L^C \subseteq [1:N] \\text{ | } |L^C| = N-\ell}} \frac{H(\tilde{A}_L, \tilde{B}_L | \tilde{A}_L, \tilde{B}_L)}{N - \ell}
\]

Now, we can apply Han’s concentration inequality [36] on conditional entropies to get

\[
H(A, B) \leq \frac{(N - \ell)}{N} H(\tilde{A}_{[1:N]}, \tilde{B}_{[1:N]}) \leq \frac{(N - \ell)}{N} \sum_{n=1}^{N} H(\tilde{A}_n, \tilde{B}_n)
\]

\[
\leq \frac{(N - \ell)}{N} \left( \sum_{n=1}^{N} H(\tilde{A}_n) + H(\tilde{B}_n) \right) \leq \frac{(N - \ell)}{N} \left( \sum_{n=1}^{N} |\tilde{A}_n| + |\tilde{B}_n| \right) \log \mathbb{F}.
\]

For independent and uniformly distributed matrices \( A \) and \( B \), \( H(A, B) = n(m + p) \log \mathbb{F} \).

For this case, rearranging above inequality gives the bound \( (27) \).
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