HIGHER DERIVATIONS
OF FINITARY INCIDENCE ALGEBRAS

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ABSTRACT. Let $P$ be a partially ordered set, $R$ a commutative unital ring and $FI(P; R)$ the finitary incidence algebra of $P$ over $R$. We prove that each $R$-linear higher derivation of $FI(P; R)$ decomposes into the product of an inner higher derivation of $FI(P; R)$ and the higher derivation of $FI(P; R)$ induced by a higher transitive map on the set of segments of $P$.

INTRODUCTION

Let $(P, \leq)$ be a preordered set and $R$ be a commutative unital ring. Assume that $P$ is locally finite, i.e. for any $x \leq y$ in $P$ there are only finitely many $z \in P$ such that $x \leq z \leq y$. The incidence algebra $I(P; R)$ of $P$ over $R$ is the set of functions

$$\{f : P \times P \to R \mid f(x, y) = 0 \text{ if } x \nleq y\}$$

with the natural structure of $R$-module and multiplication given by the convolution

$$(fg)(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y)$$

for all $f, g \in I(P; R)$ and $x, y \in P$. It would be helpful to point out that the full matrix algebra $M_n(R)$ as well as the upper triangular matrix algebra $T_n(R)$ are particular cases of incidence algebras. In addition, in the theory of operator algebras, the incidence algebra $I(P; R)$ of a finite poset $P$ is referred to as a bigraph algebra or a finite dimensional commutative subspace lattice algebra.

Incidence algebras appeared in the early work by Ward [38] as generalized algebras of arithmetic functions. Later, they were extensively used as the fundamental tool of enumerative combinatorics in the series of works “On the foundations of combinatorial theory” [29, 31, 30, 9] (see also the monograph [36]). The study of algebraic mappings on incidence algebras was initiated by Stanley [37]. Since then, automorphisms, involutions, derivations (and their generalizations) on incidence algebras have been actively investigated, see [1, 33, 7, 6, 22, 34, 17, 18, 10, 40, 20, 41, 21, 2] and the references therein.

There are many interesting generalizations of derivations (for example, see [16, 15] and their references). Another famous generalization of derivations is higher derivation. Higher derivations are an active subject of research in (not necessarily

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associative or commutative) algebra. Firstly, higher derivations have close relationship with derivations. It should be remarked that the first component \(d_1\) of each higher derivation \(D = \{d_n\}_{n=0}^{\infty}\) of an algebra \(A\) is itself a derivation of \(A\). Conversely, let \(d: A \to A\) be an ordinary derivation of an algebra \(A\) over a field of characteristic zero, then \(D = \{\frac{1}{n!}d^n\}_{n=0}^{\infty}\) is a higher derivation of \(A\). Heerema [14], Mirzavaziri [23] and Saymeh [32] independently proved that each higher derivation of an algebra \(A\) over a field of characteristic zero is a combination of compositions of derivations, and hence one can characterize all higher derivations on \(A\) in terms of the derivations on \(A\). Ribenboim systemically studied higher derivations of arbitrary rings and those of arbitrary modules in [27, 28], where some familiar properties of derivations are generalized to the case of higher derivations. Ferrero and Haetinger found in [11] the conditions under which Jordan higher derivations (or Jordan triple higher derivations) of a 2-torsion-free (semi-)prime ring are higher derivations, and in [12] the same authors studied higher derivations on (semi-)prime rings satisfying linear relations. The third author of the current article and Xiao [39] described higher derivations of triangular algebras and related mappings, such as inner higher derivations, Jordan higher derivations, Jordan triple higher derivations and their generalizations.

The objective of this paper is to investigate higher derivations of finitary incidence algebras. Many researchers have made substantial contributions to the additive mapping theory of incidence algebras. Baclawski [1] studied the automorphisms and derivations of incidence algebras \(I(P, R)\) when \(P\) is a locally finite partially ordered set. In particular, he proved that every derivation of \(I(P, R)\) can be decomposed as a sum of an inner derivation and a derivation induced by a transitive map. Koppinen [22] extended these results to the incidence algebras \(I(P, R)\) with \(P\) being a locally finite pre-ordered set. In [40], Xiao characterized the derivations of \(I(P, R)\) by a direct computation. Based on such a characterization of derivations, he proved that every Jordan derivation of \(I(P, R)\) is a derivation provided that \(R\) is 2-torsion-free. Zhang and the second author [41] considered Lie derivations of incidence algebras over 2-torsion-free commutative unital rings. They proved that each Lie derivation \(L\) of \(I(P, R)\) can be represented as \(L = D + F\), where \(D\) is a derivation of \(I(P, R)\) and \(F\) is a linear mapping from \(I(P, R)\) to its center.

More recently, special attention has been paid to additive mappings on finitary incidence algebras. Brusamarello, Fornaroli and the second author proved in [2] that each \(R\)-linear Jordan isomorphism of the finitary incidence algebra \(FI(P, R)\) of a partially ordered set \(P\) over a 2-torsion-free commutative unital ring \(R\) onto an \(R\)-algebra \(A\) is the near-sum of a homomorphism and an anti-homomorphism. Brusamarello, Fornaroli and Santulo showed in [3] that the finitary incidence algebra of an arbitrary poset \(P\) over a field \(K\) has an anti-automorphism (involution) if and only if \(P\) has an anti-automorphism (involution). A decomposition theorem for such involutions was obtained in [4]. The second author of the current article proved in [21] that each \(R\)-linear local derivation of the finitary incidence algebra \(FI(P, R)\) of a poset \(P\) over a commutative unital ring \(R\) is a derivation, generalizing (partially) a result by Nowicki and Nowosad [26].

The structure of this paper is as follows. In Section 1 we collect some basic facts about higher derivations and finitary incidence algebras. These are used in Section 2 to prove our main result Theorem 2.8.
1. Preliminaries

1.1. Higher derivations. Let $R$ be a ring. A sequence $d = \{d_n\}_{n=0}^{\infty}$ of additive maps $R \to R$ is a higher derivation of $R$ (of infinite order), if it satisfies

(i) $d_0 = \text{id}_R$;
(ii) $d_n(rs) = \sum_{i+j=n} d_i(r)d_j(s)$

for all $n \in \mathbb{N}$ and $r, s \in R$. If (ii) holds for all $0 \leq n \leq N$, then the sequence $\{d_n\}_{n=0}^{N}$ is called a higher derivation of order $N$. Evidently, $\{d_n\}_{n=0}^{\infty}$ is a higher derivation if and only if $\{d_n\}_{n=0}^{N}$ is a higher derivation of order $N$ for all $N \in \mathbb{N}$. In particular, $d_1$ is always a usual derivation of $R$.

Denote by $\text{HDer}_R$ the set of higher derivations of $R$ and consider the following operation on $\text{HDer}_R$

$$(d' * d'')_n = \sum_{i+j=n} d'_i \circ d''_j,$$

(1)

In particular,

$$(d' * d')_1 = d'_1 + d''_1.$$  

(2)

It was proved in [13] that $\text{HDer}_R$ forms a group with respect to $*$, whose identity is the sequence $\{\epsilon_n\}_{n=0}^{\infty}$ with $\epsilon_0 = \text{id}_R$ and

$$\epsilon_n = 0$$  

for $n \in \mathbb{N}$.

Given $r \in R$ and $k \in \mathbb{N}$, define

$$[r,k]_0 = \text{id}_R,$$

$$[r,k]_n(x) = \begin{cases} 0, & k \nmid n, \\ r^i x - r^{i-1} x r, & n = kl, \end{cases}$$

(3)

for all $n \in \mathbb{N}$ and $x \in R$. It was proved in [24] that $\{[r,k]_n\}_{n=0}^{\infty} \in \text{HDer}_R$, so that for any sequence $r = \{r_n\}_{n=1}^{\infty} \subseteq R$ one may define $\{\Delta_r\}_n^{\infty} = 0$ by means of

$$(\Delta_r)_n = ([r_1,1] * \cdots * [r_n,n])_n,$$

(4)

where $n \in \mathbb{N}$. Higher derivations of the form $\Delta_r$ will be called inner. By [25, Corollary 3.3] the set of inner higher derivations forms a normal subgroup in $\text{HDer}_R$, which will be denoted by $\text{IHDer}_R$. In particular,

$$(\Delta_r)_1(x) = [r_1,1]_1(x) = r_1 x - x r_1$$

(5)

is the usual inner derivation of $R$ associated with $r_1 \in R$, which we denote by $\text{ad}_{r_1}$.

We shall begin with some formulas which were used in [24] without any proof.

**Lemma 1.1.** Let $n, k \in \mathbb{N}$, such that $k < n$. Then for all $r \in R$

$$([r,n]^{-1})_k = 0.$$  

(6)

Moreover, for any $\{r_n\}_{n=1}^{\infty} \subseteq R$

(i) \([r_1,1] * \cdots * [r_n,n])_k = ([r_1,1] * \cdots * [r_k,k])_k;$$

(ii) \(([r_1,1] * \cdots * [r_n,n])^{-1})_k = (([r_1,1] * \cdots * [r_k,k])^{-1})_k.$$  

(7)
Moreover, set

\[ d = (\omega^2 - \lambda^2) \Delta_r. \]  

**Lemma 1.3.** Let \( d \in \text{Der}_R, \ r = \{r_n\}_{n=1}^\infty \subseteq R \text{ and } k \in \mathbb{N}. \) Define \( d^{(r,k)} \in \text{Der}_R \) as being

\[ d^{(r,k)} = ([r_1,1] \cdots [r_k,k])^{-1} * d. \]  

Moreover, set \( d_0^{(r)} = \text{id}_R \) and for all \( n \in \mathbb{N} \)

\[ d_n^{(r)} = d_n^{(r,n)}. \]  

Then

(i) \( d_i^{(r,k)} = d_i^{(r,l)} = d_i^{(r)} \text{ for all } 1 \leq l \leq k; \)

(ii) \( d^{(r)} \in \text{Der}_R; \)

(iii) \( d = \Delta_r * d^{(r)}. \)
Proof. For (i) observe from Lemma 1.1 (ii) that
\[
d_i^{(r,k)} = \left( \sum_{i+j=l} \left( \left[ [r_1, 1] \cdots [r_k, k] \right]^{-1} \cdot d \right) \right)_i \circ d_j
\]
and thus (iii) holds.

Then (ii) automatically follows from the fact that for each fixed \( N \in \mathbb{N} \) the sequence \( \{d_n^{(r,N)} \}_{n=0} \) coincides with the first \( N \) terms of the sequence \( d^{(r,N)} \).

Now using Lemma 1.1 (ii) and 19, 20, 12 and 13 we obtain for all \( n \in \mathbb{N} \) that
\[
(\Delta_r \cdot d^{(r)})_n = \sum_{i+j=n} (\Delta_r)_i \circ d_j^{(r)}
\]
\[
= \sum_{i+j=n} \left( \left[ [r_1, 1] \cdots [r_i, i] \right] \circ \left( \sum_{k+l=j} \left( \left[ [r_1, 1] \cdots [r_j, j] \right]^{-1} \cdot d \right) \right) \right)_i \circ d_j
\]
\[
= \sum_{i+j=n} \left( \left[ [r_1, 1] \cdots [r_i, i] \right] \circ \left( \sum_{k+l=j} \left( \left[ [r_1, 1] \cdots [r_k, k] \right]^{-1} \cdot d \right) \right) \right)_i \circ d_j
\]
\[
= \sum_{i+j=n} \left( \left[ [r_1, 1] \cdots [r_i, i] \right] \circ \left( \left[ [r_1, 1] \cdots [r_j, j] \right]^{-1} \cdot d \right) \right) \circ d_k
\]
and thus (iii) holds.

1.2. Finitary incidence algebra. Let \( P \) be a poset and \( R \) a commutative unital ring. Recall from \([19]\) that a finite series is a formal sum of the form
\[
\alpha = \sum_{x \leq y} \alpha_{xy} e_{xy},
\]
where \( x, y \in P \), \( \alpha_{xy} \in R \) and \( e_{xy} \) is a symbol, such that for any pair \( x < y \) there exists only a finite number of \( x \leq u < v \leq y \) with \( \alpha_{uv} \neq 0 \). The set of finitary series, denoted by \( FI(P, R) \), possesses the natural structure of an \( R \)-module. Moreover,
it is closed under the convolution

\[ \alpha \beta = \sum_{x \leq y} \left( \sum_{x \leq z \leq y} \alpha_{xz} \beta_{zy} \right) e_{xy}. \]

Thus, \( FI(P, R) \) is an algebra, called the \textit{finitary incidence algebra} of \( P \) over \( R \). The identity element of \( FI(P, R) \) is the series \( \delta = \sum_{x \in P} 1_x e_{xx} \). Here and in what follows we adopt the next convention. If in (14) the indices run through a subset \( X \) of the ordered pairs \((x, y)\), \( x, y \in P, x \leq y \), then \( \alpha_{xy} \) is meant to be zero for \((x, y) \notin X \).

Observe that

\[ e_{xy} \cdot e_{uv} = \begin{cases} e_{xv}, & \text{if } y = u, \\ 0, & \text{otherwise.} \end{cases} \quad (15) \]

In particular, the elements \( e_x := e_{xx}, x \in P \), are pairwise orthogonal idempotents of \( FI(P, R) \), and for any \( \alpha \in FI(P, R) \)

\[ e_x \alpha e_y = \begin{cases} \alpha_{xy} e_{xy}, & \text{if } x \leq y, \\ 0, & \text{otherwise.} \end{cases} \quad (16) \]

Given \( X \subseteq P \), we shall use the notation \( e_X \) for the idempotent \( \sum_{x \in X} 1_x e_{xx} \). In particular, \( e_x = e_{\{x\}} \). Note that \( e_X e_Y = e_{X \cap Y} \), so \( e_x e_X = e_x \) for \( x \in X \), and \( e_x e_X = 0 \) otherwise.

2. Higher derivations of \( FI(P, R) \)

**Lemma 2.1.** Let \( \{d_n\}_{n=1}^N \) be a higher derivation of \( FI(P, R) \) of order \( N \in \mathbb{N} \), such that

\[ d_n(e_x) = 0 \quad (17) \]

for all \( x \in P \) and \( 1 \leq n < N \). Then for any \( X \subseteq P \) and \( x \in X \)

\[ d_N(e_x) = e_x d_N(e_X) + d_N(e_x) e_X. \quad (18) \]

In particular, for all \( x < y \)

(i) \( d_N(e_X)_{xy} = d_N(e_x)_{xy} \), if \( x \in X \) and \( y \notin X \);

(ii) \( d_N(e_X)_{xy} = 0_{xy} \), if \( x, y \in X \).

**Proof.** Since \( e_x = e_x \cdot e_X \), we have

\[ d_N(e_x) = \sum_{i+j=N} d_i(e_x) d_j(e_X) \]

\[ = e_x d_N(e_X) + d_N(e_x) e_X \]

\[ + \sum_{i+j=N, \ i, j < N} d_i(e_x) d_j(e_X), \quad (19) \]

the sum (19) being zero by (17), whence (18). Now (i) and (ii) follow by taking the coefficients of both sides of (18) at \( e_{xy} \). \( \square \)

**Corollary 2.2.** Let \( \{d_n\}_{n=1}^N \) be a higher derivation of \( FI(P, R) \) of order \( N \in \mathbb{N} \) satisfying (17) for all \( x \in P \) and \( 1 \leq n < N \). Then

\[ d_N(e_x)_{xy} = -d_N(e_y)_{xy}. \quad (20) \]
Proof. Indeed, (20) follows from Lemma \ref{lem:2.1}\ref{lem:2.1:ii} with $X = \{x, y\}$ and the easy observation that $e_{[x, y]} = e_x + e_y$. 

\begin{lemma}
Let $d = \{d_n\}_{n=0}^\infty \in \text{HDer} F \mathcal{I}(P, R)$. Then there is $\rho = \{\rho_n\}_{n=1}^\infty \subseteq F \mathcal{I}(P, R)$ such that for all $n \in \mathbb{N}$ and $x \in P$

$$d_n^{(\rho)}(e_x) = 0,$$

where $d^{(\rho)}$ is given by \eqref{eq:21} and \eqref{eq:23}.
\end{lemma}

\begin{proof}
Define

$$(\rho_1)_{xy} = d_1(e_{xy}),$$

$$(\rho_n)_{xy} = (([\rho_1, 1] \ast \cdots \ast [\rho_{n-1}, n-1])^{-1} \ast d)_n(e_{xy}), \ n \in \mathbb{N}, \ n > 1. \tag{22}$$

We shall prove that $\rho_n \in F \mathcal{I}(P, R)$ and \eqref{eq:21} holds by induction on $n$.

Since $d_1$ is a usual derivation of $F \mathcal{I}(P, R)$ and

$$d_1 = (\Delta_\rho)_1 + d_1^{(r)} = \text{ad}_\rho + d_1^{(r)}$$

by Lemma \ref{lem:1.3}\ref{lem:1.3:iii}, \ref{lem:2.1} and \eqref{eq:7}, the case $n = 1$ is exactly \cite[Lemma 2]{18} (compare \eqref{eq:23} with formula (7) from \cite{18}).

Now assume that $\rho_n \in F \mathcal{I}(P, R)$ and \eqref{eq:21} is true for all $n < m$ and $x \in P$. In particular, $d^{(\rho, n)}$ is a well-defined higher derivation of $F \mathcal{I}(P, R)$ for each $n < m$ (see \cite{12}).

We first show that $\rho_m \in F \mathcal{I}(P, R)$. Suppose that there are $x < y$ and an infinite set $S$ of pairs $(u, v)$, such that $x \leq u < v \leq y$ and $(\rho_m)_{uv} \neq 0_{uv}$. Observe from \eqref{eq:23} that $(\rho_m)_{uv} = d^{(\rho, m-1)}_{m}(e_v)_{uv}$. Since $d^{(\rho, m-1)}_{m}(e_v)$ is a finitary series, for each $v$ there is only a finite number of $u$, such that $(u, v) \in S$. Moreover, $d^{(\rho, m-1)}_{m}(e_v) = d^{(\rho)}_{n}$ for all $n < m$ in view of Lemma \ref{lem:1.3}\ref{lem:1.3:i}. Consequently, $d^{(\rho, m-1)}_{m}(e_x) = 0$ for all $n < m$ and $x \in P$ by the induction hypothesis, and thus we may apply Corollary \ref{cor:2.2} and Lemma \ref{lem:2.1} to $\{d^{(\rho, m-1)}_{n}\}_{n=0}^m$. We have by \eqref{eq:23}

$$d^{(\rho, m-1)}_{m}(e_u)_{uv} = -d^{(\rho, m-1)}_{m}(e_v)_{uv} = -(\rho_m)_{uv} \neq 0_{uv}. \tag{24}$$

Since $d^{(\rho, m-1)}_{m}(e_u) \in F \mathcal{I}(P, R)$, it follows that for each $u$ there is only a finite number of $v$, such that $(u, v) \in S$. Therefore, as in \cite[Lemma 2]{18} we may construct an infinite $S' \subseteq S$, such that the sets

$$U = \{u \mid (u, v) \in S' \} \quad \text{and} \quad V = \{v \mid (u, v) \in S' \}$$

are infinite and disjoint. But then $d^{(\rho, m-1)}_{m}(e_u)_{uv} = d^{(\rho, m-1)}_{m}(e_u)_{uv} \neq 0_{uv}$ in view of Lemma \ref{lem:2.1}\ref{lem:2.1:i} and \eqref{eq:24}, contradicting the fact that $d^{(\rho, m-1)}_{m}(e_u) \in F \mathcal{I}(P, R)$.

Now, under the same hypothesis assumption as above, we prove \eqref{eq:21}. We have already shown that $\rho_m \in F \mathcal{I}(P, R)$. So, using \eqref{eq:8} and Corollary \ref{cor:1.2} we have

$$d^{(\rho)}_{m} = ([\rho_m, m]^{-1} \ast \cdots \ast [\rho_1, 1]^{-1} \ast d)_m$$

$$= \sum_{i+j=m} ([\rho_m, m]^{-1})_i \circ ([\rho_{m-1}, m-1]^{-1} \ast \cdots \ast [\rho_1, 1]^{-1} \ast d)_j$$

$$= ([\rho_m, m]^{-1})_m + ([\rho_{m-1}, m-1]^{-1} \ast \cdots \ast [\rho_1, 1]^{-1} \ast d)_m$$

$$= -\text{ad}_\rho + d^{(\rho, m-1)}_{m}. \tag{25}$$

\end{proof}
Notice that

\[ (d^{(m-1)}(e_x) e_x)_{uv} = \begin{cases} (d^{(m-1)}(e_x))_{ux}, & v = x, \\ 0, & v \neq x, \end{cases} \]

Moreover, since

\[ d^{(m-1)}(e_x) e_{xy} = -d^{(m-1)}(e_y) e_{xy} = -(\rho_m)_{xy}, \]

by (20), we similarly get that

\[ e_x d^{(m-1)}(e_x) = -e_x \rho_m. \]  \hspace{1cm} (27)

Thus, in view of (18), (26) and (27)

\[ d^{(m-1)}(e_x) e_{xy} = e_x d^{(m-1)}(e_x) e_{xy} = (\rho_m)_{xy}. \]

Combining this with (25) we get

\[ d^{(m)}(e_x) = 0, \] which completes the induction step and thus proves (21). \[ \square \]

Thus, it suffices to describe the higher derivations \( d \) of \( FI(P, R) \) whose terms annihilate \( e_x \) for all \( x \in P \). We shall give an equivalent characterization of such \( d \), assuming that all \( d_n \) are \( R \)-linear.

The following definition is due to Nowicky [24].

**Definition 2.4.** A sequence \( \sigma = \{\sigma_n\}_{n=0}^{\infty} \) of maps on \( I = \{(x, y) \in P \times P \mid x \leq y\} \) with values in \( R \) is called a higher transitive map, if

(i) \( \sigma_0(x, y) = 1_R \) for all \( x \leq y \);
(ii) \( \sigma_n(x, y) = \sum_{i+j=n} \sigma_i(x, z) \sigma_j(z, y) \) for all \( x \leq z \leq y \).

**Remark 2.5.** If \( \sigma \) is a higher transitive map, then

\[ \sigma_n(x, x) = 0 \]  \hspace{1cm} (28)

for all \( n \in \mathbb{N} \) and \( x \in P \).

**Proof.** Indeed, \( \sigma_1(x, x) = \sigma_1(x, x) + \sigma_1(x, x) \), so \( \sigma_1(x, x) = 0 \). Now suppose that the equality holds for all \( n < m \). Then

\[
\sigma_m(x, x) = \sum_{i+j=m} \sigma_i(x, x) \sigma_j(x, x) \\
= 1_R \cdot \sigma_m(x, x) + \sigma_m(x, x) \cdot 1_R \\
+ \sum_{i+j=m, i,j<m} \sigma_i(x, x) \sigma_j(x, x) \\
= 2 \sigma_m(x, x)
\]

by the induction hypothesis. Thus, \( \sigma_m(x, x) = 0 \). \[ \square \]

**Lemma 2.6.** Given a higher transitive map \( \sigma \), denote by \( \tilde{\sigma} \) the following sequence of maps \( FI(P, R) \to FI(P, R) \)

\[ \tilde{\sigma}_n(\alpha) = \sum_{x \leq y} \sigma(x, y) \alpha_{xy} e_{xy}, \]
where \( n \in \mathbb{N} \cup \{0\} \) and \( \alpha \in FI(P, R) \). Then \( \tilde{\sigma} \in \text{HDer } FI(P, R) \).

Proof. It is obvious that \( \tilde{\sigma}_n \) is well-defined and additive. The fact that \( \tilde{\sigma} \) satisfies \( \text{(ii)} \) of the definition of a higher derivation is easy to verify (see, for example the proof of \[24\] Lemma 3.6)).

Lemma 2.7. Let \( d = \{d_n\}_{n=0}^\infty \in \text{HDer } FI(P, R) \) be \( R \)-linear. Then

\[
d_n(e_x) = 0
\]

for all \( n \in \mathbb{N} \) and \( x \in P \) if and only if \( d = \tilde{\sigma} \) for some transitive map \( \sigma \).

Proof. Clearly, \( d = \tilde{\sigma} \) is \( R \)-linear and satisfies \( \text{(29)} \) in view of \( \text{(28)} \).

Let us prove the converse. Assume \( \text{(24)} \) and define

\[
\sigma_n(x, y) = d_n(e_{xy})_{xy}.
\]

Observe from \( \text{(ii)} \) of the definition of a higher derivation, \( \text{(16)} \) and \( \text{(29)} \) that, given \( \alpha \in FI(P, R) \) and \( x \leq y \),

\[
d_n(\alpha x e_{xy}) = d_n(e_x \alpha e_y) = \sum_{i+j+k=n} d_i(e_x) d_j(\alpha) d_k(e_y) = e_x d_n(\alpha) e_y = d_n(\alpha) y e_{xy}.
\]

Hence, using \( R \)-linearity, we conclude that

\[
d_n(\alpha)_{xy} = d_n(\alpha x e_{xy})_{xy} = \alpha y d_n(e_{xy})_{xy} = \sigma_n(x, y) \alpha_{xy},
\]

so \( d = \tilde{\sigma} \). It remains to verify \( \text{(i)} \) and \( \text{(ii)} \) of Definition 2.4. Condition \( \text{(i)} \) is simply the statement that \( (e_{xy})_{xy} = 1_R \) by \( \text{(30)} \). Now take \( x \leq z \leq y \). Then, \( e_{xy} = e_{zx} e_{zy} \) in view of \( \text{(15)} \), so that by \( \text{(30)} \) and \( \text{(ii)} \) of the definition of a higher derivation

\[
\sigma_n(x, y) = d_n(e_{xy})_{xy} = \sum_{i+j=n} d_i(e_{zx}) d_j(e_{zy}) = \sum_{i+j=n} \sigma_i(x, z) \sigma_j(z, y),
\]

proving Definition 2.4 \( \text{(ii)} \). \( \square \)

Theorem 2.8. Each \( R \)-linear higher derivation of \( FI(P, R) \) is of the form \( \Delta_\rho \ast \tilde{\sigma} \) for some \( \rho = \{\rho_n\}_{n=1}^\infty \subseteq FI(P, R) \) and a higher transitive map \( \sigma \).

Proof. This follows from \( \text{(ii)} \) and \( \text{(iii)} \) of Lemma 1.3 and Lemmas 2.3 and 2.7. \( \square \)

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