TOPOLOGICAL POSETS AND TROPICAL PHASED MATROIDS

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ABSTRACT. For a discrete poset $\mathcal{X}$, McCord proved that the natural map $|\mathcal{X}| \to \mathcal{X}$, from the order complex to the poset with the Up topology, is a weak homotopy equivalence. Much later, Živaljević defined the notion of order complex for a topological poset. For a large class of topological posets we prove the analog of McCord’s theorem, namely that the natural map from the order complex to the topological poset with the Up topology is a weak homotopy equivalence. A familiar topological example is the Grassmann poset $\mathcal{G}_n(\mathbb{R})$ of proper non-zero linear subspaces of $\mathbb{R}^{n+1}$ partially ordered by inclusion. But our motivation in topological combinatorics is to apply the theorem to posets associated with tropical phased matroids over the tropical phase hyperfield, and in particular to elucidate the tropical version of the MacPhersonian Conjecture. This is explained in Section 2.

1. INTRODUCTION

1.1. A Short Outline. In this paper we prove a fundamental theorem about topological posets. While our theorem is quite general, we are motivated by its application to matroids, specifically to a certain complex analog of oriented matroids. These analogous objects are “tropical phased matroids”, a fairly new notion supported by important foundational work of Baker and Bowler [BB]. While the name sounds exotic (it has connections with tropical geometry through the work of Viro [Vir10]), the tropical phased matroid is a complex analog of the well-loved oriented matroid. In a sense, we are simply moving from something “real” to something “complex”.

In more detail, let us start with something familiar, a discrete poset $\mathcal{X}$. Consider the associated order complex of $\mathcal{X}$, denoted by $|\mathcal{X}|$, i.e. the geometric realization of the abstract simplicial complex whose simplices are the finite totally ordered sets in $\mathcal{X}$. Long ago, McCord [McC67] proved that the natural “comparison map” $|\mathcal{X}| \to \mathcal{X}$ is a weak homotopy equivalence when $\mathcal{X}$ is given the “Up topology” whose open sets are generated by sets of the form $U_x = \{ y \mid x \leq y \}$ for all $x \in \mathcal{X}$. Much later, McCord’s theorem was used by combinatorialists, for example in [Bar11a], [Bar11b], [And19] and [AD19].

After defining topological posets, and limiting ourselves with some extra hypotheses, we prove the appropriate analog of McCord’s theorem. Of course this requires a definition of the order complex (for which we follow [Ž98]) and of a comparison map from that complex to the topological poset with the Up topology.

As explained in [AD19], McCord’s theorem is relevant to the MacPhersonian Conjecture. This conjecture says, roughly, that the homotopy type of the Grassmannian, $\text{Gr}(k, \mathbb{R}^n)$ of linear $k$-subspaces in $\mathbb{R}^n$ can be read off from the (much simpler) discrete poset of all rank $k$ oriented matroids on the set $\{1, 2, \ldots, n\}$. Until recently, it was not possible to have a complex analog of this statement because, while the complexification of $\text{Gr}(k, \mathbb{R}^n)$ is clearly $\text{Gr}(k, \mathbb{C}^n)$, it had not been clear what should serve as a suitable replacement for the discrete poset of oriented matroids. We have found that tropical phased matroids fill that gap nicely. The place of McCord’s Theorem on discrete posets is then filled by our main Comparison Theorem on topological posets.

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To give a clear account of what is hinted at in this introduction, we present in Section 2 a full discussion of the relevant aspects of tropical phased matroids. In particular, we explain the complex version of the MacPhersonian Conjecture as well as the broader place of the conjecture in the relationship between topology and combinatorics. It is our hope that the simplification afforded by the Up topology will provide new insight into the complex version of the Conjecture, as was already observed for the real case in [AD19]. The rest of the paper is then devoted to foundational material on topological posets needed for our proof of the Comparison Theorem.

We end this short outline by pointing out that this is a paper about topological posets in general, and we believe that the many technical matters discussed along the way form a contribution to the literature of such objects, quite independent of its application to matroids.

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1.2. Topological Posets.

Definition 1. A topological poset \((X, \leq, \mathcal{T})\) is a poset \((X, \leq)\) equipped with a Hausdorff topology \(\mathcal{T}\) on the set \(X\) such that the order relation \(P := \{(x, y) \in X \times X \mid x \leq y\}\) is a closed subspace of \(X \times X\).

A simple example illustrates this: Let \(X\) be the subset of the Euclidean plane consisting of the unit circle \(S^1\) and the origin \(0\); the poset inequality is fully defined by \(0 \leq x\) for all \(x \in S^1\). This becomes a topological poset if we remember the usual topology on \(X\) inherited from the plane. The order relation \(P\) is \(0 \times X\). We will have more to say about this example below in connection with realizations.

Definition 2. Following Živaljević we say that a topological poset is mirrored, and is called an M-poset, if it comes equipped with a poset map \(\mu : X \to \mathcal{R}\), where \(\mathcal{R}\) is a finite poset, satisfying

- \(x < y\) implies \(\mu(x) < \mu(y)\),
- for all \(r \in \mathcal{R}\), \(X_r := \mu^{-1}(r)\) is a non-empty closed subset of \(X\).

The map \(\mu\) is a mirror of \(X\).

In our simple example, \(\mathcal{R}\) is \(\{1, 2\}\); \(\mu(0) = 1\) and \(\mu(S^1) = 2\).

It follows that \(X\) is the topological sum \(\bigoplus_{r \in \mathcal{R}} X_r\).

A less trivial example is the real Grassmann Poset \(\mathcal{G}_n(\mathbb{R})\); here, \(X\) is the set of all proper linear subspaces of \(\mathbb{R}^{n+1}\) of positive dimension, partially ordered by inclusion, \(\mathcal{R}\) is the set \([n]\) of positive integers \(\leq n\) with the natural ordering, and \(\mu\) takes the \(k\)-dimensional subspaces to the integer \(k\).

Notation. When \(A \subseteq X\) we define

\[\uparrow A := \{x \in X \mid a \leq x \text{ for some } a \in A\} \cap X_r.\]

When \(A\) is a family of subsets of \(X\), we define \(\uparrow A := \\{\uparrow A \mid A \in A\}\), but when \(a \in X\) we write \(\uparrow a\) rather than \(\uparrow\{a\}\). For \(A \subseteq X_r\) and \(s > r\), we write \(A^{(s)} := (\uparrow A) \cap X_s\).

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Footnotes:

1For example, Proposition 4.3, Theorem 4.14 and Theorem 9.2

2The topological sum or coproduct \(\bigsqcup_{\alpha} Y_{\alpha}\) of a collection \(Y_{\alpha}\) of topological spaces is the disjoint union of the spaces \(Y_{\alpha}\) with its topology defined by: \(U\) is open in \(\bigsqcup_{\alpha} Y_{\alpha}\) if and only if for every \(\alpha\), \(U \cap Y_{\alpha}\) is open in \(Y_{\alpha}\).
Each $X_r$ inherits a topology from $\mathcal{X}$. We choose a basis $\mathcal{U}_r$ for the open sets in $X_r$.

**Definition 3.** We will assume from now on that our $M$-posets have the **Openness Property** that when $U$ is open in $X_r$ then $\uparrow U$ is open in $\mathcal{X}$; equivalently, for each $s > r$, the set $U^{(s)}$ is open in $X_s$.

**Definition 4.** The **Up topology** on $\mathcal{X}$ has basis $\bigcup_r \{\uparrow U_r \mid r \in \mathcal{R}\}$.

The Openness Property implies that this is a basis for a topology.

It is important to distinguish between the Original topology on $\mathcal{X}$ (which is Hausdorff) and the Up topology on $\mathcal{X}$ (which is $T_0$ but, in general, not $T_1$.) The $M$-posets discussed in this paper are considered to have both topologies, the “Up” being derived from the “Original”. The Openness Assumption ensures that the Up topology agrees with the Original topology on each $X_r$; i.e. the inclusion of $X_r$ into $\mathcal{X}$ with the Up topology is a topological embedding.

**Example.** Let $\mathcal{X}$ be the real Grassmann Poset $G_2(\mathbb{R})$. Here, $X_1$ is the collection of lines in $\mathbb{R}^3$ containing 0 and $X_2$ is the collection of planes in $\mathbb{R}^3$ containing 0, partially ordered by inclusion. It is instructive to consider the two topologies in this situation:

- **The Ordinary topology:** We indicate a basis. Given a line $\ell$ in $\mathbb{R}^3$ and a number $\epsilon > 0$, a basic neighborhood of $\ell$ in $X_1$ consists of all lines in $\mathbb{R}^3$ whose angle with $\ell$ is less than $\epsilon$. Given a plain $p$ in $\mathbb{R}^3$ and a number $\epsilon > 0$, a basic neighborhood of $p$ in $X_2$ consists of all plains in $\mathbb{R}^3$ whose angle with $p$ is less than $\epsilon$. Then $\mathcal{X}$ with the Ordinary topology is just $X_1 \amalg X_2$.

- **The Up topology:** Again, we indicate a basis. Given a line $\ell$ in $\mathbb{R}^3$, a basic “Up-neighborhood” $W$ of $\ell$ in $\mathcal{X}$ consists of a basic neighborhood $W_1$ of $\ell$ in $X_1$ together with all planes in $\mathbb{R}^3$ that contain a line lying in $W_1$. Given a plane $p$ in $X_2$, a basic “Up-neighborhood” of $p$ is just a basic neighborhood of $p$ in the Ordinary topology.

### 1.3. The Order Complex and the Comparison Map.

In [Z98], a simplicial space $\Delta(\mathcal{X})$, called the order complex\(^3\), is associated with each topological poset $\mathcal{X}$. For $M$-posets where the spaces $X_r$ are locally compact polyhedra (as will always be the case in this paper) there is a rather simple definition as follows:

**Definition 5.** Each element $z$ of the topological join $\ast_{r \in \mathcal{R}} X_r$ has the form $z = \sum t_i x_i$ where $x_i \in X_i$, $t_i \geq 0$, and $\sum t_i = 1$. Define the support of $z$ to be $\text{supp}(z) := \{i \in \mathcal{R} \mid t_i > 0\}$. Then the order complex is $\Delta(\mathcal{X}) := \{z \in \ast_{r \in \mathcal{R}} X_r \mid \text{supp}(z) \text{ is a chain in } \mathcal{R}, \text{ and if } i < j \in \text{supp}(z) \text{ then } x_i < x_j \in \mathcal{X}\}$. We will sometimes identify $X_r$ with the subspace of $\Delta(\mathcal{X})$ whose support is the singleton $\{r\}$.

**Example.** We return to our simple example, above, where $\mathcal{X}$ is $S^1 \amalg \{0\}$. If $X$ is considered as a topological poset, then its realization $\Delta(\mathcal{X})$ is homeomorphic to a planar disk. But if $\mathcal{X}$ is just a discrete poset then its realization $|\mathcal{X}|$ is a 1-dimensional complex consisting of uncountably many copies of the unit interval with their 0-points identified.

**Example.** Vassiliev [Vas92], [Vas91] proved that for $\mathbb{K}$ the reals, the complex numbers, or the quaternions, and $G_n(\mathbb{K})$ the Grassmann Poset of proper non-zero linear subspaces of $\mathbb{K}^{n+1}$, the corresponding order complex, $\Delta(G_n(\mathbb{K}))$, is homeomorphic to the sphere $S^m$ where $m = \binom{n+1}{2}d + n - 1$, $d$ being the dimension of $\mathbb{K}$ over $\mathbb{R}$.

\(^3\text{Strictly speaking, this is the geometric realization of a simplicial topological space.}\)
Definition 6. The *Comparison Map* $f: Δ(\mathcal{X}) \to \mathcal{X}$ is defined by $f(z) = \max\{x_i \mid i \in \text{supp}(z)\}$. It is not hard to prove that $f$ is continuous$^4$ with respect to the Up topology on $\mathcal{X}$.

Our aim is to prove that under reasonable topological and geometric assumptions, and considering $\mathcal{X}$ with the Up topology, the Comparison Map $f$ is a weak homotopy equivalence.

1.4. Polyhedral $M$-posets.

Definition 7. An $M$-poset $\mathcal{X}$ is *polyhedral* if each $X_i$ is a locally compact polyhedron$^5$ and $\mathcal{X}$ has the *Polyhedral Diagonal Property*, namely: The order relation $\mathcal{P}$ is a (closed) subpolyhedron of $\mathcal{X} \times \mathcal{X}$; here $\mathcal{X}$ has the Original topology.

1.5. Convenient Choice of Bases. When the space $X_r$ is polyhedral each point has a basic system of compact neighborhoods which are piecewise linear (PL) cones on their frontiers$^6$. We may assume that the basis $U_r$ of $X_r$ consists of the interiors of such cones. In particular, each such set is contractible and its closure is a compact contractible subpolyhedron of $X_r$.

Our basis $U = \bigcup_r \uparrow U_r$ for the topology of $\mathcal{X}$ will always be understood to consist of such sets.

1.6. Geometric Posets and Main Theorem.

Definition 8. A compact subset $C$ of a polyhedron $D$ has *trivial shape* if $C$ can be contracted to a point in any of its neighborhoods$^7$ in $D$.

This is known to be an intrinsic property of $C$, independent of $D$. Note that when $C$ is contractible then $C$ has trivial shape, but trivial shape is more general: for example, the Topologist’s Sine Curve in $\mathbb{R}^2$ has trivial shape but has two path components.

Notation. When $(M,d)$ is a metric space, the space of non-empty compact subsets of $M$ with the Hausdorff metric is denoted by $cM$.

Definition 9. A polyhedral $M$-poset $(\mathcal{X}, \leq, T)$ is *geometric* if, for each $r < s \in \mathcal{R}$ it has the following Additional Properties:

A1 : When $x \in X_r$, the set $x^{(s)}$ is compact and non-empty.
A2 : The map $\pi_{r,s}: X_r \to cX_s$ defined by $x \mapsto x^{(s)}$ is continuous.
A3 : When $U \in U_r$ and $y \in U^{(s)}$, then $\{x \in U \mid x < y\}$ has trivial shape$^8$.

We note that “geometric” only involves the Original topology.

Theorem 1.1. (Comparison Theorem) When the $M$-poset $\mathcal{X}$ is geometric and carries the Up topology, the map $f: Δ(\mathcal{X}) \to \mathcal{X}$ is a weak homotopy equivalence.

The discrete case of Theorem 1.1 is due to McCord$^9$. See Section 8 for a discussion.

Example. The Grassmann Posets $\mathcal{G}_n(\mathbb{R})$ are geometric, so Theorem 1.1 and Vassiliev’s results give us the singular homology of $\mathcal{G}_n(\mathbb{R})$ with the Up topology.

1.7. Known theorem on weak homotopy equivalences. Our proof of Theorem 1.1 uses the following theorem of McCord$^{10}$, reproved independently$^{11}$ by May$^{12}$.

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$^4$The word “map” is used in this paper to mean that the given function is continuous.
$^5$In Section 3.1 we go into detail about our use of polyhedral language.
$^6$See Section 3.1.
$^7$Shape Theory is discussed in Appendix 7.
$^8$This implies that for every such $y$ there is some $x < y$. This is because the empty space does not have trivial shape.
$^9$May’s paper appears to date from the late 1970’s though the publication year is 2006.
**Theorem 1.2.** Let \( g : C \to D \) be a map between topological spaces, and let \( \mathcal{B} \) be a basis for the open subsets of \( D \). If for all \( B \in \mathcal{B} \) the restriction \( g| : g^{-1}(B) \to B \) is a weak homotopy equivalence, then \( g \) is a weak homotopy equivalence.

We will be prove that for \( U \in \mathcal{U} \), both \( \uparrow U \) and \( f^{-1}(\uparrow U) \) are weakly contractible\(^0\) so that we can apply Theorem 1.2.

2. Matroid Examples

2.1. Introduction. In this section we describe some examples of geometric \( M \)-posets and applications of Theorem 1.1. These lead up to a discussion of what is often called the MacPhersonian Conjecture. To present these examples we draw on the work of Baker and Bowler \([BB]\), but we try to include enough basic information that the section is more or less self-contained.

2.2. The tropical phase hyperfield. Let \( T\Phi \) denote the poset \( \{0\} \cup S^1 \subseteq \mathbb{C} \), where the partial ordering is fully defined by \( 0 < x \) for all \( x \in S^1 \). This is an \( M \)-poset over \( \{1,2\} \), where the mirror \( \mu \) takes \( 0 \) to \( 1 \) and all of \( S^1 \) to \( 2 \). The Openness Property holds and this is easily seen to be a geometric poset.

\( T\Phi \) also carries the structure of a hyperfield. A hyperfield is similar in definition to a field (associative, commutative etc.) except that the addition operation is allowed to be multivalued. In the case of \( T\Phi \), multiplication is inherited from the complex plane and the sum \( \oplus \) is defined by

- \( x \oplus 0 = \{x\} \),
- \( x \oplus -x = \{0\} \cup S^1 \) whenever \( x \neq 0 \),
- \( x \oplus y \) is the smallest closed arc in \( S^1 \) joining the points \( x \) and \( y \) when \( y \neq -x \),
- For subsets \( A \) and \( B \), \( A \oplus B = \bigcup_{a \in A, b \in B} a \oplus b \).

\( T\Phi \) is the tropical phase hyperfield introduced by Viro in \([Vir10]\). It is a topological hyperfield in the sense of \([AD19]\).

**Remark.** The hyperfield \( T\Phi \) is the complex analog of the “sign hyperfield” \( \mathbb{S} := T\Phi \cap \mathbb{R} \). Just as \( T\Phi \) will lead us to a discussion of tropical phased matroids, \( \mathbb{S} \) would lead to an analogous discussion of oriented matroids. However, unlike \( T\Phi \), the original topology on \( \mathbb{S} \) is discrete, so the analog of the topological posets discussed here would be discrete posets, and everything would be much simpler.

2.3. Various relevant \( M \)-posets.

2.3.1. Products. The product space \( T\Phi^n \) (using the Original topology) is a topological poset, where the comparison \( \leq \) is defined coordinate-wise, i.e. \( (x_1, \ldots , x_n) \leq (y_1, \ldots , y_n) \) if and only if \( x_i \leq y_i \) for all \( i \). This has a minimal element \( \mathbf{0} \). More interesting are products with zero deleted. Consider \( T\Phi^n - \{0\} \). The mirror map \( \mu : T\Phi^n - \{0\} \to [n] \) is defined by\(^1\) \( \mu(v) = |\text{support}(v)| \). This \( M \)-poset is polyhedral because the pre-image of each \( i \) is homeomorphic to a disjoint union of tori. That it has the Polyhedral Diagonal Property is then clear in view of the following remark.

**Remark 2.1.** One usually checks the Polyhedral Diagonal Property as follows: There is a known polyhedral \( M \)-poset \( \mathcal{X} \) with (polyhedral) order relation \( \mathcal{P} \), and one is interested in a subset \( \mathcal{Y} \) such that each \( Y_i \) is a polyhedral subset of \( X_i \). Then the order relation for \( \mathcal{Y} \) is the polyhedron \( \mathcal{P} \cap (\mathcal{Y} \times \mathcal{Y}) \), so \( \mathcal{Y} \) is a polyhedral \( M \)-poset.

\(^0\)i.e. all homotopy groups are trivial.

\(^1\)Not to be confused with the phase hyperfield where \( x \oplus -x = \{0, x, -x\} \) and “closed arc” is replaced by “open arc”.

\(^2\)In the context of product posets, the support of an \( n \)-tuple is the set of its non-zero entries.
Proposition 2.2. The polyhedral $M$-poset $T\Phi^n - \{0\}$ is geometric.

Proof. The Additional Properties A1 and A2 are clear. Recall that Property A3 refers to our convenient choice of basis. Write $\mathcal{X} := \Phi^n - \{0\}$. Let $r < s$, let $U$ be a basic open set in $X_r$, and let $y \in U^{(s)}$. There is a sequence $y_m$ in $U^{(s)}$ converging to $y$, so (renaming points as necessary) there is a corresponding sequence $x_m$ in $U$ converging to some $x \in U$ where, for each $m$, $x_m < y_m$. Since the order relation $\mathcal{P}$ is closed, and contains each pair $(x_m, y_m)$ it must also contain $(x, y)$. Thus $\{x \in U \mid x < y\}$ is non-empty. There are only finitely many points $z \in X_r$ satisfying $z < y$, each being in a different component of $X_r$, so $x$ is the only one in the component of $X_r$ containing $U$. So $\{z \in U \mid z < y\}$ is a single point. \qed

To prove polyhedrality in our other $M$-posets we will use the following well-known theorem:

Theorem 2.3. \[(\text{Hir75})\] Let $S \subseteq \mathbb{R}^n$ be a semialgebraic set (i.e. the solution set of a finite number of polynomial equalities and polynomial inequalities, or a finite union of such). Then $S$ admits a (canonical) PL triangulation.

2.3.2. Covectors. For an $n$-vector $v = (v_1, \ldots, v_n) \in T\Phi^n - \{0\}$, its covector is

$$v^\perp := \{(x_1, \ldots, x_n) \in T\Phi^n - \{0\} \mid 0 \in v_1 x_1 \oplus \cdots \oplus v_n x_n\}.$$ 

The set $v^\perp$ is a subset of the $M$-poset $T\Phi^n - \{0\}$, so it is also an $M$-poset, provided the mirror map is taken to be $\mu : v^\perp \rightarrow \{2, 3, \ldots, n\}$ (because, with this restriction, $\mu^{-1}(1)$ is empty).

Theorem 2.4. The $M$-poset $v^\perp$ is geometric.

Proof. We first prove polyhedrality. Because $(S^1)^n$ acts transitively it is enough to consider the case $v = 1$, i.e. to show that the set $S := \{(x_1, \ldots, x_n) \mid 0 \in x_1 \oplus \cdots \oplus x_n\}$ is polyhedral. By induction we may assume the points $x_1, \ldots, x_n$ lie in $S^1$ and are distinct, since otherwise, the case would have been dealt with for a value lower than $n$. As we have said, the case $n = 1$ does not occur. When $n = 2$ the set $S$ is a copy of $S^1$ and hence is polyhedral. When $n = 3$ a point $(x_1, x_2, x_3)$ lies in $S$ if and only if the closed convex hull in $\mathbb{C}$ having these points as vertices contains $0$. To use Theorem 2.3 we regard the points $x_i$ as vectors in the plane $\mathbb{C}$. For each $i$, let $y_i$ be a unit vector orthogonal to $x_i$. The condition then is that any two of the following hold (the third being a consequence of the other two): \[13\]

1. $(y_1 \cdot x_2)(y_1 \cdot x_3) \leq 0$;
2. $(y_2 \cdot x_1)(y_2 \cdot x_3) \leq 0$;
3. $(y_3 \cdot x_1)(y_3 \cdot x_2) \leq 0$.

When $n > 3$ the condition must hold for some triple of points, so polyhedrality is proved for all $n$. The Additional Properties A1 and A2 clearly hold, and A3 holds for the same reasons as in the proof of Proposition 2.2 \qed

Remark. In [Alv], the first-named author proves that the order complex $\Delta(v^\perp)$ is homeomorphic to the sphere $S^{2n-3}$. Together with Theorems 1.1 and 2.4 this implies that the singular homology of $v^\perp$ with the $\up$ topology is that of $S^{2n-3}$.

\[13\] Geometrically this says that if we denote by $d_i$ the diameter $[x_i, -x_i]$ in the unit disk, then $x_j$ and $x_k$ must be on opposite sides of $d_i$, and this must hold for all three choices of subscripts, though if two hold then so does the third.
2.4. **Strong matroids over hyperfields.** Other than fields, the only hyperfield occurring in this paper is $T\Phi$. The general definition of a hyperfield can be found, for example, in [BB] or [BB19]. To simplify notation in this subsection it is convenient to state things in terms of an arbitrary hyperfield $F$, keeping in mind that, for us, $F$ will always be $\mathbb{C}$ or $T\Phi$.

Given positive integers $n$ and $r$ and a hyperfield $F$, a **strong Grassmannian-Plücker function** of rank $r$ on $[n]$ with coefficients in $F$ is a function $\varphi: [n]^r \to F$ such that

1. $\varphi$ is not identically zero;
2. $\varphi$ is alternating, i.e. $\varphi(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_r) = -\varphi(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_r)$;
3. $\varphi$ satisfies the strong Grassman-Plücker relations: When $\{x_1, \ldots, x_{r+1}\} \subseteq [n]$ and $\{y_1, \ldots, y_{r-1}\} \subseteq [n]$ then
   \[ 0 \in \mathbb{H}_{k=1}^{r+1}(-1)^k \varphi(x_1, \ldots, \hat{x}_k, \ldots, x_{r+1}), \varphi(x_k, y_1, \ldots, y_{r-1}). \]

Two such functions $\varphi_1$ and $\varphi_2$ are **equivalent** if $\varphi_1 = t \cdot \varphi_2$ where $t \in F - \{0\}$, i.e. one is a non-zero scalar multiple of the other.

Strong $F$-matroids are defined in [BB]. For our purposes we will instead use the following Theorem 3.13 of [BB] as our **definition** of “strong $F$-matroid”.

**Theorem 2.5.** There is a natural bijection between the set of equivalence classes of strong Grassman-Plücker functions of rank $r$ on $[n]$ with coefficients in $F$ and strong $F$-matroids of rank $r$ on $[n]$.

A parallel discussion is possible for “weak Grassman-Plücker functions”; see [BB] and [BB19] for the definitions.

2.5. **The space of strong Grassmann-Plücker functions.** A map $\varphi: [n]^r \to T\Phi$ assigns an element $\varphi(x_1, \ldots, x_r) \in T\Phi$ to each $r$-tuple $(x_1, \ldots, x_r)$ of elements of $[n]$. When we order the $r$-tuples lexicographically, $\varphi$ is encoded by an element of $T\Phi^m$ where $m = \frac{n!}{(n-r)!}$. We write GP$_s(r, T\Phi^n)$ for the space of all strong Grassmann-Plücker functions of rank $r$ on the set $[n]$ with coefficients in $T\Phi$. This is the subset of $T\Phi^m$ consisting of elements satisfying the non-zero and alternating conditions as well as (3) in the definition above.

**Proposition 2.6.** The $M$-poset GP$_s(r, T\Phi^n)$ is semialgebraic.

**Proof.** We prove this using Theorem 2.3. The first two conditions are clearly semialgebraic. For fixed $\{x_1, \ldots, x_{r+1}\} \subseteq [n]$ and $\{y_1, \ldots, y_{r-1}\} \subseteq [n]$ we consider the condition

\[ 0 \in \mathbb{H}_{k=1}^{r+1}(-1)^k \varphi(x_1, \ldots, \hat{x}_k, \ldots, x_{r+1}), \varphi(x_k, y_1, \ldots, y_{r-1}). \]

for all $\varphi$ mapping $[n]^r$ into $S^1$. The set of such $\varphi$ is semialgebraic for the reasons given in the proof of Theorem 2.4. As $\{x_1, \ldots, x_{r+1}\} \subseteq [n]$ and $\{y_1, \ldots, y_{r-1}\} \subseteq [n]$ vary this gives a finite union of semialgebraic sets. This discussion can easily be adapted to the cases where some of the $\varphi$-values are $0 \in T\Phi$. \hfill $\square$

2.6. **Strong tropical phased Grassmannians.** The **strong Grassmannian**, Gr$_s(r, T\Phi^n)$, is defined in the literature to be the set of all strong tropical phased matroids of rank $r$ on the set $[n]$ with coefficients in $T\Phi$. In other words, Gr$_s(r, T\Phi^n)$ is the quotient space GP$_s(r, T\Phi^n)/S^1$.

**Theorem 2.7.** The $M$-poset Gr$_s(r, T\Phi^n)$ is geometric.

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14Note that the passage from $u_k := \varphi(x_1, \ldots, \hat{x}_k, \ldots, x_{r+1})$ and $v_k := \varphi(x_k, y_1, \ldots, y_{r-1})$ to $u_k v_k$ is semialgebraic.

15See [AD19] for more details.
Proof. We first prove that $\text{Gr}_s(r, T\Phi^n)$ is semialgebraic hence polyhedral.

Once an ordering is chosen for the set $[n]^r$, a point of $\text{GP}_s(r, T\Phi^n)$ is an ordered set of $r$-tuples. In this case it is more convenient to forget the $r$-tuple partition, and to consider that point as just an element of $T\Phi^m \subseteq \mathbb{C}^m$ where $m = \frac{n!}{(n-r)!}$. By Proposition 2.6, $\text{GP}_s(r, T\Phi^n)$ can be considered as a semialgebraic subset of $\mathbb{C}^m$. We are to prove that the quotient of this by $S^1$-multiplication does not destroy the semialgebraic property.

Let $C_1$ denote the subset consisting of all points whose last entry lies in $S^1$. This is a union of components of a semialgebraic set, and hence is semialgebraic. For each $p \in C_1$ there is $t \in S^1$ such that $t.p$ has last entry $1 \in S^1$. We denote by $C_{1,1}$ the subset of $C_1$ consisting of points whose last entry is 1. This is easily seen to be a semialgebraic set, hence polyhedral.

It follows that $C_1$, being homeomorphic to $S^1 \times C_{1,1}$, is polyhedral. We consider $S^1 \times C_{1,1}$ to be an $S^1$-space where the $S^1$-action is by rotation on the first factor and is trivial on the second factor; and, of course, $C_1$ is an $S^1$-space under multiplication by scalars. Writing an arbitrary element of $C_1$ as $(b, x)$, where $b$ involves all the entries except the last, and $x \in S^1$ is the last entry, an $S^1$-equivariant homeomorphism $C_1 \to S^1 \times C_{1,1}$ is given by $(b, x) \mapsto (x, (x^{-1}b, 1))$.

Next, let $C_2$ denote the subset consisting of all points whose last entry is 0 and whose second-to-last entry lies in $S^1$. By the same argument, $C_2$ is polyhedral, and is equivariantly homeomorphic to $S^1 \times C_{2,1}$. The pattern is now clear. The polyhedral sets $C_i$ are clopen and pairwise disjoint. Since all the points of $\text{GP}_s(r, T\Phi^n)$ have some non-zero entry we see that $\bigcup_i C_i = \text{GP}_s(r, T\Phi^n)$ which is equivariantly homeomorphic to $\bigcup_i S^1 \times C_{i,1}$. Factoring out the $S^1$-actions gives a homeomorphism between $\text{Gr}_s(r, T\Phi^n)$ and the semialgebraic set $\bigcup_i C_{i,1}$. This proves polyhedrality.

The Additional Property A3 holds for the same reason as in the proof of Proposition 2.2. The “non-empty” requirement in A1 holds because the condition $0 \in x_1 \oplus \cdots \oplus x_n$ continues to hold if a 0-entry $x_i$ is replaced by a non-zero entry. The proof of Property A2 is left to the reader. \qed

The ordinary complex Grassmannian $\text{Gr}(r, \mathbb{C}^n)$ is the space of $r$-dimensional linear subspaces of $\mathbb{C}^n$, here $1 \leq r \leq n - 1$. But it can also be viewed in the above terms, since every field is a hyperfield (in which case $a + b = \{a + b\}$). Thus $\text{Gr}(r, \mathbb{C}^n)$ can be identified with the quotient space, modulo scalar multiplication, of the space of Grassmann-Pfaffian functions of rank $r$ on $[n]$ with coefficients in $\mathbb{C}$. The map $\mathbb{C} \to T\Phi$ sending non-zero $z$ to $\frac{z}{|z|}$ and 0 to 0 is continuous with respect to the Up topology. It induces the map $\rho_s : \text{Gr}(r, \mathbb{C}^n) \to \text{Gr}_s(r, T\Phi^n)$, which is also continuous when the $M$-poset $\text{Gr}_s(r, T\Phi^n)$ has the Up topology.

2.7. The Tropical MacPhersonian Conjecture. The following is the tropical phased analog of a well-known conjecture for oriented matroids:

Question: Is the map $\rho_s : \text{Gr}(r, \mathbb{C}^n) \to \text{Gr}_s(r, T\Phi^n)$ a weak homotopy equivalence?

Remarks. (i) There is a homotopy commutative diagram

\[
\begin{array}{ccc}
\Delta(\text{Gr}_s(r, T\Phi^n)) & \xrightarrow{\Delta(\rho_s)} & \\
\downarrow f & & \\
\text{Gr}(r, \mathbb{C}^n) & \xrightarrow{\rho_s} & \text{Gr}_s(r, T\Phi^n)
\end{array}
\]

\[16\] Over a field, strong and weak are the same.
In view of Theorem 1.1, the map $\rho_*$ is a weak homotopy equivalence if and only if $\Delta(\rho_*)$ is. The traditional form of the Question is for $\Delta(\rho)$. We believe the problem will be more tractable using the poset itself with the Up topology.

(ii) The analogous “MacPhersonian Conjecture” for oriented matroids (still open having been incorrectly dealt with in [Bis03]) says that the analogous “real” map $\Delta(\rho) : \text{Gr}(r, \mathbb{R}^n) \to \Delta(\text{Gr}(r, \mathbb{S}^n))$ is a weak homotopy equivalence. Since $\text{Gr}(r, \mathbb{S}^n)$ is a finite discrete $M$-poset, the relevant analog of Theorem 1.1 was proved long ago by McCord (see Section 8), and it was already pointed out in [AD19] that the MacPhersonian Conjecture is the same whether worded for $\rho$ or for $\Delta(\rho)$.

2.8. The place of the MacPhersonian Conjecture. The Conjecture lies at the border between topology and combinatorics. Let $\mathbb{K}$ denote either of the fields $\mathbb{R}$ or $\mathbb{C}$. The classifying space for $r$-dimensional $\mathbb{K}$-vector bundles is $\text{Gr}(r, \mathbb{K}^\infty)$, the direct limit over $n$ of spaces $\text{Gr}(r, \mathbb{K}^n)$. This means that there is a natural bijection between the set of isomorphism classes of $r$-dimensional $\mathbb{K}$-vector bundles and the set of homotopy classes of maps from $B$ to $\text{Gr}(r, \mathbb{K}^\infty)$. The point of the MacPhersonian Conjecture, if true, is that $\text{Gr}(r, \mathbb{K}^n)$ is faithfully modeled by a combinatorial object, namely $\text{Gr}(r, F^n)$, where the hyperfield $F$ is $T\Phi$ in the complex case, or the sign hyperfield $S$ in the real case, leading, in the limit, to a combinatorial model for the classifying space.

The rest of this paper is concerned with the proof of Theorem 1.1.

3. Preliminaries

3.1. Polyhedra. This is for reference. We need to be precise about our polyhedral terminology.

A rectilinear simplicial complex consists of a locally finite, finite-dimensional, abstract simplicial complex $K$, whose geometric realization is denoted by $|K|$. The space $|K|$ is embedded as a subset of a Euclidean space $E$ in such a way that each simplex is a convex subset of $E$. Moreover, the embedding is a closed proper map $|K| \to E$. The choice of $E$ and of the embedding will be suppressed, and from now on we will denote the image of $|K|$ simply by $K$, omitting the vertical bars.

The Euclidean space $E$ should carry either the $\ell_1$ metric or the $\ell_\infty$ metric; this is to ensure that metric balls in $K$ will be polyhedral. The “length metric” on $K$ is the metric which measures the distance between points $p$ and $q$ as the infimum of lengths of piecewise linear paths joining $p$ to $q$, where each linear part of the path lies in, and is measured in, a simplex of $K$. The corresponding metric topology on $K$ is the same as both the weak topology and the topology inherited from the Euclidean space $E$.

A locally compact space $A$ is a polyhedron if it is equipped with a piecewise linear (PL) structure. This means that the space $A$ has been identified, via a homeomorphism chosen once and for all, with a rectilinear simplicial complex $K$. The PL structure defined by $A$ consists of all rectilinear simplicial complexes $L$ occupying the same space as $K$, such that $L$ and $K$ have a common rectilinear simplicial subdivision. We say that any such complex $L$ triangulates $A$ or is a triangulation of $A$.

A subpolyhedron of $A$ is a closed subset $B$ such that some triangulation of $A$ has a subcomplex which triangulates $B$. We will often describe a space $A$ as polyhedral when we mean

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17A map is proper if the pre-image of every compact subset of its codomain is compact.

18References for polyhedra are [RS72], [Hud69] and Section 3.1 of [Spa95]. Here, we use nothing of that subject beyond basic definitions and elementary properties.
that $A$ is a polyhedron in the above sense; and when $B$ is a subpolyhedron of $A$ we may simply say that $B$ is “polyhedral”.

As an extension of this, we may consider an open subset $V$ of the polyhedron $A$. It is well-known that $V$ has a polyhedral structure compatible with that of $A$; this means: $V$ can be triangulated in such a way that each finite subcomplex is a subpolyhedron of $A$, and the diameters of simplexes approach zero as the simplexes approach the frontier of $V$.

The polyhedron $A$ has many length metrics, one for each triangulation. In particular, small metric balls in polyhedra are cones on their frontiers. Once a triangulation is chosen, and hence a length metric, then, for small enough $\eta > 0$, two maps from a compact domain into $A$ which are distant at most $\eta$ apart pointwise are $\eta$-homotopic. This is because when $\eta$ is small and $d(p,q) < \eta$ then that distance is realized by a unique path which varies continuously with $p$ and $q$.

The following variant on the last sentence will be important for us:

**Proposition 3.1.** When $B$ is compact (i.e is triangulated by a finite complex, say $L$) then there is $\eta_0 > 0$ such that when $\eta < \eta_0$ any two maps from a (not necessarily compact) polyhedron into $B$ which are distant at most $\eta$ apart pointwise are $\eta$-homotopic.

**Definition 10.** We call $\eta_0$ the fineness of $L$.

**Remarks.** See [RS72] or [Hud69] for the following:
1. The class of polyhedra is closed under coproducts and finite cartesian products.
2. If $B$ is a compact polyhedral subset of a finite product $A$ of polyhedra, then projections of $B$ to sub-products of $A$ are also compact polyhedral.
3. The join of two compact polyhedra is a compact polyhedron.
4. Smooth manifolds are homeomorphic to polyhedra.
5. The topological join of two locally compact spaces might not be locally compact, so if the spaces are polyhedra, their join does not satisfy our definition of “polyhedron”. However, the join of two abstract simplicial complexes has an obvious meaning even when they are not locally finite. Therefore the topological join of two (hence of finitely many) polyhedra can be triangulated by a simplicial complex which might not be locally finite. Occasionally, we will allow ourselves to extend the word “polyhedral” to include this case.

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19Maps $a, b: A \to B$ are $\eta$-homotopic if there is a homotopy between $a$ and $b$ such that for each point $x \in A$ the diameter of the image of $x \times I$ is $< \eta$. 

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3.2. Some Consequences of the Additional Properties. We now return to the world of geometric $M$-posets and the accompanying notational conventions introduced in Section 1.2.

**Proposition 3.2.** When $r < s$ and $A$ is a compact subset of $X_r$ then $A^{(s)}$ is a compact subset of $X_s$.

For this we need notation and a lemma.

**Notation.** When $(M,d)$ is a metric space and $S \subseteq cM$, we define $\bigcup(S) := \{m \in M \mid m \in s \text{ for some } s \in S\}$, i.e. $\bigcup(S)$ is the union of the sets in $S$. (The notation $cX$ was defined in Section 1.6.)

**Lemma 3.3.** Let $C$ be a compact subset of $c\mathbb{R}^n$. Then $\bigcup(C)$ is a compact subset of $\mathbb{R}^n$.

**Proof.** $C$ is totally bounded in the Hausdorff metric, so $\bigcup(C)$ is a bounded subset of $\mathbb{R}^n$. If $x$ is a limit point of $\bigcup(C)$ there exists $(x_n)$, a sequence in $\bigcup(C)$, converging to $x$. Let $x_n \in c_n$ where $c_n \in C$. Without loss of generality, assume $\{c_n\}$ converges to $c \in C$ in the Hausdorff metric. So $x$ must lie in $c$. Thus $\bigcup(C)$ is a closed and bounded subset of $\mathbb{R}^n$. \hfill \square

**Proof of Proposition 3.2.** By the definition of a geometric poset the set $\pi_{r,s}(A)$ is a compact subset of $cX_s$. The polyhedron $X_s$ can be regarded as a subset of a Euclidean space, so Lemma 3.3 implies $A^{(s)} = \bigcup(\pi_{r,s}(A))$ is compact. \hfill \square

**Proposition 3.4.** Let $A$ be a compact subset of $X_r$ and let $r \leq s$. If $A$ is polyhedral then $A^{(s)}$ is polyhedral.

**Proof.** Define $B := (A \times X_s) \cap \mathcal{P}$, where $\mathcal{P}$ is the order relation in the $M$-poset $\mathcal{X}$. Then $B = \{(a,x) \mid a < x\} \subseteq A \times A^{(s)}$ is polyhedral (being the intersection of polyhedra). The $X_s$ projection of $B$ is $\{x \mid a < x \text{ for some } a \in A\}$. This is $A^{(s)}$, and is compact by Proposition 3.2. There are compact polyhedra $R \subseteq X_r$ and $S \subseteq X_s$ such that $A \times A^{(s)} \subseteq R \times S$. Since $B$ is polyhedral and is a subset of $R \times S$, $B$ is compact polyhedral. Thus the projection of $B$, namely $A^{(s)}$, is polyhedral. \hfill \square

3.3. Tools from geometric topology. Here we state three theorems, well-known in geometric topology but perhaps less well-known in other mathematical communities.

Recall that a space $Z$ is locally $n$-connected if for each $z \in Z$ and each neighborhood $U$ of $z$ there is a neighborhood $V$ of $z$ such that, for all $k \leq n$, every map from the $k$-sphere $S^k$ into $V$ extends to a map of the $(k+1)$-ball $B^{k+1}$ into $U$. For us, the important point is that polyhedra are locally $n$-connected for all $n$.

**Definition 11.** When $K$ is a simplicial complex and $L$ is a subcomplex containing all the vertices of $K$, a partial realization of $K$ in a space $Z$ relative to a cover $\mathcal{A}$ of $Z$ is a map $\psi : L \to Z$ such that, for every simplex $\sigma$ of $K$, there is a set $A \in \mathcal{A}$ with $\psi(\sigma \cap L) \subseteq A$. When $L = K$ this is a full realization.

**Theorem 3.5.** Let $Z$ be a locally $(n-1)$-connected metrizable space. Every open cover $\mathcal{A}$ of $Z$ has an open refinement $\mathcal{B}$ such that every partial realization $j_0 : L \to Z$ relative to $\mathcal{B}$ extends to a full realization $j : K \to Z$ relative to $\mathcal{A}$.

**Proof.** This is Theorem V.4.1 of [Hu65]. \hfill \square

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20See Definition 6.
Recall that a map is\textit{ proper} if the pre-image of every compact subset of its codomain is compact. A proper map $q : A \to B$ between locally compact polyhedra is\textit{ cell-like} if $q^{-1}(b)$ has trivial shape\footnote{See Section 1.6} for every $b \in B$.

\textbf{Theorem 3.6.} Let $A$ and $B$ be locally compact metric spaces, $K$ a finite-dimensional, locally finite simplicial complex, $L$ a subcomplex of $K$, $\theta : A \to B$ a (proper) cell-like map, $\varphi : K \to B$ a proper map, $\psi : L \to A$ a proper map such that $\theta \circ \psi = \varphi \mid L$, and $\epsilon : B \to (0, \infty)$ a function. Then there exists a proper map $\bar{\varphi} : K \to A$ such that $d(\theta \circ \bar{\varphi}(x), \varphi(x)) \leq \epsilon(\varphi(x))$ for every $x \in K$.

\textit{Proof.} This is Lemma 2.3 of [Lac69]. \hfill $\Box$

\textbf{Theorem 3.7.} A cell-like map between locally compact polyhedra is a proper homotopy equivalence.

\textit{Proof.} This is Theorem 1.5 of [Lac69]. \hfill $\Box$

\section{The special case $\mathcal{R} = [2]$}

\textbf{Notation.} In this section, when $A \subseteq X_1$ we write $A'$ rather than $A^{(2)}$.

Nearly all the ideas needed for our proof of Theorem 1.1 occur when dealing with two special cases, $\mathcal{R} = [2]$ and $\mathcal{R} = [3]$, where notation is less cluttered. We treat $\mathcal{R} = [2]$ here, and $\mathcal{R} = [3]$ in the following section.

When $\mathcal{R} = [2]$, the geometric $M$-poset is $X = X_1 \bigsqcup X_2$; $\mu(X_1) = 1$ and $\mu(X_2) = 2$. It is assumed throughout that $X$ is a geometric poset.

In this case, the join $X_1 \ast X_2$ consists of a copy of $X_1$, a copy of $X_2$, and for each $x \in X_1$ and $y \in X_2$ a linearly parametrized path $\omega$ with $\omega(0) = x$ and $\omega(1) = y$. We call such a path (or its image) a\textit{ segment} from $x$ to $y$. The point $\omega(t)$ on a segment from $x$ to $y$ is often more conveniently denoted by $(1 - t)x + ty$ where $0 \leq t \leq 1$. The map $q : X_1 \times X_2 \times I \to X_1 \ast X_2$ taking $(x, y, t)$ to $(1 - t)x + ty$ defines the quotient topology on $X_1 \ast X_2$.

\subsection{Weak contractibility of basis elements.}

Recall that throughout we use the Convenient Basis of Section 1.5. Basis elements lying in $X_2$ are contractible. Here we prove that when $U$ is a (contractible) basis element in $X_1$ then $\uparrow U$ is weakly contractible; i.e.

\textbf{Theorem 4.1.} For all $n$, $\pi_n(\uparrow U, U)$ is trivial.

\textit{Notation:} $\overline{U}$ denotes the closure of $U$ in $X_1$.

Theorem 4.1 is a consequence of the following variant:

\textbf{Theorem 4.2.} For a basis element $U$, $\uparrow \overline{U}$ is weakly contractible.

\textit{Proof that Theorem 4.2 implies Theorem 4.1.} Each basis element $U$ contains basis elements $V_k$ such that $V_k \subseteq V_{k+1}$, and $\bigcup_k V_k = U$. It follows that $\uparrow U = \bigcup_k \uparrow V_k = \bigcup_k V_k$.

A singular sphere $\varphi$ in $\uparrow U$ has compact image, so it lies in some $\uparrow V_k$. Thus, by Theorem 4.2 $\varphi$ is homotopically trivial in $\uparrow U$. \hfill $\Box$

The rest of this subsection is devoted to proving Theorem 4.2.
**Definition 12.** Let $Z$ be a space, and let $A \subseteq X_1$. A map $s': Z \to \uparrow A$ lies over a map $s: Z \to A$ if $s(z) \leq s'(z)$ for every $z \in Z$.

Fundamental to understanding homotopies in this context is the following

**Proposition 4.3.** If $s'$ lies over $s$ and $C := (s')^{-1}(A)$ then, as maps $Z \to \uparrow A$, $s'$ and $s$ are homotopic rel $C$.

**Proof.** Define $F: Z \times I \to \uparrow A$ by

1. $F = s'$ on $Z \times [0,1)$;
2. $F = s$ on $Z \times \{1\}$.

We check continuity of $F$. When $W$ is open in $A'$ then $F^{-1}(W) = s^{-1}(W) \times [0,1)$ which is open in $Z \times I$. Whenever $V$ is open in $A$ we have

$$F^{-1}(V \cup V') = F^{-1}(V) \cup F^{-1}(V')$$

$$= [s^{-1}(V) \times \{1\}] \cup [s^{-1}(V') \times [0,1)$$

Since $s^{-1}(V) \subseteq s^{-1}(V')$ we conclude that $F^{-1}([V)$ is open in $Z \times I$. Clearly, the homotopy is stationary on $C \times I$. \(\square\)

**Proof of Theorem 4.2.** We abbreviate the $n$-ball $B^n$ to $B$. Recall that $U$ is a basis element in $X_1$. We consider an arbitrary map

$$g: (B, \partial B) \to (\uparrow U, \overline{U}).$$

We write $C := g^{-1}(\overline{U})$. This $C$ is a compact subset of $B$ containing $\partial B$, and $\overline{U}$ is closed in $\uparrow U$.

**Overall Strategy for the proof:** We will show that there are maps $h'$ and $h$ from $B$ to $\uparrow \overline{U}$ satisfying:

1. $h' = h = g$ on $C$;
2. $h'$ maps $B - C$ into $\overline{U}'$;
3. $h$ maps $B$ into $\overline{U}$;
4. $h'$ lies over $h$;
5. $h'$ is homotopic (in $\uparrow \overline{U}$) to $g$ rel $C$.

Proposition 4.3 will then imply that $h'$ is homotopic to $h$ rel $C$ and hence that $g$ represents the trivial element of $\pi_n(\uparrow \overline{U}, \overline{U})$.

We need some preparatory lemmas:

**Lemma 4.4.** If $\{V_m\}_{m=1}^\infty$ is a sequence of neighborhoods of $x \in X_1$ such that $\bigcap_{m=1}^\infty V_m = \{x\}$, then $\bigcap_{m=1}^\infty V'_m = x'$.

**Proof.** It is clear that $x' \subseteq \bigcap_{m=1}^\infty V'_m$. To show containment in the other direction, let $y \in \bigcap_{m=1}^\infty V'_m$. By Property A3, for each $m$, there exists $x_m \in V_m$ such that $x_m < y$ in $X$. Since $\bigcap_{m=1}^\infty V_m = \{x\}$, $x_m$ converges to $x$ as $m \to \infty$. Thus, $(x_m, y)$ converges to $(x, y)$. Since $\{(x_m, y)\} \subseteq P$ and the order relation $P$ is closed in $X^2$, $(x, y) \in P$. \(\square\)

---

\footnote{Lemma 4.4 and Lemma 4.6 are used in proving Lemma 4.7 and Lemma 4.9. Lemma 4.7 is for Lemma 4.6}
Lemma 4.5. Let $x \in \overline{U}$. Given $\epsilon > 0$, there exists $\eta(x) > 0$ such that whenever$^{23} y$ and $z$ lie in $B_\eta(x)$ then $z' \subseteq N_\epsilon(y')$.

Proof. This is an immediate consequence of Property A2. \hfill \Box

Lemma 4.6. Given $\epsilon > 0$ there exists $\delta > 0$ such that whenever $p \in C$ and $q \in B_\delta(p) - C$ then $g(q) \in N_\epsilon(g(p'))$.

Proof. We write $C := \{ \uparrow B_{\eta(x)}(x) \mid x \in \overline{U} \}$, where $\eta(x)$ is as in Lemma 4.5. Then $g^{-1}(C)$ is an open cover of the compact set $C$. Let $\delta$ be a Lebesgue number for this cover. For some $x \in \overline{U}$, $g(B_\delta(p)) \subseteq \uparrow B_{\eta(x)}(x)$. This means that for some $w \in B_{\eta(x)}(x)$, $g(q) \in w'$. By Lemma 4.5, $g(q) \in N_\epsilon(g(p'))$. \hfill \Box

Now we can proceed with the proof. While it looks complicated, it mainly consists of $\epsilon - \delta$ arguments.

Let $Q := (\overline{U} \times \overline{U'}) \cap \mathcal{P}$. By Propositions 3.2 and 3.4, $Q$ is a compact polyhedron. We choose triangulations $L$ for $\overline{U}$ and $M$ for $\overline{U'}$. Then $L \times M$ has a subdivision triangulating $Q$. Each of $L$ and $M$ carries its length metric, and the metric on $\overline{U} \times \overline{U'}$ is taken to be the sum of those separate metrics. Let $\eta_0 > 0$ be a number less than the fineness of both length metrics (see Proposition 3.1).

For each integer $i \geq 0$ we choose an open cover $\mathcal{A}_i$ of $Q$ whose members have diameter $< \frac{1}{i}$ ($\frac{1}{0} = \infty$) and let $\mathcal{B}_i$ be the refinement given by Theorem 3.5. Let $4\lambda_i$ be a Lebesgue number for $\mathcal{B}_i$, where $\lambda_i$ decreases to 0 as $i \to \infty$. It will be important later that $\lambda_0$ be chosen to be less than $\eta_0$.

For $x \in B - C$ we choose $r(x) \in C$ such that$^{24}$ $d(x, r(x)) = d(x, C)$. We write $k_0(x) = g \circ r(x) \in \overline{U}$.

By Property A1, the set $k_0(x)'$ is compact and non-empty. By Lemma 4.6, there is a sequence $(\gamma_i)$, decreasing to 0, with the following property: when $x \in B - C$ satisfies $\gamma_{i-1} \leq d(x, C) \leq \gamma_i$ there exists a point $k'_0(x) \in k_0(x)'$ such that $d(k'_0(x)), g(x)) \leq \lambda_i$. As $i$ varies, this defines a function $k'_0 : B - C \to \overline{U}$. Note that $k_0(x) < k'_0(x)$ in the poset. We write $j_0 := (k_0, k'_0) : B - C \to Q$.

\footnotesize
\begin{itemize}
  \item $^{23}$B$_\delta$ denotes the ball of radius $\delta$; $N_\epsilon$ denotes the $\epsilon$-neighborhood.
  \item $^{24}$It is not claimed that $r$ is continuous. We will only be interested in its restriction to the (discrete) 0-skeleton of a chosen triangulation of $B - C$.
\end{itemize}
We define \( I_i := \{ x \in B - C \mid \gamma_{i+1} \leq d(x, C) \leq \gamma_i \} \), and \( N_i := I_{i-1} \cup I_i \cup I_{i+1} \). These sets are compact but might not be polyhedral. The map \( g \) is uniformly continuous on \( N_i \), so there is a sequence \( (\theta_i) \), decreasing and converging to 0, such that when \( S \) is a subset of \( N_i \) of diameter < \( \theta_i \) then \( g(S) \) has diameter < \( \lambda_i \).

These choices give us control over how the 0-skeleton of a suitable triangulation of \( B - C \) is mapped into \( Q \), made precise in the following:

**Lemma 4.7.** Let \( K \) be a triangulation of \( B - C \) such that each \( I_i \) is covered by a finite subcomplex \( K_i \) lying in \( N_i \), and every simplex of \( K_i \) that meets \( I_i \) has diameter < \( \theta_i \). There is an increasing sequence of positive integers \( \ell(i) \geq i \) such that the complexes \( K_{\ell(i)} \) are pairwise disjoint, and when the simplex \( \sigma \) of \( K_{\ell(i)} \) meets \( I_{\ell(i)} \) then

(i) \( k_0(\sigma^0) \) has diameter < \( \lambda_i \)
(ii) \( k'_0(\sigma^0) \) has diameter < \( 3\lambda_i \).

**Proof.** When \( \ell > 0 \) is an integer and \( \sigma \) is a simplex of \( K_{\ell+1} \) that meets \( I_{\ell+1} \), then \( \sigma^0 \) has diameter \( \leq \theta_{\ell+1} \).

For any \( u \) and \( v \) in \( \sigma^0 \), we have \( d(u, v) \leq \theta_{\ell+1} \), \( d(u, r(u)) \leq \gamma_{\ell} \), and \( d(v, r(v)) \leq \gamma_{\ell} \). So \( d(r(u), r(v)) \leq 2\gamma_{\ell} + \theta_{\ell+1} \). This is an upper bound for the diameter of \( r(\sigma^0) \).

The restriction \( g : C \to \overline{U} \) is uniformly continuous, so there is a sequence \( (\rho_i) \), decreasing and converging to 0, such that when \( r(\sigma^0) \) has diameter < \( \rho_i \) then \( g \circ r(\sigma^0) \) (i.e. \( k_0(\sigma^0) \)) has diameter < \( \lambda_i \).

So, given \( i \), when \( \ell \) is large enough that \( 2\gamma_{\ell} + \theta_{\ell+1} < \rho_i \) then \( k_0(\sigma^0) \) has diameter \( \leq \lambda_i \). For each \( i \), pick \( \ell = \ell(i) \) to be large enough in the above sense, and such that the complexes \( K_{\ell(i)} \) are pairwise disjoint, while the sequence \( (\ell(i)) \) is increasing. Then (i) is satisfied.

When \( \sigma \) meets \( I_i \), the above proof gives \( d(u, r(u)) \leq \gamma_{i} \), and \( d(v, r(v)) \leq \gamma_{i} \), hence \( d(k'_0(u), g(u)) \leq \lambda_i \) and \( d(k'_0(v), g(v)) \leq \lambda_i \). Since \( d(g(u), g(v)) \leq \lambda_i \), we see that \( k'_0(\sigma^0) \) has diameter < \( 3\lambda_i \), so (ii) is satisfied. \( \square \)

The “annuli” \( K_{\ell(i)} \) in \( B - C \) bear down on \( C \) but there will be gaps between them. The larger complex \( J_{\ell(i)} := \bigcup_{\ell(i) \leq m \leq \ell(i+1)} K_m \) contains the \( i \)th gap. Each \( J_{\ell(i)} \) is a subpolyderon of \( B - C \). We may therefore assume
that it is triangulated as a subcomplex of $K$. Since the sequences $(\gamma_i), (\theta_i)$, and $(\rho_i)$ are all decreasing, we can extend the lemma as follows:

**Addendum 4.8.** The inequalities (i) and (ii) in Lemma 4.7 continue to hold when $\sigma$ is a simplex of $J_{ℓ(i)}$. □

We consider the function $j_0 := (k_0, k'_0) : B - C \rightarrow Q$ but only on the 0-skeleton of $J$ (so continuity is not a problem). By Addendum 4.8, for every simplex $\sigma$ of $J_{ℓ(i)}$ the set $j_0(\sigma^0)$ has diameter $< 4\lambda_i$ and thus lies in an element of the cover $B_i$ of $Q$.

The sequence $\{ℓ(i)\}$ starts with $i = 0$ (recall that $\frac{1}{0} = ∞$). The definition of the covers $A_i$ allows us to assume that $A_0$ is the singleton $\{Q\}$. The refinement $B_0$ has Lebesgue number $4\lambda_0$.

Let $J := \bigcup_{i>0} J_{ℓ(i)}$, a subcomplex of $K$. We wish to extend $j_0 \mid: J^0 \rightarrow Q$ to a map $j : J \rightarrow Q$. We do this by induction. On $J_{ℓ(0)}$, $j_0$ maps the 0-skeleton of each simplex to a set of diameter $< 4\lambda_0$. Theorem 3.5 gives an extension $j$ on $J_{ℓ(0)}$. Next, we consider $j_0^+$ on $J_{ℓ(1)}$, where $j_0^+ = j_0$ on the vertices of $J_{ℓ(1)}$ and $j_0^+$ agrees with (the previously defined) $j$ on $J_{ℓ(0)} \cap J_{ℓ(1)}$. We subdivide this intersection so that the $j$-image of each simplex in it has diameter $< 4\lambda_1$, and we extend this subdivision to $J_{ℓ(0)} \cup J_{ℓ(1)}$ without adding further vertices. At this point we have a partial realization$^{26}$ of $J_{ℓ(1)}$ in $Q$ relative to $B_1$, and Theorem 3.5 gives an extension of $j$ to all of $J_{ℓ(1)}$ extending the map $j$ previously defined on $J_{ℓ(0)}$. The $j$-image of each simplex in $J_{ℓ(1)}$ lies in an element of $B_1$. We continue in this way so that on $J_{ℓ(i)} - int J_{ℓ(i-1)}$ the $j$-image of the 0-skeleton of each simplex has diameter $< 4\lambda_i$, each time using Theorem 3.5.

We point out that $J$ does not include all of $B - C$, but $N := J \cup C$ is a polyhedral neighborhood of $C$ in $B$. The map $j = (k, k')$ has components $k : J \rightarrow \overline{U}$ and $k' : J \rightarrow \overline{U}'$. We extend both of these maps$^{26}$ (without change of label) to agree with $g$ on $C$. Thus $k$ and $k'$ now have domain $N$.

**Lemma 4.9.** The extended functions $k : N \rightarrow \overline{U}$ and $k' : N \rightarrow \overline{U} \cup \overline{U}'$ are continuous.

**Proof.** The only issue is continuity at any point $p \in C$. Let $\uparrow T$ be a basic (open) neighborhood of $g(p)$. Since $g$ is continuous $g^{-1}(\uparrow T)$ is an open neighborhood of $p$. It is enough to find a smaller neighborhood, $W$, of $p$ such that $k'(W) \subseteq T'$ and $k(W) \subseteq T$.

If $\{\overline{V_m}\}$ is a basic sequence of compact neighborhoods of $g(p)$ in $U$, then $\cap \{\overline{V_m}\} = g(p)'$ by Lemma 4.4. By Property A1, $g(p)'$ is a compact non-empty subset of the open set $T'$, so for sufficiently large $m$ the compact sets $\overline{V_m}$ and the (closed) frontier of $T'$ are disjoint. Choose $ε$ so that $2ε = d(g(p)', fr(T'))$. For this choice of $ε$, let $δ$ be as in Lemma 4.6. The following shows that $B_δ(p)$ is the required neighborhood $W$.

When $x$ is a point of $K$ such that $d(x, p) < δ$ then $d(x, r(x)) < δ$. By Property A1, $g(r(x))'$ is compact and non-empty. Since $d(x, r(x)) < δ$, $d(g(x), g(r(x))') < ε$, hence $d(g(x), k'(x)) < ε$. And since $d(x, p) < δ$, $d(g(x), g(p)') < ε$. Hence $d(k'(x), g(p)') < 2ε$. This implies $k'(x) \in T'$.

Continuity of $k$ at $p$ follows from the fact that $g$ is continuous at $p$: i.e. given $ε > 0$ let $δ > 0$ be such that $g$ maps the $δ$-neighborhood of $p$ into the $ε$-neighborhood of $g(p)$ in $U$. If $d(p, x) < δ$ then $d(p, r(x)) < δ$, so $d(g(p), g(r(x))) < ε$; i.e. $d(k(p), k(x)) < ε$. Thus $k$ is continuous at $p$. □

---

$^{26}$See Section 9.3.

$^{26}$Previously, we defined $k_0$ and $k'_0$ on all of $K$. We now confine the domains of $k_0$ and $k'_0$ to $J$. In what follows we need to be free to extend $j$, i.e. $(k, k')$, to all of $K \cup C$ in a different way.
The required maps (see Strategy, above) $h$ and $h'$ are defined to agree with $k$ and $k'$ respectively on $N$. Then $h = h' = g$ on $C$, and $h'$ lies over $h$ (on $N$).

Because the $\lambda_i$ are less than the fineness of our triangulation of $\overline{U}$, Proposition 3.1 implies that the maps $h'|N - C$ and $g|N - C$ are homotopic. Moreover, the homotopy gets smaller and smaller as $i$ increases, and so it extends to $g \times$ identity on $C$. In other words, $h'$ and $g$ are homotopic on all of $N$.

It remains to extend $h$ and $h'$ to the rest of $K$, maintaining the properties (i)-(v) of the Strategy.

Recall that $Q := (\overline{U} \times \overline{U}) \cap \mathcal{P}$. The projection map $\pi : (\overline{U} \times \overline{U}) \to \overline{U}'$ is easily seen to restrict to a proper map $Q \to \overline{U}'$, and, by Property A3 (of the Additional Properties) this map is cell-like.

Let $L$ be the finite subcomplex of $K$ covering the closure of the complement of $N$ in $B$; i.e. $L \cup J = K$. Then $L$ is disjoint from the interior of the neighborhood $N$. We write $\hat{L} = L \cap N$, the subcomplex of $L$ such that $\hat{L}$ is the frontier of $N$ in $B$. We have a map $(L \times 0) \cup (\hat{L} \times I) \to \overline{U}'$ agreeing with $g$ on $L \times 0$ and being a homotopy between $g|\hat{L}$ and $h'|\hat{L}$ on $\hat{L} \times I$. The Homotopy Extension Property gives a map $\tilde{g} : L \to \overline{U}'$ homotopic to $g$ and agreeing with $h'$ on $\hat{L}$. Now, $j| : \hat{L} \to Q$ satisfies $\pi \circ j| = h'| : \hat{L} \to \overline{U}'$. Theorem 3.6 then allows us to extend $j$ continuously to all of $L$.

Where the domains overlap, this $j$ agrees with that previously defined on $N$, so, gluing the two maps together, we have $j : K \to Q$. The components of $j$ are the required maps $h$ and $h'$. They extend the previously defined $h$ and $h'$. By Theorem 3.6, the component $h'(= \pi \circ j)$ can be as close as we please to $\tilde{g}$, and hence is homotopic to it. Thus, by Proposition 4.3, $g$ is homotopic, rel $\partial B$, to a map $h$ whose image misses $\overline{U}'$.

This completes the proof of Theorem 4.2 hence also of Theorem 4.1.

---

Recall our convention of using the letter $L$ for both the simplicial complex and its geometric realization.
4.2. Pre-images of compact polyhedral sets. Here we prove Theorem 4.14 relating \( \Delta(\uparrow A) \) to \( f^{-1}(\uparrow A) \) when \( A \) is a compact polyhedral subset of \( X_1 \).

Let \( d \) be a length metric on \( X_1 \). Define \( \varphi : X_1 \times (0, 1] \to I \) by \( \varphi(x, u) = \min \left\{ \frac{d(x, A)}{u}, 1 \right\} \). Writing \( \varphi_u \) for \( \varphi(\cdot, u) \) we note

(i) \( \varphi_u = 0 \) on \( A \);
(ii) \( \varphi_u = 1 \) on \( X_1 - N_u(A) \);
(iii) \( 0 < \varphi_u < 1 \) elsewhere.

Let \( B \) be a compact polyhedral subset of \( X_2 \). For each \( u \in (0, 1] \), define \( \psi_u : X_1 \times B \times I \to X_1 \times B \times I \) by

\[
\psi_u(x, y, t) = \begin{cases} 
(x, y, \varphi_u(x)) & \text{if } t \leq \varphi_u(x) \\
(x, y, t) & \text{if } \varphi_u(x) \leq t.
\end{cases}
\]

We note that \( \psi_u \) fixes \( A \times B \times I \) and that \( \psi_u(x, y, t) = (x, y, 1) \) when \( x \in X_1 - N_u(A) \) and \( y \in B \).

We also note the homotopy \( h_u : X_1 \times B \times I \times [0, 1] \to X_1 \times B \times I \) between \( \psi_u \) and the identity map defined by

\[
h_u(x, y, t, v) = \begin{cases} 
(x, y, v\varphi_u(x)) & \text{if } t \leq v\varphi_u(x) \\
(x, y, t) & \text{if } v\varphi_u(x) \leq t.
\end{cases}
\]

and that the image of \( \psi_u \) is the subset \( \{(x, y, t) \in X_1 \times B \times I \mid \varphi_u(x) \leq t \leq 1\} \).

Factoring by the canonical quotient \( X_1 \times B \times I \to X_1 \ast B \) we see that, while \( \psi_u \) does not induce a well-defined map \( X_1 \ast B \to X_1 \ast B \), it does induce a map \( \Psi_u : (X_1 \ast B) - (X_1 - A) \to X_1 \ast B \). Similarly, the homotopy \( h_u \) induces a homotopy \( H_u : (X_1 \ast B) - (X_1 - A) \times [0, 1] \to X_1 \ast B \).

The image of \( \Psi_u \) is the compact set

\[ I_u := \{(1 - t)x + ty \in X_1 \ast B \mid x \in X_1, y \in B, t \geq \varphi_u(x)\} \] .

Proposition 4.10. Given \( u \in (0, 1] \) and a neighborhood \( M \) of \( N_u(A) \ast B \) in \( X_1 \ast X_2 \), we can deform \( (X_1 \ast B) - (X_1 - A) \) within itself onto the compact set \( I_u \subseteq M \) by a deformation that is continuous in the variable \( u \).

By careful choice of the metric \( d \) it can be arranged that for \( u = \frac{1}{n} \) the above functions are PL, and in particular that \( I_\frac{1}{n} \) is polyhedral. Thus we get:

Proposition 4.11. Given \( u = \frac{1}{n} \) and a neighborhood \( M \) of \( N_u(A) \ast B \) in \( X_1 \ast X_2 \), we can deform \( (X_1 \ast B) - (X_1 - A) \) within itself onto the compact polyhedral set \( I_\frac{1}{n} \subseteq M \).

Since the order relation \( \mathcal{P} \) is polyhedral, this proposition can be restricted as follows, where we now take \( B \) to be \( A' \) (compact and polyhedral by Proposition 3.4):

Corollary 4.12. Given \( u = \frac{1}{n} \) and a neighborhood \( M \) of \( \Delta(\uparrow A) \) in \( \Delta(\mathcal{X}) \), \( f^{-1}(\uparrow A) \) can be deformed within itself onto a compact polyhedral set \( J_n \subseteq M \).

Proof. The set \( f^{-1}(\uparrow A) \) consists of \( \Delta(\uparrow A) \) together with line segments ending in \( A' \) whose intersections with \( X_1 \) have been deleted. Here, we write \( J_n \) for the relevant part of \( I_\frac{1}{n} \). \( \Box \)

We note that \( J_{n+1} \) is a strong deformation retract of \( J_n \).
Lemma 4.13. The set $\Delta(\uparrow A)$ is a compact polyhedral subset of $X_1 \ast X_2$.

Proof. By Lemmas 3.2 and 3.4, $A'$ is compact and polyhedral. Thus $C := ((A \times A') \cap \mathcal{P}) \times I$ is compact and polyhedral. The projection maps $A \times A' \to A$ and $A \times A' \to A'$ define PL maps $\pi_A : ((A \times A') \cap \mathcal{P}) \times \{0\} \to A$ and $\pi_{A'} : ((A \times A') \cap \mathcal{P}) \times \{1\} \to A'$. The space $\Delta(\uparrow A)$ is clearly decomposable as the union of three compact polyhedra: $C$ and the mapping cylinders of $\pi_A$ and $\pi_{A'}$, where the mapping cylinders are glued to the 0- and 1- ends of $C$ in the obvious way. Since mapping cylinders of PL maps between compact polyhedra are compact polyhedra, and the union of compact polyhedra glued along compact polyhedral subsets is again a compact polyhedron, this completes the proof.

Theorem 4.14. When $A$ is a compact polyhedral subset of $X_1$, the inclusion map $\Delta(\uparrow A) \hookrightarrow f^{-1}(\uparrow A)$ is a homotopy equivalence.

Proof. $\Delta(\uparrow A)$ is the intersection of the nested sequence of polyhedra $J_n$, each of which is a strong deformation retract of $f^{-1}(\uparrow A)$ by Corollary 4.12. Since the inclusion maps $J_{n+1} \to J_n$ are homotopy equivalences, shape theory (see Section 7 for details) implies that $\Delta(\uparrow A)$ is shape equivalent to each of the polyhedra $J_n$. Moreover, since $\Delta(\uparrow A)$ is itself polyhedral by Lemma 4.13, this means that $\Delta(\uparrow A)$ is actually homotopy equivalent to each of the $J_n$, hence also to $f^{-1}(\uparrow A)$.

4.3. Contractibility of pre-images of basis elements. The order complex is the subset $\Delta(\mathcal{X}) \subseteq X_1 \ast X_2$ which consists of $X_1$, $X_2$, and the union of all segments joining $x$ to $y$ such that $x < y$. The Comparison Map $f : \Delta(\mathcal{X}) \to \mathcal{X}$ is defined by $f(x) = x$ when $x \in X_1$ (i.e. when $t = 0$) and $f((1-t)x + ty) = y$ when $t > 0$.

As before, we consider the two kinds of basis elements in the Up topology: $U \in \mathcal{U}_2$ and $\uparrow U$ where $U \in \mathcal{U}_1$. To apply Theorem 1.2 we show that the pre-image under $f$ of each of these is weakly contractible.

When $U \in \mathcal{U}_2$, the set $f^{-1}(U)$ consists of the contractible set $U \subseteq X_2 \subseteq X_1 \ast X_2$, together with half-open segments ending in $U$. Thus $U$ is a strong deformation retract of $f^{-1}(U)$ which is therefore contractible.

Now we consider the case where $U \in \mathcal{U}_1$. It is convenient to first deal with $f^{-1}(\uparrow U)$, which Theorem 4.14 tells us is homotopy equivalent to $\Delta(\uparrow U)$.

Notation. We write $\mathcal{C}A$ for the topological cone on the space $A$, i.e. $A \times I/A \times \{0\}$.

Proposition 4.15. $\Delta(\uparrow U)$ is homotopy equivalent to $\mathcal{C}U'$, and is therefore contractible.

Proof. Let $g : \Delta(\uparrow U) \to \mathcal{C}U'$ be the map which takes $U$ to the cone point $p$, is the identity on $U'$, and maps the point $(1-t)x + ty$ to the segment $(1-t)p + ty$ in the cone. By Lemma 4.13 this is a map between compact polyhedra. Consider $g^{-1}((1-t)p + ty)$. When $t = 0$ this pre-image is the contractible set $U$; when $t = 1$ this pre-image is a single point; when $0 < t < 1$ this pre-image is homeomorphic to $\{x \in U \mid x < y\}$ which has trivial shape (actually, is contractible) by Property A3. Thus $g$ is a cell-like map, and is therefore a homotopy equivalence by Theorem 3.7.

Corollary 4.16. $f^{-1}(\uparrow U)$ is contractible.

Just as Proposition 4.2 implies Theorem 4.1, this implies that (for basis elements $U$), $f^{-1}(\uparrow U)$ is weakly contractible. Since it is an open subset of a polyhedron, it is actually contractible.
Theorem 1.1 for the case \( \mathcal{R} = [2] \) now follows by combining Theorem 4.2, Corollary 4.16 and Theorem 1.2.

5. The special case \( \mathcal{R} = [3] \)

**Notation.** In this section, when \( A \subseteq X_1 \) we write \( A' \) rather than \( A^{(2)} \) and \( A'' \) rather than \( A^{(3)} \).

5.1. Weak contractibility of basis elements. Here \( \mathcal{K} = X_1 \amalg X_2 \amalg X_3 \) and \( U \subseteq X_1 \) is a basis element. We are to show that \( \uparrow U \) is weakly contractible. If \( U \subseteq X_3 \) this is trivial, and if \( U \subseteq X_2 \) this follows from what was done in Section 4. So we assume \( U \subseteq X_1 \).

Once again, it is easier to begin with \( \overline{U} \). We consider an arbitrary map

\[
g: (B, \partial B) \to (\uparrow U, U \cup U').
\]

where, once again, \( B \) is an abbreviation of \( B^n \). We write \( C := g^{-1}(\overline{U} \cup U') \). As before, this is a closed subset of \( B \) containing \( \partial B \). We wish to produce maps \( h'' \) and \( h \) from \( B \) to \( \uparrow U \) satisfying:

(i) \( h'' = h = g \) on \( C \);
(ii) \( h'' \) maps \( B - C \) into \( U'' \);
(iii) \( h \) maps \( B \) into \( \overline{U} \cup U' \);
(iv) \( h'' \) lies over \( h \);
(v) \( h'' \) is homotopic to \( g \) rel \( C \).

Proposition 4.3 will then imply that \( h'' \) is homotopic to \( h \) rel \( C \) and hence that the map \( g \) represents the trivial element of \( \pi_n(\uparrow U, U \cup U') \); hence that group is trivial. By Proposition 4.2 it will follow that \( \pi_n(\uparrow U, U) \) is trivial. As before, we can deduce the same when \( U \) is replaced by \( \overline{U} \).

The argument here is more complicated than in Section 4 because the obvious replacement for \( \overline{U} \times U' \) in the definition of \( Q \), namely \( (\overline{U} \cup U') \times U' \), is not polyhedral (with the \( U \) topology on \( U \cup U' \)). For this reason we must construct the map \( h'' \) in two stages, one for \( U \), the other for \( U' \).

Let \( Q_1 := (\overline{U} \times U') \cap \mathcal{P} \). As with \( Q \), this is a compact polyhedron. Let \( \mathcal{A}_i \) be an open cover of \( Q_1 \) by sets of diameter \( < \frac{1}{i} \), and let \( \mathcal{B}_i \) be the refinement given by Theorem 3.5. Let \( 4\lambda_i \) be a Lebesgue number for \( \mathcal{B}_i \).

We write \( C_1 := g^{-1}(\overline{U}) \); this is a closed subset of \( C \). Define \( Y := \{ x \in B - C \mid d(x, C) = d(x, C_1) \} \), a closed subset of \( B - C \).

We now proceed as in Section 4.1 replacing \( B - C \) by \( Y \) and \( C \) by \( C_1 \). The triangulation \( K \) and its subcomplex \( J \) are replaced by a triangulation \( K_1 \) of \( Y \) and its subcomplex \( J_1 \). We define \( N_1 := J_1 \cup C_1 \), a polyhedral neighborhood of \( C_1 \) in \( Y \). We get a map \( j_1 := (k_1, k''_1) \); with components \( k_1 : J_1 \to \overline{U} \) and \( k''_1 : J_1 \to U'' \). Both of these maps are extended to agree with \( g \) on \( C_1 \). As before, the extended function \( j_1 : N_1 \to Q_1 \) is continuous.

We first define the maps \( h \) and \( h'' \) on \( N_1 \), where they agree with \( k_1 \) and \( k''_1 \) respectively. Then \( h'' \) lies over \( h \) and \( h = h'' = g \) on \( C_1 \). As in Section 4.1 we may assume \( h'' \) and \( g\mid N_1 \) are homotopic rel \( C_1 \). The extension of \( h \) and \( h'' \) to the rest of \( K_1 \) maintaining the properties (i)-(v) is achieved as in that section. We further extend \( h \) and \( h'' \) to agree with \( g \) on all of
At this point, the Homotopy Extension Property implies that \( g : B \to \uparrow U \) is homotopic rel \( C \) to a map \( g_1 : B \to \uparrow U \), where \( g_1 \) agrees with \( h'' \) on \( Y \cup C \).

Now comes the part that is different from Section 4.1. We wish to alter \( g_1 \) rel \( Y \cup C \) to have the desired covering property on the rest of \( B - C \). The set \( Z := B - (C_1 \cup Y) \) is an open polyhedral subset of \( B \) whose frontier is the compact set \( \dot{Z} := \text{fr}_B(Z) \).

Basically, we repeat what has been described above, with \( K_2 \) (triangulating \( Z \)), \( \dot{Z} \cup \bar{C} - C_1 \), \( g_1 \), and \( Q_2 := (\overline{U} \times \overline{U}'') \cap \mathcal{P} \) playing the respective roles of \( K_1 \), \( C_1 \), \( g \) and \( Q_1 \). There is a subcomplex \( J_2 \) of \( K_2 \), and a neighborhood \( N_2 \) of \( \dot{Z} \cup \bar{C} - C_1 \) in \( \dot{Z} \) playing the role previously played by \( J_1 \) and \( N_1 \). We get a map \( j_2 = (k''_2, k''_2) \) with components \( k''_2 : J_2 \to \overline{U}' \) and \( k''_2 : J_2 \to \overline{U}'' \). The construction (imitating what is given in detail in Section 4.1) means that this map \( j_2 \) extends continuously to \( \dot{Z} \cup \bar{C} \) agreeing with \( j_1 \) on \( \dot{Z} \cup \bar{C} \). The extension to the rest of \( \dot{Z} \) works as before. Thus we get maps \( h \) and \( h'' \) defined on all of \( B \) satisfying (i)-(v).

This shows that \( g \) is homotopic, rel \( \partial B \), to a map whose image misses \( \overline{U}'' \). Summarizing:

**Theorem 5.1.** For all \( n \), \( \pi_n(\uparrow \overline{U}, \overline{U} \cup \overline{U}') \) is trivial.

As in Section 4.1 this leads to:

**Corollary 5.2.** The set \( \uparrow U \) is weakly contractible.

**Remark.** It was important to handle \( \overline{U} \) before \( \overline{U}' \) since we needed \( C_1 \) to be compact. This is a simple instance of what is a more serious issue in the general case, treated in Section 6, where we need a careful ordering of \( \mathcal{R} \) in order to ensure that at each stage we are dealing with a compact set.

**5.2. Contractibility of pre-images of basis elements.** As a special case of what was defined in Section 1.3, recall that the order complex is the set \( \Delta(\mathcal{X}) \subseteq X_1 \ast X_2 \ast X_3 \) consisting of all points \( z = \sum_{i=1}^3 t_i x_i \) such that when \( i, j \in \text{supp} \ z \) then \( x_i < x_j \) in \( \mathcal{X} \). In particular, every
$x_i$ lies in this set. The Comparison Map $f : \Delta(\mathcal{X}) \to \mathcal{X}$ is defined by $f(z) = \max \{ x_i \mid i \in \text{supp } (z) \}$.

We first consider $U \subseteq X_1$. We describe the set $f^{-1}(\uparrow U) \subseteq X_1 \ast X_2 \ast X_3$ in some detail. Certainly it includes $\Delta(\uparrow U)$. Imitating what was done in Section 4.3, we let $R^o$ be the union of a certain collection of half-open segments, where the missing point of each segment is its initial point as measured by the poset $\mathcal{R} = [3]$. These segments are described as follows. In every case it is to be understood that $x_1 < x_2 < x_3$ whenever these comparisons make sense.

(i) $(x_1, x_2)$ where $x_1 \notin U, x_2 \in U^{(2)}$, and $x_2$ is not comparable to any member of $U^{(3)}$;
(ii) $(x_1, x_3)$ where $x_1 \notin U, x_3 \in U^{(3)}$, and there is no $x_2 \in U^{(2)}$ with $x_1 < x_2 < x_3$;
(iii) $(x_1, [x_2, x_3])$ where $x_1 \notin U, x_2 \in U^{(2)}$, and $x_3 \in U^{(3)}$;
(iv) $([x_1, x_2], x_3)$ where $x_1 \notin U, x_2 \notin U^{(2)}$, and $x_3 \in U^{(3)}$;
(v) $(x_2, x_3)$ where $x_2 \notin U^{(2)}, x_3 \in U^{(3)}$, and there is no $x_1 \in U$ with $x_1 < x_2$.

Then
\[
f^{-1}(\uparrow U) = \Delta(\uparrow U) \cup R^o.
\]

The set $X_2 \bigsqcup X_3$ is a sub-poset of $\mathcal{X}$, and we denote its order complex by $\Delta_{23}$. Recall that $\Delta_{23}$ consists of copies of $X_2$ and $X_3$ together with a segment $t_2x_2 + t_3x_3$ joining $x_2$ to $x_3$ whenever $x_2 < x_3$.

This gives rise to another poset $\mathcal{Y} := X_1 \bigsqcup \Delta_{23}$ with the partial ordering $x_1 < t_2x_2 + t_3x_3$ when $x_1 < x_2 < x_3$; $\mathcal{Y}$ is given the Up topology. It is a geometric poset because $\mathcal{X}$ is. We denote the corresponding Comparison Map by $g : \Delta(\mathcal{Y}) \to \mathcal{Y}$. Then, as subsets of $X_1 \ast X_2 \ast X_3$, we have
\[
f^{-1}(\uparrow U) = g^{-1}(\uparrow U).
\]

Since $g^{-1}(\uparrow U)$ is contractible by Proposition 4.16, we conclude that $f^{-1}(\uparrow U)$ is contractible.

\footnote{\textit{Notation:} $[x, y]$ stands for all the points $(1 - t)x + ty$ on the segment joining $x$ to $y$.}
\footnote{In all such expressions, the sum of coefficients is understood to be 1, and a segment such as this will require an obvious adjustment when $t_2 = 0$ or 1.}
Next we consider the case where $U \subseteq X_2$. We have the Comparison Map $f_{23} : \Delta_{23} \to X_2 \bigsqcup X_3$. By Proposition $4.16$, $f_{23}(\uparrow U)$ is contractible. But $f^{-1}(\uparrow U)$ is larger in general, as it includes deleted segments and 2-simplexes having a vertex in $f_{23}^{-1}(\uparrow U)$, the deletion being the (non-empty) part lying in $X_1$. However, as before, $f_{23}^{-1}(\uparrow U)$ is a strong deformation retract of $f^{-1}(\uparrow U)$, so the latter is contractible.

Finally, there is the case where $U \subseteq X_3$. In that case, just as in the previous paragraph, $\uparrow U$ is a strong deformation retract of $f^{-1}(\uparrow U)$, and therefore the latter is contractible. Just as Theorem $4.2$ implies Theorem $4.1$ they so we conclude:

**Proposition 5.3.** For basis elements $U$, the space $f^{-1}(\uparrow U)$ is contractible.

### 6. The General Case

The general case involves a geometric $M$-poset $\mathcal{X}$ equipped with a mirror map $\mu : \mathcal{X} \to \mathcal{R}$, where $\mathcal{R}$ is a finite poset.

#### 6.1. Weak contractibility of basis elements.

In Section $5.1$ we showed that when $\mathcal{U}(i)$ is the closure of a basis element of $X_1$ then $\uparrow \mathcal{U}(i)$ is weakly contractible. The method was to consider a map $g : (B, \partial B) \to (\uparrow \mathcal{U}(1), \mathcal{U}(1) \cup \mathcal{U}(2))$, define $C = g^{-1}(\mathcal{U}(1) \cup \mathcal{U}(2))$ and $C_1 = g^{-1}(\mathcal{U}(1))$. We picked a closed subset $Y$ of $B - C$ associated with $C_1$, and first adjusted $g$ on $Y \cup C$. Then we made a further adjustment on $Z = B - (C \cup Y)$.

If we were dealing with the case $\mathcal{R} = [4]$ we would do the same thing with one extra step: $C_1$ would be $g^{-1}(\mathcal{U}(1))$, $C_2$ would be $g^{-1}(\mathcal{U}(1) \cup \mathcal{U}(2))$ and $C$ would be $g^{-1}(\mathcal{U}(1) \cup \mathcal{U}(2) \cup \mathcal{U}(3))$. There would be subspaces $Y := \{x \in B - C \mid d(x, C) = d(x, C_1)\}$ and $Z :=$ the closure of $\{x \in B - C \mid x \notin Y$ and $d(x, C) = d(x, C_2)\}$ associated with $C_1$ and $C_2$. The adjustment of $g$ would be made first on $Y \cup C$, then on $Z \cup C$ and finally on the rest of $B - C$. Thus $g$ would be homotopic, rel $\partial B$, to a map whose image misses $\mathcal{U}(4)$.

Enough has been said here to indicate that the same is true for $\mathcal{R} = [m]$; namely, if $U(1)$ is a basis element of $X_1$ then $\uparrow U(1)$ is weakly contractible.

To get the same result for an arbitrary finite poset $\mathcal{R}$, we need to order its elements with care. Let $\mathcal{R}_0$ denote the set of minimal elements of $\mathcal{R}$. For $i > 0$ let

$$\mathcal{R}_i := \{r \in \mathcal{R} \mid \text{the longest chain connecting r to an element of } \mathcal{R}_0 \text{ has length } i\}.$$  

We choose a total ordering of $\mathcal{R}$ so that, for all $i$, every element of $\mathcal{R}_i$ comes before any element of $\mathcal{R}_{i+1}$. We call this the *useful ordering* to distinguish it from the given partial ordering on $\mathcal{R}$. The useful ordering ensures a key point: that for any $r$, $\bigcup\{X_s \mid s \text{ precedes } r \text{ in the useful ordering}\}$ is closed in the Up topology on $\mathcal{X}$; this is because elements indexed by the same $i$ are incomparable.

We are to show that, for any basis element $U^{(s)}$ in $X_s$, $\uparrow U^{(s)}$ is weakly contractible. There is no loss of generality in assuming $s \in \mathcal{R}_0$. Let $m$ be the maximal element in the useful ordering such that $s \leq m$. We consider a map $g : (B, \partial B) \to (\uparrow U^{(s)}, \uparrow U^{(s)} - U^{(m)})$. For the argument it is important that $g^{-1}(\uparrow U^{(s)} - U^{(m)})$ be compact, and this is assured by the “key point” in the previous paragraph.

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30 Compare the discussion of $U \in U_2$ in Section 4.3.

31 Compare the discussion preceding Proposition 4.15.
If the pieces $U^{(t)}$ with $s \leq t < m$ are taken in ascending order with respect to the useful ordering, then a generalization of the above discussion for $R = [3]$ and $R = [4]$ adjusts $g, \text{rel } \partial B$, to a map whose image misses $U^{(m)}$. In other words, for all $n$, $\pi_n(\uparrow U^{(s)}, \uparrow U^{(s)} - U^{(m)}) = 0$. The homotopy exact sequence then implies $\pi_n(\uparrow U^{(s)} - U^{(m)}) = 0$. By induction on $|R|$ this gives

**Theorem 6.1.** For all $n$ and $s$, $\pi_n(\uparrow U^{(s)}) = 0$.

Just as in Section 4.1 this leads to

**Corollary 6.2.** For all $s \in R$ the set $\uparrow U^{(s)}$ is weakly contractible.

### 6.2. Contractibility of pre-images of basis elements.

The Comparison Map $f : \Delta(\mathcal{X}) \to \mathcal{X}$ is, as before, $f(z) = \max\{x_i \mid i \in \text{supp}(z)\}$. Let $r \in R$ and let $U \subseteq X_r$ be a basis element.

**Proposition 6.3.** $f^{-1}(\uparrow U)$ is contractible.

**Proof.** We imitate Section 5 inducting on the cardinality of $R$. When $R$ is a singleton the proposition is trivially true. When $R$ has two elements the proposition has been proved. We assume $R$ has at least three elements.

There are two cases

First assume $r$ is a minimal element of $R$. Then

$$f^{-1}(\uparrow U) = \Delta(\uparrow U) \bigcup R^o$$

where $R^o$ is the union of deleted simplexes, each having some of its vertices outside $\uparrow U$ and the remaining vertices (at least one) in $\uparrow U$, the deletions being the faces lying outside $\uparrow U$.

Now, $\mathcal{Z} := \mathcal{X} - X_r$ is a sub-poset. From it we can construct a new poset $\mathcal{Y} := X_r \bigsqcup \Delta(\mathcal{Z})$ where the details imitate the construction given explicitly in Section 5.2. The Comparison Map is $g : \Delta(\mathcal{Y}) \to \mathcal{Y}$. Then

$$f^{-1}(\uparrow U) = g^{-1}(\uparrow U).$$

By induction and Corollary 4.16, $g^{-1}(\uparrow U)$ is contractible, so we conclude that $f^{-1}(\uparrow U)$ is contractible.

In the other case, $r$ is not minimal. Then there is a proper sub-$M$-poset $\uparrow X_r$ indexed by a proper subset of $R$. By induction on the cardinality of $R$, $g^{-1}(\uparrow U)$ is contractible, where $g$ denotes the Comparison Map for $\uparrow X_r$. Then

$$f^{-1}(\uparrow U) = g^{-1}(\uparrow U) \bigcup R^o$$

where $R^o$ is the union of deleted simplexes, each having some of its vertices outside $\uparrow U$ and at least one vertex in $\uparrow U$, the deletions being the faces lying outside $\uparrow U$. These deleted simplexes deform to $g^{-1}(\uparrow U)$. So $f^{-1}(\uparrow U)$ is contractible. \(\square\)

As before, we conclude:

**Corollary 6.4.** For basis elements $U$, $f^{-1}(\uparrow U)$ is contractible.

Together, Corollaries 6.2 and 6.4 conclude the proof of Theorem 1.1.

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32See “Proof of this implication” following the statement of Theorem 4.2.
7. **Appendix on the shape theory used in this paper.**

Shape theory is a variant of homotopy theory which works better than regular homotopy theory when the spaces are not locally nice. Here we give a brief review with the sole purpose of amplifying our use of the theory in Section 4.3.

Shape theory considers a compact metrizable space $X$ by dealing instead with an arbitrary inverse sequence of compact polyhedra and maps whose inverse limit is homeomorphic to $X$.

$$X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \cdots$$

In the following infinite diagram of space and maps, the spaces are compact polyhedra. We denote the inverse limit of the top line [resp. bottom line] by $X$ [resp.$Y$]. The outer squares are assumed to be homotopy commutative. If there exist diagonal maps making the entire diagram homotopy commutative then $X$ and $Y$ are said to be *shape equivalent*.

$$
\begin{array}{c}
X_1 \\
X_2 \\
X_3 \\
\vdots
\end{array}
\begin{array}{c}
Y_1 \\
Y_2 \\
Y_3 \\
\vdots
\end{array}
\begin{array}{c}
X_1 \xleftarrow{h_1} X_2 \xleftarrow{h_2} X_3 \xleftarrow{h_3} \cdots
\end{array}
$$

Now consider the special case in which all the maps $h_i$ are homotopy equivalences. Then we have the following homotopy commutative diagram, where $h_i^{-1}$ stands for a homotopy inverse of $h_i$:

$$
\begin{array}{c}
X_1 \xleftarrow{id} X_1 \\
\downarrow h_1 \downarrow h_1 \\
X_1 \xleftarrow{id} X_2 \xleftarrow{id} X_3 \xleftarrow{id} \cdots
\end{array}
$$

We conclude that in this case $X$ is shape equivalent to the compact polyhedron $X_1$. Moreover, in the application in Section 4.3, $X$ itself is known to be a compact polyhedron.

It is a basic theorem of shape theory that two polyhedra are shape equivalent if and only if they are homotopy equivalent.

In the context of Theorem 4.14 this discussion explains why, for $A$ compact, $\Delta(\uparrow A)$ is homotopy equivalent to each $X_n$ and hence to $f^{-1}(\uparrow A)$.

8. **Appendix on the discrete case and McCord’s work.**

A quick review of the present paper shows that the proof of the Comparison Theorem 1.1 becomes very easy in the case when each $X_r$ is discrete. Confining ourselves to the special case dealt with in Section 4, the relevant version of Proposition 4.14 becomes almost trivial, since the issue of limit points (which required a shape theoretic argument) is not present, and the need for Proposition 4.15 is gone since pre-images of basic sets $\uparrow x$ are cones. The issues dealt with in Section 4.1 become trivial because Proposition 4.3 implies that each singleton $\{x\}$ is a strong deformation retract of the set $\uparrow x$.

McCord’s Theorem 3 in [McC66] implies our Comparison Theorem 1.1 in this discrete case, and his proof is indicated by the contents of the previous paragraph. It should be added, however, that his definition of $\Delta(\mathcal{X})$ is superficially different from ours. In his work,

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33 This is not the general definition but is enough for our purposes. For more details on what is discussed in this Appendix, see [Gui16].
the role of $\Delta(\mathcal{X})$ is played by $|K(\mathcal{X})|$, where $K(\mathcal{X})$ is the classical order complex defined by the poset $\mathcal{X}$, namely: simplexes are finite chains in the poset. However, it is clear that $|K(\mathcal{X})|$ and $\Delta(\mathcal{X})$ are the same in our setting when $\mathcal{X}$ is discrete.

McCord uses the “Down topology” rather than the “Up topology” but that is of no consequence as the two are dual to one another and order complexes are preserved under duality.

It should be added that McCord’s Theorem 3 covers cases outside the setting of the present paper.

9. Appendix on homotopy equivalence

In Theorem 4.1 we showed that for geometric $M$-posets $\mathcal{X}$ the Comparison Map $f: \Delta(\mathcal{X}) \to \mathcal{X}$ is a weak homotopy equivalence. The Whitehead Theorem implies that $f$ is a homotopy equivalence if and only if $\mathcal{X}$ has the homotopy type of a $CW$ complex. In this appendix we show that every member of an important class of topological posets (with the Up topology) does not have the homotopy type of a $CW$ complex.

Let $\mathcal{X}$ be a topological poset, not necessarily geometric; it is considered with the Up topology. Define an equivalence relation on $\mathcal{X}$ by $p \sim q$ if and only if there is a finite sequence $(x_i)$ in $\mathcal{X}$ with $x_0 = p$, $x_n = q$, and, for each $i \geq 0$, either $x_i \leq x_{i+1}$ or $x_i \geq x_{i+1}$. By a proof similar to that of Proposition 4.3 each equivalence class is seen to lie in a path component of $\mathcal{X}$. We say that $\mathcal{X}$ has discrete type if each equivalence class is an entire path component.

Example. For each $n$ the real Grassmann Poset $G_n^p\mathbb{R}^q$ has discrete type.

Lemma 9.1. Each equivalence class is a closed subset of $\mathcal{X}$.

Proof. The relation $\mathcal{P}$ in the definition of topological poset is closed in $\mathcal{X} \times \mathcal{X}$, so the condition of not being related is an open condition. It follows that every limit point of an equivalence class lies in that equivalence class.

For any space $Y$ the space of path components, $\pi_0(Y)$, occurs as a quotient space in a natural way. When $Y$ is a $CW$ complex this is a discrete space.

Theorem 9.2. Let $\mathcal{X}$ have discrete type. Then $\mathcal{X}$ has the homotopy type of a $CW$ complex if and only if each path component is contractible, and the space of path components $\pi_0(\mathcal{X})$ is discrete.

Proof. Let $h: K \to \mathcal{X}$ be a homotopy equivalence, where $K$ is a $CW$ complex. Suppose there is a homotopy inverse $g$ for $h$. The fact that $g$ is continuous shows that if $x < y$ then $g(y) = g(x)$; hence $g$ is constant on equivalence classes (i.e. on path components). The map $g$ induces a continuous bijection $\pi_0(\mathcal{X}) \to \pi_0(K)$, so $\pi_0(\mathcal{X})$ is discrete. And since $h \circ g$ is homotopic to the identity map, each path component of $\mathcal{X}$ must be contractible. Conversely, the hypotheses imply that $\mathcal{X}$ has the homotopy type of a discrete space.

Remark. It follows from Theorem 9.2 that equivalence classes are closed. We include Lemma 9.1 because this is true whether or not $\mathcal{X}$ has discrete type. We wonder whether Theorem 9.2 holds for all topological posets.

34The topological poset $\mathcal{X}$ has an underlying discrete poset which we denote by $\mathcal{X}_d$, equipped with the Up topology. Comparing $\mathcal{X}_d$ with its (classical) order complex, one easily sees that each equivalence class is a path component of $\mathcal{X}_d$. This explains the term “discrete type”.

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References

[AD19] Laura Anderson and James F. Davis, Hyperfield Grassmannians, Adv. Math. 341 (2019), 336–366. MR 3872850

[Alv] Ulysses Alvarez, On the topology of corank 1 tropical phased matroids – PhD dissertation, Binghamton University, 2022.

[And22] Laura Anderson, Vectors of matroids over tracts, J. Combin. Theory Ser. A 161 (2019), 236–270. MR 3861778

[Bar11a] Jonathan Ariel Barmak, Algebraic topology of finite topological spaces and applications, Lecture Notes in Mathematics, vol. 2032, Springer, Heidelberg, 2011. MR 3024764

[Bar11b] ______, On Quillen's Theorem A for posets, J. Combin. Theory Ser. A 118 (2011), no. 8, 2445–2453. MR 2834186

[BB] M. Baker and N. Bowler, Matroids over hyperfields, arXiv:1601.01204, 2017.

[BB19] Matthew Baker and Nathan Bowler, Matroids over partial hyperstructures, Adv. Math. 343 (2019), 821–863. MR 3891757

[Bis03] Daniel K. Biss, The homotopy type of the matroid Grassmannian, Ann. of Math. (2) 158 (2003), no. 3, 929–952. MR 2031856

[Gui16] Craig R. Guilbault, Ends, shapes, and boundaries in manifold topology and geometric group theory, Topology and geometric group theory, Springer Proc. Math. Stat., vol. 184, Springer, [Cham], 2016, pp. 45–125. MR 3598162

[Hir75] Heisuke Hironaka, Triangulations of algebraic sets, Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974), Amer. Math. Soc., Providence, R.I., 1975, pp. 165–185. MR 0374131

[Huf65] Sze-tsen Hu, Theory of retracts, Wayne State University Press, Detroit, 1965. MR 0181977

[Hud69] J. F. P. Hudson, Piecewise linear topology, University of Chicago Lecture Notes prepared with the assistance of J. L. Shaneson and J. Lees, W. A. Benjamin, Inc., New York-Amsterdam, 1969. MR 0248844

[Lac69] R. C. Lacher, Cell-like mappings. I, Pacific J. Math. 30 (1969), 717–731. MR 251714

[May06] J. Peter May, Weak equivalences and quasifibrations, Lecture Notes in Math. 1425 (2006), 91–101.

[McC66] Michael C. McCord, Singular homology groups and homotopy groups of finite topological spaces, Duke Math. J. 33 (1966), 465–474. MR 196744

[McC67] ______, Homotopy type comparison of a space with complexes associated with its open covers, Proc. Amer. Math. Soc. 18 (1967), 705–708. MR 216499

[RS72] C. P. Rourke and B. J. Sanderson, Introduction to piecewise-linear topology, Springer-Verlag, New York-Heidelberg, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 69. MR 0350744

[Spa95] Edwin H. Spanier, Algebraic topology, Springer-Verlag, New York, [1995?], Corrected reprint of the 1966 original. MR 1325242

[Vas91] V. A. Vassiliev, Geometric realization of the homology of classical Lie groups, and complexes that are S-dual to flag manifolds, Algebra i Analiz 3 (1991), no. 4, 113–120. MR 1152604

[Vas92] ______, Complements of discriminants of smooth maps: topology and applications, Translations of Mathematical Monographs, vol. 98, American Mathematical Society, Providence, RI, 1992, Translated from the Russian by B. Goldfarb. MR 1168473

[Vir10] Oleg Viro, Hyperfields for tropical geometry I. hyperfields and dequantization, 2010, arXiv:1006.3034.

[Z98] Rade T. Zivaljević, Combinatorics of topological posets: homotopy complementation formulas, Adv. in Appl. Math. 21 (1998), no. 4, 547–574. MR 1652178

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