Abstract. In this paper, we focus on the following question. Assume \( \phi \) is a discrete Gaussian free field (GFF) on \( \Lambda \subset \mathbb{1} \mathbb{Z}^2 \) and that we are given \( e^{iT\phi} \), or equivalently \( \phi \mod \frac{2\pi}{T} \). Can we recover the macroscopic observables of \( \phi \) up to \( o(1) \) precision? We prove that this statistical reconstruction problem undergoes the following Kosterlitz-Thouless type phase transition:

- If \( T < T_{\text{rec}}^{-} \), one can fully recover \( \phi \) from the knowledge of \( \phi \mod \frac{2\pi}{T} \). In this regime our proof relies on a new type of Peierls argument which we call \textit{annealed} Peierls argument and which allows us to deal with an unknown \textit{quenched} groundstate.

- If \( T > T_{\text{rec}}^{+} \), it is impossible to fully recover the field \( \phi \) from the knowledge of \( \phi \mod \frac{2\pi}{T} \). To prove this result, we generalise the delocalisation theorem by Fröhlich-Spencer to the case of integer-valued GFF in an inhomogeneous medium. This delocalisation result is of independent interest and an interesting connection with Riemann-theta functions is drawn along the proof.

This statistical reconstruction problem is motivated by the two-dimensional XY and Villain models. Indeed, at low-temperature \( T \), the large scale fluctuations of these continuous spin systems are conjectured to be governed by a Gaussian free field. It is then natural to ask if one can recover the underlying macroscopic GFF from the observation of the spins of the XY or Villain model.

Another motivation for this work is that it provides us with an “integrable model” (the GFF) subjected to a KT transition.

1 Introduction

Figure 1. If you are given the values of a function \( f \) modulo 1 (left), can you reconstruct what \( f \) is (right)? If \( f \) is smooth as in this example, sure you may. But what if \( f \) is an instance of a 2d Gaussian free field? Analyzing this statistical reconstruction problem is the aim of this paper.

1.1 Main result We work on the graph \( \Lambda_n := [-1,1]^2 \cap \frac{1}{n} \mathbb{Z}^2 \) and for functions \( f, g : \Lambda_n \mapsto \mathbb{R} \), we denote

\[
\langle f, g \rangle := \sum_{x \in \Lambda_n} f(x)g(x).
\]
For each $n \geq 1$, $\phi_n$ will denote a GFF\footnote{With either free or Dirichlet boundary condition. We introduce all the relevant definitions in Section 2.} on $\Lambda_n$. Recall that for any smooth function $f : [-1,1]^2 \mapsto \mathbb{R}$
\begin{equation}
\frac{1}{n^2} (\phi_n, f) \rightarrow (\Phi , f) \quad \text{in law as } n \to \infty ,
\end{equation}
where $\Phi$ is a continuous GFF in $[-1,1]^2$, and $(\Phi , f) := \int \Phi(x) f(x) dx$. This tells us that the macroscopic observables related to $\phi_n$, are random variables of the form $n^{-2} (\phi_n, f)$.

The main focus of this paper is to understand when we can recover the full macroscopic information of $\phi_n$ by just knowing $\exp(iT\phi_n)$, or equivalently, $\phi_n$ (mod $\frac{2\pi}{n}$). We will give several motivations which lead us to consider this problem later in Section 1.3. We now state our main result which shows that this statistical reconstruction problem undergoes a phase transition as $T$ varies, which is reminiscent of the Berezinskii-Kosterlitz-Thouless transition (BKT transition) (see Section 2.3).

**Theorem 1.1.** Let $\phi_n$ be a GFF on $\Lambda_n$ with Dirichlet boundary conditions. Then, there exists $0 < T_{\text{rec}}^- \leq T_{\text{rec}}^+ < \infty$ such that

(a) If $T < T_{\text{rec}}^-$, there exists a (deterministic) reconstruction function $F_T$ such that for any continuous function $f : [-1,1]^2 \mapsto \mathbb{R}$ and any $\varepsilon > 0$,
\[ \mathbb{P} \left[ |n^{-2}(F_T(\exp(iT\phi_n)) - \phi_n, f)| \geq \varepsilon \right] \rightarrow 0, \quad \text{as } n \to \infty . \]
Furthermore, uniformly in $n$ there exists a constant $C > 0$ s.t. for any point $x \in \Lambda_n \subset [-1,1]^2$
\[ \mathbb{E} \left[ (F_T(\exp(iT\phi_n)) - \phi_n(x))^2 \right] \leq C . \]

(b) If $T > T_{\text{rec}}^+$, for any (deterministic) function $F$ and any continuous non-zero function $f$, there exists $\delta > 0$ such that
\[ \liminf_{n \to \infty} \mathbb{P} \left[ |n^{-2}(F(\exp(iT\phi_n)) - \phi_n, f)| \geq \delta \right] > 0 . \]

Also, for any $x \in (-1,1)^2$, there exists $c = c(T,x) > 0$ s.t. for any $F$
\[ \liminf_{n \to \infty} \mathbb{E} \left[ (F(\exp(iT\phi_n))(x) - \phi_n(x))^2 \right] \geq c(T,x) \log n . \]

The same result holds for a free boundary condition GFF.

**Theorem 1.2.** Let $\phi_n$ be a GFF on $\Lambda_n := [-1,1]^2 \cap \frac{1}{n} \mathbb{Z}^2$ with free boundary conditions and rooted at a vertex $x_0 \in \Lambda_n$. Then, there exists $0 < T_{\text{rec}}^- \leq T_{\text{rec}}^+ < \infty$ such that

(a) If $T < T_{\text{rec}}^-$, there exists a reconstruction function $F_T$, s.t. for any smooth function $f$ with 0-mean (i.e., $\int_{[-1,1]^2} f = 0$) and any $\varepsilon > 0$,
\[ \mathbb{P} \left[ |n^{-2}(F_T(\exp(iT\phi_n)) - \phi_n, f)| \geq \varepsilon \right] \rightarrow 0, \quad \text{as } n \to \infty . \]

(b) If $T > T_{\text{rec}}^+$, for any function $F$ and any smooth non-zero function $f$ with 0-mean there exists $\delta > 0$ such that
\[ \liminf_{n \to \infty} \mathbb{P} \left[ |n^{-2}(F(\exp(iT\phi_n)) - \phi_n, f)| \geq \delta \right] > 0 . \]

We now state two Corollaries of the above theorems. The first one rephrases this phase-transition in terms of the continuum GFF. The second one (which will give support to conjectures 3 and 4) shows that one can recover macroscopic interfaces from $\phi$ (mod $\frac{2\pi}{n}$) when $T < T_{\text{rec}}^-$.
Corollary 1.3. Let $\phi_n$ be a sequence of GFF in $\Lambda_n$, such that, in probability, $\phi_n \to \Phi$ a continuum GFF in $[-1,1]^2$. Then, if $T < T_{\text{rec}}^*$, the function $F_T(e^{iT\phi_n})$ converges in probability to $\Phi$. Furthermore, if $T > T_{\text{rec}}^*$ there is no deterministic function $F$ such that $F(e^{iT\phi_n}) \to \Phi$.

Corollary 1.4. Let $\phi_n$ be a sequence of GFF in $\Lambda_n$, let $\eta^{(n)}$ be the Schramm-Sheffield level line$^2$ of $\Phi^{(n)}$. Then, there exists a deterministic function $L_T$, such that the Hausdorff distance between $L_T(\exp(iT\phi_n))$ and $\eta^{(n)}$ is $o(1)$. In particular, $L_T(\exp(iT\phi_n))$ converges in law to an SLE$_4$.

Our work naturally belongs to the class of statistical reconstruction problems which have been the subject of an intense activity recently. For example it shares similarities with the statistical reconstruction problems analyzed in [HS13, PS14, AMM$^+$17]. In particular in the later work, Groups synchronization on grids, the authors analyze the following problem: Imagine that each site $x \in \mathbb{Z}^d$ carries a spin or group element $\theta_x \in \mathcal{G}$, a compact group (for example $\{\pm 1\}$ or $O(n)$). The question they are interested in is the following one: what macroscopic information on $\{\theta_x\}_{x \in \mathbb{Z}^d}$ can be recovered from the knowledge of

$$\{\theta_i \theta_j^{-1} + \text{noise}\}_{1 \leq i \neq j, \text{edges of } \mathbb{Z}^d}$$

where observations of neighboring spins $\theta_i \theta_j^{-1}$ are subjected to a small noise. Our setting is very similar in flavour as we also have access to $\phi_n(i) - \phi_n(j)$ when $i \sim j$ except the noise term is replaced in our case by the modulo operation $(\mod \frac{2\pi}{n})$. Similarly as adding a noise term, applying $(\mod \frac{2\pi}{n})$ is also reducing the information we have on $\phi_n(i) - \phi_n(j)$, except it cannot be analyzed as a convolution effect. The second difference with [AMM$^+$17] is that our spins belong to $\mathbb{R}$ instead of a compact group $\mathcal{G}$.

1.2 Fluctuations for integer-valued fields. Our present statistical reconstruction problem is intimately related to a generalization of the integer-valued Gaussian free field which plays a key role in the proof of the BKT transition for the Villain and XY models in [FS81]. Let us briefly recall the classical integer-valued GFF before introducing its generalisation.

For simplicity, in this subsection as well as in Sections 2.3, 4 and Appendix A, we will consider an arbitrary finite subset $\Lambda \subset \mathbb{Z}^2$, instead of the scaled box $\Lambda_n = \frac{1}{n} \mathbb{Z}^2 \cap [-1,1]^2$. This way, it matches the setup in [FS81, KP17].

Definition 1.5. Let $\Lambda \subset \mathbb{Z}^d$ be a finite domain$^3$. The integer-valued GFF (IV-GFF) on $\Lambda$ with Dirichlet-boundary conditions, i.e. 0 on $\partial \Lambda$, and inverse-temperature $\beta$ is the $\beta$-GFF $\{\phi(i)\}_{i \in \Lambda}$ conditioned on the singular event $\{\phi(i) \in \mathbb{Z}, \forall i \in \Lambda\}$. Equivalently, it can be defined as the probability measure $\mathbb{P}^{\text{IV}}_{\beta,\Lambda}$ on $\mathbb{Z}^\Lambda$ defined as follows:

$$\mathbb{P}^{\text{IV}}_{\beta,\Lambda}(d\phi) := \frac{1}{Z} \sum_{\mathbf{m} \in \mathbb{Z}^\Lambda : m_{\partial\Lambda} = 0} \delta_{\mathbf{m}}(d\phi) \exp\left(-\frac{\beta}{2} \langle \nabla \phi, \nabla \phi \rangle \right) \quad (1.4)$$

or also, to avoid any possible confusion, for any $\mathbf{m} \in \mathbb{Z}^\Lambda$ with 0 boundary conditions, we have that $\mathbb{P}^{\text{IV}}_{\beta,\Lambda}(\phi = \mathbf{m}) = \frac{1}{Z} \exp(-\frac{\beta}{2} \langle \nabla \mathbf{m}, \nabla \mathbf{m} \rangle)$.

The IV-GFF with free boundary-conditions is defined in the same manner except we replace $\mathbf{m}_{\partial\Lambda} \equiv 0$ by $\mathbf{m}(x_0) = 0$ for any choice of root vertex $x_0 \in \Lambda$.

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$^2$For the definition and the context used in this conjecture see Section 6.2

$^3$Again, the GFF as well as the graph notations $\partial\Lambda$ etc. are defined in Section 2.
This integer-valued undergoes a roughening-phase transition as $T$ increases (i.e. as $\beta$ decreases) as it was proved by Fröhlich-Spencer in [FS81] (see also the very useful survey [KP17]). Fröhlich-Spencer proved this striking phase transition for periodic and free boundary conditions on large square boxes $\Lambda$ and explained in [FS81, Appendix D.] how to adapt their proof to the case of Dirichlet boundary conditions. Very recently, Wirth has written carefully in [Wir19, Appendix A] the details of this extension to Dirichlet boundary conditions. We will come back to it later in Section 2.3. (See also Figure 2 for an illustration of the IV-GFF in $d = 1$).

**Theorem 1.6 (Fröhlich-Spencer [FS81]).** There exists $0 < \beta^+ \leq \beta^- < \infty^4$ such that for any square $\Lambda \subset \mathbb{Z}^2$, if we consider the IV-GFF with free boundary conditions rooted at $x_0 \in \Lambda$ then we have the following dichotomy:

- **Delocalised regime (rough regime).** If $\beta < \beta^+$, then for any $f : \Lambda \to \mathbb{R}$, $\sum_{i \in \Lambda} f(i) = 0$,
  
  \[ \mathbb{E}^{IV}_{\beta,\Lambda} \left[ e^{\langle \phi, f \rangle} \right] \geq e^{\frac{C}{\beta} T \left( -\Delta \right)^{-1} f} \]

  (N.B. it is not hard to extract from this Laplace transform estimate, fluctuations bounds such as $\mathbb{E}^{IV}_{\beta,\Lambda} \left[ (\phi(x))^2 \right] \geq \frac{\beta}{2} \log \| x \|_2$ for any $x \in \{-n, \ldots, n\}^2$, see for example [KP17]).

- **Localised regime.** If $\beta > \beta^-$, then for any $x \in \Lambda$,
  
  \[ \mathbb{E}^{IV}_{\beta,\Lambda, x_0} \left[ (\phi(x))^2 \right] \leq \frac{C}{\beta}. \]

The relationship between our statistical reconstruction problem and integer-valued fields is due to the following explicit structure of the conditional law of a GFF given its values modulo $\frac{2\pi}{T}$. We stick for simplicity to the case of Dirichlet boundary conditions. Let us fix $a \in [0, 1)^3$ satisfying $a_{i \partial \Lambda} \equiv 0$. We will see in Lemma 2.8 that the conditional law of the GFF $\phi$ on $\Lambda$ given $\phi$ (mod $\frac{2\pi}{T}$) is a multiple of the following **generalized integer-valued GFF** with $\beta = \beta_T := (2\pi)^2 T^{-2}$. (See Lemma 2.8 for a precise statement).

**Definition 1.7.** Let $\beta > 0$ and $a = \{a_i\}_{i \in \Lambda}$ s.t. $a_{i \partial \Lambda} \equiv 0$ be any collection of real-valued numbers. We define the $a$-$IV$-$GFF$ on $\Lambda$ to be the GFF $\{\phi(i)\}_{i \in \Lambda}$ (with Dirichlet b.c.s) conditioned to take its values in the shifted-fibers $\{a_i + \mathbb{Z}\}_{i \in \Lambda}$ for any $i \in \Lambda$. It corresponds to the following discrete probability measure on fields:

\[ \mathbb{P}^{a,IV}_{\beta,\Lambda}[\phi] := \frac{1}{Z} \sum_{\mathbf{m} \in \mathbb{Z}^\Lambda, \mathbf{m}_{i \partial \Lambda} = 0} \delta_{\mathbf{m}+a}(\phi) \exp\left( -\beta \frac{1}{2} \langle \nabla(\phi), \nabla(\phi) \rangle \right). \]

Equivalently, for any $\mathbf{m} \in \mathbb{Z}^\Lambda$ with $\mathbf{m}_{i \partial \Lambda} \equiv 0$,

\[ \mathbb{P}^{a,IV}_{\beta,\Lambda}[\phi = \mathbf{m} + a] = \frac{1}{Z} \exp\left( -\beta \frac{1}{2} \langle \nabla(\mathbf{m} + a), \nabla(\mathbf{m} + a) \rangle \right). \] \hspace{1cm} (1.5)

Notice that if $a \in \mathbb{Z}^\Lambda$, then the $a$-$IV$-$GFF$ is nothing but the standard IV-GFF. See Figures 2.3. Finally, this definition extends readily to the case of free boundary conditions in which case $a \in \mathbb{R}^\Lambda$ with $a_{x_0} = 0$.

The proof of Fröhlich-Spencer ([FS81]) readily extends to some specific choices of the shift $a$ which are sufficiently symmetric. (i.e. any $a \in \{0, \frac{1}{2}\}^3$). See Figure 2 for an illustration (in $d = 1$ only) of the cases which can be analyzed using the techniques from [FS81] and Figure 3 for the cases which need further analysis. See

\footnote{This choice $\beta^+ \leq \beta^-$ is made to highlight that these are inverse temperatures related to $T_{rec} \leq T_{rec}$.}
Figure 2. The picture on the left represents an instance of an IV-GFF in $d = 1$ on a unit interval $\{0, \ldots, n\}$ with Dirichlet Boundary conditions. In $d = 2$, the proof of the roughening phase transition for the IV-GFF in [FS81] (and [Wir19] for the extension to Dirichlet b.c.) easily generalizes to certain shifts and scaled versions on the vertical fibers $\mathbb{Z}$ as illustrated in the other pictures. In the middle one, some fibers are $\mathbb{Z}$, some others are $2\mathbb{Z}$ and one may also add fibers with arbitrarily fine meshes $2^{-k}\mathbb{Z}$ along the interval. On the right, some fibers are $\mathbb{Z}$ while some others are $\frac{1}{2} + \mathbb{Z}$. It is easy to check that in $d = 2$, any of these can be handled with the techniques from [FS81].

also Remark 2.7. Our main result on such integer-valued fields is the following extension of the above theorem of Fröhlich and Spencer [FS81].

Theorem 1.8. There exists $\beta_{IV} > 0$ and a constant $C > 0$, s.t. for any square domain $\Lambda \subset \mathbb{Z}^2$, any $\beta < \beta_{IV}$, then uniformly$^5$ in $a \in \mathbb{R}^\Lambda$, if $\phi^a \overset{\text{law}}{\sim} P_{a,\beta,\Lambda}^{\mathbb{IV}}$ (with either Dirichlet or free b.c.s) we have

- For any function $f \in \mathbb{R}^\Lambda$
  $$\text{Var}[(\phi^a, f)] \geq \frac{C}{\beta}(f, (-\Delta)^{-1}f),$$
  where the inverse of the Laplacian is taken here according to the b.c.
- If $\Lambda = \{-n, \ldots, n\}^2$ with Dirichlet boundary conditions, the variance of the field $\phi^a$ at the origin satisfies
  $$\text{Var}[\phi^a(0, 0)] \geq \frac{C}{\beta} \log n.$$

The analogous statement also holds for free b.c.

Remark 1.9. We wish to stress that the low-temperature regime ($\beta \gg 1$) happens to be much less universal in the choice of the shift $a$. Indeed, when $\Lambda = \{-n, \ldots, n\}^2$ equipped with Dirichlet boundary conditions, and if $\phi^a \overset{\text{law}}{\sim} P_{a,\beta,\Lambda}^{\mathbb{IV}}$ then we expect that the following different scenarios may happen (by tuning suitably $a$ in each case) as $n \to \infty$. (See for example Figure 5 for the scenario (1)).

1. $E[\phi^a(0, 0)] \geq 0.49n$
2. $\text{Var}[\phi^a(0, 0)] \geq (0.49)^2n$
3. $\text{Var}[\phi^a(0, 0)] \leq O(1)$ and $\text{Cov}[\phi^a(x), \phi^a(y)] \leq e^{-cn\|x-y\|^2}$.

Remark 1.10. We do not obtain a lower bound on the Laplace-transform of $P_{a,\beta}^{\mathbb{IV}}$ only on its $L^2$ behaviour which is sufficient to detect localisation v.s. delocalisation.

$^5$with $a_{\partial \Lambda} \equiv 0$ if Dirichlet b.c. and $a_{\partial \Lambda} = 0$ for free b.c.
Figure 3. As opposed to the examples in Figure 2, those choices of fibers do not satisfy the sufficient symmetries to be readily analyzed by the techniques from [FS81]. The simplest such example is the picture on the left where \( \mathbb{Z} \) fibers are shifted by \( \{0, \frac{1}{3}, \frac{2}{3}\} \) (here \( a := \left( \frac{k}{3} \right)_{k \in \mathbb{Z}} \)). In such a case, \( \sin(k\phi) \) functions appear in the Fourier transform of the periodic distribution \( \sum_{i \in \mathbb{Z}} \delta_{i} \) and this breaks the parity. For such linear shifts, Wirth obtained in [Wir19] some related bounds using a nice symmetrization trick (however a different control from the one we need here, (see Remark 2.7). A key property used in [Wir19] is the fact that \( x \mapsto ax \) needs to be harmonic in \( \Lambda \setminus \partial \Lambda \). Otherwise the symmetrization technique breaks and one cannot rely anymore on Jensen’s inequality, a key step in the proof [FS81]. For example the picture on the right where fibers are shifted by a quadratic curve requires an additional analysis w.r.t \( [FS81] \).

This is also the case in the recent works mentioned below on localisation/delocalisation of integer-valued random surfaces.

As we will see in Section 4 and particularly in Appendix A, our proof of Theorem 1.8 involves an exact identity (Proposition (4.1)) which is closely related to the modular invariance identity for Riemann-theta functions. We briefly mention this connection here as it is interesting in its own and it allows us to rephrase Fröhlich-Spencer Theorem as well as our Theorem 1.8 easily in terms of those Riemann-theta functions.

Indeed, the following function of \( a \in \mathbb{R}^{\Lambda \setminus \partial \Lambda} \):

\[
\tilde{\theta}_\Lambda(a) := \sum_{m \in \mathbb{Z}^{\Lambda \setminus \partial \Lambda}} \exp\left( -\frac{\beta}{2} \langle m, (-\Delta) m \rangle \right) \exp(-\beta m \cdot a)
\]

can be easily written in terms of the classical Riemann-theta function \( \theta(z \mid \Omega) \) (see (A.4)). Furthermore, one can check that for any \( f : \Lambda \to \mathbb{R} \) and if \( \phi^a \sim \mathcal{P}_{\beta,\Lambda} \), we have

\[
\text{Var}[\langle \phi^a, f \rangle] = [\sigma \cdot \nabla a \cdot \nabla a] \log \tilde{\theta}_\Lambda,
\]

where \( \sigma := \frac{1}{\beta} (-\Delta)^{-1} f \). This expression clarifies the effect of the shift-vector \( a \in \mathbb{R}^{\Lambda} \) and reveals that it plays the role of an external magnetic field. We may now rephrase Fröhlich-Spencer as well as our main result from this Section as follows:

- (Theorem 1.6). If \( \beta \) is small enough, then uniformly in \( \Lambda = \{-n, \ldots, n\}^2 \),

\[
[\sigma \cdot \nabla \sigma \cdot \nabla]_{a=0} \log \tilde{\theta}_\Lambda \geq \frac{C}{\beta} \langle f, (-\Delta)^{-1} f \rangle
\]
• (Theorem 1.8). If $\beta$ is small enough, then uniformly in $\Lambda = \{-n, \ldots, n\}^2$,
\[
\inf_{a \in \mathbb{R}, \Lambda} [\sigma \cdot \nabla \sigma \cdot \nabla] a \log \tilde{\theta}_\Lambda \geq \frac{C}{\beta} \langle f, -\Delta^{-1} f \rangle
\]
Finally, let us point out that over the last few years, there have been several important works which analyzed the roughening phase transition (i.e. localization/delocalisation) for other natural models of integer-valued random fields, such as the square-ice model, uniform Lipschitz functions $\mathbb{Z}^2 \to \mathbb{Z}$ etc: see in particular the recent works [DCGPS17, CPST18, GM18, DCHL+19]. These works do not rely on the Coulomb-gas techniques from [FS81] but rather on geometric techniques such as RSW.

1.3 Motivations behind this statistical reconstruction problem.
As we will see below, one of the main reasons which lead us to consider this statistical reconstruction problem on the GFF has to do with the statistical analysis of the XY and Villain models in $d = 2$. Each of these are celebrated models with continuous $O(2)$-symmetry. We briefly define what they are and we refer the reader to [FS81, Bau16, KP17, FV17] for useful background on these models.

**Definition 1.11 (Villain and XY models).** Let us fix a finite graph $\Lambda \subset \mathbb{Z}^2$ and $\beta > 0$ to be the inverse temperature. Both models are Gibbs measures on the state-space $(S^1)^\Lambda$. Let us parametrise this spin-space via its canonical identification with $[0, 2\pi)^\Lambda$.

- **XY model (or plane rotator model)**
  \[
dP^\text{XY}_\beta [\{\theta_x\}_{x \in \Lambda}] \propto \prod_{i \sim j} \exp \left( \beta \cos(\theta_i - \theta_j) \right) \prod d\theta_i.
  \]

- **Villain model**
  \[
dP^\text{Villain}_\beta [\{\theta_x\}_{x \in \Lambda}] \propto \prod_{i \sim j} \sum_{m \in \mathbb{Z}} \exp \left( -\frac{\beta}{2} (2\pi m + \theta_i - \theta_j)^2 \right) \prod d\theta_i.
  \]

We may now list what are the main motivations which guided our work.

1. Extracting macroscopic random structures from XY and Villain spins.
   For spins systems such as the Ising model, Potts models or also percolation which all have discrete symmetries, it is clear how to associate natural macroscopic fluctuating objects such as interfaces which may then converge to suitable SLE$_\kappa$ as the mesh goes to zero. On the other hand, for spin systems with continuous symmetry such as XY or Villain models, given a realization of the Gibbs measure, say $\{\sigma_x\}_{x \in \Lambda}$ with $\sigma_x \in S^1$, it is much less clear what macroscopic objects one may assign to $\{\sigma_x\}$.

   One consequence of our present statistical reconstruction problem is that it gives strong evidence to the fact that it is possible to extract a macroscopic GFF $\phi_n$ from the observation of the spins $\{\sigma_x\}_{x \in \{-n, \ldots, n\}^2}$ (up to small microscopic errors).

   Indeed, at least in the case of the Villain model, it has been conjectured by Fröhlich-Spencer in [FS83, Section 8.1] that at low temperature ($\beta \gg 1$), then up to “microscopic errors”, one should have

   \[
   \{\sigma_x\}_{x \in \{-n, \ldots, n\}^2} \overset{\text{law}}{\sim} dP^\text{Villain}_\beta \approx \{ \exp \left( \frac{1}{\sqrt{\beta}} \phi_n(x) \right) \}_{x \in \{-n, \ldots, n\}^2},
   \]

   where $\beta' = \beta(\beta)$ satisfies $|\beta' - \beta| \leq e^{-C\beta}$ and where $\phi_n$ is a GFF on $\{-n, \ldots, n\}^2$ with either free or 0 b.c.
Figure 4. Conjectures 3 and 4 in Section 6.2 predict that at low temperature, the level lines of a Villain model with $e^{i/10}$ on the right boundary and $e^{-i/10}$ on the left boundary should converge to SLE($\kappa = 4, \rho$) processes. This conjecture is supported by the present statistical reconstruction problem as well as by the techniques we have used.

Once one realises that one may extract a GFF out of the spins $\{\sigma_x\}_{x \in \Lambda}$, it is then natural to extract level lines and flow lines from this GFF studied in [She05, Dub09, MS16a]. Corollary 1.4 is a proof of this concept. We discuss this further in Section 6.2 where we highlight how our work lead us to conjecture that when $\beta$ is high enough, then the natural interface for the Villain model pictured in Figure 4 should converge to an SLE($4, \rho$) process.

(2) A different interpretation of the KT transition.

The classical way of understanding the KT transition for spins systems such as the XY model is to notice that vortices ($\equiv$ discrete 2-forms) come into the energy-balance when analyzing the Gibbs measure (1.6). This present work gives the following different interpretation of the role of the $S^1$-geometry within the BKT transition which does not explicitly involve vortices. When the temperature $T$ is low, spins wiggle slowly around $S^1$ and one should be able to recover a macroscopic GFF as we have seen in the above item (1). If instead, the temperature $T$ is large, the spins start wiggling too quickly around $S^1$ so that one cannot extract the whole macroscopic fluctuating Gaussian field $\phi$ which leaves on the top of the spin field.

(3) An integrable model for Integer-valued GFF.

The main tool we use for the regime $T > T_{\text{rec}}^+$ is the proof of delocalisation for the generalized IV-GFF $\phi^a \sim P_A^{\text{IV}}$ from Definition 1.7. We think of $a$ as the random vector $\{a_i\}_{i \in \Lambda} \in [0, 1)^\Lambda$ defined by

$$a_i := \frac{T}{2\pi} \phi_i \pmod{1}, \quad \forall i \in \Lambda,$$

where $\phi$ is a GFF in $\Lambda$. As such, we may view the random measure $P_A^{\text{IV}}$ as a quenched measure on (shifted) integer-valued fields. Interestingly these highly non-trivial quenched measures have (by construction) a very
simple anealed measure. Indeed Lemma 2.8 readily implies that
\[
\int \mathbb{P}^\beta_{\mu,\Lambda}(d\phi) \mathbb{P}_T(da) = \mathbb{P}^{\text{GFF}}[d\phi],
\]
where we denoted by \(\mathbb{P}_T(da)\) the law of the above random shift \(a\) and 
\(\beta_T = \frac{(2\pi)^2}{T^2}\). If one now assumes that some properties (such as fluctuations) 
are not very sensitive to \(a\), this identity gives a “useful laboratory” to analyze 
the classical integer-valued GFF (i.e. \(a \equiv 0\)). An illustration of this is given 
in Section 5 where we provide a new insight on the \(\varepsilon = \varepsilon(\beta)\) correction in 
the bound of Fröhlich-Spencer.

(4) Imaginary multiplicative chaos.

In this work we focus on lattice fields \(\phi: \Lambda \to \mathbb{R}\) or \(\Lambda_n \to \mathbb{R}\), but the 
question in the continuum is also interesting. Namely, given a Gaussian 
free field \(\Phi\) on \([-1,1]^2\) with 0-boundary conditions, can one recover
\(\Phi\) from \(\exp(i\phi)\)? This is the complex analog of the reconstruction procedure 
\(\exp(i\phi) \mapsto \Phi\) studied in [BSS14]. We discuss this further in Section 6.1, where 
we show that the existence of a continuous reconstruction process in this 
case implies the existence of a discrete reconstruction process. However, 
let us highlight that even if this continuum process does exist, the discrete 
reconstruction process coming from it will converge much slower \((o(1))\) than 
the one we obtain in Theorem 1.1 using statistical mechanics \((O(1/n^2))\).

1.4 Idea of the proof.

The first choice one needs to made in the proof is the reconstruction function
\(F_T\). We have essentially two natural choices here (see Figure 5 for an illustration of both).

(1) First, if \(a := \frac{T}{2\pi} (\phi (\text{mod } 2\pi))\), then there is an a.s. unique ground-state 
for \(\mathbb{P}_\beta^\mu\) which we may call
\[
m(\exp(iT\phi)) = \hat{m}(\exp(2i\pi a))
\]
\[
:= \arg\min_{m \in \mathbb{Z}} \exp \left( -\frac{\beta}{2} (m + a, -\Delta(m + a)) \right)
\].

It is reasonable to guess that when \(T\) is small, the field \(\phi\) should not fluctuate 
much around \(\hat{m}(\exp(iT\phi))\).

(2) A second natural choice is to consider instead the conditional expectation of the field given \(\exp(iT\phi)\).

The quenched groundstate \(\hat{m}(\exp(iT\phi))\) does not have enough symmetries to 
apply classical tools from Peierls theory and they are too far from the perturbative 
regime where Pirogov-Sinai theory can be used (see [FV17, Chapter 7]).

Therefore, for the low-temperature regime in the proofs of Theorem 1.1 and 
1.2, we will see that one way to recover the GFF given its phase is to take the 
function second choice, i.e,
\[
F_T(\exp(iT\Phi))(x) := \mathbb{E}[\phi(x) \mid \exp(iT\phi)]
\].

It is not so easy to study this function \(F\) directly. However, for any test function \(f\) 
we can use Markov’s inequality to see that
\[
\mathbb{P}(|\phi - F(\exp(iT\Phi)), f) | \geq \varepsilon) \leq \frac{\mathbb{E}[\text{Var}(\phi, f) \mid \exp(iT\phi)]}{\varepsilon^2}.
\]

This implies that to understand how well \(F\) approximates \(\phi\) it is enough to bound 
the conditional variance of \((\phi, f)\) given \(\exp(iT\phi)\). Working with the conditional 
variance is much easier than to work with \(F\), as one can study it by coupling
two GFFs \((\phi_1, \phi_2)\) such that \(\exp(iT\phi_1) = \exp(iT\phi_2)\) in such a way that they are conditionally independent given \(\exp(iT\phi)\) (see Definition 3.1). This is useful because

\[
\mathbb{E}[\text{Var}[\langle \phi, f \rangle | \exp(iT\phi)]] = \frac{1}{2}\mathbb{E}\left[\langle \phi_1 - \phi_2, f \rangle^2\right].
\]  

As this function does not involve any estimate of the function \(F\), and both \(\phi_1\) and \(\phi_2\) have the law of a GFF, we set up an appropriate \textit{annealed Peierl’s argument} to show, in Section 3 that (1.9) is small when \(T\) is small.

The second part of Theorems 1.1 and 1.2, also follows from similar ideas with a “statistical flavour”. In fact, we are going to show that for any \(f\) there exists an \(\varepsilon > 0\) such that for all \(n\) big enough

\[
\mathbb{E}[\text{Var}[\langle \phi, f \rangle | \exp(iT\phi)]] \geq \varepsilon \text{Var}[\langle \phi, f \rangle].
\]  

This, together with some basic tension argument, implies that the probability that \(\langle \phi_1, f \rangle\) is macroscopically different from \(\langle \phi_2, f \rangle\) is uniformly positive.

To obtain equation (1.10), we need to modify the work of Fröhlich and Spencer [FS81]. In this seminal paper, the authors showed that the integer-valued GFF has variance similar to that of the GFF when the temperature is high enough. In our case, in Section 4 we will prove a result with a similar taste (Theorem 1.8) that will uniformly show that when \(T\) is high enough, for any realisation of \(\exp(iT\phi)\)

\[
\text{Var}[\langle \phi, f \rangle | \exp(iT\phi)] \geq \varepsilon \text{Var}[\langle \phi, f \rangle].
\]  

This is read, in the context of [FS81], as the study of \textit{integer-valued GFF in an inhomogeneous medium}.

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\textit{Figure 5.} Here \(a_k := \frac{k}{3}\) on the L.H.S of the picture and then the slope is \(-k/3\). It is easy to check that the ground-state \(\hat{m}\) for the GFF conditioned to have its values in these shifted fibers is given by the purple curve \(\beta = \infty\). Then, as \(\beta\) decreases, the expectation \(x \rightarrow \mathbb{E}_\beta^{a,iV}[\hat{a}(x)]\) can be seen to decrease to 0.
models of complex and disordered systems\footnote{for repeated invitation to Santiago were a part of this paper was written.}

2 Preliminaries

2.1 Discrete differential calculus. We start the preliminaries by discussing the basics of discrete differential calculus. As the only graph we work with in this paper is $\Lambda_n := [-1, 1] \cap \frac{1}{n} \mathbb{Z}^2$ with its canonical edge set, we only discuss the needed results in this framework. For simplicity we identify $\Lambda_n$ with its vertex set and we call and $E_{\Lambda_n}$ its edge set. For a deeper discussion on discrete differential calculus, we refer the reader to [Cha18].

In this section, we study two types of functions. Functions on vertices $S : \Lambda_n \rightarrow \mathbb{R}$ and functions on directed edges $A : \tilde{E} \rightarrow \mathbb{R}$. Functions on vertices can take any values, however functions on directed edges have to always satisfy that

$$A(\tilde{xy}) = -A(\tilde{yx}).$$

(2.1)

Let us now present two canonical differential operators

$$\nabla S(\tilde{xy}) = S(y) - S(x),$$

(2.2)

$$\nabla \cdot A(x) = \sum_{\tilde{xy}} A(\tilde{xy}).$$

(2.3)

Then, one can write the Laplacian of $S$ as follows

$$\Delta S(x) = \nabla \cdot \nabla S(x) = \sum_{y \sim x} S(y) - S(x).$$

(2.4)

For a pair of functions on vertices $S_1, S_2 : \Lambda_n \rightarrow \mathbb{R}$ and on edges $A_1, A_2 : \tilde{E} \rightarrow \mathbb{R}$, we define

$$\langle S_1, S_2 \rangle := \sum_{x \in V} S_1(x)S_2(x),$$

$$\langle A_1, A_2 \rangle := \frac{1}{2} \sum_{\tilde{xy}} A_1(\tilde{xy})A_2(\tilde{xy}).$$

Furthermore, we define

$$\langle S_1, S_2 \rangle_{\nabla} = \langle \nabla S_1, \nabla S_2 \rangle.$$

Let us remark that the differentials $\nabla$ and $-\nabla \cdot$ are dual between them, i.e.,

$$\langle \nabla S, A \rangle = \langle S, -\nabla \cdot A \rangle.$$  

(2.5)

Thanks to this, we can see easily that $-\Delta$ is a positive definite operator.

**Definition 2.1 (Inverse of the Laplacian).** We fix $\partial \Lambda_n \subseteq \Lambda_n$ and we call it boundary. If $\partial \Lambda_n \neq \emptyset$, then for any function $S_1 : \Lambda_n \setminus \partial \Lambda_n \rightarrow \mathbb{R}$ there is a unique function on the vertices $S_2 : \Lambda_n \rightarrow \mathbb{R}$ such that

$$-\Delta S_2(x) = S_1(x), \quad \forall x \in V \setminus \partial \Lambda_n$$

$$S_2(x) = 0, \quad \forall x \in \partial \Lambda_n.$$

In this case, we call $S_2 := (-\Delta^{-1})S_1$.

The inverse of the Laplacian operator can be understood thanks to the Green’s function

$$G(x, \cdot) = -\Delta^{-1}(1_x).$$

(2.6)

When it is needed, we will add a superscript to make explicit the boundary conditions of $G$. Let us recall a classical bound result for the Green’s function in dimension 2.
Proposition 2.2. We have that for the graph $\Lambda_n$ and for both free and 0 boundary condition and for any $x, y \in \Lambda_n$

$$G(x, x) = C \log(d(x, \partial \Lambda_n)) + O(1),$$

(2.7)

where $C$ does not depend on any other parameter.

2.2 The Gaussian free field. In this subsection, we introduce the GFF and some of the properties we use throughout the paper. For a more detailed discussion of the GFF, we refer the reader to [She07, Szn12].

Let us fix a boundary set $\partial \Lambda_n$, the GFF with 0-boundary condition is the random function $\phi : V \mapsto \mathbb{R}$ such that

$$P((\phi(x_v) \in dx_v)_{v \in \Lambda_n}) \propto \exp\left(-\frac{\langle \phi, \phi \rangle_{\nabla^2}}{2}\right) \prod_{v \in \Lambda_n \setminus \partial \Lambda_n} dx_v \prod_{\Lambda_n \cap \partial \Lambda_n} \delta_0(dx_v).$$

We say that $\phi$ is a GFF with free-boundary condition if $\partial \Lambda_n = \{x_0\}$, for some $x_0 \in \Lambda_n$. We say that $\phi$ is a GFF with 0 (or Dirichlet) boundary condition if

$$\partial \Lambda_n := \{x \in \Lambda_n : |Re(x)| = 1 \text{ or } |Im(x)| = 1\},$$

in other words, the points in $\Lambda_n$ that are in the boundary of $[-1, 1]^2$.

An important equivalent characterisation of the GFF, is as the centred Gaussian process with covariance

$$E[\phi(x)\phi(y)] = G(x, y),$$

where the boundary values of the Green’s function are associated with the boundary values of the GFF.

A key property to understand the GFF is its Markov property.

Proposition 2.3 (Weak Markov property). Let $\phi$ be a GFF in $\Lambda_n$ with 0-boundary condition in $\partial \Lambda_n$. Furthermore, let $B$ be a subset of the vertices of $\Lambda_n$. Then, there are two independent random function $\phi_B$ and $\phi_B^B$ such that $\phi = \phi_B + \phi_B^B$ and

1. $\phi_B$ is harmonic in $\Lambda_n \setminus B$.
2. $\phi_B^B$ is a GFF in $\Lambda_n$ with 0-boundary condition in $\partial \Lambda_n \cup B$.

Let us, now, define a white noise on the edges of $\Lambda_n$.

Definition 2.4 (White noise). We denote $W$ a white noise, if $W$ is a function on the directed edges of $\Lambda_n$ such that $(W(e'))$ is a standard normal random variable independent of all other $W(e''')$ with $e \neq e'''$.

The discrete gradient of the GFF has an interesting relationship with the white noise. (See [Aru15] as well as [AKM19] for the same decomposition in the continuous setting).

Proposition 2.5. Let $\phi$ be a GFF in $\Lambda_n$, then there exists a Gaussian process $\zeta(e')$ such that

$$W := \nabla \phi + \zeta$$

is a white noise in $E$. Furthermore,

$$\phi = \Delta^{-1} \nabla \cdot W.$$
2.3 Integer-valued Gaussian free field and the KT transition. In this section, we briefly explain how Fröhlich and Spencer proved their delocalisation Theorem 1.6 as we will rely on the technology they developed (an expansion into Coulomb charges) later in Section 4. We refer the reader to the excellent review [KP17] from which we borrow the notations. See [KP17] for the relevant definitions.

For simplicity, we fix a square domain $\Lambda \subset \mathbb{Z}^2$ and we consider the case of free boundary conditions rooted at some vertex $v \in \Lambda$.

The proof by Fröhlich-Spencer can essentially be decomposed into the following successive steps:

1) The first step is to view the singular conditioning $\{\phi_i \in 2\pi \mathbb{Z}, \forall i \in \Lambda\}$ using Fourier series thanks to the identity

$$2\pi \sum_{m \in \mathbb{Z}} \delta_{2\pi m}(\phi) \equiv 1 + 2\sum_{q=1}^{\infty} \cos(q\phi).$$

To avoid dealing with infinite series, proceeding as in [KP17], we consider the following approximate IV-GFF

$$P_{\beta, \Lambda, v}[d\phi] := \frac{1}{Z_{\beta, \Lambda, v}} \prod_{i \in \Lambda} \left(1 + 2\sum_{q=1}^{N} \cos(q\phi(i))\right) P_{\text{GFF}}[d\phi].$$

In fact, more general measures are considered in [FS81, KP17]: they fix a family of trigonometric polynomials $\lambda_i := (\lambda_i)_i \in \Lambda$ attached to each vertex $i \in \Lambda$. These trigonometric polynomials are parametrized as follows: for each $i \in \Lambda$,

$$\lambda_i(\phi) = 1 + 2\sum_{q=1}^{N} \hat{\lambda}_i(q) \cos(q\phi(i)).$$

Now given a family of trigonometric polynomials $\lambda_i$, they define

$$P_{\beta, \Lambda, \lambda_i, v}[d\phi] := \frac{1}{Z_{\beta, \Lambda, \lambda_i, v}} \prod_{i \in \Lambda} \lambda_i(\phi(i)) P_{\text{GFF}}[d\phi].$$

We mention this degree of generality to keep the same notations as in [FS81, KP17] and also so that the reader will not get confused when opening these references. Yet, in our present case, we will stick to the case where $\hat{\lambda}_i(q) = 1$ for all $i \in \Lambda$ and $1 \leq q \leq N$.

2) The second step in the proof is to fix a test function $f : \Lambda \to \mathbb{R}$ such that $\sum_{i \in \Lambda} f(i) = 0$ and to consider the Laplace transform of $\langle \phi, f \rangle$: $\mathbb{E}_{\beta, \Lambda, \lambda_i, v}[e^{\langle \phi, f \rangle}]$. As $N \to \infty$ with our choice of trigonometric polynomials, this will converge to the Laplace transform $\mathbb{E}_{\beta, \Lambda}[e^{\langle \phi, f \rangle}]$. (Except instead of conditioning the GFF to be in $\mathbb{Z}^\Lambda$ as in Definition 1.5, our convention in this Section is to condition it to be in $(2\pi \mathbb{Z})^\Lambda$. (Besides changing constants, this does not make much difference).

By a simple change of variables, this Laplace transform can be rewritten

$$\mathbb{E}_{\beta, \Lambda, \lambda_i, v}[e^{\langle \phi, f \rangle}] = \frac{1}{Z_{\beta, \Lambda, \lambda_i, v}} \exp\left(\frac{1}{2\beta} (f, -\Delta^{-1} f)\right) \mathbb{E}_{\beta, \Lambda}[\prod_{i \in \Lambda} \lambda_i(\phi(i) + \sigma(i))],$$

where the function $\sigma = \sigma_f$ will be used throughout and is defined by

$$\sigma := \frac{1}{\beta} [-\Delta]^{-1} f \quad (2.8)$$

\[6\] It is slightly more convenient to consider the GFF conditioned to leave in $(2\pi \mathbb{Z})^\Lambda$ rather than $\mathbb{Z}^\Lambda$. Following [FS81, KP17], we will stick to this convention here as well as in Section 4 and Appendix A.
The main difficulty in the proof in [FS81] is in some sense to show that the effect induced by the shift σ does not have a dramatic effect compared to the exponential term \( \exp\left(\frac{1}{2\beta}(f, -\Delta^{-1}f)\right) \) so that ultimately,

\[
\mathbb{E}^{IV}_{\beta,\Lambda}[e^{(\phi,f)}] \geq \exp\left(\frac{1}{2\beta(1 + \varepsilon)}(f, -\Delta^{-1}f)\right).
\]

From such a lower bound on the Laplace transform, one can easily extract delocalisation properties of the IV-GFF.

3) The third (and by far most difficult) step is to control the effect of the shift σ via a highly non-trivial expansion into Coulomb charges which enables to rewrite the partition function as follows:

\[
Z_{\beta,\Lambda\lambda,v} = \sum_{N \in \mathcal{F}} c_N \int \prod_{\rho \in N}[1 + z(\beta, \rho, N) \cos(\langle \phi, \rho \rangle)]d\mu_{\beta,\Lambda\lambda,v}(\phi).
\]

We refer to [FS81, KP17] for the notations used in this expression and in particular for the concept of charges (i.e. \( \rho : \Lambda \to \mathbb{R} \), ensembles (i.e. sets \( N \) of mutually disjoint charges \( \rho \)) etc.

One important feature of this expansion into charges is the fact that under some (very general) assumptions on the growth of the Fourier coefficients \( |\lambda_i| \) (see (5.35) in [FS81]), it can be shown that the effective activities \( z(\beta, \rho, N) \) decay fast. Namely (see (1.14) in [KP17]),

\[
|z(\beta, \rho, N)| \leq \exp\left(-\frac{\varepsilon}{\beta}(\|\rho\|^2 + \log_2(\text{diam}(\rho) + 1))\right).
\]

As such we see that at high temperature, the partition function corresponds to a sum of positive measures. (Also the weights \( c_N \) are positive and s.t. \( \sum c_N = 1 \).

Remark 2.6. In [KP17], the authors have introduced a slightly different definition of the free b.c. GFF which makes the analysis behind this decomposition into charges more pleasant (their definition cures the presence of non-neutral charges \( \rho \) very easily). One can switch to their more convenient definition in our setting since in the limit \( N \to \infty \), both give the same integer-valued GFF.

This crucial third step thus allows us to rewrite the Laplace transform \( \mathbb{E}_{\beta,\Lambda\lambda,v}[e^{(\phi,f)}] \) as follows:

\[
e^{\frac{1}{2\beta}(f, -\Delta^{-1}f)} \sum_{N \in \mathcal{F}} c_N \int \prod_{\rho \in N}[1 + z(\beta, \rho, N) \cos(\langle \phi, \rho \rangle)]d\mu_{\beta,\Lambda\lambda,v}(\phi)
\]

We rewrite this ratio as follows (thus defining \( Z_N(\sigma) \) and \( Z_N(0) \)):

\[
\mathbb{E}_{\beta,\Lambda\lambda,v}[e^{(\phi,f)}] = e^{\frac{1}{2\beta}(f, -\Delta^{-1}f)} \sum_{N \in \mathcal{F}} c_N Z_N(\sigma) \sum_{N \in \mathcal{F}} c_N Z_N(0).
\]

4) The fourth step is an analysis for each fixed ensemble \( N \in \mathcal{F} \) of the above ratio \( \frac{Z_N(\sigma)}{Z_N(0)} \). Trigonometric inequalities are used here to in order to obtain for each \( N \):

\[
\frac{Z_N(\sigma)}{Z_N(0)} \geq \exp\left[-D_k \sum_{\rho \in N}\frac{z(\beta, \rho, N)}{(\sigma, \rho)^2}\right]\times \int e^{S(\rho, \phi)} \prod_{\rho \in N}[1 + z(\beta, \rho, N) \cos(\langle \phi, \rho \rangle)]d\mu_{\beta,\Lambda\lambda,v}(\phi).
\]
where
\[ S(N, \phi) := - \sum_{\rho \in N} \frac{z(\beta, \rho, N) \sin(\langle \phi, \bar{\rho} \rangle) \sin(\langle \sigma, \rho \rangle)}{1 + z(\beta, \rho, N) \cos(\langle \phi, \bar{\rho} \rangle)} \]  
(2.9)

Two crucial observations are made at this stage:

1. The functional \( \phi \mapsto S(N, \phi) \) is odd in \( \phi \).
2. The measure \( \prod_{\rho \in N}[1 + z(\beta, \rho, N) \cos(\langle \phi, \bar{\rho} \rangle)]d\mu_{\beta, \Lambda, v}^{GFF}(\phi) \) is invariant under \( \phi \mapsto -\phi \).

All together this simplifies tremendously the above lower bound, as by using Jensen, one obtains readily
\[ Z_N(\sigma) = \exp \left[ - D_4 \sum_{\rho \in N} |z(\beta, \rho, N)|\langle \sigma, \rho \rangle^2 \right]. \]

From this lower bound together with the specific construction of the ensembles of charges \( N \), it is then not very difficult to conclude the proof with the desired lower bound
\[ \mathbb{E}_{\beta, \Lambda, \lambda, \psi}[e^{\langle \phi, f \rangle}] \geq \exp \left( \frac{1}{2\beta(1+\varepsilon)} (f, -\Delta^{-1} f) \right). \]

As we will see in Section 4, the effect of shifting the \( Z \) fibers by \( a \in \mathbb{R}^\Lambda \) will translate as follows:
\[ \mathbb{E}_{\beta, \Lambda, \lambda, \psi}^a[e^{\langle \phi, f \rangle}] = e^{\frac{1}{2\varepsilon} (f, -\Delta^{-1} f)} \sum_{N \in \mathcal{C}^\Lambda} \prod_{\rho \in N} [1 + z(\beta, \rho, N) \cos(\langle \phi, \bar{\rho} \rangle + \langle \sigma - a, \rho \rangle)]d\mu_{\beta, \Lambda, v}^{GFF}(\phi) \]
\[ \sum_{N \in \mathcal{C}^\Lambda} \prod_{\rho \in N} [1 + z(\beta, \rho, N) \cos(\langle \phi, \bar{\rho} \rangle - \langle a, \rho \rangle)]d\mu_{\beta, \Lambda, v}^{GFF}(\phi). \]

The difficulty for us will be that, generically, \( a \) is much less regular than \( \sigma \) (defined in (2.8)) which thus makes the Dirichlet energy \( \langle \nabla(\sigma - a), \nabla(\sigma - a) \rangle \) typically huge. Because of this, we will not be able anymore to rely on the two symmetries above (in particular the use of Jensen is not longer possible except for very specific choices of \( a \), see the discussion after Definition 1.7). We will come back to this in Section 4.

**Remark 2.7.** The case of Dirichlet boundary conditions has been outlined in the appendix D. in [FS81] and the details of the proof appeared very recently in the appendix in [Wir19]. The proof structure highlighted above for free b.c. still holds except the decomposition into charges needs to be adapted to the presence of a boundary. See the Appendix in [Wir19].

We also point out that the nice symmetrization argument used in [Wir19] does not apply to our case (as \( a \) is far from being harmonic) and also because the symmetrized measure in most cases does not provide informations on the fluctuations we need.

### 2.4 Link with the a-shifted integer-valued GFF

In this section, we precise the link between our statistical reconstruction problem and the a-shifted IV-GFF introduced earlier (in Definition 1.7).

**Lemma 2.8.** Let \( \Lambda \subset \mathbb{Z}^2 \), \( T > 0 \) and \( a \in \mathbb{R}^\Lambda \) with \( a|_{\partial \Lambda} \equiv 0 \). If \( \phi \) is a 0-boundary GFF (with inverse temperature \( \beta = 1 \)) on \( \Lambda \), then its conditional law given \( \phi \mod \frac{2\pi}{T} = \frac{2\pi}{T} a \) is given by \( \frac{2\pi}{T} \psi \), where \( \psi \sim \mathbb{P}^{a,IV}_{\beta_T, \Lambda} \) and where the \( T \)-dependent inverse temperature \( \beta_T \) is given by
\[ \beta_T := \frac{(2\pi)^2}{T^2}. \]
Equivalently, for any functional $F : \mathbb{R}^\Lambda \to \mathbb{R}$,

$$
E[F(\phi) \mid e^{iT\phi} = e^{2\pi ia}] = E^{\beta=\beta_T,\Lambda}_{\mathbb{R}^\Lambda} \left[ F \left( \frac{2\pi}{T} \psi \right) \right].
$$

Proof.

Recall from Definition 1.7,

$$
P^{\beta,\Lambda}_{\mathbb{R}^\Lambda} [d\phi] := \frac{1}{Z} \sum_{m \in \mathbb{Z}^\Lambda, a_{\beta}\lambda = 0} \delta_{m+a}(\phi) \exp(-\frac{\beta}{2} \langle \nabla (m+a), \nabla (m+a) \rangle)
$$

Now, by desintegration, for any $C^0$ functional $F : \mathbb{R}^\Lambda \to \mathbb{R}$, one has

$$
E[F(\phi) \mid e^{iT\phi} = e^{2\pi ia}] = \frac{\sum_{m \in \mathbb{Z}^\Lambda} \exp \left( -\frac{\beta}{2} \langle \frac{2\pi}{T} (m+a), (\Delta) \frac{2\pi}{T} (m+a) \rangle \right) F(\frac{2\pi}{T} (m+a))}{\sum_{m \in \mathbb{Z}^\Lambda} \exp \left( -\frac{\beta}{2} \langle \frac{2\pi}{T} (m+a), (\Delta) \frac{2\pi}{T} (m+a) \rangle \right)}
$$

$$
= \frac{\sum_{m \in \mathbb{Z}^\Lambda} \exp \left( -\frac{2\pi^2}{2T} \langle (a+m), (\Delta)(m+a) \rangle \right) F(\frac{2\pi}{T} (m+a))}{\sum_{m \in \mathbb{Z}^\Lambda} \exp \left( -\frac{2\pi^2}{2T} \langle (a+m), (\Delta)(m+a) \rangle \right)}
$$

$$
= E^{\beta=\beta_T,\Lambda}_{\mathbb{R}^\Lambda} \left[ F \left( \frac{2\pi}{T} \phi \right) \right]
$$

where we have made the slightly unusual choice $\beta_T := (2\pi)^2 T^{-2}$ (in order to avoid dealing with $\sqrt{T}$ in most of the introduction).

3 Localisation regime

In this section, we prove the first part of Theorems 1.1 and 1.2. That is to say, we show that one can recover a GFF knowing $\exp(iT\phi)$, in fact the recovery function is fairly straightforward:

$$
F(\exp(iT\phi))(x) := E[\phi(x) \mid \exp(iT\phi)].
$$

To show that this is the right function, we need to recall (1.8). It says that to prove the first part of Theorems 1.1 and 1.2, it is enough to show that if $\phi$ is a GFF in $\Lambda_n$ and $f$ a fixed smooth function in $f$

$$
E \left[ \text{Var} [(\phi, f) \mid e^{iT\phi}] \right] = o \left( n^4 \right).
$$

(3.1)

Let us note that this approach still does not look very useful, as to bound this conditional variance we need to compute the conditional expectation, which is a non-trivial function of $\exp(iT\phi)$. To circumvent this issue, we write the conditional variance as follows.

$$
\text{Var} [(\phi, f) \mid e^{iT\phi}] = \frac{1}{2} E \left[ (\langle \phi_1, f \rangle - \langle \phi_2, f \rangle)^2 \mid e^{iT\phi} \right],
$$

where $\phi_1, \phi_2$ are conditionally independent given $e^{iT\phi}$. Let us be more explicit about this law.

**Definition 3.1.** Let us take $\phi$ a GFF in $\Lambda_n$ with any given boundary. We denote $(\phi_1, \phi_2)$ a pair of GFF in $\Lambda_n$ with the same boundary condition such that a.s.

$e^{iT\phi_1} = e^{iT\phi_2} = e^{iT\phi}$ and $\phi_1$ is conditionally independent of $\phi_2$ given $e^{iT\phi}$. In other words

$$
P \left[ (d\phi_1, d\phi_2) \mid e^{iT\phi} \right] \propto \prod_{i=1,2} \prod_{x \in \Lambda_n \setminus \partial \Lambda_n} \left( \sum_{k \in \mathbb{Z}} \delta_{2k} (\phi(x) - d\phi_1(x)) \right).
$$

To prove (3.1), we use an averaged Peierls argument.
3.1 Large gradients are costly for a GFF. The first stage to implement Peierls argument, is to show that it is costly for a GFF to have many edges with big gradients. To do this we are going to use the Markov property, i.e. Proposition 2.3. In fact for a given deterministic set $B \subseteq \Lambda_n$, we need to understand what is the law of the norm of $\phi_B$.

**Lemma 3.2.** Let us work in the context of Proposition 2.3 with $B \cap \partial \Lambda_n = \emptyset$.

1. The law of $\|\phi_B\|_2^2$ is that of a $\chi^2$ with $|B|$ degrees of freedom.
2. The law of $\|\phi_B\|_2^2$ is that of a $\chi^2$ with $|\Lambda_n \setminus (\partial \Lambda_n \cup B)|$ degrees of freedom.

**Proof.** We start defining $\text{Harm}(B)$ as the set of functions $\Lambda_n \mapsto \mathbb{R}$ that are harmonic in $\Lambda_n \setminus (B \cup \partial \Lambda_n)$ and take value 0 in $\partial \Lambda_n$. In fact, we have that $\Phi_B$ is the orthogonal projection of $\phi$ to $\text{Harm}(B)$ under the inner product $\langle \cdot, \cdot \rangle_{\nabla \phi}$ (see for example Section 2.6 of [She07]). One can, now, check that the subspace $\text{Harm}(B)$ has dimension $|B|$ from which (1) follows. As $\phi^B$ is the orthogonal projection under $\text{Harm}(B)^\perp$, (2) follows by a similar reason, as the space of functions with 0 boundary condition on $B \cup \partial \Lambda_n$ has dimension $|\Lambda_n \setminus (\partial \Lambda_n \cup B)|$. 

We can now use this proposition to obtain the basic input we need for a Peierl’s argument.

**Lemma 3.3.** Let $\phi$ be a GFF in $\Lambda_n$ with either 0 or free boundary condition. Then, there exist constants $\alpha, C, u_0 > 0$ independent of $\Lambda_n$ such that for all a finite set $F$ of edges and all $u > u_0$

$$\mathbb{P} \left[ |\phi(x) - \phi(y)| \geq u, \forall xy \in F \right] \leq Ce^{-\alpha u^2|F|}.$$ 

**Proof.** We use the Markov property of the GFF (Proposition 2.3) with the subset of vertices $B$ such that $x \in B$ if there exists $xy \in F$. Let us note that $|B| \leq 2|F|$. We have that

$$\|\phi_B\|_2^2 = \sum_{xy \in B} (\phi_B(y) - \phi_B(x))^2 \geq \sum_{xy \in F} (\phi(y) - \phi(x))^2.$$ 

has the law of a $\chi^2$ with $|B|$ degrees of freedom. Let us note that thanks to Proposition 2.3 (1), we have that $\phi_B(y) - \phi_B(x)$ is equal to $\phi(y) - \phi(x)$. Thus, using that $|B| \leq 2|F|$

$$\mathbb{P} \left[ |\phi(x) - \phi(y)| \geq u, \forall xy \in F \right] \leq \mathbb{P} \left[ \|\phi_B\|_2^2 \geq u^2|F| \right] \leq \mathbb{P} \left[ \|\phi_B\|_2^2 \geq \frac{u^2|B|}{2} \right].$$

We can now use Lemma 3.2 (1), to continue and see that when $u$ is big enough

$$\mathbb{P} \left[ |\phi(x) - \phi(y)| \geq u, \forall xy \in F \right] \leq C \exp(-4\alpha u^2|B|) \leq C \exp(-4\alpha u^2|F|),$$

where we used that $|F| \leq 4|B|$. 

\[\] 

3.2 The GFFs $\phi_1$ and $\phi_2$ agree on a dense percolating set.

3.2.1 The 0-boundary case. Take $\phi$ a 0-boundary GFF in $\Lambda_n$ and assume we are given an instance of $e^{iT\phi}$. Let us sample two conditionally independent copies $\phi_1, \phi_2$ given $e^{iT\phi}$ as in Definition 3.1. Let us now introduce the following definition

**Definition 3.4 (1).** We denote $I := I(\phi_1, \phi_2)$ the connected component connected to the boundary, $\partial \Lambda_n$, of the random set $\{x \in \Lambda_n, \phi_1(x) = \phi_2(x)\}$. 

Recall that by definition, \( \phi_1, \phi_2 \) are GFF with zero boundary conditions and as such one needs to have \( \phi_1 \equiv \phi_2 \) on \( \partial \Lambda_n \).

Our goal in this subsection is to show via an annealed Peierls argument, that with high probability when \( T \) is small, the random set \( I \) is percolating inside \( \Lambda_n \). To study this, for any \( x \in \Lambda_n \) we define \( O(x) \) as the empty set if \( x \in I \) and as the connected component containing \( x \) of \( \Lambda \setminus I \) if \( x \not\in I \).

Our main observation is that having an edge connecting \( O(x) \) with \( \partial \Lambda_n \) \( O(x) \) is costly in the sense that it forces either \( |\nabla \phi_1(e)| \) or \( |\nabla \phi_2(e)| \) to be larger than \( \pi/T \). Indeed the values of \( \phi_1 \) and \( \phi_2 \) are fixed modulo \( 2\pi T \), in other words for any \( x \in \Lambda_n \) and \( i \in \{1, 2\} \),

\[
\phi_i(x) \in \phi(x) + \frac{2\pi}{T} \mathbb{Z}.
\]

This way, if \( \phi_1, \phi_2 \) agree on \( x \) but disagree on \( y \sim x \), this means that either \( |\phi_1(x) - \phi_2(y)| \geq \frac{\pi}{T} \) which implies that at least one gradient must be larger than or equal to \( \pi/T \). We then have the following proposition.

**Proposition 3.5.** Using the definitions introduced above, for all \( T \) small enough there exists \( \varpi(T) > 0 \) and \( C > 0 \) such that

\[
\mathbb{P} (\text{diam}(O(x)) \geq L) \leq C \exp(-\varpi(T)L)
\]

**Proof.** Let us note that if \( \text{diam}(O(x)) \geq L \) there is a subset of edges \( \eta \) of length at least \( L \) such that its dual is a connected path surrounding \( x \) and for every \( e \in \eta \) either \( |\nabla \phi_1(e)| \geq \pi/T \) or \( |\nabla \phi_2(e)| \geq \pi/T \). This implies that

\[
\mathbb{P} (\text{diam}(O(x)) \geq L) \leq \sum_{\eta \text{ surrounds } x} \mathbb{P} (|\nabla \phi_1(e)| \geq \pi/T \text{ or } |\nabla \phi_2(e)| \geq \pi/T, \forall e \in \eta)
\]

\[\tag{3.4}
(3.4)
\]

Let us fix \( \eta \) and suppose that for all \( e \in \eta \), either \( |\nabla \phi_1(e)| \geq \pi/T \) or \( |\nabla \phi_2(e)| \geq \pi/T \). This implies that there exists a \( F \subseteq \eta \) and \( i \in \{1, 2\} \) such that for all \( e \in F \) we have that \( |\nabla \phi_i(e)| \geq \pi/T \). This implies that

\[
\mathbb{P} \left( |\nabla \phi_1(e)| \geq \frac{\pi}{T} \text{ or } |\nabla \phi_2(e)| \geq \frac{\pi}{T}, \forall e \in \eta \right) \leq 2 \sum_{j=\lfloor |\eta|/2 \rfloor}^{\lfloor |\eta| \rfloor} \binom{|\eta|}{j} \mathbb{P} \left( |\nabla \phi_i(e)| \geq \frac{\pi}{T}, \forall e \in F \right)
\]

\[
\leq 2^{|\eta|+1} \sum_{j=\lfloor |\eta|/2 \rfloor}^{\lfloor |\eta| \rfloor} \exp \left( -2\tilde{\alpha} \frac{j}{T}^2 \right),
\]

where we used Lemma 3.3 and that both \( \phi_1 \) and \( \phi_2 \) have the law of a GFF in \( \Lambda \). Additionally, \( \tilde{\alpha} := \alpha \pi^2/2 \). Thus, (3.4) is less than or equal to

\[
C \sum_{k \geq L} \sum_{\eta \text{ surrounds } x} 2^k \exp \left( -\frac{\tilde{\alpha}}{T^2} k \right) \leq C \sum_{k \geq L} \exp(-k(\tilde{\alpha} T^{-2} - \log 2 - \log 3))
\]

\[
\leq \tilde{C} \exp(-L(\tilde{\alpha} T^{-2} - \log 2 - \log 3))
\]

where we used that the amount of \( \eta \) such that \( |\eta| = k \) and \( \eta \) surrounds \( x \) is less than or equal to \( C3^k \), and that \( T \) is such that

\[\tag{3.5}
\tilde{\alpha} T^{-2} - \log 6 > 0.
\]
3.2.2 The free boundary case. We need to modify significantly the above definitions in order to analyze the free boundary case. We assume the free boundary GFF is rooted at some vertex $x_0 \in \Lambda_n$. As in the Dirichlet case, $(\phi_1, \phi_2)$ will still denote two conditionally independent copies of the GFF given $e^{iT\phi}$.

The main difference w.r.t. Dirichlet is that when $T$ is small, it is no longer true that with high probability $\phi_1$ and $\phi_2$ will agree on a large percolating set. Instead, we will find a large set, which we will call $I$ again together with a random integer $m_I \in \mathbb{Z}$ such that

$$\phi_1(x) = \phi_2(x) + m_I \frac{2\pi}{T}, \quad \forall i \in I$$

Let us then introduce the following sets: for each $m \in \mathbb{Z}$, let

$$I_m := \text{Largest connected component of } \{x \in \Lambda_n, \phi_1(x) = \phi_2(x) + m \frac{2\pi}{T}\}.$$

If there are two of the same size, we choose one in a deterministic way. From these subsets $I_m$, we define the set $I$ as well as the connected components $\{O(x)\}_{x \in \Lambda_n}$ as follows:

- If there is a unique $m_0 \in \mathbb{Z}$ s.t. $I_{m_0}$ has (graph) diameter larger than $\frac{\bar{\eta}}{2}$, then we define
  $$I := I_{m_0}$$
  and for any $x \in \Lambda_n$, we define $O(x)$ to be empty if $x \in I$ and to be the connected component of $x$ in $\Lambda_n \setminus I$ otherwise.

- If on the other hand, one can find two integers $m_1, m_2$ s.t. both $I_{m_1}$ and $I_{m_2}$ have diameter greater than $\frac{\bar{\eta}}{2}$, then we define
  $$I := \emptyset$$
  and $O(x) := \Lambda_n$, $\forall x \in \Lambda_n$.

We can now state the analogue of Proposition 3.5 for free b.c.

**Proposition 3.6.** Let $\phi_1, \phi_2$ two free-boundary GFF such that $\exp(T\phi_1) = \exp(T\phi_2)$ and conditionally independent given $\exp(T\phi_1)$. Then using the above definitions (for free b.c.), for all $T$ small enough there exists $\omega(T) > 0$ and $C > 0$ such that for all $x \in \Lambda_n$,

$$\mathbb{P}(\text{diam}(O(x)) \geq L) \leq C \exp(-\omega(T)L)$$

*Proof.* The proof follows the same lines as in the Dirichlet case, as Lemma 3.3 does not care about the boundary conditions. The only difference is that we need to deal with the dichotomy entering into the definition of the set $I$. (Dichotomy which does not exist for Dirichlet). For this, note that in order to have two sets $I_{m_1}, I_{m_2}$ with $m_1 \neq m_2$ and both have diameter $\geq n/2$, there must exist at least one path $\eta$ in the dual graph $(\mathbb{Z}^2)^*$ which has diameter greater than $n/2$ and which satisfies the constraint that any $e \in \eta$ is such that either $|\nabla \phi_1(e)| \geq \frac{\bar{\eta}}{2}$ or $|\nabla \phi_2(e)| \geq \frac{\bar{\eta}}{2}$. By Lemma 3.3 and the same argument that in the Dirichlet case, such a case only happens with probability less than $O(n) \exp(-\omega(T)n)$. Note that the same argument implies that there is at most one connected component of $\{x \in \Lambda_n, \phi_1(x) = \phi_2(x) + m \frac{2\pi}{T}\}$ with diameter at least $n/2$.

Let us now see that with high probability there is one $I_m$ with size at least $n/2$. If this were not the case, there would be a dual path from the top to the bottom of $\Lambda_n$ such that any $e \in \eta$ is such that either $|\nabla \phi_1(e)| \geq \frac{\bar{\eta}}{2}$ or $|\nabla \phi_2(e)| \geq \frac{\bar{\eta}}{2}$. As before, this case only happens with probability less than $O(n) \exp(-\omega(T)n)$.

Note that after defining $I$, the argument of Proposition 3.5 implies that the possibility for any point $x$ that $O(x)$ is huge only arises with exponentially decaying probability in the diameter $n$. Because the event of existence of $I$ does not hurt the uniform bound (in $L \leq n$) in the proposition. □
3.3 The conditional variance is small for 0-boundary GFF. We will now prove (3.1) for a 0-boundary GFF. Let us now study the law of \((\phi_1, \phi_2)\) conditionally on \(I\) and the values of \(\phi_1\) on \(I\). We fix \(e^{IT\phi}, I\), and the values of \(\phi_1\) on \(I\) and take \((\varphi_1, \varphi_2)\) a possible value of \((\phi_1, \phi_2)\) that satisfy the conditioning. Note that to check whether \((\varphi_1, \varphi_2)\) is a possible realisation, one just needs to check that \((\varphi_1)_I = (\varphi_2)_I = (\phi_1)_I\), and that for any \(O\) connected component of \(\Lambda_n \setminus I\), the pair \((\varphi_1, \varphi_2)\) restricted to \(O\) locally satisfies the conditions. Furthermore, if we define \(O\) the graph induced by all the edges in \(\Lambda_n\) that have at least one vertex in \(O\) we have that
\[
P \left( (\phi_1, \phi_2) = (\varphi_1, \varphi_2) \mid e^{IT\phi}, I, (\phi_1)_I \right) \propto \prod_O e^{-\frac{1}{2} \left( \langle (\varphi_1)_O, (\varphi_1)_O \rangle + \langle (\varphi_2)_O, (\varphi_2)_O \rangle \right)}.
\]
(3.6)

As a consequence of (3.6), we have that under this conditional law the law of \((\phi_1, \phi_2)\) restricted to \(O\) is independent of the law of \(O'\) if \(O \neq O'\). Thus, 
\[
\mathbb{E} \left[ (\phi_1 - \phi_2, f)^2 \right] \text{ is equal to }
\sum_{x,y \in \Lambda_n} f(x)f(y) \mathbb{E} \left[ (\phi_1 - \phi_2)(x)(\phi_1 - \phi_2)(y)1_{O(x) = O(y)} \right]
\leq \sum_{x,y \in \Lambda_n} |f(x)||f(y)| \mathbb{E} \left[ (\phi_1 - \phi_2)^2(x)(\phi_1 - \phi_2)^2(y) \right]^{1/2} \mathbb{P} \left( 1_{O(x) = O(y)} \right)^{1/2}.
\]
(3.7)

We can now just bound
\[
\mathbb{E} \left[ (\phi_1 - \phi_2)^4(x) \right] \leq 16 \mathbb{E} \left[ \phi_1(x)^4 \right] = 48G^2(x,x).
\]

Note that on the event \(O(x) = O(y)\) the diameter of \(O(x) \geq \|x-y\|\). Thus, we have that there exists an absolute constant \(C, \varpi(T) > 0\) such that
\[
\mathbb{P} \left( O(x) = O(y) \right) \leq C \exp(-\varpi(T)\|x-y\|).
\]

From the fact that \(\exp(-\varpi(T)\|x-y\|)\) decreases exponentially as \(\|x-y\|\) goes to infinity, we have that
\[
\mathbb{E} \left[ (\phi_1 - \phi_2, f)^2 \right] \leq C\|f\|_\infty^2 \sup_x G_n(x,x)n^2 \leq C\|f\|_\infty^2 n^2 \log(n)n^2,
\]
(3.8)
which proves (3.1).

3.4 The conditional variance is small enough for free boundary Gaussian free field. We will now prove (3.1) for a free-boundary GFF. The proof is very similar to that of the 0-boundary condition so we are going to sketch the proof only highlighting the difference with the Dirichlet boundary case.

Let us take \((\phi_1, \phi_2)\) a pair of GFF with 0-boundary condition in \(\{x_0\}\) coupled as in Definition 3.1. Thanks to Proposition 3.6, we have that there exist a (random) set \(I\) and a (random) integer \(n\) such that for all \(y \in I\), \(\Phi_1(y) = \Phi_2(y) + 2\pi n/I\), and furthermore for any \(x\) if we define \(O(x)\) as the connected component of \(\Lambda_n \setminus I\) containing \(x\), we have that
\[
P(\text{diam}(O(x)) \geq L) \leq \exp(-\varpi(T)L).
\]
(3.9)

Let us note that the same argument as in Subsection 3.3 together with the estimate of Proposition 3.7 implies that for any smooth function \(f : [-1,1]^2 \mapsto \mathbb{R}\) we have that
\[
\mathbb{E} \left[ \left( \phi_1 - \phi_2 - \frac{2\pi n}{I}, f \right)^2 \right] \leq C\|f\|_\infty^2 n^2 \log(n).
\]
Let us, now, note that for any continuous function $f$ with $\int f = 0$, we have that $\frac{1}{n^2} f_n = o(1)$. Thus, defining $\bar{f}$ as $f - \bar{f}$

$$E \left[ (\phi_1 - \phi_2, f)^2 \right] \leq 2E \left[ (\phi_1 - \phi_2, \bar{f})^2 \right] + (n^{-2} f, 1)^2 E \left[ (\phi_1 - \phi_2, 1)^2 \right] \leq C\|f\|_{\infty}^2 \log(n)n^2 + C\|f\|_{\infty}o(1)n^4. \tag{3.10}$$

Which finishes the proof.

### 3.5 The conditional variance at a given point is bounded.

In this subsection, we are going to improve the result of (3.8) for the case $f = 1_x$.

**Proposition 3.7.** Let $\phi_1$ and $\phi_2$ two zero boundary (or free-boundary) GFF taken by the law of Definition 3.1. Then, for all $T$ small enough there exists $K > 0$ such that for all $n \in \mathbb{N}$

$$E \left[ (\phi_1 - \phi_2)^2(x) \right] \leq K. \tag{3.11}$$

**Proof.** We start with (3.11) and with $\phi$ a 0-boundary GFF as in Subsection 3.3. Let $\gamma$ be a horizontal in $\Lambda_n$ from $\partial[−1, 1]^2$ to $x$. We say that the edge $e$ belongs to $\gamma \cap O(x)$ if $e \in \gamma$ and $e \cap O(x) \neq \emptyset$. We then have that

$$(\phi_1 - \phi_2)(x) = \sum_{e \in \gamma \cap O(x)} \nabla(\phi_1 - \phi_2)(e). \tag{3.12}$$

Thus,

$$(\phi_1 - \phi_2)^2(x) \leq \sum_{e, e' \in E} \nabla(\phi_1 - \phi_2)(e) \nabla(\phi_1 - \phi_2)(e') 1_{e, e' \in O(x) \cap \gamma}.$$ 

We can now upper bound $E \left[ (\phi_1 - \phi_2)^2(x) \right]$ by

$$\sum_{e, e' \in E} E \left[ \nabla(\phi_1 - \phi_2)(e) \nabla(\phi_1 - \phi_2)(e') 1_{e, e' \in O(x) \cap \gamma} \right] \leq K \sup_{e} E \left[ (\nabla(\phi_1 - \phi_2)(e))^4 \right]^{1/2} \sum_{e, e' \in \gamma} P \left[ e, e' \in O(x) \right]^{1/2}.$$ 

We conclude (3.11) by first noting that $\text{Var}(\nabla(\phi_1 - \phi_2)(e)) \leq 4$ thanks to Proposition 2.5, and by the fact that $P \left[ e, e' \in O(x) \right] \leq \exp(-\omega(T) \max\{d(e, x), d(e', x)\})$.

To prove (3.11) in the free-boundary case with 0 value in $z$, we need to control the value of $n := (\phi_1 - \phi_2)(z)/(2\pi T)$ at a point $z \in I$ (recall that this value is a constant in $I$). This is done by using the same technique as above to prove that for any $x$

$$\text{Var} \left[ (\phi_1 - \phi_2)(x) - 2\pi m_I T \right] \leq K. \tag{3.13}$$

As $\phi_1(x_0) = \phi_2(x_0) = 0$, this implies that

$$\text{Var} \left[ m_I \right] \leq K T. \tag{3.14}$$

We conclude, by putting together (3.13) and (3.14). □

**Proposition 3.8.** Note that Proposition 3.7, improves the result of (3.1) and (3.8), by showing that when the temperature is low enough

$$E \left[ \frac{1}{n^2} \langle \phi, f \rangle \mid e^{\mathcal{T} h} \right] \leq K \frac{\|f\|_{\infty}^2}{n^2}. \tag{3.15}$$
\[ \mathbb{E} \left[ (\phi_1 - \phi_2)^2(y) 1_{D(x) = O(y)} \right] \leq K \exp \left( -\frac{\alpha(T)}{2} ||x - y|| \right). \]

The proof of (3.16), is close to that of (3.11). The only difference is that one needs to choose \( \gamma \) a straight line going from \( \partial[-1,1]^2 \) to \( y \) in such a way that the distance from \( x \) to \( \gamma(t) \) is a decreasing function (in particular the distance from \( \gamma \) to \( x \) is equal to that of \( y \) to \( x \)). \( \square \)

## 4 Delocalisation regime

We start by proving the roughening transition for generalized integer-valued fields (Theorem 1.8) and then, as a corollary, extract the delocalisation regime for our statistical reconstruction problem.

### 4.1 Proof of Theorem 1.8.

In this proof, we focus on the case of Free boundary conditions (as in [FS81, KP17]), however following the Appendix D. from [FS81] or the recent [Wir19] (see Remark 2.7), our proof works in the exact same way in the Dirichlet case.

Recall from Subsection 2.3 and from (1.13) in [KP17] the following series expansion for the Laplace transform of the discrete GFFs with periodic weights \( \lambda_\Lambda = (\lambda_j)_{j \in \Lambda} \) (we assume the same hypothesis as in Theorem 1.6 from [KP17])

\[
\mathbb{E}_{\beta,\Lambda,v} \left[ e^{\langle \phi, f \rangle} \right] = e^{\frac{\beta}{2} \langle f, -\Delta^{-1} f \rangle} \sum_{N \in \mathcal{F}} c_N^{GFF} \frac{\prod_{\rho \in \mathcal{N}} [1 + z(\beta, \rho, N) \cos(\langle \phi, \rho \rangle + \langle \sigma, \rho \rangle)]}{\sum_{N \in \mathcal{F}} c_N^{GFF} \prod_{\rho \in \mathcal{N}} [1 + z(\beta, \rho, N) \cos(\langle \phi, \rho \rangle)]} \tag{4.1}
\]

We will denote by \( \mathbb{E}_{\beta,\Lambda,v}^{a} \) or \( \mu_{\beta,\Lambda,v}^{a} \) the discrete GFF whose periodic weights are shifted by environment \( a \), namely:

\[
d\mu_{\beta,\Lambda,v}^{a}(\phi) := \frac{1}{Z_{\beta,\Lambda,v}^{a}} \prod_{j \in \Lambda} \lambda_j(\phi_j - a_j) d\mu_{\beta,\Lambda,v}(\phi) \tag{4.2}
\]

The shift by \( a \) easily translates into the following expression for the Laplace transform under \( \mu_{\beta,\Lambda,v}^{a} \):

\[
\mathbb{E}_{\beta,\Lambda,v}^{a} \left[ e^{\langle \phi, f \rangle} \right] = e^{\frac{\beta}{2} \langle f, -\Delta^{-1} f \rangle} \sum_{N \in \mathcal{F}} c_N^{GFF} \frac{\prod_{\rho \in \mathcal{N}} [1 + z(\beta, \rho, N) \cos(\langle \phi, \rho \rangle + \langle \sigma - a, \rho \rangle)]}{\sum_{N \in \mathcal{F}} c_N^{GFF} \prod_{\rho \in \mathcal{N}} [1 + z(\beta, \rho, N) \cos(\langle \phi, \rho \rangle - \langle a, \rho \rangle)]} \tag{4.3}
\]

As the shift \( a \) is fixed once and for all in this proof, let us introduce the shifted partition functions \( \{ Z_N^{a}(\sigma) \}_{N,\sigma} \). For any \( \sigma : \Lambda \to \mathbb{R} \) and any collection of charges \( \mathcal{N} \in \mathcal{F} \),

\[
Z_N^{a}(\sigma) := Z_N(\sigma - a) = \int \prod_{\rho \in \mathcal{N}} [1 + z(\beta, \rho, N) \cos((\phi, \rho) + (\sigma - a, \rho))] d\mu_{\beta,\Lambda,v}(\phi) \tag{4.4}
\]
Following the same analysis as in Section 3 from [KP17] (or also Section 5 in [FS81]), we obtain the following lower bound on the ratio of partition functions,

\[
\frac{Z_{\mathcal{N}}(\sigma)}{Z_{\mathcal{N}}(0)} \geq \exp \left[ - D_4 \sum_{\rho \in \mathcal{N}} |z(\beta, \rho, \mathcal{N})|(|\sigma, \rho|^2) \right] 
\times \int e^{S(\mathcal{N}, a, \phi)} \prod_{\rho \in \mathcal{N}} \left[ 1 + z(\beta, \rho, \mathcal{N}) \cos((\phi, \rho) - \langle a, \rho \rangle) \right] d\mu_{\mathcal{GFF}, a}(\phi),
\]

where

\[
S(\mathcal{N}, a, \phi) := - \sum_{\rho \in \mathcal{N}} \frac{z(\beta, \rho, \mathcal{N}) \sin((\phi, \rho) - \langle a, \rho \rangle) \sin((\sigma, \rho))}{1 + z(\beta, \rho, \mathcal{N}) \cos((\phi, \rho) - \langle a, \rho \rangle)}.
\]

As mentioned in Subsection 2.3, one major observation in [FS81] is that \( S(\mathcal{N}, \phi) := S(\mathcal{N}, a \equiv 0, \phi) = - S(\mathcal{N}, -\phi) \). Indeed this property together with the fact that the probability measure

\[
d\mathbb{P}_\mathcal{N}(\phi) := \frac{1}{Z_{\mathcal{N}}(0)} \prod_{\rho \in \mathcal{N}} [1 + z(\beta, \rho, \mathcal{N}) \cos((\phi, \rho) - \langle a, \rho \rangle)] d\mu_{\mathcal{GFF}, a}(\phi)
\]

is invariant under \( \phi \mapsto -\phi \) avoids controlling terms such as \( e^{S(\mathcal{N}, \phi)} \) thanks to Jensen:

\[
\frac{Z_{\mathcal{N}}(\sigma)}{Z_{\mathcal{N}}(0)} \geq \exp \left[ - D_4 \sum_{\rho \in \mathcal{N}} |z(\beta, \rho, \mathcal{N})|(|\sigma, \rho|^2) \right] \times \int e^{S(\mathcal{N}, a, \phi)} d\mathbb{P}_\mathcal{N}(\phi)
\]

\[
\geq \exp \left[ - D_4 \sum_{\rho \in \mathcal{N}} |z(\beta, \rho, \mathcal{N})|(|\sigma, \rho|^2) \right] \times \exp \left( \int S(\mathcal{N}, \phi) d\mathbb{P}_\mathcal{N}(\phi) \right)
\]

\[
= \exp \left[ - D_4 \sum_{\rho \in \mathcal{N}} |z(\beta, \rho, \mathcal{N})|(|\sigma, \rho|^2) \right].
\]

Claim 3.2. in [KP17] then shows that when \( \beta \) is sufficiently small,

\[
\frac{Z_{\mathcal{N}}(\sigma)}{Z_{\mathcal{N}}(0)} \geq \exp \left( - \frac{\varepsilon \beta}{2(1 + \varepsilon)} \sum_{j=1}^{d} (\sigma_j - \sigma_j')^2 \right) = \exp \left( - \frac{\varepsilon}{2(1 + \varepsilon)} \varepsilon \beta (f_j - \Delta^{-1} f_j) \right),
\]

which thus ended the proof in [FS81, KP17].

In our present setting, the functional \( \phi \mapsto S(\mathcal{N}, a, \phi) \) introduced in (4.6) is no longer an odd functional of \( \phi \). Furthermore, the Lower-bound (4.5) suggests introducing the following \( a \)-rewighted probability measure

\[
d\mathbb{P}_\mathcal{N}^a(\phi) := \frac{1}{Z_{\mathcal{N}}^a(0)} \prod_{\rho \in \mathcal{N}} [1 + z(\beta, \rho, \mathcal{N}) \cos((\phi, \rho) - \langle a, \rho \rangle)] d\mu_{\mathcal{GFF}, a}(\phi)
\]

which is no longer invariant under \( \phi \mapsto -\phi \). This lack of symmetry does not allow us to rely on Jensen and we are left with analyzing the quantity

\[
\int e^{S(\mathcal{N}, a, \phi)} d\mathbb{P}_\mathcal{N}^a(\phi)
\]

We will not succeed in controlling the full Laplace transform but will instead extract bounds on the first and second moments from the series expansion near \( \alpha \sim 0 \) of the Laplace transform \( \alpha \mapsto \mathbb{E}_b^{a} \alpha [e^{\alpha(\phi, f)}] \).
For any $\alpha \in \mathbb{R}$, we have (recall (4.3), (4.5) and (4.7)) the lower bound
\[
\mathbb{E}^a_{\beta,\lambda,a,v}[e^{\alpha(\phi,f)}] = e^{\alpha^2(\phi, -\Delta^{-1}f)\sum_{N \in F}c_NZ_N^a(0)}/\sum_{N \in F}c_NZ_N^a(0)
\]
(4.8)
\[
\geq e^{\alpha^2(\phi, -\Delta^{-1}f)\sum_{N \in F}c_NZ_N^a(0)}/\sum_{N \in F}c_NZ_N^a(0)
\]
(4.9)
where now
\[
S_\alpha(N, a, \phi) = -\sum_{\rho \in N} \frac{z(\beta, \rho, N)\sin((\phi, \rho) - (a, \rho))\sin(\alpha(\rho))}{1 + z(\beta, \rho, N)\cos((\phi, \rho) - (a, \rho))}
\]
(4.10)
\[
\geq -\alpha \sum_{\rho \in N} \frac{z(\beta, \rho, N)\sin((\phi, \rho) - (a, \rho))\sin(\alpha(\rho))}{1 + z(\beta, \rho, N)\cos((\phi, \rho) - (a, \rho))} + O(\alpha^3).
\]
(4.11)
This Taylor expansion holds first because we are in the regime where $\beta$ can be chosen small enough so that the denominators are uniformly $\geq 1/2$ (see [FS81, KP17]), and second because our parameters $\Lambda, \beta$ etc. are fixed as $\alpha$ is going to zero.

**First order analysis.** At first order in $\alpha$, we obtain combining (4.8) and (4.10) that for any $f : \Lambda \to \mathbb{R}$ and as $\alpha \to 0$,
\[
1 + \alpha\mathbb{E}^a_{\beta,\lambda,a,v}[\langle \phi, f \rangle] + O(\alpha^2)
\]
\[
\geq (1 + O(\alpha^2))\sum_{N \in F}c_NZ_N^a(0) \int [1 + S_\alpha(N, a, \phi) + O(\alpha^2)]d\mathbb{P}_N^a(\phi)
\]
\[
= 1 - \alpha \sum_{N \in F}c_NZ_N^a(0) \mathbb{E}_N^a[\sum_{\rho \in N} \frac{z(\beta, \rho, N)\sin((\phi, \rho) - (a, \rho))\sin(\alpha(\rho))}{1 + z(\beta, \rho, N)\cos((\phi, \rho) - (a, \rho))} + O(\alpha^2)]
\]
In particular, identifying order 1 terms (and recalling that $\sigma := \frac{1}{2}(-\Delta)^{-1}f$, see (2.8)), we thus have for any $f : \Lambda \to \mathbb{R}$,
\[
\mathbb{E}^a_{\beta,\lambda,a,v}[\langle \phi, f \rangle] \geq -\frac{\sum_{N \in F}c_NZ_N^a(0) \sum_{\rho \in N} z(\beta, \rho, N)\sin((\phi, \rho) - (a, \rho))(-\Delta)^{-1}f, \rho)\sin(\alpha(\rho))}{\sum_{N \in F}c_NZ_N^a(0)}
\]
The key observation at this stage is that for each collection of charges $N$, the functional
\[
f \mapsto \hat{S}(N, a, \phi, f) := -\sum_{\rho \in N} \frac{z(\beta, \rho, N)\sin((\phi, \rho) - (a, \rho))(-\Delta)^{-1}f, \rho)}{1 + z(\beta, \rho, N)\cos((\phi, \rho) - (a, \rho))}
\]
is linear in $f$. Obviously the functional $f \mapsto \mathbb{E}^a_{\beta,\lambda,a,v}[\langle \phi, f \rangle]$ is linear as well. Now by using this linearity and plugging $-f$ into the above inequality, we obtain a rather surprising exact expression for the mean value of $(\phi, f)$ under the measure $\mu^a_{\beta,\lambda,a,v}$.

**Proposition 4.1 (Modular invariance identity).** For any function $f$ and any weights $\lambda_\Lambda = (\lambda_i)_{i \in \Lambda}$ satisfying the same hypothesis as in (5.35) in [FS81] (or equivalently (1.9) in [KP17]), we have
\[
\mathbb{E}^a_{\beta,\lambda,a,v}[\langle \phi, f \rangle] = -\frac{\sum_{N \in F}c_NZ_N^a(0) \sum_{\rho \in N} z(\beta, \rho, N)\sin((\phi, \rho) - (a, \rho))(-\Delta)^{-1}f, \rho)}{\sum_{N \in F}c_NZ_N^a(0)}
\]
(4.12)
Remark 4.2. This exact identity, as we shall see below, is a key step in our proof. Because it is so central and since it does not look like anything familiar, we added Appendix A to give a longer but more natural second derivation of this identity. It should not come as a surprise that our second derivation is longer as the above one relies in fact on several key parts of the proof of Fröhlich-Spencer [FS81]. Appendix A gives a complementary interpretation/explanation of the origin of such an identity. In particular in Appendix A, we shall view the shift vector $\mathbf{a} = \{a_z\}_{z \in \Lambda}$ as an exterior magnetic field and we will also explain why we call this identity “modular invariance” due to a relationship with the functional equation for Riemann-theta functions.

Second order analysis. The above identity for the first moment will be instrumental in bounding from below the desired second moment as we shall now see.

Again by combining (4.8) and (4.10), we find that

$$1 + \alpha \mathbb{E}_\beta,\lambda,\psi^a ((\phi, f)] + \frac{1}{2} \alpha^2 \mathbb{E}_{\beta,\lambda,\psi}^a [(\phi, f)]^2 + O(\alpha^3)$$

$$\geq e^{-2 \varepsilon (1 + \beta)} \sum_{N \in F} c_N Z_N^a (0) \mathbb{E}_N^a \left[ S_N (N, \mathbf{a}, \phi) \right]$$

$$\geq e^{-2 \varepsilon (1 + \beta)} \sum_{N \in F} c_N Z_N^a (0) \mathbb{E}_N^a \left[ 1 - S (N, \phi, f) + \frac{1}{2} [S (N, \phi, f)]^2 + O(\alpha^3) \right]$$

$$= 1 - \alpha \sum_{N \in F} c_N Z_N^a (0) \mathbb{E}_N^a \left[ \hat{S} (N, \mathbf{a}, \phi, f) \right]$$

$$+ \frac{\alpha^2}{2} \left[ \frac{1}{(1 + \varepsilon) \beta} (f, -\Delta^{-1} f) + \sum_{N \in F} c_N Z_N^a (0) \mathbb{E}_N^a \left[ \hat{S} (N, \mathbf{a}, \phi, f) \right]^2 \right] + O(\alpha^3).$$

The first order term are equal by Proposition 4.1 and from the second order terms, we extract the following lower bound

$$\mathbb{E}_\beta,\lambda,\psi^a ((\phi, f)]^2 \geq \frac{1}{(1 + \varepsilon) \beta} (f, -\Delta^{-1} f) + \frac{\sum_{N \in F} c_N Z_N^a (0) \mathbb{E}_N^a \left[ \hat{S} (N, \mathbf{a}, \phi, f) \right]^2}{\sum_{N \in F} c_N Z_N^a (0)}$$

(4.13)

$$\geq \frac{1}{(1 + \varepsilon) \beta} (f, -\Delta^{-1} f) + \left( \frac{\sum_{N \in F} c_N Z_N^a (0) \mathbb{E}_N^a \left[ \hat{S} (N, \mathbf{a}, \phi, f) \right]}{\sum_{N \in F} c_N Z_N^a (0)} \right)^2$$

(4.14)

$$= \frac{1}{(1 + \varepsilon) \beta} (f, -\Delta^{-1} f) + \mathbb{E}_{\beta,\lambda,\psi}^a [(\phi, f)]^2$$

(4.15)

first by applying Cauchy-Schwarz inequality to a suitable probability measure on the coupling $(N, \phi)$. And then we used Proposition 4.1 for the last equality, i.e. the modular invariance identity (4.12). This ends our proof. □

4.2 Non-recovery phase ($T > T^\ast_{\text{IV}}$). In this subsection, we complete the non-recovery phases of Theorem 1.1 and 1.2.

As in Definition 3.1, let $(\phi_1, \phi_2)$ be two conditionally independent instances of $\phi$ given $\phi \equiv \phi \mod \frac{2\pi}{T} = \frac{2\pi}{T} \mathbf{a}$. By Lemma 2.8, the law of $(\phi_1, \phi_2)$ is given by $\mathbb{P} (\psi_1, \psi_2)$ where $\psi_1, \psi_2$ are independently sampled according to $\mathbb{P}^a_{\beta_T = 2\pi T}$. 
Thanks to this, we have that for any continuous function \( f : [-1, 1]^2 \mapsto \mathbb{R} \)
\[
\mathbb{E} \left[ (\phi_1 - \phi_2, f)^2 \right] = \left( \frac{2\pi}{T} \right)^2 \mathbb{E} \left[ \mathbb{E}_{1 \sim \mathcal{D}} \left[ (\psi_1 - \psi_2, f)^2 \right] \right] \geq \frac{1}{1 + \varepsilon} (f, (-\Delta)^{-1} f).
\] (4.16)
Furthermore, let us note that because both \( \phi_1 \) and \( \phi_2 \) are two GFF we have that
\[
\mathbb{E} \left[ (\phi_1 - \phi_2, f)^4 \right] \leq 4 \mathbb{E} \left[ (\phi_1, f)^2 \right]^2 = 4 (f, (-\Delta)^{-1} f)^2.
\]
Therefore, using Paley-Zygmund inequality we have that
\[
\mathbb{P} \left( (\phi_1 - \phi_2, f)^2 \geq \frac{1}{2} (f, (-\Delta)^{-1} f) \right) \geq \frac{1}{16(1 + \varepsilon)} > 2^{-5}.
\] (4.17)
Now, we use that for any deterministic function \( F \) depending only on \( \exp(iT\phi) \), we have that \( F(\exp(iT\phi)) = F(\exp(iT\phi_2)) \). Using this we can compute
\[
2^{-5} \leq \mathbb{P} \left( (\phi_1 - \phi_2, f)^2 \geq \frac{1}{2} (f, (-\Delta)^{-1} f) \right) \leq \mathbb{P} \left[ (F(\exp iT\phi_1) - \phi_i, f)^2 \geq \frac{1}{8} (f, (\Delta)^{-1} f), \text{ for some } i \in \{1, 2\} \right] \leq 2 \mathbb{P} \left[ (F(\exp iT\phi_1) - \phi_1, f)^2 \geq \frac{1}{8} (f, (\Delta)^{-1} f) \right].
\] (4.18)
We conclude by noting that for any continuous non-zero function \( f \), we have that \( (f, (-\Delta)^{-1} f) \geq C n^4 \).

To finish, let us show (1.3). We start by noting that
\[
\mathbb{E} \left[ (\phi_1(x) - \phi_2(x))^2 \right] = \left( \frac{2\pi}{T} \right)^2 \mathbb{E} \left[ \mathbb{E}_{1 \sim \mathcal{D}} \left[ (\psi_1(x) - \psi_2(x))^2 \right] \right] \geq \frac{1}{(1 + \varepsilon)^2} G(x, x) \geq 4c(T, x) \log(n).
\]
We now see that
\[
2 \mathbb{E} \left[ (\phi_1(x) - F(\phi_1)(x))^2 \right] = \mathbb{E} \left[ (\phi_1(x) - F(\phi_1)(x))^2 + (\phi_2(x) - F(\phi_1)(x))^2 \right] \geq \frac{1}{2} \mathbb{E} \left[ (\phi_1(x) - \phi_2(x))^2 \right] \geq 2c(T, x) \log(n).
\]

We now complete this Section by proving Corollary 1.3.

### 4.3 Proof of Corollary 1.3

Let us take \( \phi_n \to \Phi \) in probability for the topology of the space of generalised functions. Let us now analyze the two regimes \( T \ll 1 \) and \( T \gg 1 \).

#### 4.3.1 Small \( T \)

Let us note that thanks to part (1) of Proposition 1.1, we have that for any smooth function \( f \) (with 0-mean if we are in the free boundary case). We have that
\[
\frac{1}{n^2} \langle F_T(e^{i T \phi_n}), f \rangle \to \langle \phi_n, f \rangle + \langle F_T(e^{i T \phi_n}) - \phi_n, f \rangle \to \langle \Phi, f \rangle.
\]
From this we see that \( F_T(e^{i T \phi_n}) \) also converges in probability to \( \Phi \).
4.3.2 Big $T$. Let us reason by contradiction and assume such a function $F$ exists. Let $(\phi_1^{(n)}, \phi_2^{(n)})$ be two GFF coupled as in Definition 3.1, we see that in this case

$$F(\exp(iT\phi_1^{(n)})) = F(\exp(iT\phi_2^{(n)})). \quad (4.20)$$

Because both $\phi_1^{(n)}$ and $\phi_2^{(n)}$ have the law of a GFF in $\Lambda_n$, we see that the pair $(\phi_1^{(n)}, \phi_2^{(n)})$ is tight. We can now take a subsequence of $(\phi_1^{(n)}, \phi_2^{(n)})$, that we denote the same way such that

$$(\phi_1^{(n)}, \phi_2^{(n)}) \to (\Phi_1, \Phi_2).$$

The second part of Theorems 1.1 and 1.2 imply that $\Phi_1 \neq \Phi_2$. However, we have that (by the contradiction hypothesis)

$$F(\exp(iT\phi_1^{(n)})) \to \Phi_1 \neq \Phi_2 \not\to F(\exp(iT\phi_2^{(n)})),$$

which is a contradiction with (4.20).

5 There is always information left

The objective of this section is to prove that for any $T > 0$, $\exp(iT\phi)$ gives non-trivial (macroscopic) information of $\phi$. More precisely, in this section we quantify how much information is preserved under the operation $\phi \mapsto \phi \pmod{\frac{2\pi}{T}}$.

Let us note that Theorem 1.8, implies that for all possible values of $\exp(iT\phi)$ and for all $T$ big enough there exists $\varepsilon(T)$ such that

$$\text{Var} [\langle \phi, f \rangle \exp(iT\phi)] \geq (1 - \varepsilon(T)) E [\langle \phi, f \rangle^2].$$

At the same time, it is clear that

$$E [\text{Var} [\langle \phi, f \rangle \exp(iT\phi)]] \leq E [\langle \phi, f \rangle^2].$$

Let us remark that it is not clear whether this $\varepsilon(T)$ is a technical constant coming from the proof or whether it is telling us something meaningful about the model. In the following proposition we show that in the average case the existence of this $\varepsilon(T)$ is not technical. In fact, in Remark 5.2 below we give an interpretation of its meaning. See also Remark 5.3 for the link with the $\varepsilon = \varepsilon(T)$ correction in Fröhlich-Spencer.

Proposition 5.1. Let $T > 0$ and $\phi$ be a GFF with either free or 0 boundary condition in $\Lambda_n$. Then, there exists $\varepsilon'(T) > 0$ such that

$$E [\text{Var} [\langle \phi, f \rangle^2 \exp(iT\phi)]] \leq (1 - \varepsilon'(T)) E [\langle \phi, f \rangle^2]. \quad (5.1)$$

Furthermore, we have the following upper bound for $\varepsilon'(T)$ when $T \gg 1$

$$\varepsilon'(T) \gtrsim \frac{(2\pi)^2}{T^2} e^{-\frac{\pi^2}{2T^2}} = \beta_T e^{-\frac{\pi}{T}}. \quad (5.2)$$

Remark 5.2. Proposition 5.1 should be interpreted in the following way:

The field $\exp(iT\phi_n)$ gives non-trivial information of the GFF $\phi_n$. This is because, if this were not the case we would have that for any continuous function $f : [-1,1]^2 \to \mathbb{R}$

$$n^{-4} \text{Var} [\langle \phi_n, f \rangle \exp(iT\phi_n)] = \iint_{[-1,1]^2} f(x)G(x,y)f(y)dx\,dy \Rightarrow \lim_{n \to \infty} n^{-4} E [\langle \phi_n, f \rangle],$$

where $G$ is the continuous Green's function in $[-1,1]^2$. In the Statistics world, we would say that Proposition 5.1 means that $\exp(iT\phi)$ explains at least $\varepsilon'(T)$ of the variance of $\phi$. 

Proof. Let us write $F(x) := \mathbb{E} [\phi(x) | \exp(iT\phi_n)]$ and $\phi = \phi_n$. We are going to prove that
\[
\mathbb{E} [(F, f)^2] \geq \varepsilon \mathbb{E} [(\phi, f)^2].
\] (5.3)
This suffices as
\[
\mathbb{E} [\text{Var} (\phi, f | \exp(iT\phi))] = \mathbb{E} [(F - \phi, f)^2] = \mathbb{E} [(\phi, f)^2] - \mathbb{E} [(F, f)^2].
\]

To prove equation (5.3). Let us take $W = \nabla \phi + \zeta$ as in Proposition 2.5, let us bound the following
\[
\mathbb{E} (\langle \phi, f \rangle^2) = \mathbb{E} \left[ \mathbb{E} (\langle \phi, f \rangle | \exp(iT\phi))^2 \right]
\]
\[
\geq \mathbb{E} \left[ \mathbb{E} (\langle \phi, f \rangle | \exp(iT\phi))^2, \exp(iTW) \right],
\]

where we have used Cauchy-Schwartz and the fact that $\exp(iT\phi)$ is independent of the pair $(\langle \phi, f \rangle, \exp(iTW))$. As such, to end it only remains to show that
\[
\mathbb{E} \left[ \mathbb{E} [\langle \phi, f \rangle | \exp(iTW)]^2 \right] \geq \varepsilon \mathbb{E} [(\phi, f)^2].
\] (5.4)

Now, recall from Proposition 2.5 that $\phi = \Delta^{-1} \nabla \cdot W$ and compute
\[
\mathbb{E} [\langle \phi, f \rangle | \exp(iTW)] = \mathbb{E} [\langle W, -\nabla \Delta^{-1} f \rangle | \exp(iTW)]
\]
\[
= -\frac{1}{2} \sum_{\vec{e} \in E} \mathbb{E} [W(\vec{e}) | \exp(iTW)] \nabla \Delta^{-1} f(\vec{e})
\]
\[
= \frac{1}{2} \sum_{\vec{e} \in E} \mathbb{E} [W(\vec{e}) | \exp(iTW(\vec{e}))] \nabla \Delta^{-1} f(\vec{e}),
\]
where the last line comes from the independence between the value of $W$ in different edges, and the fact that $W(\vec{e}) = -W(\vec{e})$. The equality of this last line may seem innocent but it is the main reason why the problem simplifies when we work with the white noise.

Let us note that the random variable $\mathbb{E} [W(\vec{e}) | \exp(iTW(\vec{e}))]$ is centred and has the same law for all $\vec{e}$. Furthermore, it is independent for all $e \neq e'$. Let us define
\[
\sigma(T) = \text{Var} [\mathbb{E} [W(\vec{e}) | \exp(iTW(\vec{e}))]], > 0.
\] (5.5)

We can now compute
\[
\mathbb{E} \left[ \mathbb{E} [\langle \phi, f \rangle | \exp(iTW)]^2 \right] = 2\pi \sigma(T) (\nabla \Delta^{-1} f(e), \nabla \Delta^{-1} f(e))
\]
\[
= 2\pi \sigma(T) (f, -\Delta^{-1} f) = \sigma(T) \mathbb{E} [(\phi, f)],
\]
from where we obtain (5.1).

To obtain (5.2), we remark that we set $\varepsilon'(T) = \sigma(T)$. When $T \gg 1$, one can get (5.2) by estimating (5.5) using (A.1) and Lemma 2.8. □

Remark 5.3. Proposition 5.1 is one of the reasons that this model is a laboratory for the IV-GFF. In this case, it allows us to conjecture that the best $\varepsilon(T)$ that one can obtain in the result of Fröhlich and Spencer at inverse temperature $\beta \ll 1$ is $\Omega(\beta e^{-1/\beta})$. 

6 Conjectures on \( T_{\text{rec}} \) and the Interfaces of the Models.

The main focus of this section is to state several conjectures. However, we also prove some intermediate results which are interesting on their own and which will give support to each of these predictions.

6.1 Lower bound on the value of \( T_{\text{rec}} \). The objective of this part is to justify the following conjecture:

**Conjecture 1.** We have that \( T_{\text{rec}} \geq 2\sqrt{\pi} \).

We have two reasons to believe this conjecture, both of them related to the continuum Gaussian free field. The first reason concerns the so-called imaginary chaos and the second one is related to the flow lines of the continuum GFF.

6.1.1 Reason 1: Imaginary chaos. We will not introduce all the definitions here. We refer to [LRV14, JSW18] for context and the definition. Take \( \Phi_0 \)-boundary continuum Gaussian free field in a domain \( D \subseteq \mathbb{C} \) and let \( \nu_{x, \varepsilon} \) be the uniform measure on \( \partial B(x, \varepsilon) \). We normalise \( \Phi \) so that if \( d(x, y) \geq \varepsilon \)

\[
E[(\Phi, \nu_{x, \varepsilon})(\Phi, \nu_{y, \varepsilon})] = G_D(x, y).
\]

Note that in our normalisation \( G_D(x, y) \sim \frac{1}{\pi} |\log(|x - y|)| \).

We can now prove the following result.

**Proposition 6.1.** Assume

(H1) There exists \( \hat{\alpha} \) such that for all \( \alpha < \hat{\alpha} \) the GFF \( \Phi \) can be measurably recovered from \( V^\alpha(\Phi) \), i.e., that there exists a deterministic measurable function \( F \) such that a.s \( F(V^\alpha) = \Phi \).

Then, we have that \( T_{\text{rec}} \geq \hat{\alpha} \).

In fact, we expect that (H1) holds for \( \hat{\alpha} = 2\sqrt{\pi} \). Indeed, it is conceivable that a technique similar to the one developed by [BSS14] may apply in this case. However, let us emphasise that this case is more subtle as one needs to connect the local fluctuations all the way to the values of the boundary.

To prove proposition 6.1, we need to show that the discrete imaginary chaos is converging to the continuous one.

**Proposition 6.2.** Let \( \phi^{(n)} \) be a discrete \( 0 \)-boundary GFF in \( \Lambda_n \) and let

\[
V^\alpha_n(\cdot) := \lim_{\varepsilon \to 0} \exp \left( i\alpha \phi_\varepsilon(\cdot) + \frac{\alpha^2}{2} E \phi_\varepsilon^2(\cdot) \right),
\]

then for all \( \alpha < 2\sqrt{\pi} \), as \( n \to \infty \)

\[
(\phi^{(n)}, V^\alpha_n) \to (\Phi, V^\alpha(\Phi)), \quad \text{in law,}
\]

for the topology of generalised functions. Here \( \Phi \) is a \( 0 \)-boundary GFF in \([-1, 1]^2\).
As this section is concerned mostly with conjectures we will only do a sketch of the proof of this result. The main input is the fact that Theorem 1.3 of [JSW18] which states that \((\Phi, \mathcal{V}_n(\Phi))\) is characterised by its moments.

Proof. We start by recalling that, thanks to Theorem 1.3 of [JSW18], the field \((\Phi, \mathcal{V}_n(\Phi))\) is characterised by its moments. By this, we mean that it is characterised by

\[
\mathbb{E} \left[ \left( \prod_{i} (\Phi_i, f^1_i) \right) \left( \prod_{j} (\mathcal{V}_n, f^2_j) \right) \left( \prod_{k} (\mathcal{V}_n, f^3_k) \right) \right] = \int \left( \prod_{i} f^1_i(x_i) dx_i \right) \left( \prod_{j} f^2_j(y_j) dy_j \right) \left( \prod_{k} f^3_k(z_k) dz_k \right) C((x_i), (y_j), (z_k)_k),
\]

where all \(f_i\) are smooth functions in \([-1, 1]^2\) (with 0-mean if \(\Phi\) is a free-boundary GFF). The function \(C(\cdot, \cdot, \cdot)\) is called the correlation function of this model. By a simple (but lengthy) computation one can see that (6.1) also appears from the discrete setting

\[
\mathbb{E} \left[ \left( \prod_{i} n^{-2}(\phi_n, f^1_i) \right) \left( \prod_{j} n^{-2}(\mathcal{V}_n, f^2_j) \right) \left( \prod_{k} n^{-2}(\mathcal{V}_n, f^3_k) \right) \right] \to \int \left( \prod_{i} f^1_i(x_i) dx_i \right) \left( \prod_{j} f^2_j(y_j) dy_j \right) \left( \prod_{k} f^3_k(z_k) dz_k \right) C((x_i), (y_j), (z_k)_k),
\]

at least when all function \(f_s\) have different support. This can be proven by noting that \(C\) is obtained only from the Green’s function and that the discrete Green’s function is converging to the continuum one (Corollary 3.11 of [CS11]). To finish, one needs to show that (6.2) is true for all possible \(f_s\). This can be done using the dominated convergence theorem. To see that the sum coming from the LHS of (6.2) is uniformly dominated one uses Theorem 2.5 of [CS11], i.e., that

\[
G(x, y) = -(2\pi)^{-1} \log \left( \frac{\|x - y\|}{n} \right) + O(1),
\]

and uses the same techniques as Section 3.2 of [JSW18]. \(\square\)

We can now prove Proposition 6.1.

Proof of Proposition 6.1. Take \(\phi_1^{(n)}\) and \(\phi_2^{(n)}\) two 0-boundary GFF coupled as in Definition 3.1. Thanks to Proposition 6.2, we have that the 4-tuple

\[
(\phi_1^{(n)}, \mathcal{V}_{1,n}^n, \phi_2^{(n)}, \mathcal{V}_{2,n}^n)
\]

is tight. Take \((\Phi_1, \mathcal{V}_{1}^n, \Phi_2, \mathcal{V}_{2}^n)\), any accumulation point of the sequence and note that because for all \(n \in \mathbb{N}\), a.s. \(\mathcal{V}_{1,n}^n = \mathcal{V}_{2,n}^n\) we have that \(\mathcal{V}_{1}^n = \mathcal{V}_{2}^n\). This equality implies, thanks to Assumption (H1) that a.s. \(\Phi_1 = \Phi_2\). Then, as all accumulation points are the same we have that, in fact, as \(n \to \infty\)

\[
(\phi_1^{(n)}, \mathcal{V}_{1,n}^n, \phi_2^{(n)}, \mathcal{V}_{2,n}^n) \to (\Phi_1, \mathcal{V}_1^n, \Phi_1, \mathcal{V}_1^n) \quad \text{in distribution.}
\]

Let us, now, take any smooth function \(f\), we have that for all \(j \in \{1, 2\}\)

\[
\sup_n \mathbb{E} \left[ \left( \frac{1}{n^2} (\Phi_j, f) \right)^4 \right] < K,
\]
which implies that
\[ \mathbb{E} \left[ \left( \frac{1}{n^2} (\Phi_1 - \Phi_2, f) \right)^2 \right] \to \mathbb{E} \left[ (\Phi_1 - \Phi_1, f)^2 \right] = 0. \]

As this implies that \( \mathbb{E} \left[ \text{Var} (\phi_1^{(n)}, f) | \exp(i\alpha\phi_1^{(n)}) \right] = o(n^4) \), we conclude as in the beginning of Section 3.

\[ \square \]

Remark 6.3. Let us note that even if we Assumption \((H_1)\) is proven, this only shows that \( \mathbb{E} \left[ \text{Var} (\phi_1^{(n)}, f) \right] = o(n^4) \), which is a weaker result than the one in Proposition 3.8 in which we showed that \( \mathbb{E} \left[ \text{Var} (\phi_1^{(n)}, f) \right] = O(n^2) \).

6.1.2 Reason 2: Flow lines. Flow lines of the Gaussian free field were introduced in \([\text{She05}, \text{Dub09}]\) and were studied in depth in \([\text{MS16a}, \text{MS16b}, \text{MS16c}, \text{MS17}]\). Informally, they can be described as the curve which is the solution of
\[ \eta'(t) = e^{i(\sqrt{2\pi} \, \pm \, u)}, \quad \eta(0) = z \in \partial D, \]
where \( \Phi \) is a GFF in a simply connected domain \(D\) and \( u \) is a harmonic function. For us it is important to note that the curve \( \eta \) should only be determined by \( e^{i(\sqrt{2\pi} \, \pm \, u)} \). This will motivate Assumption \((H_2)\).

Flow lines can be defined using the concept of local sets \([\text{SS13}, \text{Wer16}]\). In other words, \( \eta \) is a flow line of a GFF \( \Phi \) if for any stopping time \( \tau \) of the natural filtration of \( \eta \) we have
\[ \Phi = \Phi^{\eta_\tau} + h_{\eta_\tau}, \]
where \( \eta_\tau = \eta([0, \tau]) \). \( \Phi^{\eta_\tau} \) has the law of a GFF of \( D \setminus \eta_\tau \) and \( h_{\eta_\tau} \) is a harmonic function in \( D \setminus \eta_\tau \). Let us remark that in this case the function \( h_{\eta_\tau} \) is, in fact, a measurable function of \( \eta_\tau \). In fact, it can be found in Theorem 1.1 of \([\text{MS16a}]\).

A generalisation of flow-lines is given by the angle-varying flow lines defined in Section 5.2 of \([\text{MS16a}]\), which can be roughly described as running a flow line with initial angle \( \theta_1 \) until a stopping time \( \tau_1 \), and then continue with an angle \( \theta_2 \) until a stopping time \( \tau_2 \), and continue until finitely many iterations. This lines are called \( \eta_{\theta_1 \ldots \theta_r} \) and they are a measurable function of \( \Phi \), the GFF they are coupled with (Lemma 5.6 of \([\text{MS16a}]\)).

In fact, Proposition 5.9 of \([\text{MS16a}]\), shows that if \( \chi \geq 1/\sqrt{2} \), there exists a countable set of angle-varying flow lines \( (\eta_{\theta_1 \ldots \theta_r})_{k \in \mathbb{N}} \) such that a.s.
\[ \bigcup_n \eta_{\theta_1 \ldots \theta_r} \]
is dense (because \( \text{SLE}_8 \) is a space-filling curve). Now, define \( \mathbb{F}_n \) as the \( \sigma \)-algebra generated by \( \eta_{\theta_1 \ldots \theta_r} \). The discussion in the paragraph before and the fact that \( h_{\eta_{\theta_1 \ldots \theta_r}} \) is a measurable function of the set \( \eta_{\theta_1 \ldots \theta_r} \) implies that the \( \mathbb{F} = \bigcup_n \mathbb{F}_n \) is equal to the sigma algebra generated by \( \Phi \) (see for example Lemma 2.3 of \([\text{ALS19}]\)). In other words, \( \Phi \) is a deterministic function of \( (\eta_{\theta_1 \ldots \theta_r})_{n \in \mathbb{N}} \).

This allows us to show the following proposition.

Proposition 6.4. Take \( \phi_n \) a 0-boundary GFF in \( \Lambda_n \), assume
\[ \tau_{n}^{w.r.t.} \text{the natural filtration of } \eta \]
\((H_2)\) There exists \(\tilde{\chi} \geq 1/\sqrt{2}\) such that for all \(\chi > \tilde{\chi}\) and for any \(\eta^{\tau_1...\tau_k}_{\theta_{\ell_1}...\theta_{\ell_k}}(\Phi)\) an angle-varying flow line, there exists an approximated angle-varying flow line \(\eta^{(n)}\) depending on \(\exp(i\sqrt{2\pi} \phi_n/\chi)\) such that \((\phi_n, \eta^{(n)}(\Phi))\) converges in law to \((\Phi, \eta^{\tau_1...\tau_k}_{\theta_{\ell_1}...\theta_{\ell_k}}(\Phi))\).

Then \(T_{\text{rec}}^- \geq \sqrt{2\pi}/\chi\).

Before proving the proposition, let us recall that it is expected that the flow lines related to the discrete GFF are converging to the flow lines of the continuum GFF, as this is already the case for \(\chi = \infty\), the SLE\(_4\) case [SS09]. If this were the case, Proposition 6.4 implies that \(T_{\text{rec}}^- \geq 2\sqrt{\pi}\).

Proof. Let us take \(\Phi\) a continuous GFF with 0-boundary condition. Thanks to Assumption \((H_2)\), we can define \(\eta^{(n)}_k\) such that as \(n \to \infty\)

\[(\phi_n, \eta^{(n)}_k) \to \left(\Phi, \eta^{\tau_1...\tau_k}_{\theta_{\ell_1}...\theta_{\ell_k}}(\Phi)\right), \quad (6.4)\]

in law. We then have that

\[(\phi_n, (\eta^{(n)}_k)_{k \in \mathbb{N}}) \to \left(\Phi, (\eta^{\tau_1...\tau_k}_{\theta_{\ell_1}...\theta_{\ell_k}}(\Phi))_{k \in \mathbb{N}}\right) \quad (6.5)\]

in law for the product topology. This follows because \((6.4)\) implies that \((\phi_n, (\eta^{(n)}_k)_{k \in \mathbb{N}})\) is tight for the product topology. We can then check, again thanks to \((6.4)\), that any accumulation point \((\Phi, (\eta^{(n)}_k)_{k \in \mathbb{N}})\) has to be such that

\[(\Phi, \eta^{\infty}_k) = \left(\Phi, \eta^{\tau_1...\tau_k}_{\theta_{\ell_1}...\theta_{\ell_k}}(\Phi)\right)\]

As a consequence, we have that

\[(\Phi, (\eta^{(n)}_k)_{k \in \mathbb{N}}) = \left(\Phi, (\eta^{\tau_1...\tau_k}_{\theta_{\ell_1}...\theta_{\ell_k}}(\Phi))_{k \in \mathbb{N}}\right), \quad (6.5)\]

which implies \((6.5)\).

We can now conclude in a similar way as in Proposition 6.1. We take \((\phi_1, \phi_2)\) coupled as in Definition 3.1 and we study the 4-tuple

\[
(\phi^{(1)}_1, (\eta^{(n)}_k(\phi^{(1)}_1))_{k \in \mathbb{N}}, \phi^{(2)}_2, (\eta^{(n)}_k(\phi^{(2)}_2))_{k \in \mathbb{N}}) \]

Again, we have that this 4-tuple is tight and that any accumulation point is such that

\[
(\Phi_1, (\eta^{\tau_1...\tau_k}_{\theta_{\ell_1}...\theta_{\ell_k}}(\Phi_1))_{k \in \mathbb{N}}, \Phi_2, (\eta^{\tau_1...\tau_k}_{\theta_{\ell_1}...\theta_{\ell_k}}(\Phi_2))_{k \in \mathbb{N}}) \]

as for all \(n, k \in \mathbb{N}\), we have that \(\eta_n(\phi^{(1)}_1) = \eta^{(n)}_k(\phi^{(2)}_2)\), we have that in this accumulation point a.s.

\[
\eta^{\tau_1...\tau_k}_{\theta_{\ell_1}...\theta_{\ell_k}}(\Phi_1) = \eta^{\tau_1...\tau_k}_{\theta_{\ell_1}...\theta_{\ell_k}}(\Phi_2) .
\]

As \(\Phi_1\) is a function of this \((\eta^{\tau_1...\tau_k}_{\theta_{\ell_1}...\theta_{\ell_k}}(\Phi_1))_{k \in \mathbb{N}}\), we see that \(\Phi_1 = \Phi_2\), which implies that \((\phi_1^{(n)}, \phi_2^{(n)})\) converges in law to \((\Phi_1, \Phi_1)\). By the same reasoning as the end of 6.1 we have that for any continuous function \(f\), \(\mathbb{E} \left[\var(\phi^{(n)}, f) \mid e^{\sqrt{2\pi} \phi^{(n)} / \chi}\right]\) is \(o(n^2)\). \(\square\)
6.2 Level line of $\exp(iT\phi)$. In [SS09], the authors showed that the level line of a zero boundary GFF with a special boundary condition converges in law to an SLE$_4$. We believe a similar story holds for both $\exp(iT\phi)$, and more importantly for the Villain model. Let us be more explicit.

We define $u_n$ as the bounded harmonic function in $\Lambda_n \setminus \partial \Lambda_n$ with boundary condition $\lambda = \sqrt{\pi/8}$ in $\partial \Lambda_n \cap \{ x : \text{Re}(x) \geq 0 \}$ and $-\lambda = -\sqrt{\pi/8}$ in $\partial \Lambda_n \cap \{ x : \text{Re}(x) < 0 \}$. It is shown in [SS09], that if $\phi_n$ is a GFF in $\Lambda_n$ with 0-boundary condition and $\eta$ the level line of $\phi + u_n$. That is to say $\eta_n(\cdot)$ is a path in the dual of $\Lambda_n$ that has the following properties (see Figure 6):

- It goes from the dual of the edge $(-i + 1/n, -i)$ to the dual of the edge $(i - 1/n, i)$.
- The primal edge associated to a dual edge in the path is such that $\phi_n$ is negative to its left and positive to its right.

Let us now prove that the set $L_n$ converges in probability to $\eta$ the level line of $\phi + u_n$, note that $\phi + u_n$ takes positive values to the left and negative to the right.

**Figure 6.** The image of the left depicts the boundary values of the harmonic function $u_n$. The image to the right represents the level line of $\phi + u_n$, note that $\phi + u_n$ takes positive values to the left and negative to the right.

Theorem 1.4 of [SS09] is that $\eta^{(n)}(\cdot)$ parametrised by capacity converges in the uniform topology to an SLE$_4$. This result is improved in [SS13] by showing that as $n \to \infty$

$$(\phi_n, \eta^{(n)}) \to (\Phi, \eta) \quad \text{in law}.$$  

Here $(\Phi, \eta)$ are such that $\Phi$ is a GFF in $[-1, 1]$ and $\eta$ is the so-called level line of the continuous GFF. More precisely, $\eta$ is a measurable function of $\Phi$ and the law of $\Phi$ conditioned on $\eta$ is such that

$$\Phi + u_{\infty} = \Phi^L + \Phi^R,$$

where $\Phi^L$, resp. $\Phi^R$, is a GFF in the domain to the left, resp. right, of $\eta$ with $-\lambda$, resp. $\lambda$, boundary condition (see Figure 7).

We now have the tools to prove Corollary 1.4.

**Proof of Corollary 1.4.** We assume that $\phi_n \to \Phi$ a continuum GFF and define $L_n = L^{(n)}_\exp(iT\phi_n)$ as a set parametrised by $q$, where

$$L^{(n)}(q) = \mathbb{E} \left[ \eta^{(n)}(q) \mid \exp(iT\phi) \right].$$

Let us now prove that the set $L_n$ converges in probability to $\eta$ the level line of $\phi$. To do this, it is enough to show that for all $q$, $L^{(n)}(q)$ converges in probability to $\eta(q)$. Thanks to Theorem 1.4 of [SS09] we have that $\eta^{(n)}(q)$ converges in law to $\eta(q)$, now it suffices to show that as $n \to \infty$

$$\text{Var}[\eta^{(n)}(q) \mid \exp(iT\phi)] \to 0, \quad \text{in probability.} \quad (6.6)$$
To do this, we use the same trick as always. Let \((\phi_1^{(n)}, \phi_2^{(n)})\) be two GFFs coupled as in Definition 3.1, we know that thanks to (1) of Theorem 1.1 \((\phi_1^{(n)}, \eta_1^{(n)}, \phi_2^{(n)}, \eta_2^{(n)})\) converges in law to \((\Phi, \eta, \Phi, \eta)\). Here the topology on the curves is that of the uniform distance for continuous curves. As a consequence of the convergence we have that for any \(q \in \mathbb{Q}_P\)

\[
\mathbb{P}(\|\eta_1^{(n)}(q) - \eta_2^{(n)}(q)\| \geq \delta) \to 0 \quad \text{as } n \to \infty.
\]

Due to the fact that the set \(\Lambda_n\) is bounded, we conclude that

\[
\mathbb{E}\left[\|\eta_1^{(n)}(q) - \eta_2^{(n)}(q)\|^2\right] \to 0, \quad \text{as } n \to \infty.
\]

This concludes the proof, as it proves (6.6).

Corollary 1.4 gives us a explicit way to recover the level line of the GFF given its \(e^{iT\phi_n}\). However, this recovery does not locally depend on the field. We also believe that it is possible to recover the level line via an explicit local function of the \(e^{iT(\phi_n + u_n)}\), its own level line.

Now, we let \(T\) be small enough such that \(T\lambda < \pi\), in this way the imaginary part of \(\exp(iTu_n(x))\) has the same as sign as the real part of \(x\). We also define \(\eta^{(n),T}(\cdot)\), the level line of the imaginary part \(\exp(iT(\phi_n + u_n))\). We conjecture the following.

**Conjecture 2.** There exists a small enough \(T_c\) such that for all \(T < T_c\), \(\eta^{(n),T}\) converges in law to a SLE\(_4\). Furthermore, \(\eta^{(n)}\) and \(\eta^{(n),T}\) converge to the same limit.

A part from Corollary 1.4, we have two other reasons to believe in this conjecture. The first one is the fact that the gradient of \(\phi_n\) in its level line \(\eta^{(n)}\) is, in mean, upper and lower bounded (see Lemma 3.1 of \([SS09]\)). Thus, one could expect that most edges in \(\eta\) have corresponding primal edges for which \(\text{Im}(\exp(iT\phi_n))\) is negative on its left vertex and positive on its right one.

The second reason is that level lines do not get close to each other, neither to itself. This can be seen in Section 3.4 and 3.5 of \([SS09]\), or by understanding their scaling limit as in Remark 1.5 of \([WW16]\).

As we said before, we conjecture that we have a similar result for the Villain model. In fact, Fröhlich and Spencer conjectured that the Villain model at low temperature \(T\) is close to the imaginary exponential of a GFF with a slightly different
temperature $T_{\text{Vil}} := T_{\text{Vil}}(T) > T$ (see Section 8.1 of [FS83]). This allows us to interpret Conjecture 2 as follows.

**Conjecture 3.** Take $T$ small enough and let $\psi_n$ be a Villain model in $\Lambda_n$ with temperature $T$ and boundary values given by $\exp(-i\lambda \sqrt{T_{\text{Vil}}})$ in the left side of the boundary, i.e. $\partial \Lambda_n \cap \{x : \text{Re}(x) \geq 0\}$, and $\exp(-i\lambda \sqrt{T_{\text{Vil}}})$ in $\partial \Lambda_n \cap \{x : \text{Re}(x) \geq 0\}$. If we take $\eta^{(n)}$ to be the level line of the imaginary part of $\psi$, then $\eta^{(n)}$ converges in law to an SLE$_4$ (see Figure 8).

![Figure 8](image_url)

**Figure 8.** The image of the left depicts the boundary values of the Villain model. The image to the right represents the level line of the imaginary part of the Villain model. We believe that this level line converges in law to an SLE$_4$ when the temperature of the system is low enough.

In fact, the result should hold for a more generally boundary values.

**Conjecture 4.** Take $T$ small enough and let $\psi_n$ be a Villain model in $\Lambda_n$ with temperature $T$ and boundary values given by $\exp(-ia)$ in the left side of the boundary, i.e. $\partial \Lambda_n \cap \{x : \text{Re}(x) \geq 0\}$, and $\exp(a)$ in $\partial \Lambda_n \cap \{x : \text{Re}(x) \geq 0\}$. Then for a small enough, we take $\eta^{(n)}$ to be the level line of the imaginary part of $\psi$, then $\eta^{(n)}$ converges in law to an SLE$_4(\rho)$, with $\rho = a/(\lambda \sqrt{T}) - 1$.

**6.3 Upper bound on the value of $T_{\text{rec}}^+$.** In fact, the analysis of level lines of the GFF, makes us believe the following conjecture.

**Conjecture 5.** We have that $T_{\text{rec}}^+ \leq 2\sqrt{2\pi}$.

Let us note that the value $2\sqrt{2\pi}$, it is the smallest value of $T$ so that $\exp(iT\lambda) = \exp(-iT\lambda)$. This is to say that this is the value for which we could not expect to recognize the macroscopic difference between the left and the right side of the level line $\eta$ introduced in Section 6.2.

The level line $\eta$ is fundamental to be able to recover the GFF. This is shown, for example, in the construction of the free-boundary GFF given in [QW18].

There is another reason why we believe that one cannot recover $\phi$ when $T = 2\sqrt{2\pi}$. It has to do with the level set of the GFF.

Although the GFF is not a function, one can still define $\mathbb{A}_{-a,b}$. This is informally, the (connected component connected to the boundary of the) preimage of $[-a,b]$. 
These sets were introduced\(^8\) in [AS17, ALS19] and their existence is conditional on the size of the interval \([-a, b] \] 

The set \(\mathcal{A}_{-a, b}\) if and only if \(a, b > 0\) and \(a + b \geq 2\lambda\).

The case \(a + b = 2\lambda\) is special. These are the values such that \(\exp(-iTa) = \exp(iTb)\). Furthermore, in [AS18], it is shown that these are the only values of \(a\) and \(b\) such that the following happens

Fix two-points \(x, y \in [-1, 1]\) and let \(O(x)\) and \(O(y)\) be the connected component of \([-1, 1]^2 \setminus \mathcal{A}_{-a, b}\) containing \(x\) and \(y\) respectively. Then, there is a positive probability that \(O(x) \neq O(y)\) and \(\partial O(x) \cap \partial O(y)\) is a continuous curve.

This property implies that the places where the GFF takes values \(-a\) and the ones where it takes values \(b\) are mesoscopically separated, i.e. they are not macroscopically far apart. As the function \(x \mapsto \exp(i2\sqrt{2\pi}x)\) cannot distinguish between \(-a\) and \(b\), we believe it is not possible to recover \(\mathcal{A}_{-a, b}\) just by knowing \(\exp(i2\sqrt{2\pi}\phi)\). This would make impossible to recover all the macroscopic information of the GFF.

### A Viewing the shift \(a = \{a_i\}_{i \in A}\) as an exterior magnetic field

The goal of this appendix is to provide a different proof of Proposition 4.1. The idea of this proof was inspired to us by an inspection of this exact identity in the simplest possible case of a Gaussian free field on a single point \(\{x\}\) with Dirichlet boundary condition, namely a Gaussian \(\mathcal{N}(0, \frac{1}{\beta})\). The appendix is organized as follows, first we investigate the case of one point, then we make a link with Riemann-theta functions (thus explaining the name modular invariance) and finally we give a second proof of Proposition 4.1.

### A.1 Warm up: GFF with one point and Jacobi-theta function.

Let us consider the GFF on a graph with two points \(\{x, y\}\) with 0-boundary conditions in \(y\). The partition function of the \(a\)-shifted integer valued field (here the vector \(a\) is just one parameter which we call \(a\)) reads as follows:

\[
Z(\beta, a) = \int \left( \sum_{n \in \mathbb{Z}} \delta_{2\pi n + a}(\phi) \right) \frac{1}{\sqrt{2\pi/\beta}} e^{-\frac{\phi^2}{2}} d\phi
\]

\[
= \frac{1}{\sqrt{2\pi/\beta}} \sum_{n \in \mathbb{Z}} \exp(-\beta (2\pi n + a)^2).
\]

In the limiting case where we plug the following infinite Fourier series

\[
1 + 2 \sum_{q=1}^{\infty} \cos(q(\phi - a)) = 2\pi \sum_{n \in \mathbb{Z}} \delta_{2\pi n + a}(\phi)
\]

into the Fröhlich-Spencer expansion on one point, it can be checked that the identity \((4.12)\) reads as follows

\[
\frac{\sum_{n \in \mathbb{Z}} \exp(-\frac{\beta}{2}(2\pi n + a)^2) \cdot (2\pi n + a)}{\sum_{n \in \mathbb{Z}} \exp(-\frac{\beta}{2}(2\pi n + a)^2)} = \frac{1}{4} \sum_{q \in \mathbb{Z}} e^{-\frac{q^2}{2\pi}} \sin(q \cdot a) \cdot q = \frac{1}{4} \sum_{q \in \mathbb{Z}} e^{-\frac{q^2}{2\pi}} \cos(q \cdot a), \quad (A.1)
\]

which is correct for any \(\beta > 0\) and any real \(a \in (-\pi, \pi)\). (Note interestingly that it is degenerate for the L.H.S as \(\beta \to 0\) but not for the R.H.S!)

---

\(^{8}\)See [?] to better understand the relationship between \(\mathcal{A}_{-a, b}\) and the imaginary chaos.
One way to prove this identity is to notice its link with Jacobi’s theta function. Indeed the later is classically defined as follows (see for example [Mum83]).

\[ \theta(z | \tau) := \sum_{n \in \mathbb{Z}} \exp(i \pi n^2 \tau + 2i \pi nz), \]
defined for all \( z \in \mathbb{C}, \tau \in \mathbb{H} \). Now if one plugs
\[ z := i \beta a, \quad \tau := 2i \pi \beta \]
into \( \theta \), we find
\[ \theta(z | \tau) = e^{\frac{\beta}{2} a^2} \sum_{n \in \mathbb{Z}} \exp(-\frac{\beta}{2}(2\pi n + a)^2). \]

Jacobi’s first modular identity states that
\[ \theta(z | -1/\tau) = \alpha \theta(z | \tau), \] (A.2)
where \( \alpha = (-i \tau)^{1/2} \exp(\frac{1}{2} i z^2) = \sqrt{2 \pi \beta} \exp(-\frac{\beta}{2} a^2) \). This identity gives us:
\[ \sum_{q \in \mathbb{Z}} e^{-\frac{\beta}{2}(2\pi n + a)^2} \cos(qa) = \sqrt{2 \pi \beta} \sum_{n \in \mathbb{Z}} \exp(-\frac{\beta}{2}(2\pi n + a)^2) = \sqrt{2 \pi \beta} Z(\beta, a), \]
from which one can prove the identity (A.1) by taking a log-derivative in \( a \). Note that one may also avoid using Jacobi’s identity and reprove things using a Poisson summation formula. We indicate the link here as our shift-parameter \( a \) which is central to our work is naturally associated to the first argument \( z \) of the theta function (while the second argument \( \tau \) is related to the inverse temperature).

The argument we just outlined bares some resemblance with the fact that, in an Ising model with an exterior magnetic field \( h \), one can compute the average magnetization as a derivative w.r.t \( h \) of the free energy \( \log Z \). In our context, we used that
\[ \left\langle \sigma, \nabla_a \log Z \right\rangle = \left\langle -\frac{1}{2} (\Delta)^{-1} f, \nabla_a \log Z \right\rangle. \]
This suggests that our key identity (4.12) for a general domain \( \Lambda \subset \mathbb{Z}^2 \) should be reminiscent of the way to recover the average magnetic field of the Ising model from the derivative in \( h \) of its Free energy \( \log Z \). We implement this idea in the rest of the appendix by viewing the vector shift \( a = \{a_i\}_{i \in \Lambda} \) acting as an external magnetic field. We prove Proposition 4.1 along these lines in two steps:

A.2) First, as in the case of one-point, we work in the limiting case of infinite Fourier series at each vertex \( x \in \Lambda \). This makes the analogy with an Ising-model clearer and makes a connection with the modular invariance of certain Riemann-theta functions. From the intuition gathered here, we notice that the key identity (4.12) is an appropriate log-derivative w.r.t \( a \), namely
\[ -\langle \sigma, \nabla_a \log Z \rangle = \frac{1}{2} \nabla_a \left( \mathbb{E}_{\beta, \Lambda, \lambda \Lambda, v} \left[ \langle \phi, f \rangle \right] \right). \]

A.3) In the second part, we work in the finite cut-off case. Here it is not so clear how to recognise the integral against \( (f, \phi) \) on the R.H.S of the identity (4.12). The reason comes from the fact that expansion into charges from [FS81] (and particularly the effect of the complex translation under spin-waves) somehow obfuscates the readability of \( \mathbb{E}_{\beta, \Lambda, \lambda \Lambda, v} \left[ \langle \phi, f \rangle \right] \). To end the proof, we first get around the blurring effect caused by the expansion into charges from [FS81] (using the matching of partition functions before and after expansions into charges) and then connect to an actual average of \( \langle \phi, f \rangle \) by running Gaussian integration by parts.
A.2 Riemann-theta function and a-shifted integer valued GFF.

In this section, we implicitly rely on expansions into infinitely many charge configurations in [FS81, KP17] by attaching to each vertex $i \in \Lambda$ the following infinite trigonometric series

$$\lambda_i(\phi_i) = 1 + 2 \sum_{k=1}^{\infty} \cos(k(\phi_i - a_i)).$$

We will not properly justify here that the series are well defined as our goal is to justify properly in the next section the key identity (4.12) which holds in the finite cut-off case

$$\lambda_i(\phi_i) = 1 + 2 \sum_{k=1}^{N} \cos(k(\phi_i - a_i)),$$

with $N$ large.

Let us introduce the following two partition functions in the general case of $\Lambda \subset \mathbb{Z}^2$ with, say, Dirichlet boundary conditions.

$$Z(\beta, \mathbf{a}) := \frac{1}{\sqrt{(2\pi\beta^{-1})^{|\Lambda|} \det(-\Delta)^{-\frac{1}{2}}}} \sum_{m \in \mathbb{Z}^\Lambda} \exp\left(-\frac{\beta}{2} \sum_{i<j}(2\pi(m_i - m_j) + a_i - a_j)^2\right)$$

$$\tilde{Z}(\beta, \mathbf{a}) = \sum_{\mathcal{N} \in \mathcal{F}} c_N Z_N^a(0)$$

$$= \sum_{\mathcal{N} \in \mathcal{F}} c_N \int \prod_{\rho \in \mathcal{N}} [1 + z(\beta, \rho, \mathcal{N}) \cos((\phi, \rho) - \langle \mathbf{a}, \rho \rangle)] d\mu_{\beta, \mathcal{N}, v}(\phi).$$

(N.B. As hinted above $\mathcal{F}$ must be an infinite set of charge configurations here).

The expansion from Fröhlich-Spencer (in this limiting case) reads as follows: for all $\beta < \beta_0$ and $\mathbf{a} \in \mathbb{R}^\Lambda$

$$Z(\beta, \mathbf{a}) = \tilde{Z}(\beta, \mathbf{a}).$$

(A.3)

Inspired by the analogy with Ising, we now compute for any $g : \Lambda \to \mathbb{R}$,

$$\langle g, \nabla_{\mathbf{a}} \log Z(\beta, \mathbf{a}) \rangle$$

$$= \sum_{i \in \Lambda} g_i \partial_{a_i} \log Z(\beta, \mathbf{a})$$

$$= -\beta \sum_{i \in \Lambda} g_i \sum_{m \in \mathbb{Z}^\Lambda} e^{-\frac{\beta}{2} \sum_{i \neq j}(2\pi(m_i - m_j) + a_i - a_j)^2} \left(\sum_{i \neq j}(2\pi(m_i - m_j) + a_i - a_j)\right)$$

$$= \beta \sum_{m \in \mathbb{Z}^\Lambda} \sum_{i \in \Lambda} e^{-\frac{\beta}{2} \sum_{i \neq j}(2\pi(m_i - m_j) + a_i - a_j)^2} \left[\Delta(2\pi m + \mathbf{a})\right] g_i$$

$$= \beta \sum_{m \in \mathbb{Z}^\Lambda} e^{-\frac{\beta}{2} \sum_{i \neq j}(2\pi(m_i - m_j) + a_i - a_j)^2} \left(g, \Delta(2\pi m + \mathbf{a})\right).$$

Choose (as in [FS81, KP17]),

$$g = \sigma := \frac{1}{\beta} \Delta^{-1} f.$$

This gives us

$$\langle \sigma, \nabla_{\mathbf{a}} \log Z(\beta, \mathbf{a}) \rangle = -\sum_{m \in \mathbb{Z}^\Lambda} e^{-\frac{\beta}{2} \sum_{i \neq j}(2\pi(m_i - m_j) + a_i - a_j)^2} \langle f, 2\pi m + \mathbf{a} \rangle$$

$$= -\mathbb{E}_\beta^{\Lambda, \mathbf{a}}[\langle \phi, f \rangle].$$
Now, from (A.3), we know that for any function $g: \Lambda \to \mathbb{R}$, we have
\[
(g, \nabla a \log Z(\beta, a)) = (g, \nabla a \log \tilde{Z}(\beta, a)).
\]
As such, this implies with $g = \sigma$ the following formula for $E_{\beta,\Lambda,a}^{\text{IV}}[(\phi, f)]$:
\[
E_{\beta,\Lambda,a}^{\text{IV}}[(\phi, f)] = -\langle \sigma, \nabla_a \log \tilde{Z}(\beta, a) \rangle.
\]
Let us then compute this gradient $\nabla_a$ and check that it gives us the desired identity:
\[
\langle \sigma, \nabla_a \log \tilde{Z}(\beta, a) \rangle = \frac{1}{Z(\beta, a)} \sum_i \sum_{\mathcal{N} \in \mathcal{F}} c_N \sum_{i \in \rho = \rho_i \in \mathcal{N}} \int \left[z_i(\beta, \rho, \mathcal{N}) \left( -\sin(\langle \phi, \bar{\rho} \rangle - \langle a, \rho \rangle) - \rho_i \right) \right.
\]
\[
\times \prod_{\rho \in \mathcal{N} \setminus \{\rho_i\}} \left[ 1 + z(\beta, \rho, \mathcal{N}) \cos(\langle \phi, \bar{\rho} \rangle - \langle a, \rho \rangle) \right] d\mu^{\text{GFF}}_{\beta,\Lambda,v}(\phi) \nabla \Lambda
\]
\[
\times \prod_{\rho \in \mathcal{N}} \left[ 1 + z(\beta, \rho, \mathcal{N}) \cos(\langle \phi, \bar{\rho} \rangle - \langle a, \rho \rangle) \right] d\mu^{\text{GFF}}_{\beta,\Lambda,v}(\phi)
\]
and we thus recover the R.H.S of (4.12) in the limiting case of infinite trigonometric polynomials at each site.

Let us briefly highlight now the link with Riemann-theta functions which we believe illustrates what is beneath the identity (4.12). It is not hard to rewrite the partition function
\[
Z(\beta, a) = \frac{1}{\sqrt{(2\pi\beta^{-1})^{|\mathcal{N}|} \det(-\Delta)}} \sum_{m \in \mathbb{Z}^g} \exp \left( -\beta \sum_{i<j} (2\pi(m_i - m_j) + a_i - a_j)^2 \right)
\]
as a theta function in several variables (i.e., the Riemann-theta function). The later generalized theta functions may be defined as follows (see for example [Mum83]): for any $g \geq 1$, $z = (z_1, \ldots, z_g) \in \mathbb{C}^g$ and $\Omega$ a symmetric $g \times g$ complex matrix whose imaginary part is positive definite, set
\[
\theta(z \mid \Omega) := \sum_{m \in \mathbb{Z}^g} \exp(\pi i m^T \Omega m + 2\pi i m \cdot z). \tag{A.4}
\]
The Riemann-theta functions therefore match exactly with our model when
\[
\begin{align*}
\mathbf{z} &:= i\beta(-\Delta) a \\
\Omega &= 2\pi \beta(-\Delta).
\end{align*}
\]
We claim that the identity (4.12) is reminiscent of the suitable log-derivative (i.e., taking $F(a) \mapsto -\langle \sigma, \nabla_a F(a) \rangle$) of the modular invariance identity for Riemann-theta function (see for example 5.1 in [Mum83]) which states that
\[
\theta(\Omega^{-1} z, -\Omega^{-1}) = \sqrt{\det(-\Omega)} \exp(i z^T \Omega^{-1} z) \theta(z, \Omega). \tag{A.5}
\]

A.3 Blurring effect of the decomposition into charges. In this subsection, we work with finite cut-off Fourier series (and therefore do not need to worry with convergences of series) and we end our alternative proof of Proposition 4.1.

By running the same computation as the one outlined above for the infinite trigonometric series, we have that the R.H.S in the identity (4.12) is given by
\[
-\langle \sigma, \nabla_a \log \tilde{Z}_N(\beta, a) \rangle
\]
We now wish to compare this with an expression for which ends our proof as we obtained, as desired, the same expression as for with the initial expression of the partition function before subtle expansions into charges are made. Namely, we consider

\[ \hat{Z}_N(\beta, a) := \int \prod_{x \in \Lambda} \left( 1 + 2 \sum_{k=1}^{N} \cos(k(\phi_x - a_x)) \right) d\beta, a \]

and we then compute

\[-\langle \sigma, \nabla_a \log \hat{Z}(\beta, a) \rangle = -\langle \sigma, \nabla_a \log \hat{Z}_N(\beta, a) \rangle = - \sum_{i \in \Lambda} \sigma_i \frac{\mathbb{E}^{\text{GFF}}_{\beta, \Lambda} \left[ 2 \left( \sum_{k=1}^{N} k \sin(k(\phi_i - a_i)) \right) \prod_{x \in \Lambda \setminus i} \left( 1 + 2 \sum_{k=1}^{N} \cos(k(\phi_x - a_x)) \right) \right]}{\hat{Z}_N(\beta, a)}.\]

We now wish to compare this with an expression for \( \mathbb{E}^{a}_{\beta, \Lambda, \lambda, v} \left[ \langle \phi, f \rangle \right] \):

\[
\mathbb{E}^{a}_{\beta, \Lambda, \lambda, v} \left[ \langle \phi, f \rangle \right] = \mathbb{E}^{\text{GFF}}_{\beta, \Lambda} \left[ \prod_{x \in \Lambda} \left( 1 + 2 \sum_{k=1}^{N} \cos(k(\phi_x - a_x)) \right) \langle \phi, f \rangle \right] = \sum_{i \in \Lambda} f_i \mathbb{E}^{\text{GFF}}_{\beta, \Lambda} \left[ \sum_{j \in \Lambda} \langle \phi_i, \phi_j \rangle / \beta \mathbb{E}^{\text{GFF}}_{\beta, \Lambda} \left[ \partial_j \prod_{x \in \Lambda} \left( 1 + 2 \sum_{k=1}^{N} \cos(k(\phi_x - a_x)) \right) \right] \right],
\]

by Gaussian integration by parts. Continuing, this gives us

\[
\mathbb{E}^{a}_{\beta, \Lambda, \lambda, v} \left[ \langle \phi, f \rangle \right] = - \sum_{i \in \Lambda} f_i \frac{1}{\beta} (-\Delta)^{-1} (i, j) \mathbb{E}^{\text{GFF}}_{\beta, \Lambda} \left[ \sum_{k=1}^{N} \sin(k(\phi_j - a_j)) k \prod_{x \in \Lambda \setminus j} \left( 1 + 2 \sum_{k=1}^{N} \cos(k(\phi_x - a_x)) \right) \right] = - \frac{1}{\beta} \langle f_i, (-\Delta)^{-1} \rangle \Psi(i),
\]

where

\[
\Psi(j) := 2 \mathbb{E}^{\text{GFF}}_{\beta, \Lambda} \left[ \sum_{k=1}^{N} \sin(k(\phi_j - a_j)) k \prod_{x \in \Lambda \setminus j} \left( 1 + 2 \sum_{k=1}^{N} \cos(k(\phi_x - a_x)) \right) \right].
\]

\[
\mathbb{E}^{a}_{\beta, \Lambda, \lambda, v} \left[ \langle \phi, f \rangle \right] = - \sum_{i \in \Lambda} f_i \frac{1}{\beta} (-\Delta)^{-1} \Psi(i)
= - \frac{1}{\beta} \langle f_i, (-\Delta)^{-1} \rangle \Psi = - \frac{1}{\beta} (-\Delta)^{-1} f, \Psi
= - \sum_{i \in \Lambda} \sigma_i \mathbb{E}^{\text{GFF}}_{\beta, \Lambda} \left[ 2 \sum_{k=1}^{N} \sin(k(\phi_i - a_i)) k \prod_{x \in \Lambda \setminus i} \left( 1 + 2 \sum_{k=1}^{N} \cos(k(\phi_x - a_x)) \right) \right],
\]

which ends our proof as we obtained, as desired, the same expression as for \(-\langle \sigma, \nabla_a \log \hat{Z}_N(\beta, a) \rangle\).
REFERENCES

[AKM19] Scott Armstrong, Tuomo Kuusi, and Jean-Christophe Mourrat. Quantitative stochastic homogenization and large-scale regularity. 2019.

[ALS19] Juhan Aru, Titus Lupu, and Avelio Sepúlveda. The first passage sets of the 2D Gaussian free field. Probab. Theory Related Fields, 2019. https://doi.org/10.1007/s00440-019-00941-1.

[AMM+17] Emmanuel Abbe, Laurent Massoulie, Andrea Montanari, Allan Sly, and Nikhil Srivastava. Group synchronization on grids. arXiv preprint arXiv:1706.08561, 2017.

[Aru15] Juhan Aru. The geometry of the Gaussian free field combined with SLE processes and the KPZ relation. PhD thesis, Ecole Normale Supérieure de Lyon, 2015.

[AS18] Juhan Aru and Avelio Sepúlveda. Two-valued local sets of the 2D continuum Gaussian free field: connectivity, labels, and induced metrics. Electron. J. Probab., 23(61), 2018.

[ASW17] Juhan Aru, Avelio Sepúlveda, and Wendelin Werner. On bounded-type thin local sets of the two-dimensional Gaussian free field. J. Inst. Math. Jussieu, pages 1–28, 2017.

[Bau16] Roland Bauerschmidt. Ferromagnetic spin systems. Lecture notes available at http://www.statslab.cam.ac.uk/~rb812/doc/spin.pdf, 2016.

[BSS14] Nathanaël Berestycki, Scott Sheffield, and Xin Sun. Equivalence of liouville measure and gaussian free field. arXiv preprint arXiv:1410.5407, 2014.

[Cha18] Sourav Chatterjee. Wilson loops in ising lattice gauge theory. arXiv preprint arXiv:1811.09770, 2018.

[CPST18] Nishant Chandgotia, Ron Peled, Scott Sheffield, and Martin Tassy. Delocalization of uniform graph homomorphisms from $\mathbb{Z}^2$ to $\mathbb{Z}$. arXiv preprint arXiv:1810.10124, 2018.

[CS11] Dmitry Chelkak and Stanislav Smirnov. Discrete complex analysis on isoradial graphs. Adv. Math., 228(3):1590–1630, 2011.

[Cha18] Sourav Chatterjee. Wilson loops in ising lattice gauge theory. arXiv preprint arXiv:1811.09770, 2018.

[DCGPS17] Hugo Duminil-Copin, Alexander Glazman, Ron Peled, and Yinon Spinka. Macroscopic loops in the loop $O(n)$ model at Nienhuis’ critical point. arXiv preprint arXiv:1707.09335, 2017.

[DCHL+19] Hugo Duminil-Copin, Matan Harel, Benoît Laslier, Aran Raoufi, and Gourab Ray. Logarithmic variance for the height function of square-ice. arXiv preprint arXiv:1911.00092, 2019.

[Dub09] Julien Dubédat. SLE and the free field: partition functions and couplings. J. Inst. Math. Jussieu, 2010. arXiv preprint.

[FS81] Jürg Fröhlich and Thomas Spencer. The Kosterlitz-Thouless transition in two-dimensional abelian spin systems and the Coulomb gas. Communications in Mathematical Physics, 81(4):527–602, 1981.

[FS83] Jürg Fröhlich and Thomas Spencer. The Berezinskii-Kosterlitz-Thouless transition (energy-entropy arguments and renormalization in defect gases). In Scaling and Self-Similarity in Physics, pages 29–138. Springer, 1983.

[FV17] Sacha Friedli and Yvan Velenik. Statistical mechanics of lattice systems: a concrete mathematical introduction. Cambridge University Press, 2017.

[GM18] Alexander Glazman and Ioan Manolescu. Uniform lipschitz functions on the triangular lattice have logarithmic variations. arXiv preprint arXiv:1810.05592, 2018.

[HS13] Alexander Holroyd and Terry Soo. Insertion and deletion tolerance of point processes. Electronic Journal of Probability, 2013.

[JSW18] Janne Junnila, Eero Saksman, and Christian Webb. Imaginary multiplicative chaos: Moments, regularity and connections to the Ising model. arXiv preprint arXiv:1806.02118, 2018.

[KP17] Vital Kharash and Ron Peled. The Fröhlich-Spencer proof of the Berezinskii-Kosterlitz-Thouless transition. arXiv preprint arXiv:1711.04720, 2017.

[LRV14] Hubert Lacoin, Rémi Rhodes, and Vincent Vargas. Large deviations for random surfaces: the hyperbolic nature of Liouville Field Theory. arXiv preprint arXiv:1401.6001, 2014.

[MS16a] Jason Miller and Scott Sheffield. Imaginary geometry I: interacting SLEs. Probab. Theory Related Fields, 164(3-4):553–705, 2016.

[MS16b] Jason Miller and Scott Sheffield. Imaginary geometry II: reversibility of SLE$_{\kappa}(\rho_1;\rho_2)$ for $\kappa \in (0, 4)$. Ann. Probab., 44(3):1647–1722, 2016.

[MS16c] Jason Miller and Scott Sheffield. Imaginary geometry III: reversibility of SLE$_{\kappa}(\rho_1;\rho_2)$ for $\kappa \in (4, 8)$. Ann. Math., 184(2):455–486, 2016.

[MS17] Jason Miller and Scott Sheffield. Imaginary geometry IV: interior rays, whole-plane reversibility, and space-filling trees. Probab. Theory Related Fields, 169:729–869, 2017.
[Mum83] D Mumford. Tata lectures on theta I, jacobian theta functions and differential equations., Progress in mathematics, 28, 1983.

[PS14] Yuval Peres and Allan Sly. Rigidity and tolerance for perturbed lattices. arXiv preprint arXiv:1409.4490, 2014.

[QW18] Wei Qian and Wendelin Werner. Coupling the gaussian free fields with free and with zero boundary conditions via common level lines. Communications in Mathematical Physics, 361(1):53–80, 2018.

[She05] Scott Sheffield. Local sets of the Gaussian free field. Slides and audio, 2005.

[She07] Scott Sheffield. Gaussian free fields for mathematicians. Probab. Theory Related Fields, 139(3):521–541, 2007.

[SS09] Oded Schramm and Scott Sheffield. Contour lines of the two-dimensional discrete Gaussian free field. Acta Math., 202(1):21, 2009.

[SS13] Oded Schramm and Scott Sheffield. A contour line of the continuum Gaussian free field. Probab. Theory Related Fields, 157(1-2):47–80, 2013.

[Szn12] Alain-Sol Sznitman. Topics in occupation times and Gaussian free field. Zur. Lect. Adv. Math. European Mathematical Society, 2012.

[Wer16] Wendelin Werner. Topics on the GFF and CLE(4), 2016.

[Wir19] Mateo Wirth. Maximum of the integer-valued Gaussian free field. arXiv preprint arXiv:1907.08868, 2019.

[WW16] Menglu Wang and Hao Wu. Level lines of Gaussian free field I: zero-boundary GFF. Stochastic Process. Appl., 2016.