A cost-scaling algorithm for computing the degree of determinants

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Abstract

In this paper, we address computation of the degree \( \text{deg} \, \text{Det} \, A \) of Dieudonné determinant \( \text{Det} \, A \) of
\[
A = \sum_{k=1}^{m} A_k x_k t^{c_k},
\]
where \( A_k \) are \( n \times n \) matrices over a field \( \mathbb{K} \), \( x_k \) are noncommutative variables, \( t \) is a variable commuting \( x_k \), \( c_k \) are integers, and the degree is considered for \( t \).
This problem generalizes noncommutative Edmonds’ problem and fundamental combinatorial optimization problems including the weighted linear matroid intersection problem. It was shown that \( \text{deg} \, \text{Det} \, A \) is obtained by a discrete convex optimization on a Euclidean building. We extend this framework by incorporating a cost scaling technique, and show that \( \text{deg} \, \text{Det} \, A \) can be computed in time polynomial of \( n, m, \log_2 C \), where \( C := \max_k |c_k| \). We apply this result to an algebraic combinatorial optimization problem arising from a symbolic matrix having \( 2 \times 2 \)-submatrix structure.

Keywords: Edmonds’ problem, noncommutative rank, Dieudonné determinant, Euclidean building, discrete convex analysis, partitioned matrix.

1 Introduction

Edmonds’ problem [4] asks to compute the rank of a matrix of the following form:
\[
A = \sum_{k=1}^{m} A_k x_k,
\]
where \( A_k \) are \( n \times n \) matrices over field \( \mathbb{K} \), \( x_k \) are variables, and \( A \) is considered as a matrix over rational function field \( \mathbb{K}(x_1, x_2, \ldots, x_k) \). This problem is motivated by a linear
algebraic formulation of the bipartite matching problem and other combinatorial optimization problems. For a bipartite graph $G = ([n] \sqcup [n], E)$, consider $A = \sum_{ij \in E} E_{ij} x_{ij}$, where $E_{ij}$ denotes the 0,1 matrix having 1 only for the $(i,j)$-entry. Then $\text{rank } A$ is equal to the maximum size of a matching of $G$. Other basic classes of combinatorial optimization problems have such a rank interpretation. For example, the linear matroid intersection problem corresponds to $A$ with rank-1 matrices $A_k$, and the linear matroid matching problem corresponds to $A$ with rank-2 skew symmetric matrices $A_k$; see [22].

Symbolical treatment of variables $x_k$ makes the problem difficult, whereas the rank computation after substitution for $x_k$ is easy and it provides the correct value in high probability. A randomized polynomial time algorithm is obtained by this idea [21]. A deterministic polynomial time algorithm for Edmonds’ problem is not known, and is one of important open problems in theoretical computer science.

A recent approach to Edmonds’ problem, initiated by Ivanyos et al. [9], is to consider variables $x_k$ to be noncommutative. That is, the matrix $A$ is regarded as a matrix over noncommutative polynomial ring $\mathbb{K}(x_1, x_2, \ldots, x_m)$. The rank of $A$ is well-defined via embedding $\mathbb{K}(x_1, x_2, \ldots, x_m)$ to the free skew field $\mathbb{K}((x_1, x_2, \ldots, x_m))$. The resulting rank is called the noncommutative rank (nc-rank) of $A$ and is denoted by nc-rank $A$. Interestingly, nc-rank $A$ admits a deterministic polynomial time computation:

**Theorem 1.1** ([7, 10]). nc-rank $A$ for a matrix $A$ of form (1.1) can be computed in time polynomial of $n, m$.

The algorithm by Garg et al. [7] works for $\mathbb{K} = \mathbb{Q}$, and the algorithm by Ivanyos et al. [10] works for an arbitrary field $\mathbb{K}$. Another polynomial time algorithm for nc-rank is obtained by Hamada and Hirai [11], while the bit-length of this algorithm may be unbounded if $\mathbb{K} = \mathbb{Q}$. By the formula of Fortin and Reutenauer [5], nc-rank $A$ is obtained by an optimization problem defined on the family of vector subspaces in $\mathbb{K}^n$. The above algorithms deal with this new type of an optimization problem. It holds rank $A \leq$ nc-rank $A$, where the inequality can be strict in general. For some class of matrices including linear matroid intersection, rank $A =$ nc-rank $A$ holds, and the Fortin-Reutenauer formula provides a combinatorial duality relation. This is basically different from the usual derivation by polyhedral combinatorics and LP-duality.

In the view of combinatorial optimization, rank computation corresponds to cardinality maximization. The degree of determinants is an algebraic correspondent of weighted maximization. Indeed, the maximum-weight of a perfect matching of a bipartite graph is equal to the degree $\text{deg det } A$ of the determinant $\text{det } A$ of $A = \sum_{ij \in E} E_{ij} x_{ij} t^{c_{ij}}$, where $t$ is a new variable, $c_{ij}$ are edge-weights, and the degree is considered in $t$. Therefore, the weighed version of Edmonds’ problem is computation of the degree of the determinant of a matrix $A$ of form (1.1), where each $A_k = A_k(t)$ is a polynomial matrix with variable $t$.

Motivated by this observation and the above-mentioned development, Hirai [12] introduced a noncommutative formulation of the weighted Edmonds’ problem. In this setting, the determinant $\text{det } A$ is replaced by the Dieudonné determinant $\text{Det } A$ [3] — a determinant concept of a matrix over a skew field. For our case, $A$ is viewed as a matrix over the skew field $\mathbb{F}(t)$ of rational functions with coefficients in $\mathbb{F} = \mathbb{K}((x_1, x_2, \ldots, x_m))$. Then the degree with respect to $t$ is well-defined. He established a formula of $\text{deg Det } A$ generalizing the Fortin-Reutenauer formula for nc-rank $A$, a generic algorithm (Deg-Det) to compute $\text{deg Det } A$, and $\text{deg det } A = \text{deg Det } A$ relation for weighted linear
matroid intersection problem. In particular, deg Det is obtained in time polynomial of \( n, m \), the maximum degree \( d \) of matrix \( A \) with respect to \( t \), and the time complexity of solving the optimization problem for nc-rank. Although the required bit-length is unknown for \( K = \mathbb{Q} \), Oki [23] showed another polynomial time reduction from deg Det to nc-rank with bounding bit-length.

In this paper, we address the deg Det computation of matrices of the following special form:

\[
A = \sum_{k=1}^{m} A_k x_k t^{c_k},
\]

where \( A_k \) are matrices over \( K \) and \( c_k \) are integers. This class of matrices is natural from the view of combinatorial optimization. Indeed, the weighted bipartite matching and weighted linear matroid intersection problems correspond to deg Det of such matrices. Now exponents \( c_k \) of variable \( t \) work as weights or costs. In this setting, the above algorithms [12, 23] are pseudo-polynomial. Therefore, it is natural to ask for deg Det computation with polynomial dependency in \( \log |c_k| \). The main result of this paper shows that such a computation is indeed possible.

**Theorem 1.2.** Suppose that arithmetic operations over \( K \) are done in constant time. Then deg Det \( A \) for a matrix \( A \) of (1.2) can be computed in time polynomial of \( n, m, \log C \), where \( C := \max_k |c_k| \).

For a more general setting of “sparse” polynomial matrices, such a polynomial time deg Det computation seems difficult, since it can solve (commutative) Edmonds’ problem [23].

Our algorithm for Theorem 1.2 is based on the framework of [12]; hence the required bit-length is unknown for \( K = \mathbb{Q} \). In this framework, deg Det \( A \) is formulated as a discrete convex optimization on the Euclidean building for \( GL_n(K(t)) \). The Deg-Det algorithm is a simple descent algorithm on the building, where discrete convexity property (\( L \)-convexity) provides a sharp iteration bound of the algorithm via geometry of the building. We incorporate cost scaling into the Deg-Det algorithm, which is a standard idea in combinatorial optimization. To obtain the polynomial time complexity, we need a polynomial sensitivity estimate for how an optimal solution changes under the perturbation \( c_k \to c_k - 1 \). We introduce a new discrete convexity concept, called \( N \)-convexity, that works nicely for such cost perturbation, and show that the objective function enjoys this property, from which a desired estimate follows. This method was devised by [14] in another discrete convex optimization problem on a building-like structure.

As an application, we consider an algebraic combinatorial optimization problem for a symbolic matrix of form

\[
A = \begin{pmatrix}
A_{11} x_{11} & A_{12} x_{12} & \cdots & A_{1n} x_{1n} \\
A_{21} x_{21} & A_{22} x_{22} & \cdots & A_{2n} x_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n1} x_{n1} & A_{n2} x_{n2} & \cdots & A_{nn} x_{nn}
\end{pmatrix},
\]

where \( A_{ij} \) is a \( 2 \times 2 \) matrix over \( K \) for \( i, j \in [n] \). We call such a matrix a \( 2 \times 2 \)-partitioned matrix. Rank computation of this matrix is viewed as a “2-dimensional” generalization of the bipartite matching problem. The duality theorem by Iwata and
Murota [17] implies rank $A = \text{nc-rank } A$ relation. Although rank $A$ can be computed by the above-mentioned nc-rank algorithms, the problem has a more intriguing combinatorial nature. Hirai and Iwamasa [15] showed that rank $A$ is equal to the maximum size of a certain algebraically constrained 2-matching ($A$-consistent 2-matching) on a bipartite graph, and they developed an augmenting-path type polynomial time algorithm to obtain a maximum $A$-consistent 2-matching. We apply our cost-scaling framework for a $2 \times 2$-partitioned matrix $A$ with $x_{ij}$ replaced by $x_{ij}t_{ij}$, and obtain a polynomial time algorithm to solve the weighted version of this problem and to compute $\text{deg det } A (= \deg \text{Det } A)$. This result sheds an insight on polyhedral combinatorics, since it means that linear optimization over the polytope of $A$-consistent 2-matchings can be solved without knowledge of its LP-formulation.

Related work. A matrix $A$ of (1.1) corresponding to the linear matroid matching problem (i.e., each $A_k$ is a rank-2 skew symmetric matrix) is a representative example in which rank and nc-rank can be different. Accordingly, $\text{deg det}$ and $\deg \text{Det}$ can differ for a matrix $A$ of (1.2) with rank-2 skew symmetric matrices $A_k$. The computation of $\text{deg det}$ of such a matrix is precisely the weighted linear matroid matching problem. Camerini et al. [1] utilized this $\text{deg det}$ formulation and random substitution to obtain a random pseudo-polynomial time algorithm solving the weighted linear matroid matching, where the running time depends on $C$. Cheung et al. [2] speeded up this algorithm, and also obtained a randomized FPTAS by using cost scaling. Recently, Iwata and Kobayashi [16] finally developed a polynomial time algorithm solving the weighted linear matroid matching problem, where the running time does not depend on $C$. The algorithm also uses a similar (essentially equivalent) $\text{deg det}$ formulation, and is rather complicated. A simplified polynomial time algorithm, possibly using cost scaling, is worthy to be developed, in which the results in this paper may help.

Organization. The rest of this paper is organized as follows: In Section 2, we give necessary arguments on nc-rank, Dieudonné determinant, Euclidean building, and discrete convexity. Our argument is elementary; no prior knowledge is assumed. In Section 3, we present our algorithm for Theorem 1.2. In Section 4 we describe the results on $2 \times 2$-partitioned matrices.

2 Preliminaries

Let $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$ denote the sets of reals, rationals, and integers, respectively. Let $e_i \in \mathbb{Z}^n$ denote the $i$-th unit vector. For $s \in [n]$, let $1_s \in \mathbb{Z}^n$ denote the 0,1 vector in which the first $s$ components are 1 and the others are zero, i.e., $1_s := \sum_{i=1}^n e_i$. For a ring $R$, let $GL_n(R)$ denote the set of $n \times n$ matrices over $R$ having inverse $R^{-1}$. The degree of a polynomial $p(t) = a_k t^k + a_{k-1} t^{k-1} + \cdots + a_0$ with $a_k \neq 0$ is defined as $k$. The degree of a rational $p/q$ with polynomials $p, q$ is defined as $\deg p - \deg q$. The degree of the zero polynomial is defined as $-\infty$.

2.1 Nc-rank and the degree of Dieudonné determinant

Instead of giving necessary algebraic machinery, we simply regard the following formula by Fortin and Reutenauer [5] as the definition of the nc-rank.
Theorem 2.1 ([5]). Let $A$ be a matrix of form (1.1). Then $\text{nc-rank} A$ is equal to the optimal value of the following problem:

$$
\begin{align*}
(\text{R}) & \quad \text{Min.} & & 2n - r - s \\
& & & \text{s.t.} SAT \text{ has an } r \times s \text{ zero submatrix}, \\
& & & S, T \in GL_n(\mathbb{K}).
\end{align*}
$$

Theorem 2.2 ([10]). An optimal solution $S, T$ in (R) can be computed in polynomial time.

Notice that the algorithm by Garg et al. [7] obtains the optimal value of (R) but does not obtain optimal $(S, T)$, and that the algorithm by Hamada and Hirai [11] obtains optimal $(S, T)$ but has no guarantee of polynomial bound of bit-length when $\mathbb{K} = \mathbb{Q}$.

Next we consider the degree of the Dieudonné determinant. Again we regard the following formula as the definition.

Theorem 2.3 ([12]). Let $A$ be a matrix of form (1.2). Then $\deg \det A$ is equal to the optimal value of the following problem:

$$
\begin{align*}
(\text{D}) & \quad \text{Min.} & & -\deg \det P - \deg \det Q \\
& & & \text{s.t.} \deg((PA_kQ)_{ij} + c_k \leq 0 \quad (i, j \in [n], k \in [m]), \\
& & & P, Q \in GL_n(\mathbb{K}(t)).
\end{align*}
$$

A pair of matrices $P, Q \in GL_n(\mathbb{K}(t))$ is said to be feasible (resp. optimal) for $A$ if it is feasible (resp. optimal) to (D) for $A$.

A matrix $M = M(t)$ over $\mathbb{K}(t)$ is written as a formal power series as

$$
M = M^{(d)} t^d + M^{(d-1)} t^{d-1} + \cdots,
$$

where $M^{(d)}$ is a matrix over $\mathbb{K}$ ($\ell = d, d - 1, \ldots$) and $d \geq \max_{ij} \deg M_{ij}$. For solving (D), the leading term $(PAQ)^{(0)} = \sum_k (PA_k t^s Q)^{(0)} x_k$ plays an important role.

Lemma 2.4 ([12]). Let $(P, Q)$ be a feasible solution for $A$.

1. $(P, Q)$ is optimal if and only if $\text{nc-rank}(PAQ)^{(0)} = n$.

2. If $\text{rank}(PAQ)^{(0)} = n$, then $\deg \det A = \deg \det A = -\deg \det P - \deg \det Q$.

A self-contained proof (for regarding (D) as the definition of $\deg \det$) is given in the appendix.

Notice that the optimality condition (1) does not imply a good characterization (NP $\cap$ co-NP characterization) for $\det A$, since the size of $P, Q$ (e.g., the number of terms) may depend on $c_k$ pseudo-polynomially.

Lemma 2.5. $\deg \det \sum_{k=1}^m A_k x_k t^{c_k} > -\infty$ if and only if $\text{nc-rank} \sum_{k=1}^m A_k x_k = n$.

Proof. We observe from (D) that $\deg \det A t^b = nb + \deg \det A$ and $\deg \det$ is monotone in $c_k$. In particular, we may assume $c_k \geq 0$.

Suppose that $\text{nc-rank} \sum_{k=1}^m A_k x_k < n$ Then we can choose $S, T \in GL_n(\mathbb{K})$ such that $S \sum_{k=1}^m A_k x_k T$ has an $r \times s$ zero submatrix with $r + s > n$ in the upper right corner. Then, for every $\kappa > 0$, $((t^{\kappa t^1})S, T(t^{-\kappa t^{1-n}})t^{-C})$ is feasible in (D) with objective value $-\kappa (r + s - n) + nC$, where $C := \max_k c_k$. This means that (D) is unbounded. Suppose that $\text{nc-rank} \sum_{k=1}^m A_k x_k = n$. By monotonicity, we have $\deg \det \sum_{k=1}^m A_k x_k t^{c_k} \geq \deg \det \sum_{k=1}^m A_k x_k$. Now $(\sum_{k=1}^m A_k x_k)^{(0)} = \sum_{k=1}^m A_k x_k$ has $\text{nc-rank} n$, and $(I, I)$ is optimal by Lemma 2.4 (1). Then we have $\deg \det \sum_{k=1}^m A_k x_k = 0$. \qed
2.2 Euclidean building

Here we explain that the problem (D) is regarded as an optimization over the so-called Euclidean building. See e.g., [8] for Euclidean building. Let $\mathbb{K}(t)^-$ denote the subring of $\mathbb{K}(t)$ consisting of elements $p/q$ with $\deg p/q \leq 0$. Let $GL_n(\mathbb{K}(t)^-)$ be the subgroup of $GL_n(\mathbb{K}(t))$ consisting of matrices over $\mathbb{K}(t)^-$ invertible in $\mathbb{K}(t)^-$. The degree of the determinant of any matrix in $GL_n(\mathbb{K}(t)^-)$ is zero. Therefore transformation $(P, Q) \mapsto (LP, QM)$ for $L, M \in GL_n(\mathbb{K}(t)^-)$ keeps the feasibility and the objective value in (D). Let $\mathcal{L}$ be the set of right cosets $GL_n(\mathbb{K}(t)^-)P$ of $GL_n(\mathbb{K}(t)^-)$ in $GL_n(\mathbb{K}(t))$, and let $\mathcal{M}$ be the set of left cosets.

Then (D) is viewed as an optimization over $\mathcal{L} \times \mathcal{M}$. The projection of $P \in GL_n(\mathbb{K}(t))$ to $\mathcal{L}$ is denoted by $\langle P \rangle$, which is identified with the submodule of $\mathbb{K}(t)^n$ spanned by the row vectors of $P$ with coefficients in $\mathbb{K}(t)^-$. In the literature, such a module is called a lattice. We also denote the projections of $Q$ to $\mathcal{M}$ by $\langle Q \rangle$ and of $(P, Q)$ to $\mathcal{L} \times \mathcal{M}$ by $\langle P, Q \rangle$.

The space $\mathcal{L}$ (or $\mathcal{M}$) is known as the Euclidean building for $GL_n(\mathbb{K}(t))$. We will utilize special subspaces of $\mathcal{L}$, called apartments, to reduce arguments on $\mathcal{L}$ to that on $\mathbb{Z}^n$. For integer vector $\alpha \in \mathbb{Z}^n$, denote by $(t^\alpha)$ the diagonal matrix with diagonals $t^{\alpha_1}, t^{\alpha_2}, \ldots, t^{\alpha_n}$, that is,

$$
(t^\alpha) = \begin{pmatrix} t^{\alpha_1} & & \\ & t^{\alpha_2} & \\ & & \ddots \\ & & & t^{\alpha_n} 
\end{pmatrix}.
$$

An apartment of $\mathcal{L}$ is a subset $\mathcal{A}$ of $\mathcal{L}$ represented as

$$
\mathcal{A} = \{ \langle (t^\alpha)P \rangle \mid \alpha \in \mathbb{Z}^n \}
$$

for some $P \in GL_n(\mathbb{K}(t))$. The map $\alpha \mapsto \langle (t^\alpha)P \rangle$ is an injective map from $\mathbb{Z}^n$ to $\mathcal{L}$. The following is a representative property of a Euclidean building.

**Lemma 2.6** (See [8]). For $\langle P \rangle, \langle Q \rangle \in \mathcal{L}$, there is an apartment containing $\langle P \rangle, \langle Q \rangle$.

Therefore $\mathcal{L}$ is viewed as an amalgamation of integer lattices $\mathbb{Z}^n$. An apartment in $\mathcal{M}$ is defined as a subset of form $\{ \langle Q(t^\alpha) \rangle \mid \alpha \in \mathbb{Z}^n \}$. An apartment in $\mathcal{L} \times \mathcal{M}$ is the product of apartments in $\mathcal{L}$ and in $\mathcal{M}$.

Restricting (D) to an apartment $\mathcal{A} = \{ \langle (t^\alpha)P, Q(t^\beta) \rangle \}_{(\alpha, \beta) \in \mathbb{Z}^{2n}}$ of $\mathcal{L} \times \mathcal{M}$, we obtain a simpler integer program:

$$(D_{\mathcal{A}}) \quad \text{Min.} \quad -\sum_{i \in [n]} \alpha_i - \sum_{j \in [n]} \beta_j + \text{constant}$$

s.t. $\alpha_i + \beta_j + c^{k}_{ij} \leq 0 \quad (k \in [m], i, j \in [n]),$

$$\alpha, \beta \in \mathbb{Z}^n,$$

where $c^{k}_{ij} := \deg(PA_kQ)_{ij} + c_k$. This is nothing but the (discretized) LP-dual of a weighted perfect matching problem.

We need to define a distance between two solutions $\langle P, Q \rangle$ and $\langle P', Q' \rangle$ in (D). Let the $\ell_\infty$-distance $d_\infty(\langle P, Q \rangle, \langle P', Q' \rangle)$ defined as follows: Choose an apartment $\mathcal{A}$ containing
\((P, Q)\) and \((P', Q')\). Now \(\mathcal{A}\) is regarded as \(\mathbb{Z}^{2n} = \mathbb{Z}^n \times \mathbb{Z}^n\), and \((P, Q)\) and \((P', Q')\) are regarded as points \(x\) and \(x'\) in \(\mathbb{Z}^{2n}\), respectively. Then define \(d_\infty((P, Q), (P', Q'))\) as the \(\ell_\infty\)-distance \(\|x - x'\|_\infty\).

The \(\ell_\infty\)-distance \(d_\infty\) is independent of the choice of an apartment, and satisfies the triangle inequality. This fact is verified by applying a canonical retraction \(\mathcal{L} \times \mathcal{M} \rightarrow \mathcal{A}\), which is distance-nonincreasing; see [12].

### 2.3 N-convexity

The Euclidean building \(\mathcal{L}\) admits a partial order in terms of inclusion relation, since lattices are viewed as submodules of \(\mathbb{K}(t)^n\). By this ordering, \(\mathcal{L}\) becomes a lattice in poset theoretic sense; see [12, 13]. Then the objective function of (D) is a submodular-type discrete convex function on \(\mathcal{L} \times \mathcal{M}\), called an \(L\)-convex function [12]. Indeed, its restriction to each apartment \((\simeq \mathbb{Z}^{2n})\) is an \(L\)-convex function in the sense of discrete convex analysis [20]. This fact played an important role in the iteration analysis of the Deg-Det algorithm.

Here, for analysis of cost scaling, we introduce another discrete convexity concept, called \(N\)-convexity. Since arguments reduce to that on an apartment \((\simeq \mathbb{Z}^n)\), we first introduce \(N\)-convexity on integer lattice \(\mathbb{Z}^n\). For \(x, y \in \mathbb{Z}^n\), let \(x \rightarrow y\) defined by

\[
x \rightarrow y := x + \sum_{i: y_i > x_i} e_i - \sum_{i: x_i > y_i} e_i.
\]

Let \(x \rightarrow^{i+1} y := (x \rightarrow^i y) \rightarrow y\), where \(x \rightarrow^1 y := x \rightarrow y\). Observe that \(\ell_\infty\)-distance \(\|x - y\|_\infty\) decreases by one when \(x\) moves to \(x \rightarrow y\). In particular, \(x \rightarrow^d y\) if \(d = \|x - y\|_\infty\). The sequence \((x, x \rightarrow^1 y, x \rightarrow^2 y, \ldots, y)\) is called the normal path from \(x\) to \(y\). Let \(y \rightarrow x\) be defined by

\[
y \rightarrow x := x \rightarrow^{d-1} y = y + \sum_{i: x_i - y_i = d > 0} e_i - \sum_{i: x_i - y_i = -d < 0} e_i,
\]

where \(d = \|x - y\|_\infty\).

A function \(f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{\infty\}\) is called \(N\)-convex if it satisfies

\[
\begin{align}
f(x) + f(y) &\geq f(x \rightarrow y) + f(y \rightarrow x), \\
f(x) + f(y) &\geq f(x \rightarrow y) + f(y \rightarrow x)
\end{align}
\]

for all \(x, y \in \mathbb{Z}^n\).

**Lemma 2.7.**  
1. \(x \mapsto a^\top x + b\) is \(N\)-convex for \(a \in \mathbb{R}^n, b \in \mathbb{R}\).
2. \(x \mapsto \max(x_i + x_j, 0)\) is \(N\)-convex for \(i, j \in [n]\).
3. If \(f, g\) are \(N\)-convex, then \(cf + dg\) is \(N\)-convex for \(c, d \geq 0\).
4. Suppose that \(\sigma : \mathbb{Z}^n \rightarrow \mathbb{Z}^n\) is a translation \(x \mapsto x + v\), a transposition of coordinates \((x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) \mapsto (x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)\), or the sign change of some coordinate \((x_1, \ldots, x_i, \ldots, x_n) \mapsto (x_1, \ldots, -x_i, \ldots, x_n)\). If \(f\) is \(N\)-convex, then \(f \circ \sigma\) is \(N\)-convex.
Thus, (2.1) and (2.2) hold for all cases.

Let \( f(x) := \max(x_1 + x_2, 0) \). Choose distinct \( x, y \in \mathbb{Z}^2 \). Let \( x' := x \rightarrow y \) (or \( x \rightarrow y \)), and let \( y' := y \rightarrow x \) (or \( y \rightarrow x \)); our argument below works for both \( \rightarrow \) and \( \rightarrow \). We may consider the case \( f(x) < f(x') \in \{ f(x) + 1, f(x) + 2 \} \). We may assume \( x'_1 = x_1 + 1 \). Then \( y_1 \geq x'_1 > x_1 \). If \( f(x') = f(x) + 2 \), then \( x_1 + x_2 \geq 0 \), \( y_2 \geq x'_2 > x_2 \), and \( y' = y - (1, 1) \), implying \( f(y') = f(y) - 2 \). Suppose that \( f(x') = f(x) + 1 \). If \( x_2 = x_2 \), then \( y'_2 = y - (1, 0) \) and \( |y_2 - x_2| < y_1 - x_1 \), implying \( y_1 + y_2 > x_1 + x_2 \geq 0 \) and \( f(y') = f(y) - 1 \). If \( x_2 = x_2 + 1 \), then \( x_1 + x_2 = -1 \), \( y > x \), and \( y' = y - (1, 1) \). If \( x + (1, 1) = y \), then \( x' = y \) and \( y' = x \). Otherwise \( y_1 + y_2 \geq 2 \), implying \( f(y') = f(y) - 2 \). Thus, (2.1) and (2.2) hold for all cases.

Finally we consider the case \( n \geq 3 \). Let \( p : \mathbb{Z}^n \rightarrow \mathbb{Z}^2 \) be the projection \( x \mapsto (x_1, x_j) \). Then \( f = f \circ p \). Also it is obvious that \( p(x \rightarrow y) = p(x) \rightarrow p(y) \). Hence \( f(x) + f(y) = f(p(x)) + f(p(y)) \geq f(p(x) \rightarrow p(y)) + f(p(y) \rightarrow p(x)) = f(p(x \rightarrow y)) + f(p(y \rightarrow x)) = f(x \rightarrow y) + f(y \rightarrow x) \). Also observe that \( (p(x \rightarrow y), p(y \rightarrow x)) \) is equal to \( (p(x), p(y)) \) or \( (p(x) \rightarrow p(y), p(y) \rightarrow p(x)) \). From this we have (2.2).

Observe that the objective function of (D), \( (\alpha, \beta) \mapsto -\sum_{i=1}^{n} \alpha_i - \sum_{i=1}^{n} \beta_i + \text{const} \) if (\( \alpha, \beta \)) is feasible, and \( \infty \) otherwise, is N-convex. A slightly modified version of this fact will be used in the proof of the sensitivity theorem (Section 3.4).

N-convexity is definable on \( L \times M \) by taking apartments. That is, \( f : L \times M \rightarrow \mathbb{R} \cup \{ \infty \} \) is called N-convex if the restriction of \( f \) to every apartment is N-convex. Hence we have the following, though it is not used in this paper explicitly.

**Proposition 2.8.** The objective function of (D) is N-convex on \( L \times M \).

In fact, operators \( \rightarrow \) and \( \rightarrow \) are independent of the choice of apartments, since they can be written by lattice operators on \( L \times M \).

## 3 Algorithm

In this section, we develop an algorithm in Theorem 1.2. In the following, we assume \( \deg Det A > -\infty \). Indeed, by Lemma 2.5 it can be decided in advance by nc-rank computation. Also we may assume that each \( c_k \) is a positive integer, since \( \deg Det t^b A = nb + \deg Det A \).

### 3.1 Deg-Det algorithm

We here present the Deg-Det algorithm [12] for (D), which is a simplified version of Murota’s combinatorial relaxation algorithm [18] designed for \( \deg det \); see also [19 Section 7.1]. The algorithm uses an algorithm of solving (R) as a subroutine. For simplicity, we assume (by multiplying permutation matrices) that the position of a zero submatrix in (R) is upper right.

**Algorithm: Deg-Det**

**Input:** \( A = \sum_{k=1}^{m} A_k x_k t^{c_k} \), where \( A_k \in \mathbb{K}^{n \times n} \) and \( c_k \geq 1 \) for \( k \in [m] \), and an initial feasible solution \( P, Q \) for \( A \).
**Output:** deg Det $A$.

1: Solve the problem (R) for $(PAQ)^{(0)}$ and obtain optimal matrices $S, T$.

2: If the optimal value $2n - r - s$ of (R) is equal to $n$, then output $- \text{deg det } P - \text{deg det } Q$.

Otherwise, letting $(P, Q) \leftarrow (t^1SP, QT(t^{-1}n-r))$, go to step 1.

The mechanism of this algorithm is simply explained: The matrix $SPAQT$ after step 1 has a negative degree in each entry of its upper right $r \times s$ submatrix. Multiplying $t$ for the first $r$ rows and $t^{-1}$ for the first $n - s$ columns does not produce the entry of degree $> 0$. This means that the next solution $(P, Q) := ((t^{1r})SP, QT(t^{-1}n-r))$ is feasible for $A$, and decreases $- \text{deg det } P - \text{deg det } Q$ by $r + s - n(>0)$. Then the algorithm terminates after finite steps, where Lemma 2.3 (1) guarantees the optimality.

In the view of Euclidean building, the algorithm moves the point $(P, Q) \in \mathcal{L} \times \mathcal{M}$ to an “adjacent” point $\langle P', Q' \rangle = ((t^{1r})SP, QT(t^{-1}n-r))$ with $d_\infty(\langle P, Q \rangle, \langle P', Q' \rangle) = 1$. Then the number of the movements (= iterations) is analyzed via the geometry of the Euclidean building. Let $\text{OPT}(A) \subseteq \mathcal{L} \times \mathcal{M}$ denote the set of (the image of) all optimal solutions for $A$. Then the number of iterations of $\text{Deg-Det}$ is sharply bounded by the following distance between from $\langle P, Q \rangle$ to $\text{OPT}(A)$:

$$
\hat{d}_\infty(\langle P, Q \rangle, \text{OPT}(A)) := 
\min\{d_\infty(\langle P, Q \rangle, (P^*, Q^*)) | (P^*, Q^*) \in \text{OPT}(A) : \langle P \rangle \subseteq \langle P^* \rangle, \langle Q \rangle \supseteq \langle Q^* \rangle\},
$$

where we regard $\langle P \rangle$ (resp. $\langle Q \rangle$) as a $\mathbb{K}(t)^{-}$-submodule of $\mathbb{K}(t)^n$ spanned row (resp. column) vectors. Observe that $(P, Q) \mapsto (tP, Qt)$ does not change the feasibility and objective value, and hence an optimal solution $(P^*, Q^*)$ with $\langle P \rangle \subseteq \langle P^* \rangle, \langle Q \rangle \supseteq \langle Q^* \rangle$ always exists.

**Theorem 3.1** ([12]). The number of executions of step 1 in $\text{Deg-Det}$ with an initial solution $(P, Q)$ is equal to $\hat{d}_\infty(\langle P, Q \rangle, \text{OPT}(A)) + 1$.

This property is a consequence of L-convexity of the objective function of (D). Thus $\text{Deg-Det}$ is a pseudo-polynomial time algorithm. We will improve $\text{Deg-Det}$ by using a cost-scaling technique.

### 3.2 Cost-scaling

In combinatorial optimization, cost-scaling is a standard technique to improve a pseudo-polynomial time algorithm $A$ to a polynomial one. Consider the following situation: Suppose that an optimal solution $x^*$ for costs $\lfloor c_k/2 \rfloor$ becomes an optimal solution $2x^*$ for costs $2\lfloor c_k/2 \rfloor$, and that the algorithm $A$ starts from $2x^*$ and obtains an optimal solution for costs $c_k \approx 2\lfloor c_k/2 \rfloor$ within a polynomial number of iterations. In this case, a polynomial time algorithm is obtained by $\log \max_k c_k$ calls of $A$.

Motivated by this scenario, we incorporate a cost scaling technique with $\text{Deg-Det}$ as follows:

**Algorithm: Cost-Scaling**

**Input:** $A = \sum_{k=1}^m A_kx_k t^{c_k}$, where $A_k \in \mathbb{K}^{n \times n}$ and $c_k \geq 1$ for $k \in [m]$.

**Output:** deg Det $A$. 

9
0: Let $C \leftarrow \max_{i \in [m]} c_i$, $N \leftarrow \lceil \log_2 C \rceil$, $\theta \leftarrow 0$, and $(P, Q) \leftarrow (t^{-1}I, I)$.

1: Let $c_k^{(0)} \leftarrow \lceil c_i/2^{N-\theta} \rceil$ for $k \in [m]$, and let $A^{(0)} \leftarrow \sum_{k=1}^m A_k x_k t^{c_k^{(0)}}$.

2: Apply Deg-Det for $A^{(0)}$ and $(P, Q)$, and obtain an optimal solution $(P^*, Q^*)$ for $A^{(0)}$.

3: If $\theta = N$, then output $\deg det P^* - \deg det Q^*$. Otherwise, letting $(P, Q) \leftarrow (P^*(t^2), Q^*(t^2))$ and $\theta \leftarrow \theta + 1$, go to step 1.

For the initial scaling phase $\theta = 0$, it holds $c_k^{(0)} = 1$ for all $k$ and $(P, Q) = (t^{-1}I, I)$ is an optimal solution for $A^{(0)}$ (by Lemma 2.4 and the assumption $\text{nc-rank} \sum_{k=1}^m A_k x_k = n$).

Lemma 3.3. $(P^*(t^2), Q^*(t^2))$ is an optimal solution for $A^{(0)}(t^2) = \sum_{k=1}^m A_k x_k t^{2c_k^{(0)}}$, and is a feasible solution for $A^{(0+1)}$.

The former statement follows from the observation that the optimality (Lemma 2.4(1)) keeps under the change $(P, Q) \leftarrow (P(t^2), Q(t^2))$ and $c_k \leftarrow 2c_k$. The latter statement follows from the fact that $c_k^{(0+1)}$ is obtained by decreasing $2c_k^{(0)}$ (at most by 1). The correctness of the algorithm is clear from this lemma.

To apply Theorem 3.1, we need a polynomial bound of the distance between the initial solution $(P^*(t^2), Q^*(t^2))$ of the $\theta$-th scaling phase and optimal solutions for $A^{(\theta)}$.

The main ingredient for our algorithm is the following sensitivity result.

Proposition 3.3. Let $(P, Q)$ be the initial solution in the $\theta$-th scaling phase. Then it holds $\bar{d}_\infty((P, Q), \text{OPT}(A^{(\theta)})) \leq n^2 m$.

The proof is given in Section 3.4, in which N-convexity plays a crucial role. Thus the number of iterations of Deg-Det in step 2 is bounded by $O(n^2 m)$, and the number of the total iterations is $O(n^2 m \log C)$.

3.3 Truncation of low-degree terms

Still, the algorithm is not polynomial, since a naive calculation makes $(P, Q)$ have a pseudo-polynomial number of terms. Observe that $(S, T)$ in step 1 of Deg-Det depends only on the leading term of $PAQ = (PAQ)^{(0)} + (PAQ)^{(-1)} t^{-1} + \cdots$. Therefore it is expected that terms $(PAQ)^{(-\ell)} t^{-\ell}$ with large $\ell > 0$ do not affect on the subsequent computation. Our polynomial time algorithm is obtained by truncating such low degree terms. Note that in the case of the weighted linear matroid intersection, i.e., each $A_k$ is rank-1, such a care is not needed; see [6, 12] for details.

First, we present the cost-scaling Deg-Det algorithm in the form that it updates $A_k$ instead of $P, Q$ as follows:

Algorithm: Deg-Det with Cost-Scaling

Input: $A = \sum_{k=1}^m A_k x_k t^{c_k}$, where $A_k \in \mathbb{K}^{n \times n}$ and $c_k \geq 1$ for $k \in [m]$.

Output: $\deg Det A$.

0: Let $C \leftarrow \max_{i \in [m]} c_i$, $N \leftarrow \lceil \log_2 C \rceil$, $\theta \leftarrow 0$, $B_k \leftarrow A_k$ for $k \in [m]$, and $D^* \leftarrow n$. 
1: Letting $B \leftarrow \sum_{k=1}^{m} B_k x_k$, solve the problem (R) for $B^{(0)}$ and obtain an optimal solution $S, T$.

2: Suppose that the optimal value $2n - r - s$ of (R) is less than $n$. Letting $B_k \leftarrow (t^{1'})SB_k T(t^{-1}n^{1-\omega})$ for $k \in [m]$ and $D^* \leftarrow D^* + n - r - s$, go to step 1.

3: Suppose that the optimal value $2n - r - s$ of (R) is equal to $n$. If $\theta = N$, then output $D^*$. Otherwise, letting

$$B_k \leftarrow \left\{ \begin{array}{ll}
B_k(t^2) & \text{if } \left[ c_i/2^{N-\theta-1} \right] = 2\left[ c_i/2^{N-\theta} \right], \\
t^{-1}B_k(t^2) & \text{if } \left[ c_i/2^{N-\theta-1} \right] = 2\left[ c_i/2^{N-\theta} \right] - 1,
\end{array} \right.$$ 

$$D^* \leftarrow 2D^*, \text{ and } \theta \leftarrow \theta + 1, \text{ go to step 1.}$$

Notice that each $B_k$ is written as the following form:

$$B_k = B_k^{(0)} + B_k^{(-1)}t^{-1} + B_k^{(-2)}t^{-2} + \cdots,$$

where $B_k^{(-\ell)}$ is a matrix over $\mathbb{K}$. We consider to truncate low-degree terms of $B_k$ after step 1. For this, we estimate the magnitude of degree for which the corresponding term is irrelevant to the final output. In the modification $B_k \leftarrow (t^{1'})SB_k T(t^{-1}n^{1-\omega})$ of step 2, the term $B_k^{(-\ell)}t^{-\ell}$ splits into three terms of degree $-\ell + 1, -\ell$, and $-\ell - 1$. By Proposition 3.3 this modification is done at most $L := mn^2$ time in each scaling phase. In the final scaling phase $\theta = N$, the results of this phase only depend on terms of $B_k$ with degree at least $-L$. These terms come from the first $L/2$ terms of $B_k$ in the end of the previous scaling phase $\theta = N - 1$, which come from the first $L/2 + L$ terms of $B_k$ at the beginning of the phase. They come from the first $(L/2 + L)/2 + L$ terms of the phase $s = N - 2$. A similar consideration shows that the final result is a consequence of the first $L(1 + 1/2 + 1/4 + \cdots + 1/2^{N-\theta}) < 2L$ terms of $B_k$ at the beginning of the $\theta$-th scaling phase. Thus we can truncate each term of degree at most $-2L$: Add to Deg-Det with Cost-Scaling the following procedure after step 1.

**Truncation:** For each $k \in [m]$, remove from $B_k$ all terms $B_k^{(-\ell)}t^{-\ell}$ for $\ell \geq 2n^2m$.

Now we have our main result in an explicit form:

**Theorem 3.4. Deg-Det with Cost-Scaling** computes deg Det $A$ in $O((\gamma(n, m) + n^{2+\omega}m^2)n^2m \log C)$ time, where $\gamma(n, m)$ denotes the time complexity of solving (R) and $\omega$ denotes the exponent of the time complexity of matrix multiplication.

**Proof.** The total number of calls of the oracle solving (R) is that of the total iterations $O(n^2m \log C)$. By the truncation, the number of terms of $B_k$ is $O(n^2m)$. Hence the update of all $B_k$ in each iteration is done in $O(n^2+\omega m^2)$ time.

### 3.4 Proof of the sensitivity theorem

Let $A = \sum_{k=1}^{m} A_k x_k t^{c_k}$ and let $A' = A_1 x_1 t^{c_1-1} + \sum_{k=2}^{m} A_k x_k t^{c_k}$.

**Lemma 3.5.** Let $(P, Q)$ be an optimal solution for $A$. There is an optimal solution $(P', Q')$ for $A'$ such that $(P) \subseteq (P')$, $(Q) \supseteq (Q')$, and $d_{\infty}((P', Q'), (P, Q)) \leq n^2$. 

11
Proposition 3.3 follows from this lemma, since \( A^{(\theta)} \) is obtained from \( A^{(\theta-1)}(t^2) \) by \( O(m) \) decrements of \( 2c_k^{(\theta-1)} \).

Let \((P', Q')\) be an optimal solution for \( A' \) such that \( \langle P \rangle \subseteq \langle P' \rangle \), \( \langle Q \rangle \supseteq \langle Q' \rangle \), and 
\[ d := d_\infty((P', Q'), (P, Q)) \] 
is minimum. Suppose that \( d > 0 \). By Lemma 2.6 choose an apartment \( A \) of \( L \times M \) containing \( (P, Q) \) and \( (P', Q') \). Regard \( A \) as \( Z^n \times Z^n \). Then \( \langle P, Q \rangle \) and \( \langle P', Q' \rangle \) are regarded as points \((\alpha, \beta)\) and \((\alpha', \beta')\) in \( Z^n \times Z^n \), respectively. The inclusion order \( \subseteq \) becomes vector ordering \( \leq \). In particular, \( \alpha \leq \alpha' \) and \( \beta \geq \beta' \).

Consider the problem \((D_A)\) on this apartment. We incorporate the constraints \( x_i + y_i + c_{ij} \) \( \leq 0 \) to the objective function as barrier functions. Let \( M > 0 \) be a large number. Define \( h : Z^n \times Z^n \rightarrow \mathbb{R} \) by
\[ h(x, y) := -\sum_i x_i - \sum_y y_i + M \sum_{i,j,k} \max\{x_i + y_i + c_{ij}^k, 0\} \quad ((x, y) \in Z^n \times Z^n), \]
where \( i, j \) range over \([n]\) and \( k \) over \([m]\). Similarly define \( h' : Z^n \times Z^n \rightarrow \mathbb{R} \) with replacing \( c_{ij}^k \) by \( c_{ij}^k - 1 \) for each \( i, j \) in \([n]\).

Since \( M \) is large, \((\alpha, \beta)\) is a minimizer of \( h \) and \((\alpha', \beta')\) is a minimizer of \( h' \). Note that \((\alpha, \beta)\) is not a minimizer of \( h' \).

Consider the normal path \((z = z^0, z^1, \ldots, z^d = z')\) from \( z = (\alpha, \beta) \) to \( z' = (\alpha', \beta') \).
Since \( z \) and \( z' \) satisfy \( x_i + y_j + c_{ij}^1 \leq 1 \) and \( x_i + y_j + c_{ij}^k \leq 0 \) \((k \neq 1)\) for all \( i, j \in [n] \), by \( N\)-convexity \((\text{Lemma 2.7 (2)})\) all points \( z^\ell = (x^\ell, y^\ell) \) in the normal path satisfies these constraints. Let \( N_\ell \) be the number of the indices \((i, j)\) such that \( z^\ell = (x^\ell, y^\ell) \) satisfies \( x_i + y_j + c_{ij}^k = 1 \). Then
\[ h'(z^\ell) = h(z^\ell) - MN_\ell \quad (\ell = 0, 1, 2, \ldots, d), \]
where \( N_0 = 0 \) holds (since \( z \) is a feasible solution for \( A \)).

Next we show the monotonicity of \( h, h' \) through the normal path:
\[ h(z) \leq h(z^1) \leq \cdots \leq h(z^{d-1}) \leq h(z'), \]
\[ h'(z) > h'(z^1) > \cdots > h'(z^{d-1}) > h'(z'). \]

Since \( h \) is \( N \)-convex and \( z \) is a minimizer of \( h \), we have \( h(z) + h(z^\ell) \geq h(z \rightarrow z^\ell) + h(z^{\ell-1}) \) and \( h(z) \leq h(z \rightarrow z^\ell) \), implying \( h(z^\ell) \geq h(z^{\ell-1}) \). Similarly, since \( h' \) is \( N \)-convex, it holds \( h'(z^\ell) + h'(z') \geq h'(z^{\ell+1}) + h'(z' \rightarrow z^\ell) \). Here \( z' \rightarrow z^{\ell+1} = (\tilde{x}, \tilde{y}) \) is closer to \( z = (\alpha, \beta) \) than \( z' \), with \( \alpha \leq \tilde{x}, \beta \geq \tilde{y} \). Since \( z' \) is a minimizer of \( h' \) nearest to \( z \), we have \( h'(z') < h'(z' \rightarrow z^\ell) \). Thus \( h'(z^\ell) > h'(z^{\ell+1}) \).

By (3.1), (3.2), (3.3), we have
\[ 0 = N_0 < N_1 < \cdots < N_{d-1} < N_d \leq n^2. \]
Thus we have \( d \leq n^2 \).

## 4 Algebraic combinatorial optimization for 2 \times 2\text{-partitioned matrix}

In this section, we consider an algebraic combinatorial optimization problem for a \( 2 \times 2 \)-partitioned matrix \((L, M)\). As an application of the cost-scaling \textbf{Deg-Det} algorithm, we extend the combinatorial rank computation in [15] to the deg-det computation.
We first present the rank formula due to Iwata and Murota [17] in a suitable form for us.

**Theorem 4.1** ([17]). rank $A$ for a matrix $A$ of form (1.3) is equal to the optimal value of the following problem:

$$(R_{2\times 2}) \quad \min \quad 4n - r - s \quad \text{s.t.} \quad SAT \text{ has an } r \times s \text{ zero submatrix},$$

where $S,T$ are written as

$$S = \begin{pmatrix}
S_1 & O & \cdots & O \\
O & S_2 & \cdots & \vdots \\
\vdots & \ddots & \ddots & O \\
O & \cdots & O & S_n
\end{pmatrix}, \quad T = \begin{pmatrix}
T_1 & O & \cdots & O \\
O & T_2 & \cdots & \vdots \\
\vdots & \ddots & \ddots & O \\
O & \cdots & O & T_n
\end{pmatrix}$$

for $S_i, T_i \in GL_2(\mathbb{K})$ ($i \in [n]$).

Namely, (R$_{2\times 2}$) is a sharpening of (R) for 2 $\times$ 2-partitioned matrices, where $S, T$ are taken as a form of (4.1). This was obtained earlier than the Fortin-Reutenauer formula (Theorem 2.1). From the view, this theorem implies rank $A = nc$-rank $A$ for a 2 $\times$ 2-partitioned matrix $A$. Therefore, by Theorem 1.1, the rank of $A$ can be computed in a polynomial time.

Hirai and Iwamasa [15] showed that the rank computation of a 2 $\times$ 2-partitioned matrix can be formulated as the cardinality maximization problem of certain algebraically constraint 2-matchings in a bipartite graph. Based on this formulation and partly inspired by the Wong sequence method [9, 10], they gave a combinatorial augmenting-path type $O(n^4)$-time algorithm to obtain a maximum matching and an optimal solution $S, T$ in (R$_{2\times 2}$).

Here, for simplicity of description, we consider a weaker version of this 2-matching concept. Let $G_A = ([n] \sqcup [n], E)$ be a bipartite graph defined by $ij \in E \iff A_{ij} \neq O$. A multiset $M$ of edges in $E$ is called a 2-matching if each node in $G_A$ is incident to at most two edges in $M$. For a (multi)set $F$ of edges in $E$, let $A_F$ denote the matrix obtained from $A$ by replacing $A_{ij}$ ($ij \notin F$) by the zero matrix. Observe that a nonzero monomial $p$ of a subdeterminant of $A$ gives rise to a 2-matching $M$ by: An edge $ij \in E$ belongs to $M$ with multiplicity $m \in \{1, 2\}$ if $x_{ij}^m$ appears in $p$. Indeed, by the 2 $\times$ 2-partition structure of $A$, index $i$ appears at most twice in $p$. The monomial $p$ also appears in a subdeterminant of $A_M$. Motivated by this observation, a 2-matching $M$ is called $A$-consistent if it satisfies

$$|M| = \text{rank}(A_M),$$

where the cardinality $|M|$ is considered as a multiset.

**Proposition 4.2** ([15]). rank $A$ is equal to the maximum cardinality of an $A$-consistent 2-matching.

We see Lemma 4.4 below for an essence of the proof. In [15], a stronger notion of a (2-)matching is used, and it is shown that $|M| = \text{rank}(A_M)$ is checked in $O(n^2)$-time (by assigning a valid labeling (VL)). An $A$-consistent 2-matching is called maximum if it has the maximum cardinality over all $A$-consistent 2-matchings.
**Theorem 4.3** ([15]). A maximum $\mathcal{A}$-consistent 2-matching and an optimal solution in $(R_{2 \times 2})$ can be computed in $O(n^4)$-time.

Now we consider a weighted version. Suppose that each $x_{ij}$ has integer weight $c_{ij}$, that is, consider

$$A = \begin{pmatrix}
A_{11}x_{11}t^{c_{11}} & A_{12}x_{12}t^{c_{12}} & \cdots & A_{1n}x_{1n}t^{c_{1n}} \\
A_{21}x_{21}t^{c_{21}} & A_{22}x_{22}t^{c_{22}} & \cdots & A_{2n}x_{2n}t^{c_{2n}} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n1}x_{n1}t^{c_{n1}} & A_{n2}x_{n2}t^{c_{n2}} & \cdots & A_{nn}x_{nn}t^{c_{nn}}
\end{pmatrix}.$$ \hspace{1cm} (4.2)

The computation of $\operatorname{deg det} A$ corresponds to the maximum-weight $\mathcal{A}$-consistent 2-matching problem. We suppose that $\operatorname{rank} A = 2n$, and $\operatorname{deg det} A > -\infty$. An $\mathcal{A}$-consistent matching $M$ (defined for (4.3)) is called perfect if $|M| = 2n (= \operatorname{rank} A)$; necessarily such an $M$ is the disjoint union of cycles. The weight $c(M)$ is defined by

$$c(M) = \sum_{ij \in M} c_{ij}.$$ 

Note that $c_{ij}$ contributes to $c(M)$ twice if the multiplicity of $ij$ in $M$ is 2.

**Lemma 4.4.** $\operatorname{deg det} A$ is equal to the maximum weight of a perfect $\mathcal{A}$-consistent 2-matching.

**Proof.** Consider the leading term $q \cdot t^{\deg \det A}$ of $\det A$, where $q$ is a nonzero polynomial of variables $x_{ij}$. Choose any monomial $p$ in the polynomial $q$. As mentioned above, the set $M$ of edges $ij$ (with multiplicity $m = 1, 2$) for which $x_{ij}^m$ appears in $p$ forms a 2-matching. It is necessarily perfect and $\mathcal{A}$-consistent. Its weight $c(M)$ is equal to $\operatorname{deg det} A$. Thus $\operatorname{deg det} A$ is at most the maximum weight of a perfect $\mathcal{A}$-consistent 2-matching.

We show the converse. Choose a maximum-weight perfect $\mathcal{A}$-consistent 2-matching $M$. It suffices to show that $\det A_M$ has a nonzero term with degree $c(M)$; such a term also appears in $\det A$. Now $M$ is a disjoint union of cycles, where a cycle of two (same) edges $ij, ij$ can appear. We may consider the case where $M$ consists of a single cycle, from which the general case follows. Suppose that $M = \{ij, ij\}$. Then $A_{ij}$ must be nonsingular, and $\det(A_{ij}x_{ij}t^{c_{ij}}) = 2c_{ij} = c(M)$. Suppose that $M$ is a simple cycle of length $2n$. Then $M$ is the disjoint union of two perfect matchings $M_1, M_2$. If $A_{ij}$ is nonsingular for all edges $ij$ in the cycle $M$, then $M_1$ and $M_2$ are regarded as perfect $\mathcal{A}$-consistent 2-matchings by defining the multiplicity of all edges by 2 uniformly. By maximality and $c(M) = (c(M_1) + c(M_2))/2$, it holds $c(M_1) = c(M_2) = c(M)$. Replace $M$ by $M_1$. Then $\det A_M$ has a single term with degree $c(M)$. Suppose that $M_1$ has an edge $ij$ for which rank $A_{ij} = 1$. As in [15] (2.6)–(2.9), we can take $S_i, T_i \in GL_2(\mathbb{K})$ such that for each $ij \in M$, $A'_{ij} = S_iA_{ij}T_j$ is a $2 \times 2$ diagonal matrix with $(A'_{ij})_{\kappa \kappa} \neq 0$ if $ij \in M_\kappa$ for $\kappa = 1, 2$. From $(A'_{ij})_{22} = 0$ for an edge $ij \in M_1$ with rank $A_{ij} = 1$, we see that the term of $t^{c(M)}$ (obtained by choosing the $(\kappa, \kappa)$-element of $A'_{ij}x_{ij}t^{c_{ij}}$ for $ij \in M_\kappa, \kappa = 1, 2$) does not vanish in $\det SAT = \text{const} \cdot \det A$, where $S, T$ are block diagonal matrices with diagonals $S_i, T_j$ as in (1.1).

Corresponding to Theorem 4.1, the following holds:
Lemma 4.5. \( \deg \det A \) is equal to \( \deg \det \det A \), which is given by the optimal value of

\[
(D_{2\times 2}) \quad \text{Min.} \quad \sum_{i=1}^{n} \deg \det P_{i} - \sum_{i=1}^{n} \deg \det Q_{i}
\]

s.t. \( \deg(P_{i}A_{ij}Q_{j})_{i,j} + c_{ij} \leq 0 \quad (i,j \in [n], \kappa, \lambda = 1, 2) \),

\( P_{i}, Q_{j} \in GL_{2}(K) \quad (i,j \in [n]). \)

Proof. When we apply \textbf{Deg-Det} algorithm to \( A \) of (\ref{12}), \( (S,T) \) in the step 1 is of form of (\ref{11}). Therefore \( PAQ^{(0)} \) is always of form (\ref{13}), and \( P \) and \( Q \) are always block diagonal matrices with \( 2 \times 2 \) block diagonal matrices \( P_{1}, \ldots, P_{n} \) and \( Q_{1}, \ldots, Q_{n} \), respectively. Since \( \text{rank} PAQ^{(0)} = \text{nc-rank} PAQ^{(0)} \) (by Theorem 4.1), the output is equal to \( \deg \det A \) (Lemma 2.4 (2)). \( \square \)

Now we arrive at the goal of this section.

Theorem 4.6. Suppose that arithmetic operations on \( \mathbb{K} \) are done in constant time. A maximum-weight perfect \( A \)-consistent 2-matching (and \( \deg \det A \)) can be computed in \( O(n^{6} \log C) \)-time, where \( C := \max_{i,j \in [n]} |c_{ij}| \).

Proof. Apply \textbf{Deg-Det with Cost-Scaling} to the matrix \( A \). Since \( A_{ij} \) is \( 2 \times 2, N_{A} \) in the proof of the sensitivity theorem (Section 3.3) can be taken to be 4 (constant), whereas \( m \) is \( n^{2} \). Therefore, in each scaling phase, the number of iterations is bounded by \( n^{2} \). Then the degree bound for truncation is chosen as \( 2n^{2} \). The time complexity for matrix update is \( O(n^{2} \times n^{2}) \); this is done by matrix multiplication of \( 2 \times 2 \) matrices. By Theorem 4.3, \( \gamma(n,m) = O(n^{4}) \). The total time complexity is \( O(n^{6} \log C) \).

Next we find a maximum-weight perfect \( A \)-consistent 2-matching from the final \( (0) \) for \( B = B^{(0)} + B^{(-1)} t^{-1} + \cdots \). Consider a maximum \( B^{(0)} \)-consistent 2-matching \( M \) for \( 2 \times 2 \)-partitioned matrix \( B^{(0)} \) (of form (\ref{13})). Necessarily \( M \) is perfect (since \( B^{(0)} \) is nonsingular). We show that \( M \) contains a maximum-weighted \( A \)-consistent 2-matching. Indeed, \( B^{(0)} \) is equal to \( (PAQ)^{0} \) for \( P, Q \in GL_{n}(\mathbb{K}(t)) \), where \( P \) and \( Q \) are block diagonal matrices with \( 2 \times 2 \) block diagonals \( P_{1}, P_{2}, \ldots, P_{n} \) and \( Q_{1}, Q_{2}, \ldots, Q_{n} \). Notice that \( P_{i}, Q_{j} \) are an optimal solution of \( (D_{2\times 2}) \). Observe \( B_{M}^{(0)} = (PA_{M}Q)^{0} \). From this, we have \( \deg \det PAQ \geq \deg \det PA_{M}Q = \deg \det A_{M} + \sum_{i} \deg \det P_{i} + \sum_{j} \deg \det Q_{i} = \deg \det B_{M}^{(0)} = 0 \). This means that \( \deg \det A_{M} \) is equal to \( \deg \det A \), which is the maximum-weight of a perfect \( A \)-consistent 2-matching (Lemma 4.4). Therefore, \( M \) must contain a maximum-weight perfect \( A \)-consistent 2-matching. It is easily obtained as follows. Consider a simple cycle \( C = C_{1} \cup C_{2} \) of \( M \), where \( C_{1} \) and \( C_{2} \) are disjoint matchings in \( C \). For \( \kappa \in \{1, 2\} \), if \( C_{\kappa} \) consists of edges \( ij \) with rank \( A_{ij} = 2 \) and \( c(C_{\kappa}) \geq c(C) \), then replace \( C \) by \( C_{\kappa} \) in \( M \). Apply the same procedure to each cycle. The resulting \( M \) satisfies \( c(M) = \deg \det A_{M} \), as desired. \( \square \)

From the view of polyhedral combinatorics, it is a natural question to ask for the LP-formulation describing the polytope of \( A \)-consistent 2-matchings. One possible approach to this question is to clarify the relationship between the LP-formulation and (\( R_{2\times 2} \)).

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15
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Appendix: Proof of Lemma 2.4

(1). We have seen the only-if part in the explanation of $\text{Deg-Det}$. So we show the if part. We first extend $\text{deg Det} B$ for matrix $B = \sum_{k=1}^{m} B_k(t)x_k$, where $B_k(t)$ are matrices over $\mathbb{K}(t)$. This is naturally defined by (D) in replacing the constraint by $\text{deg}(P B_k Q)_{ij} \leq 0$. In this setting, it obviously holds that $\text{deg Det} P B Q = \text{deg det} P + \text{deg Det} B Q + \text{deg Det} B$. Therefore it suffices to show $\text{deg Det} B = 0$ if $\text{deg} B_{ij} \leq 0$ for all $i, j$ and nc-rank $B^{(0)} = n$.

Let $(P, Q)$ be any feasible solution for $B$. Recall the Smith-McMillan form that $P, Q$ are written as $P = S(t^n) S', Q = T(t^{-\beta}) T'$ for $S, S', T, T' \in GL_n(\mathbb{K}(t)^-)$. From deg$(P B Q)_{ij} \leq 0$, it must hold that $\alpha_i > \beta_j$ implies $(S^{(0)} B^{(0)} T^{(0)})_{ij} = 0$. Let $0 =: \gamma_0 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_{\ell}$ so that $\{\gamma_1, \gamma_2, \ldots, \gamma_{\ell}\} = \{\alpha_i\}_{i=1}^{n} \cup \{\beta_j\}_{j=1}^{n}$. For each $p = 1, 2, \ldots, \ell$, define the indices $r_p := \max\{i \mid \alpha_i \geq \gamma_p\}$ and $u_p = \min\{j \mid \gamma_p - 1 \geq \beta_j\}$. Then $S^{(0)} B^{(0)} T^{(0)}$ must have an $r_p \times (n - u_p + 1)$ zero submatrix in its upper right corner. Since nc-rank $B^{(0)} = n$, it holds $-r_p + u_p - 1 \geq 0$.

Also, $\alpha, \beta$ are written as

$$\alpha = \sum_{p=1}^{\ell} (\gamma_p - \gamma_{p-1}) 1_{r_p}, \quad \beta = \sum_{p=1}^{\ell} (\gamma_p - \gamma_{p-1}) 1_{u_p-1}.$$
Now \(- \deg \det P - \deg \det Q\) is equal to

\[- \sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{n} \beta_j = \sum_{p=1}^{\ell} (\gamma_p - \gamma_{p-1})(-r_p + u_p - 1) \geq 0.\]

This means that every feasible solution has the objective value at least 0, and \((I, I)\) is an optimal solution for \(B\), implying \(\deg \det B = 0\).

(2). It holds \(\deg \det PAQ = \deg \det P + \deg \det Q + \deg \det A\). If \(\text{rank}(PAQ)^{(0)} = n\) and \(\deg(PAQ)_{ij} \leq 0\) for \(i, j\), it holds \(\deg \det PAQ = 0\). In this case, it also holds \(\text{nc-rank}(PAQ)^{(0)} = n\), and hence \(\deg \det PAQ = 0\), implying \(\deg \det A = - \deg \det P - \deg \det Q\).