Weak Values, the Reconstruction Problem, and the Uncertainty Principle

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Abstract. Closely associated with the notion of weak value is the problem of reconstructing the post-selected state: this is the so-called reconstruction problem. We show that the reconstruction problem can be solved by inversion of the cross-Wigner transform, using an ancillary state. We thereafter show, using the multidimensional Hardy uncertainty principle, that maximally concentrated cross-Wigner transforms corresponds to the case where a weak measurement reduces to an ordinary von Neumann measurement.

1. Introduction
In time-symmetric quantum mechanics (TSQM) the state of a system is represented by a two-state vector $\langle \phi \rangle$ where the state $\langle \phi \rangle$ evolves backwards from the future and the state $\psi$ evolves forwards from the past [1–6]. The interpretation of the two-state vector goes as follows: assume that at an initial time $t_i$ an observable $A$ is measured and a non-degenerate eigenvalue was found: $\psi(t_i) = |A = \alpha\rangle$; similarly at a later (final) time $t_f$ a measurement of another observable $B$ yields $|\phi(t_f)\rangle = |B = \beta\rangle$. The two-time state $\langle \phi | \psi \rangle$ can be constructed as follows: Alice prepares the state $|\psi(t_i)\rangle$ at initial time $t_i$. She then sends the system to Bob (the observer), who is free to perform any measurement he wants. The system is then returned to Alice, who then performs a strong measurement with the state $|\phi(t_f)\rangle$ as one of the outcomes. Only if this outcome is obtained, does Bob keep the results of his measurement. Let now $t$ be some intermediate time: $t_i < t < t_f$. Following the time-symmetric approach to quantum mechanics at this intermediate time the system is described by the two wavefunctions

$$\psi = U_i(t, t_i)\psi(t_i), \phi = U_f(t, t_f)\phi(t_f)$$

where $U_i(t, t_i) = e^{-i\hat{H}_i(t-t_i)/\hbar}$ and $U_f(t, t_f) = e^{-i\hat{H}_f(t-t_f)/\hbar}$ are the unitary operators governing the evolution of the state before and after time $t$; $|\psi\rangle$ is the pre-selected state and $|\phi\rangle$ the post-selected state. By definition, the complex number

$$\langle \hat{A} \rangle_{\phi,\psi} = \frac{\langle \phi | \hat{A} | \psi \rangle}{\langle \phi | \psi \rangle}$$

is the weak value of $\hat{A}$ in the two-time state $\langle \phi | \psi \rangle$; it is well-defined as long as the vectors $\phi$ and $\psi$ are non-orthogonal (which we assume from now on). Notice that when $\phi = \psi$ then $\langle \hat{A} \rangle_{\phi,\psi}$ is just the usual mean $\langle \hat{A} \rangle_{\psi}$ value of $\hat{A}$ in the pure state $|\psi\rangle$. It is easy to show, using Wigner’s
phase space formalism [7–10], that formula (2) can be written [11–13]
\[
\langle \hat{A} \rangle_{\phi,\psi} = \frac{1}{\langle \phi | \psi \rangle} \int a(x,p) W(\psi,\phi)(x,p) d^n x
\]
where \(a = a(x,p)\) is the dequantization of the operator \(\hat{A}\) (i.e. the classical observable corresponding to \(\hat{A}\)) and \(W(\psi,\phi)\) is the cross-Wigner transform of \(\psi\) and \(\phi\):
\[
W(\psi,\phi)(x,p) = \left( \frac{1}{2\pi \hbar} \right)^n \int e^{-\frac{i}{\hbar} p y} \psi(x + \frac{1}{2} y) \phi^*(x - \frac{1}{2} y) d^n y
\]
(it reduces to the usual Wigner distribution [8, 10] \(W_{\psi}\) when \(\psi = \phi\).

Now, as was already shown by Lundeen et al. [14] in 2011 (also see Lundeen and Bambe [15]), weak values can be used to reconstruct the state \(|\psi\rangle\); they considered the following experiment on a particle: a weak measurement of the real position variable \(x\) is performed; this amounts to applying the projection operator \(\hat{P}_x = |x\rangle \langle x|\) to the pre-selected state \(|\psi\rangle\); thereafter they perform a strong measurement of momentum, which yields the value \(p_0\), that is \(\phi(x) = \left( \frac{1}{2\pi \hbar} \right)^{1/2} e^{\frac{i}{\hbar} p_0 x}\). The result of the weak measurement is thus
\[
\langle \hat{P}_x \rangle_{\psi,\phi} = \frac{\langle p_0 | x \rangle \langle x | \psi \rangle}{\langle p_0 | \psi \rangle} = \left( \frac{1}{2\pi \hbar} \right)^{1/2} e^{-\frac{i}{\hbar} p_0 x} \psi(x) \psi^*(p_0)
\]
(\(\hat{\psi}\) the Fourier transform of \(\psi\)). Since the value of \(p_0\) is known we get
\[
\psi(x) = \frac{1}{k} e^{\frac{i}{\hbar} p_0 x} \langle \hat{P}_x \rangle_{\psi,\phi}
\]
where \(k = (2\pi \hbar)^{n/2} \psi(p_0)\); formula (6) thus allows to determine \(\psi(x)\) by scanning through the values of \(x\). In our previous work [11–13] we have shown that this procedure boils down to the following question:

**Knowing \(W(\psi,\phi)\) and the postselected state \(\phi\) can we determine the preselected state \(\psi\)?**

The effective determination of \(W(\psi,\phi)\) is essentially a problem of quantum tomography, see [16–25]).

For instance, in the case of Lundeen’s experiment we have \(a(x,p) = \delta(x - x_0)\) and
\[
W(\psi,\phi)(x,p) = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} e^{-\frac{i}{\hbar} p y} \psi(x + \frac{1}{2} y) e^{-\frac{i}{\hbar} p_0 (x - \frac{1}{2} y)} dy
\]

hence, by formula (3),
\[
\langle \hat{P}_{x_0} \rangle_{\psi,\phi} = \left( \frac{1}{2\pi \hbar} \right)^{1/2} \frac{1}{\psi(p_0)} e^{-\frac{i}{\hbar} p_0 x_0} \psi(x_0)
\]
which is (6) if one replaces \(x\) with \(x_0\). We are going to show that it is indeed always possible to reconstruct the preselected state \(|\psi\rangle\) (in Lundeen et al.’s example, the wavefunction) if one knows the classical observable \(a\) of which the operator \(\hat{A}\) is the (Weyl) quantization together with the cross-Wigner distribution of the two-time state \(\langle \phi | \psi \rangle\); a symmetry argument will show that one can actually reverse the situation and determine the post-selected state as well; more intriguingly we will see that, in fact, that the whole two-time state can be reconstructed (up to
a factor) if one knows \( W(\psi, \phi) \). This actually raises another question, intimately related to the uncertainty principle.

**Notation.** We will work with systems having \( n \) degrees of freedom. Position (resp. momentum) variables are denoted \( x = (x_1, \ldots, x_n) \) (resp. \( p = (p_1, \ldots, p_n) \)). The corresponding phase space variable is \( z = (x, p) \). The scalar product \( p_1 x_1 + \cdots + p_n x_n \) is denoted by \( px \). When integrating we will use, where appropriate, the volume elements \( dx^n = dx_1 \cdots dx_n \), \( dp^n = dp_1 \cdots dp_n \). The unitary Fourier transform of a square-integrable function \( \psi \) of \( x \) is by definition

\[
F\psi(p) = \hat{\psi}(p) = \left( \frac{1}{2\pi\hbar} \right)^{n/2} \int e^{-\frac{i}{\hbar}px}(x)dx^n.
\]  

We denote by \( \hat{x} = (\hat{x}_1, \ldots, \hat{x}_n) \) and \( \hat{p} = (\hat{p}_1, \ldots, \hat{p}_n) \) the (vector) operators defined by \( \hat{x}_j \psi = x_j \psi \), \( \hat{p}_j \psi = -i\hbar \partial_{x_j} \psi \). We will assume from now on that \( |\psi\rangle \) and \( |\phi\rangle \) are two normalized non-orthogonal states: \( \langle \psi|\psi \rangle = \langle \phi|\phi \rangle = 1, \langle \phi|\psi \rangle \neq 0 \).

### 2. A General Reconstruction Formula

We begin by recalling some well-known results from harmonic analysis. Let \( \hat{T}(x_0, p_0) = e^{-\frac{i}{\hbar}(p_0 \cdot \hat{x} - x_0 \cdot \hat{p})} \) be the Heisenberg operator [7,9,26]; it is a unitary operator whose action on a wavefunction \( \psi \) is given by

\[
\hat{T}(x_0, p_0)\psi(x) = e^{\frac{i}{\hbar}(p_0 x - \frac{1}{2} p_0 x_0)}\psi(x-x_0).
\]  

An associated operator is the Grossmann–Royer reflection operator \( \hat{\Pi}(x_0, p_0) \) whose definition is due to Grossmann [27] and Royer [28] (see [7,29] for a detailed study of these operators):

\[
\hat{\Pi}(x_0, p_0) = \hat{T}(x_0, p_0)\hat{\Pi}\hat{T}(x_0, p_0)^\dagger
\]

where \( \hat{\Pi} \) is the parity operator: \( \hat{\Pi}\psi(x) = \psi(-x) \); the explicit action of \( \hat{\Pi}(x_0, p_0) \) on wavefunctions is easily obtained using formula (8) and one finds

\[
\hat{\Pi}(x_0, p_0)\psi(x) = e^{\frac{2i}{\hbar}p_0(x-x_0)}\psi(2x_0 - x).
\]  

Now, a straightforward calculation shows that the cross-Wigner transform is given by

\[
W(\psi, \phi)(x,p) = \left( \frac{1}{2\pi \hbar} \right)^n \langle \hat{\Pi}(x,p)\phi|\psi \rangle.
\]  

This formalism allows us to prove a basic reconstruction result. Before that we note that, using the Fourier inversion formula and the definition (4) of \( W(\psi, \phi) \), we get the formulas

\[
\psi(x')\phi^*(x'') = \int e^{\frac{i}{\hbar}p(x'-x'')}W(\psi, \phi)\left( \frac{1}{2}(x' + x''), p \right)dp^n.
\]  

\[
\hat{\phi}(p')\phi^*(p'') = \int e^{-\frac{i}{\hbar}(p'-p'')x}W(\psi, \phi)(x, \frac{1}{2}(p' + p''), p)dp^n x
\]

(the second formula is obtained by applying the first to the Fourier transforms of \( \psi, \phi \) and noting that \( W(\hat{\phi}, \hat{\phi})(x,p) = W(\psi, \phi)(-p, x) \)). Observe that when \( \psi = \phi \) these formulas reduce, setting respectively \( x' = x'' \) and \( p' = p'' = p \) to the familiar [7,8,10] marginal properties

\[
\int W\psi(x,p)dp^n = |\psi(x)|^2, \quad \int W\psi(x,p)dp^n x = |\phi(x)|^2
\]

of the Wigner distribution \( W\psi = W(\psi, \psi) \).

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1 Hence the cross-Wigner transform can be viewed as a transition amplitude.
Proposition 1 Let \( \lambda \) be an arbitrary vector in \( L^2(\mathbb{R}^n) \) such that \( \langle \phi | \lambda \rangle \neq 0 \). We have
\[
\psi(x) = \frac{2^n}{\langle \phi | \lambda \rangle} \int \int W(\psi, \phi)(y, p) \hat{\Pi}(y, p) \lambda(x) d^n p d^n y.
\]  \hspace{1cm} (14)

Similarly, if \( \langle \phi | \lambda' \rangle \neq 0 \) then
\[
\phi(x) = \frac{2^n}{\langle \phi | \lambda' \rangle} \int \int W(\psi, \phi^*)(y, p) \hat{\Pi}(y, p) \lambda'(x) d^n p d^n y.
\]  \hspace{1cm} (15)

Proof. (Cf. de Gosson and de Gosson [11]). Multiplying both sides of (12) by \( \lambda(x^n) \) and integrating, we have
\[
\psi(x)\langle \phi | \lambda \rangle = \int e^{2\pi i p(x-x')} W(\psi, \phi)(\frac{1}{2}(x + x'), p) \lambda(x) d^n p d^n x'.
\]

Setting \( y = \frac{1}{2}(x + x') \) and using the explicit formula (10) for the Grossmann–Royer parity operator we get (14). Formula (15) follows, swapping \( \psi \) and \( \phi \) and noting that \( W(\psi, \phi)^* = W(\phi, \psi) \).

This result is remarkable, because it shows that the knowledge of the cross-Wigner transform suffices to determine the two-state vector \( \langle \phi | \psi \rangle \) (and hence the weak value \( \langle A | \psi \rangle \)).

We are going to apply this result to study the case where \( W(\phi, \psi) \) is maximally concentrated around a phase space point.

3. Hardy’s Uncertainty Principle

3.1. Statement of the HUP

Hardy’s uncertainty principle (HUP) is a precise statement of the folk wisdom following which a function and its Fourier transform cannot be simultaneously sharply located. Hardy [30] proved that if \( \psi \) and its Fourier transform \( \widehat{\psi} \) satisfy estimates of the type
\[
|\psi(x)| \leq C_a e^{-\frac{a}{2\pi} x^2}, \quad |\widehat{\psi}(p)| \leq C_b e^{-\frac{b}{2\pi} p^2}
\]  \hspace{1cm} (16)

where \( a, b, C_a, C_b \) are positive constants then the following holds true:

- If \( ab > 1 \) then we must have \( \psi = 0 \);
- If \( ab = 1 \) we have \( \psi(x) = Ce^{-\frac{1}{2\pi} x^2} \) for some complex constant \( C \);
- If \( ab < 1 \) there exists a whole space of \( S \) solutions, containing the functions \( \psi(x) = Q(x)e^{-\frac{1}{2\pi} x^2} \) where \( Q \) is a polynomial (equivalently, \( S \) contains the finite linear combinations of Hermite polynomials).

Note that in the limiting case \( ab = 1 \) the function \( \psi \) is a squeezed Gaussian, hence a minimum-uncertainty state (It saturates the Robertson–Schrödinger inequalities).

We are going to use here the following multi-dimensional variant of Hardy’s result (de Gosson and Luef [31, 32]) which we state for functions defined on phase space:

Proposition 2 Let \( A \) and \( B \) be two real positive definite \( 2n \times 2n \) matrices and \( \Psi \) a square integrable function on phase space \( \mathbb{R}^{2n}, \Psi \neq 0 \). Assume that
\[
|\Psi(z)| \leq C_A e^{-\frac{1}{2\pi} A z^2} \quad \text{and} \quad |F\Psi(\zeta)| \leq C_B e^{-\frac{1}{2\pi} B \zeta^2}
\]  \hspace{1cm} (17)

for some constants \( C_A, C_B > 0 \). Then:

(i) The eigenvalues \( \lambda_j, j = 1, \ldots, 2n, \) of the \( 2n \times 2n \) matrix \( AB \) are all \( \leq 1 \);
(ii) If \( \lambda_j = 1 \) for all \( j \), then \( \psi(x) = Ce^{-\frac{1}{2\pi} A x^2} \) for some complex constant \( C \);
(iii) If \( \lambda_j < 1 \) for some \( j \) then the space of functions satisfying (17) contains every \( \psi(x) = Q(x)e^{-\frac{1}{2\pi} A x^2} \) where \( Q \) is a complex polynomial.
In (17) $F\Psi$ is the $\hbar$-Fourier transform (7) where $n$ is replaced with $2n$. Notice that the eigenvalues of $AB$ are real since $AB$ is equivalent to the symmetric matrix $A^{1/2}BA^{1/2}$.

This result allows us to prove the following Lemma, which we will need in next Subsection to prove our main result:

**Lemma 3** Assume that there exists a real positive-definite $2n \times 2n$ matrix $M$ such that

$$|W(\psi, \phi)(z)| \leq Ce^{-\frac{1}{\hbar}Mz^2}$$

for some constant $C > 0$. Assume that $\phi$ is an even state: $\phi(-x) = \phi(x)$. Then the eigenvalues $\mu_1, \ldots, \mu_{2n}$ of $M$ are all $\leq 1$.

**Proof.** Set $\Psi = W(\psi, \phi)$ and consider the Fourier transform $F\Psi$; the function

$$A(\psi, \phi)(x, p) = F\Psi(p, -x)$$

is the cross-ambiguity function familiar from radar theory [26]; it is related to the Wigner transform by [7, 26, 33] the formula

$$A(\psi, \phi)(z) = W(\psi, \phi')(\frac{1}{2}z)$$

where $\phi'(x) = \phi(-x)$. Since we are assuming that $\phi$ is even we thus have here

$$A(\psi, \phi)(z) = W(\psi, \phi)(\frac{1}{2}z)$$

and hence

$$F\Psi(x, p) = W(\psi, \phi)(-\frac{1}{2}p, \frac{1}{2}x)$$

so that the inequality (18) implies that

$$|F\Psi(x, p)| \leq Ce^{-\frac{1}{\hbar}Mx^2}. $$

Setting $A = 2M$ and $B = \frac{1}{\hbar}M$ Proposition 2 implies that the eigenvalues $\mu_1^2, \ldots, \mu_{2n}^2$ of $AB = M^2$ must be $\leq 1$, hence the result since $M$ is positive definite.

This result implies – in particular – that the (cross-)Wigner transform $W(\psi, \phi)$ can never be a compactly supported phase space function, because if it were the case we could dominate $W(\psi, \phi)$ by an arbitrarily sharply peaked phase space Gaussian.

Notice that if all the eigenvalues of $M$ are equal to one, then $M$ is the identity matrix, hence (18) becomes

$$|W(\psi, \phi)(x, p)| \leq Ce^{-\frac{1}{\hbar}(|x|^2 + |p|^2)}. $$

This inequality corresponds to *minimum uncertainty in phase space*. We are going to draw an interesting consequence of this below.

3.2. Application to weak values

We are going to see that a true weak measurement can never be associated with a maximally peaked cross-Wigner distribution (19); in fact that such a situation corresponds to an ordinary measurement, in which case we have $\psi = \phi$ and $\langle \hat{A} \rangle_{\phi, \psi} = \langle \hat{A} \rangle_\psi$ (the ordinary mean value). More precisely:
Proposition 4 Assume that the cross-Wigner transform of post- and pre-selected states satisfies (19). Then
\[ \psi(x) = \phi(x) = N\phi_0(x)e^{-\frac{1}{2\pi}|x|^2} \]
for some constant \(N\), the function \(\phi_0(x) = (\pi\hbar)^{-n/4} e^{-\frac{1}{2\pi}|x|^2}\) being the fiducial coherent state.

Proof. In view of formula (12) we have
\[ \psi(x')\phi^*(x'') = \int e^{\frac{i}{\hbar}(p'-p'')} W(\psi, \phi)(\frac{1}{2}(x' + x''), p) d^n p \]
for all \(x', x''\). Since \(\phi\) is not equal to zero everywhere there exists a value \(x''\) such that \(\phi'(x'') \neq 0\); shifting if necessary \(\phi\) it is no restriction to assume that \(x'' = 0\) so that we can rewrite, changing \(x'\) into \(x\), the equality above as
\[ \psi(x)\phi^*(0) = \int e^{\frac{i}{\hbar}p'x} W(\psi, \phi)(\frac{1}{2}x, p) d^n p \]
and hence, taking the inequality (19) into account,
\[ |\psi(x)| \leq \left| \frac{1}{|\phi'(0)|} \right| \int |W(\psi, \phi)(\frac{1}{2}x, p)| d^n p \]
\[ \leq C e^{-\frac{1}{\hbar}|x|^2} \int e^{-\frac{1}{\hbar}|p|^2} dp. \]

hence, since the integral in \(p\) is trivially convergent,
\[ |\psi(x)| \leq C' e^{-\frac{1}{\hbar}|x|^2} \quad (20) \]
for some constant \(C' > 0\). Similarly, using (13), there exists \(C'' > 0\) such that
\[ |\hat{\psi}(p)| \leq C'' e^{-\frac{1}{\hbar}|p|^2}. \quad (21) \]

In view of the multidimensional Hardy uncertainty principle these inequalities imply that \(\psi(x)\) must be proportional to \(\phi_0(x)\). ■

4. Discussion
In our treatment of the reconstruction problem we have explicitly assumed that the “right” phase space picture of quantum mechanics is obtained by using the traditional Wigner formalism which is closely associated with Weyl’s quantization of classical observables [7,9]. But life is perhaps not that simple. It turns out that there are strong indications that one should replace Weyl quantization by that introduced by Born and Jordan [34,35] in 1925, prior to Weyl’s work; we refer to de Gosson [29,36] for a discussion of these rules. If this proved to be the case, the reconstruction formulae in Proposition 1 would no longer hold. It turns out, as we already discussed in [11], that the reconstruction problem in general is much more difficult than in the Wigner case.
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References
[1] Aharonov Y, Bergmann P G and Lebowitz J L 1964 Time symmetry in the quantum process of measurement Phys. Rev. 134(6B) B1410-B1416
[2] Aharonov Y, Cohen E and Elitzur A C 2015 Can a future choice affect a past measurement’s outcome? Ann. Phys. 355 258–268 (Preprint arXiv:1206.6224 [quant-ph])
[3] Aharonov Y and Vaidman L 1990 Properties of a quantum system during the time interval between two measurements Phys. Rev. A 41(1) 11–20
[4] Aharonov Y and Vaidman L 1991 Complete description of a quantum system at a given time J. Phys. A: Math. Gen. 24(10) 2315–2328
[5] Aharonov Y and Vaidman L 2008 The two-state vector formalism: An updated review (Lecture Notes in Physics vol 734), ed Muga J G, Mayato R S and Egusquiza I L Time in Quantum Mechanics (Berlin, Heidelberg: Springer Berlin Heidelberg) pp 399–447 ISBN 978-3-540-73472-7
[6] Cohen O and Hiley B J 1995 Retrodiction in quantum mechanics, preferred Lorentz frames, and nonlocal measurements Found. Phys. 25(12) 1669–1698
[7] de Gosson M A 2011 Symplectic Methods in Harmonic Analysis and in Mathematical Physics (Basel: Birkhäuser) ISBN 978-3-7643-9991-7
[8] Hillery M, O’Connell R F, Scully M O and Wigner E P 1984 Distribution functions in physics: Fundamentals Phys. Rep. 106(3) 121–167
[9] Littlejohn R G 1986 The semiclassical evolution of wave packets Phys. Rep. 138(4-5) 193–291
[10] Wigner E 1932 On the quantum correction for thermodynamic equilibrium Phys. Rev. 40(5) 749–759
[11] de Gosson C and de Gosson M A 2014 The phase space formulation of time-symmetric quantum mechanics Quanta 4(1) 27–34
[12] de Gosson M A and de Gosson S M 2012 The reconstruction problem and weak quantum values J. Phys. A: Math. Theor. 45(11) 115305
[13] de Gosson M A and de Gosson S M 2011 Weak values of a quantum observable and the cross-Wigner distribution Phys. Lett. A 376(4) 293–296 (Preprint arXiv:1109.3665v1 [quant-ph])
[14] Lundeen J S, Sutherland B, Patel A, Stewart C and Bamber C 2011 Direct measurement of the quantum wavefunction Nature 474(7350) 188–191 (Preprint arXiv:1112.3575v1 [quant-ph])
[15] Lundeen J S and Bamber C 2012 Procedure for direct measurement of general quantum states using weak measurement Phys. Rev. Lett. 108(7) 070402
[16] Artiles L M, Gill R D and Guta M I 2005 An invitation to quantum tomography J. R. Statist. Soc. B 67(1) 109–134 (Preprint arXiv:math/0405059)
[17] Asoerey M, Ibert A, Marmo G and Ventriglia F 2015 Quantum tomography twenty years later Phys. Scr. 90(7) 074031
[18] Dunn T J, Walmsley I A and Mukamel S 1995 Experimental determination of the quantum-mechanical state of a molecular vibrational mode using fluorescence tomography Phys. Rev. Lett. 74(6) 884–887
[19] Fischbach J and Freyberger M 2012 Quantum optical reconstruction scheme using weak values Phys. Rev. A 86(5) 052110 (Preprint arXiv:1211.2185 [quant-ph])
[20] Leonhardt U 1997 Measuring the quantum state of light (Cambridge studies in modern optics no 22) (Cambridge: Cambridge Univ. Press) ISBN 978-0-521-49730-5
[21] Leonhardt U and Paul H 1995 Measuring the quantum state of light Prog. Quantum Electron. 19(2) 89–130
[22] Lutterbach B G and Davidovich L 1997 Method for direct measurement of the Wigner function in cavity QED and ion traps Phys. Rev. Lett. 78(13) 2547–2550
[23] Lvovsky A I and Raymer M G 2009 Continuous-variable optical quantum-state tomography Rev. Mod. Phys. 81(1) 299–332
[24] Roy A S and Roy S M 2014 Optimum phase space probabilities from quantum tomography J. Math. Phys. 55(1) 012102 (Preprint arXiv:1304.2857 [quant-ph])
[25] Smithey D T, Beck M, Raymer M G and Faridani A 1993 Measurement of the Wigner distribution and the density matrix of a light mode using optical homodyne tomography: Application to squeezed states and the vacuum Phys. Rev. Lett. 70(9) 1244–1247
[26] Folland G B 1989 Harmonic analysis in phase space (Annals of mathematics studies no 122) (Princeton, NJ: Princeton Univ. Press) ISBN 978-0-691-08527-2
[27] Grossmann A 1976 Parity operator and quantization of θ-functions Commun. Math. Phys. 48(3) 191–194
[28] Royer A 1977 Wigner function as the expectation value of a parity operator Phys. Rev. A 15(2) 449–450
[29] de Gosson M A 2016 Born–Jordan Quantization: Theory and Applications (Fundamental Theories of Physics no 182) (Springer) ISBN 978-3-319-27902-2
[30] Hardy G H 1933 A theorem concerning Fourier transforms J. London Math. Soc. s1-8(3) 227–231
[31] de Gosson M and Luef F 2007 Quantum states and Hardy’s formulation of the uncertainty principle: a symplectic approach Lett. Math. Phys. 80(1) 69–82
[32] de Gosson M and Luef F 2009 Symplectic capacities and the geometry of uncertainty: The irruption of symplectic topology in classical and quantum mechanics Phys. Rep. 484(5) 131–179
[33] Gray J 2010 An interpretation of Woodward’s ambiguity function and its generalization, 2010 IEEE Radar Conference pp 859–864
[34] Born M and Jordan P 1925 Zur Quantenmechanik Z. Phys. 34 858–888
[35] Born M, Heisenberg W and Jordan P 1925 Zur Quantenmechanik II Z. Phys. 35 557–615
[36] de Gosson M A 2014 Born–Jordan quantization and the equivalence of the Schrödinger and Heisenberg pictures Found. Phys. 44(10) 1096–1106