MODELS FOR KNOT SPACES AND ATIYAH DUALITY

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Abstract. Let \( \text{Emb}(S^1, M) \) be the space of smooth embeddings from the circle to a closed manifold \( M \) of dimension \( \geq 4 \). We study a cosimplicial model of \( \text{Emb}(S^1, M) \) in stable categories, using a spectral version of Poincaré-Lefschetz duality called Atiyah duality. We actually deal with a notion of a comodule instead of the cosimplicial model, and prove a comodule version of the duality as in Theorem 1.1. As an application, we introduce a new spectral sequence converging to \( H^*(\text{Emb}(S^1, M)) \) for simply connected \( M \) and for major coefficient rings as in Theorem 1.2. Using this, we compute \( H^*(\text{Emb}(S^1, S^k \times S^l)) \) in low degrees with some conditions on \( k \) and \( l \). We also prove the inclusion \( \text{Emb}(S^1, M) \to \text{Imm}(S^1, M) \) to the immersions induces an isomorphism on \( \pi_1 \) for some simply connected 4-manifolds, related to a question posed by Arone and Szumiłk. We also prove an equivalence of singular cochain complex of \( \text{Emb}(S^1, M) \) and a homotopy colimit of chain complexes of a Thom spectrum of a bundle over a comprehensible space as in Theorem 1.4. Our key ingredient is a structured version of the duality due to R. Cohen.

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1. Introduction

In [35, 36] Sinha constructed cosimplicial models of spaces of knots in a manifold of dimension \( \geq 4 \), based on the Goodwillie-Klein-Weiss embedding calculus [18, 19]. Sinha’s model was crucially used in the affirmative solution to Vassiliev’s conjecture for a spectral sequence for the space of long knots in \( \mathbb{R}^d \) (\( d \geq 4 \)) for rational coefficient in [23] (see [7] for other coefficients). In the present paper, we study a version of Sinha’s model in stable categories.

Precisely speaking, our situation is different from [35, 36] in that we consider the space \( \text{Emb}(S^1, M) \) of smooth embeddings from the circle \( S^1 \) to a closed manifold \( M \) without any base point condition while in [35, 36] spaces of embeddings from \([0,1]\) to a manifold with boundary with some endpoints conditions are dealt with. This is for technical simplicity and a similar theory will be valid for Sinha’s spaces. The space \( \text{Emb}(S^1, M) \) is studied in [11, 10] and study of embedding spaces including the knot space is a motivation of [11, 23].

In the rest of the paper, \( M \) denotes a connected closed smooth manifold of dimension \( d \geq 4 \). We endow the space \( \text{Emb}(S^1, M) \) with the \( C^\infty \)-topology. For our situation, we can construct a cosimplicial model similar to Sinha’s, which is called Sinha’s cosimplicial model and denoted by \( C^\bullet([M]) \). This model relates the knot space and the configuration space as follows.

- There exists a weak homotopy equivalence \( \text{holim} \Delta C^\bullet([M]) \simeq \text{Emb}(S^1, M) \), where \( \text{holim} \Delta \) denotes the homotopy limit over the category \( \Delta \) of standard simplices (see subsection 2.1).
- \( C^n([M]) \) is homotopy equivalent to the configuration space \( C_{n+1}(M) \) of \( n+1 \) points with a tangent vector defined as the pullback of the following diagram.

\[
\begin{align*}
\widehat{M}^{\times n+1} & \longrightarrow M^{\times n+1} \leftarrow C_{n+1}(M).
\end{align*}
\]

Here \( \widehat{M} \) denotes the tangent sphere bundle of \( M \) and \( C_{n+1}(M) \) denotes the ordered configuration space of \( n+1 \)-points in \( M \). The left arrow is the product of the projection and the right arrow is the inclusion. Actually, \( C^n([M]) \) is a version of the Fulton-MacPherson compactification.

To state our main theorems, we need some notations. Fix an embedding \( e_0 : \widehat{M} \rightarrow \mathbb{R}^K \) and a tubular neighborhood \( \nu \) of the image \( e_0(\widehat{M}) \) in \( \mathbb{R}^K \). We regard the product \( \nu^{\times n} \) as a disk bundle over \( \widehat{M}^{\times n} \) and let \( \nu^{\times n} \mid_X \) denote the restriction of the base to a subspace \( X \subset \widehat{M}^{\times n} \). For a positive integer \( n \), let \( G(n) \) be the set of graphs \( G \) with set of vertices \( V(G) = \underline{n} = \{1, \ldots, n\} \) and set of edges \( E(G) \subset \{(i, j) \mid i, j \in \underline{n}, i < j\} \). Let \( \pi_0(G) \) be the set of connected components of \( G \). A map \( M^{\times \pi_0(G)} \rightarrow M^{\times n} \) is induced by the quotient map \( \underline{n} \rightarrow \pi_0(G) \), considering \( M^{\times \Delta} \) as the set of maps \( A \rightarrow M \). Let \( \Delta_G \) be the pullback of the following diagram

\[
\begin{align*}
\widehat{M}^{\times n} & \xrightarrow{\text{projection}} M^{\times n} \xleftarrow{\Delta} M^{\times \pi_0(G)}.
\end{align*}
\]

\( \Delta_G \) is naturally regarded as a subspace of \( \widehat{M}^{\times n} \) via the projection of the pullback. \( \Delta_G \) is a rather comprehensible space, comparing to the space \( C^{n-1}([M]) \) (or \( C_n(M) \)). For example, its cohomology ring is computed in Lemmas 6.5, 6.6 under some assumption. Define a subspace \( \Delta_{\text{fat}}(n) \subset \widehat{M}^{\times n} \) as the union \( \cup_{G \in G(n)} \Delta_G \). We use a notion of right modules over an operad, which is similar to the notion in Loday-Vallette [26]. Let \( D_1 \) be a version of the little intervals operad (see Definition 3.4). We define a right \( D_1 \)-module \( \mathcal{C}([M]) \) which shares a large part of structure with \( C^\bullet([M]) \) and is quite analogous to the modules considered in [2, 4] (see Definitions 2.10, 4.11). For example, \( \mathcal{C}([M])(n) = C^{n-1}([M]) \) by definition. \( \mathcal{C}([M]) \) also has information enough to reconstruct \( \text{Emb}(S^1, M) \). We work with the category of symmetric spectra \( \mathcal{S} \mathcal{P}^\Sigma \). \( \mathcal{C}([M])(n) \) denotes a left \( D_1 \)-comodule in \( \mathcal{S} \mathcal{P}^\Sigma \) given by \( (\mathcal{C}([M])(n))^\vee = (\mathcal{C}([M])(n))^\vee \), where \( (\_)^\vee \) denotes the Spanier-Whitehead dual. See subsection 2.1 and Definition 4.3 for other notations in the following theorem.
Theorem 1.1 (Theorem 4.14). There exists a left \(\mathcal{D}_1\)-comodule \(TH_M\) of non-unital commutative symmetric ring spectra as follows.

1. The object \(TH_M(n)\) at arity \(n\) is a natural model of the Thom spectrum
   \[\Sigma^{-nK}Th(\nu^\infty)/Th(\nu^\infty|_X)\quad\text{with}\quad X = \Delta_{\text{fat}}(n),\]
   where \(\Sigma\) denotes the suspension equivalence and \(Th(-)\) the associated Thom space. Concretely speaking, \(TH_M(n)\) is a relative version of R. Cohen’s non-unital model in [12].
2. There exists a zigzag of \(\pi_\ast\)-isomorphisms of left \(\mathcal{D}_1\)-comodules
   \[\left(C\langle\{M\}\rangle\right)^\ast \simeq TH_M.\]

Theorem 1.1 is a structured version of the Poincaré-Lefschetz duality
\[H^\ast(C_n(M)) \cong H_\ast(\widetilde{M}^\infty, \Delta_{\text{fat}}(n)) \cdots (\ast),\]
deduced from the equality \(C_n(M) = \widetilde{M}^\infty - \Delta_{\text{fat}}(n)\). (We are loose on degrees.) If we do not consider the (non-unital) commutative multiplications, an analogue to Theorem 1.1 holds in the category of prespectra (in the sense of [28]), a more naive, non-symmetric monoidal category of spectra and it is enough to prove Theorem 1.2 below, but the multiplications may be useful for future study and our constructions hardly become easier for prespectra.

Throughout this paper, we fix a coefficient ring \(k\) and suppose \(k\) is either of a subring of the rationals \(\mathbb{Q}\) or the field \(\mathbb{F}_p\) of \(p\) elements for a prime \(p\). All normalized singular (co)chains \(C\ast, C\ast\) and singular (co)homology \(H\ast, H\ast\) are supposed to have coefficients in \(k\), unless otherwise stated. As an application of Theorem 1.1, we introduce a new spectral sequence converging to \(H^\ast(\text{Emb}(S^1, M))\).

Theorem 1.2 (Theorems 5.15, 5.16 and 6.10). Suppose \(M\) is simply connected. There exists a second-quadrant spectral sequence \(\{\tilde{E}^p_q\}_r\) converging to \(H^{p+q}(\text{Emb}(S^1, M))\) such that

1. its \(E_2\)-page is isomorphic to the total homology of the normalization of a simplicial commutative differential bigraded algebra \(A_{\ast\ast}(M)\) which is defined in terms of the cohomology ring \(H^\ast(\Delta_G)\) for various \(G\) and maps between them,
   \[\tilde{E}^p_q \cong H(NA_{\ast\ast}(M)) \Rightarrow H^{p+q}(\text{Emb}(S^1, M)),\]
   where the bidegree is given by \(\ast = p\), \(\ast - \bullet = q\),
2. and moreover, if \(H^\ast(M)\) is a free \(k\)-module, and the Euler number \(\chi(M)\) is zero or invertible in \(k\), the object \(A_{\ast\ast}(M)\) is determined by the ring \(H^\ast(M)\).

We call this spectral sequence Čech spectral sequence or in short, Čech s.s. A feature of Čech s.s. is that its \(E_1\) page and differential \(d_1\) are explicitly determined by the cohomology of \(M\). As spectral sequences for \(H^\ast(\text{Emb}(S^1, M))\), we have the Bousfield-Kan type cohomology spectral sequence converging to \(H^\ast(\text{Emb}(S^1, M))\), see Definition 2.7 and Vassiliev’s spectral sequence converging to the relative cohomology \(H^\ast(\Omega_f(M), E\text{mb}(S^1, M))\), where \(\Omega_f(M)\) is the space of smooth maps \(S^1 \rightarrow M\). But no small (i.e. degree-wise finite dimensional) page of these spectral sequences have been computed in general. \(E_1\)-page of the Bousfield-Kan type s.s. described by the cohomology of the ordered configuration spaces of points with a vector in \(M\), which is difficult to compute, and Vassiliev’s first term is also interesting but complicated. By this feature, we can compute examples, see section 7. We obtain new computational results in the case of the product of two spheres. While we only do elementary computation in the present paper, one of potential merits of Čech s.s. is that computation of higher differentials will be relatively accesible since we deal with fat diagonals and Čech complex instead of configuration spaces. The other is that we will be able to enrich it with operations such as the cup product and square, and relate them to those on \(H^\ast(M)\). We will deal with these subjects in a future work.
Arone and Szymik studied $\text{Emb}(S^1, M)$ for the case of dimension $d = 4$ in [1]. Let $\text{Imm}(S^1, M)$ be the space of smooth immersions $S^1 \to M$ with the $C^\infty$-topology and $i_M : \text{Emb}(S^1, M) \to \text{Imm}(S^1, M)$ be the inclusion. Among other results, they proved that $i_M$ is 1-connected, so in particular, surjective on $\pi_1$ in general. (They proved interesting results for the non-simply connected case $M = S^1 \times S^3$, see also Budney and Gabai [10].) They asked whether there is a simply connected 4-manifold $M$ such that $i_M$ has non-trivial kernel on $\pi_1$. Using Theorem [1.2], we give a restriction to this question.

**Corollary 1.3.** Suppose that $M$ is simply connected, $d = 4$, $H_2(M; \mathbb{Z}) \neq 0$, and the intersection form on $H_2(M; \mathbb{F}_2)$ is represented by a matrix of which the inverse has at least one non-zero diagonal component. Then, the inclusion $i_M$ induces an isomorphism on $\pi_1$. In particular, $\pi_1(\text{Emb}(S^1, M)) \cong H_2(M; \mathbb{Z})$.

The assumption does not depend on choices of a matrix. For example, $M = \mathbb{C}P^2 \# \mathbb{C}P^2$, the connected sum of complex projective planes, satisfies the assumption while $M = S^2 \times S^2$ does not. For the case of $H_2(M) = 0$, By Proposition 5.2 of [1], $\text{Emb}(S^1, M)$ is simply connected. We cannot prove this completely similarly to Corollary 1.3. We might need a relation of Čech s.s. and the space of long knots. The case of all of the diagonal components of the matrix being zero is unclear for the author.

Sinha’s cosimplicial model can be considered as a resolution of $\text{Emb}(S^1, M)$ into simpler spaces. We resolve it into further simpler pieces in the category of chain complexes as an application of Theorems 1.1 and 1.2. To state the result, we need additional notations. Let $\Psi^o_n$ be the category of planer trees given in [35] (see Definition 8.1). We regard the set $G(n)$ as a category (poset) having a morphism $G \to H$ for $E(G) \subset E(H)$. We denote by $\emptyset$ the graph in $G(n)$ with no edge. Let $G(n)^+$ be the poset made by adding an object $*$ to $G(n)$ and a morphism $* \to G$ for each graph $G \neq \emptyset$. Let $\tilde{\Psi}$ be a category of pairs $(T,G)$ of a tree $T \in \cup_n \Psi^o_n$ and an object $G \in G(|v_r|^{-1})^+$, where $|v_r|$ denotes the valence of the root vertex of $T$ (see Definition 8.1). Let $\mathcal{CH}_k$ be the category of chain complexes and chain maps, and hocolim be the homotopy colimit, and $C^S_* : SP^\Sigma \to \mathcal{CH}_k$ be a functor, see subsection 2.1 and Definition 5.1.

**Theorem 1.4 (Theorem 8.3).** There exists a functor $\text{Th}^M : (\tilde{\Psi})^{op} \to SP^\Sigma$ such that

1. for each $(T,G) \in \tilde{\Psi}$, if $G \neq *$, $\text{Th}^M(T,G)$ is a natural model of the Thom spectra

$$\Sigma^{-m}KTh(v^{-m}|_G) \quad \text{with} \quad m = |v_r| - 1,$$

and if $G = *$, $\text{Th}^M(T,G) = *$ and

2. if $M$ is simply connected, there exists a zigzag of quasi-isomorphisms of chain complexes

$$C^S_*(\text{Emb}(S^1, M)) \simeq \text{hocolim}_{(\tilde{\Psi})^{op}} C^S_* \circ \text{Th}^M.$$

We give intuitive explanation for this theorem. There is a standard quasi-isomorphism $C_*(\Delta_{\text{fat}}(n)) \simeq \text{hocolim}_{G \in C_1} C_*(\Delta_G)$ where $C_1 = G(n)^{op} - \{\emptyset\}$. Since the relative complex $C_*(\tilde{M}^{\times n}, \Delta_{\text{fat}})$ is a homotopy cofiber of the inclusion $C_*(\Delta_{\text{fat}}) \to C_*(\tilde{M}^{\times n})(= C_*(\Delta_\emptyset))$, we have quasi-isomorphisms

$$C^*(C^{n-1}(\{M\})) \simeq C_*(\tilde{M}^{\times n}, \Delta_{\text{fat}}(n)) \simeq \text{hocolim}_{G \in C_2} C_*(\Delta_G),$$

where $C_2 = (G(n)^+)^{op}$ and we set $C_*(\Delta_\emptyset) = 0$. We regard this presentation as a resolution of $C^*(C^{n-1}(\{M\}))$. The category $\cup_n \Psi^o_n$ is a lax analogue of the category $\Delta$. Actually, homotopy limits over these categories are weak equivalent. So, intuitively speaking, existence of the functor $\text{Th}^M$ means potential compatibility of the resolution and the cosimplicial structure.
We shall explain why the author uses spectra, which also serves as an outline of the arguments in the paper. The author’s motivation is to derive a new spectral sequence from Sinha’s cosimplicial model. The idea is to combine the cosimplicial model and a procedure of constructing a spectral sequence for $H^*(C_\oplus(M))$ due to Bendersky and Gitler [3]. So we consider the above duality ($\ast$), and describe the chain complex $C_\ast(\hat{M}_{\text{fat}}(n))$ by an augmented Čech complex as follows.

$$C_\ast(\Delta_\emptyset) \xleftarrow{\partial} \bigoplus_{G \in G(n,1)} C_\ast(\Delta G) \xleftarrow{\partial} \bigoplus_{G \in G(n,2)} C_\ast(\Delta G) \xleftarrow{\partial} \bigoplus_{G \in G(n,3)} C_\ast(\Delta G) \xleftarrow{\partial} \cdots,$$

where $G(n,p) \subset G(n)$ denotes the subset of graphs with $p$ edges. We want to extend this diagram to the following commutative diagram (**) of semisimplicial chain complexes by defining suitable face maps $d_i$. (Here, ‘semi’ means lack of degeneracy maps.)

$$C^\ast(C^n([M])) \xrightarrow{P.D.} C_\ast(\Delta_\emptyset) \bigoplus_{G \in G(n+1,1)} C_\ast(\Delta G) \bigoplus_{G \in G(n+1,2)} C_\ast(\Delta G) \bigoplus_{G \in G(n+1,3)} C_\ast(\Delta G) \bigoplus \cdots \xrightarrow{d_i} C^\ast(C^{n-1}([M])) \xrightarrow{P.D.}$$

where $d^i$ is the coface map of $C^\ast([M])$, and $P.D.$ actually denotes the zigzag of the cap product with the fundamental class and the quotient map. If we could construct a semisimplicial double complex in the right hand side of $P.D.$ in the diagram (**), by taking the total complex, we have certain triple complex $C_{\ast\ast\ast}$, where $\bullet$ (resp. $\ast$, $\ast$) denotes the cosimplicial (resp. Čech, singular) degree. Then by filtering with $\ast + \ast$, we would obtain a spectral sequence as in Theorem [1, 2].

Unfortunately, it is difficult for the author to define degeneracy maps $d_i$ fitting into the diagram (**). This difficulty is essentially analogous to the one in construction of certain chain-level intersection product on $C_\ast(M)$. We shall explain this point more precisely. The coface map $d^i : C^n([M]) \rightarrow C^{n+1}([M])$ is a deformed diagonal and the usual diagonal induces the intersection product on homology. So the maps $d_i$ should be something like a deformed intersection product. The simplicial identities for $d_i$ are analogous to the associativity of an intersection product. In addition, the map $(d^i)^\ast$ on the cochain is analogous to the cup product. So construction of $d_i$ is analogous to construction of chain-level intersection product which is associative and compatible with the cup product through the duality. The author could not find such a product in the literature.

A nice solution to this is found in a construction due to R. Cohen and Jones [12, 13] in string topology. They used spectra to give a homotopy theoretic realization of the loop product, which led to a proof of an isomorphism between the loop product and a product on Hochschild cohomology (see [30] for a detailed account). Their key notion is the Atiyah duality which is an equivalence of the Spanier-Whitehead dual $M^\gamma$ and the Thom spectrum $M^{-TM} = \Sigma^{-KTh}(\nu)$. To prove their isomorphism, Cohen [12] introduced a model of $M^{-TM}$ in the category $\mathcal{SP}_{\Sigma}$, and refine the duality to an equivalence of (non-unital) commutative symmetric ring spectra. This equivalence can be regarded as a multiplicative version of the Poincaré duality. In fact, the multiplication on the model of $M^{-TM}$ works as an analogue of a chain level intersection product in their theory. So the author considers that it is efficient to construct necessary semisimplicial objects (or comodules) and their equivalence in $\mathcal{SP}_{\Sigma}$, then send them to $CH_k$ by certain chain functor, and derive a spectral sequence. This is why we use spectra.

Even if we use spectra, the (co)simplicial object is too rigid, and we use a laxer notion of a
left comodule over an $A_{\infty}$-operad.

As demonstrated in this paper, the duality is very useful to transfer structures on a configuration space to a Thom spectrum (or complex) of fat diagonal, which is homotopically more accessible, and may be applied in many researches on configuration spaces. In future work, we will study collapse of Sinha’s (or Vassiliev’s) spectral sequence for the space of long knots in $\mathbb{R}^d$ using the duality.

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## 2. Preliminaries

In this section, we fix notations and introduce basic notions. Nothing is essentially new.

### 2.1. Notations and terminologies.

- We borrow the following notations from [35]. $\Delta$ denotes the category of standard simplices in [35] Definition 4.21. Its objects are the finite ordered sets $[n] = \{0, \ldots, n\}$ and morphisms are weakly order preserving maps. $\Delta_n$ denotes the full subcategory of $\Delta$ consists of objects $[k]$ with $k \leq n$. $P_n$ is the category (poset) of non-empty subsets $S \subset \mathbb{N}$ in [35] Definition 3.2. A functor $G_n : P_n(n+1) \to \Delta_n$ is defined in [35] Definition 6.3. $G_n$ sends a set $S$ to $\#S - 1$ and an inclusion $S \subset S'$ to the composition $\#S - 1 \cong S \subset S' \cong \#S' - 1$ where $\cong$ denotes the order preserving bijection.

- For a category $\mathcal{C}$, a morphism of $\mathcal{C}$ is also called a map of $\mathcal{C}$. A symmetric sequence in $\mathcal{C}$ is a sequence $\{X_k\}_{k \geq 0}$ (or $\{X(k)\}_{k \geq 1}$) of objects in $\mathcal{C}$ equipped with an action of the $k$-th symmetric group $\Sigma_k$ on $X_k$ (or $X(k)$) for each $k$. The group $\Sigma_k$ acts from the right throughout this paper.

- Let $\mathcal{G}(n)$ be the set of graphs defined in Introduction. For a graph $G \in \mathcal{G}(n)$, we regard $E(G)$ as an ordered set with the lexicographical order. To ease notations, we use the notation $(i, j)$ with $i > j$ to denote the edge $(j, i)$ of a graph in $\mathcal{G}(n)$. For a map $f : \mathcal{G}(n) \to \mathcal{G}(m)$ of finite sets, denote by the same symbol $f$ the map $\mathcal{G}(n) \to \mathcal{G}(m)$ defined by

$$E(f(G)) = \{(f(i), f(j)) \mid (i, j) \in E(G), f(i) \neq f(j)\}.$$ 

- $f$ also denotes the natural map $\pi_0(G) \to \pi_0(f(G))$.

- Our notion of a model category is that of [22]. $\text{Ho}(\mathcal{M})$ denotes the homotopy category of a model category $\mathcal{M}$.

- $\mathcal{CG}$ denotes the category of all compactly generated spaces and continuous maps, see [22] Definition 2.4.21. $\mathcal{CG}_*$ denotes the category of pointed compactly generated spaces and pointed maps. $\wedge$ denotes the smash product of pointed spaces.

- For a category $\mathcal{C}$, a cosimplicial object $X^\bullet$ in $\mathcal{C}$ is a functor $\Delta \to \mathcal{C}$. A map of cosimplicial object is a natural transformation. $X^n$ denotes the object of $\mathcal{C}$ at $[n]$. We define maps

$$d^i : [n] \to [n+1] \quad (0 \leq i \leq n + 1), \quad s^i : [n] \to [n-1] \quad (0 \leq i \leq n - 1)$$

by

$d^i(k) = \begin{cases} k & (k < i) \\ k + 1 & (k \geq i) \end{cases}, \quad s^i(k) = \begin{cases} k & (k \leq i) \\ k - 1 & (k > i) \end{cases}$.
• Our notion of a symmetric spectrum is that of Mandell-May-Schwede-Shipley [28]. A symmetric spectrum consists of a symmetric sequence \( \{X_k\}_{k \geq 0} \) in \( \mathcal{C}G \) and a map \( \sigma_k : X^1 \wedge X_k \to X_{k+1} \) for each \( k \geq 0 \) which subject to certain conditions. The category of symmetric spectra is denoted by \( SP^\Sigma \). We denote by \( \wedge = \wedge_S \) the canonical symmetric monoidal product on \( SP^\Sigma \) given in [28] and by \( S \) the sphere spectrum, the unit for \( \wedge \). In the rest of the paper the term 'spectrum' means symmetric spectrum. For a spectrum we refer to the numbering of the underlying sequence as the level.

• For \( K \in \mathcal{C}G \) and \( X \in SP^\Sigma \), we define a tensor \( K \hat{\otimes} X \in SP^\Sigma \) by \( (K \hat{\otimes} X)_k = (K_+) \wedge X_k \) where \( K_+ \) is \( K \) with disjoint basepoint. This tensor is extended to a functor \( \mathcal{C}G \times SP^\Sigma \to SP^\Sigma \) in an obvious manner. For \( K, L \in \mathcal{C}G \) and \( X, Y \in SP^\Sigma \), there are natural isomorphisms

\[
K \hat{\otimes} (L \hat{\otimes} X) \cong (K \times L) \hat{\otimes} X, \quad K \hat{\otimes} (X \wedge Y) \cong (K \hat{\otimes} X) \wedge Y
\]

which we call the associativity isomorphisms. A natural isomorphism \( (K \times L) \hat{\otimes} (X \wedge Y) \cong (K \hat{\otimes} X) \wedge (L \hat{\otimes} Y) \) is defined by successive composition of the associativity isomorphisms and the symmetry one for \( \wedge \). We define a mapping object \( \text{Map}(K, X) \in SP^\Sigma \) by \( \text{Map}(K, X)_k = \text{Map}_*(K_+, X_k) \), where the right hand side is the usual internal hom object (mapping space) of \( \mathcal{C}G_+ \). This defines a functor \( \mathcal{C}G^\times SP^\Sigma \to SP^\Sigma \). \( K \hat{\otimes} (-) \) and \( \text{Map}(K, -) \) forms an adjoint pair. We set \( K^\vee = \text{Map}(K, S) \) for \( K \in \mathcal{C}G \).

• We use the stable model structure on \( SP^\Sigma \), see [28]. This is only used in subsection 5.1 and section 5. Weaker equivalences in this model structure are called stable equivalences. Level equivalences and \( \pi_* \)-isomorphisms are more restricted classes of maps in \( SP^\Sigma \) (see [28]). The former are levelwise weak homotopy equivalences and the latter are maps which induce an isomorphism between (naive) homotopy groups defined as the colimit of the sequence of canonical maps \( t_k : \pi_*(X_k) \to \pi_{*+1}(X_{k+1}) \).

• We say a spectrum \( X \) is strongly semistable if there exists a number \( \alpha > 1 \) such that for any sufficiently large \( l \), the map \( t_l : \pi_*(X_l) \to \pi_{*+1}(X_{l+1}) \) is an isomorphism for each \( k \leq \alpha l \). A strongly semistable spectra is semistable in the sense of [38] so a stable equivalence between strongly semistable spectrum is a \( \pi_*\)-isomorphism.

• A non-unital commutative symmetric ring spectrum (in short, \( NUCSRS \)) is a spectrum \( A \) with a commutative associative multiplication \( A \wedge A \to A \) (but possibly without unit). A map of NUCSRS is a map of spectra preserving the multiplications.

• \( \mathcal{CH}_k \) denotes the category of (possibly unbounded) chain complexes over \( k \) and chain maps. Differentials raise the degree (see next item for our degree rule). We endow \( \mathcal{CH}_k \) the model structure where weak equivalences are quasi-isomorphisms and fibrations are surjections. \( \otimes = \otimes_k \) denotes the standard tensor product of complexes.

• We deal with modules with multiple degrees (gradings). For modules having superscript(s) and/or subscript(s), its total degree is given by the following formula.

\[
\text{(total degree)} = (\text{sum of superscripts}) - (\text{sum of subscripts}).
\]

For example, singular chains in \( C_p(M) \) have degree \(-p\), and the total degree of a triple graded module \( A_\bullet^\bullet^\ast \) is \( * + * - \bullet \). \( [a] \) denotes the (bi)degree of \( a \). We sometimes omit super or subscripts if unnecessary.
• For a simplicial chain complex $C^\bullet_\ast$ (i.e., a functor $(\Delta)^{\text{op}} \to CH_k$) the normalized complex (or normalization) $NC^\bullet_\ast$ is a double complex defined by taking the normalized complex of a simplicial $k$-module in each chain degree.
• For a small category $C$ and a cofibrantly generated model category $\mathcal{M}$ (in the sense of [22]), we denote by $\Fun(C, \mathcal{M})$ the category of functors $C \to \mathcal{M}$ and natural transformations, which is endowed with the projective model structure (see [21]). The colimit functor $\text{colim}_C : \Fun(C, \mathcal{M}) \to \mathcal{M}$ is a left Quillen functor. Its left derived functor is denoted by hocolim and called the homotopy colimit over $C$.
• A commutative differential bigraded algebra (in short, CDBA) is a bigraded module $A^{\ast \ast}$ equipped with a unital multiplication which is graded commutative for the total degree, and preserves the bigrading, and a differential $\partial : A^{\ast \ast} \to A^{\ast + 1, \ast}$ which satisfy the Leibniz rule for the total degree. A map of CDBA is a map of differential graded algebra preserving bigrading.

2.2. Čech complex and homotopy colimit.

**Definition 2.1.** Let $\mathcal{M}$ be a cofibrantly generated model category. We define a functor

$$\check{C} : \Fun(P_\nu(n+1)^{\text{op}}, \mathcal{M}) \to \Fun(\Delta^{\text{op}}, \mathcal{M})$$

by

$$\check{C}X[k] = \bigoplus_{f : [k] \to n+1} X_f([k]),$$

where $f$ runs through all weakly order-preserving maps. For order preserving map $\alpha : [l] \to [k] \in \Delta$, the map $\check{C}X[k] \to \check{C}X[l]$ is the sum of the maps $X_f([k]) \to X_f\alpha([l])$ induced by the inclusion $f \circ \alpha([l]) \subset f([k])$.

The following lemma is clear.

**Lemma 2.2.** We use the notations of Definition 2.1. Let $X \in \Fun(P_\nu(n+1)^{\text{op}}, \mathcal{M})$ be a functor.

1. There exists an isomorphism $\text{hocolim}_{P_\nu(n+1)^{\text{op}}} \check{C}X \cong \text{hocolim}_{\Delta^{\text{op}}} \check{C}X$ in $\text{Ho}(\mathcal{M})$ which is natural for $X$.
2. $X$ is cofibrant in $\Fun(P_\nu(n+1)^{\text{op}}, \mathcal{M})$ if the following canonical map is a cofibration in $\mathcal{M}$ for each $S \in P_\nu(n+1)$.

$$\colim_{S' \supsetneq S \neq S} X_{S'} \to X_S.$$

**Proof.** Let $(i_n \circ G_n)^* : \Fun(\Delta^{\text{op}}, \mathcal{M}) \to \Fun(P_\nu(n+1)^{\text{op}}, \mathcal{M})$ be the pullback by the composition of $G_n$ and the inclusion $i_n : \Delta_n \to \Delta$. Clearly, the pair $(\hat{\check{C}}, (i_n \circ G_n)^*)$ is a Quillen adjoint pair, and it is also clear that $\text{colim}_{P_\nu(n+1)^{\text{op}}} X$ and $\text{colim}_{\Delta^{\text{op}}} \check{C}X$ is naturally isomorphic. The part 1 follows from these observations. The part 2 is also clear. □

2.3. Goodwillie-Weiss embedding calculus and Sinha’s cosimplicial model. In this subsection, we give the definition of the cosimplicial space $C^\bullet([M])$ modeling Emb($S^1, M$), and state its property. We begin with an analogue of the punctured knot model in [35, Definition 3.4], which is an intermediate object between Emb($S^1, M$) and $C^\bullet([M])$.

**Definition 2.3.**

1. Let $S^1 = [0, 1]/0 \sim 1$ and $J_i \subset S^1$ be the image of the interval $(1 - \frac{1}{i}, 1 - \frac{1}{i+1}, 1 - \frac{1}{i})$ by the quotient map $[0, 1] \to S^1$.
2. We fix an embedding $M \to \mathbb{R}^{N+1}$ for sufficiently large $N$. We endow $M$ with the Riemannian metric induced by the Euclidean metric on $\mathbb{R}^{N+1}$ via this embedding. Let $\hat{M} = STM$ denote the total space of the unit sphere tangent bundle of $M$.
3. For a subset $S \subset \mathbb{R}$, let $E_S(M)$ be the space of embeddings $S^1 - \bigcup_{i \in S} J_i \to M$ of constant speed.
• Define a functor $E_n(M) : P_n(n+1) \to C^G$ by assigning to a subset $S$ the space $E_S(M)$ and set 
$$P_n\text{Emb}(S^1, M) := \text{holim}_{P_n(n+1)} E_n(M).$$

Let $\alpha_n : \text{Emb}(S^1, M) \to P_n\text{Emb}(S^1, M)$ be the map induced from the restriction of domain. The category $P_n(n)$ is a subcategory of $P_n(n+1)$ via the standard inclusion $n \to n+1$. By our choice of $J_i$, we have a canonical restriction map $r_n : P_n\text{Emb}(S^1, M) \to P_{n-1}\text{Emb}(S^1, M)$. The maps $\alpha_n$ induces a map 
$$\alpha_\infty : \text{Emb}(S^1, M) \to \text{holim}_{n} P_n\text{Emb}(S^1, M)$$

where the right hand side is the homotopy limit of the tower $\cdots \xrightarrow{r_{n+1}} P_n\text{Emb}(S^1, M) \xrightarrow{r_n} P_n\text{Emb}(S^1, M) \xrightarrow{r_{n-1}} \cdots \xrightarrow{r_2} P_1\text{Emb}(S^1, M)$.

**Remark 2.4.** Our choice of $J_i$ is different from [35] since we adopt the reverse labeling of coface and codegeneracy maps of the cosimplicial model to [35] for the author’s preference. This does not cause any new problem.

**Lemma 2.5.** The map $\alpha_n : \text{Emb}(S^1, M) \to P_n\text{Emb}(S^1, M)$ is $(n-1)(d-3)$-connected. In particular, $\alpha_\infty$ is a weak homotopy equivalence.

**Proof.** Let $p : \text{Emb}(S^1, M) \to \hat{M}$ be the evaluation of value and tangent vector at $0 \in S^1$. As is well-known, $p$ is a fibration. Let $D$ be a closed subset on $M$ diffeomorphic to closed $d$-dimensional disk. Let $\text{Emb}([0, 1], M-\text{Int}(D))$ be the space of embeddings $[0, 1] \to M-\text{Int}(D)$ whose value and tangent vector at endpoints is a fixed value in $\partial D$ and vector. If we take a point of $\hat{M}$, for some choice of the disk $D$, fixed endpoints, and embedded path between the points in $D$, we have the inclusion from $\text{Emb}([0, 1], M-\text{Int}(D))$ to the fiber of $p$ at the point. This inclusion is a weak homotopy equivalence. Its homotopy inverse is given by shrinking the disk $D$ to the point. Thus, we have a homotopy fiber sequence 
$$\text{Emb}([0, 1], M-\text{Int}(D)) \to \text{Emb}(S^1, M) \to \hat{M}$$

Restricting the domain, we have similar fiber sequence $E_S(M-\text{Int}(D)) \to E_S(M) \to \hat{M}$, where the left hand side is the space defined in [35] Definition.3.1] with the obvious modification for $J_i$. (In [35], $M$ denotes a manifold with boundary so we apply the definitions to $M-\text{Int}(D)$ instead of our closed $M$.) Passing to homotopy limits, we have the following diagram 
$$\begin{array}{ccc}
\text{Emb}([0, 1], M-\text{Int}(D)) & \xrightarrow{\alpha_n} & \text{Emb}(S^1, M) \\
\downarrow & & \downarrow id \\
P_n\text{Emb}([0, 1], M-\text{Int}(D)) & \xrightarrow{\alpha_{n-1}} & P_n\text{Emb}(S^1, M) \\
\end{array}$$

where the both horizontal sequence are homotopy fiber sequences and the left bottom corner is the punctured knot model in [35] Definition.3.4] (with the obvious modification for $J_i$). As in [35] Theorem.3.5], by theorems of Goodwillie, Klein, and Weiss, the left vertical arrow is $(n-1)(d-3)$-connected, and so is the middle. \qed

**Remark 2.6.** Let $T_n\text{Emb}(S^1, M)$ be the $n$-th stage of Taylor tower (or polynomial approximation). Restriction of the domain induces a map $P_n\text{Emb}(S^1, M) \to T_n\text{Emb}(S^1, M)$ which is compatible with canonical maps from $\text{Emb}(S^1, M)$, but the author does not know this map is weak homotopy equivalence.

Our cosimplicial space is analogous to the well-known cosimplicial model of a free loop space just like Sinha’s original space is analogous to that of a based loop space. So the space $C^n([M])$ is related to a configuration space of $n+1$ points (not $n$ points).
Definition 2.7. Let $||-||$ denote the standard Euclidean norm in $\mathbb{R}^{N+1}$.

- Let $C_n(M) = \{(x_0, \ldots, x_{n-1}) \in M^{\times n} \mid x_k \neq x_l \text{ if } k \neq l\}$ be the ordered configuration space of $n$ points in $M$. Similarly, we set $C_2([n]) = \{(k, l) \in [n]^{\times 2} \mid k \neq l\}$.

- Let $C_n([M])$ be the closure of the image of the map
  \[ C_n(M) \to M^{\times n} \times (S^n)^{\times C_2([n-1])} \quad (x_k) \mapsto (x_k, u_{kl})_{kl} \]
  where $u_{kl} = \frac{x_k - x_l}{||x_k - x_l||}$.

$C_n([M])$ is the same as the space in Definition 1.3 of [35], though our labeling of points begins with 0. Define a space $C^n([M])$ by the following pullback diagram:
\[
\begin{array}{ccc}
C^n([M]) & \to & M^{\times n+1} \\
\downarrow & & \downarrow \\
C_{n+1}([M]) & \to & M^{\times n+1}
\end{array}
\]

Here, the right vertical arrow is the product of standard projection and the bottom horizontal one is the composition of the canonical inclusion $C_{n+1}([M]) \to M^{\times n+1} \times (S^n)^{\times C_2([n])}$ and the projection.

- Let $\tau : T_xM \to \mathbb{R}^{N+1}$ be the linear monomorphism from the tangent space induced by the differential of the embedding fixed in Definition 2.3 and the identification $T_x\mathbb{R}^{N+1} \cong \mathbb{R}^{N+1}$ by the standard basis. Set $A'_{n+1}([M]) := M^{\times n+1} \times (S^n)^{\times ([n]^{\times 2})}$.

Let $\beta'_{n+1} : C^n([M]) \to A'_{n+1}([M])$ be the map given by
\[
\beta'_{n+1}(x_k, u_{kl}, y_k) = (x_k, u'_{kl}), \quad u'_{kl} = \begin{cases} u_{kl} & (k \neq l) \\ \tau(y_k) & (k = l) \end{cases},
\]
where $y_k$ is a unit tangent vector at $x_k$. This is clearly a monomorphism. For an integer $i$ with $0 \leq i \leq n+1$, we define maps $d_i : [n+1] \to [n]$ by
\[
d_i(k) = \begin{cases} k & (k \leq i) \\ k-1 & (k > i) \end{cases}, \quad d_{n+1} = d_0 \circ \sigma,
\]
where $\sigma$ is the cyclic permutation $\sigma(k) = k + 1 (\text{modulo } n+2)$. (This $d_i$ is the same as $s^i$ in subsection 2.1 but we use the different notation to avoid confusion.) We define the map $d^i : A'_{n+1}([M]) \to A'_{n+2}([M])$ by
\[
d^i(x_k, u_{kl})_{0 \leq k,l \leq n} = (x_{f(k)}, u_{f(k), f(l)})_{0 \leq k,l \leq n+1}, \quad (f = d_i).
\]
This map restricts to the map $d^i : C^n([M]) \to C^{n+1}([M])$ via $\beta'_{n+1}, \beta'_{n+2}$. Similarly, we define the codegeneracy map $s^i : C^n([M]) \to C^{n-1}([M])$ $(0 \leq i \leq n-1)$ as the pullback by the map $s_i : [n-1] \to [n], s_i(k) = \begin{cases} k & (k \leq i) \\ k+1 & (k > i) \end{cases}$. The collection $C^*([M]) = \{C^n([M]), d^i, s^i\}$ forms a cosimplicial space. Well-definedness of this is verified in Lemma 2.3 below.

- We call the Bousfield-Kan type cohomology spectral sequence associated to $C^*([M])$ the Sinha spectral sequence for $M$ in short, Sinha s.s., and denote it by $\{E_r\}_r$.

Intuitively, an element of $C_n([M])$ is a configuration of $n$ points in $M$ some points of which are allowed to collide, or in other words, to be infinitesimally close, and the direction of collision is recorded as the unit vector $u_{kl}$ if $k$-th and $l$-th points collide. An element of $C^n([M])$ is an element of $C_{n+1}([M])$, each point of which has a tangent vector. For $0 \leq i \leq n$, the map $d_i$ replaces the $i$-th point in a configuration with two points colliding at the point along its vector. These points are labeled by $i$, $i+1$. Their vectors are copies of the original vector.
Lemma 2.8.  
(1) The map $C_n(M) \to M \times \Sigma_2 \times C_2([n-1])$ given in Definition 2.7 restricts to homotopy equivalence $C_n(M) \to C_n([M])$.

(2) The cosimplicial space $C^\bullet([M])$ is well-defined.

Proof. The part 1 is proved in [37] Corollary 4.5, Theorem 5.10. For the part 2, by [37] Proposition 6.6 the image of $d^i$ and $s^i$ is contained in $C^{n\pm 1}([M])$ ($C'_n([M])$ in the proposition is the same as $C^{n-1}([M])$ in our notation). Confirmation of the cosimplicial identities is a routine work. For example, to confirm $d^{n+2}d^i = d^i d^{n+1} : C^n([M]) \to C^{n+2}([M])$ ($i < n + 2$), it is enough to confirm the dual identity $d_i d_{n+2} = d_{n+1} d_i : [n+2] \to [n]$. The both sides are equal to the map

$$j \mapsto \begin{cases} 
  k & (k \leq i) \\
  k - 1 & (i < k < n + 2) \\
  0 & (k = n + 2)
\end{cases}$$

and

$$k \mapsto \begin{cases} 
  k & (k \leq n) \\
  0 & (k = n + 1, n + 2)
\end{cases}$$

Lemma 2.9. Let $G_n C^\bullet([M])$ be the composition functor $P_\nu(n+1) \xrightarrow{G_n} \Delta_n C^\bullet([M]) \xrightarrow{G} C_G$.

(1) The homotopy limits of $E_n(M)$ and $G_n C^\bullet([M])$ are connected by a zigzag of weak homotopy equivalences which are compatible with the inclusion $n \to n + 1$.

(2) The homotopy limit of $C^\bullet([M])$ over $\Delta_n$ and that of $G_n C^\bullet([M])$ over $P_\nu(n+1)$ are connected by a zigzag of weak homotopy equivalences which are compatible with the inclusion $n \to n + 1$.

(3) The homotopy limit of $C^\bullet([M])$ over $\Delta$ and $\text{Emb}(S^1, M)$ are connected by a zigzag of weak homotopy equivalences.

Proof. The proof of the part 1 is completely analogous to the proof of Lemma 5.19 of [35] so we omit details. Idea of the proof is consider the two space $C^{\#S^{-1}}([M])$ and $E_S(M)$ as subspaces of a common space, where one can “shrink components of embeddings until they become tangent vector” as in Definition 5.14 of [35]. The space is a subspace of the space of compact subspaces of $C^{\#S^{-1}}([M])$ with the Hausdorff metric. This space and the inclusions can be chosen compatible with maps in $P_\nu(n+1)$. For example, the restriction $E_S(M) \to E_{S'}(M)$ corresponding to the inclusion $S = n + 1 \subset S' = n + 2$ divides the component including the image of $0 \in S^1$ into two components since the image of $J_{n+2}$ is removed. At the limit of shrinking components, this is consistent with the coface map $d^{n+1}$. These inclusions to the common space gives rise to zigzag of natural transformations which is a weak homotopy equivalence at each $S \subset n + 1$. This induces the claimed zigzag. The part 2 follows from the fact that the functor $G_n$ is left cofinal (see Theorem 6.7 of [35]). The part 3 follows from the part 1, 2 and Lemma 2.5.

2.4. Operads, comodules, and Hochschild complex. The term operad means non-symmetric (or non-$\Sigma$) operad (see [23, 31]). An operad $\mathcal{O} = \{O(n)\}_{n \geq 1}$ in a symmetric monoidal category $(\mathcal{C}, \otimes)$ is a symmetric sequence equipped with maps $(- \circ_i -)$:
\(\mathcal{O}(m) \otimes \mathcal{O}(n) \to \mathcal{O}(m + n - 1)\) in \(\mathcal{C}\), called partial compositions, subject to certain conditions \((1 \leq i \leq m)\). \(\mathcal{O}(n)\) is called the object at arity \(n\). (We do not consider the object at arity 0 so the sequence begins with \(\mathcal{O}(1)\).) We mainly consider operads in \(\mathcal{CG}\) (resp. in \(\mathcal{CH}_k\)), which are called topological operads (resp. chain operads), where the monoidal product is the standard cartesian product (resp. tensor product). Let \(\mathcal{O}\) be a topological operad. \(C_*(\mathcal{O})\) denotes the chain operad given by \(C_*(\mathcal{O})(n) = C_*(\mathcal{O}(n))\) with the induced structure. We equip the sequence \(\{\mathcal{O}(n) \otimes \mathbb{S}\}_n\) of spectra with a structure of an operad in \(\mathcal{SP}^\Sigma\) as follows. The \(i\)-th partial composition is given by

\[
(\mathcal{O}(m) \otimes \mathbb{S}) \triangleleft (\mathcal{O}(n) \otimes \mathbb{S}) \cong (\mathcal{O}(m) \times \mathcal{O}(n)) \otimes (\mathbb{S} \otimes \mathbb{S}) \cong (\mathcal{O}(m) \times \mathcal{O}(n)) \otimes (\mathbb{S} \otimes \mathbb{S}),
\]

see subsection 2.1 for the isomorphisms. The action of \(\Sigma_n\) is the naturally induced action. We denote this operad by the same symbol \(\mathcal{O}\). We let \(\mathcal{A}\) denote the both of (discrete) topological and \(k\)-linear versions of the associative operad by abuse of notations. For the \(k\)-linear version, we fix a generator \(\mu \in \mathcal{A}(2)\) throughout this paper. \(\mathcal{K}\) denotes the Stasheff’s associahedral operad, and \(\mathcal{A}_\infty\) the cellular chain operad of \(\mathcal{K}\). Precisely speaking, \(\mathcal{A}_\infty\) is generated by a set \(\{\mu_k \in \mathcal{A}_\infty(k)\}_{k \geq 2}\) \((|\mu_k| = -k + 2\) ) with partial compositions. The differential is given by the following formula.

\[
d\mu_k = \sum_{l, p, q} (-1)^\zeta \mu_l \circ_{p+1} \mu_q
\]

where \(\zeta = \zeta(l, p, q) = p + q(l - p - 1)\).

In the following definition, we adopt the point-set description as if a category \(\mathcal{C}\) were the category of sets for simplicity.

**Definition 2.10.** Let \(\mathcal{O}\) be an operad over a symmetric monoidal category \(\mathcal{C}\). A (left) \(\mathcal{O}\)-comodule in \(\mathcal{C}\) is a symmetric sequence \(X = \{X(n)\}_{n \geq 1}\) in \(\mathcal{C}\) equipped with a map

\[
(- \circ_i) : \mathcal{O}(m) \otimes X(m + n - 1) \to X(n) \in \mathcal{C},
\]

called a partial composition, for each \(m \geq 1\), \(n \geq 1\), \(1 \leq i \leq n\) which satisfy the following conditions.

1. For \(a \in \mathcal{O}(m), b \in \mathcal{O}(l), \) and \(x \in X(l + m + n - 2)\),

\[
a \circ_i (b \circ_j x) = \begin{cases} b \circ_{j-l-1} (a \circ_{i-l-1} x) & \text{if } j < i \\ (a \circ_{j-i+1} b) \circ_{i-l} x & \text{if } i \leq j \leq i + m - 1 \\ b \circ_{j-m+1} (a \circ_{i-l} x) & \text{if } i + m - 1 < j \end{cases}
\]

2. For the unit 1 \(\in \mathcal{O}(1)\) and \(x \in X(n)\), 1 \(\circ_i x = x\).

3. For \(a \in \mathcal{O}(m), x \in X(m + n - 1)\), and \(\sigma \in \Sigma_n\),

\[
(a \circ_i x)^\sigma = a \circ_{\sigma^{-1}(i)} (x^{\sigma^1})
\]

where \(\sigma_1 \in \Sigma_{m+n-1}\) is the permutation induced by \(\sigma\) with replacing a letter \(\sigma^{-1}(i)\) with \(m\)-letters \(\sigma^{-1}(i), \ldots, \sigma^{-1}(i) + m - 1\). In other words,

\[
\sigma_1(k) = \begin{cases} \sigma(k) & (k < \sigma^{-1}(i) \text{ and } \sigma(k) < i) \\ \sigma(k) + m - 1 & (k < \sigma^{-1}(i) \text{ and } \sigma(k) > i) \\ i + k - \sigma^{-1}(i) & (\sigma^{-1}(i) \leq k \leq \sigma^{-1}(i) + m - 1) \\ \sigma(k - m + 1) & (k > \sigma^{-1}(i) + m - 1 \text{ and } \sigma(k - m + 1) < i) \\ \sigma(k - m + 1) + m - 1 & (k > \sigma^{-1}(i) + m - 1 \text{ and } \sigma(k - m + 1) > i) \end{cases}
\]

A map \(f : X_1 \to X_2\) of \(\mathcal{O}\)-comodules is a sequence of maps in \(\mathcal{C}\) \(\{f_n : X_1(n) \to X_2(n)\}_n\) which is compatible with the actions of symmetric groups and the partial compositions.
• A (right) \( \mathcal{O} \)-module in \( \mathcal{C} \) is a symmetric sequence \( Y = \{ Y(n) \}_{n \geq 1} \) equipped with a set of partial compositions \( Y(n) \otimes \mathcal{O}(m) \to Y(m+n-1) \) which satisfy the following conditions.

1. For \( a \in \mathcal{O}(m), b \in \mathcal{O}(l) \), and \( y \in y(n) \),
\[
(y \circ_j a) \circ_l b = \begin{cases} 
(y \circ_i b) \circ_{j+l-1} a & \text{if } i < j \\
y \circ_j (a \circ_{i-j+1} b) & \text{if } j \leq i \leq j + m - 1 \\
y \circ_{i+m-1} b \circ_j a & \text{if } i > j + m - 1
\end{cases}
\]

2. For the unit \( 1 \in \mathcal{O}(1) \) and \( y \in X(n) \), \( y \circ_i 1 = y \)
3. For \( a \in \mathcal{O}(m), y \in X(n) \), and \( \sigma \in \Sigma_n \),
\[
y^\sigma \circ_i a = (y \circ_{\sigma(i)} a)^{\sigma_2}
\]
where \( \sigma_2 \in \Sigma_{m+n-1} \) is the permutation induced by \( \sigma \) with replacing a letter \( i \) with \( m \)-letters \( i, \ldots, i + m - 1 \).

A map of modules is defined similarly to that of comodules.

• For a topological operad \( \mathcal{O} \) (regarded as an operad in \( SP^\Sigma \)), a \( \mathcal{O} \)-comodule of NUCSRS is a \( \mathcal{O} \)-comodule \( X \) in \( SP^\Sigma \) such that each \( X(n) \) is equipped with a structure of a NUCSRS and the action of \( \Sigma_n \) on \( X(n) \) and the partial composition \( (a \circ_i -) : X(n+m-1) \to X(n) \) is a map of NUCSRS for each \( a \in \mathcal{O}(m) \). A map of comodules of NUCSRS is a map of comodules which is also a map of NUCSRS at each arity.

• For a topological operad \( \mathcal{O} \) and a \( \mathcal{O} \)-module \( Y \), we define a \( \mathcal{O} \)-comodule \( Y^\vee \) of NUCSRS as follows.

1. We set \( Y^\vee(n) = Y(n)^\vee \) (see subsection 2.4).
2. For \( f \in Y^\vee(n) \) and \( \sigma \in \Sigma_n \), we define an action \( f^\sigma \) by \( f^\sigma(y) = f(y^{\sigma^{-1}}) \) for each \( y \in Y(n) \).
3. For \( a \in \mathcal{O}(m) \) and \( f \in Y^\vee(m+n-1) \), we define a partial composition \( a \circ_i f \) by \( a \circ_i f(y) = f(y \circ_i a) \) for each \( y \in Y(n) \).
4. We define a multiplication \( Y^\vee(n) \wedge Y^\vee(n) \to Y^\vee(n) \) as the pushforward by the multiplication of \( \mathbb{S} \). (This is actually unital.)

This construction is natural for maps of \( \mathcal{O} \)-modules.

• A \( \mathcal{A} \)-comodule \( X \) of CDBA is a \( \mathcal{A} \)-comodule (in \( CH_k \)) such that each \( X(n) \) is a CDBA and the partial composition \( \mu \circ_i (-) : X(n) \to X(n-1) \) (with the fixed generator \( \mu \in \mathcal{A}(2) \)) and the action of \( \sigma \in \Sigma_n \) preserves the differential, bigrading, multiplication, and unit.

The axioms for the partial composition of modules in Definition 2.10 are the standard ones, which is naturally interpreted in terms of concatenation of trees. The action of \( \sigma \in \Sigma_n \) is interpreted as replacement of labels \( i \) on leaves with labels \( \sigma^{-1}(i) \), and the axioms is the natural one with this interpretation. The axioms for comodule is simply dual to those for module. The comodule in Example 2.13 may give some intuition for it.

Composing the unity and associativity isomorphisms, we obtain a natural isomorphism \( K \otimes X \cong (K \otimes \mathbb{S}) \wedge X \) in \( SP^\Sigma \). Let \( \mathcal{O} \) be a topological operad. Via this isomorphism, a structure of \( \mathcal{O} \)-comodule in \( SP^\Sigma \) on a symmetric sequence \( X \) is equivalent to a set of maps
\[
\mathcal{O}(m) \otimes X(m+n-1) \to X(n)
\]
which satisfy the conditions completely similar to those given in Definition \[2.10\]. We also call these maps partial compositions, and in the rest of the paper, we will define comodules in \( \mathcal{SP}^\Sigma \) with these maps.

**Remark 2.11.** Precisely speaking, the notion of comodule in Definition \[2.10\] should be named as contracomodule because our comodule to module is the same as contramodule to comodule in [32], but for simplicity we adopt the terminology.

The following definition is essentially due to [17] though we adopt a different sign rule.

**Definition 2.12.** Let \( X^\ast \) be a \( A_{\infty} \)-comodule in \( \mathcal{CH}_k \). We define a chain complex \( (CH_\ast X^\ast, \tilde{d}) \) called *Hochschild complex of \( X \), as follows. Set \( CH_n X^\ast = X^\ast (n + 1) \). By our convention, the total degree is \( * - 1 \). The differential \( \tilde{d} \) is given as a map

\[
\tilde{d} = d - \delta : \bigoplus_{a-n=k} CH_n X^a \longrightarrow \bigoplus_{a-n=k+1} CH_n X^a.
\]

Here \( d \) is the internal (original) differential on \( X^a(n + 1) \) and \( \delta \) is given by the following formula

\[
\delta(x) = \sum_{i=0}^{n} \sum_{k=2}^{n-i+1} (-1)^{\epsilon} \mu_k \circ_{i+1} x + \sum_{s=1}^{n} \sum_{k=s+1}^{n+1} (-1)^{\theta} \delta \mu_k \circ_1 x^s
\]

for \( x \in X^a(n + 1) \), where \( \epsilon = \epsilon(a, i, k) = (a + i)(k + 1) \), and \( \theta = \theta(s, n, k, a) = sn + (k + 1)a \), and \( x^s \) denotes the image of \( x \) by the action of permutation in \( \Sigma_{n+1} \) which transposes first \( n-s+1 \) letters and last \( s \) letters.

The following example gives some intuition of definitions of comodule and Hochschild complex, while is not used later.

**Example 2.13.** Let \( C \) denote the category of \( k \)-modules, and \( A \) denote \( k \)-algebra. Let \( m_n \in A(n) \) be the element defined by successive partial compositions of the generator \( \mu \in A(2) \). Define a \( A \)-comodule \( X_A \) by

\[
X_A(n) = A^\otimes n, \quad m_k \circ_i (x_1 \otimes \cdots \otimes x_{k+n-1}) = x_1 \otimes \cdots \otimes x_{i-1} \otimes (x_i \cdots x_{i+k-1}) \otimes x_{i+k} \otimes \cdots \otimes x_{k+n-1},
\]

where \( x_1 \cdots x_{i+k-1} \) is the product in \( A \). We regard \( X_A \) as a \( A_{\infty} \)-comodule via a map \( A_{\infty} \rightarrow A \) of operads. The Hochschild complex of \( X_A \) is the usual (unnormalized) Hochschild complex of the associative algebra \( A \).

**Lemma 2.14.** With the notation of Definition \[2.12\], \( (\tilde{d})^2 = 0 \).

**Proof.** Roughly writing,

\[
(\tilde{d})^2(x) = \tilde{d}(dx - \delta x) = (ddx - d\delta x - \delta dx - d\delta x)
\]

\[
= d(\mu_k \circ_{i+1} x + \mu_k \circ_1 x^s) + (\mu_k \circ_{i+1} dx + \mu_k \circ_1 dx^s)
\]

\[
- \mu_i \circ_{j+1} (\mu_k \circ_{i+1} x) + \mu_i \circ (\mu_k \circ_1 x^s) + \mu_i \circ_1 (\mu_k \circ_{i+1} x)^t + \mu_i \circ_1 (\mu_k \circ_1 x^s)^t
\]

\[
= (d\mu_k) \circ_{i+1} x + (d\mu_k) \circ_1 x^s
\]

\[
- \mu_i \circ_{j+1} (\mu_k \circ_{i+1} x) + \mu_i \circ (\mu_k \circ_1 x^s) + \mu_i \circ_1 (\mu_k \circ_{i+1} x)^t + \mu_i \circ_1 (\mu_k \circ_1 x^s)^t
\]

(Here we already canceled the terms containing \( dx \) since the cancellation of signs is obvious.)

So we have six types of terms. To see which terms cancel with each other, we divide these terms into the following smaller classes.

1. \( (d\mu_k) \circ_{i+1} x, \quad d\mu_k = \sum \mu_k \circ_{p+1} \mu_q \)
2. \( (d\mu_k) \circ_1 x^s, \quad d\mu_k = \sum \mu_k \circ_p \mu_q : \quad (a) \ s < p + 1 \quad (b) \ p + q \leq s \quad (c) \ p = 0 \quad (d) \ p > 0 \quad (e) \ p + q > s \geq p + 1 \)
3. \( \mu_i \circ_{j+1} (\mu_k \circ_{i+1} x) : \quad (a) \ i < j \quad (b) \ j + l - 1 < i \quad (c) \ j \leq i \leq j + l - 1 \)
4. \( \mu_i \circ_{j+1} (\mu_k \circ_1 x^s) : \quad (a) \ j = 0 \quad (b) \ j > 0 \)
(5) \( \mu_1 (\mu_k \circ_{i+1} x) \): (a) \( i + 1 < n - k - t + 3 \) and \( l < s + i + 1 \) (b) \( i + 1 < n - k - t + 3 \) and \( l \geq s + i + 1 \) (c) \( i + 1 \geq n - k - t + 3 \)

(6) \( \mu_1 (\mu_k \circ x) \)

Now we claim that the terms in (1) cancel with the terms in (3-c), (2-a) with (5-b), (2-b) with (5-c), (2-c) with (4-a), (2-d) with (6), (3-a) with (3-b), and (4-b) with (5-a).

We shall verify the first and third cancellations. Other verifications are similar and omitted. For the first one, the coefficient of \( (\mu_1 \circ_{p+1} \mu_q) \circ_{i+1} x \) in the terms in (1) is \((-1)^{a_1}\) where

\[
\alpha_1 = \zeta(l, p, q) + \epsilon(a, i, l + q + 1) + 1.
\]

For the terms in (3-c), by the rules of the partial composition, we have \( \mu_1 \circ_{j+1} (\mu_k \circ_{i+1} x) = (\mu_1 \circ_{i-j+1} \mu_k) \circ_{j+1} x \). In order to match these terms with the terms in (1), we set \( q' = k \), \( p' + 1 = i - j + 1 \), and \( i' + 1 = j + 1 \). This change of subscripts implies \( \mu_1 \circ_{j+1} (\mu_k \circ_{i+1} x) = (\mu_1 \circ_{p'+1} \mu_q') \circ_{i'+1} x \). Clearly we have \( j = i' \), \( i = p' + i' \). The coefficient of \( \mu_1 \circ_{j+1} (\mu_k \circ_{i+1} x) \) in the terms (3-c) is \((-1)^{a_2}\) where

\[
\alpha_2 = \epsilon(a, i, k) + 1 + \epsilon(a + k - 2, j, l) + 1 = \epsilon(a, p' + i', q') + \epsilon(a - q' + 2, i', l) + 2.
\]

When we substitute \( q' = q \), \( p' = p \), \( i' = i \) in the last expression, elementary computation shows \( \alpha_1 + \alpha_2 \equiv 1 \) (mod 2). Thus the terms (1) cancel with the terms (3-c).

For the third case, the coefficient of \( (\mu_1 \circ_{p+1} \mu_q) \circ x^s \) in the terms (2-b) is \((-1)^{b_1}\), where

\[
\beta_1 = \zeta(l, p, q) + \theta(s, n, l + q - 1, a) + 1.
\]

For the terms (5-c), the condition \( i + 1 \geq n - k - t + 3 \) implies that \( \mu_k \) acts on a part of the last \( t \) letters. By this and the rule of the partial composition, we have

\[
\mu_1 \circ_{i+1} x^t = \mu_1 (\mu_k \circ_{i-n+k-1} (a^{t+k-1})) = (\mu_1 \circ_{i-n+k+1} \mu_k) \circ x^{t+k-1}.
\]

In order to match these terms with the terms in (2-b), we set \( p' + 1 = i - n + k + t - 1 \), \( q' = k \) and \( s' = t + k - 1 \). This change of subscripts implies \( \mu_1 \circ_{i+1} x^t = (\mu_1 \circ_{p'+1} \mu_q') \circ x^{s'} \). Clearly we have \( t = s' - q' + 1 \) and \( i = p' + n - s' + 1 \). The coefficient of \( \mu_1 \circ_{i+1} x^t \) is \((-1)^{b_2}\), where

\[
\beta_2 = \epsilon(a, i, k) + 1 + \theta(t, a - k + 2, n - k + 1, l) + 1 = \epsilon(a, p' + n - s' + 1, q') + \theta(s' - q' + 1, n - q' + 1, a - q' + 2, l) + 2.
\]

When we substitute \( q' = q \), \( p' = p \), \( s' = s \) in the last expression, elementary computation shows \( \beta_1 + \beta_2 \equiv 1 \) (mod 2). Thus the terms (2-b) cancel with the terms (5-c). \( \square \)

3. Comodule \( \mathcal{T}H_M \)

The purpose of this section is to define the comodule \( \mathcal{T}H_M \).

3.1. A model of a Thom spectrum. We introduce a model of the Thom spectrum \( N^{-TN} \) as a symmetric spectrum for a closed manifold \( N \). This model is essentially due to Cohen [12], and slightly different from Cohen’s original non-unital model mainly in that we use expanding embeddings.

**Definition 3.1.** Let \( N \) be a closed manifold. We fix a Riemannian metric on \( N \) and denote by \( d_N(-, -) \) the distance on \( N \) induced by the metric. The standard Euclidean norm on \( \mathbb{R}^k \) is denoted by \( ||-|| \). The distance in \( \mathbb{R}^k \) is induced by \( ||-|| \).

- For a smooth embedding \( e : N \to L \) to a Riemannian manifold \( L \) we set a number

\[
r(e) = \inf \left\{ \frac{d_L(e(x), e(y))}{d_N(x, y)} \mid x, y \in N, \ x \neq y \right\}.
\]

It is easy to see \( r(e) > 0 \). We say \( e \) is *expanding* if the inequality \( r(e) \geq 1 \) holds. \( \text{Emb}^{\text{exp}}(N, L) \) denotes the space of all expanding embeddings from \( N \) to \( L \) with the topology induced by the \( C^\infty \)-topology.
• For a smooth embedding $e : N \to \mathbb{R}^k$, we define a number $|e|$ as the number of $i \in \mathbb{N}$ such that the composition
\[
N \xrightarrow{e} \mathbb{R}^k \xrightarrow{i\text{-th projection}} \mathbb{R}
\]
is not a constant map.

• Let $e : N \to \mathbb{R}^k$ be a smooth embedding. For $\epsilon > 0$, we denote by $\nu_\epsilon(e)$ the open subset of $\mathbb{R}^k$ consisting of the points whose Euclidean distance from $e(N)$ are smaller than $\epsilon$. Let $L(e)$ denote the minimum of 1 and the least upper bound of $\epsilon > 0$ such that there exists a retraction $\pi_\epsilon : \nu_\epsilon(e) \to e(N)$ satisfying the following conditions.
  - For any $u \in \nu_\epsilon(e)$ and any $y \in N$, $||\pi_\epsilon(u) - u|| \leq ||e(y) - u||$ and the equality holds if and only if $\pi_\epsilon(u) = e(y)$.
  - For any $y \in N$, $\pi_\epsilon^{-1}(\{e(y)\}) = B_\epsilon(e(y)) \cap (e(y) + (T_y N)^\perp)$. Here $B_\epsilon(e(y))$ is the open ball with center $e(y)$ and radius $\epsilon$.
  - The closure $\overline{\nu_\epsilon(e)}$ of $\nu_\epsilon(e)$ is a smooth submanifold of $\mathbb{R}^k$ with boundary. (Such a retraction exists for a sufficiently small $\epsilon > 0$ by a version of tubular neighborhood theorem, see [27].) The retraction $\pi_\epsilon$ satisfying the above three conditions is unique. We regard the map $\pi_\epsilon : \nu_\epsilon(e) \to e(N)$ as a disk bundle over $N$, identifying $N$ and $e(N)$.

• Let $\tilde{N}_k^{-\tau}$ be the subspace of $\text{Emb}^{\epsilon^x}(N, \mathbb{R}^k) \times \mathbb{R} \times \mathbb{R}^k$ consisting of the triples $(e, \epsilon, u)$ with $0 < \epsilon < L(e)$. Define a subspace $\partial \tilde{N}_k^{-\tau} \subset \tilde{N}_k^{-\tau}$ by $(e, \epsilon, u) \in \partial \tilde{N}_k^{-\tau} \iff u \notin \nu_\epsilon(e)$. We put
\[
N_k^{-\tau} = \tilde{N}_k^{-\tau} / \partial \tilde{N}_k^{-\tau}
\]
We define a structure of a symmetric spectrum on $N^{-\tau}$ as follows.
  - We let $\Sigma_k$ act on $\mathbb{R}^k$ and $\text{Emb}^{\epsilon^x}(N, \mathbb{R}^k)$ by the standard permutation on components. The action of $\Sigma_k$ on $N_k^{-\tau}$ is given by $[e, \epsilon, u]^\sigma = [e^\sigma, \epsilon, u^\sigma]$.
  - The map $S^1 \times N_k^{-\tau} \to N_{k+1}^{-\tau}$ is given by $t \times [e, \epsilon, u] \mapsto [0 \times e, \epsilon, (t, u)]$ where we regard $S^1 = \mathbb{R} \cup \{\infty\}$, and $0 \times e : M \to \mathbb{R}^{k+1}$ is given by $(0 \times e)(x) = (0, e(x))$.

• We shall define a structure of NUCSRS on $N^{-\tau}$. An element of $(N^{-\tau} \times N^{-\tau})_k$ is represented by a data $[(e_1, e_1, u_1), [e_2, e_2, u_2]; \sigma]$ consisting of $[e_i, e_i, u_i] \in N_k^{-\tau}$ for $i = 1, 2$, and $\sigma \in \Sigma_k$. We define a commutative associative multiplication $\mu : N^{-\tau} \times N^{-\tau} \to N^{-\tau}$ by
\[
\mu([(e_1, e_1, u_1), [e_2, e_2, u_2]; \sigma]) = [e_{12}, e_{12}, (u_1, u_2)]^\sigma.
\]
Here, $e_{12} = (e_1 \times e_2) \circ \Delta$ where $\Delta : N \to N \times N$ is the diagonal map, and
\[
e_{12} = \min \left\{ \frac{e_1}{8|e_1|}, \frac{e_2}{8|e_2|}, \frac{L(e_1)}{8|e_1| - |e_1|}, \ldots, \frac{L(e_m)}{8|e_1| - |e_m|} \right\} \quad m \geq 2, e_i : N \to \mathbb{R}^{l_i}, e_m : N \to \mathbb{R}^{l_m}
\]
where the finite sequence $(e'_1, \ldots, e'_m)$ runs through the sequence of expanding embeddings satisfying $(e'_1 \times \cdots \times e'_m) \circ \Delta^m = (e_{12})^\tau$ for a permutation $\tau \in \Sigma_{k_1 + k_2}$, where $\Delta^m : N \to N^{\times m}$ denotes the diagonal map.

Lemma 3.2. The structure of NUCSRS on $N^{-\tau}$ given in Definition 3.1 is well-defined.

Proof. Most part of the proof is the same as the proof of [12 Theorem 3]. We shall only verify the associativity of the number $e_{12}$. Let $[e_i, e_i, u_i]$ be an element of $N_i^{-\tau}$ for $i = 1, 2, 3$. We denote by $e_{123}$ (resp. $e_{123}$ ) the number for the result of the elements of $i = 1, 2$ (resp. $i = 2, 3$) being multiplied at first. By definition, we have
\[
e_{123} = \min \left\{ \frac{e_{12}}{8|e_{12}|}, \frac{e_3}{8|e_{12}|}, \frac{L(e_{12})}{8|e_{12}| - |e_{12}|}, \ldots, \frac{L(e'_m)}{8|e_{12}| - |e'_m|} \right\} \quad m \geq 2, e'_1 : N \to \mathbb{R}^{l_1}, \ldots, e'_m : N \to \mathbb{R}^{l_m}
\]
We shall define a map \( \rho \). After Lemma 3.7, we impose an additional assumption on 
\( \hat{a} \) point of 
\( M \)

\[ \Delta(2) = (\tau) \Delta(1) = (\tau) \Delta(3) \]

where \( \epsilon_{123} = (e_1 \times e_2 \times e_3) \circ \Delta^3 \), and the finite sequence \( (e'_1, \ldots, e'_m) \) runs through the sequence of expanding embeddings satisfying \( (e'_1 \times \cdots \times e'_m) \circ \Delta^m = (e_{123})^\tau \) for some \( \tau \in \Sigma_{k_1+k_2+k_3} \).

By the obvious equality \( |e_{123}| = |e_1| + |e_2| \), we have

\[ \epsilon(123)^\beta = \min \left\{ \frac{\epsilon_1}{8|e_2|+|e_3|}, \frac{\epsilon_2}{8|e_3|+|e_1|}, \frac{\epsilon_3}{8|e_1|+|e_2|}, L(123), \frac{L(e'_1)}{8|e_{123}|-|e'_1|}, \ldots, \frac{L(e'_m)}{8|e_{123}|-|e'_m|} \right\}, \]

where the finite sequence \( (e'_1, \ldots, e'_m) \) runs through the same set as above. The number \( \epsilon_{1(23)} \) is also seen to be equal to the number of the right hand side. \( \square \)

3.2. Construction of a comodule \( \widehat{TH}_M \).

**Definition 3.3.**
* For a closed interval \( c = [a, b] \), we set \( |c| = b - a \), and call a point \( (a+b)/2 \in c \) the center of \( c \).
* We define a version of little interval operad, denoted by \( D_1 \) as follows. Let \( D_1(n) \) be the set of \( n \)-tuples \((c_1, c_2, \ldots, c_n)\) of closed subintervals \( c_i \subset [-1/2, 1/2] \) such that \( c_1 \cup \cdots \cup c_n = [-1/2, 1/2] \) and \( c_i \cap c_j \) is a one-point set or empty if \( i \neq j \), and the labeling of \( 1, \ldots, n \) is consistent with the usual order of the real line \( \mathbb{R} \) (so \(-1/2 \in c_1 \) and \(1/2 \in c_n \)). We topologize \( D_1(n) \) as a subspace of \( \mathbb{R}^n \) by the inclusion sending each interval to its center. The partial composition is given by the way completely analogous to the usual little interval operad.
* We identify \( H_0(D_1(2)) \) with \( A(2) \) by sending the generator represented by a topological point to the generator \( \mu \).

Recall that we fixed a Riemannian metric on \( M \) in Definition 2.7. In the rest of the paper, we equip the space \( \widehat{M} \) with the Sasaki metric, and the product \( \widehat{M}^\times \) the product metric. We fix a so small positive number \( \rho \) that a geodesic of length \( \rho \) exists for any initial value in \( M \). After Lemma 3.7 we impose an additional assumption on \( \rho \).

**Definition 3.4.** We shall define a map

\[ \Delta' = \Delta[\delta, c; i] : \widehat{M} \to \widehat{M}^\times \]

for each \( \delta = (d_1, \ldots, d_m) \in D_1(n), c = (c_1, \ldots, c_m) \in D_1(m) \) and \( 1 \leq i \leq n \). Let \( (x, y) \) denote a point of \( \widehat{M} \) with \( x \in M \) and \( y \in ST_x M \). Let \( s : [-\rho/2, \rho/2] \to M \) denote the geodesic segment with length parameter such that \( s(0) = x \) and the tangent vector of \( s \) at 0 is \( y \). Let \( t_j \in [-1/2, 1/2] \) be the center of \( c_j \) and put \( x_j = s(\rho \cdot |d_i| \cdot t_j) \) and set \( y_j \) to be the tangent vector of \( s \) at \( \rho \cdot |d_i| \cdot t_j \). We set \( \Delta'(x, y) = ((x_1, y_1), \ldots, (x_m, y_m)) \), see Figure 2.

The following lemma is clear from the definition of \( \Delta[\delta, c; i] \).
Lemma 3.5. For any configurations $d, c_1, c_2$ and numbers $i, j$, the following equality holds.

$$\Delta[d, c_1 \circ_j c_2 ; i] = \Delta[d, c_1 ; i] \circ \Delta[d \circ_i c_1, c_2 ; i + j - 1].$$

\[\square\]

Lemma 3.6. For any sufficiently small positive number $\rho$, the following condition holds. The map $\Delta[d, c; i]$ is expanding for any numbers $n \geq 1$, $m \geq 1$, and $i$ with $1 \leq i \leq n$, and elements $d \in D_1(n)$, $c \in D_1(m)$,

Proof. It is enough to prove the case of $m = 2$ since for $m \geq 3$, $\Delta'$ is equal to a successive composition of $(\Delta')'$ of arity 2 by Lemma 3.5. We set $\rho_0 = |d|/\rho$. We shall consider the case that $M$ is a metric vector space $V$ as a local model. Take points $(x, y), (v, w) \in \tilde{V} = V \times SV$, where $SV$ is the unit sphere in $V$. Put $c = (c_1, c_2)$. Let $-s, t$ be the centers of $c_1, c_2$ respectively $(0 < s, t < 1/2$, $s + t = 1/2)$. By definition, we have $\Delta'(x, y) = [(x - \rho_0 sy, y), (x + \rho_0 ty, y)]$.

When we set $a = ||x - v||$ and $b = ||y - w||$, we easily see

$$||\Delta'(x, y) - \Delta'(v, w)||^2 \geq 2a^2 - \rho_0|s - t|ab + \{\rho_0^2(s^2 + t^2)/4 + 2\}b^2$$

$$\geq 2a^2 - \rho_0|s - t|(a^2 + b^2)/2 + \{\rho_0^2(s^2 + t^2)/4 + 2\}b^2.$$

So we have

$$\frac{||\Delta'(x, y) - \Delta'(v, w)||}{||(x, y) - (v, w)||} \geq \frac{\sqrt{7}}{2} \quad \text{for} \quad \rho < 1 \quad \cdots (*)$$

There exists a number $r > 0$ such that for sufficiently small $\rho$, for any point $p \in M$ and any pair $(x, y), (v, w) \in T_p M \times ST_p M$ with $||x||, ||v|| \leq r$, the following inequality holds.

$$\frac{d(\Delta'_p \exp x, \exp y), \Delta'_p \exp v, \exp w)}{d(\Delta'_p M(x, y), \Delta'_p M(v, w))} > 1 - \frac{1}{100} \quad \cdots (**)$$

where $\exp$ is the exponential map at $p$ and $\exp'$ is its differential. Combining the inequalities (*) and (**), for $(x, y), (v, w) \in \tilde{M}$, we see

$$d_{\tilde{M}^2}(\Delta'(x, y), \Delta'(v, w)) > d_{\tilde{M}}((x, y), (v, w)) \text{ if } d_{M}(x, v) \leq r.$$ For the case of $d_{M}(x, v) > r$, if we take $\rho$ sufficiently small relative to $r$, the following inequality holds.

$$d(\Delta'(x, y), \Delta'(v, w)) > 1 - \frac{1}{100} \quad \text{for} \quad (x, y), (v, w) \in \tilde{M} \text{ with } d(x, v) > r.$$ Here, $\Delta: \tilde{M} \to \tilde{M} \times \tilde{M}$ is the usual diagonal. Then, if $d_{M}(x, v) > r$, we have the inequality

$$d(\Delta'(x, y), \Delta'(v, w)) > \left(1 - \frac{1}{100}\right)\sqrt{2} d((x, y), (v, w)).$$

Thus, we have shown the lemma. \[\square\]

The following lemma is an exercise of Riemannian geometry.

Lemma 3.7. For any sufficiently small positive number $\rho$ the following condition holds. For any $n \geq 2$, $G \in G(n)$ and set of positive numbers $\{\epsilon_{ij} \mid i < j, (i, j) \in E(G)\}$ satisfying

$$\sum_{(i, j) \in E(G)} \epsilon_{ij} < \rho,$$

the inclusion of subspaces of $M^{\times n}$

$$\{(x_1, \ldots, x_n) \mid \forall (i, j) \in E(G) \quad x_i = x_j \} \rightarrow \{(x_1, \ldots, x_n) \mid \forall (i, j) \in E(G) \quad d(x_i, x_j) \leq \epsilon_{ij} \}$$

is a homotopy equivalence. \[\square\]

Assumption: In the rest of paper, we fix the number $\rho$ so that Lemmas 3.6 and 3.7 hold.

We shall define a $D_1$-comodule $\tilde{T}_H M$ of NUCRS. We set

$$\tilde{M}^{-\tau}(n) = N^{-\tau} \quad \text{for} \quad N = \tilde{M}^{\times n},$$
Definition 3.9. We first define a subspectrum $\tilde{\mathcal{H}}_M(c) \subset \tilde{M}^{-\tau}(n)$ as follows.

$$\tilde{\mathcal{H}}_M(c)_k = \left\{ [e, e, u] \in \tilde{M}^{-\tau}(n)_k \mid \epsilon < \frac{p}{2} \cdot \min\{|c_1|, \ldots, |c_n|\} \right\}$$

We define a subspectrum $\tilde{\mathcal{H}}_M(n) \subset \text{Map}(D_1(n), \tilde{M}^{-\tau}(n))$ as follows.

$$\phi \in \tilde{\mathcal{H}}_M(n)_k \iff \phi(c) \in \tilde{\mathcal{H}}_M(c)_k.$$  

It is clear that the inclusion $\tilde{\mathcal{H}}_M(n) \to \text{Map}(D_1(n), \tilde{M}^{-\tau}(n))$ is a level-equivalence for any $n \geq 1$. We denote the sequence $\{\tilde{\mathcal{H}}_M(n)\}$ by $\tilde{\mathcal{H}}_M$.

We shall define an action of $\Sigma$ on $\tilde{\mathcal{H}}_M(n)$, with which we regard $\tilde{\mathcal{H}}_M$ as a symmetric sequence. For $c = (c_1, \ldots, c_n) \in D_1(n)$ and $\sigma \in \Sigma$, we define $c^\sigma \in D_1(n)$ to be the configuration of the sub-intervals of length $|c_{\sigma(1)}|, |c_{\sigma(2)}|, \ldots, |c_{\sigma(n)}|$ placed from the side of $-1/2$ to the side of $1/2$. For $[c, e, u] \in \tilde{M}^{-\tau}(n)_k$ and $\sigma \in \Sigma$, we set $[c, e, u]_\sigma = [c \circ_G e, e, u]$ where $G : \tilde{M}^x \rightarrow \tilde{M}^x$ is given by $(z_1, \ldots, z_n) \mapsto (z_{\sigma^{-1}(1)}, \ldots, z_{\sigma^{-1}(n)})$. (To distinguish the action of $\Sigma_k$ which is a part of structure of spectrum, we use the subscript $[-]_\sigma$.)

**Definition 3.8.** With the above notations, for $\phi \in \tilde{\mathcal{H}}_M(n)_k$ and $\sigma \in \Sigma$, we define an element $\phi^\sigma \in \tilde{\mathcal{H}}_M(n)_k$

$$\phi^\sigma(c) = \{\phi(c^{\sigma^{-1}})\}_\sigma.$$  

Clearly, $\phi \mapsto \phi^\sigma$ gives a $\Sigma_n$-action on $\tilde{\mathcal{H}}_M(n)$.

In order to define a partial composition on $\tilde{\mathcal{H}}_M$, we shall define a map

$$\Xi = \Xi[d, c; i] : \tilde{M}^{-\tau}(n + m - 1) \rightarrow \tilde{M}^{-\tau}(n).$$

For an element $[e, e, u] \in \tilde{M}^{-\tau}(n + m - 1)_k$, we put

- $e' = e \circ (1_{i-1} \times \Delta' \times 1_{n-i}) : \tilde{M}^x \rightarrow \mathbb{R}^k$, where $\Delta' = \Delta[0, c; i]$ and $1_i$ is the identity on $\tilde{M}^x$, and

- $e' = \frac{1}{8m-\tau} \min\{\epsilon, L(e, \varnothing, c, i)\}$, where $L(e, c')$ is the minimum of the numbers

$L(e \circ \Delta[c_1, c_2; j])$ where the triple $(c_1, c_2, j)$ runs through those which satisfy

$e' = (c_1 \circ c_2) \circ c_3$ for some configuration $c_3$ and number $l$.

By Lemma 3.6, $e'$ is expanding. We set $\Xi([e, e, u]) = [e', e', u]$. Clearly, $\Xi$ is well-defined map of spectra.

**Definition 3.9.** We use the above notations.

- We define a partial composition

$$(- \circ i) : D_1(m) \hat{\otimes} \tilde{\mathcal{H}}_M(n + m - 1) \rightarrow \tilde{\mathcal{H}}_M(n)$$

on $\tilde{\mathcal{H}}_M$ by setting

$$(c \circ_i \phi)(d) = \Xi(\phi(d \circ_i c)), \quad \text{where} \quad \Xi = \Xi[d, c; i],$$

for elements $\phi \in \tilde{\mathcal{H}}_M(n + m - 1)$, $c \in D_1(m)$, $d \in D_1(n)$.

- We define a multiplication $\hat{\mu} : \tilde{\mathcal{H}}_M(n) \wedge \tilde{\mathcal{H}}_M(n) \rightarrow \tilde{\mathcal{H}}_M(n)$ by

$$\hat{\mu}((\phi_1, \phi_2, \sigma))(d) = \mu(\phi_1(d), \phi_2(d); \sigma)$$

where $\mu$ denotes the multiplication given in Definition 3.1.

With these operations and the action of $\Sigma_n$ in Definition 3.8, we regard $\tilde{\mathcal{H}}_M$ as a $D_1$-comodule of NUCSRS.
Lemma 3.10. The structure of $\mathcal{D}_1$-comodule of NUCSRS on $\hat{\mathcal{H}}_M$ given in Definition 3.9 is well-defined.

Proof. By Lemma 3.5, we see the equality of (1) in Definition 2.10 holds. The equality in (2) in the same definition is clear.

We shall prove the equality in (3). Take elements $c \in \mathcal{D}_1(m), d \in \mathcal{D}_1(n), \phi \in \hat{\mathcal{H}}_M(m + n - 1)$, and $\sigma \in \Sigma_n$. By definition,

\[(c \circ_i \phi)^\sigma(d) = \{c \circ_i \phi(d^{\sigma^{-1}})\}_\sigma,\]

where $\Xi_1 = \Xi[d^{\sigma^{-1}}, c; i]$, and $\Xi_2 = \Xi[d, c; \sigma^{-1}(i)]$. It is easy to check the following equalities:

\[\phi(c) = (\phi \circ_{\sigma^{-1}(i)} c)^{\sigma^{-1}}, \quad \{\Xi_1(x)\}_\sigma = \Xi_2(x_{\sigma_i}).\]

By these equalities, we have verified the desired equality. Compatibility of the multiplication with the partial composition is obvious. \hfill \Box

3.3. Construction of the comodule $\mathcal{H}_M$. For an element $[e, \epsilon, u] \in \hat{M}^{-\tau}(n)$ which is not the base point, put $\pi(e) = ((x_1, y_1), \ldots, (x_n, y_n))$ with $x_i \in M$ and $y_i \in ST_{x_i}M$. Then we define $\mathcal{H}_{pq}(c) \subset \hat{\mathcal{H}}_M(c)$ by the following equivalence.

\[\{e, \epsilon, u\} \in \mathcal{H}_{pq}(c) \iff [e, \epsilon, u] = *, \text{ or } d_M(x_p, x_q) \leq \delta_{pq}(c, \epsilon),\]

where

\[\delta_{pq}(c, \epsilon) = \frac{p}{2}(|c_p| + |c_q|) - \epsilon,\]

which is a positive number by the definition of $\hat{\mathcal{H}}_M(c)$. Define a subspectrum $\mathcal{H}_{pq}(n) \subset \hat{\mathcal{H}}_M(n)$ by

\[\phi \in \mathcal{H}_{pq}(n) \iff \forall c \in \mathcal{D}_1(n), \phi(c) \in \mathcal{H}_{pq}(c)\]

The following lemma is a key to define the comodule $\mathcal{H}_M$. Most of the technical definitions by now are necessary to make this lemma hold.

Lemma 3.11. (1) For any numbers $n \geq 1, m \geq 2$ and element $c \in \mathcal{D}(m)$, let $c \circ_i \mathcal{H}_{pq}(n + m - 1)$ denote the subspectrum of $\hat{\mathcal{H}}_M(n)$ consisting of the images of $c \circ_i (-)$. We have the following inclusion at each level $k$.

\[c \circ_i \mathcal{H}_{pq}(n + m - 1) \subset \begin{cases} \{\ast\} & (i \leq p < q \leq i + m - 1) \\ \mathcal{H}_{p,i}(n) & (p < i \leq q \leq i + m - 1) \\ \mathcal{H}_{p,q-m+1}(n) & (p < i, i + m - 1 < q) \\ \mathcal{H}_{i,q-m+1}(n) & (i \leq p \leq i + m - 1 < q) \\ \mathcal{H}_{p-m+1,q-m+1}(n) & (i + m - 1 < p < q) \end{cases}\]

More precisely, for example, the second inclusion means $c \circ_i \mathcal{H}_{pq}(n + m - 1)_k \subset \mathcal{H}_{p,i}(n)_k$ for each $k$.

(2) The image of $\mathcal{H}_{pq}(n) \land \hat{\mathcal{H}}_M(n)$ by the multiplication $\hat{\mu}$ given in Definition 3.9 is contained in $\mathcal{H}_{pq}(n)$.

Proof. We shall show the part 1. The proof of the part 2 is similar and omitted. Let $c \in \mathcal{D}_1(m)$ and $d \in \mathcal{D}_1(n)$ and $\phi \in \hat{\mathcal{H}}_M(n + m - 1)_k$. Let $(e, \epsilon, u)$ be a representative of $\phi(d \circ_i c)$. Put

\[(x_1', y_1'), \ldots, (x_{n+m-1}', y_{n+m-1}')) = ((1_{i-1}) \times \Delta' \times (1_{n-i}))(\pi_e(u))\]
This inequality means \( \phi \)

where \( d \)

By this inequality and the definition of the map \( \Delta' \)

we have proved the first inclusion.

\[ \delta_{pq}(\partial \circ_i \varepsilon, \varepsilon) \]

and \( x \)

We shall give the formal proof. Since the image of \( \pi_e \) sends \( u \) to its closest point in \( e(M) = M \), we have

\[ ||u - e(\pi_e(u))|| \leq ||u - e'(\pi_e(u))|| < \varepsilon'. \]

As \( e' = e \circ (1_{-1}) \times \Delta' \times (1_{n-i}) \), and \( e \) is expanding, we have

\[ d_1((x_1', y_1'), \ldots, (x_{n+m-1}, y_{n+m-1})), ((x_1, y_1), \ldots, (x_{n+m-1}, y_{n+m-1})) < 2 \varepsilon' \]

where \( d \) denotes the distance in \( (\bar{M})^{\times n+m-1} \). So we have

\[ d_M(x_j, x_j') \leq d_M((x_j, y_j), (x_j, y_j')) < 2 \varepsilon' \text{ for } j = 1, \ldots, n + m - 1 \]

By this inequality and the definition of the map \( \Delta' \), we have the following inequality.

\[
\begin{align*}
    d_M(x_p, x_q) &\geq d_M(x_p', x_q') - d_M(x_p, x_p') - d_M(x_q, x_q') \\
                  &\geq d_M(x_p', x_q') - 4 \varepsilon' \\
                  &\geq \frac{\rho}{2} |d_i(|c_{p-i+1}| + |c_{q-i+1}|) - 4 \varepsilon' \\
                  &= \frac{\rho}{2} (|\partial \circ_i c|_p + |\partial \circ_i c|_q) - 4 \varepsilon' \\
                  &\geq \frac{\rho}{2} (|\partial \circ_i c|_p + |\partial \circ_i c|_q) - \varepsilon/2 \\
                  &> \delta_{pq}(\partial \circ_i \varepsilon, \varepsilon)
\end{align*}
\]

This inequality means \( \phi(\partial \circ_i c) \notin \mathcal{T} \mathcal{H}_{pq}(\partial \circ_i c) \), which is a contradiction. So \( (\phi \circ_i c)(\partial) = * \) and we have proved the first inclusion.
We shall show the second inclusion, the case of \( p < i \leq q \leq i + m - 1 \). Let \((x', y') \in \widehat{M}\) be the \( i\)-th component of \( \pi_{i'}(u) \). Clearly, we have

\[
((x', y'_1), \ldots, (x'_{i+m-1}, y'_{i+m-1})) = \Delta'(x', y').
\]

By an argument similar to the above, we have the following inequality.

\[
d_M(x'_p, x') \leq d_M(x'_p, x_p) + d_M(x_p, q) + d_M(x_q, x'_q) + d_M(x'_q, x')
\]

\[
\leq 2e' + \delta_{pq}(d \circ_i c, e) + 2e' + \frac{p^2}{2} |d_i| (1 - |c_{q-i+1}|)
\]

\[
= \frac{p^2}{2} (|d_{pq}| + |d_i| |c_{q-i+1}|) - \epsilon + 2e' + \frac{p^2}{2} |d_i| (1 - |c_{q-i+1}|)
\]

\[
\leq \frac{p^2}{2} (|d_{pq}| + |d_i|) - \epsilon/2
\]

\[
< \delta_{pq}(d, e')
\]

This implies the second inclusion. The other cases are similar to the first and second cases. \( \Box \)

Let \( TH_{fat}(n) \) be the subspectrum of \( \widehat{TH}_M(n) \) whose space at level \( k \) is given by

\[
TH_{fat}(n)_k = \bigcup_{1 \leq p < q \leq n} TH_{pq}(n)_k.
\]

Since \( \{TH_{pq}(n)\}^* = TH_{\sigma^{-1}(p), \sigma^{-1}(q)}(n) \), \( TH_{fat}(n) \) is stable under action of \( \Sigma_n \). By Lemma 3.11, the sequence \( \{\Delta_{fat}^n(n)\}_{n \geq 0} \) is stable under partial compositions and is ideal for the multiplication \( \mu \). So the sequence \( \{\Delta_{fat}^n(n)\}_{n \geq 0} \) inherits structure of a comodule from \( \widehat{TH}_M \), we can define the quotient comodule as follows.

**Definition 3.12.** We define a spectrum \( TH_M(n) \) by the quotient (collapsing to *).

\[
TH_M(n)_k = \widehat{TH}_M(n)_k / TH_{fat}(n)_k
\]

for each \( k \geq 0 \). We regard the sequence \( TH_M = \{TH_M(n)\}_{n \geq 1} \) as a comodule of NUCSRS with the structure induced by that on \( \widehat{TH}_M \).

4. **Atiyah Duality for Comodules**

**Definition 4.1.** We shall define the following zigzag consisting of \( D_1 \)-comodules of NUCSRS and maps between them.

\[
(C^{(M)})^\vee \xrightarrow{(i_n)^\vee} (\hat{F}M)^\vee \xrightarrow{(i_1)^\vee} (F^M)^\vee \xleftarrow{\Phi} F_M \xrightarrow{\Phi} \widehat{TH}_M
\]

- Set \( C^{(M)}(n) = C^{n-1}([M]) \). When we regard a configuration as an element of \( C^{(M)}(n) \), we label its points by \( 1, \ldots, n \) instead of \( 0, \ldots, n - 1 \). We give the sequence \( C^{(M)}_n = \{C^{(M)}_n\}_{n \geq 1} \) a structure of an \( A \)-module as follows. For the unique element \( \mu \in A(2) \) and an element \( x \in C^{(M)}(n) \), we set \( x \circ_i \mu = d_i^{-1}(x) \) where, \( d_i^{-1} \) is the coface operator of \( C^*([M]) \). The action of \( \Sigma_n \) on \( C^{(M)}(n) \) is given by permutations of labels. \( (C^{(M)})^\vee \) is the \( A \)-comodule of NUCSRS given in Definition 2.10. By pulling back the action by the unique operad morphism \( D_1 \to A \), we also regard \( (C^{(M)})^\vee \) as a \( D_1 \)-comodule.

- Let \( F^M(n) \) be the subspace of \( D_1(n) \times \hat{M}^{\times n} \) defined by the following condition. For an element \( (c; (x_1, y_1), \ldots, (x_n, y_n)) \in D_1(n) \times \hat{M}^{\times n} \) with \( x_i \in M \) and \( y_i \in ST x_i M \),

\[
(c; (x_1, y_1), \ldots, (x_n, y_n)) \in F^M(n)
\]

\[
\iff d(x_i, x_j) \geq \frac{\rho}{2} (|c_i| + |c_j|) \text{ for each pair } (i, j) \text{ with } i \neq j,
\]

where \( \rho \) is the number fixed in subsection 3.2.
• The sequence \( \{F^M(n)\} \) has a structure of a \( D_1 \)-module. For \( c \in D_1(n) \) and \( (d; z_1, \ldots, z_n) \in F^M(n) \), we set \( (d; z_1, \ldots, z_n) \circ c = (d \circ c; z_1, \ldots, \Delta'(z_i), \ldots, z_n) \), where \( \Delta' = \Delta[d; c; i] \) is given in Definition 3.3. The symmetric group acts on \( F^M(n) \) by permutations of little intervals and components. The \( D_1 \)-comodule of NUCSRS \( (F^M)^\vee \) is the one induced by \( F^M \).

• We shall define a symmetric sequence of spectra \( \{S_M(n)\}_n \). Set \( \tilde{S}_M(n)k = \tilde{N}_k^{-\tau} \) for \( N = \tilde{M}^{\times n} \) (see Definition 3.1). Define a subspace \( \partial(\tilde{S}_M(n))k \subset \tilde{S}_M(n)k \) by \( (e, \epsilon, v) \in \partial \tilde{S}_M(n)k \iff \|v\| \geq \epsilon \). We put \( S_M(n)k = \tilde{S}_M(n)k / \partial \tilde{S}_M(n)k \).

We regard \( S_M(n) \) as a NUCSRS by a multiplication defined similarly to the one of \( N^{-\tau} \) given in Definition 3.1.

• Set \( F_M(n) := \text{Map}(F^M(n), S_M(n)) \). We give the sequence \( \{F_M(n)\}_n \) a structure of \( D_1 \)-comodule as follows: For \( c \in D_1(n) \) and \( f \in F_M(n + m - 1) \), set \( c \circ f \) to be the following composition

\[
\begin{align*}
F^M(m) \xrightarrow{(\epsilon, \epsilon, o)} F^M(n + m - 1) \xrightarrow{f} S_M(n + m - 1) \xrightarrow{\alpha} S_M(n) \,.
\end{align*}
\]

Here, \( \alpha \) is given by

\[
\alpha([e, \epsilon, v]) = [e', \epsilon', v] ,
\]

where \( e' \) and \( \epsilon' \) are those defined in the paragraph above Definition 3.3. Similarly to \( (C_{(\{M\})})^\vee \), we define a multiplication on \( F_M(n) \) as the pushforward by the multiplication on \( S_M(n) \).

• We define a map \( \bar{\Phi}_n : \bar{TH}_M(n) \to F_M(n) \) of spectra by

\[
\bar{\Phi}_n(\phi)((c; z_1, \ldots, z_n)) = [e, \bar{c}, u - e(z_1, \ldots, z_n)]
\]

Here, we write \( \phi(c) = [e, u, c] \) and we set \( \bar{c} = c/4 \). \( \bar{\Phi}_n \) induces a morphism \( \Phi_n : \bar{TH}_M(n) \to F_M(n) \) which forms a morphism of comodules, as is proved in Lemma 4.2 below.

• We shall define a \( D_1 \)-module \( \vec{F}^M \). Set \( \vec{F}^M_1(n) = [0, 1] \times D_1(n) \times C_{(\{M\})}(n) / \sim \),

where the equivalence relation is generated by the relation \( (t, c, z) \sim (s, d, z') \iff (s = t = 0 \text{ and } z = z') \). \( \vec{F}^M(n) \) is the subspace of \( \vec{F}^M_1(n) \) consisting of elements \( (t, c, z = (x_k, u_k, y_k)) \) satisfying the condition that

\[
t \neq 0 \Rightarrow z \in \text{Int}(C_{(\{M\})}(n)) \text{ and } d_M(x_i, x_j) \geq t \cdot \frac{\rho}{2}(|c_i| + |c_j|) .
\]

Here, \( \text{Int}(C_{(\{M\})}(n)) \) is the subspace consisting of the elements \( (x_k, u_k, y_k) \) such that \( x_k \neq x_l \) if \( k \neq l \), or equivalently, \( (x_k, u_k) \) belongs to \( C_n(M) \) via the canonical inclusion \( C_n(M) \subset C_{(\{M\})}(M) \). We endow the sequence \( \{\vec{F}^M(n)\}_n \) with the \( D_1 \)-module structure analogous to that of \( F^M \). The difference is that we use the number \( t \rho \) instead of \( \rho \) in the definition of \( \Delta' \) for \( t > 0 \) and use the module structure on \( C_{(\{M\})}(M) \) for \( t = 0 \). The obvious inclusions \( i_0 : C_{(\{M\})}(n) \to \vec{F}^M(n) \) and \( i_1 : F^M(n) \to \vec{F}^M(n) \) to \( t = 0, 1 \) give rise to morphisms of \( D_1 \)-modules \( i_0 : C_{(\{M\})} \to \vec{F}^M, \ i_1 : F^M \to \vec{F}^M \).

• In order to define \( F_M^e \), and \( p_\ast, q_\ast \), we shall define a symmetric sequence of symmetric spectra \( \{S_M^e(n)\}_n \). Let \( S_M^e(n) \) be the subspace of \( \text{Emb}(\vec{M}, \mathbb{R}^k) \times \mathbb{R} \times S^k \) consisting of triples \((e, \epsilon, v)\) with \( 0 < \epsilon < L(e) \). We put

\[
S_M^e(n)k = \tilde{S}_M(n)k / \{(e, \epsilon, \infty) \mid e, \epsilon \text{ arbitrary} \} .
\]
where we regard $S^k = \mathbb{R}^k \cup \{\infty\}$ We regard $S'_M(n)$ as a symmetric spectrum analogously to $S_M(n)$. Let $p : S'_M(n) \to S_M(n)$ be the map induced by the collapsing map $S^k \to \mathbb{R}^k / \{v \mid ||v|| \geq \epsilon\}$ and $q : S'_M \to S$ be the map forgetting the data $(e, \epsilon)$. Set $F'_M(n) = \text{Map}(F^M(n), S'_M(n))$. We regard $\{F'_M(n)\}$ as a $\mathcal{D}_1$-comodule of NUCSRS analogously to $F_M$, the pushforwards $p_*$ and $q_*$ are clearly morphisms of comodules of NUCSRS.

Verification of well-definedness of the objects defined in Definition 4.1 is routine work. For example, the associativity of the composition of $C([[M]])$ follows from the cosimplicial identities of $C^*([M])$, and that of $F^M$ can be verified similarly to the associativity of little cubes operads. We omit details.

**Lemma 4.2.** $\tilde{\Phi}_n$ uniquely factors through a map $\Phi_n : \mathcal{T}H_M(n) \to F_M(n)$, and the sequence $\{\Phi_n\}$ is a map of $\mathcal{D}_1$-comodules of NUCSRS.

**Proof.** We shall show that for any element $\phi \in \mathcal{T}H_{pq}(n)$, $\tilde{\Phi}_n(\phi) = \ast$. Suppose that there exists an element $(c; z_1, \ldots, z_n) \in F_M(n)$ such that $\tilde{\Phi}_n(\phi)(c; z_1, \ldots, z_n) \neq \ast \in S_M(n)$. If we put $\phi(c) = [e, \epsilon, u]$, the inequality $||u - e(z_1, \ldots, z_n)|| < \epsilon/4$ holds. So we have $||u_1 - e(\pi_1 u)|| < \epsilon/4$. Thus we have

$$||e(\pi_1 u) - e(z_1, \ldots, z_n)|| \leq ||e(\pi_1 u) - u|| + ||u - e(z_1, \ldots, z_n)|| < \epsilon/2$$

As $\epsilon$ is expanding, we have $d(\pi_1(u), (z_1, \ldots, z_n)) < \epsilon/2$ where $d$ denotes the distance in $\mathcal{M}^{\times n}$. If we write $z_i = (x_i, y_i)$ and $\pi_1(u) = ((\bar{x}_1, \bar{y}_1), \ldots, (\bar{x}_n, \bar{y}_n))$ as pairs of a point of $M$ and a tangent vector, it follows that $d_M(\bar{x}_i, x_i) < \epsilon/2$, and the following inequality.

$$d(x_p, \bar{x}_q) \geq d(x_p, x_q) - d(x_p, \bar{x}_p) - d(x_q, \bar{x}_q)$$

$$> \frac{\rho}{2}(|c_p| + |c_q|) - \epsilon = \delta_{pq}(\epsilon, \epsilon).$$

This inequality contradicts the assumption $\phi \in \mathcal{T}H_{pq}(n)$. Thus we have proved $\tilde{\Phi}_n(\mathcal{T}H_{pq}(n)) = \ast$. This implies the former part of the lemma. The latter part is obvious. $\Box$

**Definition 4.3.** A $\mathcal{D}_1$-comodule of NUCSRS is **strongly semistable** if the spectrum $X(n)$ is strongly semistable for each $n$. A map $f : X \to Y$ of $\mathcal{D}_1$-comodules of NUCSRS is a $\pi_*$-isomorphism if each map $f_n : X(n) \to Y(n)$ is a $\pi_*$-isomorphism.

The following is a version of Atiyah duality which respects our comodules. We devote the rest of this section to its proof.

**Theorem 4.4.** As $\mathcal{D}_1$-comodule of NUCSRS, $(C([[M]]))^\vee$ and $\mathcal{T}H_M$ are $\pi_*$-isomorphic. Precisely speaking, all the comodules in the zigzag in Definition 4.1 are strongly semistable and all the maps in the same zigzag are $\pi_*$-isomorphisms.

**Definition 4.5.** For $G \in \mathcal{G}(n)$, and $c \in D_1(n)$, we define two subspectra $\mathcal{T}H_G(c), \mathcal{T}H_{\text{fat}}(c) \subset \mathcal{T}H_M(c)$ by

$$\mathcal{T}H_G(c) = \bigcap_{(p,q) \in E(G)} \mathcal{T}H_{pq}(c) \quad (G \neq \emptyset)$$

$$\mathcal{T}H_M(c) \quad (G = \emptyset).$$

Similarly, we define a subspectrum $\mathcal{T}H_G \subset \mathcal{T}H_M(n)$ by

$$\mathcal{T}H_G = \bigcap_{(p,q) \in E(G)} \mathcal{T}H_{pq} \quad (G \neq \emptyset)$$

$$\mathcal{T}H_M(n) \quad (G = \emptyset).$$

Here, the union and intersections are taken in the level-wise manner.
• We fix an expanding embedding $e_0 : \tilde{M} \to \mathbb{R}^k$ and a positive number $\epsilon_0 < L(e_0)$ and a configuration $c_0 \in \mathcal{D}_1(n)$ such that $\epsilon_0 < \frac{1}{4} \min \{|c_1|, \ldots, |c_n|\}$. We set $\nu = \nu_{e_0}(c_0)$. We impose an additional condition on $e_0$ in Definition 5.7 which is satisfied by any sufficiently small $\epsilon_0$, and we will assume $K$ is a multiple of 4 in the proof of Theorem 5.15. (We may impose the assumption on $K$ from the beginning, but for the convenience of verification of signs, we do not do so.)

• Consider $\nu^{\times n} \subset \mathbb{R}^{nK}$ as a disk bundle over $\tilde{M}^{\times n}$ and denote by $\nu_G$ the restriction $\nu^{\times n}|_{\Delta_G}$ (see Introduction). Let $\lambda_G : T\nu_G \to \mathcal{T}H_G(c_0)nK$ be the map $[u] \mapsto [(e_0)^{\times n}, \epsilon, u]$. $\lambda_G$ induces a morphism $\lambda_G : \Sigma^nK T\nu_G \to \mathcal{T}H_G(c_0)$ in $\text{Ho}(\mathcal{SP}^E)$, where $\Sigma$ denotes the suspension.

**Lemma 4.6.** For any closed smooth manifold $N$ and $k \geq 1$, the inclusion $I : \text{Emb}^e(N, \mathbb{R}^k) \to \text{Emb}(N, \mathbb{R}^k)$ is a homotopy equivalence.

**Proof.** Let $f : \mathbb{R}_{>0} \to \mathbb{R}$ be a $C^\infty$-function which satisfies the following inequality

$$f(x) > \frac{1}{x} \quad (x < 1), \quad f(x) \geq 1 \quad (x \geq 1)$$

We define a continuous map $F : \text{Emb}(N, \mathbb{R}^k) \to \text{Emb}^e(N, \mathbb{R}^k)$ by $e \mapsto f(r(e)) \cdot e$ where $r(e)$ is the number given in Definition 3.1 and denotes the component-wise scaler multiplication. A homotopy from $F \circ I$ to $id$ is given by $(t, e) \mapsto \{t + (1 - t)f(r(e))\} \cdot e$, and a homotopy from $I \circ F$ to $id$ is given by the same formula. \qed

**Lemma 4.7.** We use the notations in Definition 4.6. For each $n \geq 1$ and $G \in G(n)$, $\mathcal{T}H_M(n)$ and $\mathcal{T}H_G$ are strongly semistable, and each map in the following zigzags in $\text{Ho}(\mathcal{SP}^E)$ is an isomorphism.

$$\Sigma^nK T\nu_G \xrightarrow{\lambda_G} \mathcal{T}H_G(c_0) \xrightarrow{\Sigma^n\{ T\nu_G/T(\nu^{\times n}|_{\Delta_{fat}(n)}) \}} \mathcal{T}H_G(c_0)/\{\mathcal{T}H_{fat}(c_0)\} \xleftarrow{\Sigma^n\mathcal{T}H_G(c_0)} \mathcal{T}H_M(n).$$

Here, see Introduction for $\Delta_{fat}(n)$, and the right maps are the evaluations at $c_0$.

**Proof.** For simplicity, we shall prove the claim for the maps in the first line for the case of $G = \emptyset$. The same proof works for general $G$ thanks to the assumption on $\rho$ given in subsection 3.2. Set $N = (\tilde{M})^{\times n}$. The evaluation at $c_0$ and the inclusion $\mathcal{T}H_G(c_0) \subset N^{-\tau}$ are clearly level equivalences. So all we have to prove is that $\mathcal{T}H_0$ is strongly semistable and that the composition of $\lambda_G$ and the inclusion, which is also denoted by $\lambda_G : \Sigma^nK T\nu_G \to N^{-\tau}$, is an isomorphism in $\text{Ho}(\mathcal{SP}^E)$. We define a space $\mathcal{E}_k$ by

$$\mathcal{E}_k = \{(e, \epsilon) \mid e \in \text{Emb}^e(N, \mathbb{R}^k), \ 0 < \epsilon < L(e)\}.$$ 

By Lemma 4.6 and Whitney’s theorem, $\mathcal{E}_k$ is $(k/2 - n(2d - 1) - 1)$-connected. Let $P : \tilde{N}^{-\tau}_k \to \mathcal{E}_k$ be the fiber bundle obtained from the obvious projection $\tilde{N}^{-\tau}_k \to \mathcal{E}_k$ by collapsing the complements of $\nu(e)$’s in a fiberwise manner (see Definition 3.1). So each fiber of the map $P$ is a Thom space homeomorphic to $T\nu_G$. $P$ has a section $s : \mathcal{E}_k \to \tilde{N}^{-\tau}_k$ to the basepoints, and there is an obvious homeomorphism

$$\tilde{N}^{-\tau}_k/s(\mathcal{E}_k) \cong N^{-\tau}_k.$$ 

With this observation, by observing the Serre spectral sequence for $P$, we see the composition

$$S^{k-nK} \wedge T\nu_G \xrightarrow{\lambda_G} S^{k-nK} \wedge N^{-\tau}_{nK} \xrightarrow{\text{action of } S} N^{-\tau}_k$$

is $(3k/2 - 2n(2d - 1) - 2)$-connected. This implies $N^{-\tau}$ is strongly semistable and $\lambda_G$ is an isomorphism. The same proof works for general $G$ and the maps in the second line thanks to the assumption on $\rho$ given in subsection 3.2. \qed
Proof of Theorem 4.4. Similarly to the proof of Lemma 4.7 it is easy to see \( S_M \) and \( S'_M \) are strongly semistable, which implies each comodule in the zigzag in Definition 4.11 is strongly semistable, combined with the fact that the spaces \( F^M(n), \tilde{F}^M(n), C([M],(n) \) have homotopy types of finite CW complexes. It is clear that \( p \) and \( q \) are \( \pi_* \)-isomorphisms, so are \( p_* \) and \( q_* \). \( i_0 \) and \( i_1 \) are homotopy equivalences for each \( n \) since \( \tilde{F}^M(n) \) is homotopy equivalent to the mapping cylinder of the inclusion \( C_n([M]) \subset C_n([M]) \) which is also a homotopy equivalence, so \( (i_0)^\vee \) and \( (i_1)^\vee \) are \( \pi_* \)-isomorphisms. \( \Phi_n \) is a \( \pi_* \)-isomorphism since it reduces the equivalence of the original Atiyah duality in the (homotopy) category of classical spectra via Lemma 4.7 (see [8]).

5. Spectral sequences

5.1. A chain functor.

Definition 5.1. • For a chain complex \( C_* \), \( C[k]_* \) is the chain complex given by \( C[k]_* = C_{k-1} \) with the same differential as \( C_* \) (without extra sign.).

• Fix a fundamental cycle \( w_{S^1} \in C_1(S^1) \). We shall define a chain complex \( C^S_*(X) \) for a symmetric spectrum \( X \). Define a chain map \( i^X_k : C_*(X)[k] \to C_*(X_{k+1})[k+1] \) by \( i^X_k(x) = (-1)^k \sigma x(w_{S^1} \times x) \) for \( x \in C_1(X_k) \), where \( \sigma : S^1 \times X \to X_{k+1} \) is the structure map of \( X \). We define \( C^S_*(X) \) as the colimit of the sequence \( \{ C_*(X)[k]; i^X_k \}_{k \geq 0} \). Clearly the procedure \( X \mapsto C^S_*(X) \) is extended to a functor \( SP^S \to CH_k \) in an obvious manner.

• We denote by \( H^S_*(X) \) the homology group of \( C^S_*(X) \).

• Let \( fCW \) denote the full subcategory of \( CH \) spanned by finite CW complexes. We define a functor \( C^S_* \) : \( (fCW)^{op} \to CH_k \) by \( C^S_*(X) = C^S_-(X^o) \).

Lemma 5.2. (1) If \( f : X \to Y \) is a stable equivalence between strongly semistable spectra, the induced map \( f_* : C^S_*(X) \to C^S_*(Y) \) is a quasi-isomorphism.

(2) If \( X \to Y \to Z \) is a homotopy cofiber sequence of strongly semistable spectra, \( C^S_*(X) \to C^S_*(Y) \to C^S_*(Z) \) is a homotopy cofiber sequence in \( CH_k \).

Proof. The first part clearly follows from the Hurewicz Theorem and the fact that stable equivalences between strongly semistable spectra are \( \pi_* \)-isomorphisms. To prove the second part, we take the following replacements, using the factorization of the model category \( SP^S \).

\[
\begin{array}{ccc}
\emptyset & \xrightarrow{f} & X' \\
& \downarrow & \downarrow \phi \\
& X & \to Y
\end{array}
\]

Here, the top horizontal arrows are cofibrations and the vertical arrows are trivial fibrations. Since trivial fibrations in \( SP^S \) are level trivial fibrations, \( X' \) and \( Y' \) are strongly semistable. This implies the cofiber \( Y'/X' \) is strongly semistable and the canonical map \( C^S_*(Y'/X') \to C^S_*(Z) \) is a quasi-isomorphism by the first part. We have completed the proof of the second part.

Lemma 5.3. There exists a zigzag of natural transformations between \( C_* \) and \( C^S_* \) : \( (fCW)^{op} \to CH_k \), in which each natural transformation is an objectwise quasi-isomorphism.

Proof. All we have to do is to verify \( C^S_* \) satisfies the axioms of cochain theory in [29]. (In [29], it was proved that any functor satisfying the axioms can be connected with the singular cochain by a zigzag as in the claim.) The only non-trivial axiom is the Extension/Excision axiom which states if \( X \in fCW \) and \( A \subset X \) is an inclusion of a subcomplex, the map from the homotopy fiber \( F(T(X/A) \to T(\ast)) \) to the homotopy fiber \( F(T(X) \to T(A)) \) is a quasi-isomorphism (see [29] for the notations). We shall verify this axiom for \( T = C^S_* \). For a pointed
space $Y$. Let $Y^\vee$ denote the spectrum given by $(Y^\vee)_k = \text{Map}_*(Y, S^k)$ where $\text{Map}_*$ is the space of based maps. The fiber sequence

$$(X/A)_*^\vee \longrightarrow (X)_*^\vee \longrightarrow (A)_*^\vee$$

is a homotopy fiber sequence in $\mathcal{SP}^\Sigma$ because $A \subset X$ is an inclusion of finite CW-complexes and $S$ is strongly semistable, and also a homotopy cofiber sequence because the two sequences are the same in $\mathcal{SP}^\Sigma$. By (2) of Lemma 5.2, the induced sequence

$$C_\Sigma^S((X/A)_*^\vee) \longrightarrow C_\Sigma^S(X) \longrightarrow C_\Sigma^S(A)$$

is a homotopy cofiber sequence and also a fiber one since the two sequences are the same in $\mathcal{CH}$. By a similar argument, we see the sequence

$$C_\Sigma^S((X/A)_*^\vee) \longrightarrow C_\Sigma^*(X/A) \longrightarrow C_\Sigma^*(\ast)$$

is a homotopy fiber sequence. Thus, we have verified the axiom.

The functor $C_\Sigma^S$ does not have any compatibility with symmetry isomorphisms of the monoidal products $\wedge$ in $\mathcal{SP}^\Sigma$, but it have some compatibility with the tensor $\hat{\otimes}$ with a space.

**Lemma 5.4.** (1) For $U \in \mathcal{CG}$ and $X \in \mathcal{SP}^\Sigma$, the collection of Eilenberg-Zilber shuffle map $\{EZ : C_\Sigma(U \otimes C_\Sigma(X_k))[k] \rightarrow C_\Sigma((U_+) \wedge X_k)[k]\}_k$ induces a quasi-isomorphism

$$C_\Sigma(U) \otimes C_\Sigma^S(X) \rightarrow C_\Sigma^S(U \hat{\otimes} X).$$

(2) Let $O$ be a topological operad and $Y$ be a $O$-comodule in $\mathcal{SP}^\Sigma$. A natural structure of a chain $C_\Sigma(O)$-comodule on the collection $C_\Sigma^S Y = \{C_\Sigma^S(Y(n))\}_n$ is defined as follows. The partial composition is given by the composition

$$C_\Sigma(O(m)) \otimes C_\Sigma^S(Y(m + n - 1)) \rightarrow C_\Sigma^S(O(m) \hat{\otimes} Y(m + n - 1)) \rightarrow C_\Sigma^S(Y(n)),$$

where the left map is the one defined in the first part and the right map is induced by the partial composition on $Y$. The action of $\Sigma_n$ on $C_\Sigma^S(Y)(n)$ is the natural induced one.

**Proof.** The cross product $w_S1 \times x$ is equal to $EZ(w_S1 \otimes x)$ by definition, and the the shuffle maps are associative and compatible with symmetry isomorphisms of monoidal products without any chain homotopy for normalized singular chain, so the maps $EZ$ are compatible with the maps $i^S_k$ in Definition 5.1 (the sign commuting an element of $C_\Sigma(U)$ and $w_S1$ is canceled with the sign attached in the definition of $i^S_k$). This imply the first part. The second part follows from commutativity of the following diagram which is clear from the property of the shuffle map mentioned above.

$$C_\Sigma(U) \otimes C_\Sigma(V) \otimes C_\Sigma^S(X) \longrightarrow C_\Sigma(U) \otimes C_\Sigma^S(V \hat{\otimes} X)$$

$$\downarrow \quad \downarrow$$

$$C_\Sigma(U \times V) \otimes C_\Sigma^S(X) \longrightarrow C_\Sigma^S((U \times V) \hat{\otimes} X),$$

where $U, V \in \mathcal{CG}$ and $X \in \mathcal{SP}^\Sigma$ and the left vertical arrow is induced by the EZ shuffle map and other arrows are given by the first part. 

\qed
5.2. Construction of Čech spectral sequence.

**Definition 5.5.** We shall define a $C_*(D_1)$-comodule $\tilde{T}^M_{*,*}$ of double complexes, which is consisting of the following data.

- A sequence of double complexes $\{\tilde{T}^M_{*,*}(n)\}$ with two differentials $d$ and $\partial$ of degree $(0,1)$ and $(1,0)$ respectively,
- an action of $\Sigma_n$ on $\tilde{T}^M_{*,*}(n)$ which preserves the bigrading, and
- a partial composition $(- \circ_i -) : C_k(D_1(m)) \otimes \tilde{T}^M_{*,*}(m+n-1) \to \tilde{T}^M_{*,*+k}(n)$ which satisfy the following compatibility conditions in addition to the conditions in Definition 2.10

\[
d\partial = \partial d, \quad d(\alpha \circ_i x) = da \circ_i x + (-1)^{|\alpha|} \alpha \circ_i dx, \quad \partial(\alpha \circ_i x) = \alpha \circ_i \partial x.
\]

We define the double complex $\tilde{T}^M_{*,*}(n)$ by

\[
\tilde{T}^M_{p,*}(n) = \bigoplus_{G \in G(n,p)} C^S_n(\mathcal{T}H_G)
\]

for $p \geq 0$, and $\tilde{T}^M_{p,*}(n) = 0$ for $p < 0$, where $G(n,p) \subset G(n)$ is the set of graphs with $p$ edges (see Definition 4.3 for $\mathcal{T}H_G$). The differential $d$ is the original differential of $C^S_n(\mathcal{T}H_G)$. The other differential $\partial$ is given by the signed sum

\[
\partial = \sum_{t=1}^p (-1)^{t+1} \partial_t
\]

where $\partial_t$ is induced by the inclusion $\mathcal{T}H_G \to \mathcal{T}H_{G_t}$, where the graph $G_t$ is defined by removing the $t$-th edge from $G$ (in the lexicographical order). The action of $\sigma$ on $\tilde{T}_{*,*}^M(n)$ restricts to a map $\sigma : \mathcal{T}H_G \to \mathcal{T}H_{\sigma^{-1}(G)}$ (see subsection 2.11 for $\mathcal{T}H_G$). This map induces a chain map $\sigma_* : C^S_*(\mathcal{T}H_G) \to C^S_*(\mathcal{T}H_{\sigma^{-1}(G)})$ by the standard pushforward of chains. For $G \in G(n,p)$, let $\sigma_G \in \Sigma_p$ denote the following composition

\[
P \cong E(\sigma^{-1}(G)) \to E(G) \cong P,
\]

where $\cong$ denotes the order preserving bijection and the middle map is given by $(i,j) \mapsto (\sigma(i), \sigma(j))$. We define the action of $\sigma$ on $\tilde{T}^M_{*,*}(n)$ as $\sigma(\sigma^M) \cdot \sigma_*$ on each summand. We shall define the partial composition. Let $f_i : m+n-1 \to \mathbb{N}$ be the order preserving surjection which satisfy $f(i+t) = f(i)$ for $t = 1, \ldots, m-1$. For elements $\alpha \in C_*(D_1(m))$ and $x \in C^S_{n,m}(\mathcal{T}H_G)$ with $G \in G(n+m-1)$, if $\#E(f_i(G)) = \#E(G)$, the partial composition $\alpha \circ_i x \in C^S_n(\mathcal{T}H_{f_i,G})$ is defined similarly to Lemma 5.4 with the map $(- \circ_i -) : D_1(m) \otimes \mathcal{T}H_G \to \mathcal{T}H_{f_i,G}$, and if $\#E(f_i(G)) < \#E(G)$, $\alpha \circ_i x$ is zero. This partial composition is well-defined by Lemma 3.11.

We have completed the definition of $\tilde{T}^M$. The compatibility between $d$, $\partial$, and $(- \circ_i -)$ is obvious.

Let $\text{Tot} \tilde{T}^M_{*,*}(n)$ denote the total complex. Its differential is given by $d + (-1)^p \partial$ on $\tilde{T}^M_{*,*}(n)$. We regard the sequence $\text{Tot} \tilde{T}^M_{*,*} = \{\text{Tot} \tilde{T}^M_{*,*}(n)\}$ as a chain $C_*(D_1)$-comodule with the induced structure. We fix an operad map $f : A_\infty \to C_*(D_1)$, and regard $\text{Tot} \tilde{T}^M$ as an $A_\infty$-comodule by pulling back the partial compositions by $f$. We consider the Hochschild complex $\mathcal{CH}_* (\text{Tot} \tilde{T}^M_{*,*})$ associated to this $A_\infty$-comodule, see Definition 2.12. The total degree of elements of $\mathcal{CH}_* (\text{Tot} \tilde{T}^M_{*,*})$ is $-\circ_* -\circ_* -$. We define two filtrations $\{F^{-p}\}$ and $\{F^{-p}\}$ on this complex as follows. $F^{-p}$ (resp. $F^{-p}$) is generated by the homogeneous parts whose degree satisfies $\circ_* + \circ_* \leq p$ (resp. $\circ_* \leq p$). We call the spectral sequence associated to $\{F^{-p}\}$ the Čech spectral sequence, in short, Čech s.s. and denote it by $\{\overline{E}^{-p,q}\}_r$. The spectral sequence associated to $\{\overline{E}^{-p}\}_r$ is denoted by $\{\overline{E}^{-p,0}\}_r$.

**Lemma 5.6.** The spectral sequence $\overline{E}_r$ in Definition 5.2 and Sinha spectral sequence $E_r$ in Definition 2.7 are isomorphic after the $E_1$-page.
Proof. Put \( N_0 = \#\{(i, j) \mid i, j \in \mathbb{N}, i < j \} \). By applying Lemma \( \ref{lem:iso} \) to the functor \( X : P_\nu(N_0) = G(n) - \{\emptyset\} \to \mathcal{SP}_\Sigma \) given by \( X_G = TH_G \), we see the map \( \text{Tot} \mathcal{T}_M(n) \to C^S(\mathcal{TH}_M(n)) \) induced by the collapsing (quotient) map \( \mathcal{TH}_M(n) \to \mathcal{TH}_M(n) \) is a quasi-isomorphism. Combining this with Theorem \( \ref{thm:main} \) and Lemma \( \ref{lem:comodules} \) two comodules \( C_\omega \) and \( \mathcal{TH}_M(n) \) are quasi-isomorphic. Clearly, \( \text{CH}_n C^S(\mathcal{C}_n([M])) \) is quasi-isomorphic to the normalized complex of \( C^S(\mathcal{C}_n([M])) \), which is quasi-isomorphic to the normalized total complex of \( C^*(\mathcal{C}_n([M])) \) by Lemma \( \ref{lem:normalized} \). Thus, \( \text{CH}_n \text{Tot} \mathcal{T}_M(n) \) and the normalized total complex of \( C^*(\mathcal{C}_n([M])) \) are connected by a zigzag of quasi-isomorphisms which preserve the filtration. This zigzag induces a zigzag of morphisms of spectral sequences which are isomorphisms after the \( E_1 \)-page because the homology of \( \text{Tot} \mathcal{T}_M(n+1) \) is isomorphic to \( H^*(C^n([M])) \) under the zigzag. \( \square \)

5.3. Convergence. In this subsection, we assume \( M \) is orientable. We shall prepare some notations and terminologies which is necessary to analyze the \( E_1 \)-page of \( 
abla \text{ech} \) s.s.

**Definition 5.7.**

- We fixed an embedding \( e_0 : \widehat{M} \to \mathbb{R}^K \) and a number \( e_0 \) in Definition \( \ref{def:embedding} \). We also fix an isotopy \( \iota_t : \widehat{M} \to \mathbb{R}^2K \) with \( \iota_t = 0 \times e_0 \) and \( \iota_t = \Delta_{2K} \circ e_0 \) where \( 0 \times e_0 : \widehat{M} \to \mathbb{R}^2K \) is given by \( (0 \times e_0)(z) = (0, e_0(z)) \), and \( \Delta_{2K} \) is the diagonal map on \( \mathbb{R}^K \). We impose the additional condition that \( e_0 \) is smaller than \( \min\{L(\iota_t) \mid 0 \leq t \leq 1\} \).

We also fix 1-parameter family of bundle map \( \kappa_t : \nu_{e_0}(0 \times e_0) \to \nu_{e_0}(\iota_t) \) with \( \kappa_0 = \text{id} \).

- We fix the following classes.

\[
\hat{w} \in H_{2d-1}(\widehat{M}), \quad \omega_\Delta \in H^{2d-1}(\widehat{M} \times \widehat{M}, \Delta(\widehat{M})^c), \quad w_{SK} \in H_K(S^K), \quad \omega_{SK} \in H^K(S^K), \quad \omega_\nu \in H^{K-2d+1}(Th(\nu)), \quad \omega(n) \in H^{n(K-2d+1)}(Th(\nu \times n)), \quad \gamma \in H^d(\widehat{M} \times \widehat{M}, (\widehat{M} \times M \widehat{M})^c).
\]

Here, \( \hat{w} \) is a fundamental class of \( \widehat{M} \), and \( \omega_\Delta \) is a diagonal class such that the equality

\[
(\hat{w} \times \hat{w}) \cap \omega_\Delta = \Delta_*(\hat{w}) \in H_{2d-1}(\widehat{M} \times \widehat{M})
\]

holds (\( \Delta(\widehat{M})^c \) is the complement of the tubular neighborhood of the (standard, non-deformed) diagonal), and \( w_{SK} \) is the \( K \) times product \( (w_{SK})^{\times n} \) of the class \( w_{SK} \) fixed in Definition \( \ref{def:sk} \) and \( \omega_{SK} \) is the class such that \( w_{SK} \cap \omega_{SK} \) is the class represented by a point. \( \omega_\nu \) is the Thom class satisfying the equality

\[
\kappa_1^*(\omega_\Delta \cdot (\omega_\nu \times \omega_\nu)) = \omega_{SK} \times \omega_\nu,
\]

where \( \omega_\Delta \cdot (\omega_\nu \times \omega_\nu) \) is naturally regarded as a Thom class for the bundle \( \nu_{e_0}(\Delta_{2K} \circ e_0) \).

We set \( \omega(n) = \omega_\nu \times n \). \( \gamma \) is also a Thom class of a tubuler neighborhood of \( \widehat{M} \times M \widehat{M} \) in \( \widehat{M} \times \widehat{M} \).

- We call a graph in \( G(n) \) which does not contain a cycle (a closed path) a tree. For a graph \( G \in G(n) \), vertices \( i \) and \( j \) are said to be disconnected in \( G \) if \( i \) and \( j \) belong to different connected components of \( G \).

- For \( i < j \), let \( \pi_{ij} : \widehat{M} \times n \to \widehat{M} \times 2 \) be the projection given by \( \pi_{ij}(z_1, \ldots, z_n) = (z_i, z_j) \). Set \( \Delta_{ij} = \Delta_G \) for \( E(G) = \{(i, j)\} \), and

\[
\gamma_{ij} = \pi^{-1}_{ij}(\gamma) \in H^d(\widehat{M} \times (\Delta_{ij})^c).
\]

For a tree \( G \in G(n) \), write \( E(G) \) as \( \{(i_1, j_1) < \cdots < (i_r, j_r)\} \) with \( i_t < j_t \) for \( t = 1, \ldots, r \). We put

\[
w_G = \hat{w} \times n \cap \gamma_{i_1, j_1} \cdots \gamma_{i_r, j_r} \in H_{n(2d-1)-rd}(\Delta_G).
\]

Clearly, \( w_G \) is a fundamental class of \( \Delta_G \).
• Let $G \in G(n, r)$ be a tree. Suppose $i$ and $i + 1$ are disconnected in $G$. Let $d_i : n \to n - 1$ be the map given by

$$d_i(j) = \begin{cases} j & (j \leq i) \\ j - 1 & (j \geq i + 1) \end{cases}$$

and set $H = d_i(G) \in G(n - 1)$. We define maps

$$\phi_G : \hat{H}_*(Th(\nu_G)) \to H^S_{* - nK}(\mathcal{T}H_G), \quad \mu_i : H_*^S(\mathcal{T}H_G) \to H_*^S(\mathcal{T}H_H),$$

$$\zeta_G : H_*^S(\mathcal{T}H_G) \to H_*^{* - d\nu}(\mathcal{T}H_G), \quad m_i : H_*^i(\Delta_G) \to H_*^i(\Delta_H).$$

$\phi_G$ is the composition

$$\hat{H}_*(Th(\nu_G)) \xrightarrow{\lambda_G} \hat{H}_*(\mathcal{T}H_G(\theta_0)nK) \to H^S_{* - nK}(\mathcal{T}H_G(\theta_0)) \to H^S_{* - nK}(\mathcal{T}H_G),$$

where $\lambda_G$ is the map defined in Definition [5,5] and the second map is the canonical one and the third is the inverse of evaluation at $\theta_0$. Clearly $\phi_G$ is an isomorphism. $\zeta_G$ is the composition $(w_G \cap -)^{-1} \circ (- \cap \omega(n)) \circ \phi^{-1}_G$ consisting of

$$H_*^S(\mathcal{T}H_G) \xrightarrow{\mu_i} H^S_{* - nK}(Th(\nu_G)) \xrightarrow{\cap \omega(n)} H^S_{* - 2n + 1}(\Delta_G) \xrightarrow{\nu_G(\cap -)^{-1}} H_*^{* - d\nu}(\mathcal{T}H_G).$$

$\mu_i$ is the map induced by the partial composition $\mu \circ \iota$ where $\mu \in H_0(D_1(2)) = A(2)$ is the fixed generator. $m_i$ is given by $(-1)^A\Delta_i^*$ where $A = * + d\nu + n$ with $r = \#E(G)$, and $\Delta_i^*$ denotes the pullback by the restriction to $\Delta_H$ of the diagonal

$$\Delta_i : \bar{M}^{\times n - 1} \to \bar{M}^{\times n}, \quad (z_1, \ldots, z_{n-1}) \mapsto (z_1, \ldots, z_i, z_{i+1}, \ldots, z_{n-1}).$$

• We denote by $H^S\mathcal{T}^M_{**}(n)$ the bigraded chain complex obtained taking homology of $\mathcal{T}^M_{**}(n)$ for the differential $d$, see Definition [5,5]. Its differential is induced by the differential $-\partial_r$ on $\mathcal{T}^M_{**}(n)$. We regard the collection $H^S\mathcal{T}^M = \{H^S\mathcal{T}^M(n)\}$ as an $A$-comodule with the structure induced by $\mathcal{T}^M$. As a $k$-module, $H^S\mathcal{T}^M(n)$ is the direct sum $\bigoplus_{G \in G(n)} H_*^S(\mathcal{T}H_G)$. We denote by $aG$ the element of $H^S\mathcal{T}^M(n)$ corresponding to $a \in H_*^S(\mathcal{T}H_G)$.

• The homology of the Hochschild complex $CH_*(H^S\mathcal{T}^M_{**})$ has the bidegree $(-, -, -)$. We denote the homogeneous part of bidegree $(p, q)$ by $H_{-p, -q}(CH(H^S\mathcal{T}^M_{**}))$.

• For two graphs $G, H \in G(n)$ with $E(G) \cap E(H) = \emptyset$, the product $GH \in G(n)$ denotes the graph with $E(GH) = E(G) \cup E(H)$. Let $i, j, k \in \mathbb{N}$ be distinct vertices, and $[ijk] \in G(n)$ denote the graph with $E([ijk]) = \{(i, j), (j, k)\}$ for a graph $G \in G(n)$, the products $G[ijk]$ and $G[jki]$ and $G[kij]$ have the same connected component (if they are defined) so we have $\nu_G[ijk] = \nu_G[jki] = \nu_G[kij]$. Using these equalities, and the isomorphisms $\phi_G^r$ for $G' = G[ijk], G[jki], \text{ and } G[kij]$; we identify the three groups $H_*^S(\mathcal{T}H_G[ijk], H_*^S(\mathcal{T}H_G[jki]), \text{ and } H_*^S(\mathcal{T}H_G[kij])$ with one another. Under this identification, let $I(n) \subset H^S\mathcal{T}^M(n)$ be the sub-module generated by

- the summands of graphs which are not trees, and
- the elements of the form $aG[jki] + (-1)^s aG[ijk] + (-1)^{s+t} aG[kij]$ for $(i, j), (j, k), (i, k) \notin E(G)$, where $a \in H_*^S(\mathcal{T}H_G[ijk])$ and $s + 1$ is the number of edges of $G$ between $(i, j)$ and $(i, k)$, and $t + 1$ is the one between $(i, k)$ and $(j, k)$.

• We say a graph $G \in G(n)$ with an edge set $E(G) = \{(i_1, j_1), \ldots, (i_r, j_r)\}$ is distinguished if the following inequalities hold.

$$i_1 < j_1, \ldots, i_r < j_r, \quad i_1 \leq \cdots \leq i_r.$$
Lemma 5.8. With the notations in Definition 5.7, the $E_2$-page of Čech s.s. is isomorphic to the homology of the Hochschild complex of $H^S \hat{T}$. More precisely, there exists an isomorphism of $k$-modules

$$\hat{E}_2^{p,q} \cong H_{-p,-q}(\text{CH}(H^S \hat{T}^M)) \text{ for each } (p,q).$$

\[ \square \]

Lemma 5.9. With the notations in Definition 5.7, $I(n)$ is acyclic, i.e., $H_0(I(n)) = 0$, and the sequence $\{I(n)\}_n$ is closed under the partial compositions and symmetric group actions.

Proof. Since $G(n)_{\text{dis}}$ is stable under removing edges, the submodule $\bigoplus_{G \in G(n)_{\text{dis}}} H^S_G(\mathcal{TH}_G)$ of $H^S \hat{T}^M(n)$ is a subcomplex. By the argument similar to (the dual of) Lemma 5.8, we see the inclusion $\tilde{T}(G(n)_{\text{dis}}) := \bigoplus_{G \in G(n)_{\text{dis}}} H^S_G(\mathcal{TH}_G) \subset H^S \hat{T}^M(n)$ is a quasi-isomorphism. We easily see the map $\tilde{T}(G(n)_{\text{dis}}) \to \tilde{T}(n)/I(n)$ induced by the inclusion is an isomorphism (see the proof of Lemma 5.8).

\[ \square \]

Lemma 5.10. Let $\bar{e}_t : \hat{M} \to \mathbb{R}^{2k}$ be an isotopy with $\bar{e}_0 = 0 \times e_0$ and $\bar{e}_1 = e_0 \times 0$ and $F_t : \nu_\alpha(\bar{e}_0) \to \nu_\alpha(\bar{e}_1)$ be an isotopy which is also a bundle map covering $\bar{e}_t$, and satisfy $F_0 = \text{id}$. Then we have the equality

$$(F_1)^* (\omega_\nu \times \omega_{SK}) = (-1)^K \omega_{SK} \times \omega_\nu.$$

Here $\omega_\nu \times \omega_{SK}$ is considered as a class of $H^{2k-2d+1}(\text{Th}(\nu_\alpha(\bar{e}_1)))$ via the map collapsing the subset $\nu_\alpha(e) \times \mathbb{R}^k - \nu_\alpha(e_1)$ and $\omega_{SK} \times \omega_\nu$ is similarly understood.

Proof. Since the only problem is the orientation, it is enough to see a variation of a basis via a local model. Let $e_0 : \mathbb{R}^{2d-1} \to \mathbb{R}^K$ be the inclusion to the subspace of elements with the last $K - 2d + 1$ coordinates being zero. A covering isotopy is given by $F_t(u,v) = ((1 - t)u - tv, tu + (1 - t)v)$ for $u,v \in \mathbb{R}^K$. Since $F_1(u,v) = (-v,u)$, the derivative $(F_1)^*$ maps a basis $\{a,b\}$ of the tangent space of $\mathbb{R}^{2k}$ to $\{b,-a\}$, where $a$ and $b$ denote basis of $T_{\mathbb{R}^K} \times 0$ and $0 \times T\mathbb{R}^K$ respectively. This implies $(F_1)^*(\omega_\nu \times \omega_{SK}) = (-1)^K (1)^K (-1)^{K-2d+1} \omega_{SK} \times \omega_\nu = (1)^K \omega_{SK} \times \omega_\nu.$

\[ \square \]

Lemma 5.11. We use the notations in Definition 5.7. Let $G \in G(n,r)$ be a tree with $i$ and $i+1$ are disconnected in $G$, and set $H = d_i(G) \in G(n-1)$. Then, the following diagram is commutative.

$$
\begin{array}{cccc}
H^S_G(\mathcal{TH}_G) & \xrightarrow{\mu_i} & H^S_H(\mathcal{TH}_H) \\
\downarrow{\psi_G} & & \downarrow{\varepsilon_1 \psi_H} \\
H_{*-d^r}(\Delta_G) & \xrightarrow{m_i} & H_{*-d^r}(\Delta_H),
\end{array}
$$

where $\varepsilon_1 = (-1)^B$ with $B = K(i + 1 + (K - 1)/2)$.

Proof. The claim follows from the commutativity of the following diagram (x).

$$
\begin{array}{cccc}
H^S_G(\mathcal{TH}_G) & \xrightarrow{\mu_i} & H^S_H(\mathcal{TH}_H) \\
\phi_G & & \phi_H & \phi_H \\
\tilde{H}_{*+N}(\text{Th}(\nu_G)) & \xrightarrow{\mu'} & \tilde{H}_{*+N}(\text{Th}(\nu_H)) \\
\omega(n) & & \omega' & \varepsilon_1 \omega(n-1) \\
H_{*+N(2d-1)}(\Delta_G) & \xrightarrow{\mu''} & H_{*+(n-1)(2d-1)}(\Delta_H) \\
\omega_G & & \omega_H \\
H_{*-d^r}(\Delta_G) & \xrightarrow{m_i} & H_{*-d^r}(\Delta_H)
\end{array}
$$
Here,

- $\nu'$ is the disk bundles over $\Delta_H$ of fiber dimension $nK - (n - 1)(2d - 1)$ defined by
  $$\nu' = \nu_0(e^n_0 \circ \Delta_i)|_{\Delta_H},$$
  where the restriction is taken as a disk bundle over $\tilde{M}^{x \times n - 1}$ and see Definition [5.7] for $\Delta_i$.
- $\omega'$ is defined based on the definitions in [20]. With these rules, the commutativity of the usual diagonal. We shall prove commutativity of the two triangles in the same diagram. The compatibility of cross and cap products for which we use the following rule
  $$\frac{\omega' = (-1)^{C}\omega_{\nu}}{\nu} \times (\omega_{\nu} \times \omega_{\nu}) \times (\omega_{\nu})^{x \times n - i - 1} \text{ with } C = (n + i + 1)K.$$
- $\phi'_H$ is defined using the following map $\lambda'_H$ similarly to $\phi_H$.
  $$\lambda'_H : \nu^{x \times n} \ni u \mapsto (e^n_0 \circ \Delta_i, e^0_0, u) \in TH(c_0)_{nK}.$$
- $\mu'$ is the map collapsing the subset $\nu_G - \nu'$, where $\nu'$ and $\nu_G$ are regarded as subsets in $\mathbb{R}^{nK}$.
- $\mu''$ is the composition
  $$H_s(\Delta_G) \to H_s(\Delta_G, \Delta_i(\Delta_H)^c) \to H_{s-2d+1}(\Delta_i(\Delta_H)) \cong H_{s-2d+1}(\Delta_H).$$
  Here, the first map is the standard quotient map, and the third is the inverse of the diagonal, and the second is the cap product with the class
  $$(-1)^{i+1+n}1 \times \cdots \times \omega_{\Delta} \times \cdots \times 1 \quad \omega_{\Delta} \text{ in the $i$-th factor }.$$
- $\alpha$ is the composition $(1 \times \kappa_1 \times 1)_s \circ T \circ (\varepsilon_2 w_{SK} \times -)$ of the maps
  $$\tilde{H}_s(Th(\nu_H)) \xrightarrow{\varepsilon_2 w_{SK} \times -} \tilde{H}_{s+K}(S^K \wedge Th(\nu_H)) \xrightarrow{T} \tilde{H}_{s+K}(Th(\nu'')) \xrightarrow{(1 \times \kappa_1 \times 1)_s} \tilde{H}_{s+K}(Th(\nu')),$$
  where $\lambda''$ is the disk bundle over $\Delta_H$ of the same fiber dimension as $\nu'$ given by
  $$\nu'' = \nu_0(e''|_{\Delta_H}), \text{ with } e'' = e^n_0 \times (0 \times e) \times e^{x \times n - i} : \tilde{M}^{x \times n - 1} \to \mathbb{R}^{nK},$$
  and
  $$\varepsilon_2 = (-1)^D, \quad D = K(\ast + (K - 1)/2 + i + 1),$$
  and $T$ is the composition of the transposition of $S^K$ from the first to the $i$-th component with the map induced by the map collapsing the subset $(\nu^{x \times i} \times \mathbb{R}^K \times \nu^{x \times n - i - 1})|_{\Delta_H - \nu''}$, and $1 \times \kappa_1 \times 1$ is induced by the restriction of the product map
  $$1 \times \kappa_1 \times 1 : \mathbb{R}^{(i-1)K} \times \nu_0(0 \times e_0) \times \mathbb{R}^{(n-i-1)K} \to \mathbb{R}^{(i-1)K} \times \nu_0(\Delta_{RK} \circ e_0) \times \mathbb{R}^{(n-i-1)K}$$
  with $\kappa_1$ in $i$-th component.
- The arrows to which a (co)homology class assigned denote the cap products with the class.

Our sign rules for graded products are the usual graded commutativity except for the compatibility of cross and cap products for which we use the following rule
  $$(a \times b) \cap (x \times y) = (-1)^{|a||x||y|} (a \cap x) \times (b \cap y).$$

These are the rules based on the definitions in [20]. With these rules, the commutativity of the squares in the diagram (*s) is clear since the map $\Delta'$ defined in subsection 3.2 is isotopic to the usual diagonal. We shall prove commutativity of the two triangles in the same diagram. The commutativity of the upper triangle follows from the commutativity of the following diagram.

$$\begin{align*}
\tilde{H}_1(Th(\nu_H)) & \xrightarrow{T(e_2 w_{SK} \times -)} \tilde{H}_{1+K}(Th(\nu'')) \\
& \xrightarrow{(1 \times \kappa_1 \times 1)_s} \tilde{H}_{1+K}(Th(\nu')) \\
& \xrightarrow{(\lambda''_H)_s} \tilde{H}_1(Th_{H}(c_0)_{n-1}K) \\
& \xrightarrow{e_2 w_{SK} \times -} \tilde{H}_{1+K}(Th_{H}(c_0)_{nK}).
\end{align*}$$
Here, $\lambda_H''$ is given by $u \mapsto \left( e_0^{x_i} \times (0 \times e_0) \times e_0^{x_n-i}, e_0, u \right)$. Commutativity of the left trapezoid follows from Lemma 5.10 (the sign $\varepsilon_2$ is the product of the sign in $i^X_k$ in Definition 5.1 and the sign in Lemma 5.10, and that of the right triangle follows from homotopy between $\lambda_H' \circ \kappa_1$ and $\lambda_H''$ constructed from the isotopy $\kappa_t$ in Definition 5.5. We shall show the lower triangle is commutative. We see

$$
\varepsilon_1 \alpha(x) \cap \omega' = \left\{ (\kappa_1)_* T_* (w_{SK} \times x) \right\} \cap (\omega \times \cdots \omega \Delta (\omega \times \omega) \cdots \times \omega) \\
= \left\{ (\kappa_1)_* T_* (w_{SK} \times x) \right\} \cap (\omega \times \cdots (\kappa_1^{-1})^*(\omega_{SK} \times \omega) \cdots \times \omega) \\
= (\kappa_1)_* T_* \left\{ (w_{SK} \times x) \cap (\omega \times \cdots (\omega_{SK} \times \omega) \cdots \times \omega) \right\} \\
= (\kappa_1)_* T_* \left\{ (w_{SK} \times x) \cap \omega_{SK} \times \omega \times \cdots \times \omega \right\} \\
= x \cap \omega(n-1).
$$

Here, $(\kappa_1)_*$ is abbreviation of $(1 \times \kappa_1 \times 1)_*$ and $\omega$ of $\omega_{ij}$. All capped classes are considered as elements of the homology of the base space $\Delta_H$ of involved disk bundles by projections. The second equality follows from the definition of $\omega_{ij}$. As an endomorphism on the base space, $T_*$ and $(1 \times \kappa_1 \times 1)_*$ are the identity hence the sixth equality holds.

The following lemma is easily verified and a proof is omitted.

**Lemma 5.12.** Let $G \in G(n, r)$ be a tree and $K \in G(n, r - 1)$ be the tree made by removing the $t$-th edge $(i, j)$ from $G$. Under the notations in Definition 5.7, the following diagram is commutative.

$$
\begin{array}{ccc}
H^*_S(TH_G) & \xrightarrow{\zeta_G} & H^*_S(TH_K) \\
\downarrow & & \downarrow \zeta_K \\
H^{-s-dr}(\Delta_G) & \xrightarrow{H^{-s-d(r-1)}(\Delta_K)} & H^{-s-d(r-1)}(\Delta_K)
\end{array}
$$

where the top horizontal arrow is induced by the inclusion and the bottom one is given by $(-1)^{(r-t)d} \Delta_{ij}^1$ with $\Delta_{ij}(x) = \gamma_{ij} \cdot x$. □

**Definition 5.13.** In the following, we deal with modules $X$ which is a direct sum of submodules labeled by graphs in $G(n)$ we denote by $X^{tr} \subset X$ the modules consisting of the summands labeled by trees in $G(n)$.

- We define a $A$-comodule $\tilde{A}_M^*$ of CDBA (see Definition 2.1). Put $H^*_G = H^*(\Delta_G)$. Let $\wedge (g_{ij})$ be the free bigraded commutative algebra generated by elements $g_{ij}$ ($1 \leq i < j \leq n$) with bidegree $(-1, d)$. For notational convenience, we set $g_{ij} = (-1)^d g_{ji}$ for $i > j$ and $g_{ii} = 0$. For $G \in G(n)$ with $E(G) = \{(i_1, j_1) \cdots \cdots (i_r, j_r)\}$, we set $g_G = g_{i_1, j_1} \cdots \cdots g_{i_r, j_r}$. Put

$$
\tilde{A}_M^*(n) = \bigoplus_{G \in G(n)} H^*_G g_G.
$$

Here, $H^*_G g_G$ is a copy of $H^*_G$ with degree shift. For $G \in G(n, r)$ and $a \in H^*_G$, the bidegree of the element $ag_G \in \tilde{A}_M(n)$ is $(-r, l + dr)$. We give a graded commutative multiplication on $\tilde{A}_M(n)$ as follows. For $a \in H^*_G, b \in H^*_H$, we set

$$
(ag_G) \cdot (bg_H) = \begin{cases} 
(\varepsilon)^{mr(d-1)+s} (a \cdot b) g_{GH} & \text{if } (E(G) \cap E(H) = \emptyset) \\
0 & \text{otherwise}
\end{cases}
$$

Here, we set $r = \#E(G)$, and $a$ is regarded as an element of $H^*_G$ by pulling back by the map $i_G : \Delta_{GH} \rightarrow \Delta_G$ induced by the quotient map $\pi_0(G) \rightarrow \pi_0(GH)$, and
similarly for \(b\), and the product \(a \cdot b\) is taken in \(H^*_GH\). \(s\) is the number determined by the equality \(\gamma_G: g_H = (-1)^s g_H\) for the product in \(\wedge\).

Let \(J(n) \subset \tilde{A}_M(n)\) be the ideal generated by the following element

\[
a(g_{ij}g_{jk} + g_{kj}g_{ki} + g_{ki}g_{ij})g_G, \quad bg_K
\]

where \(G, K \in G(n), \ a \in H^*_G[ijk], \ b \in H^*_K\) are elements such that \((i, j), (j, k), (k, i) \notin E(G)\), and \(K\) is not a tree. Here, by definition, \(\Delta_G\) depends only on \(\pi_0(G)\), so we have \(\Delta_G[ijk] = \Delta_{G[jki]} = \Delta_{G[kij]}\). With these identities, we regard \(a\) as an element of \(H^*_G[ijk] = H^*_G[kij]\), and the first type of the generators as an element of \(H^*_G[ijk]g_G[ijk] \oplus H^*_G[jki]g_G[jki] \oplus H^*_G[kij]g_G[kij]\).

We define an algebra \(A^*_M(n)\) as the following quotient.

\[
A^*_M(n) = \tilde{A}^*_M(n)/J(n)
\]

Since the restriction of the quotient map \(\tilde{A}_M(n)^{tr} \to A_M(n)\) is surjective, we may define a differential, a partial composition, and an action of \(\Sigma_n\) on the sequence \(A_M = \{A_M(n)\}_n\) through \(\tilde{A}_M(n)^{tr}\). We define a map \(\hat{\partial}: \tilde{A}_M(n)^{tr} \to A_M(n)^{tr}\) by

\[
\hat{\partial}(ag_G) = \sum_{t=1}^r (-1)^{(l+t-1)(d-1)}\Delta^t_{ij} \cdot (a)g_{i1,j1} \cdots g_{it,jt} \cdots g_{tr,jr} \quad (G \in G(n), \ a \in H^I(n))
\]

where \(\Delta^t_{ij}(a) = \gamma_{ij} \cdot a \) and \(\hat{g}_{ij}\) means removing \(g_{ij}\). It is easy to see \(\hat{\partial}(\tilde{A}_M(n)^{tr} \cap J(n)) \subset \tilde{A}_M(n)^{tr} \cap J(n)\). We denote the differential \(\partial\) on \(A_M(n)\) to be the map induced by \(\hat{\partial}\). For the generator \(\mu \in A(2)\) fixed in Definition 5.7, and an element \(ag_G \in \tilde{A}_M(n)^{tr}\), we define the partial composition \(\mu \circ_i (ag_G)\) by

\[
\mu \circ_i (ag_G) = \begin{cases} \Delta^t_i(a)g_H & \text{if } i \text{ and } i + 1 \text{ are disconnected in } G, \\ 0 & \text{otherwise}, \end{cases}
\]

where \(H = d_1(G)\) (see Definition 5.7). The action of \(\sigma \in \Sigma_n\) on \(\tilde{A}_M(n)^{tr}\) is given by \((ag_G)^\sigma = a^\sigma (g_G)^\sigma\) where \(a^\sigma\) is the pullback of \(a\) by \((\sigma_G)^{-1}\) (see Definition 5.5) and \((g_G)^\sigma\) denotes \(g_{r1} \circ_r \cdots g_{ri} \circ_r (\sigma_r)\) with \(r = \sigma^{-1}\). The partial composition and the action of \(\Sigma_n\) on \(\{\tilde{A}_M(n)^{tr}\}_n\) are easily seen to preserve the submodule \(\{J(n) \cap \tilde{A}_M(n)^{tr}\}_n\) and induces a structure of a \(A\)-comodule on \(A_M\).

• Let \(s_i: n \rightarrow n+1\) denote the order preserving monomorphism skipping \(i + 1\) for \(1 \leq i \leq n\). \(s_i\) naturally induces the monomorphism \(s_i: \pi_0(G) \rightarrow \pi_0(s_iG)\) (see subsection 2.1), which in turn, induces \((s_i)^*: \Delta^{s_i}_G \rightarrow \Delta_G\). Let \(s_i\) also denote the induced map \((s_i)^*: H^*(\Delta_G) \rightarrow H^*(\Delta^{s_i}_G)\). By further abuse of notations, we also denote by \(s_i\) the map \(A_M(n) \rightarrow A_M(n+1)\) given by \(s_iag_G = s_i(a)g_{siG}\).

• Define a simplicial CDBA \(A^{**}_n(M)\) (a functor from \((\Delta)^{op}\) to the category of CDBA’s) as follows. We set

\[
A^{**}_n(M) = A^{**}_n(n+1).
\]

If we consider an element of \(A_M(n+1)\) as an element of \(A_n(M)\), we relabel its subscripts with \(0, 1, \ldots, n\) instead of \(1, 2, \ldots, n + 1\). For example, \(g_{01} \in A_n(M)\) corresponds to \(g_{12} \in A_M(n+1)\). The partial compositions and the maps \(s_i\) (defined in the previous item) are also considered as beginning with \((-o_0-)\) and \(s_0\) (originally written as \((-o_1-)\) and \(s_1\)). The face map \(d_1: A_n(M) \rightarrow A_{n-1}(M)\) \((0 \leq i \leq n)\) is given by \(d_1 = \mu \circ_{i}(-)\) \((i < n)\) and \(d_n = \mu \circ_0(-)\) \((0 \leq i \leq n)\). The degeneracy map \(s_i: A_n(M) \rightarrow A_{n+1}(M)\) \((0 \leq i \leq n)\) is the map defined in the previous item.

**Lemma 5.14.** Let \(i, j, k\) be numbers with \(i < j < k\). The equalities \(\gamma_{ij} \gamma_{ik} = \gamma_{ij} \gamma_{jk} = \gamma_{ik} \gamma_{jk}\) hold.
Proof. The three classes are Thom class in \( H^*(\tilde{M}^{\times n}, \Delta_{ijk}^c) \). So to prove the equality, it is enough to identify the corresponding orientations. This is easily done by observing the corresponding basis.

**Theorem 5.15.** Suppose \( M \) is orientable.

1. The two \( A \)-comodule \( H^*\tilde{T}^M_n \) and \( A^*_n \) of differential bigraded \( k \)-modules are quasi-isomorphic in a manner where \( H^*\tilde{T}^M_{-p,-q} \) and \( A^p_q \) correspond for integers \( p, q \). (For \( H^*\tilde{T}^M \), see Definition 5.7.)

2. The Eq-2-page of Čech s.s. in Definition 5.13 is isomorphic to the total homology of the normalized complex \( NA^*_n(M) \). The homogeneous part \( \tilde{E}'_{p,q} \) consists of the summands whose degrees satisfy \( p = * - \bullet, \ q = * \).

Proof. For the part 1, we consider the composition

\[
H^*(\mathcal{T}H_G) \xrightarrow{\varepsilon_3} H^*_G \xrightarrow{d_G} H^*_G \ .
\]

The right map is given by \( a \mapsto \varepsilon_3 ag_G \) with the sign

\[
\varepsilon_3 = (-1)^E, \quad E = E(s', n, r) = s'(n + dr) + d(r) + n(n + 1)/2 + dr(r + 1)/2
\]
on \( H^*_G \). This composition defines an isomorphism as bigraded \( k \)-modules between \( H^*\tilde{T}^M(n)^{tr} \) and \( A^*_n(n)^{tr} \). By Lemma 5.14 this isomorphism maps \( H^*\tilde{T}^M_{-s,-s}(n)^{tr} \cap I(n) \) into \( A^*_n(n)^{tr} \cap J(n) \) isomorphically. A quasi-isomorphism \( H^*\tilde{T}^M(n) \rightarrow A^*_n(n) \) is defined by the following composition

\[
H^*\tilde{T}^M_{-s,-s}(n) \rightarrow H^*\tilde{T}^M_{-s,-s}(n)/I(n) \cong H^*\tilde{T}^M_{-s,-s}(n)^{tr}/\{H^*\tilde{T}^M_{-s,-s}(n)^{tr} \cap I(n)\} \\
\cong A^*_n(n)^{tr}/\{A^*_n(n)^{tr} \cap J(n)\} \cong A^*_n(n) .
\]

where the first map is the quotient map which is a quasi-isomorphism by Lemma 5.9, the second and forth maps are induced by inclusions, the third map is the isomorphism defined above. For the above number \( E \), we see

\[
E(s', n - 1, r) - E(s', n, r) \equiv s' + dr + n \\
E(s' + d, n, r - 1) - E(s', n, r) \equiv (s' + 1)d
\]

module 2. Now we shall take the integer \( K \) as a multiple of 4. With this assumption and the above equalities for \( E \), compatibility of the quasi-isomorphism with the partial composition follows from the Lemma 5.11 as \( \varepsilon_1 = 1 \). Compatibility with the (Čech) differentials follows from Lemma 5.12. Compatibility with the actions of \( \Sigma_n \) is clear. The sign \( \text{sgn}(\sigma_G) \) in Definition 5.5 the sign occurring in permutation of \( \gamma_{ij} \) and the sign occurring in permutation of \( g_{ij} \) are cancelled. Thus the isomorphism is a morphism of \( A \)-comodule. For the part 2, by part 1, the Eq2-page is isomorphic to the homology of Hochschild complex \( CH_*(A_M) \), which is isomorphic to unnormalized total complex of \( A_*(M) \) so is quasi-isomorphic to the normalized complex.

Sinha proved the convergence of his spectral sequences using the Cohen-Taylor spectral sequences. Here, we prove the convergence of our spectral sequences simultaneously by an independent method.

**Theorem 5.16.** If \( M \) is simply connected, both of the Čech s.s. and Sinha s.s. for \( M \) converge to \( H^*(\text{Emb}(S^1, M)) \).

Proof. We set a number \( s_d \) by \( s_d = \min \{d/2 \} \). If \( d \geq 4 \), clearly \( s_d > 1 \). Recall that \( \{\tilde{E}_r\} \) denotes the Čech s.s. By Lemma 5.6 we identify the Sinha-s.s. with the spectral sequence \( \tilde{E}_r \). We shall first show the claim that \( \tilde{E}_2^{p,q} = 0 \) if \( q/p < s_d \). If a graph \( G \in G(n + 1) \) has \( k \) discrete vertices, \( H^*(\Delta_G) \) is isomorphic to \( H^*(\tilde{M})^{\otimes k} \otimes H^*(\Delta_{G'}) \otimes \{\text{torsions}\} \), where \( G' \in G(n + 1 - k) \) is the graph made by removing discrete vertices. With this observation and
simple connectivity of $M$, we see that generators of the normalization $NA_n(M)$ are presented as $a_1 \cdots a_k b g G$ where $G$ is a graph in $G(n + 1)$ with $r$ edges and $k$ discrete vertices except for the vertex 0, and $a_i \in H^{\geq 2}(\hat{M})$ considered as $t$-th discrete component, and $b \in H^r_{G}$. We may ignore the torsion part in estimation of degree by the universal coefficient theorem. The bidegree $(-p, q)$ of this element satisfies conditions $p = n + r$, and $q \geq 2k + rd$. Clearly we have $k + 2r \geq n + \epsilon$ with $\epsilon = 0$ or 1 according to whether the vertex 0 has valence 0 in $G$. With this, if $d \leq 5$, we have the following estimation.

\[
\frac{q}{p} - \frac{d}{3} \geq \frac{(6k + 3r - p)d}{3(n + r)} \geq \frac{(6 - d)k + d\epsilon}{3(n + r)} \geq 0
\]

If $d \geq 6$, we have the following estimation.

\[
\frac{q}{p} - 2 = \frac{2\epsilon + (d - 6)r}{n + r} \geq 0
\]

We have shown the claim. Since the filtration $\{F^{-p}\}$ of Čech s.s. is exhaustive, and the total homology of each $F^{-p}$ is of finite type, the Čech s.s. $\{E_r\}_r$ converges to the total homology $H(NA_\bullet(M))$ of the normalized complex. By the same reason, $\{E_r\}$ also converges to $H(NA_\bullet(M))$. We shall show $\hat{E}^{-p,q}_r = 0$ if $q/p < s_d$ for sufficiently large $r$ Suppose there exists a non-zero element $x \in \bigoplus_{k, p} E^{-p,q}_k$ with $q/p < s_d$. $x$ is considered as an element of $(\hat{F}^{-p}/\hat{F}^{-p+1})H(NA_\bullet(M))$. Take a class $x'$ in $\hat{F}^{-p}H(NA_\bullet(M))$ representing $x$. Take the smallest $p'$ such that $\hat{F}_{p'}H(NA_\bullet(M))$ contains $x'$. so $\hat{E}_{\infty,-q+p'-p}'$ is not zero and $p' \geq p$ as $\hat{F}^{-p} \supset \hat{F}^{-p}$. In the coordinate plane of bidegree, $x'$ and $x$ are on the same line of the form $-p + q = \text{constant}$. This and $p' \geq p$ imply the 'slope' of $x'$ is smaller than $s_d$, which contradicts to the claim. This vanishing result on $\hat{E}_r$ and (cohomology version of) Theorem 3.4 in [5] imply the convergence of $\hat{E}_r$ and $\hat{E}_r$ to $H^*(\text{Emb}(S^1, M))$. □

6. Algebraic presentations of $E_2$-page of Čech spectral sequence

In this section, we assume $M$ is oriented and simply connected and $H^*(M)$ is a free $k$-module.

**Definition 6.1.**  
- A **Poincaré algebra of dimension** $d$ is a graded commutative algebra $\mathcal{H}^{\ast}$ with a linear isomorphism $\epsilon : \mathcal{H}^d \to k$ such that the bilinear form defined as the composition

\[
\mathcal{H}^{\ast} \otimes \mathcal{H}^{\ast} \xrightarrow{\text{multiplication}} \mathcal{H}^{\ast} \xrightarrow{\text{projection}} \mathcal{H}^d \xrightarrow{\epsilon} k,
\]

induces an isomorphism $\mathcal{H}^{\ast} \cong (\mathcal{H}^{d-\ast})^\vee$. We call $\epsilon$ the orientation of $\mathcal{H}$.

- We denote by $\Delta_{\mathcal{H}}$ the diagonal class for $\mathcal{H}^{\ast}$ given by

\[
\sum_i (-1)^{|a|^d} a_i \otimes a_i^* \in (\mathcal{H} \otimes \mathcal{H})^d
\]

where $\{a_i\}$ and $\{a_i^*\}$ are two basis of $\mathcal{H}^{\ast}$ such that $\epsilon(a_i \cdot a_j^*) = \delta_{ij}$, the Kronecker delta. This definition does not depend on a choice of a basis $\{a_i\}$.

- Let $\mathcal{H}$ be a Poincaré algebra $\mathcal{H}$ of dimension $d$ with $H^1 = 0$. We set $\mathcal{H}^{\leq d-2} = \bigoplus_{p \leq d-2} \mathcal{H}^p$ and $\mathcal{H}^{\geq 2} = \bigoplus_{p \geq 2} \mathcal{H}^p$ and define a graded $k$-module $\mathcal{H}^{\geq 2}[d-1]$ by $(\mathcal{H}^{\geq 2}[d-1])^p = X_p^{-d+1}$ with $X^d = \mathcal{H}^{\geq 2}$. We denote by $a$ the element in $(\mathcal{H}^{\geq 2}[d-1])^p$ corresponding to $a \in \mathcal{H}^{p-d+1}$. We define a Poincaré algebra $S\mathcal{H}$ of dimension $2d - 1$ as follows. As graded $k$-module, we set

\[
S\mathcal{H}^{\ast} = \mathcal{H}^{\leq d-2} \oplus \mathcal{H}^{\geq 2}[d-1].
\]

For $a, b \in \mathcal{H}^{\leq d-2}$ the multiplication $a \cdot b$ in $S\mathcal{H}$ is the one in $\mathcal{H}$ except for the case $|a| + |b| = d$, in which we set $a \cdot b = 0$. We set $a \cdot b = ab$ for $a \in \mathcal{H}^{\leq d-1}$, $b \in \mathcal{H}^{\geq 2}$, and
\[ a \cdot b = 0 \] for \( a, b \in \mathcal{H}^{2d} \). We give the same orientation on \( S\mathcal{H} \) as on \( \mathcal{H} \) via the identity \( S\mathcal{H}^{2d-1} = \mathcal{H}^d \).

- We regard \( \mathcal{H} = H^*(M) \) as a Poincaré algebra with the orientation

\[ H^d(M) \xrightarrow{w_M} H_0(M) \cong k, \]

where \( w_M \) is the fundamental class of \( M \) determined by the orientation on \( M \) and the latter isomorphism sends the class represented by a point to 1.

The following lemma is obvious.

**Lemma 6.2.** With the notations of Definition 6.7 let \((b_{ij})_{ij}\) denote the inverse of the matrix \((\epsilon(a_i \cdot a_j))_{ij}\). Then we have

\[ \Delta_{\mathcal{H}} = \sum_{i,j} (-1)^{|a_i|} b_{ij} a_i \otimes a_j. \]

\[ \square \]

Under some assumption, \( S\mathcal{H} \) is isomorphic to \( H^*(\tilde{M}) \) (see the proof of Lemma 6.6), and the algebras \( A_{\mathcal{H},G}, B_{\mathcal{H},G} \) are isomorphic to \( H^*(\Delta_G) \).

**Definition 6.3.** For a Poincaré algebra \( \mathcal{H} \) of dimension \( d \) and graph \( G \in \mathbb{G}(n) \), Define a graded commutative algebra \( A_{\mathcal{H},G} \) as follows.

\[ A_{\mathcal{H},G} = \mathcal{H} \otimes \pi_0(G) \otimes \bigwedge \{ y_1, \ldots, y_n \}, \quad \deg y_i = d - 1 \]

Here we regard \( \pi_0(G) \) as an ordered set by the minimum in each component, and the tensor product is taken in this order. Furthermore, we also define a graded commutative algebra \( B_{\mathcal{H},G} \) as follows

\[ B_{\mathcal{H},G} = S\mathcal{H} \otimes^n \bigwedge \{ y_{ij} \mid 1 \leq i, j \leq n, \ i \sim_G j \} / J_G, \quad \deg y_{ij} = d - 1 \]

Here, \( i \sim_G j \) means the vertices \( i \) and \( j \) belong to the same connected component of \( G \), and \( J_G \) is the ideal generated by the following relation.

\[ \left\{ \begin{array}{ll} e_i(a) - e_j(a), & e_i(a) - e_j(a) - ay_{ij}, \\ e_i(b) - e_j(b), & y_{ii}, y_{ij} + y_{jk} - y_{ik} & \end{array} \right. \]

where \( a \in \mathcal{H}^{\leq d-2} \) and \( b \in \mathcal{H}^d \). We set \( i < j \) if \( a \in \mathcal{H}^0 \).

For \( i < j \), let \( f_{ij} : \mathcal{H}^{\otimes 2} \to \mathcal{H}^{\otimes n} \) denote the map given by

\[ f_{ij}(a \otimes b) = 1 \otimes \cdots \otimes a \otimes \cdots \otimes b \otimes \cdots \otimes 1 \]

(\( a \) is the \( i \)-th factor, \( b \) is the \( j \)-th factor and the other factors are 1). We set

\[ \Delta_{\mathcal{H}}^{ij} = f_{ij}(\Delta_{\mathcal{H}}) \in \mathcal{H}^{\otimes n}. \]

We sometimes regard \( \Delta_{\mathcal{H}}^{ij} \) as an element of \( (S\mathcal{H})^{\otimes n} \) via the projection and inclusion \( \mathcal{H} \to \mathcal{H}^{\leq d-1} \subset S\mathcal{H} \). We also regard it as an element of \( A_{\mathcal{H},G} \) for a graph \( G \) via the map \( \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes \pi_0(G)} \) given by multiplication of factors in the same components with the standard commuting signs. Similarly, \( \Delta_{\mathcal{H}}^{ij} \) and \( \Delta_{S\mathcal{H}}^{ij} \) are regarded as elements of \( B_{\mathcal{H},G} \).

As a graded algebra, \( B_{\mathcal{H},G} \) is isomorphic to \( (S\mathcal{H})^{\otimes \pi_0(G)} \bigwedge \{ y_{ij} \mid i \sim_G j \} / (y_{ii}, y_{ij} + y_{jk} - y_{ik}) \) but we need the presentation to describe maps induced by identifying vertices and removing edges.

Proof of the following lemma is easy and omitted.
**Lemma 6.4.** Consider the Serre spectral sequence for a fibration

\[ F \to E \to B \]

with the base simply connected and the cohomology groups of the fiber and base finitely generated in each degree. If for each \( k \), there exists at most single \( p \) such that \( E^p_{\infty} \neq 0 \), the quotient map \( F^p \to F^p/F^{p+1} \) has a unique section which preserves cohomological degree. Gathering these sections for all \( p \), one can define an isomorphism of graded algebra \( E_\infty \to H^*(E) \). We refer to this isomorphism the canonical isomorphism. The canonical isomorphisms are natural for maps between fibrations satisfying the above assumption. \( \square \)

In the rest of paper, we regard the Euler number \( \chi(M) \) as an element of the base ring \( k \) via the ring map \( \mathbb{Z} \to k \), and \( k^\times \subset k \) denotes the subsets of the invertible elements.

**Lemma 6.5.** We use the notations \( \Delta_i \), \( \Delta_{ij} \), \( s_i \), and \( \Delta_{ij}^1 \) given in Definitions 5.7 and 5.13. Suppose \( \chi(M) = 0 \in k \). Set \( H^* = H^*(M) \). There exists a family of isomorphisms of graded algebras

\[ \{ \varphi_G : A_{H,G} \xrightarrow{\cong} H^*(\Delta_G) \mid n \geq 1, G \in G(n) \} \]

which satisfies the following conditions.

1. Let \( G \in G(n) \) be a tree with \( i \) and \( i + 1 \) disconnected. Set \( H = d_i(G) \). The following diagram is commutative.

\[
\begin{array}{ccc}
A_{H,G}^* & \xrightarrow{\varphi_G} & H^*(\Delta_G) \\
\downarrow{\Delta_i} & & \downarrow{\Delta_i^1} \\
A_{H,H}^* & \xrightarrow{\varphi_H} & H^*(\Delta_H).
\end{array}
\]

Here, the algebra map \( \tilde{\Delta}_i \) is defined as follows. For \( a_1 \otimes \cdots \otimes a_p \in H_{\circ \pi_0(G)} \), we set \( \tilde{\Delta}_i(a_1 \otimes \cdots \otimes a_p) = \pm a_1 \otimes \cdots \otimes a_s \cdot a_t \otimes \cdots a_p, \) \( \tilde{\Delta}_i(y_j) = y_{j'} \) with \( j' = d_i(j) \).

Here, \( s, t \in \pi_0(G) \) are the connected components containing \( i \), \( i + 1 \) respectively, and \( \pm \) is the standard sign in transposing graded elements.

2. For a graph \( G \in G(n) \), set \( S = s_i(G) \). The following diagram is commutative.

\[
\begin{array}{ccc}
A_{H,G}^* & \xrightarrow{\varphi_G} & H^*(\Delta_G) \\
\downarrow{s_i} & & \downarrow{s_i} \\
A_{H,S}^* & \xrightarrow{\varphi_S} & H^*(\Delta_S).
\end{array}
\]

Here, \( \tilde{s}_i \) is given by insertion of the unit 1 as the factor of \( H_{\circ \pi_0(G)} \) which corresponds to the component containing \( i + 1 \) and skip of \( i + 1 \) in the subscripts.

3. For a graph \( G \in G(n) \) and a permutation \( \sigma \in \Sigma_n \), the following diagram is commutative.

\[
\begin{array}{ccc}
A_{H,G}^* & \xrightarrow{\varphi_G} & H^*(\Delta_G) \\
\downarrow{\bar{\sigma}} & & \downarrow{\sigma} \\
A_{H,\tau(G)}^* & \xrightarrow{\varphi_{\tau(G)}} & H^*(\Delta_{\tau(G)}).
\end{array}
\]

Here \( \tau = \sigma^{-1} \), and the right vertical arrow is induced by the natural permutation of factors of the product \( \Delta_{\tau(G)} \to \Delta_G \) and for the left vertical arrow, \( \bar{\sigma} \) is the algebra map given by the permutation of tensor factors and subscripts.
(4) For an edge \((i, j)\) of a tree \(G \in \mathbb{G}(n)\) with \(i < j\), we define \(K \in \mathbb{G}(n)\) by \(E(K) = E(G) - \{(i, j)\}\). The following diagram is commutative.

\[
\begin{array}{ccc}
A_{H,G}^{*} & \xrightarrow{\phi_G} & H^{*}(\Delta_G) \\
\Delta_{ij} & & \Delta_{ij} \\
A_{H,K}^{*} & \xrightarrow{\phi_K} & H^{*+d}(\Delta_K).
\end{array}
\]

Here, \(\Delta_{ij}^{1}\) is the right-\(A_{H,K}^{*}\)-module homomorphism determined by \(\Delta_{ij}^{1}(1) = \Delta_{ij}^{0}\). \(A_{H,G}^{*}\) is considered as a \(A_{H,K}^{*}\)-module via the natural algebra map \(A_{H,K}^{*} \rightarrow A_{H,G}^{*}\).

**Proof.** In the following proof, we fix a generator \(y\) of \(H^{d-1}(S^{d-1})\), and we denote by \(y_i\) (or \(\bar{y}_i\)) the image of \(y\) by the inclusion to the \(i\)-th factor \(H^{d-1}(S^{d-1}) \rightarrow H^{d-1}(S^{d-1}) ^{\otimes n}\). We consider Serre spectral sequence for the fibration

\[
(S^{d-1})^{\times n} \rightarrow \Delta_G \rightarrow M \times \pi_0(G),
\]

where the projection is the restriction of that of sphere tangent bundle. The first possibly non-trivial differential is \(d_d : H^{d-1}((S^{d-1})^{\times n}) \rightarrow E_d^{0,d-1} \rightarrow E_d^{d,0} = H^d(M)\). This differential takes \(y_i\) to the generator of \(H^d(M)\) multiplied by \(\chi(M)\). As \(\chi(M) = 0\), we have \(d_d = 0\), and latter differentials on \(y_i\) is zero by degree reason, we see that \(y_i\) survives eternally, which implies \(E_2 \cong E_\infty\). Clearly \(E_\infty\) satisfies the assumption of Lemma 6.4. We define \(\varphi_G\) as the composition of the following maps

\[
A_{H,G} \rightarrow E_2 = E_\infty \rightarrow H^{*}(\Delta_G)
\]

where the left map is the isomorphism given by identifying \(y_i\) in the both sides and \(H^{\otimes \pi_0(G)}\) with \(H^{*}(M^{\otimes \pi_0(G)})\) by the Kunneth isomorphism, and the right map is the canonical isomorphism defined in Lemma 6.4. The parts 1, 2 and 3 obviously follow from naturality of the canonical isomorphisms. For the part 4, \(H^{*}(\Delta_G)\) is regarded as \(H^{*}(\Delta_K)\)-module via the pullback \(\Delta_{ij}^{1} : H^{*}(\Delta_K) \rightarrow H^{*}(\Delta_G)\) by the inclusion \(\Delta_G \rightarrow \Delta_K\). This module structure is compatible with the \(A_{H,K}^{*}\)-module structure on \(A_{H,G}^{*}\) via \(\varphi_G\) and \(\varphi_K\) by naturality of canonical isomorphism. By a general property of a shriek map, the map \(\Delta_{ij}^{1}\) is \(H^{*}(\Delta_K)\)-module homomorphism. So to prove the compatibility, we have only to check the image of 1. For simplicity, we may assume \(n = 2\) and \((i, j) = (1,2)\). We may write \(\Delta_G\) as \(\hat{M} \times_M \hat{M}\). The following diagram is commutative.

\[
\begin{array}{ccc}
H_{d-s}(M) & \xrightarrow{P.D.} & H^{*}(M) \xrightarrow{\text{proj}^{*}} H^{*}(\hat{M} \times_M \hat{M}) \\
\Delta_{s} & & \Delta_{12} \\
H_{d-s}(M \times M) & \xrightarrow{P.D.} & H^{*+d}(M \times M) \xrightarrow{\text{proj}^{*}} H^{*+d}(\hat{M} \times_M \hat{M})
\end{array}
\]

Here, P.D. denotes the cap product with the fundamental class. By the commutativity of the left square, we see \(\Delta_{s}^{1}(1)\) is the Poincaré dual class in \(H^{*}(M \times M)\) of the diagonal \(\Delta(M)\), which corresponds to \(\Delta_{H}\) by Kunneth isomorphism. By the commutativity of the right square, we see \(\Delta_{12}^{1}(1)\) corresponds to \(f_{ij}\Delta_{H}\). This completes the proof.

\[\square\]

**Lemma 6.6.** We use the notations \(d_i\), \(\Delta_{i}\), \(s_{i}\), and \(\Delta_{ij}^{1}\) given in Definitions 5.7 and 5.13. Suppose \(\chi(M) \in \mathbb{k}^{*}\). Set \(H = H^{*}(M)\). There exists a family of isomorphisms of graded algebras

\[
\{ \varphi_G : B_{H,G} \xrightarrow{\sim} H^{*}(\Delta_G) \mid n \geq 1, \ G \in \mathbb{G}(n) \}
\]

which satisfies the following conditions.
Proof. As in the proof of Lemma 6.5, we fix a generator even as $χ$ Serre spectral sequence for the sphere tangent fibration. The only non trivial differential is $d_{E}^{d}$ invertible, $\bar{\Delta}_{i}$ is defined by$$\bar{\Delta}_{i}(e_{j}(x)) = e_{j'}(x) \text{ for } x \in S^{d-1}$$. As $\chi(M) \neq 0$. We first show an isomorphism of algebras $S^{d-1} \rightarrow \tilde{M}$. Consider the Serre spectral sequence for the sphere tangent fibration $S^{d-1} \rightarrow \tilde{M} \rightarrow M$. The only non trivial differential is $d_{d} : E^{d-1,d-1} = H^{d-1}(S^{d-1}) \rightarrow H^{d}(M)$. As $\chi(M)$ is invertible, $d_{d}$ is an isomorphism. Since all other differentials vanish by degree reason, we have $E_{\infty} \cong E_{d+1} \cong S^{d}$, where the second isomorphism is given by $E_{d+1}^{p,0} = H^{p}(M) \subset H^{d-2} \subset S^{d}$
for \( p \leq d - 2 \) and \( E^p,d - 1 = H^{d - 1}(S^{d - 1}) \otimes H^p(M) \ni y \otimes a \mapsto \bar{a} \in SH \) for \( p \geq 2 \). Since \( H^1(M) = 0 \) and \( H^*(M) \) is free, \( H^{d - 1}(M) = 0 \) which implies the fibration satisfies the condition of Lemma 6.4. Composing this isomorphism with the canonical isomorphism \( E_\infty \to H^*(\bar{M}) \), we have an isomorphism

\[
SH^* \cong H^*(\bar{M}) \cdots (*)
\]

If necessary, we modify \( y \) so that the composition \( SH^{2d - 1} \to H^{2d - 1}(\bar{M}) \to k \) of the map \( (*) \) and the cup product with the fundamental class \( \bar{w} \) in Definition 5.7 coincides with the orientation given in Definition 6.1 by multiplying a scalar. We shall define the isomorphism \( \psi_G \). We may assume that \( G \in G(n) \) is connected as in disconnected case, everything involved is a tensor product of the objects corresponding to connected subgraphs. Consider the Serre spectral sequence for the fibration

\[
(S^{d - 1})^{x^n - 1} \to \Delta_G \to \bar{M}
\]

given by the projection to the first component. As \( E_0^{d,0} = SH^d = 0 \), elements in \( E_2^{0,d - 1} \cong H^1(S^{d - 1}) \otimes \mathbb{R}^{n - 1} \) survive eternally. As in the proof of Lemma 6.5 \( y_j \) denotes the copy of \( y \) living in the \( j \)-th factor of \( H^*(S^{d - 1}) \otimes \mathbb{R}^{n - 1} \), which is also regarded as a generator of \( E_2^{0,d - 1} \).

We construct an isomorphism \( \psi_G : SH^* \otimes \langle y_1, \ldots, y_{n - 1} \rangle \cong E_\infty \cong H^*(\Delta_G) \) using the isomorphism \( (*) \), similarly to the construction of the isomorphism \( (*) \). Consider the Serre spectral sequence \( \{ E_2^{p,q} \} \) for the fibration

\[
(S^{d - 1})^{x^n - 1} \to \Delta_G \to M
\]

given by the projection of the sphere bundle. Let \( \bar{y}_j \) be the copy of \( y \) in the \( j \)-th factor of \( E_2^{0,d - 1} \cong (H^*(S^{d - 1}) \otimes \mathbb{R}^{n - 1})^{d - 1} \). For any \( i, j \), since \( d_j(\bar{y}_j) = d_i(\bar{y}_j) = (a \text{ a multiple of } \chi(M)w_M) \), \( \bar{y}_i - \bar{y}_j \) survives eternally by degree reason. Clearly \( E_\infty \) satisfies the assumption of Lemma 6.4 we can take the canonical isomorphism \( E_\infty^* \to H^*(\Delta_G) \). We define an algebra map

\[
\varphi'_G : (SH)^{\otimes n} \otimes \bigwedge \{ y_{ij} \mid 1 \leq i, j \leq n \} \to E_\infty^*
\]

by \( e_i(a) \mapsto a \in E_\infty^{0,0} \) for \( a \in H^{d - 2} \), \( e_i(b) \mapsto \bar{y}_j \in E_\infty^{0,d - 1} \), for \( b \in H^{\geq 2} \), and \( y_{ij} \mapsto \bar{y}_i - \bar{y}_j \). We see \( \varphi'_G(J_G) = 0 \) where \( J_G \) is the ideal in Definition 6.3. For example, \( d_d(\bar{y}_i, \bar{y}_j) = \chi(M)(\bar{y}_i - \bar{y}_j)w_M \) (up to \( k^* \)) and \( \chi(M) \) is invertible, \( (\bar{y}_i - \bar{y}_j)w_M = 0 \) in \( E_{d + 1}^{d - 1} \), which implies \( \varphi'_G(e_i(b) - e_j(b)) = 0 \) for \( b \in H^d \). Annihilation of other elements in \( J_G \) is obvious. We define \( \varphi_G \) to be the unique map which makes the following diagram commutative.

\[
\begin{array}{ccc}
(SH)^{\otimes n} \otimes \bigwedge \{ y_{ij} \} & \longrightarrow & (SH)^{\otimes n} \otimes \bigwedge \{ y_{ij} \}/J_G \longrightarrow B_{H,G}^* \\
\varphi'_G & & \varphi_G \\
E_\infty^* & \text{can.iso} & H^*(\Delta_G)
\end{array}
\]

Since \( G \) is connected, \( e_1 : SH \to SH^{\otimes n} \) induces an isomorphism \( \alpha_G : SH \otimes \bigwedge \{ y_{12}, \ldots, y_{1n} \} \cong B_{H,G}^* \). It is easy to see the composition

\[
SH \otimes \bigwedge \{ y_{12}, \ldots, y_{1n} \} \overset{\alpha_G}{\longrightarrow} B_{H,G}^* \overset{\varphi_G}{\longrightarrow} H^*(\Delta_G) \overset{\psi_G^{-1}}{\longrightarrow} SH \otimes \bigwedge \{ y_1, \ldots, y_n \}
\]

identifies the subalgebra \( SH \) in the both side and the sub \( k \)-module \( k\langle y_{12}, \ldots, y_{1n} \rangle \) with \( k\langle y_1, \ldots, y_n \rangle \) (since these are both isomorphic to \( H^{d - 1}(\Delta_G) \)), which implies the composition is isomorphism and we conclude \( \varphi_G \) is an isomorphism.

The parts 1, 2 and 3 obviously follow from naturality of the canonical isomorphism. We shall show the part 4. Since \( \varphi_G \) is an isomorphism, we may define \( \Delta_{ij} \) to be the map which makes the square in the part 4 commutes. As in the proof of Lemma 6.5 \( \Delta_{ij} \) is \( B_{H,K}^* \)-module homomorphism and we have \( \Delta_{ij}(1) = f_{ij}(\Delta_H) \). We shall show the equality \( \Delta_{ij}(y_{ij}) =
Definition 6.7. Let $\Delta y$.

We may assume $n = 2$ and $G = (1, 2)$. In this case clearly $\Delta_G = \hat{M} \times_M \hat{M}$. We consider the following commutative diagram.

\[
\begin{array}{ccc}
H^0(Sd-1) & \longrightarrow & H^0(M) \quad \text{P.D.} \quad H_{2d-1}(\hat{M}) \\
\downarrow \Delta_1 & & \downarrow \Delta_2 \\
H^{d-1}(Sd-1 \times Sd-1) & \longrightarrow & H^{d-1}(\hat{M} \times_M \hat{M}) \quad \text{P.D.} \quad H_{2d-1}(\hat{M} \times_M \hat{M}) \\
\downarrow \Delta_{12} & & \downarrow (\Delta_{12})_* \\
H^{2d-1}(\hat{M} \times M) & \quad \text{P.D.} \quad & H_{2d-1}(\hat{M} \times M)
\end{array}
\]

where the left horizontal arrows are induced by the fiber restriction, and the right ones are capping with the fixed fundamental classes, and $\Delta_1$ and $\Delta_2$ are the shriek maps induced by the diagonals. As $d$ is even, we have $\Delta_1(1) = \bar{y}_1 - \bar{y}_2$. As $\bar{y}_1 - \bar{y}_2$ coincides with the image of $\varphi_G(y_{12})$ by the fiber restriction which induces an isomorphism in degree $d - 1$, we have $\Delta_2^1(1) = \varphi_G(y_{12})$. So we have $\Delta_{12}(\varphi_G(y_{12})) = (\Delta_{12} \circ \Delta_2)^1(1)$. By the commutativity of the right hand side square, $(\Delta_{12} \circ \Delta_2)^1(1)$ is the diagonal class for $\hat{M}$. Thanks to the modification of $y$ after the definition of the isomorphism (*), the diagonal class corresponds to $\Delta_{S\mathcal{H}}$ by $\varphi_G$. This implies $\Delta_{12}(y_{12}) = \Delta_{S\mathcal{H}}$.

\[\square\]

**Definition 6.7.** Let $\mathcal{H}$ be a Poincaré algebra of dimension $d$.

- We define a CDBA $A^{\ast}_{\mathcal{H}}(n)$ by the equality

  \[A^{\ast}_{\mathcal{H}}(n) = \mathcal{H}^{\otimes n} \otimes \{ y_i, g_{ij} \mid 1 \leq i, j \leq n \}/I.\]

  Here, for the bidegree, we set $|a| = (0, l)$ for $a \in (\mathcal{H}^{\otimes n})^* = l$ and $|y_i| = (0, d - 1)$, and $|g_{ij}| = (-1, d)$. The ideal $I$ is generated by the elements

  \[g_{ij} - (-1)^d g_{ji}, \quad (g_{ij})^2, \quad g_{ii}, \quad (e_i(a) - e_j(a))g_{ij}, \]

  \[g_{ij}g_{jk} + g_{jk}g_{ki} + g_{ki}g_{ij} \quad (1 \leq i, j, k \leq n, \quad a \in \mathcal{H}).\]

  We call the last relation the \textit{3-term relation for $g_{ij}$}. The differential is given by $\partial(a) = 0$ for $a \in \mathcal{H}^{\otimes n}$ and $\partial(g_{ij}) = \Delta^{ij}_{\mathcal{H}}$, see Definition 6.3.

- Suppose $d$ is even. We define a CDBA $B^{\ast}_{\mathcal{H}}(n)$ by the equality

  \[B^{\ast}_{\mathcal{H}}(n) = (S\mathcal{H})^{\otimes n} \otimes \{ g_{ij}, h_{ij}, \mid 1 \leq i, j \leq n \}/J.\]

  Here, for the bidegree, we set $|a| = (0, l)$ for $a \in (S\mathcal{H}^{\otimes n})^* = l$ and $|g_{ij}| = (-1, d)$ and $|h_{ij}| = (-1, 2d - 1)$. The ideal $J$ is generated by the following elements.

  \[g_{ij} - g_{ji}, \quad (g_{ij})^2, \quad g_{ii}, \quad h_{ij} + h_{ji}, \quad (h_{ij})^2, \quad h_{ii}, \]

  \[e_{ij}(a)g_{ij}, \quad e_{ij}(a)h_{ij}, \quad e_{ij}(b)g_{ij} - e_{i}(b)h_{ij}, \quad e_{ij}(b)h_{ij}, \]

  \[g_{ij}g_{jk} + g_{jk}g_{ki} + g_{ki}g_{ij}, \quad h_{ij}h_{jk} + h_{jk}h_{ki} + h_{ki}h_{ij} \quad (1 \leq i, j, k \leq n, \quad a \in \mathcal{H}^{d-2}, b \in \mathcal{H}^{2})\]

  where we regard $e_{i}(b)$ as $0$ for $b \in \mathcal{H}^{d}$, and $e_{ij} : S\mathcal{H} \rightarrow (S\mathcal{H})^{\otimes n}$ is the map given by $e_{ij} = e_{i} - e_{j}$. The differential is given by $\partial(x) = 0$ for $x \in S\mathcal{H}^{\otimes n}$ and $\partial(g_{ij}) = \Delta_{S\mathcal{H}}^{ij}$ and $\partial(h_{ij}) = \Delta_{S\mathcal{H}}^{ij}$, see Definition 6.3.

- We equip the sequences $A_{\mathcal{H}} = \{ A_{\mathcal{H}}(n) \}_n$ and $B_{\mathcal{H}} = \{ B_{\mathcal{H}}(n) \}_n$ with structures of $A$-comodules of CDBA as follows. For $B_{\mathcal{H}}$, we define a partial composition and an action.
of $\Sigma_n$ by the equalities
\[
\mu \circ_i e_j(x) = e_j(x), \quad \mu \circ_i (h_j k) = h_j k', \quad \mu \circ_i (g_j k) = g_j k', \quad e_j(x)^{ij} = e_{\tau(j)}(x),
\]
\[
h_{ij}^\sigma = h_{\tau(j), \tau(k)}, \quad g_{ij}^\sigma = g_{\tau(j), \tau(k)} \quad (x \in S \mathcal{H}, \ \sigma \in \Sigma_n)
\]
where $j'$ and $k'$ are numbers given by $j' = d_i(j)$ and $k' = d_i(k)$ and $\tau = \sigma^{-1}$ (see Definition 5.7 for $d_i$ and $\mu$). The definition of $A_H$ is similar.

- We define simplicial CDGA's $A_\Sigma^*_\big(\mathcal{H}\big)$ and $B_\Sigma^*_\big(\mathcal{H}\big)$ as follows. For $B_\Sigma^*_\big(\mathcal{H}\big)$, we set
\[
B_\Sigma^*_n(\mathcal{H}) = B_\Sigma^{n*}(n + 1).
\]
As in Definition 5.13, we relabel the involved subscripts with $0, \ldots, n$. The face map $d_i : B_\Sigma^{n*}(\mathcal{H}) \to B_\Sigma^{n-1*}(\mathcal{H})$ is given by $d_i = \mu \circ (-) \quad (i < n)$ and $d_n = \mu \circ (-)^\sigma$ where $\sigma = (n, 0, 1, \ldots, n - 1)$. The degeneracy map $s_i : B_\Sigma^{n*}(\mathcal{H}) \to B_\Sigma^{n+1*}(\mathcal{H})$ is given by insertion of 1 as the $i$-1-th factor of $S \mathcal{H}^{\otimes n+1}$ and skip of the subscript $i+1$. $A_\Sigma^*_\big(\mathcal{H}\big)$ is defined similarly using $A_\Sigma^*$.

In the rest of this section, we prove that $A_H$ and $B_H$ are isomorphic to $A_M$ as an $A$-comodule of CDGA under different assumptions, and also prove similar statements for the simplicial CDGA's. We mainly deal with the case of $B_H$. The case of $A_H$ is similar.

**Lemma 6.8.** The map
\[
\bigoplus_{G \in \text{G(n)}^{\text{dis}}} H^*_G \to A_M
\]
defined by the composition of the inclusion and quotient map is an isomorphism of $k$-modules (see Definition 5.7 for $\text{G(n)}^{\text{dis}}$).

**Proof.** Let $\Pi$ be the set of partitions of $\underline{n}$. The ideal $J(\underline{n})$ in Definition 5.13 has a decomposition $J(\underline{n}) = \bigoplus_{\pi \in \Pi} J(\underline{n})_{\pi}$ such that $J(\underline{n})_{\pi} \subset \bigoplus_{\pi(0) = \pi} H_G$ since generators of $J(\underline{n})$ are sums of monomials which have the same connected components. If $\pi_0(G) = \pi_0(H) = \pi$, clearly $H^*_E = H^*_G$. We denote this module by $H^*_G$. We have $\bigoplus_{\pi(0)(G) = \pi} H_G = H^*_E \oplus \bigoplus_{\pi(0)(G) = \pi} k G$. Similarly, we have $J(\underline{n})_{\pi} = H^*_E \otimes J(\underline{n})_{\pi}'$ where $J(\underline{n})_{\pi}'$ is the sub $k$-module of $\bigoplus_{\pi(0)(G) = \pi} k G$ generated by multiples of 3-term relations, $g_{ij}^2$ and $g_{ij} - (-1)^d g_{ij}$. We have
\[
A^*_M = \bigoplus_{\pi(0)} \{(\bigoplus_{\pi(0) = \pi} H_G g G) / J(\underline{n})_{\pi}\} \bigoplus_{\pi(0)} \{(\bigoplus_{\pi(0) = \pi} k G) / J(\underline{n})_{\pi}\}
\]
Note that $\bigoplus_{\pi(0)} \{(\bigoplus_{\pi(0) = \pi} k G) / J(\underline{n})_{\pi}\}$ is isomorphic to $H^*(C_n(\mathbb{R}^d))$, whose basis is $\{g_G \mid G \in \text{G(n)}^{\text{dis}}\}$. So $\bigoplus_{\pi(0) = \pi} k G / J(\underline{n})_{\pi}$ has a basis $\{g_G \mid G \in \text{G(n)}^{\text{dis}}, \ \pi_0(G) = \pi\}$, which implies the lemma.

Under the assumptions and notations of Lemma 6.6, we identify $H^*_G$ with $B_{\text{H,G}}$ by the isomorphism $\varphi_G$ so $A^*_M(n)$ is regarded as a quotient of $\bigoplus_{G \in \text{G(n)}} B^*_M(G)$ with this identification, we set $h_{ij} = y_{ij} g_{ij} \in A_M(n)$. $A_M(n)$ contains $S \mathcal{H}^{\otimes n}$ as the subalgebra $H \mathcal{G} \mathcal{B}_g$, the summand corresponding to the graph $\emptyset \in \mathcal{G}$. We regard $A_M(n)$ as a left $S \mathcal{H}^{\otimes n}$-module via the multiplication by $h \mathcal{G} \mathcal{B}_g$. In the following lemma and its proof, $h_G, h^*_G$ and $g_G$ are notations similar to $g_G$. For example, $h_G = h_{i_1 j_1} \cdots h_{i_r j_r}$ for $E(G) = \{(i_1, j_1) < \cdots < (i_r, j_r)\}$.

**Lemma 6.9.** Under the assumptions of Lemma 6.6 and the above notations, as $S \mathcal{H}^{\otimes n}$-module, $A_M(n)$ is generated by the set $S = \{g_G \mathcal{H} | G, H \in \text{G(n)}\}$, $E(G) \cap E(H) = \emptyset$, $G \in \text{G(n)}^{\text{dis}}$, and $B_H(n)$ is generated by the set $S' = \{g_{G,H} \mid G, H \in \text{G(n)}, E(G) \cap E(H) = \emptyset, G \in \text{G(n)}^{\text{dis}}\}$.

**Proof.** $A_M(n)$ is generated by elements $y_{H} g_{H}$ for graphs $G$ an $H$ such that each connected component of $H$ is contained in some connected component of $G$. We can express $g_G$ as a sum of monomials $g_{G_1}$ with $G_1 \in \text{G(n)}^{\text{dis}}$ and $\pi_0(G) = \pi_0(G_1)$ using the 3-term relation and the relation $g_{ij} = g_{ji}$ (This is standard procedure in the computation of $H^*(C_n(\mathbb{R}^d))$).
So we may assume $G$ is distinguished. For a sequence of edges $(i, k_1), (k_1, k_2), \ldots, (k_s, j)$ in $G$, we have $y_{ij} = y_{ik_1} + \cdots + y_{k_s j}$. By successive application of this equality, $y_H$ is expressed as a sum of monomials $y_{H_1}$ with $H_1$ being a subgraph of $G$. Thus we see any element of $A_M(n)$ is expressed as a $S\mathcal{H}^{\otimes n}$-linear combination of monomials $y_{H\mathcal{G}}$ with $G \in \mathcal{G}(n)^{dis}$ and $E(H) \subset E(G)$. Clearly, we see $y_{H\mathcal{G}} = \pm y_{H'} h_H$. Thus we have seen the set $S$ generates $A_M(n)$. Proof for the assertion for $B_H(n)$ is similar when one use 3-term relations for $g_{ij}$ and $h_{ij}$, and the last relation for $g_{ij}$ and $h_{ij}$ in the ideal $J$ in Definition 6.7.

To prove $B_H(n)$ and $A_M(n)$ are isomorphic, we define a structure of $B_HG$-module on $B_H(n)$ as follows. We first define two graded algebras $\tilde{B}_H G$ and $\tilde{B}_H(n)$. For a graph $G \in \mathcal{G}(n)$, we set

$$\tilde{B}_H G = S\mathcal{H}^{\otimes n} \otimes T\{y_{ij} \mid i < j, i \sim G j\}, \quad \tilde{B}_H(n) = S\mathcal{H}^{\otimes n} \otimes \bigwedge\{g_{ij}, h_{ij}, 1 \leq i < j \leq n\}$$

where $T\{y_{ij}\}$ denotes the tensor algebra generated by $y_{ij}$'s. For convenience, we set $y_{ij} = -y_{ji}$, $g_{ij} = g_{ji}$, and $h_{ij} = -h_{ji}$ for $i > j$. The degrees are the same as the elements of the same symbols in $B_{H,G}$ and $B_H(n)$. We shall define a map of graded $k$-modules

$$(\cdots : \tilde{B}_H G \otimes_k \tilde{B}_H(n) \to B_H(n).$$

We define $y_{ij} \cdot xg_Hh_H = x \in S\mathcal{H}^{\otimes n}$, $G, H \in \mathcal{G}(n)$) as follows. If $E(G) \cap E(H) \neq \emptyset$, we set $y_{ij} \cdot xg_Hh_H = 0$. Suppose $E(G) \cap E(H) = \emptyset$. If $(i, j) \in E(G)$ is the $t$-th edge (in the lexicographical order), we set $y_{ij} \cdot xg_Hh_H = (-1)^{|i+t+1|}h_{ij}xg_Hh_H$ with $E(K) = E(G) - \{(i, j)\}$. If $(i, j) \in E(H)$ is an edge, we set $y_{ij} \cdot xg_Hh_H = 0$. If $i \sim_G j$, we take a sequence of edges $(k_0, k_1), \ldots, (k_s, k_{s+1})$ of $G$ with $k_0 = i$ and $k_{s+1} = j$ and set $y_{ij} \cdot xg_Hh_H = \sum_{l=0}^s y_{kl} xg_Hh_H$. This does not depend on the choice of the sequence because $g_{ij}h_{kl} = 0$ if $G$ is not a tree, which is proved by using the last three relations in the definition of $J$ in Definition 6.7. If $i$ and $j$ are disconnected in $G$, we set $y_{ij} \cdot xg_Hh_H = 0$. For $z \in S\mathcal{H}^{\otimes n}$, we set $z \cdot xg_Hh_H = xzg_Hh_H$, the multiplication in $B_H(n)$. We shall show the map $(\cdots : \tilde{B}_H G \otimes_k \tilde{B}_H(n) \to B_H(n)$.

We define $y_{ij} \cdot xg_Hh_H = x \in S\mathcal{H}^{\otimes n}$, $G, H \in \mathcal{G}(n)$) as follows. If $E(G) \cap E(H) \neq \emptyset$, we set $y_{ij} \cdot xg_Hh_H = 0$. Suppose $E(G) \cap E(H) = \emptyset$. If $(i, j) \in E(G)$ is the $t$-th edge (in the lexicographical order), we set $y_{ij} \cdot xg_Hh_H = (-1)^{|i+t+1|}h_{ij}xg_Hh_H$ with $E(K) = E(G) - \{(i, j)\}$. If $(i, j) \in E(H)$ is an edge, we set $y_{ij} \cdot xg_Hh_H = 0$. If $i \sim_G j$, we take a sequence of edges $(k_0, k_1), \ldots, (k_s, k_{s+1})$ of $G$ with $k_0 = i$ and $k_{s+1} = j$ and set $y_{ij} \cdot xg_Hh_H = \sum_{l=0}^s y_{kl} xg_Hh_H$. This does not depend on the choice of the sequence because $g_{ij}h_{kl} = 0$ if $G$ is not a tree, which is proved by using the last three relations in the definition of $J$ in Definition 6.7. If $i$ and $j$ are disconnected in $G$, we set $y_{ij} \cdot xg_Hh_H = 0$. For $z \in S\mathcal{H}^{\otimes n}$, we set $z \cdot xg_Hh_H = xzg_Hh_H$, the multiplication in $B_H(n)$. We shall show the map $(\cdots : \tilde{B}_H G \otimes_k \tilde{B}_H(n) \to B_H(n)$.

Theorem 6.10. Suppose $M$ is simply connected and oriented, and $H^*(M)$ is a free $k$-module. Set $H = H^*(M)$.

1. Suppose $\chi(M) = 0 \in k$. Two $A$-comodules of $CDBA A^{\bullet}_M$ and $A^{\bullet}_H$ are isomorphic, and two simplicial CDBA $A^{\bullet}_M(n)$ and $A^{\bullet}_H(n)$ are isomorphic. In particular, the $E_2$-page of Čech s.s. is isomorphic to the total homology of the normalization $NA^{\bullet}_H(n)$ as a bigraded $k$-module. The bigrading is given by $(\bullet , \bullet , \bullet )$.

2. Suppose $\chi(M) \in k^\times$. Two $A$-comodules of $CDBA A^{\bullet}_M$ and $B^*_H$ are isomorphic, and two simplicial CDBA $A^{\bullet}_M(n)$ and $B^*_H(n)$ are isomorphic. In particular, the $E_2$-page of Čech s.s. is isomorphic to the total homology of the normalization $NB^*_H(n)$ as a bigraded $k$-module. The bigrading is given by $(\bullet , \bullet , \bullet )$.

Proof. The part 1 obviously follows from Theorem 5.13 and Lemma 6.5. We shall prove the part 2. We define a map $\Phi_n : B_H(n) \to A_M(n)$ of algebras by identifying the subalgebra
$S\mathcal{H}^{\otimes n}$ and elements $g_{ij}$ in both sides and taking $h_{ij}$ to $\bar{h}_{ij}$ (see the paragraph above Lemma 6.9). We easily verify $\Phi_n$ is well-defined. $\Phi_n$ fits into the following commutative diagram.

Here the vertical arrow is induced by the inclusion of a submodule $H_{G,GG} = B_{\mathcal{H},GGG} \subset B_{\mathcal{H}}(n)$ given by the isomorphism $\varphi_G$ in Lemma 6.6 and the module structure defined above, and the slanting arrow is given in Lemma 6.8. The vertical arrow and $\Phi_n$ are epimorphisms by Lemma 6.9 and the slanting arrow is an isomorphism by Lemma 6.8 so $\Phi_n$ is an isomorphism. By the definition of $\Phi_n$ and Lemma 6.6 the collection $\{\Phi_n\}_n$ commutes with the structures of $A$-comodule and degeneracy maps. The assertion for the $E_2$-page immediately follows from the isomorphism of simplicial objects. □

Remark 6.11. The Euler number $\chi(M)$ can be recovered from the Poincaré algebra $\mathcal{H}^\ast = H^\ast(M)$. It is the image of $\Delta_{\mathcal{H}}$ by the composition

$$(\mathcal{H}^{\otimes 2})^\ast \xrightarrow{\text{multiplication}} \mathcal{H}^d \xrightarrow{\epsilon} k$$

So under the assumption of Theorem 6.10 the $E_2$-page of Čech s.s. is determined by the cohomology algebra $H^\ast(M)$. (Different orientations give apparently different presentations but they are isomorphic.)

7. Examples

In this section, we compute some part of $E_2$-page of Čech s.s. for spheres and products of two spheres $S^k \times S^l$ for $(k, l) = (\text{odd, even})$ or $(\text{even, even})$ and deduce some results on cohomology groups for the products of spheres. We also prove Corollary 1.3. Our computation is restricted to low degrees and consists of only elementary linear algebra on differentials and degree argument based on Theorem 6.10. We briefly state the results for the case of spheres since in these cases, the Čech s.s. only gives less information than the combination of Vassiliev’s (or Sinha’s) spectral sequence for long knots and the Serre spectral sequence for a fibration $\text{Emb}(S^1, S^d) \to \text{STS}^d$ (see the proof of Proposition 7.2) gives, at least in the degrees which we have computed. We give concrete descriptions of the differentials in the case of $M = S^k \times S^l$ with $k$ odd and $l$ even. In the rest of this section, we set $\mathcal{H} = H^\ast(M)$ for a fixed orientation.

7.1. The case of $M = S^d$ with $d$ odd. In this case $A_n^\ast(\mathcal{H})$ is described as follows.

$$A_n^\ast(\mathcal{H}) = \bigwedge \{x_i, y_i, g_{ij} \mid 0 \leq i, j \leq n\}/\mathcal{I}$$

where $|x_i| = (0, d)$, $|y_i| = (0, d - 1)$, $|g_{ij}| = (-1, d)$, and $\mathcal{I}$ is the ideal generated by

$$(x_i)^2, (y_i)^2, (g_{ij})^2, g_{ii}, g_{ij} + g_{ji},$$

$$(x_i - x_j)g_{ij},$$

and the 3-term relation for $g_{ij}$.

The diagonal class is given by $\Delta_{\mathcal{H}} = x_0 - x_1 \in \mathcal{H} \otimes \mathcal{H}$. 

Proposition 7.1. Consider the Čech s.s. $\mathbb{E}_{pq}^d$ for the sphere $S^d$ with odd $d \geq 5$. We abbreviate $\mathbb{E}_{pq}^d$ as $(p,q)$. The following equalities hold.

\begin{align*}
(-3,d) &= k\langle g_{12} \rangle, \quad (-1,d-1) = k\langle y_1 \rangle, \\
(0,d-1) &= k\langle y_0 \rangle, \quad (0,d) = k\langle x_0 \rangle, \\
(-6,2d) &= k\langle g_{13}g_{24}, -g_{12}g_{34} + g_{14}g_{23} \rangle, \quad (-4,2d-1) = k\langle y_1g_{23} - y_2g_{13} + y_3g_{12} \rangle, \\
(-5,2d) &= k\langle g_{01}g_{23} + g_{02}g_{13} + g_{13}g_{23} \rangle, \quad (-3,2d-1) = k\langle y_0g_{12} \rangle, \\
(-3,2d) &= k\langle x_0g_{12} \rangle, \quad (-1,2d-1) = k\langle x_0y_1, x_1y_0, x_1y_1 \rangle, \\
(0,2d-1) &= k\langle x_0y_0 \rangle.
\end{align*}

For other $(p,q)$ with $p+q \leq 2d-1$, we have $(p,q) = 0$. \hfill \Box

Proposition 7.2. Let $d$ be an odd number with $d \geq 5$.
\begin{enumerate}
    \item $\text{Emb}(S^1, S^d)$ is $d-2$-connected.
    \item The Čech s.s. for $S^d$ does not collapse at $E_2$-page in any coefficient ring.
\end{enumerate}

Proof. For the part 1, consider the fiber sequence

$$\text{Emb}_c(\mathbb{R}, \mathbb{R}^d) \to \text{Emb}(S^1, S^d) \to \text{STS}^d$$

where the left hand side map is given by taking the tangent vector at a fixed point and the right space is the space of long knots. As is well known, $\text{STS}^d$ is $d-2$-connected and $\text{Emb}_c(\mathbb{R}, \mathbb{R}^d)$ is $2d-7$-connected. As $d \geq 5$, we have the claim. The part 2 follows from the part 1 and Proposition [7.1]. (There is non-zero elements in total degrees $d-3$ and $d-2$.) \hfill \Box

Remark 7.3. The reader may find inconsistency between [9, Proposition 3.9 (3)] and Proposition [7.2] (1). This is just a notational matter. $n-j-2$ should be replaced with $n-j-1$ (and $n-j-1$ with $n-j$) in the proposition, see its proof.

7.2. The case of $M = S^d$ with $d$ even. In this subsection, we assume $2 \in k^\times$. $B_n^*\star(\mathcal{H})$ is described as follows.

$$B_n^*\star(\mathcal{H}) = \bigwedge \{z_i, g_{ij}, h_{ij} \mid 0 \leq i,j \leq n\}/\mathcal{J}$$

where $|z_i| = (0,2d-1)$, $|g_{ij}| = (-1,d)$, $|h_{ij}| = (-1,2d-1)$, and $\mathcal{J}$ is the ideal generated by

\begin{align*}
(z_i)^2, (g_{ij})^2, (h_{ij})^2, g_{ii}, h_{ii}, g_{ij} - g_{ji}, h_{ij} + h_{ji}, \\
(z_i - z_j)g_{ij}, (z_i - z_j)h_{ij}, (h_{ij} + h_{ki})g_{jk} - (h_{ij} + h_{jk})g_{ki},
\end{align*}

and the 3-term relation for $g_{ij}$ and $h_{ij}$.

The diagonal classes are given by $\Delta_{\mathcal{H}} = 0 \in S\mathcal{H} \otimes S\mathcal{H}$ and $\Delta_{S\mathcal{H}} = z_0 - z_1 \in S\mathcal{H} \otimes S\mathcal{H}$.

Proposition 7.4. Suppose $2 \in k^\times$. Consider the Čech s.s. $\mathbb{E}_{pq}^d$ for $S^d$ with even $d \geq 4$. We abbreviate $\mathbb{E}_{pq}^d$ as $(p,q)$. The following equalities hold.

\begin{align*}
(-6,2d) &= k\langle g_{13}g_{24} \rangle, \quad (-5,2d) = k\langle g_{01}g_{23} + 3g_{02}g_{13} + 3g_{03}g_{12} \rangle, \\
(-3,2d-1) &= k\langle h_{12} \rangle, \quad (0,2d-1) = k\langle z_0 \rangle.
\end{align*}

For other $(p,q)$ with $p+q \leq 2d-1$, we have $(p,q) = 0$. \hfill \Box

For the case of $k = \mathbb{F}_2$, the same statement as Proposition 7.1 holds, except that ” odd $d \geq 5$” is replaced with ” even $d \geq 4$“.
7.3. The case of $M = S^k \times S^l$ with $k$ odd and $l$ even. We fix generators $a \in H^k(S^k)$ and $b \in H^l(S^l)$. $\mathcal{H}$ is presented as $\wedge \{a, b\}$. We fix an orientation $\epsilon$ on $\mathcal{H}$ by $\epsilon(ab) = 1$. We write $a_i$ for $e_i(a)$ (similarly for $e_i(b)$) and $A_n(\mathcal{H})$ is presented as

$$A_n(\mathcal{H}) = \bigwedge \{a_i, b_i, y_i, g_{ij} \mid 0 \leq i, j \leq n\}/\mathcal{I}$$

where $|y_i| = (0, k + l - 1)$, $|g_{ij}| = (-1, k + l)$, and $\mathcal{I}$ is the ideal generated by

$$(a_i)^2, (b_i)^2, (y_i)^2, (g_{ij})^2, g_{ii}, g_{ij} + g_{ji},$$

$$(a_i - a_j)g_{ij}, (b_i - b_j)g_{ij}$$

and the 3-term relation for $g_{ij}$.

The diagonal class is given by $\Delta_\mathcal{H} = a_0b_0 - a_1b_0 + a_0b_1 - a_1b_1 \in \mathcal{H} \otimes \mathcal{H}$. The module $NA_n(\mathcal{H})$ is generated by the monomials of the form $a_{p_1} \cdots a_{p_s} b_{q_1} \cdots b_{q_t} g_{i_1j_1} \cdots g_{i_rj_r}$ such that the set of subscripts $\{p_1, \ldots, p_s, q_1, \ldots, q_t, i_1, \ldots, i_r, j_1, \ldots, j_r\}$ contains the set $\{1, \ldots, n\}$.

We shall present the total differential $\tilde{d}$ on

$$\tilde{\mathbb{E}}_1^{p,q} = \bigoplus_{s + t = p} NA_{s,t}(\mathcal{H})$$

up to $p + q \leq \max\{2k + l, k + 2l\}$. For $(p, q) = (-1, k), (-1, l), (-1, k + l - 1), (-1, k + l), (-1, 2k), (-1, 2l), (-1, 2k + l), (-1, k + 2l), (-1, 2k + l - 1), (-1, 2k), (-2, 2l), (-2, 3k), (-2, 3l)$, $\tilde{d}$ is zero.

For $(p, q) = (-3, k + l)$, $\tilde{d}$ is presented by the following matrix

\[
\begin{array}{c|ccc}
  & g_{12} \\
\hline
  g_{01} & 0 \\
  a_1b_2 & 1 \\
  a_2b_1 & -1 \\
\end{array}
\]

This is read as $\tilde{d}(g_{12}) = a_1b_2 - a_2b_1$. For $(p, q) = (-2, k + l)$,

\[
\begin{array}{c|ccc}
  & g_{12} \\
\hline
  a_0b_1 & 1 & 1 & 1 \\
  a_1b_0 & -1 & 1 & 1 \\
  a_1b_1 & -1 & -1 & -1 \\
\end{array}
\]

For $(p, q) = (-4, 2k + l)$,

\[
\begin{array}{c|ccc}
  & a_{12}g_{23} & a_{23}g_{12} \\
\hline
  a_0g_{12} & 1 & 0 & -1 \\
  a_1g_{02} & 1 & 1 & 0 \\
  a_1g_{12} & -1 & 0 & 1 \\
  a_2g_{01} & 0 & 1 & 1 \\
  a_1a_2b_3 & -1 & 1 & 0 \\
  a_1a_3b_2 & 1 & 0 & 1 \\
  a_2a_3b_1 & 0 & 1 & -1 \\
\end{array}
\]

For $(p, q) = (-3, 2k + l)$,

\[
\begin{array}{c|cccccc}
  & a_0g_{12} & a_1g_{02} & a_1g_{12} & a_2g_{01} & a_1a_2b_3 & a_1a_3b_2 & a_2a_3b_1 \\
\hline
  a_0g_{01} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  a_0a_1b_2 & -1 & 1 & 0 & 0 & -1 & -1 & 0 \\
  a_0a_2b_1 & 1 & 0 & 0 & 1 & 0 & -1 & -1 \\
  a_1a_2b_0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 \\
  a_1a_2b_1 & 0 & 0 & 1 & -1 & 0 & 1 & 1 \\
  a_1a_2b_2 & 0 & 1 & 1 & 0 & -1 & -1 & 0 \\
\end{array}
\]
For \((p, q) = (-2, 2k + l)\),

\[
\begin{array}{ccccccc}
| & a_0 a_1 b_0 & a_0 a_1 b_2 & a_0 a_2 b_1 & a_1 a_2 b_0 & a_1 a_2 b_1 & a_1 a_2 b_2 \\
\hline
a_0 a_1 b_0 & 1 & -1 & -1 & 0 & -1 & 1 \\
a_0 a_1 b_1 & 1 & 1 & 1 & 0 & 1 & -1 \\
\end{array}
\]

For \((p, q) = (-2, 2k + l - 1)\),

\[
\begin{array}{cccc}
| & a_1 y_2 & a_2 y_1 \\
\hline
a_0 y_1 & 1 & 1 \\
a_1 y_0 & 1 & 1 \\
a_1 y_1 & -1 & -1 \\
\end{array}
\]

For \((p, q) = (-4, k + 2l)\),

\[
\begin{array}{cccc}
| & b_0 g_{12} & b_2 g_{13} & b_3 g_{12} \\
\hline
b_0 g_{12} & -1 & 0 & 1 \\
b_1 g_{02} & -1 & -1 & 0 \\
b_1 g_{12} & 1 & 0 & -1 \\
b_2 g_{01} & 0 & -1 & -1 \\
a_1 b_2 b_3 & 0 & 1 & 1 \\
a_2 b_1 b_3 & 1 & 0 & -1 \\
a_3 b_1 b_2 & -1 & -1 & 0 \\
\end{array}
\]

For \((p, q) = (-3, k + 2l)\),

\[
\begin{array}{cccccccc}
| & b_0 g_{12} & b_1 g_{02} & b_1 g_{12} & b_2 g_{01} & a_1 b_2 b_3 & a_2 b_1 b_3 & a_3 b_1 b_2 \\
\hline
b_0 g_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_0 b_1 b_2 & 0 & 1 & 0 & 1 & 1 & 0 & -1 \\
a_1 b_0 b_2 & 1 & 0 & 0 & -1 & -1 & 1 & 0 \\
a_1 b_1 b_2 & 0 & 0 & 1 & -1 & -1 & -1 & 0 \\
a_2 b_0 b_1 & -1 & -1 & 0 & 0 & 0 & -1 & 1 \\
a_2 b_1 b_2 & 0 & -1 & -1 & 0 & 0 & 1 & 1 \\
\end{array}
\]

For \((p, q) = (-2, k + 2l + 1)\),

\[
\begin{array}{cccccccc}
| & b_0 g_{01} & a_0 b_2 b_2 & a_1 b_0 b_2 & a_1 b_1 b_2 & a_2 b_0 b_1 & a_2 b_1 b_2 \\
\hline
b_0 g_{01} & 1 & 2 & 1 & 1 & 1 & 1 \\
a_0 b_0 b_1 & -1 & 0 & -1 & 1 & -1 & 1 \\
a_1 b_0 b_1 & -1 & 0 & -1 & 1 & -1 & 1 \\
\end{array}
\]

By direct computation based on the above presentation, we obtain the following result. Let \(k_2\) (resp. \(k^2\)) denote the module \(k/2k\) (resp. \(k \oplus k\)).

**Proposition 7.5.** Suppose \(k\) is either of \(\mathbb{Z}\) or \(\mathbb{F}_p\) with \(p\) prime. Let \(k\) be an odd number and \(l\) be an even numbers with \(k + 5 \leq l \leq 2k - 3\) and \(|3k - 2l| \geq 2\), or \(l + 5 \leq k \leq 2l - 3\) and
We abbreviate $\mathbb{E}_2^{pq}$ for $S^k \times S^l$ as $(p,q)$. We have the following isomorphisms.

(0, k) = (−1, k) = (0, l) = (−1, l) = (−1, 2k) = (−2, 2k) = (−1, 2l) = (−2, 2l) = k

(−2, 3k) = (−3, 3k) = (−2, 3l) = (−3, 3l) = (0, k + l − 1) = (−1, k + l − 1) = k,

(0, k + l) = k, (−1, k + l) = k ⊕ k^2 or k^2, (−2, k + l) = 0 or k,

(0, 2k + l − 1) = k, (−1, 2k + l − 1) = k^2, (−2, 2k + l − 1) = k,

(−1, 2k + l) = k^2 or k, (−2, 2k + l) = k^2 or k^2, (−3, 2k + l) = k^2 or k^2,

(−4, 2k + l) = 0 or k

(0, k + 2l − 1) = k, (−1, k + 2l − 1) = k^2, (−2, k + 2l − 1) = k,

(−1, k + 2l) = k^2 or k, (−2, k + 2l) = k or k^2, (−3, k + 2l) = k^2, (−4, 2k + l) = k

Here "(p, q) = A or B" means (p, q) = A if $k = \mathbb{Z}$ or $\mathbb{F}_p$ with $p \neq 2$, (p, q) = B if $k = \mathbb{F}_2$. For other (p, q) with $p + q \leq \max\{k + 2l, 2k + l\}$, (p, q) = 0.

The isomorphisms of proposition 7.5 holds under milder conditions on $k$ and $l$. It suffices to ensure the bidegrees presented above are pairwise distinct. By degree argument, we obtain the following corollary.

**Corollary 7.6.** Suppose $k$ is either of $\mathbb{Z}$ or $\mathbb{F}_p$ with $p$ prime. Let $k$ be an odd number and $l$ be an even number with $k + 5 \leq l \leq 2k + 3$ and $|3k + 2l| \geq 2$, or $l + 5 \leq k \leq 2l + 3$ and $|3l + 2k| \geq 2$. We set $H^* = H^*(\text{Emb}(S^1, S^k \times S^l))$.

1. We have isomorphisms

   $$H^i = k \ (i = k - 1, k, 2k - 2, 2k - 1, k + l).$$

2. If $k = \mathbb{F}_p$ with $p \neq 2$, we have isomorphisms

   $$H^i = k^2 \ (i = k + l - 2, k + l - 1, 2k + l - 3, 2k + l - 2, 2k + l - 1).$$

**Proof.** By an argument similar to the proof of Theorem 5.16, we see $\mathbb{E}_r^{−pq} = 0$ if $q/p < (k+l)/3$. We shall show any differential $d_r : \mathbb{E}_r^{−p+r+q+r−1} \to \mathbb{E}_r^{−pq}$ going into the term contained in the cohomology of the claim is zero. It is enough to the case of $(-p, q) = (0, 2k + l - 1)$ and $q + r - 1 \geq k + 2l - 1$ since other cases are obvious or follow from this case. We see

$$\frac{p + r - 1}{q + r - 1} = \frac{q - 1}{r} + 1 \leq \frac{2k + l - 2}{l - k + 1} + 1 = \frac{k + 2l - 1}{l - k + 1} < \frac{k + l}{3}.$$ 

So $\mathbb{E}_r^{−p-r,q+r−1} = 0$ and $d_r = 0$. □

### 7.4. The case of $M = S^k \times S^l$ with $k, l$ even

We fix generators $a \in H^k(S^k)$ and $b \in H^l(S^l)$. $\mathcal{H}$ is presented as $\Lambda\{a, b\}$. We fix an orientation $\epsilon$ on $\mathcal{H}$ by $\epsilon(ab) = 1$. We set $c = \bar{a} \in S\mathcal{H}$, $d = b \in S\mathcal{H}$. We write $a_i$ for $e_i(a)$ (similarly for $e_i(b)$ etc.) and $B_n(\mathcal{H})$ is presented as

$$B_n(\mathcal{H}) = \bigwedge{\{a_i, b_i, c_i, d_i, g_{ij}, h_{ij} \mid 0 \leq i, j \leq n\}} / J$$

where $|g_{ij}| = (−1, k + l)$, $|h_{ij}| = (−1, 2(k + l) − 1)$, and $J$ is the ideal generated by

$$(a_i)^2, (b_i)^2, (c_i)^2, (d_i)^2, a_ib_i, a_ic_i, b_id_i, c_id_i, a_i d_i - b_i c_i$$ $$(g_{ij})^2, (h_{ij})^2, g_{ii}, h_{ii}, g_{ij} - g_{ji}, h_{ij} + h_{ji}$$ $$(a_i - a_j)g_{ij}, (b_i - b_j)g_{ij}, (c_i - c_j)g_{ij} - a_i h_{ij}, (d_i - d_j)g_{ij} - b_i h_{ij}$$ $$(a_i - a_j)h_{ij}, (b_i - b_j)h_{ij}, (c_i - c_j)h_{ij}, (d_i - d_j)h_{ij}$$ $$(h_{ij} + h_{ik})g_{jk} - (h_{ij} + h_{jk})g_{ki},$$

and the 3-term relations for $g_{ij}$ and $h_{ij}$. 

The diagonal classes are given by \( \Delta_{\mathcal{H}} = a_0 b_1 + a_1 b_0 \in S\mathcal{H} \otimes S\mathcal{H} \) and \( \Delta_{S\mathcal{H}} = a_0 d_0 + a_1 d_0 + b_1 c_0 - b_0 c_1 - a_0 d_1 - a_1 d_1 \).

By an argument similar to the proof of Corollary 7.6, we obtain the following corollary.

**Corollary 7.7.** Suppose \( 2 \in k^\times \). Let \( k \) and \( l \) be two even numbers with \( k + 2 \leq l \leq 2k - 2 \) and \( |3k - 2l| \geq 2 \). We set \( H^* = H^*(\text{Emb}(S^1, S^k \times S^l)) \). We have isomorphisms

\[
H^i = k \quad (i = k - 1, k, l - 1, l, k + l - 3, k + l - 2, k + l - 1, 3k).
\]

For any other degree \( i \leq 2k + l \), \( H^i = 0 \). \( \square \)

7.5. **The case of 4-dimensional manifolds.** In this subsection, we prove Corollary 1.3. We assume that \( M \) is a simply connected 4-dimensional manifold. So, as is easily observed, \( \mathcal{H} \) is a free \( k \)-module for any \( k \).

**Definition 7.8.** Set \( \chi = \chi(M) \).

- We define a map \( \alpha : (\mathcal{H}^2)^{\otimes 2} \oplus k_{g_{01}} \to (\mathcal{H}^2)^{\otimes 2} \oplus \mathcal{H}^4/\chi \mathcal{H}^4 \) by

\[
\alpha(a \otimes b) = (-a \otimes b - b \otimes a) + ab, \quad \alpha(g_{01}) = pr_1(\Delta_{\mathcal{H}})
\]

Here, \( g_{01} \) is a formal free generator (which will correspond to the element of the same symbol in \( E^{-2,4}_1 \)) and \( pr_1 \) is the projection

\[
(\mathcal{H}^2)^{\otimes 2} \to (\mathcal{H}^2)^{\otimes 2} \oplus (1 \otimes \mathcal{H}^4) \to (\mathcal{H}^2)^{\otimes 2} \oplus \mathcal{H}^4/\chi \mathcal{H}^4.
\]

The following proposition follows from direct computation and degree argument based on Theorem 6.10.

**Lemma 7.9.** We use the notations in Definition 7.8. Suppose \( k \) is a field and \( \mathcal{H}^2 \) is not zero.

1. When \( p + q = 1 \), \( \hat{E}_{r}^{p,q} \) is stationary after \( E_2 \). In particular, \( \hat{E}_{2}^{p,q} \cong \hat{E}_{r}^{p,q} \). We have isomorphisms

\[
\hat{E}_{2}^{p,q} = \begin{cases} 
\mathcal{H}^2 & (p, q) = (-1, 2) \\
0 & \text{(otherwise)}
\end{cases}
\]

2. There exists an isomorphism

\[
\hat{E}_{-2,4} \cong \text{Ker}(\alpha)/k(pr_2(\Delta_{\mathcal{H}}) + 2g_{01}).
\]

Here \( pr_2 \) is the projection \( (\mathcal{H}^2)^{\otimes 2} \to (\mathcal{H}^2)^{\otimes 2} \). The differential \( d_r \) coming into this term is zero for \( r \geq 2 \). \( \square \)

**Remark 7.10.** Actually, Lemma 7.9 holds even when \( k \) is not a field since the torsions for Künneth theorem does not affect the range.

**Proof of Corollary 1.3.** In this proof, we suppose \( k \) is a field. Set \( H_2^Z = H_2(M; \mathbb{Z}) \). As is well-known, there is a weak homotopy equivalence between \( \text{Imm}(S^1, M) \) and the free loop space \( L\tilde{M} \), and there is an isomorphism \( \pi_1(L\tilde{M}) \cong \pi_1(M) \oplus \pi_2(M) \). As \( M \) is simply connected, we have \( \pi_1(M) \cong \pi_2(M) \cong H_2^Z(M) \).

By Goodwillie-Klein-Weiss convergence theorem, the connectivity of the standard projection \( \text{holim}_{\Delta} C^*([M]) \to \text{holim}_{\Delta_n} C^*([M]) \) increases as \( n \) increases. Since \( \Delta_n \) is a compact category in the sense of [14], and \( C^n([M]) \) is simply connected for any \( n \), by [14] Theorem 2.2, we see \( \text{Emb}(S^1, M) \) is \( \mathbb{Z} \)-complete. In particular, \( \pi_1(\text{Emb}(S^1, M)) \) is a pro-nilpotent group. So, by a theorem of Stallings [39], we only have to prove the composition

\[
\text{Emb}(S^1, M) \xrightarrow{i_{M}} \text{Imm}(S^1, M) \xrightarrow{\sim} L\tilde{M} \xrightarrow{cl_1} K(H_2^Z, 1)
\]

induces an isomorphism on \( H_1(-; \mathbb{Z}) \) and a surjection on \( H_2(-; \mathbb{Z}) \). Here the rightmost map \( cl_1 \) is the classifying map. See [16].
Consider the spectral sequence $E^{p,q}_r$ associated to the Hochschild complex of $C^S(T\overline{\mathcal{H}}_M)$. This spectral sequence is isomorphic to the Bousfield-Kan type cohomology spectral sequence associated to the well-known cosimplicial model for $L\overline{\mathcal{M}}$ given by $[n] \mapsto \overline{\mathcal{M}}^{km+1}$. The quotient map $\overline{\mathcal{H}}_M \to \mathcal{H}_M$ induces a map $f_r : E^{p,q}_r \to E^{p,q}_r$ of spectral sequences. For $r = \infty$, this map is identified with the map on the associated graded induced by the inclusion $i_M$. For $p + q = 1$, by Lemma 7.9 (and similar computation for $E^{p,q}_r$), $f_2$ is an isomorphism for any field $k$. Since $\pi_1(\text{Emb}(S^1, M))$ is the same as $\pi_1$ of a finite stage of Taylor tower which is finite homotopy limit of simply connected finite cell complex, it is finitely generated, and so is $H_1$. By the universal coefficient theorem, we see $i_M$ induces an isomorphism on $H_1(-; \mathbb{Z})$. For the part of $p + q = 2$, we see $E^{2,q}_2 = 0$ for $p < -2$ and $E^{-2,4} \cong \text{Ker}(\alpha) \cap (H^2)^{\otimes 2}$. Consider the following zigzag

$$L\overline{\mathcal{M}} \xrightarrow{L(c_2)} \text{LK}(H^2_{\mathbb{Z}}, 2) \overset{i_K}{\leftarrow} \Omega K(H^2_{\mathbb{Z}}, 2),$$

where the left map is induced by the classifying map $c_2 : \overline{\mathcal{M}} \to K(H^2_{\mathbb{Z}}, 2)$ and the right one is the inclusion from the based loop space. Clearly, the composition $c_1 \circ i_K : \Omega K(H^2_{\mathbb{Z}}, 2) \to K(H^2_{\mathbb{Z}}, 1)$ is a weak homotopy equivalence. Observe spectral sequences associated to the standard cosimplicial models of the above three spaces. Since the maps $L(c_2)$ and $i_K$ are induced by cosimplicial maps, they induce maps on spectral sequences. In the part of total degree 2, we see the filter $F^{-2}$ for each of three spectral sequences is the entire cohomology group, and the filter $F^{-1}$ for the one for $\Omega K(H^2_{\mathbb{Z}}, 2)$ is zero. With these observations, we see that the image of $H^2(K(H^2_{\mathbb{Z}}, 1))$ in $H^2(L\overline{\mathcal{M}})$ by the map $c_1$ is sent to a subspace $V$ of $F^{-2}/F^{-1} \cong E^{2,4}_2 \subset E^{2,4}_2$ isomorphically, and a basis of $V$ is given by $\{ a_i \otimes a_j - a_j \otimes a_i \mid i < j \}$ as elements of $E^{2,4}_2$, where $\{a_i\}$ denotes a basis of $H^2$. (We also see these elements must be stationary.) If $k \neq \mathbb{F}_2$, or if $k = \mathbb{F}_2$ and the inverse of the intersection matrix has at least one non-zero diagonal component, we see the restriction of $f_2$ to $V$ is a monomorphism by Lemmas 6.2, 7.9. (Otherwise, the elements of the basis of $V$ have the relation $pr_0(\Delta_N) = 0$.) This implies $i_M$ induces a surjection on $H_2$ for any field $k$ under the assumption of the theorem. By the universal coefficient theorem, we obtain the desired assertion on $H_2(-; \mathbb{Z})$. \hfill $\square$

**Remark 7.11.** If all of the diagonal components of the inverse of intersection matrix on $H_2(M; \mathbb{F}_2)$ is zero, the map $f_2 : V \to E^{2,4}_r$ in the proof is not a monomorphism for $k = \mathbb{F}_2$ but this does not necessarily imply the original (non-associated graded) map is not a monomorphism. So in this case, it is still unclear whether $i_M$ is an isomorphism on $\pi_1$.

### 8. Precise statement and proof of Theorem 1.4

**Definition 8.1.** A planer rooted $n$-tree $(T, \epsilon)$ consists a 1-dimensional finite cell complex $T$ and a continuous monomorphism $\epsilon$ from its realization $|T|$ to the half plane $y \geq 0$ such that

- $T$ is connected and $\pi_1(T)$ is trivial.

- the intersection of the image of $\epsilon$ and the $x$-axis consisting of the image of $n$ univalent vertices called leaves and labeled by $1, \ldots, n$ in the manner consistent with the standard order on the axis,

- $T$ has a unique distinguished vertex called the root which is at least bivalent, and

- any vertex except for the leaves and root is at least trivalent.

An isotopy between $n$-trees $(T_1, \epsilon_1) \to (T_2, \epsilon_2)$ is an isotopy of the half plane onto itself which maps $\epsilon_1(|T_1|)$ onto $\epsilon_2(|T_2|)$ and the root to the root. (So an isotopy preserves the leaves including the labels.) We will denote an isotopy class of planer rooted $n$-trees simply by $T$. The root vertex of a tree is usually denoted by $v_r$. For a vertex $v$ of a tree, $|v|$ denotes the number which is the valence minus 1 if $v \neq v_r$, and equal to the valence if $v = v_r$. 


Let $\Psi_n^o$ be a category defined as follows. An object of $\Psi_n^o$ is an isotopy class of the planer rooted $n$-trees. There is a unique morphism $T \to T'$ if $T'$ is obtained from $T$ by a successive contraction of internal edges (i.e. edges not adjacent to leaves).

Let $\mathsf{Cat}$ be the category of small categories and functors. Let $i_n : \Psi_0^o \to \Psi_{n+1}^o$ be a functor which sends $T$ to the tree made from $T$ by attaching two edges to the $n$-leaf of $T$ and labeling the new leaves with $n$ and $n+1$. We define a category $\Psi$ to be the colimit of the sequence $\Psi_1^o \xrightarrow{i_1} \Psi_2^o \xrightarrow{i_2} \cdots$ taken in $\mathsf{Cat}$. $F_n : \Psi_{n+1}^o \to P_n(v)$ denotes the functor given in Definition 4.14 of [35], which sends a tree $T \in \Psi_{n+1}^o$ to the set of the numbers $i$ such that the shortest paths from $i$ and $i+1$ to the root in $T$ intersects only at the root. For the functor $G_n : P_n(n+1) \to \Delta_n$, see subsection 2.1. The following square is clearly commutative.

\[
\begin{array}{ccc}
\Psi_{n+2}^o & \xrightarrow{F_n \circ G_{n+1}} & \Delta_n \\
\downarrow i_n & & \downarrow i_n \\
\Psi_{n+3}^o & \xrightarrow{F_{n+1} \circ G_{n+2}} & \Delta_{n+1},
\end{array}
\]

where the right vertical arrow is the natural inclusion, so we have the induced functor $F \circ G : \Psi^o \to \Delta$.

- In the rest of the paper, for a symmetric sequence $X$ and a vertex $v$ of a tree in $\Psi^o$, $X(v)$, $X(v-1)$ and $v-1$ denote $X(|v|)$, $X(|v| - 1)$ and $|v| - 1$ respectively. ($|v|$ is the number of the ‘out going edges’)

- For a $\mathcal{K}$-comodule $X$ in $\mathcal{SP}_\Sigma$, We shall define a functor $F^o_X : (\Psi_{n+2})^{op} \to \mathcal{SP}_\Sigma$. The definition is similar to (a dual of) the construction of $D_n[M]$ in Definition 5.6 of [35]. For a tree $T \in \Psi_{n+2}$, define a space $K_T^{nr}$ by

\[
K_T^{nr} = \prod_v \mathcal{K}(v)
\]

Here, $v$ runs through all the non-root and non-leaf vertices of $T$. This is denoted by $K_T^{nr}$ in [35]. We set $F^o_X(T) = \operatorname{Map}(K_T^{nr}, X(v_r - 1))$.

For a morphism $T \to T'$ given by the contraction of a non-root edge $e$ (an edge not adjacent to the root), the map $e^* : F^o_X(T') \to F^o_X(T)$ is pull-back by the inclusion $K_T^{nr} \to K_{T'}^{nr}$ to a face corresponding to the edge contraction (see Definition 4.26 of [35]). For the $i$-th root edge $e$, the corresponding map is given by the following composition.

\[
\operatorname{Map} \left( \prod_{v \in T'} \mathcal{K}(v), X(v'_r - 1) \right) = \operatorname{Map} \left( \prod_{v \in T \atop v \neq v_r} \mathcal{K}(v), X(v'_r - 1) \right)
\]

\[
\to \operatorname{Map} \left( \prod_{v \in T \atop v \neq v_r} \mathcal{K}(v), \operatorname{Map}(\mathcal{K}(v_r), X(v_r - 1)) \right)
\]

\[
\cong \operatorname{Map} \left( \left( \prod_{v \in T \atop v \neq v_r} \mathcal{K}(v) \right) \times \mathcal{K}(v_r), X(v_r - 1) \right)
\]

\[
= \operatorname{Map} \left( \prod_{v \in T} \mathcal{K}(v), X(v_r - 1) \right).
\]
Here \( v_t \) is the vertex of \( e \) which is not the root. For \( 1 \leq i \leq \vert v_r \vert - 1 \), the arrow in the second line is the pushforward by the adjoint of the partial composition \((- \circ_i -) : \mathcal{K}(v_t) \otimes X(v'_r - 1) \to X(v_r - 1)\), and for \( i = \vert v_r \vert \) it is the pushforward by the adjoint of the composition

\[
\mathcal{K}(v_t) \otimes X(v'_r - 1) \xrightarrow{id \otimes (-)_{\sigma}} \mathcal{K}(v_t) \otimes X(v'_r - 1) \xrightarrow{(- \circ_{\sigma} -)} X(v_r - 1)
\]

where \( \sigma \) is the transposition of the first \( \vert v'_r \vert - |v_t| \) and the last \( \vert v_t \vert - 1 \) letters. The functors \( \{ F_{\Psi} \}_{n} \) is compatible with the inclusion \( i_n : \Psi_{n+2}^0 \to \Psi_{n+3}^0 \). Precisely speaking, there exists an obviously defined natural isomorphism \( j_n : F_{\Psi}^n X \cong F_{\Psi}^{n+1} X \vert_{\Psi_{n+2}^0} \) because the inclusion does not change \( \vert v_r \vert \). We define a functor \( F_{\Psi} X : \Psi^o \to SP^\Sigma \) by \( F_{\Psi} X(T) \) being the colimit of the sequence \( F_{\Psi}^n X(T) \xrightarrow{\cong} F_{\Psi}^{n+1} X(T) \xrightarrow{\cong} F_{\Psi}^{n+2} X(T) \xrightarrow{\cong} \ldots \).

- We shall define a functor \( \omega : (\Psi_{n+2}^o)^{op} \to Cat \). We set \( \omega(T) = G(\vert v_r \vert - 1)^{+} \). For the contraction \( T \to T' \) of an edge \( e \), we define a map \( e^* : v'_r - 1 \to v_r - 1 \) as follows. If \( e \) is a non-root edge, \( e^* \) is the identity. If \( e \) is the \( i \)-th root edge, for \( 1 \leq i \leq \vert v_r \vert - 1 \), \( e^* \) is the order-preserving surjection with \( e^*(j) = i \) \( (i \leq j \leq i + \vert v_r \vert - 1) \). For \( i = \vert v_r \vert \), \( e^* \) is the composition

\[
v'_r - 1 \xrightarrow{(-)_{\sigma}} v'_r - 1 \xrightarrow{(e')_{\sigma}} v_r - 1, \quad \text{where} \quad (e')_{\sigma}(j) = \begin{cases} 1 & (1 \leq j \leq \vert v_r \vert) \\ j - \vert v_t \vert + 1 & (\vert v_t \vert + 1 \leq j \leq \vert v'_r \vert - 1), \end{cases}
\]

and \( \sigma \) is given in the previous item. For \( G \in G(\vert v'_r \vert - 1)^{+} \), we define an object \( e^*(G) \in G(\vert v_r \vert - 1)^{+} \) by

\[
e^*(G) = \begin{cases} \{ (e^*(s), e^*(t)) \mid \langle s, t \rangle \in E(G), \ e^*(s) \neq e^*(t) \} & \text{if} \ G \neq * \ \text{and} \ \langle j, k \rangle \notin E(G) \ \text{for any pair} \ j, k \in \{ i, \ldots, i + \vert v_t \vert - 1 \}, \\
* & \text{otherwise}
\end{cases}
\]

Here, the set \( \{ i, \ldots, i + \vert v_t \vert - 1 \} \) is considered modulo \( \vert v'_r \vert - 1 \) if \( i = \vert v_r \vert \).

- We define a category \( \tilde{\Psi}_{n+2} \) as the Grothendieck construction for the (non-lax) functor \( \omega \)

\[
\tilde{\Psi}_{n+2} = \int_{\Psi_{n+2}^o} \omega^*.
\]

An object of \( \tilde{\Psi}_{n+2} \) is a pair \( (T, G) \) with \( T \in \Psi_{n+2}^o \) and \( G \in \omega(T) \). A map \( (T, G) \to (T', G') \) is a pair of a map \( e : T \to T' \in \Psi_{n+2}^o \) and a map \( G \to e^*(G') \in \omega(T) \).

- We fix a map \( K \to D_1 \) of operads and regard \( \tilde{\mathcal{H}}_M \) as a \( K \)-comodule via this map.

- We shall define a functor \( \text{Th}_n^M : (\Psi_{n+2})^{op} \to SP^\Sigma \). We set

\[
\text{Th}_n^M(T, G) = \begin{cases} \text{Map}(K^n_T, \mathcal{H}_G) & (G \in G(\vert v_r \vert - 1)) \\
* & (G = *)
\end{cases}
\]
Lemma 8.2. Let $\mathcal{M}$ be a cofibrantly generated model category.

1. The pair $(\text{fcolim}, \text{fc})$ is a Quillen adjoint pair.

2. The restriction $\text{Fun}(\omega(T)^{\text{op}}, \mathcal{M}) \to \text{Fun}(\omega(T)^{\text{op}}, \mathcal{M}), \ X \mapsto X_T$ preserves weak equivalences and cofibrations. In particular, the natural map $\text{hocolim}_{\omega(T)} X_T \to \text{Lfcolim}_{\omega(T)} X(T) \in \text{Ho}(\mathcal{M})$ is an isomorphism.

3. For any functor $X \in \text{Fun}(\omega(\Psi)^{\text{op}}, \mathcal{M})$, there is a natural isomorphism in $\text{Ho}(\mathcal{M})$

$$\text{hocolim}_{\omega(\Psi)} \text{Lfcolim}_{\omega(\Psi)} X \cong \text{hocolim}_{\omega(\Psi)} X.$$
Theorem 8.3.  
(1) There exists an isomorphism in $\text{Ho}(\mathcal{F}un((\Psi^o)^{op}, \mathcal{S}\mathcal{P}^\Sigma))$

$$(G \circ F)^*(\mathcal{C}^*([M]))^\Sigma \cong \text{Lcolim}_\omega \text{Th}^M.$$

(2) There exists an isomorphism in $\text{Ho}(\mathcal{C}\mathcal{H}_k)$

$$C^*(\text{Emb}(S^1, M)) \cong \text{hocolim}_\Psi C^S_\ast \circ \text{Th}^M.$$

Proof. By definition, $\mathcal{T}\mathcal{H}_M(n) = \text{colim}_{G \in \omega(T)} \mathcal{T}\mathcal{H}_G$. We shall show the natural map

$$\text{hocolim}_{G \in \omega(T)} \mathcal{T}\mathcal{H}_G \rightarrow \text{colim}_{G \in \omega(T)} \mathcal{T}\mathcal{H}_G = \mathcal{T}\mathcal{H}_M(n) \in \text{Ho}(\mathcal{S}\mathcal{P}^\Sigma)$$

is an isomorphism. By abuse of notations, we denote by $P_\nu(N_0)$ the subcategory of $\nu(T)$ consisting of non-empty graphs, which is actually isomorphic to $P_\nu(N_0)$ for $N_0 = \# \{(i, j) \mid i, j \in \mathbb{N}, \ i < j\}$. Clearly the functor $P_\nu(N_0) \ni G \mapsto \mathcal{T}\mathcal{H}_G \in \mathcal{S}\mathcal{P}^\Sigma$ satisfies the assumption of the part 2 of Lemma 8.2 so the natural map $\text{hocolim} \mathcal{T}\mathcal{H}_G \rightarrow \text{colim} \mathcal{T}\mathcal{H}_G$ is an isomorphism.

As $\mathcal{T}\mathcal{H}_M(n)$ is a cofiber of the natural map $\mathcal{T}\mathcal{H}_G \rightarrow \mathcal{T}\mathcal{H}_M$, which is also a homotopy cofiber, we have the assertion. We define a natural transformation $\text{Th}^M \rightarrow \text{fc}_G \mathcal{T}\mathcal{H}_M$ by the pushforward by the constant map $\mathcal{T}\mathcal{H}_G \rightarrow \{1\} \subseteq \mathcal{T}\mathcal{H}_M(v_{r-1})$ for $G \neq \emptyset \in \omega(T)$, and by the quotient map $\mathcal{T}\mathcal{H}_\emptyset \rightarrow \mathcal{T}\mathcal{H}_M(v_{r-1})$ for $G = \emptyset$. By the assertion and the part 2 of Lemma 8.2, the derived adjoint of the natural transformation $\text{Lcolim}\ \text{Th}^M \rightarrow \text{fc}_G \mathcal{T}\mathcal{H}_M$ is an isomorphism in $\text{Ho}(\mathcal{F}un((\Psi^o)^{op}, \mathcal{S}\mathcal{P}^\Sigma))$. It is clear that $F_\Psi$ preserves weak equivalences so by Theorem 4.4, we have isomorphisms in $\text{Ho}(\mathcal{F}un((\Psi^o)^{op}, \mathcal{S}\mathcal{P}^\Sigma))$

$$F_\Psi(C^\Sigma_\ast([M])) \cong F_\Psi \mathcal{T}\mathcal{H}_M \cong \text{Lcolim}_\omega \text{Th}^M.$$

We define a natural transformation $(G \circ F)^*(\mathcal{C}^*([M]))^\Sigma \rightarrow F_\Psi(C^\Sigma_\ast([M]))$ by the inclusion $C^{v_r-1}([M]) = C_\ast([M]) \subseteq \text{Map}(K_{2r}, C_\ast([M])); (v_{r-1})$ onto constant maps. This is clearly weak equivalence, so we have proved the part 1.

For the part 2, since the functor $C^S_\ast$ preserves homotopy colimits (of strongly semistable spectra), by the part 1, Lemma 8.2 (3), and Lemma 5.3, we have isomorphisms in $\text{Ho}(\mathcal{C}\mathcal{H}_k)$

$$\text{hocolim}_\Psi (G \circ F)^* C^S_\ast (\mathcal{C}^*([M]))^\Sigma \cong \text{hocolim}_\Psi \text{Lcolim} C^S_\ast \circ \text{Th}^M \cong \text{hocolim}_\Psi C^S_\ast \circ \text{Th}^M.$$

By Lemma 5.3, Theorem 5.16 and the fact that $G \circ F : (\Psi^o)^{op} \rightarrow \Delta^{op}$ is (homotopy) right cofinal (see Proposition 4.15 and Theorem 6.7 of [35]), we have isomorphisms in $\text{Ho}(\mathcal{C}\mathcal{H}_k)$

$$C^*(\text{Emb}(S^1, M)) \cong \text{hocolim}_\Delta C^*(\mathcal{C}^*([M]))$$

$$\cong \text{hocolim}_\Delta C^S_\ast (\mathcal{C}^*([M]))^\Sigma \cong \text{hocolim}_\Psi (G \circ F)^* C^S_\ast (\mathcal{C}^*([M]))^\Sigma.$$ 

Thus, we have an isomorphism $C^*(\text{Emb}(S^1, M)) \cong \text{hocolim}_\Psi C^S_\ast \circ \text{Th}^M$. 

\[\square\]

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