AN $\hbar$-EXPANSION OF THE TODA HIERARCHY:
A RECURSIVE CONSTRUCTION OF SOLUTIONS

KANEHISA TAKASAKI AND TAKASHI TAKEBE

Abstract. A construction of general solutions of the $\hbar$-dependent Toda hierarchy is presented. The construction is based on a Riemann-Hilbert problem for the pairs $(L, M)$ and $(\bar{L}, \bar{M})$ of Lax and Orlov-Schulman operators. This Riemann-Hilbert problem is translated to the language of the dressing operators $W$ and $\bar{W}$. The dressing operators are set in an exponential form as $W = e^{X/\hbar}$ and $\bar{W} = e^{\phi/\hbar} e^{\bar{X}/\hbar}$, and the auxiliary operators $X, \bar{X}$ and the function $\phi$ are assumed to have $\hbar$-expansions $X = X_0 + \hbar X_1 + \cdots$, $\bar{X} = \bar{X}_0 + \hbar \bar{X}_1 + \cdots$ and $\phi = \phi_0 + \hbar \phi_1 + \cdots$. The coefficients of these expansions turn out to satisfy a set of recursion relations. $X, \bar{X}$ and $\phi$ are recursively determined by these relations. Moreover, the associated wave functions are shown to have the WKB form $\Psi = e^{S/\hbar}$ and $\bar{\Psi} = e^{\bar{S}/\hbar}$, which leads to an $\hbar$-expansion of the logarithm of the tau function.

0. Introduction

This paper is a continuation of our previous work [TT3, TT4] on a quasi-classical or $\hbar$-dependent (where $\hbar$ is the Planck constant) formulation of the KP hierarchy [TT2]. We presented therein a recursive construction of general solutions to the $\hbar$-dependent KP hierarchy. The construction starts from a Riemann-Hilbert problem for the pair $(L, M)$ of Lax and Orlov-Schulman operators. This Riemann-Hilbert problem can be translated to the language of the underlying dressing operator $W$. Assuming the exponential form $W = e^{\hbar^{-1}X}$ and an $\hbar$-expansion of the operator $X$, one can derive a set of recursion relations that determine the operator $X$ order-by-order of the $\hbar$-expansion from the lowest part (namely, a solution of the dispersionless KP hierarchy [TT2]). Thus the Lax, Orlov-Schulman and dressing operators are obtained. Furthermore, borrowing an idea from Aoki’s exponential calculus of microdifferential operators [A], one can show that the wave function has the WKB form $\Psi = e^{\hbar^{-1}S}$. This leads to an $\hbar$-expansion of the associated tau function as a generalisation of the “genus expansion” of partition functions in string theories and random matrices [D, KT, Mo, dFGZ]. The goal of this paper is to generalise these results to an $\hbar$-dependent formulation of the Toda hierarchy [TT2].

The Toda hierarchy is built from difference operators

$$a(s, e^{\partial_s}) = \sum_m a_m(s) e^{m \partial_s}$$

on a one-dimensional lattice (with coordinate $s \in \mathbb{Z}$) rather than microdifferential operators on a continuous line. Even in the $\hbar$-independent case [TT], the formulation of the hierarchy itself is more complicated than that of the KP hierarchy. The
hierarchy has two sets of time evolutions for time variables $t = (t_n)$ and $\bar{t} = (\bar{t}_n)$. These time evolutions are formulated with two Lax operators $L$ and $\bar{L}$. Orlov-Schulman operators, dressing operators and wave functions, too, are prepared in pairs. In the $\hbar$-dependent formulation \cite{TT2}, the Planck constant $\hbar$ plays the role of lattice spacing, which shows up in the shift operators as $\hbar \partial_s$. Difference operators in the Lax formalism are linear combinations

$$a(\hbar, s, e^{\hbar \partial_s}) = \sum_m a_m(\hbar, s) e^{m \hbar \partial_s}$$

of these shift operators with $\hbar$-dependent coefficients $a_m(\hbar, s)$.

To construct a general solution of the $\hbar$-dependent Toda hierarchy, we start from a Riemann-Hilbert problem for the pairs $(L, M)$ and $(\bar{L}, \bar{M})$ of Lax and Orlov-Schulman operators. This problem can be converted to a problem for the dressing operators $W$ and $\bar{W}$. We seek $W$ and $\bar{W}$ in the exponential form

$$W = e^{\hbar^{-1}X}, \quad \bar{W} = e^{\hbar^{-1}\phi} e^{\hbar^{-1}\bar{X}},$$

where $X$ and $\bar{X}$ are difference operators and $\phi$ is a function of $(h, s, t, \bar{t})$. Assuming that these operators and functions have $\hbar$-expansions, we can derive a set of recursion relations for the coefficients of these expansions. The lowest part of this expansion turns out to be the dressing function of the dispersionless Toda hierarchy \cite{TT1, TT2}. Thus the description of the Lax, Orlov-Schulman and dressing operators are mostly parallel to the case of the $\hbar$-dependent KP hierarchy.

The construction of the associated wave functions exhibits a new feature. To formulate an analogue of Aoki’s exponential calculus for difference operators, we define the “symbol” of a difference operator $a(\hbar, s, e^{\hbar \partial_s})$ to be $a(\hbar, s, \xi)$. The operator product $a(\hbar, s, e^{\hbar \partial_s}) b(\hbar, s, e^{\hbar \partial_s})$ induces the $\circ$-product

$$a(\hbar, s, \xi) \circ b(\hbar, s, \xi) = e^{\hbar \xi \partial_s \partial_{s'} a(\hbar, s, \xi) b(\hbar, s', \xi')} |_{s' = s, \xi' = \xi} = \sum_{n=0}^\infty \frac{\hbar^n}{n!} (\xi \partial_{s'})^n a(\hbar, s, \xi) \partial_{s'}^n b(\hbar, s', \xi') |_{s' = s, \xi' = \xi}$$

for those symbols. Although looking very similar, this product structure is slightly different from the $\circ$-product of the symbols $a(\hbar, x, \xi)$ of $\hbar$-dependent microdifferential operators $a(\hbar, x, \hbar \partial_x)$ in that $\partial_x$ is now replaced with $\xi \partial_x$. This tiny difference, however, has a considerable effect; unlike $\partial_x$, $\xi \partial_x$ does not lower the order with respect to $\xi$. Because of this, we are forced to modify our previous method \cite{TT3}.

We admit that our construction of solutions is extremely complicated. The recursive procedure is illustrated in Appendix for a special case that is related to $c = 1$ string theory at self-dual radius \cite{DMP, EK, HOP}. Even in this relatively simple case, we have been unable to derive an explicit form of the solution unless a half of the full time variables are set to zero. This is a price to pay for treating general solutions. In this sense, our method cannot be directly compared with the method in random matrix theory \cite{Me, dFGZ}, in particular, Eynard and Orantin’s topological recursion relations \cite{EO}. Their recursion relations stem from the “loop equations” for random matrices, which are constraints to single out a class of special solutions of an underlying integrable hierarchy, while our method does not use any extra structure other than the integrable hierarchy itself.

This paper is organised as follows. Section 1 is a review of the $\hbar$-dependent formulation of the Toda hierarchy. The Riemann-Hilbert problem is also formulated therein. Section 2 presents the recursive solution of the Riemann-Hilbert problem.
The method is a rather straightforward generalisation of the case of the \( h \)-dependent KP hierarchy. Section 3 deals with the \( h \)-expansion of the wave function. Aoki’s exponential calculus is reformulated for difference operators. Relevant recursion relations are thereby derived, and shown to have a solution. Section 4 mentions the \( h \)-expansion of the tau function.

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1. \( h \)-dependent Toda hierarchy: review

In this section we recall several facts on the Toda hierarchy depending on a formal parameter \( h \) in [TT2], §2.7. Throughout this paper all functions are formal power series.

The \( h \)-dependent Toda hierarchy is defined by the Lax representation

\[
\begin{align*}
\hbar \frac{\partial L}{\partial n} &= [B_n, L], & \hbar \frac{\partial L}{\partial n} &= [\bar{B}_n, L], \\
\hbar \frac{\partial \bar{L}}{\partial \bar{n}} &= [B_n, \bar{L}], & \hbar \frac{\partial \bar{L}}{\partial \bar{n}} &= [\bar{B}_n, \bar{L}], \\
B_n &= (L^n)_{\geq 0}, & \bar{B}_n &= (\bar{L}^{\bar{n}})_{\leq -1}, & n = 1, 2, \ldots,
\end{align*}
\]

where the Lax operators \( L, \bar{L} \) are difference operators with respect to the discrete independent variable \( s \in \hbar \mathbb{Z} \) of the form

\[
\begin{align*}
L &= e^{h\partial_s} + \sum_{n=0}^{\infty} u_{n+1}(\hbar, s, \bar{t}, \bar{t}) e^{-n\hbar \partial_s}, \\
\bar{L}^{-1} &= \bar{u}_0(\hbar, t, \bar{t}, \bar{t}) e^{-h\partial_s} + \sum_{n=0}^{\infty} \bar{u}_{n+1}(\hbar, t, \bar{t}, \bar{t}) e^{n\hbar \partial_s}
\end{align*}
\]

and \( (\quad)_{\geq 0} \) and \( (\quad)_{\leq -1} \) are projections onto a linear combination of \( e^{n\hbar \partial_s} \) with \( n \geq 0 \) and \( \leq -1 \), respectively. Note that \( e^{h\partial_s} \) is a difference operator with step \( h: e^{h\partial_s} f(s) = f(s + nh) \). The coefficients \( u_n(\hbar, t, \bar{t}, s), \bar{u}_n(\hbar, t, \bar{t}, s) \) of \( L, \bar{L} \) are assumed to be formally regular with respect to \( h: u_n(\hbar, t, \bar{t}, s) = \sum_{m=0}^{\infty} h^m u_n^{(m)}(t, \bar{t}, s) \), \( \bar{u}_n(\hbar, t, \bar{t}, s) = \sum_{m=0}^{\infty} \bar{h}^m \bar{u}_n^{(m)}(t, \bar{t}, s) \) as \( h \to 0 \).

We define the \( h \)-order of the difference operator by

\[
\text{ord}_h \left( \sum_{n,m} a_{n,m}(t, \bar{t}, s) \hbar^n e^{m\hbar \partial_s} \right) \overset{\text{def}}{=} \max \left\{ -n \left\vert \sum_{m} a_{n,m}(t, \bar{t}, s) e^{m\hbar \partial_s} \neq 0 \right\} \right. ,
\]

where
In particular, \( \text{ord}^\hbar \hbar = -1 \), \( \text{ord}^\hbar e^{\hbar \partial_s} = 0 \). For example, the condition which we imposed on the coefficients \( u_{n}(\hbar, t, \bar{t}, s) \) and \( u_{n}(\hbar, t, \bar{t}, s) \) can be restated as \( \text{ord}^\hbar (L) = \text{ord}^\hbar (\bar{L}) = 0 \).

The principal symbol (resp. the symbol of order \( l \)) of a difference operator \( A = \sum a_{n,m}(t, \bar{t}, s) \hbar^n e^{m \hbar \partial_s} \) with respect to the \( \hbar \)-order is

\[
\sigma^\hbar(A) \overset{\text{def}}{=} \sum_m a_{-\text{ord}^\hbar(A),m}(t, \bar{t}, s) \xi^m,
\]

and it is defined as

\[
\sigma^\hbar(A) \overset{\text{def}}{=} \sum m a_{-l,m}(t, \bar{t}, s) \xi^m.
\]

When it is clear from the context, we sometimes use \( \sigma^\hbar \) instead of \( \sigma^\hbar_l \).

The Lax operators \( L \) and \( \bar{L} \) are expressed by dressing operators \( W \) and \( \bar{W} \):

\[
(1.7) \quad L = \text{Ad} \: W(e^{\hbar \partial_s}) = W e^{\hbar \partial_s} W^{-1}, \quad \bar{L} = \text{Ad} \: \bar{W}(e^{\hbar \partial_s}) = \bar{W} e^{\hbar \partial_s} \bar{W}^{-1},
\]

The operators \( W \) and \( \bar{W} \) should have specific forms:

\[
(1.8) \quad W = e^{\hbar^{-1}X(\hbar, t, \bar{t}, s, e^{\hbar \partial_s})} e^{-\hbar^{-1} \alpha(\hbar) e^{\hbar \partial_s}},
\]

\[
(1.9) \quad X(\hbar, t, \bar{t}, s, e^{\hbar \partial_s}) = \sum_{k=1}^{\infty} \chi_k^\hbar(t, \bar{t}, s) e^{-k \hbar \partial_s},
\]

\[
(1.10) \quad \bar{W} = e^{\hbar^{-1}\bar{\phi}(\hbar, t, \bar{t}, s)} e^{\hbar^{-1}X(\hbar, t, \bar{t}, s, e^{\hbar \partial_s})} e^{-\hbar^{-1} \bar{\alpha}(\hbar) e^{\hbar \partial_s}},
\]

\[
(1.11) \quad \bar{X}(\hbar, t, \bar{t}, s, e^{\hbar \partial_s}) = \sum_{k=1}^{\infty} \bar{\chi}_k^\hbar(t, \bar{t}, s) e^{k \hbar \partial_s},
\]

\[
(1.12) \quad \text{ord}^\hbar(\phi(\hbar, t, \bar{t}, s)) = \text{ord}^\hbar(X(\hbar, t, \bar{t}, s, e^{\hbar \partial_s})) = \text{ord}^\hbar \alpha(\hbar) = \text{ord}^\hbar \bar{\alpha}(\hbar) = 0,
\]

and \( \alpha(\hbar) \) and \( \bar{\alpha}(\hbar) \) are constants with respect to \( t, \bar{t} \) and \( s \). (In \textbf{TT2} we did not introduce \( \alpha, \bar{\alpha} \), which will be necessary in Section \textbf{2}.)

Note that the set of operators of the form

\[
(1.13) \quad ah \frac{\partial}{\partial s} + \sum_k \chi_k(s) e^{k \hbar \partial_s},
\]

where \( a \) does not depend on \( s \) and \( \chi_k(s) \) are functions of \( s \), is closed under the commutator bracket. Hence any theorem or formula for Lie algebras can be applied to such operators. In particular, using the Campbell-Hausdorff formula, we can rewrite \( W \) and \( \bar{W} \) in the following form, which will be more convenient in the later
discussion:

\[(1.14) \quad W = e^{\hbar^{-1}X(h,t,\bar{t},s,e^{\hbar \sigma_3})} \]

\[(1.15) \quad X(h,t,\bar{t},s,e^{\hbar \sigma_3}) = \alpha(h)\hbar \frac{\partial}{\partial s} + \sum_{k=1}^{\infty} \chi_k(h,t,\bar{t},s)e^{-k\hbar \sigma_3}, \]

\[(1.16) \quad \bar{W} = e^{\hbar^{-1}\phi(h,t,\bar{t},s)}e^{\hbar^{-1}X(h,t,\bar{t},s,e^{\hbar \sigma_3})}, \]

\[(1.17) \quad \bar{X}(h,t,\bar{t},s,e^{\hbar \sigma_3}) = \bar{\alpha}(h)\hbar \frac{\partial}{\partial s} + \sum_{k=1}^{\infty} \bar{\chi}_k(h,t,\bar{t},s)e^{k\hbar \sigma_3}, \]

\[(1.18) \quad \text{ord}^h(\phi(h,t,\bar{t},s)) = \text{ord}^h(X(h,t,\bar{t},s,e^{\hbar \sigma_3})) = \text{ord}^h \alpha(h) = 0. \]

Here we define the \(h\)-order and the principal symbol of operators of the form (1.13), in particular those of \(X\) and \(\bar{X}\), by defining \(\text{ord}^h(h \partial_s) = 0\) and \(\sigma^h(h \partial_s) = \log \xi\), which are consistent with the former definitions (1.4) and (1.5).

The wave functions \(\Psi(h,t,\bar{t},s;z)\) and \(\bar{\Psi}(h,t,\bar{t},s;\bar{z})\) are defined by

\[(1.19) \quad \Psi(h,t,\bar{t},s;z) = Wz^{s/h}e^{\zeta(t,z)/h}, \quad \bar{\Psi}(h,t,\bar{t},s;\bar{z}) = \bar{W}\bar{z}^{s/h}e^{\zeta(\bar{t},\bar{z}^{-1})/h}, \]

where \(\zeta(t,z) = \sum_{n=1}^{\infty} t_n z^n, \quad \zeta(\bar{t},\bar{z}^{-1}) = \sum_{n=1}^{\infty} \bar{t}_n \bar{z}^{-n}\). They are solutions of linear equations

\[L\Psi = z\Psi, \quad \hbar \frac{\partial \Psi}{\partial t_n} = B_n \Psi, \quad \hbar \frac{\partial \Psi}{\partial \bar{t}_n} = \bar{B}_n \Psi, \quad (n = 1, 2, \ldots), \]

\[\bar{L}\bar{\Psi} = \bar{z}\bar{\Psi}, \quad \hbar \frac{\partial \bar{\Psi}}{\partial \bar{t}_n} = B_n \bar{\Psi}, \quad \hbar \frac{\partial \bar{\Psi}}{\partial t_n} = \bar{B}_n \bar{\Psi}, \quad (n = 1, 2, \ldots), \]

and have the WKB form (3.3), as we shall show in Section 3. Moreover they are expressed by means of the tau function \(\tau(h,t,\bar{t},s)\) as follows:

\[(1.20) \quad \Psi(h,t,\bar{t};z) = \frac{\tau(h,t-h[z^{-1}],\bar{t},s)}{\tau(h,t,s)} z^{a(h)/h} z^{s/h} e^{\zeta(t,z)/h}, \]

\[\bar{\Psi}(h,t,\bar{t};\bar{z}) = \frac{\tau(h,t,\bar{t}-h,\bar{z}^{-1},s+h)}{\tau(h,t,s)} \bar{z}^{a(h)/h} \bar{z}^{s/h} e^{\zeta(\bar{t},\bar{z}^{-1})/h} \]

where \([z^{-1}] = (1/z, 1/2z^2, 1/3z^3, \ldots), \quad [\bar{z}] = (\bar{z}, \bar{z}^2/2, \bar{z}^3/3, \ldots).\) We shall study the \(h\)-expansion of the tau function in Section 4.

The Orlov-Schulman operators \(M\) and \(\bar{M}\) [OS] are defined by

\[(1.21) \quad M = \text{Ad} \left( W \exp \left( h^{-1} \zeta(t,e^{\hbar \sigma_3}) \right) \right) s = W \left( \sum_{n=1}^{\infty} nt_n e^{n\hbar \sigma_3} + s \right) W^{-1} \]

\[(1.22) \quad \bar{M} = \text{Ad} \left( \bar{W} \exp \left( h^{-1} \zeta(\bar{t},e^{-\hbar \sigma_3}) \right) \right) s = \bar{W} \left( -\sum_{n=1}^{\infty} nt_n e^{-n\hbar \sigma_3} + s \right) \bar{W}^{-1} \]
where \( \zeta(t, e^{\hbar \partial_x}) = \sum_{n=1}^{\infty} t_n e^{\hbar \partial_x} \) and \( \zeta(\bar{t}, e^{-\hbar \partial_x}) = \sum_{n=1}^{\infty} \bar{t}_n e^{-\hbar \partial_x} \). It is easy to see that \( M \) and \( \bar{M} \) have forms

\[
(1.23) \quad M = \sum_{n=1}^{\infty} n t_n L^n + s + \alpha(h) + \sum_{n=1}^{\infty} v_n(h, t, \bar{t}, s) L^{-n},
\]

\[
(1.24) \quad \bar{M} = -\sum_{n=1}^{\infty} n \bar{t}_n L^{-n} + s + \bar{\alpha}(h) + \sum_{n=1}^{\infty} \bar{v}_n(h, t, \bar{t}, s) \bar{L}^n.
\]

and satisfies the following properties:

- \( \text{ord}^h(M) = \text{ord}^h(\bar{M}) = 0; \)
- the canonical commutation relation: \( [L, M] = hL \) and \( [\bar{L}, \bar{M}] = h\bar{L} \);
- the same Lax equations as \( L, \bar{L} \):

\[
(1.25) \quad \frac{\hbar}{\partial t_n} \partial M = [B_n, M], \quad \frac{\hbar}{\partial t_n} \partial \bar{M} = [\bar{B}_n, \bar{M}],
\]

\[
\frac{\hbar}{\partial t_n} \partial M = [B_n, \bar{M}], \quad \frac{\hbar}{\partial t_n} \partial \bar{M} = [\bar{B}_n, M], \quad n = 1, 2, \ldots ;
\]

- another linear equation for the wave function \( \Psi \):

\[
M \Psi = h \bar{z} \frac{\partial \Psi}{\partial \bar{z}}, \quad \bar{M} \Psi = \bar{z} \frac{\partial \Psi}{\partial z}.
\]

Remark 1.1. As in the KP case (Remark 1.2 in [TT1]), if an operator \( M \) of the form (1.23) and an operator \( M \) of the form (1.24) satisfy the Lax equations (1.25) and the canonical commutation relation \( [L, M] = hL \) and \( [\bar{L}, \bar{M}] = h\bar{L} \) with the Lax operator \( L \) and \( \bar{L} \) of the Toda lattice hierarchy, then \( \alpha(h) \) and \( \bar{\alpha}(h) \) in the expansions (1.23) and (1.24) do not depend on any \( t_n, \bar{t}_n \) nor \( s \). In fact, suppose that \( \alpha(h) \) depends on \( s; \alpha(h) = \alpha(h, s) \). Then, the canonical commutation relation \( [L, M] = hL \) is expanded as

\[
(h + \alpha(h, s + h) - \alpha(h, s)) e^{\hbar \partial_x} + \text{(difference operators of lower order)} = hL,
\]

which implies \( \alpha(h, s + h) = \alpha(h, s) \). Similarly, from (1.25) follows \( \frac{\partial \alpha}{\partial t_n} = \frac{\partial \bar{\alpha}}{\partial \bar{t}_n} = 0 \) with the help of (1.11) and \( [L^n, M] = nhL^n \). For \( \bar{\alpha}(s) \) the proof is the same.

The following proposition (Proposition 2.7.11 of [TT2]) is a “dispersionful” counterpart of the theorem for the dispersionless Toda hierarchy found earlier (cf. Section 4 of [TT1] and Proposition 1.3 below).

**Proposition 1.2.** (i) Suppose that operators \( f(h, s, e^{\hbar \partial_x}), g(h, s, e^{\hbar \partial_x}), \tilde{f}(h, s, e^{\hbar \partial_x}), \tilde{g}(h, s, e^{\hbar \partial_x}), \bar{L}, \bar{\bar{L}}, M \) and \( \bar{M} \) satisfy the following conditions:

- \( \text{ord}^h f = \text{ord}^h g = \text{ord}^h \tilde{f} = \text{ord}^h \tilde{g} = 0; \quad [f, g] = h f, \quad [\tilde{f}, \tilde{g}] = h \bar{f}; \)
- \( L, \bar{L}, M \) and \( \bar{M} \) are of the form (1.22), (1.23), (1.22) and (1.24) respectively.

They are canonically commuting: \( [L, M] = hL, \quad [\bar{L}, \bar{M}] = h\bar{L}; \)

- Equations

\[
(1.26) \quad f(h, M, L) = \tilde{f}(h, \bar{M}, \bar{L}), \quad g(h, M, L) = \tilde{g}(h, \bar{M}, \bar{L}) \]

hold.

Then the pair \( (L, \bar{L}) \) is a solution of the Toda lattice hierarchy (1.1) and \( M \) and \( \bar{M} \) are the corresponding Orlov-Schulman operators.

(ii) Conversely, for any solution \( (L, \bar{L}, M, \bar{M}) \) of the \( h \)-dependent Toda lattice hierarchy there exists a quadruplet \((f, \tilde{f}, g, \tilde{g})\) satisfying the conditions in (i).
The leading term of the $h$-dependent Toda lattice hierarchy with respect to the $h$-order gives the *dispersionless Toda hierarchy*. Namely,

\begin{align}
\mathcal{L} &:= \sigma^h(L) = \xi + \sum_{n=0}^{\infty} u_{0,n+1} \xi^{-n}, \quad (u_{0,n+1} := \sigma^h(u_{n+1})), \\
\bar{\mathcal{L}}^{-1} &:= \sigma^h(\bar{L}^{-1}) = \bar{u}_0 \xi^{-1} + \sum_{n=0}^{\infty} u_{0,n+1} \xi^n, \quad (\bar{u}_{0,n+1} := \sigma^h(\bar{u}_{n+1}))
\end{align}

satisfy the dispersionless Lax type equations

\begin{align}
\frac{\partial \mathcal{L}}{\partial t_n} &= \{B_n, \mathcal{L}\}, \quad \frac{\partial \bar{\mathcal{L}}}{\partial t_n} = \{\bar{B}_n, \mathcal{L}\}, \\
\frac{\partial \mathcal{L}}{\partial t_n} &= \{B_n, \bar{\mathcal{L}}\}, \quad \frac{\partial \bar{\mathcal{L}}}{\partial t_n} = \{\bar{B}_n, \bar{\mathcal{L}}\}, \\
\mathcal{B}_n &= (\mathcal{L}^n)_{\geq 0}, \quad \bar{\mathcal{B}}_n = (\bar{\mathcal{L}}^{-n})_{\leq 0}, \quad n = 1, 2, \ldots,
\end{align}

where $(\_ \geq 0)$ and $(\_ \geq 0)$ are the truncation of Laurent series to the polynomial part and to the negative order part respectively. The Poisson bracket $\{,\}$ is defined by

\begin{equation}
\{a(s, \xi), b(s, \xi)\} = \xi \left( \frac{\partial a}{\partial \xi} \frac{\partial b}{\partial s} - \frac{\partial a}{\partial s} \frac{\partial b}{\partial \xi} \right).
\end{equation}

The dressing operation \[\text{(1.7)}\] for $L$ and $\bar{L}$ becomes the following dressing operation for $\mathcal{L}$ and $\bar{\mathcal{L}}$:

\begin{align}
\mathcal{L} &= \exp(\text{ad}_{\{,\}} X_0) \xi, \quad X_0 := \sigma^h(X), \\
\bar{\mathcal{L}} &= \exp(\text{ad}_{\{,\}} \phi_0) \exp(\text{ad}_{\{,\}} \bar{X}_0) \xi, \quad \phi_0 := \sigma^h(\phi), \quad \bar{X}_0 := \sigma^h(\bar{X}),
\end{align}

where $\text{ad}_{\{,\}}(f)(g) := \{f, g\}$.

The principal symbol of the Orlov-Schulman operators are *Orlov-Schulman functions*,

\begin{align}
\mathcal{M} &= \sum_{n=1}^{\infty} n t_n \mathcal{L}^n + s + \alpha_0 + \sum_{n=1}^{\infty} v_{0,n} \mathcal{L}^{-n}, \\
(v_{0,n} := \sigma^h(v_n), \quad \alpha_0 := \sigma^h(\alpha(h)))
\end{align}

\begin{align}
\bar{\mathcal{M}} &= -\sum_{n=1}^{\infty} n \bar{t}_n \bar{\mathcal{L}}^{-n} + s + \bar{\alpha}_0 + \sum_{n=1}^{\infty} \bar{v}_{0,n} \bar{\mathcal{L}}^n, \\
(\bar{v}_{0,n} := \sigma^h(\bar{v}_n), \quad \bar{\alpha}_0 := \sigma^h(\bar{\alpha}_0(h)));
\end{align}

which are equal to

\begin{align}
\mathcal{M} &= \exp(\text{ad}_{\{,\}} X_0) \exp(\text{ad}_{\{,\}} \zeta(t, \xi)) s, \\
\bar{\mathcal{M}} &= \exp(\text{ad}_{\{,\}} \phi_0) \exp(\text{ad}_{\{,\}} \bar{X}_0) \exp(\text{ad}_{\{,\}} \zeta(\bar{t}, \xi^{-1})) s
\end{align}

where $\zeta(t, \xi) = \sum_{n=1}^{\infty} t_n \xi^n$ and $\zeta(\bar{t}, \xi^{-1}) = \sum_{n=1}^{\infty} \bar{t}_n \xi^{-n}$. The series $\mathcal{M}$ satisfies the canonical commutation relation with $\mathcal{L}$, $\{\mathcal{L}, \mathcal{M}\} = \mathcal{L}$, while $\bar{\mathcal{M}}$ satisfies the canonical commutation relation with $\bar{\mathcal{L}}$, $\{\bar{\mathcal{L}}, \bar{\mathcal{M}}\} = \bar{\mathcal{L}}$. The principal symbols of
Then the pair \( (\mathcal{M}, L) \) hold, or equivalently, the following equations hold:

\[
\frac{\hbar}{\partial t_n} \mathcal{M} = \{\mathcal{B}_n, \mathcal{M}\}, \quad \frac{\hbar}{\partial t_n} \mathcal{M} = \{\mathcal{B}_n, \mathcal{M}\}, \quad n = 1, 2, \ldots,
\]

The Riemann-Hilbert type construction of the solution is essentially the same as Proposition 1.2 (Proposition 2.5.1 of [TT2]; We do not need to assume the canonical commutation relation \( \{\mathcal{L}, \mathcal{M}\} = 0 \) and \( \{\mathcal{L}, \mathcal{M}\} = \mathcal{L} \)).

**Proposition 1.3.** (i) Suppose that functions \( f_0(s, \xi), g_0(s, \xi), \tilde{f}_0(s, \xi), \tilde{g}_0(s, \xi), \mathcal{L}, \mathcal{L}, \mathcal{M} \) and \( \mathcal{M} \) satisfy the following conditions:

- \( \{f_0, g_0\} = f_0, \{\tilde{f}_0, \tilde{g}_0\} = \tilde{f}_0; \)
- \( \mathcal{L}, \mathcal{L}^{-1}, \mathcal{M} \) and \( \mathcal{M} \) have the form (1.27), (1.28), (1.32) and (1.33) respectively.

Equations

\[
f_0(\mathcal{M}, \mathcal{L}) = \tilde{f}_0(\tilde{\mathcal{M}}, \tilde{\mathcal{L}}), \quad g_0(\mathcal{M}, \mathcal{L}) = \tilde{g}_0(\tilde{\mathcal{M}}, \tilde{\mathcal{L}}).
\]

hold.

Then the pair \( (\mathcal{L}, \tilde{\mathcal{L}}) \) is a solution of the dispersionless Toda hierarchy (1.29) and \( \tilde{\mathcal{M}} \) and \( \tilde{\mathcal{M}} \) are the corresponding Orlov-Schulman functions.

(ii) Conversely, for any solution \( (\mathcal{L}, \tilde{\mathcal{L}}, \mathcal{M}, \tilde{\mathcal{M}}) \) of the dispersionless Toda hierarchy, there exists a quadruplet \( (f_0, g_0, \tilde{f}_0, \tilde{g}_0) \) satisfying the conditions in (i).

If \( (f, g, \tilde{f}, \tilde{g}), (L, \tilde{L}, M, \tilde{M}) \) are as in Proposition 1.2 then \( (f_0 = \sigma^h(f), g_0 = \sigma^h(g), \tilde{f}_0 = \sigma^h(f), \tilde{g}_0 = \sigma^h(g)) \), \( (\mathcal{L} = \sigma^h(L), \tilde{\mathcal{L}} = \sigma^h(L), \mathcal{M} = \sigma^h(M), \tilde{\mathcal{M}} = \sigma^h(M)) \) satisfy the conditions in Proposition 1.3. In other words, \( (f, g, \tilde{f}, \tilde{g}) \) and \( (L, \tilde{L}, M, \tilde{M}) \) are quantisation of \( (f_0, g_0, \tilde{f}_0, \tilde{g}_0) \) and \( (\mathcal{L}, \mathcal{M}, \tilde{\mathcal{L}}, \tilde{\mathcal{M}}) \) respectively. (See, for example, [S] for quantised canonical transformations.)

2. Recursive Construction of the Dressing Operator

In this section we prove that the solution of the \( \hbar \)-dependent Toda lattice hierarchy corresponding to \( (f, g, \tilde{f}, \tilde{g}) \) in Proposition 1.2 is recursively constructed from its leading term, i.e., the solution of the dispersionless Toda hierarchy corresponding to the Riemann-Hilbert data \( (\sigma^h(f), \sigma^h(g), \sigma^h(f), \sigma^h(g)) \).

Given the quadruplet \( (f, g, \tilde{f}, \tilde{g}) \), we have to construct the dressing operator \( W \) and \( \tilde{W} \), or, in other words, \( X \) in (1.14) and \( \phi, \tilde{X} \) in (1.15), such that equations (1.26) hold, or equivalently, the following equations hold:

\[
\begin{align*}
\Ad (W \exp (\hbar^{-1} \zeta(t, e^{h \partial s})) ) f(h, s, e^{h \partial s}) &= \Ad (W \exp (\hbar^{-1} \zeta(t, e^{h \partial s})) ) \tilde{f}(h, s, e^{h \partial s}), \\
Ad (W \exp (\hbar^{-1} \zeta(t, e^{h \partial s})) ) g(h, s, e^{h \partial s}) &= Ad (W \exp (\hbar^{-1} \zeta(t, e^{h \partial s})) ) \tilde{g}(h, s, e^{h \partial s}).
\end{align*}
\]
Let us expand $X$, $\bar{X}$ and $\phi$ with respect to the $h$-order as follows:

\begin{align}
(2.2) \quad X &= \sum_{n=0}^{\infty} h^n X_n, \quad X_n = X_n(t, \bar{t}, s, e^{h\partial_s}) = \alpha_n h \frac{\partial}{\partial s} + \sum_{k=1}^{\infty} \chi_{n,k}(t, \bar{t}, s)e^{-kh\partial_s}, \\
(2.3) \quad \bar{X} &= \sum_{n=0}^{\infty} h^n \bar{X}_n, \quad \bar{X}_n = \bar{X}_n(t, \bar{t}, s, e^{h\partial_s}) = \bar{\alpha}_n h \frac{\partial}{\partial s} + \sum_{k=1}^{\infty} \bar{\chi}_{n,k}(t, \bar{t}, s)e^{kh\partial_s}, \\
(2.4) \quad \phi &= \sum_{n=0}^{\infty} h^n \phi_n(t, \bar{t}, s), \quad \phi_n = \phi_n(t, \bar{t}, s),
\end{align}

where $\alpha_n$, $\bar{\alpha}_n$, $\chi_{n,k}$ and $\bar{\chi}_{n,k}$ do not depend on $h$, and hence $\alpha$, $\chi_k$ in $[14.15]$ and $\bar{\alpha}$, $\bar{\chi}_k$ in $[14.17]$ are expanded as $\alpha = \sum_{n=0}^{\infty} h^n \alpha_n$, $\chi_k = \sum_{n=0}^{\infty} h^n \chi_{n,k}$, $\bar{\alpha} = \sum_{n=0}^{\infty} h^n \bar{\alpha}_n$ and $\bar{\chi}_k = \sum_{n=0}^{\infty} h^n \bar{\chi}_{n,k}$.

Assume that a solution of the dispersionless Toda hierarchy corresponding to $(\sigma^h(f), \sigma^h(g), \sigma^h(\bar{f}), \sigma^h(\bar{g}))$ is given. In other words, assume that symbols $X_0 = \alpha_0 \log \xi + \sum_{k=1}^{\infty} \chi_{0,k}(t, \bar{t}, s)\xi^k$, $\bar{X}_0 = \bar{\alpha}_0 \log \xi + \sum_{k=1}^{\infty} \bar{\chi}_{0,k}(t, \bar{t}, s)\xi^k$ and $\phi_0 = \phi_0(t, \bar{t}, s)$ are given such that

$$\sigma^h(f)(\mathcal{M}, \mathcal{L}) = \sigma^h(\bar{f})(\mathcal{M}, \mathcal{L}),$$

namely,

\begin{align*}
&\exp(\text{ad}_{\xi\lambda}) X_0 \exp(\text{ad}_{\xi\lambda}) \zeta(t, \xi)) \sigma^h(f)(s, \xi) \\
&= \exp(\text{ad}_{\xi\lambda}) \phi_0 \exp(\text{ad}_{\xi\lambda}) \bar{X}_0 \exp(\text{ad}_{\xi\lambda}) \zeta(\bar{t}, \xi^{-1})) \sigma^h(\bar{f})(s, \xi),
\end{align*}

and

$$\sigma^h(g)(\mathcal{M}, \mathcal{L}) = \sigma^h(\bar{g})(\mathcal{M}, \mathcal{L}),$$

namely,

\begin{align*}
&\exp(\text{ad}_{\xi\lambda}) X_0 \exp(\text{ad}_{\xi\lambda}) \zeta(t, \xi)) \sigma^h(g)(s, \xi) \\
&= \exp(\text{ad}_{\xi\lambda}) \phi_0 \exp(\text{ad}_{\xi\lambda}) \bar{X}_0 \exp(\text{ad}_{\xi\lambda}) \zeta(\bar{t}, \xi^{-1})) \sigma^h(\bar{g})(s, \xi).
\end{align*}

(See Proposition $[13.3]$)

We are to construct $X_n$, $\bar{X}_n$ and $\phi_n$ recursively, starting from $X_0$, $\bar{X}_0$ and $\phi_0$.

For this purpose expand both sides of equations (2.1) as follows:

\begin{align}
(2.6) \quad P := \text{Ad} \left( \exp(h^{-1}X) \right) f_t &= \sum_{k=0}^{\infty} h^k P_k, \\
(2.7) \quad Q := \text{Ad} \left( \exp(h^{-1}X) \right) g_t &= \sum_{k=0}^{\infty} h^k Q_k, \\
(2.8) \quad \bar{P} := \text{Ad} \left( \exp(h^{-1}\phi) \exp(h^{-1}\bar{X}) \right) \bar{f}_t &= \sum_{k=0}^{\infty} h^k \bar{P}_k, \\
(2.9) \quad \bar{Q} := \text{Ad} \left( \exp(h^{-1}\phi) \exp(h^{-1}\bar{X}) \right) \bar{g}_t &= \sum_{k=0}^{\infty} h^k \bar{Q}_k,
\end{align}

where

\begin{align}
(2.10) \quad f_t := \text{Ad} \left( e^{h^{-1}\zeta(t, e^{h\partial_s})} \right) f, & \quad g_t := \text{Ad} \left( e^{h^{-1}\zeta(t, e^{h\partial_s})} \right) g, \\
(2.11) \quad \bar{f}_t := \text{Ad} \left( e^{h^{-1}\zeta(\bar{t}, e^{-h\partial_s})} \right) \bar{f}, & \quad \bar{g}_t := \text{Ad} \left( e^{h^{-1}\zeta(\bar{t}, e^{-h\partial_s})} \right) \bar{g},
\end{align}
and $P_i$’s, $Q_i$’s, $\bar{P}_i$’s and $\bar{Q}_i$’s are difference operators of the $h$-order 0:
\[
\text{ord}^h P_i = \text{ord}^h Q_i = \text{ord}^h \bar{P}_i = \text{ord}^h \bar{Q}_i = 0.
\]
Suppose that we have chosen $X_0, \ldots, X_{i-1}, \bar{X}_0, \ldots, \bar{X}_{i-1}$ and $\phi_0, \ldots, \phi_{i-1}$ so that $P_j = \bar{P}_j$ ($0 \leq j \leq i - 1$) and $Q_j = \bar{Q}_j$ ($0 \leq j \leq i - 1$). If operators $X_i$, $\bar{X}_i$ and a function $\phi_i$ are constructed from these given $X_j$, $\bar{X}_j$ and $\phi_j$ ($0 \leq j \leq i - 1$) so that equations $P_i = \bar{P}_i$ and $Q_i = \bar{Q}_i$ hold, this procedure gives recursive construction of $X$, $\bar{X}$ and $\phi$ in question.

We can construct such $X_i$, $\bar{X}_i$ and $\phi_i$ as follows. (Details and meaning shall be explained in the proof of Theorem 2.1):

- **(Step 0)** Assume $X_j$, $\bar{X}_j$ and $\phi_j$ ($0 \leq j \leq i - 1$) are given and set
\[
X^{(i-1)} := \sum_{n=0}^{i-1} h^n X_n, \quad \bar{X}^{(i-1)} := \sum_{n=0}^{i-1} h^n \bar{X}_n, \quad \phi^{(i-1)} := \sum_{n=0}^{i-1} h^n \phi_n.
\]

- **(Step 1)** Set
\[
P^{(i-1)} := \text{Ad} \left( \exp h^{-1} X^{(i-1)} \right) f_t, \quad Q^{(i-1)} := \text{Ad} \left( \exp h^{-1} X^{(i-1)} \right) g_t,
\]
\[
\bar{P}^{(i-1)} := \text{Ad} \left( \exp h^{-1} \phi^{(i-1)} \right) \text{Ad} \left( \exp h^{-1} \bar{X}^{(i-1)} \right) \bar{f}_t,
\]
\[
\bar{Q}^{(i-1)} := \text{Ad} \left( \exp h^{-1} \phi^{(i-1)} \right) \text{Ad} \left( \exp h^{-1} \bar{X}^{(i-1)} \right) \bar{g}_t,
\]
and expand $P^{(i-1)}$ and $Q^{(i-1)}$ with respect to the $h$-order as
\[
P^{(i-1)} = \sum_{k=0}^{\infty} h^k P_k^{(i-1)}, \quad Q^{(i-1)} = \sum_{k=0}^{\infty} h^k Q_k^{(i-1)},
\]
\[
\bar{P}^{(i-1)} = \sum_{k=0}^{\infty} h^k \bar{P}_k^{(i-1)}, \quad \bar{Q}^{(i-1)} = \sum_{k=0}^{\infty} h^k \bar{Q}_k^{(i-1)}.
\]

$(\text{ord}^h P_k^{(i-1)} = \text{ord}^h Q_k^{(i-1)} = \text{ord}^h \bar{P}_k^{(i-1)} = \text{ord}^h \bar{Q}_k^{(i-1)} = 0.)$

- **(Step 2)** Put
\[
\mathcal{P}_0 := \sigma^h (P_0^{(i-1)}), \quad \mathcal{Q}_0 := \sigma^h (Q_0^{(i-1)}),
\]
\[
\bar{P}_0 := \sigma^h (\bar{P}_0^{(i-1)}), \quad \bar{Q}_0 := \sigma^h (\bar{Q}_0^{(i-1)}),
\]
\[
\mathcal{P}_i := \sigma^h (P_i^{(i-1)}), \quad \mathcal{Q}_i := \sigma^h (Q_i^{(i-1)}),
\]
\[
\bar{P}_i := \sigma^h (\bar{P}_i^{(i-1)}), \quad \bar{Q}_i := \sigma^h (\bar{Q}_i^{(i-1)}),
\]
and define series $\tilde{X}_i(t, \bar{t}, s, \xi) = \alpha_i \log \xi + \sum_{k=1}^{\infty} \tilde{X}_{i,k}(t, \bar{t}, s) \xi^{-k}$, $\tilde{\bar{X}}_i(t, \bar{t}, s, \xi) = \bar{\alpha}_i \log \xi + \sum_{k=1}^{\infty} \tilde{\bar{X}}_{i,k}(t, \bar{t}, s) \xi^{-k}$ and a function $\phi_i(t, \bar{t}, s)$ by one of the following integrals. (The integrand of the first integral in the right hand side of each equation is considered as a series of $\xi$ around $\xi = \infty$ and the integrand
of the second integral is considered as a series around $\xi = 0$.)

\begin{align}
(2.19) \quad -\tilde{X}_i + \phi_i + \tilde{X}_i &= \int_0^\xi p_0^{-1} \left( - \frac{\partial Q_0}{\partial \xi} P_{i}^{(i-1)} + \frac{\partial P_0}{\partial \xi} Q_{i}^{(i-1)} \right) d\xi \\
&\quad - \int_0^\xi p_0^{-1} \left( - \frac{\partial Q_0}{\partial s} P_{i}^{(i-1)} + \frac{\partial P_0}{\partial s} Q_{i}^{(i-1)} \right) ds,
\end{align}

(2.20)

\begin{align}
-\tilde{X}_i + \phi_i + \tilde{X}_i &= \int_0^s p_0^{-1} \left( - \frac{\partial Q_0}{\partial s} P_{i}^{(i-1)} + \frac{\partial P_0}{\partial s} Q_{i}^{(i-1)} \right) ds \\
&\quad - \int_0^s p_0^{-1} \left( - \frac{\partial Q_0}{\partial s} P_{i}^{(i-1)} + \frac{\partial P_0}{\partial s} Q_{i}^{(i-1)} \right) ds,
\end{align}

In fact they give the same $\tilde{X}_i$, $\phi_i$ and $\tilde{X}_i$. Exactly speaking, the coefficients of $\xi^n$ ($n \in \mathbb{Z}$, $n \neq 0$) and log $\xi$ are determined by the first equation (2.19) and its integral constant $\phi_i$ is determined by the second equation (2.20) up to an arbitrary additive constant.

Equations (2.19) and (2.20) determine the combination $-\alpha_i + \bar{\alpha}_i$ as the coefficient of log $\xi$ but not $\alpha_i$ nor $\bar{\alpha}_i$ separately, which can be chosen arbitrarily so far as $-\alpha_i + \bar{\alpha}_i$ is fixed.

\begin{itemize}
  \item (Step 3) Define a series $X_i(t, \bar{t}, s, \xi) = \alpha_i \log \xi + \sum_{k=1}^\infty \chi_i(k)(t, \bar{t}, s)\xi^{-k}$ and $\tilde{X}_i(t, \bar{t}, s, \xi) = \bar{\alpha}_i \log \xi + \sum_{k=1}^\infty \tilde{\chi}_i(k)(t, \bar{t}, s)\xi^{-k}$ by
    \begin{align}
    X_i &= \tilde{X}_i - \frac{1}{2} \{ \sigma^h(X_0), \tilde{X}_i \} + \sum_{p=1}^\infty K_{2p}(\text{ad}_{L}) (\sigma^h(X_0))^{2p} \tilde{X}_i, \\
    \tilde{X}_i &= \tilde{X}_i - \frac{1}{2} \{ \sigma^h(X_0), \tilde{X}_i \} + \sum_{p=1}^\infty K_{2p}(\text{ad}_{L}) (\sigma^h(X_0))^{2p} \tilde{X}_i,
    \end{align}

Here $K_{2p}$ is determined by the generating function

\begin{equation}
(2.22) \quad \frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{p=1}^\infty K_{2p} z^{2p},
\end{equation}

i.e., $K_{2p} = B_{2p}/(2p)!$, where $B_{2p}$'s are the Bernoulli numbers.

\begin{itemize}
  \item (Step 4) The operators $X_i(t, \bar{t}, s, e^{ih\sigma_s})$ and $\tilde{X}_i(t, \bar{t}, s, e^{ih\sigma_s})$ are defined as the operators with the principal symbols $X_i$ and $\tilde{X}_i$:
    \begin{equation}
    (2.23) \quad X_i = \sum_{k=1}^\infty \chi_{i,k}(t, \bar{t}, s) e^{-ikh\sigma_s}, \quad \tilde{X}_i = \sum_{k=1}^\infty \tilde{\chi}_{i,k}(t, \bar{t}, s) e^{ikh\sigma_s}.
    \end{equation}
\end{itemize}

The main theorem is the following:

**Theorem 2.1.** Assume that $X_0$, $\tilde{X}_0$ and $\phi_0$ satisfy (2.5) and construct $X_i$'s, $\tilde{X}_i$'s and $\phi_i$'s by the above procedure recursively. Then $X$, $\tilde{X}$ and $\phi$ defined by (2.22), (2.3) and (2.4) satisfy (1.26). Namely $W = \exp(X/h)$ and $\tilde{W} = \exp(\phi/h) \exp(X/h)$ are dressing operators of the $h$-dependent Toda hierarchy corresponding to the data $(f, g, \bar{f}, \bar{g})$.

The rest of this section is the proof of Theorem (2.1) by induction. The essential idea of the proof is almost the same as the proof of Theorem 2.1 in [TT3].
Let us denote the “known” part of $X$, $\tilde{X}$ and $\phi$ by $X^{(i-1)}$, $\tilde{X}^{(i-1)}$ and $\phi^{(i-1)}$ as in (2.12) and, as intermediate objects, consider $P^{(i-1)}$, $Q^{(i-1)}$, $\tilde{P}^{(i-1)}$ and $\tilde{Q}^{(i-1)}$ defined by (2.13) and (2.14), which are expanded as (2.17) and (2.18).

If $X$, $\tilde{X}$ and $\phi$ are expanded as (2.2), (2.3) and (2.4), the dressing operators $W = \exp(X/h)$ and $\tilde{W} = \exp(\phi/h)\exp(X/h)$ are factorised as follows by the Campbell-Hausdorff theorem:

\begin{align}
W &= \exp \left( h^{-1} \tilde{X}_i + h\tilde{X}_{>i} \right) \exp \left( h^{-1} X^{(i-1)} \right), \\
\tilde{W} &= \exp \left( h^{-1} \phi_i + h^i \phi_{>i} \right) \exp \left( h^{-1} \tilde{X}_i + h\tilde{X}_{>i} \right) \times \exp \left( h^{-1} \phi^{(i-1)} \right) \exp \left( h^{-1} X^{(i-1)} \right),
\end{align}

where $\tilde{X}_i$, $X_{>i}$, $\phi_i$, $\tilde{\phi}_{>i}$, $\tilde{X}_i$ and $\tilde{X}_{>i}$ have $h$-order not more than 0 and the principal symbols of $\tilde{X}_i$ and $\tilde{X}_i$ are defined by

\begin{align}
\sigma^h(\tilde{X}_i)(s, \xi) &= \sum_{n=1}^{\infty} \frac{(\text{ad}_{(1)} \sigma^h(0))^n - 1}{n!} \sigma^h(X_i), \\
\sigma^h(\tilde{X}_i)(s, \xi) &= e^{\text{ad}_{(1)} \phi_0} \left( \sum_{n=1}^{\infty} \frac{(\text{ad}_{(1)} \sigma^h(0))^n - 1}{n!} \sigma^h(X_i) \right)
\end{align}

Note that the log terms in (2.26) and (2.27) are $\alpha_1 \log \xi$ and $\tilde{\alpha}_1 \log \xi$ respectively.

The other terms in (2.26) (resp. (2.27)) are negative (resp. positive) powers of $\xi$. The principal symbol of $\tilde{X}_i$ is recovered from $\tilde{X}_i$ by the formula

\begin{align}
\sigma^h(X_i) = \sigma^h(\tilde{X}_i) - \frac{1}{2} \{ \sigma^h(X_0), \sigma^h(\tilde{X}_i) \} + \sum_{p=1}^{\infty} K_{2p}(\text{ad}_{(1)}(\sigma^h(0)))^{2p} \sigma^h(\tilde{X}_i),
\end{align}

Here coefficients $K_{2p}$ are defined by (2.22). Similarly the principal symbol of $\tilde{X}_i$ is recovered from $\tilde{X}_i$ by

\begin{align}
\sigma^h(X_i) = \sigma^h(\tilde{X}_i) - \frac{1}{2} \{ \sigma^h(X_0), \sigma^h(\tilde{X}_i) \} + \sum_{p=1}^{\infty} K_{2p}(\text{ad}_{(1)}(\sigma^h(0)))^{2p} \sigma^h(\tilde{X}_i),
\end{align}

where $\sigma^h(\tilde{X}_i) := e^{-\text{ad}_{(1)} \phi_0} \sigma^h(\tilde{X}_i)$.

These inversion relations are the origin of (2.21). (Note that the principal symbol determines the operators $X_i$ and $\tilde{X}_i$, since they are homogeneous terms in the expansions (2.2) and (2.3).) The factorisation formula (2.24) and the inversion formula (2.28) are proved in Appendix A of [TT3]. The formulae (2.25) and (2.29) are derived in the same way.

The factorisation (2.24) implies

\begin{align}
P = \text{Ad} \left( \exp \left( h^{-1} \tilde{X}_i + h\tilde{X}_{>i} \right) \right) P^{(i-1)}
= P^{(i-1)} + h^{-1} \{ \tilde{X}_i + hX_{>i}, P^{(i-1)} \} + \text{(terms of h-order < } -i).
\end{align}

Thus, substituting the expansion (2.17) in the Step 1, we have

\begin{align}
P = P^{(i-1)} + hP_1^{(i-1)} + \cdots + h^i P_i^{(i-1)} + \cdots \\
+ h^{-1} \{ \tilde{X}_i, P_0^{(i-1)} \} + \text{(terms of h-order < } -i).
\end{align}
Comparing this with the $h$-expansion (2.6) of $P$, we can express $P_j$’s in terms of $P_j^{(i-1)}$’s and $\tilde{X}_i$ as follows:

\begin{align}
P_j &= P_j^{(i-1)} \quad (j = 0, \ldots, i - 1), \\
\sigma_0(P_i) &= \sigma_0(P_i^{(i-1)} + h^{-1}[\tilde{X}_i, P_0^{(i-1)}]).
\end{align}

Similar equations for $Q$ are obtained in the same way. For the operators $\tilde{P}$ the corresponding equations are

\begin{align}
\tilde{P}_j &= \tilde{P}_j^{(i-1)} \quad (j = 0, \ldots, i - 1), \\
\sigma_0(\tilde{P}_i) &= \sigma_0(\tilde{P}_i^{(i-1)} + h^{-1}[\phi_i, \tilde{P}_0^{(i-1)}] + h^{-1}[\tilde{X}_i, \tilde{P}_0^{(i-1)}]).
\end{align}

The corresponding equations for $\tilde{Q}$ are the same.

The equations (2.31), (2.33) and corresponding equations for $Q$ and $\tilde{Q}$ show that the terms of $h$-order greater than $-i$ in (2.6) are already fixed by $X_0, \ldots, X_{i-1}$, which justifies the inductive procedure. That is to say, we are assuming that $X_0, \ldots, X_{i-1}$ have been already determined so that $P_j = P_j^{(i-1)}$ and $Q_j = Q_j^{(i-1)}$ for $j = 0, \ldots, i - 1$ coincide with $\tilde{P}_j = \tilde{P}_j^{(i-1)}$ and $\tilde{Q}_j = \tilde{Q}_j^{(i-1)}$ respectively.

The operators $X_i$, $\tilde{X}_i$ and the function $\phi_i$ should be chosen so that the right hand sides of (2.32) and (2.34) coincide and the corresponding expressions for $Q$ and $\tilde{Q}$ coincide. Taking equations $P_0^{(i-1)} = P_0$, $Q_0^{(i-1)} = Q_0$, $\tilde{P}_0^{(i-1)} = \tilde{P}_0$ and $\tilde{Q}_0^{(i-1)} = \tilde{Q}_0$ into account, we define

\begin{align}
\tilde{P}_i^{(i)} &= P_i^{(i-1)} + h^{-1}[\tilde{X}_i, P_0], \\
\tilde{Q}_i^{(i)} &= Q_i^{(i-1)} + h^{-1}[\tilde{X}_i, Q_0], \\
\tilde{P}_i^{(i)} &= \tilde{P}_i^{(i-1)} + h^{-1}[\phi_i, \tilde{P}_0] + h^{-1}[\tilde{X}_i, \tilde{P}_0], \\
\tilde{Q}_i^{(i)} &= \tilde{Q}_i^{(i-1)} + h^{-1}[\phi_i, \tilde{Q}_0] + h^{-1}[\tilde{X}_i, \tilde{Q}_0].
\end{align}

Then the condition for $X_i$, $\tilde{X}_i$ and $\phi_i$ is written in the following form of equations for symbols:

\begin{align}
\sigma_0^h(\tilde{P}_i^{(i)}) &= \sigma_0^h(\tilde{P}_i^{(i)}), \\
\sigma_0^h(\tilde{Q}_i^{(i)}) &= \sigma_0^h(\tilde{Q}_i^{(i)}).
\end{align}

(The parts of $h$-order less than $-1$ should be determined in the next step of the induction.) To simplify notations, we denote the symbols $\sigma_0^h(\tilde{P}_i^{(i)}), \sigma_0^h(P_i^{(i-1)})$ and so on by the corresponding calligraphic letters as $\tilde{P}_i^{(i)}$, $P_i^{(i-1)}$ etc. By this notation we can rewrite the equations (2.36) in the following form:

\begin{align}
\tilde{P}_i^{(i)} &= \tilde{P}_i^{(i)}, \\
\tilde{Q}_i^{(i)} &= \tilde{Q}_i^{(i)}, \\
\tilde{P}_i^{(i)} := P_i^{(i-1)} + \{\tilde{X}_i, P_0\}, \\
\tilde{Q}_i^{(i)} := Q_i^{(i-1)} + \{\tilde{X}_i, Q_0\}, \\
\tilde{P}_i^{(i)} := \tilde{P}_i^{(i-1)} + \{\phi_i, \tilde{P}_0\} + \{\tilde{X}_i, \tilde{P}_0\}, \\
\tilde{Q}_i^{(i)} := \tilde{Q}_i^{(i-1)} + \{\phi_i, \tilde{Q}_0\} + \{\tilde{X}_i, \tilde{Q}_0\}.
\end{align}
In the matrix form, these equations are encapsulated in the following equation.

\[
\begin{pmatrix}
P^{(i-1)}_i \\ Q^{(i-1)}_i 
\end{pmatrix} + \xi \begin{pmatrix}
\frac{\partial P_0}{\partial s} & -\frac{\partial P_0}{\partial \xi} \\
\frac{\partial Q_0}{\partial s} & -\frac{\partial Q_0}{\partial \xi} 
\end{pmatrix} \begin{pmatrix}
\frac{\partial}{\partial \xi} \tilde{X}_i \\
\frac{\partial}{\partial s} \tilde{X}_i 
\end{pmatrix} = \begin{pmatrix}
P^{(i-1)}_i \\ Q^{(i-1)}_i 
\end{pmatrix} + \xi \begin{pmatrix}
\frac{\partial P_0}{\partial s} & -\frac{\partial P_0}{\partial \xi} \\
\frac{\partial Q_0}{\partial s} & -\frac{\partial Q_0}{\partial \xi} 
\end{pmatrix} \begin{pmatrix}
\frac{\partial}{\partial \xi} (\phi_i + \tilde{X}_i) \\
\frac{\partial}{\partial s} (\phi_i + \tilde{X}_i) 
\end{pmatrix}
\]  
(2.38)

Recall that operators \(P^{(i-1)}\) and \(Q^{(i-1)}\) are defined by acting adjoint operation to the canonically commuting pair \( (f, g) \) in (2.13), (2.14) and (2.10). Hence they also satisfy the canonical commutation relation: \([P^{(i-1)}, Q^{(i-1)}] = \hbar P^{(i-1)}\). The principal symbol of this relation gives

\[
\{P^{(i-1)}_0, Q^{(i-1)}_0\} = \{P_0, Q_0\} = P_0,
\]

which means that the determinants of the 2 \(\times 2\) matrices in both sides of (2.38) are equal to \(\xi^{-1}P_0\). (Recall that those matrices are equal because of the induction hypothesis, \(P_0 = P_0, Q_0 = Q_0\).) Hence its inverse matrix is easily computed and we have

\[
P_0^{-1} \begin{pmatrix}
\frac{\partial Q_0}{\partial \xi} & \frac{\partial P_0}{\partial \xi} \\
\frac{\partial Q_0}{\partial s} & \frac{\partial P_0}{\partial s} 
\end{pmatrix} \begin{pmatrix}
P^{(i-1)}_i \\ Q^{(i-1)}_i 
\end{pmatrix} = \begin{pmatrix}
P^{(i-1)}_i \\ Q^{(i-1)}_i 
\end{pmatrix} + \begin{pmatrix}
\frac{\partial}{\partial \xi} (\phi_i + \tilde{X}_i) \\
\frac{\partial}{\partial s} (\phi_i + \tilde{X}_i) 
\end{pmatrix}
\]
(2.40)

Note that the left hand side (the first line) is a series of \(\xi\) around \(\xi = \infty\), while the right hand side (the second line) is a series around \(\xi = 0\). Equation (2.40) is rewritten as

\[
\begin{pmatrix}
\frac{\partial}{\partial \xi} (-\tilde{X}_i + \phi_i + \tilde{X}_i) \\
\frac{\partial}{\partial s} (-\tilde{X}_i + \phi_i + \tilde{X}_i) 
\end{pmatrix} = P_0^{-1} \begin{pmatrix}
\frac{\partial Q_0}{\partial \xi} & \frac{\partial P_0}{\partial \xi} \\
\frac{\partial Q_0}{\partial s} & \frac{\partial P_0}{\partial s} 
\end{pmatrix} \begin{pmatrix}
P^{(i-1)}_i \\ Q^{(i-1)}_i 
\end{pmatrix} - P_0^{-1} \begin{pmatrix}
\frac{\partial Q_0}{\partial \xi} & \frac{\partial P_0}{\partial \xi} \\
\frac{\partial Q_0}{\partial s} & \frac{\partial P_0}{\partial s} 
\end{pmatrix} \begin{pmatrix}
P^{(i-1)}_i \\ Q^{(i-1)}_i 
\end{pmatrix},
\]

which determines \(-\tilde{X}_i + \phi_i + \tilde{X}_i\). According to the above remark, the first term in the right hand side is a series of \(\xi\) around \(\xi = \infty\) and the second term is a series of \(\xi\) around \(\xi = 0\).
The system (2.41) is solvable thanks to Lemma 2.2 below. Hence, integrating the first element of the right hand side with respect to $\xi$, we obtain $-\hat{X}_i + \phi_0 + \hat{X}_i$ up to an integration constant which does not depend on $\xi$. Integrating the second element of the above equation, we can determine this integration constant up to a constant which does not depend on $s$. This is Step 2, (2.19) and (2.20), which determine the symbols $\hat{X}_i$, $\hat{X}_i$ and the function $\phi_i$. (The ambiguity of $\phi_i$ is harmless.)

In the end, the principal symbols of $X_i$ and $\hat{X}_i$ are determined by (2.21) or (2.23) in Step 3. Operators $X_i$ and $\hat{X}_i$ are defined as in Step 4. This completes the construction of $X_i$, $\hat{X}_i$ and $\phi_i$ and the proof of the theorem.

**Lemma 2.2.** The system (2.41) is compatible.

**Proof.** We check the compatibility condition,

\[
\frac{\partial}{\partial s} \left( P_0^{-1} \left( \frac{\partial Q_0}{\partial \xi} P_i^{(i-1)} + \frac{\partial P_0}{\partial \xi} Q_i^{(i-1)} \right) \right) - \frac{\partial}{\partial s} \left( \bar{P}_0^{-1} \left( \frac{\partial \bar{Q}_0}{\partial \xi} \bar{P}_i^{(i-1)} + \frac{\partial \bar{P}_0}{\partial \xi} \bar{Q}_i^{(i-1)} \right) \right) = \frac{\partial}{\partial \xi} \left( P_0^{-1} \left( \frac{\partial Q_0}{\partial s} P_i^{(i-1)} + \frac{\partial P_0}{\partial s} Q_i^{(i-1)} \right) \right) - \frac{\partial}{\partial \xi} \left( \bar{P}_0^{-1} \left( \frac{\partial \bar{Q}_0}{\partial s} \bar{P}_i^{(i-1)} + \frac{\partial \bar{P}_0}{\partial s} \bar{Q}_i^{(i-1)} \right) \right).
\]

(2.42)

Using the relation $\{P_0, Q_0\} = \bar{P}_0$ (2.39), this equation reduces to

\[
P_0^{-1} \xi^{-1} (-P_i^{(i-1)} + \{P_i^{(i-1)}, Q_0\} + \{P_0, Q_i^{(i-1)}\}) - \bar{P}_0^{-1} \xi^{-1} (-\bar{P}_i^{(i-1)} + \{\bar{P}_i^{(i-1)}, \bar{Q}_0\} + \{\bar{P}_0, \bar{Q}_i^{(i-1)}\}) = \frac{\partial}{\partial \xi} \left( P_0^{-1} \left( \frac{\partial Q_0}{\partial s} P_i^{(i-1)} + \frac{\partial P_0}{\partial s} Q_i^{(i-1)} \right) \right) - \frac{\partial}{\partial \xi} \left( \bar{P}_0^{-1} \left( \frac{\partial \bar{Q}_0}{\partial s} \bar{P}_i^{(i-1)} + \frac{\partial \bar{P}_0}{\partial s} \bar{Q}_i^{(i-1)} \right) \right).
\]

(2.43)

Defined from canonically commuting pair $(f, g)$ by adjoint action (2.10), (2.13) and (2.14), the pair of operators $(P_i^{(i-1)}, Q_i^{(i-1)})$ is canonically commuting: $[P_i^{(i-1)}, Q_i^{(i-1)}] = \hbar P_i^{(i-1)}$. Similarly we have $[\bar{P}_i^{(i-1)}, \bar{Q}_i^{(i-1)}] = \hbar \bar{P}_i^{(i-1)}$ and thus

\[
[P_i^{(i-1)}, Q_i^{(i-1)}] - \hbar P_i^{(i-1)} = [\bar{P}_i^{(i-1)}, \bar{Q}_i^{(i-1)}] - \hbar \bar{P}_i^{(i-1)}.
\]

(2.44)

Substituting the expansions (2.41) and (2.43) in it and noting that $P_j^{(i-1)} = \bar{P}_j^{(i-1)}$ and $Q_j^{(i-1)} = \bar{Q}_j^{(i-1)}$ for $j = 0, \ldots, i - 1$ by the induction hypothesis, the terms of $\hbar$-order higher than $-i - 1$ in (2.41) cancel. Thus (2.44) becomes

\[
[h P_i^{(i-1)}, Q_0] + [\bar{P}_0, h Q_i^{(i-1)}] - h^{i+1} P_i^{(i-1)} + (\hbar\text{-order} < -i - 1)
\]

\[
= [h P_i^{(i-1)}, \bar{Q}_0] + [\bar{P}_0, h \bar{Q}_i^{(i-1)}] - h^{i+1} \bar{P}_i^{(i-1)} + (\hbar\text{-order} < -i - 1).
\]

Taking the symbol of $\hbar$-order $-i - 1$ of this equation, we have (2.43) because $P_0 = \bar{P}_0$. \hfill $\square$

3. **Asymptotics of the wave function**

In this section we prove that the dressing operator of the form (1.14) or (1.8) (with $\alpha = 0$), i.e.,

\[
(3.1) \quad W(\hbar, t, \tilde{t}, s, e^{\hbar \partial s}) = \exp(X(\hbar, t, \tilde{t}, s, e^{\hbar \partial s})/\hbar),
\]

\[
(3.2) \quad X(\hbar, t, \tilde{t}, s, e^{\hbar \partial s}) = \sum_{k=1} \chi_k(\hbar, t, \tilde{t}, s) e^{-k \hbar \partial s}, \quad \text{ord}^\hbar X \leq 0,
\]
gives the wave function of the WKB form
\begin{equation}
\Psi(h, t, \bar{t}, s; z) = Wz^{s/h}e^{\zeta(t, z)/h} = e^{S(h, t, \bar{t}, s, z)/h}z^{s/h}, \quad \text{ord}^h S \leq 0,
\end{equation}
\begin{equation}
S(h, t, \bar{t}, s; z) = \sum_{n=0}^{\infty} \hbar^n S_n(t, \bar{t}, s; z) + \zeta(t, z), \quad \zeta(t, z) := \sum_{n=1}^{\infty} t_n z^n,
\end{equation}
and vice versa. As the factor $e^{\hbar^{-1}a(h)/h}$ becomes a constant factor $z^{\alpha(h)/h}$ when it is applied to $z^{s/h}$, we omit it here.

By changing the sign of $s$ and replacing $z$ by $\tilde{z}^{-1}$, we can deduce the formula for the wave function $\Psi$ corresponding to the dressing operator $\tilde{W}$ of the form $\tilde{W}$ or $\tilde{W}$ immediately from the above results: if $\tilde{W}$ has the form
\begin{equation}
\tilde{W}(h, t, \bar{t}, s, e^{\tilde{\alpha}h}) = \exp(\phi(h, t, \bar{t}, s)/h)\exp(\tilde{X}(h, t, \bar{t}, e^{\tilde{\alpha}h})/h),
\end{equation}
gives the wave function of the WKB form
\begin{equation}
\tilde{\Psi}(h, t, \bar{t}, s; \tilde{z}) = \tilde{W}\tilde{z}^{s/h}\tilde{e}^{\tilde{\zeta}(\tilde{t}, \tilde{z}^{-1})/h} = e^{\tilde{S}(h, t, \bar{t}, s, \tilde{z})/h}\tilde{z}^{s/h}, \quad \text{ord}^h S \leq 0,
\end{equation}
\begin{equation}
\tilde{S}(h, t, \bar{t}, s; \tilde{z}) = \sum_{n=0}^{\infty} h^n \tilde{S}_n(t, \bar{t}, s; \tilde{z}) + \tilde{\zeta}(\tilde{t}, \tilde{z}^{-1}), \quad \tilde{\zeta}(\tilde{t}, \tilde{z}^{-1}) := \sum_{n=1}^{\infty} \tilde{t}_n \tilde{z}^{-n}.
\end{equation}
Since the time variables $t_n$ and $\bar{t}_n$ do not play any role in this section, we set them to zero.

Let $A(h, s, e^{\alpha h}) = \sum_n a_n(h, s)e^{nh\alpha}$ be a difference operator. The total symbol of $A$ is a power series of $\xi$ defined by
\begin{equation}
\sigma_{\text{tot}}(A)(h, s, \xi) := \sum_n a_n(h, s)\xi^n.
\end{equation}
Actually, this is the factor which appears when the operator $A$ is applied to $z^{s/h}$:
\begin{equation}
A z^{s/h} = \sigma_{\text{tot}}(A)(h, s, z)z^{s/h}.
\end{equation}
Using this terminology, what we show in this section is that a operator of the form $e^{X/h}$ has a total symbol of the form $e^S/h$ and that an operator with total symbol $e^S/h$ has a form $e^{X/h}$. Exactly speaking, the main results in this section are the following two propositions.

**Proposition 3.1.** Let $X = X(h, s, e^{\alpha h})$ be a difference operator of the form $\tilde{W}$, which has the $h$-order 0: $\text{ord}^h X = 0$. Then the total symbol of $e^{X/h}$ has such a form as
\begin{equation}
\sigma_{\text{tot}}(\exp(h^{-1}X(h, s, e^{\alpha h}))) = e^{S(h, s, \xi)/h},
\end{equation}
where $S(h, s, \xi)$ is a power series of $\xi^{-1}$ without non-negative powers of $\xi$ and has an $h$-expansion
\begin{equation}
S(h, s, \xi) = \sum_{n=0}^{\infty} h^n S_n(s, \xi).
\end{equation}
Moreover, the coefficient $S_n$ is determined by $X_0, \ldots, X_n$ in the $h$-expansion (2.2) of $X = \sum_{n=0}^{\infty} h^n X_n$.

Explicitly, $S_n$ is determined as follows:
• (Step 0) Assume that $X_0, \ldots, X_n$ are given. Let $X_i(s, \xi)$ be the total symbol $\sigma_{\text{tot}}(X_i(s, e^{\hbar \partial_x}))$.

• (Step 1) Define $Y^{(l)}_{k,m}(s, s', \xi, \xi')$ and $S^{(l)}(s, \xi)$ by the following recursion relations:

\begin{align}
Y^{(l)}_{k,-1} &= 0 \\
S^{(0)}_m &= 0, \\
Y^{(l)}_{0,m}(s, s', \xi, \xi') &= \delta_{l,0}Y_m(s, \xi) \\
Y^{(l)}_{k+1,m}(s, s', \xi, \xi') &= \frac{1}{k+1} \left( \xi \partial_{s'} Y^{(l)}_{k-1,m}(s, s', \xi, \xi') + \sum_{\substack{0 \leq l' \leq l-1 \\ 0 \leq m' \leq m}} \xi \partial_{s'} Y^{(l')}_{k,m'}(s, s', \xi, \xi') \partial_{s'} S^{(l-l')}(s', \xi') \right) \\
S^{(l+1)}_m(s, \xi) &= \frac{1}{l+1} \sum_{k=0}^{l+m} Y^{(l)}_{k,m}(s, s, \xi, \xi).
\end{align}

(We shall prove that $Y^{(l)}_{k,m} = 0$ if $k > l + m$.) Schematically this procedure goes as follows:

$$
Y^{(l)}_{0,0} = \delta_{l,0}X_0 \quad Y^{(l)}_{0,1} = \delta_{l,0}X_1 \quad Y^{(l)}_{0,2} = \delta_{l,0}X_2
$$

$$
Y^{(l)}_{k,-1} = 0 \quad Y^{(l)}_{k,0} \quad Y^{(l)}_{k,1} \quad Y^{(l)}_{k,2} \quad \ldots
$$

$$
S^{(l+1)}_0 = 0 \quad S^{(l+1)}_1 \quad S^{(l+1)}_2
$$

• (Step 2) $S_n(s, \xi) = \sum_{l=1}^{\infty} S^{(l)}_n(s, \xi)$. (The sum makes sense as a power series of $\xi$.)

**Proposition 3.2.** Let $S(h, s, \xi) = \sum_{n=0}^{\infty} h^n S_n(s, \xi)$ be a power series of $\xi^{-1}$ without non-negative powers of $\xi$. Then there exists a difference operator $X(h, s, e^{\hbar \partial_x})$ of the form (5.2) such that $\text{ord}^h X \leq 0$ and

$$
\sigma_{\text{tot}}(\exp(h^{-1}X(h, s, e^{\hbar \partial_x}))) = e^{S(h, s, \xi)/\hbar}.
$$

Moreover, the coefficient $X_n(s, \xi)$ in the $h$-expansion $X = \sum_{n=0}^{\infty} h^n X_n$ of the total symbol $X = X(h, s, \xi)$ is determined by $S_0, \ldots, S_n$ in the $h$-expansion of $S$.

Explicit procedure is as follows:

• (Step 0) Assume that $S_0, \ldots, S_n$ are given. Expand them into homogeneous terms with respect to powers of $\xi$: $S_n(s, \xi) = \sum_{j=1}^{\infty} S_{n,j}(s, \xi)$, where $S_{n,j}$ is a term of degree $-j$.

• (Step 1) Define $Y^{(l)}_{k,n,j}(s, s', \xi, \xi')$ as follows:

\begin{align}
Y^{(l)}_{k,-1,j}(s, s', \xi, \xi') &= 0, \\
Y^{(l)}_{k,m,j}(s, s', \xi, \xi') &= \delta_{l,0} \delta_{k,0} S_{m,1}(s, \xi)
\end{align}
for $m = 0, \ldots, n$, $k \geq 0$, $l \geq 0$ and
\begin{equation}
Y_{0,m,j}^{(l)} = 0
\end{equation}
for $m = 0, \ldots, n$, $l > 0$, $j \geq 1$. For other $(l, k, m, j)$, $(l, k) \neq (0, 0)$, $Y_{k,m,j}^{(l)}$
are determined by the recursion relation:
\begin{equation}
Y_{k+1,m,j}^{(l)}(s', \xi, \xi') = \frac{1}{k+1} \left( \xi \partial_s \partial_s' Y_{k,m-1,j}^{(l)}(s, s', \xi, \xi') + \sum_{0 \leq l' \leq l-1} \frac{1}{l-l'} \partial_{s'} Y_{k,m',j'}^{(l')} (s, s', \xi, \xi') \partial_{s'} Y_{k',m-m',j-j'}^{(l'-1)} (s, s, \xi, \xi) \right).
\end{equation}
The remaining $Y_{0,m,j}^{(0)}$ is determined by:
\begin{equation}
Y_{0,m,j}^{(0)}(s', \xi, \xi') = S_{m,j}(s, \xi) - \sum_{0 \leq l \leq m, (l,k) \neq (0,0)} \frac{1}{l+1} Y_{k,m,j}^{(l)}(s, s, \xi, \xi).
\end{equation}
(We shall show that $Y_{k,m,j}^{(l)} = 0$ for $k > l + m$ or $j \leq l$.) Schematically this procedure goes as follows:
\begin{align*}
Y_{k,m,1}^{(l)} & \downarrow \\
Y_{k',m',1}^{(l)}(m' < m) & \rightarrow Y_{k,m,2}^{(l)} (k, l \neq 0) \rightarrow Y_{0,m,2}^{(0)} \leftarrow S_{m,2} \\
Y_{k',m',2}^{(l)}(m' < m) & \rightarrow Y_{k,m,3}^{(l)} (k, l \neq 0) \rightarrow Y_{0,m,3}^{(0)} \leftarrow S_{m,3} \\
\vdots
\end{align*}

(Step 2) $X_n(s, \xi) = \sum_{j=1}^{\infty} Y_{0,n,j}^{(0)}(s, s, \xi, \xi)$. (The infinite sum is the homogeneous expansion in terms of powers of $\xi$.)

Combining these propositions (and the corresponding statements for $\bar{X}$ and $\bar{S}$ (3.7)) with the results in Section 2, we can, in principle, make a recursion formula for $S_0$ and $\bar{S}_0$ ($n = 0, 1, 2, \ldots$) of the wave functions of the solution of the Toda lattice hierarchy corresponding to $(f, g, \bar{f}, \bar{g})$ by Proposition 1.2 (i) as follows: let $S_0, \ldots, S_{i-1}$, $\bar{S}_0, \ldots, \bar{S}_{i-1}$ and $\phi_0, \ldots, \phi_{i-1}$ be given.

(1) By Proposition 3.2 and its variant with the opposite sign of $s$ we have $X_0, \ldots, X_{i-1}$ and $X_0, \ldots, \bar{X}_{i-1}$.

(2) We have a recursion formula for $X_i$, $\bar{X}_i$ and $\phi_i$ by Theorem 2.1.

(3) Proposition 3.1 (with its variant) gives a formula for $S_i$, $\bar{S}_i$.

If we take the factor $e^{\hbar^{-1} a(h)b(h) \partial_{\alpha}(\hbar \partial_{\alpha})}$ into account, this process becomes a little bit more complicated, but essentially the same.

The rest of this section is devoted to the proof of Proposition 3.1 and Proposition 3.2.

To avoid confusion, the commutative multiplication of total symbols $a(h, s, \xi)$ and $b(h, s, \xi)$ as power series is denoted by $a(h, s, \xi) b(h, s, \xi)$ and the non-commutative
multiplication corresponding to the operator product is denoted by \( a(h, s, \xi) \circ b(h, s, \xi) \). Recall that the latter multiplication is expressed (or defined) as follows:

\[
a(h, s, \xi) \circ b(h, s, \xi) = e^{\hbar \xi \partial_s} a(h, s, \xi) b(h, s', \xi') \big|_{s'=s, \xi'=\xi}
\]

(3.22)

\[
= \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} (\xi \partial_s)^n a(h, s, \xi) \partial_s^n b(h, s', \xi') \big|_{s'=s, \xi'=\xi}.
\]

(This corresponds to Equation (3.21) of \[TT3\] for microdifferential operators.) The order of the symbol \( a(h, s, \xi) \) (the order with respect to \( \xi \) as a power series of \( \xi \)) is denoted by \( \text{ord}_\xi a(h, s, \xi) \):

\[
\text{ord}_\xi \left( \sum a_m(h, s) \xi^m \right) \overset{\text{def}}{=} \max \{ m \mid a_m(h, s) \neq 0 \}.
\]

The \( \hbar \)-order is the same as that of operators: \( \text{ord}_h s = \text{ord}_h \xi = 0, \text{ord}_h \hbar = -1 \).

The main idea of proof of propositions is the same as those in Section 3 of \[TT3\], which is a formal version of Aoki’s exponential calculus of microdifferential operators, \[A\]. Since the Euler operator \( \xi \partial_s \) does not lower the order with respect to \( \xi \) in contrast to the differential operator \( \partial_s \), proof of convergence of series like \( (3.22) \) as a formal power series is different from that in \[TT3\].

First, we prove the following lemma.

**Lemma 3.3.** Let \( a(h, s, \xi), b(h, s, \xi), p(h, s, \xi) \) and \( q(h, s, \xi) \) be symbols such that \( \text{ord}_\xi a(h, s, \xi) = M, \text{ord}_h a(h, s, \xi) \leq 0, \text{ord}_\xi b(h, s, \xi) = N, \text{ord}_h b(h, s, \xi) \leq 0, \text{ord}_\xi p(h, s, \xi) \leq -1, \text{ord}_h p(h, s, \xi) \leq 0, \text{ord}_\xi q(h, s, \xi) \leq -1, \text{ord}_h q(h, s, \xi) \leq 0 \).

Then there exist symbols \( c(h, s, \xi) \) (\( \text{ord}_\xi c(h, s, \xi) = N + M, \text{ord}_h c(h, s, \xi) \leq 0 \)) and \( r(h, s, \xi) \) (\( \text{ord}_\xi r(h, s, \xi) \leq 0, \text{ord}_h r(h, s, \xi) \leq 0 \)) such that

\[
(a(h, s, \xi) e^{p(h,s,\xi)/\hbar}) \circ (b(h, s, \xi) e^{q(h,s,\xi)/\hbar}) = c(h, s, \xi) e^{r(h,s,\xi)/\hbar}.
\]

In the proof of Proposition 3.1 and Proposition 3.2 we use the construction of \( c \) and \( r \) in the proof of Lemma 3.3.

**Proof.** Following \[A\], we introduce a parameter \( t \) and consider

\[
\pi(t) = \pi(t; h, s, s', \xi, \xi') := e^{b t \xi \partial_s} a(h, s, \xi) b(h, s', \xi') e^{\left( p(h,s,\xi) + q(h,s',\xi') \right)/\hbar}.
\]

If we set \( t = 1 \), \( s' = s \) and \( \xi' = \xi \), this reduces to the operator product of \( (3.22) \).

The series \( \pi(t) \) is the unique solution of an initial value problem:

\[
\partial_t \pi = h \xi \partial_t \partial_s \pi, \quad \pi(0) = a(h, s, \xi) b(h, s', \xi') = \left( p(h,s,\xi) + q(h,s',\xi') \right)/\hbar.
\]

We construct its solution in the following form:

\[
\pi(t) = \psi(t) e^{w(t)/\hbar},
\]

\[
\psi(t) = \psi(t; h, s, s', \xi, \xi') = \sum_{n=0}^{\infty} \psi_n t^n,
\]

\[
w(t) = w(t; h, s, s', \xi, \xi') = \sum_{k=0}^{\infty} w_k t^k.
\]
Later we set $t = 1$ and prove that $\psi(1)$ and $w(1)$ are meaningful as a formal power series of $\xi$ and $\xi'$. The differential equation (3.25) is rewritten as

$$\frac{\partial \psi}{\partial t} + \hbar^{-1} \psi \frac{\partial w}{\partial t} = h \xi \partial_{\xi'} \psi + \xi \partial_{\xi'} \partial_{s'} \psi + \psi \left( \xi \partial_{\xi'} \partial_{s'} \psi + h^{-1} \xi \partial_{\xi'} \partial_{s'} \psi \right).$$

Hence it is sufficient to construct $\psi(t) = \psi(t; h, s, s', \xi, \xi')$ and $w(t) = w(t; h, s, s', \xi, \xi')$ which satisfy

$$\frac{\partial w}{\partial t} = h \xi \partial_{\xi'} \partial_{s'} \psi + \xi \partial_{\xi'} \partial_{s'} \psi, \quad (3.28)$$

$$\frac{\partial \psi}{\partial t} = h \xi \partial_{\xi'} \partial_{s'} \psi + \xi \partial_{\xi'} \partial_{s'} \psi + \xi \partial_{\xi'} \partial_{s'} \psi.$$ 

(This is a sufficient condition but not a necessary condition for $\pi = \psi e^{w/\hbar}$ to be a solution of (3.25). The solution of (3.25) is unique, but $\psi$ and $w$ satisfying (3.27) are not unique at all.)

To begin with, we solve (3.28) and determine $w(t)$. Expanding it as

$$w(t) = \sum_{k=0}^{\infty} w_k t^k,$$

we have a recursion relation and the initial condition

$$w_{k+1} = \frac{1}{k+1} \left( h \xi \partial_{\xi'} w_k + \sum_{\nu=0}^{k} \xi \partial_{\xi'} w_{\nu} \partial_{s'} w_{k-\nu} \right),$$

$$w_0 = p(s, \xi) + q(s', \xi'),$$

which determine $w_k = w_k(h, s, s', \xi, \xi')$ inductively. In order to show that $\sum_{k=0}^{\infty} w_k$ converges as a formal power series, let us expand each $w_k$ as follows:

$$w_k(h, s, s', \xi, \xi') = \sum_{n=0}^{\infty} h^n w_{k,n}(s, s', \xi, \xi').$$

Expanding (3.30) as a series of $h$, we obtain a recursion relation of $w_{k,n}$ and the initial condition

$$w_{k+1,n} = \frac{1}{k+1} \left( \xi \partial_{\xi'} w_{k,n-1} + \sum_{\nu=0}^{n-1} \xi \partial_{\xi'} w_{k,n-\nu} \right),$$

$$w_0 = p(s, \xi) + q(s', \xi').$$

Because of the assumption $\operatorname{ord}_{\xi} p \leq -1$ and $\operatorname{ord}_{\xi} q \leq -1$, $w_0$ also has the order $\leq -1$ and consequently

$$\operatorname{ord}_{\xi} w_{0,n} \leq -1. \quad (3.33)$$

(Here $\operatorname{ord}_{\xi}$ means the order with respect to both $\xi$ and $\xi'$: $\operatorname{ord}_{\xi} (\sum a_{m,n}(h, s) \xi^m \xi'^n) \stackrel{\text{def}}{=} \max \{m + n \mid a_{m,n}(h, s) \neq 0 \} \). We show

$$\operatorname{ord}_{\xi} w_{k,n} \leq \min(-1, -k + n - 1) \quad (3.34)$$

by induction on $k$.

- First, when $k = 0$, (3.34) holds for any $n \geq 0$ because of (3.33).
• Assume that (3.34) holds for any pair \((k, n)\) with \(n \geq 0\) and \(k = 0, \ldots, k_0\).
Then the right hand side of (3.32) with \(k = k_0\) has the order (with respect to \(\xi\) and \(\xi'\)) less than or equal to \(-1\) and \(-k_0 + (n - 1) - 1 = (k' + n' - 1) + (-k'' + n'') - 1) = -k_0 + n - 2\), since \(\xi \partial_\xi\) does not change the order.

• Hence (3.35) is true for \(k = k_0 + 1\).

Thus the estimate (3.34) has been proved for all \(k\) and \(n\).

This shows that \(w(1) = \sum_{k=0}^{\infty} w_k = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} h^n w_{k,n}\) makes sense as a formal series of \(h, \xi\) and \(\xi'\). Moreover it is obvious that \(w_k\) and \(w(1)\) are formally regular with respect to \(h\).

As a next step, we expand \(\psi(t)\) as \(\psi(t) = \sum_{k=0}^{\infty} \psi_k t^k\) and rewrite (3.29) into a recursion relation and the initial condition:

\[
\psi_{k+1} = \frac{1}{k+1} \left( h \xi \partial_\xi \partial_{s'} \psi_k + \sum_{\nu=0}^{k} (\xi \partial_\xi \psi_{\nu} \partial_{s'} w_{k-\nu} + \xi \partial_\xi w_{k-\nu} \partial_{s'} \psi_{\nu}) \right),
\]

\[
\psi_0 = a(s, \xi) b(s', \xi')
\]

To prove the convergence as a formal power series, we expand \(\psi_k\) as

\[
\psi_k(h, s, s', \xi, \xi') = \sum_{n=0}^{\infty} h^n \psi_{k,n}(s, s', \xi, \xi'),
\]

and rewrite the recursion relation as follows.

\[
\psi_{k+1,n} = \frac{1}{k+1} \left( \xi \partial_\xi \partial_{s'} \psi_{k,n-1} + \sum_{k'+k''=k \atop n'+n''=n} (\xi \partial_\xi \psi_{k',n'} \partial_{s'} w_{k''} + \partial_{s'} \psi_{k',n'} \xi \partial_\xi w_{k''}) \right),
\]

\[
\psi_0 = a(s, \xi) b(s', \xi')
\]

Our assumption being \(\text{ord}_\xi a(s, \xi) = M\) and \(\text{ord}_\xi b(s', \xi') = N\), we have

\[
\text{ord}_\xi \psi_{0,n} \leq M + N.
\]

As in the estimate of \(w_{k,n}\), we prove

\[
\text{ord}_\xi \psi_{k,n} \leq \min(M + N, M + N - k + n)
\]

by induction.

• First, (3.39) holds for \(k = 0\) and any \(n \geq 0\) because of (3.38).

• Assume that (3.39) holds for \(k = 0, \ldots, k_0\) and \(n \geq 0\). The right hand side of (3.37) with \(k = k_0\) has the order with respect to \(\xi\) and \(\xi'\) not more than \(M + N - k_0 + (n - 1) = (M + N - k' + n') + (-k'' + n'') - 1) = M + N - k_0 + n - 1\), nor \(M + N\) because of the induction hypothesis and (3.34).

• This proves (3.39) for \(k = k_0 + 1\).

Thus we have proved (3.39) for any \(k\) and \(n\), which shows that the infinite sum \(\psi(1) = \sum_{k=0}^{\infty} \psi_k = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} h^n \psi_{k,n}\) makes sense. The regularity of \(\psi_k\) and \(\psi(1)\) is also obvious.

We have constructed \(\pi(t) = \pi(t; h, s, s', \xi, \xi') = \psi(t; h, s, s', \xi, \xi')e^{w(t; h, s, s', \xi, \xi')},\) which is meaningful also at \(t = 1\). Hence the product \(a(h, s, \xi) \circ b(h, s, \xi) = \pi(1; h, s, s, \xi, \xi)\) is expressed in the form \(c(h, s, \xi)e^{r(h, s, \xi)/h}\), where \(c(h, s, \xi) = \psi(1; h, s, s, \xi, \xi)\), \(r(h, s, \xi) = w(1; h, s, s, \xi, \xi)\).
Proof of Proposition 3.1. We make use of differential equations satisfied by the operator
\[
E(t) = E(t; \hbar, s, e^{\hbar \partial_s}) := \exp \left( \frac{t}{\hbar} X(h, s, e^{\hbar \partial_s}) \right),
\]
depending on a parameter $t$. The total symbol of $E(t)$ is defined as
\[
E(t; \hbar, s, \xi) = \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} X^{(k)}(h, s, \xi), \quad X^{(0)} = 1, \quad X^{(k+1)} = X \circ X^{(k)}.
\]
Taking the logarithm (as a function, not as an operator) of this, we can define $S(t) = S(t; \hbar, s, \xi)$ by
\[
E(t; \hbar, s, \xi) = e^{\hbar^{-1} S(t; \hbar, s, \xi)}
\]
What we are to prove is that $S(t)$, constructed as a series, makes sense at $t = 1$ and formally regular with respect to $\hbar$.

Differentiating (3.42), we have
\[
X(h, s, \xi) \circ E(t; \hbar, s, \xi) = \frac{\partial S}{\partial t} e^{S(t; \hbar, s, \xi)/\hbar}
\]
By Lemma 3.3 (where $a \rightarrow X$, $b \rightarrow 1$, $p \rightarrow 0$, $q \rightarrow S$) and the technique in its proof, we can rewrite the left hand side as follows. (Hereafter we sometimes omit the argument $\hbar$ of functions for brevity):
\[
X(s, \xi) \circ E(t; s, \xi) = Y(t; s, s', \xi, \xi') e^{S(t; s, \xi)/\hbar}
\]
where $Y(t; s, s', \xi, \xi') = \sum_{k=0}^{\infty} Y_k$ and $Y_k(t; s, s', \xi, \xi')$ are defined by
\[
Y_{k+1}(t; s, s', \xi, \xi') = \frac{1}{k+1} (\hbar \xi \partial_{\xi'} Y_k(t; s, s', \xi, \xi') + \xi \partial_{\xi} Y_k(t; s, s', \xi, \xi') \partial_{s} S(t; s, \xi'))
\]
$Y_0(t; s, s', \xi, \xi') = X(s, \xi)$.

$Y_k(t)$ corresponds to $\psi_k$ in the proof of Lemma 3.3 while $w_k$ there is $\delta_{k,0} S(t)$. (Recall that the role of $t$ is different. The parameter $t$ in the proof of Lemma 3.3 is set to 1 here.) On the other hand, substituting (3.44) into the left hand side of (3.43), we have
\[
\frac{\partial S}{\partial t}(t; s, \xi) = Y(t; s, s, \xi)
\]
We rewrite the system (3.45) and (3.46) in terms of expansion of $S(t; s, \xi) = S(t; \hbar, s, \xi)$ and $Y_k(t; s, s', \xi, \xi') = Y_k(t; \hbar, s, s', \xi, \xi')$ in powers of $t$ and $\hbar$:
\[
S(t; \hbar, s, \xi) = \sum_{l=0}^{\infty} S^{(l)}(\hbar, s, \xi) t^l = \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} S^{(l)}_n(s, \xi) \hbar^n t^l,
\]
\[
Y_k(t; \hbar, s, s', \xi, \xi') = \sum_{l=0}^{\infty} Y_k^{(l)}(\hbar, s, s', \xi, \xi') t^l = \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} Y_k^{(l)}_{k,n}(s, s', \xi, \xi') \hbar^n t^l,
\]
\footnote{Of course this parameter $t$ does not have any relation with the time variables of the Toda lattice hierarchy. It is not the same $t$ in the proof of Lemma 3.3 either.}
The coefficient of $h^n t^l$ in the recursion relation (3.45) is

$$Y_{k+1,n}^{(l)}(s, s', \xi, \xi') = \frac{1}{k+1} \left( \xi \partial_{\xi} \partial_{s'} Y_{k,n-1}^{(l)}(s, s', \xi, \xi') + \sum_{l' + l'' = l, n' + n'' = n} \xi \partial_{\xi} Y_{k,n'}^{(l')}(s, s', \xi, \xi') \partial_{s'} S_{n''}^{(l'')} (s', \xi') \right)$$

$(Y_{k,-1}^{(l)} = 0)$ while (3.46) gives

$$S_{n}^{(l+1)}(s, \xi) = \frac{1}{l + 1} \sum_{k=0}^{\infty} Y_{k,n}^{(l)}(s, s, \xi, \xi)$$

We first show that these recursion relations consistently determine $Y_{k,n}^{(l)}$ and $S_{n}^{(l)}$. Then we prove that the infinite sum in (3.49) is finite.

Fix $n \geq 0$ and assume that $Y_{k,0}^{(l)}, \ldots, Y_{k,n-1}^{(l)}$ and $S_{0}^{(l)}, \ldots, S_{n-1}^{(l)}$ have been determined for all $(l, k)$. (When $n = 0$, $Y_{k,-1}^{(l)} = 0$ as mentioned above and $S_{-1}^{(l)}$ can be ignored as it does not appear in the induction.)

1. Since $E(t = 0) = 1$ by the definition (3.40), we have $S_{n}^{(0)} = 0$. Hence

$$S_{n}^{(0)} = 0.$$

2. From the initial condition in (3.45) we have

$$Y_{0,n}^{(l)}(s, s', \xi, \xi') = \delta_{l,0} X_n(s, \xi).$$

It follows from this equation and the assumption (3.2) that

$$\text{ord}_{\xi} Y_{0,n}^{(0)} \leq -1.$$

3. When $l = 0$, the second sum in the right hand side of the recursion relation (3.48) is absent because of (3.50). Hence if $n \geq k + 1$, we have

$$Y_{k+1,n}^{(0)} = \frac{1}{k+1} \xi \partial_{\xi} \partial_{s'} Y_{k,n-1}^{(0)} = \cdots = \frac{1}{(k+1)!} (\xi \partial_{\xi} \partial_{s'})^{k+1} Y_{0,n-k-1}^{(0)} = 0$$

since $Y_{0,n-k-1}^{(0)}$ does not depend on $s'$ thanks to (3.51). If $n < k + 1$, the above expression becomes zero by $Y_{k-n+1,-1}^{(0)} = 0$. Hence together with (3.51), we obtain

$$Y_{k,n}^{(0)} = \delta_{k,0} X_n.$$

4. By (3.49) we can determine $S_{n}^{(1)}$:

$$S_{n}^{(1)} = \sum_{k=0}^{\infty} Y_{k,n}^{(0)} = Y_{0,n}^{(0)} = X_n.$$ 

In particular,

$$\text{ord}_{\xi} \partial_{s'} S_{n}^{(1)} = \text{ord}_{\xi} \partial_{s'} X_n \leq -1.$$
(5) Fix \( l_0 \geq 1 \) and assume that for all \( l = 0, \ldots, l_0 - 1 \) and for all \( k = 0, 1, 2, \ldots \), we have determined \( Y_{k,n}^{(l)} \) and that for all \( l = 0, \ldots, l_0 \) we have determined \( S_n^{(l)} \). (The steps (3) and (4) are for \( l_0 = 1 \).)

Since \( S_n^{(0)} = 0 \) by (3.50), the index \( l' \) in the right hand side of the recursion relation (3.48) (with \( l = l_0 \)) runs essentially from 0 to \( l_0 - 1 \). Hence this relation determines \( Y_{k+1,n}^{(l_0)} \) from known quantities for all \( k \geq 0 \).

Because of the initial condition \( Y_0(t; s, s', \xi, \xi') = X(s, \xi) \) (cf. (3.45)), \( Y_0 \) does not depend on \( t \), which means that its Taylor coefficients \( Y_{0,n}^{(l_0)} \) vanish for all \( l_0 \geq 1 \):

\[
Y_{0,n}^{(l_0)} = 0.
\]

Thus we have determined all \( Y_{k,n}^{(l_0)} \) \((k = 0, 1, 2, \ldots)\).

(6) We shall prove below that \( Y_{k,n+1}^{(l_0+1)} = 0 \) if \( k > l_0 + n + 1 \). Hence the sum in (3.49) is finite and \( S_{n+1}^{(l_0+1)} \) is determined. The induction proceeds by incrementing \( l_0 \) by one.

In this way induction proceeds and all \( Y_{k,n}^{(l)} \) and \( S_n^{(l)} \) are determined.

Let us prove that \( Y_{k,n}^{(l)} \) is determined above satisfy

\[
Y_{k,n}^{(l_0)} = 0, \quad \text{if } k > l + n,
\]

\[
\text{ord}_\xi Y_{k,n}^{(l_0)} \leq -l - 1, \quad \text{if } 0 \leq k \leq l + n,
\]

(We define that \( \text{ord}_\xi 0 = -\infty \).) In particular, the sum in (3.49) is well-defined and

\[
\text{ord}_\xi S_n^{(l+1)} \leq -l - 1.
\]

If \( n = -1 \), both (3.57) and (3.58) are obvious. Fix \( n_0 \geq 0 \) and assume that we have proved (3.57) and (3.58) for \( n < n_0 \) and all \((l,k)\).

When \( n = n_0 \) and \( l = 0 \), (3.57) and (3.58) are true for all \( k \) because of (3.53) and (3.52).

Fix \( l_0 \geq 0 \) and assume that we have proved (3.57) and (3.58) for \( n = n_0 \), \( l \leq l_0 \) and all \( k \). As a result (3.59) is true for \( l \leq l_0 \).

For \((n, l, k) = (n_0, l_0 + 1, 0)\) (3.57) is void and (3.58) is true because of (3.51) and \( \text{ord}_\xi X_{n_0}(s, \xi) \leq -1 \).

Put \( n = n_0 \) and \( l = l_0 + 1 \) in (3.48) and assume that \( k + 1 > (l_0 + 1) + n_0 \). Then \( k > (l_0 + 1) + (n_0 - 1) \), which guarantees that \( Y_{k,n-1}^{(l)} = Y_{k,n_0-1}^{(l_0+1)} = 0 \) by the induction hypothesis on \( n \). As we mentioned in the step (5) above, \( l' \) in the right hand side of (3.48) runs from 0 to \( l - 1 = l_0 \). Hence, as we are assuming that \( k > l_0 + n \), we have \( k > l' + n' \), which leads to \( Y_{k,n'}^{(l')} = 0 \) by the induction hypothesis on \( l' \) and \( n' \). Therefore all terms in the right hand side of (3.48) vanish and we have \( Y_{k+1,n}^{(l_0+1)} = 0 \). The induction on \( k \) for (3.57) is completed, namely it is proved for \( n = n_0 \), \( l = l_0 + 1 \) and \( k \geq 1 \).

The estimate (3.58) is easy to check for \( n = n_0 \), \( l = l_0 + 1 \) and \( k \geq 1 \) by the recursion relation (3.48). (Recall once again that \( \xi \partial_\xi \) does not change the order.)

The step \( l = l_0 + 1 \) being proved, the induction proceeds with respect to \( l \) and consequently with respect to \( n \).
In summary we have constructed \(Y(t; s, s', \xi, \xi')\) and \(S(t; s, \xi)\) satisfying (3.44) and (3.46). Moreover, thanks to (3.57), the sum over \(k\) for each fixed \((n, l)\) in

\[
Y(1; s, s', \xi, \xi') = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} Y_{k,n}^{(l)}(s, s', \xi, \xi') h^n,
\]
is finite and the sum over \(l\) is meaningful as a power series of \(\xi\) because of (3.58). The series

\[
S(1; s, \xi) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} S_n^{(l)}(s, \xi) h^n,
\]
is also meaningful as a power series of \(\xi\) thanks to (3.59).

Thus Proposition 3.2 is proved. \(\square\)

**Proof of Proposition 3.2.** We reverse the order of the previous proof. Namely, given \(S(h, s, \xi)\), we shall construct \(X(h, s, \xi)\) such that the corresponding \((1; h, s, \xi)\) in the above proof coincides with it.

Suppose we have such \(X(h, s, \xi)\). Then the above procedure determine \(Y_{k,n}^{(l)}\) and \(S_n^{(l)}\). We expand them as follows:

\[
S(h, s, \xi) = \sum_{n=0}^{\infty} S_n(s, \xi) h^n = \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} S_{n,j}(s, \xi) h^n,
\]

\[
X(h, s, \xi) = \sum_{n=0}^{\infty} X_n(s, \xi) h^n = \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} X_{n,j}(s, \xi) h^n,
\]

\[
S(t; h, s, \xi) = \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} S_n^{(l)}(s, \xi) h^n t^l = \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} S_{n,j}^{(l)}(s, \xi) h^n t^l,
\]

\[
Y_k(t; h, s, s', \xi, \xi') = \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} Y_{k,n}^{(l)}(s, s', \xi, \xi') h^n t^l
= \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} Y_{k,n,j}^{(l)}(s, s', \xi, \xi') h^n t^l.
\]

Here terms with index \(j\) are homogeneous terms of degree \(-j\) with respect to \(\xi\) and \(\xi'\).

At the end of this proof we shall determine \(X_n\) by (3.51),

\[
X_n(s, \xi) = Y_{0,n}^{(0)}(s, s', \xi, \xi').
\]

(In particular, \(Y_{0,n}^{(0)}(s, s', \xi, \xi')\) should not depend on \(s'\) and \(\xi'\)). For this purpose, \(Y_{0,n}^{(0)}\) should be determined by

\[
Y_{0,n}^{(0)}(s, s', \xi, \xi') = S_n(s, \xi) - \sum_{(l,k) \neq (0,0), l,k \geq 0} \frac{1}{l+1} Y_{k,n}^{(l)}(s, s, \xi, \xi)
\]
because of (3.49) and \(S_n(s, \xi) = S_n(t = 1; s, \xi) = \sum_{n=0}^{\infty} S_n^{(l)}(s, \xi)\).

Since \(\mathrm{ord}_Y Y_{k,n}^{(l)}\) should be not more than \(-l-1\) (cf. (3.58)), we expect \(Y_{k,n,1}^{(l)} = 0\) for \(l > 0\). For \(l = 0\) and \(k > 0\) \(Y_{k,n,1}^{(0)} = 0\) follows from (3.59). Hence picking
up homogeneous terms of degree \(-1\) with respect to \(\xi\) from (3.63), the following equation should hold:

\[
Y_{k,n,1}^{(l)} = \delta_{l,0}\delta_{k,0}S_{n,1}
\]

All \(Y_{k,n,1}^{(l)}\) are determined by this condition. Note also that

\[
Y_{0,n,j}^{(l)} = 0 \quad \text{for} \quad l \neq 0
\]

because \(Y_0\) should not depend on \(s\) because of (3.51).

Having determined initial conditions in this way, we shall determine \(Y_{k,n,j}^{(l)}\) inductively. To this end we rewrite the recursion relation (3.48) by (3.49) and pick up homogeneous terms of degree \(-j\):

\[
Y_{k+1,n,j}^{(l)}(s,s',\xi,\xi') = \frac{1}{l+1} \left( \xi \frac{\partial \xi}{\partial s'} Y_{k,n-1,j}^{(l)}(s,s',\xi,\xi') \right)
\]

\[
+ \sum_{l'+l''=l, \; l'' \geq 1, \; j'+j'' \geq 1, \; n'+n''=n, \; 0 \leq k''} \frac{1}{l''} \xi \frac{\partial \xi}{\partial s'} Y_{k,n'',j''}^{(l'')} (s,s',\xi,\xi') \partial_{s'} Y_{k',n'',j''}^{(l''-1)} (s',s'',\xi',\xi''')
\]

(As before, terms like \(Y_{k-1,j-1}^{(l)}\) appearing the above equation for \(n = 0\) can be ignored.)

Fix \(n_0 \geq 0\) and assume that \(Y_{k,0,j}, \ldots, Y_{k,n_0-1,j}\) are determined for all \((l,k,j)\). (This is consistent with the recursion relation (3.66).)

(2) Fix \(j_0 \geq 2\) and assume that \(Y_{k,n_0,j}^{(l)}\) are determined for \(j = 1, \ldots, j_0 - 1\) and all \((l,k)\). (The above step is for \(j_0 = 2\).)

Since all the quantities in the right hand side of the recursion relation (3.66) with \(j = j_0\) are known by the induction hypothesis, we can determine \(Y_{k,n_0,j_0}^{(l)}\) for \(l = 0, 1, 2, \ldots\) and \(k = 1, 2, \ldots\)

(3) Together with (3.65), \(Y_{0,n_0,j_0}^{(l)} = 0\) for \(l = 1, 2, \ldots\), we have determined all \(Y_{k,n_0,j_0}^{(l)}\) except for the case \((l,k) = (0,0)\).

(4) It follows from (3.66) and (3.64) by induction that all \(Y_{k,n,j}^{(l)}\) determined in (1), (2) and (3) satisfy the following properties:

- if \(k > l + n\), then

\[
Y_{k,n,j}^{(l)} = 0,
\]

which corresponds to (3.57) in the proof of Proposition 3.1

- if \(0 \leq k \leq l + n\) and \(j \leq l\), then

\[
Y_{k,n,j}^{(l)} = 0,
\]

which corresponds to (3.58) in the proof of Proposition 3.1

(5) We determine \(Y_{0,n_0,j_0}^{(l)}\) by

\[
Y_{0,n_0,j_0}^{(l)} = S_{n_0,j_0} - \sum_{(l,k) \neq (0,0)} \frac{1}{l+1} Y_{k,n_0,j_0}^{(l)}(s,s,\xi,\xi)
\]
which is the homogeneous part of degree \( -j_0 \) in (3.63). The sum in the right hand side is finite because of (3.67) and (3.68).

(6) The induction with respect to \( j \) proceeds by incrementing \( j_0 \).

Thus all \( Y^{(j)}_{k, n_0, j} \) are determined and \( X_{n_0} \) is determined by (3.92), namely, \( X_n(x, \xi) = \sum_{j=1}^{\infty} Y^{(0)}_{0, n_0, j} \) (cf. (3.69)), which completes the proof of Proposition 3.2.

4. ASYMPTOTICS OF THE TAU FUNCTION

In this section we derive an \( \hbar \)-expansion

\[ \log \tau(h, t, \tilde{t}, s) = \sum_{n=0}^{\infty} \hbar^{n-2} F_n(t, \tilde{t}, s) \]

of the tau function (cf. (1.20)) from the \( \hbar \)-expansion of the \( S \)-functions \( S(h, t, \tilde{t}, s; z) \) (3.3) and \( \hat{S}(h, t, \tilde{t}, s; \tilde{z}) \) (3.4).

Let us recall the fundamental relations (1.20) between the wave functions and the tau function again:

\[ \Psi(h, t, \tilde{t}, z) = \frac{\tau(h, t-h[z^{-1}], \tilde{t}, \tilde{z})}{\tau(h, t, \tilde{t}, \tilde{z})} e^{\frac{1}{\hbar} \zeta(t, z)} / \hbar, \]

\[ \tilde{\Psi}(h, t, \tilde{t}, z) = \frac{\tau(h, t, \tilde{t}-h[\tilde{z}], s+\hbar)}{\tau(h, t, \tilde{t}, \tilde{z})} e^{\frac{1}{\hbar} \zeta(\tilde{t}, \tilde{z})}, \]

where \([z^{-1}] = (1/2, 1/2z^2, 1/3z^3, \ldots)\), \( \zeta(t, z) = \sum_{n=1}^{\infty} t_n z^n \) etc. (Here we again omit inessential constants, \( \alpha(h) \) and \( \tilde{\alpha}(\hbar) \)). This implies that

\[ \hbar^{-1} \hat{S}(h, t, \tilde{t}, s; z) = \left( e^{-\hbar D(z)} - 1 \right) \log \tau(h, t, \tilde{t}, s), \]

\[ \hbar^{-1} \hat{S}(h, t, \tilde{t}, s; \tilde{z}) = \left( e^{-\hbar \tilde{D}(\tilde{z})} e^{\hbar \tilde{\alpha}_s} - 1 \right) \log \tau(h, t, \tilde{t}, s), \]

where

\[ \hat{S}(h, t, \tilde{t}, s; z) = S(h, t, \tilde{t}, s; z) - \zeta(t, z), \]

\[ \hat{S}(h, t, \tilde{t}, s; \tilde{z}) = \tilde{S}(h, t, \tilde{t}, s; \tilde{z}) - \zeta(\tilde{t}, \tilde{z}), \]

and

\[ D(z) = \sum_{j=1}^{\infty} \frac{z^{-j} \partial}{\partial t_j}, \quad \hat{D}(\tilde{z}) = \sum_{j=1}^{\infty} \tilde{z}^{-j} \frac{\partial}{\partial t_{j}}, \]

\[ \hbar^{-1} \frac{\partial}{\partial z} \hat{S}(h, t, \tilde{t}, s; z) = -\hbar D'(z) e^{-\hbar D(z)} \log \tau(h, t, \tilde{t}, s) \]

\[ = -\hbar D'(z)(h^{-1} \hat{S}(h, t, \tilde{t}, s; z) + \log \tau(h, t, \tilde{t}, s)), \]

where \( D'(z) := \frac{\partial}{\partial z} D(z) = -\sum_{j=1}^{\infty} z^{-j-1} \frac{\partial}{\partial t_j}. \) Hence

\[ -\hbar D'(z) \log \tau(h, t, \tilde{t}, s) = h^{-1} \left( \frac{\partial}{\partial z} + \hbar D'(z) \right) \hat{S}(h, t, \tilde{t}, s; z) \]

Multiplying \( z^n \) to this equation and taking the residue, we obtain a system of differential equations

\[ \hbar \frac{\partial}{\partial t_n} \log \tau(h, t, \tilde{t}, s) = h^{-1} \text{Res}_{z=\infty} z^n \left( \frac{\partial}{\partial z} + \hbar D'(z) \right) \hat{S}(h, t, \tilde{t}, s; z) \]
and (2.4) into (4.8), (4.10) and (4.12), we have
\begin{equation}
(4.14)
\end{equation}
and
\begin{equation}
(4.15)
\end{equation}
for }n = 1, 2, \ldots\). In the same way we have
\begin{equation}
(4.10)
\end{equation}
and
\begin{equation}
(4.11)
\end{equation}
for }n = 1, 2, \ldots\). By putting }z = 0 in (4.14) we have a difference equation for the tau function:
\begin{equation}
(4.16)
\end{equation}
In fact, it follows from (1.16), (1.19) and (3.7) that }\hat{\tau}(h, t, i, s) = 0. Hence we have
\begin{equation}
(4.17)
\end{equation}
As is shown in [UT], the system (4.9), (4.11) and (4.12) is compatible and determines the tau function up to multiplicative constant.

By substituting the }h\text{-expansions}
\begin{equation}
(4.13)
\end{equation}
and (2.3) into (4.8), (4.10) and (4.12), we have
\begin{equation}
(4.16)
\end{equation}
and
\begin{equation}
(4.17)
\end{equation}
and
\begin{equation}
(4.18)
\end{equation}
It is obvious from these equations that }F_n = \text{const. for } n < 0. Therefore we can conclude that log }\tau has the expansion (4.1).

Let us expand }S_n(t; z) and }\bar{S}_n(t; \bar{z}) into a power series of }z^{-1} and }\bar{z}:
\begin{equation}
(4.19)
\end{equation}
(The notation is chosen so that it is consistent with our previous work, e.g., [TT2].) Comparing the coefficients of }z^{-j-1}h^{n-1} in (4.16) and the coefficients of }\bar{z}^{j-1}h^{n-1}
in (4.17), we have the equations

\[ \frac{\partial F_n}{\partial t_j} = v_{n,j} + \sum_{k+l=j \atop k \geq 1, l \geq 1} \frac{1}{l} \frac{\partial v_{n-1,l}}{\partial t_k} \quad (v_{-1,j} = 0), \]

\[ -\frac{\partial F_n}{\partial \bar{t}_j} = \bar{v}_{n,j} + \frac{\partial \phi_n}{\partial \bar{t}_j} + \sum_{k+l=j \atop k \geq 1, l \geq 1} \frac{1}{l} \frac{\partial \bar{v}_{n-1,l}}{\partial \bar{t}_k} \quad (\bar{v}_{-1,j} = 0), \]

for \( n = 0, 1, 2, \ldots \).

From the equation (4.18) it is easy to see that \( \frac{\partial F_n}{\partial s} \) is determined recursively. Let us rewrite it in more explicit way. First arrange the coefficients of \( \hbar^{n-1} \) in (4.18) in the vector form:

\[ \begin{pmatrix} \partial_s \\ \frac{1}{2!} \partial_s^2 \\ \frac{1}{3!} \partial_s^3 \\ \frac{1}{4!} \partial_s^4 \\ \ddots \end{pmatrix} \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \vdots \end{pmatrix}. \]

The matrix in the left hand side is

\[ \sum_{n=0}^{\infty} \frac{\partial_s^{n+1}}{(n+1)!} \Lambda^{-n} = \frac{e^T - 1}{T} \bigg|_{T=\partial_s \Lambda^{-1}} \partial_s \]

where \( \Lambda^{-n} \) is the shift matrix \((\delta_{i-n,j})_{i,j=1}^{\infty}\). Hence, applying the matrix

\[ \frac{T}{e^T - 1} \bigg|_{T=\partial_s \Lambda^{-1}} \]

to (4.22), we have

\[ \frac{\partial}{\partial s} \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ \vdots \end{pmatrix} \bigg|_{T=\partial_s \Lambda^{-1}} \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \vdots \end{pmatrix}, \]

or equivalently,

\[ \frac{\partial F_n}{\partial s} = \phi_n - \frac{\phi_{n-1}}{2} + \sum_{p=1}^{[n/2]} K_{2p} \phi_{n-2p}, \]

where \( K_{2p} \) is determined by (2.22). The system of first order differential equations (4.20), (4.21) and (4.24) may be understood as defining equations of \( F_n(t, \bar{t}, s) \). This system is integrable and determines \( F_n \) up to integration constants, because the system (4.9), (4.11) and (4.12) are compatible.

Remark 4.1. Tau functions in string theory and random matrices are known to have a genus expansion of the form

\[ \log \tau = \sum_{g=0}^{\infty} \hbar^{2g-2} \mathcal{F}_g, \]

where \( \mathcal{F}_g \) is the contribution from Riemann surfaces of genus \( g \). In contrast, general tau functions of the \( \hbar \)-dependent Toda hierarchy is not of this form, namely, odd
powers of $\hbar$ can appear in the $\hbar$-expansion of $\log \tau$. To exclude odd powers therein, we need to impose conditions

$$0 = v_{2m+1,j} + \sum_{k+l=j \atop k \geq 1, l \geq 1} \frac{1}{l} \frac{\partial v_{2m,l}}{\partial t_k} = \bar{v}_{2m+1,j} + \frac{\partial \phi_{2m+1}}{\partial t_j} + \sum_{k+l=j \atop k \geq 1, l \geq 1} \frac{1}{l} \frac{\partial \bar{v}_{2m,l}}{\partial t_k}$$

$$= \phi_{2m+1} - \frac{\phi_{2m}}{2} + m \sum_{p=1} K_{2p} \phi_{2m+1-2p},$$

on $v_{n,j}, \bar{v}_{n,j}$ and $\phi_n$ or

$$0 = \frac{\partial S_{2m+1}}{\partial z} - \sum_{j=1}^{\infty} z^{-j-1} \frac{\partial S_{2m}}{\partial t_j} = \frac{\partial S_{2m+1}}{\partial z} + \sum_{j=1}^{\infty} z^{-j-1} \frac{\partial S_{2m}}{\partial t_j}$$

(4.26)

$$= \phi_{2m+1} - \frac{\phi_{2m}}{2} + m \sum_{p=1} K_{2p} \phi_{2m+1-2p},$$

on $S_n, \bar{S}_n$ and $\phi_n$.

**Appendix A. Example ($c = 1$ string theory)**

In this appendix, we apply our algorithm to the compactified $c = 1$ string theory at a self-dual radius, following the formulation in [T1]. We use the notations in Section 2.

According to (4.10) in [T1] ($\beta = 1$), the string equation for this case is

$$L = (-M - \hbar + 1) \bar{L},$$

(A.1)

$$\bar{L}^{-1} = (-M + 1) L^{-1}.$$

Multiplying the left and right hand side of the second equation to the right and left hand side of the first equation from the right, we have

$$L(-M + 1) L^{-1} = -M - \hbar + 1,$$

and using the canonical commutation relation $[L, M] = \hbar L$, we have $M = \bar{M}$, namely

(A.2)

$$L = (1 - \bar{M} - \hbar) \bar{L}, \quad M = \bar{M}.$$

Hence the data $(f, g, \bar{f}, \bar{g})$ for Proposition 1.2 in this case are

(A.3)

$$f(h, s, e^{\hbar \partial_s}) = e^{\hbar \partial_s}, \quad g(h, s, e^{\hbar \partial_s}) = s,$$

$$\bar{f}(h, s, e^{\hbar \partial_s}) = (1 - s - \hbar) e^{\hbar \partial_s}, \quad \bar{g}(h, s, e^{\hbar \partial_s}) = s.$$

The corresponding dispersionless data $(f_0, g_0, \bar{f}_0, \bar{g}_0)$ for Proposition 1.3 are

(A.4)

$$f_0(s, \xi) = \xi, \quad g_0(s, \xi) = s,$$

$$\bar{f}_0(s, \xi) = (1 - s) \xi, \quad \bar{g}_0(s, \xi) = s.$$

For the sake of simplicity, we fix the time variables $\bar{t}_n$ ($n = 1, 2, \ldots$) to 0, which makes it possible to determine all $X_n$'s explicitly, (A.18). If we turn on $\bar{t}_n$'s, we need to proceed perturbatively.

To begin with, let us determine the leading terms of $X$, $\bar{X}$ and $\phi$ with respect to the $\hbar$-order, namely $X_0$, $\bar{X}_0$ and $\phi_0$ in (2.2), (2.3) and (2.4).
The Riemann-Hilbert type problem for \((L, M, \bar{L}, \bar{M})\) is
\[(A.5)\]
\[L = (1 - \bar{M}) \bar{L}, \quad M = \bar{M}.\]
Recall that \(L, M, \bar{L}\) and \(\bar{M}\) have the following form by \((1.27), (1.28), (1.32)\) and \((1.33)\) when \(\bar{t} = 0\).

\[
L = \xi + \sum_{n=0}^{\infty} u_{0,n} \xi^{-n},
\]
\[
\bar{L} = \sum_{n=0}^{\infty} \bar{u}_{0,n} \xi^{n+1},
\]
\[
M = \sum_{n=1}^{\infty} n t_n L^n + s + \alpha_0 + \sum_{n=1}^{\infty} v_{0,n} \bar{L}^{-n},
\]
\[
\bar{M} = s + \bar{\alpha}_0 + \sum_{n=1}^{\infty} \bar{v}_{0,n} \bar{L}^n,
\]

Therefore \((1 - \bar{M}) \bar{L}\) is a Taylor series with positive powers of \(\xi\), while \(L\) does not have a positive power of \(\xi\) except for the first term, \(\xi\). Therefore the first equation in \((A.5)\) implies that
\[(A.6)\]
\[L = \xi.\]
From this and the second equation \(M = \bar{M}\) in \((A.5)\), it follows that \(M\) and \(\bar{M}\) do not have negative powers of \(\xi\) and \(\alpha_0 = \bar{\alpha}_0\). Hence we may assume that \(\alpha_0 = \bar{\alpha}_0 = 0\) and
\[(A.7)\]
\[M = \bar{M} = s + \sum_{n=1}^{\infty} n t_n \xi^n.\]
Substituting this into the first equation of \((A.5)\), we have
\[
(A.8) \quad \bar{L} = \xi \left( 1 - s - \sum_{n=1}^{\infty} n t_n \xi^n \right)^{-1}, \quad \text{or} \quad \bar{L}^{-1} = \xi^{-1} \left( 1 - s - \sum_{n=1}^{\infty} n t_n \xi^n \right).\]

Next, let us determine the leading terms \(X_0, \bar{X}_0\) and \(\phi_0\) of the dressing operators \(X, \bar{X}\) and \(\phi\). We denote the symbols of \(X_0\) and \(\bar{X}_0\) by \(X_0 = X_0(t, s, \xi)\) and \(\bar{X}_0 = \bar{X}_0(t, s, \xi)\).

Since \(L = \exp(\text{ad}_{\{,\}} X_0) \xi = X_0\), \(X_0\) does not depend on \(s\). On the other hand, since \(M = \exp(\text{ad}_{\{,\}} X_0) \xi + s + \sum_{n=1}^{\infty} n t_n \xi^n, \bar{X}_0\) does not depend on \(\xi\), either, which means that \(X_0 = 0\).

Note that \(\text{ad}_{\{,\}} \phi_0(s)\) does not change the degree of homogeneous terms with respect to \(\xi\), since \(\phi_0(s)\) does not contain \(\xi\). Hence \(\bar{L}\) has the following asymptotic behaviour around \(\xi = 0\).

\[
\bar{L} = e^{\text{ad}_{\{,\}} \phi_0} \xi + e^{\text{ad}_{\{,\}} \phi_0} \sum_{N=1}^{\infty} \frac{1}{N!} (\text{ad}_{\{,\}} \bar{X}_0)^N \xi = e^{\text{ad}_{\{,\}} \phi_0} \xi + O(\xi^2),
\]
because \(\bar{X}_0\) is a Taylor series of \(\xi\) with positive powers. Comparing this expansion with \((A.8)\), we have
\[
(A.9) \quad e^{\text{ad}_{\{,\}} \phi_0} \xi = (1 - s)^{-1} \xi.
\]
It is easy to see that the left hand side is equal to \( e^{-\phi_0(s)} \xi \), where \( \xi \) denotes the derivation by \( s \). Thus we obtain

\[
\phi_0(s) = \int^s \log(1-s) \, ds = -(1-s) \log(1-s) + (1-s).
\]

It remains to determine \( \bar{X}_0 \). Operating \( e^{-\text{ad}_{(\cdot)}} \phi_0 \) to \( \bar{X}^{-1} \) (A.8) and \( \bar{M} \) (A.7) and using the formula

\[
e^{-\text{ad}_{(\cdot)}} \phi s \text{derivation by } \phi = (A.12)
\]

\( (A.10) \), which follows directly from (A.9), we have two equations characterising \( \bar{X}_0 \):

\[
e^{\text{ad}_{(\cdot)}} \bar{X}_0 \xi^{-1} = \xi^{-1} - \sum_{n=1}^{\infty} nt_n (1-s)^{n-1} \xi^{n-1},
\]

\( (A.12) \)

\[
e^{\text{ad}_{(\cdot)}} \bar{X}_0 s = s + \sum_{n=1}^{\infty} nt_n (1-s)^n \xi^n.
\]

In fact we can determine \( \bar{X}_0 \) explicitly as follows.

\[
\bar{X}_0 = \sum_{n=1}^{\infty} n (1-s)^n \xi^n.
\]

Indeed, since

\[
\{\bar{X}_0, \xi^{-1}\} = -\sum_{n=1}^{\infty} nt_n (1-s)^{n-1} \xi^{n-1} \text{ and } \{\bar{X}_0, s\} = \sum_{n=1}^{\infty} nt_n (1-s)^n \xi^n
\]

commute with \( \bar{X}_0 \) itself (this is a direct consequence of a trivial fact \( \{(1-s)^k \xi^k, (1-s)^l \xi^l\} = 0 \)), the exponentials in \( e^{\text{ad}_{(\cdot)}} \bar{X}_0 \xi^{-1} \) and \( e^{\text{ad}_{(\cdot)}} \bar{X}_0 s \) are truncated up to the first order, namely

\[
e^{\text{ad}_{(\cdot)}} \bar{X}_0 \xi^{-1} = \xi^{-1} + \{\bar{X}_0, \xi^{-1}\} = \xi^{-1} - \sum_{n=1}^{\infty} nt_n (1-s)^{n-1} \xi^{n-1},
\]

\( (A.14) \)

\[
e^{\text{ad}_{(\cdot)}} \bar{X}_0 s = s + \{\bar{X}_0, s\} = s + \sum_{n=1}^{\infty} nt_n (1-s)^n \xi^n,
\]

which proves that \( \bar{X}_0 \) in (A.13) satisfies (A.12).

Thus we have determined the leading terms of \( X, \bar{X} \) and \( \phi \) as follows:

\[
X_0 = 0, \quad \bar{X}_0 = \sum_{n=1}^{\infty} t_n (1-s)^n e^{nh \partial_s}, \quad \phi_0 = -(1-s) \log(1-s) + (1-s).
\]

(A.15)

Having determined \( X_0, \bar{X}_0 \) and \( \phi_0 \), we can start the algorithm discussed in Section 2. Following the procedure by straightforward computation (actually, not so straightforward, as we shall see later), we obtain as the first and the second steps,

\[
X_1 = 0, \quad \bar{X}_1 = -\sum_{n=1}^{\infty} t_n \frac{n(n+1)}{2} (1-s)^{n-1} e^{nh \partial_s}, \quad \phi_1 = \frac{1}{2} \log(1-s),
\]

(A.16)

and

\[
X_2 = 0, \quad \bar{X}_2 = -\sum_{n=1}^{\infty} t_n \frac{n(n^2 - 1)(3n + 2)}{24} (1-s)^{n-2} e^{nh \partial_s}, \quad \phi_2 = -\frac{1}{12} (1-s)^{-1}.
\]

(A.17)
From these results we can infer the Ansatz for all $n \geq 2$:

\[ X_n = 0, \quad \bar{X}_n = \sum_{m=1}^{\infty} t_m c_{n,m} (1 - s)^{m-n} e^{\hbar \bar{\partial} s}, \quad \phi_n = c_{n,0} (1 - s)^{-n+1}, \]

with suitable constants $c_{n,m}$ ($n, m \geq 1$) and $c_{n,0}$ ($n \geq 2$). Eventually these constants are determined recursively as follows:

\[ c_{n,m} = \frac{1}{n} \sum_{j=0}^{n-1} (-1)^{n-j} \binom{m - j + 1}{k - j + 1} c_{j,m}, \]

\[ c_{n,0} = \frac{1}{-n + 1} \left( \frac{1}{n + 1} - \frac{c_{1,0}}{n} - \sum_{j=2}^{n-1} (-1)^{n-j} \binom{-j + 1}{k - j + 1} c_{j,0} \right), \]

with the initial values $c_{0,m} = 1$ and $c_{1,0} = 1/2$.

Let us prove that the above Ansatz is true. To do this, we have only to check that it is consistent with the algorithm in Section 2.

It is easy to compute the intermediate objects $P^{(i-1)}$ and $Q^{(i-1)}$, since the operators $X_0, \ldots, X_{i-1}$ are zero. Using the notations (2.10) and (2.12), we have

\[ P^{(i-1)} = \exp \left( \frac{X^{(i-1)}}{\hbar} \right) f_t = f_t = e^{\hbar \bar{\partial} s}, \]

\[ Q^{(i-1)} = \exp \left( \frac{X^{(i-1)}}{\hbar} \right) g_t = g_t = s + \sum_{n=1}^{\infty} n t_n e^{\hbar \partial s}. \]

Hence the terms in the expansion (2.17) vanish except for $P^{(i-1)}_0$ and $Q^{(i-1)}_0$:

\[ P^{(i-1)}_0 = e^{\hbar \bar{\partial} s}, \quad P^{(i-1)}_1 = P^{(i-1)}_2 = \cdots = P^{(i-1)}_i = 0, \]

\[ Q^{(i-1)}_0 = s + \sum_{n=1}^{\infty} n t_n e^{\hbar \partial s}, \quad Q^{(i-1)}_1 = Q^{(i-1)}_2 = \cdots = Q^{(i-1)}_i = 0. \]

Their symbols are

\[ P^{(i-1)}_0 = \xi, \quad P^{(i-1)}_1 = P^{(i-1)}_2 = \cdots = P^{(i-1)}_i = 0, \]

\[ Q^{(i-1)}_0 = s + \sum_{n=1}^{\infty} n t_n \xi^n, \quad Q^{(i-1)}_1 = Q^{(i-1)}_2 = \cdots = Q^{(i-1)}_i = 0. \]

To compute $\tilde{P}^{(i-1)}$, let us consider $\exp(\text{ad}(X^{(i-1)}/\hbar))\bar{f}_t$ first. Note that

\[ \hbar^{-1} [\bar{X}^{(i-1)}, \bar{f}] = \sum_{n=0}^{i-1} \hbar^{n-1} [\bar{X}_n, \bar{f}] \]

\[ = \sum_{n=0}^{i-1} \hbar^{n-1} \sum_{m=1}^{\infty} t_m c_{n,m} [(1 - s)^{m-n} e^{\hbar \bar{\partial} s}, (1 - s - \hbar) e^{\hbar \bar{\partial} s}] \]

Substituting

\[ [(1 - s)^{m-n} e^{\hbar \bar{\partial} s}, (1 - s - \hbar) e^{\hbar \bar{\partial} s}] = \left( -n \hbar (1 - s)^{m-n} - \sum_{r=2}^{m-n+1} \binom{m-n+1}{r} (-\hbar)^r (1 - s)^{m-n+1-r} \right) e^{(m+1)\hbar \bar{\partial} s}, \]
Because of the definition of $c_{n,m}$ \[ (A.19)\], the coefficients of $\hbar^n t_m$ ($n = 0, \ldots, i - 1$, $m = 1, 2, \ldots$) in the right hand side vanish and the coefficient of $\hbar^i$ is equal to $i c_{i,m}$. (Actually this is why $c_{n,m}$ is defined by the recursion relation \[ (A.19)\].) This means

\[ (A.28) \quad \hbar^{-1}[\bar{X}^{(i-1)}, \bar{f}] = \hbar^i \sum_{m=1}^{\infty} t_m i c_{i,m} (1 - s)^{m-i} e^{(m+1)\hbar \partial_x} + O(\hbar^{i+1}). \]

Further application of $\text{ad}(\hbar^{-1}\bar{X}^{(i-1)})$ changes the symbol of terms of $\hbar$-order $-i$ (i.e., coefficients of $\hbar^i$) by application of $\text{ad}_{\bar{X}_0}$, as $\text{ad}\hbar^{-1}\bar{X}_j$ ($j = 1, \ldots, i - 1$) lowers the $\hbar$-order. Hence for $N \geq 1$ we have

\[ (A.29) \quad (\text{ad}\hbar^{-1}\bar{X}^{(i-1)})^N \bar{f} = \hbar^i (\text{ad}_{\bar{X}_0})^{N-1} \left( \sum_{m=1}^{\infty} t_m i c_{i,m} (1 - s)^{m-i} \xi^{m+1} \right)_{\xi \to e^{\hbar \partial_x}} + O(\hbar^{i+1}). \]

Next we compute the conjugation of $\bar{f}$ by $e^{\hbar^{-1}\phi^{(i-1)}(s)}$.

\[ e^{\text{ad}\hbar^{-1}\phi^{(i-1)}(s)} \bar{f} = e^{\hbar^{-1}\phi^{(i-1)}(s)} (1 - s - \hbar) e^{\hbar \partial_x} e^{-\hbar^{-1}\phi^{(i-1)}(s)} \]

\[ = e^{\hbar^{-1}(\phi^{(i-1)}(s) - \phi^{(i-1)}(s + \hbar))} (1 - s - \hbar) e^{\hbar \partial_x} \]

\[ = \exp \left( \frac{1}{\hbar} (\phi_0(s) - \phi_0(s + \hbar)) + \log (1 - s - \hbar) \right) + \sum_{j=1}^{i-1} \hbar^{-1}(\phi_j(s) - \phi_j(s + \hbar)) \right) e^{\hbar \partial_x}. \]

By \[ (A.15), (A.16) \] and \[ (A.18)\], we have

\[ \frac{1}{\hbar} (\phi_0(s) - \phi_0(s + \hbar)) + \log (1 - s - \hbar) = -\sum_{k=1}^{\infty} \frac{\hbar^k}{k+1} (1 - s)^{-k}, \]

\[ \phi_1(s) - \phi_1(s + \hbar) = \sum_{k=1}^{\infty} \frac{\hbar^k c_{1,0}}{k} (1 - s)^{-k}, \]

\[ \phi_j(s) - \phi_j(s + \hbar) = \sum_{k=1}^{\infty} \hbar^k (-1)^{k+1} c_{j,0} \binom{-j + 1}{k} (1 - s)^{-j-k+1}. \]
Therefore

\[
\frac{1}{\hbar}(\phi_0(s) - \phi_0(s + \hbar)) + \log(1 - s - \hbar) + \sum_{j=1}^{i-1} \hbar^{j-1} (\phi_j(s) - \phi_j(s + \hbar)) = (i - 1)c_{i,0}\frac{\hbar^i}{(1 - s)^i} + O(\hbar^{i+1}).
\]

Thus we have

\[(A.31)\]

\[e^{ad h^{-1} \phi^{(i-1)}} \bar{f} = (1 + h^i(i - 1)c_{i,0}(1 - s)^{-i} + O(h^{i+1})) e^{\hbar \partial_s},\]

for \(i \geq 2\). When \(i = 1\), we should replace \((1 - s)^{-1}\) by \(\log(1 - s)\) but the rest is the same.

Summarising the above results \((A.29)\) and \((A.31)\), we obtain the following expansion of \(P^{(i-1)}\).

\[(A.32)\]

\[
P^{(i-1)} = e^{ad h^{-1} \phi^{(i-1)}} e^{ad h^{-1} X^{(i-1)}} \bar{f} = e^{ad h^{-1} \phi^{(i-1)}} \bar{f} + e^{ad h^{-1} \phi^{(i-1)}} \sum_{N=1}^{\infty} \frac{(ad h^{-1} X^{(i-1)})^N}{N!} \bar{f}
\]

\[
e^{\hbar \partial_s} + h^i(i - 1)c_{i,0}(1 - s)^{-i} e^{\hbar \partial_s}
\]

\[+ h^i e^{ad_{(i)}} \phi_0 \sum_{N=1}^{\infty} \frac{(ad_{(i)}) X_0)^N}{N!} \left( \sum_{m=1}^{\infty} t_m i c_{i,m}(1 - s)^{m-i} \xi^{m+1} \right) \bigg|_{\xi \to e^{\hbar \partial_s}}.
\]

Therefore

\[(A.33)\]

\[\bar{P}^{(i-1)}_0 = e^{\hbar \partial_s}, \quad \bar{P}^{(i-1)}_1 = \ldots = \bar{P}^{(i-1)}_{i-1} = 0,
\]

which coincide with \((A.22)\), and

\[(A.34)\]

\[\bar{P}^{(i-1)}_i = (i - 1)c_{i,0}(1 - s)^{-i} \xi
\]

\[+ e^{ad_{(i)}} \phi_0 \sum_{N=1}^{\infty} \frac{(ad_{(i)}) X_0)^N}{N!} \left( \sum_{m=1}^{\infty} t_m i c_{i,m}(1 - s)^{m-i} \xi^{m+1} \right).
\]

From formulae

\[ad_{(i)} X_0((1 - s)^{-j}) = j(1 - s)^{-j-1} \left( \sum_{k=1}^{\infty} k t_k (1 - s)^k \xi^k \right),
\]

\[\{ X_0, (1 - s)^k \xi^k \} = 0 \quad \text{and} \quad (A.9)\] it follows that

\[(A.35)\]

\[e^{ad_{(i)}} \phi_0 (ad_{(i)} X_0)^{N-1} ((1 - s)^{m-i} \xi^{m+1})
\]

\[= (i + 1) \cdots (i + N - 1)(1 - s)^{-i-N} \xi^{m+1} \left( \sum_{k=1}^{\infty} k t_k \xi^k \right)^{N-1}.
\]
Substituting it to (A.34), we have

\[(A.36) \quad \bar{P}_i^{(i-1)} = (i - 1)c_{i,0}(1 - s)^{-i} \xi \]
\[+ \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{m=1}^{\infty} t_m c_{i,m} \xi^{m+1}(i + 1) \cdots (i + N - 1) \times \]
\[\times (1 - s)^{-i-N} \left( \sum_{k=1}^{\infty} k t_k \xi^k \right)^{N-1}.\]

Now let us compute (2.20). Although we have not computed \(\bar{Q}^{(i-1)}\), thanks to (A.24) and (A.25), integrals in (2.20) are simplified to

\[-\tilde{\lambda}_i + \phi_i + \tilde{\lambda}_i = \int_s \xi^{-1} \bar{P}_i^{(i-1)} ds,\]

which is computable without information of \(\bar{Q}^{(i-1)}\). By the explicit formula (A.36), we obtain

\[(A.37) \quad -\tilde{\lambda}_i + \phi_i + \tilde{\lambda}_i = c_{i,0}(1 - s)^{-i+1}
+ \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{m=1}^{\infty} t_m c_{i,m} \xi^m(i + 1) \cdots (i + N - 2)(1 - s)^{-i-N+1} \left( \sum_{k=1}^{\infty} k t_k \xi^k \right)^{N-1}.\]

In this formula there is no term with negative powers of \(\xi\), which means \(\tilde{\lambda}_i = 0\), i.e., \(\lambda_i = 0\). The constant term with respect to \(\xi\) is \(c_{i,0}(1 - s)^{-i+1}\), which is \(\phi_i(s)\), as was expected. The remaining part is \(\tilde{\lambda}_i\).

In general, it is almost hopeless to compute \(\tilde{\lambda}_i\) from \(\bar{X}_i\) by (2.21). However, quite fortunately, in the present case we are able to find the explicit answer. Using (A.35), we can rewrite \(\tilde{\lambda}_i\) as follows.

\[(A.38) \quad \tilde{\lambda}_i = e^{\text{ad}_{(\cdot)}} \phi_0 \left( \sum_{N=1}^{\infty} \frac{(\text{ad}_{(\cdot)}) \bar{X}_0}{N!} \left( \sum_{m=1}^{\infty} t_m c_{i,m}(1 - s)^{m-i} \xi^m \right)^{N-1} \right).\]

Recall that equations in (2.21) are the inversion formulae of (2.26) and (2.27). Comparing (A.38) and (2.27), we can conclude that

\[\bar{X}_i = \sum_{m=1}^{\infty} t_m c_{i,m}(1 - s)^{m-i} \xi^m,\]

which finally proves the Ansatz (A.18).

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Graduate School of Human and Environmental Studies, Kyoto University, Yoshida, Sakyo, Kyoto, 606-8501, Japan

E-mail address: takasaki@math.h.kyoto-u.ac.jp

Faculty of Mathematics, National Research University – Higher School of Economics, Vavilova Street, 7, Moscow, 117312, Russia

E-mail address: ttakebe@hse.ru