Quantum Black Hole Evaporation.

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Abstract

We investigate a recently proposed model for a full quantum description of two-dimensional black hole evaporation, in which a reflecting boundary condition is imposed in the strong coupling region. It is shown that in this model each initial state is mapped to a well-defined asymptotic out-state, provided one performs a certain projection in the gravitational zero mode sector. We find that for an incoming localized energy pulse, the corresponding out-going state contains approximately thermal radiation, in accordance with semi-classical predictions. In addition, our model allows for certain acausal strong coupling effects near the singularity, that give rise to corrections to the Hawking spectrum and restore the coherence of the out-state. To an asymptotic observer these corrections appear to originate from behind the receding apparent horizon and start to influence the out-going state long before the black hole has emitted most of its mass. Finally, by putting the system in a finite box, we are able to derive some algebraic properties of the scattering matrix and prove that the final state contains all initial information.
1. Introduction

Classically, black holes are absolutely black. They do not allow any signals to leave their surface and forever hide themselves behind a horizon without any means of communication with the outside world. In quantum mechanics, however, this classical notion of a black hole needs a fundamental revision. Hawking’s discovery that quantum black holes can evaporate by emitting thermal radiation [1] shows that they are neither absolutely black nor can hide themselves forever. Nevertheless, the very mechanism by which black holes can lose their mass does not seem to allow for any information to be carried out with it. However drastic, it seems at present an almost foregone conclusion that black hole evaporation leads to information loss and to the evolution of pure states into mixed states. Before accepting such a far reaching conclusion, however, it seems only appropriate to make sure that other, more conservative, logical possibilities are ruled out.

A simple toy model for studying some of these issues is two-dimensional dilaton gravity coupled to $N$ massless scalar fields [2]. This model exhibits an instability against gravitational collapse very similar to the Einstein theory, but at the same time it is simple enough to allow an explicit analysis of its quantum properties. In this paper we will further investigate a recently proposed method for quantizing dilaton gravity with $N = 24$ matter fields [3]. Our goal is to use this model to obtain a full quantum description of two-dimensional black hole evaporation and to investigate whether the problem of information loss can be avoided while retaining the correspondence with semi-classical physics. This last requirement means concretely that for physically reasonable initial states, a significant part of the out-going state should describe almost thermal radiation, as this would be a true signature that a black hole was actually formed.

This paper is organized as follows. In section 2 we use a simple gravitational shock-wave picture to draw a parallel between dilaton gravity and ’t Hooft’s model for a 3+1-dimensional black hole $S$-matrix. The quantization of $N = 24$ dilaton gravity is reviewed in section 3. We describe the physical spectrum and formulate the boundary condition that leads to the definition of the $S$-matrix. In section 4 we summarize the interactions between the in- and out-going modes by means of a certain ‘exchange algebra’. We use this algebra to analyze some physical properties of the $S$-matrix and to make contact with the semi-classical theory and to discuss the corrections to it. In section 5 we use the analogy with open string theory to put the model in a finite box. This allows us to prove that the $S$-matrix indeed maps every in-state to a pure out-state. In the last section we address some questions concerning the unitarity and physical interpretation of the scattering matrix.
2. The Black Hole Problem Miniaturized

In this section we will use the two-dimensional dilaton gravity model to illustrate some aspects of the black hole information paradox. In particular we will summarize ’t Hooft’s argument [4] for the form of the black hole scattering matrix in this simplified context (see also [5]). The purpose of this discussion is to provide a physical context for some of the calculations in the coming sections, and to motivate the need for and the form of the boundary conditions. Moreover, it turns out that this simplified version of ’t Hooft’s $S$-matrix appears as an overall factor in the full quantum scattering matrix of dilaton gravity, and therefore similar issues arise when one tries to find their proper physical interpretation. The reasonings in this section are semi-classical and somewhat intuitive, as the full quantum treatment of the model will be given later.

The two-dimensional dilaton gravity model is described by the action

$$S = \frac{1}{2\pi} \int d^2x \sqrt{-g} \left[ e^{-2\phi} \left( R + 4(\nabla \phi)^2 + 4\lambda^2 \right) - \frac{1}{2} \sum_{i=1}^{N} (\nabla f_i)^2 \right].$$  \hspace{1cm} (2.1)

Here $\phi$ is the dilaton, which forms together with the metric $g = ds^2$ the gravitational sector of the model, and the $f_i$ are free massless matter fields. The classical and semi-classical properties of this model have been the subject of many recent papers [2], [6]-[9], so we will only give a brief description here.

The classical equations of motion of (2.1) can be integrated exactly for arbitrary in- and out-flux of energy. The general solution describing the gravitational collapse of an incoming amount of massless matter forming a black hole is given by

$$ds^2 = e^{2\rho} dx^+ dx^-,$$

$$e^{-2\rho} = e^{-2\phi} = -\lambda^2 x^+ [x^- + \frac{1}{\lambda^2} P_+(x^+)] + M(x^+).$$  \hspace{1cm} (2.2)

Here $P_+(x^+)$ and $M(x^+)$ are expressed in terms of the incoming energy-momentum flux $T_{++}(x^+)$ as

$$P_+(x^+) = \int_0^{x^+} dy^+ T_{++},$$  \hspace{1cm} (2.3)

$$M(x^+) = \int_0^{x^+} dy^+ y^+ T_{++}.$$  \hspace{1cm} (2.4)
Figure 1a and 1b. Two representations of the same classical geometry describing the shockwave and subsequent black hole formation. In the left figure the dilaton and metric are continuous across the shockwave, and rightmoving light rays are straight lines. On the right the lightray trajectories undergo a discontinuous shift when crossing the shockwave. The horizon is indicated in both figures as the last ray that reaches $I^+_R$.

The form of this geometry is depicted in fig 1a for the case in which the incoming matter pulse is localized in the form of an (approximate) shockwave. The solution is obtained by gluing together, along the shockwave trajectory, the linear dilaton vacuum solution

$$e^{-2\rho} = e^{-2\phi} = -\lambda^2 x^+ x^-$$  \hspace{1cm} (2.5)

for small $x^+$, to a static black hole solution \[10, 11\] for large $x^+$

$$e^{-2\rho} = e^{-2\phi} = -\lambda^2 x^+ (x^- + \frac{1}{\lambda^2} P_+) + M$$  \hspace{1cm} (2.6)

with $P_+ = P_+(\infty)$ the total incoming Kruskal momentum and $M = M(\infty)$ the total energy of the incoming matter wave. The matching condition that determines the form of this solution is that the dilaton and metric are continuous along the shockwave. Classical trajectories of out-going massless particles are therefore straight lines without any discontinuity at the shockwave.

Observers that do not decide to throw themselves into the black hole have only access to that part of the space-time which is outside of the horizon. Since we will in the following only be interested in what these outside observers can see, it will be convenient to redefine
the $x^+, x^-$ coordinate system such that the observable part of space time is always given by the region $x^+ > 0$ and $x^- < 0$. In these coordinates, the geometry describing the formation of a black hole looks as in fig 1b. This geometry becomes physically equivalent to the previous one, provided we take into account that the $x^-$ coordinate of a right-moving particle undergoes a shift

$$x^- \rightarrow x^- + \delta x^- \quad \delta x^- = \frac{1}{\lambda^2} P_+$$

(2.7)

when it crosses the shockwave trajectory. Thus particles for which initially $x^- > -\lambda^{-2} P_+$ will get shifted to positive $x^-$ values, i.e. inside the black hole region, and thus become invisible to outside observers.

Now let us consider the effect of this shockwave on the quantum mechanical wave function of the right-moving $f$-particles. The coordinate shift (2.7) is associated to the (formally unitary) quantum operator

$$U = \exp\left(\frac{i}{2\pi} \delta x^- P_\right)$$

(2.8)

where $P_- = \int_{-\infty}^{0} dy^- T_-$ is the right-moving total Kruskal momentum. After inserting that $\delta x^- = \lambda^{-2} P_+$, this operator takes a form that is symmetric between the in-going and out-going fields, as [3]

$$U = \exp\left(\frac{i}{2\pi \lambda^2} P_+ P_\right).$$

(2.9)

So far, this quantum operator just represents the shockwave interaction between the left- and right-moving $f$-fields.

A qualitative difference between two-dimensional dilaton gravity and, for example, the spherical symmetric reduction of Einstein gravity is that the linear dilaton vacuum does not provide a natural (timelike) boundary analogous to the origin $r = 0$. On the other hand, the effective coupling constant $\kappa = e^\phi$ becomes infinite near the left null boundaries of the linear dilaton vacuum, and it is therefore not appropriate to treat these as asymptotic regions. Instead it is more natural to try to impose suitable boundary conditions in this strong coupling regime, which will then prescribe the initial conditions for the right-moving fields. An important consequence of this way of setting up the model is that initial data only need to be specified in the right in-region. The black hole information problem can be formulated as the question whether (there exists a physically reasonable choice of boundary conditions such that) all the initial information will be contained in the final data that arrive in the right out-region.
In the presence of a reflecting boundary condition, the operator (2.9) describing the interaction between the left- and right-moving modes acquires a new meaning. To illustrate this, let us for the moment imagine that the matter is described by only one single quantum mechanical particle. The operator $P_+$ then measures the incoming Kruskal momentum of this particle, and is conjugate to its coordinate $x^+$. Similarly, $P_-$ measures the out-going momentum conjugate to $x^-$. Now, because of the reflection condition, we are asked to identify the in- and out Hilbert spaces, and this must be done in a way that takes into account the shockwave interaction between the particles. This is achieved if, following [4], we promote the operator (2.9) to the $S$-matrix relating the in-state and the out-state

$$\langle P_+, \text{out} | P_-, \text{in} \rangle = \exp \left( \frac{i}{2\pi \lambda^2} P_+ P_- \right).$$

(2.10)

This one-particle $S$-matrix is the dimensional reduction of the black hole scattering matrix proposed by 't Hooft in 3+1-dimensions [3].

The above formula implies that, as a quantum operator, the out-going coordinate $x^-$ is identified with the incoming momentum operator via

$$\lambda^2 x^- = P_+ .$$

(2.11)

This relation may look somewhat mysterious, but note that classically, as seen in fig. 1b, this is precisely where the in-coming particle ends up! It leads, however, to an important difficulty with the interpretation of (2.10) as an $S$-matrix. Namely, while (2.9) indeed looks like a unitary operator in the single particle Hilbert space, this is true only if the Hilbert space contains all eigenstates of $P_\pm$. If, however, we are interested in what an outside observer sees, we should restrict ourselves to in- and out-states that have support only in the outside region $x^+ > 0$ and $x^- < 0$, and after this restriction the $S$-matrix (2.10) is no longer unitary. For example, a basis of states that has support only in the outside region is provided by the eigenstates of the asymptotic energy operator $M$, given by

$$\langle x^+ | \omega, \text{in} \rangle = (\lambda x^+)^{-i\omega - \frac{1}{2}} \theta(x^+),$$

$$\langle x^- | \omega, \text{out} \rangle = (-\lambda x^-)^{i\omega - \frac{1}{2}} \theta(-x^-).$$

(2.12)

The $S$-matrix is diagonal on this basis, with matrix elements (up to a phase)

$$\langle \omega', \text{out} | \omega, \text{in} \rangle = \frac{1}{\sqrt{2\pi}} e^{-\frac{i}{2} \omega} \Gamma \left( \frac{1}{2} - i \omega \right) \delta_{\omega, \omega'} .$$

(2.13)
The fact that these matrix elements are not phase factors, but instead satisfy \(|S|^2 < 1\), shows that part of the wavefunction has disappeared behind the horizon. This is of course no surprise, since classically an in-going particle will never reach \(I_+^R\). The surprise is in fact that in quantum mechanics a part of the final wavefunction does end up in the right out-region. The form of this asymptotic out-state follows in an essentially unique way from the shockwave interaction described above.

It is evident that (2.10) cannot be used as an \(S\)-matrix in a second quantized model, as it only knows about the total Kruskal momentum \(P_{\pm}\). Instead, if one would naively generalize the above semi-classical reasoning to the second quantized theory, one will almost inevitably end up with the standard conclusion that the out-going state becomes a mixed state that only depends on the total incoming mass \(M\) and momentum \(P_{\pm}\) (see section 4.2). The fact that for first quantized matter a part of the wave-function disappears behind the horizon, would translate in second quantized language to loss of coherence. However, in this reasoning one assumes that the only interaction between the in- and out-fields is described by (2.9). This is in fact not true: the fields also communicate with each other via the reflecting boundary condition, and since this interaction takes place in the strong coupling region, it cannot be ignored or treated semi-classically.

In the following we will define a simple model for this boundary interaction, and taking its quantum effects into account, we will construct a generalization of the above one-particle \(S\)-matrix that does keep track of the full structure of the quantum states. This \(S\)-matrix will contain (2.10) as an overall factor acting in the zero-mode space of the dilaton gravity fields. As a consequence it will have the same property that it is unitary only in an enlarged Hilbert space that includes a sector that is unobservable from outside the black hole. However, since this concerns only the zero-mode sector, the asymptotic state in the right out-region will be pure and will contain all initial information.

3. Quantization of \(N = 24\) Dilaton Gravity

We will now describe the free field formulation of dilaton gravity coupled to \(N = 24\) massless scalar fields given in \[3\], see also \[12\]. This formulation will reveal a close correspondence with critical open string theory, which we will further exploit in section 5.

\*It should be mentioned that the above dimensional reduction is of course a slightly misleading caricature of ’t Hooft’s \(S\)-matrix, as in 3+1-dimensions the Kruskal momenta depend on the angular coordinates.
In the conformal gauge \( ds^2 = e^{2\rho} du dv \), two-dimensional dilaton gravity is described by the action

\[
S = \frac{2}{\pi} \int du dv \left[ e^{-2\phi}(2\partial_u \partial_v \rho - 4\partial_u \phi \partial_v \phi + \lambda^2 e^{2\rho}) + \frac{1}{2} \sum_{i=1}^{N} \partial_u f_i \partial_v f_i \right].
\] (3.1)

In the following we will choose the number of matter fields to be \( N = 24 \), in which case the quantization of the theory is most straightforward. In particular, in this case the classical dilaton gravity theory of the \( \rho \) and \( \phi \) fields does not receive any one-loop corrections due to the conformal anomaly. The gravitational and matter sector each separately define a conformal field theory, with respective energy-momentum tensors

\[
T_{uu}^g = (4\partial_u \partial_v \phi - 2\partial_u^2 \phi) e^{-2\phi},
\] (3.2)

\[
T_{uu}^m = \sum_{i=1}^{N} \frac{1}{2} (\partial_u f_i)^2.
\] (3.3)

These energy momentum tensors each generate a Virasoro algebra of central charge \( c = 2 \) and \( c = N = 24 \), respectively. The coupling between the two sectors is described via the Virasoro constraint

\[
T_{uu}^g + T_{uu}^m = 0
\] (3.4)

which supplements the equations of motion of (3.1) and ensures the general covariance of the combined theory.

The general solution to the classical equation of motion of (3.1) can be parametrized in terms of pure left or right-moving fields as follows

\[
e^{2\rho-2\phi} = \partial_u X^+(u) \partial_v X^-(v)
\] (3.5)

\[
e^{-2\phi} = -\lambda^2 X^+(u) X^-(v) + \Omega^+(u) + \Omega^-(v)
\] (3.6)

\[
f_i = f_i^+(u) + f_i^-(v).
\] (3.7)

The first equation shows that in the classical theory one can choose special light-cone coordinates in which \( \rho = \phi \). The required conformal transformation \((u, v) \rightarrow (X^+, X^-)\) are dynamical variables in the quantum theory. The only mechanism by which the left and right-moving fields interact is via the boundary conditions in the strong coupling regime. These boundary conditions are necessary to prescribe the initial conditions for
the right-moving fields in terms of the incoming fields, but also to effectively implement the restriction on the chiral fields that follows from the condition that $e^{-2\phi} > 0$.

To set up the quantum theory, we can therefore first concentrate on the two chiral sectors separately, and impose the boundary conditions afterwards. In the following we will only write the formulas for the left-moving fields. It is convenient to introduce the field variable $P_+(u)$ via

$$\partial_u \Omega^+ = P_+ \partial_u X^+. \quad (3.8)$$

In this parametrization the gravitational energy-momentum tensor takes the simple form

$$T^g_{uu} = \partial_u P_+ \partial_u X^+. \quad (3.9)$$

In the quantum theory, this operator should generate conformal transformations $u \rightarrow \tilde{u}(u)$ on the fields $X^+$ and $P_+$. Hence it is a reasonable procedure to identify $P_+$ with the canonical conjugate to the chiral coordinate $X^+$, and define the quantum theory by postulating the commutation relation

$$[\partial_u P_+(u_1), X^+(u_2)] = -2\pi i \delta(u_{12}), \quad (3.10)$$

where $u_{12} = u_1 - u_2$. In this way we arrive at a free field formulation of the pure dilaton gravity theory away from the boundary.

Because of the somewhat unconventional asymptotic conditions on the $X$ fields, we can not simply use the standard mode-expansion. We find that the only mode-expansion that is consistent with the required asymptotic behaviour is of the form

$$\partial_u X^+(u) = x^+ e^{\lambda u} + e^{\lambda u} \int d\omega \: x^+(\omega) e^{-i\lambda\omega u}, \quad (3.11)$$

$$\partial_u P_+(u) = p_+ e^{-\lambda u} + e^{-\lambda u} \int d\omega \: p_+(\omega) e^{-i\lambda\omega u}, \quad (3.12)$$

where $x^+$ and $p_+$ are $c$-numbers and the other modes satisfy the algebra

$$[x^+(\omega_1), p_+(\omega_2)] = (\omega_1 + i)\delta(\omega_1 + \omega_2). \quad (3.13)$$

The $c$-number coefficients in front of the first terms in (3.11) can be fixed by requiring that the conformally normal ordered energy-momentum tensor $T^g_{uu}$ has no vacuum expectation value. One finds that this implies the condition

$$x^+ p_+ = -\frac{\lambda^2}{2}. \quad (3.14)$$
A possible criticism of the above quantization procedure is that, by treating the dynamical coordinates $X^\pm$ as free fields, we have given up the restriction that they should be invertible functions of $u$ and $v$, respectively. This could create some potential problems with the correspondence principle. However, we are helped here by the fact that the quantum dynamics of the whole model is constrained by the Virasoro conditions (3.4) and thus invariant under conformal transformations. This invariance can be used, if one wants, to choose a light-cone gauge and fix $X^+$ or $X^-$ to be specific regular functions of the coordinates. One thereby ensures, at least asymptotically, that no degenerations occur. In the strong coupling region, however, large quantum fluctuations can still lead to possible acausal behaviour, but it seems a reasonable assumption that this does not lead to unacceptable physical consequences in the classical region.

3.2. Physical Operators

As in any theory of quantum gravity, only operators that have a coordinate invariant definition correspond to physical observable quantities. In the quantum theory this corresponds to the requirement that the operators must commute with the Virasoro conditions. The most convenient way to obtain these physical operators is by using the dynamical fields $X^\pm$ as a reference coordinate system. This procedure corresponds to choosing the gauge $\rho = \phi$. It is further important to note that the asymptotic light-cone coordinates defined by the physical metric are given by $\tau^\pm = \pm \log(\lambda X^\pm)$. These coordinates are what asymptotic observers identify as their proper time, and we must therefore use them to define physical quantities such as energy, etc.

In the left-moving sector, we can thus associate a local physical operator to each $f_i$-field as follows

$$f_i^{(in)}(x^+) = i \int \frac{d\omega}{\omega} (\lambda x^+)^{-i\omega} \alpha_i(\omega)$$  \hspace{1cm} (3.15)

where $x^+$ is a c-number and $\alpha_i(\omega)$ the operator

$$\alpha_i(\omega) = \frac{1}{2\pi} \int du (\lambda X^+(u))^{i\omega} \partial_u f_i(u).$$  \hspace{1cm} (3.16)

These operators $\alpha_i(\omega)$ manifestly commute with the Virasoro operators, and thus create/annihilate physical modes of the $f$-field of energy $\omega$ (measured in units of $\lambda$). They satisfy the algebra $[\alpha_i(\omega_1), \alpha_j(\omega_2)] = \omega_1 \delta_{ij} \delta(\omega_1 + \omega_2)$, and are the direct analogues of the DDF operators that span the physical Hilbert space in critical string theory [13]. We can
similarly define a complete basis for the right-moving sector via

\[ f_i^{(\text{out})}(x^-) = -i \int \frac{d\omega}{\omega} (\lambda x^-)^{i\omega} \beta_i(\omega) \]  

(3.17)

where \( x^- \) is a \( c \)-number and \( \beta_i(\omega) \) the operator

\[ \beta_i(\omega) = \frac{1}{2\pi} \int dv \left( -\lambda X^-(v) \right)^{-i\omega} \partial_v f_i(v). \]  

(3.18)

The fact that these physical in- and out-modes have such a simple expression in terms of the fundamental field variables is a major advantage of the present formulation of dilaton gravity.

In the following it will be crucial to know for sure that the modes \( \alpha_i \) and \( \beta_i \) indeed define a complete basis of the left- and right-moving sectors of the physical Hilbert space. In particular, this would mean that any other physical quantity in our model can be expressed in terms of these modes. In section 5 we will show that this is indeed the case. As an example, let us discuss here the physical operator associated with the dilaton field \( \phi \) at a given point \((x^+, x^-)\) in the \( \rho = \phi \) coordinate system. Using the field redefinitions (3.6) and (3.8), we find that this operator is expressed in terms of the \( X \) and \( P \) variables as follows

\[ e^{-2\phi(x^+, x^-)} = M - \lambda^2 x^+ x^- - \int_{x^+}^{\infty} dy^+ P_+(y^+) - \int_{-\infty}^{x^-} dy^- P_-(y^-), \]  

(3.19)

where the constant \( M \) is identified with the black hole mass and e.g. \( P_+(x^+) \) is defined via

\[ P_+(x^+) = -\int \frac{d\omega}{1 + i\omega} (\lambda x^+)^{-1-i\omega} \hat{P}_+(\omega), \]

\[ \hat{P}_+(\omega) = \frac{1}{2\pi} \int du (\lambda X^+)^{1+i\omega} \partial_u P_+(u). \]  

(3.20)

In the quantum theory, the composite operator in the integral in (3.20) has to be normal ordered in a conformally invariant fashion. The precise procedure will be described in section 5. The resulting operators \( \hat{P}_+(\omega) \) are then physical operators, analogous to the operators introduced by Brower in his proof of the no-ghost theorem in string theory [14]. From their form (3.20) it can be seen that they generate diffeomorphisms of the physical coordinate \( X^+ \), and it can indeed be verified that they satisfy a Virasoro algebra with
central charge $c = 24$. These two facts are sufficient to show that within the BRST-cohomology (i.e. modulo spurious physical fields) the field $P_+(x^+)$ is identified with the integral

$$P_+(x^+) = \int_{x^+}^{\infty} dy^+ T_{++}(y^+)$$

(3.21)

of the left-moving physical energy-momentum tensor

$$T_{++}(x^+) = \frac{1}{2} \sum_i (\partial_+ f_i(x^+))^2,$$

(3.22)

which can be expressed directly in terms of the $\alpha_i(\omega)$. A similar formula holds for $P_-(x^-)$.

We have thus found that the classical statement that the matter energy momentum flux uniquely determines the form of the dilaton and the metric, as expressed by the eqns. (2.2) - (2.4), has led to the quantum identification (3.21) of operators within the physical Hilbert space. Due to this, a state in the physical Hilbert space is uniquely characterized once we specify its in-going and out-going matter content, so that the $\alpha_i$ and $\beta_j$-modes each form a complete basis for the physical Hilbert space. The $\alpha_i$ and $\beta_j$-operators are independent as long as we do not impose any reflection condition at strong coupling. However, we will now formulate the boundary condition that will lead to an identification between the left- and right-moving Hilbert space sectors and thus to a relation between both types of physical modes.

3.3. The boundary condition

Since we would like to set up the model in such a way that initial data need to be specified only in the right in-region, we must introduce some reflection condition in the strong coupling regime to prescribe the initial conditions for the right-moving fields. However, since the classical model is unstable against black hole formation, it is a non-trivial problem to find a boundary condition that satisfies all reasonable physical requirements. Namely, on the one hand it should lead to a consistent quantum model, while on the other hand it should not lead to major modifications of the classical physics. In combination, these two requirements are very restrictive.

Perhaps the most natural procedure is to pick a line on which the dilaton $\phi$ takes a large but finite value, and define this line to be a reflecting boundary. In the vacuum this

*Note that the coordinate $x^-$ has been shifted with an amount $P_+/\lambda^2$.  

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line of constant $\phi$ is of the form $\lambda^2 x^+ x^- = -\epsilon$, with $\epsilon$ some infinitesimal positive constant. In general, however, the boundary trajectory is more complicated, since it depends in a dynamical way on the incoming energy flux. We will now define a simple model for the boundary dynamics, which will enable us to take this backreaction into account quantum mechanically. The main justification for the specific choice we will make is that it is the simplest one that appears to lead to the correct physics to an outside observer. In particular, as we will see, it preserves the property that a black hole can be formed.

Now let us describe the boundary condition. (The following boundary condition was first proposed and studied for large $N$ dilaton gravity by Russo, Susskind and Thorlacius [8].) We choose the $(u, v)$-coordinate system in such a way that the boundary becomes identified with the line $u = v$, and denote the parameter along this boundary by $s$. As suggested above, we first require that the dilaton field is constant along the boundary. In terms of the $X$ and $P$-fields this condition reads

$$\partial_s e^{-2\phi} = -\left(\lambda^2 X^- - P_+\right)\partial_s X^+ - \left(\lambda^2 X^+ + P_-\right)\partial_s X^- = 0.$$  \hspace{1cm} (3.23)

In addition we will require that the incoming energy momentum flux gets directly reflected off the boundary. We demand this for the gravitational and matter components separately

$$\partial_s X^+ \partial_s P_+ = -\partial_s X^- \partial_s P_-$$
$$\sum_i \frac{1}{2} (\partial_s f_i^+)^2 = \sum_i \frac{1}{2} (\partial_s f_i^-)^2.$$ \hspace{1cm} (3.24)

The simplest (but certainly not unique) way to ensure that all these boundary conditions are satisfied is to make the following identifications

$$\lambda^2 X^- = P_+$$
$$\lambda^2 X^+ = -P_- \quad \text{at } u = v.$$ \hspace{1cm} (3.25)

These equations should be read as quantum identifications between operators in the left- and right-moving sector of the theory. In the remainder of this paper we will adopt these reflection conditions and study their physical consequences.

An unusual property of the above reflection equations (3.23) is that they do not have an obvious classical interpretation. In the first place, since the boundary lies in the strong coupling regime, strong quantum fluctuations will produce a large uncertainty in its precise location. Still, one can try to describe the reflection process semi-classically.
by first solving the boundary trajectory in terms of the incoming energy flux and then determining the out-going fields via direct reflection off it. This boundary trajectory can be specified as a relation between the coordinates $X^\pm$, which from (3.25) and (3.21) is seen to take the form

$$\lambda^2 X^-(x^+) = \int_{x^+}^{\infty} dy^+ T_{++}(y^+) .$$

(3.26)

Combined with the reflection equation

$$f^{(in)}(x^+) = f^{(out)}(X^-(x^+))$$

(3.27)

this procedure leads to an explicit non-linear scattering equation between the in- and out-going fields \[ [3]. \] This scattering equation, however, will be of a rather unusual type. The incoming energy-flux $T_{++}$ is classically always greater or equal to zero, which means that the boundary trajectory (3.26) is always space-like or light-like. Moreover, the right-hand side of (3.26) is always positive, whereas the out-going fields are supposed to come out at negative $x^-$. The identifications (3.25) can therefore not simply be interpreted as a direct classical reflection of some time-like boundary. Instead, they show that the classical boundary trajectory always stays behind the global event horizon, and thus its presence does not modify the properties of the classical theory as seen by an outside observer. In a way, this is what we want, because it shows that we are still dealing with a theory in which infalling matter leads to black hole formation.

To get a better idea of the physical role of the boundary condition, let us for the moment imagine that the energy momentum flux $T_{++}$ in (3.26) contains an infinitesimal negative ‘vacuum’ contribution $\langle T_{++} \rangle = -\epsilon/(x^+)^2$. The line (3.26) will then be time-like in the vacuum, and thus the reflection will in particular identify the left- and right-moving vacuum states. During collisions with the incoming particles, however, the boundary trajectory will still become space-like, and from this point on the classical description of the theory breaks down. Indeed, in the classical dilaton gravity model, the boundary then goes over in a black hole singularity. In our parametrization of the dilaton gravity fields, what happens instead is that the mapping from the $(u, v)$ parameter space to the physical coordinate space $(X^+, X^-)$ becomes non-invertible. If we assume that the $X^+$ is a monotonic function of $u$, then it is easy to show that the $X^-$ coordinate will turn around at the point where normally the singularity is formed and go backwards in time.

\[ ^1 \text{In fact, semi-classical arguments } \[ 8, 3 \] suggest that the quantum energy momentum tensor $T_{++}$ indeed receives a constant negative contribution from vacuum fluctuations. We will comment further on the role of this vacuum energy in section 4.3. \]
for a while. This implies that, in our model, the propagation in the physical coordinate space can become acausal near the singularity.

\[ u = v \]
\[ x^+ = 0 \]
\[ x = 0 \]

**Fig 2.** The complete black hole formation process takes place within the region of the \((u, v)\) plane where the fields \(X^\pm\) provide an invertible parametrization. The acausal strong coupling effects take place in the shaded region, \textit{i.e.} near the black hole singularity.

It is important to note, however, that up to this point our model is classically completely equivalent to the standard dilaton gravity theory. In particular, as indicated in fig. 2, it can be seen that the complete black hole formation process takes place within the region of the \((u, v)\) plane where the fields \(X^\pm\) are non-degenerate, invertible functions. The acausal strong coupling effects take place in the shaded region, which is after the black hole is formed, and do not modify the classical physics as seen by an out-side observer. The present model should therefore not be viewed as a \textit{modification} of the original dilaton gravity theory, but rather as a \textit{completion} of it, in which a certain \textit{Ansatz} is made about what happens near the singularity. This \textit{Ansatz} is motivated and strongly constrained by the observable properties of the model.

Finally, it is instructive to make a comparison between the above model and the two-dimensional reduction of 't Hooft's black hole \(S\)-matrix discussed in section 2. Indeed, the operator identifications (3.27) are the simplest generally covariant generalization of the relation (2.11) between the out-going coordinate \(x^-\) and in-going momentum \(P_+\) of the matter particle considered there. In that case we saw that, although (2.11) classically states that the particle ends up behind the event horizon, quantum mechanically it uniquely predicts the form of the out-going wavefunction on \(I_R^+\). In the following sections we will show that the identification (3.23) leads in the same way to a unique out-going wavefunction in the full dilaton gravity theory.
4. Physical Properties of the $S$-matrix

The reflection equations (3.25) prescribe the initial conditions for the right-moving modes by identifying the right- and left-moving sectors of the dilaton gravity Hilbert space. Thus we can now describe the full Hilbert space in terms of, say, only the left-moving matter fields $f_i(u)$ and the coordinate fields $X^\pm(u)$, which become each other’s canonical conjugate

$$[\partial_u X^\pm(u_1), X^\mp(u_2)] = -\frac{2\pi i}{\lambda^2} \delta(u_{12}).$$

(4.1)

This allows us to define the $S$-matrix elements between physical in- and out-states as an expectation value of the corresponding product of in- and out-creation- and annihilation operators

$$\langle out|in \rangle = \langle 0| \prod_m \beta_{jm}(\omega_m) \prod_n \alpha_{in}(\omega_n) |0 \rangle$$

(4.2)

where $\alpha_i$ and $\beta_j$ are defined in (3.16)-(3.18). Because the $\alpha_i$ and $\beta_j$ modes each form a complete basis of the physical Hilbert space, the collection of $S$-matrix elements (4.2) define a unique and invertible mapping from a given in-state to an out-state. A more precise algebraic description of this mapping will be given in section 5. The purpose of this section is to investigate the physical properties of the above $S$-matrix. We will show that it indeed largely behaves as one would expect from semi-classical considerations. This correspondence provides an important justification for our simple choice of boundary condition.

4.1. The Exchange Algebra.

The observation that will enable us to make contact with semi-classical results obtained in previous studies of dilaton gravity \[3\] is that the interaction between the in- and out-modes can be naturally separated into two parts, namely the shockwave interaction discussed in section 2 and the reflection off the dynamical boundary. The key property that distinguishes the shockwave interaction from the other interactions is that it already affects the form of the out-state before the incoming particles have reflected. Furthermore, as we will see, it is directly responsible for the presence of thermal radiation in the asymptotic out-state. The other interactions, describing the reflection, will of course lead to important corrections to the Hawking spectrum and are responsible for restoring the coherence of the out-state.
A natural but naive attempt to compute the expectation value \((4.2)\) is to try to commute the operators \(\beta_j\) through the operators \(\alpha_i\), until they annihilate the opposite vacuum. This approach suggests that it may be useful to summarize the interaction between the in- and out-modes \(\alpha_i(\omega)\) and \(\beta_j(\omega)\) in terms of an exchange algebra. Now it is well-known from conformal field theory that the two vertex operators \(e^{-\frac{i}{2\pi}p_+X^+}\) and \(e^{-\frac{i}{2\pi}p_-X^-}\) indeed have simple exchange-properties: if we interchange their order one simply picks up a phase that only depends on the sign of their relative position.\(^*\)

\[
e^{-\frac{i}{2\pi}p_-X^-(u_1)} e^{-\frac{i}{2\pi}p_+X^+(u_2)} = e^{-\frac{1}{2\pi\lambda^2}p_+ p_- \theta(-u_{12})} e^{-\frac{i}{2\pi}p_+X^+(u_2)} e^{-\frac{i}{2\pi}p_-X^-(u_1)}.
\](4.3)

We would now like to use the above formula \((4.3)\) to determine an exchange algebra of the physical in and out fields.

To this end, let us first introduce the following physical operators

\[
\begin{align*}
\hat{A}_i(p_+) &= \frac{1}{2\pi} \int du e^{-\frac{i}{2\pi}p_+X^+(u)} \partial_u f_i(u), \\
\hat{B}_j(p_-) &= \frac{1}{2\pi} \int du e^{-\frac{i}{2\pi}p_-X^-(u)} \partial_u f_j(u),
\end{align*}
\](4.4)

which create in-coming and out-going particles with a definite Kruskal-momentum \(p_+\) and \(p_-\) respectively. These operators can be expressed as linear combinations of the energy eigenmodes \(\alpha_i(\omega)\) resp. \(\beta_j(\omega)\). For example, for \(p_+ > 0\) we have

\[
\hat{A}_i(p_+) = \int d\omega e^{-\frac{\omega}{2\pi\lambda}} \Gamma(-i\omega) \left(\frac{p_+}{2\pi\lambda}\right)^{i\omega} \alpha_i(\omega).
\](4.5)

This equation, which can also be read as the definition of \(\hat{A}_i(p_+)\), shows that these Kruskal modes are in fact somewhat singular operators, because they contain \(\alpha_i(\omega)\) modes of arbitrarily high frequency. In the following we will mostly ignore this singularity, as it will not affect the main conclusions.

The Kruskal modes \((4.4)\) are very convenient for our purposes, since from \((4.3)\) we find that they satisfy an exchange algebra of the following form

\[
\hat{B}_j(p_-) \hat{A}_i(p_+) = e^{-\frac{i}{2\pi\lambda}p_- p_+} \hat{A}_i(p_+) \hat{B}_j(p_-) + R_{ij}(p_+, p_-).
\](4.6)

Here in the first term on the right-hand side we recognize the two-particle S-matrix

\(^*\)Here we used that \([X^-(u_1), X^+(u_2)] = 2\pi i \lambda^{-2} \theta(-u_{12})\).
described in section 2, representing the shockwave interaction between the in- and out-mode. The remaining term takes the form

\[
R_{ij}(p_+, p_-) = \int du_1 du_2 e^{-\frac{i}{2\pi p_-} X^-(u_1)} e^{\frac{i}{2\pi p_+} X^+(u_2)} \left[ 2\pi i \delta_{ij} \delta'(u_{12}) \right. \\
+ \left. (1 - e^{-\frac{i}{2\pi x} p_+ p_-}) f'_i(u_2) f'_j(u_1) \theta(u_{12}) \right],
\]

(4.7)

and can be seen to describe the effects due to the presence of the dynamical boundary. Namely, the integrand in (4.7) has support only for \( u_1 \geq u_2 \) and this operator \( R_{ij}(p_+, p_-) \) therefore only contains interactions that take place at the moment of reflection or afterwards. The first term in the integrand represents the direct reflection off the boundary, while the second term corrects for the fact that the shockwave interaction between the in and out-modes takes place only when the in- and out-mode cross each other before the reflection has taken place.

The physical interpretation of (4.6) becomes a little more clear when we go to a coordinate representation for the outgoing modes. In terms of the coordinate field \( f^{\text{out}}(x^-) \), defined in (3.17), the exchange algebra takes the form

\[
f_j^{\text{out}}(x^-) \hat{A}_i(p_+) = \hat{A}_i(p_+) f_j^{\text{out}}(x^- - \frac{p_+}{\lambda^2}) + \tilde{R}_{ij}(p_+, x^-),
\]

(4.8)

and we explicitly see that the in-mode shifts the out-fields by an amount proportional to the incoming Kruskal-momentum. The reflection term \( \tilde{R}_{ij}(p_+, x^-) \) is the \( x^- \) Fourier transform of (4.7). By construction, this term only affects the form of the out-going wave function after the incoming wave has reflected.

### 4.2. Hawking radiation.

We would like to use the above results to obtain some physical insight into what the out-state will look like for a given instate. Specifically, we wish to consider an incoming state in which the matter is localized in a finite time interval \( x_0^+ < x^+ < x_0^+ + \Delta x^+ \) and carries a large total energy \( E \pm \Delta E \). This in-state may be represented as a sum of eigenstates of the total Kruskal-momentum with eigenvalues are concentrated around \( P_+ = E/x_0^+ \). Before these incoming high energy modes will reflect off the boundary, they will first interact with the outgoing fields via the gravitational shockwave. In this subsection we will study the physical effect of this shockwave.
Since the incoming wave is localized, it is reasonable to assume that the quantum reflection will take place within some finite outgoing time interval concentrated around some reflection time \( x_0 \). In the following we will be interested in the structure of the out-state before this reflection time, which means that we are allowed to ignore the presence of the reflection term \( R_{ij}(p_+, x^-) \). (A more detailed justification for this procedure will be given in subsection 4.3.) Thus, for the purpose of the following discussion, we are left with the pure exchange algebra

\[
 f_j^{(\text{out})}(x^-) \hat{A}_i(p_+) = \hat{A}_i(p_+) f_j^{(\text{out})}(x^- - \frac{p_+}{\lambda^2}), \quad x^- < x_0. \tag{4.9}
\]

When the incoming energy \( p_+ \) is large enough, this shift-interaction can produce large physical effects in the out-region. Indeed, the shockwave due to an incoming particle can classically lead to the formation of an event horizon, and, as is well known, the resulting distortion of the out-modes produces Hawking radiation. In the following we will rederive this result in our model, using equation (4.9) as a starting point. The derivation requires only a small adaptation of the standard reasoning. Some calculations done in [16] are also useful.

If the coordinate \( x^- \) were a normal Minkowski coordinate, ranging from \( -\infty \) to \( +\infty \), a constant shift in \( x^- \) would have had no physical effect whatsoever: it could simply be absorbed by shifting the Minkowski vacuum. In our case, however, it is crucial that \( x^\pm \) parametrizes only a Rindler wedge \( \pm x^\pm > 0 \) and that the vacuum of the \( f \)-fields is defined accordingly in terms of the Rindler type modes \( \alpha(\omega) \) and \( \beta(\omega) \). To see how the outgoing vacuum state is affected by the coordinate shift (4.8), let us rewrite the exchange algebra in terms of the \( \beta \)-modes. One finds

\[
 \beta_j(\omega) \hat{A}_i(p_+) = \hat{A}_i(p_+) \int d\xi B_{\omega \xi}(p_+) \beta_j(-\xi) \tag{4.10}
\]

with

\[
 B_{\omega \xi}(p_+) = \frac{1}{2\pi} \left( \frac{p_+}{\lambda} \right)^{-i(\xi+\omega)} \frac{\Gamma(1-i\omega)\Gamma(i(\xi+\omega))}{\Gamma(1+i\xi)}. \tag{4.11}
\]

The linear combination of \( \beta \)-modes on the right-hand-side contains both creation- and annihilation- operators. Consistency of the algebra (4.10) further requires that these combinations again satisfy canonical commutation relations, so we see that exchanging a \( \beta_j(\omega) \)-oscillator with \( \hat{A}_i(p_+) \) leads to a Bogoliubov-transformation. Note that the transformed modes occurring on the right-hand side of (4.10) do not form a complete basis
of all β-modes, since they cover only the interval \( x^- < -\lambda^{-2}p_+ \). The exchange property (4.10) can therefore in general only be used in one direction.

Now let us consider an in-state with definite total Kruskal momentum \( P_+ \), and let us further assume it can be written in the form

\[
|\psi\rangle = \prod_a \hat{A}_{ia}(p_a)|0\rangle,
\]

(4.12)

with \( \sum_a p_a = P_+ \). To determine the properties of the corresponding out-state we can act on \( |\psi\rangle \) with the β\(_j\)-modes and repeatedly use (4.10) until we can act on the vacuum. These manipulations are of course the direct quantum counterpart of Hawking’s original semi-classical calculation. The repeated use of the exchange-algebra describes the propagation of the outgoing particles through the infalling matter, while taking into account the gravitational interaction between the two. Now, it is clear from (4.9) that in this procedure only the total momentum \( P_+ \) plays a role, so the exchange relation between the β\(_j\)-modes and the product \( \prod_a \hat{A}_{ia}(p_a) \) is again of the same form (4.10). In this way we find that, just as in the semi-classical calculation, the asymptotic out-state is no longer equal to the vacuum-state, but given by the Bogoliubov transform (4.10)-(4.11) of the vacuum.

Although at this point we could simply refer to the standard analysis [1, 16], let us continue to show that the spectrum of out-going radiation indeed looks approximately thermal. To this end, let us compute the expectation value of the particle number operator

\[
\langle N_j(\omega) \rangle = \frac{1}{\omega} \langle \psi| \beta_j^*(\omega) \beta_j(\omega)|\psi\rangle.
\]

(4.13)

Inserting the definition of \( |\psi\rangle \) we find that the right-hand-side can be written as

\[
\langle N_j(\omega) \rangle = \frac{1}{\omega} \langle 0| \prod_b \hat{A}^*(p_b) \beta_j^*(\omega) \beta_j(\omega) \prod_a \hat{A}(p_a)|0\rangle
\]

\[
= \int_0^\infty d\xi \frac{\xi}{\omega} |B_{\omega\xi}(P_+)|^2,
\]

(4.14)

where we made use of the exchange algebra (4.10), and the fact that the vacuum-state is annihilated by \( \beta_j(\omega) \) for \( \omega > 0 \). We further assumed that the state \( |\psi\rangle \) is normalized. Applying some standard formulas about Γ-functions then gives

\[
\langle N_j(\omega) \rangle = C \cdot \int_0^\infty d\xi \frac{\xi}{\omega + \xi} \frac{\sinh \pi \xi}{\sinh \pi(\omega + \xi) \sinh \pi \omega}.
\]

(4.15)
It can be seen that the dominant contribution in the integral comes from large $\xi$, which allows us to approximate the integrand. In this way we find that the expectation value of the particle number operator is indeed given by a thermal distribution

$$\langle N_j(\omega) \rangle = C' \frac{e^{-2\pi\omega}}{1 - e^{-2\pi\omega}}. \quad (4.16)$$

The above reasoning can be extended in a straightforward way to analyze arbitrary expectation values of products of $\beta_j$-modes.

The integral in (4.15) as it stands would actually lead to an infinite constant $C'$ in (4.16), which would imply that the out-state contains thermal radiation for arbitrary late times. However, it is important to note that above we have of course dealt with an idealized situation. In the first place we assumed that $|\psi\rangle$ is an exact eigen state of $P_+$, and as noted before, such states contain arbitrarily high energy modes and infinite total energy. If the in-coming energy were bounded, the resulting black hole would of course radiate for only a finite amount of time. Moreover, after a certain time $x_0^{-}$ the reflection term $R_{ij}$ in the algebra (4.8) will become important and this will produce important corrections to the Hawking spectrum. We will now discuss the resulting modifications of the above semi-classical picture.

### 4.3. Quantum Reflection.

As discussed in section 3.3, the classical boundary trajectory always stays behind the global event horizon $x^- = 0$, and thus remains invisible to an outside observer. In the quantum theory, however, fluctuations of the boundary will be able to produce observable effects outside the horizon. These effects are represented by reflection term $R_{ij}$ in the exchange algebra (4.8) between the in- and out-fields. In the following it will be useful to write this algebra in the coordinate representation

$$f_j^{(\text{out})}(x^-) f_i^{(\text{in})}(x^+) = e^{\frac{2\pi i}{\lambda^2 \theta_+ \theta_-}} f_i^{(\text{in})}(x^+) f_j^{(\text{out})}(x^-) + \tilde{R}_{ij}(x^+, x^-) \quad (4.17)$$

where the reflection term $\tilde{R}_{ij}(x^+, x^-)$ is the Fourier transform of (4.7). We would now like to determine the regime in which the presence of this term will become important. The idea is to use the fact that the integrand in (4.7) vanishes for $u_2 \geq u_1$ to show that the reflection term is negligible to the right of some critical line in the physical $x^\pm$-plane.
For simplicity, we will restrict our discussion to only the first term in (4.7). In coordinate space it reads

$$\Delta_{ij}(x^+, x^-) = 2\pi i \delta_{ij} \int du \delta(x^+ - X^+(u)) \partial_u \delta(x^- - X^-(u))$$  \hspace{1cm} (4.18)$$

where we performed one of the $u$-integrations compared to (4.7). We will ignore the issue of normal ordering, as our discussion will be semi-classical.

The above operator (4.18) describes the direct reflection off the dynamical boundary, and it will therefore be mainly responsible for the information transfer from the in-going to the out-going modes. Using the first $\delta$-function to perform the remaining integral over $u$, we can rewrite the right-hand side of (4.18) as

$$\Delta_{ij}(x^+, x^-) = 2\pi i \delta_{ij} \partial_+ \delta(x^- - X^-(x^+))$$  \hspace{1cm} (4.19)$$

Here $X^-(x^+)$ is the physical operator representing the boundary trajectory in the $x^\pm$-plane, which by the equations of motion is given in terms of the incoming energy flux $T_{++}$ via equation (3.26). From (4.18) we see explicitly that the operator $\Delta_{ij}(x^+, x^+)$ indeed represents the direct reflection, and furthermore, that it contributes only when its argument $(x^+, x^-)$ defines a point on the boundary line. In a similar way it can be shown that the other term in $\hat{R}_{ij}(x^+, x^-)$ has its support behind this same line.

So, from this we are led to conclude that physical effect of the complete reflection term is negligible as long as we remain to the right of the semi-classical boundary trajectory

$$\hat{R}_{ij}(x^+, x^-) \simeq 0 \quad x^- < X^-(x^+)$$  \hspace{1cm} (4.20)$$

with $X^-(x^+)$ given in (3.26). What we mean here by the semi-classical boundary trajectory is the outermost line that the boundary can reach when we include the effect of quantum fluctuations. A semi-classical estimate of the magnitude of these quantum effects can be given by considering the contribution of vacuum fluctuations to the energy-momentum tensor $T_{++}$. Namely, it is well-known that the definition of the quantum tensor $T_{++}$ depends on the normal ordering prescription. A natural choice is to normal order with respect to the local Kruskal coordinates $x^\pm$, and in this case the physical vacuum state will in fact contain a negative vacuum energy, given by $\langle T_{++} \rangle = -\frac{N}{24(x^+)^2}$. Although we do not want to assign any real physical meaning to this negative vacuum energy — as it depends on the choice of normal ordering — it does give an indication of how large an effect quantum fluctuations can have on the position of the boundary.
Fig 3. Again the collapsing black hole geometry, where we have now indicated the critical line behind which the quantum effects of the boundary can become visible. These quantum effects give corrections to the Hawking spectrum after the critical time $x^-$ and are responsible for the information transfer.

Thus a reasonable estimate of the critical line behind which virtual boundary effects can be expected to take place is obtained by including this negative vacuum contribution to $T_{++}$ in the classical equation of motion (3.26) for the boundary. The trajectory of this line in the $x^\pm$-plane is indicated in fig. 3. Before the shockwave has passed, it coincides with the line at which in the semi-classical theory the dilaton takes its critical value $\phi = \phi_{cr}$ [2, 8], and afterwards goes over in the apparent horizon of the black hole formed by the incoming matter wave [8]. We wish to emphasize, however, that this critical line should not be confused with the location of the reflecting boundary, as the region behind it is still present in our model (see fig 2.). It only borders the region of space-time from which the quantum gravitational corrections appear to originate to an asymptotic observer.

The corrections to the Hawking spectrum will thus become visible after the time when the out-state starts to depend on the physics that takes place in the small region between the critical line and the event horizon $x^- = 0$. From the above discussion we see that the reflection time $x^0_-$ on $I^{-\frac{1}{2}}$ at which this first happens is related to the incoming time $x^+_0$ via $x^-_0 \simeq -1/(\lambda^2 x^+_0)$. This time must be compared with the initial time at which the Hawking radiation starts to come out, which is around $x^- \simeq -P_+/\lambda^2$. Thus, provided the incoming energy $E = P_+x^+_0$ is much larger than 1 (measured in units of $\lambda$), there is a semi-classical regime in our model where the black hole will emit thermal radiation.

\[1^{\text{Note that the distance between the critical line and the event horizon } x^- = 0 \text{ is } 1/E \text{ in ‘Planck units’.}}\]
The energy carried out during this time interval proportional to \( \log(E) \), and so the corrections to the Hawking spectrum will already start to occur when the black hole has lost only a small fraction of its total mass. This results clearly contradicts the usual supposition that strong coupling physics starts playing a role only after the black hole has reached a mass of the order of the Planck mass. As we will now argue, this supposition may indeed be false in general. Namely, it can be seen that, due to the exponentially growing redshift, any question about the form of the asymptotic state on \( I^+_R \) at late times translates back into a question about the form of the state near the horizon at extremely short distances. It is indeed true that, and Hawking calculation is based on this fact, the event horizon looks like a perfectly regular part of space-time to a local inertial observer, so that the state there looks approximately like the vacuum. This changes, however, when this inertial observer wants to make observations concerning the structure of this state at very short distances. In that case he will clearly notice the presence of the infalling matter, and this short distance structure will thus depend on strong coupling physics. So the fact that we find corrections already at early out-going times does not contradict any (well-established) semi-classical result. Instead it shows explicitly that to determine what happens after a critical time on \( I^+_R \) one must go beyond the semi-classical approximation, as from that point on Planck scale physics will play a role. Our dynamical boundary provides a simple model for these strong coupling effects.

5. Algebraic Properties of the \( S \)-Matrix

It is useful to make a comparison between the present formulation of dilaton gravity and critical string theory. There is evidently a large similarity between the two if we identify the matter and \((x^+, x^-)\)-fields with the transverse and light-cone string coordinates, respectively. In fact, this correspondence can be made exact when we compare our model with critical open string theory in a constant electric field. In that case the two light-cone string fields have an expansion similar to (3.11), except that the frequency sum is in general discrete \([17]\). However, it can be seen that in the limit in which the electrical field strength goes to the critical value \([17]\) the open string spectrum becomes effectively continuous. Intuitively, this happens because at this value the electric field stretches the string to cover a full quadrant of the Minkowski plane. This correspondence with open strings in an electric field can thus be used to introduce an infrared cut-off in our model, namely by taking the electric field just below the critical value. After this,
standard techniques of string theory become available in studying the properties of the Hilbert space.

In this section we will use this finite volume regularization to make a number of precise statements about the dilaton gravity scattering matrix defined in section 3.3. In particular, we prove that the in- and out-modes $\alpha_i(\omega)$ and $\beta_j(\omega)$ each provide a complete basis for the physical Hilbert space and establish the existence of a unitary transformation between these two bases. The physical $S$-matrix is obtained from this transformation by performing a suitable projection within the zero-mode sector. We also show how the computation of the $S$-matrix elements can be streamlined by using the Virasoro decomposition of the space of physical states. As an example, we explicitly compute the amplitude for a process that involves particle production in the strong coupling region.

5.1. Dilaton Gravity in a Finite Volume.

Following the above discussion, we now put the fields $f_i(u)$ and $X^\pm(u)$ in a finite volume by identifying $u$ modulo $2\pi L$. We use the following mode expansions

$$\partial_u f_i(u) = \sum_{m=-\infty}^{\infty} f_i^m e^{-i\frac{m}{L}u},$$

$$\partial_u X^\pm(u) = \pm \sum_{m=-\infty}^{\infty} x^\pm_m e^{-i\frac{(m\pm\mu)}{L}u},\quad (5.1)$$

where we introduced the dimensionless parameter $\mu = \lambda L$. The canonical commutation relations of the fields (5.1) lead to

$$[f_m^i, f_n^j] = m \delta_{m+n} \delta^{ij}, \quad [x^+_m, x^-_n] = (m + i\mu) \delta_{m+n}.$$  \quad (5.2)

The Virasoro generators $L_m$ for $m \neq 0$ take the usual form; for $m = 0$ there is a non-standard shift, which we can fix by defining $L_0 = \frac{1}{2}[L_1, L_{-1}]$. The result is

$$L_m = \frac{1}{2} \sum_{i,n} f^i_m f^i_{-m-n} + \sum_{n} x^+_n x^-_{m-n} + \frac{1}{4} \mu^2 \delta_m,0,$$ \quad (5.3)\n
where the normal ordering signs imply that we take the symmetric product of the zero-modes in $L_0$. In string theory the parameter $\mu$ measures the strength of the electric field,

*For recent work closely related to this section, see [12].
and the limit of critical field strength corresponds to the limit $\mu \to \infty$. If at the same time one takes $n \to \infty$ with $\omega = n/\mu$ finite one recovers the infinite volume theory.

In order to describe the Hilbert space we shall pick in and out vacuum states that are annihilated by all positive frequency modes $f^+_m, x^\pm_m, m > 0$. They are then also annihilated by the all $L_n$ with $n > 0$. We will assume that there is no momentum in the matter sector, so that the modes $f^+_0$ annihilate these vacua. The remaining zero modes $x^+_0$ and $x^-_0$ are canonically conjugate and need to be treated with some care. To avoid a cluttering of indices lateron, we will write

$$ x^+_0 \equiv x ; \quad x^-_0 \equiv y. \quad (5.4) $$

The algebra is thus

$$ [x, y] = i\mu. \quad (5.5) $$

We will look for in and out vacuum states that are eigenstates of the operator $(xy + yx)$. The corresponding eigenvalue can be obtained from the condition that the vacua are annihilated by the total $L_0$-operator, which is a sum of a matter-, ghost- and gravitational part. Just as in string theory, the ghost- and matter vacuum have $L_0$-eigenvalue $-1$ and $0$, hence for the gravitational $L_0$-operator we must have

$$ L_0 |0\rangle = |0\rangle. \quad (5.6) $$

Working out the oscillator algebra gives the following condition for the zero modes

$$ (xy + yx)|0\rangle = (-\mu^2 + 2)|0\rangle. \quad (5.7) $$

In the coordinate representation for $x$, where we write $y = -i\mu d/dx$, we can represent the vacuum by the following wave function

$$ \psi_0(x) = x^{-i\delta - \frac{1}{2}}, \quad \text{with} \quad \delta = \frac{1}{2}\mu - \frac{1}{\mu}. \quad (5.8) $$

This wave function is well-defined for $x > 0$. For the out-going states, where we use the coordinate $y$, we can similarly use the wavefunction $\psi_0(y) = (-y)^{i\delta - \frac{1}{2}}$, which is well-defined for $y < 0$.

Although at the classical level the linear dilaton vacuum is completely covered by the coordinate ranges $X^+ > 0, X^- < 0$, it will turn out to be inconsistent to restrict the range
of the zero modes $x$ and $y$ the corresponding values $x > 0$ and $y < 0$. This means that we need to worry about the fact that, due to the branch cut at $x = 0$, the wavefunction $\psi_0(x)$ is not uniquely defined away from the positive real axis. We should either give a prescription for how to continue to negative values of $x$, or more generally, introduce two independent in-vacuum states $|0, in\rangle_\pm$, and similarly two out-vacuum states $|0, out\rangle_\pm$, via

\[
\langle x | 0, in\rangle_\pm = \frac{1}{\sqrt{2\pi}} |x|^{-i\delta - \frac{1}{2}} \theta(\pm x) ,
\]

\[
\langle y | 0, out\rangle_\pm = \frac{1}{\sqrt{2\pi}} |y|^{i\delta - \frac{1}{2}} \theta(\pm y) .
\]

These in and out vacuum states are linearly dependent, and the change of basis from the one to the other will lead to factors in the $S$-matrix, which we will describe later. We will also postpone the discussion of the physical significance of this vacuum-doubling.

5.2. Completeness of the Lightcone Bases

The following states form a basis for the Fock space in the in notation

\[
|\{\lambda\}, in\rangle_\pm = \prod_{i,m>0} (f^{-}_m)^{\lambda^i_m} \prod_{m>0} (x^{-}_m)^{\lambda^+_m} \prod_{m>0} (x^{+}_m)^{\lambda^-_m} x^\beta - i\omega |0, in\rangle_\pm .
\]

(5.10)

Similarly, the states $|\{\lambda\}, out\rangle_\pm$ form a basis in the out notation.

As in string theory, we define physical states to be those states that are annihilated by the positive frequency Virasoro generators $L_m, m > 0$, and that are eigenstates of $L_0$ with eigenvalue 1. The latter condition fixes the value of the zero-mode exponents $\beta, \omega$ in (5.10) according to

\[
\beta(\lambda^i_m, \lambda^+_m, \lambda^-_m) = - \sum_m \lambda^+_m + \sum_m \lambda^-_m ,
\]

\[
\omega(\lambda^i_m, \lambda^+_m, \lambda^-_m) = \frac{1}{\mu} \sum_m m(\lambda^i_m + \lambda^+_m + \lambda^-_m) .
\]

(5.11)

In this section we will only be concerned with states that satisfy these conditions or the corresponding conditions for the states in the out basis.

We discussed before that the physical in-states for the 24 scalar fields are created by the dressed oscillators $\alpha^i(\omega)$. In a finite volume we write these as $A^i_n$, defined as

\[
A^i_n = \frac{1}{2\pi^2} \int_0^{2\pi L} du \partial_u f_i(u) (\lambda X^+)^{in/\mu} .
\]

(5.12)
Note that the integrand in here is periodic of period $2\pi L$. The physical out-states are created by the oscillators

$$B^i_n = \frac{1}{2\pi L} \int_0^{2\pi L} du \partial_u f_i(u) (\lambda X^-)^{-in/\mu}$$

(5.13)

In this section we will show that every in-state, created by repeated application of the $A$-oscillators, can be decomposed as a linear combination of out-states written in terms the $B$-oscillators. This decomposition is modulo spurious physical states, i.e. physical states that are orthogonal to every physical state. This result implies that the $S$-matrix elements defined as $\langle out | in \rangle$ define a unitary $S$-matrix, provided we include the Hilbert space sectors corresponding to both vacua.

The line of reasoning that we will follow for establishing the completeness of the $B$-basis of physical modes is closely analogous to Brower’s proof of the no-ghost theorem for critical strings in $D = 26$ dimensions \[\text{(14)}\]. The plan of action is as follows. Working in the out-language, where mode expansions are defined using powers of the field $X^- (u)$, we will first define operators $\tilde{X}^+_n$ and $\Phi^-_n$ which together with the dressed oscillators $B^i_n$ span the complete Hilbert space of states with $L_0 = 1$. This means that a general state can be written as a linear combination of states $|\{\mu\}, out\rangle_{\pm}$ of the form

$$|\{\mu\}, out\rangle_{\pm} = \prod_{i,n>0} (B^i_{-n})^{\mu_i} \prod_{n>0} (\tilde{X}^+_n)^{\mu^+_n} \prod_{n>0} (\Phi^-_n)^{\mu^-_n} |0, out\rangle_{\pm}.$$ 

(5.14)

We then go on to show that a state of the form (5.14) is physical if and only if $\mu^-_n = 0$ for all $n > 0$, i.e. if there are no factors $\Phi^-_n$. In addition we find that such a physical state is spurious as soon as it contains one or more of the operators $\tilde{X}^+_n$. This will establish that every physical state can be written as a state created by using the oscillators $B^i_n$ only, denoted by $|B, out\rangle_{\pm}$, plus a state that is spurious and physical,

$$|\text{physical}\rangle = |B, out\rangle_+ + |B, out\rangle_- + |\text{spurious physical}\rangle.$$ 

(5.15)

Putting a physical in-state, created by the $A^i_n$, on the left hand side of this relation will then establish the desired result.

Before we come to the operators $\tilde{X}^+_n$ we define operators $\check{X}^+_n$ by

$$\check{X}^+_n = -\frac{\mu^2}{2\pi L} \int_0^{2\pi L} du (\lambda X^-)^{-in/\mu} + \frac{\mu^2}{2\pi L} (1 - \frac{in}{\mu}) \int_0^{2\pi L} du (\lambda X^-)^{-\frac{in}{\mu}}$$

$$-\frac{\mu^2}{2\pi} (1 - \frac{in}{\mu}) \int_0^{2\pi L} du \partial_u \log(\lambda X^-) \frac{\mu}{\mu - in}.$$ 

(5.16)
This expression has been chosen such that it commutes with the Virasoro generators $L_m$,

\[ [L_m, \hat{X}_n^+] = 0 . \tag{5.17} \]

This guarantees that the action of \( \hat{X}_n^+ \) on a physical state gives another physical state. A remarkable property is that the modes \( \hat{X}_n^+ \) define a Virasoro algebra among themselves

\[ [\hat{X}_m^+, \hat{X}_n^+] = (m - n)\hat{X}_{m+n}^+ + 2m(m^2 + \mu^2)\delta_{m+n} . \tag{5.18} \]

We now define the operators \( \check{X}_n^+ \) and \( \Phi_n^- \) as

\[
\check{X}_n^+ = \hat{X}_n^+ + (\mu^2 + 1)\delta_n - \frac{1}{2} \sum_{i,m} :B^i_{n-m}B^i_m:\nn = \Phi_n^- = \frac{1}{2\pi L} \int_0^{2\pi L} du (\lambda X^-)^{-\frac{1}{2}} . \tag{5.19} \]

These operators commute with the oscillators \( B_m^i \) and satisfy the following commutation relations

\[ [\check{X}_m^+, \check{X}_n^+] = (m - n)\check{X}_{m+n}^+, \]

\[ [\check{X}_m^+, \Phi_n^-] = -n\Phi_{m+n}^- , \quad [\Phi_m^-, \Phi_n^-] = 0 . \tag{5.20} \]

Note that the vanishing of the central term in the Virasoro algebra satisfied by the \( \check{X}_n^+ \) is a consequence of our choice of taking \( N = 24 \) scalar fields for the matter system.

We will now argue that all states of the form \( |\{\mu^i_n\}, out\rangle_{\pm} \) are linearly independent. We start by picking a set of oscillators \( B_m^i \) and applying them to the vacuum to create the state \( |\{\mu^i_n\}, out\rangle_{\pm} \). (In the corresponding analysis of light cone gauge string theory such states are usually called DDF states.) As a first result, we claim all the states that we can create from \( |\{\mu^i_n\}, out\rangle_{\pm} \) by acting with the operators \( \check{X}_n^+ \) and \( \Phi_n^- \), \( n \leq 0 \), are linearly independent. At a given level \( N \) in the module, where \( N = \sum n(\mu^+_n + \mu^-_n) \), this follows from the fact that the determinant of the matrix of inner products of all states at this level is non-vanishing. This can be established by using only the commutation relations \( (5.20) \), the fact that the positive modes of \( \check{X}_n^+ \) and \( \Phi_n^- \) annihilate the state \( |\{\mu^i_n\}, out\rangle_+ \) and the fact that \( \Phi_0^- \) is non-vanishing on \( |\{\mu^i_n\}, out\rangle_{\pm} \). An elegant derivation of this result was presented by Thorn in \cite{18} (see also \cite{19}). A second result, which again follows simply from the algebraic properties of the generators \( B_m^i \), \( \check{X}_n^+ \) and \( \Phi_n^- \), is that all DDF states are mutually orthogonal and that states created by acting with \( \check{X}_n^+ \) and \( \Phi_n^- \) on different
DDF states are also mutually orthogonal. Combining these observations shows that all the states \( (5.14) \) are linearly independent. An easy counting argument then shows that all \textit{out} states \(|\{\lambda\},\text{out}\rangle_\pm\) with \(L_0 = 1\) (compare with \( (5.10) \), \( (5.11) \)) can be written as linear combinations of the states \(|\{\mu\},\text{out}\rangle_\pm\).

The fact that the commutator \([L_m,\Phi^-_m]\) is non-vanishing for \(m > 0\) directly implies that a state of the form \( (5.14) \) is not physical as soon as one of the \(\mu^+_n\) is nonzero. This shows that the dressed oscillators \(B^+_n\) together with the \(\tilde{X}^+_n\), which all commute with the \(L_m\), create all the physical states in the Hilbert space. The fact that the \(\tilde{X}^+_n\) form a centerless Virasoro algebra and that the \(\tilde{X}^+_n\) with \(n \geq 0\) annihilate the DDF states implies that every physical state \( (5.14) \) containing one or more of the \(\tilde{X}^+_n\) is a spurious state. This finally brings us to the relation \( (5.13) \), which guarantees the existence of a unitary transformation between the physical in-states and out-states.

### 5.3. The Zero Mode Part of the S-matrix

The calculation of the \(S\)-matrix elements is in principle straightforward as far as the oscillator modes \(f^+\) and \(x^+_n\), \(n \neq 0\), are concerned. By using the expansion

\[
\left(X^\pm(u)\right)^{i\omega} = e^{-i\omega u} \sum_n \frac{1}{n!} \frac{\Gamma(i\omega + 1)}{\Gamma(i\omega - n + 1)} (x^+_0)^{i\omega - n} \left(\sum_m x^+_m e^{-i\frac{\omega}{n} u}\right)^n
\]

one can perform the \(u\)-integrals in \( (5.12) \) and \( (5.13) \) and explicitly write the physical in- and out-states in terms of integer powers of the oscillator modes. The change of basis from in-states to out-states (as in \( (5.15) \)) can then be done by evaluating the appropriate inner products.\(^\dagger\) For this one uses the commutator algebra \( (5.2) \) and the fact that the vacuum is annihilated by all negative frequency modes. Working out the algebra, one eventually is left with an expression that only involves the zero-modes. We shall first explain how to compute the relevant zero mode expectation values and after that discuss the structure of more general \(S\)-matrix elements.

To evaluate the expressions for the \(S\)-matrix elements we need to know the expectation values

\[
\pm \langle 0, \text{out}| y^{-i\omega'} - n |x^{-i\omega - m}|0, \text{in}\rangle_\pm
\]

\(^\dagger\)Note that, although this calculation looks similar to that of open string amplitudes, our \(S\)-matrix does \textit{not} coincide with the open string \(S\)-matrix.
where $\omega$ and $\omega'$ are real, and $n$ and $m$ are integers (in the finite volume $\omega$ and $\omega'$ will be of the form $N/\mu$, where $N$ is a positive integer). It easily seen that the expectation values (5.22) are only non-zero when $\omega = \omega'$ and $m = n$. This follows from the property (5.7) of the physical vacuum states, which is sufficient to fix the expectation value of integer powers of $x$ and $y$.

Let us first put $m = n = 0$ and evaluate the matrix expectation value (5.22). The transition from $x$ to $y$ is just a Fourier transformation, and it is easily checked that this leads to the following relation

$$b\langle 0,\text{out} | |y|^{-i\omega'}|x|^{-i\omega} |0,\text{in}\rangle_a = S^b_a(\omega)$$ (5.23)

where $a, b = \pm$ and the coefficients are given by

$$S^\pm_\pm(\omega) = \frac{1}{\sqrt{2\pi}} \mu^{-i(\omega+\delta)} e^{\pm \frac{\pi i}{4}} e^{-\frac{i\pi}{2}(\omega+\delta)} \Gamma\left(\frac{1}{2} - i(\omega + \delta)\right),$$

$$S^\pm_\mp(\omega) = \frac{1}{\sqrt{2\pi}} \mu^{-i(\omega+\delta)} e^{\pm \frac{\pi i}{4}} e^{\frac{i\pi}{2}(\omega+\delta)} \Gamma\left(\frac{1}{2} - i(\omega + \delta)\right).$$ (5.24)

The relation

$$|S^\pm_\pm(\omega)|^2 + |S^\pm_\mp(\omega)|^2 = 1$$ (5.25)

shows that the transition from in to out vacuum states preserves probability.

By repeatedly using the zero-mode algebra (5.5) we can finally derive

$$b\langle 0,\text{out} | |y|^{-i\omega'-n}|x|^{-i\omega-m} |0,\text{in}\rangle_a = (i\mu)^{-m} \frac{\Gamma\left(\frac{1}{2} - m - i(\omega + \delta)\right)}{\Gamma\left(\frac{1}{2} - i(\omega + \delta)\right)} S^b_a(\omega) \delta_{\omega,\omega'} \delta_{m,n}.$$ (5.26)

It turns out that not all these zero mode overlaps are relevant for computing the $S$-matrix elements between asymptotic in- and out-states of the infinite volume theory. For observers who do not enter the strong coupling region of space-time, but instead perform their measurements in the asymptotic regions $I^-_R$ and $I^+_R$, only part of the full zero mode wavefunction is accessible (cf. the discussion at the end of section 2). Therefore, to describe experiments that involve measurements in those two regions, we are forced to project the above zero mode $S$-matrix onto the component $S^\pm_-$. Due to this projection, which can be thought of as a single $Z_2$-measurement of the sign of $x^-$, the physical observable part of the full dilaton gravity $S$-matrix will no longer be represented by a unitary operator. However, it is clear from the results of this section that no information
will be lost. We will comment further on this point in section 6. In the following subsection we will continue work with the full unitary zero mode $S$-matrix (5.24).

### 5.4. Virasoro Decomposition.

We are now ready to discuss the general structure of the $S$-matrix elements between physical in- and out-states. We first remark that the space of physical in-states, which are created by the dressed oscillators $A^i_m$, can be organized in terms of representations of the $c = 24$ Virasoro algebra generated by

$$L^A_m = \frac{1}{2} \sum_{i,n} : A^n_i A^i_{m-n} : .$$  \hspace{1cm} (5.27)

Every physical in-state can be written as a linear combination of states of the form

$$(L^A_{-1})^{l_1} (L^A_{-2})^{l_2} \cdots (L^A_{-M})^{l_M} | M, \Delta; i in \rangle \pm ,$$  \hspace{1cm} (5.28)

where $| M, \Delta; in \rangle \pm$ is a highest weight state (primary state) of the Virasoro algebra. The highest weight state carries energy $\Delta$ and it has $SO(24)$ quantum numbers that are determined by the structure of the tensor $M$.

In the previous section we defined the operators $\hat{X}^+_m$ and $\tilde{X}^+_m$, and we established that the latter create spurious physical states in the Hilbert space. We can similarly define operators $\hat{X}^-_m$ and $\tilde{X}^-_m$ in the in-sector and show that modulo spurious physical states we can write

$$L^A_m = \hat{X}^-_n + (\mu^2 + 1) \delta_n .$$  \hspace{1cm} (5.29)

This relation is nothing else than the quantum implementation of the statement that the combined energy momentum tensor of the matter fields $f^i$ and the dilaton gravity fields $X^\pm$ should vanish. The relation (5.29) implies that, modulo spurious physical states, the descendant states in each highest weight module can be created by acting with the $\hat{X}^-_n$, i.e., with operators that are written entirely in terms of the dynamical dilaton gravity fields $X^+$ and $X^-$. This observation can be used to simplify the computation of the $S$-matrix elements.

We now consider the scattering matrix for an incoming state that is primary under
the $L_m^A$. We write it as

$$| M, \Delta; in \rangle_\pm = \sum_{I(1), \ldots, I(k)} M_{I(1), \ldots, I(k)} (A-1)^{I(1)} (A-2)^{I(2)} \ldots (A-k)^{I(k)} |0, \text{in} \rangle \pm,$$  \hspace{1cm} (5.30)

where $I(j)$ are multi-indices: $I(j) = \{i^1_j, i^2_j, \ldots, i^{\mu_j}_j\}$ and we write $A^{I(j)} = A^1_{-j} A^2_{-j} \ldots A^{\mu_j}_{-j}$. The conformal dimension of the state is expressed as $\Delta = \sum l\mu_l$. Explicitly working out the expressions for the $A^i_m$ oscillators for a general incoming state leads to a complicated sum of terms, each of which is written as a product over negative modes of the (bare) $f^i$ and $x^+$ oscillators and an appropriate power of $x = x^+_0$. However, if the incoming state is primary the expression simplifies and reduces to a form similar to (5.30), with the $A^i_m$ replaced by $f^i_m$ and an overall factor $x^{-i\omega}$, with $\omega = \frac{\Delta}{\mu}$. (The fact that all other terms drop out can for example be seen by working out the condition that the state is physical; the sum of the leading terms that we just described is physical by itself and there are no candidates for sub-leading corrections involving only $f^i_m$ and $x^+_m$.) Writing the same state in the out-basis of course gives a similar result, this time with an overall factor $y^{i\omega}$. It will be clear that the $S$-matrix on these states is simply diagonal, the only non-trivial effect being the change of basis in the zero-mode sector described above. In formula

$$| \Delta, M; in \rangle_a = S^b_a \left( \frac{\Delta}{\mu} \right) | \Delta, M; out \rangle b,$$  \hspace{1cm} (5.31)

with the factor $S^b_a(\omega)$ as given in (5.24). Obviously, the scattering phase only depends on the conformal weight (energy) $\Delta$ of the incoming state and not on any other details of the group theoretical factor $M$.

The situation becomes more interesting if we consider an incoming state that is a descendant as in (5.28). We will find that in that case the scattering matrix shows a mixing of different states at a certain level of a given highest weight module. We will first illustrate this phenomenon with a few examples and later discuss the more general structure.

The simplest example of a descendant state is a state of the form $L_{-1}^A | \Delta; in \rangle$. At this level the module has a single state, which means that the scattering will again be diagonal. However, the $S$-matrix elements are not simply given by the zero mode overlaps $S^b_a((\Delta + 1)/\mu)$, but instead take the form

$$L_{-1}^A | \Delta; in \rangle_a = S^b_a \left( \frac{\Delta+1}{\mu} \right) (\mu - i)(\mu^2 + i\mu + 2\Delta) \frac{\Delta}{\mu + i}(\mu^2 - i\mu + 2\Delta) L_{-1}^B | \Delta; out \rangle b,$$  \hspace{1cm} (5.32)
as can be checked by direct computation. For $\Delta = 0$ this expression is somewhat meaningless, since in that case the incoming and outgoing states are spurious physical states.

At level 2 the general highest weight module has two independent descendant states, generated by $(L_A^+)^2$ and $L_-^2$, respectively. These provide the first example of matter states that show non-trivial mixing of in the process of scattering off the strong coupling region of the dilaton gravity system. We have worked out the corresponding $S$-matrix elements for these particular descendants in the module with primary state $A_i^i|0\rangle$ with $\Delta = 1$. The following states provide an orthonormal basis for the in-states

$$(e_A)_a = \frac{1}{\sqrt{13}} \left( L_{-2}^+ - \frac{1}{2}(L_{-1}^+)^2 \right) A_i^i|0, in\rangle_a = \frac{1}{2\sqrt{13}} \sum_j (A_j^j A_j^j) A_i^i|0, in\rangle_a ,$$

$$(f_A)_a = \frac{1}{\sqrt{12}} (L_{-1}^+)^2 A_i^i|0, in\rangle_a = \frac{1}{\sqrt{3}} A_i^{i-3}|0, in\rangle_a . \tag{5.33}$$

The explicit change of basis from the incoming states $(e_A, f_A)_a$ to the outgoing states $(e_B, f_B)_b$ yields the following $S$-matrix at this level

$$\begin{pmatrix} e_A \\ f_A \end{pmatrix}_a = S_{ab}^b \begin{pmatrix} A/\mu \\ \lambda B - \lambda^* C \end{pmatrix} \begin{pmatrix} e_B \\ f_B \end{pmatrix}_b , \tag{5.34}$$

where

$$\lambda = \frac{\mu^5 - 2i\mu^4 + 13\mu^3 - 32i\mu^2 - 4\mu - 20i}{(\mu + 2i)(\mu^2 - 3i\mu + 4)(\mu^2 - i\mu + 4)} ,$$

$$\nu = -2i\sqrt{39} \frac{(\mu - i)(\mu^2 + 2)}{(\mu + i)(\mu + 2i)(\mu^2 - 3i\mu + 4)(\mu^2 - i\mu + 4)} . \tag{5.35}$$

It is easily checked that this level two $S$-matrix describes a unitary transformation. The physical $S$-matrix elements between asymptotic in- and out-states are obtained by projecting onto the right zero mode sectors. The off-diagonal amplitudes proportional to $\nu$ describe the process where a single particle of discretized momentum 3 approaches the strong coupling region of the dilaton gravity system and gets scattered into 3 out-going particles, each of momentum 1.

With this Virasoro structure at hand, we can try to sharpen a bit the physical picture that we developed in earlier sections. If we substitute for a moment the term ‘$SO(24)$ quantum numbers’ for the term ‘information’, we see immediately that our $S$-matrix preserves information. This is because, as we saw, the $S$-matrix respects the decomposition
into Virasoro modules, which each carry a specific representation of $SO(24)$. An in-going Virasoro primary state is simply reflected to an out-going state with identical structure, picking up an energy dependent phase factor. However, descendant states do mix when scattering off the strong coupling region of the dilaton gravity system.

The in-state created by $\hat{A}_i(p_\pm)$ (see (4.4)), is a linear combination, as in (4.5), of states created by the $\alpha_i(\omega)$. The latter correspond to the finite volume states created by the $A^i_{-n}$ with $n$ large (of order $L$), which can be written as $(L_{-1}^{A_i})^{n-1}A_{-1}^i |0, in\rangle_a$. Since these states are ‘deep’ descendants, they will scatter into out-states which contain many, less energetic, particles. This is in accordance with the result of section 4.3, that a considerable fraction of the out-going state describes thermal radiation. It further suggest that the information will typically come out in the form of low energy modes.

6. Discussion

In this paper we have presented an exact quantization procedure for two-dimensional dilaton-gravity, and we used it to construct a scattering matrix between asymptotic in- and out-states. We have further given strong evidence that this $S$-matrix indeed describes the formation and evaporation of two-dimensional black holes. In particular, we have made contact with the semi-classical theory and the standard derivation of Hawking radiation.

Our results also give an indication when important quantum corrections to the Hawking spectrum can be expected to appear. Somewhat surprisingly we find that this already occurs after a relatively short time $t = \log(M/\lambda)$ after the black hole, with mass $M$, has been formed. As we explained, by this time signals become visible to the asymptotic observer that originated from sub-Planckian distances from the event horizon. To understand the structure of the asymptotic outgoing state it is then no longer reliable to assume that the state near the horizon is given by the vacuum. The reason is that the infalling matter distorts this local vacuum state in a slight way via quantum gravitational interactions, and although this effect is hardly noticeable to a local inertial observer, it has important consequences for an asymptotic observer. Since time translations at infinity correspond to boosts near the horizon, the asymptotic state will depend on the ultra-local fine-structure of the state in that region. In a way, the asymptotic observer is looking at the strong coupling physics near the horizon through a Planckian magnifying glass. Applying the same argumentation to the 3+1 dimensional situation leads to the
prediction that already after a time
\[ t = 2M \log(\frac{M}{M_{pl}}) \]
quantum gravitational effects can start to modify the thermal spectrum of Hawking radiation. For a macroscopic black hole this is well before the time that its mass is reduced to \( M_{pl} \). Thus the instant at which quantum gravitational corrections start to play a role depends not only on the mass of the black hole at a given time, but also on its history.

Our dilaton gravity \( S \)-matrix contains the one-particle \( S \)-matrix discussed in section 2 as an overall factor acting in the space of zero modes, and as a consequence it requires that we perform a projection within this zero mode sector to ensure that the final state can be properly interpreted as an asymptotic state in the right out-region. Although we have shown that this projection does not lead to any information loss, the resulting \( S \)-matrix does not define a unitarity operator in the usual sense. It is an important question to what extent the need for this projection is an artifact of the present model, or whether it is a general feature of all quantum gravitational models containing black holes. It seems to indicate that the dilaton gravity model has a quantum instability against decaying into a state that is unobservable from outside the black hole, and that the only stabilizing mechanism is to project, as if one is performing a measurement, on the observable part of the wave function. While the model is not complete without a better understanding of this issue, it seems to us that this apparent problem is relatively mild compared to some of the problems that arise in other proposed ways of dealing with quantum black holes. However, a real comparison can only be made by trying to see if each of the other possible scenarios can be realized in the form of a fully quantum mechanical treatment of dilaton gravity.

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