ABSENCE OF SINGULARITIES IN A COSMOLOGICAL PERFECT-FLUID SOLUTION

F.J.CHINEA and L.FERNÁNDEZ-JAMBRINA
Departamento de Física Teórica II, Facultad de Ciencias Físicas
Ciudad Universitaria, 28040 Madrid, Spain
and
Instituto de Física Fundamental
Ciudad Universitaria, 28040 Madrid, Spain

and

J.M.M.SENOVILLA
Departament de Física Fonamental, Universitat de Barcelona
Diagonal 647, 08028 Barcelona, Spain
and
Laboratori de Física Matemàtica
Societat Catalana de Física, I.E.C., Barcelona, Spain

ABSTRACT

In this lecture we will show some properties of a singularity-free solution to Einstein’s equations and its accordance with some theorems dealing with singularities. We will also discuss the implications of the results.

1. Introduction

The occurrence of singularities in solutions to Einstein’s equations has been largely studied in the literature (see, for instance, [1] and references therein). Recently, one of us presented a cosmological, perfect-fluid solution whose curvature scalar polynomials were finite [2] and this raised the question whether there was a singularity of other kind. It has already been proven [3] that this space-time is singularity-free. So, in this lecture, we would like to give an account of the properties that follow from this fact. More details can be found in [3].

In section 2 we deal with the geodesic completeness of the solution, whereas in section 3 we show some of its nice properties which are useful to see how it agrees with the singularity theorems. This we will show in section 4. As there are many singularity theorems in the literature, we shall only consider two of them: We shall deal with that of Penrose [1,6] and the most powerful one, proven by Hawking and Ellis [1,7]. Section 5 is devoted to a brief discussion of the results.
2. Geodesic completeness

The solution presented in [2] describes a cosmological space-time corresponding to a radiation-dominated universe exhibiting cylindrical symmetry:

\[
ds^2 = \cosh^4(at)cosh^2(3ar)(-dt^2 + dr^2) + \\
(9a^2)^{-1}\cosh^4(at)\sinh^2(3ar)\cosh^{-2/3}(3ar)d\phi^2 + \\
cosh^{-2}(at)\cosh^{-2/3}(3ar)dz^2
\] (1)

The density and pressure of the fluid take the following form:

\[
\chi \rho = 15a^2\cosh^{-4}(at)\cosh^{-4}(3ar)
\] (2)

\[
p = \rho/3
\] (3)

where \( \chi \) is the gravitational constant.

Using standard techniques, we get the following equations for the geodesic motion:

\[
\ddot{t} + 2a \tanh(at)(\dot{t}^2 + \dot{r}^2) + 6a \tanh(3ar)\dot{t}\dot{r} + \\
(2/9a)\tanh(at)\tanh^2(3ar)\cosh^{-8/3}(3ar)\dot{\phi}^2 - \\
a \cosh^{-7}(at)\sinh(at)\cosh^{-8/3}(3ar)\dot{z}^2 = 0
\] (4)

\[
\ddot{r} + 3a \tanh(3ar)(\dot{t}^2 + \dot{r}^2) + 4a \tanh(at)\dot{t}\dot{r} - \\
(1/9a)\sinh(3ar)\cosh^{-5/3}(3ar)[3 - \tanh^2(3ar)]\dot{\phi}^2 + \\
a \cosh^{-6}(at)\sinh(3ar)\cosh^{-11/3}(3ar)\dot{z}^2 = 0
\] (5)

\[
(9a)^{-1}\cosh^4(at)\sinh^2(3ar)\cosh^{-2/3}(3ar)\dot{\phi} = K
\] (6)

\[
cosh^{-2}(at)\cosh^{-2/3}(3ar)\dot{z} = L
\] (7)

\[
cosh^4(at)cosh^2(3ar)(-\dot{t}^2 + \dot{r}^2) + \\
(9a^2)^{-1}\cosh^4(at)\sinh^2(3ar)\cosh^{-2/3}(3ar)\dot{\phi}^2 + \\
cosh^{-2}(at)\cosh^{-2/3}(3ar)\dot{z}^2 = -\delta
\] (8)

Here a dot means derivative with respect to the affine parameter, \( K \) and \( L \) are constants of motion along the geodesics and \( \delta \) takes the value one or zero for timelike or null geodesics respectively. As all functions involved are non-singular, the solutions exist and are unique.
It can be easily seen [3] that the geodesics do not grow faster than their tangents for positive time coordinate in the vicinity of the axis (for small enough radius) or for large $r$ or $t$. So the geodesics can be extended to arbitrarily large values of the affine parameter.

Two families will be particularly interesting for the discussion: One is formed by the congruences of ingoing and outgoing radial null geodesics through every point of the manifold, whose motion is unbounded and its radial speed never changes sign. The other one comprises the null geodesics on the hyperplanes $z = \text{const.}$ The geodesic motion along the latter is bounded in the radial coordinate [3].

3. Properties of the solution

1.1. Global hyperbolicity

It follows from section 2 that every maximally extended null geodesic meets any of the hypersurfaces $t = \text{const.}$. This means [4] that every non-spacelike curve intersects the mentioned hypersurfaces only once (they are global Cauchy surfaces) and so the solution is globally hyperbolic and fulfills the whole hierarchy of causality conditions under it [5], for instance the chronology condition (there are no closed timelike curves), which will be needed in the next section.

1.2. Singularity-free

Since the $t$ is a cosmic time and the hypersurfaces $t = \text{const.}$ are Cauchy surfaces, every non-spacelike curve can be extended to arbitrarily large values of the affine parameter. This means that the solution is bundle-complete [1], which is the usual definition for lack of singularities.

1.3. Strong energy and generic conditions

From the formulae (2,3) it is obvious that the energy-momentum tensor never vanishes and $R_{ab}v^a v^b > 0$, so both conditions are satisfied [1,5].

4. Accordance with the singularity theorems

4.1 Penrose’s theorem

We have already proven that the solution is null geodesically complete, globally hyperbolic and that the strong energy condition holds. Obviously, if it is to be in accordance with the theorem [1,6], it must have no closed trapped surface (a spacelike surface in which the traces of the two null second fundamental forms have the same sign). Computing both traces in the point where the radial coordinate reaches its maximum (it has got one since the surface is compact), we get that they have opposite signs, so there is no closed trapped surface.
4.1 Hawking and Penrose’s theorem

Since the space-time is geodesically complete and every other condition of the theorem holds (generic, strong energy and chronology conditions), the whole set of three alternative conditions must not be fulfilled\cite{1,7}. We know that there’s no trapped surface, so we only have to check that the other two do not hold:

1. Existence of a point \( q \) such that on every past (future) null geodesic from \( q \) the expansion becomes negative: We can see such point does not exist in this solution just remembering the two families of geodesics pointed out in section 2. The radial null geodesics through any point in the manifold diverge towards the future (past) if they are outgoing (ingoing), so their expansion cannot be negative. Another way to see that the point \( q \) does not exist is to remember that through any \( q \) there are null geodesics with \( z = \text{const.} \) which are bounded above and below in \( r \). Thus, these geodesics can never converge with the radial ones, which are unbounded in \( r \).

2. The existence of a compact achronal set without edge: Suppose there is one. Take a point \( q \) in the set. By using the radial geodesics we can always choose points \( q_- \in I^-(q) \) (the chronological past of \( q \)) and \( q_+ \in I^+(q) \) (the chronological future of \( q \)) such that \( r(q_-) = r(q_+) > r(q) \), where for any point \( s \) we denote by \( r(s) \) the value of the coordinate \( r \) at \( s \). Since \( q_+ \in I^+(q_-) \) and \( r(q_+) = r(q_-) \), we can join \( q_- \) and \( q_+ \) with a future-directed worldline of the fluid congruence. As the required achronal set has no edge, this worldline intersects the set, and it will do it at a point \( \tilde{q} \) with \( r(\tilde{q}) = r(q_-) = r(q_+) > r(q) \). This proves that the coordinate \( r \) cannot be bounded for any achronal set without edge. It is obvious then that any achronal set in the manifold cannot be both compact and without edge.

5. Discussion

We have shown that the solution \cite{2} is singularity-free and in agreement with some of the main singularity theorems. This illustrates the importance of the initial conditions (existence of a trapped surface or an achronal set without boundary) for the development of singularities, since the energy and causality conditions are not determinant for their appearance. There is no reason then why there should not be singularity-free solutions with little or even no symmetry at all if they do not have the initial conditions required by the singularity theorems.

6. Acknowledgements

The present work has been supported in part by DGICYT Project PB89-0142 (F.J.C.) and CICYT Project AEN90-0061 (J.M.M.S); L.F.J. is supported by a FPI Predoctoral Scholarship from Ministerio de Educación y Ciencia (Spain). J.M.M.S. wishes to thank E. Ruiz for discussions.
7. References

1. S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, Cambridge Univ. Press, Cambridge (1973).
2. J.M.M. Senovilla, *Phys. Rev. Lett.* 64, 2219 (1990).
3. F.J. Chinea, L. Fernández-Jambrina and J.M.M. Senovilla, *Phys. Rev. D* 45, 481 (1992) [arXiv: gr-qc/0403075]
4. R. Geroch, *J. Math. Phys.* 11, 437 (1970).
5. J. Beem and P. Ehrlich, *Global Lorentzian Geometry*, Dekker, New York (1981).
6. R. Penrose, *Phys. Rev. Lett.* 14, 57 (1965).
7. S.W. Hawking and R. Penrose, *Proc. Roy. Soc. London A* 314, 529 (1970).