A New Probabilistic Representation of the Alternating Zeta Function
and a New Selberg-like Integral Evaluation

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Abstract
In this paper, we present two new representations of the alternating Zeta function. We show that for any $s \in \mathbb{C}$ this function can be computed as a limit of a series of determinant. We then express these determinants as the expectation of a functional of a random vector with Dixon-Anderson density. The generalization of this representation to more general alternating series allows us to evaluate a Selberg-type integral with a generalized Vandermonde determinant.

1 Introduction
The Dirichlet eta function is defined by the following Dirichlet series, which converges for any complex number having real part greater than 0

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots$$

(1)

This Dirichlet series is the alternating sum corresponding to the Dirichlet series expansion of the Riemann zeta function $\zeta(s)$ and for this reason the Dirichlet eta function is also known as the alternating zeta function. The following relation holds:

$$\eta(s) = (1 - 2^{1-s}) \zeta(s).$$
The starting point of our work is a result from [2] (p. 456). In their paper, they show that the sum (1) can be approximated using an array of coefficients \( a_{n,N}, 1 \leq n \leq N \). Let

\[
a_{n,N} \overset{\text{def}}{=} \frac{1}{2} \prod_{j=1, j \neq n}^{N} \frac{j^{2} - n^{2}}{j^{2}} = (-1)^{n-1} \frac{2N}{N - n}.
\] (2)

then

**Theorem 1** (Biane and al.). For \( \varepsilon_{i}, 1 \leq i \leq N \) independent standard exponential variables, and \( \Re(s) > -2N \)

\[
\mathbb{E} \left( \left( \sum_{n=1}^{N} \frac{\varepsilon_{n}}{n^{2}} \right)^{s} \right) = s! \left( \frac{s}{2} \right)^{N} \sum_{n=1}^{N} \frac{a_{n,N}}{n^{s}}
\] (3)

where the \( a_{n,N} \) are defined by (2), and

\[
\eta_{N}(s) \overset{\text{def}}{=} \sum_{n=1}^{N} \frac{a_{n,N}}{n^{s}} \rightarrow \eta(s) \text{ as } N \rightarrow \infty
\] (4)

uniformly on every compact subset of \( \mathbb{C} \).

**Remark 1.** The signs of the coefficients \( a_{n,N} \) have been modified with respect to those given in the previously cited article, in order to obtain convergence in equation (4) towards \( \eta(s) \) rather than \( -\eta(s) \).

The second part of the theorem shows that, for any \( s \in \mathbb{C} \) the alternating Zeta function can be obtained by weighting the first \( N \) terms of the original series which is defined only for \( \Re(s) > 0 \). In this paper, we will show that the weighted finite series \( \eta_{N}(s) \) defined in (1) can be written as a determinant (section 2.1 proposition 2). We will then show that this determinant can be written as the expectation of a functional of a Dixon-Anderson random vector (section 2.2 theorem 3). This result is new (up to our knowledge) and seems to show that there is a relation between the Zeta function and the theory of random matrices.

In section 3 we give a generalization of the representations given in section 2 for general series and, by computing the expectation of these representations, we obtain the evaluation of two Selberg integrals involving a generalized Vandermonde determinant (theorem 6).

## 2 New Representations of the Alternating Zeta Function

### 2.1 Determinant Representation

We start with the following result

**Proposition 2.** With the notations and conditions given in theorem 1 we have

\[
\eta_{N}(s) = \frac{1}{2} \begin{vmatrix}
1 & \frac{1}{3!} & \cdots & \frac{1}{(2N-1)!} \\
2^{1-s} & \frac{2}{3!} & \cdots & \frac{2}{(2N-1)!} \\
\vdots & \vdots & \ddots & \vdots \\
N^{1-s} & \frac{N}{3!} & \cdots & \frac{N^{2N-1}}{(2N-1)!}
\end{vmatrix}.
\]
Proof. Let \( a_n = n^2 \), \( Q_N(x) = \prod_{n=1}^{N} (1 - x/a_n) \), and \( P_N \) a polynomial of degree \( N - 1 \) over \( \mathbb{C}[X] \), with the convention \( P_N(x) = \sum_{n=1}^{N} c_{n,N} x^{n-1} \). An adaptation of the arguments given in annex A shows that

\[
\frac{P_N(x)}{Q_N(x)} = \sum_{n=1}^{N} P_n(a_n) \prod_{j=1 \atop j \neq n}^{N} \left( \frac{1}{1 - \frac{a_n}{a_j}} \right) \left( \frac{1 - \frac{x}{a_j}}{1 - \frac{x}{a_n}} \right).
\]

Setting \( x = 0 \), we have \( Q_N(0) = 1 \) and we get

\[
P_N(0) = 2 \sum_{n=1}^{N} a_{n,N} P_n \left( n^2 \right).
\]

We choose \( P_N \) as the polynomial of degree \( N - 1 \) such that \( P_N(a_n) = n^{-s} \) for \( n = 1, \ldots, N \). The coefficients \( (c_{n,N})_{n=1}^{N} \) of \( P_N \) are solutions of the Vandermonde system

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & 2^2 & \cdots & 2^{2(N-1)} \\
\vdots & \vdots & \ddots & \vdots \\
1 & N^2 & \cdots & N^{2(N-1)}
\end{pmatrix}
\begin{pmatrix}
c_{1,N} \\
c_{2,N} \\
\vdots \\
c_{N,N}
\end{pmatrix}
= \begin{pmatrix} 1 \\ 2^{-s} \\ \vdots \\ N^{-s} \end{pmatrix}.
\]

(5)

The Vandermonde matrix is invertible (see annex B) showing that the polynomial \( P_N \) is uniquely determined. We are only interested in \( c_{1,N} = P_N(0) \). Let us denote \( V_N \) the Vandermonde matrix in (5) and \( V_N^{(s)} \) the matrix obtained by replacing the first column of \( V_N \) by the right-end term.

Using Cramer’s rule, we get

\[
c_{1,N} = \frac{\text{det} \left( V_N^{(s)} \right)}{\text{det} \left( V_N \right)}.
\]

(6)

The Vandermonde Determinant of \( V_N \) is

\[
\text{det} \left( V_N \right) = \prod_{1 \leq i < j \leq N} \left( j^2 - i^2 \right) = \prod_{1 \leq i \leq j \leq N} (j - i)(j + i) = \prod_{j=2}^{N} (j-1)! \frac{(2j-1)!}{j!} = \frac{1}{N!} \prod_{n=1}^{N-1} (2n + 1)!
\]

(7)

and thus, we have

\[
c_{1,N} = \frac{1}{N!} \frac{1}{2} \frac{1}{3} \cdots \frac{1}{(2N-1)} \frac{1}{2} \frac{1}{3} \cdots \frac{1}{(2N-1)} \frac{1}{2} \frac{1}{3} \cdots \frac{1}{(2N-1)}
\]

ending the proof.

\[
\boxed{\text{Remark 2. Observe that using the expression (6), it is immediate that } \eta_N(s) = \frac{1}{2} \text{ if } s = 0 \text{ and } \eta_N(s) = 0 \text{ if } s = -2, -4, \ldots, -2(N-1).}
\]

\[
\boxed{\text{Remark 3. An alternative (and more direct) proof could have be used using the representation of the generalized Vandermonde determinant given in lemma (4).}}
\]

3
2.2 Probabilistic Representation

We observe that $\det \left( V_N^{(s)} \right)$ (equation [5]) is a generalized Vandermonde determinant. Using the argument used in ([8]), we get the following lemma

**Lemma 2.1.** For $N > 1$, let

$$\Gamma_{N-1} \left( \frac{s}{N} \right) = \frac{2(N-1)!}{s \Gamma_{N-1} \left( \frac{s}{N} \right)} \int_1^{2^2} dx_1 \int_1^{2^3} dx_2 \ldots \int_1^{2^N} \prod_{1 \leq i < j \leq N-1} (x_j - x_i) \prod_{n=1}^{N-1} \left( \frac{x_n}{n(n+1)} \right)^{s/2} \, dx_N \ldots dx_{N-1} \ldots dx_1 . \quad (8)$$

**Proof.** Observe first that $(s \Gamma_{N-1} \left( \frac{s}{N} \right))^{-1}$ is defined for all $N > 1$ and all $s \in \mathbb{C}$. Next, we have

$$\det \left( V_N^{(s)} \right) = \prod_{n=2}^{N-1} \frac{1}{n^s} \times \begin{vmatrix} 1 & 1 & \ldots & 1 \\ 2^{-s} & 2^2 & \ldots & 2^{2(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ N^{-s} & N^2 & \ldots & N^{2(N-1)} \end{vmatrix} = \prod_{n=2}^{N-1} \frac{1}{n^s} \times \begin{vmatrix} 1 & 1 & \ldots & 1 \\ 1 & 2^{2(1+s/2)} & \ldots & 2^{2(N-1+s/2)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & N^{2(1+s/2)} & \ldots & N^{2(N-1+s/2)} \end{vmatrix}$$

$$= \prod_{n=1}^{N-1} \frac{n+s/2}{(n+1)^s} \times \begin{vmatrix} \int_1^{2^2} x_1^{s/2} \, dx_1 & \cdots & \int_1^{2^2} x_1^{2(N-2+s/2)} \, dx_1 \\ \vdots & \ddots & \vdots \\ \int_1^{2^2} x_1^{N-2+s/2} \, dx_1 & \cdots & \int_1^{2^2} x_1^{N-2+s/2} \, dx_1 \\ \int_1^{2^2} x_N^{s/2} \, dx_N & \cdots & \int_1^{2^2} x_N^{N-2+s/2} \, dx_N \end{vmatrix}$$

$$= \prod_{n=1}^{N-1} \frac{n+s/2}{(n+1)^s} \times \begin{vmatrix} \frac{x_1^{s/2}}{x_N^{s/2}} & \cdots & \frac{x_1^{N-2+s/2}}{x_N^{N-2+s/2}} \\ \vdots & \ddots & \vdots \\ \frac{x_N^{s/2}}{x_N^{s/2}} & \cdots & \frac{x_N^{N-2+s/2}}{x_N^{N-2+s/2}} \end{vmatrix}$$

$$= \prod_{n=1}^{N-1} \frac{n+s/2}{(n+1)^s} \int_1^{2^2} dx_1 \int_1^{2^2} dx_2 \ldots \int_1^{2^2} dx_N \prod_{1 \leq i < j \leq N-1} (x_j - x_i) \prod_{n=1}^{N-1} \left( \frac{x_n}{n(n+1)} \right)^{s/2} \, dx_N \ldots dx_{N-1} \ldots dx_1 . \quad \square$$

Let $\mathcal{D}_{N-1}(x; \alpha, a)$, with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N), \alpha_0 > 0$ and $a = (a_1, a_2, \ldots, a_N)$ with $a_1 < a_2 < \ldots < a_N$, denote the Dixon-Anderson probability density function (pdf) over the domain $\mathcal{X}_{N-1} = \{a_1 < x_1 < a_2 < \ldots < x_{N-1} < a_N\}$
\(Y = \) uniformly on every compact of \(C\) formations. More precisely if \(X\) is a pdf over \(C\), we get that
\[
D_{N-1}(x; \alpha, a) = (N-1)! \frac{\prod_{1 \leq i < j \leq N-1} (x_j - x_i)}{\prod_{1 \leq i < j \leq N} (j^2 - i^2)}
\]
is a pdf over \(X_{N-1} = \{ 1 < x_1 < 2^2 < \ldots < x_{N-1} < N^2 \}\). From the previous lemma, we obtain the following theorem

**Theorem 3.** Let \(X = (X_1, \ldots, X_{N-1})\) be a random vector with Dixon-Anderson distribution given by (10), then
\[
\frac{1}{s} \frac{1}{\Gamma_{N-1} \left( \frac{s}{2} \right)} \mathbb{E} \left[ \prod_{n=1}^{N-1} \left( \frac{X_n}{n(n+1)} \right)^{s/2} \right] \rightarrow \eta(s) \text{ as } N \rightarrow \infty
\]
uniformly on every compact subset of \(C\).

**Proof.** Using the expression of \(\det(V_N)\) given in equation (7) we find the expectation given in (11) for \(N\) fixed. The Gamma function can be defined as an infinite product for all complex numbers \(z\) except the non-positive integers
\[
\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right)
\]
and for any \(s \in \mathbb{C}\) (the case \(s = 0\) is handled by continuity)
\[
\frac{2}{s \Gamma_{N-1} \left( \frac{s}{2} \right)} \xrightarrow{N \rightarrow \infty} \frac{1}{\Gamma \left( 1 + \frac{s}{2} \right)}
\]
uniformly on every compact of \(C\). For any \(s \in \mathbb{C}\), we have
\[
\eta_N(s) = \frac{\det(V_N^{(s)})}{\det(V_N)} = \frac{1}{s} \frac{1}{\Gamma_{N-1} \left( \frac{s}{2} \right)} \mathbb{E} \left[ \prod_{n=1}^{N-1} \left( \frac{X_n}{n(n+1)} \right)^{s/2} \right]
\]
and thus the conclusion of theorem 1 occurs as well. \(\square\)

### 2.3 A Result Related to Theorem 3

One interesting fact about the Dixon-Anderson distribution given in (10) is that it is invariant under some linear transformations. More precisely if \(X\) is a Dixon-Anderson random vector with pdf \(D_N(x; \alpha, a)\) and \((u, v) \in \mathbb{R}^* \times \mathbb{R}\) then \(Y = uX + v1_N\) is a Dixon-Anderson random vector with pdf \(D_N(x; \alpha, au + v1_N)\). Using this property we can renormalize the random vector \(X\) over \([0, 1]\) by using the change of variable \(Y = (X - 1_{N-1})/(N^2 - 1)\) giving us the identity
\[
\eta_N(s) = \frac{1}{s \Gamma_{N-1} \left( \frac{s}{2} \right)} \mathbb{E} \left[ \prod_{n=1}^{N-1} \left( \frac{Y_n + \frac{1}{N^2 - 1}}{N!} \right)^{s/2} \right]
\]
We have the following theorem
Theorem 4. Let $\psi_N(x; s)$ denote the application

$$\psi_N(x; s) = \frac{2}{\Gamma(1 + \frac{s}{2})} \left( \frac{(N^2 - 1)(N-1)!}{N!(N-1)!} \right)^{\frac{s}{2}} \mathbb{E} \left[ \prod_{n=1}^{N-1} \frac{Y_n - x^2 - 1}{N^2 - 1} \right].$$

Then for all $n \in \mathbb{N}^*$

$$\psi_N(n; s) \xrightarrow{N \to \infty} \frac{1}{n^s}$$

and for $n = 0$

$$\psi_N(0; s) \xrightarrow{N \to \infty} \eta(s).$$

Proof. The case $n = 0$ is a consequence of the theorem 3. Let $b_{n,N} = \frac{n^2 - 1}{N^2 - 1}$ for $n = 1, \ldots, N$ and $\gamma_{N-1} = \{0 = b_{1,N} < y_1 < b_{2,N} < y_2 < \cdots < y_{N-1} < b_{N,N} = 1\}$. Taking $x = n$ we can compute the value of $\psi_N(x; s)$. We have

$$\mathbb{E} \left[ \prod_{n=1}^{N-1} |Y_n - b_{n,N}|^s \right] = \prod_{1 \leq i < j \leq N} (b_{j,N} - b_{i,N}) \int_{\gamma_{N-1}} dy \prod_{1 \leq i < j \leq N-1} (y_j - y_i) \prod_{k=1}^{N-1} |y_k - b_{n,N}|^{s/2}.$$  

The integral of the right hand side is (Consider the pdf given in (9) with $\alpha_k = 1$ if $k \neq n$ and $\alpha_k = 1 + s/2$ otherwise)

$$\int_{\gamma_{N-1}} dy \prod_{1 \leq i < j \leq N-1} (y_j - y_i) \prod_{k=1}^{N-1} (y_k - b_{n,N})^{s/2}$$

$$= \frac{\Gamma(1 + \frac{s}{2})}{\Gamma(N + \frac{s}{2})} \prod_{1 \leq i < j \leq N} (b_{j,N} - b_{i,N}) \prod_{k=1}^{N-1} (b_{n,N} - b_{k,N})^{s/2} \prod_{k=n+1}^{N} (b_{k,N} - b_{n,N})^{s/2}$$

From this we deduce that when $n \neq 1$ we have

$$\psi_N(n; s) = \left( \prod_{k=1}^{n-1} \frac{n^2 - k^2}{N^2 - 1} \prod_{k=n+1}^{N} \frac{k^2 - n^2}{N^2 - 1} \right)^{s/2} \frac{\Gamma(N)}{\Gamma(N + \frac{s}{2})}$$

$$= \left( \frac{(N - n)!(N + n)!}{n^2 N!(N-1)!} \right)^{s/2} \frac{\Gamma(N)}{\Gamma(N + \frac{s}{2})}$$

$$= \left( \frac{1}{[a_{n,N}]^2} \right)^{s/2} N^{s/2} \frac{\Gamma(N)}{\Gamma(N + \frac{s}{2})}$$

with $a_{n,N}$ defined in (2). In the case $n = 1$ we find directly

$$\psi_N(1; s) = N \frac{\Gamma(N)}{\Gamma(N + \frac{s}{2})}$$

Taking the limit and observing that $\Gamma(N + \frac{s}{2}) \sim \Gamma(N) N^{s/2}$ as $N \to +\infty$ end the proof.
3 Averaged Alternating Random Series

3.1 A generalization of proposition and theorem

Let \( s \in \mathbb{C} \) with \( s \neq -2, -4, \ldots, -2(N-1) \) and let \( u_1 < u_2 < \ldots < u_N \) be an increasing sequence of real numbers in \( \mathbb{R}^* \). From this sequence, we define the \( N \times N \) generalized Vandermonde determinant

\[
V_{N/2}^{(s/2)}(u) = \begin{vmatrix}
    u_1^{-s/2} & u_1 & u_1^2 & \cdots & u_1^{N-1} \\
    u_2^{-s/2} & u_2 & u_2^2 & \cdots & u_2^{N-1} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    u_N^{-s/2} & u_N & u_N^2 & \cdots & u_N^{N-1}
\end{vmatrix}
\]

with \( u \) denoting the ordered vector \( (u_1, \ldots, u_N) \).

**Lemma 3.1.** Let \( u_1 < u_2 < \ldots < u_N \) be an increasing sequence of real numbers in \( \mathbb{R}^* \). The following hold

\[
\frac{V_{N/2}^{(s/2)}(u)}{V_N^{(0)}(u)} = \sum_{n=1}^{N} (-1)^{n-1} \frac{1}{u_n^{s/2}} \prod_{1 \leq j \leq n, j \neq n} u_j \frac{u_j - u_n}{|u_j - u_n|}.
\]

**Proof.** Observe that \( V_N^{(0)}(u) \) denotes the usual determinant of a Vandermonde matrix. Let us denote by \( V_N^{-n} \) the following determinant

\[
V_n^{-n} = \left( \prod_{1 \leq j \leq N} u_j \right) V_N^{(0)}(u_1, \ldots, u_{n-1}, u_{n+1}, \ldots, u_N).
\]

Then, it is obvious that

\[
V_n^{-n} = \left( \prod_{1 \leq j \leq N} u_j \right) V_N^{(0)}(u_1, \ldots, u_{n-1}, u_{n+1}, \ldots, u_N).
\]

By looking closely at the missing products, we obtain that

\[
V_n^{-n} = \left( \prod_{1 \leq j \leq N} u_j \right) \frac{V_N^{(0)}(u)}{\prod_{1 \leq j \leq N} (u_n - u_j) \prod_{n+1}^{N} (u_j - u_n)} = (-1)^{n-1} V_N^{(0)}(u) \prod_{1 \leq j \leq N, j \neq n} u_j \frac{u_j - u_n}{|u_j - u_n|}.
\]

We have thus

\[
\frac{V_{N/2}^{(s/2)}(u)}{V_N^{(0)}(u)} = \sum_{n=1}^{N} (-1)^{n-1} \frac{1}{u_n^{s/2}} \prod_{1 \leq j \leq n, j \neq n} u_j \frac{u_j - u_n}{|u_j - u_n|} = \sum_{n=1}^{N} \frac{1}{u_n^{s/2}} \prod_{1 \leq j \leq n, j \neq n} u_j \frac{u_j - u_n}{|u_j - u_n|}.
\]

As the sequence \((u_n, n = 1, \ldots, N)\) is strictly increasing, the sum \((12)\) is alternating as announced.  

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\footnote{The reader will be aware that in section \( V_N^{(s/2)} \) represented a matrix, whereas from now the notation \( V_N^{(s/2)}(u) \) represents a determinant}
We have the following result which generalize theorem 3.

**Proposition 5.** Let \( 0 < u_1 < u_2 < \ldots < u_N \) be an arbitrary increasing sequence of positive real number, let \( s \in \mathbb{C} \) with \( s \neq -2, -4, \ldots, -2(N - 1) \) and let \( \mathbf{X} = (X_1, \ldots, X_N) \) denote a random vector with Dixon-Anderson density \( \mathcal{D}_{N-1}(\mathbf{x}; \mathbf{1}_{N-1}, \mathbf{u}) \) then

\[
\frac{2}{s \Gamma_{N-1} \left( \frac{3}{2} \right)} N^{s/2} \mathbb{E} \left[ \prod_{n=1}^{N-1} \frac{X_n^{s/2}}{\prod_{n=1}^{N} u_n^{s/2}} \right] = \sum_{n=1}^{N} \frac{(-1)^{n-1}}{u_n} \prod_{1 \leq j \leq N, j \neq n} \frac{u_j}{|u_j - u_n|}, \tag{13}
\]

The proof follows the same steps as in lemma 2.4 and is left to the reader. The reader can also note that taking \( u_n = n^2 \) we get the expression obtained in section 2.2.

**Remark 4.** It is clear that if the left hand side of the equation (13) converges in some sense as \( N \to \infty \) to a well defined function in \( s \), then this function will be equal to 1 when \( s = 0 \) and equal to \( t^{-s/2} \) when \( s = -2k, k \in \mathbb{N}^* \).

Finally we note that a similar lemma have been proved in [2] using exponential random variables

**Lemma 3.2.** For \( (\varepsilon_n, 1 \leq n \leq N) \) independent standard exponential variables, and \( u_1, u_2, \ldots, u_N \) an arbitrary sequence of numbers all distincts and strictly positive, if \( \mathfrak{R}(s/2) > -N \)

\[
\mathbb{E} \left[ \left( \sum_{n=1}^{N} \frac{\varepsilon_n}{u_n} \right)^{s/2} \right] = \Gamma \left( 1 + \frac{s}{2} \right) \sum_{n=1}^{N} \frac{1}{u_n^{s/2}} \prod_{1 \leq j \leq N, j \neq n} \frac{u_j}{u_j - u_n}.
\]

### 3.2 A family of joint density probability

Let \( \mathbf{u} \in \mathbb{R}^N \) and \( \mathbf{x} \in \mathbb{R}^{N-1} \) be two interlacing vectors in the sense that they lie in the region \( \mathcal{X}_N' \) defined as

\[ \mathcal{X}_N' = \{ 0 < u_1 < x_1 < u_2 < \ldots < u_{N-1} < x_{N-1} < u_N \}. \]

Let \( g \) denote a positive function over \( \mathbb{R}^+ \) to be precised hereafter. We define a joint density probability over \( \mathcal{X}_N' \) by putting

\[
f_{\mathbf{x}, \mathbf{u}}(\mathbf{x}, \mathbf{u}) = \frac{(N - 1)! N_1}{Z_N} V_N^{(0)}(\mathbf{x}) \prod_{n=1}^{N} g(u_n) \]

\[
= \mathcal{D}_{N-1}(\mathbf{x}; \mathbf{1}_{N-1}, \mathbf{u}) \frac{N_1}{Z_N} \left( V_N^{(0)}(\mathbf{u}) \right)^2 \prod_{n=1}^{N} g(u_n). \tag{14}
\]

It is quite evident that if \( (\mathbf{X}, \mathbf{U}) \) are two random interlacing vectors with such distribution, then the distribution of \( \mathbf{X} \) conditional to \( \mathbf{U} = \mathbf{u} \) is a Dixon-Anderson random vector of density \( \mathcal{D}_{N-1}(\mathbf{x}; \mathbf{1}_{N-1}, \mathbf{u}) \). The marginal distributions of \( \mathbf{X} \) and \( \mathbf{U} \) are obtained by integrating the probability density function (13) with respect to \( \mathbf{u} \) and \( \mathbf{x} \) respectively. Integrating with respect to \( \mathbf{x} \), we find that the density of \( \mathbf{U} \) is

\[
f_{\mathbf{U}}(\mathbf{u}) = \frac{N_1}{Z_N} \left( V_N^{(0)}(\mathbf{u}) \right)^2 \prod_{n=1}^{N} g(u_n). \tag{15}
\]

over the domain \( \mathcal{U}_N = \{ 0 < u_1 < u_2 < \ldots < u_N \} \). Note that, as \( f_{\mathbf{U}} \) is invariant under permutation, we have

\[
Z_N = \int_{0}^{\infty} du_1 \ldots \int_{0}^{\infty} du_n \left( V_N^{(0)}(\mathbf{u}) \right)^2 \prod_{n=1}^{N} g(u_n)
\]
assuming the integral exists. Thus if \((\mathbf{X}, \mathbf{U})\) are random vectors with joint probability density function (13) and \(s \neq -2, -4, \ldots, -2(N - 1)\) then it follows from identity given in (13) that

\[
\frac{2}{s \Gamma_{N-1} \left( \frac{s}{2} \right)} N^{s/2} \mathbb{E} \left[ \prod_{n=1}^{N} \frac{X_n^{s/2}}{U_n^{s/2}} \right] = \mathbb{E} \left[ \frac{V_N^{(s/2)}(\mathbf{U})}{V_N^{(0)}(\mathbf{U})} \right]
\]

assuming again that the expectations involved in this equality exist and are finite.

There is two obvious choices for \(g\) allowing us to compute these expectations: the Jacobi ensemble and the Laguerre ensemble.

### 3.2.1 The Jacobi Ensemble

We set \(g(u) = u^{a-1}(1-u)^{b-1} \mu_{0,1}(u)\) with \(a, b > 0\). In this case, the distribution of \(\mathbf{U}\) conditional to \(\mathbf{X} = \mathbf{x}\) is a Dixon-Anderson random vector of density \(D_N(\mathbf{u}; (a, 1_{N-1}, b), (0, \mathbf{x}, 1))\) and the marginal distribution of \(\mathbf{U}\) is a Selberg density \(S_N(\mathbf{u}; a, b, \lambda)\) (see [4]) with

\[
S_N(\mathbf{u}; a, b, \lambda) = \frac{N!}{S_N(a, b, \lambda)} \left( V_N^{(0)}(\mathbf{u}) \right)^{2 \lambda} \prod_{n=1}^{N} u_n^{a-1}(1-u_n)^{b-1}
\]

when supported on \(\mathcal{U}_N = \{0 < u_1 < u_2 < \ldots < u_N < 1\}\). \(S_N(a, b, \lambda)\) denotes the Selberg’s integral formula (see [1], chapitre 8). We choose the definition given in this reference rather than the one given in [4]). We have thus

\[
Z_N = S_N(a, b, 1) = \prod_{n=0}^{N-1} \frac{\Gamma(a + n) \Gamma(b + n) \Gamma(2 + n)}{\Gamma(a + b - 1 + N + n)}
\]

Integrating (13) with respect to \(\mathbf{u}\) gives the marginal density of \(\mathbf{X}\)

\[
f_{\mathbf{X}}(\mathbf{x}; a, b) = \frac{(N-1)! N!}{S_N(a, b, 1) \Gamma(a + b - 1 + N)} \left( \frac{V_N^{(0)}(\mathbf{x})}{V_N^{(0)}(\mathbf{X})} \right)^{2 \lambda} \prod_{n=1}^{N} x_n^{a-1}(1-x_n)^{b}
\]

\[
= \frac{(N-1)!}{S_{N-1}(a + 1, b + 1, 1)} \left( V_N^{(0)}(\mathbf{x}) \right)^{2 \lambda} \prod_{n=1}^{N} x_n^{a-1}(1-x_n)^{b}
\]

i.e. the marginal density of \(\mathbf{X}\) is the Selberg density \(S_{N-1}(\mathbf{x}; a + 1, b + 1, 1)\) supported over \(\mathcal{X}_{N-1} = \{0 < x_1 < x_2 < \ldots < x_{N-1} < 1\}\).

### 3.2.2 The Laguerre Ensemble

We set now \(g(u) = u^{a-1} e^{-u/b} \mu_{0,1}(u)\) with \(a, b > 0\). The joint density of \((\mathbf{X}, \mathbf{U})\) can be obtained as a limit of the Jacobi ensemble case by changing variables \(u_n = v_n/L, x_n = y_n/L\), replacing \(b - 1\) by \(L/\theta\) and by taking the limit \(L \to \infty\). We have in this case

\[
Z_N = W_N(a, \theta) = \lim_{L \to \infty} \frac{S_N(a, L/\theta + 1, 1)}{L^{(a+N)N}} = \theta^{(a+N)N} \prod_{n=0}^{N-1} \Gamma(a + n) \Gamma(2 + n).
\]

The marginal distribution of \(\mathbf{U}\) is a Laguerre density

\[
L(\mathbf{u}; a, \theta) = \frac{N!}{W_N(a, \theta)} \left( V_N^{(0)}(\mathbf{u}) \right)^{2 \lambda} \prod_{n=1}^{N} u_n^{a-1} e^{-u_n/\theta}
\]
supported on \( U_N = \{0 < u_1 < u_2 < \ldots < u_N < 1\} \). Integrating (14) with respect to \( u \) gives the marginal density of \( X \)

\[
f_X(x;a,\theta) = \frac{(N-1)!N^a}{W_N(a,\theta)}g^a+N\Gamma(a) \left(V_{N-1}^{(0)}(x)\right)^2 \prod_{n=1}^{N-1} x_n^a e^{-x_n/\theta}
\]

\[
= \frac{(N-1)!}{W_{N-1}(a+1,\theta)} \left(V_{N-1}^{(0)}(x)\right)^2 \prod_{n=1}^{N-1} x_n^a e^{-x_n/\theta}
\]

(19)

i.e. the marginal density of \( X \) is a Laguerre density \( L_{N-1}(x; a + 1, \theta) \).

3.3 Main Result

**Theorem 6.** Let \( a, b, \theta > 0 \) and \( \Re(a - s/2) > 0 \) and let \( U \) be a random vector of \( \mathbb{R}^N \). If the distribution of \( U \) is the Selberg density \( S_N(u; a, b, 1) \) supported on \( U_N = \{0 < u_1 < u_2 < \ldots < u_N < 1\} \) then

\[
\mathbb{E} \left[ \frac{V_N^{(s/2)}(U)}{V_N^{(0)}(U)} \right] = \frac{2N^{s/2}}{s\Gamma_N-1(1/2)} \frac{\Gamma(a - \frac{s}{2} + b - 1 + N)}{\Gamma(a - \frac{s}{2} + b - 1 + N)} \frac{\Gamma(a)}{\Gamma(a)}.
\]

If the distribution of \( U \) is the Laguerre ensemble density \( L(u; a, \theta) \) supported on \( U'_N = \{0 < u_1 < u_2 < \ldots < u_N\} \) then

\[
\mathbb{E} \left[ \frac{V_N^{(s/2)}(U)}{V_N^{(0)}(U)} \right] = \frac{2N^{s/2}}{s\Gamma_N-1(1/2)} \frac{\Gamma(a - \frac{s}{2} + b - 1 + N)}{\Gamma(a - \frac{s}{2} + b - 1 + N)} \frac{\Gamma(a)}{\Gamma(a)}.
\]

**Proof.** We have

\[
\mathbb{E} \left[ \frac{V_N^{(s/2)}(U)}{V_N^{(0)}(U)} \right] = \frac{2N^{s/2}}{s\Gamma_N-1(1/2)} \frac{\Gamma(a - \frac{s}{2} + b - 1 + N)}{\Gamma(a - \frac{s}{2} + b - 1 + N)} \frac{\Gamma(a)}{\Gamma(a)}.
\]

We integrate with respect to \( u \). In the Jacobi ensemble case, we get

\[
\mathbb{E} \left[ \frac{V_N^{(s/2)}(U)}{V_N^{(0)}(U)} \right] = \frac{2N^{s/2}}{s\Gamma_N-1(1/2)} \frac{\Gamma(a - \frac{s}{2} + b - 1 + N)}{\Gamma(a - \frac{s}{2} + b - 1 + N)} \frac{\Gamma(a)}{\Gamma(a)}.
\]

Finally, we have the following corollary

**Corollary 7.** Let \( a, b, \theta > 0 \) and \( \Re(a - s/2) > 0 \). Then

\[
\int_0^1 du_1 \ldots \int_0^1 du_n V_N^{(s/2)}(u)V_N^{(0)}(u) \prod_{n=1}^{N} u_n^{a-1}(1-u_n)^{b-1}
\]

\[
= \frac{2N^{s/2}S_N(a,b,1)}{s\Gamma_N-1(1/2)} \frac{\Gamma(a + b - 1 + N)}{\Gamma(a - \frac{s}{2} + b - 1 + N)} \frac{\Gamma(a - \frac{s}{2} + b - 1 + N)}{\Gamma(a)}
\]
\[ \int_0^\infty du_1 \ldots \int_0^\infty du_n V_{N}^{(s/2)}(u) V_{N}^{(0)}(u) \prod_{n=1}^{N} u_n^{a-1} e^{-u_n/\theta} = \prod_{n=0}^{N-1} \frac{\Gamma(a+n)\Gamma(2+n) 2^{(a+N)N}}{s^{N} \Gamma_N(\theta)}} ^{s/2} \frac{\Gamma(a-\frac{aN}{2})}{\Gamma(a)} .\]

4 Conclusion

It is well-known, even if it is not well understood, that there is a connection between the random matrix theory and the Zeta function. For example, Keating and its co-authors [5] successfully use the characteristic polynomials \( Z(U, \theta) \) of matrices \( U \) in the Circular Unitary Ensemble (CUE) to study the behavior of the correctly renormalized integral

\[ \int_0^T |\zeta(1/2 + it)|^2 dt \]

Our work seems to be a first step in explaining this connection. The Selberg integral plays a fundamental role in the theory of the various \( \beta \)-ensembles (see [3]) and the Dixon-Anderson probability distribution function is an intermediate step to the Selberg’s integral evaluation.

We hope that the results presented in this article will pave the way for a deeper understanding of the links between these two fields.

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A  Partial fraction decomposition

Let $Q$ be a polynomial of degree $N$ such that $Q(x) = \prod_{n=1}^{N} (x - x_n)$ with $x_i \neq x_j$ if $i \neq j$ and let $L_n(x)$, $n = 1, \ldots N$ denote the Lagrange's polynomials

$$L_n(x) = \prod_{j=1, j \neq n}^{N} \frac{x - x_j}{x_n - x_j}.$$ 

If $P$ is a polynomial of degree strictly less than $N$ then by the Lagrange interpolation formula, it can be written as

$$P(x) = \sum_{n=1}^{N} P(x_n) L_n(x).$$ 

From this, we deduce the partial fraction expansion of $P/Q$

$$\frac{P(x)}{Q(x)} = \sum_{n=1}^{N} \frac{P(x_n)}{Q(x)} L_n(x) = \sum_{n=1}^{N} \frac{c_n}{x - x_n}$$

with

$$c_n = \frac{P(x_n)}{\prod_{j=1, j \neq n}^{N} (x_n - x_j)} = \frac{P(x_n)}{Q'(x_n)}.$$ 

B  Inverse of the Vandermonde’s matrix

The proof of this formula can be found at https://proofwiki.org/wiki/Inverse_of_Vandermonde_Matrix and is proposed as Exercise 40 from section 1.2.3 in [6]. Consider the Vandermonde’s matrix

$$V_N(x_1, \ldots, x_N) = \begin{pmatrix}
1 & x_1 & x_1^2 & \cdots & x_1^{N-1} \\
1 & x_2 & x_2^2 & \cdots & x_2^{N-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & x_N & x_N^2 & \cdots & x_N^{N-1}
\end{pmatrix}$$ (20)

Assume $x_i \neq x_j$ if $i \neq j$, then the Vandermonde Determinant of $V_N$ is

$$\det (V_N(x_1, \ldots, x_N)) = \prod_{1 \leq i < j \leq N} (x_j - x_i) \neq 0.$$ 

Since this is non-zero, the inverse matrix, denoted $W_N = [w_{ij}]$, is guaranteed to exist. Using the definition of the matrix product and the inverse matrix

$$\sum_{k=1}^{N} x_i^{k-1} w_{kj} = \delta_{ij},$$

For $1 \leq n \leq N$, if $P_n(x)$ is the polynomial

$$P_n(x) := \sum_{k=1}^{N} w_{kn} x^{k-1}$$

then $P_n(x_1) = 0, \ldots, P_n(x_{n-1}) = 0, P_n(x_n) = 1, P_n(x_{n+1}) = 0, \ldots, P_n(x_N) = 0$. By the Lagrange’s interpolation formula, the $n$th column of $W_N$ is composed of the coefficients of the $n$th Lagrange basis polynomial
\[ P_n(x) = \sum_{k=1}^{N} w_{kn} x^{k-1} = \prod_{1 \leq j \leq N, j \neq n} \frac{x - x_j}{x - x_n}. \]

We can identify the terms \( w_{ij} \) by expanding the product. In particular, setting \( x = 0 \), we get that the constant of the polynomials are

\[ w_{1n} = \prod_{1 \leq j \leq N, j \neq n} \frac{-x_j}{x_n - x_j} = \prod_{1 \leq j \leq N, j \neq n} \frac{x_j}{x_n - x_j}, \quad n = 1, \ldots, N. \]

\[ \eta_N(s) = \frac{1}{2} \begin{bmatrix} 1 + \frac{\lambda_{2,N}}{\lambda_{1,N}} & -1 & 0 & \ldots & 0 \\ -\frac{\lambda_{2,N}}{\lambda_{1,N}} & 1 + \frac{\lambda_{3,N}}{\lambda_{2,N}} & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & 1 + \frac{\lambda_{N-2,N}}{\lambda_{N-1,N}} & -1 \\ 0 & \ldots & 0 & \frac{\lambda_{N-1,N}}{\lambda_{N,N}} & 1 + \frac{\lambda_{N-1,N}}{\lambda_{N,N}} \end{bmatrix} \]

with \( \lambda_{n,N}^{-1} = 2a_{n,N}(n^{-s} - 1) \).

**Proposition 8.** If \( s \neq 0 \), then

\[ \eta_N(s) = \frac{1}{2} \prod_{n=2}^{N} 2a_{n,N}(n^{-s} - 1) \begin{bmatrix} 1 + \frac{2a_{2,N}(2^{-s} - 1)}{1} & 1 & \ldots & 1 \\ 1 & 1 + \frac{2a_{3,N}(3^{-s} - 1)}{1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & 1 & \ldots & 1 + \frac{2a_{N,N}(N^{-s} - 1)}{1} \end{bmatrix}. \]
Setting $\lambda_{n,N}^{-1} = 2a_{n,N}(n^{-s} - 1)$ and using Gaussian elimination method we get

$$\eta_N(s) = \frac{1}{2} \prod_{n=2}^{N} \lambda_n^{-1} \begin{bmatrix} 1 + \lambda_{2,N} & -\lambda_{2,N} & 0 & \ldots & \ldots & 0 \\ -\lambda_{2,N} & \lambda_{2,N} + \lambda_{3,N} & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & & \ddots & \ddots & \ddots & \ddots \\ 0 & \ldots & 0 & -\lambda_{N-1,N} & \lambda_{N-1,N} + \lambda_{N,N} \end{bmatrix},$$

ending the proof.

**Remark 5.** It is tempting to find a condition on the coefficients of the determinant (22) in order to find when it is zero. Lemma D.1 is equivalent to $\eta(s) = 0$ if and only if $2 \sum_{n=2}^{N} a_{n,N}(n^{-s} - 1) = -1$. As $\sum_{n=1}^{N} a_{n,N} = 1/2$, this equality can be rewritten $\sum_{n=2}^{N} a_{n,N} - s = -a_{1,N}$ which is not really helpful!

Assume $s \neq 0$, then, setting $\Delta_{0,N} = 0, \Delta_{1,N} = 1$, the determinant appearing in equation (21) can be computed using the recurrence relation

$$\Delta_{n,N} = \left( 1 + \frac{\lambda_{n-1,N}}{\lambda_{n,N}} \right) \Delta_{n-1,N} - \frac{\lambda_{n-1,N}}{\lambda_{n,N}} \Delta_{n-2,N}, \quad n = 2, \ldots, N \quad \text{(D)}$$

with the convention $\lambda_{1,N} = 1$.

**Proposition 9.** Let

$$\beta_{2,N} = \frac{1}{\lambda_{2,N}} \quad \text{and} \quad \beta_{n,N} = \frac{\lambda_{n-1,N}}{\lambda_{n,N}}, \quad n = 3, \ldots, N$$

and let $\tilde{\Delta}_{n,N}$ denote the solution of the recurrence equation (D) with initial conditions $\tilde{\Delta}_{0,N} = 1$ and $\tilde{\Delta}_{1,N} = 0$. Then (see [?], p. 15 for the continued fraction representation) for $n = 2, \ldots, N$, we have

$$\Delta_{n,N} = 1 + \sum_{k=0}^{n-2} \beta_{n-k} \ldots \beta_2 = 1 + \sum_{k=2}^{n} \lambda_{k,N}^{-1}, \quad n = 2, \ldots, N,$$

and

$$\tilde{\Delta}_{n,N} = -\sum_{k=0}^{n-2} \beta_{n-k} \ldots \beta_2 = -\sum_{k=2}^{n} \lambda_{k,N}^{-1}, \quad n = 2, \ldots, N.$$

Thus $\Delta_{N,N} = 2\eta_N(s), \tilde{\Delta}_{N,N} = 1 - 2\eta_N(s)$ and

$$\frac{1}{2\eta_N(s)} = 1 + \frac{1 - \beta_{2,N} + \beta_{3,N} + \ldots + 1 - \beta_{N-1,N}}{\beta_{2,N} + \beta_{3,N} + \ldots + \beta_{N,N}}.$$
Proof. \(|A| = 0\) if and only if there exists \(K\) numbers \((\alpha_1, \ldots, \alpha_N)\) such that for some \(i\), \(\alpha_i \neq 0\) and such that \(\sum_{j=1}^{N} \alpha_j A^j = 0\), where \(A^j\) denotes the \(j\)th column of \(A\). Thus we have

\[
\alpha_i \lambda_i = -\sum_{j=1}^{K} \alpha_j, \quad i = 1, \ldots, K.
\]

By assumption, all \(\lambda_i\) are different from zero and for some \(i \in \{1, \ldots, K\}\) there exists \(\alpha_i \neq 0\). This implies first that \(\sum_{j=1}^{K} \alpha_j \neq 0\) and second that in fact all \(\alpha_j\) are different from 0. We get the identity

\[
\frac{-\alpha_i}{\sum_{j=1}^{K} \alpha_j} = \frac{1}{\lambda_i}, \quad i = 1, \ldots, K.
\]

Summing we obtain the announced result.

**Lemma D.2.** Let \(A\) is a \(K \times K\) matrix with coefficients \(a_{ij} = 1\) if \(i \neq j\) and \(a_{ii} = 1 + \lambda_i\) with \(\lambda_i \neq 0\) otherwise. If \(S = 1 + \sum_{i=1}^{K} \lambda_i^{-1} \neq 0\) then \(B = A^{-1}\) exists with coefficients \(b_{ij} = -\frac{1}{S \lambda_i \lambda_j}\) if \(i \neq j\) and \(b_{ii} = \frac{1}{\lambda_i} - \frac{1}{S \lambda_i^2}\) otherwise.

Proof. Let \(S_j = \sum_{i=1}^{K} b_{ij}\), if \(B^j\) denotes the \(j\)th column of \(B\), identity \(AB^j = e_j\) show that

\[
\begin{cases}
S_j + \lambda_i b_{ij} = 0 & \text{if } i \neq j \\
S_j + \lambda_j b_{jj} = 1 & \text{otherwise}.
\end{cases}
\]

Thus \(b_{ij} = -S_j/\lambda_i\) if \(i \neq j\) and \(b_{jj} = (1 - S_j)/\lambda_j\). Summing and equating, we find that \(S_j = 1/(S \lambda_j)\) giving the announced result.