SHARP ESTIMATES OF TRANSITION PROBABILITY DENSITY FOR BESSEL PROCESS IN HALF-LINE

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Abstract. In this paper we study the Bessel process $R^{(\mu)}_t$ with index $\mu \neq 0$ starting from $x > 0$ and killed when it reaches a positive level $a$, where $x > a > 0$. We provide sharp estimates of the transition probability density $p^{(\mu)}_a(t, x, y)$ for the whole range of space parameters $x, y > a$ and every $t > 0$.

1. Introduction

Let $R^{(\mu)}_t$ be the Bessel process with index $\mu \neq 0$. The transition probability density (with respect to the Lebesgue measure) of the process is expressed by the modified Bessel function in the following way

$$p^{(\mu)}(t, x, y) = \frac{1}{t} \left( \frac{y}{x} \right)^\mu \exp \left( -\frac{x^2 + y^2}{2t} \right) I_{|\mu|} \left( \frac{xy}{t} \right), \quad x, y, t > 0. \quad (1.1)$$

Our main goal is to describe behaviour of densities of the transition probabilities for the process $R^{(\mu)}_t$ killed when it leaves a half-line $(a, \infty)$, where $a > 0$. Note that if the process starts from $x > a$ then the first hitting time $T^{(\mu)}_a$ of a level $a$ is finite a.s. when $\mu < 0$ but it is infinite with positive probability when $\mu > 0$. The density kernel of the killed semi-group is given by the Hunt formula

$$p^{(\mu)}_a(t, x, y) = p^{(\mu)}(t, x, y) - E^{(\mu)}_x \left[ t > T^{(\mu)}_a; p^{(\mu)}(t - T^{(\mu)}_a, R^{(\mu)}_{T^{(\mu)}_a}, y) \right], \quad (1.2)$$

where $x, y > a$ and $t > 0$. The main result of the paper is given in

Theorem 1. Let $\mu \neq 0$ and $a > 0$. For every $x, y > a$ and $t > 0$ we have

$$p^{(\mu)}_a(t, x, y) \approx \left[ \begin{array}{c} 1 \wedge \frac{(x-a)(y-a)}{t} \\ \frac{xy}{t} \end{array} \right] \left( \begin{array}{c} x^2 + y^2 \\ 2 \end{array} \right)^{\mu+\frac{1}{2}} \frac{1}{\sqrt{t}} \exp \left( -\frac{(x-y)^2}{2t} \right). \quad (1.3)$$

Here $f(t, x, y) \approx g(t, x, y)$ means that there exist positive constants $c_1$ and $c_2$ depending only on the index $\mu$ such that $c_1 \leq f/g \leq c_2$ for every $x, y > a$ and $t > 0$. Since the constants are independent of $a > 0$, one can pass to the limit with $a \to 0^+$ and obtain the well-known estimates of $p^{(\mu)}(t, x, y)$. Since the function $I_{|\mu|}(z)$ behaves as a power function at zero and that some exponential term appears in the asymptotic expansion at infinity (see Preliminaries for the details), the behaviour of $p^{(\mu)}(t, x, y)$ depends on the ratio $xy/t$. Note that similar situation takes place in the case of $p^{(\mu)}_a(t, x, y)$, which depends on $xy/t$ as well. It can be especially seen in the proof of Theorem 1 where different methods and arguments are applied to obtain estimates (1.3), whenever $xy/t$ is large or small. Finally, taking into account the behaviour of $p^{(\mu)}(t, x, y)$, one can rewrite the statement

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of Theorem 1 in the following way

\[ \frac{p_\mu(t, x, y)}{p(t, x, y)} \approx \left( 1 \wedge \frac{(x-a)(y-a)}{t} \right) \left( 1 \lor \frac{t}{xy} \right), \quad x, y > a, \quad t > 0, \quad (1.4) \]

where the expression on the right-hand side of (1.4) should be read as the description of the behaviour of \( p_\mu(t, x, y) \) near the boundary \( a \).

There are several ways to define the function \( p_\mu(t, x, y) \) hence our result and its applications can be considered from different points of view. It seems to be the most classical approach to define the heat kernel \( p_\mu(t, x, y) \) as the fundamental solution of the heat equation \((\partial_t - L^{(\mu)}) u = 0\), where \( L^{(\mu)} \) is the Bessel differential operator. In the most classical case, i.e. when the operator \( L^{(\mu)} \) is replaced by the classical Laplacian, the problem of finding description of the heat kernel has a very long history (see for example [18] and the references within) and goes back to 1980s and the works of E.B. Davies (see [7], [4], [5], [6]). However, the known results for Dirichlet Laplacian on the subsets of \( \mathbb{R}^n \) (see [19]) or in general on Riemannian manifolds (see [18] for the references) are only qualitatively sharp, i.e. the constants appearing in the exponential terms in the upper and lower estimates are different. Note that in our result these constants are the same and consequently, the exponential behaviour of the density is very precise. Such sharp estimates seems to be very rare.

Note also that the operator \( L^{(\mu)} \) plays an important rôle in harmonic analysis. However, since the set \((a, \infty)\) is unbounded, our consideration corresponds to the case when the spectrum is continuous. This operator on the set \((0, 1)\) and the estimates of the corresponding Fourier-Bessel heat kernel were studied recently in [16] and [17], but once again the results presented there are only qualitatively sharp, i.e. the estimates are not sharp whenever \(|x-y|^2 > t\). Another essential difference between the case of bounded sets and our case is that in the first one, we can limit our considerations to \( t \leq 1 \), by the application of the intrinsic ultracontractivity. However, the most interesting part of Theorem 1 (with difficult proof) seems to be when \( t \) is large.

The third and our principal motivation comes from the theory of stochastic processes and the interpretation of \( p_\mu(t, x, y) \) as a transition density function of the killed semigroup related to the Bessel process \( R^{(\mu)}_t \). From this point of view, the present work is a natural continuation of the research started in [3] (see also [1]), where the integral representation of the density \( q^(\mu)(x) \) of \( T^{(\mu)}_a \) were provided together with its some asymptotics description. The sharp estimates of the density for the whole range of parameters with the explicit description of the exponential behaviour was given in [2]. For the in-depth analysis of the asymptotic behaviour of \( q^{(\mu)}(x) \) see [12], [11], [10].

The case \( \mu = 0 \) is excluded from our consideration and it will be addressed in the subsequent work. As it is very common in this theory, this case requires different methods and should be considered separately. In particular, some logarithmic behavior is expected whenever \( xy < t \).

The paper is organized as follows. In Preliminaries we introduce some basic notation and recall properties and known results related to modified Bessel functions as well as Bessel processes, which are used in the sequel. In particular, using scaling property and absolute continuity of the Bessel processes we reduced our consideration only to the case \( \mu > 0 \) and \( a = 1 \). After that we turn to the proof of Theorem 1 which is split into two main parts, i.e. in Section 3 we provide estimates whenever \( xy/t \) is large and in Section 4 we prove (1.3) for \( xy/t \) small. In both cases the result is given in series of propositions.
2. Preliminaries

2.1. Notation. The constants depending on the index \( \mu \) and appearing in theorems and propositions are denoted by capitals letters \( C_1^{(\mu)}, C_2^{(\mu)}, \ldots \). We will denote by \( c_1, c_2, \ldots \) constants appearing in the proofs and to shorten the notation we will omit the superscript \( (\mu) \), however we will emphasize the dependence on the other variables, if such occurs.

2.2. Modified Bessel function. The modified Bessel function of the first kind is defined as (see [8] 7.2.2 (12))

\[
I_{\mu}(z) = \sum_{k=0}^{\infty} \left( \frac{z}{2} \right)^{\mu+2k} \frac{1}{k!\Gamma(k+\mu+1)}, \quad z > 0, \quad \mu > -1.
\]

It is well-known that whenever \( z \) is real the function is a positive increasing real function. Moreover, by the differentiation formula (see [8] 7.11 (20))

\[
\frac{d}{dz} \left( \frac{I_{\mu}(z)}{z^{\mu}} \right) = \frac{I_{\mu+1}(z)}{z^{\mu}}, \quad z > 0
\]

and positivity of the right-hand side of (2.1) we obtain that \( z \rightarrow z^{-\mu}I_{\mu}(z) \) is also increasing.

The asymptotic behavior of \( I_{\mu}(z) \) at zero follows immediately from the series representation of \( I_{\mu}(z) \)

\[
I_{\mu}(z) = \left( \frac{z}{2} \right)^{\mu} \frac{1}{\Gamma(\mu+1)} + O(z^{\mu+2}), \quad z \rightarrow 0^+,
\]

where the behaviour at infinity is given by (see [8] 7.13.1 (5))

\[
I_{\mu}(z) \sim \frac{e^{z}}{\sqrt{2\pi z}} (1 + O(1/z)), \quad z \rightarrow \infty.
\]

Some parts of the proof strongly depends on the estimates of the ratio of two modified Bessel functions with different arguments. Here we recall the results of Laforgia given in Theorem 2.1 in [13]. For every \( \mu > -1/2 \) we have

\[
\frac{I_{\mu}(y)}{I_{\mu}(x)} < \left( \frac{y}{x} \right)^{\mu} e^{y-x}, \quad y \geq x > 0.
\]

Moreover, whenever \( \mu \geq 1/2 \), the lower bound of similar type holds, i.e. we have

\[
\frac{I_{\mu}(y)}{I_{\mu}(x)} \geq \left( \frac{x}{y} \right)^{\mu} e^{y-x}, \quad y \geq x > 0.
\]

2.3. Bessel processes. In this section we introduce basic properties of Bessel processes. We follow the notation presented in [14] and [15], where we refer the reader for more details.

We write \( P_{x}^{(\mu)} \) and \( P_{x}^{(\mu)} \) for the probability law and the corresponding expected value of a Bessel process \( R_{t}^{(\mu)} \) with an index \( \mu \in \mathbb{R} \) on the canonical path space with starting point \( R_0 = x > 0 \). The filtration of the coordinate process is denoted by \( \mathcal{F}_t^{(\mu)} = \sigma\{R_s^{(\mu)} : s \leq t\} \). The laws of Bessel processes with different indices are absolutely continuous and the corresponding Radon-Nikodym derivative is described by

\[
\frac{dP_{x}^{(\mu)}}{dP_{x}^{(\nu)}} \left|_{\mathcal{F}_t} \right. = \left( \frac{w(t)}{x} \right)^{\mu-\nu} \exp \left( -\frac{\mu^2 - \nu^2}{2} \int_0^t \frac{ds}{w^2(s)} \right),
\]

where \( x > 0, \mu, \nu \in \mathbb{R} \) and the above given formula holds \( P_{x}^{(\nu)}\)-a.s on \( \{T_{0}^{(\nu)} > t\} \). Here \( T_{0}^{(\mu)} \) denotes the first hitting time of 0 by \( R_{t}^{(\mu)} \). The behaviour of \( R_{t}^{(\mu)} \) at zero depends
on $\mu$. Since we are interested in a Bessel process in a half-line $(a, \infty)$, for a given strictly positive $a$, the boundary condition at zero is irrelevant from our point of view. However, for completeness of the exposure we impose killing condition at zero for $-1 < \mu < 0$, i.e. in the situation when 0 is non-singular. Then the density of the transition probability (with respect to the Lebesgue measure) is given by (1.1).

For $x > 0$ we define the first hitting of a given level $a > 0$ by

$$T_a^{(\mu)} = \inf\{t > 0 : R_t^{(\mu)} = a\}.$$ 

Notice that for $\mu \geq 0$ we have $T_a^{(\mu)} < \infty$ a.s., but for $\mu < 0$ the variable $T_a^{(\mu)}$ is infinite with positive probability. We denote by $q^{(\mu)}_{x,a}(s)$ the density function of $T_a^{(\mu)}$. The sharp estimates of $q^{(\mu)}_{x,a}(s)$ were obtained in [2]. We recall this result for $a = 1$, which implies the result for every $a > 0$, due to the scaling property of Bessel processes. More precisely, it was shown that for every $x > 1$ and $t > 0$ we have

$$q^{(\mu)}_{x,1}(s) \approx (x - 1) \left( 1 \wedge \frac{1}{x^{2\mu}} \right) \frac{e^{-(x-1)^2/2t}}{t^{3/2}} \frac{x^{2|\mu|-1}}{t^{|\mu|} + x^{|\mu|} - 1}, \quad \mu \neq 0. \quad (2.7)$$

The above-given bounds imply the description of the survival probabilities (see Theorem 10 in [2])

$$P_x^{(\mu)}(T_1^{(\mu)} > t) \approx \frac{x - 1}{\sqrt{x} + t + x - 1} \frac{1}{t^\mu + x^{2\mu}}, \quad x > 1, \quad t > 0. \quad (2.8)$$

The main object of our study is the density of the transitions probabilities for the Bessel process starting from $x > a$ killed at time $T_a^{(\mu)}$. Taking into account the Hunt formula (1.2) and the fact that continuity of the paths implies $R_t^{(\mu)} = a$ a.s., we can represent $p_a^{(\mu)}(t, x, y)$ in terms of $p^{(\mu)}(t, x, y)$ and $q^{(\mu)}_{x,a}(s)$ in the following way

$$p_a^{(\mu)}(t, x, y) = p^{(\mu)}(t, x, y) - r_a^{(\mu)}(t, x, y) = p^{(\mu)}(t, x, y) - \int_0^t p^{(\mu)}(t - s, a, y)q^{(\mu)}_{x,a}(s)ds. \quad (2.9)$$

The scaling property of a Bessel process together with (2.10) imply that

$$p_a^{(\mu)}(t, x, y) = \frac{1}{a^{2\mu}} p_1^{(\mu)}(t/a^2, x/a, y/a), \quad x, y > a, \quad t > 0. \quad (2.11)$$

Moreover, the absolute continuity property (2.6) applied for $\mu > 0$ and $\nu = -\mu$ gives

$$p_1^{(-\mu)}(t, x, y) = \left( \frac{x}{y} \right)^{2\mu} p_1^{(\mu)}(t, x, y), \quad x, y > 1, \quad t > 0. \quad (2.12)$$

These two properties show that it is enough to prove Theorem 11 only for $a = 1$ and $\mu > 0$. To shorten the notation we will write $q^{(\mu)}_x(s) = q^{(\mu)}_{x,1}(s)$. Since we consider the densities with respect to the Lebesgue measure (not with respect to the speed measure $m(dx) = 2x^2\mu+1dx$) the symmetry property of $p_1^{(\mu)}(t, x, y)$ in this case reads as follows:

$$p_1^{(\mu)}(t, x, y) = \left( \frac{y}{x} \right)^{2\mu+1} p_1^{(\mu)}(t, y, x), \quad x, y > 1, \quad t > 0. \quad (2.12)$$

Finally, for $\mu = 1/2$ one can compute $p_1^{(\mu)}(t, x, y)$ explicitly from (2.10), by using $I_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sinh(z)$ and the fact that $q^{(1/2)}_x(s)$ is a density of 1/2-stable subordinator. More
Making the substitution

\[ w = \frac{x-1}{x} \sqrt{2\pi t^3} \exp \left( -\frac{(x-1)^2}{2t} \right), \]

we obtain

\[ r_1^{(1/2)}(t, x, y) = \int_0^t q_x^{(1/2)}(s)p^{(1/2)}(t-s, 1, y)ds \]

\[ = \frac{x-1}{2\pi} \left( H(t, (x-1)^2, (y-1)^2) - H(t, (x-1)^2, (y+1)^2) \right), \]

where

\[ H(t, a, b) = \int_0^t \frac{1}{\sqrt{t-s}} \exp \left( -\frac{a}{2s} \right) \exp \left( -\frac{b}{2(t-s)} \right) ds, \quad a, b > 0. \]

Making the substitution \( w = 1/s - 1/t \) and using formula 3.471.15 in [9] we get

\[ H(t, a, b) = \frac{1}{\sqrt{t}} \exp \left( -\frac{a+b}{2t} \right) \int_0^\infty w^{-1/2} \exp \left( -\frac{a}{2w} - \frac{b}{2w} \right) dw \]

\[ = \sqrt{\frac{2\pi}{ta}} \exp \left( -\frac{(\sqrt{a} + \sqrt{b})^2}{2t} \right). \]

Hence we have

\[ r_1^{(1/2)}(t, x, y) = \frac{1}{\sqrt{2\pi t x}} \left[ \exp \left( -\frac{(x+y-2)^2}{2t} \right) - \exp \left( -\frac{(x+y)^2}{2t} \right) \right] \]

which together with (2.10) and (2.14) give

\[ p_1^{(1/2)}(t, x, y) = \frac{1}{\sqrt{2\pi t x}} \left( \exp \left( -\frac{(x-y)^2}{2t} \right) - \exp \left( -\frac{(x+y-2)^2}{2t} \right) \right). \]

One can also obtain this formula using the relation between 3-dimensional Bessel process (i.e. with index \( \mu = 1/2 \)) and 1-dimensional Brownian motion killed when leaving a positive half-line. Note also that

\[ p_1^{(1/2)}(t, x, y) \approx \left( 1 - \frac{(x-1)(y-1)}{t} \right) \frac{y}{x} \frac{1}{\sqrt{t}} \exp \left( -\frac{(x-y)^2}{2t} \right). \]

which is exactly (1.3) for \( \mu = 1/2 \).

We end this section providing very useful relation between densities \( q_x^{(\mu)}(t) \) with different indices, which once again follows from the absolute continuity property.

**Lemma 1.** For every \( x > 1 \) and \( t > 0 \) we have

\[ x^{\mu-1/2}q_x^{(\mu)}(t) \leq q_x^{(1/2)}(t) \leq x^{\nu-1/2}q_x^{(\nu)}(t), \]

whenever \( \nu \leq 1/2 \leq \mu \).

**Proof.** The second inequality in (2.19) was given in Lemma 4 in [2]. To deal with the right-hand side of (2.19) we use (2.6) to obtain for every \( \delta > 0 \) and \( 0 < \varepsilon \leq \delta^2/2 \wedge 1 \)

\[ x^{\mu-1/2}E_x^{(\mu)}[t - \varepsilon \leq T_1^{(\mu)} \leq t] \leq E_x^{(1/2)}[t - \varepsilon \leq T_1^{(1/2)} \leq t; (R_t)^{\mu-1/2}] \leq (1 + \delta)^{\mu-1/2} E_x^{(1/2)}[t - \varepsilon \leq T_1^{(1/2)} \leq t] + F_\varepsilon(x, t), \]
where, by Strong Markov property

\[ F_\varepsilon(x, t) = \mathbb{E}^{(1/2)}_{x}[t - \varepsilon \leq T^{(1/2)}_1 \leq t, R_t \geq 1 + \delta; (R_t)^{\mu-1/2}] \]

\[ = \mathbb{E}^{(1/2)}_{x}[t - \varepsilon \leq T^{(1/2)}_1 \leq t; \mathbb{E}^{(1/2)}_{x}[R_{t-T^{(1/2)}_1} \geq 1 + \delta; (R_{t-T^{(1/2)}_1})^{\mu-1/2}]] \]

\[ = \int_{t-\varepsilon}^t q^{(1/2)}(u, \int_{u+\delta}^\infty y^{\mu-1/2}p^{(1/2)}(t-u, 1, y) dy du. \]

By (2.14), for every \( r \in (0, \varepsilon) \) we have

\[ \int_{1+\delta}^\infty y^{\mu-1/2}p^{(1/2)}(r, 1, y) dy \leq \frac{1}{\sqrt{2\pi r}} \int_{1+\delta}^\infty \exp \left( -\frac{(y-1)^2}{2r} \right) y^{\mu+1/2} dy \]

\[ \leq \frac{1}{\sqrt{2\pi r}} \exp \left( -\frac{\delta^2}{4r} \right) \int_{1+\delta}^\infty \exp \left( -\frac{(y-1)^2}{4} \right) y^{\mu+1/2} dy \]

\[ \leq \frac{1}{\sqrt{2\pi \varepsilon}} \exp \left( -\frac{\delta^2}{4\varepsilon} \right) \int_{1+\delta}^\infty \exp \left( -\frac{(y-1)^2}{4} \right) y^{\mu+1/2} dy, \]

where the last inequality follows from \( \varepsilon \leq \delta^2/2 \). It implies that \( F_\varepsilon(t, x)/\varepsilon \) vanishes when \( \varepsilon \to 0 \). Consequently, dividing both sides of (2.20) by \( \varepsilon \) and taking a limit when \( \varepsilon \to 0 \), we arrive at

\[ x^{\mu-1/2}q^{(\mu)}_{x}(t) \leq (1 + \delta)^{\mu-1/2}q^{(1/2)}_{x}(t). \]

Since \( \delta \) was arbitrary, the proof is complete. \( \square \)

3. Estimates for \( xy/t \) Large

We begin this Section with the application of the absolute continuity property of Bessel processes and the formula (2.17) which give the upper bounds for \( \mu \geq 1/2 \) and lower bounds for \( \nu \leq 1/2 \). These bounds are sharp whenever \( xy \geq t \).

**Proposition 1.** Let \( \mu \geq 1/2 \geq \nu > 0 \). For every \( x, y > 1 \) and \( t > 0 \) we have

\[ \left( \frac{x}{y} \right)^{\mu-1/2} p^{(\mu)}_{1}(t, x, y) \leq p^{(1/2)}_{1}(t, x, y) \leq \left( \frac{x}{y} \right)^{\nu-1/2} p^{(\nu)}_{1}(t, x, y). \]  

(3.1)

**Proof.** From the absolute continuity property (2.6) we get that for every \( \mu \geq \nu > 0 \) and every Borel set \( A \subset (1, \infty) \) we have

\[ \int_{A} p^{(\mu)}_{1}(t, x, y) dy = \frac{1}{x^{\mu-\nu} \mathbb{E}^{(\nu)}_{x}[T^{(\nu)}_1 > t, R_t \in A; (R_t)^{\mu-\nu} \exp \left( -\frac{\mu^2 - \nu^2}{2} \int_{0}^{t} ds \frac{1}{R_s^2} \right)]] \]

\[ \leq \frac{1}{x^{\mu-\nu} \mathbb{E}^{(\nu)}_{x}[T^{(\nu)}_1 > t, R_t \in A; (R_t)^{\mu-\nu}] = \int_{A} \left( \frac{y}{x} \right)^{\mu-\nu} p^{(\nu)}_{1}(t, x, y) dy. \]

Hence

\[ p^{(\mu)}_{1}(t, x, y) \leq \left( \frac{y}{x} \right)^{\mu-\nu} p^{(\nu)}_{1}(t, x, y). \]

(3.2)

Taking \( \mu \geq 1/2 \) and \( \nu = 1/2 \) gives the left-hand side of (3.1) and taking \( \nu \leq 1/2 \) and \( \mu = 1/2 \) gives the right-hand side of (3.1). \( \square \)

The absolute continuity can also be used to show the estimates for small times \( t \) in a very similar way. Note that if \( t < 1 \) then we always have \( xy > t \). The proof of the main Theorem will be provided in subsequent propositions without the assumption that \( t \) is bounded, but we present this simple proof to show that for \( xy \geq t \) the estimates for small \( t \) are just an immediate consequence of the absolute continuity of Bessel processes.
Proposition 2. Let $\mu > 0$. For every $x, y > 1$ and $t \in (0, 1]$ we have

$$p_1^{(\mu)}(t, x, y)^{\mu} \approx \left(1 \wedge \frac{(x - 1)(y - 1)}{t}\right) \left(\frac{y}{x}\right)^{\mu + 1/2} \frac{1}{\sqrt{t}} \exp\left(-\frac{(x - y)^2}{2t}\right). \quad (3.3)$$

Proof. Let $\mu \geq \nu > 0$. Taking Borel set $A \subset (1, \infty)$ and $t \leq 1$ we have

$$\int_A p_1^{(\mu)}(t, x, y)dy = \frac{1}{x^{\mu - \nu} E_x} \left[T_1^{(\nu)} > t; R_t \in A; (R_t)^{\mu - \nu} \exp\left(-\frac{\mu^2 - \nu^2}{2t}\right)\right].$$

Since $\inf\{R_s : s < t\} > 1$ on $\{T_1^{(\nu)} > t\}$ we can write

$$\int_A p_1^{(\mu)}(t, x, y)dy \geq \frac{1}{x^{\mu - \nu} E_x} \left[T_1^{(\nu)} > t; R_t \in A; (R_t)^{\mu - \nu} \exp\left(-\frac{\mu^2 - \nu^2}{2t}\right)\right]$$

$$\geq \exp\left(-\frac{\mu^2 - \nu^2}{2}\right) \int_A \left(\frac{y}{x}\right)^{\mu - \nu} p_1^{(\nu)}(t, x, y)dy.$$

Hence we get

$$p_1^{(\mu)}(t, x, y) \geq \exp\left(-\frac{\mu^2 - \nu^2}{2}\right) \left(\frac{y}{x}\right)^{\mu - \nu} p_1^{(\nu)}(t, x, y).$$

Now taking $\mu \geq 1/2$ and $\nu = 1/2$ together with (2.17) and the result of Proposition 1 gives the proof of (3.3) for $\mu \geq 1/2$. Analogous argument applied for $\mu < 1/2$ ends the proof. \(\square\)

Next proposition together with Proposition 1 provide the estimates for $x, y$ bounded away from 1. Notice that if $x, y > c > 1$ and $xy > t$ then

$$\frac{(x - 1)(y - 1)}{t} \geq \left(1 - \frac{1}{c}\right)^2 \frac{xy}{t} \geq \left(1 - \frac{1}{c}\right)^2. \quad (3.4)$$

and consequently the right-hand side of (1.4) is comparable with a constant which means that $p_1^{(\mu)}(t, x, y)$ is comparable with $p^{(\mu)}(t, x, y)$.

Proposition 3. Let $\mu \geq 1/2 \geq \nu > 0$. Then there exist constants $C_1^{(\nu)}, C_2^{(\mu)} > 0$ and $C_3^{(\mu)} > 1$ such that

$$C_1^{(\nu)} \left(\frac{x}{y}\right)^{\nu + 1/2} p_1^{(\nu)}(t, x, y) \leq \frac{1}{\sqrt{t}} \exp\left(-\frac{(x - y)^2}{t}\right) \leq C_2^{(\mu)} \left(\frac{x}{y}\right)^{\mu + 1/2} p_1^{(\mu)}(t, x, y),$$

whenever $xy > t$ and the lower bounds holds for $x, y > 2$ and the upper bounds are valid for $x, y > C_3^{(\mu)}$.

Proof. Taking $0 < \nu \leq 1/2$ and using the description of the behaviour of $I_\nu(z)$ at infinity (2.3) together with general estimate $p_1^{(\nu)}(t, x, y) \leq p^{(\nu)}(t, x, y)$ (which is an immediate consequence of the definition (2.2)) we get

$$p_1^{(\nu)}(t, x, y) \leq p^{(\nu)}(t, x, y) \approx \frac{1}{\sqrt{t}} \left(\frac{y}{x}\right)^{\nu + 1/2} \exp\left(-\frac{(x - y)^2}{t}\right).$$

This ends the proof for small indices.

Now let $\mu \geq 1/2$. Since the modified Bessel function $I_\mu(z)$ is positive, continuous and behaves like $(2\pi z)^{-1/2}e^z$ at infinity (see (2.3)) there exists constant $c_1 > 1$ such that

$$I_\mu\left(\frac{xy}{t}\right) \geq \frac{1}{c_1 \sqrt{2\pi xy}} \exp\left(\frac{xy}{t}\right).$$
whenever \( xy \geq t \). One can show that it is enough to take \( c_1 = (I_\mu(1)e^{-\sqrt{2\pi}})^{-1} \). Consequently, applying above given estimate to (1.1) we arrive at
\[
(y/x)^{\mu-1/2} p^{(1/2)}(t, x, y) \geq p^{(\mu)}(t, x, y) \geq \frac{1}{c_1 \sqrt{2\pi t}} (y/x)^{\mu+1/2} \exp \left(-\frac{(x-y)^2}{2t}\right), \quad xy \geq t,
\]
where the first inequality is just (3.2). Moreover, by (2.19), we have
\[
\frac{q_x^{(\mu)}(t)}{x^{\mu-1/2}} = \frac{x-1}{x^{\mu-1/2}} \frac{1}{\sqrt{2\pi t}^{3/2}} \exp \left(-\frac{(x-1)^2}{2t}\right), \quad t > 0, x > 1.
\]
and it together with left-hand side of (3.5) and (2.16) imply
\[
\begin{align*}
r_1^{(\mu)}(t, x, y) &= \int_0^t q_x^{(\mu)}(s)p^{(\mu)}(t-s, 1, y) \, ds \leq \frac{1}{2} \int_0^t q_x^{(1/2)}(s) p^{(1/2)}(t-s, 1, y) \, ds \\
&= \left(y/x\right)^{\mu-1/2} r_1^{(1/2)}(t, x, y) \\
&= \frac{1}{\sqrt{2\pi t}} \left(y/x\right)^{\mu+1/2} \left(\exp \left(-\frac{(x+y-2)^2}{2t}\right) - \exp \left(-\frac{(x+y)^2}{2t}\right)\right).
\end{align*}
\]
Let \( C_3^{(\mu)} = \left(1 - \frac{2c_1}{2c_1+1}\right)^{-1} \) and taking into account right-hand side of (3.5) and (3.4) we obtain for \( x, y > C_3^{(\mu)} \) that
\[
\begin{align*}
\frac{r_1^{(\mu)}(t, x, y)}{p^{(\mu)}(t, x, y)} &\leq c_1 \exp \left(-\frac{(x-y)^2}{2t}\right) \left(\exp \left(-\frac{(x+y-2)^2}{2t}\right) - \exp \left(-\frac{(x+y)^2}{2t}\right)\right) \\
&= c_1 \left(\exp \left(-\frac{2(x-1)(y-1)}{t}\right) - \exp \left(-\frac{2xy}{t}\right)\right) \\
&\leq c_1 \left(\exp \left(-c_2 \frac{2xy}{t}\right) - \exp \left(-\frac{2xy}{t}\right)\right),
\end{align*}
\]
where
\[
c_2 = \left(1 - \frac{1}{C_3^{(\mu)}}\right)^2 = \frac{2c_1}{2c_1+1} < 1.
\]
Taking into account the general estimate
\[
e^{-c_2z} - e^{-z} \leq \frac{1 - c_2}{c_2}, \quad z > 0, c_2 < 1
\]
we arrive at
\[
\frac{r_1^{(\mu)}(t, x, y)}{p^{(\mu)}(t, x, y)} \leq c_1 \frac{1 - c_2}{c_2} = \frac{1}{2}.
\]
Consequently
\[
\begin{align*}
p_1^{(\mu)}(t, x, y) &\geq \frac{1}{2} p^{(\mu)}(t, x, y) \geq \frac{1}{2c_1} \frac{1}{\sqrt{2\pi t}} \left(y/x\right)^{\mu+1/2} \exp \left(-\frac{(x-y)^2}{2t}\right).
\end{align*}
\]
Now we turn our attention to the case when \( x \) and \( y \) are bounded. The next proposition, however, is much more general.
Proposition 4. For fixed $m > 0$ and $\mu \geq 1/2 \geq \nu > 0$ there exist constants $C_4^{(\mu)}$, $C_4^{(\nu)} > 0$ such that
\[
C_4^{(\mu)} \left(\frac{x}{y}\right)^{\mu + 1/2} p_1^{(\mu)}(t, x, y) \geq \left(1 \wedge \frac{(x-1)(y-1)}{t}\right) \frac{1}{\sqrt{t}} \exp \left(-\frac{(x-y)^2}{2t}\right)
\]
and
\[
\left(1 \wedge \frac{(x-1)(y-1)}{t}\right) \frac{1}{\sqrt{t}} \exp \left(-\frac{(x-y)^2}{2t}\right) \geq C_4^{(\nu)} \left(\frac{x}{y}\right)^{\nu + 1/2} p_1^{(\nu)}(t, x, y)
\]
whenever $(x \wedge y)^2 \geq mt$.

Proof. Without lost of generality we can assume that $1 < x < y$. We put $b = (x+1)/2$ and take $\mu \geq 1/2$. Using (2.6) and the fact that $T_b^{(1/2)} \leq T_1^{(1/2)}$ we can write for every Borel set $A \subset (1, \infty)$ that
\[
\int_A p_1^{(\mu)}(t, x, y) dy \geq E_{x^{(1/2)}} \left[ t < T_b^{(1/2)}, R_t \in A; \left(\frac{R_t}{x}\right)^{\mu-1/2} \exp \left(-\frac{\mu^2 - 1/4}{2} \int_0^t ds \right) \right]
\]
Since up to time $T_b^{(1/2)}$ we have
\[
\int_0^t ds \leq \frac{4t}{(x+1)^2} \leq \frac{4t}{x^2} \leq \frac{4}{m},
\]
we obtain
\[
\int_A p_1^{(\mu)}(t, x, y) dy \geq \exp \left(-\frac{4\mu^2 - 1}{2m}\right) E_{x^{(1/2)}} \left[ t < T_b^{(1/2)}, R_t \in A; \left(\frac{R_t}{x}\right)^{\mu-1/2} \right],
\]
which gives
\[
p_1^{(\mu)}(t, x, y) \geq \exp \left(-\frac{4\mu^2 - 1}{2m}\right) \left(\frac{y}{x}\right)^{\mu-1/2} p_b^{(1/2)}(t, x, y).
\]
(3.6)

From the other side, the scaling property (2.11) and the formula (2.18) give
\[
p_b^{(1/2)}(t, x, y) = \frac{1}{b} p_1^{(1/2)} \left(\frac{t}{b^2}; \frac{x}{b}, \frac{y}{b}\right)
\]
\[
\approx \frac{1}{\sqrt{t}} \exp \left(-\frac{(x-y)^2}{2t}\right) \left(1 \wedge \frac{(x-b)(y-b)}{t}\right)
\]
\[
\approx \frac{1}{\sqrt{t}} \exp \left(-\frac{(x-y)^2}{2t}\right) \left(1 \wedge \frac{(x-1)(y-1)}{t}\right),
\]
where the last equalities follows from
\[
x - b = \frac{x-1}{2}, \quad \frac{y-1}{2} \leq y - b \leq y - 1.
\]
It ends the proof for $\mu \geq 1/2$.

For $1/2 \geq \nu > 0$ we similarly write
\[
\int_A p_b^{(1/2)}(t, x, y) dy \leq E_{x^{(\nu)}} \left[ t < T_1^{(\nu)}, R_t \in A; \left(\frac{R_t}{x}\right)^{1/2 - \nu} \exp \left(-\frac{\nu^2 - 1/4}{2} \int_0^t ds \right) \right]
\]
and we obtain
\[
p_b^{(1/2)}(t, x, y) \leq \exp \left(-\frac{4\nu^2 - 1}{2m}\right) \left(\frac{y}{x}\right)^{1/2 - \nu} p_1^{(\nu)}(t, x, y).
\]
This together with the above-given estimates for $p_b^{(1/2)}(t, x, y)$ finish the proof. $\square$
Corollary 1. For every $m$ applying the results of Proposition 4 (with where the last equality follows from (2.15). Using (2.4) we obtain

whenever $x, y < C$ and $xy \geq t$.

Finally, we end this section with two propositions related to the case when one of the space variables is close to 1 and the other is large. We deal with this case separately for $\mu < 1/2$ and $\mu \geq 1/2$.

Proposition 5. For every $\nu \in (0, 1/2)$ there exists constant $C_{5,\nu} > 0$ such that

for $1 < x \leq 2 \leq y$ and $xy \geq t$.

Proof. By monotonicity of $I_\nu(z)$, for every $s \in (0, t)$ we have

\[
\frac{1}{t-s} \geq \frac{1}{\sqrt{t} \sqrt{t-s}}, \quad I_\nu \left( \frac{y}{t-s} \right) \geq I_\nu \left( \frac{y}{t} \right).
\]

Hence, using the right-hand side of (2.19), we get

\[
q_\nu^x(s) \geq \frac{x-1}{\sqrt{2\pi s^3}} \exp \left( -\frac{(x-1)^2}{2s} \right), \quad 0 \leq \nu < 1/2, \ s > 0,
\]

and the formula (1.1) we get

\[
r_1(\nu)(t, x, y) = \int_0^t q_\nu^x(s) \frac{y^{1+\nu}}{t-s} \exp \left( -\frac{1+y^2}{2(t-s)} \right) I_\nu \left( \frac{y}{t-s} \right) ds
\]

where the last equality follows from (2.15). Using (2.14) we obtain

\[
p(\nu)(t, x, y) = \frac{y^{\nu+1}}{t} \exp \left( -\frac{x^2+y^2}{2t} \right) \frac{1}{x^{\nu}} I_\nu \left( \frac{xy}{t} \right)
\]

which together with previously given estimates, (2.9) and finally (2.3) give

\[
p_1(\nu)(t, x, y) \leq \frac{y^{\nu+1}}{t} \exp \left( -\frac{(x-y)^2}{2t} \right) \exp \left( -\frac{y}{t} \right) I_\nu \left( \frac{y}{t} \right) f_{y,t}(x)
\]

\[
\leq c_1 \frac{y^{\nu+1/2}}{\sqrt{t}} \exp \left( -\frac{(x-y)^2}{2t} \right) f_{y,t}(x),
\]

Since for $x, y < C$ and $xy \geq t$, for some fixed $C > 1$, we have

\[
\frac{(x \land y)^2}{t} \geq \frac{xy}{Ct} \geq \frac{1}{C},
\]
where 
\[ f_{y,t}(x) = 1 - \frac{1}{x^{\nu+1/2}} \exp \left( -\frac{(x-1)(\sqrt{y^2+1}+y-1)}{t} \right). \]

By elementary computation we can see that
\[ -f'_{y,t}(x) = \frac{1}{x^{\nu+3/2}} \exp \left( -\frac{(x-1)(\sqrt{y^2+1}+y-1)}{t} \right) \left( \frac{\sqrt{y^2+1}+y-1}{t} \right)^{\nu+1/2} \frac{1}{\sqrt{t}} \exp \left( -\frac{(x-y)^2}{2t} \right). \]

Here we have used the following inequalities
\[ \sqrt{y^2+1}+y-1 < 2y, \quad xy \geq t, \quad 1 < x \leq 2 \leq y. \]

Thus, by the mean value theorem, there exists \( d = d_{x,y,t} \in (1, x) \) such that
\[ f_{y,t}(x) = (1-x)f'_{y,t}(d) \leq 16 \left( \frac{x-1}{t} \right)^{\nu+1/2} \frac{1}{\sqrt{t}} \exp \left( -\frac{(x-y)^2}{2t} \right). \]

**Proposition 6.** For every \( \mu \geq 1/2 \) and \( c > 1 \) there exists constant \( C_6^{(\mu)}(c) > 0 \) such that for every \( 1 < x \leq c \) and \( y \geq 5c(\mu+1) \) we have
\[ p_{1}^{(\mu)}(t, x, y) \geq C_6^{(\mu)}(c) \frac{1}{\sqrt{t}} \left( \frac{y}{x} \right)^{\nu+1/2} \exp \left( -\frac{(x-y)^2}{2t} \right) \left( 1 - \frac{(x-1)(y-1)}{t} \right), \]
whenever \( xy \geq t \).

**Proof.** Let us fix \( \mu \geq 1/2 \). For every \( 0 < s < t \), using (2.4), we have
\[ I_{\mu} \left( \frac{y}{t-s} \right) < I_{\mu} \left( \frac{y}{t} \right) \left( \frac{t}{t-s} \right)^{\mu} \exp \left( \frac{y}{t} \right) \exp \left( -\frac{y}{t} \right) \]
and consequently
\[ \frac{p_{1}^{(\mu)}(t-s, 1, y)}{p_{1}^{(\mu)}(t, 1, y)} < \left( \frac{t}{t-s} \right)^{\mu+1} \exp \left( -\frac{(y-1)^2}{2t} \left( \frac{1}{t-s} - \frac{1}{t} \right) \right) = \frac{g_y(t-s)}{g_y(t)}, \]
where
\[ g_y(w) = \left( \frac{1}{w} \right)^{\mu+1} \exp \left( -\frac{(y-1)^2}{2w} \right), \quad w > 0. \]

Note that
\[ g_y'(w) = \left( \frac{1}{w} \right)^{\mu+2} \exp \left( -\frac{(y-1)^2}{2w} \right) \left( \frac{(y-1)^2}{2w} - (\mu+1) \right). \]

Since \( x \leq c, \ y \geq 5c(\mu+1) > 2 \) and \( xy \geq t \) we have \( 4(y-1) \geq 2y \geq 2t/c. \) Moreover \( y-1 \geq 4c(\mu+1). \) Thus
\[ \frac{(y-1)^2}{2t} \geq \frac{4c(\mu+1)(y-1)}{2t} \geq \mu+1. \]
It means that under our assumptions on $x$, $y$ and $t$ the function $g_y(w)$ is increasing on $(0, t)$ and consequently $g_y(t - s) \leq g_y(t)$ for every $0 < s < t$.

$$r_1^{(\mu)}(t, x, y) = \int_0^t q_2^{(\mu)}(s)p^{(\mu)}(t - s, 1, y)ds \leq p^{(\mu)}(t, 1, y)\int_0^t q_2^{(\mu)}(s)ds = x^{-\mu}\exp\left(\frac{x^2 - 1}{2t}\right)\frac{I_{\mu}(y/t)}{I_{\mu}(xy/t)}p^{(\mu)}(t, x, y).$$

The above-given ratio of modified Bessel functions can be estimated from above by using (2.5) as follows

$$I_{\mu}\left(\frac{y}{t}\right) \leq I_{\mu}\left(\frac{xy}{t}\right)\exp\left(-\frac{(x - 1)y}{t}\right)x^{\mu}.$$

Consequently

$$r_1^{(\mu)}(t, x, y) \leq p^{(\mu)}(t, x, y)\exp\left(-\frac{(x - 1)(2y - x - 1)}{2t}\right).$$

Finally observe that $2y - x - 1 > y - 1$ and we arrive at

$$p_1^{(\mu)}(t, x, y) \geq \left(1 - \exp\left(-\frac{(x - 1)(y - 1)}{2t}\right)\right)p^{(\mu)}(t, x, y)$$

$$\geq \frac{1\wedge \frac{(x - 1)(y - 1)}{t}}{\sqrt{t}}\left(\frac{y}{x}\right)^{\mu + 1/2}\exp\left(-\frac{(x - y)^2}{2t}\right).$$

This ends the proof.

The proof of (1.3) in the case $xy \geq t$ can be deduced from above-given propositions in the following way. Let $\mu \geq 1/2$ and without any loss of generality we assume that $x \leq y$. The upper bounds for every $x, y > 1$ are given in Proposition 1. From Proposition 3 we know that the lower bounds are valid for $x, y > C^{(\mu)}$. If $x \leq C^{(\mu)}_3$ and $y \geq 5C^{(\mu)}_3(\mu + 1)$ then the lower bounds are given in Proposition 6. Finally, taking $C = 5C^{(\mu)}_3(\mu + 1)$ in Corollary 1 we get the lower bounds in the remaining range of the parameters $x$ and $y$. The proof for $\nu \leq 1/2$ is obtained in the same way.

4. Estimates for $xy/t$ small

In this section we provide estimates of $p_1^{(\mu)}(t, x, y)$ whenever $xy < t$. Note also that (1.3) can be written in the following shorter way

$$p_1^{(\mu)}(t, x, y) \geq \frac{x - 1}{x}\frac{y - 1}{y}\left(\frac{y}{t}\right)^{\mu + 1/2}\frac{1}{\sqrt{t}}\exp\left(-\frac{x^2 + y^2}{2t}\right),$$

whenever $xy < t$. The main difficulty is to obtain the estimates when one of the space parameters is close to 1 and the other is large, i.e. tends to infinity. In this case we have to take care of cancellations of two quantities appearing in (2.4) but also not to lose a control on the exponential behaviour. We begin with the upper bounds.

**Proposition 7.** For every $\mu > 0$, there exists constant $C^{(\mu)}_7 > 0$ such that

$$p_1^{(\mu)}(t, x, y) \leq C^{(\mu)}_7\frac{x - 1}{x}\frac{y - 1}{y}\left(\frac{y}{t}\right)^{\mu + 1/2}\frac{1}{\sqrt{t}}\exp\left(-\frac{x^2 + y^2}{2t}\right),$$

whenever $xy \leq t$. 

Proof. If $x, y > 2$ the result follows immediately from the general estimate $p_{1}^{(\mu)}(t, x, y) \leq p^{(\mu)}(t, x, y)$ and (2.2) which gives

$$p^{(\mu)}(t, x, y) \approx \left(\frac{y^{2}}{t}\right)^{\mu+1/2} \frac{1}{\sqrt{t}} \exp\left(-\frac{x^{2} + y^{2}}{2t}\right), \quad \frac{xy}{t} \leq 1. \tag{4.1}$$

Note that for every $x, y > 0$ and $t > 0$ there exists $c_{1} > 0$ such that

$$p_{1}^{(\mu)}(t, x, y) \leq c_{1} \frac{y^{2\mu+1}}{t^{\mu+1}}. \tag{4.2}$$

If $xy < t$, then it immediately follows from (4.1) by estimating the exponential term by 1. For $xy \geq t$ we use the asymptotic behaviour (2.3) to show that

$$p^{(\mu)}(t, x, y) \approx \frac{1}{\sqrt{t}} \left(\frac{y}{x}\right)^{\mu+1/2} \exp\left(-\frac{|x - y|^{2}}{2t}\right) \leq \frac{y^{2\mu+1}}{t^{\mu+1}} \left(\frac{t}{xy}\right)^{\mu+1/2} \leq \frac{y^{2\mu+1}}{t^{\mu+1}} \tag{4.3}$$

In particular, for all $z, w > 1$ and $1 < y < 2$ there exists $c_{2} > 0$ such that

$$p^{(\mu)}(t/3, z, w) \leq c_{2} \left(\frac{w}{y}\right)^{2\mu+1} \frac{1}{t^{\mu+1}}. \tag{4.4}$$

The Chapman-Kolmogorov equation and estimating the middle term using (4.4) give

$$p_{1}^{(\mu)}(t, x, y) = \int_{1}^{\infty} \int_{1}^{\infty} p_{1}^{(\mu)}(t/3, x, z) p_{1}^{(\mu)}(t/3, z, w) p_{1}^{(\mu)}(t/3, w, y) dz dw \leq c_{3} \frac{\mu+1}{t^{\mu+1}} \int_{1}^{\infty} \int_{1}^{\infty} p_{1}^{(\mu)}(t/3, x, z) dz \int_{1}^{\infty} \left(\frac{w}{y}\right)^{2\mu+1} p_{1}^{(\mu)}(t/3, w, y) dw \approx c_{3} \frac{\mu+1}{t^{\mu+1}} P_{x}^{(\mu)}(T_{1}^{(\mu)} > t/3) P_{y}^{(\mu)}(T_{1}^{(\mu)} > t/3).$$

Here the last equality follows from the symmetry property (2.12). Since, by (2.8), whenever $xy < t$ and $1 < x, y < 2$ we have

$$P_{x}^{(\mu)}(T_{1}^{(\mu)} > t/3) = P_{x}^{(\mu)}(\infty > T_{1}^{(\mu)} > t/3) + P_{x}^{(\mu)}(T_{1}^{(\mu)} = \infty) \approx \frac{x - 1}{x^{2\mu}} + 1 - \frac{1}{x^{2\mu}} \approx x - 1,$$

which ends the proof of the upper-bound in this case.

Now assume that $y \geq 2$, $1 < x \leq 2$ and $xy \leq t$. The other case $x \geq 2$, $1 < y \leq 2$ follows from the symmetry condition mentioned above. Using the fact that $\int_{0}^{\infty} q_{x}^{(\mu)}(u) du = x^{-2\mu}$ and (2.9), we can write

$$p_{1}^{(\mu)}(t, x, y) \leq p^{(\mu)}(t, x, y) - \int_{0}^{1/2} q_{x}^{(\mu)}(u)p^{(\mu)}(t - u, 1, y) du = J_{1}(t, x, y) + J_{2}(t, x, y) + J_{3}(t, x, y),$$

where

$$J_{1}(t, x, y) = p^{(\mu)}(t, x, y) - \frac{1}{x^{2\mu}} p^{(\mu)}(t, x, y) + P_{x}^{(\mu)}(\infty > T_{1}^{(\mu)} > 1/2) p^{(\mu)}(t, x, y),$$

$$J_{2}(t, x, y) = \mathbf{P}_{x}^{(\mu)}(T_{1}^{(\mu)} \leq 1/2)(p^{(\mu)}(t, x, y) - p^{(\mu)}(t, 1, y)),$$

$$J_{3}(t, x, y) = \int_{0}^{1/2} q_{x}^{(\mu)}(u)(p^{(\mu)}(t, 1, y) - p^{(\mu)}(t - u, 1, y)) du.$$
To deal with $J_2(t, x, y)$ note that the differentiation formula (2.1), the asymptotic behavior (2.2) and positivity of $I_{\mu}(\varepsilon)$ give
\[
\frac{d}{dt}\left[ e^{-x^2/2t} \left( \frac{t}{xy} \right)^\mu I_{\mu} \left( \frac{xy}{t} \right) \right] = -\frac{x}{t} e^{-x^2/2t} \left( \frac{t}{xy} \right)^\mu I_{\mu} \left( \frac{xy}{t} \right) + e^{-x^2/2t} \frac{y}{t} \left( \frac{t}{xy} \right)^\mu I_{\mu+1} \left( \frac{xy}{t} \right) \\
\leq c_5 e^{-x^2/2t} \left( \frac{xy}{t} \right)^2 \leq c_5,
\]
whenever $xy < t$. Consequently, by mean value theorem, we obtain
\[
J_2(t, x, y) \leq (p^{(\mu)}(t, x, y) - p^{(\mu)}(t, 1, y)) \leq c_5(x - 1) \left( \frac{y^2}{t} \right)^{\mu+1/2} \frac{1}{\sqrt{t}} e^{-y^2/2t}.
\]
Finally, the bounds of $J_3(t, x, y)$ follow from the estimates for the derivative of $p^{(\mu)}(t, 1, y)$ in $t$. Using once again (2.1) and skipping the negative components we have
\[
h(t, y) = \frac{d}{dt} \left( \frac{1}{\mu+1} e^{-x^2/2t} \left( \frac{t}{y} \right)^\mu I_{\mu} \left( \frac{y}{t} \right) \right) \\
= e^{-(1+y^2)/(2t)} I_{\mu}(y/t) t y^{-\mu} \left( \frac{\mu + 1}{t} + \frac{1 + y^2}{2t^2} - \frac{y}{t} I_{\mu+1}(y/t) \right) \\
\leq e^{-(1+y^2)/(2t)} I_{\mu}(y/t) \frac{1 + y^2}{2t^2} \leq c_6 e^{-(1+y^2)/(2t)} \frac{1}{\mu+1},
\]
whenever $y < t$. Thus, there exists $c = c_{\mu, u, y} \in (t - u, t)$ such that
\[
J_3(t, x, y) = \int_0^{1/2} q_{x}^{(\mu)}(u) y^{2\mu+1} h(c_{\mu, u, y}, y) du \leq c_6 y^{2\mu+1} \int_0^{1/2} q_{x}^{(\mu)}(u) c e^{-(1+y^2)/(2c)} \frac{1}{\mu+1} du \\
\leq c_6 e^{-(1+y^2)/(2t)} \frac{y^{2\mu+1}}{(t/2)^{\mu+1}} \int_0^{1/2} c u^{(\mu)}(u) du.
\]
Taking into account the upper bounds given in (2.7) we get
\[
\int_0^{1/2} c u^{(\mu)}(u) du \leq c_7 \frac{x - 1}{x^{\mu+1/2}} \int_0^{1/2} e^{-(x-\mu)(u)} \frac{du}{u^{1/2}} \leq c_8(x - 1).
\]
This ends the proof. \hfill \Box

The proof of the lower bounds is split into two parts. Next proposition corresponds to the case when $y > x > 1$ and $(y - 1)^2/t$ is large. Moreover, we enlarge the region and assume that $xy < mt$ for a given $m \geq 1$. It is forced by the lower bounds given in Proposition 6 where it is required to have $xy/t$ sufficiently large but also by the proof of Proposition 6.

**Proposition 8.** For every $\mu > 0$ and $m \geq 1$, there exists constant $C_8^{(\mu)}(m) > 0$ such that
\[
\frac{p_{1}^{(\mu)}(t, x, y)}{p^{(\mu)}(t, x, y)} \geq C_8^{(\mu)}(m) \frac{x - 1}{x}, \quad y > x > 1
\]
whenever $xy < mt$ and $\frac{(y-1)^2}{t} \geq 2(\mu + 1)$.

**Proof.** Since
\[
\frac{p_{1}^{(\mu)}(t, x, y)}{p^{(\mu)}(t, x, y)} = 1 - \frac{p^{(\mu)}(t, 1, y)}{p^{(\mu)}(t, x, y)} \frac{p_{1}^{(\mu)}(t, x, y)}{p^{(\mu)}(t, 1, y)},
\]
using (2.4) for every $\mu > 0$ and $(y - 1)^2/t \geq 2(\mu + 1)$, we have
\[
\frac{p_1^{(\mu)}(t, x, y)}{p^{(\mu)}(t, x, y)} = \int_0^t q_x^{(\mu)}(s) \frac{p^{(\mu)}(t-s, 1, y)}{p^{(\mu)}(t, 1, y)} \, ds \\
= \int_0^t q_x^{(\mu)}(s) \left( \frac{t}{t-s} \exp \left( -\frac{1 + y^2}{2t}, \frac{s}{t-s} \right) \right) \left( \frac{t}{t-s} \right)^{\mu+1} \exp \left( -\frac{(y-1)^2}{2t}, \frac{s}{t-s} \right) ds.
\]
For every $s < t$ we can write
\[
\left( \frac{t}{t-s} \right)^{\mu+1} \exp \left( -\frac{(y-1)^2}{2t}, \frac{s}{t-s} \right) = \frac{f_y(t-s)}{f_y(t)}, \tag{4.5}
\]
where $f_y(w) = w^{-\mu-1}e^{-(y-1)^2/2w}$. Then by simple calculation we get $f'_y(w) = w^{-\mu-2}e^{-(y-1)^2/2w} \left( \frac{(y-1)^2}{2w} - \mu + 1 \right)$ and consequently $f_y(w)$ is increasing on $\left( 0, \frac{(y-1)^2}{2(\mu+1)} \right)$. It implies that right-hand side of (4.5) is smaller than $1$ whenever $\frac{(y-1)^2}{t} \geq 2(\mu + 1)$ and
\[
\frac{p_1^{(\mu)}(t, x, y)}{p^{(\mu)}(t, x, y)} \geq 1 - x^\mu \exp \left( \frac{x^2 - 1}{t} \right) \frac{I_{\mu}(y/t)}{I_{\mu}(xy/t)} \int_0^t q_x^{(\mu)}(s) \, ds.
\]
Since the function $z^{-\mu}I_{\mu}(z)$ is increasing on $(0, \infty)$
\[
\frac{I_{\mu}(y/t)}{I_{\mu}(xy/t)} \leq \frac{1}{x^\mu}, \quad x, y > 1 \quad t \geq 0.
\]
This, together with $P_x^{(\mu)}(T_1^{(\mu)} < \infty) = x^{-2\mu}$, gives
\[
\frac{p_1^{(\mu)}(t, x, y)}{p^{(\mu)}(t, x, y)} \geq 1 - \frac{1}{x^{2\mu}} \exp \left( \frac{x^2 - 1}{t} \right). \tag{4.6}
\]
Now we assume that $1 < x < (2e^m)^{1/(2\mu)}$ and $t > \frac{2(e^m)^{1/\mu}}{\mu}$. Then
\[
\frac{p_1^{(\mu)}(t, x, y)}{p^{(\mu)}(t, x, y)} \geq 1 - \frac{1}{x^{2\mu}} \exp \left( \frac{\mu(x^2 - 1)}{2(2e^m)^{1/\mu}} \right).
\]
The mean value theorem ensures the existence of a constant $d \in (1, x)$ such that
\[
1 - \frac{1}{x^{2\mu}} \exp \left( \frac{\mu(x^2 - 1)}{2(2e^m)^{1/\mu}} \right) = \frac{2\mu(x-1)}{d^{2\mu+1}} \exp \left( \frac{\mu(d^2 - 1)}{2(2e^m)^{1/\mu}} \right) \left( 1 - \frac{d^2}{2(2e^m)^{1/\mu}} \right) \geq c_1(m)(x-1)
\]
where the last inequality comes from the fact that $1 < d < x < (2e^m)^{1/(2\mu)}$.

The next step is to take $x \geq (2e^m)^{1/(2\mu)}$ and $t > \frac{2(e^m)^{1/\mu}}{\mu}$. Since $x^2 < xy < mt$ using (4.6) we get
\[
\frac{p_1^{(\mu)}(t, x, y)}{p^{(\mu)}(t, x, y)} \geq 1 - \frac{1}{x^{2\mu}} e^m \geq 1 - \frac{1}{2} \approx \frac{x-1}{x}.
\]
Finally, we consider the case when \( x > 1 \), \( xy/m < t \leq \frac{2(2e^{-m})^{1/\mu}}{t} =: t_0 \) and \( \frac{(y-1)^2}{t} \geq 2(\mu+1) \). Using absolute continuity property (2.6) and (2.17), we can write

\[
p_1^{(\mu)}(t, x, y) \geq \left( e^{-t_0(x^2/2-1/8)} \wedge 1 \right) \left( \frac{y}{x} \right)^{\mu-1/2} p_1^{(1/2)}(t, x, y)
\]

\[
\mu^m \approx \left( 1 \wedge \left( \frac{x-1}{t} \right) \right) \left( \frac{y}{x} \right)^{\mu+1/2} \frac{1}{\sqrt{t}} \exp \left( -\frac{x^2+y^2}{2t} \right)
\]

\[
\geq \left( 1 \wedge \left( \frac{x-1}{t} \right) \right) \left( \frac{y^2}{t} \right)^{\mu+1/2} \frac{1}{\sqrt{t}} \exp \left( -\frac{x^2+y^2}{2t} \right)
\]

\[
\mu^m \approx \frac{x-1}{x} p^{(\mu)}(t, x, y).
\]

This ends the proof. \( \square \)

We end this section with the proof of the lower bounds, whenever \((y \vee x) - 1)^2/t\) is small. Note that in the proof of the next proposition we use the lower bounds of \(p_1^{(\mu)}(t, x, y)\) for \(xy \geq t\) obtained previously in Section 3 as well as the result of Proposition 6. As previously, due to the symmetry, it is enough to assume that \(y > x > 1\).

**Proposition 9.** For every \( \mu > 0 \) there exists constant \( C_9^{(\mu)} > 0 \) such that

\[
\frac{p_1^{(\mu)}(t, x, y)}{p^{(\mu)}(t, x, y)} \geq C_9^{(\mu)} \frac{x-1}{x} \frac{y-1}{y}, \quad y > x > 1,
\]

whenever \( xy < t \) and \( \frac{(y-1)^2}{t} \leq 2(\mu+1) \).

**Proof.** Let \( xy < t \) and \( y > x > 1 \). At the beginning we additionally assume that \( t \geq 4 \). Note that there exists \( c_1 > 0 \) such that for every \( s > 1/2 \) we have \( e^{-s} \geq c_1 s^{\mu+1/2} e^{-2s} \).

This, together with the lower bounds of \(p_1^{(\mu)}(t, z, w)\) for \(z, w \geq \sqrt{t} \) (then \(zw \geq t\)) obtained in Section 3 enable us to write

\[
p_1^{(\mu)}(t, z, w) \geq c_2 \left( 1 \wedge \left( \frac{z-1}{t} \right) \right) \left( \frac{w}{z} \right)^{\mu+1/2} \frac{1}{\sqrt{t}} \exp \left( -\frac{|z-w|^2}{2t} \right)
\]

\[
\geq c_2 \frac{w}{z} \left( \frac{w}{z} \right)^{\mu+1/2} \frac{1}{\sqrt{t}} \exp \left( -\frac{z^2}{2t} \right) \exp \left( -\frac{w^2}{2t} \right)
\]

\[
\geq c_2 \frac{w}{z} \left( \frac{w}{z} \right)^{\mu+1/2} \frac{1}{\sqrt{t}} \exp \left( -\frac{z^2}{2t} \right) \exp \left( -\frac{w^2}{2t} \right)
\]

\[
\geq c_3 \left( \frac{w^2}{t} \right)^{\mu+1/2} \frac{1}{\sqrt{t}} \exp \left( -\frac{z^2}{2t} \right) \exp \left( -\frac{w^2}{2t} \right)
\]

Consequently, using the Chapmann-Kolmogorov equation and (2.12), we get

\[
p_1^{(\mu)}(3t, x, y) = \int_{1}^{\infty} \int_{1}^{\infty} p_1^{(\mu)}(t, x, z)p_1^{(\mu)}(t, z, w)p_1^{(\mu)}(t, w, y)dzdw
\]

\[
\geq \int_{\sqrt{t}}^{\infty} \int_{\sqrt{t}}^{\infty} p_1^{(\mu)}(t, x, z)p_1^{(\mu)}(t, z, w)p_1^{(\mu)}(t, w, y)dzdw
\]

\[
\geq c_3 \left( \frac{y^2}{t} \right)^{\mu+1/2} \frac{1}{\sqrt{t}} \int_{\sqrt{t}}^{\infty} p_1^{(\mu)}(t, x, z)e^{-z^2/t}dz \int_{\sqrt{t}}^{\infty} \left( \frac{w}{y} \right)^{2\mu+1} p_1^{(\mu)}(t, w, y)e^{-w^2/t}dw
\]

\[
= c_3 \left( \frac{y^2}{t} \right)^{\mu+1/2} \frac{1}{\sqrt{t}} L_1^{(\mu)}(x)L_1^{(\mu)}(y),
\]
where
\[ F_t^{(\mu)}(x) := \int_{\sqrt{t}}^\infty p_1^{(\mu)}(t, x, z)e^{-z^2/\mu}dz. \]

Since for \( t \geq 4 \) and \( \frac{(y-1)^2}{t} \leq 2(\mu + 1) \) we have
\[ \frac{x^2}{t} \leq \frac{y^2}{t} \leq \left( 2 \wedge 4 \frac{(y-1)^2}{t} \right) \leq c_4 \]
and consequently
\[ p^{(\mu)}(3t, x, y) \approx \left( \frac{y^2}{t} \right)^{-\mu+1/2} \frac{1}{\sqrt{t}}, \quad xy < t, \]
it is enough to show that \( F_t^{(\mu)}(x) \geq c_6 \frac{x-1}{x} \) for every \( x > 1 \). However, for \( z \geq b\sqrt{t} \), with \( b = 2\sqrt{2(\mu + 1)} \), and \( t \geq 4 \) we have \( \frac{x^2}{t} \geq 1 \frac{x^2}{t} \geq 2(\mu + 1) \). We can use the lower bounds given in Proposition 8 with \( m = 2b \) and obtain
\[ F_t^{(\mu)}(x) \geq \int_{b\sqrt{t}}^{2bt/x} p_1^{(\mu)}(t, x, z)e^{-z^2/\mu}dz \]
\[ \geq c_6 \frac{x-1}{x} \int_{b\sqrt{t}}^{2bt/x} \left( \frac{z^2}{t} \right)^{\mu+1/2} e^{-z^2/2\mu}e^{-z^2/\mu}dz \geq c_7 \frac{x-1}{x\sqrt{t}} \int_{b\sqrt{t}}^{2bt/x} e^{-z^2/\mu}dz \]
\[ = c_7 \frac{x-1}{x} \int_{b}^{2b} e^{-2u^2}du \geq c_7 \frac{x-1}{x} \int_{b}^{2b} e^{-2u^2}du. \]

Finally, for \( t \leq 4 \), the same computations as in the end of the proof of the previous Proposition (but with \( t_0 = 4 \)) gives
\[ p_1^{(\mu)}(t, x, y) \geq c_7(e^{-4(\mu^2/2-1/8)} \wedge 1) \left( 1 \wedge \frac{(x-1)(y-1)}{t} \right) \left( \frac{y^2}{t} \right)^{\mu+1/2} \frac{1}{\sqrt{t}} \exp \left( -\frac{x^2 + y^2}{2t} \right) \]
\[ \approx \frac{x-1}{x} \frac{y-1}{y} p^{(\mu)}(t, x, y), \]
where the last approximation follows from the fact that \( (x-1)(y-1) < xy \leq t \leq 4 \) which gives
\[ 1 \wedge \frac{(x-1)(y-1)}{t} = \frac{(x-1)(y-1)}{t} = \frac{(x-1)(y-1)}{xy} \approx \frac{(x-1)(y-1)}{xy}. \]

\[ \square \]

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**References**

[1] T. Byczkowski, P. Graczyk, and A. Stos. Poisson kernels of half-spaces in real hyperbolic spaces. *Rev. Mat. Iberoamericana*, 23(1):85–126, 2007.
[2] T. Byczkowski, J. Malecki, and M. Ryznar. Hitting times of Bessel processes. *Potential Anal.*, 38:753–786, 2013.
[3] T. Byczkowski and M. Ryznar. Hitting distribution of geometric Brownian motion. *Studia Math.*, 173(1):19–38, 2006.
[4] E. B. Davies. The equivalence of certain heat kernel and Green function bounds. *J. Funct. Anal.*, 71:88–103, 1987.
[5] E. B. Davies. *Heat kernels and spectral theory (Cambridge Tracts in Mathematics)*, volume 92. Cambridge University Press, Cambridge, 1990.
[6] E. B. Davies. Intrinsic ultracontractivity and the dirichlet laplacian. *J. Funct. Anal.*, 100:162–180, 1991.

[7] E. B. Davies and B. Simon. Ultracontractivity and heat kernels for Schrödinger operators and Dirichlet Laplacians. *J. Funct. Anal.*, 59:335–395, 1984.

[8] Erdelyi et al. *Higher Transcendental Functions*, volume II. McGraw-Hill, New York, 1953.

[9] I. S. Gradstein and I. M. Ryzhik. *Table of integrals, series and products*. 7th edition. Academic Press, London, 2007.

[10] Y. Hamana and H. Matsumoto. Hitting times of Bessel processes, volume of Wiener sausages and zeros of Macdonald functions. *arXiv:1302.4526*.

[11] Y. Hamana and H. Matsumoto. The probability densities of the first hitting times of Bessel processes. *J. Math-for-Ind.*, 4B:91–95, 2012.

[12] Y. Hamana and H. Matsumoto. The probability distributions of the first hitting times of Bessel processes. *Trans. Amer. Math. Soc.*, 365:5237–5257, 2013.

[13] A. Laforgia. Bounds for modified Bessel functions. *J. Comput. Appl. Math.*, 34:263–267, 1991.

[14] H. Matsumoto and M. Yor. Exponential functionals of Brownian motion, I: Probability laws at fixed time. *Probability Surveys*, 2:312–347, 2005.

[15] H. Matsumoto and M. Yor. Exponential functionals of Brownian motion, II: Some related diffusion processes. *Probability Surveys*, 2:348–384, 2005.

[16] A. Nowak and L. Roncal. On sharp heat and subordinated kernel estimates in the Fourier-Bessel setting. *arXiv:1111.5700*.

[17] A. Nowak and L. Roncal. Sharp heat kernel estimates in the Fourier-Bessel setting for a continuous range of the type parameter. *arXiv:1208.5199*.

[18] L. Saloff-Coste. The heat kernel and its estimates. *Adv. Stud. Pure Math.*, 57:405–436, 2010.

[19] Q. S. Zhang. The boundary behavior of heat kernels of Dirichlet Laplacians. *J. Differential Equations*, 182:416–430, 2002.

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