The Moutard Transformation for the Davey–Stewartson II Equation and Its Geometrical Meaning

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Abstract—The Moutard transformation for the solutions of the Davey–Stewartson II equation is constructed. It is geometrically interpreted using the spinor (Weierstrass) representation of surfaces in four-dimensional Euclidean space. Examples of solutions that have smooth fast decaying initial data and lose regularity in finite time are constructed by using the Moutard transformation and minimal surfaces.

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1. MAIN RESULTS

In this paper, we construct the Moutard transformation for the solutions of the Davey–Stewartson II (DS-II) equation [1]

\[ U_t = i(U_{zz} + U \bar{U} + (V + \nabla)U), \quad \nabla = 2(|U|^2)_z, \]  (1.1)

which is the compatibility condition for the linear problems

\[ \mathcal{D}\Psi = 0, \quad \partial_t \Psi = A\Psi \]

where

\[ \mathcal{D} = \begin{pmatrix} 0 & \partial \\ -\bar{U} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & \bar{U} \end{pmatrix} \]  (1.2)

is the two-dimensional Dirac operator with complex-valued potential \( U \) and

\[ \partial = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \]

as well as

\[ A = i \begin{pmatrix} -\partial^2 - V & \bar{U} \partial - U \bar{\partial} \\ U \partial - U \bar{\partial} & \partial^2 + \nabla \end{pmatrix} \]  (1.3)

and

\[ \Psi = \begin{pmatrix} \psi_1 \\ -\psi_2 \end{pmatrix} \]

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Equation (1.1) is also obtained as the compatibility condition for the following linear problems related to the previous ones:

\[ D^\vee \Phi = 0, \quad \Phi_t = A^\vee \Phi, \]

where

\begin{align*}
D^\vee &= \left( \begin{array}{cc} 0 & \partial \\ -\overline{\partial} & 0 \end{array} \right) + \left( \begin{array}{cc} U & 0 \\ 0 & U \end{array} \right), \\
A^\vee &= -i \left( \begin{array}{cc} -\partial^2 - V & U\overline{\partial} - U \overline{z} \\ U\partial - U_z & \overline{\partial}^2 + \overline{V} \end{array} \right),
\end{align*}

(1.4)

and

\[ \Phi = \left( \begin{array}{c} \varphi_1 \\ -\overline{\varphi}_2 \\ \varphi_2 \\ \overline{\varphi}_1 \end{array} \right). \]

It is reasonable to consider \( \Psi \) and \( \Phi \) as \( \mathbb{H} \)-valued functions, where \( \mathbb{H} \) is the linear space of quaternions formed by all matrices of the form

\[ \left( \begin{array}{cc} a & b \\ -\overline{b} & \overline{a} \end{array} \right), \quad \text{where} \quad a, b \in \mathbb{C}. \]

For each pair \( \Psi \) and \( \Phi \) of \( \mathbb{H} \)-valued functions, the 1-form

\[ \omega(\Phi, \Psi) = -\frac{i}{2} (\Phi^\top \sigma_3 \Psi + \Phi^\top \Psi) dz - \frac{i}{2} (\Phi^\top \sigma_3 \Psi - \Phi^\top \Psi) d\overline{z} \]

is constructed, where \( X \rightarrow X^\top \) is the transpose of the matrix \( X \) and

\[ \sigma_3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \]

is the Pauli matrix. If \( \Psi \) and \( \Phi \) satisfy the Dirac equations (1.2) and (1.4), then the forms \( \omega(\Phi, \Psi) \) and \( \omega(\Psi, \Phi) \) are closed and, in particular, the \( \mathbb{H} \)-valued function

\[ S(\Phi, \Psi)(z, \overline{z}) = \Gamma \int \omega(\Phi, \Psi) \]

(1.6)

is defined. Here

\[ \gamma = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) = i\sigma_2, \]

where \( \sigma_2 \) is the Pauli matrix. Formula (1.6) defines the spinor (Weierstrass) representation of a surface in four-dimensional space \( \mathbb{R}^4 = \mathbb{H} \) with the conformal parameter \( z \) on it. This surface is determined up to a shift, i.e., up to a constant of integration \( q \in H \).

We define the \( \mathbb{H} \)-valued function

\[ K(\Phi, \Psi) = \Psi S^{-1}(\Phi, \Psi) \Gamma \Phi^\top \Gamma^{-1} = \left( \begin{array}{cc} i\overline{W} & a \\ -\pi & -iW \end{array} \right). \]

(1.7)

The following was shown in [2].

**Statement.** Given solutions \( \Psi_0 \) and \( \Phi_0 \) of equations (1.2) and (1.4), for each pair \( \Psi \) and \( \Phi \) of solutions of the same equations, the \( \mathbb{H} \)-valued functions

\[ \widetilde{\Psi} = \Psi - \Psi_0 S^{-1}(\Phi_0, \Psi_0) S(\Psi_0, \Phi), \quad \widetilde{\Phi} = \Phi - \Phi_0 S^{-1}(\Psi_0, \Phi_0) S(\Psi_0, \Phi) \]

(1.8)

satisfy the Dirac equations

\[ \overline{D} \widetilde{\Psi} = 0, \quad \overline{D}^\vee \widetilde{\Phi} = 0 \]
for the operators $\bar{D}$ and $\bar{D}^\vee$ with potential

$$\bar{U} = U + W,$$

(1.9)

where $W$ is defined by formula (1.7) for $K(\Phi_0, \Psi_0)$. Here it is assumed that

$$\Gamma S^{-1}(\Phi_0, \Psi_0) \Gamma = (S^{-1}(\Psi_0, \Phi_0))^\top,$$

(1.10)

and this is achieved by a suitable choice of the constant of integration in the definition of $S(\Psi_0, \Phi_0)$ given by formula (1.6).

This Moutard-type transformation for Dirac operators is a generalization of the transformation derived in [3] for the case $U = \ov{\Psi} \Psi$ and $\Psi_0 = \Phi_0$. In [3], the Moutard transformation was extended to the transformation of solutions of the modified Novikov–Veselov equation.

Just as in the case of the modified Novikov–Veselov equation, the transformation (1.8), (1.9) is extended to the transformation of the solutions of the DS-II equation as follows. We replace the function $S(\Phi, \Psi)$ in the definition of $K(\Phi, \Psi)$ by

$$S(\Phi, \Psi)(z, \bar{z}, t) = \Gamma \int \omega(\Phi, \Psi) + \Gamma \int \omega_1(\Phi, \Psi),$$

where

$$\omega_1(\Phi, \Psi) = \left[ \Phi_\bar{z} \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) + \Phi_z \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \right] \Psi - \Phi_\bar{z} \left[ \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \Psi_z + \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \Psi_\bar{z} \right] dt.$$

The following statement is valid.

**Theorem 1.** If $U$ satisfies the Davey–Stewartson II equation (1.1) and $\Psi$ and $\Phi$ satisfy the equations $D\Psi = 0$, $\Psi_t = A\Psi$, $D\Psi \Phi = 0$, $\Phi_t = A^\Psi \Phi$, then the Moutard transformation (1.9) of the function $U$ yields the solution $\bar{U}$ of the DS–II equation

$$\bar{U}_t = i(\bar{U}_{zz} + \bar{U}_{\bar{z}\bar{z}} + 2(\bar{V} + \bar{V})\bar{U}), \quad \bar{V} = V + 2ia_z,$$

(1.11)

and $a$ is given by (1.7).

The geometrical meaning of this transformation is the following: for each given $t$, the spinors $\Psi$ and $\Phi$ define a surface $\Sigma_t$ in $\mathbb{R}^4$ by means of the (spinor) Weierstrass representation [4]–[6] and $U$ is the potential of this representation. The family of surfaces $\Sigma_t$ evolves according to the DS–II equation [7], [8]. For each surface from this family, we apply the composition of the inversion centered at the origin and of the reflection

$$(x_1, x_2, x_3, x_4) \rightarrow (-x_1, -x_2, -x_3, x_4)$$

and obtain a new surface $\Sigma_t$ for which $\bar{U}$ is the potential of its spinor representation [2]. The resulting family of surfaces also evolves by the DS–II equation.

Starting with a family of smooth surfaces and their corresponding smooth potentials $U$, we can construct singular solutions of equation (1.1). Indeed, when $\Sigma_t$ passes through the origin, the function $\bar{U}$ loses continuity or smoothness, because the origin is mapped by the inversion to the infinitely distant point.

One of the simplest applications of Theorem 1 is the construction of exact solutions of equation (1.1) using holomorphic functions. In that case, we start from the trivial solution $U = V = 0$, for which $\Psi$ and $\Phi$ are given by holomorphic data and, using Theorem 1, we construct the nontrivial solutions of equation (1.1). For example, the following theorem is valid.
**Theorem 2.** Let $f(z,t)$ be a function that is holomorphic with respect to $z$ and satisfies the equation

$$\frac{\partial f}{\partial t} = i \frac{\partial^2 f}{\partial z^2}.$$ 

Then

$$U = \frac{i(zf' - f)}{|z|^2 + |f|^2}, \quad V = 2ia_z,$$

where $a = \frac{i(z + f')(\overline{f})}{|z|^2 + |f|^2}$, satisfies the Davey–Stewartson II equation (1.1).

Geometrically, we have a deformation of the graphs $w = f(z,t)$, which are minimal surfaces in $\mathbb{R}^4 = \mathbb{C}^2$, and when $f(z,t)$ vanishes at the point $z = 0$, the graph passes through the origin and the solution $\hat{U}$ loses continuity or smoothness. A few explicit examples are given in Sec. 4.2.

Let us make the following change of variables: $X = 2y$, $Y = 2x$. Now the Davey–Stewartson II equation (1.1) will take the form

$$iU_t - U_{XX} + U_{YY} = -4|U|^2 U + 8\varphi_X U,$$

where $\Re V = 2|U|^2 - 4\varphi_X$, $\varphi_X = \partial \varphi / \partial X$. We must distinguish between this version of the DS-II equation, called a focusing equation, from the defocusing case, for which the right-hand side has an opposite sign.

Ozawa constructed a blow-up solution of equation (1.2), for which the initial data are smooth and fast decaying and for which $||U||^2$, the quadratic $L_2$–norm, tends to the Dirac distribution [9].

Theorem 2 allows us to construct solutions that lose regularity in finite time. For example, the simplest of them has the form

$$U = \frac{i(z^2 - 2it + iT)}{|z|^2 + |z|^2 + 2it - iT|^2},$$

It is regular for $t \neq T/2$ and $U \sim i e^{2it} \Phi$ for $t = T/2$, where $z = re^{i\phi}$.

We will discuss these solutions in more detail in Secs. 4.2 and 5.

This paper can be regarded as a continuation of the papers [10], [11], in which similar ideas were applied to the modified Novikov–Veselov equation. In [10], we found the geometric interpretation of the Moutard transformation from [3] in terms of Mobius geometry in $\mathbb{R}^3$ and, in [11], used this transformation to construct solutions to the modified Novikov–Veselov equation that lose regularity in finite time. In [2] and in this paper, we consistently implement this program for the DS–II equation.

## 2. SURFACES IN $\mathbb{R}^4$ OBTAINED BY MEANS OF SPINORS AND THEIR SOLITON DEFORMATIONS

### 2.1. Weierstrass (Spinor) Representation of Surfaces in Four-Dimensional Space

For solutions $\psi$ and $\varphi$ of the Dirac equation $\mathcal{D} \Psi = \mathcal{D}^\vee \varphi = 0$, the formulas

$$x^k = x^k(0) + \int \eta_k, \quad k = 1, 2, 3, 4,$$

where

$$\eta_k = f_k dz + \overline{f_k} d\overline{z}, \quad k = 1, 2, 3, 4,$$

$$f_1 = \frac{i}{2}(\overline{\varphi_2} \psi_2 + \varphi_1 \psi_1), \quad f_2 = \frac{i}{2}(\overline{\varphi_2} \overline{\psi_2} - \varphi_1 \psi_1),$$

$$f_3 = \frac{i}{2}(\overline{\varphi_2} \psi_1 + \varphi_1 \overline{\psi_2}), \quad f_4 = \frac{i}{2}(\overline{\varphi_2} \psi_1 - \varphi_1 \overline{\psi_2}).$$
define (up to a shift) the embedded surface in \( \mathbb{R}^4 \) [8]. The induced metric has the form
\[ e^{2\alpha} \, dz \, d\zeta = (|\psi_1|^2 + |\psi_2|^2)(|\varphi_1|^2 + |\varphi_2|^2) \, dz \, d\zeta \]
and the potential \( U \) of the Dirac operators is related to the mean curvature vector \( H \) as follows:
\[ |U| = \frac{|H|e^{\alpha}}{2}, \quad H = 2e^{-2\alpha}x_2\zeta, \quad x = (x^1, x^2, x^3, x^4) \in \mathbb{R}^4. \]
The integrals in (2.1) are defined up to constants of integration, which means that the surface is defined up to shifts.

For \( \psi = \varphi \) and a real-valued potential \( U \), we have \( x^4 = \text{const} \), and these formulas define a surface in \( \mathbb{R}^3 \) [4].

In fact, these formulas are general and give a representation of any surface with given conformal parameter \( z \). For topologically nontrivial surfaces, the functions \( \Psi \) and \( \Phi \) are sections of spinor bundles, and the integral of \( |U|^2 \) over a surface, up to a constant factor, is a Willmore functional [5], [7] (see also the survey [6]).

For surfaces in \( \mathbb{R}^3 \), the spinor \( \Psi \) is uniquely defined up to a multiplication by \( \pm 1 \), but, for surfaces in \( \mathbb{R}^4 \), the representations of the same surface are related by gauge transformations of the form
\[ (\psi_1, \psi_2) \rightarrow (e^f \psi_1, e^f \psi_2), \quad (\varphi_1, \varphi_2) \rightarrow (e^{-f} \varphi_1, e^{-f} \varphi_2), \]
where \( f \) is a holomorphic function.

### 2.2. Geometrical Meaning of the Moutard Transformation

Following [2], [10], we identify our four-dimensional space with the space of quaternions \( \mathbb{H} \) and rewrite the spinor representation as
\[
S(\Phi, \Psi) = \int \left[ i \begin{pmatrix} \psi_1 \overline{\varphi_2} & -\overline{\psi_2} \overline{\varphi_2} \\ \psi_1 \varphi_1 & -\psi_2 \varphi_1 \end{pmatrix} dz + i \begin{pmatrix} \psi_2 \overline{\varphi_1} & \overline{\psi_1} \overline{\varphi_1} \\ -\psi_2 \varphi_2 & -\psi_1 \varphi_2 \end{pmatrix} d\zeta \right]
\]
\[ = \int d \begin{pmatrix} ix^3 + x^4 & -x^1 + ix^2 \\ x^1 - ix^2 & -ix^3 + x^4 \end{pmatrix}, \tag{2.2} \]
where the spinors \( \Psi \) and \( \Phi \) satisfy the Dirac equations (1.2) and (1.4). It is easy to see that \( S(\Phi, \Psi) \) is the same as in (1.6).

The meaning of Moutard transformations (1.8) and (1.9) is the following: consider \( S \) as a surface in \( S^4 = \mathbb{R}^4 \cup \{\infty\} \) given by the spinors \( \Psi_0 \) and \( \Phi_0 \). Apply to \( S \) the conformal transformation
\[ S \rightarrow \tilde{S} = S^{-1}. \]
The resulting surface is given by the spinors
\[ \tilde{\Psi}_0 = \Psi_0 S^{-1}(\Phi_0, \Psi_0), \quad \tilde{\Phi}_0 = \Phi_0 S^{-1}(\Psi_0, \Phi_0), \]
which satisfy the Dirac equations with potential \( \tilde{U} \). Since \( S \) is defined up to constants of integration, we assume that \( S(\Phi_0, \Psi_0) \) and \( S(\Psi_0, \Phi_0) \) are related by (1.10).

For real-valued potentials \( U \) corresponding to surfaces in \( \mathbb{R}^3 \), the geometry of the Moutard transformation was described in [10] and later generalized to the case of general potentials in [2].

It is useful to state all operations on \( S \) in terms of quaternions. Let us introduce the following basis in \( \mathbb{H} \):
\[ 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad k = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \]
The surface \( S \) of the form (2.2) is written in coordinates as
\[ S(\Phi, \Psi) = x^1 i + x^2 j + x^3 k + x^4 1. \]
and the standard operations are written as
\[ S^{-1} = \frac{1}{|x|^2}(-x^1 i - x^2 j - x^3 k + x^4 1), \quad S^\top = -x^1 i + x^2 j + x^3 k + x^4 1. \]

Hence it is clear that the transformation \( S \to S^{-1} \) in the language of surfaces is the composition of the inversion \( x \to x/|x|^2 \) and reflection:
\[ (x_1, x_2, x_3, x_4) \to (-x_1, -x_2, -x_3, x_4). \]

Note that \( \gamma = -i \), and if the functions \( S(\Psi, \Phi) \) and \( S(\Phi, \Psi) \) defined up to constants are normalized so that they both vanish at the same point, then
\[ S(\Psi, \Phi) = x^1 i + x^2 j + x^3 k - x^4 1, \]
which implies (1.10).

2.3. Soliton Deformation of Surfaces by Means of the Davey–Stewartson II Equation

The Dirac operators \( D \) and \( D^\vee \) are included in \( L, A, B \)-triples for Davey–Stewartson equations. This means that there are differential operators \( A \), \( B \) and \( A^\vee \), \( B^\vee \) such that the compatibility conditions for systems
\[ D\Psi = 0, \quad \psi_t = A\psi, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \]
\[ D^\vee \varphi = 0, \quad \varphi_t = A^\vee \varphi, \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \]
take the form of a scalar equation from the Davey–Stewartson hierarchy. This equation is represented by the triples
\[ D_t + [D, A] - BD = 0, \quad D^\vee_t + [D^\vee, A^\vee] - B^\vee D^\vee = 0. \]

Therefore, if \( \psi \) and \( \varphi \) evolve according to this equation, then, by the spinor representation, they determine the soliton deformation of the corresponding surfaces.

Such deformations were introduced in [4] for surfaces in three-dimensional space (and they correspond to the modified Novikov–Veselov hierarchy) and later, in [8], for surfaces in \( \mathbb{R}^4 \). In the latter case, the corresponding integrable systems are equations from the Davey–Stewartson hierarchy.

Since these equations contain nonlocal terms, it is necessary to carefully resolve all the constraints so that the Willmore functional is a first integral of the system [7]:
\[ \frac{d}{dt} \int |U|^2 dx \wedge dy = 0. \]

Therefore, we only consider the DS–II equation, which corresponds to the \( A \)-operators (1.3) and (1.5) and takes the form (1.1).

3. MOUTARD TRANSFORMATION FOR THE DS–II EQUATION: PROOF OF THEOREM 1

The DS–II deformations of surfaces are defined by the deformations of the corresponding spinors, which do not yield a surface, but only yield its Gauss mapping. Therefore, the DS–II deformation is defined up to shifts. Assume that the shifts are smooth functions of the time variable. Then the deformation can be written as
\[ S(\Phi, \Psi)(Z, \overline{Z}, t) = \int_{(P,0)}^{(Z, \overline{Z}, t)} (M dz + N d\overline{z} + Q dt), \]
where \( z \) is the conformal parameter in a (singly connected) domain \( U \subseteq \mathbb{C} \), \( P \in U \) is a fixed point, and \( (Z, \overline{Z}) \in U \).
By the definition of a DS-II deformation, we have
\[ M = i \begin{pmatrix} \psi_1 \varphi_2 & -\bar{\psi}_2 \varphi_2 \\ \psi_1 \varphi_1 & -\bar{\psi}_2 \varphi_1 \end{pmatrix}, \quad N = i \begin{pmatrix} \psi_2 \varphi_1 & \bar{\psi}_1 \varphi_1 \\ -\psi_2 \varphi_2 & -\bar{\psi}_1 \varphi_2 \end{pmatrix}. \]

Since the 1-form \( M \, dz + N \, d\bar{z} + Q \, dt \) must be closed, we have
\[ \frac{\partial M}{\partial \bar{z}} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial t} = \frac{\partial Q}{\partial z}, \quad \frac{\partial N}{\partial t} = \frac{\partial Q}{\partial \bar{z}}. \]

The first equation follows from (1.2) and (1.4) and, from the other two equations, we can determine \( Q \) up to functions of \( t \).

**Proposition 1.** Let the spinors \( \Psi \) and \( \Phi \) satisfy the Dirac equations \( D \Psi = D^\ast \Phi = 0 \) and evolutionary equations \( \Psi_t = A \Psi \) and \( \Phi_t = A^\ast \Phi \), where \( U \) satisfies the DS-II equation (1.1). Then the DS-II deformation of the surface given by the spinors \( \Phi \) and \( \Psi \) has the form
\[ S(\Phi, \Psi) = \int \left[ i \begin{pmatrix} \psi_1 \varphi_2 & -\bar{\psi}_2 \varphi_2 \\ \psi_1 \varphi_1 & -\bar{\psi}_2 \varphi_1 \end{pmatrix} \, dz + i \begin{pmatrix} \psi_2 \varphi_1 & \bar{\psi}_1 \varphi_1 \\ -\psi_2 \varphi_2 & -\bar{\psi}_1 \varphi_2 \end{pmatrix} \, d\bar{z} + Q_0(t) \, dt \]
\[ + \left( \psi_{1z} \varphi_{2z} - \psi_{1z} \varphi_{2z} + \psi_{2z} \varphi_{1z} - \psi_{2z} \varphi_{1z} - \psi_{1z} \varphi_{1z} + \psi_{1z} \varphi_{1z} - \psi_{2z} \varphi_{2z} - \psi_{2z} \varphi_{2z} \right) dt \], \quad (3.1) \]

where \( Q_0(t) \) is a smooth \( \mathbb{H} \)-valued function of \( t \).

**Proof.** Since \( M_{11} = -\bar{N}_{22}, M_{22} = \bar{N}_{11}, M_{12} = -\bar{N}_{21}, M_{21} = \bar{N}_{12} \) and \( Q \) is an \( \mathbb{H} \)-valued function, we need only to consider the equation
\[ \frac{\partial M}{\partial t} = \frac{\partial Q}{\partial z}. \]

Consider \( M_{11} = i \psi_1 \varphi_2 \). By (1.3) and (1.5), we have
\[ (\psi_1 \varphi_2)_t = i(\psi_{1z} - V \psi_1) + (U \bar{\psi}_2 \varphi_2 + U \psi_2 \varphi_2) \bar{\varphi}_2 + i(\psi_{1z} - V \psi_1) + (U \psi_2 \varphi_2 + U \bar{\psi}_2 \varphi_2) \varphi_2. \]

Let us split the right-hand side into three components and consider each one separately. It is obvious that
\[ \psi_{1z} \varphi_{1z} - \psi_{1z} \varphi_{2z} = (\psi_1 \varphi_{2z} - \psi_1 \varphi_{2z})_z. \]

Let us rewrite (1.2) and (1.4) in the form
\[ U \psi_1 = -\psi_{2z}, \quad U \bar{\psi}_2 = \psi_{1z}, \quad U \varphi_2 = \varphi_{1z}, \quad U \varphi_1 = -\varphi_{2z}. \]

and consider the other two components. From (3.2), we deduce that
\[ (U \psi_2 \bar{\varphi}_2 - \bar{U} \psi_2 \varphi_2) \varphi_2 = (U \bar{\psi}_2 \varphi_2 - \bar{U} \psi_2 \varphi_2) \varphi_2 = 2(|U \psi_2 \bar{\varphi}_2 - \bar{U} \psi_2 \varphi_2| \varphi_2 - (\psi_{12} \varphi_{1z}) \bar{\varphi}_2). \]

Similarly, we conclude that
\[ U \psi_1 \bar{\varphi}_1 - \bar{U} \psi_1 \varphi_1 = U \psi_1 \bar{\varphi}_1 - (U \psi_1 \bar{\varphi}_1) \bar{\varphi}_1 + U \psi_1 \varphi_1 \bar{\varphi}_1 = (U \psi_1 \bar{\varphi}_1 + U \psi_1 \varphi_1 \bar{\varphi}_1) \bar{\varphi}_1 = -\psi_{2z} \varphi_{1z} - \bar{\psi}_{2z} \psi_{1z} \bar{\varphi}_1. \]

Combining the results of the calculations for all three components, we deduce that
\[ (\psi_1 \varphi_2)_t = i(\psi_{1z} - \psi_{1z} + \psi_{2z} \varphi_{1z} - \psi_{2z} \varphi_{1z})_z. \]

Hence
\[ Q_{11} = -(\psi_{12} \varphi_{2z} - \psi_{1z} \varphi_{2z} + \psi_{2z} \varphi_{1z} + \bar{\psi}_{2z} \psi_{1z} \varphi_{1z} + Q_{0,11}(\bar{z}, t)). \]

It follows from the equation \( \partial N/\partial t = \partial Q/\partial z \) that \( Q_{0,11} = Q_{0,11}(t) \). Thus, we have established (3.1) for \( M_{11} \). Formulas for the other elements of the matrices \( M \) and \( N \) are derived in a similar way, so we will omit these calculations. The proposition is proved.
Consider the derivative $S_T$ in more detail:

\[
\begin{pmatrix}
\psi_{1z} \varphi_2 - \psi_1 \varphi_{2z} + \psi_2 \varphi_{1z} - \psi_2 \varphi_2 \\
\varphi_{1z} - \psi_1 \varphi_{1z} + \psi_2 \varphi_2 - \psi_2 \varphi_2
\end{pmatrix}
= \begin{pmatrix}
(\psi_1 - \psi_2) \varphi_2 + (\varphi_2 - \varphi_1) \\
(\varphi_1 - \varphi_2) \varphi_2 + (\psi_2 - \psi_1)\varphi_2
\end{pmatrix}
= \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
\Phi_+^T (1 & 0 \\
0 & 0)
\end{pmatrix} + \begin{pmatrix}
\Phi_+^T (0 & 1)
\end{pmatrix} \Phi_+ \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
\Psi_z + (0 & 0)
\end{pmatrix} \Psi.
\]

Similarly, we obtain

\[
S_z = -i \gamma \Phi^T \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} \Psi, \quad S_T = -i \Phi^T \begin{pmatrix}
0 & 0 \\
0 & -1
\end{pmatrix} \Psi.
\]

Consider the surface $\tilde{S} = S^{-1}(\Phi, \Psi)$. According to [2, Proposition 1], it is obtained from $S = S(\Phi, \Psi)$ by the composition of the inversion and reflection

\[
(x^1, x^2, x^3, x^4) \rightarrow (x^1, x^2, x^3, x^4)
\]

and its spinor representation is given by by the spinors

\[
\Phi = \Phi S^{-1}(\Psi, \Phi), \quad \Psi = \Psi S^{-1}(\Phi, \Psi) = \Psi S^{-1}.
\]

Following (1.10), we rewrite the transformation $\Phi$ as

\[
\Phi^T = \Gamma S^{-1} \Gamma \Phi^T.
\]

Substituting these formulas for $\Phi$ and $\Psi$ into (3.3), we obtain the formula for $\tilde{S}_t$ modulo the function $Q_0(t)$. If we compare the formulas with $S_t^{-1} = -S^{-1} S t S^{-1}$, then, by direct calculations, we deduce that these two expressions coincide if and only if

\[
\tilde{Q}_0(t) = -S^{-1}(z, \bar{z}, t) Q_0(t) S^{-1}(z, \bar{z}, t).
\]

For a general surface, there are no such nontrivial functions $Q_0$ and $\tilde{Q}_0$.

Hence we can assert the following.

**Proposition 2.** Suppose that the assertions of Proposition 1 hold. Then, for $Q_0 = 0$,

\[
\tilde{S}_t = \Gamma \left[ \Phi_+^T \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} + \Phi_+^T \begin{pmatrix}
0 & 1
\end{pmatrix} \right] \psi - \Gamma \Phi^T \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} \psi + \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix} \psi.
\]

and the surface $\tilde{S} = S^{-1}$ of the form (3.1) is also deformed according to the DS-II equation.

As in [2], $\tilde{\Psi}$ and $\tilde{\Phi}$ satisfy the Dirac equations with potential $\tilde{U} = U + W$, where $W$ is given by formula (1.7), which we will rewrite as

\[
K = \tilde{\Psi} \Gamma \Phi^T \Gamma^{-1} = \begin{pmatrix} i \bar{W} & a \\
-\bar{a} & -i W
\end{pmatrix}.
\]

We now need to find operators $\tilde{A}$ and $\tilde{A}^\vee$ of the form (1.3) and (1.5) such that

\[
\tilde{\Psi}_t = \tilde{A} \tilde{\Psi}, \quad \tilde{\Phi}_t = \tilde{A}^\vee \tilde{\Phi}.
\]

Since the potential $\tilde{U}$ is known, it remains to find $\tilde{V}$. To do this, we will write

\[
\tilde{\Psi}_t = (\Psi \tilde{S})_t = \Psi_t \tilde{S} + \Psi \tilde{S}_t = A \Psi \cdot \tilde{S} + \Psi \tilde{S}_t.
\]
The matrix $\tilde{\Lambda}$ and hence $\tilde{\Psi}_t$. We have deduced that

$$\tilde{\Psi}_t = A\tilde{\Psi} + \Lambda\tilde{\Psi} + \Theta,$$

where $\lambda$ is the matrix, which will be considered in what follows, and

$$\Theta = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \Psi\Gamma\tilde{\Phi}_z^\top \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tilde{\Psi}_z + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \Psi\Gamma\tilde{\Phi}_z^\top \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tilde{\Psi}_\tau$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} K\Gamma \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tilde{\Psi}_z + \begin{pmatrix} 1 \\ 0 \end{pmatrix} K\Gamma \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tilde{\Psi}_\tau$$

$$= \begin{pmatrix} 0 & k_{11} \partial \\ -k_{22} \partial & 0 \end{pmatrix} \tilde{\Psi} = i \begin{pmatrix} 0 \\ W\partial \\ 0 \end{pmatrix} \tilde{\Psi}.$$

The matrix $\Lambda$ decomposes into the following sum:

$$\Lambda = \Psi\Gamma \begin{pmatrix} \tilde{\Phi}_z^\top \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \tilde{\Phi}_\tau^\top \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ 0 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Psi_z\Gamma\tilde{\Phi}_z^\top \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Psi_z\Gamma\tilde{\Phi}_z^\top \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Psi\Gamma\tilde{\Phi}_z^\top \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + i\Upsilon \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Psi\Gamma\tilde{\Phi}_z^\top \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$+ i\Upsilon \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Psi\Gamma\tilde{\Phi}_z^\top \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

We have deduced that

$$\tilde{\Psi}_t = i \begin{pmatrix} -\partial^2 - V \\ U\partial - U_z \end{pmatrix} \tilde{\psi} + \Lambda\tilde{\psi},$$

and hence $\tilde{V} = V + i\Lambda_{11}$. It is easy to see that not all the terms of (3.4) make a contribution to the element $\Lambda_{11}$, which is equal to $M_{11}$, where

$$M = \Psi\Gamma\tilde{\Phi}_z^\top + 2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Psi_z\Gamma\tilde{\Phi}_z^\top + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Psi\Gamma\tilde{\Phi}_z^\top$$

$$= 2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} (\Psi\Gamma\tilde{\Phi}_z^\top)z + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Psi\Gamma\tilde{\Phi}_z^\top = -2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} (K\Gamma)_z + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Psi\Gamma\tilde{\Phi}_z^\top.$$

From the structure of the matrix $K$, we conclude that $\Lambda_{11} = M_{11} = 2a_z$, and thus obtain (1.11):

$$\tilde{V} = V + 2ia_z.$$

This completes the proof of Theorem 1.
Minimal surfaces are defined as surfaces with a zero average curvature $H = 0$ and, in our case, these are exactly the surfaces given by (2.2), where the functions $\psi_1, \overline{\psi}_2, \varphi_1,$ and $\overline{\varphi}_2$ are holomorphic. This means that $\Psi$ and $\Phi$ satisfy the Dirac equations $\mathcal{D}\Psi = 0$ and $\mathcal{D}^\dagger \Phi$ for $\cal U = 0$.

Consider such functions as the initial data (for $t = 0$) for the solutions of the equations
\[
\psi_{1t} = -i\partial^2 \psi_1, \quad \overline{\psi}_{2t} = -\partial \overline{\psi}_2, \quad \varphi_{1t} = i\partial^2 \varphi_1, \quad \overline{\varphi}_{2t} = i\partial^2 \overline{\varphi}_2.
\]
These are exactly the solutions of the equations $\Psi_t = A \Psi$ and $\Phi_t = A^\dagger \Phi$ for $\cal U = \cal V = 0$. The solutions $\psi(z, \overline{\tau}, t)$, and $\Phi(z, \overline{\tau}, t)$ of these equations define a one-parameter family of minimal surfaces $S(z, \overline{\tau}, t)$. The inversion $S^{-1}$ of these surfaces yields the Moutard transformation (Theorem 1) of the trivial solution $\cal U = \cal V = 0$ of the DS-II equation.

Thus, we have a simple method for finding nontrivial solutions of the DS-II equation that depend on four functional parameters that are holomorphic functions.

We will omit general formulas and demonstrate this method by using a one-parameter family of solutions.

Let us identify $\mathbb{R}^4$ with $\mathbb{C}^2$ with coordinates $z = x_1 + ix_2, w = x_3 + ix_4$. For each holomorphic function $f$, its graph $w = f(z)$ defines the minimal surface in $\mathbb{R}^4$. The representation (2.2) of this surface is given by the spinors
\[
\Psi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \Phi = \begin{pmatrix} f' & i \\ i & \overline{f}' \end{pmatrix}
\]
and takes the form
\[
S = \begin{pmatrix} if & -z \\ \overline{z} & -if \end{pmatrix}.
\]
Suppose that $f$ also depends on $t$ and satisfies the equation $f_t = i\partial^2 f$. By Proposition 1 and (3.3), the transformation $\tilde{S} = S^{-1}$ of the one-parameter family of minimal surfaces
\[
S = \int \begin{pmatrix} 0 & -1 \\ 0 & -if' \end{pmatrix} dz + \begin{pmatrix} if' & 0 \\ 0 & 0 \end{pmatrix} d\overline{\tau} + \begin{pmatrix} f'' & 0 \\ 0 & 0 \end{pmatrix} dt
\]
defines a DS-II deformation of the surface in $\mathbb{R}^4$. The matrix (1.7) for these data takes the form
\[
K = \Psi S^{-1} \Gamma \Phi^\dagger \Gamma^{-1} = \frac{1}{|z|^2 + |f|^2} \begin{pmatrix} zf' - f & -iz - if'f' \\ -i(z - if')f' & zf' - f \end{pmatrix}.
\]
By Theorem 1, this transformation of surfaces induces the Moutard transformation of the trivial solution $\cal U = \cal V = 0$ of the DS-II equation to the solution
\[
U = \frac{i(zf' - f)}{|z|^2 + |f|^2}, \quad V = 2iaz, \quad a = -\frac{i(z + f')f}{|z|^2 + |f|^2}.
\]
(4.1)

Theorem 2 is proved.
4.2. Examples

Let us consider the simplest examples of solutions of the form (4.1). They correspond to the case where \( \varphi_1 \) is a polynomial of \( z \) of degree \( n \leq 3 \). In what follows, we will denote by \( c \) a constant that can take any complex values, and by \( r \) we will denote \( |z|, z \in \mathbb{C} \).

1) \( n = 0 \). We have

\[
\varphi_1 = 1, \quad f = z + c, \quad U = -\frac{ic}{|z|^2 + |z + c|^2}, \quad V = -\frac{2(\tau + c)^2}{(|z|^2 + |z + c|^2)^2}.
\]

This solution is time-independent and, for \( c \neq 0 \), is nonsingular.

2) \( n = 1 \). We have

\[
\varphi_1 = 2z, \quad f = z^2 + 2ict + c, \quad U = \frac{i(z^2 - 2ict - c)}{|z|^2 + |z^2 + 2ict + c|^2}, \quad V = \frac{4(\tau^2 - 2ict + \tau)}{|z|^2 + |z^2 + 2ict + c|^2} - \frac{2(2\tau(z^2 - 2ict + \tau) + \tau)^2}{(|z|^2 + |z^2 + 2ict + c|^2)^2}.
\]

The function \( |U| \) decreases as \( O(1/r^2) \), \( r \to \infty \). If \( c \) is not purely imaginary, then the solution is smooth everywhere. If \( c = it\tau, \tau \in \mathbb{R} \), then, for \( t = -\tau/2 \), the function \( U \) has a singularity at \( z = 0 \) of the form

\[
U \sim ie^{2i\Phi} \quad \text{as} \quad r \to 0, \quad \text{where} \quad z = re^{i\Phi}.
\]

Note that \( U \in L^2(\mathbb{R}^2) \) for all values of \( t \) and \( c \). Since the small variations of \( c \) eliminate singularities, it follows that they are unstable.

For brevity, we will omit formulas for \( V \) in the following cases.

3) \( n = 2 \). We have

\[
\varphi_1(z, t) = 3z^2 + 6it, \quad f = z^3 + 6it^2z + c, \quad U = \frac{i(2z^3 - c)}{|z|^2 + |z^3 + 6it^2z + c|^2}.
\]

If \( c \neq 0 \), then the solution is smooth everywhere. If \( c = 0 \), then it has a singularity of the form

\[
U \sim \frac{2i}{1 + 36t^2}re^{3i\Phi} \quad \text{for} \quad r = 0 \quad \text{for all} \quad t.
\]

4) \( n = 3 \). We have

\[
\varphi_1 = 4z^3 + 24it^2z, \quad f = z^4 + 12it^2z^2 - 12t^2 + c, \quad U = \frac{i(3z^4 + 12it^2z^2 + 12t^2 - c)}{|z|^2 + |z^4 + 12it^2z^2 - 12t^2 + c|^2}.
\]

This solution becomes singular for \( c = 12t^2 \), which is only possible if \( c \) is real and positive. In this case, it has singularities of the form \( U \sim -12te^{2i\Phi} \) at the point \( z = 0 \) for \( t = \pm\sqrt{c/12} \).

5. REMARKS

1) Ozawa constructed a blow-up solution for the DS-II equation in the form of (1.2) as follows: he took

\[
U(X, Y, 0) = \frac{e^{-ib(4a)^{-1}(X^2 - Y^2)}}{a(1 + ((X/a)^2 + (Y/a)^2)/2)}
\]

as the initial data at \( t = 0 \) and showed that, for constants \( a \) and \( b \) such that \( ab < 0 \),

\[
\|U\|^2 \to 2\pi \cdot \delta \quad \text{as} \quad t \to T = -\frac{a}{b}
\]
in $\mathcal{S}'$, where

$$\|U\|^2 = \int_{\mathbb{R}^2} |U|^2 \, dx \, dy$$

is the quadratic $L_2$-norm of the solution and $\delta$ is the Dirac distribution centered at the origin. Note that $\|U\|^2 = 2\pi$ and that the solution is extended for $T > -a/b$, becoming regular. A survey dealing with the problem of blow-up solutions for the DS-II equation was given in [12, Sec. 5].

2) The Willmore functional $\|U\|^2$ is a first integral of the system. For the solution (4.2), it is equal to $2\pi$ everywhere, except at time $T_{\text{sing}}$ when the solution becomes singular. For $t = T_{\text{sing}}$, the Willmore functional is equal to $\pi$. Similarly, for the solution (4.3), the Willmore functional is $4\pi$ for values of $t$ such that $U$ is nonsingular and equal to $3\pi$ for $t = T_{\text{sing}}$.

A similar effect was observed for the solutions of the modified Novikov–Veselov equation constructed in [11]. It turns out that the functional is a multiple of $\pi$, which is due to the fact that, in both cases, the surfaces $S$ are immersed Willmore spheres (with singularities at singular times).

3) A Moutard-type transformation for the DS-II equation was constructed earlier in [13]. However, it was constructed by using only one spinor $\Psi$ and, therefore, has no geometric interpretation in terms of surface theory. The solutions constructed in [13] have some other interesting properties. Note that, in [14], just as in [13], a Moutard-type transformation was called a Darboux-type (binary) transformation.

The original Moutard transformation for the two-dimensional Schrödinger operator was used to construct the potentials and solutions of the Novikov–Veselov equation with some interesting analytical properties, for example, in [15]–[17].

For similar purposes, Moutard-type transformations of two-dimensional Dirac operators were successfully applied in [11] and also in [18]–[20].

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