The Hamiltonian formulation of tetrad gravity: three dimensional case

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(Dated: May 14, 2010)
Abstract

The Hamiltonian formulation of the tetrad gravity in any dimension higher than two, using its first order form when tetrads and spin connections are treated as independent variables, is discussed and the complete solution of the three dimensional case is given. For the first time, applying the methods of constrained dynamics, the Hamiltonian and constraints are explicitly derived and the algebra of the Poisson brackets among all constraints is calculated. The algebra of the Poisson brackets among first class secondary constraints locally coincides with Lie algebra of the ISO(2,1) Poincaré group. All the first class constraints of this formulation, according to the Dirac conjecture and using the Castellani procedure, allow us to unambiguously derive the generator of gauge transformations and find the gauge transformations of the tetrads and spin connections which turn out to be the same found by Witten without recourse to the Hamiltonian methods [Nucl. Phys. B 311 (1988) 46]. The gauge symmetry of the tetrad gravity generated by Lie algebra of constraints is compared with another invariance, diffeomorphism. Some conclusions about the Hamiltonian formulation in higher dimensions are briefly discussed; in particular, that diffeomorphism invariance is not derivable as a gauge symmetry from the Hamiltonian formulation of tetrad gravity in any dimension when tetrads and spin connections are used as independent variables.

PACS: 11.10.Ef, 11.30.Cp
I. INTRODUCTION

The Hamiltonian formulation of General Relativity (GR) has a history going back a half-century. On one hand, the Hamiltonian formulation of such a highly non-trivial theory as GR is a good laboratory where general methods of constrained dynamics [1–3] can be studied and some subtle points that cannot be even seen in simple examples, can be found, investigated and lead to further development of the method itself. On the other hand, the correct Hamiltonian formulation of a theory is a prerequisite to its successful canonical quantization. In this paper we consider the Hamiltonian formulation of the tetrad gravity. Nowadays this is a more popular formulation of GR, in particular, because it is used in Loop Quantum Gravity [4–6]. The accepted Hamiltonian formulation of tetrad gravity leads to the so-called “diffeomorphism constraint”, or more precisely, the “spatial diffeomorphism constraint” [4–6] (though the word “spatial” is often omitted in the literature).

Recently it was demonstrated [7–9] that the long-standing problem of having only spatial diffeomorphism in the Hamiltonian formulation of metric GR [10] is just a consequence of a non-canonical change of variables. Without making such changes, the full diffeomorphism invariance of the metric tensor is derivable from the Hamiltonian formulation in all dimensions higher than two ($D > 2$) [7, 8]. This result suggests the necessity of reconsidering also the Hamiltonian formulation of tetrad gravity, especially because the accepted Hamiltonian formulation was performed using a change of variables for tetrads [11] similar to metric gravity and this has led to the same “diffeomorphism constraint” which is only spatial.

Moreover, the three dimensional case of tetrad gravity poses additional questions. For instance, what is the gauge symmetry of the tetrad gravity in three dimensions? In some papers it is written that the gauge symmetry is Poincaré symmetry [12], in others that it is Lorentz symmetry plus diffeomorphism [13]1, or that there exists various ways to define the constraints of tetrad gravity leading to different gauge transformations [16]. According to [17], two symmetries are present and we have to decide “what is a gauge symmetry and what is not”. We think that this is the right question but the answer should not depend on our decision or desire. The Hamiltonian method is the perfect instrument to find the

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1 In some papers Lorentz symmetry plus diffeomorphism are even called the Poincaré gauge symmetry (see, for example, [14, 15]).
unique answer to the question what the gauge symmetry is.

To the best of our knowledge, despite the existence of numerous review articles, living reviews, and books, e.g. [18], [19], and [20], there is no complete Hamiltonian formulation of tetrad gravity in three dimensions. The only papers that somehow related to tetrad gravity in three dimensions are: the work due to Blagojević and Cvetković [21] where all steps of the Dirac procedure were performed, but for the three dimensional Mielke-Baekler model; and Blagojević in [22] performed the Hamiltonian analysis but for the Chern-Simons action. A complete Hamiltonian formulation means that all steps of the Dirac procedure should be performed [1–3]: (i) momenta are introduced to all variables leading to the primary constraints, (ii) the Hamiltonian is found, (iii) the time development of constraints is considered until (iv) the closure of the Dirac procedure is reached, (v) all constraints are classified as being first and second class, (vi) second class constraints are eliminated (Hamiltonian reduction) [23], (vii) a gauge generator, according to the Dirac conjecture [1], is constructed from all the first class constraints using one of the available methods [24–26], and (viii) this gauge generator is used to derive unambiguously the gauge transformations of all fields. If some of these steps are missing or implemented incorrectly then we cannot be sure that the correct gauge symmetry has been found.

The first attempts to interpret GR as a gauge theory started from work of Utiyama [28]. In his approach, in the same way as was done in Yang-Mills theory [30], by postulating the invariance of a system under a certain group of transformations it is possible to introduce a new compensating field, determine the form of interaction, and construct the modified Lagrangian which makes the action invariant under the given transformations. Utiyama applied his method to the gravitational field using local Lorentz transformations for vierbein fields. Later Kibble [31] extended Utiyama’s prescription by considering the group of inhomogeneous Lorentz transformations, Poincaré group, (though he switched from the translational parameters of the Poincaré symmetry to the parameters which “specify a general coordinate transformation”, e.g. diffeomorphism transformation). Localization of Poincaré symmetry leads to the Poincaré gauge theory of gravity (PGTG) (see, e.g. [32] and for the Poincaré-Weyl theory [33]). All these approaches have the same feature: the aim to construct a

\footnote{We would like to note that the methods of [25, 26] should be applied with a great caution (see [27]).}

\footnote{Actually, before Utiyama’s attempt, Weyl introduced what we know now as a principle of gauge invariance in his attempt to unify electricity and gravitation [29].}
theory from a given gauge symmetry rather than to derive a gauge symmetry for a given Lagrangian. This is the main disadvantage of such methods as they cannot be used for the systems with unknown \textit{a priori} gauge invariance.

In this paper we explore another approach. We do not relate our analysis to the Chern-Simons action (as was done by Witten in [12]), we do not perform any change of variables (even canonical), and do not use any formulation which is specific to a particular dimension (like Plebanski’s [34] for dimension $D = 4$). Our goal is to start from the first order action of the tetrad gravity, in which tetrads and spin connections are treated as independent variables, follow all steps of the Dirac procedure, without any assumption of what the gauge symmetry should be, and see what gauge transformations will be derived (or what “decision” the procedure will make). The structure of our paper follows steps (i)-(viii) of the procedure outlined above.

In the next Section we apply the Dirac procedure to the first order formulation of tetrad GR in any dimensions ($D > 2$). The first steps are independent of the dimension until we reach the point where a peculiarity of three dimensions appears. From this stage onwards, where we must consider the elimination of second class constraints, we restrict our analysis to three dimensional case. In Section III we perform the Hamiltonian reduction by eliminating second class primary constraints and the corresponding pairs of canonical variables, and then derive the explicit expressions for the secondary first class constraints and the Hamiltonian. The closure of the Dirac procedure and Poisson brackets (PB) among all constraints are given in Section IV where it is demonstrated that PB algebra of secondary first class constraints coincides with Lie algebra of the ISO(2,1) Poincaré group. In Section V, using the Dirac conjecture and the Castellani procedure, we derive the gauge generator from all the first class constraints and their PB algebra which was found in the previous Section. Both gauge parameters presented in the generator turn out to have only internal (“Lorentz”) indices and describe rotations and translations in the tangent space, not a diffeomorphism. When this gauge generator acts on fields, it gives gauge transformations which are equivalent to Witten’s result [12] obtained for $D = 3$ without the use of the Dirac procedure. This unique gauge symmetry which has been derived for the tetrad gravity in its first order form is compared with another non-gauge symmetry, diffeomorphism, in Section VI. Our consideration is based on the original variables, tetrads and spin connections, without making even a canonical change of variables and without specialization of either the variables or the
form of the original action to any particular dimension. This allows us to use the three
dimensional case as a guide for higher dimensions and to draw some conclusions about
the Hamiltonian formulation of the tetrad gravity in higher dimensions. This discussion is
presented in Section VII. The preliminary results on the analysis of the tetrad gravity in
higher dimensions are reported in [35–37].

II. THE HAMILTONIAN AND CONSTRAINTS

To compare our results with previous incomplete attempts of the Hamiltonian for-
mlation, we start our analysis from the Einstein-Cartan (EC) Lagrangian of tetrad gravity
written in its first order form (found, e.g., in [38, 39])

\[
L = -e \left( e^{\mu(\alpha)} e^{\nu(\beta)} - e^{\nu(\alpha)} e^{\mu(\beta)} \right) \left( \omega_{\nu(\alpha\beta),\mu} + \omega_{\mu(\alpha\gamma)} \omega^{(\gamma)}_{\nu(\beta)} \right),
\]

where the covariant tetrads \( e_{\gamma(\rho)} \) and the spin connections \( \omega_{\nu(\alpha\beta)} \) are treated as independent
fields, and \( e = \text{det}(e_{\gamma(\rho)}) \). We assume that the inverse \( e^{\mu(\alpha)} \) of the tetrad field \( e_{\gamma(\rho)} \) exists
and \( e^{\mu(\alpha)} e_{\mu(\beta)} = \delta^{(\alpha)}_{(\beta)} \), \( e^{\mu(\alpha)} e_{\nu(\alpha)} = \delta^{\mu}_{\nu} \). Indices in brackets (..) denote the internal (“Lorentz”)
indices, whereas indices without brackets are external or “world” indices. Internal and
external indices are raised and lowered by the Minkowski tensor \( \eta^{(\alpha)(\beta)} = (-, +, +, ...) \) and
the metric tensor \( g_{\mu\nu} = e_{\mu(\alpha)} e^{(\alpha)}_{\nu} \), respectively. For the tetrad gravity, the first order form of
(1) and second order formulations are equivalent in all dimensions, except of \( D = 2 \). (On the
Hamiltonian formulation of tetrad gravity when \( D = 2 \) see [40].) Because we are interested
in obtaining a formulation valid in all dimensions, we will not specialize our notation to a
particular dimension (as, e.g., [12, 34]), imposing only one restriction: \( D > 2 \).

To make the analysis more transparent, we rewrite the Lagrangian using integration by
parts and introducing a few short notations:

\[
L = eB^{\gamma(\rho)\mu(\alpha)\nu(\beta)} e_{\gamma(\rho),\mu} \omega_{\nu(\alpha\beta)} - eA^{\mu(\alpha)\nu(\beta)} \omega_{\mu(\alpha\gamma)} \omega^{(\gamma)}_{\nu(\beta)},
\]

where the coefficients \( A^{\mu(\alpha)\nu(\beta)} \) and \( B^{\gamma(\rho)\mu(\alpha)\nu(\beta)} \) are

\[
A^{\mu(\alpha)\nu(\beta)} \equiv e^{\mu(\alpha)} e^{\nu(\beta)} - e^{\mu(\beta)} e^{\nu(\alpha)}
\]

(3)
and

\[ B_{\gamma(\rho)\mu(\alpha)\nu(\beta)} \equiv e^{\gamma(\rho)} A_{\mu(\alpha)\nu(\beta)} + e^{\gamma(\alpha)} A_{\mu(\beta)\nu(\rho)} + e^{\gamma(\beta)} A_{\mu(\rho)\nu(\alpha)}. \]  (4)

The symmetries of \( A_{\mu(\alpha)\nu(\beta)} \) and \( B_{\gamma(\rho)\mu(\alpha)\nu(\beta)} \) follow from their definitions: e.g., \( A_{\mu(\alpha)\nu(\beta)} = A_{\nu(\beta)\mu(\alpha)} \), \( A_{\mu(\alpha)\nu(\beta)} = -A_{\nu(\alpha)\mu(\beta)} \), \( A_{\mu(\alpha)\nu(\beta)} = -A_{\mu(\beta)\nu(\alpha)} \). Similar antisymmetry properties hold for \( B_{\gamma(\rho)\mu(\alpha)\nu(\beta)} \). In (4) the second and third terms can be obtained by a cyclic permutation of the internal indices \( \rho\alpha\beta \rightarrow \alpha\beta\rho \rightarrow \beta\rho\alpha \) (keeping external indices in the same position). \( B_{\gamma(\rho)\mu(\alpha)\nu(\beta)} \) can also be presented in different form with cyclic permutations of external indices (keeping internal indices in the same position)

\[ B_{\gamma(\rho)\mu(\alpha)\nu(\beta)} = e^{\gamma(\rho)} A_{\mu(\alpha)\nu(\beta)} + e^{\gamma(\beta)} A_{\mu(\rho)\nu(\alpha)}. \]  (5)

These form, (4) and (5), are useful in calculations. As follows from their antisymmetry, \( A_{\mu(\alpha)\nu(\beta)} \) and \( B_{\gamma(\rho)\mu(\alpha)\nu(\beta)} \) equal zero when two external or two internal indices have the same value. The properties of \( A, B \) and similar functions are collected in Appendix A.

As in any first order formulation, the Hamiltonian analysis of first order tetrad gravity leads to primary constraints equal in number to the number of independent fields. Introducing momenta for all fields

\[ \pi_{\mu(\alpha)} = \frac{\delta L}{\delta e_{\mu(\alpha),0}}, \quad \Pi_{\mu(\alpha)\beta} = \frac{\delta L}{\delta \omega_{\mu(\alpha)\beta},0}, \]

we obtain the following set of primary constraints

\[ \pi^{\gamma(\rho)} - \frac{\delta}{\delta e^{\gamma(\rho),0}} (e B^{\gamma(\rho)\mu(\alpha)\nu(\beta)} e_{\gamma(\rho)\mu} \omega_{\nu(\alpha)\beta}) = \pi^{\gamma(\rho)} - e B^{\gamma(\rho)\mu(\alpha)\nu(\beta)} \omega_{\nu(\alpha)\beta} \approx 0, \]  (6)

\[ \Pi_{\mu(\alpha)\beta} \approx 0. \]  (7)

The fundamental Poisson brackets are

\[ \{ e_{\mu(\alpha)}(x), \pi^{\gamma(\rho)}(y) \} = \delta^{\gamma(\rho)}_{\mu(\alpha)} \delta(x - y), \quad \{ \omega_{\lambda(\alpha)\beta}(x), \Pi^{\rho(\mu\nu)}(y) \} = \tilde{\Delta}^{(\rho\nu)}_{(\alpha\beta)} \delta^{\rho}_{(\mu)} \delta(x - y) \]  (8)

where

\[ \tilde{\Delta}^{(\rho\nu)}_{(\alpha\beta)} = \frac{1}{2} \left( \delta^{(\rho)}_{(\alpha)} \delta^{(\nu)}_{(\beta)} - \delta^{(\nu)}_{(\alpha)} \delta^{(\rho)}_{(\beta)} \right). \]
(Note that in the text we often write a PB without the factor $\delta(x - y)$).

From the antisymmetry of $B^{\gamma(\rho)\mu(\alpha)\nu(\beta)}$ we immediately obtain for $\pi^{0(\rho)}$

$$\pi^{0(\rho)} \approx 0,$$

and for $\pi^{k(\rho)}$

$$\pi^{k(\rho)} - eB^{k(\rho)0(\alpha)\nu(\beta)}\omega_{\nu(\alpha\beta)} \approx 0.$$  \hfill (10)

Here and everywhere below in our paper we shall apply the usual convention: Greek letters for “spacetime” (both internal and external) indices, e.g. $\alpha = 0, 1, 2, \ldots, D - 1$, $\beta = 0, 1, 2, \ldots, D - 1$, and Latin letters for “space” indices $k = 1, 2, \ldots, D - 1$, $m = 1, 2, \ldots, D - 1$, etc.

All primary constraints are now identified and the Hamiltonian density takes the form

$$H = \pi^{0(\rho)}\dot{\epsilon}_{0(\rho)} + \left(\pi^{k(\rho)} - eB^{k(\rho)0(\alpha)\nu(\beta)}\omega_{\nu(\alpha\beta)}\right)\dot{\epsilon}_{k(\rho)} + \Pi^{\mu(\alpha\beta)}\dot{\omega}_{\mu(\alpha\beta)}$$

$$- eB^{\gamma(\rho)k(\alpha)\nu(\beta)}\epsilon_{\gamma(\rho),k}\omega_{\nu(\alpha\beta)} + eA^{\mu(\alpha)\nu(\beta)}\omega_{\mu(\alpha\gamma)}\omega^{(\gamma)\nu}_{\beta}.$$ \hfill (11)

There should be second class among these primary constraints because the constraint $\Pi^{10}$ contains connections $\omega_{\nu(\alpha\beta)}$ and the PBs of at least some of them with $\Pi^{\mu(\alpha\beta)}$ are not zero (in particular, $\{\pi^{k(\rho)}, \Pi^{m(\alpha\beta)}\} = -eB^{k(\rho)0(\alpha)m(\beta)}$). To clarify this and to see what connections are present in $\Pi^{10}$, we further separate $\pi^{k(\rho)}$ into components (using antisymmetry of $B^{\gamma(\rho)\mu(\alpha)\nu(\beta)}$)

$$\pi^{k(m)} - 2eB^{k(m)0(\rho)p(0)}\omega_{p(0)} - eB^{k(m)0(\rho)n(0)}\omega_{n(0)} \approx 0,$$

$$\pi^{k(0)} - eB^{k(0)0(\rho)m(0)}\omega_{m(0)} \approx 0.$$ \hfill (12)

This form shows the explicit appearance of particular connections ($\omega_{m(0)}$ or $\omega_{p(0)}$) in the primary constraints. There are no connections with the “temporal” external index in $\Pi^{12}$ and $\Pi^{13}$ and, correspondingly, the primary constraints $\Pi^{0(\alpha\beta)}$ commute with the rest of primary constraints, and therefore, the constraints $\Pi^{0(\alpha\beta)}$ are first class at this stage.

One group of constraints, which has the same form in all dimensions $D > 2$,
\[ \pi^{k(m)} - 2eB^{k(m)0(q)p(0)}\omega_{p(0)} - eB^{k(m)0(p)n(q)}\omega_{n(pq)} \approx 0, \quad \Pi^{p(q0)} \approx 0, \] (14)

form a second class subset, and using them one pair of canonical variables \((\omega_{p(0)}, \Pi^{p(q0)})\) can be now eliminated. These constraints, (14), are not of a special form \([3]\), but they are linear in \(\omega_{p(0)}\) and \(\Pi^{p(q0)}\) and the coefficient in front of \(\omega_{p(0)}\) in (14) does not depend either on \(\omega_{p(0)}\) or \(\Pi^{p(q0)}\), so after their elimination, the PBs among the remaining canonical variables will not change (i.e., they are the same as the Dirac brackets).

To eliminate this pair, \((\omega_{p(0)}, \Pi^{p(q0)})\), we have to solve (12) for \(\omega_{p(0)}\) in terms of \(\omega_{n(pq)}\) and \(\pi^{k(m)}\), and substitute this solution, as well as \(\Pi^{p(q0)} = 0\), into the total Hamiltonian.

The solution to equation (12) for \(\omega_{p(0)}\) exists in all dimensions \(D > 2\). In fact, it becomes especially simple if one notices that

\[ B^{k(m)0(q)p(0)} = -e^{0(0)}E^{k(m)p(q)} \] (15)

where

\[ E^{k(m)p(q)} \equiv \gamma^{k(m)}\gamma^{p(q)} - \gamma^{k(q)}\gamma^{p(m)} \] (16)

and

\[ \gamma^{k(m)} \equiv e^{k(m)} - \frac{e^{k(0)}e^{0(m)}}{e^{0(0)}} \] (17)

with properties

\[ \gamma^{m(p)}e_{m(q)} = \delta^{(p)}_{(q)}, \quad \gamma^{n(q)}e_{m(q)} = \delta^{n}_m. \] (18)

\(E^{k(m)p(q)}\) is also antisymmetric (i.e. \(E^{k(m)p(q)} = -E^{p(m)k(q)} = -E^{k(q)p(m)}\)) and equals to zero if \(k = p\) or \((m) = (q)\).

For any dimension \((D > 2)\) we can define the inverse of \(E^{k(m)p(q)}\)

\[ I_{m(q)a(b)} \equiv \frac{1}{D - 2}e_{m(q)}e_{a(b)} - e_{m(b)}e_{a(q)}. \] (19)

It is easy to check that

\[ I_{m(q)a(b)}E^{a(b)n(p)} = E^{n(p)a(b)}I_{a(b)m(q)} = \delta^{n}_m\delta^{(p)}_{(q)}. \] (20)
Using the above notation, the solution of (12) can be written in the form

$$\omega_{k(q0)} = -\frac{1}{2e^{\delta(0)}}I_{k(q)m(p)}\pi_{m(p)} + \frac{1}{2e^{\delta(0)}}I_{k(q)m(p)}B_{m(p)0(0)}^{0(a)n(b)}\omega_{n(ab)}.$$  \hspace{1cm} (21)

Hence we see that the constraint (12) can be solved for $\omega_{p(q0)}$ in any dimension $D > 2$, because of the existence of the inverse $I_k(q)m(p)$ and because of the same number of equations and unknowns in (12), i.e. $[\pi^{m(p)}] = [\omega_{k(q0)}] = (D - 1)^2$ (where $[X]$ indicates the number of components of a field $X$).

The second constraint, (13), cannot be solved unambiguously for $\omega_{k(pq)}$ because the number of equations, $[\pi^{k(0)}] = D - 1$, and the number of unknowns, $[\omega_{m(pq)}] = \frac{1}{2}(D - 1)^2(D - 2)$, are, in general, different. This difference depends on the dimension of spacetime:

$$[\omega_{m(pq)}] - [\pi^{k(0)}] = \frac{1}{2}(D - 1)^2(D - 2) - (D - 1) = \frac{1}{2}(D - 1)D(D - 3).$$  \hspace{1cm} (22)

In dimensions $D > 3$ we can choose only a subset of these variables for elimination which is not unique and, more importantly, this procedure will destroy the covariant form of constraints. It also creates difficulties in a consistent elimination of these fields. The components of momenta (primary constraints) that would be left after this elimination presumably would lead to secondary constraints that could be eliminated at the next stage of the Hamiltonian reduction (solving this problem in different order or mixing a primary second class pair with pairs of primary and secondary constraints is a difficult task). The detail of this analysis for $D > 3$ can be found in [33, 37].

When $D = 3$, the difference in (22) is zero. We have $[\pi^{k(0)}] = [\omega_{m(pq)}] = 2$, or two equations in two unknowns in (13). This drastically simplifies calculations. What is important, we have unambiguously one more pair of second class primary constraints, and all connections with “spatial” external indices and their conjugate momenta are eliminated at this stage leading immediately to the Hamiltonian and primary constraints which have vanishing PBs. In next Section we analyze this case ($D = 3$).
III. THE HAMILTONIAN ANALYSIS OF THE TETRAD GRAVITY IN D=3

From this point onwards, we specialize to the case of $D = 3$. The same number of equations and unknowns in (12) and (13) allows us to eliminate all connections with “spatial” external indices by solving the primary second class constraints. Moreover, there are additional simplifications that occur only for $D = 3$. First of all, in this dimension the constraint (12) becomes simpler because the second term is zero (there are three “spatial” internal indices in $B_{\gamma}^{\mu}B_{\nu}(q)$ and in when $D = 3$ at least two of them have to be equal which, based on the properties of $B_{\gamma}^{\mu}B_{\nu}(\alpha\beta)$, gives $B_{\gamma}^{\mu}\omega_{\nu}(\alpha\beta) = 0$). This leads also to a separation of the components of the spin connections among the primary constraints. Equations (12) and (13) become

$$\pi^{k(m)} - 2\epsilon B_{\gamma}^{\mu}(q)p^{(0)}\omega_{p(0)} \approx 0,$$  
(23)

$$\pi^{k(0)} - \epsilon B_{\gamma}^{\mu}(0)m(q)\omega_{m(pq)} \approx 0.$$  
(24)

The Hamiltonian in this case is

$$H = \pi^{0(\rho)}\dot{e}_{0(\rho)} + (\pi^{k(0)} - \epsilon B_{\gamma}^{\mu}(0)m(q)\omega_{m(pq)}) \dot{e}_{k(0)} + (\pi^{k(m)} - 2\epsilon B_{\gamma}^{\mu}(q)p^{(0)}\omega_{p(0)}) \dot{e}_{k(m)}$$

$$\quad + \Pi^{m(\alpha\beta)}\omega_{m(\alpha\beta)} + \Pi^{0(\alpha\beta)}\omega_{0(\alpha\beta)} - \epsilon B_{\gamma}^{\mu}(q)\omega_{\nu}(\alpha\beta) + \epsilon A_{\mu(\alpha\beta)}\omega_{\nu}(\gamma)\omega_{\nu}(\beta).$$  
(25)

One group of constraints allows us to perform the Hamiltonian reduction

$$\Pi^{m(pq)} = 0,$$  
(26)

$$\omega_{k(q)} = -\frac{1}{2\epsilon e^{0(0)}}I_{k(q)m(p)}\pi^{m(p)}.$$  
(27)

Similarly, for the second group of constraints we have

$$\Pi^{m(pq)} = 0$$  
(28)
and we need to solve (24) for \( \omega_{m(pq)} \). Using the symmetries of \( B^{\gamma(\rho)\mu(\alpha)\nu(\beta)} \) and (15) we can rewrite (24) in the form

\[
\pi^{k(0)} + ee^{0(0)}E^{k(q)m(p)}\omega_{m(qp)} = 0. \tag{29}
\]

When \( D = 3 \) there are only two independent components of \( \omega_{m(pq)} \): \( \omega_{1(12)} \) and \( \omega_{2(12)} \). Writing explicitly (29) in components and using the antisymmetry of \( E^{k(q)m(p)} \), the solution of this equation can be found and presented in a short, manifestly “covariant”, form

\[
\omega_{m(pq)} = -\frac{1}{2ee^{0(0)}}I_{m(p)k(q)}\pi^{k(0)}. \tag{30}
\]

Note that (30) is the result of peculiarities of the three dimensional case, contrary to (12) which is valid in any dimension \( D > 2 \).

Substitution of (26), (27) and (28), (30) into (11) gives us the reduced Hamiltonian with a fewer number of canonical variables and primary constraints

\[
H = \pi^{0(\rho)}\dot{\epsilon}_{0(\rho)} + \Pi^{0(\alpha\beta)}\dot{\omega}_{0(\alpha\beta)} - L(\omega_{m(pq)} = \omega_{m(pq)}(\pi^{k(0)}), \omega_{k(qp)} = \omega_{k(qp)}(\pi^{m(p)})) \tag{31}
\]

where we have explicitly separated terms proportional to \( \omega_{0(\alpha\beta)} \) in the canonical Hamiltonian

\[
H_c = -L \text{ (without "velocities")} = -2eB^{\gamma(\rho)k(p)m(0)}e_{\gamma(\rho),k}\omega_{m(p0)}
- eB^{\gamma(\rho)k(p)m(q)}e_{\gamma(k),\omega_{m(pq)}} + eA^{n(p)m(0)}\omega_{n(p0)\omega_{m}^{0(q)}} + 2eA^{n(0)m(q)}\omega_{n(0r)\omega_{m}^{0(q)}}
+ \left(2eA^{0(\alpha)m(\gamma)}\omega_{m}^{(\beta)(\gamma)} - eB^{\gamma(\rho)k(\alpha)0(\beta)}e_{\gamma(\rho),k}\right)\omega_{0(\alpha\beta)}. \tag{32}
\]

Note that in (32) there are no terms quadratic in connections with “spatial” external indices when \( D = 3 \). This is the result of antisymmetry of \( A^{\mu(\alpha)\nu(\beta)} \) and spin connections \( \omega_{\gamma(\alpha\beta)} \) in \( \alpha \) and \( \beta \), and since when \( D = 3 \) the “spatial” index can take only two values, 1 and 2.

The reduced total Hamiltonian is

\[
H_T = \pi^{0(\rho)}\dot{\epsilon}_{0(\rho)} + \Pi^{0(\alpha\beta)}\dot{\omega}_{0(\alpha\beta)} + H_c \left( e_{\mu(\alpha)}, \pi^{m(\alpha)}, \omega_{0(\alpha\beta)} \right) \tag{33}
\]
where, after a few simple rearrangements, the canonical Hamiltonian when \( D = 3 \) becomes

\[
H_c = e_0(\rho,k)\pi^k(\rho) - e_0(\rho) + \frac{1}{4e e_0(0)} I_{m(q)n(r)} \pi^m(q) \pi^n(r) \left( \eta^{(\rho)(0)} \pi^{n(0)} - 2\eta^{(\rho)(r)} \pi^{n(0)} \right)
\]

\[
- \frac{1}{2} \left( \pi^k(\alpha) \pi^{k(\beta)} - \pi^k(\beta) \pi^{k(\alpha)} \right) + 2eB\gamma(\rho)k(\alpha)0(\beta) e_\gamma(\rho),k \omega_\gamma(\alpha,\beta).
\]

(34)

To summarize, after the reduction, we have the Hamiltonian with simple primary constraints \( \pi^0(\rho) \) and \( \Pi^{0(\alpha\beta)} \) and all PBs among them are zero (as they are just fundamental, canonical, variables of this formulation). With such a simple Hamiltonian and a trivial PB algebra of primary constraints, the secondary constraints follow immediately from conservation of primary constraints

\[
\dot{\pi}^0(\rho) = \{ \pi^0(\rho), H_T \} = -\frac{\delta H_c}{\delta \pi^0(\rho)} \equiv \chi^0(\rho),
\]

(35)

\[
\dot{\Pi}^{0(\alpha\beta)} = \{ \Pi^{0(\alpha\beta)}, H_T \} = -\frac{\delta H_c}{\delta \omega^{0(\alpha\beta)}} \equiv \chi^{0(\alpha\beta)}.
\]

(36)

The explicit expressions for \( \chi^0(\rho) \) in (35) is

\[
\chi^0(\rho) = \pi^k(\rho) + \frac{1}{4e e_0(0)} I_{m(q)n(r)} \pi^m(q) \left( \eta^{(\rho)(0)} \pi^{n(0)} - 2\eta^{(\rho)(r)} \pi^{n(0)} \right)
\]

(37)

(note that, because of (30), the form of \( \chi^0(\rho) \) is also specific for \( D = 3 \) only).

The last constraint, (36), is obviously

\[
\chi^{0(\alpha\beta)} = \frac{1}{2} \pi^k(\alpha) \pi^{k(\beta)} - \frac{1}{2} \pi^k(\beta) \pi^{k(\alpha)} + eB\gamma(\rho)k(\alpha)0(\beta) e_\gamma(\rho),k.
\]

(38)

We will call (37) and (38) the “translational” and “rotational” constraints, respectively, for reasons that will become clear at the end of the analysis.

**IV. CLOSURE OF THE DIRAC PROCEDURE**

To prove that the Dirac procedure closes, we have to find the time development of secondary constraints, and check whether they produce new constraints or not. If tertiary constraints arise, we have to continue the procedure until no new constraints appear. If the PBs of secondary constraints with the total Hamiltonian are zero or proportional to
constraints already present, then the procedure stops \[1\]. The time development of the first class constraints and the PBs amongst them and with \(H_T\) are sufficient to find the gauge transformations of all canonical variables \[24\].

We first compute the PBs amongst the constraints. The primary constraints \(\pi^{0(\rho)}\) and \(\Pi^{0(\alpha\beta)}\) have vanishing PBs amongst themselves

\[\{\pi^{0(\rho)}, \Pi^{0(\alpha\beta)}\} = 0.\] (39)

The rotational constraint has obviously a zero PB with the primary constraint that generates it

\[\{\Pi^{0(\mu\nu)}, \chi^{0(\alpha\beta)}\} = 0.\] (40)

The PB of this constraint, \(\chi^{0(\alpha\beta)}\), with the second primary constraint is also zero

\[\{\pi^{0(\rho)}, \chi^{0(\alpha\beta)}\} = 0.\] (41)

With this result it is obvious that the only contribution to the secondary translational constraint comes from variation of that part of the Hamiltonian \[34\] which is not proportional to the spin connections with a “temporal” external index. Because there are no contributions proportional to the connection \(\omega_{0(\mu\nu)}\) in the secondary translational constraints \[37\], its PB with the primary rotational constraint is zero

\[\{\chi^{0(\alpha)}, \Pi^{0(\mu\nu)}\} = 0.\] (42)

The PB among the secondary and primary translational constraints has to be calculated. The result is

\[\{\pi^{0(\rho)}, \chi^{0(\alpha)}\} = 0.\] (43)

These vanishing PBs among all primary and secondary constraints simplify the analysis. We almost immediately can express the canonical Hamiltonian as a linear combination of secondary constraints plus a total spatial derivative

\[H_c = -e_0(\rho)\chi^{0(\rho)} - \omega_{0(\alpha\beta)}\chi^{0(\alpha\beta)} + (e_0(\rho)\pi^{k(\rho)})_{,k}.\] (44)
Taking into account the PBs among primary and secondary constraints, constraints (35) and (36) follow from the variation of $H_c$.

Calculation of PBs among secondary constraints is straightforward though tedious and the presence of derivatives of $e_{n(\rho)}$ requires the use of test functions (e.g., see [41]). We obtain

$$\{\chi^0(\rho), \chi^0(\gamma)\} = 0,$$

(45)

$$\{\chi^{0(\alpha\beta)}, \chi^0(\rho)\} = \frac{1}{2} \eta^{(\beta)(\rho)} \chi^{0(\alpha)} - \frac{1}{2} \eta^{(\alpha)(\rho)} \chi^0(\beta),$$

(46)

$$\{\chi^{0(\alpha\beta)}, \chi^{0(\mu\nu)}\} = \frac{1}{2} \eta^{(\beta)(\mu)} \chi^{0(\alpha\nu)} - \frac{1}{2} \eta^{(\alpha)(\mu)} \chi^{0(\beta\nu)} + \frac{1}{2} \eta^{(\beta)(\nu)} \chi^{0(\alpha\mu)} - \frac{1}{2} \eta^{(\alpha)(\nu)} \chi^{0(\beta\mu)}.$$

(47)

Note, in calculations of (45) and (46) we also used the fact that the form of $\chi^{0(\rho)}$, (30), is peculiar to $D = 3$. However, the PBs of $\chi^{0(\alpha\beta)}$ among themselves, (47), and with primary constraints, (40) and (41), are found without reference to $D = 3$, and, as was shown in [35], these PBs remain valid in all dimensions ($D > 2$). In papers on group theory the brackets (46) and (47) usually appear without $\frac{1}{2}$. It is easy to remove this factor if we replace $\chi^{0(\alpha\beta)}$ by $2\chi^{0(\alpha\beta)}$. However, we do not make this replacement here, because when deriving gauge transformations using the method of [24] it is important to find out what secondary constraint is produced exactly by the time development of the corresponding primary constraint, (38).

It is very simple to calculate time development of the secondary constraints $\chi^{0(\rho)}$ and $\chi^{0(\alpha\beta)}$, because $H_c$ is proportional to these constraints (44), and we have only simple local PB (there are no derivatives of $\delta$-functions among them and this allows us to use the associative properties of the PB). The result is:

$$\dot{\chi}^{0(\gamma)} = \{\chi^{0(\gamma)}, H_c\} = \frac{1}{2} \omega^{0(\alpha\beta)} \left(\eta^{(\beta)(\gamma)} \chi^{0(\alpha)} - \eta^{(\alpha)(\gamma)} \chi^0(\beta)\right),$$

(48)

$$\dot{\chi}^{0(\mu\nu)} = \{\chi^{0(\mu\nu)}, H_c\} = -\frac{1}{2} \epsilon^{0(\rho)} \left(\eta^{(\nu)(\rho)} \chi^{0(\mu)} - \eta^{(\mu)(\rho)} \chi^0(\nu)\right)$$

(49)

$$-\frac{1}{2} \omega^{0(\alpha\beta)} \left(\eta^{(\alpha)(\mu)} \chi^{0(\beta\nu)} - \eta^{(\beta)(\mu)} \chi^{0(\alpha\nu)} + \eta^{(\alpha)(\nu)} \chi^{0(\mu\beta)} - \eta^{(\beta)(\nu)} \chi^{0(\mu\alpha)}\right).$$
The above relations, (48) and (49), show that no new constraints appear. This completes the proof that the Dirac procedure is closed. All constraints \((\pi^0(\rho), \Pi^0(\alpha\beta), \chi^0(\rho), \chi^0(\alpha\beta))\) are first class after elimination of \(\omega_k(pq)\) and \(\omega_p(q0)\) and, moreover, the PBs of the secondary constraints, (45)-(47), form an ISO(2,1) Poincaré algebra. This is a well defined Lie algebra: there are only structure constants, no non-local PBs, and it is closed “off-shell”. (In this respect, it is similar to the gauge invariance of Yang-Mills theory.) The same result was obtained by Witten [12], though he did not use the Dirac procedure, instead he constructed the theory starting from the Poincaré algebra.

Now let us evaluate the degree of freedom (DOF) in the case of 3D tetrad gravity after eliminating \(\omega_k(pq)\) and \(\omega_p(q0)\). Using \(DOF = \#(\text{fields}) - \#(\text{FC constraints})\) we obtain

\[
\text{DOF} = [\varepsilon_{\mu}(\rho)] + [\omega_0(\alpha\beta)] - ([\pi^0(\rho)] + [\Pi^0(\alpha\beta)] + [\chi^0(\rho)] + [\chi^0(\alpha\beta)]) = 0 \tag{50}
\]

as expected for the \(D = 3\) case.

In the literature there are some discussions about the Hamiltonian formulation of pure three dimensional tetrad gravity, but, in fact, a complete Hamiltonian analysis has never before been performed. In particular, in [17] the Poincaré algebra is given but there is no explicit form of either the constraints or the Hamiltonian. In [12] the analysis was done by comparing the three dimensional tetrad gravity with Chern-Simons theory, the gauge transformations were given but without a derivation. The Hamiltonian analysis of the Chern-Simons action was done in [22]. We have not found any work (including reviews and books which are dedicated to \(D = 3\) case, e.g. [20]) where the gauge transformations of the three dimensional tetrad gravity were derived from the first class constraints according to one of the known procedure [24-26] (see footnote 2). Such a derivation is the subject of next Section.

V. GAUGE TRANSFORMATIONS FROM THE CASTELLANI PROCEDURE

We will derive the gauge transformations arising from the first class constraints by using the Castellani procedure. This procedure [24] is based on a derivation of gauge generators which are defined by chains of first class constraints. One starts with primary first class constraint(s), \(i = 1, 2, \ldots\), and construct the chain(s) \(\varepsilon_i^{(m)} G_{(m)}\) where the \(\varepsilon_i^{(m)}\) are time derivatives of order \(n\) of \(i\)th gauge parameter (the maximum value of \(n\) is fixed by the
length of the chain). In the case under consideration (the three dimensional tetrad gravity) with only primary and secondary constraints \( n = 0, 1 \). The number of independent gauge parameters is equal to the number of first class primary constraints. Note that chains are unambiguously constructed once the primary first class constraints are determined.

For tetrad gravity, we have two chains of constraints starting from the translational \((\pi^0(\rho))\) and rotational \((\Pi^0(\alpha\beta))\) primary first class constraints. According to the Castellani procedure, the generator is given by

\[
G = G^{(\rho)}_{(1)} t(\rho) + G^{(\rho)}_{(0)} t(\rho) + G^{(\alpha\beta)}_{(1)} r(\alpha\beta) + G^{(\alpha\beta)}_{(0)} r(\alpha\beta).
\]

Here \( t(\rho) \) and \( r(\alpha\beta) \) are gauge parameters which, as we will show later, parametrize the translational and rotational gauge symmetries, respectively. Note that these gauge parameters have internal indices only. It is clear even now that from the first class constraints of tetrad gravity it is impossible to derive a generator of the diffeomorphism invariance for tetrads and spin connections.\(^4\) The diffeomorphism gauge parameter \( \xi_{\mu} \) is of a very different nature. It is a “world” vector because it has an external index, whereas \( t(\rho) \) and \( r(\alpha\beta) \) are “world” scalars. We will discuss the relation between the gauge symmetry of tetrad gravity and diffeomorphism invariance in the next Section.

The functions \( G_{(1)} \) in (51) are the primary constraints

\[
G^{(\rho)}_{(1)} = \pi^{0(\rho)}, \quad G^{(\alpha\beta)}_{(1)} = \Pi^{0(\alpha\beta)}
\]

and \( G_{(0)} \) are defined using the following relations \[24\]

\[
G^{(\rho)}_{(0)} (x) = - \{ \pi^{0(\rho)} (x), H_T \} + \int \left[ \alpha^{(\rho)}_{(\gamma)} (x, y) \pi^{0(\gamma)} (y) + \alpha^{(\rho)}_{(\alpha\beta)} (x, y) \Pi^{0(\alpha\beta)} (y) \right] d^2y,
\]

\[
G^{(\alpha\beta)}_{(0)} (x) = - \{ \Pi^{0(\alpha\beta)} (x), H_T \} + \int \left[ \alpha^{(\alpha\beta)}_{(\gamma)} (x, y) \pi^{0(\gamma)} (y) + \alpha^{(\alpha\beta)}_{(\nu\mu)} (x, y) \Pi^{0(\nu\mu)} (y) \right] d^2y,
\]

where the functions \( \alpha^{(\cdot)}_{(\cdot)} (x, y) \) have to be chosen in such a way that the chains end at primary constraints

\(^4\) Diffeomorphism was also not derivable from the constraint structure of the Chern-Simons action \[22\].
\{G^{\alpha}_{(0)}(x), H_T\} = \text{primary.} \quad (55)

To construct the generator (51), we have to find \(\alpha_{(\alpha \beta)}^{(\gamma)}(x, y)\) using condition (55). This calculation, because of the simple PBs among the constraints when \(D = 3\), is straightforward:

\[
\{G^{(\rho)}_{(0)}(x), H_T\} = \left\{-\chi^{0(\rho)}(x) + \int \left[\alpha^{(\rho)}_{(\gamma)}(x, y) \pi^{0(\gamma)}(y) + \alpha^{(\omega)}_{(\rho \beta)}(x, y) \Pi^{0(\omega \beta)}(y)\right] d^2y, H_T\right\} = 0, \quad (56)
\]

\[
\{G^{(\alpha \beta)}_{(0)}(x), H_T\} = \left\{-\chi^{0(\alpha \beta)}(x) + \int \left[\alpha^{(\alpha \beta)}_{(\gamma)}(x, y) \pi^{0(\gamma)}(y) + \alpha^{(\omega \nu)}_{(\nu \mu)}(x, y) \Pi^{0(\omega \nu \mu)}(y)\right] d^2y, H_T\right\} = 0 \quad (57)
\]

where \(H_T\) can be replaced by \(H_c = -e_0(\sigma) \chi^{0(\sigma)} - \omega_{0(\sigma \lambda)} \chi^{0(\sigma \lambda)},\) because PBs among primary constraints themselves and among primary and secondary constraints are zero.

From (56) and (57) and the PBs among first class constraints we find all the functions \(\alpha_{(\gamma)}^{(\rho)}(x, y)\) in (53), (54),

\[
\alpha^{(\rho)}_{(\alpha \beta)}(x, y) = 0, \quad (58)
\]

\[
\alpha^{(\rho)}_{(\alpha \beta)}(x, y) = \omega_{0(\rho)} \delta(x - y), \quad (59)
\]

\[
\alpha^{(\alpha \beta)}_{(\gamma)}(x, y) = \frac{1}{2} \left( e_0^{(\alpha)} \delta^{(\beta)}_{(\gamma)} - e_0^{(\beta)} \delta^{(\alpha)}_{(\gamma)} \right) \delta(x - y), \quad (60)
\]

\[
\alpha^{(\alpha \beta)}_{(\nu \mu)}(x, y) = \left( \omega_{0(\nu)}^{(\alpha)} \delta^{(\beta)}_{(\mu)} - \omega_{0(\mu)}^{(\alpha)} \delta^{(\beta)}_{(\nu)} \right) \delta(x - y). \quad (61)
\]

This completes the derivation of the generator (51) as now

\[
G^{(\rho)}_{(0)} = -\chi^{0(\rho)} + \omega_{0(\gamma)} \delta^{0(\gamma)} \quad (62)
\]

and

\[
G^{(\alpha \beta)}_{(0)} = -\chi^{0(\alpha \beta)} + \frac{1}{2} \left( e_0^{(\alpha)} \delta^{(\beta)}_{(\gamma)} - e_0^{(\beta)} \delta^{(\alpha)}_{(\gamma)} \right) \pi^{0(\gamma)} + \omega_{0(\nu)}^{(\alpha)} \Pi^{0(\beta \nu)} - \omega_{0(\mu)}^{(\beta)} \Pi^{0(\alpha \mu)}. \quad (63)
\]

Substitution of (52), (62), and (63) into (51) gives
\[ G = \pi^0(\rho)\dot{t}(\rho) + \left( -\chi^0(\rho) + \omega_0(\rho) \pi^0(\gamma) \right) t(\rho) + \Pi^0(\alpha \beta) \dot{r}(\alpha \beta) + \]

\[
\left[ -\chi^{0(\alpha \beta)} + \frac{1}{2} \left( e^{(\alpha) \delta(\beta)} - e^{(\beta) \delta(\alpha)} \right) \pi^{0(\gamma)} + \omega_0^{(\alpha \mu)} \Pi^0(\beta \mu) - \omega_0^{(\beta \mu)} \Pi^0(\alpha \mu) \right] r(\alpha \beta). \tag{64}
\]

Now using

\[ \delta (field) = \{ G, field \} \tag{65} \]

we can find the gauge transformations of fields.

For example, for \( \delta \omega_0(\sigma \lambda) \) one finds

\[ \delta \omega_0(\sigma \lambda) = -\dot{r}(\sigma \lambda) - \left( \omega_0^{(\alpha \lambda)} \delta^{(\beta)}(\sigma) - \omega_0^{(\alpha \sigma)} \delta^{(\beta)}(\lambda) \right) r(\alpha \beta). \tag{66} \]

This result is the same as Witten’s \cite{12} for spin connections with the “temporal” external index, \( \omega_0(\sigma \lambda) \). Witten used a different notation which is specific to 3D, while we will present the transformations of the fields in covariant form. Note that \( \delta \omega_0(\sigma \lambda) \) depends only on the rotational parameter. In the gauge transformation of tetrads both parameters are present

\[ \delta e_0(\lambda) = -\dot{t}(\lambda) - \omega_0(\rho) t(\rho) - \frac{1}{2} \left( e^{(\alpha)} \delta(\beta) - e^{(\beta)} \delta(\alpha) \right) r(\alpha \beta). \tag{67} \]

Equation (65) gives for \( \delta e_k(\lambda) \):

\[ \delta e_k(\lambda) = \frac{\delta \chi^{0(\rho)}}{\delta \pi^k(\lambda)} t(\rho) + \frac{\delta \chi^{0(\alpha \beta)}}{\delta \pi^k(\lambda)} r(\alpha \beta) = -t(\lambda,k) - \omega_k(\rho) t(\rho) - \frac{1}{2} \left( e^k(\alpha) \delta(\beta) - e^k(\beta) \delta(\alpha) \right) r(\alpha \beta). \tag{68} \]

(Here we substituted \( \pi^k(\lambda) \) in term of \( \omega_k(\alpha \beta) \) from (23) and (24)).

We can write together (67) and (68) as one covariant equation

\[ \delta e_\gamma(\lambda) = -t(\lambda,\gamma) - \omega_\gamma(\rho) t(\rho) - \frac{1}{2} \left( e^\gamma(\alpha) \delta(\beta) - e^\gamma(\beta) \delta(\alpha) \right) r(\alpha \beta). \tag{69} \]

This gauge transformation, (69), also confirms the result of Witten \cite{12}, but in \cite{12} it was not derived.

To obtain \( \delta \omega_k(\sigma \lambda) \), we need first to find \( \delta \pi^\gamma(\lambda) \). Equation (65) gives for \( \delta \pi^\gamma(\lambda) \)

\[ \delta \pi^0(\lambda) = \frac{1}{2} \left( \eta^{(\alpha)(\lambda)} \pi^{0(\beta)} - \eta^{(\beta)(\lambda)} \pi^{0(\alpha)} \right) r(\alpha \beta), \tag{70} \]
\[ \delta \pi^m(\lambda) = \frac{1}{2} \left( \eta^{(\alpha)(\lambda)} \pi^m(\beta) - \eta^{(\beta)(\lambda)} \pi^m(\alpha) \right) r_{(\alpha\beta)} + e B^m(\lambda) \pi^{0(\beta)} r_{(\alpha\beta),n}. \]  

Note that both equations, (70) and (71), can be written in one covariant form

\[ \delta \pi^\gamma(\lambda) = \frac{1}{2} \left( \eta^{(\alpha)(\lambda)} \pi^\gamma(\beta) - \eta^{(\beta)(\lambda)} \pi^\gamma(\alpha) \right) r_{(\alpha\beta)} + e B^\gamma(\lambda) \pi^{0(\beta)} r_{(\alpha\beta),n}. \]  

Using (27) and (30), together with the transformation properties of \( e^\gamma(\lambda) \) (69) and \( \pi^\gamma(\lambda) \) (72), we can obtain the gauge transformation of \( \delta \omega_k(\sigma\lambda) \):

\[ \delta \omega_k(\sigma\lambda) = -r_{(\sigma\lambda),k} - \left( \omega_k^{(\alpha)} \delta^{(\beta)}(\sigma) - \omega_k^{(\beta)} \delta^{(\alpha)}(\sigma) \right) r_{(\alpha\beta)}. \]  

Now combining (66) and (73), we get the covariant equation for \( \delta \omega_{\gamma(\sigma\lambda)} \)

\[ \delta \omega_{\gamma(\sigma\lambda)} = -r_{(\sigma\lambda),\gamma} - \left( \omega_{\gamma}^{(\alpha)} \delta^{(\beta)}(\sigma) - \omega_{\gamma}^{(\beta)} \delta^{(\alpha)}(\sigma) \right) r_{(\alpha\beta)}. \]  

The gauge transformations of field variables \( e \gamma(\lambda) \) and \( \omega\gamma(\sigma\lambda) \) have been expressed in covariant form.

To summarize, our analysis has confirmed Witten’s result: when \( D = 3 \), we have obtained the same gauge transformations for \( e \gamma(\lambda) \) and \( \omega\gamma(\sigma\lambda) \) as in [12]. From our analysis, which is based on the Dirac procedure, we derived the gauge transformations (69) and (74) generated by the first class constraints for the tetrad gravity when \( D = 3 \). The PBs of the secondary first class constraints form the Poincaré algebra ISO(2,1). This is not surprising, as equivalent formulations of the same theory should produce the same result, e.g. as Lagrangian and Hamiltonian formulations. When \( D = 3 \), the canonical Hamiltonian (44) is a linear combination of the secondary first class constraints which we have called translational and rotational, and this is consistent with there being zero degrees of freedom (50). We see that the notorious “diffeomorphism constraint” (neither the full nor “spatial” one) does not arise in the course of the Hamiltonian analysis of tetrad gravity in \( D = 3 \). We would like to also mention that Blagojević [22] performing the Hamiltonian analysis of the Chern-Simons action and using the Castellani procedure to find the gauge invariance stated that “the diffeomorphisms are not found” and concluded: “Thus, the diffeomorphisms are not an independent symmetry [Italic is of M.B.]”. In the next Section we will compare the gauge invariance found here using the Hamiltonian analysis with diffeomorphism invariance.
The last step is to check the invariance of the Lagrangian under the gauge transformations. Actually, it is not necessary as the derivation of gauge transformations is performed in such a way that Lagrangian should be automatically invariant, however, we will check this for consistency. It is not difficult to show that the transformations under rotation, the part in (69) proportional to \( r_{(\alpha \beta)} \) and (74), give \( \delta_r L = 0 \) in all dimensions \( (D > 2) \). Note that in derivation of rotational constraints we did not use any peculiarity of the \( D = 3 \) case. We also confirm in [36] that using the Lagrangian methods the same transformations under rotation arise in all dimensions \( (D > 2) \) and they leave the Lagrangian invariant. As we have shown in [36], the transformations under translation are different in dimensions \( D > 3 \). It is not evident that the Lagrangian (1) which has the same form in all dimensions \( (D > 2) \) is invariant under translational transformations that are specific to \( D = 3 \). From (69) and (74) we can see that in the \( D = 3 \) case they are

\[
\delta_t e_{\gamma(\lambda)} = -t_{(\lambda),\gamma} - \omega_{\gamma(\lambda)}^\rho t_{(\rho)}, \quad \delta_t \omega_{\gamma(\sigma\lambda)} = 0.
\]  

(75)

The proof that the EC Lagrangian (1) is invariant under (75) is given in Appendix A.

VI. GAUGE INVARIANCE VERSUS DIFFEOMORPHISM INVARIANCE FOR TETRAD GRAVITY IN D=3

We would like to mention again that we will call the “gauge symmetry” the invariance that follows from the structure of the first class constraints of the Hamiltonian formulation of a theory. In particular, in the Hamiltonian analysis of the EC action in \( D = 3 \) we found that the gauge symmetry is translation and rotation in the tangent space. But we know that the Lagrangian (1) is also invariant under diffeomorphism. Let us compare these invariances.

We will use the particular form of a diffeomorphism transformation given by [42, 43]  
\[
\delta g_{\mu\nu} = -\xi_{\mu,\nu} - \xi_{\nu,\mu}, \quad (76)
\]

or by another equivalent form

\[5\] In the mathematical literature, the term diffeomorphism refers to a mapping from one manifold to another which is differentiable, one-to-one, onto, and with a differentiable inverse.
\[ \delta g_{\mu\nu} = -\xi_{\mu,\nu} - \xi_{\nu,\mu} + g^{\alpha\beta} (g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta}) \xi_{\alpha}, \]  

(77)

where \( \xi_\mu \) is the diffeomorphism parameter (which is a “world” vector) and the semicolon \( ; \) signifies the covariant derivative. In the literature on the Hamiltonian formulation of metric General Relativity the word “diffeomorphism” is often used as being equivalent to the transformation (76), which is similar to gauge transformations in ordinary field theories. It is in exactly this sense that diffeomorphism invariance was derived in the Hamiltonian analysis of the Einstein-Hilbert action (metric gravity) when \( D > 2 \) for the second order [7, 9] and the first order [27] forms, without any need for a noncovariant and/or a field dependent redefinition of the parameter \( \xi_\mu \).

The transformation similar to (77) can also be derived for the tetrad field \( e_{\gamma(\lambda)} \). One way is to use the relation between the metric tensor \( g_{\mu\nu} \) and the tetrads \( e_{\gamma(\lambda)} \)

\[ g_{\mu\nu} = e_{\mu(\lambda)} e^{(\lambda)}_{\nu}. \]  

(78)

From (78) it follows that

\[ \delta e_{\nu(\lambda)} = \frac{1}{2} e^{(\lambda)}_{\rho} \delta g_{\rho\nu}. \]  

(79)

If we substitute (77) into (79) and use \( \xi^\rho = g^{\rho\alpha} \xi_\alpha \), we obtain

\[ \delta e_{\nu(\lambda)} = -e_{\rho(\lambda)} \xi^\rho_{,\nu} - e_{\nu(\lambda),\rho} \xi^\rho. \]  

(80)

Another way to derive the transformation (80) is to use the fact that \( e_{\nu(\lambda)} \) is a “world” vector and transforms under a general coordinate transformations as

\[ e^{\nu}_{(\lambda)} (x') = \frac{\partial x'^{\nu}}{\partial x^\gamma} e^{(\gamma)}_{(\lambda)} (x). \]  

(81)

For infinitesimal transformations

\[ x^\mu \rightarrow x'^{\mu} = x^\mu + \xi^\mu (x) \]  

(82)

equation (81) can be written as

\[ e^{\mu}_{(\lambda)} (x') = e^{\mu}_{(\lambda)} (x) + \xi^\mu_{,\gamma} e^{(\gamma)}_{(\lambda)} (x) + O (\xi^2). \]  

(83)
Combining the Taylor expansion of $e^\mu_{(\lambda)}(x')$

$$e^\mu_{(\lambda)}(x') = e^\mu_{(\lambda)}(x^\gamma + \xi^\gamma(x)) = e^\mu_{(\lambda)}(x) + e^\mu_{(\lambda),\gamma}\xi^\gamma + O(\xi^2) \quad (84)$$

with (83) and replacing $e^\mu_{(\lambda),\gamma}$ by $e^\mu_{(\lambda),\gamma}$, the transformation $\delta e^\mu_{(\lambda)}(x)$ follows

$$\delta e^\mu_{(\lambda)}(x) = e^\mu_{(\lambda)}(x) - e^\mu_{(\lambda)}(x) = e^\gamma_{(\lambda)}\xi^\mu - e^\mu_{(\lambda),\gamma}\xi^\gamma. \quad (85)$$

Using $\delta \left(e^{\lambda(\gamma)}_{(\lambda)}e^\mu(\gamma)\right) = 0$ it is easy to obtain the transformation $\delta e^\nu_{(\lambda)}$ given by (80).

This perpetrated “gauge” transformation of $e^\nu_{(\lambda)}$, (80), can be found in many papers on the tetrad gravity, e.g. [11, 20, 38], as well as in Witten’s paper [12]. However, the only gauge transformation of $e^\nu_{(\lambda)}$ following from the Hamiltonian formulation is given by (69) and is not a diffeomorphism.

As stated in [12], the gauge transformation (69) and the diffeomorphism (80) “are equivalent”. However, this equivalence needs an imposition of severe additional conditions: (i) a field-dependent redefinition of a gauge parameter $\xi^\beta = e^{\beta(\rho)}t_{(\rho)}$; (ii) keeping only the translational invariance and disregarding the rotational invariance; (iii) using the equations of motion (“on-shell” invariance). It is difficult to accept such an “equivalence” and voluntarily replace the derived ISO(2,1) gauge symmetry of tetrad gravity by diffeomorphism plus Lorentz invariance or, even worse, with only a “spatial” diffeomorphism, as is often presented in the literature. As we have already shown, a gauge invariance of a theory can be found exactly if one follows the Dirac procedure in which one casts the theory into a Hamiltonian form, finds all constraints, the PBs among them and classifies them as first class or second class, derives the gauge generator from the first class constraints, and finally uses this gauge generator to find gauge transformations of variables in the theory. Using this procedure, we have derived the gauge transformations of tetrads $e^\gamma_{(\lambda)}$ (69) and spin connections $\omega_{\rho(\alpha\beta)}$ (74). Moreover, the algebra of secondary first class constraints gives unambiguously the Poincaré algebra ISO(2,1), (45)-(47), not an algebra of diffeomorphism and Lorentz rotations. We have to conclude that the gauge invariance of the tetrad gravity in three dimensions is a Poincaré symmetry. Diffeomorphism (80) is the symmetry of the Einstein-Cartan action, but it is NOT A GAUGE SYMMETRY derived from the first class constraints in the Hamiltonian formulation of tetrad gravity [36].

It is not surprising that metric and tetrad gravity theories have different gauge symmetries
as they are not equivalent. Einstein in his article on tetrad (n-bein) gravity wrote [44]: “The n-bein field is determined by \( n^2 \) functions \( h^\mu_\nu \) [tetrads \( \epsilon_\alpha^{\mu} \), in our notation], whereas the Riemannian metric is determined by \( \frac{n(n+1)}{2} \) quantities. According to (3) \( [g_{\mu\nu} = h_{\mu\alpha}h^{\alpha}_{\nu}] \), the metric is determined by the n-bein field but not vice versa”. So, the attempt to deduce a gauge transformation of \( \epsilon_{\gamma(\lambda)} \) from the diffeomorphism invariance of \( g_{\mu\nu} \) is the wrong way. But it should be possible to deduce a transformation of \( g_{\mu\nu} \) from that of \( \epsilon_{\gamma(\lambda)} \) (the “vice versa” of Einstein).

To compare the results of (69) and (80), without forcing an equivalence by imposing the restrictions (i)-(iii) mentioned after (85), we rewrite (69) in a slightly different form. First of all, from the equation of motion \( \frac{\delta \eta}{\delta \omega_{\mu(\alpha,\beta)}} = 0 \) it follows

\[
B^{\varepsilon(\alpha)(\mu(\lambda))\varepsilon(\rho)}_{\varepsilon(\alpha)\mu(\lambda)} - A^{\sigma(\lambda)(\mu(\beta))}_{\sigma(\lambda)\mu(\beta)} + A^{\sigma(\rho)(\nu)}_{\sigma(\rho)\nu} = 0. \tag{86}
\]

Solving (86) for \( \omega^{(\mu(\lambda))}_{\nu(\beta)} \), we can express it in terms of \( \epsilon_{\rho(\lambda)} \) and its derivatives

\[
\omega^{(\mu(\lambda))}_{\nu(\beta)} = \frac{1}{2} \varepsilon_{\rho(\lambda)}^{\sigma(\mu(\lambda))\mu(\lambda)} \left( \eta^{(\mu(\lambda))}_{\sigma(\mu(\lambda))\mu(\lambda)} \right) + \eta^{(\rho(\lambda))}_{\sigma(\rho(\lambda))\sigma(\rho(\lambda))} \epsilon_{\rho(\lambda)}^{\sigma(\rho(\lambda))} \epsilon_{\rho(\lambda)}^{\mu(\lambda)} \left( \epsilon_{\rho(\lambda)}^{\mu(\lambda)} - \mu_{\rho(\lambda)} \right), \tag{87}
\]

or in more familiar form

\[
\omega^{(\mu(\lambda))}_{\nu(\beta)} = \frac{1}{2} \left[ \varepsilon^{\rho(\lambda)}_{\rho(\lambda)} \left( \epsilon^{\rho(\lambda)}_{\rho(\lambda)} - \epsilon^{\rho(\lambda)}_{\rho(\lambda)} \right) - \varepsilon^{\rho(\lambda)}_{\rho(\lambda)} \left( \epsilon^{\rho(\lambda)}_{\rho(\lambda)} - \epsilon^{\rho(\lambda)}_{\rho(\lambda)} \right) - \varepsilon^{\rho(\lambda)}_{\rho(\lambda)} \epsilon^{\rho(\lambda)}_{\rho(\lambda)} \epsilon^{\mu(\lambda)}_{\rho(\lambda)} \left( \epsilon_{\rho(\lambda)}^{\mu(\lambda)} - \epsilon_{\rho(\lambda)}^{\mu(\lambda)} \right) \right]. \tag{88}
\]

Substitution of (87) into (69) gives

\[
\delta \epsilon_{\gamma(\lambda)} = - t_{\gamma(\lambda)} \epsilon_{\gamma(\lambda)} - \frac{1}{2} \left( \epsilon_{\gamma(\lambda)} - \epsilon_{\gamma(\lambda)} \right) \epsilon^{(\mu)}_{\rho(\lambda)} t^{(\mu)}_{\rho(\lambda)} - \frac{1}{2} \left( \epsilon^{(\mu)}_{\rho(\lambda)} - \epsilon^{(\mu)}_{\rho(\lambda)} \right) t^{(\mu)}_{\rho(\lambda)}
\]

\[
- \frac{1}{2} \left( \epsilon^{(\mu)}_{\rho(\lambda)} - \epsilon^{(\mu)}_{\rho(\lambda)} \right) t^{(\mu)}_{\rho(\lambda)} - \frac{1}{2} \left( \epsilon^{(\mu)}_{\rho(\lambda)} - \epsilon^{(\mu)}_{\rho(\lambda)} \right) \epsilon^{(\beta)}_{\rho(\lambda)} r_{\rho(\lambda)}. \tag{89}
\]

From (78) we can find that

\[
\delta g_{\mu\nu} = e^{(\lambda)}_{\mu} \delta \epsilon_{\nu(\lambda)} + e^{(\lambda)}_{\nu} \delta \epsilon_{\mu(\lambda)}. \tag{90}
\]

Now from \( \delta \epsilon_{\mu(\lambda)} \), (89), and from (90) we can obtain \( \delta g_{\mu\nu} \)

\[
\delta g_{\mu\nu} = - \left( \epsilon^{(\mu)}_{\rho(\lambda)} t^{(\mu)}_{\rho(\lambda)} \right)_{\nu(\lambda)} - \left( \epsilon^{(\mu)}_{\nu(\lambda)} t^{(\mu)}_{\nu(\lambda)} \right)_{\mu(\lambda)} + (g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta}) \epsilon^{(\beta)}_{\rho(\lambda)} t^{(\rho)}_{\nu(\lambda)}. \tag{91}
\]
that after the redefinition $\xi^\beta = e^\beta(\rho) t(\rho)$ leads to (77). Note, that contributions with the rotational parameter $r_{(\alpha\beta)}$ from (89) completely cancel out in (91), as well as some terms proportional to $t(\rho)$, without imposing any conditions. We do not need the additional restrictions (ii)-(iii); only field-dependent redefinition of parameters (i) is needed. Thus it is possible to obtain the diffeomorphism invariance of $g_{\mu\nu}$ from the Poincaré symmetry of $e_{\gamma(\lambda)}$, but not vice versa. A field-dependent redefinition of the gauge parameter is necessary but this is a consequence of the non-equivalence of metric and tetrad gravities. A similar redefinition had to be introduced when we considered the two dimensional metric and tetrad gravities in [40], despite that the gauge symmetry of two dimensional gravity (both metric and tetrad) is very different from higher dimensions.

VII. CONCLUSION

In his book [1] Dirac wrote “I [Dirac] feel that there will always be something missing from them [non-Hamiltonian methods] which we can only get by working from a Hamiltonian”. We feel that the Hamiltonian method not only allows one to find something that can be missed when using other methods but also protects us from “finding” something that might be attributed to a theory but really not there. In particular, the gauge invariance of a theory should follow from the Dirac procedure and any guess (even an intelligent one) has to be supported by calculations.

The results reported in this paper confirm Dirac’s old conjecture [1]. The Hamiltonian formulation of the tetrad gravity considered here using the Dirac procedure leads to self-consistent and unambiguous results. In particular, this approach gives a unique answer to the question of what is the true gauge invariance of the tetrad gravity and eliminates any possibility of being able to “choose” a gauge invariance based on either a belief, desire or common wisdom, because the gauge invariance should be derivable from the unique constraint structure of this (or any) theory. This constraint structure can be modified only by a non-canonical change of variables that immediately destroys any connection with an original theory and, at best, can be considered as some model not related to the tetrad gravity (see the discussion on non-canonical change of variables for tetrad gravity in Section 5 of [27]).

After Hamiltonian reduction, solving the second class constraints and eliminating non-
physical (redundant) variables, the remaining first class constraints form a Lie algebra. This is exactly what one can expect for the local field theory and this is precisely what one needs to quantize it. The results presented in this paper were mainly obtained for $D = 3$ case which has been treated completely. The Hamiltonian analysis of the EC action in higher dimensions is in progress \cite{35,37} and further developments will be reported elsewhere.

However, even results of this paper provide enough information to form some conclusions about the Hamiltonian formulation of tetrad gravity in higher dimensions, based on the following reasoning:

- the Lagrangian of the first order formulation of the tetrad gravity (2) with tetrads and spin connections treated as independent fields gives the equivalent equations of motion as its second order counterpart, and this can be demonstrated in any dimensions $D > 2$ in covariant form and without any recourse to a particular dimension and does not show any peculiarities in $D = 3$;

- the primary first class constraints are a part of the generator and unambiguously define the gauge parameters in (51) in any dimension, because of the antisymmetry of $B_{\gamma(\rho)\mu(\alpha)\nu(\beta)}$, there is a translational constraint $\pi^{0(\rho)}$ and the corresponding gauge parameter cannot have a “world” index that we would need if diffeomorphism were a gauge invariance derivable from the first class constraints;

- the explicit form of the rotational constraint $\chi^{0(\alpha\beta)}$ (38) is not peculiar to being $D = 3$ and written in covariant form, that is why it remains unchanged in dimensions $D > 3 \ [35,37]$;

- the PBs among the rotational constraints (47) were also calculated without using any specific property of $D = 3$;

- the final form of gauge transformations of the tetrad gravity when $D = 3$ can be cast into a covariant dimension-independent form \cite{69,74};

- the secondary translational constraint $\chi^{0(\rho)}$ has specific to $D = 3$ form (the second term in (37)) but it does not mean that it will be modified in higher dimensions in such a way that it will destroy the translational invariance in the internal space.

Based on the above arguments, we can conclude that diffeomorphism invariance is not a gauge symmetry derived from the first class constraints of the tetrad gravity, neither in $D = 3$ nor in higher dimensions, and the translational and rotational invariance are expected in all dimensions ($D > 2$).
These conclusions for $D > 3$ can be called conjectures. The only way to prove or disprove them is to apply the Dirac procedure and explicitly find the Hamiltonian, eliminate all second class constraints and use all remaining first class constraints to build the generators that will produce the true gauge invariance of the tetrad gravity in higher dimensions.\textsuperscript{6} Some preliminary results are reported in [35–37].

**Acknowledgements**

We would like to thank D.G.C. McKeon for helpful discussions during the preparation of the paper and reading the manuscript.

**Appendix A: $ABC$ properties and translational invariance of the EC Lagrangian**

Here we collect properties of the $ABC$ functions that are useful in the Hamiltonian analysis [35, 37] and also in the Lagrangian approach [36] to the Einstein-Cartan action.

These functions are generated by consecutive variation of the $n$-bein density $ee^{\mu(\alpha)}$:

\[
\frac{\delta}{\delta e_{\nu(\beta)}} (ee^{\mu(\alpha)}) = e \left( e^{\mu(\alpha)} e^{\nu(\beta)} - e^{\mu(\beta)} e^{\nu(\alpha)} \right) = e A^{\mu(\alpha)\nu(\beta)},
\]

(A1)

\[
\frac{\delta}{\delta e_{\lambda(\gamma)}} (eA^{\mu(\alpha)\nu(\beta)}) = eB^{\lambda(\gamma)\mu(\alpha)\nu(\beta)},
\]

(A2)

\[
\frac{\delta}{\delta e_{\tau(\sigma)}} (eB^{\lambda(\gamma)\mu(\alpha)\nu(\beta)}) = eC^{\tau(\sigma)\lambda(\gamma)\mu(\alpha)\nu(\beta)}, \ldots
\]

(A3)

The first important property of these density functions is their total antisymmetry: interchange of two indices of the same nature (internal or external), e.g.

\[
A^{\mu(\beta)\mu(\alpha)} = -A^{\nu(\alpha)\mu(\beta)} = -A^{\mu(\beta)\nu(\alpha)}
\]

(A4)

with the same being valid for $B$, $C$, etc. In calculations presented here, nothing is needed beyond $C$.

The second important property is their expansion using an external index

\textsuperscript{6} Of course, a non-canonical change of variables has to be excluded at any step of calculations, and any manipulation that destroys the equivalence with the original theory are not permissible.
\[ B^\tau(\rho)\mu(\alpha)\nu(\beta) = e^\tau(\rho) A^\mu(\alpha)\nu(\beta) + e^\tau(\alpha) A^\mu(\beta)\nu(\rho) + e^\tau(\beta) A^\mu(\rho)\nu(\alpha), \quad \text{(A5)} \]

\[ C^\tau(\rho)\lambda(\sigma)\mu(\alpha)\nu(\beta) = e^\tau(\rho) B^{\lambda(\sigma)\mu(\alpha)\nu(\beta)} - e^\tau(\sigma) B^{\lambda(\alpha)\mu(\beta)\nu(\rho)} + e^\tau(\alpha) B^{\lambda(\beta)\mu(\rho)\nu(\sigma)} - e^\tau(\beta) B^{\lambda(\rho)\mu(\sigma)\nu(\alpha)} \quad \text{(A6)} \]

or an internal index

\[ B^\tau(\rho)\mu(\alpha)\nu(\beta) = e^\tau(\rho) A^\mu(\alpha)\nu(\beta) + e^\mu(\rho) A^{\nu(\alpha)\tau(\beta)} + e^\nu(\rho) A^{\tau(\alpha)\mu(\beta)}, \quad \text{(A7)} \]

\[ C^\tau(\rho)\lambda(\sigma)\mu(\alpha)\nu(\beta) = e^\tau(\rho) B^{\lambda(\sigma)\mu(\alpha)\nu(\beta)} - e^\lambda(\rho) B^{\lambda(\alpha)\nu(\beta)\tau(\lambda)} + e^\mu(\rho) B^{\nu(\sigma)\tau(\alpha)\mu(\beta)} - e^\nu(\rho) B^{\tau(\sigma)\lambda(\alpha)\mu(\beta)}. \quad \text{(A8)} \]

The third property involves their derivatives

\[ (eA^{\nu(\beta)\mu(\alpha)})_{,\sigma} = \frac{\delta}{\delta e_{\lambda(\gamma)}} (eA^{\nu(\beta)\mu(\alpha)}) e_{\lambda(\gamma),\sigma} = eB^{\lambda(\gamma)\nu(\beta)\mu(\alpha)} e_{\lambda(\gamma),\sigma}, \quad \text{(A9)} \]

\[ (eB^{\tau(\rho)\nu(\beta)\mu(\alpha)})_{,\sigma} = \frac{\delta}{\delta e_{\lambda(\gamma)}} (eB^{\tau(\rho)\nu(\beta)\mu(\alpha)}) e_{\lambda(\gamma),\sigma} = eC^{\tau(\rho)\lambda(\gamma)\nu(\beta)\mu(\alpha)} e_{\tau(\rho),\sigma}. \quad \text{(A10)} \]

Upon using the antisymmetry of \( B \) both in the external and internal indices and the antisymmetry of \( \omega \) in its internal indices leads to

\[ B^{\tau(\rho)\mu(\alpha)\nu(\beta)} \omega^{(\gamma}_{\mu(\alpha\gamma)} \omega^{(\sigma)_{\nu(\sigma)}} \omega^{(\beta)} = 0 \quad \text{(A11)} \]

and

\[ eB^{\tau(\rho)\mu(\alpha)\nu(\beta)} \omega^{(\nu(\alpha\beta))} = 0. \quad \text{(A12)} \]

The above properties considerably simplify calculations. The list of \( ABC \) properties can be extended, but for our purpose the above relations are adequate.

Let us check the invariance of the Einstein-Cartan Lagrangian \[ \text{(1)} \] under the translational transformations in \( D = 3 \) \[ \text{(75)} \].
\[ \delta_t L = \delta_t \left[ -e A^{\mu(\alpha)\nu(\beta)} \left( \omega_{\nu(\alpha\beta),\mu} + \omega_{\mu(\alpha\sigma)} \omega^{\sigma(\beta)}_{\nu} \right) \right] = -\delta_t \left( e A^{\mu(\alpha)\nu(\beta)} \left( \omega_{\nu(\alpha\beta),\mu} + \omega_{\mu(\alpha\sigma)} \omega^{\sigma(\beta)}_{\nu} \right) \right) \\
= -e B^{\gamma(\lambda)\mu(\alpha)\nu(\beta)} \delta_t e_{\gamma(\lambda)} \left( \omega_{\nu(\alpha\beta),\mu} + \omega_{\mu(\alpha\sigma)} \omega^{\sigma(\beta)}_{\nu} \right) \\
= -e B^{\gamma(\lambda)\mu(\alpha)\nu(\beta)} \left( -t(\lambda,\gamma) - \omega_{\gamma(\lambda)} t(\rho) \right) \left( \omega_{\nu(\alpha\beta),\mu} + \omega_{\mu(\alpha\sigma)} \omega^{\sigma(\beta)}_{\nu} \right). \]

After extracting the total derivative it leads to the following

\[ \delta_t L = - \left[ e B^{\gamma(\lambda)\mu(\alpha)\nu(\beta)} t(\lambda) \left( \omega_{\nu(\alpha\beta),\mu} + \omega_{\mu(\alpha\sigma)} \omega^{\sigma(\beta)}_{\nu} \right) \right]_{\gamma} \tag{A13} \]

The first term in (A13) is the total derivative, the second term is zero in \( D = 3 \). This is because the derivative \( (e B^{\gamma(\lambda)\mu(\alpha)\nu(\beta)})_{\gamma} \) is proportional to \( C \) which is antisymmetric in four internal and four external indices, but in \( D = 3 \) there are only three distinct indices: 0, 1, 2. The third term in (A13) is also zero because of (A12). It is not evident that the fourth and fifth terms cancel. From the antisymmetry of \( B \) it follows that \( \lambda \neq \alpha \neq \beta \) in these terms; in addition, from the antisymmetry of \( \omega \) in internal indices and from the fact that in \( D = 3 \) there are only three distinct indices we can conclude that in the fifth term \( \rho \) should be equal either \( \alpha \) or \( \beta \) (note that this is a peculiarity of \( D = 3 \) only). If we consider both cases, \( \rho = \alpha \) and \( \rho = \beta \), and relabel dummy indices in the fifth term, we find that the fifth and fourth terms cancel. The last, sixth, term is not exactly in the form of \( (A11) \). However, if we use again antisymmetry of \( B \) and \( \omega \) and peculiarity of \( D = 3 \) (only three distinct indices), then we have also only two cases: either \( \rho = \alpha \) or \( \rho = \beta \). In both cases we can apply (A11) for the sixth term. Finally, under translational transformations of (75) in \( D = 3 \) we have

\[ \delta_t L = - \left[ e B^{\gamma(\lambda)\mu(\alpha)\nu(\beta)} t(\lambda) \left( \omega_{\nu(\alpha\beta),\mu} + \omega_{\mu(\alpha\sigma)} \omega^{\sigma(\beta)}_{\nu} \right) \right]_{\gamma} \]

or, using definition of \( B \) (A5) the Ricci tensor \( R_{\mu(\alpha)(\beta)} = \omega_{\nu(\alpha\beta),\mu} - \omega_{\mu(\alpha\beta),\nu} + \omega_{\mu(\alpha\sigma)} \omega^{\sigma(\beta)}_{\nu} - \omega_{\nu(\alpha\sigma)} \omega^{\sigma(\beta)}_{\mu} \),

\[ \delta_t L = - \left[ e \left( e^{\gamma(\lambda)} e^{\mu(\alpha)} e^{\nu(\beta)} + e^{\gamma(\alpha)} e^{\mu(\beta)} e^{\nu(\lambda)} + e^{\gamma(\beta)} e^{\mu(\lambda)} e^{\nu(\alpha)} \right) t(\lambda) R_{\mu(\alpha)(\beta)} \right]_{\gamma}. \]
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