The emission of electromagnetic radiation from a quantum system interacting with an external noise: a general result

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Abstract
We compute the spectrum of radiation emitted by a generic quantum system interacting with an external classic noise. Our motivation is a wish to understand this phenomenon within the framework of collapse models. However, the computation is general and applies to practically any situation in which a quantum system interacts with a noise. The computation is carried out at a perturbative level. This poses problems as regards the correct way of performing the analysis, as repeatedly discussed in the literature. We will also clarify this issue.

Keywords: radiation emission, collapse models, perturbation theory

1. Introduction
In this paper we analyze the dynamics of a nonrelativistic charged quantum system under the influence of an external classical random scalar field. We focus on the spectrum of the radiation emitted by the system, due to the interaction with the field.

The study of the radiation emission is particularly important in the context of collapse models, because up to now this process has set the strongest bound on the possible values that the phenomenological parameters defining these models can take [1, 2]. The idea is the following: collapse models modify the standard quantum evolution given by the Schrödinger equation, by adding a nonlinear interaction with a random scalar field, which induces the collapse of the wavefunction [3–6]. In the case of a charged particle, the random field forces the particle to emit radiation, whereas standard quantum theory predicts no emission (if the particle is free). Therefore this situation represents a case study for testing collapse models against the standard theory [7–11].
This calculation finds application also in the theory of open quantum systems. A charged particle in a bath, typically interacting with the other particles via a position-dependent force, would also emit radiation.

The problem of radiation emission has already been studied in the literature of collapse models. In [12] a calculation for the free particle was carried out to the first perturbative order using the continuous spontaneous localization (CSL) model [4]. This result was confirmed and generalized to the case of a noise that was not a white noise in [13], where the analysis was also extended to the case of a hydrogenic atom. Later, in [14], the formula for the emission rate was computed by using the simpler quantum mechanics with universal position localizations (QMUPL) model [15, 16] for the case of a free particle and a harmonic oscillator. However, the formula found in [14] did not agree with the one found in [12, 13] as it should have: in the case of a white noise, the rate of emission from a free particle found in [14] turned out to be twice that of [12, 13]. The origin of this discrepancy was clarified in [17], where the perturbative calculation in the CSL model was repeated. The result, in the white noise case, was found to be in agreement with that of [14]. However, it was also shown that, when the computation is generalized to the case of a noise that is not a white noise, an unphysical contribution appears. In fact, the formula for the rate was found to be [17]

\[
\frac{d\Gamma}{dk} = \frac{\lambda h^2}{4\pi^2\epsilon_0 cm_0^2 r_C^2 k} \left[ \tilde{f}(0) + \tilde{f}(\omega_k) \right],
\]

where \( h, c \) and \( \epsilon_0 \) have the usual meaning, \( \lambda \) and \( r_C \) are two parameters characterizing the CSL model [4], \( m_0 \) is the mass of a nucleon, \( k = |k| \) with \( k \) the photon wavevector, and

\[
\tilde{f}(\omega) := \int_{-\infty}^{+\infty} f(s) e^{i\omega s} ds,
\]

is the noise spectral density, where \( f(s) \) is the noise time correlation function. In the white noise case, \( f(s) = \delta(s) \), which implies that \( \tilde{f}(\omega) = 1 \) for any \( \omega \). The second term in equation (1) tells us that the probability of emitting a photon with wavevector \( k \) is proportional to the spectral density of the noise at the frequency \( \omega_k = kc \). This is an expected contribution. On the other hand, the first term is proportional to the spectral density of the noise at zero energy and is physically suspicious. In fact, the zero-energy component of the noise is not expected to contribute to the emission of photons with an arbitrarily high energy. More precisely, as we will show, this energy nonconserving term is exactly what in standard perturbation theory are known as ‘nonresonant terms’ [18]. In general, these terms should not give any important contribution when one is computing the emission rate. This is not the case for equation (1). This shows that there are problems when standard perturbative techniques are used to find the emission rate. A first way out of the problem was found in [17]: it was shown that when the computation is repeated taking wave packets as final states and confining the noise, then the unphysical contribution proportional to \( \tilde{f}(0) \) is no longer present. However, even if this prescription leads to a satisfactory result, it does not really clarify the origin of the unphysical terms.

A deeper insight into the problem was obtained using the QMUPL model, where an exact treatment of the problem is possible [19]. It was shown that in the case of a free particle, the unphysical term proportional to \( \tilde{f}(0) \) is still present. However, for a harmonic oscillator with frequency \( \omega_0 \), the unphysical term is suppressed by an exponential damping factor \( e^{-\Lambda t} \) with \( \Lambda = \frac{\omega_0^2 \beta}{2m} \) and \( \beta = \frac{\epsilon_0}{4\pi m_0 c^2} \). It is important to note that treating the electromagnetic interaction at the lowest order is equivalent to setting \( \beta = 0 \), meaning that \( \Lambda = 0 \), in which case the unphysical term \( \tilde{f}(0) \) is no longer suppressed. The same problem arises when \( \omega_0 = 0 \), which
is the free particle case. Therefore, the analysis done in [19] proved that in order to get a physically meaningful result, first, the particle cannot be treated as completely free and, second, the electromagnetic interaction cannot be treated at the lowest perturbative order. Using the above result, the emission rate for a harmonic oscillator within the CSL model was computed in [20]. The interaction with the noise was treated perturbatively and that with the electromagnetic field exactly. It was found that in the free particle limit, the emission rate is equal to that given in equation (1) without the unphysical term. However, this analysis lacks generality: the calculations required solving the Heisenberg equations while treating the electromagnetic interaction exactly, and this can be done only for simple systems.

The aim of this work is to derive a very general result, which can be applied to a large variety of systems and interactions. Instead of considering a specific model, as was done in the works previously cited, we derive a general result for a generic interaction with an external classic noise of the type described in equation (4). Moreover, our result will hold for any generic bounded system, contrary to those from the previous analyses where only the free particle, harmonic oscillator and hydrogenic atom cases were considered. Since we are considering generic systems, the calculation of the emission rate is too complicated to be done exactly, so we need to resort to perturbation theory. According to the results found in [19, 20], in order to avoid the presence of unphysical contributions, we have to find a way of including the effect of the higher order electromagnetic contributions to the interaction. A first-order analysis would once again lead to the problem that one encounters with equation (1). A first attempt might be to consider all the diagrams at the next relevant order. However, this would require too long a calculation, since the number of such diagrams is very large (of the order of 70). As we will show, there is a cleverer way to take into account the effects of the relevant higher order contributions. The key point is the observation that, as mentioned before, the unphysical term is exactly what in standard perturbation theory is known as the ‘nonresonant’ term [18]. The most general and elegant way of avoiding the presence of these terms is to take into account the decay of the propagator. In fact, because of the electromagnetic interaction, the propagator is not stable and can decay3 [18, 21]. We will show that when this effect is taken into account, the unphysical term is no longer present. As a result, we will be able to find a formula for the rate of emission from a generic system. The result applies to all known collapse models and, as we mentioned before, also to the open quantum system, where the effect of the environment is modeled by the interaction with a random potential. Moreover, as mentioned before, this result is also valid for any quantum system interacting with an external classic noise field.

2. The model

The starting point is the Schrödinger equation:

\[ i\hbar \frac{d}{dt} \psi_t = H_{\text{TOT}} \psi_t, \tag{3} \]

with

\[ H_{\text{TOT}} := H - \hbar \gamma \sum_{i} N_i w_{i,t}, \tag{4} \]

where \( H \) is the standard Hamiltonian of the system, the \( N_i \) are a set of commuting self-adjoint operators, \( \gamma \) is a coupling constant and the \( w_{i,t} \) are a set of independent noises such that

3 In principle, there is also a similar effect due to the noise, but here we are not interested in computing it.
with \( \mathbb{E} \) denoting the average over the noise and \( f \) a generic correlation function. In many cases, the index \( 'r' \) is replaced by the coordinate \( 'x' \) and the set of noises become a random classic scalar field in space (and time) \([4–6]\).

We will consider a generic system composed of \( N_p \) charged particles. Since the number of particles is fixed, they can be described by using the first-quantization formalism. In contrast, the electromagnetic field will be described by using the second-quantization formalism.

The standard Hamiltonian \( H \) contains three terms:

\[
H = H_P + H_R + H_{\text{INT}}.
\]

The first term is the Hamiltonian of the particles, which has the form

\[
H_P = \sum_{j=1}^{N_p} \left( \frac{p_j^2}{2m_j} + V(x_j) + \sum_{i<j=1}^{N_p} U(x_j - x_i) \right),
\]

with \( m_j \) the mass of the \( j \)th particle of the system, \( V(x) \) an external potential and \( U(x_j - x_i) \) the interaction potential, between the \( j \)th particle and the \( i \)th particle. We are not making any assumption on the form of the potentials \( V \) and \( U \), so they are generic. The second term is the free Hamiltonian of the electromagnetic field (we are working in the Coulomb gauge):

\[
H_R = \int dx \frac{1}{2} \left( \epsilon_0 E^2 + \frac{B^2}{\mu_0} \right),
\]

where \( \epsilon_0 \) and \( \mu_0 \) are, respectively, the vacuum permittivity and permeability, and \( E_\perp = -\frac{\partial A}{\partial t} \) is the transverse part of \( E \). The last term describes the interaction between the electromagnetic field and the particles:

\[
H_{\text{INT}} = \sum_{j=1}^{N_p} \left( -\frac{e_j}{m_j} \right) \mathbf{A}(x_j) \cdot \mathbf{p}_j + \sum_{j=1}^{N_p} \frac{e_j^2}{2m_j} \mathbf{A}^2(x_j).
\]

Here \( e_j \) is the charge of the \( j \)th particle of the system and \( \mathbf{A}(x) \) is the vector potential, which can be expanded in plane waves as

\[
\mathbf{A}(x) = \int d\mathbf{k} \sum_{l} a_{l} \left[ \mathbf{e}_{k,l} a_{k,l} e^{i \mathbf{k} \cdot \mathbf{x}} + \mathbf{e}_{k,l}^{\dagger} a_{k,l}^{\dagger} e^{-i \mathbf{k} \cdot \mathbf{x}} \right],
\]

with \( a_{l} = \sqrt{\hbar/2\epsilon_0 a_{l}(2\pi)^3} \), \( a_{l} = ke_l \), \( \epsilon_{k,l} \) the polarization vectors, and \( a_{k,l} \) and \( a_{k,l}^{\dagger} \), respectively, the annihilation and creation operators of a photon with wavelength \( \lambda \) and polarization \( \lambda \).

Here we are assuming that the evolution is unitary and driven by the total Hamiltonian \( H \) in equation (4). The noise models the interaction with an external environment \([22, 23]\) or, through the ‘imaginary noise trick’ \([24]\); see also references therein), the collapse of the

4 In principle, one could also consider different forms of interaction between the system and the noises. We focus on the one given in equation (4) because this is the coupling between the noises and the system taken in the collapse models, which are the models that we are mainly interested in studying.

5 To be more precise, in principle there is a restriction on \( V \) and \( U \). As we will see, in the derivation of the emission rate (equation (65)) a fundamental role is played by the fact that the eigenstates of \( H_P \) are not stable and decay because of vacuum fluctuations of the electromagnetic field. This is true only when the imaginary part of the energy shift \( \Delta E_i \) defined in equation (B.1) is different from zero. As one can easily realize from inspecting equation (B.1), \( \Delta E_i \) vanishes only in specific and rather pathological cases, which do not arise for typical bounded systems.
wavefunction. In this second case, which is the one that we are primarily interested in, on setting \( \gamma \to \lambda \) and \( N_{\ell} \to q_{\ell} \) with \( \ell = 1, 2, 3 \) labeling the three space directions, one recovers the ‘imaginary noise’ version of the QMUPL model \([15, 16]\). If instead one replaces the discrete index \( \ell \) with the continuous parameter \( x \), then the discrete sum over \( \ell \) becomes an integral over \( x \), and on making the substitutions

\[
N_{\ell} \to \sum_{j=1}^{N_p} \frac{m_j}{m_0} \left( \frac{\sqrt{2\pi} r_c}{\lambda} \right)^3 \exp \left( -\left( \mathbf{r}_j - \mathbf{x} \right)^2 / 2 \lambda^2 \right).
\]

with \( N_p \) the number of particles of the system and therefore \( w_{\ell}(t) \to w(x, t) \), the ‘imaginary noise’ version of the first-quantization version of the CSL model is obtained \([4]\).

In view of the perturbative expansion, we write the Hamiltonian \( H_{\text{TOT}} \) in equation (4) as the sum of two contributions:

\[
H_{\text{TOT}} = H_0 + H_1(t)
\]

where \( H_0 = H_P + H_R \) is the unperturbed Hamiltonian with known eigenvalues \( E_n \) and relative eigenvectors \( n \), while the remaining term describes the interactions with the electromagnetic field and the noise:

\[
H_1(t) = \sum_{j=1}^{N_p} \left( -\frac{e_j}{m_j} \right) \mathbf{A}(\mathbf{x}_j) \cdot \mathbf{p}_j + \sum_{j=1}^{N_p} \frac{e_j^2}{2m_j} \mathbf{A}^2(\mathbf{x}_j) - \sqrt{\hbar} \sum_{\ell} N_{\ell} w_{\ell}(t).
\]

In the following calculation we neglect the term containing \( \mathbf{A}^2 \) since we only need a result at the lowest perturbative order in \( \epsilon \) and \( \gamma \). Then equation (12) becomes

\[
H_1(t) = \sum_{j=1}^{N_p} \left( -\frac{e_j}{m_j} \right) \mathbf{A}(\mathbf{x}_j) \cdot \mathbf{p}_j - \sqrt{\hbar} \sum_{\ell} N_{\ell} w_{\ell}(t).
\]

### 3. The emission rate at the lowest perturbative order

In order to compute the emission rate, we need the probability of transition from the initial state \( |i; \Omega \rangle = |i \rangle \otimes |\Omega \rangle \) to the final state \( |f; \lambda \rangle = |f \rangle \otimes |\mathbf{k}, \lambda \rangle \). Here \( |i \rangle \) and \( |f \rangle \) denote, respectively, the initial and final states of the system that are eigenvectors of \( H_0 \), while \( |\Omega \rangle \) and \( |\mathbf{k}, \lambda \rangle \) denote respectively the vacuum state of the electromagnetic field and the state with one photon with the wavevector \( \mathbf{k} \) and polarization \( \lambda \). The transition probability is given by

Figure 1. The lowest order contributions to the emission rate, represented in terms of Feynman diagrams. Here ‘C.C.’ denotes the complex conjugate of the term in the second line. The solid lines represent the charged fermion, the wavy lines the photon, and the dashed lines the noise field. In the above diagrams each electromagnetic vertex gives a factor proportional to \( \epsilon \) while each noise vertex gives a factor proportional to \( \sqrt{\hbar} \).
\[ P_{\Omega} = \mathcal{E} \left[ \langle f ; k, \lambda | U(t, t_i) | i \rangle | \Omega \rangle \right] \] (14)

where \( U(t, t_i) \) is the time evolution operator, from the initial time \( t_i \) to the time \( t \). Using the standard perturbative approach, we compute the relevant contribution at the lowest order, which is given by the diagrams in figure 1.

As shown in appendix A, the corresponding amplitudes are:

\[
A_1 = \left( \frac{-i}{\hbar} \right)^2 (-h \sqrt{\gamma}) \int_{t_i}^{t_f} dt_1 \int_{t_i}^{t_f} dt_2 \times \sum_n \sum_{l,m} e^{i(\Delta_n + \omega_m) h t} e^{i\Delta_n t} W_F(t_2) \langle f | R_k | n \rangle \langle n | N_F | i \rangle; \]
(15)

\[
A_2 = \left( \frac{-i}{\hbar} \right)^2 (-h \sqrt{\gamma}) \int_{t_i}^{t_f} dt_1 \int_{t_i}^{t_f} dt_2 \times \sum_n \sum_{l,m} e^{i(\Delta_n + \omega_m) h t} e^{i\Delta_n t} W_F(t_1) \langle f | N_F | n \rangle \langle n | R_k | i \rangle; \]
(16)

\[
B = \left( \frac{-i}{\hbar} \right) \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt_2 \int_{t_i}^{t_f} dt_3 \int_{t_i}^{t_f} dt_4 \times \sum_{n,m,l} \sum_{l,m} e^{i(\Delta_n + \omega_m) h t} e^{i\Delta_n t} W_F(t_4) \langle f | R_k | n \rangle \langle n | N_F | m \rangle \langle m | N_F | i \rangle; \]
(17)

\[
C_1 = \left( \frac{-i}{\hbar} \right)^3 h^2 \gamma \int_{t_i}^{t_f} dt_1 \int_{t_i}^{t_f} dt_2 \int_{t_i}^{t_f} dt_3 \times \sum_{n,m,l} \sum_{l,m} e^{i(\Delta_n + \omega_m) h t} e^{i\Delta_n t} W_F(t_1) W_F(t_3) \times \langle f | N_F | n \rangle \langle n | R_k | m \rangle \langle m | N_F | i \rangle; \]
(18)

\[
C_2 = \left( \frac{-i}{\hbar} \right)^3 h^2 \gamma \int_{t_i}^{t_f} dt_1 \int_{t_i}^{t_f} dt_2 \int_{t_i}^{t_f} dt_3 \times \sum_{n,m,l} \sum_{l,m} e^{i(\Delta_n + \omega_m) h t} e^{i\Delta_n t} W_F(t_1) W_F(t_3) \times \langle f | N_F | n \rangle \langle n | R_k | m \rangle \langle m | N_F | i \rangle; \]
(19)

\[
C_3 = \left( \frac{-i}{\hbar} \right)^3 h^2 \gamma \int_{t_i}^{t_f} dt_1 \int_{t_i}^{t_f} dt_2 \int_{t_i}^{t_f} dt_3 \times \sum_{n,m,l} \sum_{l,m} e^{i(\Delta_n + \omega_m) h t} e^{i\Delta_n t} W_F(t_1) W_F(t_3) \times \langle f | N_F | n \rangle \langle n | N_F | m \rangle \langle m | R_k | i \rangle. \]
(20)

Here we have introduced the radiation matrix element:

\[
R_k := \sum_{j=1}^{N_E} \left( -\frac{e_j}{m_j} \right) e^{-ik \cdot p_j} \cdot \mathbf{r}_j, \quad \text{with} \quad \alpha_k \equiv \sqrt{\frac{\hbar}{2 \varepsilon_0 m_k (2\pi)^3}}. \]
(21)
The formula for the transition probability then reads

\[ P_{fi} = \mathbb{E} \left\{ |A_1 + A_3|^2 + 2 \text{Re} \left[ \left( B^*C_1 + B^*C_2 + B^*C_3 \right) \right] \right\} \]  

(22)

As indicated earlier, we are interested in computing the emission rate:

\[ \frac{dI}{dk} = \sum_i \int d\Omega_k \frac{d}{dt} \sum_f P_{fi}, \]  

(23)

where, apart carrying out the summation over the possible final states of the system, we also integrate over the possible directions (\( \int d\Omega_k \)) and polarizations (\( \sum_\lambda \)) of the emitted photon. Using equation (22), the emission rate becomes

\[ \frac{dI}{dk} = \sum_i \int d\Omega_k \frac{d}{dt} \sum_f \mathbb{E} \left\{ |A_1 + A_2|^2 + 2 \text{Re} \left[ \left( B^*C_1 + B^*C_2 + B^*C_3 \right) \right] \right\}. \]  

(24)

In the following sections we will focus on computing the terms introduced in equations (15)-(20). As discussed in the introduction, a direct computation of these terms leads to the wrong result. In the next section we analyze the term \( A_1 \), pointing out where problems arise and how to avoid them.

4. The unphysical terms: how to avoid them

In this section we show that the unphysical contributions to the emission rate arise because of the presence of nonresonant terms in the transition amplitude. We will show how to avoid these terms by taking into account the decay of the propagator, due to the electromagnetic self-interaction.

4.1. The connection with the ‘nonresonant terms’

In order to understand the origin of the undesired terms, we study the contribution \( A_1 \) coming from the diagram ‘\( A_1 \)’ in figure 1. The corresponding transition amplitude for this diagram is given by equation (15). In the following analysis it will be convenient to expand the noises \( w_{\ell}(t) \) in Fourier components:

\[ w_{\ell}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\nu \ e^{-i\nu t} w_{\ell}(\nu), \]  

(25)

such that

\[ A_1 = \frac{\sqrt{\rho}}{2\pi} \sum_n \sum_\ell \langle f | R_k | n \rangle \langle n | N_\ell | i \rangle \int_{-\infty}^{+\infty} d\nu \ w_{\ell}(\nu) T, \]  

(26)

with

\[ T := \int_{t_1}^{t_2} dt_1 \int_{t_1}^{t_2} dt_2 \ e^{i(\Delta u + o_\ell) t_1} e^{i(\Delta u - o_\ell) t_2}; \]

\[ = \int_{t_1}^{t_2} dt_2 \int_{t_1}^{t_2} dt_1 \ e^{i(\Delta u + o_\ell) t_2} e^{i(\Delta u - o_\ell) t_1}. \]  

(27)

When only the term \( A_1 \) is considered, the emission rate is proportional to the time derivative of the transition probability \( P_{fi} = \mathbb{E} \left| A_1 \right|^2 \). Using the relation
\[ E \left[ w_f(\nu)^n w_f(\omega) \right] = 2\pi \delta_{\nu}\delta(\nu - \omega)\tilde{f}(\nu), \]

with \( \tilde{f}(\nu) \) defined in equation (2), we can write \( P_{fi} \) as

\[
P_{fi} = \frac{\gamma}{4\pi^2\hbar^2} \int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} d\nu \sum_{n} \sum_{n} \left( \langle n | R_k | n \rangle \langle n | N_i | i \rangle \right)^2.
\]

Let us focus on \( T \), which contains the time dependence of \( P_{fi} \), which is the source of the problems. Taking \( t_i = 0 \) we get

\[
T = \frac{-1}{i(\Delta_{fi} + \omega_k)} \left[ \frac{e^{i(\Delta_{fi} + \omega_k)t} - 1 - e^{i(\Delta_{fi} - \nu)t}}{i(\Delta_{fi} - \nu)} \right] .
\]

When taking the square modulus, in the large time limit the crossed terms oscillate and do not contribute to the emission rate. In contrast, the square modulus of each term in equation (30) has the form

\[
\left| \frac{e^{i\nu} - 1}{ix} \right|^2 \xrightarrow{t \to \infty} 2\pi \delta(x).
\]

The first term in equation (30), called the resonant term, gives the relevant contribution for when the energy is conserved, i.e. when \( \nu = \Delta_{fi} + \omega_k \). In contrast, the second term in equation (30), called the nonresonant term, becomes relevant when \( \nu = \Delta_{ni} \). It is because of this term that, in the case of a free particle, one gets the unphysical contribution to the rate, proportional to \( \tilde{f}(0) \) (see equation (1)). Notice that the presence of nonresonant terms is not related to the fact that our interaction is a noise: they appear also with a generic potential [25]. In the next subsection we introduce the decay of the propagator and we show why, on taking this effect into account, the nonresonant terms can be neglected.

### 4.2. Decay of the propagator

In [14, 20] it was shown that the higher order contributions of the electromagnetic interaction play a fundamental role in avoiding the presence of the unphysical terms. This suggests that their effect should be introduced also in the perturbative calculations. In particular, it was shown that the role of higher order contributions of the electromagnetic interaction is exponentially damping the unphysical terms. This exponential damping resembles the exponential decay of the eigenstates of the unperturbed Hamiltonian \( H_0 \) due to the electromagnetic interaction. In fact, it is well known that, due to the electromagnetic interaction, the eigenstates of the free Hamiltonian \( H_0 \) are not stable and can decay [21, 25]. More precisely, the higher order contributions of the electromagnetic interaction add a complex shift \( \Delta E = \Delta E_i + i\Delta E_i \) to the eigenenergies. The real part \( \Delta E_i \) is a shift of the energy levels (the Lamb shift), while the imaginary part \( \Delta E_i \) describes the decay rate (or natural broadening) of the state. As a check that this decay is related to the exponential damping factors found in [14, 20], in appendix B we compute \( \Delta E_i \) for a harmonic oscillator. The broadening is shown to be proportional to the decay rate \( \Lambda = \frac{\alpha^2 \hbar}{2m} \) found in
[14, 20], responsible for suppressing the terms proportional to \( \tilde{f}(0) \). This strongly suggests that the decay of the eigenstates due to electromagnetic interactions plays a fundamental role in ways of avoiding the presence of the unphysical terms\(^6\).

Therefore, we compute again \( T \) of equation (27), taking into account the possibility that the propagator decays. This is equivalent to replacing, in the integral, \( e^{\frac{E_i}{\hbar}(t_1-t_2)} \) with \( e^{\frac{i}{\hbar}(E_i+E_R)\Delta t} \). In such a case,

\[
T = \int_0^t dt_1 \int_0^N dt_2 e^{i\left(\Delta_{f,n}+\Delta_{f,k}\right)t_1} e^{i\left(\Delta_{f,n}+\Delta_{f,k}\right)t_2} e^{-\frac{1}{\hbar}\left[\frac{\Delta_{f,n}+\Delta_{f,k}}{2}\right] t_1} e^{-\frac{1}{\hbar}\left[\frac{\Delta_{f,n}+\Delta_{f,k}}{2}\right] t_2} \frac{1}{i\left(\Delta_{f,n}+\Delta_{f,k}\right)-\frac{i}{\hbar}}.
\]  
 (32)

The first term is the same as in equation (30). Therefore we still have a contribution in the large time limit, when \( \nu = \Delta_{f,n} + \Delta_{f,k} \). However, because of the damping \( e^{-\frac{1}{\hbar}t} \), the second term no longer contributes to the emission rate for large times. This shows that, taking into account the decay of the propagator, one can avoid the presence of the non-resonant term and therefore the unphysical factor \( \tilde{f}(0) \). The great advantage of this method is that it is quite general and does not depend on the form of the matrix elements \( \langle f \mid R_i \mid n \rangle \) and \( \langle n \mid N_{\nu} \mid i \rangle \).

In the rest of the article we apply this method to compute all contributions in equations (15)–(20). We will show that, with a proper use of the decay of the propagator, the unphysical term is no longer present in the final formula.

5. Contributions of the amplitudes \( A_1 \) and \( A_2 \) to the emission rate

The contribution of the amplitudes \( A_1 \) and \( A_2 \) to the emission rate (equation (24)) can be written as the sum of three terms:

\[
\sum_k \int d\Omega_k \frac{d^2 \Omega}{d\epsilon} |A_1 + A_2|^2 = \frac{\gamma}{\hbar^2} \sum_k \int d\Omega_k \left[ R_{11} + 2 \text{Re}(R_{12}) + R_{22} \right].
\]  
 (33)

where we have introduced:

\[
R_{11} = \frac{d^2 \Omega}{d\epsilon} \sum_k \left| \sum_f \sum_n \sum_{\ell} \int_{t_0}^{t_1} dt_1 \int_{t_0}^{t_2} dt_2 e^{i\left(\Delta_{f,n}+\Delta_{f,k}\right)\ell} e^{i\Delta_{f,n}t_2} \langle f \mid R_k \mid n \rangle \langle n \mid N_{\nu} \mid i \rangle \right|^2;
\]  
 (34)

\[
R_{12} = \frac{d^2 \Omega}{d\epsilon} \left( \sum_k \sum_f \sum_n \sum_{\ell} \int_{t_0}^{t_1} dt_1 \int_{t_0}^{t_2} dt_2 e^{i\left(\Delta_{f,n}+\Delta_{f,k}\right)\ell} e^{i\Delta_{f,n}t_2} \langle f \mid R_k \mid n \rangle \langle n \mid N_{\nu} \mid i \rangle \right) \times \left( \sum_k \sum_f \sum_n \sum_{\ell} \int_{t_0}^{t_1} dt_1 \int_{t_0}^{t_2} dt_2 e^{i\Delta_{f,n}t_1} \langle f \mid N_{\nu} \mid n' \rangle \langle n' \mid R_k \mid i \rangle \right);\]
 (35)

\(6\) This is also discussed in [18], for Compton scattering.
\[ R_{22} = \frac{d}{dt} \sum_f \mathbb{E} \left[ \sum_n \sum_r \int_{t_1}^{t} dt_1 \int_{t_1}^{t} dt_2 e^{i\Delta_n t_1} e^{i(\Delta_n + \omega_1 + \omega_2) t_2} w_r(t_1) \right] \times \left\langle f \right| N_r \left| n \right\rangle \left\langle n \right| R_k \left| i \right\rangle^2. \] (36)

\( R_{11} \) \((R_{22})\) is the transition probability corresponding to the amplitude represented by diagram \( A_1 \) \((A_2)\) in figure 1. The effects of the interference between these two transition amplitudes are contained in the term \( R_{12} \). We compute the three terms separately.

### 5.1. Computation of \( R_{11} \)

Since we will focus on the time-dependent part, it is convenient to write \( R_{11} \) in the following way:

\[ R_{11} = \sum_f \sum_{n,m} \left( \sum_{\ell, \ell'} \left\langle f \right| R_k \left| n \right\rangle \left\langle n \right| N_r \left| i \right\rangle \left\langle f \right| R_k \left| m \right\rangle^* \left\langle m \right| N_r \left| i \right\rangle^* \right) \frac{d}{dt} T_{11}. \] (37)

where

\[ T_{11} = \int_{t_1}^{t_1'} dt_1 \int_{t_1}^{t_1'} dt_2 \int_{t_2}^{t_1} dt_3 \int_{t_1}^{t_1'} dt_4 e^{i(\Delta_n + \omega_1) t_1} e^{i(\Delta_n + \omega_1) t_2} e^{-i(\Delta_n + \omega_1) t_3} e^{-i\Delta_n t_4} \times \mathbb{E} \left[ w_r(t_2) w_r(t_4) \right]. \] (38)

Until now we have never introduced decay of the propagator. If one computes \( T_{11} \) as defined in equation (38), then the unphysical term is present. Taking into account the decay of the propagator, i.e., the fact that the intermediate states \( |m\rangle \) and \( |n\rangle \) decay respectively with rates \( \Gamma_m \) and \( \Gamma_n \), amounts to introducing the exponentials \( e^{-\Gamma_m (t_2 - t_1)} \) and \( e^{-\Gamma_n (t_4 - t_1)} \) in equation (38).

Then, setting for simplicity \( t_1 = 0 \) and using \( \mathbb{E} \left[ w_r(t_2) w_r(t_4) \right] = \delta_{\ell, l'} f(t_2 - t_4) \), equation (38) becomes

\[ T_{11} = \delta_{\ell, l'} \int_{t_1}^{t_1'} dt_2 \int_{t_2}^{t_1} dt_4 e^{i(\Delta_n + \omega_1) t_2} e^{i(\Delta_n + \omega_1 + \omega_2)} e^{-i(\Delta_n + \omega_1 + \omega_2) t_4} f(t_2 - t_4) \times \mathbb{E} \left[ e^{i(\Delta_n + \omega_1) t_2} e^{i(\Delta_n + \omega_1 + \omega_2) t_4} \right]. \]

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In the large time limit $t \to \infty$ only the last term survives. Notice that without the introduction of the decay of the propagator, the first term in equation (39) would also not have been negligible, giving rise to a term proportional to $\tilde{f}(0)$. Using the relation

$$\frac{d}{dt} e^{-\alpha t} \left( \int_0^t dt_1 \int_0^{t_1} dt_2 \ e^{\beta t_2} f(t_2 - t_1) \right) = e^{-\alpha t} \left( \int_0^t dx \ e^{\alpha x} \phi(x) + \int_0^t dx \ e^{\beta x} \phi(x) \right)$$

and the fact that in the fourth line of equation (39) we have $a = -b = \Delta_\beta + \omega_k$, we get, in the large time limit,

$$\frac{d}{dt} T_1 = \left[ i\left( \Delta_{fn} + \omega_k \right) - \Gamma_n \right] \left[ -i\left( \Delta_{fm} + \omega_k \right) - \Gamma_m \right] \tilde{f}(\Delta_\beta + \omega_k),$$

with $\tilde{f}(k)$ defined in equation (2). Coming back to equation (37), we have

$$R_{11} = \sum_{f, n, m} \sum_{\ell} \left\langle f \mid R_k \mid n \right\rangle \left\langle n \mid N = i \right\rangle \left\langle f \mid R_k \mid m \right\rangle \left\langle m \mid N = i \right\rangle \frac{d}{dt} T_1,$$

5.2. Computation of $R_{12}$

Like in the computation of $R_{11}$, we start by splitting the time-dependent part of $R_{12}$ from the rest:

$$R_2 = \sum_{f, n, m} \sum_{\ell} \left\langle f \mid R_k \mid n \right\rangle \left\langle n \mid N = i \right\rangle \left\langle f \mid R_k \mid m \right\rangle \left\langle m \mid N = i \right\rangle \frac{d}{dt} T_2,$$

where

$$T_2 := \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 \ e^{i\Delta_{fn} t_4} e^{i\Delta_{fm} t_3} e^{-\Delta_\beta t_2} e^{-i\Delta_{fn} t_1} e^{-i\Delta_{fm} t_2} \phi(t_2 - t_3).$$

As before, we introduce the decay of the propagator by including the exponentials $e^{-\Gamma_n(t_t - t_f)}$ and $e^{-\Gamma_m(t_t - t_f)}$ in the equation above (we also set $t_t = 0$):

$$T_2 = \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 \ e^{i\Delta_{fn} t_4} e^{i\Delta_{fm} t_3} e^{-\Delta_\beta t_2} e^{-i\Delta_{fn} t_1} e^{-i\Delta_{fm} t_2} \phi(t_2 - t_3)$$
The first two terms of equation (45) have the same structure as in equation (40) and, in the large time limit, they go to zero. The third term and the fourth term of equation (45) have the structure

\[ I(a, b, t) := \int_0^t dt_1 \int_0^t dt_2 e^{at_1} e^{bt_2} f(t_1 - t_2) \]

\[ = \frac{1}{a + b} \left( e^{(a+b)t} \int_0^t dx \ e^{-bx} f(x) - \int_0^t dx \ e^{ax} f(x) \right) + e^{(a+b)t} \int_0^t dx \ e^{-ax} f(x) - \int_0^t dx \ e^{bx} f(x) \]  

(46)

The emission rate is proportional to the time derivative of \( I(a, b, t) \). In the large time limit the only terms which survive are those with \( a = -b \). In fact, in such a case \( I(a, b, t) \) increases linearly with time:

\[ I(a, -a, t) = \int_0^t dx \ e^{ax} (t - x) f(x) + \int_0^t dx \ (t - x) e^{-ax} f(x) \]

\[ \sim \int_{-t}^t dx \ e^{ax} f(x) \]

(47)

while for \( a \neq -b \) it oscillates or goes to zero. Then only the third term in equation (45) survives, so

\[ \frac{dT_2}{dt} \xrightarrow{t \to \infty} \frac{\hat{I} (\Delta_B + \omega_k)}{\hat{I}^2 \left( i(\Delta_B + \omega_k) - \Gamma_n \right) \left[ -i(\Delta_m + \omega_k) + \Gamma_m \right]} \]  

(48)
Coming back to equation (43), we have

$$R_{12} = \sum_{f,n,m} \sum_{\ell} \left\{ \langle f | R_k | n \rangle \langle n | N | i \rangle \langle f | N | m \rangle \langle m | R_k | i \rangle \right\} e^{i(\Delta_{n} + \omega_k) \ell} \int (\Delta_{\beta} + \omega_k).$$

(49)

5.3. Computation of $R_{22}$

As in the previous cases, we write $R_{22}$ as

$$R_{22} = \sum_{f,m,n} \sum_{\ell} \left\{ \langle f | N | n \rangle \langle n | R_k | i \rangle \langle f | N | m \rangle \langle m | R_k | i \rangle \right\} \frac{d}{dt} T_3,$$

(50)

where

$$T_3 := \int_{t_1}^{t_2} dt_1 \int_{t_1}^{t_2} dt_2 \int_{t_2}^{t_3} dt_3 \int_{t_3}^{t_4} dt_4 e^{i(\Delta_{n} + \omega_k) \ell} f(t_1 - t_2) e^{-i(\Delta_{m} + \omega_k) t_2} \n \times f(t_2 - t_3) e^{-i(\Delta_{m} + \omega_k) t_3} \n \times f(t_3 - t_4) e^{-i(\Delta_{m} + \omega_k) t_4} \n \times f(t_4 - t_5) e^{-i(\Delta_{n} + \omega_k) t_5}.$$  

(51)

On adding the decay of the propagator and setting $t_i = 0$ we have

$$T_3 = \int_{0}^{t} dt_1 \int_{0}^{t} dt_2 \int_{0}^{t} dt_3 \int_{0}^{t} dt_4 e^{i(\Delta_{n} - \Delta_{m}) \ell} \left( \int_{0}^{t_2} dt_2 e^{i(\Delta_{m} + \omega_k) t_2} f(t_1 - t_2) e^{-i(\Delta_{n} + \omega_k) t_2} \right) \n \times f(t_2 - t_3) e^{-i(\Delta_{n} - \Delta_{m}) t_3} \n \times f(t_3 - t_4) e^{-i(\Delta_{n} - \Delta_{m}) t_4} \n \times f(t_4 - t_5) e^{-i(\Delta_{m} + \omega_k) t_5} \n \times \left( \int_{0}^{t_5} dt_5 e^{-i(\Delta_{m} + \omega_k) t_5} \right).$$

(52)

All four terms in equation (52) contain integrals with the same structure of $I(a,b,t)$ as was given in equation (46). As already discussed, the only relevant contribution in the large time limit is that with $a = -b$. Then only the first term in equation (52) contributes, which implies that

$$\frac{d}{dt} T_3 \quad \longrightarrow \quad \int_{0}^{t} \frac{d\ell}{t} e^{i(\Delta_{n} - \Delta_{m}) \ell} \left\{ \frac{1}{i(\Delta_{m} + \omega_k) + \Gamma_m} \right\} \left( \int_{0}^{t_2} dt_2 e^{i(\Delta_{n} + \omega_k) t_2} f(t_1 - t_2) \right) \n \times f(t_2 - t_3) \left[ e^{-i(\Delta_{n} - \Delta_{m}) t_3} \right] \n \times f(t_3 - t_4) \left[ e^{-i(\Delta_{n} - \Delta_{m}) t_4} \right] \n \times f(t_4 - t_5) \left[ e^{-i(\Delta_{m} + \omega_k) t_5} \right].$$

(53)
Thus we have

\[ R_{22} = \sum_{f} \sum_{m,n} \sum_{\ell} \left\langle f \left| N_{f} \right| n \right\rangle \left\langle n \left| R_{k} \right| i \right\rangle \left\langle f \left| N_{m} \right| m \right\rangle \left\langle m \left| R_{k} \right| i \right\rangle \left[i(\Delta_{ni} + i\omega) + \Gamma_{n}\right] \left[-i(\Delta_{ni} + i\omega) + \Gamma_{m}\right] f\left(\Delta_{fi} + \omega_{k}\right). \]

(54)

6. The contribution due to the mixed terms

In this section we study the contribution to the rate due to the mixed terms \(B^{i}C_{1}\), \(B^{i}C_{2}\) and \(B^{i}C_{3}\) (see equation (24)).

6.1. Computation of \(B^{i}C_{1}\)

We start from the contribution from the terms \(B\) and \(C_{1}\) given in equations (17) and (18). The \(B\) term is simply

\[ B = \left( -\frac{i}{\hbar} \right) \left\langle f \left| R_{k} \right| i \right\rangle T_{B}. \]

(55)

where, setting \(t_{i} = 0\),

\[ T_{B} = \int_{0}^{t} dt_{1} e^{i(\Delta_{fi} + i\omega_{k})t_{1}} = \frac{e^{i(\Delta_{fi} + i\omega_{k})t_{1}} - 1}{i(\Delta_{fi} + i\omega_{k})}. \]

(56)

The term \(C_{1}\) can be written as

\[ C_{1} = \frac{i}{\hbar} \sum_{n,m,\ell,\ell'} \left\langle f \left| R_{k} \right| n \right\rangle \left\langle n \left| N_{f} \right| m \right\rangle \left\langle m \left| N_{\ell} \right| \ell \right\rangle T_{C_{1}}. \]

(57)

Upon taking the average over the noise and introducing the decay of the propagators, \(T_{C_{1}}\) becomes

\[
T_{C_{1}} = \delta_{\ell,\ell'} \int_{0}^{t} dt_{1} \int_{0}^{t_{1}} dt_{2} \int_{0}^{t_{2}} dt_{3} e^{i(\Delta_{fi} + i\omega_{k})t_{1}} \\
\times e^{i\Delta_{ni}t_{2}} e^{i\Delta_{ni}t_{3}} f(t_{2} - t_{3}) e^{-\Gamma_{n}(t_{2} - t_{3})} e^{-\Gamma_{n}(t_{2} - t_{3})} \\
= \int_{0}^{t} dt_{1} e^{i(\Delta_{fi} + i\omega_{k})t_{1}} \int_{0}^{t_{1}} dt_{2} \int_{0}^{t_{2}} dt_{3} \theta(t_{2} - t_{3}) \\
\times e^{i\Delta_{ni}t_{2}} e^{i\Delta_{ni}t_{3}} f(t_{2} - t_{3}) e^{-\Gamma_{n}(t_{2} - t_{3})}. \]

(58)

The integral can be computed like in the previous cases and the final result is

\[ T_{C_{1}} = \frac{\delta_{\ell,\ell'}}{(i\Delta_{ni} + \Gamma_{n})i(\Delta_{fi} + \omega_{k})} \]

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\[
\begin{align*}
\times \left( e^{i(\Delta_{\beta} + \omega_k) t} & \int_0^t \delta_{L,L'} \left[ e^{i(\Delta_{\alpha} - \Gamma_0)} f(x) - \int_0^x \delta_{L,L'} \left[ e^{i(\Delta_{\alpha} + \omega_k) - \Gamma_0)} f(x) \right] \right] \right) \\
& - \frac{\delta_{L,L'}}{(i\Delta_{\alpha} + \bar{\Gamma}_0)} \left[ e^{i(\Delta_{\beta} + \omega_k) - \Gamma_0)} f(x) \right] \\
\times \left( e^{i(\Delta_{\beta} + \omega_k) - \Gamma_0)} f(x) \right) \int_0^x \delta_{L,L'} \left[ e^{i(\Delta_{\alpha} + \omega_k) - \Gamma_0)} f(x) \right] - \int_0^t \delta_{L,L'} \left[ e^{i(\Delta_{\alpha} + \omega_k) - \Gamma_0)} f(x) \right].
\end{align*}
\]

(59)

Since in the formula for the rate (equation (24)) we need to compute \( \mathbb{E} \{ 2 \Re (B^*C_1) \} \), we focus on

\[
T_{BC_1} = T_B \cdot T_C = \frac{T_{C_1} - T_{C_1} e^{-i(\Delta_{\beta} + \omega_k) t}}{i(\Delta_{\beta} + \omega_k)},
\]

(60)

and in particular on its time derivative:

\[
\frac{d}{dt} T_{BC_1} = \frac{d}{dt} T_{C_1} \frac{d}{dt} \frac{T_{C_1} e^{-i(\Delta_{\beta} + \omega_k) t}}{i(\Delta_{\beta} + \omega_k)}.
\]

(61)

It is straightforward to see that, in the large time limit,

\[
\frac{dT_{C_1}}{dt} \sim \delta_{L,L'} \frac{e^{i(\Delta_{\beta} + \omega_k) t}}{(i\Delta_{\alpha} + \bar{\Gamma}_0)} \int_0^x \delta_{L,L'} \left[ e^{i(\Delta_{\alpha} - \Gamma_0)} f(x) \right] \\
\frac{d}{dt} \left[ T_{C_1} e^{-i(\Delta_{\beta} + \omega_k) t} \right] \sim -\delta_{L,L'} \frac{e^{-i(\Delta_{\beta} + \omega_k) t}}{i(\Delta_{\beta} + \omega_k) - \Gamma_0)} \int_0^x \delta_{L,L'} \left[ e^{i(\Delta_{\alpha} + \omega_k) - \Gamma_0)} f(x) \right].
\]

Then in the large time limit,

\[
\frac{d}{dt} T_{BC_1} = \delta_{L,L'} \frac{1}{i(\Delta_{\beta} + \omega_k)} \times \left\{ \frac{e^{i(\Delta_{\beta} + \omega_k) t}}{(i\Delta_{\alpha} + \bar{\Gamma}_0)} \int_0^x \delta_{L,L'} \left[ e^{i(\Delta_{\alpha} - \Gamma_0)} f(x) \right] \\
+ \frac{e^{-i(\Delta_{\beta} + \omega_k) t}}{i(\Delta_{\beta} + \omega_k) - \Gamma_0)} \int_0^x \delta_{L,L'} \left[ e^{i(\Delta_{\alpha} + \omega_k) - \Gamma_0)} f(x) \right] \right\}.
\]

(62)

The important point is that equation (62) contains oscillating terms, which average to zero for \( t \to \infty \). The only exception is when \( \Delta_{\beta} + \omega_k = 0 \). However, as long as we study systems whose initial state is the ground state, the condition \( \Delta_{\beta} + \omega_k = 0 \) is never fulfilled.

When the initial state is not the ground state, the effect of the spontaneous emission due only to the vacuum fluctuations of the electromagnetic field is expected to be bigger than the one given in equation (62), which is due to the presence of the noise. This can be qualitatively understood by considering the fact that vacuum fluctuations are of order \( e^2 \), while the effect given by equation (62) is of order \( e^{2\gamma} \). Therefore, before having any emission due to the noise, the system is already decayed in its ground state because of the vacuum fluctuations of the electromagnetic field. Since we are interested only in the radiation emission induced by the noise, this contribution can be neglected.

The same conclusions hold true also for the contributions \( B^*C_2 \) and \( B^*C_3 \), since they behave in a similar way to \( B^*C_1 \).
7. The final result

We have seen that the terms $B_n C_n$ with $n = 1, 2, 3$ give a null contribution to the emission rate, while $A_1$ and $A_2$ contribute as

$$\frac{d\Gamma}{dt} = \sum_n \int d\Omega_k \frac{\gamma}{\hbar^2} \left[ R_{11} + 2 \text{Re}(R_{12}) + R_{22} \right],$$

with $R_{11}$, $R_{12}$ and $R_{22}$ given respectively by equations (42), (49) and (54). The term in the square brackets can be rewritten as

$$\sum_n \sum_\ell \left[ \frac{\langle f | R_k | n \rangle \langle n | N_\ell | i \rangle}{i(\Delta_{\beta_n} + \omega_k) - \Gamma_n} \right]^2 \times \tilde{f}(\Delta_{\beta_n} + \omega_k).$$

Then the formula for the emission rate becomes

$$\frac{d\Gamma}{dt} = \sum_n \int d\Omega_k \frac{\gamma}{\hbar^2} \sum_f \sum_\ell \left[ \sum_n \frac{\langle f | R_k | n \rangle \langle n | N_\ell | i \rangle}{i(\Delta_{\beta_n} + \omega_k) - \Gamma_n} - \frac{\langle f | N_\ell | n \rangle \langle n | R_k | i \rangle}{i(\Delta_{\beta_n} + \omega_k) + \Gamma_n} \right]^2 \times \tilde{f}(\Delta_{\beta_n} + \omega_k).$$

As expected, in equation (65) unphysical terms proportional to $\tilde{f}(0)$ are not present. This result agrees with that of [13], with two differences. First, it holds for any situation where a quantum system interacts with an external noise; second and more importantly, its rigorous derivation clarifies a long debated issue about the origin of the unphysical terms in the emission rate formula and how to eliminate them.

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Appendix A. Derivation of equations (15)–(20)

In this appendix we derive equations (15)–(20) using the standard perturbative approach. We want to compute perturbatively the transition probability given in equation (14):

$$P_{fi} = \mathbb{E} \left[ \left| \langle f; \mathbf{k}, \lambda | U(t, t_i) | i; \Omega \rangle \right|^2 \right] = \mathbb{E} \left[ \left| \langle f; \mathbf{k}, \lambda | U(t, t_i) | i; \Omega \rangle \right|^2 \right]$$

where $U(t, t_i) = e^{iH_0(t - t_i)} U(t, t_i) e^{-iH_0(t - t_i)}$ is the time evolution operator in the interaction picture. We recall that here $|i\rangle$ and $|f\rangle$ represent the initial and final state of the system while $|\Omega\rangle$ and $|\mathbf{k}, \lambda\rangle$ represent, respectively, the vacuum state of the electromagnetic field and the state with
one photon with wavevector $k$ and polarization $\lambda$. We expand $U_i(t, t_i)$ according to the Dyson series [25]:

$$U_i(t, t_i) = 1 + \sum_{n=1}^{\infty} \left( \frac{-i}{\hbar} \right)^n \int_{t_i}^{t} dt_1 \int_{t_i}^{t} dt_2 \cdots \int_{t_i}^{t_{n-1}} dt_n H_1(t_1)H_1(t_2)\cdots H_1(t_n)$$  \hspace{1cm} (A.2)

where $H_1(t) = e^{iH_0t}H(t)e^{-iH_0t}$ with $H(t)$ defined in equation (13). For the following calculation, it is convenient to rewrite $H_1(t)$, inserting the plane wave expansion for the potential vector $A(x)$ as given in equation (10):

$$H_1(t) = \int dk \sum_{\lambda} R_{k,\lambda}^i a_{k,\lambda}^i + \int dk \sum_{\lambda} R_{k,\lambda} f a_{k,\lambda}^f - \sqrt{\hbar} \sum_l N_l w_l(t),$$  \hspace{1cm} (A.3)

with $R_l$ defined in equation (21). The first term in equation (A.3) contributes to the processes where a photon is destroyed, the second term to the processes where a photon is created and the last term to the processes where the number of photons is conserved.

We now focus on computing the transition amplitude:

$$T_{ji} := \langle f; k, \mu | U_i(t, t_i) | i; \Omega \rangle.$$  \hspace{1cm} (A.4)

We will show that, when the Dyson series equation (A.2) is substituted in equation (A.4), the lowest order terms correspond to the contributions given in equations (15)–(20). We start with the zero-order term of the Dyson series, which gives a null contribution:

$$\langle f; k, \mu | i; \Omega \rangle = 0,$$  \hspace{1cm} (A.5)

since the initial and the final states are orthogonal because they contain different numbers of photons. The next term is the term of equation (A.2) corresponding to $n = 1$:

$$\left( \frac{-i}{\hbar} \right)^1 \int_{t_i}^{t} dt_1 \langle f; k, \mu | H_1(t_1) | i; \Omega \rangle = -\frac{i}{\hbar} \int_{t_i}^{t} dt_1 e^{i(E_f + \hbar \omega_1 - E_i)n}$$

$$\times \langle f; k, \mu | H_1(t_1) | i; \Omega \rangle.$$ \hspace{1cm} (A.6)

Only the second term of $H_1(t_1)$ gives a contribution in the matrix element $\langle f; k, \mu | H_1(t_1) | i; \Omega \rangle$, since the other two terms lead to an initial state and a final state with different numbers of photons. Then we are left with

$$-\frac{i}{\hbar} \int_{t_i}^{t} dt_1 \langle f; k, \mu | H_1(t_1) | i; \Omega \rangle = -\frac{i}{\hbar} \int_{t_i}^{t} dt_1 e^{i(E_f + \hbar \omega_1 - E_i)n} \langle f | R_k | i \rangle,$$ \hspace{1cm} (A.7)

which is exactly the term ‘B’ of equation (17).

We then proceed to studying the terms of the Dyson expansion corresponding to $n = 2$ (which are the ones corresponding to two-vertex diagrams):

$$\left( \frac{-i}{\hbar} \right)^2 \int_{t_i}^{t} dt_1 \int_{t_i}^{t} dt_2 \langle f; k, \mu | H_1(t_1)H_1(t_2) | i; \Omega \rangle$$

$$= \left( \frac{-i}{\hbar} \right)^2 \sum_n \int_{t_i}^{t} dt_1 \int_{t_i}^{t} dt_2 \langle f; k, \mu | H_1(t_1) | n; \Omega \rangle \langle n; \Omega | H_1(t_2) | i; \Omega \rangle$$

$$+ \left( \frac{-i}{\hbar} \right)^2 \sum_n \int \frac{dk'}{2} \sum_{\mu'} \int_{t_i}^{t} dt_1 \int_{t_i}^{t} dt_2 \langle f; k, \mu | H_1(t_1) | n; k', \mu' \rangle$$

$$\times \langle n; k', \mu' | H_1(t_2) | i; \Omega \rangle.$$ \hspace{1cm} (A.8)
In the above equation we inserted the completeness over the system states \( |n\rangle \) and that over the Fock space of photons. The latter involves only completeness on states with zero or one photon, since for any state involving two or more photons the matrix elements in equation (A.8) are null. The first term of equation (A.8) is the term \( A_1 \) of equation (15); in fact,

\[
\left( \frac{-i}{\hbar} \right)^2 \sum_n \int_{t_1}^{t_2} dt_1 \int_{t_1}^{0} dt_2 \langle f; k, \mu \mid H_{II}(t_1) \mid n; \Omega \rangle \langle n; \Omega \mid H_{II}(t_2) \mid i; \Omega \rangle \\
= \left( \frac{-i}{\hbar} \right)^2 \sum_n \int_{t_1}^{t_2} dt_1 \int_{t_1}^{0} dt_2 \ e^{i(E_f + \hbar \omega_k - E_h) t_1} e^{i(E_f - E_h) t_2} \\
\times \langle f; k, \mu \mid H_{II}(t_1) \mid n; \Omega \rangle \langle n; \Omega \mid H_{II}(t_2) \mid i; \Omega \rangle \\
= \left( \frac{-i}{\hbar} \right)^2 \sum_n \sum_{\ell} \int_{t_1}^{t_2} dt_1 \int_{t_1}^{0} dt_2 \ e^{i(E_f + \hbar \omega_k - E_h) t_1} e^{i(E_f - E_h) t_2} W_\ell(t_1) \\
\times \langle f; \{ R_{k} \mid n \rangle \langle n \mid N_{\ell} \mid i \rangle. \tag{A.9} \]

In the same way, we can show that the term \( A_2 \) introduced in equation (16) is given by the second term of equation (A.8):

\[
\left( \frac{-i}{\hbar} \right)^2 \sum_n \int d k' \sum_{\ell} \int_{t_1}^{t_2} dt_1 \int_{t_1}^{0} dt_2 \langle f; k', \mu' \mid H_{II}(t_1) \mid n; \Omega \rangle \langle n; \Omega \mid H_{II}(t_2) \mid i; \Omega \rangle \\
= \left( \frac{-i}{\hbar} \right)^2 \sum_n \int d k' \sum_{\ell} \int_{t_1}^{t_2} dt_1 \int_{t_1}^{0} dt_2 \ e^{i(E_f + \hbar \omega_k - E_h) t_1} e^{i(E_f - E_h) t_2} \\
\times \langle f; k, \mu \mid H_{II}(t_1) \mid n; \Omega \rangle \langle n; \Omega \mid H_{II}(t_2) \mid i; \Omega \rangle \\
= \left( \frac{-i}{\hbar} \right)^2 \sum_n \int d k' \sum_{\ell} \int_{t_1}^{t_2} dt_1 \int_{t_1}^{0} dt_2 \ e^{i(E_f + \hbar \omega_k - E_h) t_1} e^{i(E_f - E_h) t_2} W_\ell(t_1) \\
\times \langle f; \{ N_{\ell} \mid n \rangle \langle n \mid R_k \mid i \rangle. \tag{A.10} \]

In a similar way, one can show that the terms \( C_1, C_2 \) and \( C_3 \) of equations (18)–(20), which are represented by diagrams containing three vertices, can be obtained from the term of the Dyson expansion equation (A.2) corresponding to \( n = 3 \).

Appendix B. Calculation of the natural broadening for a harmonic oscillator

In this appendix we compute the natural broadening for a harmonic oscillator. The starting point is the equation for the imaginary part of the energy shift which, in the case of non-relativistic electromagnetic interactions, is given in [18] (page 67):
\[ \Delta E_i = -\pi \sum \int d\mathbf{k} \sum_n \left| \langle \mathbf{k}, \lambda; n | H_{\text{INT}} | \Omega; i \rangle \right|^2 \delta(E_i - E_n - \hbar \omega_k). \] (B.1)

where \(|i\rangle\) and \(|n\rangle\) are the eigenstates of the harmonic oscillator Hamiltonian \(H_0\) with eigenvalues \(E_i\) and \(E_n\) and \(|\Omega \rangle\) and \(|\mathbf{k}, \lambda\rangle\) represent, respectively, the vacuum state of the electromagnetic field and the state with one photon with wavevector \(\mathbf{k}\) and polarization \(\lambda\). We work in the dipole approximation, so the interaction Hamiltonian becomes

\[ H_{\text{INT}} = -\frac{e}{m} \mathbf{A} \cdot \mathbf{p} = -\frac{e}{m} \sum \int d\mathbf{k} \alpha_k (a_{\mathbf{k}, \lambda} + a_{\mathbf{k}, \lambda}^\dagger)(c_{\mathbf{k}, \lambda} \cdot \mathbf{p}). \] (B.2)

Then equation (B.1) becomes

\[ \Delta E_i = -\left(\frac{e}{m}\right)^2 \pi \sum \int d\mathbf{k} \alpha_k^2 \sum_n \left| \langle \mathbf{k}, \lambda; n | \mathbf{p} | i \rangle \right|^2 \delta(E_i - E_n - \hbar \omega_k) \]

\[ = -\left(\frac{e}{m}\right)^2 \pi \sum \int d\mathbf{k} \alpha_k^2 \sum_n \left\{ i \left| \langle \mathbf{k}, \lambda; n | \mathbf{p} | i \rangle \right| \delta(E_i - H_0 - \hbar \omega_k) \right\} \]

\[ = -\left(\frac{e}{m}\right)^2 \pi \sum \int d\mathbf{k} \alpha_k^2 \left\{ i \left| \langle \mathbf{k}, \lambda; \mathbf{p} \delta(E_i - H_0 - \hbar \omega_k) \mathbf{c}_{\mathbf{k}, \lambda} \cdot \mathbf{p} | i \rangle \right| \right\} \] (B.3)

where in the third line we used the completeness over the states \(|n\rangle\). Note that \(H_0\) is an operator; therefore the delta function cannot be brought out of the scalar product.

It is now convenient to write the momentum operator \(\mathbf{p}\) in terms of the raising and lowering operators \(b_j^\dagger\) and \(b_j\), where \(j\) labels the three spatial components. Then we get

\[ \epsilon_{\mathbf{k}, \lambda} \cdot \mathbf{p} = \sum_{j=1}^{3} \epsilon_{\mathbf{k}, j} i \sqrt{\frac{m \omega_0 \hbar}{2}} \left[ b_j^\dagger - b_j \right]. \] (B.4)

Substituting this expression in equation (B.3) and using the relation

\[ \sum \int d\mathbf{k} \epsilon_{\mathbf{k}, j} \epsilon_{\mathbf{k}, j} = \frac{8}{3} \pi \delta_{jj}, \] (B.5)

we obtain

\[ \Delta E_i = \frac{4e^2 \pi^2 \omega_0 \hbar}{3m} \int d\mathbf{k} k^2 \alpha_k^2 \sum_{j=1}^{3} \left\{ i \left| (b_j^\dagger - b_j) \delta(E_i - H_0 - \hbar \omega_k) (b_j^\dagger - b_j) \right| \right\}. \] (B.6)

It is straightforward to show that the matrix element is equal to

\[ \left\{ i \left| (b_j^\dagger - b_j) \delta(E_i - H_0 - \hbar \omega_k) (b_j^\dagger - b_j) \right| \right\} \]

\[ = -(i_j + 1) \delta(\hbar \omega_0 + \hbar \omega_k) - i_j \delta(\hbar \omega_0 - \hbar \omega_k), \] (B.7)

where \(i_j\) with \(j = 1, 2, 3\) is one of the three quantum numbers which identify the initial energy state, i.e. \(E_i = \hbar \omega_0 \left(\frac{3}{2} + i_1 + i_2 + i_3\right)\). Therefore equation (B.6) becomes

\[ \Delta E_i = \frac{4e^2 \pi^2 \omega_0 \hbar}{3mc^3} \int d\omega_k \omega_k^2 \left( \frac{\hbar}{2 \epsilon_0 \omega_k (2\pi)^3} \right) \]

\[ \times \sum_j \left[ -(i_j + 1) \delta(\hbar \omega_0 + \hbar \omega_k) - i_j \delta(\hbar \omega_0 - \hbar \omega_k) \right], \] (B.8)
where we used $\alpha_c^2 = \frac{\hbar}{2\omega_0(2\pi)^3}$ and we performed the change of variable $k \rightarrow \alpha_k = kc$. The first delta function never contributes; thus equation (B.8) becomes

$$ \Delta E_i = -\hbar \left( \frac{\beta \omega_0^2}{2m} \right) \sum_j i_j,$$

(B.9)

where we introduced $\beta = \frac{\epsilon^2}{6 \omega_0^2}$. The decay is proportional to $-\frac{\Delta E_i}{\hbar} = \left( \frac{\beta \omega_0^2}{2m} \right) \sum_j i_j$, which is proportional to the decay rate $\Lambda = \frac{\alpha_c^2 \beta}{2m}$ found in [19, 20].

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