Bias Mitigated Learning from Differentially Private Synthetic Data: A Cautionary Tale

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Abstract

Increasing interest in privacy-preserving machine learning has led to new models for synthetic private data generation from undisclosed real data. However, mechanisms of privacy preservation introduce artefacts in the resulting synthetic data that have significant impact on downstream tasks such as learning predictive models or inference. In particular, bias can affect all analyses as the synthetic data distribution is an inconsistent estimate of the real-data distribution. We propose several bias mitigation strategies using privatized likelihood ratios that have general applicability to differentially private synthetic data generative models. Through large scale empirical evaluation we show that bias mitigation provides simple and effective privacy-compliant augmentation for general applications of synthetic data. However, the work highlights that even after bias correction significant challenges remain on the usefulness of synthetic private data generators for tasks such as prediction and inference.

1 INTRODUCTION

The prevalence of sensitive datasets, such as electronic health records, continues to grow in research and real-world contexts contributing to an increasing concern for the implications and violations this could have on an individual’s privacy. In recent years, the notion of Differential Privacy (DP) [Dwork et al., 2006] has gained popularity as a privacy metric offering statistical theoretical guarantees. This framework bounds how much the likelihood of any output can differ under a neighboring real dataset. We say two datasets $D$ and $D'$ are neighboring when they differ by at most one observation. An algorithm $f$ satisfies $(\varepsilon, \delta)$-differential privacy for two non-negative numbers $\varepsilon$ and $\delta$ if and only if for all neighboring datasets $D, D'$ and all subsets $S$ of $f$’s range, we have

$$\Pr(f(D) \in S) \leq \delta + e^\varepsilon \Pr(f(D') \in S).$$

The parameter $\varepsilon$ is referred to as the privacy ‘budget’ and is often the main quantity of interest.

Datasets can be published under this framework in a private manner by injecting noise into sensitive data prior to publication. With the rise of deep generative modelling approaches, such as Generative Adversarial Networks (GANs) [Goodfellow et al., 2014], there has been a surge of literature proposing generative models for differentially private synthetic data release e.g. [Jordon et al., 2019, Xie et al., 2018]. However, these generative models often fail to capture the true underlying distribution of the real data, often due to flawed parametric assumptions and the injection of noise to their training and release mechanisms. The constraints imposed by privacy-preservation can lead to significant differences between Nature’s true data generating process (DGP) and the induced synthetic data generating process (SDGP) [Wilde et al., 2020]. This increases the bias of estimators trained on data from the SDGP which reduces their utility.

Recent literature has proposed techniques to decrease this bias by modifying the training processes of private algorithms [Zhang et al., 2018, Neunhoeffer et al., 2020, Fregierio et al., 2019]. These approaches are, however, specific to a particular synthetic data generating approach and are thus not generally applicable nor do they offer higher level insights as to the quality and nature of the synthetic data these methods produce with respect to its non-private source. In this work we propose several post-processing approaches that aid in mitigating the bias induced by working with DP synthetic data.

This motivates the following main contributions:

- Privatized importance sampling: a method to inject noise into importance sampling weights such that the resulting analysis retains unbiasedness, with a formulation of the resulting estimator’s variance.
• An adjustment to DP Stochastic Gradient Descent’s sampling probability and noise injection to facilitate its use in the training of DP-compliant neural network-based classifiers to compute importance weights from mixes of real and synthetic data.
• Importance weighting for DP GAN architectures that do not require any additional privacy budget.
• An assessment of how hold-out data from the true DGP can be used for post-hoc calibration of likelihood ratio estimation to better compute importance weights.
• An extension of importance weighted learning to Bayesian posterior belief updating.

All proofs are detailed in the Supplementary Material.

2 BACKGROUND

This work uses ideas from the field of synthetic data generation in the context of differential privacy. We first provide some background on bias mitigation in non-private synthetic data generation.

2.1 DENSITY RATIOS FOR NON-PRIVATE GANS

Since their introduction, GANs have become a popular tool for synthetic data generation in semi-supervised and unsupervised settings. GANs work to produce realistic synthetic data by trading off the learning of a generator \( G \) to produce synthetic observations, with that of a classifier \( D \) learning to correctly classify the training and generated data as real or fake. The generator \( G \) takes samples from the prior \( u \sim p_u \) as an input and generates samples \( G(u) \in X \). The discriminator \( D \) takes an observation \( x \in X \) as input and outputs the probability \( D(x) \) of this observation being drawn from the true DGP. The classification network \( D \) distinguishes between samples from the DGP with label \( y = 1 \) and distribution \( p_y \), and data from the SDGP with label \( y = 0 \) and distribution \( p_G \).

Following Bayes’ rule we can show that the output of \( D(x) \), namely the probabilities \( p(y = 1|x) \) and \( p(y = 0|x) \), can be used for likelihood ratio estimation:

\[
\frac{p_D(x)}{p_G(x)} = \frac{p(x|y = 1)}{p(x|y = 0)} \cdot \frac{p(y = 1|x)}{p(y = 0|x)} = \frac{p(y = 1)}{p(y = 0)}.
\] (1)

This observation has been exploited in a stream of literature focusing on importance weighting based sampling approaches for GANs. Grover et al. [2019] analyze how importance weighting of the GAN’s outputs can lead to performance gains; extensions include their proposed usage in rejection sampling on the GAN’s outputs [Azadi et al., 2018], and Metropolis-Hastings sampling from the GAN alongside improvements to the robustness of this sampling via calibration of the discriminator [Turner et al., 2019]. Up to now, no one has leveraged these discriminator based importance weighting approaches in DP settings where the weights can mitigate the increased bias introduced by the training of a privatized generative model.

2.2 DIFFERENTIAL PRIVACY IN SYNTHETIC DATA GENERATION

Private synthetic data generation through DP GANS is built upon the post processing theorem: Let \( D \) be \((\varepsilon, \delta)\)-DP. Then \( G \circ D \) is also \((\varepsilon, \delta)\)-DP [Dwork et al., 2014]. This result allows us to train private GANs by only privatizing the discriminator, see e.g. Hyland et al. [2018]. Xie et al. [2018] propose a Wasserstein GAN which is trained by noise injections on the gradients of the parameters of the discriminator. In contrast, Jordon et al. [2019] privatize the GAN discriminator by using the Private Aggregation of Teacher Ensembles algorithm [Papernot et al., 2016]. Other generative approaches include among others Zhang et al. [2017], Chen et al. [2018], Acs et al. [2018], Abay et al. [2018] provide an extensive overview of private synthetic data generative models.

In this paper we offer an augmentation to the usual release procedure for synthetic data by leveraging true and estimated importance weights. Elkan [2010] and Ji and Elkan [2013] train a regularized logistic regression model and assign weights, based on the by Laplace noise contaminated coefficient of the logistic regression, to all public samples. In follow up work, Ji et al. [2014] propose to instead modify the update step of the Newton-Raphson optimization algorithm used in fitting a logistic regression classifier to achieve DP. However, neither of these generalize well to more complex and high dimensional settings. Further, the authors assume the existence of a public dataset. We consider instead the case where we first generate synthetic data under DP and then weight it \textit{a posteriori}, providing a generic approach to the problem. This distinction necessitates a more principled analysis of privacy.

In other work, Abay et al. [2020] analyze importance weighting strategies in the domain of federated learning, focusing on fairness when multiple parties share data. Hardt and Rothblum [2010] consider multiplicative weighting in a query based setting which is not related to bias mitigation for synthetic data. Their approach serves to maintain a private dataset with which a large amount of online queries can be answered.

3 DIFFERENTIAL PRIVACY AND IMPORTANCE WEIGHTING

From a decision theory perspective [Berger, 2013], the goal of statistics is estimating expectations of functions \( h : X \rightarrow \mathbb{R} \), e.g. loss or utility functions, with respect to the
distribution of future uncertainties $x \sim p_D$. Given data from $x_{1:N} \overset{i.i.d.}{\sim} p_D$ the data analyst can estimate these expectations consistently via the strong law of large numbers as

$$
\mathbb{E}_{x \sim p_D}(h(x)) \approx \frac{1}{N_D} \sum_{i=1}^{N_D} h(x_i).
$$

However, under DP constraints the data analyst is no longer presented with a sample from the true DGP $x_{1:N} \overset{i.i.d.}{\sim} p_D$ but with a synthetic data sample $x_{1:N_G}$ from the SDGP $p_G$. Applying the naive estimator in this scenario biases the downstream tasks as $\frac{1}{N_G} \sum_{i=1}^{N_G} h(x_i) \to \mathbb{E}_{x \sim p_G}(h(x))$ almost surely. One particularly interesting example is when $h(x_i) = \log f(x_i|\theta)$, the log-likelihood of observation $x_i$ under parameter $\theta$, whose expectation we wish to maximize in order to best capture the DGP $p_D$ using model $f(\cdot|\theta)$. It was pointed out in Wilde et al. [2020] that naively using the synthetic data to do so and targeting

$$
\theta_{PD}^{KLD} := \arg \max_{\theta} \mathbb{E}_{x \sim p_D}[\log f(x|\theta)]
= \arg \min_{\theta} \text{KLD}(p_G(\cdot)||f(\cdot|\theta))
$$

is not necessarily useful for capturing $p_D$, KLD being the Kullback–Leibler divergence.

This bias can be mitigated using a standard Monte Carlo method known as importance sampling. If $p_G(\cdot) > 0$ whenever $h(\cdot)p_D(\cdot) > 0$, then importance sampling relies on

$$
\mathbb{E}_{x \sim p_D}[h(x)] = \mathbb{E}_{x \sim p_G}[w(x)h(x)], \text{ for } w(x) := \frac{p_D(x)}{p_G(x)}, \tag{2}
$$

so we have almost surely for $x_{1:N} \overset{i.i.d.}{\sim} p_G$

$$
I_N(h|w) := \frac{1}{N} \sum_{i=1}^{N} w(x_i)h(x_i) \xrightarrow{N \to \infty} \mathbb{E}_{x \sim p_D}[h(x)].
$$

**Bayesian Updating.** For $h(x) = \log f(x|\theta)$, such an approach naturally extends to the Bayesian paradigm. Standard Bayesian updating

$$
\pi(\theta|x) \propto \pi(\theta) \prod_{i=1}^{n} p(x_i|\theta) = \pi(\theta) \exp \left( - \sum_{i=1}^{n} \log f(x_i|\theta) \right)
$$

can be seen as a procedure which learns model parameter $\theta_{PD}^{KLD}$ when $x_i \sim p_D$ [Berk, 1966, Bissiri et al., 2016].

Given only synthetic data $x_i$ from ‘proposal distribution’ $p_G$, we can use the importance weights defined in Equation (2) to construct a loss function, $\ell_{IW}(x_i; \theta) := -w(x_i)\log f(x_i|\theta)$, allowing for a coherent general Bayesian updating of beliefs

$$
\pi_{IW}(\theta|x) \propto \pi(\theta) \exp \left( - \sum_{i=1}^{n} \ell_{IW}(x_i; \theta) \right) \tag{3}
$$

according to the notion of Bissiri et al. [2016] where we use data from the synthetic $p_G$ to continue targeting $\theta_{PD}^{KLD}$ even if $p_G$ differs from $p_D$. The next theorem shows that such a posterior given observations from $p_G$ is asymptotically calibrated to the standard posterior obtained from real data from $p_D$. A precise statement of Theorem 1 is given in the Supplementary Material.

**Theorem 1.** The importance weighted Bayesian posterior $\pi_{IW}(\theta|x_{1:N})$, defined in (3) for $x_{1:N} \overset{i.i.d.}{\sim} p_G$, admits the same limiting Gaussian distribution as the Bayesian posterior $\pi(\theta|x_{1:N})$ where $x_{1:N} \overset{i.i.d.}{\sim} p_D$, under regularity conditions as in [Chernozhukov and Hong, 2003, Lyddon et al., 2018].

Importance weights and Bayesian updating have previously been combined [e.g. Chen, 1994, Perrakis et al., 2014] in order to improve computations of the original posterior under study, not a reformulation of the target. From a learning perspective, a constant (down)-weighting, $w(x_i) = w$ has been widely considered [e.g. Lyddon et al., 2018, Holmes and Walker, 2017], but while this affects the ‘rate’ at which learning is done, it will not change the limiting parameter. The importance weighted Bayesian updating is closer to that of Agostinelli and Greco [2012], who consider a re-weighting of observations based on kernel density estimates in order to robustify against outliers. The IW-Bayes update however, is not designed to adjust only for outliers but for the whole data generating distribution of the synthetic data being ‘misspecified’ relative to $p_D$.

**Remark 1.** Further, the MAP estimator of Equation (3) can be viewed as an $M$-estimator with clear relations to the standard MAP/MLE as seen in the Supplementary Material.

### 3.1 Estimating the Importance Weights

The previous section shows that importance sampling can be used to re-calibrate inference from synthetic data. Unfortunately, both the DGP $p_D$ and SDGP $p_G$ densities (due to the intractability of GAN generation for example) are typically unknown and thus the ‘perfect’ weight $w(x)$ cannot be calculated. Instead we must rely on estimates of these weights, $\hat{w}(x)$.

One approach to such an estimation, as discussed in Section 2.1, is the observation that the ratio of the probabilities of a converged GAN discriminator equals the likelihood ratio up to a multiplicative constant. Using the same line of reasoning, we can in fact argue that any calibrated classification method that learns to distinguish between data from the DGP $y = 1$ and from the SDGP $y = 0$ can be used to estimate the likelihood ratio [Sugiyama et al., 2012]. Using (1), we can compute

$$
\hat{w}(x) = \frac{\hat{p}(y = 1|x)}{\hat{p}(y = 0|x)} \frac{N_D}{N_G}
$$
where $\hat{p}$ are the probabilities estimated by such a classification algorithm. To increase numerical stability, we can also express the log weights as

$$\log \hat{w}(x) = \log \frac{1 + \exp(-\sigma^{-1}(\hat{p}(y = 1|x)))^{-1}}{1 - (1 + \exp(-\sigma^{-1}(\hat{p}(y = 1|x)))^{-1}} + \log \frac{N_D}{N_G}$$

$$= \sigma^{-1}(\hat{p}(y = 1|x)) + \log \frac{N_D}{N_G}$$

where $\sigma(x) := (1 + \exp(-x))^{-1}$ is the logistic function and $\sigma^{-1}(\hat{p}(y = 1|x))$ are the logits of the classification method.

We will later discuss two such classifiers: the logistic regression and neural networks.

### 3.2 NOISY IMPORTANCE SAMPLING

In theory an arbitrarily complex classifier ought to be able to learn these weights exactly. However, in so doing the estimation of this classifier has queried the real dataset $x_i | N_p$, and therefore these weights cannot be released for fear that they compromise the real dataset’s privacy.

A simple approach to ensure DP of an algorithm is to add noise [Dwork et al., 2006] to its output, that is here the estimated importance weights of the synthetic data. We establish general results under which such a noise perturbation of an unbiased non-private weights algorithm $w(x)$ preserves the unbiasedness of importance sampling estimation.

**Theorem 2.** Let $\sigma^2(h)/N$ denote the variance of the importance sampling estimate $I_N(h|w)$ defined in (2). Then the importance sampling estimator $I_N(h|w^\ast)$ using noise perturbed importance weights $w^\ast(x_i) = w(x_i) + \zeta_i$ where $\zeta_i$ are i.i.d. and $\mathbb{E}[\exp(\zeta_i)] = 1$, is unbiased and has variance $\sigma^2(h)/N$ where

$$\sigma^2(h) = \sigma^2(h) + \text{Var} [\exp(\zeta) \mathbb{E}_{p_G} [(w(x)h(x))^2]]$$  

(4)

In the following we will analyze how the noise $\zeta$ has to be chosen to ensure DP.

### 3.3 PRIVATIZING LOGISTIC REGRESSION

The scale of the noise that has to be added to the classification algorithm $f$ is determined by how much the algorithm differs when one observation of the dataset changes. In more formal terms, the sensitivity of $f$ with respect to the norm $|\ |$ is defined by the smallest number $S(f)$ such that for any two neighboring datasets $D$ and $D'$ it holds that

$$|f(D) - f(D')| \leq S(f).$$

Dwork et al. [2006] show that to ensure the differential privacy of $f$, it suffices to add Laplace noise with standard deviation $S(f)/\varepsilon$ to each coordinate of $f$.

Possibly the most naive classifier $f$ one could use to estimate the importance weights is logistic regression. It turns out this also has a convenient form for its sensitivity. If the data is scaled to a range from 0 to 1 such that $X \subset [0, 1]^d$, Chaudhuri et al. [2011] show that the sensitivity of the optimal $\beta$ in a regularized logistic regression with model

$$\hat{p}(y = 1|x_i) = \sigma(\beta^T x_i)$$

is $2\sqrt{d}/(N_D\lambda)$ where $\lambda$ is the coefficient of the $L_2$ regularization term added to the loss function during training.

In fact, Ji and Elkan [2013] then propose to compute differentially private importance weights by training such an $L_2$ regularized logistic classifier on the private and the synthetic data, and perturbing the coefficient vector $\beta$ with Laplacian noise. For $\zeta \sim \text{Laplace}(2\sqrt{d}/(N_D\varepsilon))$, the private regression coefficient is then $\tilde{\beta} = \beta + \zeta$. We notice that perturbing the regression coefficients corresponds to adding heteroscedastic noise to the log weights computed by a non-private logistic regression as

$$\log w(x_i) = \tilde{\beta}^T x_i = \beta^T x_i + \zeta x_i.$$  

(5)

This approach can however be shown to add bias to the estimates of the importance weights.

**Proposition 1.** Let $\overline{w}$ denote the importance weights computed by noise perturbing the regression coefficients as in Equation 5 [Ji and Elkan, 2013, Algorithm 1]. The importance sampling estimator $I_N(h|\overline{w})$ is biased.

Introducing bias on downstream estimators of sensitive information is undesirable. Especially in health care applications, biased estimators cannot be interpreted without caution. To address this problem, we instead propose to privatize the importance weights computed by a regularized logistic regression by perturbing them directly.

**Corollary 1.** The importance sampling estimator with importance weights defined by

$$\log w^\ast(x_i) = \beta^T x_i + \zeta_i$$  

(6)

for $\zeta_i \sim \text{Laplace}(\log(1 - \rho^2), \rho)$ and $\rho = \frac{2\sqrt{d}}{N_D\varepsilon} < 1$

is $(N_D\varepsilon, \delta)$-differentially private. It is further unbiased and for $\rho < \frac{1}{4}$ has variance as defined in equation 4 with

$$\text{Var}[\exp(\zeta)] = \text{exp}(2\log(1 - \rho^2)) \left( \frac{1}{1 - 4\lambda^2} - \frac{1}{(1 - \lambda^2)^2} \right).$$

Note that privacy budget is additive. If we want to release $N_S$ DP weights, we thus have to scale the noise proportional to $N_S$. Even though this approach thus increases the variance of the estimator, it can be shown to be unbiased.
This approach is further limited by the restrictions on the scale parameter $\rho$. Alternatively, Blum et al. [2005] show that adding Gaussian noise $\zeta' \sim N(0, \frac{D}{2}\epsilon^2 s(\mathbf{f})^2 \log \frac{2}{\delta})$ to an algorithm $f$ ensures $(\epsilon, \delta)$-DP for $\delta > 0$. From our analysis it follows that we could replace the noise distribution defined in Corollary 1 in the following way.

**Corollary 2.** The importance sampling estimator with importance weights defined by

$$
\log w^*(x_i) = \beta^T x_i + \zeta_i
$$

for $\zeta_i \sim N(-\frac{\gamma^2}{2}, \gamma^2)$ and $\gamma = \sqrt{\frac{8d}{(N_D\lambda\epsilon)^2} \log \frac{2}{\delta}}$

is $(N_S\epsilon, \delta)$-differentially private with $\delta > 0$. It is further unbiased and has variance as defined in equation 4 with $\text{Var} [\exp(\zeta')] = \gamma^2$.

**Sources of Bias and Variance.** This analysis gives us insights on two sources of bias and variance. The first one is the bias and/or variance introduced by privatizing the weights. The estimator of Ji and Elkan [2013] is biased but as a result adds noise with a smaller variance, whereas to be unbiased by noising the weights we have to pay a price of increasing the variance, e.g., by adding more noise or by releasing fewer samples. The second source is the bias and variance introduced by estimating the weights through the classifier. The importance weighting procedure is only unbiased when we know exactly how to estimate the true weights. Using a logistic regression to estimate these cannot reasonably be considered as unbiased for any complicated data. However, using an arbitrarily complex classifier such as a classification neural network could arguably be considered as less biased at estimating the density ratio if it converges, but possibly increases the variance of the estimators due to the increased number of parameters to learn.

### 3.4 Privatizing Neural Networks

The last 30 years of machine learning research has seen an explosion in methods to classify complex data, with the Neural Network being a particular success. We can train DP classification neural networks for the aim of likelihood ratio estimation with stochastic gradient decent (SGD) by clipping the gradients and adding calibrated noise at each step of the SGD, see e.g. Abadi et al. [2016]. The noised gradients are then added up in a lot before the descent step. Technically, a lot resembles a mini-batch. However, as averaging gradients increases privacy, the term lot is used to stress that these batches are not formed to save computational resources (the common purpose of mini-batches).

These optimization algorithms are commonly formulated for the case when the complete dataset is private. However in our setting $N_D$ observations are private and $N_G$ observations are non-private. Thus, we can define a relaxed version of DP SGD. Please refer to Algorithm 1 for an overview of our proposed method. We highlighted the modifications to Algorithm 1 from Abadi et al. [2016] in blue.

**Algorithm 1: Relaxed differentially private SGD**

**Input:** Examples $x_1:N_D, y_1:N_D$ from the DGP and $x_{N_D+1:N_D+G}, y_{N_D+1:N_D+G}$ from the SDGP, loss function $L'(\theta) = \frac{1}{N_D+N_G} \sum_i L'(\theta, x_i)$.

**Parameters:** learning rate $\eta$, noise scale $\sigma$, group size $L$, gradient norm bound $C$.

1. **Initialize** $\theta_0$ randomly
2. **for** $t \in [T]$ **do**
   3. **Take a random sample** $L_t$ **with sampling probability** $L/(N_D+N_G)$
   4. **Compute gradient**
   5. For each $i \in L_t$, compute $g_i(x_i) \leftarrow \Delta_{\theta_t} L'(\theta_t, x_i)$
   6. **Clip gradient**
   7. $\bar{g}_i(x_i) \leftarrow g_i(x_i) / \max(1, ||g_i(x_i)||_2)$
   8. **Add noise**
   9. $\tilde{g}_i \leftarrow \frac{1}{L} \sum_i (\bar{g}_i(x_i) + N(0, \sigma^2 C^2 I)(y_i = 1))$
   10. **Descent**
    $\theta_{t+1} \leftarrow \theta_t + \eta \tilde{g}_i$
   **Output:** $\theta_T$ and compute the overall privacy cost $(\epsilon, \delta)$ using the moment’s accountant of Abadi et al. [2016] with sampling probability $q = \frac{L}{N_D+N_G}$.

The differential privacy with respect to a lot results directly from the observation that the gradients of the synthetic data are already private. Further, the labels of the synthetic data are public knowledge. Lastly, the differential privacy with respect to the dataset follows from the amplification theorem [Kasiviswanathan et al., 2011], the fact that sampling one particular private observation within a lot of size $L$ is $q = \frac{L}{N_D+N_G}$ and the reasoning behind the moment accountant of Abadi et al. [2016]. Note that we still clip the gradients of the public dataset. Otherwise their influence will be overproportional under strong maximum norm assumptions.

**Proposition 2.** Each step in the SGD outlined in Algorithm 1 is $(\epsilon, \delta)$-differentially private with respect to the lot and $O(\epsilon, \delta)$ differentially private with respect to the full dataset where $q = \frac{L}{N_D+N_G}$ and $\sigma = \sqrt{2 \log \frac{(1+5)}{\delta}} / \epsilon$.

Even less noise would be required when training the classification network using the framework of private aggregated teacher ensembles proposed by Papernot et al. [2016]. The same reasoning of Proposition 2 applies to the modification of PATE’s moment’s accountant.

Instead of multilayer perceptrons (MLPs), other approaches of density estimation include for instance support vector machines [Chaudhuri et al., 2011] or conditional normalizing flows [Papamakarios et al., 2017, Waites and Cummings, 2020]. In our experiments we identify the importance of the
classifier being able to capture the ‘true’ density ratio, and encourage further work in calibrating classifiers to this task.

**Discriminator Weights of GANS.** The downside of the aforementioned likelihood ratio estimators is that their training requires an additional privacy budget which has to be added to the privacy budget used to learn the SDGP. If we however use a GAN such as DPGAN or PATEGAN for private synthetic data generation, we can use the GAN discriminators for the computation of the importance weights. According to the post processing theorem, these importance weights can be released without requiring an additional privacy budget. In contrast to the weights computed from DP classification networks, this approach is more robust and requires less hyperparameter tuning (confer to Section 4). However, the effect of the privacy on the estimation of these weights is more opaque.

### 3.5 POST-PROCESSING OF THE LIKELIHOOD RATIO ESTIMATORS

The performance of importance weighting can suffer from a heavy right tailed distribution of the likelihood ratio estimates which increases the variance of downstream estimators. A simple remedy is tempering: for a $\tau \in [0,1]$ the weights $\{w(x_i)^\tau\}_{i \in \{1,\ldots,N_q\}}$ are less extreme.

Alternatively, Vehtari et al. [2015] propose Pareto smoothed importance sampling (PSIS). This procedure requires to fit a generalized Pareto distribution to the upper tail of the distribution of the simulated importance ratios. Their algorithm does not only post-hoc stabilize importance sampling, but also reports a warning when the estimated shape parameter of the Pareto distribution exceeds a certain threshold. Similarly, Koopman et al. [2009] propose a test to detect whether importance weights have finite variance. In both warnings, there are certain characteristics of the DGP which are not captured by the SDGP and the resulting importance sampling estimates are likely to be unstable. This warning can thus be understood as a general indicator for unsuitable proposal distributions. For large shape parameters the data owner should not release the SDGP. It is also computationally more efficient than comparable distribution divergences such as maximum mean discrepancy or Wasserstein distance. We must also consider that unlike traditional importance sampling where the importance weights are known (at least up to normalization), here they are being estimated from data, providing further motivation for regularization.

Aside from unstable likelihood ratios, the computed importance weights can suffer from the inability of the classification method to correctly capture the density ratios. To mitigate this problematic, Turner et al. [2019] propose post-calibration of the likelihood ratios in a non-private setting. Even in our setting, we can assume that the data analyst has access to a small dataset of the DGP, as e.g. in Wilde et al. [2020]. We can then make use of post-calibration methods, such as beta calibration [Kull et al., 2017].

### 4 EXPERIMENTS

We compared the performance of the different weighting schemes explained in this paper on both simulated data and real world data. The code and data for all experiments will be made available online. Because of the large scale of our experimental study, we only present the most important results in the main paper. Please refer to the Supplementary Material for a complete overview of the results. We note that our results are not comparable to those presented by Jordon et al. [2019] as we decided against hyperparameter tuning since it is not a differentially private operation. What is more Jordon et al. [2019] train their algorithm 50 times, and choose the model that performs best on a validation data set. Such a procedure is also not differentially private. In general we note that differentially private generative approaches still under-perform considerably compared to non-private algorithms. Importance sampling provides one such improvement.

We start by scaling the observations of all datasets to a range of 0 to 1 [Chaudhuri et al., 2011, Ji and Elkan, 2013]. Then, we run all experiments for five seeds using a train test split of 80%. The train data of size $N_B$ is used to fit the DP SDGP from which we then sample $N_G = N_B$ observations, unless otherwise specified. Based on both sets of observations, we apply one of the following importance weighting approaches: weights computed from a privatized logistic regression with noise perturbed coefficients according to Ji and Elkan [2013] (beta_noised), the noise perturbed variants introduced in Corollaries 1 and 2 (output_lapl and output_norm), likelihood ratios estimated by a non-private MLP with two dense layers and RELU activation layers (mlp), and eventually, weights from the same MLP privatized with our relaxed DP SGD procedure implemented in tensorflow (priv_mlp). Uniform weights implementing existing current methods (current). For the logistic regression alternatives we use an adaption of the sklearn implementation. With logreg, we refer to weights from a non-private logistic regression. Please refer to the Supplementary Material for an exhaustive description of the hyperparameters. Since hyperparameter tuning decreases the privacy budget available for the task, we did not optimise over the parameter of any model which explains the difference in reported results compared to, e.g., Jordon et al. [2019].

To evaluate the divergence of the SDGP and DGP, we compute a weighted version of the Wasserstein-1 distance w.r.t. the test data. Note that we can compute the distances between weighted samples by weighting the sums over the observations (see supplements). In all experiments we fix $\delta$ to $10^{-5}$ following Jordon et al. [2019] and $\varepsilon$ to 6 [Lee and
Table 1: Results for parameters ($\epsilon = 6.0, \delta = 10^{-5}$): Wasserstein-1 distance (WST), maximum mean discrepancy (MMD), support vector classifier AUC (SVM), random forest classifier AUC (RF), multi-layer perceptron classifier AUC (MLP)

Figure 1: Kernel density plots of a $5 \times 5$ Gaussian mixture model as DGP and a mixture of uniform distributions as SDGP (bottom right). The first row depicts histograms of the computed un-normalized weights starting with the true importance weights (generator) and the weights from an un-privatized logistic regression (logreg) and also with no weighting (none)

Figure 2: Wasserstein-1 distance and 95% confidence intervals over 5 seeds on the Gaussian Mixture Model

Figure 3: Mean Square Distance between posterior sample and generating parameter and Mahalanobis distance between posterior mean on private and synthetic data
If the weight computation procedure requires a separate privacy budget (e.g. if the weights are computed by a separate MLP or logistic regression), we spend half the budget on fitting the SDGP \((\epsilon = 3)\) and half of it on the weight computation whereas the complete budget can be spent on GAN training if the weights of the discriminator are used instead.

4.1 GAUSSIAN MIXTURE MODEL

We start our analysis with the standard toy example of 10,000 samples from a 5 \times 5 grid of two-dimensional Gaussians used i.e. by Azadi et al. [2018]. The means are arranged on a grid of \([-2, -1, 0, 1, 2]\) and the standard deviation is 0.05. For illustration purposes, we implement a mixture of uniform distributions as a possible SDGP.

In Figure 1, we notice that the output_noised synthetic data resembles the data weighted with the true weights more than the beta_noised alternative. Especially the neural network based methods manage to move probability mass from the over-weighted half of the SDGP to the under-weighted half. We can thus argue that the additional complexity of neural networks is already useful in the simple setting of this toy example. In Figure 2, we see that the output noised weights achieve a much closer Wasserstein-1 distance to the true distribution than the beta noised alternative. The disadvantage of output noising is however that it is not applicable for larger values of \(N_G\) as the increased sensitivity of the weights results in an explosion of \(E[\exp(\zeta)]\).

The priv_mlp are unstable in training, especially in the GMM example. Careful hyperparameter tuning is required to obtain a classification accuracy of more than 50%. We found that setting the micro-batch size to 20 and using a learning rate of approximately 0.01 gives the best results in general. We thus recommend the use of output-perturbed logistic regression for small data release.

4.2 REAL WORLD DATASETS

For the analysis of the proposed methods on real datasets, we performed all of our experiments on three UCI datasets of different characteristics: breast (569 samples of 30 continuous features), spam (4,601 observations of 57 continuous observations), and credit (30,000 samples of 14 continuous and 9 categorical features) [Lichman et al., 2013]. For performance evaluation we follow Jordon et al. [2019] and train a downstream classifier (using logistic regression, support vector classification, MLP and random forest classification) on the synthetic data which is assessed by the AUC on the private test dataset. We implement all benchmark models with default parameters in sklearn. To evaluate the divergence of the SDGP and DGP, we compute a weighted version the Wasserstein-1 distance and the MMD with respect to the test data. In Table 1, we see that the performance of the models improved when weighted with any type of estimated weights. For the smaller breast dataset the beta_noised weights outperform the deep learning based methods. However for the large credit dataset, we clearly see that the deep learning based models outperform the beta_noised alternative. The competitive performance of the weights of the discriminator (either from DPGAN or PATEGAN) speak for them considering that no additional computational cost is required. In the Supplementary Material we experimentally extend the results of Vehtari et al. [2015] and Kull et al. [2017] and show that PSIS and \(\beta\)-calibration also improve upon the performance of the un-processed importance weights in a DP setting, especially for larger datasets. Further, we note as an interesting observation that in some cases classification based on the SDGP seems to perform worse than a coin flip. Even though importance weighting increases the performance, the gains are smaller compared to the case where we knew the true weights.

4.3 BAYESIAN UPDATING

For simplicity, we consider two examples with explicit DGPs in downstream IW Bayesian inference task. The first one is a logistic regression with a two dimensional feature space where the private data is drawn from a multivariate Gaussian \(x \sim p_D = N(\mu_D = (-1.25, 1.25), \Sigma_D = ((1, 0.5)^T, (0.5, 1)^T))\). For the SDGP, we draw instead from a normal with \(\mu_C = (2, -2.5)\) and \(\Sigma_C = ((4, 0.2)^T, (0.2, 4)^T)\). We consider binary labels \([-1, 1]\) that are generated according to \(p_D(y \mid X = x) = \sigma(\theta_D^T x)\) with \(\theta_D = (1.5, 1.0, 2.5)\) and accordingly with \(\theta_C = (-1.5, 1.0, -2.5)\). We investigate the performance of our methods along a scale \(\gamma\) of a convex combination of the DGP and SDGP. In other words, we sample the synthetic data from \(p_{G,T} = \gamma p_G + (1 - \gamma)p_D\) instead of \(p_G\). We generate 10,000 data items and fit a Bayesian logistic regression where we place a prior of \(p_\theta = N(0, 10 \cdot I)\) on \(\theta\). We compare the resulting posterior across 40 seeds. In Figure 3, we report the difference of the posterior means in Mahalanobis distance (posterior covariance adjusted distance) and the Mean Square Distance between posterior sample and generating parameter. Additionally, we consider the treatment effect estimation based on a realistic synthetic DGP with explicit treatment effect as described in Loh et al. [2019] which also considers the problem of subgroup identification. The results can be found in the supplements, Table 4. They demonstrate that noise in the weight estimation in a real world example can be too large for the bias to be mitigated.

5 DISCUSSION

In this work, we analyzed different approaches to rebalance synthetic data based on importance weights in a differentially private manner. We established conditions under
which this technique can lead to unbiasedness of downstream estimators and discussed the resulting bias-variance trade-off. We also propose a method for re-weighting based on the discriminator of a DP GAN which does not increase the privacy budget.

In future work, we will analyze how the importance weights computed by MLPs can be privatized without inducing additional bias and how the use of boosting [Dwork et al., 2010] can decrease the scale of the noise added to the weights. Note that importance sampling in general does not perform well in high-dimensional spaces. Future work will consider approaches to remedy this problem, such as an MCMC on a low-dimensional projection of the data [Che et al., 2020].

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A PROOFS

A.1 BIAS AND VARIANCE OF ALGORITHM 1 OF JI & ELKAN (2013)

Consider Ji and Elkan [2013] Algorithm 1, where under the assumption that \( \frac{p(y=1)}{p(y=0)} \approx \frac{N_D}{N_S} = 1 \) the unprivatized importance weights are estimated using logistic regression

\[
w(x_i) = \frac{p^*(y=1|x_i)}{p^*(y=0|x_i)} = \exp (\beta^T x_i),
\]

(1)

and then the privacy preserving process adds noise to the \( \beta \) coefficients of this logistic regression \( \beta^* = \beta + \zeta \) with \( \zeta \sim \text{Laplace}(2\sqrt{d}/(N_D\lambda e)) \), a vector of length \( d \), to generate privatized estimates of the importance weights

\[
\overline{w}(x_i) = \exp (\beta^T x_i) = \exp (\beta^T x_i) \cdot \exp (\zeta x_i).
\]

(2)

The following proposition proves that \( \overline{w}(x_i) \) is a biased estimate \( w(x_i) \), the consequences being that if the ‘true’ importance weight really is given by a logistic regression then the procedure of Ji and Elkan [2013] will be biased.

**Proposition 1.** Let \( \overline{w} \) denote the importance weights computed by noise perturbing the regression coefficients as in Equation 2 [Ji and Elkan, 2013, Algorithm 1]. The importance sampling estimator \( I_N(h|\overline{w}) \) is biased.

**Proof.** Firstly, we show that \( \overline{w}(x_i) \) is not an unbiased estimate of \( w(x_i) \)

\[
\mathbb{E}_\zeta [\overline{w}(x_i)] = \mathbb{E}_\zeta [\exp (\beta^T x_i) \cdot \exp (\zeta x_i)]
\]

\[
= \mathbb{E}_\zeta [w(x_i) \cdot \exp (\zeta x_i)]
\]

\[
\neq w(x_i).
\]

(3)

As a consequence, we show that even if the true density ratio can be captured by a logistic regression, i.e. there exists \( \beta_0 \) such that \( \frac{p_D(x)}{p_G(x)} = \exp (\beta_0^T x) \) then the importance sampling estimator

\[
I_N(h|\overline{w}) = \frac{1}{N} \sum_{i=1}^{N} \overline{w}(x_i)h(x_i), \quad x_i \sim p_G(\cdot),
\]

(4)

with \( \overline{w}(\cdot) \) calculated using ‘privatised’ \( \beta^* = \beta_0 + \zeta \), \( \zeta \) distributed as above, is a biased estimate of \( \mathbb{E}_{p_D}[h(x)] \). Indeed, we
have

\[
\mathbb{E}_{x_1:N \sim p_G} \left[ \frac{1}{N} \sum_{i=1}^{N} w(x_i) h(x_i) \right] = \mathbb{E}_{x_1:N \sim p_G} \left[ \frac{1}{N} \sum_{i=1}^{N} \exp (\beta_0^T x_i) \cdot \exp (\xi x_i) h(x_i) \right]
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}_{x_i \sim p_G} \left[ w(x_i) \cdot \exp (\xi x_i) h(x_i) \right]
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}_{x_i \sim p_D} \left[ \exp (\xi x_i) h(x_i) \right]
\]

\[
\neq \mathbb{E}_{x_i \sim p_D} \left[ h(x_i) \right].
\]

(5)

The proof of Proposition 1 provides several insights on what is required for an unbiased estimator:

- Firstly, the fact that the bias depends explicitly on the observation suggests adding noise to the weights themselves rather than the process of how they are calculated.
- Secondly, even if all of the \(x_i\)’s were constant and equal to 1, these would not be unbiased as \(\mathbb{E}_\xi [\exp (\xi)] \neq 1\). However we acknowledge that this is a trivial situation.

Ji and Elkan [2013] compute the variance of the estimator \(\beta^* = \beta + \xi\) where \(\xi \sim \text{Laplace}(\frac{4(d+1)d}{(ND\lambda^2)^2})\) as

\[
\text{Var}(\beta^*) = \text{Var}(\beta) + \text{Var}(\xi) = \text{Var}(\beta) + \frac{4(d+1)d}{(ND\lambda^2)^2}.
\]

They show that the asymptotic variance of importance sampling with the unperturbed weights obtained from the logistic regression \(w_{\text{logreg}}\) can be upper bounded by

\[
\text{Var}(I_N(h, w_{\text{logreg}})) = \alpha^T \text{Var}(\beta) \alpha = \alpha^T \frac{dI_d}{ND\lambda^2} \alpha
\]

with

\[
\alpha = \sum_{x_i, x_j \in \mathcal{D}} e^{\beta_0^T (x_i + x_j)} \left( h(x_i) - h(x_j) \right) (x_i - x_j)
\]

\[
\sum_{x_i, x_j \in \mathcal{E}} e^{\beta_0^T (x_i + x_j)}
\]

where \(\beta_0\) optimizes the loss function of logistic regression on fixed \(G\) and the true distribution of \(D\). The asymptotic variance of the importance sampling estimator with the weights \(w_{\text{logreg}}^*\) from the logistic regression with parameter \(\beta^*\) is then

\[
\text{Var}(I_N(h, w_{\text{logreg}}^*)) = \alpha^T \text{Var}(\beta^*) \alpha = \alpha^T \left( \frac{dI_d}{ND\lambda^2} + \frac{4(d+1)d}{(ND\lambda^2)^2} \right) \alpha.
\]

### A.2 Noisy Importance Sampling

For privacy purposes, we want to be able to noise the importance weights as in

\[
\log w^*(x) = \log w(x) + \xi, \quad \text{for} \quad \xi \sim g \text{ drawn from a noise distribution}
\]

(6)

but we would like to still preserve the consistency properties of importance sampling estimates.

To achieve this, we expand the original target in importance sampling as follows

\[
p_D^*(x, \xi) = p_D(x) \exp(\xi) g(\xi)
\]

where \(\xi \in \mathbb{R}\) will correspond to some additive noise on the log weights, and \(g(\xi)\) is a probability density on \(\mathbb{R}\) such that by assumption

\[
\int \exp(\xi) g(\xi) d\xi = 1,
\]
So, in particular, this implies that

\[ \int p^*_D(x, \zeta) d\zeta = p_D(x). \]

Now, we can use a proposal density \( p^*_G(x, \zeta) = p_G(x)g(\zeta) \) targeting \( p^*_D(x, \zeta) \) and the resulting importance weight is indeed

\[ w^*(x, \zeta) = \frac{p^*_D(x, \zeta)}{p^*_G(x, \zeta)} = w(x) \exp(\zeta), \]

i.e. the importance weight in this extended space is a noisy version of the original weight \( w(x) \). We thus have

\[ \mathbb{E}_{p_D}[h(x)] = \mathbb{E}_{p_G}[h(x)w(x)] = \mathbb{E}_{p_G^*}[h(x)w^*(x, \zeta)] = \mathbb{E}_{p_G^*}[h(x)w(x)\exp(\zeta)]. \]

It follows that for i.i.d. \((x_i, \zeta_i) \sim p_G^*\), i.e. \( x_i \sim p_G \) and \( \zeta_i \sim g \), then

\[ I_N(h|w^*) = \frac{1}{N} \sum_{i=1}^{N} h(x_i)w(x_i) \exp(\zeta_i) \]

is an unbiased and consistent estimator of \( \mathbb{E}_{p_G}[h(x)] \). Its variance is

\[ \text{Var}[I_N(h|w^*)] = \frac{1}{N} \text{Var}_{p_G}[h(x)w(x)\exp(\zeta)] = \frac{\sigma^2(h)}{N}. \]

By the variance decomposition formula, we have

\[
\sigma^2(h) = \text{Var}_{p_G}[h(x)w(x)\exp(\zeta)] = \mathbb{E}_{\exp(\zeta)}[\text{Var}_{p_G}[h(x)w(x)]]
+
\text{Var}_{\exp(\zeta)}[\mathbb{E}_{p_G}[(h(x)w(x))^2]]
= \sigma^2(h) + \text{Var}_{\exp(\zeta)}[\mathbb{E}_{p_G}[(h(x)w(x))^2]],
\]

as \( \mathbb{E}_{\exp(\zeta)} = 1 \) by assumption and \( \text{Var}[I_N(h|w)] = \frac{1}{N} \text{Var}_{p_G}[h(x)w(x)] \). The variance of our estimator is inflated as expected by the introduction of noise.

### A.3 Differential Privacy of Log-Laplace Noised Importance Weights

Following Kozubowski and Podgórski [2003], the (symmetric) log-Laplace distribution is the distribution of random variable \( x \) such that \( y = \log(x) \) has a Laplace density with location parameter \( \mu \) and scale \( \lambda \). The density of a log-Laplace \((\mu, \lambda)\) random variable is

\[ f_X(x|\mu, \lambda) = \frac{1}{2\lambda x} \exp \left( -\frac{1}{\lambda} |\log x - \mu| \right). \quad (7) \]

Note this is recovered from the asymmetric log-Laplace in Kozubowski and Podgórski [2003] with \( \alpha = \beta = \frac{1}{2} \). Kozubowski and Podgórski [2003] further provide forms for the expectation and variance of the log-Laplace distribution as

\[ \mathbb{E}[X] = \frac{\exp(\mu)}{1 - \lambda^2} \text{ for } \lambda < 1, \quad (8) \]

\[ \text{Var}[X] = \exp(2\mu) \left( \frac{1}{1 - 4\lambda^2} - \frac{1}{(1 - \lambda^2)^2} \right) \text{ for } \lambda < \frac{1}{2}. \]

Next we wish to investigate the differential privacy provided by using the Laplace mechanism [Dwork et al., 2006] to noise importance weights. Adding Laplace noise to the log-weights, as in equation 6, is equivalent to multiplying the importance weights by log-Laplace noise. In order for the importance sampling to remain unbiased, the log-Laplace noise must have expectation 1. From equation (8) this will be the case for all \( \lambda < 1 \) if we set \( \mu = \log \left( 1 - \lambda^2 \right) \).
A.4 DIFFERENTIAL PRIVACY OF LOGISTIC REGRESSION

A binary logistic-regression classifier specifies class probabilities

\[ p(y = 1|x, \beta) = \frac{1}{1 + \exp(-x\beta)}, \quad p(y = 0|x, \beta) = \frac{\exp(-x\beta)}{1 + \exp(-x\beta)}. \]

We denote by \( z_{1:N}\) the private data sampled from the DGP, and by \( x_{1:N}\) the synthetic data sampled from the SDGP. Let \( z'_{1:N}\) be the neighboring data set of \( z_{1:N}\). The importance weights estimated by such a classifier become

\[
\begin{align*}
  w(x_i|x_{1:N}, z_{1:N}) &= \frac{\tilde{p}(y_i = 1|x_i, \hat{\beta}(x_{1:N}, z_{1:N})) N_D}{\tilde{p}(y_i = 0|x_i, \hat{\beta}(x_{1:N}, z_{1:N})) N_G} \\
  &= \frac{1}{1 + \exp(-x_i\hat{\beta}(x_{1:N}, z_{1:N}))} \frac{1 + \exp(-x_i\hat{\beta}(x_{1:N}, z_{1:N}))}{\exp(-x_i\hat{\beta}(x_{1:N}, z_{1:N}))} N_D N_G \\
  &= \exp(x_i\hat{\beta}(x_{1:N}, z_{1:N})) \frac{N_D}{N_G},
\end{align*}
\]

and as a result

\[
\begin{align*}
  |\log w(x_i|x_{1:N}, z_{1:N}) - \log w(x_i|x_{1:N}, z'_{1:N})| &
  = |x_i\hat{\beta}(x_{1:N}, z_{1:N}) + \log \frac{N_D}{N_G} - (x_i\hat{\beta}(x_{1:N}, z'_{1:N}) + \log \frac{N_D}{N_G})| \\
  &\leq |x_i| \sum_{j=1}^d \left| \hat{\beta}(x_{1:N}, z_{1:N})_j - \hat{\beta}(x_{1:N}, z'_{1:N})_j \right| \\
  &\leq \frac{2\sqrt{d}}{N_D} \lambda
\end{align*}
\]

if the features are minmax scaled using the sensitivity computed by Chaudhuri et al. [2011].

A.5 ASYMPTOTIC POSTERIOR DISTRIBUTION OF IMPORTANCE WEIGHTED BAYESIAN UPDATING

Section 3.5 of the paper considers the importance weighted Bayesian updating as a special case of general Bayesian updating where the loss function is specifically chosen to account for the fact that inference is being done with samples from \( p_G\) while trying to approximate \( p_D\). We henceforth write

\[\pi_{\text{IW}}(\theta|x_i \in \{1, \ldots, N_G\}) \propto \pi(\theta) \exp \left( - \sum_{i=1}^{N_G} w(x_i) \log f(x_i|\theta) \right) \]

\[= \pi(\theta) \exp \left( - \sum_{i=1}^{N_G} \ell_{\text{IW}}(x_i; \theta) \right),\]

for \( \ell_{\text{IW}}(x_i; \theta) := -w(x_i) \log f(x_i|\theta) \) and \( w(x_i) = p_D(x_i)/p_G(x_i) \). The next theorem shows that such a posterior given observations from \( p_G \) has the same asymptotic distribution as the standard Bayes posterior given samples from \( p_D \) would have, and therefore we consider this posterior to be asymptotically calibrated.

We give here the formal statement of Theorem ???. Below \( \overset{D}{\longrightarrow} \) denotes convergence in distribution.

**Theorem 1.** Let the regular conditions in [Chernozhukov and Hong, 2003, Lyddon et al., 2018] hold. Consider \( \hat{\theta}^{(N)}_{\text{IW}} := \arg\min_{\theta \in \Theta} \sum_{i=1}^{N} \ell_{\text{IW}}(x_i; \theta), x_i \overset{i.i.d.}{\sim} p_G \) and \( \hat{\theta}^{(N)}_{0} := \arg\min_{\theta \in \Theta} \sum_{i=1}^{N} \ell_0(x_i; \theta), x_i \overset{i.i.d.}{\sim} p_D \) where \( \ell_0(x; \theta) := -\log f(x; \theta) \). Then
both $\hat{\theta}^{(N)}_0$ and $\hat{\theta}^{(N)}_{IW}$ are consistent estimates of $\theta^*_0 := \arg\min_{\theta \in \Theta} \int \ell_0(x; \theta)dP_D(x)$. Moreover there exists a non-singular matrix $J^{-1}$ such that we have under the importance weighted Bayesian posterior $\pi_{IW}(\theta|x_{1:N})$

$$\sqrt{N} \left( \theta - \hat{\theta}^{(N)}_{IW} \right) \xrightarrow{D} \mathcal{N} \left( 0, J^{-1} \right),$$

almost surely w.r.t. $x_{1:n}^\text{i.i.d.}$ while under the standard Bayesian posterior $\pi(\theta|x_{1:N})$

$$\sqrt{N} \left( \theta - \hat{\theta}^{(N)}_0 \right) \xrightarrow{D} \mathcal{N} \left( 0, J^{-1} \right),$$

almost surely w.r.t. $x_{1:n}$.

**Proof.** Firstly, define

$$\theta^{(N)}_{IW} := \arg\min_{\theta \in \Theta} \int \ell_{IW}(x; \theta)dP_G(x), \quad J_{IW}(\theta) := \int \nabla^2_\theta \ell_{IW}(x; \theta)dP_G(x).$$

Then Chernozhukov and Hong [2003], Lyddon et al. [2018] show that under regularity conditions the following asymptotic result

$$\sqrt{N} \left( \theta - \hat{\theta}^{(N)}_{IW} \right) \xrightarrow{D} \mathcal{N} \left( 0, J_{IW}(\theta^*_0)^{-1} \right)$$

as $N \to \infty$ when $\theta$ is distributed according to the general Bayesian posterior almost surely w.r.t. $x_{1:n}$. Similarly, if we define

$$J_0(\theta) := \int \nabla^2_\theta \ell_{0}(x; \theta)dP_D(x),$$

then we have that under the standard Bayesian posterior [Chernozhukov and Hong, 2003, Kleijn et al., 2012, Lyddon et al., 2018]

$$\sqrt{N} \left( \theta - \hat{\theta}^{(N)}_0 \right) \xrightarrow{D} \mathcal{N} \left( 0, J_0(\theta^*_0)^{-1} \right)$$

almost surely w.r.t. $x_{1:n}$. Now it follows from the importance sampling identity that

$$\theta^{(N)}_{IW} = \arg\min_{\theta \in \Theta} \int \ell_{IW}(x; \theta)dP_G(x) = \arg\min_{\theta \in \Theta} \int \ell_0(x; \theta)dP_D(x) = \theta^*_0,$$

$$J_{IW}(\theta) = \int \nabla^2_\theta \ell_{IW}(x; \theta)dP_G(x) = \int w(x) \nabla^2_\theta \ell_0(x; \theta)dP_G(x) = \int \nabla^2_\theta \ell_0(x; \theta)dP_D(x) = J_0(\theta)$$

Moreover $\hat{\theta}^{(N)}_0$ and $\hat{\theta}^{(N)}_{IW}$ are also consistent estimates of $\theta^*_0$ under the same regularity conditions. This establishes the result.

A.6 THE IMPORTANCE-WEIGHTED LIKELIHOOD AND M-ESTIMATION

Remark 1 of the paper points out the the connection between the MAP of the general Bayesian Importance Weighted posteriors and the standard maximum likelihood estimator which can be seen through the lens of M-estimation. We exemplify this below.

Following Van der Vaart [2000], the $M$-estimate of parameter

$$\beta^*_m := \arg\max_{\beta} \mathbb{E}_{x \sim p_D} [m(x, \beta)]$$

is given by

$$\hat{\beta}^{(m)}_m := \arg\max_{\beta} \sum_{i=1}^n m(x_i, \beta).$$

---

1 $\pi_{IW}(\theta|x_{1:N})$ and $\pi(\theta|x_{1:N})$ are here interpreted as random probability measures, function of the random observations $x_{1:N}$. 

5
The estimator \( \hat{\beta}^{(n)} \) is consistent and is asymptotically normal, i.e.

\[
\sqrt{n} \left( \hat{\beta}^{(n)} - \beta^* \right) \xrightarrow{D} \mathcal{N} \left( 0, \tilde{V} \right)
\]

where

\[
\tilde{V}(\beta) := \left( \mathbb{E} \left[ \nabla^2 p(x, \beta) \right] \right)^{-1} \cdot \text{Var} \left[ \nabla p(x, \beta) \right] \cdot \left( \mathbb{E} \left[ \nabla^2 p(x, \beta) \right] \right)^{-1}.
\]

M-estimators generalises the case of MLE under model misspecification and the variance calculation collapses to the standard inverse Fisher’s information if the likelihood is correctly specified for the DGP.

The maximum a posteriori (MAP) estimate of the general Bayesian Importance Weighted posteriors can be considered an M-estimate with the following form

\[
\hat{\theta}^{(n)} = \arg \max \left\{ -\ell_{IW}(z; \theta) + \frac{1}{n} \log \pi(\theta) \right\} = \arg \max \left\{ w(z) \log f(z; \theta) + \frac{1}{n} \log \pi(\theta) \right\}.
\]

We can therefore consider the asymptotics of our MAP estimator, from a frequentist standpoint, noting that we can we can disregard the \( \frac{1}{n} \log \pi(\theta) = O(1/n) \). In particular, given \( z_{1:n} \sim P_G \) the asymptotic Gaussian variance,

\[
\begin{align*}
\tilde{V}_{IW}(\theta^*_G) &= \left( \mathbb{E}_{P_G} \left[ \nabla^2 \ell_{IW}(z, \theta^*_G) \right] \right)^{-1} \cdot \text{Var}_{P_G} \left[ \nabla \ell_{IW}(z, \theta^*_G) \right] \cdot \left( \mathbb{E}_{P_G} \left[ \nabla^2 \ell_{IW}(z, \theta^*_G) \right] \right)^{-1} \\
&= \left( \mathbb{E}_{P_D} \left[ \nabla^2 \ell_0(z, \theta^*_G) \right] \right)^{-1} \cdot \text{Var}_{P_D} \left[ \nabla \ell_0(z, \theta^*_G) \right] \cdot \left( \mathbb{E}_{P_D} \left[ \nabla^2 \ell_0(z, \theta^*_G) \right] \right)^{-1} \\
&= \left( \mathbb{E}_{P_D} \left[ \nabla^2 \ell_0(z, \theta^*_G) \right] \right)^{-1} \cdot \mathbb{E}_{P_G} \left[ (\nabla \ell_0(z, \theta^*_G)) (\nabla \ell_0(z, \theta^*_G))^T \right] \cdot \left( \mathbb{E}_{P_D} \left[ \nabla^2 \ell_0(z, \theta^*_G) \right] \right)^{-1}
\end{align*}
\]

given then \( \mathbb{E}_{P_G} \left[ (\nabla \ell_0(z, \theta^*_G)) (\nabla \ell_0(z, \theta^*_G))^T \right] = 0 \).

Further we can write the variance of the IW-Bayes MAP in terms of the variance of the standard MLE given the same number of observations \( x_{1:n} \sim P_D \) as follows:

\[
\begin{align*}
\tilde{V}_{IW}(\hat{\theta}^{(n)}_G) &= \mathbb{E}_{P_G} \left[ (\nabla \ell_{IW}(z, \hat{\theta}^{(n)}_G)) (\nabla \ell_{IW}(z, \hat{\theta}^{(n)}_G))^T \right] = \mathbb{E}_{P_D} \left[ (\nabla \ell_0(z, \hat{\theta}^{(n)}_G)) (\nabla \ell_0(z, \hat{\theta}^{(n)}_G))^T \right] \\
&= \mathbb{E}_{P_D} \left[ (\nabla \ell_0(z, \hat{\theta}^{(n)}_G)) (\nabla \ell_0(z, \hat{\theta}^{(n)}_G))^T \right]
\end{align*}
\]

It is interesting to observe that although the IW-Bayes posterior and the standard posterior have the same asymptotic distribution, their MAPs do not. This echoes a standard discrepancy between Bayesian and frequentist asymptotics when the likelihood is not correctly specified for the DGP [Müller, 2013]. Further we can use such notions to produce an idea of the effective sample size of synthetic data.

### A.6.1 The effective sample size of synthetic data

When constructing traditional Importance Sampling estimates it is typical to talk about the ‘effective sample’ size of the sample from the proposal density. The effective sample size is number of independent samples from the true target that gives an unbiased estimator with the same variance as the importance sampling estimator using \( N_G \) samples from proposal density. When using importance weights to adjust the likelihood for Bayesian updating we are not directly seeking to estimate an expectation, but minimize an (expected) loss to produce a parameter estimates.

Therefore, in this scenario we can define the a notion of the effective sample size of the synthetic data as the number of samples, \( N_G^{(e)} \), from true DGP \( P_D \) that would provide an unbiased maximum likelihood estimate (MLE) with the same variance as the Importance-Weighted MLE (IW-MLE), i.e.

\[
N_G^{(e)} := \left\{ n : \mathbb{V} \left[ \hat{\theta}^{(n)}_{IW} \right] = \mathbb{V} \left[ \hat{\theta}^{(n)}_0 \right] \right\},
\]

where the function \( \mathbb{V} \) corresponds to the asymptotic variance of hat estimator, and \( |\cdot| \) is a norm summary of the matrix values covariance of the estimator. Given the asymptotic analysis presented above for the importance-weighted likelihood we have
that
\[
N_G^{(e)} = \left( \frac{\sqrt{N_G} \tilde{V} (\hat{\theta}_0^{(n)})}{\tilde{V} (\hat{\theta}^{(N_G)}_W)} \right)^2
\]

where
\[
\frac{\tilde{V} (\hat{\theta}_0^{(n)})}{\tilde{V} (\hat{\theta}^{(N_G)}_W)} = \frac{E_{PD} \left[ w(z) (\nabla_{\theta} \ell_0(z, \theta_W)) (\nabla_{\theta} \ell_0(z, \theta_W^*))^T \right]}{E_{PD} \left[ (\nabla_{\theta} \ell_0(z, \theta_0^*)) (\nabla_{\theta} \ell_0(z, \theta_0^*)) \right]^T} = \frac{E_{PG} \left[ (\nabla_{\theta} \ell_W(z, \theta_W^*)) (\nabla_{\theta} \ell_W(z, \theta_W^*)) \right]^T}{E_{PG} \left[ w(z) (\nabla_{\theta} \ell_0(z, \theta_0^*)) (\nabla_{\theta} \ell_0(z, \theta_0^*)) \right]^T}.
\]

We note that for multidimensional parameter vectors the V’s are covariance matrices and therefore we need to take a scalar summary of these matrices in order to provide an integer effective sample size \(N_G^{(e)}\). Faced with a similar problem Lyddon et al. [2018] consider the matrix trace for example.

Lastly, given a sample \(z_{1:N_G} \sim P_G\) the effective sample size can be estimated by using empirical expectations
\[
\frac{\tilde{V} (\hat{\theta}_0^{(n)})}{\tilde{V} (\hat{\theta}^{(N_G)}_W)} \approx \frac{\sum_{i=1}^{N_G} (\nabla_{\theta} \ell_W(z_i, \hat{\theta}_W^{(n)})) (\nabla_{\theta} \ell_W(z_i, \hat{\theta}_W^{(n)}))^T}{\sum_{i=1}^{N_G} w(z_i) (\nabla_{\theta} \ell_0(z_i, \hat{\theta}_0^{(n)})) (\nabla_{\theta} \ell_0(z_i, \hat{\theta}_0^{(n)}))^T}.
\]

### A.7 FINITE SAMPLE IMPORTANCE-WEIGHTED BAYESIAN POSTERIOR

To complement the asymptotic results connecting the importance weighted general Bayesian posterior given data from \(p_G\) and the standard Bayesian \(p_D\) we can consider the difference between these two for finite \(n = m\). This is formulated in the following proposition.

**Proposition 2.** The expected KLD between standard Bayesian posterior \(\pi(\theta|x_{1:n})\) and its importance weighted approximation \(\pi_W(\theta|z_{1:m})\) in expectation over the generating distributions for \(x_{1:n} \sim P_D\) and \(z_{1:m} \sim P_G\), for \(n = m\) is
\[
E_{x \sim P_D} \left[ E_{z \sim P_G} \left[ \text{KLD}(\pi(\theta|x_{1:n})||\pi_W(\theta|z_{1:m})) \right] \right] = n E_{x \sim P_D} \left[ E_{\theta \sim \pi(\cdot|x_{1:n})} \left[ (\log f(x; \theta) - m E_{x' \sim P_D} [\log f(x'; \theta)]) \right] \right]
\]

**Proof.** We have
\[
E_{x \sim P_D} \left[ E_{z \sim P_G} \left[ \text{KLD}(\pi(\theta|x_{1:n})||\pi_W(\theta|z_{1:m})) \right] \right] = E_{x \sim P_D} \left[ E_{z \sim P_G} \left[ \int \pi(\theta|x_{1:n}) \log \frac{\pi(\theta|x_{1:n})}{\pi_W(\theta|z_{1:m})} d\theta \right] \right] = E_{x \sim P_D} \left[ E_{z \sim P_G} \left[ \sum_{i=1}^{n} \log f(x_i; \theta) - m \sum_{j=1}^{m} w(z_j) \log f(z_j; \theta) \right] \right].
\]

Now by Fubini we can reorder these integrals assuming that they all exist
\[
= E_{x \sim P_D} \left[ E_{z \sim \pi(\cdot|x_{1:n})} \left[ \left( \sum_{i=1}^{n} \log f(x_i; \theta) - \sum_{j=1}^{m} w(z_j) \log f(z_j; \theta) \right) \right] \right] = E_{x \sim P_D} \left[ E_{\theta \sim \pi(\cdot|x_{1:n})} \left[ \left( \sum_{i=1}^{n} \log f(x_i; \theta) - m E_{x' \sim P_D} [\log f(x'; \theta)] \right) \right] \right].
\]
Now assuming \( n = m \), we have

\[
= \mathbb{E}_{x \sim p_D} \left[ \mathbb{E}_{\theta \sim \pi(\cdot|x_{1:n})} \left[ \sum_{i=1}^{n} \left( \log f(x_i; \theta) - \mathbb{E}_{x' \sim p_D} \left[ \log f(x'; \theta) \right] \right) \right] \right]
\]

\[
= n \mathbb{E}_{x \sim p_D} \left[ \mathbb{E}_{\theta \sim \pi(\cdot|x_{1:n})} \left[ \left( \log f(x; \theta) - \mathbb{E}_{x' \sim p_D} \left[ \log f(x'; \theta) \right] \right) \right] \right].
\]

\[
E_{\theta \sim \pi(\cdot|x_{1:n})} \left[ \left( \log f(x; \theta) - \mathbb{E}_{x' \sim p_D} \left[ \log f(x'; \theta) \right] \right) \right]
\]

\[
\sum_{i=1}^{n} \left( \log f(x_i; \theta) - \mathbb{E}_{x' \sim p_D} \left[ \log f(x'; \theta) \right] \right)
\]

\[
\text{MMD}^2(p_D, p_G) = \frac{1}{N(N-1)} \sum_{i \neq j}^{N} (k(x_i, x_j) + k(z_i, z_j) - k(x_i, z_j) - k(x_j, z_i)),
\]

where \( k \) is a continuous kernel function; we use a standard normal kernel. The higher the value the more likely the two samples are drawn from different distributions. We can include the normalized weights in this equation in such a form:

\[
\text{MMD}^2(p_D, p_G) = \frac{1}{N(N-1)} \sum_{i \neq j}^{N} (w(x_i)w(x_j)k(x_i, x_j) + k(z_i, z_j) - w(x_i)k(x_i, z_j) - w(x_j)k(x_j, z_i)).
\]

The Wasserstein is computed by weighting the distance matrix analogously.

\[
B \text{ EXPERIMENTS}
\]

\[
B.1 \text{ EVALUATION METRICS}
\]

Given \( N \) i.i.d. samples \( x_i \) from the \( p_G \) and \( N \) i.i.d. samples \( z_i \) from the \( p_D \), an unbiased estimate of the MMD is:

\[
\text{MMD}^2(p_D, p_G) = \frac{1}{N(N-1)} \sum_{i \neq j}^{N} (k(x_i, x_j) + k(z_i, z_j) - k(x_i, z_j) - k(x_j, z_i)),
\]

where \( k \) is a continuous kernel function; we use a standard normal kernel. The higher the value the more likely the two samples are drawn from different distributions. We can include the normalized weights in this equation in such a form:

\[
\text{MMD}^2(p_D, p_G) = \frac{1}{N(N-1)} \sum_{i \neq j}^{N} (w(x_i)w(x_j)k(x_i, x_j) + k(z_i, z_j) - w(x_i)k(x_i, z_j) - w(x_j)k(x_j, z_i)).
\]

The Wasserstein is computed by weighting the distance matrix analogously.

\[
B.2 \text{ IMPLEMENTATION}
\]

The implementation will be made available online. PATE and DPGAN were implemented using the open-source implementation of Li et al. [2014] with default parameters (1000 iterations, a noise multiplier of 1.0, laplace scale of 1e-2, a moments order of 100, 1,000 observations per teacher).

The MLP for likelihood ratio estimation was computed based on the tensorflow and tensorflow_privacy package. To ensure the privacy of the MLP, we started with a configuration of one epoch, a batch size of 1, an L2 norm clip of 1, a noise multiplier of 5.2, 20 microbatches and a learning rate of 0.1. We computed the \( \epsilon \) using built-in functions and increased/decreased the noise multiplier and the number of epochs until the desired privacy level was reached. The logistic regression for density ratio estimation is learned with L2 regularization of 0.1 using sklearn. We chose \( N_3 = N_D \) unless otherwise mentioned. To compute the output-noised weights we computed the largest \( N_3 \) such that the scale restriction was satisfied and conducted the downstream analysis on this smaller dataset.

The downstream classification algorithms were trained using sklearn with default parameters.

\[
B.3 \text{ ADDITIONAL RESULTS}
\]

In Table 1, we see that PSIS and calibration processed discriminator weights improve upon the unprocessed weights. Note that the post-processing was only applied on the weights from the GAN discriminator to extend the results others has already proven given space limitations. For a more extensive overview of all results obtained look at Tables 6 to 8 where we abbreviate the PSIS processed weights by appending a \(-p\) and post-calibrated weights by appending a \(-c\).

Next, we present some more results on the GMM toy example. Please refer to Table 5 for all results for the case where we have a mixture of uniform distributions as described in Section ?? as SDGP and a GMM as DGP (see Figure 1).

We extend this experiment by assigning labels of 0 to 1 randomly to the observations of the SDGP. For the DGP we sampled a label of 1 if the first coordinate was smaller than 0.5 with a probability of 80%. Otherwise we sampled it with a probability
| weight       | PATEGAN     | DPGAN       |
|--------------|-------------|-------------|
|              | WST  | MMD  | SVM  | RF   | MLP  | WST  | MMD  | SVM  | RF   | MLP  |
| Breast       |      |      |      |      |      |      |      |      |      |      |
| discriminator| 1.5194 | 0.0670 | 0.4867 | 0.1923 | 0.0898 | 1.4975 | 0.0592 | 0.5260 | 0.2592 | 0.1021 |
| PSIS         | 1.5890 | 0.0754 | 0.5978 | 0.2992 | 0.1307 | 1.5209 | 0.0613 | 0.4416 | 0.2365 | 0.1159 |
| calibrated   | 1.6098 | 0.0754 | 0.5985 | 0.3156 | 0.0718 | 1.5223 | 0.0613 | 0.4417 | 0.2349 | 0.1306 |
| Spam         |      |      |      |      |      |      |      |      |      |      |
| discriminator| 2.9582 | 0.1963 | 0.4945 | 0.4150 | 0.4867 | 0.6185 | 0.0043 | 0.4938 | 0.4781 | 0.4148 |
| PSIS         | 2.9598 | 0.1960 | 0.4760 | 0.3611 | 0.5284 | 2.3378 | 0.0988 | 0.5997 | 0.3953 | 0.5784 |
| calibrated   | 3.0072 | 0.1960 | 0.4771 | 0.3566 | 0.5095 | 2.3060 | 0.0982 | 0.5998 | 0.3972 | 0.5589 |
| Credit       |      |      |      |      |      |      |      |      |      |      |
| discriminator| 0.9247 | 0.0549 | 0.4597 | 0.5208 | 0.4935 | 1.0555 | 0.0474 | 0.5006 | 0.5030 | 0.4388 |
| PSIS         | 0.8803 | 0.0505 | 0.4507 | 0.5395 | 0.5284 | 0.8723 | 0.0473 | 0.5060 | 0.6121 | 0.4444 |
| calibrated   | 0.8860 | 0.0505 | 0.4508 | 0.5365 | 0.4872 | 0.8123 | 0.0003 | 0.5059 | 0.6121 | 0.5101 |

Table 1: Results for the parameters ($\varepsilon = 6.0, \delta = 1e-5$) (Wasserstein distance, maximum mean discrepancy, support vector classifier AUC, random forest classifier AUC, multi-layer perceptron classifier AUC)

Figure 1: A KDE plot of the scaled data sampled for the GMM toy example

| weight         | WST    | MMD    | AUC logreg | AUC SVM | AUC RF | AUC MLP |
|----------------|--------|--------|------------|---------|--------|---------|
| generator      | 0.0193 | 0.0006 | 0.7962     | 0.6786  | 0.5113 | 0.7926  |
| logreg         | 0.0699 | 0.0305 | 0.4592     | 0.5222  | 0.5052 | 0.3260  |
| beta_noised    | 0.0699 | 0.0305 | 0.4593     | 0.5458  | 0.5100 | 0.5356  |
| priv_mlp       | 0.1018 | 0.0243 | 0.5867     | 0.6715  | 0.5121 | 0.5290  |
| none           | 0.1296 | 0.0657 | 0.4643     | 0.5205  | 0.5055 | 0.6182  |

Table 2: Results for the ($\varepsilon = 6.0, \delta = 1e-5$) mixture of uniform model as SDGP with randomly assigned labels (coin flip) and the GMM as DGP with labels $y_i \sim Bernoulli(0.8)$ if $x_{i,0} < 0.5$ and $y_i \sim Bernoulli(0.2)$ if $x_{i,0} > 0.5$
Table 3: Results for the \((\varepsilon = 6.0, \delta = 1e - 05)\) mixture of uniform distributions as SDGP model trained on the GMM toy example

| weight     | Beta MSE | WST   | MMD   | logreg | SVM   | RF   | MLP   |
|------------|----------|-------|-------|--------|-------|------|-------|
| generator  | 0.2984   | 0.0036| 0.0984| 0.2203 | 0.5278| 0.5197| 0.2759|
| logreg     | 1.1631   | 0.0682| 0.0356| 0.5628 | 0.4956| 0.5172| 0.3923|
| beta_noised| 1.1634   | 0.0682| 0.0357| 0.5628 | 0.4881| 0.5178| 0.3802|
| priv_mlp   | 1.1468   | 0.0207| 0.0624| 0.6707 | 0.5172| 0.5178| 0.5480|
| none       | 1.1650   | 0.1376| 0.0642| 0.5863 | 0.5293| 0.5172| 0.5096|

Table 4: Results for the parameters \((\varepsilon = 6.0, \delta = 1e - 5)\) on the modelLoh et al. [2019] repeated over 25 seeds (Log Score, Maximum Mean Discrepancy, Mahalhobis Distance, Absolute Error in \(\theta_1\) (the first parameter of interest in the model), Absolute Error in \(\theta_2\) (the second parameter of interest in the model))

| weight     | PATEGAN | DPGAN  |
|------------|---------|--------|
|            | LogS    | MahaD  | \(\theta_1E\) | \(\theta_2E\) | LogS    | MahaD  | \(\theta_1E\) | \(\theta_2E\) |
| none       | 10831   | 0.194  | 0.582  | 2.23   | 9013   | 0.188  | 0.612  | 1.92   |
| beta_noised| 9480    | 0.178  | 0.487  | 2.03   | 19668  | 0.309  | 1.13   | 2.46   |
| output_lapl| 22716   | 0.379  | 0.680  | 3.84   | 35212  | 0.457  | 1.36   | 3.70   |
| priv_mlp   | 10541   | 0.172  | 0.534  | 1.52   | 15606  | 0.284  | 1.01   | 2.25   |
| discriminator| 10875  | 0.195  | 0.584  | 2.24   | 9148   | 0.185  | 0.610  | 1.90   |

of 20%. Please refer to Table 2 for the results and to Figure 5a for a visualization of the unweighted two data set, and Figure 5b for the weighted SDGP alternatives.

For a more sophisticated experiment, we assigned the labels to the DGP according to a logistic regression with \(\beta = (0.2, 1)\). The data is represented in 4a. Again we can see a visualization of the weighted synthetic data in Figure 4b. We see in Table 3 that the MSE for the beta computation is smallest for the weighted alternatives, especially the priv_mlp. We however also note that the MSE of the generator is by far smaller than the other values which makes the need for better density estimators apparent.

In the context of Section ??, we compute the SDGP as the convex combination of the DGP (with weight \(1 - \gamma\)) and the a second different distribution (with weight \(\gamma\)). Please refer to Figure 3a for a visualization of the data when the SDGP is equal to the DGP (\(\gamma = 0\)). Figure 3b visualizes the data when \(\gamma = 0.4\) and Figure 3c depicts the situation when the SDGP does not consist of a mixture of the DGP anymore. We clearly see the departure of the not weighted posterior from the true parameter to the synthetic wrong parameter when \(\gamma\) gets larger. For large \(\gamma\) the effect of importance weighting improves. We see that the Laplace weighted posterior suffers from a high variance as expected.

We also consider the treatment effect estimation as described in Loh et al. [2019] which also considers the problem of subgroup identification. They introduce a collection of synthetic DGP with explicit treatment effect before they consider examples in breast cancer. The results are in Table 4 they demonstrate that noise in the weight estimation can be too large for the bias to be mitigated. More precisely, we consider the DGP of Loh et al. [2019] and PATEGAN and DPGAN as SDGP. In this the improvement if any is minimal - likely due to low quality weight estimates in the first place.
Figure 2: Posterior distribution of $\theta$ in convex combination toy example for different values of $\gamma$. The red dashed line is the synthetic parameter whereas the green dashed line represents the value of the true theta.
(a) Convex combination toy example of multivariate Gaussians for $\gamma = 0$
(b) Convex combination toy example of multivariate Gaussians for $\gamma = 0.4$
(c) Convex combination toy example of multivariate Gaussians for $\gamma = 1$
(a) KDE plots for mixture of uniform model as SDGP with randomly assigned labels (coin flip) (lower figure) and the GMM as DGP with labels drawn according to mixture of Bernoulli variables (upper figure).

(b) Weighted KDE plots where labels of the DGP were sampled according to a conditional mixture of Bernoulli distributions.

Figure 4

(a) KDE plots of the DGP were sampled according to the probabilities of a logistic regression with $\beta = (0.2, 1)$ and SDGP

(b) Weighted KDE plots where the labels of the DGP were sampled according to the probabilities of a logistic regression with $\beta = (0.2, 1)$

Figure 5
Table 5: Results for the \((\epsilon = 6.0, \delta = 1e^{-5})\) model trained on the GMM toy example

| weight | PATEGAN | DPGAN |
|--------|---------|-------|
|        | WST     | MMD   | WST     | MMD   |
| none   | 0.0330  | 0.0157| 0.0882  | 0.0177|
| generator | 0.0331  | 0.0157| 0.0880  | 0.0177|
| LR     | 0.0388  | \textbf{0.0007} | 0.1248  | 0.0488|
| beta   | 0.0388  | \textbf{0.0007} | 0.1249  | 0.0486|
| lapl   | 0.0041  | 0.0034| 0.0122  | 0.0546|
| norm   | \textbf{0.0001} | 0.0198 | \textbf{0.0001} | 0.0653|
| priv_mlp | 0.0312  | 0.0100| 0.1295  | 0.0649|
| mlp    | 0.0342  | 0.0112| 0.2624  | 0.0658|

| generator-p | 0.0314  | 0.0192| 0.2563  | 0.0824|
| LR-p        | 0.0388  | \textbf{0.0007} | 0.1326  | 0.0371|
| beta-p      | 0.0388  | \textbf{0.0007} | 0.1327  | 0.0369|
| lapl-p      | 0.0041  | 0.0019| 0.0133  | 0.0370|
| norm-p      | 0.0023  | 0.0130| \textbf{0.0001} | 0.0518|
| priv_mlp-p  | 0.0312  | 0.0100| 0.1327  | 0.0686|
| mlp-p       | 0.0342  | 0.0112| 0.2618  | 0.0669|

| generator-c | 0.0312  | 0.0194| 0.2564  | 0.0824|
| LR-c        | 0.0367  | 0.0042| 0.2300  | 0.0399|
| beta-c      | 0.0367  | 0.0042| 0.2303  | 0.0401|
| lapl-c      | 0.0249  | 0.0132| 0.1698  | 0.0659|
| norm-c      | 0.0258  | 0.0169| 0.1967  | 0.0785|
| priv_mlp-c  | 0.0343  | 0.0127| 0.2498  | 0.0700|
| mlp-c       | 0.0335  | 0.0133| 0.2730  | 0.0688|

Table 6: Results for the \((\epsilon = 6.0, \delta = 1e^{-5})\) model trained on breast

| weight | PATEGAN | DPGAN |
|--------|---------|-------|
|        | WST     | MMD   | WST     | MMD   |
| none   | 1.5472  | 0.0670| 0.5799  | 0.4876|
| generator | 1.5194  | 0.0670| 0.5804  | 0.4867|
| LR     | \textbf{0.0012} | 0.0462| 0.5450  | 0.4804|
| beta   | 0.0023  | 0.0462| 0.5825  | 0.5172|
| norm   | 0.0001  | 0.1818| 0.3651  | 0.4570|
| priv_mlp | 0.1769  | 0.0495| \textbf{0.5912} | 0.6196|
| mlp    | 1.0956  | 0.3844| 0.5740  | 0.3893|

| generator-p | 0.0514  | 0.0592| 0.5142  | 0.5260|
| LR-p        | 0.0042  | 0.0375| 0.5800  | 0.4429|
| beta-p      | 0.0050  | 0.0375| 0.5762  | 0.4450|
| lapl-p      | 0.0001  | 0.1549| 0.5483  | 0.4762|
| norm-p      | 0.0744  | 0.0466| 0.5039  | 0.3994|
| priv_mlp-p  | 0.8989  | 0.0523| 0.4597  | 0.4360|
| mlp-p       | 1.5209  | 0.0754| 0.4009  | 0.2992|

| generator-c | 1.5209  | 0.0613| 0.5292  | 0.4416|
| LR-c        | 0.0042  | 0.0375| 0.5777  | 0.4430|
| beta-c      | 0.0050  | \textbf{0.0374} | 0.5765  | 0.4450|
| lapl-c      | 0.0797  | 0.0438| 0.5039  | 0.4041|
| norm-c      | 0.8989  | 0.0523| 0.4597  | 0.4360|
| priv_mlp-c  | 1.5223  | 0.0613| 0.5286  | 0.4417|
| mlp-c       | 1.2320  | 0.0681| 0.4573  | 0.3818|

Table 6: Results for the \((\epsilon = 6.0, \delta = 1e^{-5})\) model trained on breast
| weight         | PATEGAN | DPGAN     |
|----------------|---------|-----------|
|                | WST     | MMD       | LR  | SVM | RF  | MLP | WST     | MMD       | LR  | SVM | RF  | MLP |
| none           | 3.0221  | 0.4569    | **0.4966** | 0.4508 | 0.4269 |
| generator      | 2.9582  | 0.4562    | 0.4945 | 0.4150 | 0.4485 |
| LR             | 0.0109  | 0.1134    | 0.4830 | 0.4731 | 0.4368 | 0.4651 |
| beta           | 0.1863  | 0.1163    | 0.4910 | 0.4751 | 0.4237 | 0.4783 |
| norm           | **0.0001** | 0.2768    | 0.4858 | 0.4771 | 0.3936 | 0.3689 |
| priv_mlp       | 0.0117  | 0.1249    | 0.5094 | 0.4564 | 0.4230 | 0.4959 |
| mlp            | 0.0109  | 0.1320    | 0.4575 | 0.4398 | 0.4076 | 0.3908 |
| generator-p    | 2.9598  | 0.1960    | **0.5610** | 0.4760 | 0.3611 | 0.5284 |
| LR-p           | 0.0111  | 0.1140    | 0.4808 | 0.4737 | 0.4084 | 0.4354 |
| beta-p         | 0.1911  | 0.1170    | 0.4882 | 0.4756 | 0.4012 | 0.4719 |
| norm-p         | 0.0023  | 0.3088    | 0.4994 | 0.4790 | 0.3950 | 0.3553 |
| priv_mlp-p     | 0.0124  | 0.1263    | 0.5051 | 0.4560 | 0.4385 | 0.4948 |
| mlp-p          | 0.0113  | 0.1334    | 0.4588 | 0.4414 | 0.4343 | 0.3918 |
| generator-c    | 3.0072  | 0.1960    | 0.5599 | 0.4771 | 0.3565 | 0.5095 |
| LR-c           | 0.1155  | 0.1292    | 0.5050 | 0.4717 | 0.3581 | 0.4751 |
| beta-c         | 0.7920  | 0.1376    | 0.5126 | 0.4702 | 0.3606 | 0.4706 |
| norm-c         | 2.8678  | 0.1948    | 0.5587 | 0.4725 | 0.3708 | **0.5101** |
| priv_mlp-c     | 0.1285  | 0.1356    | 0.5210 | 0.4676 | 0.4037 | 0.4910 |
| mlp-c          | 0.1149  | 0.1355    | 0.4654 | 0.4519 | 0.3756 | 0.4231 |

Table 7: Results for the \((\epsilon = 6.0, \delta = 1e^{-5})\) model trained on spam

| weight         | PATEGAN | DPGAN     |
|----------------|---------|-----------|
|                | WST     | MMD       | LR  | SVM | RF  | MLP | WST     | MMD       | LR  | SVM | RF  | MLP |
| none           | 0.9406  | 0.0548    | 0.5041 | 0.4594 | 0.5196 | 0.4910 |
| generator      | 0.9247  | 0.0549    | 0.5033 | 0.4597 | 0.5208 | 0.4935 |
| LR             | 0.0007  | **0.0155** | 0.5252 | 0.4919 | 0.5522 | 0.4916 |
| beta           | **0.0001** | 0.0155    | 0.5253 | 0.4919 | 0.5519 | 0.4878 |
| norm           | 0.0090  | 0.0769    | 0.4894 | **0.5250** | 0.5390 | 0.4856 |
| priv_mlp       | **0.0001** | 0.0202    | 0.5095 | 0.5078 | 0.5661 | 0.4788 |
| mlp            | 0.0000  | 0.0229    | 0.5189 | 0.5103 | 0.5417 | 0.5055 |
| generator-p    | 0.8803  | 0.0505    | 0.5288 | 0.4507 | 0.5395 | **0.5284** |
| LR-p           | 0.0001  | 0.0157    | 0.5255 | 0.4923 | 0.5556 | 0.4967 |
| beta-p         | **0.0001** | 0.0157    | 0.5258 | 0.4923 | 0.5560 | 0.4966 |
| norm-p         | 0.0002  | 0.0566    | 0.4852 | 0.5204 | 0.5254 | 0.4678 |
| priv_mlp-p     | 0.0003  | 0.0216    | 0.5258 | 0.5081 | **0.5845** | 0.4930 |
| mlp-p          | 0.0002  | 0.0218    | 0.5194 | 0.5116 | 0.5400 | 0.5124 |
| generator-c    | 0.8890  | 0.0505    | **0.5292** | 0.4508 | 0.5365 | 0.4872 |
| LR-c           | **0.0001** | 0.0171    | 0.5033 | 0.4831 | 0.5627 | 0.4799 |
| beta-c         | 0.0984  | 0.0176    | 0.5083 | 0.4810 | 0.5656 | 0.4973 |
| norm-c         | 0.8807  | 0.0454    | 0.5259 | 0.4577 | 0.5564 | 0.4814 |
| priv_mlp-c     | 0.0002  | 0.0181    | 0.4956 | 0.5043 | 0.5552 | 0.4907 |
| mlp-c          | 0.0048  | 0.0186    | 0.5120 | 0.4912 | 0.5503 | 0.5018 |

Table 8: Results for the \((\epsilon = 6.0, \delta = 1e^{-5})\) model trained on credit
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