MARTINGALE REPRESENTATION AND LOGARITHMIC-SOBOLEV INEQUALITY FOR THE FRACTIONAL ORNSTEIN-UHLENBECK MEASURE∗

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Abstract In this paper, we consider the measure determined by a fractional Ornstein-Uhlenbeck process. For such a measure, we establish an explicit form of the martingale representation theorem and consequently obtain an explicit form of the Logarithmic-Sobolev inequality. To this end, we also present the integration by parts formula for such a measure, which is obtained via its pull back formula and the Bismut method.

Key words Fractional Ornstein-Uhlenbeck measure; integration by parts formula; martingale representation theorem; Logarithmic-Sobolev inequality

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1 Introduction

A stochastic process \((X_t)_{0 \leq t \leq 1}\) is called a fractional Ornstein-Uhlenbeck process if it satisfies the stochastic differential equation

\[
dX_t = -\alpha X_t dt + dB_t^H, \quad X_0 = 0,
\]

where \(\alpha > 0\) and \((B_t^H)_{0 \leq t \leq 1}\) is an \(n\)-dimensional fractional Brownian motion with Hurst parameter \(H > 1/2\). Fractional Ornstein-Uhlenbeck processes are widely used to describe the long memory property for some time series in fields such as finance, hydrology, telecommunications, insurance and computer networks. The measure determined by a fractional Ornstein-Uhlenbeck process is called the fractional Ornstein-Uhlenbeck measure. In this paper, we investigate the martingale representation and the Logarithmic-Sobolev inequality for this measure.

Quite a lot of interest has been paid to the study of the martingale representation and the Logarithmic-Sobolev inequalities for different measures. It is proved in [13] that Logarithmic-Sobolev inequality holds for the Wiener measure on the path space over a connected Lie

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group. For the Wiener measure on the path space over a Riemannian manifold, [17] gives the Logarithmic-Sobolev inequality with a bounding constant which can be estimated in terms of the Ricci curvature. Moreover, for such a measure, the Logarithmic-Sobolev inequality can also be obtained by embedding the manifold into a Euclidean space [3] and the martingale representation [5]. For the Brownian bridge measure on loop space, the martingale representation and the Logarithmic-Sobolev inequality are investigated in [2, 12, 14].

The Logarithmic-Sobolev inequalities for measures can be obtained via their martingale representations, which in turn can be established by their integration by parts formulas. The integration by parts formulas for different measures are important in infinite dimensional analysis, and have been well studied. For instance, the integration by parts formula is investigated for the Wiener measure on the path space in [7, 11, 15], for the Brownian bridge measure on the loop space in [1, 8, 10, 16], for the fractional Wiener measure under different integrals in [6, 9], and for the fractional Ornstein-Uhlenbeck measure in [21].

Contributions We establish the pull back formula (Proposition 3.1) and an integration by parts formula for the fractional Ornstein-Uhlenbeck measure (Theorem 3.2). We give an explicit form of martingale representation theorem (Theorem 4.1) by the corresponding integration by parts formula. Consequently, we derive an explicit form of the Logarithmic-Sobolev inequality (Theorem 4.2) by the martingale representation theorem.

The paper is organized as follows: in Section 2, we give some preliminaries about fractional Brownian motions. We present in Section 3 the pull back formula and the integration by parts formula. In Section 4, we obtain the martingale representation theorem and the Logarithmic-Sobolev inequality for the fractional Ornstein-Uhlenbeck measure.

2 Preliminaries

Let \((\Omega, \mathcal{F}, \mathcal{F}_t, \nu)\) be a filtered probability space, where \(\Omega\) is the space of \(\mathbb{R}^n\)-valued continuous functions on \([0, 1]\) with the initial value zero, \(\nu\) is the fractional Ornstein-Uhlenbeck measure such that coordinate process \((X_t(\omega))_{0 \leq t \leq 1} = (\omega_t)_{0 \leq t \leq 1}\) satisfies (1.1), \(\mathcal{F}\) is the \(\nu\)-completion of the Borel \(\sigma\)-algebra of \(\Omega\), and \(\mathcal{F}_t\) is the \(\nu\)-completed natural filtration of \(\omega\). The space \(\Omega\) is a metric space with the uniform metric

\[
d(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)|, \quad f, g \in \Omega,
\]

and the topology of uniform convergence is derived from this metric.

In what follows we consider fractional integral with \(H > 1/2\). Let

\[
\mathcal{H} = \left\{ h : \Omega \to L^2([0, 1]; \mathbb{R}^n) \mid h \text{ is adapted processes and } \mathbb{E}_\nu \left[ \int_0^1 |h_t|^2 dt \right] < \infty \right\}.
\]

We denote \(I_{0+}^{H+1/2}(\mathcal{H})\) as the \((H + 1/2)\)-Hölder left fractional Riemann-Liouville integral operator. By [6], the isomorphism operator \(K : \mathcal{H} \to I_{0+}^{H+1/2}(\mathcal{H})\) is defined by

\[
(Kh)_t = \int_0^t K(t, s)h_s ds,
\]

where \(h \in \mathcal{H}\) and

\[
K(t, s) = c_H s^{\frac{1}{2} - H} \int_s^t u^{\frac{1}{2} - H} (u - s)^{H - \frac{1}{2}} du I_{[0, t]}(s), \quad (2.1)
\]
in which, for beta function $B(\cdot, \cdot)$,

$$c_H = \sqrt{\frac{H(2H - 1)}{B(2 - 2H, H - \frac{1}{2})}}. \quad (2.2)$$

By the definition of $K\nu$, for $H > 1/2$, we have

$$(K^{-1}h)_t = t^{H - \frac{1}{2}} D^{H - \frac{1}{2}}_t t^{\frac{1}{2} - H} h'$$

$$= \frac{1}{\Gamma(\frac{3}{2} - H)} \left( t^{\frac{1}{2} - H} h'_t + (H - \frac{1}{2}) t^{H - \frac{1}{2}} \int_0^t \frac{t^{\frac{1}{2} - H} h'_t - u^{\frac{1}{2} - H} h'_u}{(t - u)^{\frac{1}{2} + H}} du \right), \quad (2.3)$$

where $D^{H - 1/2}$ is the $(H - 1/2)$-order left-sided Riemann-Liouville derivative and $h'$ is the derivative of $h$. Inspired by [6, Theorem 3.3], the vector field on $\Omega$ can be defined as

$$\mathcal{H}_H = \{ Kh | h \in \mathcal{H} \},$$

with scalar product

$$\langle Kh, Kg \rangle_{\mathcal{H}_H} = \langle h, g \rangle_{L^2(\Omega; \nu)} = \mathbb{E}_\nu \left[ \int_0^1 \langle h_t, g_t \rangle dt \right].$$

In fact, it is easy to check that $\mathcal{H}_H$ is a Hilbert space. For $Kh \in \mathcal{H}_H$, the directional derivative of $F$ along $Kh$ is

$$D_h F(\omega) = \lim_{\delta \to 0} \frac{1}{\delta} (F(\omega + \delta(Kh)) - F(\omega)).$$

We denote all the smooth cylindrical functions on $\Omega$ by

$$\mathcal{F}^\infty C(\Omega) = \{ F | F(\omega) = f(\omega_{t_1}, \ldots, \omega_{t_n}), 0 < t_1 \leq t_2 \leq \ldots \leq t_n \leq 1, f \in C^\infty(\mathbb{R}^n) \}. $$

For $F \in \mathcal{F}^\infty C(\Omega)$, the directional derivative of $F$ along $Kh$ is

$$D_h F(\omega) = \sum_{i=1}^n \langle \nabla^i F, (Kh)_{i,} \rangle_{\mathbb{R}^n},$$

where

$$\nabla^i F = \nabla^i f(\omega_{t_1}, \ldots, \omega_{t_n})$$

is the gradient with respect to the $i$ variable of $f$. For $F \in \mathcal{F}^\infty C(\Omega)$, the gradient $DF : \Omega \to \mathcal{H}_H$ is determined by

$$\langle DF, Kh \rangle_{\mathcal{H}_H} = D_h F.$$ 

Then, through the integration by parts formula (3.4), we can show that the gradient operator $D$ is closable in $L^p(\nu)$ for all $p$ and that $\mathcal{F}^\infty C(\Omega)$ is a core. We denote again its smallest closure by $D$. Furthermore, $\text{Dom}(D)$ denotes the domain of the smallest closure of $D$.

### 3 Integration by Parts Formula for Fractional Ornstein-Uhlenbeck Measure

To obtain the integration by parts formula for the fractional Ornstein-Uhlenbeck measure, inspired by the idea in [4], for any $H \in \mathcal{H}$, we first construct an $\mathbb{R}^n$-valued function $(\beta_t)_{0 \leq t \leq 1}$ such that for any $r \in (-\epsilon, \epsilon)$, the stochastic differential equation

$$dX_t(r) = -\alpha X_t(r)dt + dB_t^H(r) \quad (3.1)$$

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has a solution \((X_t(r))_{0 \leq t \leq 1}\) satisfying

Condition 1. that \((X_t(r))_{0 \leq t \leq 1} \in \Omega\) for any \(r\),

Condition 2. that \(\frac{d}{dr}X_t(r)|_{r=0}\) exists and \((Kh)_t = \frac{d}{dr}X_t(r)|_{r=0}\) for \(((Kh)_t)_{0 \leq t \leq 1} \in \mathcal{H}_H\),

where \(B^H_t(r)\) is defined by

\[
B^H_t(r) = B^H_t + r \int_0^t \beta_s ds.
\]

Note that \((X_t(0))_{0 \leq t \leq 1} = (X_t)_{0 \leq t \leq 1}\) and \((B^H_t(0))_{0 \leq t \leq 1} = (B^H_t)_{0 \leq t \leq 1}\).

**Proposition 3.1** Suppose that \(h \in \mathcal{H}\). If \((\beta_t)_{0 \leq t \leq 1}\) satisfies Condition 3 and Condition 3, then we have

\[
\beta_t = (Kh)'_t + \alpha(Kh)_t. \tag{3.2}
\]

**Proof** Differentiating (3.1) with respect to \(r\) at \(r = 0\), we get

\[
\frac{d}{dr}X_t(r)|_{r=0} = -\alpha \frac{d}{dr}X_t(r)|_{r=0} dt + \frac{d}{dr}B^H_t(r)|_{r=0}.
\]

By Condition 2, we have \(\frac{d}{dr}X_t(r)|_{r=0} = (Kh)'_t dt\). Then, it holds that \((Kh)'_t dt = -\alpha(Kh)_t dt + \beta_t dt\), which yields \(\beta_t = (Kh)'_t + \alpha(Kh)_t\). \(\Box\)

Suppose that \(B^H\) is a fractional Brownian motion with the integral representation

\[
B^H_t = \int_0^t K(t, s) dB_s, \tag{3.3}
\]

where \((B_t)_{0 \leq t \leq 1}\) is an \(n\)-dimensional Brownian motion under \(\nu\) (see [6]). In the following we establish the integration by parts formula for the fractional Ornstein-Uhlenbeck measure \(\nu\) via the pull back formula given in Proposition 3.1:

**Theorem 3.2** For \(F \in \text{Dom}(D)\) and \(Kh \in \mathcal{H}_H\), the integration by parts formula for the fractional Ornstein-Uhlenbeck measure \(\nu\) is

\[
\mathbb{E}_\nu \left[ F \int_0^1 \left( \left( K^{-1} \int_0^t \beta_u du \right)_t, dB_t \right) \right] = \mathbb{E}_\nu[D_h F], \tag{3.4}
\]

where \(\beta_t = (Kh)'_t + \alpha(Kh)_t\).

**Proof** By (3.3) and Proposition 3.1, we obtain

\[
B^H_t(r) = \int_0^t K(t, s) dB_s + r \int_0^r \left( K^{-1} \int_0^s \beta_u du \right)_v dv.
\]

For \(t \in [0, 1]\), we set

\[
\rho_t(r) = \exp \left\{ -r \int_0^t \left( K^{-1} \int_0^s \beta_u du \right)_v dB_s - \frac{r^2}{2} \int_0^t \left( K^{-1} \int_0^s \beta_u du \right)_v^2 ds \right\}. \tag{3.5}
\]

For \(H > 1/2\), by (2.3) and (3.2), we have

\[
\left( K^{-1} \int_0^s \beta_u du \right)_v = h_s + \alpha \left( K^{-1} \int_0^s (Kh)_u du \right)_v
\]

\[
= h_s + \frac{\alpha}{\Gamma(\frac{3}{2} - H)} \left\{ s^{\frac{1}{2} - H}(Kh)_s + (H - \frac{1}{2}) s^{H - \frac{1}{2}} \int_0^s \frac{s^{\frac{1}{2} - H}(Kh)_u - u^{\frac{1}{2} - H}(Kh)_u}{(s-u)^{\frac{3}{2} + H}} du \right\}
\]

\[
= h_s + \frac{\alpha s^{\frac{1}{2} - H}}{\Gamma(\frac{3}{2} - H)} (Kh)_s + \frac{\alpha(H - \frac{1}{2}) s^{H - \frac{1}{2}}}{\Gamma(\frac{3}{2} - H)} \int_0^s \frac{s^{\frac{1}{2} - H} - u^{\frac{1}{2} - H}}{(s-u)^{\frac{3}{2} + H}} (Kh)_u du
\]
In fact, by replacing \( u \) with \( sv \), we get
\[
\int_0^s \frac{s^{\frac{1}{2} - H} - u^{\frac{1}{2} - H}}{(s-u)^{\frac{1}{2} + H}} \, du = C_2 s^{1-2H}.
\] (3.9)

By Hölder inequality,
\[
|(Kh)_s| \leq \left( \int_0^s K^2(s,u) \, du \right)^{\frac{1}{2}} \left( \int_0^1 |h|^2 \, du \right)^{\frac{1}{2}} \leq \frac{C_1 s^{1-H}}{\sqrt{2-2H}} \left( \int_0^1 |h|^2 \, du \right)^{\frac{1}{2}},
\]
where \( C_1 \) satisfies
\[
|K(s,t)| \leq C_1 t^{\frac{1}{2}-H},
\] (3.7)

which is due to Theorem 3.2 in [6]. Hence, for \( s \in [0,1] \), we have
\[
|I_1| \leq \frac{\alpha C_1 s^{\frac{3}{2}-2H}}{\Gamma(\frac{3}{2} - H)\sqrt{2-2H}} \left( \int_0^1 |h|^2 \, du \right)^{\frac{1}{2}}.
\] (3.8)

By [19, Theorem 3], there is a constant \( C_2 \) such that
\[
\int_0^s \frac{s^{\frac{1}{2} - H} - u^{\frac{1}{2} - H}}{(s-u)^{\frac{1}{2} + H}} \, du = C_2 s^{1-2H}.
\] (3.9)

In fact, by replacing \( u \) with \( sv \), we get
\[
\int_0^s \frac{s^{\frac{1}{2} - H} - u^{\frac{1}{2} - H}}{(s-u)^{\frac{1}{2} + H}} \, du = s^{1-2H} \int_0^1 \frac{1 - v^{\frac{1}{2} - H}}{(1-v)^{\frac{1}{2} + H}} \, dv,
\]
Since
\[
\lim_{v \to 1} (1-v)^H \frac{1 - v^{\frac{1}{2} - H}}{(1-v)^{\frac{1}{2} + H}} = 0 \quad \text{and} \quad \lim_{v \to 0} v^H \frac{1 - v^{\frac{1}{2} - H}}{(1-v)^{\frac{1}{2} + H}} = 0,
\]
by the comparison test for improper integrals, (3.9) holds. Therefore,
\[
|I_2| \leq -\frac{(H - \frac{1}{2}) \alpha C_1 C_2 s^{\frac{1}{2}-2H}}{\Gamma(\frac{3}{2} - H)\sqrt{2-2H}} \left( \int_0^1 |h|^2 \, du \right)^{\frac{1}{2}}.
\] (3.10)

Since \( Kh \in L^2_u \), by [20, Theorem 3.6], \( Kh \) is \( H \)-Hölder continuous. Therefore, there exists a constant \( C_3 \) such that
\[
|I_3| \leq \frac{C_3 \alpha (H - \frac{1}{2})}{\Gamma(\frac{3}{2} - H)} \int_0^s \frac{(s-u)^H}{(s-u)^{\frac{1}{2} + H}} \, du \left( \int_0^1 |h|^2 \, du \right)^{\frac{1}{2}} = \frac{C_3 \alpha (H - \frac{1}{2})}{2\Gamma(\frac{3}{2} - H)} \left( \int_0^1 |h|^2 \, du \right)^{\frac{1}{2}}.
\] (3.11)

By (3.6), (3.8), (3.10) and (3.11), we conclude that
\[
\int_0^1 \left( K^{-1} \int_0^s \beta_u \, du \right)_s^2 \, ds \\
\leq 4 \int_0^1 |h_s|^2 \, ds + 4 \int_0^1 |I_1|^2 \, ds + 4 \int_0^1 |I_2|^2 \, ds + 4 \int_0^1 |I_3|^2 \, ds \\
\leq 4 \left( 1 + \frac{\alpha C_1}{\Gamma(\frac{3}{2} - H)\sqrt{(2-2H)(4-4H)}} \right)^2 + \left( \frac{(H - \frac{1}{2}) \alpha C_1 C_2}{\Gamma(\frac{3}{2} - H)\sqrt{(2-2H)(4-4H)}} \right)^2 \\
+ \left( \frac{C_3 \alpha (H - \frac{1}{2})}{2\Gamma(\frac{3}{2} - H)} \right)^2 \left( \int_0^1 |h|^2 \, du \right).
\] (3.12)
Suppose that $h$ is bounded adapted process, then, by (3.12), $\mathbb{E}_\nu[\rho_1(r)] = 1$. It is easy to see that
\[
\int_0^t \beta_u du \in L^{H+\frac{3}{2}}(L^2(H)).
\]
Therefore, by [19, Theorem 2],
\[
\int_0^t K(t,s) d \left( B_s + \left( K^{-1} \int_0^s \beta_u du \right)_s \right)
\]
is a fractional Brownian motion on $\Omega$ under $\rho_1(r)\nu$. Hence, $(B^H_t(r))_{0 \leq t \leq 1}$ and $(B^H_t)_{0 \leq t \leq 1}$ have the same distribution under $\rho_1(r)\nu$ and $\nu$, respectively, which implies that $(X_t(r))_{0 \leq t \leq 1}$ and $(X_t)_{0 \leq t \leq 1}$ have the same distribution under $\rho_1(r)\nu$ and $\nu$, respectively. Therefore, for $F = f(X_{t_1}, \ldots, X_{t_n}) \in \mathcal{FC}^\infty(\Omega)$,
\[
\mathbb{E}_{\rho_1(r)\nu}[f(X_{t_1}(r), \ldots, X_{t_n}(r))] = \mathbb{E}_\nu[f(X_{t_1}, \ldots, X_{t_n})].
\]
Differentiating the above equation with respect to $r$, we obtain
\[
\frac{d}{dr} \mathbb{E}_\nu[\rho_1(r)f(X_{t_1}(r), \ldots, X_{t_n}(r))] \bigg|_{r=0} = \mathbb{E}_\nu \left[ \frac{d}{dr} \rho_1(r) \bigg|_{r=0} f(X_{t_1}, \ldots, X_{t_n}) \right] + \mathbb{E}_\nu \left[ \frac{d}{dr} f(X_{t_1}(r), \ldots, X_{t_n}(r)) \bigg|_{r=0} \right] = -\mathbb{E}_\nu \left[ F \int_0^1 \left( K^{-1} \int_0^s \beta_u du \right)_t dB_t \right] + \mathbb{E}_\nu[D_h F] = 0.
\]
Hence, for the bounded adapted process $h$, we have the following integration by parts formula:
\[
\mathbb{E}_\nu \left[ F \int_0^1 \left( K^{-1} \int_0^s \beta_u du \right)_t dB_t \right] = \mathbb{E}_\nu[D_h F]. \tag{3.13}
\]
By (3.12), it is easy to know that $(K^{-1} \int_0^t \beta_u du)_{0 \leq t \leq 1} \in \mathcal{H}$ for the adapted process $h \in \mathcal{H}$.

For any $h \in H$, we can find a sequence of bounded processes $h^n$ such that
\[
\lim_{n \to \infty} \mathbb{E}_\nu \left[ \int_0^1 |h^n_t - h_t|^2 dt \right] = 0. \tag{3.14}
\]
Then,
\[
\lim_{n \to \infty} \mathbb{E}_\nu[|D_h F - D_{h^n} F|^2] = 0. \tag{3.15}
\]
Let
\[
dt(h) = \left( K^{-1} \int_0^t \beta_u du \right)_t.
\]
By (3.12) and (3.14), there exist two constants $K_1$ and $K_2$ such that
\[
\lim_{n \to \infty} \mathbb{E}_\nu \left[ F \int_0^1 (\dt(h^n) - \dt(h), dB_t)^2 \right] \leq K_1 \lim_{n \to \infty} \mathbb{E}_\nu \left[ \int_0^1 |\dt(h^n) - \dt(h)|^2 dt \right] 
\]
\[
\leq K_2 \lim_{n \to \infty} \mathbb{E}_\nu \left[ \int_0^1 |h^n_t - h_t|^2 dt \right] = 0. \tag{3.16}
\]
By (3.13),
\[
\mathbb{E}_\nu[D_{h^n} F] = \mathbb{E} \left[ F \int_0^1 (\dt(h^n), dB_t) \right].
\]
holds for bounded processes $h^n$, then by (3.15) and (3.16), the integration by parts formula (3.13) holds for any adapted process $h \in \mathcal{H}$. Moreover, since $D$ is a closable operator, the integration by parts formula (3.13) holds for any $F \in \text{Dom}(D)$.

Now we have established the integration by parts formula for the fractional Ornstein-Uhlenbeck measure $\nu$. In the next section we will study the martingale representation and the Logarithmic-Sobolev inequality for $\nu$ via the integration by parts formula using an idea similar to that employed in [12, 18].

4 Martingale Representation Theorem and Logarithmic-Sobolev Inequality

To prove the Logarithmic-Sobolev inequality, we generalize the classical Clark-Ocone martingale representation theorem for Wiener measure to the fractional Ornstein-Uhlenbeck measure $\nu$. The classical Clark-Ocone martingale representation theorem states that every martingale adapted to the filtration $\mathcal{F}_t$ is a stochastic integral with respect to Brownian motion $B$. Supposing that $F \in L^2(\Omega; \nu)$, there exists a $\mathcal{F}_t$-predictable process $\eta$ such that

$$F = \mathbb{E}_\nu[F] + \int_0^1 \langle \eta_t, dB_t \rangle.$$

The following theorem gives the explicit form of $\eta$ for the fractional Ornstein-Uhlenbeck measure:

**Theorem 4.1** Suppose that $F \in \text{Dom}(D)$. There exists a $L^2(\nu \times dt)$-integrable and $\mathcal{F}_t$-predictable process $(\eta_t)_{0 \leq t \leq 1}$ such that

$$F = \mathbb{E}_\nu[F] + \int_0^1 \langle \eta_t, dB_t \rangle,$$

where

$$\eta_t = \mathbb{E}_\nu\left[ (K^{-1}DF)_t + \int_t^1 K(s,t) \left( \int_s^1 \alpha^2 e^{-\alpha u} \alpha^\alpha P_u du - \alpha P_s \right) ds \bigg| \mathcal{F}_s \right],$$

in which

$$P_t = A_t(K^{-1}DF)_t + \frac{1}{\Gamma(\frac{3}{2} - H)} \int_t^1 (H - \frac{1}{2})u^{H - \frac{1}{2}} \frac{-t^{\frac{3}{2} - H}(K^{-1}DF)_u du}{u^{H + \frac{1}{2} - (H - \frac{1}{2})t}},$$

$$A_t = \frac{1}{\Gamma(\frac{3}{2} - H)} \left( t^{\frac{3}{2} - H} + (H - \frac{1}{2}) \int_0^t \frac{1}{(t - u)^{H + \frac{1}{2}}} du \right).$$

**Proof** By the definition of $D_h F$, we have

$$\mathbb{E}_\nu[D_h F] = \mathbb{E}_\nu[(DF,Kh)_{\gamma_h}] = \mathbb{E}_\nu\left[ \int_0^1 \langle (K^{-1}DF)_t, h_t \rangle dt \right].$$

At the same time, by the integration by parts formula (3.4), it holds that

$$\mathbb{E}_\nu[D_h F] = \mathbb{E}_\nu\left[ \int_0^1 \langle \eta_t, dB_t \rangle \int_0^1 \left\langle (K^{-1} \int_0^t \beta_u du)_t, dB_t \right\rangle \right]$$

$$= \mathbb{E}_\nu\left[ \int_0^1 \langle \eta_t, (K^{-1} \int_0^t \beta_u du)_t \rangle dt \right].$$

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Let 

\[ j_t = \left( K^{-1} \int_0^t \beta_u du \right)_t. \]

Then, 

\[ (Kh)_t + \alpha \int_0^t (Kh)_s ds = (Kj)_t, \]

which implies that 

\[ \int_0^t (Kh)_s ds = e^{-\alpha t} \int_0^t e^{\alpha s} (Kj)_s ds. \]

Thus 

\[ h_t = \left( K^{-1} \left( -\alpha e^{-\alpha} \int_0^t e^{\alpha u} (Kj)_u du + (Kj)_t \right) \right)_t. \] (4.4)

Combining (4.2), (4.3) and (4.4), we have 

\[ E_t \left[ \int_0^t \left( (K^{-1}DF)_t, \left( K^{-1} \left( -\alpha e^{-\alpha} \int_0^t e^{\alpha u} (Kj)_u du + (Kj)_t \right) \right)_t \right) dt \right] = \]

\[ E_t \left[ \int_0^1 \langle \eta_t, j_t \rangle dt \right]. \] (4.5)

By (2.3) and the linearity of \( K^{-1} \), 

\[
\left( K^{-1} \left( -\alpha e^{-\alpha} \int_0^t e^{\alpha u} (Kj)_u du + (Kj)_t \right) \right)_t = \\
\frac{1}{\Gamma\left(\frac{\nu}{2} - H\right)} \left( t^{\frac{\nu}{2} - H} + (H - \frac{1}{2}) \int_0^t \frac{1}{(t-u)^{\frac{\nu}{2} + H}} du \right) \left( \alpha^2 e^{-\alpha t} \int_0^t e^{\alpha u} (Kj)_u du - \alpha (Kj)_t \right) + \\
+ \frac{(H - \frac{1}{2}) t^{H - \frac{1}{2}}}{\Gamma\left(\frac{\nu}{2} - H\right)} \int_0^t \frac{-u^{\frac{\nu}{2} - H} \delta_u}{(t-u)^{\frac{\nu}{2} + H}} du + j_t, \\
\]

where 

\[ \delta_t = \alpha^2 e^{-\alpha t} \int_0^t e^{\alpha u} (Kj)_u du - \alpha (Kj)_t, \]

\[ A_t = \frac{1}{\Gamma\left(\frac{\nu}{2} - H\right)} \left( t^{\frac{\nu}{2} - H} + (H - \frac{1}{2}) \int_0^t \frac{1}{(t-u)^{\frac{\nu}{2} + H}} du \right). \] (4.6)

Hence, the left side of (4.5) can be written as 

\[ E_t \left[ \int_0^1 \left( (K^{-1}DF)_t, \left( K^{-1} \left( -\alpha e^{-\alpha} \int_0^t e^{\alpha u} (Kj)_u du + (Kj)_t \right) \right)_t \right) dt \right] = \\
E_t \left[ \int_0^1 \langle (K^{-1}DF)_t, j_t \rangle dt \right] + E_t \left[ \int_0^1 \langle (K^{-1}DF)_t, A_t \delta_t \rangle dt \right] + \\
E_t \left[ \int_0^1 \langle (K^{-1}DF)_t, \frac{(H - \frac{1}{2}) t^{H - \frac{1}{2}}}{\Gamma\left(\frac{\nu}{2} - H\right)} \int_0^t \frac{-u^{\frac{\nu}{2} - H} \delta_u}{(t-u)^{\frac{\nu}{2} + H}} du \rangle dt \right]. \] (4.7)

By calculation, the third term of the above equation is 

\[ E_t \left[ \int_0^1 \langle (K^{-1}DF)_t, \frac{(H - \frac{1}{2}) t^{H - \frac{1}{2}}}{\Gamma\left(\frac{\nu}{2} - H\right)} \int_0^t \frac{-u^{\frac{\nu}{2} - H} \delta_u}{(t-u)^{\frac{\nu}{2} + H}} du \rangle dt \right] \]
Then (4.7) equals

\[
= \mathbb{E}_{\nu} \left[ \int_0^1 \left( \left( t, \frac{1}{2} \right) \Gamma(-H) \left( \frac{1}{2} \right)^{1-H} \left( -u \right)^{1-H} \right) \left( K^{-1} \right) \left( (1 - H) \gamma \right) \right] du
\]

\[
= \mathbb{E}_{\nu} \left[ \int_0^1 \left( \left( t, \frac{1}{2} \right) \gamma \right) \left( -u \right)^{1-H} \left( K^{-1} \right) \left( (1 - H) \gamma \right) \right]\]

Then (4.7) equals

\[
\mathbb{E}_{\nu} \left[ \int_0^1 \left( \left( t, \frac{1}{2} \right) \gamma \right) \left( -u \right)^{1-H} \left( K^{-1} \right) \left( (1 - H) \gamma \right) \right] du
\]

\[
+ \mathbb{E}_{\nu} \left[ \int_0^1 \left( \left( t, \frac{1}{2} \right) \gamma \right) \left( -u \right)^{1-H} \left( K^{-1} \right) \left( (1 - H) \gamma \right) \right] du
\]

\[
= \mathbb{E}_{\nu} \left[ \int_0^1 \left( \left( t, \frac{1}{2} \right) \gamma \right) \left( -u \right)^{1-H} \left( K^{-1} \right) \left( (1 - H) \gamma \right) \right] du,
\]

where

\[
P_t = A_t \left( K^{-1} DF \right)_t - \int_0^1 \left( H - \frac{1}{2} \right) u^{1-H} (K^{-1} DF)_u \, du.
\]

By (4.6), we infer that

\[
\mathbb{E}_{\nu} \left[ \int_0^1 \left( P_t, \delta_t \right) dt \right] = \mathbb{E}_{\nu} \left[ \int_0^1 \left( P_t, \alpha e^{-\alpha t} \int_0^t e^{\alpha u} (K_j)_u \, du \right. \right.
\]

\[
\left. \left. \left( \alpha \right) \right) \right] dt
\]

\[
= \mathbb{E}_{\nu} \left[ \int_0^1 \left( P_t, \alpha e^{-\alpha t} \int_0^t e^{\alpha u} (K_j)_u \, du \right. \right.
\]

\[
\left. \left. \left( \alpha \right) \right) \right] dt - \mathbb{E}_{\nu} \left[ \int_0^1 \left( P_t, \alpha (K_j)_t \right) dt \right]
\]

\[
= \mathbb{E}_{\nu} \left[ \int_0^1 \left( \int_0^t e^{-\alpha t} e^{\alpha u} \, P_t, (K_j)_u \right) du \right]
\]

\[
- \mathbb{E}_{\nu} \left[ \int_0^1 \left( P_t, \alpha (K_j)_t \right) dt \right]
\]

\[
= \mathbb{E}_{\nu} \left[ \int_0^1 \left( \int_0^t e^{-\alpha t} e^{\alpha u} \, P_t, (K_j)_u \right) du \right]
\]

\[
- \mathbb{E}_{\nu} \left[ \int_0^1 \left( P_t, \alpha (K_j)_t \right) dt \right]
\]

\[
= \mathbb{E}_{\nu} \left[ \int_0^1 \left( \int_0^t e^{-\alpha t} e^{\alpha u} \, P_t, (K_j)_u \right) du \right]
\]

\[
- \mathbb{E}_{\nu} \left[ \int_0^1 \left( P_t, \alpha (K_j)_t \right) dt \right]
\]

\[
= \mathbb{E}_{\nu} \left[ \int_0^1 \left( \int_0^t e^{-\alpha t} e^{\alpha u} \, P_t, (K_j)_u \right) du \right]
\]

\[
- \mathbb{E}_{\nu} \left[ \int_0^1 \left( P_t, \alpha (K_j)_t \right) dt \right]
\]

\[
= \mathbb{E}_{\nu} \left[ \int_0^1 \left( \int_0^t e^{-\alpha t} e^{\alpha u} \, P_t, (K_j)_u \right) du \right]
\]

\[
- \mathbb{E}_{\nu} \left[ \int_0^1 \left( P_t, \alpha (K_j)_t \right) dt \right]
\]

\[
= \mathbb{E}_{\nu} \left[ \int_0^1 \left( \int_0^t e^{-\alpha t} e^{\alpha u} \, P_t, (K_j)_u \right) du \right]
\]

\[
- \mathbb{E}_{\nu} \left[ \int_0^1 \left( P_t, \alpha (K_j)_t \right) dt \right]
\]

\[
= \mathbb{E}_{\nu} \left[ \int_0^1 \left( \int_0^t e^{-\alpha t} e^{\alpha u} \, P_t, (K_j)_u \right) du \right]
\]

\[
- \mathbb{E}_{\nu} \left[ \int_0^1 \left( P_t, \alpha (K_j)_t \right) dt \right]
\]

\[
= \mathbb{E}_{\nu} \left[ \int_0^1 \left( \int_0^t e^{-\alpha t} e^{\alpha u} \, P_t, (K_j)_u \right) du \right]
\]

\[
- \mathbb{E}_{\nu} \left[ \int_0^1 \left( P_t, \alpha (K_j)_t \right) dt \right]
\]

Combining (4.5), (4.8) and (4.10), we obtain

\[
\mathbb{E}_{\nu} \left[ \int_0^1 \left( (K^{-1} DF)_t + \int_0^1 K(s, t) \left( \int_s^1 \alpha^2 e^{-\alpha u} e^{\alpha s} P_u \right) du \right. \right.
\]

\[
\left. \left. \left( \alpha \right) \right) \right] \, ds, j_t \right) dt \right]\]

\[
= \mathbb{E}_{\nu} \left[ \int_0^1 \left( \eta_t, j_t \right) dt \right]
\]

which implies that

\[
\mathbb{E}_{\nu} \left[ \int_0^1 \left( (K^{-1} DF)_t + \int_0^1 K(s, t) \left( \int_s^1 \alpha^2 e^{-\alpha u} e^{\alpha s} P_u \right) du \right. \right.
\]

\[
\left. \left. \left( \alpha \right) \right) \right] \, ds, j_t \right) dt \right] = 0.
\]

Note that the \((F_t)\)-predictable projection of the process

\[
(K^{-1} DF)_t + \int_0^1 K(s, t) \left( \int_s^1 \alpha^2 e^{-\alpha u} e^{\alpha s} P_u \right) du \right. \]

\[
\left. \left( \alpha \right) \right) \right] \, ds - \eta_t
\]
is
\[ E \left[ (K^{-1}DF)_t + \int_t^1 K(s,t) \left( \int_s^1 \alpha^2 e^{-\alpha u} e^{\alpha s} P_u \, du - \alpha P_s \right) \, ds - \eta_t \big| \mathcal{F}_t \right] . \]

In what follows, we show that
\[ E \left[ (K^{-1}DF)_t + \int_t^1 K(s,t) \left( \int_s^1 \alpha^2 e^{-\alpha u} e^{\alpha s} P_u \, du - \alpha P_s \right) \, ds \big| \mathcal{F}_t \right] \in L^2(\nu \times dt). \]

By the Hölder inequality and the inequality \((a + b)^2 \leq 2a^2 + 2b^2\), we get
\[
\begin{align*}
&\leq E_\nu \left[ (K^{-1}DF)_t + \int_t^1 K(s,t) \left( \int_s^1 \alpha^2 e^{-\alpha u} e^{\alpha s} P_u \, du - \alpha P_s \right) \, ds \big| \mathcal{F}_t \right] \\
&\leq 2E_\nu \left[ |(K^{-1}DF)_t|^2 + 2 \left( \int_t^1 \alpha^2 K(s,t) e^{\alpha s} \int_s^1 e^{-\alpha u} P_u \, du \, ds \right)^2 \\
&\quad + 2\alpha^2 \left( \int_t^1 K(s,t) P_s \, ds \right)^2 \big| \mathcal{F}_t \right] \\
&\leq 2E_\nu \left[ |(K^{-1}DF)_t|^2 + 2\alpha^4 e^{2\alpha} \int_t^1 (K(s,t))^2 \, ds \int_t^1 \left( \int_s^1 e^{-\alpha u} P_u \, du \right)^2 \, ds \\
&\quad + 2\alpha^2 \left( \int_t^1 K(s,t) P_s \, ds \right)^2 \big| \mathcal{F}_t \right] . \tag{4.12}
\end{align*}
\]

It is obvious that
\[
\begin{align*}
\int_t^1 \left( \int_s^1 e^{-\alpha u} P_u \, du \right)^2 \, ds &= \int_t^1 \left( - \int_s^1 e^{-\alpha u} \int_u^1 P_v \, dv \right)^2 \, ds \\
&\leq \int_0^1 \left( e^{-\alpha s} \int_s^1 P_v \, dv - \alpha \int_s^1 \int_u^1 P_v \, dv e^{-\alpha u} \right)^2 \, ds \\
&\leq 2 \int_0^1 \left( \int_s^1 P_v \, dv \right)^2 + \alpha^2 \left( \int_s^1 P_v \, dv \right)^2 \, ds. \tag{4.13}
\end{align*}
\]

Hence, by (3.7), (4.12) and (4.13), we gain that
\[
\begin{align*}
&\leq E_\nu \left[ |(K^{-1}DF)_t|^2 + 4\alpha^4 e^{2\alpha} C_\alpha^2 t^{1-2H} (1 + \alpha^2) \int_0^1 \left( \int_s^1 P_v \, dv \right)^2 \, ds \\
&\quad + 2\alpha^2 \left( \int_t^1 K(s,t) P_s \, ds \right)^2 \big| \mathcal{F}_t \right] . \tag{4.14}
\end{align*}
\]

In what follows, we estimate \( \left( \int_t^1 P_s \, ds \right)^2 \) and \( \left( \int_t^1 K(s,t) P_s \, ds \right)^2 \) for (4.14).

Firstly, the expression of \( P \) implies that
\[
\begin{align*}
\int_t^1 P_s \, ds &= \int_t^1 \frac{1}{(\frac{1}{2} - H) s^{\frac{1}{2} - H}} (K^{-1}DF)_s \, ds + \frac{(H - \frac{1}{2})}{\Gamma(\frac{1}{2} - H)} \int_t^1 \frac{1}{(s - u)^{\frac{1}{2} + H}} du (K^{-1}DF)_s \, ds \\
&\quad + \frac{(H - \frac{1}{2})}{\Gamma(\frac{1}{2} - H)} \int_t^1 \int_t^1 u^{H - \frac{1}{2} - \frac{1}{2} - H} (K^{-1}DF)_u \, du \, ds. \tag{4.15}
\end{align*}
\]
It is obvious that
\[
\begin{align*}
&\frac{(H - \frac{1}{2})}{\Gamma(\frac{3}{2} - H)} \int_t^1 \int_s^1 u^{H-\frac{1}{2}} \frac{-s^{\frac{1}{2}-H} - (u-s)^{\frac{1}{2}+H}(K^{-1}DF)_u}{(u-s)^{\frac{1}{2}+H}} du \, ds \\
&= \frac{(H - \frac{1}{2})}{\Gamma(\frac{3}{2} - H)} \int_t^1 \int_s^u u^{H-\frac{1}{2}} \frac{-s^{\frac{1}{2}-H} - (u-s)^{\frac{1}{2}+H}(K^{-1}DF)_u}{(u-s)^{\frac{1}{2}+H}} du \, ds \\
&= \frac{(H - \frac{1}{2})}{\Gamma(\frac{3}{2} - H)} \int_t^1 \int_s^u s^{H-\frac{1}{2}} \frac{-u^{\frac{1}{2}-H}}{(s-u)^{\frac{1}{2}+H}} du (K^{-1}DF)_s \, ds.
\end{align*}
\]
Then (4.15) becomes
\[
\begin{align*}
&\int_t^1 P_s ds = \int_t^1 \frac{1}{\Gamma(\frac{5}{2} - H)} s^{\frac{3}{2}-H} (K^{-1}DF)_s \, ds + \frac{(H - \frac{1}{2})}{\Gamma(\frac{3}{2} - H)} \int_t^1 \int_0^t \frac{1}{(s-u)^{\frac{1}{2}+H}} du \, (K^{-1}DF)_s \, ds \\
&\quad + \frac{(H - \frac{1}{2})}{\Gamma(\frac{3}{2} - H)} \int_t^1 \int_0^s \frac{1}{(s-u)^{\frac{1}{2}+H}} du \, (K^{-1}DF)_s \, ds \\
&\quad + \frac{(H - \frac{1}{2})}{\Gamma(\frac{3}{2} - H)} \int_t^1 \int_s^1 \frac{1}{s^{\frac{1}{2}-H} - (s-u)^{\frac{1}{2}+H}} du \, (K^{-1}DF)_s \, ds \\
&\quad + \frac{(H - \frac{1}{2})}{\Gamma(\frac{3}{2} - H)} \int_t^1 \int_s^u \frac{1}{s^{\frac{1}{2}-H} - (u-s)^{\frac{1}{2}+H}} du \, (K^{-1}DF)_s \, ds \\
&\quad + \frac{(H - \frac{1}{2})}{\Gamma(\frac{3}{2} - H)} \int_t^1 \int_s^u \frac{1}{s^{\frac{1}{2}-H} - (u-s)^{\frac{1}{2}+H}} du \, (K^{-1}DF)_s \, ds.
\end{align*}
\]
(4.16)
It holds that
\[
\begin{align*}
&\left( \int_t^1 P_s ds \right)^2 \leq \frac{1}{\Gamma^2(\frac{5}{2} - H)(2-2H)} \int_t^1 \left| (K^{-1}DF)_s \right|^2 ds \\
&\quad + \frac{(H - \frac{1}{2})^2}{\Gamma^2(\frac{3}{2} - H)} \int_t^1 \left( \int_0^t \frac{1}{(s-u)^{\frac{1}{2}+H}} du \right)^2 ds \int_0^1 \left| (K^{-1}DF)_s \right|^2 ds \\
&\quad + \frac{(H - \frac{1}{2})^2}{\Gamma^2(\frac{3}{2} - H)} \int_0^1 \left( \int_t s^{\frac{1}{2}-H} - (s-u)^{\frac{1}{2}+H} du \right)^2 ds \int_0^1 \left| (K^{-1}DF)_s \right|^2 ds.
\end{align*}
\]
(4.17)
Since
\[
\int_0^t \frac{1}{(s-u)^{\frac{1}{2}+H}} du = \frac{1}{\frac{1}{2} - H} (s^{\frac{1}{2}-H} - (s-t)^{\frac{1}{2}-H}),
\]
we get
\[
\begin{align*}
&\frac{(H - \frac{1}{2})^2}{\Gamma^2(\frac{3}{2} - H)} \int_t^1 \left( \int_0^t \frac{1}{(s-u)^{\frac{1}{2}+H}} du \right)^2 ds \\
&\leq \frac{(H - \frac{1}{2})^2}{\Gamma^2(\frac{3}{2} - H)} \int_t^1 \frac{2}{(\frac{1}{2} - H)^2} (s^{1-2H} + (s-t)^{1-2H}) ds \\
&= \frac{2}{\Gamma^2(\frac{3}{2} - H)(2-2H)}(1 + (1-t)^{2-2H} - t^{2-2H}) \\
&\leq \frac{4}{\Gamma^2(\frac{3}{2} - H)(2-2H)}.
\end{align*}
\]
(4.18)
By (3.9), we obtain

\[
\frac{(H - \frac{1}{2})^2}{\Gamma^2(\frac{3}{2} - H)} \int_0^1 \left( \int_t^s \frac{s^{\frac{1}{2} - H} - u^{\frac{1}{2} - H}}{(s - u)^{\frac{1}{2} + H}} \, du \right) \, ds \\
\leq \frac{(H - \frac{1}{2})^2}{\Gamma^2(\frac{3}{2} - H)} \int_0^1 \left( \int_0^s \frac{u^{\frac{1}{2} - H} - s^{\frac{1}{2} - H}}{(s - u)^{\frac{1}{2} + H}} \, du \right) \, ds \\
= \frac{(H - \frac{1}{2})^2}{\Gamma^2(\frac{3}{2} - H)} \int_0^1 C^2 s^{-2H} \leq \frac{C^2(H - \frac{1}{2})^2}{\Gamma^2(\frac{3}{2} - H)(2 - 2H)}. \tag{4.19}
\]

Combining (4.17), (4.18) and (4.19), it holds that

\[
\left( \int_1^t P_s \, ds \right)^2 \leq C \int_0^1 |(K^{-1}DF)_s|^2 \, ds,
\tag{4.20}
\]

where

\[
C = \frac{1}{\Gamma^2(\frac{3}{2} - H)(2 - 2H)} + \frac{4}{\Gamma^2(\frac{3}{2} - H)(2 - 2H)} + \frac{C^2(H - \frac{1}{2})^2}{\Gamma^2(\frac{3}{2} - H)(2 - 2H)}.
\]

Secondly, by (4.9), we have

\[
\int_t^1 K(s, t) P_s \, ds = \int_t^1 \frac{1}{\Gamma(\frac{3}{2} - H)} s^{\frac{1}{2} - H} K(s, t)(K^{-1}DF)_s \, ds \\
+ \frac{(H - \frac{1}{2})}{\Gamma(\frac{3}{2} - H)} \int_t^1 \left( \int_0^s \frac{1}{(s - u)^{\frac{1}{2} + H}} \, du \right) K(s, t)(K^{-1}DF)_s \, ds \\
+ \frac{(H - \frac{1}{2})}{\Gamma(\frac{3}{2} - H)} \int_t^1 \left( \int_s^1 u^{\frac{1}{2} - H} \frac{-s^{\frac{1}{2} - H}}{(u - s)^{\frac{1}{2} + H}} (K^{-1}DF)_u \right) \, du K(s, t) \, ds. \tag{4.21}
\]

It is obvious that

\[
\frac{(H - \frac{1}{2})}{\Gamma(\frac{3}{2} - H)} \int_t^1 \int_0^s \frac{u^{\frac{1}{2} - H} - s^{\frac{1}{2} - H}}{(u - s)^{\frac{1}{2} + H}} (K^{-1}DF)_u \, du K(s, t) \, ds \\
= \frac{(H - \frac{1}{2})}{\Gamma(\frac{3}{2} - H)} \int_t^1 \int_s^u \frac{u^{\frac{1}{2} - H} - s^{\frac{1}{2} - H}}{(u - s)^{\frac{1}{2} + H}} (K^{-1}DF)_u \, du K(s, t) \, ds \, du \\
= \frac{(H - \frac{1}{2})}{\Gamma(\frac{3}{2} - H)} \int_t^1 \int_t^s \frac{s^{\frac{1}{2} - H} - u^{\frac{1}{2} - H}}{(s - u)^{\frac{1}{2} + H}} K(u, t) \, du \, ds. (K^{-1}DF)_s \, ds.
\]

Hence, by (4.21),

\[
\int_t^1 K(s, t) P_s \, ds = \int_t^1 \frac{1}{\Gamma(\frac{3}{2} - H)} s^{\frac{1}{2} - H} K(s, t)(K^{-1}DF)_s \, ds \\
+ \frac{(H - \frac{1}{2})}{\Gamma(\frac{3}{2} - H)} \int_t^1 \left( \int_0^s \frac{1}{(s - u)^{\frac{1}{2} + H}} \, du \right) K(s, t)(K^{-1}DF)_s \, ds \\
+ \frac{(H - \frac{1}{2})}{\Gamma(\frac{3}{2} - H)} \int_t^1 \left( \int_t^s \frac{s^{\frac{1}{2} - H} K(s, t) - u^{\frac{1}{2} - H} K(u, t)}{(s - u)^{\frac{1}{2} + H}} \, du \right) s^{\frac{1}{2} - H} K(s, t)(K^{-1}DF)_s \, ds \\
= \int_t^1 \frac{1}{\Gamma(\frac{3}{2} - H)} s^{\frac{1}{2} - H} K(s, t)(K^{-1}DF)_s \, ds \\
+ \frac{(H - \frac{1}{2})}{\Gamma(\frac{3}{2} - H)} \int_t^1 \left( \int_0^t \frac{1}{(s - u)^{\frac{1}{2} + H}} \, du \right) K(s, t)(K^{-1}DF)_s \, ds \\
+ \frac{(H - \frac{1}{2})}{\Gamma(\frac{3}{2} - H)} \int_t^1 \left( \int_t^s \frac{s^{\frac{1}{2} - H} (K(s, t) - K(u, t))}{(s - u)^{\frac{1}{2} + H}} \, du \right) s^{\frac{1}{2} - H} K(s, t)(K^{-1}DF)_s \, ds.
\]
In a fashion similar to (4.18), it holds that
\[ B \]
Then, by Hölder’s inequality, we get
\[
\left( \int_t^1 K(s, t) P_s ds \right)^2 \leq \frac{C^2 t^{1-2H}}{\Gamma^2(\frac{3}{2} - H)(2 - 2H)} \int_0^1 |(K^{-1}DF)_s|^2 ds
\]
\[
+ \frac{(H - \frac{1}{2})^2}{\Gamma^2(\frac{3}{2} - H)} \int_t^1 \left( \int_0^1 \frac{1}{(s - u)^{\frac{3}{2} + H}} duK(s, t) \right)^2 ds \int_0^1 |(K^{-1}DF)_s|^2 ds
\]
\[
+ \frac{(H - \frac{1}{2})^2}{\Gamma^2(\frac{3}{2} - H)} \int_t^1 \left( \int_0^1 \frac{1}{(s - u)^{\frac{3}{2} + H}} duK(s, t) \right)^2 ds \int_0^1 |(K^{-1}DF)_s|^2 ds
\]
\[
+ \frac{(H - \frac{1}{2})^2}{\Gamma^2(\frac{3}{2} - H)} \int_t^1 \left( \int_0^1 \frac{1}{(s - u)^{\frac{3}{2} + H}} duK(s, t) \right)^2 ds \int_0^1 |(K^{-1}DF)_s|^2 ds. \quad (4.23)
\]
In a fashion similar to (4.18), it holds that
\[
\int_t^1 \left( \int_0^1 \frac{1}{(s - u)^{\frac{3}{2} + H}} duK(s, t) \right)^2 ds \leq \frac{2C^2 t^{1-2H}}{(\frac{3}{2} - H)^2(2 - 2H)(1 + (1 - t)^{2-2H} - t^{2-2H})} \leq \frac{4C^2 t^{1-2H}}{(\frac{3}{2} - H)^2(2 - 2H)}. \quad (4.24)
\]
By the expression of $K(s, t)$ in (2.1), we have
\[
\left( \int_t^s \frac{s^{\frac{1}{2} - H}}{(s - u)^{\frac{3}{2} + H}} duK(s, t) \right)^2 = \left( cHt^{\frac{1}{2} - H} \int_t^s \frac{v^{H - \frac{1}{2}}(v - t)^{H - \frac{1}{2}}}{(s - u)^{\frac{3}{2} + H}} dvdu \right)^2
\]
\[
= \left( cHt^{\frac{1}{2} - H} \int_t^s v^{H - \frac{1}{2}}(v - t)^{H - \frac{1}{2}} \int_t^v \frac{1}{(s - u)^{\frac{3}{2} + H}} dudu \right)^2
\]
\[
= \left( \frac{cHt^{\frac{1}{2} - H}}{(H - \frac{1}{2})^{1-2H}} \int_t^s v^{H - \frac{1}{2}}(v - t)^{H - \frac{1}{2}} \left( (s - v)^{\frac{3}{2} - H} - (s - t)^{\frac{3}{2} - H} \right) dv \right)^2
\]
\[
\leq \frac{2c^2 H^2}{(\frac{3}{2} - H)^2} \left( \left( \int_t^s v^{H - \frac{1}{2}}(v - t)^{-H - \frac{1}{2}}(s - v)^{\frac{3}{2} - H} dv \right)^2
\]
\[
+ \left( \int_t^s v^{H - \frac{1}{2}}(v - t)^{-H - \frac{1}{2}}(s - t)^{\frac{3}{2} - H} dv \right)^2 \right)
\]
\[
\leq \frac{2c^2 H^2}{(\frac{3}{2} - H)^2} \left( B(H - \frac{1}{2}, \frac{3}{2} - H) \right)^2 + \frac{1}{(H - \frac{3}{2})^2} t^{1-2H}, \quad (4.25)
\]
where $B(\cdot, \cdot)$ is a beta function. By (3.7) and (3.9),
\[
\int_t^1 \left( \int_t^s \frac{s^{\frac{1}{2} - H} - u^{\frac{1}{2} - H}}{(s - u)^{\frac{3}{2} + H}} K(u, t) du \right)^2 ds
\]
\[ \leq \int_t^1 C_1^2 C_2^2 s^{2-4H} t^{1-2H} s^{2H-1} ds \leq \frac{C_1^2 C_2^2}{2-2H}s^{1-2H}. \] (4.26)

Then, combining (4.23), (4.24), (4.25) and (4.26) yields that
\[ \left( \int_t^1 K(s,t)P_s ds \right)^2 \leq \frac{C_1^2 t^{1-2H}}{\Gamma^2(\frac{3}{2} - H)(2 - 2H)} \int_0^1 |(K^{-1}DF)_s|^2 ds \]
\[ + \frac{4C_1^2}{\Gamma^2(\frac{3}{2} - H)(2 - 2H)} \int_0^1 |(K^{-1}DF)_s|^2 ds \]
\[ + \frac{2C_2^2 (B(H - \frac{1}{2}, \frac{3}{2} - H))^2 + \frac{1}{(H-\frac{1}{2})^2}}{\Gamma^2(\frac{3}{2} - H)} \int_0^1 |(K^{-1}DF)_s|^2 ds \]
\[ + \frac{C_1^2 C_2^2 (H - \frac{1}{2})^2 t^{1-2H}}{\Gamma^2(\frac{3}{2} - H)(2 - 2H)} \int_0^1 |(K^{-1}DF)_s|^2 ds. \]

It follows that
\[ \left( \int_t^1 K(s,t)P_s ds \right)^2 dt \leq \hat{C} t^{1-2H} \int_0^1 |(K^{-1}DF)_s|^2 ds, \] (4.27)

where
\[ \hat{C} = \frac{C_1^2}{\Gamma^2(\frac{3}{2} - H)(2 - 2H)} + \frac{4C_1^2}{\Gamma^2(\frac{3}{2} - H)(2 - 2H)} \]
\[ + \frac{2C_2^2 (B(H - \frac{1}{2}, \frac{3}{2} - H))^2 + \frac{1}{(H-\frac{1}{2})^2}}{\Gamma^2(\frac{3}{2} - H)} + \frac{C_1^2 C_2^2 (H - \frac{1}{2})^2}{\Gamma^2(\frac{3}{2} - H)(2 - 2H)}. \]

Therefore, by (4.14), (4.20) and (4.27),
\[ \mathbb{E}_\nu \left[ \int_0^1 \mathbb{E}_\nu \left[ (K^{-1}DF)_t + \int_t^1 K(s,t) \left( \int_s^1 \alpha^2 e^{-\alpha u} e^{\alpha s} P_s du - \alpha P_s \right) ds \bigg| \mathcal{F}_t \right] \right] ^2 dt \]
\[ \leq 2 \mathbb{E}_\nu \left[ \int_0^1 |(K^{-1}DF)_t|^2 dt + 4\alpha^4 e^{2\alpha} C_1^2 (1 + \alpha^2) \int_0^1 t^{-2H} dt C \int_0^1 |(K^{-1}DF)_s|^2 ds \right. \]
\[ + 2\alpha^2 \hat{C} \int_0^1 t^{1-2H} dt \int_0^1 |(K^{-1}DF)_s|^2 ds \]
\[ = 2 \left( 1 + \frac{4\alpha^4 e^{2\alpha} C_1^2 (1 + \alpha^2)}{2 - 2H} + \frac{2\alpha^2 \hat{C}}{2 - 2H} \right) \mathbb{E}_\nu \left[ \int_0^1 |(K^{-1}DF)_s|^2 ds \right] \]. (4.28)

Thus, we obtain
\[ \mathbb{E}_\nu \left[ (K^{-1}DF)_t + \int_t^1 K(s,t) \left( \int_s^1 \alpha^2 e^{-\alpha u} e^{\alpha s} P_s du - \alpha P_s \right) ds - \eta_t \bigg| \mathcal{F}_t \right] \in L^2(\nu \times dt). \] (4.29)

Note that any \( L^2(\nu \times dt) \)-integrable and \((F_t)\)-predictable process can be approximated w.r.t. the \( L^2(\nu \times dt) \) norm by processes \( j \in \mathcal{H} \). Hence, by (4.29) and (4.11), we have
\[ \mathbb{E}_\nu \left[ (K^{-1}DF)_t + \int_t^1 K(s,t) \left( \int_s^1 \alpha^2 e^{-\alpha u} e^{\alpha s} P_s du - \alpha P_s \right) ds - \eta_t \bigg| \mathcal{F}_t \right] = 0, \]
which implies that
\[ \eta_t = \mathbb{E}_\nu \left[ (K^{-1}DF)_t + \int_t^1 K(s,t) \left( \int_s^1 \alpha^2 e^{-\alpha u} e^{\alpha s} P_s du - \alpha P_s \right) ds \bigg| \mathcal{F}_t \right] = 0. \]
Hence, by (4.30), we obtain the Logarithmic-Sobolev inequality

$$G_{\nu} \left( 1 + \frac{4\alpha^2 \epsilon_{1} C_1^2 (1 + \alpha^2) C}{2 - 2H} + \frac{2\alpha^2 \tilde{C}}{2 - 2H} \right) \langle DF, DF \rangle_{\mathcal{H}, \nu} + \mathbb{E}_{\nu} [F^2 \ln \mathbb{E}_{\nu} [F^2]],$$

where

$$C = \frac{1}{\Gamma^2(\frac{3}{2} - H)(2 - 2H)} + \frac{4}{\Gamma^2(\frac{3}{2} - H)(2 - 2H)} + \frac{2C_2^2 (H - \frac{1}{2})^2}{\Gamma^2(\frac{3}{2} - H)(2 - 2H)},$$

$$\tilde{C} = \frac{4C_2^2}{\Gamma^2(\frac{3}{2} - H)(2 - 2H)} + \frac{2C_2^2 (\frac{1}{2} - 3 + \frac{1}{2})^2}{\Gamma^2(\frac{3}{2} - H)(2 - 2H)} + \frac{C_1^2 C_2^2 (H - \frac{1}{2})^2}{\Gamma^2(\frac{3}{2} - H)(2 - 2H)},$$

with $C_1$ and $C_2$ satisfying (3.7) and (3.9), respectively.

**Proof** Let $G = F^2$ and let $G_t$ be a right continuous version of $\mathbb{E}_{\nu} [G|\mathcal{F}_t]$. Then, by Theorem 4.1,

$$dG_t = (\eta_t, dB_t),$$

where $\eta$ satisfies (4.1), in which $F$ is replaced by $G$. By Itô’s formula, we obtain

$$d(G_t \ln(G_t)) = (1 + \ln(G_t))dG_t + \frac{1}{2} |\eta_t|^2 dt = \langle 1 + \ln(G_t) \rangle \eta_t dB_t + \frac{1}{2} |\eta_t|^2 dt.$$}

Due to $G_1 = \mathbb{E}_{\nu} [G|\mathcal{F}_1] = G$ and $G_0 = \mathbb{E}_{\nu} [G|\mathcal{F}_0] = \mathbb{E}_{\nu} [G]$, we get

$$\mathbb{E}_{\nu} [G \ln G] - \mathbb{E}_{\nu} [G] \ln \mathbb{E}_{\nu} [G] = \frac{1}{2} \mathbb{E}_{\nu} \left[ \int_0^1 |\eta_t|^2 dt \right]. \quad (4.30)$$

Since $DF^2 = 2FDF$, it holds that

$$\eta_t = \mathbb{E}_{\nu} \left[ 2F \left((K^{-1}DF)_t + \int_t^1 (K(s, t) \int_s^1 \alpha^2 e^{-\alpha u} e^{\alpha s} dP_u - \alpha P_s) ds \right) \bigg| \mathcal{F}_t \right].$$

By Hölder’s inequality and the inequality $(a + b)^2 \leq 2a^2 + 2b^2$, we get that

$$|\eta_t|^2 \leq 4 \mathbb{E}_{\nu} \left[ F^2 |\mathcal{F}_t \right] \mathbb{E}_{\nu} \left[ (K^{-1}DF)_t + \int_t^1 (K(s, t) \left( \int_s^1 \alpha^2 e^{-\alpha u} e^{\alpha s} dP_u - \alpha P_s \right) ds \right] \bigg| \mathcal{F}_t \right].$$

Then, by (4.28), we have

$$\mathbb{E}_{\nu} \left[ \int_0^1 |\eta_t|^2 dt \right] \leq 8 \left( 1 + \frac{4\alpha^4 \epsilon_{1} C_1^2 (1 + \alpha^2) C}{2 - 2H} + \frac{2\alpha^2 \tilde{C}}{2 - 2H} \right) \mathbb{E}_{\nu} \left[ \int_0^1 |(K^{-1}DF)_s|^2 ds \right].$$

Hence, by (4.30), we obtain the Logarithmic-Sobolev inequality

$$\mathbb{E}_{\nu} [F^2 \ln F^2] \leq 4 \left( 1 + \frac{4\alpha^4 \epsilon_{1} C_1^2 (1 + \alpha^2) C}{2 - 2H} + \frac{2\alpha^2 \tilde{C}}{2 - 2H} \right) \mathbb{E}_{\nu} \left[ \int_0^1 |(K^{-1}DF)_s|^2 ds \right] + \mathbb{E}_{\nu} [F^2] \ln \mathbb{E}_{\nu} [F^2].$$

This completes the proof. \qed
Remark 4.3 We should point out that the Logarithmic-Sobolev inequality for $\nu$ in Theorem 4.2 can be deduced directly from a Gross’s logarithmic Sobolev inequality in [18, Theorem 8.6.1], since such inequalities are dimension free. The proof given in this paper can be seen as an application for the Martingale Representation Theorem in Theorem 4.1 and this idea is useful when we consider the Logarithmic-Sobolev inequality for the fractional Brownian motion over a Riemannian manifold (c.f. [18, Theorem 8.7.1] and [12, Theorem 5.2]).

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