RELATIVISTIC PLANCK-SCALE POLYMER

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Polymer quantum mechanics has been studied as a simplified picture that reflects some of the key properties of Loop Quantum Gravity; however, while the fate of relativistic symmetries in Loop Quantum Gravity is still not established, it is usually assumed that the discrete polymer structure should lead to a breakdown of relativistic symmetries. We here focus for simplicity on a one-spatial-dimension polymer model and show that relativistic symmetries are deformed, rather than being broken. The specific type of deformed relativistic symmetries which we uncover appears to be closely related to analogous descriptions of relativistic symmetries in some noncommutative spacetimes. This also contributes to an ongoing effort attempting to establish whether the “quantum-Minkowski limit” of Loop Quantum Gravity is a noncommutative spacetime.

I. INTRODUCTION

For more than a decade now the study of the fate of relativistic symmetries in the quantum-gravity realm has been very intense. The present understanding is based on three possible scenarios: relativistic symmetries might preserve their ordinary structure, unaffected by Planck-scale effects [1], or they might be broken by Planck-scale effects, with the emergence of a preferred frame [2] or they could be “Planck-scale deformed”, with a novel role for the Planck scale in the transformation rules among relativistic observers, but still no preferred frame [3, 4]. Frustratingly the situation remains unclear in Loop Quantum Gravity [10–13], one of the most ambitious and rich approaches to the quantum-gravity problem: because of our current limitations in the analysis of the Hamiltonian constraint any one of the three mentioned outcomes for the fate of relativistic symmetries in Loop Quantum Gravity remains possible, with different authors formulating different intuition for what that outcome might turn out to be.

In light of the complexity of full-fledged quantum-gravity theories, it is being appreciated that simplified models, capturing some aspects of the more ambitious models, can serve the purpose of both shedding light on the relevant conceptual issues and possibly providing guidance to Planck-scale phenomenology [14]. Polymer Quantization (PQ) [15] is relevant from this perspective: it is believed [15–20] to provide a simplified perspective: it is believed [15–20] to provide a simplified description of Polymer Quantization in terms of Planck-scale-deformed relativistic symmetries, and interestingly these deformed relativistic symmetries are rather similar to the ones encountered in the analysis of some noncommutative spacetimes. From that perspective our analysis might also contributes to an ongoing effort [23–28] attempting to establish whether the “quantum-Minkowski limit” of Loop Quantum Gravity is a noncommutative spacetime.

II. PRELIMINARY ON POLYMER QUANTIZATION

In PQ the Hilbert space realizes a representation of the basic observables of the theory which is unitarily inequivalent to the familiar Schrödinger’s one [29]. It is not just that the state vectors, the inner product and the operators take different form, but rather they also lead to different physical predictions in their domain of applicability.

In order to introduce the polymer representation it is convenient to start from some aspects of the Schrödinger representation. We introduce the exponentiated versions of position q and momentum π operators, the one-parameter family of unitary operators [15, 29]:

\[ U_\mu = e^{i\mu q}, \quad V_\lambda = e^{i\lambda \pi}. \]  (1)

In standard quantum mechanics, the operators \( U_\mu \) and \( V_\lambda \) are well-defined on the Hilbert space of square integrable functions \( \mathcal{H}_{Sch} = L^2(\mathbb{R}, dq) \) and their action on a state \( \psi \in \mathcal{H}_{Sch} \) is given by:

\[ U_\mu \psi(q) = e^{i\mu q} \psi(q), \quad V_\lambda \psi(q) = \psi(q + \lambda). \]  (2)

It is easy to see, using the canonical commutation relation, \([\pi, q] = i\), that \( U_\mu \) and \( V_\lambda \) satisfy the following composition rules [15]:

\[ U_{\mu_1} U_{\mu_2} = e^{i\mu_1 q} e^{i\mu_2 q} = e^{i(\mu_1 + \mu_2)q} = U_{\mu_1 + \mu_2}, \]

\[ V_{\lambda_1} V_{\lambda_2} = e^{i\lambda_1 \pi} e^{i\lambda_2 \pi} = e^{i(\lambda_1 + \lambda_2)\pi} = V_{\lambda_1 + \lambda_2}. \]
\[ U_\mu V_\lambda = e^{i\mu q} e^{i\lambda \pi} = e^{-i\mu \lambda} e^{i\lambda \pi} e^{i\mu q} = e^{-i\mu \lambda} V_\lambda U_\mu, \tag{3} \]
which comprise the Weyl algebra \[23\).

According to the Stone-von Neumann theorem \[30\] all the representations of the Weyl algebra that fulfill certain conditions of regularity and irreducibility are unitarily equivalent to the Schrödinger’s one. In the polymer representation one of the assumptions of the Stone-von Neumann theorem, namely that the operator \( V_\lambda \) is weakly continuous in the \( \lambda \) parameter, is no longer valid, thus leading to a unitarily inequivalent representation.

The polymer Hilbert space is characterized by a non-countable orthonormal basis \(|q\rangle\), labelled by real numbers. The inner product of the Hilbert space is defined as
\[
\langle q'|q \rangle = \delta_{q,q'}. \tag{4} \]
The failure of continuity of \( V_\lambda \) in the polymer context is evident if we notice that
\[
\lim_{\lambda \to 0} \langle q|V_\lambda|q \rangle = \lim_{\lambda \to 0} \langle q|q + \lambda \rangle = \lim_{\lambda \to 0} \delta_{q,q+\lambda} = 0. \tag{5} \]
It does not matter how small \( \lambda \) is, \(|q\rangle\) will be always orthogonal to \( V_\lambda |q\rangle\), while \( \langle q|V_{\lambda=0}|q \rangle = 1 \). Due to such lack of continuity, the momentum operator \( \pi \), as seen as the derivative of \( V_\lambda \) evaluated in \( \lambda = 0 \) is not well defined in the polymer Hilbert space, and thus \( V_\lambda \) acquires the status of a fundamental observable. In order to define an operator that plays the role of the momentum in the polymer description, one can get help from the usual description of the operator \( V_\lambda \), where \( V_\lambda = e^{i\lambda \pi} \). Considering the semiclassical limit \( \lambda \pi \ll 1 \) \cite{29, 61}, one can define a momentum operator \( P = \frac{1}{\lambda} \sin (\lambda \pi) \approx \pi + \mathcal{O}(\pi^3) \). Then, in terms of the fundamental observable \( V_\lambda \), one has
\[
P = \frac{4}{\pi} \frac{V_\lambda - V_{\lambda=0}}{2i}. \tag{6} \]
The polymer Hilbert space is comprised by wavefunctions which take values different from zero in points of the real line which are regularly spaced: \( q = q_0 + n\lambda \) for a given point \( q_0 \in \mathbb{R} \) and \( n \in \mathbb{Z} \). Such functions with support on the lattice \( \mathcal{L} = \{ q = q_0 + n\lambda, q \in \mathbb{R} \} \), which we denote with \( \psi_{q_0} \), belong to a separable Hilbert space which is a superselected sector of the full polymer Hilbert space. States belonging to such a sector cannot be mapped to states in other sectors by any physical operator. Hence a state belonging to the full Hilbert space can be written as a linear superposition of all the functions indexed by \( q_0 \) belonging to the continuous interval \( q_0 \in [0,\lambda) \) \cite{32, 33}. This leads to the following characterization of the polymer Hilbert space as a direct sum of superselected sectors \( \mathcal{H}_{q_0} \)
\[
\mathcal{H}_{poly} = \bigoplus_{q_0 \in [0,\lambda)} \mathcal{H}_{q_0}, \tag{6} \]
and thus the most general element of \( \mathcal{H}_{poly} \) is not contained in just one superselected Hilbert space \( \mathcal{H}_{q_0} \). However, to gain some physical intuition it is customary to restrict the attention to a specific superselected Hilbert space \( \mathcal{H}_{q_0} \) and therefore to just work on a fixed regular lattice. We shall also follow this approach, taking for simplicity the point used to fix the lattice as \( q_0 = 0 \).

The presence of fundamental discreteness in the polymer framework has led most authors to assume that relativistic symmetries are broken \cite{21, 22}, with emergence of a preferred frame; however, the study we here report provides evidence in support of the possibility that the polymer framework, contrary to what is commonly expected, can provide the basis for a relativistic picture.

## III. PRELIMINARIES ON THE MANIFESTLY-COVARIANT FORMULATION OF QUANTUM MECHANICS

For our purposes it is natural to work within the manifestly-covariant formulation of Quantum Mechanics, in which one starts from a “kinematical Hilbert space” where both the time and the spatial coordinates are self-adjoint operators \[33, 50\]. The Heisenberg algebra of observables is then obtained via the imposition of a suitable constraint \[32\], and states that satisfy that constraint are said to be in the physical Hilbert space.

Readers will of course find elsewhere more detailed introductions to the manifestly-covariant formulation of Quantum Mechanics. We shall be here satisfied with illustrating the logic of this setup by considering a free special-relativistic particle in a \((1+1)\)-dimensional spacetime. In that case on the kinematical Hilbert space one has spacetime coordinates \((q, \pi)\) and momenta \((\pi_t, \pi)\) satisfying the canonical commutation relations
\[
[\pi_t, \pi] = 0, \quad [q_t, q] = 0, \quad [\pi_t, q_t] = -i, \quad [\pi_t, q] = 0, \quad [\pi, q] = i, \tag{7} \]
that are represented on the Hilbert space of square-integrable functions \( L^2(\mathbb{R}^2, dq_t dq) \sim L^2(\mathbb{R}^2, d\pi_t d\pi) \) \cite{52}.

States on the physical Hilbert space are those satisfying the Hamiltonian constraint, which of course for a free special-relativistic is an on-shell constraint:
\[
H \psi = [\pi_t^2 - \pi^2 - m^2] \psi = 0. \tag{8} \]

Basically the kinematical Hilbert space describes abstract points of spacetime (no particles, no physics), whereas the physical Hilbert space describes on-shell particles (i.e. worldlines, rather than points).

## IV. POLYMER SYMMETRIES

Relevant to the Loop-Quantum-Gravity perspective on PQ is the fact that the usual Schrödinger representation can be seen as an approximate description of PQ via a continuous limit \cite{29}. Here we are interested in the special-relativistic version of PQ, for which, as mentioned, we shall consider a superselected \cite{12} sector of the full polymer Hilbert space. We shall focus on a free relativistic particle in \((1+1)\) dimensions and polymerize...
only the spatial coordinate $\xi$ while the temporal coordinate remains continuous. This is the polymer picture usually adopted in particular in loop quantum cosmology $^{38}$. The kinematical Hilbert space will be described by the tensor product $\mathcal{H}_{\text{Sch}} \otimes \mathcal{H}_q$.

In order to relate the polymer picture and the Schrödinger picture, it is useful to explicitly write the polymer operators of the relativistic system in terms of the operators that characterize the covariant formulation of quantum mechanics $^{54, 55}$ (time coordinate $q_t$, spatial coordinate $q$, time momentum $\pi_t$, and spatial momentum $\pi$), to which we shall refer as the pregeometric representation of the polymer operators (in the same spirit of the pregeometric representations in use in studies of spacetime noncommutativity $^{37, 40, 41}$). We start by introducing, consistently with what is frequently done in the polymer literature $^{31}$, the translation generator $P$ with a time-translation generator $P_t$ and a boost generator $B$ defined as follows:

$$P = \frac{\sin(\lambda \pi)}{\lambda},$$  \hspace{1cm} (9)

where $\lambda$ is a fixed parameter with dimensions of length. In our derivation of a (deformed-) relativistic description of the polymer picture, also taking as guidance analogous studies of other quantum spacetimes $^{54, 55}$, we combine the translation generator $P$ with a time-translation generator $P_t$ and a boost generator $B$ defined as follows:

$$P_t = \pi_t e^{-i\lambda \pi/2}, \quad B = e^{-i\lambda \pi} \eta,$$  \hspace{1cm} (10)

where

$$\eta = \left( \frac{e^{2i\lambda \pi} - 1}{2i\lambda} + i\frac{\lambda}{2\pi^2} \right) q_t - \pi t q.$$  \hspace{1cm} (11)

We shall show that this ansatz for the description of relativistic symmetries has several properties suggesting that $P$, $P_t$, and $B$ are indeed generators of the (deformed) relativistic symmetries of the 1-1-dimensional polymer.

We start by verifying that these generators take a state in $\mathcal{H}_{\text{Sch}} \otimes \mathcal{H}_q$ and map it into another state still in $\mathcal{H}_{\text{Sch}} \otimes \mathcal{H}_q$, i.e.

$$\psi \in \mathcal{H}_{\text{Sch}} \otimes \mathcal{H}_q \rightarrow (S \triangleright \psi) \in \mathcal{H}_{\text{Sch}} \otimes \mathcal{H}_q,$$  \hspace{1cm} (12)

with $S$ any combination of $P$, $P_t$ and $B$. For this purpose we describe a general wave function in the polymer picture in the following way:

$$\int dq_t f(q_t, j) e^{iq_t k_t} e^{-iq_j k},$$  \hspace{1cm} (13)

and therefore by linearity one can focus on the action of the operators on the product of exponentials

$$f(q_t, q_j) \equiv e^{iq_t k_t} e^{-iq_j k}.$$  

For the operator $P$ one finds

$$P \triangleright f(q_t, q_j) = P \triangleright (e^{iq_t k_t} e^{-iq_j k}) = \frac{f(q_t, q_j + \lambda) - f(q_t, q_j - \lambda)}{2i\lambda},$$  \hspace{1cm} (14)

while for $B$

$$B \triangleright f(q_t, q_j) = B \triangleright (e^{iq_t k_t} e^{-iq_j k}) = q_t \left( \frac{f(q_t, q_j) - f(q_t, q_j - 2\lambda)}{2i\lambda} - i\frac{\lambda}{2} (\partial_0) f(q_t, q_j) \right) + iq \partial_0 f(q_t, q_j).$$  \hspace{1cm} (15)

Therefore the operators $P$ and $B$ satisfy the criteria $^{12}$, the action of $P$ and $B$ on a function on $\mathcal{H}_{\text{Sch}} \otimes \mathcal{H}_q$ gives a function which is still on $\mathcal{H}_{\text{Sch}} \otimes \mathcal{H}_q$. The same evidently holds also for $P_t$.

Let us then notice that the algebra described by $(P_t, P, B)$, to which we shall refer as the “polymer algebra”, is characterized by:

$$[P_t, P] = 0,$$

$$[B, P_t] = iP(\sqrt{1 - \lambda^2 q_t^2} - i\lambda P)^{1/2},$$

$$[B, P] = iP \sqrt{1 - \lambda^2 q^2} (\sqrt{1 - \lambda^2 q^2} - i\lambda P)^{1/2},$$  \hspace{1cm} (16)

so we have a deformed relativistic algebra (a “DSR-relativistic algebra” $^{5, 6}$) which recovers the classical Poincaré algebra in the limit $\lambda \rightarrow 0$.

Our next task is to show that our candidate algebra of relativistic symmetries also correctly “predicts” the Hamiltonian that for independent reasons has been adopted in the literature for the description of a polymer particle in the Galilean regime. For this purpose we start by noticing that the Casimir of the polymer algebra is given by

$$C = 2 \frac{\lambda^2}{\lambda^2 - 1} \left( 1 - \sqrt{1 - \lambda^2} P_t^2 \right) - P_t^2,$$  \hspace{1cm} (17)

which in terms of $\pi$ takes the form

$$C = 4 \frac{\lambda^2}{\lambda^2 - 1} \sin^2 \left( \frac{\lambda \pi}{2} \right) - P_t^2.$$  \hspace{1cm} (18)

In preparation for considering the Galilean regime we make the replacements

$$C \rightarrow -m^2 c^2, \quad P_t \rightarrow \frac{E}{c},$$  \hspace{1cm} (19)

$m$ being the mass and $c$ the speed of light, so that $^{18}$ takes the form

$$E = mc^2 \sqrt{1 + \frac{4}{\lambda^2 m^2 c^2} \sin^2 \left( \frac{\lambda \pi}{2} \right)}.$$  \hspace{1cm} (20)

Thus, in the Galilean regime ($c \rightarrow \infty$) one has

$$E = \frac{1 - \cos(\lambda \pi)}{\lambda^2 m},$$  \hspace{1cm} (21)

where of course we dropped the constant contribution to energy, $mc^2$, which is unnoticeable in the Galilean regime (since there is no particle production). Eq. (21) is indeed exactly the Hamiltonian that is often used in the literature $^{22, 31}$ in the description of a free Galilean-regime polymer particle.
V. A POSSIBLE CONNECTION WITH THE $\kappa$-POINCARÉ ALGEBRA

We have provided evidence in support of the thesis that PQ, rather than breaking relativistic symmetries as usually assumed [21, 22], may be characterized by a deformation of relativistic symmetries (without a preferred frame). Since the polymer picture is being considered as a simplified version of Loop Quantum Gravity, our results strengthen the case (already suggested by several authors [23–28]) for the emergence of DSR-relativistic symmetries in the “Minkowski regime” of Loop Quantum Gravity. Our next objective is to consider a possible connection between our results and the most studied possibility for the description of these Minkowski-regime relativistic symmetries, which involves the $\kappa$-Poincaré Hopf algebra [23–28].

It had already been observed more than 20 years ago [42] that phonons in certain condensed matter systems, with atomic-structure discreteness, are governed by equations of motion that are $\kappa$-Poincaré invariant. On the technical side some evidence for a role of discreteness in the structure of $\kappa$-Poincaré has been uncovered so far only in studies of an associated differential calculus [40, 41, 42]. We here contribute to these investigations by observing that our polymer algebra is related to the $\kappa$-Poincaré algebra, a simplified version of Loop Quantum Gravity, our results may contribute both to the debate of the fate of relativistic symmetries in Loop Quantum Gravity and to the investigations of the possibility that the “quasi-Minkowski regime” of Loop Quantum Gravity might be described in terms of spacetime noncommutativity.

Let us consider, within our polymer algebra, the following relationships between the operators $P_t, P, B$, generators of our polymer algebra, and some other operators $\mathcal{P}_t, \mathcal{P}, B$:

$$P_t = \mathcal{P}_t e^{\lambda \mathcal{P}/2},$$

$$P = \frac{\sin(\lambda \mathcal{P})}{\lambda}, \quad B = B.$$

It is straightforward to verify that upon these replacements the commutators of our polymer algebra take the form

$$[\mathcal{P}_t, P] = 0, \quad [B, P] = i\mathcal{P}_t,$$

$$[B, P_t] = \frac{1 - e^{-2i \lambda \mathcal{P}}}{2\lambda} + \frac{\lambda}{2} P_t^2.$$

VI. OUTLOOK

We have provided results in support of a rather unexpected scenario for the quantum-gravity realm, in which spacetime discretization and relativistic covariance (however deformed) coexist. As already stressed above we feel that our results are particularly intriguing since PQ is viewed as a toy model for Loop Quantum Gravity and our results may contribute both to the debate of the fate of relativistic symmetries in Loop Quantum Gravity and to the investigations of the possibility that the “quasi-Minkowski regime” of Loop Quantum Gravity might be described in terms of spacetime noncommutativity.

Some further checks are needed in order to reach a fully established picture: we verified a particular significant subset of expected properties of a relativistic theory, but surely more tests would be needed in order to be confident of the overall consistency of the scenario. Among these we feel particular interest is deserved by the introduction of interactions among particles, to be handled consistently within a quantum-field-theory formulation of the polymer.

While we might be near to establishing this picture for the 2D polymer toy model (with only the spatial coordinate polymerized), of course ultimately we would want to see all this at work in more realistic models. We expect that already the generalization to a 4D polymer structure should be rather nontrivial.

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