Learning to Emulate an Expert Projective Cone Scheduler

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Abstract—Projective cone scheduling defines a large class of rate-stabilizing policies for queueing models relevant to several applications. While there exists considerable theory on the properties of projective cone schedulers, there is little practical guidance on choosing the parameters that define them. In this paper, we propose an algorithm for designing an automated projective cone scheduling system based on observations of an expert projective cone scheduler. We show that the estimated scheduling policy is able to emulate the expert in the sense that the average loss realized by the learned policy will converge to zero. Specifically, for a system with $n$ queues observed over a time horizon $T$, the average loss for the algorithm is $O\left(\ln(T)\sqrt{\ln(n)/T}\right)$. This upper bound holds regardless of the statistical characteristics of the system. The algorithm uses the multiplicative weights update method and can be applied online so that additional observations of the expert scheduler can be used to improve an existing estimate of the policy. This provides a data-driven method for designing a scheduling policy based on observations of a human expert. We demonstrate the efficacy of the algorithm with a simple numerical example and discuss several extensions.

I. INTRODUCTION

In a variety of application areas, processing systems are dynamically scheduled to maintain stability and to meet various other objectives. Indeed, the basic problem in scheduling theory has been to find and study policies that accomplish this task under different modeling assumptions. In practice however, while human experts may be able to manage real-world processing systems, it is typically non-trivial to precisely quantify the costs and objectives that govern expert schedulers. For example, in operating room scheduling, ad hoc metrics have been applied in an attempt to model the cost of delays, e.g. [1], but these metrics are largely subjective. The Delphi Method is commonly used in management science to quantitatively model expert opinions but such methods have no algorithmic guarantees and are not always reliable [2].

In this paper, we present an online algorithm that allows us to emulate an expert scheduler based on observations of the backlog of the queues in the system and observations of the expert's scheduling decisions. We use the term “emulate” to mean that while the parametric form of the learned policy may not converge to the parametric form of the expert policy in all cases, it will always yield scheduling decisions that on average converge to the expert's decisions. This offers a data-driven way for designing autonomous scheduling systems. We specifically consider a projective cone scheduling (PCS) model which has applications in manufacturing, call/service centers, and in communication networks [3], [4].

The algorithm in this paper uses the multiplicative weight update (MWU) method [5]. The MWU method has been used in several areas including solving zero-sum games [6], solving linear programs [7], and inverse optimization [8]. Because the PCS policy can be written as a maximization, our techniques are most similar to those used in [8]. In [8], the authors apply an MWU algorithm over a fixed horizon to learn the objective of an expert who is solving a sequence of linear programs. Our results differ from [8] in several ways. One is that because of the queueing dynamics that we consider, our expert’s objective will vary over time whereas in [8] the objective is constant. A related issue is that in [8], when the expert has a decision variable of dimension $n$, the dimension of the parameter being learned is also $n$. In our case, when the expert has a decision variable of dimension $n$ (i.e. there are $n$ queues in the system), we need to estimate $\Theta(n^2)$ parameters. We also note that in this paper we provide an algorithm that can be applied even when the horizon is not known a priori.

The goal of inferring parts of an optimization model from data is a well-studied problem in many other applications. For example, genetic algorithm heuristics have been applied to estimate the objective and constraints of a linear program in a data envelopment analysis context [9]. The goal of imputing the objective function of a convex optimization problem has also been considered in the optimization community, e.g. [10], [11]. These papers rely heavily on the convexity of the objective and the feasible set. This approach does not apply in a PCS context because the set of feasible scheduling actions is discrete and hence non-convex.

This paper is also related to inverse reinforcement learning. Inverse reinforcement learning is the problem of estimating the rewards of a Markov decision process (MDP) given observations of how the MDP evolves under an optimal policy [12]. Inverse reinforcement learning can be used to emulate expert decision makers (referred to as “apprenticeship learning” in the machine learning community) as long as the underlying dynamics are Markovian [13]. In the PCS model, no such assumption is made and so our results naturally do not require Markovian dynamics.

The remainder of this paper is organized as follows. Section II specifies the PCS model that we consider. Section III presents our algorithms and the relevant guarantees. Because we take a MWU approach to the problem, our guarantees are bounds on the average loss. However, we also provide a concentration bound which gives guarantees on the tail of the loss distribution. In Section IV, we provide a simple
numerical demonstration of our algorithms, and compare our algorithms with the algorithm presented in [8]. In Section V we discuss some extensions of our results and we conclude in Section VI.

II. PROJECTIVE CONE SCHEDULING DYNAMICS

In this section we summarize the PCS model presented in [4] and comment on the connection to the model presented in [3]. The PCS model has \( n \) queues each with infinite waiting room following an arbitrary queueing discipline. Time is discretized into slots \( t \in \mathbb{Z}_+ \). The backlog in queue \( i \) at the beginning of time slot \( t \) is \( x_i(t) \). The backlog across all queues can be written as a vector \( x_t \in \mathbb{Z}_+^n \). The number of customers that arrive at queue \( i \) at the end of time slot \( t \) is \( a_i(t) \). The arrivals across all queues can be written as a vector \( a_t \in \mathbb{Z}_+^n \). Scheduling configurations are chosen from a finite set \( S \subseteq \mathbb{Z}_+^n \). If configuration \( s_t \in S \) is chosen in time slot \( t \) then for each queue \( i \), \( \min \{a_i(t), x_i(t)\} \) customers are served at the beginning of the time slot. We take the departure vector as \( d_t = \min \{s_{t}, x_{t}\} \in \mathbb{Z}_+^n \) where the minimum is taken component-wise. This gives us the following dynamics

\[
x_{t+1} = x_{t} - d_{t} + a_{t}
\]

where \( x_0 \in \mathbb{Z}_+^n \) is arbitrary. Note that the arrival vector is allowed to depend on previous scheduling configurations, previous arrivals, and previous backlogs in an arbitrary way.

The scheduling configurations are dynamically chosen by solving the maximization

\[
\max_{s \in S} \langle s, Bx_t \rangle = \max_{s \in S} \sum_{i,j} s(i) B(i,j) x_t(j)
\]

where \( B \in \mathbb{R}^{n \times n} \) is symmetric and positive-definite with non-positive off-diagonal elements. We assume that \( S \) is endowed with some arbitrary ordering used for breaking ties. This PCS policy defines a broad class of scheduling policies and in particular we note that by taking \( B \) as the identity matrix, we return to the typical maximum weight matching scheduling algorithm.

Although \( B \) is a matrix, because \( B \) is symmetric, there are only \( p = n(n+1)/2 \) rather than \( n^2 \) free parameters that need to be learned. Consequently, we will represent the projective cone scheduler with an upper-triangular array rather than a matrix. In particular, take \( b(i,i) \propto B(i,i) \) for \( i \in [n] \) and \( b(i,j) \propto -B(i,j) \) for \( (i,j) \in \{(i,j) \in [n] \times [n] : i < j\}^2 \). We can also assume without loss of generality that \( \sum_i \sum_j b(i,j) = 1 \). Then we can write

\[
s_t = \arg \max_{s \in S} \langle s, Bx_t \rangle
\]

\[
= \arg \max_{s \in S} \sum_{i} \sum_{j} B(i,j) s(i) x_t(j)
\]

\[
= \arg \max_{s \in S} \left\{ \sum_{i} B(i,i) s(i) x_t(i) + \sum_{i < j} B(i,j) \left( s(i) x_t(j) + s(j) x_t(i) \right) \right\}
\]

\[
= \arg \max_{s \in S} \left\{ \sum_{i} b(i,i) s(i) x_t(i) - \sum_{i < j} b(i,j) \left( s(i) x_t(j) + s(j) x_t(i) \right) \right\}
\]

\[
\triangleq \mu(x_t; b).
\]

For convenience, let us define

\[
\sigma(i, j) = \begin{cases} 1 & i = j \\ -1 & i \neq j \end{cases}
\]

so that we can write \( \mu \) more compactly as follows:

\[
\mu(x_t; b) = \arg \max_{s \in S} \sum_{i \leq j} \sigma(i, j) b(i, j) (s(i) x_t(j) + s(j) x_t(i))
\]

Note that if we define \( y_t \), a normalized version of \( x_t \), as follows,

\[
y_t = \begin{cases} x_t / \|x_t\|_1 & \|x_t\|_1 > 0 \\ x_t / \|x_t\|_1 & \|x_t\|_1 = 0 \\ 0 & \text{otherwise} \end{cases}
\]

then we have that \( s_t = \mu(x_t; b) = \mu(y_t; b) \).

Modeling each customer as having a uniform deterministic service time is motivated largely by applications in computer systems and in particular, packet switch scheduling. However, PCS models with non-deterministic service times have also been considered in the literature [3]. However, the results in [3] only apply to the case where \( B \) is diagonal. We have opted to present our algorithms in the context of the non-diagonal case because we feel that having \( \Theta(n^2) \) parameters is more interesting than having only \( \Theta(n) \) parameters. Our algorithms can still be applied in the case of stochastic service times; this is discussed along with other extensions in Section V.

Finally, we note that previous literature on PCS [3, 4] has required a variety of additional assumptions. For example, in [4] it is assumed that the arrival process is mean ergodic. We do not require such an assumption and moreover, while the results in [3] and [4] are primarily stability guarantees, we make no assumptions on the stability of the system.

III. THE LEARNING ALGORITHM

In this section we present our algorithm. We first present a finite horizon algorithm and then leverage this to present an infinite horizon algorithm. For both algorithms, we show that the average error is \( O(\ln(T)\sqrt{\ln(p)}/T) \), where \( T \) is the time horizon. We also provide a bound on the fraction of

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1We use the notation \( \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \).

2We use the notation \( [k] = \{1, 2, \ldots, k\} \).
observations for which the error exceeds our average case bound.

These algorithms are applied causally in an online fashion. Although we do not focus on computational issues, we note that computing \(\mu(y_t; b)\) is generally a difficult problem. However, there are local search heuristics that allow efficient computation of \(\mu(y_{t+1}; b)\) based on the solution to \(\mu(y_t; b)\) [14]. Our algorithms require the computation of \(\mu(y_t; \hat{b}_t)\) where \(\hat{b}_t\) is the estimate of \(b\) at time \(t\) and so an online algorithm is appropriate if we want to use the previous solution as a warm start.

Before presenting the algorithms, we consider the loss function of interest. Since the expert we are trying to emulate is specified by the array \(b\), it may seem reasonable to want our estimates \(\hat{b}_t\) to converge to \(b\). However, this goal is not as reasonable as it may seem. Because \(S\) is discrete, it is possible that two different values of \(b\) can render the same scheduling decisions. Consequently, the goal of exactly recovering \(b\) may be ill-posed. We aim to emulate the expert scheduler; therefore, we want \(\hat{s}_t = \mu(y_t; \hat{b}_t)\), the scheduling decision induced by the estimate \(\hat{b}_t\), to be the same as \(s_t\), the expert’s scheduling decision. Hence, the loss should directly penalize discrepancies between \(s_t\) and \(\hat{s}_t\) so that the loss at time \(t\) is

\[
\ell_t = \sum_{i \leq j} \sigma(i, j)(\hat{b}_t(i, j) - b(i, j))(\delta_t(i) y_t(j) + \delta_t(j) y_t(i))
\]  

(7)

where \(\delta_t = \hat{s}_t - s_t\). When \(b = \hat{b}_t\), we have that \(s_t = \hat{s}_t\) and \(\ell_t = 0\). In addition, when \(s_t = \hat{s}_t\) we have that \(\ell_t = 0\) even if \(b \neq \hat{b}_t\). The definition of \(\mu\) will allow us to show below that \(\ell_t \geq 0\).

Another advantage to this loss function is that it allows us to give guarantees that are independent of the statistics of the arrival process. For example, suppose that there are no arrivals at some subset of the queues. In this case, it would be unreasonable to expect to be able to estimate the rows and columns of \(b\) relevant to those queues. More generally, the arrival process may not sufficiently excite all modes of the system. By considering \(\hat{s}_t\) and \(\hat{b}_t\) simultaneously, we can provide bounds that apply even in the presence of pathological arrival processes.

A. A Finite Horizon Algorithm

We first present Algorithm 1, a finite horizon algorithm that requires knowledge of the horizon. Algorithm 1 is a multiplicative weights update algorithm [5] and this time horizon is used to set the learning rate.

**Theorem 1.** Let

\[
D = \max_{(u, v) \in S^2} \|u - v\|_{\infty}
\]

and \(p = n(n + 1)/2\). If \(T > 4\ln(p)\) then the output of Algorithm 1 satisfies the following inequality:

\[
0 \leq \frac{1}{T} \sum_{t=1}^{T} \ell_t \leq 2D \sqrt{\frac{\ln(p)}{T}}
\]  

(8)

**Algorithm 1:**

**Online Parameter Learning with a Fixed Horizon**

**Input:** \([(y_1, s_1), \ldots, (y_T, s_T)]\) # observations  
**Output:** \((b_1, \ldots, b_T)\) # parameter estimates  
**Output:** \((\hat{s}_1, \ldots, \hat{s}_T)\) # scheduling estimates  
1. \(\eta \leftarrow \sqrt{\ln(p)/T}\)  
2. \(w_t \leftarrow\) upper triangular array of 1s  
3. for \(t = 1, \ldots, T\) do  
4. \(\hat{b}_t \leftarrow w_t/\sum_{i \leq t} w_t(i, j)\)  
5. \(\hat{s}_t \leftarrow \mu(y_t; \hat{b}_t)\)  
6. \(m_t \leftarrow\) upper triangular array of 0s  
7. \(\delta_t = \hat{s}_t - s_t\)  
8. if \(\hat{s}_t \neq s_t\) then  
9. \(z_t = \delta_t/\|\delta_t\|\)  
10. for \((i, j) \in [n]^2 : i \leq j\) do  
11. \(m_t(i, j) \leftarrow \sigma(i, j)(z_t(i)y_t(j) + z_t(j)y_t(i))\)  
12. end  
13. end  
14. \(w_{t+1} \leftarrow w_t(1 - \eta m_t)\) # component-wise
15. end

**Proof:** Note that because \(m_t \in [-1, 1]^p\) and \(\eta < \frac{1}{2}\) we can directly apply [5, Corollary 2.2.]:

\[
\sum_{t=1}^{T} \sum_{i \leq j} m_t(i, j)\hat{b}_t(i, j) 
\leq \sum_{t=1}^{T} \sum_{i, j} (m_t(i, j) + \eta |m_t(i, j)|) b(i, j) + \frac{\ln(p)}{\eta}
\]

(9)

Since \(|m_t(i, j)| \leq 1\) and \(\sum_{i \leq j} \hat{b}_t(i, j) = 1\) we have the following:

\[
\sum_{t=1}^{T} \sum_{i \leq j} m_t(i, j)\hat{b}_t(i, j) 
\leq \sum_{t=1}^{T} \sum_{i, j} m_t(i, j) b(i, j) + \eta T + \frac{\ln(p)}{\eta}
\]

(10)

A straightforward calculation shows that this upper bound is minimized when \(\eta = \sqrt{\ln(p)/T}\). Rearranging the inequality and applying this fact give us the following:

\[
\frac{1}{T} \sum_{t=1}^{T} \sum_{i \leq j} m_t(i, j)\hat{b}_t(i, j) - \frac{1}{T} \sum_{t=1}^{T} \sum_{i \leq j} m_t(i, j) b(i, j) 
\leq 2\sqrt{\frac{\ln(p)}{T}}
\]

(11)

Now we apply the specifics of \(m_t\). By definition of \(D\),
This gives us the following:
\[
\frac{1}{T} \sum_{t=1}^{T} \sum_{i \leq j} \sigma(i, j) (\delta_t(i) y_t(j) + \delta_t(j) y_t(i)) \hat{b}_t(i, j) \\
+ \frac{1}{T} \sum_{t=1}^{T} \sum_{i \leq j} \sigma(i, j) (-\delta_t(i) y_t(j) - \delta_t(j) y_t(i)) b(i, j) \\
\leq 2D \sqrt{\frac{\ln(p)}{T}} \tag{12}
\]

Note that \( s_t = \mu(y_t^i; \hat{b}_t) \) and \( \mu \) is defined in terms of a maximization. Therefore,
\[
\sum_{i \leq j} \sigma(i, j) (s(i) y_t(j) + s_t(j) y_t(i)) \hat{b}_t(i, j) \\
\geq \sum_{i \leq j} \sigma(i, j) (s(i) y_t(j) + s(j) y_t(i)) \hat{b}_t(i, j)
\]
for any \( s \in S \). This shows that each term in the first Cesàro sum in (12) is non-negative. Similarly, each term in the second Cesàro sum in (12) is non-negative. This gives us a lower bound of zero. Rearranging the terms leaves us with the desired results. \( \blacksquare \)

B. An Infinite Horizon Algorithm

We now present Algorithm 2, an infinite horizon algorithm that dynamically changes the learning rate. Algorithm 2 applies the “doubling trick” to Algorithm 1. The idea is that we define epochs \([T_k, T_{k+1}]\) where \( T_k = 2^k(4\ln(p)) \) for \( k \geq 0 \) with \( T_{-1} = 0 \). The duration of the \( k^{th} \) epoch is \( T_k \) and in this epoch we apply Algorithm 1. Up to poly-logarithmic factors of \( T \), this gives us the same convergence rate that we had for Algorithm 1.

**Algorithm 2:**

Online Parameter Learning with an Unknown Horizon

**Input:** \((y_1, s_1), (y_2, s_2), \ldots\) # observations

**Output:** \((\hat{b}_1, \hat{b}_2, \ldots)\) # parameter estimates

**Output:** \((\hat{s}_1, \hat{s}_2, \ldots)\) # scheduling estimates

1. \( T_{-1} \equiv 0 \)
2. \( T_k \equiv 2^k(4\ln(p)) \) for \( k \in \{0, 1, 2, \ldots\} \).
3. for \( t = 1, 2, \ldots \) do
4. 4. if \( T_k < t \leq T_{k+1} \) then
5. 4. 5. Apply Algorithm 1
6. 4. 6. with \( T \equiv T_k \) and without re-initializing \( w_t \)
7. end
8. end

**Theorem 2.** Suppose \( T \geq T_0 \). Define \( \lg(\cdot) \) as \( \lfloor \log_2(\cdot) \rfloor \). Then the output of Algorithm 2 satisfies the following inequality:
\[
0 \leq \frac{1}{T} \sum_{t=1}^{T} \ell_t \leq 2\sqrt{2}D \lg \left( \frac{2T}{T_0} \right) \sqrt{\frac{\ln(p)}{T}} \tag{13}
\]

Note that these are the same bounds as in Theorem 1 but with an additional factor of \( \sqrt{2} \lg(2T/T_0) \).

**Proof:** First note that the proof of [5, Corollary 2.2.] does not require the initial weights to be uniform so Theorem 1 still applies even without the initialization on line 2 of Algorithm 1. For convenience, let \( U_k = 2D \sqrt{\ln(p)/T_k} \) and take \( K = \lg(T/T_0) \). Applying Theorem 1 to each stage of Algorithm 2 gives us the following:
\[
0 \leq \sum_{t=1}^{T} \ell_t \leq \sum_{k=0}^{K-1} \sum_{t=1}^{T_k} U_k \leq \sum_{k=0}^{K} 2D \sqrt{\ln(p)/T_k} \leq 2\sqrt{2}D \sqrt{(K+1) \ln(p)/T}
\]
(14)
The first inequality follows from the fact that \( \ell_t \geq 0 \); the second inequality follows by extending the sum from \( T \) up to \( T_K \); the third and fourth inequalities follow from Theorem 1. The penultimate inequality follows from the fact that \( \{T_k\} \) is an increasing sequence and the final inequality follows because \( T_K \) can be no more that \( 2T \).

Dividing by \( T \) gives the desired result. \( \blacksquare \)

C. A Concentration Bound

Our previous results provided bounds on the average loss of our algorithms. In this section, we provide bounds for the tail of the distribution of the loss. This gives us the guarantee that the fraction of observations for which the loss exceeds our average case bound tends to zero.

**Theorem 3.** Let
\[
f_T(\epsilon) = \left\{ t \leq T : \ell_t > 2\sqrt{2}D \lg \left( \frac{2T}{T_0} \right) \sqrt{\frac{\ln(p)}{T}} + \epsilon \right\}
\]
be the fraction of observations up to time \( T \geq T_0 \) for which the loss exceeds the average-case bound by at least \( \epsilon \). Then for any \( \epsilon > 0 \) we have that
\[
f_T(\epsilon) \leq 1 - \frac{\epsilon}{2\sqrt{2}D \lg \left( \frac{2T}{T_0} \right) \sqrt{\frac{\ln(p)}{T}} + \epsilon}
\]
and hence,
\[
\lim_{T \to \infty} f_T(\epsilon) = 0.
\]
(17)

**Proof:** The observed loss sequence \( \{\ell_t\}_{t=1}^{T} \) defines a point measure on \( \mathbb{R}_+ \) where each point has mass \( 1/T \). Applying Markov’s Inequality to this measure gives us that
\[
f_T(\epsilon) \leq \frac{1}{2\sqrt{2}D \lg \left( \frac{2T}{T_0} \right) \sqrt{\frac{\ln(p)}{T}} + \epsilon} \cdot \frac{1}{T} \sum_{t=1}^{T} \ell_t \\
\leq \frac{1}{2\sqrt{2}D \lg \left( \frac{2T}{T_0} \right) \sqrt{\frac{\ln(p)}{T}} + \epsilon} \cdot 2\sqrt{2}D \lg \left( \frac{2T}{T_0} \right) \sqrt{\frac{\ln(p)}{T}} \]
Rearranging the upper bound gives the first result. For the second result we simply take the limit and note that
\[
\lim_{T \to \infty} 2\sqrt{2}D \lg \left( \frac{2T}{T_0} \right) \sqrt{\frac{\ln(p)}{T}} = 0. \quad \blacksquare
\]
IV. A Numerical Demonstration

We now demonstrate Algorithm 2 on a small example of \( n = 2 \) queues. In each time slot, the number of arriving customers is geometrically distributed on \( \mathbb{Z}_+ \). For queues 1 and 2 the mean number of arriving customers is 1 and 2 respectively. The arrivals are independent across time slots as well as across queues. We take

\[
b = \begin{bmatrix} 0.5 & 0.3 \\ 0.2 & \end{bmatrix},
\]

and

\[
S = \{[0 0]', [1 0]', [2 1]', [0 2]'
\}.
\]

This choice of \( b \) shows that the expert scheduler prioritizes queue 1 over queue 2 and the expert also has a preference to not serve both queues simultaneously. We simulate the system and run Algorithm 2 for \( T = 10^6 \) time slots with \( x_0 = \begin{bmatrix} 0 & 0 \end{bmatrix}' \). The results are shown in Figure 1.

First note that Figure 1a shows that the \( \hat{b}_t \) does not converge to \( b \). We see that (to 4 decimal places)

\[
\hat{b}_T = \begin{bmatrix} 0.4998 & 0.3018 \\ 0.1984 & \end{bmatrix}
\]

and for the majority of the simulation these parameter estimates do not change. The reason is that (as shown in Figure 1b) \( \hat{b}_T \) yields the same scheduling decisions as \( b \). The algorithm learns to emulate the expert scheduler so the loss becomes zero and the weights stop updating. This possibility was discussed at the beginning of Section III.

Figure 1c shows that while the average loss does indeed tend to zero, the upper bound proved in Theorem 2 is quite loose in this situation. This is expected due to the generality of the theorem. This also means that the concentration bound in Theorem 3 is quite conservative. Indeed, for this simulation we see that \( f_T(\epsilon) = 0 \) for all \( t \) and for any \( \epsilon > 0 \), i.e. no observed loss ever exceeds the average case bound.

We also compared our algorithm to the inverse reinforcement learning algorithm of Bärmann et al [8]. In order to allow this algorithm to operate based on an unknown horizon, we modified it in a similar fashion as in Algorithm 2. We used the same values for \( b, S, \) and \( x_0 \), meaning that the two algorithms had the same sequence of normalized observations \((y_t, s_t)\) for the experiment.

Importantly, the Bärmann algorithm uses a vector of weights \( w \) to model the underlying expert, as opposed to the upper triangular array used in Algorithm 2. Since the loss function in Equation 7 is ill-defined for the Bärmann algorithm, we instead compare the algorithms based on the cumulative average of the error metric plotted in Figure 1b, i.e.

\[
e(t) = \frac{1}{t} \sum_{t' = 1}^{t} \| \hat{s}_{t'} - s_{t'} \|_{\infty}
\]

The average cumulative error for the two algorithms is shown in Figure 2. The final weights found by the Bärmann

Fig. 1: Output of Algorithm 2 for the example in Section IV. Although the estimates \( \hat{b} \) do not converge to \( b \), the algorithm perfectly emulates the expert from \( t = 0.2 \times 10^6 \) onwards, and the average cumulative loss tends to zero.
algorithm were
\[
\frac{w_t}{\|w_t\|} = \begin{bmatrix} 0.5349 \\ 0.4651 \end{bmatrix}.
\]
These values indicate that the Bärmann algorithm did uncover the preference of queue 1 over queue 2, but it had no means of modeling the expert's preference not to serve both queues simultaneously (i.e., the off-diagonal element of \(b\)). This results in the average error remaining relatively constant over the entire simulation. In contrast, the average error for Algorithm 2 begins to decrease after the algorithm has learned to emulate the optimal action (which is also seen in Figure 1b).

V. EXTENSIONS

We now discuss some extensions to our algorithms. We first note that we could replace line 14 in Algorithm 1 with
\[
w_{t+1} \leftarrow w_t \cdot \exp(-\eta w_t).
\]

The new algorithm would be a Hedge-style algorithm, and we would be able to apply other results (e.g., [5, Theorem 2.4]) to obtain similar theoretical upper bounds on the average loss.

As noted in Section II, a continuous-time PCS model with heterogenous and stochastic service times was considered in [3]. Our algorithms could be applied in this setting as well by updating the algorithm immediately after customer arrivals and departures rather than in discrete time slots. Our theorems would still hold because they do not require the state update to happen at regularly spaced intervals – the algorithms merely require a stream of observed backlogs and observed scheduling actions.

VI. CONCLUSIONS AND FUTURE WORK

In this paper we have proposed an algorithm that learns a scheduling policy that emulates the behavior of an expert projective cone scheduler. This offers a data-driven way of designing automated scheduling policies that achieve the same goals as a human manager. We have provided several theoretical guarantees on the loss of the proposed algorithm, and have numerically demonstrated the efficacy of the algorithm on a simple numerical example when compared to a recent inverse reinforcement learning algorithm.

This paper opens the door for a few areas of future work. One idea is to provide tighter bounds that depend on the statistical properties of the arrival process. A benefit of the current approach is that it does not require any assumptions on the arrival process, but the clear downside is that the resulting bounds are quite loose. An algorithm that uses information about the arrival process could have faster convergence rates and tighter bounds.

Another future direction is to investigate the impact of an approximate computation of \(\mu\). As mentioned in Section III, in large-scale problems, exactly computing \(\mu(y; b)\) is generally a difficult problem and heuristic approaches are typically taken in practice. Future work could explore several heuristic methods of approximating \(\mu(y; b)\) and consider how the approximation "noise" of these methods affects our ability to emulate the expert scheduler.

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