Research Article
A Note on Small Amplitude Limit Cycles of Liénard Equations Theory

Yassine Bouattia,1 Djalil Boudjehem2, Ammar Makhlouf3, Sulima Ahmed Zubair4, and Sahar Ahmed Idris5

1Department of Mathematics, College of Sciences, Université 8 Mai 1945 Guelma, Box 401, Guelma 24000, Algeria
2Department of Electronics and Telecommunication, College of Sciences and Technology, University of Guelma, 8 Mai 1945Box 401, Guelma 24000, Algeria
3Department of Mathematics, College of Sciences, Annaba University, Annaba 23000, Algeria
4Department of Mathematics, College of Sciences and Arts, ArRass, Qassim University, Buraydah, Saudi Arabia
5College of Industrial Engineering, King Khalid University, Abha, Saudi Arabia

Correspondence should be addressed to Sulima Ahmed Zubair; sulimaa2021@gmail.com

Received 3 August 2021; Accepted 13 September 2021; Published 29 September 2021

1. Introduction

A lot of previous works consider studies on a limit cycles’ existence for Liénard systems [1–3]. It represents a very important class of nonlinear systems due to its appearance in some branches of science and engineering as well as in some ecological models, planar physical models, and even in some chemical models, where using a suitable transformation can change these systems into nonlinear Liénard systems. However, an extensive attention has been also devoted to the question of its uniqueness [4–6]; this uniqueness can be verified using different ways of methods based on Poincare–Bendixson theorem. In [4], Zhou et al. proposed a set of theorems for the limit cycles’ uniqueness for the Liénard systems; the proposed theorems represent a guarantee to complete the proof of some previous works’ propositions. In [7], Sabatini and Gabriele studied the uniqueness of limit cycles for a class of planar dynamical systems taking into account those which are equivalent to Liénard systems, and they have also proved a theorem for limit cycles of a class of plane differential systems. In the paper proposed by Li and Llibre [8], the authors proved that for any classical Liénard differential equation of degree four, there exists at most one hyperbolic limit cycle. In [9], a sufficient condition for the existence and the uniqueness of limit cycles for Liénard systems has been proposed for some applications.

In the theory of small amplitude limit cycles, Liénard systems have n solutions, However, in this paper, we use a counterexample to demonstrate that the existence of n solutions for some systems is not true unless we add an extra condition. A new condition is derived for some specific Liénard systems where a violation of the small amplitude limit cycles theorem takes place.
2. Bendixon Criterion

We consider the following autonomous system:

\[
\begin{align*}
\dot{x} &= P(x, y), \\
\dot{y} &= Q(x, y).
\end{align*}
\]

Let \( X = (P, Q) \) be the vector field and \( \operatorname{div} X = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \).

**Theorem 1.** Let \( D \) be a simply connected open subset of \( \mathbb{R} \). If \( \operatorname{div} X = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \) is of constant sign and not identically zero in \( D \), then the system defined by 2 has no periodic orbit lying entirely in the region \( D \).

**Proof.** If \( y \) is a periodic orbit in \( D \), then \( P(x, y)\, dy - Q(x, y)\, dx = 0 \) on \( y \). Since the interior \( U \) of \( y \) is simply connected, we can apply Green's theorem to obtain the following:

\[
0 = \oint (P(x, y)\, dy - Q(x, y)\, dx) = \iint \left( \frac{\partial P}{\partial x} (x, y) + \frac{\partial Q}{\partial y} (x, y) \right) \, dx \, dy.
\]

This is a contradiction since our hypothesis implies that the integral on the right cannot be zero. \( \square \)

**Proof.** If we suppose that the system given by 2 has a periodic solution of a period \( T \), then it has a closed orbit \( \Gamma \) in \( D \). Let \( G \) be the interior of \( \Gamma \), we can apply Greens theorem to obtain the following:

\[
I = \iint_G \left( \frac{\partial P}{\partial x} (x, y) + \frac{\partial Q}{\partial y} (x, y) \right) \, dx \, dy
\]

\[
= \oint \left( P(x, y)\, dy - Q(x, y)\, dx \right),
\]

\[
= \int_0^T \left( P(x(t), t) \frac{dy}{dt} - Q(x(t), t) \frac{dx}{dt} \right) \, dt,
\]

\[
= \int_0^T (P(x(t), t)Q(x(t), t) - Q(x(t), t)P(x(t), t)) \, dt = 0.
\]

Since \( \operatorname{div} X \) is either \( > 0 \) or \( < 0 \), then \( \iint_G \operatorname{div} X \, dx \, dy \) will not be zero; therefore, there are no periodic solutions. \( \square \)

3. A Note on Liénard Equations Theory

We consider the following system:

\[
\begin{align*}
\dot{x} &= y + c_2 y^2 + \cdots + c_L y^L - (a_1 x + a_3 x^3 + \cdots + a_{2m+1} x^{2m+1}), \\
\dot{y} &= - (x + b_2 x^2 + \cdots + b_N x^N),
\end{align*}
\]

where \( c_2, c_3, \ldots, c_L, a_1, a_3, \ldots, a_{2m+1}, b_2, b_3, \ldots \) and \( b_N \) are real coefficients.

**Theorem 2** (see [1]). For the system of form (2), there are at most \( n \) small-amplitudes limit cycles. If \( a_1, a_3, \ldots, a_{2m+1} \) are so chosen that

\[
|a_1| \ll |a_3| \ll \cdots \ll |a_{2m+1}|, \quad a_{2j-1}a_{2j+1} < 0 \quad (j = 1, \ldots, n),
\]

then there are exactly \( n \) small-amplitudes limit cycles.

**Proof.** (counterexample).

We suppose the following system:

\[
\begin{align*}
\dot{x} &= X = \phi(y) - F(x), \\
\dot{y} &= Y = -g(x),
\end{align*}
\]

where

\[
F(x) = \sum_{j=0}^n \frac{(-1)^j}{(2j + 1)10^{10(n-j)}} x^{2j+1}, \quad n = 2m,
\]

\[
g(x) = x + b_2 x^2 + \cdots + b_N x^N,
\]

\[
\phi(y) = y + c_2 y^2 + \cdots + c_L y^L.
\]

By putting \( a_{2j+1} = (-1)^j / (2j + 1)10^{10(n-j)} \), we obtain

\[
\left| \frac{a_{2j-1}}{a_{2j+1}} \right| = \frac{2j + 1}{2j - 1} 10^{-10} \leq 3 \times 10^{-10} \ll 1, \quad \text{for } j = 1, \ldots, n,
\]

then

\[
|a_1| \ll |a_3| \ll \cdots \ll |a_{2m+1}|, \quad a_{2j-1}a_{2j+1} < 0, \quad \text{for } j = 1, \ldots, n.
\]

However,

\[
\operatorname{div}(X, Y) = \frac{\partial (\phi(y) - \sum_{j=0}^n a_{2j+1} x^{2j+1})}{\partial x} + \frac{\partial (-g(x))}{\partial y}
\]

\[
= -\sum_{j=0}^n \frac{(-1)^j}{10^{10(n-j)} x^{2j}}
\]

\[
= -\prod_{j=0}^{n+1} \left( x^2 - \frac{1}{10^{10} \cos \frac{2j+1}{n+1} \pi} \right)^2 + \frac{1}{10^{10} \sin \frac{2j+1}{n+1} \pi} < 0,
\]

because
\[ f(x) = \sum_{j=0}^{n} \frac{(-1)^j}{10^{10(n-j)}} x^{2j}, \]
\[ = \prod_{j=0}^{n/2-1} \left( x - 10^{-5} e^{i(2j+1)\pi/2n+2} \right) \left( x + 10^{-5} e^{-i(2j+1)\pi/2n+2} \right) \left( x + 10^{-5} e^{-i(2j+1)\pi/2n} \right). \] (13)

\[ \text{Theorem 3 (see [2]).} \]

We consider the following equation:
\[ \dot{r} = r \left( v_0 + v_1 r^2 + v_2 r^4 + \cdots + v_n r^{2n} \right). \] (14)

If the focus values \( v_j \) given in equation (3) satisfy the following conditions:
\[ v_j v_{j+1} < 0, \text{ and } |v_j| \ll |v_{j+1}| \ll 1, \text{ for } j = 0, 1, 2, \ldots, n-1, \] (15)

then the polynomial equation given by \( r' = 0 \) in equation (3) has \( n \) positive real roots for \( r' \).

**Proof.** (counterexample).

We consider the following equation:
\[ \dot{r} = f(r) = r \sum_{j=0}^{n} \frac{(-1)^j}{10^{10(n-j)+10^2j}}. \] (16)

By putting \( v_j = (-1)^j/10^{10(n-j)+10} \), we obtain
\[ v_j v_{j+1} < 0, \text{ and } |v_j| \ll |v_{j+1}| \ll 1, \text{ for } j = 0, 1, 2, \ldots, n-1, \] (17)

because
\[ \frac{v_j}{v_{j+1}} = \frac{10^{10(n-j-1)+10}}{10^{10(n-j)+10}} = 10^{-10} \ll 1, \text{ for } j = 0, 1, 2, \ldots, n-1, \text{ and } |v_n| = 10^{-10} \ll 1. \] (18)

However,
\[ \frac{f(r)}{r} = \sum_{j=0}^{n} \frac{(-1)^j}{10^{10(n-j)+10^2j}} \]
\[ = 10^{-10} \prod_{j=0}^{n/2-1} \left( \frac{r^2 - 10^{-10} \cos \frac{2j + 1}{n + 1} \pi}{n + 1} \right)^2 + 10^{-10} \sin \frac{2j + 1}{n + 1} \pi \right) \neq 0, \forall r \in \mathbb{R}. \] (19)
4. Examples

In this section, by using the counterexample, we can demonstrate that Theorems 2 and 3 are not true. However, the previous theorems will be true if we add the following condition: \( a_1/a_2 \ll a_2/a_4 \ll a_4/a_6 \ll a_6/a_8 \ll \cdots \ll a_{2j-2}/a_{2j} \), \( j = 1, \ldots, n \).

Example 1. We consider the following equation:

\[
r' = f(r),
\]

or

\[
f(r) = \sum_{j=0}^{4} v_j r^{2j+1} = r \left( \frac{1}{10^{10}} - \frac{1}{10^{20}} r^2 + \frac{1}{10^{25}} r^4 - \frac{1}{10^{30}} r^6 + \frac{1}{10^{35}} r^8 \right).
\]  

(20)

We have

\[
v_j v_{j+1} < 0, \quad \text{and} \quad |v_j| \ll |v_{j+1}| \ll 1, \quad \text{for } j = 0, 1, \ldots, 3,
\]

(21)

because

\[
\frac{v_j}{v_{j+1}} = 10^{-10} \ll 1, \quad \text{for } j = 0, 1, \ldots, 3, \quad \text{and} \quad |v_4| = 10^{-10} \ll 1.
\]

(22)

However,

\[
f(r) = 10^{-10} \left( r - \frac{1}{10^3} \left( \cos \frac{\pi}{10} + i \sin \frac{\pi}{10} \right) \right) \left( r - \frac{1}{10^3} \left( \cos \frac{3\pi}{10} + i \sin \frac{3\pi}{10} \right) \right) \left( r - \frac{1}{10^3} \left( \cos \frac{9\pi}{10} + i \sin \frac{9\pi}{10} \right) \right) \left( r - \frac{1}{10^3} \left( \cos \frac{13\pi}{10} + i \sin \frac{13\pi}{10} \right) \right) \left( r - \frac{1}{10^3} \left( \cos \frac{19\pi}{10} + i \sin \frac{19\pi}{10} \right) \right) \neq 0, \quad \forall r \in \mathbb{R},
\]

where the system roots are given by

\[
\begin{align*}
r_1 &= \frac{1}{10^5} \left( \cos \frac{\pi}{10} + i \sin \frac{\pi}{10} \right) = 9.5106 \times 10^{-6} + 3.0902 \times 10^{-6} i, \\
r_2 &= \frac{1}{10^5} \left( \cos \frac{3\pi}{10} + i \sin \frac{3\pi}{10} \right) = 5.8779 \times 10^{-6} + 8.0902 \times 10^{-6} i, \\
r_3 &= \frac{1}{10^5} \left( \cos \frac{7\pi}{10} + i \sin \frac{7\pi}{10} \right) = -5.8779 \times 10^{-6} + 8.0902 \times 10^{-6} i, \\
r_4 &= \frac{1}{10^5} \left( \cos \frac{9\pi}{10} + i \sin \frac{9\pi}{10} \right) = -9.5106 \times 10^{-6} + 3.0902 \times 10^{-6} i, \\
r_5 &= \frac{1}{10^5} \left( \cos \frac{11\pi}{10} + i \sin \frac{11\pi}{10} \right) = -9.5106 \times 10^{-6} - 3.0902 \times 10^{-6} i, \\
r_6 &= \frac{1}{10^5} \left( \cos \frac{13\pi}{10} + i \sin \frac{13\pi}{10} \right) = -5.8779 \times 10^{-6} - 8.0902 \times 10^{-6} i, \\
r_7 &= \frac{1}{10^5} \left( \cos \frac{17\pi}{10} + i \sin \frac{17\pi}{10} \right) = 5.8779 \times 10^{-6} - 8.0902 \times 10^{-6} i, \\
r_8 &= \frac{1}{10^5} \left( \cos \frac{19\pi}{10} + i \sin \frac{19\pi}{10} \right) = 9.5106 \times 10^{-6} - 3.0902 \times 10^{-6} i.
\end{align*}
\]  

(24)
Example 2. Let us consider now the following system:

\[
\dot{r} = f(r) = r \sum_{j=0}^{4} \frac{(-1)^j}{10^{4j+1}} r^{2j} = r \left( \frac{1}{10^{1024}} - \frac{1}{10^{1025}} r^2 + \frac{1}{10^{64}} r^4 - \frac{1}{10^{16}} r^6 + \frac{1}{10^8} r^8 \right),
\]

with positive roots such as

\[
\begin{pmatrix}
    r_1 = 10^{-6} \\
    r_2 = 10^{-24} \\
    r_3 = 10^{-96} \\
    r_4 = 10^{-384}
\end{pmatrix},
\]

because

\[
\begin{align*}
    v_j & < 0, & & \text{and } |v_j| |v_{j+1}| < 1, \text{ for } j = 0, 1, \ldots, 3, \\
    v_j & \ll v_{j+1} & & \text{for } j = 0, \ldots, 2.
\end{align*}
\]

Example 3. We suppose the following system:

\[
\begin{align*}
\dot{x} &= y - \varepsilon \left( b_1 x + b_2 x^3 + b_3 x^5 + b_4 x^7 + b_5 x^9 \right), \\
\dot{y} &= -x,
\end{align*}
\]

or

\[
b_1 = 2 \left( 10^{-196} \right),
\]

\[
b_2 = \frac{8}{3} \left( 10^{-192} + 10^{-152} + 10^{-132} + 10^{-112} \right),
\]

\[
b_3 = \frac{16}{5} \left( 10^{-148} + 10^{-128} + 10^{-110} + 10^{-88} + 10^{-68} + 10^{-48} \right),
\]

\[
b_4 = \frac{128}{35} \left( 10^{-4} + 10^{-44} + 10^{-64} + 10^{-84} \right),
\]

\[
b_5 = \frac{256}{63}
\]

By putting \( a_{2j+1} = \varepsilon b_{2j+1}, \ j = 0, 4 \), we obtain

\[
\begin{align*}
\dot{x} &= y - \left( a_1 x + a_2 x^3 + a_3 x^5 + a_4 x^7 + a_5 x^9 \right), \\
\dot{y} &= -x,
\end{align*}
\]

and by applying the first-order averaging method \([13, 14]\) on (14), we obtain

\[
f^0(r) = f^0 \left( r^8 - \left( 10^{-4} + 10^{-44} + 10^{-64} + 10^{-84} \right) r^6 + \left( 10^{-148} + 10^{-128} + 10^{-110} + 10^{-88} + 10^{-68} + 10^{-48} \right) r^4 \right) \]

\[
\frac{-10^{-192} + 10^{-152} + 10^{-132} + 10^{-112} r^2 + 10^{-196}}{10^{-196}}.
\]

\[
f^0(r) = 0 \text{ implied } r_1 = 10^{-2}, r_2 = 10^{-22}, r_3 = 10^{-32}, \text{ and } r_4 = 10^{-42}, \text{ then there are exactly 4 small-amplitude limit cycles } r_i, \ i = 1, 4.
\]

Note that \( a_{2j+1}/a_{2j+3} = b_{2j+1}/b_{2j+3} \ll 1 \) for \( j = 0, \ldots, 3 \) and \( b_{2j+1}/b_{2j+3} \ll b_{2j+3}/b_{2j+5} \) for \( j = 0, \ldots, 2. \)

5. Conclusion

In this work, by using a counterexample for a theorem of the small amplitude limit cycles in some Liénard systems, we have shown that that there will be no solutions unless an
extra condition is added. In addition, a new condition is derived for some specific Liénard systems where a violation of the small amplitude limit cycles theorem takes place. However, these theorems will be true if we add the following condition: $a_0/a_2 \ll a_2/a_4 \ll a_4/a_6 \ll \cdots \ll a_{2j-2}/a_{2j}$, $j = 1, \ldots, n$.

Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

Acknowledgments
The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through Research Group Project under Grant no. (R.G.P.-2/53/42).

References
[1] J. Yang and W. Ding, “Limit cycles of a class of Liénard systems with restoring forces of seventh degree,” Applied Mathematics and Computation, vol. 316, pp. 422–437, 2018.
[2] H. Zhu, M. Wei, S. Yang, and C. Jiang, “Bifurcation of limit cycles from a Liénard system of degree 4,” Acta mathematica scientia, Series A, vol. 41, no. 4, pp. 936–953, 2021.
[3] F. Jiang, Z. Ji, and Y. Wang, “On the number of limit cycles of discontinuous Liénard polynomial differential systems,” International Journal of Bifurcation and Chaos, vol. 28, no. 14, Article ID 1850175, 2018.
[4] Y. Zhou, C. Wang, and D. Blackmore, “The uniqueness of limit cycles for Liénard system,” Journal of Mathematical Analysis and Applications, vol. 304, no. 2, pp. 473–489, 2005.
[5] Z. Daoxiang and P. Yan, “On the uniqueness of limit cycles in a generalized Liénard system,” Qualitative theory of dynamical systems, vol. 18, no. 3, pp. 1191–1199, 2019.
[6] G. Villari and F. Zanolin, “On the uniqueness of the limit cycle for the Liénard equation with f(x) not sign-definite,” Applied Mathematics Letters, vol. 76, pp. 208–214, 2018.
[7] M. Sabatini and G. Villari, “Limit cycle uniqueness for a class of planar dynamical systems,” Applied Mathematics Letters, vol. 19, no. 11, pp. 1180–1184, 2006.
[8] C. Li and J. Llibre, “Uniqueness of limit cycles for Liénard differential equations of degree four,” Journal of Differential Equations, vol. 232, no. 4, pp. 3142–3162, 2007.
[9] T. Carletti and G. Villari, “On the existence and uniqueness of limit cycles for Liénard systems,” Journal of Mathematical Analysis and Applications, vol. 307, no. 2, pp. 763–773, 2005.
[10] S. Lynch, “Liénard systems and the second part of Hilbert’s sixteenth problem,” Nonlinear Analysis: Theory, Methods & Applications, vol. 30, no. 3, pp. 1395–1403, 1997.
[11] P. Yu and M. Han, “Limit cycles in generalized Liénard systems,” Chaos, Solitons & Fractals, vol. 30, no. 5, pp. 1048–1068, 2006.
[12] N. G. Lloyd and S. Lynch, “Small-amplitude limit cycles of certain Liénard systems,” Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences, vol. 418, no. 1854, pp. 199–208, 1988.
[13] J. Llibre and A. Makhlouf, “Limit cycles of a class of generalized Liénard polynomial equations,” Journal of Dynamical and Control Systems, vol. 21, no. 2, pp. 189–192, 2015.
[14] A. Menaceur, S. M. Boulaaras, A. Makhlouf, K. Rajagopal, and M. Abdalla, “Limit cycles of a class of perturbed differential systems via the first-order averaging method,” Dynamic Analysis, Learning, and Robust Control of Complex Systems, vol. 2021, Article ID 5581423, 6 pages, 2021.