Detecting entanglement with partial state information

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We introduce a sequence of numerical tests that can determine the entanglement or separability of a state even when there is not enough information to completely determine its density matrix. Given partial information about the state in the form of linear constraints on the density matrix, the sequence of tests can prove that either all states satisfying the constraints are entangled, or there is at least one separable state that satisfies them. The algorithm works even if the values of the constraints are only known to fall in a certain range. If the states are entangled, an entanglement witness is constructed and lower bounds on entanglement measures and related quantities are provided; if a separable state satisfies the constraints, a separable decomposition is provided to certify this fact.

I. INTRODUCTION

Entanglement is one of the central features in quantum information processing (QIP). It has been identified as a key ingredient in many useful QIP tasks such as teleportation, quantum key distribution, superdense coding and quantum computation \cite{1}. It is also remarkable that even when a complete description of a quantum state is given (in the form of a density matrix), it can be extremely difficult to computationally decide whether such a state is entangled or not. This is due to the fact that this problem (usually called “the separability problem”) is known to be NP-Hard \cite{2}. The problem is even more difficult when we consider experimental tests of entanglement, since measurements may not provide a full description of the state, and when they do (such as in quantum state tomography \cite{3}) the reconstructed density matrix may be unphysical (i.e., not positive semidefinite (PSD)).

A key problem with important practical applications is to determine the entanglement characteristics of a state when only a limited amount of information is available. If this information comes from measuring a set of observables, it takes the form of a set of linear constraints on the elements of the density matrix. In this article we will introduce a sequence of numerical tests that can decide whether all states that satisfy a set of linear constraints are entangled, or if there is at least one separable state that satisfies those same constraints. The approach is based on an extension of the PPT Symmetric Extension (PPTSE) criterion \cite{4} and its dual introduced in \cite{5}. If the states are shown to be entangled, the algorithm constructs an entanglement witness (EW) that certifies this fact for all such states and such a witness can be used to provide lower bounds on certain entanglement measures and related quantities. If a separable state satisfying the constraints exists, the algorithm finds it and provides a separable decomposition as a proof.

The paper is organized as follows: in Section II we review the PPTSE criterion and its dual; Section III shows how to extend these two criteria to the case where only partial information about the state $\rho$ is available in the form of a set of linear constraints; Section IV shows how to construct and entanglement witness if all states satisfying the constraints are shown to be entangled; if the state is shown to be entangled, Section V provides lower bounds on entanglement measures and other related quantities; Section VI shows an example of the application of this technique; Section VII discusses some basic features of the approach and our conclusions are presented in Section VIII.

II. THE PPT SYMMETRIC EXTENSION CRITERION AND ITS DUAL

To determine the entanglement or separability of a state $\rho$ we will use the PPTSE criterion \cite{4} and a dual approach introduced by Navascués et al. \cite{5}. When used together, these two criteria can conclusively determine if a state is separable or entangled in a finite number of steps (however, the number of steps and the computational resources required to implement them can be arbitrarily high for some states). Let us start with some definitions. If $\rho$ is a state in $\mathcal{H}_A \otimes \mathcal{H}_B$, we will call $\tilde{\rho}$ in $\mathcal{H}_A^{\otimes k} \otimes \mathcal{H}_B$ a PPT symmetric extension of $\rho$ to $k$ copies of subsystem $A$ if: (i) $\rho = \text{Tr}_{A^{k-1}}[\tilde{\rho}]$, (ii) $\tilde{\rho}$ is symmetric under exchanges of copies of subsystem $A$, and (iii) $\tilde{\rho}$ has positive partial transposes for any bipartite arrangement of the subsystems $A$ and $B$. The key point is that, since any separable state in $\mathcal{H}_A \otimes \mathcal{H}_B$ can be written as $\rho = \sum p_i |\psi_i\rangle \langle \psi_i| \otimes |\phi_i\rangle \langle \phi_i|$, it trivially has such an extension given by $\tilde{\rho} = \sum p_i |\psi_i\rangle \langle \psi_i|^{\otimes k} \otimes |\phi_i\rangle \langle \phi_i|$. For each value of $k$, the non-existence of a PPTSE provides a sufficient (but not necessary) condition for entanglement. In the limit $k \to \infty$ the condition becomes necessary. The practical value of this approach is that searching for such extensions or proving their impossibility can be cast as a semidefinite program (SDP).

An SDP is a type of convex optimization problem that...
has a broad range of applications and has been widely applied in quantum information. An SDP has both a primal and a dual form. A typical SDP in its primal form reads

\[
\begin{align*}
\text{minimize} \quad & c^T x \\
\text{subject to} \quad & F_0 + \sum_i x_i F_i \succeq 0, \\
& x \geq 0,
\end{align*}
\]

where \(c\) is a given vector, \(x = (x_1, \ldots, x_n)\), and \(F_0\) and \(F_i\) are some fixed Hermitian matrices. The minimization in the second line means that the affine combination of the \(F\) matrices must be positive semidefinite. The minimization is performed over the vector \(x\), whose components are the variables of the problem. The dual of this SDP takes the form

\[
\begin{align*}
\text{maximize} \quad & -\text{Tr}[F_0 Z] \\
\text{subject to} \quad & Z \succeq 0, \\
& \text{Tr}[F_i Z] = c_i,
\end{align*}
\]

where the dual variables are the components of the matrix \(Z\).

Each test in the PPTSE hierarchy provides a sufficient but not necessary condition for entanglement. The hierarchy is complete in the limit: any entangled state is guaranteed to be detected by one of the tests. But a separable state will pass all tests, leading to a non-terminating algorithm. Fortunately, a dual approach was developed by Navascués et al.,\cite{5}, that applies a sequence of tests that can certify separability in a finite number of steps (although that number can be very high for some states). Geometrically, the PPTSE hierarchy of tests works by monotonically approximating the cone of separable states from the outside with a sequence of cones associated with states having PPT symmetric extensions to a certain number of copies of one of the subsystems. The dual approach in \cite{5} constructs a similar approximation to the cone of separable states, but from the inside: it provides sufficient (but not necessary) conditions for separability. By interleaving the two sequences of tests we can, in a finite number of steps, determine if a state is entangled (and give an entanglement witness as a proof), or separable (and provide an explicit separable decomposition). We will now briefly describe the test in \cite{5}.

Let \(S_p\) be the set of states in \(\mathcal{H}_A \otimes \mathcal{H}_B\) that have a PPT symmetric extension to \(N\) copies of \(A\). In \cite{5} it was shown that a small perturbation in \(\mathcal{H}_B\) makes these states separable. More precisely we have that \(S_p^N = \{(1-\epsilon_N)\omega_{AB} + \epsilon_N \omega_A \otimes \frac{1}{\sqrt{N}} : \omega_{AB} \in S_p^N\}\) satisfies \(S_p^N \subset S\), for all \(N\), where \(S\) is the set of separable states in \(\mathcal{H}_A \otimes \mathcal{H}_B\), \(\omega_A = \text{Tr}_B[\omega_{AB}]\), and \(\epsilon_N \equiv d_B/(2(d_B - 1))\min (1 - x : P^{(d_B - 2, N \mod 2)}_{\lfloor N/2 \rfloor + 1}(x) = 0)\), with \(P^{(\alpha, \beta)}(x)\) the Jacobi polynomials. Since \(\tilde{S}_p^N \rightarrow S_p^N\) for \(N\) going to infinity, and \(S \subset S_p^N\) for all \(N\), we have that \(\tilde{S}_p^N \rightarrow S\) \((N \rightarrow \infty)\). This result can be easily transformed into a SDP like \cite{5} that tests if a given state is separable \cite{5}. If that is the case, the output of the SDP can be used to construct an explicit separable decomposition (the details of this construction can be found in \cite{5}).

### III. ENTANGLEMENT TESTING WITH PARTIAL STATE INFORMATION

The PPTSE criterion and its dual discussed above require as input the complete density matrix, and so cannot be applied directly if we only have access to partial information about the state. We will now show that they can be extended so that they can be applied in this more general case.

Consider a situation in which we are given partial information of the state of a quantum system in the form of \(L\) linear constraints on the elements of its density matrix

\[
\text{Tr}[\rho M_l] = m_l, \quad l = 1, \ldots, L
\]

where the operators \(M_l\) are arbitrary. Our goal is to determine if all the states satisfying these constraints are entangled, or if there is a separable state that satisfies them. The constraints in \cite{3} are nothing but a linear system of equations for the elements of the density matrix \(\rho\). If this system is incompatible it means that these constraints do not describe a physical state. If the system is invertible, then the Hermitian matrix \(\rho\) can be completely determined from the equations, and once we have an explicit expression we can check if it corresponds to a state (i.e., it is PSD and normalized), and then apply the PPTSE criterion and its dual. But the situation that is the most interesting (and typically more common) corresponds to the case in which the linear system \cite{3} is underdetermined, and we do not have enough information to uniquely define the state. This situation corresponds naturally to being able to measure only the expectation values of a limited number of observables (the operators \(M_l\) in \cite{3} are then Hermitian matrices). We will show that in this case, the linear system defines an affine subspace in the space of Hermitian matrices, and the PPTSE criterion and its dual can be applied to either prove that all states in that affine subspace are entangled, or to show that a separable state exists that satisfies \cite{3}.

Let us start with the linear system \cite{3}. The most general solution \(\rho\) of this system can be written as

\[
\rho = \rho^{\text{part}} + \sum_{a=1}^{D_K} y_a \mu^{(a)},
\]

where \(\rho^{\text{part}}\) is a particular solution of \cite{3} (i.e., \(\text{Tr}[\rho^{\text{part}} M_l] = m_l, \quad l = 1, \ldots, L\)), the matrices \(\{\mu^{(a)}\}\) form a basis of the subspace of solutions of the homogeneous system (i.e., \(\text{Tr}[\mu^{(a)} M_l] = 0, \quad l = 1, \ldots, L\)), \(D_K\) is the dimension of this subspace, and \(y_a\) are real variables. Note that \(\rho^{\text{part}}\) is just a Hermitian matrix and not necessarily a state since it need not be PSD or normalized. The question then reduces to whether there are values of the real variables \(y_a\) such that the resulting Hermitian matrix is a normalized, separable state. If there are not, then all the states of the form \cite{3}, that is all normalized PSD Hermitian matrices satisfying \cite{3}, must be entangled.
To test the entanglement of a state of the form (2), we can apply the PPTSE criterion for any value of $k$. We will present in detail how this works for $k = 2$ (the general case is straightforward). So we need to check if, for some values of the variables $y_a$, the resulting matrix is PSD, normalized and has a PPTSE. Let $\{\sigma_i^A\}_{i=1}^{d_A^2}$, $\{\sigma_j^B\}_{j=1}^{d_B^2}$ be bases for the spaces of Hermitian matrices that operate on $\mathcal{H}_A$ and $\mathcal{H}_B$, of dimensions $d_A$ and $d_B$ respectively, such that they satisfy $\text{Tr}[\sigma_i^A \sigma_j^B] = \alpha_{ij}$ and $\text{Tr}[\sigma_i^A \sigma_i^A] = \delta_{i1}$ (where $X$ stands for $A$ or $B$), and $\alpha_X$ is some constant. Then we can expand $\rho$ in the basis $\{\sigma_i^A \otimes \sigma_j^B\}$, and write $\rho = \sum_{i,j} \rho_{ij} \sigma_i^A \otimes \sigma_j^B$, with $\rho_{ij} = \alpha_{ij}^{-1} \alpha_{ij}^{-1} \text{Tr}[\rho \sigma_i^A \otimes \sigma_j^B]$. In the same way, we can expand the extension $\hat{\rho}$ in $\mathcal{H}_A^\otimes 2 \otimes \mathcal{H}_B$ as

$$
\hat{\rho} = \sum_{i<k} \hat{\rho}_{ikj} \{\sigma_i^A \otimes \sigma_k^A \otimes \sigma_j^B + \sigma_k^A \otimes \sigma_i^A \otimes \sigma_j^B\} +
+ \sum_{k j} \hat{\rho}_{kkj} \sigma_k^A \otimes \sigma_k^A \otimes \sigma_j^B,
$$

where we made explicit use of the swapping symmetry between the two copies of $A$. To satisfy the condition that $\hat{\rho}$ is an extension of $\rho$, we need to impose $\text{Tr}_A[\hat{\rho}] = \rho$. This implies $\hat{\rho}_{1j} = \rho_{1j}$. From (2) we have that $\rho_{ij} = \rho_{ij}^{part} + \sum_{a=1}^{D_k} y_a \mu_{ij}^{(a)}$, which fixes some of the components of the extension (6). We then have

$$
\hat{\rho} = \sum_{i>1} \rho_{ij}^{part} \{\sigma_i^A \otimes \sigma_i^A \otimes \sigma_j^B + \sigma_i^A \otimes \sigma_i^A \otimes \sigma_j^B\} +
+ \sum_{j} \rho_{ij}^{part} \sigma_i^A \otimes \sigma_i^A \otimes \sigma_j^B +
+ \sum_{a=1}^{D_k} y_a \left( \sum_{i=1} \rho_{ij}^{(a)} \{\sigma_i^A \otimes \sigma_i^A \otimes \sigma_j^B + \sigma_i^A \otimes \sigma_i^A \otimes \sigma_j^B\} +
+ \sum_{i<j} \mu_{ij}^{(a)} \sigma_i^A \otimes \sigma_i^A \otimes \sigma_j^B \right) +
+ \sum_{i<k} \hat{\rho}_{ikj} \{\sigma_i^A \otimes \sigma_k^A \otimes \sigma_j^B + \sigma_k^A \otimes \sigma_i^A \otimes \sigma_j^B\} +
+ \sum_{k j} \hat{\rho}_{kkj} \sigma_k^A \otimes \sigma_k^A \otimes \sigma_j^B,
$$

(6)

If we define a vector of variables $x = (y, \hat{\rho}_{ikj})$ (with $2 \leq k \leq i \leq d_A^2$, $1 \leq j \leq d_B^2$), we can see that the most general form of the extension (6) has the form $G_0 + \sum_i x_i G_i$, where the expressions for $G_0$ and $G_i$ can be easily extracted from it.

The first condition we need to impose on this extension is that it represents a state, i.e., that it is PSD and normalized. The normalization condition can always be assumed to be contained in the set of linear equations (5), by adding another constraint with $M = 1$ and expectation value equal to 1. Requiring that the extension is PSD means imposing the linear matrix inequality (LMI) $G_0 + \sum_i x_i G_i \succeq 0$. And finally, imposing the positivity of the partial transposes requires two more LMIs, namely $G_0^{TA} + \sum_i x_i G_i^{TA} \succeq 0$ and $G_0^{TB} + \sum_i x_i G_i^{TB} \succeq 0$ (due to the swapping symmetry, these are the only two independent partial transposes). We can combine these three LMIs into a single one by defining matrices $F_0 = G_0 \oplus G_0^{TA} \oplus G_0^{TB}$ and $F_i = G_i \oplus G_i^{TA} \oplus G_i^{TB}$ (a block diagonal matrix is PSD if and only if all of its blocks are PSD). So searching for a PPTSE of a state of the form (2) corresponds to a SDP of the form (3) with $c = (0, \ldots, 0)$. If there are values of $y_a$ such that (3) is separable, then there must exist values of $\hat{\rho}_{ikj}$, $(2 \leq k \leq i \leq d_A^2, 1 \leq j \leq d_B^2)$ such that the SDP is feasible (because separable states always have PPTSE). But if the SDP is infeasible, it means that there is no set of values $y_a$ for which the resulting state $\rho$ has a PPTSE, and hence all states of the form (2) must be entangled.

If the SDP is feasible it means that there is a state with the required PPT symmetric extension that is compatible with (3). Furthermore, the output of the SDP provides us the values of the variables $y_a$ of such state, so we can completely determine its density matrix using (2).

### A. Extension to more general constraints

We can take this approach a little bit further and consider the case in which the expectation values of the operators $M_i$ (the right-hand side of (3)) are known only approximately. Assume that instead of (3) we have

$$
m_{i}^{\text{min}} \leq \text{Tr}[\rho M_i] \leq m_{i}^{\text{max}} \quad l = 1, \ldots, L.
$$

(7)

Now consider the set of matrices $\{\tau_p : \text{Tr}[\tau_p M_i] = \delta_{l0}\}$. We can use these matrices to write any particular solution of the linear system $\text{Tr}[\rho M_i] = z_l$ as $\rho^{part} = \sum_{l=1}^{L} \tau_p \tau_l$. Combining this with (3) we have

$$
\rho = \sum_{l=1}^{L} \sum_{l=1}^{D_k} z_l \tau_p \mu_{ij}^{(a)},
$$

as the most general solution of (7) provided that $z_l \in [m_{l}^{\text{min}}, m_{l}^{\text{max}}]$. If we apply the PPTSE criterion to (8) as before, we will obtain a linear combination of matrices representing the PPT symmetric extension of a state satisfying this set of equations. We just need to once again construct the required LMIs to impose the positive semidefiniteness of the extension and its partial transposes, and solve the resulting SDP satisfying the constraints $z_l \in [m_{l}^{\text{min}}, m_{l}^{\text{max}}]$. These constraints can be imposed by another LMI, namely $\text{diag}(z_l - m_{l}^{\text{min}}, m_{l}^{\text{max}} -$
where \( z_1, \ldots, z_L \) are such that
\[
(z_1 - m_L^{\min}, \ldots, z_L - m_L^{\max}) \geq 0,
\]
showing that constraining the range of the variables does not change the SDP structure.

**B. Alternative SDP formulation**

We can formulate the search for a PPTSE as a slightly different SDP that has the advantage of performing better numerically and providing a connection with entanglement measures (as discussed in Section V). We will replace the feasibility SDP discussed above by the following:

\[
\begin{align*}
\text{minimize} & \quad t \\
\text{subject to} & \quad F_0 + \sum_i x_i F_i + t \mathbf{1} \succeq 0. \\
\end{align*}
\]

Here, we have added a term proportional to the identity \( \mathbf{1} \) to the affine combinations of the \( F_i \) matrices, and we minimize its coefficient \( t \). The purpose of this is to make the SDP feasible: the LMI can always be satisfied if we choose \( t \) large enough. This makes the SDP solvers perform better in practice. To connect this SDP with the feasibility problem, we just need to realize that \( F_0 + \sum_i x_i F_i \succeq 0 \) is feasible if and only if \( t_{\text{opt}} \leq 0 \). If the optimal value of \( t \) is positive and bounded away from zero, the original SDP is infeasible and the state does not have a PPTSE (and hence it is entangled).

**C. Extension to the multipartite case**

Even though the approach described in this section considers only the bipartite case, the technique can be extended to the multipartite case. In [8] it was shown that requiring the existence of PPTSE to any number of copies of any subset of parties of a multipartite state, gives a complete characterization of the set of fully separable states. Again, the search for these extensions can be cast as an SDP, and failure to find one implies entanglement of the state. The PPTSE algorithm was used to show entanglement of a \( 2 \otimes 2 \otimes 2 \) state that has the property of being separable under any bipartition. The algorithm presented here can thus also detect multipartite entanglement, although it will not distinguish between inequivalent forms of multipartite entanglement (like \( W \) and GHZ entanglement in the case of three qubits).

**IV. ENTANGLEMENT WITNESSES**

Another useful feature of the PPTSE criterion is that, if the primal SDP is infeasible (i.e., there is no separable state satisfying the constraints), the dual SDP provides a certificate of this fact in the form of an entanglement witness [4]. Let us recall that an entanglement witness (EW) for a state \( \rho \) is a Hermitian operator \( W \) that satisfies

\[
\text{Tr}[\sigma_{\text{sep}} W] \geq 0 \quad \text{and} \quad \text{Tr}[\rho W] < 0,
\]

where \( \sigma_{\text{sep}} \) is any separable state. These operators provide a proof of the entanglement of a given state. Entanglement witnesses are a consequence of the separating hyperplane theorem (or Hahn-Banach theorem) of convex geometry: if two closed convex sets are disjoint and one of them is compact, there is a hyperplane that separates them. In the context of checking separability of linearly constrained states, the convex sets in question are the set of separable states and the affine subspace spanned by all the solutions of the linear system \( (3) \) (see Figure 1). If these two sets are disjoint, it means that no separable state satisfies \( (3) \); on the other hand, the separating hyperplane theorem assures us that there is an entanglement witness that can certify the entanglement of every state of the form \( (4) \).

The dual SDP to \( (4) \) takes the form

\[
\begin{align*}
\text{maximize} & \quad -\text{Tr}[F_0 Z] \\
\text{subject to} & \quad Z \succeq 0 \\
& \quad \text{Tr}[F_0 Z] = 0 \\
& \quad \text{Tr}[Z] = 1.
\end{align*}
\]

In [8] it was shown how to use the solution of the dual SDP to construct an EW. Without going into a detailed derivation we can point out the main elements of the correspondence between the operator \( Z \) and the corresponding EW. First, note that \( F_0 \) lies in a vector space that is the direct sum of three copies of the space \( \mathcal{H}_A \otimes \mathcal{H}_B \), so the operator \( Z \) lies in the same space. From \( (10) \) we can see that \( F_0 \) is a linear function of the matrix \( \rho^{\text{part}} \), so we can write \( F_0 = \Lambda(\rho^{\text{part}}) \) for some linear map \( \Lambda \) that operates on matrices in \( \mathcal{H}_A \otimes \mathcal{H}_B \). If \( \Lambda^* \) is the adjoint map, we can write \( \text{Tr}[F_0Z] = \text{Tr}[\Lambda(\rho^{\text{part}}) Z] = \text{Tr}[\rho^{\text{part}} \Lambda^*(Z)] \), so the objective of the dual SDP is minimizing the expectation value of the operator \( \Lambda^*(Z) \) on the matrix \( \rho^{\text{part}} \). By applying the same line of reasoning, it is not difficult to see that the constraints \( \text{Tr}[F_i Z] = 0 \) in \( (11) \) allows us also to write \( \text{Tr}[F_0 Z] = \text{Tr}[\rho^{\text{part}} + \sum_{\alpha=1}^{D_K} y_{\alpha} \mathbf{1}^{(\alpha)} \Lambda^*(Z)] \). This is the expectation value of \( \Lambda^*(Z) \) on all states compatible with \( (3) \). The other constraints in \( (11) \) can be used to show that \( Z = \Lambda^*(Z) \) is actually positive on all pure product states as required for an EW [8]. The dual SDP can be interpreted as minimizing the expectation value of \( Z \) on \( \rho^{\text{part}} + \sum_{\alpha=1}^{D_K} y_{\alpha} \mathbf{1}^{(\alpha)} \) over a particular subset of EWs. Figure 1 gives a simple pictorial representation of the basis for this technique. The key point in our case, where the state is only partially determined, is that if the affine space defined by \( (11) \) does not intersect the set of separable states (i.e., all such states are entangled), the Hahn-Banach theorem guarantees the existence of an EW that separates the set of separable states from every state in this affine subspace. The dual SDP is used to construct one such EW. Consequently, this approach
FIG. 1: The affine subspace defined by the linear constraints, which can either intersect the set of separable states $\mathcal{S}$ or not. In the former case, there are separable states compatible with the constraints so no conclusion can be drawn about the entanglement of the state. In the latter case, all states compatible with the constraints are entangled, and an entanglement witness $W$ exists that certifies this fact.

is not plagued by the “fake entanglement” problem that can arise when using the maximum entropy method to infer the most probable state associated with $\mathcal{S}$.

V. LOWER BOUNDS ON ENTANGLEMENT MEASURES

In the case where we are able to prove entanglement using the PPTSE criterion, we can use the output of both the primal and dual SDPs to provide lower bounds on certain entanglement measures and other related quantities. Consider the primal problem and let $t_{opt}$ be the optimal value. If $t_{opt} > 0$ then all the states are entangled. But then $d_A^2 d_B t_{opt}$ is a lower bound on the minimum amount of the maximally mixed state we need to add to a state satisfying to make it separable (in $2 \otimes 2$ and $2 \otimes 3$ this bound is tight). This is known as the random robustness of entanglement $R_r(\rho)$, and quantifies how robust the entanglement is against white noise. It also provides a lower bound on a geometric measure of quantum discord.

The entanglement witness constructed from the dual SDP can also be used to quantify the entanglement of the states satisfying $\mathcal{S}$. Any entanglement measure that can be expressed as

$$E(\rho) = \max\{0, -\min_{W \in \mathcal{M}} \text{Tr}[W \rho]\}$$

with $\mathcal{M}$ a subset of entanglement witnesses, is referred to as witnessed entanglement. The set $\mathcal{M}$ determines which particular measure this expression represents. Several well-known measures are of this form, such as the best separable approximation $\text{BSA}(\rho)$, the negativity $N(\rho)$, and the concurrence $C(\rho)$. Clearly, any $W \in \mathcal{M}$ that satisfies $\text{Tr}[W \rho] < 0$ provides a lower bound to $E(\rho)$. In particular, the quantities

$$E_{n,m}(\rho) = \max\{0, -\min_{W \in \mathcal{M}_{n,m}} \text{Tr}[W \rho]\}$$

$(n, m \geq 0)$ with $\mathcal{M}_{n,m} = \{W : -n1 \leq W \leq m1\}$ a subset of entanglement witnesses, are entanglement monotones, and satisfy $E_{n,m}(\rho) \to n\text{BSA}(\rho)$ when $m \to \infty$, where $\text{BSA}(\rho)$ is the best separable approximation to $\rho$. Since $E_{n,m}(\rho)$ is obviously monotonically increasing with $m$ (for fixed $n$) and any entanglement witness $W$ must be in some $\mathcal{M}_{n,m}$, $\text{Tr}[W \rho]$ provides a lower bound on $\text{BSA}(\rho)$, which is an entanglement measure. This analysis is just an illustration of the connection between the PPTSE criterion and entanglement measures, and does not pretend to give the best bounds possible.

VI. EXAMPLE

Let us use a simple example to illustrate the power of this approach. Consider a system that produces two photons and we want to determine if they are entangled in the polarization basis. One possible approach is to do quantum state tomography. This can be accomplished by measuring the 16 observables given by $\hat{\mu}_i \otimes \hat{\mu}_j (i,j = 0, 1, 2, 3)$ with

$$\hat{\mu}_0 = |H\rangle\langle H| + |V\rangle\langle V|$$
$$\hat{\mu}_1 = |H\rangle\langle V|$$
$$\hat{\mu}_2 = |D\rangle\langle D|$$
$$\hat{\mu}_3 = |R\rangle\langle R|$$

with $|D\rangle = (|H\rangle - |V\rangle)/\sqrt{2}$ and $|R\rangle = (|H\rangle - i|V\rangle)/\sqrt{2}$. Note that these operators are all positive on pure product states and so they are good candidates to be entanglement witnesses. Assume that we measure these observable and we obtain

$$0.48 \leq \text{Tr}[(\hat{\mu}_1 \otimes \hat{\mu}_1)\rho] \leq 0.5$$
$$0.24 \leq \text{Tr}[(\hat{\mu}_1 \otimes \hat{\mu}_2)\rho] \leq 0.25$$
$$0.48 \leq \text{Tr}[(\hat{\mu}_2 \otimes \hat{\mu}_2)\rho] \leq 0.5$$
$$0 \leq \text{Tr}[(\hat{\mu}_3 \otimes \hat{\mu}_3)\rho] \leq 0.02.$$  

Note that all expectation values are non-negative, so they cannot show entanglement by themselves. However, applying our test we find that there is an entanglement witness given by

$$Z = 0.1343|HH\rangle\langle HH| + 0.3977|HV\rangle\langle HV| +$$
$$0.234(|VV\rangle\langle VH| + |VV\rangle\langle VV|) +$$
$$+\{(0.0658 + 0.1583)(|HH\rangle\langle VH| + |HV\rangle\langle VV| +$$
$$+|VH\rangle\langle VV|) + h.c.) +$$
$$+\{-0.2242|HH\rangle\langle VV| + 0.0925|HV\rangle\langle VH| + h.c.$$  

(16)
such that \( \text{Tr}[Z\rho] < -0.0168 \) for all states satisfying the constraints \([15]\). Moreover, \( Z \in \mathcal{M}_{4,1} \), so this result provides a lower bound on the best separable approximation, i.e., \( BSA(\rho) \geq 1.68 \times 10^{-2} \). The primal SDP also computes a lower bound on the random robustness, \( R_r(\rho) \geq 8 \times 0.0168 = 0.1344 \). Additional information about the state can improve these bounds. For example, if we add \( \text{Tr}[(\hat{\mu}_1 \otimes \hat{\mu}_2)\rho] \in [0.24,0.25] \) to the constraints we now obtain a new entanglement witness \( Z' \) such that \( \text{Tr}[Z'\rho] < -0.021 \), which translates to \( BSA(\rho) \geq 2.1 \times 10^{-2} \), and \( R_r(\rho) \geq 0.168 \) (the MATLAB code used is available online from the author [14]).

VII. SOME IMPORTANT FEATURES OF THE APPROACH

Having described the idea behind this method for detecting entanglement from partial state information, we can now shift our attention to more general features regarding its usefulness and limitations. First, we want to stress an important feature of this technique: independently of the number of constraints available, if we are allowed enough computational resources we are guaranteed to arrive at a definite answer to the question “Are all states satisfying these constraints entangled, or is there at least one such state that is separable?” This is accomplished by applying successive steps in the PPTSE hierarchy and its dual. Furthermore, if the answer is that all such states are entangled, this affirmation is free of the “fake entanglement” issue that appears in maximum entropy inference approaches.

The second question that arises is whether there is a clear correlation between the number of constraints (i.e., the amount of information about the state) and the number of steps in the PPTSE hierarchy and its dual we need to apply. One could naively expect that if the number of constraints is very small compared to the number of parameters in the density matrix, there would have to be some separable state that satisfies them. However, it is easy to see that this is not the case: if the partial information is the expectation value of a single observable that happens to be an entanglement witness, and its value is negative, we know for sure that the state is entangled.

Let us assume that a set of constraints is shown to be compatible with a separable state. One may consider the question of which extra observables we should add to further constraint the state and determine its entanglement. Without extra assumptions, this question does not have a definite answer: if the unknown state is actually separable, no matter which observables we choose, their expectation values will always be compatible with a separable state. On the other hand, if a set of constraints is sufficient to prove entanglement and provides lower bounds on some entanglement measures, an extra constraint may be useful to improve that bound and so it may be worth performing that extra measurement. If for some reason we had some additional information, for example, the state is being drawn from some distribution, we may use it to find an observable that maximizes the probability of detecting the entanglement (if it exists). But the approach considered in this paper is aimed precisely at the situation where we do not have any more information than the one provided by the linear constraints associated with the measured observables.

From this discussion we can start to form a picture of when this approach would fail in practice, i.e., it would not produce a definite answer after a reasonable number of steps. If we look at the PPTSE part of the method (proving entanglement), we can see that if the affine space defined by \( \mathcal{S}_p \) intersects sets \( \mathcal{S}_N^p \) with \( N \) large \( (\mathcal{S}_N^p \) is the set of states with PPT symmetric extensions to \( \tilde{N} \) copies of \( A \)), even if the states are actually entangled it will take at least \( N \) steps to prove this fact. On the other hand, if we consider the dual approach and \( \mathcal{S}_p \) intersects the set of separable states but does not intersect \( \tilde{S}_N^p \) for \( N \) small, the procedure will take a long time to provide the required separable decomposition. Since \( \mathcal{S}_N^p \) and \( \tilde{S}_N^p \) approach \( S \) from the outside and the inside respectively, the most challenging situation for this procedure occurs when \( \mathcal{S}_p \) is “almost” tangent to the set of separable states, either intersecting it or not. For a given instance, if this approach fails to provide an answer in a reasonable number of steps, adding extra constraints may be of some help (although this is not guaranteed in general).

VIII. CONCLUSIONS

In this article we have introduced a sequence of tests that can determine the entanglement or separability of a state when only partial information is available. Using a set of linear constraints on the density matrix, such as the ones associated with the expectation values of a set of observables, we can apply the PPTSE separability criterion and its dual to determine whether all the states satisfying these constraints are entangled or if there is one such state that is separable. When entanglement is proven by this method, the algorithm constructs an entanglement witness that can be shown to certify the entanglement of all states that satisfy the constraints. On the other hand, when a separable state is found that is compatible with the constraints, a separable decomposition is also constructed to prove this fact.

Even though this approach is technically very similar to the original PPTSE criterion, its range of applicability is radically different. The original PPTSE criterion requires as input the complete density matrix of the state that we want to analyze. This is very useful for theoretical considerations, when the state is explicitly constructed to accomplish some particular task (such as some communication protocol or a particular scheme for quantum key distribution), but it is not as helpful when the state comes from actual experimental measurements on a physical system. Before applying the PPTSE crite-
rion in this case the state must be reconstructed using a procedure like quantum state tomography, but this has the disadvantage of typically requiring a large number of measurements and it is not even guaranteed to provide a consistent answer. In contrast, the sequence of tests in this paper can be applied directly to experimental data, and in the case where entanglement if proven it also provides lower bounds on entanglement measures and other related quantities.

The method introduced here avoids performing quantum state tomography and analyzes what can be said about the entanglement of the state using only the partial information provided. When this information is in the form of a set of linear constraints on the elements of the density matrix, the state in question belongs to an affine subspace in the space of density matrices. If this affine subspace does not intersect the set of separable states (see Figure 1), the Hahn-Banach theorem guarantees the existence of a separating hyperplane that is associated with an entanglement witness that certifies entanglement for all states in such an affine subspace. The dual SDP of the criterion presented here can be interpreted as a search for such a separating hyperplane over a restricted set of entanglement witnesses. The completeness of the PPTSE criterion can be extended to this case to guarantee that if all states compatible with the constraints are entangled, such an entanglement witness will be found. On the other hand, if the subspace intersects the set of separable states the dual test will eventually provide a state in that intersection. These two central features imply that it is not possible for the method to certify entanglement if there is a single separable state that satisfies the constraints. Thus, this approach is free from the “fake entanglement” issue common to maximum entropy inference based methods. Given that experimental data can be used as input to these tests with basically no preprocessing, this technique could be a very useful and practical tool for experimentally certifying entanglement of real physical systems.

IX. ACKNOWLEDGMENTS

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