Brane-bulk energy exchange and cosmological acceleration

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Abstract: The consequences for the brane cosmological evolution of energy exchange between the brane and the bulk are analyzed. A rich variety of brane cosmologies is obtained, depending on the precise mechanism of energy transfer, the equation of state of brane-matter and the spatial topology. An accelerating era is generically a feature of the solutions.

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1. Introduction

My review lecture has focused on some recent progress in the physics of brane-worlds. Both model building and cosmological implications were considered. Since there is already a published review on such issues \[\text{[1]},\] and since space here is scarce, I have decided to develop a particular aspect of brane cosmology, which is both important and little studied, namely that of brane-bulk energy exchange. I will review recent progress and present some new results. There has been some study of cosmological aspects of brane-bulk energy exchange in the recent past \[\text{[2]}-\text{[7]}\]. Here, I will mostly focus on the approach described in \[\text{[6]}\].

In any theory, where gravity is higher-dimensional classically (around flat space), a mechanism must be invoked to explain why observable gravity is four-dimensional at experimentally verifiable distances. The easiest and most popular mechanism is compactification, but alternatives have been also considered lately, namely RS localisation (4d in the IR) \[\text{[8]}\] and brane-induced gravity (4d in the UV). In all such realizations, the 4d graviton is accompanied by KK modes that propagate in the bulk and can thus mediate energy-exchanging processes between brane and bulk matter. Such processes are important because on the one hand they provide stringent
constraints on brane-world models and on the other as it was argued in \cite{kkttz} they may generate interesting cosmological effects.

In \cite{ktt} the cosmological brane-bulk energy exchange processes, due to KK gravitons were studied in the simplest RS cosmological evolution supplemented with a four-dimensional (induced) Einstein term on the brane. It was shown that, in agreement with intuition, such processes do not affect the cosmological evolution during 4d eras ($H \sim \sqrt{\rho}$), while they are important in higher-dimensional eras ($H \sim \rho$). Moreover, in many important cases, the rate of energy loss can be written as a power of the driving brane-energy density, something that can be used in more general situations.

To generalize, we consider more bulk fields and more general bulk-brane couplings. Although this can be formulated by the standard action principle and the relevant exact equations derived and studied we will take a short-cut. We will analyze the regime where the bulk energy, at the brane position, can be consistently neglected from the equations. Moreover, we parameterize appropriately the energy exchange as a specific power of the matter density of the brane.

The effects found, include accelerated evolution with standard $w = 0$ matter and outflow, inflationary fixed points in the presence of inflow, and tracking behavior between observable and mirage (dark) energy.

2. The model

We shall be interested in the model described by the action

$$S = \int d^5 x \sqrt{-g} \left( M^3 R - \Lambda + \mathcal{L}_{B}^{\text{mat}} \right) + \int d^4 x \sqrt{-\hat{g}} \left( -V + \mathcal{L}_b^{\text{mat}} \right), \quad (2.1)$$

where $R$ is the curvature scalar of the five-dimensional metric $g_{AB}$, $A, B = 0, 1, 2, 3, 5$, $\Lambda$ is the bulk cosmological constant, and $\hat{g}_{\alpha\beta}$, with $\alpha, \beta = 0, 1, 2, 3$, is the induced metric on the 3-brane. We identify $(x, z)$ with $(x, -z)$, where $z \equiv x_5$. However, following the conventions of \cite{rs} we extend the bulk integration over the entire interval $(-\infty, \infty)$. The quantity $V$ includes the brane tension as well as quantum contributions to the four-dimensional cosmological constant.

We consider an ansatz for the metric of the form

$$ds^2 = -n^2(t, z) dt^2 + a^2(t, z) \gamma_{ij} dx^i dx^j + b^2(t, z) dz^2, \quad (2.2)$$

where $\gamma_{ij}$ is a maximally symmetric 3-dimensional metric. We use $\tilde{k}$ to parameterize the spatial curvature.

The non-zero components of the five-dimensional Einstein tensor are

$$G_{00} = 3 \left\{ \frac{\dot{a}}{a} \left( \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right) - \frac{n^2}{b^2} \left( \frac{a''}{a} + \frac{a'}{a} \left( \frac{a'}{a} - \frac{b'}{b} \right) \right) + \tilde{k} \frac{n^2}{a^2} \right\}, \quad (2.3)$$
\[
G_{ij} = \frac{a^2}{b^2} \gamma_{ij} \left\{ \frac{a'}{a} \left( \frac{a'}{a} + \frac{n'}{n} \right) - \frac{b'}{b} \left( \frac{n'}{n} + \frac{2a'}{a} \right) + \frac{2a''}{a} + \frac{n''}{n} \right\} \\
+ \frac{a^2}{n^2} \gamma_{ij} \left\{ \frac{\dot{a}}{a} \left( -\frac{\dot{a}}{a} + \frac{\dot{n}}{n} \right) - \frac{\ddot{a}}{a} + \frac{\dot{b}}{b} \left( -\frac{2\dot{a}}{a} + \frac{\dot{n}}{n} \right) - \frac{\ddot{b}}{b} \right\} - \tilde{k} \gamma_{ij}, \tag{2.4} \]
\]

\[
G_{05} = 3 \left( \frac{n' a'}{n a} + \frac{a' b'}{a b} - \frac{\dot{a}'}{a} \right), \tag{2.5} \]

\[
G_{55} = 3 \left( \frac{a'}{a} \left( \frac{a'}{a} + \frac{n'}{n} \right) - \frac{b^2}{n^2} \left( \frac{\dot{a}}{a} \left( \frac{\dot{a}}{a} - \frac{\dot{n}}{n} \right) + \frac{\ddot{a}}{a} \right) - \frac{\tilde{k} b^2}{a^2} \right). \tag{2.6} \]

Primes indicate derivatives with respect to \(z\), while dots derivatives with respect to \(t\).

The five-dimensional Einstein equations take the usual form

\[
G_{AC} = \frac{1}{2M^3} T_{AC}, \tag{2.7} \]

where \(T_{AC}\) denotes the total energy-momentum tensor.

Assuming a perfect fluid on the brane and, possibly an additional energy-momentum \(T^A_{|m,B}\) in the bulk, we write

\[
T^A_C = T^A_C|_{v,b} + T^A_C|_{m,b} + T^A_C|_{v,B} + T^A_C|_{m,B} \tag{2.8} \]

\[
T^A_C|_{v,b} = \frac{\delta(z)}{b} \text{diag}(-V,-V,-V,-V,0) \tag{2.9} \]

\[
T^A_C|_{v,B} = \text{diag}(-\Lambda,-\Lambda,-\Lambda,-\Lambda,-\Lambda) \tag{2.10} \]

\[
T^A_C|_{m,b} = \frac{\delta(z)}{b} \text{diag}(-\tilde{\rho},\tilde{p},\tilde{p},\tilde{p},0), \tag{2.11} \]

where \(\tilde{\rho}\) and \(\tilde{p}\) are the energy density and pressure on the brane, respectively. The behavior of \(T^A_{C|m,B}\) is in general complicated in the presence of flows, but we will not specify it further at this point.

We wish to solve the Einstein equations at the location of the brane following \cite{10}. We indicate by the subscript \(o\) the value of various quantities on the brane. Integrating equations (2.3), (2.4) with respect to \(z\) around \(z = 0\) gives the known jump conditions

\[
a''_o = -a'_o = -\frac{1}{12M^3} b_o a_o (V + \tilde{\rho}) \tag{2.12} \]

\[
n''_o = -n'_o = \frac{1}{12M^3} b_o n_o (-V + 2\tilde{\rho} + 3\tilde{p}). \tag{2.13} \]

The other two Einstein equations (2.5), (2.6) give

\[
\frac{n'_o a_o}{a_o} + \frac{a'_o b_o}{a_o b_o} - \frac{\dot{a}'_o}{a_o} = \frac{1}{6M^3} T_{o5} \tag{2.14} \]
where $T_{05}, T_{55}$ are the 05 and 55 components of $T_{AC}|_{m,B}$ evaluated on the brane. Substituting (2.12), (2.13) in equations (2.14), (2.15) one obtains

$$\dot{\rho} + 3 \frac{\dot{a}}{a} (\tilde{\rho} + \tilde{\rho}) = - \frac{2n_o^2}{b_o} T_5^0. \quad (2.16)$$

$$\frac{1}{n_o^2} \left( \frac{\ddot{a}}{a_o} + \left( \frac{\dot{a}}{a_o} \right)^2 - \frac{\dot{a}}{a_o} \frac{n_o}{a_o} n_o \right) + \frac{k}{a_o^2} = \frac{1}{6M^3} \left( \Lambda + \frac{1}{12M^3} V^2 \right)$$

$$- \frac{1}{144M^6} (V(3\tilde{\rho} - \tilde{\rho}) + \tilde{\rho}(3\tilde{\rho} + \rho)) - \frac{1}{6M^3} T_5^5. \quad (2.17)$$

In the model that reduces to the Randall-Sundrum vacuum [8] in the absence of matter, the first term on the right hand side of equation (2.17) vanishes. A new scale $K$ may be defined through the relations $V = -\Lambda/K = 12M^3K$. Sometimes, we will relax this condition.

At this point we find it convenient to employ a coordinate frame in which $b_o = n_o = 1$ in the above equations. This can be achieved by using Gauss normal coordinates with $b(t, z) = 1$, and by going to the temporal gauge on the brane with $n_o = 1$. The assumptions for the form of the energy-momentum tensor are then specific to this frame. Using $\beta \equiv M^{-6}/144$ and $\gamma \equiv VM^{-6}/144$, and omitting the subscript o for convenience in the following, we rewrite equations (2.16) and (2.17) in the equivalent first order form

$$\dot{\rho} + 3(1 + w) \frac{\dot{a}}{a} \tilde{\rho} = -\tilde{T} \quad , \quad \frac{\dot{a}}{a^2} = \beta \tilde{\rho}^2 + 2\gamma (\tilde{\rho} + \tilde{\chi}) - \frac{k}{a^2} + \tilde{\lambda} \quad (2.18)$$

$$\dot{\tilde{\chi}} + 4 \frac{\dot{a}}{a} \tilde{\chi} = \left( \frac{\beta}{\gamma} \tilde{\rho} + 1 \right) \tilde{T} - \frac{1}{6\gamma M^3 a} \frac{\dot{a}}{a} T_5^5, \quad (2.19)$$

where $\tilde{\rho} = w\tilde{\rho}, \tilde{T} = 2T_5^0$ is the discontinuity of the zero-five component of the bulk energy-momentum tensor, and $\tilde{\lambda} = (\Lambda + V^2/12M^3)/12M^3$ the effective cosmological constant on the brane.

In the equations above, Eq. (2.18) is the definition of the auxiliary density $\tilde{\chi}$. With this definition, the other two equations are equivalent to the original system (2.16) (2.17). As we will see later on, in the special case of no-exchange ($\tilde{T} = 0$) $\tilde{\chi}$ represents the mirage radiation reflecting the non-zero Weyl tensor of the bulk.

The second order equation (2.17) for the scale factor becomes

$$\frac{\ddot{a}}{a} = -(2 + 3w)\beta \tilde{\rho}^2 - (1 + 3w)\gamma \tilde{\rho} - 2\gamma \tilde{\chi} + \tilde{\lambda}. \quad (2.20)$$

As mentioned above, in the Randall-Sundrum model the effective cosmological constant $\tilde{\lambda}$ vanishes, and this is the value we shall assume for most of the rest.
We may now pass to dimensionless quantities. The AdS scale is defined as \( V = M^3 K \) and the dimensionless densities as

\[
\rho = \frac{\tilde{\rho}}{72 M^3 K}, \quad \chi = \frac{\tilde{\chi}}{72 M^3 K}, \quad \lambda = \frac{\tilde{\lambda}}{K^2} = \frac{\Lambda}{12 M^3 K^2} + \frac{1}{144} \quad (2.21)
\]

\[
T = \frac{\tilde{T}}{72 M^3 K^2}, \quad T_5 = \frac{T_5}{3 M^3 K^2} \quad (2.22)
\]

as well as a rescaled time \( \tau = K t \). The combination \( k = \tilde{k}/K^2 \) is then dimensionless and by scaling \( a \) we can set it to \( k = 0, \pm 1 \). Then the cosmological equations become

\[
\dot{\rho} + 3(1 + w) \frac{\dot{a}}{a} \rho = -T, \quad \frac{\dot{a}}{a^2} = 36 \rho^2 + \rho + \chi - \frac{k}{a^2} + \lambda \quad (2.23) \tag{rho1}
\]

\[
\dot{\chi} + 4 \frac{\dot{a}}{a} \chi = (72 \rho + 1) T - \frac{\dot{a}}{a} T_5, \quad q \equiv \frac{\dot{a}}{a} = -36(2 + 3w) \rho^2 - \frac{3w + 1}{2} \rho - \chi - \frac{1}{2} T_5 + \lambda \quad (2.24) \tag{qq}
\]

where now dots stand for derivatives with respect to \( \tau \).

We may separate the dynamics associated with \( T_0^5 \) and \( T_5^5 \) by introducing a new energy density \( \sigma \) so that

\[
\frac{\dot{a}}{a^2} = 36 \rho^2 + \rho + \chi + \sigma - \frac{k}{a^2} + \lambda, \quad \dot{\chi} + 4 \frac{\dot{a}}{a} \chi = (72 \rho + 1) T, \quad \dot{\sigma} + 4 \frac{\dot{a}}{a} \sigma = -\frac{\dot{a}}{a} T_5 \quad (2.25)
\]

In order to derive a solution that is largely independent of the bulk dynamics, the \( T_5^5 \) term on the right hand side of the same equation must be negligible relative to the second one. This is possible if we assume that the diagonal elements of the various contributions to the energy-momentum tensor satisfy the schematic inequality

\[
\frac{|T|_{\text{diag}}}{|T|_{\text{off-diag}}} \ll \frac{|T|_{\text{diag}}}{|T|_{\text{off-diag}}} \quad (2.26) \tag{vacdom}
\]

Our assumption is that the bulk matter relative to the bulk vacuum energy is much less important than the brane matter relative to the brane vacuum energy. In this case the bulk is largely unperturbed by the exchange of energy with the brane. When the off-diagonal term \( T_0^5 \) is of the same order of magnitude or smaller than the diagonal ones, the inequality \( (2.26) \) implies \( T \ll \tilde{\rho} K \).

Thus, under this assumption, we may neglect \( T_5 \) from \( (2.24) \). Moreover, we will parameterize \( T \) as a function of the driving energy density \( \rho \). The precise form depends on the details of the interaction between brane and bulk. In the AdS scaling region this has typically a power dependence \( T \sim \rho^\nu \).

2.1 Other forms of the cosmological equations
Combining equations (2.23) we obtain

\[
\frac{d\rho}{da} = -3(1 + w)\rho - \varepsilon T(\rho) \left(36\rho^2 + \rho + \chi - \frac{k}{a^2} + \lambda \right)^{-1/2}.
\] (2.27)  

Similarly, equations (2.23) and (2.24) give

\[
\frac{d\chi}{da} = -4\chi + \varepsilon (36\rho + 1)T(\rho) \left(36\rho^2 + \rho + \chi - \frac{k}{a^2} + \lambda \right)^{-1/2},
\] (2.28)  

where \(\varepsilon = 1\) refers to expansion, while \(\varepsilon = -1\) to contraction. These two equations form a two-dimensional dynamical system. The function \(\chi(\rho)\) is obtained from the equation

\[
\left(3(1 + w)\rho \sqrt{36\rho^2 + \rho + \chi - \frac{k}{a^2} + \lambda + \varepsilon T(\rho)} \right) \frac{d\chi}{d\rho} = 4\chi \sqrt{36\rho^2 + \rho + \chi - \frac{k}{a^2} + \lambda - \varepsilon (36\rho + 1)T(\rho)}. \] (2.29)  

Note that the equations of contraction are those of expansion with the roles of outflow and influx interchanged.

For \(k = 0\), \(\hat{T} = \rho^{-3/2}T\) and \(\zeta = \sqrt{36\rho + \frac{k}{\rho} + 1 + \frac{\lambda}{\rho}}\) the equation can be simplified

\[
2\rho\zeta\zeta' = \frac{(1 - 3w)(\zeta^3 - \zeta) + 72(1 + 3w)\rho\zeta - \hat{T}\zeta^2 + 36\rho\hat{T} - 4\frac{\lambda\zeta}{\rho}}{3(1 + w)\zeta + \hat{T}} \] (2.30)

If we further define

\[
\zeta = \rho^{\frac{1 + 3w}{3(1 + w)}}\xi
\] (2.31)

we obtain

\[
2\rho\xi\xi' = -\left(\frac{\hat{T}}{3(1 + w)}\right) \rho^{\frac{1 + 3w}{3(1 + w)}}\xi^2 + \left[3w - 1 + 72(1 + 3w)\rho - 4\frac{\lambda}{\rho}\right] \rho^{\frac{1 + 3w}{3(1 + w)}}\xi + 36\rho\hat{T}
\] (2.32)

Finally the acceleration satisfies

\[
-\frac{dq}{d\rho} = \frac{72(1 + 3w)(2 + 3w)\rho^2 + \frac{9w^2 - 1}{2}\rho + 4\lambda - 4q}{3(1 + w)\rho R + \hat{T}} \bigg[72(1 + 3w)(2 + 3w)\rho^2 + \frac{9w^2 - 1}{2}\rho + 4\lambda - 4q\bigg] R + 108(2w + 1)\rho T + \frac{3w - 1}{2}T
\] (2.33)

where

\[
R = \sqrt{\frac{1 - 3w}{2}\rho - 36(1 + 3w)\rho^2 + 2\lambda - q - \frac{k}{a^2}}
\] (2.34)
2.2 The four-dimensional regime

When $\rho << 1$ we are in the 4d regime ($H^2 \sim \rho$). Here the cosmological equations simplify

\[
\dot{\rho} + 3(1 + w) \frac{\dot{a}}{a} \rho = -T \quad , \quad \frac{\dot{a}^2}{a^2} = \rho + \chi - \frac{k}{a^2} + \lambda \tag{2.35}
\]

\[
\dot{\chi} + 4 \frac{\dot{a}}{a} \chi = T \quad , \quad q = \frac{\ddot{a}}{a} = -\frac{3w + 1}{2} \rho - \chi + \lambda \tag{2.36}
\]

For $k = 0$, $\hat{T} = \rho^{-3/2} T$ and $\zeta = \sqrt{\frac{\chi}{\rho} + 1 + \frac{\lambda}{\rho}}$. Then

\[
2\rho \zeta' = \frac{(1 - 3w)(\zeta^2 - 1) - \hat{T} \zeta - 4\frac{\lambda}{\rho}}{3(1+w)\zeta + \hat{T}} \tag{2.37}
\]

while (2.35) becomes

\[
a \frac{d \log \rho}{da} = -3(1 + w) - \frac{\hat{T}}{\zeta} \tag{2.38}
\]

3. Inflating fixed points

An interesting feature of the cosmological equations is the possible presence of accelerating cosmological solutions. We may look for exponential expansion with a constant Hubble parameter $H$, even if the brane content is not pure vacuum energy. We will restrict ourselves for simplicity to the 4d regime. For the non-linear analysis we refer the reader to [6].

For a fixed point equations (2.35), (2.36) must have a time-independent solution, without necessarily requiring $w = -1$ ($\lambda = k = 0$). The possible fixed points (denoted by $*$) of these equations for $k = 0$ satisfy

\[
3H_*(1 + w)\rho_* = -T(\rho_*) \quad , \quad H_*^2 = \rho_* + \chi_* \quad , \quad 4H_* \chi_* = T(\rho_*) \tag{3.1}
\]

It is clear from equation (3.1) that, for positive matter density on the brane ($\rho > 0$), flow of energy into the brane ($T(\rho) < 0$) is necessary.

The accretion of energy from the bulk depends on the dynamical mechanism that localizes particles on the brane. Its details are outside the scope of our discussion. However, it is not difficult to imagine scenarios that would lead to accretion. If the brane initially has very low energy density, energy can be transferred onto it by bulk particles such as gravitons. An equilibrium is expected to set in if the brane energy density reaches a limiting value. As a result, a physically motivated behavior for the function $T(\rho)$ is to be negative for small $\rho$ and cross zero towards positive values for larger densities. In the case of accretion it is also natural to expect that the energy transfer approaches a negative constant value for $\rho \to 0$. 

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The solution of equations (3.1) satisfies
\[ T(\rho^*) = -\frac{3}{2}(1 + w)\sqrt{1 - 3w} \rho_*^{3/2} , \quad H_*^2 = \frac{1 - 3w}{4} \rho_* , \quad \chi_* = -\frac{3(1 + w)}{4} \rho \] (3.2)

For a general form of \( T(\rho) \) equation (3.2) is an algebraic equation with a discrete number of roots. For any value of \( w \) in the region \(-1 < w < 1/3\) a solution is possible. The corresponding cosmological model has a scale factor that grows exponentially with time. The energy density on the brane remains constant due to the energy flow from the bulk. This is very similar to the steady state model of cosmology [11]. The main differences are that the energy density is not spontaneously generated, and the Hubble parameter receives an additional contribution from the “mirage” field \( \chi \) (see equation (3.1)).

The stability of the fixed point (3.1) determines whether the exponentially expanding solution is an attractor of neighboring cosmological flows. If we consider a small homogeneous perturbation around the fixed point \( (\rho = \rho_* + \delta \rho, \chi = \chi_* + \delta \chi) \) we find that \( \delta \rho, \delta \chi \) satisfy
\[ \frac{d}{dt} \begin{pmatrix} \delta \rho \\ \delta \chi \end{pmatrix} = \frac{T(\rho_*)}{\rho_*} M \begin{pmatrix} \delta \rho \\ \delta \chi \end{pmatrix}, \quad (3.3) \]

where
\[ M = \begin{pmatrix} -\nu + 3(1 - w)/(1 - 3w) & 2(1 - 3w) \\ \nu - 2/(1 - 3w) & -2(1 + 9w)/[3(1 + w)(1 - 3w)] \end{pmatrix} \] (3.4)
\[ \nu = \frac{d \ln |T|}{d \ln \rho}(\rho_*), \quad (3.5) \]

and we have employed the relations (3.1) and \( T(\rho) \propto \rho^\nu \). The eigenvalues of the matrix \( M \) are
\[ M_{1,2} = \frac{7 + 3w - 3\nu(1 + w) \pm \sqrt{24(-3 + 2\nu)(1 + w)[7 + 3w - 3\nu(1 + w)]^2}}{6(1 + w)}. \]

For \(-1 < w < 1/3, 0 \leq \nu < 3/2\) they both have a positive real part. As we have assumed \( T(\rho) < 0 \), the fixed point is stable in this case. The approach to the fixed-point values depends on the sign of the quantity under the square root. If this is negative the energy density oscillates with diminishing amplitude around its fixed-point value.

4. Tracking solutions

We will now analyze the case \( \nu = 3/2 \) which lies at the boundary of the stability region discussed above. We will thus assume that \( T = A \rho^{3/2} \), and that the universe
expands and is dominated by non-relativistic matter (w=0). Then, in the 4d regime, equation (2.37) becomes (λ = 0)

\[ 2 \rho \zeta' = \frac{\zeta^2 - A \zeta - 1}{3 \zeta + A} \] (4.1)

We will parameterize the dimension-less coefficient \( A \) as \( A = \mu - \frac{1}{\mu} \). \( A \) is determined by the details of the microscopic cross section that gives rise to this type of energy exchange. \( \mu \) running on non-negative real numbers parameterizes all possible values of \( A \). When we have expansion, \( A > 0 \) means outflow. When we have contraction \( A < 0 \) means outflow.

The general solution of equation (4.1) is

\[ (\zeta - \mu)^{2 \mu + 8 \mu} \left( \zeta + \frac{1}{\mu} \right)^{-2 \mu + \frac{8}{\mu}} = C' \rho^{\mu + \frac{1}{\mu}} \] (4.2)

Since \( H^2 = \rho + \chi \) the equation above can be re-written, in terms of \( \zeta = H/\sqrt{\rho} \). Then equation (4.1) becomes

\[ \frac{a \, d\rho}{\rho \, da} = -\frac{3 \zeta + A}{\zeta} \] (4.3)

and can be integrated as a function of \( a \) with the result

\[ (\zeta - \mu)^{2 \mu^2} \left( \zeta + \frac{1}{\mu} \right)^{2} = C' a^{-(\mu^2 + 1)} \] (4.4)

\( \rho(a) \) can be obtained by solving (4.4) and substituting into (4.3). Finally \( \chi(a) = \rho(\zeta^2 - 1) \).

We will first study a few special cases:

(i) \( \mu = 1 \). Here we obtain

\[ \zeta^2 = 1 + C \rho^{1/3} \Rightarrow H^2 = \rho + C \rho^{4/3} \quad , \quad \rho = \frac{C^{\epsilon_3}}{C^3} \frac{1}{a^3} \quad , \quad \chi = \frac{C^{\epsilon_4}}{C^4} \frac{1}{a^4} \] (4.5)

compatible with the absence of energy exchange in this case and consequent independence of the evolution of \( \rho, \chi \).

(ii) \( \mu = -1/2 \).

\[ \zeta = 2 + C \rho^{1/2} \Rightarrow H^2 = \rho(2 + C \rho^{1/6})^2 \] (4.6)

For large \( a \)

\[ \rho = \left( \frac{2C'}{5C^2} \right)^{3/2} a^{-15/4} + \mathcal{O} \left( a^{-15/2} \right) \] (4.7)

There is a similar solution for \( \mu = 1/2 \) with asymptotic contraction. The cases \( \mu = \pm 2 \) are equivalent to the above.
(iii) $1/2 < \mu$. It corresponds to $-\frac{3}{2} < A$. Asymptotically ($a \to \infty$) we obtain the tracking solution

$$\zeta = \mu + \tilde{C} \rho^{\frac{\mu^2+1}{4(\mu^2-1)}} + \ldots, \quad H^2 = \mu^2 \rho + 2 \mu \tilde{C} \rho^{\frac{\mu^2+1}{4(\mu^2-1)}} + \ldots$$

(4.8)

$$\rho \sim \tilde{C}' a^{\frac{1}{\mu^2-4}} + \ldots, \quad \chi = (\mu^2 - 1) \rho + \ldots$$

(4.9)

Here, although the initial conditions for the real $\rho$ and mirage $\chi$ energy density are arbitrarily different (parameterized by the independent integration constants $C, C'$), at late times the scale similarly with the scale factor

$$\rho \sim \tilde{C}' a^{\frac{1}{\mu^2-4}} + \ldots, \quad \chi = (\mu^2 - 1) \rho + \ldots$$

(4.10)

Thus, the dark energy behaves as the visible energy, and such a mechanism could be used so that bulk energy simulates dark matter.

(iv) $-2 < \mu < 0$ It corresponds to $-\frac{3}{2} < A$. For asymptotically small $\rho$ we obtain

$$\zeta = \frac{1}{|\mu|} + \tilde{C} \rho^{\frac{\mu^2+1}{4(\mu^2-1)}} + \ldots, \quad H^2 = \frac{\rho}{\mu^2} + 2 \tilde{C} \rho^{\frac{\mu^2+1}{4(\mu^2-1)}} + \ldots$$

(4.11)

On the other hand

$$\rho \sim \tilde{C}' a^{\frac{1}{\mu^2-4}} + \ldots, \quad \chi = \frac{1 - \mu^2}{\mu^2} \rho + \ldots$$

(4.12)

Here we have again tracking behavior.

All other ranges have asymptotic $\zeta$ which is negative and thus unphysical.

Finally, in the case of outflow, there is a fixed point in the 5d regime, when $A^2 < 9/4$ with

$$\rho_* = \frac{9 - 4A^2}{8.81}, \quad H_* = \frac{A}{108} \sqrt{\frac{9 - 4A^2}{2}}$$

(4.13)

This is a saddle point

It is interesting to note that a similar tracking behavior has been observed in matter interacting with the dilaton in $[\text{ref}].$

5. Fixed points in the non-linear regime

We will consider solutions to the non-linear system \(2.23\), \(2.24\) with $H = H_*$ constant. The equations also imply that $\rho = \rho_*, T = T_*, \chi = \chi_*$ are also constant. We will see that although there may be a leftover cosmological constant $\lambda$ on the brane, the cosmological acceleration because of energy inflow, may be much smaller than $\sqrt{\lambda}$.

From the equations we obtain

$$\rho_*^+ = \frac{1}{144(1+3w)} \left[ (1 - 3w) \pm \sqrt{(1 - 3w)^2 + 1152(1 + 3w)(\lambda - H_*^2)} \right]$$

(5.1)
\[ \chi_* = -\frac{3}{4}(1 + w)\rho_* [72\rho_* + 1] , \quad T_* = -3(1 + w)H_*\rho_* , \quad q_* = H_*^2 \quad (5.2) \]

Assuming the rate of expansion \( H_* \) to be small compared to the cosmological constant \( \lambda \), we have the following two possibilities

(i) \( \lambda \) dominates in the square root. In this case \( \rho_* \simeq \sqrt{\frac{\lambda}{18(1+3w)}} \). There is still space for this approximation to be correct and \( \rho_* \ll 1 \) so that we are in the 4d period.

(ii) In the opposite case \( \rho_* \simeq \frac{1-3w}{72(1+3w)} \) and we can be either in the 4d or the 5d regime.

In either case, energy exchange can mask a leftover brane cosmological constant

6. Other accelerating solutions

We will present here two different families of solutions that are characteristic in their classes.

\[ \frac{\ddot{a}}{a} \]

\[ \rho \]

**Figure 1:** Outflow, \( k = 0, \ w = 0, \ \nu = 1 \). The arrows show the direction of increasing scale factor

A global phase portrait of \( q \equiv \ddot{a}/a \) with respect to \( \rho \) during expansion in the outflow case for \( k = 0, \ w = 0, \ \nu = 1 \) is shown in Figure 1. All solutions are below the limiting parabola \( q < \frac{1-3w}{2}\rho - 36(1+3w)\rho^2 \).

One recognizes two families of solutions: The first have \( q < 0 \) for all values of \( \rho \), while the second start with a deceleration era for large \( \rho \), enter an acceleration era and then return to deceleration for small enough values of \( \rho \).

Solutions corresponding to initial conditions with positive \( q \) (always under the limiting parabola shown with the dotted line), necessarily had a deceleration era in the past, and are going to end with an eternal deceleration era also. The straight dashed line represents the standard FRW solution without the effects of energy exchange.
The global phase portrait of $q \equiv \ddot{a}/a$ with respect to $\rho$ during expansion for the case $k = 0, w = 0, \nu = 1$ is shown in Figure 2. The presence of the limiting parabola as in the outflow case is apparent. However, new characteristics appear. For example, $\rho_*^{(-)}$ attracts to eternal acceleration a whole family of solutions which start their evolution at either very low or very high densities. There is another family of solutions which are attracted to acceleration by $\rho_*^{(+)}$ and which eventually exit to a deceleration era. Finally, there is a family of solutions, near the limiting parabola, which start with acceleration at very low densities, and eventually exit to eternal deceleration, while their density increases monotonically with time because of the influx.

Figure 2: Influx, $k = 0, w = 0, \nu = 1$.

For $\nu \neq 1$ one expects a different set of fixed points with varying behaviors around them.

7. Conclusions

An approximate phenomenological analysis of the cosmological effects of brane-bulk energy exchange indicates the presence of several interesting phenomena ranging from acceleration (and its exit) from energy outflow, to inflationary fixed points in the case of inflow to tracking solutions.

This makes imperative the detailed study of the associated cosmological dynamics. In particular the role of the "small-back-reaction approximation" must be understood. It is expected that this is ok at late times in the evolution but it breaking at large densities should be quantified.

A particularly interesting direction is the holographic formulation of this problem. This should correspond to the observable matter, interacting directly to classical four-dimensional gravity as well as to a "hidden" four-dimensional gauge theory (a perturbation of N=4 super Yang-Mills). Brane-bulk energy exchange corresponds
to the interaction between observable matter and "hidden" matter. A quantitative study should clarify several aspects of the problem.

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References

[1] E. Kiritsis, “D-branes in standard model building, gravity and cosmology,” arXiv:hep-th/0310001.
[2] C. van de Bruck, M. Dorca, C. J. Martins and M. Parry, Phys. Lett. B 495 (2000) 183 [arXiv:hep-th/0009056];
   P. Brax, C. van de Bruck and A. C. Davis, JHEP 0110 (2001) 026 [arXiv:hep-th/0108215].
[3] U. Ellwanger, Eur. Phys. J. C 25 (2002) 157 [arXiv:hep-th/0001126];
   A. Hebecker and J. March-Russell, Nucl. Phys. B 608 (2001) 375 [arXiv:hep-ph/0103214].
[4] D. Langlois, L. Sorbo and M. Rodriguez-Martinez, Phys. Rev. Lett. 89 (2002) 171301 [arXiv:hep-th/0206146];
   D. Langlois and L. Sorbo, Phys. Rev. D 68 (2003) 084006 [arXiv:hep-th/0306281].
[5] E. Kiritsis, N. Tetradis and T. N. Tomaras, JHEP 0203 (2002) 019 [arXiv:hep-th/0202037].
[6] E. Kiritsis, G. Kofinas, N. Tetradis, T. N. Tomaras and V. Zarikas, JHEP 0302 (2003) 035 [arXiv:hep-th/0207060];
   N. Tetradis, Phys. Lett. B 569 (2003) 1 [arXiv:hep-th/0211200].
[7] Y. S. Myung and J. Y. Kim, Class. Quant. Grav. 20 (2003) L169 [arXiv:hep-th/0304033].
[8] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 4690 [arXiv:hep-th/9906064].
[9] G. R. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B 485 (2000) 208 [arXiv:hep-th/0005016].

[10] P. Binetruy, C. Deffayet and D. Langlois, Nucl. Phys. B 565 (2000) 269 [arXiv:hep-th/9905012];
    J. M. Cline, C. Grojean and G. Servant, Phys. Rev. Lett. 83 (1999) 4245 [arXiv:hep-ph/9906523];
    P. Binetruy, C. Deffayet, U. Ellwanger and D. Langlois, Phys. Lett. B 477 (2000) 285
    [arXiv:hep-th/9910219].

[11] F. Hoyle and J.V. Narlikar, Proc. Royal Soc. A277 (1964) 1; Ann. Phys. 54 (1969)
    207.

[12] T. Damour, F. Piazza and G. Veneziano, Phys. Rev. D 66 (2002) 046007 [arXiv:hep-th/0205111].