Abstract: In 2006, A. Stakhov introduced a new coding/decoding process based on generating matrices of the Fibonacci \( p \)-numbers, which he called the Fibonacci coding/decoding method. Stakhov’s papers have motivated many other scientists to seek certain generalizations by introducing new additional coefficients into recurrence of Fibonacci \( p \)-numbers. In 2013, I. Włoch et al. studied \((2, q)\)-distance Fibonacci numbers \( F_2(q, n) \) and found some of their combinatorial properties. In this paper, we state a new coding theory based on the sequence \( T_q(n) \), which is an extension of Włoch’s sequence \( (F_2(q, n))_{n=0}^{\infty} \).

Keywords: fibonacci numbers; generalizd fibonacci numbers; characteristic equation; coding theory

MSC: primary 11A63, 11B39; secondary 11J86

1. Introduction

Let \((F_n)_{n \geq 0}\) be the sequence of Fibonacci numbers given by recurrence relation \( F_n = F_{n-1} + F_{n-2} \), for \( n \geq 2 \), with initial conditions \( F_0 = 0 \) and \( F_1 = 1 \). These numbers possess many interesting and amazing properties (see [1,2] together with its very extensive annotated bibliography for additional references and history). For example, it is well-known that, for the Fibonacci numbers hold the following Binet’s formula,

\[
F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}},
\]

where \( \alpha := (1 + \sqrt{5})/2 \) and \( \beta := (1 - \sqrt{5})/2 \) are the roots of the characteristic equation

\[
x^2 - x - 1 = 0. \tag{1}
\]

The connection between the Fibonacci numbers and the Golden ratio \( \alpha \) is similarly famous,

\[
\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \alpha, \tag{2}
\]

which follows from Binet’s formula and the relation \( |\alpha| > |\beta| \) between the roots of characteristic polynomial \( f(x) = x^2 - x - 1 \). This limit was probably firstly studied by Johannes Kepler in 1619 (see English translation [3]) as he formulated there the approximation of the golden ratio \( \alpha \) by the proportions of consecutive Fibonacci numbers.

There are many types of generalizations of the Fibonacci numbers (see [4–13]); e.g., changing their recurrence to the form \( P_n = P_{n-2} + P_{n-3} \), for \( n \geq 3 \), with initial conditions \( P_0 = P_1 = P_2 = 1 \) we get Padovan numbers, which are named after R. Padovan, but in 1991 he attributed their discovery to Dutch architect Hans van der Laan and the sequences with the same recurrence were studied in 1899 by R. Perrin and in 1924 by Cordonnier (see [14–16]).
In 1876, Lucas realized that Fibonacci numbers appear as the sums of the northeast diagonals in Pascal’s triangle (see Figure 1), thus the following identity holds for any non-negative integer \( n \)

\[
F_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}.
\]  

(3)

In 1989, Stakhov [17] considered reformatted Pascal’s triangle (see Table 1), where the sum of all numbers in the \( j \)th column, \( j \geq 0 \), is equal to the Fibonacci number \( F_{j+1} \).

![Figure 1. Pascal’s triangle with added north-east diagonals.](image)

**Table 1.** Pascal’s triangle with transformed north-east diagonals to the columns.

\[
\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \ldots \\
1 & 3 & 6 & 10 & 15 & 21 & 28 & 36 & \ldots \\
1 & 4 & 10 & 20 & 35 & 56 & \ldots \\
1 & 5 & 15 & 35 & \ldots \\
1 & 6 & \ldots \\
\vdots \\
1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & 55 & 89 & 144 & \ldots \\
\end{array}
\]

He introduced, for any non-negative \( p \), Fibonacci \( p \)-numbers, as sums of columns in a transformed Pascal’s triangle (he denoted these triangles as Pascal’s \( p \)-triangles), which is creating from Pascal’s triangle by shifting all numbers in each row such way, that the first number in the \( i \)th row, \( i \geq 0 \), is in the \( i(p+1) \)th column. Stakhov showed that these Fibonacci \( p \)-numbers \( \phi_p(n) \) are given by the following recurrence relation

\[
\phi_p(n) = \begin{cases} 
0, & n < 0; \\
1, & n = 0; \\
\phi_p(n-1) + \phi_p(n-p-1), & n > 0.
\end{cases}
\]  

(4)

On the fourth annual meeting of the Pacific Northwest Section of the Mathematical Association of America in 1950, Brenner [18] gave a lecture “Lucas’ matrices” in which he presented usage of Fibonacci’s matrix and Lucas’ matrix

\[
\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \alpha + \beta & -\alpha \beta \\ 1 & 0 \end{pmatrix},
\]  

and showed that their \( n \)th power are
\[
\begin{pmatrix}
F_{n+1} & F_n \\ F_n & F_{n-1}
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
U_{n+1}(a, b) & U_n(a, b) \\ U_n(a, b) & U_{n-1}(a, b)
\end{pmatrix},
\]

respectively, where \((U_n(a, b))_{n \geq 0}\) is a non-degenerate Lucas sequence of the first kind (clearly \(F_n = U_n(1, 1)\)), defined, for any \(n \geq 2\), by the recurrence relation \(U_n(a, b) = a U_{n-1}(a, b) + b U_{n-2}(a, b)\), with \(U_0(a, b) = 0\), \(U_1(a, b) = 1\) and \(a^2 + 4b \neq 0\). Its characteristic equation \(x^2 - ax - b = 0\) has discriminant \(D = a^2 + 4b\) and roots \(\alpha = \frac{1}{2}(a + \sqrt{D})\), \(\beta = \frac{1}{2}(a - \sqrt{D})\), thus, for any \(n \geq 0\), the following generalized Binet formula holds
\[
U_n(a, b) = \frac{\alpha^n - \beta^n}{\alpha - \beta}.
\]

The first matrix in (5) is usually called Fibonacci “Q-matrix”, as King in his master’s thesis [19] originated this notation (more on the history and some applications of Q-matrices can be found in [20,21]).

In 1999, Stakhov [22] introduced a new class of square matrices \(Q_p\) of the order \(p + 1\), where \(p\) is a non-negative integer, which are a generalization of Fibonacci Q-matrix
\[
Q_p =
\begin{pmatrix}
1 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 \\
1 & 0 & 0 & \ldots & 0 & 0
\end{pmatrix}_{(p+1) \times (p+1)}.
\]

He defined the Fibonacci \(p\)-numbers for integers \(n \leq 0\) by the different way then in (4), thus these Fibonacci \(p\)-numbers \(F_p(n)\) are defined for positive integer \(p\) and any integer \(n\) by the following way
\[
F_p(n) = \begin{cases} 
F_p(n - 1) + F_p(n - p - 1), & n > p + 1; \\
F_p(n + p + 1) - F_p(n + p), & n \leq 0; \\
1, & 1 \leq n \leq p + 1.
\end{cases}
\]

Then, he showed that elements of \(Q_p\)-matrices are connected to Fibonacci \(p\)-numbers by the following way
\[
Q_p =
\begin{pmatrix}
F_p(2) & F_p(1) & \ldots & F_p(3 - p) & F_p(2 - p) \\
F_p(2 - p) & F_p(1 - p) & \ldots & F_p(3 - 2p) & F_p(2 - 2p) \\
F_p(3 - p) & F_p(2 - p) & \ldots & F_p(4 - 2p) & F_p(3 - 2p) \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
F_p(0) & F_p(-1) & \ldots & F_p(1 - p) & F_p(-p) \\
F_p(1) & F_p(0) & \ldots & F_p(2 - p) & F_p(1 - p)
\end{pmatrix}.
\]

In 2006, Stakhov [23] developed a theory of the \(Q_p\)-Fibonacci matrices and he introduced a generalization of Cassini formula based on these matrices for any non-negative integers \(p, n > 0\), as he proved that:
1. For the $n$th power of matrix $Q_p$ holds:

$$Q_p^n = \begin{pmatrix}
F_p(n+1) & F_p(n) & \ldots & F_p(n+2-p) & F_p(n+1-p) \\
F_p(n+1-p) & F_p(n-p) & \ldots & F_p(n+2-2p) & F_p(n+1-2p) \\
F_p(n+2-p) & F_p(n+1-p) & \ldots & F_p(n+3-2p) & F_p(n+2-2p) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
F_p(n-2) & F_p(n-3) & \ldots & F_p(n-1-p) & F_p(n-2-p) \\
F_p(n-1) & F_p(n-2) & \ldots & F_p(n-p) & F_p(n-1-p) \\
F_p(n) & F_p(n-1) & \ldots & F_p(n+1-p) & F_p(n-p)
\end{pmatrix}. \quad (9)$$

2. For the determinant of matrix $Q_p$ holds:

$$|Q_p^n| = (-1)^{np}.$$

3. For the inverse of $Q_p^n$ the following extremely interesting identity holds:

$$\left(Q_p^n\right)^{-1} = \begin{pmatrix}
F_p(-n+1) & F_p(-n) & \ldots & F_p(-n+1-p) \\
F_p(-n+1-p) & F_p(-n-p) & \ldots & F_p(-n+1-2p) \\
F_p(-n+2-p) & F_p(-n+1-p) & \ldots & F_p(-n+2-2p) \\
\vdots & \vdots & \ddots & \vdots \\
F_p(-n-2) & F_p(-n-3) & \ldots & F_p(-n-2-p) \\
F_p(-n-1) & F_p(-n-2) & \ldots & F_p(-n-1-p) \\
F_p(-n) & F_p(-n-1) & \ldots & F_p(-n-p)
\end{pmatrix}. \quad (10)$$

He also describes the original Fibonacci coding/decoding method based on the $Q_p$ Fibonacci matrix. Stakhov’s discovery attracted much attention, and so many scientists have tried to more generalize this result. In 2009, Kocer et al. [24] generalized the Fibonacci $p$-numbers $F_p(n)$ by introducing a new parameter $m$ and defined the $m$-extension of Fibonacci $p$-numbers, $F_{p,m}(n)$, by the following way

$$F_{p,m}(n) = \begin{cases}
mF_{p,m}(n-1) + F_{p,m}(n-p-1), & n > p + 1; \\
F_{p,m}(n+p+1) - mF_{p,m}(n+p), & n \leq 0; \\
m^{n-1}, & 1 \leq n \leq p + 1,
\end{cases} \quad (11)$$

where $m$ is any positive real number. In 2009, Basu and Prasad [25] defined the matrix $G_{p,m}$ as a generalization of the matrix $Q_p$, but there is a mistake in this paper, so we write the right form here, by the following way

$$G_{p,m} = \begin{pmatrix}
m & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 \\
1 & 0 & 0 & \ldots & 0 & 0
\end{pmatrix}_{(p+1) \times (p+1)}.$$

Furthermore, they noted that $G_{p,m}$ can be written using $m$-extension of Fibonacci $p$-numbers $F_{p,m}(n)$, when we replace $F_p(n)$ by $F_{m,p}(n)$ in the matrix $Q_p$ in (8). Then, they showed that $|G_{p,m}| = (-1)^p$ and that analogous identities as (9) and (10) hold for $(G_{p,m})^n$ after replacing $F_p(n)$ by $F_{m,p}(n)$. In 2011, Tuglu et al. [26] and Basu et al. [27] (independently and in different notation)
generalized $m$-extension of Fibonacci $p$-numbers to $(m,t)$-extension of Fibonacci numbers by the following way

$$F_{m,t,p}(n) = \begin{cases} m F_{m,t,p}(n-1) + t F_{m,t,p}(n-p-1), & n > p + 1; \\ \frac{1}{2} F_{m,t,p}(n+p+1) - \frac{t}{2} F_{m,t,p}(n+p), & n \leq 0; \\ m^{n-1}, & 1 \leq n \leq p + 1, \end{cases}$$  \hspace{1cm} (12)

where $m$ and $t$ are any positive real numbers. They introduced the matrix $G_{m,t,p}$ as a generalization of the matrix $Q_p$ by the following way

$$G_{m,t,p} = \begin{pmatrix} m & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ t & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}_{(p+1) \times (p+1)}$$

which is again the matrix in the form (6), when we imagine $F_p(n)$ replaced by $F_{m,t,p}(n)$. Recently, many other papers that modify Stakhov’s method in some way have been published (see, e.g., [28–33]). We showed that much effort was devoted to a different generalization of (4) by introducing some new coefficients in the Fibonacci $p$-numbers recurrence for the purpose to design a coding/decoding system in the last ten years. In this paper we will continue in the described direction of research aimed at possible generalizations of recurrence (4) for use in information security. In our generalization, we were motivated by the paper of Włoch et al. [34], which dealt with the combinatorial properties of the terms of the sequence $(F_2(q,n))_{n \geq 0}$, where $q \geq 1$ and $n > 0$ are any integers, defined by the following recurrence

$$F_2(q,n) = F_2(q,n-2) + F_2(q,n-q) \quad \text{for} \quad n \geq q, \hspace{1cm} (13)$$

with initial conditions $F_2(q,i) = 1$ for $i = 0, 1, \ldots, q - 1$. The authors called this sequence as the sequence of $(2,q)$-distance Fibonacci numbers. Clearly, Stakhov’s $p$-Fibonacci numbers $\phi_p(n)$ can be recalled as $(1,p + 1)$-distance Fibonacci numbers in this terminology. Probably independently, but four years later, the sequence with the recurrence (13) was studied in Deveci and Karaduman [35]. They called this sequence as Padovan $p$-sequence, as clearly it is generalization of Padovan sequence $(P_q(n))_n$, considered $q = p + 2$ in (13); denoted this sequence as $(Pap(n))_n$; and used the initial conditions $Pap(i) = 0$ for $i = 1, 2, \ldots, p$ and $Pap(p + 1) = 1, Pap(p + 2) = 0$.

In this paper, to construct a new coding/decoding process, we first prove that the sequence $(T_q(n))_n$ has the Kepler limit for any odd integer $q \geq 3$ (we prove the existence of dominant root of the characteristic polynomial of this sequence and we set a new criterion for test of “non-omitted root summand” in Binet-like formulas). Then, we construct the generating matrix of the sequence $T_q(n)$ in a different way than it was defined in the already mentioned Włoch’s paper to be suitable for constructing a coding method. Further, we derive the relations among the code matrix elements, but our method is new and much simpler than the method introduced by Stakhov (the authors of all subsequent papers only took over his approach). Next, we discuss the correction ability of our encoding method and we show an example of concrete construction of coding/decoding system based on $(2,5)$-distance Fibonacci numbers.

2. Auxiliary Results

For our main purpose, which is to design a coding/decoding system, we will need some known results, which we will only recall here, but also other auxiliary results, which we derive in this section.
2.1. Binet-Like Form of General Linear Recurrence

A fundamental result in the theory of recurrence sequences asserts that:

**Lemma 1.** Let \( q \) be any positive integer, \( q > 1 \). Let \( (u_n)_n \) be a linear recurrence sequence of the order \( q \) whose characteristic polynomial \( f(x) \) splits as

\[
 f(x) = (x - \lambda_1)^{m_1}(x - \lambda_2)^{m_2} \cdots (x - \lambda_q)^{m_q},
\]

where the \( \lambda_i \)'s are distinct non-zero complex numbers and \( \sum_{j=1}^{\ell} m_j = q \). Then, there exist uniquely determined polynomials \( g_1, g_2, \ldots, g_\ell \in \mathbb{Q}((\lambda_j)_{j=1}^{\ell})[x] \), with \( \deg g_j \leq m_j - 1 \) (\( m_j \) is the multiplicity of \( \lambda_j \) as zero of \( f(x) \)), for \( j \in [1, \ell] \), such that for all \( n \) holds

\[
 u_n = g_1(n)\lambda_1^n + g_2(n)\lambda_2^n + \cdots + g_\ell(n)\lambda_\ell^n. \tag{14}
\]

The proof of this result can be found in [36], Theorem C.1, but we need only the following special case.

**Corollary 1.** Let \( q \) be any positive integer, \( q > 1 \). Let \( a_i \) and \( c_i \), \( i = 0, 1, \ldots, q - 1 \), be any complex numbers. Let the sequence \( (u_n)_n \) be defined by recurrence relation

\[
 u_{n+q} = a_0u_n + a_1u_{n+1} + \cdots + a_{q-2}u_{n+q-2} + a_{q-1}u_{n+q-1}, \tag{15}
\]

with initial conditions \( u_0 = c_0, u_1 = c_1, \ldots, u_{q-1} = c_{q-1} \). Let \( \lambda_1, \lambda_2, \ldots, \lambda_q \) be any distinct non-zero roots of its characteristic polynomial \( f(x) \). Then, there exist uniquely determined complex numbers \( g_1, g_2, \ldots, g_q \) such that for all integers \( n \) holds

\[
 u_n = g_1(n)\lambda_1^n + g_2(n)\lambda_2^n + \cdots + g_q(n)\lambda_q^n. \tag{16}
\]

2.2. The Sequence \( F_2(q, n) \) and Its Extension \( T_q(n) \)

The relation (3), which shows a connection between the Pascal’s triangle and the Fibonacci sequence, led to the search for a connections of the Pascal’s triangle with other linear recurrent sequences, which are a generalization of the Fibonacci sequence. Clearly, Stakhov’s sequences \( (\phi_p(n))_{n \geq 0} \) and \( (F_p(n))_{n \geq 0} \), defined in (4) and (7), respectively, belong to these sequences, as the following holds for any integers \( n \geq 0, p \geq 0 \)

\[
 \phi_p(n) = \sum_{j=0}^{\lfloor \frac{n}{p} \rfloor} \binom{n-jp}{j} \quad \text{and} \quad F_p(n) = \sum_{j=0}^{\lfloor \frac{n-1}{p} \rfloor} \binom{n-1-jp}{j}. \tag{17}
\]

Many mathematicians, working primarily in the field of combinatorics and discrete mathematics, began to deal with the other sequences which are a similar generalization of the Fibonacci sequence (see [34,37–45]). In [34], the authors noted that from recurrence relation (13) of the sequence \( (F_2(q, n))_{n \geq 0} \) immediately follows that:

- the sequence \( (F_2(1, n))_{n \geq 0} \) (thus the sequence of \((2, 1)\)-distance Fibonacci numbers) coincides with the Fibonacci sequence \( (F_n)_{n \geq 0} \);
- the sequence \( (F_2(2, n))_{n \geq 0} \) is the known sequence with powers of 2 which double up; and
- for even \( q \), the terms of the sequence \( (F_2(q, n))_{n \geq 0} \) satisfy the relation \( F_2(q, 2n) = F_2(q, 2n+1) \) for any non-negative integer \( n \) (consequently, for even \( q \), we can talk about “double” \((2, q)\)-distance Fibonacci sequences).
Further, they showed that the sequence \((F_2(q,n))_{n \geq 0}\) is connected to the Pascal’s triangle too, as for any integers \(n \geq 2, q \geq 2\) holds
\[
F_2(q,n) = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \left( i + \left\lfloor \frac{n-iq}{2} \right\rfloor \right).
\]

Next, they dealt with the companion matrix of the sequence \((F_2(q,n))_{n \geq 0}\), but for our purpose we find different generating matrices of this sequence that more closely match Stakhov’s procedure for the sequence \((F_p(n))_n\).

To use the sequence \((F_2(q,n))_{n \geq 0}\) to design a coding and decoding system, we need to shift terms of the sequence \((F_2(q,n))_{n \geq 0}\) one place left and extend the definition this sequence to all integers \(n\) in the following way (we denote this extended sequence as the sequence \((T_q(n))_n\) to simplify the notation in all subsequent text).

**Definition 1.** Let \(q\) be any positive integer, \(q \neq 2\), and let \(n\) be any integer. We define the \((2,q)\)-distance Fibonacci numbers \(T_q(n)\) by
\[
T_q(n) = \begin{cases} 
1, & 1 \leq n \leq \max(2, q); \\
T_q(n-2) + T_q(n-q), & n > \max(2,q); \\
T_q(n + \max(2,q)) - T_q(n-2 + \max(3,q)), & n \leq 0.
\end{cases} \tag{18}
\]

The sequence of integers in (18) is clearly a generalization of Fibonacci numbers as recurrence (18) would define for \(q = 1\) and \(q = 3\) exactly the sequence of Fibonacci numbers \(F_n\) and Padovan numbers \(P_n\) for any integer \(n\), respectively (see Table 2).

| \(n\)     | -8 | -7 | -6 | -5 | -4 | -3 | -2 | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-----------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| \(T_1(n)\)| -21| 13 | -8 | 5  | -3 | 2  | -1 | 1  | 1  | 0  | 1  | 1  | 2  | 3  | 5  | 8  | 13 | 21 | 34 |
| \(T_2(n)\)| -3 | 2  | 1  | -1 | 1  | 0  | 0  | 2  | 2  | 2  | 2  | 3  | 3  | 5  | 5  | 7  | 9  | 9  | 9  |
| \(T_3(n)\)| 1  | -1 | -1 | 0  | 1  | 1  | 0  | 0  | 0  | 0  | 1  | 1  | 1  | 0  | 2  | 3  | 3  | 4  | 4  |
| \(T_4(n)\)| -1 | 0  | 0  | 1  | 1  | 0  | 0  | 0  | 0  | 0  | 1  | 1  | 1  | 1  | 1  | 2  | 2  | 3  | 3  |
| \(T_5(n)\)| 0  | 0  | 1  | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 1  | 1  | 1  | 1  | 0  | 2  | 3  | 3  | 3  |
| \(T_6(n)\)| 0  | 1  | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 2  | 2  |
| \(T_7(n)\)| 1  | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 2  |

Firstly, we prove an elementary fact for the sequence \((T_q(n))_n\), which we need for proof of some special cases in our main theorem.

**Lemma 2.** Let \(q, n\) be any integers, \(q \geq 4\). Then,
\[
T_q(2 - q) = T_q(1 - q) = 1
\]
and
\[
T_q(n) = 0
\]
for \(5 - 2q \leq n \leq -q\) or \(3 - q \leq n \leq 0\).
Proof. Using the initial conditions \( T_q(n) = 1 \) for \( n \in [1, q] \) and recurrence (18), we have \( T_q(n) = 0 \) for \( n \in [3-q, 0] \). Further,

\[
\begin{align*}
T_q(2-q) &= T_q(2-q+q) - T_q(2-q+q-2) = T_q(2) - T_q(0) = 1, \\
T_q(1-q) &= T_q(1-q+q) - T_q(1-q+q-2) = T_q(1) - T_q(-1) = 1.
\end{align*}
\]

Finally, let us consider \( n \in [5-2q, -q] \) (for \( q = 4 \) such \( n \) does not exist, so we can consider \( q \geq 5 \)). Thus we can set \( n = 5-2q+i \), for \( i = 0, 1, \ldots, q-5 \), and using recurrence (18) we have

\[
\begin{align*}
T_q(5-2q+i) &= T_q((5-2q+i) + q) - T_q((5-2q+i) + q-2) \\
&= T_q(5-q+i) - T_q(3-q+i) = 0 - 0 = 0.
\end{align*}
\]

\[ \square \]

3. The Generalized Kepler Limit of \( T_q(n) \)

In this section, we prove that the sequence \( (T_q(n))_n \) has the Kepler limit for any odd integer \( q \geq 3 \).

It is well known that the existence of the Kepler limit is related to the existence of a dominant root between the roots \( \alpha_1, \alpha_2, \ldots, \alpha_q \) of the characteristic polynomial of \( (T_q(n))_n \) (namely that there exists a root \( \alpha_q \) such that \( |\alpha_q| > |\alpha_j| \) for every \( j \neq q \) (see, e.g., [46, 47]). It is less known that the existence of a dominant root is not a sufficient condition for the existence of the Kepler limit. Fiorenza and Vincenzi [48, 49] addressed this problem and formulated a number of examples of problematic initial conditions, but they have not found any criteria, which we will do here.

For this purpose, we need to show two facts:

- the characteristic polynomial \( f_q(x) \) of the sequence \( (T_q(n))_n \) has a dominant root \( \alpha_q \); and

- the summand, containing the power of the dominant root \( \alpha_q \), is actually occurred among the summands in the Binet-like formula of \((2,q)\)-distance Fibonacci numbers.

Theorem 1. Let \( q > 2 \) be any integer. Let \( f_q(x) \) denote the characteristic polynomial of the sequence \( (T_q(n))_n \), thus \( f_q(x) = x^q - x^{q-2} - 1 \). Then,

(i) \( f_q(x) \) does not have multiple roots; and

(ii) \( f_q(x) \) has a dominant root for all positive odd integers \( q > 1 \).

Proof. Case (i) was proved in [35], thus we prove Case (ii) only. By the Descartes’ sign rule, we have the existence of only one positive real root \( \alpha \) of \( f_q(x) \). In fact, \( \alpha \in [1,3/2] \), since \( f_q(1) = -1 \) and \( f_q(3/2) = (3/2)^{q-2} \cdot (5/4) - 1 > 0 \). Note that, we also have \( f_q(x) > 0 \), for all \( x > \alpha \). Let \( z \) be a complex root of \( f_q(x) \) with \( |z| \geq \alpha \). Then, \( f_q(|z|) \geq 0 \) and so \( |z| \geq |z^{q-2}| + 1 \). On the other hand, by the triangle inequality, we have \( |z^q| \leq |z^{q-2}| + 1 \) and so \( |z^q| = |z^{q-2}| + 1 \). Thus, \( 1, z^{q-2} \) and \( z^q \) become to the same ray. This implies that there exists a real number \( t_0 \) such that \( z^q = 1 + t_0(z^{q-2} - 1) \). Since \( z^q = z^{q-2} + 1 \), we deduce that \( |z| = \alpha \) and also that \( z^{q-2}/(z^{q-2} - 1) = t_0 \) is a real number and so is \( z^{q-2} \). It follows that \( z^q \) is also a real number and moreover, \( z^{q-2} \) and \( z^q \) are both positive (because the relation \( z^q = z^{q-2} + 1 \) together with the fact that \( |z| > 1 \)). Now, since \( z^2 = z^q/z^{q-2} > 0 \), then \( z \) is a nonzero real number. However, \( z \) is not negative, since in this case \( z^q \) would be also negative (since \( q \) is odd). Thus, \( z \) is a positive real number which is a root of \( f_q(x) \) which yields that \( z = \alpha \) (since \( \alpha \) is the only positive real root of \( f_q(x) \)). This completes the proof. \[ \square \]

Remark 1. In Theorem 1, we have proved that the dominant root exists for odd \( q \). It is easy to realize that for even \( q \), due to the same parity of numbers \( q \) and \( q-2 \), the characteristic polynomial \( f_q(x) \) cannot have a dominant root, because if \( \alpha \) is its root, then its root is also \(-\alpha \).
Theorem 2. Let \( q \) be any positive integer, \( q > 1 \). Let the sequence \((u_n)\_n\) be defined by a linear recurrence of order \( q \) by (15), whose characteristic polynomial \( f(x) \) has distinct non-zero roots \( \lambda_1, \lambda_2, \ldots, \lambda_q \). If determinant of the following matrix \( U_{q,n} \)

\[
\begin{pmatrix}
    u_{n-1} & u_{n-2} & \cdots & u_{n-(q-1)} & u_{n-q} \\
    u_{n-2} & u_{n-3} & \cdots & u_{n-(q-1)} & u_{n-1-q} \\
    u_{n-3} & u_{n-4} & \cdots & u_{n-(q+1)} & u_{n-(q+2)} \\
    u_{n-4} & u_{n-5} & \cdots & u_{n-(q+2)} & u_{n-(q+3)} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    u_{n-(q-1)} & u_{n-q} & \cdots & u_{n-(2q-3)} & u_{n-(2q-2)} \\
    u_{n-q} & u_{n-(q+1)} & \cdots & u_{n-(2q-2)} & u_{n-(2q-1)}
\end{pmatrix}
\]

is non-zero, then all \( g_{ir}, i = 1, 2, \ldots, q, \) in (16) are non-zero.

Proof. Using Binet-like form for the linear recurrence \((u_n)\_n\), given by (16), we can write the matrix \( U_{q,n} \) as

\[
\begin{pmatrix}
    \sum_{i=1}^{q} \lambda_1^{n-1} & \sum_{i=1}^{q} \lambda_1^{n-2} & \cdots & \sum_{i=1}^{q} \lambda_1^{n-(q-1)} & \sum_{i=1}^{q} \lambda_1^{n-q} \\
    \sum_{i=1}^{q} \lambda_2^{n-1} & \sum_{i=1}^{q} \lambda_2^{n-2} & \cdots & \sum_{i=1}^{q} \lambda_2^{n-(q-1)} & \sum_{i=1}^{q} \lambda_2^{n-q} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    \sum_{i=1}^{q} \lambda_q^{n-1} & \sum_{i=1}^{q} \lambda_q^{n-2} & \cdots & \sum_{i=1}^{q} \lambda_q^{n-(q-1)} & \sum_{i=1}^{q} \lambda_q^{n-q}
\end{pmatrix}_{q \times q}
\]

and using additive property of determinants with only one distinct column we can write determinant \( |U_{q,n}| \) as sum of \( q^q \) determinants, but \( q^q - q! \) determinants is clearly zero as they have at least two columns created by powers of the same root \( \lambda_i, i = 1, 2, \ldots, q \). Thus,

\[
|U_{q,n}| = \sum_{(i_1, i_2, \ldots, i_q) \in S_n} \begin{vmatrix}
    g_{i_1} \lambda_{i_1}^{n-1} & g_{i_2} \lambda_{i_2}^{n-2} & \cdots & g_{i_{q-1}} \lambda_{i_{q-1}}^{n-(q-1)} & g_{i_q} \lambda_{i_q}^{n-q} \\
    g_{i_1} \lambda_{i_1}^{n-2} & g_{i_2} \lambda_{i_2}^{n-3} & \cdots & g_{i_{q-1}} \lambda_{i_{q-1}}^{n-q} & g_{i_q} \lambda_{i_q}^{n-(q+1)} \\
    \cdots & \cdots & \cdots & \cdots & \cdots \\
    g_{i_1} \lambda_{i_1}^{n-(q-1)} & g_{i_2} \lambda_{i_2}^{n-(q-1)} & \cdots & g_{i_{q-1}} \lambda_{i_{q-1}}^{n-(2q-2)} & g_{i_q} \lambda_{i_q}^{n-(2q-2)} \\
    g_{i_1} \lambda_{i_1}^{n-q} & g_{i_2} \lambda_{i_2}^{n-(q+1)} & \cdots & g_{i_{q-1}} \lambda_{i_{q-1}}^{n-(2q-2)} & g_{i_q} \lambda_{i_q}^{n-(2q-1)}
\end{vmatrix}_{q \times q},
\]

where \( S_n \) is the symmetric group on elements \( 1, 2, \ldots, n \). Then, we can write

\[
|U_{q,n}| = \prod_{i=1}^{q} g_{i} \sum_{(i_1, i_2, \ldots, i_q) \in S_n} \lambda_{i_1}^{n-q} \lambda_{i_2}^{n-q-1} \cdots \lambda_{i_q}^{n-2q+1} V(\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_q}),
\]

where \( V(\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_q}) \) are Vandermonde determinants. \( \square \)

By Theorems 1, 2, and 3 (ii) and Corollary 1, we can write a \((2, q)\)-distance Fibonacci sequence \((T_q(n))\_n\) in its asymptotic form

\[
T_q(n) = a^n q(1 + o(1)),
\]
where \(a_q\) is the dominant root of the sequence \((T_q(n))_n\), \(g_q\) is a nonzero constant and \(o(1)\) is a function which tends to 0 as \(n \to \infty\). Hence the following holds

**Corollary 2.** Let \(q\) be any odd integer, \(q \geq 3\), let \(\ell\) be any positive integer. Then, there is a so-called generalized Kepler limit

\[
\lim_{n \to \infty} \frac{T_q(n + \ell)}{T_q(n)} = a_q^\ell
\]

where \(a_q\) is the dominant root of the sequence \((T_q(n))_n\).

### 4. Generating Matrices of \(T_q(n)\)

We showed that Q-matrix, which generates the Fibonacci sequence, was already constructed for many kinds of generalized Fibonacci sequences. Now, we would like to note that these generating matrices can be created in slightly different forms (usually based on certain rearrangements of the indices of the studied sequence), e.g., we can find more than ten forms of generating matrices of Padovan sequence in papers [50–54]. Similarly, in [34,35], the authors constructed generating matrices of the sequence \((F_2(q, n))_n\) and \((Pap(n))_n\), respectively, but, to apply the extended sequence \((T_q(n))_n\) for encoding and decoding, we consider another form of generating matrix, we denote it by \(R_q\), whose \(n\)th power, denoted by \(M_{q,n}\), corresponds to the form of the matrix obtained by Stakhov in (9).

**Definition 2.** Let \(q\) be any positive integer, \(q \geq 3\), and let \(n\) be any integer. Let \(M_{q,n} = (m_{i,j}(q,n))_{1 \leq i,j \leq q}\), where

\[
m_{i,j}(q,n) = \begin{cases} T_q(n - j), & i = 1 \land 1 \leq j \leq q; \\ T_q(n - 1 + i - j - q), & 2 \leq i \leq q \land 1 \leq j \leq q, \end{cases}
\]

thus, \(M_{q,n}\) is equal to

\[
\begin{pmatrix}
T_q(n - 1) & T_q(n - 2) & \cdots & T_q(n - (q - 1)) & T_q(n - q) \\
T_q(n - q) & T_q(n - (q + 1)) & \cdots & T_q(n - (2q - 2)) & T_q(n - (2q - 1)) \\
T_q(n - (q - 1)) & T_q(n - q) & \cdots & T_q(n - (2q - 3)) & T_q(n - (2q - 2)) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
T_q(n - 4) & T_q(n - 5) & \cdots & T_q(n - (q + 2)) & T_q(n - (q + 3)) \\
T_q(n - 3) & T_q(n - 4) & \cdots & T_q(n - (q + 1)) & T_q(n - (q + 2)) \\
T_q(n - 2) & T_q(n - 3) & \cdots & T_q(n - q) & T_q(n - (1 - q))
\end{pmatrix}.
\]

**Example 1.** Let us consider \(n = 4\) and \(q = 3\) (see Table 2). Then, the matrix \(M_{3,4}\) has the following special form

\[
M_{3,4} = \begin{pmatrix}
T_3(3) & T_3(2) & T_3(1) \\
T_3(1) & T_3(0) & T_3(-1) \\
T_3(2) & T_3(1) & T_3(0)
\end{pmatrix} = \begin{pmatrix}
1 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}.
\]

**Lemma 3.** Let \(q\) be any positive integer, \(q \geq 3\). The matrix \(M_{q,4} = (m_{i,j}(q,4))_{1 \leq i,j \leq q}\) has the following special form

\[
m_{i,j}(q,4) = \begin{cases} 1, & (i,j) \in \{(1,1), (q - 1,1), (q,1), (q,2)\}; \\ 1, & j - i \in \{1,2\}; \\ 0, & \text{otherwise}, \end{cases}
\]
thus,

\[
M_{q,4} = \begin{pmatrix}
1 & 1 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 1 & \cdots & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & \ddots & \ddots & 0 & \ddots & \ddots & 0 \\
1 & 1 & 0 & \cdots & \cdots & 0 & 0
\end{pmatrix}_{q \times q}.
\]

**Proof.** By (20) \( M_{q,4} \) is equal to

\[
\begin{pmatrix}
T_q(3) & T_q(2) & T_q(1) & T_q(0) & \cdots & T_q(5-q) & T_q(4-q) \\
T_q(4-q) & T_q(3-q) & T_q(2-q) & T_q(1-q) & \cdots & T_q(6-2q) & T_q(5-2q) \\
T_q(5-q) & T_q(4-q) & T_q(3-q) & T_q(2-q) & \cdots & T_q(7-2q) & T_q(6-2q) \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
T_q(-1) & T_q(-2) & T_q(-3) & T_q(-4) & \cdots & T_q(1-q) & T_q(-q) \\
T_q(0) & T_q(-1) & T_q(-2) & T_q(-3) & \cdots & T_q(2-q) & T_q(1-q) \\
T_q(1) & T_q(0) & T_q(-1) & T_q(-2) & \cdots & T_q(3-q) & T_q(2-q) \\
T_q(2) & T_q(1) & T_q(0) & T_q(-1) & \cdots & T_q(4-q) & T_q(3-q)
\end{pmatrix}_{q \times q}
\]

and using Lemma 2 and with respect to Example 1 we get the assertion. \( \square \)

**Lemma 4.** Let \( a \) be any positive integer. Let \( A \) denote a matrix such that

\[ A = ((-1)^{i+j})_{1 \leq i, j \leq 2a} \]

thus

\[
A = \begin{pmatrix}
1 & -1 & 1 & \cdots & \cdots & -1 & 1 & -1 \\
-1 & 1 & -1 & \cdots & \cdots & -1 & 1 & -1 \\
1 & -1 & 1 & \cdots & \cdots & -1 & 1 & -1 \\
-1 & 1 & -1 & \ddots & \ddots & \ddots & -1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
-1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & -1 & \cdots & 1 & \cdots & -1 & 1 & 1 \\
-1 & 1 & \cdots & \cdots & -1 & 1 & -1 & 1
\end{pmatrix}.
\]

Then,

(i) \( A \) is a Toeplitz matrix (thus, a diagonal-constant matrix).

(ii) Every element on the main diagonal of \( A \) is 1 and all adjacent diagonals have opposite signs.

(iii) The addition of the \( i \)th row to the \((i+1)\)th row is equal to the zero vector.

(iv) The addition of the first row and the last row is equal to the zero vector.

**Proof.** The proof is trivial by the definition of matrix \( A \). \( \square \)

**Theorem 3.** Let \( q \) be any positive integer, \( q \geq 3 \), and let \( n \) be any integer. Then, the following hold

(i)

\[ M_{q,n} = R_q \cdot M_{q,n-1}, \]
where \( R_q = (r_{i,j})_{1 \leq i,j \leq q} \), with

\[
  r_{i,j} = \begin{cases} 
    1, & (i,j) \in \{(1,q),(q,1)\}; \\
    1, & j - i = 1; \\
    0, & \text{otherwise}, 
  \end{cases}
\]

thus

\[
  R_q = \begin{pmatrix} 
    0 & 1 & 0 & \cdots & 0 & 1 \\
    0 & 0 & 1 & \cdots & \cdots & 0 \\
    0 & 0 & 0 & \cdots & \cdots & \cdots \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    \vdots & \vdots & \vdots & \cdots & 0 & 1 \\
    0 & \cdots & \cdots & \cdots & 0 & 0 \\
    1 & 0 & \cdots & \cdots & 0 & 0
  \end{pmatrix}.
\]

(ii)

\[
  |M_{q,n}| = (-1)^{n(q+1)}. \]

(iii) \( R_q^{-1} = (\rho_{i,j})_{1 \leq i,j \leq q} \), with

\[
  \rho_{i,j} = \begin{cases} 
    1, & (i,j) = (1,q); \\
    -1, & (i,j) = (2,q-1); \\
    1, & j = i - 1 \land 2 \leq i \leq q, \\
    0, & \text{otherwise}, 
  \end{cases}
\]

thus

\[
  R_q^{-1} = \begin{pmatrix} 
    0 & 0 & \cdots & 0 & 0 & 0 & 1 \\
    1 & 0 & \cdots & \cdots & 0 & -1 & 0 \\
    0 & 1 & \cdots & \cdots & \cdots & 0 & 0 \\
    \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
    \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots \\
    0 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
    0 & 0 & \cdots & \cdots & 0 & 1 & 0
  \end{pmatrix}.
\]

(iv) \( M_{q,4}^{-1} = (\mu_{i,j})_{1 \leq i,j \leq q} \), with

\[
  \mu_{i,j} = \begin{cases} 
    1, & (i,j) = (3,1); \\
    1 + (-1)^q, & (i,j) = (2,q); \\
    0, & 3 \leq i \leq q \land i - 1 \leq j \leq q - 1, \\
    (-1)^{i+j}, & (i,j) = (1,q) \lor i - j \geq 2 \\
    (-1)^{q+i+j}, & \text{otherwise}, 
  \end{cases}
\]
Thus $\mathcal{M}_q^{-1}$ is equal to
\[
\begin{pmatrix}
(-1)^{q+2} & (-1)^{q+3} & (-1)^{q+4} & \cdots & (-1)^{2q-1} & (-1)^{2q} & (-1)^{q+1} \\
(-1)^{q+3} & (-1)^{q+4} & (-1)^{q+5} & \cdots & (-1)^{2q+1} & 1+(-1)^q \\
1 & 0 & 0 & \cdots & 0 & 0 & (-1)^{2q+3} \\
(-1)^{q+4} & (-1)^{q+6} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
(-1)^{2q-4} & (-1)^{2q-3} & (-1)^{2q-2} & 0 & (-1)^q
\end{pmatrix}.
\]

Proof. 
Proof of (i). We compute the product of matrices $R$ and $\mathcal{M}_{q,n-1}$, thus
\[
(R_q \cdot \mathcal{M}_{q,n-1})_{ij} = \sum_{k=1}^{q} r_{ik} \cdot m_{kj}(n, n-1),
\]
for $1 \leq i, j \leq q$.

Case 1 Let $i = 1$. Then,
\[
(R_q \cdot \mathcal{M}_{q,n-1})_{1j} = r_{1j} \cdot m_{2j}(n, n-1) + r_{1, q} \cdot m_{qj}(n, n-1),
\]
\[
= T_q(n - 1 - 1 + 2 - j - q) + T_q(n - 1 - 1 - j - 2),
\]
\[
= T_q(n - j - q) + T_q(n - j - 2),
\]
\[
= T_q(n - j) = m_{1,j}(q, n).
\]

Case 2 Let $i \in [2, q - 1]$. Then,
\[
(R_q \cdot \mathcal{M}_{q,n-1})_{ij} = r_{ij+1} \cdot m_{i+1,j}(q, n-1) = m_{i+1,j}(q, n-1),
\]
\[
= T_q(n - 1 - 1 + i - 1 + j - q) = T_q(n - 1 + i - j - q) = m_{ij}(q, n).
\]

Case 3 Let $i = q$. Then,
\[
(R_q \cdot \mathcal{M}_{q,n-1})_{qj} = r_{q1} \cdot m_{q1}(n, n-1) = m_{q1}(q, n-1),
\]
\[
= T_q(n - 1 - 1 + q - j - q) = T_q(n - 1 - j) = m_{qj}(q, n).
\]

Thus, we show that $\mathcal{M}_{q,n} = R_q \cdot \mathcal{M}_{q,n-1}$ for any integer $n$.

Proof of (ii). By item (i), we have
\[
\mathcal{M}_{q,n} = R_q^{n-4} \cdot \mathcal{M}_{q,4}
\]
for any positive integer $n > 4$ (similarly it holds $\mathcal{M}_{q,n} = R_q^n \cdot \mathcal{M}_{q,0}$, for any positive integer $n$, but the matrix $\mathcal{M}_{q,0}$ is not as simple as matrix $\mathcal{M}_{q,4}$).

We calculate the determinant of the matrix $\mathcal{M}_{q,n}$ with the help of identity (22), i.e., firstly we find the determinant of the matrix $R$ and then the determinant of the matrix $\mathcal{M}_{q,4}$.

Determinant of the matrix $R_q$ We use the Laplace theorem. We expand the determinant of matrix $R$ along the first column, thus
\[\det R_q = (-1)^{q+1},\]

where we use the fact that the resulting matrix is an upper triangular matrix.

**Determinant of the matrix** \(\mathcal{M}_{q,4}\)  We again use the Laplace theorem and we expand it along the last row, thus

\[
\det \mathcal{M}_{q,4} = (-1)^{q+1} \cdot 1 + (-1)^{q} \cdot 1 + (-1)^{2(q-1)} \cdot 1 = (-1)^{q+1} + 1 + (-1)^{q} = 1,
\]

where we use the fact that the last two resulting matrices are triangular matrices with the main diagonal containing only ones.

Finally, using well-known formula for determinant of product of matrices, we have by (22)

\[
\det \mathcal{M}_{q,n} = \det \left( R_q^{n-4} \cdot \mathcal{M}_{q,4} \right) = (\det R_q)^{n-4} \cdot \det \mathcal{M}_{q,4} = (-1)^{n(q+1) - 2n - 4(q-1)} = (-1)^{n(q+1)}.
\]

**Proof of (iii).** Let us denote by \(\tilde{R}_q = (\tilde{r}_{ij})_{1 \leq i, j \leq q}\) a matrix such that

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 & 1 \\
0 & 1 & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \cdots & 1 & 0 \\
0 & \cdots & 0 & 1 & 1
\end{bmatrix}
\]
\[
\tilde{r}_{ij} = \begin{cases} 
1, & (i,j) = (1,q); \\
-1, & (i,j) = (2,q-1); \\
1, & j = i-1 \land 2 \leq i \leq q; \\
0, & \text{otherwise}.
\end{cases}
\]

Thus, we wish to prove that \( \tilde{R}_q \cdot R_q = R_q \cdot \tilde{R}_q = I_q \). To avoid unnecessary repetition, we prove only that \( \tilde{R}_q \cdot R_q = I_q \). Thus, we must prove for \( 1 \leq i,j \leq q \) that

\[
(\tilde{R}_q \cdot R_q)_{ij} = \delta_{ij} = \begin{cases} 
1, & i = j; \\
0, & \text{otherwise}.
\end{cases}
\]  

(23)

We split the proof of (iii) into the following cases

**Case 1** Let \( i = 1 \). Then,

\[
(\tilde{R}_q \cdot R_q)_{1,j} = \sum_{k=1}^{q} \tilde{r}_{1k} r_{kj} = r_{qj} = \begin{cases} 
1, & j = 1; \\
0, & \text{otherwise}.
\end{cases}
\]

**Case 2** Let \( i = 2 \). Then,

\[
(\tilde{R}_q \cdot R_q)_{2,j} = \sum_{k=1}^{q} \tilde{r}_{2k} r_{kj} = \tilde{r}_{2,1} r_{1,j} + \tilde{r}_{2,q-1} r_{q-1,j} = \\
= \begin{cases} 
1 \cdot 1 + (-1) \cdot 1 = 0, & j = q; \\
1 \cdot 1 + (-1) \cdot 0 = 1, & j = 2; \\
1 \cdot 0 + (-1) \cdot 0 = 0, & \text{otherwise}.
\end{cases}
\]

**Case 3** Let \( i > 2 \). Then,

\[
(\tilde{R} \cdot R)_{ij} = \sum_{k=1}^{q} \tilde{r}_{ik} r_{kj} = \tilde{r}_{i,j-1} r_{i-1,j} = r_{i-1,j} = \begin{cases} 
1, & j = i; \\
0, & \text{otherwise}.
\end{cases}
\]

Thus, \( \tilde{R} \) coincides with the inverse matrix \( R^{-1} \) of the matrix \( R \).

**Proof of (iv).** Assuming that \( q \) is even (thus, let \( q = 2a \), where \( a \geq 2 \) is any integer), the proof is analogous for odd \( k \). We show that the matrix \( \tilde{M}_{2a} \) defined by (21), with \( q = 2a \), is the inverse matrix to the matrix \( M_{2a} \). We have that

\[
M_{2a} = (\tilde{m}_{ij})_{1 \leq i,j \leq 2a}, \text{ with}
\]

\[
\tilde{m}_{ij} = \begin{cases} 
2, & (i,j) = (2a); \\
0, & 3 \leq i \leq 2a \land i-1 \leq j \leq 2a-1; \\
(-1)^{i+j}, & \text{otherwise},
\end{cases}
\]  

(24)

thus
Let 

\[
\begin{pmatrix}
1 & -1 & 1 & \cdots & \cdots & -1 & 1 & -1 \\
-1 & 1 & -1 & \cdots & \cdots & 1 & -1 & 2 \\
1 & 0 & 0 & \cdots & \cdots & 0 & 0 & -1 \\
-1 & 1 & 0 & \vdots & \vdots & \vdots & 0 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
-1 & 1 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & -1 & \cdots & 1 & 0 & 0 & -1 \\
-1 & 1 & -1 & \cdots & -1 & 1 & 0 & 1
\end{pmatrix}
\]

\[\tilde{M}_{2a,A} = \begin{pmatrix}
1 & -1 & 1 & \cdots & \cdots & -1 & 1 & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
-1 & 1 & 0 & \vdots & \vdots & \vdots & 0 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
-1 & 1 & -1 & \cdots & -1 & 1 & 0 & 1
\end{pmatrix}.\]

It is clear that the matrix \( \tilde{M}_{2a,A} \) arises from the matrix \( A \), defined in Lemma 4, on the basis of change the element in the position \((2, 2a)\) to 2 and the “triangle of elements in the center part” to 0. The matrix \( A \) is a Toeplitz matrix (thus, diagonal-constant matrix) with the main diagonal with the plus signs and side by side diagonals with opposite signs. Hence, the addition of any side by side rows leads to the zero vector.

Therefore, we have to prove that \( \tilde{M}_{2a,A} \cdot M_{2a,A} = M_{2a,A} \cdot \tilde{M}_{2a,A} = I_{2a} \). To avoid unnecessary repetitions, here we only provide the proof that \( M_{2a,A} \cdot \tilde{M}_{2a,A} = I_{2a} \). Thus, we have to prove for \( 1 \leq i, j \leq 2a \) that

\[
(M_{2a,A} \cdot \tilde{M}_{2a,A})_{i,j} = \delta_{i,j} = \begin{cases} 1, & i = j; \\ 0, & \text{otherwise}. \end{cases}
\] (25)

We split the proof of (25) to the following cases

**Case 1** Let \( i = 1 \). By Lemma 4 (c) and as \( \tilde{m}_{3,j} = 0 \) for \( 2 \leq j \leq 2a - 1 \) we have

\[
(M_{2a,A} \cdot \tilde{M}_{2a,A})_{1,j} = \sum_{k=1}^{2a} m_{1k} \tilde{m}_{kj} = 1 \cdot \tilde{m}_{1j} + 1 \cdot \tilde{m}_{2j} + 1 \cdot \tilde{m}_{3j} = \begin{cases} 1 + (-1) + 1 = 1, & j = 1; \\ (-1)^{j+1} + (-1)^{j} + 0 = 0, & 2 \leq j \leq 2a - 1; \\ -1 + 2 - 1 = 0, & j = 2a; \end{cases}
\]

\[
= \begin{cases} 1, & j = 1; \\ 0, & \text{otherwise}. \end{cases}
\]

**Case 2** Let \( 2 \leq i \leq 2a - 2 \). Then,

\[
(M_{2a,A} \cdot \tilde{M}_{2a,A})_{i,j} = \sum_{k=1}^{2a} m_{ik} \tilde{m}_{kj} = m_{i,i+1} \tilde{m}_{i+1,j} + m_{i,i+2} \tilde{m}_{i+2,j} = \tilde{m}_{i+1,j} + \tilde{m}_{i+2,j} \]

\[
= \begin{cases} (-1)^{i} + (-1)^{i+1} = 0, & 1 \leq j \leq i - 1; \\ 0 + 1 = 1, & j = i; \\ 0 + 0 = 0, & i + 1 \leq j \leq 2a - 1; \\ (-1)^{i+1} + (-1)^{i} = 0, & j = 2a; \end{cases}
\]

\[
= \begin{cases} 1, & i = j; \\ 0, & \text{otherwise}. \end{cases}
\]
Case 3 Let \( i = 2a - 1 \). Then,
\[
(M_{2a,4} \cdot \bar{M}_{2a,4})_{2a-1,j} = \sum_{k=1}^{2a} m_{2a-1,k} \bar{m}_{k,j} = m_{2a-1,1} \bar{m}_{1,j} + m_{2a-1,2a} \bar{m}_{2a,j}
\]
\[
= 1 \cdot \bar{m}_{1,j} + 1 \cdot \bar{m}_{2a,j}
\]
\[
= \begin{cases} 
(-1)^{i+1} + (-1)^{i} = 0, & 1 \leq j \leq 2a - 3; \\
-1 + 2 = 1, & j = 2a.
\end{cases}
\]

Case 4 Let \( i = 2a \). Then,
\[
(M_{2a,4} \cdot \bar{M}_{2a,4})_{2a,j} = \sum_{k=1}^{2a} m_{2a,k} \bar{m}_{k,j} = m_{2a,1} \bar{m}_{1,j} + m_{2a,2} \bar{m}_{2,j}
\]
\[
= 1 \cdot \bar{m}_{1,j} + 1 \cdot \bar{m}_{2,j}
\]
\[
= \begin{cases} 
(-1)^{i+1} + (-1)^{i} = 0, & 1 \leq j \leq 2a - 1; \\
-1 + 2 = 1, & j = 2a.
\end{cases}
\]

\[\square\]

5. Coding/Decoding System Based on \((2,q)\)-Distance Fibonacci Numbers

Let us consider that we have a message \( M \) (represented by a string of digits), which we want to secretly send to a certain unique recipient through a communication channel. There are various methods for this secret transmission of a message \( M \), but, in this paper, we deal with only one of these methods based on matrix multiplication. For this reason we rewrite the original message \( M \) by grouping of its digits to elements of a matrix \( \bar{M} = (m_{q(i-1)+j})_{1 \leq i,j \leq q} \) of order \( q \), \( q > 2 \) is an integer (in this step, we have many possibilities, so we use such grouping to be matrix \( \bar{M} \) a non-singular matrix), which we want to transform into a code matrix \( E = (e_{q(i-1)+j})_{1 \leq i,j \leq q} \) (this transformation is called coding process) and send it to the recipient. We can use an invertible matrix \( M \) as a coding matrix and its inverse \( M^{-1} \) as a decoding matrix, thus we get the code matrix \( E = \bar{M} : M \) with coded original message \( M \) (if we rewrite elements of matrix \( E \) as the string of digits, we get the coded message \( E \)). Then, we can decode \( E \) by using the inverse matrix \( M^{-1} \), thus \( E : M^{-1} = \bar{M} \) (this transformation is called decoding process).

To design a fully functional coding/decoding process based on matrix multiplication Stakhov [23] defined the matrix \( M \) on the base of terms of the sequence \( (F_p(n))_n \) (defined in (7)), we set \( M := Q^n_p \) (see identity (9)) and propose the following requirements:

(i) There is the Kepler limit for the sequence of Fibonacci \( p \)-numbers \( F_p(n) \), which allows an error detection and correction.

(ii) Determinant of the matrix \( Q^n_p\) needs to be \( \pm 1 \), as we want all entries of the inverse matrix \( (Q^n_p)^{-1} \) to be integers. Further, determinant \( |Q^n_p| \) plays a role of the “checking element” in the coding process, as it is sent immediately after the coded message through the transmission channel.

In previous sections, we construct the sequence of the matrices \( M_{q,n} = (m_{ij}(q,n))_{1 \leq i,j \leq q} \) based on the \((2,q)\)-distance Fibonacci numbers \( T_q(n) \) and now we show that matrices \( M_{q,n} = (m_{ij}(q,n))_{1 \leq i,j \leq q} \) can be used effectively in coding/decoding process. Clearly, our sequence of \((2,q)\)-distance Fibonacci numbers \( T_q(n) \), with an odd value of \( q \), fulfills the previous Stakhov’\'s Requirements (i) and (ii), as, by Corollary 2 we know, that the sequence \( (T_q(n))_n \) satisfies the generalized Kepler limit with the dominant root \( a_q \), for any odd \( q \geq 3 \), and by Theorem 3 determinant of the matrix \( M_{q,n} \) is equal to 1, respectively.

Hence, we can use matrix \( M_{q,n} \) as a coding matrix and its inverse \( M_{q,n}^{-1} \) as a decoding matrix and we get the code matrix \( E = \bar{M} : M_{q,n} \) with coded original message \( M \). Then, we can decode \( E \) by using the inverse matrix \( M_{q,n}^{-1} \), thus \( E : M_{q,n}^{-1} = \bar{M} \). With respect to Theorem 3, we can use in decoding process these facts:
• For an odd \( q \geq 3 \), we have \( |\mathcal{M}_{q,n}| = 1 \), thus the “checking element” if an error occurs during transmission in coding process is very suitable as \( |E| = |\mathcal{M}| \) must be satisfied.

• For finding the message matrix \( \mathcal{M} \) from the code matrix \( E \), we have the following computationally much faster way

\[
\mathcal{M} = E \cdot \mathcal{M}_{q,n}^{-1} = E \cdot \left( R_q^{-n-4} \cdot \mathcal{M}_{q,n} \right)^{-1} = E \cdot (\mathcal{M}_{q,n})^{-1} \cdot \left( R_q^{-1} \right)^{n-4}.
\]

5.1. Relation among Code Matrix Elements and Code Rate of This Method

When a message is transmitted from a sender to a recipient, the message may be distorted in the communication channel. Hence, some errors can occur in the code matrix \( E \) and we must be able to identify these errors and subsequently correct them. We show that the basis of an error detection as well as its correction is the value of the determinant \( |E| \) and the following relation among elements of code matrix \( E \).

**Lemma 5.** Let \( q \) be any integer, \( q \geq 3 \), let \( \ell, k \) be any positive integers, \( \ell < q \) and \( k \in [1, q^2], k \not\equiv 0, 1, \ldots, \ell - 1 \) (mod \( q \)). Let \( a_q \) be the strict dominant root of the sequence \( (T_q(n))_n \). Then, for elements of the matrix \( E \) holds

\[
\lim_{n \to \infty} \frac{c_k}{c_{k+\ell}} = a_q^{\ell}.
\]

**Proof.** With respect to defining identity (20) of the matrix \( \mathcal{M}_{q,n} = (m_{i,j}(q,n))_{1 \leq i,j \leq q} \) the following statements immediately hold:

(a) All elements \( m_{i,j} \), \( (i,j) \in [1,q]^2 \), of the matrix \( \mathcal{M}_{q,n} \) are in the form \( T_q(n - \ell) \) with \( \ell \in [1, 2q - 1] \).

(b) If \( m_{i,j} = T_q(n - \ell) \), \( (i,j) \in [1,q] \times [1,q - 1] \) and \( \ell \in [1, 2q - 1] \), then \( m_{i,j+1} = T_q(n - \ell - 1) \).

(c) The \( j \)th column \( C_j \), \( j \in [1,q] \), of \( \mathcal{M}_{q,n} \) consists of elements of the set \( \{ T_q(n - j - r); r \in [0,q - 1] \} \), has as its first element \( T_q(n - j) \) and the others are arranged in descending order according to the index in round brackets. Thus, it has the form

\[
C_j = (T_q(n - j), T_q(n - j - (q - 1)), T_q(n - j - (q - 2)), \ldots, T_q(n - j - 1))^T.
\]

As \( E = \mathcal{M} \cdot \mathcal{M}_{q,n} \), we get elements of the code matrix \( E = (e_{q(i-1)+j})_{1 \leq i,j \leq q} \) with respect to the previous Statements (b) and (c) by the following way

\[
e_{q(i-1)+j} = m_{i} \cdot C_j
\]

\[
= (m_{i-1}q+1, m_{i-1}q+2, \ldots, m_{iq}) \cdot (T_q(n - j), T_q(n - j - (q - 1)), T_q(n - j - (q - 2)), \ldots, T_q(n - j - 1))^T
\]

\[
= m_{i-1}q+1 T_q(n - j) + \sum_{r=1}^{q-1} m_{i-1}q+r+1 T_q(n - j - q + r).
\]

Now, we prove the case for \( \ell = 1 \).

When we denote \( k := q(i-1) + j \), \( (i,j) \in [1,q] \times [1,1-q] \), thus \( k \in [1,q^2], k \not\equiv 0 \) (mod \( q \)), we obtain
\[ \frac{e_k}{e_{k+1}} = \frac{m_i \cdot C_j}{m_i \cdot C_{j+1}} \]
\[ = \frac{m_{(i-1)q+1} T_q(n-j) + \sum_{r=1}^{q-1} m_{(i-1)q+r+1} T_q(n-j-q+r)}{m_{(i-1)q+1} T_q(n-j-1) + \sum_{r=1}^{q-1} m_{(i-1)q+r+1} T_q(n-j-1-q+r)} \]
\[ = \frac{m_{(i-1)q+1} T_q(n-j) + \sum_{r=1}^{q-1} m_{(i-1)q+r+1} T_q(n-j-q+r)}{m_{(i-1)q+1} T_q(n-j-1) + \sum_{r=1}^{q-1} m_{(i-1)q+r+1} T_q(n-j-1-q+r)} \]

and with respect to Corollary 2

\[ \lim_{n \to \infty} \frac{e_k}{e_{k+1}} = \frac{m_{(i-1)q+1} a_q + \sum_{r=1}^{q-1} m_{(i-1)q+r+1} a_q^{-q+r-1}}{m_{(i-1)q+1} + \sum_{r=1}^{q-1} m_{(i-1)q+r+1} a_q^{-q+r-1}} = a_q. \quad (27) \]

Cases for \(1 < \ell < q\).

Using proved identity (27), we clearly get the assertion with respect to the following trivial identity

\[ \frac{e_k}{e_{k+\ell}} = \frac{e_k}{e_{k+1}} \cdot \frac{e_{k+1}}{e_{k+2}} \cdot \ldots \cdot \frac{e_{k+\ell-2}}{e_{k+\ell-1}} \cdot \frac{e_{k+\ell-1}}{e_{k+\ell}}. \]

□

Let us consider that we send by the communication channel the code matrix \(E\) and right after the determinant \(|M|\) of the message matrix \(M\). As already mentioned, determinants of the message matrix \(|M|\) and the code matrix \(|E|\) are equal in our case; thus, clearly, by comparing the determinant of the matrix obtained by the recipient from the communication channel with the value of determinant \(|M|\), the recipient can decide whether the code matrix \(|E|\) is damaged or not, but he cannot determine which element of the code message is damaged. To find the damaged element (or more elements), the recipient necessarily needs both the value of the determinant and the approximation properties among elements of code matrix \(|E|\), which we found in Lemma 5.

It is possible to show, the same way as done, e.g., in [23,55], that by these properties can be corrected all cases except for the case with all error elements of the code matrix \(M\), namely \((q^2)\) cases with one error element, \((q^2)\) cases with any two error elements, \(\ldots, (q^2)\) cases with \((q^2 - 1)\) error elements. Thus, the correction ability \(S_{\text{cor}}\) of this coding/decoding method is done by the formula

\[ S_{\text{cor}} = \frac{\sum_{j=1}^{q^2-1} (q^2)}{\sum_{j=1}^{q^2} (q^2)} = \frac{2q^2 - 2}{2q^2 - 1} \approx 100\%. \]

5.2. Example of Coding/Decoding by Matrix Based on \((2,5)\)-Distance FIBONACCI Numbers

Now, we consider the special case of our method for \(q = 5\) (the case for \(q = 3\), thus based on the sequence of Padovan numbers \((P_n)_n\), is separately discussed, e.g., in [54], and for \(q = 4\) our sequence \(T_q(n)\) does not have the dominant root, with respect to Theorem 1). Therefore, we represent the initial message \(M\) by the matrix \(M = (m_{5(i-1)+j})_{1 \leq i,j \leq 5}\), thus
we easily can show that the previous identity for \( R_5^{-1/4} \) holds for any integer \( n \), thus we can write

\[
(R_5^{-1})^{n-4} = \begin{pmatrix}
  l_{n+2} - l_{n+1} & l_{n+1} - l_n & l_n - l_{n-1} & l_{n-1} - l_{n-2} & l_{n-3} - l_{n+2} \\
  l_{n+2} - l_{n+1} & l_{n+1} - l_n & l_n - l_{n-1} & l_{n-1} - l_{n-2} & l_{n-3} - l_{n+2} \\
  l_{n+2} - l_{n+1} & l_{n+1} - l_n & l_n - l_{n-1} & l_{n-1} - l_{n-2} & l_{n-3} - l_{n+2} \\
  l_{n+2} - l_{n+1} & l_{n+1} - l_n & l_n - l_{n-1} & l_{n-1} - l_{n-2} & l_{n-3} - l_{n+2} \\
  l_{n+2} - l_{n+1} & l_{n+1} - l_n & l_n - l_{n-1} & l_{n-1} - l_{n-2} & l_{n-3} - l_{n+2} \\
\end{pmatrix}
\]
for any integer $n$. Hence,

$$
\mathcal{M} = \mathcal{E} \cdot (\mathcal{M}_{5,4})^{-1} \left( R_5^{-1} \right)^{n-4},
$$

where

$$(\mathcal{M}_{5,4})^{-1} = \begin{pmatrix}
-1 & 1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 0 \\
1 & 0 & 0 & 0 & -1 \\
-1 & 1 & 0 & 0 & 1 \\
1 & -1 & 1 & 0 & -1
\end{pmatrix}.
$$

If we denote $v_n := t_{n-2} - t_{n-5} + 2(t_{n-4} - t_{n-3})$ we get by (28) and (30) the following very computationally effective form of the decoding matrix

$$(\mathcal{M}_{5,4})^{-1} \cdot \left( R_5^{-1} \right)^{n-4} = \begin{pmatrix}
\alpha_5 & v_{n-1} & v_{n-2} & v_{n-3} & v_{n+1} \\
v_{n+1} & v_{n} & v_{n-1} & v_{n-2} & v_{n+2} \\
v_{n-3} & v_{n-4} & v_{n-5} & v_{n-6} & v_{n-2} \\
v_{n-2} & v_{n-3} & v_{n-4} & v_{n-5} & v_{n-1} \\
v_{n-1} & v_{n-2} & v_{n-3} & v_{n-4} & v_{n}
\end{pmatrix}.
$$

In this case, we get the following form of relation among code matrix elements and code rate of this coding/decoding method by Lemma 5.

**Corollary 3.** Let $\alpha_5$ be dominant root of the sequence $(T_5(n))_n$. Then, for elements of the code matrix $\mathcal{E}$ the following asymptotic properties (the asymptotic values of all fractions in the table body are in the table header) hold

| $a_5$ | $a_5$ | $a_5$ | $a_5$ | $a_5^2$ | $a_5^2$ | $a_5^2$ | $a_5^2$ | $a_5^2$ | $a_5^2$ |
|-------|-------|-------|-------|---------|---------|---------|---------|---------|---------|
| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_1$   | $e_2$   | $e_3$   | $e_4$   | $e_5$   | $e_5$   |
| $e_6$ | $e_7$ | $e_8$ | $e_9$ | $e_6$   | $e_7$   | $e_8$   | $e_9$   | $e_10$  | $e_10$  |
| $e_{11}$ | $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{11}$ | $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{15}$ | $e_{15}$ |

where

$$a_5 \approx 1.236505703391499024337574800976 \ldots$$

and for the correction ability $S_{cor}$ holds

$$S_{cor} = \frac{2^{5^2} - 2}{2^{5^2} - 1} = \frac{33554430}{33554431} = 99.999997\%.$$

### 6. Conclusions

In this paper, we are interested in construction of a new coding/decoding system based on the sequence $(T_q(n))_n$, which is the extension of $(2,q)$-distance Fibonacci numbers introduced by Wloch et al. [34]. We show that the coding/decoding process can be based on sequences of a higher type than $(1,q)$-distance Fibonacci numbers, which was used in all previous papers, which immediately leads to the conclusion that our method significantly expands the group of currently used methods. Our proposal thus reduces the probability of an error due to noise or intentional modification of the message by a person who would enter the transmission channel without permission. Our research was motivated by Stakhov [23], who introduced the construction of so-called Fibonacci coding method.
His paper aroused great interest among many mathematicians who have designed their own encoding systems, mostly constructed on the basis of certain generalizations by introducing new additional coefficients into $p + 1$ order linear recurrence of Stakhov’s Fibonacci $p$-numbers. All of these papers contain the full construction of the coding system based on matrix multiplication, but they contain one very significant gap, as their authors only assumed the existence of the dominant root of the generalized Fibonacci $p$-numbers and polynomials, but did not prove their existence. Therefore, we first prove that our sequence $(T_q(n))_n$ has a dominant root for every odd integers $q \geq 3$ and that the criterion for “non omitted root summand” in Binet-like formula, what is used for the proof of existence of the generalized Kepler limit. Then, we construct a new type of generating matrix of the sequence $(T_q(n))_n$ by a new way to be more suitable for constructing of the coding/decoding method. We find the general relations among the code matrix elements $E$, which are analogous with the relations founded in previous papers, but our method is new and much simpler than the method introduced by Stakhov and replicated by his followers (see, e.g., [55]). Finally, we discuss the correction ability of our coding/decoding method and we construct an example of concrete coding/decoding system based on $(2, 5)$-distance Fibonacci numbers $T_5(n)$.

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