Some Energy Properties of Yang-Mills Connections

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Abstract

We consider a vector bundle $E$ over a compact Riemannian manifold $M = M^n$, $n \geq 4$, and $A$ is a Yang-Mills connection with $L^2$ curvature $F_A$ on $E$. Then we prove a mean value inequality for the density $|F_A|^2$. This inequality gives rise to an energy concentrate principle for sequences of solutions that have bounded energy. We also prove that the energy must be bounded from below by some positive constant unless $E$ is a flat bundle.

1 Introduction

Let $M$ be a compact $n$-dimensional Riemannian manifold and let $E$ be a vector bundle of rank $r$ over $M$ with structure group $G$, where $G$ is a compact Lie group. Let $A$ denote a connection on $E$. The Yang-Mills energy of $A$ is

$$YM(A) := \|F_A\|^2$$

where $F_A$ denotes the curvature of $A$, $\| \cdot \|$ denotes the $L^2$ norm. The critical points of $YM(A)$ are called Yang-Mills connections, they satisfy the Yang-Mills equation:

$$d_A^* F_A = 0 \quad (1.1)$$

We consider energy gap and energy concentrate phenomena for the Yang-Mills connections. The energy gap phenomena first time considered by Bourguignon and Lawson in [2]. They proved in ([2], Theorem C) for any Yang-Mills connection that over $S^n$, $(n \geq 3)$, the curvature of the connection satisfies the pointwise estimate

$$F^2 = -tr(F_{\alpha\lambda}F^{\mu\lambda}) < n(n - 1)/2 \quad (1.2)$$

then the connection is flat. In [4], Gerhardt considered a compact Riemannian manifold, $M$, with a metric which satisfied the condition

$$R_{\alpha\beta}\Lambda_\lambda^\lambda \Lambda_\beta^\lambda - \frac{1}{2} R_{\alpha\beta\mu\lambda} \Lambda_\alpha^\beta \Lambda_\mu^\lambda \geq c_0 \Lambda_\alpha^\beta \Lambda_\alpha^\beta \quad (1.3)$$

for all skew-symmetric $\Lambda_\alpha^\beta \in T^{0,2}(M)$, where $R_{\alpha\beta}$, $R_{\alpha\beta\mu\lambda}$ are the Ricci and Riemann curvature tensors and $0 < c_0$. Then Gerhardt proved the following theorem.
**Theorem 1.1.** (4, Theorem 1.2) Let $M$ be a compact manifold for which the condition (1.3) with $c_0 > 0$ holds. Then the Yang-Mills connections over $M$ with compact, semi-simple Lie group either are flat or satisfy

$$\left(\int_M |F|^2 \right)^{\frac{n}{2}} \geq k_0$$

(1.4)

for some constant $k_0 > 0$ depending only on the Sobolev constants of $M$, $n$, $c_0$ and the dimension of the Lie group $G$.

In this paper, we provide an alternative proof to Theorem 1.1. When we consider an arbitrary compact Riemannian manifold, we cannot obtain similar result, however we can prove either any Yang-Mills connection satisfies (1.4) or the vector bundle $E$ is a flat bundle, i.e. there exists a flat connection over the bundle.

**Theorem 1.2.** Let $M = M^n$, $n \geq 4$, be a compact Riemannian manifold and $E$ be a vector bundle over $M$. Then either any Yang-Mills connection over $M$ with compact, semi-simple Lie group satisfies (1.4) for some constant $k_0 > 0$ depending on $M$, $n$ or the bundle $E$ is smoothly isomorphic to a flat bundle.

## 2 Preliminaries and basic estimates

First, we recall some standard notations and definitions.

Let $T^*M$ be the cotangent bundle of $M$ and for $1 \leq p \leq n$, let $\Lambda^p(M)$ be the $p$-form bundles on $M$ with $T^*M = \Lambda^1M$. One can form the associated bundle $E \otimes \Lambda^p$. Let $\Omega^p(E)$ be the set of sections of $E \otimes \Lambda^p$. Let $g$ be the Lie algebra of $G$, $Ad : G \to Aut(g)$ be the adjoint representation and $adE$ be the associated adjoint vector bundle.

Denote $\Omega^p(ad(E)) = \Gamma(adE \otimes \Lambda^p(M))$. For a connection $A$ on $E$, we have exterior derivatives

$$d_A : \Omega^p(adE) \to \Omega^{p+1}(adE).$$

These are uniquely determined by the properties (see [3], p.35):

1. $d_A = \nabla_A$ on $\Omega^0(adE)$
2. $d_A(\alpha \wedge \beta) = d_A\alpha \wedge \beta + (-1)^p \alpha \wedge d_A\beta$

for any $\alpha \in \Omega^p(adE)$, $\beta \in \Omega^q(adE)$.

The curvature $F_A \in \Omega^2(ad(E))$ of the connection $A$ is defined by

$$d_A d_A u = F_A u$$

for any section $u \in \Gamma(E)$. If $A$ is a connection on $E$, we can define covariant derivatives

$$\nabla_A : \Omega^p(E) \to \Gamma(\Lambda^pT^*M \otimes T^*M \otimes E)$$
For $\nabla_A$ and $d_A$, we have adjoint operators $\nabla_A^*$ and $d_A^*$. We also have Weitzenböck formula (2), Theorem 3.10

$$ (d_A d_A^* + d_A^* d_A) \varphi = \nabla_A^* \nabla_A \varphi + \varphi \circ (Ric \wedge g + 2R) + \mathcal{R}^A(\varphi) $$ (2.1)

where $\varphi \in \Omega^2(\text{ad}(E))$, $Ric$ is the Ricci tensor and $R$ denotes the Riemannian curvature tensor.

The operator of $Ric \wedge g + 2R$ and $\varphi \circ (Ric \wedge g + 2R)$ are defined by Bourguignon and Lawson [2]. They are

$$ (Ric \wedge g)_X Y = Ric(X) \wedge Y + X \wedge Ric(Y) $$

and

$$ \varphi \circ (Ric \wedge g + 2R)(X, Y) = \frac{1}{2} \sum_{j=1}^{n} \varphi(e_j, (Ric \wedge g + 2R)_X Y(e_j)) $$

In a local orthonormal frame $(e_1, \ldots, e_n)$ of $TM$, the quadratic term $\mathcal{R}^A(F_A) \in \Omega^2(\text{ad}(E))$ can be expressed as

$$ \mathcal{R}^A(F_A)(X, Y) = 2 \sum_{j=1}^{n} [F_A(e_j, X), F_A(e_j, Y)] $$

**Lemma 2.1.** Let $M$ be a compact Riemannian manifold and $\lambda$ be the minimal eigenvalue of the operator $Ric \wedge g + 2R$. Assume that $\lambda$ is positive. If $A$ is a Yang-Mills connection and $\|F_A\|_{L^\infty}$ is sufficiently small then $A$ is flat.

**Proof.** From the Weitzenböck formula (2.1), we have

$$ (d_A d_A^* + d_A^* d_A)F_A = \nabla_A^* \nabla_A F_A + F_A \circ (Ric \wedge g + 2R) + \mathcal{R}^A(F_A) $$

The left hand side vanishes by (1.1) and the Bianchi identity $d_A F_A = 0$. Taking inner product with $F_A$ in $L^2$ we get

$$ 0 = \|\nabla_A F_A\|_{L^2}^2 + \langle F_A, F_A \circ (Ric \wedge g + 2R) \rangle + \langle F_A, \mathcal{R}^A(F_A) \rangle $$

$$ \geq \|\nabla_A F_A\|_{L^2}^2 + (\lambda - 4\|F_A\|_{L^\infty})\|F_A\|_{L^2}^2 $$ (2.2)

If $\|F_A\|_{L^\infty}$ is sufficiently small, then $A$ is flat. Here we have used our assumption

$$ \langle F_A, F_A \circ (Ric \wedge g + 2R) \rangle \geq \lambda\|F_A\|_{L^2}^2 $$

and the fact

$$ |\langle F_A, \mathcal{R}^A(F_A) \rangle| \leq 4\|F_A\|_{L^\infty}\|F_A\|_{L^2}^2. $$

$\square$
In fact, the condition (1.3) with $c_0 > 0$ is equivalent to the positivity of $\lambda$ which is the minimal eigenvalue of the operator $\text{Ric} \wedge g + 2R$. Thanks to Uhlenbeck’s work ([7], Theorem 3.5) one can control the $L^\infty (M)$ norms of the curvature $F_A$ by the $L_2^n$ norms. Then by Lemma 2.1, we provided another proof of Theorem 1.1.

**Remark 2.2.** There are $M$ such that $\text{Ric} \wedge g + 2R$ is a positive operator, for example,

1. $S^n$, where $\lambda \equiv 2(n - 1)$;
2. $M$ with the positive curvature operator, $\text{Ric} \wedge g + 2R$ must be positive;
3. $M$ with section curvature $\bar{R}$ which satisfy

$$\alpha \bar{R} \max \leq \bar{R} \leq \bar{R} \max (\alpha \geq 1 - \frac{2}{2n - 2}) \ (\text{see [1]}, \ p. 79)$$

From the Weitzenböck formula ([2], Theorem 3.10), we can also obtain a differential inequality for $|F_A|^\frac{2}{n}$. The proof is similar to the case $n = 4$ (see [3]).

**Lemma 2.3.** Let $M$ be a compact $n$-dimensional Riemannian manifold, $n \geq 4$ and $A$ be a Yang-Mills connection, then $|F_A|^\frac{2}{n}$ satisfies

$$\Delta |F_A|^\frac{2}{n} \leq C_1 |F_A|^\frac{2}{n} + c |F_A|^\frac{n+2}{n} \quad (2.3)$$

where $C_1, c$ only depend on the metric on $M$.

**Proof.** Form the Weitzenböck formula 2.1, we have

$$(d_A d_A^* + d_A^* d_A) F_A = \nabla_A \nabla_A F_A + F_A \circ (\text{Ric} \wedge g + 2R) + R^A(F_A)$$

The left hand side vanishes form 1.1 and $d_A F_A = 0$. The quadratic term $R^A(F_A) \in \Omega^2(\text{ad}(E))$ can be expressed with the help of a local orthonormal frame $(e_1, \ldots, e_n)$ of $TM$ as

$$R^A(F_A)(X, Y) = 2 \sum_{j=1}^n [F_A(e_j, X), F_A(e_j, Y)]$$

The estimate for the Laplacian now follow from

$$-\nabla_A \nabla_A |F_A|^\frac{2}{n} = \frac{n}{2} \langle \nabla_A \nabla A F_A, F_A \rangle \langle F_A, F_A \rangle^{-\frac{4}{n}-1} - \frac{n}{2} \langle \nabla_A F_A, \nabla A F_A \rangle \langle F_A, F_A \rangle^{-\frac{4}{n}-1}$$

$$\leq \frac{n}{2} \langle \nabla_A \nabla A F_A, F_A \rangle \langle F_A, F_A \rangle^{-\frac{4}{n}-1}$$

$$\leq \frac{n}{2} \langle (F_A, F_A \circ (\text{Ric} \wedge g + 2R)) + (F_A, R^A(F_A)) \rangle \langle F_A, F_A \rangle^{-\frac{4}{n}-1}$$

$$\leq C|F_A|^\frac{2}{n} + c|F_A|^\frac{n+2}{n} \quad (2.4)$$
Here the constant $C$ depends on the Ricci transform $\text{Ric}$ and the scalar curvature $R$ of the metric on $M$. The constant $c$ only depends on the metric.

**Theorem 2.4.** ([8], Theorem 7) Let $M$ be a compact Riemannian $n$-manifold. Let $U = \{U_\alpha\}_{\alpha \in I}$ be a finite open covering of $M$ such that any two points $x, y$ in a nonempty intersection $U_\alpha \cap U_\beta$ can be connected by a $C^1$ curve within $U_\alpha \cap U_\beta$ with length $\leq l$, a uniform constant. Let $\{g_{\alpha\beta}\}$ be a set of smooth transition functions with respect to $U$. Then there exist a constant $\varepsilon_1 = \varepsilon_1(M, l, U) > 0$, such that if

$$\sup_{x \in U_{\alpha\beta}} |\nabla g_{\alpha\beta}(x)| \leq \varepsilon_1, \forall \alpha, \beta \in I$$

then the bundle defined by $\{g_{\alpha\beta}\}$ is smoothly isomorphic to a flat bundle.

### 3 Proof of The Main Theorem

Let $\{\frac{\partial}{\partial x_i}\}_{i=1}^n$ and $\{dx_i\}_{i=1}^n$ denote respectively the basis of the tangent bundle $TM$ and cotangent bundle $T^*M$ on $B_r$ with $r \leq i(M)$, where $i(M)$ is the infimum of the injectivity radius of each point $x \in M$. Let $(g_{ij})$ be a Riemannian metric of $M$ by

$$\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle = g_{ij}, \quad \langle dx_i, dx_j \rangle = g^{ij}$$

where $(g^{ij}) = (g_{ij})^{-1}$. For any $x_0 \in M$, there exists normal coordinates in the geodesic ball $B_{r_0}(x_0)$ at the center $x_0$ with radius $r_0 \leq i(M)$ and for some constant $C$, we have

$$|g_{ij} - \delta_{ij}| \leq C|x|^2, \quad |\frac{\partial g_{ij}}{\partial x_k}| \leq C|x|, \forall x \in B_{i(M)}$$

**Proposition 3.1.** For every $n \in \mathbb{N}$ there exist constants $C_0$ and $\delta > 0$ such that the following holds for all $0 < r \leq 1$ and all metrics $g$ on $\mathbb{R}^n$ with $\|g_{ij} - \delta_{ij}\|_{W^{1,\infty}} \leq \delta$. If $v \in C^2(B_r(0))$ and $v \geq 0$ satisfies $\Delta v \leq 0$, then

$$v(0) \leq C_0 r^{-n} \int_{B_r(0)} v$$

The above proposition is a special case of Theorem 2.1 in [5]. The starting point for the proof is Morrey’s [6] mean value inequality for subharmonic functions.

**Lemma 3.2.** Let $B_r(0)$ be a geodesic ball of radius $r$, $0 < r \leq 1$, which is sufficiently small. Then there exists constant $C_2, \mu > 0$, we have either

$$\int_{B_r(0)} |F_A|^\frac{n}{2} \geq \mu c^{-\frac{n}{2}}$$
or

\[ |F_A|^2 (0) \leq C_2 (C_1^2 + r^{-n}) \int_{B_r(0)} |F_A|^2 \]

where \( c \) and \( C_1 \) are the same constants as in Lemma 2.3.

Proof. We denote \( e = |F_A|^2 \). Consider the function \( f(\rho) = (1-\rho)^n \sup_{B_{\rho r}(0)} e \) for \( \rho \in [0,1] \). It attains its maximum at some \( \bar{\rho} < 1 \). Let \( \bar{a} = \sup_{B_{\rho r}(0)} e = e(\bar{x}) \) and \( \delta = \frac{1}{2} (1 - \bar{\rho}) < \frac{1}{2} \), then

\[ e(0) = f(0) \leq f(\bar{\rho}) = 2^n \delta^n \bar{a} \]

Moreover, we have for all \( x \in B_{\bar{\rho} r}(\bar{x}) \subset B_r(0) \)

\[ e(x) \leq \sup_{B(\bar{\rho}+\delta r)(0)} e = (1 - \bar{\rho} - \delta)^{-n} f(\bar{\rho} + \delta) \leq 2^n (1 - \bar{\rho})^{-n} f(\bar{\rho}) = 2^n \bar{a} \]

From lemma 2.3, \( \Delta e \leq C_1 e + ce^{\frac{n+2}{2}} \), we have

\[ \Delta e \leq 2^n C_1 \bar{a} + 2^{n+2} c \bar{a}^{\frac{n+2}{n}} \]

Now we define the function

\[ v(x) := e(x) + \frac{1}{n} (2^n \bar{a} (C_1 + 4n \bar{a} \frac{2}{n})) |x - \bar{x}|^2 \]

with the Euclidean norm \( |x - \bar{x}| \). It is nonnegative and subharmonic on \( B_{\bar{\rho} r}(\bar{x}) \) if the metric \( g_{ij} \) is sufficiently \( C^1 \)-close to \( \delta_{ij} \). This can be seen as follows \( \Delta_0 |x - \bar{x}|^2 = -2n \), where \( \Delta_0 = - \sum_{i=1}^n \partial^2_i \), and \( |x - \bar{x}| \leq \delta r \leq 1 \) is bounded, so \( \Delta |x - \bar{x}|^2 \leq n \) whenever \( \|g_{ij} - \delta_{ij}\|_{W^{1,\infty}} \leq \epsilon \) is sufficiently small. If not, from (3.1) we can choose a smaller radial so it is true. The control of the metric also ensures that the integral \( \int_{B_{\rho r}(\bar{x})} |x - \bar{x}|^2 \) is bounded by the following integral over the Euclidean ball \( B_{2\rho r}(\bar{x}) \): With the constant \( C_3 = 2^{n+2} Vol S^{n-1} / (n + 2) \)

\[ 2 \int_{B_{2\rho r}(\bar{x})} |x - \bar{x}|^2 = 2 \int_{0}^{2\rho r} t^{n+1} Vol S^{n-1} \, dt = C_2 (\rho r)^{n+2} \]

So from (3.1), then for function \( v(x) \), we have

\[ v(x) \leq C_0 ( \rho r )^{-n} \int_{B_{\rho r}(\bar{x})} v \]

(3.2)

Let \( C_4 = \max\{C_0, \frac{1}{n} 2^n C_1 C_2\} \). The for all \( 0 < \rho \leq \delta \), from (3.2), we get

\[ \bar{a} = v(\bar{x}) \leq C_4 \bar{a} (C_1 + 4n \bar{a} \frac{2}{n} ) ( \rho r )^2 + C_4 ( \rho r )^{-n} \int_{B_{\rho r}(\bar{x})} e \]

(3.3)
If $C_4(C_1 + 4c\tilde{a}^\frac{2}{n})(\rho r)^2 \leq \frac{1}{2}$, then $\tilde{a} \leq 2C_4(\rho r)^{-n} \int_{B_r(0)} e$. So if $C_4(C_1 + 4c\tilde{a}^\frac{2}{n})(\delta r)^2 \leq \frac{1}{4}$ then $\rho = \delta$ proves the assertion,

$$e(0) \leq 2^n \delta^n \tilde{a} \leq 2^{n+1} C_4 r^{-n} \int_{B_r(0)} e$$

Otherwise we can choose $0 < \rho < \delta$ such that $(\rho r)^{-2} = 2C_4(C_1 + 4c\tilde{a}^\frac{2}{n})$. Then we obtain with $C_5 = (2C_4)^{1+\frac{n}{2}}$

$$e(0) \leq \tilde{a} \leq C_5(C_1 + 4c\tilde{a}^\frac{2}{n}) \frac{1}{2} \int_{B_{\rho r}(\tilde{a})} e$$

Again, we have to distinguish two cases: Firstly, if $4c\tilde{a}^\frac{2}{n} \leq C_1$ then this yields

$$e(0) \leq C_5(2C_1) \frac{1}{2} \int_{B_{\rho r}(\tilde{a})} e$$

Secondly, if $C_1 < 4c\tilde{a}^\frac{2}{n}$ then $\tilde{a} < aC_5(8c)^{\frac{1}{2}} \int_{B_{\rho r}(\tilde{a})} e$ and thus with $\mu = 8^{-\frac{1}{2}} C_5^{-1} > 0$ we have

$$\int_{B_r(0)} e > \mu e^{-\frac{1}{2}}$$

So we either have the above or with some constant $C_2 = \max\{2^{n+1} C_4, 2^{\frac{2}{n}} C_5\}$

$$e(0) \leq C_2(C_1^{\frac{2}{n}} + r^{-n}) \int_{B_r(0)} e$$

Remark 3.3. By using local geodesic coordinates the above lemma also implies a mean value inequality on closed Riemannian manifolds with uniform constants $C_2, \mu$, and for all geodesic balls radius less than a uniform constant.

Theorem 3.4. Let $M = M^n$, $n \geq 4$, be a compact Riemann manifold and $A_i$ be a sequence Yang-Mills connection, we denote $e_i = |F_{A_i}|^{\frac{n}{2}}$, assume that there a uniform bounded $\int_M e_i \leq E < \infty$. Then there exists finitely many points, $x_1, x_2, \ldots, x_N \in M$ (with $N \leq E/\nu$) and a sequence such that the $e_i$ are uniformly bounded on every compact subset of $M \setminus \{x_1, x_2, \ldots, x_N\}$, and there is a concentration of energy $\nu$ at each $x_j$: For every $r > 0$ there exists $N_{j,r} \in \mathbb{N}$ such that

$$\int_{B_r(x_j)} e_i \geq \nu, \quad \forall i \geq N_{j,r} \quad (3.4)$$

where $\nu$ is a constant only depending on $n, M$. 

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Proof. Suppose that for some \( x \in M \) there is no neighbourhood on which the \( e_i \) are uniformly bounded. Then there exists a subsequence (again denoted \( e_i \)) and \( M \ni y_i \to x \) such that \( e_i(y_i) = R_i^n \) with \( R_i^n \to \infty \). We can then apply the lemma 3.2 on the balls \( B_{r_i}(y_i) \) of the radius \( r_i = R_i^{-\frac{1}{2}} > 0 \). For sufficiently large \( i \in \mathbb{N} \), there lie with appropriate coordinates charts of \( M \). The lemma 3.2 provide uniform constant \( C_2 \) and \( \nu = \mu c^{-\frac{\pi}{2}} > 0 \) such that for every \( i \in \mathbb{N} \) either
\[
\int_{B_{r_i}(y_i)} e_i > \nu
\] (3.5)
or \( \int_{B_{r_i}(y_i)} e_i \leq \nu \) and hence
\[
R_i^n = e(y_i) \leq C_2(C_1^n + r_i^{-n}) \int_{B_{r_i}(y_i)} e_i
\]
In the latter case multiplication by \( r_i^n = R_i^{-\frac{n}{2}} \) implies
\[
R_i^n \leq C_2 \nu(C_1^n R_i^{-\frac{n}{2}} + 1)
\] (3.6)
As \( i \to \infty \), the left hand side diverges to \( \infty \), where the right hand side converges to \( C_2 \nu \). Thus the alternative (3.5) must hold for all sufficiently large \( i \in \mathbb{N} \). In particular, this implies the energy concentration (3.4) at \( x_j = x \).

Now we can go through the same argument for any other point \( x \in M \) at which the present subsequence \( e_i \) is not locally uniformly bounded. That way we iteratively find points \( x_j \in M \) such that iteration yields \( N \leq E/\nu \) distinct points \( x_1, x_2, \ldots, x_N \) (and might not even terminate after that). Then we have a subsequence \( e_i \) for which at least energy \( \nu > 0 \) concentrates near each \( x_j \). Since the points are distinct, this contradicts the energy bound \( \int_M e_i \leq E \). Hence this iteration must stop after at most \( E/\nu \) steps, when the present subsequence \( e_i \) is locally uniformly bounded in the complement of the finitely many points, where we found the energy concentration before. \( \square \)

To prove Theorem 1.2 we need only consider the case where \( A \) is a Yang-Mills connection on \( E \) with \( \|F_A\|_{L^\frac{1}{2}} \) sufficiently small. We have the following theorem proved by Uhlenbeck.

**Theorem 3.5.** ([7], Theorem 3.5) There exists a constant \( \varepsilon_1 \) such that if \( F_A \) is Yang-Mills field in \( B_{a_0}(x_0) \) and \( \int_{B_{2a_0}(x_0)} |F_A|_{L^\frac{1}{2}} < \varepsilon_2 \), then \( |F_A(x)| \) is uniformly bounded in the interior of \( B_{2a_0}(x_0) \) and
\[
|F_A(x)|^2 \leq C_6(a^{-n} \int_{B_a(x)} |F_A|_{L^\frac{1}{2}})\frac{4}{\pi}
\]
for all \( B_a(x) \subset B_{a_0}(x_0) \).
Lemma 3.6. Let \( \rho \) be the injectivity radius of \( M \).

Assume that \( \|F_A\|_{L^\infty(M)} \) is sufficiently small. Then from the above Theorem 3.5, we have

\[
\|F_A\|_{L^\infty} = \sup_{x \in M} |F_A|(x) \leq C_6 \rho^{-4} \|F_A\|_{L^2}
\]

here \( \rho \) is the injectivity radius of \( M \).

**Lemma 3.6.** Let \( M \) be a compact \( n \)-dimensional Riemannian manifold and \( E \) is smooth vector bundle over \( M \). Let \( A \) be a smooth connection on \( E \), then there exists \( \varepsilon_3 = \varepsilon_3(M) > 0 \), such that if

\[
\|F_A\|_{L^\infty} \leq \varepsilon_3
\]

then \( E \) is smoothly isomorphic to a flat bundle.

**Proof.** We cover \( M \) with coordinate balls \( \{U_\alpha\} \) such that any two points \( x, y \) in a nonempty intersection \( U_\alpha \cap U_\beta \) can be connected by a \( C^1 \) curve within \( U_\alpha \cap U_\beta \) with length \( \leq \text{diam}(M) \). Let \( \phi_\alpha : E|_{U_\alpha} \to B_1(0) \times \mathbb{R}^r \) be trivializations on \( U_\alpha \) and \( A_\alpha \) be the \( g \)-value 1-form on \( U_\alpha \) corresponding to \( A \) under \( \phi_\alpha \).

Let \( x_\alpha \) be the center of the \( U_\alpha \). For any point \( x \in U_\alpha \), we let \( \gamma^x_\alpha \) be the shortest geodesic from \( x_\alpha \) to \( x \) inside \( U_\alpha \) and define \( h_\alpha(x) \in G \) to be the parallel transport of the bundle from \( x_\alpha \) to \( x \) along \( \gamma^x_\alpha \), using the trivialization of the bundle.

Note that \( h_\alpha(x_\alpha) = \text{Id} \). We regard \( h_\alpha^{-1} \) as gauge transformations on \( U_\alpha \), and denote \( h_\alpha^{-1}(A) \) by \( \tilde{A}_\alpha \). We use the normal spherical coordinates \( \{r, \theta^j\}_{j=1, \ldots, n-1} \). Let us assume that

\[
\tilde{A}_\alpha = \tilde{A}_{\alpha, r} dr + \tilde{A}_{\alpha, j} d\theta^j, \text{ on } U_\alpha
\]

and

\[
F_{\tilde{A}_\alpha} = F_{\tilde{A}_{\alpha, r}} dr \wedge d\theta^j + F_{\tilde{A}_{\alpha, j}} d\theta^i \wedge d\theta^j, \text{ on } U_\alpha
\]

Then by the definition of \( h_\alpha \), we have \( \tilde{A}_{\alpha, r} \equiv 0 \) on \( U_\alpha \). Hence

\[
\partial_r(\tilde{A}_{\alpha, j}) = F_{\tilde{A}_{\alpha, j}}, \quad j = 1, \ldots, n-1 \tag{3.7}
\]

By integrating (3.7) and \( \tilde{A}_\alpha(0) = 0 \), we have

\[
|\tilde{A}_\alpha|(x) \leq |x| \int_0^1 |F_{\tilde{A}_\alpha}|(tx) \, dt \leq \varepsilon_3 r_\alpha \tag{3.8}
\]

We define \( h_{\alpha\beta} = h_\alpha^{-1}(\phi_\alpha \cdot \phi_\beta^{-1})h_\beta \) on \( U_\alpha \cap U_\beta \) and we can check that \( \{h_{\alpha\beta}\} \) is a set of transition functions. Now we have

\[
dh_{\alpha\beta} = dh_\alpha^{-1}(\phi_\alpha \cdot \phi_\beta^{-1})h_\beta + h_\alpha^{-1} d(\phi_\alpha \cdot \phi_\beta^{-1})h_\beta + h_\alpha^{-1}(\phi_\alpha \cdot \phi_\beta^{-1}) dh_\beta
\]

\[
= dh_\alpha^{-1}h_{\alpha\beta} + h^{-a}(A_\alpha\phi_\alpha \cdot \phi_\beta^{-1})h_\beta
\]

\[
- h^{-a}(\phi_\alpha \cdot \phi_\beta^{-1})A_\beta h_\beta + h_\alpha^{-1}A_\alpha(\phi_\alpha \cdot \phi_\beta^{-1}) d h_\beta
\]

\[
= h_\alpha^{-1}(A_\alpha) \circ h_{\alpha\beta} - h_{\alpha\beta} \circ h^{-1}_\beta(A_\beta) \tag{3.9}
\]
where we using $h^{-1}(A) = h^{-1}A h + h^{-1} d h$, $h^{-1}(A) = h^{-1}A h + h^{-1} d h$ and $d(\phi \cdot \phi^{-1}) = A \phi \cdot \phi^{-1} - (\phi \cdot \phi^{-1}) A$. Hence from (3.5), we have

$$|\nabla h_{\alpha\beta}| \leq \varepsilon_3, \text{ on } U_{\alpha} \cup U_{\beta}$$

By taking $\varepsilon_3$ sufficiently small from Lemma 2.4 we establish the lemma. ☐

Remark 3.7. We cannot conclude that the Yang-Mills connection, $A$ on $E$, in Lemma 3.6 with $L^2$-small curvature, $F_A$, is itself flat, but rather just that $E$ supports some flat connections and thus is a flat bundle.

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