INTEGRAL TQFT FOR A ONE-HOLED TORUS

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ABSTRACT. We give new explicit formulas for the representations of the mapping class group of a genus one surface with one boundary component which arise from Integral TQFT. Our formulas allow one to compute the $h$-adic expansion of the TQFT-matrix associated to a mapping class in a straightforward way. Truncating the $h$-adic expansion gives an approximation of the representation by representations into finite groups. As a special case, we study the induced representations over finite fields and identify them up to isomorphism. The key technical ingredient of the paper are new bases of the Integral TQFT modules which are orthogonal with respect to the Hopf pairing. We construct these orthogonal bases in arbitrary genus, and briefly describe some other applications of them.

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1. Introduction

Consider the integral TQFT studied in [G1, GM1]. Associated to a compact oriented surface $\Sigma$ and an odd prime $p$, there is a free lattice (i.e., a free finitely generated module) $S_p^+(\Sigma)$ over the ring of cyclotomic integers $O_p^+ = \mathbb{Z}[\zeta_p]$ formed by adjoining to $\mathbb{Z}$ a primitive $p$-th root of unity $\zeta_p$. Tensoring $S_p^+(\Sigma)$ with the cyclotomic field $\mathbb{Q}(\zeta_p)$ yields a version of the Reshetikhin-Turaev TQFT associated with the Lie group $SO(3)$, and we think of $S_p^+$ as an integral refinement of that theory (see [GM1] for more details).

There is a natural representation $\rho_p$ of a certain central extension of $\Gamma_\Sigma$, the mapping class group of $\Sigma$, on the lattice $S_p^+(\Sigma)$. Let $h$ denote $1 - \zeta_p$; this is a prime in $O_p^+$. For every $N \geq 0$, the representation $\rho_p$ induces a representation $\rho_{p,N}$ on

$$S_{p,N}^+(\Sigma) = S_p^+(\Sigma)/h^{N+1}S_p^+(\Sigma),$$

which is a free module over the quotient ring $O_p^+/h^{N+1}O_p^+$. Note that for $N = 0$ this ring is the finite field $\mathbb{F}_p$, so that $\rho_{p,0}$ is a representation on a finite-dimensional

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Fp-vector space. The representation ρp,0 factors through the mapping class group ΓΣ [GM2], but this is not the case for ρp,N when N > 0.

Loosely speaking, we refer to the sequence of representations ρp,N as the h-adic expansion of the TQFT representation ρp. Note that each ρp,N factors through a finite group, since S p,N(Σ) is of finite rank over O p/hN+1O p, which itself is finite. Moreover, the natural map
\[ O_p^+ = \mathbb{Z}[\zeta_p] \to \lim_{\leftarrow} O_p^+/h^{N+1}O_p^+ \cong \mathbb{Z}_p[\zeta_p] \]

is injective. (Here Zp denotes the p-adic integers.) The last isomorphism comes from the fact that \((h^{p-1}) = (p)\) as ideals in Zp[ζp].

Thus the h-adic expansion approximates the TQFT-representation ρp by representations into bigger and bigger finite groups.

We believe these representations into finite groups deserve further study. Unfortunately, the known formulas for ρp are not well suited to computing the h-adic expansion explicitly in practice, and new techniques are needed. In this paper we complete the first step in this direction. That is: we compute explicit matrices for the h-adic expansion in the case when Σ = Σ1,1 is a genus one surface with one boundary component. Our formulas give partial information in higher genus as well, but as we don’t yet have a complete answer, we won’t discuss formulas for the h-adic expansion in the higher genus case in this paper.

We refer to Σ1,1 as a one-holed torus. As is customary in the TQFT world, when dealing with surfaces with boundary, in practice one caps off the boundary circles by disks containing one colored banded point each in their interiors. (A banded point is a small oriented interval.) When doing this for Σ1,1, we denote the resulting closed torus by Tc, where 2c is the color of the banded point; here 2c can be any even integer satisfying 0 ≤ 2c ≤ p−3. Mapping classes have to preserve the banded point, so that the Dehn twist tδ about a simple closed curve δ encircling the colored banded point is non-trivial. Thus we are indeed dealing with Γ1,1, the mapping class group of the one-holed torus. This group has presentation
\[ \Gamma_{1,1} = \langle t_\alpha, t_\beta | t_\alpha t_\beta t_\alpha = t_\beta t_\alpha t_\beta > \]

where tα and tβ are Dehn twists tα and tβ about essential simple closed curves α and β which avoid the colored banded point and intersect exactly once. One has
\[ t_\delta = (t_\alpha t_\beta)^{2c}. \]

We denote by t and t* the endomorphisms of S p+(Tc) given as ρp of certain lifts of tα and tβ to the extended mapping class group. The notation t* is motivated by the fact that t and t* are adjoint with respect to a symmetric bilinear form \((, )\) called the Hopf pairing which will play a crucial role in this paper. To describe the TQFT representation ρp in our situation, it suffices to write down matrices for t and t*.

In the TQFT defined over the cyclotomic field Q(ζp), it is well-known how to do this, because the endomorphism t can be diagonalized over Q(ζp) using a certain basis of ‘colors’, and the entries of tt* in this basis can then be computed from an appropriate (generalized) S-matrix. However this basis in which t is diagonal is not an O p+-basis for the lattice S p+(Tc). For example, the coefficients of the matrix of

\[^1\text{Here, we mean the h-adic expansion at a fixed p-th root of unity, not to be confused with an Ohsuki-style expansion when p goes to infinity (hence q = ζ_p goes to 1) which is discussed in [M3].}\]
$t^*$ in this basis do not lie in $\mathcal{O}^+_{p}$. Hence we cannot compute the $h$-adic expansion of the representation $\rho_p$ using this basis.

On the other hand, for an arbitrary surface $\Sigma$ equipped with colored banded points, a $\mathcal{O}^+_{p}$-basis for the lattice $\mathcal{S}^+_{p}(\Sigma)$ was constructed explicitly in our paper [GM1]. In this lollipop basis, all mapping classes are represented by matrices with coefficients in $\mathcal{O}^+_{p}$. However, the explicit formulas we get from [GM1] are still not good enough to compute the $h$-adic expansion in a nice way.

In the present paper, we will describe a modification of the construction of [GM1]. The crucial additional property of the bases obtained here is that they are orthogonal with respect to a Hopf-like pairing $((\cdot,\cdot))$ which generalizes the one mentioned above in the genus one case. We refer to this new basis as the orthogonal lollipop basis. We perform its construction in the general situation of a surface $\Sigma$ equipped with colored banded points, as the higher genus case presents no extra difficulty, and give two immediate applications (see Remarks 3.5 and 3.6). Moreover, [G4] and work in progress [M3,GM3] make use of this basis in higher genus as well.

Let us end this introduction with a brief description of how the TQFT-representation $\rho_t$ looks like in the orthogonal lollipop basis for the one-holed torus, and discuss an application we find interesting. The lattice $\mathcal{S}^+_{p}(\mathcal{T}_c)$ has rank $d-c$, where $d = (p-1)/2$. The basis we’ll construct is denoted by

$$Q_n^{(c)} = (0 \leq n \leq d - c - 1).$$

One of its main features is that $t$ is upper triangular and $t^*$ is lower triangular in this basis, as will be shown in Section 4. This fact is also the main reason why the new basis is better than the one described in [GM1], where only one of $t$ and $t^*$ would be given by triangular matrix.

The matrix coefficients of $t$ and $t^*$ in this basis will be denoted by $a_{m,n}^{(c)}$ and $b_{m,n}^{(c)}$, respectively, and we will give explicit formulas for them in Section 4. These formulas can be used to compute the $h$-adic expansion of $a_{m,n}^{(c)}$ and $b_{m,n}^{(c)}$; in particular, they make it evident that $a_{m,n}^{(c)}$ and $b_{m,n}^{(c)}$ lie in $\mathbb{Z}[\zeta_p]$.

The fact that $t$ is upper triangular and $t^*$ is lower triangular is reminiscent of the action of $SL(2,\mathbb{Z})$ on homogeneous polynomials in two variables. Motivated by this observation, we will show in Section 5 that the representation $\rho_{p,0}$ on the $\mathbb{F}_p$-vector space $\mathcal{S}^+_{p,0}(\mathcal{T}_c)$ factors through $SL(2,\mathbb{F}_p)$ and is, in fact, isomorphic to the representation of $SL(2,\mathbb{F}_p)$ on the space of homogeneous polynomials over $\mathbb{F}_p$ in two variables of the appropriate degree $D = d - c - 1$.

In order to put this last result into perspective, we remark that when $c = 0$, it is known [LW,FKI,G3] that the representation $\rho_p$ itself factors through (a finite central extension of) $SL(2,\mathbb{F}_p)$, as the situation then reduces to the classical case of the closed torus. In contrast, when $c > 0$, there is no reason to believe that the representation should factor through $SL(2,\mathbb{F}_p)$. Indeed, we conjecture that the representation $\rho_p$ does not factor through any finite group, although, of course, each $\rho_{p,N}$ does, as was explained above. It would follow that the quotient group of the extended mapping class group which acts effectively under the representation $\rho_{p,N}$ must get bigger and bigger as $N$ increases, and one of our motivations for this paper was to develop techniques to compute the representations $\rho_{p,N}$ explicitly and efficiently.

The contrast between the cases $c = 0$ and $c > 0$ are, of course, expected to be related to geometry: a closed torus is Euclidean, whereas a torus with non-empty
boundary is hyperbolic. We remark that it is shown in unpublished work of the second author [M2] that for \( c = d - 2 \), any pseudo-Anosov mapping class \( \varphi \) in \( F_{1,1} \) is represented by a matrix of infinite order under \( \rho_p \) if \( p \) is big enough. (See [AMU] for a proof of the analogous result for the mapping class group of a four-holed sphere.) In this situation, it follows that the order of \( \rho_{p,N} (\varphi) \) must go to infinity as \( N \) increases, showing in particular that the above-mentioned conjectural picture is correct at least in this case.

2. THE ORTHOGONAL LOLLIPOP BASIS FOR A ONE-HOLED TORUS

Let \( p \geq 5 \) be a prime, and \( \zeta_p \) be a primitive \( p \)-th root of unity. We denote the cyclotomic ring \( \mathbb{Z}[\zeta_p] \) by \( \mathcal{O}_p^+ \). We also let \( \mathcal{O}_p \) denote \( \mathcal{O}_p^+ \) if \( p \equiv 3 \pmod{4} \) and \( \mathcal{O}_p^+[i] \) if \( p \equiv 1 \pmod{4} \). Throughout the paper, we use the notation \( h = 1 - \zeta_p \) and \( d = (p - 1)/2 \). When referring to quantum integers or to skein-theoretical constructions, we will often write \( q \) for \( \zeta_p \), and we put \( A = -q^{d+1} \); this is a primitive \( 2p \)-th root of unity, and a square root of \( q = \zeta_p \).

Here is some more notation that we will need. For \( 0 \leq n < p \), let \( \{n\} = (-A)^n - (-A)^{-n} \), \( \{n\}^+ = (-A)^n + (-A)^{-n} \). (Warning: the minus signs make this different notation than commonly used.) We also let \( \{n\}_q = q^n - q^{-n} \). As usual, we put \( \{n\}! = \{1\}\{2\} \cdots \{n\} \), and \( \{0\}! = 1 \). If \( n \) is negative, let \( \{n\}! = 0 \). Also let \( \{n\}!! = \{n\}\{n-2\} \cdots \) ending in \( \{1\} \) or \( \{2\} \). Interpret \( \{n\}^+!, \{n\}^+!! \), \( \{n\}_q \) similarly.

We refer the reader to section 2 of [GMI] for the definitions of the TQFT-module \( V_p(\Sigma) \), defined over \( \mathcal{O}_p[h^{-1}] \), and the integral TQFT-module \( S_p(\Sigma) \), defined over \( \mathcal{O}_p \). In this paper, we are mainly interested in the refined integral TQFT-module \( S^+_p(\Sigma) \), with coefficients in \( \mathcal{O}_p^+ \); it is defined in [GMI] Section 13]. Here \( \Sigma \) is a compact oriented surface equipped with a (possibly empty) set of colored banded points. The set of allowed colors is \( \{0, 1, \cdots, p-2\} \), but the sum of the colors of all the banded points must be even. The modules \( V_p(\Sigma) \) and \( S_p(\Sigma) \) are obtained from \( S^+_p(\Sigma) \) by tensoring with \( \mathcal{O}_p[h^{-1}] \) and \( \mathcal{O}_p \), respectively; thus, all three modules are free of the same finite rank over their respective coefficient rings, and we have natural inclusions

\[ S^+_p(\Sigma) \subset S_p(\Sigma) \subset V_p(\Sigma) \]

coming from the inclusions \( \mathcal{O}_p^+ \subset \mathcal{O}_p \subset \mathcal{O}_p[h^{-1}] \). The \( V_p \)-theory is a version of the Reshetikhin-Turaev \( SO(3) \)-TQFT, the only difference being that the latter is usually considered with coefficients in the quotient field of \( \mathcal{O}_p \) (or even with coefficients in \( \mathbb{C} \)).

Let \( T_c \) denote a torus with one banded point colored \( 2c \). If \( c = 0 \), we can forget the banded point, and we simply write \( T \) for \( T_0 \). Elements of \( V_p(T_c) \) are represented skein-theoretically as linear combinations of colored banded graphs in a solid torus, which meet the boundary nicely in the colored banded point. See [BHMV2] for the skein-theoretical construction of TQFT we are using here.
The small graph basis [GM1 Prop. 3.2] of $V_p(T_c)$ is $L_{c,n}$ ($0 \leq n \leq d - c - 1$), where

$$L_{c,n} = \begin{array}{c}
\text{\tiny \#} \\
\text{\tiny \#} \\
\end{array}$$

is a lollipop graph with stick color $2c$, and color $c + n$ on the loop edge. The Hopf pairing is the bilinear form $(( , ))$ on $V_p(T_c)$ defined as follows:

$$((L_{c,n},L_{c,m})) = \langle \begin{array}{c}
c+n \\
c+m \\
\end{array} \rangle,$$

where $\langle \rangle$ denotes the evaluation of a colored banded graph in $S^3$, see [MV, KL]. (There is a choice of the type of clasp. We insist that the clasp be as drawn.) The Hopf pairing restricts to a symmetric bilinear form

$$( ( , ) : S^+_p(T_c) \times S^+_p(T_c) \to O^+_p.$$ 

If $c = 0$, the Hopf pairing is the same as the pairing $\langle , \rangle$ studied in [BHMVI]. (In this case, the choice of clasp does not matter.) As shown there, an orthogonal basis for $V_p(T)$ with respect to this pairing is \{Q_0, Q_1, \cdots Q_{d-1}\}, where $Q_0 = 1$, and

$$Q_n = \prod_{i=0}^{n-1} (z - \lambda_i).$$

Here \( \lambda_i = -q^{i+1} - q^{-i-1} \), and $z$ denotes the core of the solid torus with color 1. (In our current notation, $z$ is $L_{0,1}$ and its framing is as drawn, i.e., the blackboard framing.) The product uses the commutative algebra structure on $V_p(T)$ where $z^n$ means $n$ parallel copies of $z$.

We now generalize this to the case when $c \neq 0$. Observe that $V_p(T_c)$ is a module over $V_p(T)$. Since $V_p(T)$ is commutative, we may view this both as a left and as a right module structure, whatever is notationally more convenient. For example, one has

$$L_{c,n}z = zL_{c,n} = \begin{array}{c}
\text{\tiny \#} \\
\text{\tiny \#} \\
\end{array}$$

where the right hand side can then be expressed as a linear combination of $L_{c,n+1}$ and $L_{c,n-1}$ in the usual way, see [MV, KL].
Definition 2.1. For $0 \leq n \leq d - c - 1$, we set

$$Q_n^{(c)} = L_{c,0} \prod_{i=c}^{c+n-1} (z - \lambda_i),$$

and $Q_n^{(c)} = (\{n\}!)^{-1} Q_n^{(c)}$.

Here is the main result of this section.

Theorem 2.2. \{$Q_0^{(c)},Q_1^{(c)},\ldots,Q_d^{(c)}$\} is an orthogonal basis for $S_p^+(T_c)$ with respect to the Hopf pairing. Moreover

$$((Q_n^{(c)},Q_n^{(c)})) = q^{-c(c+1)/2} \frac{(2c + 2n + 1)!}{n!} \left(\frac{2}{q}\right)^2 \left(\frac{2c}{q}\right)^{n} = \frac{q^{-c(c+1)/2}}{(n!)^2} \left(\frac{2}{q}\right)^2 \left(\frac{2c}{q}\right)^{n}.$$

We use the notation $x \sim y$ to mean that $x = uy$ for some unit $u$ in $O_p$. Here $x, y$ can be numbers, or elements of some module such as $V_p(T_c)$. For $0 < n < p$, one has both \{n\} $\sim h$ and \{n\} $\sim h$, and (hence) \{n\} $= \{n\} / \{n\} \sim 1$. The following corollary is easily checked.

Corollary 2.3. \((Q_n^{(c)},Q_n^{(c)}) \sim h^c.

Proof of Theorem 2.2: First, we show that the $Q_n^{(c)}$ are a basis for $S_p^+(T_c)$. The basis of $S_p^+(T_c)$ constructed in [GM] is \{L_{c,0}v^n | 0 \leq n \leq d - c - 1\} where $v = h^{-1}(z + 2) \in S_p^+(T_c)$. Note that $z - \lambda_i = (z + 2) - (2 + \lambda_i)$ and $2 + \lambda_i \sim h^2$. Thus $h^{-1}(z - \lambda_i) = v + (a scalar in M)$, which will form a basis. Since \{n\} $\sim h^n$, it follows that the $Q_n^{(c)}$ are also a basis for $S_p^+(T_c)$.

Next, we show that this basis is orthogonal with respect to the Hopf pairing. It is enough to show that the $Q_n^{(c)}$ are orthogonal. The idea of proof is the same as showing orthogonality of the $Q_n$ in [BHMV]. Each $Q_n^{(c)}$ is a linear combination of $L_{c,m}$ where $0 \leq m \leq n$, and the coefficient of $L_{c,n}$ is one. We have that \((Q_n^{(c)},Q_n^{(c)}) = 0\) when $k > 0$, as $Q_n^{(c)}$ contains a $z - \lambda_{c+m}$ factor for $0 \leq m \leq n + k - 1$ which we may use to annihilate the $L_{c,m}$ terms in $Q_n^{(c)}$. Since the form is symmetric, this shows that the basis is orthogonal. We remark that if $(c,c)$ is non-degenerate, this argument also shows that $Q_n^{(c)} = 0$ for $n \geq d - c$, which will be used in the proof of Theorem 4.3. If $k = 0$, we can eliminate all but the $L_{c,n}$ term in the expansion of one copy of $Q_n^{(c)}$ when computing $((Q_n^{(c)},Q_n^{(c)}))$. Thus

$$((Q_n^{(c)},Q_n^{(c)})) = ((Q_n^{(c)},L_{c,n})) = ((L_{c,0},L_{c,n})) \prod_{i=c}^{c+n-1} (\lambda_{c+n-i} - \lambda_i).$$

But $((L_{c,0},L_{c,n}))$ can be simplified using the first formula in the proof of [MI] Lemma(4.1). One gets

$$((L_{c,0},L_{c,n})) = C_{c+n,c,c} \begin{pmatrix} 2c & c+n & c+n \end{pmatrix} \begin{pmatrix} n & c & c \end{pmatrix},$$

where $C_{c+n,c,c}$ is given in [MI], p. 550, and the other symbol is a tetrahedral symbol in the notation of [MV]. The formula for a tetrahedral symbol is a sum of products,
but in this case there is only one term in this sum. Simplification gives

\[ (Q_n^{(c)}, Q_n^{(c)}) = q^{-c(c+1)/2} \{2c + 2n + 1\}^{n+1} \frac{\{c\}_q!^2}{\{2c\}_q!^2} \]

This easily implies (4) and completes the proof.

3. ORTHOGONAL LOLLIPOP BASIS IN HIGHER GENUS

In this section, we define the Hopf pairing and construct an orthogonal basis for $S^+_p(\Sigma)$ where $\Sigma$ has arbitrary genus $g$. The reader interested only in the one-holed torus case may proceed directly to Section 4.

For the remainder of this section, we assume the reader is familiar with the concepts and notations of [GM1, p. 820-822] where a basis of $S^+_p(\Sigma)$ is defined in terms of certain colorings of a lollipop tree. The same construction actually yields a basis of $S^+_p(\Sigma)$, see [GM1, p. 837].

Suppose $\Sigma$ is equipped with a set $X$ of colored banded points. Fix a handlebody $H$ with boundary $\Sigma$, and a banded lollipop tree $G$ in $H$ with boundary vertices equal to $X$, so that $H$ retracts onto $G$. The small graph basis $\mathcal{G}$ of $V_p(\Sigma)$ consists of the elements $g(a, b, c)$ represented by small colorings of $G$ [GM1 Prop. 3.2]. See Fig. 1 for an example representing the basis element $g((a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3, c_4))$ of $V_p(\Sigma)$, where $\Sigma$ has genus 3 and $|X| = 2$, with colors $c_3$ and $c_4$ on the two banded points. The basis $\mathcal{B} = \{v(a, b, c)\}$ of $S^+_p(\Sigma)$ given in [GM1 p. 822] is indexed by the same set of small colorings. It is obtained from $\mathcal{G}$ by a triangular (but not unimodular!) basis change. We will show how to modify the basis $\mathcal{B}$ to make it orthogonal with respect to the Hopf pairing which we discuss next.

![Figure 1. Lollipop graph $G$ in good position, with some coloring.](image-url)

In order to define the Hopf pairing in our context, assume that $G$ and $H$ have been isotoped so that $G$ is presented by a half-plane diagram with no crossings, with the banded points on the boundary of the half-plane, and one edge from each loop near the boundary of the half-plane. The graph in Fig. 1 is already drawn in this way. One may produce a dual $G'$ by rotating the half plane $180^\circ$ around its boundary line and then bringing the loops together and inserting clasps (linking pairs of loops) as illustrated in Fig. 2. Gluing $G$ and $G'$ along their boundary vertices produces a banded trivalent graph which we denote by $GG'$. Here, as in the genus one case, we insist that all the clasps be as drawn.

As in the genus one case, although the Hopf pairing can be defined in a basis-independent way, it is convenient to define it by its value on basis elements, as
follows: \( (\mathbf{g}(a,b,c), \mathbf{g}(a',b',c')) \) is the evaluation of \( GG' \) with colors \((a,b,c)\) on the \( G \) part of \( GG' \), and colors \((a',b',c')\) on the \( G' \) part, where we use the 180° rotation to transport \((a',b',c')\) from \( G \) to \( G' \). As before, this restricts to a symmetric bilinear form

\[
( , ) : S^p_+ (\Sigma) \times S^p_+ (\Sigma) \to O^p_+ .
\]

The basis \( \mathcal{B} \) of \( S^p_+ (\Sigma) \) found in [GM1, p. 822] is given as follows:

\[
b(a,b,c) = h^{-|\frac{1}{2} (e + \sum_i a_i)|} g(a,0,c) \prod_{j=1}^{g} \frac{(2 + z_j)^{b_j}}{h^{b_j}} .
\]

Here \( 2e \) is the color of the trunk edge of \( G \), see [GM1, p. 821], and \( z_j \) is a zero-framed circle colored 1 around the \( j \)-th hole. The following modification will be our orthogonal basis:

\[
\tilde{b}(a,b,c) = h^{-|\frac{1}{2} (e + \sum_j a_j)|} g(a,0,c) \prod_{j=1}^{g} \frac{\prod_{i=a_j}^{a_j+b_j-1} (z_j - \lambda_i)}{\{b_j\}!} .
\]

**Theorem 3.1.** The set \( \tilde{\mathcal{B}} = \{ \tilde{b}(a,b,c) \} \) where \((a,b,c)\) runs through the small colorings of \( G \), is a basis of \( S^p_+ (\Sigma) \) which is orthogonal with respect to the Hopf pairing. Moreover,

\[
((\tilde{b}(a,b,c), \tilde{b}(a,b,c))) = h^{-2|\frac{1}{2} (e + \sum_i a_i)|} \langle DG(a,c) \rangle \prod_{i=1}^{g} \frac{\langle (Q_{b_i}^{(a_i)}, Q_{h_i}^{(a_i)}) \rangle \{1\}_q}{\{2a_i + 1\}_q} ,
\]

where \( 2e \) is the color of the trunk edge, \( \langle DG(a,c) \rangle \) is the evaluation of the colored banded graph \( DG(a,c) \) defined below, and \( \langle (Q_{b_i}^{(a_i)}, Q_{h_i}^{(a_i)}) \rangle \) is given in Eq. (4).

The proof is the same as in the one-holed torus case. To define \( DG(a,c) \), assume that \( G(a,b,c) \) is a colored lollipop graph in good position, as pictured in Fig. 1, representing the small graph basis element \( g(a,b,c) \). Then \( DG(a,c) \) is the colored graph in the plane obtained by removing the loop edges and then adjoining a reflected copy, as in Fig. 3.

Using formulas in [MV], it is easy to see that \( \langle DG(a,c) \rangle \) is always a unit in \( O^p_+ \). Therefore one has the following
Corollary 3.2.

\[
((\tilde{b}(a, b, c), \tilde{b}(a, b, c))) \sim \begin{cases} 
  h^e & \text{if } \sum_j a_j \equiv e \pmod{2} \\
  h^{e+1} & \text{if } \sum_j a_j \not\equiv e \pmod{2}
\end{cases}
\]

Remark 3.3. The dual lattice \( S^+_p(\Sigma) \) and its refined version \( S^{+\sharp}_p(\Sigma) \) are defined in [GMI Def. 8.1] and [GMI Sec. 13], respectively, using certain hermitian forms. It is not hard to see that \( S^{+\sharp}_p(\Sigma) \) is also the lattice dual to \( S^+_p(\Sigma) \) with respect to the Hopf pairing:

\[
S^{+\sharp}_p(\Sigma) = \{ x \in S^+_p(\Sigma) \otimes \mathbb{Q}(\zeta_p) \mid ((x, y)) \in \mathcal{O}_p^+ = \mathbb{Z}[\zeta_p] \text{ for all } y \in S^+_p(\Sigma) \}.
\]

Using orthogonality of the basis \( \tilde{B} \), Cor. 3.2 easily implies that

Corollary 3.4. The following rescaling

\[
\tilde{b}^+(a, b, c) = \begin{cases} 
  h^{-e} \tilde{b}(a, b, c) & \text{if } \sum_j a_j \equiv e \pmod{2} \\
  h^{-e-1} \tilde{b}(a, b, c) & \text{if } \sum_j a_j \not\equiv e \pmod{2}
\end{cases}
\]

\[
= h^{-\frac{1}{2}(e+\sum_j a_j)} g(a, 0, c) \prod_{j=1}^g \frac{\prod_{i=a_j}^{a_j+b_j-1} (z_j - \lambda_i)}{b_j}!
\]

is a basis \( \tilde{B}^\sharp = \{ \tilde{b}^+(a, b, c) \} \) of \( S^{+\sharp}_p(\Sigma) \) which is orthogonal with respect to the Hopf pairing.

Remark 3.5. In the above, we deduced the bases \( \tilde{B} \) of \( S^+_p(\Sigma) \) and \( \tilde{B}^\sharp \) of \( S^{+\sharp}_p(\Sigma) \) from the basis \( B \) constructed in [GMI]. Here is a sketch of a more direct proof. First, observe that both \( \tilde{B} \) and \( \tilde{B}^\sharp \) are bases of \( V_p(\Sigma) \). Now use the 3-ball lemma and the lollipop lemma of [GMI] to show that \( \tilde{B} \subset S^+_p(\Sigma) \) and \( \tilde{B}^\sharp \subset S^{+\sharp}_p(\Sigma) \). Then use orthogonality and Cor. 3.2 to show that

\[
((\tilde{b}(a, b, c), \tilde{b}^+(a', b', c')) \sim \delta_a^a \delta_b^{b'} \delta_c^{c'}
\]

and deduce that \( \tilde{B} \) generates \( S^+_p(\Sigma) \) and \( \tilde{B}^\sharp \) generates \( S^{+\sharp}_p(\Sigma) \). This proof avoids the index counting argument of [GMI].

Remark 3.6. Recall \( J^+_p(N) \), the Frohman Kania-Bartoszynska ideal [FKB, GMI, G2] of a compact connected oriented 3-manifold \( N \) with connected boundary \( \Sigma \). Using our orthogonal basis, we can give generators for this ideal as follows. The
TQFT associates to $N$ a vector $[N] \in S^+_p(\Sigma)$ which is well defined up to multiplication by a root of unity. Write

$$[N] = \sum_{(a,b,c)} x_{(a,b,c)} \tilde{b}(a,b,c),$$

where the coefficients $x_{(a,b,c)}$ lie in $O^+_p$, and the sum is over small admissible colorings of a lollipop graph in a handlebody with boundary $\Sigma$. Then by Theorem 3.1 and Corollary 3.2 together with [GM1, Thm. 16.5], one has that $J^+_p(N)$ is the $O^+_p$-ideal generated by the numbers $h^\varepsilon(a)x_{(a,b,c)}$ where $\varepsilon(a)$ is zero or one accordingly as $\sum_j a_j$ is even or odd.

4. Matrices for $t$ and $t^*$

Recall that elements of $V_p(T_c)$ are represented skein-theoretically as linear combinations of colored banded graphs in a solid torus which meet the boundary nicely in the colored banded point. We refer to such linear combinations as skein elements.

**Definition 4.1.** We let $t$ be the endomorphism of $V_p(T_c)$ induced by a full positive twist of the solid torus. It follows from [GM1] that $t$ preserves the lattice $S^+_p(T_c)$.

In the basis (3) of $V_p(T_c)$, $t$ is diagonal:

$$t(L_{c,n}) = \mu_{c+n} L_{c,n},$$

where $\mu_k = (-1)^k A^{k(k+2)}$ is the twist eigenvalue [BHMV1].

Let $\omega_+ \in V_p(T)$ be defined as

$$\omega_+ = \sum_{m=0}^{d-1} \gamma_m \frac{Q_m}{m!} \text{ where } \gamma_m = \frac{(-A)(m^2+5m)/2}{\prod_{k=1}^m (A^{2k+1} - 1)} \sim 1.$$  

(To see that $\gamma_m \sim 1$ one uses that $1-A$ is a unit in $O^+_p$, see [GMW, Lemma 4.1.(i)].) It follows from Theorem 2.2 (for $c = 0$) that $\omega_+ \in S^+_p(T)$.

**Theorem 4.2.** If $x \in V_p(T_c)$ is represented by a skein element (which we also denote by $x$), then $t(x)$ is represented by the union of $x$ with a zero-framed $\omega_+$ placed on a meridian of the solid torus pushed slightly into the interior. (See Fig. 4)

$$\omega_+ = \omega +$$

**Figure 4.** Encircling a strand with $\omega_+$ has the same effect in TQFT as giving that strand a positive twist.

**Proof.** The existence of an $\omega_+$ with this property is well-known in TQFT [BHMV2]. The expression (8) for it can be deduced from [BHMV1], as follows. Let $\omega_{BHMV1}$ be
the skein element of \([BHMV1]\). By \([BHMV1] 4.3(2)\], encircling with \(t(\omega_{BHMV1})\)
produces a \textit{negative} twist, and
\[
t(\omega_{BHMV1}) = \sum_{m=0}^{d-1} \frac{(-1)^m A^{-2m}}{\prod_{j=2}^{2m+1} (A^j - 1)} \mu_m Q_m = \sum_{m=0}^{d-1} \frac{A^{m^2}}{\prod_{j=2}^{2m+1} (A^j - 1)} Q_m.
\]
A skein expression for \(\omega_+\) can now be obtained from this formula for \(t(\omega_{BHMV1})\)
by taking mirror images of the skein elements and conjugating the scalars (using \(\overline{A} = A^{-1}\)). (Note that this operation leaves \(Q_m\) invariant.) Formula \(8\) follows. □

**Definition 4.3.** We let \(t^*\) be the endomorphism of \(V_p(\mathcal{T}_c)\) given by
\[
t^*(x) = x \omega_+.
\]
It follows from \([GM1]\) that \(t^*\) preserves \(S^+_p(\mathcal{T}_c)\) (since \(\omega_+ \in S^+_p(\mathcal{T})\)).

Here the multiplication on the right hand side is the module structure discussed in Section 2. Observe that \(t^*(x)\) is represented by the union of \(x\) with a zero-framed \(\omega_+\) placed on the standard longitude of the solid torus pushed slightly into the interior. (See the last figure in Section 2.)

Recall that the mapping class group \(\Gamma_{1,1}\) is generated by the two Dehn twists about the meridian and the longitude of the torus. It follows from standard results in the skein-theoretical construction of the TQFT-representation \(\rho_p\) \([BHMV2]\) that \(t\) and \(t^*\) are \(\rho_p\) of certain lifts of these twists to the extended mapping class group. Thus we have the following

**Theorem 4.4.** The endomorphisms \(t\) and \(t^*\) describe the TQFT representation \(\rho_p\) of the extended mapping class group on \(S^+_p(\mathcal{T}_c)\).

**Remark 4.5.** The lifts chosen here are the ‘geometric’ lifts as discussed in \([MR]\). Choosing different lifts would multiply \(t\) and \(t^*\) by well-known scalar factors. Adapting the techniques of \([MR]\), one can show that \(tt^*t = t^*tt\) and
\[
(tt^*t)^4 = q^{-6+2(c+1)-p(p+1)/2}.
\]
We will discuss our method to do computations like these in a more general context in \([GM2]\).

We now set out to compute matrix coefficients for \(t\) and \(t^*\). The following proposition tells us that it will be good to use the basis constructed in Section 2 which is orthogonal for the Hopf pairing.

**Proposition 4.6.** The endomorphisms \(t\) and \(t^*\) are adjoint for the Hopf pairing:
\[
((t(x), y)) = ((x, t^*(y))).
\]

**Proof.** We need to show that
\[
((x \cup \omega_+(\text{along the meridian}), y)) = ((x, y \cup \omega_+(\text{along the longitude})))
\]
But this is clear, since in the standard decomposition of \(S^3\) into two solid tori, the meridian of one solid torus is the longitude of the other. □

\(^2\text{Warning: the} \omega \text{ of} \ [BHMV1] \text{ is not the same as the one of} \ [BHMV2].}
It follows that if we denote the matrix coefficients for $t$ and $t^*$ in the basis $(Q_n^{(c)})$ by $a_{m,n}^{(c)}$ and $b_{n,m}^{(c)}$, respectively:

\[ t(Q_n^{(c)}) = \sum_{m=0}^{d-1-c} a_{m,n}^{(c)} Q_m^{(c)} \]

\[ t^*(Q_m^{(c)}) = \sum_{n=0}^{d-1-c} b_{n,m}^{(c)} Q_n^{(c)} , \]

then we have that:

\[ a_{m,n}^{(c)}((Q_m^{(c)}, Q_m^{(c)})) = ((t(Q_n^{(c)}), Q_m^{(c)})) = ((Q_n^{(c)}, t^*(Q_m^{(c)}))) = b_{n,m}^{(c)}((Q_n^{(c)}, Q_n^{(c)})). \]

Hence

\[ a_{m,n}^{(c)} = \frac{((Q_n^{(c)}, Q_n^{(c)}))}{((Q_m^{(c)}, Q_m^{(c)}))} b_{n,m}^{(c)} = R_{n,m}^{(c)} b_{n,m}^{(c)}, \]

where

\[ R_{n,m}^{(c)} = \frac{\{m\}!\{2c+2n+1\}!\{2c+n+1\}!}{\{n\}!\{2c+2m+1\}!\{2c+m+1\}!} \sim 1 . \]

Thus we are reduced to compute the $b_{n,m}^{(c)}$. To do so, we need the coefficients defined in the following proposition, which describes the module structure of $V_p(T_c)$ over the algebra $V_p(T)$ in terms of the $Q$-bases.

**Proposition 4.7.**

\[ Q_m Q_n^{(c)} = \sum_{l=0}^{\min\{m,n+c\}} C_{m,n+c}^l Q_{m+n-l}^{(c)} \]

where

\[ C_{m,n}^l = (-1)^l \frac{\{m\}!\{n\}!\{m+n+1\}!}{\{m+l\}!\{n-l\}!\{m+n+1-l\}!\{l\}!} \sim h^{2l} . \]

We defer the proof to the end of this section. We are now ready to state the main result of this section.

**Theorem 4.8.** The matrix for $t^*$ is lower triangular, and the matrix for $t$ is upper triangular:

\[ b_{n,m}^{(c)} = a_{m,n}^{(c)} = 0 \text{ if } m > n . \]

Moreover, if $m \leq n$, then

\[ b_{n,m}^{(c)} = \sum_{l=0}^{m+c} b_{n,m,l}^{(c)}, \quad a_{m,n}^{(c)} = \sum_{l=0}^{m+c} a_{n,m,l}^{(c)} , \]

where

\[ b_{n,m,l}^{(c)} = C_{l+n-m,m+c}^l \frac{\{n\}!}{\{l+n-m\}!\{m\}!} h^l \sim h^l \]

\[ a_{m,n,l}^{(c)} = C_{l+n-m,m+c}^l \frac{\{n\}!}{\{l+n-m\}!\{m\}!} h^l \sim h^l . \]
Proof. We compute
\[
\tau^*(Q^{(c)}_m) = \omega^* Q^{(c)}_m = \sum_{c=0}^{d-1} \frac{\gamma_k}{k!} \frac{Q_k}{\{m\}!} Q^{(c)}_m
\]
\[
= \sum_{c=0}^{d-1} \frac{\gamma_k}{k!} \min\{k,m+c\} \sum_{l=0}^{m+c} C_{d+c,m+c}^{l} Q^{(c)}_{m+c}
\]
\[
= \sum_{l=0}^{m+c} \sum_{k=l}^{d-1} \frac{\gamma_k}{k!} Q^{(c)}_{m+c} Q^{(c)}_{m+c-l}
\]
\[
= \sum_{l=0}^{m+c} \sum_{k=l}^{d-1} \frac{\gamma_k}{k!} \{l+n-m\}! \{m\}! C_{l+n-m,m+c}^{l} Q^{(c)}_{n}(\text{letting } n = k + m - l)
\]
\[
= \sum_{l=0}^{m+c} \sum_{k=l}^{d-1} \frac{\gamma_k}{k!} \{l+n-m\}! \{m\}! C_{l+n-m,m+c}^{l} Q^{(c)}_{n}(\text{as } Q^{(c)}_{n} = 0 \text{ for } n \geq d - c)
\]
\[
= \sum_{l=0}^{d-1} \sum_{k=l}^{m+c} \beta^{(c)}_{n,m,l} Q^{(c)}_{n}
\]
where \(\beta^{(c)}_{n,m,l}\) is as in the statement of the theorem. This proves the result for the matrix coefficients \(\beta^{(c)}_{n,m,l}\) of \(\tau^*\). To get the matrix coefficients for \(\tau\), it now suffices to apply Eq. (10).

It remains to give the

Proof of Prop. 4.4
Let us first consider the case \(c = 0\). It is easy to see that

\[
(14) \quad Q_m Q_n = \sum_{l=0}^{m+n} C_{n,m}^{l} Q_{m+n-l}
\]

for some coefficients \(C_{n,m}^{l}\) in the range \(0 \leq l \leq \min\{m, n\}\). We must show that \(C_{n,m}^{l}\) satisfies Eq. (13). Observe that if \(Q_n = \beta^{0}_{n,m} Q_n\) holds for \(n \leq m\), then it holds in general, by symmetry. So we may assume \(n \leq m\). We now fix \(m\), and prove the formula by induction on \(n\), for all \(0 \leq l \leq n\) at once. If \(n = 0\), \(C_{0,0}^{0} = 1\), and formula (13) agrees. Assume inductively the formula is proven for \(n\), and consider it for \(n + 1\). Again, we may assume that \(n + 1 \leq m\).

To perform the inductive step, let \(\beta^{0}_{n,m} = \lambda_m - \lambda_n = \{n-m\} \{m+n+2\}\). Since \(Q_{n+1} = (z - \lambda_n) Q_n\), after multiplying (14) by \(z - \lambda_n\) we get the following recursive relation:

\[
C_{n,m,n+1}^{l} = \begin{cases} 
C_{n,m,n}^{0} & \text{if } l = 0 \\
C_{n,m,n}^{l} + \beta_{n,m+1,n} C_{n,m,n}^{l-1} & \text{if } 1 \leq l \leq n \\
\beta_{n,m,n} C_{n,m,n}^{n} & \text{if } l = n + 1
\end{cases}
\]

It is not hard to check that formula (13) satisfies these recursion relations. Thus, the \(c = 0\) case of Prop. 4.7 is proved.

For the general case, let

\[
Q_{n,c} = \prod_{i=c}^{c+n-1} (z - \lambda_i) = \frac{Q_{n+c}}{Q_c}
\]
so that $Q_n^{(c)} = Q_{n,c}L_{c,0}$. We have

$$Q_m Q_{n,c} = \sum_{l=0}^{\min\{m,n+c\}} C_{m,n+c}^{l} Q_{m+n-l,c},$$

as follows from applying (14) with $n + c$ in place of $n$, and dividing by $Q_c$. Multiplying both sides of this equation by $L_{c,0}$, we get (12), completing the proof. □

We remark that Prop. 4.7 and its proof also work in the context of the Kauffman Bracket skein module of a solid torus (relative to a 2-colored banded point on its boundary if $c > 0$) over the Laurent polynomial ring $\mathbb{Z}[A^{\pm 1}]$ where $A$ is generic (i.e. not necessarily a root of unity).

5. Irreducibility of the representation over $\mathbb{F}_p$

Recall that $\mathbb{Z}[\mathfrak{g}]/(h)$ is the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. The TQFT-representation $\rho_p$ on $S_p^+(\mathcal{T}_c)$ induces a representation $\rho_{p,0}$ on

$$S_{p,0}^+(\mathcal{T}_c) = S_p^+(\mathcal{T}_c)/hS_p^+(\mathcal{T}_c),$$

which is an $\mathbb{F}_p$-vector space of dimension $d - c$. (Recall $d = (p - 1)/2$.) This representation is generated by the two matrices $t$ and $t^*$ considered modulo $h$. Recall from Remark 4.5 that $tt^*t = t^*tt^* = (tt^*)^2 \equiv 1 \mod h$. Thus $\rho_{p,0}$ factors through a representation of

$$(15) \quad SL(2, \mathbb{Z}) = \langle T, T^* | TT^*T = T^*TT^*, (TT^*)^4 = 1 \rangle,$$

where $T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $T^* = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$.

Note that $T$ and $T^*$ are respectively the matrices for the action of a meridional and a longitudinal Dehn twist on the homology of the torus $T$ with respect to the meridian-longitude basis.

We now come to the main result of this section.

**Theorem 5.1.** For every $0 \leq c \leq d - 1$, the representation $\rho_{p,0}$ factors through $SL(2, \mathbb{F}_p)$. Moreover, the induced representation of $SL(2, \mathbb{F}_p)$ is isomorphic to the representation of $SL(2, \mathbb{F}_p)$ on the vector space $H_{p,D}$ of homogeneous polynomials in two variables over $\mathbb{F}_p$ of total degree $D$, where $D + 1 = d - c$.

Here we consider the left action of $SL_2(\mathbb{F}_p)$ on polynomials in variables $x$ and $y$ over $\mathbb{F}_p$ given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} x^m y^n = (ax + cy)^m(bx + dy)^n.$$

This restricts to a representation $r_{p,D}$ of $SL_2(\mathbb{F}_p)$ on the vector space $H_{p,D}$ of homogeneous polynomials of a given degree $D$. The dimension of this representation is $D + 1$. It is irreducible if and only if $0 \leq D \leq p - 1$ [11, pp. 31-32]. These are the only irreducible representations of $SL_2(\mathbb{F}_p)$ over $\mathbb{F}_p$.

**Corollary 5.2.** The representation $\rho_{p,0}$ on the $\mathbb{F}_p$-vector space $S_{p,0}^+(\mathcal{T}_c)$ is irreducible for all $0 \leq c \leq d - 1$.

**Remark 5.3.** The matrices $t$ and $t^*$ have order $p$, as follows from (7). But this is not enough to ensure that $\rho_{p,0}$ factors through $SL(2, \mathbb{F}_p)$, because adding the relation $T^p = (T^*)^p = 1$ to the presentation (15) defines still an infinite group if $p \geq 7$ (it is a double cover of the $(2,3,p)$ triangle group).
Remark 5.4. The reader may wonder whether $S^+_{p,q}(\Sigma)$ is an irreducible representation of the mapping class group for general surfaces $\Sigma$. In fact, if the genus of $\Sigma$ is at least 3, $S^+_{p,q}(\Sigma)$ is not irreducible, as already observed in \cite{GM3} Section 14. The situation can be understood completely using the orthogonal lollipop basis; one finds that $S^+_{p,0}(\Sigma)$ has a composition series with at most two irreducible pieces (at least when $\Sigma$ has at most one colored point). Details will be given elsewhere \cite{GM3}.

We remark that over the complex numbers, the representation $S^+_{p}(\Sigma) \otimes \mathbb{C}$ is always irreducible when $p$ is prime and when $\Sigma$ has at most one colored point \cite{Ro}. Irreducibility over the complex numbers implies irreducibility over the cyclotomic field $\mathbb{Q}(\zeta_p)$, but does not, of course, imply irreducibility over the finite field $\mathbb{F}_p$. Also, Roberts’ argument does not apply over $\mathbb{F}_p$, because it uses the fact that the twist eigenvalues $\mu_k = (-A)^{k(k+2)} = \zeta_p^{(d+1)k(c+2)}$ for $0 \leq k \leq d-1$ are all distinct. But these eigenvalues become equal to 1 in $\mathbb{F}_p$ (since $\zeta_p = 1$ in $\mathbb{F}_p$).

Proof of Theorem 5.1. Consider the basis
\[(x^{D-n}y^n)_{n=0,1,...,D}\]
of $H_{p,D}$. In this basis, $T$ is upper triangular, $T^*$ is lower triangular, and the matrix coefficients of $T$ and $T^*$ are given by
\[
T(x^{D-n}y^n) = \sum_{m=0}^{n} \alpha_{m,n} x^{D-m} y^m \quad \text{where} \quad \alpha_{m,n} = \binom{n}{m},
\]
\[
T^*(x^{D-n}y^n) = \sum_{m=0}^{n} \beta_{m,n} x^{D-m} y^m \quad \text{where} \quad \beta_{m,n} = (-1)^{m-n} \binom{D-n}{m-n}.
\]

The matrix coefficients $t$ and $t^*$ acting on $S^+_{p,0}(\mathcal{T}_c)$ are given by $\alpha^{(c)}_{m,n}$ and $\beta^{(c)}_{m,n}$ (mod $h$), which we denote by $\hat{a}^{(c)}_{m,n}$ and $\hat{b}^{(c)}_{m,n}$. Recall that they lie in the finite field $\mathbb{F}_p$.

Lemma 5.5.
\[
\hat{a}^{(c)}_{m,n} = \frac{(-1)^{n-m}(2c+2n+1)!!}{(n-m)!!(2c+2m+1)!!}
\]
\[
\hat{b}^{(c)}_{m,n} = (-2)^{n-m} \binom{m}{n}.
\]

Proof. This follows from Theorem 4.8 using that $q \equiv 1$ (mod $h$), $A \equiv -1$ (mod $h$), $\gamma_m \equiv (-1)^m 2^{-m}$ (mod $h$), $\{n\}^+ \equiv 2$ (mod $h$), and $\{n\}/\{1\} \equiv n$ (mod $h$). \qed

Now assume $D = d - c - 1$ and define a vector space isomorphism $\Phi : H_{p,D} \rightarrow S^+_{p,0}(\mathcal{T}_c)$ by
\[
\Phi(x^{D-n}y^n) = (-1)^n \frac{n!}{(2c+2n+1)!!} Q^{(c)}_n.
\]

We claim that $\Phi$ intertwines the representations $r_{p,D}$ on $H_{p,D}$ and $\rho_{p,0}$ on $S^+_{p,0}(\mathcal{T}_c)$. To see this, it suffices to check that (in $\mathbb{F}_p$) one has
\[
(-1)^m \frac{m!}{(2c+2m+1)!!} \alpha_{m,n}^{(d-c-1)} = (-1)^n \frac{n!}{(2c+2n+1)!!} \beta_{m,n}^{(c)}.
\]
\[ (-1)^m \frac{m!}{(2c + 2m + 1)!!} \hat{g}^{(d-c-1)}_{m,n} = (-1)^n \frac{n!}{(2c + 2n + 1)!!} \hat{h}^{(c)}_{m,n}. \]

The first is immediate. The second reduces to
\[ (-2)^{-n}(d-c-1-n)!(2c + 2n + 1)!! = (-2)^{-m}(d-c-1-m)!(2c + 2m + 1)!! \]
which is true due to the following lemma.

**Lemma 5.6.** For \( 0 \leq k \leq d-1 \),
\[ (-2)^{-k}(d-k-1)!(2k + 1)!! = (d-1)! \pmod{p}. \]

**Proof.** We prove by induction on \( k \). Let \( u_k \) be the expression on the left hand side. We have that \( u_0 = (d-1)! \), and also \( u_{k+1} = (2k+3)u_k/(-2)(d-k-1) = u_k \in \mathbb{F}_p \), as \( (-2)(d-k-1) = 2k + 3 \in \mathbb{F}_p \).

This completes the proof of Theorem [5.1]. \( \square \)

**Remark 5.7.** With van Wamelen [GMW] p. 264, we gave a formula for the action of \( t \) on \( v^n \) in the Kauffman bracket skein module of \( S^1 \times D^2 \). This formula gives an expression for the matrix coefficients \( \hat{a}^{(0)}_{m,n} \), which looks quite different (and more complicated) than the \( c = 0 \) case of the expression in Lemma 5.5. Comparing the two expressions leads to a proof of the following identity:

For integers \( n, m, \) and \( i \) with \( n \geq m \geq 1 \), and \( i \geq 0 \),
\[ \sum_{k=m}^{n} (-1)^{k-n} \frac{k^2 - 1}{n} \binom{k - 1}{n - m - i} \binom{2n}{k - m} = \begin{cases} \frac{(2n - 1)!}{(n-m)!(2m-1)!} & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases} \]

Conversely, if one has an independent proof of this identity for \( i = 0 \), then one can deduce the formula for \( \hat{a}^{(0)}_{m,n} \) in Lemma 5.5 from the formula for \( t(v^n) \) in [GMW].

We thank C. Krattenthaler [K] for providing such an independent proof of the identity. Although the sum is not hypergeometric (due to the \( k^2 \) in the binomial coefficient), Krattenthaler was able to reduce the proof to applications of Dixon’s identity [S] (2.3.3.6); Appendix (III.9)).

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