Free vibration model and theoretical solution of the tympanic membrane

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ABSTRACT
Myringoplasty is one of the routine surgeries in the treatment of tympanic membrane (TM) perforation. Since the gross anatomical structure of the middle ear cannot be simulated in clinical practice, the surgery is mainly performed by experience and expertise. Based on the mechanical properties of TM, four hypotheses are presented where TM is simplified as a sectorial annulus membrane. This paper proposes a free vibration model of TM whose natural frequency of free vibration and the analytical expressions of corresponding natural vibration mode are obtained by variables separation method and Bessel function. Compared with the ANSYS numerical results, it shows that natural frequency calculated by finite element (FE) method is slightly higher because of the increase of model stiffness by ignoring high-order quantity. Compared with the experimental data from volunteers, it shows that the first-order, second-order and third-order principal resonances in the test are the combined effects of multiple natural frequencies and natural vibration modes instead of the single one. The theoretical model deduced in this paper is in higher precision with comparatively fewer parameters. It provides more precise mechanical reference to myringoplasty by calculating the response of the normal human ear.

1. Introduction
Human ear is one of the extremely small and complex biological structures in the living body.[1] Conduction deafness is one of the types of hearing loss due to external acoustic impediment in the process of transferring sound to the inner ear because of congenital or acquired diseases of outer or middle ear.[2,3] In clinical treatment, myringoplasty and ossiculoplasty are used to restore hearing for patients with conduction deafness. Due to practical difficulty involved in simulation of the complicated anatomical structure and acoustic mechanism of the middle ear, surgeries are mainly experience-oriented.[4] Though the surgery is targeted for complete cure, the reconstructed conduction structure fails to return to its original physiological state and therein obtain ideal hearing.[5] Therefore, it is crucial to study the biomechanics behavior of the ear structure to understand the cognitive acoustic, phonosensitive and pathological mechanisms of the human ear more clearly.

Human ear is a typical strong nonlinear flow-solid coupled biological dynamic system with acoustic excitation. The acoustic signals are collected through the pinna to the external auditory canal, followed by TM vibration. Then the vibration is carried through malleus, incus and finally stapes. Vibration energy of the floor of stapes is transmitted to the inner ear perilymph through the vestibular window, which then leads to the fluctuation of its basement membrane, and passes the impulses to the central nervous system, by the cochlear nerve fibers.[6] It is too tiny of the ear structure to accurately test integrated mechanical characteristics as a whole, whilst the observation of limited quantity of elements is possible. Funnell et al.’s pioneering research[7] was to implement finite element (FE) method to establish TM models of cats. The model used spring and damping to represent TM impedance by the floor of stapes and cochlea. Lesser et al. [8,9] used FE method to study the mechanical properties of human by setting up a FE model of 2-D cross section on TM and the malleus to analyze the static displacement of those two. Subsequently, Williams et al. [10,11] changed variable parameters of TM in FE model to study middle ear model characteristics under external forces in a fixed frequency. Koike et al. [12] extended the middle ear model by establishing FE model of the external ear canal, middle ear cavity, ossicular chain, muscle and ligament as a whole.
Furthermore, they also [13] measured the mechanical properties of the middle ear tendon. Then, the researchers used more accurate means to build FE models. For example, Gan et al. [14] implemented tissue section to build a 3-D FE model of the complete human middle ear, similar to Koike et al., adding on features like straightened simplified cochlear model containing the scala tympani, scala vestibuli and basement membrane. Liu et al. [15] and Yao et al. [16] established the middle ear FEM with complete boundary confirmations delineated by computed tomography (CT).

Due to the introduction of the mechanical analysis and numerical simulation methods, the structure and acoustic mechanism of the human ear have been further interpreted in spite of its complicated structure since the late twentieth century. These studies provide the theoretical basis for the research of the biomechanical mechanism of clinical diseases. Prendergast et al. [17] and Vard et al. [18] studied the influence of tympanostomy tube insertion on TM vibration and stress, as well as the effects of size characteristics of tympanostomy tube on sound conduction. Their research implemented FE method regardless of the middle ear cavity structure. Utilizing FE models of the external ear, middle ear structure and the gas of the middle ear cavity, Gan et al. [19] concluded that the changes of sound pressure in the middle ear are related to TM perforation positions and types, which are sensitive to frequency variation. Venkatesh et al. [20] investigated the effect of ear canal pressure on the dynamic behavior of the outer and middle ear in newborns. The research used the sweep frequency impedance (SFI) technology with and without a conductive condition. Luis et al. [21] designed a totally new ossicular replacement prosthesis and implemented FE model of the human middle ear to assess its theoretical acoustic mechanical behavior. Based on the Petrov-Galerkin projection, Ihrle et al. [22] built a nonlinear model of the middle ear as an elastic multi-body system by applying model order reduction to acousto-structural coupled systems. Greef et al. [23] studied the connection between TM and malleus. The results confirmed that a large individual variability exists in the properties of the TM-malleus connection in humans in terms of its dimensions, tissue composition and configuration.

In this paper, the influence of the pathological changes on the hearing system is studied from the perspective of mechanics. Since TM is the key component to convert sound waves to vibration, it is crucial to establish a free vibration model of TM for studying the whole structure vibration. This paper proposes a free vibration model of TM according to conduction vibration rules of the ear structure. Adding on, it obtains the theoretical solution of TM free vibration, including the natural frequency and the analytical expressions of natural vibration mode through the methods of variables separation and Bessel function. Finally, it compares and verifies natural frequencies and natural vibration modes of several orders obtained by the theoretical method with ANSYS FE calculation results and the experimental data drawn from volunteers.

2. The free vibration model of the TM

Tympanic membrane, also called as the eardrum, comprises of a membrane body and annular ligament as shown in Figure 1. The annular ligament connects the membrane body and bony tympanic ring, hence fixing the eardrum. The manubrium of malleus embeds in TM from its edge to center. The umbo is located in the central part. The thickness of TM is between 50 and 100 µm, and the average thickness is 74 µm. The horizontal axis length is 8–9 mm and the vertical axis length is 8.5–10 mm. The total area is 85 mm², while the actual area which takes part in the physiological activity is about 55 mm².

Therefore, four hypotheses are proposed according to the structural characteristics of TM:

1. The TM is soft and springy, which cannot resist moment, and the tension is in the tangent plane at any time.
2. The malleus vibration caused by TM is a small amplitude vibration. The maximum displacement of the malleus is about 1/3×10⁻⁵ mm under the conditions of 100–10,000 Hz sound frequency and 90 dB sound pressure.
3. The influence of TM thickness variation in vibration process can be neglected.
4. The annular ligament is the hinge support of TM.

![Figure 1. The TM structure diagram.](image-url)
According to the above hypotheses, TM can be simplified as a sectorial annulus membrane which cannot resist bending and shear deformation. It completely depends on annular ligament tension to balance the horizontal load. Therefore, TM is set as a completely flexible and constant thickness sheet, subjected to a uniform tension in any direction, in Figure 2(a).

Set up a $xoy$ plane as the TM plane without deformation, $z$ axis being perpendicular to the plane, and constitute a right-handed coordinate system in Figure 2(b). $T$ is the TM tension. $d$ is the thickness of TM. $q$ is the quality on the unit area and $w$ is the displacement along the $z$ axis. Suppose the inner diameter of TM is $R_1$, the outer diameter of TM is $R_2$, and the angle of sectorial annulus membrane is $\alpha$.

The geometrical equation of large deformation of the circular membrane in polar coordinates is given by

$$e_r = \frac{d}{dr}(r_{\theta}) + \frac{1}{2} \left( \frac{dw}{dr} \right)^2$$

$$\varepsilon_{\theta} = \frac{u_r}{r}$$

where $e_r$ represents the radial strain, $\varepsilon_{\theta}$ represents the circumferential strain, $u_r$ denotes the radial displacement.

Substituting Equation (2) into Equation (1), the equation of compatibility is

$$e_r = \frac{d}{dr}(r_{\theta}) + \frac{1}{2} \left( \frac{dw}{dr} \right)^2$$

The physical equations are

$$\varepsilon_r = \frac{1}{E} (\sigma_r - \mu \sigma_{\theta}) = \frac{1}{E} \left( \frac{d}{dr} \right) \left( N_r - \mu N_{\theta} \right)$$

$$\varepsilon_{\theta} = \frac{1}{E} (\sigma_{\theta} - \mu \sigma_r) = \frac{1}{E} \left( \frac{d}{dr} \right) \left( N_{\theta} - \mu N_r \right)$$

where $E$ represents Young’s modulus; $\mu$ represents Poisson’s ratio.

Substituting Equation (4) and Equation (5) into Equation (3), yield

$$r^2 \frac{d^2}{dr^2} (r N_r) + r \frac{d}{dr} (r N_{\theta}) - r N_r = -\frac{1}{2} E \delta \left( \frac{dw}{dr} \right)^2$$

The equation of transverse free vibration of membranes is:

$$\nabla^2 w = \frac{\rho \partial^2 w}{T \partial t^2}$$

Since the boundary is a circle, we transform Equation (6) to polar coordinates $(r, \theta)$:

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} = \frac{\rho \partial^2 w}{T \partial t^2}$$

The boundary conditions are as follows:

$$w(R_1, \theta, t) = w(R_2, \theta, t) = 0, \quad w(r, 0, t) = w(r, \alpha, t) = 0$$

According to variables separation method, let:

$$w(r, \theta, t) = W(r, \theta) \times q(t)$$

Substituting Equation (10) into Equation (6), we obtain two differential equations of variable separation:

$$\ddot{q}(t) + \omega^2 q(t) = 0$$

$$\frac{\partial^2 W}{\partial r^2} + \frac{1}{r} \frac{\partial W}{\partial r} + \frac{1}{r^2} \frac{\partial^2 W}{\partial \theta^2} + \frac{\omega^2 \rho}{T} W = 0$$

where $\omega$ is the constant. According to the general solution of differential equation, solution of Equation (11) can be expressed as:

$$q(t) = c_1 \sin(\omega t + \phi)$$
This means harmonic oscillation that natural frequency is \( \omega \). Solve Equation (12) by using the variable separation method, let:

\[
W(r, \theta) = L(r) \times \Phi(\theta) \tag{14}
\]

Substituting Equation (14) into Equation (12), we obtain two differential equations of variable separation:

\[
d^2\Phi \over dr^2 + n^2\Phi = 0 \tag{15}
\]

\[
r^2d^2L \over dr^2 + r dL \over dr + r^2\lambda^2 = n^2 \tag{16}
\]

where

\[
\lambda^2 = \omega^2 \rho / T \tag{17}
\]

Solution of Equation (15) can be calculated as:

\[
\Phi(\theta) = c_2 \sin(n\theta + \phi) \tag{18}
\]

From Equation (9), we get:

\[
\Phi(0) = \Phi(\pi) = 0 \tag{19}
\]

Substituting Equation (19) into 18, we obtain:

\[
\phi = 0 \tag{20}
\]

Let

\[
n = k\pi / \alpha (k, n \text{ are integers}) \tag{21}
\]

Equation (16) can be transmitted to:

\[
r^2d^2L \over dr^2 + r dL \over dr + (r^2\lambda^2 - n^2)L = 0 \tag{22}
\]

This is an \( n \)-order Bessel equation, the solution is:

\[
L(r) = c_3 J_n(\lambda r) + c_4 Y_n(\lambda r) \tag{23}
\]

where \( J_n(\lambda r) \) is Bessel function of the first kind, \( Y_n(\lambda r) \) is Bessel function of the second kind. They can be expressed as:

\[
J_n(\lambda r) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(n + m + 1)} \left( \frac{\lambda r}{2} \right)^{n + 2m} \tag{24}
\]

\[
Y_n(\lambda r) = \lim_{\beta \to -n} \frac{J_\beta(\lambda r) \cos \beta \pi - J_{-\beta}(\lambda r) \sin \beta \pi}{\sin \beta \pi} \tag{25}
\]

According to boundary conditions that the TM inner and outer edge is fixed, we obtain:

\[
\begin{cases}
L(R_1) = c_3 J_n(\lambda R_1) + c_4 Y_n(\lambda R_1) = 0 \\
L(R_2) = c_3 J_n(\lambda R_2) + c_4 Y_n(\lambda R_2) = 0
\end{cases} \tag{26}
\]

If \( c_2 \) and \( c_4 \) exist as non-zero solutions, then there must be:

\[
J_n(\lambda R_1)Y_n(\lambda R_2) - J_n(\lambda R_2)Y_n(\lambda R_1) = 0 \tag{27}
\]

Then, \( \lambda_m^{(n)} \) for \( m \)-th zero can be calculated. The natural frequency \( \omega_{mn} \) is:

\[
\omega_{mn} = \frac{\lambda_m^{(n)} \sqrt{T/\rho}} \tag{28}
\]

The corresponding natural vibration modes are:

\[
W_{mn}(r, \theta) = \left[ J_n(\lambda_m^{(n)} r) - J_n(\lambda_m^{(n)} R_2) \right] Y_n(\lambda_m^{(n)} r) \sin \left( \frac{k\pi \theta}{\alpha} \right) \tag{29}
\]

Finally, we get the general solution for free vibration of sectorial annulus TM with fixed boundary:

\[
w(r, \theta, t) = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} A_{mn} W_{mn}(r, \theta) \sin(\omega_{mn} t + \phi) \tag{30}
\]

where \( A_{mn} \) and \( \phi \) are determined by initial conditions.

3. Discussion and validation

In order to verify the validity and reliability of the model, a test example is calculated for the natural frequency and natural mode. The material parameters and geometric parameters of TM are taken as:

\[
\begin{align*}
R_1 &= 0.5 \text{mm}, R_2 = 4.5 \text{mm}, \delta = 0.1 \text{mm}, \\
E &= 33.4 \text{N/mm}^2, \rho = 0.00012 \text{g/mm}^2, \\
T &= 35 \text{N/mm}^2, \mu = 0.3, \alpha = 340^\circ
\end{align*}
\]

The results are compared with ANSYS results and the result from the experiment to check the validity of theoretical model.

Table 1 shows that the vibration mode obtained by theoretical method has proven to be well-matched with ANSYS numerical solutions, and the error of natural frequency is below 4%. The natural frequency calculated by ANSYS is slightly higher than the theoretical result. It might be due to the reason that, TM deformation in ANSYS is calculated with the given shape function, which is based on small deflection theory and truncated with high-order components. It leads to the increase of TM stiffness, with the decrease of displacement field solution. Thus, more the rigid the TM model, more is the enlargement of the natural frequency.

Model parameter \( m \) represents the number of extremum in the TM radial direction. Model parameter \( k \) represents the number of extremum in the TM circumferential direction. If \( k \) is invariant, the natural frequency of the different order is linearly increasing with \( m \), as shown in Figure 3. Similarly, if \( m \) is invariant, the natural frequency of the different order is linearly increasing with \( k \), as shown in Figure 4. The natural frequency with fixed \( m \) is in a small scope. We speculate the natural frequency with fixed \( m \) shows a main response peak in sweep frequency curve.
Table 1. Natural frequency and natural vibration mode of TM from theoretical solution.

| m | k | Natural frequency (MATLAB) (Hz) | Natural vibration mode (MATLAB) | Natural frequency (ANSYS) (Hz) | Natural vibration mode (ANSYS) |
|---|---|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| 1 | 1 | 412.837                       |                               | 426.829                       |                               |
| 2 | 1 | 822.142                       |                               | 850.719                       |                               |
| 3 | 1 | 1228.744                      |                               | 1274.587                      |                               |
| 1 | 2 | 471.096                       |                               | 484.158                       |                               |
| 2 | 2 | 869.287                       |                               | 894.337                       |                               |
| 3 | 2 | 1261.575                      |                               | 1308.523                      |                               |
| 1 | 3 | 541.622                       |                               | 557.315                       |                               |

(continued)
and the order of the resonance frequency is the parameter $m$ in the theoretical model.

Literature [6] truncates transient response to render sweep frequency curve of TM under the different sound power according to the test results of volunteers TM response in Figure 5. It can be clearly observed that first-order principle resonance is 380–500 Hz, second-order is 720–900 Hz and third-order is 1100–1240 Hz. The present solutions agree well with the experimental data, which proves the reliability of the present results. In Figure 5, Position (a) is the first principle resonance for the different FRFs of a TM measurement. There is approximately 50 Hz offset range ($\Delta f$) existing in the first-order principle resonance frequency of TM under different sound pressures.

The natural frequency obtained by theoretical method and by experimental data is shown in Table 2. The peak range of 1st pole is about 120 Hz. Compared with the model solution, the range covers two natural frequencies: $m = 1$, $k = 1$ and $m = 1$, $k = 2$. Thus, we conclude that the main components of the first-order principle resonance are composed of the above two natural frequencies, and the amplitude of natural vibration mode appears the largest corresponding to the two natural frequencies. Similarly, the peak range of 2nd pole is about 180 Hz, including two natural frequencies: $m = 2$, $k = 1$ and $m = 2$, $k = 2$. Whilst the peak range of 3rd pole is about 200 Hz, including only one natural frequency; $m = 3$, $k = 1$, and other two frequencies are nearby: $m = 3$, $k = 2$ and $m = 3$, $k = 3$. Hypothesis 3 sets the malleus vibration as a small amplitude vibration that is unable to influence the resonance of TM in the theoretical model. During the experiment, a slight displacement of the malleus occurs in different sound pressures to change the tension of TM (model parameter $T$), which results in the shift of solutions in the TM theoretical model.
4. Conclusions

Based on the mechanical properties of tympanic membrane as mentioned in its anatomy, four hypotheses are presented. This paper proposes a free vibration model of TM, which simplifies TM as a sectorial annulus membrane. TM’s natural frequency of free vibration and the analytical expressions of corresponding natural vibration mode are obtained by variables separation method and Bessel function. The numerical results show that:

1. The vibration mode obtained by theoretical method has proven to be well-matched with ANSYS numerical solutions, and the error of natural frequency is below 4%. The natural frequency calculated by ANSYS is slightly higher than the theoretical result because the truncated high-order components lead to the increase of TM stiffness.

2. The theoretical model is proved to be valid since the natural frequency of the model is consistent with the experimental data. The parameters \( m \) and \( k \) represent peak times in radial and circumferential of TM, respectively, and \( k \) represents the principle resonance frequency order as well.

3. In comparison with the test data, it shows that the first-order, second-order and third-order principle resonance always consist of multiple natural frequencies and natural vibration mode of TM instead of a single one. Since the theoretical model neglects the changes of TM tension caused by the vibration from manubrium of the malleus, the theoretical solutions appear offset in the high frequency phase compared to the experiment results.

The theoretical model deduced in this paper is a free vibration simplified model of the tympanic membrane. The model can estimate the response of the eardrum vibration under the specific sound frequency. It provides mechanical reference to myringoplasty and also provides a theoretical basis for preparing an artificial tympanic membrane.

Disclosure statement

The authors declare that there is no conflict of interests regarding the publication of this paper.

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