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FREE MARTINGALE POLYNOMIALS FOR STATIONARY JACOBI PROCESSES

N. DEMNI

Abstract. We generalize a previous result concerning free martingale polynomials for the stationary free Jacobi process of parameters \( \lambda \in [0,1], \theta = 1/2. \) Hopelessly, apart from the case \( \lambda = 1, \) the polynomials we derive are no longer orthogonal with respect to the spectral measure. As a matter of fact, we use the multiplicative renormalization method to write down its corresponding orthogonal polynomials as well as the orthogonality measure associated with the martingale polynomials. We finally give a realization of the spectral measure of the free stationary Jacobi process by means of the corresponding one mode interacting Fock space.

1. Preliminaries

Let \((\mathcal{A}, \phi)\) a \(W^\star\)-non commutative probability space. Easily speaking, \(\mathcal{A}\) is a unital von Neumann algebra and \(\phi\) is a tracial faithful linear functional (state). In a previous work ([8]), we defined, via matrix theory, and studied a two parameters-dependent self-adjoint free process, called free Jacobi process. Our focus will be on a particular case called the stationary Jacobi process since its spectral distribution does not depend on time. It is defined as \(J_t := PUY_t QY_t^* U^* P\) where

- \((Y_t)_{t \geq 0}\) is a free multiplicative Brownian motion (see [4]).
- \(U\) is a Haar unitary operator in \((\mathcal{A}, \Phi)\).
- \(P\) is a projection with \(\Phi(P) = \lambda \theta \leq 1, \ \theta \in [0,1]\).
- \(Q\) is a projection with \(\Phi(Q) = \theta\).
- \(QP = PQ = \begin{cases} P & \text{if} \ \lambda \leq 1 \\ Q & \text{if} \ \lambda > 1 \end{cases}\)
- \(\{U, U^*\}\) and \(\{P, Q\}\) are free (see [12] for freeness).

Thus the process takes values in the compressed space \((P \mathcal{A} P, (1/\phi(P))\phi)\). The spectral distribution has the following decomposition:

\[
\mu_{\lambda, \theta}(dx) = \frac{1}{2\pi \lambda \theta} \frac{\sqrt{(x_+ - x)(x_+ - x_-)}}{x(1-x)} 1_{[x_-, x_+]}(x) dx + a_0 \delta_0(dx) + a_1 \delta_1(dx)
\]

where \(\delta_y\) stands for the Dirac mass at \(y\) with corresponding weight \(a_y, y \in \{0,1\}\) and

\[
x_{\pm} = \left(\sqrt{\theta(1-\lambda \theta)} \pm \sqrt{\lambda \theta(1-\theta)}\right)^2
\]
Its Cauchy transform writes

\[
G_{\mu,s}(z) = \frac{(2 - (1/\lambda\theta))z + (1/\lambda - 1) + \sqrt{Az^2 - Bz + C}}{2z(z - 1)}, \quad z \in \mathbb{C} \setminus [0, 1]
\]

with \(A = 1/(\lambda\theta)^2, \ B = 2((1/\lambda\theta)(1 + 1/\lambda) - 2/\lambda)\) and \(C = (1 - 1/\lambda)^2\). It was shown in [8] that if \(\lambda \in [0, 1], 1/\theta \geq \lambda + 1\) then the process is injective in \(P_{\mathcal{M}}\), that is \(a_0 = a_1 = 0\). Moreover, \(\mu_{1,1/2}(dx)\) fits the Beta distribution \(B(1/2, 1/2)\):

\[
\mu_{1,1/2}(dx) = \frac{1}{\pi \sqrt{x(1 - x)}} 1_{[0,1]}(x)dx
\]

Recall that the Tchebycheff polynomials of the first kind are defined by

\[
T_n(x) = \cos(n \arccos x), \quad n \geq 0, \quad |x| \leq 1.
\]

and that they are orthogonal with respect to \(\mu_{1,1/2}(dx)\). Their generating function is given by:

\[
g(u, x) = \sum_{n \geq 0} T_n(x)u^n = \frac{1 - ux}{1 - 2ux + u^2}, \quad |u| < 1.
\]

In [8], we proved that for \(r > 0\)

\[
g(re^t, J_t) = ((1 + re^t)P - 2e^tJ_t)((1 + re^t)^2P - 4re^tJ_t)^{-1}, \quad t < -\ln r
\]

defines a free martingale with respect to the natural filtration of \(J\), say \(\mathcal{F}_t\), the unit of the compressed space being the projection \(P\). It follows that \((e^{nt}T_n(2J_t - P))_{t \geq 0}, n \geq 1\)

is a family of free martingale polynomials. Note also that

\[
h(re^t, J_t) := 2g(re^t, J_t) - P = \frac{(1 - re^t)^2}{(1 + re^t)^2}(P - \frac{4re^t}{(1 + re^t)^2}J_t)^{-1}
\]

\[
= \frac{1 - re^t}{1 + re^t}(P - \frac{4re^t}{(1 + re^t)^2}J_t)^{-1}
\]

\[
= (1 - (re^t)^2)(P - 2re^t(2J_t - P) + (re^t)^2)^{-1}
\]

is also a free martingale. Let \(U_n\) denote the \(n\)-th Tchebycheff polynomial of the second kind defined by

\[
U_n(\cos \alpha) = \frac{\sin((n + 1)\alpha)}{\sin \alpha}, \quad \alpha \in \mathbb{R}
\]

with generating function given by

\[
\sum_{n \geq 0} U_n(x)u^n = \frac{1}{1 - 2ux + u^2}, \quad |x| \leq 1, \quad |u| < 1.
\]

Then, one deduces either from the above generating function or from the relation

\[
2T_n = U_n - U_{n-2}, U_{-1} := 0 \quad \text{that} \quad \{M_t^n := e^{nt}(U_n - U_{n-2})(2J_t - P), n \geq 1\}_{t \geq 0}\]

is a family of free martingale polynomials. The aim of this work is to extend this claim to the range \(\theta = 1/2, \lambda \in [0, 1]\). The motivation originates from [10] where the author determines the family of orthogonal polynomials with respect to \(\mu_{\lambda, \theta}\). Our first guess was that these will be free martingales polynomials for all \(\lambda \in [0, 1], \theta \leq 1/(\lambda + 1)\). Yet, things turn to be more complicated: not only the range is restricted but the martingale polynomials we derive are not orthogonal with respect to \(\mu_{\lambda, 1/2}\) except for \(\lambda = 1\). As
a matter of fact, we will on one hand derive the orthogonal polynomials corresponding to $\mu_{\lambda,1/2}$ and compute on the other hand the appropriate orthogonality measure for our martingales polynomials. The last part of the paper is devoted to a realization of the free stationary Jacobi process using the Accardi-Bozejko isomorphism (see [1]) as well as some comments.

Remark. From a matrix theory point of view, the choice $\theta = 1/2$ corresponds to the ultraspherical multivariate Beta distribution (see [8]). Moreover, to our level of Knowledge, there is only one result concerning martingale polynomials for the stationary (classical) Jacobi process, which is restricted to the one dimensional case. More precisely, pick a vector $(x_1, \ldots, x_d)$ belonging to the sphere $S^{d-1}, d \geq 2$ distributed according to the uniform (Haar) measure, then form the discrete process defined by

$$s_p = \sum_{i=0}^{p} x_i^2, \quad 1 \leq p \leq d - 1.$$  

It is a known that each random variable has the Beta distribution $B(\frac{d-p}{2}, p/2)$. It was shown in [11] that $M_n^{d}(p) = \frac{1}{((d-p)/2)_n} P_{n,\alpha,\beta}^{\alpha,\beta}(2s_p - 1)$, where $P_{n,\alpha,\beta}^{\alpha,\beta}$ denotes the $n$-th Jacobi polynomial of parameters $\alpha = \frac{d-p}{2} - 1, \beta = \frac{p}{2} - 1$, is a martingale with respect to the natural filtration of the process. To relate this to our work, we rewrite $s_p$ in the matrix form

$$s_p = P_1 U_d Q_p U_d^* P_1,$$

where $U_d$ is a $d \times d$ Haar unitary matrix, $P_1$ is a $d \times d$ projection with only one non vanishing coefficient ($P_1)_{11} = 1$ and $Q_p$ is a $d \times d$ projection with only $p$ non vanishing term $(Q_p)_{11} = \cdots = (Q_p)_{pp} = 1$. For $d = 2p$, we get the ultraspherical polynomials of parameter $\lambda = (p - 1)/2$.

2. Main result

One has for $\lambda \in ]0, 1], \theta = 1/2$

$$x_0 = \left(\frac{\sqrt{2} - \lambda}{2} - \frac{\sqrt{\lambda}}{2}\right)^2 \leq x \leq x_0 = \left(\frac{\sqrt{2} - \lambda}{2} + \frac{\sqrt{\lambda}}{2}\right)^2 \Rightarrow -1 \leq \frac{2x - 1}{\sqrt{\lambda(2-\lambda)}} \leq 1$$

and our main result is stated as follows:

**Proposition 2.1.** Set

$$a(\lambda) = \frac{(1 - \lambda)}{\sqrt{\lambda(2-\lambda)}}, \quad x_{t,\lambda} = \frac{2J_t - P}{\sqrt{\lambda(2-\lambda)}}$$

For each $n \geq 1$, the process defined by

$$M_n^t := [U_n(x_{t,\lambda}) - 2a(\lambda)U_{n-1}(x_{t,\lambda}) - U_{n-2}(x_{t,\lambda})] \left(\frac{e^t}{\lambda(2-\lambda)}\right)^n, t \geq 0$$

is a $(\mathcal{F}_t)$-free martingale.
3. Proof of the main result

The proof consists of two parts: the first one consists in deriving a martingale function for all values of $\lambda \in [0, 1]$, $\theta \leq 1/2 \leq 1/(\lambda+1)$. In the second one, we specialize for $\theta = 1/2$ and show that this function corresponds to the generating function of the polynomials stated above.

First step: inspired by the above expression of $h(re^t, J_t)$, we will look for martingales of the form

$$R_t := K_t(P - Z_tJ_t)^{-1} = K_t \sum_{n \geq 0} Z_t^n J_t^n := K_tH_t$$

where $K, Z$ are differentiable functions of the variable $t$ lying in some interval $[0, t_0]$ such that $0 < Z_t < 1$ for $t \in [0, t_0]$. The finite variation part of $dR_t$ is given by

$$FV(dR_t) = K'_tH_t dt + K_t FV(dH_t)$$

Our main tool is the free stochastic calculus and more precisely the free stochastic differential equation already set for $J^n_t, n \geq 1$ (8):

$$dJ^n_t = dM_t + n(\theta P - J_t)J^{n-1}_t dt + \lambda \theta \sum_{l=1}^{n-1} l[m_{n-l}(P - J_t)J^{l-1}_t + (m_{n-l-1} - m_{n-l})J^l_t]dt$$

where $dM$ stands for the martingale part and $m_n$ is the $n$-th moment of $J_t$ in $P_{\lambda P}$:

$$m_n := \tilde{\phi}(J^n_t) = \frac{1}{\phi(P)} \phi(J^n_t)$$

The finite variation part $FV(dJ^n_t)$ of $J^n_t$ transforms to:

$$FV(dJ^n_t) = n(\theta P - J_t)J^{n-1}_t dt + \lambda \theta \left[ \sum_{l=1}^{n-1} l[m_{n-l}J^{l-1}_t + \sum_{l=1}^{n-1} l(m_{n-l-1} - 2m_{n-l})J^l_t] \right] dt$$

$$= n(\theta P - J_t)J^{n-1}_t dt + \lambda \theta \sum_{l=1}^{n-1} l[m_{n-l}J^{l-1}_t + \sum_{l=1}^{n-1} (l - 1)(m_{n-l} - 2m_{n-l+1})J^l_t] dt$$

$$= n(\theta P - J_t)J^{n-1}_t dt + \lambda \theta \sum_{l=1}^{n} [m_{n-l} + (l - 1)(m_{n-l} - 2m_{n-l+1})]J^{l-1}_t dt - n\lambda J^n_t dt$$

$$= n\theta(1 - \lambda)J^{n-1}_t dt - nJ^n_t dt + \lambda \theta \sum_{l=1}^{n} [m_{n-l} + 2(l - 1)(m_{n-l} - m_{n-l+1})]J^{l-1}_t dt$$
Thus
\[ FV(dH_t) = \sum_{n \geq 1} n Z_t^n J_t^n dt + \sum_{n \geq 1} Z_t FV(J_t^n) \]
\[ = \sum_{n \geq 1} n Z_t^n J_t^{n-1} dt + \sum_{n \geq 0} n Z_t^n J_t^n dt + \theta(1 - \lambda) \sum_{n \geq 1} n Z_t^n J_t^{n-1} dt \]
\[ + \lambda \theta \sum_{n \geq 1} \sum_{l=0}^{n} Z_t^{n+l} m_{n-l} J_t^{l-1} dt + 2 \lambda \theta \sum_{n \geq 1} \sum_{l=1}^{n} (l - 1) Z_t^n (m_{n-l} - m_{n-l+1}) J_t^{l-1} dt \]
\[ = \sum_{n \geq 1} n [Z_t^n J_t^{n-1} - Z_t^n] J_t^n dt + \theta(1 - \lambda) \sum_{n \geq 0} n Z_t^{n+1} J_t^n dt \]
\[ + \lambda \theta \sum_{n \geq 0} \sum_{l \geq 0} Z_t^{n+l+1} m_n J_t^l dt + 2 \lambda \theta \sum_{n \geq 0} \sum_{l \geq 0} l Z_t^{n+l+1} (m_n - m_{n+1}) J_t^l dt \]
\[ = [Z_t^n / Z_t - 1 + \theta(1 - \lambda) Z_t] \sum_{n \geq 1} n Z_t^n J_t^n dt + \theta(1 - \lambda) Z_t \sum_{n \geq 0} Z_t^n J_t^n dt \]
\[ + \lambda \theta \sum_{n \geq 0} Z_t^{n+1} m_n \sum_{l \geq 0} Z_t^l J_t^l dt + 2 \lambda \theta \sum_{n \geq 0} Z_t^{n+1} (m_n - m_{n+1}) \sum_{l \geq 0} l Z_t^l J_t^l dt \]

Recall that the Cauchy transform of a measure on the real line is defined by
\[ G_\nu(z) = \int_{\mathbb{R}} \frac{1}{z - x} \nu(dx) = \sum_{n \geq 0} \frac{1}{z^{n+1}} \int_{\mathbb{R}} x^n \nu(dx) \]
for some values of z for which both the integral and the infinite sum make sense. Then, since
\[ 0 < Z < 1 \]
and \( \mu_{\lambda, \theta} \) is supported in \([0, 1]\), it is easy to see that
\[ \sum_{n \geq 0} Z_t^{n+1} (m_n - m_{n+1}) = \left( 1 - \frac{1}{Z_t} \right) G_{\mu_{\lambda, \theta}} \left( \frac{1}{Z_t} \right) + 1 \]
with \( G_{\mu_{\lambda, \theta}} \) given by (9). This gives
\[ 2 \lambda \theta (1 - z) G_{\mu_{\lambda, \theta}}(z) = \frac{(1 - 2 \lambda \theta) z - \theta(1 - \lambda) - \sqrt{z^2 - (\lambda \theta)^2 B z + (\lambda \theta)^2 C}}{z} \]
so that
\[ 2 \lambda \theta (1 - Z_t^{-1}) G_{\mu_{\lambda, \theta}}(Z_t^{-1}) + 2 \lambda \theta = 1 - \theta(1 - \lambda) Z_t - \sqrt{1 - (\lambda \theta)^2 B Z_t + (\lambda \theta)^2 C Z_t}, \]
We finally get:
\[ FV(dH_t) = [Z_t^n / Z_t - \sqrt{1 - (\lambda \theta)^2 B Z_t + (\lambda \theta)^2 C Z_t}] \sum_{n \geq 1} n Z_t^n J_t^n dt \]
\[ + \left[ \lambda \theta G_{\mu_{\lambda, \theta}} \left( \frac{1}{Z_t} \right) + \theta(1 - \lambda) Z_t \right] \sum_{n \geq 0} Z_t^n J_t^n dt \]

In order to derive free martingales, we shall pick \( Z \) such that \( Z_t' = Z_t \sqrt{1 - (\lambda \theta)^2 B Z_t + (\lambda \theta)^2 C Z_t} \). This shows that \( Z \) is an increasing function and one can solve the above non linear differential equation as follows: use the variables change \( u = Z_t, t < t_0 \), then integrate to
The discriminant equals to 
\[ \Delta = 16r \]

Remark. Let \( c_1 = 2\theta(1 + \lambda - 2\lambda\theta), c_2 = \theta^2(1 - \lambda)^2 \). Then, the function \( u \mapsto 1 - c_1u + c_2u^2 \) is decreasing for \( u \in [0, 1] \); in fact,
\[
2c_2u - c_1 < 2c_2 - c_1 = 2\theta^2(1 - \lambda)^2 - 2\theta(1 + \lambda - 2\lambda\theta) \\
= 2\theta(1 + \lambda^2) - (1 + \lambda) \leq 2\theta \left( \frac{1 + \lambda^2}{1 + \lambda} - (1 + \lambda) \right) = -\frac{4\lambda\theta}{1 + \lambda} < 0
\]
which yields \( 1 - c_1u + c_2u^2 > 1 - c_1 + c_2 = (1 - \theta(1 + \lambda))^2 \geq 0 \).

Next, use the variable change \( 1 - vu = \sqrt{1 - c_1u + c_2u^2} \). This gives
\[
u = \frac{2v - c_1}{v^2 - c_2}, \quad du = -\frac{v^2 + c_2 - c_1v}{(v^2 - c_2)^2} dv, \quad 1 - vu = -\frac{v^2 + c_2 - c_1v}{v^2 - c_2}
\]
Moreover
\[
u = \frac{1 - \sqrt{1 - c_1u + c_2u^2}}{u}, \quad 0 < u < 1
\]
is an increasing function: in fact the numerator of its derivative writes
\[
c_1u - 2c_2u^2 + 2(1 - c_1u + c_2u^2) - 2\sqrt{1 - c_1u + c_2u^2} = (2 - c_1u) - 2\sqrt{1 - c_1u + c_2u^2}
\]
Since \( 2 - c_1u > 2 - c_1 = 2(1 - \theta(1 + \lambda)) + 4\lambda\theta^2 > 0 \), our claim follows from the fact that \( c_1^2 - 4c_2 = 16\lambda\theta^2(1 - \lambda\theta)(1 - 2\theta) \geq 0 \).

Finally, the integral transforms to
\[
\int_{[0, v_1]} \frac{2dv}{2v - c_2} = \log \left| \frac{2v - c_1}{2v_0 - c_1} \right| = t
\]
where \( 1 - Z_t v_t = \sqrt{1 - c_1Z_t + c_2Z_t^2} \), \( 1 - Z_0v_0 = \sqrt{1 - c_1Z_0 + c_2Z_0^2} \). Note also that \( c_1^2 - 4c_2 \geq 0 \) implies that for all \( u \in [Z_0, Z_t] \subset [0, 1] \)
\[
v - \frac{c_1}{2} = \frac{1 - \sqrt{1 - c_1u + c_2u^2} - c_1}{u} = \frac{(1 - c_1u/2) - \sqrt{1 - c_1u + c_2u^2}}{u(1 - c_1u/2) + \sqrt{1 - c_1u + c_2u^2}} \geq 0
\]
since \( 1 - c_1/2u \geq 1 - c_1/2 \geq 0 \). Thus \( v \geq c_1/2 \geq \sqrt{c_2} \).

\[
v_t = \left[ (2v_0 - c_1)e^t + c_1 \right]/2 \Leftrightarrow \sqrt{1 - c_1Z_t + c_2Z_t^2} = 1 - \frac{(2v_0 - c_1)e^{zt} + c_1}{2} Z_t
\]
We finally get
\[
Z_t = \frac{4(2v_0 - c_1)e^{zt}}{((2v_0 - c_1)e^t + c_1)^2 - 4c_2}, \quad t \leq t_0
\]
where \( t_0 \) is the first time such that \( Z_{t_0} = 1 \Leftrightarrow (2v_0 - c_1)e^{t_0} + c_1 - 4c_2 - 4(2v_0 - c_1)e^{t_0} \).
Set \( r = r(\lambda, \theta) := (2v_0 - c_1) \) and \( x_0 = e^{t_0} > 0 \), then \( r^2x_0^2 + 2(c_1 - r)x_0 + c_1^2 - 4c_2 = 0 \).
The discriminant equals to \( \Delta = 16r^2(1 + c_2 - c_1) = 16r^2(1 - (1 + \lambda)^2) \). Thus
\[
x_0 = \frac{-(c_1 - 2) - 2(1 - (1 + \lambda))}{r} = \frac{2(1 - (1 + \lambda)) + 4\lambda\theta^2 - 2(1 - (1 + \lambda))}{r} = \frac{4\lambda\theta^2}{r} \geq 1
\]
The last inequality follows from the fact that \(1 - \sqrt{c_2}u \geq 1 - \theta(1 + \lambda) \geq 0\) and from
\[
\rho - 4\lambda\theta^2 = 2v_0 - c_1 - 4\lambda\theta^2 = 2(v_0 - \theta(1 + \lambda)) = 2(v_0 - \sqrt{c_2}) \leq 0.
\]
It gives \(t_0 = -\ln(r/4\lambda\theta^2)\). Note also that the denominator is well defined for all \(t \leq t_0\) since \(c_1^2 \geq 4c_2\) and \(2v_0 - c_1 \geq 0\).

For the remaining terms, we shall choose \(K\) such that
\[
K_t' + K_t \left[ \lambda \theta G_{\mu,\sigma} \left( \frac{1}{Z_t} \right) + \theta(1 - \lambda)Z_t \right] = 0
\]
An easy computation shows that this equals to
\[
K_t' + K_t \left[ \frac{Z_t^2}{Z_t - 1} + (1 - 2\theta) \frac{Z_t}{Z_t - 1} - \frac{Z_t\sqrt{1 - c_1Z_t + c_2Z_t^2}}{Z_t - 1} \right] = 0
\]
Remembering the choice of the function \(Z\), this writes
\[
K_t' - \frac{K_t}{2} \left[ \frac{Z_t'}{Z_t - 1} - (1 - 2\theta) \frac{Z_t}{Z_t - 1} - \theta(1 - \lambda) \frac{Z_t^2}{Z_t - 1} \right] = 0
\]
or equivalently
\[
K_t' - \frac{K_t}{2} \left[ \frac{Z_t'}{Z_t - 1} - (1 - \theta - \lambda\theta) \frac{Z_t}{Z_t - 1} - \theta(1 - \lambda)Z_t \right] = 0
\]
If \(K_t \neq 0\), then
\[
\log K_t = \frac{1}{2} \log(1 - Z_t) - \frac{1 - \theta - \lambda\theta}{2} \int \frac{Z_s}{Z_s - 1} ds - \frac{\theta(1 - \lambda)}{2} \int Z_s ds + C
\]
If \(\lambda \neq 1\), then the last term is given by
\[
-\frac{\theta(1 - \lambda)}{2} \int Z_s ds = \frac{\theta(1 - \lambda)}{\sqrt{c_2}} \int \frac{(r/2\sqrt{c_2})e^t}{1 - \left(\frac{re^t + c_1}{2\sqrt{c_2}}\right)^2} = \arg \tanh \left( \frac{re^t + c_1}{2\sqrt{c_2}} \right)
\]
where \(\arg \tanh(u) = (1/2) \log((u + 1)/(u - 1)), |u| > 1\). The second term writes
\[
\frac{Z_t}{Z_t - 1} = \frac{4re^t}{4c_2 + 4re^t - (re^t + c_1)^2} = \frac{4re^t}{c_2 + 1 - c_1 - (re^t + c_1 - 2)^2} = \frac{1}{c_2 + 1 - c_1 - (re^t + c_1 - 2)^2}
\]
\[
= \frac{2}{\sqrt{c_2 + 1 - c_1} - \frac{(re^t + c_1 - 2)}{2\sqrt{c_2 + 1 - c_1}}}
\]
Observe that \(2 - c_1 - re^t > 2 - c_1 - re^0 = 2(1 - \theta(1 + \lambda)) \geq 0\). Thus, if \(\theta(1 + \lambda) \neq 1\)
\[
\frac{1 - \theta(1 + \lambda)}{2} \int \frac{Z_s}{Z_s - 1} ds = \arg \tanh \left( \frac{2 - c_1 - re^t}{2\sqrt{c_2 + 1 - c_1}} \right)
\]
Thus, if $\lambda \neq 1 (\theta \leq 1/2 < 1/(\lambda + 1))$,

$$K_t = C(1 - Z_t)^{1/2} \left( \frac{re^t + c_1 + 2\sqrt{c_2}}{re^t + c_1 - 2\sqrt{c_2}} \right)^{1/2} \left( \frac{2 - c_1 - 2c_3 - re^t}{2 - c_1 + 2c_3 - re^t} \right)^{1/2}$$

where $c_3 := \sqrt{c_2 + 1 - c_1} = 1 - \theta(\lambda + 1)$. Note that for $\lambda = 1, \theta = 1/2$, $c_1 = 1, c_2 = 0, c_3 = 0$ and

$$K_t = C\frac{1 - re^t}{1 + re^t}, \quad t < t_0 = -\ln r.$$

The case $\theta = 1/2, \lambda \neq 1$: free martingales polynomials: one has

$$c_1 = 1, c_2 = \frac{(1 - \lambda)^2}{4}, c_3 = \sqrt{c_2} = \frac{1 - \lambda}{2}, Z_t = \frac{4re^t}{(re^t + 1)^2 - (1 - \lambda)^2}$$

$$c_1 + 2\sqrt{c_2} = 2(1 + c_3) - c_1 = 2 - \lambda, c_1 - 2\sqrt{c_2} = 2(1 - c_3) - c_1 = \lambda.$$

$$1 - Z_t = \frac{(re^t - 1)^2 - (1 - \lambda)^2}{(re^t + 1)^2 - (1 - \lambda)^2} = \frac{(re^t + \lambda - 2)(re^t - \lambda)}{(re^t + 2 - \lambda)(re^t + \lambda)}$$

Thus, for $t < -\ln(r/\lambda)$,

$$K_t = C_\lambda \frac{\lambda - re^t}{\lambda + re^t}$$

so that

$$R_t = C_\lambda \frac{\lambda - re^t}{\lambda + re^t} (P - \frac{4re^t}{(re^t + 1)^2 - (1 - \lambda)^2} J_t)^{-1}$$

$$= C(\lambda - re^t)(2 - \lambda + re^t)(\lambda(2 - \lambda)P + (re^t)^2P - 2re^t(2J_t - P))^{-1}$$

$$= C(\lambda - re^t)(2 - \lambda + re^t) \left( P - \frac{2re^t}{\sqrt{\lambda(2 - \lambda)}} \frac{(2J_t - P)}{\sqrt{\lambda(2 - \lambda)}} + (\frac{re^t}{\sqrt{\lambda(2 - \lambda)}})^2 P \right)^{-1}$$

$$= C \left( 1 - 2 \frac{(1 - \lambda)re^t}{\sqrt{\lambda(2 - \lambda)}} + \frac{(re^t)^2}{\lambda(2 - \lambda)} \right) \left( P - \frac{2re^t}{\sqrt{\lambda(2 - \lambda)}} \frac{(2J_t - P)}{\sqrt{\lambda(2 - \lambda)}} + (\frac{re^t}{\sqrt{\lambda(2 - \lambda)}})^2 P \right)^{-1}$$

is a free martingale with respect to the natural filtration $\mathcal{F}_t$. Besides, since $\lambda \in [0, 1]$, then $\lambda \leq \sqrt{\lambda(2 - \lambda)}$, hence $(re^t)/\sqrt{\lambda(2 - \lambda)}) < 1$ for all $t < -\ln(r/\lambda)$. Now, let us consider the following generating function

$$g(u, x) = \frac{1 - 2au - u^2}{1 - 2ux + u^2}, \quad 0 < a, u < 1, \ |x| \leq 1.$$ 

It follows that

$$g(u, x) = U_0(x) + (U_1(x) - 2a)u + \sum_{n \geq 2} [U_n(x) - 2aU_{n-1}(x) - U_{n-2}(x)]u^n$$

Setting

$$u_{t, \lambda} := \frac{re^t}{\sqrt{\lambda(2 - \lambda)}}, \quad t < t_0,$$

then

$$R_t = C[P + (x_{t, \lambda} - 2a(\lambda)P)u_{t, \lambda} + \sum_{n \geq 2} [U_n(x_{t, \lambda}) - 2a(\lambda)U_{n-1}(x_{t, \lambda}) - U_{n-2}(x_{t, \lambda})]u_{t, \lambda}^n]$$
Setting \( U_{-1} = U_{-2} = 0 \), it can be written as
\[
R_t = C \sum_{n=0}^{\infty} [U_n(x_{t,\lambda}) - 2a(\lambda)U_{n-1}(x_{t,\lambda}) - U_{n-2}(x_{t,\lambda})]w^n_{t,\lambda}
\]

**Remark.** The case \( \lambda = 1 \).

\( c_1 = 4\theta(1 - \theta) \), \( c_2 = 0 \) and \( Z_t \) writes
\[
Z_t = \frac{4re^t}{(re^t + 4\theta(1 - \theta))^2}
\]

Moreover, \( c_3 = \sqrt{1 - c_1} = (1 - 2\theta) \), \( 2 - 2c_3 - c_1 = 4\theta^2 \), \( 2 + 2c_3 - c_1 = 4(1 - \theta)^2 \). \( K_t \) then writes
\[
K_t = \frac{\sqrt{(re^t + 4\theta(1 - \theta))^2 - 4re^t}}{re^t + 4\theta(1 - \theta)} \sqrt{\frac{4\theta^2 - re^t}{4(1 - \theta)^2 - re^t}}
\]

4. one-parameter measures family and Orthogonal polynomials

Let \( \mu \) be a measure on the real line which is not supported by a finite set. Assume that \( \mu \) has finite moments of all orders. Applying the Gram-Schmidt orthogonalization method to the basis \((1, x, x^2, \ldots)\), there exist a unique family of monic orthogonal polynomials with respect to \( \mu \), say \((P_n)_{n \geq 0}\). These polynomials satisfy the three-terms recurrence relation
\[
(x - \alpha_n)P_n(x) = P_{n+1}(x) + \omega_n P_{n-1}(x), \quad n \geq 0, P_{-1} := 0.
\]

where \( \alpha_n \in \mathbb{R}, \omega_n > 0 \). \((\alpha_n, \omega_n)_{n \geq 0}\) are called the Jacobi-Szegö parameters of \( \mu \). It is known that \( \mu \) is symmetric if and only if \( \alpha_n = 0 \), \( n \geq 0 \). Another way to derive the family \((P_n)_{n}\) is the multiplicative renormalization method ([3], [4], [5], [6]) that we shall recall here: a nice function \((u, x) \mapsto \psi(u, x)\) is a generating function for the measure \( \mu \) if \( \psi \) has the expansion
\[
\psi(u, x) = \sum_{n \geq 0} c_n P_n(x) u^n, \quad c_n \in \mathbb{R}
\]

where \( P_n \) are orthogonal with respect to \( \mu \). Of course, there is more than one generating function corresponding to a given measure and in order to claim whether a function is a generating function or not, authors in [3] provided a necessary and sufficient condition. For a particular form of \( \psi \) which fits our need, their result is formulated as follows:

**Theorem 4.1.** Define
\[
\theta(u) := \int_{\mathbb{R}} \frac{1}{1 - ux} \mu(dx), \quad \theta(u, v) := \int_{\mathbb{R}} \frac{1}{(1 - ux)(1 - vx)} \mu(dx).
\]

Let \( \rho \) analytic around 0 such that \( \rho(0) = 0 \) and \( \rho'(0) \neq 0 \). Then
\[
(2)
\psi(u, x) := \frac{(1 - \rho(u)x)^{-1}}{\theta(\rho(u))}
\]

is a generating function for \( \mu \) if and only if
\[
\Theta_\rho(u, v) := \frac{\theta(\rho(u), \rho(v))}{\theta(\rho(u))\theta(\rho(v))}
\]

is a function of \( uv \).
We will apply this result to the measures family \( \nu_\lambda, \lambda \in [0,1] \) which is the image of 
\[
\mu_{\lambda,1/2} = \frac{1}{\pi \lambda} \frac{\sqrt{(x_+ - x)(x - x_+)} 1_{[x_-,x_+]}(x)}{x(1-x)} dx, \quad x_\pm = \frac{(\sqrt{\lambda} \pm \sqrt{2-\lambda})^2}{4}
\]
by the map
\[
x \mapsto \frac{2x - 1}{\sqrt{\lambda}(2-\lambda)}
\]
Then,
\[
\nu_\lambda(dx) = \frac{(2-\lambda)}{\pi} \frac{\sqrt{1-x^2}}{1-\lambda(2-\lambda)x^2} 1_{[-1,1]}(x) dx
\]
Our scheme is the almost the same used in [9] except the computation of \( \theta(u) \) which follows easily from \( G_{\mu_{\lambda,1/2}} \). More precisely, authors considered the one-parameter measures family
\[
\mu_a(dx) = \frac{a \sqrt{1-x^2}}{a^2 + (1-2a)x^2} 1_{[-1,1]}(x) dx, \quad a > 0.
\]
It is forward that \( \mu_{1/(2-\lambda)} = \nu_\lambda \) almost everywhere for \( 0 < \lambda \leq 1 \iff 1/2 < a \leq 1 \).

**Proposition 4.1.**

\[
\theta(u) = \theta_\lambda(u) = \frac{2-\lambda}{1-\lambda + \sqrt{1-u^2}}, \quad |u| < 1
\]

Using
\[
\frac{1}{(1-ux)(1-vx)} = \frac{1}{u-v} \left( \frac{u}{1-ux} - \frac{v}{1-vx} \right)
\]
it follows that \( \theta(u, v) = (u\theta(u) - v\theta(v))/(u-v) \) from which we deduce

**Corollary 4.1.**

\[
\theta(u, v) = \theta_\lambda(u, v) = \frac{1}{2-\lambda} \left[ 1 - \lambda + \frac{u + v}{u \sqrt{1-u^2} + v \sqrt{1-v^2}} \right]
\]

**Proof:** from the definition of \( \nu_\lambda \), one writes for \( 0 < u < \lambda(2-\lambda) \leq 1 \):

\[
\int_{\mathbb{R}} \frac{1}{1-ux} \nu_\lambda(dx) = \int_{\mathbb{R}} \frac{1}{1-u} \frac{\mu_{\lambda,1/2}(dx)}{\sqrt{\lambda}(2-\lambda)} = \frac{\sqrt{\lambda(2-\lambda)}}{2u} G_{\mu_{\lambda,1/2}} \left( \frac{\sqrt{\lambda(2-\lambda)} + u}{2u} \right)
\]
The result follows from
\[
G_{\mu_{\lambda,1/2}}(z) = \frac{(1-\lambda)(2z - 1) - \sqrt{4z^2 - 4z + (1-\lambda)^2}}{2\lambda z(1-z)}, \quad z \in \mathbb{C} \setminus [0,1]
\]
Let \( \rho(u) = 2u/(1 + u^2) \), then
\[
\frac{\rho(u) + \rho(v)}{\rho(u) \sqrt{1 - \rho^2(v)} + \rho(v) \sqrt{1 - \rho^2(u)}} = \frac{1 + uv}{1 - uv}
\]
so that Theorem [1] applies and claims that
\[
\psi_\lambda(u, x) = \frac{1 - \lambda/(2-\lambda)u^2}{1 - 2ux + u^2}
\]
is a generating function for $\nu_\lambda$ corresponding to the polynomials

$$Q_n^\lambda(x) = U_n(x) - \frac{\lambda}{2-\lambda} U_{n-2}(x), \quad n \geq 0, \quad U_{-1} = U_{-2} := 0.$$  

Using the recurrence relation

(3) \hspace{1cm} 2xU_n(x) = U_{n+1}(x) + U_{n-1}(x), \quad U_{-1} := 0,

These polynomials satisfy

$$2xQ_n^\lambda(x) = Q_n^\lambda_0(x),$$

$$2xQ_n^\lambda_1(x) = Q_n^\lambda_0(x) + \left(1 + \frac{\lambda}{2-\lambda}\right) Q_n^\lambda_0(x)$$

$$2xQ_n^\lambda_2(x) = Q_n^\lambda_1(x) + Q_n^\lambda_{n-1}(x), \quad n \geq 2.$$  

Setting $Q_{-1}^\lambda := 0$ and since the coefficient of the leading power in $Q_n^\lambda(x)$ is $2^n$, then one deduces that the Jacobi-Szegö parameters are given by : $\alpha_n = 0, \quad n \geq 0, \quad w_1 = 1/(2(2 - \lambda)), \quad w_n = 1/4, \quad n \geq 2.$

Remark. In [10], authors characterize the absolutely continuous measures for which the multiplicative renormalization method is applicable with the generating function given by (2). They derived a two-parameters densities family written as

$$f(x) = \frac{c \sqrt{1-x^2}}{\pi [b^2 + c^2 - 2b(1-c)x + (1-2c)x^2]} 1_{[-1,1]}(x), \quad |b| < 1-c, \quad 0 < c \leq 1.$$  

These densities fit the image of absolutely continuous part of $\mu_{\lambda, \theta}$ by the map

$$u = \frac{2x - s}{d} \in [-1, 1]$$  

with $d = d(\lambda, \theta) = x_+ - x_- = 4\theta \sqrt{\lambda(1-\theta)(1-\lambda\theta)}$, $s = s(\lambda, \theta) = x_+ + x_- = 2\theta(1 + \lambda - 2\lambda\theta)$. One gets

$$\nu_{\lambda, \theta}(dx) = \frac{d^2}{2\pi \lambda \theta s(2-s)} \frac{\sqrt{1-x^2}}{2d(1-s)x - d^2 x^2} dx$$  

which provides the following relations

(4) \hspace{1cm} c = \frac{1}{2(1-\lambda\theta)}, \quad b = \sqrt{\frac{\lambda}{(1-\theta)(1-\lambda\theta)}(2\theta - 1)}$$

As a result, one can derive the corresponing orthogonal polynomials for $\lambda \in [0,1], \theta \leq 1/(\lambda + 1)$ from the generating function (14):

(5) \hspace{1cm} \phi(u, x) = \frac{1 - 2bu + (1-2c)u^2}{1 - 2ux + u^2}.
5. MORE ORTHOGONAL POLYNOMIALS

Consider the polynomials $P_n^\lambda$ defined by

$$P_n^\lambda(x) = U_n(x) - 2a(\lambda)U_{n-1}(x) - U_{n-2}(x), \quad U_1 = U_2 := 0$$

with generating function

$$g(u, x) = \frac{1 - 2a(\lambda)u - u^2}{1 - 2ux + u^2}, \quad a(\lambda) = \frac{1 - \lambda}{\lambda(2 - \lambda)}, \quad 0 < u < 1.$$ 

The $P_n^\lambda$’s appear in [2] as a limiting case of the $q$-Pollaczek polynomials. The coefficient of the highest monomial is equal to $2^n$. Using (3), one deduces that

$$2|x - a(\lambda)|P_0^\lambda(x) = P_1^\lambda(x)$$

$$2xP_0^\lambda(x) = P_2^\lambda(x) + 2P_1^\lambda(x)$$

$$2xP_n^\lambda(x) = P_{n+1}^\lambda(x) + P_{n-1}^\lambda(x), \quad n \geq 2.$$

Thus the Jacobi-Szegő parameters are given by

$$\alpha \lambda / \rho$$

for which the $P_n^\lambda$’s are orthogonal. Since $\alpha \lambda \neq 0$, then $\xi_\lambda$ is not symmetric. Indeed, keeping the same function $\rho$ previously defined, then the function $\theta$ must be equal to

$$\theta(\rho(u)) = \frac{1 + u^2}{1 - 2a(\lambda) - u^2}$$

so that

$$\theta(u) = \frac{1}{\sqrt{1 - u^2 - a(\lambda)u}}$$

From the definition of $\theta$, one deduces that

$$G_{\xi_\lambda}(u) := \int_{\mathbb{R}} \frac{1}{u - x} \xi_\lambda(dx) = \frac{1}{u} \theta \left( \frac{1}{u} \right) = \frac{\sqrt{u^2 - 1} + a(\lambda)}{u^2 - (1 + a^2(\lambda))}$$

for $|u| > 1$, $u \neq \pm \sqrt{1 + a(\lambda)^2}$. Thus, $\xi_\lambda$ has two atoms $a_\pm$ at $\pm \sqrt{a^2(\lambda) + 1}$ and an absolutely continuous part given by

$$a_\pm = - \lim_{y \to 0^+} y \Im G_{\xi_\lambda}(\pm \sqrt{a^2(\lambda) + 1} + iy), \quad g(x) = -\frac{1}{\pi} \lim_{y \to 0^+} \Im G_{\xi_\lambda}(x + iy)$$

Using that the Cauchy transform maps $\mathbb{C}^+$ to $\mathbb{C}^-$, one finally gets

$$\xi_\lambda(dx) = \frac{a(\lambda)}{\sqrt{a^2(\lambda) + 1}} \delta_{a^2(\lambda) + 1}(dx) + \frac{1}{\pi a^2(\lambda) + 1 - x^2} 1_{|x| < 1} dx$$

**Remark.** To see that this defines a probability measure for $\lambda \neq 1$, it suffices to write

$$\frac{1}{\pi} \int_{-1}^1 \sqrt{1 - x^2} dx = \frac{1}{\pi} \int_0^1 \frac{\sqrt{1 - x}}{\sqrt{a^2(\lambda) + 1 - x}} dx$$

$$= \frac{1}{2(a^2(\lambda) + 1)^2} F_1 \left( \frac{1}{2}, \frac{1}{2}; \frac{1}{a^2(\lambda) + 1} \right)$$
where \( \binom{2}{1} \) denotes the Gauss hypergeometric function given by
\[
\binom{2}{1}(c, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-c}dx, \quad \Re(b) \land \Re(c-b) > 0
\]
for \(|u| < 1\). Then, one uses the identity
\[
\binom{2}{1}(1, b, 2; z) = \frac{1-(1-z)^{1-b}}{(1-b)z}
\]
to get
\[
\frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-x^2}}{a^2(\lambda) + 1-x^2} dx = 1 - \frac{a(\lambda)}{\sqrt{a^2(\lambda) + 1}}
\]

6. ONE MODE INTERACTING Fock space

In the sequel, we give a realization of \( \nu_{\lambda, \theta} \), image of the spectral measure \( \mu_{\lambda, \theta} \) for \( \lambda \in ]0,1[ \), \( \theta \leq 1/(\lambda + 1) \) so that the support is \([ -1, 1] \). In the quantum scope, it is known as the quantum decomposition of \( \nu_{\lambda, \theta} \). We only need the Jacobi-Szegő parameters in order to apply Accardi-Bozejko Theorem \([1]\). We first write down from the generating function \([\Phi]\) the orthogonal polynomials \((\nu_{\lambda, \theta})\) corresponding to \( \nu_{\lambda, \theta} \):
\[
Q_{\nu_{\lambda, \theta}} = U_n - 2bU_{n-1} + (1-2c)U_{n-2}, \quad U_{-1} = U_{-2} = 0,
\]
where \( b = b(\lambda, \theta), c = c(\lambda, \theta) \) are given by \([1]\). It follows that \( \alpha_0 = b, \alpha_n = 0 \) for \( n \geq 1 \) and \( \omega_1 = c/2, \omega_n = 1/4 \) for \( n \geq 1 \). In order to use Accardi-Bozejko Theorem \([1]\), we shall introduce the so-called one-mode interacting Fock space: let \( \mathcal{H} \) be a one dimensional separable complex Hilbert space \( \sim \mathbb{C} \). Then the \( n \)-th tensor product \( \mathcal{H}^{\otimes n} \) is one dimensional: indeed \( z_1 \otimes \cdots \otimes z_n = (z_1 \ldots z_n) 1 \otimes \cdots \otimes 1 \in \mathbb{C} \Phi_n \). The one-mode interacting Fock space associated to \( \nu_{\lambda, \theta} \) is defined as \( \Gamma(\mathbb{C} \Phi_n, (\lambda_n)) \) as the infinite orthogonal sum of \( \mathbb{C} \Phi_n \) equipped with the weighted scalar product
\[
(z_1 \Phi_n, z_2 \Phi_n) := \lambda_n z_1 \overline{z}_2, \quad z_1, z_2 \in \mathbb{C},
\]
where \( \lambda_n = \omega_1 \ldots \omega_n \). Then \( \nu_{\lambda, \theta} \) is the vacuum distribution (in the vacuum state \( \Phi_1 \)) of any extension of the operator \( a^+ + a + \alpha_N \) where
\[
a^+ \Phi_n = \Phi_{n+1} \quad \text{(creation operator)}
\]
\[
a \Phi_{n+1} = \omega_{n+1} \Phi_n = \frac{\lambda_{n+1}}{\lambda_n} \Phi_n, \text{ } a \Phi_1 = 0, \quad \text{(annihilation operator)}
\]
\[
N \Phi_n = n \Phi_n \quad \text{(Number operator)}, \quad aa^+ \Phi_n = \frac{\lambda_{n+1}}{\lambda_n} \Phi_n,
\]
and \( \alpha_N \) is defined by the spectral Theorem, that is \( \alpha_N \Phi_n = \alpha_n \Phi_n \).

Remark. The concept of one mode interacting Fock space (IFS) is purely algebraic as the reader can see from \([1]\) and is fully characterized by both the commutation relations between creation and annihilation operators and \( a \Phi_1 = 0 \). The most important feature of Accardi-Bozejko Theorem is illustrated in the canonical isomorphism between one mode IFS and the \( L^2 \)-space of a given measure \( \mu \) of all order moments. It is noteworthy that only the \( \omega_n s \) are involved in the commutation relations (thus in both one mode IFS and \( L^2(\mu) \)) while the \( \alpha_n s \) reflect only the symmetry of \( \mu \).
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