Surjective $L^p$-isometries on Grassmann spaces

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Abstract Let $\phi : \mathcal{P}_c(\mathfrak{M}_1) \to \mathcal{P}_c(\mathfrak{M}_2)$ be a surjective $L^p$-isometry between Grassmann spaces of projections with the trace value $c$ in semifinite factors $\mathfrak{M}_1$ and $\mathfrak{M}_2$. Based on the characterization of surjective $L^p$-isometries of unitary groups in finite factors, we show that $\phi$ or $I - \phi$ can be extended to a $\ast$-isomorphism or a $\ast$-anti-isomorphism. In particular, $\phi$ is given by a $\ast$-(anti-)isomorphism unless $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are finite and $c = \frac{1}{2}$.

Keywords surjective $L^p$-isometry, unitary group, Grassmann space, semifinite factor

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1 Introduction and statement of the main results

The research of isometries between substructures of operator algebras has gained much attention recently. One central issue in such a study is investigating if the isometries reflect the algebraic information of the underlying algebra encoded by the substructure, or, more specifically, if the isometries can be described by using algebra homomorphisms. The celebrated unitary-antiunitary theorem of Wigner [28] is one of the most classical results in this direction. Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators on the Hilbert space $\mathcal{H}$. The Wigner’s theorem asserts that every surjective isometry for the operator norm or the $L^2$-norm on the Grassmann space of rank-one projections in $\mathcal{B}(\mathcal{H})$ can be extended to a $\ast$-automorphism or a $\ast$-anti-automorphism of $\mathcal{B}(\mathcal{H})$. Due to its importance in the mathematical formulation of quantum mechanics, Wigner’s theorem has been studied and generalized by many authors (see, e.g., [3, 15, 20, 23]).

In recent years, there have been several remarkable developments in the research on isometries of Grassmann spaces. In [8, 9], Gehér and Šemrl described the general form of surjective isometries for the operator norm between Grassmann spaces in $\mathcal{B}(\mathcal{H})$. Based on Gehér and Šemrl’s work, Mori [19] characterized the surjective isometries for the operator norm between projection lattices of von Neumann algebras using Jordan $\ast$-isomorphisms. As for the isometries with respect to the $L^2$-norm, Gehér [7]...
generalized the early work of Molnár [16] and proved that every (not necessarily surjective) $L^2$-isometry on Grassmann spaces in $B(H)$ is induced by a Jordan $*$-homomorphism of $B(H)$. This result was later generalized to the case of $L^2$-isometries on Grassmann spaces in semifinite factors [21].

Most recently, Shi et al. [24] studied surjective isometries for the $L^p$-norm on Grassmann spaces in a factor of type II. Let $\mathfrak{M}$ be a semifinite factor equipped with a normal semifinite faithful (n.s.f.) tracial weight $\tau$. For every $A$ in the two-sided ideal $\mathcal{J}$ of operators with finite range projections in $\mathfrak{M}$, its $L^p$-norm is defined by $\|A\|_p := \tau(|A|^p)^{1/p}$ where $0 < p < \infty$. It is well known that $\|\cdot\|_p$ is a norm on $\mathcal{J}$ for $p \in [1, \infty)$. If $p \in (0, 1)$, then $\|\cdot\|_p$ is a quasi-norm on $\mathcal{J}$ such that

$$\|A_1 + A_2\|_p^p \leq \|A_1\|_p^p + \|A_2\|_p^p$$

for every $A_1, A_2 \in \mathcal{J}$. We direct the readers to [25–27] for a general reference on the theory of von Neumann algebras. A map $\phi$ between two subsets of $\mathcal{J}$ is called an $L^p$-isometry if

$$\|B_1 - B_2\|_p = \|\phi(B_1) - \phi(B_2)\|_p$$

for every $B_1$ and $B_2$ in the domain of $\phi$. The $L^p$-isometry between two subsets in different semifinite factors can be defined similarly. Let $\mathcal{P}(\mathfrak{M})$ be the set of all the projections in $\mathfrak{M}$. For $c \in (0, \tau(I))$, we use $\mathcal{P}_c(\mathfrak{M})$ to denote the Grassmann space of all the projections in $\mathfrak{M}$ with trace $c$, i.e.,

$$\mathcal{P}_c(\mathfrak{M}) := \{P \in \mathcal{P}(\mathfrak{M}) : \tau(P) = c\}.$$

To avoid unnecessary complexity, we always assume that $c$ is chosen so that $\mathcal{P}_c(\mathfrak{M})$ is nonempty, and $\tau$ is a tracial state whenever $\mathfrak{M}$ is a finite factor. It is not hard to check that $L^p$-isometries on $\mathcal{P}_c(\mathfrak{M})$ preserve orthogonality in both directions (see Lemma 2.5). By the structural result for orthogonality preserving maps on $\mathcal{P}_c(B(H))$ proved in [8], we know that every surjective $L^p$-isometry on $\mathcal{P}_c(B(H))$ extends to a $*$-(anti-)automorphism of $B(H)$ if $c \neq \tau(I)/2$. When $\mathfrak{M}$ is a type II factor and $c \neq 1/2$, the same result holds and is proved in [24]. Although the method used in [24] can be modified to give an alternative proof for the case $\mathfrak{M} = B(H)$, it fails when $\mathfrak{M}$ is a finite factor and $c = 1/2$. In this paper, we solve this remaining case and describe the general form of surjective $L^p$-isometries of Grassmann spaces in semifinite factors.

We point out that the surjective $L^p$-isometries on the Grassmann spaces of trace 1/2 projections in a finite factor are closely related to the surjective $L^p$-isometries of unitary groups. This claim can be backed up by the following observation. Let $\mathfrak{M} = \mathfrak{N} \otimes M_2(\mathbb{C})$, where $\mathfrak{N}$ is a finite factor. Then the set

$$\mathcal{J} := \left\{ \frac{1}{2} \begin{pmatrix} I & U \\ U^* & 1 \end{pmatrix} : U \in \mathcal{U}(\mathfrak{N}) \right\}$$

is contained in $\mathcal{P}_{1/2}(\mathfrak{M})$, where $\mathcal{U}(\mathfrak{N})$ is the group of unitaries in $\mathfrak{N}$. If $\phi$ is a surjective $L^p$-isometry on $\mathcal{P}_{1/2}(\mathfrak{M})$ fixing the projection

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

then $\phi$ restricts to a surjective $L^p$-isometry on $\mathcal{J}$ and induces a surjective $L^p$-isometry on $\mathcal{U}(\mathfrak{N})$ (see Lemma 3.2). Therefore, the characterization of surjective $L^p$-isometries on $\mathcal{U}(\mathfrak{N})$ is essential for the study of surjective $L^p$-isometries on $\mathcal{P}_{1/2}(\mathfrak{M})$.

In Section 2, we describe the general form of surjective $L^p$-isometries between unitary groups in finite factors. More explicitly, we prove the following theorem, which is one of the main results in this paper.

**Theorem 1.1.** Let $\mathfrak{N}_1$ and $\mathfrak{N}_2$ be two finite factors. If $\psi : \mathcal{U}(\mathfrak{N}_1) \to \mathcal{U}(\mathfrak{N}_2)$ is a surjective $L^p$-isometry, then there exist a unitary $W \in \mathfrak{N}_2$ and a $*$-isomorphism or a $*$-anti-isomorphism $\varphi : \mathfrak{N}_1 \to \mathfrak{N}_2$ such that $\psi$ has one of the following forms:

1. \( \psi(U) = W \varphi(U) \) for every $U \in \mathcal{U}(\mathfrak{N}_1)$;
2. \( \psi(U) = W \varphi(U^*) \) for every $U \in \mathcal{U}(\mathfrak{N}_1)$. 
When $\mathcal{M}_1 = \mathcal{M}_2 = M_n(\mathbb{C})$, the above result is proved in [17] (see also [12, 18] for some related results). For the other cases, there are two important ingredients in the proof. One is the results on geodesics between unitaries given in [1] (see Lemma 2.4). The other one is the fact that every unitary in $M_{2n}(\mathbb{C})$ can be written as the product of self-adjoint unitaries and a complex number of magnitude 1 (see Lemma 2.12).

In Section 3, based on Theorem 1.1, we describe all the surjective $L^p$-isometries on the Grassmann spaces of trace $1/2$ projections of a finite factor. We then reduce the problem of characterizing surjective $L^p$-isometries between Grassmann spaces in different algebras to the characterization of surjective $L^p$-isometries on Grassmann spaces in a factor. Finally, combining our result with the characterization of surjective $L^p$-isometries on Grassmann spaces given in [24], we prove the following two theorems that describe the general form of surjective $L^p$-isometries between Grassmann spaces of projections with trace $c$.

**Theorem 1.2.** Let $\phi$ be a surjective $L^p$-isometry from $\mathcal{P}_c(\mathcal{M}_1)$ to $\mathcal{P}_c(\mathcal{M}_2)$, where $c, p \in (0, \infty)$ and $\mathcal{M}_i$ is a properly infinite semifinite factor equipped with an n.s.f. tracial weight $\tau_i$ ($i = 1, 2$). Then there is a $*$-isomorphism or a $*$-anti-isomorphism $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$ such that $\tau_1 = \tau_2 \circ \varphi$ and $\phi(P) = \varphi(P)$ for every projection $P \in \mathcal{P}_c(\mathcal{M}_1)$.

**Theorem 1.3.** Let $\phi$ be a surjective $L^p$-isometry from $\mathcal{P}_c(\mathcal{M}_1)$ to $\mathcal{P}_c(\mathcal{M}_2)$, where $c \in (0, 1)$, $p \in (0, \infty)$ and $\mathcal{M}_i$ is a finite factor equipped with a tracial state $\tau_i$ ($i = 1, 2$). If $c \neq 1/2$, there is a $*$-isomorphism or a $*$-anti-isomorphism $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$ such that $\phi$ is of the following form:

$$\phi(P) = \varphi(P), \quad \forall P \in \mathcal{P}_c(\mathcal{M}_1).$$

In the case $c = 1/2$, we have either (1.1), or the following additional possibility:

$$\phi(P) = I - \varphi(P), \quad \forall P \in \mathcal{P}_{1/2}(\mathcal{M}_1).$$

2 Surjective $L^p$-isometries of unitaries in finite factors

This section is devoted to proving Theorem 1.1. Let $\mathfrak{M}$ be a finite factor with the tracial state $\tau$. We use $\mathcal{U}(\mathfrak{M})$ to denote the set of unitaries in $\mathfrak{M}$. In [17], Molnár showed that every isometry on $\mathcal{U}(M_n(\mathbb{C}))$ with respect to a unitarily invariant norm is induced by a $*$-automorphism or a $*$-anti-automorphism on $M_n(\mathbb{C})$. For $p \in [1, \infty)$, $\| \cdot \|_p$ is a unitarily invariant norm on $\mathcal{U}(M_n(\mathbb{C}))$. We thus know that every $L^p$-isometry $\psi$ on $\mathcal{U}(M_n(\mathbb{C}))$ satisfying $\psi(I) = I$ can be extended to a $*$-automorphism or a $*$-anti-automorphism if $p \geq 1$. We now show that the same result holds for $L^p$-isometries on $\mathcal{U}(M_n(\mathbb{C}))$ when $0 < p < 1$.

**Lemma 2.1.** Let $\psi$ be a surjective $L^p$-isometry on $\mathcal{U}(M_n(\mathbb{C}))$. Then there exist a unitary $W \in \mathcal{U}(M_n(\mathbb{C}))$ and a $*$-automorphism or a $*$-anti-automorphism $\varphi$ on $M_n(\mathbb{C})$ such that $\psi$ is of one of the following forms:

1. $\psi(U) = W\varphi(U)$ and $U \in \mathcal{U}(M_n(\mathbb{C}))$;
2. $\psi(U) = W\varphi(U^*)$ and $U \in \mathcal{U}(M_n(\mathbb{C}))$.

**Proof.** Without loss of generality, we may assume that $\psi(I) = I$. Otherwise, we can consider the map $\psi(I)^*\psi(\cdot)$ instead. The lemma will be proved if we can show that there exists a $*$-automorphism or a $*$-anti-automorphism $\varphi$ such that $\psi(U) = \varphi(U)$ or $\psi(U) = \varphi(U^*)$ for every unitary $U$. By [17, Theorem 3], we only need to prove this result for the case $0 < p < 1$.

Since $\|A\| = \|\mathcal{N}_p\|$, we have

$$\|A\|_p \leq \|A\| \leq \sqrt[2p]{\|A\|_p}, \quad \forall A \in M_n(\mathbb{C}).$$

In particular, $\psi$ is continuous with respect to the operator norm. By [17, Proposition 10], we only need to show that $\psi$ preserves the inverted Jordan triple product, i.e.,

$$\psi(VW^*V) = \psi(V)\psi(W)^*\psi(V), \quad \forall V, W \in \mathcal{U}(M_n(\mathbb{C})).$$

(2.1)
Let $c := \frac{1}{2q}$ and $V, W \in \mathcal{U}(M_n(\mathbb{C}))$ such that $\|V - W\| \leq c^2/2$. We claim that
\[
\sqrt{3}\| VX^* - I \|_p \leq \| V X^* V X^* - I \|_p
\]  
(2.2)
for every unitary $X$ satisfying
\[
\| X - W \|_p = \| V W^* V - X \|_p = \| V - W \|_p.
\]
Let $U = VX^*$. Note that
\[
\|U - I\|_p = \|V - X\|_p \leq 2 \max\{\|V - W\|, \|W - X\|\} \leq \frac{2}{c} \|V - W\|_p \leq c.
\]
Therefore, $\|U - I\| \leq 1$. This implies that the spectrum of $U$ is contained in the set
$$\{ \lambda : |\lambda| = 1, |\lambda - 1| \leq 1 \}.$$ 
Then it is not hard to see that $U + I$ is invertible and $\|(U + I)^{-1}\| \leq 1/\sqrt{3}$. Consequently, we have
\[
\sqrt{3}\| U - I \|_p = \sqrt{3}\|(U + I)^{-1}(U^2 - I)\|_p \leq \sqrt{3}\|(U + I)^{-1}\|\|U^2 - I\|_p \leq \| U^2 - I \|_p,
\]
and the claim is proved.

By [11, Corollary 3.9], we know that (2.1) holds for every pair of unitaries $V$ and $W$ satisfying $\|V - W\| \leq c^2/2$. Proceeding as in the proof of [12, Theorem 8], we can show that (2.1) holds for every pair of unitaries. Hence the lemma is proved.

Remark 2.2. As $\mathcal{U}(M_n(\mathbb{C}))$ is a compact connected manifold, according to the invariance domain theorem, the $L^p$-isometry on $\mathcal{U}(M_n(\mathbb{C}))$ must be surjective. We give the surjective assumption in the above lemma for the consistency.

Let $\mathcal{R}_1$ be a finite factor with the tracial state $\tau_i$ ($i = 1, 2$). We next show that $\mathcal{R}_1$ and $\mathcal{R}_2$ are either $*$-isomorphic or $*$-anti-isomorphic if there exists an $L^p$-isometry from $\mathcal{U}(\mathcal{R}_1)$ onto $\mathcal{U}(\mathcal{R}_2)$. We prepare the proof by establishing three lemmas.

Lemma 2.3. Let $U \in \mathcal{U}(\mathcal{R}_i)$, where $\mathcal{R}_i$ is a finite factor. Assume that $p \neq 2$. Then $\|I - U\|_p^p + \|I + U\|_p^p = 2^p$ if and only if $U = U^*$.

Proof. If $\|I - U\|_p^p + \|I + U\|_p^p = 2^p$, then
\[
2(|I - U|_p^p + |I + U|_p^p) = \|(I - U) + (I + U)|_p^p + \|(I - U) - (I + U)|_p^p.
\]
Since $I$ and $U$ are contained in a commutative von Neumann subalgebra of $\mathcal{R}_i$, we have $(I + U)^*(I - U) = 0$ by [14, Corollary 2.1]. Thus $U = U^*$.

Conversely, if $U = U^*$, then
\[
\|I - U\|_p^p + \|I + U\|_p^p = 2^p \tau(P) + 2^p \tau(Q) = 2^p,
\]
where $P = (I + U)/2$ and $Q = (I - U)/2$ are orthogonal projections.  

Lemma 2.4. Let $\psi : \mathcal{U}(\mathcal{R}_1) \to \mathcal{U}(\mathcal{R}_2)$ be a surjective $L^p$-isometry such that $\psi(I) = I$, where $\mathcal{R}_1$ and $\mathcal{R}_2$ are two finite factors. For every $U \in \mathcal{U}(\mathcal{R}_1)$, we have $U^* = U$ if and only if $\psi(U)^* = \psi(U)$.

Proof. If $p \neq 2$, the lemma is proved by Lemma 2.3. Assume that $p = 2$. Since $\psi$ is an $L^2$-isometry, we have
\[
\text{Re}(\tau_2(U^* V)) = \text{Re}(\tau_2(\psi(U)^* \psi(V))), \quad \forall U, V \in \mathcal{U}(\mathcal{R}_1).
\]

Thus it is easy to check that $\sum_{i=1}^n \lambda_i U_i = \sum_{i=1}^n \lambda_i \psi(U_i)$, where $\lambda_i \in \mathbb{R}$ and $U_i \in \mathcal{U}(\mathcal{R}_1)$. Recall that for every $A \in \mathcal{R}_1$, there exist two unitaries $W_1$ and $W_2$ in $\mathcal{R}_1$ such that $A = \frac{1}{\sqrt{2}}(W_1 + W_2)$. Then $\psi$ has a unique real-linear extension $\Psi : \mathcal{R}_1 \to \mathcal{R}_2$, i.e.,
\[
\Psi \left( \sum_{i=1}^n \lambda_i U_i \right) := \sum_{i=1}^n \lambda_i \psi(U_i).
It is clear that \( \Psi \) is a surjective \( L^2 \)-isometry.

Let \( U = 2P - I \), where \( P \) is a projection in \( \mathfrak{M}_1 \). Then \( \psi(U)^* = \psi(U) \) if and only if \( \Psi(P)^* = \Psi(P) \). By [1, Lemma 3.1], the curve \( t \rightarrow e^{itP}, t \in [0,1] \), which is \( C^\infty \) as a curve in the Hilbert space \( L^2(\mathfrak{M}_1, \tau_1) \), has minimal length among piecewise \( C^1 \) curves of unitaries jointing \( I \) and \( e^{itP} \) in the \( L^2 \)-metric. Since \( \Psi \) is a real-linear \( L^2 \)-isometry, \( f(t) := \psi(e^{itP}) \) is a \( C^\infty \) curve in \( L^2(\mathfrak{M}_2, \tau_2) \) with

\[
\frac{df}{dt}(t) = \Psi(iPe^{itP}), \quad \frac{d^2f}{dt^2}(t) = -\Psi(Pe^{itP}).
\]

Since \( \Psi \) is bijective, the curve \( f \) has minimal length among piecewise \( C^1 \) curves of unitaries jointing \( I \) and \( \psi(e^{itP}) \). By [1, Theorem 2.4 and Proposition 2.2], there exists a self-adjoint operator \( H \in \mathfrak{M}_2 \) such that \( f(t) = e^{itH} \). Therefore \( \Psi(P) = -\frac{d^2f}{dt^2}(0) = H^2 \). In particular, \( \Psi(P) \) is self-adjoint.

The following lemma is a special case of the noncommutative Clarkson inequality (see [2]).

**Lemma 2.5.** Let \( \mathfrak{M} \) be a semifinite factor with an n.s.f. tracial weight \( \tau \) and \( P_1 \) and \( P_2 \) be two finite projections in \( \mathfrak{M} \), i.e., \( \tau(P_i) < \infty \). Then \( P_1 \perp P_2 \) if and only if

\[
\|P_1 - P_2\|_\rho = \tau(P_1) + \tau(P_2).
\]

**Theorem 2.6.** Let \( \psi: \mathcal{H} \rightarrow \mathcal{H} \) be a surjective \( L^p \)-isometry such that \( \psi(I) = I \), where \( \mathfrak{M}_1 \) and \( \mathfrak{M}_2 \) are two finite factors. Then there exists a \( * \)-isomorphism or \( * \)-anti-isomorphism \( \varphi: \mathfrak{M}_1 \rightarrow \mathfrak{M}_2 \) such that \( \psi(U) = \varphi(U) \) for every self-adjoint unitary \( U \).

**Proof.** Let

\[
\phi(P) := \frac{\psi(2P - I) + I}{2}, \quad \forall P \in \mathcal{P}(\mathfrak{M}_1).
\]

By Lemma 2.4, \( \phi \) is an \( L^p \)-isometry \( \phi \) on \( \mathcal{P}(\mathfrak{M}_1) \). Since \( \psi \) is surjective, \( \phi \) is also surjective. Noting that \( \phi(I) = I \), we have

\[
1 - \tau_1(P) = \|I - P\|_\rho = \|I - \phi(P)\|_\rho = 1 - \tau_2(\phi(P)),
\]

where \( \tau_i \) is the tracial state on \( \mathfrak{M}_i \). Therefore, \( \tau_2(\phi(P)) = \tau_1(P) \). By Lemma 2.5, \( \phi \) preserves orthogonality in both directions.

If \( \mathfrak{M}_1 \ncong M_2(\mathbb{C}) \), then there exists a \( * \)-isomorphism or \( * \)-anti-isomorphism \( \varphi: \mathfrak{M}_1 \rightarrow \mathfrak{M}_2 \) such that \( \phi(P) = \varphi(P) \) for every \( P \in \mathcal{P}(\mathfrak{M}_1) \) by [6, Corollary] (see also [13]). Thus \( \psi(U) = \varphi(U) \) for every self-adjoint unitary.

If \( \mathfrak{M}_1 \cong M_2(\mathbb{C}) \), then we have \( \mathfrak{M}_2 \cong M_2(\mathbb{C}) \). Then Lemma 2.1 implies the result.

By Lemma 2.1 and Theorem 2.6, we only need to prove Theorem 1.1 under the assumption that \( \mathfrak{M}_1 = \mathfrak{M}_2 = \mathfrak{M} \), where \( \mathfrak{M} \) is a type \( \Pi_1 \) factor, and \( \psi \) is a surjective \( L^p \)-isometry on \( \mathcal{H}(\mathfrak{M}) \). From now on, we use \( \mathfrak{M} \) to denote a type \( \Pi_1 \) factor and \( \psi \) to denote a surjective \( L^p \)-isometry on \( \mathcal{H}(\mathfrak{M}) \). The proof of Theorem 1.1 is divided into two cases: \( p = 2 \) and \( p \neq 2 \).

**2.1 The \( p = 2 \) case**

**Proof of Theorem 1.1 (\( p = 2 \)).** By Lemma 2.1 and Theorem 2.6, we only need to consider the case \( \mathfrak{M}_1 = \mathfrak{M}_2 = \mathfrak{M} \), where \( \mathfrak{M} \) is a type \( \Pi_1 \) factor. Without loss of generality, we may assume that \( \psi(I) = I \). Otherwise, we can consider the map \( \psi(I)^* \psi(\cdot) \) instead. By Theorem 2.6, there exists a \( * \)-automorphism or a \( * \)-anti-automorphism \( \varphi \) on \( \mathfrak{M} \) such that \( \psi(U) = \varphi(U) \) for every self-adjoint unitary \( U \). By considering the map \( \varphi^{-1} \circ \psi \) instead, we may further assume that \( \psi(U) = U \) for every self-adjoint unitary \( U \).

By the proof of Lemma 2.4, we know \( \psi \) has a unique real-linear extension \( \Psi \) which is a surjective \( L^2 \)-isometry on \( \mathfrak{M} \). Moreover, \( \Psi(P) = P \) for every projection \( P \) and

\[
\text{Re}(\tau(A^*B)) = \text{Re}(\tau(\psi(A)^*\psi(B))).
\]
for every pair of elements $A, B \in \mathcal{R}$, where $\tau$ is the tracial state on $\mathcal{R}$. It is clear that the conclusion will be proved if we can show that $\Psi(A) = A$ or $\Psi(A) = A^*$ for every $A \in \mathcal{R}$.

Note that $\Psi((U + iI)/\sqrt{2}) = (U + \psi(iI))/\sqrt{2}$ is a unitary for every self-adjoint unitary $U$. Thus $\psi(iI)^* = -\psi(iI)$ and $\psi(iI)U = U\psi(iI)$. Since $\mathcal{R}$ is a factor, $\psi(iI) = iI$ or $\psi(iI) = -iI$. We claim that $\Psi(iP) = \psi(iI)P$ for every projection $P$ in $\mathcal{R}$. Since

$$(I - P) + \Psi(iP) = \psi((I - P) + iP) \quad \text{and} \quad (I - P) - \Psi(iP) = \psi((I - P) - iP)$$

are unitaries, we have $\Psi(iP)^*\Psi(iP) = \Psi(iP)\Psi(iP)^* = P$. Note that

$$\tau(P) = \text{Re}(\tau((iI)^*(I - P) + iP))) = \text{Re}(\tau(\psi(iI)^*\psi((I - P) + iP))) = \text{Re}(\tau(\psi(iI)^*\Psi(iP))).$$

Therefore, $\Psi(iP) = \psi(iI)P$. Since every operator can be approximated by linear combinations of projections, we have $\Psi(A) = A$ or $\Psi(A) = A^*$ for every $A \in \mathcal{R}$. \qed

2.2 The $p \neq 2$ case

In the rest of this subsection, we assume that $p \neq 2$.

Lemma 2.7. Let $U \in \mathcal{U}(\mathcal{R})$ and $\theta \in (-\pi, \pi)$. Then

$$\|U - 2P + I\|_p^p = \tau(P)(2 - 2\cos \theta)^{p/2} + \tau(I - P)(2 + 2\cos \theta)^{p/2}$$

(2.3)

for every projection $P \in \mathcal{R}$ if and only if $U$ equals either $e^{i\theta}I$ or $e^{-i\theta}I$.

Proof. We only need to prove the necessity. Note that $-I$ is the only unitary $U$ satisfying $\|U - I\|_p = 2^p$ and $I$ is the only unitary $U$ satisfying $\|U + I\|_p = 2^p$. Then the statement of the lemma is true if $\theta$ equals $0$ or $\pi$.

Let $\theta \in (-\pi, 0) \cup (0, \pi)$. We need to show that (2.3) implies that $U = e^{i\theta}I$ or $U = e^{-i\theta}I$. Without loss of generality, we could assume that $0 < \theta < \pi$. Let $\chi$ be the characteristic function of the set $\{e^{it} : t \in [0, 2\pi - \theta]\}$. Note that $1 - \cos t_1 > 1 - \cos \theta$ for every $t_1 \in (\theta, 2\pi - \theta)$ and $1 + \cos t_2 > 1 + \cos \theta$ for every $t_2 \in (-\theta, \theta)$. By the spectral theorem, we have

$$\|U - 2\chi(U) + I\|_p^p \geq \tau(\chi(U))(2 - 2\cos \theta)^{p/2} + \tau(I - \chi(U))(2 + 2\cos \theta)^{p/2}.$$

Furthermore, the equality holds if and only if there exists a projection $E$ in $\mathcal{R}$ such that $U = e^{i\theta}E + e^{-i\theta}(I - E)$. Thus we may assume that $U = e^{i\theta}E + e^{-i\theta}(I - E)$.

We now show that $E = 0$ or $E = I$. Assume that $E \neq 0$ and $E \neq I$. We only consider the case $E \leq I - E$. The proof of the case $I - E \leq E$ is similar, which is omitted. Let $W$ be a partial isometry such that $W^*W = E$ and $WW^* \leq I - E$. Let $V = (W + W^*) - (I - E - WW^*)$. Note that $V$ is a self-adjoint unitary and the trace of the spectral projection of $V$ corresponding to $\{1\}$ equals $\tau(E)$. By (2.3), we have

$$\|U - V\|_p^p = \tau(E)(2 - 2\cos \theta)^{p/2} + \tau(I - E)(2 + 2\cos \theta)^{p/2}.$$

Since

$$(U - V)^*(U - V) = 2(E + WW^* - e^{i\theta}W - e^{-i\theta}W^*) + (2 + 2\cos \theta)(I - E - WW^*)$$

and $\frac{1}{2}(E + WW^* - e^{i\theta}W - e^{-i\theta}W^*)$ is a projection whose trace equals $\tau(E)$, we have

$$\|U - V\|_p^p = \tau(I - 2E)(2 + 2\cos \theta)^{p/2} + \tau(E)2^p.$$

Therefore, $(2 - 2\cos \theta)^{p/2} + (2 + 2\cos \theta)^{p/2} = 2^p$. Recall that $p \neq 2$ and $|\cos \theta| < 1$. We have

$$\left[\left(\frac{1 - \cos \theta}{2}\right)^{\frac{p}{2}} + \left(\frac{1 + \cos \theta}{2}\right)^{\frac{p}{2}}\right] \neq 1,$$

which leads to a contradiction. \qed
Lemma 2.8.  Let $\psi$ be a surjective $L^p$-isometry on $\mathcal{U}(\mathfrak{N})$ such that $\psi(I) = I$. Then we have either $\psi(e^{i\theta}I) = e^{i\theta}I$ or $\psi(e^{i\theta}I) = e^{-i\theta}I$ for every $\theta \in \mathbb{R}$.

Proof.  By Theorem 2.6, we may assume that $\psi(U) = U$ for every self-adjoint unitary. Then Lemmas 2.1 and 2.7 imply the result. □

Lemma 2.9.  Let $\psi$ be a surjective $L^p$-isometry on $\mathcal{U}(\mathfrak{N})$. If $\psi(U) = U$ for every self-adjoint unitary $U$ and $\psi(e^{i\theta}I) = e^{i\theta}I$ for every $\theta \in \mathbb{R}$, then $\psi(e^{i\theta}U) = e^{i\theta}U$ for every self-adjoint unitary $U$ and $\theta \in \mathbb{R}$.

Proof.  Let $P$ be a projection in $\mathfrak{N}$. Note that $\psi'(-) := (2P - I)\psi((2P - I)(-))$ is a surjective $L^p$-isometry on $\mathcal{U}(\mathfrak{N})$ satisfying $\psi'(I) = I$. By Lemma 2.8, we have either $\psi(e^{i\theta}(2P - I)) = e^{i\theta}(2P - I)$ or $\psi(e^{i\theta}(2P - I)) = e^{-i\theta}(2P - I)$ for every $\theta \in \mathbb{R}$. If we can show that

$$\psi(i(2P - I)) = i(2P - I),$$

then the lemma is proved.

Assume that $\psi(i(2P - I)) = -i(2P - I)$. Let $Q$ be a projection in $\mathfrak{N}$. If $\psi(i(2Q - I)) = i(2Q - I)$, we have

$$\|2(P - Q)\|_p = \|i(2P - I) - i(2Q - I)\|_p = \|\psi(i(2P - I)) - \psi(i(2Q - I))\|_p = \|2(P + Q - I)\|_p.$$ 

Recall that

$$\|P - Q\|_p + \|P + Q - I\|_p \geq \|2P - I\|_p = 1, \quad p \geq 1,$$

$$\|P - Q\|_p^p + \|P + Q - I\|_p^p \geq \|2P - I\|_p^p = 1, \quad 0 < p < 1$$

(see [2, Chapter 5]). Thus $\psi(i(2Q - I)) = -i(2Q - I)$ if $\|P - Q\|_p$ is small enough. Since the set of projections in $\mathfrak{N}$ is connected with respect to the topology induced by the quasi-norm $\|\cdot\|_p$, we have

$$\psi(i(2E - I)) = -i(2E - I)$$

for every projection $E$ in $\mathfrak{N}$. Recall that $\psi(I) = iI$. We have a contradiction. Therefore, $\psi(i(2P - I)) = i(2P - I)$. □

Lemma 2.10.  Let $\psi$ be a surjective $L^p$-isometry on $\mathcal{U}(\mathfrak{N})$ such that $\psi(e^{i\theta}U) = e^{i\theta}U$ for every self-adjoint unitary $U \in \mathfrak{N}$ and $\theta \in \mathbb{R}$. Then we have

$$\psi(e^{i\theta}UV) = e^{i\theta}UV$$

for every pair of self-adjoint unitaries $U, V \in \mathfrak{N}$.

Proof.  Let $P$ be a trace 1/2 projection. Then $(2P - I)\psi((2P - I)(-))$ is a surjective $L^p$-isometry on $\mathcal{U}(\mathfrak{N})$. Note that $(2P - I)\psi(e^{i\theta}(2P - I)) = e^{i\theta}I$. By Theorem 2.6 and Lemma 2.9, there exists a $^*$-automorphism or a $^*$-anti-automorphism $\varphi$ on $\mathfrak{N}$ such that $(2P - I)\psi(e^{i\theta}(2P - I)V) = e^{i\theta}\varphi(V)$ for every self-adjoint unitary $V$ and $\theta \in \mathbb{R}$. We claim that $\varphi$ is the identity $^*$-isomorphism. Indeed, note that $\varphi(P) = P$ and

$$\varphi(W) = \frac{1}{2}(\varphi(W + W^*) + \varphi(W - W^*)) = \frac{1}{2}(2P - I)(\psi(W - W^*) + \psi(W + W^*)) = W$$

for every partial isometry $W$ satisfying $W^*W = I - P$ and $WW^* = P$. Note that $W = \varphi(PW) \neq \varphi(W)\varphi(P) = WP = 0$.

Then $\varphi$ can only be a $^*$-isomorphism. Since $\mathfrak{N} = \{W, W^* : W \in \mathfrak{N}, W^*W = I - P, WW^* = P\}'$, $\varphi$ is the identity $^*$-isomorphism. Therefore, we conclude that

$$\psi(e^{i\theta}(2P - I)V) = e^{i\theta}(2P - I)V$$
for every trace 1/2 projection $P$ and self-adjoint unitary $V$.

Since $\psi'(\cdot) := e^{-i\theta}\psi(e^{i\theta}(\cdot)V)V$ is a surjective $L^p$-isometry such that $\psi'(I) = I$, there exists a $*$-automorphism or a $*$-anti-automorphism $\varphi'$ on $\mathcal{R}$ such that $\psi'(U) = \varphi'(U)$ for every self-adjoint unitary $U \in \mathcal{R}$ by Theorem 2.6. Based on the above discussion, we have

$$\varphi'(2P - I) = \psi'(2P - I) = 2P - I$$

for every trace 1/2 projection $P$ in $\mathcal{R}$. Recall that the linear span of the set of trace 1/2 projections is dense in $\mathcal{R}$ (see [21, Lemma 3.7]). Therefore, $\varphi'$ is the identity $*$-isomorphism, and the lemma is proved. □

Lemma 2.11. Let $\psi$ be a surjective $L^p$-isometry on $\mathcal{R}^{\mathcal{R}}$ such that $\psi(U) = U$ for every self-adjoint unitary $U \in \mathcal{R}$ and $\psi(e^{i\theta}I) = e^{i\theta}I$ for every $\theta \in \mathbb{R}$. Then we have

$$\psi(e^{i\theta}U_1 \cdots U_n) = e^{i\theta}U_1 \cdots U_n$$

for self-adjoint unitaries $U_1, \ldots, U_n \in \mathcal{R}$ and $\theta \in \mathbb{R}$ with $n \geq 1$.

Proof. We prove the lemma by induction. Let $V = U_1 \cdots U_n$, where $U_i$ is a self-adjoint unitary in $\mathcal{R}$ and $i = 1, \ldots, n$. Note that $\psi'(\cdot) := V^*\psi(V(\cdot))$ is a surjective $L^p$-isometry such that $\psi'(e^{i\theta}U_i) = e^{i\theta}U_i$ for every self-adjoint unitary $U_i$. By Lemma 2.10, we have

$$(U_1 \cdots U_{k-1})^* \psi(e^{i\theta}U_1 \cdots U_{k+1}) = e^{i\theta}U_kU_{k+1},$$

and the lemma is proved. □

The following lemma is well known to experts (see, for example, [4, 5, 22]), and we include the proof for the sake of completeness.

Lemma 2.12. Every unitary $V$ in $M_{2^n}(\mathbb{C})$ ($n \geq 1$) can be written as

$$V = e^{i\theta}U_1 \cdots U_1,$$

where $U_i$’s are self-adjoint unitaries in $M_{2^n}(\mathbb{C})$.

Proof. Let $E_{ij}$ ($i, j = 1, \ldots, 2^n$) be the canonical matrix units of $M_{2^n}(\mathbb{C})$. We only need to show that there exist self-adjoint unitaries $U_1, \ldots, U_{2^n}$ and $\theta \in \mathbb{R}$ such that

$$e^{i\beta}E_{11} + (I - E_{11}) = e^{i\theta}U_1 \cdots U_{2^n}$$

for every $\beta \in \mathbb{R}$. We prove this fact by induction on $n$.

For $n = 1$, we have

$$e^{i\beta}E_{11} + E_{22} = e^{i\beta/2}(iE_{12} - iE_{21})(ie^{-i\beta/2}E_{12} - ie^{i\beta/2}E_{21}).$$

Assume that the fact is true for $n = k \geq 1$. Let $m = 2^k$. Then there exist self-adjoint unitaries $U_1, \ldots, U_{2m} \in M_{2^{k+1}}(\mathbb{C})$ and $\theta \in \mathbb{R}$ such that

$$e^{i\beta}E_{11} + (E_{22} + \cdots + E_{mm}) + e^{i\theta}(I - E_{11} - \cdots - E_{mm}) = e^{i\theta}U_1 \cdots U_{2m}.$$  

By (2.4), there exist two self-adjoint unitaries $W_1, W_2 \in M_{2^{k+1}}(\mathbb{C})$ such that

$$(E_{11} + E_{22} + \cdots + E_{mm}) + e^{-i\theta}(I - (E_{11} + E_{22} + \cdots + E_{mm})) = e^{-i\theta}W_1W_2.$$

Therefore, $e^{i\beta}E_{11} + (I - E_{11}) = e^{i\theta}U_1 \cdots U_{2m}W_1W_2.$ □
Proof of Theorem 1.1 \((p \neq 2)\). By Theorem 2.6 and Lemma 2.1, we only need to consider the case \(\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}\), where \(\mathcal{R}\) is a type \(\text{II}_1\) factor. Without loss of generality, we may assume that \(\psi(I) = I\). By Theorem 2.6, there exists a \(*\)-automorphism or a \(*\)-anti-automorphism \(\phi\) on \(\mathcal{R}\) such that \(\psi(U) = \phi(U)\) for every self-adjoint unitary \(U\). Let

\[
\psi'(V) := \begin{cases} 
\phi^{-1} \circ \psi(V), & \text{if } \psi(\theta I) = \theta I, \\
\phi^{-1} \circ \psi(V^*), & \text{if } \psi(\theta I) = e^{-\theta I}.
\end{cases}
\]

By Lemma 2.9, \(\psi'\) is a surjective \(L^p\)-isometry such that \(\psi'(\theta U) = \theta U\) for every self-adjoint unitary \(U\). We now show that \(\psi'\) is the identity map on \(\mathcal{R}\).

Let \(V \in \mathcal{W}(\mathcal{R})\). For every \(k \in \mathbb{N}\), there exist an integer \(n\) and a subfactor \(\mathcal{R}_k\) of \(\mathcal{R}\) such that \(\mathcal{R}_k \simeq M_{2^n}(\mathbb{C})\) and a unitary \(V_k \in \mathcal{R}_k\) satisfying \(\|V - V_k\|_p < 1/k\). By Lemmas 2.11 and 2.12, \(\psi'(V_k) = V_k\). Since

\[
\lim_k \|\psi'(V) - V_k\|_p = \lim_k \|V - V_k\|_p = 0,
\]

we have \(\psi'(V) = \lim_k V_k = V\). \(\square\)

3 Surjective \(L^p\)-isometries of the Grassmann spaces

Let \(\mathcal{M}\) be a semifinite factor equipped with an n.s.f. tracial weight \(\tau\). If \(\mathcal{M}\) is a finite factor, we require that \(\tau\) be the tracial state on \(\mathcal{M}\), i.e., \(\tau(I) = 1\). In the following, we always assume that \(\mathcal{P}_1(\mathcal{M}) \neq \emptyset\). Let \(\phi\) be a surjective \(L^p\)-isometry on \(\mathcal{P}_1(\mathcal{M})\). Then \(\phi\) preserves the orthogonality in both directions by Lemma 2.5. By [8, Theorem 1.2] and [24, Theorem 4.9], \(\phi\) can be extended to a \(*\)-automorphism or a \(*\)-anti-automorphism on \(\mathcal{M}\) if \(c \neq \tau(I)/2\). If \(\mathcal{M}\) is a finite factor and \(c = 1/2\), it is clear that \(\varphi(\cdot)\) and \(I - \varphi(\cdot)\) are both surjective \(L^p\)-isometries on \(\mathcal{P}_{1/2}(\mathcal{M})\), where \(\varphi\) is a \(*\)-automorphism or a \(*\)-anti-automorphism on \(\mathcal{M}\). We now show that every surjective \(L^p\)-isometry on \(\mathcal{P}_{1/2}(\mathcal{M})\) arises in the above manners.

Lemma 3.1. Let \(\mathcal{M}\) be a finite factor equipped with a tracial state \(\tau\) and \(P, Q \in \mathcal{P}_{1/2}(\mathcal{M})\). If \(p \neq 2\), then

\[
\|P - Q\|_p = \|(I - P) - Q\|_p = \frac{1}{\sqrt{2}}
\]

if and only if there exists a partial isometry \(U \in \mathcal{M}\) such that \(U^*U = P\), \(UU^* = I - P\) and \(Q = (I + U + U^*)/2\).

Proof. Without loss of generality, we may assume that \(\mathcal{M} = \mathcal{R} \otimes M_2(\mathbb{C})\) and

\[
P = I \otimes E_{11}, \quad Q = H \otimes E_{11} + \sqrt{H(I - H)} \otimes (E_{12} + E_{21}) + (I - H) \otimes E_{22},
\]

where \(H\) is a positive contraction in the finite factor \(\mathcal{R}\) and \(\{E_{ij}\}\) is the canonical matrix units of \(M_2(\mathbb{C})\). By a direct computation, we have

\[
\|I - H\|_{p/2}^2 = \|P - Q\|_p^2 = \|(I - P) - Q\|_p^2 = \|H\|_{p/2}^2 = 1/2,
\]

where \(\|\cdot\|_{p/2}\) is the \(L^{p/2}\)-norm on \(\mathcal{R}\) induced by the tracial state on \(\mathcal{R}\). Therefore,

\[
\|(I - H) + H\|_{p/2} = \|I - H\|_{p/2} + \|H\|_{p/2}.
\]

Note that \(p/2 \neq 1\). By the condition for Minkowski’s inequality to be an equality (see [10, Subsection 6.13]), we know that there exists a non-negative number \(\lambda\) such that \(H = \lambda(I - H)\). Since \(\|I - H\|_{p/2} = \|H\|_{p/2}\), we have \(\lambda = 1\). The lemma is proved. \(\square\)
Lemma 3.2. Let $\mathcal{M}_i = \mathcal{R}_i \otimes M_2(\mathbb{C})$, where $\mathcal{R}_i$ ($i = 1, 2$) is a finite factor. If $\phi$ is a surjective $L^p$-isometry from $\mathcal{P}_{1/2}(\mathcal{M}_1)$ to $\mathcal{P}_{1/2}(\mathcal{M}_2)$ satisfying $\phi(I \otimes E_{11}) = I \otimes E_{11}$, then there exists a surjective $L^p$-isometry $\psi: \mathcal{R}(\mathcal{M}_1) \to \mathcal{R}(\mathcal{M}_2)$ such that

$$\phi\left(\frac{1}{2}(I \otimes (E_{11} + E_{22}) + V \otimes E_{12} + V^* \otimes E_{21})\right) = \frac{1}{2}(I \otimes (E_{11} + E_{22}) + \psi(V) \otimes E_{12} + \psi(V)^* \otimes E_{21})$$

(3.1)

for every unitary $V \in \mathcal{R}_1$, where $\{E_{ij}\}_{i,j}$ are the canonical matrix units of $M_2(\mathbb{C})$.

Proof. We only need to show that for every unitary $V \in \mathcal{R}_1$, there exists a unitary $\psi(V) \in \mathcal{R}_2$ such that (3.1) holds. By Lemma 2.5, $\phi(I \otimes E_{22}) = I \otimes E_{22}$. If $p \neq 2$, then Lemma 3.1 implies the result, since $\psi$ is a surjective $L^p$-isometry.

Assume that $p = 2$. Proceeding as in the proof of [16, Lemma 1] (or the proof of [21, Lemma 2.2]), we can show that $\phi$ has a unique linear extension $\Phi$ on the linear span of $\mathcal{P}_{1/2}(\mathcal{M}_1)$. More explicitly, $\Phi$ is defined as follows:

$$\Phi\left(\sum_{i=1}^{n} \lambda_i Q_i\right) := \sum_{i=1}^{n} \lambda_i \phi(Q_i),$$

where $\lambda_i \in \mathbb{C}$ and $Q_i \in \mathcal{P}_{1/2}(\mathcal{M}_1)$. Let $W := V \otimes E_{12} + V^* \otimes E_{21}$. Note that

$$\cos(t)I \otimes (E_{11} - E_{22}) + \sin(t)W$$

is a self-adjoint unitary such that

$$\tau_1(\cos(t)I \otimes (E_{11} - E_{22}) + \sin(t)W) = 0,$$

where $\tau_1$ is the tracial state on $\mathcal{M}_1$. Therefore there exists a $P \in \mathcal{P}_{1/2}(\mathcal{M}_1)$ such that

$$\cos(t)I \otimes (E_{11} - E_{22}) + \sin(t)W = 2P - I,$$

and

$$2\phi(P) - I = \Phi(\cos(t)I \otimes (E_{11} - E_{22}) + \sin(t)W) = \cos(t)I \otimes (E_{11} - E_{22}) + \sin(t)\Phi(W)$$

is a self-adjoint unitary for every $t \in \mathbb{R}$. In particular, $\Phi(W)$ is a self-adjoint unitary. Since $[\cos(t)I \otimes (E_{11} - E_{22}) + \sin(t)\Phi(W)]^2 = I$, we have

$$[I \otimes (E_{11} - E_{22})]\Phi(W)[I \otimes (E_{11} - E_{22})] = -\Phi(W).$$

Therefore, there exists a unitary $\psi(V) \in \mathcal{R}_2$ such that

$$\Phi(W) = \psi(V) \otimes E_{12} + \psi(V)^* \otimes E_{21}.$$

Then it is easy to check that (3.1) holds. $\square$

Lemma 3.3. Let $\mathfrak{M} = \mathcal{R} \otimes M_2(\mathbb{C})$, where $\mathcal{R}$ is a finite factor. If $\phi$ is an $L^p$-isometry on $\mathcal{P}_{1/2}(\mathfrak{M})$ such that $\phi(I \otimes E_{11}) = I \otimes E_{11}$ and

$$\phi\left(\frac{1}{2}(I \otimes I_2 + V \otimes E_{12} + V^* \otimes E_{21})\right) = \frac{1}{2}(I \otimes I_2 + V \otimes E_{12} + V^* \otimes E_{21})$$

(3.2)

for every unitary $V \in \mathfrak{M}$, then $\phi(Q) = Q$ for every $Q \in \mathcal{P}_{1/2}(\mathfrak{M})$. 

Proof. There exist a unitary $W_0$ and a positive contraction $H$ in $\mathcal{R}$ such that
\[
Q = H \otimes E_{11} + \sqrt{H(I-H)}W_0 \otimes E_{12} + W_0^* \sqrt{H(I-H)} \otimes E_{21} + W_0^*(I-H)W_0 \otimes E_{22}.
\]
Let $W := \frac{1}{\sqrt{2}}(I \otimes E_{11} + iW_0 \otimes E_{12} - i\otimes E_{21} - W_0 \otimes E_{22})$. It is easy to check that $W$ is a unitary and
\[
Q = \frac{1}{2}W^*(I \otimes I_2 + V_0 \otimes E_{12} + V_0^* \otimes E_{21})W,
\]
where $V_0 = -2\sqrt{H(I-H)} - i(I - 2H)$ is a unitary in $\mathcal{R}$.

Let $\phi_W$ be the surjective $L^p$-isometry on $\mathcal{P}_{1/2}(\mathcal{R})$ defined as follows:
\[
\phi_W(E) := W\phi(W^*EW)W^*, \quad \forall E \in \mathcal{P}_{1/2}(\mathcal{R}).
\]
By a direct computation and (3.2), we have
\[
\phi_W(I \otimes E_{11}) = I \otimes E_{11},
\]
\[
\phi_W\left(\frac{1}{2}(I \otimes I_2 + iI \otimes E_{12} - iI \otimes E_{21})\right) = \frac{1}{2}(I \otimes I_2 + iI \otimes E_{12} - iI \otimes E_{21})
\]
and
\[
\phi_W\left(\frac{1}{2}(I \otimes I_2 + U \otimes E_{12} + U \otimes E_{21})\right) = \frac{1}{2}(I \otimes I_2 + U \otimes E_{12} + U \otimes E_{21})
\]
for every self-adjoint unitary $U \in \mathcal{R}$. By Lemma 3.2 and Theorem 1.1, we know that
\[
\phi_W\left(\frac{1}{2}(I \otimes I_2 + V \otimes E_{12} + V^* \otimes E_{21})\right) = \frac{1}{2}(I \otimes I_2 + V \otimes E_{12} + V^* \otimes E_{21})
\]
for every unitary $V \in \mathcal{R}$. In particular,
\[
\phi(Q) = W^*\phi_W\left(\frac{1}{2}(I \otimes I_2 + V_0 \otimes E_{12} + V_0^* \otimes E_{21})\right)W = Q.
\]
This completes the proof. 

Theorem 3.4. Let $\mathcal{R}$ be a finite factor and $\phi$ be a surjective $L^p$-isometry on $\mathcal{P}_{1/2}(\mathcal{R})$. Then there exists a $*$-automorphism or a $*$-anti-automorphism $\varphi$ on $\mathcal{R}$ such that $\phi$ has one of the following forms:
1. $\phi(P) = \varphi(P)$ and $P \in \mathcal{P}_{1/2}(\mathcal{R})$;
2. $\phi(P) = I - \varphi(P)$ and $P \in \mathcal{P}_{1/2}(\mathcal{R})$.

Proof. Without loss of generality, we assume that $\mathcal{R} = \mathcal{R} \otimes M_2(\mathcal{C})$, where $\mathcal{R}$ is a finite factor and $\phi(I \otimes E_{11}) = I \otimes E_{11}$. By Lemma 3.2, there exists a surjective $L^p$-isometry $\psi$ on $\mathcal{H}(\mathcal{R})$ such that
\[
\psi\left(\frac{1}{2}(I \otimes I_2 + V \otimes E_{12} + V^* \otimes E_{21})\right) = \frac{1}{2}(I \otimes I_2 + \psi(V) \otimes E_{12} + \psi(V)^* \otimes E_{21})
\]
for every $V \in \mathcal{H}(\mathcal{R})$. We can further assume that $\psi(I) = I$.

By Theorem 1.1, there exists a $*$-automorphism or a $*$-anti-automorphism $\varphi$ on $\mathcal{R}$ such that $\psi(V) = \varphi(V)$ or $\psi(V) = \varphi(V^*)$ for every unitary $V$. Let $\sigma$ be the $*$-anti-automorphism on $M_2(\mathcal{C})$ defined by $\sigma(A) = A^\dagger$, where $A^\dagger$ is the transpose of $A$. We define a new surjective $L^p$-isometry $\phi_1$ on $\mathcal{P}_{1/2}(\mathcal{R})$ as follows:
1. If $\varphi$ is a $*$-automorphism and $\psi(V) = \varphi(V)$ for every unitary $V$, let
   \[
   \phi_1(Q) := (\varphi^{-1} \otimes \text{Id}) \circ \phi(Q).
   \]
2. If $\varphi$ is a $*$-automorphism and $\psi(V) = \varphi(V^*)$ for every unitary $V$, let
   \[
   \phi_1(Q) := I \otimes I_2 - [I \otimes (E_{12} - E_{21})](\varphi^{-1} \otimes \text{Id}) \circ \phi(Q)[I \otimes (E_{21} - E_{12})].
   \]
(3) If $\varphi$ is a $*$-anti-automorphism and $\psi(V) = \varphi(V)$ for every unitary $V$, let
\[
\phi_1(Q) := I \otimes I_2 - [I \otimes (E_{12} - E_{21})](\varphi^{-1} \otimes \sigma) \circ \phi(Q)[I \otimes (E_{21} - E_{12})].
\]
(4) If $\varphi$ is a $*$-anti-automorphism and $\psi(V) = \varphi(V^*)$ for every unitary $V$, let
\[
\phi_1(Q) := (\varphi^{-1} \otimes \sigma) \circ \phi(Q).
\]
It is not hard to check that $\phi_1(I \otimes E_{11}) = I \otimes E_{11}$ and
\[
\phi_1\left(\frac{1}{2}(I \otimes I_2 + V \otimes E_{12} + V^* \otimes E_{21})\right) = \frac{1}{2}(I \otimes I_2 + V \otimes E_{12} + V^* \otimes E_{21})
\]
for every unitary $V \in \mathfrak{M}$. Then Lemma 3.3 implies that $\phi_1(Q) = Q$ for every $Q \in \mathcal{P}_{1/2}(\mathfrak{M})$, and the theorem is proved.

Let $\mathfrak{M}_1$ be a semifinite factor equipped with an n.s.f. tracial weight $\tau_i$ ($i = 1, 2$). Assume that there exists a surjective $L^p$-isometry $\phi : \mathcal{P}_c(\mathfrak{M}_1) \to \mathcal{P}_c(\mathfrak{M}_2)$. Recall that $\phi$ preserves orthogonality in both directions. Therefore, $\mathfrak{M}_1$ is properly infinite if and only if $\mathfrak{M}_2$ is properly infinite. Moreover, it can be shown that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are either $*$-isomorphic or $*$-anti-isomorphic if there exists a surjective $L^p$-isometry $\mathcal{P}_c(\mathfrak{M}_1) \to \mathcal{P}_c(\mathfrak{M}_2)$.

Lemma 3.5. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be two properly infinite semifinite factors, i.e., $\tau_i(I) = \infty$. If there exists a surjective $L^p$-isometry $\phi : \mathcal{P}_c(\mathfrak{M}_1) \to \mathcal{P}_c(\mathfrak{M}_2)$, then there exists a $*$-isomorphism or a $*$-anti-isomorphism $\varphi : \mathfrak{M}_1 \to \mathfrak{M}_2$ such that $\tau_1 = \sigma_2 \circ \varphi$.

Proof. Let $E_1, E_2 \in \mathcal{P}_c(\mathfrak{M}_1)$ such that $E_1 \perp E_2$. Since $\phi$ is surjective and preserves orthogonality in both directions, there exists a Hilbert space $\mathcal{H}$ such that
\[
\mathfrak{M}_1 \simeq E_1 \mathfrak{M}_1 E_1 \otimes B(\mathcal{H}), \quad \mathfrak{M}_2 \simeq \phi(E_1) \mathfrak{M}_1 \phi(E_1) \otimes B(\mathcal{H}).
\]
To prove the lemma, we only need to show that there exists a $*$-isomorphism or a $*$-anti-isomorphism $\varphi_0 : E_1 \mathfrak{M}_1 E_1 \to \phi(E_1) \mathfrak{M}_2 \phi(E_1)$. Invoking Lemma 2.5, we obtain
\[
\phi(\{E \in \mathcal{P}_c(\mathfrak{M}_1) : E \perp (E_1 + E_2)\}) = \{E' \in \mathcal{P}_c(\mathfrak{M}_2) : E' \perp (\phi(E_1) + \phi(E_2))\}.
\]
Therefore, $\phi(F) < \phi(E_1) + \phi(E_2)$ for every projection $F \in \mathcal{P}_c(\mathfrak{M}_1)$ satisfying $F < E_1 + E_2$ and $\phi$ restricts to a surjective $L^p$-isometry from $\mathcal{P}_{1/2}(E_1 + E_2)\mathfrak{M}_1(E_1 + E_2)$ to
\[
\mathcal{P}_{1/2}(\phi(E_1) + \phi(E_2))\mathfrak{M}_2(\phi(E_1) + \phi(E_2)).
\]
By Lemma 3.2 and Theorem 1.1, there exists a $*$-isomorphism or a $*$-anti-isomorphism $\varphi_0 : E_1 \mathfrak{M}_1 E_1 \to \phi(E_1) \mathfrak{M}_2 \phi(E_1)$.

Proof of Theorem 1.2. By Lemma 3.5, we may assume that $\mathfrak{M}_1 = \mathfrak{M}_2$ and $\tau_1 = \tau_2$. Recall that $\phi$ preserves orthogonality in both directions. Then the theorem is an immediate consequence of [8, Theorem 1.2] and [24, Theorem 4.9].

Lemma 3.6. Let $\mathfrak{M}_i$ be a finite factor with the tracial state $\tau_i$ ($i = 1, 2$). If there exists a surjective $L^p$-isometry $\phi : \mathcal{P}_c(\mathfrak{M}_1) \to \mathcal{P}_c(\mathfrak{M}_2)$, then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are either $*$-isomorphic or $*$-anti-isomorphic.

Proof. We only need to prove the lemma under the assumption that $c \in (0, \frac{1}{2}]$. It is well known that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are $*$-(anti)-isomorphic if there exist $s \in (0, 1)$ and nonzero projections $P_1 \in \mathcal{P}_c(\mathfrak{M}_1)$ and $P_2 \in \mathcal{P}_c(\mathfrak{M}_2)$ such that $P_1 \mathfrak{M}_1 P_1$ and $P_2 \mathfrak{M}_2 P_2$ are $*$-(anti)-isomorphic.

Let $P \in \mathcal{P}_c(\mathfrak{M}_1)$. If $c \in (0, \frac{1}{2}] \cup \{\frac{1}{2}\}$, then we can show that $P \mathfrak{M}_1 P$ and $\phi(P) \mathfrak{M}_2 \phi(P)$ are either $*$-isomorphic or $*$-anti-isomorphic by the same method employed in the proof of Lemma 3.5.

We now assume that $c \in (\frac{1}{2}, \frac{1}{2})$. Let $f$ be the continuous function on $(-\infty, 1)$ defined by $f(x) = \frac{1-2x}{1-x}$. By Lemma 2.5, we have
\[
\phi(\{Q \in \mathcal{P}_c(\mathfrak{M}_1) : QP = 0\}) = \{Q' \in \mathcal{P}_c(\mathfrak{M}_2) : Q'\phi(P) = 0\}.
\]
Let $c_1 = f(c)$. Note that $0 < c_1 < \frac{1}{2}$. It is clear that $E \in \mathcal{P}_{c_1}((I - P)\mathfrak{M}_1(I - P))$ if and only if $E \in \mathcal{P}_{1-2c}(\mathfrak{M}_1)$ and $E < I - P$. The map $\phi_1$ defined as follows is a surjective $L^p$-isometry from $\mathcal{P}_{c_1}((I - P)\mathfrak{M}_1(I - P))$ to $\mathcal{P}_{\frac{1}{2}}((I - \phi(P))\mathfrak{M}_2(I - \phi(P)))$:

$$\phi_1(E) := I - \phi(P) - \phi(I - P - E).$$

If $c_1 \notin \left(\frac{1}{3}, \frac{1}{2}\right)$, then the discussion above implies that $E\mathfrak{M}_1E$ and $\phi_1(E)\mathfrak{M}_2\phi_1(E)$ are either $*$-isomorphic or $*$-anti-isomorphic.

Note that the only fixed point of $f$ in $\left(\frac{1}{3}, \frac{1}{2}\right)$ is $\frac{3 - \sqrt{5}}{2}$. We claim that if $c \notin \frac{3 - \sqrt{5}}{2}$, then there exists an $n \in \mathbb{N}$ such that $c_n := f^n(c) \notin \left(\frac{1}{3}, \frac{1}{2}\right)$, where $f^n$ is the $n$-fold composition of $f$ with itself. We prove the claim by contradiction. Assume that $c \notin \frac{3 - \sqrt{5}}{2}$ and $c_n \in \left(\frac{1}{3}, \frac{1}{2}\right)$ for every $n \in \mathbb{N}$. Since

$$\begin{align*}
\{c_{2k}\}_k \text{ is a monotone sequence in } \left[\frac{1}{3}, \frac{1}{2}\right]. \quad \text{Then } \lim_k c_{2k} \text{ is a fixed point of } f^2. \quad \text{Since } \lim_k c_{2k} \neq \frac{3 - \sqrt{5}}{2} \quad \text{and } \frac{3 - \sqrt{5}}{2} \text{ is the only fixed point of } f^2 \text{ in } \left[\frac{1}{3}, \frac{1}{2}\right], \text{ we have a contradiction and the claim is proved.}
\end{align*}$$

If $c \notin \frac{3 - \sqrt{5}}{2}$, let $n$ be the number such that $c_n \notin \left(\frac{1}{3}, \frac{1}{2}\right)$ and $c_i \in \left(\frac{1}{3}, \frac{1}{2}\right)$ for every $i < n$. Then we can repeat the above argument $n$ times and show that there exist $s \in (0, 1)$, $P_1 \in \mathcal{P}_s(\mathfrak{M}_1)$ and $P_2 \in \mathcal{P}_s(\mathfrak{M}_2)$ such that $P_1 \mathfrak{M}_1 P_1$ and $P_2 \mathfrak{M}_2 P_2$ are either $*$-isomorphic or $*$-anti-isomorphic.

From now on, $c = \frac{3 - \sqrt{5}}{2}$. Note that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are type II$_1$ factors since $\frac{3 - \sqrt{5}}{2}$ is an irrational number. By the same argument used in the proof of [24, Proposition 4.5], we can show that $(I - P)\mathfrak{M}_1(I - P)$ and $(I - \phi(P))\mathfrak{M}_1(I - \phi(P))$ are either $*$-isomorphic or $*$-anti-isomorphic. We therefore only sketch the proof and refer the interested readers to [24] for the complete argument.

Let $\mathcal{F}_1 = \{E \in \mathcal{P}(\mathfrak{M}_1) : \tau_1(E) \geq c, E \leq I - P\}$ and $\mathcal{F}_2 = \{F \in \mathcal{P}(\mathfrak{M}_2) : \tau_2(F) \geq c, F \leq I - \phi(P)\}$. The map $\phi^+ : \mathcal{F}_1 \to \mathcal{F}_2$ defined as follows is a bijection which preserves the orthogonality and the order in both directions (see [24, Proposition 3.1] for the proof):

$$\phi^+(E) := \vee\{\phi(Q) : Q \in \mathcal{P}_s(\mathfrak{M}_1), Q \leq E\}.$$ 

Note that $\phi^+(I - P) = I - P$. We have

$$\phi^+(E) := I - \phi(P) - \phi^+(I - P - E)$$

is a map from $\{E \in \mathcal{P}(\mathfrak{M}_1) : \tau_1(E) \leq 1 - 2c, E < I - P\}$ to $\{F \in \mathcal{P}(\mathfrak{M}_2) : \tau_2(F) \leq 1 - 2c, F < I - \phi(P)\}$. Proceeding as in the proof of [24, Proposition 4.5], one can show that $\phi^+$ is a bijection which preserves the orthogonality and the order in both directions. By [24, Remark 3.3], $\phi^+$ can be extended to an ortho-isomorphism from $\mathcal{P}((I - P)\mathfrak{M}_1(I - P))$ to $\mathcal{P}((I - \phi(P))\mathfrak{M}_2(I - \phi(P)))$. Then [6, Corollary] implies that $(I - P)\mathfrak{M}_1(I - P)$ and $(I - \phi(P))\mathfrak{M}_2(I - \phi(P))$ are either $*$-isomorphic or $*$-anti-isomorphic.

**Proof of Theorem 1.3.** We only need to prove the theorem under the further assumption that $c \in (0, 1/2)$. By Lemma 3.6, we may assume that $\mathfrak{M}_1 = \mathfrak{M}_2$ and $\tau_1 = \tau_2$. Recall that $\phi$ preserves orthogonality in both directions. Then the theorem is an immediate consequence of [8, Theorem 1.2], [24, Theorem 4.9] and Theorem 3.4.

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