RIEMANN HYPOTHESIS AND SOME NEW INTEGRALS
CONNECTED WITH THE INTEGRAL NEGATIVITY OF THE
REMAINDER IN THE FORMULA FOR THE PRIME-COUNTING
FUNCTION $\pi(x)$

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Abstract. In this paper a new integral for the remainder of $\pi(x)$ is obtained. It is proved that there is an infinite set of the formulae containing miscellaneous parts of this integral.

1. The result

Let us remind that

$$\pi(x) = \int_0^x \frac{dt}{\ln t} + P(x) = \text{li}(x) + P(x)$$

where $\pi(x)$ is the prime-counting function. In 1914 Littlewood proved that the remainder $P(x)$, $x \geq 2$ changes the sign infinitely many times (see [1]). However, on the Riemann hypothesis, the inequality

$$\int_2^Y P(x)dx < 0, \ Y \to \infty$$

(1.1)

holds true (see [2], p. 106), i.e. the remainder $P(x)$ is negative in mean. In this direction the following theorem holds true.

Theorem. Let $0 < \Delta$ is a sufficiently small fixed number. Then, on Riemann hypothesis, there is a positive function $N(\delta)$, $\delta \in (0, \Delta)$ such that

$$\int_2^{N(\delta)} P(x)dx = -\frac{a}{\delta} + \mathcal{O}(1), \ \delta \in (0, \Delta), \ a = e^{9/2}$$

(1.2)

where $\mathcal{O}(1)$ stands for a continuous and bounded function.

Remark 1. The formula (1.2) is the new asymptotic formula in the direction of (1.1).

Remark 2. Since (see (1.2)

$$\liminf_{\delta \to 0^+} \int_2^{N(\delta)} P(x)dx = -\infty \Rightarrow \liminf_{\delta \to 0^+} N(\delta) = +\infty$$

then

$$\lim_{\delta \to 0^+} N(\delta) = \infty.$$  

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2. Some properties of asymptotic multiplicability for the integral (1.2)

Let \( 0 < \delta_k, \ k = 1, \ldots, l \). Since (see (1.2))

\[
\int_2^{N(\delta)} P(x)dx = -\frac{a}{\delta_k} \{ 1 + o(1) \}, \ \delta_k \to 0^+
\]

then we have

**Corollary 1.** If \( \delta_k \to 0^+, \ k = 1, \ldots, l \) then

\[
\prod_{k=1}^{l} \int_2^{N(\delta_k)} P(x)dx \sim (-a)^{l-1} \int_2^{N(\prod_{k=1}^{l} \delta_k)} P(x)dx,
\]

i.e.

\[
\prod_{k=1}^{l} \int_2^{N(\delta_k)} \frac{P(x)}{a}dx \sim (-1)^{l-1} \int_2^{N(\prod_{k=1}^{l} \delta_k)} \frac{P(x)}{a}dx,
\]

for arbitrary fixed \( l \in \mathbb{N} \).

**Remark 3.** Let, for example,

\[
n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_l^{\alpha_l}, \ p_k^{\alpha_k} \to \infty, \ k = 1, \ldots, l
\]

be the canonical product for \( n \). Then (see (2.2))

\[
\prod_{k=1}^{l} \int_2^{N(p_k^{-\alpha_k})} \frac{P(x)}{a}dx \sim (-1)^{l-1} \int_2^{N(\prod_{k=1}^{l} p_k^{-\alpha_k})} \frac{P(x)}{a}dx.
\]

3. The division in the set of some integrals

The simple algebraic formula

\[
\left( \frac{1}{\delta_2} \right)^n - \left( \frac{1}{\delta_1} \right)^n = \left( \frac{1}{\delta_2} - \frac{1}{\delta_1} \right) \sum_{k=0}^{n-1} \left( \frac{1}{\delta_1} \right)^k \left( \frac{1}{\delta_2} \right)^{n-k-1}, \ \delta_2 < \delta_1,
\]

implies due to (1.2) the following dual formula.

**Corollary 2.**

\[
\frac{\int_2^{N(\delta_2)} P(x)dx}{\int_2^{N(\delta_1)} P(x)dx} \sim -\sum_{k=0}^{n-1} \int_2^{N(\delta_1 \delta_2^{-k-1})} \frac{P(x)}{a}dx,
\]

for arbitrary fixed \( n \in \mathbb{N}, \ \delta_1, \delta_2 \to 0^+ \).

**Remark 4.** The asymptotic division

\[
\frac{\int_2^{N(\delta_2)} P(x)dx}{\int_2^{N(\delta_1)} P(x)dx}
\]

is defined by the formula (3.1).
4. THE DIVISION OF SOME INTEGRALS ON ASYMPTOTICALLY EQUAL PARTS

First of all we have (see (1.2))

\[ \int_{\frac{N(a\delta)}{n}}^{N(a\delta)/n} P(x)dx \sim -\frac{n}{\delta}, \quad n \in \mathbb{N}, \quad \delta \in (0, \Delta). \]

Then from (4.1) the following corollary follows.

**Corollary 3.**

\[ \int_{\frac{N(a\delta)}{n}}^{N(a\delta)/n} P(x)dx \sim \int_{\frac{N(a\delta)}{m}}^{N(a\delta)/m} P(x)dx, \]

for every \( k, l \in \mathbb{N}, \quad \delta \to 0^+. \)

**Remark 5.** Thus the sequence

\[ \left\{ N\left(\frac{a\delta}{n}\right) \right\}_{n=1}^{\infty} \]

subdivides each of the integrals

\[ \int_{\frac{N(a\delta)}{n}}^{N(a\delta)/n} P(x)dx, \quad \forall \, n, m \in \mathbb{N} \]

on the asymptotically equal parts (see (4.1)).

**Remark 6.** For every sequence of the type

\[ \delta_k(\alpha_0, r) = \frac{a\delta}{\alpha_0 + kr}, \quad k = 0, 1, 2, \ldots, \alpha_0, r > 0 \]

we obtain the similar result.

**Remark 7.** Thus we see that there is an infinite set of the formulae containing miscellaneous parts of the integral

\[ \int_{\frac{N(a\delta)}{n}}^{N(a\delta)/n} P(x)dx \]

(see for example (1.3), (2.1)-(2.3), (4.2)).

5. PROOF OF THE THEOREM

Let us remind the formula

\[ \int_{2}^{\infty} \frac{\ln(e^{-2x})}{x^{3/2+\delta}} P(x)dx = -\frac{1}{\delta} + O(1), \quad \delta \in (0, \Delta) \]

where \( O(1), \quad \delta \in (0, \Delta) \) is a continuous and bounded function (see (4)). Using the von Koch estimate \( P(x) = O(\sqrt{x \ln x}) \), which follows from the Riemann hypothesis, we obtain (comp. (4), part 3)

\[ \left| \int_{Y}^{\infty} \frac{\ln(e^{-2x})}{x^{3/2+\delta}} P(x)dx \right| < \frac{A}{\delta} Y^{-\delta/2} \leq 1 \]

for

\[ Y \geq T(\delta) = \left( \frac{A}{\delta} \right)^{2/\delta}, \quad \delta \in (0, \Delta). \]
Then we have (see (5.1))

\[
\int Y \frac{\ln(e^{-2x})}{x^{3/2+\delta}} P(x)dx = -\frac{1}{\delta} + O(1), \quad \delta \in (0, \Delta),
\]

where \(\int_{Y}^{3} = O(1)\). Let us remind further the Bonet’s form of the second mean-value theorem

\[
\int_{a}^{b} f(x)g(x)dx = f(a + 0) \int_{a}^{\xi} g(x)dx, \quad \xi \in (a, b)
\]

where \(f(x), g(x)\) are integrable function on \([a, b]\), and \(f(x)\) is a non-negative and non-increasing function (in our case \(g(x) = P(x)\)). We have as a consequence the following formula (comp. [3])

\[
\int^{Y} \frac{\ln(e^{-2x})}{x^{3/2+\delta}} P(x)dx = \frac{1}{e^{9/2+3\delta}} \int_{e^{3}}^{M(Y, \delta)} P(x)dx
\]

where \(M \in (e^{3}, Y)\). Let \(Y = T(\delta)\), i.e.

\[
M(Y, \delta) = M[T(\delta), \delta] = N(\delta).
\]

Then from (5.2) by (5.3) and (5.4) the formula

\[
\int_{e^{3}}^{Y} \frac{\ln(e^{-2x})}{x^{3/2+\delta}} P(x)dx = \frac{1}{e^{9/2+3\delta}} \int_{e^{3}}^{M(Y, \delta)} P(x)dx
\]

follows. Since

\[
e^{3\delta} = 1 + O(\delta), \quad \int_{e^{3}}^{Y} P(x)dx = O(1), \quad \delta \in (0, \Delta)
\]

we obtain our formula (1.2) from (5.5).

6. Concluding remarks: on the Littlewood’s point and on the point of simple discontinuity of the function \(N(\delta)\)

Since the right-hand side of the formula (5.5) contains the continuous function \((Y = T(\delta))\) then

\[
\int_{T(\delta)}^{\infty} \frac{\ln(e^{-2x})}{x^{3/2+\delta}} P(x)dx
\]

is also the continuous function. Let us remind that the continuity of the two functions \(F(u)\) and \(F(f(x))\) does not imply the continuity of the function \(f(x)\), (comp. \(\sin\{\pi[x]\}, x \in \mathbb{R}\)).

If \(\delta_{0} \in (0, \Delta)\) is the point of the simple discontinuity of \(N(\delta)\) (another kind of discontinuity of \(N(\delta)\) is excluded), i.e. if \(N(\delta_{0} + 0) \neq N(\delta_{0} - 0)\) then we have (see (1.2))

\[
\int_{N(\delta_{0} - 0)}^{N(\delta_{0} + 0)} P(x)dx = 0,
\]

and from this

\[
\mu[P(x)] \{N(\delta_{0} + 0) - N(\delta_{0} - 0)\} = 0 \Rightarrow \mu[P(x)] = 0
\]

follows, where \(\mu\) is the mean-value of \(P(x)\) relative to the segment generated by the points \(N(\delta_{0} - 0), N(\delta_{0} + 0)\).
Let us remind that the remainder $P(x)$ changes its sign at the Littlewood’ sequence points $\{L_k\}_{k=1}^{\infty}$.

**Remark 8.** If $\delta_0 \in (0, \Delta)$ is the point of the simple discontinuity of $N(\delta)$ (if any), then from (6.1) we have: there is $L_k$ for which

$$L_k \in (N(\delta_0 - 0), N(\delta_0 + 0)), \ [N(\delta_0 - 0), N(\delta_0 + 0)]$$

holds true.

**References**

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