IFP-based Distributed Optimization With Event-triggered Communication

Mengmou Li, Lanlan Su, and Tao Liu

Abstract—In this work, we address the distributed optimization problem with event-triggered communication by introducing the notion of input feedforward passivity (IFP). First, we analyze a distributed continuous-time algorithm over uniformly jointly strongly connected balanced digraphs and show its exponential convergence over strongly connected digraphs. Then, we propose an event-triggered communication mechanism for this algorithm. Next, we discretize the continuous-time algorithm by the forward Euler method and show that the discretization can be seen as a stepsize dependent passivity degradation of the input feedforward passivity. The discretized system preserves IFP property and enables the same event-triggered communication mechanism but without Zeno behavior due to its sampling nature. Finally, a numerical example is presented to illustrate our results.

I. INTRODUCTION

DISTRIBUTED optimization problem aims to optimize the sum of objective functions of agents cooperatively, where each agent estimates the optimal solution based on local information and information obtained from its neighbors through a communication network. It has been widely studied in recent years, and numerous algorithms have been proposed, which can be basically categorized into two groups, i.e., discrete-time algorithms [1]–[4] and continuous-time algorithms [5]–[7].

An important issue in distributed optimization is the relaxation of communication graph conditions, since the communication network may be unidirectional or even time-varying in practice. The work [5] generalizes the well-known proportional-integral (PI) algorithm to weight-balanced and strongly connected digraphs. A fully distributed adaptive algorithm for the design of parameters is proposed in [6]. The work [8] proposes a modified PI algorithm over weight-balanced and strongly connected switching digraphs. Recently, [9] incorporates a continuous-time push-sum algorithm to address the problem of general directed graphs. In addition, [10] proposes fully distributed algorithms over weight-balanced and uniformly jointly strongly connected digraphs based on input feedforward passivity (IFP). There are also many related works in the discrete-time scheme (e.g., [1]–[4] and references therein), while most of them adopt diminishing stepsize or require global information.

Event-triggered based communication for distributed optimization is of critical importance due to practical issues of the communication network such as network congestion, limited bandwidth, and energy consumption, especially in large-scale networks. A centralized event-triggered condition is proposed in [8]. An encoder-decoder event-trigger communication mechanism is introduced in [11]. An edge-based event-triggered method is proposed in [12]. However, most of the abovementioned works only consider cases with undirected and non-switched communication networks and cannot apply directly to directed or switching networks. Recently, a periodic sampling communication mechanism is proposed in [13] over weight-balanced and uniformly jointly strongly connected digraphs for resource allocation problem. Considering distributed algorithms under event-triggered control over uniformly jointly strongly connected digraphs is of great significance, since the communication effort can be greatly reduced due to the lack of graph connectivity and consecutive communication, which has never been addressed yet.

In this work, we address the distributed optimization problem by exploiting the notion of IFP. First, we analyze a distributed continuous-time algorithm over uniformly jointly strongly connected balanced digraphs by introducing IFP, and show its exponential convergence over strongly connected digraphs. Then, we propose an event-triggered communication mechanism for this algorithm. Next, we discretize the continuous-time algorithm by the forward Euler method and show that the discretization can be seen as a stepsize dependent passivity degradation of the IFP. The discretized system preserves IFP property and enables the same event-triggered communication mechanism but without Zeno behavior due to its sampling nature.

II. PRELIMINARIES

A. Notation

Let $\mathbb{R}$ be the set of real numbers. The Kronecker product is denoted as $\otimes$. Let $\|\cdot\|$ denote the 2-norm of a vector and also the induced 2-norm of a matrix. Given a symmetric matrix $M \in \mathbb{R}^{m \times m}$, $M > 0$ ($M \geq 0$) means that $M$ is positive definite (positive semi-definite). Denote the eigenvalues of $M$ as $s_1(M) \leq s_2(M) \leq \ldots \leq s_m(M)$. Let $I$ and $0$ denote the identity matrix and zero matrix of proper dimensions, respectively. $1_m := (1, \ldots, 1)^T \in \mathbb{R}^m$ denotes the vector with all ones. We denote a vector $v$ without subscript as $v = \text{col}(v_1, \ldots, v_m) := (v_1^T, \ldots, v_m^T)^T$ as a compact vector of vectors $v_1, \ldots, v_m$ if not otherwise specified.

B. Convex Analysis

A differentiable function $f : \mathbb{R}^m \to \mathbb{R}$ is convex over a convex set $\mathcal{X} \subset \mathbb{R}^m$ if and only if $(\nabla f(x) - \nabla f(y))^T (x - y)$.
\( y \geq 0, \forall x, y \in \mathcal{X}, \) and is strictly convex if and only if the strict inequality holds for any \( x \neq y. \) It is \( \mu \)-strongly convex if and only if \( \langle \nabla f(x) - \nabla f(y) \rangle^T (x - y) \geq \mu \| x - y \|^2, \) or equivalently, \( f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{\mu}{2} \| y - x \|^2, \) \( \forall x, y \in \mathcal{X}. \) An operator \( f : \mathbb{R}^m \rightarrow \mathbb{R}^m \) is \( l \)-Lipschitz continuous over a set \( \mathcal{X} \subset \mathbb{R}^m \) if \( \| f(x) - f(y) \| \leq l \| x - y \|, \) \( \forall x, y \in \mathcal{X}. \)

C. Communication Network and Graph Theory

The communication network is represented by a graph \( G = (\mathcal{N}, \mathcal{E}) \), where \( \mathcal{N} = \{1, \ldots, N\} \) is the node set of all agents, \( \mathcal{E} \subset \mathcal{N} \times \mathcal{N} \) is the edge set. The edge \((i, j) \in \mathcal{E}\) means that agent \( j \) can send information to agent \( i. \) The graph \( G \) is said to be undirected if \((i, j) \in \mathcal{E} \iff (j, i) \in \mathcal{E} \) and directed otherwise. The adjacency matrix of \( G \) is defined as \( A = [a_{ij}], \) where \( a_{ii} = 0; a_{ij} > 0 \) if \((i, j) \in \mathcal{E}, \) and \( a_{ij} = 0, \) otherwise. \( G \) is said to be strongly connected if there exists a sequence of successive edges between any two agents. The in-degree and out-degree of the \( i \)th agent are \( d^i_{in} = \sum_{j=1}^N a_{ij} \) and \( d^i_{out} = \sum_{j=1}^N a_{ji}, \) respectively. The graph \( G \) is said to be weight-balanced if \( d^i_{in} = d^i_{out}, \forall i \in \mathcal{N}. \) The Laplacian matrix of \( G \) is defined as \( L = \text{diag}\{\mathbf{A}_N\} - \mathbf{A}. \) Clearly, \( \mathbf{L}_N = \mathbf{0}. \) If \( G \) is weight-balanced, then \( \mathbf{1}_N^T \mathbf{L} = \mathbf{0}. \) Further, a time-varying graph \( \mathcal{G}(t) \) with fixed nodes is said to be uniformly jointly strongly connected (UJSC) if there exists a \( T > 0 \) such that for any \( t_k, \) the union \( \cup_{t \in [t_k, t_k + T]} \mathcal{G}(t) \) is strongly connected.

D. Passivity

Consider a nonlinear system \( \Sigma \) described by

\[
\begin{align*}
\dot{x}^+ &= F(x, u) \\
y &= H(x, u)
\end{align*}
\]

where \( x \in \mathcal{X} \subset \mathbb{R}^n, u \in \mathcal{U} \subset \mathbb{R}^m \) and \( y \in \mathcal{Y} \subset \mathbb{R}^m \) are the state, input and output, respectively, and \( \mathcal{X}, \mathcal{U} \) and \( \mathcal{Y} \) are the state, input and output spaces, respectively. \( x^+ \) denotes the derivative of the state in the continuous-time (CT) case and the state at the next time step in the discrete-time (DT) case. The nonlinear functions \( F : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}^n, H : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}^m \) represent system and output dynamics, respectively, and are assumed to be sufficiently smooth.

System \( \Sigma \) is said to be passive if there exists a continuously differentiable positive semi-definite function \( V(x) \), called the storage function, such that \( \dot{V}(x) \leq u^T y - \nu u^T u \) for all \( u \in \mathcal{U} \) in CT case (or, \( V(x(k + 1)) \leq V(x(k)) - u^T y - \nu u^T u \) in DT case). Moreover, it is said to be input feedforward passive (IFP) if \( \dot{V}(x) \leq u^T y - \nu u^T u \) for CT case (or, \( V(x(k + 1)) \leq V(x(k)) - u^T y - \nu u^T u \) for DT case), for some \( \nu \in \mathbb{R} \), denoted as IFP(\( \nu \)). The sign of the IFP index \( \nu \) denotes an excess or shortage of passivity.

E. Problem Formulation

Consider a distributed optimization problem among a group of agents in \( \mathcal{N} = \{1, \ldots, N\} \)

\[
\min_{x} \sum_{i=1}^N f_i(x) \tag{2}
\]

where \( f_i : \mathbb{R}^m \rightarrow \mathbb{R} \) is the local objective function for agent \( i \) and \( x \in \mathbb{R}^m \) is the decision variable. We consider problem (2) with the following assumptions.

Assumption 1. Each local function \( f_i \) is sufficiently smooth, \( \mu_i \)-strongly convex and has \( l_i \)-Lipschitz continuous gradient.

Assumption 2. The time-varying communication digraph \( \mathcal{G} \) is weight-balanced and UJSC with bounded Laplacian matrix.

III. CONTINUOUS-TIME ALGORITHM

A. IFP-based Distributed Algorithm

We adopt the following distributed algorithm

\[
\begin{align*}
\dot{x}_i &= -\alpha \nabla f_i(x_i) - \lambda_i \\
\dot{\lambda}_i &= -\beta \sum_{j=1}^N a_{ij}(t)(x_j - x_i)
\end{align*}
\]

\[
\tag{3}
\]

where \( x_i \in \mathbb{R}^m \) is the decision variable for agent \( i, \lambda_i \in \mathbb{R}^m \) is an auxiliary state for agent \( i \) to track the difference between neighboring agents and satisfies \( \sum_{i=1}^N \lambda_i(0) = 0, \) \( \alpha > 0, \) \( \beta > 0 \) are parameters to be designed. The compact form of (3) is written as

\[
\begin{align*}
\dot{x} &= -\alpha \nabla f(x) - \lambda \\
\dot{\lambda} &= \beta \mathbf{L}(t)x
\end{align*}
\]

where \( \mathbf{L}(t) = \mathbf{L}(t) \otimes I_N \) and \( \mathbf{L}(t) \) is the Laplacian matrix.

Note that algorithm (3) is a simplified version, whose equilibrium and optimality have been reported in [8], [10]. In this work, we first analyze passivity and provide further results on convergence properties, and then propose an event-triggered mechanism for the algorithmic dynamics.

Define \( (x_i^*, \lambda_i^*) \) as the equilibrium point to system (3). Obviously, \( x_i^* = x_j^*, \forall i, j \in \mathcal{N}, \) with \( x_i^* \) being the optimal solution to problem (2) [10]. Denote \( \Delta x_i = x_i - x_i^*, \Delta \lambda_i = \lambda_i - \lambda_i^* \).

The error subsystem from the optimal point is

\[
\begin{align*}
\Delta \dot{x}_i &= -\alpha \nabla f_i(x_i) - \nabla f_i(x_i^*) - \Delta \lambda_i \\
\Delta \dot{\lambda}_i &= -u_i \\
u_i &= \beta \sum_{j=1}^N a_{ij}(t)(\Delta x_j - \Delta x_i)
\end{align*}
\]

\[
\tag{5}
\]

where \( u_i \) is the system input and \( \Delta x_i \) is defined as the output. Let us rewrite \( \nabla f_i(x_i) - \nabla f_i(x_i^*) = B_{x_i} \Delta x_i \) where \( B_{x_i} \) is defined as \( B_{x_i} = \int_0^1 \nabla^2 f_i(x_i^* + \tau(x_i - x_i^*))d\tau. \) It follows that \( \Delta \dot{x}_i = -\alpha B_{x_i} \Delta x_i - \Delta \lambda_i \). Under Assumption 1 we have \( \mu_i I \geq B_{x_i} \geq l_i I. \)

Lemma 1. Under Assumption 2 the error subsystem (5) is IFP(\( \nu \)) from \( u_i \) to \( \Delta x_i \) with index \( \nu_i \geq -\frac{1}{\alpha l_i} \) with respect to the storage function

\[
V_i = \frac{1}{\alpha l_i} \| \Delta x_i \|^2 - \Delta x_i^T \Delta \lambda_i + \alpha \langle f_i(x_i^*) - f_i(x_i) \rangle
\]

Moreover, \( V_i \) is radially unbounded and there exists a constant \( \xi > 0 \) such that

\[
\xi \left\| \left( \alpha B_{x_i} \Delta x_i, \Delta \lambda_i \right) \right\|^2 \leq V_i \leq \frac{2}{\alpha l_i} \left\| \left( \alpha B_{x_i} \Delta x_i, \Delta \lambda_i \right) \right\|^2.
\]

(6)
Proof. To prove (7), let us observe that the strong convexity of \( f_i(x_i) \) provides that
\[
\alpha (f_i(x_i^*) - f_i(x_i)) + \alpha \nabla f_i(x_i)^T \Delta x_i \\
\geq \Delta x_i^T \left( -\alpha B_{x_i} + \frac{\alpha n}{2} \right) \Delta x_i,
\]
and \( \alpha (f_i(x_i^*) - f_i(x_i)) + \alpha \nabla f_i(x_i)^T \Delta x_i \leq -\frac{\alpha n}{2} \| \Delta x_i \|^2. \)
Therefore, it can be derived that
\[
V_i \geq \left( \frac{\alpha B_{x_i} \Delta x_i}{\Delta x_i} \right)^T \left( \frac{\alpha B_{x_i} \Delta x_i}{\Delta x_i} \right) = R_i \\
where R_i = \left( \frac{1}{\alpha} - \frac{\alpha B_{x_i}}{2} \frac{B_{x_i}}{2} \left( \frac{1}{\alpha} - \frac{\alpha B_{x_i}}{2} \frac{B_{x_i}}{2} \right) \right) > 0 \quad \text{and} \quad V_i \leq \left( \frac{\alpha B_{x_i} \Delta x_i}{\Delta x_i} \right)^T \left( \frac{\alpha B_{x_i} \Delta x_i}{\Delta x_i} \right)
\]
where \( R_i = \left( \frac{1}{\alpha} - \frac{\alpha B_{x_i}}{2} \frac{B_{x_i}}{2} \left( \frac{1}{\alpha} - \frac{\alpha B_{x_i}}{2} \frac{B_{x_i}}{2} \right) \right) > 0. \) By some calculations, we obtain that \( R_i \geq \frac{\epsilon_i}{\alpha} = \frac{\epsilon_i}{\alpha} \) and \( R_i < \frac{\epsilon_i}{\alpha}. \) Hence, (7) is verified.

Next, we show that with the storage function \( V_i \), the system (5) is IFP(\( u_i \)) from \( u_i \) to \( \Delta x_i \). It can be obtained that
\[
\dot{V}_i = \frac{1}{\alpha} \cdot \left( d_i \| \Delta x_i \|^2 \right)^2 + \left( d_i \| \Delta x_i \|^2 \right)^2 + \alpha \cdot \left( d_i - d_i \right) \| \Delta x_i \|^2 + \alpha \cdot \left( \| \Delta x_i \|^2 \right)^2 \\
\leq -2 \| \Delta x_i \|^2 + \frac{2}{\alpha} \| \Delta x_i \|^2 \| u_i \|^2 + \| \Delta x_i \|^2 \| u_i \|^2 \\
- \left( \alpha e_{x_i} \Delta x_i + \Delta x_i \right) \| \Delta x_i \|^2 \\
\leq \| \Delta x_i \|^2 \| u_i \|^2 + \frac{2}{\alpha} \| \Delta x_i \|^2 \| u_i \|^2 + \frac{1}{\alpha^2} \| u_i \|^2
\]
Thus, it can be summarized that \( \dot{V}_i \leq \| \Delta x_i \|^2 \| u_i \|^2 + \frac{1}{\alpha^2} \| u_i \|^2 \), which completes the proof. \( \square \)

Lemma 2 (10). Under Assumption 7 and 2 the states of algorithm (4) with initial condition \( \lambda_i(0) \) satisfying \( \sum_{i=1}^{N} \lambda_i(0) = 0 \) will converge to the optimal solution to problem (2) if \( \alpha, \beta \) satisfy
\[
\beta d_i(t) < \frac{1}{\alpha}, \quad \forall t \geq 0
\]
where \( d_i(t) \) denotes the in-degree of the ith agent.

Readers can refer to [10] for the proof. This lemma characterizes the design of parameters \( \alpha, \beta \) through input-feedback passivity.

B. Exponential Convergence Over Strongly Connected Diagraphs

In this subsection, we analyze the exponential convergence of algorithm (4) when the communication graph is strongly connected. Denote \( w = \text{col}(\Delta x, \Delta \lambda), w^* = 0. \)

Lemma 3 (Exponential Convergence). Suppose Assumption 2 and the condition in Lemma 2 are satisfied. In addition, if the graph \( G(t) \) is strongly connected with \( 0 < \delta \leq d_i(t) \leq \bar{d} \) for some constant \( \delta, \bar{d} \), then the states of system (4) will exponentially converge to the optimal solution, i.e., \( \| w(t) - w^* \| \leq C \| w(0) - w^* \| e^{-\frac{\delta}{\min}} \), for any \( t > 0 \) where \( C \geq 1 \) and \( \epsilon > 0 \) are constants.

Proof. Adopt a new Lyapunov function candidate \( V_c := V + \frac{2}{\alpha} \Delta x^T \Delta x \) where \( V = \sum_{i=1}^{N} V_i \) and \( \gamma > 0 \) is to be designed. Denote \( \mu = \min_{\| \xi \| \leq 1} \| \xi \| \), \( l = \max \{ l_i \}. \) Since \( \| \Delta x \|^2 \leq \frac{1}{\gamma^2} \| \Delta x \|^2 \) by (7), we have that \( V_c \) is radially unbounded and satisfies
\[
\| (\alpha B_{\Delta x} \Delta x) \|^2 \leq V_c \leq \left( \frac{2}{\alpha} + \frac{\gamma}{2 \alpha^2} \right) \| (\alpha B_{\Delta x} \Delta x) \|^2
\]
where \( \epsilon = \min_{i} \{ \xi_i \}, B_{\Delta x} := \nabla f(x) - \nabla f(x) \) and \( B_{\Delta x} \in \mathbb{R}^{m \times N \times \mathbb{R}^N} \) is positive definite. First, denote \( \phi = 1 - \frac{1}{1 + \alpha^2 \mu^2} \), where \( \rho > 0 \) is a constant to be designed, then the derivative of \( V_c \) satisfies
\[
\dot{V}_c \leq -\phi \Delta x^T \Delta x - \frac{1}{1 + \alpha^2 \mu^2} \Delta x^T \Delta x + 2 \| \Delta x \| \| u \| + \Delta x^T u \\
- \sum_{i=1}^{N} \left( \frac{1}{\alpha^2 \mu^2} + \rho \right) u_i^T u_i + \sum_{i=1}^{N} \left( \frac{1}{\alpha^2 \mu^2} + \rho \right) u_i^T T u_i \\
\leq -\phi \Delta x^T \Delta x \\
- \sum_{i=1}^{N} \left( \frac{1}{\alpha^2 \mu^2} + \rho \right) d_i(t) \sum_{i=1}^{N} a_{ij} \| x_j - x_i \|^2
\]
where \( \epsilon = \beta \min_{i} \left\{ \left( \frac{1}{2} - \beta \left( \frac{1}{1 + \alpha^2 \mu^2} + \rho \right) \right) \right\} \) and it can be observed that \( \rho < \min_{i} \left\{ \frac{\beta \| d_i(t) \|^2}{\alpha^2 \mu^2} \right\} \) such that \( \epsilon > 0. \) The second inequality follows from the perfect square formula and proof of Lemma 2 (10). Consequently, the derivative of \( V_c \) satisfies
\[
\dot{V}_c \leq -\phi \Delta x^T \Delta x - \epsilon \Delta x^T L(t) \Delta x - \gamma \Delta x^T B_{\Delta x} \Delta x \\
- \gamma \Delta x^T \Delta \lambda \\
\leq -\phi \Delta x^T \Delta x - \gamma \Delta x^T B_{\Delta x} \Delta x - \epsilon s_2 \| x - \bar{x} \|^2 \\
- \gamma (x - \bar{x})^T \Delta \lambda \\
\leq -\phi \Delta x^T \Delta x - \gamma \Delta x^T B_{\Delta x} \Delta x - \epsilon s_2 \| x - \bar{x} \|^2 \\
+ \gamma \left( \frac{\delta}{2} \| x - \bar{x} \|^2 + 1 \right) \| \Delta \lambda \|^2 \\
\leq -\left( \alpha e_{x_i} \Delta x \right)^T \left( \phi \frac{1}{\alpha} + \phi \frac{1}{\alpha} \right) \left( \alpha e_{x_i} \Delta x \right) \\
- \epsilon s_2 \gamma \frac{\delta}{2} \| x - \bar{x} \|^2
\]
where \( x := 1_N \otimes \left( \frac{1}{\alpha} \right) \) is the stacked vector of the average value of \( x_i, \forall i, s_2 := \min_{i \geq 0} s_2 \left( L(t) + L(t) \right) \), the second inequality follows from the null space of \( L(t) \) and \( 1_N \Delta \lambda = 0, \theta > 0 \) is a constant for Young’s inequality. Observe from the above that \( V_c \) is negative definite if the following conditions hold,
\[
2 \epsilon s_2 - \gamma \theta > 0, \quad 2 \theta - \alpha l > 0, \\
2 \phi \theta - \gamma > 0, \quad (2 \theta - \alpha l) \phi - \gamma > 0.
\]
Choose \( \theta = \alpha l, \) then apparently, there exists a \( \gamma \in (0, \min_{i} \{ \alpha d_i \}) \) such that the above conditions are satisfied and \( V_c \) is negative definite. By calculations,
\[
\dot{V}_c \leq -\frac{2}{\alpha^2 \mu^2} \left( \phi \frac{1}{\alpha} + \phi \frac{1}{\alpha} \right) \left( \alpha e_{x_i} \Delta x \right) \\
\leq -\frac{2}{\alpha^2 \mu^2} \left( \phi \frac{1}{\alpha} + \phi \frac{1}{\alpha} \right) \left( \alpha e_{x_i} \Delta x \right)
\]
where $\epsilon > 0$. Then, we have $V_e(t) \leq V_e(0)e^{-\epsilon t}$ and 
\[ \left\| \left( \frac{\alpha^2 + \alpha}{2} \right) \Delta x \right\| \leq \left( \frac{\alpha^2 + \alpha}{2} \right) \left\| \left( \frac{\alpha^2 + \alpha}{2} \right) \Delta x(0) \right\| e^{-\frac{\epsilon}{2}} \quad [14]. \]
Recall that $\|\Delta x\| \leq \frac{1}{\mu} \|\Delta_B x\| \leq \frac{\epsilon}{\mu} \|\Delta x\|$ and $B_z \Delta x = 0$ if and only if $\Delta x = 0$. Finally, we obtain $\|w(t) - w^*\| \leq \frac{\epsilon}{\mu} \left( \frac{\alpha^2 + \alpha}{2} \right) \|w(0) - w^*\| e^{-\frac{\epsilon}{2}}$, for any $t \geq 0$, which completes the proof. 

**Remark 1.** Here, we analyze the convergence from the perspective of passivity. Since algorithm (3) is a simplified version of the algorithm reported in [8], [10], the exponential convergence can also hold for the original one. Moreover, our main contribution lies in the event-trigger communication mechanism for both continuous-time and discrete-time algorithms.

### C. Event-triggered Mechanism

In this subsection, we reconsider the algorithm in (3) by incorporating an event-triggered communication mechanism, i.e.,
\[
\dot{x}_i = -\alpha \nabla f_i(x_i) - \lambda_i \\
\dot{\lambda}_i = -\beta \sum_{j=1}^N a_{ij}(t) (\hat{x}_j - \hat{x}_i) 
\]
where $\hat{x}_j$, $i \in N$ denotes the last transmitted state of agent $i$ that has been transmitted to its neighbors.

The following theorem presents a triggering condition for each agent to update its output while the convergence to the global optimal solution is ensured.

**Theorem 1.** Under Assumption [12] if $\alpha$, $\beta$ are designed such that (9) holds, and the triggering instant for agent $i$, $i \in N$ to transmit its current information of $x_i$ is chosen whenever $\Delta^d_{in}(t) > 0$ and the following condition is satisfied
\[
\|e_i(t)\|^2 \geq \frac{c_i}{d_i^2(t)} \left( \frac{1}{2} - \frac{\beta d^2_{in}(t)}{\alpha^2 \mu_i^2} \right) \sum_{j=1}^N a_{ij}(t) \|\hat{x}_j - \hat{x}_i\|^2
\]
where $e_i(t) = x_i(t) - \hat{x}_i(t)$ and $c_i \in (0, 1)$, then the states of algorithm (10) with initial condition $\lambda_i(0) = 0$ will converge to the optimal solution to problem (2).

**Proof.** Consider the Lyapunov function candidate $V = \sum_{i=1}^N V_i \geq 0$, where $V_i$ is defined in (6), and its derivative yields
\[
\dot{V} \leq \sum_{i=1}^N \Delta x_i^T u_i + \frac{1}{\alpha^2 \mu_i^2} u_i^T u_i \\
= \sum_{i=1}^N \left( \beta \Delta x_i^T \sum_{j=1}^N a_{ij}(t) (\Delta \hat{x}_j - \Delta \hat{x}_i) \right) \\
+ \sum_{i=1}^N \frac{\beta^2}{\alpha^2 \mu_i^2} \sum_{j=1}^N a_{ij}(t) (\Delta \hat{x}_j - \Delta \hat{x}_i) \right\|^2
\]

Next, we have
\[
= \sum_{i=1}^N \left( \beta (\Delta \hat{x}_i + e_i) \sum_{j=1}^N a_{ij}(t) (\Delta \hat{x}_j - \Delta \hat{x}_i) \right) \\
+ \sum_{i=1}^N \frac{\beta^2}{\alpha^2 \mu_i^2} \sum_{j=1}^N a_{ij}(t) (\Delta \hat{x}_j - \Delta \hat{x}_i) \right\|^2
\]

Finally, we obtain
\[
\|w(t) - w^*\| \leq \frac{\epsilon}{\mu} \left( \frac{\alpha^2 + \alpha}{2} \right) \|w(0) - w^*\| e^{-\frac{\epsilon}{2}}, \text{ for any } t \geq 0, \text{ which completes the proof.}
\]

**Remark 2.** Under the event triggering condition (11), each agent broadcasts its current state $x_i$ to its out-neighbors when
a local error signal exceeds a threshold depending on its own cost function and the last received state of $x_j$ from its in-neighbors. Hence, the triggering condition is fully distributed. The triggering condition (11) might not avoid Zeno behavior when $\| \hat{x}_j - \hat{x}_i \|$ gets infinitely small. To avoid this, one can implement the following triggering condition instead,

$$\| e_i(t) \|^2 \geq \max\left\{ \frac{c_i}{\alpha \mu_i}, \left( \frac{1}{2} - \frac{\beta d_i^*(t)}{\mu_i^2} \right)^2 \sum_{j=1}^N a_{ij}(t) \|\hat{x}_j - \hat{x}_i\|^2 \right\}, \zeta$$

where $\zeta > 0$ is a small predefined error. It can be inferred that only practical consensus of $x_i, i \in \mathcal{N}$ can be reached, and a smaller $\zeta$ will result in a more accurate solution.

IV. DISCRETE-TIME ALGORITHM

In this section, we study the discretization of the continuous-time algorithm (5). By applying the forward Euler method to algorithm (5) with respect to a constant stepsize $\delta > 0$, we can obtain the following discrete-time algorithm

$$x_i(k+1) = x_i(k) - \delta \left( \alpha \nabla f_i(x_i(k)) + \lambda_i(k) \right)$$

$$\lambda_i(k+1) = \lambda_i(k) - \delta \beta \sum_{j=1}^N a_{ij}(k) (x_j(k) - x_i(k)).$$

A. IFP Preservation

It is known that the Euler discretization of an exponentially stable dynamical system can achieve convergence given a sufficiently small stepsize [13, 16]. Nevertheless, to ensure convergence under uniformly jointly strongly connected digraphs, we analyze the discrete-time algorithm from the perspective of passivity in this subsection. The associated discrete-time error system with respect to $(x_i^*, \lambda_i^*)$ is

$$\Delta x_i(k+1) = \Delta x_i(k) - \delta \left( \alpha \nabla f_i(x_i(k)) + \Delta \lambda_i(k) \right)$$

$$\Delta \lambda_i(k+1) = \Delta \lambda_i(k) - \delta u_i(k)$$

$$u_i(k) = \beta \sum_{j=1}^N a_{ij}(k) (\Delta x_j(k) - \Delta x_i(k))$$

where $\nabla f_i(x_i(k)) := \nabla f_i(x_i(k)) - \nabla f_i(x_i^*)$, $u_i$ is the system input and the output of system (13) is $\Delta x_i(k)$.

Before characterizing the passivity of (13), let us observe the storage function $V_i(w_i)$ defined in (6) where $w = \text{col}(\Delta x, \Delta \lambda)$. Since $f_i$ is smooth, $\nabla V_i(w_i)$ is locally Lipschitz continuous. Define a sufficiently large constant $D_i$. Then $\| \nabla V_i(w_i) \| \leq M_i \| w_i - w_i' \|$, for all $w_i, w_i' \in \Omega_{D_i} := \{ w \| w \| \leq D_i \}$ where $M_i$ is the Lipschitz constant in $\Omega_{D_i}$. It holds that [17] Theorem 2.1.5

$$V_i(w_i) - V_i(w_i') - (w_i - w_i')^T \nabla V_i(w_i') \leq \frac{M_i}{2} \| w_i - w_i' \|^2$$

for all $w_i, w_i' \in \Omega_{D_i}$.

Lemma 4 (IFP preservation). By selecting a proper stepsize $\delta < \frac{2}{M_i}$, $\forall i \in \mathcal{N}$, the input feedforward passivity is preserved in (13). Namely, system (13) is IFP($\hat{v}_i$) from $u_i$ to $\Delta x_i$ with $\hat{v}_i \geq -\frac{\alpha \mu_i^2}{1 - \frac{\beta}{\mu_i^2}} \frac{\delta}{2}$.

Proof. Adopt the storage function $\hat{V}_i = \frac{1}{2}V_i$, where $V_i$ is defined in (6). By substituting $w_i = w_i(k+1), w_i' = w_i(k)$ into (14), we have

$$\hat{V}_i(w_i(k+1)) - \hat{V}_i(w_i(k)) \leq -\frac{\alpha}{\mu_i}(\frac{1}{2} - \frac{\beta}{\mu_i^2}) \sum_{j=1}^N a_{ij}(k) \|\hat{x}_j - \hat{x}_i\|^2 + \frac{M_i}{2} \| u_i(k) \|^2$$

where $z_i = \alpha \nabla \hat{f}_i(x_i(k)) + \Delta \lambda_i(k)$.

It can be observed that Lemma 4 characterizes the passivity degradation over discretization. The lower bound of the IFP index decreases as the stepsize grows larger. If the stepsize $\delta$ is infinitely small, then $\lim_{\delta \to 0^+} \left( \frac{1}{\alpha \mu_i^2} - \frac{1}{\mu_i^2} - \frac{M_i}{2} \right)^{-1} = -\frac{\beta}{\mu_i^2}$, which recovers the IFP index for the continuous-time system.

Theorem 2. Under Assumption 7 and 2, the states of algorithm (13) with initial condition $\lambda_i(0)$ satisfying $\sum_{i=1}^N \lambda_i(0) = 0$ will converge to the optimal solution to problem 2 if the stepsize $\delta < \frac{2}{M_i}, \forall i \in \mathcal{N}$, and $\alpha, \beta$ satisfy

$$\left( \frac{1}{\alpha \mu_i^2} - \frac{1}{\mu_i^2} - \frac{M_i}{2} \right)^{-1} \beta d_{in}(k) < \frac{1}{2}, \forall i \in \mathcal{N}, \forall k \geq 0$$

where $d_{in}(k)$ denotes the in-degree of the $i$th agent.

The proof is similar to Lemma 4 by considering the discrete-time Lyapunov function candidate $V = \sum_{i=1}^N \hat{V}_i$ where $V_i$ is defined in the proof of Lemma 4.

Remark 3 (Semiglobal Convergence). Theorem 2 provides a semiglobal convergence result [8]. Since $\nabla V$ is continuously differentiable everywhere, one can always find a large enough compact set $\Omega_D$ with constant Lipschitz index $M$ and a small enough stepsize $\delta$ such that the compact set is contained in the region of attraction of the equilibrium point.

B. Discrete-time Event-triggered Mechanism

Similarly, let us consider the discrete-time algorithm incorporating the same event-triggered mechanism, i.e.,

$$x_i(k+1) = x_i(k) - \delta \left( \alpha \nabla f_i(x_i(k)) + \lambda_i(k) \right)$$

$$\lambda_i(k+1) = \lambda_i(k) - \delta \beta \sum_{j=1}^N a_{ij}(k) (\hat{x}_j(k) - \hat{x}_i(k))$$

where $\hat{x}_i(k), i \in \mathcal{N}$ denotes the last sampled state of agent $i$ sent to its neighbors until time $k$. Then we have the following theorem on the convergence of discrete-time algorithm under event-triggered communication.

Theorem 3. Under Assumption 7 and 2 if the stepsize satisfies $\delta < \frac{2}{M_i}, \forall i \in \mathcal{N}, \alpha, \beta$ are designed such that (15) holds, and the triggering instant for agent $i, i \in \mathcal{N}$ to transmit its neighbor.
current information of $x_i$ is chosen whenever $d_{in}^i(k) > 0$ and the following condition is satisfied
\[
\|e_i(k)\|^2 \geq \frac{\delta c_i}{d_{in}^i(k)} \left[ \frac{1}{2} - \left( \frac{1}{\alpha \mu^2} \frac{1}{1 - \mu^2} + \frac{M_i \lambda}{2} \right) \beta d_{in}^i(k) \right]^2 \sum_{j=1}^{N} a_{ij}(k) \|\hat{x}_j(k) - \hat{x}_i(k)\|^2
\]
where $e_i(k) = x_i(k) - \hat{x}_i(k)$ and $c_i \in (0, 1)$, then the states of algorithm (16) with initial condition $\lambda_i(0)$ satisfying $\sum_{i=1}^{N} \lambda_i(0) = 0$ will converge to the optimal solution to problem (2).

V. NUMERICAL EXAMPLE

In this section, we provide a numerical example to illustrate the proposed event-triggered mechanism over continuous-time algorithm. Consider the distributed optimization problem (2) among 5 agents over a weight-balanced and uniformly jointly strongly connected digraph that is switching every two seconds among two modes, as shown in Figure 1. The local objective functions are
\[
\begin{align*}
    f_1(x) &= \frac{1}{2} x^2 + 3x + 1, \\
    f_2(x) &= \frac{1}{2} x^2 - x, \\
    f_3(x) &= x^2 + \sin x, \\
    f_4(x) &= \ln(e^{2x} + 1) + 0.5 x^2, \\
    f_5(x) &= \ln(e^{2x} + e^{-0.2x}) + 0.6 x^2.
\end{align*}
\]
We obtain that these functions are strongly convex with $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1$, $\mu_5 = 1.2$ and have Lipschitz gradient with $l_1 = l_2 = 1$, $l_3 = 3$, $l_4 = 2$ and $l_5 = 2.41$. Let $\alpha = 1$, then we obtain from Lemma 2 that $0 < \beta < 0.5$. Select $\beta = 0.2$, $c_i = 0.99$, initial conditions $x_i \in [0, 1]$, $\lambda_i(0) = 0$ and apply the continuous-time algorithm (10) under event-triggered control laws (11) in MATLAB. The trajectories of $x_i(t)$ and trigger instant of $x_i$ under event-triggered communication are shown in Figure 2. It can be observed that the states converge to the optimal solution while the communication effort is greatly reduced due to both the jointly strongly connected graph and the event-triggered mechanism.

VI. CONCLUSION

We have first analyzed a distributed continuous-time algorithm over uniformly jointly strongly connected balanced digraphs and shown its exponential convergence over strongly connected digraphs. Then, an event-trigger communication mechanism for the distributed continuous-time algorithms and its discrete-time counterpart has been proposed via the property of IFP.

REFERENCES

[1] A. Nedić and A. Olshevsky, “Distributed optimization over time-varying directed graphs,” IEEE Transactions on Automatic Control, vol. 60, no. 3, pp. 601–615, 2014.
[2] P. Xie, K. You, R. Tempo, S. Song, and C. Wu, “Distributed convex optimization with inequality constraints over time-varying unbalanced directed graphs,” IEEE Transactions on Automatic Control, vol. 63, no. 12, pp. 4331–4337, 2018.
[3] H. Li, Q. Lü, and T. Huang, “Distributed projection subgradient algorithm over time-varying general unbalanced directed graphs,” IEEE Transactions on Automatic Control, vol. 64, no. 3, pp. 1309–1316, 2018.
[4] G. Scutari and Y. Sun, “Distributed nonconvex constrained optimization over time-varying digraphs,” Mathematical Programming, vol. 176, no. 1-2, pp. 497–544, 2019.
[5] B. Gharesifard and J. Cortés, “Distributed continuous-time convex optimization on weight-balanced digraphs,” IEEE Transactions on Automatic Control, vol. 59, no. 3, pp. 781–786, 2013.
[6] Z. Li, Z. Ding, J. Sun, and Z. Li, “Distributed adaptive convex optimization on directed graphs via continuous-time algorithms,” IEEE Transactions on Automatic Control, vol. 63, no. 5, pp. 1434–1441, 2017.
[7] M. Li, “Generalized Lagrange multiplier method and KKT conditions with an application to distributed optimization,” IEEE Transactions on Circuits and Systems II: Express Briefs, vol. 66, no. 2, pp. 252–258, 2019.
[8] S. S. Kia, J. Cortés, and S. Martínez, “Distributed convex optimization via continuous-time coordination algorithms with discrete-time communication,” Automatica, vol. 55, pp. 254–264, 2015.
[9] B. Touri and B. Gharesifard, “A modified saddle-point dynamics for distributed convex optimization on general directed graphs,” IEEE Transactions on Automatic Control, 2019.
[10] M. Li, G. Chesi, and Y. Hong, “Input-feedforward-passivity-based distributed optimization over jointly connected balanced digraphs,” arXiv preprint arXiv:1905.03468, 2019.
[11] S. Liu, L. Xie, and D. E. Quevedo, “Event-triggered quantized communication-based distributed convex optimization,” IEEE Transactions on Control of Network Systems, vol. 5, no. 1, pp. 167–178, 2016.
[12] Y. Kajiyama, N. Hayashi, and S. Takai, “Distributed subgradient method with edge-based event-triggered communication,” IEEE Transactions on Automatic Control, vol. 63, no. 7, pp. 2248–2255, 2018.
[13] L. Su, M. Li, V. Gupta, and G. Chesi, “Distributed resource allocation over time-varying balanced digraphs with discrete-time communication,” arXiv preprint arXiv:1907.13003, 2019.
[14] H. K. Khalil, “Nonlinear systems,” Prentice-Hall, New Jersey, 1996.
[15] G. Qi and N. Li, “On the exponential stability of primal-dual gradient dynamics,” IEEE Control Systems Letters, vol. 3, no. 1, pp. 43–48, 2018.
[16] H. J. Stetter, Analysis of discretization methods for ordinary differential equations. Springer, 1973, vol. 23.
[17] Y. Nesterov, Introductory lectures on convex optimization: A basic course. Springer Science & Business Media, 2013, vol. 87.