Free-Boundary Problems for Holomorphic Curves in the 6-Sphere

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Abstract

We remark on two different free-boundary problems for holomorphic curves in nearly-Kähler 6-manifolds. First, we observe that a holomorphic curve in a geodesic ball $B$ of the round 6-sphere that meets $\partial B$ orthogonally must be totally geodesic. Consequently, we obtain rigidity results for reflection-invariant holomorphic curves in $S^6$ and associative cones in $\mathbb{R}^7$.

Second, we consider holomorphic curves with boundary on a Lagrangian submanifold in a strict nearly-Kähler 6-manifold. By deriving a suitable second variation formula for area, we observe a topological lower bound on the Morse index. In both settings, our methods are complex-geometric, closely following arguments of Fraser-Schoen and Chen-Fraser.

1 Introduction

The 6-sphere is the only $n$-sphere for $n \geq 3$ that admits an almost complex structure. Viewing $S^6$ as the unit sphere in $\mathbb{R}^7$, the standard almost-complex structure $J$ at $p \in S^6$ is given by $J_p(v) = p \times v$, where $\times$ is the usual the 7-dimensional cross product on the imaginary octonions $\text{Im}(\mathbb{O}) = \mathbb{R}^7$. While $J$ is not integrable, it does act by isometries with respect to the round metric and is invariant under the compact Lie group $G_2$. An immersed surface $u: \Sigma^2 \rightarrow S^6$ is a holomorphic curve (or complex curve) if each tangent space is $J$-invariant:

$$J(T_p\Sigma) = T_p\Sigma, \quad \forall p \in \Sigma.$$ 

Holomorphic curves in $S^6$ are minimal surfaces. One way to see this is via $G_2$ geometry. Namely, a surface $\Sigma \subset S^6$ is a holomorphic curve if and only if its (3-dimensional) cone $C(\Sigma) := \{rp \in \mathbb{R}^7: \ r > 0, \ p \in \Sigma\} \subset \mathbb{R}^7$ is an associative 3-fold, one of the four classes of calibrated geometries discovered in the pioneering work of Harvey and Lawson [14]. Associative 3-folds are fundamental objects in $G_2$ geometry, and holomorphic curves in $S^6$ model their conical singularities. In sum, if $\Sigma \subset S^6$ is a holomorphic curve, then $C(\Sigma) \subset \mathbb{R}^7$ is an associative 3-fold, so $C(\Sigma)$ is homologically volume-minimizing, and thus $\Sigma$ is a minimal surface.

There are by now very many studies of holomorphic curves in the 6-sphere (see the surveys [3, §19] and [15, §12.1-§12.3]), though research in this direction is still ongoing (see, e.g., the recent works [20], [5], [19]). Thus far, global questions have primarily concerned closed surfaces. In this paper, by contrast, we study holomorphic curves with boundary.

Now, the last decade has seen great advances in the subject of minimal surfaces with boundary, particularly in the context of the free-boundary condition. A minimal surface $u: \Sigma^2 \rightarrow B$ in a domain $B$ is free-boundary if $u(\partial \Sigma) \subset \partial B$ and $u(\Sigma)$ intersects $\partial B$ orthogonally. The orthogonality requirement here arises naturally from the first variation of area. We will not attempt to survey this rapidly developing field of study — an excellent recent overview is [16] — but simply mention a particular result of interest.
In 2015, Fraser and Schoen [9] proved a remarkable rigidity theorem. They showed that any free-boundary minimal 2-disk $u: D^2 \to B$ in a geodesic ball $B$ of a real space form must be totally geodesic. Thus, in geodesic balls $B$ of round spheres $S^n$, any free-boundary minimal surface must have non-trivial topology. In the case of $S^6$, it is natural to ask what happens if the topological assumption ($\Sigma$ is a 2-disk) is replaced by a geometric one (viz., $u(\Sigma)$ holomorphic). In §3, we show:

**Theorem 1.1.** Let $u: \Sigma^2 \to B$ be a compact immersed surface in a geodesic ball $B$ of the round 6-sphere with $u(\partial \Sigma) \subset \partial B$. If $u$ is a holomorphic curve, and if $u(\Sigma)$ meets $\partial B$ orthogonally, then $u(\Sigma)$ is totally geodesic.

Note that Theorem 1.1 makes no topological assumptions about $\Sigma$; that $\Sigma$, if connected, is homeomorphic to a 2-disk is part of the conclusion. Moreover, Theorem 1.1 immediately yields a rigidity result for closed, connected holomorphic curves $\Sigma$ in $S^6 \subset \mathbb{R}^7$ that are reflection-invariant across a hyperplane, say $P = \{x_7 = 0\} \subset \mathbb{R}^7$. Indeed, if such $\Sigma$ meets $P$ transversely, then $\Sigma \cap \{x_7 \geq 0\}$ is free-boundary in the upper hemisphere, and hence is totally-geodesic. That is:

**Corollary 1.2.** Let $u: \Sigma^2 \to S^6$ be a holomorphic curve, where $\Sigma^2$ is a closed connected surface, and let $P$ denote a totally geodesic 5-sphere in $S^6$. If $u(\Sigma)$ is invariant under reflection in $P$, and if the intersection of $u(\Sigma)$ with $P$ is transverse, then $u(\Sigma)$ is a totally geodesic 2-sphere.

**Corollary 1.3.** Let $f: N^3 \to \mathbb{R}^7$ be an associative cone whose link in $S^6$ is a closed connected surface, and let $P \subset \mathbb{R}^7$ be a 6-plane. If $f(N)$ is invariant under reflection in $P$, and if the intersection of $f(N)$ with $P$ is transverse, then $f(N)$ is a 3-plane.

We now turn to the Lagrangian free-boundary problem. For holomorphic curves in symplectic manifolds, this is a well-studied boundary condition with a wealth of applications [13], [7], [23]. While the 6-sphere is not symplectic, the pair $(\langle \cdot, \cdot \rangle, J)$ consisting of the round metric and standard almost-complex structure is, in fact, the prototype of a “strictly nearly-Kähler” structure.

In general, recall that an almost-Hermitian 6-manifold $(M^6, \langle \cdot, \cdot \rangle, J, \omega)$, consisting of a metric $\langle \cdot, \cdot \rangle$, compatible almost-complex structure $J$, and 2-form $\omega(X, Y) := \langle JX, Y \rangle$ is nearly-Kähler if

$$\langle \nabla_X J \rangle(Y) = -\langle \nabla_Y J \rangle(X), \quad \forall X, Y \in \Gamma(TM),$$

where $\nabla$ is the Levi-Civita connection. By definition, a Kähler 6-manifold ($\nabla J = 0$) is nearly-Kähler. A nearly-Kähler 6-manifold is strict if it is not Kähler. Strict nearly-Kähler 6-manifolds are of great importance to $G_2$-geometry as models of $G_2$-holonomy cones [1].

In §4, we study compact holomorphic curves $u: \Sigma \to M^6$ with boundary $u(\partial \Sigma) \subset L$ on a fixed Lagrangian 3-fold $L \subset M$ in a nearly-Kähler 6-manifold $M$. For normal variations $\eta \in \Gamma(T\Sigma)$ with $\eta$ tangent to $L$ along $\partial \Sigma$, we prove the second variation of area formula:

$$(\delta^2 A)(\eta) = \int_\Sigma \frac{1}{2} \| \mathcal{D} \eta \|^2 + \frac{1}{3} d\omega(e, J\eta, \mathcal{D} e \eta) - 2\lambda^2 \| \eta \|^2. \quad (1.1)$$

Here, $\mathcal{D}$ is the $J$-antilinear part of the normal connection, $e \in \Gamma(T\Sigma)$ is any (local) unit vector field, and $\lambda \geq 0$ is a constant (the type of $M$) having $\lambda > 0$ if and only if $M^6$ is strict. A remarkable feature of this formula is that it contains no boundary integral, a consequence of the trinity of conditions “$M$ nearly-Kähler,” “$u$ holomorphic,” and “$L$ Lagrangian.” Using (1.1), together with a suitable index formula, we deduce the following topological lower bound on the Morse index:

**Theorem 1.4.** Let $u: \Sigma \to M^6$ be a holomorphic curve in a strictly nearly-Kähler 6-manifold, where $\Sigma$ is compact orientable surface with boundary, such that $u(\partial \Sigma) \subset L$ for some Lagrangian submanifold $L \subset M$. Then the Morse index of $u$ for the area functional satisfies

$$\text{Ind}(u) \geq \mu(u^*(TM), TL), \quad (1.2)$$
where $\mu(u^*(TM), TL) \in \mathbb{Z}$ is the boundary Maslov index of the bundle pair $(u^*(TM), TL) \to (\Sigma, \partial \Sigma)$.

**Remark.** In the edge case of $\partial \Sigma = \emptyset$, the bound (1.2) gives no information, as that situation has $\mu(u^*(TM), \emptyset) = 2c_1(u^*(TM)) = 0$. More generally, as the boundary Maslov index is integer-valued, the bound (1.2) is vacuous whenever $\mu(u^*(TM), TL) \leq 0$. It is not clear to the author how often this occurs. In fact, while there are by now plenty of examples of holomorphic curves and Lagrangian submanifolds (see [3, §19], [15, §12], [18], [26] and references therein) in strict nearly-Kähler 6-manifolds, there presently seem to be few explicit examples of pairs $(u(\Sigma), L)$ of holomorphic curves $u(\Sigma)$ with Lagrangian boundary $L$. Constructing such pairs is an interesting direction for future research.

**Remark.** Recently, Pacini [24] discovered a beautiful Chern-Weil type integral formula for the boundary Maslov index. This formula may yield more precise information on the bound (1.2) in various special cases (e.g., if $M^6 = S^6$, say). We leave this to the interested reader.

This work is organized as follows. In §2, we establish conventions and fix notation. We also review the basics of nearly-Kähler manifolds and holomorphic curves for readers unfamiliar with these subjects. In §3, we study holomorphic curves in geodesic balls of $S^6$. We prove Theorem 1.1 by a short Hopf differential argument analogous to that of Fraser–Schoen [9].

In §4 we study holomorphic curves with Lagrangian boundary. In §4.1, we derive the second variation formula (1.1), and in §4.3 we deduce Theorem 1.4 by appealing to a suitable generalization of the Riemann-Roch formula. Our arguments follow those of Chen–Fraser [4] who study holomorphic curves with Lagrangian boundary in (Kähler) complex projective spaces. Note that sections 3 and 4 can be read independently of one another, although both rely on §2.

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## 2 Preliminaries

### 2.1 Nearly-Kähler $2n$-Manifolds

Let $M^{2n}$ be an *almost-Hermitian* $2n$-manifold of real dimension $2n \geq 6$, meaning that $M^{2n}$ is equipped with a triple $(\langle \cdot, \cdot \rangle, J, \omega)$ consisting of a Riemannian metric $\langle \cdot, \cdot \rangle$, an orthogonal almost-complex structure $J$, and a non-degenerate 2-form $\omega \in \Omega^2(M)$ given by $\omega(X, Y) = \langle JX, Y \rangle$. We say that $M^{2n}$ is Kähler if any of the following equivalent conditions are satisfied:

$$\nabla J = 0 \iff \nabla \omega = 0 \iff J \text{ integrable and } \omega \text{ closed},$$

where $\nabla$ is the Levi-Civita connection of $\langle \cdot, \cdot \rangle$. The obstruction to $M^{2n}$ being Kähler is measured by the *intrinsic torsion* tensor $P = \nabla J \in \Gamma(T^*M \otimes T^*M \otimes TM)$ given by

$$P(X, Y) = (\nabla_X J)(Y).$$

Note that $P$ satisfies the symmetries

$$P(X, JY) = -JP(X, Y) \quad \quad \langle P(X, Y), Z \rangle = -\langle P(X, Z), Y \rangle. \quad \quad (2.1)$$

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Note also that $P = \nabla J$ and $\nabla \omega$ carry essentially equivalent information, in view of the fact that

$$
(\nabla_X \omega)(Y, Z) = \langle (\nabla_X J)(Y), Z \rangle = \langle P(X, Y), Z \rangle.
$$

Fix $m \in M$, abbreviate $T = T_m M$, and note that the standard $U(n)$-representation on $T \cong \mathbb{R}^{2n}$ induces a $U(n)$-representation on $T^* \otimes T^* \otimes T$, and hence on the $U(n)$-invariant subspace $E \subset T^* \otimes T^* \otimes T$ consisting of tensors with the symmetries (2.1). In a now classical paper [12], Gray and Hervella showed (for $n \geq 3$) that $E$ decomposes into four irreducible $U(n)$-submodules

$$
E \cong V_1 \oplus V_2 \oplus V_3 \oplus V_4
$$

of real dimensions

$$
\frac{1}{3} n(n-1)(n-2), \quad \frac{2}{3} n(n-1)+1, \quad n(n+1)(n-2), \quad 2n
$$

respectively.

Letting $V_1, \ldots, V_4 \subset T^*M \otimes T^*M \otimes TM$ denote the vector subbundles over $M$ corresponding to the subspaces $V_1, \ldots, V_4 \subset T^* \otimes T^* \otimes T$, respectively, Gray and Hervella further showed that:

\begin{align*}
\nabla J &\in \Gamma(V_1) \iff (\nabla_X J)(Y) = -(\nabla_Y J)(X), \quad \forall X, Y \in TM \\
\nabla J &\in \Gamma(V_2) \iff d\omega = 0 \\
\nabla J &\in \Gamma(V_3 \oplus V_4) \iff J \text{ integrable.}
\end{align*}

We say that $M$ is nearly-Kähler if $\nabla J \in \Gamma(V_1)$, i.e.:

$$
P(X, Y) = -P(Y, X), \quad \forall X, Y \in TM.
$$

An equivalent condition [12] is that $\nabla \omega = \frac{1}{4} d\omega$. In particular, every Kähler manifold is nearly-Kähler. More precisely, the Kähler manifolds are exactly those nearly-Kähler manifolds with $J$ integrable (or, equivalently, with $d\omega = 0$).

A nearly-Kähler manifold is said to be strict if $\nabla_X J \neq 0$ for all non-zero $X \in TM$. In particular, every strict nearly-Kähler manifold is not Kähler. In real dimension 6, Gray showed [11, Theorem 5.2] that the converse holds: If $M^6$ is nearly-Kähler and not Kähler, then there exists a constant $\lambda > 0$, called the type, for which

$$
\| (\nabla_X J)(Y) \|^2 = \lambda^2 \left[ \| X \|^2 \| Y \|^2 - \langle X, Y \rangle^2 - \langle JX, JY \rangle^2 \right] \quad (2.2)
$$

for all $X, Y \in TM$, and hence $M$ is strict. Of course, if $M^6$ is Kähler, then equation (2.2) is still true with $\lambda = 0$.

If $M^{2n}$ is a nearly-Kähler $2n$-manifold, its Riemann curvature tensor $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z$ enjoys several symmetries with respect to the almost-complex structure $J$. A convenient list of identities is given in [11, §2], though we caution that Gray uses the opposite sign convention for $R$. In this work, we will only need the identity

$$
R(X, Y, Z, X) + R(X, JY, JZ, X) + R(X, JX, Y, JZ) = 2 \langle P(X, Y), P(X, Z) \rangle
$$

given in [11, (2.4)]. In particular, setting $Y = Z$ yields the relation

$$
R(X, Y, Y, X) + R(X, JY, JY, X) + R(X, JX, Y, JY) = 2 \| P(X, Y) \|^2. \quad (2.3)
$$
2.2 Nearly-Kähler 6-Manifolds and $G_2$ Geometry

Among nearly-Kähler manifolds, the strict nearly-Kähler 6-manifolds $M^6$ are particularly special. In that case, the 3-form $\frac{1}{3\lambda} d\omega$ gives rise to a complex volume form on $M^6$, which in turn leads to a relationship with $G_2$-holonomy cones. The purpose of this section is to explain this point.

2.2.1 Elements of $G_2$ Geometry

Let $X^7$ be a smooth 7-manifold. A $G_2$-structure on $X^7$ is a 3-form $\phi \in \Omega^3(X)$ such that at each $x \in X$, the symmetric bilinear form $B_{\phi} : \text{Sym}^2(T_x X) \to \Lambda^7(T^*_x X)$ given by

$$B_{\phi}(v,w) := \frac{1}{6} (v \lrcorner \phi) \wedge (w \lrcorner \phi) \wedge \phi$$

is definite. It is well-known that $X$ admits a $G_2$-structure if and only if $X$ is orientable and spin.

Moreover, a $G_2$-structure induces a Riemannian metric $g_{\phi}$ and orientation $\text{vol}_{\phi}$ according to

$$g_{\phi}(v,w) \text{vol}_{\phi} = \frac{1}{6} (v \lrcorner \phi) \wedge (w \lrcorner \phi) \wedge \phi \quad \text{vol}_{\phi} = \phi \wedge *\phi.$$

A $G_2$-structure $\phi$ is called torsion-free if it is closed and co-closed — i.e., if $d\phi = 0$ and $d(*\phi) = 0$. The relationship between torsion-free $G_2$-structures and Riemannian metrics with $G_2$ holonomy is given by the following theorem of Fernández and Gray:

**Theorem 2.1** ([6]). Let $X^7$ be orientable and spin. If $\phi \in \Omega^3(X)$ is a torsion-free $G_2$-structure on $X$, then its induced metric $g_{\phi}$ has $\text{Hol}(g_{\phi}) \leq G_2$. Conversely, if $g$ is a Riemannian metric on $X$ with $\text{Hol}(g) \leq G_2$, then $g = g_{\phi}$ for some torsion-free $G_2$-structure $\phi$ on $X$.

Let $X$ have a $G_2$-structure $\phi \in \Omega^3(M)$. A 3-dimensional submanifold $N^3 \subset X^7$ is called associative if

$$\phi|_N = \text{vol}_N.$$

If $d\phi = 0$, then $\phi$ is a calibration and its calibrated submanifolds are precisely the associative 3-folds. Thus, in this case, associatives are homologically volume-minimizing. Associative 3-folds are fundamental objects in $G_2$-geometry, and are in many ways analogous to holomorphic curves in symplectic geometry.

2.2.2 Strict Nearly-Kähler 6-Manifolds and $G_2$ Holonomy Cones

On an almost-Hermitian manifold $(M^{2n}, \langle \cdot, \cdot \rangle, J, \omega)$, a complex volume form is an $(n,0)$-form $\Upsilon$ satisfying the normalization

$$(-1)^{n(n-1)/2} \left( \frac{i}{2^n} \right)^n \Upsilon \wedge \overline{\Upsilon} = \text{vol}_M.$$

If $(M^6, J, \omega)$ is a strict nearly-Kähler 6-manifold, then it is well-known [25, §4.3] that each of the 3-forms (for a constant $\theta \in [0, 2\pi]$)

$$\Upsilon_\theta(X,Y,Z) := \frac{e^{i\theta}}{3\lambda} (d\omega(X,Y,Z) - i d\omega(X,Y,JZ))$$

is a complex volume on $M$. Moreover, the pair $(\omega, \Upsilon_\theta)$ satisfies the differential equations

$$d\omega = 3\lambda (\cos(\theta) \text{Re}(\Upsilon_\theta) + \sin(\theta) \text{Im}(\Upsilon_\theta))$$

$$d\text{Re}(\Upsilon_\theta) = 2\lambda \sin(\theta) \omega \wedge \omega$$

$$d\text{Im}(\Upsilon_\theta) = -2\lambda \cos(\theta) \omega \wedge \omega.$$
Remark. In §2.1, we defined “nearly-Kähler” as a particular class of almost-Hermitian structures \((\langle \cdot, \cdot \rangle, J, \omega)\). This is the original definition used by Gray [10]. However, in real dimension 6, it has recently become common for authors to use the term “nearly-Kähler 6-manifold” to mean a “strict nearly-Kähler 6-manifold together with a fixed \(\Upsilon_0\) (usually \(\Upsilon_0\) or \(\Upsilon_{\pi/2}\) and scaled to have \(\lambda = 1\).”

For a Riemannian manifold \((M, g)\), recall that its metric cone is the Riemannian manifold
\[
C(M) := (\mathbb{R}^+ \times M, dr^2 + r^2 g).
\]
We will routinely identify \(M\) with the subset \(\{1\} \times M \subset C(M)\) and call \(M\) the link of \(C(M)\). The relationship between strict nearly-Kähler 6-manifolds and \(G_2\) geometry is given by:

**Theorem 2.2** ([1]). If \(M^6\) is a strict nearly-Kähler 6-manifold, scaled to have \(\lambda = 1\), then \(C(M)\) admits a torsion-free \(G_2\)-structure. Conversely, if \(C(M)\) is a 7-dimensional metric cone with a torsion-free \(G_2\)-structure, then its link \(M^6\) admits a strict nearly-Kähler structure with \(\lambda = 1\).

Since manifolds with holonomy contained in \(G_2\) are Ricci-flat, and since the links of Ricci-flat cones are Einstein of positive scalar curvature, it follows that:

**Corollary 2.3** ([11]). Every strict nearly-Kähler 6-manifold is Einstein of positive scalar curvature.

Here is a sketch of Theorem 2.2. If \((M^6, \langle \cdot, \cdot \rangle, J, \omega)\) is strictly nearly-Kähler with \(\lambda = 1\), then define a 3-form \(\phi \in \Omega^3(C(M))\) and 4-form \(\psi \in \Omega^4(C(M))\) via
\[
\phi := r^2 dr \wedge \omega + r^3 \text{Re}(\Upsilon_0) \quad \text{and} \quad \psi := -r^3 dr \wedge \text{Im}(\Upsilon_0) + \frac{1}{2} r^4 \omega \wedge \omega.
\]
Linear algebra shows that \(\phi\) is a \(G_2\)-structure on \(C(M)\) and that \(\psi = \ast \phi\). Equations (2.4) for \((\omega, \Upsilon_0)\) imply that \(\phi = d\left(\frac{r^4}{3} \omega\right)\) and \(\psi = d\left(-\frac{r^4}{4} \text{Im}(\Upsilon_0)\right)\), so \(\phi\) is torsion-free.

Conversely, if \(\phi \in \Omega^3(C(M))\) is a torsion-free \(G_2\)-structure, then one can define a 2-form \(\omega \in \Omega^2(M)\) and a complex 3-form \(\Upsilon_0 \in \Omega^3(M; \mathbb{C})\) via
\[
\omega := (\partial_r \ast \phi)|_M \quad \text{and} \quad \Upsilon_0 := (\phi - i \partial_r \ast \phi)|_M.
\]
Linear algebra shows that \(\omega\) is a non-degenerate 2-form that is \(\langle \cdot, \cdot \rangle\)-compatible, so that defining \(J\) via \(\langle X, JY \rangle = \omega(X, Y)\) makes \((\langle \cdot, \cdot \rangle, J, \omega)\) an almost-Hermitian structure on \(M\). Moreover, \(\Upsilon_0\) is a complex volume form with respect to \(J\). The torsion-free condition can then be used to calculate \(d\omega = 3 \text{Re}(\Upsilon_0)\), from which one can deduce that \((\langle \cdot, \cdot \rangle, J, \omega)\) is strictly nearly-Kähler with \(\lambda = 1\). This concludes the sketch. For more details, see [1, §7], [22], [27, §2].

### 2.3 Holomorphic Curves in Nearly-Kähler 6-Manifolds

We now define our primary objects of interest. In an almost-Hermitian manifold \((M^{2n}, \langle \cdot, \cdot \rangle, J, \omega)\), a holomorphic curve is an immersion \(u: \Sigma^2 \to M^{2n}\) such that each tangent space is \(J\)-invariant:
\[
J(T_p \Sigma) = T_p \Sigma, \quad \forall p \in \Sigma.
\]
An equivalent condition is that
\[
\omega|_\Sigma = \text{vol}_\Sigma
\]
where \(\text{vol}_\Sigma\) is the volume form on \(\Sigma\). If \(d\omega = 0\), then \(\omega\) is a calibration, and hence holomorphic curves are homologically area-minimizing.

Now, if \(M^6\) is a strict nearly-Kähler 6-manifold, then \(\omega\) is not closed, and hence not a calibration. Nevertheless, the holomorphic curves in \(M^6\) are rather special geometric objects, as the following well-known fact shows:
Proposition 2.4. Let $M^6$ be a strict nearly-Kähler 6-manifold. Let $\Sigma^2 \subset M^6$ be an immersed oriented surface. Then $\Sigma^2 \subset M^6$ is a holomorphic curve if and only if $C(\Sigma) \subset C(M)$ is an associative 3-fold.

Consequently, if $\Sigma^2 \subset M^6$ is a holomorphic curve, then $C(\Sigma) \subset C(M)$ is a (3-dimensional) calibrated submanifold, and hence $\Sigma$ is a minimal surface in $M$.

We now set our conventions for immersed submanifolds $u: \Sigma^k \to M^n$ of a Riemannian manifold $(M^n, \langle \cdot, \cdot \rangle)$. In general, we let $\nabla$ denote the Levi-Civita connection on $M$, and split $u^*(TM) = T\Sigma \oplus N\Sigma$ into tangential and normal parts. For $X, Y \in \Gamma(T\Sigma)$ and $\eta \in \Gamma(N\Sigma)$, we have the decompositions

$$
\nabla_X Y = \nabla^\top_X Y + \mathbb{I}(X,Y) \\
\nabla_X \eta = W_X \eta + \nabla^\perp_X \eta
$$

where $\nabla^\top$ is the Levi-Civita connection on $\Sigma$, where $\nabla^\perp$ is the normal connection, where $\mathbb{I}$ is the second fundamental form, and where $W$ is the shape operator satisfying the Weingarten equation

$$
(\langle W_X N, Y \rangle) = -\langle \mathbb{I}(X,Y), N \rangle.
$$

The curvature tensors of $\nabla, \nabla^\top, \nabla^\perp$ will be denoted by $R, R^\top, R^\perp$, respectively. In particular, we recall the Ricci equation

$$
R(X,Y,\eta,\xi) = R^\perp(X,Y,\eta,\xi) + \langle W_X \eta, W_Y \xi \rangle - \langle W_X \xi, W_Y \eta \rangle
$$

for $X, Y \in T\Sigma$ and $\eta, \xi \in N\Sigma$.

Suppose now that $u: \Sigma^2 \rightarrow M^6$ is an immersed holomorphic curve in a nearly-Kähler 6-manifold $(M^6, \langle \cdot, \cdot \rangle, J, \omega)$. In this case, it is easy to check that the second fundamental form enjoys the symmetries

$$
\mathbb{I}(X,JY) = \mathbb{I}(JX,Y) \\
\mathbb{I}(X,JY) = J\mathbb{I}(X,Y).
$$

When written in terms of the shape operator, these symmetries take the equivalent form

$$
W_{JX}\eta = -JW_X\eta \\
W_X(J\eta) = J(W_X\eta)
$$

for $X \in T\Sigma$ and $\eta \in N\Sigma$. Note that the identity $\mathbb{I}(X,JY) = \mathbb{I}(JX,Y)$ is true more generally for any (2-dimensional) oriented minimal surface in a Riemannian manifold, where $J$ is the complex structure on $T\Sigma$ induced from the metric and orientation.

Finally, we remark for future use that the restriction of $P: TM \times TM \to TM$ to $T\Sigma \times N\Sigma$ maps into $N\Sigma$. Indeed, by virtue of the symmetry $W_X(J\eta) = J(W_X\eta)$, the restricted map

$$
P: T\Sigma \times N\Sigma \to N\Sigma
$$

satisfies

$$
P(X,\eta) = (\nabla_X J)(\eta) = (\nabla^\perp_X J)(\eta)
$$

for $X \in T\Sigma$ and $\eta \in N\Sigma$.

3 Free-Boundary Holomorphic Curves in $S^6$

In this short section, we recall the standard strict nearly-Kähler structure on the 6-sphere, set up the moving frame for holomorphic curves in $S^6$, and prove Theorem 1.1. Our argument relies on various holomorphic bundle structures that first appeared in [2], though our notation follows [19].
3.1 The Round 6-Sphere

To begin, let us view $\mathbb{R}^7 = \text{Im}(\mathbb{O})$ as the imaginary octonions, and let $g_0$ denote the standard euclidean inner product on $\mathbb{R}^7$. The so-called “7-dimensional cross product” is the alternating bilinear map $\times : \text{Im}(\mathbb{O}) \times \text{Im}(\mathbb{O}) \to \text{Im}(\mathbb{O})$ given by

$$x \times y := \frac{1}{2}(xy - yx).$$

Using the metric $g_0$ to lower an index, we can recast $\times$ as a tensor $\phi_0 : \text{Im}(\mathbb{O}) \times \text{Im}(\mathbb{O}) \times \text{Im}(\mathbb{O}) \to \mathbb{R}$ via

$$\phi_0(x, y, z) := g_0(x \times y, z).$$

The properties of octonionic multiplication show that $\phi_0$ is alternating. In fact, $\phi_0 \in \Omega^3(\mathbb{R}^7)$ is the flat $G_2$-structure on $\mathbb{R}^7$. It is clear that $\phi_0$ is torsion-free and that $\text{Hol}(g_0) = \{\text{Id}\}$.

Let $(S^6, \langle \cdot, \cdot \rangle)$ denote the round 6-sphere with constant curvature 1, and let us embed $S^6 \subset \mathbb{R}^7$ in the standard way. For each $p \in S^6$, the linear map $J_p : T_p S^6 \to T_p S^6$ given by

$$J_p(x) = p \times x$$

satisfies $(J_p)^2 = -\text{Id}$ and $\langle J_p x, J_p y \rangle = \langle x, y \rangle$. The map $J : TS^6 \to TS^6$ is the standard ($G_2$-invariant, orthogonal) almost-complex structure on the round $S^6$. Defining $\omega \in \Omega^2(S^6)$ via

$$\omega(x, y) := \langle Jx, y \rangle$$

the triple $(\langle \cdot, \cdot \rangle, J, \omega)$ is an almost-Hermitian structure on $S^6$. In fact, $(\langle \cdot, \cdot \rangle, J, \omega)$ is a $G_2$-invariant, strict nearly-Kähler structure with $\lambda = 1$. Moreover, the 3-form $\Upsilon \in \Omega^3(S^6)$

$$\Upsilon := \omega \wedge \omega$$

is a complex volume form that satisfies $d\omega = 3 \text{Im}(\Upsilon)$ and $d\text{Re}(\Upsilon) = 2 \omega \wedge \omega$.

Let $\nabla$ denote the Levi-Civita connection on $S^6$. Since $(\langle \cdot, \cdot \rangle, J, \omega)$ is strictly nearly-Kähler, we have both $\nabla J \neq 0$ and $\nabla \omega \neq 0$. Thus, as far as the nearly-Kähler structure of $S^6$ is concerned, the Levi-Civita connection $\nabla$ is perhaps not the most geometrically natural one. Indeed, in many applications, it is more convenient to use the nearly-Kähler connection $\overline{\nabla}$ on $TS^6$ defined by

$$\overline{\nabla}XY := \nabla XY + \frac{1}{2}P(X, JY),$$

where we recall $P(X, Y) = (\nabla_X J)(Y)$. The advantage of $\overline{\nabla}$ is that both $J$ and $\omega$ are $\overline{\nabla}$-parallel.

For computations, we extend both $\nabla$ and $\overline{\nabla}$ to the complexified tangent bundle $TS^6 \otimes_{\mathbb{R}} \mathbb{C}$ by $\mathbb{C}$-linearity. As usual, we decompose $TS^6 \otimes_{\mathbb{R}} \mathbb{C}$ into types with respect to $J$:

$$TS^6 \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}S^6 \oplus T^{0,1}S^6.$$ 

Since $\overline{\nabla}$ preserves $J$, its restriction to $T^{1,0}S^6$ is a well-defined connection, which we continue to denote $\overline{\nabla}$. We caution that the restriction of $\nabla$ to $T^{1,0}S^6$, by contrast, does not yield a well-defined connection.
3.2 Holomorphic Curves in $S^6$

Let $u: \Sigma^2 \to S^6$ be a holomorphic curve in the round 6-sphere, where $S^6$ carries its standard strict nearly-Kähler structure discussed above. Recalling the connection $\nabla$ on $T^{1,0}S^6 \to S^6$, we use the same symbol to denote its pullback to $u^*(T^{1,0}S^6) \to \Sigma$. Since $\Sigma$ is a Riemann surface, we may endow $u^*(T^{1,0}S^6) \to \Sigma$ with the Koszul-Malgrange holomorphic structure for $\nabla$. Since $u$ is an immersion, the complex line subbundle $T^{1,0}\Sigma \hookrightarrow u^*(T^{1,0}S^6)$ is then a holomorphic line subbundle, and the quotient bundle $Q := u^*(T^{1,0}S^6)/T^{1,0}\Sigma$ inherits a natural holomorphic structure as well.

For computations, let $(e_1, \ldots, e_6)$ be a local $SU(3)$-frame for $TS^6$. Concretely, this means that $(e_1, \ldots, e_6)$ is an oriented orthonormal frame, that
\[ Je_1 = e_2 \quad Je_3 = e_4 \quad Je_5 = e_6 \]
and that
\[ \Upsilon = (\omega_1 + i\omega_2) \wedge (\omega_3 + i\omega_4) \wedge (\omega_5 + i\omega_6), \]
where $(\omega_1, \ldots, \omega_6)$ is the dual coframe to $(e_1, \ldots, e_6)$. We also let
\[ f_1 = \frac{1}{2}(e_1 - ie_2) \quad f_2 = \frac{1}{2}(e_3 - ie_4) \quad f_3 = \frac{1}{2}(e_5 - ie_6), \]
so that $(f_1, f_2, f_3)$ is a local complex $SU(3)$-frame for $T^{1,0}S^6$. Finally, note that
\[ \zeta_1 = \omega_1 + i\omega_2 \quad \zeta_2 = \omega_3 + i\omega_4 \quad \zeta_3 = \omega_5 + i\omega_6 \]
is the local $SU(3)$-frame of $\Lambda^{1,0}(S^6)$ dual to $(f_1, f_2, f_3)$.

We now adapt frames to the holomorphic curve $u: \Sigma \to S^6$. Say that a local $SU(3)$-frame along $u(\Sigma)$ is an adapted $U(2)$-frame if
\[ T\Sigma = \text{span}(e_1, e_2) \quad N\Sigma = \text{span}(e_3, e_4, e_5, e_6). \]
An equivalent requirement is that $T^{1,0}\Sigma = \text{span}_C(f_1)$. With respect to a (local) adapted $U(2)$-frame, the second fundamental form of $u$ can be expressed as
\[ \Pi(e_1, e_1) = \kappa_1 e_3 + \kappa_2 e_4 + \mu_1 e_5 - \mu_2 e_6 \]
\[ \Pi(e_1, e_2) = -\kappa_2 e_3 + \kappa_1 e_4 + \mu_2 e_5 + \mu_1 e_6 \]
for some (frame-dependent) functions $\kappa = \kappa_1 + i\kappa_2$ and $\mu = \mu_1 + i\mu_2$. In [2, Lemma 4.3], Bryant shows that
\[ \Phi \in \Gamma(\Lambda^{1,0}\Sigma \otimes \Lambda^{1,0}\Sigma \otimes Q) \]
\[ \Phi = \kappa \zeta_1 \otimes \zeta_1 \otimes (f_2) + \mu \zeta_1 \otimes (f_1) \]
is a well-defined (frame-independent) holomorphic section, where the projection of a vector $v \in u^*(T^{1,0}S^6)$ to the quotient bundle is being denoted $(v) \in Q$. Notice that, as remarked in [2, Lemma 4.4], we have that $\Phi = 0$ identically if and only if $u$ is totally-geodesic.

With the above structures in place, it is now simple to prove Theorem 1.1, which we restate for convenience.
Theorem 3.1. Let $u: \Sigma^2 \to B$ be a compact immersed surface in a geodesic ball $B$ of the round 6-sphere with $u(\partial \Sigma) \subset \partial B$. If $u$ is a holomorphic curve, and if $u(\Sigma)$ meets $\partial B$ orthogonally, then $u(\Sigma)$ is totally geodesic.

Proof. Let $B \subset S^6$ denote a geodesic ball, let $S := \partial B$ denote its boundary sphere, and let $\nu$ denote the outward-pointing unit normal vector field to $S$. Let $A: TS \to TS$ denote the shape operator of $S \subset S^6$, i.e.:

$$A(X) = \nabla_X \nu$$

The crucial point is that geodesic spheres $S \subset S^6$ are totally umbilic, meaning that $A = c \text{Id}$ for some constant $c > 0$ (namely, the principal curvature).

Let $u: \Sigma^2 \to B$ be a holomorphic curve in $B$ with $u(\partial \Sigma) \subset S$ and $u(\Sigma)$ meeting $S$ orthogonally. For $X \in u^*(TS^6) = T\Sigma \oplus N\Sigma$, we will write $X = X^\Sigma + X^N$ for its decomposition into tangential and normal parts. Now, let $(e_1, e_2)$ be a local oriented orthonormal frame for $T\Sigma$ defined on a neighborhood of a (possibly small) arc $C \subset \partial \Sigma$ such that $\nu = e_1$ along $C$. Complete $(e_1, e_2)$ to a $U(2)$-adapted frame $(e_1, e_2, \ldots, e_6)$. Along $C \subset \partial \Sigma$, we now compute

$$\mathbb{II}(e_1, e_2) = (\nabla_{e_2} \nu)^N = [A(e_2)]^N = (ce_2)^N = 0.$$ 

Consequently, $\Phi = 0$ along $C \subset \partial \Sigma$, so by holomorphicity, $\Phi = 0$ on all of $\Sigma$, so $u$ is totally-

to geodesic. \hfill \Box

4 Holomorphic Curves with Lagrangian Boundary

We now turn to holomorphic curves with Lagrangian boundary in nearly-Kähler 6-manifolds. In §4.1, we derive a second variation of area formula (Theorem 4.4). Separately, in §4.2 we recall a generalization of the Riemann-Roch formula (Theorem 4.7) that is suited to holomorphic curves with boundary in almost-complex manifolds. Finally, in §4.3, we deduce the lower bound for the Morse index of area stated in Theorem 1.4. For the Morse index of energy, the interested reader may consult [17]. This section is independent of §3, but draws on the notation set in §2.

4.1 The Second Variation of Area

Let $u: \Sigma^2 \to M^6$ be a compact oriented minimal surface in a Riemannian manifold $M$ such that $u(\partial \Sigma) \subset L$ for some fixed submanifold $L \subset M$. An admissible variation vector field is a normal vector field $\eta \in \Gamma(N\Sigma)$ such that $\eta$ is tangent to $L$ along $\partial \Sigma$. Let $\nu \in \Gamma(T\Sigma|_{\partial \Sigma})$ denote the outer unit conormal, i.e., the unit vector field that is orthogonal to $\partial \Sigma$ and outward-pointing.

It is well-known [8] that the second variation of area of $u$ at an admissible variation vector field $\eta \in \Gamma(N\Sigma)$ is

$$\langle \delta^2 A(\eta) \rangle = \int_{\Sigma} \|\nabla^\perp \eta\|^2 - \|W \eta\|^2 - \mathcal{R}(\eta, e_i, e_i, \eta) + \int_{\partial \Sigma} \langle \overline{\nabla}_\eta \eta, \nu \rangle$$

where $(e_1, e_2)$ is any oriented orthonormal frame for $T\Sigma$. Here, we recall our conventions from §2.3 that $\nabla$ is the ambient covariant derivative, that $\nabla^T, \nabla^\perp$ are the induced tangential and normal connections, and the corresponding curvature tensors are denoted $\mathcal{R}, R^T, R^\perp$, respectively.

We now specialize to the case where $u: \Sigma^2 \to M^6$ is a compact holomorphic curve in a nearly-Kähler 6-manifold $M^6$ having boundary $u(\partial \Sigma) \subset L$ for some fixed submanifold $L \subset M$. For the moment, we make no assumptions about $L$. To begin, we consider the zeroth-order terms $\|W \eta\|^2$ and $\mathcal{R}(\eta, e_i, e_i, \eta)$. We calculate:
Lemma 4.1. We have
\[ \|W\eta\|^2 + R(e_1, e_2, \eta, J\eta) = -R^\perp(e_1, e_2, \eta, J\eta) + 2\lambda^2 \|\eta\|^2. \]

Proof. First, the nearly-Kähler curvature identity (2.3) and the constant type equation (2.2) give
\[ R(e_1, e_2, \eta, J\eta) = 2\|P(e_1, \eta)\|^2 = 2\lambda^2 \|\eta\|^2. \]
Now, the Ricci equation (2.5), followed by the shape operator symmetries (2.6), shows that
\[ R(e_1, e_2, \eta, J\eta) = R^\perp(e_1, e_2, \eta, J\eta) + \langle W_{e_1} \eta, W_{e_2} (J\eta) \rangle - \langle W_{e_1} (J\eta), W_{e_2} \eta \rangle \\
= R^\perp(e_1, e_2, \eta, J\eta) + \langle W_{e_1} \eta, W_{e_2} \eta \rangle + \langle W_{e_2} \eta, W_{e_2} \eta \rangle \\
= R^\perp(e_1, e_2, \eta, J\eta) + \|W\eta\|^2. \]

In view of Lemma 4.1, our second variation formula (4.1) now reads:
\[ (\delta^2 A)(\eta) = \int_{\Sigma} \|\nabla^\perp \eta\|^2 + R^\perp(e_1, e_2, \eta, J\eta) - 2\lambda^2 \|\eta\|^2 + \int_{\partial\Sigma} \langle \nabla \eta, \nu \rangle. \]

To handle the first two terms, we require a few pieces of notation. First, we recall the tensor 
\[ P(X, \eta) = (\nabla_X^\perp J)(\eta) = (\overline{\nabla}_X J)(\eta). \]
Second, for a fixed \( \eta \in \Gamma(\Sigma) \), we let \( \alpha_\eta \in \Omega^1(\Sigma) \) denote the 1-form on \( \Sigma \) given by
\[ \alpha_\eta(X) = \langle \nabla_X^\perp \eta, J\eta \rangle. \]
Finally, we let \( \mathcal{D}: \Gamma(\Sigma) \to \Omega^1(\Sigma) \otimes \Gamma(\Sigma) \) denote the differential operator
\[ \mathcal{D}_X \eta = \nabla^\perp_X \eta + J\nabla_X^\perp \eta. \]

We now calculate:

Lemma 4.2. We have
\[ \|\nabla^\perp \eta\|^2 + R^\perp(e_1, e_2, \eta, J\eta) = \frac{1}{2} \|\mathcal{D}\eta\|^2 + \langle P(e_1, J\eta), \mathcal{D}_{e_1} \eta \rangle + d\alpha_\eta(e_1, e_2). \]

Proof. First, from
\[ \frac{1}{2} \|\mathcal{D}\eta\|^2 = \|\mathcal{D}_{e_1} \eta\|^2 = \|\nabla_{e_1}^\perp \eta\|^2 + \|\nabla_{e_2}^\perp \eta\|^2 + 2\langle \nabla_{e_1}^\perp \eta, J\nabla_{e_2}^\perp \eta \rangle \]
we observe that
\[ \|\nabla^\perp \eta\|^2 = \frac{1}{2} \|\mathcal{D}\eta\|^2 - 2\langle \nabla_{e_1}^\perp \eta, J\nabla_{e_2}^\perp \eta \rangle. \]

Second, we calculate
\[ e_1(\alpha_\eta(e_2)) = \langle \nabla_{e_1}^\perp \nabla_{e_2}^\perp \eta, J\eta \rangle + \langle \nabla_{e_2}^\perp \eta, P(e_1, \eta) \rangle - \langle \nabla_{e_1}^\perp \eta, J\nabla_{e_2}^\perp \eta \rangle \\
e_2(\alpha_\eta(e_1)) = \langle \nabla_{e_2}^\perp \nabla_{e_1}^\perp \eta, J\eta \rangle + \langle \nabla_{e_1}^\perp \eta, P(e_2, \eta) \rangle + \langle \nabla_{e_1}^\perp \eta, J\nabla_{e_2}^\perp \eta \rangle \]
and hence
\[ d\alpha_\eta(e_1, e_2) = e_1(\alpha_\eta(e_2)) - e_2(\alpha_\eta(e_1)) - \alpha_\eta([e_1, e_2]) \]
\[ = \left\langle (\nabla^\perp e_1 \nabla^\perp e_2 - \nabla^\perp e_2 \nabla^\perp e_1)\eta, J\eta \right\rangle - \alpha_\eta([e_1, e_2]) - 2\left\langle \nabla^\perp e_1 \eta, J\nabla^\perp e_2 \eta \right\rangle \\
+ \left\langle \nabla^\perp_{\partial e_1} \eta, P(e_1, \eta) \right\rangle - \left\langle \nabla^\perp_{e_1} \eta, P(Je_1, \eta) \right\rangle \]
\[ = R^\perp(e_1, e_2, \eta, J\eta) - 2\left\langle \nabla^\perp e_1 \eta, J\nabla^\perp e_2 \eta \right\rangle + \left\langle \nabla^\perp_{\partial e_1} \eta - J\nabla^\perp_{e_1} \eta, P(e_1, \eta) \right\rangle \]
\[ = R^\perp(e_1, e_2, \eta, J\eta) - 2\left\langle \nabla^\perp e_1 \eta, J\nabla^\perp e_2 \eta \right\rangle - \left\langle \partial_{e_1} \eta, P(e_1, J\eta) \right\rangle \]
so that
\[ -2\left\langle \nabla^\perp e_1 \eta, J\nabla^\perp e_2 \eta \right\rangle = -R^\perp(e_1, e_2, \eta, J\eta) + \left\langle P(e_1, J\eta), \partial_{e_1} \eta \right\rangle + d\alpha_\eta(e_1, e_2). \]  
From (4.4) and (4.5), we obtain the result.

**Remark.** The formula of Lemma 4.2 implies that the quantity \( \left\langle P(e_1, J\eta), \partial_{e_1} \eta \right\rangle \) is independent of the choice of orthonormal frame \((e_1, e_2)\). Here is another way to see this. For \( \eta \in \Gamma(N\Sigma) \), consider the tensor \( G_\eta : T\Sigma \otimes T\Sigma \rightarrow \mathbb{R} \) via
\[ G_\eta(X, Y) = \left\langle P(X, J\eta), \partial_{e_1} \eta \right\rangle = \frac{1}{2} d\omega(X, J\eta, \partial_{e_1} \eta). \]
The symmetries of \( P \) and \( \partial \) show that \( G_\eta(X, Y) = G_\eta(JX, JY) \), so \( G_\eta(e_1, e_1) = G_\eta(e_2, e_2) \). It follows that \( \left\langle P(e_1, J\eta), \partial_{e_1} \eta \right\rangle = \frac{1}{2} tr(G_\eta) \), which is clearly independent of \((e_1, e_2)\).

Using Lemma 4.2 in (4.2), followed by Stokes' Theorem, our second variation formula is
\[ (\delta^2 A)(\eta) = \int_{\Sigma} \frac{1}{2} \|\partial \eta\|^2 + \left\langle P(e_1, J\eta), \partial_{e_1} \eta \right\rangle - 2\lambda^2 \|\eta\|^2 + \int_{\partial \Sigma} \left\langle \nabla_{\eta} \eta, \nu \right\rangle + \left\langle \nabla^\perp T \eta, J\eta \right\rangle \]
where \( T \in \Gamma(T\Sigma|_{\partial\Sigma}) \) is the positively-oriented unit vector field tangent to \( \partial\Sigma \). Thus far, we have not imposed any conditions on the submanifold \( L \). We now suppose that \( L \) is Lagrangian and again exploit the fact that \( u \) is holomorphic and \( M \) is nearly-Kähler:

**Lemma 4.3.** If \( L \) is Lagrangian, then
\[ \left\langle \nabla_{\eta} \eta, \nu \right\rangle + \left\langle \nabla^\perp T \eta, J\eta \right\rangle = 0. \]

**Proof.** Along \( \partial\Sigma \), we have that \( T \) and \( \eta \) are tangent to \( L \), so that \([T, \eta]\) must be tangent to \( L \). On the other hand, since \( L \) is Lagrangian, \( J\eta \) is normal to \( L \). Therefore,
\[ \left\langle \nabla_{T} \eta - \nabla_{\eta} T, J\eta \right\rangle = \left\langle [T, \eta], J\eta \right\rangle = 0. \]
Consequently,
\[ \left\langle \nabla^\perp T \eta, J\eta \right\rangle = \left\langle \nabla_{T} \eta, J\eta \right\rangle = \left\langle \nabla_{\eta} T, J\eta \right\rangle = -\left\langle T, \nabla_{\eta}(J\eta) \right\rangle = -\left\langle T, J\nabla_{\eta} \eta \right\rangle = \left\langle JT, \nabla_{\eta} \eta \right\rangle \]
\[ = -\left\langle \nu, \nabla_{\eta} \eta \right\rangle, \]
where the second equality uses (4.7), the third equality uses \( \eta([T, \eta]) = 0 \), the fourth equality uses the nearly-Kähler condition \( \left\langle \nabla_{X} J \right\rangle(X) = 0 \), and the last equality uses that the holomorphicity of \( u \) implies \( JT = -\nu \).

From Lemma 4.3 and (4.6), and recalling that \( \left\langle P(X, Y), Z \right\rangle = \left\langle \nabla_{X} \omega \right\rangle(Y, Z) = \frac{1}{3} d\omega(X, Y, Z) \) for all \( X, Y, Z \in TM \), we arrive at our desired formula:
Theorem 4.4. Let \( u: \Sigma^2 \to M^6 \) be a compact holomorphic curve in a nearly-Kähler 6-manifold with boundary \( u(\partial \Sigma) \subset L \) for a Lagrangian submanifold \( L \subset M \). For any normal vector field \( \eta \in \Gamma(N\Sigma) \) that is tangent to \( L \) along \( \partial \Sigma \), the second variation of area is

\[
(\delta^2 A)(\eta) = \int_{\Sigma} \frac{1}{2} \|\mathcal{D}\eta\|^2 + \frac{1}{3} d\omega(e, J\eta, \mathcal{D}_{e\eta}) - 2\lambda^2 \|\eta\|^2
\]

where \( \mathcal{D} \) is the operator defined in (4.3), where \( \lambda \geq 0 \) is the type constant (2.2), and where \( e \in \Gamma(T\Sigma) \) is any unit tangent vector field.

4.2 The Riemann-Roch Theorem

From Theorem 4.4, we see that any admissible vector field \( \eta \in \Gamma(N\Sigma) \) with \( \mathcal{D}\eta = 0 \) will satisfy \( (\delta^2 A)(\eta) \leq 0 \), with equality only if \( \lambda = 0 \) or \( \eta = 0 \). Therefore, if \( \lambda \neq 0 \), then the Morse index of \( u \) is at least the Fredholm index of \( \mathcal{D} \) on admissible variations. This latter quantity can be computed by means of a well-known index formula for real Cauchy-Riemann operators. Recalling this formula (Theorem 4.7) is the aim of this section. The material here is by now standard; our rather brief discussion is drawn from [21, Appendix C] and [28, §2.3, §3.4].

4.2.1 Connections on Complex Vector Bundles

Let \( E \to \Sigma \) be a real vector bundle over a real manifold \( \Sigma \). Recall that a \((real) \ connection\) on \( E \) is an \( \mathbb{R} \)-linear operator

\[
\nabla: \Gamma(E) \to \Gamma(\text{Hom}_R(T\Sigma, E)) = \Gamma(\Lambda^1(\Sigma; \mathbb{R}) \otimes_\mathbb{R} E)
\]

that satisfies the Leibniz rule

\[
\nabla(f\xi) = f\nabla\xi + df \otimes \xi
\]

for all \( f \in C^\infty(\Sigma; \mathbb{R}) \) and all \( \xi \in \Gamma(E) \). From now on, let us suppose that \( E \to \Sigma \) has the structure of a complex bundle, say \( J: E \to E \). A real connection \( \nabla \) is said to preserve \( J \) if \( \nabla_X(J\xi) = J\nabla_X\xi \) for all \( X \in T\Sigma \) and \( \xi \in \Gamma(E) \).

Let \( \text{Hom}_\mathbb{C}(T\Sigma^\mathbb{C}, E) \to \Sigma \) denote the vector bundle whose fiber at \( p \in \Sigma \) consists of the \((i, J)\)-linear maps \( T_p\Sigma^\mathbb{C} \to E_p \). Each fiber carries an obvious \( \mathbb{C} \)-module structure via \( i \cdot \alpha := J \circ \alpha = \alpha \circ i \).

With this understood, we recall that a \textit{complex connection} on \( (E, J) \) is a \( \mathbb{C} \)-linear operator

\[
\nabla: \Gamma(E) \to \Gamma(\text{Hom}_\mathbb{C}(T\Sigma^\mathbb{C}, E)) = \Gamma(\Lambda^1(\Sigma; \mathbb{C}) \otimes_\mathbb{C} E)
\]

that satisfies the Leibniz rule (4.8) for all \( f \in C^\infty(\Sigma; \mathbb{C}) \) and \( \xi \in \Gamma(E) \). Of course, there is a natural bijection between complex connections and \( J \)-preserving real connections.

We now suppose that \((\Sigma, j)\) is a Riemann surface. Let \( \text{Hom}^+(T\Sigma, E) \) and \( \text{Hom}^-(T\Sigma; E) \) denote the vector bundles over \( \Sigma \) whose fiber at \( p \in \Sigma \) consists of the \((j, J)\)-linear and \((j, J)\)-antilinear maps \( T_p\Sigma \to E_p \), respectively. Explicitly,

\[
\text{Hom}^+(T\Sigma, E)|_p = \{ A \in \text{Hom}_R(T_p\Sigma, E_p) : J \circ A = A \circ j \}
\]

\[
\text{Hom}^-(T\Sigma; E)|_p = \{ A \in \text{Hom}_R(T_p\Sigma, E_p) : J \circ A = -A \circ j \}.
\]

Given a real connection \( \nabla \) on \((E, J) \to (\Sigma, j)\), we define the operators

\[
\partial^\nabla: \Gamma(E) \to \Gamma(\text{Hom}^+(T\Sigma, E)) \quad \bar{\partial}^\nabla: \Gamma(E) \to \Gamma(\text{Hom}^-(T\Sigma, E))
\]

\[
\partial^\nabla \xi := \frac{1}{2} (\nabla\xi - J \circ \nabla\xi \circ j) \quad \bar{\partial}^\nabla \xi := \frac{1}{2} (\nabla\xi + J \circ \nabla\xi \circ j)
\]
In the case of the trivial complex line bundle $(\mathbb{C},i) \to \Sigma$, the exterior derivative $d: C^\infty(\Sigma; \mathbb{C}) \to \Omega^1(\Sigma; \mathbb{C}) = \Gamma(L^1(\Sigma; \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}) = \Gamma(\text{Hom}_\mathbb{R}(T\Sigma, \mathbb{C}))$ is a real connection that preserves $i$. The corresponding operators are denoted

\[
\partial: C^\infty(\Sigma; \mathbb{C}) \to \Gamma(\text{Hom}^+(T\Sigma, \mathbb{C})) \quad \bar{\partial}: C^\infty(\Sigma; \mathbb{C}) \to \Gamma(\text{Hom}^-(T\Sigma, \mathbb{C}))
\]

\[
\partial f := \frac{1}{2} (df - i df \circ j) \quad \bar{\partial} f := \frac{1}{2} (df + i df \circ j)
\]

Finally, we remark that $\text{Hom}^+(T\Sigma, \mathbb{C}) \cong \Lambda^{1,0}(\Sigma)$ and $\text{Hom}^-(T\Sigma, \mathbb{C}) \cong \Lambda^{0,1}(\Sigma)$. More generally,

\[
\text{Hom}^+(T\Sigma, E) \cong \text{Hom}_\mathbb{C}(T^{1,0}\Sigma, E^{1,0}) \cong \Lambda^{1,0}(\Sigma) \otimes_\mathbb{C} E
\]

\[
\text{Hom}^-(T\Sigma, E) \cong \text{Hom}_\mathbb{C}(T^{0,1}\Sigma, E^{1,0}) \cong \Lambda^{0,1}(\Sigma) \otimes_\mathbb{C} E.
\]

### 4.2.2 Cauchy-Riemann Operators

Given a complex bundle $(E, J) \to \Sigma$ over a Riemann surface $(\Sigma, j)$, the operators of the form $\bar{\partial}^\nabla$ for some complex connection $\nabla$ are particularly important. They are called complex Cauchy-Riemann operators (or $\bar{\partial}$-operators or holomorphic structures), and their Fredholm indices are the subject of the classical Riemann-Roch Theorem. To be precise:

**Definition 4.5.** An $\mathbb{R}$-linear operator $\mathcal{D}_0: \Gamma(E) \to \Gamma(\text{Hom}^-(T\Sigma, E))$ is called a complex Cauchy-Riemann operator if either of the following equivalent conditions hold:

(i) For all $f \in C^\infty(\Sigma; \mathbb{R})$ and all $\xi \in \Gamma(E)$:

\[
\mathcal{D}_0(f\xi) = f \mathcal{D}_0\xi + \bar{\partial} f \otimes \xi \quad \mathcal{D}_0(J\xi) = J \mathcal{D}_0\xi.
\]

(ii) There exists a $J$-preserving real connection $\nabla$ on $E$ for which $\mathcal{D}_0 = \bar{\partial}^\nabla$.

For certain geometric applications — especially those involving holomorphic curves in (non-integrable) almost-complex manifolds — one often needs to consider the wider class of $\bar{\partial}^\nabla$ operators for which $\nabla$ does not necessarily preserve the complex structure $J$. Such operators are called:

**Definition 4.6.** An $\mathbb{R}$-linear operator $\mathcal{D}: \Gamma(E) \to \Gamma(\text{Hom}^-(T\Sigma, E))$ is called a real Cauchy-Riemann operator if any of the following equivalent conditions hold:

(i) For all $f \in C^\infty(\Sigma; \mathbb{R})$ and $\xi \in \Gamma(E)$:

\[
\mathcal{D}(f\xi) = f \mathcal{D}\xi + \bar{\partial} f \otimes \xi.
\]

(ii) There exists a real connection $\nabla$ on $E$ for which $\mathcal{D} = \bar{\partial}^\nabla$.

(iii) There exists a section $\alpha \in \Gamma(E^* \otimes_\mathbb{R} \text{Hom}^-(T\Sigma, E))$ and a complex Cauchy-Riemann operator $\mathcal{D}_0$ on $E$ for which $\mathcal{D} = \mathcal{D}_0 + \alpha$.

Finally, recall that a subbundle $F \subset E|_{\partial \Sigma}$ is totally real if each fiber is a totally real subspace, meaning that $J(F_p) \cap F_p = \{0\}$ for all $p \in \partial \Sigma$. In this case, we define

\[
W^{k,p}_F(E) := \{\xi \in W^{k,p}(E): \xi(\partial \Sigma) \subset F\}. \tag{4.9}
\]

Now, on a complex vector bundle $(E, J) \to (\Sigma, j)$ equipped with a totally real subbundle $F \subset E|_{\partial \Sigma}$ and a Cauchy-Riemann operator $\mathcal{D}$, we let $\mathcal{D}_F: W^{k,p}_F(E) \to W^{k-1,p}(\text{Hom}^-(T\Sigma, E))$ denote the restriction of $\mathcal{D}$ to $W^{k,p}_F(E) \subset W^{k,p}(E)$. It turns out that the boundary condition (4.9) is Fredholm, and the Fredholm index of $\mathcal{D}_F$ can be calculated via the following Riemann-Roch formula:
Theorem 4.7 ([21]). Let $E \to \Sigma$ be a complex vector bundle of complex rank $r$ over a compact Riemann surface $\Sigma$ with boundary, and let $F \subset E|_{\partial \Sigma}$ be a totally real subbundle. Let $\mathcal{D}$ denote a real Cauchy-Riemann operator on $E$ of class $W^{\ell-1,p}$, where $\ell \in \mathbb{Z}^+$ and $p > 1$ satisfy $\ell p > 2$. Then the real Fredholm index of $\mathcal{D}_F$ is

$$\text{Ind}(\mathcal{D}_F) = r \chi(\Sigma) + \mu(E, F),$$

where $\chi(\Sigma)$ is the Euler characteristic and $\mu(E, F)$ is the boundary Maslov index.

4.3 The Morse Index

We now return to the study of holomorphic curves $u: \Sigma^2 \to M^6$ with boundary in a Lagrangian $L \subset M$ of a nearly-Kähler 6-manifold $M$. Recall Theorem 4.4: For a normal vector field $\eta \in \Gamma(N\Sigma)$ that is tangent to $L$ along $\partial \Sigma$, we have

$$\left(\delta^2 A\right)(\eta) = \int_\Sigma \frac{1}{2} \|\mathcal{D}\eta\|^2 + \frac{1}{3} d\omega(e, J\eta, \mathcal{D}e\eta) - 2\lambda^2 \|\eta\|^2$$

(4.10)

The operator $\mathcal{D}: \Gamma(N\Sigma) \to \Gamma(\text{Hom}^-(T\Sigma, N\Sigma))$ appearing in this formula, defined in (4.3), is a real Cauchy-Riemann operator. The proof of Theorem 1.4 now follows quickly:

Proof. Suppose $M^6$ is a strict nearly-Kähler 6-manifold, so that $\lambda \neq 0$. Let us orthogonally decompose $TL|_{\partial \Sigma} = T(\partial \Sigma) \oplus F$. In other words, $F = TL|_{\partial \Sigma} \cap N\Sigma|_{\partial \Sigma}$, and hence is a totally real subbundle of $N\Sigma|_{\partial \Sigma}$. In this language, the set admissible normal vector fields is

$$A := \{\eta \in \Gamma(N\Sigma): \eta|_{\partial \Sigma} \in TL\} = \{\eta \in \Gamma(N\Sigma): \eta(\partial \Sigma) \subset F\}.$$

Let $\mathcal{H} := \{\eta \in A: \mathcal{D}\eta = 0\}$. If $\eta \in \mathcal{H}$ and $\eta \neq 0$, then (4.10) shows that

$$\left(\delta^2 A\right)(\eta) = -2\lambda^2 \int_\Sigma \|\eta\|^2 < 0.$$

Thus, the Riemann-Roch Theorem 4.7 implies that the Morse index of $u$ satisfies

$$\text{Ind}(u) \geq \dim_R(\mathcal{H}) \geq \text{Ind}(\mathcal{D}_F) = 2\chi(\Sigma) + \mu(N\Sigma, F) = \mu(T\Sigma, T(\partial \Sigma)) + \mu(N\Sigma, F) = \mu(u^*(TM), TL).$$

Here, we used that $\mu(T\Sigma, T(\partial \Sigma)) = 2\chi(\Sigma)$ for any compact surface with boundary [24, §5]. In the last equality, we used the splittings $u^*(TM) = T\Sigma \oplus N\Sigma$ and $TL|_{\partial \Sigma} = T(\partial \Sigma) \oplus F$ together with the additivity [21, Appendix C.3] of the boundary Maslov index.

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