The point equivalence problem for ordinary differential equations of the second order

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Abstract. We use É. Cartan’s method to solve the problem of equivalence of the second order ordinary differential equations with respect to the pseudogroup of point transformations.

Key words: Lie pseudogroups, ordinary differential equations.

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1. Introduction

The problem of finding the necessary and sufficient conditions of equivalence of ordinary differential equations of the second order

\[ u_{xx} = F(x, u, u_x) \]  

(1)

with respect to the pseudogroup of local diffeomorphisms

\[ \tilde{x} = \varphi(x, u), \quad \tilde{u} = \psi(x, u) \]  

(2)

has a long history. S. Lie showed, [15], that equations of the form

\[ u_{xx} = A_3(x, u) u_x^3 + A_2(x, u) u_x^2 + A_1(x, u) u_x + A_0(x, u) \]  

(3)

generate an invariant subclass in the class (1) with respect to the changes of variables (2). He also showed that equation (3) is linearizable by means of the transformation (2) whenever the following system

\[
\begin{align*}
U_x &= U + A_0 A_3 - \frac{1}{3} A_{1,u} + \frac{1}{3} A_{2,u}, \\
V_x &= V - A_0 U + A_1 V - A_{0,u} + A_0 A_2, \\
U_u &= U^2 - A_2 U + A_3 V + A_{3,x} + A_1 A_3,
\end{align*}
\]

is compatible. The compatibility condition of this system is equivalent to the system

\[
\begin{align*}
L_1 &= 3 A_{0,u} u_x - 2 A_{1,x} u_x + 3 A_{3,0} u_x - 3 A_{2,0} u_x - 3 A_1 A_{1,u} \\
&\quad - A_{1,2} u_x - 3 A_{0,2,u} + 6 A_0 A_{3,x}, \\
L_2 &= A_{1,u} u_x - 2 A_{2,2} u_x + 3 A_{3,2} u_x - 6 A_{3,0} u_x + 3 A_{3,1} u_x + 2 A_2 A_{1,u} \\
&\quad - 2 A_{2,2} u_x + 3 A_{1,3} u_x - 3 A_0 A_{3,u}.
\end{align*}
\]

(4)

These functions were found by R. Liouville in the first systematic study of the equivalence problem for equations (3) with respect to transformations (2). Liouville found series of relative and absolute invariants and pointed out the procedure for generating of invariants of higher orders.

A. Tresse used S. Lie’s infinitesimal method to find differential invariants of equations (3) and (1), [22, 23]. Papers [24] and [14] are devoted to the modern exposition of Tresse’s approach. Let us note one of the results of [22]: if either \( L_1 \) or \( L_2 \) is not equal to zero, then there exists a change of variables (2) that maps equation (3) into equation with

\[ L_1 \equiv 0, \quad L_2 \equiv 0 \]

for the functions

\[
\begin{align*}
\tilde{x} = \varphi(x), \quad \tilde{u} = \psi(x, u).
\end{align*}
\]

(6)

É. Cartan developed the equivalence method [31–37] and apilied it to equation (3) in the paper [8], however he studied there dierenential geometry of projective connections, not equations themselves, (see [8 § 8]).

The paper [9] applies Cartan’s method to the equivalence problem for equations (1) with respect to the pseudogroup (6).

Subclass (3) contains Painlevé’s equations [18, 19], which are of interest since they often appear in study of invariant solutions of integrable nonlinear equations and in other applications of physical importance [21]. In [12, 9, 11] Cartan’s method was used to find the necessary and sufficient conditions of equivalence of equation (3) to the first or to the second Painlevé equation. In [11] it was proven that all the Painlevé equations can be transformed to the form

\[ u_{xx} = A_0(x, u). \]

(7)

In this paper it was also established that the point transformations that preserve subclass (3) have the form \( \tilde{x} = \varphi(x), \tilde{u} = \sqrt{|\varphi'(x)|} u + \chi(x) \). In [2] R. Liouville’s results were applied to study equations (3) that are reductions of systems of differential equations describing chaotic dynamics in physics of the atmosphere and and in chemical kynetics.

In the present paper we use Cartan’s method [3–7, 10, 11, 17] to solve the equivalence problem for equations (1) with respect to transformations (2).
2. The solution of the equivalence problem in the case $F_{u_x u_x u_x u_x} \neq 0$.

All considerations in the paper are local, all maps are supposed to be real-analytic.

Equation (1) is a submanifold in the bundle $J^2(\pi)$ of the second order jets of local sections of the bundle $\pi: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\pi(x, u) \mapsto x$. Local coordinates on $J^2(\pi)$ are $(x, u, u_x, u_{xx})$. The pseudogroup of local diffeomorphisms $\Phi: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$, $\Phi(x, u) \mapsto (\tilde{x}, \tilde{u})$, acts on $\mathbb{R} \times \mathbb{R}$. The second prolongation $\Phi^{(2)}: J^2(\pi) \to J^2(\pi)$, $\Phi^{(2)}: (x, u, u_x, u_{xx}) \mapsto (\tilde{x}, \tilde{u}, \tilde{u}_x, \tilde{u}_{xx})$ of a diffeomorphism $\Phi$ is defined as follows:

\[
(\Phi^{(2)})^* \left( \begin{array}{c} d\tilde{x} \\ d\tilde{u} - \tilde{u}_x d\tilde{x} \\ d\tilde{u}_x - \tilde{u}_{xx} d\tilde{x} \end{array} \right) = B \left( \begin{array}{c} dx \\ du - u_x dx \\ du_x - u_{xx} dx \end{array} \right), \quad B \in \mathcal{H} = \left\{ \begin{pmatrix} b_1 & b_2 & 0 \\ 0 & b_3 & 0 \\ 0 & b_5 & b_4 \end{pmatrix} \in \text{GL}(3) \right\}.
\]

Diffeomorphisms $\Phi^{(2)}$ constitute the pseudogroup $\text{Cont}_0(J^2(\pi))$ of point transformations of the bundle $J^2(\pi)$. When a superposition of two local diffeomorphisms from pseudogroup $\text{Cont}_0(J^2(\pi))$ is defined, this superposition belongs to $\text{Cont}_0(J^2(\pi))$ as well. Therefore the forms

\[
\begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} = B \cdot \begin{pmatrix} dx \\ du - u_x dx \\ du_x - u_{xx} dx \end{pmatrix}
\]

are invariant with respect to the lifts $\Psi: J^2(\pi) \times \mathcal{H} \to J^2(\pi) \times \mathcal{H}$ of diffeomorphisms from the pseudogroup $\text{Cont}_0(J^2(\pi))$. Two equations (1) are locally equivalent with respect to $\text{Cont}_0(J^2(\pi))$ whenever the restrictions $\omega_i = \Omega_i|_{u_x = F(x, u, u_x)}$ of forms $\Omega_i$ onto these equations are equivalent with respect to a diffeomorphism $\Psi: J^2(\pi) \times \mathcal{H} \to J^2(\pi) \times \mathcal{H}$: $\Psi^*(\tilde{\omega}_i) = \omega_i$, $i \in \{1, 2, 3\}$. Hence we obtain $\mathcal{H}$-valued equivalence problem for the collection of 1-forms $\omega = \{\omega_1, \omega_2, \omega_3\}$ (see [17, Def. 9.5]). In accordance with Cartan’s method, to solve this problem we analyze the structure equations for forms $\omega_i$, that is, the expressions for the exterior differentials $d\omega_i$ via $\omega$.

The structure equation for form $\omega_2$ is

\[
d\omega_2 = \eta \wedge \omega_2 + b_3 b_1^{-1} b_4^{-1} \omega_1 \wedge \omega_3,
\]

where $\eta = db_3 b_1^{-1} - b_3 b_1^{-1} b_4^{-1} \omega_1 + r \omega_2 + b_2 b_1^{-1} b_4^{-1} \omega_3$, and $r$ is an arbitrary constant. Since forms $\omega_i$ and their differentials are invariant with respect to $\Psi$, function $b_3 b_1^{-1} b_4^{-1}$ is invariant as well. We can normalize it, that is, to put it equal to any non-zero constant, see [17, Prop. 9.11]. In the case $b_3 b_1^{-1} b_4^{-1} = 1$ we get $b_3 = b_1 b_4$. After this normalization the structure equations acquire the form

\[
\begin{align*}
d\omega_1 &= \eta_1 \wedge \omega_1 + \eta_2 \wedge \omega_2, \\
d\omega_2 &= \eta_3 \wedge \omega_2 + \omega_1 \wedge \omega_3, \\
d\omega_3 &= \eta_4 \wedge \omega_2 + (\eta_3 - \eta_1) \wedge \omega_3, \\
d\eta_1 &= -2 \eta_4 \wedge \omega_1 + \eta_5 \wedge \omega_2 - \eta_2 \wedge \omega_3, \\
d\eta_2 &= (\eta_1 - \eta_3) \wedge \eta_2 + \eta_5 \wedge \omega_1 + \frac{1}{6} F_4 b_1^{-1} b_4^{-1} \omega_2 \wedge \omega_3,
\end{align*}
\]

where forms $\eta_1, ..., \eta_4$ depend on differentials of the remaining non-normalized parameters of group $\mathcal{H}$, form $\eta_5$ is obtained by means of the procedure of prolongation of the structure equations [17, Ch. 12], and where we use the notation $F_k = \left( \frac{\partial}{\partial u} \right)^k F$, $k \in \mathbb{N}$.

The further analysis divides on two cases: case $\mathcal{A}$ corresponds to the condition $F_4 \neq 0$, and case $\mathcal{B}$ corresponds to $F_4 \equiv 0$.

In case $\mathcal{A}$ we can shrink, if it is necessary, the domain of diffeomorphism $\Psi$, therefore we can assume that $F_4 \neq 0$. Then normalization $F_4 b_1^{-1} b_4^{-1} = 1$ yields $b_1 = F_4 b_4^{-1}$. After this we get

\[
\begin{align*}
d\omega_1 &= \eta_1 \wedge \omega_1 + \eta_2 \wedge \omega_2 + \frac{1}{2} (5 b_2 b_1^{-1} + F_5) b_4^{-1} F_4^{-1} \omega_1 \wedge \omega_3, \\
d\omega_2 &= \frac{2}{3} \eta_1 \wedge \omega_2 + \omega_1 \wedge \omega_3,
\end{align*}
\]
with new forms \( \eta_1 \) and \( \eta_2 \). Then normalization \( b_2 = -\frac{1}{6} F_5 b_4^{-3} \) gives the following structure equations
\[
\begin{align*}
dw_1 &= \eta_1 \wedge \omega_1 + \frac{1}{32} (5 F_4 F_6 - 6 F_5^2) F_4^{-2} b_4^{-2} \omega_2 \wedge \omega_3, \\
dw_2 &= \frac{2}{3} \eta_1 \wedge \omega_2 + \omega_1 \wedge \omega_3, \\
dw_3 &= \eta_2 \wedge \omega_2 - \frac{1}{3} \eta_1 \wedge \omega_3 + \frac{1}{2} b_4^2 F_4^{-2} (F_4 b_5 + (D_x(F_4) + 2 F_1 F_4) b_4) \omega_1 \wedge \omega_3,
\end{align*}
\]
where we denote \( D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + F \frac{\partial}{\partial \eta} \). We put the coefficient at \( \omega_1 \wedge \omega_3 \) in the third equation equal to zero and obtain \( b_5 = -b_4 F_4^{-1} F (D_x(F_4) + 2 F_1 F_4) \).

The further analysis divides on two subcases: case \( \mathcal{A}_1 \) corresponds to the conditions \( 5 F_4 F_6 - 6 F_5^2 \neq 0 \), while case \( \mathcal{A}_2 \) corresponds to the condition \( 5 F_4 F_6 - 6 F_5^2 \equiv 0 \).

In the case \( \mathcal{A}_1 \) we can assume without loss of generality that \( 5 F_4 F_6 - 6 F_5^2 \neq 0 \). Then we can put the coefficient at \( \omega_2 \wedge \omega_3 \) equal to \( \frac{1}{32} \). This yields \( b_4 = F_4^{-1} \sqrt{5 F_4 F_6 - 6 F_5^2} \). After this normalization all the parameters of group \( \mathcal{H} \) are defined as functions on \( J^2(\pi) \), and the structure equations acquire the form
\[
\begin{align*}
dw_1 &= G_1 \omega_1 \wedge \omega_2 + (G_2 \omega_1 + \frac{1}{32} \omega_3) \wedge \omega_3, \\
dw_2 &= G_3 \omega_1 \wedge \omega_2 + (\omega_1 + \frac{3}{2} G_2 \omega_2) \wedge \omega_3, \\
dw_3 &= G_4 \omega_1 \wedge \omega_2 + (G_5 \omega_2 - \frac{1}{2} G_3 \omega_1) \wedge \omega_3,
\end{align*}
\]
where \( G_1, ... , G_5 \) are defined as
\[
\begin{align*}
G_1 &= \frac{3}{10} F_5 F_4^{-4} V_{1,x} + \frac{3}{10} (u_x F_5 + 5 F_4) F_4^{-4} V_{1,u} - \frac{4}{5} V_1 F_4^{-4} (F_5, u_x F_5, u) + \frac{8}{5} G_2 F_5 F_4^{-5} V_1^{3/2} \\
&\quad + \frac{4}{5} V_1 F_4^{-5} \left( \left( 5 G_2 V_1^{1/2} - 3 F_5 \right) F_4, u + \left( 5 (u_x G_2 V_1^{1/2} - 4 F_4) - 3 u_x F_5 \right) F_4, u \right) \\
&\quad + 2 G_2 F_1 V_4^{3/2} F_4^{-4} - \frac{3}{2} F_1 V_4^{-2}, \\
G_2 &= \frac{3}{10} V_1^{3/2} (5 F_4 V_{1,u} - 14 F_5 V_1), \\
G_3 &= V_1^{1/2} F_4^{-5} \left( 4 V_1 (F_4, u_x F_4, u) - F_4 (V_{1,x} + u_x V_{1,u}) + \frac{3}{2} F_5 F_5 V_1 + 2 F_1 F_4 V_1 - \frac{2}{3} G_2 F_4 V_1^{3/2} \right), \\
G_4 &= -V_1^{3} F_4^{-9} \left( F_5 (D_x(F) + 3 F_4 F_4) + 2 F_4 (F_5, u_x F_5, u) + 3 F_1 F_4 F_4, x \right) \\
&\quad + (3 u_x F_1 + F) F_4, u_x + 2 F_4 D_x(F) + u_x^2 F_{4,uu} - F_4, u + 2 F_1 F_4 + F_4, x + 2 u_x F_4, u_x \right) \\
&\quad - \frac{3}{7} F^2 V_1^3 F_4^{-10} (6 F_5^2 - V_1), \\
G_5 &= -5 G_1 + \frac{8}{5} F_5 F_4^{-4} (V_{1,x} + u_x V_{1,u}) + 8 F_4^{-3} V_{1,u} + \frac{4}{5} V_1 F_4^{-5} F_4, x (4 G_2 V_1^{1/2} - 3 F_5) \\
&\quad - \frac{4}{5} V_1 F_4^{-5} F_{4,u} (u_x (3 F_5 - 4 G_2 V_1^{1/2}) + 15 F_4) + \frac{8}{5} F_5 F_4^{-5} V_1 F_4^{-5} \\
&\quad + \frac{8}{5} V_3 F_5 F_4^{-5} (F_1 F_4 + 4 G_2 F_4 V_1^{1/2}) + 2 F_2 V_1 F_4^{-3} + \frac{2}{5} F_1 V_1^{3/2} F_4^{-4} (8)
\end{align*}
\]
with \( V_1 = \left| 5 F_4 F_6 - 6 F_5^2 \right| \). The functions \( G_1, ... , G_5 \) are invariants of equation (1) with respect to \( \text{Cont}_0(J^2(\pi)) \). All the other invariants can be obtained from \( G_1, ... , G_5 \) by applying the invariant derivatives
\[
\begin{align*}
D_1 &= V_1^{-3/2} F_4^{-4} D_x, \\
D_2 &= \frac{4}{5} V_1 \left( F_3 F_4^{-4} D_x + 5 F_4^{-3} \frac{\partial}{\partial u} + 5 F_4^{-4} (D_x(F_4) + 2 F_1 F_4) \frac{\partial}{\partial \eta} \right), \\
D_3 &= F_4 V_1^{-1/2} \frac{\partial}{\partial \eta}.
\end{align*}
\]
The operators \( D_1 \) and \( D_2 \) are defined by the requirement that \( dZ = D_1(Z) \omega_1 + D_2(Z) \omega_2 + D_3(Z) \omega_3 \) hold for an arbitrary function \( Z(x, u, u_x) \).

The \( s^{th} \) order classifying manifold associated with forms \( \omega \), in the case \( \mathcal{A}_1 \) has the form
\[
C^{(s)}_{\mathcal{A}_1}(\omega, U) = \left\{ D_1^i D_2^j D_3^k G_{mn}(x, u, u_x) \mid 0 \leq i + j + k \leq s, 1 \leq m \leq 5, (x, u, u_x) \in U \right\},
\]
where \( U \subset J^1(\pi) \) is an open subset such that \( 5 F_4 F_6 - 6 F_5^2 \neq 0 \) everywhere in \( U \), and operators \( D_i \) are defined by equations (7). Since all the invariants (8) depend on three variables \( x, u, u_x \), the number of functionally-independent invariants is no more than 3, and to formulate the solution of the equivalence problem it is enough to consider 3rd order classifying manifolds.
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The analysis of the case $\mathcal{A}_2$ disparts on two subcases depending on whether one of the two conditions $F_3 \equiv 0$ or $F_3 \not\equiv 0$ hold. The first condition gives equations

$$u_{xx} = A_4(x, u) u_x^4 + A_3(x, u) u_x^3 + A_2(x, u) u_x^2 + A_1(x, u) u_x + A_0(x, u),$$  \hspace{1cm} (11)

with $A_4 = F_1 \not\equiv 0$, the second condition together with the defining identity $5F_4F_6 - 6F_5^2 \equiv 0$ of subcase $\mathcal{A}_2$ gives equations

$$u_{xx} = (B_1(x, u) u_x + B_0(x, u))^{-1} + A_3(x, u) u_x^3 + A_2(x, u) u_x^2 + A_1(x, u) u_x + A_0(x, u)$$  \hspace{1cm} (12)

with $B_1 \neq 0$. However, the analysis of the second subcase can be reduced to the first one.

**Lemma 1.** Each equation (12) can be mapped into equation (11) by means of a transformation from $\text{Cont}_0(J^2(\pi))$.

**Proof:** Consider 1-form $\mu = B_1 du + B_0 dx$. For its differential we have

$$d\mu = (B_{1x} - B_{0,u}) dx \land du = (B_{1x} - B_{0,u}) B_1^{-1} dx \land \mu.$$  

This implies that form $\mu$ meets conditions of Frobenius’ theorem [20, Th. 2.4.2]. Therefore there exists function $U = U(x, u)$ such that $\mu \equiv 0 \text{ mod } dU$. The direct check shows that the change of variables $\hat{x} = U(x, u), \hat{u} = x$ maps equation (12) into equation of the form (11).

QED

Moreover, without loss of generality we can put $A_3 \equiv 0$ in equation (11).

**Lemma 2:** Each equation from the class (11) can be mapped into equation from the same class with $A_3 \equiv 0$ via a transformation from $\text{Cont}_0(J^2(\pi))$.

**Proof:** Consider 1-form $\nu = du - 3A_3 A_4^{-1} dx$. If $(A_3 A_4^{-1})_u = 0$, then $d\nu = 0$, otherwise $d\nu = 3(A_3 A_4^{-1}) u dx \land du = 3(A_3 A_4^{-1}) u dx \land \nu$. In both cases form $\nu$ meets the conditions of Frobenius’ theorem, hence there exists function $U = U(x, u)$ such that $\nu \equiv 0 \text{ mod } dU$. The direct check shows that the change of variables $\hat{x} = U(x, u), \hat{u} = x$ maps equation from the class (11) into equation from the same class with $A_3 \equiv 0$.

QED

For equation

$$u_{xx} = A_4(x, u) u_x^4 + A_2(x, u) u_x^2 + A_1(x, u) u_x + A_0(x, u)$$  \hspace{1cm} (13)

after the above normalizations we get the structure equations for the forms $\omega_i$

$$d\omega_1 = \eta_1 \land \omega_1,$$

$$d\omega_2 = \frac{\eta_1}{4} \land \omega_2 + \omega_1 \land \omega_3,$$

$$d\omega_3 = -\frac{1}{4} \eta_1 \land \omega_3 + \ldots \omega_1 \land \omega_2 + \frac{1}{18} b_4^2 u_x^2 Z_0^{-2} \omega_2 \land \omega_3,$$

where $Z_0 = A_4 u_x | 18 A_2^2 u_x^2 + 3 A_2 A_4 + A_4 |^{-1/2}$. We normalize the coefficient at $\omega_2 \land \omega_3$ in the last equation by putting $b_4 = Z_0 u_x^{-1}$. Then all the parameters of group $\mathcal{K}$ are defined as functions on $J^2(\pi)$. We obtain

$$\omega_1 = 24 Z_0^{-3} A_4 u_x^3 dx,$$

$$\omega_2 = 24 Z_0^{-2} A_4 u_x^2 (du - u_x dx),$$

$$\omega_3 = Z_0 u_x^{-1} (du_x - (A_2^2 u_x^2 + A_2 A_4 u_x + 2 A_1 A_4 + A_4 u_x) du - (A_0 A_4 - (A_1 A_4 + A_4 x) u_x) dx),$$

and $d\omega_1 = 54 Z_0 \omega_1 \land \omega_2 + \ldots \omega_1 \land \omega_3$, that is, function $Z_0$ is an invariant of equations (13) with respect to the transformations from the pseudogroup $\text{Cont}_0(J^2(\pi))$. Since forms $\omega$ and function $Z_0$ are invariant, we can take forms $\hat{\omega}_1 = \frac{1}{22} Z_3^3 \omega_1, \hat{\omega}_2 = \frac{1}{22} Z_3^2 \omega_2, \hat{\omega}_2 = Z_0^{-1} \omega_3$ instead of forms (14).

In what follows we return to the previous notation, that is we will write $\omega_i$ instead of $\hat{\omega}_i$. The structure equations for the new forms are

$$d\omega_1 = - (3 + Z_1 u_x^{-2} + 3 Z_2 u_x^{-3}) \omega_1 \land \omega_2 - 3 \omega_1 \land \omega_3,$$

$$d\omega_2 = (3 + Z_1 u_x^{-2} + 2 Z_3 u_x^{-3} + 2 Z_3 u_x^{-3}) \omega_1 \land \omega_2 + (\omega_1 - 2 \omega_2) \land \omega_3,$$

$$d\omega_3 = (Z_2 u_x^{-3} - 3 Z_3 u_x^{-4} + Z_5 u_x^{-5} + Z_4 u_x^{-6}) \omega_1 \land \omega_2 + (2 - Z_2 u_x^{-3} - Z_3 u_x^{-4}) \omega_1 \land \omega_3$$

\hspace{1cm} + (2 - Z_2 u_x^{-3}) \omega_2 \land \omega_3,
where
\[ Z_1 = A_{4u}A_4^{-1} + 3 A_2, \]
\[ Z_2 = A_{4x}A_4^{-2} + 2 A_1A_4^{-1}, \]
\[ Z_3 = A_0A_4^{-1}, \]
\[ Z_4 = (Z_{3u} - Z_{2x} + 3 A_1 Z_2 - A_2 Z_3) A_4^{-1} + Z_3 A_{4u}A_4^{-2} - 2 Z_2^2, \]
\[ Z_5 = (A_{1u} - A_{2x}) A_4^{-2} - (Z_{2u} - 3 A_2 Z_2) A_4 - Z_1 Z_2. \]

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The invariant derivatives
\[ = I \]

The case \( \mathfrak{A}_2 \) corresponds to condition \( Z_1 \neq 0 \). In this case the structure equations acquire the form
\[ d\omega_1 = -(3 H_1 F_1 + P_1^3 + 3) \omega_1 \wedge \omega_2 - 3 \omega_1 \wedge \omega_3, \]
\[ d\omega_2 = (2 H_2 P_1^3 + 2 H_1 F_1 + P_1^2 + 3) \omega_1 \wedge \omega_2 + (\omega_1 - 2 \omega_2) \wedge \omega_3, \]
\[ d\omega_3 = P_1^3 (H_3 F_1^3 + H_4 P_1^2 - 3 H_2 F_1 + H_1) \omega_1 \wedge \omega_2 - (H_2 P_1^3 + H_1 P_1^3 - 2) \omega_1 \wedge \omega_3 - (H_1 P_1^3 - 2) \omega_2 \wedge \omega_3, \]

where the unique invariant that depends on \( x, u \), and \( u_x \) is defined as \( P_1 = Z_1^{1/2} u_x^{-1} \), and its differential is
\[ dP_1 = - \frac{1}{2} P_1 (2 H_2 P_1^3 - H_6 P_3^3 - H_5 P_2^2 + 2) \omega_1 - \frac{1}{2} P_1 (2 H_1 P_3^3 - H_5 P_2^2 + 2) \omega_2 - P_1 \omega_3. \]

All the invariants that depend on \( x \) and \( u \) only are
\[ H_1 = Z_2 Z_1^{-3/2}, \]
\[ H_2 = Z_3 Z_1^{-2}, \]
\[ H_3 = Z_4 Z_1^{-3}, \]
\[ H_4 = Z_5 Z_1^{-5/2}, \]
\[ H_5 = (Z_{1u} - 2 A_2 Z_1) Z_1^{-2} A_4^{-1}, \]
\[ H_6 = (Z_{1x} - 2 A_1 Z_1) Z_1^{-5/2} A_4^{-1}. \]

The invariant derivatives \( \mathcal{D}_1 = A_1^{-1} Z_1^{-3/2} \frac{\partial}{\partial x} \) and \( \mathcal{D}_2 = A_4^{-1} Z_1^{-1} \frac{\partial}{\partial u} \) are defined by the requirement that equation \( dY = P_1^2 (HP_1^4 + H_2 P_1^2 (Y)) \omega_1 + \mathcal{D}_1 (Y) \omega_2 \) holds for every function \( Y(x, u) \). In the case \( \mathfrak{A}_2 \) the second order classifying manifold associated with forms \( \omega \) has the form
\[ C^{(2)}_{\mathfrak{A}_{21}}(\omega, \mathcal{V}) = \left\{ \mathcal{D}_1 \mathcal{D}_2 (H_m(x, u)) \mid 0 \leq i + j \leq 2, 1 \leq m \leq 6, (x, u) \in \mathcal{V} \right\}, \tag{15} \]

where \( \mathcal{V} \subset J^0(\pi) \) is an open subset such that \( F_4 \neq 0, F_5 \equiv 0 \) and \( Z_1 \neq 0 \) everywhere on it.

The case \( \mathfrak{A}_2 \) is defined by \( Z_1 \equiv 0 \) and \( Z_2 \equiv 0 \). In this case the structure equations
\[ d\omega_1 = 3 (P_2^3 + 1) \omega_2 \wedge \omega_3 \]
\[ d\omega_2 = (2 I_1 P_2^3 + 2 P_2^3 + 3) \omega_1 \wedge \omega_2 + (\omega_1 - 2 \omega_2) \wedge \omega_3, \]
\[ d\omega_3 = P_2^3 (I_2 P_2^3 + I_3 P_2^2 - 3 I_2 P_2 + 1) \omega_1 \wedge \omega_2 - ((I_1 P_2^4 + P_2^3 - 2) \omega_1 + (P_2^3 - 2) \omega_2) \wedge \omega_3 \]

contain invariant \( P_2 = Z_2^{1/3} u_x^{-1} \) that depends on \( x, u, \) and \( u_x \), with the differential
\[ dP_2 = - \frac{1}{3} P_2 (3 I_1 P_2^4 - I_5 P_2^3 - I_4 P_2^2 + 3) \omega_1 - \frac{1}{3} P_3 (3 P_2^3 + (I_3 - I_4) P_2^2 + 3) \omega_2 - P_2 \omega_3. \]

Invariants that depend on \( x \) and \( u \) only have the form
\[ I_1 = Z_3 Z_2^{-4/3}, \]
\[ I_2 = Z_4 Z_2^{-2}, \]
\[ I_3 = Z_5 Z_2^{-5/3}, \]
\[ I_4 = (A_{1u} - A_{2x}) Z_2^{-5/3} A_4^{-2}, \]
\[ I_5 = (Z_{2x} - 3 A_1 Z_2) Z_2^{-2} A_4^{-1}. \]
The invariant derivatives $\mathbb{D}_1 = A_4^{-1} Z_5^{-8} \frac{\partial}{\partial x}$ and $\mathbb{D}_2 = A_4^{-1} Z_5^{-2/3} \frac{\partial}{\partial u}$ are defined by the requirement that equation $dY = P_4^2 ((P_5 \mathbb{D}_1(Y) + \mathbb{D}_2(Y)) \omega_1 + \mathbb{D}_1(Y) \omega_2)$ holds for every function $Y(x,u)$. In the case $\mathcal{A}_{22}$ the second order classifying manifold associated with forms $\omega$ has the form
\[
C^{(2)}_{\mathcal{A}_{22}}(\omega, V) = \left\{ \mathbb{D}_1 \mathbb{D}_2 (J_m(x,u)) \mid 0 \leq i + j \leq 2, 1 \leq m \leq 5, (x,u) \in V \right\},
\]
where $V \subset J^0(\pi)$ is an open subset such that $F_4 \neq 0$, $F_5 \equiv 0$, $Z_1 \equiv 0$, and $Z_2 \neq 0$ hold everywhere in it.

The subcase $\mathcal{A}_{23}$ is defined by conditions $Z_1 \equiv 0$, $Z_2 \equiv 0$, $Z_3 \neq 0$. In this case the structure equations have the form
\[
d\omega_1 = 3 (\omega_2 + \omega_3) \wedge \omega_1,
\]
\[
d\omega_2 = 2 P_3^4 + 3 \omega_1 \wedge \omega_2 + (\omega_1 - 2 \omega_2) \wedge \omega_3,
\]
\[
d\omega_3 = P_3^4 (J_1 P_3^2 + J_3 P_3 - 3) \omega_1 \wedge \omega_2 - (P_3^4 - 2) \omega_1 - 2 \omega_2 \wedge \omega_3,
\]
where $P_3 = Z_3^{1/4} u_x^{-1}$ and
\[
dP_3 = -\frac{1}{4} P_3 (4 P_3^4 - J_3 P_3^3 - J_1 P_3^2 + 4) \omega_1 + \frac{1}{4} P_3 (J_1 P_3^3 - 4) \omega_2 - P_3 \omega_3.
\]
In this case the invariants that depend on $x$ and $u$ are
\[
J_1 = Z_4 Z_3^{-3/2},
\]
\[
J_2 = Z_5 Z_3^{-5/4},
\]
\[
J_3 = (A_4 Z_3, u + 2 Z_4 A_4) Z_3^{-7/4} A_4^{-2}.
\]
The invariant derivatives $\mathbb{D}_1 = A_4^{-1} Z_5^{-3/4} \frac{\partial}{\partial x}$ and $\mathbb{D}_2 = A_4^{-1} Z_5^{-1/2} \frac{\partial}{\partial u}$ are defined by equation $dY = P_4^2 ((P_5 \mathbb{D}_1(Y) + \mathbb{D}_2(Y)) \omega_1 + \mathbb{D}_1(Y) \omega_2)$. In the case $\mathcal{A}_{23}$ the second order classifying manifold associated with forms $\omega$ has the form
\[
C^{(2)}_{\mathcal{A}_{23}}(\omega, V) = \left\{ \mathbb{D}_1 \mathbb{D}_2 (J_m(x,u)) \mid 0 \leq i + j \leq 2, 1 \leq m \leq 3, (x,u) \in V \right\},
\]
where $V \subset J^0(\pi)$ is an open subset such that $F_1 \neq 0$, $F_5 \equiv 0$, $Z_1 \equiv 0$, $Z_2 \equiv 0$, and $Z_3 \neq 0$ in all its points.

The subcase $\mathcal{A}_{24}$ is defined by conditions $Z_1 \equiv 0$, $Z_2 \equiv 0$, $Z_3 \equiv 0$, which imply $Z_4 \equiv 0$, but $Z_5 \neq 0$. In this case the structure equations have the form
\[
d\omega_1 = 3 (\omega_2 + \omega_3) \wedge \omega_1,
\]
\[
d\omega_2 = 3 \omega_1 \wedge \omega_2 + (\omega_1 - 2 \omega_2) \wedge \omega_3,
\]
\[
d\omega_3 = -P_3^5 \omega_1 \wedge \omega_2 + (\omega_1 + \omega_2) \wedge \omega_3.
\]
They contain only one invariant $P_4 = Z_5^{1/5} u_x^{-1}$ that depends on $x$, $u$, and $u_x$, and has differential
\[
dP_4 = \frac{1}{4} P_4 (K_2 P_4^4 + K_1 P_4^2 - 5) \omega_1 + \frac{1}{4} P_4 (K_1 P_4^3 - 5) \omega_2 - P_4 \omega_3,
\]
which contains invariants
\[
K_1 = (3 A_4 Z_5, u + 5 Z_5 A_4, u) Z_5^{-7/5} A_4^{-2},
\]
\[
K_2 = (3 A_4 Z_3, u + 5 Z_5 A_4, x) Z_5^{-8/5} A_4^{-2},
\]
that depend on $x$ and $u$. The invariant derivatives $\mathbb{D}_1 = A_4^{-1} Z_5^{-3/5} \frac{\partial}{\partial x}$ and $\mathbb{D}_2 = A_4^{-1} Z_5^{-2/5} \frac{\partial}{\partial u}$ are defined by
\[
dY = P_4^2 ((P_4 \mathbb{D}_1(Y) + \mathbb{D}_2(Y)) \omega_1 + \mathbb{D}_1(Y) \omega_2).
\]
In the case $\mathcal{A}_{24}$ the second order classifying manifold associated with forms $\omega$ has the form
\[
C^{(2)}_{\mathcal{A}_{24}}(\omega, V) = \left\{ \mathbb{D}_1 \mathbb{D}_2 (K_m(x,u)) \mid 0 \leq i + j \leq 2, 1 \leq m \leq 2, (x,u) \in V \right\},
\]
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where \( V \subset J^0(\pi) \) is an open subset such that \( F_4 \neq 0, F_5 \equiv 0, Z_1 \equiv 0, Z_2 \equiv 0, Z_3 \equiv 0, Z_4 \equiv 0, \) and \( Z_5 \neq 0 \) everywhere in it.

Finally, the case \( A_{25} \) is defined by the requirements \( Z_1 \equiv Z_2 \equiv Z_3 \equiv Z_4 \equiv Z_5 \equiv 0 \). In this case the structure equations have the form

\[
\begin{align*}
d\omega_1 &= 3(\omega_2 + \omega_3) \wedge \omega_1, \\
d\omega_2 &= 3 \omega_1 \wedge \omega_2 + (\omega_1 - 2 \omega_2) \wedge \omega_3, \\
d\omega_3 &= 2(\omega_1 + \omega_2) \wedge \omega_3.
\end{align*}
\]

Their coefficients are constant. The same structure equations for forms \( \omega \) has the symmetry pseudogroup of equation

\[ u_{xx} = u_x^4. \quad (19) \]

Therefore the class of equations \( (\mathcal{A}) \) such that \( F_{u_x u_x u_x u_x} \neq 0 \) is divided on invariant subclasses \( \mathcal{A}_1, \mathcal{A}_{21}, \ldots, \mathcal{A}_{25}, \) and we construct invariants coframes \( \omega \) for each subclass. Hence the solution of the equivalence problem for equations \( (\mathcal{A}) \) is reduced to the restricted equivalence problem for invariant coframes, see \([17, \text{Ch. 8, Ch. 14}]\). The results of the above computations together with Theorem 14.24 from \([17]\) give the following theorem:

**Theorem 1:** Each equation \( (\mathcal{A}) \) such that \( F_{u_x u_x u_x u_x} \neq 0 \) can be mapped by means of a diffeomorphism from the pseudogroup of point transformations \( (\mathcal{B}) \) into an equation from one of the invariant subclasses \( \mathcal{A}_1, \mathcal{A}_{21}, \ldots, \mathcal{A}_{25}. \)

The invariant subclass \( \mathcal{A}_1 \) contains equations \( (\mathcal{A}) \) such that

\[ 5 F_{u_x u_x u_x u_x} F_{u_x u_x u_x u_x u_x u_x} - 6 F_{u_x u_x u_x u_x u_x u_x u_x}^2 \neq 0. \]

Two equations from the invariant subclass \( \mathcal{A}_1 \) are locally equivalent whenever their classifying manifolds \( (\mathcal{I}) \) are locally congruent.

Equations \( (\mathcal{A}) \) with \( F_{u_x u_x u_x u_x} \neq 0, 5 F_{u_x u_x u_x u_x} F_{u_x u_x u_x u_x u_x u_x} - 6 F_{u_x u_x u_x u_x u_x u_x u_x}^2 \equiv 0 \) can be mapped into an equation of the form \( (\mathcal{I}) \).

The invariant subclass \( \mathcal{A}_{21} \) contains equations \( (\mathcal{I}) \) such that \( Z_1 \neq 0 \), the invariant subclass \( \mathcal{A}_{22} \) contains equations \( (\mathcal{I}) \) such that \( Z_1 \equiv 0, Z_2 \neq 0 \), the invariant subclass \( \mathcal{A}_{23} \) contains equations \( (\mathcal{I}) \) such that \( Z_1 \equiv Z_2 \equiv 0, Z_3 \neq 0 \), the invariant subclass \( \mathcal{A}_{24} \) contains equations \( (\mathcal{I}) \) such that \( Z_1 \equiv Z_2 \equiv Z_3 \equiv Z_4 \equiv 0, Z_5 \neq 0 \), the invariant subclass \( \mathcal{A}_{25} \) contains equations \( (\mathcal{I}) \) such that \( Z_1 \equiv Z_2 \equiv Z_3 \equiv Z_4 \equiv Z_5 \equiv 0 \).

Equations from the invariant subclasses \( \mathcal{A}_{21}, \ldots, \mathcal{A}_{24} \) are locally equivalent with respect to the pseudogroup of point transformations \( (\mathcal{B}) \) whenever their classifying manifolds \( (\mathcal{I}), (\mathcal{I}), (\mathcal{I}), (\mathcal{I}), \) and \( \mathcal{I} \) are locally congruent. Equations from the invariant subclass \( \mathcal{A}_{25} \) are locally equivalent to equation \( (19) \).

3. The solution of the equivalence problem in the case \( F_{u_x u_x u_x u_x} = 0. \)

After the normalization \( b_3 = b_1 b_4 \) and applying the procedure of prolongation we get the structure equations of forms \( \omega_i \) for equation \( (\mathcal{A}) \):

\[
\begin{align*}
d\omega_1 &= \eta_1 \wedge \omega_1 + \eta_2 \wedge \omega_2, \\
d\omega_2 &= \eta_3 \wedge \omega_2 + \omega_1 \wedge \omega_3, \\
d\omega_3 &= (\eta_3 - \eta_1) \wedge \omega_3 + \eta_4 \wedge \omega_2, \\
d\eta_1 &= -2 \eta_4 \wedge \omega_1 + \eta_5 \wedge \omega_2 - \eta_2 \wedge \omega_3, \\
d\eta_2 &= \eta_5 \wedge \omega_1 + (\eta_1 - \eta_3) \wedge \eta_2, \\
d\eta_3 &= -\eta_4 \wedge \omega_1 + 2 \eta_5 \wedge \omega_2 + \eta_2 \wedge \omega_3, \\
d\eta_4 &= \eta_4 \wedge \eta_1 + \eta_5 \wedge \omega_3 - \frac{1}{3} b_1^{-3} b_4^{-1} (L_1 + L_2 u_x) \omega_1 \wedge \omega_2,
\end{align*}
\]

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where

\[ L_1 = 3 (A_{0,uu} + A_3 A_{0,x} - A_2 A_{0,u}) - 2 A_{1,uu} + A_{2,xx} - A_1 (3 A_{1,u} + A_{2,x}) - 3 A_0 (A_{2,u} - 2 A_{3,x}), \]

\[ L_2 = A_{1,uu} - 2 A_{2,xx} + 3 (A_{3,xx} - A_3 (A_{1,x} - 2 A_{0,u}) + A_1 A_{3,x} - A_0 A_{3,u}) + 2 A_2 (A_{1,u} - A_{2,x}). \]

The further analysis depends on whether condition \( L_1 \equiv L_2 \equiv 0 \) holds.

In the case \( \mathcal{B}_1 \), when \( L_1 \equiv L_2 \equiv 0 \), the structure equations after the prolongation acquire the form

\[
\begin{align*}
d\omega_1 &= \eta_1 \wedge \omega_1 + \eta_2 \wedge \omega_2, \\
d\omega_2 &= \eta_3 \wedge \omega_2 + \omega_1 \wedge \omega_3, \\
d\omega_3 &= (\eta_3 - \eta_1) \wedge \omega_3 + \eta_4 \wedge \omega_2, \\
d\eta_1 &= -2 \eta_4 \wedge \omega_1 + \eta_5 \wedge \omega_2 - \eta_2 \wedge \omega_3, \\
d\eta_2 &= \eta_5 \wedge \omega_1 + (\eta_1 - \eta_3) \wedge \eta_2, \\
d\eta_3 &= -2 \eta_4 \wedge \omega_1 + 2 \eta_5 \wedge \omega_2 + \eta_2 \wedge \omega_3, \\
d\eta_4 &= \eta_4 \wedge \eta_1 + \eta_5 \wedge \omega_3, \\
d\eta_5 &= \eta_2 \wedge \eta_4 + \eta_5 \wedge \eta_3.
\end{align*}
\]

The direct check shows that any second-order linear ODE \( u_{xx} = a_2(x) u_x + a_1(x) u + a_0(x) \), in particular, equation \( u_{xx} = 0 \), has the same structure equations. Hence equations (4) with \( L_1 \equiv L_2 \equiv 0 \) are equivalent to equation \( u_{xx} = 0 \), in agreement with S. Lie’s and R. Liouville’s results.

We now turn to the analysis of the case \( \mathcal{B}_2 \), in which one of the functions \( L_1 \) or \( L_2 \) is not equal to zero. We put the coefficient at \( \omega_1 \wedge \omega_2 \) in the last equation of (20) equal to \( \frac{1}{3} \). This yields \( b_4 = b_1^{-3} (L_1 + L_2 u_x) \). Then we have

\[
\begin{align*}
d\omega_1 &= \eta_1 \wedge \omega_1 + \eta_2 \wedge \omega_2, \\
d\omega_2 &= -2 \eta_2 \wedge \omega_2 + \omega_1 \wedge \omega_3 + b_1^2 (L_1 + L_2 u_x)^{-2} \left( b_2 (L_1 + L_2 u_x) - b_1 L_2 \right) \omega_2 \wedge \omega_3.
\end{align*}
\]

Further, we put the coefficient at \( \omega_2 \wedge \omega_3 \) in the last equation equal to zero and obtain \( b_2 = b_1 L_2 (L_1 + L_2 u_x)^{-1} \). Now we have \( \omega_1 = b_1 (L_1 + L_2 u_x)^{-1} (L_1 dx + L_2 du) \) and \( d\omega_1 = \eta_1 \wedge \omega_1 \). Therefore \( \omega_1 \) meets the conditions of Frobenius’ theorem, so there are functions \( X(x,u) \) and \( \bar{b}_1(x,u,x,b_1,b_5) \) such that \( \omega_1 = \bar{b}_1 dX \). Let \( U(x,u) \) be any function such that \( dX \wedge dU \neq 0 \). Then we have \( \omega_1 = \bar{b}_1 d\bar{x} \) after the change of variables \( \bar{x} = X(x,u), \bar{u} = U(x,u) \). In the new variables, equality \( \omega_2 = \bar{b}_1 \left( \bar{L}_1 + \bar{L}_2 \bar{u}_x \right)^{-1} \left( \bar{L}_1 d\bar{x} + \bar{L}_2 d\bar{u} \right) \) should hold. This implies \( \bar{L}_2 = 0 \), that is, we get Tresse’s result, \( \bar{L}_1 \neq 0 \) and \( \bar{L}_2 \equiv 0 \) in the new coordinates \( \bar{x}, \bar{u} \). If it is necessary, we make this change of variables, therefore without loss of generality we assume that conditions \( L_1 \neq 0, L_2 \equiv 0 \) hold for equation (3).

Now the structure equations have the form

\[
\begin{align*}
d\omega_1 &= \eta_1 \wedge \omega_1, \\
d\omega_2 &= -2 \eta_1 \wedge \omega_2 + \omega_1 \wedge \omega_3, \\
d\omega_3 &= \eta_2 \wedge \omega_2 + \frac{1}{3} b_1^{-1} L_1^{-1} \left( 5 b_1^2 b_5 + 6 A_3 L_1 u_x^2 + 4 A_2 L_1 - L_1 u_x + 2 A_1 L_1 - L_1 \omega_1 \wedge \omega_3 - 3 \eta_1 \wedge \omega_3 \right)
\end{align*}
\]

We can assume \( b_5 = -\frac{1}{3} b_1^{-3} (6 A_3 L_1 u_x^2 + 4 A_2 L_1 - L_1 u_x + 2 A_1 L_1 - L_1 \omega_1 \wedge \omega_3) \). Then we obtain

\[
\begin{align*}
d\omega_1 &= \eta_1 \wedge \omega_1, \\
d\omega_2 &= -2 \eta_1 \wedge \omega_2 + \omega_1 \wedge \omega_3, \\
d\omega_3 &= -3 \eta_1 \wedge \omega_3 + \frac{4}{3} b_1^{-2} L_1^{-2} \left( 3 A_3 L_1 u_x + A_2 L_1 + L_1 \omega_2 \wedge \omega_3 + \ldots \omega_1 \wedge \omega_2 \right).
\end{align*}
\]

The further analysis separates on two cases: the case \( \mathcal{B}_2 \) such that \( A_3 \neq 0 \), and the case \( \mathcal{B}_2 \) such that \( A_3 \equiv 0 \).
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In the case $\mathcal{B}_{21}$ we put

$$b_1 = L_1 \left| 3 A_3 L_1 u_x + A_2 L_1 + L_{1,u} \right|^{-1/2}. \quad \text{(21)}$$

This yields $d\omega_1 = \frac{3}{5} M_0^2 \omega_1 \wedge \omega_3 + \ldots \omega_1 \wedge \omega_2$, where $M_0 = A_3^{1/5} L_1^{3/5} \left| 3 A_3 L_1 u_x + A_2 L_1 + L_{1,u} \right|^{-1/2}$, and $\omega_1 = M_0 A_3^{-1/3} \frac{L_1^{2/5}}{dx}$, $\omega_2 = M_0^{-2} A_3^{2/5} L_1^{1/5} (du - u_x dx)$, $\omega_3 = M_0^{-3} A_3^{5/7} L_1^{-1/5} \left( du_x - \frac{1}{3} \left( 6 A_3 u_x^2 + (4 A_2 - L_{1,u} L_1^{-1}) u_x + 2 A_1 - L_{1,x} L_1^{-1} \right) du + \frac{2}{3} (3 A_3 u_x^2 - (A_2 + L_{1,u} L_1^{-1} u_x^2 - (3 A_1 + L_{1,x} L_1^{-1}) u_x - 5 A_0) dx \right). \quad \text{(22)}$

Since forms $\omega$ and function $M_0$ are invariant with respect to a diffeomorphism $\Psi$, we can multiply, without loss of generality, the right hand sides of forms $\omega_1$, $\omega_2$, and $\omega_3$ by $M_0^{-1}$, $M_0^2$, and $M_0^3$, respectively. We denote the obtained forms as $\omega_1$, $\omega_2$, and $\omega_3$ again. These forms have the following structure equations:

$$d\omega_1 = M_1 \omega_1 \wedge \omega_2,$$
$$d\omega_2 = \frac{2}{5} \left( P_2^2 + 5 M_1 P_3 + M_2 \right) \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3,$$
$$d\omega_3 = - \frac{1}{675} \left( 2 P_3^2 + 25 M_1 P_3^2 + 3 \left( 10 M_2 - M_3 \right) P_2^2 - M_1 P_5 - M_3 \right) \omega_1 \wedge \omega_2 + \frac{1}{5} \left( P_2^2 + 5 M_1 P_5 + M_2 \right) \omega_1 \wedge \omega_3 + \frac{1}{5} \left( 4 P_5 + 15 M_1 \right) \omega_2 \wedge \omega_3,$$

where $P_5 = A_3^{-2/5} L_1^{-6/5} \left( 3 A_3 L_1 u_x + A_2 L_1 + L_{1,u} \right)$ is the only differential invariant that depends on $x$, $u$, and $u_x$, while

$$dP_3 = \frac{1}{90} \left( 10 P_3^2 + 90 M_1 P_3^2 + 6 \left( 3 M_2 + M_3 \right) P_5 + 15 M_5 \right) \omega_1 + 3 \omega_3 + \frac{1}{5} \left( 2 P_2^2 + 15 M_1 P_5 + M_3 \right) \omega_2,$$

and invariants $M_1$, $\ldots$, $M_6$ depend on $x$ and $u$:

$$M_1 = \frac{1}{7} A_3^{-7/5} L_1^{-6/5} (L_1 A_{3,u} - 2 A_3 L_{1,u}),$$
$$M_2 = - A_3^{-4/5} L_1^{12/5} \left( L_1^{2,u} + (2 A_2 + 5 M_1 A_3^{2/5} L_1^{1/5}) L_1 L_{1,u} + L_1^2 (A_2 - 3 A_1 A_3 - 3 A_{3,x} + 5 M_1 A_3^{2/5} L_1^{1/5}) \right),$$
$$M_3 = A_3^{-4/5} L_1^{-7/5} \left( 5 L_{1,u} - 12 L_1^{-1} L_1^{2,u} - (9 A_2 + 25 M_1 A_3^{2/5} L_1^{1/5}) L_{1,u} - 3 A_3 L_{1,x} - L_1 (2 A_2^2 - 6 A_1 A_3 - 5 A_{2,u} + 25 M_1 A_3^{2/5} L_1^{1/5}) \right),$$
$$M_4 = A_3^{-1/5} L_1^{-6/5} \left( 90 L_{1,,u} + 18 A_3^{-1} L_1^{-2} L_{1,u} + 3 A_3^{-1} L_1^{-1} (8 A_2 + 25 M_1 A_3^{2/5} L_1^{1/5}) L_{1,u} - 18 L_1^{-1} (6 L_{1,u} + A_2 L_1) L_{1,x} + 135 L_1 A_{1,u} - 180 L_1 A_{2,u} + 6 A_3^{-1} (4 A_2^2 + 6 A_1 A_3 + 25 M_1 A_2 A_3^{2/5} L_1^{1/5}) L_{1,u} + A_3^{-1} L_1 (8 A_2^2 + 36 A_1 A_2 A_3 - 540 A_0 A_3^2 + 6 (10 M_2 - M_3) A_2 A_3^{4/5} L_1^{1/5} + 75 M_1 A_2 A_3^{2/5} L_1^{1/5}) \right),$$
$$M_5 = A_3^{2/5} L_1^{-9/5} \left( 135 L_{1,,x} + 2 A_3^{-2} L_{1}^3 L_{1,u} + A_3^{-2} L_{1}^{-3} (8 A_2 + 25 M_1 A_3^{2/5} L_1^{1/5}) L_{1,u} - 27 A_1 L_{1,x} + 3 A_3^{-2} L_1^{-1} (4 A_2^2 + (10 M_2 - M_3) A_3^{4/5} L_1^{2/5} + 25 M_1 A_2 A_3^{2/5} L_1^{1/5}) L_{1,u} - 162 L_1^{-1} L_{1,x} + A_3^{-2} (8 A_2^2 + 135 A_0 A_3^2 + 6 (10 M_2 - M_3) A_2 A_3^{4/5} L_1^{2/5} + 75 M_1 A_2 A_3^{2/5} L_1^{1/5} - M_4 A_3^{2/5} L_1^{1/5}) L_{1,u} - 270 L_1 A_{2,x} + 675 L_1 A_{0,u} + 2 A_2 A_3^{-2} L_1 + 25 M_1 A_2 A_3^{2/5} L_1^{-8/5} L_1^{1/5} + 3 (10 M_2 - M_3) A_2 A_3^{6/5} L_1^{-7/5} + A_2 L_1 (M_4 A_3^{4/5} L_1^{3/5} + 540 A_0) + 162 L_1^2 L_{1,x} \right),$$
$$M_6 = - A_3^{-6/5} L_1^{-18/5} \left( 2 L_1^3 + 3 (2 A_2 L_1 + 5 M_1 A_3^{2/5} L_1^{6/5}) L_{1,u} + 6 L_1^2 (A_2^2 + M_1 A_2 A_3^{2/5} L_1^{1/5} + M_2 A_3^{4/5} L_1^{2/5}) L_{1,u} + 9 A_3 L_1^3 (A_{1,u} - 2 A_2 x) + 2 L_1^3 (A_2^2 - 27 A_0 A_3^2) + 15 M_1 A_2 A_3^{2/5} L_1^{16/5} + 6 M_2 A_2 A_3^{4/5} L_1^{17/5} \right).
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The invariant derivatives $\mathbb{D}_1 = A_1^{1/5} L_1^{-2/5} \frac{\partial}{\partial x} - A_3^{-4/5} L_1^{-7/5} (L_{1,u} + A_2 L_1) \frac{\partial}{\partial u}$ and $\mathbb{D}_2 = A_3^{-2/5} L_1^{-1/5} \frac{\partial}{\partial u}$ are defined by the requirement that $dY = \frac{1}{5} (\mathbb{D}_1(Y) + P_b \mathbb{D}_2(Y)) \omega_1 + \mathbb{D}_2(Y) \omega_2$ holds for any function $Y(x,u)$. The second order classifying manifold associated with forms $\omega$ in the case $\mathcal{B}_{221}$ has the form

$$C_{\mathcal{B}_{221}}^{(2)}(\omega, V) = \left\{ \mathbb{D}_1 \mathbb{D}_2 (M_m(x,u)) \mid 0 \leq i + j \leq 2, \ 1 \leq m \leq 6, \ (x,u) \in V \right\},$$

where $V \subset J^0(\pi)$ is an open subset such that $A_3 \neq 0$ in all its points.

The case $\mathcal{B}_{22}$ is defined by the requirement $A_3 = 0$. This case is separated on two subcases:

- subcase $\mathcal{B}_{221}$ corresponds to the condition $N_0 = L_{1,u} + A_1 L_1 \neq 0$, while subcase $\mathcal{B}_{222}$ is defined by $N_0 \equiv 0$.

In the case $\mathcal{B}_{221}$ normalization (21) has the form $b_1 = L_1 |N_0|^{-1/2}$. This gives the structure equations

$$d\omega_1 = N_1 \omega_1 \wedge \omega_2,$$

$$d\omega_2 = \left( \frac{6}{5} (5 N_2 + 6) P_6 + N_2 \right) \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3,$$

$$d\omega_3 = \left( \frac{6}{5} (5 N_1 + 6) P_6^2 + \frac{1}{5} (2 N_2 + 3 N_4) P_6 + N_3 \right) \omega_1 \wedge \omega_2 + \frac{3}{5} (5 N_1 + 4) \omega_2 \wedge \omega_3 + \frac{1}{10} (2 (5 N_1 + 6) P_6 + 15 N_2) \omega_1 \wedge \omega_3,$$

where for the invariant $P_6 = N_0^3 L_1^{-1} u_x$ the following equation holds

$$dP_6 = \frac{6}{5} ((5 N_1 + 4) P_6 - N_5) \omega_2 + 3 \omega_3 + ((N_1 + 1) P_6^2 + \frac{1}{10} (15 N_2 - 2 N_5) P_6 + N_6) \omega_1,$$

while the other invariants

$$N_1 = 3 N_0^{-3} (L_1 N_{0,u} + A_2 L_1 N_0) - 3,$$

$$N_2 = 2 L_1^{-1} (N_{0,x} - A_1 N_0) - \frac{6}{5} N_5,$$

$$N_3 = \frac{1}{5} N_0 L_1^{-1} N_{5,x} + N_0^2 L_1^{-1} A_{0,u} + \frac{1}{5} N_0^2 L_1^{-3} A_0 (N_5^2 - 5 A_2 L_1) + \frac{1}{25} N_5^2 - \frac{1}{10} N_2 N_5,$$

$$N_4 = N_0^{-1} (A_{1,u} - 2 A_{2,x}),$$

$$N_5 = N_0 L_1^{-2} (L_{1,u} - 2 A_1 L_1),$$

$$N_6 = A_0 N_0^3 L_1^{-3}$$

depend on $x$ and $u$, the invariant derivatives $\mathbb{D}_1 = N_0 L_1^{-1} \frac{\partial}{\partial x}$ and $\mathbb{D}_2 = N_0^{-2} L_1 \frac{\partial}{\partial u}$ are defined by the equation $dY = (\mathbb{D}_1(Y) + \mathbb{D}_2(Y) P_b) \omega_1 + 3 \mathbb{D}_2(Y) \omega_2$. The second order classifying manifold associated with forms $\omega$ in the case $\mathcal{B}_{221}$ has the form

$$C_{\mathcal{B}_{221}}^{(2)}(\omega, V) = \left\{ \mathbb{D}_1 \mathbb{D}_2 (M_m(x,u)) \mid 0 \leq i + j \leq 2, \ 1 \leq m \leq 6, \ (x,u) \in V \right\},$$

where $V \subset J^0(\pi)$ is an open subset such that $A_3 \equiv 0$ and $N_0 \neq 0$ in all its points.

In the case $\mathcal{B}_{222}$ which is defined by the conditions $A_3 \equiv 0$ and $N_0 \equiv 0$ we get the structure equations

$$d\omega_1 = \eta_1 \wedge \omega_1,$$

$$d\omega_2 = -2 \eta_1 \wedge \omega_2 + \omega_1 \wedge \omega_3,$$

$$d\omega_3 = -3 \eta_1 \wedge \omega_1 + \frac{1}{5} b_1^{-2} \left( 3 (A_{1,u} - 2 A_{2,x}) u_x + 5 Q_0 L_1^{-2} \right) \omega_1 \wedge \omega_2,$$

where we denote

$$Q_0 = \frac{1}{5} (L_1 L_{1,xx} - 2 A_{1,x} L_1^2) + \frac{1}{25} (6 A_0^2 L_1^2 - A_1 L_1 L_{1,x} - 6 L_1^2) + L_1^2 (A_{0,u} - A_0 A_2).$$

We consider two cases in the further analysis, the case $\mathcal{B}_{2221}$ is defined by the condition $V_0 = A_{1,u} - 2 A_{2,x} \neq 0$, while the case $\mathcal{B}_{2222}$ is defined by the condition $A_{1,u} - 2 A_{2,x} \equiv 0$.

In the case $\mathcal{B}_{2221}$ the normalization $b_1 = |15 (A_{1,u} - 2 A_{2,x}) u_x + 25 Q_0 L_1^{-2}|^{1/2}$ gives the structure equation $d\omega_1 = -\frac{25}{2} P_7^{1/2} \omega_1 \wedge \omega_2 + \ldots \omega_1 \wedge \omega_3$, with the invariant $P_7 = \frac{1}{5} V_0^2 L_1^{-4} (3 V_0^2 L_1^2 u_x + 5 Q_0)$. 

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We have

\[ \begin{align*}
\omega_1 &= \frac{1}{5} P_7^{1/2} L_1 V_0^{-1} \, dx, \\
\omega_2 &= \frac{1}{7} P_7^{-1} L_1^{-1} V_0 (du - u_x \, dx), \\
\omega_3 &= \frac{1}{375} P_7^{-3/2} L_1^{-3} V_0^3 \left( L_1 \, du_x - \frac{1}{5} (5 A_2 L_1 u_x + 2 A_1 L_1 - L_1, x) \, du \right. \\
&\hspace{1cm} \left. - \frac{1}{7} ((3 A_1 L_1 + L_1, x) u_x + 5 A_0 L_1) \, dx \right).
\end{align*} \]

Since forms \( \omega \) and function \( P_7 \) are invariant, we can divide, without loss of generality, the right-hand sides of forms \( \omega_1, \omega_2, \) and \( \omega_3 \) by \( \frac{1}{5} P_7^{1/2}, \frac{1}{7} P_7^{-1}, \) and \( \frac{1}{375} P_7^{-3/2} \), respectively. The obtained forms satisfy the structure equations

\[ \begin{align*}
d\omega_1 &= 0, \\
d\omega_2 &= Q_1 \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3, \\
d\omega_3 &= P_7 \omega_1 \wedge \omega_2 + \frac{3}{5} Q_1 \omega_1 \wedge \omega_3,
\end{align*} \]

while the differential of the invariant \( P_7 \) acquires the form

\[ dP_7 = \frac{1}{18} ((31 Q_1 + 10) P_7 + 18 Q_2) \omega_1 + \frac{1}{18} (2 Q_1 + 5) \omega_2 + \frac{3}{5} \omega_3, \]

and invariants

\[ \begin{align*}
Q_1 &= 2 V_{0, x} L_1^{-1} - \frac{2}{3} V_0 L_1^{-2} (3 L_{1, x} - A_1 L_1), \\
Q_2 &= \frac{2}{3} A_0 V_0^4 L_1^{-3} + \frac{1}{5} V_0^3 L_1^{-6} (5 L_{1, x} Q_0 - 2 Q_0 (A_1 L_1 + 7 L_{1, x})) - \frac{1}{18} Q_0 V_0^2 L_1^{-4} (13 Q_1 + 10)
\end{align*} \]

depend on \( x \) and \( u \). Then equation \( dY = (D_1 (Y) + \frac{2}{3} P_7 D_2 (Y)) \omega_1 + D_2 (Y) \omega_2 \) defines the invariant derivatives \( D_1 = V_0 L_1^{-1} \frac{\partial}{\partial u} - \frac{2}{3} V_0 L_1^{-3} \frac{\partial}{\partial u} \) and \( D_2 = V_0^{-3} L_1 \frac{\partial}{\partial u} \). The second order classifying manifold associated with forms \( \omega \) in the case \( \mathcal{P}_{2222} \) has the form

\[ C^{(2)}_{\mathcal{P}_{2222}} (\omega, \mathcal{J}_2) = \left\{ (D_1 D_2 (Q_m (x, u))) \mid 0 \leq i + j \leq 2, 1 \leq m \leq 2, (x, u) \in \mathcal{V} \right\}, \]

where \( \mathcal{V} \subset J^9(\pi) \) is an open subset such that \( A_3 \equiv N_0 \equiv 0 \) and \( V_0 \neq 0 \) in all its points.

In the case \( \mathcal{P}_{2222} \), which is defined by the requirement \( A_{1, u} - 2 A_{2, x} \equiv 0 \), there exists a function \( B(x, u) \) such that \( A_1 = 2 B_x, A_2 = B_u \), that is, equation (31) in this case has the form

\[ u_{xx} = B_u \omega^2 + 2 B_x u_x + A_0. \]

We use the following result of [1].

**Lemma 3:** Each equation (27) can be mapped into equation of the form

\[ u_{xx} = A_0 (x, u), \]

by means of a transformation from \( \text{Cont}_0 (J^2(\pi)) \).

**Proof:** Let \( U = U(x, u) \) be a function such that \( U_u = \exp (-B) \). Then the direct check shows that the change of variables \( \bar{x} = x, \bar{u} = U(x, u) \) transforms equation (26) into equation (27). QED

For equation (27) the structure equations acquire the form

\[ \begin{align*}
d\omega_1 &= \eta_1 \wedge \omega_1, \\
d\omega_2 &= -2 \eta_1 \wedge \omega_2 + \omega_1 \wedge \omega_3, \\
d\omega_3 &= -3 \eta_1 \wedge \omega_1 + \ldots \omega_1 \wedge \omega_2 + \frac{1}{5} b_1^2 A_{0, uu} A_{0, uu}^{-2} \omega_2 \wedge \omega_3.
\end{align*} \]

The further analysis depends on whether condition \( A_{0, uu} \neq 0 \) holds.

In the case \( \mathcal{P}_{2222} \), which is defined by this condition, the normalization \( b_1 = A_{0, uu} \left| A_{0, uu} \right|^{-1/2} \) gives the structure equations

\[ \begin{align*}
d\omega_1 &= R_1 \omega_1 \wedge \omega_2, \\
d\omega_2 &= \left( \frac{2}{5} (5 R_1 + 2) P_8 + R_2 \right) \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3, \\
d\omega_3 &= \left( \frac{2}{25} (5 R_1 + 2) P_8^2 + \frac{2}{5} R_2 P_8 + R_3 \right) \omega_1 \wedge \omega_2 + \frac{3}{10} (2 (5 R_1 + 2) P_8 + 5 R_2) \omega_1 \wedge \omega_3 \\
&\hspace{1cm} + (3 R_1 + \frac{2}{5}) \omega_2 \wedge \omega_3,
\end{align*} \]
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where the invariant $P_8 = |A_{0,uuu}|^{3/2} A_{0,uu}^{-2} u_x$ depends on $x, u, u_x$, while

$$dP_8 = ((3 R_1 + 1) P_8^2 + (R_4 + \frac{3}{2} R_2) P_8 + R_5) \omega_1 + ((3 R_1 + \frac{4}{5}) P_8 + R_4) \omega_2 + \omega_3,$$

and the invariants

$$R_1 = \frac{1}{2} A_{0,uuu} A_{0,xu} A_{0,xxx} - 1,$$
$$R_2 = (5 R_1 + 1) R_4 - 5 R_{4,uuu} A_{0,uuu} A_{0,xxx},$$
$$R_3 = A_{0,uuu} A_{0,xxx} - R_{4,uuu} |A_{0,uuu}|^{1/2} A_{0,xxx}^{-1} A_{0,xxx} + \frac{5}{2} R_5 + \frac{1}{2} R_4 (R_2 + 2 R_4),$$
$$R_4 = - \frac{1}{2} A_{0,uuu} |A_{0,uuu}|^{1/2} A_{0,xxx}^{-2},$$
$$R_5 = A_0 A_{0,uuu} A_{0,xxx}^{-3}$$

depend on $x, u$. The identity $dY = (D_1(Y) + D_2(Y)) P_1 + D_2(Y) \omega_2$ defines the invariant derivatives $D_1 = |A_{0,uuu}|^{1/2} A_{0,uuu} \frac{\partial}{\partial y}$ and $D_2 = A_{0,uuu} A_{0,xx} \frac{\partial}{\partial u}$. The second order classifying manifold associated with forms $\omega$ in the case $\mathcal{B}_{22222}$ has the form

$$C^{(2)}_{\mathcal{B}_{22222}}(\omega, V) = \left\{ D_1(D_2(R_m(x, u))) \mid 0 \leq i + j \leq 2, 1 \leq m \leq 5, (x, u) \in V \right\},$$

where $V \subset J^0(\pi)$ is an open subset such that $A_{0,uuu} \neq 0$ in all its points.

In the case $\mathcal{B}_{22222}$, when $A_{0,uuu} \equiv 0$, the form of equation (27) can be specified.

**Lemma 4:** Each non-linearizable equation (27) with $A_{0,uuu} \equiv 0$ can be mapped into equation of the form

$$u_{xx} = u^2 + a_0(x)$$

by means of a transformation from the pseudogroup 4.

**Proof:** If $A_{0,uuu} \equiv 0$, then a non-linearizable equation (27) has the form $u_{xx} = a_2(x) u^2 + a_1(x) u + a_0(x)$ with $a_2 \neq 0$. The change of variables $\bar{x} = \varphi(x)$, $\bar{u} = (a_2(x))^{2/5} u + b_0(x)$, where $\varphi(x)$ is a function such that $\varphi_x = a_2^{2/5}$, while $b_0 = \frac{1}{50} a_2^{-14/5} (5 a_2 a_{2,xx} - 6 a_{2,xx}^2 + 25 a_1 a_2^2)$, maps this equation into equation $\bar{u}_{\bar{x}\bar{x}} = \bar{u}^2 + a_0(\bar{x})$ with $a_0(\varphi(x)) = (a_2(x))^{-4/5} (b_0(x) + a_0(x)) - \frac{5}{6} (a_2(x))^{-9/5} a_{2,xx} b_0(x) - (b_0(x))^2$.

QED

For equation (29) the structure equations are of the form

$$d\omega_1 = - \frac{i}{2} \omega_1 \wedge \omega_2,$$
$$d\omega_2 = - P_3 \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3,$$
$$d\omega_3 = 2 \omega_1 \wedge \omega_2 - \frac{3}{2} P_3 \omega_1 \wedge \omega_3 - \frac{3}{2} \omega_2 \wedge \omega_3,$$

where $P_3 = u_x u^{-3/2}$.

In the case $\mathcal{B}_{222222}$ such that $a_0 \equiv 0$, that is, in the case when equation (29) is $u_{xx} = u^2$, we have

$$dP_3 = \left(1 - \frac{3}{2} P_3^2\right) \omega_1 - \frac{3}{2} P_3 \omega_2.$$

In the case $\mathcal{B}_{222222}$ such that $a_0 \neq 0$ we obtain

$$dP_3 = (S_1 + 1 - \frac{3}{2} P_3^2) \omega_1 - \frac{3}{2} P_3 \omega_2,$$

where $S_1 = a_0 u^{-2}$. Consider the subcase $\mathcal{B}_{2222222}$ such that $a_{0,x} \equiv 0$. In this case we get

$$dS_1 = -2 S_1 (P_3 \omega_1 + \omega_2).$$

Therefore all the equations (29) in this case have the same structure equations and are equivalent to each other, in particular, all of them are equivalent to equation $u_{xx} = u^2 + 1$.

Finally, in the last subcase $\mathcal{B}_{2222222}$ such that $a_{0,x} \neq 0$ we obtain

$$dS_1 = S_1 (T_1 S_1^{1/4} - 2 P_3) \omega_1 - 2 S_1 \omega_2,$$
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where \( T_1 = a_{0,x}a_0^{-5/4} \), and

\[
dT_1 = \left( T_2 - \frac{5}{4} S_2 \right) \omega_1,
\]

where \( T_2 = a_{0,x}a_0^{-3/2} \). Therefore the first order classifying manifold associated with forms \( \omega \) in the case \( \mathcal{B}_{2222222} \) for equation (29) can be taken in the form

\[
C^{(1)}_{\mathcal{B}_{2222222}}(\omega, I) = \left\{ (a_0(x)(a_0(x))^{-5/4}, a_{0,xx}(x)(a_0(x))^{-3/2}) \mid x \in I \right\}, \tag{30}
\]

where \( I \subset \mathbb{R} \) is an open interval such that \( a_{0,x} \neq 0 \) in all its points.

Combining the results of the above computations and applying Theorem 14.24 from [17], we get the following theorem.

**Theorem 2:** Each equation (3) can be transformed to an equation from one of the invariant subclasses \( \mathcal{B}_1, \ldots, \mathcal{B}_{2222222} \) by a diffeomorphism for the pseudogroup of point transformations (2).

- **Subclass** \( \mathcal{B}_1 \) contains equations (3) with \( L_1 \equiv L_2 \equiv 0 \). These equations are locally equivalent to equation \( u_{xx} = 0 \).

  - Equations (3) such that one of the functions \( L_1 \) or \( L_2 \) is not equal to zero, can be mapped to an equation with \( L_1 \neq 0, L_2 \equiv 0 \). These equations are divided on invariant subclasses \( \mathcal{B}_{21}, \ldots, \mathcal{B}_{2222222} \).

  - **Subclass** \( \mathcal{B}_{21} \) contains equations (3) with \( L_1 \neq 0, L_2 \equiv 0, A_3 \neq 0 \). Two equations from this subclass are locally equivalent with respect to the pseudogroup (3) whenever their classifying manifolds (29) locally overlap.

  - **Subclass** \( \mathcal{B}_{221} \) contains equations (3) with \( L_1 \neq 0, L_2 \equiv A_3 \equiv 0, N_0 \neq 0 \). Two equations from this subclass are locally equivalent with respect to the pseudogroup (3) whenever their classifying manifolds (29) locally overlap.

  - **Subclass** \( \mathcal{B}_{2221} \) contains equations (3) with \( L_1 \neq 0, L_2 \equiv A_3 \equiv N_0 \equiv 0, V_0 \neq 0 \). Two equations from this subclass are locally equivalent with respect to the pseudogroup (3) whenever their classifying manifolds (29) locally overlap.

  - Equations (3) with \( L_1 \neq 0, L_2 \equiv A_3 \equiv N_0 \equiv V_0 \equiv 0 \) can be mapped into equations of the form (27).

  - **Subclass** \( \mathcal{B}_{22221} \) contains equations (27) with \( A_{0,uuu} \neq 0 \). Two equations from this subclass are locally equivalent with respect to the pseudogroup (3) whenever their classifying manifolds (29) locally overlap.

    - Equations (27) with \( A_{0,uuu} \equiv 0 \) can be mapped into equations of the form (29).

    - **Subclass** \( \mathcal{B}_{222221} \) contains one equation \( u_{xx} = u^2 \).

    - **Subclass** \( \mathcal{B}_{2222221} \) contains equations \( u_{xx} = u^2 + \alpha, \alpha = \text{const}, \alpha \neq 0 \). All these equations are equivalent to each other, in particular, all of them are equivalent to equation \( u_{xx} = u^2 + 1 \).

    - **Subclass** \( \mathcal{B}_{2222222} \) contains equations (29) with \( a_{0,x} \neq 0 \). Two equations from this subclass are locally equivalent with respect to the pseudogroup (3) whenever their classifying manifolds (30) locally overlap.

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