Search of fractal space-filling curves with minimal dilation

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Abstract

We introduce an algorithm for a search of extremal fractal curves in large curve classes. It heavily uses SAT-solvers — heuristic algorithms that find models for CNF boolean formulas. Our algorithm was implemented and applied to the search of fractal surjective curves $\gamma: [0, 1] \to [0, 1]^d$ with minimal dilation

$$\sup_{t_1 < t_2} \frac{\|\gamma(t_2) - \gamma(t_1)\|^d}{t_2 - t_1}.$$

We report new results of that search in the case of Euclidean norm. We have found a new curve that we call “YE”, a self-similar (monofractal) plane curve of genus $5 \times 5$ with dilation $5^{43}_{73} = 5.5890\ldots$. In dimension 3 we have found facet-gated bifractals (that we call “Spring”) of genus $2 \times 2 \times 2$ with dilation $< 17$. In dimension 4 we obtained that there is a curve with dilation $< 62$.

Some lower bounds on the dilation for wider classes of cubically decomposable curves are proved.

1 Introduction

A $d$-dimensional space-filling curve is a continuous map $\gamma: [0, 1] \to \mathbb{R}^d$ having image with non-empty interior. The first examples of such
curves for \( d = 2,3 \) were constructed by G. Peano in 1890. In the computer era space-filling curves have found many practical applications, see [B13]. The point is that these curves allow one to efficiently order multidimensional lattices, which is important in some computations.

The constructions of space-filling curves are based on the idea of self-similarity. Let us consider curves with cubic image \([0, 1]^d\). Suppose that the cube is divided on a number of sub-cubes and the curve on each sub-cube (“fraction”) is similar to the whole curve. Such curve is a monofractal and the number of fractions is curve’s genus. In this paper we consider also multifractal curves. We say that a tuple of curves is a multifractal, if each fraction of each curve is similar to one of the curves (the precise definition is given below). This class is rich enough to contain many interesting examples, and simple enough to make computations with it.

An important property of Peano curves is their Hölder continuity. It is easy to see that space-filling curves \( \gamma: [0, 1] \to \mathbb{R}^d \) are not \( \alpha \)-Hölder for \( \alpha > 1/d \), but fractal curves are indeed \( (1/d) \)-Hölder. See [S10] for more details. So we have a natural measure of curve smoothness, or “locality”:

\[
\sup_{t_1 < t_2} \frac{\|\gamma(t_2) - \gamma(t_1)\|^d}{t_2 - t_1},
\]

that we call the “dilation”. In this paper we treat the case of the Euclidean norm \( \|x\| = \|x\|_2 \).

We introduce an algorithm for a search of extremal fractal curves in large curve classes. It relies on SAT-solvers — heuristic algorithms that find models for CNF boolean formulas. We used our algorithm to find some new curves with small dilation.

The monofractal Peano curve of genus \( 2 \times 2 \) is unique (up to isometry). This is the most popular plane-filling curve — the Hilbert curve. It has dilation 6, as proved by K. Bauman [B06]. Peano monofractals of genus \( 3 \times 3 \) were studied in detail in the papers of E.V. Shchepin and K. Bauman [SB08, B11]. It was proven that the minimal dilation equals \( 5\frac{2}{3} \) and the unique minimal diagonal curve was found. The same curve was found independently by H. Haverkort and F. Walderveen [HW10] and named “Meurthe”. On the other hand, E.V. Shchepin [S04] and K. Bauman [B14] proved that any Peano monofractal has the dilation at least 5. The best known plane Peano multifractal is the \( \beta \Omega \) curve, which has dilation 5.
One of our results is a new curve that we call “YE”, a monofractal plane curve of genus $5 \times 5$ with dilation $\frac{43}{73} = 5.5890\ldots$, which is best-known among plane monofractals. Moreover, the YE-curve is the unique minimal curve among monofractals of genus $\leq 6 \times 6$. We give a proof of minimality, which relies both on computations and theoretical results. This result (together with the algorithm) was announced in [MS20].

In the 3D case much of the research was done by H. Haverkort. We refer to his papers [H11] and [H17].

We consider classes of facet-gated multifractals, i.e. multifractals with entrances and exits on facets. These classes are rigid enough to allow an efficient search, and contain many curves with small dilation. In dimension 3 we have found facet-gated bifractals of genus $2 \times 2 \times 2$ with dilation less than 17. We give a name “Spring” for this family of curves. In dimension 4 we obtained that there is a curve with dilation less than 62. As we are aware, these new curves are the best known (in terms of the dilation) in both dimensions 3 and 4.

Some lower bounds on the dilation for wider classes of cubically decomposable curves are proved.

Let us outline the structure of the paper. In the next section we give necessary definitions. In Section 3 we state the results of the paper and compare them with known results, see Table 1. In Section 4 we describe our algorithm, give a computer-assisted proof of the minimality of the YE-curve and provide statistics of runs of our algorithm. Section 5 is devoted to the proofs of lower bounds for the dilation. In the end of our paper we discuss possible directions of further work.

### 2 Definitions

**Dilation.** Recall that a map $f$ between metric spaces $(T, \rho)$ and $(X, \sigma)$ is called Hölder continuous with exponent $\alpha$ (or $\alpha$-Hölder, for short) if for some $M$ the inequality holds

$$\sigma(f(t_1), f(t_2)) \leq M \rho(t_1, t_2)^\alpha, \quad \forall t_1, t_2 \in T.$$

The smallest such $M$ is called Hölder coefficient, or Hölder seminorm (in case of normed $X$), it is denoted as $|f|_{H^\alpha(T, X)}$.

Let us consider the $\ell_p^d$-norms in $\mathbb{R}^d$: $\|x\|_p = (|x_1|^p + \ldots + |x_d|^p)^{1/p}$, $1 \leq p < \infty$, $\|x\|_\infty = \max |x_i|$. It is convenient to raise the correspond-
ing \((1/d)\)-Hölder seminorm of \(d\)-dimensional space-filling curves to the \(d\)-th power.

**Definition.** The \(\ell_p\)-dilation of a curve \(\gamma: [a, b] \to \mathbb{R}^d\) is the quantity

\[
WD_p(\gamma) := \sup_{a \leq t_1 < t_2 \leq b} \frac{\|\gamma(t_1) - \gamma(t_2)\|_p^d}{t_2 - t_1}.
\]

(1)

It is obvious that

\[
WD_p(\gamma) = |\gamma|_{H^{1/(d \cdot p)}([a, b]; \ell_p^d)}.
\]

Let us look at a slightly more geometrical approach to measure curve locality. We say that a curve \(\gamma: [0, 1] \to \mathbb{R}^d\) is volume-parametrized, if for any \(t_1 < t_2\) we have \(\text{vol} \gamma([t_1, t_2]) = t_2 - t_1\). Define the \(\ell_p\)-dilation of a set \(S \subset \mathbb{R}^d\) as

\[
dil_p(S) := \frac{\text{diam}_p(S)^d}{\text{vol}(S)}.
\]

(2)

It is easy to see that for a curve with volume parametrization we have

\[
WD_p(\gamma) = \sup_{t_1 < t_2} \text{dil}_p(\gamma([t_1, t_2])).
\]

The last quantity depends only on the scanning order, not on the parametrization of the curve.

The quantity \(WD_p\) was proposed as a measure of curve quality in the paper [GL96]. The notation \(WD_p\) is not well-established. Gotsman and Lindenbaum in [GL96] introduced notation \(L_p(\gamma)\); Bader [B13] writes \(C_p\) for Hölder coefficient and \(\tilde{C}_p\) for its \(d\)-th power (so, \(\tilde{C}_p = WD_p\)). Haverkort and Walderveen [HW10] suggested the notation \(WL_p\) (Worst-case Locality) for (1); Haverkort [H11] added the notation \(WD_p\) (Worst-case Diameter ratio) for the equivalent quantity with diameter instead of distance. Later, in [H17], he called \(WL_p\) the \(L_p\)-dilation. We decided to stick to the “dilation” term and the \(WD_p\) notation (Worst-case Dilation).

In this paper we consider only the Euclidean case, i.e. the Euclidean dilation \(WD_2\), but our algorithms apply to any \(p \in [1, \infty]\).

**Peano multifractals.** We say that curves \(\gamma_1: I_1 \to Q_1\) and \(\gamma_2: I_2 \to Q_2\) are isometric, if there exist isometries \(\alpha: I_1 \to I_2\) and \(\beta: Q_1 \to Q_2\), such that \(\beta \circ \gamma_1 = \gamma_2 \circ \alpha\). Curves are similar with a coefficient \(s\), if \(s\gamma_1(t/s^d)\) is isometric to \(\gamma_2(t)\).
Fix two parameters: a dimension \( d \in \mathbb{N} \) and a scale \( s \in \mathbb{N}, \ s \geq 2. \) The unit cube \([0,1]^d\) is divided into the regular grid of \( g := s^d \) so-called \( s \)-cubes
\[
\left[ \frac{j_1}{s}, \frac{j_1+1}{s} \right] \times \cdots \times \left[ \frac{j_d}{s}, \frac{j_d+1}{s} \right].
\]
Moreover, we divide \([0,1]\) into \( g \) equal segments \( I^g_k := [k/g, (k+1)/g] \).

Given a curve \( \gamma \), each of the \( g \) subcurves \( \gamma|_{I^g_k} \) is called a fraction of a curve.

**Definition.** A tuple of surjective curves \( \gamma_1, \ldots, \gamma_m : [0,1] \to [0,1]^d \) forms a Peano multifractal of genus \( g = s^d \) if for all \( r = 1, 2, \ldots, m \) each of the \( g \) fractions \( \gamma_r|_{I^g_k} \) maps \( I^g_k \) to some \( s \)-cube \( Q \) and is similar to one of the \( \gamma_l \) with the coefficient \( s \).

The parameter \( m \) is called the multiplicity of a multifractal. For \( m = 1 \) we get a Peano mono-fractal, for \( m = 2 \) a bifractal, etc. A related but broader class of polyfractal curves was introduced in [S15] for dimension 3.

Components of Peano multifractals are volume-parametrized curves. They are \((1/d)\)-Hölder continuous and the maximum in (1) is attained. If a multifractal \((\gamma_1, \ldots, \gamma_m)\) is irreducible, i.e. does not contain a non-trivial subset \((\gamma_i)_{i \in I}\) that is a multifractal itself, then \( \text{WD}_p(\gamma_1) = \ldots = \text{WD}_p(\gamma_m) \). For the generic case, one may formally define the dilation of a multifractal as \( \max_j \text{WD}_p(\gamma_j) \).

Let us give some more definitions. A Peano mono-fractal of genus \( s^d \) defines the order (of traversal) of \( s \)-cubes. The sequence of \( s \)-cubes in that order is called a prototype. Peano multifractal \((\gamma_1, \ldots, \gamma_m)\) has \( m \) prototypes, accordingly. We remark that a prototype is a special case of a polycubic chain, see Section 5.2.

The entrance of a curve \( \gamma : [0,1] \to \mathbb{R}^d \) is the point \( \gamma(0) \), and the exit is \( \gamma(1) \). They are also called entrance and exit gates. A pointed prototype is a prototype with specified entrance/exit points for each cube.

Let us give the definition, following [H17].

**Definition.** A Peano multifractal \((\gamma_1, \ldots, \gamma_m)\) is called facet-gated if all entrances \( \gamma_i(0), \ldots, \gamma_f(0) \) and exits \( \gamma_i(1), \ldots, \gamma_m(1) \) lie on the cube facets (i.e., on \((d-1)\)-dimensional faces) and not on faces of lower dimension.

Each fraction \( \gamma_r|_{I^g_k} \) of a multifractal \((\gamma_1, \ldots, \gamma_m)\) of genus \( g \) is similar to some of the \( \gamma_l \), \( l = l_{r,k}. \) That similarity is defined by a pair of
isometry of the cube $[0,1]^d$ (space orientation) and isometry of $[0,1]$ (time orientation). We will refer to this pair as a base orientation. So, a multifractal is completely defined by $m$ prototypes, $mg$ base orientations and $mg$ numbers $l_{r,k}$. Note that base orientations form a group isomorphic to $\mathbb{Z}_2^d \times S_d \times \mathbb{Z}_2$. We remark that base orientations are not completely defined by a multifractal, if it has symmetries.

Each curve in the multifractal of genus $g$ is divided into $g$ fractions; they are called fractions of order 1. If we divide each of the fractions into $g$ sub-fractions, we will obtain $g^2$ fractions of order 2, etc.

Often we identify fractions (curves) with their images (cubes). A junction $F \rightarrow G$ of a curve is a pair of adjacent fractions of some order $k$. We call $F \rightarrow G$ a facet junction if $F$ and $G$ intersect by a common facet. The derived junction $F' \rightarrow G'$ is defined as the junction of $(k+1)$-th order formed by the last fraction of the subdivision of $F$ and the first fraction of the subdivision of $G$. The depth of a curve (see [S04]) is the maximal order $k$ that generates new junctions.

An important observation made by E. Shchepin [S15] is that the maximum in (1) is attained at a pair of points from some non-adjacent sub-fractions $F_1 \subset F, G_1 \subset G$ of some junction $F \rightarrow G$ (i.e., a pair from $F \times G \setminus F' \times G'$, where $F' \rightarrow G'$ is the derived junction), or from some non-adjacent first-order fractions $F_1, G_1$.

**Cubically decomposable curves.** We will require one more definition, from [KS18].

**Definition.** Consider a surjective curve $\gamma : [0,1] \rightarrow [0,1]^d$ and a number $g = s^d$. We say that $\gamma$ is decomposed into $g$ fractions if each of the $g$ fractions $\gamma_m |_{I_k}$ maps $I_k^d$ onto some $s$-cube $Q$. The curve is called cubically decomposable if it can be decomposed into an arbitrarily large number of fractions.

Cubically decomposable curves are volume-parametrized. Obviously, Peano multifractal curves are cubically decomposable.

It is proven in [KS18] that cubically decomposable curves have a recursive structure: there is some minimal number $s_1$ such that curve is decomposable into $s_1^d$ first-order fractions; then there is some $s_2$ such that each of the first-order fractions is decomposed into $s_2^d$ second-order fractions, etc. The definition of a junction is the same as for multifractals: it is a pair of order-$k$ adjacent fractions, for some $k$. 6
3 Results

3.1 Results on the $\ell_2$-dilation of monofractals.

One of the main results of this paper is a Peano monofractal with $\text{WD}_2 < \frac{2}{3}$.

**Theorem 1.** There exists a unique (up to isometry) plane Peano monofractal curve $\gamma : [0, 1] \to [0, 1]^2$ of genus $5 \times 5$ such that $\text{WD}_2(\gamma) = 5 \frac{43}{73}$. Moreover, it minimizes $\text{WD}_2$ over monofractals of genus less or equal $6 \times 6$.

![Figure 1: YE minimal curve](image)
The fractal curve that we have found is shown in the Figure 1. More precisely, the figure shows the central broken lines of its prototype and prototypes of its fractions; that uniquely defines it. The prototype of this curve resembles stucked together letters Y and E, so we suggest the name YE-shaped for curves of this prototype. It is proved that the dilation of the curve shown in the figure is minimal among monofractals of genus $\leq 36$. So we will call it minimal YE-shaped curve, or just YE-curve.

3.2 Computations for facet-gated multifractals.

Recall that the best known plane Peano multifractal, $\beta \Omega$ curve, has entrances and exits on the sides of the square $[0,1]^2$ and not in vertices. So, facet-gated curves are candidates for small $\ell_2$-dilation.

Facet-gated curves are facet-continuous, i.e., adjacent fractions intersect by facets. But the gate condition is, of course, more rigid (restrictive) and allows us to make an effective search among such curves.

We suppose that facet-gated curves minimize $\ell_2$-dilation in many cases. This is true for the class of 2D bifractals of genus $2 \times 2$ and for 3D monofractals of genus $2 \times 2 \times 2$. Let us consider 3D bifractals.

**Definition.** We call any 3D facet-gated Peano bifractal of genus $2 \times 2 \times 2$ having $WD_2 < 17$ a Spring curve.

The following computation shows that Spring curves exist.

**Computation 1.** The minimal $\ell_2$-dilation of 3D facet-gated Peano bifractals of genus $2 \times 2 \times 2$ satisfies $16.9912 < WD_2 < 16.9913$.

Our hypothesis is that Spring curves are minimal among all bifractals of genus $2^3$. Let us describe some properties of that curves.

Our calculations show that all Spring curves have dilation less than 16.9913; we suppose that their dilations are equal to $1533\sqrt{6}/221$. Moreover, all non-Spring facet-gated bifractals have $WD_2 > 17.04$. There are 9 pointed prototypes containing Spring curves.

The entrance/exit points and prototypes of Spring curves are uniquely determined (up to isometry, of course). Spring bifractal is a pair $(\gamma_1, \gamma_2)$; for short, we denote the first curve as (a) and the second one as (b). Entrance of (a) is $(1/3, 1/3, 0)$ and exit is $(2/3, 0, 1/3)$; entrance of (b) is $(1/3, 1/3, 0)$ and exit: $(1, 1/3, 1/3)$. 

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To describe prototypes for (a) and (b), we use chain codes (see [KS18]). Fix a standard basis \( \{i,j,k\} \) in \( \mathbb{R}^3 \) and let \( I = -i, J = -j, K = -k \).

![Figure 2: Standard basis i, j, k.](image)

Then, we can define a sequence of cubes \( Q_1, \ldots, Q_g \) by sequence of vectors \( \delta_l \) such that \( Q_{l+1} = Q_l + \delta_l \). For facet-continuous curves that sequence may be represented as a word in the alphabet \( \{i,j,k,I,J,K\} \), called chain code.

In our case, the chain code for prototype (a) is \( jikIJiK \) and for prototype (b) is \( jkJijKJ \).

To give an example of a Spring curve, we have to specify base orientations and (a)/(b) letters for each of the fractions. A space orientation (an isometry of \([0,1]^3\)) is completely defined by its linear part, which is determined by images of basis vectors. Say, \( KIJ \) is the isometry with linear part that maps \( i \to K, j \to i, k \to J \). A time orientation is either identity or time-reversal; we denote by \( (\overline{a}) \) time-reversed curve \( (a) \) and similarly for \( (b) \).

In our example we have the following fractions (see Fig. 3):
(a): \( KIJ(\overline{b}), KIJ(a), kji(b), Jki(\overline{\pi}), JkI(b), kIJ(a), KJi(b), JiK(b) \).
(b): \( KIJ(\overline{b}), jkJ(\overline{\pi}), lkj(\overline{\pi}), KJi(\overline{\pi}), kji(\overline{\pi}), Ijk(\overline{\pi}), IjK(\overline{a}), IjK(a), ikJ(b) \).

We also have a result in the 4D case.

**Computation 2.** The minimal \( \ell_2 \)-dilation of 4D facet-gated Peano monofractals of genus 2 satisfies \( 61.935 < WD_2 < 61.936 \).

Let us give an example of such curve. We use basis \( \{i,j,k,l\} \).
Prototype: \( kljIKlkJI \). Base orientations: \( iKLJ(a), kLiJ(a), 1Jki(a), jklI(a), 1KjI(\overline{\pi}), LKIj(a), Iklj(\overline{\pi}), ikLj(a), Lkij(\overline{\pi}), 1kji(a), kjlI(a), 1JKi(\overline{\pi}), LkIJ(a), Lkji(\overline{\pi}), 1ljk(\overline{\pi}), jlK(\overline{\pi}) \). Entrance is \( (0,1/3,1/3,1/3) \) and exit is \( (1/3,0,1/3,2/3) \). The dilation is at least \( 61\frac{20}{31} \) and we suppose that this is sharp.
Figure 3: Example of a Spring curve.

### 3.3 Lower bounds on the dilation.

The proof of the YE-curve optimality relies on the following result.

**Theorem 2.** If a plane Peano monofractal curve $\gamma$ has a diagonal junction, then $\text{WD}_2(\gamma) \geq 5\frac{2}{3}$.

We also prove some lower bounds on the dilation for cubically decomposable curves.

**Theorem 3.** There exists some $\varepsilon > 0$, such that for any plane cubically decomposable curve $\gamma: [0, 1] \rightarrow [0, 1]^2$ one has $\text{WD}_2(\gamma) \geq 4 + \varepsilon$.

Note that the bound $\text{WD}_2 \geq 4$ (and even stronger bound $\text{WD}_\infty \geq 4$) was proven in [HW10, Th.4].
**Theorem 4.** Let $\gamma: [0, 1] \to [0, 1]^3$ be a cubically decomposable curve. Then $WD_2(\gamma) \geq 7\sqrt{14}/2 > 13.09$. If $\gamma$ has a non-facet junction, then $WD_2(\gamma) \geq 17.17$.

From this Theorem and Computation[1] we obtain the following.

**Corollary.** 3D Peano bifractals of genus $2 \times 2 \times 2$ with minimal $\ell_2$-dilation have only facet junctions.

We conjecture that minimal bifractals are facet-gated, but we do not have a proof.

**Theorem 5.** Let $\gamma: [0, 1] \to [0, 1]^4$ be a cubically decomposable curve. Then $WD_2(\gamma) \geq 42.25$. If $\gamma$ has a non-facet junction, then $WD_2(\gamma) \geq 62.25$.

**Corollary.** 4D Peano monofractals of genus $2^4$ with minimal $\ell_2$-dilation have only facet junctions.

Finally, we prove a bound on the dilation for arbitrary surjective curves in the 4D case.

**Theorem 6.** For any surjective curve $\gamma: [0, 1] \to [0, 1]^4$ we have $WD_2(\gamma) \geq 30$.

Table[1] summarizes known results on the minimal $\ell_2$-dilation of surjective curves $\gamma: [0, 1] \to [0, 1]^d$. In the last column we normalize the curve dilation by the dilation of the unit cube; this allows to compare dilations across different dimensions and metrics.

## 4 Search algorithm and implementation

### 4.1 Algorithm

The algorithms in this section work for arbitrary Peano multifractals, but for simplicity we describe them for monofractals.

Recall that the prototype of a Peano monofractal of genus $s^d$ is the sequence of $s$-cubes in the curve scanning order and the pointed prototype is the prototype with specified entry/exit points for each cube.

To define a fractal curve with given pointed prototype, you must specify valid base orientations for all fractions, i.e., such orientations
| d | genus | curve type            | minimal curve and references | $\text{WD}_2$ | $\text{wd}_{d/4}^{1/d}$ |
|---|-------|-----------------------|-----------------------------|--------------|-------------------------|
| 2 | $2 \times 2$ | monofractal           | Hilbert [B06]               | 5$\frac{2}{3}$ | 173%                    |
|    | $3 \times 3$ |                       | Meurthe [SB08]              | 5$\frac{13}{24}$ | 168%                    |
|    | $\leq 6 \times 6$ |                       | YE (5 $\times$ 5)          |              | 167%                    |
|    | any     | monofractal           | Haverkort’s $\beta\Omega$  | 5.000        | 158%                    |
|    | any     | bifractal             | Wierum’s $\beta\Omega$     | 5.000        | 158%                    |
|    | any     | any cubically decomposable | Theorem $\frac{1}{3}$ | $\gtrsim 4$ | $\gtrsim 141\%$        |
|    | any     | special triangle fractal | Serpinski [NRS02]      | 4            | 141%                    |
|    | any     | any surjective curve  | Theorem $\frac{1}{3}$ $\times$ $\frac{1}{3}$ | $\gtrsim 4$ | $\gtrsim 141\%$        |
|    | any     | any cubically decomposable | Theorem $\frac{1}{3}$ | $\gtrsim 4$ | $\gtrsim 141\%$        |
|    | any     | any surjective curve  | Theorem $\frac{1}{3}$ $\times$ $\frac{1}{3}$ | $\gtrsim 4$ | $\gtrsim 141\%$        |
| 3 | $2 \times 2 \times 2$ | monofractal           | Haverkort’s Beta [H17]     | 18.566       | 153%                    |
|    | $2 \times 2 \times 2$ | facet-gated bifractal | Spring (Computation 1)     | 16.991       | 148%                    |
|    | any     | any cubically decomposable | Theorem $\frac{1}{3}$ | $\gtrsim 4$ | $\gtrsim 141\%$        |
|    | any     | any surjective curve  | Theorem $\frac{1}{3}$ $\times$ $\frac{1}{3}$ | $\gtrsim 4$ | $\gtrsim 141\%$        |
| 4 | $2^4$   | facet-gated monofractal | Computation 2              | 61.93        | 140%                    |
|    | any     | any cubically decomposable | Theorem $\frac{1}{3}$ | $\gtrsim 4$ | $\gtrsim 141\%$        |
|    | any     | any surjective curve  | Theorem $\frac{1}{3}$ $\times$ $\frac{1}{3}$ | $\gtrsim 4$ | $\gtrsim 141\%$        |

Table 1: Known results on $\ell_2$-dilation
that transform entrance/exit points as required (note that time reversal swaps entrance and exit). An important property of that structure is that the choices for different fractions are independent.

Our computations are based on the algorithm for the estimation of the minimal dilation within a class of curves with given pointed prototype.

Let us give an example to show the efficiency of such approach. Consider curves of genus $5 \times 5$ with entrance $(0, 0)$ and exit $(1, 1)$. Fix some pointed prototype. It is clear that there are four possible base orientations (two with time reversal and two without) in each fraction. This gives $2^{50}$ possible curves. Our algorithm allows us to consider all that curves simultaneously.

**Bisection of the curve dilation.** It is advisable to consider the case of completely defined curve first.

We describe an algorithm that, given a monofractal curve $\gamma$ and thresholds $\omega_- < \omega_+$, either outputs pair of points on a curve providing $WD_p(\gamma) \geq \omega_-$, or guarantees that $WD_p(\gamma) \leq \omega_+$.

The dilation of the curve is equal to the maximum of the dilation for all possible pairs of non-adjacent sub-fractions $F \subset F^\circ, G \subset G^\circ$ of curve junctions $F^\circ \rightarrow G^\circ$, or non-adjacent first-order fractions $F, G$:

$$WD_p(F, G) := \max \left\{ \frac{\| \gamma(t_1) - \gamma(t_2) \|_p}{|t_2 - t_1|} : \gamma(t_1) \in F, \gamma(t_2) \in G \right\}.$$ 

There is a finite number of such pairs (because the number of distinct junctions is finite), and for each pair we can get upper $U$ and lower $L$ bounds for the quantity $WD_p(F, G)$:

$$U = \frac{\max\{\| \gamma(t_1) - \gamma(t_2) \|_p : \gamma(t_1) \in F, \gamma(t_2) \in G\}}{\min\{|t_2 - t_1| : \gamma(t_1) \in F, \gamma(t_2) \in G\}},$$

$$L = \frac{\max\{\| \gamma(t_1) - \gamma(t_2) \|_p : \gamma(t_1) \in F, \gamma(t_2) \in G\}}{\max\{|t_2 - t_1| : \gamma(t_1) \in F, \gamma(t_2) \in G\}}.$$ 

Three cases are possible:

(A) If $L \geq \omega_-$, then $WD_p(\gamma) \geq \omega_-$ and the work is done.

(B) If $U \leq \omega_+$, then we can throw away (do not consider further) that pair.

(C) If $L < \omega_- < \omega_+ < U$, then we subdivide both fractions and continue to evaluate pairs of sub-fractions.
Eventually, \( U - L \) will be small enough to avoid the case (C). So, we either will get (A), or we will throw away all pairs — it will mean that \( \text{WD}_p(\gamma) \leq \omega_+ \).

**Bisection of the minimal dilation for a pointed prototype.**

Now we describe an algorithm that, given a pointed prototype and thresholds \( \omega_- < \omega_+ \), either guarantees that for all curves with that pointed prototype \( \text{WD}_p(\gamma) \geq \omega_- \), or finds a curve with \( \text{WD}_p(\gamma) \leq \omega_+ \).

The difference from the previous algorithm is the following: in the case (C), when subdividing fractions, we need to know the base orientations on the corresponding fractions. Therefore, this step is branching in accordance with all possible base orientations. Next, at the case (A), when we arrive at a configuration with \( L \geq \omega_- \), we cannot state that \( \text{WD}_p(\gamma) \geq \omega_- \) for all curves in the class. Instead, this only gives us a “ban” on all curves compatible with orientations we selected in (C) that led to this configuration.

How to check that the forbidden configurations obtained at (A) cases exhaust all curves of the class? We can describe the curves using boolean variables. Namely, for each index of the fraction \( i \) and for each valid base orientation \( b \) we introduce a boolean variable \( x_{i,b} \), which is true if and only if the base orientation at fraction \( i \) is \( b \). Our restrictions can be written as a CNF formula over these boolean variables. If this formula is unsatisfiable, it means that all curves have the dilation at least \( \omega_- \). Otherwise, if all pairs are processed and the resulting formula has a model, then it will give us a curve with \( \text{WD}_p \leq \omega_+ \).

The satisfiability problem of Boolean formulas (SAT-problem) is well studied (see \( \text{[BHM09]} \)). There are many software libraries available to solve SAT tasks (so-called SAT-solvers). Because the SAT belongs to the NP class, there is no proven polynomial estimates for SAT-solvers. The empirical fact we observed is that modern SAT-solvers work well on CNF formulas, arising in the problem of finding optimal curves.

**Technical details.** Let us give some more details of our algorithm that may have significant impact on it’s performance. When we subdivide pairs of fractions, we pick the pair with the highest upper bound on the dilation. And when we do the subdivision, we divide only one fraction in a pair (and keep pairs “balanced”, i.e. the depth of subdivision differs at most by 1). If a pair falls into both cases (A) and (B),
i.e. \( \omega_- \leq L < U \leq \omega_+ \), then we choose (B), i.e. the pair is discarded (not to enlarge CNF formula).

The SAT-solver is used in the simplest way: every time we call the solve method, we rebuild the CNF formula in it from scratch. To reduce computational costs of this, we do this every \( \lfloor \frac{1}{3} k \rfloor \)-th subdivision step.

To estimate the minimal dilation of a pointed prototype we use the bisection algorithm many times. Suppose that we know some lower and upper bounds \((L, U)\) on the minimal dilation. We put \( \omega_- := \frac{2}{3}L + \frac{1}{3}U \) and \( \omega_+ := \frac{1}{3}L + \frac{2}{3}U \). If there is a curve with \( \text{WD} \leq \omega_+ \), then the bounds are updated as \( L' := L, \quad U' := \frac{1}{3}L + \frac{2}{3}U \). If every curve has \( \text{WD} \geq \omega_- \), then \( L' = \frac{2}{3}L + \frac{1}{3}U, \quad U' = U \).

We estimate the minimal dilation for a sequence of pointed prototypes in several “epochs”. In the first epoch we estimate the minimal dilation of each pointed prototype with some relative error \( \varepsilon \) and keep only possibly minimal prototypes. In the second epoch the relative error is taken smaller, e.g. \( \varepsilon' = \varepsilon / 4 \). And so on.

### 4.2 Computer-assisted proof of Theorem 1

In this section we will give a computer-assisted proof of Theorem 1 that consists of two steps: finding the exact value of the dilation and proving that it is minimal.

Our main tool that allows to find the exact value of the dilation is the following result.

**Theorem A** ([SB08], Theorem 7). If a plane Peano monofractal curve \( \gamma \) of genus \( g \) satisfies \( \omega_{\infty} < \omega_2 \), where \( \omega_p := \text{WD}_p(\gamma) \), then the \( \ell_2 \)-dilation of \( \gamma \) is attained on a pair of vertices of fractions of order \( k \), where

\[
  k < \text{depth}(\gamma) + 3 + \log_2 \frac{\omega_2^2}{4(\omega_2 - \omega_{\infty})}.
\]

(3)

Recall that the depth of a curve is the largest number of the curve subdivision that contains a junction that is not similar to junctions of less order. The minimal YE-curve \( \gamma_{\text{YE}} \), see Fig.1, has depth 1 because derivatives of it’s junctions of first subdivision are similar to that junctions. For our curve we have \( g = 25 \), \( \omega_2 = 5 \frac{13}{73} \) and \( \omega_{\infty} = 5 \frac{1}{3} \). Hence from (3) it follows that \( k \leq 5 \) and we can calculate exact value of \( \text{WD}_2 \) using 5-th subdivision only.

The rigorous proof of the equality \( \omega_2 = \text{WD}_2(\gamma_{\text{YE}}) = 5 \frac{43}{73} = 5.5890... \) goes in the following way: first, we compute approximations
for $\omega_2$ and $\omega_\infty$ and prove that they differ from presumed values $5\frac{43}{73}$ and $5\frac{1}{3}$, say, less than 0.01. These approximate computations allow us to estimate the logarithm in (3) and to find the value of dilation using 5-th subdivision.

Let us describe the computations that show that YE curve has minimal $\ell_2$-dilation among all Peano monofractal curves of genus $\leq 6 \times 6$. For brevity we will write $W = W_2$. As we already mentioned, there is one $2 \times 2$ curve (Hilbert curve), it has $\ell_2$-dilation 6; the minimal $3 \times 3$ curve has $W = 5\frac{2}{3}$. Our computations show that minimum $\ell_2$-dilation of plane monofractal curves of genus $4 \times 4$ is greater than $5$.

It was proven in [B14] that plane Peano monofractal curves fall into three categories:

- side: curves with entrance and exit in the vertices of the same side;
- diagonal: curves with entrance and exit in the opposite vertices;
- median: curves with entrance in some vertex and exit in the middle of the opposite side (or vice versa).

In the cases of genus $5 \times 5$ or $6 \times 6$ we have processed only curves without diagonal steps in prototype; other curves have diagonal junctions and by Theorem 2 they satisfy $W \geq 5\frac{2}{3}$. Our computations provide the following estimates: median $6 \times 6$ curves have $W > 6.5$; side $6 \times 6$ curves have $W > 5.6$; diagonal $6 \times 6$ curves do not exist; median $5 \times 5$ curves have $W > 6.5$; diagonal $5 \times 5$ curves have $W > 5.9$.

The remaining case is side $5 \times 5$ curves. Such curves with non-YE prototype have dilation at least 5.7. There are $2^{25}$ YE-shaped curves. To prove that our curve is minimal, for each of the 25 fractions of the YE-prototype we have fixed the base orientation that does not match our curve. It turned out that for curves with such constraints $W > 6.6$.

### 4.3 Software and running stats

Our code is a Python package `peano` which can be found on the GitHub, [gitYM]. We have used the library Glucose3 [ALS13] and the `python-sat` (PySAT) wrapper [IMM18] for SAT-solving. All calculations are based on rational arithmetic, with `quicktions` implementation.
The **peano** package implements Peano multifractals of arbitrary dimension, multiplicity and genus. Let us list main capabilities of our library:

- define Peano multifractals in terms of prototypes and base orientations;
- find entrance and exit points;
- find vertex moments: \( \{ t : \gamma(t) \in \{0,1\}^n \} \) and, more generally, first/last face moments for cube faces of arbitrary dimension;
- generate entrance/exit configurations given dimension, multiplicity and genus;
- effectively generate all entrance/exit configurations on facets;
- generate pointed prototypes given entrance/exit configuration;
- estimate dilation of a Peano multifractal for any coordinate-monotone norm (possibly using face moments);
- estimate minimal dilation for a sequence of pointed prototypes.

The code was thoroughly tested on a set of known Peano curves. The main source of curves in the 3D case was the paper \[H11\] with many detailed descriptions. We have used also \[HW10\] for a bunch of 2D curves and \[KS18\] for the Tokarev curve.

In Table 2 we give some basic statistics for three search runs: Computation 1, the search of a minimal curve among facet-gated 3D bifractals of genus \(2 \times 2 \times 2\); Computation 2, the search among facet-gated 4D monofractals of genus \(2 \times 2 \times 2 \times 2\); and the proof of minimality of YE curve, Theorem 1. The commands to perform these runs are given in the README file of the package. Let us give some comments on Table 2.

The row “Peano curves” contains the numbers of all Peano curves with pointed prototypes that were processed during the search. This number is rather small for the Computation 1; it seems that the Spring curve can be found without SAT solvers at all. The number of curves in the YE proof is huge; main contribution comes from 11 diagonal \(5 \times 5\) pointed prototypes (without diagonal steps), each of them contains \(4^{25}\) curves.

The size of a SAT problem for a CNF formula is determined by three parameters: number of variables, number of clauses in the formula and total number of literals in clauses. We show the SAT problem having the most literals.
Table 2: Running stats

|                          | Comp. 1 | Comp. 2 | YE proof |
|--------------------------|---------|---------|----------|
| Gate configurations      | 35      | 3       | 3        |
| Pointed prototypes       | 909     | 1761    | 2763     |
| Peano curves             | 122955  | 115 × 10^6 | 1.24 × 10^16 |
| Bisect steps             | 3301    | 8345    | 8472     |
| Pairs of fractions       | 10.4 × 10^6 | 143 × 10^6 | 204 × 10^6 |
| Banned pairs             | 40840   | 815466  | 1.41 × 10^6 |
| SAT solve calls          | 35764   | 98164   | 47984    |
| SAT largest problem:     |         |         |          |
| literals                 | 19515   | 7825    | 480792   |
| clauses                  | 7019    | 2405    | 82918    |
| variables                | 819     | 275     | 1713     |
| Running time             | 4 min   | 45 min  | 1 h 20 min |

5 Lower bounds

In this section we denote \(|x| := \|x\|_2\) for brevity. Given two points \(\gamma(t_1), \gamma(t_2)\) on a curve, we define the cube-to-linear ratio as \(|\gamma(t_1) - \gamma(t_2)|^d/|t_2 - t_1|\). The curve dilation equals the supremum of cube-to-linear ratios over all pairs.

5.1 Proof of Theorem 2

Theorem 2. If a plane Peano monofractal curve \(\gamma\) has a diagonal junction, then \(\text{WD}_2(\gamma) \geq 5\frac{4}{5}\).

Suppose that consecutive fractions \(F\) and \(G\) of the curve \(\gamma\) form a diagonal junction. Let us put the transition point from \(F\) to \(G\) to the origin \(O = (0, 0)\) and suppose that \(G\) lies in the first, positive quadrant (right and above \(O\)) and \(F\) lies in the third, negative quadrant (left and below \(O\)). We also consider fractions \(F_1\) and \(G_1\) of the next subdivision, such that \(F \supset F_1 \ni O\) and \(G \supset G_1 \ni O\). Moreover, let \(G_2, G_3, G_4\) be consecutive fractions that go after \(G_1\) and let \(F_1, F_3, F_2\) be consecutive fractions before \(F_1\) (so, \(F_2\) meets \(F_1\)). For convenience, we assume that \(F_i\) and \(G_i\) have unit side length (and passed in the unit time). See Figure 4 for an example. We will treat several cases, illustrated in Figure 5. Some of them will require the following Lemma.
Lemma 1. Let $A$ be the left bottom vertex of $F_i$, and $B$ be the right upper vertex of $G_j$ and the conditions hold true:

- $i + j = 6$,
- the distance from $A$ to the entrance of $F_i$ is at least 1;
- the distance from $B$ to the exit of $G_j$ is at least 1;
- $|A - B| \geq \sqrt{32}$.

Then $WD_2(\gamma) \geq 5 \frac{2}{3}$.

Proof. Suppose that $WD_2(\gamma) \leq 5 \frac{2}{3}$. Then the curve spends at least $\frac{3}{17}$ units of time to pass one unit of length. So the time interval between $A$ and $B$ is at most $6 - \frac{6}{17} = \frac{6 \cdot 17}{17}$ and the cube-to-linear ratio for $(A, B)$ is at least

$$\frac{|A - B|^2}{t_B - t_A} \geq \frac{32 \cdot 17}{6 \cdot 16} = \frac{17}{3} = 5 \frac{2}{3}.$$ 

Let us return to the proof of the theorem.
Diagonal curves. If \( \gamma \) is diagonal, then the farthest (from origin) point \( A \) of \( F_3 \cup F_4 \) has \( \|A - O\|_1 \geq 5 \). The same holds for the farthest point \( B \) of \( G_3 \cup G_4 \) and hence \( \|A - B\|_1 \geq 10 \), so \( |A - B|^2 \geq 5^2 + 5^2 = 50 \). The time interval from \( A \) to \( B \) is at most 8, so the cube-to-linear ratio for this pair is at least \( 6 \frac{1}{4} \).

Median curves. If four fractions \( G_1 \ldots G_4 \) do not lie in a \( 2 \times 2 \) square, then the \( \ell_1 \)-distance from the farthest point \( B \) of \( G_1 \cup \ldots \cup G_4 \) to \( O \) is at least 5. And in any case the \( \ell_1 \)-distance from the farthest point \( A \) of \( F_1 \cup F_2 \cup F_3 \) to \( O \) is at least 4. Hence \( \|A - B\|_1 \geq 9 \) and \( |A - B|^2 \geq 4^2 + 5^2 = 41 \). The time interval from \( A \) to \( B \) is at most 7, so the cube-to-linear ratio is \( \geq 41 \frac{1}{7} > 5 \frac{2}{3} \). If fractions \( F_1 \ldots F_4 \) do not lie in a \( 2 \times 2 \) square, the same arguments apply.

Finally, if both quads \( F_1 \ldots F_4 \) and \( G_1 \ldots G_4 \) lie in \( 2 \times 2 \) squares, then the dilation bound follows from the Lemma 1 with \( i = j = 3 \).

Side curves. If the directions (from entrance to exit) of consecutive fractions coincide, then we say that the corresponding junction keeps the direction.

We will start with the case when \( \gamma \) has no diagonal junctions that keep the direction.

W.l.o.g. let \( G_1 \) has the horizontal direction, so \( F_1 \) is vertical. It follows that \( F_2 \) is below \( F_1 \) and \( G_2 \) is to the right from \( G_1 \).

Case 1. \( G_2 \) is horizontal. Then \( G_3 \) is to the right from \( G_2 \). Let us consider fraction \( F_3 \): it is below \( F_2 \) or to the left from \( F_2 \); it cannot be to the left from \( F_1 \) because in that case \( F_3 \rightarrow F_2 \) will be a diagonal junction that keeps the direction. In both cases we can apply Lemma 1 for \( F_3 \) and \( G_3 \).

If \( F_2 \) is vertical, the reasoning is the same as in the Case 1.

Case 2. \( G_2 \) is vertical, and \( F_2 \) is horizontal. Then \( G_3 \) must be above \( G_2 \) and \( F_3 \) — to the left from \( F_2 \). We apply Lemma 1 for \( F_3 \) and \( G_3 \).

So, it remains to consider the case of a side curve with a diagonal junction keeping direction. We may assume that the junction \( F \rightarrow G \) keeps the direction.

Denote by \( \tau_1(\gamma), \tau_2(\gamma) \) and \( \tau_3(\gamma) \) the time intervals from the curve entrance to the next visited vertex, then to the third vertex and, finally, from it to the exit. We say that a curve is accelerating if \( \tau_1(\gamma) > \tau_3(\gamma) \) and is decelerating if \( \tau_1(\gamma) < \tau_3(\gamma) \). We will apply these definitions to the fractions of \( \gamma \).
Figure 5: Cases in the proof of Theorem 2
Lemma 2. If a diagonal junction $F \to G$ keeps the direction and $\text{WD}_2(\gamma) \leq \frac{5}{3}$, then the first fraction $F$ is decelerating and the second, $G$, is accelerating. Moreover, we have

$$\tau_1(F) = \tau_3(G) \leq \frac{5}{17} < \frac{6}{17} \leq \tau_3(F) = \tau_1(G).$$

Proof. Let us consider first the case when vertices are visited along the edges, as in Figure 5. Denote $\tau_1 := \tau_1(F)$, $\tau_3 := \tau_3(F)$, $\tau_1' := \tau_1(G)$, $\tau_3' := \tau_3(G)$. Condition $\text{WD}_2(\gamma) \leq \frac{5}{3}$ gives us inequalities $\tau_3 + \tau_1' \geq \frac{24}{17}$ and $2 - \tau_1 - \tau_3' \geq \frac{8}{17} = \frac{24}{17}$, so

$$\tau_1 + \tau_3' \leq \frac{10}{17} < \frac{12}{17} \leq \tau_3 + \tau_1'.$$

This excludes case $\tau_1 = \tau_1'$, $\tau_3 = \tau_3'$. Since $\gamma$ is a monofractal, the only possible case is $\tau_1 = \tau_3' \leq \frac{5}{17} < \frac{6}{17} \leq \tau_3 = \tau_1'$.

Now, suppose that the first and the second vertex visited in $F$ lie diagonally opposite of each other (and therefore, so do the third and the fourth vertex, that is, $O$). Then we have $\tau_3 + \tau_1' \geq \frac{24}{17}$ and $\tau_1 \geq \frac{6}{17}$. If $\tau_1 = \tau_1'$, $\tau_3 = \tau_3'$, then $\tau_3 + \tau_1' + \tau_3' + \tau_1 \geq \frac{48}{17} > 2$, which is impossible. If $\tau_1 = \tau_3'$, $\tau_3 = \tau_1'$, then $\tau_3 \geq \frac{12}{17}$ and $\tau_1 + \tau_3 \geq \frac{18}{17}$, which is also impossible. So, this case cannot take place.

From Lemma we derive that the junction $F \to G$ is centrally-symmetric. Since our junction keeps the direction, we may assume that $F_1$ and $G_1$ are both horizontal.

If $G_2$ is also horizontal, then $F_2$ has the same direction and the squared diameter of $F_3 \cup \ldots \cup G_3$ is at least 40, which gives $\text{WD}_2(\gamma) \geq \frac{40}{6}$.

If $G_2$ is vertical, then $\gamma$ runs the distance at least 2 in time $1 + \tau$, where $\tau := \tau_1(G_2)$. Since the whole $F \to G$ junction is centrally-symmetric, $\gamma$ runs the distance $\geq 4$ in time $2 + 2\tau$. Hence $\text{WD}_2(\gamma) \geq \frac{16}{2 + 2\tau}$. If $\text{WD}_2(\gamma) \leq \frac{5}{3}$, then it gives $\tau \geq \frac{7}{17}$. Lemma 2 implies that $G_2$ is accelerating. It also follows that junction $G_2 \to G_3$ cannot be diagonal, so $G_3$ is above $G_2$. Finally, application of Lemma 1 for $F_3$ and $G_3$ finishes the proof.
5.2 Polycubic chains

In this section we will prove theorems 3, 4 and 5.

Definition. A sequence $Q_1, \ldots, Q_n$ of distinct $d$-dimensional cubes in $d$-dimensional Euclid space if called a polycubic chain if in some coordinates we have $Q_1 = [0,1]^d$ and $Q_{i+1} = Q_i + \delta_i$, $1 \leq i \leq n-1$, for some translation vectors $\delta_i \in \{-1, 0, 1\}^d$.

The length of a chain is the number of elements (cubes) in it. Elements of a chain are called chain’s fractions. The union $Q_1 \cup Q_2 \cup \ldots \cup Q_n$ of all fractions is called the body of a chain. A subsequence of cubes $Q_i, \ldots, Q_{i+k}$ is called a segment of a chain (of course, it is a polycubic chain itself). A junction if a pair of adjacent fractions (i.e., a segment of length 2).

A chain is facet-continuous if adjacent fractions intersect by facets. Note that by definition polycubic chains are continuous, i.e., adjacent fractions have non-empty intersection.

The dilation of a polycubic chain is the maximal dilation of bodies of its segments. (Recall the definition (2) of the set dilation.)

It is obvious that the sequence of fractions (of some order) of a cubically decomposable curve is a polycubic chain and the dilation of the curve is at least the dilation of this chain.

We say that fractions of a chain are collinear (coplanar), if the centers of them are collinear (coplanar).

Dimension 2. We will use the following result.

Theorem B ([MS21], Theorem 1). If a plane surjective curve $\gamma: [0,1] \to [0,1]^2$ has entrance $\gamma(0)$ and exit $\gamma(1)$ on the opposite sides of $[0,1]^2$, then $\text{WD}_2(\gamma) > 4$.

Lemma 3. A 2D facet-continuous polycubic chain $S_1, \ldots, S_n$ with more than 4 fractions and square body has a collinear segment of length 3.

Proof. Consider the square $P = S_1 \cup \ldots \cup S_n$. It has size $s \times s$ with $s^2 = n \geq 9$, so $P$ has has a vertex that is not contained in $(S_1 \cup S_2) \cup (S_{n-1} \cup S_n)$. This vertex is contained in some fraction $S_i$ with $3 \leq i \leq n - 2$. Denote by $S_j$ the only fraction that intersects $S_i$ by a vertex. If $j > i$ then one can take triple $S_{i-2}, S_{i-1}, S_i$ and if $j < i$ we take $S_i, S_{i+1}, S_{i+2}$. \qed
**Theorem 3.** There exists some $\varepsilon > 0$, such that for any plane cubically decomposable curve $\gamma : [0, 1] \to [0, 1]^2$ one has $\text{WD}_2(\gamma) \geq 4 + \varepsilon$.

**Proof of Theorem 3.** If all junctions of a cubically decomposable curve $\gamma$ are facet, then one can take a decomposition into $n \geq 9$ squares and apply Lemma 3 to find a collinear segment $S_{i-1}, S_i, S_{i+1}$. Then the middle fraction $S_i$ has its entrance and exit on the opposite sides on the square. We apply Theorem B to obtain that $\text{WD}(\gamma) > 4$.

Suppose that there is a diagonal junction $F \to G$. Consider two last sub-fractions $F_2, F_1$ of $F$ and two first sub-fractions $G_1, G_2$ of $G$ (so, $F_1$ meets $G_1$). Then

$$\text{diam}(F_2 \cup F_1 \cup G_1 \cup G_2) \geq \min\{|(3, 3)|, |(4, 2)|\} = \sqrt{18},$$

which gives $\text{WD} \geq 18/4 = 4.5$.

We have proved that $\text{WD}(\gamma) > 4$. The bound $\text{WD}(\gamma) \geq 4 + \varepsilon$ with some absolute $\varepsilon > 0$ follows from the compactness argument. \qed

**Dimension 3.**

**Lemma 4.** A 3D facet-continuous 4-cubic chain with empty intersection of its fractions has dilation at least $\frac{7\sqrt{14}}{2} = 13.0958\ldots$.

**Proof.** Note that if the intersection of fractions of our chain $Q_1, \ldots, Q_4$ is empty, then it is coplanar. In appropriate coordinates, the vector between two farthest points of $Q_1$ and $Q_4$ (the “diameter vector”) equals $(3, 2, 1)$ or $(4, 1, 1)$ and the diameter of the body $Q_1 \cup \ldots Q_4$ at least $|(3, 2, 1)| = \sqrt{14}$, which gives the dilation bound $\geq \frac{7\sqrt{14}}{2}$.

**Lemma 5.** If a polycubic chain with cubic body has more than 8 fractions then it contains a 5-cubic segment with empty intersection.

**Proof.** Let $Q_1, \ldots, Q_n$ be a chain with a cubic body $P$. If $n > 8$, then $P$ is a $s \times s \times s$ cube with $s \geq 3$. Any vertex of $P$ belongs to some fraction; let $Q_i$ be the fraction that contains the 4-th vertex of $P$ (in the chain traverse order). Then $7 \leq i \leq n - 8$.

Let $k$ be the maximal number such that $Q_{i-1}, Q_{i-2}, \ldots, Q_{i-k}$ intersect $Q_i$, and $l$ be the maximal number such that $Q_{i+l}, \ldots, Q_{i+l}$ intersect $Q_i$. There are 7 fractions intersecting $Q_i$, so $\min(k, l) \leq 3$. Let, e.g., $l \leq 3$. Then the chain $Q_i, \ldots, Q_{i+l+1}$ has at most 5 fractions and empty intersection. \qed
Statement 1. Any facet-continuous 3-dimensional polycubic chain with cubic body and more than 8 fractions has the dilation at least \( 7\sqrt{14}/2 \).

Proof. If our chain contains a 4-cubic segment with empty intersection, then Lemma 3 provides the required bound.

Otherwise, consider a 5-cubic chain \( Q_1, \ldots, Q_5 \) with empty intersection, given by Lemma 5. Fractions \( Q_2, Q_3, Q_4 \) must intersect by an edge. \( Q_1 \) and \( Q_5 \) must intersect that edge (otherwise a 4-cubic chain with empty intersection appears), but they cannot contain that edge. Instead of that, they contain opposite endpoints of it. It follows that the diameter of the body \( Q_1 \cup \ldots \cup Q_5 \) is at least \( \sqrt{3^2 + 2^2 + 2^2} \) and the dilation is at least \( 17\sqrt{17}/5 > 14 \).

Now we are ready to prove lower bounds for the dilation of 3D cubically decomposable curves.

Theorem 4. Let \( \gamma : [0, 1] \to [0, 1]^3 \) be a cubically decomposable curve. Then \( WD_2(\gamma) \geq 7\sqrt{14}/2 > 13.09 \). If \( \gamma \) has a non-facet junction, then \( WD_2(\gamma) \geq 17.17 \).

Proof of Theorem 4. If all junctions of a cubically-decomposable curve \( \gamma \) are facet, then we apply Statement 1.

Suppose that \( \gamma \) has a non-facet junction \( F \to G \). If \( F \) and \( G \) intersect by a vertex, then we have the dilation bound: \( WD(\gamma) \geq \text{diam}(F \cup G)^{3/2} / 2 \geq 12^{3/2} / 2 > 20 \). So we may suppose that \( \gamma \) has no such junctions.

Let \( F \) and \( G \) intersect by an edge. Consider sub-fractions \( F_3, F_2, F_1 \subset F \) and \( G_1, G_2, G_3 \subset G \), such that \( F_1 \) meets \( G_1 \). Let us choose coordinates \( x, y, z \), such that the edge \( F \cap G \) is parallel to the third (vertical) axis, the projections of \( F \) and \( G \) onto horizontal plane \( \{ z = 0 \} \) lie in the quadrants \( \{ x, y \leq 0 \} \) and \( \{ x, y \geq 0 \} \), correspondingly. For definiteness, let \( G_1 = [0, 1]^3 \) and \( F_1 = G_1 + (-1, -1, 0) \).

Consider the fraction \( G_2 \). If \( G_1 \) and \( G_2 \) are on the same height, then

\[
\text{diam}(F_2 \cup F_1 \cup G_1 \cup G_2) \geq \min\{|(3, 2, 2)|, |(3, 3, 1)|, |(4, 2, 1)|\} = \sqrt{17},
\]

hence \( WD(\gamma) \geq 17\sqrt{17}/4 > 17.5 \). If \( G_1 \) and \( G_2 \) intersect by an edge, then the diameter of \( F_1 \cup G_2 \) is already too large. So, we may assume that \( G_2 \) is above: \( G_2 = G_1 + (0, 0, 1) \).

Consider the fraction \( G_3 \). If the junction \( G_2 \to G_3 \) is facet, then one can show that \( \text{diam}(F_1 \cup G_3) \geq |(3, 2, 2)| \), which is sufficient for
us. It remains one (up to isometry) “difficult” case, when $G_3$ and $G_2$ intersect by an edge: $G_3 = G_1 + (1, 0, 0)$.

Consider the “difficult” case, illustrated in Figure 6. Take points $A_0 = (-1, -1, 1) \in F_1$, $A_1 = (1, 1, 0) \in G_1$, $A_2 = (0, 0, 2) \in G_2$, $A_3 = (2, 1, 0) \in G_3$. There exists some moments $t_0 < t_1 < t_2 < t_3$, such that $\gamma(t_i) = A_i$. Then by definition of WD, we have

$$\sum_{i=0}^{2} |A_{i+1} - A_i|^3 \leq WD(\gamma) \sum_{i=0}^{2} |t_{i+1} - t_i| \leq 4 WD(\gamma),$$

$$WD(\gamma) \geq \frac{1}{4} \sum_{i=0}^{2} |A_{i+1} - A_i|^3 = \frac{1}{4} (27 + 6\sqrt{6} + 27) > 17.17.$$
Lemma 6. If a 4D facet-continuous 7-cubic chain $Q_1, \ldots, Q_7$ has a collinear subchain $Q_3, Q_4, Q_5$, then the dilation of $Q_1, \ldots, Q_7$ is more than 64.

Proof. Let us reduce the problem to a planar situation. Suppose that 5-chain $Q_2, \ldots, Q_6$ is not coplanar. Then the directions $Q_2Q_3$ and $Q_5Q_6$ are orthogonal to each other and to the direction of $Q_3, Q_4, Q_5$. Then, in appropriate coordinates, the vector between two farthest points of $Q_2$ and $Q_6$ (the “diameter vector”) equals $(3, 2, 2, 1)$ and the dilation is $18^2/5 = 64.8$.

So, $Q_2, \ldots, Q_6$ is coplanar to some plane $\pi$. If one of the directions $Q_2Q_3$ or $Q_5Q_6$ is parallel to $Q_3, Q_4, Q_5$, then be have a collinear 4-chain. It has dilation $19^2/4 > 90$.

If $Q_1, \ldots, Q_5$ or $Q_3, \ldots, Q_7$ are not coplanar to $\pi$, then the proof goes the same way as for $Q_2, \ldots, Q_7$. So, we may assume that the whole 7-chain is coplanar to $\pi$. The projection onto $\pi$ of one of the 5-chains $Q_1 \ldots Q_5$, $Q_2 \ldots Q_6$, $Q_3 \ldots Q_7$ does not fit into a $2 \times 3$ rectangle and it gives a diameter vector $(3, 3, 1, 1)$ or $(4, 2, 1, 1)$ and the dilation at least $20^2/5 = 80$.

Statement 2. The dilation of 4D facet-continuous polycubic chain with cubic body and more than one fraction is at least 42.25.

Proof. Denote our chain as $Q_1, \ldots, Q_n$. If there is a collinear 3-cubic chain $Q_i, Q_{i+1}, Q_{i+2}, 3 \leq i \leq n - 4$, then by Lemma 6 the dilation at least 64. Assume that such 3-cubic chains are not collinear. Let us prove that there is a 4-cubic chain $Q_i, Q_{i+1}, Q_{i+2}, Q_{i+3}$ that is not coplanar. It this is not the case, then the chain $Q_2, \ldots, Q_{n-1}$ is coplanar, because all adjacent 4-chains in it are coplanar to the same plane (defined by their three common fractions). Obviously, our chain cannot have cubic body in that case.

If $Q_i, Q_{i+1}, Q_{i+2}, Q_{i+3}$ is not coplanar, then $Q_{i+3}$ is a translation of $Q_i$ by the vector $(1, 1, 1, 0)$ (in appropriate coordinates). So, $diam(Q_i \cup \ldots \cup Q_{i+3}) \geq \sqrt{13}$ and we have the dilation bound $13^2/4 = 42.25$. 

Theorem 5. Let $\gamma : [0, 1] \to [0, 1]^4$ be a cubically decomposable curve. Then WD$_2(\gamma) \geq 42.25$. If $\gamma$ has a non-facet junction, then WD$_2(\gamma) \geq 62.25$.

Proof of Theorem 5. If a cubically-decomposable curve $\gamma$ has only facet junctions, then we apply Statement 2.
Suppose that \( \gamma \) has a non-facet junction \( F \to G \). If \( F \) and \( G \) intersect by an edge, then we have the dilation bound:

\[
WD(\gamma) \geq \frac{1}{2} \text{diam}(F \cup G)^4 = |(2, 2, 2, 1)|^4/2 = 84.5
\]

and the proof is done. If \( F \) and \( G \) intersect by a vertex, there bound is even stronger.

Let \( F \) and \( G \) intersect by a 2-dimensional face. Consider subfractions \( F_3, F_2, F_1 \subset F \) and \( G_1, G_2, G_3 \subset G \), such that \( F_1 \) meets \( G_1 \). Let us choose coordinates \( x, y, z, w \), such that \( F \cap G \) is parallel to the third (vertical) axis \( Oz \) and the fourth axis \( Ow \), the projections of \( F \) and \( G \) onto horizontal plane \( \{z = w = 0\} \) lie in the quadrants \( \{x, y \leq 0\} \) and \( \{x, y \geq 0\} \). For definiteness, let \( G_1 = [0, 1]^4 \) and \( F_1 = G_1 + (-1, -1, 0, 0) \).

If the projections of \( G_1 \) and \( G_2 \) onto \( Oxy \) do not coincide, then \( \text{diam}(F_1 \cup G_2) \geq |(3, 2, 1, 1)| = \sqrt{15} \), which gives the dilation bound \( 15^2/3 = 75 \). So, \( G_2 \) may differ from \( G_1 \) only in the coordinates \( z \) and \( w \). If it differs in both coordinates, e.g., \( G_2 = G_1 + (0, 0, 1, 1) \), then we have the bound \( WD \geq \text{diam}(F_1 \cup G_2)^4/3 \geq |(2, 2, 2, 2)|^4/3 = 85 \frac{1}{7} \). So, we may assume that \( G_2 \) differs in 3-rd coordinate: \( G_2 = G_1 + (0, 0, 1, 0) \). Analogously, \( F_2 \) differs from \( F_1 \) in either 3-rd or 4-th coordinate. Suppose, \( F_2 \) differs from \( F_1 \) in the 4-th coordinate, say, \( F_2 = F_1 + (0, 0, 0, 1) \) — then \( WD \geq \text{diam}(F_2 \cup G_2)^4/4 \geq |(2, 2, 2, 2)|^4/4 = 64 \). So, \( F_2 = F_1 + (0, 0, 0, \pm 1, 0) \). The case “-1” is impossible, since then \( \text{diam}(F_2 \cup G_2) \geq |(3, 2, 2, 1)| = \sqrt{18} \). We obtain that \( F_2 = F_1 + (0, 0, 0, 1, 0) \).

Let us summarize the configuration that we have: \( G_1 = [0, 1]^4 \), \( G_2 = G_1 + (0, 0, 1, 0) \), \( F_1 = G_1 + (-1, -1, 0, 0) \), \( F_2 = G_1 + (-1, -1, 1, 0) \).

Now we have to consider \( G_3 \) and the translation vector \( \delta : G_3 = G_2 + \delta \). Note that \( \delta_1, \delta_2 \geq 0 \).

The first case: \( \delta_3 = 1 \): we have \( \text{diam}(F_1 \cup G_3) \geq |(3, 2, 2, 1)| = \sqrt{18} \), which gives \( WD \geq 18^2/4 = 81 \).

The second case: \( \delta_3 = 0 \) (i.e., \( G_3 \) and \( G_2 \) are on the same “height”). As \( \delta \neq 0 \), we have \( \delta_1 = 1 \) or \( \delta_2 = 1 \) or \( \delta_4 = \pm 1 \). In any case, we have \( \text{diam}(F_1 \cup G_3) \geq \min\{|(3, 2, 2, 1)|, |(2, 2, 2, 2)|\} = 4 \), \( WD \geq 4^4/4 = 64 \).

The last case: \( \delta_3 = -1 \). If \( \delta_1 = 1 \) or \( \delta_2 = 1 \), then \( \text{diam}(F_2 \cup G_3) \geq |(3, 2, 2, 1)| \), \( WD \geq 18^2/5 = 64.8 \) and the proof is done. So we may assume that \( \delta_1 = \delta_2 = 0 \). As \( G_3 \neq G_1 \), we have \( \delta_4 \neq 0 \); so, w.l.o.g. we may assume that \( \delta = (0, 0, -1, 1) \) and \( G_3 = G_1 + (0, 0, 0, 1) \). This
“difficult” case needs special treatment, as in the proof of the previous theorem.

Take the following points: \( A_0 = (-1, -1, 1, 0) \in F_1, A_1 = (1, 1, 0, 1) \in G_1, A_2 = (0, 0, 2, 0) \in G_2, A_3 = (1, 1, 0, 2) \in G_3. \) There exist some moments \( t_0 < t_1 < t_2 < t_3 \) such that \( \gamma(t_i) = A_i. \) By the definition of WD, we have

\[
\sum_{i=0}^{2} |A_{i+1} - A_i|^4 \leq WD(\gamma) \sum_{i=0}^{2} |t_{i+1} - t_i| \leq 4 WD(\gamma),
\]

so \( WD(\gamma) \geq \frac{1}{4}(10^2 + 7^2 + 10^2) = 62.25. \)

5.3 Proof of Theorem 6

**Theorem 6.** For any surjective curve \( \gamma: [0, 1] \to [0, 1]^4 \) we have \( WD_2(\gamma) \geq 30. \)

Vertices of the cube \([0, 1]^4\) form the boolean cube \(\{0, 1\}^4\). In fact we will obtain the dilation bound for any curve whose image contains \(\{0, 1\}^4\). We say that vertices \(v, v' \in \{0, 1\}^4\) are **3-antipodes** if they differ in exactly 3 coordinates, and **4-antipodes**, if the differ in all 4 coordinates.

**Lemma 7.** Any set \(S\) of 6 elements of the boolean cube \(\{0, 1\}^4\) contains a pair of 4-antipodes or two pairs of 3-antipodes.

**Proof.** We may assume that there are no 4-antipodes in \(S\).

Let us prove that there is at least one pair of 3-antipodes. We may assume that \(v = (0, 0, 0, 0) \in S.\) If \(S\) has no 3-antipodes, then all points in \(S\) have at most two non-zero coordinates. So, \(S\) contains a point, say, \(w = (1, 1, 0, 0)\) with two non-zero coordinates (because there are only 4 vertices with one coordinate). But \(w\) has 3-antipodes \((0, 0, 1, 0)\) and \((0, 0, 0, 1)\), so that points can’t lie in \(S.\) Then \(S\) contains one more point \(u\) with two non-zero coordinates; as \(u \neq (0, 0, 1, 1)\), we may assume \(u = (0, 1, 1, 0)\). Then vertices \((1, 0, 0, 0)\) and \((0, 0, 0, 1)\) are forbidden, so \(S\) must contain at least 4 points with two non-zero coordinates. These points contain a pair of 4-antipodes, that contradicts our assumption.

So, we have a pair of 3-antipodes, say \(v = (0, 0, 0, 0) \in S\) and \(v' = (1, 1, 1, 0) \in S.\) If there are no other 3-antipodes, then all vertices in \(S\) have at most two non-zero coordinates. But if the last coordinate
of such point is non-zero, then this point is 3-antipode to \( v' \). So, \( S \) lies in the 3-dimensional sub-cube \( \{ (x_1, x_2, x_3, x_4): x_4 = 0 \} \). And in the 3-dimensional case the statement of Lemma is obvious.

Now we can finish the proof of Theorem 6. Let \( A_0 := \gamma(t_0), \ldots, A_{15} := \gamma(t_{15}) \) be the sequence of points of \( \{0, 1\}^4 \) in the traversal order. Note that \( (t_5 - t_0) + (t_{10} - t_5) + (t_{15} - t_{10}) \leq 1 \), so one of the summands is at most 1/3. Hence we have a sub-sequence \( A_s, \ldots, A_{s+5} \) of six points with \( t_{s+5} - t_s \leq 1/3 \).

We apply Lemma to points \( A_s, \ldots, A_{s+5} \) and arrive to three possible cases: 1) there is a pair of 4-antipodes 2) there are three points that give two pairs of 3-antipodes, 3) there are four points that give two pairs of 3-antipodes.

Denote \( \omega := WD_2(\gamma) \). In the case 1) we have a pair \( A_i, A_j \) of 4-antipodes,

\[
16 = |A_i - A_j|^4 \leq \omega(t_j - t_i) \leq \omega/3,
\]

In the case 2) we have \( A_i, A_j, A_k, i < j < k \) and two of three pairwise distances equal \( \sqrt{3} \), so

\[
10 \leq |A_i - A_j|^4 + |A_j - A_k|^4 \leq \omega(t_k - t_i) \leq \omega/3.
\]

Consider the last case of four points \( A_i, A_j, A_k, A_l \). If the antipode \( B \) of \( A_i \) is one of \( A_j, A_k \), then

\[
10 \leq |A_i - B|^4 + |B - t_l|^4 \leq \omega(t_l - t_i) \leq \omega/3.
\]

If \( A_i \) and \( A_l \) are antipodes, then \( A_j \) and \( A_k \) are antipodes too,

\[
11 \leq |A_i - A_j|^4 + |A_j - A_k|^4 + |A_k - A_l|^4 \leq \omega/3.
\]

In any case \( \omega \geq 30 \) and the Theorem is proven.

### 6 Further work

There remain a lot of problems on the minimal \( \ell_p \)-dilation of Peano multifractals, both theoretical and computational. Let us discuss some of them.
Promising directions. Our library allows to estimate minimal dilation for Peano multifractals of arbitrary dimension $d$, multiplicity $m$ and genus $g$. The number of possible variants grows rapidly as we increase any of the parameters $d$, $m$, $g$; this raises two difficulties. When we make a brute force search over all pointed prototypes in some class, the number of pointed prototypes corresponds to the “width” of the search, it is CPU-bounded. Given a pointed prototype, we use a SAT-solver to approximate minimum dilation and the number of possible curves corresponds to the “depth” of the search, it is memory (RAM) bounded. For example, for edge 3D trifractals of genus $2 \times 2 \times 2$ there are not much pointed prototypes, but they are very “deep”. Conversely, there are too many pointed prototypes for edge 2D monofractals of genus $6 \times 6$; we have succeeded in the search among them only because we could throw away prototypes with diagonal steps (and these prototypes are not so deep).

Still it is possible to use the library in many interesting cases. Here is a list of configurations to start with:

- plane $7 \times 7$ monofractals ($\ell_2$-dilation);
- 3D-bifractals of genus $2^3$ ($\ell_1$-dilation);
- 3D facet-gated trifractals of genus $2^3$ ($\ell_2$-dilation)
- 3D-monofractals of genus $3^3$;
- 4D facet-gated monofractals of genus $2^4$ ($\ell_1$, $\ell_\infty$-dilation);

Some of them will require optimizations in the algorithm or additional constraints (on prototypes, base orientations, junctions, gates, etc) to reduce the number of curves. A good example is the narrow class of facet-gated curves. Another interesting constraint is the symmetry of a curve.

SAT-solvers may be applied in broader optimization, e.g., in the classes of curves with fixed prototype — just add the clauses for adjacent fractions that ensure the continuity of a curve.

Other metrics. Our algorithm for minimal $\ell_p$-dilation estimation in fact works for any norms and other curve functionals. This allows, e.g., to investigate curves in boxes instead of cubes: $\gamma: [0,1] \to [a_1,b_1] \times \cdots \times [a_d,b_d]$; it is equivalent to the consideration of usual Peano curves in metric $\|x\| := \|(b-a)x\|_p$.

Another interesting family of metrics is based on the bounding boxes of the sets $\gamma([t_1,t_2])$, see [HW10]. The structure of our algorithm
allows to minimize them analogously.

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Data availability statement. The datasets generated during and/or analysed during the current study are available in the github repository [gitYM].

**References**

[MS21] E.V. Shchepin, E.Yu. Mychka, “Lower Bounds for the Square-to-Linear Ratio for Plane Peano Curves”, Math. Notes, 110:2 (2021), 267–272.

[B13] M. Bader, *Space-Filling Curves: An Introduction with Applications in Scientific Computing* (Springer, Berlin, 2013).

[GL96] C. Gotsman, M. Lindenbaum, “On the metric properties of discrete space-filling curves”, *IEEE Trans. Image Process.*, 5 (5), 794–797 (1996).

[S04] E.V. Shchepin, “On Fractal Peano Curves”, *Proc. Steklov Inst. Math.*, 247 (2004), 272–280.

[S10] E.V. Shchepin, “On Hölder maps of cubes”, *Mathematical Notes*, 87:5 (2010), 757–767.

[B06] K.E. Bauman, “The dilation factor of the Peano–Hilbert curve”, *Math. Notes*, 80:5 (2006), 609–620.

[B14] K.E. Bauman, “Lower estimate of the square-to-linear ratio for regular Peano curves”, *Discrete Math. Appl.*, 24:3 (2014), 123–128.

[SB08] E.V. Shchepin, K.E. Bauman, “Minimal Peano Curve”, *Proc. Steklov Inst. Math.*, 263 (2008), 236–256.

[B11] K.E. Bauman, “One-side Peano curves of fractal genus 9” *Proc. Steklov Inst. Math.*, 275 (2011), 47–59.

[KS18] A.A. Korneev, E.V. Shchepin, “$L_\infty$-Locality of Three-Dimensional Peano Curves”, *Proc. Steklov Inst. Math.*, 302 (2018), 217–249.

[S15] E.V. Shchepin, “Attainment of maximum cube-to-linear ratio for three-dimensional Peano curves”, *Mathematical Notes*, 98:6 (2015), 971–976.
Appendices

A Complexity of the search

Fix three main parameters: dimension $d$, multiplicity $m$ and genus $g$. Let us consider the complexity of search of minimal curves among Peano multifractals $(\gamma_1, \ldots, \gamma_m)$. For simplicity, we suppose that all
entrance/exit pairs \((\gamma_i(0), \gamma_i(1))\) are equivalent (i.e., may be transformed into each other by a base orientation) to some pair \(e = (p_0, p_1)\); we also assume that \(e\) is equivalent to \(\tilde{e} = (p_1, p_0)\). How many such curves are there for fixed \(e\)?

Let \(N\) be the number of pointed prototypes with entrance/exit pairs equivalent to \(e\). Let there be \(K\) ways to make the next step from \(p_1 \in [0, 1]^d\), i.e. to choose a neighbour cube and a next exit in it. For example, if \(e\) is an edge of a square, then \(K = 6\) — there are 3 neighbour squares and 2 possible exits in each of them. If we consider only facet-continuous edge curves, then \(K = 4\). So, the pointed prototype is defined by \(m\) words of length \((g - 1)\) in \(K\)-alphabet. Hence, the pointed prototype complexity is bounded above by \(m(g - 1) \log_2 K\), so, \(N \leq K^m(g - 1)\). This is of course a very rough bound, but to keep things simple we will use it to measure the complexity.

Denote by \(H\) the subgroup of base orientations that keep \(e\). Given a pointed prototype, on each fraction there are \(m|H|\) ways to choose an appropriate base orientation and curve index. For \(e\) being an edge of a square we have \(|H| = 2\), because there is an identity map that keeps \(e\) and there is a base orientation that consists of an appropriate reflection and time reversal, that also keeps \(e\). Hence, curves with fixed pointed prototype are defined by equations written as a word of length \(mg\) in alphabet of \(m|H|\) symbols and the equations complexity equals \(mg(\log_2 m + \log_2 |H|)\). Note that as \(H\) is a subgroup of \(\mathbb{Z}_2^d \times S_d \times \mathbb{Z}_2\), we have \(|H| \leq d!2^{d+1}\).

Total number of curves equals \(N(m|H|)^mg \leq K^m(g - 1) \cdot (m|H|)^mg\) and the overall complexity may be written as

\[
\text{complexity} = \underbrace{m(g - 1) \log_2 K}_{\text{prototype}} + \underbrace{mg(\log_2 m + \log_2 |H|)}_{\text{equations}} \text{ bits.}
\]

The prototype complexity and equations complexity may serve as two indicators of the complexity of the curve class and it makes sense to treat them separately. In our algorithm, we make a brute force search over all pointed prototypes, hence the prototype complexity roughly corresponds to the “width” of the search, which is CPU-bounded. Given a pointed prototype, we use a SAT-solver to approximate minimum dilation in memory, so the equations complexity corresponds to “depth” of the search, which is memory (RAM) bounded.

Let us consider some configurations that we are capable to deal with.
Plain monofractal edge curves of genus $5 \times 5$. We have $K = 4$ for facet-continuous curves, and $K = 6$ for arbitrary curves. In both cases $|H| = 2$. It gives the complexity $48 + 25$ bits in the facet-continuous case, and $24\log_2 6 + 25 \approx 62 + 25$ in the general case.

Diagonal $5 \times 5$ monofractals have $K = 2$ in the facet-continuous case and $K = 3$ in the general case. In both cases $|H| = 4$. So we obtain $24 + 50$ bits and $24\log_2 3 + 50 \approx 38 + 50$ bits, correspondingly.

To prove Theorem 1 we had to process facet-continuous $6 \times 6$ curves. The edge curves have complexity $70 + 36$. The diagonal curves would have large equations complexity – 72; luckily enough, they do not exist!

Let us compare this with some cases that are too complex to process.

Edge $6 \times 6$ curves have prototype complexity $\approx 90$, this is beyond our capabilities. (So we had to use theoretical bound, Theorem 2.)

The situation is even worse in dimension 3. Consider 3D monofractals of genus $3 \times 3 \times 3$ with entrance end exit on the same edge. We have $K = 21$, because there are 7 neighbour cubes and 3 possible next exits in each of them. That gives us prototype complexity $m(g-1)\log_2 K = 27\log_2 21 \approx 119$; this is too large for the brute-force search.

Another example is 3D edge trifractals of genus $2 \times 2 \times 2$. We have $|H| = 4$, that gives equations complexity $mg \cdot \log_2 12 \approx 86$ and it is too large for our algorithm.