EXAMPLES OF HOLOMORPHIC FUNCTIONS VANISHING TO INFINITE ORDER AT THE BOUNDARY

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Abstract. We present examples of holomorphic functions that vanish to infinite order at points at the boundary of their domain of definition. They give rise to examples of Dirichlet minimizing $Q$-valued functions indicating that ”higher”-regularity boundary results are difficult. Furthermore we discuss some implication to branching and vanishing phenomena in the context of minimal surfaces, $Q$-valued functions and unique continuation.

1. Introduction

In general branching phenomena are of interest in geometric measure theory and geometry, and are strongly related to vanishing phenomena in the context of PDE’s. There is some literature on branching in the interior and one has unique continuation results for PDE’s in the interior of their domains of definition. A more robust quantity then analyticity that seems to capture the structure and properties of branching and unique continuation seems to be Almgren’s frequency function. Little seems to be known about the branching phenomena and the behaviour of the frequency function towards the boundary. We present examples of holomorphic functions that vanish to infinite order at boundary points of their domain of definition. In these points the monotonicity of Almgren’s frequency function fail in general. Thereafter we discuss some implication in the context of minimal surfaces, $Q$-valued functions and unique continuation. These might be an invitation and motivation to the study on boundary behaviour.

Let me shortly explain how I got motivated to this approach, looking for holomorphic functions vanishing to infinite order with a ”large” zero set. My own attempts trying to understand the boundary regularity of $Q$-valued Dirichlet minimizing imposed the question: ”Can one say something about the structure of the singular set towards the boundary?”

Almgren’s frequency function is a key tool to study the singular set in the interior. It is monotone quantity that enables a stratification procedure, compare for example [10] section 3.4 - 3.6] or the work of N. Wickramasekera et al. Such a stratification procedure built on a monotone quantity had been successfully applied as well in other context. (In some sense they can be considered refinements of the ”dimension reducing” argument of Federer [4].) Unfortunately Almgren’s frequency function is in general only monotone in the interior, so a direct extension to the boundary is not possible.

An inspiring discussion with N. Wickramasekera about possible expectations about the structure of the singular set towards the boundary made it apparent that a first impression could be obtained by looking at harmonic or holomorphic functions with zeros accumulating towards the boundary. This link was motivated by the fact that Almgren’s frequency functions has been successfully applied in the context of unique continuation (e.g. [3]) where the vanishing order is measured with the frequency function. The examples presented are perhaps as well of interest in other context such as minimal surfaces and unique continuation. This is discussed
in more detail in section 4.

To give a first impression we state here an implication to $Q$-valued Dirichlet minimizers heuristically to avoid introducing the relevant terminology. The precise statement can be found in corollary 4.5.

**Corollary**: Given $0 < s \leq 1$, an integer $Q \geq 2$ there is a $Q$-valued function $u$, Dirichlet minimizing with respect to compact perturbations satisfying the additional properties:

(i) the trace $u|_{\partial \mathbb{R}^2}$ is "smooth";

(ii) if $s < 1$ then $\mathcal{H}^s(\text{sing}(u)) = 1$ and if $s = 1$ then $\dim_{\mathcal{H}}(\text{sing}(u)) = 1$.

We state now the underlying properties of the holomorphic functions. We present examples of holomorphic functions on the half plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \Re(z) > 0\}$ that admit $C^\infty$-extension to $\overline{\mathbb{C}^+}$ and vanish to infinite order at boundary points. Their properties are:

**Lemma 1.1.** Let $0 < s \leq 1$ be given. There exist

(i) a nowhere dense compact Cantor type subset $E_s \subset [0, 1]$ with $\mathcal{H}^s(E_s) = 1$ if $0 < s < 1$ and $\dim_{\mathcal{H}}(E_1) = 1$;

(ii) holomorphic functions $F(z)$, $G(z)$ on $\mathbb{C}^+$ with the property that $f(z) = e^{-F(z)}$, $g(z) = G(z)e^{-F(z)}$ admit $C^\infty$-extensions to $\overline{\mathbb{C}^+}$. Moreover, $f$, $g$ vanish to infinite order at any $z \in -iE_s$ and for every $z \in -iE_s$ there is a sequence $z_k \in \mathbb{C}^+$ with $z_k \to z$ and $g(z_k) = 0$ for all $k$.

The functions are constructed similar to the Weierstrass’ function, an example of a non-differentiable function. Instead of an infinite series we use infinite products of the following holomorphic building blocks:

(1.1) $a(z) = e^{-z^{\alpha}}$ for $0 < \alpha < 1$

$b(z) = \cos(\ln(z))e^{-z^{\alpha}}$ for $0 < \alpha < 1$.

This note has the following structure:

the results of sections 2 and 3 combined prove lemma 1.1. Section 4 presents some first implications to branching of minimal surfaces, $Q$-valued functions and unique continuation.

**Acknowledgements**

My most sincere thanks to my supervisor Camillo De Lellis and Emanuele Spadaro for their insights and stimulating discussions. Their knowledge and expertise on a vast number of topics was invaluable. Moreover I want to thank Neshan Wirchameskra for the initiative idea. Finally, thank you to the many others who I have failed to mention in this abbreviated list who helped me along the way.

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2. Construction and Properties of the Set $E_s$

The construction is a classical Cantor type construction. Nonetheless for the sake of completeness and to fix certain parameters we present the construction in detail. We follow closely an approach of Falconer in [3, Theorem 1.15].

Lemma 2.1. Let $0 < s \leq 1$ be given. Then there is a nowhere dense compact subset $E_s \subset [0,1]$ s.t. $\mathcal{H}^s(E_s) = 1$ if $0 < s < 1$ and $\dim(E_1) = 1$.

Proof. The set $E_s$ is obtained classically as the intersection of a decreasing family of compact sets

$$E_s = \bigcap_{k=1}^{\infty} \bigcup_{l=1}^{2^k} E_{k,l}.$$ 

The compact subintervals $E_{k,l}$ are defined inductively.

We fix a sequence of parameters by

$$\sigma_k = \begin{cases} \frac{1}{2}, & \text{if } 0 < s < 1 \\ \frac{1}{2^k} - \frac{1}{\sigma_{k+1}} = \frac{1}{2^k} - (k+1)^{-\frac{s}{2}} - (k-1)^{-\frac{s}{2}}, & \text{if } s = 1 \end{cases}$$

In both cases we have $\sigma_k \leq \sigma_{k+1}$ for all $k$. If $s = 1$ we have $\frac{1}{2^k} - \frac{1}{\sigma_{k+1}} = \frac{1}{2^k} - (k+1)^{-\frac{s}{2}} - (k-1)^{-\frac{s}{2}} < 0$ due to concavity of $t \mapsto t^{-\frac{s}{2}}$.

We choose $E_{0,1} = [0,1]$ and proceed inductively. Suppose $E_{k-1,l}, l = 1,\ldots, 2^{k-1}$ defined, then $E_{k,2l-1}, E_{k,2l}$ are the closed subintervals obtained by removing an open interval in the middle of $E_{k-1,l}$ with

$$|E_{k,2l-1}|^{\sigma_k} = |E_{k,2l}|^{\sigma_k} = \frac{1}{2}|E_{k-1,l}|^{\sigma_k}.$$ 

We obtained $2^k$ closed intervals $E_{k,l}$ of equal length

$$|E_{k,l}| = 2^{-\frac{k}{2}}|E_{k-1,l'}| = \begin{cases} 2^{-\frac{k}{2}}, & \text{if } 0 < s < 1 \\ 2^{-k-k^2}, & \text{if } s = 1 \end{cases}$$

where we used that $\sum_{l=1}^{k} \sigma_k^{-1} = \frac{k}{2}$ if $0 < s < 1$ and $\sum_{l=1}^{k} \sigma_k^{-1} = k + k^2$ if $s = 1$.

In a first step we will check that $\mathcal{H}^s(E_s) \leq 1$ ($\mathcal{H}^1(E_1) = 0$). To do so, let $\delta > 0$ be given. Due to (2.2) there is $k_0 > 0$ with $|E_{k_0,l}| < \delta$. Hence $\{E_{k,l}\}_{l=1}^{2^k}$ is an admissible $\delta$-cover for $E_s$ for any $k \geq k_0$. With (2.2) in mind we have

$$\mathcal{H}^s_\delta(E_s) \leq \sum_{l=1}^{2^k} |E_{k,l}|^s = \begin{cases} 2^k \left(2^{-\frac{k}{2}}\right)^s = 1, & \text{if } 0 < s < 1 \\ 2^k \left(2^{-k-k^2}\right)^s \rightarrow 0, & \text{if } s = 1, k \rightarrow \infty \end{cases}$$

Now in the second step we check that $\mathcal{H}^s(E_s) \geq 1$ if $s < 1$ and $\mathcal{H}^s(E_1) = +\infty$ for all $s < 1$ if $s = 1$. Equivalently we have to show that for any $\epsilon > 0$ there is a
\[ \delta > 0 \text{ with the property that for any } \delta-\text{cover } U \text{ of } E_s \text{ we have} \]

\[\sum_{C \in U} \text{diam}(C)^s \geq \mathcal{H}^s_\delta(E_s) > 1 - \epsilon, \quad \text{if } 0 < s < 1 \]

\[\sum_{C \in U} \text{diam}(C)^\sigma \geq \mathcal{H}^\sigma_\delta(E_1) > \frac{1}{\epsilon}, \quad \text{if } s = 1 \text{ i.e. } \sigma < 1 \]

Let \( \epsilon > 0, \sigma < 1 \) be given. We fix \( k_0 > 0 \) large, determined later s.t. at least \( \sigma_k > \sigma \) and \( 0 < \delta < |E_{k_0,l}| \).

Fix an admissible \( \delta-\text{cover } U \) by intervals \( E_{k,l} \). Hence \( k > k_0 \) for any of these intervals. The compact intervals \( E_{k,l} \) are relative open to the compact set \( E_s \), so that the cover can assumed to be finite. Removing all intervals that are contained in some other of the collection we can even assume that they are mutually disjoint. Let \( E_{k,2l-1} \) (or \( E_{k,2l} \)) be one of the shortest intervals in \( U \). Its companion \( E_{k,2l} \) (respectively \( E_{k,2l-1} \)) has to be in \( U \) as well because all intervals are disjointed and they are one of shortest. The sums in (2.4) do not increase if we replace these two intervals by its precessor \( E_{k-1,l} \supset E_{k,2l-1} \cup E_{k,2l} \) because

\[|E_{k,2l-1}|^s + |E_{k,2l}|^s = |E_{k-1,l}|^s, \quad \text{if } 0 < s < 1 \]

\[|E_{k,2l-1}|^\sigma + |E_{k,2l}|^\sigma = 2^{1-\frac{s}{\sigma}}|E_{k-1,l}|^\sigma \geq |E_{k-1,l}|^\sigma, \quad \text{if } s = 1 \text{ i.e. } \sigma < 1 \]

where we used (2.3) and \( \sigma_k \geq \sigma_k > \sigma \). We may proceed in this way, replacing the shortest intervals by larger ones without increasing the value of the sums, until we reach that all intervals are of same size i.e. \( U \rightarrow \{E_{k_1,l}\}_{l=1}^{V_1} \) for some \( k_1 > k_0 \). We conclude

\[\sum_{C \in U} \text{diam}(C)^s = \sum_{l=1}^{k_1} |E_{k_1,l}|^s = 1, \quad \text{if } 0 < s < 1 \]

\[\sum_{C \in U} \text{diam}(C)^\sigma = \sum_{l=1}^{k_1} |E_{k_1,l}|^\sigma = 2^{(1-\sigma)k_1-\sigma k_0^s} > \frac{1}{\epsilon}, \quad \text{if } s = 1 \text{ i.e. } \sigma < 1 \]

where we used (2.3) and \( k_1 > k_0 \) with \( k_0 > 0 \) sufficient large s.t. \( 2^{(1-\sigma)k_0-\sigma k_0^s} > \frac{1}{\epsilon} \).

It remains to argue that the assumption that the \( \delta \)-cover is made out of intervals \( E_{k,l} \) is no real restriction. Fix any \( \delta \)-cover \( V \). We can assume that it consists of open intervals without changing the value in (2.4) significantly. Since \( E_s \) is compact the cover can assumed to be finite.

Firstly let us argue for \( E_1 \). Any interval \( I \in V \) intersects at most three intervals \( E_{k_1,l} \) with \( |E_{k_1,l}| \leq |I| < |E_{k_1-1,l}| \). Otherwise \( I \) would need to contain an interval of length at least \( |E_{k_1-1,l}| \) due to the Cantor type construction. This is impossible by the choice of \( k_1 \). Replacing \( I \) by these at most three intervals \( E_{k_1,l} \) and the same for any other interval in \( T \) we obtain an open cover \( U \) by intervals \( E_{k,l} \). Furthermore

\[\sum_{E_{k_1,l} \in U} |E_{k_1,l}|^s \leq 3 \sum_{I \in V} |I|^s. \]

We had just shown that the left hand side is larger then \( \frac{1}{\epsilon} \), so (2.4) holds for \( s = 1 \). If \( 0 < s < 1 \) we transform the \( \delta \)-cover \( V \) iteratively without increasing the sum in (2.4) to a \( \delta \)-cover \( U \) by sets in \( E_{k,l} \). At first contracting each interval \( I \in V \) we pass to a cover \( V_I \) by closed intervals \( J \) with endpoints that are the endpoints of some \( E_{k,l} \). This process ensures \( \sum_{I \in V} |J|^s \geq \sum_{J \in V_I} |J|^s \). Let \( J \) be any such closed
By construction we ensured this process terminates after finitely many steps till we reach the desired cover. Based on this construction, we define the index set:

\[
J = \{0 < s < 1\}
\]

because \(|E_{k,2l-1}|^s + |E_{k,2l}|^s = |E_{k-1}|^s\) and the left-hand side of (2.5) increases faster than the right-hand side. If either \(J \cap E_{k,2l-1} \neq E_{k,2l-1}\) or \(J \cap E_{k,2l} \neq E_{k,2l}\), we repeat the process, replacing \(J \cap E_{k,2l-1}\) and \(J \cap E_{k,2l}\) by smaller intervals. This process terminates after finitely many steps till we reach the desired cover \(U\). By construction we ensured \(\sum_{s \in J \cap U} |J|^s \geq \sum_{s \in J \cap U} |E_{k,2l}|^s = 1\). This proves (2.4) if \(0 < s < 1\).

\[\square\]

3. Construction of the holomorphic functions

The Cantor set \(E_s\) was obtained as

\[
E_s = \bigcap_{k=1}^{\infty} \bigcup_{l=1}^{2^k} E_{k,l}.
\]

Based on this construction, we define the index set:

\[
I = \{(k,l): k = 1, \ldots, \infty, l = 1, \ldots, 2^k\}
\]

with \(\tau = (k,l) \in I\).

Recall that the enumeration had been chosen s.t. \(E_{k,2l-1} \cup E_{k,2l} \subset E_{k-1} \cup \{l\}\). The Cantor set \(E_s\) constructed in lemma 2.1, i.e. (2.2), had the property that

\[
|E_\tau| = |E_{k,l}| = \begin{cases} 
2^{-\frac{k}{2}}, & \text{if } 0 < s < 1 \\
2^{-k-l}, & \text{if } s = 1
\end{cases} \quad \forall \tau \in I.
\]

We denote with \(y_\tau\) the left boundary point of the compact interval \(E_\tau\). Furthermore it is useful to fix some terminology. \(\mathbb{R}_- = \{z = x + iy: x < 0\}\) denotes the negative real axis. We will use \(z + iy_\tau = r_\tau e^{i\theta_\tau}\) for any \(\tau \in I\). And for any \(y \in \mathbb{R}\) let \(\mathbb{R}_- iy\) be the by \(-iy\) translated negative real axis i.e. the set \(\{x - iy: x < 0\}\). And we will use

\[
\mathbb{R}_- - iy = \bigcup_{y \in E_s} (\mathbb{R}_- - iy) = \{x - iy: x \in \mathbb{R}_-, y \in E_s\}.
\]

The proof to lemma 1.1 is split into two parts. In the next paragraph we construct holomorphic functions \(F,G\) based on the Cantor set \(E_s\) and then in the subsequent paragraph the \(C^\infty\) extension is proven.

3.1. Holomorphy. On the slit plane \(\mathbb{C} \setminus \mathbb{R}_-\) the principal value of the logarithmic function \(\ln: \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{C} \cap \{-\pi < \Im(z) < \pi\}\) is single valued and holomorphic. So will be all roots for \(\alpha \in \mathbb{R}\) defined as \(z^\alpha = e^{\alpha \ln(z)}\).

As composition of holomorphic functions on \(\mathbb{C} \setminus \mathbb{R}_-\) the building blocks, \(a(z) = e^{-z^{-\alpha}}\), \(b(z) = \cosh(\ln(z)) e^{-z^{-\alpha}}\) are clearly holomorphic on \(\mathbb{C} \setminus \mathbb{R}_-\). \((z + iy_\tau)^{-\alpha} = r_\tau e^{-i\alpha \theta_\tau}\) is single valued and holomorphic on \(\mathbb{C} \setminus (\mathbb{R}_- - iy_\tau) \subset \mathbb{C} \setminus (\mathbb{R}_- - iE_s)\) for every \(\tau \in I\), \(\alpha_k \in \mathbb{R}\).

**Lemma 3.1.** Given a sequence of complex numbers \(a_k \in \mathbb{C}\) with \(\sum_{k=0}^{\infty} 2^k |a_k| < \infty\) and a sequence of real numbers \(0 < \alpha_k \leq 1\) then

\[
F(z) = \sum_{\tau \in I} a_k (z + iy_\tau)^{-\alpha_k}
\]

is holomorphic on \(\mathbb{C} \setminus \{\mathbb{R}_- - iE_s\}\) and so is \(e^{-F(z)}\).
Proof. For a fixed $0 < d < 1$ we have for any $z \in \{z \in C: \text{dist}(z, -iE_a) > d\}$ satisfies $|(z + iy_r)^{-\alpha_k}| = r^{-\alpha_k} \leq d^{-1}$. So that the sum $\sum_{k=1}^{\infty} 2^k |a_k| < \infty$ converges absolutely. $F$ is therefore the uniform limit of holomorphic functions on $\{z \in C: \text{dist}(z, -iE_a) > d\}$ and so itself holomorphic. $d$ has been arbitrary and therefore $F$ is holomorphic on $C \setminus \{R_- - iE_a\}$. $e^{-F(z)}$ is the composition of two holomorphic functions and so itself holomorphic on the same set.

Lemma 3.2. Given a sequence of non-negative real numbers $b_k \in \mathbb{R}_+$ that satisfies $\sum_{k=0}^{\infty} 2^k b_k < \infty$, then for any subset $J \subset I$

$$G_J(z) = \prod_{\tau \in J} \cos(b_k \ln(z + iy_\tau))$$

is holomorphic on $C \setminus (R_- - iE_a)$ and uniformly bounded by

$$|G_J(z)| \leq e^{\sum_{\tau \in J} b_k |\theta_\tau|} \leq e^{\pi \sum_{k=0}^{\infty} 2^k b_k}.$$ 

Proof. As a composition of holomorphic functions $\cos(b_k \ln(z + iy_\tau))$ is holomorphic on $C \setminus (R_- - iE_a)$ for every $\tau \in I$. Since $\cos(z + iy) = \cos(x) \cosh(y) - i \sin(x) \sinh(y)$ we have $\cos(b_k \ln(z + iy_\tau)) = \cos(b_k \ln(r_\tau)) \cosh(b_k \theta_\tau) + i \sin(b_k \ln(r_\tau)) \sinh(b_k \theta_\tau)$. So we got that for every $\tau \in I$

$$|\cos(b_k \ln(r_\tau)) \cosh(b_k \theta_\tau)| \leq |\cos(b_k \ln(z + iy_\tau))| \leq \cosh(b_k \theta_\tau)$$

and

$$\sinh(b_k \theta_\tau)$$

To show that $3.1$ is well defined and holomorphic, fix $0 < d < 1$ and $k_0 \in \mathbb{N}$ sufficient large s.t. $0 \leq -b_k \ln(d) \leq \frac{\pi}{4}$ for all $k \geq k_0$. This ensures that for any $z \in \{d < \text{dist}(z, -iE_a) < \frac{\pi}{2}\}$ and $\tau \in I \cap \{k \geq k_0\}$ we have $-\frac{\pi}{4} < b_k \ln(r_\tau) < \frac{\pi}{4}$. Hence $\ln(\cosh(b_k \ln(z + iy_\tau)))$ is a holomorphic function on $\{d < \text{dist}(z, -iE_a) < \frac{\pi}{2}\}$ if $\tau \in I \cap \{k \geq k_0\}$.

(3.4) $\ln(\cosh(b_k \theta_\tau)) + \ln(\cos(b_k \ln(r_\tau))) \leq \ln(\cosh(b_k \ln(z + iy_\tau)))$ where we used (3.2). This is the real part of $\ln(\cos(b_k \ln(z + iy_\tau)))$. Its imaginary part, the argument of $\cos(b_k \ln(z + iy_\tau))$ can be estimated by $\frac{\pi}{2} b_k \ln(r_\tau)$ taking into account that $|\tan| < 1$ and $-\frac{\pi}{4} < b_k \ln(r_\tau) < \frac{\pi}{4}$. Combining both we deduce

$$|\ln(\cos(b_k \ln(z + iy_\tau)))| \leq \ln(\cos(b_k \theta_\tau)) - \ln(\cos(b_k \ln(r_\tau))) + |b_k \ln(r_\tau)|.$$ 

One checks that $h(x) = -\ln(\cos(x))$ is monotone increasing on $[0, \frac{\pi}{2}]$, hence for $|x| \leq \frac{\pi}{4}$ we have $-\ln(\cos(x)) \leq C|x|$ with $C = h(\frac{\pi}{8})$. Consequently we have

$$|\ln(\cos(b_k \ln(z + iy_\tau)))| \leq b_k |\theta_\tau| + (C + 2)|b_k \ln(r_\tau)| \leq (\pi - \ln(d)(C + 1)) b_k;$$

$\sum_{\tau \in I \cap \{k \geq k_0\}} |\ln(\cos(b_k \ln(z + iy_\tau)))| < (\pi - \ln(d)(C + 1)) \sum_{k=k_0}^{\infty} 2^k b_k$ converges uniformly on $\{d < \text{dist}(z, -iE_a) < \frac{\pi}{2}\}$ so that

$$G_1(z) = e^{\sum_{\tau \in J, k \geq k_0} \ln(\cos(b_k \ln(z + iy_\tau)))}$$

is holomorphic on $\{d < \text{dist}(z, -iE_a) < \frac{\pi}{4}\}$. (3.4) showed that $\Re(\ln(\cos(b_k \ln(z + iy_\tau)))$ converges uniformly to $\ln(\cos(b_k \theta_\tau))$ and therefore

$$|G_1(z)| = e^{\sum_{\tau \in J, k \geq k_0} \Re(\ln(\cos(b_k \ln(z + iy_\tau))))} \leq e^{\sum_{\tau \in J, k \geq k_0} b_k |\theta_\tau|}.$$

$$G_2(z) = \prod_{\tau \in J, k < k_0} \cos(b_k \ln(z + iy_\tau))$$
is the product of finitely many holomorphic functions on $\mathbb{C} \setminus (\mathbb{R}_{-} - iE_{a})$ and so itself holomorphic with
\[
|G_{2}(z)| \leq \prod_{\tau \in \mathcal{J}} \left| \cos(b_{\tau} \ln(z + iy_{\tau})) \right| \leq \prod_{\tau \in \mathcal{J}} \cosh(b_{\tau} \theta_{\tau}) \leq e^{\sum_{\tau \in \mathcal{J}, k < k_{0}} b_{k} |\theta_{\tau}|}
\]
where we used (3.2). Multiplication of $G_{1}$ and $G_{2}$ closes the argument. \hfill \Box
\[
\cos(b_{\tau} \ln(z + iy_{\tau})) = 0 \text{ for } z = -iy_{\tau} + e^{-\frac{m\pi - \frac{\alpha}{2}}{a_{k}}} \text{ for any } \tau = (k,l) \in \mathcal{I} \text{ and } m \in \mathbb{N},
\]
so that
\[
G(z) = G_{1}(z) = 0 \text{ for all } z = -iy_{\tau} + e^{-\frac{m\pi - \frac{\alpha}{2}}{a_{k}}}, \tau = (k,l) \in \mathcal{I}, m \in \mathbb{N}.
\]
Consequently we got the following:

**Corollary 3.3.** Let $a_{k}, b_{k}, a_{k}$ be sequences of non-negative real numbers, that satisfies $0 \leq a_{k} \leq 1$ and $\sum_{k=1}^{\infty} 2^{k} a_{k}, \sum_{k=1}^{\infty} 2^{k} b_{k} < \infty$ then
\[
f(z) = e^{-F(z)}, \quad g(z) = G(z)e^{-F(z)}
\]
are holomorphic on $\mathbb{C} \setminus (\mathbb{R}_{-} - iE_{a})$. Furthermore
\[
g(z) = 0 \text{ for } z = -iy_{\tau} + e^{-\frac{m\pi - \frac{\alpha}{2}}{a_{k}}}, \tau = (k,l) \in \mathcal{I}, m \in \mathbb{N}.
\]

**3.2. $C^{\infty}$-extension.** In this section we will show that one can choose sequences $a_{k}, b_{k}, a_{k}$ appropriately (satisfying the conditions of corollary 3.3) such that $f, g$ are holomorphic on $\mathbb{C}_{+}$ and admit a $C^{\infty}$-extension to $\mathbb{C}_{+} = \{ z \in \mathbb{C} : \Re(z) > 0 \}$.

Firstly we check that the building blocks, $a, b$, introduced in (1.1), admit such a $C^{\infty}$-extension to $\mathbb{C}_{+}$ and are vanishing to infinite order in 0 i.e.
\[
(3.5) \quad \lim_{|z| \to 0, z \in \mathbb{C}_{+}} \left| \frac{d^{m}}{dz^{m}} a(z) \right|, \left| \frac{d^{m}}{dz^{m}} b(z) \right| = 0.
\]
By induction one shows that there are constants $C = C(m), D = D(m) > 0$ and $\mu = \mu(m), \nu = \nu(m) \in \mathbb{R}$ (depending only on $m$) s.t. for any $0 < \alpha < 1, z = re^{i\theta} \in \mathbb{C} \setminus \mathbb{R}_{-}, r < 1$
\[
\left| \frac{d^{m}}{dz^{m}} e^{-z^{-\alpha}} \right| \leq C |z^{-2m}| |e^{-z^{-\alpha}}| = Cr^{-2m}e^{-\Re(z^{-\alpha})}
\]
and
\[
\left| \frac{d^{m}}{dz^{m}} \cos(\ln(z)) \right| = \left| \mu \frac{\cos(\ln(z))}{z^{m}} + \nu \frac{\sin(\ln(z))}{z^{m}} \right| \leq Dr^{-m} \cos(\theta).
\]
Hence (3.5) holds if $r^{-m}e^{-\Re(z^{-\alpha})} \to 0$ as $r \to 0$ for every $m \in \mathbb{N}$. This is equivalent to $\Re(z^{-\alpha}) + m \ln(r) \to +\infty$ as $r \to 0$. For $z \in \mathbb{C}_{+} \setminus \{0\}$ we have $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and so
\[
\Re(z^{-\alpha}) + m \ln(r) = r^{-\alpha} \cos(\alpha \theta) + m \ln(r) \geq r^{-\alpha} \cos(\frac{\alpha \pi}{2}) + m \ln(r) \to \infty (r \to 0).
\]
Similarly we can conclude the extension for $f, g$:

**Lemma 3.4.** Let the sequences be $a_{k} = b_{k} = \frac{z^{-k}}{a_{k}}$ and
\[
\alpha_{k} = \begin{cases} 
\alpha, & \text{if } 0 < s < 1 \text{ for some } s < \alpha < 1, \\
1 - \frac{1}{2}k^{-\frac{1}{2}} & \text{otherwise}.
\end{cases}
\]
Then the function \( f, g \) of corollary 3.3 are holomorphic on \( \mathbb{C} \setminus (\mathbb{R}_- - iE_a) \) and admit \( C^\infty \) extensions to \( \overline{\mathbb{C}}_+ \) with

\[
\lim_{z \to \infty} \left| \frac{d^m}{dz^m} f(z) \right|, \left| \frac{d^m}{dz^m} g(z) \right| = 0.
\]

**Proof.** That \( f, g \) are well-defined and holomorphic is the content of corollary 3.3. It remains to check the \( C^\infty \)-extension.

Due to the general Leibnitz rule \( \frac{d^m}{dz^m} f(z) = \sum_{n=0}^{m} \binom{m}{n} G^{(m-n)}(z)(e^{-F(z)})^{(n)} \) it is sufficient to check that for any \( m, n \in \mathbb{N} \),

\[
\lim_{z \to \infty} \left| G^{(m)}(z)(e^{-F(z)})^{(n)} \right| = 0.
\]

Firstly we note that \( F \) is holomorphic on \( \mathbb{C}_+ \), \( (e^{-F(z)})' = -F'(z)e^{-F(z)} \) and

\[
|F^{(m)}(z)| \leq \sum_{\tau \in I} a_k \frac{d^m}{dz^m} (z + iy)^{-\alpha_k} \leq \frac{m!d^{-m-1}}{\sum_k 2^k}
\]

for \( z \in \mathbb{C}_+, \text{dist}(z, -iE_a) \geq d \), so that by induction we deduce

\[
\left| \frac{d^m}{dz^m} e^{-F(z)} \right| \leq C d^{-m-1} |e^{-F(z)}| \text{ for } z \in \{ \text{dist}(z, -iE_a) \geq d \}.
\]

for a constant \( C > 0 \) that depends only on \( m \) and \( \sum_{k=1}^\infty a_k 2^k = \frac{a_0}{b} \). Secondly, Cauchy’s integral formula

\[
G^{(m)}(z) = \frac{m!}{2\pi i} \oint_{\partial B_d(z)} \frac{G(w)}{(w - z)^{m+1}} dw
\]

applies since \( G \) is holomorphic on \( B_d(z) \). Combining it with the uniform bound on \( |G| \) (lemma 3.2) gives

\[
|G^{(m)}(z)| \leq \frac{m!}{d^m} \sup_{w \in B_d(z)} |G(w)| \leq \frac{C m!}{d^m}.
\]

Considering (3.6), (3.7) and the general Leibniz rule the \( C^\infty \) lemma follows if for every \( m \in \mathbb{N} \)

\[
d^{-m}|e^{-F(z)}| = e^{-(\Re(F(z)) + m \ln(d))} \to 0 \text{ for } d = \text{dist}(z, -iE_a) \to 0.
\]

This is equivalent to

\[
\Re(F(z)) + m \ln(d) \to +\infty \text{ as } d \to 0.
\]

To check it, let \( z \in \overline{\mathbb{C}}_+ \) with \( d = \text{dist}(z, -iE_a) > 0 \) be given. Fix \( y \in E_s \) with \( d = |z - iy| \) and \( \tau_k = (k, l) \in I \) with \( y \in E_{\tau_k} \) for each \( k \in \mathbb{N} \). Take \( k_0 \in \mathbb{N} \) with

\[
|E_{k_0+1, \cdot}| < d \leq |E_{k_0, \cdot}|.
\]

Hence for \( k \leq k_0 \) we have \( r_{\tau_k} \leq d + |E_{\tau_k}| \leq 2|E_{\tau_k}| \) and so

\[
\Re(F(z)) = \sum_{\tau \in I} a_k \cos(\alpha_k \theta_\tau) r_{\tau_k}^{-\alpha_k} \geq \sum_{k=1}^{k_0} a_k \cos(\alpha_k \pi k) r_{\tau_k}^{-\alpha_k}
\]
\[
\geq \frac{1}{2} \sum_{k=1}^{k_0} a_k \cos(\alpha_k \pi k)|E_{\tau_k}|^{-\alpha_k}.
\]
Federer observed that (4.1)\[ V \] i.e. \([\llbracket \]

one defines the irreducible holomorphic variety \( V \subset k \subset C \).

If we take \( 0 < s < 1 \) we have \( a_k \cos(\alpha_k \pi/2)|E_{r_k}|^{-\alpha_k} = k^{-2} \cos(\alpha_k \pi)\zeta^k \) where \( \zeta = 2^{\frac{s}{2}} - 1 > 1 \).

We combine this with

\[
(\zeta - 1) \sum_{k=1}^{k_0} k^{-2} \zeta^k = k_0^{-2} \zeta^{k_0+1} - \zeta + \sum_{k=1}^{k_0-1} (k^{-2} - (k+1)^{-2}) \zeta^{k+1} \geq k_0^{-2} \zeta^{k_0+1} - \zeta
\]

to conclude that

\[
\Re(F(z)) + m \ln(d) \geq c k_0^{-2} \zeta^{k_0+1} + m \ln(d) - c \zeta
\]

\[
\geq c k_0^{-2} \zeta^{k_0+1} - \frac{m \ln(2)}{s}(k_0 + 1) - c \zeta \to +\infty \quad (k_0 \to \infty)
\]

where \( c = \frac{\cos(\alpha_k \pi)}{2(k \pi)} \). This is equivalent to (3.8) since due to (3.9), \(-\frac{\ln(2)}{s}(k_0 + 1) < \ln(d) \leq -\ln(2)k_0 \).

If \( s = 1 \), we have

\[
(3.10) \quad a_k \cos(\alpha_k \pi/2)|E_{r_k}|^{-\alpha_k} \geq \frac{1}{2} \cdot \frac{2^{\frac{s}{2}} - 1}{k_0^{2}}
\]

(3.10) holds because firstly \( |E_{r_k}| = 2^{-k - k^{2/3}} \), \( \alpha_k = 1 - \frac{1}{2} k^{-\frac{2}{3}} \) and therefore

\[
\frac{\ln(d^{2^k}|E_{r_k}|^{-\alpha_k})}{\ln(2)} = (1 - \frac{1}{2} k^{-\frac{2}{3}}(k + k^{2/3})) - k \geq \frac{k^{\frac{2}{3}}}{4} \quad \text{for} \quad k \geq 9.
\]

Secondly, \( \cos(\alpha_k \pi) \geq 1 - (1 - \alpha_k) = \frac{2^{\frac{s}{2}} - 1}{2} \) because \( \cos((1 - t)\pi) \geq t \) for \( 0 \leq t \leq 1 \).

Similar as before we have

\[
(3.11) \quad (2^{\frac{s}{2}} - 1) \sum_{k=9}^{k_0} \frac{2^{\frac{s}{2}}}{k_0^{2}} = \frac{2^{\frac{s}{2}}}{k_0^{2}} - \frac{2^{\frac{s}{2}}}{9^{2}} + \sum_{k=9}^{k_0-1} \frac{2^{\frac{s}{2}}}{k^{2}} - \frac{2^{\frac{s}{2}}}{(k+1)^{2}} \geq \frac{2^{\frac{s}{2}}}{k_0^{2}} - 1,
\]

where we used that \( k^{\frac{2}{3}} + \frac{2}{3} \geq (k + 1)^{\frac{2}{3}} \) to conclude that the sum in the middle is non-negative. We combine (3.10) and (3.11) to conclude

\[
\Re(F(z)) + m \ln(d) \geq \sum_{k=9}^{k_0} a_k \cos(\alpha_k \pi/2)|E_{r_k}|^{-\alpha_k} + m \ln(d)
\]

\[
\geq c \frac{2^{\frac{s}{2}}}{k_0^{2}} - c - m \ln(2)(k_0 + 1 + (k_0 + 1)^{\frac{2}{3}}) \to +\infty \quad (k_0 \to \infty)
\]

where \( c = \frac{1}{4(2^{\frac{s}{2}} - 1)} \). As before it is equivalent to (3.8) because of (3.9), which is equivalent to \(-\ln(2)(k_0 + 1 + (k_0 + 1)^{\frac{2}{3}}) < \ln(d) \leq -\ln(2)(k_0 + k_0^{\frac{2}{3}}) \).

\[ \square \]

4. Applications

4.1. Minimal surfaces. Given a holomorphic function \( h \) on \( \Omega \subset C \) open, \( Q \in \mathbb{N} \) one defines the irreducible holomorphic variety \( V \subset \Omega \times C \) by

\[
(4.1) \quad V = \{(z,u) \in \Omega \times C : u^Q = h(z)\}.
\]

Following Federer we associate to \( V \) an integer rectifiable current of real dimension two denoted by \( [V] \). It is given by integration over the manifold part of \( V \), \( V_{reg} \), i.e. \( V_{reg} = \{(z,u) : u^Q = h(z), h(z) \neq 0\} \).

Federer observed that \( [V] \) is a mass-minimizing cycle, since \( V \) as a complex submanifold of \( C^2 \) is calibrated by the Kähler form (Wirtinger’s form).

If we take \( h = g \), \( \Omega = C^+ \) in (4.1) we get the following example:
Example 4.1. Given $0 < s \leq 1$ and an integer $Q \geq 2$ there is a mass-minimizing cycle $\mathcal{V} \subset \mathbb{C}_+ \times \mathbb{C}$ with the additional property that if $s < 1$ then $\mathcal{H}^s(\partial \mathcal{V} \setminus \mathcal{V}_{\text{reg}}) = 1$ and if $s = 1$ then $\dim \mathcal{H}^1(\partial \mathcal{V} \setminus \mathcal{V}_{\text{reg}}) = 1$.

The additional property holds since $\mathcal{V} \setminus \mathcal{V}_{\text{reg}} = \{(z,0) \in \mathbb{C}_+ \times \mathbb{C} : G(z) = 0\}$ and therefore $\mathcal{V} \setminus \mathcal{V}_{\text{reg}} = \{(z,0) \in \mathbb{C}_+ \times \mathbb{C} : G(z) = 0\} \cup -iE_s \cup \{z, 0) \in \mathbb{C}_+ \times \mathbb{C} : G(z) = 0\}$ is countable so that the claim follows by the properties of $E_s$.

Remark 4.2. For two-dimensional minimal surfaces in $\mathbb{R}^3$ R. Osserman had shown in [11] that true branching points can be ruled out in the interior. If the boundary curve is real analytic the existence branching points at the boundary can be ruled out as well. This was shown by R. Gulliver and F. Leslie in [7] for two-dimensional surfaces in $\mathbb{R}^3$.

R. Gulliver presents in [6, Theorem 1.6] the following example:

Theorem 4.1. There is a smooth minimal immersion $X(\Omega) \subset \mathbb{R}^3$, $\Omega \subset \mathbb{C}_+$ simply connected with the following property: $X$ maps $\partial \Omega$ diffeomorphically onto a regular $C^\infty$ Jordan curve $\Gamma \subset \mathbb{R}^3$ and has a true branch point at $z = 0 \in \Gamma$. The set of self intersections of $X$ consists of the union of an infinite sequence of disjoint real analytic arcs, each which joins two points of $\Gamma$ lying on opposite sides of the branch point.

His construction uses the Weierstrass representation with a holomorphic vector-field that comes from a perturbation of the building block $a(z) = e^{-z^2}$, [11], with $\alpha = \frac{1}{4}$. It could be of interest to see if one can follow his analysis using one of the holomorphic functions $f$ or $g$ (lemma 1.1) to construct a minimal immersion $X$ in $\mathbb{R}^3$ with $C^\infty$ boundary curve and a large set of true branching points on the boundary.

4.2. Dirichlet minimizing $Q$-valued functions. One of the implications of lemma 1.1 in the context of $Q$-valued functions had been stated heuristically in the introduction.

F. Almgren developed in his pioneering work [1] the theory of multivalued functions to prove a regularity result on area minimizing rectifiable currents. He introduced them as $Q$-valued functions. $Q \in \mathbb{N}$, fixed, indicates the number of values the function takes, counting multiplicity. We will refer to them from now on as $Q$-valued functions. We assume that the reader is familiar with the most basic definitions and results concerning the theory of $Q$-valued functions with focus on Dirichlet minimizers. We follow mainly the notation and terminology introduced by C. De Lellis and E. Spadaro in [10]. It differs slightly from Almgren’s original work e.g. $(A_Q(\mathbb{R}^n), G)$ denotes the metric space of unordered $Q$-tuples in $\mathbb{R}^n$, $W^{1,2}(\Omega, A_Q(\mathbb{R}^n))$ the Sobolev space of $Q$-valued functions on a domain $\Omega \subset \mathbb{R}^N$. A recollection of the most general definitions and results omitting the actual proofs can be found in [8, section 1]. C. De Lellis and E. Spadaro gave a modern revision of Almgren’s original theory and results concerning Dirichlet minimizers in [10].

The holomorphic functions $f, g$ generate examples of $Q$-valued functions that are Dirichlet minimizing with respect to compact perturbations. Furthermore these examples are defined on $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : x > 0\} \cong \mathbb{C}_+$ and have ”large” singular set towards the boundary. As we mentioned before the classical theory of Dirichlet minimizing $Q$-valued functions had been developed in [1] and revisited with modern methods in [10].

Before we are going to state the precise properties of the examples we recall the definition of the singular set and related results thereafter the definition of $C^k(\Omega, A_Q(\mathbb{R}^m))$ for a domain $\Omega \subset \mathbb{R}^n$. 

Theorem 4.4. the following:

\[ \text{Definition of the singular set:} \]
Given a Dirichlet minimizer \( u \in W^{1,2}(\Omega, A_Q(\mathbb{R}^m)) \), \( \Omega \subset \mathbb{R}^N \) open, a point \( y \in \Omega \) is called a regular point of \( u \) if \( \exists U \subset \Omega \) open neighborhood of \( y \), \( u_i \in C^\infty(U, \mathbb{R}^m) \) harmonic with

\[
u(x) = \sum_{i=1}^Q u_i(x) \] for a.e. \( x \in U \)

and \( u_i(x) \neq u_j(x), \forall x \in U \) or \( u_i \equiv u_j \). The open set (by definition) of all regular points is denoted by \( \text{reg}(u) \). \( \text{sing}(u) \) then denotes the relative closed complement \( \Omega \setminus \text{reg}(u) \).

An outcome of Almgrens original work is an estimate on the size of the singular set in the interior, compare [10, Theorem 0.11].

Theorem 4.2. \( u \in W^{1,2}(\Omega, A_Q(\mathbb{R}^m)) \) Dirichlet minimizing has \( \text{dim}_H(\text{sing}(u)) \leq N - 2 \). In the case of \( N = 2 \), \( \text{sing}(u) \) is countable.

This estimate had been improved by C. De Lellis and E. Spadaro, [10, Theorem 0.12].

Theorem 4.3. \( u \) as above and \( N = 2 \) then \( \text{sing}(u) \) consists of isolated points.

That the upper bound on the Hausdorff dimension is sharp is a consequence of the following:

Theorem 4.4. Let \( V \subset \mathbb{C}^N \times \mathbb{C}^m \simeq \mathbb{R}^{2N} \times \mathbb{R}^{2m} \) be an irreducible holomorphic variety with the property that \( \exists \Omega \subset \mathbb{C}^N \) open, \( C^1 \)-regular, \( V \) is is a \( Q : 1 \) cover of \( \Omega \) under the orthogonal projection and \( \mathcal{M}(V \cap (\Omega \times \mathbb{C}^m)) < \infty \). Then \( \exists u \in W^{1,2}(\Omega, A_Q(\mathbb{R}^m)) \) Dirichlet minimizing with \( \text{graph}(u) = V \cap (\Omega \times \mathbb{C}^m) \).

This was original be proven by Almgren, [1, Theorem 2.20]. E. Spadaro found a very elegant more elementary proof, [12, Theorem 0.1].

Hence the holomorphic varieties \( V = V_h \) defined in (4.1) generate examples of Dirichlet minimizers:

\begin{align}
(4.2) \quad u_h(z) = \sum_{v \in \mathbb{C}} \sum_{v^Q = h(z)} [v] \text{ for } z \in \Omega.
\end{align}

Definition of \( C^k(\Omega, A_Q(\mathbb{R}^m)) \):
Let \( k \in \mathbb{N} \) and \( \Omega \subset \mathbb{R}^N \), \( u \in C^0(\Omega, A_Q(\mathbb{R}^m)) \) is said to be \( C^k(\Omega, A_Q(\mathbb{R}^m)) \) if there exists a \( Q \)-valued map \( U \),

\[
x \mapsto U_x(y) = \sum_{i=1}^Q [P_x^i(y)], \quad P_x^i \text{ is a polynomial with degree } \leq k
\]
such that the following properties hold

(a) \( U_x(x) = \sum_{i=1}^Q [P_x^i(x)] = u(x) \) for all \( x \in \Omega \);
(b) \( P_x^i = P_y^j \) if \( u_i(x) = u_j(x) \);
(c) whenever \( K \subset \subset \Omega \), compact, \( \delta > 0 \) let

\[
\rho_K(\delta) = \sup_{x, y \in K} \inf_{\sigma \in S_Q} \sum_{i=1}^Q \sum_{|\alpha| \leq k} |D^\alpha P_x^i(y) - D^\alpha P_y^i(y)||x-y|^{k-|\alpha|}(k-|\alpha|)!)
\]

then \( \rho_K(\delta) \to 0 \) as \( \delta \to 0 \).
We want to remark, that condition (b) is not always assumed, compare [9] Definition 3.6] and [10] Definition 1.9. Let $u_1, \ldots, u_Q$ be a collection of single valued $C^k$-functions on $\Omega$.

$$u(x) = \sum_{i=1}^{Q} |u_i(x)|$$

defines a $Q$-valued $C^k$-function (including property (b)), if $D_\alpha u_i(x) = D_\alpha u_j(x)$ for all $|\alpha| \leq k$ whenever $u_i(x) = u_j(x)$. The function $U_2$ is given by

$$U_2(y) = \sum_{i=1}^{Q} |P^i(y)|$$

where $P^i(y) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D_\alpha u_i(x)(y-x)^\alpha$ is the $k$th-order Taylor polynomial of $u_i$. Property (c) follows from the properties of the Taylor polynomials and (b) by the assumption on the order of contact.

Now we are able to state properly the properties of the examples:

**Corollary 4.5.** Let $0 < s \leq 1$ and an integer $Q \geq 2$ be given, then there is $u \in W^{1,2}(\mathbb{R}_+, A_Q(\mathbb{R}^2))$, Dirichlet minimizing with respect to compact perturbations of $\mathbb{R}^2$ and the additional properties

(i) $u|_{\partial \mathbb{R}^2_+} \in C^k(\partial \mathbb{R}^2_+, A_Q(\mathbb{R}^2))$ for all $k \in \mathbb{N}$;

(ii) if $s < 1$ then $H^s(\text{sing}(u)) = 1$ and if $s = 1$ then $\dim_{H^s}(\text{sing}(u)) = 1$.

**Proof of lemma 4.3** Let $0 < s \leq 1$ be fixed and $g(z) = G(z)e^{-F(z)}$ be the holomorphic function on $\mathbb{C}_+$ constructed in lemma 1.1.

$$u(z) = \sum_{\nu \in \mathbb{C}} [\nu] z \in C_+$$

is Dirichlet minimizing and an element of $W^{1,2}(\Omega, A_Q(\mathbb{R}^2))$ for any $C^1$-regular bounded subset $\Omega \subset \mathbb{C}_+$ as a consequence of theorem 4.4.

It remains to check the $C^\infty$-regularity at the boundary and the property of the singular set.

We start with the regularity of the trace. By construction we had $g(z) = G(z)e^{-F(z)}$ is holomorphic on $\mathbb{C}_+ \setminus (\mathbb{R}_- - iE_s)$ and $g|_{\mathbb{C}_+}$ has an $C^\infty$ extension to $\mathbb{C}_+$. Furthermore $G(z) \neq 0$ for all $z \in \mathbb{C}_+ \setminus (\mathbb{R}_- - iE_s)$, $|G(z)| < C$ uniformly on $\mathbb{C}_+ \setminus (\mathbb{R}_- - iE_s)$. So that for any $B_r(z_0) \subset \mathbb{C}_+ \setminus (\mathbb{R}_- - iE_s)$ there exists a holomorphic branch $\psi : G(B_r(z_0)) \to \mathbb{C}$ of the $Q$-th root. $u$ is then explicitly given by

$$u(z) = \sum_{l=0}^{Q-1} \left[ \xi^l (\psi \circ G)(z) e^{-\frac{F(z)}{\psi}} \right] \forall z \in B_r(z_0), \xi = e^{i\frac{\pi}{2}}.$$

Hence we are in the situation of (??) on $B_r(z_0)$. The $k$-jet of $u$ is

$$U^k_{B} = \sum_{l=0}^{Q} [\langle \xi^l (\psi \circ g)(z), \xi^l (\psi \circ g)^{(1)}(z), \ldots, \xi^l (\psi \circ g)^{(k)}(z) \rangle]$$

where we write $\psi \circ g(z)$ for $(\psi \circ G)(z) e^{-\frac{F(z)}{\psi}}$. The $C^\infty$-regularity will follow from (4.3)

$$|\psi \circ g|^{(m)}(-iy) = O(\text{dist}(y, E_s)) \quad \text{for all } m \in \mathbb{N}.$$
It remains to check the properties of the singular set. By construction of \( u \) we have
\[ k \]
for all \( i \in \mathbb{N} \). In the planar case C. De Lellis and E. Spadaro determined the spectrum of Dirichlet minimizer \( u \). So Cauchy’s integral formula gives
\[
(\psi \circ G)(m) = \frac{m!}{2\pi i} \oint_{\partial B_d(z)} \frac{\psi \circ G(w)}{(w-z)^{m+1}} \, dw
\]
and therefore
\[
| (\psi \circ G)(m)(z) | \leq \frac{m!}{d^m} \sup_{w \in \partial B_d(z)} | G(w) | \frac{1}{d^m} \leq C m! d^{-m}.
\]
We used the uniform bound on \( |G| \). Hence we deduce
\[
| (\psi \circ G)(m)(z) e^{-\frac{i}{d} F(z)} | \leq C \left( d^{-Qm} e^{-\Re(F(z))} \right)^{\frac{1}{d}} \quad \forall z \in \{ \text{dist}(z, \mathbb{R} - iE_s) > d \}.
\]
So (4.3) follows from (3.8) where we showed that for any \( m \in \mathbb{N} \)
\[
\Re(F(z)) + m \ln(d) \to +\infty \quad \text{as} \quad d \to 0.
\]

It remains to check the properties of the singular set. By construction of \( u \) we have
\[
\text{sing}(u) = \{ z \in \mathbb{C}_+ : g(z) = 0 \} \cup -iE_s
\]
because \( g \) has the property that to any \( z \in -iE_s \) there exists \( z_k \in \mathbb{C}_+, z_k \to 0 \) and \( g(z_k) = 0 \). Set \( A_k = \{ z \in \mathbb{C}_+ : g(z) = 0, 2^k \leq \Re(z) < 2^{k+1} \} \) for any \( k \in \mathbb{Z} \). \( A_k \)
consists of isolated points since \( g \) is holomorphic on \( \mathbb{C}_+ \) and therefore \( \mathcal{H}^s(A_k) = 0 \)
for all \( k \in \mathbb{Z} \) and \( s > 0 \). Hence we deduce
\[
\mathcal{H}^s(-iE_s) \leq \mathcal{H}^s(\text{sing}(u)) \leq \mathcal{H}^s(-iE_s) + \sum_{k \in \mathbb{Z}} \mathcal{H}^s(A_k) = \mathcal{H}^s(-iE_s).
\]

This example, corollary 4.5, shows that the singular set can behave very badly
borderside the boundary. In the interior a blow-up analysis together with a Federer
reduction argument is used to study the singular set, compare [10, section 3]. With
the following calculation we want to show that this procedure cannot directly trans-
ferral to the boundary.

Almgren’s celebrated frequency function is the major tool to carry out the blow-up
analysis. For \( u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^m)) \) with \( \Omega \subset \mathbb{R}^N \) open it is defined as
\[
I(u, y, r) = \frac{D(u, y, r) - H(u, y, r)}{H(u, y, r)} = e^{2-N} \int_{B_r(y) \cap \Omega} |Du|^2 - r^{1-N} \int_{B_r(y)} |u|^2.
\]
Its essential property is, compare [10, Theorem 3.15]

**Theorem 4.6.** Let \( u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^m)) \) be Dirichlet minimizing, then for any \( y \in \Omega \) either \( \exists 0 < R < \text{dist}(y, \partial \Omega) \) s.t. \( u\big|_{u(y) = 0} \) or \( r \in 0, \text{dist}(y, \partial \Omega) \to I(u, y, r) \)
is absolutely continuous, nondecreasing and positive.

Consequently the following limit is well-defined in the interior of \( \Omega \)
\[
I(u, y) = \lim_{r \to 0} I(u, y, r)
\]
In the planar case C. De Lellis and E. Spadaro determined the spectrum of \( y \to I(u, y) \) to be \( \{ \frac{p}{Q} : P \in \mathbb{N} \} \cup \{ 0 \} \). [10, Proposition 5.1].

The following examples show that this may fail at boundary points.

**Corollary 4.7.** Let \( Q \geq 2, P > 0 \) be two divisor free integers then there exists a
Dirichlet minimizer \( u \in W^{1,2}_\text{loc}(\mathbb{R}^m_+, \mathcal{A}_Q(\mathbb{R}^2)) \) with
\[
(i) \quad u\big|_{\partial \mathbb{R}^2_+} \in C^k(\partial \mathbb{R}^2_+, \mathcal{A}_Q(\mathbb{R}^2));
\]
(ii) for all \( k \in \mathbb{N} \), \( z_k = (e^{-k\pi + \frac{i\pi}{2}}, 0) \) is a branch point of "order" \( \frac{P}{Q} \) i.e. \( I(u, z_k) = \frac{P}{Q} \);

(iii) \( \lim_{r \to 0} I(u, 0, r) = +\infty \).

**Corollary 4.8.** Let \( Q > 2 \) be an integer, \( 0 < s < 1 \) be given there is a Dirichlet minimizer \( u \in W^{1,2}_{\text{loc}}(\mathbb{R}^2, A_Q(\mathbb{R}^2)) \) with

(i) \( u|_{\partial \Omega} \in C^k(\partial \Omega^2 \setminus \{u\}, A_Q(\mathbb{R}^2)) \);

(ii) \( \text{sing}(u) = \emptyset \), but \( u(z) = Q[0] \) \( \forall z \in -iE_u \) with \( \mathcal{H}^s(E_u) = 1 \);

(iii) \( \lim_{n \to \infty} I(u, -iy, R_n) = +\infty \) for a countable subset \( \{y_k\}_{k \in \mathbb{N}} \subset E_u \) and a sequence \( R_n \to 0 \).

Before we are given the proofs, we collect two observations to calculate energy and \( L^2 \)-norm for multivalued functions arising from the holomorphic varieties defined in (4.3).

\( A_Q(\mathbb{C}) \simeq A_Q(\mathbb{R}^2) \) enables us to define a \( Q \)-root "globally", i.e. an "inverse" to the holomorphic function \( z \mapsto z^Q \) by

\[
\Pi(w) = \sum_{v^Q = w} [v] = \sum_{l=0}^{Q} [\xi^l v_0]
\]

for \( \xi = e^{i\frac{k\pi}{Q}} \) and an arbitrary choice of \( v_0 \in \mathbb{C} \) with \( v_0^Q = w \). Furthermore we observed already before that for \( y \in \Omega \) with \( h(y) \neq 0 \) there is an open neighborhood \( U \) with \( |h(z) - h(y)| < |h(y)|, \forall z \in U \). There is an holomorphic branch \( \psi \) of the \( Q \)-root on \( |w - h(y)| < |h(y)| \) so that \( \Pi(w) = \sum_{l=0}^{Q-1} [\xi^l \psi(w)] \) on \( B_{|h(y)|}(h(y)) \) showing that \( \Pi \) is continuous on all of \( \mathbb{C} \). Furthermore

\[
u(z) = \Pi \circ h(z) = \sum_{l=0}^{Q-1} [\xi^l (\psi \circ h)(z)] \quad \forall z \in U.
\]

Hence \( u \in C^k(U, A_Q(\mathbb{R}^2)) \) for all \( k \) since we are in the situation mentioned in (4.3) with

\[
U_k^\mathbb{C} = \sum_{l=0}^{Q-1} [\xi^l (\psi \circ h)(z), \xi^l (\psi \circ h)^{(1)}(z), \ldots, \xi^l (\psi \circ h)^{(k)}(z)] \quad \forall z \in U.
\]

We note that \( U_k^\mathbb{C} \) does not depend on the particular choice of the branch. As an immediate consequence of (4.7) the \( L^2 \) norm of \( u \) is given by

\[
\int_{V \cap \Omega} |u|^2 = Q \int_{V \cap \Omega} |h|^\frac{2}{Q}
\]

for any \( V \subset \mathbb{C} \). The energy of \( u \) on \( V \cap \Omega \) due to (4.8) is then

\[
\int_{V \cap \Omega} |Du|^2 = 2Q \int_{V \cap \Omega \setminus \{h \neq 0\}} |(\psi \circ h')|^2 = \frac{2}{Q} \int_{V \cap \Omega \setminus \{h \neq 0\}} |h|^\frac{2}{Q} - 2|h'|^2
\]

where \( \psi \) is any local choice of a branch \( \psi \) to the \( Q \)-root.

For instance we can use it to calculate the value of the frequency at interior branch points.

**Example 4.3.** Let \( h \) be holomorphic on \( \Omega \subset \mathbb{C} \) and \( u \) the related Dirichlet minimizer (see (4.2)). Let \( z_0 \in \Omega \) be a zero of order \( P \geq 1 \) then

\[
I(u, z_0) = \frac{P}{Q}.
\]

\( z_0 \) is a zero of order \( P \), hence there is \( k \) holomorphic on \( \{z : |z| < \delta\} \), \( k_0 = k(0) \neq 0 \) s.t. \( h(z_0 + z) = z^P k(z) \). We may assume that \( |k(z)| > \frac{1}{2}|k_0|^2 \) for all \( |z| < \delta \).
\[ h'(z_0 + z) = P z^{P-1} k(z)(1 + \frac{k'(z)}{k(z)}) = \frac{P}{h(z_0 + z)(1 + o(z))} \] and so we may use
\[ |h|^k \approx h'(z_0 + z) = P|z|^{k-1}|k_0|^k(1 + o(z)) \] in (4.10) to deduce
\[
\int_{B_r(z_0)} |Du|^2 \frac{2P^2}{Q} \int_{B_r(0)} |z|^\frac{k-2}{k} |k_0|^k (1 + o(z)) = 2\pi P |k_0|^k r^{\frac{k-2}{k}} (1 + o(r))
\]
for any \( 0 < r < \delta \). Similarly, using (4.9) we have
\[
\frac{1}{r} \int_{\partial B_r(z_0)} |u|^2 = \frac{Q}{r} \int_{\partial B_r} |z|^\frac{k-2}{k} |k_0|^k (1 + o(z)) = 2\pi Q |k_0|^k r^{\frac{k-2}{k}} (1 + o(r)).
\]
We conclude the claim:
\[ I(u, z_0, r) = \frac{P}{Q}(1 + o(r)). \]

For boundary points \( z_0 \in \partial \Omega \) we are facing two problems to estimate \( I(u, z_0, r) \) and possible limits. Firstly \( r \mapsto I(u, z_0, r) \) is a priori not a monotone quantity as it is in the interior. Secondly, even restricting ourselves to minimizers of the type (4.2), \( h(z) \) does not necessarily have a convergent Taylor series at \( z_0 \). The strategy will be to use the mean value theorem for integration in the radial variable to estimate \( D(u, z_0, r) = \int_{B_r(z_0) \cap \Omega} |Du|^2 \) from below by a multiple of \( H(u, z_0, r) = \frac{1}{r} \int_{\partial B_r(z_0) \cap \Omega} |u|^2 \). The strategy is motivated by the following observation.

Given a function \( k \) holomorphic in a neighbourhood of \( z \in \mathbb{C} \) and \( k(z) \neq 0, \gamma > 0 \), for any \( \xi = e^{i\theta} \) one has
\[
D_{k_0} |k|^2 = 2\Re k^\gamma = 2|k_0|^2 \Re \left( \frac{k'}{k} \right)
\]
and so \( D_{k_0} |k|^2 = \frac{2}{\gamma} |k_0|^2 - 2 D_{k_0} |k|^2 = \gamma |k_0|^2 \Re \left( \frac{k'}{k} \right) \). This gives
\[
(4.11) \quad \gamma |k|^2 - 2 |k'|^2 = \gamma |k_0|^2 \Re \left( \frac{k'}{k} \right)
\]
The strategy is illustrated in the following example:

**Example 4.4.** Let \( h(z) = e^{-z^{-\alpha}}, 0 < \alpha < 1 \) (\( h(z) = a(z) \) of (1.3)) in (4.2), i.e.
\[
u(z) = \sum v_{e^R = h(z)} \quad \text{with } z \in \Omega = \mathbb{C}^+, \quad \text{then } u \text{ satisfies}
\]
\[
\lim_{R \to 0} I(u, 0, R) = +\infty.
\]
We will use the classic radial notation \( z = re^{i\theta} \). We define
\[
\varphi(z) = r \Re \left( \frac{h'(z)}{h(z)} e^{i\theta} \right) = \alpha \Re(z^{-\alpha}) = \alpha r^{-\alpha} \cos(\alpha \theta).
\]
Combining (4.10) with (4.11) (\( h(z) \neq 0 \forall \in C^+ \)) gives
\[
\int_{B_{R} \cap \mathbb{C}^+} |Du|^2 = \int_{B_{R} \cap \mathbb{C}^+} \frac{2}{Q} |h(z)|^{k-2} |k_0|^k (1 + o(z)) \geq \int_{B_{R} \cap \mathbb{C}^+} \frac{\varphi(z)}{r} \frac{\partial}{\partial r} |h|^k \hat{z}
\]
\[
\frac{1}{-z} \int_{0}^{R} \varphi(re^{i\theta}) \left( \frac{\partial}{\partial r} |h|^k \hat{z} \right) (re^{i\theta}) dr d\theta
\]
Since \( \varphi(z) \geq \alpha r^{-\alpha} \cos(\alpha \hat{z}) > 0, (4.11) \) implies that \( \frac{\partial}{\partial r} |h|^k \hat{z} \geq 0 \). Thus we apply the 1-dimensional mean value theorem to deduce that to every \( |\theta| \leq \frac{\pi}{2} \) there is
Let $\theta$ be a Borel regular measure on a path-connected space $X$, $\nu$ a measure on some space $Y$ and $\mu \times \nu$ the product measure on $X \times Y$. Given $f, g$ with the properties that

(i) $f, g, fg$ are $\mu \times \nu$ summable, i.e. $f, g, fg \in L^1(X \times Y, \mu \times \nu)$;
(ii) $x \mapsto f(x, y)$ is continuous for a.e. $y$.

Then there exists a map $\chi : Y \to X$ s.t.

\[ y \mapsto f(\chi(y), y) \int_X g(x, y) \, d\mu(x) = \int_X fg(x, y) \, d\mu(x) \text{ is } \nu \text{-integrable and} \]

\[ f(\chi(y), y) \int_X g(x, y) \, d\mu(x) = \int_X fg(x, y) \, d\mu(x) \text{ for a.e. } y \]

Indeed, let $A \subset Y$ be the set of $y \in Y$ s.t.

(a) $x \mapsto f(x, y)$ is continuous and $|f|$ is finite;
(b) $x \mapsto g(x, y), fg(x, y)$ are $\mu$-summable ($g(\cdot, y), fg(\cdot, y) \in L^1(X, \mu)$).

We have $\nu(Y \setminus A) = 0$ since (a) holds for a.e. $y$ by assumption and (b) holds for a.e. $y$ by general measure theory. The 1-dimensional mean value theorem tells that for $y \in A$ there exists $\chi(y) \in X$ s.t. the identity (4.13) holds. Indeed let $y \in A$ be fixed, then $z \mapsto f(z, y) \int_X g(x, y) \, d\mu(x)$ is continuous and since $|\int_X f(x, y) g(x, y) \, d\mu(x)| < \infty$ we can find $x_0, x_1 \in X$ s.t.

\[ \inf_{z \in X} f(z, y) \int_X g(x, y) \, d\mu(x) \leq f(x_0, y) \int_X g(x, y) \, d\mu(x) \]

\[ \leq \int_X f(x, y) g(x, y) \, d\mu(x) \]

\[ \leq f(x_1, y) \int_X g(x, y) \, d\mu(x) \leq \sup_{z \in X} f(z, y) \int_X g(x, y) \, d\mu(x). \]

By assumption there is a continuous path $\gamma$ connecting $x_0$ with $x_1$. Now we may apply the 1-dimensional mean value theorem to $t \mapsto f(\gamma(t), y) \int_X g(x, y) \, d\mu(x)$ to find a point $\chi(y)$. Since $\int_X f(y) g(x, y) \, d\mu(x)$ is $\nu$-integrable and for all $y \in A$ (4.13) is satisfied (4.13) holds. If in addition $\int_X f(x, y) \, d\mu(x) \neq 0$ for a.e. $y$ then $y \mapsto f(\chi(y), y)$ is $\nu$-measurable.
Proof of corollary 4.7. We claim that the minimizer \(u(z) = \sum_{\nu \in \mathbb{C}} e^{\nu z} [v]\) with \(b(z) = \cos(\ln(z)) e^{-z^{-\alpha}}\) (compare (1.1)) has the desired properties.

(i) follows from the same arguments presented in the proof of corollary 4.5 so we omit the details here.

(ii) corresponds to example 4.3. Since \(\{z \in \mathbb{C}_+: b(z) = 0\} = \{e^{-\frac{\pi (2k+1)}{4}}: k \in \mathbb{Z}\}\), \(b'(e^{-\frac{\pi (2k+1)}{4}}) = (-1)^{k+1} e^{-\alpha e^{-\frac{\pi (2k+1)}{4}}} \neq 0\) and so \(e^{-\frac{\pi (2k+1)}{4}}\) is a zero of order \(P\) to \(b(z)\).

(iii) remains to be proven. We want to do it similarly to the example 4.3 As before we define

\[
\varphi(z) = \Re \left( \frac{b'(z)}{b(z)} z \right) = \Re \left( \alpha z^{-\alpha} - \frac{\sin(\ln(z))}{\cos(\ln(z))} \right).
\]

\(\Re(\tan(ln(re^{i\theta})))\) is not uniformly bounded as \(|\theta| \to 0\), hence we can not conclude directly \(\varphi(re^{i\theta}) \geq 0\) for \(r > 0\) sufficient small. But \(|\tan(ln(re^{i\theta}))|^2 \leq \frac{1}{\tan^2(\theta)}\) is bounded on \(\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}\) and so

\[
\varphi(re^{i\theta}) \geq \alpha r^{-\alpha} \cos(\frac{\pi}{2}) - \frac{1}{\tanh(\frac{\pi}{4})} \geq 0
\]

for \(\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}\) and \(0 < r \leq R, R > 0\), \(r > 0\) sufficient small.

\(\lambda \mapsto |b(\alpha e^{i\lambda \theta})|^2 = |\cos(ln(\alpha e^{i\lambda \theta}))|^2 e^{-2\alpha^{-\alpha} \cos(\alpha \lambda \theta)}\)

is monotone increasing. \(\lambda \mapsto e^{-\alpha \lambda \theta} \cos(\alpha \lambda \theta)\) is monotone increasing for \(|\lambda \alpha \theta| \leq \frac{\pi}{4}\) and \(\lambda \mapsto |\cos(ln(\alpha e^{i\lambda \theta}))|^2\) because \(\frac{2\pi}{\alpha} |\cos(ln(\alpha e^{i\lambda \theta}))|^2 = \sinh(2\lambda \theta) > 0\). Combine it with (4.9) \(|h|^2 = |b|^{2P}Q\) gives

\[
\frac{1}{R} \int_{B_R \cap \mathbb{C}_+} |u|^2 = Q \int_{\frac{\pi}{4} < \theta < \frac{\pi}{2}} |b(Re^{i\theta})|^2 \frac{Q}{R} d\theta \leq Q \int_{\frac{\pi}{4} < \theta < \frac{\pi}{2}} |b(Re^{i\theta})|^2 \frac{Q}{R} d\theta + Q \int_{|\theta| < \frac{\pi}{4}} |b(Re^{i\theta})|^2 \frac{Q}{R} d\theta = \frac{3Q}{2} \int_{\frac{\pi}{4} < \theta < \frac{\pi}{2}} |b(Re^{i\theta})|^2 \frac{Q}{R} d\theta.
\]

(4.15) together with (4.11) gives with \(h = b^P, h' = Pb^{P-1}b', |h|^2 = P^2|b|^{2P-2}|b'|^2\)

\[
\int_{B_R \cap \mathbb{C}_+} |Du|^2 \geq \frac{2P}{Q} \int_{\frac{\pi}{4} < \theta < \frac{\pi}{2}} \varphi(z) \left| \frac{\partial}{\partial r} |b|^{2P} \right| d\theta.
\]

(4.11) (i.e. \(\frac{\partial}{\partial r} |b|^{2P} = \frac{2P}{Q} \frac{\varphi(z)}{r} |b|^{2P}\) and (4.15) show that \(\frac{\partial}{\partial r} |b|^{2P}(re^{i\theta}) \geq 0\) for \(\frac{\pi}{4} \leq |\theta| \leq \frac{\pi}{2}, 0 < r < R\). Hence we apply the 1-dimensional mean value theorem to deduce that to every \(\frac{\pi}{4} \leq |\theta| \leq \frac{\pi}{2}\) there is \(0 < r_0 \leq R\) with

\[
\int_{\frac{\pi}{4} \leq |\theta| \leq \frac{\pi}{2}} \varphi(z) \left| \frac{\partial}{\partial r} |b|^{2P} \right| d\theta = \int_{\frac{\pi}{4} \leq |\theta| \leq \frac{\pi}{2}} \varphi(r_0 e^{i\theta}) \int_0^R \left| \frac{\partial}{\partial r} |b|^{2P} (re^{i\theta}) \right| dr d\theta \geq \left( \alpha R^{-\alpha} \cos(\frac{\pi}{2}) - \frac{1}{\tanh(\frac{\pi}{4})} \right) \int_{\frac{\pi}{4} \leq |\theta| \leq \frac{\pi}{2}} |b|^{2P} (Re^{i\theta}) d\theta.
\]

(Again we can avoid measurability questions using the bound (4.14), nonetheless compare the previous remark 4.8) Recall (4.15) to deduce (iii) in total since for \(R > 0\) sufficient small

\[
I(u, 0, R) \geq \frac{P}{Q} \left( \alpha R^{-\alpha} \cos(\frac{\pi}{2}) - \frac{1}{\tanh(\frac{\pi}{4})} \right) \to \infty \quad (R \to 0).
\]

\(\square\)
Proof of corollary 4.8. We claim that for the choice \( f(z) = e^{-F(z)} \) of lemma 4.1 with a fixed \( 0 < s < 1 \) the minimizer \( u(z) = \sum_{s \in \mathbb{C}} e^f(z) \) has the desired properties.

(i) follows as before by similar arguments presented in the proof to corollary 4.5 and so we omit the details.

(ii) corresponds with \( f(z) \neq 0 \) for all \( z \in \mathbb{C}_+ \).

(iii) remains to be proven. We define

\[
R_n = |E_{n,|} + \frac{2}{3} (|E_{n-1,-} - 2|E_{n,|}) = \frac{2}{3} |E_{n-1,-} - \frac{1}{3} |E_{n,|} = \frac{2}{3} (2^{1 + \frac{1}{2}} - 1) 2^{-\frac{n}{2}}
\]

\[
R_n = |E_{n,|} + \frac{1}{3} (|E_{n-1,-} - 2|E_{n,|}) = \frac{1}{3} |E_{n-1,-} + \frac{1}{3} |E_{n,|} = \frac{1}{3} (2^{\frac{n}{2}} + 1) 2^{-\frac{n}{2}}
\]

and set \( \delta = \frac{1}{4} (2^{\frac{n}{2}} - 2) > 0 \). We will show that (iii) holds for the countable set \( \{y_r\}_{r \in \mathbb{Z}} \) and the sequence \( R_n \).

Let \( y_{r_0} \) be given and fixed from now on. Set

\[
\mathcal{I}_0 = \{ r \in \mathbb{Z} : y_r = y_{r_0} \};
\]

hence for any \( \exists k_0 \in \mathbb{N} \) s.t. \( \forall r = (k,l) \) with \( k < k_0, y_r \neq y_{r_0} \) and \( \forall k > k_0 \) \( \exists r = (k,l) \in \mathcal{I}_0 \). We may assume that \( n_0 = (k_0, l_0) \). We partition \( \mathcal{I} \setminus \mathcal{I}_0 \) as follows:

\[
\mathcal{I}_1 = \{ r \in \mathcal{I} : y_r \notin E_{r_{r_0}} \}
\]

and for any \( r = (k,l) \in \mathcal{I}_0 \) (i.e. \( l \) is odd and \( k > k_0 \)) set

\[
\mathcal{I}_r = \{ r' \in \mathbb{Z} : y_{r'} \in E_{k,l+1_0} \cap E_{r_{r_0}} \}.
\]

Observe that then for each such \( r = (k,l) \in \mathcal{I}_0 \), \( \tilde{k} \geq k > k_0 \) one has

\[
\{|r' = (k',l') \in I_r : k' = \tilde{k}\} = 2^{k-k}.
\]

Define

\[
\varphi(z + iy_{r_0}) = \mathbb{R}(-F'(z)(z + iy_{r_0})).
\]

To simplify notation we will set \( r = r_{r_0}, \theta = \theta_{r_0} \) i.e. \( z + iy_{r_0} = re^{i\theta} \). In our case corresponds to

\[
(4.16) \quad \frac{\partial}{\partial y_r} |f|^{\frac{2}{3}} = \frac{2 \varphi(re^{i\theta})}{Q} \frac{1}{r} |f|^{\frac{2}{3}}.
\]

Recall from lemma 4.1 that \( \frac{1}{Q} F'(z)(z + iy_{r_0}) = \sum_{r \in I} a_k(z + iy_{r_0})^{\alpha-1} \) converging absolutely and \( \mathbb{R}((z + iy_{r_0})^{\alpha-1} + r \cos((a + 1)\theta_{r} - \theta)) \). For \( r \in \mathcal{I}_0 \) we have \( z + iy_{r_0} = re^{i\theta_0} \) and so

\[
\mathbb{R} \left( \sum_{r \in \mathcal{I}_0} a_k(z + iy_{r})^{\alpha-1} \right) = r^{-\alpha} \cos(a\theta) \sum_{r \in \mathcal{I}_0} \sum_{k=0} c_k \geq c_{r} r^{-\alpha}
\]

with \( c_0 = \cos(a\frac{\pi}{2}) \sum_{k=0} a_k > 0 \).

For \( r \in \mathcal{I}_1, 0 < r < R, R > 0 \) sufficient small we have \( r \geq \delta |E_{k_0}| \) because \( r \geq |E_{k_0} - 2|E_{k_0} - r \). Therefore we found

\[
\mathbb{R} \left( \sum_{r \in I_1} a_k(z + iy_{r})^{\alpha-1} \right) \geq -(\delta |E_{k_0}|)^{-\alpha-1} r \sum_{r \in I_1} a_k \geq -c_1 r
\]

In the rest of the argument we restrict us to \( R_n \leq r \leq R_n \) and \( n > N \) for some large \( N \in \mathbb{N} \). If \( r = (k,l) \in \mathcal{I}_0 \) with \( k_0 < k \leq n \) and \( r' \in \mathcal{I}_r \) then \( r \geq |y_{r'} - y_{r_0}| + r \geq \)
\(|E_{k-1}| - |E_k| - R_n \geq \delta |E_k|\), so that

\[
\sum_{\tau = (k,l) \in \mathcal{T}_0} \sum_{k_0 < k \leq n} a_{k_0} r_{\tau, k}^{-\alpha-1} r \cos((\alpha + 1)\theta_{\tau} - \theta) \geq - \sum_{k_0 < k \leq n} (\delta |E_k|)^{\alpha-1} r \sum_{k' = k}^{\infty} \frac{2^{-k}}{(k')^2}
\]

\[
\geq - \frac{r}{\delta^{\alpha+1}} \sum_{k_0 < k \leq n} \frac{M^k}{k - 1} \geq -r M^{n+1} - M^{k_0+1} \frac{(M - 1)}{\delta^{\alpha+1} k_0} \geq - \frac{\epsilon_2^\alpha}{\epsilon_2} M^n
\]

where \(M = (2^{\frac{n+1}{2} - 1}) > 1\). If \(\tau = (k, l) \in \mathcal{T}_0\) with \(n < k\) and \(\tau' \in \mathcal{T}_\tau\) then \(r_\tau \geq r - |y_{\tau'} - y_\tau| \geq R_n - |E_{k-1}| \geq R_n - |E_{n-1}| = \delta |E_{n-1}|\) hence

\[
\sum_{\tau = (k,l) \in \mathcal{T}_0} \sum_{\tau' \in \mathcal{T}_\tau} a_{k_0} r_{\tau, k}^{-\alpha-1} r \cos((\alpha + 1)\theta_{\tau'} - \theta) \geq - (\delta |E_{n-1}|)^{-\alpha-1} r \sum_{k = n+1}^{\infty} \sum_{k' = k}^{\infty} \frac{2^{-k}}{(k')^2}
\]

\[
\geq - (\delta |E_{n-1}|)^{-\alpha-1} r \sum_{k = n+1}^{\infty} \sum_{k' = k}^{\infty} \frac{2^{-k}}{(k')^2} \geq -r \frac{1}{\delta^{\alpha+1} n} M^n = - \frac{\epsilon_2^\alpha}{n} M^n.
\]

Summarizing for \(R_n \leq r \leq R_n\) and \(n \geq N = N(k_0)\), we have

(4.17) \[
\frac{1}{\alpha} \varphi(re^{-i\theta}) \geq r^{-\alpha} \left( c_0 - c_1 r^{1+\alpha} + \frac{\epsilon_2}{k_0} + \frac{\epsilon_2}{n} M^{n+\alpha} \right) \geq \frac{c_0}{2} r^{-\alpha}
\]

because \(M^{n+\alpha} \leq M^n R_1^{n+\alpha} = \left(\frac{1}{2} (2^{1+\frac{1}{2}} - 1) 2^{-\alpha} \right)^{1+\alpha} 2^{-n} \to 0\) (as \(n \to \infty\)).

(4.10) and (4.17) gives for \(R_n \leq r \leq R_n\)

\[
\frac{\partial}{\partial r} \ln(|f|^{\frac{2}{Q}}) = \frac{2\alpha}{Q} \varphi(re^{i\theta}) \geq \frac{c_0}{Q} r^{-\alpha}
\]

or integrated

(4.18) \[
\ln \left( \frac{|f|^{\frac{2}{Q}} R_n e^{i\theta}}{|f|^{\frac{2}{Q}} R_n e^{i\theta}} \right) \geq c R_n^{-\alpha}
\]

with \(c = \frac{\varphi}{Q} \left( \left( \frac{R_n}{R_n} \right)^{\alpha} - 1 \right) > 0\) (independent of \(n\)).

Now we combine the just established with (4.10)

\[
\int_{B_{R_n}(-iy_{\tau_0} \cap \mathbb{C}^+)} |Du|^2 \geq \int_{(B_{R_n} \cap \{ |z + iy_{\tau_0} | \leq R_n \}) \cap \mathbb{C}^+} |Du|^2
\]

\[
= \frac{2}{Q} \int_{B_{R_n} \cap \{ |z + iy_{\tau_0} | \leq R_n \}) \cap \mathbb{C}^+} \left| -F^r \right|^2 d\theta \geq \frac{2}{Q} \frac{\varphi(re^{i\theta})}{r} \frac{\partial}{\partial r} |f|^{\frac{2}{Q}} d\theta.
\]

(4.10) and (4.17) show that \(\frac{\partial}{\partial r} \ln |f|^{\frac{2}{Q}} > 0\) for \(R_n \leq r \leq R_n\). We apply as before the 1-dimensional mean value theorem to deduce that to every \(|\theta| \leq \frac{\pi}{2}\) there is \(0 < \tau_0 \leq R\) with

\[
\int_{B_{R_n} \cap \{ |z + iy_{\tau_0} | \leq R_n \}) \cap \mathbb{C}^+} \frac{2 \varphi(r e^{i\theta})}{r} \frac{\partial}{\partial r} |f|^{\frac{2}{Q}} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \varphi(|r e^{i\theta} \cap B_{R_n}) \frac{\partial}{\partial r} |f|^{\frac{2}{Q}} d\theta
\]

\[
= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \varphi(r e^{i\theta}) \left( |f|^{\frac{2}{Q}} (-iy_{\tau_0} + R_n e^{i\theta}) - |f|^{\frac{2}{Q}} (-iy_{\tau_0} + R_n e^{i\theta}) \right) d\theta
\]

\[
\geq \frac{\alpha c_0}{2} R_n^{-\alpha} \left( 1 - e^{-c R_n^{-\alpha}} \right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |f|^{\frac{2}{Q}} (-iy_{\tau_0} + R_n e^{i\theta}) d\theta
\]
Let $0 < s \leq 1$ there exists $u \in C^\infty(\mathbb{R}^2_+), u \neq 0$ with
\[ \Delta u = 0 \text{ on } \mathbb{R}^2_+ \text{ (i.e. harmonic)} \]
and a set $E_s \subset \partial \mathbb{R}^2_+$ with $\mathcal{H}^s(E_s) = 1 \text{ (0 < s < 1)}, \dim_H(E_s) = 1 \text{ (s = 1)}$ such that $u$ vanishes to infinite order for all $z \in -iE_s$.

Observe that $\Delta$ satisfies the conditions of theorem 4.9 and therefore has the strong unique continuation property in the interior of $\mathbb{R}^2_+$.

**Proof of example 4.6.** Let $0 < s \leq 1$ be given and $f$ the related holomorphic function of lemma 4.3. Since $f$ is $C^\infty$ on $\overline{C_+}$ and $C_+$ convex we have by 1-dimensional analysis (4.21)
\[ f(z) = \sum_{l=1}^{k-1} \frac{1}{l!} f^{(l)}(z_0)(z-z_0)^l + \frac{1}{(k-1)!} \int_0^1 (1-s)^{k-1} f^{(k)}(z_0+s(z-z_0))(z-z_0)^k ds. \]
The function
\[ u(z) = \Re(f(z)) \]
is harmonic and non-constant on $\mathbb{R}^2_+, C^\infty$ on $\overline{\mathbb{R}^2}$ and has the desired property since for $z_0 \in -iE_s, f^{(l)}(z_0) = 0$ for all $l \neq 1$ and therefore by (4.21)
\[ |u(z)| \leq \frac{1}{k!} \sup_{w \in \overline{C_+} \cap B_1(z_0)} |f^{(k)}(w)||z-z_0|^k \text{ for all } z \in \overline{C_+} \cap B_1(z_0). \]
This implies that $u$ satisfies (4.20).

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