OPERATIONAL QUANTUM LOGIC: AN OVERVIEW

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The term quantum logic has different connotations for different people, having been considered as everything from a metaphysical attack on classical reasoning to an exercise in abstract algebra. Our aim here is to give a uniform presentation of what we call operational quantum logic, highlighting both its concrete physical origins and its purely mathematical structure. To orient readers new to this subject, we shall recount some of the historical development of quantum logic, attempting to show how the physical and mathematical sides of the subject have influenced and enriched one another.

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1This paper is a slightly modified version of the introductory chapter to the volume B. Coecke, D.J. Moore and A. Wilce (Eds.), Current Research in Operational Quantum Logic: Algebras, Categories, Languages, Fundamental Theories of Physics series, Kluwer Academic Publishers, 2000. Consequently, there is a particular focus in this paper on the (survey) articles of that volume, which are dedicated to D.J. Foulis in honor of his seminal contributions to our field.

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1. Introduction

The subject of operational quantum logic — lying somewhere at the crossroads of mathematics, physics and philosophy — has a long and complicated history, and has generated a large, scattered and unruly literature. It is not an easy thing to explain, in a few words, what it is supposed to be about! Our best attempt is to say that operational quantum logic involves

(a) the fact that the structure of the 2-valued observables in orthodox quantum mechanics may usefully be regarded as a non-classical propositional logic,

(b) the attempt to give independent motivation for this structure, as part of a general programme to understand quantum mechanics, and

(c) the branch of pure mathematics that has grown out of (a) and (b), and now concerns itself with a variety of “orthomodular” structures generalizing the logic of 2-valued quantum observables.

Whatever else it may be, quantum logic is a living, growing part of contemporary mathematics and theoretical physics — one that has continued to hold the interest of a body of mathematicians, physicists and philosophers of science. This sustained interest reflects in part the fact that the basic ideas and language of quantum logic inform most discussions of the vexing foundational problems of quantum mechanics (indeed, to a degree that is often not recognized even by the discussants). It reflects also the fact that quantum logic has spawned an autonomous and fascinating branch of pure mathematics concerned with a variety of structures — orthomodular lattices and posets, orthoalgebras, partial Boolean algebras, etc. — generalizing $\mathcal{P}(\mathbf{H})$. Finally, the recent advent of quantum computation and quantum information theory offers a field of practical applications for quantum logic which has yet to be fully explored.

Historically, quantum logic derives from von Neumann’s (more than casual) observation that the 2-valued observables, represented in his formulation of quantum mechanics by projection operators, constitute a sort of “logic” of experimental propositions. This idea was further pursued by Birkhoff and von Neumann. After two decades of neglect, interest in quantum logic was revived, due in large part to Mackey’s analysis of the probabilistic calculus of standard quantum theory coupled with his theory of induced representations. The further development of the subject has occurred at several levels and in a number of directions. Mackey’s work was extended significantly by Piron, whose representation theorem and axiomatic framework provided much impetus for further development. At the same time, dissatisfaction with Mackey’s axiomatic framework led to a search for more primitive, and more concretely operational, foundations. Notable here is the work of Foulis and Randall, and also that of Ludwig and his colleagues at Marburg. Work in foundational physics has also stimulated, and been stimulated by, purely mathematical research, notably in the development of an abstract theory of orthomodular lattices and, in recent years, more general structures such as orthoalgebras and effect algebras. More recently still, the subject has seen the application
of powerful category-theoretic techniques. Given the somewhat overwhelming variety of these developments, in this introductory essay, we are going to attempt an outline of quantum logic that will help readers who are not already experts in the subject to understand the various papers that are produced — and also, to see them as belonging to a common subject. We start by discussing the seminal work of Birkhoff and von Neumann and its development by Mackey. We then turn to a brief exposition of Piron’s representation theorem and his axiomatic framework, referring the reader to [Coecke and Moore 2000] and [Valckenborgh 2000] for expositions in a more contemporary, categorical idiom. Next, we discuss the work of Foulis and Randall, in particular their introduction of the notion of an orthoalgebra, and their observation that tensor products of orthomodular posets generally exist only as orthoalgebras. The Foulis-Randall formalism is discussed in detail in [Wilce 2000]. We follow with a general exposition of the pure mathematical theory of orthomodular structures. For more details on orthomodular lattices we refer to [Bruns and Harding 2000], for observables on orthomodular posets to [Ptak 2000] and for group representations on (interval) effect algebras to [Foulis 2000]. Finally, we mention the notion of categorical enrichment and the theory of quantales, surveyed respectively in [Borceux and Stubbe 2000] and [Paseka and Rosický 2000], before introducing computational and linguistic aspects, treated respectively in [Resende 2000] and [Gudder 2000].

2. Von Neumann’s quantum mechanics

While precise mathematical treatments of quantum mechanics existed before Johann von Neumann’s monumental treatise [1932], it may reasonably be argued that this work fixed once and for all the theoretical framework of standard quantum theory, in which each quantum mechanical system is associated with a Hilbert space $H$, each unit vector $\psi \in H$ determines a state of the system, and each observable physical quantity associated with the system is represented by a self-adjoint operator $A$ on $H$. The spectral theorem tells us that such an operator is associated with a spectral measure

$$ P_A : B(\mathbb{R}) \rightarrow \mathcal{P}(H) $$

assigning to each real Borel set $B$ a projection operator $P_A(B)$ on $H$. For any unit vector $\psi \in H$, the quantity

$$ \mu_{A,\psi}(B) := \langle P_A(B)\psi, \psi \rangle $$

then defines a probability measure on the line, which von Neumann regards as giving the probability that the observable (represented by) $A$ has a value in the set $B$ when the system’s state is (represented by) $\psi$. If the identity function has finite variance in $\mu_{A,\psi}$, then $\psi$ is in the domain of $A$, and the expectation value of $A$ relative to $\psi$ is given by

$$ \text{Exp}(A, \psi) = \int_\mathbb{R} s d\mu_{A,\psi}(s). $$
One easily checks that this works out to $\text{Exp}(A, \psi) = \langle A\psi, \psi \rangle$.

2.1. THE LOGIC OF PROJECTIONS

If the *Mathematische Grundlagen der Quantenmechanik* signalled the passage into maturity of quantum mechanics, it also signalled the birth of quantum logic. Evidently, it is the projection-valued measure $P_A$, more than the operator $A$, that most directly carries the statistical interpretation of quantum mechanics outlined above. Now, as von Neumann notes, each projection $P \in \mathcal{P}(H)$ itself defines an observable — one with values 0 and 1. If $P = P_A(B)$ is the spectral projection associated with an observable $A$ and a Borel set $B$, we may construe this observable as “testing” whether or not $A$ takes a value in $B$. Von Neumann regards $P$ as representing a physical property of the system (or rather, of the system’s states). He remarks that

“the relation between the properties of a physical system on the one hand, and the projections on the other, makes possible a sort of logical calculus with these. However, in contrast to the concepts of ordinary logic, this system is extended by the concept of ‘simultaneous decidability’ which is characteristic for quantum mechanics.” [von Neumann 1932, p.253]

Indeed, if $P$ and $Q$ are commuting projections, then their meet $P \land Q$ and join $P \lor Q$ in the lattice $\mathcal{P}(H)$ may be interpreted classically as representing the conjunction and disjunction of the properties encoded by $P$ and $Q$; further, the projection $P' = 1 - P$ serves as a sort of negation for $P$. If $P$ and $Q$ do not commute, however, then they are not “simultaneously decidable”, and the meaning of $P \land Q$ and $P \lor Q$ is less clear. Nevertheless, $\mathcal{P}(H)$ retains many features of a Boolean algebra, and may be regarded as an algebraic model for a non-classical propositional logic. In particular, $\mathcal{P}(H)$ is orthocomplemented and so enjoys analogues of the de Morgan laws; more sharply, the sub-ortholattice generated by any commuting family of projections is a Boolean algebra.

2.2. THE LOGIC OF QUANTUM MECHANICS

It is noteworthy that von Neumann speaks of the simultaneous “decidability” (i.e., testability) of properties, but does not distinguish between decidable and undecidable properties *per se*. Classically, of course, any subset of the state-space counts as a *categorical* property of the system, and nothing in principle prevents us from taking the same view in quantum mechanics. However, only those subsets of the state space corresponding to closed linear subspaces of the Hilbert space are associated with observables, and so “decidable” by measurement. If one adopts a rather severe positivism, according to which no undecidable proposition is meaningful at all, one is led to the seemingly strange doctrine that, for a quantum...
mechanical system, the set of meaningful properties forms, not a Boolean algebra, but rather the lattice $\mathcal{P}(\mathcal{H})$ of projections of a Hilbert space.

This idea was further developed by von Neumann in a joint paper with Garrett Birkhoff entitled *The Logic of Quantum Mechanics* [Birkhoff and von Neumann 1936]. Birkhoff and von Neumann observe that $\mathcal{P}(\mathcal{H})$ retains a number of the familiar features of the algebra of classical propositional logic — in particular, it is orthocomplemented and hence satisfies de Morgan’s laws. It is not, however, Boolean — that is, the distributive law fails. Birkhoff and von Neumann go so far as to suggest that

“whereas logicians have usually assumed that properties L71-L73 of negation were the ones least able to withstand a critical analysis, the study of mechanics point to the *distributive identities* L6 as the weakest link in the algebra of logic.” [Birkhoff and von Neumann 1936, p.839]

As we shall see in section 7, this remark is rather deeper than one may imagine, being interpretable in terms of the fundamental difference between Heyting algebras and orthomodular lattices considered as generalizations of Boolean algebra.

This suggestion that the projection lattice may be viewed as a propositional logic has been understood in a number of rather different ways. Some have seen it as calling into question the *correctness* of classical logic. Others have seen it as entailing a less drastic modification of classical probability theory. As we have seen, von Neumann himself [1932 §3.5] is rather cautious, remarking that the equivalence between subspaces and projections induces a sort of logical calculus. Similarly, Birkhoff and von Neumann [1936 §0] conclude that by heuristic arguments one can reasonably expect to find a calculus of propositions for quantum mechanical systems which is *formally* indistinguishable from the calculus of subspaces and resembles the usual logical calculus.

More radical is the view of Finkelstein [1968, 1972] that logic is in a certain sense empirical, a view championed by such philosophical luminaries as Putnam [1968, 1976]. Finkelstein highlighted the abstractions we make in passing from mechanics to geometry to logic, and suggested that the dynamical processes of fracture and flow already observed at the first two levels should also arise at the third. Putnam, on the other hand, argued that the metaphysical pathologies of superposition and complementarity are nothing more than artifacts of logical contradictions generated by an indiscriminate use of the distributive law.

This view of the matter, which remains popular in some quarters, depends on a reading of the projection $\mathcal{P}$ as encoding a *physical property* of the quantal system, and on the assumption that only physical properties are ultimately to count as

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4For instance, Bamberg and Sternberg [1990, pp.833-835] write: “In fact, [quantum mechanics] represents the most profound revolution in the history of science, because it modifies the elementary rules of logic. . . . [T]he distributive law does not hold in quantum logic. As we mentioned above, the validity of quantum mechanics has been experimentally demonstrated over and over again during the past sixty years. So experiment has shown that one must abandon one of the most cherished principles of logic when dealing with quantum observables.”
meaningful (or at any rate, as fundamental). There is, however, a different way of construing $P$, namely, that it encodes a statement about the possible result of some “measurement”. Thus, if $A$ is the self-adjoint operator corresponding to the observable $\mathcal{A}$ and $P = P_A(B)$ is the spectral projection of $A$ corresponding to the Borel set $B$, we might construe $P$ as encoding the proposition that a measurement of $A$ would yield a value in $E$ if made. This construal, usually dubbed operational, guided Mackey in his reconstruction of von Neumann’s quantum mechanics, to which we now turn.

3. Mackey’s programme

In an influential paper [1957], subsequently expanded into the monograph [1963], George Mackey argued (a) that one could reconstruct most if not all of the apparatus of von Neumann’s quantum mechanics from the premise that the experimental propositions form an ortholattice isomorphic to $\mathbb{P}(\mathcal{H})$, and (b) that this premise itself could be independently motivated by very general considerations about how probabilistic models of physical systems ought to look.

3.1. QUANTUM MECHANICS AS A PROBABILITY CALCULUS

Mackey construed quantum mechanics as simply being a non-classical probability calculus, in which the Boolean algebra of events of classical probability theory is replaced by the lattice $\mathbb{P}(\mathcal{H})$. More exactly, Mackey stressed that both the states and the observables of a quantum mechanical system can be defined purely in terms of $\mathbb{P}(\mathcal{H})$. First, any statistical state $W$ determines a probability measure on $\mathbb{P}(\mathcal{H})$, namely the mapping $\omega_W : \mathbb{P}(\mathcal{H}) \to [0, 1];\ P \mapsto \text{tr}(PW)$. A deep theorem by Gleason [Gleason 1957; Dvurečenskij 1993] shows that, conversely, every $\sigma$-additive probability measure on $\mathbb{P}(\mathcal{H})$ has this form. Second, an observable with values in the measurable space $(\mathcal{S}, \mathcal{F})$ may be represented by a projection-valued measure $M : \mathcal{F} \to \mathbb{P}(\mathcal{H})$ where, for each measurable set $B \in \mathcal{F}$, the projection $M(B)$ is taken to encode the “experimental proposition” that a measurement of the observable yields a value in the set $B$. Evidently, we may pull probability measures on $\mathbb{P}(\mathcal{H})$ back along $M$ to obtain a classical probability measure on $\mathcal{F}$. We interpret $M^*(\omega) = \omega \circ M$ as giving the statistical distribution of values of $M$ (in $\mathcal{S}$) when the system is in the state represented by $\omega$. In other words,

\[ \omega_W(M(B)) = \text{tr}(M(B)W) \]

represents the probability that the observable represented by $M$ will yield a value in the set $B$, when measured, when the state of the system is represented by $W$. 
This connects with von Neumann’s operator-theoretic representation of observables in a natural manner, as follows: if $f : S \to \mathbb{R}$ is any bounded classical real-valued random variable defined on $S$, we may define the self-adjoint operator

$$A_f := \int_S f(s) \, dM(s)$$

in the usual way. Then, for any probability measure $\mu$ on $\mathcal{P}(\mathcal{H})$, we have

$$E_{M^\ast(\mu)}(f) = \int_S f(s) \, dM^\ast(\mu)(s) = \text{tr}(A_f W)$$

where $W$ is the density operator corresponding to $\mu$. This view of quantum mechanics is strikingly powerful. Gleason’s theorem, together with the spectral theorem, the classical results of Stone, Wigner, Weyl and von Neumann, and Mackey’s own work on induced unitary representations, allow one essentially to derive the entire apparatus of non-relativistic quantum mechanics (including its unitary dynamics, the CCRs, etc.), from the premise that the logic of experimental propositions is represented by the projection lattice $\mathcal{P}(\mathcal{H})$. For an outline of this reconstruction, see [Mackey 1963] or [Beltrametti and Cassinelli 1981]; for a detailed account, see [Varadarajan 1968].

3.2. MACKEY’S AXIOMS

Its success notwithstanding, Mackey’s account of quantum mechanics as a probability calculus still rests on one undeniably ad hoc element: the Hilbert space $\mathcal{H}$ itself. Indeed, once one entertains the idea that the testable propositions associated with a physical system need not form a Boolean algebra, the door is opened to a huge range of other possibilities. It then becomes a matter of urgency to understand why nature (or we) should choose to model physical systems in terms of projection lattices of Hilbert spaces, rather than anything more general. Mackey outlined an ambitious programme to do just this, by deducing the Hilbert space model from a set of more primitive and, ideally, more transparently plausible axioms for a calculus of events.

The framework Mackey adopts is an abstract structure $(O, S, p)$, where $O$ is understood to represent the set of real-valued “observables” and $S$ the set of “states” of a physical system. These are connected by a mapping

$$p : O \times S \to \Delta : (A, s) \mapsto p_A(\cdot | s),$$

where $\Delta$ is the set of Borel probability measures on the line. The intended interpretation is that $p_A(\cdot | s)$ gives the statistical distribution of values of a measurement of the observable $A \in O$, when the system is in the state $s \in S$. We may take the pair $(A, B)$, where $A \in O$ and $B$ is a real Borel set, to represent the “experimental proposition” that a measurement of $A$ yields (would yield, has yielded) a value $\leq f$.

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If $f$ is non-negative, then $A_f$ is given by the supremum of the operators $A_g = \sum_i g_i M(B_i)$ where $g = \sum_i g_i \chi_{B_i}$ is a simple random variable with $0 \leq g \leq f$. 

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in $B$. Mackey considers two such propositions equivalent iff they have the same probability in every state — in other words, $(A_1, B_1)$ and $(A_2, B_2)$ are equivalent iff the associated mappings $P_{A_i,B_i} := p_{A_i}(B_i|·)$ are the same. The set $L$ of such mappings $P_{A,B}$, which he calls questions, is Mackey’s quantum logic.

Now, ordered pointwise on $S$, the set $L$ is an orthocomplemented poset with unit 1 given by $P_{A,R}$ for any observable $A$, whose orthocomplementation is given by $P'_{A,B} = 1 - P_{A,B} = P_{A,R\setminus B}$. Let us say that the questions $P, Q \in L$ are compatible iff $P = P_{A,B}$ and $Q = P_{A,C}$ for some common observable $A$ and some pair of Borel sets $B$ and $C$. Then we may consider $P$ and $Q$ to be “simultaneously measurable”.

Further, let us say that the questions $P, Q \in L$ are orthogonal (or “disjoint”, in Mackey’s language) iff $P \leq Q'$. In this case, we write $P \perp Q$. Mackey at this point imposes his Axiom V: If $P_i$ is any countable family of pairwise orthogonal elements of $P$, then there exists an element $P \in L$ with $P_1 + P_2 + \cdots = P$.

This axiom guarantees that $L$ is a $\sigma$-orthomodular poset — that is, $L$ satisfies the two conditions

(a) Any countable family of pairwise orthogonal elements $P_i \in L$ have a join (least upper bound) $\bigvee_i P_i$ in $L$, and

(b) If $P \leq Q$, then $(Q \land P') \lor P = Q$.

On any such poset $L$, one can define probability measures on $L$ to be mappings $\mu : L \to [0, 1]$ such that $\mu(1) = 1$ and, for any countable pairwise orthogonal family of elements $P_i \in L$ we have $\mu(\bigvee_i P_i) = \sum_i \mu(P_i)$. We can also define, given any two $\sigma$-OMPs $L$ and $M$, an $M$-valued measure on $L$ to be a mapping $\alpha : L \to M$ such that $\alpha(1_L) = 1_M$ and, for any countable pairwise orthogonal family $P_i \in L$ we have $\alpha(\bigvee_i P_i) = \bigvee_i \alpha(P_i)$. For a general discussion of such maps in terms of observables see [Pták 2000]. Returning now to the OMP $L$ of questions, Mackey observes that

(a) Each state $s \in S$ defines a probability measure $\hat{s} : L \to [0, 1]$ by evaluation: $\hat{s}(P_{A,B}) = P_{A,B}(s) = p_A(B|s)$.

(b) Each observable $A \in O$ defines an $L$-valued measure $P_A : B(\mathbb{R}) \to L$ via $P_A(B) = P_{A,B}$ on the real Borel sets (which, constituting a $\sigma$-Boolean algebra, certainly constitute a $\sigma$-OMP).

Conversely, suppose $L$ is any $\sigma$-OMP $L$, and that $S$ is any order-determining set of probability measures on $L$ — i.e., $\mu(p) \leq \mu(q)$ for all $\mu \in S$ implies that $p \leq q$. Let $O$ be the set of all $L$-valued Borel measures on the line, and define $p : O \times S \to \Delta(\mathbb{R})$ by $p_\alpha(B|\mu) = \mu(\alpha(B))$. Then the structure $(O, S, p)$ satisfies Mackey’s axioms, and, furthermore, the OMP of questions constructed from it is canonically isomorphic to $L$. 
As remarked by [Foulis 1962; Gudder 1965], Mackey’s axioms then define the theory of the following class of structures: pairs \((L, \Delta)\) where \(L\) is a \(\sigma\)-OMP and \(\Delta\) is an order-determining family of probability measures on \(L\). Such pairs (routinely referred to as quantum logics in the mathematical literature in the 1960s and 1970s) have been studied intensively by many authors. For detailed discussions of orthomodular posets in the quantum logical context see [Beltrametti and Cassinelli 1981; Gudder 1985; Pták 2000; Pták and Pulmannová 1991]. Of course such quantum logics are still a far cry from the standard quantum logic \(\mathcal{P}(H)\). Among other things, the orthomodular poset \(\mathcal{P}(H)\) is a complete lattice: arbitrary joins exist, not just countable orthogonal joins. Still, one might hope that a deeper analysis — perhaps involving additional axioms — might lead to a meaningful characterization, and, ideally, a motivation for the standard quantum logics. This was Mackey’s expressed goal:

“Ideally, one would like to have a list of physically plausible assumptions from which one could deduce [the Hilbert space model]. Short of this, one would like a list from which one could deduce a set of possibilities ..., all but one of which could be shown to be inconsistent with suitably planned experiments. At the moment, such lists are not available...” [Mackey 1963, p.72]

This topic lies at the heart of Piron’s original axiomatization, to be discussed in the next section. On the other hand, as we shall discuss in sections 5 and 6, the autonomous study of such structures leads naturally to further generalizations, notably, to orthoalgebras and effect algebras.

Before turning to a rapid survey of some of the major developments which have occurred since Mackey’s foundational work, let us make a few comments. First, and foremost, the major feature which separates Mackey’s formalism from current tendencies in operational quantum logic is the former’s reliance on probability as a primitive concept. While important advances in this context have been made, for example in [Pulmannová 1986a,b; Gudder and Pulmannová 1987; Pulmannová and Gudder 1987], most contemporary work relegates probability to a derived notion. This is not to say that statistical states are unimportant in operational quantum logic. However, they have passed from the status of a rather vaguely construed primitive concept to that of a well defined structural tool. Here mention may be made of the characterization of the state spaces of standard quantum logics, culminating in Navara’s proof of the independence of the automorphism group, center, and state space of a quantum logic [Navara 1992].

A notable exception to this trend is the theory of decision effects introduced

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6For example, while in [Piron 1964 §7] generalized probability is discussed as a useful physical heuristic, in [Jauch and Piron 1969 §5] states are defined as maximal sets of actual properties of the system. Similarly, while the formalism introduced in [Randall and Foulis 1970; Foulis and Randall 1972; Randall and Foulis 1973] is explicitly concerned with operational statistics, in [Foulis, Piron and Randall 1983; Randall and Foulis 1983; Foulis, Greechie and Rüttimann 1992, 1993] emphasis is placed on the conception of states in terms of supports in the outcome space associated to the system.
by Günther Ludwig during the revision of his classic text [1954, 1955]. This work is based on the classification of macroscopic notions into preparative and effective parts which both participate in measurement interactions mediated by action carriers. We shall not enter into the details of Ludwig’s axiomatic scheme, successively refined in [Ludwig 1964, 1967, 1968; Dähn 1968; Mielnik 1968, 1969; Stolz 1969, 1971; Dähn 1972; Ludwig 1972] and codified in the monumental treatise [Ludwig 1985, 1987], but will content ourselves with some general remarks. The primitive notion in this theory is that of a probability relation defined on the Cartesian product of the set of ensembles and the set of effects, these two sets being taken as embedded in a suitable pair of Banach spaces. In some sense, then, the work of Ludwig and his collaborators runs parallel to the field of operational quantum logic as we have introduced it, focusing more on the functional analytic structure of the problem than its ordered algebraic aspects. As such it perhaps bears more formal relation to the algebraic quantum theory of Segal [1947] and Haag and Kastler [1964] than the operational theories of Piron and Foulis–Randall to be discussed next. Nevertheless, an important physical feature of Ludwig’s work is that it attempts to deal with the notion of non-ideal measurements by exploiting subprojective operators. Note that such operators appear naturally in discussions of generalized localisability [Jauch and Piron 1967; Amrein 1969]. For general surveys of the different approaches to operational quantum mechanics see [Gudder 1977, 1979, 1981; Ludwig and Neumann 1981], for a detailed analysis of the model relationship between the approaches of Piron and Ludwig see [Cattaneo and Laudisa 1994; Cattaneo and Nisticò 1993], and for an overview of the application of POVM-measures to questions in the foundations of quantum mechanics see [Busch, Lahti and Mittelstaedt 1991; Schroeck 1996].

4. The work of Piron

Significant progress at both ends of the problem of completing and extending Mackey’s programme was made by Constantin Piron [1964] and further developed in what has become known as the Geneva School approach to quantum physics. Piron characterized abstractly those complete orthomodular lattices representable as the lattices of closed subspaces of generalized Hilbert spaces. He also supplied a deep analysis of the basic physical ideas of quantum mechanics that helped to motivate the assumptions needed in his representation theorem as reasonable, general axioms. In this section we describe a formalized version of these axioms in the spirit of [Piron 1976], before making some remarks on more recent developments.
4.1. THE REPRESENTATION THEOREM

The projection lattice $\mathcal{P}(H)$ has a much more regular structure than the general OMP provided by Mackey’s axioms. In particular, $\mathcal{P}(H)$

(a) is a complete lattice — that is, the meet and join of any subset of $L$ exist,

(b) is atomistic — that is, every element of $\mathcal{P}(H)$ is the join of the atoms (here, the one-dimensional projections) beneath it,

(c) satisfies the atomic covering law: if $P \in \mathcal{P}(H)$ is an atom and $Q \in \mathcal{P}(H)$ is arbitrary, then $P \lor Q$ covers $Q$, i.e., is an atom in the lattice $\{M \in L | Q \leq M \}$,

(d) is irreducible — that is, it cannot be factored as a non-trivial direct product. Equivalently, no element of $\mathcal{P}(H)$, other than 0 and 1, commutes with all other elements.

In his thesis [Piron 1964], Piron proved a partial converse, namely that all such lattices (of sufficient length) $L$ may be realised as the set of biorthogonal subspaces of a generalized Hilbert space. Explicitly, by considering the (essentially) unique meet and atom preserving embedding of $L$ in a projective geometry, and exploiting the standard vector space realization of projective geometries of dimension at least three, he showed that the image of the original lattice could be characterised by a definite hermitian form.

Now, for an arbitrary inner product space $V$, the complete atomistic ortholattice $L(V)$ of biorthogonal subspaces need not be orthomodular. When it is, $V$ is termed a generalized Hilbert space. This terminology is motivated by another striking result, namely that if $V$ is an inner product space over one of the standard division rings (i.e., $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$), then $L(V)$ is orthomodular iff $V$ is complete. This was first proved by Piron, using a hypothesis on measure extensions which turned out to be independent of ZF set theory; under prompting by Stone, a geometric

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7Of course, not every quantum mechanical system is irreducible, but in general decomposes into a family of purely quantum systems indexed by superselection rules. For example, [Piron 1964] shows that each orthomodular lattice satisfying axioms (a)–(c) is the direct union of a family of irreducible lattices, its corresponding projective geometry being the direct union of the corresponding geometries. Abstractly, systems with discrete superselection rules may be treated by taking projection-valued measures with values in an appropriate von Neumann algebra $\mathfrak{N}$. If the induced orthomodular lattice $L(\mathfrak{N})$ does not contain a summand of type $I_2$ then Gleason’s theorem continues to apply: every $\sigma$-additive probability measure on $L(\mathfrak{N})$ extends uniquely to a normal state on $\mathfrak{N}$ [Christensen 1982; Yeadon 1983]. For further discussion see, for example, [Bunce and Hamhalter 1994; Bunce and Wright 1994; Hamhalter 1993, 1995].

8This development has become much more physically transparent and mathematically elegant since the seminal work of Faure and Frölicher [1993, 1994, 1995], where the construction of linear representations for projective geometries and their morphisms is carried through in a categorically natural manner. For example, an orthogonality relation determines a morphism from the projective geometry to its dual and so a quasilinear map from the underlying vector space to its dual. In this way the inner product of quantum mechanics finds a rigorous and neat foundation.
proof was later obtained by Amemiya and Araki [1965]. Finally, let us remark that the Geneva School formalism that was inspired by this theorem has been extensively applied to several problems of a more or less concrete nature, for example, symmetries [Emch and Piron 1962, 1963], superselection rules [Piron 1965, 1969], observables [Piron 1971; Giovannini and Piron 1979; Giovannini 1981a,b,c], the a priori probability [Piron 1972], and irreversible processes [Gisin and Piron 1981; Gisin 1981, 1982a,b, 1983a,b].

4.2. PIRON’S AXIOMS

Mackey’s axioms produce only a \(\sigma\)-complete orthomodular poset \(L\) — a far cry from the complete, atomistic OML figuring in Piron’s Theorem. Piron was able to motivate the necessary extra structure in the context of an axiomatic framework similar to Mackey’s, but differing from it in taking as basic not the concept of probability, but a concept of physical property based on the certainty of obtaining an experimental outcome. Here Piron consciously exploits the work of Dirac [1930 §1.2], who gives an operational discussion of light polarisation in terms of the certainty or otherwise of passage through an appropriate crystal, and the conception of Einstein, Podolsky and Rosen [1935] that elements of reality are sufficient conditions that one be able to predict a physical quantity with certainty and without disturbing the system.

Piron begins with a primitive set \(Q\) of questions — understood to represent definite experimental projects having just two possible outcomes, which we designate as yes and no. For ease of presentation let us consider given a set \(P\) of preparation procedures. For \(P \in P\) and \(\alpha \in Q\) we write \(P \vdash \alpha\) to indicate that the preparation \(P\) is such that the answer to the question \(\alpha\) can be predicted with certainty to be yes. We can then associate, to every question \(\alpha\), the proposition

\[
[\alpha] = \{ P \in P \mid P \vdash \alpha \}
\]

Let \(L := \{ [\alpha] \mid \alpha \in Q \}\) be the set of all such propositions, considered as a poset under set inclusion. Note that \([\alpha] \subseteq [\beta]\) iff every preparation making \(\alpha\) certain also makes \(\beta\) certain. Piron proceeds to adduce several axioms the force of which is to make \(L\) a complete, atomistic OML satisfying the covering law.

\(L\) is a complete lattice The first, and probably the most novel, of these axioms

\[\text{Note that necessary and sufficient conditions for the underlying division ring to be standard have recently been found — one of the simplest statements in the infinite dimensional case being that the vector space admit an infinite orthonormal sequence [Solèr 1995; Holland 1995; Prestel 1995]; for an example of a nonstandard generalized Hilbert space see [Keller 1980], for a detailed discussion of the geometry of generalized Hilbert spaces see [Gross 1979, 1990], and for a survey of other completeness results see [Dvurečenskij 1992].}\]

\[\text{Note that this is not strictly necessary, but is just an expedient to avoid locutions such as ‘if the system is, or has been prepared, in such a way that . . .’. Similarly, the usual identification of propositions with equivalence classes of questions is made for ease of exposition and should not be taken too seriously as a definition.}\]
involves the notion of a product question. Given a non-empty set \( A \) of questions, their product is the question \( \alpha = \Pi A \) defined as follows: to pose \( \alpha \), one selects, in any way one will, a question \( \beta \in A \) and, posing this question, attributes to \( \alpha \) the answer obtained. Piron’s first axiom requires that \( Q \) be closed under the formation of arbitrary product questions. A moment’s reflection reveals that \( \Pi A = \bigcap_{\beta \in A} [\beta] \). Hence, \( L \) is closed under arbitrary intersections and thus a complete lattice. 

Orthocomplementation If \( \alpha \) is any question, we may define an inverse question \( \alpha^\sim \) by interchanging the roles of yes and no. Piron requires that \( Q \) be closed under the formation of inverses. The intended interpretation requires us to suppose that \( [\alpha] \cap [\alpha^\sim] = \emptyset \). In order to secure an orthocomplementation on \( L \), Piron introduces another axiom, namely, that for any question \( \alpha \), there exists some compatible complement \( \beta \in [\alpha] \) satisfying \( [\beta^\sim] \lor [\alpha] = 1 \).

Orthomodularity On the face of it, this does not rule out the possibility that there may exist several inequivalent compatible complements for a given \( \alpha \). However, this is remedied by a third axiom, Piron’s

Axiom P: If \( b < c \) and \( b' \) and \( c' \) are compatible complements for \( b \) and \( c \), respectively, then the sub-lattice of \( L \) generated by \( b, b', c, c' \) is distributive.

It follows that compatible complements are unique, thereby defining an orthocomplementation. Moreover, Axiom P dictates that \( L \) be orthomodular: if \( b < c \), then \( (c \land b') \lor b = c \) by the distributivity of \( \{b, c, b', c'\} \).

Atomicity and the Covering Law Piron enforces the atomicity of the lattice with an ad-hoc axiom (A1) requiring that \( L \) be atomic — i.e., every element dominates at least one atom. The covering law is also imposed directly (as axiom A2), but with some substantial motivation, as follows. Let us denote by \( \Sigma_L \) the set of atoms of \( L \). Now, in any orthocomplemented poset \( L \), the Sasaki mapping \( \phi : L \times L \rightarrow L \) is given by \( \phi(a, b) = b \land (b' \lor a) \). If \( b \) is fixed, we write \( \phi_b : L \rightarrow L \) for the mapping \( \phi_b(a) = \phi(a, b) = b \land (b' \lor a) \). Note that \( L \) is orthomodular iff \( \phi_b(a) = a \) for all \( a \leq b \), in which case \( \phi_b(a) \lor b' = a \lor b' \). Using these remarks, it is not hard to prove that an OML \( L \) satisfies the atomic covering law iff, for all \( a \in L \), we have \( a \in \Sigma_L \land b \not\perp a \Rightarrow \phi_b(a) \in \Sigma_L \). We then have an alternative formulation of the covering law, namely, that Sasaki projections map atoms either to atoms, or to 0.

Piron defines the state of the system to be the set of all propositions \( p = [\alpha] \). 

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11 The product operation was first introduced in [Jauch and Piron 1969]. Earlier, the meet had been introduced via either semantic conjunction [Piron 1964] or limit filters [Jauch 1968].

12 The fact that an axiom must be postulated guaranteeing the existence of an orthocomplementation is due to the fact that the inverses of equivalent questions need not themselves be equivalent. For example, \( 0 \cdot I = 0 \) however \( (0 \cdot I)^\sim = 0^\sim \cdot I^\sim = I \cdot 0 = 0 \) and \( 0^\sim = I \). For a discussion of some confusions on this point see [Foulis and Randall 1984].
that are certain (at a given time, in a given situation). We naturally require that
the state be closed under intersection and enlargement, i.e., that it be a complete
filter in the lattice \( L \). Such a filter is principal, and is generated by an atom.
Hence, states may be represented by atoms. Finally, call \( a, b \) in \( L \) compatible iff
\( \{a, b, a', b'\} \) is distributive, iff \( \phi_b(a) = a \land b \). Piron calls the question

(a) ideal iff every proposition compatible with \([\alpha]\) that is certain before a mea-
surement of \( \alpha \) is also certain again afterwards when the result of that mea-
surement is yes.

(b) first-kind iff the answer to \( \alpha \) immediately after securing the answer yes is
certain to be again yes.

In the presence of axiom A2 (i.e., the covering law), one can then prove that, for
\( \beta \) an ideal first-kind measurement of \( b \in L \), if \( a \) is the state before measurement,
then the state after securing yes upon measurement of \( \alpha \) is \( \phi_b(a) \).

Many people have found the physical reasoning that motivates Piron’s axioms
compelling. However, this framework turns out to have some sharp limitations. In
particular, a system consisting of two “separated” systems, in the sense of Aerts
[1981, 1982], each of which individually obeys Piron’s axioms, will as a whole con-
form to these axioms if and only if one of the systems is classical. To prove this key
result, Aerts exploited the notion of an orthogonality relation, where two states
are orthogonal if there exists a question which is certain for the first and impos-
sible for the second. The use of this relation has become central in more recent
axiomatizations of the Geneva School approach, such as [Piron 1990; Moore 1999].
For a detailed analysis see [Valckenborgh 2000]. Note that in these works attention
is focussed on complete atomistic ortholattices as models for the most direct
axiomatizations based on the physical duality between the state and property de-
scriptions of a physical system. Somewhat paradoxically, then, Piron’s approach
keeps one axiom rejected in the OMP approach — namely completeness — and
rejects another which the latter keeps — namely (some form of) orthomodularity.
As we shall see in the next section, this cleavage is symptomatic of the fact that
one should distinguish conceptually the property lattice of a system from its logic,
even when they turn out to be isomorphic.

5. The work of Foulis and Randall

Contemporary with these developments was the work of Dave Foulis and the late
Charlie Randall on empirical logic, a happy synthesis of ideas coming from their
respective doctoral dissertations, on abstract lattice theory [Foulis 1958] and con-
crete operational statistics [Randall 1966]. Not only does this formalism provide

\[13\] Conversely, if \( p \neq 0 \), then \( p = \lceil \alpha \rceil \) where \( \alpha \) is a question that is certain for at least one prepa-
ration. Hence, there exists at least one state (i.e., any state compatible with that preparation)
that contains \( p \). Hence, there must be sufficiently many states so that for every \( p \in L \), there is
a state/atom \( a \leq p \). But this easily implies that every atom is a state. Thus, states correspond
exactly to atoms of \( L \).
a powerful general heuristic, but as we shall see it has also laid the ground for several of the purely mathematical developments to be discussed in the following.

5.1. TEST SPACES

Both Mackey and Piron begin with a primitive structure in which experimental propositions of the form “observable $A$ takes value in set $B$” are unrelated for distinct observables. In effect, each observable $A$ is associated with a Boolean algebra $B_A$ of possible events (isomorphic to the Borel field in Mackey’s scheme, and to $\{0, 1\}$ in Piron’s), these Boolean algebras being initially disjoint from one another. Identifications are then made between the Boolean algebras corresponding to different observables. In Mackey’s scheme, the primitive propositions $(A_1, B_1)$ and $(A_2, B_2)$ are identified iff they are equiprobable in every state; in Piron’s, iff they are certain in exactly the same situations. Both approaches to the construction of quantum logics have been the object of some criticism. In particular, as a number of authors point out, both become problematic when one considers compound or iterated measurements.\footnote{A simple example is offered in [Cooke and Hilgevoord 1981]. The point is a familiar one — even in orthodox Hilbert space quantum mechanics, one must keep track of phase relations in discussing iterated experiments, and these are lost when one identifies experimental propositions according to either the Mackey or the Piron scheme.}

In a long series of papers (e.g., [Foulis and Randall 1972, 1974, 1978, 1981a; Randall and Foulis 1970, 1973, 1978, 1983a]), Foulis and Randall developed an extensive theory — which they termed empirical logic — in which such identifications are given a priori, with no prior reference to any concept of state or property. Their formalism is based on the primitive notion of an operation or test — that is, a definite set of mutually exclusive alternative possible outcomes. The Foulis-Randall theory focusses on test spaces, i.e., collections $A$ of overlapping tests. The identification of outcomes between distinct tests is understood to be given, i.e., Foulis and Randall lay down no doctrine as to how such identifications must be made. Letting $X = \bigcup A$ stand for the outcome space of $A$, a statistical state on $A$ is defined to be a mapping $\omega : X \to [0, 1]$ such that $\sum_{x \in E} \omega(x) = 1$ for every test $E \in A$, and a realistic state is represented [Foulis, Piron and Randall 1983] by a certain kind of subset of $X$ called a support, representing the totality of outcomes possible in that state. Note that, quite apart from its own merits, this notion can be used to give a perspicuous mathematical treatment of Piron’s axiomatics; see [Randall and Foulis 1983b] and [Wilce 1997].

A number of algebraic, analytic and order-theoretic objects can be attached to a test space $A$, each serving in a slightly different way as a sort of “logic”. Under simple normative conditions on the combinatorial structure of $A$, these turn out to coincide with more familiar structures. In particular, if $A$ is “algebraic”\footnote{A test space is algebraic if any two elements sharing a common complement share exactly the same complements, where $A$ and $B$ are complements if they are disjoint and $A \cup B$ is a test.} one can construct from the events of $A$ a rather well-behaved ordered partial
algebraic structure $\Pi(\mathfrak{A})$, called an orthoalgebra. These can be defined abstractly: an orthoalgebra is a pair $(L, \oplus)$ where $L$ is a set and $\oplus$ is a commutative, associative partial binary operation on $L$ satisfying the three additional conditions:

(a) There exists a neutral element $0 \in L$ such that, for every $p \in L$, $p \oplus 0 = p$,

(b) There exists a unit element $1 \in L$ such that, for every $p \in L$, there is a unique $q \in L$ with $p \oplus q = 1$,

(c) If $p \oplus p$ exists, then $p = 0$.

Orthoalgebras then generalize orthomodular posets, which can be defined as orthoalgebras in which, given that $p \oplus q$, $q \oplus r$ and $r \oplus p$ all exist, the element $p \oplus q \oplus r$ also exists. This axiom, called orthocoherence, is in fact a finitistic version of Mackey’s axiom V. Conversely, dropping condition (c) we obtain what is called an effect algebra, called generalized orthoalgebras by [Giuntini and Greuling 1989] and D-posets by [Köpka 1992].

5.2. ORTHOALGEBRAS

Orthoalgebras and effect algebras are sufficiently regular objects to have an interesting mathematical theory (one that is only beginning to be explored). In particular, nearly all of the conceptual apparatus of OMP-based quantum logic, such as centers [Greechie, Foulis and Pulmannová 1995] and Sasaki projections [Bennett and Foulis 1998; Wilce 2000], can be rather easily extended to this more general context. On the other hand, because of their simplicity, test spaces are often much easier to manipulate than their associated “logics”. They also have the heuristic advantage that the operational interpretation is, so to say, right on the surface, with the logics serving only as useful invariants. In particular, while it is completely straightforward to combine test spaces sequentially, the various “logics” rarely respect such combinations. Finally, if $\mathfrak{A}$ is algebraic, there exists a canonical order-preserving mapping $L \rightarrow \mathcal{L}$ from the logic of $\mathfrak{A}$ into the property lattice associated with any entity $(\mathfrak{A}, \Sigma)$ over $\mathfrak{A}$. In both classical and quantum mechanical examples, this mapping is in fact an isomorphism, so that $L$ inherits from $\mathcal{L}$ the structure of a complete lattice, and $\mathcal{L}$ inherits from $L$ an orthocomplementation and orthomodularity. This isomorphism is, however, the exception rather than the rule. As stressed by [Foulis, Piron and Randall 1983], the tendency to identify $\mathcal{L}$ and $L$ — even when they are isomorphic — has caused a great deal of unnecessary confusion in discussions of the foundations and interpretation of quantum mechanics.

Besides its awkwardness in dealing with sequential measurements, another difficulty that arises with the Mackey scheme of quantum logic, again recognized first by Foulis and Randall [1979], is that it is not stable under the formation of any reasonable sort of tensor product. Given quantum logics $(L, \Delta)$ and $(L', \Delta')$, each understood to represent some “physical” system, one wants to construct a
model \((M, \Gamma)\) of the coupled system in which \(L\) and \(L'\) may display correlations, but do not directly interact. Minimal requirements would be that

(a) there exists a map \(L \times L' \to M\) carrying \(p, q\) to some representative proposition \(p \otimes q \in M\), and

(b) for every pair of states \(\mu \in \Delta, \nu \in \Delta'\), we be able to form a state \(\mu \otimes \nu \in \Gamma\) such that \((\mu \otimes \nu)(p \otimes q) = \mu(p)\nu(q)\).

However, Foulis and Randall produce a simple example showing that this is in general impossible: a small, finite OML \(L\) with a full set of states such that no such “tensor product” exists for two copies of \(L\).

The culprit turns out to be Mackey’s Axiom V — or, more precisely, orthocoherence. Indeed, one can show, under the very mild assumption that the orthoalgebras involved each carry a unital family of states, that tensor products of orthoalgebras can be formed in such a way that desiderata (a) and (b) are satisfied [Foulis and Randall 1981b; Randall and Foulis 1981]. However, as the example just discussed illustrates, orthocoherence is not stable under this tensor product. Combining these results with the above mentioned negative results of Aerts for property lattices, what emerges is that the isomorphism between logic and property lattice, characteristic of both quantum and classical systems, breaks down when one forms tensor products unless the systems in question are classical. This is not to say that the results were entirely negative. Later research into the structure of tensor products [Klécza, Randall and Foulis 1987; Golfin 1987; Wilce 1990, 1992; Bennett and Foulis 1993; Dvurečenskij and Pulmannová 1994; Dvurečenskij 1995] revealed that Foulis–Randall tensor products of quantum-mechanical entities, while no longer strictly quantum, still retain a rich geometric structure.

These results gave substantial impetus to the study of orthoalgebras, test spaces and other structures more general than those considered by Mackey and Piron (some of which will be discussed below). The theory of test spaces, in particular, has developed in several directions in the past decade. A number of authors (e.g., [Dvurečenskij and Pulmannová 1994b; Pulmannová and Wilce 1995; Gudder 1997]) have discussed generalized test spaces in which outcomes are permitted to occur with some multiplicity or intensity, and have used these to provide an operational semantics for effect algebras that parallels the test-space semantics for orthoalgebras. Measure theory on orthoalgebras has been discussed by [Habil 1993]. Nishimura [1993, 1995] has generalized the idea of a test space by replacing discrete outcome sets by complete Boolean algebras and locales. [Wilce 2000] gives an up-to-date survey of the Foulis-Randall theory; for a personal view of the historical development of this strand of operational quantum logic see [Foulis 1998, 1999].

6. Orthomodular structures

Thus far, we have focussed on quantum logic as a foundational or interpretive
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programme in physics. But the subject has other, quite independent roots in pure mathematics. Von Neumann himself had stressed the importance of order theoretic methods in studying infinite-dimensional analogues of projective geometry. Loomis [1955] and Maeda [1955] independently recognized that a fair bit of the dimension theory of von Neumann algebras could be carried over into a purely lattice-theoretic setting, namely, that of an orthomodular lattice equipped with a suitable equivalence relation. This stimulated a number of mathematicians to begin investigating orthomodular lattices in abstracto. It soon became apparent that such lattices occur with some naturality in a wide range of mathematical contexts.

If \((S, \ast)\) is any involutive semigroup, call an element \(e \in S\) satisfying \(p = p^2 = p^*\) a projection. If \(S\) contains a (two-sided) zero element, the right annihilator of \(x \in S\) is the right ideal \(\{a \in S \mid ax = 0\}\). Foulis [1958, 1960, 1962] defined a Baer \(*\)-semigroup to be an involutive semigroup \(S\) with zero having the property that the right-annihilator of any element \(x \in S\) is the right ideal generated by a (necessarily, unique) projection \(x'\). He showed that the set \(L(S)\) of closed projections \(p = p''\) in \(S\) always forms an orthomodular lattice. Conversely, every orthomodular lattice can be represented as \(L(S)\) for some Baer \(*\)-semigroup. Indeed, while this representation is not unique, there is a canonical choice for \(S\), namely the semigroup \(S(L)\) of residuated self-mappings of \(L\), i.e., mappings \(\phi : L \to L\) for which there exists a mapping \(\psi : L \to L\) satisfying \(\psi(x) \leq y' \iff x \leq \phi(y)'\). Among these mappings are the Sasaki projections \(\phi_b\), discussed in section 4, which turn out to be exactly the closed projections in \(S(L)\).

Over the following decades, a substantial pure theory of orthomodular lattices was developed by Foulis and others. The state of this theory as of the early 1980s is represented by the book of Kalmbach [1983]. [Bruns and Harding 2000] discuss more recent developments, of which there have been many. Particularly striking is the recent discovery of Harding [1996, 1998] that one can organize the set of direct-product decompositions of essentially any algebraic object into an orthomodular poset. On the other hand, the on-going work on Mackey’s programme also produced a variety of structures more general than orthomodular lattices and posets — orthoalgebras, the still more general effect algebras, and, in a different direction, the partial Boolean algebras of Kochen and Specker [1967]. All of these are primarily partial algebraic, and only secondarily order theoretic, objects. All have attracted, especially during the past few years, significant mathematical interest.

The theory of effect algebras, much of which is due to the pioneering work of Foulis and the late M. K. Bennett [Bennett and Foulis, 1995, 1997; Foulis and Bennett 1994], continues to develop rapidly. Of particular interest here is their recent reformulation of a large part of the theory of effect algebras (and thus, of quantum logic) as a branch of the theory of ordered abelian groups, also discussed in [Foulis, Bennett and Greechie 1996; Foulis, Greechie and Bennett 1998; Wilce 1995, 1998]. This is the subject of [Foulis 2000].

Finally, orthomodular lattices have also been studied in detail in the purely

\[16\] One notable result is that of Kochen and Conway [Kochen 1996], that very small sets of projections in \(P(H)\) generate a partial Boolean algebra that is dense in the full projection lattice.
logical context, and in particular the possibility of defining reasonable implication connectives. One of the basic results in this direction is that of Kalmbach [1974] who, exploiting the characterisation of free orthomodular lattices on two generators [Bruns and Kalmbach 1973], was able to show that there are exactly five lattice polynomials \( a \rightarrow b \) satisfying the primitive implicative condition \( a \leq b \iff (a \rightarrow b) = 1 \). Note that here orthomodularity is essential, a simple consideration of the non-orthomodular “benzene ring” showing that such connectives do not exist in the non-orthomodular case [Moore 1993]. For an analysis of the weaker exportation condition \( a \leq b \Rightarrow (a \rightarrow b) = 1 \) conjoined with modus ponens see [Herman, Marsden and Piziak 1975], and for a detailed investigation of the deduction theorem see Malinowski [1990, 1992]. On the other hand, defining a Kripkean accessibility relation induced from non-orthogonality, an idea having its origins in Foulis and Randall’s work on lexicographic orthogonality [1971], has allowed the introduction of modal quantum logic [Dalla Chiara 1977, 1983; Goldblatt 1974, 1975]. Of course there has been much other work on implications in quantum logic; for general overviews see, for example, [Dalla Chiara 1986; van Fraassen 1981; Hardegree and Frazer 1981].

7. Dynamical, categorical and computational aspects

We close by considering the categorical reformulation of the basic notions of order structures and its application to operational quantum theory, a subject with strong links to various recent developments in enriched category theory and computational semantics. The basic tool of this theory is that of pairs \( f \dashv g \), where \( f : L \rightarrow M \) and \( g : M \rightarrow L \) are isotone maps between posets, satisfying the adjunction condition \( f(a) \leq b \iff a \leq g(b) \). For a pedestrian development of the theory of adjunctions with a particular focus on its operational applications we refer to [Coecke and Moore 2000]. It is amusing to note that this notion may be used to shed some light on Birkhoff and von Neumann’s remark cited above that while philosophers have tended to focus on the nature of negation in non-classical logics, the study of quantum mechanics highlights the distributive law as the weak link in operational quantum logic. To see this, let us note that Heyting algebras, considered as models for intuitionistic logic, may be defined as those lattices admitting an implication connection \( \rightarrow \) satisfying the adjunction condition \( (x \land a) \leq b \iff x \leq (a \rightarrow b) \) [Birkhoff 1940 §161; Birkhoff 1942 §27]. Since the condition \( f \dashv g \) implies that \( f \) preserves existing joins and \( g \) preserves existing meets, any Heyting algebra is distributive. On the other hand, much of the structure theory of orthomodular lattices rests on the so-called Sasaki adjunction \( \phi_a \dashv \phi^a \), where

\[
\phi_a(x) = a \land (a' \lor x) \quad \text{and} \quad \phi^a(x) = a' \lor (a \land x)
\]

[Nakamura 1957; Sasaki 1955]. In a certain sense, then, we may consider Heyting algebras as a class of distributive lattices where those elements possessing a complement may be simply characterised, and orthomodular lattices as a class of ortholattices where the set of complements of any given element may be simply
computed. For discussions of Heyting algebras and the more general semicomplemented lattices, see [Curry 1963; Frink 1962; Kähler 1981; Nemitz 1965].

One of the earliest, and most important, applications of residuations in orthomodular lattices was Foulis’ pioneering work on Baer *-semigroups, described above. This research has not only led to a deeper understanding of the notion of residuation [Blyth and Janowitz 1972; Derdérian 1967], but was also crucial in the development of dynamical aspects of operational quantum logic. One of the first of these developments was the work of Pool [1968a,b], who sought a phenomenological interpretation of Baer *-semigroups via the notion of conditional probability supplied by the conventional quantum theory of measurement. A more distinctly operational approach to evolutions in general was provided by Daniel [1982, 1989] and extended by Faure, Moore and Piron [1995], the latter leading to a general study of the categories of state spaces and property lattices [Moore 1995, 1997]. Here an externally imposed evolution is modeled by pulling back definite experimental projects defined at the final time to their images defined at the initial time. By physical arguments, this map must preserve the product operation and so the lattice meet. Hence, under suitable stability conditions, its join preserving left adjoint then describes the propagation of the state of the system. These observations have been generalized by Amira, Coecke and Stubbe [1998], who explicate the structure of operational tests derived from the notions of free choice and composition. Note that the latter of these notions in particular plays a fundamental role in the heuristics of Foulis and Randall mentioned above. Finally, the abstract structure of operational resolutions has been analysed by Coecke and Stubbe [1999a,b, 2000], allowing, for example, an analysis of the physical notions of compoundness [Coecke 2000] and the duality between causality and propagation [Coecke, Moore and Stubbe 2000].

Mathematically, the structure induced from operational resolutions is that of a quantaloid, namely a category whose Hom-sets are join complete lattices such that composition distributes on both sides over joins. One thus obtains a simple example of an enriched category, in which Hom-sets are objects in some base category and composition is realised by natural transformations satisfying coherence criteria. This has become a central notion in category theory, and is treated in the standard text [Borceux 1994]; for a specialised treatment see [Kelly 1982] and for a pedagogical development see [Borceux and Stubbe 2000]. Restricting attention to categories with a single object, we then recover quantales, introduced by Mulvey [1986] as a non-commutative generalization of locales. Here mention may be made of the recent extension of the localic notions of simplicity and spatiality to the context of quantales [Kruml 2000; Paseka 1997; Paseka and Kruml 2000; 18].

17 The name quantaloid was introduced by Rosenthal [1991], although much of the basic conceptual development had already been made by Joyal and Tierney [1984] and Pitts [1988] in their studies of Grothendieck topoi.

18 Note that historical precedents for these notions can be traced back to the work of Ward and Dilworth [Ward 1937, 1938; Ward and Dilworth 1939a,b], who used such multiplications to study ideals in rings, a technique which has recently been applied to non-commutative $C^*$-algebras [Borceux, Rosický and Van Den Bossche 1989; Rosický 1989].
Rosický 1995], a subject treated in some detail in [Paseka and Rosický 2000]. It is interesting to observe that similar structures have also been exploited in computer science. An important example is the so-called observational logic of Abramsky and Vickers [Abramsky 1991; Abramsky and Vickers 1993; Vickers 1989]. Here it is observed that the possibility that observation induces a change of state formally leads to a passage from frames to quantales. This line of thought has been extended by Resende [1999, 2000], who describes general systems on the basis of their observable behaviour independently of any supposed state space. For an overview see the [Resende 2000]. It should be noted, however, that these considerations are rather different from the contemporary notion of quantum computation (see for example [Gudder 2000]'s general theory of quantum languages, i.e., languages accepted by quantum automata).

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