Long range \textit{p}-wave proximity effect into a disordered metal

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We use quasiclassical methods of superconductivity to study the superconducting proximity effect from a topological \textit{p}-wave superconductor into a disordered one-dimensional metallic wire. We demonstrate that the corresponding Eilenberger equations with disorder reduce to a closed nonlinear equation for the superconducting component of the matrix Green’s function. Remarkably, this equation is formally equivalent to a classical mechanical system (i.e., Newton’s equations), with the Green function corresponding to a coordinate of a fictitious particle and the coordinate along the wire corresponding to time. This mapping allows to obtain exact solutions in the disordered nanowire in terms of elliptic functions. A surprising result that comes out of this solution is that the \textit{p}-wave superconductivity proximity-induced into the disordered metal remains long-range, decaying as slowly as the conventional \textit{s}-wave superconductivity. It is also shown that impurity scattering leads to the appearance of a zero-energy peak.

Introduction. – Superconducting heterostructures have attracted a lot of attention recently as possible hosts of Majorana fermions \cite{majorana}. One of the important outstanding questions in the studies of these heterostructures is the interplay between topological superconductivity and disorder \cite{disorder}. Here we explore this issue focusing on the leakage of \textit{p}-wave superconductivity into a disordered metal. Naïvely, it may not appear to be a particularly meaningful question, because unconventional superconductivity is known to be suppressed by disorder per Anderson’s theorem \cite{anderson}. However, Anderson’s theorem is only relevant to an intrinsic superconductor and has little to do with a leakage of superconductivity.

The linearized Usadel equations are standard tools in studies of proximity effects \cite{usadel}. Their derivation, however, assumes that an anisotropic component of the superconducting condensate’s wave-function is small compared to the isotropic one, which is not the case in the systems we are interested in. Here, we focus on the more general Eilenberger equations \cite{eilenberger}, which allow us to straightforwardly model systems with complicated geometries, and varying degree of disorder. (In the context of topological superconductivity, similar approach has been used in Refs. \cite{topo}.) We obtain exact solutions of these equations, and study superconducting correlations induced by proximity in a metallic wire. In particular, we demonstrate that the \textit{p}-wave correlations can be surprisingly long-ranged, even in the presence of disorder. We also show that impurity scattering produces a zero-energy peak in the density of states (DOS).

Solution for \textit{s}-wave and \textit{p}-wave order parameters. – We study the quasiclassical Green’s function \(\hat{\gamma}\), which is a matrix in both Nambu and spin space \cite{spin}. It is obtained from the full microscopic Green’s function by integrating over the energies close to the Fermi surface, and it faithfully captures the long lengthscale features of the system \cite{long}. In one-dimensional systems, \(\hat{\gamma}\) depends on the Matsubara frequency \(\omega\), the center-of-mass coordinate of the pair \(\langle x \rangle\), and the direction of the momentum at the Fermi points \(\zeta = p_x/p_F = +1/−1\) for right/left going particles. The Green’s function obeys the Eilenberger equation \cite{eilenberger}:

\begin{equation}
\zeta v_F \partial_x \hat{\gamma} = -[\omega \tau_3, \hat{\gamma}] + i [\Delta, \hat{\gamma}] - \frac{1}{2 \tau_{\text{imp}}} \langle \hat{\gamma}, \hat{\gamma} \rangle.
\end{equation}

The effect of impurities enters the equation through the mean time between collisions \(\tau_{\text{imp}}\), and \(\langle \cdot \rangle\) denotes an average over the Fermi surface. We ignore self-consistency, and assume that the order parameter \(\Delta\) is constant throughout the wire. (We believe enforcing self-consistency would not change our results qualitatively.)

We consider \textit{s}-wave and \textit{p}-wave order parameters in parallel, even though the appropriate Eilenberger equations differ significantly. First, we decompose the Green’s function in Nambu space using the Pauli matrices \(\tau_3\): \(\hat{\gamma} = -i q_1 \tau_1 + g_3 \tau_2 + g_3 \tau_3\). The scalar functions \(g_3\) have to satisfy the normalization condition \(-g_3^2 + g_3^2 + g_3^2 = 1\) (This will be referred to as the norm of \(\hat{\gamma}\), from here on). Note that the DOS of the system can be obtained from the diagonal component \(g_3\) \cite{dos}.

In the case of an \textit{s}-wave superconductor, \(\Delta\) is a spin-singlet and, ignoring the spin indices, it can be written as \(\Delta_0 i \tau_2\). The diagonal component \(g_3\) contains the particle-hole correlations. The function \(g_3\) encodes the \textit{s}-wave pairing, whereas \(g_1\) describes the \textit{p}-wave, odd-frequency superconducting correlations, induced by boundaries or other inhomogeneities (it disappears in the bulk uniform state \cite{bcs}). In the case of a \textit{p}-wave wire we consider spinless fermions, and the order parameter can be written as \(\zeta \Delta_0 i \tau_2\). The difference from the \textit{s}-wave case arises...
from the fact that now $g_2$ is $p$-wave, and $g_1$ contains the secondary $s$-wave (odd-frequency) correlations 20 23.

The components of $\hat{g}$ obey three coupled differential equations. These equations, however, differ for the $s$-wave and the $p$-wave cases, due to the Fermi surface averaging: in the $s$-wave case we have $\langle g_1 \rangle = 0$, $\langle g_2 \rangle = g_2$, whereas in the $p$-wave case $\langle g_1 \rangle = g_1$, $\langle g_2 \rangle = 0$. In both cases $\langle g_3 \rangle = g_3$ applies (particle-hole correlations are $s$-wave-like). We use an index $j = (1, 2)$ that allows us to write the component equations in a unified way: in the $s$-wave case we have $j = 1$, and $j = 2$ pertains to the $p$-wave case. This index will be used for the rest of the paper, unless the state is explicitly indicated with a subscript $s$ or $p$. For the order parameters we have $\Delta_{(1)} \equiv \Delta_s = \Delta_0$ for $s$-wave, and $\Delta_{(2)} \equiv \Delta_p = \zeta \Delta_0$ for $p$-wave. In $g_j$ the subscript denotes the Nambu space components $- g_1$ and $g_2$ for $s$-wave and $p$-wave cases respectively. With these, and using the Kronecker delta $\delta_{ij}$, we write the Eilenberger equation as:

\[
\begin{align*}
\zeta v_F \partial_x g_1 &= -2\omega g_2 + 2\Delta_{(j)} g_3 - \left( \frac{1}{\tau_{\text{imp}}} g_2 g_3 \right) \delta_{j2}, \tag{2a} \\
\zeta v_F \partial_x g_2 &= -2\omega g_1 - \left( \frac{1}{\tau_{\text{imp}}} g_1 g_3 \right) \delta_{j1}, \tag{2b} \\
\zeta v_F \partial_x g_3 &= 2\Delta_{(j)} g_1 - (-1)^j \left( \frac{1}{\tau_{\text{imp}}} g_1 g_2 \right). \tag{2c}
\end{align*}
\]

In the clean case, these equations become linear and are easily solved 21 24 25. Impurities introduce nonlinear coupling, proportional to $1/\tau_{\text{imp}}$. Nevertheless, as we will demonstrate, these equations can still be treated analytically.

To be integrable, this system (either for $s$-wave or $p$-wave state) should have two constants of integration. The norm of $\hat{g}$ is one of them, and it can be shown that another constant is given by:

\[
C_{(j)} = \frac{(-1)^j - 1}{2\tau_{\text{imp}}} g_j^3 + 2\Delta_{(j)} g_2 + 2\omega g_3. \tag{3}
\]

This can be seen from equations Eq. 2, by verifying that $\partial_x C_{(j)} = 0$, for both $s$- and $p$-wave cases. Using $C_{(j)}$ we can derive from the system (Eqs. 2) a second-order equation for a single component. In the $s$-wave case we proceed by differentiating Eq. 2a. Using $C_s \equiv C_{(1)}$ we obtain the following equation:

\[
\zeta v_F^2 \partial_x^2 g_1 = 4\alpha_s g_1 - \frac{g_1^3}{2\tau_{\text{imp}}}, \tag{4}
\]

where we have defined $\alpha_s = \Omega^2 + C_s/(4\tau_{\text{imp}})$, with $\Omega^2 = \omega^2 + \Delta_0^2$. In the case of a $p$-wave order parameter we differentiate Eq. 2b. and by using $C_p \equiv C_{(2)}$, and defining $\alpha_p = \Omega^2 + C_p/(4\tau_{\text{imp}})$, the resulting equation is:

\[
\zeta v_F^2 \partial_x^2 g_2 = -2\zeta \Delta_0 C_p + 4\alpha_p g_2 - \frac{3\Delta_0}{\tau_{\text{imp}}} g_2^2 + \frac{g_2^3}{2\tau_{\text{imp}}} \tag{5}
\]

Either of these equations can now be integrated on its own, without explicit reference to the other two components. However, once $g_j$ is determined the other components follow from $C_{(j)}$ and $\partial_x g_j$.

We also consider the case of a normal metallic segment in contact with a superconductor with order parameter $\Delta_0$ (for $s$-wave) or $\zeta \Delta_0$ (for $p$-wave). To study the superconducting correlations induced in the normal part we can use the Eilenberger equation with the order parameter in the metal set to zero. The constant of integration becomes $C'_{(j)} = (-1)^j - 1/2(2\tau_{\text{imp}}) + 2\omega g_3$. To streamline notation we introduce the dimensionless constants $\tilde{C}_{(j)} = C_{(j)}/2\Delta_0$, $\tilde{\alpha}_{(j)} = \alpha_{(j)}/\Delta_0^2$, and $\beta = 1/(2\tau_{\text{imp}} \Delta_0)$. In addition, we define the coherence length, $\xi_0 = v_F/\Delta_0$. (Note that in these definitions $\Delta_0$ is introduced only as an energy scale.) With these, we can write, for the $g_1$ component in a normal segment in contact with $s$-wave wire, the following equation:

\[
\xi_0^2 \partial_x^2 g_1 = 4\tilde{\alpha}_s g_1 - 2\beta^2 g_1^3. \tag{6}
\]

In the case of a normal wire in contact with a $p$-wave superconductor we have equation for $g_2$:

\[
\xi_0^2 \partial_x^2 g_2 = 4\tilde{\alpha}_p g_2 + 2\beta^2 g_2^3. \tag{7}
\]

Notice the difference in the sign between the $\beta^2$ terms in the two equations.

**FIG. 1.** The potential landscape of a classical particle with motion describing the Green’s function, for a normal metallic segment, in contact with a superconductor. Depending on the superconductor ($s$ or $p$-wave) potential is either $V_s$ or $V_p$. In the clean limit both converge to $V_{\text{clean}}$.

**Classical particle analogy.**— Equations 4 5 6 or 7 can be integrated analytically. Before we do this, however, it is instructive to interpret them as equations of motion for a classical particle with one degree of freedom, moving in an external potential. The “position” of this particle is $g_j$ and the “time” $\tilde{t}$, is given by $\zeta 2\tilde{x}/\xi_0$, hence its “momentum” is $\partial_x g_j$. In both $s$-wave and $p$-wave cases the external potential is described by a quartic polynomial function. For example, from Eqs. 6 and 7 we can write $V_{(j)}(g_j) = -\tilde{\alpha}_{(j)} g_j^2/2 + (-1)^j - 1/2\beta^2 g_j^4/8$. Note that $V_j$ describes a double well for the $s$-wave case ($j = 1$), and a hill for the $p$-wave case ($j = 2$). In the clean limit we
have $\beta \to 0$ and the potential energy becomes an inverted parabola, $V_j(g_j) = -\alpha_j g_j^2/2$, for both the s-wave and the p-wave cases (see Fig. 1).

We denote the dimensionless “energy” of the classical system by $\tilde{E}_j$. It is a constant of integration, and can be determined from the boundary conditions for $g_j$.

Since we want to study proximity effects, we concentrate on Eqs. [6] and [7]. After multiplying both sides with $v_F \partial_x g_j$, we integrate the equations two times. The result is the following elliptic integral, where the variable $x$ spans the length of the wire that starts at $x = 0$ and ends at $L$:

$$g_j(x) = \int_{g_j(0)} \frac{dg_j}{\left(\tilde{\alpha}_j g_j^2 + (-1)^j \frac{\beta^2}{4} g_j^4 + 2\tilde{E}_j \right)^{1/2}} = \pm \frac{2x}{\zeta_0}.$$  \hspace{1cm} (8)

The $\pm$ sign before the right hand side of Eq. [8] is to ensure that $x$ is positive, and it depends on the choice of the integration contour in the complex $g_j$ plane. We will denote the poles of the integrand as $\rho_j^\pm$. The integral can be written in terms of the inverse Jacobi elliptic function $sn^{-1}$, with elliptic parameter $m = \rho_j^+ / \rho_j^-$. The monontonic solution is given by

$$sn^{-1}\left(\frac{g_j(x')}{\rho_j^+}\right) = \pm \frac{\zeta_0 x}{\beta^2 \rho_j^- (-1)^j \rho_j^+} \left[-1, \frac{\beta^2}{4}\tilde{E}_j \right]^{1/2},$$  \hspace{1cm} (9a)

$$\rho_j^\pm = \frac{\beta^2}{\rho_j^-} \left[(-1)^j \rho_j^+ \pm \left[\rho_j^+ - (-1)^j \tilde{E}_j \beta^2\right]^{1/2}\right].$$  \hspace{1cm} (9b)

It is important to note that another choice of the integration contour may lead to non monotonic, and/or oscillatory solutions. We can understand this by considering the classical particle in one of the potentials shown on Fig 1. In the s-wave case, the potential is a double well, hence the motion is generally periodic. However, the non-monotonic solutions are unphysical and we have to discard them, since the turning points of the trajectories scale as $\pm (\omega \tau_{imp})$ at high frequency, and for both $\omega \to \infty$ or $\tau_{imp} \to \infty$ the periodic motion has unbounded amplitude. In the p-wave case, the period of the elliptic function is imaginary, as $V_p$ does not lead to periodic motion. We conclude that in both of the s-wave and p-wave cases the only physically acceptable solutions are monontonic (given by Eq. [9]). They can be visualized by imagining the motion of a particle, with initial position $g_j(0)$ and velocity directed towards the origin $g_j = 0$, climbing a non-harmonic hill potential $V_j(g_j)$. The amount of “time”, for the particle to reach its final position represents the length of the wire $L$. For example, if $L$ is infinite the particle is coming to a stop at the origin (no superconducting correlations at infinity means vanishing velocity), hence should have zero “energy”, $\tilde{E} = 0$.

$p$-wave wire with normal segment. Let us use the solution of the Eilenberger equation to study the leakage of superconductivity in a metallic wire. We consider an infinite wire extending along the $x$-axis with two segments that meet at $x = 0$. The semi-infinite segment on the left ($x < 0$) is made of clean $p$-wave superconductor. The segment on the right ($x > 0$) is made of a diffusive normal metal (the order parameter is zero).

We obviously want a solution that, in the limit $x \to -\infty$ reproduces the mean field result for a uniform clean $p$-wave superconductor. Introducing the parameter $B$ and the dimensionless variables $\Omega = \Omega / \Delta_0$, $\tilde{\omega} = \omega / \Delta_0$, we can write such a solution [21, 29, 30]:

$$g_1(x) = (1/\tilde{\omega}) [1 - \tilde{\Omega} B \exp(2\tilde{\Omega} x / \xi_0)],$$  \hspace{1cm} (10a)

$$g_2(x) = \frac{\zeta}{1 - [1 - \tilde{\Omega} B] \exp(2\tilde{\Omega} x / \xi_0)}, \quad (10b)$$

$$g_3(x) = \left\{1 - [1 - \tilde{\Omega} B] / (\tilde{\Omega} \tilde{\omega})\right\} \exp(2\tilde{\Omega} x / \xi_0) + \tilde{\omega} / \tilde{\Omega}.$$  \hspace{1cm} (10c)

$B$ has to be determined from the boundary conditions at $x = 0$. For simplicity, we will consider the case of perfectly transparent boundary there, which guarantees the continuity of the Green’s functions [31].

Now we consider the diffuse normal segment with infinite length. Then, for $x \to \infty$ we have $g_1 \to 0$, $g_2 \to 0$ and $g_3 \to \sgn(\omega)$. The constant of integration is $\tilde{C}_p = -\beta g_0^2 / (2 + \tilde{\omega} g_0)$, when normalized to $2\Delta_0$. Using the fact that $\tilde{C}_p(0) = \tilde{C}_p(x \to \infty) = \tilde{\omega} g_0$, we immediately obtain $B = (1/\beta)[1 + (1/\beta) \tilde{\Omega} / \tilde{\omega} g_0]$, with $\tilde{\omega} g_0 = \omega^2 + \beta^2 \tilde{\omega} g_0$.

We can understand intuitively the behavior of $g_2$ by again invoking the classical analogy. The particle in potential $V_p$, with “position” $g_2$ where time is $\tilde{t} = 2x / \xi_0$, starts at $g_2(0) = \zeta B$, with velocity $\partial_t g_2(0) = -\omega \tilde{g}_1(0) = -\zeta (1 - \tilde{\Omega} B)$, and moves towards its unstable equilibrium point $g_2(+\infty) = 0$, gradually slowing down until $\partial_t g_2(+\infty) = 0$. Thus, the trajectory of $g_2$ satisfies $\tilde{E}_p = 0$. The integral in Eq. [8] is now straightforward, and defining the dimensionless constant $\kappa = [1 + \beta^2 B^2 / (4\tilde{g}_p)]^{1/2}$, we can write the solution for $g_2$:

$$g_2(x) = \frac{\zeta B}{\cosh(x/\xi')} + \kappa \sinh(x/\xi'), \quad (11)$$

Here $\xi' = \xi_0 / (2\tilde{g}_p^{1/2})$ gives the effective decay length of the solution (at $T = 0$). In physical units it is

$$\xi' = \frac{v_F}{\sqrt{4\omega^2 + 2|\Delta_0\tilde{\omega}|/\tau_{imp}}}.$$  \hspace{1cm} (12)

In the dirty limit we have $\xi' = \sqrt{D|\tilde{\omega}|}$, where $D$ is the diffusion coefficient. Finally, in the clean limit $g_2$ converges to $\zeta B \exp(-2|\tilde{\omega}| x / \xi_0)$, as expected [21].

The other two components of the Green’s function can be derived from $g_2$ using $\tilde{C}_p$ and the Eilenberger equations: $g_1 = -\zeta_0 \partial_x g_2 / (2\tilde{\omega})$ and $g_3 = \sgn(\tilde{\omega}) + \beta g_2^2 / (2\tilde{\omega})$. As expected, impurities suppress $g_2$ relative to $g_1$. However, they both decay in the normal segment over the
same lengthscale, given by Eq. [12] This decay is long-range, and furthermore, with exactly the same lengthscale we obtain for the case of s-wave order parameter (see below). Thus, the naïve expectation of strong suppression of the p-wave correlations is misleading in this case. This is one of the main points of our paper.

![Image](Image)

FIG. 2. Contour plot of the DOS of an infinite wire. There is moderate disorder (β = 1) in the normal segment (x > 0). The solid yellow marks the regions that are beyond the plot range (where N/N₀ > 3.5). Notice the zero-energy peak in the normal segment.

We can now obtain the DOS of the system, which is proportional to the real part of g3(ω → −iε + δ). On Fig. 2, we show the DOS for a system with moderate amount of disorder. Several things are apparent from this plot. First, for energies below Δ₀ there is a significant decrease in the DOS of the normal segment, caused by the proximity effect; however, it is not a real gap, since the DOS stays finite everywhere. This decrease is entirely due to impurities – in the clean case the DOS is constant for x > 0 [21]. The impurity-induced term in g₃ also has a divergence in the limit of small frequencies (g₃ ∼ 1/ω), which leads to an infinite peak in the DOS. This zero-energy peak has the same origin as the Majorana edge state (namely, the sign change in the order parameter [29, 30, 33]). Thus, in the infinite wire case, impurity scattering creates zero-energy peak, but it is not sufficient to localize it exponentially.

As a side note, if the p-wave superconductor was replaced by an s-wave superconductor, the solution to Eq. [6] would be g₁ = ζA(cosh(x/ξ) + κₚ sinh(x/ξ))⁻¹. Here, κₚ = [1 − β²A²/(4AΩ)]¹/² and ζA is the value of g₁ at the junction, and is determined by the boundary values at the infinities in a way similar to that in the p-wave case. However, unlike the p-wave case, the g₁ component at the boundary is proportional to ω. This dependence on ω changes the zero energy behavior of the DOS as follows. From g₃ = sgn(ω) − β/(2ω)g₂², we see that the low frequency limit is finite and thus there is no zero energy peak in the s-wave case [32].

If the normal segment has finite length L, we impose the condition g₂(L) = 0, since the p-wave component is suppressed by the reflection from the boundary. Then the solution follows immediately from Eq. [2] as g₂(x) = ζ(ρₚ⁺)¹/² sin[β(ρₚ⁻)¹/²(x − L)/ξ₀], with elliptic parameter m = ρₚ⁺/ρₚ⁻. However, this expression has limited practical value. The unknown constant B_L, which should be obtained from matching the two solutions for g₂ at x = 0, enters the expression through the parameters ρₚ⁺, which makes it difficult to solve. Fortunately, an approximate analytic form for B_L can be obtained. In the limit L → ∞, B_L converges to B, that was previously calculated for the infinite wire case. In the opposite limit, L → 0, B_L vanishes. Numerical investigation suggests that B_L as a function of L can be approximated by B[1 − exp(−2L/λ_B)], where the length scale λ_B controls how quickly B_L approaches to the infinite wire limit with increasing L. By expanding the integral in Eq. [8] around B = 0 and matching it with the approximate expression, we obtain λ_B = Bξ₀. Once we have B_L, we can use ad-

![Image](Image)

FIG. 3. Components of ̂g₃ (g₁: blue, g₂: purple, g₃: red) for a wire with infinite p-wave section and finite disordered section of length L = 5ξ₀. Top panel: weak disorder (β = 1/(2τ_{imp}Δ₀) = 0.1). Middle panel: moderate disorder (β = 1). And bottom panel: strong disorder (β = 10). The Matsubara frequency is set to ω = Δ₀/2.

...dion and transformation rules for elliptic functions [34] to write g₂ in a form that manifestly converges to that of the L = ∞ case. To save space, we shorten the common argument of elliptic functions, β|ρₚ⁻|¹/²x/ξ₀ as (.). The common elliptic parameter of the elliptic functions is (ρₚ⁺ − ρₚ⁻)/ρₚ⁻, and it lies in the interval [0, 1]. With
can write the zero-energy limit as:

\[
g_2(x) = \zeta \frac{B_L \text{dn}(.) - \text{sn}(.) \text{cn}(.) \sqrt{|\rho_p|^2 + B_L^2 \sqrt{1 + B_L^2 / |\rho_p|}}}{\text{cn}(.) - (B_L^2 / |\rho_p|) \text{sn}(.)}.
\] (13)

We can again obtain the two other components from \(g_2\) by using: \(g_1 = -\zeta \theta x g_2 / (2 \omega)\) and \(g_3 = (\alpha - \omega^2) / (2 \beta \omega g_2 / (2 \omega))\). As \(L \to \infty\), \(\tilde{E}_p\) tends to zero, the elliptic functions are replaced by their hyperbolic counterparts, and we recover the solution for the infinite wire case (Eq. 11).

As \(L \to \infty\), the elliptic functions are replaced by their hyperbolic counterparts, and we recover the solution for the infinite wire case (Eq. 11).

Again, it is the impurity-induced contribution to \(g_3\) that is of most interest. After analytic continuation we can write the zero-energy limit as:

\[
g_3(x) = \frac{1}{\pi} \delta(\epsilon) M(x).
\] (14)

The function \(M(x)\) describes the \(x\)-dependent weight of the zero energy mode, and we can extract it from Eq. [13]. Its values at the junction point and at the end of the wire are \(M(0) = 1 - B_L\) and \(M(L) = M(0) - \beta B_L^2 / 2\) respectively. It can be approximated by a decaying exponent with decay length \(\lambda_M = \epsilon_0 \beta B_L / (4 \alpha_p + 2 \beta^2 B_L^2)\). Thus, in sharp contrast with the \(L = \infty\) case, the zero-energy peak of a finite wire is exponentially localized. Figure 4 shows \(M(x)\) in the normal section with length \(L = 5 \xi_0\), for various disorder strengths. As can be seen, \(M(x)\) (i.e., the zero-energy peak) becomes more localized as the disorder in the normal section increases.

![FIG. 4. The weight of the zero energy mode \(M(x)\) in a normal section with length \(L = 5 \xi_0\) for three disorder strengths (blue: \(\beta = 1/(2 \tau_{\text{emp}} \Delta_0) = 0.1\); purple: \(\beta = 1\); red: \(\beta = 10\)). The expression in the inset is deduced from Eq. [13], and (. ) stands for \((2 \sigma_p / \beta)^{1/2} x / \xi_0\).](image)

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