SU(3) Baryon Resonance Multiplets in Large $N_c$ QCD

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Abstract

We extend the recently developed treatment of baryon resonances in large $N_c$ QCD to describe resonance multiplets collected according to the SU(3) flavor symmetry that includes strange quarks. As an illustration we enumerate the SU(3) partners of a hypothetical $J^P = \frac{1}{2}^\pm$ resonance in the SU(3) representation that reduces to $\mathbf{10}$ when $N_c = 3$, and reproduce results hitherto obtained only in the context of a large $N_c$ quark picture. While these specific quantum numbers represent one favored set for the possible pentaquark state $\Theta^+(1540)$, the method is applicable to baryon resonances with any quantum numbers.

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I. INTRODUCTION

A recent series of papers [1, 2, 3] by a collaboration led by the current authors shows how baryon resonances may be properly treated in the context of large $N_c$ QCD. For reviews of this literature, see Ref. [4].

In the standard treatment of baryons at arbitrary $N_c$, the ground-state spin-flavor multiplet is taken to be the completely symmetric $N_c$-tableau representation, which is the analog to the SU(6) 56. Notationally, we denote such arbitrary-$N_c$ generalizations of $N_c=3$ representations with quotes, as in “56". It should be pointed out that the spin-flavor symmetry of the ground state is not a rigorous result following directly from manipulations of the QCD action, but rather an assumption whose phenomenological predictions concur with all available experimental measurements. Turning this result around, one can show that the successes of the old baryon SU(6) spin-flavor symmetry, such as $\mu_p = -\frac{3}{2}\mu_n$ or the relative closeness of $N$ and $\Delta$ masses, are actually consequences of the $1/N_c$ expansion [5, 6, 7]. Equivalently, assuming that the baryon masses and the $\pi N$ axial-current coupling $g_A$ scale as $N_c^3$ (as naturally arises in quark and Skyrme models) leads to a degenerate “56" multiplet through an analysis using “consistency relations" in $\pi N$ scattering, which are obtained by the imposition of unitarity order-by-order in $N_c$ [5].

The ground-state band “56" decomposes into multiplets with spins $S_B = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{N_c}{2}$, with corresponding SU(3) representations [in the Dynkin weight notation ($p, q$)] $(2S_B, \frac{N_c}{2} - S_B)$. The first two members of this series are denoted, as expected, “8" and “10", while we may label the $S_B^{PB} = \frac{5}{2}^+, \frac{1}{2}^+, \text{ etc.}$ members (those that disappear as $N_c \to 3$) as “large-$N_c$ exotic." The mass splittings between the multiplets with $J = O(N_c^0)$ are only $O(\Lambda_{\text{QCD}}/N_c)$, which for sufficiently large $N_c$ are smaller than $m_\pi$; hence, such members of the ground-state multiplet are stable against strong interactions and have widths that vanish in the large $N_c$ limit. One may argue that it is a fluke of our universe that the chiral limit is more closely realized than the large $N_c$ limit, allowing the decay $\Delta \to \pi N$.

Once baryon states are determined to be stable in this way, they may be analyzed using a Hamiltonian formalism, in which the spin-flavor symmetry is broken perturbatively in powers of $1/N_c$ by operators with specific quantum numbers under spin and flavor. This method has a long history and been dubbed the “operator approach" [1].

Baryon resonances, on the other hand, are completely different entities. Appearing in
meson-baryon scattering amplitudes, whose generic size is $O(N_c^0)$, resonances arise as poles at complex values of energy in the analytic continuation of these amplitudes. Of course, the real and imaginary part of each such value represents the excitation mass and width, respectively, of the resonance, both of which are typically $O(N_c^0)$. Resonances are unstable against strong decay even in the large $N_c$ limit, and require a treatment distinct from that of the stable baryons.

Nevertheless, if a mechanism other than the $1/N_c$ expansion can be invoked to suppress the creation of quark-antiquark pairs—the mechanism through which a baryon resonance decays—then the treatment of resonances as almost stable baryons becomes more reasonable. For example, if the quarks comprising the baryon are heavy compared to $\Lambda_{\text{QCD}}$ then pair creation is suppressed, and an approach analogous to that used for the ground-state baryons applies. The analysis of ordinary baryon resonances treated as almost stable in the $1/N_c$ expansion, using the operator approach, has been carried out in great detail in the literature [8].

However, there exists a model-independent treatment [1, 2] using the $1/N_c$ expansion in which the resonances have natural $[O(N_c^0)]$ widths and yet retains a good deal of predictive power. This “scattering approach” was originally inspired [9] by the observation that numerous results obtained in chiral soliton models such as the Skyrme model appeared to be purely group-theoretical in origin. A series of papers in the 1980s by Mattis and collaborators [10, 11] showed that such results are in fact independent of any dynamical details of the models. Indeed, in the case of two light quark flavors the dominant $S$ matrix amplitudes were found to be precisely those with t-channel exchange quantum numbers $I_t = J_t$ [11]. This result can in turn be shown to follow directly from the analysis of consistency relations derived from scattering processes in large $N_c$ [1].

Exploiting crossing relations, the $I_t = J_t$ rule can be used to express observable meson-baryon scattering amplitudes in terms of a smaller set of reduced amplitudes labeled in the s channel by eigenvalues $K$ of the “grand spin” $K = I + J$. In particular, a resonant pole appearing in one scattering amplitude must appear in at least one of the reduced amplitudes, which in turn appears in other scattering amplitudes. Baryon resonances therefore appear in multiplets degenerate in both mass and width for large $N_c$.

The group theory for the three-flavor case relevant to hyperon physics is of course more involved, but the necessary exercise was carried out [12] by Mattis and Mukerjee in 1989 in
the context of soliton models. When three flavors are included, the $I_t = J_t$ rule no longer holds in its original form, but as we discuss below a more complicated set of constraints applies. In fact, in repeating the derivation of Ref. [12], we find small discrepancies, and discuss them below. Nevertheless, the (suitably modified) Mattis-Mukerjee relation is the proper SU(3) generalization of the SU(2) scattering relation at large $N_c$. It predicts degenerate SU(3) multiplets of baryon resonances at large $N_c$ and in the SU(3) limit. This observation and its practical implementation (which required the computation of relevant SU(3) Clebsch-Gordan coefficients (CGC) [3]) are the purposes of this short paper.

As a first illustration of the power of the scattering method, we consider the SU(3) partners to a hypothetical $J^P = \frac{1}{2}^+$ or $\frac{1}{2}^-$ isosinglet baryon resonance in an SU(3) “10”. These are none other than two theoretically favored sets of quantum numbers for the purported pentaquark state $\Theta^+(1540)$. We discerned the partners of this $I = 0$ state using the two-flavor formalism in Ref. [2]. We of course make no claims whether this state does indeed exist, but rather conclude that if any baryon resonance with these quantum numbers exists, then it must have partners degenerate in mass and width at leading order in the $1/N_c$ expansion that carry specific $J^P$ and SU(3) quantum numbers.

Moreover, we show below that the particular pattern of partners to this state (for $P = +$) is precisely the one recently derived by Jenkins and Manohar [13]. They employed a $1/N_c$ operator approach originally inspired by the rigid rotor Skyrme model, but consistent with the group theory of quark models as well. One might have thought that such a coincidence is trivial. However, since the physical picture in Ref. [13] is based upon stable states at large $N_c$, one can easily imagine the possibility that when the widths of the states become order unity, mixing between the multiplets given in Ref. [13] could occur. This is not unheard of in the scattering approach; indeed, the old nonrelativistic SU($2N_f$)×O(3) quark model multiplets were shown [1] to form reducible collections of complete distinct multiplets in the scattering approach. It is therefore quite heartening that the two methods agree so well in this case.

This paper is organized as follows. In Sec. III, we present the master expression for three-flavor meson-baryon scattering and explain its origin and relation to previous work. Section III presents a specific example: the enumeration of quantum numbers of resonances degenerate in the large $N_c$ limit with a hypothetical isoscalar, strangeness +1 resonance in the SU(3) “10” representation, for the $J^P$ values $\frac{1}{2}^-$ and $\frac{1}{2}^+$. These are of course the
favored theoretical preferences for the purported pentaquark \( \Theta^+(1540) \) state; however, even if this state should turn out not to survive the current experimental scrutiny, the example presented here should be viewed as an indication of the power of the method. In Sec. \textbf{IV} we summarize, indicate future directions of research, and conclude.

\section{The SU(3) Amplitude Relation}

We now present the expression for the \( S \)-matrix amplitude in the meson-baryon scattering process $\phi(S_\phi, R_\phi, I_\phi, Y_\phi) + B(S_B, R_B, I_B, Y_B) \rightarrow \phi'(S_{\phi'}, R_{\phi'}, I_{\phi'}, Y_{\phi'}) + B'(S_{B'}, R_{B'}, I_{B'}, Y_{B'})$, where \( S, R, I, \) and \( Y \) stand, respectively, for the spin, SU(3) representation, isospin, and hypercharge of the mesons \( \phi \) and \( \phi' \) and the baryons \( B \) and \( B' \). Primes here indicate final-state quantum numbers. The total spin angular momentum (added vectorially) among the meson-baryon pairs are denoted by \( S \) and \( S' \), and the relative angular momenta between the meson-baryon pairs are denoted by \( L \) and \( L' \). The amplitude is described in terms of s-channel angular momentum \( J_s \), SU(3) representation \( R_s \), isospin \( I_s \), and hypercharge \( Y_s \).

In addition, multiple copies of \( R_s \) can arise in the products \( R_B \otimes R_\phi \) and \( R_{B'} \otimes R_{\phi'} \), and the quantum numbers defined to lift this degeneracy are labeled by \( \gamma_s \) and \( \gamma'_s \) (which need not be equal). That the other s-channel quantities are conserved can be demonstrated explicitly (e.g., \( R_s = R'_s \)), and thus the primes on such quantities are suppressed. The amplitude is reduced, in the sense of the Wigner-Eckart theorem, in that the SU(2) quantum numbers \( J_{sz} \) and \( I_z \) do not appear explicitly. The notation \([X]\) refers to the dimension of a given representation, whether \( X \) is labeled by \( I \) or \( J \) in SU(2), or by the actual dimension in SU(3) (i.e., \([J=1] = 3\), but \([R=8] = 8\)). The master expression for such scattering amplitudes in the large-\( N_c \) limit then reads

$$
S_{LL' SS' JJ_R s \gamma_s' I_s Y_s} = (-1)^{S_B - S_{B'}} ([R_B] [R'_{B'}] [S][S'])^{1/2} / [R_s] \sum_{I \in R_s, I' \in R_{\phi'}, I'' \in R_s, Y \in R_{\phi} \cap R_{\phi'}} (-1)^{I + I' + Y} [I''] \\
\times \left( \begin{array}{c|c|c|c|c|c|c|c|c} R_B & R_\phi & R_s & \gamma_s & I'' & Y + \frac{N_s}{3} \\
S_B & \frac{N_s}{3} & I' & \gamma_s' & I_s & Y_s \end{array} \right) \left( \begin{array}{c|c|c|c|c|c|c|c|c} R_B & R_\phi & R_s & \gamma_s & I'' & Y + \frac{N_s}{3} \\
I_B Y_B & I_{\phi} Y_{\phi} & I_s & Y_s \end{array} \right)
\times \left( \begin{array}{c|c|c|c|c|c|c|c|c} R_{B'} & R_{\phi'} & R_s & \gamma_s' & I'' & Y + \frac{N_s}{3} \\
S_{B'} & \frac{N_s}{3} & I' & \gamma_s' & I_s & Y_s \end{array} \right) \left( \begin{array}{c|c|c|c|c|c|c|c|c} R_{B'} & R_{\phi'} & R_s & \gamma_s' & I'' & Y + \frac{N_s}{3} \\
I_B Y_{B'} & I_{\phi'} Y_{\phi'} & I_s & Y_s \end{array} \right)
,$$

5
\[ \times \sum_{K,K',\bar{K}} [K][\bar{K}][\bar{K}']^{1/2} \left\{ \begin{array}{ccc} L & I & \bar{K} \\ S & S_B & S_\phi \\ J_s & I'' & K \end{array} \right\} \left\{ \begin{array}{ccc} L' & I' & \bar{K}' \\ S' & S_{B'} & S_{\phi'} \\ J_s & I'' & K \end{array} \right\} \tau^{(I''Y)}_{KK\bar{K}'LL'} . \tag{1} \]

The quantities containing double vertical bars are SU(3) isoscalar CGC, while those in braces are ordinary SU(2) $9j$ symbols. This expression should be compared with the original Mattis-Mukerjee result [Ref. [12] Eq. (12)]. Since its derivation was a primary result of that paper, we do not present a detailed rederivation here, but merely discuss its structure, and then detail differences between the present expression and that of Ref. [12] (in particular, why expression in Ref. [12] is not suitable for physical processes).

The two-flavor scattering formula is derived in the earlier chiral soliton-type treatment [10] by starting with a fundamental soliton in the conventional hedgehog configuration, which is an eigenstate of the grand spin $K \equiv I+J$. Scattering is accomplished by the standard linear expansion of the soliton in terms of pion field fluctuations. However, physical hadrons are of course specified not by $K$ but by $I$ and $J$, and hence one must allow for multiple values of $K$ in a full physical scattering process in order to form a linear superposition that is an eigenstate of $I$ and $J$; nevertheless, one may treat $K$ as a hidden degree of freedom conserved in the underlying scattering processes, which therefore attaches as a label to the reduced scattering amplitudes $\tau$. In generalizing the process to allow for mesons $\phi, \phi'$ of arbitrary isospin $I$ and spin $S$, one requires also the intermediate quantum numbers $\bar{K} \equiv I_\phi + L$ and $\bar{K}' \equiv I_{\phi'} + L'$ (so that $K = \bar{K} + S_\phi = \bar{K}' + S_{\phi'}$) used in Eq. (1). The $9j$ symbols simply arise through the combination of the numerous SU(2) CGC that arise in this procedure from the vector addition of multiple SU(2)-valued quantities.

The three-flavor generalization is conceptually quite straightforward, if mathematically more cumbersome: One simply rotates the full initial and final states into their nonstrange partners in the same irreducible SU(3) representation, and uses the two-flavor expression for the nonstrange scattering process. The inclusion of SU(3) rotation matrices of course introduces the SU(3) CGC in Eq. (1).

In repeating the derivation of Ref. [12] Eq. (12) to obtain our Eq. (1), we find a few small but significant differences. First, the overall phase of the original result lacks our phase $(-1)^{S_{B'}-S_B}$. Second, Ref. [12] appears to average over baryons and mesons in the external states with all possible quantum numbers within the given SU(3) multiplets (rendering their
expression phenomenologically less useful); if we do the same with Eq. (11), two of our SU(3) CGC are absorbed through an orthogonality relation, matching the older result. Finally, their explicit unity values for the nonstrange baryon hypercharges must be modified to \( \frac{N_c}{3} \), in light of the proper quantization [14] of the Wess-Zumino term for arbitrary \( N_c \). While these agree for \( N_c = 3 \), it is important to keep the general form so that consistency in \( N_c \) scaling can be verified.

A difference in our interpretation relative to Ref. [12] also helps resolve a paradox of that work. In Ref. [12] it was noted that the \( I_t = J_t \) rule does not hold for meson-baryon scattering in soliton models with SU(3) flavor at leading order in \( N_c \), even for processes with no exchange of strangeness. This is worrying, since as noted in Refs. [1, 2] the \( I_t = J_t \) rule can be derived for such processes directly from large \( N_c \) QCD with no additional model assumptions. The origin of this perplexing discrepancy is that the SU(3) representations used for the baryons of interest in Ref. [12] are those which occur for \( N_c = 3 \) (e.g., the literal 8 and 10). However, as noted in the Introduction, the appropriate representations in a large \( N_c \) world are not these but rather the “8”, “10”, and so on. Strictly speaking the relations derived here hold for the large \( N_c \) world, and thus one expects the \( I_t = J_t \) rule to hold for meson-baryon scattering with strangeness and the baryons in their large \( N_c \) representations. Thus, by using the \( N_c = 3 \) representations, Ref. [12] implicitly includes specific \( 1/N_c \) corrections. However, since this was done with \( N_c \) set to 3, one could not cleanly isolate the numerically small violation of the \( I_t = J_t \) rule as a \( 1/N_c \) correction. Using the proper large \( N_c \) representations and formulae derived here, the \( I_t = J_t \) rule indeed holds.

For phenomenological purposes, the most interesting special case of Eq. (11) is that in which the baryons belong to the parity-positive ground-state “56” multiplet, and the mesons are both pseudoscalar SU(3)-octet pseudo-Nambu-Goldstone bosons. In this case, parity \((P)\) conservation demands that \( L - L' \) is an even integer, and the 9j symbols collapse to 6j symbols. The master scattering amplitude for the process \( \phi(S_\phi = 0, R_\phi = 8, I_\phi, Y_\phi) + B(S_B, R_B, I_B, Y_B) \to \phi'(S_{\phi'} = 0, R_{\phi'} = 8, I_{\phi'}, Y_{\phi'}) + B'(S_{B'}, R_{B'}, I_{B'}, Y_{B'}) \) then reads

\[
S_{LL'SB'SB'}J_sR_s\gamma'_sI_sY_s = (-1)^{S_{B'}-S_{B}}([R_B][R'_B])^{1/2}/[R_s] \sum_{I,I',Y \in 8, I'' \in R_s} (-1)^{I+I'+Y}[I'']
\]
\begin{align}
&\times \left( \begin{array}{cc}
R_B & 8 \\
S_B \frac{N_c}{3} & I_Y \\
& I'' Y + \frac{N_c}{3}
\end{array} \right) \
&\times \left( \begin{array}{cc}
R_{s'} & 8 \\
S_{B'} \frac{N_c}{3} & I'' I_Y \\
& I'' Y + \frac{N_c}{3}
\end{array} \right) \\
&\times \sum_K \left\{ \left[ \begin{array}{ccc}
K & I'' & J_s \\
S_B & L & I
\end{array} \right] \right\} \tau_{KKKLL'}^{\{II'Y\}}.
\end{align}

While these expressions were originally derived within the context of a chiral soliton picture, they are model-independent consequences of QCD in the large $N_c$ limit. By employing crossing relations, one first shows [11] that the conservation of $K$ in $s$-channel processes for the two-flavor case is equivalent to the rule $I_t = J_t$ in the $t$ channel; and as discussed in Sec. II this rule is a direct large $N_c$ consequence [1]. The three-flavor generalization, in turn, is simply an SU(3) rotation of the SU(2) result with no additional dynamics, and therefore is also a model-independent large $N_c$ result.

III. EXPLICIT EXAMPLE

To illustrate the utility of the approach, we apply it to find SU(3) partners of the reported narrow $\Theta^+$ exotic pentaquark resonance. A few caveats are useful before proceeding. First, there is considerable controversy as to whether these states are real; in this work we take an agnostic position. The issue addressed here is that if the resonance is real, then at large $N_c$ and in the SU(3) limit it must have degenerate partners, which at finite $N_c$ would correspond to some nearly degenerate partners; our goal is to enumerate them. Second, the analysis is based on exact SU(3) symmetry. While small SU(3) violations can be accounted for perturbatively, if for some reason large SU(3) violations occur then the present formalism breaks down. For example, it has been suggested by Jaffe and Wilczek [15] in the context of a diquark model that nearly ideal mixing may occur between different SU(3) multiplets (such as for $\phi$ and $\rho$ mesons), which could lead to large SU(3) violations. The present work assumes that such a scenario does not occur; indeed, sound phenomenological arguments [16] oppose it. Third, if the $\Theta^+$ does in fact exist, then apart from its strangeness and isospin we do not know its quantum numbers directly from experiment. Since predictions of SU(3) partners depend upon the quantum numbers, we assume here that the $\Theta^+$ is a spin-$\frac{1}{2}$ isoscalar, which
seems to be the most natural possibility from a theoretical perspective. The parity of the \( \Theta^+ \) is unknown, and various plausible theoretical arguments can be made to suggest either parity; accordingly, we consider both cases. Finally, it remains possible that the \( \Theta^+ \) exists and has different quantum numbers. In such a case one could perform an analysis entirely analogous to the one considered here.

A. “Seed” Amplitudes

We suppose that a pole corresponding to a baryon resonance appears in an \( NK \) partial wave (\( N: S_B = S_{B'} = \frac{1}{2}, P_B = P_{B'} = +, I_B = I_{B'} = \frac{1}{2}, Y_B = Y_{B'} = \frac{N_c}{3}, R_B = R_{B'} = "8" \); \( K: I_\phi = I_{\phi'} = \frac{1}{2}, Y_\phi = Y_{\phi'} = 1, R_\phi = R_{\phi'} = 8 \)) with quantum numbers \( I_s = 0, Y_s = \frac{N_c}{3} + 1, J_s = \frac{1}{2}, R_s = "10" \), and examine the consequences of Eq. (2). The first task is to determine which reduced amplitudes \( \tau \) contribute to partial waves carrying these quantum numbers, and therefore act as “seed” amplitudes to produce poles in other partial waves. As we now show, only \( \tau_{\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2}} \) appears (with \( L = 0, 1 \) for \( P_s = \mp \)), implying that the assumed resonant pole must lie in that reduced amplitude; had several amplitudes arisen, it would have necessary to perform a more delicate analysis to look for degenerate poles in multiple partial waves in order to determine which reduced amplitudes they have in common.

The triangle rules imposed by Eq. (2) are \( \delta(S_BII''), \delta(S_{B'}I'I''), \delta(KI''I_s), \delta(KLI), \delta(KL'I'), \delta(S_BLJ_s), \) and \( \delta(S_{B'}L'J_s) \). Imposing the substitutions listed above, the last two imply \( L = L' = 0 \) or 1 for \( P_s = - \) or +, respectively. If \( L = L' = 0 \), then \( I'' \) equals either \( \pm \frac{1}{2} \) added to the common value \( I = I' = K \). On the other hand, if \( L = L' = 1 \), then satisfying the triangle rules requires that each of \( I, I' \), and \( K \) differ from \( I'' \) by \( \pm \frac{1}{2} \). The sums in Eq. (2) are truncated by the requirement that \( (I, Y) \) and \( (I', Y) \) must be the quantum numbers of states within a literal SU(3) octet, which are \((\frac{1}{2}, 1), (1, 0), (0, 0), \) and \((\frac{1}{2}, -1)\).

The current case is simplified considerably by noting that \( "10" = [0, (N_c+3)/2] \) contains only singly-degenerate states; in particular, one finds using the variables of Eq. (2) that \( 2I'' + Y = 1 \). To each isospin multiplet \( (I, Y) \) or \( (I', Y) \) within the \( 8 \) one therefore identifies a unique value \( I'' = (1-Y)/2 \). The required SU(3) CGC then assume the form

\[
\begin{pmatrix}
\begin{array}{c}
\text{"8"} \\
\frac{1}{2}, \frac{N_c}{3} \\
\frac{1}{2}, \frac{1-Y}{2}, Y + \frac{N_c}{3}
\end{array}
\end{pmatrix}
\begin{pmatrix}
\begin{array}{c}
{I, I'}, Y
\end{array}
\end{pmatrix}
\begin{pmatrix}
\text{"10"}
\end{pmatrix}
\]

(3)

All of these CGC are compiled in Table I of Ref. 3. From this source one readily determines
that the only such coefficient nonvanishing in the large $N_c$ limit has $(\{I, I'\}, Y) = (\frac{1}{2}, 1)$, for which the CGC equals $-1$. But then also $I'' = 0$, which forces not only $I = I' = \frac{1}{2}$ and $Y = 1$, but also $K = \frac{1}{2}$. It follows that the unique reduced amplitude contributing in each of the $L = 0$ and $L = 1$ cases is, as promised, $\tau_{\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}}^{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}}$. 

To complete the simplification of Eq. (2) for this case, we note that $[K] = 2$, the dimension of any SU(3) representation $(p, q)$ assumes the usual value $\frac{1}{2}(p+1)(q+1)(p+q+2)$: $[\text{"8"}] = \frac{1}{4}(N_c+5)(N_c+1)$ and $[\text{"10"}] = \frac{1}{8}(N_c+7)(N_c+5)$, and the 6j symbols give

$$\begin{bmatrix}
\frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & \{0, 1\} & \frac{1}{2}
\end{bmatrix} = \mp \frac{1}{2}.
$$

(4)

In the large $N_c$ limit, one then finds the numerical coefficient of $\tau_{\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}}^{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}}$ to be unity.

B. Degrees of “Exoticness”

Before continuing, it is important to point out that the concept of “exoticness” can be used in three different but related senses here: First, we label the members of the “$\text{56}$” with $S_B > \frac{3}{2}$ as “large-$N_c$ exotic” because their corresponding SU(3) representations $(2S_B, \frac{N_c}{2} - S_B)$ are not allowed for $N_c = 3$. We denote processes in which the initial ground-state baryon is large-$N_c$ exotic by $\mathcal{E}$, nonexotic by $\mathcal{N}$. Second, the product representation in which the baryon resonances appear may be nonexotic ($\mathcal{N}^*$) or exotic in one of two distinct ways: Either it is a perfectly ordinary SU(3) representation that cannot be produced through a $qqq$ state, such as the “$\text{10}$” (denoted by $\mathcal{E}_1^*$), or it may also be large-$N_c$ exotic (denoted by $\mathcal{E}^*_1$). All 6 types of scattering process, $\mathcal{NN}^*$, $\mathcal{NE}_0^*$, $\mathcal{NE}^*_1$, $\mathcal{EN}^*$, $\mathcal{EE}_0^*$, and $\mathcal{EE}^*_1$, may occur in a large $N_c$ world.

The possibility of some of the mixed $\mathcal{N}$-$\mathcal{E}$ combinations may come as a bit of a surprise. As an example of an $\mathcal{NE}_1^*$ process, note that the product “$\text{8}$ $\otimes$ $\text{8}$” contains the large-$N_c$ exotic SU(3) representation $[2, (N_c - 5)/2]$. On the other hand, $\mathcal{EN}^*$ also can occur: The $\frac{5}{2}^+$ ground-state baryon can scatter a pseudoscalar $\text{8}$ meson to give a “$\text{10}$” resonance. For our present purposes, we are interested in $\mathcal{E}_0^*$ processes, both “singly exotic” ($\mathcal{NE}_0^*$) and “doubly exotic” ($\mathcal{EE}_0^*$). That is, we are interested in exotic resonances that lie in SU(3) representations existing at $N_c = 3$, but allow for the possibility that the ground-state baryon representation needed to produce them in scattering with a pseudoscalar $\text{8}$ meson might itself not occur for $N_c = 3$. We make this choice to mirror the terminology of Ref. [13].
One can readily show that there exists an upper limit to ground-state baryon spins $S_B$ in “$56$” allowing $\mathcal{E}_0^*$ processes via scattering with $8$ mesons (beyond which only $\mathcal{E}_1^*$ occurs). As $S_B$ increases, the second row of its SU(3) tableau (length $\frac{N_c}{2} - S_B$) becomes so short that the 3 boxes in the $8$ are insufficient to produce a large-$N_c$ nonexotic resonance. By direct computation, one finds that the amplitudes contribute. The triangle rules imposed by Eq. (2) with 

$$B = \frac{3}{2} = \pm$$

force each of $L$, $L'$ to equal either 0 or 1; and the fact that all baryons in the ground-state “$56$” have $P_B = +$ again forces $L = L'$. Note in particular that this procedure obtains only degenerate partners all carrying the same parity.

1. Negative Parity

We first analyze the case $P_s = -$ case, for which $L = L' = 0$. Then $S_B = S_{B'} = J_s$, so that $R_B = R_{B'}$, and the only remaining nontrivial triangle rule is $\delta(S_B \frac{1}{2} I'')$; Eq. (2) collapses to

$$S_{00}S_{B'}J_sR_s\gamma_s\gamma_s'I_sY_s = \delta R_B R_{B'} \delta S_B S_{B'} \delta S_B J_s \frac{[R_B]}{[R_s]}\frac{[R_B]}{[S_B]} \sum_{I'' \in R_s} \tau_{I''} \sum_{I''} [I''] \times \left( \begin{array}{c|c} \frac{R_B}{S_B} & \frac{8}{2} I'' \frac{N_c}{3} \frac{1}{2} + 1 \frac{R_s}{S_B} \gamma_s \frac{8}{2} I'' \frac{N_c}{3} + 1 \end{array} \right) \left( \begin{array}{c|c} \frac{R_B}{I_B Y_B} & \frac{8}{2} I'' \frac{N_c}{3} \frac{1}{2} + 1 \frac{R_s}{S_B} \gamma_s' \frac{8}{2} I'' \frac{N_c}{3} + 1 \end{array} \right).$$

In order to study only $\mathcal{E}_0^*$ processes, as discussed above we limit $S_B \leq \frac{7}{2}$. The CGC for $S_B = \frac{1}{2} = (R_B = "8")$ and $\frac{3}{2} = (R_B = "10")$ again all appear in Ref. [3]. One finds that the only CGC surviving as $N_c \to \infty$ for $S_B = \frac{1}{2}$ have either $R_s = "10", I'' = 0$ (giving $-1$), or $R_s = "27", I'' = 1 (±1)$. For $S_B = \frac{3}{2}$, we have either $R_s = "27", I'' = 1 (-1)$, or $R_s = "35", I'' = 2 (±1)$. Reference [3] does not compile CGC for $S_B = \frac{5}{2}$ or $\frac{7}{2}$ baryons, but for our purposes
it is only necessary to know that there exist \( O(N_c^0) \) couplings for \( S_B = \frac{5}{2} \) to \( R_s = \text{“35”} \) and \( \text{“28”} \), and for \( S_B = \frac{7}{2} \) to \( R_s = \text{“28”} \) and the \( \mathcal{E}_1^* \) representation \([8, (N_c-5)/2]\). Indeed, for the states of maximal hypercharge in \( R_B \left( \frac{N_c}{2} \right) \), 8 (1+), and \( R_s \left( \frac{N_c}{2} + 1 \right) \) [as required by the first CGC in Eq. (2)], it is straightforward to show that only one SU(3) representation occurs for \( I'' = S_B + \frac{1}{2} \) \((p, q) = (2S_B + 1, \frac{N_c}{2} - S_B + 1)\), and only one occurs for \( I'' = S_B - \frac{1}{2} \) \((p, q) = (2S_B - 1, \frac{N_c}{2} - S_B + 2)\), which are the representations listed above. The CGC in each of these cases must therefore be either +1 or −1.

Collecting these results, one then finds the set of degenerate multiplets \((R_s, J_s^-)\) to be \((\text{“10”}, 1/2), (\text{“27”}, 1/2), (\text{“27”}, 3/2), \) and \((\text{“35”}, 3/2)\) (singly exotic, via \( \mathcal{N} \mathcal{E}_0^* \) processes), and \((\text{“35”}, 5/2), (\text{“28”}, 5/2), \) and \((\text{“28”}, 7/2)\) (doubly exotic, via \( \mathcal{E}_0^* \) processes).

2. Positive Parity

The case \( P_s = + \), for which \( L = L' = 1 \), is only a bit more complicated. Now one may have \( S_B \neq S_{B'} \) and \( R_B \neq R_{B'} \), and \( J_s \) must be separately specified. In this case, Eq. (2) becomes

\[
S_{11SBS_{B'}J_sR_s\gamma_s\gamma_iI_sY_s} = (-1)^{S_B-S_{B'}} 2\left( \frac{|R_B||R_{B'}|}{|R_s|} \right)^{1/2} \sum_{I'' \in R_s} [I''] \times \begin{pmatrix} R_B & 8 & |R_s\gamma_s| \\ S_B & \frac{N_c}{2} & \frac{1}{2} & 1 & I'' & \frac{N_c}{2} & +1 \end{pmatrix} \begin{pmatrix} R_B & 8 & |R_s\gamma_s| \\ I_B Y_B & \tilde{I}_\phi Y_\phi & I_s Y_s \end{pmatrix} \\
\times \begin{pmatrix} R_{B'} & 8 & |R_s\gamma_s'| \\ S_{B'} & \frac{N_c}{2} & \frac{1}{2} & 1 & I'' & \frac{N_c}{2} & +1 \end{pmatrix} \begin{pmatrix} R_{B'} & 8 & |R_s\gamma_s'| \\ I_{B'} Y_{B'} & \tilde{I}_\phi Y_{\phi'} & I_s Y_s \end{pmatrix} \\
\times \begin{pmatrix} \frac{1}{2} & I'' & J_s \\ S_B & 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & I'' & J_s \\ S_{B'} & 1 & \frac{1}{2} \end{pmatrix} .
\]

(6)

Note that precisely the same set of SU(3) CGC are relevant to the case \( P_s = + \), meaning that the enumeration of SU(3) representations carries over verbatim from the case \( P_s = - \); only the angular momenta need be considered more carefully. The remaining independent triangle rules imposed by the 6j symbols are \( \delta(S_B \frac{1}{2} I''), \delta(S_{B'} \frac{1}{2} I''), \) and \( \delta(\frac{1}{2} I'' J_s) \). One then finds the following combinations. From \( S_B = \frac{1}{2}, R_B = \text{“8”} \): \( I'' = 0 \to R_s = \text{“10”}, J_s = \frac{1}{2} \) and \( I'' = 1 \to R_s = \text{“27”}, J_s = \frac{1}{2}, \frac{3}{2} \). From \( S_B = \frac{3}{2}, R_B = \text{“10”} \): \( I'' = 1 \to R_s = \text{“27”}, J_s = \frac{1}{2}, \frac{3}{2} \) and \( I'' = 2 \to R_s = \text{“35”}, J_s = \frac{3}{2}, \frac{5}{2} \). From \( S_B = \frac{5}{2}, R_B = [5, (N_c-5)/2] \): \( I'' = 2 \to R_s = \text{“35”}, J_s = \frac{3}{2}, \frac{5}{2} \) and \( I'' = 3 \to R_s = \text{“28”}, J_s = \frac{5}{2}, \frac{7}{2} \). And from \( S_B = \frac{7}{2}, R_B = [7, (N_c-7)/2] \): \( I'' = 3 \to R_s = \text{“28”}, J_s = \frac{5}{2}, \frac{7}{2} \) and \( I'' = 4 \to R_s = [8, (N_c-5)/2], J_s = \frac{7}{2}, \frac{9}{2} \).
Collecting these results, one then finds the set of degenerate multiplets \((R_s, J^+_s)\) to be \(\left(\{\overline{10}\}, \frac{1}{2}^+\right), \left(\{27\}, \frac{1}{2}^+\right), \left(\{27\}, \frac{3}{2}^+\right), \left(\{35\}, \frac{3}{2}^+\right)\), and \(\left(\{35\}, \frac{5}{2}^+\right)\) (singly exotic, via \(\mathcal{NE}_0^*\) processes), and \(\left(\{28\}, \frac{5}{2}^+\right)\), and \(\left(\{28\}, \frac{7}{2}^+\right)\) (doubly exotic, via \(\mathcal{EE}_0^*\) processes). As promised, these multiplets precisely match those obtained in Ref. \([13]\) via counting using Young tableaux, once a consistent definition of degree of exoticness is included.

**IV. CONCLUSIONS**

We have generalized to three flavors the two-flavor large \(N_c\) meson-baryon scattering method that relates different scattering partial waves. In particular, resonant poles occurring in one such amplitude appear in others, creating multiplets of resonances degenerate in both mass and width at leading order in the \(1/N_c\) expansion limit. We illustrated the method by finding the partners of a resonance carrying the quantum numbers suggested by the purported \(\Theta^+\)(1540) particle, and found that the results (for \(J^P = \frac{1}{2}^+\) agree with those obtained using a large \(N_c\) method that does not recognize the instability of the resonances.

One may immediately apply this formalism to numerous other problems involving three-flavor baryon resonances. For example, in it has been shown \([1]\) that the suppression of the \(N(1535)\) \(\pi N\) partial width is due to the fact that the reduced amplitude appearing in the \(S_{11}\) partial wave couples in the large \(N_c\) limit does not couple to spinless isovector mesons. What interesting analogous consequences arise for the \(\Lambda\) and \(\Xi\) resonances?

Finally, the \(I_t = J_t\) rule was used \([1]\) to parametrize \(1/N_c\) corrections to the leading-order large \(N_c\) results, an absolute must for useful phenomenological studies. As discussed above, the crossing constraint for three flavors cannot be described so simply. We defer to future work the description and application of this important concept.

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