Characterizations of Certain Doubly Truncated Distribution Based on Order Statistics
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Introduction

The order statistics arise naturally in many real life applications and it is considered as an increasingly important subject. Articles relating to this area have appeared in numerous different publications. Many authors have studied order statistics; for example, David [1], Balakrishnan and Cohen [2], Arnold et al. [3], David [4], David and Nagaraja [5] and Mahmoud et al. [6,7]. Several authors discussed conditional expectations, for example, Balakrishnan and Sultan [8], Mohie El-Din et al. [9], Abu-Yousef [10], Abd-El-Mougod [11], Shawky and Abu-Zinadah [12], Shawky and Bakoban [13] and Pushkarna et al. [14].

Let \( X_{(r)} \leq X_{(s)} \leq \ldots \leq X_{(n)} \) be the first n order statistics based on distribution with probability density function (pdf) \( f(x) \) and cumulative distribution function (cdf) \( F(x) \). Then the pdf of the \( r \)th order statistics, \( X_{(r)} \), \( 1 \leq r \leq n \), is given by (see David (1981))

\[
f_r(x) = nf_r(x) = \frac{n!}{(r-1)!(n-r)!} F^{-1}(r)\left(F^{-1}(r+1)\right)^{r-1} f(x), \tag{1.1}
\]

and the joint pdf of two order statistics \( X_{(r)} \) and \( X_{(s)} \), \( 1 \leq r < s \leq n \), is given by

\[
f_{r,s}(x,y) = C_{r,s} f_r(x) f_s(y) \left(\frac{1}{F(x)}\right)^{r-1} \left(\frac{1}{F(y)}\right)^{s-1} f(x) f(y), \quad -\infty < x < y < \infty, \tag{1.2}
\]

where

\[
C_{r,s} = \binom{n}{r,s} = \frac{n!}{(r-1)!(s-1)!(n-r-s)!}.
\]

Main Results

In this section, we characterize three general classes of distributions,

\[
F(x) = 1 - \frac{1}{\beta} e^{-\alpha x}, \quad \alpha < x < \beta, \tag{1.3}
\]

where \( \beta = Q - P \), \( G(x) = 0 \) and \( G(y) = 1 \).

The conditional density function of \( X_{(r)} \) given that \( X_{(r)} = x \) is given by

\[
\frac{f_{X_{(r)}}(y \mid X_{(r)} = x)}{f_X(x)} = \frac{(n-r)!}{(s-r-1)!((n-s-1))\left(1-\frac{G(x)}{G(y)}\right)^{n-r-s}} G(y), \quad \varepsilon < x < y < \gamma. \tag{1.4}
\]

The conditional density function of \( X_{(r)} \) given that \( X_{(s)} = y \) is given by

\[
\frac{f_{X_{(r)}}(x \mid X_{(s)} = y)}{f_X(y)} = \frac{s!}{(n-s)!}\frac{(s-r)!}{(s-r-1)!\left(1-\frac{G(y)}{G(x)}\right)^{s-r-1}} G(x), \quad \varepsilon < x < y < \gamma. \tag{1.5}
\]

Also, the conditional density function of \( X_{(r)} \) given that \( X_{(s)} = y \) is given by

\[
\frac{f_{X_{(r)}}(x \mid X_{(s)} = y)}{f_X(y)} = \frac{(s-r)!}{(s-r-1)!\left(1-\frac{G(y)}{G(x)}\right)^{s-r-1}} G(x), \quad \varepsilon < x < y < \gamma. \tag{1.6}
\]

Let

\[
\mu_{SP} = E[\varphi(X_{(r)}) \mid X_{(s)} = y] \quad \text{and} \quad \mu_{FP} = E[\varphi(X_{(r)}) \mid X_{(s)} = y],
\]

where \( \varphi(.) \) is a monotonic, continuous and differentiable function on the interval \((\alpha, \beta)\). For abbreviation, we will denote

\[
\mu_{SP} = E[\varphi(X_{(r)}) \mid X_{(s)} = y] \quad \text{and} \quad \mu_{FP} = E[\varphi(X_{(r)}) \mid X_{(s)} = y]. \tag{1.7}
\]

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\[ G(x) = \frac{1}{L} \left[ b - a e^{-G(x)} \right] - v], e < x < y, \]  
where
\[ v = 1 - P, l = P - Q, P = F(e), Q = F(y), G(e) = 0, G(y) = 1, \]
\[ F(a) = 0, F(b) = 1, \]
a, b are finite constants.
\[ F(x) = 1 - [a - b \Theta(x)], a < x < b, \text{i.e.}, \]
\[ G(x) = \frac{1}{l} \left[ b - a e^{-G(x)} \right] - v], e < x < y, \]
where
\[ v = 1 - P, l = P - Q, P = F(e), Q = F(y), G(e) = 0, G(y) = 1, \]
\[ F(a) = 0, F(b) = 1, \]
b, c, e, a are finite constants.
\[ F(x) = 1 - [b - a e^{-G(x)}], a < x < b, \text{i.e.}, \]
\[ G(x) = \frac{1}{l} \left[ b - a e^{-G(x)} \right] - v], e < x < y, \]
where
\[ v = 1 - P, l = P - Q, P = F(e), Q = F(y), G(e) = 0, G(y) = 1, \]
\[ F(a) = 0, F(b) = 1, \]
and (2.12), after some simplifying (2.9), we get (2.4). Thus, the theorem is proved.

Theorem 2

Referring to (1.6), (1.7), then (2.1) if and only if
\[ \mu_{P+1} + \frac{1}{ac} \sum_{i=0}^{\infty} \left( -1 \right)^{i+1} \left( \frac{1}{v} \right)^{i+1} \]
\[ \left[ \frac{G(y)}{v + i + 1} \right]^{i+1} \left[ \frac{b e^{G(x)}}{v + i + 1} \right] \left[ e^{v(x) - a} \right] \left[ X_{r+i+1} = y - a \right]. \]

Proof

It is clear that
\[ \mu_{P+1} = \frac{\phi(y)}{[G(y)]^{i+1}} g(x) dx. \]

Integrating by parts, we get
\[ \mu_{P+1} = \phi(y) - \frac{1}{[G(y)]^{i+1}} \left[ \frac{g(x)}{G(x)} \right]^{i+1} \phi(x) [G(x)]^{i} dx. \]

Compensation for (2.8) in (2.11), we have
\[ \mu_{P+1} = \phi(y) - \frac{1}{ac(G(y))^{i+1}} \left[ \frac{g(x)}{G(x)} \right]^{i+1} \left[ \frac{b e^{G(x)}}{v + G(x)} \right] [G(x)]^{i} dx. \]

Expand \[ \frac{r}{G(y)} \phi(y) \] and compensation for (2.12), after some simplification, we get (2.10). Thus (2.1) implies (2.10). Now from (1.6) and (2.10), we obtain
\[ \phi(y) + \sum_{i=0}^{\infty} \left( -1 \right)^{i} \left( \frac{1}{v} \right)^{i+1} \left[ G(y) \right]^{i+1} \left[ b e^{G(x)} \right] \left[ e^{v(x) - a} \right] g(x) dx. \]

Taking the derivative, we get
\[ \phi(y) = \frac{\left( \frac{1}{v} \right) g(y)}{1 + \frac{b}{ac} e^{G(y)} - \frac{1}{e}}, \]

which gives
\[ \frac{ac e^{G(y)} - e^{G(y)}}{b - ac e^{G(y)}} = \frac{lg(y)}{v + G(y)} \]  
(2.14)

Integrate (2.14), hence G(y) has the form (2.1), and so (2.10) implies (2.1).

Special case:

Return to the (2.10), if we put \( l = -1, v = 1 \) we get
\[ \mu_{P+1} = \phi(y) + \frac{1}{ac} \sum_{i=0}^{\infty} \left( -1 \right)^{i+1} \left( \frac{1}{v} \right)^{i+1} \left[ b e^{G(x)} \right] \left[ e^{v(x) - a} \right] \left[ X_{r+i+2} = y - a \right], \]
\[ \alpha < x < y < \beta \]  
(2.15)

the relation (2.15) is before doubly truncated case.

Theorem 3

Referring to (1.5), (1.7) and (2.1), then
\[ \mu_{P+1} = \phi(y) + \frac{1}{ac(n - s + 1)} \left[ b e^{G(x)} \right] \left[ e^{v(x) - a} \right] \left[ X_{r+i+2} = y - a \right], \]
\[ a e^{G(x)} N(x, x) X_{r+i+2} = x - b e^{G(x)} N(x, x) X_{r+i+2} = x \]  
(2.16)
where
\[ N(y) = \frac{[1-G(y)]}{[G(y)+v]} \]  
(2.17)

It is clear from (1.5) and (1.7) that
\[ \mu_{1,*} = \frac{(n-r)!}{(s-r-1)(n-s)!} \int_{y} \phi(y)(G(y) - G(x))^{r-s-1} [1-G(x)]^{s-r-1} g(x) dy. \]
Integrating by parts, we get
\[ \mu'_{1,*} = \frac{(n-r)!}{(s-r-1)(n-s)!} \int_{y} \phi(y)g(y)(1-G(x))^{s-r-1} dy. \]
(2.18)

Substituting (2.7) in (2.18), we get
\[ \mu'_{1,*} = \frac{(n-r)!}{(s-r-1)(n-s)!} \int_{y} \phi(y)g(y)(1-G(x))^{s-r-1} \left[ \frac{a}{[G(x)+v]} - \frac{b}{[1-G(x)]^r} \right] dx. \]
(2.20)

After some simplification, we get (2.16).

**Theorem 4**

Referring to (1.5), (1.7), then (2.1) if and only if
\[ \mu'_{1,*} = \phi(x) + \frac{b}{ac(n-r)} E_{\nu} \left[ N(y) \right] X_{\nu} = x \]
(2.20)

where \( N(y) \) is defined in (2.17).

**Proof**

It is clear that
\[ \mu_{1,*} = \frac{(n-r)!}{(s-r-1)(n-s)!} \int_{y} \phi(y)(G(y) - G(x))^{r-s-1} [1-G(x)]^{s-r-1} g(x) dy. \]
Integrating by parts, we get
\[ \mu'_{1,*} = \phi(x) + \frac{1}{ac(n-r)} \int_{y} \phi(y)(1-G(x))^{s-r-1} dx. \]
(2.21)

Compensation for (2.8) in (2.22), we have
\[ \mu_{1,*} = \phi(x) + \frac{b}{ac(n-r)} E_{\nu} \left[ N(y) X_{\nu} = x \right] \]
(2.22)

Simplifying (2.23), we obtain (2.20). Thus (2.1) implies (2.20), i.e. the necessary condition is proved. To prove the sufficient condition, from (2.20) and (1.7), we have
\[ \mu_{1,*} = \frac{(n-r)!}{(s-r-1)(n-s)!} \int_{y} \phi(y)(1-G(x))^{r-s-1} g(x) dy = \frac{b}{ac(n-r)} E_{\nu} \left[ N(y) X_{\nu} = x \right]. \]
(2.24)

Taking the derivative of (2.24) with respect to \( x \), we get (2.8), and integrate it we have (2.1), thus (2.20) implies (2.1). Then, the Theorem is proved.

**Special case**

Return to (2.17), if we put \( l=1, v=1 \) we get
\[ \mu_{1,*} = \phi(x) + \frac{b}{ac(n-r)} E_{\nu} \left[ N(y) X_{\nu} = x \right] + \frac{1}{c(n-r)}, \]
(2.26)

it is before doubly truncated case (Table 1).

**Theorem 5**

Referring to (1.6), (1.7) and (2.2), then
\[ \mu'_{1,*} = \phi(x) + \frac{b}{ac(n-r)} E_{\nu} \left[ N(y) X_{\nu} = x \right] - \frac{b}{ac(n-r)} E_{\nu} \left[ N(y) X_{\nu} = x \right], \]
(2.25)

where \( V(x) \) is defined in (2.5).

**Proof**

As before in Theorem (1), differentiate (2.2) with respect to \( x \), we have
\[ \phi(x) = \frac{b \psi(x) - a}{b c(x) + a c(x)}. \]
(2.26)

Compensation for (2.26) in (2.7), we get
\[ \mu'_{1,*} = \mu_{1,*} - \frac{b}{ac(n-r)} E_{\nu} \left[ (l(y)-v) X_{\nu} = x \right] + \frac{b}{ac(n-r)} E_{\nu} \left[ (l(y)-v) X_{\nu} = x \right]. \]
(2.27)

Simplifying (2.27), we obtain (2.25). Thus, the Theorem is proved.

### Table 1: Example of \( G(x) = \frac{1}{2} [1 - ae^{-cx}]^{-v} \) distributions.

| Name          | \( [1G(x)+v] \) | \( \phi(x) \) | \( \nu \) | \( [a,b,c] \) |
|---------------|-----------------|---------------|---------|---------------|
| Weibull       | \( e^{-ax} \), \( a \leq x < \gamma \leq \beta \) | \( G(x) \) | \( (c,-0,0) \) | \( (-1,0,0) \) |
| Pareto        | \( \theta x^\gamma \), \( 0 \leq x < \gamma \leq \beta \) | \( \ln(x) \) | \( \theta(\gamma-1), \theta^\gamma \) | \( (-\theta,0,0) \) |
| Power function | \( 1-\theta \psi, \theta \leq x < \gamma \leq \beta \) | \( \ln(x) \) | \( \theta^\gamma, \theta^{\gamma-1} \) | \( (1,1,1) \) |
| Rayleigh      | \( e^{\lambda x^\gamma} \), \( a \leq x < \gamma < \beta \) | \( x^\beta \) | \( e^{\lambda x^\gamma} + e^{\lambda x^\gamma} \) | \( (-1,0,0) \) |
| Inverse Weibull | \( e^{-\lambda x^\gamma} \), \( a \leq x < \gamma < \beta \) | \( x^\beta \) | \( e^{\lambda x^\gamma} + e^{\lambda x^\gamma} \) | \( (-1,0,1) \) |
Theorem 6
Referring to (1.6), (1.7), then (2.2) if and only if
\[
\mu_{k} = \phi(k) - \frac{1}{bc} \int_{bc}^{x} \frac{bG(x)}{G(x) + v} \left[ g(x) \right]^{-1} g(x) \, dx
\]
(2.28)
Where \( V(x) \) is defined in (2.5).

Proof
As before in Theorem (2), from (2.26) and (2.11), we have
\[
\mu_{k} = \phi(k) - \frac{1}{bc} \int_{bc}^{x} \frac{bG(x)}{G(x) + v} \left[ g(x) \right]^{-1} g(x) \, dx
\]
(2.29)
Therefore, we get (2.28), then (2.2) implies (2.28). To prove the sufficient condition, from (2.28) and (1.7), we obtain
\[
\mu_{k} = \phi(k) - \frac{1}{bc} \int_{bc}^{x} \left[ bG(x) \right]^{-1} g(x) \, dx
\]
(2.30)
Taking the derivative, we get (2.26) and we obtain, after integration, (2.2). Thus (2.28) implies (2.2).

Special case
Return to (2.28), then put \( l = 1, v = 1 \), we get
\[
\mu_{k} = \phi(k) - \frac{1}{bc} \int_{bc}^{x} \left[ bG(x) \right]^{-1} g(x) \, dx
\]
(2.31)
where \( N(y) \) is defined in (2.17).

Theorem 7
Referring to (1.5), (1.7) and (2.2), then
\[
\mu_{k} = \mu_{k-1} - \frac{l}{bc(n-k+1)} \left\{ bE_{k} \left[ \phi(X_{n-x}) \{ N(X_{n-x}) = x \} \right] - aE_{k} \left[ N(X_{n-x}) = x \} \right] \right\}
\]
(2.32)
where \( N(y) \) is defined in (2.17).

Proof
As before in Theorem (3), compensation for (2.26) in (2.18), we have
\[
\mu_{k} = \mu_{k-1} - \frac{l}{bc(n-k+1)} \left\{ bE_{k} \left[ \phi(X_{n-x}) \{ N(X_{n-x}) = x \} \right] - aE_{k} \left[ N(X_{n-x}) = x \} \right] \right\}
\]
(2.33)
After simplification, we get (2.31). Then (2.2) implies (2.31).

Table 2: Example of \( G(x) = \frac{1}{\gamma} \left[ b + \alpha \phi(x) \right]^{-\gamma} \) distributions.

| Name             | \( \Phi(x) \)                          | \( L(V) \)                          | \( \alpha, \beta, \gamma \) |
|------------------|---------------------------------------|------------------------------------|----------------------------|
| Weibull          | \( e^{(-\beta x)^{\gamma} - e^{-\alpha x^{\gamma}}} \) | \( e^{-\beta x} \)                 | (-1,0,0)                   |
| Power function   | \( 1 - \theta^{x}, \text{if} \ 0 \leq x < \theta \leq \beta, \theta \neq 0 \) | \( \frac{x^{\gamma}}{\theta^{\gamma}} \) | \( (1,1,1) \) (\( \theta > 1 \)) |
| Rayleigh         | \( e^{-\gamma x^{\gamma}}, \alpha \leq x < y \leq \beta, e = 0, y \rightarrow \infty \) | \( e^{-\theta x} \) | \( (-1,0) \) |
| Inverse Weibull  | \( e^{(-\gamma x^{\gamma})^{\gamma} + \alpha e^{-\gamma x^{\gamma}}}, e < x \leq \beta, e = 0, y \rightarrow \infty \) | \( e^{-\theta x^{\gamma}} \) | \( (0,-1,1) \) |
Now from (1.7) and (2.36), we obtain

\[ (2.36) \]

which gives (2.36). Thus, (2.3) implies (2.36).

### Theorem 10

Referring to (1.6) and (1.7), then (2.3) if and only if

\[ (2.37) \]

Proof

As given previously in Theorems (2) and (6), substituting from (2.26) in (2.11), we have

\[ (2.38) \]

After some simplification, we get (2.36). Then (2.3) implies (2.36). Now from (1.7) and (2.36), we obtain

\[ (2.39) \]

Taking the derivative with respect to \( y \), we get

\[ (2.40) \]

Integrate (2.41), we obtain (2.3).

### Special case

Return to (2.36), then put \( l=1 \), \( v=1 \), we get

\[ (2.41) \]

it is before doubly truncated case.

### Theorem 11

Referring to (1.5), (1.7) and (2.3), then

\[ (2.42) \]

Proof

As previously in Theorems (3) and (7), from (2.37) in (2.18), we have
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