THE LAX INTEGRABILITY OF A TWO-COMPONENT HIERARCHY OF THE BURGERS TYPE DYNAMICAL SYSTEMS WITHIN ASYMPTOTIC AND DIFFERENTIAL-ALGEBRAIC APPROACHES

DENIS L. BLACKMORE, ANATOLIJ K. PRYKARPATSKI, EMIN ÖZÇAĞ, AND KAMAL SOLTANOV

Abstract. The Lax type integrability of a two-component polynomial Burgers type dynamical system within a differential-algebraic approach is studied, its linear adjoint matrix Lax representation is constructed. A related recursion operator and infinite hierarchy of Lax integrable nonlinear dynamical systems of the Burgers-Korteweg-de Vries type are derived by means of the gradient-holonomic technique, the corresponding Lax type representations are presented.

1. Introduction

Recently a great deal of research articles [32, 13, 18, 19, 29] were devoted to classification of polynomial integrable dynamical systems on smooth functional manifolds. In particular, in the articles [32] there was presented a wide enough list of two-component polynomial dynamical systems of Burgers and Korteweg-de Vries type, which either reduce by means of some, in general nonlocal, change of variables to the respectively separable triangle Lax type integrable forms or transform to the completely linearizable flows. Amongst these systems the authors of [32] singled out the following two-component Burgers type dynamical system

\begin{equation}
\begin{aligned}
  u_t &= u_{xx} + 2uu_x + v_x \\
  v_t &= u_xv + uv_x
\end{aligned}
\end{equation}

on a smooth of the Schwartz type functional manifold \( M \subset C^\infty(\mathbb{R};\mathbb{R}^2) \), where \((u,v)^T \in M\), the subscripts "\( x \)" and "\( t \)" denote, respectively, the partial derivatives with respect to the variables \( x \in \mathbb{R} \) and \( t \in \mathbb{R}_+ \), the latter being the evolution parameter. Within this work we will assume that the dynamical system \((1.1)\) possesses smooth enough solutions for an evolution parameter \( t \in \mathbb{R}_+ \), as it follows from the standard functional-analytic compactness principle considerations from [10].

It is mentioned in [32] (p. 7706) that the Burgers type dynamical system \((1.1)\) was before extensively studied in [5, 17], where "... the symmetry integrability of \((1.1)\) as well the existence of a recursion operator has already been demonstrated..." for it. The dynamical system \((1.1)\) appears to have interesting applications, as its long-wave limit reduces to the well-known Leroux system [6, 33], describing dynamical processes in two-component hydro- and lattice gas dynamics. As there is claimed in [32] (p. 7726), by now the integrability of \((1.1)\) remains still unproven, and having found no new result devoted to this problem available in literature, we undertaken this challenge to close it by means of the gradient-holonomic [26, 1], linear adjoint mapping [25] approaches and recently devised [22, 23] differential algebraic integrability testing tools. As a general result we have proved the following theorem.

Theorem 1.1. The two-component polynomial Burgers type dynamical system \((1.1)\) possesses only two local conserved quantities \( \int dxu \) and \( \int dxv \) and no other infinite affine ordered local conserved quantities. Moreover, on the functional manifold \( M \) the Burgers type

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dynamical system (1.1) is linearizable by means of a Hopf-Cole type transformation and a dual adjoint mapping to the matrix Lax type representation

\[
D_x \left( \frac{f}{\dot{f}} \right) = \begin{pmatrix}
(u + \lambda^{-1}v - \lambda)/2 & 0 \\
\lambda^{-1}v - u)/2 & 1
\end{pmatrix} \begin{pmatrix} f \\ \dot{f} \end{pmatrix},
\]
and

\[
D_t \left( \frac{f}{\dot{f}} \right) = \begin{pmatrix}
D_x(u + \lambda)|/2 + 
\lambda^{-1}v - \lambda)/2 & 0 \\
-(u + \lambda)|/2 + 
\lambda^{-1}v - \lambda)/2 & (u + \lambda)
\end{pmatrix} \begin{pmatrix} f \\ \dot{f} \end{pmatrix},
\]

compatible for all \( \lambda \in \mathbb{C} \setminus \{0\} \) with \((f, \dot{f})^T \in \Lambda^0(\mathcal{K}\{u, v; D_x^{-1}\sigma[N]\})^2 \), where \(\mathcal{K}\{u, v; D_x^{-1}\sigma[N]\}\) denotes some \([34]\) finitely extended differential ring \(\mathcal{K}\{u, v\}\). The related with the Lax operator (1.2) infinite hierarchy of generalized Burgers type dynamical systems allows the following compact representation:

\[
D_{t_n}(u + \lambda^{-1}v - \lambda) = D_x[D_x \alpha_n(x; \lambda) + (u + \lambda^{-1}v - \lambda)\alpha_n(x; \lambda)],
\]
where the evolution parameters \(t_n \in \mathbb{R}_+\) and, by definition,

\[
\alpha_n(x; \lambda) := (\lambda^n \alpha_n(x; \lambda))_+ \quad \text{for all natural } n \in \mathbb{N}
\]

is the corresponding nonnegative degree polynomial part generated by the asymptotic local functional solution \(\alpha_n(x; \lambda) \sim \sum_{j \in \mathbb{Z}_+} \lambda^{-j} \alpha_j[u, v] \) as \(|\lambda| \to \infty\) to the differential functional equation

\[
D_x^2 \alpha_n(x; \lambda) + D_x((u + \lambda^{-1}v - \lambda)\alpha_n(x; \lambda)) = 0.
\]

As a simple consequence of the Theorem (1.1) one finds that the Burgers type dynamical system (1.1) does not allow on the functional manifold \(M\) a Hamiltonian formulation, and the corresponding recursion operator

\[
\Phi := \begin{pmatrix}
D_x + D_x u D_x^{-1} & 1 \\
D_x v D_x^{-1} & 0
\end{pmatrix},
\]

satisfying the determining commutator equation

\[
D_t \Phi = [K', \Phi],
\]
for the two-component Burgers type dynamical system (1.1) and found before in \([5, 17]\), proves to be not factorizable by means of compatible Poissonian structures, as they on the whole do not exist.

The scalar Lax type representation (1.2) can be reduced by means of the nonlocal change of variables \(\tilde{g} = f^2 \exp[\lambda^2 t - D_x^{-1}(\lambda^{-1} v - \lambda)] \in \Lambda^0(\mathcal{K}\{u, v; D_x^{-1}\sigma[N]\})\) to a more simpler linear form

\[
D_{t} \tilde{g} = D_{xx} \tilde{g} + v \tilde{g} = 0, \quad D_{x} \tilde{g} = u \tilde{g},
\]
compatible for \(\tilde{g} \in \Lambda^0(\mathcal{K}\{u, v; D_x^{-1}\sigma[N]\})\) and not depending on the parameter \(\lambda \in \mathbb{C} \setminus \{0\}\). The above representation (1.10) can be easily generalized to the following higher-order evolution equation case:

\[
D_{t} \tilde{g} = D_{xx} \tilde{g} + v \tilde{g} = 0, \quad D_{x} \tilde{g} = u \tilde{g},
\]

where \(n \in \mathbb{Z}_+\). Making use of the mentioned above nonlocal change of variables \(\hat{g} = \tilde{g}\exp(\lambda^{-2} D_x^{-1}v) \in \Lambda^0(\mathcal{K}\{u, v; D_x^{-1}\sigma[N]\})\), one can obtain a new infinite hierarchy of two-component Lax type integrable polynomial Burgers type dynamical systems, generalizing those discussed before in \([31, 32]\). For instance, at \(n = 3\) we find the following dynamical Burgers type dynamical system of the third order:

\[
u_t = (u_{xx} + 3 (u u_x)_x + 3u^2 u_x + v, \quad v_t = (u v, u v)_{xx},
\]

where \(r : J[u, v] \to C^\infty(\mathbb{R}^2; \mathbb{R}^2)\) is a polynomial mapping on the jet-space \(J(\mathbb{R}^2; \mathbb{R}^2)\) of elements \((x, t; u, v, D_x u, D_x v, D_t u, D_t v, D_x^2 u, D_x^2 v, \ldots) \in J(\mathbb{R}^2; \mathbb{R}^2)\), suitably determined by the relationship (1.5) at \(n = 2\). Its scalar Lax type representation easily obtains, respectively,
either from (1.2), (1.3) or from (1.0). The latter at n = 3 easily gives rise to the scalar
Lax type representation
\[ D_x g = \left( u + \lambda^{-1}v \right) \hat{g}, \]
\[ D_t \hat{g} = D_{xx} \hat{g} + 3\lambda^{-1} v D_{xx} \hat{g} + 3\left( v u / \lambda + v^2 / \lambda^2 \right) D_x \hat{g} + \]
\[ + (v_{xx} / \lambda + 3v v_{x} / \lambda^2 + v^3 / \lambda^3 - (u \eta / \lambda v) / \lambda) \hat{g} = 0, \]
compatible for all \( \lambda \in \mathbb{C} \setminus \{0\} \) and \( \hat{g} \in \Lambda^0(\mathbb{K}\{u, v, D_x^{-1} \sigma [N] \})^2 \), which can be suitably extended by means of the related adjoint mapping to the matrix representation.

2. Differential-algebraic preliminaries

As our consideration of the integrability problem, discussed above, will be based on some differential-algebraic techniques, to be for further more precise, we need to involve here some additional differential-algebraic preliminaries [11, 8, 9, 10, 12, 11].

Take the ring \( \mathcal{K} := \mathbb{R} \{ t \} \setminus \mathbb{R} \times (0, T) \), of convergent germs of real-valued smooth functions from \( C^\infty(\mathbb{R}^2; \mathbb{R}) \) and construct the associated differential quotient ring \( \mathcal{K}\{u, v\} := \text{Quot}(\mathcal{K}[\Theta u, \Theta v]) \) with respect to two functional variables \( u, v \in \mathcal{K} \), where \( \Theta \) denotes [14, 27, 7, 8, 11] the standard monoid of all commuting differentiations \( D_x \) and \( D_t \), satisfying the standard Leibniz condition, and defined by the natural conditions
\[ (2.1a) \quad D_x(x) = 1 = D_t(t), \quad D_t(x) = 0 = D_x(t), \]
The ideal \( I\{u, v\} \subset \mathcal{K}\{u, v\} \) is called differential if the condition \( I\{u, v\} = \Theta I\{u, v\} \) holds. In the differential ring \( \mathcal{K}\{u, v\} \), interpreted as an invariant differential ideal in \( \mathcal{K} \), there are two naturally defined differentiations
\[ (2.2) \quad D_t, \quad D_x : \mathcal{K}\{u, v\} \to \mathcal{K}\{u, v\}, \]
satisfying the commuting relationship
\[ (2.3) \quad [D_t, D_x] = 0. \]
Consider the ring \( \mathcal{K}\{u, v\}, u, v \in \mathcal{K} \), and the exterior differentiation \( d: \mathcal{K}\{u, v\} \to \Lambda^1(\mathcal{K}\{u, v\}) \): \( d: \Lambda^p(\mathcal{K}\{u, v\}) \to \Lambda^{p+1}(\mathcal{K}\{u, v\}) \) for \( p \in \mathbb{Z}_+ \), acting in the freely generated Grassmann algebras \( \Lambda(\mathcal{K}\{u, v\}) = \oplus_{p \in \mathbb{Z}_+} \Lambda^p(\mathcal{K}\{u, v\}) \) over the field \( \mathbb{C} \), where by definition,
\[ (2.4) \quad \Lambda^1(\mathcal{K}\{u, v\}) := \mathcal{K}\{u, v\} dx + \mathcal{K}\{u, v\} dt + \sum_{j, k \in \mathbb{Z}_+} \mathcal{K}\{u, v\} dA_{(j, k)} + \sum_{j, k \in \mathbb{Z}_+} \mathcal{K}\{u, v\} dA_{(j, k)}, \]
The triple \( \mathcal{A} := (\mathcal{K}\{u, v\}, \Lambda(\mathcal{K}\{u, v\}); d) \) will be called the Grassmann differential algebra with generatrices \( u, v \in \mathcal{K} \). In the algebra \( \mathcal{A} \), generated by \( u, v, \in \mathcal{K} \), one naturally defines the action of differentiations \( D_t, D_x \) and \( \partial / \partial u_{(j, k)}, \partial / \partial v_{(j, k)}: \mathcal{A} \to \mathcal{A}, j, k \in \mathbb{Z}_+ \), as follows:
\[ (2.5) \quad D_t u_{(j, k)} = u_{(j+1, k)}, \quad D_x u_{(j, k)} = u_{(j, k+1)}, \]
\[ D_t v_{(j, k)} = v_{(j+1, k)}, \quad D_x v_{(j, k)} = v_{(j, k+1)}, \]
\[ D_t du_{(j, k)} = du_{(j+1, k)}, \quad D_x du_{(j, k)} = du_{(j, k+1)}, \]
\[ D_t dv_{(j, k)} = dv_{(j+1, k)}, \quad D_x dv_{(j, k)} = dv_{(j, k+1)}, \]
\[ dP[u, v] = \sum_{j, k \in \mathbb{Z}_+} dA_{(j, k)} \wedge \partial P[u, v] / \partial u_{(j, k)} + \sum_{j, k \in \mathbb{Z}_+} dA_{(j, k)} \wedge \partial P[u, v] / \partial v_{(j, k)} = \]
\[ = \sum_{j, k \in \mathbb{Z}_+} (\pm) \partial P[u, v] / \partial u_{(j, k)} \wedge dA_{(j, k)} + \sum_{j, k \in \mathbb{Z}_+} (\pm) \partial P[u, v] / \partial v_{(j, k)} \wedge dA_{(j, k)} := < P'[u, v], \Lambda(du, dv) > \wedge R^2, \]
where the sign " \( \wedge " \) denotes the standard [12] exterior multiplication in \( \Lambda(\mathcal{K}\{u, v\}) \), and for any \( P[u, v] \in \Lambda(\mathcal{K}\{u, v\}) \) the mapping
\[ (2.6) \quad P[u, v] : \Lambda^0(\mathcal{K}\{u, v\})^2 \to \Lambda(\mathcal{K}\{u, v\}), \]
is linear. Moreover, the commutation relationships
\[ (2.7) \quad D_x d = d D_x, \quad D_t d = d D_t \]
hold in the Grassmann differential algebra \( \mathcal{A} \). The following remark [11] is also important.
Remark 2.1. Any Lie derivative $L_V : \mathcal{K}\{u,v\} \to \mathcal{K}\{u,v\}$, satisfying the condition $L_V : \mathcal{K} \subseteq \mathcal{K}$, can be uniquely extended to the differentiation $L_V : \mathcal{A} \to \mathcal{A}$, satisfying the commutation condition $L_V d = d L_V$.

The variational derivative, or the functional gradient $\nabla P[u,v] \in \Lambda(\mathcal{K}\{u,v\})^2$ with respect to the variables $u, v \in \mathcal{K}$, is defined for any $P[u,v] \in \Lambda(\mathcal{K}\{u,v\})$ by means of the following expression:

$$\text{(2.8)} \quad \text{grad} P[u,v] = P^* [u,v] (1),$$

where a mapping $P^* [u,v] : \Lambda^0(\mathcal{K}\{u,v\}) \to \Lambda^0(\mathcal{K}\{u,v\})^2$ is the formal adjoint mapping for that of \text{(2.7)}. The latter is strongly based on the following important lemma, stated for a special case in \text{[11, 12, 8, 9, 10, 21]}.

Lemma 2.2. Let the differentiations $D_x$ and $D_t : \Lambda(\mathcal{K}\{u,v\}) \to \Lambda(\mathcal{K}\{u,v\})$ satisfy the conditions \text{(2.5)}. Then the mapping

$$\text{Ker grad} / (\text{Imd} \oplus \mathbb{C}) \simeq H^1(\mathcal{A}) := $$

$$= \text{Ker} \{ d : \Lambda^1(\mathcal{K}\{u,v\}) \to \Lambda^2(\mathcal{K}\{u,v\}) \} / d \Lambda^0(\mathcal{K}\{u,v\})$$

is a canonical isomorphism, where $H^1(\mathcal{A})$ is the corresponding cohomology class of the Grassmann complex $\Lambda(\mathcal{K}\{u,v\})$.

It is well known [27] that in the case of the differential ring $\mathcal{K}\{u,v\}$ not all of the cohomology classes $H^j(\mathcal{A}), j \in \mathbb{Z}_+$, are trivial. Nonetheless, one can impose on the functions $u, v \in \mathcal{K}$ some additional restrictions, which will give rise to the condition $H^1(\mathcal{A})$, or equivalently, to the relationship $\text{Ker} \nabla = \text{Im} D_x \oplus \text{Im} D_t \oplus \mathbb{C}$. In addition, the following simple relationship will hold:

$$\text{grad} (\text{Im} D_x \oplus \text{Im} D_t) = 0. \quad \text{(2.9)}$$

Based on Lemma 2.2, one can define the equivalence class $\tilde{\mathcal{A}} := \mathcal{A} / \{ \text{Im} D_x \oplus \text{Im} D_t \oplus \mathbb{R} \}$, whose elements will be called functionals, that is any element $\gamma \in D(\mathcal{A} ; dxdt)$ can be represented as a suitably defined integral $\gamma := \int \int dxdt \gamma \{u,v\} \in D(\mathcal{A} ; dxdt)$ for some $\gamma \in \Lambda(\mathcal{K}\{u,v\})$ with respect to the Lebesgue measure $dxdt$ on $\mathbb{R}^2$.

Consider now our two-component dynamical system \text{(1.1)} as a polynomial differential constraint

$$\text{(2.10)} \quad D_t \{u,v\} = \mathcal{K}\{u,v\},$$

imposed on the ring $\mathcal{K}\{u,v\}$. The following definitions will be useful for our further analysis.

Definition 2.3. Let the reduced ring $\tilde{\mathcal{K}}\{u,v\} := \mathcal{K}\{u,v\} |_{D_t \{u,v\} = \mathcal{K}\{u,v\}}$. Then the triple $\tilde{\mathcal{A}} := (\tilde{\mathcal{K}}\{u,v\}, \Lambda(\tilde{\mathcal{K}}\{u,v\}), d)$ be called a reduced Grassmann differential algebra over the reduced ring $\tilde{\mathcal{K}}\{u,v\}$.

Definition 2.4. Any pair of elements $(\gamma \{u,v\}, \rho \{u,v\}) \in \Lambda^0(\tilde{\mathcal{K}}\{u,v\})^2$, satisfying the relationship

$$\text{(2.11)} \quad D_t \gamma \{u,v\} + D_x \rho \{u,v\} = 0,$$

is called a scalar conservative quantity with respect to the differentiations $D_x$ and $D_t$.

Based on the differential-algebraic setting, described above, one can naturally define the spaces of functionals $D(\tilde{\mathcal{A}} ; dx) := \tilde{\mathcal{A}} / \{ D_x \tilde{\mathcal{A}} \}$ and $D(\tilde{\mathcal{A}} ; dt) = \tilde{\mathcal{A}} / \{ D_t \tilde{\mathcal{A}} \}$ on the reduced Grassmann differential algebra $\tilde{\mathcal{A}}$. From the functional point of view these factor spaces $D(\tilde{\mathcal{A}} ; dx)$ and $D(\tilde{\mathcal{A}} ; dt)$ can be understood more classically as the corresponding spaces of suitably defined integral expressions subject to the measures $dx$ and $dt$, respectively. Then the relationship \text{(2.12)} means equivalently that the functional $\gamma := \int dx \gamma \{u,v\} \in D(\tilde{\mathcal{A}} ; dx)$ is a conserved quantity for the differentiation $D_t$, and the functional $\gamma := \int dt \rho \{u,v\} \in D(\tilde{\mathcal{A}} ; dt)$ is a conserved quantity for the differentiation $D_x$.

Since the differential relationship \text{(2.11)} naturally defines \text{[11, 12]} on the reduced ring $\tilde{\mathcal{K}}\{u,v\}$ a smooth vector field $K : \tilde{\mathcal{K}}\{u,v\} \to T(\tilde{\mathcal{K}}\{u,v\})$, one can construct the corresponding Lie derivative $L_K : \mathcal{A} \to \mathcal{A}$ along this vector field and calculate the differential Lax type \text{[15]} expression.
\[ \frac{\partial \varphi[u,v]}{\partial t} + L_K \varphi[u,v] = 0 \]

for the element \( \varphi[u,v] := \text{grad} \varphi[u,v] \in \Lambda^0(\hat{K}\{u,v;D_{-1}^{-1}\sigma[N]\})^2 \), where \( \hat{K}\{u,v;D_{-1}^{-1}\sigma[N]\} \) denotes some finitely extended differential ring \( \hat{K}\{u,v\} \) and \( \gamma \in \mathcal{D}(\mathcal{A};dx) \) is an arbitrary scalar conserved quantity with respect to the differentiation \( D_t \). The following classical Noether-Lax lemma \([15, 21, 1, 26, 21]\), inverse to the Lax relationship \((2.13)\), holds.

**Lemma 2.5.** (E.Noether-P.Lax) Let a quantity \( \varphi[u,v] \in \Lambda^0(\hat{K}\{u,v;D_{-1}^{-1}\sigma[N]\})^2 \) be such that the following equation

\[ D_t \varphi[u,v] + K^\ast[u,v] \varphi[u,v] = 0, \]

equivalent to \((2.13)\), holds in the ring \( \hat{K}\{u,v;D_{-1}^{-1}\sigma[N]\} \) satisfying the differential constraint \((2.11)\). Then, if the Volterra condition \( \varphi^\ast[u,v] = \varphi[u,v] \) is satisfied in the ring \( \hat{K}\{u,v;D_{-1}^{-1}\sigma[N]\} \), the constructed homology type functional

\[ \gamma := \int_0^1 d\lambda \int dx \varphi[\lambda u, \lambda v], \quad (u,v)^\top >_{\mathbb{R}} \in \mathcal{D}(\mathcal{A};dx) \]

is a scalar conserved quantity with respect to the differentiation \( D_t \).

Assume now that the nonlinear two-component polynomial dynamical system \((2.11)\) possesses a nontrivial compatible differential Lax type representation in the form

\[ D_x f(x,t;\lambda) = l[u,v;\lambda] f(x,t;\lambda), \quad D_t f(x,t;\lambda) = p[u,v;\lambda] f(x,t;\lambda) \]

for some matrices \( l[u,v;\lambda], p[u,v;\lambda] \in \text{End} \Lambda^0(\hat{K}\{u,v\})^q, \quad f(x,t;\lambda) \in \Lambda^0(\hat{K}\{u,v;D_{-1}^{-1}\sigma[N]\})^q \), analytically depending on a parameter \( \lambda \in \mathbb{C} \), where \( q \in \mathbb{Z}_+ \setminus \{0,1\} \) is finite. Then the following important proposition, based on the gradient-holonomic approach, devised before in \([1, 26]\), holds.

**Proposition 2.6.** The Lax type integrable dynamical system \((2.11)\) possesses a set (either finite or infinite) of naturally ordered functionally independent scalar conserved differential quantities

\[ D_t \sigma_j[u,v] + D_x \rho_j[u,v] = 0, \]

where the pairs \( (\sigma_j[u,v], \rho_j[u,v])^\top \in \Lambda^0(\hat{K}\{u,v\})^2, j \in \mathbb{Z}_+ \).

**Proof.** Assume that the Lax type integrable dynamical system \((2.11)\) possesses a set (either finite or infinite) of naturally ordered functionally independent scalar conserved differential quantities \((2.17)\). Let \( \hat{K}\{u,v;D_{-1}^{-1}\sigma[N]\} \) denote the finitely extended differential ring \( \hat{K}\{u,v;D_{-1}^{-1}\sigma[N]\} \) for arbitary finite integer \( N \in \mathbb{Z}_+ \) under the constraints \((2.11)\). Then the Lax equation \((2.14)\), if considered on the invariant functional submanifold

\[ M_N : = \{(u,v)^\top \in M : \text{grad} < c^{(N)}, \int dx \Sigma^{(N)} >_{\mathbb{C}^N} = 0, \]

\[ c^{(N)} \in \mathbb{C}^{N+1} \setminus \{0\}, \Sigma^{(N)} := (\sigma_0, \sigma_1, \ldots, \sigma_N)^\top \in \Lambda^0(\hat{K}\{u,v\})^{N+1} \}, \]

allows \([1, 26]\) as \( |\lambda| \to \infty \) an asymptotic solution \( \varphi(x;\lambda) \in \Lambda^0(\hat{K}\{u,v;D_{-1}^{-1}\sigma[N]\})^2 \) in the form

\[ \varphi(x;\lambda) \sim \psi(x,t;\lambda) \exp(\omega(x,t;\lambda) + D_{-1}^{-1}\sigma(x,t;\lambda)), \]

with a scalar analytical "dispersion" function \( \omega(x,t;\cdot) : \mathbb{C} \to \mathbb{C} \) determined for all \( (x,t) \in \mathbb{R} \times [0,T) \), and the compatible local functionals expansions

\[ \Lambda^0(\hat{K}\{u,v\})^2 \ni \sigma(x,t;\lambda) \sim \sum_{j \in \mathbb{Z}_+} \sigma_j[u,v] \lambda^{-j+\sigma}, \]

\[ \Lambda^0(\hat{K}\{u,v\})^2 \ni \psi(x,t;\lambda) \sim \sum_{j \in \mathbb{Z}_+} \psi_j[u,v] \lambda^{-j+\psi} \]

for some fixed integers \( |\sigma|, |\psi| \in \mathbb{Z}_+ \). Moreover, owing to the Lax equation \((2.14)\), all of the scalar functionals

\[ \gamma_j := \int dx \sigma_j[u,v] \]
for $j \in \mathbb{Z}_+$ are conserved quantities with respect to the differentiation $D_t$. Now, vice versa, if the Lax equation \eqref{2.14} possesses an asymptotic as $|\lambda| \to \infty$ solution in the form \eqref{2.19} $\varphi[u, v; \lambda] \in L^0(\tilde{\mathcal{K}}[u, v; D_x^{-1}\sigma[N]])^2$ with compatible expansions \eqref{2.20}, then all of the scalar functionals \eqref{2.21} are, a priori, the conserved quantities with respect to the differentiation $D_t$, that is there exist such scalar quantities $\rho_j[u, v] \in L^0(\tilde{\mathcal{K}}[u, v]), j \in \mathbb{Z}_+$, satisfying the relationships \eqref{2.17}. □

The analytical expressions for representation \eqref{2.19} and asymptotic expansions \eqref{2.20} for a Lax type integrable dynamical system \eqref{2.11} easily enough follow from the general theory of asymptotic solutions \cite{3, 28} to linear differential equations, applied to a linear differential system \eqref{2.16} and from an important fact \cite{20, 04, 05, 20}, that the trace functional $\Delta[\lambda]$ with \eqref{3.4} $\lambda \in \tilde{\mathcal{K}}[u, v; D_x^{-1}\sigma[N]]$ with any constant matrix $C(\lambda) \in \text{End } \mathbb{C}^q$ is for almost all $\lambda \in \mathbb{C}$ a conserved quantity with respect to both differentiations $D_t$ and $D_x$, where $F(x, t; \lambda)$ and $\tilde{F}(x, t; \lambda), (x, t) \in \mathbb{R} \times \mathbb{R}_+$, are, respectively, the fundamental solutions to the linear Lax type equation
\begin{equation}
D_x f(x, t; \lambda) = l[u, v; \lambda]f(x, t; \lambda)
\end{equation}
and its adjoint version
\begin{equation}
D_x \bar{f}(x, t; \lambda) = -\bar{f}(x, t; \lambda)l[u, v; \lambda],
\end{equation}
where $f(x, t; \lambda), \bar{f}(x, t; \lambda) \in L^0(\tilde{\mathcal{K}}[u, v; D_x^{-1}\sigma[N]])^2$. Thereby, the corresponding gradient
\begin{equation}
\text{grad} \Delta[u, v; \lambda] := \varphi[u, v; \lambda] \in L^0(\tilde{\mathcal{K}}[u, v; D_x^{-1}\sigma[N]])^2,
\end{equation}
owing to Lemma \ref{2.3} a priori satisfies the Lax equation \eqref{2.14}. Having assumed that $|\lambda| \to \infty$, from the asymptotic properties of linear equations \eqref{2.22} and \eqref{2.23} one obtains the result of Proposition \ref{2.6}.

3. The two-component polynomial Burgers type dynamical system integrability analysis

Proceed now to analyzing the Lax type integrability of the two-component polynomial Burgers type dynamical system \eqref{1.1}. To do this, owing to the approach described above, it is necessary to prove to the Lax equation \eqref{2.14} possesses an asymptotic solution of the form \eqref{2.19} in $L^0(\tilde{\mathcal{K}}[u, v; D_x^{-1}\sigma[N]])^2$. Concerning the dynamical system \eqref{1.1} the following proposition holds.

**Proposition 3.1.** The Lax equation \eqref{2.14} with the differential matrix operator
\begin{equation}
K^t,\star[u, v] = \begin{pmatrix}
D_x^2 - 2uD_x & -vD_x \\
-D_x & -uD_x
\end{pmatrix},
\end{equation}
possesses the asymptotic as $|\lambda| \to \infty$ solution
\begin{equation}
\varphi(x; \lambda) = (1, 1/\lambda)^T g(x; \lambda) \exp[-\lambda^2 t - \lambda x + D_x^{-1}(u + \lambda^{-1}v)],
\end{equation}
where the scalar invertible local functional element
\begin{equation}
g(x; \lambda) := \exp(-u + \sum_{j \in \mathbb{Z}_+ \setminus \{0,1\}} D_x^{-1}\sigma_j[u, v]/\lambda^j) \in L^0(\tilde{\mathcal{K}}[u, v]).
\end{equation}
The solution \eqref{3.2} corresponds to the local conservative quantity $\Delta(\lambda) := \int dx (u + \lambda^{-1}v) \in D(\mathcal{A}, dx)$ in the extended ring $\tilde{\mathcal{K}}[u, v; D_x^{-1}\sigma[N]]^2$:
\begin{equation}
\text{grad} \Delta(\lambda)[u, v] = \varphi(x; \lambda) \in L^0(\tilde{\mathcal{K}}[u, v; D_x^{-1}\sigma[N]])^2.
\end{equation}

**Proof.** Assume that the Lax equation \eqref{2.14} possesses the asymptotic as $|\lambda| \to \infty$ solution \eqref{2.19}, where $\omega(x, t; \lambda) = -\lambda x - \lambda^2 t$,
\begin{equation}
L^0(\tilde{\mathcal{K}}[u, v; D_x^{-1}\sigma[N]])^2 \ni \varphi(x; \lambda) = \psi(x, t; \lambda) \exp\{-\lambda^2 t - \lambda x + D_x^{-1}\sigma(x, t; \lambda)\}
\end{equation}
and
\begin{equation}
L^0(\tilde{\mathcal{K}}[u, v])^2 \ni \psi(x, t; \lambda) = (1, a(x, t; \lambda))^T
\end{equation}
which reduces to an equivalent system of the differential-functional relationships
\begin{equation}
D_x^{-1} \sigma_t - \lambda^2 + \sigma_x + (-\lambda + \sigma)^2 - (2u + va)(-\lambda + \sigma) - va_x = 0,
\end{equation}
\begin{equation}
a_t + a(-\lambda^2 + D_x^{-1} \sigma_t) - ua_x - (au + 1)(-\lambda + \sigma) = 0.
\end{equation}

The coefficients of the corresponding asymptotic expansions
\begin{equation}
\Lambda^0(\mathcal{K}\{u,v\}) \ni \sigma(x,t;\lambda) \sim \sum_{j \in \mathbb{Z}_+} a_j[u,v] \lambda^{-j},
\end{equation}
should satisfy two infinite hierarchies of recurrent relationships
\begin{equation}
D_x^{-1} \sigma_{j-1,t} + \sigma_{j-1,x} + 2 \sigma_j + \sum_{k \in \mathbb{Z}_+} \sigma_{j-1-k} \sigma_k - 2u \delta_{j-1,1} - 2u \sigma_{j-1} = 0,
\end{equation}
\begin{equation}
a_{j-2,t} - a_j + \sum_{k \in \mathbb{Z}_+} a_{j-2-k} D_x^{-1} \sigma_{k,t} - ua_{j-2,x} - \delta_{j-2,1} = 0,
\end{equation}

compatible for all \( j \in \mathbb{Z}_+ \). It is easy to calculate from (3.9) the corresponding coefficients
\begin{align*}
\sigma_0 &= u, \quad \sigma_1 = u_x + v, \quad \sigma_2 = v_x + u_{xx} + uu_x, \\
\sigma_3 &= D_x(u^3/3 + uv + u_{xx} + 2uu_x + v_x), \ldots, \sigma_j = D_x(...), \\
a_0 &= 0, \quad a_1 = 1, \quad a_2 = 0, \quad a_3 = 0, \ldots, a_j = 0, \ldots
\end{align*}

and to get convinced that only two functionals
\begin{equation}
\gamma_0 := \int dx \sigma_0[u,v] = \int dx, \quad \gamma_1 := \int dx \sigma_1[u,v] = \int dx,
\end{equation}

are nontrivial conservation laws with respect to the differentiation \( D_t \), since all other functionals
\begin{equation}
\gamma_j := \int dx \sigma_j[u,v] = \int dx D_x(...) = 0
\end{equation}

are trivial in the ring \( \mathcal{K}\{u,v\} \). Equivalently, it means that the gradient \( \varphi(x;\lambda) := \text{grad} \int dx(u + \lambda^{-1}v) = (1, 1/\lambda)^T \in \Lambda^0(\mathcal{K}\{u,v\})^2 \) satisfies the Lax equation \( (2.14) \) in the ring \( \mathcal{K}\{u,v\} \) and thus it should coincide with the expression (3.5). As a result one easily obtains that
\begin{equation}
(1, 1/\lambda)^T g(x;\lambda) \exp[-\lambda^2 t - \lambda x + D_x^{-1}(u + \lambda^{-1}v)],
\end{equation}

where the scalar invertible element
\begin{equation}
g(x;\lambda) := \exp(-u + \sum_{j \in \mathbb{Z}_+ \setminus \{0,1\}} D_x^{-1} \sigma_j[u,v]/\lambda^j) \in \Lambda^0(\mathcal{K}\{u,v\}),
\end{equation}
giving rise to the expressions (3.2) and (3.3). The latter proves the proposition. \( \square \)

As a consequence of Proposition 3.1 we can formulate the following theorem.

**Theorem 3.2.** The two-component polynomial Burgers type dynamical system (1.1) possesses only two local conserved quantities \( \int dx u \) and \( \int dx v \) and no other infinite affinely ordered conserved quantities (either local or nonlocal). Moreover, on the functional manifold \( M \) the Burgers type dynamical system (1.1) is linearizable by means of a Hopf-Cole type transformation and the related linear adjoint mapping to the matrix Lax type representation

\begin{equation}
D_x \left( \begin{array}{c} f \\ \hat{f} \end{array} \right) = \left[ \begin{array}{cc} (u + \lambda^{-1}v - \lambda)/2 \\ 1 \end{array} \right] \left( \begin{array}{c} f \\ \hat{f} \end{array} \right),
\end{equation}

and

\begin{equation}
\left( \begin{array}{c} f \\ \hat{f} \end{array} \right) = \left[ \begin{array}{cc} -(u + \lambda)(u + \lambda^{-1}v - \lambda)/2 \\ (u + \lambda) \end{array} \right] \left( \begin{array}{c} f \\ \hat{f} \end{array} \right),
\end{equation}

compatible for all \( \lambda \in \mathbb{C} \setminus \{0\} \), where vector function \( (f, \hat{f})^T \in \Lambda^0(\mathcal{K}\{u,v;D_x^{-1} \sigma[N]\})^2, \ N = 2. \)
Since the mutual compatibility condition of relationships (3.27) reduces to the expression
\begin{equation}
\tag{3.27}
(3.22a)
\end{equation}
holds, where we have put, by definition, that
\begin{equation}
\tag{3.20}
D_x f = D_x(\bar{u} + \bar{v}) + \bar{v}, \quad D_x \bar{g} = u \bar{g},
\end{equation}
which easily reduces to the following system of differential relationships:
\begin{equation}
\tag{3.21}
D_t \hat{f} = (u_x + u^2 + v) \hat{f}/2 = 0, \quad D_x \hat{f} = (u/2) \hat{f},
\end{equation}
if to make the change of variables \( \hat{f} := \bar{g}^{1/2} \in \Lambda^0(\tilde{K}(u, v; D^{-1}_x \sigma |N)) \). The system (3.20) can be specified further, if to make use of the substitution \( \hat{f} := f \exp(\lambda x - \lambda^{-1} D^{-1}_x v) \), giving rise to the next scalar operator Lax type representation
\begin{equation}
\tag{3.21}
D_t f = D_x(u + \lambda) f/2 + (u + \lambda) D_x f, \quad D_x f = (u + \lambda^{-1} v - \lambda) f/2,
\end{equation}
compatible for all \( \lambda \in \mathbb{C} \setminus \{0\} \).

Now we will proceed to constructing a suitably linearly extended adjoint differential relationships [25] for the system of equations (3.21). Doing the standard way, one easily obtains that the following linearly adjoint relationship compatible with the second equation of (3.21)
\begin{equation}
\tag{3.22a}
D_x \hat{f} = D_x(\bar{f}/f) = -f^{-1} \hat{f} D_x f + f^{-1} D_x(\hat{f} f) =
\end{equation}
holds, where we have put, by definition, that \( D_x(\hat{f} f) := \chi[u, v; \lambda] f^2 \) for some arbitrarily chosen element \( \chi[u, v; \lambda] \in \tilde{K}(u, v) \). The compatibility of (3.22a) with the first equation of (3.21) and its suitable extension gives rise to the condition \( \chi[u, v; \lambda] = 1 \). Thus, we have obtained that the linearly adjoint relationship compatible with the second equation of (3.21) reads as
\begin{equation}
\tag{3.23}
D_x \hat{f} = -(\lambda - \lambda^{-1} v - u) \hat{f}/2 + f.
\end{equation}
The respective adjoint linear relationship compatible with the first equation of (3.21) obtains easily as
\begin{equation}
\tag{3.24}
D_t \hat{f} = -(u + \lambda) \hat{f}/2 + (u + \lambda) D_x \hat{f},
\end{equation}
and is compatible with (3.22a) for all \( \lambda \in \mathbb{C} \setminus \{0\} \). It is now dexterous to rewrite equations (3.21), (3.23) and (3.24) as the following two equivalent matrix systems:
\begin{equation}
\tag{3.25}
D_x \begin{pmatrix}
\hat{f} \\
\bar{f}
\end{pmatrix} = \begin{pmatrix}
0 & (u + \lambda^{-1} v - \lambda)/2 \\
(\lambda - \lambda^{-1} v - u)/2 & 1
\end{pmatrix} \begin{pmatrix}
\hat{f} \\
\bar{f}
\end{pmatrix},
\end{equation}
and
\begin{equation}
\tag{3.26}
D_t \begin{pmatrix}
\hat{f} \\
\bar{f}
\end{pmatrix} = \begin{pmatrix}
D_x(u + \lambda)/2 + \\
(u + \lambda)(u + \lambda^{-1} v - \lambda)/2
\end{pmatrix} \begin{pmatrix}
0 \\
-\lambda x^{-1} v - \lambda/2 + (u + \lambda)(\lambda - u - \lambda^{-1} v)/2
\end{pmatrix} \begin{pmatrix}
\hat{f} \\
\bar{f}
\end{pmatrix},
\end{equation}
where, by construction, elements \((f, \hat{f})^T \in \Lambda^0(\tilde{K}(u, v; D^{-1}_x \sigma |N))^2, \quad N = 2 \). Based on the systems (3.25) and (3.26) one can easily calculate that
\begin{equation}
\tag{3.27}
D_x(\hat{f} f) = f^2, \quad D_t(\hat{f} f) = (u + \lambda)f^2,
\end{equation}
Since the mutual compatibility condition of relationships (3.27) reduces to the expression
\begin{equation}
\tag{3.28}
D_t f = |D_x(u + \lambda)/2|f + (u + \lambda) D_x f,
\end{equation}
exactly coinciding with the first equation of the system \((3.26)\), we now can interpret both systems \((3.25)\) and \((3.26)\) as the corresponding matrix Lax type representation for the Burgers type system \((1.1)\). This proves the theorem.

The scalar representation \((3.19)\), as it can be easily observed, can be generalized to the following higher-order evolution equation:

\[
(3.29) \quad D_\lambda \tilde{g} = D_\lambda^2 \tilde{g} + v \tilde{g} = 0, \quad D_\lambda \tilde{g} = u \tilde{g},
\]

where \(n \in \mathbb{N} \setminus \{1, 2\}, \tilde{g} \in \Lambda^0(\tilde{K}\{u, v; D_\lambda^{-1} \sigma[N]\})\) and there is imposed no \textit{a priori} constraint on the function \(v \in \tilde{K}\{u, v\}\) except the functional \(\int dx v \in D(\mathcal{A}; dx)\) has to be a conserved quantity with respect to the differentiation \(D_t\). Applying to \((3.29)\) the nonlocal change of variables \(u := 2D_\lambda \ln \tilde{f}[u, v; \lambda] \) for \(\tilde{f} \in \Lambda^0(\tilde{K}\{u, v; D_\lambda^{-1} \sigma[N]\}), N = 2\), one can obtain a new infinite hierarchy of two-component integrable polynomial Burgers type dynamical systems, generalizing the systems studied before in \([31, 30]\). For instance, at \(n = 3\) we find the following dynamical Burgers-Korteweg-de Vries type dynamical system of the third order:

\[
(3.30) \quad \begin{align*}
    u_t &= u_{3x} + 3(uu_x)_x + 3u^2 u_x + v_x, \\
    v_t &= u r[u,v]_{x},
\end{align*}
\]

where \(r[u,v] \in \tilde{K}\{u,v\}\) is for the present an arbitrary element. To choose from those for which the dynamical systems of type \((3.30)\) will possess suitably extended matrix Lax type representations, it is natural to take the first pair of Lax type equation \((3.15)\):

\[
(3.31) \quad \begin{align*}
    D_x f &= (u + \lambda^{-1} v - \lambda) f/2, \\
    D_x \hat{f} &= -(u + \lambda^{-1} v - \lambda) \hat{f}/2 + f
\end{align*}
\]

for \((f, \hat{f}) \in \Lambda^0(\tilde{K}\{u, v; D_\lambda^{-1} \sigma[N]\})^2, N = 2\), and to supplement it by means of the following systems of evolution equations, naturally generalizing that of \((3.26)\) with respect to the temporal parameters \(t_n, n \in \mathbb{R}\):

\[
(3.32) \quad \begin{align*}
    D_{t_n} f &= D_x \alpha_n(x; \lambda) f/2 + \alpha_n(x; \lambda) D_x f, \\
    D_{t_n} \hat{f} &= -D_x \alpha_n(x; \lambda) \hat{f}/2 + \alpha_n(x; \lambda) D_x \hat{f},
\end{align*}
\]

which are, by construction, compatible for all \(\lambda \in \mathbb{C} \setminus \{0\}\) for a polynomial in \(\lambda \in \mathbb{C}\) element \(\alpha_n(x; \lambda) \in \tilde{K}\{u, v\}, n \in \mathbb{N}\), satisfying the standard determining relationship

\[
(3.33) \quad D_{t_n}(u + \lambda^{-1} v - \lambda) = D_x[D_x \alpha_n(x; \lambda) + (u + \lambda^{-1} v - \lambda) \alpha_n(x; \lambda)].
\]

It is also easy to check that for \(n = 1\) the choice

\[
(3.34) \quad \alpha_1(x; \lambda) = u + \lambda
\]

entails exactly the Burgers type dynamical system \((1.1)\).

The general algebraic structure of the whole infinite hierarchy of resulting dynamical systems \((3.33)\) can be extracted easily from the matrix spectral Lax pair \((3.25)\):

\[
(3.35) \quad D_x \left( \begin{array}{c}
    f \\
    \hat{f}
\end{array} \right) = \left( \begin{array}{cc}
    (u + \lambda^{-1} v - \lambda)/2 & 0 \\
    (\lambda - u - \lambda^{-1} v)/2 & 1
\end{array} \right) \left( \begin{array}{c}
    f \\
    \hat{f}
\end{array} \right) = l[u, v; \lambda] \left( \begin{array}{c}
    f \\
    \hat{f}
\end{array} \right),
\]

which allows by means of the gradient-holonomic scheme \([26, 1]\) to obtain successfully from the corresponding differential commutator equation

\[
(3.36) \quad D_x S = [l, S], \quad S = \left( \begin{array}{cc}
    S_{11} & S_{12} \\
    S_{21} & S_{22}
\end{array} \right),
\]

for the related "monodromy" matrix \(S := S(x; \lambda) \in sl(2; \mathbb{C})\) the resulting \([20, 26]\) canonical differential relationships for the gradient \((\varphi_1, \varphi_2)^T := \varphi := \text{grad}(\text{tr}S) \in \mathbb{R}^2\).
\[ \Lambda^0(\mathcal{K}\{u, v; D_x^{-1}\sigma[N]\})^2 \] of the dynamical \( D_x \) and \( D_x \)-invariant trace functional \( \text{tr } S(x; \lambda) \in \mathcal{D}(\mathfrak{A}; dx) \):

\[
\begin{align*}
&\begin{cases}
-D_x \varphi_1 + D_x^{-1}uD_x \varphi_1 + D_x^{-1}vD_x \varphi_2 = \lambda \varphi_1, \\
\varphi_1 = \lambda \varphi_2,
\end{cases}
\end{align*}
\]

(3.37)

where the component \( \varphi_1 \in \mathcal{K}\{u, v\} \) possesses, owing to the construction, the following differential-algebraic representation:

(3.38)

\[ \varphi_1 = (u + \lambda^{-1}v - \lambda)S_{21} + D_x S_{21} \]

for some polynomial expression \( S_{21} := S_{21}(x; \lambda) \in \mathcal{K}\{u, v\} \). The differential expressions (3.37) can be rewritten in the following useful matrix form:

(3.39)

\[ \Lambda \varphi = \lambda \varphi, \quad \Lambda := \begin{pmatrix} -D_x + D_x^{-1}uD_x & D_x^{-1}vD_x \\ 1 & 0 \end{pmatrix}, \]

where the recursion operator \( \Lambda : T^*(\mathcal{K}\{u, v\}) \to T(\mathcal{K}\{u, v\}) \) satisfies the determining operator equation

(3.40)

\[ D_t \Lambda = [\Lambda, K^{[*]}], \]

easily following from the Noether-Lax condition (3.14) and the adjoint linear spectral relationship (3.39).

Recall now that our Burgers type dynamical system (1.1) possesses only two conservations laws: \( \gamma_0 = \int du \) and \( \gamma_1 = \int dv \in \mathcal{D}(\mathfrak{A}; dx) \). This means that the expression (3.38) exactly equals \( \varphi_1 = \text{grad}_u \gamma_0[u, v] = 1 \), or equivalently the condition

(3.41)

\[ D_x[D_xS_{21}(x; \lambda) + (u + \lambda^{-1}v - \lambda)S_{21}(x; \lambda)] = 0 \]

should be satisfied for some element \( S_{21}(x; \lambda) \in \Lambda^0(\mathcal{K}\{u, v\}) \) and all \( \lambda \in \mathbb{C}\backslash\{0\} \). The following proposition characterizes asymptotic as \( |\lambda| \to \infty \) solutions to (3.41) and their relationships to the generalized dynamical systems (3.33).

**Proposition 3.3.** The nonnegative degree polynomial part of the asymptotic, as \( |\lambda| \to \infty \), solution \( S_{21}(x; \lambda) \sim \sum_{j \in \mathbb{Z}_+} \lambda^{-j}S_{21}^{(j)}[u, v; \lambda] \) to the differential relationship (3.41) makes it possible to represent the generating elements \( \alpha_n(x; \lambda) \in \Lambda^0(\mathcal{K}\{u, v\}) \) of the generalized dynamical systems (3.33) as

(3.42)

\[ \alpha_n(x; \lambda) = (\lambda^nS_{21}(x; \lambda))_+ \]

for any \( n \in \mathbb{Z}_+ \).

**Proof.** Taking into account that the whole hierarchy of the generalized Burgers type dynamical systems (3.33) can be represented in the recursive form

(3.43)

\[ D_t^n(u, v)^T = \Phi^n(D_xu, D_xv)^T, \quad \Phi := \Lambda^* = \begin{pmatrix} D_x + D_x^{-1}uD_x^{-1} & 1 \\ D_xvD_x^{-1} & 0 \end{pmatrix}, \]

we can rewrite it equivalently as

(3.44)

\[ D_t^n(u + \lambda^{-1}v - \lambda) = D_x[D_x \alpha_n(x; \lambda) + (u + \lambda^{-1}v - \lambda)\alpha_n(x; \lambda)], \]

where, by definition,

(3.45)

\[ \alpha_n(x; \lambda) := (\lambda^n\alpha(x; \lambda))_+ \]

is the corresponding nonnegative degree polynomial part generated by the asymptotic solution \( \alpha(x; \lambda) \sim \sum_{j \in \mathbb{Z}_+} \lambda^{-j}\alpha_j[u, v; \lambda] \) as \( |\lambda| \to \infty \) to the differential functional equation

(3.46)

\[ D^2_x\alpha(x; \lambda) + D_x((u + \lambda^{-1}v - \lambda)\alpha(x; \lambda)) = 0, \]

exactly equivalent to the dual to (3.39) symmetry relationship

(3.47)

\[ \Phi (D_x\alpha, D_x\beta)^T = \lambda(D_x\alpha, D_x\beta)^T \]

with the generalized symmetry of the flow (1.1)

(3.48)

\[ (D_x\alpha, D_x\beta)^T := \sum_{j \in \mathbb{Z}_+} \lambda^{-j}\Phi^j(D_xu, D_xv)^T. \]

The observation that the differential functional equation (3.46) coincides exactly with that of (3.41) proves the proposition. \( \square \)
From Theorem 1.1 we also can derive that the Burgers type dynamical system (1.1) does not allow on the functional manifold $M$ a Hamiltonian formulation. This means that the recursion operator (3.39) constructed above and found before in [5, 17] for the two-component Burgers type dynamical system (1.1), proves to be not factorizable by means of suitably defined compatible Poissonian structures, as they on the whole, eventually do not exist. It is strongly related with the fact that the dynamical system (1.1) possesses no infinite hierarchy of local conservation laws, whose existence is responsible for the factorization mentioned above. Nonetheless, similar to the situation happened in the work [30], if one to succeed to state that the Burgers type dynamical system (1.1) does possess another infinite hierarchy of nonlocal conservation laws, then some degree of the found before symmetry recursion operator (5.33) will be already factorized by means of the respectively constructed Poissonian structures. Yet, by now, this problem remains still open.

4. Conclusion

Having based on the differential-algebraic approach [1, 26, 22, 23, 30] to testing the Lax type integrability of nonlinear dynamical systems on functional manifolds, we stated that the two component polynomial Burgers type dynamical system (1.1) does possess an adjoint matrix Lax type representation and the corresponding recursion operator, which does not allow a bi-Poissonian factorization and makes it possible to construct only two local conserved quantities. A problem to construct a generalized bi-Poissonian factorization of a suitably powered recursion operator, similarly to that of the work [30], is left for the future analysis. Thus, the differential-algebraic approach, jointly with considerations based on the symplectic geometry, can serve as simple enough as effective tool for analyzing the Lax type integrability of a wide class of polynomial nonlinear dynamical systems on functional manifolds. Moreover, as it was recently demonstrated in [24], this approach also appears to be useful in the case of nonlocal polynomial dynamical systems.

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The Department of Mathematical Sciences at the NJIT, Newark, USA
E-mail address: denblac@gmail.com

The Department of Applied Mathematics at AGH University of Science and Technology of Krakow, Poland, and the Ivan Franko State Pedagogical University of Drohobych, Lviv region, Ukraine

The Department of Mathematics at the Hacettepe University of Ankara, Turkey
E-mail address: sultan_kamal@hotmail.com, ozcag1@hacettepe.edu.tr

The Department of Mathematics at the Hacettepe University of Ankara, Turkey
E-mail address: sultan_kamal@hotmail.com, ozcag1@hacettepe.edu.tr
E-mail address: pryk.anat@ua.fm, prykanat@cybergal.com