CONVERGENCE RATE OF THE PRESCRIBED CURVATURE FLOW

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Abstract. The prescribed scalar curvature flow was introduced to study the problem of prescribing scalar curvature on manifolds. Carlotto, Chodosh and Rubinstein have studied the convergence rate of the Yamabe flow. Inspired by their result, we study in this paper the convergence rate of the prescribed scalar curvature flow.

1. Introduction

Let \((M, g_0)\) be a closed (i.e. compact without boundary) Riemannian manifold of dimension \(n \geq 3\). As a generalization of the Uniformization Theorem, the Yamabe problem is to find a metric conformal to \(g_0\) such that its scalar curvature \(R_g\) is constant. This was solved by Aubin [2], Trudinger [37] and Schoen [32].

The Yamabe flow is a geometric flow introduced to study the Yamabe problem. It is defined as

\[
\frac{\partial}{\partial t} g(t) = - (R_{g(t)} - \overline{R}_{g(t)}) g(t),
\]

where \(\overline{R}_{g(t)}\) is the average of the scalar curvature of \(g(t)\):

\[
\overline{R}_{g(t)} = \frac{\int_M R_{g(t)} dV_{g(t)}}{\int_M dV_{g(t)}}.
\]

The existence and convergence of the Yamabe flow has been studied in [5, 6, 14, 34, 38]. See also [3, 12, 16, 18, 19, 20, 29, 33] and references therein for results related to the Yamabe flow.

In [7], Carlotto, Chodosh and Rubinstein studied the rate of convergence of the Yamabe flow (1.1). They proved the following: (see Theorem 1 in [7])

**Theorem 1.1.** Assume \(g(t)\) is a solution of the Yamabe flow (1.1) that converges in \(C^{2,\alpha}(M, g_\infty)\) to \(g_\infty\) as \(t \to \infty\) for some \(\alpha \in (0, 1)\). Then there is a \(\delta > 0\) depending only on \(g_\infty\) such that:

(i) If \(g_\infty\) is an integrable critical point, then the convergence occurs at an exponential rate, that is

\[
\|g(t) - g_\infty\|_{C^{2,\alpha}(M, g_\infty)} \leq C e^{-\delta t}
\]

for some constant \(C > 0\) depending on \(g(0)\).

(ii) In general, the rate of convergence cannot be worse than polynomial, that is

\[
\|g(t) - g_\infty\|_{C^{2,\alpha}(M, g_\infty)} \leq C (1 + t)^{-\delta}
\]

for some constant \(C > 0\) depending on \(g(0)\).

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**Theorem 1.2.** Assume that \( g_\infty \) is a nonintegrable critical point of the Yamabe energy with order of integrability \( p \geq 3 \). If \( g_\infty \) satisfies the Adams-Simon positive condition \( AS_p \), then there exists metric \( g(0) \) conformal to \( g_\infty \) such that the solution \( g(t) \) of the Yamabe flow (1.1) starting from \( g(0) \) exists for all time and converges in \( C^\infty(M,g_\infty) \) as \( t \to \infty \). The convergence occurs “slowly” in the sense that

\[
C^{-1}(1 + t)^{-\frac{1}{p-1}} \leq \|g(t) - g_\infty\|_{C^{2,\alpha}(M,g_\infty)} \leq C(1 + t)^{-\frac{1}{p-2}}
\]

for some constant \( C > 0 \).

We refer the readers to [7, Definition 8] and [7, Definition 10] respectively for the precise definitions of integrable critical point and Adams-Simon positive condition \( AS_p \).

As a generalization of the Yamabe problem, the problem of prescribing scalar curvature is to find a metric \( g \) conformal to \( g_0 \) such that its scalar curvature \( R_g \) is equal to a given smooth function \( f \) in \( M \). In particular, when \( (M,g_0) = (S^n,g_{S^n}) \) is the \( n \)-dimensional unit sphere equipped with standard metric \( g_{S^n} \), the problem of prescribing scalar curvature is called Nirenberg’s problem. These problems have been studied extensively and it is impossible to include all the references here. We refer the readers to [8, 9, 15, 17, 21, 22, 30, 31] and the references therein for results related to the problem of prescribing scalar curvature and the Nirenberg’s problem.

Given a positive smooth function \( f \) in \( M \), we consider the prescribed scalar curvature flow:

\[
\frac{\partial}{\partial t} g(t) = (\alpha(t)f - R_{g(t)})g(t),
\]

where \( R_{g(t)} \) is the scalar curvature of \( g(t) \), and \( \alpha(t) \) satisfies

\[
\alpha(t) = \frac{\int_M R_{g(t)}dV_{g(t)}}{\int_M f dV_{g(t)}}.
\]

The flow (1.2) was introduced by Chen and Xu [11] in studying the Nirenberg problem. See also [10, 24] for results related to the flow (1.2). Note that when \( f \equiv 1 \), it follows from (1.3) that the flow (1.2) reduces to the Yamabe flow (1.1).

Inspired by the results of Carlotto, Chodosh and Rubinstein about the convergence rate of Yamabe flow, i.e. Theorems 1.1 and 1.2 mentioned above, we study in this paper the rate of convergence of the prescribed scalar curvature flow (1.2).

The following theorems are the main results in this paper:

**Theorem 1.3.** Assume that \( g(t) \) is a solution to the prescribed scalar curvature flow that is converging in \( C^{2,\alpha}(M,g_\infty) \) to \( g_\infty \) as \( t \to \infty \) for some \( \alpha \in (0,1) \). Then, there is \( \delta > 0 \) depending only on \( f \) and \( g_\infty \) such that

(i) If \( g_\infty \) is an integrable critical point, then the convergence occurs at an exponential rate

\[
\|g(t) - g_\infty\|_{C^{2,\alpha}(M,g_\infty)} \leq Ce^{-\delta t},
\]

for some constant \( C > 0 \) depending on \( g(0) \).

(ii) In general, the convergence cannot be worse than a polynomial rate

\[
\|g(t) - g_\infty\|_{C^{2,\alpha}(M,g_\infty)} \leq C(1 + t)^{-\delta},
\]

for some constant \( C > 0 \) depending on \( g(0) \).
Theorem 1.4. Assume that $g_\infty$ is a non-integrable critical point of the functional $E_f$ with order of integrability $p \geq 3$. If $g_\infty$ satisfies the Adam-Simon positivity condition $AS_p$, then there exists a metric $g(0)$ conformal to $g_\infty$ such that the prescribed scalar curvature flow $g(t)$ starting from $g(0)$ exists for all time and converges in $C^\infty(M, g_\infty)$ to $g_\infty$ as $t \to \infty$. The convergence occurs “slowly” in the sense that

$$C(1 + t)^{-\frac{4}{n-2}} \leq \|g(t) - g_\infty\|_{C^2(M, g_\infty)} \leq C(1 + t)^{-\frac{4}{n-2}}$$

for some constant $C > 0$.

The precise definitions of integrable critical point and the Adam-Simon positivity condition $AS_p$ can be found in Section 2. In Section 5, we construct example Riemannian manifolds which satisfies the condition $AS_3$. This allows us to conclude that there exists a prescribed scalar curvature flow converges exactly at a polynomial rate described in Theorem 1.4.

2. Prescribed scalar curvature flow

For a positive smooth function $f$ on $M$, we define the functional

$$E_f(g) = \frac{\int_M R_g dV_g}{\left(\int_M f dV_g\right)^{\frac{n-2}{2}}}$$

where $R_g$ is the scalar curvature of $g$. If $g = w^{\frac{4}{n-2}} g_\infty$ for some $0 < w \in C^2(M)$ and smooth metric $g_\infty$, then an alternative expression for the function $E_f$ (restricted to the conformal class of $g$) is

$$E_f(w) = \frac{\int_M \left(\frac{4(n-1)}{n-2} |\nabla g_\infty w|^2 + R_{g_\infty} w^2\right) dV_{\tilde{g}_\infty}}{\left(\int_M f w^{\frac{2n}{n-2}} dV_{\tilde{g}_\infty}\right)^{\frac{n-2}{n}}}$$

since

$$-\frac{4(n-1)}{n-2} \Delta_{g_\infty} w + R_{g_\infty} w = R_{w^{\frac{4}{n-2}} g_\infty} w^{\frac{n+2}{n-2}}.$$

Along the flow (1.2), the volume $\int_M dV_{g(t)}$ is preserved. Indeed, it follows from (1.2) and (1.3) that

$$\frac{d}{dt} \left( \int_M dV_{g(t)} \right) = \int_M \frac{\partial}{\partial t} (dV_{g(t)}) = \frac{n}{2} \int_M (\alpha(t)f - R_{g(t)}) dV_{g(t)} = 0.$$

Since the flow (1.2) preserves the conformal structure, we can write $g(t) = u(t)^{\frac{4}{n-2}} g_\infty$. Therefore, (1.2) can be written as

$$\frac{\partial}{\partial t} u(t) = \frac{n}{4} (\alpha(t)f - R_{g(t)}) u(t).$$

Using this, we find (see Lemma 2.1 in [11])

$$\frac{d}{dt} E_f(u(t)) = -\frac{n-2}{2} \sum_{i=0}^{n-2} \frac{\int_M (R_{g(t)} - \alpha(t)f)^2 dV_{g(t)}}{\left(\int_M f dV_{g(t)}\right)^{\frac{n-2}{n}}}$$

along the flow.

Consider the following set of conformal metrics associated to $g_\infty$:

$$[g_\infty] = \left\{ w^{\frac{4}{n-2}} g_\infty : 0 < w \in C^{2,\alpha}(M) \right\}.$$
For \( k \in \mathbb{N} \), we denote the \( k \)-th differential of the functional \( E_f \) on \([g_\infty] \) at the point \( w \) in the directions \( v_1, \ldots, v_k \) by

\[
D^k E_f(w)[v_1, \ldots, v_k].
\]

As we will see below, the functional \( v \mapsto D^k E_f(w)[v_1, \ldots, v_{k-1}, v] \) is in the image of \( L^2(M, g_\infty) \) under the natural embedding onto \( C^{2,\alpha}(M, g_\infty) \). Therefore, we will also write

\[
D^k E_f(w)[v_1, \ldots, v_{k-1}]
\]

for this element of \( L^2(M, g_\infty) \). When \( k = 1 \), we will drop the (second) brackets, and thus consider \( DE_f(w) \in L^2(M, g_\infty) \).

We may write the differential of \( E_f \) restricted to \([g_\infty] \) as

\[
\frac{1}{2} DE_f(w)[v] = \left. \frac{d}{dt} E_f(w + tv) \right|_{t=0}
\]

\[
= \int_M \left( \frac{4(n-1)}{n-2} \langle \nabla_{g_\infty} w, \nabla_{g_\infty} v \rangle + R_{g_\infty} w v \rangle \right) dV_{g_\infty} - \frac{E_f(w)}{\int_M f w^{\frac{n+2}{n-2}} dV_{g_\infty}} \int_M f w^{\frac{n+2}{n-2}} dV_{g_\infty}
\]

\[
= \int_M \left( -\frac{4(n-1)}{n-2} \Delta_{g_\infty} w + R_{g_\infty} w - \alpha(w) w^{\frac{n+2}{n-2}} \right) dV_{g_\infty}
\]

\[
= \int_M \left( R_{w^{\frac{n}{n-2}} g_\infty} - \alpha(w) f \right) w^{\frac{n+2}{n-2}} dV_{g_\infty}
\]

where \( \alpha(w) \) is defined as

\[
\alpha(w) = \frac{\int_M R_{w^{\frac{n}{n-2}} g_\infty} w^{\frac{2n}{n-2}} dV_{g_\infty}}{\int_M f w^{\frac{2n}{n-2}} dV_{g_\infty}} = \frac{\int_M \left( \frac{4(n-1)}{n-2} \langle \nabla_{g_\infty} w, \nabla_{g_\infty} v \rangle + R_{g_\infty} w v \rangle \right) dV_{g_\infty}}{\int_M f w^{\frac{n+2}{n-2}} dV_{g_\infty}}.
\]

Thus, a metric \( g_\infty \) is a critical point for the energy \( E_f \) restricted to \([g_\infty] \) exactly when the scalar curvature of \( g_\infty \) satisfies \( R_{g_\infty} = \alpha f \) in \( M \), where

\[
\alpha = \alpha(1) = \frac{\int_M R_{g_\infty} dV_{g_\infty}}{\int_M f dV_{g_\infty}}.
\]

Regarded as an element of \( L^2(M, g_\infty) \), we have that

\[
\frac{1}{2} DE_f(w) = -\frac{4(n-1)}{n-2} \Delta_{g_\infty} w + R_{g_\infty} w \alpha f w^{\frac{n+2}{n-2}}
\]

\[
\left( \int_M f w^{\frac{2n}{n-2}} dV_{g_\infty} \right)^{-\frac{n-2}{n}}
\]

From now on, we assume that \( g_\infty \) satisfies

\[
\int_M f dV_{g_\infty} = 1 \quad \text{and} \quad R_{g_\infty} = \alpha f \quad \text{in} \quad M
\]

where \( \alpha \) is as in (2.6). We denote by \( \mathcal{FSC} \) the set

\[
\mathcal{FSC} = \left\{ w^{\frac{4}{n-2}} g_\infty \in [g_\infty] : \int_M f w^{\frac{2n}{n-2}} dV_{g_\infty} = 1 \text{ and } R_{w^{\frac{4}{n-2}} g_\infty} = \alpha(w) f \quad \text{in} \quad M \right\}.
\]
It follows from (2.7) that \( g_\infty \in \mathcal{FSC} \). We find

\[
\frac{1}{2} D^2 E_f(g_\infty)[v, w] = \frac{1}{2} \left. \frac{d}{dt} (DE_f(1 + tw)[v]) \right|_{t=0}
\]

\[
= \frac{d}{dt} \left[ \int_M (\frac{4(n-1)}{n-2} (\nabla g_\infty (1 + tw), \nabla g_\infty v) + R g_\infty (1 + tw)v) dV_{g_\infty} - \int_M (1 + tw)^{-\frac{n+2}{2}} vdV_{g_\infty} \right]_{t=0}
\]

\[
= \frac{\int_M (\frac{4(n-1)}{n-2} (\nabla g_\infty w, \nabla g_\infty v) + R g_\infty vw - \frac{n+2}{n-2} \alpha f vw) dV_{g_\infty}}{\left( \int_M f dV_{g_\infty} \right)^{-\frac{n-2}{2}}}
\]

where we have used (2.7) in the last equality. Therefore, if we define \( \mathcal{L}_\infty \) by means of the formula

\[
\mathcal{L}_\infty v = (n-1) \Delta g_\infty v + R g_\infty v = (n-1) \Delta g_\infty v + fv,
\]

then, by (2.8), we can obtain

\[
\mathcal{L}_\infty v = (n-1) \Delta g_\infty v + R g_\infty v = (n-1) \Delta g_\infty v + fv.
\]

We define \( \Lambda_0 \) by \( \ker \mathcal{L}_\infty \subset L^2(M, g_\infty) \).

It follows from a classical theorem of spectral theory (cf. [23]) that \( \Lambda_0 \) is finite dimensional, since it is the eigenspace of the Schrödinger operator \( (n-1) \Delta g_\infty + f \) for the zero eigenvalue. We will write \( \Lambda_0^+ \) for the \( L^2(M, g_\infty) \)-orthogonal complement.

It is crucial throughout this work that the functional \( E_f \) is an analytic map in the sense of [39] Definition 8.8. More precisely, one can easily prove the following by expanding the denominator of \( E_f \) in a power series: fix a metric \( g_\infty \) then the functional \( E_f \) is an analytic functional on \( \{ u \in C^{2,\alpha}(M, g_\infty) : u > 0 \} \) in the sense that for each \( w_0 \in C^{2,\alpha}(M, g_\infty) \) with \( w_0 > 0 \), there is an \( \epsilon > 0 \) and bounded multilinear operators

\[
E_f^{(k)} : C^{2,\alpha}(M, g_\infty)^t \to \mathbb{R}
\]

such that if \( \| w - w_0 \|_{C^{2,\alpha}} < \epsilon \), then \( \sum_{k=0}^\infty \| E_f^{(k)} \| \cdot \| w - w_0 \|_{C^{2,\alpha}}^k < \infty \) and

\[
E_f(w) = \sum_{k=0}^\infty E_f^{(k)}(w - w_0, \cdots, w - w_0) \text{ in } C^{2,\alpha}(M, g_\infty).
\]

We need the following Proposition from [36] Section 3; which can be established with the help of the implicit function theorem:
Proposition 2.1. There is $\epsilon > 0$ and an analytic map $\Phi : \Lambda_0 \cap \{ v : \|v\|_{L^2} < \epsilon \} \to C^{2,\alpha}(M, g_\infty) \cap \Lambda^1_\delta$ such that $\Phi(0) = 0$, $D\Phi(0) = 0$, and
\[
\sup_{\|v\|_{L^2} < \epsilon, \|w\|_{L^2} \leq 1} \|D\Phi(v)[w]\|_{L^2} < 1,
\]
and so that defining $\Psi(v) = 1 + v + \Phi(v)$, we have that $\Psi(v) > 0$, $\int_M f\Psi(v)^{\frac{2}{n-2}}dV_{g_\infty} = 1$ and
\[
\text{proj}_{\Lambda^1_\delta}[DE_f(\Psi(v))] = \text{proj}_{\Lambda^1_0} \left[ \left( R_{\Psi(v)}^{\frac{4}{n-2}}g_\infty - \alpha(\Psi(v))f \right)\Psi(v)^{\frac{n+2}{n-2}} \right] = 0.
\]
Furthermore,
\[
\text{proj}_{\Lambda_0}[DE_f(\Psi(v))] = \text{proj}_{\Lambda_0} \left[ \left( R_{\Psi(v)}^{\frac{4}{n-2}}g_\infty - \alpha(\Psi(v))f \right)\Psi(v)^{\frac{n+2}{n-2}} \right] = DF,
\]
where $F : \Lambda_0 \cap \{ v : \|v\|_{L^2} \leq \epsilon \} \to \mathbb{R}$ is defined by $F(v) = E_f(\Psi(v))$. Finally, the intersection of $\mathcal{FSC}$ with a small $C^{2,\alpha}(M, g_\infty)$-neighborhood of 1 coincides with
\[
S_0 := \{ \Psi(v) : v \in \Lambda_0, \|v\|_{L^2} < \epsilon, DF(v) = 0 \},
\]
which is a real analytic subvariety (possible singular) of the following $(\dim \Lambda_0)$-dimensional real analytic submanifold of $C^{2,\alpha}(M, g_\infty)$:
\[
S : \{ \Psi(v) : v \in \Lambda_0, \|v\|_{L^2} < \epsilon \}.
\]

We will refer to $S$ as the natural constraint for the problem.

Definition 2.2. For $g_\infty \in \mathcal{FSC}$, we say that $g_\infty$ is integrable if for all $v \in \Lambda_0$, there is a path $w(t) \in C^2((-\epsilon, \epsilon) \times M, g_\infty)$ such that $w(t)^{\frac{4}{n-2}}g_\infty \in \mathcal{FSC}$ and $w(0) = 1$, $w'(0) = v$. Equivalently, $g_\infty$ is integrable if and only if $\mathcal{FSC}$ agrees with $S$ in a small neighborhood of 1 in $C^{2,\alpha}(M, g_\infty)$.

We remark that the integrability defined in Definition 2.2 is equivalent to the functional $F$ (as defined in Proposition 2.1) being constant in a neighborhood of 0 inside $\Lambda_0$ [I, Lemma 1].

Definition 2.3. If $\Lambda_0 = 0$, i.e. if $\mathcal{L}_\infty$ is injective, then we call $g_\infty$ a nondegenerate critical point. On the other hand, if $\Lambda_0$ is nonempty, we call $g_\infty$ degenerate.

Note that if $g_\infty$ is a nondegenerate critical point, then $g_\infty$ is automatically integrable in the above sense.

Now suppose that $g_\infty$ is a nonintegrable critical point. Because $F(v) = E_f(\Psi(v))$, defined in Proposition 2.1, is analytic, we may expand it in a power series
\[
F(v) = F(0) + \sum_{j \geq p} F_j(v)
\]
where $F_j$ is a degree-$j$ homogeneous polynomial on $\Lambda_0$ and $p$ is chosen so that $F_p$ is nonzero. We will call $p$ the order of integrability of $g_\infty$. We will also need a further hypothesis for nonintegrable critical points introduced in [I].

Definition 2.4. We say that $g_\infty$ satisfies the Adams-Simon positivity condition, $\mathcal{A}_{S^k}$ for short (here $p$ is the order of integrability of $g_\infty$), if it is nonintegrable and $F_{p+q}$ attains a positive maximum for some $\hat{v} \in S^k \subset \Lambda_0$. Recall that $F_p$ is the lowest-degree nonconstant term in the power series expansion of $F(v)$ around 0 and $S^k$ is the unit sphere in $\Lambda_0$. 

An important observation is that when the order of integrability \( p \) is odd, the Adams-Simon positivity condition is always satisfied. Moreover the order of integrability (at a critical point of \( E_f \)) always satisfies \( p \geq 3 \). Furthermore, we will show in the Appendix that

\[
F_3(v) = 2 \left( \frac{n+2}{n-2} \right) \left( \frac{4}{n-2} \right) \int_M R_{g_\infty} v^3 dV_{g_\infty}.
\]

(2.10)

### 3. Proof of Theorem 3.3

One of the tools for controlling the rate of convergence of the prescribed scalar curvature flow will be the Lojasiewicz-Simon inequality.

**Proposition 3.1.** Suppose that \( g_\infty \) satisfies (2.7). There is \( \theta \in (0, \frac{1}{2}], \epsilon > 0 \) and \( C > 0 \) (both depending only on \( n, f \) and \( g_\infty \)) such that for \( u \in C^{2,\alpha}(M, g_\infty) \) with \( \|u - 1\|_{C^{2,\alpha}(M, g_\infty)} < \epsilon \), then

\[
|E_f(u) - E_f(1)|^{1-\theta} \leq C\|DE_f(u)\|_{L^2(M, g_\infty)}.
\]

If \( g_\infty \) is an integrable critical point, then \( \theta = \frac{1}{2} \). If \( g_\infty \) is non-integrable, then this holds for some \( \theta \in (0, \frac{1}{2}], \) where \( p \) is the order of integrability of \( g_\infty \).

**Proof.** See Appendix (Section 6). \( \square \)

**Proof of Theorem 1.3.** We consider the prescribed scalar curvature flow \( g(t) = u(t) \frac{d}{dt} g_\infty \) which converges to \( g_\infty \) in \( C^2(M, g_\infty) \) as \( t \to \infty \). In Proposition 3.1 we have shown that there is a Lojasiewicz-Simon inequality near \( g_\infty \) for some \( \theta \in (0, \frac{1}{2}] \).

We emphasize that if we regard \( DE_f(g(t)) \) as an element of \( L^2(M, g_\infty) \), then

\[
DE_f(g(t)) = \frac{2(R_{g(t)} - \alpha(u(t)) f) u(t)^{\frac{n+2}{2}}}{\left( \int_M f dV_{g(t)} \right)^{\frac{n}{2}}}. 
\]

(3.1)

Since \( g(t) \to g_\infty \) as \( t \to \infty \), it follows from (2.7) that

\[
0 < m_1 \text{Vol}(M, g_\infty) \leq \int_M f dV_{g(t)} \leq m_2 \text{Vol}(M, g_\infty).
\]

(3.2)

where \( m_1 = \min_M f \) and \( m_2 = \max_M f \). Choose \( t_0 \) so that for \( t \geq t_0 \), \( \|u(t) - 1\|_{C^0(M, g_\infty)} \leq \frac{1}{2} \). This together with (2.3), (3.2) and Proposition 3.1 implies that

\[
\frac{d}{dt}(E_f(u(t)) - E_f(1)) = -\frac{n-2}{2} \int_M (R_{g(t)} - \alpha(t) f) u(t)^{\frac{2n+4}{2}} dV_{g_\infty} 
\]

\[
\leq -c \int_M (R_{g(t)} - \alpha(t) f) u(t)^{\frac{2(n+2)}{2}} dV_{g_\infty} = -c\|DE_f(u(t))\|_{L^2(M, g_\infty)}^2
\]

(3.3)

\[
\leq -c\|E_f(u(t)) - E_f(1)\|^{2-2\theta},
\]

where \( c > 0 \) is a constant depending only on \( n, f \) and \( g_\infty \) (that we let change from line to line). Let us first assume that the Lojasiewicz-Simon inequality is satisfied with \( \theta = \frac{1}{2} \), i.e. we are in the integrable case. Then (3.3) yields \( E_f(u(t)) - E_f(1) \leq C e^{-2\delta t} \), for some \( \delta > 0 \) depending only on \( n, f \) and \( g_\infty \), and \( C > 0 \) depending on \( g(0) \) (chosen so that this actually holds for all \( t \geq 0 \)). On the other hand, if
Lojasiewicz-Simon inequality holds with \( \theta \in (0, \frac{1}{2}) \), then the same argument shows that \( E_f(u(t)) - E_f(1) \leq C(1 + t)^{-\frac{\theta}{2}} \).

Exploiting the fact that the flow converges in \( C^2 \), we may use the Lojasiewicz-Simon inequality to compute

\[
\frac{d}{dt} (E_f(u(t)) - E_f(1))^\theta = \theta (E_f(u(t)) - E_f(1))^\theta - 1 \frac{d}{dt} (E_f(u(t)) - E_f(1))
\]

\[
\leq -c \theta (E_f(u(t)) - E_f(1))^\theta \| DE_f(u(t)) \|^2_{L^2(M, g_\infty)}
\]

\[
\leq -c \theta \left\| \frac{\partial u(t)}{\partial t} \right\|_{L^2(M, g_\infty)}
\]

where we have used (2.2) and (3.1) in the last equality. Thus, if \( \theta = \frac{1}{2} \) (recall \( \lim_{t \to \infty} u(t) = 1 \)), then

\[
\| u(t) - 1 \|_{L^2(M, g_\infty)} \leq \int_t^\infty \left\| \frac{\partial u(s)}{\partial s} \right\|_{L^2(M, g_\infty)} ds
\]

\[
\leq -c \int_t^\infty \frac{d}{ds} \left[ (E_f(u(s)) - E_f(1))^\frac{1}{2} \right] ds
\]

\[
= c (E_f(u(t)) - E_f(1))^{\frac{1}{2}} \leq C e^{-\delta t}.
\]

A similar computation if \( \theta \in (0, \frac{1}{2}) \) yields \( \| u(t) - 1 \|_{L^2(M, g_\infty)} \leq C(1 + t)^{-\frac{\theta}{2}} \).

To obtain \( C^2 \) estimates, we may interpolate between \( L^2(M, g) \) and \( W^{k,2}(M, g) \) for \( k \) large enough: interpolation [4] Theorem 6.4.5] and Sobolev embedding yields some constant \( \eta \in (0, 1) \) so that

\[
\| u(t) - 1 \|^{2,\alpha}_{W^{k,2}(M, g_\infty)} \leq \| u(t) - 1 \|_{L^2(M, g_\infty)}^{\eta} \| u(t) - 1 \|_{W^{k,2}(M, g_\infty)}^{1-\eta}.
\]

Because \( u(t) \) converges to 1 in \( C^{2,\alpha} \) (and thus in \( C^\infty \) by parabolic Schauder estimates and bootstrapping), the second term is uniformly bounded. Thus, exponential (polynomial) decay of the \( L^2 \) norm gives exponential (polynomial) decay of the \( C^{2,\alpha} \) norm as well.

\[
\text{\square}
\]

4. Slowly converging prescribed scalar curvature flow

In this section, we show that given a nonintegrable critical point \( g_\infty \) satisfying a particular hypothesis, there exists a prescribed scalar curvature flow \( g(t) \) such that \( g(t) \) converges to \( g_\infty \) exactly at a polynomial rate.

This section is organized as follows: In section 4.1, we show that the prescribed scalar flow can be represented by two different flows. To be more specific, we will project the flow equation to the kernel \( \Lambda_0 \) of \( \mathcal{L}_\infty \) and its orthogonal complement \( \Lambda_0^\bot \), respectively. In section 4.2, we solve the kernel-projected flow. In section 4.3, we solve the kernel-orthogonal projected flow. In section 4.4, we combine all the previous results to prove Theorem 1.4.

4.1. Projecting the prescribed scalar curvature flow with estimates. Here and in the sequel we will always use \( f'(t) \) to denote the time derivative of a function \( f(t) \). We will skip the proof of the following lemma, for its proof is the same as that of [7] Lemma 15.
Lemma 4.1. Assume that $g_\infty$ satisfies $AS_p$, as defined in Definition 3.4, i.e., $F_p|_{\mathbb{S}^k}$ achieves a positive maximum for some point $\hat{v}$ in the unit sphere $\mathbb{S}^k \subset \Lambda_0$. Then, for any fixed $T \geq 0$, the function

$$
\varphi(t) := \varphi(t, T) = (T + t)^{-\frac{8}{n-2}} \left( \frac{8}{(n-2)p(p-2)} F_p(\hat{v}) \right)^{\frac{p-2}{p}} \hat{v}
$$

solves $\frac{\partial}{\partial t} \varphi' + DF_p(\varphi) = 0$.

In the next result and subsequently in this section, we will always denote by $\|f(t)\|_{C^{k,\alpha}}$ the parabolic $C^{k,\alpha}$ norm on $(t, t+1) \times M$. More precisely, for $\alpha \in (0, 1)$, we define the seminorm

$$
\|f(t)\|_{C^{0,\alpha}} = \sup_{(s, x) \in (t, t+1) \times M} \frac{|f(s_1, x_1) - f(s_2, x_2)|}{(d_{g_\infty}(x_1, x_2)^2 + |t_1 - t_2|^2)^{\alpha}}
$$

and for $k \geq 0$ and $\alpha \in (0, 1)$, we define the norm

$$
\|f(t)\|_{C^{k,\alpha}} = \sum_{|\beta| + 2 \leq k} \sup_{(t, x) \in (t, t+1) \times M} |D^\beta D_t^2 f| + \sum_{|\beta| + 2 = k} |D^\beta D_t^2 f|_{C^{0,\alpha}}
$$

where the norm and derivatives in the sum are taken with respect to $g_\infty$. When we mean an alternative norm, we will always indicate the domain.

Lemma 4.2. For the functional $E_f$, there holds

$$
\|D^3 E_f(w)[u, v]\|_{C^{0,\alpha}} \leq C \|u\|_{C^{2,\alpha}} \|v\|_{C^{2,\alpha}}
$$

for some uniform constant $C > 0$. Furthermore, for $w_1, w_2$ such that $\|w_i - 1\|_{C^{2,\alpha}} < 1$, we have

$$
\|D^3 E_f(w_1)[v, w] - D^3 E_f(w_2)[u, u]\|_{C^{0,\alpha}} \leq C(\|w_1\|_{C^{2,\alpha}} + \|w_2\|_{C^{2,\alpha}}) \times (\|u\|_{C^{2,\alpha}} + \|v\|_{C^{2,\alpha}}) \|u - v\|_{C^{2,\alpha}}
$$

for some uniform constant $C > 0$.

Proof. This follows from the arguments in [7, Appendix A].

Lemma 4.3. There exists $T_0 > 0$, $\epsilon_0 > 0$ and $c > 0$, all depending on $g_\infty$ and $\hat{v}$, such that the following holds: Fix $T > T_0$. Then, for $\varphi(t)$ as in Lemma 4.1 and $w \in C^{2,\alpha}(M \times [0, \infty))$, and $u := 1 + \varphi + w^\top + \Phi(\varphi + w^\top) + w^\top$, the function

$$
E_0^\top(w) := \proj_{\Lambda_0} \left[ D E_f(u) u^{-\frac{1}{n-2}} - DE_f(u) \right]
$$

satisfies

$$
\|E_0^\top(w)\|_{C^{0,\alpha}} \leq c \left\{ (T + t)^{-\frac{2}{p-2}} + \|w^\top\|_{C^{0,\alpha}}^{p-1} + \|w^\top\|_{C^{2,\alpha}} \right\} \left\{ (T + t)^{-\frac{2}{p-2}} + \|w\|_{C^{2,\alpha}} \right\} ,
$$

$$
\|E_0^\top(w_1) - E_0^\top(w_2)\|_{C^{0,\alpha}} \leq c \left\{ (T + t)^{-\frac{2}{p-2}} + \|w_1^\top\|_{C^{0,\alpha}}^{p-1} + \|w_2^\top\|_{C^{0,\alpha}}^{p-1} + \|w_1^\top\|_{C^{2,\alpha}} + \|w_2^\top\|_{C^{2,\alpha}} \right\} \times \|w_1 - w_2\|_{C^{2,\alpha}}
$$

$$
+ c \left\{ (T + t)^{-\frac{2}{p-2}} + \|w_1\|_{C^{2,\alpha}} + \|w_2\|_{C^{2,\alpha}} \right\} \left\{ \|w_1^\top\|_{C^{0,\alpha}}^{p-2} + \|w_2^\top\|_{C^{0,\alpha}}^{p-2} \right\} \times \|w_1^\top - w_2^\top\|_{C^{0,\alpha}}
$$

$$
+ c \left\{ (T + t)^{-\frac{2}{p-2}} + \|w_1\|_{C^{2,\alpha}} + \|w_2\|_{C^{2,\alpha}} \right\} \|w_1^\top - w_2^\top\|_{C^{2,\alpha}}.$$
Identical estimates hold for $E_d^0(w) := \text{proj}_{\Lambda_0} \left[ D E_f(u) u^{-\frac{3}{n-2}} - D E_f(u) \right]$. Here, we are using the parabolic Hölder norms on $(t, t + 1) \times M$ as defined above; the bounds hold for each fixed $t \geq 0$, with the constants independent of $T$ and $t$.

Proof. First, we have

$$u^{-\frac{3}{n-2}} = 1 + \int_0^1 \frac{d}{ds} \left\{ 1 + s (\varphi + w^\top + \Phi(\varphi + w^\top) + w^\top) \right\}^{\frac{3}{n-2}} ds = 1 - \frac{4}{n-2} \int_0^1 (1 + s\phi)\,\frac{c}{n-2} \phi ds.$$ 

So, letting $E_0(w) := D E_f(u) u^{-\frac{3}{n-2}} - D E_f(u)$, we have

$$\|E_0(w)\|_{C^{0,\alpha}} \leq c \|DE_f(u)\|_{C^{0,\alpha}} \left( \|\varphi\|_{C^{0,\alpha}} + \|w^\top\|_{C^{0,\alpha}} + \|\Phi(\varphi + w^\top)\|_{C^{0,\alpha}} + \|w^\top\|_{C^{0,\alpha}} \right),$$

where we have used the fact that $\Phi(0) = 0$ and $\Phi$ is an analytic map. It follows from Taylor’s theorem that, for $\psi_{s, r} = 1 + r \left[ \varphi + w^\top + \Phi(\varphi + w^\top) + s w^\top \right]$, $D E_f(u) = D E_f(1 + \varphi + w^\top + \Phi(\varphi + w^\top)) + \int_0^1 D^2 E_f(\psi_{s, 1})[w^\top] ds$.

$$= D E_f(\Psi(\varphi + w^\top)) - \frac{8}{n-2} L_{\infty} w^\top + \int_0^1 \int_0^s D^3 E_f(\psi_{s, s})[w^\top, \varphi + w^\top + \Phi(\varphi + w^\top) + s w^\top] d\sigma ds$$

$$= \text{proj}_{\Lambda_0} D E_f(\Psi(\varphi + w^\top)) + \text{proj}_{\Lambda_0} D E_f(\Psi(\varphi + w^\top)) - \frac{8}{n-2} L_{\infty} w^\top + \int_0^1 \int_0^s D^3 E_f(\psi_{s, s})[w^\top, \varphi + w^\top + \Phi(\varphi + w^\top) + s w^\top] d\sigma ds$$

$$= D F(\varphi + w^\top) - \frac{8}{n-2} L_{\infty} w^\top + \int_0^1 \int_0^s D^3 E_f(\psi_{s, s})[w^\top, \varphi + w^\top + \Phi(\varphi + w^\top) + s w^\top] d\sigma ds$$

where the last equality follows from Proposition 2.1. Now, observe that $D F(0) = D^2 F(0) = \ldots D^{p-1} F(0) = 0$, where $p$ is the order of integrability. Therefore, by Taylor’s theorem, we have

$$\|D F(\varphi + w^\top)\|_{C^{0,\alpha}} \leq c \|\varphi + w^\top\|_{C^{p-1,\alpha}} \leq c \left( (T + t)^{-1 - \frac{3}{p-2}} + \|w^\top\|_{C^{p-1,\alpha}} \right).$$

By using the bound (4.3) on $D^3 E_f$, we can bound the other two terms in the above expression for $D E_f(u)$. Combining these, we have

$$\|D E_f(u)\|_{C^{0,\alpha}} \leq c \left( (T + t)^{-1 - \frac{3}{p-2}} + \|w^\top\|_{C^{p-1,\alpha}} + \|w^\top\|_{C^{2,\alpha}} \right).$$

We define

$$E_0^+(w) := \text{proj}_{\Lambda_0} E_0(w), \quad E_0^-(w) := \text{proj}_{\Lambda_0} E_0(w).$$
Now the asserted bounds for $E_0^\top(w)$ follow from the bound (4.4) on $E_0(w)$, the estimate (4.5) and the continuity of the map $\text{proj}_{\Lambda_0} : C^{0,\alpha}(M, g_\infty) \to \Lambda_0$,
\[ \|\text{proj}_{\Lambda_0} f\|_{C^{0,\alpha}(M,g_\infty)} \leq c \|\text{proj}_{\Lambda_0} f\|_{L^2(M,g_\infty)} \leq c \|f\|_{L^2(M,g_\infty)} \leq c \|f\|_{C^{0,\alpha}(M,g_\infty)}, \]
where the first inequality follows because of the finite dimensionality of $\Lambda_0$. Note that this is a spatial bound, so it does not include the $t$-Hölder norm, but the desired space-time norm bound follows easily from it: if $t$ is time dependent,
\[ \|\text{proj}_{\Lambda_0} f(t_1) - \text{proj}_{\Lambda_0} f(t_2)\|_{C^{0,\alpha}(M,g_\infty)} = \|\text{proj}_{\Lambda_0} (f(t_1) - f(t_2))\|_{C^{0,\alpha}(M,g_\infty)} \leq c \|f(t_1) - f(t_2)\|_{C^{0,\alpha}(M,g_\infty)}. \]
Dividing by $|t_1 - t_2|^p$ and taking the supremum over all such $t_1, t_2 \in (t, t + 1)$, the asserted bound follows. The bound for $E_0^\top(w_1) - E_0^\top(w_2)$ follows similarly. This together with the bound (4.4) on $E_0(w)$ and the estimate (4.5) gives the estimates for $E_0^\top(w)$.

The next result reduces the prescribed scalar curvature flow to two flows, one on $\Lambda_0$ and the other on $\Lambda_0^1$.

**Proposition 4.4.** There exists $T_0 > 0$, $\varepsilon_0 > 0$ and $c > 0$, all depending on $g_\infty$ and $\hat{v}$, such that the following holds: Fix $T > T_0$. Then, for $\varphi(t)$ as in Lemma 4.4, and $w \in C^{0,\alpha}(M \times [0, \infty))$, there are functions $E_0^\top(w)$ and $E_1^\top(w)$ such that $u := 1 + \varphi + \varphi^\top + \Phi(\varphi + \varphi^\top) + w^\perp$ is a solution to the prescribed scalar curvature flow if and only if
\[ \frac{8}{n-2} (w^\top)' + D^2 F_p(\varphi) w^\top = E_0^\top(w), \]
\[ (w^\top)' - \frac{1}{2} L_\infty w^\top = E_1^\top(w). \]
Here, as long as $\|w\|_{C^{2,\alpha}} \leq \varepsilon_0$, the error terms $E_0^\top$ and $E_1^\top$ satisfy
\[ \|E_0^\top(w)\|_{C^{0,\alpha}} \leq c \left( (T + t)^{-\frac{n-2}{2}} + \|w^\top\|_{C^{0,\alpha}} + \|w^\perp\|_{C^{2,\alpha}} \right) \left( (T + t)^{-\frac{n-2}{2}} + \|w\|_{C^{2,\alpha}} \right) \]
\[ + c(T + t)^{-\frac{n-2}{2}} (\|\varphi\|_{C^{0,\alpha}} + \|\varphi^\top\|_{C^{0,\alpha}} + \|\Phi\|_{C^{0,\alpha}} + \|w\|_{C^{2,\alpha}}), \]
\[ \|E_1^\top(w_1) - E_1^\top(w_2)\|_{C^{0,\alpha}} \leq c \left( (T + t)^{-\frac{n-2}{2}} + \|w_1^\top\|_{C^{0,\alpha}} + \|w_2^\top\|_{C^{0,\alpha}} + \|w_1^\perp\|_{C^{2,\alpha}} + \|w_2^\perp\|_{C^{2,\alpha}} \right) \]
\[ \times \|w_1 - w_2\|_{C^{2,\alpha}} \]
\[ + c \left( (T + t)^{-\frac{n-2}{2}} + \|w_1\|_{C^{2,\alpha}} + \|w_2\|_{C^{2,\alpha}} \right) \left( \|w_1^\top\|_{C^{0,\alpha}} + \|w_2^\top\|_{C^{0,\alpha}} + \|w_1^\perp\|_{C^{2,\alpha}} + \|w_2^\perp\|_{C^{2,\alpha}} \right) \]
\[ \times \|w_1^\top - w_2^\top\|_{C^{0,\alpha}} \]
\[ + c \left( (T + t)^{-\frac{n-2}{2}} + \|w_1\|_{C^{2,\alpha}} + \|w_2\|_{C^{2,\alpha}} \right) \|w_1^\top - w_2^\top\|_{C^{2,\alpha}} \]
\[ + c \left( (T + t)^{-\frac{n-2}{2}} \left( \|w_1^\top\|_{C^{0,\alpha}} + \|w_2^\top\|_{C^{0,\alpha}} + \|w_1^\perp\|_{C^{2,\alpha}} + \|w_2^\perp\|_{C^{2,\alpha}} \right) \right) \]
\[ \times \|w_1^\top - w_2^\top\|_{C^{0,\alpha}} \]
\[ + c(T + t)^{-\frac{n-2}{2}} \|w_1^\top - w_2^\top\|_{C^{0,\alpha}}. \]
\[
\|E^+(w)\|_{C^{0,\alpha}} \\
\leq c \left((T + t)^{-\frac{p-1}{2}} + \|w^T\|_{C^{p-1,\alpha}} + \|w^+\|_{C^{2,\alpha}}\right) \left((T + t)^{-\frac{p-1}{2}} + \|w\|_{C^{2,\alpha}}\right) \\
+ c \left((T + t)^{-\frac{p-1}{2}} + \|w\|_{C^{2,\alpha}}\right) \|w^+\|_{C^{2,\alpha}} \\
+ c \left((T + t)^{-\frac{p-1}{2}} + \|w\|_{C^{2,\alpha}}\right) \left((T + t)^{-\frac{p-1}{2}} + \|w\|_{C^{0,\alpha}}\right)
\]

\[
\|E^+(w)_1 - E^+(w)_2\|_{C^{0,\alpha}} \\
\leq c \left((T + t)^{-\frac{p-1}{2}} + \|w^T\|_{C^{p-1,\alpha}} + \|w^T\|_{C^{p-1,\alpha}} + \|w^+\|_{C^{2,\alpha}} + \|w^T\|_{C^{2,\alpha}} \right) \\
\times \|w_1 - w_2\|_{C^{2,\alpha}} \\
+ c \left((T + t)^{-\frac{p-1}{2}} + \|w_1\|_{C^{2,\alpha}} + \|w_2\|_{C^{2,\alpha}}\right) \left(\|w^T\|_{C^{p-2,\alpha}} + \|w^T\|_{C^{p-2,\alpha}}\right) \\
\times \|w_1 - w_2\|_{C^{0,\alpha}} \\
+ c \left((T + t)^{-\frac{p-1}{2}} + \|w_1\|_{C^{2,\alpha}} + \|w_2\|_{C^{2,\alpha}}\right) \|w^+_1 - w^+_2\|_{C^{2,\alpha}} \\
+ c \left((T + t)^{-\frac{p-1}{2}} + \|w_1\|_{C^{2,\alpha}} + \|w_2\|_{C^{2,\alpha}}\right) \|w^+_1 - w^+_2\|_{C^{\alpha,\alpha}} \\
+ c \left((T + t)^{-\frac{p-1}{2}} + \|w^+_1\|_{C^{0,\alpha}} + \|w^+_2\|_{C^{0,\alpha}}\right) \|w_1 - w_2\|_{C^{2,\alpha}}.
\]

Here we are using the parabolic Hölder norms on \((t, t + 1) \times M\) as defined above; the bounds hold for each fixed \(t \geq 0\), with the constants independent of \(T\) and \(t\).

**Proof.** Recall that \(u\) is a solution to the prescribed scalar curvature flow if and only if

\[
\frac{4}{n-2} \partial_t u - \frac{1}{2} DE_f(u) u^T = -2
\]

where as always in this paper, \(E_f\) is defined on the conformal class with the condition \(\int_M f u^{\frac{4}{n-2}} dV_{g_{\infty}} = 1\) and \(D(\cdot)\) is the corresponding constrained differential. We now project the prescribed scalar curvature flow equation onto \(\Lambda_0\) and \(\Lambda_0\), so \(u\) solves the prescribed scalar curvature flow if and only if the following two equations are satisfied:

\[
\frac{8}{n-2} (\varphi + w^T)' \\
= -\text{proj}_{\Lambda_0} \left[DE_f \left(1 + \varphi + w^T + \Phi(\varphi + w^T) + w^+\right)\right] - E^+_0(w),
\]

\[
\frac{8}{n-2} (\Phi(\varphi + w^T) + w^+)' \\
= -\text{proj}_{\Lambda_0} \left[DE_f \left(1 + \varphi + w^T + \Phi(\varphi + w^T) + w^+\right)\right] - E^+_1(w),
\]

where \(E^+_0(w)\) is defined as in Lemma 1.3. Now we claim that we can use Taylor’s theorem to show that

\[
\text{proj}_{\Lambda_0} \text{DE}_f \left(1 + \varphi + w^T + \Phi(\varphi + w^T) + w^+\right) \\
= \text{proj}_{\Lambda_0} \text{DE}_f \left(1 + \varphi + w^T + \Phi(\varphi + w^T)\right) + E^+_1(w),
\]

with the bounds

\[
\|E^+_1(w)\|_{C^{0,\alpha}} \leq c \left((T + t)^{-\frac{p-1}{2}} + \|w\|_{C^{2,\alpha}}\right) \|w^+\|_{C^{2,\alpha}},
\]

\[
\|E^+_1(w_1) - E^+_1(w_2)\|_{C^{0,\alpha}} \leq c \left(\|w_1\|_{C^{2,\alpha}} + \|w_2\|_{C^{2,\alpha}}\right) \|w^+_1 - w^+_2\|_{C^{2,\alpha}}.
\]
These follow from the integral form of the remainder in Taylor’s theorem: if we define $$\psi_{s,r} = 1 + r \{ \varphi + w^\top + \Phi(\varphi + w^\top) + sw^\top \}$$, then we have

$$E_1^T(w) = \int_0^1 \frac{d}{ds} \text{proj}_{\Lambda_0} DE_f (1 + \varphi + w^\top + \Phi(\varphi + w^\top) + sw^\top) \, ds$$

$$= \int_0^1 \text{proj}_{\Lambda_0} D \frac{d}{ds} DE_f(\psi_{s,1}) \, ds$$

$$= \int_0^1 \text{proj}_{\Lambda_0} D^2 E_f(\psi_{s,1}) [w^\top] ds$$

$$= \int_0^1 \text{proj}_{\Lambda_0} D^2 E_f(\psi_{s,1}) (g_\infty) [w^\top] ds + \int_0^1 \int_0^s \text{proj}_{\Lambda_0} \frac{d}{ds} D^2 E_f(\psi_{s,1}) (g_\infty) [w^\top] d\tilde{s} \, ds$$

$$= - \frac{4}{n-2} \int_0^1 \text{proj}_{\Lambda_0} C \infty w^\top ds + \int_0^1 \int_0^s \text{proj}_{\Lambda_0} \frac{d}{ds} D^2 E_f(\psi_{s,1}) (g_\infty) [w^\top] d\tilde{s} \, ds$$

$$= \int_0^1 \int_0^s \text{proj}_{\Lambda_0} \frac{d}{ds} D^3 E_f(\psi_{s,1}) (g_\infty) [w^\top, \varphi + w^\top + \Phi(\varphi + w^\top) + sw^\top] d\tilde{s} \, ds.$$  

By Lemma 12, the $$C^{0,\alpha}$$ norm of $$D^3 E_f(\psi_{s,1}) (g_\infty) [w^\top, \varphi + w^\top + \Phi(\varphi + w^\top) + sw^\top]$$ is uniformly bounded by

$$c \left( (T + t)^{-\frac{1}{p-2}} + ||w||_{C^{2,\alpha}} \right) ||w^\top||_{C^{2,\alpha}}$$

(as long as we choose $$T \geq T_0$$ large enough and $$||w||_{C^{2,\alpha}} \leq \epsilon_0$$ small enough to ensure that $$\psi_{s,1}$$ is sufficiently close to 1 in $$C^{2,\alpha}$$).

Recall that $$F(v) := E_f(\Psi(v)) = E_f(1 + v + \Phi(v))$$, and using the Lyapunov-Schmidt reduction (Proposition 2.11),

$$\text{proj}_{\Lambda_0} DE_f (1 + \varphi + w^\top + \Phi(\varphi + w^\top)) = DF(\varphi + w^\top).$$  

Furthermore, by analyticity (Proposition 2.11), $$DF$$ has a convergent power series representation around 0 with lowest order term of order $$p - 1$$. Thus, as long as $$\varphi + w^\top$$ is small enough, we may write

$$DF(\varphi + w^\top) = DF(\varphi) + D^2 F(\varphi)(w^\top) + E_2^T(w^\top),$$

where

$$||E_2^T(w^\top)||_{C^{0,\alpha}} \leq c \left( (T + t)^{-\frac{1}{p-2}} + ||w^\top||_{C^{0,\alpha}} \right) ||w^\top||_{C^{2,\alpha}}^2,$$

$$||E_2^T(w^\top) - E_2^T(w_1^\top)||_{C^{0,\alpha}} \leq c \left( (T + t)^{-\frac{1}{p-2}} (||w_1^\top||_{C^{0,\alpha}} + ||w_2^\top||_{C^{0,\alpha}}) + ||w_1^\top||_{C^{p-2,\alpha}} + ||w_2^\top||_{C^{p-2,\alpha}} \right) \times ||w_1^\top - w_2^\top||_{C^{0,\alpha}}.$$

We prove this in the case when the dimensional of $$\Lambda_0$$ is equal to one, namely for $$k = 1$$; but the general case can also be proved through similar arguments. Again, by analyticity, we have

$$DF(z) = \sum_{j=p-1}^\infty c_j z^j.$$
Thus
\[
\|E_2^T(w^T)\|_{C^{0,\alpha}} = \|DF(\varphi + w^T) - DF(\varphi) - D^2F(\varphi)(w^T)\|_{C^{0,\alpha}}
\]
\[
= \left\| \sum_{j=p-1}^{\infty} c_j \left[ (\varphi + w^T)^j - \varphi^j - j\varphi^{j-1}w^T \right] \right\|_{C^{0,\alpha}}
\]
\[
\leq \sum_{j=p-1}^{\infty} \sum_{l=2}^{j} \left| c_j \right| \left( \|\varphi\|_{C^{0,\alpha}}^j \|w^T\|_{C^{0,\alpha}}^l \right) - 1
\]
\[
\leq \|w^T\|^2_{C^{0,\alpha}} \sum_{j=p-1}^{\infty} \sum_{l=2}^{j} \left| c_j \right| \left( \|\varphi\|_{C^{0,\alpha}}^j \|w^T\|_{C^{0,\alpha}}^l + \|w^T\|_{C^{0,\alpha}}^{j-2} \right)
\]
\[
\leq 2^{p-2}\|\varphi\|_{C^{0,\alpha}}^{p-3}\|w^T\|_{C^{0,\alpha}}^{p-1} \sum_{j=p-1}^{\infty} \left| c_j \right| 2^{j+1-p}\|\varphi\|_{C^{0,\alpha}}^{j+1-p} + 2^{p-1}\|w^T\|_{C^{0,\alpha}}^{p-1} \sum_{j=p-1}^{\infty} \left| c_j \right| 2^{j+1-p}\|w^T\|_{C^{0,\alpha}}^{j+1-p}
\]

Because \(D^{p-1}F(z)\) has an absolutely convergent power series for every \(z\) sufficiently close to 0, choosing \(c_0\) small enough, \(T_0\) large enough, and using Lemma [1,1], we may guarantee that both sums are bounded above by 1. The asserted bound on \(E_2^T(w^T)\) follows. A similar argument yields the other bound.

By the results we have obtained so far, the \(A_0\)-component of the prescribed scalar curvature flow may be written as
\[
\frac{8}{n-2}(\varphi + w^T)'
\]
\[
= -\text{proj}_{A_0} \left[ DE_f \left( 1 + \varphi + w^T + \Phi(\varphi + w^T) + w^T \right) \right] - E_0^T(w)
\]
\[
= -\text{proj}_{A_0} \left[ DE_f \left( 1 + \varphi + w^T + \Phi(\varphi + w^T) \right) \right] - E_1^T(w) - E_0^T(w)
\]
\[
= -DF(\varphi + w^T) - E_1^T(w) - E_0^T(w)
\]
\[
= -DF(\varphi) - D^2F(\varphi)(w^T) - E_2^T(w^T) - E_1^T(w) - E_0^T(w)
\]

where the second equality follows from [4,10], the third equality follows from [4,12] and the last equality follows from [4,13]. Now, expanding \(F\) in a power series, \(F = F(0) + \sum_{j=p}^{\infty} F_j\), we may write the above expression as
\[
\frac{8}{n-2}(\varphi + w^T)'
\]
\[
= -DF_p(\varphi) - D^2F_p(\varphi)(w^T) + E_3^T(w) - E_2^T(w^T) - E_1^T(w) - E_0^T(w)
\]
\[
:= E_3^T(w)
\]

where
\[
E_3^T(w) = \sum_{j \geq p+1} (DF_j(\varphi) + D^2F_j(\varphi)w^T).
\]
On the other hand, by Lemma 4.11 we have that $\frac{8}{n-2}\varphi' = -DF_p(\varphi)$. Therefore, $w^\top$ must satisfy the equation

$$\frac{8}{n-2}(w^\top)' + D^2F_p(\varphi)w^\top = E^\top(w).$$

By analyticity, $E^\top_3(w)$ converges in $C^{0,\alpha}$ for $\|\varphi\|_{C^{2,\alpha}}$ and $\|w\|_{C^{2,\alpha}}$ small enough. Because each term in the sum is a homogeneous polynomial, we get the following error bound by using the formula for $\varphi$:

$$\|E^\top_3(w)\|_{C^{0,\alpha}} \leq c \left( (T + t)^{-\frac{n}{2n-2}} + (T + t)^{-\frac{n}{2n-2}} \|w^\top\|_{C^{0,\alpha}} \right),$$

(4.15)

$$\|E^\top_3(w_1) - E^\top_3(w_2)\|_{C^{0,\alpha}} \leq c(T + t)^{-\frac{n}{2n-2}} \|w^\top_1 - w^\top_2\|_{C^{0,\alpha}}.$$  

Combining (4.11), (4.14) and (4.15), we can see that $E^\top(w)$ satisfies the asserted bounds.

Now we turn to the $\Lambda^\perp_n$ portion of the prescribed scalar curvature flow. First, recall that by Proposition 2.3

$$\text{proj}_{\Lambda^\perp_n} D\Phi_f(\Psi(w + w^\top)) = 0.$$  

(4.16)

From the fact that the operators $D$ and $\text{proj}_{\Lambda^\perp_n}$ commute, we have

$$D\left( \text{proj}_{\Lambda^\perp_n} D\Phi_f(1) \right) = \text{proj}_{\Lambda^\perp_n} D^2\Phi_f(1) = -\frac{4}{n-2} \text{proj}_{\Lambda^\perp_n} L_\infty.$$  

(4.17)

We claim that we may use Taylor’s theorem to show that

$$\text{proj}_{\Lambda^\perp_n} D\Phi_f(\Psi(w + w^\top) + w^\perp) = -\frac{4}{n-2} L_\infty w^\perp - E^\top_1(w),$$  

where

$$\|E^\top_1(w)\|_{C^{0,\alpha}} \leq c \left( (T + t)^{-\frac{n}{2n-2}} + \|w\|_{C^{2,\alpha}} \right) \|w^\perp\|_{C^{2,\alpha}},$$

(4.19)

$$\|E^\top_1(w_1) - E^\top_1(w_2)\|_{C^{0,\alpha}} \leq c(\|w_1\|_{C^{2,\alpha}} + \|w_2\|_{C^{2,\alpha}}) \|w^\perp_1 - w^\perp_2\|_{C^{2,\alpha}}.$$  

To check this, we write

$$\text{proj}_{\Lambda^\perp_n} D\Phi_f(\Psi(w + w^\top) + w^\perp) = \text{proj}_{\Lambda^\perp_n} D\Phi_f(\Psi(w + w^\top) + sw^\perp) + \int_0^1 \frac{d}{ds} \text{proj}_{\Lambda^\perp_n} D\Phi_f(\Psi(w + w^\top) + sw^\perp) ds$$

$$= \int_0^1 \text{proj}_{\Lambda^\perp_n} D^2\Phi_f(\Psi(w + w^\top) + sw^\perp) [w^\perp] ds$$

$$= -\frac{4}{n-2} L_\infty w^\perp + \int_0^1 \left[ \text{proj}_{\Lambda^\perp_n} D^2\Phi_f(\Psi(w + w^\top) + sw^\perp) [w^\perp] - \text{proj}_{\Lambda^\perp_n} D^2\Phi_f(1)[w^\perp] \right] ds$$

where the second equality follows from (4.16), and the last equality follows from (4.17). Given this, we may control the asserted $C^{0,\alpha}$ bounds by the $C^{2,\alpha}$ norm of $\varphi$ and $w$, by an argument similar to $E^\top_1$. 


We also consider \( \Psi(\varphi + w^\top) := E^\perp_2(w) \) as an error term, as it satisfies

\[
\|E^\perp_2(w)|_{C^{0,\alpha}} \leq c \left((T + t)^{-\frac{n-2}{n-2}} + \|w\|_{C^{2,0}}\right) \left((T + t)^{-\frac{n-2}{n-2}} + \|w\|_{C^{0,\alpha}}\right), \\
\|E^\perp_2(w_1) - E^\perp_2(w_2)\|_{C^{0,\alpha}} \leq c \left((T + t)^{-\frac{n-2}{n-2}} + \|w_1\|_{C^{2,0}} + \|w_2\|_{C^{2,0}}\right) \|w'_1 - w'_2\|_{C^{0,\alpha}} \\
+ c \left((T + t)^{-\frac{n-2}{n-2}} + \|w'_1\|_{C^{0,\alpha}} + \|w'_2\|_{C^{0,\alpha}}\right) \|w_1 - w_2\|_{C^{2,0}}.
\]

Here we have used (4.20) and have controlled \( \|w\|_{L^2} \) by the \( C^{2,\alpha} \) norm.

Combining the above results, we can write the second equation in (4.9) as

\[
\frac{8}{n-2}(w^\perp)' = -\frac{8}{n-2}\Psi(\varphi + w^\top)' - E^\perp_0(w) \\
- \text{proj}_{\lambda^\perp} \left[ DE_f (1 + \varphi + w^\top + \Psi(\varphi + w^\top) + w^\perp) \right] \\
= -\frac{8}{n-2}\Psi(\varphi + w^\top)' - E^\perp_0(w) - \text{proj}_{\lambda^\perp} \left[ DE_f (\Psi(\varphi + w^\top) + w^\perp) \right] \\
= -\frac{8}{n-2}E^\perp_2(w) - E^\perp_0(w) + \frac{4}{n-2}C_\infty w^\perp + E^\perp_1(w)
\]

where the last equality follows from (4.18). Thus the kernel-orthogonal component of the prescribed scalar curvature flow is

\[
(w^\perp)' = \frac{1}{2}C_\infty w^\perp + E^\perp_1(w)
\]

where \( E^\perp_1(w) := -E^\perp_2(w) - \frac{n-2}{8}(E^\perp_0(w) - E^\perp_1(w)) \) satisfies the asserted bounds.

Combining the \( \Lambda_0 \) equation with the \( \Lambda^\perp_1 \) equation finishes the proof. \( \square \)

### 4.2. Solving the kernel-projected flow with polynomial decay estimates.

In this subsection we solve the kernel-projected flow (4.10). First, from the definition of \( \varphi \) in (4.11) and the fact that \( D^2F_p \) is homogeneous of degree \( p - 2 \),

\[
D^2F_p(\varphi) = (T + t)^{-1} \left( 8 \left( \frac{n-2}{n-2}p(p-2)F_p(\hat{v}) \right) \right) D^2F_p(\hat{v}).
\]

Let \( \mu_1, \ldots, \mu_k \) be the eigenvalues of \( \mathcal{D} \) and \( e_i \) the corresponding orthonormal basis in which \( \mathcal{D} \) is diagonalized. Then the kernel-projected flow is equivalent to the following system of ODEs for \( v_i := w^\top \cdot e_i \),

\[
\frac{8}{n-2}v_i' + \frac{\mu_i}{T + t} v_i = E_i^\perp := E^\perp \cdot e_i, \quad i = 1, \ldots, k.
\]

Fix for the rest of this subsection a number \( \gamma \) with \( \gamma \notin \{ \frac{n-2}{8}\mu_1, \ldots, \frac{n-2}{8}\mu_k \} \). Define the following weighted norms:

\[
\|u\|_{C^{0,\alpha}} := \sum_{t>0} [(T + t)^\gamma \|u(t)\|_{C^{0,\alpha}}] \quad \text{and} \quad \|u\|_{C^{0,\alpha}} := \|u\|_{C^{0,\alpha}} + \|u\|_{C^{0,\alpha}}.
\]

We recall that these Hölder norms are space-time norms on the interval \((t, t+1) \times M\), as defined in (4.2).
Given $\gamma$ as above, we define $\Pi_0 = \Pi_0(\gamma)$ by
\begin{equation}
\Pi_0 := \text{span} \left\{ v \in \Lambda_0 : \nabla v = \mu v, \text{ and } \mu > \frac{8}{n-2} \right\}.
\end{equation}
Moreover, let $\text{proj}_{\Pi_0} : \Lambda \rightarrow \Pi_0$ be the corresponding linear projector.

**Lemma 4.5.** For any $T > 0$ such that $\|E^T\|_{C^{0,\alpha}_{1+\gamma}} < \infty$, there is a unique $u$ with $u(t) \in \Lambda_0$, $t \in [0,\infty)$, satisfying $\|u\|_{C^0_t} < \infty$, $\text{proj}_{\Pi_0}(u(0)) = 0$, and such that $v_i := u \cdot e_i$ solves the system (4.20). Furthermore, we have the bound
\[ \|u\|_{C^{0,\alpha}_{1+\gamma}} \leq C\|E^T\|_{C^{0,\alpha}_{1+\gamma}}. \]

**Proof.** Letting $w_j := (T + t)^{-\frac{n-2}{8} \mu_j} v_j$, the system (4.20) is equivalent to
\[ w_j' = \frac{n-2}{8} (T + t)^{\frac{n-2}{8} \mu_j} E_j^T, \quad j = 1, \ldots, k. \]
Then, we claim that we may solve the $j$-th ODE as
\[ w_j(t) = \begin{cases} \alpha_j - \frac{n-2}{8} \int_0^t (T + \tau)^{\frac{n-2}{8} \mu_j} E_j^T(\tau) d\tau, & \text{if } \gamma > \frac{n-2}{8} \mu_j; \\ \alpha_j + \frac{n-2}{8} \int_0^t (T + \tau)^{\frac{n-2}{8} \mu_j} E_j^T(\tau) d\tau, & \text{if } \gamma < \frac{n-2}{8} \mu_j. \end{cases} \]
First suppose that $j$ is such that $\gamma > \frac{n-2}{8} \mu_j$. Then the claim would imply that
\[ v_j(t) = (T + t)^{-\frac{n-2}{8} \mu_j} w_j \]
\[ = (T + t)^{-\frac{n-2}{8} \mu_j} \alpha_j - \frac{n-2}{8} (T + t)^{-\frac{n-2}{8} \mu_j} \int_t^\infty (T + \tau)^{\frac{n-2}{8} \mu_j} E_j^T(\tau) d\tau. \]
To prove the claim, we check that the integral converges under our assumption on $E^T$:
\[ (T + t)^{-\frac{n-2}{8} \mu_j} \int_t^\infty (T + \tau)^{\frac{n-2}{8} \mu_j} E_j^T(\tau) d\tau \]
\[ \leq (T + t)^{-\frac{n-2}{8} \mu_j} \|E_j^T\|_{C^{0,\alpha}_{1+\gamma}} \int_0^\infty (T + \tau)^{\frac{n-2}{8} \mu_j - \gamma} d\tau \]
\[ = \left( \gamma - \frac{n-2}{8} \mu_j \right)^{-1} (T + t)^{-\frac{n-2}{8} \mu_j} \|E_j^T\|_{C^{0,\alpha}_{1+\gamma}} (T + t)^{-\frac{n-2}{8} \mu_j - \gamma} \]
\[ = C_j (T + t)^{-\gamma} \|E_j^T\|_{C^{0,\alpha}_{1+\gamma}}. \]
The previous estimate also shows that, since by assumption $\gamma > \frac{n-2}{8} \mu_j$, to have $\|u\|_{C^0_t} < \infty$, it must hold that $\alpha_j = 0$. On the other hand, if $\gamma < \frac{n-2}{8} \mu_j$, by requiring $\text{proj}_{\Pi_0} u(0) = 0$, we see that $\alpha_j = 0$. As a result, the bounds for $\|v_j\|_{C^0_t}$ follow from a similar calculation as before. Combining these two cases proves existence, uniqueness and the $\|u\|_{C^0_t}$ bound.

It thus remains to prove the inequality $\|u\|_{C^{0,\alpha}_{1+\gamma}} \leq C\|E^T\|_{C^{0,\alpha}_{1+\gamma}}$. By finite dimensionality, the (spatial) $C^{0,\alpha}(\mathcal{M})$-Hölder norms of each basis element in $\Lambda_0$ are uniformly bounded. Thus, it remains to show that the desired inequality holds for the Hölder norms in the time direction, along with the same thing for $u'(t)$.
Suppose that \( j \) is such that \( \gamma > \frac{n-2}{8} \mu_j \). Then, we have seen above that
\[
v_j(t) = -\frac{n-2}{8} (T+t)^{-\frac{n-2}{8}} \mu_j \int_t^\infty (T+\tau)^{\frac{n-2}{8}} \mu_j E_j^\top(\tau)d\tau,
\]
which gives
\[
v_j'(t) = \mu_j (T+t)^{-\frac{n-2}{8} \mu_j} \int_t^\infty (T+\tau)^{\frac{n-2}{8} \mu_j} E_j^\top(\tau)d\tau - \frac{n-2}{8} E_j^\top(t).
\]
Thus
\[
\|v_j'\|_{C^0,\alpha} \leq C \left( (T+t)^{-\frac{n-2}{8} \mu_j} \int_t^\infty (T+\tau)^{-\frac{n-2}{8} \mu_j} E_j^\top(\tau)d\tau \right)_{C^1} + C\|E_j^\top(t)\|_{C^0,\alpha}
\]
\[
\leq C \left( (T+t)^{-\frac{n-2}{8} \mu_j} \int_t^\infty (T+\tau)^{-\frac{n-2}{8} \mu_j} E_j^\top(\tau)d\tau \right)_{C^0} + C\|E_j^\top(t)\|_{C^0,\alpha}
\]
\[
\leq C(T+t)^{-1-\gamma} \|E_j^\top\|_{C^0,\alpha}.\]

On the other hand, the case of \( \gamma < \frac{n-2}{8} \mu_j \) can be easily be obtained through a similar argument. Combining these calculations, we obtain a Hölder estimate for \( v_j \). From this the claimed inequality follows. \( \square \)

4.3. Solving the kernel-orthogonal projected flow. Define the weighted norms
\[
\|u\|_{L^2_q} = \sup_{t \in [0, \infty)} [(T+t)^q \|u(t)\|_{L^2(M)}],
\]
where the \( L^2 \) norm is the spatial norm of \( u(t) \) on \( M \), taken with respect to \( g_\infty \), and
\[
\|u\|_{C^0,\alpha} = \sup_{t \geq 0} [(T+t)^\alpha \|u(t)\|_{C^2,\alpha}],
\]
where, as usual, the Hölder norms are the space-time norms defined in (4.2). Also, let

\[
\Lambda_\downarrow := \text{span}\{ \varphi \in C^\infty(\partial M) : L_\infty \varphi + \delta \varphi = 0, \delta > 0 \}^{L^2},
\]
\[
\Lambda_\uparrow := \text{span}\{ \varphi \in C^\infty(\partial M) : L_\infty \varphi + \delta \varphi = 0, \delta < 0 \}.
\]

From the spectral theory, \( L^2(M, g_\infty) = \Lambda_\uparrow \oplus \Lambda_0 \oplus \Lambda_\downarrow \) and \( \Lambda_\uparrow \) and \( \Lambda_0 \) are finite dimensional. Write the nonnegative integers as an ordered union \( \mathbb{N} = K_\uparrow \cup K_0 \cup K_\downarrow \), where the ordering of the indices comes from an ordering of the eigenfunctions of the \( L_\infty \) and the partitioning of \( \mathbb{N} \) corresponds to which of \( \Lambda_\downarrow \), \( \Lambda_0 \), or \( \Lambda_\uparrow \) the \( k \)-th eigenfunction of \( L_\infty \) lies in.

Lemma 4.6. For any \( T > 0 \) and \( q < \infty \) such that \( \|E^\downarrow\|_{L^2_q} < \infty \), there is a unique \( u(t) \) with \( u(t) \in \Lambda_\downarrow \), \( t \in [0, \infty) \), satisfying \( \|u\|_{L^2_q} < \infty \), \( \text{proj}_{\Lambda_\downarrow}(u(0)) = 0 \) and
\[
u' = L_\infty u + E^\downarrow.
\]
Furthermore, \( \|u\|_{L^2_q} \leq C\|E^\downarrow\|_{L^2_q} \) and \( \|u\|_{C^0,\alpha} \leq C\|E^\downarrow\|_{C^0,\alpha} \).

Proof. Let \( \varphi_i \) be an eigenfunction of \( L_\infty \) with eigenvalue \( -\delta_i \) which is orthogonal to the kernel \( \Lambda_0 \). The equation (4.22) reduces to the system
\[
u_i' + \delta_i u_i = E^\downarrow, \]
where \( u_i = \langle u, \varphi_i \rangle \) and \( E^\downarrow_i = \langle E^\downarrow, \varphi_i \rangle \). This is equivalent to
\[
(e^{\delta t} u_i)' = e^{\delta t} E^\downarrow_i.
\]
Thus, we may represent the components of the solution as

$$ u_i^\perp(t) = \beta_i e^{-\delta_i t} + e^{-\delta_i t} \int_0^t e^{\delta_i \tau} E_i^\perp(\tau)d\tau \text{ for } i \in K_\perp, $$

$$ u_i^\perp(t) = \beta_i e^{-\delta_i t} - e^{-\delta_i t} \int_t^\infty e^{\delta_i \tau} E_i^\perp(\tau)d\tau \text{ for } i \in K_\perp. $$

In particular, we have

$$ u(t) = \sum_{j \in K_\perp} \left( \beta_j e^{-\delta_j t} + e^{-\delta_j t} \int_0^t e^{\delta_j \tau} E_j^\perp(\tau)d\tau \right) \varphi_j $$

$$ + \sum_{j \in K_\perp} \left( \beta_j e^{-\delta_j t} - e^{-\delta_j t} \int_t^\infty e^{\delta_j \tau} E_j^\perp(\tau)d\tau \right) \varphi_j. $$

This sum is in an $L^2$ sense (but then elliptic regularity guarantees that the sum converges uniformly on compact time intervals). We note that for $i \in K_\perp$, if $\|u\|_{L^2} < \infty$, then necessarily $\beta_i = 0$. Furthermore, by requiring that $\text{proj}_{\Lambda} u(0) = 0$, we also have $\beta_i = 0$ for $i \in K_\perp$.

The $L^2$-bound for the first term in $u$ can be estimated as:

$$ \left\| \sum_{j \in K_\perp} u_j(t) \varphi_j \right\|_{L^2}^2 = \left\| \sum_{j \in K_\perp} \int_0^t e^{\delta_j (\tau-t)} E_j^\perp(\tau)d\tau \varphi_j \right\|_{L^2}^2 $$

$$ \leq \sum_{j \in K_\perp} \left( \int_0^t e^{\delta_j (\tau-t)} E_j^\perp(\tau)d\tau \right)^2 $$

$$ \leq \sum_{j \in K_\perp} \left( \int_0^t e^{\delta_{\min} (\tau-t)} E_j^\perp(\tau)d\tau \right)^2 $$

$$ \leq \left\| \int_0^t e^{\delta_{\min} (\tau-t)} E^\perp(\tau)d\tau \right\|_{L^2}^2 $$

where $\delta_{\min} = \min_{j \in K_\perp} \delta_j$ and the last inequality follows from the Parseval identity. Taking square roots gives

$$ \left\| \sum_{j \in K_\perp} u_j(t) \varphi_j \right\|_{L^2} \leq \left\| \int_0^t e^{\delta_{\min} (\tau-t)} E^\perp(\tau)d\tau \right\|_{L^2}^2 \leq \int_0^t e^{\delta_{\min} (\tau-t)} \| E^\perp \|_{L^2} d\tau, $$

and hence we can finally make use of our decay assumption on $E^\perp$ to get

$$ \left\| \sum_{j \in K_\perp} u_j(t) \varphi_j \right\|_{L^2} \leq \| E^\perp \|_{L^2} \int_0^t e^{\delta_{\min} (\tau-t)} (T+t)^{-q} d\tau. $$
We bound the integral term in the above equation as follows:

\[
\int_0^t e^{\delta_{\min}(\tau-t)}(T+\tau)^{-q}d\tau
\]

\[
= \int_0^{\frac{t}{2}} e^{\delta_{\min}(\tau-t)}(T+\tau)^{-q}d\tau + \int_{\frac{t}{2}}^t e^{\delta_{\min}(\tau-t)}(T+\tau)^{-q}d\tau
\]

\[
\leq T^{-q} \int_0^{\frac{t}{2}} e^{\delta_{\min}(\tau-t)}d\tau + \left(T + \frac{t}{2}\right)^{-q} \int_{\frac{t}{2}}^t e^{\delta_{\min}(\tau-t)}d\tau
\]

\[
\leq \delta_{\min}^{-1} T^{-q} \left(e^{-\delta_{\min} \frac{t}{2}} - e^{-\delta_{\min} t}\right) + \delta_{\min}^{-1} \left(T + \frac{t}{2}\right)^{-q} \left(1 - e^{-\delta_{\min} \frac{t}{2}}\right).
\]

Therefore, we have

\[
\left\| \sum_{j \in K_t} u_j(t)\varphi_j \right\|_{L^2} \leq C\|E^\perp\|L^2_q(T + t)^{-q}
\]

A similar argument holds for the \(K^*_t\) terms. From this, the asserted bounds for \(\|u\|_{L^2_q}\) follow readily.

We now consider the \(C^{2,\alpha}_q\) bounds for \(u\). By interior parabolic Schauder estimates \([27]\) Theorem 4.9, we have that for \(t \geq 1\),

\[
\|u(t)\|_{C^{2,\alpha}_q} \leq C \left(\sup_{s \in (t-1,t+1) \times M} |u(s, x)| + \|E^\perp\|_{C^{0,\alpha}((t-1,t+1) \times M)}\right).
\]

We emphasize that the \(C^{2,\alpha}_q\)-norm on the left-hand side is the space-time norm on \((t, t+1) \times M\), as defined in \((4.24)\). Note that, by Arzelà-Ascoli theorem, for \(\epsilon > 0\), there exists \(c(\epsilon) > 0\) such that for any function \(\varphi \in C^{0,\alpha}(\partial M)\),

\[
\sup_{x \in M} |\varphi(x)| \leq c(\epsilon) \|\varphi\|_{L^2(M)} + \epsilon \|\varphi\|_{C^{0,\alpha}(M)}.
\]

Combining \((4.25)\) and \((4.26)\), we get

\[
\|u(t)\|_{C^{2,\alpha}_q} \leq C \left(\sup_{s \in (t-1,t+1) \times M} \|u(s, x)\|_{L^2(M)} + \|E^\perp\|_{C^{0,\alpha}((t-1,t+1) \times M)}\right)
\]

\[
+ C\epsilon \|u(t)\|_{C^{0,\alpha}((t-1,t+1) \times M)}.
\]

Multiplying it by \((T + t)^q\) and taking the supremum over \(t \geq 1\) yields

\[
\sup_{t \geq 1} [(T + t)^q \|u(t)\|_{C^{2,\alpha}_q}]
\]

\[
\leq C \left(\sup_{t \geq 0} [(T + t)^q \|u(s, x)\|_{L^2(M)}] + \sup_{t \geq 0} [(T + t)^q \|E^\perp\|_{C^{0,\alpha}((t,t+2) \times M)}]\right)
\]

\[
+ C\epsilon \sup_{t \geq 0} [(T + t)^q \|u(t)\|_{C^{0,\alpha}((t,t+2) \times M)}]
\]

\[
\leq C(\|u\|_{L^2_q} + \|E^\perp\|_{C^{0,\alpha}_q}) + C\epsilon \|u\|_{C^{0,\alpha}_q}
\]

\[
\leq C(\|E^\perp\|_{L^2_q} + \|E^\perp\|_{C^{0,\alpha}_q}) + C\epsilon \|u\|_{C^{0,\alpha}_q}
\]

\[
\leq C\|E^\perp\|_{C^{0,\alpha}_q} + C\epsilon \|u\|_{C^{0,\alpha}_q}
\]

where the third inequality follows from \(\|u\|_{L^2_q} \leq C\|E^\perp\|_{L^2_q}\), which was proved earlier. To finish the proof, it remains to extend the supremum up to \(t = 0\). The
global Schauder estimates [27, Theorem 4.28] shows that
\[ \|u(t)\|_{C^{2,\alpha}((0,1)\times M)} \leq C \left( \sup_{s \in (0,1)} \|u(s, x)\|_{L^2(M)} + \epsilon \|u\|_{C^{0,\alpha}((0,1)\times M)} \right) + \left\| E^\perp \right\|_{C^{0,\alpha}((0,1)\times M)} + \left\| u(0) \right\|_{C^{2,\alpha}(M)}. \tag{4.28} \]
Except for the last term \( \|u(0)\|_{C^{2,\alpha}(M)} \) on the right-hand side of the above expression, the rest of the terms can be bounded in a manner similar to the argument used above. Note that
\[ u(0) = - \sum_{j \in K_1} \left( \int_0^\infty e^{\delta_j \tau} E_j^\perp(\tau) d\tau \right) \varphi_j. \]
The space \( \Lambda_\tau \) is finite-dimensional, so there must be a uniform constant \( C > 0 \) such that \( \|\varphi_j\|_{C^{2,\alpha}(M)} \leq C\|\varphi_j\|_{L^2(M)} \) for all \( j \in K_1 \). Using this we have that
\[ \|u(t)\|_{C^{2,\alpha}(M)}^2 \leq C \sum_{j \in K_1} \left( \int_0^\infty e^{\delta_j \tau} E_j^\perp(\tau) d\tau \right) \|\varphi_j\|_{C^{2,\alpha}(M)}^2 \leq C \sum_{j \in K_1} \left( \int_0^\infty e^{\delta_j \tau} E_j^\perp(\tau) d\tau \right) \|\varphi_j\|_{L^2(M)}^2 = C\|u(0)\|_{L^2(M)}^2. \tag{4.29} \]
Combining (4.27), (4.28), and (4.29), we obtain that
\[ \sup_{t \geq 0} [(T + t)^q \|u(t)\|_{C^{2,\alpha}}] \leq C\|E^\perp\|_{C^{0,\alpha}_q} + C\epsilon \|u\|_{C^{0,\alpha}_q}. \]
By choosing \( \epsilon \) small, we get the desired Hölder bounds. \( \square \)

4.4. Construction of a slowly converging flow. To proceed further, we define the norm
\[ \|f\|_{\gamma}^* := \|\text{proj}_{\Lambda_\gamma} f\|_{C^{0,\alpha}_\gamma} + \|\text{proj}_{\Lambda_\delta} f\|_{C^{2,\alpha}_{1+\gamma}}. \]
Recall that
\[ \|u\|_{C^{0,\alpha}_\gamma} = \sup_{t \geq 0} [(T + t)^\gamma \|u(t)\|_{C^{0,\alpha}}] + \sup_{t \geq 0} [(T + t)^{1+\gamma} \|u(t)\|_{C^{0,\alpha}}], \]
\[ \|u\|_{C^{2,\alpha}_{1+\gamma}} = \sup_{t \geq 0} [(T + t)^{1+\gamma} \|u(t)\|_{C^{2,\alpha}}], \]
where the Hölder norms are the space-time Hölder norms defined in (4.2). For \( \gamma \) to be specified below, the Banach space \( X \) is defined as
\[ X := \{ w : \|w\|_{\gamma}^* < \infty \}. \tag{4.30} \]

Proposition 4.7. Assume that \( g_\infty \) satisfies AS\(_p\). We may thus fix a point where \( F_{p|2^{k-1}} \) achieves a positive maximum and denote it by \( \hat{v} \). Define
\[ \varphi(t) = (T + t)^{-\frac{1}{p-2}} \left( \frac{8}{(n-2)p(p-2)F_p(\hat{v})} \right)^{\frac{1}{p-2}} \hat{v} \]
as in Lemma 4.7. Then, there exists \( C > 0 \), \( T > 0 \), \( \frac{1}{p-2} < \gamma < \frac{2}{p-2} \) and \( u(t) \in C^\infty(M \times (0,\infty)) \) such that \( u(t) > 0 \) for all \( t > 0 \), \( g(t) := u(t)^{\frac{1}{n-2}} g_\infty \) is a solution to the prescribed scalar curvature flow and
\[ \|w^\top(t) + \Phi(\varphi(t) + w^\top(t)) + w^\perp(t)\|_{\gamma}^* = \|u(t) - \varphi(t) - 1\|_{\gamma}^* \leq C. \]
Proof. We fix $\frac{1}{p-2} < \gamma < \frac{2}{p-2}$ so that $\gamma \notin \{\frac{2\mu_1}{p}, \ldots, \frac{2\mu_k}{p}\}$. By Proposition 4.4, the prescribed scalar curvature flow can be reduced to two flows, i.e., kernel projected flow and kernel-orthogonal projected flow, so it is enough to solve

$$\frac{8}{n-2}(w^\top)' + D^2 F_p(\varphi)w^\top = E^\top (w), \quad (w^\perp)' - \frac{1}{2}L_\infty w^\perp = E^\perp (w),$$

for $w(t)$ with $\|w\|_\gamma^* < C$. To do so, we will use the contraction mapping method. We define a map

$$S : \{w \in X : \|w\|_\gamma^* \leq 1\} \to X$$

where $X$ is the Banach space defined in (4.30), by defining $u := \text{proj}_{\Lambda_0} S(w)$ and $v := \text{proj}_{\Lambda^0} S(w)$ to be the solution of the kernel projected flow and the kernel-orthogonal projected flow respectively, i.e.

$$\frac{8}{n-2}u' + D^2 F_p(\varphi)u = E^\top (w) \quad \text{and} \quad v' - \frac{1}{2}L_\infty v = E^\perp (w).$$

Thus, we have defined the map $S(w)$ by its orthogonal projections onto $\Lambda_0$ and $\Lambda_0^\perp$. These solutions exist, in the right function spaces, by combining the bounds for the error terms in Proposition 4.4 with Lemmas 4.5 and 4.6. Furthermore, we have the explicit bound

$$\|\text{proj}_{\Lambda_0} S(w)\|_{C^0,\alpha} \leq c \|E^\top (w)\|_{C^0,\alpha}$$

$$\leq \int_{t \geq 0} (T + t)^{1+\gamma} \left\{ (T + t)^{-\frac{\gamma}{p-2}} + \|w\|_{C^0,\alpha}^\gamma + \|w^\top\|_{C^0,\alpha}^{p-1} \right\}$$

$$\times \left\{ (T + t)^{\frac{1}{p-2}} \right\}$$

$$+ \sup_{t \geq 0} \left\{ (T + t)^{1+\gamma} \right\}$$

$$+ \sup_{t \geq 0} \left\{ (T + t)^{1+\gamma} \right\}$$

$$\leq c \left( T^{-\frac{1}{p-2}} + (T^{-\frac{1}{p-2}}) \|w\|_\gamma^* \right)$$

where the first inequality follows from Lemma 4.5, the second inequality follows from Proposition 4.4 and the last inequality follows from the definition of the norm $\|\cdot\|_\gamma^*$ and the bound

$$\|w^\top\|_{C^2,\alpha((t, t+1) \times M)} \leq c \left( \|w^\top\|_{C^0,\alpha((t, t+1) \times M)} + \|(w^\top)'\|_{C^0,\alpha((t, t+1) \times M)} \right).$$

More precisely, we have absorbed powers of $(T + t)$ into the various $w$ norms and bounded the result by $\|w\|_\gamma^*$. And the bound (4.31) is a consequence of the fact that $\Lambda_0$ is finite-dimensional and that the parabolic $C^2,\alpha$-Hölder norms only contain at most one time derivative.
By the same argument, using Proposition 4.3 and Lemma 4.6, we obtain the similar bound for the kernel-orthogonal projected part:

$$
\|\text{proj}_{\Lambda_0^+} S(w)\|_{C^{2,\alpha}_{1+\gamma}} \leq c \sup_{t \geq 0} \left\{ (T + t)^{1+\gamma} \left( (T + t)^{-\frac{\gamma}{p-2}} + \|w\|_{C^{0,\alpha}} + \|w_t\|_{C^{2,\alpha}} \right) \right. \\
+ \sup_{t \geq 0} \left\{ \left\{ (T + t)^{1+\gamma - \frac{\gamma}{p-2}} + (T + t)^{1+\gamma} \|w\|_{C^{2,\alpha}} \right\} \|w_t\|_{C^{2,\alpha}} \right. \\
+ \sup_{t \geq 0} \left\{ \left\{ (T + t)^{1+\gamma - \frac{\gamma}{p-2}} + (T + t)^{1+\gamma} \|w\|_{C^{2,\alpha}} \right\} \times \left\{ (T + t)^{-\frac{\gamma}{p-2}} + \|w_t\|_{C^{0,\alpha}} \right\} \right. \\
\left. \leq c \left( T^\gamma - \frac{\gamma}{p-2} + \left( T^{\frac{1}{p-2}} + T^{(p-2)(\frac{1}{p-2} - \gamma)} \right) \|w\|_{\gamma}^* \right). \right.
$$

Therefore, we have

$$
\|S(w)\|_{\gamma}^* = \|\text{proj}_{\Lambda_0} S(w)\|_{C^{0,\alpha}_{1+\gamma}} + \|\text{proj}_{\Lambda_0^+} S(w)\|_{C^{2,\alpha}_{1+\gamma}} \\
\leq c \left\{ T^\gamma - \frac{\gamma}{p-2} + \left( T^{\frac{1}{p-2}} + T^{(p-2)(\frac{1}{p-2} - \gamma)} \right) \right\} \|w\|_{\gamma}^*.
$$

Thus, because $\gamma \in \left( \frac{1}{p-2}, \frac{2}{p-2} \right)$, by choosing $T$ large enough we can ensure that $S$ maps $\{w : \|w\|_{\gamma}^* \leq 1\} \subset X$ into itself. Now let $w_1, w_2 \in \{w : \|w\|_{\gamma}^* \leq 1\}$. We have to calculate $\|S(w_1) - S(w_2)\|_{\gamma}^*$ to make sure that $S$ is a contraction mapping. To do so, we can obtain the following inequalities by using the same argument we have just used:

$$
\|\text{proj}_{\Lambda_0} S(w_1) - \text{proj}_{\Lambda_0} S(w_2)\|_{C^{0,\alpha}_{1+\gamma}} \leq c \left( T^{\frac{1}{p-2}} + T^{(p-2)(\frac{1}{p-2} - \gamma)} \right) \|w_1 - w_2\|_{\gamma}^*, \\
\|\text{proj}_{\Lambda_0^+} S(w_1) - \text{proj}_{\Lambda_0^+} S(w_2)\|_{C^{2,\alpha}_{1+\gamma}} \leq c \left( T^{\frac{1}{p-2}} + T^{(p-2)(\frac{1}{p-2} - \gamma)} \right) \|w_1 - w_2\|_{\gamma}^*.
$$

Therefore we have that

$$
\|S(w_1) - S(w_2)\|_{\gamma}^* \leq c \left( T^{\frac{1}{p-2}} + T^{(p-2)(\frac{1}{p-2} - \gamma)} \right) \|w_1 - w_2\|_{\gamma}^*.
$$

Thus, by enlarging $T$ if necessary, we have that $S$ is a contraction map. This finishes the proof.

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. From Proposition 4.3, we have constructed $\varphi(t)$ and $u(t)$ so that

$$
\varphi(t) = (T + t)^{-\frac{1}{2}} \left( \frac{8}{(n-2)p(p-2)F_p(\hat{v})} \right)^{\frac{1}{p-2}} \hat{v},
$$

$u(t) \to g_\infty$ is a solution to the prescribed scalar curvature flow, and $u(t) = 1 + \varphi(t) + \hat{w}(t) := 1 + \varphi(t) := 1 + \varphi(t) + w^T(t) + \Phi (\varphi(t) + w^T(t)) + w^T(t)$, where $\hat{w}(t)$ satisfies $\|\hat{w}\|_{C^0} \leq C(1 + t)^{-\gamma}$ for some $C > 0$ and all $t \geq 0$. We have arranged that $\gamma > 1/(p-2)$, which implies that $\varphi(t)$ is decaying slower than $\hat{w}(t)$. Thus

$$
\|u(t) - 1\|_{C^0} \geq C(1 + t)^{-\frac{1}{p-2}}
$$
as $t \to \infty$. From this, the assertion follows. 

5. Examples satisfying $A S\rho$

In this section, we provide a family of metrics which satisfy $A S\rho$. This allows us, via Theorem 1.4, to conclude the existence of slowly converging prescribed scalar curvature flows.

Consider the warped product metric $g = dr^2 + w(r)^2 g_{S^{n-1}}$ on $M = [0, \pi] \times S^{n-1}$ where $g_{S^{n-1}}$ is the standard metric on the $(n-1)$-dimensional unit sphere $S^{n-1}$. We will show that there exists a smooth function $\psi \in C^\infty(M)$ satisfying $\psi \in \Lambda_0$ and $F_3(\psi) \neq 0$. By the warped product structure of the metric $g$, we see that the scalar curvature $R_g$ and the Laplacian $\Delta_g$ are given as follows:

$$R_g = -2(n - 1) \frac{w''}{w} + (n - 1)(n - 2) \left( \frac{1 - (w')^2}{w^2} \right),$$

$$\Delta_g = \frac{\partial^2}{\partial r^2} + (n - 1) \frac{w'}{w} \frac{\partial}{\partial r}.$$ 

Thus, for a radial function $\psi = \psi(r) \in C^\infty(M)$, the condition $\psi \in \Lambda_0$ is equivalent to the condition that $\psi$ is a solution to the following ODE:

$$(5.1) \quad \psi'' + (n - 1) \frac{w'}{w} \psi' - 2 \frac{w''}{w} \psi + \frac{(n - 2)\psi}{w^2} (1 - (w')^2) = 0.$$

Note that $(w(r), \psi(r)) = (\sin r, \cos r)$ satisfies (5.1) with $R_g = n(n-1)$. Actually, in this case $(M, g)$ is nothing but the $n$-dimensional unit sphere $S^n$ equipped with the standard metric $g_{S^n}$. By (2.13), we have

$$F_3(\psi) = \frac{8(n + 2)}{(n - 2)^2} \int_M R_g \psi^3 dV_g$$

$$= \frac{8n(n - 1)(n + 2)}{(n - 2)^2} \int_{S^n} (\cos r)^3 dV_g$$

$$= \frac{8n^2(n - 1)(n + 2)}{(n - 2)^2} \alpha(n) \int_0^\pi r^{n-1}(\cos r)^3 dr$$

where $\alpha(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$. Now suppose $n = 3$, then

$$\int_0^\pi r^2(\cos r)^3 dr = -\frac{14\pi}{9},$$

so we have $F_3(\psi) < 0$. Therefore, $(S^3, g_{S^3})$ satisfies $A S_3$.

In the following, we are going to perturb $w(r)$ slightly to construct a new metric $g_\epsilon$ on $S^3$ whose scalar curvature is not constant. To this end, for $\epsilon > 0$, we consider a smooth positive function $w_\epsilon(r) : [0, \pi] \to \mathbb{R}$ satisfying the following conditions:

- $|w_\epsilon(r) - \sin r| < \epsilon$,
- $w_\epsilon(0) = w_\epsilon(\pi) = 0$, $w'_\epsilon(0) = 1$, $w'_\epsilon(\pi) = -1$,
- $w^{(even)}_\epsilon(0) = w^{(even)}_\epsilon(\pi) = 0$,
- $R_{g_\epsilon} = -2(n - 1) \frac{w''_\epsilon}{w_\epsilon} + (n - 1)(n - 2) \left( \frac{1 - (w'_\epsilon)^2}{w^2_\epsilon} \right) \neq \text{const.}$

Note that the conditions in the second and third lines guarantee that the metric $g_\epsilon = dr^2 + (w_\epsilon)^2 g_{S^2}$ is a smooth metric on $S^3$. Note also that, since we just perturb
\( w(r) \) slightly, it follows from \( R_g = n(n - 1) \) that the scalar curvature \( R_{g_\epsilon} \) of \( g_\epsilon \) is still positive. Let \( \psi_r(r) \) be a solution of the following ODE:

\[
\psi''_\epsilon + 2\frac{w'}{w_\epsilon} \psi'_\epsilon - 2\frac{w''}{w_\epsilon^2} \psi_\epsilon + \frac{\psi_\epsilon}{w_\epsilon^2} (1 - (w_\epsilon')^2) = 0.
\]

By a basic perturbation theory for ODE, if \( \epsilon \) is sufficiently small, then we have \( \psi_\epsilon \approx \cos r \). Hence, if we choose \( \epsilon > 0 \) small enough, we have

\[
|F^g_3(\psi_\epsilon) - F^g_{3\beta}(\cos r)|
\]

\[
= C \left| \int_M R_g \psi_\epsilon^3 dV_g - \int_M R_{g_{\beta}} (\cos r)^3 dV_{\beta} \right|
\]

\[
\leq C \left| \int_M R_g \psi_\epsilon^3 dV_g - \int_M R_{g_{\beta}} \psi_\epsilon^3 dV_{\beta} \right| + C \int_M (R_g \psi_\epsilon^3 - R_{g_{\beta}} (\cos r)^3) dV_{\beta}
\]

\[
\leq C \epsilon \int_M R_g \psi_\epsilon^3 dV_{\beta} + C \epsilon \text{Vol}(\mathbb{S}^3, g_{\beta})
\]

where the last inequality follows from the fact that for any \( \varphi \in C^\infty(M) \), we have

\[
\left| \int_M \varphi dV_{g_1} - \int_M \varphi dV_{g_2} \right| \leq C \epsilon \int_M |\varphi| dV_{g_1} \text{ if } ||g_1 - g_2||_{C^0(M)} \leq \epsilon.
\]

This shows that we can find a radial function \( \psi_\epsilon \) that makes \( F^g_3(\psi_\epsilon) \) closed to \( F^g_{3\beta}(\cos r) \) as much as we want, assuming \( \epsilon > 0 \) is small enough. Hence, we have \( F^g_3(\psi_\epsilon) < 0 \) for sufficiently small \( \epsilon \). In particular, we can conclude that \((\mathbb{S}^3, dr^2 + w_\epsilon^2 g_{\beta})\) satisfies \( \text{AS}_3 \).

6. Appendix : The Lojasiewicz-Simon Inequality

In this appendix, we prove Proposition [3,1]

**Definition 6.1** (Lojasiewicz-Simon inequality). Suppose that \( \mathcal{B} \) is a Banach space and \( U \subset \mathcal{B} \) is an open subset. Fix a functional \( E \in C^2(U, \mathbb{R}) \), and denote by \( DE \in C^1(U, \mathcal{B}') \) its first derivative. We will additionally fix a Banach space \( \mathcal{W} \) with a continuous embedding \( \mathcal{W} \hookrightarrow \mathcal{B}' \). For \( x_0 \in U \) a critical point of \( E \), i.e. \( DE(x_0) = 0 \), we say that \( E \) satisfies the Lojasiewicz-Simon inequality with exponent \( \theta \in (0, \frac{1}{2}] \) near \( x_0 \) if there exists a neighborhood \( x_0 \in V \subset U \) as well as constants \( C > 0 \) such that

\[
|E(x) - E(x_0)|^{1-\theta} \leq C \|DE(x)\|_{\mathcal{W}} \text{ for all } x \in V.
\]

Notice that if \( B = W = \mathbb{R}^n \), this reduces to the classical Lojasiewicz inequality [28]. Simon [35] showed that the classical Lojasiewicz inequality could be extended to a Banach space setting.

Our basic strategy is as follows: We will first find a necessary condition for \((\mathcal{B}, W, E)\) to satisfy the Lojasiewicz inequality. And then, by showing that our functional \( Q \) meets the condition, we will prove Theorem [1,3] by using the Lojasiewicz inequality. To this end, we need the following:

**Proposition 6.2.** [13, Theorem 3.10] Fix \( \mathcal{B}, U \subset \mathcal{B}, E \in C^2(M, \mathbb{R}), \mathcal{W} \hookrightarrow \mathcal{B}' \) and \( x_0 \in U \) with \( DE(x_0) = 0 \) as in the previous definition. We also define the second derivative \( \mathcal{L} := D^2E \in C(U, \mathcal{B}(\mathcal{B}', \mathcal{B}')) \), where \( \mathcal{B}(\mathcal{B}', \mathcal{B}') \) is the space of continuous maps between the Banach spaces \( \mathcal{B} \) and \( \mathcal{B}' \). We will suppose the following hypotheses are satisfied:
(A) The kernel \( \ker \mathcal{L}(x_0) \subset \mathcal{B} \) is complemented in \( \mathcal{B} \), i.e. there exists a projection \( P \in \mathfrak{B}(\mathcal{B}, \mathcal{B}) \) such that range \( P = \ker \mathcal{L}(x_0) \). It follows from this that \( \mathcal{B} = \ker \mathcal{L}(x_0) \oplus \ker P \) is a topological direct sum. Denote by \( P' \in \mathfrak{B}(\mathcal{B}', \mathcal{B}') \) the adjoint map.

(B1) The map \( \mathcal{W} \to \mathcal{B}' \) is a continuous embedding.
(B2) The adjoint projection \( P' \) leaves \( \mathcal{W} \) invariant.
(B3) The map \( \mathcal{W} \to \mathcal{B}' \) is in \( C^1(U, \mathcal{W}) \).
(B4) We have range range \( P = \ker P' \cap \mathcal{W} \).

Under these hypotheses, we may find a neighborhood \( U_0 \) of 0 in \( \ker \mathcal{L}(x_0) \) and a neighborhood \( U_1 \) of 0 in \( \ker P \) as well as a function \( H \in C^1(U_0, U_1) \) parameterizing the natural constraint, i.e.

\[
\{ x \in U_0 + U_1 : D\mathcal{E}(x_0 + x) \in (\ker \mathcal{L}(x_0))' \} = \{ x + H(x) : x \in U_0 \}.
\]

Recall that the natural constraint is then

\[
S := \{ x_0 + x_1 + H(x) : x \in U_0 \}.
\]

Finally, suppose that the following holds:

(C) The function \( E(x_0 + \cdot) \) satisfies the Lojasiewicz inequality on the natural constraint \( S \) with exponent \( \theta \in (0, \frac{1}{2}] \). More precisely, we assume that

\[
|E(x_0 + x + H(x)) - E(x_0)|^{1-\theta} \leq C\|D\mathcal{E}(x_0 + x + H(x))\|_W
\]

for all \( x \in U_0 \).

Then the functional \( E \) satisfies the Lojasiewicz-Simon inequality near \( x_0 \) with the same exponent \( \theta \in (0, \frac{1}{2}] \).

Proof of Proposition 3.1. To prove this, we will show that the hypotheses of Proposition 2.1 are satisfied for the functional \( E_f \). We work with the Banach space \( \mathcal{B} := C^{2,\alpha}(M, g_\infty) \) and \( \mathcal{W} := L^2(M, g_\infty) \), and fix \( U \) a small enough ball around 1 in \( C^{2,\alpha}(M, g_\infty) \) so that Proposition 2.1 is applicable in \( U \).

For hypothesis (A), one first checks that the \( L^2 \) projection map \( \text{proj}_{\Lambda_0} \) restricts to a continuous map from \( C^{2,\alpha}(M, g_\infty) \) onto \( \Lambda_0 \). From this, it follows that \( \Lambda_0' \) is complemented in the dual space \( C^{2,\alpha}(M, g_\infty)' \), and its complement \( (\Lambda_0')^\perp \) may be canonically identified with \( (\Lambda_0')' \).

Hypothesis (B) is satisfied as follows: Consider the map

\[
\mathcal{W} = L^2(M, g_\infty) \hookrightarrow C^{2,\alpha}(M, g_\infty)', \quad \psi \mapsto \left( \varphi \mapsto \int_M \psi \varphi dV_{g_\infty} \right).
\]

(B1) This map is continuous.
(B2) The map \( \text{proj}_{\Lambda_0}' \in \mathfrak{B}(C^{2,\alpha}(M, g_\infty)') \) leaves \( L^2(M, g_\infty) \) invariant; of course, here we are considering the composition

\[
\text{proj}_{\Lambda_0} C^{2,\alpha}(M, g_\infty) \to \Lambda_0 \to C^{2,\alpha}(M, g_\infty).
\]

(B3) That \( D\mathcal{E}_f \in C^1(U, L^2(M, g_\infty)) \) follows from the explicit form of \( D\mathcal{E}_f \) given in (2.4).

(B4) Finally, we must verify that range \( \mathcal{L}_\infty = (\Lambda_0')^\perp \cap L^2(M, g_\infty) \). The fact that range \( \mathcal{L}_\infty \subset (\Lambda_0')^\perp \cap L^2(M, g_\infty) \) is obvious because \( \mathcal{L}_\infty \) is formally self-adjoint on \( L^2 \). The other inclusion follows from the \( L^2 \) spectral decomposition of \( \mathcal{L}_\infty \).
Thus to prove the Lojasiewicz-Simon inequality with exponent $\theta \in (0,1/2]$, it is enough to check (C). Recall that in Proposition 2.1 we have defined $F(v) = E_f(\Psi(v))$. In the integrable case, clearly $F(v) = F(0)$, so $F$ satisfies the Lojasiewicz-Simon inequality for $\theta = 1/2$. In general, $F$ is an analytic function whose power series has its first nonzero term of degree $p$, by definition. In [13 Proposition 2.3], Chill proved that if $f : U \subset \mathbb{R} \to \mathbb{R}$ is a $C^k$-function satisfying $f^{(j)}(a) = 0$ for $1 \leq j \leq k - 1$, and $f^{(k)}(a) \neq 0$, then $f$ satisfies the Lojasiewicz inequality near $a$ with exponent $\theta = 1/k$. Thus we may conclude that $F$ satisfies the Lojasiewicz-Simon inequality with exponent $\theta \in (0,1/2]$.

\[ \square \]

7. Appendix : Computing $F_3$

In this Appendix, we prove (2.10) by computing the term $F_3$ at a metric $g_\infty$ satisfying (2.7). First we will show that $F_1(v) = F_2(v) = 0$. To check this, notice that $DF(w)[v] = DE_f(\Psi(w))[\Psi_f(w)[v]]$. Thus $DF(0) = 0$ as $DE_f(1) = 0$ as $1$ is a critical point of the functional $E_f$ (by assumption, $g_\infty \in FSC$) and $\Psi(0) = 1$. Therefore, $F_1(0) = 0$. Similarly, $D^2F(w)[v,u] = D^2E_f(\Psi(w))[\Psi_f(w)[u],\Psi_f(w)[v]] + \langleDE_f(\Psi(w)),D^2\Psi(w)[v,u]\rangle$. When setting $w = 0$, $\Psi(0) = 1$, $D\Psi(0) = id$, and

\[
D^2F(0)[v,u] = D^2E_f(1)[u,v] + \langle DE_f(1), D^2\Psi(0)[v,u]\rangle \\
= -\frac{8}{n-2} \langle L_\infty u, v \rangle + \langle DE_f(1), D^2\Psi(0)[v,u]\rangle.
\]

As before, the second term vanishes. The first term vanishes because $u$ is in the kernel of $L_\infty$ by assumption.

To compute $D^3F(0)$, we may in fact compute $D^3\tilde{F}(0)$, where $\tilde{F} : \Lambda_0 \to \mathbb{R}$ is defined by $\tilde{F}(v) = E_f(1+v)$. We first compute $D^3F$: 

\[
D^3F(w)[v,u,z] = D^3E_f(\Psi(w))[\Psi_f(w)[v],\Psi_f(w)[u],\Psi_f(w)[z]] \\
+ D^2E_f(\Psi(w))[D^2\Psi(w)[u,z],\Psi_f(w)[v]] \\
+ D^2E_f(\Psi(w))[\Psi_f(w)[u],D^2\Psi(w)[v,z]] \\
+ D^2E_f(\Psi(w))[\Psi_f(w)[z],D^2\Psi(w)[v,u]] \\
+ \langle DE_f(\Psi(w)),D^3\Psi(w)[v,u,z]\rangle.
\] (7.1)

Setting $w = 0$, and using similar considerations as before (in particular noting that $D^2E_f(1)[\cdot]$ is self-adjoint), we obtain $D^3F(0)[v,u,z] = D^3E_f(1)[v,u,z]$. Performing the same computation for $D^3\tilde{F}(0)$ yields the same result. Next, we compute
On the other hand, it follows from (2.4)-(2.7) that
\[
\frac{1}{2}D^3F(0)[v, u, z] = \frac{d}{dt}\left[\int_M \left(\frac{4(n-1)}{n-2}(\nabla g_\infty v, \nabla g_\infty u) + R_{g_\infty} vu - \frac{n+2}{n-2}\alpha(1+tz)f v u\right) dV_{g_\infty}\right]_{t=0}
\]
(7.2) = \int_M \left(\frac{4(n-1)}{n-2}(\nabla g_\infty v, \nabla g_\infty u) + R_{g_\infty} vu - \frac{n+2}{n-2}\alpha f v u\right) dV_{g_\infty}
\]
\[
\frac{d}{dt}\left(\int_M f(1+tz)\frac{d\alpha}{dt} dV_{g_\infty}\right)_{t=0} - \frac{n+2}{n-2}\int_M f v u \frac{d\alpha}{dt}(1+tz) dV_{g_\infty} \cdot \frac{1}{\left(\int_M f dV_{g_\infty}\right)^{\frac{n-2}{n}}}
\]

We emphasize that here we are regarding \(\frac{d}{dt}\alpha(1+tz)\) as an element of \(L^2(M, g_\infty)\).

It follows from (2.7) that
\[
\int_M \left(\frac{4(n-1)}{n-2}(\nabla g_\infty v, \nabla g_\infty u) + R_{g_\infty} vu - \frac{n+2}{n-2}\alpha v u\right) dV_{g_\infty}
\]
\[
= \int_M \left(-\frac{4(n-1)}{n-2}\Lambda_{g_\infty} v - \frac{4}{n-2}R_{g_\infty} v\right) dV_{g_\infty} = -\frac{4}{n-2}\int_M u\Lambda_{g_\infty} v dV_{g_\infty}.
\]

Hence, the first term on the right hand side of (7.2) is zero, since \(v \in \Lambda_0 = \ker \Lambda_\infty\).

On the other hand, it follows from (2.4)-(2.7) that
\[
\frac{d}{dt}\alpha(1+tz)\left|_{t=0}\right. = \frac{2\int_M R_{g_\infty} z dV_{g_\infty}}{\int_M f dV_{g_\infty}} - \frac{2n}{n-2}\left(\int_M R_{g_\infty} dV_{g_\infty}\right) \left(\int_M f z dV_{g_\infty}\right) \left(\int_M f dV_{g_\infty}\right)^{\frac{n-2}{n}}
\]
\[
= -\frac{4}{n-2}\int_M R_{g_\infty} z dV_{g_\infty}.
\]

Hence, if we are regarding \(\frac{d}{dt}\alpha(1+tz)\) as an element of \(L^2(M, g_\infty)\), we have
\[
\int_M f v u \frac{d\alpha}{dt}(1+tz) dV_{g_\infty} = -\frac{4}{n-2}\left(\int_M R_{g_\infty} dV_{g_\infty}\right) \int_M f v u z dV_{g_\infty}.
\]

Combining these with (2.5) and (2.7), we can rewrite (7.2) as
\[
D^3E_f(1)[v, u, z] = 2\left(\frac{n+2}{n-2}\right)\left(\frac{4}{n-2}\right)\alpha \int_M f v u z dV_{g_\infty}
\]
\[
= 2\left(\frac{n+2}{n-2}\right)\left(\frac{4}{n-2}\right)\int_M R_{g_\infty} v u z dV_{g_\infty}.
\]

This together with (7.1) implies that
\[
D^3F(0)[v, u, z] = 2\left(\frac{n+2}{n-2}\right)\left(\frac{4}{n-2}\right)\int_M R_{g_\infty} v u z dV_{g_\infty},
\]
which proves (2.10).
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