Entropic characteristics of subset of states.

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1 Introduction

This paper is devoted to systematic study of the properties of the quantum entropy and of the Holevo capacity (in what follows the $\chi$-capacity) considered as a function of a set of quantum states.

It is known that the quantum entropy is concave lower semicontinuous function on the set of all quantum states with the range $[0; +\infty]$, but it has bounded and even continuous restrictions to some nontrivial closed subsets of states [11], [23]. The problem of characterization of such subsets of states arises in many applications, in particular, in the condition of existence of an optimal measure for constrained quantum channel [7]. In this paper we consider this and some other problems related to the quantum entropy.

By the HSW theorem the $\chi$-capacity of a set of states defines the maximal rate of transmission of classical information, which can be achieved by using this set as an alphabet and nonentangled encoding in the transmitter followed by entangled measurement-decoding procedure in the receiver [5], [19]. Usually the notion of the $\chi$-capacity is related to the notion of a quantum channel. But it is easy to see that the $\chi$-capacity of a channel is uniquely defined by the output set of this channel. So, we may consider the $\chi$-capacity as a function of a set of states [20]. Despite some limitations this approach provides a convenient way to study the $\chi$-capacity. Namely, treating the $\chi$-capacity as a function of a set of states we obtain a certain flexibility in studying its properties since in this case we may speak about the $\chi$-capacity of an arbitrary set of states, not necessary of an output set of a particular channel. From this point of view the $\chi$-capacity is a nonnegative nonadditive
function of a set ("nonadditive measure") possessing many interesting properties, which detailed investigation seems to be useful for the development of the infinite dimensional quantum information theory.

We begin in section 3 with considering the conditions of boundedness and of continuity of the restriction of the quantum entropy to subsets of quantum states as well as the conditions of existence of the Gibbs state of these sets (propositions 1a, 3a, 4, 6a and corollaries 1,2,3). It is also shown that the quantum entropy is continuous at a particular state with respect to the convergence defined by the relative entropy if and only if this state has the sufficient rate of decreasing of the spectrum (proposition 2). The relations between several properties of sets of states and the corresponding properties of so called "classical projections" of these sets are considered (proposition 5). The obtained observations show, in particular, that discontinuity and unboundedness of the quantum entropy has purely classical nature (the note at the end of the section).

In section 4 the definition of the $\chi$-capacity of an arbitrary set of states and its general properties are considered.

First of all in subsection 4.1 the notion of the optimal average state as the unique state inheriting the most important properties of the average state of an optimal ensemble in the finite dimensional case is introduced (theorem 1 and corollary 4).

Then in subsection 4.2 the general properties of the $\chi$-capacity as a function of a set of states are considered (theorem 2 and corollaries 8,9). In particular, it is shown that every set with finite $\chi$-capacity is relatively compact and is contained in the maximal set with the same $\chi$-capacity. This compactness result implies many interesting observations concerning continuity of the $\chi$-capacity with respect to monotonous families of sets and to the problem of existence of the minimal closed set with given $\chi$-capacity. It also implies the following result related to quantum channels: if the $\chi$-capacity of an infinite dimensional channel constrained by a particular set is finite then the image of this set under this channel is relatively compact, in particular, every unconstrained channel with finite $\chi$-capacity has relatively compact output (corollary 10).

The lower and the upper bounds for the $\chi$-capacity of finite unions is obtained (proposition 7, remark 7).

It turns out that the obtained results concerning the $\chi$-capacity imply several observations concerning general properties of sets of states and of the quantum entropy (corollaries 5,6,7, remark 6, the note after corollary 8).
Finally, in subsection 4.3 the notion of an optimal measure of a set of states is considered and the generalized "maximal distance property" (cf.[20]) is proved (proposition 8), which implies necessary condition of existence of an optimal measure (corollary 11). Sufficient condition of existence of an optimal measure is obtained (theorem 3).

The general results of sections 3 and 4 are illustrated in section 5, where different types of sets of states are considered and their properties are explored.

The conditions of boundedness and of continuity of the restriction of the entropy to the several sets of states as well as the conditions of existence of the Gibbs state of these sets are obtained (propositions 1a,3a,6a,9a,10,12 and corollary 12).

The $\chi$-capacity and the optimal average state of the several sets of states are determined and the related properties (existence of an optimal measure, regularity) are explored (propositions 1b,3b,6b,9b,11,12).

The following examples of sets with finite $\chi$-capacity are constructed (in subsections 5.1, 5.2, 5.3 and 5.5 correspondingly):

- the closed countable set having no optimal measure;
- the closed set having no minimal closed subset with the same $\chi$-capacity;
- the decreasing sequence of closed sets with the same positive $\chi$-capacity, having the intersection with zero $\chi$-capacity;
- the closed set having optimal measure, but having no atomic optimal measure.

Section 6 is devoted to the "constructive" approach to the definition of the $\chi$-capacity and of the optimal average state for an arbitrary set of quantum states. It is shown that both these notions can be defined by a finite dimensional construction and a limiting procedure similarly to the case of the entropy and of the relative entropy (theorem 4). This provides a principal possibility of numerical approximation of the $\chi$-capacity and of the optimal average state of a set of general quantum states.

## 2 Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space, $\mathfrak{B}(\mathcal{H})$ - the set of all bounded operators in $\mathcal{H}$ with the cone $\mathfrak{B}_+(\mathcal{H})$ of all positive operators, $\mathfrak{S}(\mathcal{H})$ - the Banach
space of all trace-class operators with the trace norm $\| \cdot \|_1$ and $\mathcal{S}(\mathcal{H})$ - the closed convex subset of $\mathcal{T}(\mathcal{H})$ consisting of all density operators in $\mathcal{H}$, which is complete separable metric space with the metric defined by the trace norm. Each density operator uniquely defines a normal state on $\mathcal{B}(\mathcal{H})$ [2], so, in what follows we will also for brevity use the term ”state”. Note that convergence of a sequence of states to a state in the weak operator topology is equivalent to convergence of this sequence to this state in the trace norm [3]. We will use the following compactness criterion for subsets of states: a closed subset $\mathcal{K}$ of states is compact if and only if for any $\varepsilon > 0$ there is a finite dimensional projector $P$ such that $\text{Tr} \rho P \geq 1 - \varepsilon$ for all $\rho \in \mathcal{K}$. [15],[7]

In what follows $\log$ denotes the function on $[0, +\infty)$, which coincides with the natural logarithm on $(0, +\infty)$ and vanishes at zero. Let $A$ and $B$ be positive trace class operators. Let $\{|i\rangle\}$ be a complete orthonormal set of eigenvectors of $A$. The entropy is defined by $H(A) = -\sum_i \langle i | A \log A | i \rangle$ while the relative entropy – as $H(A \parallel B) = \sum_i \langle i | A \log A - A \log B + B - A | i \rangle$, provided $\text{ran} A \subseteq \text{ran} B$,\footnote{ran denotes the closure of the range of an operator in $\mathcal{H}$} and $H(A \parallel B) = +\infty$ otherwise (see [9] for more detailed definition). The entropy and the relative entropy are nonnegative lower semicontinuous (in the trace-norm topology) concave and convex functions of their arguments correspondingly [9],[11],[23]. We will use the following inequality

$$H(\rho \parallel \sigma) \geq \frac{1}{2} \| \rho - \sigma \|_1^2, \quad (1)$$

which holds for arbitrary states $\rho$ and $\sigma$ in $\mathcal{S}(\mathcal{H})$ [11].

The relative entropy $H(\rho \parallel \sigma)$ for two states $\rho$ and $\sigma$ can be considered as a measure of divergence of these states which classical analog is called Kullback-Leibler divergence. Despite the fact that this measure is not a metric it is possible to introduce the notion of convergence of a sequence of states $\{\rho_n\}$ to a particular state $\rho_*$ defined by the condition $\lim_{n \to +\infty} H(\rho_n \parallel \rho_*) = 0$. This type of convergence plays an important role in this paper and it will be called $H$-convergence. By inequality (1) the $H$-convergence is stronger than the convergence defined by the trace norm.

For arbitrary set $\mathcal{A}$ let $\text{co}(\mathcal{A})$ and $\overline{\text{co}}(\mathcal{A})$ be the convex hull and the convex closure of the set $\mathcal{A}$ correspondingly, let $\text{Ext}(\mathcal{A})$ be the set of all extreme points of the set $\mathcal{A}$ [14].

Speaking about continuity of a particular function on some set of states we mean continuity of the restriction of this function to this set.
Arbitrary finite collection \( \{ \rho_i \} \) of states in \( \mathcal{S}(\mathcal{H}) \) with corresponding set of probabilities \( \{ \pi_i \} \) is called ensemble and is denoted by \( \{ \pi_i, \rho_i \} \). The state \( \bar{\rho} = \sum_i \pi_i \rho_i \) is called the average state of the ensemble. Following \[7\] we treat an arbitrary Borel probability measure \( \mu \) on \( \mathcal{S}(\mathcal{H}) \) as generalized ensemble and the barycenter of the measure \( \mu \) defined by the Pettis integral

\[
\bar{\rho}(\mu) = \int_{\mathcal{S}(\mathcal{H})} \rho \mu(d\rho)
\]
as the average state of this ensemble. In this notations the conventional ensembles correspond to measures with finite support. For arbitrary closed subset \( \mathcal{A} \) of \( \mathcal{S}(\mathcal{H}) \) we denote by \( \mathcal{M}(\mathcal{A}) \) the set of all probability measures supported by the set \( \mathcal{A} \) \[21\].

In what follows an arbitrary ensemble \( \{ \pi_i, \rho_i \} \) is considered as a particular case of probability measure and is also denoted by \( \mu \), especially in the cases in which the specific features of an ensemble are not essential. In particular, a convex mixture of ensembles is defined as a convex mixture of the corresponding probability measures.

Consider the functionals

\[
\chi(\mu) = \int H(\rho \| \bar{\rho}(\mu)) \mu(d\rho) \quad \text{and} \quad \hat{H}(\mu) = \int H(\rho) \mu(d\rho).
\]

In \[7\] (proposition 1 and the proof of the theorem) it is shown that both these well defined functionals are lower semicontinuous on \( \mathcal{M}(\mathcal{S}(\mathcal{H})) \) and

\[
\chi(\mu) = H(\bar{\rho}(\mu)) - \hat{H}(\mu) \quad \text{for arbitrary } \mu \text{ such that } H(\bar{\rho}(\mu)) < +\infty.
\]

If \( \mu = \{ \pi_i, \rho_i \} \) then

\[
\chi(\{ \pi_i, \rho_i \}) = \sum_{i=1}^n \pi_i H(\rho_i \| \bar{\rho}) \quad \text{and} \quad \hat{H}(\{ \pi_i, \rho_i \}) = \sum_{i=1}^n \pi_i H(\rho_i).
\]

In analysis of the \( \chi \)-capacity we shall use Donald’s identity \[4\],[11]

\[
\sum_{i=1}^n \pi_i H(\rho_i \| \bar{\rho}) = \sum_{i=1}^n \pi_i H(\rho_i \| \bar{\rho}) + H(\bar{\rho} \| \bar{\rho}), \quad \text{(3)}
\]
which holds for arbitrary ensemble \( \{ \pi_i, \rho_i \} \) of \( n \) states with the average state \( \bar{\rho} \) and arbitrary state \( \hat{\rho} \).

We shall also use the generalized integral version of Donald’s identity \([7]\)

\[
\int H(\rho \| \hat{\rho}) \mu(\rho \| \hat{\rho}) = \int H(\rho \| \bar{\rho} \mu) \mu(\rho \| \bar{\rho}) + H(\bar{\rho} \mu \| \hat{\rho}), \tag{4}
\]

which holds for arbitrary probability measure \( \mu \) with the barycenter \( \bar{\rho}(\mu) \) and arbitrary state \( \hat{\rho} \).

The generalized Donald’s identity (4) implies the following observation.

**Lemma 1.** Let \( \{ \mu_k \}_{k=1}^m \) be a finite set of probability measures on \( \mathcal{S}(\mathcal{H}) \) and \( \{ \lambda_k \}_{k=1}^m \) be a probability distribution. Then

\[
\chi \left( \sum_{k=1}^m \lambda_k \mu_k \right) = \sum_{k=1}^m \lambda_k \chi \left( \mu_k \right) + \chi \left( \{ \lambda_k, \bar{\rho}(\mu_k) \}_{k=1}^m \right).
\]

In the case \( m = 2 \) for arbitrary \( \lambda \in [0; 1] \) the following inequality holds

\[
\chi (\lambda \mu_1 + (1 - \lambda) \mu_2) \geq \lambda \chi (\mu_1) + (1 - \lambda) \chi (\mu_2) + \frac{\lambda(1-\lambda)}{2} \| \bar{\rho}(\mu_2) - \bar{\rho}(\mu_1) \|_1^2.
\]

**Proof.** Let \( \mu = \sum_{k=1}^m \lambda_k \mu_k \). By definition

\[
\chi (\mu) = \sum_{k=1}^m \lambda_k \int H(\rho \| \bar{\rho}(\mu)) \mu_k(\rho).\]

Applying generalized Donald’s identity (4) to each inner integral in the right side of the above expression we obtain the main identity of the lemma.

To prove the inequality in the case \( m = 2 \) it is sufficient to apply inequality (1) for the estimation of the relative entropies in the main identity of the lemma:

\[
\lambda H(\bar{\rho}_1 \| \lambda \bar{\rho}_1 + (1 - \lambda) \bar{\rho}_2) + (1 - \lambda) H(\bar{\rho}_2 \| \lambda \bar{\rho}_1 + (1 - \lambda) \bar{\rho}_2)
\]

\[
\geq \frac{1}{2} \lambda \| (1 - \lambda)(\bar{\rho}_2 - \bar{\rho}_1) \|_1^2 + \frac{1}{2} (1 - \lambda) \| \lambda (\bar{\rho}_2 - \bar{\rho}_1) \|_1^2
\]

\[
= \frac{1}{2} \lambda (1 - \lambda) \| \rho_2 - \rho_1 \|_1^2, \quad \square
\]

Note that lemma 1 implies the following inequality

\[
H(\rho_1 + (1 - \lambda) \rho_2) \geq \lambda H(\rho_1) + (1 - \lambda) H(\rho_2) + \frac{\lambda(1-\lambda)}{2} \| \rho_2 - \rho_1 \|_1^2, \quad (5)
\]

valid for arbitrary states \( \rho_1 \) and \( \rho_2 \). To show this it is sufficient to consider spectral decompositions of these states as probability measures on \( \mathcal{S}(\mathcal{H}) \).
3 On properties of the quantum entropy

In this section the properties of restrictions of the quantum entropy to sets of quantum states are considered.

Let $\mathcal{A}$ be a closed set of states with finite $\sup_{\rho \in \mathcal{A}} H(\rho)$. If this supremum is achieved at a particular state in $\mathcal{A}$ then this state is usually called the Gibbs state \[23\]. We will denote it by $\Gamma(\mathcal{A})$. Inequality (5) implies the following simple observation.

**Lemma 2.** Let $\mathcal{A}$ be a closed convex subset of states and let $\{\rho_n\}$ be an arbitrary sequence of states in $\mathcal{A}$ such that

$$
\lim_{n \to +\infty} H(\rho_n) = \sup_{\rho \in \mathcal{A}} H(\rho) < +\infty.
$$

Then this sequence converges to the uniquely defined state $\rho_\star(\mathcal{A})$ in $\mathcal{A}$.\[^2\]

If the Gibbs state $\Gamma(\mathcal{A})$ exists then it coincides with the state $\rho_\star(\mathcal{A})$ and the restriction of the entropy to the set $\mathcal{A}$ is continuous at the state $\Gamma(\mathcal{A})$.

**Proof.** By the assumption for arbitrary $\varepsilon > 0$ there exists $N_\varepsilon$ such that $H(\rho_n) > \sup_{\rho \in \mathcal{A}} H(\rho) - \varepsilon$ for all $n \geq N_\varepsilon$. Inequality (5) with $\lambda = 1/2$ implies

$$
\sup_{\rho \in \mathcal{A}} H(\rho) - \varepsilon \leq \frac{1}{2} H(\rho_{n_1}) + \frac{1}{2} H(\rho_{n_2})
$$

and hence $\|\rho_{n_2} - \rho_{n_1}\|_1 < \sqrt{8}\varepsilon$ for all $n_1 \geq N_\varepsilon$ and $n_2 \geq N_\varepsilon$. Thus the sequence $\{\rho_n\}$ is a Cauchy sequence and hence it converges to a particular state $\rho_\star$ in $\mathcal{A}$. It is easy to see that this state $\rho_\star$ does not depend on the choice of the sequence $\{\rho_n\}$, so, it is determined only by the set $\mathcal{A}$. Denote this state by $\rho_\star(\mathcal{A})$.

If the Gibbs state $\Gamma(\mathcal{A})$ exists then by the above observation it coincides with the state $\rho_\star(\mathcal{A})$. The continuity assertion follows from lower semicontinuity of the entropy. □

Following [7] an unbounded positive operator $H$ in $\mathcal{H}$ with discrete spectrum of finite multiplicity will be called $\mathcal{H}$-operator. Let $Q_n$ be the spectral projector of $H$ corresponding to the lowest $n$ eigenvalues. In accordance with [6] we shall denote

$$
\text{Tr} \rho H = \lim_{n \to \infty} \text{Tr} \rho Q_n H,
$$

\[^2\]By using the arguments from the proof of theorem 1 in section 4 it possible to show $H$-convergence of the sequence $\{\rho_n\}$ to the state $\rho_\star(\mathcal{A})$. By using this and proposition 2 below we conclude that $\rho_\star(\mathcal{A}) = \Gamma(\mathcal{A})$ if there exists $\lambda < 1$ such that $\text{Tr}(\rho_\star(\mathcal{A}))^\lambda < +\infty$.  

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where the sequence on the right side is monotonously nondecreasing. In [6,7] it is shown that any compact subset \( K \) of \( \mathcal{S}(\mathcal{H}) \) is contained in the convex compact set \( K_{H,h} = \{ \rho \in \mathcal{S}(\mathcal{H}) \mid \text{Tr} \rho H \leq h \} \) defined by a particular \( \mathcal{H} \)-operator \( H \) and by a positive number \( h \). Let \( h_m(H) \) be the minimal eigenvalue of \( H \) and \( \mathcal{H}_m(H) \) be the corresponding (finite dimensional) eigen subspace.

Note that \( K_{H,h} \) is empty if \( h < h_m(H) \), \( K_{H,h} = \mathcal{S}(\mathcal{H}_m(H)) \) if \( h = h_m(H) \) and \( K_{H,h} \) necessarily contains infinite dimensional states if \( h > h_m(H) \).

As it is shown in the following proposition properties of the restriction of the quantum entropy to the set \( K_{H,h} \) is determined by the increase coefficient \( ic(H) \) of the \( \mathcal{H} \)-operator \( H \) defined as

\[
ic(H) = \inf\{ \lambda > 0 \mid \text{Tr} \exp(-\lambda H) < +\infty \}
\]

with \( ic(H) = +\infty \) if \( \text{Tr} \exp(-\lambda H) = +\infty \) for all \( \lambda > 0 \).

It is known [1,23] that under the condition \( ic(H) = 0 \) the entropy is continuous on the compact set \( K_{H,h} \) and achieves its (finite) maximum on this set at the Gibbs state having the form \( (\text{Tr} \exp(-\lambda H))^{-1} \exp(-\lambda H) \). The following proposition generalizes this observation. It also provides necessary and sufficient condition of existence of the Gibbs state of the set \( K_{H,h} \) and reveals another sense of the term "increase coefficient" for \( ic(H) \). Let

\[
h_*(H) = \frac{\text{Tr} H \exp(-ic(H)H)}{\text{Tr} \exp(-ic(H)H)} \quad \text{if} \quad \text{Tr} \exp(-ic(H)H) < +\infty \quad \text{and} \quad h_*(H) = +\infty \quad \text{otherwise}.
\]

**Proposition 1a.** Let \( H \) be a \( \mathcal{H} \)-operator in the Hilbert space \( \mathcal{H} \) and \( h \) be a positive number such that \( h > h_m(H) \).

The entropy is bounded on the set \( K_{H,h} \) if and only if \( ic(H) < +\infty \).

The entropy is continuous on the set \( K_{H,h} \) if and only if \( ic(H) = 0 \).

If \( h \leq h_*(H) \) then \( \sup_{\rho \in K_{H,h}} H(\rho) = \lambda^* h + \log \text{Tr} \exp(-\lambda^* H) \), where \( \lambda^* = \lambda^*(H,h) \geq ic(H) \) is uniquely defined by the equation

\[
\text{Tr} H \exp(-\lambda H) = h \text{Tr} \exp(-\lambda H),
\]

and there exists the Gibbs state \( \Gamma(K_{H,h}) = (\text{Tr} \exp(-\lambda^* H))^{-1} \exp(-\lambda^* H) \) of the set \( K_{H,h} \).

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3Existence of an \( \mathcal{H} \)-operator with finite \( h_*(H) \) is verified by the following example: \( H = \sum_{k=1}^{+\infty} \log((k+1) \log^3(k+1)) |k\rangle \langle k| \).

4The assertions of this proposition have classical nature and are probably obtained somewhere in literature. The author would be grateful for any references.
If \( h > h_s(H) \) then \( \sup_{\rho \in K_{H,h}} H(\rho) = \text{ic}(H)h + \log \text{Tr} \exp(-\text{ic}(H)H) \) and there exists no state \( \rho \) in \( K_{H,h} \) such that \( H(\rho) = \sup_{\rho \in K_{H,h}} H(\rho) \).

In the all cases \( \sup_{\rho \in K_{H,h}} H(\rho) = \inf_{\lambda \in (\text{ic}(H),+\infty)} (\lambda h + \log \text{Tr} \exp(-\lambda H)) \).

The function \( F_{H}(h) = \sup_{\rho \in K_{H,h}} H(\rho) \) has the following properties:

- the function \( F_{H}(h) \) is a continuous increasing function on \([h_m; +\infty)\) such that \( F_{H}(h_m) = \log \dim \mathcal{H}_m(H) \) and \( \lim_{h \to +\infty} F_{H}(h) = +\infty \);

- the function \( F_{H}(h) \) has a continuous derivative
  \[
  \frac{dF_{H}(h)}{dh} = \begin{cases} 
  \lambda^*(H,h), & h \in (h_m(H), h_s(H)) \\
  \text{ic}(H), & h \in [h_s(H), +\infty),
  \end{cases}
  \]
  such that
  \[
  \frac{dF_{H}(h)}{dh} \bigg|_{h=h_m+0} = \lim_{h \to h_m(H)+0} \frac{dF_{H}(h)}{dh} = +\infty \quad \text{and} \quad \lim_{h \to +\infty} \frac{dF_{H}(h)}{dh} = \text{ic}(H);
  \]

- the function \( F_{H}(h) \) is strictly concave on \([h_m(H), h_s(H))\) and linear on \([h_s(H), +\infty)\) if \( h_s(H) < +\infty \).

In fig.1 the result of numerical calculations of \( \sup_{\rho \in K_{H,h}} H(\rho) \) as a function of \( h \) for the \( \mathcal{H} \)-operator \( H = -\log \sigma \) with finite \( h_s(H) \) and \( h = c \) is shown.

**Proof.** Through this proof we will assume that \( H = \sum_{k=1}^{+\infty} h_k|k\rangle\langle k| \), where \( \{|k\rangle\}_{k \in \mathbb{N}} \) is an orthonormal basis in the space \( \mathcal{H} \) and \( \{h_k\} \) is a non-decreasing sequence of positive numbers converging to the infinity. Let \( d = \dim \mathcal{H}_m(H) \) so that \( h_k = h_m, \ k = 1, d \) and \( \{|k\rangle\}_{k=1}^{d} \) is a basis of \( \mathcal{H}_m(H) \).

Begin with the proof of the first part of the proposition.

Suppose \( \text{ic}(H) < +\infty \). Then there exists \( \lambda > 0 \) such that
\[
\sigma = (\text{Tr} \exp(-\lambda H))^{-1} \exp(-\lambda H)
\]
is a state. By using nonnegativity of relative entropy and the definition of the set \( K_{H,h} \) we obtain
\[
H(\rho) = \lambda \text{Tr} \rho H + \log \text{Tr} \exp(-\lambda H) - H(\rho\|\sigma) \leq \lambda h + \log \text{Tr} \exp(-\lambda H) < +\infty
\]
for all \( \rho \) in \( K_{H,h} \), which means boundedness of \( H(\rho) \) on \( K_{H,h} \).

Suppose \( \sup_{\rho \in K_{H,h}} H(\rho) < +\infty \). Show first that the equation
\[
\sum_{k=1}^{n} h_k \exp(-\lambda h_k) = h \sum_{k=1}^{n} \exp(-\lambda h_k).
\]
has the unique positive solution \( \lambda_n \) for all sufficiently large \( n \) and that the sequence \( \{ \lambda_n \} \) is increasing. Note that equation (8) is equivalent to the equation \( f_n(\lambda) = 0 \), where \( f_n(\lambda) = \sum_{k=1}^n (h_k - h) \exp(-\lambda(h_k - h)) \). Since \( f'_n(\lambda) = -\sum_{k=1}^n (h_k - h)^2 \exp(-\lambda(h_k - h)) < 0 \) the function \( f_n(\lambda) \) is strictly decreasing on \([0; +\infty)\). It is easy to see that

\[
f_n(0) = \sum_{k=1}^n h_k - nh \quad \text{and} \quad \lim_{\lambda \to +\infty} f_n(\lambda) = -\infty \quad \text{provided} \quad h > h_m.
\]

Since the sequence \( \{h_k\} \) is nondecreasing and unbounded \( \sum_{k=1}^n h_k > nh \) for all sufficiently large \( n \) and the above observation imply existence of the unique positive solution \( \lambda_n \) of the equation \( f_n(\lambda) = 0 \). To show that \( \lambda_{n+1} > \lambda_n \) it is sufficient to note that \( f_{n+1}(\lambda) > f_n(\lambda) \) for all \( \lambda \) in \([0; +\infty)\) and for all \( n \) such that \( h_n > h \).

For each sufficiently large \( n \) consider the state

\[
\rho_n = \left( \sum_{k=1}^n \exp(-\lambda_n h_k) \right)^{-1} \sum_{k=1}^n \exp(-\lambda_n h_k) |k\rangle \langle k| \quad (9)
\]

in \( \mathcal{K}_{H,h} \). This state is the maximum point of the entropy \( H(\rho) \) on the subset \( \mathcal{K}_{H,h}^n \) of \( \mathcal{K}_{H,h} \), consisting of states supported by the linear hull of the vectors \( \{|k\rangle\}_{k=1}^n \). Indeed, by using nonnegativity of the relative entropy and definition of the state \( \rho_n \) it is easy to see that

\[
H(\rho) = \lambda_n \text{Tr} \rho H + \log \sum_{k=1}^n \exp(-\lambda_n h_k) - H(\rho||\rho_n) \leq \lambda_n h + \log \sum_{k=1}^n \exp(-\lambda_n h_k)
\]

for all \( \rho \in \mathcal{K}_{H,h}^n \) and that the equality in this inequality takes place if and only if \( \rho = \rho_n \). By using this and monotonicity of logarithm we obtain

\[
H(\rho_n) = \lambda_n h + \log \sum_{k=1}^n \exp(-\lambda_n h_k) \geq \lambda_n (h - h_m). \quad (10)
\]

Since \( h > h_m \), the assumption \( \sup_{\rho \in \mathcal{K}_{H,h}} H(\rho) < +\infty \) implies boundedness of the sequence \( \{\lambda_n\} \). By this and due to the mentioned above monotonicity of this sequence we conclude that there exists \( \lim_{n \to +\infty} \lambda_n = \lambda^* < +\infty \). Since \( \lambda_n \leq \lambda^* \) for all \( n \) the first equality in (10) implies

\[
\sum_{k=1}^n \exp(-\lambda^* h_k) \leq \sum_{k=1}^n \exp(-\lambda_n h_k) = \exp \left( \sup_{\rho \in \mathcal{K}_{H,h}} H(\rho) \right) < +\infty \quad (11)
\]
for all $n$ and hence
\[ \sum_{k=1}^{+\infty} \exp(-\lambda^* h_k) < +\infty. \] (12)

This shows that $\text{ic}(H) \leq \lambda^* < +\infty$.

Since $\mathcal{K}_{H,h} = \bigcup_{n} \mathcal{K}_{H,h}^n$ and $\sup_{\rho \in \mathcal{K}_{H,h}^n} H(\rho) = H(\rho_n)$ lower semicontinuity of the entropy implies
\[ \sup_{\rho \in \mathcal{K}_{H,h}} H(\rho) = \lim_{n \to +\infty} H(\rho_n). \]

By lemma 2 the sequence of states $\{\rho_n\}$ converges to the state $\rho_*(\mathcal{K}_{H,h})$.

Since $\lim_{n \to +\infty} \lambda_n = \lambda^*$ the sequence $\{A_n = \sum_{k=1}^{n} \exp(-\lambda_n h_k) |k\rangle \langle k|\}_n$ of operators in $\mathfrak{T}(\mathcal{H})$ converges to the operator $A_* = \sum_{k=1}^{+\infty} \exp(-\lambda^* h_k) |k\rangle \langle k|$ in $\mathfrak{T}(\mathcal{H})$ in the weak operator topology. By combining these observations it is easy to see that
\[ \lim_{n \to +\infty} \text{Tr} A_n = \lim_{n \to +\infty} \sum_{k=1}^{n} \exp(-\lambda_n h_k) = \sum_{k=1}^{+\infty} \exp(-\lambda^* h_k) = \text{Tr} A_* \] (13)

and that
\[ \rho_*(\mathcal{K}_{H,h}) = \lim_{n \to +\infty} \rho_n = \left(\sum_{k=1}^{+\infty} \exp(-\lambda^* h_k) \right)^{-1} \sum_{k=1}^{+\infty} \exp(-\lambda^* h_k) |k\rangle \langle k|. \] (14)

By using (10) and (13) we obtain
\[ \sup_{\rho \in \mathcal{K}_{H,h}} H(\rho) = \lim_{n \to +\infty} H(\rho_n) = h\lambda^* + \log \sum_{k=1}^{+\infty} \exp(-\lambda^* h_k). \] (15)

Lower semicontinuity of the entropy implies
\[ H(\rho_*(\mathcal{K}_{H,h})) = \lambda^* \frac{\sum_{k=1}^{+\infty} h_k \exp(-\lambda^* h_k)}{\sum_{k=1}^{+\infty} \exp(-\lambda^* h_k)} + \log \sum_{k=1}^{+\infty} \exp(-\lambda^* h_k) \leq \lim_{n \to +\infty} H(\rho_n). \]

It follows from (15) that this inequality is equivalent to the inequality
\[ \sum_{k=1}^{+\infty} h_k \exp(-\lambda^* h_k) \leq h \sum_{k=1}^{+\infty} \exp(-\lambda^* h_k). \] (16)
Note that equality in this inequality implies that $\rho_*(\mathcal{K}_{H,h})$ is the Gibbs state $\Gamma(\mathcal{K}_{H,h})$. Conversely, by lemma 2 if the Gibbs state $\Gamma(\mathcal{K}_{H,h})$ exists then it coincides with $\rho_*(\mathcal{K}_{H,h})$ and hence equality holds in (16). Thus existence of the Gibbs state $\Gamma(\mathcal{K}_{H,h})$ is equivalent to equality in (16). So, to complete the proof of this part of the proposition it is sufficient to show that the inequality $h \leq h_*(H)$ is equivalent to equality in (16).

Show first that $\lambda^* > \text{ic}(H)$ implies equality in (16). Consider the function

$$f(\lambda) = \lim_{n \to +\infty} f_n(\lambda) = \sum_{k=1}^{+\infty} (h_k - h) \exp(-\lambda(h_k - h)).$$

Since the series $\sum_{k=1}^{+\infty} h_k^p \exp(-\lambda h_k)$ converges uniformly on $[\text{ic}(H) + \varepsilon; +\infty)$ for arbitrary $p \in \mathbb{N}$ and $\varepsilon > 0$ the function $f(\lambda)$ has a continuous derivative $f'(\lambda) = -\sum_{k=1}^{+\infty} (h_k - h)^2 \exp(-\lambda(h_k - h)) < 0$ on $(\text{ic}(H); +\infty)$. By the construction $f(\lambda_n) > f_n(\lambda_n) = 0$ for all sufficiently large $n$. This and continuity of the function $f(\lambda)$ at the point $\lambda^* \in (\text{ic}(H); +\infty)$ imply $f(\lambda^*) \geq 0$. Since (16) implies the converse inequality we obtain $f(\lambda^*) = 0$, which means equality in (16).

If $h < h_*(H)$ then (finite or infinite) $f(\text{ic}(H)) > 0$. Since (16) implies $f(\lambda^*) \leq 0$ this means $\lambda^* > \text{ic}(H)$ and by the above observation $f(\lambda^*) = 0$.

If $h = h_*(H)$ then $f(\text{ic}(H)) = 0$ and hence $\lambda^* = \text{ic}(H)$. Indeed, if $\lambda^* > \text{ic}(H)$ then by the above observation $f(\lambda^*) = 0 = f(\text{ic}(H))$ contradicting to the strict decreasing property of the function $f(\lambda)$.

If $h > h_*(H)$ then $f(\text{ic}(H)) < 0$. Since the function $f(\lambda)$ is decreasing this implies $f(\lambda^*) < 0$ and hence equality does not hold in (16).

Let us prove the second part of the proposition. If $\text{ic}(H) = 0$ then the entropy is continuous on the set $\mathcal{K}_{H,h}$ by the observation in [23]. It follows also from the implication $(1) \Rightarrow (2)$ in the below proposition 4.

To prove the converse implication consider the sequence of states

$$\{\sigma_n = (1 - q_n)|1\rangle\langle 1| + q_n n^{-1} \sum_{k=2}^{n+1} |k\rangle \langle k|\},$$

where $\{q_n = (h - h_m) \left(n^{-1} \sum_{k=2}^{n+1} h_k - h_m\right)^{-1}\}$ is a sequence of positive numbers obviously converging to zero.$^5$ Since the sequence $\{\sigma_n\}$ lies in $\mathcal{K}_{H,h}$ and

$^5$We assume that $n$ is sufficiently large so that $q_n \leq 1$. 12
converges to the pure state $|1\rangle\langle 1|$ continuity of the entropy on the set $\mathcal{K}_{H,h}$ implies convergence of the sequence

$$\{H(\sigma_n) = h_2(q_n) + q_n \log n = h_2(q_n) + \frac{(h - h_m) \log n}{n^{-1} \sum_{k=2}^{n+1} h_k - h_m}\}$$

to zero. By the obvious estimation $n^{-1} \sum_{k=2}^{n+1} h_k \leq h_{n+1}$ it follows that the sequence $\{\nu_n = h_{n+1}^{-1} \log n\}$ converges to zero. Therefore for arbitrary $\lambda > 0$ we have

$$\text{Tr} \exp(-\lambda H) = \sum_{n=0}^{+\infty} \exp(-\lambda h_{n+1}) = \sum_{n=1}^{+\infty} n^{-\frac{\lambda}{n}} < +\infty$$

and hence $\text{ic}(H) = 0$.

The general expression for $\sup_{\rho \in \mathcal{K}_{H,h}} H(\rho)$ can be deduced from the previous observation by noting that the infimum in this expression is achieved at $\lambda^*$ if $h \leq h^*_s(H)$ and at $\text{ic}(H)$ if $h \geq h^*_s(H)$.

The proof of the properties of the function $F_H(\rho)$ is based on the implicit function theorem and is presented in the Appendix. □

Let $\sigma$ be an arbitrary state. In what follows we will use the decrease coefficient $dc(\sigma)$ of the state $\sigma$ defined as

$$dc(\sigma) = \inf\{\lambda > 0 \mid \text{Tr} \sigma^\lambda < +\infty\} \in [0; 1].$$

If $\sigma$ is a full rank state then $-\log \sigma$ is an $\mathcal{F}$-operator and $dc(\sigma) = \text{ic}(- \log \sigma)$.

It is easy to see that $dc(\sigma) < 1$ implies finiteness of the entropy $H(\sigma)$ but there exist states $\sigma$ with finite entropy such that $dc(\sigma) = 1$. The special role of these states is shown in the following proposition.

**Proposition 2.** Let $\sigma$ be a state with finite entropy.

If $dc(\sigma) < 1$ then

$$\lim_{n \to +\infty} H(\rho_n) = H(\sigma)$$

for arbitrary sequence $\{\rho_n\}$ of states $H$-converging to the state $\sigma$.\(^6\)

If $dc(\sigma) = 1$ then for arbitrary $h \geq H(\sigma)$ there exists a sequence $\{\rho_n\}$ of states with finite support $H$-converging to the state $\sigma$ such that

$$\lim_{n \to +\infty} H(\rho_n) = h.$$\(^7\)

---

\(^6\)For example, the state with the spectrum $\{a((k + 1) \log^2(k + 1))^{-1}\}$, where $a$ is a coefficient.

\(^7\)This means that $\lim_{n \to +\infty} H(\rho_n\|\sigma) = 0$. 

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Remark 1. Proposition 2 shows that the set \( \{ \sigma \in \mathcal{S}(H) \mid \text{dc}(\sigma) < 1 \} \) is the maximal set of continuity of the entropy with respect to the \( H \)-convergence.

The proof of the proposition is based on the following lemma.

Lemma 3. If \( \sigma \) is a state with \( \text{dc}(\sigma) < 1 \) then for arbitrary state \( \rho \) such that \( H(\rho || \sigma) < +\infty \) the entropy \( H(\rho) \) is finite and for all \( \lambda > \text{dc}(\sigma) \) the following identity holds

\[
H(\rho || (\text{Tr} \sigma^\lambda)^{-1} \sigma^\lambda) = \lambda H(\rho || \sigma) + \log \text{Tr} \sigma^\lambda - (1 - \lambda)H(\rho).
\]

If \( \text{Tr} \sigma^{\text{dc}(\sigma)} < +\infty \) then this identity holds for \( \lambda = \text{dc}(\sigma) \).

Proof. Let \( \{P_n\} \) be an increasing sequence of spectral projectors of the state \( \sigma \). Let \( A_n = P_n \rho P_n \) and \( B_n = P_n \sigma \) be positive trace class operators.

By definition we have

\[
H(A_n || B_n^\lambda) = \text{Tr}(A_n \log A_n - A_n \log B_n^\lambda + B_n^\lambda - A_n)
\]

\[
= \text{Tr}((\lambda + (1 - \lambda))A_n \log A_n - \lambda A_n \log B_n + B_n^\lambda - A_n)
\]

\[
= \lambda H(A_n || B_n) + \text{Tr}B_n^\lambda - \lambda \text{Tr}B_n - (1 - \lambda)\text{Tr}A_n - (1 - \lambda)\text{Tr}A_n(- \log A_n).
\]

Since \( B_n^\lambda = P_n \sigma^\lambda \) Lindblad’s results [9] imply

\[
\lim_{n \to +\infty} \text{Tr}A_n(- \log A_n) = H(\rho) \quad \text{and} \quad \lim_{n \to +\infty} H(A_n || B_n^\lambda) = H(\rho || \sigma^\lambda)
\]

for all \( \lambda > \text{dc}(\sigma) \). So, passing to the limit in the above equality we obtain

\[
H(\rho || \sigma^\lambda) = \lambda H(\rho || \sigma) + \text{Tr} \sigma^\lambda - 1 - (1 - \lambda)H(\rho).
\]

Thus finiteness of \( H(\rho || \sigma) \) implies finiteness of \( H(\rho) \) and of \( H(\rho || \sigma^\lambda) \) for all \( \lambda > \text{dc}(\sigma) \). By noting that

\[
H(\rho || (\text{Tr} \sigma^\lambda)^{-1} \sigma^\lambda) = H(\rho || \sigma^\lambda) + \log \text{Tr} \sigma^\lambda - \text{Tr} \sigma^\lambda + 1
\]

we obtain the identity of the lemma. \( \square \)

Proof of proposition 2. Let \( \text{dc}(\sigma) < 1 \). Then lemma 3 implies

\[
\frac{H(\rho_n || (\text{Tr} \sigma^\lambda)^{-1} \sigma^\lambda) - \lambda H(\rho_n || \sigma)}{1 - \lambda} = \frac{\log \text{Tr} \sigma^\lambda}{1 - \lambda} - H(\rho_n) \quad (17)
\]
for all $\lambda > \text{dc}(\sigma)$. Suppose $\liminf_{n \to +\infty} H(\rho_n) - H(\sigma) = \Delta > 0$. Since the first term in the right side of (17) tends to $H(\sigma)$ as $\lambda \to 1$ there exists $\lambda' < 1$ such that the right side of (17) is less than $-\Delta/2$ for this $\lambda'$ and sufficiently large $n$ while by nonegativity of the relative entropy the left side of (17) is greater than $\frac{\lambda' H(\rho_n\|\sigma)}{1 - \lambda'}$, which tends to zero as $n \to +\infty$.

Let $\text{dc}(\sigma) = 1$ and let $h > H(\sigma)$. Without loss of generality we may assume that $\sigma$ is a full rank state so that $-\log \sigma$ is a $\mathfrak{H}$-operator such that $\text{ic}(-\log \sigma) = 1$ and $h = H(\sigma) < +\infty$. By proposition 1a

$$\sup_{\rho \in \mathcal{K}_{-\log \sigma, h}} H(\rho) = h$$

for all $h > h = (-\log \sigma)$ in the proof of proposition 1a the sequence $\{\rho_n\}$ of states defined by (9) and converging to the state $\rho_* = (\mathcal{K}_{-\log \sigma, h}) = \sigma$ was constructed. By this construction

$$\lim_{n \to +\infty} H(\rho_n) = \sup_{\rho \in \mathcal{K}_{-\log \sigma, h}} H(\rho) = h$$

and

$$\lim_{n \to +\infty} H(\rho_n\|\sigma) = 0.$$

Consider the set $\mathcal{V}_{\sigma, c} = \{\rho \in \mathfrak{S}(\mathcal{H}) \mid H(\rho\|\sigma) \leq c\}$ defined by a particular state $\sigma$ and by a nonnegative number $c$. By the properties of the relative entropy the set $\mathcal{V}_{\sigma, c}$ is a nonempty closed and convex subset of $\mathfrak{S}(\mathcal{H})$ for arbitrary $\sigma$ and $c$. We may consider the set $\mathcal{V}_{\sigma, c}$ as a $c$-pseudovicinity of the state $\sigma$ with respect to the pseudometric defined by the relative entropy. We will see in the next section that this set plays the special role related with the notion of the $\chi$-capacity of a set of states.

Let $c_*(\sigma) = H((\text{Tr}\sigma_{\text{dc}(\sigma)})^{-1}\sigma_{\text{dc}(\sigma)}\|\sigma)$ if $\text{Tr}\sigma_{\text{dc}(\sigma)} < +\infty$ and $c_*(\sigma) = +\infty$ otherwise. The properties of the restriction of the entropy to the set $\mathcal{V}_{\sigma, c}$ as well as the necessary and sufficient condition of existence of the Gibbs state of this set are considered in the following proposition.

**Proposition 3a.** Let $\sigma$ be an arbitrary state and $c$ be a positive number.

The set $\mathcal{V}_{\sigma, c}$ is a compact convex subset of $\mathfrak{S}(\mathcal{H})$.

The entropy is bounded on the set $\mathcal{V}_{\sigma, c}$ if and only if $\text{dc}(\sigma) < 1$.

The entropy is continuous on the set $\mathcal{V}_{\sigma, c}$ if and only if $\text{dc}(\sigma) = 0$.

If $\text{dc}(\sigma) < 1$ and $c \leq c_*(\sigma)$ then

$$\sup_{\rho \in \mathcal{V}_{\sigma, c}} H(\rho) = \frac{\lambda^* c + \log \text{Tr}\sigma_{\lambda^*}}{1 - \lambda^*}$$

where
\[ \lambda^* = \lambda^*(\sigma, c) \geq \text{dc}(\sigma) \] is uniquely defined by the equation\(^8\)

\[ (\lambda - 1)\text{Tr}(\sigma^\lambda \log \sigma) = (c + \log \text{Tr}\sigma^\lambda)\text{Tr}\sigma^\lambda \]

and there exists the Gibbs state \( \Gamma(\mathcal{V}_{\sigma,c}) = (\text{Tr}\sigma^{\lambda^*})^{-1}\sigma^{\lambda^*} \) for the set \( \mathcal{V}_{\sigma,c} \).

If \( \text{dc}(\sigma) < 1 \) and \( c > c^*_*(\sigma) \) then \( \sup_{\rho \in \mathcal{V}_{\sigma,c}} \text{H}(\rho) = \frac{\text{dc}(\sigma)c + \log \text{Tr}\sigma^\text{dc}(\sigma)}{1 - \text{dc}(\sigma)} \) and there exists no state \( \rho \) in \( \mathcal{V}_{\sigma,c} \) such that \( \text{H}(\rho) = \sup_{\rho \in \mathcal{V}_{\sigma,c}} \text{H}(\rho) \).

If \( \text{dc}(\sigma) < 1 \) then \( \sup_{\rho \in \mathcal{V}_{\sigma,c}} \text{H}(\rho) = \inf_{\lambda \in (\text{dc}(\sigma), 1]} \frac{\lambda c + \log \text{Tr}\sigma^\lambda}{1 - \lambda} \) for arbitrary \( c \).

In fig.1 the result of numerical calculations of \( \sup_{\rho \in \mathcal{V}_{\sigma,c}} \text{H}(\rho) \) as a function of \( c \) for the state \( \sigma \) with \( \text{dc}(\sigma) < 1 \) and finite \( c^*_*(\sigma) \) is shown.

**Proof.** Without loss of generality we may assume that \( \sigma \) is a full rank state so that \( -\log \sigma \) is a \( \mathcal{H} \)-operator.\(^9\)

The proof of the compactness assertion is based on the compactness criterion, described in section 2, and the inequality

\[ \text{H}(\rho||\sigma) \geq \text{H}(P\rho||P\sigma P) \geq \text{Tr}(P\rho) \log \frac{\text{Tr}(P\rho)}{\text{Tr}(P\sigma)} + \text{Tr}(P\sigma) - \text{Tr}(P\rho), \quad (18) \]

valid for arbitrary states \( \rho, \sigma \) and arbitrary projector \( P \). This inequality follows from lemma 3 in [9] and the monotonicity property of the relative entropy [10], applied to the completely positive trace preserving map \( \Phi(A) = (\text{Tr}A)\tau \), where \( \tau \) is an arbitrary state.

For given \( \sigma \) let \( \{P_n\} \) be a sequence of finite rank projectors such that \( \text{Tr}P_n\sigma > 1 - n^{-1} \). Suppose, \( \mathcal{V}_{\sigma,c} \) is not compact. By compactness criterion for arbitrary \( n \) there exists a state \( \rho_n \) in \( \mathcal{V}_{\sigma,c} \) such that \( \text{Tr}(I - P_n)\rho_n > \varepsilon \) for some positive \( \varepsilon \). By this and using inequality (18) with \( P = I - P_n \) we have

\[ \text{H}(\rho_n||\sigma) \geq \text{Tr}((I - P_n)\rho_n) \log \frac{\text{Tr}((I - P_n)\rho_n)}{\text{Tr}((I - P_n)\sigma)} + \text{Tr}((I - P_n)\sigma) - \text{Tr}((I - P_n)\rho_n) \geq \varepsilon \log(\varepsilon n) - 1 \]

---

\(^8\)This equation means that \( H((\text{Tr}\sigma^{\lambda^*})^{-1}\sigma^{\lambda^*}||\sigma) = c \).

\(^9\)This assumption and infinite dimensionality of the space \( \mathcal{H} \) used in the proof imply that \( \sigma \) is a state with infinite rank. But it is possible to show that the all assertions of proposition 3a are valid for an arbitrary state \( \sigma \) with finite rank.
for sufficiently large $n$ and hence $H(\rho_n\|\sigma)$ tends to the infinity as $n \to +\infty$, contradicting to the definition of the set $V_{\sigma,c}$.

If $dc(\sigma) = 1$ then by the second part of proposition 2 the entropy is unbounded on the set $V_{\sigma,c}$.

If $dc(\sigma) < 1$ then lemma 3 implies

$$H(\rho) = \frac{\lambda H(\rho\|\sigma) + \log \text{Tr} \sigma^\lambda - H(\rho\|\sigma_\lambda)}{1 - \lambda} \leq \frac{c\lambda + \log \text{Tr} \sigma^\lambda}{1 - \lambda}$$

(19)

for all $\lambda$ in $(dc(\sigma); 1)$ and all $\rho$ in $V_{\sigma,c}$. This implies $\sup_{\rho \in V_{\sigma,c}} H(\rho) < +\infty$.

If $dc(\sigma) > 0$ then by proposition 1a the entropy is not continuous on the set $K_{-\log \sigma,c}$, which is contained in $V_{\sigma,c}$.

If $dc(\sigma) = 0$ then by the above observation $\sup_{\rho \in V_{\sigma,c}} H(\rho) = d < +\infty$ and hence the set $V_{\sigma,c}$ is contained in $K_{-\log \sigma,c,d}$. By proposition 1a the entropy is continuous on the set $K_{-\log \sigma,c,d}$.

To prove the next part of the proposition denote the state $\left(\text{Tr} \sigma^\lambda\right)^{-1} \sigma^\lambda$ by $\sigma_\lambda$ and note that the continuous function $f(\lambda) = H(\sigma_\lambda\|\sigma)$ is decreasing on $(dc(\sigma); 1)$. Indeed, it is easy to see by direct calculation that this function has a derivative

$$f'(\lambda) = -(1 - \lambda) \left(\text{Tr} \sigma_\lambda \log^2 \sigma - (\text{Tr} \sigma_\lambda \log \sigma)^2\right) < 0$$

for each $\lambda$ in $(dc(\sigma); 1)$. Note also that

$$\lim_{\lambda \to dc(\sigma)+0} f(\lambda) = c_* \leq +\infty \quad \text{and} \quad f(1) = 0.$$

Suppose that $c \leq c_*$. Then the above observation implies existence of the unique solution $\lambda^*$ of the equation $f(\lambda) = c$. Thus $H(\sigma_\lambda^*\|\sigma) = c$ and hence

$$H(\sigma_\lambda^*) = \frac{c\lambda^* + \log \text{Tr} \sigma^{\lambda^*}}{1 - \lambda^*}.$$ 

The inequality (19) implies $H(\rho) \leq H(\sigma_\lambda^*)$ for all $\rho$ in $V_{\sigma,c}$.

Suppose $c_*$ is finite and $c > c_*$. Then

$$h = \frac{dc(\sigma)c + \log \text{Tr} \sigma^{dc(\sigma)}}{1 - dc(\sigma)} \geq \frac{dc(\sigma)c_* + \log \text{Tr} \sigma^{dc(\sigma)}}{1 - dc(\sigma)} = H(\sigma_{dc(\sigma)}) \quad \forall \rho \in V_{\sigma,c}.$$ 

Since $dc(\sigma_{dc(\sigma)}) = 1$ it follows from proposition 2 that for each sufficiently large $m$ there exists a sequence $\{\rho_n^m\}$ of states such that

$$\lim_{n \to +\infty} H(\rho_n^m\|\sigma_{dc(\sigma)}) = 0 \quad \text{and} \quad \lim_{n \to +\infty} H(\rho_n^m) = h - 1/m.$$ (20)
By lemma 3 we have
\begin{equation}
\lim_{n \to +\infty} H(\rho^n_m \| \sigma) = \lim_{n \to +\infty} \frac{H(\rho^n_m \| \sigma_{dc(\sigma)}) - \log \text{Tr} \sigma_{dc(\sigma)} + (1 - dc(\sigma))H(\rho^n_m)}{dc(\sigma)} \\
= \frac{(1 - dc(\sigma))h - \log \text{Tr} \sigma_{dc(\sigma)} - 1 - dc(\sigma)}{dc(\sigma)m} = c - \frac{1 - dc(\sigma)}{dc(\sigma)m}.
\end{equation}

Thus for each $m$ there exists $N(m)$ such that $\rho^n_m \in V_{\sigma, c}$ for all $n \geq N(m)$. This and (20) implies possibility to extract from the family $\{\rho^n_m\}_{n,m}$ a sequence $\{\hat{\rho}_n\}$ of states in $V_{\sigma, c}$ converging to the state $\sigma_{dc(\sigma)}$ such that
\begin{equation}
\lim_{n \to +\infty} H(\hat{\rho}_n) = h.
\end{equation}
This shows that $\sup_{\rho \in V_{\sigma, c}} H(\rho) \geq h$. Since the converse inequality follows from (19) we obtain $\sup_{\rho \in V_{\sigma, c}} H(\rho) = h > H(\sigma_{dc(\sigma)})$, which by lemma 2 implies nonexistence of the Gibbs state of the set $V_{\sigma, c}$ in this case.

The general expression for $\sup_{\rho \in V_{\sigma, c}} H(\rho)$ can be deduced from the previous observation by noting that the infinum in this expression is achieved at $\lambda^*$ if $c \leq c_*(\sigma)$ and at $dc(\sigma)$ if $c \geq c_*(\sigma)$.

\begin{remark}
The last assertion of this proposition implies that the properties (i) -- (iii) remain valid for the set $\overline{\sigma}{A, \sigma}$.\end{remark}

\begin{proof}
(ii) $\Rightarrow$ (iii) Since every continuous function is finite we have
\begin{equation}
H(\rho \| \sigma) = -H(\rho) + \text{Tr} \rho(- \log \sigma), \quad \forall \rho \in A.
\end{equation}

\end{proof}

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By proposition 3a the set $A$ is compact and hence the entropy is bounded on $A$. Thus the conditions of $(ii)$ and (21) imply continuity and boundedness of the function $\text{Tr}\rho(-\log\sigma)$ on the set $A$. Hence $(iii)$ holds with $\tilde{H} = -\log\sigma$.

$(iii) \Rightarrow (ii)$ For given $\lambda > \text{ic}(\tilde{H})$ let $\sigma = (\text{Tr}\exp(-\lambda\tilde{H}))^{-1}\exp(-\lambda\tilde{H})$ be a state in $\mathcal{G}(\mathcal{H})$ with finite entropy. Then $(iii)$ means continuity and boundedness of the function $\text{Tr}\rho(-\log\sigma)$ on the set $A$. By lower semicontinuity of the entropy and of the relative entropy this and (21) imply continuity and boundedness of the functions $H(\rho)$ and $H(\rho\|\sigma)$ on the set $A$.

$(i) \Rightarrow (iii)$ By the assumption $\sum_k \exp(-\lambda h_k) < +\infty$ for all $\lambda > 0$ and hence $\sum_k h_k \exp(-\lambda h_k) < +\infty$ for all $\lambda > 0$. This implies existence of a sequence $\{\lambda_k\}$ of positive numbers monotonously converging to zero and such that $\sum_k h_k \exp(-\lambda_k h_k) < +\infty$. This sequence can be constructed as follows. For arbitrary natural $m$ let $N(m)$ be the minimal number such that $\sum_{k=N(m)}^{+\infty} h_k \exp(-h_k/m) < 2^{-m}$. Consider a sequence

$$
\lambda_k = \begin{cases} 
 1, & k < N(2) \\
 1/m, & N(m) \leq k < N(m+1), m \geq 2.
\end{cases}
$$

It is easy to see that this sequence satisfies the above condition. Since $\text{Tr}\rho H = \sum_k h_k \langle k|\rho|k\rangle \leq h$ for all $\rho$ in $A$ the series $\sum_k \lambda_k h_k \langle k|\rho|k\rangle$ converges uniformly on $A$. This implies continuity of the function $\text{Tr}\rho(-\log\sigma)$, where $\sigma = (\sum_k \exp(-\lambda_k h_k))^{-1}\sum_k \exp(-\lambda_k h_k)|k\rangle\langle k|$. Note that the condition $\sum_k h_k \exp(-\lambda_k h_k) < +\infty$ implies $\text{Tr}\sigma \tilde{H} < +\infty$ and $H(\sigma) < +\infty$. Thus $(iii)$ holds with $\tilde{H} = -\log\sigma$.

$(iii) \Rightarrow (i)$ Let $\tilde{H} = \sum_k \tilde{h}_k |k\rangle\langle k|$, where $\{|k\rangle\}$ is an orthonormal basis in $\mathcal{H}$. Since $(iii)$ means $(ii)$ proposition 3a implies compactness of the set $A$. By the assumption the series $\sum_k \tilde{h}_k \langle k|\rho|k\rangle$ converges on the compact set $A$ to the continuous function $\text{Tr}\rho \tilde{H}$. By Dini’s lemma it converges uniformly on $A$. This implies existence of a sequence $\{\lambda_k\}$ of positive numbers monotonously converging to the infinity and such that $\sum_k \lambda_k \tilde{h}_k \langle k|\rho|k\rangle \leq h < +\infty$ for all $\rho$ in $A$. It is easy to see that the $\tilde{\mathcal{H}}$-operator $\tilde{H} = \sum_k \lambda_k \tilde{h}_k |k\rangle\langle k|$ has the all properties stated in $(i)$.

The last assertion of the proposition follows from the above construction.

$\square$

Propositions 1a and 4 imply the following observation.

**Corollary 1.** If $H$ is a $\tilde{\mathcal{H}}$-operator with $\text{ic}(H) = 0$ then there exist a state $\sigma$ in $\mathcal{G}(\mathcal{H})$ and a $\tilde{\mathcal{H}}$-operator $\tilde{H}$ with $\text{ic}(\tilde{H}) < +\infty$ such that the relative
entropy \( H(\rho\|\sigma) \) and the linear functional \( \text{Tr}\rho\tilde{H} \) are continuous on the set \( \mathcal{K}_{H,h} \).

Since the set \( \mathcal{K}_{H,h} \) is convex by definition propositions 1a and 4 also provide the following result.

**Corollary 2.** If the entropy is continuous on the closed set \( \mathcal{A} \) and there exists a state \( \sigma \) in \( \mathcal{S}(\mathcal{H}) \) such that the relative entropy \( H(\rho\|\sigma) \) is continuous and bounded on the set \( \mathcal{A} \) then the entropy is continuous on the set \( \overline{\mathcal{A}} \).

**Remark 3.** The assumption of existence of the state \( \sigma \) in the statement \((ii)\) of proposition 4 and in corollary 2 is essential. Indeed, let \( \mathcal{A} \) be the closed subset of all pure states in \( \mathcal{S}(\mathcal{H}) \). Then the entropy is trivially continuous on this set \( \mathcal{A} \), but it is not continuous on \( \overline{\mathcal{A}} = \mathcal{S}(\mathcal{H}) \). There exists compact countable set \( \mathcal{A} \) of pure states such that the entropy is unbounded on the set \( \overline{\mathcal{A}} \) (see the example in subsection 5.1 below). □

The implication \((iii) \Rightarrow (ii)\) in proposition 4 makes possible to show continuity of the entropy on some nontrivial subsets of states, which will be used in subsection 5.5.

**Corollary 3.** Let \( \lambda \mapsto U_\lambda \) be a continuous mapping from some compact set \( \Lambda \) into the set of all unitaries (antiunitaries) in \( \mathcal{H} \) and let \( \omega \) be a state in \( \mathcal{S}(\mathcal{H}) \) such that \( U_\lambda \omega U^*_\lambda = \omega \) for all \( \lambda \in \Lambda \). Then for arbitrary state \( \sigma \) such that \( \text{Tr}\sigma(-\log \omega) < +\infty \) the functions \( H(\rho) \) and \( H(\rho\|\omega) \) are continuous on the set \( \overline{\{U_\lambda \sigma U^*_\lambda\}_{\lambda \in \Lambda}} \).

For an arbitrary orthonormal basis \( \{|k\rangle\} \subset \mathcal{H} \) consider the expectation
\[
\Pi_{\{|k\rangle\}} : \rho \mapsto \sum_k \langle k|\rho|k\rangle |k\rangle\langle k|.
\]
Note that the output states of \( \Pi_{\{|k\rangle\}} \) can be considered as classical states (probability distributions). So, we may call the set \( \Pi_{\{|k\rangle\}}(\mathcal{A}) \) classical projection of the set \( \mathcal{A} \), corresponding to the basis \( \{|k\rangle\} \).

The following proposition shows, roughly speaking, that properties of sets of quantum states are closely related to the properties of classical projections of these sets.

**Proposition 5.** Let \( \mathcal{A} \) be an arbitrary closed subset of \( \mathcal{S}(\mathcal{H}) \).

A) **The set \( \mathcal{A} \) is compact if the set \( \Pi_{\{|k\rangle\}}(\mathcal{A}) \) is compact for at least one basis \( \{|k\rangle\} \).**

B) **If the set \( \mathcal{A} \) is compact then the set \( \Pi_{\{|k\rangle\}}(\mathcal{A}) \) is compact for arbitrary basis \( \{|k\rangle\} \).**
C) The entropy is bounded on the set $\mathcal{A}$ if it is bounded on the set $\Pi_{\{k\}}(\mathcal{A})$ for at least one basis $\{|k\rangle\}$.

D) If the entropy is bounded on the set $\mathcal{A}$ and the set $\mathcal{A}$ is convex then it is bounded on the set $\Pi_{\{k\}}(\mathcal{A})$ for at least one basis $\{|k\rangle\}$.

E) The entropy is continuous on the set $\mathcal{A}$ if it is continuous on the set $\Pi_{\{k\}}(\mathcal{A})$ for at least one basis $\{|k\rangle\}$.

F) If the entropy is continuous on the set $\mathcal{A}$ and there exists a state $\sigma$ in $\mathcal{S}(\mathcal{H})$ such that the relative entropy $H(\rho\|\sigma)$ is continuous and bounded on the set $\mathcal{A}$ then the entropy is continuous on the set $\Pi_{\{k\}}(\mathcal{A})$ for at least one basis $\{|k\rangle\}$.

Proof. If the set $\Pi_{\{k\}}(\mathcal{A})$ is compact then by the compactness criterion for subsets of classical states for arbitrary $\varepsilon > 0$ there exists $N_\varepsilon$ such that

$$\text{Tr} P_\varepsilon \rho = \sum_{k=1}^{N_\varepsilon} \langle k | \rho | k \rangle \geq 1 - \varepsilon, \quad \forall \rho \in \mathcal{A},$$

where $P_\varepsilon = \sum_{k=1}^{N_\varepsilon} |k\rangle\langle k|$ is a finite rank projector. By the compactness criterion for subsets of $\mathcal{S}(\mathcal{H})$ this implies compactness of the set $\mathcal{A}$.

If the set $\mathcal{A}$ is compact then for arbitrary basis $\{|k\rangle\}$ the set $\Pi_{\{k\}}(\mathcal{A})$ is compact as an image of a compact set under a continuous mapping.

In the proof of the following statements we will use the following identity valid for arbitrary state $\rho$ in $\mathcal{S}(\mathcal{H})$ with finite $H(\mathcal{A})$.

$$H(\rho\|\Pi_{\{k\}}(\rho)) = H(\Pi_{\{k\}}(\rho)) - H(\rho), \quad (22)$$

valid for arbitrary state $\rho$ in $\mathcal{S}(\mathcal{H})$ with finite $H(\Pi_{\{k\}}(\rho))$. If the entropy is bounded on the set $\Pi_{\{k\}}(\mathcal{A})$ then it is bounded on the set $\mathcal{A}$ since identity (22) and nonnegativity of the relative entropy implies $H(\rho) \leq H(\Pi_{\{k\}}(\rho))$ for arbitrary $\rho$ in $\mathcal{A}$.

If the entropy is bounded on the convex set $\mathcal{A}$ then by corollary 5 below this set $\mathcal{A}$ is contained in the set $\mathcal{K}_{H,h}$ defined by a particular $\mathcal{H}$-operator $H$ with $\text{ic}(H) < +\infty$. Let $\{|k\rangle\}$ be the basis of eigenvectors for the $\mathcal{H}$-operator $H$. Then $\Pi_{\{k\}}(\mathcal{A})$ also is contained in the set $\mathcal{K}_{H,h}$ and hence the entropy is bounded on the set $\Pi_{\{k\}}(\mathcal{A})$ by proposition 1a.

Suppose the entropy is continuous on the set $\Pi_{\{k\}}(\mathcal{A})$. Then the entropy is finite on this set and by (22) it is finite on the set $\mathcal{A}$. Let $\rho$ be a state in
$\mathcal{A}$ and $\{\rho_n\}$ be a sequence of states in $\mathcal{A}$ converging to the state $\rho$. By the assumption, lower semicontinuity of the relative entropy and \((22)\) we have

$$\limsup_{n \to +\infty} H(\rho_n) = \lim_{n \to +\infty} H(\Pi\{|k\}\{(\rho_n)\}) - \liminf_{n \to +\infty} H(\rho_n \| \Pi\{|k\}\{(\rho_n)\})$$

$$\leq H(\Pi\{|k\}\{\rho\}) - H(\rho \| \Pi\{|k\}\{\rho\}) = H(\rho).$$

This and lower semicontinuity of the entropy imply $\lim_{n \to +\infty} H(\rho_n) = H(\rho)$.

If the entropy is continuous on the set $\mathcal{A}$ and there exists a state $\sigma$ in $\mathcal{S}(\mathcal{H})$ such that the relative entropy $H(\rho \| \sigma)$ is continuous and bounded on the set $\mathcal{A}$ then by proposition 4 the set $\mathcal{A}$ is contained in the set $\mathcal{K}_{H,h}$ defined by a particular $\mathcal{H}$-operator $H$ with $\text{ic}(H) = 0$. Let $\{|k\}\}$ be the basis of eigenvectors for $H$. Then $\Pi\{|k\}\{(\mathcal{A})$ also is contained in the set $\mathcal{K}_{H,h}$ and hence the entropy is continuous on the set $\Pi\{|k\}\{(\mathcal{A})$ by proposition 1a. □

**Remark 4.** Note that the expression "for at least one" in the statements D and F of the proposition 5 can not be changed to "for arbitrary" in contrast to the statement B. Indeed, it is easy to find a pure state $\rho$ and a basis $\{|k\}\}$ such that $H(\Pi\{|k\}\{\rho\}) = +\infty$. □

Let $\sigma$ be a state with the basis of eigenvectors $\{|k\}\}$. The set $\Pi^{-1}\{|k\}\{(\sigma)$ of all states having the same diagonal values in the basis $\{|k\}\}$ as the state $\sigma$ will be called layer, corresponding to the state $\sigma$ and denoted by $\mathcal{L}(\sigma)$.\(^{10}\) In a sense a layer can be considered as the simplest purely quantum subset of states.

By \((22)\) we have

$$H(\rho) \leq H(\sigma), \quad \forall \rho \in \mathcal{L}(\sigma)$$

and hence the quantum entropy is bounded on the layer corresponding to the state $\sigma$ if and only if $H(\sigma) < +\infty$. The above proposition implies that boundedness of the entropy on a layer means its continuity.

**Proposition 6a.** Let $\sigma$ be an arbitrary state.

*The set $\mathcal{L}(\sigma)$ is a compact convex subset of $\mathcal{S}(\mathcal{H})$.\(^{10}\)*

\(^{10}\)If the state has different eigenvalues then the basis $\{|k\}\}$ is (essentially) unique and the set $\mathcal{L}(\sigma)$ depends only on the state $\sigma$. If there are multiple eigenvalues then the set $\mathcal{L}(\sigma)$ depends also on the choice of the basis $\{|k\}\}$. Since in the last case all "variants" of the set $\mathcal{L}(\sigma)$ are isomorphic to each other we will assume that one of them is chosen.
The entropy $H(\rho)$ is continuous on the set $\mathcal{L}(\sigma)$ if and only if

$$\sup_{\rho \in \mathcal{L}(\sigma)} H(\rho) = H(\sigma) < +\infty$$

If $H(\sigma) < +\infty$ then $H(\rho\|\sigma) = H(\sigma) - H(\rho)$ for arbitrary state $\rho$ in $\mathcal{L}(\sigma)$.

If $H(\sigma) = +\infty$ then $H(\rho\|\sigma) = +\infty$ for arbitrary pure state $\rho$ in $\mathcal{L}(\sigma)$.

Proof. The first and the second assertions follows from the statements A and E of proposition 5 correspondingly since $\Pi_{\{|k\rangle\}}(\mathcal{L}(\sigma)) = \{\sigma\}$ if $\{|k\rangle\}$ is the basis of eigenvectors for the state $\sigma$.

The expression for the relative entropy in the case $H(\sigma) < +\infty$ is a reformulation of (22).

Let $H(\sigma) = +\infty$ and $\rho$ be an arbitrary pure state in $\mathcal{L}(\sigma)$. Consider the sequences of states $\{\sigma_n = (\text{Tr} P_n \sigma)^{-1} P_n \sigma\}$ and $\{\rho_n = (\text{Tr} P_n \rho)^{-1} P_n \rho P_n\}$, where $P_n$ be a spectral projector of the state $\sigma$ corresponding to its $n$ maximal eigen values.

Since for each $n$ pure state $\rho_n$ lies in $\mathcal{L}(\sigma_n)$ by using (22) we obtain

$$H(\rho_n\|\sigma_n) = H(\sigma_n) - H(\rho_n) = H(\sigma_n).$$

By Lindblad’s results [9] the left and right sides of this equality tends to $H(\rho\|\sigma)$ and to $H(\sigma) = +\infty$ correspondingly as $n \to +\infty$.□

Propositions 5 and 6a imply the following observation: Absence of such properties of the quantum entropy as finiteness and continuity in the infinite dimensional case has purely classical nature. Indeed, the set of all quantum states can be considered as a union of the layers corresponding to all states diagonalizable in a particular basis. The set of these states can be identified with the set of all classical states - probability distributions while a single layer - with a set of purely quantum states. Proposition 6a shows that the entropy is continuous on the whole layer if it is finite on the corresponding classical state. By proposition 5 possible discontinuity of the quantum entropy is connected with transitions between layers corresponding to a set of classical states, on which the entropy is not continuous.
4 The $\chi$-capacity

4.1 The optimal average state

Let $\mathcal{A}$ be an arbitrary subset of $\mathcal{S}(\mathcal{H})$. Consider the $\chi$-capacity of the set $\mathcal{A}$ defined by

$$\bar{C}(\mathcal{A}) = \sup_{\{\pi_i, \rho_i\}} \chi(\{\pi_i, \rho_i\}).$$

(24)

where the supremum is over all ensembles $\{\pi_i, \rho_i\}$ of states in $\mathcal{A}$.

If the entropy is bounded on the set $\text{co}(\mathcal{A})$ then

$$\bar{C}(\mathcal{A}) = \sup_{\{\pi_i, \rho_i\}} \left( H\left(\sum_i \pi_i \rho_i\right) - \sum_i \pi_i H(\rho_i) \right) \leq \sup_{\rho \in \text{co}(\mathcal{A})} H(\rho) < +\infty.$$

But boundedness of the entropy is not a necessary condition for finiteness of the $\chi$-capacity, as it follows from the examples in the next section.

In accordance to [17] a sequence of ensembles $\{\{\pi^n_i, \rho^n_i\}\}_n$ of states in $\mathcal{A}$ such that

$$\lim_{n \to +\infty} \chi(\{\pi^n_i, \rho^n_i\}) = \bar{C}(\mathcal{A})$$

is called approximating sequence for the set $\mathcal{A}$.

If $\mathcal{A}$ is a set of states in finite dimensional Hilbert space then there exists ensemble $\{\pi_i, \rho_i\}$ - optimal ensemble for the set $\mathcal{A}$ - at which the supremum in the definition (24) of the $\chi$-capacity is achieved [20]. If $\mathcal{A}$ is a set of states in infinite dimensional Hilbert space then we can not assert existence of optimal ensemble but we can assert existence of the unique state, possessing the properties of the average state of the optimal ensemble in the finite dimensional case.

**Theorem 1.** Let $\mathcal{A}$ be a set with finite $\chi$-capacity $\bar{C}(\mathcal{A})$. Then there exists the unique state $\Omega(\mathcal{A})$ in $\mathcal{S}(\mathcal{H})$ such that

$$H(\rho\|\Omega(\mathcal{A})) \leq \bar{C}(\mathcal{A}) \quad \text{for all } \rho \text{ in } \mathcal{A}.$$

The state $\Omega(\mathcal{A})$ lies in $\text{co}(\mathcal{A})$ and for arbitrary approximating sequence of ensembles $\{\{\pi^n_i, \rho^n_i\}\}_n$ for the set $\mathcal{A}$ the corresponding sequence $\{\bar{\rho}_n\}$ of their average states $H$-converges to the state $\Omega(\mathcal{A})$.\footnote{This means that $\lim_{n \to +\infty} H(\bar{\rho}_n\|\Omega(\mathcal{A})) = 0.$}
The $\chi$-capacity $\bar{C}(A)$ can be defined by the expression

$$\bar{C}(A) = \inf_{\sigma \in \mathcal{S}(H)} \sup_{\rho \in A} H(\rho\|\sigma) = \inf_{\sigma \in \mathcal{S}(H)} \sup_{\rho \in A} H(\rho\|\sigma) = \sup_{\rho \in A} H(\rho\|\Omega(A)), \quad (25)$$

in which the first two equalities remain valid in the case $\bar{C}(A) = +\infty$.

**Proof.** Show first that for arbitrary approximating sequence of ensembles

$$\{\mu_n = \{\pi^n_i, \rho^n_i\}_{i=1}^{N(n)}\}$$

for the set $A$ the corresponding sequence of the average states $\{\bar{\rho}_n\}$ converges to a particular state in $\mathcal{S}(H)$. By definition of an approximating sequence for arbitrary $\varepsilon > 0$ there exists $N_\varepsilon$ such that $\chi(\mu_n) > C(A) - \varepsilon$ for all $n \geq N_\varepsilon$. By lemma 1 with $m = 2$ and $\lambda = 1/2$ we have

$$\bar{C}(A) - \varepsilon \leq \frac{1}{2} \chi(\mu_{n_1}) + \frac{1}{2} \chi(\mu_{n_2})$$

and hence $\|\bar{\rho}_{n_2} - \bar{\rho}_{n_1}\|_1 < \sqrt{8\varepsilon}$ for all $n_1 \geq N_\varepsilon$ and $n_2 \geq N_\varepsilon$. Thus the sequence $\{\bar{\rho}_n\}$ is a Cauchy sequence and hence it converges to a particular state $\rho_*$ in $\mathcal{S}(H)$.

Let $\sigma$ be an arbitrary state in $A$. For each $n$ consider the ensemble

$$\mu^n_n = \{(1-\eta)\pi^n_1\rho^n_1, \ldots, (1-\eta)\pi^n_{N(n)}\rho^n_{N(n)}, \eta\sigma\}, \quad \eta \in [0, 1]$$

obtained from the ensemble $\mu_n = \{\pi^n_i, \rho^n_i\} = \mu^n_0$ of the approximating sequence by adding the state $\sigma$ with probability $\eta$.\(^{12}\) We obtain the sequence of ensembles $\{\mu^n_n\}$ with the corresponding sequence of the average states $\{\bar{\rho}^n_n = (1-\eta)\bar{\rho}_n + \eta\sigma\}_n$ converging to the state $\bar{\rho}_n = (1-\eta)\rho_* + \eta\sigma$ as $n \to +\infty$.

For arbitrary $n$ we have

$$\chi(\mu^n_n) = (1-\eta) \sum_i \pi^n_i H(\rho^n_i\|\bar{\rho}^n_n) + \eta H(\sigma\|\bar{\rho}^n_n). \quad (26)$$

By the assumption $\bar{C}(A) < +\infty$ both sums in the right side of the above expression are finite. Applying Donald’s identity (3) to the first sum in the right side we obtain

$$\sum_i \pi^n_i H(\rho^n_i\|\bar{\rho}^n_n) = \chi(\mu^n_0) + H(\bar{\rho}_n\|\bar{\rho}^n_n).$$

\(^{12}\)This trick was originally used in [20] in the finite dimensional case.
Substitution of the above expression into \( (26) \) gives
\[
\chi(\mu_n) = \chi(\mu_0) + (1 - \eta)H(\bar{\rho}_n \| \bar{\rho}_n^\eta) + \eta \left( H(\sigma \| \bar{\rho}_n^\eta) - \chi(\mu_n) \right).
\]
Due to nonnegativity of the relative entropy it follows that
\[
H(\sigma \| \bar{\rho}_n^\eta) \leq \eta^{-1} \left( \chi(\mu_n^\eta) - \chi(\mu_0^\eta) \right) + \chi(\mu_0), \quad \eta \neq 0. \tag{27}
\]
By definition of the approximating sequence we have
\[
\lim_{n \to +\infty} \chi(\mu_n^\eta) = C(A) \geq \chi(\mu_n^\eta) \tag{28}
\]
for all \( n \) and \( \eta > 0 \). It follows that
\[
\liminf_{\eta \to 0} \liminf_{n \to +\infty} \eta^{-1} \left[ \chi(\mu_n^\eta) - \chi(\mu_n) \right] \leq 0 \tag{29}
\]
Lower semicontinuity of the relative entropy with \((27),(28)\) and \((29)\) implies
\[
H(\sigma \| \rho_\star) \leq \liminf_{\eta \to 0} \liminf_{n \to +\infty} H(\sigma \| \bar{\rho}_n^\eta) \leq \bar{C}(A).
\]
This proves that
\[
\sup_{\sigma \in A} H(\sigma \| \rho_\star) \leq \bar{C}(A), \tag{30}
\]
Let \( \{\lambda_j^n, \sigma_j^n\}_n \) be an arbitrary approximating sequence of ensembles. By inequality \((30)\), we have
\[
\sum_j \lambda_j^n H(\sigma_j^n \| \rho_\star) \leq \bar{C}(A).
\]
Applying Donald’s identity \((3)\) to the left side we obtain
\[
\sum_j \lambda_j^n H(\sigma_j^n \| \rho_\star) = \sum_j \lambda_j^n H(\sigma_j^n \| \bar{\sigma}_n) + H(\bar{\sigma}_n \| \rho_\star) \tag{31}
\]
From the two above expressions we have
\[
H(\bar{\sigma}_n \| \rho_\star) \leq \bar{C}(A) - \sum_j \lambda_j^n H(\sigma_j^n \| \bar{\sigma}_n).
\]
The right side of this inequality tends to zero as \( n \to +\infty \) due to the approximating property of the sequence \( \{\lambda_j^n, \sigma_j^n\}_n \). Thus the sequence \( \{\bar{\sigma}_n\}_n \)
$H$-converges to the state $\rho_*$ and hence it converges to this state in the trace norm topology. Hence this state $\rho_*$ does not depend on the choice of an approximating sequence, so, it is determined only by the set $\mathcal{A}$. Denote this state by $\Omega(\mathcal{A})$. The above observation implies also that $\rho_* = \Omega(\mathcal{A})$ is the unique state in $\mathcal{S}(\mathcal{H})$ for which inequality (30) holds.

To prove expression (25) show first that inequality (30) is in fact equality. Indeed expression (31), valid for an approximating sequence $\{\{\lambda_j^n, \sigma_j^n\}\}_n$, and nonnegativity of the relative entropy imply

$$\sum_j \lambda_j^n H(\sigma_j^n \| \bar{\sigma}_n) \leq \sum_j \lambda_j^n H(\sigma_j^n \| \rho_*) \leq \sup_{\sigma \in \mathcal{A}} H(\sigma \| \rho_*).$$

By the approximating property of the sequence $\{\{\lambda_j^n, \sigma_j^n\}\}_n$ the left side in the above inequality tends to $\bar{C}(\mathcal{A})$ as $n \to +\infty$. This proves $"="$ in (30).

Consider the function $F(\sigma) = \sup_{\rho \in \mathcal{A}} H(\rho \| \sigma)$ on $\mathcal{S}(\mathcal{H})$. By the equality in (30) we have $F(\Omega(\mathcal{A})) = \bar{C}(\mathcal{A})$. It follows that the state $\Omega(\mathcal{A})$ is the unique minimal point of the function $F(\sigma)$ on $\mathcal{S}(\mathcal{H})$. Indeed, let $\sigma_0$ be a state in $\mathcal{S}(\mathcal{H})$ such that

$$\sup_{\rho \in \mathcal{A}} H(\rho \| \sigma_0) = F(\sigma_0) \leq F(\Omega(\mathcal{A})) = \bar{C}(\mathcal{A})$$

By the first part of the theorem this implies $\sigma_0 = \Omega(\mathcal{A})$.

If $\bar{C}(\mathcal{A}) = +\infty$ then the right side of expression (25) is equal to $+\infty$ as well. Indeed, if $\sigma'$ is a state in $\mathcal{S}(\mathcal{H})$ such that $\sup_{\rho \in \mathcal{A}} H(\rho \| \sigma') = c < +\infty$ then by using Donald’s identity and nonnegativity of the relative entropy we have

$$\sum_i \pi_i H(\rho_i \| \bar{\rho}) \leq \sum_i \pi_i H(\rho_i \| \sigma') - H(\bar{\rho} \| \sigma') \leq c.$$ 

for arbitrary ensemble $\{\pi_i, \rho_i\}$ of states in $\mathcal{A}$. This implies $\bar{C}(\mathcal{A}) \leq c < +\infty$. $\square$

**Definition 1.** The state $\Omega(\mathcal{A})$ described in theorem 1 is called the optimal average state of the set $\mathcal{A}$.

Theorem 1, Donald identity (3) and inequality (11) imply the following useful result.

**Corollary 4.** Let $\mathcal{A}$ be a set with finite $\chi$-capacity. For arbitrary ensemble $\{\pi_i, \rho_i\}$ of states in $\mathcal{A}$ with the average state $\bar{\rho}$ the following inequality holds

$$\bar{C}(\mathcal{A}) - \chi(\{\pi_i, \rho_i\}) \geq H(\bar{\rho} \| \Omega(\mathcal{A})) \geq \frac{1}{2} \| \bar{\rho} - \Omega(\mathcal{A}) \|_1^2.$$
Theorem 1 and proposition 1a provide the following observation on the properties of the entropy. 

**Corollary 5.** The entropy is bounded on a convex set $\mathcal{A}$ if and only if this set $\mathcal{A}$ is relatively compact and is contained in the set $\mathcal{K}_{H,h}$ defined by a particular $\mathcal{H}$-operator $H$ with $\text{ic}(H) < +\infty$ and positive $h$.

**Proof.** If the set $\mathcal{A}$ is contained in the set $\mathcal{K}_{H,h}$ with $\text{ic}(H) < +\infty$ then by proposition 1a $\sup_{\rho \in \mathcal{A}} H(\rho) < +\infty$.

If $\sup_{\rho \in \mathcal{A}} H(\rho) < +\infty$ then $\bar{C}(\mathcal{A}) < +\infty$ and by theorem 1

$$H(\rho\|\Omega(\mathcal{A})) = \text{Tr} \rho (-\log \Omega(\mathcal{A})) - H(\rho) \leq \bar{C}(\mathcal{A})$$

for all $\rho$ in $\mathcal{A}$. It follows

$$\text{Tr} \rho (-\log \Omega(\mathcal{A})) \leq \bar{C}(\mathcal{A}) + \sup_{\rho \in \mathcal{A}} H(\rho)$$

for all $\rho$ in $\mathcal{A}$ and hence $\mathcal{A} \subseteq \mathcal{K}_{H,h}$, where $H = -\log \Omega(\mathcal{A})$ and $h = \bar{C}(\mathcal{A}) + \sup_{\rho \in \mathcal{A}} H(\rho)$. □

By corollary 5 boundedness of the entropy on a convex set $\mathcal{A}$ means that this set $\mathcal{A}$ is contained in the set $\mathcal{K}_{H,h}$ defined by a particular $\mathcal{H}$-operator $H$ with finite $\text{ic}(H)$. By theorem 1 finiteness of the $\chi$-capacity of an arbitrary set $\mathcal{A}$ means that this set $\mathcal{A}$ is contained in the set $\mathcal{V}_{\Omega(\mathcal{A}),\bar{C}(\mathcal{A})}$, having the same $\chi$-capacity and the same optimal average state.

### 4.2 General properties

In this section we consider general properties of the $\chi$-capacity as a function of a set. We show also the special role of the optimal average state introduced in the previous subsection. It turns out that many properties of sets of states related to the $\chi$-capacity depend on validity for these sets of one of the two special continuity properties. So, it is convenient to introduce the following definition.

**Definition 2.** An arbitrary set $\mathcal{A}$ with finite $\chi$-capacity is called regular if one of the two following conditions holds:

- $H(\Omega(\mathcal{A}))$ is finite and $\lim_{n \to +\infty} H(\rho_n) = H(\Omega(\mathcal{A}))$ for arbitrary sequence $\{\rho_n\}$ of states in $\text{co}(\mathcal{A})$ $H$-converging to the state $\Omega(\mathcal{A})$;\(^{13}\)

- the relative entropy $H(\rho\|\Omega(\mathcal{A}))$ is continuous on the set $\bar{\mathcal{A}}$.

\(^{13}\)This means that $\lim_{n \to +\infty} H(\rho_n\|\Omega(\mathcal{A})) = 0$. 

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Note that continuity of the entropy on the set $\overline{\text{co}}(\mathcal{A})$ is a sufficient condition for regularity of the set $\mathcal{A}$, but it is very restrictive requirement. In a sense the conditions in the above definition are the minimal continuity requirements which guarantees the "good" properties of the $\chi$-capacity. These conditions do not imply each other: there exist sets, for which the first condition holds but the second one is not valid and vice versa. The most of the examples of sets with finite $\chi$-capacity presented in section 5 are regular. The examples of the nonregular sets with finite $\chi$-capacity and consequences of this nonregularity are considered in subsections 5.1, 5.2 and 5.3.

In the following theorem we summarize the properties of the $\chi$-capacity and of the optimal average state, which will be used later. These properties shows that $\chi$-capacity can be considered as a specific nonadditive measure of a set of quantum states.

**Theorem 2.** The following properties hold$^{14}$

A) $\bar{C}(\mathcal{A}) \geq 0$ for arbitrary set $\mathcal{A}$ and equality here takes place if and only if the set $\mathcal{A}$ consists of a single point;

B) $\bar{C}(\mathcal{A}) = \bar{C}(\overline{\text{co}}(\mathcal{A}))$ and $\Omega(\mathcal{A}) = \Omega(\overline{\text{co}}(\mathcal{A}))$ for arbitrary set $\mathcal{A}$;

C) if $\mathcal{A} \subseteq \mathcal{B}$ then $\bar{C}(\mathcal{A}) \leq \bar{C}(\mathcal{B})$ and equality here implies $\Omega(\mathcal{A}) = \Omega(\mathcal{B})$,$^{15}$

D) if $\bar{C}(\mathcal{A}) < +\infty$ then $\mathcal{A}$ is relatively compact and hence $\bar{C}(\mathcal{A}) = \bar{C}(\text{Ext}\mathcal{A})$;

E) if $\text{dc}(\Omega(\mathcal{A})) < 1$ then the set $\mathcal{A}$ is regular and the entropy is bounded on the set $\overline{\text{co}}\mathcal{A}$, if $\text{dc}(\Omega(\mathcal{A})) = 0$ then the entropy is continuous on the set $\overline{\text{co}}\mathcal{A}$;

F) let $\{\mathcal{A}_n\}$ be a sequence of sets such that $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ for all $n$ then

$$\lim_{n \to +\infty} \bar{C}(\mathcal{A}_n) = \bar{C} \left( \bigcup_n \mathcal{A}_n \right) \quad \text{and} \quad \lim_{n \to +\infty} \Omega(\mathcal{A}_n) = \Omega \left( \bigcup_n \mathcal{A}_n \right);$$

$^{14}$In all statements concerning the optimal average state of a particular set it is assumed that this set has finite $\chi$-capacity.

$^{15}$Note that $\mathcal{A} \subsetneq \mathcal{B}$ does not imply $\bar{C}(\mathcal{A}) < \bar{C}(\mathcal{B})$ even in the case of convex and closed sets $\mathcal{A}$ and $\mathcal{B}$ (see the examples in section 5).
G) let \( \{ \mathcal{A}_n \} \) be a sequence of closed sets such that \( \mathcal{A}_n \supseteq \mathcal{A}_{n+1} \) for all \( n \) then

\[
\lim_{n \to +\infty} \bar{C}(\mathcal{A}_n) = \bar{C}\left( \bigcap_n \mathcal{A}_n \right) \quad \text{and} \quad \lim_{n \to +\infty} \Omega(\mathcal{A}_n) = \Omega\left( \bigcap_n \mathcal{A}_n \right)
\]

take place if one of the following conditions holds: \(^{16}\)

- the set \( \mathcal{A}_1 \) is regular and \( \Omega(\mathcal{A}_n) = \Omega(\mathcal{A}_1) \) for all \( n \);
- the restriction of the entropy \( H(\rho) \) to the set \( \mathcal{A}_1 \) is continuous at some limit point \( \omega \) of the sequence \( \{ \Omega(\mathcal{A}_n) \} \); \(^{17}\)
- the relative entropy \( H(\rho\|\omega) \) is continuous on the set \( \mathcal{A}_1 \) for some limit point \( \omega \) of the sequence \( \{ \Omega(\mathcal{A}_n) \} \);

H) each set \( \mathcal{A} \) with finite \( \chi \)-capacity is contained in the maximal set \( \mathcal{V}_{\Omega(\mathcal{A}), C(\mathcal{A})} \) with the same \( \chi \)-capacity; \(^{18}\)

I) each regular closed set \( \mathcal{A} \) with finite \( \chi \)-capacity contains the minimal closed set with the same \( \chi \)-capacity; \(^{19}\)

J) if \( \bar{C}(\mathcal{A}) < +\infty \) and \( \bar{C}(\mathcal{B}) < +\infty \) then \( \bar{C}(\mathcal{A} \cup \mathcal{B}) < +\infty \), in particular, the coincidence \( \Omega(\mathcal{A}) = \Omega(\mathcal{B}) \) implies \( \bar{C}(\mathcal{A} \cup \mathcal{B}) = \max(\bar{C}(\mathcal{A}), \bar{C}(\mathcal{B})) \);

K) if \( \Phi : \mathcal{S}(\mathcal{H}) \mapsto \mathcal{S}(\mathcal{H}') \) is an arbitrary channel then \( \bar{C}(\Phi(\mathcal{A})) \leq \bar{C}(\mathcal{A}) \) and equality here implies \( \Omega(\Phi(\mathcal{A})) = \Phi(\Omega(\mathcal{A})) \);

L) if \( \{ \Phi_t \}_{t \in \mathbb{R}_+} \) is an arbitrary family of channels from \( \mathcal{S}(\mathcal{H}) \) into itself such that \( \lim_{t \to +0} \Phi_t(\rho) = \rho \) for all states \( \rho \) in \( \mathcal{A} \) then \(^{20}\)

\[
\lim_{t \to +0} \bar{C}(\Phi_t(\mathcal{A})) = \bar{C}(\mathcal{A}) \quad \text{and} \quad \lim_{t \to +0} \Omega(\Phi_t(\mathcal{A})) = \Omega(\mathcal{A}).
\]

\(^{16}\)These condition are essential (see remark 5 below).
\(^{17}\)By the assertion D the set of limit points of the sequence \( \{ \Omega(\mathcal{A}_n) \} \) is nonempty.
\(^{18}\)A set is called maximal set with given \( \chi \)-capacity if it is not a proper subset of a set with the same \( \chi \)-capacity.
\(^{19}\)A set is called minimal closed set with given \( \chi \)-capacity if it has no proper closed subsets with the same \( \chi \)-capacity.
\(^{20}\)This assertion can be considered as a stability property of the \( \chi \)-capacity and of the optimal average state with respect to a quantum noise.
Remark 5. The regularity and the continuity requirements in the assertions G and I are essential. Moreover, nonregularity of a particular set with finite \( \chi \)-capacity can be shown by finding a decreasing family of subset of this set for which the assertion G does not hold. This possibility is used in the proof of proposition 3b in subsection 5.3. The example of a closed set with finite \( \chi \)-capacity having no minimal closed subset with the same \( \chi \)-capacity is considered in subsection 5.2.

Proof. The assertions A, B and the first part of C directly follows from the definition of the \( \chi \)-capacity due to lower semicontinuity and convexity of the relative entropy. The second part of C is proved as follows. Let \( A \subseteq B \) and \( \bar{C}(A) = \bar{C}(B) \). Then by theorem 1 \( H(\rho \| \Omega(B)) \leq \bar{C}(B) = \bar{C}(A) \) for all states \( \rho \) in \( B \). Since \( A \subseteq B \) this inequality holds for all states \( \rho \) in \( A \). Thus the uniqueness assertion of theorem 1 implies \( \Omega(A) = \Omega(B) \).

The first part of D follows from proposition 3a since by theorem 1 each set \( A \) with finite \( \chi \)-capacity is contained in the set \( V_{\Omega(A), \bar{C}(A)} \). The second part of D is a corollary of B and the Krein-Milman theorem.

Since theorem 1 implies \( \overline{co}(A) \subseteq V_{\Omega(A), \bar{C}(A)} \) the assertion E follows from propositions 2 and 3a.

To prove F note that C implies existence of the limit and the inequality

\[
\lim_{n \to +\infty} \bar{C}(A_n) \leq \bar{C} \left( \bigcup_n A_n \right).
\]  

Let \( \{\{\pi^k_i, \rho^k_i\}\}_k \) be an arbitrary approximating sequence of ensembles for the set \( \bigcup_n A_n \), so that

\[
\lim_{k \to +\infty} \chi(\{\pi^k_i, \rho^k_i\}) = \bar{C} \left( \bigcup_n A_n \right).
\]  

Since an ensemble is a finite collection of states for each \( k \) there exists \( n(k) \) such that \( \rho^k_i \in A_{n(k)} \) for all \( i \) and hence \( \bar{C}(A_{n(k)}) \geq \chi(\{\pi^k_i, \rho^k_i\}) \). This and (33) imply “=” in (32).

Suppose \( \bar{C}(\bigcup_n A_n) = \bar{C}(\overline{co}(\bigcup_n A_n)) < +\infty \). By the assertion D the set \( \overline{co}(\bigcup_n A_n) \) is compact. It follows that the sequence \( \{\Omega(A_n)\} \) has partial limits. Let \( \omega = \lim_{k \to +\infty} \Omega(A_{n(k)}) \) for a particular subsequence \( n(k) \).

By theorem 1 for each \( n \) there exists ensemble \( \{\pi^n_i, \rho^n_i\} \) of states in \( A_n \) with the average state \( \bar{\rho}_n \) such that

\[
\chi(\{\pi^n_i, \rho^n_i\}) \geq \bar{C}(A_n) - 1/n \quad \text{and} \quad \|\bar{\rho}_n - \Omega(A_n)\|_1 \leq 1/n
\]  

(34)
By the proved equality in (32) the sequence \[\{\pi_n^i, \rho_n^i\}\] is approximating for the set \(\bigcup_n A_n\) and hence, by theorem 1, the sequence \(\{\bar{\rho}_n\}\) converges to the state \(\Omega(\bigcup_n A_n)\) as \(n \to +\infty\). By (34) the subsequence \(\{\bar{\rho}_{n(k)}\}\) converges to the state \(\omega\). So, we have \(\omega = \Omega(\bigcup_n A_n)\). Thus each partial limit of the sequence \(\{\Omega(A_n)\}\) coincides with the state \(\Omega(\bigcup_n A_n)\).

To prove G note that C implies existence of the above limit and the inequality

\[
\lim_{n \to +\infty} \bar{C}(A_n) \geq \bar{C}\left(\bigcap_n A_n\right). \tag{35}
\]

The additional conditions in G provide different ways of proving the equality in this inequality.

Consider first the second and the third conditions. Without loss of generality we may assume that

\[
\lim_{n \to +\infty} \Omega(A_n) = \omega. \tag{36}
\]

By theorem 1 for each natural \(n\) there exists a measure \(\mu_n\) (finitely) supported by the set \(A_n\) such that

\[
\chi(\mu_n) \geq \bar{C}(A_n) - 1/n \quad \text{and} \quad \|\bar{\rho}(\mu_n) - \Omega(A_n)\|_1 \leq 1/n \tag{37}
\]

The supports of all measures in the sequence \(\{\mu_n\}\) lie in the set \(A_1\), which is compact by the assertion D. Hence this sequence is compact in the weak topology and contains subsequence \(\{\mu_{n(k)}\}\) weakly converging to a particular measure \(\mu_*\). Continuity of the mapping \(\mu \mapsto \bar{\rho}(\mu)\), (35) and (37) imply \(\omega = \bar{\rho}(\mu_*) = \lim_{k \to +\infty} \bar{\rho}(\mu_{n(k)})\) By theorem 6.3 in [21] \(\text{supp} \mu_* \subseteq \bigcap_n A_n\).

Suppose the second condition in G is valid. Then there exists

\[
\lim_{k \to +\infty} H(\bar{\rho}(\mu_{n(k)})) = H(\bar{\rho}(\mu_*)) = H(\omega) < +\infty \tag{38}
\]

and by using (2) we have

\[
\chi(\mu_{n_k}) = H(\bar{\rho}(\mu_{n_k})) - \hat{H}(\mu_{n_k})
\]

for sufficiently large \(k\). By using (38) and lower semicontinuity of the functional \(\hat{H}(\mu)\) we obtain

\[
\lim_{n \to +\infty} \bar{C}(A_n) = \lim_{k \to +\infty} \sup \chi(\mu_{n_k}) = \lim_{k \to +\infty} H(\bar{\rho}(\mu_{n_k})) - \lim \inf \hat{H}(\mu_{n_k})
\]

\[
\leq H(\bar{\rho}(\mu_*)) - \hat{H}(\mu_*) = \chi(\mu_*) \leq \bar{C}\left(\bigcap_n A_n\right),
\]

32
which implies equality in (35).

Suppose the third condition in G is valid. Since this means continuity of the function $H(\rho \parallel \omega)$ on the compact set $A_1$ the definition of the weak convergence implies

$$
\lim_{k \to +\infty} \int H(\rho \parallel \omega) \mu_{n_k}(d\rho) = \int H(\rho \parallel \omega) \mu_*(d\rho) = \chi(\mu_*) \leq \bar{C} \left( \bigcap_n A_n \right).
$$

By generalized Donald’s identity (4) we have

$$
\int H(\rho \parallel \omega) \mu_{n_k}(d\rho) = \chi(\mu_{n_k}) + H(\bar{\rho}(\mu_{n_k}) \parallel \omega) \geq \chi(\mu_{n_k})
$$

and by the above inequality we obtain

$$
\bar{C} \left( \bigcap_n A_n \right) \geq \lim_{k \to +\infty} \int H(\rho \parallel \omega) \mu_{n_k}(d\rho) \geq \lim_{k \to +\infty} \chi(\mu_{n_k}) = \lim_{n \to +\infty} \bar{C}(A_n),
$$

which implies equality in (35).

To complete the consideration of the second and the third conditions in G it is sufficient to show that the limit state $\omega$ in (36) is the optimal average state of the set $\bigcap_n A_n$. By theorem 1 $H(\rho \parallel \Omega(A_n)) \leq \bar{C}(A_n)$ for arbitrary state $\rho$ in $\bigcap_n A_n$ and for arbitrary $n$. By using (36), the proved equality in (35) and lower semicontinuity of the relative entropy we obtain that

$$
H(\rho \parallel \omega) \leq \liminf_{n \to +\infty} H(\rho \parallel \Omega(A_n)) \leq \liminf_{n \to +\infty} \bar{C}(A_n) = \bar{C} \left( \bigcap_n A_n \right)
$$

for all such $\rho$. Theorem 1 implies that $\omega = \Omega(\bigcap_n A_n)$.

Now consider the first condition in G. Note that the assumed regularity of the set $A_1$ and the condition $\Omega(A_n) = \Omega(A_1)$ for all $n$ implies regularity of the sets $A_n$ for all $n$. By theorem 3 in the next section for each $n$ there exists an optimal measure $\mu_n$ supported by the set $A_n$ such that (37) holds with $0$ instead of $1/n$. If the first condition of regularity is valid then relation (38) in this case holds trivially and by repeating arguments in the proof of the second condition we complete the proof. If the second condition of regularity is valid then the arguments in the proof of the third condition are applied immediately.

The assertion H immediately follows from theorem 1.
To prove I consider the nonempty set $\mathcal{A}$ of all closed subsets of $\mathcal{A}$ having the same $\chi$-capacity endowed with the partial order $\prec$ defined by

$$B \prec C \iff B \supseteq C.$$ 

It is clear that I means existence of a maximal element in $\mathcal{A}$. By the Zorn lemma to show this it is sufficient to show that an arbitrary chain in $\mathcal{A}$ has maximal element. The role of this maximal element for a given chain can be plaid by the intersection of all elements of the chain provided that this intersection is an element of $\mathcal{A}$. Since D implies compactness of the set $\mathcal{A}$ the intersection of an arbitrary decreasing family of subsets of the set $\mathcal{A}$ coincides with the intersection of its particular countable subfamily. So, it is sufficient to show that

$$\bar{C}\left(\bigcap_n B_n\right) = \bar{C}(\mathcal{A})$$

for arbitrary monotonously decreasing sequence $\{B_n\}$ of closed subsets of $\mathcal{A}$ such that $\bar{C}(B_n) = \bar{C}(\mathcal{A})$. But this follows from regularity of the set $\mathcal{A}$ and $G$ with the first condition since $C$ implies $\Omega(B_n) = \Omega(\mathcal{A})$ for all $n$.

The first part of J follows from proposition 4 below. The second is a corollary of C and theorem 1 since it implies

$$H(\rho\|\Omega(\mathcal{A})) = \Omega(\mathcal{B})) \leq \max(\bar{C}(\mathcal{A}), \bar{C}(\mathcal{B}))$$

for all $\rho$ in $\mathcal{A} \cup \mathcal{B}$.

The first part of K is a direct corollary of the definition of the $\chi$-capacity and the monotonicity property of the relative entropy.

To prove the second suppose $\bar{C}(\Phi(\mathcal{A})) = \bar{C}(\mathcal{A})$. By using monotonicity of the relative and theorem 1 we obtain

$$H(\Phi(\rho)\|\Phi(\Omega(\mathcal{A}))) \leq H(\rho\|\Omega(\mathcal{A})) \leq \bar{C}(\mathcal{A}) = \bar{C}(\Phi(\mathcal{A}))$$

for arbitrary state $\rho$ in $\mathcal{A}$. By theorem 1 this implies $\Omega(\Phi(\mathcal{A})) = \Phi(\Omega(\mathcal{A}))$.

The assertion L follows the first part of K and lemma 4 below. □

Theorem 2E implies the following observation.

**Corollary 6.** Let $\mathcal{A}$ be a closed convex set with finite $\chi$-capacity.

If $dc(\rho) < 1$ for all $\rho$ in $\mathcal{A}$ then the set $\mathcal{A}$ is regular and the entropy is bounded on the set $\mathcal{A}$.

If $dc(\rho) = 0$ for all $\rho$ in $\mathcal{A}$ then the entropy is continuous on the set $\mathcal{A}$.
Remark 6. Corollary 6 implies, in particular, that boundedness of the entropy on a particular closed convex set of states with zero decrease coefficient (for example, Gaussian states) implies continuity of the entropy on this set. □

The assertions D and F in theorem 2 provide a sufficient condition for compactness of unions.

Corollary 7. If \( \{A_n\} \) be a sequence of sets such that \( A_n \subseteq A_{n+1} \) and \( C(A_n) \leq M < +\infty \) for all \( n \) then the set \( \bigcup_n A_n \) is relatively compact.

Theorem 2K implies the following observation.

Corollary 8. Let \( A \) be a set with finite \( \chi \)-capacity \( C(A) \). Then \( \Omega(A) \) is an invariant state for arbitrary channel \( \Phi \) such that \( \Phi(A) \subseteq \overline{\mathcal{C}}(A) \) and \( C(\Phi(A)) = C(A) \). In particular, \( \Omega(A) \) is an invariant state for arbitrary automorphism \( \alpha \) of \( \mathcal{S}(\mathcal{H}) \)\(^{21} \) such that \( \alpha(A) \subseteq \overline{\mathcal{C}}(A) \).

Let \( \mathcal{F}(A) \) be the set of all channels \( \Phi \) from \( \mathcal{S}(\mathcal{H}) \) into itself such that \( \Phi(A) \subseteq \overline{\mathcal{C}}(A) \) and \( C(\Phi(A)) = C(A) \). This set is nonempty and contains all automorphisms \( \alpha \) of \( \mathcal{S}(\mathcal{H}) \) such that \( \alpha(A) \subseteq \overline{\mathcal{C}}(A) \).

Corollary 8 implies the following observation (in the spirit of the Markov-Kakutany theorem): For arbitrary set \( A \) with finite \( \chi \)-capacity the set \( \overline{\mathcal{C}}(A) \) contains at least one common invariant state for all channels from \( \mathcal{F}(A) \).

Theorem 1 and corollary 8 provide the following result.

Corollary 9. Let \( A \) be an arbitrary set of states and \( \mathcal{F}_0 \) be an arbitrary subset of \( \mathcal{F}(A) \). Let \( \text{Inv}_{\mathcal{F}_0} \) be a set of all common invariant states for all channels from \( \mathcal{F}_0 \).

The \( \chi \)-capacity of the set \( A \) can be defined by the expression

\[
\bar{C}(A) = \inf_{\sigma \in \text{Inv}_{\mathcal{F}_0} \cap \overline{\mathcal{C}}(A)} \sup_{\rho \in A} H(\rho\|\sigma),
\]

keeping in mind that \( \bar{C}(A) = +\infty \) if \( \text{Inv}_{\mathcal{F}_0} \cap \overline{\mathcal{C}}(A) = \emptyset \).

In particular, if there exists the unique invariant state \( \sigma_0 \in \overline{\mathcal{C}}(A) \) for all channels from \( \mathcal{F}_0 \) then \( \bar{C}(A) = \sup_{\rho \in A} H(\rho\|\sigma_0) \) and if \( \bar{C}(A) < +\infty \) then \( \Omega(A) = \sigma_0 \).

Corollaries 8 and 9 provide a possibility to determine the optimal average state and to calculate the \( \chi \)-capacity of a particular set of states by finding a sufficient family \( \mathcal{F}_0 \) of channels from \( \mathcal{F}(A) \). We will use this possibility in the next section.

\(^{21}\)By Wigner’s theorem each automorphism of \( \mathcal{S}(\mathcal{H}) \) has the form \( U(\cdot)U^* \), where \( U \) is either unitary or antiunitary operator in \( \mathcal{H} \).
Theorem 2D implies the following observation concerning the $\chi$-capacity of constrained quantum channels $[7],[17]$.

**Corollary 10.** Let $\Phi : S(\mathcal{H}) \mapsto S(\mathcal{H}')$ be an arbitrary quantum channel and $\mathcal{A}$ be a subset of $S(\mathcal{H})$. If $\bar{C}(\Phi, \mathcal{A}) < +\infty$ then $\Phi(\mathcal{A})$ is a relatively compact subset of $S(\mathcal{H}')$.

**Proof.** It is easy to see by the definitions that

$$\bar{C}(\Phi(\mathcal{A})) \leq \bar{C}(\Phi, \mathcal{A}). \square$$

By this corollary the $\chi$-capacity of an unconstrained quantum channel can be finite only if the output set of this channel is relatively compact.

Now we consider the bounds for the $\chi$-capacity of finite union of sets.

**Proposition 7.** If $\{\mathcal{A}_k\}_{k=1}^n$ is a finite collection of sets then

$$\max_{\{\lambda_k\}} \left( \sum_{k=1}^n \lambda_k \bar{C}(\mathcal{A}_k) + \chi(\{\lambda_k, \Omega(\mathcal{A}_k)\}) \right) \leq \bar{C} \left( \bigcup_{k=1}^n \mathcal{A}_k \right) \leq \max_{1 \leq k \leq n} \bar{C}(\mathcal{A}_k) + \log n,$$

where the first maximum is over all probability distributions with $n$ outcomes.

In the case $\bar{C}(\mathcal{A}_k) = C$ for all $k = \frac{1}{n}$ this implies

$$C + \bar{C}(\{\Omega(\mathcal{A}_1), ..., \Omega(\mathcal{A}_n)\}) \leq \bar{C} \left( \bigcup_{k=1}^n \mathcal{A}_k \right) \leq C + \log n.$$

**Proof.** By theorem 1 for each natural $m$ and each $k = \frac{1}{n}$ there exists ensemble $\mu_k^m$ such that

$$\chi(\mu_k^m) \geq \bar{C}(\mathcal{A}_k) - 1/m \quad \text{and} \quad \|\bar{\rho}(\mu_k^m) - \Omega(\mathcal{A}_k)\|_1 \leq 1/m. \quad (39)$$

Taking arbitrary probability distribution $\{\lambda_k\}_{k=1}^n$ consider the ensemble $\mu_m = \sum_{k=1}^n \lambda_k \mu_k^m$ of states in $\bigcup_{k=1}^n \mathcal{A}_k$. By using lemma 1, lower semicontinuity of the relative entropy and (39) we obtain

$$\bar{C} \left( \bigcup_{k=1}^n \mathcal{A}_k \right) \geq \liminf_{m \to +\infty} \chi(\mu_m) = \liminf_{m \to +\infty} \left( \sum_{k=1}^n \lambda_k \chi(\mu_k^m) + \chi(\{\lambda_k, \bar{\rho}(\mu_k^m)\}) \right)$$

$$= \sum_{k=1}^n \lambda_k \bar{C}(\mathcal{A}_k) + \liminf_{m \to +\infty} \chi(\{\lambda_k, \bar{\rho}(\mu_k^m)\}) \geq \sum_{k=1}^n \lambda_k \bar{C}(\mathcal{A}_k) + \chi(\{\lambda_k, \Omega(\mathcal{A}_k)\}),$$

which implies the lower bound of the proposition.
To prove the upper bound note that arbitrary ensemble $\mu$ of states in $\bigcup_{k=1}^{n} \mathcal{A}_k$ can be represented as a convex combination $\sum_{k=1}^{n} \lambda_k \mu_k$, where $\mu_k$ is an ensemble of states in $\mathcal{A}_k$ for $k = 1, n$ and $\{\lambda_k\}_{k=1}^{n}$ is a probability distribution. By using lemma 1 and proposition 9b below we obtain

$$\chi(\mu) = \sum_{k=1}^{n} \lambda_k \chi(\mu_k) + \chi(\{\lambda_k, \bar{\rho}(\mu_k)\}) \leq \max_{1 \leq k \leq n} \bar{C}(\mathcal{A}_k) + \log n. \Box$$

**Remark 7.** Proposition 7 shows that the $\chi$-capacity of a union of sets with given $\chi$-capacities depends on relative positions of their optimal average states. By theorem 2J if all the optimal average states coincide with each other then the $\chi$-capacity of the union is minimal and is equal to the maximal $\chi$-capacity of the united sets. The greater diversity of the optimal average states the higher the $\chi$-capacity of the union. This is obvious in the case of union of two sets for which the lower bound in proposition 7 and inequality (1) imply

$$\bar{C}(\mathcal{A} \cup \mathcal{B}) \geq \max_{\lambda \in [0,1]} \left( \lambda \bar{C}(\mathcal{A}) + (1 - \lambda) \bar{C}(\mathcal{B}) + \frac{1}{2} \lambda(1 - \lambda) ||\Omega(\mathcal{A}) - \Omega(\mathcal{B})||_2^2 \right).$$

Note also that the lower and the upper bounds in proposition 7 coincides if and only if

$$\bar{C}(\mathcal{A}_i) = \bar{C}(\mathcal{A}_j) \quad \text{and} \quad \bigcup_{\rho \in \mathcal{A}_i} \text{supp}\rho \perp \bigcup_{\rho \in \mathcal{A}_j} \text{supp}\rho \quad \text{for all} \quad i \neq j.$$

To complete the proof of theorem 2 we obtain the following result, which will be also used in section 6.

**Lemma 4.** Let $\{\Psi_\lambda\}_{\lambda \in \Lambda}$ be a family of continuous mappings from $\mathcal{S}(\mathcal{H})$ into itself indexed by some ordered set $\Lambda$ and such that $\lim_\lambda \Psi_\lambda(\rho) = \rho$ for all states $\rho$ in a particular subset $\mathcal{A}$ of $\mathcal{S}(\mathcal{H})$. Then

$$\liminf_\lambda \bar{C}(\Psi_\lambda(\mathcal{A})) \geq \bar{C}(\mathcal{A}).$$

If there exists $\lim_\lambda \bar{C}(\Psi_\lambda(\mathcal{A})) = \bar{C}(\mathcal{A})$ then there exists $\lim_\lambda \Omega(\Psi_\lambda(\mathcal{A})) = \Omega(\mathcal{A})$.

**Proof.** The first assertion of the lemma easily follows from lower semi-continuity of the relative entropy. Indeed, for arbitrary $\varepsilon > 0$ there exists an ensemble $\{\pi_i, \rho_i\}$ such that

$$\chi(\{\pi_i, \rho_i\}) \geq C(\varepsilon) = \begin{cases} \bar{C}(\mathcal{A}) - \varepsilon, & \bar{C}(\mathcal{A}) < +\infty \\ \varepsilon, & \bar{C}(\mathcal{A}) = +\infty \end{cases}$$
By the assumption and due to lower semicontinuity of the relative entropy we obtain
\[
\liminf_{\lambda} \bar{C}(\Psi_{\lambda}(A)) \geq \liminf_{\lambda} \chi(\{\pi_i, \Psi_{\lambda}(\rho_i)\}) \geq \chi(\{\pi_i, \rho_i\}) \geq C(\varepsilon).
\]
Since $\varepsilon$ can be arbitrary this implies the first assertion on the lemma.

Let $\lim_{\lambda} \bar{C}(\Psi_{\lambda}(A)) = \overline{C}(A) < +\infty$. By theorem 1 for arbitrary $\varepsilon > 0$ there exists an ensemble $\{\pi_i, \rho_i\}$ such that
\[
\chi(\{\pi_i, \rho_i\}) \geq \bar{C}(A) - \varepsilon \quad \text{and} \quad \|\sum_i \pi_i \rho_i - \Omega(A)\|_1 < \varepsilon \quad (40)
\]

Applying the arguments from the first part of the proof we obtain that there exists $\lambda_1^\varepsilon$ such that
\[
\chi(\{\pi_i, \Psi_{\lambda}(\rho_i)\}) \geq \chi(\{\pi_i, \rho_i\}) - \varepsilon, \quad \forall \lambda \geq \lambda_1^\varepsilon.
\]

By the assumption there exists $\lambda_2^\varepsilon$ such that
\[
\bar{C}(\Psi_{\lambda}(A)) \leq \bar{C}(A) + \varepsilon, \quad \forall \lambda \geq \lambda_2^\varepsilon.
\]

Thus for all $\lambda \geq \max(\lambda_1^\varepsilon, \lambda_2^\varepsilon)$ we have
\[
0 \leq \bar{C}(\Psi_{\lambda}(A)) - \chi(\{\pi_i, \Psi_{\lambda}(\rho_i)\}) \leq \bar{C}(A) - \chi(\{\pi_i, \rho_i\}) + 2\varepsilon \leq 3\varepsilon
\]
and by using corollary 4 we obtain
\[
\frac{1}{2} \|\sum_i \pi_i \Psi_{\lambda}(\rho_i) - \Omega(\Psi_{\lambda}(A))\|_1^2 \leq H(\sum_i \pi_i \Psi_{\lambda}(\rho_i)\|\Omega(\Psi_{\lambda}(A)))
\leq \bar{C}(\Psi_{\lambda}(A)) - \chi(\{\pi_i, \Psi_{\lambda}(\rho_i)\}) \leq 3\varepsilon \quad (41)
\]

The continuity property of the family $\{\Psi_{\lambda}\}$ implies existence of $\lambda_3^\varepsilon$ such that
\[
\|\sum_i \pi_i \Psi_{\lambda}(\rho_i) - \sum_i \pi_i \rho_i\|_1 \leq \varepsilon, \quad \forall \lambda \geq \lambda_3^\varepsilon. \quad (42)
\]

By using (40), (41) and (42) we obtain
\[
\|\Omega(\Psi_{\lambda}(A)) - \Omega(\rho)\|_1 \leq \|\Omega(\Psi_{\lambda}(A)) - \sum_i \pi_i \Psi_{\lambda}(\rho_i)\|_1 + \|\sum_i \pi_i \Psi_{\lambda}(\rho_i) - \sum_i \pi_i \rho_i\|_1 + \|\sum_i \pi_i \rho_i - \Omega(\rho)\|_1 \leq 2\varepsilon + \sqrt{6\varepsilon}
\]
for all $\lambda \geq \max(\lambda_1^\varepsilon, \lambda_2^\varepsilon, \lambda_3^\varepsilon)$. Since $\varepsilon$ is arbitrary this implies the second statement of the lemma. □
4.3 The optimal measure

Let $A$ be a closed set with finite $\chi$-capacity. By theorem 2D the set $A$ is compact. Hence the set $\mathcal{M}(A)$ of all probability measures supported by the set $A$ is compact in the topology of weak convergence (Prokhorov’s topology). Since an arbitrary measure in $\mathcal{M}(A)$ can be weakly approximated by a sequence of measures with finite support, lower semicontinuity of the functional $\chi(\mu)$ implies

$$\bar{C}(A) = \sup_{\mu \in \mathcal{M}(A)} \chi(\mu),$$

which means that the supremum over all measures coincides with the supremum over all measures with finite support.

**Definition 3.** A measure $\mu_*$ supported by the set $A$ and such that

$$\bar{C}(A) = \chi(\mu_*) = \int H(\rho \| \bar{\rho}(\mu_*)) \mu_*(d\rho)$$

is called the optimal measure for the set $A$.

By using the arguments from the proof of proposition 1 in [7] it is easy to see that the functional $\mu \mapsto \int H(\rho \| \bar{\rho}(\mu_*)) \mu(d\rho)$ is lower semicontinuous on $\mathcal{M}(A)$. This, the above mentioned weak density of measures with finite support in $\mathcal{M}(A)$ and generalized Donald’s identity (4) imply the following generalization of theorem 1 and corollary 4: For arbitrary closed set $A$ with finite $\chi$-capacity and arbitrary measure $\mu$ from $\mathcal{M}(A)$ the following inequalities hold

$$\int_A H(\rho \| \bar{\rho}(\mu_*)) \mu(d\rho) \leq \bar{C}(A),$$

$$\bar{C}(A) - \chi(\mu) \geq H(\bar{\rho}(\mu) \| \bar{\rho}(\mu_*)) \geq \frac{1}{2} \| \bar{\rho}(\mu) - \Omega(A) \|_1^2.$$
ensemble for the set $A$ - then its the average state $\bar{\rho}$ coincides with the optimal average state $\Omega(A)$ and $H(\rho_i \| \Omega(A)) = \bar{C}(A)$ for all $i$ such that $\pi_i > 0$.

**Corollary 11.** Let $A$ be a closed set with finite $\chi$-capacity. Existence of an optimal measure for the set $A$ implies $\bar{C}(A) \leq H(\Omega(A))$.

**Proof.** It is sufficient to consider the case $H(\Omega(A)) < +\infty$ for which (2), the definition of an optimal measure $\mu_*$ and proposition 8 imply

$$\bar{C}(A) = \chi_* = H(\bar{\rho}(\mu_*)) - \hat{H}(\mu_*) \leq H(\bar{\rho}(\mu_*)) = H(\Omega(A)).$$

This corollary provides the simple way to show nonexistence of an optimal measure for a particular set of states, which will be used in the proof of proposition 1b and 3b in section 5 below.

The following theorem provides the sufficient condition for existence of an optimal measure.

**Theorem 3.** Let $A$ be a convex closed set with finite $\chi$-capacity. If $\text{Ext}(A)$ is a regular set then there exists an optimal measure for the set $A$ supported by the set $\text{Ext}(A)$.

The main ingredient of the proof of this theorem is the following lemma.

**Lemma 5.** Let $A$ be a convex closed set with finite the $\chi$-capacity. There exists a sequence of measures $\{\mu_n\}$ supported by the set $\text{Ext}(A)$ weakly converging to some measure $\mu_*$ supported by the set $\text{Ext}(A)$ with the barycenter $\Omega(A)$ such that

$$\lim_{n \to +\infty} H(\overline{\rho}(\mu_n) \| \Omega(A)) = 0 \quad \text{and} \quad \lim_{n \to +\infty} \chi(\mu_n) = \bar{C}(A).$$

**Proof.** Let $\{(\pi^n_i, \rho^n_i)\}_n$ be an approximating sequence of ensembles for the set $A$ with the corresponding sequence of the average states $\{\bar{\rho}_n\}$. Theorem 1 implies $\lim_{n \to +\infty} H(\bar{\rho}_n \| \Omega(A)) = 0$. Since by theorem 2D the set $A$ is compact the theory of barycentric decomposition [1], [2] implies existence for each $n$ and $i$ of a measure $\mu^n_i$ supported by $\text{Ext}(A)$ such that $\bar{\rho}(\mu^n_i) = \rho^n_i$. Convexity of the relative entropy and Jensen’s inequality\(^{22}\) imply

$$H(\rho^n_i \| \bar{\rho}_n) = H \left( \int \rho \mu^n_i(d\rho) \| \bar{\rho}_n \right) \leq \int H(\rho \| \bar{\rho}_n) \mu^n_i(d\rho).$$

\(^{22}\)Application of Jensen’s inequality in this case is valid since the relative entropy can be represented as a pointwise limit of a monotonously increasing sequence of continuous convex functions [3].

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By using this and \((43)\) we obtain
\[
\sum_i \pi_i^n H(\rho^n_i \| \bar{\rho}_n) \leq \sum_i \pi_i^n \int H(\rho \| \bar{\rho}_n) \mu^n_i(d\rho) = \chi \left( \sum_i \pi_i^n \mu^n_i \right) \leq \bar{C}(\mathcal{A}).
\]

Let \(\mu_n = \sum_i \pi_i^n \mu^n_i\) be a measure with the barycenter \(\bar{\rho}_n\) for each \(n\). It follows from the approximating property of the sequence \(\{\{\pi_i^n, \rho^n_i\}\}_n\) and from the above inequality that \(\lim_{n \to +\infty} \chi(\mu_n) = \bar{C}(\mathcal{A})\). Compactness of the set \(\mathcal{A}\) implies compactness of the set \(\mathcal{M}(\mathcal{A})\) in the weak topology and hence existence of a subsequence of the sequence \(\{\mu_n\}\) converging to a particular measure \(\mu_\ast\), supported by \(\text{Ext}(\mathcal{A})\) due to theorem 6.1 in [21]. Continuity of the mapping \(\mu \mapsto \bar{\rho}(\mu)\) and theorem 1 imply \(\bar{\rho}(\mu_\ast) = \Omega(\mathcal{A})\). Thus this subsequence has the all properties stated in the lemma. \(\square\)

**Proof of theorem 3.** The two regularity conditions provide two different ways to show that the limit measure \(\mu_\ast\) involved in the above lemma is an optimal measure for the set \(\mathcal{A}\).

Let \(\{\mu_n\}\) be a sequence provided by lemma 5.

By the first regularity condition
\[
\lim_{n \to +\infty} H(\bar{\rho}(\mu_n)) = H(\bar{\rho}(\mu_\ast)) = H(\Omega(\mathcal{A})) < +\infty.
\]

Hence expression \((2)\) and lower semicontinuity of the functional \(\hat{H}(\mu)\) imply
\[
\limsup_{n \to +\infty} \chi(\mu_n) = \limsup_{n \to +\infty} (H(\bar{\rho}(\mu_n)) - \hat{H}(\mu_n)) \leq H(\bar{\rho}(\mu_\ast)) - \hat{H}(\mu_\ast) = \chi(\mu_\ast).
\]

Since \(\lim_{n \to +\infty} \chi(\mu_n) = \bar{C}(\mathcal{A})\) and \(\chi(\mu_\ast) \leq \bar{C}(\mathcal{A})\) this inequality implies \(\chi(\mu_\ast) = \bar{C}(\mathcal{A})\), which means optimality of the measure \(\mu_\ast\).

The second regularity condition, compactness of the set \(\mathcal{A}\) and the definition of the weak convergence imply
\[
\chi(\mu_\ast) = \int H(\rho \| \Omega(\mathcal{A})) \mu_\ast(d\rho) = \lim_{n \to +\infty} \int H(\rho \| \Omega(\mathcal{A})) \mu_n(d\rho).
\]

By generalized Donald’s identity \((4)\) and nonnegativity of the relative entropy we have
\[
\int H(\rho \| \Omega(\mathcal{A})) \mu_n(d\rho) = \chi(\mu_n) + H(\bar{\rho}(\mu_n) \| \Omega(\mathcal{A})) \geq \chi(\mu_n).
\]

Since \(\lim_{n \to +\infty} \chi(\mu_n) = \bar{C}(\mathcal{A})\) the above expressions imply \(\chi(\mu_\ast) = \bar{C}(\mathcal{A})\), which means optimality of the measure \(\mu_\ast\). \(\square\)
Remark 8. The regularity condition in theorem 3 is essential but is not necessary. There exist nonregular sets with finite \( \chi \)-capacity having no optimal measure (see propositions 1b and 3b in subsections 5.2 and 5.3 correspondingly). It is surprising that there exist converging sequences of states with finite \( \chi \)-capacity having no optimal measure (see the example in subsection 5.1). There also exists nonregular sets having an optimal measure (see the note before lemma 6 in subsections 5.2).

5 Examples

The general results of the previous section are illustrated in this section by considering several examples of sets of states.

5.1 Finite set of states and converging sequences

By theorem 2D each set of states with finite \( \chi \)-capacity is relatively compact. In this subsection we consider the following simplest examples of relatively compact sets:

- a finite collection of states \( \{ \rho_n \}_{n=1}^N \);
- a sequence of states \( \{ \rho_n \}_{n=1}^{+\infty} \) converging to a particular state \( \rho_* \);
- a sequence of states \( \{ \rho_n \}_{n=1}^{+\infty} \) \( H \)-converging to a particular state \( \rho_* \).

The properties of the restriction of the entropy to the convex closure of the above sets are considered in the following proposition.

Proposition 9a. A) Let \( \{ \rho_n \}_{n=1}^N \) be a finite collection of states in \( \mathcal{S}(\mathcal{H}) \). The entropy is continuous on the (closed) set \( \text{co} \left( \{ \rho_n \}_{n=1}^N \right) \) if and only if

\[
H(\rho_n) < +\infty \quad \text{for all } \quad n = 1, 2, ..., N.
\]

B) Let \( \{ \rho_n \}_{n=1}^{+\infty} \) be a sequence of states converging to a state \( \rho_* \).

The entropy is bounded on the set \( \text{co} \left( \{ \rho_n \}_{n=1}^{+\infty} \right) \) if and only if there exists \( \mathcal{H} \)-operator \( H \) with \( \text{ic}(H) < +\infty \) such that

\[
\sup_n \text{Tr} \rho_n H < +\infty.
\]

\footnote{This means that \( \lim_{n \to +\infty} H(\rho_n \| \rho_*) = 0 \).}
The entropy is continuous on the set \( \overline{\text{co}}(\{\rho_n\}_{n=1}^{+\infty}) \) if one of the following equivalent conditions holds:

- \( H(\rho_n) < +\infty \) for all \( n \), \( \lim_{n \to +\infty} H(\rho_n) = H(\rho_*) < +\infty \) and there exists a state \( \sigma \) such that
  \[
  \lim_{n \to +\infty} H(\rho_n \parallel \sigma) = H(\rho_* \parallel \sigma) < +\infty;
  \]
- there exists a \( \mathfrak{H} \)-operator \( H \) with \( \text{ic}(H) = 0 \) such that
  \[
  \sup_n \text{Tr} \rho_n H < +\infty;
  \]
- there exists \( \mathfrak{H} \)-operator \( H \) with \( \text{ic}(H) < +\infty \) such that
  \[
  \text{Tr} \rho_n H < +\infty \text{ for all } n \text{ and } \lim_{n \to +\infty} \text{Tr} \rho_n H = \text{Tr} \rho_* H < +\infty.
  \]

C) Let \( \{\rho_n\}_{n=1}^{+\infty} \) be a sequence of states \( H \)-converging to a state \( \rho_* \). The entropy is bounded on the set \( \overline{\text{co}}(\{\rho_n\}_{n=1}^{+\infty}) \) if and only if

\[
\sup_n H(\rho_n) < +\infty.
\]

The entropy is continuous on the set \( \overline{\text{co}}(\{\rho_n\}_{n=1}^{+\infty}) \) if and only if

\[
H(\rho_n) < +\infty \text{ for all } n \text{ and } \lim_{n \to +\infty} H(\rho_n) = H(\rho_*) < +\infty.
\]

**Remark 9.** It is interesting to compare the boundedness and the continuity conditions for converging and for \( H \)-converging sequences. The conditions for \( H \)-converging sequence look like natural generalizations of the corresponding conditions for finite set of states while the conditions for converging sequence include some additional requirements. These requirements are essential - there exists a converging sequence \( \{\rho_n\}_{n=1}^{+\infty} \) of states for which \( H(\rho_n) \) is finite for all \( n \) and

\[
\lim_{n \to +\infty} H(\rho_n) = H\left(\lim_{n \to +\infty} \rho_n \right) < +\infty.
\]

but the entropy is unbounded on the set \( \overline{\text{co}}(\{\rho_n\}_{n=1}^{+\infty}) \) (see the example below). \( \Box \)
Proof. A) Let $\mathcal{A} = \{\rho_i\}_{i=1}^N$. Necessity of the continuity condition is obvious. To show its sufficiency note that this condition and general properties of quantum entropy \[23\] implies its boundedness on the closed set $\text{co}(\mathcal{A})$ and hence finiteness of the $\chi$-capacity of this set. By theorem 1 there exists the unique state $\Omega(\mathcal{A})$ such that

$$H(\rho_n\|\Omega(\mathcal{A})) = \text{Tr}\rho_n(-\log \Omega(\mathcal{A})) - H(\rho_n) \leq \bar{C}(\mathcal{A}) < +\infty$$

and hence $\text{Tr}\rho_n(-\log \Omega(\mathcal{A})) \leq \bar{C}(\mathcal{A}) + \max_n H(\rho_n) < +\infty$ for all $n = 1,N$. Thus the linear functional $\text{Tr}(\rho(-\log \Omega(\mathcal{A}))$ is finite and hence continuous on the finite set $\mathcal{A}$. By proposition 4 this means continuity of the entropy on the set $\text{co}(\mathcal{A})$.

B) The boundedness condition for this case follows from proposition 1a while the continuity condition - from proposition 4.

C) Let $\mathcal{A} = \{\rho_i\}_{i=1}^{+\infty}$. Necessity of the boundedness and of the continuity conditions for this case is obvious. To show sufficiency of the boundedness condition note that the $\chi$-capacity of the set $\mathcal{A}$ is finite (see proposition 9b below). By theorem 1 there exists the unique state $\Omega(\mathcal{A})$ such that

$$H(\rho_n\|\Omega(\mathcal{A})) = \text{Tr}\rho_n(-\log \Omega(\mathcal{A})) - H(\rho_n) \leq \bar{C}(\mathcal{A}) < +\infty$$

for all $n$ and hence $\sup_n \text{Tr}\rho_n(-\log \Omega(\mathcal{A})) \leq \bar{C}(\mathcal{A}) + \sup_n H(\rho_n) < +\infty$. By proposition 1a this implies boundedness of the entropy on the set $\text{co}(\mathcal{A})$. Sufficiency of the continuity condition follows from the first continuity condition for the case B with $\sigma = \rho_*$. □

The questions concerning the $\chi$-capacity of finite sets of states and of converging sequences are considered in the following proposition.

Proposition 9b. A) Let $\{\rho_n\}_{n=1}^N$ be a finite collection of states in $\mathcal{S}(\mathcal{H})$. The set $\{\rho_n\}_{n=1}^N$ is regular and

$$\bar{C}(\{\rho_n\}_{n=1}^N) \leq \log N$$

There exists optimal ensemble $\mu_* = \{\pi_n, \rho_n\}_{n=1}^N$ for the set $\{\rho_n\}_{n=1}^N$.

B) Let $\{\rho_n\}_{n=1}^{+\infty}$ be a sequence of states converging to a state $\rho_*$. The $\chi$-capacity of the set $\{\rho_n\}_{n=1}^{+\infty}$ is finite if and only if there exists a state $\sigma$ such that\(^\text{24}\)

$$\sup_n H(\rho_n\|\sigma) < +\infty.$$
Let \( \{ \rho_n \}_{n=1}^{\infty} \) be a sequence of states \( H \)-converging to a state \( \rho_* \).

The \( \chi \)-capacity of the set \( \{ \rho_n \}_{n=1}^{\infty} \) is finite and

\[
\bar{C}(\{ \rho_n \}_{n=1}^{\infty}) \leq \inf \max \left( \sup_{n>m} H(\rho_n \parallel \rho_*); \log m \right) + \log 2.
\]

In the cases A,B,C existence of an optimal measure \( \mu_* = \{ \pi_n, \rho_n \} \) for the set \( \{ \rho_n \} \) is equivalent to existence of a probability distribution \( \{ \pi_n \} \) and of a positive number \( C \) satisfying to the following system

\[
\begin{cases}
H(\rho_n \parallel \sum_k \pi_k \rho_k) = C, \quad \pi_n > 0 \\
H(\rho_n \parallel \sum_k \pi_k \rho_k) \leq C, \quad \pi_n = 0
\end{cases}
\]

(44)

If this system has a solution then \( \bar{C}(\{ \rho_n \}) = C \) and \( \Omega(\{ \rho_n \}) = \sum_n \pi_n \rho_n \).

**Proof.** A) To prove the upper bound for the \( \chi \)-capacity of the set \( \mathcal{A} = \{ \rho_n \}_{n=1}^{N} \) it is sufficient to note that \( \text{co}(\mathcal{A}) \) is an output set for the channel \( \sigma \mapsto \sum_{n=1}^{N} \langle n | \sigma | n \rangle \rho_n \) from \( N \)-dimensional Hilbert space with orthonormal basis \( \{ |n\rangle \}_{n=1}^{N} \) and to use the monotonicity property of the relative entropy. Finiteness of the \( \chi \)-capacity and theorem 1 imply finiteness of \( H(\rho_n \parallel \Omega(\mathcal{A})) \) for all \( n \) and hence regularity of the set \( \mathcal{A} \). Existence of optimal measure=optimal ensemble follows from theorem 3.

B) This directly follows from theorem 1.

C) To prove the upper bound for the \( \chi \)-capacity of the set \( \mathcal{A} = \{ \rho_n \}_{n=1}^{\infty} \) consider this set as the union of the finite set \( \mathcal{A}_1 = \{ \rho_n \}_{n=1}^{m} \) and the "tail" \( \mathcal{A}_2 = \{ \rho_n \}_{n=m+1}^{\infty} \). Proposition 7, theorem 1 and the part A of this proposition imply

\[
\bar{C}(\mathcal{A}) = \bar{C}(\mathcal{A}_1 \cup \mathcal{A}_2) \leq \max(\bar{C}(\mathcal{A}_1), \bar{C}(\mathcal{A}_2)) + \log 2
\]

\[
\leq \max (\sup_{n>m} H(\rho_n \parallel \rho_*), \log m) + \log 2.
\]

If \( \{ \pi_n \} \) is an optimal probability distribution then by proposition 8 it satisfies system (44) with \( C = \bar{C}(\{ \rho_n \}) \). Conversely, if \( (\{ \pi_n \}, C) \) is a solution of this system then by using the second part of theorem 1 it is easy to see that the ensemble \( \{ \pi_n, \rho_n \} \) is optimal for the set \( \{ \rho_n \} \) and \( C = \bar{C}(\{ \rho_n \}) \). □

Consider the case of finite set of states.

If \( N = 2 \) we have \( \Omega(\{ \rho_1, \rho_2 \}) = \pi \rho_1 + (1 - \pi) \rho_2 \), where \( \pi \) is uniquely defined by the equation

\[
H(\rho_1 \parallel \pi \rho_1 + (1 - \pi) \rho_2) = H(\rho_2 \parallel \pi \rho_1 + (1 - \pi) \rho_2)
\]
and both sides of this equality are equal to $\bar{C}(\{\rho_1, \rho_2\})$. In the case $N > 2$ the situation is more difficult in general. It may happen that there exists proper subset $\{\rho_{n_1}, \ldots, \rho_{n_N'}\}$, $N' < N$, of the set $\{\rho_1, \ldots, \rho_N\}$ such that $\bar{C}(\{\rho_{n_1}, \ldots, \rho_{n_N'}\}) = \bar{C}(\{\rho_1, \ldots, \rho_N\})$. This means that some "weights" in the above optimal probability distribution $\{\pi_n\}$ are equal to zero. Indeed, this situation takes place if we add to the set $\{\rho_1, \rho_2\}$ arbitrary state $\rho_3$ such that $H(\rho_3 || \Omega(\{\rho_1, \rho_2\})) \leq \bar{C}(\{\rho_1, \rho_2\})$. By using theorem 1 it is easy to see that $\Omega(\{\rho_1, \rho_2\}) = \Omega(\{\rho_1, \rho_2, \rho_3\})$ and $\bar{C}(\{\rho_1, \rho_2\}) = \bar{C}(\{\rho_1, \rho_2, \rho_3\})$ in this case. This provides the simplest example showing that $\mathcal{A} \subseteq \mathcal{B}$ does not imply $\bar{C}(\mathcal{A}) < \bar{C}(\mathcal{B})$ in general.

There are two cases in which the optimal average state can be easily determined as the uniform average: $\Omega(\{\rho_n\}_{n=1}^N) = N^{-1} \sum_{n=1}^N \rho_n$. The first one is the case when the states $\rho_1, \ldots, \rho_N$ form an orbit of some group of automorphisms of $\mathcal{G}(\mathcal{H})$ (see subsection 5.5). The second one is the case when the supports of the states $\rho_1, \ldots, \rho_N$ are orthogonal to each other. It is this case in which the $\chi$-capacity achieves its maximal value $\log N$ independently of types of the states $\rho_1, \ldots, \rho_N$ and of values of their entropies. Indeed, this follows from the equality

$$H(\rho_n || N^{-1} \sum_{k=1}^N \rho_k) = H(\rho_n || N^{-1} \rho_n) + 1 - N^{-1} = \log N, \quad n = 1, N,$$

obtained by using properties of relative entropy [19,23].

The case of converging sequence is illustrated by the following example, which shows in particular that system [19] defining the optimal probability distribution and the value of the $\chi$-capacity can be solved directly in some nontrivial cases.

**Example of a converging sequence of states.** Let $\{|n\rangle\}$ be an orthonormal basis in $\mathcal{H}$ and let $\{q_n\}$ be a sequence of numbers in $[0; 1]$ converging to zero. For given $\varepsilon \in [0; 1]$ consider the set $\mathcal{S}_{\{q_n\}}^\varepsilon = \{\rho_n^\pm\}$ of states

$$\rho_n^\pm = (1 - q_n)|1\rangle\langle 1| + q_n|n\rangle\langle n| \pm \eta_n(q_n, \varepsilon) \sqrt{(1 - q_n)q_n}|1\rangle\langle n| + |n\rangle\langle 1|, \quad n \geq 2,$$

where the parameter $\eta_n(q_n, \varepsilon) \in [0; 1]$ is defined by the condition

$$H(\rho_n^\pm) = (1 - \varepsilon) h_2(q_n) = -(1 - \varepsilon)((1 - q_n) \log(1 - q_n) + q_n \log q_n).$$

Thus $\varepsilon$ can be considered as a purity parameter. If $\varepsilon = 0$ then $\eta_n(q_n, \varepsilon) = 0$ and the all states $\rho_n^+ = \rho_n^-$ are diagonalizable in the basis $\{|n\rangle\}$ and have maximal entropy, if $\varepsilon = 1$ then $\eta_n(q_n, \varepsilon) = 1$ and the all states $\rho_n^\pm$ are pure.
The set $S_{\{q_n\}}$ can be considered as a sequence converging to the state $\rho_1 = |1\rangle\langle 1|$. We will establish that:

The $\chi$-capacity of the set $S_{\{q_n\}}$ is finite if and only if there exists positive $\lambda$ such that

$$\sum_n \exp \left( -\frac{\lambda}{q_n} \right) < +\infty \quad (45)$$

If condition (45) holds then the necessary and sufficient condition of existence of optimal measure = optimal ensemble $\mu_* = \{\pi^\pm_n, \rho^\pm_n\}$ for the set $S_{\{q_n\}}$ is given by the inequality

$$\sum_{n>1} q_n^{-\varepsilon} (1-q_n)^{1+(1-q_n)(1-\varepsilon)} \exp \left( -\frac{\lambda^*_{\{q_n\}}}{q_n} \right) |n\rangle\langle n| \geq 1, \quad (46)$$

where

$$\lambda^*_{\{q_n\}} = \inf \left\{ \lambda : \sum_n \exp \left( -\frac{\lambda}{q_n} \right) < +\infty \right\}.$$

If conditions (45) and (46) with given $\varepsilon$ hold for the sequence $\{q_n\}$ then

• the $\chi$-capacity of the set $S_{\{q_n\}}$ is expressed by

$$\bar{C} \left( S_{\{q_n\}} \right) = \lambda^\varepsilon_{\{q_n\}} - \log \pi^\varepsilon_{\{q_n\}},$$

• the optimal average state $\Omega(S_{\{q_n\}})$ of the set $S_{\{q_n\}}$ has the form

$$\pi^\varepsilon_{\{q_n\}} |1\rangle\langle 1| + \pi^\varepsilon_{\{q_n\}} \sum_{n>1} \left( q_n (1-q_n)^{(1-q_n)(1-\varepsilon)} \exp \left( -\frac{\lambda^\varepsilon_{\{q_n\}}}{q_n} \right) \right) |n\rangle\langle n|,$$

• the optimal probability distribution $\{\pi^\pm_n\}$ is defined as follows

$$\pi^\pm_1 = 0, \quad \pi^\pm_n = \frac{1}{2} \pi^\varepsilon_{\{q_n\}} q_n^{-\varepsilon} (1-q_n)^{(1-q_n)(1-\varepsilon)} \exp \left( -\frac{\lambda^\varepsilon_{\{q_n\}}}{q_n} \right), \quad n \geq 2,$$

where $\lambda^\varepsilon_{\{q_n\}}$ is the unique solution of the equation

$$\sum_{n>1} q_n^{-\varepsilon} (1-q_n)^{1+(1-q_n)(1-\varepsilon)} \exp \left( -\frac{\lambda}{q_n} \right) = 1,$$

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and \( \pi^\varepsilon_{\{q_n\}} = \left( \sum_{n>1} q_n^{-\varepsilon} (1 - q_n)^{(1-\varepsilon)} q_n^{-n} \exp \left( -\frac{\lambda^1_{\{q_n\}}}{q_n} \right) \right)^{-1} \in [0; 1]. \)

Condition (43) means boundedness of the entropy on the set \( \overline{\sigma}(S^\varepsilon_{\{q_n\}}) \) for arbitrary \( \varepsilon. \)

Existence of the Gibbs state \( \Gamma(\overline{\sigma}(S^\varepsilon_{\{q_n\}})) \) of the set \( \overline{\sigma}(S^\varepsilon_{\{q_n\}}) \) for some and hence for arbitrary \( \varepsilon \) is equivalent to validity of conditions (43) and (44) with \( \varepsilon = 1 \) for the sequence \( \{q_n\} \). If these conditions hold then

\[
\Gamma(\overline{\sigma}(S^\varepsilon_{\{q_n\}})) = \pi^1_{\{q_n\}} |1\rangle\langle 1| + \pi^1_{\{q_n\}} \sum_{n>1} \exp \left( -\frac{\lambda^1_{\{q_n\}}}{q_n} \right) |n\rangle\langle n|,
\]

for arbitrary \( \varepsilon \), where \( \pi^1_{\{q_n\}} \) and \( \lambda^1_{\{q_n\}} \) are the above defined parameters.\(^{25}\)

If condition (43) holds for arbitrary \( \lambda > 0 \) then the entropy is continuous on the set \( \overline{\sigma}(S^\varepsilon_{\{q_n\}}) \) for arbitrary \( \varepsilon. \)

In fig.2 the results of numerical calculation of the \( \chi \)-capacity of the set \( S^\varepsilon_{\{q_n\}} \) as a function of \( \varepsilon \) for different sequences \( \{q_n\} \) are presented.

By theorem 1 finiteness of the \( \chi \)-capacity of the set \( S^\varepsilon_{\{q_n\}} \) means existence of the optimal average state \( \Omega(S^\varepsilon_{\{q_n\}}) \) in \( \overline{\sigma}(S^\varepsilon_{\{q_n\}}) \) such that

\[
\sup_{n \geq 1} H(\rho^\pm_n || \Omega(S^\varepsilon_{\{q_n\}})) < +\infty. \tag{47}
\]

By lemma 1 in [8] the optimal average state can be represented as follows

\[
\Omega(S^\varepsilon_{\{q_n\}}) = \pi_1 \rho_1 \pm \sum_{n>1, \pm} \pi_n^{\pm} \rho_n^{\pm}. \tag{48}
\]

Since the set \( S^\varepsilon_{\{q_n\}} \) is invariant under action of the automorphism \( U(\cdot) U^* \), where \( U \) is a unitary operator diagonalizable in the basis \( \{\langle n|\} \) and having eigen values \( \pm 1 \), corollary 8 implies that the state \( \Omega(S^\varepsilon_{\{q_n\}}) \) is invariant under the action of the above automorphism and hence it is diagonalizable in the basis \( \{\langle n|\} \). This means that \( \pi_n^+ = \pi_n^- = \frac{1}{2} \pi_n \) for all \( n > 1 \) in (48), where \( \{\pi_n\}_{n=1}^{\infty} \) is a probability distribution. So we have

\[
\Omega(S^\varepsilon_{\{q_n\}}) = \pi |1\rangle\langle 1| \pm \sum_{n>1} \pi_n q_n |n\rangle\langle n|, \tag{49}
\]

\(^{25}\)It is interesting to compare this observation with the results of proposition 1a with the \( H \)-operator \( H = \sum_{n=2}^{+\infty} q_n^{-1} |n\rangle\langle n|. \)
where \( \pi = \pi_1 + \sum_{n>1} (1 - q_n) \pi_n \). Thus

\[
H(\rho_1 \| \Omega(S_{(q_n)}^1)) = -\log \pi
\]

and

\[
H(\rho_n^r \| \Omega(S_{(q_n)}^r)) = -(1 - q_n) \log \pi - q_n \log(\pi_n q_n)
\]

\[
+ (1 - \varepsilon)((1 - q_n) \log(1 - q_n) + q_n \log q_n) = -(1 - q_n) \log \pi
\]

\[
- q_n \log \pi_n - \varepsilon q_n \log q_n + (1 - \varepsilon)(1 - q_n) \log(1 - q_n), \quad n > 1.
\]

Since \( q_n \to 0 \) as \( n \to +\infty \) condition (47) means that \( \sup_{n>1} q_n (- \log \pi_n) \) is finite. It is easy to see that existence of a probability distribution \( \{\pi_n\} \) satisfying this condition is equivalent to existence of positive \( \lambda \) such that the series \( \sum_n \exp \left(-\frac{1}{q_n}\right) \) is finite.

Note that (47) and (51) imply \( \pi_n > 0 \) for all \( n > 1 \). By using this (50) and (51) system (44) can be rewritten in the form

\[
\begin{cases}
- \log \pi \leq C, & \pi_1 (C + \log \pi) = 0 \\
(1 - q_n)((1 - \varepsilon) \log(1 - q_n) - \log \pi) - q_n \log \pi_n - \varepsilon q_n \log q_n = C.
\end{cases}
\]

The second part of this system implies

\[
\pi_n = \pi q_n^{-\varepsilon}(1 - q_n) \frac{(1 - q_n)(1 - \varepsilon)}{q_n} \exp \left(-\frac{C + \log \pi}{q_n}\right), \quad n \geq 2.
\]

Since \( \pi_n \) must be arbitrary small for large \( n \) we conclude that \( - \log \pi < C \) and the first part of the above system implies \( \pi_1 = 0 \).

It is easy to see that if there exists a probability distribution \( \{\pi_n\} \) satisfying system (52) then \( \pi = \sum_{n>1} (1 - q_n) \pi_n \) and \( C \) forms a solution of the system

\[
\begin{cases}
\sum_{n>1} q_n^{-\varepsilon}(1 - q_n) \frac{1 + (1 - q_n)(1 - \varepsilon)}{q_n} \exp \left(-\frac{C + \log \pi}{q_n}\right) = 1 \\
\sum_{n>1} q_n^{-\varepsilon}(1 - q_n) \frac{(1 - q_n)(1 - \varepsilon)}{q_n} \exp \left(-\frac{C + \log \pi}{q_n}\right) = \pi^{-1}.
\end{cases}
\]

and vise versa by means of (53) any solution \((\pi, C)\) of system (54) provides a probability distribution \( \{\pi_n\} \) satisfying system (52).
Now will show that system (54) has a solution \((\pi, C)\) if and only if inequality (46) holds. Consider the functions
\[
F(x) = \sum_{n>1} q_n^{-\varepsilon} (1 - q_n)^{(1 - \varepsilon)} \frac{1 + (1 - q_n)(1 - \varepsilon)}{q_n} \exp \left( -\frac{x}{q_n} \right)
\]
and
\[
G(x) = \sum_{n>1} q_n^{-\varepsilon} (1 - q_n)^{(1 - \varepsilon)} \frac{1 + (1 - q_n)(1 - \varepsilon)}{q_n} \exp \left( -\frac{x}{q_n} \right).
\]
It is easy to see that these functions are continuous and strictly decreasing on \((\lambda^*_\{q_n\}; +\infty)\) such that \(F(x) \leq G(x)\). Hence there exist the converse functions \(F^{-1}(y)\) and \(G^{-1}(y)\), which are continuous and strictly decreasing on \(F((\lambda^*_\{q_n\}; +\infty))\) and on \(G((\lambda^*_\{q_n\}; +\infty))\) correspondingly. By means of these functions system (54) can be rewritten in the form
\[
\begin{align*}
F(C + \log \pi) &= 1 \\
G(C + \log \pi) &= \pi^{-1}.
\end{align*}
\]
It is easy to see that inequality (46) is equivalent to the following one
\[
\lim_{x \to \lambda^*_\{q_n\} + 0} F(x) \geq 1
\]
and
\[
\lim_{x \to \lambda^*_\{q_n\} + 0} G(x) \geq 1
\]
lim_{x \to \lambda^*_\{q_n\} + 0} F(x) = 1
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\lim_{x \to \lambda^*_\{q_n\} + 0} F(x) \geq 1
\]
and
\[
\lim_{x \to \lambda^*_\{q_n\} + 0} G(x) \geq 1
\]
state $\Omega(S^1_{\{q_n\}})$ coincides with the Gibbs state $\Gamma(\mathcal{C}(S^1_{\{q_n\}}))$. By noting that $\Omega(S^1_{\{q_n\}})$ lies in $\mathcal{C}(S^0_{\{q_n\}})$ and that $\mathcal{C}(S^0_{\{q_n\}}) \subseteq \mathcal{C}(S^1_{\{q_n\}})$ for arbitrary $\varepsilon$ we conclude that $\Gamma(\mathcal{C}(S^1_{\{q_n\}})) = \Omega(S^1_{\{q_n\}})$ for arbitrary $\varepsilon$ in this case.

Suppose there exists the Gibbs state $\Gamma(\mathcal{C}(S^1_{\{q_n\}}))$ for some $\varepsilon$. By using the observations in the end of section 3 it is easy to see that this implies existence the Gibbs state $\Gamma(\mathcal{C}(S^1_{\{q_n\}}))$ for arbitrary $\varepsilon$, in particular, for $\varepsilon = 1$. Since the closed set $S^1_{\{q_n\}}$ consists of pure states the Gibbs state $\Gamma(\mathcal{C}(S^1_{\{q_n\}}))$ coincides with the optimal average state $\Omega(S^1_{\{q_n\}})$. By lemma 2 the restriction of the entropy to the set $\mathcal{C}(S^1_{\{q_n\}})$ is continuous at the state $\Omega(S^1_{\{q_n\}}) = \Gamma(\mathcal{C}(S^1_{\{q_n\}}))$, which implies regularity of the set $S^1_{\{q_n\}}$. By theorem 3 there exists an optimal measure for the set $S^1_{\{q_n\}}$ and hence, by the above observation, condition (46) with $\varepsilon = 1$ holds for the sequence $\{q_n\}$.

By the second continuity condition in the part B of proposition 9a with the $\mathcal{H}$-operator $\sum_{n=2}^{+\infty}q_n^{-1}|n\rangle\langle n|$ finiteness of the series in (44) for arbitrary $\lambda$ implies continuity of the entropy on the set $\mathcal{C}(S^1_{\{q_n\}})$ for arbitrary $\varepsilon$.

We complete this subsection with the example of the sequence $\{q_n\}$ for which condition (45) holds while condition (46) with arbitrary $\varepsilon$ does not hold. Let $q_n = 1/\log(n \log^3(2n + 1))$ for $n \geq 2$. Then $\lambda^*_{\{q_n\}} = 1$ and the left side of (46) with $\varepsilon = 1$ is approximately equal to 0.89. It follows that condition (46) does not hold with arbitrary $\varepsilon$. By the above observation for arbitrary $\varepsilon$ the entropy is bounded on the set $\mathcal{C}(S^1_{\{q_n\}})$ and the $\chi$-capacity of the set $S^1_{\{q_n\}}$ is finite but the Gibbs state $\Gamma(\mathcal{C}(S^1_{\{q_n\}}))$ of the set $\mathcal{C}(S^1_{\{q_n\}})$ and the optimal measure $\mu^* = \{\pi^\pm_n, \rho^\pm_n\}$ for the set $S^1_{\{q_n\}}$ do not exist.

### 5.2 The sets $\mathcal{L}(\sigma)$ and $\mathcal{K}_{H,h}$

Let $\sigma = \sum_k \lambda_k |k\rangle\langle k|$ be an arbitrary state. The layer $\mathcal{L}(\sigma)$ is defined in section 3 as the set consisting of all states, having the same diagonal values as the state $\sigma$ in the basis $\{|k\rangle\}$. By proposition 6a the entropy is continuous on the set $\mathcal{L}(\sigma)$ if and only if $H(\sigma) < +\infty$ and $\sup_{\rho \in \mathcal{L}(\sigma)} H(\rho) = H(\sigma)$. The questions concerning the $\chi$-capacity of the set $\mathcal{L}(\sigma)$ are considered in the following proposition.

**Proposition 6b.** Let $\sigma$ be an arbitrary state.

The $\chi$-capacity of the set $\mathcal{L}(\sigma)$ is equal to $H(\sigma)$. 51
The set $\mathcal{L}(\sigma)$ is regular if and only if $H(\sigma) < +\infty$. If this condition holds then there exists an optimal measure for the set $\mathcal{L}(\sigma)$ with the barycenter $\Omega(\mathcal{L}(\sigma)) = \sigma$ supported by pure states in $\mathcal{L}(\sigma)$.

**Proof.** Suppose $\bar{C}(\mathcal{L}(\sigma))$ is finite. Let $G$ be the group of all unitaries in $\mathfrak{B}(\mathcal{H})$ diagonizable in the basis $\{|k\rangle\}$. Since the set $\mathcal{L}(\sigma)$ is invariant under the action of the automorphism $U(\cdot)U^*$ for each $U \in G$ corollary 9 implies $\Omega(\mathcal{L}(\sigma)) = \sigma$. Let $\rho$ be an arbitrary pure state in $\mathcal{L}(\sigma)$, for example, the state, corresponding to the vector $\sum k \sqrt{\lambda_k} |k\rangle$. By theorem 1 and proposition 6a we have

$$\bar{C}(\mathcal{L}(\sigma)) \geq H(\rho, \sigma) = H(\sigma).$$

Since obviously $\bar{C}(\mathcal{L}(\sigma)) \leq \sup_{\rho \in \mathcal{L}(\sigma)} H(\rho) = H(\sigma)$ there is equality here. To complete the proof of the first assertion of the proposition note that the last inequality $\bar{C}(\mathcal{L}(\sigma)) = +\infty$ implies $H(\sigma) = +\infty$.

The regularity assertion follows from proposition 6a.

Since $\bar{C}(\mathcal{L}(\sigma)) = H(\Omega(\mathcal{L}(\sigma)))$ the assertion concerning existence of optimal measure follows from theorem 3, propositions 6a and 8. □

The set $\mathcal{K}_{H,h}$ is introduced in section 3 as the set defined by the inequality $\text{Tr} \rho H \leq h$, where $H$ is a $\mathfrak{H}$-operator and $h$ is a positive number. Proposition 1a gives necessary and sufficient conditions of boundedness and of continuity of the entropy on the set $\mathcal{K}_{H,h}$ in terms of the increase coefficient $\text{ic}(H)$ of the $\mathfrak{H}$-operator $H$. This proposition also shows that existence of the Gibbs state of the set $\mathcal{K}_{H,h}$ is equivalent to the inequality $h \leq h_*(H)$. The questions concerning the $\chi$-capacity of the set $\mathcal{K}_{H,h}$ are considered in the following proposition.

**Proposition 1b.** Let $H$ be a $\mathfrak{H}$-operator on the Hilbert space $\mathcal{H}$ and $h$ be a positive number such that $h \geq h_m(H)$.

The $\chi$-capacity of the set $\mathcal{K}_{H,h}$ coincides with $\sup_{\rho \in \mathcal{K}_{H,h}} H(\rho)$ and hence it is finite if and only if $\text{ic}(H) < +\infty$. If this condition holds then

$$\Omega(\mathcal{K}_{H,h}) = \begin{cases} \Gamma(\mathcal{K}_{H,h}) = (\text{Tr} \exp(-\lambda^* H))^{-1} \exp(-\lambda^* H), & h \leq h_*(H) \\ (\text{Tr} \exp(-\text{ic}(H) H))^{-1} \exp(-\text{ic}(H) H), & h > h_*(H), \end{cases}$$

where $\lambda^*$ is uniquely defined by equation [7].

The following statements are equivalent

i) the inequality $h \leq h_*(H)$ holds;

---

27 The parameters $\text{ic}(H)$ and $h_*(H)$ are defined before proposition 1a.
ii) the set $\mathcal{K}_{H,h}$ is regular;

iii) $\overline{C}(\mathcal{K}_{H,h}) \leq H(\Omega(\mathcal{K}_{H,h}))$;\footnote{This inequality implies equality.}

iv) $\overline{C}(\mathcal{K}_{H,h}) = \overline{C}(\mathcal{K}_{H,h} \cap \mathcal{L}(\Omega(\mathcal{K}_{H,h})))$;

v) there exists an optimal measure for the set $\mathcal{K}_{H,h}$.

**Proof.** Let $H = \sum_k h_k |k\rangle \langle k|$ and $\mathcal{K}_{H,h}^c$ be the subset of $\mathcal{K}_{H,h}$ consisting of states diagonalizable in the basis $\{|k\rangle\}$. Then $\mathcal{K}_{H,h} = \bigcup_{\rho \in \mathcal{K}_{H,h}^c} \mathcal{L}(\rho)$ and hence

$$\overline{C}(\mathcal{K}_{H,h}) \geq \sup_{\rho \in \mathcal{K}_{H,h}^c} \overline{C}(\mathcal{L}(\rho)) = \sup_{\rho \in \mathcal{K}_{H,h}^c} H(\rho) = \sup_{\rho \in \mathcal{K}_{H,h}} H(\rho),$$

where the last equality follows from inequality (23). Since the converse inequality is obvious the first statement of the proposition is proved.

In the proof of proposition 1a the sequence $\{\rho_n\}$ of states in $\mathcal{K}_{H,h}^c$ such that $\lim_{n \to \infty} H(\rho_n) = \sup_{\rho \in \mathcal{K}_{H,h}} H(\rho)$ and $\lim_{n \to \infty} \rho_n = \rho^*(\mathcal{K}_{H,h})$ was constructed. By proposition 6b for each $n$ there exists optimal measure $\mu_n$ for the set $\mathcal{L}(\rho_n)$ such that $\bar{\rho}(\mu_n) = \rho_n$ and $\chi(\mu_n) = H(\rho_n)$. By the first part of the proposition the sequence of measures $\{\mu_n\}$ is an approximating sequence for the set $\mathcal{K}_{H,h}$. By theorem 1 the limit $\rho^*(\mathcal{K}_{H,h})$ of the corresponding sequence of barycenters $\{\rho_n\}$ is the optimal average state of the set $\mathcal{K}_{H,h}$.

The asserted equivalence of statements (i) – (iv) will be proved in the following order (i) $\Rightarrow$ (ii) $\Rightarrow$ (v) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). By proposition 1a (i) means that $\Omega(\mathcal{K}_{H,h})$ is the Gibbs state $\Gamma(\mathcal{K}_{H,h})$ of the set $\mathcal{K}_{H,h}$. By lemma 2 the restriction of the entropy to the set $\mathcal{K}_{H,h}$ is continuous at the state $\Omega(\mathcal{K}_{H,h})$, which implies regularity of the set $\mathcal{K}_{H,h}$.

(ii) $\Rightarrow$ (v). This directly follows from theorem 3.

(v) $\Rightarrow$ (iii). This directly follows from corollary 11.

(iii) $\Rightarrow$ (iv). This follows from proposition 6b and the first part of this proposition.

(iv) $\Rightarrow$ (i). If $h > h_*(H)$ then by propositions 1a and 6b

$$\overline{C}(\mathcal{L}(\Omega(\mathcal{K}_{H,h}))) = H(\Omega(\mathcal{K}_{H,h})) < \sup_{\rho \in \mathcal{K}_{H,h}} H(\rho) = \overline{C}(\mathcal{K}_{H,h}). \square$$

The observations in the proof of propositions 1a and 1b provide the following example, showing that the regularity condition in theorem 2I is essential.
Example of a closed set with finite $\chi$-capacity having no minimal closed subset with the same $\chi$-capacity.

Let $H$ be a $\mathcal{H}$-operator such that $h_*(H) = \frac{\text{Tr} H \exp(-i\text{c}(H)H)}{\text{Tr} \exp(-i\text{c}(H)H)} < +\infty$. For example, $H = \sum_{k=1}^{+\infty} \log((k + 1) \log^2(k + 1))|k\rangle\langle k|$. By the observation in the proof of proposition 1a for given $h > h_*(H)$ there exists natural $n_0$ such that the state $\rho_n$ is well defined by $\mathcal{H}$ for all $n \geq n_0$ and the sequence $\{\rho_n\}_{n\geq n_0}$ converges to the state $\rho_*(\mathcal{K}_{H,h})$ defined by $\mathcal{H}$. Let $\mathcal{A}_0 = \bigcup_{n\geq n_0} \mathcal{L}(\rho_n)$ and $\mathcal{A} = \bigcup_{n\geq n_0} \mathcal{L}(\rho_*(\mathcal{K}_{H,h}))$. By the observation in the proof of proposition 1a and proposition 6a

$$\bar{C}(\mathcal{A}) = \lim_{n \to +\infty} H(\rho_n) > H(\rho_*(\mathcal{K}_{H,h})) = \sup_{\rho \in \mathcal{L}(\rho_*(\mathcal{K}_{H,h}))} H(\rho).$$

We assert that the closed set $\mathcal{A}$ has no minimal closed subsets with the same $\chi$-capacity. Suppose, $\mathcal{B}$ is the minimal subset of $\mathcal{A}$. Since $\bar{C}(\mathcal{L}(\rho_*(\mathcal{K}_{H,h})))$ is less than $\bar{C}(\mathcal{A}) = \bar{C}(\mathcal{B})$ the set $\mathcal{B}$ has nonempty intersection with the set $\mathcal{L}(\rho_{n_*})$ for some $n_* \geq n_0$. We will show that the closed set $\mathcal{B}\setminus \mathcal{L}(\rho_{n_*}) \subset \mathcal{B}$ has the same $\chi$-capacity as the set $\mathcal{B}$ contradicting to the assumed minimality of this set.

Since $\sup_{\rho \in \mathcal{L}(\rho_*(\mathcal{K}_{H,h}))} H(\rho) < \bar{C}(\mathcal{B})$ there exists approximating sequence $\{\{\pi_i^k, \rho_i^k\}\}_k$ for the set $\mathcal{B}$ such that the corresponding sequence of the average states $\{\bar{\rho}_k\}_k$ has no intersection with $\mathcal{L}(\rho_*(\mathcal{K}_{H,h}))$ and hence $\Pi_{\{\pi_i^k\}}(\bar{\rho}_k) = \rho_{n(k)}$ for some sequence of natural numbers $\{n(k)\}_k$. Since $\sup_{\rho \in \mathcal{L}(\rho_{n_k})} H(\rho) < \bar{C}(\mathcal{B})$ for each $n \geq n_0$ the sequence $\{n(k)\}_k$ tends to $+\infty$. Since for arbitrary state $\rho$ in $\mathcal{B}$ the state $\Pi_{\{\pi_i^k\}}(\rho)$ is either $\rho_*(\mathcal{K}_{H,h})$ or $\rho_n$ for a particular $n$ and since $\bar{\rho}_k = \sum_i \pi_i^k \rho_i^k$ implies $\Pi_{\{\pi_i^k\}}(\bar{\rho}_k) = \sum_i \pi_i^k \Pi_{\{\pi_i^k\}}(\rho_i^k)$ for each $k$ by using $\mathcal{H}$ and $\mathcal{L}$ we conclude that $\Pi_{\{\pi_i^k\}}(\rho_i^k) = \rho_{n(k)}$ for all $i$ and $k$. Thus the states $\{\rho_i^k\}$ are not contained in $\mathcal{L}(\rho_{n_k})$ for all sufficiently large $k$ and hence the "tail" of the sequence $\{\{\pi_i^k, \rho_i^k\}\}_k$ is an approximating sequence for the set $\mathcal{B}\setminus \mathcal{L}(\rho_{n_*})$. This implies $\bar{C}(\mathcal{B}) = \bar{C}(\mathcal{B}\setminus \mathcal{L}(\rho_{n_*}))$.

### 5.3 The set $\mathcal{V}_{\sigma,c}$

The set $\mathcal{V}_{\sigma,c}$ is introduced in section 3 as the set defined by the inequality $H(\rho||\sigma) \leq c$, where $\sigma$ is a state and $c$ is a nonnegative number $c$. If $\sigma$ is a state with infinite dimensional support then the family of nonempty sets $\{\mathcal{K}_{\sigma,c}\}_{c \in \mathbb{R}_+}$ is strictly increasing and $\mathcal{K}_{\sigma,0} = \{\sigma\}$.
By theorem 1 every set $\mathcal{A}$ with finite $\chi$-capacity is contained in the compact convex set $\mathcal{V}_{\Omega(\mathcal{A}),\bar{C}(\mathcal{A})}$ such that

$$\Omega(\mathcal{V}_{\Omega(\mathcal{A}),\bar{C}(\mathcal{A})}) = \Omega(\mathcal{A}) \quad \text{and} \quad \bar{C}(\mathcal{V}_{\Omega(\mathcal{A}),\bar{C}(\mathcal{A})}) = \bar{C}(\mathcal{A}).$$

Below we consider the $\chi$-capacity of the set $\mathcal{V}_{\sigma,c}$ with arbitrary $\sigma$ and $c$.

Proposition 3a gives necessary and sufficient conditions of boundedness and of continuity of the entropy on the set $\mathcal{V}_{\sigma,c}$ in terms of the decrease coefficient $dc(\sigma)$ of the state $\sigma$. This proposition also shows that existence of the Gibbs state of the set $\mathcal{V}_{\sigma,c}$ is equivalent to the inequality $c \leq c_*(\sigma)$.\(^{29}\)

The questions concerning the $\chi$-capacity of the set $\mathcal{V}_{\sigma,c}$ are considered in the following proposition. Let $c^*(\sigma) = \frac{\text{Tr}_\sigma dc(\sigma)(-\log \sigma)}{\text{Tr}_\sigma dc(\sigma)}$ if $\text{Tr}_\sigma dc(\sigma) < +\infty$ and $c^*(\sigma) = +\infty$ otherwise. Note that $c^*(\sigma) = \frac{c_*(\sigma) + \log \text{Tr}_\sigma dc(\sigma)}{1 - dc(\sigma)} \geq c_*(\sigma)$ if $dc(\sigma) < 1$ and $c^*(\sigma) = c_*(\sigma) = H(\sigma)$ if $dc(\sigma) = 1$.

**Proposition 3b.** Let $\sigma$ be an arbitrary infinite dimensional state.

If $c \leq H(\sigma) \leq +\infty$ then

$$\bar{C}(\mathcal{V}_{\sigma,c}) = c \quad \text{and} \quad \Omega(\mathcal{V}_{\sigma,c}) = \sigma.$$ 

If $H(\sigma) < c \leq c^*(\sigma)$ then

$$\bar{C}(\mathcal{V}_{\sigma,c}) = \lambda^c + \log \text{Tr}_\lambda^\sigma \quad \text{and} \quad \Omega(\mathcal{V}_{\sigma,c}) = (\text{Tr}_\lambda^\sigma)^{-1} \sigma^\lambda,$$

where $\lambda^*$ is uniquely defined by the equation

$$\text{Tr}_\lambda(-\log \sigma) = c \text{Tr}_\lambda^\sigma.$$

If $c^*(\sigma) < +\infty$ and $c \geq c^*(\sigma)$ then

$$\bar{C}(\mathcal{V}_{\sigma,c}) = dc(\sigma)c + \log \text{Tr}_\sigma dc(\sigma) \quad \text{and} \quad \Omega(\mathcal{V}_{\sigma,c}) = (\text{Tr}_\sigma dc(\sigma))^{-1} \sigma^{dc(\sigma)}.$$

In the all cases $C(\mathcal{V}_{\sigma,c}) = \inf_{\lambda \in (dc(\sigma); 1]} (\lambda c + \log \text{Tr}_\lambda^\sigma)$.

The following statements are equivalent:

i) the inequality $c \leq c^*(\sigma)$ holds;

---

\(^{29}\)The parameters $dc(\sigma)$ and $c_*(\sigma)$ are defined before proposition 3a.
ii) $\bar{C}(\mathcal{V}_{\sigma,c}) \leq H(\Omega(\mathcal{V}_{\sigma,c}))$;

iii) $\bar{C}(\mathcal{V}_{\sigma,c}) = \bar{C}(\mathcal{V}_{\sigma,c} \cap \mathcal{L}(\Omega(\mathcal{V}_{\sigma,c})))$;

iv) there exists optimal measure for the set $\mathcal{V}_{\sigma,c}$.

The set $\mathcal{V}_{\sigma,c}$ is regular if and only if $dc(\sigma) < 1$ and $c < c^*(\sigma)$.

In fig.1 the result of numerical calculations of the $\chi$-capacity of the set $\mathcal{V}_{\sigma,c}$ as a function of $c$ for the state $\sigma$ with finite $c^*(\sigma)$ is shown.

**Proof.** Let $\sigma = \sum_k \lambda_k |k\rangle\langle k|$ be a full rank state so that $-\log \sigma$ is a $\mathcal{H}$-operator. The inequality

\[ \bar{C}(\mathcal{V}_{\sigma,c}) \leq c \tag{55} \]

follows from expression (25) in theorem 1.

Let $c \leq H(\sigma) \leq +\infty$. Consider the subset $\mathcal{T} = \mathcal{V}_{\sigma,c} \cap \mathcal{L}(\sigma)$ of $\mathcal{V}_{\sigma,c}$. By monotonicity of the $\chi$-capacity and (55) we have $\bar{C}(\mathcal{T}) \leq \bar{C}(\mathcal{V}_{\sigma,c}) \leq c < +\infty$. So, to prove that $\bar{C}(\mathcal{V}_{\sigma,c}) = c$ it is sufficient to show that $\bar{C}(\mathcal{T}) \geq c$.

Let $G$ be the group of all unitaries in $\mathfrak{B}(\mathcal{H})$ diagonizable in the basis $\{|k\rangle\}$. Since the set $\mathcal{T}$ is invariant under the action of the automorphism $U(\cdot)U^*$ for each $U \in G$ corollary 9 implies $\Omega(\mathcal{T}) = \sigma$. By expression (25) in theorem 1 to show that $\bar{C}(\mathcal{T}) \geq c$ it is sufficient to find a state $\sigma_c$ in the set $\mathcal{T}$ such that $H(\sigma_c\|\Omega(\mathcal{T})) = H(\sigma_c\|\sigma) = c$.

By proposition 6a in the case $H(\sigma) < +\infty$ the relative entropy $H(\rho\|\sigma)$ is a continuous function on $\mathcal{L}(\sigma)$ with the range $[0; H(\sigma)]$. This implies existence of the state $\sigma_c$ with the desired properties.

In the case $H(\sigma) = +\infty$ existence of the state $\sigma_c$ follows from lemma 6 below (with $n = 1$).

Thus $\bar{C}(\mathcal{V}_{\sigma,c}) = \bar{C}(\mathcal{T}) = c$ and theorem 2C implies $\Omega(\mathcal{V}_{\sigma,c}) = \Omega(\mathcal{T}) = \sigma$.

Let $c > H(\sigma)$. Since $\mathcal{K}_{-\log \sigma,c} \subset \mathcal{V}_{\sigma,c}$ monotonicity of the $\chi$-capacity implies

\[ \bar{C}(\mathcal{K}_{-\log \sigma,c}) \leq \bar{C}(\mathcal{V}_{\sigma,c}) \tag{56} \]

Note that $c^*(\sigma) = h_*(\log \sigma)$. By proposition 1b to prove the all assertions concerning the cases $H(\sigma) < c \leq c^*(\sigma)$ and $c \geq c^*(\sigma)$ it is sufficient to show that

\[ \bar{C}(\mathcal{V}_{\sigma,c}) = \bar{C}(\mathcal{K}_{-\log \sigma,c}) \tag{57} \]

since this equality and theorem 2C imply $\Omega(\mathcal{V}_{\sigma,c}) = \Omega(\mathcal{K}_{-\log \sigma,c})$. 56
Suppose \( dc(\sigma) = ic(− \log \sigma) = 1 \). Then \( c^*(\sigma) = h_*(− \log \sigma) = H(\sigma) \). By proposition 1b \( \bar{C}(\mathcal{K}_{−\log \sigma,c}) = c \) for all \( c \geq H(\sigma) \). Thus inequalities (55) and (56) imply equality (57).

Suppose \( dc(\sigma) = ic(− \log \sigma) < 1 \). Then lemma 3 implies

\[
H(\rho\|(\text{Tr} \sigma^\lambda)^{-1} \sigma^\lambda) \leq \lambda H(\rho\|\sigma) + \log \text{Tr} \sigma^\lambda \leq \lambda c + \log \text{Tr} \sigma^\lambda
\]

for all \( \rho \) in \( \mathcal{V}_{\sigma,c} \) and for all \( \lambda \in (dc(\sigma); 1] \). By the second part of theorem 1 we have

\[
\bar{C}(\mathcal{V}_{\sigma,c}) \leq \inf_{\lambda \in (dc(\sigma); 1]} \sup_{\rho \in \mathcal{V}_{\sigma,c}} H(\rho\|(\text{Tr} \sigma^\lambda)^{-1} \sigma^\lambda) \leq \inf_{\lambda \in (dc(\sigma); 1]} \left( \lambda c + \log \text{Tr} \sigma^\lambda \right).
\]

By proposition 1b \( \bar{C}(\mathcal{K}_{−\log \sigma,c}) = \inf_{\lambda \in (dc(\sigma); +\infty)} \left( \lambda c + \log \text{Tr} \sigma^\lambda \right) \) and it is easy to see that the condition \( c > H(\sigma) \) implies that the last infimum is achieved at some \( \lambda^* \leq 1 \). Thus this infimum coincides with the previous one and hence (57) holds in this case.

Equivalence of statements \((i) \Rightarrow (iv) \) will be shown by proving the following implications \((i) \Rightarrow (iv) \Rightarrow (ii) \Rightarrow (i) \) and \((i) \Rightarrow (iii) \Rightarrow (i) \).

\((i) \Rightarrow (iv) \). In the case \( H(\sigma) < +\infty \) existence of an optimal measure for the set \( \mathcal{V}_{\sigma,c} \) under the condition \( c \leq c^*(\sigma) \) is proved by considering the following subcases separately

- \( c \leq H(\sigma) \);
- \( dc(\sigma) < 1 \) and \( H(\sigma) < c \leq c^*(\sigma) \).

If \( c \leq H(\sigma) \) then by the above observation \( \bar{C}(\mathcal{V}_{\sigma,c}) = \bar{C}(\mathcal{T}) \), where \( \mathcal{T} = \mathcal{V}_{\sigma,c} \cap \mathcal{L}(\sigma) \). By proposition 6a the entropy is continuous on the set \( \mathcal{T} \), which implies its regularity. It follows from this and theorem 3 that there exists an optimal measure for the set \( \mathcal{T} \). Since \( \bar{C}(\mathcal{V}_{\sigma,c}) = \bar{C}(\mathcal{T}) \) and \( \mathcal{T} \subset \mathcal{V}_{\sigma,c} \) this measure is an optimal measure for the set \( \mathcal{V}_{\sigma,c} \).

If \( dc(\sigma) < 1 \) and \( H(\sigma) < c \leq c^*(\sigma) \) then by the above observation \( \bar{C}(\mathcal{V}_{\sigma,c}) = \bar{C}(\mathcal{K}_{−\log \sigma,c}) \) and \( c^*(\sigma) = h_*(− \log \sigma) \). By proposition 1b there exists an optimal measure for the set \( \mathcal{K}_{−\log \sigma,c} \). Since \( \bar{C}(\mathcal{V}_{\sigma,c}) = \bar{C}(\mathcal{K}_{−\log \sigma,c}) \) and \( \mathcal{K}_{−\log \sigma,c} \subset \mathcal{V}_{\sigma,c} \) this measure is an optimal measure for the set \( \mathcal{V}_{\sigma,c} \).

In the case \( H(\sigma) = +\infty \), in which \( c^*(\sigma) = +\infty \), existence of an optimal measure is verified by the following direct construction.

For given \( c \) let \( m \) and \( \rho_{c,1,m} \) be a natural number and a state provided by lemma 6. Let \( P_m = \sum_{k=1}^{m} |k\rangle\langle k| \) and \( G_m \) be the compact group of all unitaries.
in $\mathfrak{B}(P_m(\mathcal{H}))$ diagonalizable in the basis $\{|k\rangle\}_{k=1}^m$ in $P_m(\mathcal{H})$. For arbitrary $U$ in $G_m$ denote by $\hat{U}$ the unitary operator $U \oplus I_{H \oplus P_m(\mathcal{H})}$ in $\mathfrak{B}(\mathcal{H})$. By using the construction of the state $\rho_{c,1,m}$ it is easy to see that
\[
\int_{G_m} \hat{U} \rho_{c,1,m} \hat{U}^* \mu_H(dU) = \sigma,
\]
where $\mu_H$ is the Haar measure on $G_m$. Since
\[
H(\hat{U} \rho_{c,1,m} \hat{U}^* \| \sigma) = H(\rho_{c,1,m} \| \hat{U}^* \sigma \hat{U}) = H(\rho_{c,1,m} \| \sigma) = c
\]
the image of the measure $\mu_H$ under the mapping $U \mapsto \hat{U} \rho_{c,1,m} \hat{U}^*$ is an optimal measure for the set $\mathcal{V}_{\sigma,c}$, which is supported by the set $\mathcal{L}(\sigma)$ by the construction.

$(iv) \Rightarrow (ii)$. This directly follows from corollary 11.

$(ii) \Rightarrow (i)$. If $c^*(\sigma)$ is finite and $c > c^*(\sigma)$ then the proof of the previous part of the proposition, propositions 1a and 1b imply
\[
\bar{C}(\mathcal{V}_{\sigma,c}) = \bar{C}(\mathcal{K}_{-\log \sigma,c}) > H(\Omega(\mathcal{V}_{\sigma,c}) = \Omega(\mathcal{K}_{-\log \sigma,c})).
\]  

(58)  

$(i) \Rightarrow (iii)$. If $c \leq H(\sigma)$ then by the proof of the previous part of the proposition $\bar{C}(\mathcal{V}_{\sigma,c}) = \bar{C}(\mathcal{T})$ and $\Omega(\mathcal{V}_{\sigma,c}) = \sigma$, where $\mathcal{T} = \mathcal{V}_{\sigma,c} \cap \mathcal{L}(\sigma)$. If $H(\sigma) < c \leq c^*(\sigma)$ then by the proof of the previous part of the proposition, propositions 1b and 6b we have
\[
\bar{C}(\mathcal{V}_{\sigma,c}) = \bar{C}(\mathcal{K}_{-\log \sigma,c}) = H(\Omega(\mathcal{K}_{-\log \sigma,c})) = \bar{C}(\mathcal{L}(\Omega(\mathcal{K}_{-\log \sigma,c}))).
\]  

(59)  

Since $\mathcal{L}(\Omega(\mathcal{K}_{-\log \sigma,c})) \subset \mathcal{K}_{-\log \sigma,c} \subset \mathcal{V}_{\sigma,c}$ and $\Omega(\mathcal{V}_{\sigma,c}) = \Omega(\mathcal{K}_{-\log \sigma,c})$ in this case we obtain $(iii)$.

$(iii) \Rightarrow (i)$. If $c^*(\sigma)$ is finite and $c > c^*(\sigma)$ then inequality (58) holds which contradicts to $(iii)$ by proposition 6b.

If $dc(\sigma) < 1$ and $c < c^*(\sigma)$ then by the above observation $dc(\Omega(\mathcal{V}_{\sigma,c})) < 1$ and regularity of the set $\mathcal{V}_{\sigma,c}$ follows from theorem 2E.

To prove the converse assertion note that lemma 7 below and the above observation imply that the second regularity condition does not hold for the set $\mathcal{V}_{\sigma,c}$ for arbitrary infinite dimensional state $\sigma$ and $c > 0$. Thus it is sufficient to show the first regularity condition does not hold for the set $\mathcal{V}_{\sigma,c}$ if either $dc(\sigma) = 1$ or $c \geq c^*(\sigma)$.
If \( \text{dc}(\sigma) = 1 \) then by the above observation \( \Omega(\mathcal{V}_{\sigma,c}) = \sigma \) for arbitrary \( c \).
In the case \( H(\sigma) < +\infty \) proposition 2 implies existence of the sequence of states \( \{\rho_n\} \) such that
\[
\lim_{n \to +\infty} H(\rho_n | \sigma) = 0 \quad \text{and} \quad \lim_{n \to +\infty} H(\rho_n) > H(\sigma).
\]
Thus the state \( \rho_n \) lie in \( \mathcal{V}_{\sigma,c} \) for sufficiently large \( n \) and hence the first regularity condition does not hold. In the case \( H(\sigma) = +\infty \) the first regularity condition does not hold obviously.

If \( \text{dc}(\sigma) < 1 \) and \( c \geq c^*(\sigma) \) then by the above observation \( \Omega(\mathcal{V}_{\sigma,c}) = (\text{Tr}\sigma^{\text{dc}(\sigma)})^{-1}\sigma^{\text{dc}(\sigma)} \). In the proof of proposition 3a it is shown that for arbitrary \( m \) the states in the sequence \( \{\rho^m_n\} \) for which relations (20) are valid lie in the set \( \mathcal{V}_{\sigma,c} \) for all sufficiently large \( n \). Thus the first regularity condition does not hold in this case. □

The set \( \mathcal{K}_{\sigma,c} \) with \( H(\sigma) = +\infty \) is a nontrivial example of a nonregular set containing states with infinite entropy but having finite \( \chi \)-capacity and possessing the optimal measure.

**Lemma 6.** Let \( \sigma = \sum_{k=1}^{\infty} \lambda_k |k\rangle\langle k| \) be a state with infinite entropy. For arbitrary natural \( n \) let \( \mathcal{L}_n(\sigma) \) be the convex closed subset of \( \mathcal{L}(\sigma) \) consisting of all states \( \rho \) such that \( \langle i|\rho|j\rangle = 0 \) if \( i \neq j \) and either \( i < n \) or \( j < n \). Then for arbitrary \( c \geq 0 \) and \( n \in \mathbb{N} \) there exist a natural \( m \) and a state \( \rho_{c,n,m} \) in \( \mathcal{L}_n(\sigma) \) such that
\[
H(\rho_{c,n,m} | \sigma) = c
\]
and \( \langle i|\rho|j\rangle = 0 \) if \( i \neq j \) and either \( i > m \) or \( j > m \).

**Proof.** Let \( c \geq 0 \) and \( n \in \mathbb{N} \) be arbitrary. Consider the state
\[
\sigma_n = \mu_n^{-1} \sum_{k=n}^{+\infty} \lambda_k |k\rangle\langle k|,
\]
where \( \mu_n = \sum_{k=n}^{+\infty} \lambda_k \), and the sequence of states
\[
\{\rho^m_n = \mu_n^{-1} \sum_{n \leq i,j \leq m} \sqrt{\lambda_i \lambda_j} |i\rangle\langle j| + \mu_n^{-1} \sum_{k > m} \lambda_k |k\rangle\langle k| \}_m,
\]
converging in the trace norm to the pure state \( \rho^*_n = \mu_n^{-1} \sum_{i,j \geq n} \sqrt{\lambda_i \lambda_j} |i\rangle\langle j| \) as \( m \to +\infty \). Since \( H(\sigma_n) = +\infty \) proposition 6a implies \( H(\rho^*_n | \sigma_n) = +\infty \). By using this and the general properties of the relative entropy we obtain
\[
H(\rho^m_n | \sigma_n) < +\infty, \quad \forall m \in \mathbb{N} \quad \text{and} \quad \lim_{m \to +\infty} H(\rho^m_n | \sigma_n) = +\infty.
\]
\[\llap{59} \]
Thus there exists natural $m(c)$ such that $c\mu_n^{-1} \leq H(\rho_n^{m(c)}\|\sigma_n) < +\infty$. The convex lower semicontinuous function $f(\lambda) = H(\lambda\rho_n^{m(c)} + (1 - \lambda)\sigma_n\|\sigma_n)$ does not exceed $\lambda H(\rho_n^{m(c)}\|\sigma_n)$ on $[0;1]$ and hence it is continuous on $[0;1]$. Since $f(0) = 0$ and $f(1) = H(\rho_n^{m(c)}\|\sigma_n) \geq c\mu_n^{-1}$ there exists $\lambda^* \in [0;1]$ such that $f(\lambda^*) = c\mu_n^{-1}$.

Let $m = m(c)$ and $\rho_{c,n,m} = \sum_{k=1}^{n-1} \lambda_k|k\rangle\langle k| + \mu_n(\lambda^*\rho_n^{m(c)} + (1 - \lambda^*)\sigma_n)$. It is easy to see that $H(\rho_{c,n,m}\|\sigma) = \mu_n H(\lambda^*\rho_n^{m(c)} + (1 - \lambda^*)\sigma_n\|\sigma_n) = c$ and that $\rho_{c,n,m} \in \mathcal{L}_n(\sigma)$. By the construction $\langle i|\rho|j \rangle = 0$ if $i \neq j$ and either $i > m$ or $j > m$.

**Lemma 7.** Let $\sigma$ be a state with infinite dimensional support. Then the relative entropy $H(\rho\|(\text{Tr}\sigma)^{-1}\sigma)$ is not a continuous function of the state $\rho$ on the set $V_{\sigma,c}$ for arbitrary $c > 0$ and for arbitrary $\lambda$ such that $\text{Tr}\sigma^\lambda < +\infty$.

**Proof.** Without loss of generality we may assume that $\sigma$ is a full rank state. Let $\varrho$ be a pure state such that $H(\varrho\|\sigma) = +\infty$ and $P_n$ be the spectral projector of the state $\sigma$, corresponding to its maximal $n$ eigenvalues. Then the sequence of pure states $\{\varrho_n = (\text{Tr} P_n\varrho)^{-1} P_n\varrho P_n\}$ converges to the pure state $\varrho$ and by using general properties of the relative entropy we have

$$H(\varrho_n\|\sigma) < +\infty \quad \text{for all } n \quad \text{and} \quad \lim_{n \to +\infty} H(\varrho_n\|\sigma) = +\infty.$$ 

Consider the sequence $\{\eta_n = c(H(\varrho_n\|\sigma))^{-1}\}_{n \geq n_0}$, where $n_0$ is chosen to be so large that $H(\varrho_n\|\sigma) > c$ for all $n \geq n_0$. Let $\rho_n = \eta_n\varrho_n + (1 - \eta_n)\sigma$ for all $n \geq n_0$. Then by using general properties of the relative entropy we obtain

$$c - h_2(\eta_n) = \eta_n H(\varrho_n\|\sigma) - h_2(\eta_n) \leq H(\rho_n\|\sigma) \leq \eta_n H(\varrho_n\|\sigma) = c,$$

where $h_2(x) = -x \log x - (1 - x) \log(1 - x)$.

Since $\eta_n \to 0$ as $n \to 0$ this inequality implies that

$$\rho_n \in V_{\sigma,c} \quad \text{for all } n \quad \text{and} \quad \lim_{n \to +\infty} H(\rho_n\|\sigma) = c. \quad (60)$$

Let $\lambda$ be an arbitrary positive number such that $\text{Tr}\sigma^\lambda < +\infty$. By lemma 3 we have

$$H(\rho_n\|(\text{Tr}\sigma^\lambda)^{-1}\sigma) = \lambda H(\rho_n\|\sigma) + \log \text{Tr}\sigma^\lambda - (1 - \lambda) H(\rho_n). \quad (61)$$

By using general properties of the entropy we obtain

$$(1 - \eta_n) H(\sigma) \leq H(\rho_n) \leq (1 - \eta_n) H(\sigma) + h_2(\eta_n),$$
for all \( n \geq n_0 \) and hence \( \lim_{n \to +\infty} H(\rho_n) = H(\sigma) \).

Thus (60) and (61) implies

\[
\lim_{n \to +\infty} H(\rho_n\|(Tr\sigma^\lambda)^{-1}\sigma^\lambda) = c + \log Tr\sigma^\lambda - (1 - \lambda)H(\sigma).
\]

By the construction the sequence \( \{\rho_n\} \) of states in \( \mathcal{V}_{\sigma,c} \) tends to the state \( \sigma \).

Since

\[
H(\sigma\|(Tr\sigma^\lambda - 1)^{-1}\sigma^\lambda) = \log Tr\sigma^\lambda - (1 - \lambda)H(\sigma)
\]

the previous expression means

\[
\lim_{n \to +\infty} H(\rho_n\|(Tr\sigma^\lambda - 1)^{-1}\sigma^\lambda) = H(\sigma\|(Tr\sigma^\lambda - 1)^{-1}\sigma^\lambda) + c,
\]

which implies discontinuity of the function \( H(\rho\|(Tr\sigma^\lambda - 1)^{-1}\sigma^\lambda) \) on the set \( \mathcal{V}_{\sigma,c} \).

\[ \square \]

Nonregularity of the set \( \mathcal{V}_{\sigma,c} \) for arbitrary state with infinite entropy are illustrated by the following example.

**Example of a decreasing sequence of closed sets with the same positive \( \chi \)-capacity, having the intersection with zero \( \chi \)-capacity.**

For arbitrary natural \( n \) let \( L_n(\sigma) \) be the convex closed subset of \( \mathcal{S}(H) \) introduced in lemma 6. For given \( c > 0 \) consider the monotonously decreasing sequence \( \{A_n = L_n(\sigma) \cap \mathcal{V}_{\sigma,c}\} \) of closed convex sets. Corollary 9 implies that \( \Omega(A_n) = \sigma \) - the only state in \( A_n \) invariant under the action of all automorphism from \( \mathfrak{F}(A_n) \). Lemma 6 provides existence of the state \( \rho_{c,n,m} \) in \( A_n \) such that \( H(\rho_{c,n,m}||\Omega(A_n)) = H(\rho_{c,n,m}||\sigma) = c \), which by theorem 1 implies \( \bar{C}(A_n) \geq c \). By theorem 2C we have \( \bar{C}(A_n) \leq \bar{C}(\mathcal{V}_{\sigma,c}) = c \) and hence

\[
\bar{C}(A_n) = c \text{ for all } n \text{ while } \bar{C}
\left(\bigcap_n A_n\right) = 0 \text{ since } \bigcap_n A_n = \{\sigma\}.
\]

### 5.4 The set \( \mathcal{A} \otimes \mathcal{B} \)

Let \( \mathcal{H} \) and \( \mathcal{K} \) be separable Hilbert spaces. For arbitrary sets \( \mathcal{A} \subseteq \mathcal{S}(\mathcal{H}) \) and \( \mathcal{B} \subseteq \mathcal{S}(\mathcal{K}) \) consider the set

\[
\mathcal{A} \otimes \mathcal{B} = \{\omega \in \mathcal{S}(\mathcal{H} \otimes \mathcal{K})| \omega^\mathcal{H} \in \mathcal{A}, \omega^\mathcal{K} \in \mathcal{B}\},
\]

where \( \omega^\mathcal{H} = \text{Tr}_K \omega \) and \( \omega^\mathcal{K} = \text{Tr}_H \omega \).

In [\[17\]] the following lemma was proved.

**Lemma 8.** The set \( \mathcal{A} \otimes \mathcal{B} \) is a convex subset of \( \mathcal{S}(\mathcal{H} \otimes \mathcal{K}) \) if and only if the sets \( \mathcal{A} \) and \( \mathcal{B} \) are convex subsets of \( \mathcal{S}(\mathcal{H}) \) and of \( \mathcal{S}(\mathcal{K}) \) correspondingly.
The set $A \otimes B$ is a compact subset of $\mathcal{S}(H \otimes K)$ if and only if the sets $A$ and $B$ are compact subsets of $\mathcal{S}(H)$ and of $\mathcal{S}(K)$ correspondingly.

The properties of the restriction of the entropy to the set $A \otimes B$ are also determined by the properties of the restrictions of the entropy to the sets $A$ and $B$ correspondingly.

**Proposition 10.** Let $A$ and $B$ be an arbitrary subsets of $\mathcal{S}(H)$ and of $\mathcal{S}(K)$ correspondingly.

The entropy is bounded on the set $A \otimes B$ if and only if the entropy is bounded on the sets $A$ and $B$.

The entropy is continuous on the set $A \otimes B$ if and only if the entropy is continuous on the sets $A$ and $B$.

**Proof.** If the entropy is bounded (continuous) on the set $A \otimes B$ then it is bounded (continuous) on the sets $A$ and $B$ since for every state $\rho$ in $A$ and for every state $\sigma$ in $B$ the state $\rho \otimes \sigma$ lies in $A \otimes B$ and $H(\rho \otimes \sigma) = H(\rho) + H(\sigma)$.

If the entropy is bounded on the sets $A$ and $B$ then the entropy is bounded on the set $A \otimes B$ due to its subadditivity.

Suppose, the entropy is continuous on the sets $A$ and $B$. Let $\omega_0$ be a state in $A \otimes B$ and $\{\omega_n\}$ be a sequence of states in $A \otimes B$ converging to the state $\omega_0$. Since

$$H(\omega_n) = H(\omega_n^H) + H(\omega_n^K) - H(\omega_n^H \otimes \omega_n^K)$$

the assumption and lower semicontinuity of the relative entropy imply

$$\limsup_{n \to +\infty} H(\omega_n) = \lim_{n \to +\infty} H(\omega_n^H) + \lim_{n \to +\infty} H(\omega_n^K) - \liminf_{n \to +\infty} H(\omega_n^H \otimes \omega_n^K) \leq H(\omega_0^H) + H(\omega_0^K) - H(\omega_0^H \otimes \omega_0^K) = H(\omega_0).$$

This and lower semicontinuity of the entropy implies $\lim_{n \to +\infty} H(\omega_n) = H(\omega_0)$.

□

The important example of the set $A \otimes B$ is the set consisting of all states $\omega$ in $\mathcal{S}(H \otimes K)$ with given partial traces $\omega^H = \rho$ and $\omega^K = \sigma$. Following [22], we denote this set $\mathcal{C}(\rho, \sigma)$. By lemma 8 the set $\mathcal{C}(\rho, \sigma)$ is convex and compact for arbitrary $\rho$ and $\sigma$. By subadditivity of the entropy $\sup_{\omega \in \mathcal{C}(\rho, \sigma)} H(\omega) = H(\rho) + H(\sigma)$. Similarly to the case of the set $\mathcal{L}(\sigma)$ finiteness of the entropy on the set $\mathcal{C}(\rho, \sigma)$ implies its continuity.

**Corollary 12.** The entropy is continuous on the set $\mathcal{C}(\rho, \sigma)$ if and only if the entropies $H(\rho)$ and $H(\sigma)$ are finite.

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For two arbitrary ensembles \( \{ \pi_i, \rho_i \} \) and \( \{ \lambda_j, \sigma_j \} \) of states in \( \mathcal{A} \) and in \( \mathcal{B} \) correspondingly the ensemble \( \{ \pi_i \lambda_j, \rho_i \otimes \sigma_j \} \) of states in \( \mathcal{A} \otimes \mathcal{B} \) is called the tensor product of the two above ensembles. By considering such tensor products of all possible ensembles of states in \( \mathcal{A} \) and \( \mathcal{B} \) it is easy to deduce from the definition that

\[
\bar{C}(\mathcal{A} \otimes \mathcal{B}) \geq \bar{C}(\mathcal{A}) + \bar{C}(\mathcal{B}). \tag{62}
\]

There exist nontrivial examples of sets \( \mathcal{A} \) and \( \mathcal{B} \), for which equality holds in (62). This takes place if \( \mathcal{A} \) and \( \mathcal{B} \) are sets of the types considered in subsection 5.2. But there exist examples of sets \( \mathcal{A} \) and \( \mathcal{B} \), for which strict inequality holds in (62). Moreover, if \( \mathcal{A} = \{ \rho \} \) and \( \mathcal{B} = \{ \sigma \} \) where \( \rho \) and \( \sigma \) are isomorphic states in \( \mathcal{S}(\mathcal{H}) \) and in \( \mathcal{S}(\mathcal{K}) \) with infinite entropy then by proposition 11 below the left side of (62) is equal to the infinity while the right side is obviously equal to zero.\(^{30}\)

Note that the equality in (62) implies

\[
\Omega(\mathcal{A} \otimes \mathcal{B}) = \Omega(\mathcal{A}) \otimes \Omega(\mathcal{B}). \tag{63}
\]

Indeed, if \( \{ \{ \pi_i^n, \rho^n_i \} \} \) and \( \{ \{ \lambda^n_j, \sigma^n_j \} \} \) are some approximating sequences of ensembles for the sets \( \mathcal{A} \) and \( \mathcal{B} \) correspondingly then by the assumed equality in (62) the sequence of ensembles \( \{ \{ \pi_i^n \lambda_j^n, \rho_i^n \otimes \sigma_j^n \} \} \) will be approximating sequence for the set \( \mathcal{A} \otimes \mathcal{B} \). By theorem 1 the sequences \( \{ \bar{\rho}_n \} \) and \( \{ \bar{\sigma}_n \} \) converges to the optimal average states \( \Omega(\mathcal{A}) \) and \( \Omega(\mathcal{B}) \) correspondingly. So, the sequence \( \{ \bar{\rho}_n \otimes \bar{\sigma}_n \} \) converges to the state \( \Omega(\mathcal{A}) \otimes \Omega(\mathcal{B}) \) and, hence, by theorem 1 this state is the optimal average state \( \Omega(\mathcal{A} \otimes \mathcal{B}) \) of the set \( \mathcal{A} \otimes \mathcal{B} \). The below proposition 11 shows, in particular, that (63) does not imply (62).

In the rest of this section we restrict our attention on the set \( \mathcal{C}(\rho, \sigma) \). Let \( \rho = \sum_i \pi_i |e_i\rangle\langle e_i| \) and \( \sigma = \sum_j \lambda_j |f_j\rangle\langle f_j| \), where \( \{ |e_i\rangle \} \) and \( \{ |f_j\rangle \} \) are orthonormal systems of vectors in \( \mathcal{H} \) and in \( \mathcal{K} \) correspondingly. Let \( E_{ij} = |e_i\rangle\langle e_j| \) and \( F_{kl} = |f_k\rangle\langle f_l| \) be one rank operators in \( \mathcal{B}(\mathcal{H}) \) and in \( \mathcal{B}(\mathcal{K}) \) correspondingly. For arbitrary probability distributions \( \{ \pi_i \} \) and \( \{ \lambda_j \} \) let \( \mathcal{C}(\{ \pi_i \}, \{ \lambda_j \}) \) be the set of all probability distribution \( \{ \omega_{ij} \} \) such that \( \sum_j \omega_{ij} = \pi_i \) and \( \sum_i \omega_{ij} = \lambda_j \), so that \( \mathcal{C}(\{ \pi_i \}, \{ \lambda_j \}) \) is the classical analog of the set \( \mathcal{C}(\rho, \sigma) \). Denote by \( \mathcal{C}_s(\rho, \sigma) \) the closed convex subset of \( \mathcal{C}(\rho, \sigma) \) consisting of all states

\(^{30}\)The strict inequality in (62) does not contradict to the additivity conjecture for the \( \chi \)-capacity of quantum channels. Indeed, if \( \mathcal{A} \) and \( \mathcal{B} \) are the output sets of particular channels \( \Phi \) and \( \Psi \) correspondingly then the output set of the channel \( \Phi \otimes \Psi \) is a proper subset of the set \( \mathcal{A} \otimes \mathcal{B} \).
of the form \( \sum_{ij} \omega_{ij} E_{ii} \otimes F_{jj} \), where \( \{ \omega_{ij} \} \in \mathcal{C}(\{ \pi_i \}, \{ \lambda_j \}) \). The set \( \mathcal{C}_s(\rho, \sigma) \) can be identified with the classical analog \( \mathcal{C}(\{ \pi_i \}, \{ \lambda_j \}) \) of the set \( \mathcal{C}(\rho, \sigma) \).

Let \( G \) be the group of all unitaries in \( \mathfrak{B}(\mathcal{H} \otimes \mathcal{K}) \), diagonalizable in the basis \( \{|e_i \otimes f_j\rangle\} \). We will use the following simple observation.

**Lemma 9.** Let \( \rho = \sum_i \pi_i |e_i\rangle \langle e_i| \) and \( \sigma = \sum_j \lambda_j |f_j\rangle \langle f_j| \) be two states in \( \mathcal{S}(\mathcal{H}) \) and in \( \mathcal{S}(\mathcal{K}) \) correspondingly. An arbitrary state \( \omega \) in \( \mathcal{C}(\rho, \sigma) \) can be represented by

\[
\omega = \sum_{ij} \omega_{ij} E_{ii} \otimes F_{jj} + \sum_{i \neq j, k \neq l} \eta_{ijkl} E_{ij} \otimes F_{kl},
\]

where \( \{ \omega_{ij} \} \in \mathcal{C}(\{ \pi_i \}, \{ \lambda_j \}) \).

The set \( \mathcal{C}(\rho, \sigma) \) is invariant under the action of the automorphism \( U(\cdot)U^* \) for arbitrary \( U \in G \) while \( \mathcal{C}_s(\rho, \sigma) \) is the set of all invariant states in \( \mathcal{C}(\rho, \sigma) \) for the group of the above automorphisms.

**Proof.** An arbitrary state \( \omega \) in \( \mathcal{C}(\rho, \sigma) \) can be represented by

\[
\omega = \sum_{ijkl} \eta_{ijkl} E_{ij} \otimes F_{kl}.
\]

The requirements \( \text{Tr}_\mathcal{K} \omega = \rho = \sum_i \pi_i E_{ii} \) and \( \text{Tr}_\mathcal{H} \omega = \sigma = \sum_j \lambda_j F_{jj} \) provides the first statement of the lemma.

Since an arbitrary \( U \) in \( G \) is defined by the set \( \{ \varphi_{ij}(U) \}_{ij} \) of numbers in \( [0; 2\pi) \) via the expression

\[
U = \sum_{ij} \exp(i \varphi_{ij}(U)) E_{ii} \otimes F_{jj},
\]

we have \( U E_{ii} \otimes F_{jj} U^* = E_{ii} \otimes F_{jj} \) and \( U E_{ij} \otimes F_{kl} U^* = \exp(i(\varphi_{ik} - \varphi_{jl})) E_{ij} \otimes F_{kl} \) for this \( U \). By this for the above \( \omega \in \mathcal{C}(\rho, \sigma) \) and \( U \) we obtain

\[
U \omega U^* = \sum_{ij} \omega_{ij} E_{ii} \otimes F_{jj} + \sum_{i \neq j, k \neq l} \eta_{ijkl} \exp(i(\varphi_{ik} - \varphi_{jl})) E_{ij} \otimes F_{kl},
\]

which provides the second statement of the lemma. \( \square \)

The following proposition shows that the problems of calculation of the \( \chi \)-capacity and of finding the optimal average state of the set \( \mathcal{C}(\rho, \sigma) \) are nontrivial even in the symmetrical case \( \rho \cong \sigma \).
Proposition 11. Let $\rho = \sum_i \lambda_i |e_i\rangle \langle e_i|$ and $\sigma = \sum_j \lambda_j |f_j\rangle \langle f_j|$ be two isomorphic states supported by the subspaces $H_\rho \subseteq H$ and $K_\sigma \subseteq K$ correspondingly such that $H(\rho) = H(\sigma) = -\sum_i \lambda_i \log \lambda_i = h \leq +\infty$. Then

$$h \leq \bar{C}(\mathcal{C}(\rho, \sigma)) \leq 2h,$$

where the equality in the left side holds if and only if $\rho$ and $\sigma$ are pure states.

In the case $h < +\infty$ there exists an optimal measure $\mu_*(\rho, \sigma)$ with the barycenter $\Omega(\mathcal{C}(\rho, \sigma))$ in $\mathcal{C}_s(\rho, \sigma)$ having the support $H_\rho \otimes K_\sigma$ and the following statements are equivalent:

(i) $\bar{C}(\mathcal{C}(\rho, \sigma)) = 2h$;

(ii) $\Omega(\mathcal{C}(\rho, \sigma)) = \rho \otimes \sigma$;

(iii) $\rho$ and $\sigma$ are multiples of projectors of the same finite rank;

(iv) $\mu_*(\rho, \sigma)$ is supported by pure states.

Proof. By subadditivity of the entropy $H(\omega) \leq H(\rho) + H(\sigma) = 2h$ for all $\omega$ in $\mathcal{C}(\rho, \sigma)$. This implies the upper bound for $\bar{C}(\mathcal{C}(\rho, \sigma))$.

Suppose $\bar{C}(\mathcal{C}(\rho, \sigma))$ is finite. By theorem 1 there exists the unique state $\Omega(\mathcal{C}(\rho, \sigma))$ in $\mathcal{C}_s(\rho, \sigma)$ such that

$$H(\omega \| \Omega(\mathcal{C}(\rho, \sigma))) \leq \bar{C}(\mathcal{C}(\rho, \sigma)), \quad \forall \omega \in \mathcal{C}(\rho, \sigma). \quad (64)$$

By lemma 9 and corollary 8 this state $\Omega(\mathcal{C}(\rho, \sigma))$ is invariant under automorphism $U(\cdot)U^*$ for arbitrary $U$ in $G$ and hence

$$\Omega(\mathcal{C}(\rho, \sigma)) = \sum_{ij} \omega_{ij} E_{ii} \otimes F_{jj} \quad (65)$$

for some probability distribution $\{\omega_{ij}\}$ from $\mathcal{C}(\{\lambda_i\}, \{\lambda_j\})$. All elements $\omega_{ij}$ of this distribution must be positive since otherwise it is easy to find $\omega$ in $\mathcal{C}(\rho, \sigma)$ such that $H(\omega \| \Omega(\mathcal{C}(\rho, \sigma))) = +\infty$ contradicting to (64).

Let $\omega = \sum_{ij} \sqrt{\lambda_i} \sqrt{\lambda_j} E_{ij} \otimes F_{ij}$ be a pure state in $\mathcal{C}(\rho, \sigma)$. By (64) and (65) we have

$$\bar{C}(\mathcal{C}(\rho, \sigma)) \geq H(\omega \| \Omega(\mathcal{C}(\rho, \sigma))) = -\text{Tr} \omega \log(\Omega(\mathcal{C}(\rho, \sigma)))$$

$$= -\text{Tr} \sum_{ij} \sqrt{\lambda_i} \sqrt{\lambda_j} \log \omega_{ij} E_{ij} \otimes F_{ij} = -\sum_i \lambda_i \log \omega_{ii}. \quad (66)$$

65
If \( \rho \) and \( \sigma \) are not pure states then the right side of this expression is greater than \(-\sum \lambda_i \log \lambda_i = h\) since \( \omega_{ii} + \sum_{j \neq i} \omega_{ij} = \lambda_i \) and \( \omega_{ij} > 0 \) for all \( i \) and \( j \).

The existence of optimal measure in the case \( h < +\infty \) follows from corollary 12 and theorem 3.

The asserted equivalence of statements (i) \(- (iv)\) will be proved in the following order (ii) \( \Rightarrow \) (i) \( \Rightarrow \) (iv) \( \Rightarrow \) (iii) \( \Rightarrow \) (ii).

(ii) \( \Rightarrow \) (i) Suppose, \( \Omega(\mathcal{C}(\rho, \sigma)) = \rho \otimes \sigma = \sum_{ij} \lambda_i \lambda_j E_{ii} \otimes F_{jj}. \) Let \( \omega \) be the above pure state. By using expression (66) with \( \omega_{ij} = \lambda_i \lambda_j \) we have

\[
\bar{C}(\mathcal{C}(\rho, \sigma)) \geq H(\omega\|\Omega(\mathcal{C}(\rho, \sigma))) = -\sum_i \lambda_i \log \lambda_i^2 = 2h.
\]

Since the converse inequality is already proved we obtain \( \bar{C}(\mathcal{C}(\rho, \sigma)) = 2h \).

(i) \( \Rightarrow \) (iv) Suppose \( \bar{C}(\mathcal{C}(\rho, \sigma)) = 2h = H(\rho \otimes \sigma). \) Let \( \mu_* \) be an arbitrary optimal measure for the set \( \mathcal{C}(\rho, \sigma) \). Since \( 2h \) is the maximum of the entropy on the set \( \mathcal{C}(\rho, \sigma) \) we necessarily have \( \int H(\omega)\mu_+(d\omega) = 0 \) and hence \( \mu_* \) is supported by pure states in \( \mathcal{C}(\rho, \sigma) \).

(iv) \( \Rightarrow \) (iii) Let \( \mu_* \) be an optimal measure for the set \( \mathcal{C}(\rho, \sigma) \) supported by pure states. This implies that its barycenter \( \Omega(\mathcal{C}(\rho, \sigma)) \) lies in the convex closure of pure states in \( \mathcal{C}(\rho, \sigma) \). Since by the above observation \( \Omega(\mathcal{C}(\rho, \sigma)) \) is a state in \( \mathcal{C}_s(\rho, \sigma) \) supported by \( H_{\rho} \otimes K_{\sigma} \) lemma 10 below implies that \( \rho \) and \( \sigma \) are multiples of projectors of the same finite rank.

(iii) \( \Rightarrow \) (ii) Suppose \( \rho \) and \( \sigma \) are multiples of projectors. By lemma 10 below there exists an ensemble of pure states in \( \mathcal{C}(\rho, \sigma) \) with the average state \( \rho \otimes \sigma \). Since this ensemble is obviously optimal for the set \( \mathcal{C}(\rho, \sigma) \) its the average state coincides with \( \Omega(\mathcal{C}(\rho, \sigma)) \).

\textbf{Lemma 10.} Let \( \rho \) and \( \sigma \) be two states supported by the subspaces \( H_{\rho} \subseteq H \) and \( K_{\sigma} \subseteq K \) correspondingly.

The following statements are equivalent:

i) the set \( \mathcal{C}_s(\rho, \sigma) \) contains a state with the support \( H_{\rho} \otimes K_{\sigma} \), which lies the convex closure of the set of all pure states in \( \mathcal{C}(\rho, \sigma) \);

ii) the states \( \rho \) and \( \sigma \) are multiples of projectors of the same finite rank;

iii) the state \( \rho \otimes \sigma \) in \( \mathcal{C}_s(\rho, \sigma) \) can be represented as a finite convex combination of pure states in \( \mathcal{C}(\rho, \sigma) \).

\textbf{Proof.} The all statements of the lemma imply that the states \( \rho \) and \( \sigma \) are isomorphic. Otherwise there exist no pure states in \( \mathcal{C}(\rho, \sigma) \).
It is sufficient to show \((i) \Rightarrow (ii)\) and \((ii) \Rightarrow (iii)\).

\((i) \Rightarrow (ii)\) Let \(\hat{\omega} = \sum_{ij} \omega_{ij} E_{ii} \otimes F_{jj}\) be a state in \(\mathcal{C}_s(\rho, \sigma)\), contained in the convex closure of the set of all pure states in \(\mathcal{C}(\rho, \sigma)\). By lemma 1 in \([8]\) there exists a measure \(\mu\) supported by pure states in \(\mathcal{C}(\rho, \sigma)\) such that

\[
\hat{\omega} = \int_{\mathcal{C}(\rho, \sigma)} \omega \mu(d\omega)
\]

It is sufficient to prove that the state \(\rho\) has no different positive eigenvalues. Suppose \(\lambda_i\) and \(\lambda_j\) are such eigenvalues. By using the Schmidt decomposition for any pure state \(\omega\) in \(\mathcal{C}(\rho, \sigma)\) it is easy to see that \(E_{ii} \otimes F_{jj} \omega = 0\). Hence

\[
\omega_{ij} E_{ii} \otimes F_{jj} = E_{ii} \otimes F_{jj} \hat{\omega} = \int_{\mathcal{C}(\rho, \sigma)} E_{ii} \otimes F_{jj} \omega \mu(d\omega) = 0,
\]

which implies that the support of the state \(\hat{\omega}\) does not coincide with \(\mathcal{H}_{\rho} \otimes \mathcal{K}_{\sigma}\).

\((ii) \Rightarrow (iii)\) Let \(\rho = d^{-1} P\) and \(\sigma = d^{-1} Q\), where \(P\) and \(Q\) are \(d\)-dimensional projectors in \(\mathcal{B}(\mathcal{H})\) and in \(\mathcal{B}(\mathcal{K})\) correspondingly. Let \(\{|\varphi_i\rangle\}\) is a particular basis of maximally entangled vectors in \(P(\mathcal{H}) \otimes Q(\mathcal{K})\). Then \(\rho \otimes \sigma = d^{-2} \sum_i |\varphi_i\rangle \langle \varphi_i|\). \(\Box\)

**Remark 10.** It is interesting to compare the \(\chi\)-capacity of the set \(\mathcal{C}(\rho, \sigma)\) with the \(\chi\)-capacity of the set \(\mathcal{C}_s(\rho, \sigma)\) which can be identified with the classical analog \(\mathcal{C}([\pi_1], [\lambda_j])\) of the set \(\mathcal{C}(\rho, \sigma)\). Let \(\rho\) and \(\sigma\) are multiples of \(d\)-dimensional projectors. In this case the set \(\mathcal{C}([\pi_1], [\lambda_j])\) consists of all probability distribution \(\{\omega_{ij}\}_{i,j=1}^d\) such that \(\sum_{i=1}^d \omega_{ij} = d^{-1} = \sum_{j=1}^d \omega_{ij}\). It is easy to see that the optimal ensemble for the set \(\mathcal{C}_s(\rho, \sigma) \cong \mathcal{C}([\pi_1], [\lambda_j])\) consists of \(d\) states, having one nonzero element \(d^{-1}\) in each row and in each column, with equal probabilities, so that the average state is the uniform distribution \(\{\omega_{ij} = d^{-2}\}\). Thus

\[
\bar{\mathcal{C}}(\mathcal{C}_s(\rho, \sigma)) = \log d^2 - \log d = \log d = h = \frac{1}{2} \bar{\mathcal{C}}(\mathcal{C}(\rho, \sigma)),
\]

where the last equality follows from proposition 11. So, using entangled states in \(\mathcal{C}(\rho, \sigma)\) leads to twice increasing of the \(\chi\)-capacity.

### 5.5 An orbit of a compact group of automorphisms

Let \(G\) be a compact group and \(\{U_g\}_{g \in G}\) be its unitary (projective) representation on the Hilbert space \(\mathcal{H}\). Let \(\sigma\) be an arbitrary state in \(\mathcal{S}(\mathcal{H})\).
Consider the set \( \mathcal{O}_{G,U,g,\sigma} = \{ U_g \sigma U_g^* : g \in G \} \). This set is compact as the image of the compact set \( G \) under the continuous mapping \( g \mapsto U_g \sigma U_g^* \). This and separability of the space \( \mathcal{S}(\mathcal{H}) \) implies compactness of its convex closure \( \text{co}(\mathcal{O}_{G,U,g,\sigma}) \). Let \( \omega(G,U,g,\sigma) = \int_G U_g \sigma U_g^* \mu_H(dg) \) be a state in \( \text{co}(\mathcal{O}_{G,U,g,\sigma}) \), where \( \mu_H \) is the Haar measure on \( G \).

**Proposition 12.** The entropy is bounded on the set \( \text{co}(\mathcal{O}_{G,U,g,\sigma}) \) if and only if \( H(\omega(G,U,g,\sigma)) < +\infty \). In this case the entropy is continuous on the set \( \text{co}(\mathcal{O}_{G,U,g,\sigma}) \) and achieves its maximum at the Gibbs state

\[
\Gamma(\text{co}(\mathcal{O}_{G,U,g,\sigma})) = \omega(G,U,g,\sigma).
\]

The \( \chi \)-capacity \( \bar{C}(\mathcal{O}_{G,U,g,\sigma}) \) of the set \( \mathcal{O}_{G,U,g,\sigma} \) is equal to \( H(\sigma \| \omega(G,U,g,\sigma)) \). If the \( \chi \)-capacity is finite then the image of the Haar measure \( \mu_H \) corresponding to the mapping \( g \mapsto U_g \sigma U_g^* \) is the optimal measure for the set \( \mathcal{O}_{G,U,g,\sigma} \) and

\[
\Omega(\mathcal{O}_{G,U,g,\sigma}) = \omega(G,U,g,\sigma).
\]

The set \( \mathcal{O}_{G,U,g,\sigma} \) is regular if and only if it has finite \( \chi \)-capacity.

**Proof.** Since \( \int_G U_g \rho U_g^* \mu_H(dg) = \omega(G,U,g,\sigma) \) for arbitrary state \( \rho \) in \( \text{co}(\mathcal{O}_{G,U,g,\sigma}) \) the boundedness assertion of the proposition easily follows from concavity of the entropy and Jensen’s inequality.\(^{31}\) The continuity assertion follows from corollary 3 since \( \text{Tr}\sigma(-\log\omega(G,U,g,\sigma)) = H(\omega(G,U,g,\sigma)) \).

The set \( \mathcal{O}_{G,U,g,\sigma} \) is invariant under the action of the family of automorphisms \( \{ U_g(U_g^*)^* \}_{g \in G} \) and \( \omega(G,U,g,\sigma) \) is the only invariant state in \( \text{co}(\mathcal{O}_{G,U,g,\sigma}) \) for this family. It follows from corollary 9 that \( \bar{C}(\mathcal{O}_{G,U,g,\sigma}) = H(\sigma \| \omega(G,U,g,\sigma)) \) and that \( \Omega(\mathcal{O}_{G,U,g,\sigma}) = \omega(G,U,g,\sigma) \).

The assertion concerning existence of optimal measure for the set \( \mathcal{O}_{G,U,g,\sigma} \) is obvious.

The regularity assertion follows from the above observation since it is easy to see that \( H(\rho \| \omega(G,U,g,\sigma)) = H(\sigma \| \omega(G,U,g,\sigma)) \) for all \( \rho \) in \( \mathcal{O}_{G,U,g,\sigma} \). \(\Box\)

**Example of a closed set having optimal measure, but having no atomic optimal measure.** Let \( G = \mathbb{T} \) - one dimensional rotation group represented as the interval \( [\pi, \pi] \). In this case the Haar measure is the normalized Lebesgue measure \( \frac{dx}{2\pi} \). Let \( \mathcal{H} = \mathcal{L}^2[\pi, \pi] \). We may consider

\(^{31}\)Application of Jensen’s inequality in this case is valid since the entropy can be represented as a pointwise limit of a monotonously increasing sequence of continuous concave functions.\(^{[9]}\)
elements of $L_2([\pi, \pi])$ as $2\pi$-periodic functions on $\mathbb{R}$. Let $\{U_\lambda\}_{\lambda \in \mathbb{T}}$ be unitary representation of the group $\mathbb{T}$ defined by

$$U_\lambda(\psi(x)) = \psi(x - \lambda), \quad \psi(x) \in L_2([\pi, \pi]).$$

For given $|\varphi_0\rangle$ in $L_2([\pi, \pi])$ consider the set $O_{T,U_\lambda,|\varphi_0\rangle\langle\varphi_0|}$. It this case

$$\omega(T, U_\lambda, |\varphi_0\rangle\langle\varphi_0|) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\varphi_\lambda\rangle\langle\varphi_\lambda| d\lambda,$$

where $|\varphi_\lambda\rangle = U_\lambda|\varphi_0\rangle$. Note that $\overline{co}(O_{T,U_\lambda,|\varphi_0\rangle\langle\varphi_0|})$ is the closure of the output set of the channel $\Phi$ considered in [8]. In the proof of theorem 4 in [8] it was shown that

$$C(O_{T,U_\lambda,|\varphi_0\rangle\langle\varphi_0|}) = H(\omega(T, U_\lambda, |\varphi_0\rangle\langle\varphi_0|)) = -\sum_{n=-\infty}^{\infty} c_n^2(\varphi_0) \log c_n^2(\varphi_0), \quad (67)$$

where $\{c_n(\varphi_0)\}_{n \in \mathbb{Z}}$ are the set of the Fourier coefficients of the function $\varphi_0$ with respect to trigonometric orthonormal system $\{\exp(inx)\}_{n \in \mathbb{Z}}$. By proposition 12 finiteness of the above series means continuity of the entropy on the set $\overline{co}(O_{T,U_\lambda,|\varphi_0\rangle\langle\varphi_0|})$. Proposition 12 also implies that the image of the normalized Lebesgue measure $\frac{dx}{2\pi}$ corresponding to the mapping $\lambda \mapsto U_\lambda|\varphi_0\rangle\langle\varphi_0|U_\lambda^*$ is an optimal measure for the set $O_{T,U_\lambda,|\varphi_0\rangle\langle\varphi_0|}$. This measure is nonatomic, but its existence does not mean that there is no purely atomic optimal measure in this case. We will show that for a particular function $\varphi_0$ there is no purely atomic optimal measure for the set $O_{T,U_\lambda,|\varphi_0\rangle\langle\varphi_0|}$.

Let

$$\varphi_0(x) = \begin{cases} 
0, & x \in [-\pi; 0) \\
\sqrt{2}, & x \in [0; +\pi). 
\end{cases}$$

It this case $c_n(\varphi_0) \sim n^{-1}$ so that the series in (67) is finite.

By proposition 8 to prove nonexistence of an atomic optimal measure it is sufficient to show that the state $\Omega(O_{T,U_\lambda,|\varphi_0\rangle\langle\varphi_0|}) = \omega(T, U_\lambda, |\varphi_0\rangle\langle\varphi_0|)$ can not be represented as a countable convex combination of states in $O_{T,U_\lambda,|\varphi_0\rangle\langle\varphi_0|}$. For this aim it is possible to apply the method used in [8], but we will consider another approach based on the theory of generalized functions (distributions).

Suppose $\omega(T, U_\lambda, |\varphi_0\rangle\langle\varphi_0|) = \sum_{i=1}^{\infty} \pi_i |\varphi_{\lambda_i}\rangle\langle\varphi_{\lambda_i}|$. Without loss of generality we may assume that $\pi_1 \geq \pi_i$ for all $i > 1$ and that $\lambda_1 = 0$. For arbitrary
we have

\[
\sum_{i=1}^{+\infty} \pi_i \langle \varphi_\eta | \varphi_\lambda \rangle^2 = \langle \varphi_\eta | \omega(\mathbb{T}, U_\lambda, |\varphi_0\rangle \langle \varphi_0 |) \varphi_\eta \rangle
\]

(68)

\[
= \frac{1}{2\pi} \int_{-\pi}^{+\pi} \langle \varphi_\eta | \varphi_\lambda \rangle^2 d\lambda = \text{Const}(\eta).
\]

Let \(\theta(x)\) be the \(2\pi\)-periodical function equal to \((1 - \pi^{-1}|x|)^2\) on \([-\pi; +\pi]\). Then \(\langle \varphi_\eta | \varphi_\lambda \rangle^2 = \theta(\eta - \lambda)\) for all \(\lambda\) and \(\eta\). Since for each \(\lambda\) the function \(\theta_\lambda(x) = \theta_0(x - \lambda)\) is locally integrable it generates elements \(\tilde{\theta}_\lambda\) of the space \(\mathcal{D}'\) of generalized functions. \(^{32}\) Let \(\tilde{\theta}_\lambda \in \mathcal{D}'\) be the (generalized) derivative of the generalized function \(\tilde{\theta}_\lambda \in \mathcal{D}'\). Then (68) implies

\[
\mathcal{D}' - \lim_{n \to +\infty} \sum_{i=1}^{n} \pi_i \tilde{\theta}'_{\lambda_i} = 0.
\]

(69)

Let

\[
\omega_\delta(x) = \begin{cases} 
\exp(-((1 - (x/\delta)^2)^{-1}), & x \in [-\delta; +\delta] \\
0, & x \in \mathbb{R} \setminus [-\delta; +\delta]
\end{cases}
\]

be a function from the space \(\mathcal{D}\) for each \(\delta > 0\). By direct integration it is easy to see that

\[
\tilde{\theta}'_\lambda(\omega'_\delta) = \int_{-\delta}^{+\delta} \theta'_\lambda(x) \omega'_\delta(x) dx = \frac{2}{\pi} \int_{-\delta}^{\lambda} \left( 1 + \frac{x - \lambda}{\pi} \right) \omega'_\delta(x) dx
\]

\[
+ \frac{2}{\pi} \int_{\lambda}^{+\delta} \left( \frac{x - \lambda}{\pi} - 1 \right) \omega'_\delta(x) dx = \frac{4\omega_\delta(\lambda)}{\pi} - \frac{2\delta I}{\pi^2}
\]

if \(\lambda \in [-\delta; +\delta]\) and

\[
\tilde{\theta}'_\lambda(\omega'_\delta) = \int_{-\delta}^{+\delta} \theta'_\lambda(x) \omega'_\delta(x) dx = \frac{2}{\pi^2} \int_{-\delta}^{+\delta} x \omega'_\delta(x) dx = -\frac{2\delta I}{\pi^2}
\]

\(^{32}\)The space \(\mathcal{D}'\) is the linear space of continuous linear functional on the space \(\mathcal{D}\) of smooth functions with finite support.\(^{13}\)
if $\lambda \in \mathbb{R} \backslash [-\delta; +\delta]$, where $I = \delta^{-1} \int_{-\delta}^{+\delta} \omega_{\delta}(x)dx = \int_{-1}^{+1} \exp(-(1-x^2)^{-1})dx$ is a positive number.

Let $\mathcal{N}(\delta) = \{i \in \mathbb{N} \mid \lambda_i \in [-\delta; +\delta] \}$ and $\mathcal{K}_n = \{2, 3, ..., n\}$. By using the above expressions we obtain

$$\sum_{i=1}^{n} \pi_i \hat{\theta}_{\lambda_i}(\omega_{\delta}') = \pi_1 \hat{\theta}_{0}(\omega_{\delta}') + \sum_{i \in \mathcal{N}(\delta) \cap \mathcal{K}_n} \pi_i \hat{\theta}_{\lambda_i}(\omega_{\delta}') \sum_{i \in (\mathbb{N}\setminus\mathcal{N}(\delta)) \cap \mathcal{K}_n} \pi_i - \frac{2\delta I}{\pi^2}, \quad \forall n.$$  

Since $\sum_{i \in \mathcal{N}(\delta), i>1} \pi_i$ obviously tends to zero as $\delta$ tends to zero the above inequality implies $\liminf_{n \to +\infty} \sum_{i=1}^{n} \pi_i \hat{\theta}_{\lambda_i}(\omega_{\delta}') > 0$ for all sufficiently small $\delta$, which contradicts to (69).

### 6 On another definition of $\bar{C}(\mathcal{A})$ and of $\Omega(\mathcal{A})$

It is known that the entropy and the relative entropy for general quantum states can be introduced via finite dimensional definition and a limiting procedure. To show this consider the nonlinear mapping

$$\Theta_P(\rho) = (\text{Tr}P\rho)^{-1}P\rho P$$

corresponding to arbitrary finite rank projector $P$ and having the domain $\mathcal{D}(\Theta_P) = \{\rho \in \mathcal{S}(H) \mid P\rho \neq 0\}$. By the results in [9] the entropy $H(\rho)$ of an arbitrary state $\rho$ can be defined by

$$H(\rho) = \lim_{n \to +\infty} H(\Theta_{P_n}(\rho)),$$

while the relative entropy $H(\rho \parallel \sigma)$ for arbitrary states $\rho$ and $\sigma$ - by

$$H(\rho \parallel \sigma) = \lim_{n \to +\infty} H(\Theta_{P_n}(\rho) \parallel \Theta_{P_n}(\sigma)),$$

where $\{P_n\}$ is an arbitrary increasing sequence of finite rank projectors strongly converging to the identity operator $I_H$.\(^{33}\) This implies that both

\(^{33}\)It is assumed that $n$ is sufficiently large so that $\rho$ and $\sigma$ lie in $\mathcal{D}(\Theta_{P_n})$
above limits exist (finite or infinite) and do not depend on the choice of the sequence \( \{P_n\} \). Since the states \( \Theta_{P_n}(\rho) \) and \( \Theta_{P_n}(\sigma) \) are supported by finite dimensional subspaces \( P_n(\mathcal{H}) \) for all \( n \) this observation reduces the definition of the entropy and of the relative entropy to the finite dimensional case.

In this section we obtain the analogous results for the \( \chi \)-capacity and for the optimal average state of an arbitrary set of states. Since for any closed subset of states in the \( d \)-dimensional Hilbert space the supremum in the definition of the \( \chi \)-capacity can be over all ensembles of \( d^2 \) states the \( \chi \)-capacity and the optimal average state of this subset can be defined by linear programming procedure \( \text{[16]} \). So, the results of this section provides the definition of the \( \chi \)-capacity and of the optimal average state for an arbitrary set of the infinite dimensional states, which can be used (in principal) for their numerical approximations.

It is clear that for arbitrary projector \( P \) the corresponding mapping \( \Theta_P(\sigma) \) is continuous in each point of its domain. Despite nonlinearity of this mapping the following result is valid.

\textbf{Lemma 11.} For arbitrary convex subset \( \mathcal{A} \) of \( \mathcal{D}(\Theta_P) \) its image \( \Theta_P(\mathcal{A}) \) under the mapping \( \Theta_P \) is a convex subset of \( \mathcal{S}(\mathcal{H}) \).

For arbitrary ensemble \( \{\pi_i, \rho_i\}_{i=1}^m \) of states in \( \Theta_P(\mathcal{A}) \) there exists ensemble \( \{\lambda_i, \sigma_i\}_{i=1}^m \) of states in \( \mathcal{A} \) such that

\[
\Theta_P(\sigma_i) = \rho_i \quad \text{and} \quad \lambda_i \text{Tr} P \sigma_i = \pi_i \sum_{j=1}^m \lambda_j \text{Tr} P \sigma_j \quad \text{for } i = 1, m.
\]

\textbf{Proof.} It is sufficient to prove the second statement of the lemma since it implies

\[
\Theta_P \left( \sum_i \lambda_i \sigma_i \right) = \sum_i \pi_i \rho_i.
\]

For each \( i \) the state \( \rho_i \) in \( \Theta_P(\mathcal{A}) \) is an image of a particular state \( \sigma_i \) in \( \mathcal{A} \). Let \( \eta_i = \pi_i (\text{Tr} P \sigma_i)^{-1} \) be a positive number for each \( i = 1, m \) and \( \left\{ \lambda_i = \eta_i \left( \sum_{j=1}^m \eta_j \right)^{-1} \right\} \) be a probability distribution. By summing the equalities \( \lambda_i \text{Tr} P \sigma_i = \pi_i \left( \sum_{j=1}^m \eta_j \right)^{-1} \) we obtain \( \sum_{i=1}^m \lambda_i \text{Tr} P \sigma_i = \left( \sum_{j=1}^m \eta_j \right)^{-1} \).

\textbf{Lemma 12.} Let \( \mathcal{A} \) be a set with finite \( \chi \)-capacity and \( P \) be a projector such that \( \eta(\mathcal{A}, P) = \inf_{\rho \in \mathcal{A}} \text{Tr} P \rho > 0 \). Then

\[
\eta(\mathcal{A}, P) \tilde{C}(\Theta_P(\mathcal{A})) \leq \tilde{C}(\mathcal{A}).
\]
Proof. For arbitrary ensemble \( \{ \pi_i, \rho_i \} \) of states in \( \Theta_P(\mathcal{A}) \) let \( \{ \lambda_i, \sigma_i \} \) be
the corresponding ensemble of states in \( \mathcal{A} \) provided by lemma 11. It follows
that \( \eta = \sum_i \lambda_i \eta_i \), where \( \eta_i = \text{Tr} P \sigma_i \) and \( \eta = \text{Tr} P \bar{\sigma} \).

Consider the channel
\[
\Phi(\rho) = P \rho P + (\text{Tr}(I - P)\rho) \tau,
\]
where \( \tau \) is a pure state corresponding to arbitrary unit vector in \( \mathcal{H} \oplus P(\mathcal{H}) \).

By general properties of the relative entropy we obtain
\[
\chi(\{ \lambda_i, \Phi(\sigma_i) \}) = \sum_i \lambda_i H(P \sigma_i P \parallel P \bar{\sigma} P)
+ \sum_i \lambda_i H((\text{Tr}(I - P)\sigma_i) \tau \parallel (\text{Tr}(I - P)\bar{\sigma}) \tau) \geq \sum_i \lambda_i H(P \sigma_i P \parallel P \bar{\sigma} P)
= \sum_i \lambda_i H(\eta_i \rho_i \parallel \eta \bar{\rho}) \geq \sum_i \lambda_i \eta_i H(\rho_i \parallel \bar{\rho}) = \eta \sum_i \pi_i H(\rho_i \parallel \bar{\rho}) \geq \eta(\mathcal{A}, P) \chi(\{ \pi_i, \rho_i \}).
\]

By monotonicity of the relative entropy we have
\[
\chi(\{ \lambda_i, \Phi(\sigma_i) \}) \leq \chi(\{ \lambda_i, \sigma_i \}).
\]
The two above inequalities implies the statement of the lemma. \( \square \)

Remark 11. The constant \( \eta(\mathcal{A}, P) \) in lemma 12 cannot be replaced by 1 (see the example in remark 12 below). \( \square \)

Now we can prove the following approximation result.

**Theorem 4.** Let \( \mathcal{A} \) be an arbitrary subset of \( \mathfrak{S}(\mathcal{H}) \).

If the \( \chi \)-capacity of the set \( \mathcal{A} \) is finite then
\[
\lim_{n \to +\infty} \bar{C}(\Theta_{P_n}(\mathcal{A})) = \bar{C}(\mathcal{A}) \quad \text{and} \quad \lim_{n \to +\infty} \Omega(\Theta_{P_n}(\mathcal{A})) = \Omega(\mathcal{A})
\]
for arbitrary sequence \( \{ P_n \} \) of projectors strongly converging to \( I_{\mathcal{H}} \).

If there exists a sequence of projectors \( \{ P_n \} \) strongly converging to \( I_{\mathcal{H}} \)
such that the mappings in the corresponding sequence \( \{ \Theta_{P_n} \} \) are well defined
on the set \( \mathcal{A} \) and the sequence \( \{ \bar{C}(\Theta_{P_n}(\mathcal{A})) \} \) is bounded then \( \bar{C}(\mathcal{A}) \) is finite.

**Proof.** Let \( \bar{C}(\mathcal{A}) < +\infty \) and \( \{ P_n \} \) be an arbitrary sequence of projectors
strongly converging to \( I_{\mathcal{H}} \). By theorem 2D the set \( \mathcal{A} \) is compact. By compactness criterion \( \lim_{n \to +\infty} \eta(\mathcal{A}, P_n) = 1 \), where \( \eta(\mathcal{A}, P_n) = \inf_{\rho \in \mathcal{A}} \text{Tr} P_n \rho \).
Thus \( \mathcal{A} \subseteq \mathfrak{D}(\Theta_{P_n}) \) for all sufficiently large \( n \) and by lemma 12 we have
\[
\limsup_{n \to +\infty} \bar{C}(\Theta_{P_n}(\mathcal{A})) \leq \bar{C}(\mathcal{A}).
\]
Since $\Theta_{P_n}(\rho) \to \rho$ as $n \to +\infty$ the first part of lemma 4 implies
\[
\liminf_{n \to +\infty} \bar{C}(\Theta_{P_n}(\mathcal{A})) \geq \bar{C}(\mathcal{A}).
\]
By the two above inequalities we obtain the first limit expression in the theorem, the second follows from the first and the second part of lemma 4.

If $\bar{C}(\mathcal{A}) = +\infty$ and $\{P_n\}$ be a sequence of finite dimensional projectors strongly converging to $I_H$ such that $\mathcal{A} \subseteq \mathcal{D}(\Theta_{P_n})$ for all sufficiently large $n$ then the first part of lemma 4 implies
\[
\lim_{n \to +\infty} \bar{C}(\Theta_{P_n}(\mathcal{A})) = +\infty. \quad \square
\]

**Remark 12.** The convergence of the sequence $\{\bar{C}(\Theta_{P_n}(\mathcal{A}))\}$ to $\bar{C}(\mathcal{A})$ has different nature depending on the choice of the sequence $\{P_n\}$. It may seem surprising that for a particular set $\mathcal{A}$ and a sequence $\{P_n\}$ the sequence $\{\bar{C}(\Theta_{P_n}(\mathcal{A}))\}$ converges to $\bar{C}(\mathcal{A})$ strongly decreasing. Indeed, let $\mathcal{A}$ be the set consisting of two states $\{\frac{1}{2}\rho + \frac{1}{2} \sigma_i\}_{i=1,2}$, where $\rho$ is a state with infinite dimensional support $H_{\rho}$ such that $H \ominus H_{\rho}$ is a two dimensional subspace and $\sigma_1, \sigma_2$ are the states corresponding to orthonormal unit vectors in $H \ominus H_{\rho}$.

Let $\{P_n\}$ be such sequence of finite rank projectors that $P_n(H) \supseteq H \ominus H_{\rho}$ and the sequence $\{\eta_n = \text{Tr}P_n\rho\}$ is strongly increasing to 1. It is easy to obtain that
\[
\bar{C}(\Theta_{P_n}(\mathcal{A})) = \frac{1}{1 + \eta_n} \log 2 \downarrow \frac{1}{2} \log 2 = \bar{C}(\mathcal{A}) \quad \text{as} \quad n \to +\infty.
\]

7 Appendix

In this section the detailed investigation of the properties of the function $F_{H}(h) = \sup_{\rho \in \mathcal{K}_{H,h}} H(\rho)$ described in proposition 1a is presented.

Note first that by lower semicontinuity of the entropy $\lim_{n \to +\infty} \sup_{\rho \in \mathcal{K}_{H,h}} H(\rho) = +\infty$ for arbitrary value of $\text{ic}(H)$ since $\bigcup_{h \in \mathbb{R}} \mathcal{K}_{H,h} = \mathcal{G}(H)$.

Consider the function
\[
g(\lambda, h) = \sum_{k=1}^{+\infty} (h_k - h) \exp(-\lambda h_k).
\]

By using the theorem about series depending on parameters it is easy to see that this function is differentiable at any point $(\lambda, h)$ with $\lambda > \text{ic}(H)$ and
\[
\frac{\partial g(\lambda, h)}{\partial \lambda} = \sum_{k=1}^{+\infty} h_k (h_k - h_k) \exp(-\lambda h_k), \quad \frac{\partial g(\lambda, h)}{\partial h} = -\sum_{k=1}^{+\infty} \exp(-\lambda h_k). \quad (70)
\]
By the observation in the proof of proposition 1a for each $h$ in $(h_m(H); h_*(H))$ there exists the unique $\lambda^* = \lambda^*(h) > \text{ic}(H)$ such that $g(\lambda^*(h), h) = 0$. It follows from (70) that

$$\frac{\partial g(\lambda, h)}{\partial \lambda} \bigg|_{\lambda = \lambda^*(h)} = \sum_{k=1}^{+\infty} (h_k - h)^2 \exp(-\lambda^*(h) h_k) < 0.$$  

By the implicit function theorem the function $\lambda^*(h)$ is differentiable on $(h_m(H); h_*(H))$ and

$$\frac{d\lambda^*(h)}{dh} = -\left[ \frac{\partial g(\lambda, h)}{\partial \lambda} \right]^{-1} \frac{\partial g(\lambda, h)}{\partial h},$$  

(71)

Expression (15) implies

$$F_H(h) = \lambda^*(h) h + \log \sum_{k=1}^{+\infty} \exp(-\lambda^*(h) h_k)$$  

(72)

for all $h$ in $(h_m(H); h_*(H)]$.

By direct derivatives calculation we obtain

$$\frac{dF_H(h)}{dh} = \frac{d}{dh} \left[ \lambda^*(h) h + \log \sum_{k=1}^{+\infty} \exp(-\lambda^*(h) h_k) \right] = \lambda^*(h),$$  

(73)

where the equality $g(\lambda^*(h), h) = 0$ was used. This and (71) implies

$$\frac{d^2 F_H(h)}{dh^2} = \frac{d\lambda^*(h)}{dh} < 0,$$

which shows strict concavity of the function $F_H(h)$ on $(h_m(H); h_*(H))$.

Suppose $h_*(H) < +\infty$. If $h > h_*(H)$ then by the proved part of the proposition 1a

$$F_H(h) = \text{ic}(H) h + \log \sum_{k=1}^{+\infty} \exp(-\text{ic}(H) h_k)$$  

(74)
is a linear function and
\[ \frac{dF_{H}(h)}{dh} = \text{ic}(H). \] (75)

If \( h = h_{*}(H) \) then by the observation in the proof of proposition 1a \( \lambda^{*}(h) = \text{ic}(H) \) and hence representations (72) and (74) coincides in this case.

To show smoothness of the function \( F_{H}(h) \) at the point \( h_{*}(H) \) note that \( \lambda^{*}(h) \to \text{ic}(H) \) as \( h \to h_{*}(H) - 0 \). Indeed, by (71) the function \( \lambda^{*}(h) \) is decreasing on \( (h_{*}(H); h_{*}(H)) \) and for arbitrary \( \lambda > \text{ic}(H) \) there exists \( h_{\lambda} = \left[ \sum_{k=1}^{+\infty} \exp(-\lambda h_{k}) \right]^{-1} \sum_{k=1}^{+\infty} h_{k} \exp(-\lambda h_{k}) \) such that \( \lambda = \lambda^{*}(h_{\lambda}) \).

Thus (72), (73), (74) and (75) imply
\[
\lim_{h \to h_{*}(H) - 0} F_{H}(h) = F_{H}(h_{*}(H)) \quad \text{and} \quad \lim_{h \to h_{*}(H) - 0} \frac{dF_{H}(h)}{dh} = \frac{dF_{H}(h)}{dh}|_{h=h_{*}(H)+0}
\]
and hence the function \( F_{H}(h) \) has a continuous derivative at the point \( h_{*}(H) \).

To prove right continuity of the function \( F_{H}(h) \) at the point \( h_{m}(H) \) note first that
\[ \lambda^{*}(h) \to +\infty \quad \text{as} \quad h \to h_{m} + 0. \] (76)
Indeed, by (74) the function \( \lambda^{*}(h) \) is decreasing on \( (h_{m}(H); h_{*}(H)) \) and hence there exists \( \lambda^{m} = \lim_{h \to h_{m}(H) + 0} \lambda^{*}(h) \). If \( \lambda^{m} < +\infty \) then by passing to the limit as \( h \to h_{m}(H) + 0 \) in the identity
\[
\sum_{k=1}^{+\infty} h_{k} \exp(-\lambda^{*}(h)h_{k}) \equiv h \sum_{k=1}^{+\infty} \exp(-\lambda^{*}(h)h_{k}),
\]
valid for all \( h \) in \( (h_{m}(H); h_{*}(H)) \), we obtain a contradiction.

Let \( d = \text{dim} \mathcal{H}_{m}(H) \). It is easy to see that
\[
P(h) = \log \sum_{k=1}^{+\infty} \exp(-\lambda^{*}(h)h_{k}) = -\lambda^{*}(h)h_{m}(H) + Q(h), \quad (77)
\]
where \( Q(h) = \log(d + \sum_{k=d}^{+\infty} \exp(-\lambda^{*}(h)(h_{k} - h_{m}(H))) \) is a nondecreasing function on \( (h_{m}(H); h_{*}(H)) \) tending to \( \log d \) as \( h \to h_{m}(H) + 0 \).

Since the function \( F_{H}(h) \) is obviously nonnegative and nondecreasing on \( [h_{m}(H); +\infty) \) there exists \( \lim_{h \to h_{m}(H) + 0} F_{H}(h) = F_{H}(h_{m}(H)) \). This, (72) and (77) imply that there exists \( \lim_{h \to h_{m}(H) + 0} \lambda^{*}(h)(h - h_{m}(H)) = C < +\infty \) and that
\[
\lim_{h \to h_{m}(H) + 0} F_{H}(h) = C + \log d = C + F_{H}(h_{m}(H)).
\]
Thus to prove right continuity of the function $F_H(h)$ at the point $h_m(H)$ it is sufficient to show that $C = 0$. This can be done by proving that
\[
\int_{h_m(H)}^{h''} \lambda^*(h)dh = \lim_{h' \to h_m(H)+0} \int_{h'}^{h''} \lambda^*(h)dh < +\infty,
\]
for some $h'' > h_m(H)$. Indeed, finiteness of this integral and the assumption $C > 0$ imply finiteness of the integral $\int_{h_m(H)}^{h''} (h - h_m(H))^{-1}dh$.

It is easy to see that
\[
\frac{dP(h)}{dh} = -h \frac{d\lambda^*(h)}{dh}
\]
and hence
\[
- \frac{d\lambda^*(h)}{dh}(h - h_m(H)) = \frac{dQ(h)}{dh}.
\]
By direct integration we obtain
\[
Q(h'') - Q(h') = \lambda^*(h')(h' - h_m(H)) - \lambda^*(h'') (h'' - h_m(H)) + \int_{h'}^{h''} \lambda^*(h)dh.
\]
This and the mentioned before existence of $\lim_{h' \to h_m(H)+0} Q(h') = \log d$ and of $\lim_{h' \to h_m(H)+0} \lambda^*(h')(h' - h_m(H)) = C < +\infty$ imply (78).

By the above observation
\[
\frac{F_H(h) - F_H(h_m(H))}{h - h_m(H)} \geq \lambda^*(h), \quad \forall h > h_m(H),
\]
and hence (76) implies $\frac{dF_H(h)}{dh}|_{h=h_m(H)+0} = +\infty$.

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