Numerical Solution of Nonlinear Abel Integral Equations: An $hp$-Version Collocation Approach

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Abstract

This paper is concerned with the numerical solution for a class of nonlinear weakly singular Volterra integral equation of the first kind. The existence and uniqueness issue of the nonlinear Abel integral equations is studied completely. An $hp$-version collocation method in conjunction with Jacobi polynomials is introduced so as an appropriate numerical solution to be found. We analyze it properly and find an error estimation in $L^2$-norm. The efficiency of the method is illustrated by some numerical experiments.

Keywords: nonlinear operator, first kind Volterra integral equation, weakly singular operator, $hp$-version collocation method, error analysis.

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2 Introduction

This paper deals with the numerical solution of the nonlinear weakly Volterra integral equation of the first kind

$$Ku(t) := \int_0^t (t - s)^{\alpha - 1} \kappa(s, t, u(s))ds = f(t), \quad 0 < \alpha \leq 1, \quad 0 \leq t \leq T < \infty. \quad (1)$$

This equation could be expressed by an operator notation in a more general form

$$Ku(t) := \int_0^t (t - s)^{\alpha - 1} \kappa(s, t, u(s))ds = f(t), \quad t \in \Omega := [0, T], \quad (2)$$

where the operator $Ku$ is nonlinear Abel integral equation.

From Niels Henrik Abel’s endeavor to generalize the tautochrone problem till today applications of Abel integral equations such as the practical physical models originating from spectroscopy, astrophysics (cf. [16], [30], [31]) and inverse problems arising in image reconstruction [15], there is a long way which signifies the importance of these equations. In addition, the fractional differential (FD) operators in one dimensional domains are mainly defined by these operators; so, the surge of FD equations in research and application is a leading motivation for further research in the numerical solutions of the generalized Abel integral equations.

The numerical discretization of Abel integral equation in general form is studied in many researches and a variety of the methods are suggested. These methods can be categorized into global and local

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ones in which the approximation is provided on the whole domain and the grid points, respectively. The global approaches include the piecewise polynomial discontinuous Galerkin method [9], collocation [6], (quadrature) discontinuous Galerkin [20] and full discontinuous Galerkin methods [21] to solve the linear case of Eq. (1). Furthermore, the global order of convergence for the collocation method in the space of piecewise polynomials of degree \( m \geq 0 \), with jump discontinuities on the set of knots is studied in [7, 8].

The product integration and adaptive Huber methods can solve Eq. (1) locally [10, 3]. A Nyström-type method based on the trapezoidal and composite trapezoidal rules is analyzed for the Abel integral equation in [14] and [27], respectively. The majority of the aforementioned schemes are utilized for linear Abel integral equations. In [1], Branca utilizes the interpolation quadrature technique with linear and quadratic polynomials in order to approximate the nonlinear Abel integral equation. Here, we mainly attempt to develop this idea in conjunction with global methods to approximate the numerical solution of nonlinear Abel integral equation.

In above mentioned works, to obtain an efficient approximation, one should increase the number of mesh points (\( h \)-version) or the degree of polynomials in the expansion (\( p \)-version). In order to employ both beneficial features of \( h \)- and \( p \)-versions simultaneously, we investigate the \( hp \)-version collocation method to achieve an appropriate solution for nonlinear Abel integral equations. For this aim, it is necessary to express the existence and uniqueness of the solution which is investigated properly in this paper.

Due to efficiency and accuracy, the \( hp \)-version Galerkin and collocation methods have received considerable attentions. For example, the \( hp \)-version of discontinuous Galerkin and Petrov-Galerkin have been studied for integro-differential equations of Volterra types (for more details see [24, 35]). Sheng et al. have introduced a multi-step Legendre-Gauss spectral collocation method and given a comprehensive analysis of convergence in \( L^2 \)-norm for the nonlinear Volterra integral equations of the second kind [28]. This approach has been extended to the Volterra integral and integro-differential equations with vanishing delays [34, 29]. Locally varying time steps makes these methods popular for investigating the numerical solution of integral equations with weakly singular kernels [32]. Recently, the \( hp \)-version collocation method is studied for nonlinear Volterra integral equation of the first kind [25] which is our starting point to develop the results for the nonlinear Abel integral equations.

An important aspect of this method is its flexibility with respect to the step size and the order of polynomials in each sub-interval. As we will identify in the numerical experiments, the proposed collocation method works well for the approximation of the equations with non-smooth solutions. Two main features of this paper are as follows:

1. Developing the \( hp \)-version collocation method for the weakly singular integral equations of the first kind is one of the aspects of the present manuscript. In accordance with the heuristic of the scheme which converts the first kind integral equation into the second one, without any restriction on the kernel, the discretized equation becomes more regular. Not only does the scheme overcome the difficulty of the nonlinear term in these equations which are rarely investigated, but also the local viewpoint of scheme makes it a powerful tool for better approximating, specially when the unknown solution is non-smooth.

2. Jacobi polynomials are utilized to derive an exact quadrature formula for the integrals with weakly singular integrand which can be accounted as a merit of these polynomials. Being adjustable, the parameters \( M \) and \( N \), which are related to \( h \)- and \( p \)-versions, give the scheme rise to more accuracy and applicability. Furthermore, the error estimation for the presented schemes is analyzed in the sense of \( L^2 \)-norms and the numerical results are compatible with these findings.

This paper is organized in the following way. In Section 3 we give some regularity results for the nonlinear Abel integral equation of the first kind. Section 4 is devoted to the description of the \( hp \)-version collocation method for the first kind weakly singular nonlinear Volterra integral equation. In Section 5 an error analysis of the proposed method is provided in some suitable Hilbert spaces. Finally, in order to show the applicability and efficiency of the method and compare it with other methods, several examples with smooth and non-smooth solutions are illustrated in Section 6.
3 Theoretical treatment of the problem

In order to find out a suitable numerical scheme for the solution of Eq. (1), knowledge of the behavior of the exact solution is important. The kernel of the integral equation contains a singular term, for the reason that weighted Lebesgue spaces are utilized as the suitable functional spaces. For this aim, let us define the weight function \( \chi^{\alpha,\beta}(x) := (1-x)^\alpha(1+x)^\beta \) on the interval \( \Lambda := [-1, 1] \) for \( \alpha, \beta > -1 \). For \( r \in \mathbb{N} \), \( H_{r,\chi}^{\alpha,\beta}(\Lambda) \) is a weighted Sobolev space defined by

\[
H_{r,\chi}^{\alpha,\beta}(\Lambda) = \left\{ v : v \text{ is measurable and } \| v \|_{r,\chi^{\alpha,\beta}} < \infty \right\},
\]

where

\[
\| v \|_{r,\chi^{\alpha,\beta}} = \left( \sum_{k=0}^{r} |v|_{k,\chi^{\alpha,\beta}}^2 \right)^{\frac{1}{2}}.
\]

The above semi-norm is defined as \( |v|_{k,\chi^{\alpha,\beta}} = ||\partial^k v||_{\chi^{\alpha+r,\beta+r}} \), where \( \| . \|_{\chi^{\alpha,\beta}} \) is an appropriate norm for the space \( L_2^{\alpha,\beta}(\Lambda) \). For arbitrary real number \( r = [r] + \theta \) with \( \theta \in (0, 1) \), \( H_{r,\chi}^{\alpha,\beta}(\Lambda) \) can be defined by the interpolation space as

\[
H_{r,\chi}^{\alpha,\beta}(\Lambda) = [H_{r,\chi}^{[r+\theta],\alpha}(\Lambda), H_{r,\chi}^{[r+\theta+1],(\Lambda)]}_0.
\]

More details can be seen in [2, 22].

We need some definitions from fractional calculus. The Riemann-Liouville integral operator \( \mathcal{I}_x^r \) is defined as follows

\[
\mathcal{I}_x^r u(x) = \int_0^x (x-t)^{r-1} u(t) \, dt,
\]

and \( \mathcal{D}_x^r \), the Riemann-Liouville fractional derivative of order \( r \) for a function \( u \in H^n(\Omega) \) can be defined as

\[
\mathcal{D}_x^r u = D^n \mathcal{I}_x^{n-r} u,
\]

where the operator \( D^n \) denotes the classical derivative of order \( n \) [12]. In the following theorem, some adequate assumptions are given in order to have a unique solution for (1) in some weighted Sobolev spaces.

**Theorem 1** Assume that the Eq. (1) satisfies the following assumptions

i. \( f(t) \in H_{\chi^{\alpha-1,0}}^m(\Omega), \ f(0) = 0, \)

ii. \( \kappa(s,t) \in C^n(\Omega \times \Omega) \) and \( \kappa(t,t) \not= 0 \) for all \( t \in \Omega, \)

iii. \( \psi(s,u) \in H_{\chi^{\alpha-1,0}}^{m-1}(\Omega \times \mathbb{R}), \)

iv. \( \inf \left\{ \left| \frac{\partial \psi}{\partial u}(s,u) \right| : (s,u) \in \Omega \times \mathbb{R} \right\} \geq M > 0, \)

v. \( \psi(s,u) \) is Lipschitz continuous w.r. to \( u, \)

vi. \( k(t) = \int_0^t \int_0^s (t-y)^{-\alpha}(y-x)^{\alpha-1} \kappa(y,x) \psi(x,u(x)) \, dy \, dx, \) then \( k \in H_{\chi^{\alpha-1,0}}^{m-1}(\Omega). \)

Then it has a unique solution \( u \) in \( H_{\chi^{\alpha-1,0}}^{m-1}(\Omega). \)

**Proof.** In advance, let us convert the main problem [1] to the second kind integral equation by multiplying both sides into \( (x-t)^{-\alpha} \) and taking integration; hence, from the assumption (i), Eq. (1) reads that

\[
\psi(t,u(t)) + \int_0^t \frac{\mathcal{L}_x(t,s,u(s))}{\kappa(t,t)} \, ds = \frac{(\mathcal{D}_x^\alpha f)(t)}{\kappa(t,t)},
\]

(5)
where

$$\mathcal{L}(t, x, u(x)) = \int_x^t (t - y)^{-\alpha}(y - x)^{\alpha-1}\kappa(y, x)\psi(x, u(x)) \, dy.$$  

By the new variable \(y = x + \tau(t - x)\), the operator \(\mathcal{L}\) could be written as

$$\mathcal{L}(t, x, u) = \int_0^1 (1 - \tau)^{-\alpha+\alpha} \kappa(x + \tau(t - x), x)\psi(x, u(x)) \, d\tau.$$  

It is evident that \(\mathcal{L}(x, x, u) = \kappa(x, x)\psi(x, u)\) and

$$\mathcal{L}_t(t, x, u) = \psi(x, u(x)) k^*(t, x),$$  

is continuous on \(\Omega \times \mathbb{R}\) and Lipschitz continuous with respect to \(u\) with the same constant as for \(\psi\) and \(k^*(t, x) = \int_0^1 (1 - \tau)^{-\alpha+\alpha} \kappa(x + \tau(t - x), x) \, d\tau\). Conditions (i)-(iii) lead that each function \(u(t)\) is a solution of Eq. (5) if and only if it is a solution of Eq. (6). In order to prove the existence of a solution for Eq. (6), we trace [13] and define the sequence \(\{u_n(t)\}_{n \in \mathbb{N}}\) as follows:

$$\psi(0, u_0(t)) := \frac{(0D_x^\alpha f)(0)}{\kappa(0, 0)},$$  
$$\psi(t, u_{n+1}(t)) := \frac{(0D_x^\alpha f)(t)}{\kappa(t, t)} - \int_0^t \frac{\mathcal{L}_t(t, s, u_n(s))}{\kappa(t, t)} \, ds, \quad n \geq 1.$$  

By the assumptions (iii) and (iv), the function \(\psi(t, u(t))\) is strictly monotonic continuous with respect to \(u\). So by considering the Inverse Theorem [11, p. 68], \(u_0\) is well-defined and belongs to \(H^{m-1}_{\chi_{n-1,0}}(\Omega)\).

Now, using induction hypothesis, \(u_n\) is well-defined and belongs to \(H^{m-1}_{\chi_{n-1,0}}(\Omega)\). From the assumption (i), it is deduced that \(f' \in H^{m-1}_{\chi_{n-1,0}}([0, x])\). On the other hand, from the fractional calculus we get that

$$\frac{(0D_x^\alpha f)(x)}{\kappa(t, t)} = \int_0^x \frac{f(t)}{(x - t)^\alpha} \, dt = \int_0^x \frac{f'(t)}{(x - t)^\alpha} \, dt + \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{\Gamma(1 + \alpha k)} t^k.$$  

Now, utilizing \(f(0) = 0\) and Theorem 3.1 of the paper [19], it is concluded that \(\int_0^x \frac{f'(t)}{(x - t)^\alpha} \, dt \in H^{m-1}_{\chi_{n-1,0}}(\Omega)\), which is a subset of \(H^{m-1}_{\chi_{n-1,0}}(\Omega)\). From the above argument and the assumption (iv), we deduce that the function

$$\frac{(0D_x^\alpha f)(t)}{\kappa(t, t)} - \int_0^t \frac{\mathcal{L}_t(t, s, u_n(s))}{\kappa(t, t)} \, ds,$$  

belongs to \(H^{m-1}_{\chi_{n-1,0}}(\Omega)\). Hence, by the Inverse Theorem, \(u_{n+1} \in H^{m-1}_{\chi_{n-1,0}}(\Omega)\). Using the assumptions (iv) and (v), one can conclude that

$$|u_{n+1}(t) - u_n(t)| \leq \frac{(JL)^n}{M} \max_{s \in \Omega} |u_1(s) - u_0(s)|,$$  

where \(L\) is the Lipschitz constant in the assumption (iv) and \(J := \max\{|\frac{k(t, s)}{\kappa(t, t)}| \mid (t, s) \in \Omega \times \Omega\}\). Therefore, without loss of generality for \(m > n\),

$$|u_m(t) - u_n(t)| \leq \sum_{i=n}^{m-1} |u_{i+1}(t) - u_i(t)| \leq \|u_1(t) - u_0(t)\| \sum_{i=n}^{m-1} \left(\frac{JLT}{M}\right)^{\frac{1}{i}}.$$  

The term \(\sum_{i=0}^{\infty} \left(\frac{JLT}{M}\right)^{\frac{1}{i}}\) is convergent, so the Cauchy sequence \(\{u_n\}\) is convergent uniformly to

$$\lim_{n \to \infty} u_n(t) = u(t),$$  

where \(u(t)\) belongs to \(H^{m-1}_{\chi_{n-1,0}}(\Omega)\). This result follows from the fact that \(u_n(t) \in H^{m-1}_{\chi_{n-1,0}}(\Omega)\).
4 Numerical scheme

In this section, we introduce an $hp$-version Jacobi collocation method for nonlinear weakly singular integral equations of the first kind. In order to get a self-contained paper, some basic properties of the shifted Jacobi-Gauss and Legendre-Gauss-Lobatto polynomial interpolations are introduced in the following subsection.

4.1 Preliminaries

The shifted Jacobi-Gauss interpolation operator. Let us denote the standard Jacobi polynomial of degree $k$ by $J_{k}^{\alpha,\beta}(x)$, for $\alpha, \beta > -1$. It is well-known that the set of Jacobi polynomials makes a complete orthogonal system with respect to the weight function $\chi_{\alpha,\beta}(x)$ which means that

$$\int_{\Lambda} J_{k}^{\alpha,\beta}(x) J_{j}^{\alpha,\beta}(x) \chi_{\alpha,\beta}(x) dx = \gamma_{k}^{\alpha,\beta} \delta_{k,j},$$

(8)

wherein $\delta_{k,j}$ is the Kronecker function, and

$$\gamma_{k}^{\alpha,\beta} = \begin{cases} \frac{2^{\alpha+1} \Gamma(\alpha+1) \Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}, & k = 0, \\ \frac{2^{\alpha+1} \Gamma(k+\alpha+1) \Gamma(k+\beta+1)}{\Gamma(k+\alpha+\beta+1) k!}, & k \geq 1. \end{cases}$$

In order to work with these polynomials on the sub-intervals $\Omega_{n}$ properly, the shifted Jacobi polynomial of degree $k$ is also defined as follows

$$J_{n,k}^{\alpha,\beta}(t) = J_{k}^{\alpha,\beta}(\frac{2t - t_{n-1} - t_{n}}{h_{n}}), \quad t \in \Omega_{n}, \quad k \geq 0.$$  

(9)

It is worth to mention that the set of shifted Jacobi polynomials constructs a complete orthogonal system with the weight function $\chi_{n,\alpha,\beta}(t) = (t_{n} - t)^{\alpha}(t - t_{n-1})^{\beta}$. Similar to the relation (8), we can write

$$\int_{\Omega_{n}} J_{n,k}^{\alpha,\beta}(t) J_{n,j}^{\alpha,\beta}(t) \chi_{n}^{\alpha,\beta}(t) dt = \left(\frac{h_{n}}{2}\right)^{\alpha+\beta+1} \gamma_{k}^{\alpha,\beta} \delta_{k,j}. \quad (10)$$

Let $x_{n,j}^{\alpha,\beta}$ be the zeros of the standard Jacobi polynomial of degree $k$ for $0 \leq j \leq M_{n}$ and $\omega_{n,j}^{\alpha,\beta}$ be the corresponding Christoffel numbers. Then we can define the shifted Jacobi-Gauss quadrature points on the interval $\Omega_{n}$ as follows

$$t_{n,j}^{\alpha,\beta} = \frac{1}{2} (h_{n} x_{n,j}^{\alpha,\beta} + t_{n-1} + t_{n}), \quad 0 \leq j \leq M_{n}. \quad (11)$$

Let $P_{M}(\Omega)$ be the set of all polynomials of degree at most $M$ on $\Omega$. It is known from [2, 17] that for any $\phi(t) \in P_{2M+1}(\Omega_{n})$

$$\int_{\Omega_{n}} \phi(t) \chi_{n}^{\alpha,\beta}(t) dt = \left(\frac{h_{n}}{2}\right)^{\alpha+\beta+1} \sum_{j=0}^{M_{n}} \phi(t_{n,j}^{\alpha,\beta}) \omega_{n,j}^{\alpha,\beta},$$

(12)

which leads to the result

$$\sum_{j=0}^{M_{n}} J_{n,p}^{\alpha,\beta}(t_{n,j}^{\alpha,\beta}) J_{n,q}^{\alpha,\beta}(t_{n,j}^{\alpha,\beta}) \omega_{n,j}^{\alpha,\beta} = \gamma_{p}^{\alpha,\beta} \delta_{p,q}. \quad (13)$$

For any $v \in C(\Omega_{n})$, the shifted Jacobi-Gauss interpolation operator in the $t$-direction is defined as follows

$$I_{t,M_{n}}^{\alpha,\beta} v(t_{n,j}^{\alpha,\beta}) = v(t_{n,j}^{\alpha,\beta}), \quad 0 \leq j \leq M_{n}. \quad (14)$$
The shifted Legendre-Gauss interpolation. Let \( L_p(t) \) be defined as

\[
L_p(t) = \begin{cases} 
  l_p(t), & t \in \Lambda, \\
  0, & \text{otherwise},
\end{cases}
\]

where \( l_p(t) \) is the Legendre polynomial of degree \( p \). Therefore, the shifted Legendre polynomials of degree \( p \) over the subinterval \( \Omega_n \) are defined as

\[
L_{n,p}(t) = L_p\left(\frac{2t - t_{n-1} - t_n}{h_n}\right), \quad t \in \Omega_n.
\]

In the previous definition about the Jacobi interpolation operator and its properties, if we take \( \alpha = \beta = 0 \), the shifted Legendre-Gauss interpolation \( I_{t,M_n} \) can be defined as \( I_{0,0,t,M_n} \). For instance, in Eqs. (10), (12) and (13), we have

\[
\int_{\Omega_n} L_{n,p}(t)L_{n,q}(t)dt = \frac{h_n}{2p+1}\delta_{p,q}, \quad (15)
\]

\[
\int_{\Omega_n} \phi(t)dt = \frac{h_n M_n}{2} \sum_{j=0}^{M_n} \phi(t_{n,j})w_{n,j}, \quad (16)
\]

and

\[
\sum_{j=0}^{M_n} L_{n,p}(t_{n,j})L_{n,q}(t_{n,j})w_{n,j} = \frac{2}{2p+1}\delta_{p,q}, \quad (17)
\]

where \( \{t_{k,i}, w_{k,i}\}_{i=0}^{M_k} \) are the shifted Legendre-Gauss quadrature nodes and weights. These shifted functions form a complete orthogonal set for \( L^2(\Omega_n) \) functions, i.e., for any function \( g \in L^2(\Omega_n) \), it can be represented as

\[
g(t) = \sum_{p=1}^{\infty} \hat{g}_p L_{n,p}(t).
\]

Therefore, the operator \( I_{t,M_n}^t g(t) \) is stated by

\[
I_{t,M_n}^t g(t) = \sum_{p=0}^{M_n} \hat{g}_p L_p(t), \quad (18)
\]

where the coefficients \( \hat{g}_p \) can be obtained by means of the orthogonality property of Legendre polynomials as

\[
\hat{g}_p = \frac{2p+1}{2} \int_{\Omega_n} g(t)L_{n,p}(t)dt.
\]

The shifted Legendre-Gauss-Lobatto interpolation. Let \( \{x_{n,j}^L, w_{n,j}^L\}_{j=0}^{M_n} \) be the nodes and Christoffel numbers of the standard Legendre-Gauss-Lobatto interpolation on \( \Lambda \). The corresponding nodes of this interpolation on \( \Omega_n \) can be defined by

\[
s_{n,j}^L = \frac{h_n x_{n,j}^L + t_n + t_{n-1}}{2}.
\]

For the definition of the shifted Legendre-Gauss-Lobatto interpolation \( I_{s,M_n}^L \), it is easily observed that \( I_{s,M_n}^L v \in \mathcal{P}_{M_n}(\Omega_n) \) and \( I_{s,M_n}^L v(s_{n,j}^L) = v(s_{n,j}^L) \). For any function \( \phi \in \mathcal{P}_{2M_n+1}(\Lambda) \), the following identities are deduced from the main property of Legendre-Gauss-Lobatto quadrature,

\[
\int_{\Omega_n} \phi(s)ds = \frac{h_n}{2} \sum_{j=0}^{M_n} w_{n,j}^L \phi(s_{n,j}^L),
\]

and

\[
\sum_{j=0}^{M_n} L_{n,p}(s_{n,j}^L)L_{n,q}(s_{n,j}^L)w_{n,j}^L = \frac{2}{2p+1}\delta_{p,q}.
\]
Due to the presence of the weakly singular term \((t - s)^{\alpha - 1}\) in the main problem \([1]\), the weighted interpolatory quadrature formulae are utilized in the approximation procedure. For a function \(\phi \in \mathcal{P}_{M_k}(\Omega_k)\), the weighted quadrature formula is interpreted as \[33\]

\[
\int_{\Omega_k} (t - s)^{\alpha - 1} \phi(s) ds = \sum_{j=0}^{M_k} \tilde{w}_{k,j}^L(t) \phi(s_{k,j}), \quad t \in \Omega_n, \quad k < n,
\]

where \(\tilde{w}_{k,j}^L(t) = \int_{\Omega_k} (t - s)^{\alpha - 1} l_{k,j}(s) ds\) and \(\{l_{k,j}(s)\}_{j=0}^{M_k}\) are Lagrange polynomials associated with the collocation points \(\{s_{k,j}^L\}_{j=0}^{M_k}\).

### 4.2 The hp-collocation method for weakly singular integral equations

For a fixed integer \(N\), let \(\Omega_h := \{t_n : 0 = t_0 < t_1 < \cdots < t_N = T\}\) be as a mesh on \(\Omega\), \(h_n := t_n - t_{n-1}\) and \(h_{\text{max}} = \max_{1 \leq n \leq N} h_n\). Moreover, denote \(u^n(t)\) as the solution of Eq. \([1]\) on the \(n\)-th sub-interval of \(\Omega\), namely,

\[u^n(t) = u(t), \quad t \in \Omega_n := (t_{n-1}, t_n], \quad n = 1, 2, \ldots, N.\]

By the above mesh, we rewrite the Eq. \([1]\) as

\[
\int_{t_n-1}^{t} (t - s)^{\alpha - 1} \kappa(s,t) \psi(s,u(s)) ds + \int_{t_n-1}^{t} (t - s)^{\alpha - 1} \kappa(s,t) \psi(s,u(s)) ds = f(t),
\]

then for any \(t \in \Omega_n\), this equation can be written as

\[
\int_{t_n-1}^{t} (t - t_n)^{\alpha - 1} \kappa(\tau,t) \psi(\tau,u^n(\tau)) d\tau = f(t) - \sum_{k=1}^{n-1} \int_{\Omega_k} (t - s)^{\alpha - 1} \kappa(s,t) \psi(s,u^k(s)) ds.
\]

Now, we transfer the interval \((t_{n-1}, t)\) to \(\Omega_n\) by the following linear transform

\[
\tau = \sigma(\lambda,t) := t_{n-1} + \frac{(\lambda - t_{n-1})(t - t_{n-1})}{h_n},
\]

(21)

to get

\[
\left(\frac{t - t_{n-1}}{h_n}\right)^{\alpha} \int_{\Omega_n} (t_{n-\lambda})^{\alpha - 1} \kappa(\sigma(\lambda,t),t) \psi(\sigma(\lambda,t),u^n(\sigma(\lambda,t))) d\lambda = f(t)
\]

\[
- \sum_{k=1}^{n-1} \int_{\Omega_k} (t - s)^{\alpha - 1} \kappa(s,t) \psi(s,u^k(s)) ds.
\]

(22)

In the following, we mention some requirements considered in the next section. Let \(T_{\lambda,M_n}^{\alpha - 1,0} : C(\Omega_n) \rightarrow \mathcal{P}_{M_n}(\Omega_n)\) be the Jacobi-Gauss interpolation operator. Now, we define a new Legendre-Gauss interpolation operator \(T_{\gamma,M_n}^{\alpha - 1,0} : C(t_{n-1}, t) \rightarrow \mathcal{P}_{M_n}(t_{n-1}, t)\) owing to the relation \([1]\) with the following property

\[
T_{\gamma,M_n}^{\alpha - 1,0} g(\tau_{n,i}) = g(\tau_{n,i}), \quad 0 \leq i \leq M_n,
\]

where \(\tau_{n,i} := \tau_{n,i}(x) = \sigma(\lambda_{n,i}, t)\) and \(\lambda_{n,i}\) are the \(M_n + 1\) Jacobi-Gauss quadrature nodes in \(\Omega_n\). Clearly,

\[
T_{\gamma,M_n}^{\alpha - 1,0} g(\tau_{n,i}) = g(\sigma(\lambda_{n,i}, t)) = T_{\lambda,M_n}^{\alpha - 1,0} g(\sigma(\lambda_{n,i}, t)), \quad 0 \leq i \leq M_n,
\]
and by Eq. (12), we get
\[
\int_{t_{n-1}}^{t} (t - \tau)^{\alpha - 1} I_{\tau,M_n}^{\alpha - 1,0} g(\tau) d\tau = \left( \frac{t - t_{n-1}}{h_n} \right)^\alpha \int_{\Omega_n} (t_n - \lambda)^{\alpha - 1} I_{\lambda,M_n}^{\alpha - 1,0} g(\sigma(\lambda,t)) d\lambda
\]
\[
= \left( \frac{t - t_{n-1}}{2} \right)^\alpha \sum_{j=0}^{M_n} g(\sigma(\lambda_{n,j},t)) w_{n,j}
\]
\[
= \left( \frac{t - t_{n-1}}{2} \right)^\alpha \sum_{j=0}^{M_n} g(\tau_{n,j}) w_{n,j}.
\] (23)

Meanwhile, it is noticed that
\[
\int_{t_{n-1}}^{t} (t - \tau)^{\alpha - 1} (I_{\tau,M_n}^{\alpha - 1,0} g(\tau))^2 d\tau = \left( \frac{t - t_{n-1}}{2} \right)^\alpha \sum_{j=0}^{M_n} g^2(\tau_{n,j}) w_{n,j}.
\] (24)

These equations will be valid for the Legendre interpolation operator $I_{\tau,M_n}^{t}$, if we take $\alpha = 1$ and $t = t_n$.

### 4.2.1 The $hp$-version of Jacobi-Gauss collocation method

In order to seek the solution $u_{M_n}^n(t) \in \mathcal{P}_{M_n}(\Omega_n)$ of Eq. (22) by $hp$-collocation method, at the first step this equation is fully discretized as
\[
I_{\tau,M_n}^t \left( \left( \frac{t - t_{n-1}}{h_n} \right)^\alpha \int_{\Omega_n} (t_n - \lambda)^{\alpha - 1} I_{\lambda,M_n}^{\alpha - 1,0} \kappa(\sigma(\lambda,t),t) \psi(\sigma(\lambda,t), u_{M_n}^n(\sigma(\lambda,t))) d\lambda \right) = I_{\tau,M_n}^t (f(t)) - I_{\tau,M_n}^t \left( \sum_{k=1}^{n-1} \int_{\Omega_k} (t - s)^{\alpha - 1} I_{s,M_k}^{\alpha - 1,0} \kappa(s, t) \psi(s, u_{M_k}^n(s)) ds \right), \quad t \in \Omega_n,
\] (25)

where
\[
I_{\tau,M_n}^t u_{M_n}^n(t) = u_{M_n}^n(t) = \sum_{p=0}^{M_n} \hat{a}_p L_{n,p}(t),
\]
\[
\sum_{p=0}^{M_n} \sum_{q=0}^{M_k} \hat{b}_{pq} L_{n,p}(t) = \sum_{p=0}^{M_n} \sum_{q=0}^{M_k} \hat{b}_{pq} L_{n,p}(t) = \sum_{p=0}^{M_n} \sum_{k=1}^{M_k} \hat{b}_{kp} L_{n,p}(t)
\] (26)

and
\[
I_{\tau,M_n}^t f(t) = \sum_{p=0}^{M_n} \hat{f}_p L_{n,p}(t).
\] (27)
Then, we get
\[
\int_{\Omega_n} \frac{(t_n - \lambda)^{\alpha - 1}}{h_n} I_{M_n} I_{\lambda, M_n} \left( \left( \frac{t - t_n - 1}{h_n} \right)^\alpha \kappa(\sigma(\lambda, t), t) \psi(\sigma(\lambda, t), u_{M_n}(\sigma(\lambda, t))) \right) d\lambda
\]
\[
= \int_{\Omega_n} \frac{(t_n - \lambda)^{\alpha - 1}}{h_n} \sum_{p,q=0}^{M_n} a_{pq}^n L_{n,p}(t) J_{n,q}^{\alpha - 1,0}(\lambda) d\lambda
\]
\[
= \sum_{p=0}^{M_n} \hat{a}_{p0}^n L_{n,p}(t)
\]

It is evident from Eqs. (26)-(28) that
\[
\hat{a}_{p0}^n = \frac{2p + 1}{2} \sum_{i=0}^{M_n} w_{n,i}^n L_{n,p}(t) w_{n,i},
\]
\[
\hat{a}_{p0}^n = \frac{2p + 1}{2^{1+\alpha}} \sum_{i=0}^{M_n} (t_n - t - 1)^\alpha \kappa(\sigma(t_n - 1, t, t_n), t_n) \psi(\sigma(t_n - 1, t, t_n), u_{M_n}(\sigma(t_n - 1, t, t_n)))
\]
\[
L_{n,p}(t) w_{n,i} w_{n,j}^{\alpha - 1,0},
\]
\[
\hat{b}_{pq}^k = \frac{2p + 1}{2} \sum_{i=0}^{M_n} \hat{w}_{k,q}^L(t, t_n) \kappa(t_{k,q}, t_n) \psi(t_{k,q}^L, u_{M_n}(t_{k,q})) L_{n,p}(t) w_{n,i},
\]
\[
\hat{f}_{p}^n = \frac{2p + 1}{2} \sum_{i=0}^{M_n} f_{M_n}^n(t) L_{n,p}(t) w_{n,i}.
\]

With Eqs. (26)-(28), the equation (25) reads
\[
\sum_{p=0}^{M_n} \hat{a}_{p0}^n L_{n,p}(t) = \sum_{p=0}^{M_n} \hat{f}_{p}^n L_{n,p}(t) + \sum_{p=0}^{M_n} \hat{b}_{pq}^k L_{n,p}(t),
\]
where
\[
\hat{b}_{pq}^k = \sum_{k=1}^{M_n} \sum_{q=0}^{M_n} b_{pq}^k.
\]

Consequently, we compare the coefficients to obtain
\[
\hat{a}_{p0}^n = \hat{f}_{p}^n + \hat{b}_{pq}^k, \quad 0 \leq p \leq M_n.
\]

To evaluate the unknown coefficients \(u_{n}^n\) for any given \(n\), we solve the nonlinear system (30) with the Newton iteration method. Finally, the approximate solution can be obtained as
\[
u_{M}^N(t) = \sum_{n=1}^{N} \sum_{p=0}^{M_n} u_{n,p}^n L_{n,p}(t).
\]

It is worth to notice that for the linear case of Eq. (1), the unknown coefficients \(\hat{u}_{n}^n\) for any given \(n\) can be obtained by the following linear system of equations
\[
Au = b + c,
\]
where the entries of the matrix $A = [a_{p,q}]_{p,q=0}^{M}$ are defined by

$$a_{p,q} = \frac{2p + 1}{2^{1+\alpha}} \sum_{i,j=0}^{M} (t_{n,i} - t_{n,j})^{\alpha} \kappa(\sigma(t_{n,j}, t_{n,i}), t_{n,i}) L_{n,q}(\sigma(t_{n,j}, t_{n,i})) L_{n,p}(t_{n,i}) w_{n,i} u_{n,j}^{\alpha-1,0},$$

and

$$u = (\hat{u}_{0}^{n}, \ldots, \hat{u}_{M}^{n})^{T}, \quad b = (\hat{b}_{0}^{n}, \ldots, \hat{b}_{M}^{n})^{T}, \quad c = (\hat{c}_{0}^{n}, \ldots, \hat{c}_{M}^{n})^{T}.$$

## 5 Error analysis

This section is devoted to the analysis of the introduced numerical scheme. Throughout this paper, we denote $\|\cdot\|_{D}$ as $L^2$-norm on the interval $D$ and $M_{\min} = \min_{1 \leq n \leq N} M_{n}$.

**Lemma 2** ([18]) (Grönwall inequality) Assume that there are numbers $\alpha, \beta \geq 0$ ($l = 0, 1, \ldots, n-1$) and $0 \leq M_{o} < 1$ such that

$$0 \leq \varepsilon_{n} \leq \alpha + \sum_{l=0}^{n-1} \beta_{l} \varepsilon_{l} + M_{o} \varepsilon_{n}, \quad n \geq 1.$$

Then the quantities $\varepsilon_{n}$ fulfill the following estimate for $n \geq 0$

$$\varepsilon_{n} \leq \frac{\alpha}{1 - M_{o}} \exp \left( \sum_{l=0}^{n-1} \frac{\beta_{l}}{1 - M_{o}} \right).$$

In the following, some theoretical results regarding the convergence of the method are expressed from [33].

**Lemma 3** For any $v \in H_{\chi_{n}, \beta}(\Omega_{n})$ with integer $1 \leq m \leq M_{n} + 1$ and $\alpha, \beta > -1$, we get

$$\|v - T_{x,M_{n}}^{\alpha,\beta} v\|_{\chi_{n}, \beta} \leq c \sqrt{\frac{\Gamma(M_{n} + 2 - m)}{\Gamma(M_{n} + 2 + m)}} \|\partial_{x}^{m} v\|_{\chi_{n}, \beta + m}.$$

In particular, for any fixed $m$, we obtain

$$\|v - T_{x,M_{n}}^{\alpha,\beta} v\|_{\chi_{n}, \beta} \leq c (M_{n} + 1)^{-m} \|\partial_{x}^{m} v\|_{\chi_{n}, \beta + m} \leq c h_{n}^{m} (M_{n} + 1)^{-m} \|\partial_{x}^{m} v\|_{\chi_{n}, \beta}.$$

**Lemma 4** For any $v \in H^{m}(\Omega_{n})$ with integer $1 \leq m \leq M_{n} + 1$, we get

$$\|v - I_{M_{n}}^{L} v\|_{\Omega_{n}} \leq c h_{n}^{m} (M_{n} + 1)^{-m} \|\partial_{x}^{m} v\|_{\Omega_{n}}.$$

**Lemma 5** For any $v \in H^{m}(\Omega_{n})$ with integer $1 \leq m \leq M_{n} + 1$, we get

$$\|v - I_{M_{n}}^{L} v\|_{\Omega_{n}} \leq c h_{n}^{m} (M_{n} + 1)^{-m} \|\partial_{x}^{m} v\|_{\Omega_{n}}.$$

**Theorem 6** Let $u^{n}$ be the solution of Eq. (22) under the hypothesis of Theorem 1 and $u_{n}^{M_{o}}$ be the solution of Eq. (23). According the assumptions of Theorem 1 the function $\psi(\cdot, u)$ fulfills the Lipschitz condition with respect to the second variable, i.e.,

$$|\psi(\cdot, u_{1}) - \psi(\cdot, u_{2})| \leq \gamma |u_{1} - u_{2}|, \quad \gamma \geq 0.$$

Then, for any $1 \leq n \leq N$ and $m \leq M_{\min} + 1$,

$$B_{1} = B_{2} + B_{3},$$
with

\[
\|B_1\|_{\Omega_n}^2 \leq ch_n T^{2\alpha-1} \left( h_k^{2m} (M_k + 1)^{-2m} \| \partial_x^m \psi(s, u^k(s)) \|_{\Omega_n}^2 + \gamma^2 (\| e_k \|_{\Omega_k}^2 + h_k^{2m-1} (M_k + 1)^{-2m} \| \partial_x^m u \|_{\Omega_k}^2) \right)
+ ch_n^2 (M_n + 1)^{-2m} \| \partial^m f \|_{\Omega_n}^2,
\]

where

\[
B_1 = T_{M_n}^{1} \left( \frac{t-t_n}{h_n} \right)^{\alpha} \int_{\Omega_n} (t_n - \lambda)^{\alpha-1} \left( \mathcal{T}_{M_n}^{\alpha,1} (\kappa(\sigma(\lambda, t), u_{M_n}(\sigma(\lambda, t)))) - \kappa(\sigma(\lambda, t), \psi(s, t)) \right) d\lambda,
\]

\[
B_2 = f(t) - T_{M_n}^{1} f(t),
\]

\[
B_3 = \sum_{k=1}^{n-1} T_{M_k}^{1} \left( \int_{\Omega_k} (t - s)^{\alpha-1} (\kappa(s, t) \psi(s, u^k(s)) - \mathcal{T}_{s,M_k}^{L} (\kappa(s, t) \psi(s, u^k_M(s)))) ds \right),
\]

and \( e_k = u^k - u^k_{M_k} \) for \( 1 \leq k \leq N \).

**Proof.** Regarding Eq. (22), we have

\[
\mathcal{T}_{M_n}^{1} \left( \frac{t-t_n}{h_n} \right)^{\alpha} \int_{\Omega_n} (t_n - \lambda)^{\alpha-1} \kappa(\sigma(\lambda, t), t) \psi(s, t, u_{M_n}(\sigma(\lambda, t))) d\lambda = T_{M_n}^{1} (f(t))
\]

\[
- T_{M_n}^{1} \left( \sum_{k=1}^{n-1} \int_{\Omega_k} (t - s)^{\alpha-1} \kappa(s, t) \psi(s, u^k(s)) ds \right).
\]

By subtracting (25) from the above equation, we have

\[
B_1(x) = B_2(x) + B_3(x),
\]

where the above terms are defined by (33). In order to obtain an estimation for the term \( B_1 \), we need error bounds for \( \|B_i\|, i = 2, 3 \). First using Lemma 3, we infer that

\[
\|B_2\|_{\Omega_n}^2 = \| f(t) - T_{M_n}^{1} f(t) \|_{\Omega_n}^2 \leq ch_n^2 (M_n + 1)^{-2m} \| \partial^m f \|_{\Omega_n}^2.
\]

To seek an upper bound for \( \|B_3\| \), let us define,

\[
\kappa(s, t) \psi(s, u^k(s)) - \mathcal{T}_{s,M_k}^{L} (\kappa(s, t) \psi(s, u^k(s))) = \kappa(s, t) \psi(s, u^k(s)) - \mathcal{T}_{s,M_k}^{L} (\kappa(s, t) \psi(s, u^k(s)))
\]

\[
+ \mathcal{T}_{s,M_k}^{L} (\kappa(s, t) \psi(s, u^k(s))) - \mathcal{T}_{s,M_k}^{L} (\kappa(s, t) \psi(s, u^k_M(s)))
\]

\[
: = \xi(s, t) + \eta(s, t).
\]
\[
\|B_3\|^2 = \left\| \mathcal{I}_{M_n} \left( \sum_{k=1}^{n-1} \int_{\Omega_k} (t-s)^{\alpha-1} \left( \kappa(s,t) \psi(s,u^k(s)) - \mathcal{I}_{s,M_k}^L (\kappa(s,t) \psi(s,u_{M_k}^k(s))) \right) ds \right) \right\|^2 \\
\quad = \left\| \mathcal{I}_{M_n} \left( \sum_{k=1}^{n-1} \int_{\Omega_k} (t-s)^{\alpha-1} \left( \xi(s,t) + \eta(s,t) \right) ds \right) \right\|^2 \\
\quad = \int_{\Omega_n} \left[ \mathcal{I}_{M_n} \left( \sum_{k=1}^{n-1} \int_{\Omega_k} (t-s)^{\alpha-1} \left( \xi(s,t) + \eta(s,t) \right) ds \right) \right]^2 dt \\
\quad = \frac{h_n}{2} \sum_{j=0}^{M_n} w_{n,j} \left( \sum_{k=1}^{n-1} \int_{\Omega_k} (t_{n,j} - s)^{\alpha-1} \left( \xi(s,t_{n,j}) + \eta(s,t_{n,j}) \right) ds \right)^2 \\
\quad \leq \frac{h_n}{2} \sum_{j=0}^{M_n} w_{n,j} \left( \int_0^{t_{n-1}} (t_{n,j} - s)^{2\alpha-2} ds \right) \left( \int_0^{t_{n-1}} \left( \xi(s,t_{n,j}) + \eta(s,t_{n,j}) \right)^2 ds \right) \\
\quad \leq c_n h_n T^{2\alpha-1} \sum_{j=0}^{M_n} \left( \int_0^{t_{n-1}} \left( \xi(s,t_{n,j}) + \eta(s,t_{n,j}) \right)^2 ds \right).
\]

By virtue of the fact that \( \sum_{j=0}^{M_n} w_{n,j} = 2 \), we get
\[
\|B_3\|^2 \leq c_n h_n T^{2\alpha-1} \int_0^{t_{n-1}} \left( \xi(s,t_{n,j}) + \eta(s,t_{n,j}) \right)^2 ds \leq c_n h_n T^{2\alpha-1} \|\xi(s,t_{n,j}) + \eta(s,t_{n,j})\|^2_{L^2[0,t_{n-1}]}.
\]  

Minkowski inequality yields
\[
\|\xi + \eta\|^2_{L^2[0,t_{n-1}]} = \sum_{k=1}^{n-1} \|\xi_k + \eta_k\|^2_{L^2_{\Omega_k}} \leq 2 \sum_{k=1}^{n-1} (\|\xi_k\|^2_{L^2_{\Omega_k}} + \|\eta_k\|^2_{L^2_{\Omega_k}}),
\]  

where by Lemma 5
\[
\|\xi_k\|^2 \leq \|\mathcal{I}_{s,M_k}^L (\kappa(s,t_{n,j}) \psi(s,u^k(s)))\|^2 \\
\quad \leq c h_k^2 (M_k + 1)^{-2m} \|\partial^m \kappa(s,t_{n,j}) \psi(s,u^k(s))\|^2 \\
\quad \leq c h_k^2 (M_k + 1)^{-2m} \|\partial^m \psi(s,u(s))\|^2_{L^2_{\Omega_k}},
\]  

in which the last inequality follows from the assumption that \( \kappa \in C^m(\Omega \times \Omega) \). Moreover, it is deduced that
\[
\|\eta_k\|^2 \leq \mathcal{I}_{s,M_k}^L \left( \kappa(s,t_{n,j}) \left( \psi(s,u^k(s)) - \psi(s,u_{M_k}^k(s)) \right) \right)^2 \\
\quad = \frac{h_k}{2} \sum_{j=0}^{M_k} \left( \kappa(s_{k,j},t_{n,j}) \left( \psi(s_{k,j}^L,u_{k,j}^k(s_{k,j}^L)) - \psi(s_{k,j}^L,u_{M_k}^k(s_{k,j}^L)) \right) \right)^2 w_{k,j} \\
\quad \leq c \gamma^2 h_k \sum_{j=0}^{M_k} \left( u^k(s_{k,j}) - u_{M_k}^k(s_{k,j}) \right)^2 w_{k,j} \\
\quad \leq c \gamma^2 \int_{\Omega_k} \left( \mathcal{I}_{s,M_k}^L (u^k(\tau) - u_{M_k}^k(\tau))^2 \right) d\tau \\
\quad \leq c \gamma^2 \int_{\Omega_k} \left( \mathcal{I}_{s,M_k}^L u^k(\tau) - u_{M_k}^k(\tau) \right)^2 + \left( u^k(\tau) - u_{M_k}^k(\tau) \right)^2 d\tau \\
\quad \leq c \gamma^2 \|e_k\|^2_{L^2_{\Omega_k}} + h_k^2 (M_k + 1)^{-2m} \|\partial^m u\|^2_{L^2_{\Omega_k}}.
\]
Now, utilizing (41)-(43), the term (40) can be simplified as

\[ \|B_3\|_{\Omega_n}^2 \leq c h_n T^{2\alpha-1} \sum_{k=1}^{n-1} \left( \|\xi_k\|_{\Omega_k}^2 + \|\eta_k\|_{\Omega_k}^2 \right) \]

\[ \leq c h_n T^{2\alpha-1} \sum_{k=1}^{n-1} \left( h_k^{2m}(M_k + 1)^{-2m} \|\partial_s^m \psi(s, u(s))\|_{\Omega_k}^2 + \gamma^2(\|e_k\|_{\Omega_k}^2 + h_k^{2m}(M_k + 1)^{-2m} \|\partial_s^m u\|_{\Omega_k}^2) \right), \]

so, the desired result is deduced from \( \|B_1\|^2 \leq 2(\|B_2\|^2 + \|B_3\|^2). \)

**Lemma 7** Under the hypotheses of the previous theorem, the term

\[ B_0(t) = \mathcal{I}_{M_n} \left( \left( \frac{t - t_{n-1}}{h_n} \right)^\alpha \int_{\Omega_n} (t_n - \lambda)^{\alpha-1} \mathcal{I}_{\lambda M_n}^{-1,0}(\kappa(\sigma(\lambda, t), t) \psi(\sigma(\lambda, t), u^n_{\lambda}(\sigma(\lambda, t)))) d\lambda \right) \]

\[ - \left( \frac{t - t_{n-1}}{h_n} \right)^\alpha \int_{\Omega_n} (t_n - \lambda)^{\alpha-1} \kappa(\sigma(\lambda, t), t) \psi(\sigma(\lambda, t), u^n(\sigma(\lambda, t)))) d\lambda, \]

has the following error bound

\[ \|B_0\|^2 \leq c h_n T^{2\alpha-1} \sum_{k=1}^{n-1} \left( h_k^{2m}(M_k + 1)^{-2m} \|\partial_s^m \psi(s, u(s))\|_{\Omega_k}^2 + \gamma^2(\|e_k\|_{\Omega_k}^2 + h_k^{2m}(M_k + 1)^{-2m} \|\partial_s^m u\|^2) \right) \]

\[ + c h_n^{2m}(M_n + 1)^{-2m} \|\partial_s^m f\|_{\Omega_n}^2 + c_\alpha h_n^{2m+\alpha+1}(M_n + 1)^{-2m} \|\psi(\cdot, u(\cdot))\|_{L^\infty_{\alpha-1,0}(\Omega_n)}^2. \]

**Proof.** By adding and subtracting the term

\[ \mathcal{I}_{M_n} \left( \left( \frac{t - t_{n-1}}{h_n} \right)^\alpha \int_{\Omega_n} (t_n - \lambda)^{\alpha-1} \kappa(\sigma(\lambda, t), t) \psi(\sigma(\lambda, t), u^n(\sigma(\lambda, t)))) d\lambda, \]

to the term \( B_0(t), \) we have

\[ B_0(t) = B_1(t) + B_4(t), \]

where \( B_1(t) \) is defined by (35) and

\[ B_4(t) = \mathcal{I}_{M_n} - \mathcal{I} \left( \left( \frac{t - t_{n-1}}{h_n} \right)^\alpha \int_{\Omega_n} (t_n - \lambda)^{\alpha-1} \kappa(\sigma(\lambda, t), t) \psi(\sigma(\lambda, t), u^n(\sigma(\lambda, t)))) d\lambda. \]

In order to obtain an estimation for \( B_0(t), \) it suffices to seek an upper bound for \( B_4. \) Therefore, using operator norm and Lemma 4

\[ \|B_4\|_{\Omega_n}^2 \leq \|\mathcal{I}_{M_n} - \mathcal{I}\|^2 \|t^t \int_{t_{n-1}}^t (t - \tau)^{\alpha-1} \kappa(\tau, t) \psi(\tau, u^n(\tau)) d\tau\|^2 \]

\[ \leq c h_n^{2m}(M_n + 1)^{-2m} \int_{\Omega_n} \left( \int_{t_{n-1}}^t (t - \tau)^{\alpha-1} \kappa(\tau, t) \psi(\tau, u^n(\tau)) d\tau \right)^2 dt \]

\[ \leq c h_n^{2m}(M_n + 1)^{-2m} \int_{\Omega_n} \left( \int_{t_{n-1}}^t (t - \tau)^{\alpha-1} d\tau \right) \left( \int_{t_{n-1}}^t (t - \tau)^{\alpha-1} \kappa(\tau, t) \psi(\tau, u^n(\tau)) d\tau \right)^2 dt \]

\[ \leq c_\alpha h_n^{2m+\alpha+1}(M_n + 1)^{-2m} \|\psi(\cdot, u(\cdot))\|_{L^\infty_{\alpha-1,0}(\Omega_n)}^2. \]
Theorem 8 Assume that the Fréchet derivative of the operator $Ku$ with respect to $u$ is satisfied at $|K'(u)(t)| \geq \alpha > 0$, then under the hypothesis of the Theorem for sufficiently small $h_{\max}$ the following error estimate is obtained

$$
\|e_n\|^2 = \|u^n - u_{M_n}^n\|^2 \leq \frac{c_{\alpha}}{\delta^2} \exp(c T^{2\alpha}) \left( T^{2\alpha - 1} \sum_{k=1}^{n-1} \left( h_k^2 (M_k + 1)^{-2m} \left\| \partial_s \psi(s,u(s)) \right\|_{\Omega_k}^2 + \gamma h_k^2 (M_k + 1)^{-2m} \left\| \partial_s u \right\|_{\Omega_k}^2 \right) + h_n^2 (M_n + 1)^{-2m} \left\| \partial_s f \right\|_{\Omega_n}^2 + \left\| \psi(\cdot,u(\cdot)) \right\|_{H_{\chi_{\alpha-1,0}(\Omega_n)}^2}^2 \right)
$$

(47)

Proof. For convenience, let

$$
F(t,\tau,u(\tau)) := (t - \tau)^{\alpha - 1} \kappa(\tau,t) \psi(\tau,u(\tau)), \quad \tau \in (t_{n-1},t].
$$

(48)

and $G(t) := \int_{t_{n-1}}^{t} F(t,\tau,u(\tau))d\tau$. Under the mean value theorem (p. 229), we have

$$
\int_{t_{n-1}}^{t} F(t,\tau,u^n(\tau))d\tau - \int_{t_{n-1}}^{t} F(t,\tau,u_{M_n}^n(\tau))d\tau = G'(\xi)(u^n(t) - u_{M_n}^n(t)),
$$

(49)

where $\xi \in (\min\{u^n, u_{M_n}^n\}, \max\{u^n, u_{M_n}^n\})$ and $G'$ denotes the Fréchet derivative of $G$, namely,

$$
G'(u)h(t) = \int_{t_{n-1}}^{t} (t - \tau)^{\alpha - 1} \kappa(\tau,t) \frac{\partial \psi(\tau,u(\tau))}{\partial u} h(\tau)d\tau.
$$

It is well-known that

$$
K'(u)h(t) = \int_{0}^{t} (t - s)^{\alpha - 1} \kappa(s,t) \frac{\partial \psi(s,u(\tau))}{\partial u} h(s)ds.
$$

Since $|K'(u)(t)| \gg 0$, then one can deduce that $\delta := |G'(u)h(t)| \gg 0$. Therefore,

$$
\left| u^n(t) - u_{M_n}^n(t) \right| \leq \frac{1}{\delta} \int_{t_{n-1}}^{t} F(t,\tau,u^n(\tau)d\tau - \int_{t_{n-1}}^{t} F(t,\tau,u_{M_n}^n(\tau))d\tau.
$$

(50)

It is evident that

$$
\int_{t_{n-1}}^{t} F(t,\tau,u^n(\tau))d\tau = (\frac{t - t_{n-1}}{h_n})^\alpha \int_{\Omega_n} (t_n - \lambda)^{\alpha - 1} \kappa(\sigma(\lambda,t),t) \psi(\sigma(\lambda,t),u^n(\sigma(\lambda,t)))d\lambda,
$$

(51)

so from (50), we infer that

$$
\left| e_n(t) \right| = \left| u^n(t) - u_{M_n}^n(t) \right| \leq \frac{1}{\delta} \left| (\frac{t - t_{n-1}}{h_n})^\alpha \int_{\Omega_n} (t_n - \lambda)^{\alpha - 1} \kappa(\sigma(\lambda,t),t) \psi(\sigma(\lambda,t),u^n(\sigma(\lambda,t))) \right|
$$

$$
- \psi(\sigma(\lambda,t),u_{M_n}^n(\sigma(\lambda,t))) \right| d\lambda
$$

(52)

$$
\leq \frac{1}{\delta} \left| B_0(t) \right| + E_1(t) + E_2(t),
$$

where $B_0(t)$ is defined by Lemma and

$$
E_1(t) = \left| (I_{M_n}^\alpha - I)(\frac{t - t_{n-1}}{h_n})^\alpha \int_{\Omega_n} (t_n - \lambda)^{\alpha - 1} T_{\chi_{\alpha-1,0}(\Omega_n)}^{-1,0}(\kappa(\sigma(\lambda,t),t) \psi(\sigma(\lambda,t),u_{M_n}^n(\sigma(\lambda,t))))d\lambda \right|
$$

$$
E_2(t) = \left| (\frac{t - t_{n-1}}{h_n})^\alpha \int_{\Omega_n} (t_n - \lambda)^{\alpha - 1} T_{\chi_{\alpha-1,0}(\Omega_n)}^{-1,0}(\kappa(\sigma(\lambda,t),t) \psi(\sigma(\lambda,t),u_{M_n}^n(\sigma(\lambda,t))))d\lambda \right|,
$$

(53)

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Thus
\[ \|e_n\|^2_{\Omega_n} \leq \frac{3}{2^5} (\|B_0\|^2_{\Omega_n} + \|E_1\|^2_{\Omega_n} + \|E_2\|^2_{\Omega_n}). \] (54)
The error estimations for the terms \(\|E_i\|_{\Omega_n}\), with \(i = 1, 2\), as calculated in Appendix A are
\[ \|E_1\|^2 \leq c_\alpha h_n^2 \left( h_n^\alpha \gamma^2 \|e_n\|^2_{\Omega_n} + h_n^2 (M_n + 1) - 2m \left( \gamma^2 \|u^n\|_{H^m_{\chi_n-1,0}(\Omega_n)}^2 + \|\psi(\cdot, u^n(\cdot))\|_{H^m_{\chi_n-1,0}(\Omega_n)}^2 \right) \right), \] and
\[ \|E_2\|^2 \leq c_\alpha h_n^{2m+2 \alpha} (M_n + 1) - 2m \|\psi(\cdot, u^n(\cdot))\|_{H^m_{\chi_n-1,0}(\Omega_n)}^2. \] (55) (56)
Hence, the desired result follows from the relation [54]
\[ \|e_n\|^2_{\Omega_n} \leq \frac{c_\alpha}{\delta^2} \left( h_n T^{2 \alpha - 1} \sum_{k=1}^{n-1} \left( h_k^{2m} (M_k + 1) - 2m \|\partial^m f\|^2_{\Omega_k} + \gamma^2 (\|e_k\|^2_{\Omega_k} + h_k^{2m} (M_k + 1) - 2m \|\partial^m u\|^2_{\Omega_k}) \right) + h_n^2 (M_n + 1) - 2m (\|\partial^m f\|^2_{\Omega_n} + h_n^{2m} \|\psi(\cdot, u^n(\cdot))\|_{H^m_{\chi_n-1,0}(\Omega_n)}^2) \right), \] or equivalently,
\[ (1 - \frac{c_\alpha}{\delta^2} h_n^2) \|e_n\|^2 \leq \frac{c_\alpha}{\delta^2} \left( h_n T^{2 \alpha - 1} \sum_{k=1}^{n-1} \left( h_k^{2m} (M_k + 1) - 2m \|\partial^m f\|^2_{\Omega_k} + \gamma^2 (\|e_k\|^2_{\Omega_k} + h_k^{2m} (M_k + 1) - 2m \|\partial^m u\|^2_{\Omega_k}) \right) + h_n^2 (M_n + 1) - 2m \|\partial^m f\|^2_{\Omega_n} + h_n^{2m} \|\psi(\cdot, u^n(\cdot))\|_{H^m_{\chi_n-1,0}(\Omega_n)}^2 \right). \] (57) (58)
Next, assume that the step size \(h_{\text{max}}\) is sufficiently small such that
\[ dt^2 h_{\text{max}}^2 \leq \beta < 1. \]
Therefore, by means of Grönwall inequality and taking \(\varepsilon_k = h_k^{-1} \|e_k\|^2\), we have
\[ \|e_n\|^2_{\Omega_n} \leq \frac{c_\alpha}{\delta^2} \exp(c \gamma^2 T^{2 \alpha}) \left( T^{2 \alpha - 1} \sum_{k=1}^{n-1} \left( h_k^{2m} (M_k + 1) - 2m \|\partial^m f\|^2_{\Omega_k} + \gamma^2 h_k^{2m} (M_k + 1) - 2m \|\partial^m u\|^2_{\Omega_k} \right) + h_n^{2m} (M_n + 1) - 2m \|\partial^m f\|^2_{\Omega_n} + h_n^{2m} \|\psi(\cdot, u^n(\cdot))\|_{H^m_{\chi_n-1,0}(\Omega_n)}^2 \right), \] which infers the desired result. \(\blacksquare\)

**Theorem 9** Assume that \(u(t)\) be the exact solution of Eq. 4 and \(u_M^N(t)\) be the global approximate solution obtained from Eq. 31. Under the hypothesis of Theorem 8, the following error estimate can be derived for sufficiently small \(h_{\text{max}}\) as
\[ \|u - u_M^N\|_{\Omega} \leq \frac{c_\alpha}{\delta^2} \exp(c \gamma^2 T^{2 \alpha}) h_{\text{max}}^{m} (M_{\text{min}} + 1) - m \left( T^\alpha (\gamma \|\partial^m f\|_{\Omega} + \|\partial^m \psi(\cdot, u^n(\cdot))\|_{\Omega}) + \|\partial^m f\|_{\Omega} \right) + \gamma \|u\|_{H^m_{\chi_n-1,0}(\Omega)} + \|\psi(\cdot, u^n(\cdot))\|_{H^m_{\chi_n-1,0}(\Omega)} + h_{\text{max}}^2 \|\psi(\cdot, u_M^N(\cdot))\|_{H^m_{\chi_n-1,0}(\Omega)} \] (60)
Proof. The global convergence error of the approximate solution \( u_M^N(t) \) which is given by

\[
u_M^N(t)|_{\Omega_n} = u_M^n(x) \bigg|_{x = \frac{2u-t_n-1-t_n}{h_n}}, \quad 1 \leq n \leq N,
\]

and the exact solution \( u(t) \) which is fulfilled in

\[
u(t)|_{\Omega_n} = u^n(x) \bigg|_{x = \frac{2u-t_n-1-t_n}{h_n}}, \quad 1 \leq n \leq N,
\]

can be easily obtained using Theorem 8 and the following formula

\[
\|u - u_M^N\|_{H^1}^2 = \frac{1}{2} \sum_{n=1}^{N} h_n \|e_n\|_{\Omega_n}^2.
\]

Therefore,

\[
\|u - u_M^N\|_{H^1}^2 \leq C_n \exp(c\gamma^2 T^{2\alpha}) \sum_{n=1}^{N} h_n T^{2\alpha - 1} \sum_{k=1}^{n} (h_k^{2m} (M_k + 1)^{-2m} |\partial_t^m \psi(s, u(s))|_{\Omega_k}^2 + \gamma^2 h_k^{2m} (M_k + 1)^{-2m} |\partial_t^m u|_{\Omega_k}^2)
\]

\[
+ h_n^{2m} (M_n + 1)^{-2m} (|\partial_t^m f|_{\Omega_n}^2 + h_n^{2m+1} |\psi(., u_M^N(\cdot))|_{H^m_{\chi_n-1,0}(\Omega_n)}^2)
\]

\[
+ h_n^{2m+\alpha+1} (M_n + 1)^{-2m} (\gamma^2 |u|_{H^{m}_{\chi_n-1,0}(\Omega_n)}^2 + |\psi(., u(\cdot))|_{H^m_{\chi_n-1,0}(\Omega_n)}^2).
\]

(61)

All terms of the above error bound can be simplified using \( h_{\text{max}} \) and \( M_{\text{min}} \) as follows

\[
\sum_{n=1}^{N} h_n^{2m} (M_n + 1)^{-2m} |\partial_t^m f|_{\Omega_n}^2 \leq h_{\text{max}}^{2m} (M_{\text{min}} + 1)^{-2m} |\partial_t^m f|_{\Omega}^2,
\]

similarly,

\[
\sum_{n=1}^{N} h_n^{2m+2\alpha+1} (M_n + 1)^{-2m} |\psi(., u_M^N(\cdot))|_{H^m_{\chi_n-1,0}(\Omega_n)}^2 \leq h_{\text{max}}^{2m+2\alpha+1} (M_{\text{min}} + 1)^{-2m} |\psi(., u_M^N(\cdot))|_{H^m_{\chi_n-1,0}(\Omega)}^2.
\]

Also the following inequalities can be proved

\[
\sum_{n=1}^{N} h_n^{2m+\alpha+1} (M_n + 1)^{-2m} \gamma^2 |u|_{H^m_{\chi_n-1,0}(\Omega_n)}^2 \leq h_{\text{max}}^{2m+\alpha+1} (M_{\text{min}} + 1)^{-2m} \gamma^2 |u|_{H^m_{\chi_n-1,0}(\Omega)}^2,
\]

and

\[
\sum_{n=1}^{N} h_n^{2m+\alpha+1} (M_n + 1)^{-2m} |\psi(., u(\cdot))|_{H^m_{\chi_n-1,0}(\Omega_n)}^2 \leq h_{\text{max}}^{2m+\alpha+1} (M_{\text{min}} + 1)^{-2m} |\psi(., u(\cdot))|_{H^m_{\chi_n-1,0}(\Omega)}^2.
\]

Furthermore, we can obtain

\[
\sum_{n=1}^{N} h_n T^{2\alpha - 1} \sum_{k=1}^{n} \gamma^2 h_k^{2m} (M_k + 1)^{-2m} |\partial_t^m u|_{\Omega_k}^2 \leq \gamma^2 h_{\text{max}}^{2m} (M_{\text{min}} + 1)^{-2m} T^{2\alpha - 1} \sum_{n=1}^{N} h_n \sum_{k=1}^{n-1} |\partial_t^m u|_{\Omega_k}^2
\]

\[
\leq \gamma^2 h_{\text{max}}^{2m} (M_{\text{min}} + 1)^{-2m} T^{2\alpha} |\partial_t^m u|_{\Omega}^2,
\]

and

\[
\sum_{n=1}^{N} h_n T^{2\alpha - 1} \sum_{k=1}^{n-1} h_k^{2m} (M_k + 1)^{-2m} |\partial_t^m \psi(s, u(s))|_{\Omega_k}^2 \leq h_{\text{max}}^{2m} (M_{\text{min}} + 1)^{-2m} T^{2\alpha - 1} \sum_{n=1}^{N} h_n \sum_{k=1}^{n-1} |\partial_t^m \psi(s, u(s))|_{\Omega_k}^2.
\]

\[
\leq h_{\text{max}}^{2m} (M_{\text{min}} + 1)^{-2m} T^{2\alpha} |\partial_t^m \psi(s, u(s))|_{\Omega}^2.
\]
Correspondingly, the combination of the above error bounds for Eq. \([61]\) leads to

\[
\| u - u^N_{M} \|_{2_{\Omega}}^2 \leq C_\alpha \exp(\gamma^2 T^{2\alpha}) \frac{\alpha}{2} \left( T^{2\alpha} (\| \partial^m_f \|_{2_{\Omega}}^2 + \| \partial^m_f u \|_{2_{\Omega}}^2 + \| \partial^m u \|_{2_{\Omega}}^2)^2 + \| \partial^m_f f \|_{2_{\Omega}}^2 \right) + h^{\alpha+1} \max \left( \gamma^2 \| \psi(t, u(\cdot)) \|_{H^{\alpha+1}_0(\Omega)}^2 + h^{\alpha} \| \psi(t, u^N_0(\cdot)) \|_{H^{\alpha}_0(\Omega)}^2 \right),
\]

which completes the proof. \(\blacksquare\)

### 6 Numerical results

This section illustrates some numerical experiments in order to scrutinize the efficiency of the hp-collocation method for the Abel integral equations. The experiments are implemented in Mathematica\textsuperscript{\textregistered} software platform and the programs are executed on a PC with 3.50 GHz Intel(R) Core(TM) i5-4690K processor. In order to analyze the method, the following notations are introduced:

\[
E_1(u^N_{M}) = \left( \sum_{k=1}^{N} \sum_{j=0}^{M_k} \frac{h_k}{2} w_{k,j} \left( u^k(x_{k,j}) - u_{M_k}(x_{k,j}) \right) \right)^{1/2},
\]

\[
E_2(u^N_{M}) = \max_{t \in \Omega} | u(t) - u^N_{M}(t) |.
\]

The discrete \(L^2\)-norm error is denoted by \(E_1(u^N_{M})\), and also \(E_2(u^N_{M})\) indicates the infinite norm. Furthermore, the order of convergence \(\rho_N\) is defined by \(\log_2 \left( \frac{E_1(u^N_{M})}{E_1(u^{2N}_{M})} \right)\). The relation of the theoretical order of convergence stated in Theorem \([9]\) and \(\rho_N\) is derived as

\[
\rho_N = \log_2 \left( \frac{E_1(u^N_{M})}{E_1(u^{2N}_{M})} \right) \approx \log_2 \left( \frac{c h_{\text{max}}^m M_{\text{min}}^{-m}}{h_{\text{max}}^m M_{\text{min}}^{-m}} \right) = \log_2 2^m = m. \tag{63}
\]

This criterion can be utilized to check the order of convergence in practice based on the continuous injection between \(L^2(\Omega)\) and \(L^\infty(\Omega)\) \([5]\).

Let \(L\) denote the number of unknown coefficients; in this way we have \(L = \sum_{n=1}^{N} (M_n + 1)\) for the \(hp\)-collocation method and in a specific case, if all degrees of polynomials \(M_n\) are equal, i.e. \(M_n = M^*\), for \(n = 1, \ldots, N\), then according to relation \([31]\), \(L = (M^* + 1) \times N\). For convenience, we denote \(M := M^* + 1\), so \(L = M \times N\).

The nonlinear systems which arise in the formulation of the method are solved by utilizing the Newton iteration method which needs an initial guess. In these examples, all the initial points are chosen by an algorithm based on the steepest descent method.

**Remark 10** In \([25, 27]\), two adaptive schemes based on mesh refinement are introduced. A bound for the error is chosen arbitrary and then the implementation proceeds by increasing the degree of polynomials or refinement of the mesh size until the desired error bound is observed. The described scheme is called “adaptive hp-collocation method”.

**Singular solution**

**Example 1** Consider a test problem with singular solution

\[
\int_0^t (t-s)^{\alpha-1} \exp(ts) u^2(s) ds = \left( \frac{1}{t} \right)^{-2\gamma^2 + \sigma} \Gamma(3 + 2\sigma) \Gamma(\alpha) \Gamma(3 + 2\sigma, 3 + 3\sigma, t^2), \quad t \in [0, 1],
\]
where the function $F_1$ is called confluent hypergeometric function of the first kind. The exact solution $u(t) = t^{1+\alpha}$ belongs to $H^2_{\alpha-1,0}([0,1])$. By means of relation (63), $\rho_N \approx m = 2$ and is experimentally verified in the left sub-figure of Figure 1. The left sub-figure with the fixed $M = M^* + 1 = 2$ and different values of $N$ depicts $h$-version and the right sub-figure displays $p$-version with the fixed value $N = 2$ and different $M$. The right sub-figure depicts various $\alpha$ which shows that by increasing the values of $\alpha$, the convergence rate increases as verified by the result of Theorem 9. Figure 2 demonstrates the $hp$-version collocation method for each fixed $N = 1, 2, 4, 8$ when $h_n = h = \frac{1}{N}$ and various values of $M_n = M^*$ for $n = 1, \ldots, N$. Comparison of Figures 1 and 2 shows the superiority of $hp$-version against $h$- and $p$-versions.

**Figure 1**: Plots of the $E_1(u_N^M)$ error in logarithmic scale for the $h$-, $p$-version collocation methods for Example 1.

**Figure 2**: Plots of the $E_1(u_N^M)$ error in logarithmic scale for the $hp$-version collocation method for Example 1.

**Example 2** ([27]) In the following example, we consider solving the linear Abel equation

$$\int_0^t (t - s)^{-0.5} \exp(-t + s)u(s)ds = f(t), \quad t \in [0, T],$$

where $f(t) = \exp(-t)(t^4 + t^6)$ and the exact solution is $u(t) = \exp(-t)\left(\frac{4!}{\Gamma(4.5)} t^{3.5} + \frac{6!}{\Gamma(6.5)} t^{5.5}\right)$.

For the sake of good comparison, let us take into account some assumptions considered in the interesting paper [27]. Numerical experiments with $M = 2$ and step sizes $h = 1/2^q, q = 5, 6, \ldots, 11$ accompanied with the noise level $\delta = h^{2.5}$ for investigating the effect of perturbation are employed. The perturbation is added to the right hand side as $f^\delta(t) = f(t) + \delta$. Let us denote $E_3^\delta(u_N^\delta) := \max_{t \in [0,T]} |u_N^\delta(t) - u(t)|$. Table 1 demonstrates the superiority of the $hp$-collocation method against trapezoidal method. The last column shows that the order of perturbation symbol $\delta$ is not even linear. This importance verifies the well-posedness of the scheme.
On the other hand, the purpose of adaptive $hp$-collocation method is to utilize subspaces with lower dimension which yield less computational complexity and CPU time. For instance, if we take $M = 12$ and $N = 2$, then the absolute error is $2.6e-07$ for $L = 24$ whereas the best results of Table 1 are achieved by $L = 2 \times 2048$.

Table 1: The comparison of trapezoidal method [27] and $hp$-collocation method for different $N$ with fixed $M = M_n + 1 = 2$ and $h = 1/2^n, q = 5, 6, \ldots, 11$ in the sense of $E_N^q$ for Example 2

| $N$ | $\delta$ | $E_N^q$ [27] | $E_N^q$ (hp) | $E_N^q$ [27]/$E_N^q(hp)^{0.8}$ | $E_N^q(hp)^{0.85}$ |
|-----|--------|--------------|--------------|-------------------------------|-------------------|
| 32  | 1.7e-04| 2.95e-03     | 7.29e-04     | 3.02                          | 1.17              |
| 64  | 3.1e-05| 1.04e-03     | 1.83e-04     | 4.26                          | 1.24              |
| 128 | 5.4e-06| 2.03e-04     | 4.57e-05     | 3.32                          | 1.37              |
| 256 | 9.5e-07| 5.79e-05     | 1.14e-05     | 3.79                          | 1.50              |
| 512 | 1.7e-07| 1.72e-05     | 2.84e-06     | 4.50                          | 1.61              |
| 1024| 3.0e-08| 4.24e-06     | 7.09e-07     | 4.45                          | 1.75              |
| 2048| 5.3e-09| 1.09e-06     | 1.99e-07     | 4.58                          | 2.15              |

$\rho_N$ | 1.50 | 1.99 |

Smooth solution

**Example 3** (27) In this example, we apply the method to the following nonlinear weakly singular Volterra integral equation of the first kind

$$\int_0^t (t - s)^{-0.5}(1 + s + tu(s))u(s)ds = \frac{32}{45045}(1287 + 1144 + 960t^4)t^{3.5}, \quad t \in [0,1],$$

with the exact solution $u(t) = t^3$. Table 2 reports the comparison of Finite Difference Method (FDM) of the third order [4] and $hp$-collocation method with the same value of $L$. The present scheme runs for various values of $N$ with fixed step size $h_n = h = \frac{1}{N}$, uniform mode $M = M_n + 1 = 3$ for $n = 1, \ldots, N$. As expected from (63), $\rho_N$ is approximately equal to $m \leq M_{\text{max}} + 1 = 3$.

Adaptivity and capability of the present scheme to obtain the best result are provided according to Remark 10. We take our desired absolute error equal to $10^{-14}$ and hence achieve the appropriate solution with the absolute error $2.33e-15$, when $M$ and $N$ are chosen 4 and 1, respectively. As we expected, $p$-version works well for the problems with smooth solution.

Table 2: The comparison of FDM [4] and $hp$-collocation method for different $N$ with fixed $M = M_n + 1 = 3$ and $h = h_n = \frac{1}{N}$ for $n = 1, \ldots, N$ in terms of $E_2(u_n^N)$ for Example 3

| $N$ | FDM [4] | $hp$–collocation |
|-----|---------|------------------|
| 10  | 3.7e-04 | 9.42e-05         |
| 20  | 4.7e-05 | 1.08e-05         |
| 40  | 6.0e-06 | 1.12e-06         |
| 80  | 7.6e-07 | 1.18e-07         |
| 160 | 9.6e-08 | 1.21e-08         |
| 320 | 1.2e-08 | 1.32e-09         |

$\rho_N$ | 2.97 | 3.12 |
Example 4 \((\text{[23]}\) Consider the following linear Abel equation
\[
\int_0^t (t-s)^{-0.25}(t^2s^3 + s^4 + 1)u(s)ds = \frac{128t^{14}}{908523} (3933 + 256t^4(8 + 9t)), \quad t \in [0, 1],
\]
with smooth solution \(u(t) = t^2\). This example is studied in \([23]\) using mechanical quadrature method and its extrapolation by converting the above first kind integral equation to the second kind. The lowest error in terms of absolute error utilizing \(h^2\)-extrapolation and \(N = 80\) is reported as \(1.72e-8\) in \([23]\). As described in the previous example, we expect \(p\)-version to work well for such a smooth case. The appropriate approximate solution using adaptive \(hp\)-collocation for the given tolerance \(10^{-15}\) is achieved by \(M = 4\) and \(N = 1\); namely \(E_2(u_1^*) = 3.33e-16\). The superiority of the proposed method in the sense of computational complexity and accuracy is evident.

Discontinuous solution.

Example 5 In the following example, we consider solving the nonlinear Abel equation
\[
\int_0^t (t-s)^{-0.2}\kappa(t,s)u^5(s)ds = f(t), \quad t \in [0, 1],
\]
where \(\kappa(t,s) = \sin(t-s)\) and \(f(t)\) is chosen such that
\[
u(t) = \begin{cases} 
\exp(-t), & 0 \leq t < 0.5, \\
2 - t^2, & 0.5 \leq t \leq 1,
\end{cases}
\]
be the exact solution. Note that in this example, \(\kappa(t,t) = 0\) which demonstrates that the problem can not be converted into the second kind Abel equation. Secondly, the exact solution \(u(x)\) is a discontinuous function which obviously can not be solved by \(p\)-version methods. Figure 3 shows considerable results for various \(M\) with fixed step size \(h_n = h = \frac{1}{2}\) and \(M_n = M^*\) for \(n = 1, 2\). Here, we take the degree of polynomials for each \(I_k\) with \(M_n = M^*\) for all \(n = 1, ..., N\). Now, we implement the program in order to reach the best result of the scheme automatically. The best finding belongs to different degree mode \(M_1 = 9\) and \(M_2 = 4\) for the first and second subintervals. Using only 15 basis functions leads to achieve an appropriate solution with the \(L^2\)-error norm \(3.02e-8\). The best reported error in Figure 3 was for \(M_1 = M_2 = 9\) with the value \(2.12e-7\). A comparison between these two results show the adaptivity of the scheme in which a lower dimension of basis functions prevent the scheme from aggregating more errors.

Figure 3: Plots of the \(E_N^N(u_2^*)\) error in logarithmic scale for different mode \(M\) with fixed \(h_n = h = \frac{1}{2}\) and \(M_n = M^*\) for \(n = 1, 2\) for Example 5.
Unknown exact solution

In the following example, we consider an equation which has a unique solution according Theorem 1.

**Example 6** Consider the following nonlinear weakly singular integral equation

\[ \int_0^t (t-s)^{-0.35} (t^2 s - t + 1)(u - t^2)^3 \, ds = f(t), \quad t \in [0, 2]. \]

where

\[ f(t) = \begin{cases} 
  t^4, & 0 < t < 1, \\
  t^{2.65}, & 1 < t < 2.
\end{cases} \]

We know that the introduced equation has a unique solution, but the exact solution is not known. Hence for a sake of good comparison, we choose \( u_{16}^2 \) as a benchmark depicted in Figure 4. It is displayed by Figure 5 the convergence of the scheme by increasing \( M \) and \( N \) where \( M = M_n + 1 = M^* + 1 \) and fixed step size \( h_n = h = \frac{1}{N} \) for \( n = 1, \ldots, N \).

![Figure 4: The approximate solution \( u_{16}^2(t) \) for \( T = 2 \).](image)

![Figure 5: Plots of the \( E_1(u_{M}^N(t)) \) in logarithmic scale for Example 6.](image)

**Example 7** The following nonlinear weakly singular integral equation is considered

\[ \int_0^t (t-s)^{-0.4} \kappa(t,s) \left( \cos(2su(s)) - \ln u^3(s) - \sqrt{su(s)} \right) ds = t^{1.5} - |t|, \quad t \in [0, 1.5], \]

where

\[ \kappa(t,s) = \begin{cases} 
  t^2 - s + 5, & 0 < s < 0.5, \quad 0 < t < 1.5, \\
  \exp(st) + \frac{1}{s+1} - 2, & 0.5 < s < 1, \quad 0 < t < 1.5, \\
  \frac{t}{s}, & 1 < s < 1.5, \quad 0 < t < 1.5.
\end{cases} \]
The exact solution in unknown, therefore we choose \( u_{312}(t) \) with \( L = 12 \times 3 = 36 \) basis functions as a benchmark for comparison. Figure 6 depicts the convergence of the scheme by increasing \( M \) and \( N \) with \( T = 1.5 \), \( M = M_n + 1 = M^* + 1 \) and fixed step size \( h_n = h = \frac{1}{N} \) for \( n = 1, \ldots, N \). Figure 7 shows the benchmark which is the approximate solution for \( T = 1.5 \).

![Figure 6: Plots of the \( E_2(u_{NM}(t)) \) in logarithmic scale for Example 7](image1)

![Figure 7: The approximate solution \( u_{312}(t) \) for \( T = 1.5 \)](image2)

**Conclusion**

The first kind integral equations and their approximations are interesting problems from both theory and application view points. This paper concerns a nonlinear class of them so-called Abel integral equations in a general form. The existence and uniqueness of the solution have been investigated in the suitable Sobolev spaces under some assumptions. The \( hp \)-version Jacobi projection methods have been studied and a prior error analysis in \( L^2 \)-norm is developed for Abel integral equations. Numerical treatments indicate that the proposed scheme is effective and powerful to deal with smooth and non-smooth solutions.
Appendix A.

In this section, we are acquiring error bounds for the terms \( \|E_1\|_{\Omega_n} \) and \( \|E_2\|_{\Omega_n} \). In advance,

\[
\|E_2\|^2_{\Omega_n} = \int_{\Omega_n} \left( \int_{\Omega_n} (t_n - \lambda)^{m-1}(T^{\alpha-1,0}_{\lambda,M_n} - I)(\kappa(\sigma(t), t)\psi(\sigma(t), u^\alpha_{M_n}(\sigma(t))))d\lambda \right) dt \\
\leq \int_{\Omega_n} \left( \int_{\Omega_n} (t_n - \lambda)^{m-1}(I - I_{\lambda,M_n})(\kappa(\sigma(t), t)\psi(\sigma(t), u^\alpha_{M_n}(\sigma(t))))d\lambda \right) dt \\
\leq c \left( \int_{\Omega_n} (t_n - \lambda)^{m-1}d\lambda \right) \int_{\Omega_n} \left( \int_{\Omega_n} (t_n - \lambda)^{m-1}(I - I_{\lambda,M_n})(\kappa(\sigma(t), t)\psi(\sigma(t), u^\alpha_{M_n}(\sigma(t))))d\lambda \right) dt \\
\leq c a h_{\alpha}^{m-1} \int_{\Omega_n} \left( \int_{\Omega_n} (t_n - \lambda)^{m-1}\left| (I - I_{\lambda,M_n})(\kappa(\sigma(t), t)\psi(\sigma(t), u^\alpha_{M_n}(\sigma(t)))) \right| d\lambda \right) dt.
\]

(64)

where the above inequalities follow directly from the Cauchy-Schwartz inequality. Now using Lemma 3 and the relation (64), we get

\[
\|E_2\|^2_{\Omega_n} \leq c a h_{\alpha}^{m+\alpha}(M_n + 1)^{-2m} \int_{\Omega_n} \left( \int_{\Omega_n} (t_n - \lambda)^{m-1}d\lambda \right) \left( \int_{\Omega_n} \left| \partial^\alpha(\kappa(\sigma(t), t)\psi(\sigma(t), u^\alpha_{M_n}(\sigma(t)))) \right| d\lambda \right) dt \\
\leq c a h_{\alpha}^{m+\alpha}(M_n + 1)^{-2m} \int_{\Omega_n} \left( \int_{\Omega_n} (t_n - \lambda)^{m-1}d\lambda \right) \left( \int_{\Omega_n} \left| \partial^\alpha(\kappa(\sigma(t), t)\psi(\sigma(t), u^\alpha_{M_n}(\sigma(t)))) \right| d\lambda \right) dt \\
\leq c a h_{\alpha}^{2m+2\alpha}(M_n + 1)^{-2m} \sum_{i=0}^{m} \left( \int_{\Omega_n} \left| \partial^\alpha(\psi(\tau, u^\alpha_{M_n}(\tau))) \right| d\lambda \right) dt \\
= c a h_{\alpha}^{2m+2\alpha}(M_n + 1)^{-2m} \|\psi(\tau, u^\alpha_{M_n}(\tau))\|_{H^m_{\chi_{\alpha-1,0}(\Omega_n)}}^2.
\]

(65)

In order to find an upper bound for the term \( E_1 \), we notice that

\[
\|E_1\|^2_{\Omega_n} = \left\| (T^{\alpha}_{\lambda,M_n} - I) \left( \int_{t_{n-1}}^t (t - \tau)^{m-1}T^{\alpha,1,0}_{\tau,M_n}(\kappa(\sigma(t), t)\psi(\sigma(t), u^\alpha_{M_n}(\tau)))d\tau \right) \right\|^2_{\Omega_n} \\
= \left\| \int_{t_{n-1}}^t (t - \tau)^{m-1}T^{\alpha,1,0}_{\tau,M_n}(\kappa(\sigma(t), t)\psi(\sigma(t), u^\alpha_{M_n}(\tau)))d\tau \right\|^2_{\Omega_n} \\
\leq (E_{1,1} + E_{1,2} + E_{2,3} + \|B_4\|^2),
\]

where the above terms \( E_{1,i}, i = 1, 2, 3 \) will introduce in sequel with their relevant upper bounds and the error bound for \( \|B_4\|^2 \) is derived in (66). Using Eqs. (23), (24) and Cauchy-Schwartz inequality,
we get

\[
E_{1,1} = \left\| \int_{t_{n-1}}^{t} (t - \tau)^{\alpha - 1} T_{\tau, M_{n}}^{\alpha-1,0}(\kappa(\tau, t) \left[ \psi(\tau, u_{M_{n}}^{n}(\tau)) - \psi(\tau, u_{n}(\tau)) \right]) d\tau \right\|_{L^{2}(\Omega_{n})}^{2} \\
= \int_{\Omega_{n}} \left| \int_{t_{n-1}}^{t} (t - \tau)^{\alpha - 1} T_{\tau, M_{n}}^{\alpha-1,0}(\kappa(\tau, t) \left[ \psi(\tau, u_{M_{n}}^{n}(\tau)) - \psi(\tau, u_{n}(\tau)) \right]) d\tau \right|^{2} dt \\
\leq \int_{\Omega_{n}} \left( \int_{t_{n-1}}^{t} (t - \tau)^{\alpha - 1}(T_{\tau, M_{n}}^{\alpha-1,0}(\kappa(\tau, t) \left[ \psi(\tau, u_{M_{n}}^{n}(\tau)) - \psi(\tau, u_{n}(\tau)) \right])^{2} d\tau \right) dt \\
\leq c_{h}h_{n}^{\alpha} \int_{\Omega_{n}} \left( \int_{t_{n-1}}^{t} (t - \tau)^{\alpha - 1}(T_{\tau, M_{n}}^{\alpha-1,0}(\kappa(\tau, t) \left[ \psi(\tau, u_{M_{n}}^{n}(\tau)) - \psi(\tau, u_{n}(\tau)) \right])^{2} d\tau \right) dt \\
\leq c_{h}h_{n}^{\alpha} \int_{\Omega_{n}} \left( \frac{t - t_{n-1}}{2} \right)^{\alpha} \sum_{j=0}^{M_{n}} \left( \kappa(\tau_{n,j}, t) \left[ \psi(\tau_{n,j}, u_{M_{n}}^{n}(\tau_{n,j})) - \psi(\tau_{n,j}, u_{n}(\tau_{n,j})) \right] \right)^{2} w_{n,j} dt.
\]  

\text{(67)}

Moreover, due to the Lipschitz condition for the function \( \psi \) with respect to second variable, we obtain

\[
E_{1,1} \leq c_{h}h_{n}^{\alpha} \gamma^{2} \int_{\Omega_{n}} \left( \frac{t - t_{n-1}}{2} \right)^{\alpha} \sum_{j=0}^{M_{n}} \left[ u_{M_{n}}^{n}(\tau_{n,j}) - u_{n}(\tau_{n,j}) \right]^{2} w_{n,j} dt \\
\leq c_{h}h_{n}^{\alpha} \gamma^{2} \int_{\Omega_{n}} \int_{t_{n-1}}^{t} (t - \tau)^{\alpha - 1} \left[ T_{\tau, M_{n}}^{\alpha-1,0}(u_{M_{n}}^{n}(\tau) - u_{n}(\tau)) \right]^{2} d\tau dt \\
\leq c_{h}h_{n}^{\alpha} \gamma^{2} \int_{\Omega_{n}} \int_{t_{n-1}}^{t} \left[ T_{\tau, M_{n}}^{\alpha-1,0}(u_{M_{n}}^{n}(\tau) - u_{n}(\tau))^{2} + (u_{n}(\tau) - u_{M_{n}}^{n}(\tau))^{2} \right] (t - \tau)^{\alpha - 1} d\tau dt \\
\leq c_{h}h_{n}^{\alpha} \gamma^{2} \left( h_{n}^{2} \| e_{n} \|_{L^{2}(\Omega_{n})}^{2} + h_{n}^{2m+1}(M_{n} + 1)^{-2m} \| \partial_{t}^{m} u_{n} \|_{L^{2}(\alpha - 1, 0)(\Omega_{n})}^{2} \right) \\
\leq c_{h}h_{n}^{\alpha} \gamma^{2} \left( h_{n}^{2} \| e_{n} \|_{L^{2}(\Omega_{n})}^{2} + h_{n}^{2m+1}(M_{n} + 1)^{-2m} \| u_{n} \|_{H^{m-1,0}(\Omega_{n})}^{2} \right). 
\]  

\text{(68)}

In order to seek an upper bound for \( E_{1,2} \), we follow Eqs. \((16), (23), (24)\), Lipschitz condition and Hölder inequality to get that

\[
E_{1,2} = \left\| I_{M_{n}}^{t} \left( \int_{t_{n-1}}^{t} (t - \tau)^{\alpha - 1} T_{\tau, M_{n}}^{\alpha-1,0}(\kappa(\tau, t) \left[ \psi(\tau, u_{M_{n}}^{n}(\tau)) - \psi(\tau, u_{n}(\tau)) \right]) d\tau \right) \right\| \\
= \int_{\Omega_{n}} \left[ I_{M_{n}}^{t} \left( \int_{t_{n-1}}^{t} (t - \tau)^{\alpha - 1} T_{\tau, M_{n}}^{\alpha-1,0}(\kappa(\tau, t) \left[ \psi(\tau, u_{M_{n}}^{n}(\tau)) - \psi(\tau, u_{n}(\tau)) \right]) d\tau \right) \right]^{2} dt \\
= \frac{h_{n}}{2} \sum_{j=0}^{M_{n}} \left( \int_{t_{n-1}}^{t_{n,j}} (t_{n,j} - \tau)^{\alpha - 1} T_{\tau, M_{n}}^{\alpha-1,0}(\kappa(\tau, t_{n,j}) \left[ \psi(\tau, u_{M_{n}}^{n}(\tau)) - \psi(\tau, u_{n}(\tau)) \right]) d\tau \right)^{2} w_{n,j} \\
\leq c_{h} \sum_{j=0}^{M_{n}} \left( \int_{t_{n-1}}^{t_{n,j}} (t_{n,j} - \tau)^{\alpha - 1} d\tau \right) \left( \int_{t_{n-1}}^{t_{n,j}} (t_{n,j} - \tau)^{\alpha - 1} \left( T_{\tau, M_{n}}^{\alpha-1,0}(\kappa(\tau, t_{n,j}) \left[ \psi(\tau, u_{M_{n}}^{n}(\tau)) - \psi(\tau, u_{n}(\tau)) \right]) \right)^{2} d\tau \right) w_{n,j} \\
\leq c_{h}h_{n}^{\alpha+1} \gamma^{2} \sum_{j=0}^{M_{n}} \left( \frac{t_{n,j} - t_{n-1}}{2} \right)^{\alpha} \sum_{i=0}^{M_{n}} \left( \kappa(\tau_{n,i}, t_{n,j}) \left[ \psi(\tau_{n,i}, u_{M_{n}}^{n}(\tau_{n,i})) - \psi(\tau_{n,i}, u_{n}(\tau_{n,i})) \right] \right)^{2} w_{n,i} w_{n,j} \\
\leq c_{h}h_{n}^{\alpha+1} \gamma^{2} \sum_{j=0}^{M_{n}} w_{n,j} \left( \frac{t_{n,j} - t_{n-1}}{2} \right)^{\alpha} \sum_{i=0}^{M_{n}} \left[ u_{M_{n}}^{n}(\tau_{n,i}) - u_{n}(\tau_{n,i}) \right]^{2} w_{n,i} w_{n,j}.
\]  

\text{(69)}
Since $\sum_{j=0}^{M_n} w_{n,j} = 2$ and Eq. (24) and Lemma 3, the above inequality can be simplified as

$$ E_{1.2} \leq c_0 h_n^{a+1} \sum_{t_{n-1}}^{t_{n-1}} \left( T_{\tau,M_n}^{\alpha-1,0} u^n(\tau) - u_{M_n}^n(\tau) \right)^2 (t_{n,j} - \tau)^{\alpha-1} d\tau $$

$$ \leq c_0 h_n^{a+1} \sum_{t_{n-1}}^{t_{n-1}} \left( T_{\tau,M_n}^{\alpha-1,0} u^n(\tau) - u^n(\tau) \right)^2 + (u^n(\tau) - u_{M_n}^n(\tau))^2 (t_{n,j} - \tau)^{\alpha-1} d\tau $$

$$ \leq c_0 h_n^{a+1} \sum_{t_{n-1}}^{t_{n-1}} \left( T_{\tau,M_n}^{\alpha-1,0} u^n(\tau) - u^n(\tau) \right)^2 + (u^n(\tau) - u_{M_n}^n(\tau))^2 (t_{n,j} - \tau)^{\alpha-1} d\tau $$

$$ \leq c_0 h_n^{a+1} \sum_{t_{n-1}}^{t_{n-1}} \left( T_{\tau,M_n}^{\alpha-1,0} u^n(\tau) - u^n(\tau) \right)^2 + (u^n(\tau) - u_{M_n}^n(\tau))^2 (t_{n,j} - \tau)^{\alpha-1} d\tau $$

Next, we derive an estimation for $E_{1.3}$.

$$ E_{1.3} = \| (I - T_{\lambda,M_n}) \int_{t_{n-1}}^{t} (t - \tau)^{\alpha-1} (T_{\tau,M_n}^{\alpha-1,0} - I) (\kappa(\tau,t)\psi(\tau,u^n(\tau))d\tau) \|^2 $$

$$ = \int_{\Omega_n} \left( (I - T_{\lambda,M_n})(\int_{t_{n-1}}^{t} (t - \tau)^{\alpha-1} (T_{\tau,M_n}^{\alpha-1,0} - I) (\kappa(\tau,t)\psi(\tau,u^n(\tau))d\tau) \right)^2 dt $$

$$ = \int_{\Omega_n} \left( (I - T_{\lambda,M_n})(\int_{t_{n-1}}^{t} (t - \tau)^{\alpha-1} (T_{\tau,M_n}^{\alpha-1,0} - I) (\kappa(\tau,t)\psi(\tau,u^n(\tau))d\tau) \right)^2 dt $$

According to Lemma 4 with $m = 1$, we get that

$$ E_{1.3} \leq c h_n^2 (M_n + 1)^{-2} \int_{\Omega_n} \left( \int_{\Omega_n} (t_n - \lambda)^{\alpha-1} (T_{\lambda,M_n}^{\alpha-1,0} - I) \left| \frac{t - t_{n-1} \lambda}{h_n} \right|^2 \kappa(\tau,\psi(\sigma(\lambda,t),u^n(\sigma(\lambda,t))))d\lambda \right)^2 dt $$

(72)
Now, we derive the upper bound for each above terms as follows

\[
D_{3,1} \leq c_\alpha h_n^{2\alpha} \int_{\Omega_n} (t - t_n - 1)^{2\alpha - 1} \left[ \int_{\Omega_n} (t_n - \lambda)^{\alpha - 1} d\lambda \cdot \int_{\Omega_n} (t_n - \lambda)^{\alpha - 1} \left( I^{\alpha - 1,0}_{\lambda,M_n} - I \right) \right] \times \left( \kappa(\sigma(\lambda,t),t) \psi(\sigma(\lambda,t),u^n(\sigma(\lambda,t))) \right)^2 d\lambda dt \\
\leq c_\alpha h_n^{2m+\alpha} (M_n + 1)^{-2m} \|\psi(\cdot, u^n(\cdot))\|_{H^{\alpha-1,0}(\Omega_n)}^2 \int_{\Omega_n} (t - t_n - 1)^{2\alpha - 1} dt \\
\leq c_\alpha h_n^{2m+\alpha} (M_n + 1)^{-2m} \|\psi(\cdot, u^n(\cdot))\|_{H^{\alpha-1,0}(\Omega_n)}^2,
\]

Similarly,

\[
D_{3,2} \leq c_\alpha h_n^{2m+\alpha+1} (M_n + 1)^{-2m} \|\psi(\cdot, u^n(\cdot))\|_{H^{\alpha-1,0}(\Omega_n)}^2,
\]

and finally,

\[
D_{3,3} \leq c_\alpha h_n^{2m+\alpha} (M_n + 1)^{2-2m} \left[ \int_{\Omega_n} (t_n - \lambda)^{\alpha-1} \left( I^{\alpha-1,0}_{\lambda,M_n} - I \right) \left( \partial_t (\psi(\sigma(\lambda,t),u^n(\sigma(\lambda,t)))) \right)^2 d\lambda \right] dt \\
\leq c_\alpha h_n^{2m+\alpha} (M_n + 1)^{2-2m} \left[ \int_{\Omega_n} (t_n - \lambda)^{\alpha-1} \left( \partial_t (\psi(\sigma(\lambda,t),u^n(\sigma(\lambda,t)))) \right)^2 d\lambda \right] dt \\
\leq c_\alpha h_n^{2m+\alpha} (M_n + 1)^{-2m} \|\psi(\cdot, u^n(\cdot))\|_{H^{\alpha-1,0}(\Omega_n)}^2.
\]

Consequently from Eqs. (72)-(76), the upper bound for \(E_{1,3}\) can obtain as follows

\[
E_{1,3} \leq c_\alpha h_n^{2m+\alpha+1} (M_n + 1)^{-2m} \|\psi(\cdot, u^n(\cdot))\|_{H^{\alpha-1,0}(\Omega_n)}^2.
\]

Therefore, using Eqs. (66), (68), (70), (78) and (46) we have

\[
\|E_1\|^2 \leq c_\alpha h_n^\alpha \left( h_n^{-2} e_n \|\xi_n\|_{\Omega_n}^2 + h_n^{2m+1} (M_n + 1)^{-2m} \|\psi(\cdot, u^n(\cdot))\|_{H^{\alpha-1,0}(\Omega_n)}^2 \right).
\]

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