Visualization for Petrov’s odd unitary group

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ABSTRACT
In this article, we define a set of matrices analogous to Vaserstein-type matrices, which were introduced in the paper ‘Serre’s problem on projective modules over polynomial rings and algebraic $K$-theory’ by Suslin–Vaserstein in 1976. We prove that these are elementary linear matrices. Also, under some conditions, these matrices belong to Petrov’s odd unitary group, which is a generalization of all classical groups. We also prove that these matrices generate the Petrov’s odd elementary hyperbolic unitary group when the ring is commutative.

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1. Introduction

In [1], L.N. Vaserstein and A.A. Suslin studied the freeness of projective modules over polynomial rings. They proved that a positive solution exists for Serre’s problem for polynomial rings in five variables over an arbitrary field and also for polynomial rings in four variables over a principal ideal domain. For a given alternating matrix $\varphi$ of size $2n$, Vaserstein proved the existence of two elementary matrices of size $2n-1$, which can be modified to get symplectic matrices with respect to $\varphi$.

In [2,3], P. Chattopadhyay and R.A. Rao used Vaserstein’s construction for defining the relative elementary symplectic group with respect to a given invertible alternating matrix. They proved a dilation principle and a local-global principle for the relative elementary symplectic group and used these results for the study of symplectic transvection groups. They have also established the equality of the orbit space of a unimodular element under the action of the linear group, the symplectic group with respect to the standard symplectic form, and the symplectic group with respect to an invertible alternating matrix.

In [4], V.A. Petrov introduced a new type of classical-like group known as an odd unitary groups, which generalizes all the classical Chevalley groups, the groups $U_{2n+1}(R)$ of E. Abe [5], and the classical-like groups such as Bak’s hyperbolic unitary groups [6] and G. Tang’s Hermitian groups [7]. The importance of this group is that it includes the odd-dimensional orthogonal group $O_{2n+1}$ as well as the groups $U_{2n+1}(R)$ of E. Abe, which was not covered in the theory of quadratic or Hermitian groups. Later A. Bak and R. Preussler [8] studied a subclass of Petrov’s odd unitary groups, which contain all the classical Chevalley groups.
and classified the E-normal subgroups of the members of this subclass. The E-normal sub-
groups of odd unitary groups are also studied by W. Yu, Y. Li and H. Liu in [9] and also by
R. Preusser in [10].

In a recent preprint [11], A.A. Ambily and R.A. Rao defined the Vaserstein-type matrices
for DSER elementary orthogonal transformations and proved that the group generated by
these matrices is congruent to the DSER group. In this paper, we do an analogous result
for Petrov’s odd elementary hyperbolic unitary group. Since Petrov’s odd unitary group
is a generalization of all classical groups, and it also coincides with the DSER elementary
orthogonal group in the commutative ring case with pseudo-involution $a \mapsto -a$, the result
in this paper generalizes analogous results in the literature.

Here we define two matrices, $L(v)$ and $L(v)^*$, similar to the construction by Vaserstein
which reduces to Vaserstein-type matrices and the matrices defined by Ambily–Rao in
certain special cases. We prove that for arbitrary $v \in \mathbb{R}^{n+2m-1}$, these matrices belong to
$E_{n+2m}(R)$. We also prove that if $v = (a_1, \ldots, a_{n+2m-1}) \in \mathbb{R}^{n+2m-1}$ satisfies the condition
$$a_1 - a_1 = \bar{1}^{-1}(a_2, \ldots, a_{n+2m-1})(\bar{\psi}_m - \varphi)(a_2, \ldots, a_{n+2m-1})^t,$$
then $L(v) \in U_{2m}(R, \mathcal{L}_{\text{max}})$ and for those $v$ satisfying
$$\bar{1}a_1 - \bar{1}a_1 = (a_2, \ldots, a_{n+2m-1})(\bar{\psi}_m - \varphi)(a_2, \ldots, a_{n+2m-1})^t,$$
$L(v)^* \in U_{2m}(R, \mathcal{L}_{\text{max}})$, where $\bar{\psi}_m$ and $\bar{\psi}_m'$ are as per the Equations (1) and (2) in
Section 4 and $\varphi$ is the form matrix for the quadratic form $q_0 = (\langle \cdot, \cdot \rangle, \mathcal{L}_0)$. These results
involve more calculations than the case of Bak’s unitary group because of the fact that,
for a transvection to be an element in Petrov’s odd unitary group, it has to be an isometry
as well as congruent to the identity modulo the odd form parameter taken. The latter
condition is an additional requirement.

The following is the main result proved in this article.

**Theorem 1.1:** Let $R$ be a commutative ring with unity and suppose that $v = (a_1, \ldots, a_{n+2m-1}) \in \mathbb{R}^{n+2m-1}$ satisfies the conditions
$$a_1 - a_1 = \bar{1}^{-1}(a_2, \ldots, a_{n+2m-1})(\bar{\psi}_m - \varphi)(a_2, \ldots, a_{n+2m-1})^t$$
and
$$\bar{1}a_1 - \bar{1}a_1 = (a_2, \ldots, a_{n+2m-1})(\bar{\psi}'_m - \varphi)(a_2, \ldots, a_{n+2m-1})^t,$$
where $\bar{\psi}_m$ and $\bar{\psi}'_m$ are as per the Equations (1) and (2) in Section 4 and $\varphi$ is the form
matrix for the quadratic form $q_0$. Then the matrices $L(v)$ and $L(v)^*$ generate the Petrov’s odd
elementary hyperbolic unitary group $EU_{2m}(R, \mathcal{L}_{\text{max}})$.

We can study the relative version of Petrov’s odd unitary group using this result analogous
to the study done by Chattopadhyay and Rao in Ref. [2]. This theorem will help us to
visualize the odd elementary hyperbolic unitary group in a simpler form and enable us to
study the structure of that group in more depth.

2. **Preliminaries**

2.1. **Odd unitary groups**

Odd unitary groups were defined by V.A. Petrov in Ref. [3].
**Definition 2.1:** Let $R$ be an associative ring with 1. An additive map $\sigma : R \to R$ defined by $r \mapsto \tilde{r}$ satisfying the properties $\tilde{r_1 r_2} = \tilde{r_2} \tilde{r_1}^{-1}$ and $\tilde{r} = r$ for all $r, r_1, r_2 \in R$ is called a pseudo-involution on $R$.

Let $R$ be a ring with pseudo-involution and $V$ be a right $R$-module. A map $\langle \cdot, \cdot \rangle : V \times V \to R$ is called a sesquilinear form on $V$ if it is biadditive and satisfies the equation

$$\langle ur, vs \rangle = \tilde{r} \tilde{s}^{-1} \langle u, v \rangle,$$

for all $u, v \in V$ and $r, s \in R$.

A sesquilinear form is anti-Hermitian if it also satisfies $\langle u, v \rangle = -\overline{\langle v, u \rangle}$ for all $u, v \in V$.

**Definition 2.2:** Let $V$ be a right $R$-module and $\langle \cdot, \cdot \rangle$ be an anti-Hermitian form on it. The set $\mathcal{H} = V \times R$ is a group with composition defined as:

$$(u, r) \cdot (v, s) = (u + v, r + s + \langle u, v \rangle).$$

Here the identity element is $(0, 0)$ and the inverse of $(u, r)$ is $(-u, -r + \langle u, u \rangle)$, which is denoted by $-(u, r)$. This group is called the *Heisenberg group* of the form $\langle \cdot, \cdot \rangle$.

Consider the right action of $R$ on $\mathcal{H}$ defined by $(u, r) \cdot s = (us, s\tilde{r}^{-1}rs)$. The *trace* map on $\mathcal{H}$, $\text{tr} : \mathcal{H} \to R$ is a group homomorphism defined by $\text{tr}((u, r)) = r - \tilde{r} - \langle u, u \rangle$.

Define $\mathcal{L}_{\text{min}}$ and $\mathcal{L}_{\text{max}}$ as follows.

$$\mathcal{L}_{\text{min}} = \{(0, r + \tilde{r}) : r \in R\}, \quad \mathcal{L}_{\text{max}} = \{\zeta \in \mathcal{H} : \text{tr}(\zeta) = 0\}.$$ 

The subsets $\mathcal{L}_{\text{min}}$ and $\mathcal{L}_{\text{max}}$ are subgroups of $\mathcal{H}$. These subgroups are stable under the action of $R$ and satisfy $\mathcal{L}_{\text{min}} \leq \mathcal{L}_{\text{max}}$.

**Definition 2.3:** An *odd form parameter* is a subgroup $\mathcal{L}$ of $\mathcal{H}$ satisfying $\mathcal{L}_{\text{min}} \leq \mathcal{L} \leq \mathcal{L}_{\text{max}}$ and is stable under the action of $R$.

If $\mathcal{L}$ is an odd form parameter corresponding to the sesquilinear form $\langle \cdot, \cdot \rangle$, then the pair $q = (\langle \cdot, \cdot \rangle, \mathcal{L})$ is called an *odd quadratic form* and the pair $(V, q)$ is called an *odd quadratic space*.

Let $(V, q)$ be an odd quadratic space. The even part of the odd form parameter $\mathcal{L}$ is defined as $\mathcal{L}_{\text{ev}} = \{a \in R : (0, a) \in \mathcal{L}\}$ and the even part of the quadratic space is defined as

$$V_{\text{ev}} = \{u \in V : (u, a) \in \mathcal{L} \text{ for a certain } a \in R\}.$$ 

Let $V$ and $V'$ be two $R$-modules equipped with form parameters $\mathcal{L}$ and $\mathcal{L}'$, respectively. An isometry $f : V \to V'$ preserves the form parameters if $(f(v), r) \in \mathcal{L}'$ for all $(v, r) \in \mathcal{L}$. Two such isometries $f$ and $g$ are said to be equivalent modulo $\mathcal{L}'$ if $(f(v) - g(v), (g(v) - f(v), g(v))) \in \mathcal{L}'$ for all $v \in V$. It can be verified that this relation is an equivalence relation on the set of all isometries from $V$ to $V'$. If $f$ and $g$ are equivalent modulo $\mathcal{L}$, we denote it by $f \equiv g \pmod{\mathcal{L}}$.

**Definition 2.4:** The *odd unitary group* denoted by $U(V, q)$ is the group of all bijective isometries $f$ on $V$ such that $f \equiv e \pmod{\mathcal{L}}$, where $e$ is the identity map on $V$. 
Lemma 2.1 ([3], Lemma 1): The transvections $T_u,v(r)$ lie in $U(V, q)$.

A pair of vectors $(u, v)$ such that $\langle u, v \rangle = 1, (u, 0) \in \mathcal{L}$, $(v, 0) \in \mathcal{L}$ is called a hyperbolic pair. Let $\mathbb{H}$ be odd quadratic space spanned by vectors $e_1$ and $e_{-1}$ such that $\langle e_1, e_{-1} \rangle = 1$, equipped with the odd form parameter $\mathcal{L}$ generated by $e_1, 0$ and $(e_{-1}, 0)$. Denote the orthogonal sum of $m$ copies of $\mathbb{H}$ by $\mathbb{H}^m$.

Consider an odd quadratic space $V_0$ equipped with an odd quadratic form $q_0 = (\langle r \rangle_0, \mathcal{L}_0)$. The orthogonal sum $V = \mathbb{H}^m \oplus V_0$ is called the odd hyperbolic unitary space of rank $m$ corresponding to $\mathcal{L}$ and the odd unitary group in this case is called the odd hyperbolic unitary group and is denoted by $U_{2m}(R, \mathcal{L})$.

Definition 2.5: The greatest number $m$ satisfying the condition that there exist $m$ mutually orthogonal hyperbolic pairs in $(V, q)$ is called the Witt index of $(V, q)$, denoted by $\text{ind}(V, q)$.

Let $V$ be an odd quadratic space with Witt index at least $m$. Then we can choose an embedding of $\mathbb{H}^m$ to $V$. Fix any such embedding. Then we have elements $\{e_i\}_{i=1,\ldots,m,-m,\ldots,-1}$ in $V$ such that $\langle e_i, e_j \rangle = 0$ for $i \neq -j$, $\langle e_i, e_{-i} \rangle = 1$ for $i = 1, \ldots, m$ and $(e_i, 0) \in \mathcal{L}$. Let $V_0$ denote the orthogonal complement of $(e_1, e_{-1}, \ldots, e_m, e_{-m})$ in $V$ and $(\langle, \rangle_0$ and $\mathcal{L}_0$ be the restrictions of $\langle, \rangle$ and $\mathcal{L}$, respectively to $V_0$. Then $V_0$ is a quadratic space with the quadratic form $(\langle r \rangle_0, \mathcal{L}_0)$ and $V$ is isometric to the odd hyperbolic space $\mathbb{H}^m \oplus V_0$. Thus $U(V, q) = U_{2m}(R, \mathcal{L})$.

We can define the elementary subgroup $\text{EU}_{2m}(R, \mathcal{L})$ of an odd hyperbolic unitary group $U_{2m}(R, \mathcal{L})$ as follows. Define elementary transvections as transvections on $V$ of the form

$$
T_{i,j}(r) = T_{e_{-i},-e_{r}e_{j}}(0), \quad j \neq \pm i, \quad r \in R,
$$

$$
T_{i}(u, r) = T_{e_{i},ue_{-i}}(-\bar{e}_{-i}r_{e_{-i}}), \quad (u, r) \in \mathcal{L},
$$

where $i, j \in \{-m, \ldots, -1, 1, \ldots, m\}$ and $e_i = \begin{cases} 1, & i > 0 \\ -1, & i < 0 \end{cases}$. The odd elementary hyperbolic unitary group $\text{EU}_{2m}(R, \mathcal{L})$ is the subgroup of $U_{2m}(R, \mathcal{L})$ generated by all elementary transvections.

3. Matrix form of elementary transvections

Let $R$ be a commutative ring with identity. Here we write the matrices for the generators of $\text{EU}_{2m}(R, \mathcal{L})$ when the odd quadratic space $V_0$ is free of rank $n$. Let $\{e_1, e_{-1}, \ldots, e_m, e_{-m}, v_1, \ldots, v_n\}$ be a basis for the odd hyperbolic unitary space $V = \mathbb{H}^m \oplus \mathcal{L}_0$. Then $U_{2m}(\mathbb{H}^m \oplus \mathcal{L}_0)$ is represented by matrices of the form
Thus we get generated by all elements of the form $T_{e_{1},v}(a)$, where $\langle e_{1}, v \rangle = \langle e_{-1}, v \rangle = 0$ and $(v, a) \in \mathcal{L}$.

Let $u = b_{1}e_{1} + \cdots + b_{m}e_{m} + b_{m}e_{m} + t_{1}v_{1} + \cdots + t_{n}v_{n}$ be an arbitrary element of $V$. For $i \in \{1, \ldots, m\}$, we have $e_{i} = -1$ and therefore $T_{i}(u, a) = T_{e_{i},u}(-a)$. Also for $i \in \{-m, \ldots, -1\}$, we have $e_{-i} = \bar{1}^{-1}$ and therefore $T_{i}(u, a) = T_{e_{i},u}(-1)(-a)$. Thus for arbitrary $(v, a) \in \mathcal{L}$ with $\langle e_{1}, v \rangle = \langle e_{-1}, v \rangle = 0$ we have $T_{e_{1},v}(a) = T_{1}(-v, -a)$ and $T_{e_{-1},v}(a) = T_{-1}(v, -a)$. Thus we can restate the above proposition as follows.

**Proposition 3.2:** The group $\text{EU}_{2m}^{n}(R, \mathcal{L})$ coincides with the group generated by all elements of the form $T_{\pm 1}(v, a)$, where $\langle e_{1}, v \rangle = \langle e_{-1}, v \rangle = 0$ and $(v, a) \in \mathcal{L}$.

Note 1 The pseudo-involution we have taken satisfies the properties $\bar{r}_{1}r_{2} = \bar{r}_{2}\bar{1}^{-1}r_{1}$ and $\bar{r} = r$ for all $r, r_{1}, r_{2} \in R$. Therefore we have $\bar{1} \cdot \bar{1} = \bar{1}^{-1}$.

Now we write the matrices for the generators $T_{\pm 1}(v, a)$, of the Petrov’s odd elementary hyperbolic unitary group.

For computing $T_{1}(u, a) = T_{e_{1},u}(-a)$, the element $u$ should satisfy $\langle e_{1}, u \rangle = 0$ which implies that $b_{-1} = 0$. Therefore $u$ has the form

$$u = b_{1}e_{1} + b_{2}e_{2} + b_{m}e_{m} + b_{m}e_{m} + t_{1}v_{1} + \cdots + t_{n}v_{n}.$$  

We have $T_{1}(u, a)(w) = T_{e_{1},u}(-a)(w) = w + e_{1}\bar{1}^{-1}(\langle -u, w \rangle - a(e_{1}, w)) - u(e_{1}, w)$. Thus we get

$$T_{1}(u, a)(v_{k}) = v_{k} + e_{1}\bar{1}^{-1}(-v_{1}t_{1} - \cdots - v_{n}t_{n}, v_{k})$$  

$$= v_{k} - e_{1}\bar{1}^{-2}(\bar{r}_{1}(v_{1}, v_{k}) + \cdots + \bar{r}_{n}(v_{n}, v_{k})), \text{ for } k \in \{1, \ldots, n\},$$

$T_{1}(u, a)(e_{1}) = e_{1}$ and

$$T_{1}(u, a)(e_{-1}) = -e_{1}(\bar{b}_{1}\bar{1}^{-2} + a\bar{1}^{-1} + b_{1}) + e_{-1} - b_{2}e_{2} - b_{-2}e_{-2} - \cdots$$  

$$- b_{m}e_{m} - b_{-m}e_{-m} - t_{1}v_{1} - \cdots - t_{n}v_{n}.$$  

Also $T_{1}(u, a)(e_{k}) = \begin{cases} e_{k} + e_{1}\bar{1}^{-1}\bar{b}_{-k} & \text{for } k \in \{2, \ldots, m\}, \\ e_{k} - e_{1}\bar{1}^{-2}\bar{b}_{-k} & \text{for } k \in \{-m, \ldots, -2\}. \end{cases}$

Therefore the matrix of $T_{1}(u, a)$ is of the form

$$\begin{pmatrix} 1 & -\bar{b}_{1}\bar{1}^{-2} + a\bar{1}^{-1} + b_{1} \\ \bar{1}^{-1}\bar{b}_{-2} & \bar{1}^{-2}\bar{b}_{-2} & \cdots & \bar{1}^{-1}\bar{b}_{-m} & \bar{1}^{-2}\bar{b}_{-m} & -\bar{1}^{-2}(\bar{r}_{1}(v_{1}, v_{1})) & \cdots & -\bar{1}^{-2}(\bar{r}_{n}(v_{n}, v_{1})) & \cdots & -\bar{1}^{-2}(\bar{r}_{n}(v_{n}, v_{n})) \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -b_{2} & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -b_{-2} & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & -b_{m} & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & -b_{-m} & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & -t_{1} & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & -t_{n} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$
Now we consider the case when $R$ is a commutative ring and the involution is $a \mapsto \bar{a} = -a$. In this case, the above matrices reduces to the following form.

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-b_1 + a - b_{-1} & 1 & -b_2 & \bar{b}_2 & \cdots & 0 & 0 \\
0 & -b_m & \bar{b}_m & \bar{b}_{-m} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-t_1 & 0 & 0 & 0 & \cdots & 0 & 1 \\
-t_2 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\end{pmatrix}
$$
\[ T_1(u, a) = \begin{pmatrix} 
1 & a & b_{-2} & b_2 & \ldots & b_{-m} & b_m & t_1(v_1, v_1) & \ldots & t_1(v_1, v_n) \\
0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
0 & -b_2 & 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
0 & -b_{-2} & 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & -b_m & 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 \\
0 & -b_{-m} & 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
0 & -t_1 & 0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\
0 & -t_n & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 
\end{pmatrix}, \text{ and} \\
T_{-1}(u, a) = \begin{pmatrix} 
1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
a & 1 & b_{-2} & b_2 & \ldots & b_{-m} & b_m & t_1(v_1, v_1) & \ldots & t_1(v_1, v_n) \\
-b_2 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
-b_{-2} & 0 & 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-b_m & 0 & 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 \\
-b_{-m} & 0 & 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
-t_1 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-t_n & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 
\end{pmatrix}. 
\]

4. Visualization of odd elementary hyperbolic unitary group

Here for a natural number \( n \), we take \( R^n \) as the space of rows of length \( n \) with entries from \( R \). Let \( \varphi \) be an invertible alternating matrix of size \( 2n \) of the form \( \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix} \). Then \( \varphi^{-1} \) can be written in the form \( \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix} \), where \( c, d \in R^{2n-1} \). For \( v \in R^{2n-1} \), Vaserstein (in Ref. [1], Lemma 5.4) considered the matrices

\[ \alpha = I_{2n-1} + d^t v \mu \quad \text{and} \quad \beta = I_{2n-1} - \rho v^t c \]

and proved that the matrices \( \begin{pmatrix} \frac{1}{d} & 0 \\ \rho & \alpha \end{pmatrix} \) and \( \begin{pmatrix} \frac{1}{v} & 0 \\ \rho & \beta \end{pmatrix} \) belongs to \( E_{2n}(R) \cap \text{Sp}_{\psi} (R) \), where \( \text{Sp}_{\psi} (R) \) is the group of all symplectic matrices with respect to the form \( \psi \).

Here we define analogous matrices for Petrov’s odd elementary hyperbolic unitary group. The form matrix for Petrov’s group is \( \Psi = \tilde{\psi} \bot \varphi \), where \( \varphi \) is the form matrix for the quadratic space \( (V_0, L_0) \) and \( \tilde{\psi}_r \) is defined as follows.

\[ \tilde{\psi}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad \tilde{\psi}_r = \tilde{\psi}_1 \bot \tilde{\psi}_{r-1} \quad \text{for} \quad r > 1. \]
Then $\Psi$ can be written in the form $\begin{pmatrix} 0 & c \\ -1c^t & \mu \end{pmatrix}$, where $c = (1, 0, \ldots, 0) \in \mathbb{R}^{n+2m-1}$ and $\mu$ is the $(n + 2m) \times (n + 2m)$ matrix given by $\mu = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & \psi_{n+1} & 0 & \ldots & 0 \\ 0 & 0 & \psi_{n+2} & \ldots & 0 \\ 0 & 0 & 0 & \ldots & \psi_{2m} \end{pmatrix}$. Also we have $\Psi^{-1}$ is of the form $\begin{pmatrix} 0 & d \\ -1d^t & \rho \end{pmatrix}$, where $d = (-1, 0, \ldots, 0) \in \mathbb{R}^{n+2m-1}$ and $\rho = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & \psi_{n+1} & 0 & \ldots & 0 \\ 0 & 0 & \psi_{n+2} & \ldots & 0 \\ 0 & 0 & 0 & \ldots & \psi_{2m} \end{pmatrix}$, where $\psi_{2m}$ is defined as follows.

$$
\tilde{\psi}^r_1 = \begin{pmatrix} 0 & -\bar{1}^{-1} \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{\psi}_r = \tilde{\psi}^r_1 \perp \tilde{\psi}^r_{r-1}, \quad \text{for } r > 1.
$$

(2)

For $\nu = (a_1, \ldots, a_{n+2m-1}) \in \mathbb{R}^{n+2m-1}$, define the Vaserstein type matrices $L(\nu)$ and $L(\nu)^*$ as

$$
L(\nu) = \begin{pmatrix} 1 & 0 \\ \nu^t & \alpha \end{pmatrix} \quad \text{and} \quad L(\nu)^* = \begin{pmatrix} 1 & \nu \\ 0 & \beta \end{pmatrix},
$$

where $\alpha = I_{n+2m+1} + d^t \tilde{\nu} \mu$ and $\beta = I_{n+2m+1} - \bar{1}^{-1} \rho \tilde{v} c$.

By direct computations, we get the matrices $\alpha$ and $\beta$ corresponding to the form $\Psi$ as follows:

$$
\alpha = \begin{pmatrix} 1 & -\bar{1}^{-1} \bar{a}_1 & \ldots & -\bar{1}^{-1} \bar{a}_{2m-1} & -\bar{1}^{-1} (\bar{a}_{2m} \varphi_{11} + \bar{a}_{2m+1} \varphi_{21}) & \ldots & -\bar{1}^{-1} (\bar{a}_{2m} \varphi_{1n} + \bar{a}_{2m+1} \varphi_{2n}) \\ 0 & 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 1 \end{pmatrix},
$$

and

$$
\beta = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \\ -\bar{1}^{-1} \bar{a}_3 \\ \vdots \\ -\bar{1}^{-1} \bar{a}_{2m-1} \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ -\bar{1}^{-1} (\bar{a}_{2m} (\varphi^{-1})_{11} + \bar{a}_{2m+1} (\varphi^{-1})_{12}) + \cdots + \bar{a}_{2m-1+n} (\varphi^{-1})_{1n} \\ \vdots \\ \vdots \\ \vdots \\ -\bar{1}^{-1} (\bar{a}_{2m} (\varphi^{-1})_{n1} + \bar{a}_{2m+1} (\varphi^{-1})_{n2}) + \cdots + \bar{a}_{2m-1+n} (\varphi^{-1})_{nn} \end{pmatrix}.
$$

Therefore, we have

$$
L(\nu) = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ a_1 & 1 & -\bar{1}^{-1} \bar{a}_1 & \ldots & 0 & 0 & \ldots & 0 \\ a_2 & 0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\ a_3 & 0 & 0 & 1 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n+2m-1} & 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 1 \end{pmatrix}.
$$
Proof: By direct computation, we can verify that \( \bar{\nu} \mu d^t = 0 \). Therefore \( 1 + \bar{\nu} \mu d^t = 1 \). By [1, Lemma 2.2], we have \( \alpha = 1 + d^t \bar{\nu} \mu \in \text{GL}_{n+2m}R \) and \( \left( \begin{array}{c} 1 \\ \nu^t \\ 0 \end{array} \right) \in \text{E}_{n+2m}R \). Now by using [1, Lemma 2.1], we get \( \left( \begin{array}{ccc} 1 \\ \nu^t \\ 0 \end{array} \right) \in \text{E}_{n+2m}R \). Therefore, we have

\[
\left( \begin{array}{cc} 1 \\ \nu^t \\ 0 \end{array} \right) = \left( \begin{array}{ccc} 1 \\ 0 \\ 0 \\ I^{n+2m-1} \end{array} \right) = \left( \begin{array}{ccc} 1 \\ 0 \\ 0 \\ \alpha \end{array} \right) \in \text{E}_{n+2m}R. \]

The matrices \( L(v) \) and \( L(v)^* \) are defined for arbitrary \( v = (a_1, \ldots, a_{n+2m-1}) \in \mathbb{R}^{n+2m-1} \). Under some conditions on the components of \( v \), we get the following result.

**Theorem 4.2:** We have \( L(v) \in U_{2m}(R, \mathcal{L}_{\text{max}}) \), if \( v = (a_1, \ldots, a_{n+2m-1}) \in \mathbb{R}^{n+2m-1} \) satisfies the condition

\[
\bar{a}_1 - a_1 = \bar{1}^{-1}(a_2, \ldots, a_{n+2m-1})(\bar{\psi}_{m-1} \perp \varphi)(a_2, \ldots, a_{n+2m-1})^t \quad (3)
\]

and \( L(v)^* \in U_{2m}(R, \mathcal{L}_{\text{max}}) \) for those \( v \) satisfying the condition

\[
\bar{a}_1 - a_1 = (a_2, \ldots, a_{n+2m-1})(\bar{\psi}_{m-1} \perp \bar{\psi}_{m-1} \perp \bar{\psi}_{m-1}^t)^t(a_2, \ldots, a_{n+2m-1})^t, \quad (4)
\]

where \( \bar{\psi}_{m-1} \) and \( \bar{\psi}_{m-1}' \) are as per the Equations (1) and (2) and \( \varphi \) is the form matrix for the quadratic form \((\cdot, \cdot)_0, \mathcal{L}_0\).

Proof: By direct calculation we can verify that under the condition given in Equation (3), \( L(v) \) satisfies

\[
\bar{1}^{-1}L(v)^tL(v) = \Psi.
\]

Also if \( v \) satisfies the condition given in Equation (4), \( L(v)^* \) satisfies

\[
\bar{1}^{-1}L(v)^*^tL(v)^* = \Psi,
\]

which shows that \( L(v) \) and \( L(v)^* \) preserves the form matrix \( \Psi \). Now to prove that \( L(v) \) and \( L(v)^* \) belongs to \( U_{2m}(R, \mathcal{L}_{\text{max}}) \), it suffices to prove that \( L(v), L(v)^* \equiv I_{n+2m} \) (mod
\( L \). That is, for \( g \in \{ L(v), L(v)^* \} \), \((g(w) - w, \langle w - g(w), w \rangle) \in L_{\text{max}} \), for all \( w \in V \). This is always true since both \( L(v) \) and \( L(v)^* \) are isometries and

\[
(gw - w, \langle w - gw, w \rangle) \in L_{\text{max}} \\
\iff tr(gw - w, \langle w - gw, w \rangle) = 0 \\
\iff \langle w - gw, w \rangle + \langle w, w - gw \rangle = \langle gw - w, gw - w \rangle \\
\iff \langle gw, gw \rangle = \langle w, w \rangle
\]

for all \( w \) in \( V \).}

Next, we shall consider the group generated by these two types of matrices and we have the following theorem.

**Theorem 4.3:** Let \( R \) be a commutative ring with unity and and suppose that \( v \in R^{n+2m-1} \) satisfies the conditions (3) and (4) in Theorem 4.2. Then the matrices \( L(v) \) and \( L(v)^* \) generate the Petrov’s odd elementary hyperbolic unitary group \( EU_{2m}(R, L_{\text{max}}) \).

**Proof:** By comparing the matrices for \( L(v) \) and \( T_{-1}(u, a) \), we have

\[
L(v) = T_{-1}(-a_2e_2 - a_3e_{-2} - \cdots - a_{2m-2}e_m - a_{2m-1}e_{-m} - a_{2m}v_1 - a_{2m+1}v_2 - \cdots \\
- a_{n+2m-1}v_n, a_1),
\]

and

\[
L(v)^* = T_{1}(-\tilde{1}^{-2}\tilde{a}_3e_2 + \tilde{1}^{-1}\tilde{a}_2e_{-2} - \cdots - \tilde{1}^{-2}\tilde{a}_{2m-1}e_m + \tilde{1}^{-1}\tilde{a}_{2m-2}e_{-m} \\
+ \tilde{1}^{-1}(\tilde{a}_{2m}(\varphi^{-1})_{11} + \tilde{a}_{2m+1}(\varphi^{-1})_{12} + \cdots + \tilde{a}_{2m-1+n}(\varphi^{-1})_{1n})v_1 \\
+ \cdots + \tilde{1}^{-1}(\tilde{a}_{2m}(\varphi^{-1})_{n1} + \tilde{a}_{2m+1}(\varphi^{-1})_{n2} + \cdots + \tilde{a}_{2m-1+n}(\varphi^{-1})_{nn})v_n, -\tilde{1}a_1).
\]

These expressions are defined for those \( v \) satisfying the conditions (3) and (4) in Theorem 4.2, because \((u_1, a_1) \) and \((u_2, -\tilde{1}a_1) \) \( \in L_{\text{max}} \), where

\[
u_1 = -a_2e_2 - a_3e_{-2} - \cdots - a_{2m-2}e_m - a_{2m-1}e_{-m} - a_{2m}v_1 - a_{2m+1}v_2 - \cdots \\
- a_{n+2m-1}v_n \text{ and}
\]

\[
u_2 = -\tilde{1}^{-2}\tilde{a}_3e_2 + \tilde{1}^{-1}\tilde{a}_2e_{-2} - \cdots - \tilde{1}^{-2}\tilde{a}_{2m-1}e_m + \tilde{1}^{-1}\tilde{a}_{2m-2}e_{-m} \\
+ \tilde{1}^{-1}(\tilde{a}_{2m}(\varphi^{-1})_{11} + \tilde{a}_{2m+1}(\varphi^{-1})_{12} + \cdots + \tilde{a}_{2m-1+n}(\varphi^{-1})_{1n})v_1 \\
+ \cdots + \tilde{1}^{-1}(\tilde{a}_{2m}(\varphi^{-1})_{n1} + \tilde{a}_{2m+1}(\varphi^{-1})_{n2} + \cdots + \tilde{a}_{2m-1+n}(\varphi^{-1})_{nn})v_n.
\]

Thus \( L(v) \) and \( L(v)^* \) belongs to \( EU_{2m}(R, L_{\text{max}}) \).

Now to prove the reverse inclusion take an arbitrary element in \( EU_{2m}(R, L_{\text{max}}) \). By Proposition 3.2, the elements of \( EU_{2m}(R, L_{\text{max}}) \) are generated by elements of the form \( T_{\pm 1}(v, a) \) where \((v, a) \in L_{\text{max}} \) and \( \langle e_1, v \rangle = \langle e_{-1}, v \rangle = 0 \). Therefore in the expression for \( v \), both \( b_1 \) and \( b_{-1} \) are zero and \( v \) has the form

\[
v = b_2e_2 + b_{-2}e_{-2} + \cdots + b_m e_m + b_{-m} e_{-m} + t_1v_1 + \cdots + t_nv_n.
\]
Then $T_1(v, a) = L(u)^*$, where

$$
u = (-\bar{t}_1, -\bar{t}_1 b_{-2}, -\bar{t}_1^{-2} b_2, \ldots, -\bar{t}_1^{-2} b_m, -\bar{t}_1^{-2} \bar{t}_m, -\bar{t}_1^{-2} \bar{t}_m \varphi_{11} + \cdots + \bar{t}_n \varphi_{1n})$$

and $T_{-1}(v, a) = L(u)$, where $u = (a, -b_2, -b_{-2}, \ldots, -b_m, -b_{-m}, -t_1, \ldots, -t_n)$.

Thus for $v \in R^{n+2m-1}$ satisfying the conditions (3) and (4), $L(v)$ and $L(v)^*$ generate the Petrov's odd elementary hyperbolic unitary group $EU_2m(R, \mathcal{L}_{\max})$.  

**Example 4.1:** Here we illustrate Theorem 1.1 in which in the special case where $R$ is a commutative ring, and the involution is $a \mapsto \bar{a} = -a$. In this case, the anti-Hermitian form becomes Hermitian. The form matrix for Petrov's group in this case is $\Psi = \widetilde{\Psi}_m \perp \varphi$, where $\varphi$ is the form matrix for $(\langle, \rangle_0, \mathcal{L}_0)$ and $\widetilde{\Psi}_r$ for $r > 1$ is defined as follows.

Let $\widetilde{\Psi}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\widetilde{\Psi}_r = \widetilde{\Psi}_1 \perp \widetilde{\Psi}_{r-1}$ for $r > 1$. The form matrix $\Psi$ is of the form $\begin{pmatrix} 0 & c \\ c^t & \mu \end{pmatrix}$, where $c = (1 \ 0 \ \ldots \ 0) \in R^{n+2m-1}$ and $\mu$ is the $(n + 2m) \times (n + 2m)$ matrix given by $\mu = \begin{pmatrix} 0 & 0 & 0 & \varphi_{m-1} \\ 0 & \varphi_{m-1} & 0 & 0 \\ 0 & 0 & \varphi & 0 \\ 0 & \varphi & 0 & 0 \\ 0 & 0 & 0 & \varphi^{-1} \end{pmatrix}$, and $\Psi^{-1}$ is of the form $\begin{pmatrix} 0 & d \\ d^t & \rho \end{pmatrix}$, where $d = (1 \ 0 \ \ldots \ 0) \in R^{n+2m-1}$ and $\rho = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 0 & \cdots \end{pmatrix}$.

Thus for $v = (a_1, \ldots, a_{n+2m-1}) \in R^{n+2m-1}$, we get the matrices $\alpha$ and $\beta$ as follows:

$$\alpha = I_{n+2m-1} - d^t \nu \mu \quad \text{and} \quad \beta = I_{n+2m-1} - \rho v^t c.$$ 

The matrices $\alpha$ and $\beta$ are of the following form.

\[
\alpha = \begin{pmatrix}
1 & -a_1 & -a_2 & \cdots & -a_{2m-1} & -a_{2m-2} & -a_{2m} \varphi_{11} & \cdots & -a_{2m} \varphi_{1n} & -a_{2m} \varphi_{mn} - a_{2m+1} \varphi_{2n} \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix},
\]

\[
\beta = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
a_2 & 0 & 0 & \cdots & 0 \\
a_3 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\bar{a}_{2m-1} & 0 & 0 & \cdots & 0 \\
-\bar{a}_{2m-2} & 0 & 0 & \cdots & 0 \\
\bar{a}_{2m}(\varphi^{-1})_{11} + a_{2m+1}(\varphi^{-1})_{12} + \cdots + a_{2m-1+n}(\varphi^{-1})_{1n} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\bar{a}_{2m}(\varphi^{-1})_{n1} + a_{2m+1}(\varphi^{-1})_{n2} + \cdots + a_{2m-1+n}(\varphi^{-1})_{mn} & 0 & 0 & \cdots & 1
\end{pmatrix}.
\]
Therefore, we have the matrices $L(v)$ and $L(v)^*$ of the form

$$
L(v) = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
a_1 & 1 & -a_3 & -a_2 & \cdots & -a_{2m-1} & -a_{2m-2} & \cdots & -a_{2m\psi_{11}} - a_{2m+1\psi_{21}} & \cdots & -a_{2m\psi_{1n}} - a_{2m+1\psi_{2n}} \\
a_2 & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
a_3 & 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_{n+2m-1} & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1
\end{pmatrix}
$$

and $L(v)^* = \begin{pmatrix}
1 & a_1 & a_2 & \cdots & a_{n+2m-1} \\
0 & 1 & 0 & \cdots & 0 \\
0 & -a_3 & 1 & \cdots & 0 \\
0 & -a_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & -a_{2m-2} & 0 & \cdots & 0 \\
0 & -a_{2m-1} & 0 & \cdots & 0 \\
0 & a_{2m}(\varphi^{-1})_{11} + a_{2m+1}(\varphi^{-1})_{12} + \cdots + a_{2m-1+n}(\varphi^{-1})_{1n} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & a_{2m}(\varphi^{-1})_{n1} + a_{2m+1}(\varphi^{-1})_{n2} + \cdots + a_{2m-1+n}(\varphi^{-1})_{nn} & 0 & \cdots & 1
\end{pmatrix}$

Both these matrices $L(v)$ and $L(v)^*$ in this example belongs to $U_{2m}(R, L_{\text{max}})$ and we have

$$
L(v) = T_{-1}(-a_2e_2 - a_3e_2 - \cdots - a_{2m-2}e_m - a_{2m-1}e_m - a_{2m}v_1 - \cdots - a_{n+2m-1}v_n, a_1),
$$

$$
L(v)^* = T_1(a_3e_2 + a_2e_2 + \cdots + a_{2m-1}e_m + a_{2m-2}e_m + (a_{2m}(\varphi^{-1})_{11} + \cdots + a_{n+2m-1}(\varphi^{-1})_{1n}v_1 + \cdots + (a_{2m}(\varphi^{-1})_{n1} + \cdots + a_{n+2m-1}(\varphi^{-1})_{nn}v_n, a_1).
$$

Also, $(u_1, a_1)$ and $(u_2, a_1)$ belongs to $L_{\text{max}}$, where

$$
u_1 = -a_2e_2 - a_3e_2 - \cdots - a_{2m-2}e_m - a_{2m-1}e_m - a_{2m}v_1 - \cdots - a_{n+2m-1}v_n$$

and

$$
u_2 = a_3e_2 + a_2e_2 + \cdots + a_{2m-1}e_m + a_{2m-2}e_m + (a_{2m}(\varphi^{-1})_{11} + \cdots + a_{n+2m-1}(\varphi^{-1})_{1n}v_1 + \cdots + (a_{2m}(\varphi^{-1})_{n1} + \cdots + a_{n+2m-1}(\varphi^{-1})_{nn}v_n.
$$

Thus $L(v)$ and $L(v)^*$ belongs to $EU_{2m}(R, L_{\text{max}})$.

Now for the reverse inclusion take an arbitrary element in $EU_{2m}(R, L_{\text{max}})$. By Proposition 3.2, the elements of $EU_{2m}(R, L_{\text{max}})$ are generated by elements of the form $T_{\pm 1}(v, a)$ where $(v, a) \in L_{\text{max}}$ and $\langle e_1, v \rangle = \langle e_{-1}, v \rangle = 0$. Therefore in the expression for $v, b_1$ and $b_{-1}$ are zero. Thus $v = b_2e_2 + b_{-2}e_2 + \cdots + b_me_m + b_{-m}e_m + t_1v_1 + \cdots + t_nv_n$. Then we have

$$
T_1(v, a) = L(a, b_2, b_2, \ldots, b_{-m}, b_{-m}, t_1, \ldots, t_n),
$$

and

$$
T_{-1}(v, a) = L(a, b_2, b_2, \ldots, b_{-m}, b_{-m}, \varphi_{11}t_1 + \cdots + \varphi_{1n}t_n, \ldots, \varphi_{1n}t_1 + \cdots + \varphi_{mn}t_n)^*.
$$

Thus for $v \in \mathbb{R}^{n+2m-1}$ satisfying the conditions (3) and (4), the group generated by $L(v)$ and $L(v)^*$ coincides with the Petrov’s group $EU_{2m}(R, L_{\text{max}})$. 

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