TOTAL CURVATURE OF COMPLETE SURFACES IN HYPERBOLIC SPACE

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ABSTRACT. We prove a Gauss-Bonnet formula for the extrinsic curvature of complete surfaces in hyperbolic space under some assumptions on the asymptotic behaviour.

1. Introduction and main results

In this paper we prove a Gauss-Bonnet formula for the total extrinsic curvature of complete surfaces in hyperbolic space. Our result is analogous to those obtained by Dillen and Kühnel in [DK] for submanifolds of euclidean space, where the total curvature of a submanifold $S$ is given in terms of the Euler characteristic $\chi(S)$, and the geometry of $S$ at infinity (cf. also [Dut]).

Our starting point is the following well-know equality for $S \subset \mathbb{H}^3$, a compact surface with boundary in hyperbolic 3-space

$$\int_S KdS = 2\pi\chi(S) + F(S) - \int_{\partial S} k_\gamma ds$$

being $K$ the extrinsic curvature of $S$, $F(S)$ the area, and $k_\gamma$ the geodesic curvature of $\partial S$ in $S$. We plan to make $S$ expand over a complete non-compact surface, but the last two terms in (1) are likely to become infinite. To avoid an indeterminate form, we add and subtract the area enclosed by the curve $\partial S$. Such a notion was defined by Banchoff and Pohl (cf. [BP] and also [Teu]) for any closed space curve $C$ as

$$\mathcal{A}(C) := \frac{1}{\pi} \int_{\mathcal{L}} \lambda^2(\ell, C)d\ell$$

where $\mathcal{L}$ is (in our case) the space of geodesics in $\mathbb{H}^3$, $d\ell$ is the invariant measure on $\mathcal{L}$ (unique up to normalization), and $\lambda(\ell, \partial S)$ is the linking number of $\partial S$ with $\ell \in \mathcal{L}$. This definition was motivated by the Crofton formula which states

$$F(S) = \frac{1}{\pi} \int_{\mathcal{L}} \#(\ell \cap S)d\ell,$$

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where \# stands for the cardinal. Hence, we can rewrite (1) as follows

\[
\int_S KdS = 2\pi \chi(S) + \frac{1}{\pi} \int_{\mathcal{L}} (\#(\ell \cap S) - \lambda^2(\ell, \partial S))d\ell - A(\partial S) + \int_{\partial S} k_g ds
\]

Our main result is a similar formula for complete surfaces in \(H^3\) defining a smooth curve \(C\) at infinity. In that case, the last two terms of the previous equation are replaced by a conformal invariant of the geometry of \(C\) in the ideal boundary of \(H^3\). To be precise, our result applies to surfaces with cone-like ends in the sense defined next. A similar notion of cone-like ends for submanifolds in euclidean space appears in [DK].

**Definition 1.1.** Let \(f : S \rightarrow H^3\) be an immersion of a \(C^2\)-differentiable surface \(S\) in hyperbolic space. We say \(S\) has cone-like ends if

i) \(S\) is the interior of a compact surface with boundary \(\overline{S}\), and taking the Poincaré half-space model of hyperbolic space, \(f\) extends to an immersion \(f : \overline{S} \rightarrow \mathbb{R}^3\),

ii) \(C = f(\partial S)\) is a collection of disjoint simple closed curves contained in \(\partial_{\infty} H^3\), the boundary of the model, and

iii) \(f(S)\) is orthogonal to \(\partial_{\infty} H^3\) along \(C\).

In particular, such a surface is complete with the induced metric. We will see that surfaces with cone-like ends have finite total extrinsic curvature. There are also examples of complete non-compact surfaces with finite total extrinsic curvature which do not fulfill i) or ii) in the previous definition. Condition iii) however is necessary for the total curvature to be finite: the limit of the extrinsic curvature of \(S\) at an ideal point \(x \in C\) is \(\cos^2(\beta)\) where \(\beta\) is the angle between \(S\) and \(\partial_{\infty} H^3\) at \(x\).

In the Klein (or projective) model, the definition reads the same, but replacing the word ‘orthogonal’ by ‘transverse’. We will mainly work with the Poincaré half-space model. Unless otherwise stated all the metric notions (such as length, area or curvature) will refer to the hyperbolic metric. We shall use the words ‘smooth’ or ‘differentiable’ as synonymous of \(C^1\)-differentiable. For simplicity, we will abuse the notation by identifying immersed surfaces and their image.

Given an oriented curve \(C \subset \partial_{\infty} H^3 \equiv \mathbb{R}^2\), and a pair of distinct points \(x, y \in C\), let us consider the oriented angle at \(x\) from \(C\) to the oriented circle through \(x\) that is positively tangent to \(C\) at \(y\). This angle admits a unique continuous determination \(\theta : C \times C \rightarrow \mathbb{R}\) that vanishes on the diagonal. Note that \(\theta(y, x) = \theta(x, y)\) and \(\theta\) is independent of the orientation of \(C\).

We will prove the following result.

**Theorem 1.** Let \(S \subset H^3\) be a simply connected surface of class \(C^2\), embedded in the Poincaré half-space model of hyperbolic space, and with a (connected) cone-like
end \( C \subset \partial_\infty \mathbb{H}^3 \). Then, the integral over \( S \) of the extrinsic curvature \( K \) is
\[
\int_S KdS = \frac{1}{\pi} \int_L (\#(\ell \cap S) - \lambda^2(\ell, C))d\ell - \frac{1}{\pi} \int_{C \times C} \theta \sin \theta \frac{dxdy}{\|y - x\|^2}
\]
where
- \( d\ell \) is an invariant measure on the space of geodesics \( L \),
- \( \lambda^2(\ell, C) \) is 1 if the ideal endpoints of \( \ell \) are on different components of \( \partial_\infty \mathbb{H}^3 \setminus C \) and 0 otherwise, and
- \( dx, dy \) denote length elements on \( C \) with respect to an euclidean metric \( \| \cdot \| \) on \( \partial_\infty \mathbb{H}^3 \equiv \mathbb{R}^2 \).

The integrals in (3) are absolutely convergent.

The idea of the proof is roughly the following. We pull-back \( d\ell \) to the space of chords of \( S \). Integration gives the first term on the right hand side of (3). Applying Stokes theorem yields then the result. This procedure was already used in [Po], but here we use a different form having \( d\ell \) as exterior derivative. This leads to a somehow dual construction, where the total curvature instead of the area appears. This dual viewpoint could not be taken in euclidean space.

**Remark 1.** The form \( dxdy/\|y - x\|^2 \), as well as \( \theta(x, y) \), are invariant under Möbius transformations. Hence, the last term in (3) is invariant under the action of the Möbius group on \( \partial_\infty \mathbb{H}^3 \). We call this term the ideal defect of \( S \). Similar expressions for space curves appear often in the study of conformally invariant knot energies (cf. [LO]).

From the previous theorem one gets easily a formula for a general surface with cone-like ends.

**Corollary 2.** Let \( S \hookrightarrow \mathbb{H}^3 \) be a \( C^2 \)-immersed complete surface with cone-like ends \( C_1, \ldots, C_n \), the curves \( C_i \) being disjoint and connected. Then
\[
\int_S KdS = 2\pi(\chi(S) - n) + \frac{1}{\pi} \int_L (\#(\ell \cap S) - \sum_{i=1}^n \lambda^2(\ell, C_i))d\ell - \frac{1}{\pi} \sum_{i=1}^n \int_{C_i \times C_i} \theta \sin \theta \frac{dxdy}{\|y - x\|^2},
\]
and the previous integrals are absolutely convergent.

**Proof.** Take a compact set \( K \subset \mathbb{H}^3 \) with smooth boundary \( \partial K \) transverse to \( S \), and such that \( S \setminus K = S_1 \cup \ldots \cup S_n \), where each \( S_i \) is an embedded topological cylinder over \( C_i \). Applying (1) and (2) to \( R = S \cap K \) yields
\[
\int_R KdR = 2\pi\chi(R) - \int_{\partial R} k_g(s)ds + \frac{1}{\pi} \int_L (\#(\ell \cap R))d\ell
\]
where \( k_g \) is the geodesic curvature in \( R \).
Let $R_i$ be a compact surface with boundary such that $T_i = R_i \cup S_i$ is a complete embedded simply connected surface. Combining again (1) and (2),

\[ \int_{R_i} K dR_i = 2\pi - \int_{\partial R_i} k_g(s) ds + \frac{1}{\pi} \int_{\mathcal{L}} \#(\ell \cap R_i) d\ell. \]

Applying Theorem 1 to each $T_i$, and comparing with (5) yields

\[ \int_{S_i} K dS_i = -2\pi + \frac{1}{\pi} \int_{\mathcal{L}} \#(\ell \cap S_i) - \lambda^2(\ell, C_i) d\ell \]

\[ - \frac{1}{\pi} \int_{C_i \times C_i} \theta \sin \theta \frac{dy}{\|y-x\|^2} + \int_{\partial R_i} k_g(s) ds. \]

Addition of (4) and (6) finishes the proof. □

1.1. The ideal defect. The last term in (3), which we call the ideal defect, can be described in the following way.

**Proposition 3.** Let $C \subset \partial_{\infty} \mathbb{H}^3$ be a closed simple (connected) curve of class $C^2$. Let $\Omega \subset \partial_{\infty} \mathbb{H}^3$ be one of the domains bounded by $C$. Then

\[ \int_{C \times \Omega} \theta \sin \theta \frac{dy}{\|y-x\|^2} = 4 \int_{NT(\Omega)} \frac{dz}{\|z-w\|^4} \]

where $NT(\Omega) \subset \Omega \times \Omega$ is the set of point pairs $(z, w)$ such that any circle $\xi \subset \partial_{\infty} \mathbb{H}^3$ containing $z$ and $w$ intersects $\partial_{\infty} \mathbb{H}^3 \setminus \Omega$ (i.e. $z, w \in \xi \Rightarrow \xi \notin \Omega$.)

**Proof.** Let $Q \subset \mathbb{H}^3$ be the convex hull of $\Omega^c = \partial_{\infty} \mathbb{H}^3 \setminus \Omega$; i.e. $Q$ is the minimal convex set containing $\Omega^c$. Using the Klein model, $Q$ can be seen as the euclidean convex hull of $\Omega^c$. Let us consider the boundary $S = \partial Q \subset \mathbb{H}^3$, which is a surface of class $C^1$. Next we construct a sequence of convex sets $Q_n \subset \mathbb{H}^3$ such that: $Q_n \supset Q_{n+1}$, $Q = \cap_{n=1}^{\infty} Q_n$, and $S_n = \partial Q_n$ is a $C^2$ surface with cone-like end $C$. First, let $X \in \mathcal{X}(\mathbb{R}^3)$ be a vector field in the Klein model such that $X$ vanishes only at $C$, and $X|_{\Omega}$ points to the interior of the model. Then, for small $t > 0$, the flow $\varphi_t$ brings $\Omega$ to a surface $\varphi_t(\Omega)$ with a cone-like end on $C$, and bounding a convex domain $D$. On the other hand, let $Q$ be approximated by a decreasing sequence $Q'_n \subset \mathbb{H}^3$ of euclidean convex sets with smooth boundary (cf. [Sch]). Then, smoothening the corners of $D \cap Q'_n$ yields the desired sequence.

By Theorem 1

\[ \int_{S_n} K(x) dx = \frac{1}{\pi} \int_{\mathcal{L}} \#(\ell \cap S_n) - \lambda^2(\ell, C) d\ell - \int_{C \times C} \theta \sin \theta \frac{dy}{\|y-x\|^2}. \]

Using, for instance, the arguments in [LS], one can show

\[ \lim_n \int_{S_n} K(x) dx = 0. \]
On the other hand, by monotone convergence,
\[
\lim_{n} \int_{\mathcal{L}} (\#(\ell \cap S_{n}) - \lambda^{2}(\ell, C))d\ell = \int_{\mathcal{L}} (\#(\ell \cap S) - \lambda^{2}(\ell, C))d\ell.
\]
Hence,
\[
\int_{C \times C} \theta \sin \theta \frac{dxdy}{\|y - x\|^{2}} = \frac{1}{\pi} \int_{\mathcal{L}} (\#(\ell \cap S) - \lambda^{2}(\ell, C))d\ell.
\]
The right hand side above is the measure of geodesics intersecting $Q$ but not $\Omega$. We determine each geodesic $\ell \in \mathcal{L}$ by its ideal endpoints. From equation (12) below, one easily computes
\[
(7) \quad d\ell = 4 \frac{dzdw}{\|z - w\|^{4}}.
\]
Finally, we just need to note that a geodesic $\ell$ intersects the convex hull $Q$ if and only if every geodesic 2-plane containing $\ell$ intersects $\Omega$. \qed

1.2. Integral of the inverse of the chord. Next we find an alternative description of the ideal defect. Let $C \subset \partial_{\infty} \mathbb{H}^{3}$ be a connected smooth simple closed curve, and consider $S = C \times (0, \infty) \subset \mathbb{H}^{3}$. We may think of $S$ as a surface with one end by closing the top end at infinity with an infinitesimally small surface. Then, the total curvature of $S$ equals $2\pi$, and Theorem 1 applied to $S$ yields
\[
2\pi + \frac{1}{\pi} \int_{C \times C} \theta \sin \theta \frac{dxdy}{\|y - x\|^{2}} = \frac{2}{\pi} \int_{\mathbb{R}^{2} \times \mathbb{R}^{2}} (\#(\overline{zw} \cap C) - \lambda^{2}(z, w; C)) \frac{dzdw}{\|w - z\|^{4}}
\]
\[
(8) \quad = \frac{2}{\pi} \int_{A(2, 1)} \sum_{x, y \in \mathcal{L} \cap C} \frac{(-1)^{\#(x \cap y \cap C)}}{\|y - x\|} dL
\]
where $\overline{zw}$ denotes the line segment joining $z, w \in \mathbb{R}^{2}$, and $dL$ is the invariant measure on the space $A(2, 1)$ of (unoriented) lines of $\mathbb{R}^{2}$, normalized as in [Sa1]. The first equality uses (7). The second equality follows from $dzdw = \|t - s\|dtdsdt$ where $s, t$ are arc-length parameters of $z, w$ along $L$ (cf. [Sa1], equation (4.2)).

As a consequence, the integral in (8) is invariant under Möbius transformations, which was a priori not obvious. In fact, if $C$ bounds a convex domain $\Omega$, this integral is
\[
(9) \quad \frac{4}{\pi} \int_{A(2, 1)} \frac{1}{\sigma(L \cap \Omega)} dL
\]
where $\sigma(L \cap \Omega)$ is the chord length. The previous functional (9) is one of the so-called Franklin invariants of convex sets, defined by Santaló in [Sa2] as a generalization of a functional introduced by Franklin with motivations from stereology (cf. [Fr]). These functionals had the nice property of being invariant by dilatations. For instance, functional (9) could be used to estimate, by means of line sections,
the number of particles in a plane region, if these particles have the same shape
but possibly different size.

An immediate consequence of our results is that (9) is in fact invariant under the
Möbius group. An interesting question is to determine which of the Franklin func-
tionals enjoy this bigger invariance. Besides, it was conjectured that the Franklin
invariants are minimal for balls (cf.[Fr] and [Sa2]). This was shown by Franklin
among ellipsoids while Santaló obtained some general non-sharp inequalities. As
a consequence of our results, we can prove this conjecture in the plane case.

Corollary 4. For a convex set $\Omega \subset \mathbb{R}^2$ we have

\begin{equation}
\int_{A(2,1)} \frac{1}{\sigma(L \cap \Omega)} dL \geq \frac{\pi^2}{2}.
\end{equation}

where $\sigma$ is the length of the chord, and $A(2,1)$ is the space of lines. Equality holds
in (10) if and only if $\Omega$ is a round disk. Moreover, the left hand side of (10) is
invariant by Möbius transformations (keeping $\Omega$ convex).

Proof. By (8) we have

\begin{equation}
\frac{4}{\pi} \int_{A(2,1)} \frac{1}{\sigma(L \cap \Omega)} dL = 2\pi + \frac{1}{\pi} \int_{C \times C} \theta \sin \theta \frac{dxdy}{\|y - x\|^2} \geq 2\pi,
\end{equation}

and the equality occurs if and only if $\theta \equiv 0$. Indeed, since $C$ is convex it is easy
to see that $-\pi < \theta < \pi$. \hfill \Box

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2. Preliminaries

2.1. The space of geodesics. Let $\mathcal{F} = \{(x; g_1, g_2, g_3)\}$ be the bundle of positive
orthonormal frames of $\mathbb{H}^3$; i.e., each $(g_i)_{i=1,2,3}$ is a positive orthonormal basis of
$T_x \mathbb{H}^3$. We consider on $\mathcal{F}$ the dual and the connection forms

$\omega_i = \langle dx, g_i \rangle, \quad \omega_{ij} = \langle \nabla g_i, g_j \rangle,$

where $\langle , \rangle$ denotes the (hyperbolic) metric in $\mathbb{H}^3$, and $\nabla$ is the corresponding
riemannian connection. The structure equations read

\begin{equation}
d\omega_i = \omega_j \wedge \omega_{ji}, \quad d\omega_{ij} = \omega_i \wedge \omega_j + \omega_{ik} \wedge \omega_{kj}.
\end{equation}

Let $\mathcal{L}^+$ be the space of oriented geodesics of $\mathbb{H}^3$. Consider $\pi_1: \mathcal{F} \to \mathcal{L}^+$ given by
$\pi_1(x; g_1, g_2, g_3) = \ell$ with $x \in \ell$, and $g_1 \in T_x \ell$ pointing in the positive direction. The
space $\mathcal{L}^+$ can be endowed with a differentiable structure such that $\pi_1$ is a smooth
submersion. Moreover, $\mathcal{L}^+$ admits a volume form $d\ell$ invariant under isometries of
$\mathbb{H}^3$, which is unique up to normalization, and characterized by (cf.[Sa1])

\begin{equation}
\pi_1^*(d\ell) = \omega_2 \wedge \omega_3 \wedge \omega_{12} \wedge \omega_{13}.
\end{equation}
Similarly, one can consider $L_2$, the space of (unoriented) totally geodesic surfaces (geodesic planes) of $H^3$. We will use the flag space $L_{1,2} = \{(\ell, \varphi) \in L^+ \times L_2 | \ell \subset \varphi\}$, and the canonical projection $\pi : L_{1,2} \to L^+$ which makes $L_{1,2}$ a principal $S^1$-bundle over $L^+$. Let us project $\pi_{1,2} : F \to L_{1,2}$ so that $\pi_{1,2}(x; (g_i)) = (\ell, \varphi)$ with $\varphi \supset \ell = \pi_1(x; (g_i))$ and $g_3 \perp T_x \varphi$. Then $\omega_{23} = \pi_1^* \varphi$ for a certain form $\varphi \in \Omega^1(L_{1,2})$, which is an invariant global angular form (or connection) of the bundle $\pi$. Next we show some useful properties of the corresponding curvature form.

**Proposition 5.** There exists a unique 2-form $\alpha \in \Omega^2(L^+)$ such that $\pi^*(\alpha) = d\varphi \in \Omega^2(L_{1,2})$.

Moreover $\alpha \wedge \alpha = -2d\ell$, so that $d(\pi^* \alpha \wedge \varphi) = -2\pi^* d\ell$, where $\varphi$ is the global angular form of $\pi$.

**Proof.** Assuming $\alpha$ exists, structure equations (11) give
\begin{equation}
\pi^*(\alpha) = d\omega_{23} = \omega_2 \wedge \omega_3 - \omega_1 \wedge \omega_1,
\end{equation}
whence
\[\pi^*(\alpha \wedge \alpha) = -2\omega_2 \wedge \omega_3 \wedge \omega_1 = -2\pi^* d\ell.\]
Therefore $\alpha \wedge \alpha = -2d\ell$ (as $d\pi_1$ is exhaustive).

Let $X \in \mathfrak{X}(L_{1,2})$ be the tangent vector field along the fibers of $\pi$ such that $\varphi(X) = 1$. By (13), for any $\tilde{X} \in \mathfrak{X}(F)$ such that $d\pi_{1,2} \tilde{X} = X$,
\[\pi_{1,2}^*(i_X d\varphi) = i_{\tilde{X}} d\omega_{23} = 0,
\]
whence $i_X d\varphi = 0$. Then $L_X \varphi = 0$, and
\[L_X d\varphi = dL_X \varphi = 0.
\]
Hence, $d\varphi$ is constant along the fibers of $\pi$, and null on their tangent vectors, which shows the existence of $\alpha$. The uniqueness follows from the injectivity of $\pi^*$.

The forms $\varphi, \alpha$ are in a sense dual to the forms $\omega_1, dI$ used in [Po]. In fact, many of the subsequent constructions are parallel to those of [Po]. However, choosing $\alpha$ leads us to the total curvature, while $dI$ made the area appear. This choice could not be done in an euclidean ambient since there $\alpha \wedge \alpha = 0$.

The following notation will be used throughout the paper:
\[A \ltimes B := \{(x, y) \in A \times B | x \neq y\}.
\]

**Remark 2.** The following complex valued 2-form on $\mathbb{CP}^1 \ltimes \mathbb{CP}^1$, invariant under the diagonal action of $SL(2, \mathbb{C})$, was called the *infinitesimal cross-ratio* in [LO]
\[\omega_{cr} = \frac{dwdz}{(w-z)^2}.\]
By considering the ideal endpoints of each geodesic, $\mathcal{L}^+$ is identified with $\mathbb{CP}^1 \times \mathbb{CP}^1$. Then, it is not difficult to see that $\alpha/8 = \Im(\omega_\gamma)$, the imaginary part of the infinitesimal cross-ratio.

**Remark 3.** i) Consider $\Psi : \mathbb{H}^3 \times \mathbb{H}^3 \to \mathcal{L}^+$ such that $\Psi(x,y)$ goes first through $x$ and then through $y$. Given unit vectors $u \in T_x\mathbb{H}^3$, $v \in T_y\mathbb{H}^3$, and considering $\bar{\pi} = (u,0), \bar{\nu} = (0,v) \in T_x\mathbb{H}^3 \oplus T_y\mathbb{H}^3$, a computation shows

$$
\Psi^* \alpha(\bar{u}, \bar{v}) = \frac{\sin \beta_x \sin \beta_y \sin \theta \cosh r}{\sinh^2 r}
$$

where $\beta_x \in [0, \pi)$ (resp. $\beta_y$) is the angle at $x$ (resp. $y$) between $\ell = \Psi(x,y)$ and $u$ (resp. $\ell$ and $v$), and $r$ denotes the distance between $x,y$. As for $\theta \in [0, 2\pi)$, it is the oriented angle (with respect to the orientation of $\ell$) between the two oriented geodesic planes $\varphi_x, \varphi_y \in \pi^{-1}\ell$ given respectively by $u \wedge w_x$ and $v \wedge w_y$, where $w_x \in T_x\ell, w_y \in T_y\ell$ point in the positive direction of $\ell$.

ii) Similarly, given unit vectors $u, v \in T_y\mathbb{H}^3$, and denoting $\bar{\pi} = (u,0), \bar{\nu} = (v,0) \in T_x\mathbb{H}^3 \oplus T_y\mathbb{H}^3$, one has

$$
\Psi^* \alpha(\bar{u}, \bar{v}) = -\frac{\det(u,v,t)}{\sinh^2 r}
$$

where $t \in T_x \Psi(x,y)$ is the positive unit tangent vector.

### 2.2. The space of chords.

Given a $C^2$-differentiable manifold $S'$ (without boundary), the *space of chords* of $S'$ is a $C^1$-differentiable manifold $M_{S'}$, with boundary, introduced by Whitney in [WH], and described with detail in [Pa]. The interior of $M_{S'}$ is $S' \times S'$, and the boundary is $T^1S'$, the bundle of oriented tangent directions of $S'$. Next we describe the differentiable structure of $M_{S'} = (S' \times S') \cup T^1S'$ in case $S'$ has dimension two. On the one hand, the interior $S' \times S' \subset M_{S'}$ is diffeomorphically identified to the open set $S' \times S' \subset S' \times S'$. On the other hand, given local coordinates $(x_1, x_2)$ on $U \subset S'$, a neighborhood of $T^1U$ in $M_{S'}$ is diffeomorphic to the hypersurface with boundary $L \subset U \times U \times S^1$ given by

$$
L = \{(x_1, x_2), (y_1, y_2), (\cos \theta, \sin \theta) \in U \times U \times S^1 | \sin \theta(y_1 - x_1) = \cos \theta(y_2 - x_2), \cos \theta(y_1 - x_1) \geq 0\}.
$$

The diffeomorphism is as follows: $L \ni (x, y, v) \mapsto (x, y) \in M_{S'}$, for $x \neq y$, and $L \ni (x, x, \cos \theta, \sin \theta) \mapsto v \cdot \mathbb{R}^+ \in T^1S' = \partial M_{S'}$, where $v = \cos \theta \partial/\partial x_1 + \sin \theta \partial/\partial x_2 \in T_xS$. With this differentiable structure, the mappings $p_1, p_2 : M_{S'} \to S'$ given by $p_1(x_1, x_2) = x_1$ for $(x_1, x_2) \in S' \times S'$, and $p_1(x, v) = x$ for $v \in T_xS'$, are smooth submersions. We will consider $M_{S'}$ oriented so that the inclusion $S' \times S' \subset M_{S'}$ preserves the product orientation.

**Remark 4.** Given a regular $C^2$-differentiable curve $x : [0, a) \to S'$, we can define $c : [0, a) \to M_{S'}$ by $c(t) = (x(0), x(t))$ if $t \in (0, a)$, and $c(0) = \dot{x}(0) \cdot \mathbb{R}^+ \in T^1S$. 
Using suitable coordinates in the above description, it is clear that \( c \) is smooth (also at \( t = 0 \)) and meets \( \partial M_{S'} \) transversely.

Let now \( S \subset S' \) be a compact surface with boundary, embedded in \( S' \). We consider \( M_S := p_1^{-1}(S) \cap p_2^{-1}(S) \subset M_{S'} \), the subset of chords of \( S' \) having both ends in \( S \). It should be noticed that \( M_S \) is not a manifold with (smooth) boundary. Indeed, denoting \( C = \partial S \subset S' \), we can describe \( M_S \) as the intersection of two regions \( p_1^{-1}(S), p_2^{-1}(S) \) bounded respectively by the smooth hypersurfaces \( Q_1' = p_1^{-1}(C), Q_2' = p_2^{-1}(C) \). It will be convenient to denote \( Q'_0 = T^1 S' \subset M_{S'} \), and \( Q_0 = T^1 S = Q'_0 \cap M_S \). Set \( Q_1 = C \times S \subset Q'_1 \cap M_S, Q_2 = S \times C \subset Q'_2 \cap M_S \), the topological closures in \( M_{S'} \). In other words, \( Q_1 \setminus (C \times S) \) consists of the inner tangent vectors of \( S \) along \( C = \partial S \), and \( Q_2 \setminus (S \times C) \) contains exterior vectors. We shall denote \( \partial M_S := Q_0 \cup Q_1 \cup Q_2 \). In fact, \( Q_0 \cup Q_1 \cup Q_2 \) is the boundary of \( M_S \) as a topological manifold with boundary.

**Remark 5.** By Remark 11 it is easy to see that \( Q'_0 \) is transverse to both \( Q'_1 \) and \( Q'_2 \). It is even easier to check that \( Q'_1 \) and \( Q'_2 \) intersect transversely at points of \( C \times C \). However, at \( Q'_0 \cap Q'_1 \cap Q'_2 = T^1 C \), the hypersurfaces \( Q'_1 \) and \( Q'_2 \) intersect tangentially. Indeed, \( Q'_1 \cap Q'_2 = (C \times C) \cup T^1 S|_C \), so they cannot be transverse.

Again by Remark 11 the surfaces \( (C \times C) \cup T^1 C(\equiv M_C) \) and \( T^1 S|_C \) are transverse inside \( Q'_1 \), and also in \( Q'_2 \).

Let now \( S \subset \mathbb{H}^3 \) be a compact surface with boundary embedded in hyperbolic space. Then we can consider \( S \) included in an embedded surface \( S' \) without boundary. Consider \( \Phi : S' \setminus S' \to L^+ \) such that \( \Phi(x, y) \) is the oriented geodesic going first through \( x \) and then through \( y \) (in the notation of Remark 11 \( \Phi = \Psi|_{S \times S'} \)). One can extend \( \Phi \) to a smooth mapping \( \Phi : M_{S'} \to L^+ \). This was shown by Pohl for submanifolds in euclidean space (cf. [Po]). Using the Klein model of hyperbolic space it is clear that the same holds for surfaces in \( \mathbb{H}^3 \).

The following is the hyperbolic version of equation (6.5) in [Po] (with \( n = 3, m = 2 \)). We include the proof for the sake of completeness.

**Proposition 6.** Let \( S \) be a compact surface with boundary \( C = \partial S \) embedded in \( \mathbb{H}^3 \). Then

\[
\int_{M_S} \Phi^*(d\ell) = \frac{-1}{2} \int_{L^+} (\#(\ell \cap S) - \lambda^2(\ell, C)) d\ell.
\]

**Proof.** By the coarea formula

\[
\int_{M_S} \Phi^*(d\ell) = \int_{L^+} \mu(\ell) d\ell
\]

where

\[
\mu(\ell) = \sum_{z \in \Phi^{-1}(\ell)} \varepsilon(z)
\]
and $\varepsilon(z) = 1$ (resp. $-1$) if $\Phi$ locally preserves (resp. reverses) orientations at $z \in M_S$. Since a generic line $\ell$ is transverse to $S$, we can consider $z = (x, y) \in S \times S$. Then $\varepsilon(z) = \varepsilon(x) \varepsilon(y)$, being $\varepsilon(w)$ the sign at $w$ of the algebraic intersection $\ell \cdot S$. Now, let $p$ (resp. $q$) be the number of points of $\ell \cap S$ with $\varepsilon = 1$ (resp. $\varepsilon = -1$), so that

$$\#(\ell \cap S) = p + q, \quad \lambda(\ell, C) = \ell \cdot S = p - q.$$  

Then $\Phi^{-1}(\ell)$ contains $(p(p - 1) + q(q - 1))/2$ elements with $\varepsilon = 1$, and $pq$ elements with $\varepsilon = -1$. Therefore $2\mu(\ell) = p(p - 1) + q(q - 1) - 2pq = \lambda^2(\ell, C) - \#(\ell \cap S).$ □

3. Compact surfaces with boundary

From here on we assume $S \subset \Omega \subset \mathbb{H}^3$ is a compact simply connected surface with boundary, contained in a strictly convex set $\Omega$, with $C = \partial S \subset \partial \Omega$. In this case we say that $S$ has convex boundary. We consider $S \subset S'$ with $S' \subset \mathbb{H}^3$ a simply connected surface without boundary. In this section we get a Gauss-Bonnet formula for $S$. By a limit procedure, this formula will lead to Theorem [I].

3.1. Constructing the section. Consider the pull-back by $\Phi$ of the $S^1$-bundle $\pi: L_{1,2} \rightarrow L^+$; i.e. $\Phi^*(L_{1,2}) = \{(x, (\ell, \nu)) \in M_{S'} \times L_{1,2} \mid \Phi(x) = \ell\}$, and the following diagram (with the obvious mappings) commutes

$$\begin{array}{ccc}
\Phi^*(L_{1,2}) & \xrightarrow{\Phi^*} & L_{1,2} \\
\downarrow \Phi^*(\pi) & & \downarrow \pi \\
M_{S'} & \xrightarrow{\Phi} & L^+
\end{array}$$

(16)

Next we aim to construct a global section $s$ of $\Phi^*(\pi)$. Moreover, we try to adapt $s$ to $S$ by imposing some boundary conditions. After doing this, we shall apply Stokes theorem to $s^*\Phi^* \varphi \wedge \Phi^* \alpha$ on $M_S$.

Let $\Theta_0: Q_0 \rightarrow L_2$ be such that $\Theta_0(x)$ is tangent to $S$ at $p_1(x)$. For $a = 1, 2$, let

$$\Theta_a: Q_a \setminus T^1C \rightarrow L_2$$

be such that $\Theta_a(x)$ contains $\Phi(x)$, and is tangent to $C$ at $p_a(x)$. This mapping is well defined since $\Phi(\nu) \cap S \subset \Phi(T^1C) \cap \Omega = \{y\}$ for all $(y, \nu) \in T^1C$. Clearly, $\Theta_0$ and $\Theta_a$ coincide on $Q_0 \cap (Q_a \setminus T^1C) = T^1S|_C \setminus T^1C$ for $a = 1, 2$. However, $\Theta_1$ and $\Theta_2$ do not coincide on $C \cap C = Q_1 \cap Q_2 \setminus T^1C$.

We define

$$s: \partial M_S \setminus (C \cap C) \rightarrow \Phi^*(L_{1,2})$$

$$x \mapsto (x, (\Phi(x), \Theta(x))).$$

(17)

where $\Theta(x) = \Theta_a(x)$ whenever $x \in Q_a$ for some $a \in \{0, 1, 2\}$.

We are assuming $S$ and $S'$ are simply connected. This makes the topology of $\Phi^*(\pi)$ quite simple, as shown by the following proposition.
Proposition 7. \( \Phi^*(\pi) \) is a trivial principal \( S^1 \)-bundle over \( M_{S'} \). Moreover, there is a bundle isomorphism \( \tau: \Phi^*(L_{1,2}) \to M_{S'} \times S^1 \), such that \( \tau \circ s \) lifts over the covering \( q: M_S \times \mathbb{R} \to M_S \times S^1 \); i.e., exists a continuous function

\[
q: \partial M_S \setminus (C \times C) \to \mathbb{R}
\]
such that \( q(x, g(x)) = \tau \circ s(x) \) for every \( x \in \partial M_S \setminus (C \times C) \).

Proof. Consider an isotopy of embeddings \( H: S' \times [0,1] \to \mathbb{R}^3 \) such that \( H_0 = \text{id} \), and \( H_1(S') \) is contained in a plane \( \varphi \in L_2 \). We can assume moreover that \( H_t(S) \) has convex boundary. For each \( t \in [0,1] \) we have \( \Phi_t: M_{S'} \to L^+ \), and the corresponding family of \( S^1 \)-bundles \( \Phi_t^*(\pi): \Phi_t^*(L_{1,2}) \to M_{S'} \), as well as sections \( s_t \) defined on \( \partial M_S \setminus (C \times C) \) as above.

By the homotopy covering theorem (cf. [Spi], Thm.13.4), the maps \( \Phi_t \) induce bundle isomorphisms \( \sigma_t: \Phi_t^*(L_{1,2}) \to \Phi_t^*(L_{1,2}) \). Note that \( x \mapsto (\Phi(x), \varphi) \) is a global section of \( \Phi_1^*(\pi) \). Then, a bundle isomorphism \( \tau_1: \Phi_1^*(L_{1,2}) \to M_{S'} \times S^1 \) exists such that \( \tau_1 \circ s_1 \equiv 1 \in S^1 \). In particular, \( \phi_1^*(\pi) \) is trivial \( \forall t \). As for the second part, take \( \tau = \tau_1 \circ s_0 \), and note that \( \tau_1 \circ \sigma_1 \circ s_1: \partial M_S \setminus (C \times C) \to M_{S'} \times S^1 \) is a homotopy from \( \tau \circ s \) to \( \tau_1 \circ s_1 \), and the latter is constant in the second component. \( \square \)

Proposition 8. On each of the hypersurfaces with boundary \( Q_0, Q_1 \setminus Q_2, Q_2 \setminus Q_1 \), the restriction of \( g \) is smooth. Given \( x \in Q_1 \setminus Q_2 \) (resp. \( x \in Q_2 \setminus Q_1 \)), let \( \alpha(x) \in (0, \pi/2] \) be the angle between \( \Phi(x) \) and \( T_{p_1(x)}C \) (resp. \( T_{p_2(x)}C \)). Then \( \sin(\alpha) \, dg \in T^*Q_1 \setminus Q_2 \) (resp. \( T^*Q_2 \setminus Q_1 \)) is bounded with respect to any continuous norm on \( T^*Q_1 \) (resp. \( T^*Q_2 \)).

Proof. From the relation \( q(x, g(x)) = \tau \circ s(x) \), to show that \( g \) is smooth it is enough to check the smoothness of \( s \). Indeed, \( g \) and \( \tau \) are local diffeomorphisms. Now \( s \) is clearly differentiable on \( Q_0 \), also at the boundary, since \( s \) can be naturally extended over \( T^1S' \). To show \( s \) is smooth on \( Q_1 \setminus Q_2 \) it remains to check the smoothness of \( \Theta_1: Q_1 \setminus Q_2 \to L_2 \). Using the Klein model of hyperbolic space we can write

\[
\Theta_1(x) = p_1(x) + (Z(x) \wedge V(x)),
\]

where \( Z, V: Q_1 \setminus Q_2 \to \mathbb{R}P^2 \), and \( \wedge: \mathbb{R}P^2 \times \mathbb{R}P^2 \to \mathbb{R}P^2 \) are smooth mappings given by

\[
Z(x) = \Phi(x) - p_1(x), \quad V(x) = T_{p_1(x)}C, \quad a, b \perp a \wedge b.
\]

Thus, \( \Theta_1 \) and therefore \( g \) is differentiable on \( Q_1 \setminus Q_2 \). By symmetry, \( g \) is differentiable on \( Q_2 \setminus Q_1 \).

In order to prove that \( \sin adg \) is bounded, it is enough to show that \( \sin ad\Theta_1 \in T^*(Q_1 \setminus Q_2) \otimes T\mathcal{L} \) is bounded, with respect to any continuous norm on \( T^*Q_1 \otimes T\mathcal{L} \). If \( z, v \in S^2 \) are local lifts of \( Z, V \in \mathbb{R}P^2 \), then \( Z \wedge V = \pm (z \wedge v)/\|z \wedge v\| \), and

\[
d\Theta_1 = dp_1 \pm \left( \frac{dz \wedge v + z \wedge dv}{\|z \wedge v\|} - \left( \frac{dz \wedge v + z \wedge dv}{\|z \wedge v\|} \cdot \frac{z \wedge v}{\|z \wedge v\|} \right) \frac{z \wedge v}{\|z \wedge v\|} \right)
\]
Now $dp_1, dz, dv$ are bounded since $p_1, Z, V$ are differentiable all over $Q_1$ which is compact. Finally, $\|z \wedge v\| \geq A|\sin \alpha|$ for some constant $A$, since the euclidean and the hyperbolic metrics are equivalent on compact sets of the Klein model. \hfill \Box

Next we define a function on $\partial M_S$ that coincides with $g$ outside a neighborhood of $C \times C$. In order to define this neighborhood, let us consider $\psi : \mathbb{S}^1 \times [0, 1) \to S$, a smooth embedding with $\psi(\mathbb{S}^1 \times \{0\}) = C$. For every $\varepsilon \in [0, 1]$ let $F_{1, \varepsilon} : \mathbb{S}^1 \times [0, 1] \times [0, 1) \to Q_1$ be given by

$$F_{1, \varepsilon}(r, s, t) = \left\{ \begin{array}{ll}
\left( \psi(r, 0), \psi(r + s, \varepsilon ts(1 - s)) \right), & \text{if } s \in (0, 1) \\
\frac{d\psi(u, 0)}{du} \bigg|_{u=0} + \varepsilon t(1 - 2s) \frac{d\psi(v, 0)}{dv} \bigg|_{v=0} & \text{if } s = 0, 1.
\end{array} \right.$$

where we identify $\mathbb{S}^1 \equiv \mathbb{R}/\mathbb{Z}$. This mapping is smooth in the following sense: taking local coordinates in $M_S$, the resulting local expression of $F_{1, \varepsilon}$ extends to a smooth mapping in a neighborhood of $\mathbb{S}^1 \times \mathbb{R}^2$ of every point in $\mathbb{S}^1 \times [0, 1] \times [0, 1)$.

Symmetrically we consider $F_{2, \varepsilon} : \mathbb{S}^1 \times [0, 1] \times (-1, 0) \to Q_2$ given by

$$F_{2, \varepsilon}(r, s, t) = \left\{ \begin{array}{ll}
\left( \psi(r, -\varepsilon ts(1 - s)), \psi(r + s, 0) \right), & \text{if } s \in (0, 1) \\
\frac{d\psi(u, 0)}{du} \bigg|_{u=0} + \varepsilon t(1 - 2s) \frac{d\psi(v, 0)}{dv} \bigg|_{v=0} & \text{if } s = 0, 1.
\end{array} \right.$$

These two maps can be joined to give a homeomorphism

$$F_{\varepsilon} : \mathbb{S}^1 \times [0, 1] \times (-1, 1) \to U_{\varepsilon} : = imF_{1, \varepsilon} \cup imF_{2, \varepsilon}(\subset \partial M_S).$$

**Definition 3.1.** For every $\varepsilon \geq 0$, let $f_{\varepsilon} : \partial M_S \to \mathbb{R}$ be defined by the following conditions

i) $f_{\varepsilon}(x) = g(x)$ for $x \in \partial M_S \setminus U_{\varepsilon}$

ii) $f_{\varepsilon} \circ F_{\varepsilon}(r, s, t) = \frac{1}{2}((1 + \rho(t))g(F_{\varepsilon}(r, s, 1)) + (1 - \rho(t))g(F_{\varepsilon}(r, s, -1)))$ being $\rho : \mathbb{R} \to [-1, 1]$ a smooth function with $\rho(t) = -1$ for $t \leq -1$, and $\rho(t) = 1$ for $t \geq 1$.

Then $s_{\varepsilon} : \partial M_S \to \phi^*(\mathcal{L}_{1, 2})$ is defined as $s_{\varepsilon}(x) = \tau^{-1} \circ g(x, f_{\varepsilon}(x))$.

**Proposition 9.** The restriction of $f_{\varepsilon}$ is smooth on each of the hypersurfaces $Q_0, Q_1, Q_2$. Moreover $\varepsilon df_{\varepsilon}$ is uniformly bounded over $U_{\varepsilon}$; i.e. after fixing some continuous norm $\| \cdot \|$ on $T^*Q_1 \cup T^*Q_2$, exists $A > 0$ such that

$$\| \varepsilon (df_{\varepsilon})_x \| \leq A, \quad \forall \varepsilon > 0, \forall x \in U_{\varepsilon}.$$

**Proof.** The smoothness of $f_{\varepsilon}$ follows from Proposition and the smoothness of $F_{a, \varepsilon}$. Let $z, v$ be as in the proof of Proposition. Consider the signed angle $\beta$ in the Klein model between $z$ and $v$. We give $\beta$ locally a sign according to some local orientation of $S$; i.e. $\sin \beta = N \cdot (z \wedge v)$ where $N$ is a unit normal vector. Then $\beta(r, s, \varepsilon) := \beta \circ F_{\varepsilon}(r, s, 1) = \beta \circ F_1(r, s, \varepsilon)$ is a smooth function on $\mathbb{S}^1 \times [0, 1] \times [0, 1)$. Since $C$ is in the boundary of a strictly convex set, $\beta$ vanishes only at points of the form $(r_0, 0, 0)$ or $(r_0, 1, 0)$. Taylor expansion near $(r_0, 0, 0)$ yields

$$\beta(r, s, \varepsilon) = a(r_0)\varepsilon + b(r_0)s + O((r - r_0)^2 + s^2 + \varepsilon^2)$$
for some functions $a, b > 0$. After a similar analysis near $(r_0, 1, 0)$, it follows that
\[ \beta(r, s, z) > C \varepsilon \]
for some constant $C > 0$. Similarly, $\beta \circ F_\varepsilon(r, s, -1) > C' \varepsilon$ for some $C' > 0$. Finally, Proposition 8 gives the result, since $\beta/\alpha$ is bounded on compact sets of the Klein model.

**Proposition 10.** There exists a smooth function on $M_{S'}$ extending $f_\varepsilon$.

**Proof.** By remark 5, we can define $\xi \in T^*_z M_{S'}$ at every $z \in \partial M_{S'}$ so that $\xi$ coincides with $df_\varepsilon$ on each of the faces $Q_a$ concurring at $z$. Then Whitney’s extension theorem (cf. [Wh2]) applies, providing an extension of $f_\varepsilon$ to a neighborhood of every point of $\partial M_S$. Using partitions of unity, the extension exists globally on $M_{S'}$.

**Remark 6.** Abusing the notation, we denote the extension by $f_\varepsilon$. We also denote by $s_\varepsilon: M_{S'} \to \phi^*(\mathcal{L}_{1,2})$ the corresponding section of $\Phi^*(\pi)$.

### 3.2. Stokes Theorem

**Proposition 11.** If $S \subset \mathbb{H}^3$ is a compact simply connected surface with convex boundary, then

\[ \int_{\mathbb{H}^3} (\#(\ell \cap S) - \lambda^2(\ell, C))d\ell = \int_{\partial M_S} \Phi^* \alpha \wedge s_\varepsilon^* \varphi, \]

where $\varphi = \Phi^* \varphi$ is the global angular form of $\Phi^* \pi$, and $\partial M_S$ has the orientation induced by $M_{S'}$.

**Proof.** Since the diagram (16) commutes we have $\Phi^* = s_\varepsilon^* \circ \Phi^* \circ \pi^*$. Then Proposition 5 yields

\[ \Phi^*(d\ell) = -\frac{1}{2} d(\Phi^* \alpha \wedge s_\varepsilon^* \varphi). \]

Next we apply Stokes theorem for domains with piecewise smooth boundary (cf. for instance [AMR]) to $M_S \subset M_{S'}$, to get

\[ \int_{M_S} \Phi^*(d\ell) = -\frac{1}{2} \int_{\partial M_S} \Phi^* \alpha \wedge s_\varepsilon^* \varphi. \]

The proof is finished by Proposition 6.

**Theorem 12.** Let $S \subset \mathbb{H}^3$ be a compact simply connected surface with convex boundary. Then

\[ \int_S KdS = \frac{1}{2\pi} \int_L (\#(\ell \cap S) - \lambda^2(\ell, C))d\ell + \frac{1}{\pi} \int_{S \setminus C} \Phi^* \alpha \wedge s_\varepsilon^* \varphi + \frac{1}{2\pi} \int_{\partial \varepsilon C} \theta \Phi^* \alpha \]

where $\theta: C \times C \to \mathbb{R}$ is given by

\[ \theta(x, y) = \lim_{z \to y} g(x, z) - \lim_{z \to x} g(z, y) \quad x, y \in C, z \in S \setminus C, \]

being $g$ as in Proposition 7.
Proof. Starting from (19), we split \( \partial M_S = Q_0 \cup U_\varepsilon \cup (Q_1 \cup Q_2) \setminus U_\varepsilon \), and we begin by integrating over \( Q_0 = T^1 S \): given \((x, v) \in T^1 S\) we have \( \Phi \circ s(x, v) = (\Phi(x, v), \varphi) \) where \( \varphi \) is tangent to \( S \) at \( x \). Let \((x; e_1, e_2, e_3)\) be a positive orthonormal frame with \( e_1 = v \), \( e_3 \perp T_x S \). Then by (13)

\[
\int_{Q_0} \Phi^* \alpha \wedge s^* \varphi = \int_{T^1 S} \omega_{12} \wedge \omega_{13} \wedge \omega_{23} = 2\pi \int_S K dS.
\]

where we use the natural orientation on \( T^1 S \) which is opposite to the one induced by \( M_S \).

Next we integrate on \( U_\varepsilon \), and we pull-back by \( F_\varepsilon \)

\[
\int_{U_\varepsilon} \Phi^* \alpha \wedge s^* \varphi = \int_{S^1} \int_0^1 \int_{-1}^1 F_\varepsilon^* \Phi^* \alpha \wedge F_\varepsilon^* s^* \varphi \left( \frac{\partial}{\partial r}, \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right) dtdsdr.
\]

The previous integrand is uniformly bounded in the following sense: \( \exists A > 0 \) such that

\[
|\left( F_\varepsilon^* \Phi^* \alpha \wedge F_\varepsilon^* s^* \varphi \right) \left( \frac{\partial}{\partial r}, \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right) | < A
\]

everywhere on \( S^1 \times [0, 1] \times [-1, 1] \) and \( \forall \varepsilon > 0 \). Indeed, since \( \Phi \) and \( F_\varepsilon \) are smooth for \( \varepsilon \geq 0 \), and \( F_\varepsilon(r, s, t) = F_1(r, s, \varepsilon t) \), the functions

\[
F_\varepsilon^* \Phi^* \alpha \left( \frac{\partial}{\partial r}, \frac{\partial}{\partial s} \right), \quad \varepsilon^{-1} F_\varepsilon^* \Phi^* \alpha \left( \frac{\partial}{\partial r}, \frac{\partial}{\partial t} \right), \quad \varepsilon^{-1} F_\varepsilon^* \Phi^* \alpha \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right)
\]

are uniformly bounded. Since \( g \) is bounded, it follows directly from Definition 3.1 that \( F_\varepsilon^* s^* \varphi \left( \frac{\partial}{\partial t} \right) \) is uniformly bounded. From Proposition 9 \( \varepsilon df_\varepsilon \circ dF_\varepsilon(\partial/\partial r), \varepsilon df_\varepsilon \circ dF_\varepsilon(\partial/\partial s) \) are uniformly bounded. Therefore,

\[
\varepsilon F_\varepsilon^* s^* \varphi \left( \frac{\partial}{\partial r} \right), \quad \varepsilon F_\varepsilon^* s^* \varphi \left( \frac{\partial}{\partial s} \right)
\]

are uniformly bounded. Altogether shows (21), and we may apply Lebesgue’s dominated convergence theorem to the limit \( \varepsilon \rightarrow 0 \). Thus

\[
\lim_{\varepsilon \rightarrow 0} \int_{U_\varepsilon} \Phi^* \alpha \wedge s^* \varphi = \int_{S^1} \int_0^1 \int_{-1}^1 F_0^* \Phi^* \alpha(\partial r, \partial s) \lim_{\varepsilon \rightarrow 0} \varphi(d\varepsilon \circ dF_\varepsilon(\partial t)) dtdsdr
\]

Now,

\[
\lim_{\varepsilon \rightarrow 0} \varphi(d\varepsilon \circ dF_\varepsilon(\partial t)) = \frac{1}{2} \rho'(t) \theta(\psi(r, 0), \psi(r + s, 0)).
\]

Integrating with respect to \( t \) yields

\[
\lim_{\varepsilon \rightarrow 0} \int_{U_\varepsilon} \Phi^* \alpha \wedge s^* \varphi = \int_{C \times C} (\Phi^* \alpha) \theta.
\]
Finally, we just need to note that by symmetry,
\[ \lim_{\epsilon \to 0} \int_{(Q_1 \cup Q_2) \setminus U_\epsilon} \Phi^* \alpha \wedge s_\epsilon^* \varphi = \int_{(S_0 \cup C) \setminus (C_0 S)} \Phi^* \alpha \wedge s^* \varphi = 2 \int_{S_0 C} \Phi^* \alpha \wedge s^* \varphi \]

\[ \square \]

4. Complete surfaces

In this section we prove Theorem [1]. From here on, \( S \subset \mathbb{H}^3 \) will denote a complete simply connected surface with a cone-like end \( C \subset \partial_\infty \mathbb{H}^3 \). In the following \( \mathbb{H}^3 \) will denote the Poincaré half-space model. Unless otherwise stated, all the metric notions (such as length or curvature) will be referred to the hyperbolic metric \( \langle , \rangle \). For \( h > 0 \), let \( S_h = \{ x \in S | x_3 \geq h \} \) which is a compact surface with convex boundary \( C_h = \partial S_h = \{ x \in S | x_3 = h \} \). We will use the following standard notation: given two functions \( f, g \) (or sequences), \( f = O(g) \) means that \( |f| < A|g| \) for some constant \( A > 0 \). When \( f_n, g_n \) are sequences of functions, \( f_n = O(g_n) \) must be understood uniformly: \( \exists A \) such that \( |f_n| < A|g_n|, \forall n \).

**Lemma 1.** For \( x = (x_1, x_2, x_3) \in S \), let \( II(x) \) be the second fundamental form of \( S \) at \( x \). If \( \|II(x)\|_\infty \) is the maximum of \( |II(x)(u, v)| \) over all unit vectors \( u, v \in T_x S \), then

\[ \|II(x)\|_\infty = O(x_3), \quad \forall x \in S. \]

**Proof.** Fix \( y \in S \), and let \( e_1, e_2, e_3 \) be an orthonormal frame on \( y \) with \( e_3 = y_3 \partial/\partial x_3 \), and \( e_2 \in T_y S \). We can assume \( e_i = y_3 \partial/\partial x_i, \ i = 1, 2 \) and consider the global orthonormal frame \( e_i(x) = x_3 \partial/\partial x_i, \) for all \( x \in \mathbb{H}^3 \). The connection forms \( \theta_{ij} = \langle \nabla e_i, e_j \rangle \) are then given by

\[ \theta_{ij} = \frac{dx_i}{x_3}, \quad \theta_{ij} = 0 \quad \text{for} \ i, j \neq 3 \]

Let \( v_1, v_2, v_3 \) be a frame locally defined on \( S \) (around \( y \)) so that \( v_2(y) = e_2 \), and \( v_1(x), v_2(x) \in T_x S \). Then \( v_i(x) = a_{ij}(x)e_j(x) \) for an orthogonal matrix \( (a_{ij}(x)) \in O(3) \). In particular \( v_1(y) = \cos \alpha e_1 + \sin \alpha e_3, v_3(y) = -\sin \alpha e_1 + \cos \alpha e_3 \) for some \( \alpha \in [0, 2\pi) \). Then \( \omega_{i3}(y) = \langle dv_i, v_3 \rangle_y \) are given by

\[ \omega_{13}(y) = -\sin \alpha da_{11} + \cos \alpha da_{13} + \frac{dx_1}{y_3}, \]

\[ \omega_{23}(y) = -\sin \alpha da_{21} + \cos \alpha da_{23} + \cos \alpha \frac{dx_2}{y_3}. \]

Therefore \( \omega_{ij}(y) = O(y_3) \) for \( i, j = 1, 2 \), since \( \cos \alpha = \langle e_3, v_3 \rangle = O(y_3) \). The latter can be seen using a second order approximation of \( S \) at points of \( C \subset \partial_\infty \mathbb{H}^3 \), and bearing in mind that \( C \) is a curvature line of \( \overline{S} = S \cup C \). \( \square \)
Proposition 13. If $K$ denotes the extrinsic curvature of $S$, and $dS$ is the area element, then

$$\int_S K dS$$

is absolutely convergent. In particular

$$\lim_{h \to 0} \int_{S_h} K dS = \int_S K dS.$$ 

Proof. From Lemma 11 we have $K(x) = O((x_3)^2)$. Then, note that $dS = dS^e/(x_3)^2$ where $dS^e$ is the euclidean area element of $S$. □

Proposition 14. If $\lambda(\ell, C_h)$ is the linking number of a geodesic $\ell$ with the curve $C_h$ then

$$(22) \lim_{h \to 0} \int_{\mathcal{L}^+} (\#(\ell \cap S_h) - \lambda^2(\ell, C_h))d\ell = \int_{\mathcal{L}^+} (\#(\ell \cap S) - \lambda^2(\ell, C))d\ell$$

where $\lambda(\ell, C)$ is the limit of $\lambda(\ell, C_h)$ when $h \to 0$.

Proof. Let $\ell \in \mathcal{L}^+$ be transverse to $S$, which happens for almost every $\ell$. Then $\ell \cap S$ is discrete, and $\#(\ell \cap S_h) - \lambda^2(\ell, C_h)$ is a monotone function in $h$. Hence, (22) follows by monotone convergence. □

We will see a posteriori, that the limit in (22) is finite. This will follow from (3) and the finiteness of the other terms.

Proposition 15. For every $h > 0$ consider the section $s_h$ defined over $\partial M_{S_h \setminus C_h \times C_h}$ in (17). Then

$$(23) \lim_{h \to 0} \int_{S_h \setminus C_h} \Phi^* \alpha \wedge s_h^*\varphi = 0$$

Proof. Let $(y; e_1, e_2, e_3)$ be a local frame associated to $(x, y) \in S_h \times C_h$ with $e_1 \in T_y \Phi(x, y), e_3 \perp \Theta_2(x, y)$. Recall $x, y \in \Theta_2(x, y)$, and $C_h, \Theta_2(x, y)$ are tangent at $y$. Then we can write

$$\Phi^* \alpha \wedge s_h^*\varphi = \pm \omega_{12} \wedge \omega_{13} \wedge \omega_{23}.$$ 

Let $v$ be an euclidean unit tangent vector to $C_h$ at $y$, and $u_1, u_2 \in T_y S$ orthonormal (with respect to the euclidean metric) with $u_1$ tangent to the plane $\Theta_2(x, y)$. It is not difficult to see $\omega_{13}(u_1) = \omega_{13}(v) = \omega_{23}(u_1) = 0$. Then

$$\Phi^* \alpha \wedge s_h^*\varphi (u_1, u_2, v) = \pm \omega_{12}(u_1) \omega_{13}(u_2) \omega_{23}(s) = \Phi^* \alpha (u_1, u_2) s_h^*\varphi (v).$$

By (13), the latter converges to 0 when $h \to 0$ and $(x, y)$ converge to a point in $S \times C$. Moreover, Lemmas 4 and 5 of the next section give $A > 0$ such that

$$(24) \quad |\Phi^* \alpha \wedge s_h^*\varphi| \leq A \frac{|dx \wedge dy|}{\|y - x\|^2 + x_3 - y_3}, \quad \forall h > 0$$
where \(dx\) (resp. \(dy\)) denotes the euclidean area (resp. length) element of \(S\) (resp. \(C_h\)) at \(x\) (resp. \(y\)). Now we consider a sequence of diffeomorphisms \(\psi_h : S \to S_h\) with the euclidean norm of \(d\psi_h\) uniformly bounded. Set \(\Psi_h = \psi_h \times \psi_h|_C : S \times C \to S_h \times C_h\). Then

\[
|\Psi^*_h(\phi^* \alpha \wedge s^*_h \tau)| \leq A |\Psi^*_h \left( \frac{dx \wedge dy}{\|y - x\|^2 + x_3 - h} \right)| \leq A' \left| \frac{dx \wedge dy}{\|y - x\|^2 + x_3} \right|
\]

for some uniform constants \(A, A' > 0\). The right hand side of (25) has finite integral over \(S \times C\). Thus, we can apply Lebesgue’s dominated convergence theorem, and the result follows. \(\square\)

Proposition 16. Let \(\theta_h\) be the function defined on \(C_h \times C_h\) by (20), and denote \(\theta = \lim_{h \to 0} \theta_h\). Then

\[
\lim_{h \to 0} \int_{C_h \times C_h} \theta_h \Phi^* \alpha = 2 \int_{C \times C} \theta \sin \theta \frac{dx dy}{\|y - x\|^2}
\]

where \(dx, dy\) are the length elements of \(C\) at \(x, y\) respectively, with respect to the euclidean metric \(\| \cdot \|\) on \(\partial_\infty \mathbb{H}^3 \equiv \mathbb{R}^2\). Moreover, the right hand side of (26) is finite.

Proof. By Lemma 3 of the next section,

\[
|\Phi^* \alpha| \leq A \left| \frac{dx \wedge dy}{\|y - x\|} \right|
\]

for some \(A > 0\). On the other hand, it is not hard to see

\[
\theta = O \left( \frac{\tau \|y - x\|}{h} \right) = O(\|y - x\|)
\]

where \(\tau\) is the maximal (hyperbolic) torsion of \(C_h\). Hence, Lebesgue’s dominated convergence theorem applies. A computation in the model shows

\[
x_3 y_3 \sinh r = \|y_\infty - x_\infty\| R
\]

where \(x_\infty = (x_1, x_2, 0) \in \partial_\infty \mathbb{H}^3\), and \(y_\infty = (y_1, y_2, 0) \in \partial_\infty \mathbb{H}^3\), and \(R\) is the euclidean radius of the half-circle representing the geodesic \(\Phi(x, y)\). Equations (11), and (28) yield

\[
\lim_{h \to 0} \Phi^* \alpha \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) = \lim_{h \to 0} \frac{\sin \beta_x \sin \beta_y \sin \theta \cosh r}{h^2 \sinh^2 r} = \lim_{h \to 0} \frac{\sin \theta}{\|y_\infty - x_\infty\| R} = \frac{2 \sin \theta}{\|y - x\|^2}.
\]

Finally, the finiteness of the right hand side of (26) follows at once from (27). \(\square\)

Proof of Theorem 11. For every \(h\), we apply Theorem 12 to \(S_h\). Then make \(h \to 0\), and use Propositions 13, 14, 15 and 16. \(\square\)
5. Asymptotic estimations

Here we prove some estimations that have been used before.

Lemma 2. Let \( x, y \in \mathbb{H}^3 \) be two points in the half-space model, at hyperbolic distance \( r \) from each other. Then

\[
\begin{align*}
\text{(29)} & \quad i) \quad x_3 y_3 \sinh r \geq \frac{1}{2} \| y - x \|^2, \\
\text{(30)} & \quad ii) \quad \max\{x_3, y_3\} \sinh r \geq \| y - x \|.
\end{align*}
\]

Proof. We prove \( i) \) in case the geodesic through \( x, y \) is a circle of radius \( R < \infty \) (the case \( R = \infty \) is straightforward). We will use \((28)\). Given \( \| y_\infty - x_\infty \| \), and \( R \) it is clear that \( \| y_\infty - x_\infty \|/\| y - x \| \) is maximum when \( x \) or \( y \) are in \( \partial_\infty \mathbb{H}^3 \). In this case, classical plane geometry yields \( \| y_\infty - x_\infty \|/\| y - x \| = \| y - x \|/(2R) \).

To show \( ii) \) we use that the length \( h \) of an arc of horocycle from \( x \) to \( y \) is \( 2 \sinh(r/2) \). Moreover, this is bigger than the hyperbolic length \( s \) of the euclidean segment joining \( x, y \). Thus

\[
\sinh r = 2 \sinh(r/2) \cosh(r/2) \geq 2 \sinh(r/2) = h \geq s \geq \frac{\| y - x \|}{\max\{x_3, y_3\}}.
\]

Lemma 3. Given a surface \( S \subset \mathbb{H}^3 \) with cone-like ends, there exists some constant \( A > 0 \) such that

\[
|\Phi^\ast \alpha(u, v)| \leq \frac{A}{\| y - x \|} \quad \forall z = (x, y) \in S \times S, \forall u = (u, 0) \in T_1^1 S \oplus 0 \subset T_z(S \times S), \forall v = (0, v) \in 0 \oplus T_y^1 S \subset T_z(S \times S)
\]

where \( T^1 S \) denotes the euclidean unit tangent bundle of \( S \) in the model.

Proof. Assume two sequences \( (x(n), u(n)), (y(n), v(n)) \in TS \) with \( \| u(n) \| = \| v(n) \| = 1 \) such that \( |\Phi^\ast \alpha(u(n), v(n))| \| y(n) - x(n) \| \) is unbounded. We look for a contradiction. For notation simplicity we will omit the index \( n \) of these two sequences.

In case the distance \( r \) between \( x, y \) is bounded below, by \((14)\) and \((29)\) we get

\[
|\Phi^\ast \alpha(u, v)| = O \left( \frac{\sin \theta}{x_3 y_3 \sinh r} \right) \leq O \left( 2 \frac{\sin \theta}{\| y - x \|^2} \right).
\]

We show next that \( \sin \theta = O(\| y - x \|) \). Firstly, we define a sequence of isometries \( f' = f'_n \) of \( \mathbb{H}^3 \) given by \( f'(z) = \rho^{-1} z \) where \( \rho := \| y - x \| \). Then \( \| f'(x) - f'(y) \| = 1 \) which, together with \( \inf r > 0 \), implies \( y_3/\rho, x_3/\rho \in \mathbb{R} \) are bounded above. Then, taking isometries \( f = f' \circ \tau \) with suitable horizontal translations \( \tau \), will make \( f(y) \) and \( f(x) \) converge (after taking subsequences if necessary) in \( \overline{\mathbb{H}^3} = \mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3 \).
If \( P \) is the vertical plane that contains \( \lim f(x), \lim f(y) \) one can see that \( P \cap K \) is approached by \( f(S) \cap K \) for every compact neighborhood \( K \) of \( \lim x \) or \( \lim y \). This convergence is in the \( C^1 \) topology, and has order \( \rho \). In particular, \( \exists (x', u'), (y', v') \in T^1 P \) at distance of order \( \rho \) from \( (f(x), f_s(u)), (f(y), f_s(v)) \in T^1 f(S) \) respectively.

Now consider \( \theta \) as a function defined on \( \mathbb{H}^3 \times \mathbb{H}^3 \times S^2 \times S^2 \). It is easy to see that \( \theta \) is smooth, and therefore it is Lipschitz over the set \( \{(x, y) \in \mathbb{H}^3 \times \mathbb{H}^3 | \|y - x\| = 1\} \times S^2 \times S^2 \). Then

\[
\theta(x, y, u, v) = \theta(f(x), f(y), f_s(u), f_s(v)) = O(\rho)
\]

since \( \theta(x', y', u', v') = 0 \), as \( \theta \) vanishes on points and vectors of a common plane.

In case inf \( r = 0 \), we take a subsequence with \( r \to 0 \). Then, by (14) and (30),

\[
|\Phi^* (u, v)| = O \left( \frac{\sin \beta_y \sin \theta}{\sinh r \|y - x\|} \max \left\{ \frac{x_3}{y_3}, \frac{y_3}{x_3} \right\} \right)
\]

and the last factor is bounded since \( r \) is bounded. We shall see \( \sin \beta_y \sin \theta = O(\sinh r) \). To this end, we rescale by \( x_3^{-1} \) and we compose with horizontal translations so that \( x, y \) converge to some point \( q \in \mathbb{H}^3 \). Then \( \sinh r = O(\|y - x\|) \).

On the other side, if \( v' \) is the parallel transport of \( v \in T_y \mathbb{H}^3 \) to \( T_x \mathbb{H}^3 \) along the geodesic, then

\[
\sin \beta_x \sin \beta_y \sin \theta = \frac{1}{\|y - x\| \|v'\|} \det(u, v', y - x)
\]

\[
= \frac{1}{\|y - x\| \|v'\|} (\det(u, v' - (v' \cdot n)n, y - x) + (v' \cdot n) \det(u, n, y - x))
\]

\[
= O \left( \frac{(y - x) \cdot n}{\|y - x\|} \right) + O (v' \cdot n),
\]

where \( n \) is the euclidean unit normal to \( S \) at \( x \), and \( \cdot \) denotes the euclidean scalar product. The first term has the order of \( \|y - x\| \), since the euclidean normal curvatures of the rescaled surface \( S \) are uniformly bounded. As for the last term note that \( \|v' - v\| = O(\|y - x\|) \), and thus \( v' \cdot n = (v' - v) \cdot n + v \cdot n = O(\|y - x\|) \). \( \square \)

**Lemma 4.** Given a surface \( S \subset \mathbb{H}^3 \) with cone-like ends, there exists some constant \( A > 0 \) such that

\[
|\Phi^* (u, v)| \leq \frac{A}{\|y - x\|} \quad \forall z = (x, y) \in S \times S,
\]

\[
\forall u = (u, 0), v = (v, 0) \in T_x^1 S \oplus T_y^1 S \subset T_z(S \times S).
\]

**Proof.** If \( \beta \) denotes the angle at \( x \) of the geodesic \( \Phi(x, y) \) with \( T_x S \), then by (15) and inequality (30)

\[
|\Phi^* (u, v)| \leq \frac{\sin \beta}{t^2 \sinh^2 r} \leq \frac{\sin \beta}{\|y - x\|^2} = O (\|y - x\|^{-1}).
\]
The last equality follows from \( \sin \beta = O(\|y - x\|) \) which can be proven with arguments similar to those of the previous lemma.

\[\text{Lemma 5. Given a surface } S \subset \mathbb{H}^3 \text{ with cone-like ends, there exists some constant } A > 0 \text{ such that } \forall h \in (0, 1/2) \text{ and } \forall x \in S_h, \forall y \in C_h \]

\[|s_h^\nu(v)| \leq \frac{A}{\|y - x\| + \cos \varphi}, \]

where \( v \in T_y^1 C_h \) and \( \cos \beta = (y_3 - x_3)/\|y - x\|; i.e. \beta = \angle(y - x, \partial/\partial x_3) \).

\[\text{Proof. Let us consider the curve } C = S \cap (y + (\partial/\partial x_3)^\perp); \text{ so that } v \in T_y C. \text{ A couple of geometric arguments give} \]

\[|s_h^\nu(v)| = \left| \frac{K \sin \zeta}{y_3 \sin \gamma} \right|\]

where \( \zeta \) is the angle between \( \varphi_1 \), the hyperbolic osculating geodesic plane of \( C \), and \( \varphi_2 \), the geodesic plane tangent to \( C \) that contains \( x \). As for \( \gamma \), it denotes the angle between the (hyperbolic) geodesic segment \( xy \) and \( C \). Finally \( K \) is the geodesic curvature of \( C \) in \( \mathbb{H}^3 \). This is uniformly bounded so we can forget about.

Next we show \( y_3^{-1} \sin \zeta \) is uniformly bounded. Consider \( \zeta_1 \) the angle of the osculating plane \( \varphi_1 \) with the vertical direction at \( y \), and \( \zeta_2 \) the angle of the plane \( \varphi_2 \) with the vertical direction at \( y \). Then \( \sin \zeta \leq \sin \zeta_1 + \sin \zeta_2 \). A simple computation shows \( \zeta_1 = O(y_3) \). Also \( \zeta_2 = O(y_3) \), since the plane \( \varphi_2 \) has a uniformly bounded radius. Indeed, a small sphere centered at \( \partial_\infty \mathbb{H}^3 \) intersects \( S \) in a curve with only one critical point of the function \( x_3 \). Altogether yields \( \sin \zeta = O(y_3) \).

Therefore we have \( |s_h^\nu(v)| = O(1/\sin \gamma) \), and we must control the latter. After some elementary euclidean computations one finds

\[\sin \gamma \geq \sin \gamma' = \frac{d + 2y_3 \cos \beta}{\sqrt{4y_3^2 + d^2 + 4y_3d \cos \beta}}\]

where \( \gamma' \) is the angle between the geodesic segment and the horizontal plane, and \( d = \|y - x\| \). Now

\[\frac{1}{\sin^2 \gamma} \leq \frac{d^2 + 4y_3^2 \cos^2 \beta + 4y_3d \cos \beta}{d^2 + 4y_3^2 + 4y_3d \cos \beta} = 1 + \frac{4y_3^2 \cos^2 \beta}{(d + 2y_3 \cos \beta)^2} \leq 1 + \frac{4y_3^2}{(d + 2y_3 \cos \beta)^2} \leq 1 + \frac{1}{(d + \cos \beta)^2} \leq \frac{(d + \cos \beta)^2 + 1}{(d + \cos \beta)^2} \leq \frac{A^2}{(d + \cos \beta)^2} \]

for some \( A \), where we have used \( y_3 < 1/2 \), and the compacity of \( S \). □

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