Higher curvature Bianchi identities, generalised geometry and $L_\infty$ algebras

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Abstract

The Bianchi identities for bosonic fluxes in supergravity can receive higher derivative quantum and string corrections, the most well known being that of Heterotic theory $dH = \frac{1}{4} \alpha' \left( \text{tr} F^2 - \text{tr} R^2 \right)$. Less studied are the modifications at order $R^4$ that may arise, for example, in the Bianchi identity for the seven-form flux of $M$ theory compactifications. We argue that such corrections appear to be incompatible with the exceptional generalised geometry description of the lower order supergravity, and seem to imply a gauge algebra for the bosonic potentials that cannot be written in terms of an (exceptional) Courant bracket. However, we show that this algebra retains the form of an $L_\infty$ gauge field theory, which terminates at a level ten multibracket for the case involving just the seven-form flux.

I. GENERALISED GEOMETRY AND BIANCHI IDENTITIES

The formalism of generalised geometry has proven to be a very powerful tool to tackle problems in string theory and supergravity. By looking at structures on a generalised tangent space which has ‘baked in’ the much richer gauge field content of these theories, it provides a unified language for the bosonic sector that brings within reach previously intractable problems. However, precisely because the gauge fields are built into the definition of the generalised geometry, their Bianchi identities are assumed by construction and any modification of them requires a change in the formalism. As we will review shortly, including the first $\alpha'$ correction due to the Green-Schwarz mechanism in Heterotic requires relaxing the exactness condition of the Courant algebroid in ‘base’ generalised geometry, and adding the Ramond-Ramond fields (or moving to $M$ theory) requires the introduction of an exceptional Courant algebroid. In this paper we will argue that further considering $R^4$ corrections – which would be highly desirable as it could provide a path to finally obtain their supersymmetric completion and would have applications to phenomenological models that rely on perturbative effects to fix moduli in flux compactifications – implies again...

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expanding the exceptional generalised geometry, and also that the gauge algebra can no longer be captured by a bracket acting just on elements of the generalised tangent space. Finally we will show that there is nonetheless an $L_\infty$ algebra structure remaining, which we compute explicitly for a particular case.

A. Generalised geometry

Generalised geometry was originally introduced in [1, 2] as a way of combining complex and symplectic geometry, by considering structures on the generalised tangent bundle $E$

$$T^* \to E \to T,$$  \hfill (1)

so that $E$ is locally isomorphic to the sum of the tangent and cotangent bundle, $E \sim T \oplus T^*$. The generalised tangent bundle is naturally equipped with a Dorfman bracket or, equivalently, its antisymmetrisation a Courant bracket [3, 4]

$$[X_1, X_2] = [x_1, x_2] + \mathcal{L}_{x_1} \lambda_2 - \mathcal{L}_{x_2} \lambda_1 - \frac{1}{2} d(i_{x_1} \lambda_2 - i_{x_2} \lambda_1),$$  \hfill (2)

where $X_i = x_i + \lambda_i \in E$. The generalised tangent bundle is in fact an example of an exact Courant algebroid [5] and it possesses a three-form $H$ that is closed [6],

$$dH = 0,$$  \hfill (3)

which can be thought of as the curvature of a “gerbe” $B$ [7], i.e. $H = dB$ locally, which specifies a splitting of the sequence (1). Physicists quickly realised that this formalism provides a way of geometrising the NSNS sector of type II supergravity [8–11], $B$ being identified with the Kalb-Ramond field and $H$ being its flux. The Dorfman bracket along a generalised vector $L_X, X \in E$ then generates the combined (infinitesimal) bosonic symmetries of the theory: diffeomorphisms $\mathcal{L}_x$ by taking the Lie derivative along a vector field $x \in T$, and gauge $B$-shifts by $d\lambda$, exact two-forms parametrised by one-forms $\lambda \in T^*$. Introducing also a metric, it is possible to unify all the NSNS fields into a single object, and rewrite all the supergravity equations as a generalised geometry equivalent of Einstein gravity [12] (see also [13] for an overview of the closely related subject of Double Field Theory that often implies many of these results).
B. Heterotic generalised geometry

In Heterotic theory, however, the field strength $H$ is no longer closed. Supersymmetry and the Green-Schwarz anomaly cancelation mechanism \[14\] require that $H$ satisfy a more complicated Bianchi identity. This can be handled in the generalised geometry formalism by enlarging the generalised tangent space. The resulting “Heterotic generalised geometry” \[15–20\] is given in terms of a bundle which is a transitive, but not exact, Courant algebroid $E$, that can be built as a result of two extensions

$$\mathfrak{g} \rightarrow \mathcal{A} \rightarrow T,$$

$$T^* \rightarrow E \rightarrow \mathcal{A}.$$  \hspace{1cm} (4)

The first sequence defines a Lie algebroid $\mathcal{A}$ known as the Atiyah algebroid for the quadratic Lie algebra $\mathfrak{g}$, which replaces the role of the tangent bundle $T$ in the original generalised geometry.

Writing $X_i = x_i + \Lambda_i + \lambda_i \in E \sim T \oplus \mathfrak{g} \oplus T^*$, the Courant bracket takes the form

$$[X_1, X_2] = [x_1, x_2] + \mathcal{L}_{x_1} \Lambda_2 - \mathcal{L}_{x_2} \Lambda_1 + [\Lambda_1, \Lambda_2]$$

$$+ \mathcal{L}_{x_1} \lambda_2 - \mathcal{L}_{x_2} \lambda_1 - \frac{1}{2} d(i_{x_1} \lambda_2 - i_{x_2} \lambda_1) + \text{tr}(\Lambda_2 d\Lambda_1 - \Lambda_1 d\Lambda_2).$$  \hspace{1cm} (5)

The bundle $E$ then encodes the information for local gauge fields: a two-form\(^1\) $B$ and a Yang-Mills one-form $A$ taking values in $\mathfrak{g}$. These fields are not independent, they satisfy the global condition in terms of their respective field-strengths

$$dH = \text{tr} F^2.$$  \hspace{1cm} (6)

By considering a Lie group $G$ (with algebra $\mathfrak{g}$) composed of two factors, a “gravitational” Lorentz group, and the usual $SO(32)$ or $E_8 \times E_8$ (and choosing the correct normalisation of the metric in $\mathfrak{g}$), one obtains the Heterotic Bianchi identity:

$$dH = \frac{1}{4} \alpha' (\text{tr} F^2 - \text{tr} R^2),$$  \hspace{1cm} (7)

where $R$, the field strength of the Lorentz factor, is now identified with the gravitational curvature. Once more, the Courant bracket in $E$ precisely reproduces the physical infinitesimal bosonic symmetries: diffeomorphims $\mathcal{L}_x$, $B$-shifts by $d\lambda$, and now also non-Abelian gauge transformations by some parameter $\Lambda \in \mathfrak{g}$. It is then possible to show that formulating the generalised equivalent of

\(^1\) The heterotic $B$ field we are considering here is not gauge invariant under Yang-Mills transformations and is not a gerbe connection, it is a rather more complicated object \[21\].
Einstein gravity in $E$ precisely reproduces the known Heterotic supergravity to order $\alpha'$ [15]. The ‘trick’ of treating the gravitational term in (7) as if it were a Yang-Mills factor goes back to [22], though, as shown there, supersymmetry requires that the $\text{tr} \, R^2$ be given by the curvature of a specific torsionful connection $\nabla - \frac{1}{2} H$. In [19] it was shown that this is entirely consistent with the generalised geometry set-up.

C. M theory and $E_{7(7)} \times \mathbb{R}^+$ generalised geometry

In M theory, the equation of motion for the four-form flux $\mathcal{F}$ in eleven-dimensional supergravity [23] is corrected by higher order terms, starting with eight derivatives [24, 25]

$$d \ast F = -\frac{1}{2} F^2 + \kappa (\text{tr} \, R^4 - \frac{1}{4} (\text{tr} \, R^2)^2),$$

(8)

where $\kappa$ is some constant which will be set to 1 as it will not influence the rest of our discussion, and with further terms which are functions of the flux expected to appear at the same order in derivatives but whose complete form is not yet known.

In order to find four-dimensional Minkowski backgrounds of M theory, one considers field ansätze that are compatible with the external global Lorentz symmetry. This means decomposing the eleven-dimensional manifold $\mathcal{M}_{11}$ as a warped product

$$ds^2_{11}(\mathcal{M}_{11}) = e^\Delta ds^2_{4}(\mathbb{R}^{3,1}) + ds^2_7(M),$$

(9)

where $M$ is some seven-dimensional internal space and $\Delta$ the warp factor, demanding that the fields depend only on internal coordinates, and keeping the components of the $\mathcal{F}$ flux which are external scalars, i.e. the purely internal four-form $F$ and seven-form $\tilde{F}$. Their components are set in terms of the eleven-dimensional $\mathcal{F}$ simply by restricting

$$F = \mathcal{F}|_M, \quad \tilde{F} = (\ast \mathcal{F})|_M,$$

(10)

where $\ast \mathcal{F}$ is the eleven-dimensional Hodge dual. All other components of $\mathcal{F}$ are set to zero. The fact that $\mathcal{F}$ is closed in eleven dimensions together with the equation of motion (8) then imply the Bianchi identities for the internal fluxes

$$dF = 0,$$

(11)

$$d\tilde{F} = -\frac{1}{2} F^2 + \text{tr} \, R^4 - \frac{1}{4} (\text{tr} \, R^2)^2,$$

(12)
where the second equation should be taken as purely formal, since it vanishes identically in the seven-dimensional $M$. These induce internal local potentials, a three-form $C$ and a six-form $\tilde{C}$, which together with a Riemannian metric for $M$ and a warp factor $\Delta$ make up the bosonic degrees of freedom of the theory.

Ignoring the higher-curvature terms, it was shown in [26] that this supergravity set-up (together with the fermionic sector) has a very natural interpretation as the analogue of Einstein gravity when formulated in $E_7(7) \times \mathbb{R}^+$ generalised geometry, also known as exceptional generalised geometry [27–29]. One introduces a generalised tangent bundle again as a series of extensions, such that it has a local form

$$ E \sim T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^* \oplus (T^* \otimes \Lambda^7 T^*) , $$

(13)

which encodes the bosonic symmetries of the theory, namely diffeomorphism generated by vector fields $x$ and shifts by two-forms $\omega$ and five-forms $\sigma$ of the gauge fields $C$ and $\tilde{C}$ respectively. The peculiar one-form-tensor-seven-form term $\tau \in T^* \otimes \Lambda^7 T^*$ would be a charge for a “dual graviton” field which happens to vanish identically in seven-dimensional compactifications and so, while it is implied by the higher-dimensional M theoric geometry, it has no immediate physical meaning given this setup (though see [30], for example, where $\tau$ and other mixed-symmetry charges become crucial to formulate higher exceptional geometries). By construction, the generalised tangent bundle defines a global closed four-form $F$ that can locally be expressed in terms of the potential

$$ F = dC, $$

(14)

and a seven-form such that

$$ \tilde{F} = d\tilde{C} - \frac{1}{2} CF. $$

(15)

The supergravity Bianchi identities inherited from eleven-dimensions are thus automatically satisfied. The gauge algebra is then given by the natural differential structure over $E$, the (exceptional) Courant bracket of two generalised vector fields, which takes the form$^2$

$$ [X_1, X_2] = \mathcal{L}_{x_1} x_2 + \mathcal{L}_{x_1} \omega_2 - \mathcal{L}_{x_2} \omega_1 - \frac{1}{2} d(i_{x_1} \omega_2 - i_{x_2} \omega_1) $$

$$ + \mathcal{L}_{x_1} \sigma_2 - \mathcal{L}_{x_2} \sigma_1 - \frac{1}{2} d(i_{x_1} \sigma_2 - i_{x_2} \sigma_1) + \frac{1}{2} \sigma_1 d\sigma_2 - \frac{1}{2} \sigma_2 d\sigma_1 $$

$$ + \frac{1}{2} \mathcal{L}_{x_1} \tau_2 - \frac{1}{2} \mathcal{L}_{x_2} \tau_1 + \frac{1}{2} (j \sigma_1 \wedge d\omega_2 - j \sigma_2 \wedge d\omega_1) - \frac{1}{2} (j \sigma_2 \wedge d\omega_1 - j \sigma_1 \wedge d\omega_2) , $$

(16)

$^2$ The $j$-notation corresponds to a projection to the $T^* \otimes \Lambda^7 T^*$ space, see [28, 29] for its precise definition, though it will not be needed for what follows.
where we write \( X_i = x_i + \omega_i + \sigma_i + \tau_i \in E \). The bundle \( E \) has a natural \( E_7(7) \times \mathbb{R}^+ \) structure and the bracket is compatible with this structure. The bosonic degrees of freedom turn out to simply be the components of a generalised metric for the generalised tangent space, reducing the structure group to its maximal compact subgroup \( SU(8)/\mathbb{Z}_2 \), and the corresponding generalised Ricci scalar precisely reproduces the supergravity bosonic action. That eleven-dimensional supergravity admitted this larger symmetry had already been proven in [31]. This efficient rewriting has made it possible to tackle several physical problems in full generality (without needing to restrict to some subsector of the fluxes, for example), such as classifying supersymmetric backgrounds [32–40] or describing their moduli spaces and holographic duals [41, 42]. It would thus seem promising to apply the same techniques with the higher derivative corrections included [43].

### D. M theory corrections

So now let us consider adding back the higher curvature terms originating in eleven dimensions (8). These are incompatible with the \( E_7(7) \times \mathbb{R}^+ \) generalised tangent bundle previously introduced, since by construction it forces \( \tilde{F} = d\tilde{C} - \frac{1}{2}CF \). On the contrary, the corrected Bianchi identity (12) implies the local form for the flux \( \tilde{F} \)

\[
\tilde{F} = d\tilde{C} - \frac{1}{2}CF + \omega_7(A) - \frac{1}{4}\omega_3(A) \text{ tr } R^2,
\]

where \( A \) is the spin-connection for the Riemann curvature \( R \) and \( \omega_n(A) \) denotes the Chern-Simons \( n \)-form for \( A \) such that \( d\omega_{2n-1}(A) = \text{ tr } R^n \), see the appendix for their explicit form.

Following the same trick as for the Heterotic case, we may treat at first the curvature \( R \) simply as the field strength for a generic Yang-Mills gauge field \( A \) taking values in some algebra \( g \), though naturally it will eventually be necessary to identify \( g \) with \( \text{spin}(7) \) and express \( A \) in terms of gravitational degrees of freedom.\(^3\) The Heterotic generalised geometric prescription would then lead us to consider structures over a generalised tangent space of the form

\[
T \oplus g \oplus \Lambda^2 T^* \oplus \Lambda^5 T^* \oplus T^* \otimes \Lambda^7 T^*,
\]

in other words, replacing the tangent bundle component of (13) with the Atiyah algebroid.

\(^3\) Though an intriguing possibility is to consider a larger gauge group that could accommodate the flux degrees of freedom, such as taking \( g = su(8) \) and relating \( A \) to the \( SU(8) \) connections implied by supersymmetry. This could naturally give rise to a Bianchi identity which includes higher derivative flux terms.
In what follows, however, we will restrict ourselves to simpler versions of this problem, which
will still suffice to show that the situation is more complex than the one of Heterotic generalised
geometry. In particular, we will find gauge algebras that are best described in terms of higher order
$L_\infty$-algebras.

In section II we will first look at a Bianchi identity

$$d\tilde{F}_5 = \text{tr} \, R^3,$$

where $\tilde{F}_5$ is a five-form which, even though it has no immediate physical motivation, is easier to
handle and already displays the important features we wish to demonstrate. The corresponding
generalised tangent space will be of the form

$$T \oplus g \oplus \Lambda^3 T^*.$$  \hspace{1cm} (20)

We will then move on in section III to the case

$$d\tilde{F}_7 = \text{tr} \, R^4,$$

where now $\tilde{F}_7$ is genuinely a seven-form, and so this corresponds to a special case of (17). The
generalised tangent space is then

$$T \oplus g \oplus \Lambda^5 T^*.$$  \hspace{1cm} (22)

In both cases we will find that the Bianchi identities imply a gauge algebra which cannot be
expressed in terms of simply a Courant bracket. Instead it is of the type of the $L_\infty$ field theory
formalism of [44]. The analysis of the complete Bianchi identity implied by the corrected eleven-
dimensional supergravity will be left for future work.

As an aside, we expect that similar conclusions would hold for $(2n - 1)$-form fluxes $\tilde{F}_{(2n-1)}$ satisfying

$$d\tilde{F}_{(2n-1)} = \text{tr} \, R^n,$$

based on generalised tangent spaces

$$E \sim T \oplus g \oplus \Lambda^{(2n-3)} T^*,$$  \hspace{1cm} (24)

though we will not attempt to prove this here. Note as well that in all these cases the Bianchi
identities, when viewed in cohomology classes, correspond to obstructions to this construction,
namely the requirement that $n$-th Chern character of the gauge vector bundle is trivial.
We also remark that the fact that the gauge algebras we are examining fit into the $L_\infty$ setting is not surprising. It has already been shown that the “higher Courant algebroids” of the type $T \oplus \Lambda^p T^*$ have an associated $L_\infty$ algebra [45], and the extra terms we are considering arise from adding an invariant polynomial to the Bianchi, which in the context of chiral anomalies lead to the well-known “descent equations” derived from the extended Cartan homotopy [46], with many of the terms in the brackets we present here being directly related to the extra homotopy operator.

E. $L_\infty$ algebras and field theory

$L_\infty$ algebras or strong homotopy Lie algebras, introduced in [47, 48] to the physics context, have found numerous applications in both mathematics and physics, see [49] for a recent review of the field. In particular, they can be found in the theories of Courant algebroids and generalised geometry. Courant algebroids were shown to have an $L_3$-algebra in [50]. In the case of Heterotic Courant algebroids, it has recently been proven that this algebra is directly connected to the physical problem of finding the moduli of finite deformations of the Strominger-Hull system [51]. Higher Courant algebroids over a space $T \oplus \Lambda^p T^*$ were proven to have $L_\infty$ algebras for arbitrary $p$ in [45] using a derived bracket construction, and in [52] a large class of “Leibniz algebroids” (a Leibniz bracket being a generalisation of the Lie derivative that still satisfies the Leibniz identity but is not necessarily anti-symmetric), of which exceptional generalised geometries are examples, were likewise shown to admit $L_\infty$ algebras. This later point was further explored in [53, 54], where the correct $L_\infty$ algebra was demonstrated to follow from interpreting the M theory geometries as dg-symplectic manifolds. More broadly speaking, the higher structures that feature in string and M theory are known to be classified by super homotopy theory, see the review [55] and references therein. In particular, note that the Heterotic generalised geometry that we described earlier corresponds to a “string Lie 2-algebra” [56], while anomaly cancelation in M theory was examined in this formalism in [57]. There has also been much current work showing how such structures appear in the related fields of Double/Exceptional Field Theory, for example in [58–70].

Recently, in [44] (see also [71]) many of these ideas were systematised in a manner to be more immediately applicable to physics, by introducing the notion of “$L_\infty$ gauge field theories”. It is this approach that we will be following, and we start by quickly reviewing some of the concepts that will be relevant here.

There are a few alternative ways of defining an $L_\infty$ algebra. Following the conventions of [44]
we will be working with the “ℓ-picture” in terms of graded-antisymmetric multilinear brackets.

Given a \( \mathbb{Z} \)-graded vector space

\[
V = \bigoplus_{i \in \mathbb{Z}} V_i,
\]

(25)

where the subscript denotes the degree, one defines an \( L_\infty \) algebra by endowing it with a series of multilinear products \( \ell_n : \Lambda^n V \mapsto V \). These brackets are of degree \( n - 2 \), i.e. for inputs \( v_i \in V \), the total degree of \( \ell_n(v_1, \ldots, v_n) \) is

\[
\deg \ell_n(v_1, \ldots, v_n) = n - 2 + \sum_{i=1}^n \deg v_i.
\]

(26)

They are also graded antisymmetric,

\[
\ell_n(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = (-1)^{|\sigma|} \epsilon(\sigma) \ell_n(v_1, \ldots, v_n),
\]

(27)

for some permutation \( \sigma \) and where \( \epsilon \) is the Koszul sign for the given permutation and grading of \( V \). Crucially, for each \( n \) the brackets must also satisfy a Jacobi identity “up to higher homotopies”, namely the generalised Jacobi identities

\[
\sum_{i+j=n+1} (-1)^{(j-1)} \sum_{\sigma} (-1)^{|\sigma|} \epsilon(\sigma) \ell_j(\ell_i(v_{\sigma(1)}, \ldots, v_{\sigma(i)}), v_{\sigma(i+1)}, \ldots, v_{\sigma(n)}) = 0,
\]

(28)

or in abbreviated form

\[
\sum_{i+j=n+1} (-1)^{(j-1)i} \ell_j \ell_i = 0.
\]

(29)

Explicitly, this gives at level one

\[
\ell_1(\ell_1(v)) = 0,
\]

(30)

which shows that the graded vector space \( V \) of an \( L_\infty \) algebra forms a chain complex with the operator \( \ell_1 \). Level two establishes \( \ell_1 \) as a derivation on \( \ell_2 \),

\[
\ell_1(\ell_2(v_1, v_2)) = \ell_2(\ell_1(v_1), v_2) + (-1)^{|v_1|} \ell_2(v_1, \ell_1(v_2)).
\]

(31)

Level three would be the ‘traditional’ Jacobi identity, where the \( L_\infty \) algebra starts to diverge from the normal graded Lie algebras

\[
\ell_1(\ell_3(v_1, v_2, v_3)) + \ell_3(\ell_1(v_1), v_2, v_3) + (-1)^{|v_1|} \ell_3(v_1, \ell_1(v_2), v_3)
\]
\[
+ (-1)^{|v_1|+|v_2|} \ell_3(v_1, v_2, \ell_1(v_3)) + \ell_2(\ell_2(v_1, v_2), v_3)
\]
\[
+ (-1)^{|v_2|+|v_3|} \ell_2(\ell_2(v_2, v_3), v_1) + (-1)^{|v_1|+|v_2|+|v_3|} \ell_2(\ell_2(v_3, v_1), v_2) = 0,
\]

(32)
and so on.

Proceeding with the proposal of [44] for a gauge field theory, one considers spaces of type\

\[ V = \oplus_{i>0} V_i \oplus V_0 \oplus V_{-1}. \]

(33)

An important point here is that, since one allows a space with negative grading, there is a priori no guarantee that the \( L_\infty \) algebra will ever terminate even for a finite number of \( V_i \). This is in contrast to \( L_n \)-algebras, defined such that the graded vector space is concentrated in degrees 0 to \( n - 1 \) and therefore all brackets of degree higher than \( n + 1 \) vanish trivially as a consequence of (26). However, we will see that the cases we consider in the next sections do indeed truncate and there is a finite number of brackets to consider.

In order to find the physical meaning of (33), one identifies elements \( X \in V_0 \) with gauge parameters and \( \Psi \in V_{-1} \) are taken to be the gauge fields. Elements of \( \oplus_{i>0} V_i \) are to be thought of as making up a tower of trivial gauge parameters. An \( L_\infty \) gauge field theory may then be defined with the symmetries given by

\[ \delta_X \Psi = \Sigma_n \frac{1}{n!} (-1)^{\frac{n(n-1)}{2}} \ell_{n+1} (X, \Psi^n), \]

(34)

satisfying a gauge algebra

\[ [\delta_X, \delta_Y] \Psi = [X, Y] \Psi, \quad [X_1, X_2] = \Sigma_n \frac{1}{n!} (-1)^{\frac{n(n-1)}{2}} \ell_{n+2} (X_1, X_2, \Psi^n). \]

(35)

Note in particular that in this formalism the gauge algebra of the parameters is permitted to depend explicitly on the fields. In what follows we will show how the higher curvature problem we are considering fits precisely into this picture.

We also remark that another way the gauge fields may appear in Courant brackets is via their ‘twisting’. One can think of the fields as defining connections that split the exact sequences that define the generalised tangent bundle, i.e. they make explicit the isomorphism \( E \sim T \oplus \ldots \). Under that map, the Courant bracket then becomes twisted by the curvature of the connection – for example, for the exact Courant algebroid case the \( B \) field splits the defining sequence \( T^* \rightarrow E \rightarrow T \) through a map \( e^B : x + \lambda \mapsto x + \lambda + i_x B \) (see [2]), and the bracket (2) then gets modified by \( [e^B X_1, e^B X_2] = e^B [X_1, X_2] + i_{x_1} i_{x_2} H \). Note also that if \( B \) is closed, physically ‘pure gauge’, then it is a symmetry of the bracket, extending the usual diffeomorphism invariance of the Lie

\[^4\text{In [44] an extra subspace } V_{-2} \text{ is also allowed, corresponding to the equations of motion, but we will not make use of it here.}\]
bracket. One would expect that something similar for the higher order brackets we define in the following sections, it should be possible to twist them with the field strengths of the gauge fields, and pure gauge finite transformations should leave them invariant. However, we do not attempt to verify this here, a full study of the twisted structure of the higher order generalised tangent bundles, their patching rules (which would involve computing finite gauge transformations of the Chern-Simons forms), the automorphisms of these $L_\infty$ structures, etc. will be left for future work.

II. $d\tilde{F}_5 = \text{tr} \, R^3$

We begin by considering a theory with a globally defined five-form flux $\tilde{F}$ and a Yang-Mills $g$-valued potential $A$ with corresponding field strength $R$ such that

\begin{align*}
\quad d_A R &= 0, \quad (36) \\
\quad d\tilde{F}_5 &= \text{tr} \, R^3. \quad (37)
\end{align*}

We can thus define a four-form potential $\tilde{C}$ for the flux by

\begin{equation}
\tilde{F}_5 = d\tilde{C}_4 + \omega_5(A). \quad (38)
\end{equation}

Much like the $B$ field in Heterotic theory, we find that since $\tilde{F}_5$ is gauge invariant, $\tilde{C}_4$ must transform to compensate for a variation of the Chern-Simons five-form $\omega_5(A)$. That is, if $\Lambda \in g$ parametrises an infinitesimal gauge transformation, we must have that locally

\begin{equation}
\quad d\delta_X \tilde{C}_4 = -\delta_{\Lambda} \omega_5(A) = -d \omega_4^1(\Lambda, A) = -d \text{tr} \, d\Lambda \left( \text{Ad}A + \frac{1}{2} A^3 \right), \quad (39)
\end{equation}

from the properties of the Chern-Simons forms (see appendix). It is also clear that $\tilde{F}$ remains invariant under shifts of $\tilde{C}_4$ by a closed four-form, locally parametrised by the exterior derivative of some three-form $\sigma$. Together with a diffeomorphism symmetry parametrised by some vector field $x$, we have that the potentials obey the infinitesimal gauge transformations

\begin{align*}
\quad \delta_X A &= \mathcal{L}_x A - d\Lambda - [A, \Lambda], \\
\quad \delta_X \tilde{C}_4 &= \mathcal{L}_x \tilde{C}_4 - d\sigma - \text{tr} \, d\Lambda \left( \text{Ad}A + \frac{1}{2} A^3 \right), \quad (40)
\end{align*}

where $\delta_X$ denotes the combined infinitesimal diffs, gauge and shifts in terms of parameters $X = x + \Lambda + \sigma$. This therefore suggests a generalised tangent space

\begin{equation}
E = T \oplus g \oplus \Lambda^3 T^*. \quad (41)
\end{equation}
So far, this precisely matches the procedure for constructing the Heterotic generalised geometry, see for example [19]. However, let us look at how the algebra of transformations $\delta_X$ closes when acting on the fields. Taking two parameters $X_1, X_2 \in E$, we find that

\[
\begin{align*}
[\delta_{X_1}, \delta_{X_2}] A &= L_{[x_1,x_2]} A - d ([\Lambda_1, \Lambda_2] + i_{x_1} d\Lambda_2 - i_{x_2} d\Lambda_1) \\
&\quad - [A, [\Lambda_1, \Lambda_2] + i_{x_1} d\Lambda_2 - i_{x_2} d\Lambda_1],
\end{align*}
\]

\[
\begin{align*}
[\delta_{X_1}, \delta_{X_2}] \tilde{C}_4 &= L_{[x_1,x_2]} \tilde{C}_4 - d (i_{x_1} d\sigma_2 - i_{x_2} d\sigma_1 + \frac{1}{2} d i_{x_1} \sigma_2 - \frac{1}{2} d i_{x_2} \sigma_1) \\
&\quad - \text{tr} \ d ([\Lambda_1, \Lambda_2] + i_{x_1} d\Lambda_2 - i_{x_2} d\Lambda_1) (A(dA)^2 + \frac{1}{2} A^3) \\
&\quad + d \text{tr} (\Lambda_1 d\Lambda_2 dA - \Lambda_2 d\Lambda_1 dA),
\end{align*}
\]

so we have that indeed the algebra closes on a parameter $X_3$ given by

\[
\begin{align*}
[\delta_{X_1}, \delta_{X_2}] (A + \tilde{C}_4) &= \delta_{X_3} (A + \tilde{C}_4), \\
X_3 &= [x_1, x_2] + [\Lambda_1, \Lambda_2] + i_{x_1} d\Lambda_2 - i_{x_2} d\Lambda_1 \\
&\quad + i_{x_1} d\sigma_2 - i_{x_2} d\sigma_1 + \frac{1}{2} d i_{x_1} \sigma_2 - \frac{1}{2} d i_{x_2} \sigma_1 \\
&\quad - \text{tr} (\Lambda_1 d\Lambda_2 dA - \Lambda_2 d\Lambda_1 dA) \in E,
\end{align*}
\]

but note that this depends not just on $X_1$ and $X_2$ but also explicitly on the fields. Therefore, unlike the previous examples in generalised geometry, the gauge algebra does not define for us a bracket over just the space $E$. It does, nonetheless, fit into the $L_\infty$ field theory setting.

**A. An $L_\infty$ gauge algebra for $R^3$**

Let us then introduce the graded vector space:

\[
V = V_3 \oplus V_2 \oplus V_1 \oplus V_0 \oplus V_{-1},
\]

where\(^5\)

\[
\begin{align*}
V_3 &= C^\infty(M), \quad V_2 = T^*, \quad V_1 = \Lambda^2 T^*, \\
V_0 &= E = T \oplus g \oplus \Lambda^3 T^*, \quad V_{-1} = T^* \otimes g \oplus \Lambda^4 T^*,
\end{align*}
\]

\(^5\)A more ‘generalised’ treatment in the sense of [29] would presumably involve introducing a space of “generalised frames” for $E$ (that is a subspace of $\text{End}(E)$ that preserves the defining generalised structures – $O(d,d)$ in NSNS generalised geometry, $E_{7(7)}$ in exceptional generalised geometry, etc.), and identifying its ‘geometric subspace’ with $V_{-1}$ which is used to construct the physical brackets.
and we label elements in the subspaces as

$$\xi \in V_3 \oplus V_2 \oplus V_1, \quad X = x + \Lambda + \sigma \in V_0, \quad \Psi = A + \tilde{C} \in V_{-1}. \quad (46)$$

We will then endow $V$ with a series of multilinear brackets to define an $L_\infty$ algebra that will realise the gauge algebra (43). Terms in the brackets involving only elements in $V_{i>0}$ or the vector + three-form part of $V_0$ will be necessarily the ones in [45], but we must introduce new products for terms involving the Lie algebra $\mathfrak{g}$. Comparing with (35), we can directly read off some of the multibrackets, since we must insist that picking particular elements $A + \tilde{C}_4 = \Psi \in V_{-1}$ corresponds to specifying the data for the supergravity gauge fields, i.e. that they satisfy a gauge algebra

$$\delta_X \Psi = \ell_1(X) + \ell_2(X, \Psi) - \frac{1}{2} \ell_3(X, \Psi, \Psi) - \frac{1}{6} \ell_4(X, \Psi, \Psi, \Psi), \quad (47)$$

with

$$[\delta_{X_1}, \delta_{X_2}] \Psi = \delta_{X_3} \Psi, \quad X_3 = \ell_2(X_1, X_2) + \ell_3(X_1, X_2, \Psi), \quad (48)$$

such that it precisely matches (40) and (43) respectively.

Several more brackets are necessary to complete the algebra, which can be obtained from the requirement that they satisfy the generalised Jacobi identities (28). It is possible to do this exhaustively term-by-term since, due to both the grading of the vector space $V$ and the subdivisions inside $V_0$ and $V_{-1}$, many will vanish trivially. For example, we will see that all brackets $\ell_n$ of level $n > 2$ whose image is in $V_{-1}$ actually only map to the four-form subspace, i.e. they are $\tilde{C}$-type objects. On the other hand, the brackets of level $n > 2$ that take an object in $V_{-1}$ as input are all independent of $\tilde{C}$. So chaining together those sets of brackets is trivial.

Note that due to the grading and symmetry properties of the $\ell_n$ brackets (27), products involving multiple factors of $X_i$ will always have to be antisymmetrised, and products involving products of $\Psi_i$ will always have to be symmetrised. We denote this explicitly using the typical index notation of symmetrisers and antisymmetrisers.

We find that the (non-trivial) $L_\infty$ products are then:

at level one

$$\ell_1(\xi) = d\xi, \quad \ell_1(X) = -d\Lambda - d\sigma, \quad \ell_1(\Psi) = 0, \quad (49)$$
at level two

\[ \ell_2(X, \xi) = \frac{1}{2} \mathcal{L}_x \xi, \quad (50a) \]

\[ \ell_2(X_1, X_2) = [x_1, x_2] + [\Lambda_1, \Lambda_2] + \mathcal{L}_{x_1} \Lambda_2 - \mathcal{L}_{x_2} \Lambda_1 + \mathcal{L}_{x_1} \sigma_2 - \mathcal{L}_{x_2} \sigma_1 - \frac{1}{2} \mathbf{d}i_{x_1} \sigma_2 + \frac{1}{2} \mathbf{d}i_{x_2} \sigma_1, \quad (50b) \]

\[ \ell_2(X, \Psi) = \mathcal{L}_x \Psi - [A, \Lambda], \quad (50c) \]

at level three

\[ \ell_3(\xi, X_1, X_2) = -\frac{1}{6} (i_{x_1} \mathcal{L}_{x_2} + i_{[x_1, x_2]}) \xi, \quad (51a) \]

\[ \ell_3(X_1, X_2, X_3) = -\frac{1}{2} (i_{x_1} \mathcal{L}_{x_2} + i_{[x_1, x_2]} + i_{x_1} i_{x_2} d) \sigma_3, \quad (51b) \]

\[ \ell_3(X_1, X_2, \Psi) = -2 \text{tr} \Lambda_{[1} d \Lambda_{2]} d A, \quad (51c) \]

\[ \ell_3(X, \Psi_1, \Psi_2) = 2 \text{tr} d \Lambda A_{(1} d A_{2)}, \quad (51d) \]

at level four

\[ \ell_4(X_1, X_2, X_3, X_4) = -12 \text{tr} \Lambda_{[1} A_2 A_3 A_4], \quad (52a) \]

\[ \ell_4(X_1, X_2, X_3, \Psi) = 6 \text{tr} \Lambda_{[1} A_2 d A_3] A - \frac{3}{2} i_{x_1} \ell_3(X_2, X_3, \Psi), \quad (52b) \]

\[ \ell_4(X, \Psi_1, \Psi_2, \Psi_3) = 3 \text{tr} d \Lambda A_{(1} d A_{2} d A_{3]}, \quad (52c) \]

at level five

\[ \ell_5(X_1, \ldots, X_5) = -12 \text{tr} \Lambda_{[1} A_2 A_3 A_4 A_5] - \frac{5}{2} i_{x_1} \ell_4(X_2, X_3, X_4, X_5) \]

\[ - \frac{1}{3} i_{x_1} i_{x_2} \ell_3(X_3, X_4, X_5), \quad (53a) \]

\[ \ell_5(X_1, X_2, X_3, X_4, \Psi) = -2 i_{x_1} \ell_4(X_2, X_3, X_4, \Psi) - 2 i_{x_1} i_{x_2} \ell_3(X_3, X_4, \Psi), \quad (53b) \]

and finally at level six

\[ \ell_6(X_1, \ldots, X_5, \Psi) = \frac{5}{3} i_{x_1} i_{x_2} \ell_4(X_3, X_4, X_5, \Psi) + \frac{5}{2} i_{x_1} i_{x_2} i_{x_3} \ell_3(X_4, X_5, \Psi). \quad (54) \]

All other brackets vanish. Even though there is a large number of them, it should be clear from the form of the non-trivial brackets that they are straightforward to obtain by iterating through the
generalised Jacobi identities. As an example, consider the fourth level identity when the inputs are in \((V_0)^3 \otimes V_{-1}\). This will involve the bracket \(\ell_4(X_1, X_2, X_3, \Psi)\) given in (52b), which cannot be read directly from the gauge algebra (and has not been previously derived in the literature), instead we infer it from the generalised Jacobi identity. We compute from the lower order brackets

\[
\begin{align*}
\ell_3(\ell_2(X_1, X_2), X_3, \Psi) &= -2 \text{tr} \mathcal{L}_{x_1} (\Lambda_2 d \Lambda_3) dA + 2 \text{tr} \Lambda_2 \Lambda_3 dA, \\
\ell_3(X_1, X_2, \ell_2(X_3, \Psi)) &= -2 \text{tr} \Lambda_1 d \Lambda_2 (\mathcal{L}_{x_3} dA - d[A, \Lambda_3]), \\
\ell_2(X_1, \ell_3(X_2, X_3, \Psi)) &= -2 \mathcal{L}_{x_1} \text{tr} \Lambda_2 d \Lambda_3 dA + d \mathcal{L}_{x_1} \text{tr} \Lambda_2 d \Lambda_3 dA,
\end{align*}
\]

and noting that \(\ell_4(\ell_1(X_1), X_2, X_3, \Psi)\) and \(\ell_4(X_1, X_2, X_3, \ell_1(\Psi))\) vanish identically, the generalised Jacobi identity then reads

\[
\begin{align*}
\ell_1(\ell_4(X_1, X_2, X_3, \Psi)) &= 3 \ell_4(\ell_1(X_1), X_2, X_3, \Psi) + 3 \ell_2(X_1, \ell_3(X_2, X_3, \Psi)) \\
&= 6 \text{d tr} \Lambda_2 d \Lambda_3 A + 3 d \mathcal{L}_{x_1} \text{tr} \Lambda_2 d \Lambda_3 dA \\
&= 6 \text{d tr} \Lambda_2 d \Lambda_3 A - \frac{3}{2} d \mathcal{L}_{x_1} \ell_3(X_2, X_3, \Psi),
\end{align*}
\]

which is consistent with (52b).

III. \(d \tilde{F}_7 = \text{tr} R^4\)

The analysis for a seven-form flux follows in much the same way as the five-form case we just considered, it is simply more computationally intensive. We again introduce a g-valued one-form potential \(A\) with field strength \(R\), and a globally defined seven-form \(\tilde{F}\) such that

\[
d \tilde{F}_7 = \text{tr} R^4, \tag{57}
\]

and so locally we define a six-form potential \(\tilde{C}\) by

\[
\tilde{F}_7 = d \tilde{C}_6 + \omega_7(A). \tag{58}
\]

As previously remarked, this is a toy example for the supergravity theory of section IC when one truncates equation (17). Now, gauge invariance of \(\tilde{F}_7\) once again implies that \(\tilde{C}_6\) must vary as

\[
\begin{align*}
d \delta_A \tilde{C}_6 &= -\delta_A \omega_7(A) = -d \omega_6^1(A, A) \\
&= -d \text{tr} dA (A d A)^2 + \frac{2}{5} (A^3 d A + d AA^3 + A^5) + \frac{1}{5} (A^2 d AA + A d AA^2), \tag{59}
\end{align*}
\]

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for some gauge parameter $\Lambda \in g$ (from equation (A.8) in the appendix). We also have the usual diffeomorphism $L_x$ and shift symmetries $d\sigma$ generated by, respectively, a vector field $x \in T$ and a five-form $\sigma \in \Lambda^5 T^*$. We are thus led to consider a generalised tangent space

$$E = T \oplus g \oplus \Lambda^5 T^*, \quad (60)$$

$$X = x + \Lambda + \sigma \in E.$$ 

This is a close cousin of the $SL(8, \mathbb{R}) \times \mathbb{R}^+$ “half-exceptional” generalised geometry obtained by truncating the $E_{7(7)} \times \mathbb{R}^+$ case, as was described in [72]. We can then group the infinitesimal symmetries as

$$\delta X = \text{infinitesimal diffs, gauge and shifts}$$

$$\delta X A = L_x A - d\Lambda - [A, \Lambda]$$

$$\delta X \tilde{C}_6 = L_x \tilde{C}_6 - d\sigma - \text{tr} \ d\Lambda \left( A(dA)^2 + \frac{2}{5}(A^3 dA + dAA^3 + A^5) + \frac{1}{5}(A^2 dAA + AdAA^2) \right)$$

As in the $R^3$ case, we find that the gauge algebra closes on terms that explicitly depend on the gauge fields. Taking two parameters $X_1, X_2 \in E$, we have

$$[\delta X_1, \delta X_2] A = L_{[x_1,x_2]} A - d \left( [ \Lambda_1, \Lambda_2 ] + i_{x_1} d\Lambda_2 - i_{x_2} d\Lambda_1 \right)$$

$$- [A, [ \Lambda_1, \Lambda_2 ]] + i_{x_1} d\Lambda_2 - i_{x_2} d\Lambda_1],$$

$$[\delta X_1, \delta X_2] \tilde{C}_6 = L_{[x_1,x_2]} \tilde{C}_6 - d (i_{x_1} d\sigma_2 - i_{x_2} d\sigma_1 + \frac{1}{2} d\sigma_1 \sigma_2 - \frac{1}{2} d\sigma_2 \sigma_1)$$

$$- \text{tr} \ d ( [ \Lambda_1, \Lambda_2 ] + i_{x_1} d\Lambda_2 - i_{x_2} d\Lambda_1 ) \left( A(dA)^2 + \frac{2}{5}(A^3 dA + dAA^3 + A^5) + \frac{1}{5}(A^2 dAA + AdAA^2) \right)$$

$$+ d \text{tr} \left( \Lambda_1 (dA^2 dAdA + \frac{3}{5} d\Lambda_2 d(A^3) + \frac{1}{5} d(A^2 d\Lambda_2 A)) \right)$$

$$- \Lambda_2 \left( dA_1 dA_2 + \frac{3}{5} d\Lambda_1 d(A^3) + \frac{1}{5} d(A^2 d\Lambda_1 A) \right),$$

and therefore, the algebra of the gauge parameters is

$$[X_1, X_2] = [x_1, x_2] + [\Lambda_1, \Lambda_2] + i_{x_1} d\Lambda_2 - i_{x_2} d\Lambda_1$$

$$+ i_{x_1} d\sigma_2 - i_{x_2} d\sigma_1 + \frac{1}{2} d\sigma_1 \sigma_2 - \frac{1}{2} d\sigma_2 \sigma_1$$

$$- \text{tr} \left( \Lambda_1 (dA_2 dAdA + \frac{3}{5} d\Lambda_2 d(A^3) + \frac{1}{5} d(A^2 d\Lambda_2 A) \right)$$

$$- \Lambda_2 \left( dA_1 dA_2 + \frac{3}{5} d\Lambda_1 d(A^3) + \frac{1}{5} d(A^2 d\Lambda_1 A) \right) \in E.$$ 

Let us then see how this fits with the $L_{\infty}$ formalism.
A. An $L_\infty$ gauge algebra for $\mathbb{R}^4$

We start by building a seven term graded vector space

\[ V = V_5 \oplus V_4 \oplus V_3 \oplus V_2 \oplus V_1 \oplus V_0 \oplus V_{-1}, \]  

where

\[ V_5 = C^\infty(M), \quad V_4 = T^*, \quad V_3 = \Lambda^2 T^*, \quad V_2 = \Lambda^3 T^*, \quad V_1 = \Lambda^4 T^*, \]
\[ V_0 = E = T \oplus \mathfrak{g} \oplus \Lambda^5 T^*, \quad V_{-1} = T^* \otimes \mathfrak{g} \oplus \Lambda^6 T^*, \]

whose elements we will generically label as

\[ \xi \in V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus V_5, \quad X = x + \Lambda + \sigma \in V_0, \quad \Psi = A + \tilde{C} \in V_{-1}. \]  

We now construct the $L_\infty$ products as before. The terms in the products which are independent of $V_{-1}$ or the $\mathfrak{g}$ part of $V_0$ must reproduce the results of [45]. Again we read off some of the brackets by comparing (61) and (63) with (34) and (35) respectively. Then picking a specific point $\Psi$ in the space $V_{-1}$ will correspond to specifying the supergravity data by demanding that the gauge algebra obeyed by $\Psi$ is

\[ \delta_X \Psi = \ell_1(X) + \ell_2(X, \Psi) - \frac{1}{2} \ell_3(X, \Psi^2) - \frac{1}{6} \ell_4(X, \Psi^3) \]
\[ + \frac{1}{24} \ell_5(X, \Psi^4) + \frac{1}{120} \ell_6(X, \Psi^5), \]  

and

\[ [X_1, X_2] = \ell_2(X_1, X_2) + \ell_3(X_1, X_2, \Psi) - \frac{1}{2} \ell_4(X_1, X_2, \Psi^2) - \frac{1}{6} \ell_5(X_1, X_2, \Psi^3), \]

such that its components $\Psi = A + \tilde{C}$ match (61) and (63) by construction.

We can then use the generalised Jacobi conditions to complete the algebra. As in the previous section, we can verify the relations by exhaustively going through every term of (28), for each level $n$ and for each possible set of inputs for the brackets, since the extra sub-structure of the vector spaces $V_0$ and $V_{-1}$ means that many of those terms vanish trivially and thus the method becomes tractable. The full list of non-vanishing multilinear brackets is nonetheless still rather long. We find the following:

at level one (these show that $V$ is a differential chain complex)

\[ \ell_1(\xi) = d\xi, \quad \ell_1(X) = -d\Lambda - d\sigma, \quad \ell_1(\Psi) = 0, \]  

(69)
at level two (these include the normal gauge transformations)

\[
\ell_2(X, \xi) = \frac{1}{2} \mathcal{L}_X \xi, \quad (70a)
\]

\[
\ell_2(X_1, X_2) = [x_1, x_2] + [\Lambda_1, \Lambda_2] + \mathcal{L}_{x_1} \Lambda_2 - \mathcal{L}_{x_2} \Lambda_1 + \mathcal{L}_{x_1} \sigma_2 - \mathcal{L}_{x_2} \sigma_1 - \frac{1}{2} d_{x_1} \sigma_2 + \frac{1}{2} d_{x_2} \sigma_1, \quad (70b)
\]

\[
\ell_2(X, \Psi) = \mathcal{L}_X \Psi - [A, \Lambda], \quad (70c)
\]

at level three (this is the level where the usual Jacobi identity breaks and one is led to use the higher formalism)

\[
\ell_3(\xi, X_1, X_2) = -\frac{1}{6} (i_{x_1} \mathcal{L}_{x_2} + i_{[x_1, x_2]}) \xi, \quad (71a)
\]

\[
\ell_3(X_1, X_2, X_3) = -\frac{1}{2} (i_{x_1} \mathcal{L}_{x_2} + i_{[x_1, x_2]} + i_{x_1} i_{x_2} d) \sigma_3, \quad (71b)
\]

at level four

\[
\ell_4(X_1, X_2, \Psi_1, \Psi_2) = 4 \operatorname{tr} \Lambda_1 d\Lambda_2 dA_1 dA_2, \quad (72a)
\]

\[
\ell_4(X, \Psi_1, \Psi_2, \Psi_3) = 6 \operatorname{tr} d\Lambda A_1 dA_2 dA_3, \quad (72b)
\]

at level five

\[
\ell_5(\xi, X_1, X_2, X_3, X_4) = -\frac{1}{5} i_{x_1} i_{x_2} \ell_3(\xi, X_3, X_4), \quad (73a)
\]

\[
\ell_5(X_1, \ldots, X_5) = -\frac{1}{3} i_{x_1} i_{x_2} \ell_3(X_3, X_4, X_5), \quad (73b)
\]

\[
\ell_5(X_1, X_2, X_3, X_4, \Psi) = -\frac{24}{5} \operatorname{tr} (2\Lambda_1 \Lambda_2 \Lambda_3 dA_4 - \Lambda_1 \Lambda_2 d\Lambda_3 dA_4) dA, \quad (73c)
\]

\[
\ell_5(X_1, X_2, X_3, \Psi_1, \Psi_2) = -\frac{3}{2} i_{x_1} \ell_4(X_2, X_3, \Psi_1, \Psi_2)
\]

\[
- \frac{12}{5} \operatorname{tr} (2\Lambda_1 \Lambda_2 \Lambda_3 dA_1 dA_2 + 3\Lambda_1 \Lambda_2 \Lambda_3 A_1 dA_2 - \Lambda_1 d\Lambda_2 d\Lambda_3 d(A_1 A_2) - \Lambda_1 d\Lambda_2 dA_1 \Lambda_3 dA_2 + \Lambda_1 d\Lambda_2 A_1 \Lambda_3 dA_2), \quad (73d)
\]

\[
\ell_5(X_1, X_2, \Psi_1, \Psi_2, \Psi_3) = \frac{12}{5} \operatorname{tr} (3\Lambda_1 d\Lambda_2 d(A_1 A_2 A_3) + \Lambda_1 d(A_1 A_2 dA_2 A_3)), \quad (73e)
\]

\[
\ell_5(X, \Psi_1, \Psi_2, \Psi_3, \Psi_4) = -\frac{24}{5} \operatorname{tr} dA \left(2A_1 A_2 A_3 dA_4 + A_1 A_2 dA_3 A_4 + A_1 dA_2 A_3 A_4 + 2dA_1 A_2 A_3 A_4\right), \quad (73f)
\]
at level six

\[
\ell_6(X_1, \ldots, X_6) = -144 \text{tr} \Lambda_1 \Lambda_2 \Lambda_3 \Lambda_4 \Lambda_5 \Lambda_6, \quad (74a)
\]

\[
\ell_6(X_1, \ldots, X_5, \Psi) = 24 \text{tr} \left( 2\Lambda_1 \Lambda_2 \Lambda_3 \Lambda_4 \Lambda_5 + \Lambda_1 \Lambda_2 \Lambda_3 \Lambda_4 \Lambda_5 \right) A
\]

\[
- \frac{5}{2} i_{x_1} \ell_5(X_2, X_3, X_4, X_5, \Psi), \quad (74b)
\]

\[
\ell_6(X_1, X_2, X_3, X_4, \Psi_1, \Psi_2) = -\frac{48}{5} \text{tr} \left( \Lambda_1 (d \Lambda_3) \Lambda_4 A_1 A_2 \right)
\]

\[
+ \Lambda_1 \Lambda_2 d \Lambda_3 A_1 A_2 A_3 + \Lambda_1 (d \Lambda_2 A_1 A_2 A_3)
\]

\[
- 2 i_{x_1} \ell_5(X_2, X_3, X_4, \Psi_1, \Psi_2) - 2 i_{x_1} i_{x_2} \ell_4(X_3, X_4, \Psi_1, \Psi_2), \quad (74c)
\]

\[
\ell_6(X_1, X_2, X_3, \Psi_1, \Psi_2, \Psi_3) = -\frac{3}{2} i_{x_1} \ell_5(X_2, X_3, \Psi_1, \Psi_2, \Psi_3)
\]

\[
- \frac{36}{5} \text{tr} \left( 3\Lambda_1 \Lambda_2 d \Lambda_3 A_1 A_2 A_3 - \Lambda_1 \Lambda_2 A_1 A_2 d \Lambda_3 A_3 \right), \quad (74d)
\]

\[
\ell_6(X, \Psi_1, \ldots, \Psi_5) = -48 \text{tr} d \Lambda A_1 A_2 A_3 A_4 A_5, \quad (74e)
\]

at level seven (last level with a bracket acting just on the generalised tangent space \(E\), the corresponding higher Courant algebroid of [45] would terminate here)

\[
\ell_7(X_1, \ldots, X_7) = -144 \text{tr} \Lambda_1 \Lambda_2 \Lambda_3 \Lambda_4 \Lambda_5 \Lambda_6 \Lambda_7 - \frac{7}{2} i_{x_1} \ell_6(X_1, \ldots, X_7)
\]

\[
+ \frac{1}{3} i_{x_1} i_{x_2} i_{x_3} i_{x_4} \ell_3(X_5, X_6, X_7), \quad (75a)
\]

\[
\ell_7(X_1, \ldots, X_6, \Psi) = -3 i_{x_1} \ell_6(X_1, \ldots, X_6, \Psi) - 5 i_{x_1} i_{x_2} \ell_5(X_3, X_4, X_5, X_6, \Psi)
\]

\[
- \frac{10}{3} i_{x_1} i_{x_2} \ell_5(X_3, X_4, X_5, \Psi_1, \Psi_2) - \frac{5}{2} i_{x_1} i_{x_2} i_{x_3} \ell_4(X_4, X_5, \Psi_1, \Psi_2), \quad (75b)
\]

\[
\ell_7(X_1, X_2, X_3, X_4, \Psi_1, \Psi_2, \Psi_3) = -2 i_{x_1} \ell_6(X_2, X_3, X_4, \Psi_1, \Psi_2, \Psi_3)
\]

\[
- 2 i_{x_1} i_{x_2} \ell_5(X_3, X_4, \Psi_1, \Psi_2, \Psi_3), \quad (75d)
\]
at level eight

\[ \ell_8(X_{1, \ldots, X_7}, \Psi) = \frac{7}{2} i_{x[1} i_{x_2} \ell_6(X_{3, \ldots, X_7}, \Psi) + \frac{35}{4} i_{x[1} i_{x_2} i_{x_3} \ell_5(X_4, X_5, X_6, X_7, \Psi), \]  

\[ \ell_8(X_{1, \ldots, X_6}, \Psi_1, \Psi_2) = \frac{5}{2} i_{x[1} i_{x_2} \ell_6(X_{4, X_5, X_6}, \Psi_1, \Psi_2) + 5 i_{x[1} i_{x_2} i_{x_3} \ell_5(X_4, X_5, X_6, \Psi_1, \Psi_2) + \frac{5}{2} i_{x[1} i_{x_2} i_{x_3} i_{x_4} \ell_4(X_5, X_6, \Psi_1, \Psi_2), \]  

\[ \ell_8(X_{1, \ldots, X_5}, \Psi_1, \Psi_2, \Psi_3) = \frac{5}{3} i_{x[1} i_{x_2} \ell_6(X_{3, X_4, X_5}, \Psi_1, \Psi_2, \Psi_3) + \frac{5}{2} i_{x[1} i_{x_2} i_{x_3} \ell_5(X_4, X_5, \Psi_1, \Psi_2, \Psi_3), \]  

at level nine

\[ \ell_9(X_{1, \ldots, X_7}, \Psi_1, \Psi_2) = -\frac{7}{6} i_{x[1} i_{x_2} i_{x_3} i_{x_4} \ell_5(X_5, X_6, X_7, \Psi_1, \Psi_2) - \frac{7}{4} i_{x[1} i_{x_2} i_{x_3} i_{x_4} i_{x_5} \ell_4(X_6, X_7, \Psi_1, \Psi_2), \]  

\[ \ell_9(X_{1, \ldots, X_6}, \Psi_1, \Psi_2, \Psi_3) = -\frac{1}{3} i_{x[1} i_{x_2} i_{x_3} i_{x_4} \ell_5(X_5, X_6, \Psi_1, \Psi_2, \Psi_3), \]  

and finally at level ten

\[ \ell_{10}(X_{1, \ldots, X_7}, \Psi_1, \Psi_2, \Psi_3) = -\frac{7}{6} i_{x[1} i_{x_2} i_{x_3} i_{x_4} i_{x_5} \ell_6(X_5, X_6, X_7, \Psi_1, \Psi_2, \Psi_3) - \frac{7}{4} i_{x[1} i_{x_2} i_{x_3} i_{x_4} i_{x_5} \ell_5(X_6, X_7, \Psi_1, \Psi_2, \Psi_3). \]  

All other brackets vanish. As in the previous section, we observe that most of terms in the brackets can be expressed recursively, which is to be expected since they were built by explicitly iterating through the generalised Jacobi identities. Note also that, as mentioned earlier, the terms that depend only on elements \(\Lambda\) and \(A\) are simply reproducing the (polarised) \(p\)-forms that result from the descent equations of the anomaly polynomial. For example, we have that

\[ \omega_6^4(\Lambda, A) = \frac{1}{3!} \ell_4(\Lambda, A^3) - \frac{1}{4!} \ell_5(\Lambda, A^4) - \frac{1}{5!} \ell_6(\Lambda, A^5). \]  

Despite no longer being able to describe these higher order gauge algebras in terms of just a Leibniz bracket on the generalised tangent space, we thus have that the extra structure of \(E\) is still enough to ensure that we can find an \(L_\infty\) algebra, and that this algebra has a finite number of brackets. And while there should not be much difficulty in adding the extra geometrical data that make up the physical degrees of freedom such as the Riemannian metric, it will require further study to see whether this weaker differential structure will be enough to give a natural geometric description of the dynamics of higher-derivative-corrected supergravity.
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Appendix: Conventions and Chern-Simons forms

We mostly follow the conventions of [19], though we generally omit the wedge symbol for the product of differential forms.

Given a Lie algebra-valued one-form $A$ we define its curvature by

$$ R(A) = dA + A^2, \quad (A.1) $$

which satisfies

$$ d_A R = dR + [A, R] = 0, \quad (A.2) $$

and is invariant under the infinitesimal gauge transformations of the potential

$$ \delta_A A = -d_A \Lambda = -d\Lambda - [A, \Lambda]. \quad (A.3) $$

As is well known from the study of anomalies [46, 73–75], taking the trace of powers of the curvature one can define invariant polynomials

$$ d \text{ tr } R^n = \delta \text{ tr } R^n = 0, \quad (A.4) $$

from the $n$-th Chern character $\text{ tr } R^n$ of the gauge vector bundle. Poincaré’s lemma then implies that one can locally define the Chern-Simons forms $\omega(2n−1)$

$$ d\omega_{(2n−1)}(A) = \text{ tr } R^n, \quad (A.5) $$

and applying the lemma once again, now for $\delta$, gives

$$ d\omega_{(2n−2)}(\Lambda, A) = \delta_A \omega_{(2n−1)}(A), \quad (A.6) $$

where the superscript denotes the powers of the gauge parameter, since, in principle, one can continue “descending” along this chain.
We can list (up to exact terms) some of the Chern-Simons forms that will be important for us explicitly

\[
\omega_7(A) = \text{tr} \left( A(dA)^3 + \frac{8}{5} A^3 (dA)^2 + \frac{4}{5} \text{Ad} AA^2 dA + 2 A^5 dA + \frac{4}{7} A^7 \right), \quad (A.7)
\]

\[
\omega_6(\Lambda, A) = \text{tr} d\Lambda \left( A(dA)^2 + \frac{2}{5} (A^3 dA + dAA^3 + A^5) \\
+ \frac{1}{5} (A^2 dAA + \text{Ad} AA^2) \right), \quad (A.8)
\]

\[
\omega_5(A) = \text{tr} \left( A(dA)^2 + \frac{3}{2} A^3 dA + \frac{3}{5} A^5 \right), \quad (A.9)
\]

\[
\omega_4(\Lambda, A) = \text{tr} d\Lambda \left( \text{Ad} A + \frac{1}{2} A^3 \right), \quad (A.10)
\]

\[
\omega_3(A) = \text{tr} \left( \text{Ad} A + \frac{2}{3} A^3 \right), \quad (A.11)
\]

\[
\omega_2(\Lambda, A) = \text{tr} d\Lambda A. \quad (A.12)
\]

The last two are not used in this work but are the ones that are featured in Heterotic generalised geometry. These agree with the usual ones in the literature [74] up to exact terms corresponding to our convention choice of having the differential acting on the parameter \( \Lambda \).

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