BO 2.0: Plasma Wave and Instability Analysis with Enhanced Polarization Calculations

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Abstract

Besides the relation between the wave vector $k$ and the complex frequency $\omega$, wave polarization is useful for characterizing the properties of a plasma wave. The polarization of the electromagnetic fields, $\delta E$ and $\delta B$, have been widely used in plasma physics research. Here, we derive equations for the density and velocity perturbations, $\delta n_s$ and $\delta v_s$, respectively, of each species in the electromagnetic kinetic plasma dispersion relation by using their relation to the species current density perturbation $\delta J_s$. Then we compare results with those of another commonly used plasma dispersion code (WHAMP) and with those of a multi-fluid plasma dispersion relation. We also summarize a number of useful polarization quantities, such as magnetic ellipticity, orientation of the major axis of the magnetic ellipse, various ratios of field energies and kinetic energies, species compressibility, parallel phase ratio, Alfvén-ratio, etc., which are useful for plasma physics research, especially for space plasma studies. This work represents an extension of the BO electromagnetic dispersion code [H.S. Xie, Comput. Phys. Comm. 244 (2019) 343-371] to enhance its calculation of polarization and to include the capability of solving the electromagnetic magnetized multi-fluid plasma dispersion relation.

Keywords: Plasma physics, Kinetic dispersion relation, Waves and instabilities, Matrix eigenvalue

PROGRAM SUMMARY

Program Title: BO 2.0
License: BSD 3-clause
Programming Language: Matlab

Journal reference of previous version: [1] H.S. Xie, BO: A unified tool for plasma waves and instabilities analysis, Comput. Phys. Comm. 244 (2019) 343-371. [2] H.S. Xie, Y. Xiao, PDRK: A General Kinetic Dispersion Relation Solver for Magnetized Plasma, Plasma Sci. Technol. 18 (2) (2016) 97. [3] H. S. Xie, PDRF: A general dispersion relation solver for magnetized multi-fluid plasma, Comput. Phys. Comm. 185 (2014) 670-675.

Does the new version supersede the previous version?: Yes

Reasons for the new version: Enhance the code capability, especially to support the calculation of density and velocity perturbations. Also, the multi-fluid and kinetic versions are combined into one version.

Summary of revisions:* In this new version, multi-fluid model is included as one option. The density and velocity perturbations of kinetic versions are also supported. Many useful polarizations are included.

Nature of problem: The linear fluid and kinetic waves and instabilities in plasma can be described by dispersion relations. The challenges are to provide a dispersion relation as general as possible and to obtain all the solutions of it, which is the goal of BO. The BO code provides a unified numerically solvable framework for kinetic and multi-fluid plasma dispersion relations, which greatly extends the standard ones, with an arbitrary number of species.

Solution method: Transforming the dispersion relation to an equivalent matrix eigenvalue problem and find all the solutions using standard matrix eigenvalue library function.

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1. Introduction

Plasma is a combination of particles and electromagnetic fields. The electromagnetic fields are usually described by the Maxwell equations, whereas particles can be described by either a kinetic model using the distribution function, \( f_s(v, t) \), or a fluid model that employs velocity moments of the distribution function. It is well known that, for a uniform plasma, the linear plasma dispersion relation can be solved using

\[
D \cdot \delta E = 0,
\]

with

\[
|D(\omega, k)| = 0,
\]

where \( D \) is a 3-by-3 matrix tensor and \( \delta E = (\delta E_x, \delta E_y, \delta E_z) \) is the perturbed electric field. For a given wave vector \( k \), we solve the dispersion relation Eq. (2), which yields the complex frequency, \( \omega = \omega_r + i\omega_i \). Then, we obtain the matrix elements of \( D \) from \( \omega \) and solve Eq. (1) for the perturbed electric field \( \delta E \). We can then calculate the perturbed magnetic field \( \delta B \) and current density \( \delta J \) using the Maxwell equations.

Besides the perturbed electromagnetic fields, the plasma waves can carry the perturbed density \( \delta n_s \) and velocity \( \delta v_s \), which are widely used for the wave mode identification. Here “s” denotes the particle species. \( \delta n_s \) and \( \delta v_s \) were not given in the kinetic dispersion code BO v1.0 [1, 2], which has been shown to be a powerful tool for studying plasma waves and instabilities in the solar-terrestrial plasmas [3][5]. The major purpose of this work is to derive expressions for \( \delta n_s \) and \( \delta v_s \) using the kinetic dispersion relation, and check the validity of the approach by comparing results with those of another commonly used electromagnetic dispersion code (WHAMP [6]) and with those of a multi-fluid plasma model [3]. In section 2 we derive the equations and show how we implement them in the BO kinetic dispersion code [1, 2] and the PDRK fluid dispersion code [3]. In section 3 we benchmark the results using two independent kinetic solvers and with the multi-fluid solver PDRF. In section 4 we give a summary with some discussion. In the Appendices, we list useful polarization quantities calculated in BO and give a summary of the updated model used in PDRF. All of these updates are summarized to the new version BO v2.0 (https://github.com/hsxie/bo).

2. How to calculate the perturbed density, velocity and plasma current in BO

2.1. Perturbed density and velocity

In the plasma kinetic model, the density and velocity are zeroth and first order moment of the velocity distribution function, respectively. The perturbed density is given by

\[
\partial_t \delta n_s + \nabla \cdot (\delta J_s/q_s) = 0,
\]

where \( \delta J_s = n_{s0}q_s \int \delta f_s dv^3 \) is the perturbed plasma current. The perturbed plasma current can be expressed in terms of the perturbed fluid density and velocity,

\[
\delta J_s = q_s n_{s0} \delta v_s + q_s \delta n_s v_s,
\]

or

\[
\begin{align*}
\delta J_{sx} &= q_s n_{s0} \delta v_{sx} + q_s \delta n_s v_{sx}, \\
\delta J_{sy} &= q_s n_{s0} \delta v_{sy} + q_s \delta n_s v_{sy}, \\
\delta J_{sz} &= q_s n_{s0} \delta v_{sz} + q_s \delta n_s v_{sz}
\end{align*}
\]
where \( v_{ds} = (v_{dx}, v_{dy}, v_{dz}) \) denotes the zeroth order drift velocity, and directions of \( x \) and \( y \) axes are perpendicular to the background magnetic field \( B_0 = B_0 z \). It should be noted that one of advantages of BO [1] and PDRF [3] is that the three components of \( v_{ds} \) are included in these two solvers.

Using Eqs. (3) and (5), we can directly obtain \( \delta n_s \) and \( \delta v_s \), once the dispersion relation of one plasma wave mode and \( \delta J_s \) are known. Under the plane wave assumption, i.e., \( \partial_t \rightarrow -i\omega, \nabla \rightarrow ik \), one readily obtains

\[
\omega \delta n_s = \frac{1}{q_s} \mathbf{k} \cdot \mathbf{\delta J}_s. \tag{6}
\]

Since the wavevector is given as \( \mathbf{k} = (k_x, 0, k_z) \) in BO [1], the controlling equations for the perturbed density and velocity are

\[
\begin{align*}
\delta n_s &= \frac{1}{q_s^2} (k_x \delta J_{sx} + k_z \delta J_{sz}), \\
\delta v_{sx} &= \frac{1}{q_s^2} \left[ \delta J_{sx} - \frac{1}{\omega} (k_x \delta J_{sx} + k_z \delta J_{sz}) v_{dx} \right], \\
\delta v_{sz} &= \frac{1}{q_s^2} \left[ \delta J_{sz} - \frac{1}{\omega} (k_x \delta J_{sx} + k_z \delta J_{sz}) v_{dz} \right].
\end{align*} \tag{7}
\]

Here we note that \( \delta v_s = \int dv^3 v \delta f_s = \delta J_s / n_0 q_i \) in motionless plasmas where \( v_{ds} = 0 \), and \( \delta v_s \neq \int dv^3 v \delta f_s \) in a plasma where \( v_{ds} \neq 0 \).

2.2. Perturbed plasma current

In this subsection, we will discuss how to calculate the plasma current \( \delta J_s \) in BO/PDRF [1][2]. The first approach for giving \( \delta J_s \) is through \( \delta J_s = \mathbf{\sigma}_s \cdot \mathbf{\delta E} \), where the conductivity tensor \( \mathbf{\sigma}_s \) in BO/PDRF can be obtained by the following procedures. Eq. (129) in Ref. [1] gives the relation of the total plasma current and electric field in BO/PDRF

\[
\begin{pmatrix}
\delta J_{m}^m \\
\delta J_{m}^y \\
\delta J_{m}^z
\end{pmatrix} = -i\epsilon_0 \begin{pmatrix}
\frac{\mu_r}{\epsilon_0} + \sum_{m, j} \frac{b_{m11}}{\omega^2 - \epsilon_m} & \sum_{m, j} \frac{b_{m12}}{\omega^2 - \epsilon_m} & \sum_{m, j} \frac{b_{m13}}{\omega^2 - \epsilon_m} \\
\sum_{m, j} \frac{b_{m21}}{\omega^2 - \epsilon_m} & \frac{\mu_r}{\epsilon_0} + \sum_{m, j} \frac{b_{m22}}{\omega^2 - \epsilon_m} & \sum_{m, j} \frac{b_{m23}}{\omega^2 - \epsilon_m} \\
\sum_{m, j} \frac{b_{m31}}{\omega^2 - \epsilon_m} & \sum_{m, j} \frac{b_{m32}}{\omega^2 - \epsilon_m} & \frac{\mu_r}{\epsilon_0} + \sum_{m, j} \frac{b_{m33}}{\omega^2 - \epsilon_m}
\end{pmatrix} \begin{pmatrix}
\delta E_x \\
\delta E_y \\
\delta E_z
\end{pmatrix},
\]

with coefficients

\[
\begin{align*}
b_{m11} &= \sum_{\alpha} r_{\alpha} \alpha^2 p_{11m_j} / \epsilon_{m_j}, & b_{m12} &= -\sum_{\alpha} \alpha^2 p_{31m_j} / \epsilon_{m_j}, & b_{m13} &= \sum_{\alpha} \alpha^2 p_{13m_j} / \epsilon_{m_j}, \\
b_{m21} &= \sum_{\alpha} r_{\alpha} \alpha^2 p_{21m_j} / \epsilon_{m_j}, & b_{m22} &= -\sum_{\alpha} \alpha^2 p_{32m_j} / \epsilon_{m_j}, & b_{m23} &= \sum_{\alpha} \alpha^2 p_{23m_j} / \epsilon_{m_j}, \\
b_{m31} &= \sum_{\alpha} r_{\alpha} \alpha^2 p_{21m_j} / \epsilon_{m_j}, & b_{m32} &= -\sum_{\alpha} \alpha^2 p_{32m_j} / \epsilon_{m_j}, & b_{m33} &= \sum_{\alpha} \alpha^2 p_{23m_j} / \epsilon_{m_j}, \\
b_{m1j1} &= \sum_{\alpha, \beta} r_{\alpha} \alpha \beta p_{11m_j} / \epsilon_{m_j}, & b_{m2j2} &= \sum_{\alpha, \beta} r_{\alpha} \alpha \beta p_{22m_j} / \epsilon_{m_j}, & b_{m3j3} &= \sum_{\alpha, \beta} r_{\alpha} \alpha \beta p_{33m_j} / \epsilon_{m_j}.
\end{align*} \tag{8}
\]

The definition for variables in Eqs. (8) and (9) can be found in Ref. [1].

To implement Eq. (7), we need to separate the contribution from each species of \( \delta J = \sum_s \delta J_s = \sum_s \mathbf{\sigma}_s \cdot \mathbf{\delta E} \). To do this, we use a relation of \( \frac{b_{m11}}{\epsilon_0} = \sum \frac{b_{m11}}{\omega^2 - \epsilon_m} \) and then we rewrite Eqs. (8) and (9) as

\[
\begin{pmatrix}
\delta J_{m}^m \\
\delta J_{m}^y \\
\delta J_{m}^z
\end{pmatrix} = -i\epsilon_0 \begin{pmatrix}
\sum_{m, j} \frac{\mu_r}{\epsilon_0} + \sum_{m, j} \frac{b_{m11}}{\omega^2 - \epsilon_m} & \sum_{m, j} \frac{b_{m12}}{\omega^2 - \epsilon_m} & \sum_{m, j} \frac{b_{m13}}{\omega^2 - \epsilon_m} \\
\sum_{m, j} \frac{b_{m21}}{\omega^2 - \epsilon_m} & \sum_{m, j} \frac{b_{m22}}{\omega^2 - \epsilon_m} & \sum_{m, j} \frac{b_{m23}}{\omega^2 - \epsilon_m} \\
\sum_{m, j} \frac{b_{m31}}{\omega^2 - \epsilon_m} & \sum_{m, j} \frac{b_{m32}}{\omega^2 - \epsilon_m} & \sum_{m, j} \frac{b_{m33}}{\omega^2 - \epsilon_m}
\end{pmatrix} \begin{pmatrix}
\delta E_x \\
\delta E_y \\
\delta E_z
\end{pmatrix},
\]

\[
= \sum_s \mathbf{\sigma}_s^m \begin{pmatrix}
\delta E_x \\
\delta E_y \\
\delta E_z
\end{pmatrix}, \tag{10}
\]

with the coefficients

\[
\begin{align*}
\text{and } b_{s_jk} & = \sum r_{sr} \omega_{ps} p_{s_jk} / c_{m_j}, \\
b_{s_jm} & = -\omega_{ps}^2 \sum r_{sr} \omega_{ps} / c_{m_j}, \\
b_{s_jn} & = -\omega_{ps}^2 / c_{m_j}, \\
b_{s_jo} & = -\omega_{ps}^2 / c_{m_j}, \\
b_{s_jp} & = -\omega_{ps}^2 / c_{m_j}, \\
b_{s_jq} & = -\omega_{ps}^2 / c_{m_j}.
\end{align*}
\]

Consequently, we have

\[
\delta J^m_s = \sigma^m_s \delta E, \quad \sigma^m_s = -i \epsilon_0 \left( \begin{align*}
\frac{b_{s_j1}}{\sqrt{\omega}} + \sum_{m_j} b_{s_j2} / c_{m_j} + \sum_{n_j} b_{s_j3} / c_{m_j} + \sum_{o_j} b_{s_j4} / c_{m_j} + \sum_{p_j} b_{s_j5} / c_{m_j} + \sum_{q_j} b_{s_j6} / c_{m_j} \end{align*} \right).
\]

We can use Eq. (12) to obtain \( \delta J_s \). Since \( \delta E \) is known, the first approach requires solving the above 3-by-3 tensor for each species.

The second approach would be more convenient for obtaining \( \delta J_s \) based on the fact that a matrix eigenvalue method is used in BO/PDRK. For example, as given in Eq. (132) of Ref. 11, the perturbed current in \( x \) direction is

\[
i \delta J_s = \begin{align*}
& j_x + \sum v_{m_jx} + \sum v_{j_s} \quad (13)
\end{align*}
\]

where \( j_s, v_{m_jx} \) and \( v_{j_s} \) have been solved along with \( \delta E \) and \( \delta B \). \( \delta J_s \) can be directly obtained once \( j_s, v_{m_jx} \) and \( v_{j_s} \) for each species \( s \) are known. Similarly, we can obtain \( \delta J_y \) and \( \delta J_z \).

The quantities \( v_{m_j} \) and \( v_{j_s} \) are species quantities, but \( j_s \) was not in the original version of BO/PDRK. With the addition of only \( S - 1 \) matrix elements, we can replace the matrix element \( j_s \) by a sum over \( S \) matrix elements \( j_s \), where \( S \) is the number of species. To do this, we modified the BO/PDRK matrix equations in Eq. (10), i.e.,

\[
\omega j_x = b_{s_1} \delta E_x + b_{s_2} \delta E_y + b_{s_3} \delta E_z,
\]

\[
\omega j_y = b_{s_1} \delta E_x + b_{s_2} \delta E_y + b_{s_3} \delta E_z,
\]

\[
\omega j_z = b_{s_1} \delta E_x + b_{s_2} \delta E_y + b_{s_3} \delta E_z
\]

as

\[
\omega j_{sx} = b_{s_1} \delta E_x + b_{s_2} \delta E_y + b_{s_3} \delta E_z,
\]

\[
\omega j_{sy} = b_{s_1} \delta E_x + b_{s_2} \delta E_y + b_{s_3} \delta E_z,
\]

\[
\omega j_{sz} = b_{s_1} \delta E_x + b_{s_2} \delta E_y + b_{s_3} \delta E_z
\]

This separation can directly give \( j_{sx, sy, sz} \) from the BO/PDRK matrix, which then yields \( \delta J_{sx, sy, sz} \) through the following equations

\[
\begin{align*}
\delta J_{sx} &= (j_x + \sum_{m_j} v_{m_jx} + \sum_{j_s} v_{j_s}) / (i \epsilon_0), \\
\delta J_{sy} &= (j_y + \sum_{m_j} v_{m_jy} + \sum_{j_s} v_{j_s}) / (i \epsilon_0), \\
\delta J_{sz} &= (j_z + \sum_{m_j} v_{m_jz} + \sum_{j_s} v_{j_s}) / (i \epsilon_0).
\end{align*}
\]
The updated matrix equations of BO, i.e., Eq.(132) of Ref.[1], become

\[
\begin{align*}
\omega v_{m,j}^{nm} &= c_{nm}v_{m,j} + b_{nmj1}\delta E_x + b_{nmj2}\delta E_y + b_{nmj3}\delta E_z, \\
\omega v_{j,r}^{nm} &= c_{j,r}v_{j,r} + b_{j,r1}\delta E_x + b_{j,r2}\delta E_y + b_{j,r3}\delta E_z, \\
\omega j_{x} &= b_{11}\delta E_x + b_{12}\delta E_y + b_{13}\delta E_z, \\
\delta J_{x}/\epsilon_0 &= j_x + \sum_{nm} v_{m,j} + \sum_{j,r} v_{j,r}, \\
\omega v_{m,j}^{nm} &= c_{nm}v_{m,j} + b_{nmj1}\delta E_x + b_{nmj2}\delta E_y + b_{nmj3}\delta E_z, \\
\omega v_{j,r}^{nm} &= c_{j,r}v_{j,r} + b_{j,r1}\delta E_x + b_{j,r2}\delta E_y + b_{j,r3}\delta E_z, \\
\omega j_{y} &= b_{21}\delta E_x + b_{22}\delta E_y + b_{23}\delta E_z, \\
\delta J_{y}/\epsilon_0 &= j_y + \sum_{nm} v_{m,j} + \sum_{j,r} v_{j,r}, \\
\omega v_{m,j}^{nm} &= c_{nm}v_{m,j} + b_{nmj1}\delta E_x + b_{nmj2}\delta E_y + b_{nmj3}\delta E_z, \\
\omega v_{j,r}^{nm} &= c_{j,r}v_{j,r} + b_{j,r1}\delta E_x + b_{j,r2}\delta E_y + b_{j,r3}\delta E_z, \\
\omega j_{z} &= b_{31}\delta E_x + b_{32}\delta E_y + b_{33}\delta E_z, \\
\delta J_{z}/\epsilon_0 &= j_z + \sum_{nm} v_{m,j} + \sum_{j,r} v_{j,r}, \\
\omega \delta E_x &= c^2k_j\delta B_x - i\delta J_{x}/\epsilon_0, \\
\omega \delta E_y &= c^2k_j\delta B_y - c^2k_j\delta B_x - i\delta J_{y}/\epsilon_0, \\
\omega \delta E_z &= c^2k_j\delta B_z - k_j\delta E_x - k_j\delta E_y, \\
\omega \delta B_x &= k_j\delta E_x - k_j\delta E_z, \\
\omega \delta B_y &= k_j\delta E_y - k_j\delta E_z, \\
\omega \delta B_z &= k_j\delta E_z - k_j\delta E_x, \\
\end{align*}
\]

(17)

which yields a sparse matrix eigenvalue problem \( \omega X = M(k) \cdot X \). The symbols such as \( v_{m,j} \), \( j_{x,y,z} \) and \( \delta J_{x,y,z} \) used here are analogous to the perturbed velocity and current density in fluid derivations of plasma waves. The elements of the eigenvector \( (\delta E_x, \delta E_y, \delta E_z, \delta B_x, \delta B_y, \delta B_z) \) represent the perturbed electric and magnetic fields. Thus, all variables of one plasma wave mode can be obtained in a straightforward manner. In addition, the dimension of the matrix is \( N_N = 3\times(N_{S_{m,N}} + N_{S_{adj}} + S) + 6 = 3\times[(S_{m}\times(2\times N + 1) + S_{u}\times 2] \times J + S) + 6 \), where \( S_{m} \) and \( S_{u} \) are the numbers of magnetized and unmagnetized species, respectively, \( S = S_{m} + S_{u} \), \( N \) is the number of harmonics retained for magnetized species, and \( J \) is the order of the \( J \)-pole expansion used for calculation of the plasma dispersion \( Z \) function.

2.3. Benchmark strategies

In order to test whether the values of \( (\delta E, \delta B, \delta J, \delta n, \delta v) \) calculated by BO are correct, we do the following benchmarks:

- (1) Use \( \omega \delta B = k \times \delta E \) to check \( \delta B \) and \( \delta J = \frac{1}{\mu_0}i k \times \delta B + i \epsilon_0 \omega \delta E \) to check \( \delta J \).
- (2) Use \( \delta J_s = \sigma_s \cdot \delta E \) to calculate \( \delta J_s \) and \( \delta J = \sum_s \delta J_s \) to calculate \( \delta J \), and compare this with \( \delta J \) from (1).
- (3) Compare \( \delta J_s \) and \( \delta J \) in (2) with the analogous quantities calculated in the new version of BO/PDRK using \( j_{x,y,z} \) and \( v_{m,j} \).
- (4) Write out 3-by-3 tensors \( \sigma_s, Q_s, \sigma, Q, K \) and \( D(\omega, k) \) and verify that \( |D(\omega, k)| = 0 \).
- (5) Compare values of \( \delta J_s, \delta J, \delta n, \delta v \), with those calculated using jWHAMP and the multi-fluid solver PDRF.

Using the new version of BO, we have checked the above (1)-(4) in several test cases and have identified a good consistence between these two methods proposed in Subsection 2.2. In following Section, we will present benchmark results in the above (5).
Figure 1: Comparison of results from BO and jWHAMP with parameters case#1. All of the quantities calculated by these two codes agree well except for $\delta v_z(s = 3)$. BO contains the effect relating to $v_{dz}$ for species 3, whereas $v_{dz} \neq 0$ was not taken into account in the current version of jWHAMP.
3. Benchmark and comparing with multi-fluid plasma model

3.1. Benchmark with jWHAMP

jWHAMP is Dartmouth College’s java extension of the WHAMP electromagnetic dispersion code [6], which exports a number of polarization quantities such as \( \delta E, \delta B, \delta n_s, \delta v_s \). However, in jWHAMP, the drift velocity, \( v_{dz} \), was not taken into account when calculating \( \delta v_s \). To test the greatest number of features of a kinetic calculation, we consider a case (case#1) where the plasma consists of four species. We also consider both parallel and perpendicular components of the wave vector in case#1. The input species parameters for this case (specified in the ‘bo.in’ input file) are

| q_s(e) | m_s(m_unit) | n_s(m^-3) | T_zs(eV) | T_ps(eV) | v_{dz}/c |
|--------|-------------|-----------|----------|----------|-----------|
| 1      | 1           | 1e6       | 24.838e3 | 99.352e3 | 0.0       |
| -1     | 5.447e-4    | 1.1e6     | 24.838e3 | 24.838e3 | 0.0       |
| 1      | 1           | 0.01e6    | 24.838e4 | 24.838e4 | 0.0727    |
| 1      | 4           | 0.1e6     | 0.1e3    | 0.1e3    | 0.0       |

Here, we use the default normalization in BO: the mass \( m_s \) is normalized to the proton mass \( m_p \), \( k_x \) and \( k_z \) are normalized to \( k_n = \omega_{ps1}/c \) and the frequency is normalized to \( \omega_n = |\omega_{cs1}| \), where “1” indicates the first species (i.e., the proton component with \( n_s = 10^6 \) m\(^{-3}\)). The case#1 contains both anisotropic temperature and parallel drift velocity effects, which would destabilize the Alfvén/ion-cyclotron mode wave as shown in Fig. 1 that presents the distributions of the real and imaginary parts of frequency \( \omega \), \( \delta n_s \), \( \delta v_s \), and \( \delta E \) as a function of \( k_z \) under \( k_x = 0.05 \) and \( B_0 = 100 \) nT.

For all quantities presented in Fig. 1 both BO and jWHAMP give the same distributions except for the values of \( \delta v_z(s = 3) \). The reason is that the effect of the drift was not included in jWHAMP to calculate \( \delta v_z \). If we ignore \( v_{dz} \) in Eq. (7), we find that \( \delta v_z(s = 3) \) from BO agrees with the value from jWHAMP.

This benchmark indicates that the new version of BO can correctly give the perturbed density and velocity in case#1. Moreover, the comparison between BO and jWHAMP shows that it needs to take into account \( v_{dz} \) for calculating the perturbed velocity.

3.2. Comparing with multi-fluid plasma model

Here we compare the results of BO with those of a multi-fluid model. In the cold plasma limit (\( v_{zts}, v_{\perp ts} \approx 0 \) or \( \omega \gg k_z v_{zts} \)), the kinetic results and fluid results should be identical. Note that we have updated the multi-fluid plasma dispersion relation solver PDRF, and that it can be run in the new version of BO (see Appendix A).

Fig. 2 compares results from the BO kinetic and fluid models for a cold four species plasma where \( B_0 = 2000 \) nT. The results are shown as a function of \( k \) for the most unstable mode wave which has the wave normal angle \( \theta = 10^\circ \).

The input parameters for this case (case#2) are

| q_s(e) | m_s(m_unit) | n_s(m^-3) | T_zs(eV) | T_ps(eV) | v_{dz}/c | v_{dx}/c | v_{dy}/c |
|--------|-------------|-----------|----------|----------|----------|----------|----------|
| 1      | 1           | 0.8e10    | 1.0e-1   | 1.0e-1   | 0.0      | 0.0      | 0.0      |
| 1      | 1           | 0.1e10    | 1.0e-1   | 1.0e-1   | 2.0e-3   | 3.0e-3   | 1.0e-3   |
| 2      | 4           | 0.05e10   | 1.0e-1   | 1.0e-1   | 0.0      | 0.0      | 0.0      |
| -1     | 5.447e-4    | 1.0e10    | 1.0e-1   | 1.0e-1   | 2.0e-4   | 3.0e-4   | 1.0e-4   |

We consider both parallel and perpendicular beams in case#2, and the default normalization is used. Fig. 2 shows that both kinetic and fluid models give the nearly same results. For large \( k \) (\( \sim 5k_n \)), there is a slight difference between \( \omega_n \) for the kinetic and fluid models, due to Landau damping in the kinetic model. If we decrease the temperature to 0.01 eV, the deviation nearly vanishes. These results indicate that the equations in Sec. 2 and there implementation in BO are correct, even including perpendicular beams.

We further compare results from the fluid and kinetic models for a warm plasma with isotropic pressure. The species parameters for this case (case#3) are

| q_s(e) | m_s(m_unit) | n_s(m^-3) | T_zs(eV) | T_ps(eV) |
|--------|-------------|-----------|----------|----------|
| 1      | 1           | 0.36e8    | 24.838e1 | 24.838e1 |
| -1     | 5.447e-4    | 0.36e8    | 1.0      | 1.0      |


Figure 2: Comparison of results from the BO kinetic and multi-fluid models for a cold plasma with parameters case#2. The wave is the most unstable mode with the wave normal angle $\theta = 10^\circ$. All quantities calculated using these two models agree well.
Figure 3: Comparison of BO kinetic and fluid results for the fast-magnetosonic/whistler mode wave with parameters case#3.
Figure 4: Comparison of BO kinetic and fluid results for the Alfvén/ion-cyclotron mode wave with parameters case#3. This wave is strongly damped at large $k$ in the kinetic model due to wave-particle interactions.
In order to have the consistent sound speed in both kinetic and fluid models, i.e., $c_\text{f}^2 \approx v_{\text{th,f}}^2$, we choose the default adiabatic pressure closure with adiabatic coefficients $\gamma_{\text{f}} = \gamma_{\text{f},\perp} = 2.0$. We also use $B_0 = 100 \text{ nT}$ and $\theta = 30^\circ$. Figs. 3 and 4 give the results of the fast-magnetosonic/westler mode and the Alfvén/ion-cyclotron mode, respectively. Since the kinetic wave-particle interactions considerably enhance in the warm plasma, the kinetic results would be different from the fluid results. Fig. 3 shows that although the wave frequency from the kinetic and fluid models is almost the same, the quantities $(\delta E, \delta B, \delta n_s, \delta v_s)$ have much larger deviations. Fig. 4 shows that both the wave frequency and polarization have significant differences.

4. Summary and discussions

In this paper, we describe the updated BO plasma wave dispersion relation solver that can be used for both kinetic and fluid plasma models. We extend the kinetic version to obtain density and velocity perturbations for each species. In the cold plasma limit, the kinetic model yields results of $(k, \omega, \delta E, \delta B, \delta J, \delta n_s, \delta v_s)$ quite similar to those from the multi-fluid model, even for zeroth order drift beams in arbitrary directions. In a warm plasma, Landau and cyclotron wave-particle resonance effects can alter the wave frequency and the polarization, which induce the difference between the kinetic and fluid models. The extensive set of polarization quantities calculated by the updated BO (see Appendix B) could be useful for identifying and characterizing plasma waves and instabilities in space plasmas.

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Appendix A. Reduced version of multi-fluid plasma dispersion relation solver PDRF

To make the PDRF code more amenable to comparison with BO-K/PDRK, we simplify the original version of PDRF by removing density inhomogeneity, relativistic effects and collisions, and make it available as BO-F in BO code. Drifts in arbitrary directions and pressure anisotropy are retained. Some typos in Ref. [3] are also corrected here.

We consider a multi-fluid plasma in an external magnetic field $B_0 = (0, 0, B_0)$. The zero-th order flow velocity of the fluid component $s$ is $v_{ds} = (v_{dsx}, v_{dty}, v_{dsz})$. The species densities and temperatures are homogeneous, i.e., gradient effects are ignored, and the wave vector is assumed to be $k = (k_x, 0, k_z) = (k \sin \theta, 0, k \cos \theta)$.

We start with the multi-fluid equations

$$\partial_t n_s = -\nabla \cdot (n_s v_s),$$

$$\partial_t v_s = -v_s \cdot \nabla v_s + \frac{q_s}{m_s} (E + v_s \times B) - \frac{\nabla p_s}{\rho_s},$$

$$\partial_t E = c^2 \nabla \times B - J/\epsilon_0,$$

$$\partial_t B = -\nabla \times E,$$  \hspace{1cm} (A.1a, A.1b, A.1c, A.1d)

where we ignore the relativistic effects, and

$$J = \sum_s q_s n_s v_s,$$

$$\partial_t (P_{\parallel,\perp} \cdot \rho_s^{-\gamma_{\parallel,\perp}}) = 0,$$  \hspace{1cm} (A.2a, A.2b)

where the mass density is $\rho_s \equiv m_s n_s$, and the speed of light is $c = 1/\sqrt{\mu_0 \epsilon_0}$. In the above equations, we have used adiabatic model for pressure closure, with $\gamma_{\parallel,\perp}$ being the parallel and perpendicular exponents. Furthermore, $P_{\parallel,\perp} = n k_B T_{\parallel,\perp}$, $P = P_{\parallel} \hat{b}_B + P_{\perp} (I - \hat{b}_B)$ and $\hat{b} = B/B$. Different anisotropic pressure closures will yield different results. Usually, one take $\gamma_{\parallel} = \gamma = 5/3$. However, we find $\gamma_{\parallel} = \gamma = 2$ would yield closer results to those of the kinetic model. If not specified by the user, $\gamma_{\parallel} = \gamma = 2$ are the default settings.

After linearizing, (A.2) becomes

$$\delta J = \sum_s q_s (n_s \delta v_s + \delta n_s v_s),$$

$$\delta P_{\parallel,\perp} = P_{\parallel,\perp} \gamma_{\parallel,\perp} \delta n_s n_s + \frac{c^2}{\rho_s} \delta n_s,$$  \hspace{1cm} (A.3a, A.3b)
where $c_{\perp,\perp}^2 \equiv \gamma_{\perp,\perp} P_{\perp,\perp}/p_{\perp} \cdot \mathbf{P}$ and $P_{\perp} = n_0 k_b T_{\perp,0}$. We also define $c_{\perp} \equiv B_{\perp,0}^2/(\mu_0 p_{\perp} \cdot \mathbf{P})$. We have

$$
\nabla \cdot \delta \mathbf{P} = (i k_x, 0, i k_z) \cdot \begin{pmatrix} \delta P_{\perp,\perp} & 0 & \Delta \delta B_z \\ 0 & \delta P_{\perp,\perp} & \Delta \delta B_x \\ \Delta \delta B_z & \Delta \delta B_x & \delta P_{\perp,\perp} \end{pmatrix}^T,
$$

(A.4)

where $\Delta_x \equiv (P_{\perp,0} - P_{\perp,0})/B_0$ and $\beta_{\perp,\perp} = 2 \mu_0 P_{\perp,\perp}/B_0^2$. The off-diagonal terms coming from the tensor rotation from $\hat{b}_0$ to $\hat{b}$ are related to energy exchange and are important for the anisotropic instabilities.

The linearized version of (A.1) with $f = f_0 + \delta f e^{k^r - i \omega t}$, $\delta f \ll f_0$ is equivalent to a matrix eigenvalue problem

$$\omega X = MX,$n

(A.5)

where $\omega$ is the eigenvalue and $X$ is the corresponding eigenvector containing polarization information for the eigenvectors. Accordingly, we have $X = (\delta n_x, \delta v_{xz}, \delta v_{yz}, \delta E_x, \delta E_y, \delta E_z, \delta B_x, \delta B_y, \delta B_z)^T$, and the matrix

$$
M = \begin{pmatrix}
k \cdot \mathbf{v}_{dx} & k_x n_0 & 0 & k_z n_0 & 0 & 0 & 0 & 0 & 0 \\
k_x^2 e^{-q_x/m_x} & k \cdot \mathbf{v}_{dx} - i \omega_{\perp,\perp} & 0 & k_x n_0 & 0 & 0 & 0 & 0 & 0 \\
0 & -i \omega_{\perp,\perp} & k_x n_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{i q_x}{m_x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
$$

(A.6)

where the elements between ‘|’ and ‘|’ means each species s has its own matrix elements, $\omega_{\perp,\perp} = q_x B_0/m_x$, $\omega_{\perp,\perp} = -\epsilon e^q_p q_x^2/(\epsilon_0 m_x)$, and $k \cdot \mathbf{v}_{dx} = k_x \mathbf{v}_{dx} + k_z \mathbf{v}_{dz}$. For a plasma containing S species, the dimension of $M$ is $(4S + 6) \times (4S + 6)$. If we define the thermal velocity $v_{\perp,\perp} = \sqrt{2 k_b T_{\perp,\perp}/m_x}$ as in the kinetic version of BO[1], we can have $c_{\perp,\perp} = \sqrt{\gamma_{\perp,\perp}/2} v_{\perp,\perp}$, or the temperature $k_b T_{\perp,0} = m_x c_{\perp,\perp}^2/\gamma_{\perp,\perp}$.

We can also use some other pressure closures. For example, the double-polytropic laws for pressure closure

$$
d_i(p_{\perp}, \rho_{\perp}^{\gamma_i}, \mathbf{B}^{n-1}) = 0,
$$

(A.7a)

$$
d_i(p_{\perp}, \rho_{\perp}^{-1}, \mathbf{B}^{\gamma_{i} +1}) = 0,
$$

(A.7b)

with $\gamma_i$ and $\gamma_{i,\perp}$ being the parallel and perpendicular polytropic exponent has been used previously in space plasma studies, c.f., Ref.[9]. Note that $\gamma_i = 3$ and $\gamma_{i,\perp} = 2$ yield the CGL relations[10], whereas $\gamma_i = \gamma_{\perp,\perp} = 1$ yields isothermal behavior. For this pressure closure, we have

$$
\delta P_{\perp} = P_{\perp,0}[\gamma_{\perp,\perp} \delta \epsilon_{\perp} - \gamma_{\perp,\perp} \delta B_{\perp}/B_{\perp,0}] = c_{\perp,\perp,\perp}^2 m_x \delta \epsilon_{\perp} - P_{\perp,0}(\gamma_{\perp,\perp} - 1) \delta B_{\perp}/B_{\perp,0},
$$

(A.8a)

$$
\delta P_{\perp} = P_{\perp,0}[\delta \epsilon_{\perp} + \delta \delta_{\perp} - \delta B_{\perp}/B_{\perp,0}] = c_{\perp,\perp,\perp}^2 m_x \delta \epsilon_{\perp} + P_{\perp,0}(\gamma_{\perp,\perp} - 1) \delta B_{\perp}/B_{\perp,0},
$$

(A.8b)

where $c_{\perp,\perp,\perp}^2 \equiv \gamma_{\perp,\perp} P_{\perp,0}/p_{\perp} \cdot \mathbf{P}$ and $c_{\perp,\perp,\perp}^2 \equiv P_{\perp,0}/p_{\perp}$ (here is different from the $c_{\perp,\perp}^2 \equiv \gamma_{\perp} P_{\perp,0}/p_{\perp}$ in Ref.[9]), and hence the matrix elements $M_{\perp,\perp,\perp}, M_{\perp,\perp,\perp},$ and $M_{\perp,\perp,\perp}$ in $M$ would be modified accordingly. Similarly to the adiabatic pressure closure case, we can have $c_{\perp} = \sqrt{\gamma_{\perp}/2} v_{\perp,\perp}$ and $c_{\perp,\perp} = \sqrt{\gamma_{\perp,\perp}/2} v_{\perp,\perp}$, or the temperature $k_b T_{\perp,0} = m_x c_{\perp,\perp}^2/\gamma_{\perp,\perp}$ and $k_b T_{\perp,0} = m_x c_{\perp,\perp}^2/\gamma_{\perp,\perp}$.

In BO, because of the limitations of the pressure closure, we only use the above fluid version to get a rough description of the waves and instabilities and for comparison with the kinetic version. It is especially useful for studying cold plasma waves and beam modes, in which case the pressure closure is not important. The fluid closure has many limitations and leads to some un-physical results. For example, in the double-polytropic CGL case ($\gamma_{\perp,\perp} = 3$, $\gamma_{\perp,\perp} = 2$), the waves can be unstable even when $P_{\perp} = P_{\perp,\perp} \cdot \mathbf{P}$ because $c_{\perp,\perp}^2 \neq c_{\perp,\perp}^2$. The high beta anisotropic firehose and mirror mode instabilities are also difficult to calculate accurately from fluid model. For accurate results with finite pressure, we recommend the kinetic version of BO.
Appendix B. Polarizations in BO

For given real \( k_x, k_z \) and corresponding complex \( \omega \), we find the complex quantities \( \delta E_x, \delta E_y, \delta E_z, \delta B_x, \delta B_y, \delta B_z, \delta J_x, \delta J_y, \delta J_z, \delta J_{x5}, \delta J_{z5}, \delta n_x, \delta n_y, \delta n_z, \delta \nu_{x5}, \delta \nu_{y5}, \delta \nu_{z5} \). We list the comprehensive polarization quantities calculated in the new version of BO code, and summarize them in Table B.1. Note that for a given \((k_x, k_z)\), there exist multiple branches corresponding to different eigenmodes \( \omega \), and each branch has its unique polarization.

Table B.1: List of the polarization quantities in BO. The numbers before each polarizations are the default indexes of them used in the BO code. We set npf=50 and nps=50 by default. For example, since npf+nps*(s-1)+4 is the s-th perturbed density, which means 50+50*(2-1)+4=104 is the default index of the perturbed density for the 2nd species.

| 1. electric field in x-direction (V/m) | \( \delta E_x \) | 2. electric field in y-direction (V/m) | \( \delta E_y \) |
| 3. electric field in z-direction (V/m) | \( \delta E_z \) | 4. magnetic field in x-direction (T) | \( \delta B_x \) |
| 5. magnetic field in y-direction (T) | \( \delta B_y \) | 6. magnetic field in z-direction (T) | \( \delta B_z \) |
| 7. electric field energy density (J/m³) | \( \delta U_{E} \) | 8. Magnetic field energy density (J/m³) | \( \delta U_{B} \) |
| 9. fraction of field energy in the electric field | \( \frac{\delta E}{\sqrt{\delta E_x^2 + \delta E_y^2 + \delta E_z^2}} \) | 10. fraction of electric field energy in \( \delta E_x \) | \( \frac{\delta E_x}{\sqrt{\delta E_x^2 + \delta E_y^2 + \delta E_z^2}} \) |
| 11. fraction of electric field energy in \( \delta E_y \) | \( \frac{\delta E_y}{\sqrt{\delta E_x^2 + \delta E_y^2 + \delta E_z^2}} \) | 12. fraction of electric field energy in \( \delta E_z \) | \( \frac{\delta E_z}{\sqrt{\delta E_x^2 + \delta E_y^2 + \delta E_z^2}} \) |
| 13. a measure of how electrostatic is | \( \frac{\sqrt{\delta E_x^2 + \delta E_y^2 + \delta E_z^2}}{\sqrt{\delta E_x^2 + \delta E_y^2 + \delta E_z^2}} \) | 14. another measure of how electrostatic is | \( \frac{\sqrt{\delta E_x^2 + \delta E_y^2 + \delta E_z^2}}{\sqrt{\delta E_x^2 + \delta E_y^2 + \delta E_z^2}} \) |
| 15. fraction of magnetic field energy in \( \delta B_x \) | \( \frac{\delta B_x}{\sqrt{\delta B_x^2 + \delta B_y^2 + \delta B_z^2}} \) | 16. fraction of magnetic field energy in \( \delta B_y \) | \( \frac{\delta B_y}{\sqrt{\delta B_x^2 + \delta B_y^2 + \delta B_z^2}} \) |
| 17. fraction of magnetic field energy in \( \delta B_z \) | \( \frac{\delta B_z}{\sqrt{\delta B_x^2 + \delta B_y^2 + \delta B_z^2}} \) | 18. magnetic polarization ellipticity | \( \epsilon_p \) |
| 19. angle of major axis of magnetic ellipse | \( \theta_B \) | 20. magnetic polarization ratio, \( |\eta| \) | \( \frac{\delta B_z}{\delta B_x} \) |
| 21. angle of magnetic polarization ratio, \( \phi_B \) | \( \arg \left( \frac{\delta B_z}{\delta B_x} \right) \) | 22. wave group velocity in x-direction (m/s) | \( V_{g\delta} \) |
| 23. wave group velocity in z-direction (m/s) | \( v_{g\delta} \) | 24. spatial growth rate in x-direction, \( S_x \) | \( \frac{\delta U_{B}}{v_{g\delta}} \) |
| 25. spatial growth rate in z-direction (m⁻¹), \( S_z \) | \( S_z \) | 26. total spatial growth rate (m⁻¹), \( S \) | \( \frac{\delta U_{B}}{v_{g\delta}} \) |
| 27. refractive index | \( |\epsilon| \) | 28 to 36. dispersion tensor elements i, j = 1, 2, 3 | \( D_{ij} \) |

The linear polarizations can have arbitrary large magnitude. After we obtain the eigenvectors \( \delta X_0 = [\delta E_x, \delta E_y, \delta E_z, \delta B_x, \delta B_y, \delta B_z, \delta J_x, \delta J_y, \delta J_z, \delta J_{x5}, \delta J_{z5}, \delta n_x, \delta n_y, \delta n_z, \delta \nu_{x5}, \delta \nu_{y5}, \delta \nu_{z5}] \) in the code, the normalization of modes is done like this

\[
\delta X_1 = \frac{\delta X_0}{|\delta E_x|}, \quad \delta X = \delta X_1 \sqrt{\delta E_x^2 + \delta E_y^2 + \delta E_z^2},
\]

which causes \( \delta E_x \) to be real and positive, and \( |\delta E| = \sqrt{\delta E_x^2 + \delta E_y^2 + \delta E_z^2} = 1 \) V/m. This procedure will work except for some extreme cases for which \( |\delta E_x| < 10^{-10} |\delta E| \) in double precision calculations. Equations to calculate
some other relevant field quantities are

\[ |\delta E_1|^2 = \delta E_x \cdot \delta E_x^* = |\text{Re}(\delta E_1)|^2 + |\text{Im}(\delta E_1)|^2, \]

similar for \(|\delta E_2|^2, |\delta B_1|^2, |\delta B_2|^2, |\delta B_2|^2\) \hspace{1cm} (B.2)

\[ |\delta E|^2 = |\delta E|^2 = |\delta E_1|^2 + |\delta E_2|^2, \]

\[ \delta U_E = \frac{1}{2} \cdot \frac{\mathrm{Re}(|\delta E|^2)}{\rho_0}, \]

(\ref{deltaUb})

\[ |\delta B|^2 = |\delta B|^2 = |\delta B_1|^2 + |\delta B_2|^2, \]

\[ \delta U_B = \frac{1}{2} \cdot \frac{|\delta B|^2}{\rho_0}, \]

(\ref{deltaUb})

\[ |k \times \delta E|^2 = -k_z \cdot \delta E_x k_x \cdot \delta E_y - k_x \cdot \delta E_x k_z \cdot \delta E_y, \]

(\ref{deltaU})

where the asterisk denotes complex conjugation. The extra \(\frac{1}{2}\) in \(\delta U_E\) and \(\delta U_B\) is for a time average. Note that usually \(|\delta E_{x,y,z}|^2 \neq |\delta E_{x,y,z}|^2\) because \(|\delta E_{x,y,z}|^2\) is always real, whereas \(\delta E_{x,y,z}\) is usually complex. The total energy density \(\delta U_{tot} = \delta U_E + \delta U_B + \delta U_K\), where

\[ \delta U_K = \sum_i \delta U_{K_i}, \quad \delta U_{K_i} = \frac{1}{2} \cdot \frac{1}{2} m_i n_i |\delta v_i|^2 + \frac{1}{2} \cdot \frac{1}{2} m_i (v_{ixx}^2 + v_{iyy}^2 + v_{i_z}^2) |\delta n_i|. \]  

(\ref{Etot})

To get the magnetic ellipticity, we use

\[ \alpha_B = \frac{\delta B_x}{\delta B_z} = |\alpha_B| e^{i\phi_B}, \quad \text{with} \quad |\alpha_B| = \frac{\delta B_x}{\delta B_z}, \quad \phi_B = \arg \left( \frac{\delta B_x}{\delta B_z} \right). \]  

(\ref{alphaB})

\[ \delta B_L = \delta B_x + i \delta B_y, \quad \delta B_R = \delta B_x - i \delta B_y, \quad \epsilon_B = \frac{|\delta B_L|^2 - |\delta B_R|^2}{|\delta B_L|^2 + |\delta B_R|^2}. \]  

(\ref{epsilonB})

If \(\epsilon_B = 1\), the wave is right hand circularly polarized, if \(\epsilon_B = 0\), the wave is linearly polarized, and if \(\epsilon_B = -1\), the wave is left-hand circularly polarized.

The quantity \(\theta_B\) is the angle of major axis of the magnetic ellipse, and it should satisfy

\[ \theta_B = \tan^{-1} \left( \frac{|\alpha_B| \cos(\phi_B + \varphi)}{\cos \varphi} \right), \]  

(\ref{thetaB})

where \(\varphi\) is the angle for the following quantity be at maximum

\[ f(\varphi) = \cos^2 \varphi + |\alpha_B|^2 \cos^2(\phi_B + \varphi), \]  

(\ref{fphi})

i.e., its derivative vanishes

\[ \frac{\partial}{\partial \varphi} \left[ \cos^2 \varphi + |\alpha_B|^2 \cos^2(\phi_B + \varphi) \right] = -2 \sin(2\varphi) - 2|\alpha_B|^2 \sin[2(\phi_B + \varphi)] = 0, \]  

(\ref{c2phi})

\[ \frac{\partial^2}{\partial \varphi^2} \left[ \cos^2 \varphi + |\alpha_B|^2 \cos^2(\phi_B + \varphi) \right] = -4 \cos(2\varphi) - 4|\alpha_B|^2 \cos[2(\phi_B + \varphi)] < 0, \]  

(\ref{c4phi})

which yields

\[ \varphi = \frac{1}{2} \cot^{-1} \left( -\frac{1/|\alpha_B|^2 + \cos(2\phi_B)}{\sin(2\phi_B)} \right), \quad \text{and required} \quad \cos(2\varphi) + |\alpha_B|^2 \cos[2(\phi_B + \varphi)] > 0. \]  

(\ref{varphi})

If \(\cos(2\varphi) + |\alpha_B|^2 \cos[2(\phi_B + \varphi)] < 0\), we set \(\varphi \rightarrow \varphi + \pi\) since the difference between major axis and minor axis is \(\pi\). Using above equations, we can obtain \(|\alpha_B|, \phi_B\) and \(\theta_B\). If \(\phi_B = \pi/2\) or \(\in (0, \pi)\), the wave is left-hand polarization; if \(\phi_B = -\pi/2\) or \(\in (-\pi, 0)\), the wave is right-hand polarization; if \(\phi_B = 0\), or \(|\alpha_B| \approx 1\) or \(\ll 1\), the wave is linear polarization; if \(|\alpha_B| = 1\), the wave is circular polarization; other case, the wave is elliptical polarization.

The properties compressibility, parallel phase ratio and Alfven-ratio are calculated using definitions in Refs.\[7,8\].

The refractive index \(n = \frac{\omega}{v_p}\). The phase velocity \(v_p = \frac{\omega}{k}\), the group velocity \(v_g = \frac{\partial \omega}{\partial k}\). For \(F(\omega, k) = 0\), using the implicit function derivative formula, we have \(v_g = \frac{\partial \omega}{\partial k} = -\frac{\partial F}{\partial \omega} \frac{1}{\partial \omega} \frac{\partial (\omega, k)}{\partial (\omega, k)}\), where \(F(\omega, k) = \left| M(k) - \omega \right|\) or \(F(\omega, k) = \left| D(\omega, k) \right|\). However, calculating this is very complicated. Thus, we can use numerical differentiation to calculate the group velocity; i.e., after we obtain a \(\omega\) for a given \((k_x, k_y)\), we solve the dispersion relation with \((k_x, +\Delta k_x, k_y)\) and \((k_x, k_y, +\Delta k_y)\), which gives \(\omega + \Delta \omega_x\) and \(\omega + \Delta \omega_y\). The group velocity is then \(v_{g_x} = \frac{\Delta \omega_x}{\Delta k_x}\) and \(v_{g_y} = \frac{\Delta \omega_y}{\Delta k_y}\).
Appendix C. Typos or bugs fixed in BO 1.0

In BO version 1.0, some typos are found. In page 352, \( T_{zz} = \frac{1}{2} k_B m_s v_{zz}^2 \) and \( T_{\perp,rr} = \frac{1}{2} k_B m_r v_{\perp,rr}^2 \) should be corrected to \( k_B T_{zz} = \frac{1}{2} m_s v_{zz}^2 \) and \( k_B T_{\perp,rr} = \frac{1}{2} m_r v_{\perp,rr}^2 \).

In page 360,

\[ P_{\|32} = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \left( \nu_n \right) \frac{A_{\parallel,rr}}{v_{\|,rr}^2} \frac{Z_1}{k_c} + A_{\perp,rr} \frac{k_v}{v_{\perp,rr}^2} \right\} v_{d\parallel} + \left( \nu_n \right) \frac{A_{\perp,rr}}{v_{\perp,rr}^2} \frac{Z_2}{k_v} + A_{\perp,rr} \frac{k_v}{v_{\perp,rr}^2} \right\} v_{d\perp} v_{d\parallel} \]

should be corrected to

\[ P_{\|32} = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \left( \nu_n \right) \frac{A_{\parallel,rr}}{v_{\|,rr}^2} \frac{Z_1}{k_c} + A_{\perp,rr} \frac{k_v}{v_{\perp,rr}^2} \right\} v_{d\parallel} + \left( \nu_n \right) \frac{A_{\perp,rr}}{v_{\perp,rr}^2} \frac{Z_2}{k_v} + A_{\perp,rr} \frac{k_v}{v_{\perp,rr}^2} \right\} v_{d\perp} v_{d\parallel} \]

where the \( k_v \) in the numerator of \( v_{d\perp} \) term should be removed, i.e., \( k_v v_{d\perp} Z_2 \) should be \( v_{d\perp} Z_2 \). Otherwise, the dimension/unit is incorrect. The terms \( v_{d\parallel} \Pi_{\parallel,32} \) in page 360 and \( P_{\perp,32} \) in page 364 should also be updated accordingly. This will affect the final matrix in the code for \( v_{d\parallel} \neq 0 \) cases.

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