Stability Analysis and Stabilization Strategies for Linear Supply Chains

Takashi Nagatani \textsuperscript{a} Dirk Helbing \textsuperscript{b}

\textsuperscript{a}Department of Mechanical Engineering, Shizuoka University, Hamamatsu 432-8561, Japan

\textsuperscript{b}Institute for Economics and Traffic, Dresden University of Technology, 01062 Dresden, Germany

Abstract

Due to delays in the adaptation of production or delivery rates, supply chains can be dynamically unstable with respect to perturbations in the consumption rate, which is known as “bull-whip effect”. Here, we study several conceivable production strategies to stabilize supply chains, which is expressed by different specifications of the management function controlling the production speed in dependence of the stock levels. In particular, we will investigate, whether the reaction to stock levels of other producers or suppliers has a stabilizing effect. We will also demonstrate that the anticipation of future stock levels can stabilize the supply system, given the forecast horizon $\tau$ is long enough. To show this, we derive linear stability conditions and carry out simulations for different control strategies. The results indicate that the linear stability analysis is a helpful tool for the judgement of the stabilization effect, although unexpected deviations can occur in the non-linear regime. There are also signs of phase transitions and chaotic behavior, but this remains to be investigated more thoroughly in the future.

\textit{Key words:} Supply chains, bull-whip effect, stop-and-go traffic, stability analysis, non-linear dynamics, phase transitions, stabilization strategies, forecasts

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1 Introduction

Concepts from statistical physics and non-linear dynamics have been very successful in discovering and explaining dynamical phenomena in traffic flows \cite{1}. Many of these phenomena are based on mechanisms such as delayed adaptation to changing conditions and competition for limited resources, which are
relevant for production systems as well. Therefore, mathematicians [2], physicists [3], traffic scientists [4], and economists [5] have recently pointed out that methods used for the investigation of traffic dynamics are also of potential use for the study of supply networks.

In the following, we will investigate the stability of a linear supply chain. The assumed model consists of a series of $u$ suppliers $i$, which receive products from the next “upstream” supplier $i - 1$ and generate products for the next “downstream” supplier $i + 1$ (see Fig. 1). The final products are delivered to the consumers $u + 1$. Here, we will assume that the consumption rate is subject to perturbations, which may cause variations in the stock levels and deliveries of upstream suppliers. This is due to delays in their adaptation of the delivery rate. Under certain conditions, the oscillations in delivery and in the resulting inventories (stock levels of the products) grow from one producer to the next upstream one. This is called the bullwhip effect [4] and reported, for example, for beer distribution [6,7]. Similar dynamical effects are also known for a series of subsequent production processes.

In the following, we will investigate this amplification effect for a simple supply chain model by means of linear stability analysis and computer simulations. Interestingly, special cases of this model turns out to have common features with some traffic models [8,9]. However, production systems have more general features, which are still to be explored. For example,

- one may have complex supply networks rather than one-dimensional spatial interactions, and
- the adaptation strategies of the producers can be varied to a much larger extent than those of drivers.

While the first aspect is discussed in Refs. [3,10], in this contribution we will focus on the latter aspect, in particular the possibility to stabilize production by taking into account the inventories of other suppliers and by anticipating changes in the inventories. In Sec. 2, we will formulate the supply chain model, while in Sec. 3, we will derive conditions for the linear stability of it. Section 4 introduces our simulation approach, while Sec. 5 will compare different management strategies (“policies”) with regard to their effects on the time-dependent product flows. Section 6 finally summarizes our findings, discusses them in a broader context, and indicates further research directions.

2 The Supply Chain Model

We will assume $u$ suppliers $i$ delivering their product(s) to the next downstream supplier $i + 1$ [11]. The stock level (“inventory”) at supplier $i$ shall be
denoted as $N_i$. It changes in time $t$ according to the equation

$$\frac{dN_i}{dt} = \lambda_i(t) - \lambda_{i+1}(t)$$ (1)

[12]. Here, $\lambda_i$ has the meaning of the rate at which supplier $i$ receives ordered products from supplier $i-1$, while $\lambda_{i+1}$ is the rate at which he delivers products to the next downstream supplier $i+1$. Therefore, Eq. (1) is just a continuity equation which reflects the conservation of the quantity of products. (It is easy to generalize this equation to cases where products are lost. One would just have to add a term of the form $-\gamma_i N_i(t)$. It is also possible to treat cases, where products are broken down into smaller units or several units are required to produce one unit of a new product. This is discussed in Ref. [3].) Boundary conditions must be formulated for $i = 0$, which corresponds to the supplier of the raw materials (fundamental resources), and for $i = u+1$, which corresponds to the consumers. That is, $\lambda_0$ is the supply or production rate of the basic product, while $\lambda_{u+1}$ is the consumption rate. These are specified in Sec. 4.

The question remains, how the delivery rates $\lambda_i$ evolve in time. It is reasonable to assume that the temporal change of the delivery rate is proportional to the deviation of the actual delivery rate from the desired one $W_i$ (the order rate) and its adaptation takes on average some time interval $T$. According to this, we have the equation

$$\frac{d\lambda_i}{dt} = \frac{1}{T} [W_i(t) - \lambda_i(t)] .$$ (2)

The order rate $W_i$ will usually be reduced with increasing stock levels $N_j$, but their temporal changes $dN_j/dt$ may be taken into account as well, e.g. when the stock levels are forecasted. Therefore, it is natural to assume a general dependence of the form

$$W_i(t) = W_i (\{N_j(t)\} , \{dN_j(t)/dt\}) .$$ (3)

The function $W_i$ reflects the management strategy, i.e. the order policy regarding the desired delivery rate as a function of the actual stock levels $N_j(t)$ or anticipated stock levels $N_j(t) + \tau dN_j(t)/dt \approx N_j(t + \tau)$ (in first order Taylor approximation). The simplest strategy of supplier $i$ would be to react to the own stock level $N_i(t)$. However, it may be useful to consider also the stock levels of the next downstream suppliers $j > i$, as these determine the future demand, and the stock levels of the next upstream suppliers $j < i$, as they determine future deliveries or shortages of the product (“out of stock” situations). We will, therefore, assume

$$W_i(t) = W_i (\{N_j(t)\} , \{dN_j(t)/dt\}) = W(N_{ij}(t)) ,$$ (4)
where

\[ N_{(i)}(t) = \sum_{l=-n}^{n} c_l \left( N_{i+l} + \tau \frac{dN_{i+l}}{dt} \right) \]  

(5)

is a weighted mean value of the own stock level and the ones of the next \( n \) upstream and \( n \) downstream suppliers (with \( 2n + 1 \leq u \)). For \( i + l < 0 \) and \( i + l > u \), the weights \( c_l \) are always set to zero, and they are normalized to one:

\[ \sum_{l=-n}^{n} c_l = 1. \]  

(6)

For \( \tau = 0 \), the management adapts the delivery rate \( \lambda_i \) to the actual weighted stock level \( N_{(i)}(t) \), while for \( \tau > 0 \), the management orients at the anticipated weighted stock level. The parameter \( \tau \) has the meaning of the forecast time horizon, and the specification of the parameters \( c_k \) and \( \tau \) reflects the management strategy.

It is reasonable to assume that some strategies will be more favourable than others. For example, some strategies will produce oscillating supply patterns (the “bull-whip effect”), while others will stabilize the product flow. These aspects will be discussed in Sec. 5. Here, we will first discuss some potential applications: We have already mentioned beer distribution, for which the bull-whip effect has been impressively demonstrated. Similar effects are expected for all delivery chains, which contain enough levels of delivery. We actually do not have to assume a linear structure, as a hierarchical distribution system (with, for example, one producer, a few huge global suppliers, some big country-based suppliers, more medium-sized regional suppliers, and many local suppliers) will have similar properties, if the suppliers belonging to the same level behave approximately the same (see Ref. [3], Figs. 10 and 11). Linear supply systems are, however, known from transport chains. These are, for example, used in disaster management (and often made up of several dozen or hundred persons). Moreover, some production processes have similar features as well, in particular if they use bucket brigades [13]. In this case, the index \( i \) represents the different successive production steps or machines, \( \lambda_i \) describes the corresponding production rate, and the so-called production function \( W_i \) reflects the desired production rate as a function of the stock levels \( N_i \) in the respective output buffers. Although the above model does not consider effects of limited buffer sizes (full buffers) [14] or of limited transport capacities, these features can be easily added (see Ref. [3]).

Before we continue with a linear stability analysis of the above model, let us shortly point out some analogies with other systems. For the particular specifications \( T \to 0 \), \( N_{(i)}(t) = N_i(t) \), and \( W_i(t) = N_{i-1}(t)V(N_i(t)) \), the mathematical structure of the supply chain model agrees with the macroscopic traffic
model of Hilliges and Weidlich [8]:

\[
\frac{dN_i(t)}{dt} = N_{i-1}(t)V(N_i(t)) - N_i(t)V(N_{i+1}(t)). \tag{7}
\]

However, in this equation, \(N_{i-1}\) has the meaning of the traffic density in a road section \(i-1\) of length \(\Delta x\), while \(V(N_i)\) is the average velocity-density relation of vehicles on road section \(i\). In this model, vehicles in section \(i-1\) are assumed to adapt to the velocity in the next road section \(i\), which describes an anticaptive driving behavior.

Another special case is accidentally related to the so-called optimal velocity model [9]. This car-following model assumes an acceleration equation of the form

\[
\frac{dv_i(t)}{dt} = \frac{V_o(d_i(t)) - v_i(t)}{T} \tag{8}
\]

and the complementary equation

\[
\frac{dd_i(t)}{dt} = -[v_i(t) - v_{i+1}(t)]. \tag{9}
\]

In this “microscopic” traffic model, the quantities have a totally different meaning than in the above supply chain model: The index \(i\) represents single vehicles, \(v_i(t)\) is their actual velocity of motion, \(V_o\) the so-called optimal (safe) velocity, which depends on the distance \(d_i(t)\) to the next vehicle ahead, and \(T\) is an adaptation time. Comparing this equation with Eq. (2), the velocities \(v_i\) would correspond to the delivery rates \(\lambda_i\), the optimal velocity \(V_o\) to the order rate (desired delivery rate) \(W_i\), and the inverse vehicle distance \(1/d_i\) would approximately correspond to the stock level \(N_i\) (apart from a proportionality factor). This shows, that the analogy between supply chain and traffic models concerns only their mathematical structure, but not their interpretation, although both relate to transport processes. Nevertheless, this mathematical relationship can give us hints, how methods, which have been successfully applied to the investigation of traffic models before, can be generalized for the study of supply networks. While in traffic flow, the velocity-density relation is empirically given, the new feature of supply chains is that the management has a large degree of freedom to choose the order function \(W_i\). In principle, this can be specified not only as a function of the own stock level, but of the stock levels of other suppliers as well, if the information is available. Therefore, instead of the simple dependence of \(W_i\) on \(N_i\), we have to explore the considerably more complex dependence on the weighted stock level (5). This will lead to new results regarding the dynamical behavior, which are of potential relevance for the management of supply chains.
3 Linear stability analysis

Let us now study the stability of the steady-state solution of Eqs. (1) and (2). This steady-state solution is given by \( N_i = N_0 \) and \( \lambda_i = \lambda_0 = W(N_0) \). By \( \delta N_i(t) \) and \( \delta \lambda_i(t) \), we will denote small deviations from the steady state:

\[
N_i(t) = N_0 + \delta N_i(t), \tag{10}
\]

\[
\lambda_i(t) = \lambda_0 + \delta \lambda_i(t). \tag{11}
\]

Then, the linearized equations obtained from Eqs. (1) and (2) read

\[
\frac{d\delta N_i(t)}{dt} = \delta \lambda_i(t) - \delta \lambda_{i+1}(t) \tag{12}
\]

and

\[
T \frac{d\delta \lambda_i(t)}{dt} = W'(N_0) \sum_{l=-n}^{n} c_l \left( \delta N_{i+l} + \tau \frac{d\delta N_{i+l}}{dt} \right) - \delta \lambda_i(t), \tag{13}
\]

where \( W'(N_0) \) is the derivative of the control function at \( N_{(i)} = N_0 \).

With the ansatz \( \delta N_j(t) = X \exp(ik + zt) \) and \( \delta \lambda_j(t) = Y \exp(ik + zt) \), where \( i = \sqrt{-1} \) denotes the imaginary unit, \( k \) the wave number, and \( z \) a complex eigenvalue, one obtains

\[
z X = (1 - e^{ik}) Y \tag{14}
\]

and

\[
z TY = W'(N_0)(1 + z\tau) \left( \sum_{l=-n}^{n} c_l e^{ikl} \right) X - Y. \tag{15}
\]

Inserting Eq. (14) into Eq. (15), we find

\[
z^2 T = W'(N_0)(1 + z\tau)(1 - e^{ik}) \sum_{l=-n}^{n} c_l e^{ikl} - z. \tag{16}
\]

This quadratic equation in \( z \) may be solved with respect to \( z \). In order to investigate the limiting case of small wave numbers \( k \) (large wave lengths), we will use the expansion of the exponential function:

\[
e^{ikl} = \sum_{j=0}^{\infty} \frac{(ikl)^j}{j!} = \sum_{j=0}^{\infty} \frac{i^j}{j!}(k)^j. \tag{17}
\]

Inserting this into Eq. (16), one finds that the leading term of \( z \) is of the order of \( ik \), when \( ik \to 0 \), \( z \to 0 \). Let us derive the long wave expansion of \( z \), which
is determined order by order around \( i k \approx 0 \). By expanding

\[ z = \sum_{j=1}^{\infty} z_j (ik)^j, \]

the first- and second-order terms of \( i k \) are obtained as

\[ z_1 = -W'(N_0) \]

and

\[ z_2 = -z_1^2 T - W'(N_0) \left( z_1 \tau + \frac{1}{2} + \sum_{l=-n}^{n} l c_l \right), \]

If \( z_2 \) is a negative value, the steady state becomes unstable for long wavelength modes, i.e. modes with small wave numbers \( k \). However, if \( z_2 \) is a positive value, the steady state is stable. Therefore, the linear stability condition is given by

\[ T < \tau + \frac{1}{|W'(N_0)|} \left( \frac{1}{2} + \sum_{l=-n}^{n} l c_l \right), \]

where we have used \(-W'(N_0) = |W'(N_0)| > 0\) [11], as the order rate normally decreases with increasing stock levels \( N_0 \). According to Eq. (21), anticipation of the stock levels \((\tau > 0)\) has always a positive effect on stability, and it appears to be favorable to orient the order rate at downstream suppliers (with \( l > 0 \)). These points will be discussed in Sec. 5 in more detail. However, the validity of condition (21) is restricted to systems with a large number \( u \gg n \) of suppliers. For small systems, we may have corrections due to boundary effects (see Ref. [15] for their treatment). Generally, smaller systems tend to be more stable.

### 4 Simulation Approach

Apart from a linear stability analysis, we have carried out computer simulations of Eqs. (1) and (2), using Euler integration with a time discretization of \( \Delta t = 0.01 \). In order to make boundary effects small, we have chosen a large number \( u = 200 \) of suppliers, which is realistic for the transport chains mentioned in Sec. 2. Moreover, the boundary conditions have been specified as follows:

\[ \lambda_0 = W(N_0) \quad \text{and} \quad N_{u+1}(t) = N_0 + \xi(t)/2, \]

where \( \xi(t) \) is a white noise with mean value

\[ \langle \xi(t) \rangle = 0 \]
and time correlation
\[ \langle \xi(t)\xi(t') \rangle = \delta_{tt'}/4. \] (24)

Finally, the order rate (i.e. the desired supply rate) was chosen as
\[ W(N_{(i)}) = 1 - \left[ \tanh(N_{(i)} - N_c) + \tanh(N_c) \right]/2. \] (25)

This function can be viewed as representative for many other monotonously falling functions with a turning point \( N_c \). Here, the turning point was set to \( N_c = 3 \), which guarantees \( N_{(i)}(t) \geq 0 \) for small enough adaptation times \( T \). The fact that the values of \( W \) range from 1 to 0 is no restriction: Every bounded and monotonously falling function with a minimum value of 0 can be scaled so that it varies from 1 to 0. This just implies a scaling of the delivery rates \( \lambda_{i} \), i.e. a time scaling. In addition, for many order functions \( W \), the quantities \( N_{i} \) of products can be scaled in a way that they are approximated by relation (25). Although other kinds of functions are conceivable as well (see e.g. Ref. [3]), we expect quantitatively the same findings in the linear regime around the stationary state and qualitatively the same phenomena in the nonlinear regime (for many but not all monotonously falling functions).

5 Discussion of Different Control Strategies

In the following, we will investigate some reasonable management strategies (order strategies) in more detail. In particular, we will study how the dynamics of a linear supply chain can be stabilized by selecting an appropriate strategy. We will set \( c_{-l} = 0 \) for all \( l > 0 \), as positive values would reduce the stability threshold according to Eq. (21). This means, it is not helpful to take into account the stock levels of upstream suppliers \( j = i - l \) in the order strategy \( W \), as it is expected to destabilize the supply chain. Consequently, we will only discuss order strategies considering the own stock level and the ones of downstream suppliers:

(A) With strategy A we mean the case that the order policy of the management of supplier \( i \) takes into account only the own present stock level \( N_{i}(t) \). That is, strategy A does not consider anticipation \( (\tau = 0) \), and the weighted stock level agrees with the own one: \( N_{(i)}(t) = N_{i}(t) \). This is inserted into Eq. (25).

The linear stability condition for this order strategy reduces to
\[ |W'(N_0)| < \frac{1}{2T}, \] (26)

which is mathematically equivalent with the stability condition of the optimal velocity model mentioned before [9]. Correspondingly, if the change \( W'(N_0) \) of the order rate with the stock level \( N_{i} \) at the stationary point
$N_i = N_0$ is greater than half of the inverse adaptation time $1/T$, the supply chain is expected to behave unstable. In this situation, small fluctuations in the consumption rate can trigger large oscillations in the stock levels.

(B) In order to suppress these oscillations, with strategy B, which corresponds to the specification $N_{(i)}(t) = N_i(t) + \tau dN_i(t)/dt$, we have additionally taken into account the effect of anticipation of the stock levels. That is, with strategy B, the management reacts to the forecasted stock level at time $t + \tau$, which presupposes that the (sufficiently reliable) forecast time horizon is at least $\tau$. The corresponding stability condition becomes

$$T < \tau + \frac{1}{2|W'(N_0)|}.$$  \hspace{1cm} (27)

According to this, the neutral stability line is shifted to higher values by the time horizon $\tau$ of the forecast, and long enough forecast time horizons $\tau$ should always stabilize the supply chain. Due to Eq. (21), this is also true in combination with other strategies than A. However, in reality the stabilization strength is, of course, limited by the practical forecast capability. Nevertheless, anticipation should always have a stabilizing effect.

Figure 2 shows plots of the neutral stability lines against the steady-state inventories $N_0$ for $\tau = 0.0, 0.4, 0.8, 1.2$, and $1.6$, where $N_c = 3$. Note that the supply chain behaves stable below the neutral stability line, while it behaves unstable above it. With increasing $\tau$, the neutral stability line increases.

In Figure 3, we have compared strategies A and B with respect to the time evolution of the stock levels (inventories) $N_i$ of all suppliers $i$. While subfigure (a) corresponds to the dynamical pattern of strategy A with adaptation time $T = 2$ and anticipation time horizon $\tau = 0$, the dynamical pattern of strategy B for $T = 2$ and $\tau = 0.4$ is shown in (b). Without forecasting (see Fig. 3a), the stock levels of most suppliers show significant oscillation amplitudes, while with anticipation, the oscillations are weaker and concern mainly the suppliers upstream of $i \approx 150$. Note that the oscillations propagate backwards at a constant speed, which is analogous to the propagation of stop-and-go waves [17].

For strategy B, let us now study the dependence of the amplitude of the inventories $N_i$ on the anticipation time horizon $\tau$. Figure 4 shows the plot of the maximum oscillation amplitude as a function of the time horizon $\tau$ for an adaptation time $T = 2$. With increasing time horizon $\tau$, the amplitude decreases. At a critical threshold of about $\tau = 0.9$, the oscillations disappear completely, although the anticipation time horizon is considerably smaller than the adaptation time $T$ (which is due to the finite value of $|W'(N_0)|$).

For higher values than the critical threshold $\tau = 0.9$, which can be estimated by the linear stability condition, the supply chain does not oscillate and is stable.

(C) When the practical forecast time horizon is too small to stabilize the supply chain, anticipation should be combined with better strategies than A. It is
reasonable that the consideration of the stock level of the next downstream supplier can improve the stability, if the information is somehow available. Therefore, with strategy C we assume that the management also takes into account the actual inventory of the downstream (front) nearest-neighbor \( i + 1 \) it delivers to. In mathematical terms, we set \( N_{(i,j)}(t) = c_0N_i(t) + (1 - c_0)N_{i+1}(t) \) with a model parameter \( c_0 (0 \leq c_0 \leq 1) \). Instead of \( T < 0.5/|W'(N_0)| \) (see Eq. (26)), the related linear stability condition becomes

\[
T < \frac{1 + 2(1 - c_0)}{2|W'(N_0)|}. \tag{28}
\]

Therefore, the stability threshold is increased for \( c_0 < 1 \). The case \( c_0 = 1 \) corresponds exactly to strategy A (our reference strategy for the purpose of comparison).

Figure 5 shows plots of the neutral stability point against the steady-state inventory for \( c_0 = 1.0, 0.8, 0.6, 0.4, \) and 0.2, where \( N_c = 3 \). Each solid curve represents a neutral stability line. Again, the production system is stable below the neutral stability line, while the system is unstable above the curve. With decreasing \( c_0 \), the neutral stability line increases. As a consequence, the production system is stabilized by taking into account the inventory of the next downstream supplier.

We have also studied the dependence of the oscillation amplitude on the adaptation time for strategy C with \( c_0 = 0.7 \) (see Fig. 6). The oscillation of the inventories appears at about \( T = 1.8 \) and strongly grows with increasing adaptation time \( T \).

(D) As the consideration of the stock level of the next downstream supplier has a stabilization effect, we will investigate with strategies D and E, whether it is even better to take into account the stock levels \( N_j(t) \) of the second-next supplier \( (j = 2) \) or the consumer market \( (j = u + 1) \), given this information is available. For strategy D we assume that the management considers the inventory of the second-next (instead of the next) downstream supplier: \( N_{(i,j)}(t) = c_0N_i(t) + (1 - c_0)N_{i+2}(t) \). In this case, one obtains the linear stability condition

\[
T < -\frac{1 + 4(1 - c_0)}{2W'(N_0)}, \tag{29}
\]

which implies an increased stability threshold for \( c_0 < 1 \), i.e. a greater stability.

Figure 7 shows plots of the neutral stability lines against the steady-state inventory for \( c_0 = 1.0, 0.8, 0.6, 0.4, \) and 0.2, where \( N_c = 3 \). The value \( c_0 = 1 \) again corresponds to the reference strategy A. With decreasing \( c_0 \), the neutral stability line increases. For the same value of \( c_0 \), strategy D is more stable than strategy C. As a result, the production system is more efficiently stabilized by taking into account the inventory of the second-next downstream supplier.
Moreover, with strategy E, the management is assumed to take into account the consumption directly, i.e. 
\[ N_{i}(t) = c_{0}N_{i}(t) + (1 - c_{0})N_{i+1}(t) \]
control function (25), where \( N_{i+1}(t) \) represents the boundary condition. One would assume that this strategy would have a greater stabilization effect than strategies C and D. In order to judge this, let us compare for strategies C, D, and E the profiles of the inventories of all suppliers at time \( t = 20000 \), which are displayed in Figure 8. The chosen parameter values are \( T = 2 \) and \( N_{c} = 3 \). While plot (a) shows the inventories for strategy C with \( c_{0} = 0.9 \), plot (b) displays the inventories for strategy D with the same value of \( c_{0} \). Moreover, plot (c) shows the inventories for \( c_{0} = 0.8 \) and strategy D’, for which we assume for comparison that a manager considers the stock-level of the third-next downstream supplier: 
\[ N(t) = c_{0}N_{i}(t) + (1 - c_{0})N_{i+3}(t). \]
Finally, plot (d) displays the inventories for strategy E with \( c_{0} = 1 - c_{0} = 0.3 \). Obviously, taking into account the inventories of downstream suppliers can reduce the oscillations, see (a) and (b). However, with strategy D’, the supply chain may surprisingly become more unstable, see (c). That is, if the management chooses an order strategy taking into account the inventory \( N_{i+n}(t) \) of a further downstream supplier \( j = i + n \) with \( n \geq 3 \), the supply chain may be destabilized. This instability is in contrast to our expectations based on the linear stability analysis. The likely reason are resonance effects in the non-linear regime (when the vicinity of the stationary state \( N_{i} = N_{0} \) is left), so that the implications of the linear stability analysis have to be considered with care and checked by simulations.

Despite such kinds of surprises, strategy E exhibits a stabilization effect, since the oscillations are reduced, when the consumption is taken into account, see (d). However, the stabilization effect is not as strong as expected. To see this, let us investigate the stabilization of strategies C, D, and E by means of Figure 9. It shows the plots of the maximum oscillation amplitude of the inventories as a function of \( 1 - c_{0} \) for strategies C, D, and E, where \( T = 2 \) and \( N_{c} = 3 \). With increasing fraction \( 1 - c_{0} \), the maximum amplitude decreases for all three strategies. The oscillation of the inventories disappears at about \( c_{0} = 0.8 \) for strategy D. For strategies C and E, the oscillation disappears at \( c_{0} = 0.6 \) and 0.55, respectively. For strategy D, the reduction of the oscillation amplitude is strongest. Surprisingly, the stabilization effect of strategy E is less than that of strategy C, which is probably due to a resonance effect, similar to the observation for strategy D’.

6 Summary, Discussion, and Outlook

Similar to stop-and-go waves in vehicle traffic, linear supply chains sometimes suffer from the bull-whip effect, i.e. growing oscillations in the inventories
(stock levels), which are due to delays in the adaptation of the delivery rate or production speed. We have studied the impact of various reasonable management strategies with regard to the stabilization or destabilization of the dynamics of a linear supply chain. This has been judged by the linear stability condition we derived, and the dynamical behavior of the inventories has additionally been investigated by means of simulations. While we have focussed on the effect of perturbations in the consumption rate, the result of perturbations in the delivery rates of suppliers may be studied in a similar way. However, according to the linear stability analysis, we do not expect any qualitatively new results.

We have shown that a supply chain can be stabilized, i.e. oscillations in the delivery rates and stock levels can be reduced by anticipation of the temporal evolution of the inventories and by taking into account the inventories of other suppliers. In the case of perturbations in the consumption rate, our results were as follows:

- Anticipation of the own future inventory was an efficient means to stabilize the production system. Surprisingly, it turned out that anticipation time horizons considerably smaller than the adaptation time were sufficient to reach complete stability. Similar results are expected, if anticipated inventories of other suppliers are taken into account as well. Fig. 4 suggests that the transition to stability with increasing anticipation time horizon becomes discontinuous in the limit $u \to \infty$ of infinitely many suppliers. It would be interesting to investigate this in more detail in the future, as well as the role of errors in the forecast.

- According to the linear stability analysis, the adaptation to a variation in the consumption rate is better, if not only the own inventory, but also the inventories of downstream suppliers or the consumer sector itself are taken into account by so-called “pull strategies”. In contrast, considering the inventories of upstream suppliers corresponding to “push strategies” destabilized the system (cf. Ref. [16]). One would think that this is, because the oscillations of upstream inventories tend to be larger, but the same result comes out from a linear stability analysis, i.e. in the limit of vanishing oscillation amplitudes. It is rather the direction of the information flow in the system which matters: The oscillations in the consumption rate travel upstream, as in stop-and-go traffic [17] (see Fig. 3).

- The linear stability analysis gives a good idea under which conditions the oscillation amplitude in the system becomes zero. For example, the results displayed in Figs. 4 and 9 are in good agreement with our expectations. However, as for traffic systems, the implications of a linear stability analysis are limited, because the evolving oscillation amplitudes are usually large, and non-linear effects dominate. For example, the emerging wavelength in the system does often not agree with the most unstable wavelength mode, i.e. the mode with the largest growth rate [18]. Therefore, the simulation
result displayed in Fig. 8(c) was not in agreement with our expectations based on the linear stability analysis. Surprisingly, the oscillation amplitudes were larger, when inventories of further downstream suppliers were taken into account. This point is related with the different wavelength emerging in Fig. 8(c) compared to Fig. 8(a), (b), and (d), as the resulting oscillation amplitude depends on the frequency [10]. It would certainly be interesting to investigate in the future the relation with period-doubling phenomena [14], which are known to exist for traffic systems [19].

• Let us finally come back to the simulation results displayed in Fig. 9, showing the amplitude of oscillations in the inventories for different management strategies. For \( c_0 = 1 \), strategies C, D, and E agree with strategy A, and the oscillation amplitudes are the same. However, when in the weighted stock level \( N_{i(j)}(t) = c_0 N_i(t) + (1 - c_0) N_j(t) \), the weight \( 1 - c_0 \) of another inventory \( N_j(t) \) is increased in the production strategy, there is a surprise: The oscillation amplitudes are significantly reduced, when the second next downstream supplier is taken into account with \( N_j(t) = N_{i+2}(t) \) instead of the next downstream one with \( N_j(t) = N_{i+1}(t) \), as expected. However, considering the variation in the consumption rate itself with \( N_j(t) = N_{u+1}(t) \) has a very weak stabilization effect, although the consumer sector is located even further downstream. This is probably due to resonance effects, when the vicinity of the stationary state is left, and due to the fact that the characteristics of the perturbation in the consumption rate differs from those of the emerging oscillation patterns.

In conclusion, there are several non-trivial and unexpected effects in the behavior of linear supply chains. Therefore, simulation models describing the non-linear interactions and dynamics of supply chains and production processes could be relevant for their optimization. From the practical point of view it is, for example, useful that Eq. (21) allows one to estimate the maximum adaptation time \( T \) or the minimum forecast time horizon \( \tau \) supporting a stable supply chain. Moreover, the stabilizing effect of a reaction to inventories of downstream suppliers suggests to exchange these data on-line. Note that our conclusions regarding the stabilization by forecasts and the consideration of downstream stock levels are expected to be transferable to more complex systems than the linear supply chains treated here. They should be also applicable to cases where suppliers are characterized by different parameters, to situations with limited buffers and transportation capacities, or to supply networks [10].

From the physical point of view, it will be particularly interesting to study the conditions for period doubling phenomena in the future, and to investigate whether the stabilization transition (when the weight \( c_0 \), the adaptation time \( T \), or the anticipation time horizon \( \tau \) is varied) would become discontinuous in the limit \( u \to \infty \) of infinitely many suppliers. In view of the partially irregular patterns in Fig. 8, it will also be interesting to seek for conditions for chaotic
dynamics [6,20] and for concepts to control it [21].

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Figure captions

FIG. 1. Illustration of the linear supply chain treated in this paper, including the key variables of the model. Circles represent different suppliers $i$, $N_i$ their respective stock levels, and $\lambda_i$ the delivery rate to supplier $i$ or the production speed of this supplier. $i = 0$ corresponds to the resource sector generating the basic products and $i = u + 1$ to the consumers.

FIG. 2. Plots of the neutral stability line as a function of the steady-state inventory for strategy B and various values of the anticipation time horizon (ATH) $\tau = 0.0, 0.4, 0.8, 1.2, 1.6$ $(N_c = 3)$. The supply chain behaves stable below the neutral stability line, while it behaves unstable above the curve. Strategy A corresponds to $\tau = 0$.

FIG. 3(a). Dynamics of the inventories of 200 suppliers $i$. (a) Strategy A with adaptation time $T = 2$. (b) Strategy B with $T = 2$ and anticipation time horizon $\tau = 0.4$.

FIG. 4. Plot of the maximal amplitude of oscillation in the inventories as a function of the time horizon $\tau$ for strategy B with adaptation time $T = 2$.

FIG. 5. Plots of the neutral stability line as a function of the steady-state inventory for strategy C and $c_0 = 1.0, 0.8, 0.6, 0.4, 0.2$. Strategy A corresponds to $c_0 = 1$.

FIG. 6. Plot of the maximum amplitude of the inventories as a function of the adaptation time $T$ for strategy C with $c_0 = 0.7$.

FIG. 7. Plots of the neutral stability line against the steady-state inventory for strategy D and $c_0 = 1.0, 0.8, 0.6, 0.4, 0.2$. Again, strategy A corresponds to $c_0 = 1$.

FIG. 8(a). Stock levels (inventories) of 200 suppliers $i$ for various strategies at time $t = 20000$, where the adaptation time is $T = 2$ and the turning point is located at $N_c = 3$. (a) Inventories for strategy C with $c_0 = 0.9$. (b) Inventories for strategy D with $c_0 = 0.9$. (c) Inventories for strategy D’ with $c_3 = 1 - c_0 = 0.2$. (d) Inventories for strategy E with $c_{u+1} = 1 - c_0 = 0.3$. 

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FIG. 9. Plots of the maximum amplitude of oscillations in the inventories over $1 - c_0$ for strategy C (label $i + 1$), strategy D (label $i + 2$), and strategy E (label $u + 1$) with $T = 2$ and $N_c = 3$. 
Figure 1.
Figure 2.
Figure 3.
Figure 4.
Figure 5.
Figure 6.
Figure 7.
Figure 8.
Figure 9.