We present a method for tracking time-dependent entanglement between different modes of a quantum system as measured by observers in different states of relative (non-uniform) motion. By describing states on a given spacelike hypersurface, observers/detectors in different states of motion can detect different modes of excitation in a quantum field at any desired instant and thereby track various measures of entanglement as function of time. We illustrate our method for a scalar field, showing how entanglement degrades as a function of time if one observer begins in a state of inertial motion but ends in a state of uniform acceleration while the other remains inertial.

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I. INTRODUCTION

Entanglement is a key resource in quantum computational tasks [1] such as teleportation [2], communication, quantum control [3] and quantum simulations [4]. It is a property of multipartite quantum states that arises from the superposition principle and the tensor product structure of Hilbert space. It can be quantified uniquely for non-relativistic bipartite pure states by the von Neumann entropy, and several measures such as entanglement cost, distillable entanglement and logarithmic negativity have been proposed for mixed states [5].

Understanding entanglement in a relativistic setting is important both for providing a more complete framework for theoretical considerations, and for practical situations such as the implementation of quantum computational tasks performed by observers in arbitrary relative motion. For observers in uniform relative motion the total amount of entanglement is the same in all inertial frames [6], although different inertial observers may see these correlations distributed differently amongst various degrees of freedom. However for observers in relative uniform acceleration a communication horizon appears, limiting access to information about the whole of spacetime, resulting in a degradation of entanglement as demonstrated for scalars [7, 8] and fermions [9], and restricting the fidelity of processes such as teleportation [10], and other communication protocols [11].

The most general situation is that of observers in different states of non-uniform motion. This situation is relevant even for inertial observers in curved spacetime, who will undergo relative non-uniform acceleration due to the geodesic deviation equation, and therefore disagree on the degree of entanglement in a given bipartite quantum state. While it is expected such entanglement will be time-dependent, there is presently no approach for determining how the entanglement changes throughout the course of the motion.

II. SOLUTION OF THE KLEIN GORDON EQUATION

For the inertial observer Minkowski coordinates \((t, x)\) are the most suitable for describing the field, whereas for a uniformly accelerated (UA) observer Rindler coordinates \((τ, ξ)\) are appropriate. As these coordinates cover only a quadrant of Minkowski space, the UA observer remains constrained to a particular Rindler quadrant and has no access to the other Rindler sector. Both the inertial and UA observers can define a vacuum state and a Fock space for a scalar field obeying the Klein-Gordon equation, with solutions in each related by Bogoliubov transformations. A given mode seen by an inertial observer corresponds for the UA observer to a two-mode squeezed state, associated with the field observed in the two distinct Rindler regions [12]. The UA observer can
neither access nor influence field modes in the causally disconnected region and so information is lost about the quantum state, resulting in detection of a thermal state, a phenomenon known as the Unruh effect [13].

For an NUA observer consider the coordinates \((T, X)\)

\[
w(t+x) = 2 \sinh(w(T+X)) \quad w(t-x) = -e^{-w(T-X)}
\]

yielding the metric

\[
ds^2 = [\exp(-2wT) + \exp(2wX)] [dT^2 - dX^2]
\]

which covers \(t-x<0\), referred to as region (I). The remaining half-plane \(t-x>0\) (region (II)) can be covered using coordinates [14] after making the substitutions \(X \rightarrow -X\), \(T \rightarrow -T\) and \(w \rightarrow -w\). This observer’s acceleration

\[
a(T, X_0) = [e^{-2wT} + e^{2wX_0}]^{-3/2} w \exp(2wX_0)
\]

along \(X = X_0\) is non-uniform, with \(a(T \rightarrow -\infty, X_0) = 0\) and \(a(T \rightarrow +\infty, X_0) = w \exp(-wX_0)\). This observer, restricted to the region \(t-x<0\) can causally influence events in \(t-x>0\) but cannot be causally influenced by them. Consequently one expects a loss of information about the state of a quantum field, with an associated degradation of entanglement.

For both the Rindler and NUA observers entropy no longer quantifies entanglement because an entangled pure state seen by inertial observers appears mixed from their frames. However for the NUA observer this mixed state changes with time, making its description somewhat problematic. Furthermore, since \(\partial/\partial T\) is not a Killing vector there is no clear way to define a vacuum state.

Our approach is to describe the field modes on a given spacelike hypersurface at time \(T_0\). On any such hypersurface instantaneous positive and negative frequencies (with their respective annihilation/creation operators) can be defined and a solution to the Klein-Gordon equation in \((T, X)\) coordinates can be related to one in \((t, x)\) coordinates by a Bogoliubov transformation at \(T = T_0\). It is then possible to bound the entanglement of the mixed state using logarithmic negativity, which is a full entanglement monotone that bounds distillable entanglement from above [14]. This quantity will depend on \(T_0\), and so one can track the change in entanglement as \(T_0\) is varied. Similar considerations apply for mutual information [15], which can be used to quantify the state’s total correlations (classical plus quantum). We find that two modes maximally entangled in an inertial frame become less entangled as a function of \(T_0\), with the entanglement approaching a fixed value as \(T_0 \rightarrow \infty\).

The Klein Gordon equation in \((T, X)\) coordinates is

\[
\left[ \partial_T^2 - \partial_X^2 + m^2(e^{-2wT} + e^{2wX}) \right] \Phi(X, T) = 0
\]

Separating variables via \(\Phi(X, T) = F(T)G(X)\) yields the second order differential equations

\[
\frac{d^2F(T)}{dT^2} + (m^2 \exp(-2wT) + K^2)F(T) = 0
\]

\[
d^2G(X) - (m^2 \exp(2wX) - K^2)G(X) = 0
\]

where \(K^2\) is a constant of separation. Writing \(\nu = K/w\), the solutions

\[
F(T) = A_{\nu}J_{\nu}(\frac{m}{w} e^{-wT}) + B_{\nu}J_{-\nu}(\frac{m}{w} e^{-wT})
\]

\[
G(X) = C_{\nu}K_{\nu}(\frac{m}{w} e^{wX}) + D_{\nu}K_{-\nu}(\frac{m}{w} e^{wX})
\]

can be expressed as linear combinations of Bessel functions \(J_{\nu}(Z)\) and McDonald functions \(K_{\nu}(Z)\) and \(I_{\nu}(Z)\).

As \(T \rightarrow \infty\) the solutions to (3) asymptote to \(F \sim e^{\mp iKT}\), similar to those for a Rindler observer with time coordinate \(\tau = T\), whereas for \(T \rightarrow -\infty\) the solutions asymptote to \(F \sim \exp(\mp iK/T)\), similar to those for an inertial observer with time coordinate \(wt = -e^{-wT}\).

Requiring the solutions to be nonsingular in \(X\) we obtain from (7) the positive frequency solution with the former asymptotic behaviour is

\[
\Phi_{\nu}^+(X, T) = (\frac{\nu}{\pi w})^{1/2} J_{\nu}(\tilde{T}) K_{\nu}(\tilde{X})
\]

where \(\tilde{T} = \frac{m}{w} e^{-wT}, \tilde{X} = \frac{m}{w} e^{wX}\). The solution with the latter asymptotic behaviour is

\[
\Phi_{\nu}^{1+}(X, T) = \sqrt{\frac{1-e^{-2w\nu}}{2}} (\frac{\nu}{\pi w})^{1/2} H_{\nu}(\tilde{T}) K_{\nu}^{1}(\tilde{X})
\]

as can be determined from the asymptotic behavior of the Bessel functions. The solutions (9) and (10) are orthogonal

\[
\langle \Phi_{\mu}^+(X, T), \Phi_{\nu}^{1+}(X, T) \rangle = \left( we^{\pi \nu} \sqrt{e^{2\pi \nu} - 1} \right)^{-1} \delta(\mu - \nu)
\]

under the inner product

\[
\langle \Phi, \Psi \rangle = -i \int_{S} dS^{*} \Phi \bar{\partial}_{\nu} \Psi^{*}
\]
A solution for the NUA observer can be expressed in terms of positive and negative frequency solutions
\[ \Phi_k(x, t) = \frac{1}{2} \pi \exp(\pi i (ct - kx)) \quad (15) \]
of the inertial observer via the Bogoliubov transformations
\[ \alpha_{\nu, k} = \langle \Phi_{\nu}^+(X, T), \Phi_{k}^{M+}(x, t) \rangle \]
\[ \beta_{\nu, k} = -\langle \Phi_{\nu}^+(X, T), \Phi_{k}^{M-}(x, t) \rangle \]
\[ \nu = \frac{K}{w}, \quad \text{for various values of } K \text{ and } w. \]

For the positive frequency solution, the Bogoliubov coefficients are
\[ 1 \quad \text{and} \quad 2 \chi \quad \text{where} \quad \chi = \frac{\hbar}{w} \quad \text{and} \quad \epsilon = \sqrt{\frac{\hbar^2}{m^2} + \frac{n^2}{w^2}} \quad (20) \]

We then rewrite (22) in terms of a product of two-mode squeezed states of the NUA vacuum defined on the hypersurface \( T = T_0 \)
\[ |0>^M = \sqrt{1 - |q|^2} \sum_{n=0}^{\infty} q^n |n_k>_I |n_k>_{II} \quad (25) \]

where \( |n_k>_I \) and \( |n_k>_{II} \) refer to the mode decomposition in regions (I) and (II), respectively. Each Minkowski mode \( j \) has a mode expansion given by Eq. (25). We assume that all modes except for mode \( s \) for Alice and \( k \) for Vic are in the vacuum. Tracing over all of these other modes yields a pure state since each set of solutions in regions (I) and (II) are orthogonal and so different modes \( j \) and \( j' \) do not mix on the hypersurface \( T = T_0 \).

Since events in region (II) cannot causally influence those in region (I) we rewrite Eq. (25) using (26), tracing over states in region (II). This yields a mixed state between Alice (A) and Vic (V)
\[ \rho_{AV} = \frac{1 - |q|^2}{2} \sum_n q^{2n} \rho_n, \]

where \( \rho_n = |0_n>(0_n| + \frac{1 - |q|^2}{\sqrt{n + 1}} |0_n>(1_n+1|0_n) |1_n+1> + \frac{1 - |q|^2}{\sqrt{n + 1}} |1_n+1>(0_n| |0_n) + (1 - |q|^2) |n+1>(n+1| |1_n+1) |1_n+1> \)

whose elements are functions of \( T_0 \) via the parameter \( q \), where \( |n_m> = |n_s>^M|m_k>_{II} \).

From here the calculation proceeds in a manner similar to that for the Rindler case \( \frac{C}{2} \). If at least one eigenvalue of the partial transpose of \( \rho \) is negative, then the density matrix is entangled; but a state with positive partial
One eigenvalue is always negative since $|\rho|_{\mathrm{negativity}}$ [14], defined as amount of ‘almost-pure state entanglement’ that can be increasing time, its slope more pronounced close to $T_0 = 0$ where the change in acceleration is maximal. For vanishing acceleration ($q \to 0$ as $T \to -\infty$), $N(\rho_{AV}) \to 1$, and as $T_0 \to +\infty$, it approaches the values obtained for the UA case [7], with different values of $K = \nu w$ yielding different asymptotic values of $N(\rho_{AV})$.

Hence entanglement degradation increases as a function of time. This behaviour could in principle be empirically determined by Alice from an ensemble of experiments with different NUA observers each with the same value of $w$, making measurements for different choices of $T_0$ that are classically communicated to Alice.

To estimate the total amount of correlation in the state we compute the mutual information, defined as $I(\rho_{AV}) = S(\rho_A) + S(\rho_V) - S(\rho_{AV})$ where $S(\rho) = -\text{Tr}(\rho \log_2(\rho))$ is the entropy of the density matrix $\rho$. We obtain Alice’s density matrix $\rho_A = \frac{1}{2}I$ by tracing over Vic’s states and Vic’s density matrix by tracing over Alice’s states. The resultant entropies can be straightforwardly calculated along with the entropy of the joint state, yielding

$$I = 1 - \frac{1}{2} \log_2 \left( \left| q \right|^2 \right) - \frac{1}{2} \left( 1 - \left| q \right|^2 \right) \sum_{n=0}^{\infty} \left| q \right|^{2n} \mathcal{D}_n,$$

$$\mathcal{D}_n = (1 + \frac{n(1 - \left| q \right|^2)}{|q|^2}) \log_2 \left( 1 + \frac{n(1 - \left| q \right|^2)}{|q|^2} \right) - (1 + (n + 1)(1 - \left| q \right|^2) \log_2 \left( 1 + (n + 1)(1 - \left| q \right|^2) \right),$$

for the mutual information, which we plot in Fig. 2 as a function of $T_0$. For large negative values of $T_0$ the acceleration vanishes and the mutual information is 2 as expected. As the acceleration increases, the mutual information decreases reaching, as $T_0 \to +\infty$, a constant value larger than unity.

The logarithmic negativity as well as the mutual information approach for large values of $T$ a constant asymptotic value that depends on $\nu = K/w$. Fig. 1 and Fig. 2 show that, as expected from the Rindler case [7], for a given value of $K$, the asymptotes of $N$ and $I$ decrease as $w$ increases. We also see that for a given value of $w$, the asymptotic values of $N$ and $I$ increase monotonically with $K$.

**IV. CONCLUDING REMARKS**

Our method for tracking the time-dependence of measures of quantum information via a sequence of measurements on hypersurfaces where positive/negative frequencies can be defined is quite general and can be applied to different kinds of motions and fields beyond the example we consider here. For a situation in which both observers begin freely falling into a black hole with one observer increasing acceleration to avoid this fate, the distillable entanglement degrades to a finite value. The entanglement degradation is due to the increase of entanglement with the modes in the region causally undetectable by the NUA observer. In curved spacetime, we expect in general that entanglement is a time-dependent as well as an observer-dependent concept.

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