Abstract

We calculate the bubble nucleation rate in a fourth-order gravity theory whose action contains an $R^2$ term, under the thin-wall approximation, by two different methods. First the bounce solution is found for the Euclidean version of the original action. Next, we use a conformal transformation to transform the theory into general relativity with an additional scalar field and then proceed to find the bounce solution. The same results are obtained from both calculations.
1. Introduction

Consider the theory of a single scalar field defined by the action

\[ S = \int d^4x \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] \tag{1.1} \]

where \( V \) has two local minima, \( \phi_\pm \). One of these, \( \phi_- \), is a global minimum (“true vacuum”). The other corresponds to a metastable state (“false vacuum”) that decays through bubble nucleation. The decay rate from the false vacuum to the true vacuum per unit volume can be calculated in the semiclassical limit

\[ \frac{\Gamma}{V} = A e^{-B/\hbar} \left[ 1 + O(\hbar) \right]. \tag{1.2} \]

Algorithms for computing the coefficients \( B \) and \( A \) have been given by Coleman[1] and Callan and Coleman[2] respectively.

In applications to the early universe, the effect of gravity becomes interesting. When gravity is weak, the basic features of bubble nucleation are similar to those in flat space, with small modification of the decay rate and the bubble radius. If we confine ourselves to the case where both the true and false vacua have positive energy density, then the true and false vacua will be de Sitter spaces. Gravitation makes the materialization of the bubble more likely, and makes the radius of the bubble at its moment of materialization smaller. Coleman and De Luccia[3] have calculated the coefficient \( B \) in the case where gravity is described by general relativity.

The Hilbert action for general relativity contains a single term proportional to the curvature scalar, and leads to second-order field equations. For some modified theories of gravity, the action contains terms that are combinations of the curvature scalar, Ricci tensor, Weyl tensor and the full Riemann curvature tensor. This leads to higher-order field equations. Such theories have been considered (see, e.g., [7]), for example, in the context of variations of extended inflation [8, 9]. In this
paper, we will confine ourselves to the case where the gravitational action contains an arbitrary function of the curvature scalar. The method can be easily applied to the case where the action contains arbitrary functions of the curvature scalar multiplied by a function of another scalar field.

When there is only an additional $R^2$ term in the action, Whitt found a conformal transformation that transforms the theory to standard general relativity but introduces a new scalar field[5]. This method can be generalized to the case of an arbitrary function of the curvature scalar. This is done in Section II, where some general features of fourth-order gravity are discussed. To calculate the bubble nucleation rate in fourth-order gravity theory, one could use the Euclidean version of the original action. This action contains higher powers of the curvature scalar. The bounce solution[1] of this Euclidean action gives the coefficient $B$ of the bubble nucleation rate. Alternatively one could use a conformal transformation to transform the theory into general relativity with an additional scalar field. One then could proceed following Coleman and De Luccia[3]. In Section III, we calculate the coefficient $B$ when the extra terms are small, using a perturbation method. In Section IV, we will use the conformal transformation to calculate the coefficient $B$. The result of this approach is found to agree exactly with the direct approach. We have made no attempt to calculate the coefficient $A$. Even in general relativity, there is not a satisfactory method to evaluate this coefficient.
2. Fourth-Order Gravity and Conformal Transformation

Consider a theory described by the action

\[
S = \int d^4x \sqrt{-g} \left( -\frac{f(R)}{16\pi G} + \mathcal{L}_m \right)
\]  

(2.1)

where \( R \) is the curvature scalar and \( \mathcal{L}_m \) is the Lagrangian of the matter fields. General relativity is the case when \( f(R) = R \). For simplicity, we will confine ourselves to only one matter field with the matter Lagrangian

\[
\mathcal{L}_m = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi)
\]  

(2.2)

The generalization to more fields is straightforward. It is easy to show that the equations of motion for this theory are

\[
\frac{1}{2} f(R) g_{\mu\nu} - f'(R) R_{\mu\nu} + f''(R) (R_{\mu\nu} - R_{\gamma\delta} g_{\mu\nu})
+ f'''(R) (R_{\mu\nu} R_{\nu\lambda} - R_{\alpha\beta} R_{\mu\nu} - R_{\mu\nu} g_{\alpha\beta}) - 8\pi G (\partial_\mu \phi \partial_\nu \phi - \mathcal{L}_m g_{\mu\nu}) = 0
\]  

(2.3)

and

\[
\phi;_\lambda ;^\lambda = -\frac{dV(\phi)}{d\phi}.
\]  

(2.4)

Because the curvature contains second order derivatives of the metric, the above equations contain fourth-order derivatives of the metric unless \( f''(R) \equiv 0 \) or, in other words, if the theory is general relativity. However, the higher order terms enter the equation only through the derivatives of the curvature scalar. We can lower the order of the differential equations by two if we introduce a new variable \( \sigma \) and arrange for this to be equal to the curvature scalar. The equations are then second order, but we have to supply the equation

\[
\sigma(x) = R(g_{\mu\nu})
\]  

(2.5)

for the new variable.
This can also be done by looking at the action directly. The action (2.1) is equivalent to

\[ S = \int d^4x \sqrt{-g} \left\{ -\frac{1}{16\pi G} \left[ f'(\sigma)R - \sigma f'(\sigma) + f(\sigma) \right] + \mathcal{L}_m \right\}. \] (2.6)

By varying \( \sigma \) we get

\[ \sigma = R \] (2.7).

Substituting this relation back into the action (2.6), we find that it is exactly the same as the original action (2.1). To make the theory look like general relativity, we introduce the conformal transformation

\[ \tilde{g}_{\mu\nu} = f'(\sigma) g_{\mu\nu}. \] (2.8)

In the new metric the connection is

\[ \tilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \Sigma^\lambda_{\mu\nu} \] (2.9)

where the tensor \( \Sigma^\lambda_{\mu\nu} \) is

\[ \Sigma^\lambda_{\mu\nu} = \frac{f''(\sigma)}{2f'(\sigma)} \left[ \sigma,_{\mu}g^\lambda_{\nu} + \sigma,_{\nu}g^\lambda_{\mu} - \sigma,_{\lambda}g_{\mu\nu} \right]. \] (2.10)

The Ricci tensor in the new metric is found to be

\[ \tilde{R}_{\mu\nu} = R_{\mu\nu} - \frac{f''(\sigma)}{2f'(\sigma)} \left[ 2\sigma,_{\mu\nu} + \sigma,_{\lambda}g_{\mu\nu} \right] \]
\[ - \frac{f'''}{2f'} \left[ 2\sigma,_{\mu}\sigma,_{\nu} + \sigma,_{\lambda}\sigma,_{\rho}g_{\mu\nu} \right] + \frac{3}{2} \left( \frac{f''}{f'} \right) \sigma,_{\mu}\sigma,_{\nu} \] (2.11)

and the new curvature scalar is

\[ \tilde{R} = \tilde{R}_{\mu\nu}\tilde{g}^{\mu\nu} \]
\[ = \frac{1}{f'(\sigma)} \left\{ R - \frac{3f''}{f'} \sigma,_{\lambda} + \frac{3f'''}{f'} \sigma,_{\lambda}\sigma,_{\rho} + \frac{3}{2} \left( \frac{f''}{f'} \right)^2 \sigma,_{\lambda}\sigma,_{\rho} \right\} \] (2.12)
where the indices are raised and lowered by the old metric. Also,
\[
\sigma^{\lambda\lambda}_{\text{old}} = \sigma_{\mu\nu} g^{\mu\nu} = (f'\sigma^{\lambda\lambda} - f''\sigma_{\lambda\lambda})_{\text{new}}
\]
\[
\sigma_{\lambda\lambda}^{\lambda\lambda}_{\text{old}} = f'\sigma_{\lambda\lambda}^{\lambda\lambda}_{\text{new}}
\]
(2.13)

With the aid of the above relation, the action can be written as
\[
S = \int \! d^4x \sqrt{-\tilde{g}} \left\{ \frac{\tilde{R}}{16\pi G} + \frac{1}{16\pi G} \left[ \frac{3}{2} \left( \frac{f''}{f'} \right)^2 \sigma_{\lambda\lambda}^{\lambda\lambda} + \frac{\sigma}{f'} - \frac{f}{f'^2} \right] \right. \\
\left. + \frac{\partial_{\mu}\phi \partial_{\nu}\phi \tilde{g}^{\mu\nu}}{2f'} - \frac{V(\phi)}{f'^2} \right\}
\]
(2.14)
where the arguments of \( f, f' \) and \( f'' \) are \( \sigma \). Now we have the action of general relativity coupled to two scalar fields, although the matter Lagrangian is somewhat unconventional. It can be shown that the equations of motion for the new action are equivalent to those of the original action, which proves the legitimacy of the above procedure.

Before we go to the next section, let us look at the de Sitter space solution of fourth-order gravity. De Sitter space is a homogeneous space with \( O(4,1) \) symmetry. Because of this symmetry, the Ricci tensor can be written as
\[
R_{\mu\nu} = \frac{1}{4} R g_{\mu\nu}
\]
(2.15)
where \( R \) is a constant throughout the whole space-time. The matter field is also homogeneous throughout the space-time manifold, so the equation of motion (2.3) reduces to
\[
\frac{1}{4} f'(R) R - \frac{1}{2} f(R) = \kappa V(\phi_0)
\]
(2.16)
where \( \phi_0 \) is a stationary point of \( V \) and \( \kappa = 8\pi G \). This is an algebraic equation which can be solved for \( R \).
Let us consider the specific case where the action is

\[ S = \int d^4x \sqrt{-g} \left\{ -\frac{R}{16\pi G} + \alpha R^2 + \frac{1}{2} \partial_{\mu} \phi \, \partial^{\mu} \phi - V(\phi) \right\}. \tag{2.17} \]

Then \( f(\sigma) \) is

\[ f(\sigma) = \sigma - 16\pi G \alpha \sigma^2, \tag{2.18} \]

and the conformal transformation is

\[ g_{\mu\nu} = (1 - 4\alpha \kappa \sigma \sigma^2) g_{\mu\nu} \] \( \tag{2.19} \)

In this new metric the action is

\[ S = \int d^4x \sqrt{-\tilde{g}} \left\{ -\frac{\tilde{R}}{2\kappa} + \frac{12\alpha^2 \kappa \sigma \partial_{\mu} \phi \partial^{\mu} \phi}{(1 - 4\alpha \kappa \sigma)^2} + \frac{\partial_{\mu} \phi \, \partial^{\mu} \phi}{2(1 - 4\alpha \kappa \sigma)} - \frac{V(\phi) + \alpha \sigma^2}{(1 - 4\alpha \kappa \sigma)^2} \right\}. \tag{2.20} \]

For both general relativity and the theory described by (2.17), equation (2.16) gives

\[ R = -4\kappa V(\phi_0) \tag{2.21} \]

because the \( \alpha \) terms happen to cancel. If higher order terms enter the action, the result differs in general from the general relativistic result. In the case when only \( R \) and \( R^2 \) terms enter, this solution corresponds to the stationary point of the last term of (2.20),

\[ U(\sigma) = \frac{V(\phi_0) + \alpha \sigma^2}{(1 - 4\alpha \kappa \sigma)^2} \tag{2.22} \]

This stationary point is located at \( \sigma = -4\kappa V(\phi_0) \), which is just the curvature scalar in the original metric. The curvature scalar in the new metric is

\[ \tilde{R} = -\frac{4\kappa V(\phi_0)}{1 + 16\alpha \kappa^2 V(\phi_0)} \tag{2.23} \]

in agreement with Eq. (2.12).
3. Bounce Action: The Direct Approach

Consider a theory described by the action (2.1), where $V(\phi)$ has two local minima, $\phi_{\pm}$, only one of which, $\phi_-$, is an absolute minimum. Let us also assume that both $V(\phi_-)$ and $V(\phi_+)$ are positive. The classical field theory defined by this action possesses two stable homogeneous equilibrium states, $\phi = \phi_+$ and $\phi = \phi_-$. In the quantum version of the theory, though, only the second one corresponds to a truly stable state, a true vacuum. The first decays through barrier penetration; it is a false vacuum. The decay rate per unit time per unit volume can be calculated in the semiclassical approximation as

$$\Gamma/V = Ae^{-B/\hbar}(1 + O(\hbar)). \quad (3.1)$$

To calculate the coefficient $B$, let us consider the Euclidean version of the theory. The Euclidean action is defined as minus the formal analytic continuation to imaginary time of the Lorentzian action (2.17)

$$S_E = \int d^4x \sqrt{g} \left\{ -\frac{f_E(R)}{16\pi G} + \frac{1}{2} \partial_\mu \phi \partial_\nu \phi g^{\mu\nu} + V(\phi) \right\} \quad (3.2)$$

where the metric is the usual positive-definite one of Euclidean four-space. Let $(\phi, g)$ be a solution of the Euler-Lagrange equations associated with $S_E$ such that: [i] $(\phi, g)$ approaches the false vacuum solution at large Euclidean distance, [ii] $(\phi, g)$ is not identical to the false vacuum solution, and [iii] $(\phi, g)$ has Euclidean action less than or equal to that of any other solution obeying [i] and [ii]. Then the coefficient $B$ in the vacuum decay amplitude is given by

$$B = S_E(\phi, g) - S_E(\phi_+, g_+) \quad (3.3)$$

where $(\phi_+, g_+)$ is the false vacuum solution. $(\phi, g)$ is called the bounce solution.
It can be shown that in flat space the bounce is always $O(4)$-symmetric\cite{4}. In curved space, although not proven, it is very plausible that the bounce is still $O(4)$-symmetric. We will work under this assumption. The metric can then be written as

$$ds^2 = d\xi^2 + \rho^2(\xi)d\Omega^2$$ \hspace{1cm} (3.4)

where $\xi$ is a radial coordinate from the center of the $O(4)$-symmetry and $\rho(\xi)$ measures the circumference divided by $2\pi$ at radial coordinate $\xi$.

In the following, we will confine ourselves to the case where only $R$ and $R^2$ terms enter the action and the theory is described by the action (2.17). In $O(4)$-symmetric coordinates (3.4), the equations of motion are

$$\rho'' = 1 + \frac{\kappa \rho^2}{3} \left\{ \frac{1}{2} \rho'^2 - V \right\} + 36\alpha \left( \frac{1}{\rho^4} + 2\rho'' \right) \left\{ \frac{1}{\rho^3} - 2\rho'' \rho'' \rho^2 - \rho'' \rho^2 + 2\rho' \rho''' \rho^2 \right\}$$ \hspace{1cm} (3.5)

and

$$\phi'' + 3\rho \phi' = \frac{dV}{d\phi}.$$ \hspace{1cm} (3.6)

We will use perturbative methods to find the effect of small $\alpha$. To be self-consistent, the second term in the bracket should be much smaller than the first term. This is always true if

$$\alpha \frac{m^2}{M_p^2} \ll 1$$

because the second term is at most of the order $\alpha m^6/M_p^2$, where $m$ is the mass scale of the potential $V(\phi)$.

Under this condition, the Euclidean action can be split into an unperturbed
part

\[ S_0 = \int d^4 x \sqrt{g} \left\{ -\frac{R}{2\kappa} + \frac{1}{2} (\partial \phi)^2 + V(\phi) \right\} \]  

(3.7)

and a perturbation

\[ S_\alpha = \int d^4 x \sqrt{g} (-\alpha R^2). \]  

(3.8)

Let \((\rho_0, \phi_0)\) be the bounce solution when \(\alpha\) vanishes and \((\rho_1, \phi_1)\) be a small correction

\[ \rho = \rho_0 + \rho_1 \]
\[ \phi = \phi_0 + \phi_1 \]  

(3.9)

Because \((\rho_0, \phi_0)\) is a solution to the Euler-Lagrange equations of \(S_0\), the first order correction to \(S_0\) due to \((\rho_1, \phi_1)\) vanishes, so to first order the Euclidean action is

\[ S_E = S_0[\rho_0, \phi_0] + S_\alpha[\rho_0, \phi_0] + O(\alpha^2) \]  

(3.10)

and the bounce action is

\[ B = B_0 + B_\alpha \]
\[ = \{ S_0(\rho_0, \phi_0) - S_0(\text{false vacuum}) \} + \{ S_\alpha(\rho_0, \phi_0) - S_\alpha(\text{false vacuum}) \} \]  

(3.11)

We will use the thin wall approximation in the following calculation[1,3]. The thin wall approximation is valid when the energy density difference between the false and true vacua is small compared to the potential barrier. In this case the thickness of the wall is small compared to the radius of the bubble. We will also assume that gravity is weak, i.e., that the scale \(m\) of \(V(\phi)\) is much smaller than the Planck mass. The Euclidean true and false vacua with positive cosmological terms are Euclidean de Sitter spaces, which are four-spheres. We will further assume that the radii of these four-spheres are much larger than the flat-space bubble radius. Under above assumptions the bounce solution has a thin 3-dimensional spherical wall which separates an interior true vacuum region and an exterior false vacuum.
region. The radius of this wall is large compared to the wall thickness, but much smaller than the radii of the true and false vacuum 4-spheres.

First let us consider the unperturbed solution with $\alpha = 0$. Inside (outside) the bubble $(\rho_0, \phi_0)$ is just de Sitter space of true (false) vacuum

$$\phi_0 = \phi_\pm$$

$$\rho_0 = \frac{\sin(H_\pm \xi)}{H_\pm}$$

with

$$R_\pm = 12H_\pm^2 = 4\kappa V(\phi_\pm).$$

This is the exact result for the true and false vacuum 4-spheres even when $\alpha \neq 0$ because the $\alpha$ terms happen to cancel. The contributions to $B_0$ have been calculated by Coleman and De Luccia\[3\]

$$B_0|_{\text{out}} = 0$$

$$B_0|_{\text{in}} = \frac{12\pi^2}{\kappa^2 V(\phi_-)} \left\{ \left[ 1 - \frac{1}{3} \kappa \bar{\rho}_0^2 V(\phi_-) \right]^{3/2} - 1 \right\} - (\phi_- \rightarrow \phi_+)$$

(3.14)

where $2\pi \bar{\rho}_0$ is the circumference of the 3-sphere thin wall. In the thin wall approximation, the contribution to $B_0$ from the shell can be shown to be\[3\]

$$B_0|_{\text{wall}} = 4\pi^2 \bar{\rho}_0^3 \int d\xi [V(\phi) - V(\phi_+)] = 2\pi^2 \bar{\rho}_0^3 S_1$$

(3.15)

where $\bar{\rho}_0$ is determined by requiring that the bounce action

$$B_0 = B_0|_{\text{in}} + B_0|_{\text{wall}} + B_0|_{\text{out}}$$

be stationary.
We need the curvature scalar in the wall region for latter calculations. Because \((\rho_0, \phi_0)\) is a solution to the theory defined by \(S_0\) (which is standard Euclidean general relativity), \((\rho_0, \phi_0)\) satisfies Einstein’s equation

\[
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\kappa T_{\mu\nu}
\]  

(3.16)

The trace of this,

\[
R = \kappa T^\lambda_\lambda,
\]  

(3.17)

gives

\[
R = \kappa \left[ \phi_0'^2 + 4V(\phi_0) \right].
\]  

(3.18)

Because the difference between \(V(\phi_+)\) and \(V(\phi_-)\) is very small compared to \(V(\phi)\) in the thin-wall region, define

\[
V_0 = \frac{1}{2} \left[ V(\phi_+) + V(\phi_-) \right]
\]

(3.19)

\[
U(\phi) = V(\phi) - V_0
\]

Except in the thin wall region, \(\phi\) is almost a constant (either \(\phi_+\) or \(\phi_-\)) and \(V\) is well approximated by \(V_0\). Because the bubble’s circumference radius is large, we can neglect the \(\frac{3\phi'}{\rho} \phi'\) term in Eq. (3.6). We then have the “approximate conservation law”

\[
\frac{1}{2} \phi_0'^2 - V(\phi_0) = -V_0.
\]  

(3.20)

We find

\[
\phi_0' = \sqrt{2U(\phi_0)}
\]

\[
R = \kappa \left[ 6U(\phi_0) + 4V_0 \right].
\]  

(3.21)

Now consider the first order correction to the coefficient \(B\)

\[
B_\alpha = S_\alpha(\rho_0, \phi_0) - S_\alpha(\text{false vacuum})
\]  

(3.22)

In the exterior of the bubble the bounce solution coincides with the false vacuum
solution, so the contribution to $B_\alpha$ vanishes

$$B_\alpha|_{\text{out}} = 0. \quad (3.23)$$

The contribution of the wall region to $B_\alpha$ is

$$B_\alpha|_{\text{wall}} = -\alpha \int d^4x \left( R^2 - R^2_{\text{false}} \right) \quad (3.24)$$

Using (3.21) we find that

$$B_\alpha|_{\text{wall}} = (-\alpha \kappa) 2\pi^2 \rho_0^3 \int d\xi \ 6U \left( 6U + 8V_0 \right)$$

$$= (-\alpha \kappa) 2\pi^2 \rho_0^3 \int_{\phi_-}^{\phi_+} \frac{d\phi}{\sqrt{2U}} \left( 6U + 8V_0 \right)$$

$$= -12\alpha \pi^2 \kappa^2 \rho_0^3 \int d\phi \ \sqrt{2U} \left( 3U + 4V_0 \right). \quad (3.25)$$

The contribution from the interior of the bounce to $B_\alpha$ is

$$B_\alpha|_{\text{in}} = \int d^4x (-\alpha R^2)_{\text{true}} - \int d^4x (-\alpha R^2)_{\text{false}}$$

$$= -2\pi^2 \alpha \left\{ \int d\xi \rho^3 (4\kappa V)^2 \bigg|_{\text{true}} - \int d\xi \rho^3 (4\kappa V)^2 \bigg|_{\text{false}} \right\}$$

Using equations (3.12) and (3.13) we find

$$B_\alpha|_{\text{in}} = -32\alpha^2 \kappa^2 \left\{ \int d\xi \ \frac{\sin^3 \left( H_- \xi \right)}{H_-^3} V_-^2 - \int d\xi \ \frac{\sin^3 \left( H_+ \xi \right)}{H_+^3} V_+^2 \right\}$$

$$= 96\alpha^2 \left[ \sqrt{1 - \frac{\kappa V_-}{3} \rho_0^2} \left( 2 + \frac{\kappa V_-}{3} \rho_0^2 \right) - \sqrt{1 - \frac{\kappa V_+}{3} \tilde{\rho}_0^2} \left( 2 + \frac{\kappa V_+}{3} \tilde{\rho}_0^2 \right) \right] \quad (3.26)$$

where $V_-$ and $V_+$ are the energy densities of the true and false vacua respectively.
Combining the contributions from the three regions, we find that to first order in $\alpha$

\[ B = B_0 - 12\alpha \pi^2 \kappa^2 \bar{\rho}_0^3 \int_{\phi_-}^{\phi_+} d\phi \sqrt{2U} (3U + 4V_0) + 96\alpha \pi^2 \left[ \sqrt{1 - \frac{\kappa V_0}{3} \bar{\rho}_0^2} \left( 2 + \frac{\kappa V_+}{3} \bar{\rho}_0^2 \right) - \sqrt{1 - \frac{\kappa V_-}{3} \bar{\rho}_0^2} \left( 2 + \frac{\kappa V_+}{3} \bar{\rho}_0^2 \right) \right]. \]  

(3.27)

Now we want to determine the sign of the first order correction to the bounce action. If it is positive, the bubble nucleation will be less likely. First notice that

\[ \int_{\phi_-}^{\phi_+} d\phi \ 3U \sqrt{2U} > 0 \]  

(3.28)

If the cosmological term $V_0$ is positive, then $B_\alpha|_{\text{wall}}$ (3.25) has the opposite sign from $\alpha$ and

\[ \left| B_\alpha|_{\text{wall}} \right| > 48\alpha \pi^2 \kappa^2 \bar{\rho}_0^3 V_0 \int_{\phi_-}^{\phi_+} d\phi \sqrt{2U} \]  

(3.29)

Under our assumption of weak gravity, $\kappa V_0 \bar{\rho}_0^2$ is small and we expand $B_\alpha|_{\text{in}}$ to the leading order in it

\[ B_\alpha|_{\text{in}} = 8\alpha \pi^2 \kappa^2 \bar{\rho}_0^4 V_0 (V_+ - V_-) \]  

(3.30)

This has the same sign as $\alpha$ if $V_0$ is positive. Therefore the contributions to the first order correction from the wall region and from the bubble interior have opposite signs. To compare their magnitudes

\[ \left| \frac{B_\alpha|_{\text{wall}}}{B_\alpha|_{\text{in}}} \right| > \frac{6 \int d\phi \sqrt{2U}}{\bar{\rho}_0 (V_+ - V_-)} \]  

(3.31)
notice that to leading order (see [1,3])

\[ \bar{\rho}_0 = \frac{3}{(V_+ - V_-)} \int d\phi \sqrt{2U} = \frac{3S_1}{(V_+ - V_-)} \tag{3.32} \]

So we conclude

\[ \left| \frac{B_\alpha}{\bar{\rho}_0} \right|_{\text{wall}} > 2 \]
\[ \left| \frac{B_\alpha}{\bar{\rho}_0} \right|_{\text{in}} < 0 \tag{3.33} \]

Similarly, for

\[ B_\alpha' = \frac{dB_\alpha(\bar{\rho}_0)}{d\bar{\rho}_0} \tag{3.34} \]

we have

\[ \left| \frac{B_\alpha'}{\bar{\rho}_0} \right|_{\text{wall}} > \frac{3}{2} \]
\[ \left| \frac{B_\alpha'}{\bar{\rho}_0} \right|_{\text{in}} < 0 \tag{3.35} \]

The bubble radius \( \bar{\rho} \) is determined by requiring that \( B \) be stationary at that point. Let \( \bar{\rho}_0 \) be the unperturbed bubble radius and \( \Delta \bar{\rho} \) the first order correction

\[ \bar{\rho} = \bar{\rho}_0 + \Delta \bar{\rho} + O(\alpha^2). \tag{3.36} \]

We require that

\[ 0 = \frac{\partial B}{\partial \rho} \bigg|_{\bar{\rho}} = \frac{\partial B_0}{\partial \rho} \bigg|_{\bar{\rho}} + \frac{\partial B_\alpha}{\partial \rho} \bigg|_{\bar{\rho}} \tag{3.37} \]

To first order in \( \alpha \), we have

\[ \frac{\partial B_0}{\partial \rho} \bigg|_{\bar{\rho}} \approx \frac{\partial B_0}{\partial \rho} \bigg|_{\bar{\rho}_0} + \frac{\partial^2 B_0}{\partial \rho^2} \bigg|_{\bar{\rho}_0} \Delta \bar{\rho} = \frac{\partial^2 B_0}{\partial \rho^2} \bigg|_{\bar{\rho}_0} \Delta \bar{\rho} \tag{3.38} \]

and

\[ \frac{\partial B_\alpha}{\partial \rho} \bigg|_{\bar{\rho}} \approx \frac{\partial B_\alpha}{\partial \rho} \bigg|_{\bar{\rho}_0} \tag{3.39} \]
Then equation (3.37) requires
\[ \Delta \bar{\rho} = - \frac{B_{\alpha}'}{B_0''} \left| \frac{\bar{\rho}}{\bar{\rho}_0} \right| \tag{3.40} \]

As mentioned above, \( B_{\alpha}' \) is negative when \( \alpha \) is positive. From the work by Coleman and De Luccia[3] and the work by Parke[6], we know that \( B_0'' \) is negative. Therefore when \( \alpha \) is positive, \( \Delta \bar{\rho} \) is negative, which means that bubble has a smaller circumference radius than it does when the \( R^2 \) term is absent in the Lagrangian.

The above conclusion can be understood intuitively. Because of the gradient term in the Lagrangian, the inhomogeneity in the wall region creates surface tension and increases the bubble energy. In order to be energetically balanced, the bubble interior should have a lower potential energy density to compensate the energy increase due to surface tension. For small \( \alpha \), the theory can be treated as general relativity with the effective potential
\[ V_{\text{eff}} = V - \alpha R^2 \tag{3.41} \]

The presence of the \( R^2 \) term lowers the potential. Both the true and false vacua are de Sitter spaces and the curvature scalar is proportional to the energy density. For the false vacuum the energy density is larger than for the true vacuum, so the relative energy difference is reduced by the presence of the \( R^2 \) term. However in the thin wall region the curvature scalar becomes fairly big when the field changes rapidly. This means that in the wall region the potential energy decreases at a much larger rate than the potential energy inside the bubble. So both the surface tension and the interior potential energy difference decrease, but the surface tension decreases at a larger rate. So to be balanced energetically, the volume inside the bubble should decrease. This is why the bubble radius decreases. Generally, it is easier for a small bubble to nucleate, so the bounce action decreases and the bubble nucleation rate increases.
4. Bounce Action: Conformal Transformation Method

In this section, we are going to use the conformal transformation of Section II to study the bubble nucleation rate in fourth-order gravity. If the Lorentzian action is Eq. (2.17), then the Euclidean action is

\[ S_E = \int d^4x \sqrt{g} \left\{ -\frac{R}{16\pi G} - \alpha R^2 + \frac{1}{2}(\partial\phi)^2 + V(\phi) \right\}. \tag{4.1} \]

The conformal transformation described in Section II leads to the action

\[ S_E = \int d^4x \sqrt{\tilde{g}} \left\{ -\frac{m^2}{16\pi} \tilde{R} + \frac{12\alpha^2 \kappa \phi \partial^\nu \sigma}{(1 + 4\alpha \kappa \sigma)^2} + \frac{\partial_\mu \phi \partial^\nu \phi}{2(1 + 4\alpha \kappa \sigma)} + \frac{V(\phi) + \alpha \sigma^2}{(1 + 4\alpha \kappa \sigma)^2} \right\}. \tag{4.2} \]

Assuming $O(4)$ symmetry, we use the metric (3.4). The Euclidean action then becomes

\[ S_E = 2\pi^2 \int d\xi \left\{ \rho^3 \left[ \frac{\phi'^2}{2(1 + 4\alpha \kappa \sigma)} + \frac{12\alpha^2 \kappa \sigma'^2}{(1 + 4\alpha \kappa \sigma)^2} + \frac{V + \alpha \sigma^2}{(1 + 4\alpha \kappa \sigma)^2} \right] \right. \]
\[ + \left. \frac{3}{\kappa} (\rho' \rho'' + \rho \rho'^2 - \rho) \right\}. \tag{4.3} \]

The Einstein equation is

\[ \rho'^2 = 1 + \frac{1}{3}\kappa \rho^2 \left[ \frac{\phi'^2}{2(1 + 4\alpha \kappa \sigma)} + \frac{12\alpha^2 \kappa \sigma'^2}{(1 + 4\alpha \kappa \sigma)^2} - \frac{\alpha \sigma^2 + V}{(1 + 4\alpha \kappa \sigma)^2} \right]. \tag{4.4} \]

The other equations of motion are

\[ \phi'' + \frac{3\rho'}{\rho} \phi' = \frac{dV}{d\phi} + \frac{4\alpha \kappa \sigma' \phi'}{1 + 4\alpha \kappa \sigma} \tag{4.5} \]

and

\[ \sigma'' + \frac{3\rho'}{\rho} \sigma' = \frac{8\alpha \kappa \sigma'^2}{1 + 4\alpha \kappa \sigma} - \frac{\phi'^2}{6\alpha} + \frac{1}{6\alpha} \frac{\sigma - 4\kappa V}{1 + 4\alpha \kappa \sigma}. \tag{4.6} \]

When $\alpha = 0$, these equations of motion agree with Eqs. (3.5) and (3.6).
The second derivative of $\rho$ can be eliminated by integration by parts. (The surface term from the parts integration is harmless because we are only interested in the action difference between two solutions that agree at the boundary.) We thus obtain

$$S_E = 2\pi^2 \int d\xi \left\{ \rho^3 \left[ \frac{\phi^2}{2(1 + 4\alpha\kappa\sigma)} + \frac{12\alpha^2\kappa^2\sigma^2}{(1 + 4\alpha\kappa\sigma)^2} + \frac{V + \alpha\sigma^2}{(1 + 4\alpha\kappa\sigma)^2} \right] - \frac{3}{\kappa}(\rho\rho'^2 + \rho) \right\}. \quad (4.7)$$

We now use Eq. (4.4) to eliminate $\rho'$. We find

$$S_E = 4\pi^2 \int d\xi \left[ -\frac{3}{\kappa}\rho + \rho^3\frac{V + \alpha\sigma^2}{(1 + 4\alpha\kappa\sigma)^2} \right]. \quad (4.8)$$

Now we evaluate the above integral in three regions. Outside the bubble the bounce solution agrees with the false vacuum and the contribution to the bounce action is zero:

$$B_{\text{out}} = 0 \quad (4.9)$$

The interior of the bubble is de Sitter space and from the equations of motion we know

$$\sigma = 4\kappa V$$
$$\rho = \frac{\sin(H\xi)}{H} \quad (4.10)$$

with

$$H^2 = \frac{\kappa V}{3(1 + 16\alpha\kappa^2 V)}. \quad (4.11)$$

The contribution to the bounce action is the difference of the integral (4.8) between
of the true vacuum and false vacuum. It is easy to find that

$$B_{in} = \frac{12\pi^2}{\kappa^2 V} (1 + 16\alpha^2 V) \left\{ \left[ 1 - (H \bar{\rho})^2 \right]^{3/2} - 1 \right\}_{true} - (false) \quad (4.12)$$

Expanding this to first order in $\alpha$, we find

$$B_{in} = B_{0in}(\bar{\rho}) + 96\pi^2 \left[ \sqrt{1 - \frac{\kappa V_0}{3} \bar{\rho}^2} \left( 2 + \frac{\kappa V_0}{3} \bar{\rho}^2 \right) - \sqrt{1 - \frac{\kappa V_+}{3} \bar{\rho}^2} \left( 2 + \frac{\kappa V_+}{3} \bar{\rho}^2 \right) \right]$$

which is exactly the same result as in the previous section.

In the wall region, we expand the integral (4.8) to first order in $\alpha$. This gives

$$B_{wall} = 4\pi^2 \bar{\rho}^3 \int d\xi (V_b - V_f) + 4\pi^2 \alpha^2 \bar{\rho}^3 \int d\xi \left[ (\sigma^2 - 8\kappa \sigma V)_{bounce} - (false) \right] \quad (4.14)$$

At first sight one might think that the first integral is the unperturbed contribution with $\alpha = 0$ and the second integral is the first order correction. However, the first integral also contains a first order correction because $V$ is a function of $\phi$ and when $\alpha \neq 0$ the dependence of $\phi$ on $\xi$ changes. Let us investigate the equations of motion as a power series in $\alpha$. From Eqs. (4.6) and (4.5), the equations of motion in the thin-wall approximation are

$$\sigma = \kappa (4V + \phi'^2) + O(\alpha)$$
$$\phi'' = \frac{dV}{d\phi} + 4\alpha \kappa \sigma' \phi' - 4\alpha \kappa \sigma \frac{dV}{d\phi} + O(\alpha^2) \quad (4.15)$$

Multiplying the second equation by $\phi'$, we get

$$\left[ \frac{1}{2} \phi'^2 - V \right]' = 4\alpha \kappa \left[ \sigma' \phi'^2 - \sigma V' \right] + O(\alpha^2) \quad (4.16)$$

To first order in $\alpha$, this implies

$$\frac{1}{2} \phi'^2 = V - V_0 + O(\alpha) \quad (4.17)$$
Together with the first equation in (4.15), this leads to

\[ \sigma' \phi'^2 - \sigma V' = \kappa (3U^2 - 4V_0 U)' + O(\alpha) \]  

(4.18)

Putting this back in (4.16) and integrating, we get

\[ \phi'^2 = 2U (1 + 12 \alpha \kappa^2 U - 16 \alpha \kappa^2 V_0) + O(\alpha^2) \]  

(4.19)

The first term of (4.14) is

\[ 4\pi^2 \bar{\rho}^3 \int d\xi (V_b - V_f) = 4\pi^2 \bar{\rho}^3 \int \frac{d\phi}{\phi'} (V_b - V_f) \]  

(4.20)

Using Eq. (4.19), we find the first order correction to this is

\[ 4\pi^2 \bar{\rho}^3 \int d\phi \alpha \kappa^2 \sqrt{2U} (-3U + 4V_0) \]  

(4.21)

The contribution to the first order correction from the second integral is easily found to be

\[ 4\pi^2 \bar{\rho}^3 \alpha \kappa^2 \int d\phi \sqrt{2U} (-6U - 16V_0) \]

Therefore the first order correction due to the wall region is given by

\[ B_{\alpha|\text{wall}} = -12\pi^2 \alpha \kappa^2 \bar{\rho}^3 \int d\phi \sqrt{2U} (3U + V_0) \]  

(4.22)

Again, this is exactly the same result as in last Section. Although \( \bar{\rho} \) is determined in the new metric now, the difference between the old metric and the new one is first order in \( \alpha \). So it does not invalidate our conclusion.

Therefore, we get exactly the same results by using two completely different methods. This also proves the validity of the conformal transformation we discussed in Section II.

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