Analysis of optimal portfolio model under ambiguity

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Abstract. Portfolio theory is one of the important directions of financial research nowadays, its purpose is to achieve maximum profit or minimum risk. Since the founding of the Von Neuman and Morgenstern (1947), the Expected Utility theory has been widely used in the decision-making of investors. However, in the process of portfolio, decision makers often aren’t sure the probability of one or more of the interests. This uncertainty (or ambiguity) may affect the preferences of decision makers. Therefore, decision makers may want to add for ambiguity aversion to analysis. In this paper, the smooth ambiguity model is introduced to study the problem of portfolio, which can solve multiple problems of the subjective expected utility and smooth ambiguity. And the specific model is given under CARA utility function.

1. Introduction
Now the problem of optimal investment is one of the important research contents of microeconomics, which is indispensable to the development of economics. With the development of the society and the progress of the times, the economy has been developing more and more rapidly, and the market competition has become increasingly fierce. People are no longer limited to the simple way of money management, and they began to invest in a variety of products. But now, the investment projects are also beginning to diversified development, economic agents are facing the choice of gradually increased. At the same time, the degree and types of the risks will increase accordingly. A portfolio is a method used to diversify risk. The economic agents always expect limited wealth distribution in the different investment products, in order to obtain the biggest benefit. Therefore, the optimal portfolio is the key to economic agent. And the portfolio choice problem plays a guiding role in decisions. Nowadays, investors are demanding more and more circumspective in the identification of risks and loss, and more and more attention is paid to the problem of optimal portfolio.

Von Neumann and Morgenstern (1946) determined the expected utility theory (EU). Then, the economic agent began model attitude to risk. Since the new portfolio theory of Markowitz (1952), the quantitative method has been used to explain the optimal portfolio. He substituted the mean of the portfolio for the return, and substituted the variance of the portfolio for the risk. This is the most traditional portfolio model: the mean - variance framework. When decision makers are making decisions, they want to get the maximum return, or the minimizing risk. With the rapid development, the optimal portfolio framework connects the real economy and the virtual economy, which plays a guiding role in realizing the dynamic balance of investment, savings and consumption, and also drives the operation of securities market and the development of economic society. In the field of traditional
portfolio problem, common decision-making methods include mean-variance framework, VaR (Value at Risk) framework, stochastic integral framework, martingale analysis and so on.

The biggest characteristic of portfolio is the uncertainty of risk and return, so the core problem of portfolio is the balance between return and risk. How to determine the investors’ attitude towards risk and return, and realize the maximization of return and minimization of risk is the focus of decision maker. When investor require more and more precision of risk quantification, it is found that characterizing risk with variance is not well. And then scholars start to look for more convincing framework. As economists pay more and more attention to the study of non-deterministic economic behavior, one of the most important developments is to enhance the ability to quantify risk. Then the theory of expected utility is more and more popularized. Some qualitative differences that are difficult to quantify can be quantified through the utility function. If the risks in the financial market can be quantified accurately, the risky consumption investment can be measured relatively accurately and social resources can be better allocated.

Uncertainty has always been a hot topic in academic circles. However, it is not enough to only focus on the risk. The ambiguity of the model is gradually found by economists and distinguished from the risk. Knight determined a clear concept that could distinguish between ambiguity and risk for the first time. He believed that risks were knowable uncertainties and could be described by a single probability distribution, while ambiguity could not be described by a single probability distribution. In practical application, ambiguity seems to more realistic than traditional risk.

In this paper, the investment problem will be solved numerically through the model establishment and utilization. Because of the complexity of the financial market, in most of the returns and risk of investment products are uncertain. Therefore, this paper will consider the uncertainty when the economic agent making decision. And we try to introduce risk and ambiguity in the model. Then we hope to give reasonable advice to consumers on investment decisions.

2. Basic theory

2.1. Theory of utility

In the financial market, different investors have different needs. They will make a portfolio choice according to their own wishes. The portfolio standard is to rank the returns and risks of the financial products, which are chosen according to the preferences of investors. Although preference is very simple and intuitive in understanding, it is too abstract in expression and cannot be analyzed quantitatively. Therefore, a measure is needed to measure the degree, and the utility function is a more effective and convenient means to describe the investor’s preference relationship. The utility function is usually used to represent the quantity relationship between the utility obtained by consumers in consumption and the commodity combination consumed, in order to measure the satisfaction of consumers obtained from the consumption given commodity combination. Different consumers have different consumption functions.

The function of utility function is to materialize the consumption set of investors or the elements in the decision set in a certain case, so as to assign a value to the preference of investors. But in real life, there's a lot of uncertainty, and the utility function doesn't apply in the case of risk and uncertainty, because the probabilities aren't exactly available.

2.2 Expected utility function

In order to deal with the unknown probability distribution in the uncertain environment, in the 1950s, Von Neumann and Morgenstern established the framework for the analysis of rational person selection, namely the expected utility function, by using logic, mathematical and physical tools on the basis of the axiomatic hypothesis. This is also the core method to solve the uncertainty problem at present.

2.3 Risk and ambiguity
In 1921, Knight\cite{4}, an American economist, first proposed that there would often be risks in the financial market that could not be measured by a single probability measure, which became the risk of Knight uncertainty. For the first time, Knight defined risk and ambiguity and distinguished them. In order to make a clear distinction between risks and ambiguity, Knight divided the probability into three cases to explain: a priori probability, statistical probability, and estimate. The probabilities in the first two states are measurable. Knight calls them "risk". The third scenario is completely unknowable and can only be predicted subjectively, what Knight calls "ambiguity". That is to say, "risk" is in fact able to determine the objective probability distribution, and "ambiguity" indicates the uncertainty of its objective probability distribution.

2.4 The smooth ambiguity model

The traditional expected utility model is a Bayesian decision, and people usually use prior probability to make the decision, which is essentially derived from historical data or decision-making experience. Therefore, when people cannot get prior probabilities, they cannot make decisions. This is the famous Ellsberg paradox\cite{5}. To explain the inconsistency between ambiguity and SEU (subjectively expected utility) model, scholars evolved SEU model into MEU (Maxmin Expected Utility) model, CEU (Choquet Expected Utility) model\cite{6}, Expected Utility function recursively, female-meu model and KMM (smooth ambiguity) model.

The KMM model is used in this paper. The KMM model is used to model the ambiguity, which reflects the sensitivity of the model. This model is a smooth and non-bending model established by Klibanoff, Marnacci and Mukerji\cite{7}, which expresses the preference of decision makers with the functional form of double expectations below:

\[
V(f) = \int_\Delta \Phi \left( \int_S u(f) d\pi \right) d\mu = E_\mu \Phi( E_\pi u \circ f )
\]

where \( f \) is a function on a state space \( S \), \( u \) is a VNM utility function, \( \pi \) is the probability measure on \( S \), \( \Phi \) expressed with ambiguous attitude, so as to make the KMM model will separate the ambiguity from the ambiguous attitude. One advantage of the KMM model is that it can distinguish the attitude to risk from the attitude to ambiguity. Another advantage is that it is smooth and not a curve like many other models of ambiguity. The KMM model can analyses the behavior under different ambiguous attitudes, that is, it can track different behaviors. When the decision maker is neutral to the ambiguous attitudes, the KMM model is simplified to the original SEU model. After that, Klibanoff\cite{8} extended the KMM model so that it could handle cases across time periods.

3. Model construction

This paper uses the KMM model to solve the portfolio problem:

\[
V(f) = \int_\Delta \Phi \left( \int_S u(f) d\pi \right) d\mu = E_\mu \Phi( E_\pi u \circ f )
\]

Suppose that in frictionless financial market conditions, short selling is not allowed. Consider an investor with an initial wealth of \( W_0 \) over a defined investment period. The goal is to achieve the ultimate wealth utility \( U(W_t) \) in the \( t \) period to maximize the investment portfolio. There's a riskless asset and \( n \) risky assets, Let \( R_t \) be the return of riskless asset, \( R_t=(r_{1t}, r_{2t}, \ldots, r_{nt})' \) is the return vector of \( n \) risky assets in \( t \) time, \( x=(x_1, x_2, \ldots, x_n)' \) is the weight of \( n \) risk assets, the portfolio investment in \( t \) period of the total return of \( \bar{R}_t(x)=R_t+x_1'(R_1-R_t), x_1'(R_1-R_t) \), so the total wealth at the end of period \( t \) is going to be \( W_t=W_0\bar{R}_t \). For the convenience of calculation and observation, and do not break general, to simplify the \( W_0 \) is 1, so, the total wealth is \( W_t=\bar{R}_t+X_1'(R_1-R_t) \).

Therefore, the model is:
\[ \max_{x} E_{\mu} \Phi(E_{x}U(W_{t})) \]
\[ s.t. X_{t}^{i} I \leq 1 \]
\[ X_{t}^{i} \geq 0, i = 1, 2, ..., n \]

where, 1 is a unit vector of \( N \) dimensions. \( U(W_{t}) \) is a utility function that captures risk attitude, and \( \Phi(\cdot) \) is a continuous and increasing map that captures ambiguity attitudes.\(^9\)

This paper considers the portfolio problem when the utility function is exponential function (CARA case) and we from the perspective of expected utility maximization.

\[ U(W_{t}) = -e^{-aw_{t}}, a > 0 \]

where, \( a \) is used to measure risk aversion. Do not break in general, this article assumes that the distribution of returns is normal distribution \( N(\mu, \sigma^{2}) \), then the expected utility \( E_{x}U(W_{t}) \) can be transformed into:

\[ \int_{0}^{1} \left( -e^{-[\mu + \sigma(x_{t} - \mu)]} \right) f(x, \mu, \sigma^{2}) \ dx \]  

(5)

In traditional expected utility model, the distribution of returns is known, that is, \( f \) is prior distribution. The type of \( \mu, \sigma \) is palpable, but in actual life, the investor to invest in, for the future income distribution is not so sure, so \( \mu, \sigma \) actually as unknowns to calculate more persuasive. In order to calculate the convenient, only take \( \sigma \) for unknown in this paper. Thus, the expected utility of loss (that is, the first layer integral) after solving out is an expression with parameter \( \sigma \).

This paper assumes that the distribution of return rate of risk assets is all subject to normal distribution:

\[ f(x, \sigma) = \frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{(x-\mu)^{2}}{2\sigma^{2}}} \]  

(6)

And the rate of return of each asset is considered to be independent, so their joint distribution density function is:

\[ f(x, \sigma) = \frac{1}{(\sqrt{2\pi \sigma})^{n}} e^{-\frac{\sum_{i=1}^{n}(x_{i}-\mu_{i})^{2}}{2\sigma^{2}}} \]  

(7)

Therefore, \( E_{x}U(W_{t}) \) can be converted to:

\[ \int_{0}^{1} \left( -e^{-[\mu + \sigma(x_{t} - \mu)]} \right) \frac{1}{(\sqrt{2\pi \sigma})^{n}} e^{-\frac{\sum_{i=1}^{n}(x_{i}-\mu_{i})^{2}}{2\sigma^{2}}} \ dx \]

\[ = \frac{1}{(\sqrt{2\pi \sigma})^{n}} \int_{0}^{1} \left( -e^{-[\mu + \sigma(x_{t} - \mu)]} \right) e^{-\frac{\sum_{i=1}^{n}(x_{i}-\mu_{i})^{2}}{2\sigma^{2}}} \ dx \]  

(8)

After the investor’s expectation of utility function for loss is obtained, the type of \( \sigma \) as a parameter. And the expectation of utility functions for ambiguous is an expectation by \( \sigma \), therefore need to estimated the distribution of \( \sigma \). Due to the probability model about the \( \sigma \) is not known, this paper uses the nonparametric kernel density estimation to calculate the return estimate of its density function:

\[ \hat{p}(\sigma) = \frac{1}{Th} \sum_{i=1}^{T} k \left( \frac{\sigma - \sigma_{i}}{h} \right) \]  

(9)

where, \( \sigma_{i} = (\sigma_{1}, \sigma_{2}, ..., \sigma_{T}) \) is the sample data set of volatility, the kernel function \( k(\cdot) \) is a weight function, \( h \) is window width, is a parameter related to the sample size, The kernel function adopted is gaussian kernel function:
The method of selecting the optimal window width $h$ is Rule of thumb:

$$h \approx 1.06 \sigma T^{\frac{1}{4}}$$  \hspace{1cm} (11)

where, $\sigma$ is the standard deviation of the sample data. And using the following formula can be obtained volatility of sample data matrix:

$$\sigma_j = |R_j - R_{\mu}|
\quad R_{\mu} = \frac{1}{T} \sum_{i=1}^{T} R_j$$  \hspace{1cm} (12)

where, $R_j$ is the sample data of the return and $R_{\mu}$ is the average return of the JTH stock.

Assuming that $\sigma(T,n)$ is the sample data set of volatility of $n$ stocks in $T$ period, then the kernel density function of each stock is respectively:

$$\hat{p}(\sigma_j) = \frac{1}{T h_j} \sum_{i=1}^{n} k \left( \frac{\sigma_j - \sigma_y}{h_j} \right)$$  \hspace{1cm} (13)

As the $\sigma$ comes from the rate of return, the previous paper thinks that the rate of return of each stock is independent distribution, so the $\sigma$ of each stock is independent distribution, and its joint distribution is:

$$\hat{p}(\sigma) = \prod_{j=1}^{n} \hat{p}(\sigma_j)$$  \hspace{1cm} (14)

Plug this density into the expected utility function, and then:

$$\int_{\Delta} \Phi \left( \int_{\mathbb{R}} U(W) f(x,\sigma) dx \right) \hat{p}(\sigma) d\sigma
\quad = \int_{\Delta} \Phi \left( -\frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} \left( e^{-\frac{1}{2\sigma} \sum_{i=1}^{n} (x_i - \bar{x}_i)^2} \right) dx \right) \prod_{j=1}^{n} \left( \frac{1}{T h_j} \sum_{k=1}^{T} \left( \frac{\sigma_j - \sigma_y}{h_j} \right) \right) d\sigma$$  \hspace{1cm} (15)

Put the gaussian kernel $k(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2}$ into the above formula, to get:

$$\int_{\Delta} \Phi \left( \int_{\mathbb{R}} U(W) f(x,\sigma) dx \right) \hat{p}(\sigma) d\sigma
\quad = \frac{1}{T \sum_{j=1}^{n} h_j} \int_{\Delta} \Phi \left( -\frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} \left( e^{-\frac{1}{2\sigma} \sum_{i=1}^{n} (x_i - \bar{x}_i)^2} \right) dx \right) \prod_{j=1}^{n} \left( \frac{1}{T h_j} \sum_{k=1}^{T} \left( \frac{\sigma_j - \sigma_y}{h_j} \right) \right) d\sigma$$  \hspace{1cm} (16)
In this paper, $\Phi(\cdot)$ also take CRRA: $\Phi(x) = -e^{-bx}, b > 0$, then:

$$\int_\Delta \Phi \left( \int_\Delta U(W) f(x, \sigma) dx \right) \hat{\rho} (\sigma) d\sigma$$

$$= \frac{1}{(T \sqrt{2\pi})^n \prod_j h_j} \int_\Delta e^{-\frac{1}{2 \varnothing} \left( 1 - \frac{\sigma - \mu_i}{\sqrt{\sigma}} \right)^2} \prod_{j=1}^n \left( \sum_{i=1}^n e^{\frac{1}{2} \frac{\sigma - \mu_i}{\sqrt{\sigma}}} \right)^{\gamma_j} d\sigma \quad (17)$$

The model can be obtained by finishing:

$$\max \int_\Delta \Phi \left( \int_\Delta U(W) f(x, \sigma) dx \right) \hat{\rho} (\sigma) d\sigma$$

s.t. $\int_\Delta \Phi \left( \int_\Delta U(W) f(x, \sigma) dx \right) \hat{\rho} (\sigma) d\sigma$

$$= \frac{1}{(T \sqrt{2\pi})^n \prod_j h_j} \int_\Delta e^{-\frac{1}{2 \varnothing} \left( 1 - \frac{\sigma - \mu_i}{\sqrt{\sigma}} \right)^2} \prod_{j=1}^n \left( \sum_{i=1}^n e^{\frac{1}{2} \frac{\sigma - \mu_i}{\sqrt{\sigma}}} \right)^{\gamma_j} d\sigma \quad (18)$$

$$X^T \leq 1$$

$$x_i \geq 0, i = 0, 1, 2, ..., n$$

4. Conclusion

First, this paper introduces the development of the portfolio, and then discusses the modeling of decision analysis for optimal portfolio in the case of risk and ambiguity. Based on the classical vNM theoretical framework, the smooth ambiguity (KMM) model is studied. And the parameter significance of solving the portfolio with this model is determined.

There are always unknown parameters in the model of financial decision, and we need to estimate them. However, the distribution of some parameters is unknown, so this paper focuses on the part of ambiguity. It can be seen from this paper that part of the influence on investors' decision is from risk aversion, part from ambiguity aversion. By using the method of quadratic integration of smooth ambiguity (KMM) model, the risk and ambiguity can be distinguished. Through clear expression of ambiguity attitude, we can systematically evaluate the impact of decision makers' preference for ambiguity and risk. Then we can understand the ambiguous, and gain a quantitative understanding of the impact of ambiguity and our attitudes. With the goal of maximizing expected utility, the optimal portfolio is obtained, which combines the subjective expectation of decision maker with the actual data of the market. In the hope of providing investors with a feasible investment strategy and effective guidance in reality.

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