On Bayesian inference for the Extended Plackett-Luce model

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Abstract

The analysis of rank ordered data has a long history in the statistical literature across a diverse range of applications. In this paper we consider the Extended Plackett-Luce model that induces a flexible (discrete) distribution over permutations. The parameter space of this distribution is a combination of potentially high-dimensional discrete and continuous components and this presents challenges for parameter interpretability and also posterior computation. Particular emphasis is placed on the interpretation of the parameters in terms of observable quantities and we propose a general framework for preserving the mode of the prior predictive distribution. Posterior sampling is achieved using an effective simulation based approach that does not require imposing restrictions on the parameter space. Working in the Bayesian framework permits a natural representation of the posterior predictive distribution and we draw on this distribution to address the rank aggregation problem and also to identify potential lack of model fit. The flexibility of the Extended Plackett-Luce model along with the effectiveness of the proposed sampling scheme are demonstrated using several simulation studies and real data examples.

Keywords: Markov chain Monte Carlo; MC³; permutations; predictive inference; rank aggregation; rank ordered data.

1 Introduction

Rank ordered data arise in many areas of application and a wide range of models have been proposed for their analysis; for an overview see Marden (1995) and Alvo and Yu (2014). In this paper we focus on the Extended Plackett-Luce (EPL) model proposed by Mollica and Tardella (2014); this model is a flexible generalisation of the popular Plackett-Luce model (Luce, 1959; Plackett, 1975) for permutations. In the Plackett-Luce model, entity \( k \in \mathcal{K} = \{1, \ldots, K\} \) is
assigned parameter $\lambda_k > 0$, and the probability of observing the ordering $x = (x_1, x_2, \ldots, x_K)'$ (where $x_j$ denotes the entity ranked in position $j$) given the entity parameters $\lambda = (\lambda_1, \ldots, \lambda_K)'$ is

$$\Pr(X = x | \lambda) = \prod_{j=1}^{K} \frac{\lambda_{x_j}}{\sum_{m=j}^{K} \lambda_{x_m}}. \quad (1)$$

We refer to (1) as the standard Plackett-Luce probability. This probability is constructed via the so-called “forward ranking process” (Mollica and Tardella, 2014), that is, it is assumed that a rank ordering is formed by allocating entities from most to least preferred. This is a rather strong assumption. It is easy to imagine a scenario where an individual ranker might assign entities to positions/ranks in an alternative way. For example, it is quite plausible that rankers may find it easier to identify their most and least preferred entities first rather than those entities they place in the middle positions of their ranking (Mollica and Tardella, 2018).

In such a scenario rankers might form their rank ordering by first assigning their most and then least preferred entities to a rank before filling out the middle positions through a process of elimination using the remaining (unallocated) entities, that is, they use a different ranking process. The Extended Plackett-Luce model relaxes the assumption of a fixed and known ranking process.

It is somewhat natural to recast the underlying ranking process in terms of a “choice order” where the choice order is the order in which rankers assign entities to positions/ranks. For example, suppose a ranker must provide a preference ordering of $K$ entities; a choice order of $\sigma = (1, K, 2, 3, \ldots, K - 1)$ corresponds to the ranking process where the ranker first assigns their most preferred entity, then their least preferred entity before then assigning the remaining entities in rank order from second down. Note that the choice order $\sigma$ is simply a permutation of the ranks 1 to $K$.

Whilst the EPL model is motivated in terms of a choice order as described above, we find this justification is not always appropriate. For example, the notion of a choice order clearly does not apply in the analysis of the Formula 1 data in Section 6, where the data are simply the finishing orders of the drivers in each race. We prefer to view the EPL model as a flexible probabilistic model for rank ordered data; ultimately all such probabilistic models induce a discrete distribution $P_x$ over the set of all $K!$ permutations $S_K$ and we wish this distribution to provide a flexible model for the observed data.
We adopt a Bayesian approach to inference which we find particularly appealing and natural as we focus heavily on predictive inference for observable quantities. We also make three main contributions, as outlined below. When the number of entities is not small, choosing a suitable prior distribution for $\sigma$, the permutation of the ranks 1 to $K$, is a somewhat daunting task. We therefore propose to use the (standard) Plackett-Luce model to define the prior probability of each permutation, although we note that our inference framework is sufficiently general and does not rely on this choice. We also address the thorny issue of specifying informative prior beliefs about the entity parameters $\lambda$ by proposing a class of priors that preserve the modal prior predictive rank ordering under different choice orders $\sigma$. Constructing suitable posterior sampling schemes for the Extended Plackett-Luce model is challenging due to multi-modality of the marginal posterior distribution for $\sigma$, with local modes separated by large distances within permutation space. To the best of our knowledge, the only current solution is given by Mollica and Tardella (2018) but this relies on a restricted parameter space for $\sigma$. In this paper we appeal to Metropolis coupled Markov chain Monte Carlo (MC$^3$) to overcome the difficult sampling problem when the full parameter space for $\sigma$ is considered.

The remainder of the paper is structured as follows. In Section 2 we outline the Extended Plackett-Luce model and our associated notation, and in Section 2.2 we provide some guidance on interpreting the model parameters. In Section 3 we propose our Bayesian approach to inference. In particular we discuss suitable choices for the prior distribution and describe our simulation based scheme for posterior sampling. A simulation study illustrating the efficacy of the posterior sampling scheme and the performance of the EPL model over a range of values for the number of entities and number of observations is considered in Section 4; with further details also given in Section 2 of the supplementary materials. Section 5 outlines how we use the posterior predictive distribution for inference on observable quantities and for assessing the appropriateness of the model. Two real data analyses are considered in Section 6 to illustrate the use of the (unrestricted) EPL model. Section 7 offers some conclusions.
2 The Extended Plackett-Luce model

We now present the Extended Plackett-Luce model along with our associated notation and also discuss the interpretation of the model parameters in terms of the preferences of entities.

2.1 Model and notation

Recall that there are \( K \) entities to be ranked and that the collection of all entities is denoted \( \mathcal{K} = \{1, \ldots, K\} \). The Extended Plackett-Luce model is only well defined for complete rank orderings in which all entities are included. Thus a typical observation is \( x_i = (x_{i1}, \ldots, x_{iK}) \) where \( x_{ij} \) denotes the entity ranked in position \( j \) in the \( i \)th rank ordering.

The choice order is represented by \( \sigma = (\sigma_1, \ldots, \sigma_K) \), where \( \sigma_j \) denotes the rank allocated at the \( j \)th stage. Conditional on \( \sigma \), each entity has a corresponding parameter \( \lambda_k > 0 \) for \( k = 1, \ldots, K \); let \( \lambda = (\lambda_1, \ldots, \lambda_K)' \). Crucially, the meaning and interpretation of \( \lambda \) depends on \( \sigma \) and this is addressed shortly.

The probability of a particular rank ordering under the Extended Plackett-Luce model (Mollica and Tardella, 2014) is defined as

\[
\Pr(X_i = x_i|\lambda, \sigma) = \frac{\prod_{j=1}^{K} \lambda_{x_{i\sigma_j}}}{\sum_{m=j}^{K} \lambda_{x_{i\sigma_m}}}. \tag{2}
\]

Therefore, the Extended Plackett-Luce probability (2) is simply the standard Plackett-Luce probability (1) evaluated at “permuted data” \( x_i^* \) where \( x_{ij}^* = x_{i\sigma_j} \) for \( j = 1, \ldots, K \) with entity parameters \( \lambda \). Here \( x_{ij}^* \) denotes the entity chosen at the \( j \)th stage of the \( i \)th ranking process and therefore receiving rank \( \sigma_j \).

Indeed, both the (forward ranking) standard Plackett-Luce model and (backward ranking) reverse Plackett-Luce model are special cases of (2) and are recovered when \( \sigma = (1, \ldots, K) \equiv \mathcal{I} \), the identity permutation, and \( \sigma = (K, K-1, \ldots, 1) \), the reverse of the identity permutation, respectively. We use the notation \( X_i|\lambda, \sigma \sim \text{EPL}(\lambda, \sigma) \) to denote that the probability of rank ordering \( i \) is given by (2). Note that here and throughout we have adopted different notation from that in Mollica and Tardella (2014) and Mollica and Tardella (2018) but the essential components of the model remain unchanged.
It is clear that the EPL probability (2) is invariant to scalar multiplication of the entity parameters $\lambda$. This identifiability issue is not of great concern as the parameters can be normalised as required. However, the parameter identifiability issue can lead to potential mixing problems for MCMC algorithms and this is revisited in Section 3.3.

2.2 Interpretation of the entity parameters $\lambda$

A key aspect of analysing rank ordered data using Plackett-Luce type models is the interpretation of the entity parameters $\lambda$. Moreover, it is essential to understand the interpretation of the $\lambda$ parameters if one is to specify informative prior beliefs about the likely preferences of the entities.

For the Extended Plackett-Luce model, $\lambda_k$ is proportional to the probability that entity $k$ is selected at the first stage of the ranking process and therefore ranked in position $\sigma_1$ of the rank ordering $x$. Then, conditional on an entity being assigned to position $\sigma_1$ in the rank ordering, the entity with the largest parameter of those remaining is that most likely to be assigned to position $\sigma_2$, and so on. For the standard Plackett-Luce model, arising from the forward ranking process with $\sigma = (1, \ldots, K)$, we have that $\lambda_k$ is proportional to the probability that entity $k$ is assigned rank $\sigma_1 = 1$ (and is thus the most preferred entity), and so on. Therefore, for the standard Plackett-Luce model, entities with larger values are more likely to be given a higher rank. In other words, the $\lambda$ parameters for the standard Plackett-Luce model correspond directly with preferences for entities. A consequence is that ordering the entities in terms of their values in $\lambda$, from largest to smallest, will give the modal ordering $\hat{x}$, that is, the permutation of the entities which yields the maximum Plackett-Luce probability (1), given $\lambda$. Specifically, $\hat{x} = \text{order}_\downarrow(\lambda)$, where $\text{order}_\downarrow(\cdot)$ denotes the ordering operation from largest to smallest. This makes specifying a prior distribution for $\lambda$, when $\sigma = (1, \ldots, K)$, relatively straightforward based on entity preferences. The interpretation of the $\lambda$ parameters directly in terms of preferences can also be achieved in a straightforward manner with the backward ranking process ($\sigma = (K, \ldots, 1)$) of the reverse Plackett-Luce model. Apart from these special cases, however, the interpretation of the $\lambda$ parameters in terms of preferences is not at all transparent for other choices of $\sigma$. For example, suppose that $\lambda_i > \lambda_j$ and $\sigma = (2, 3, 1)$. Here
entity $i$ is more likely to be ranked in second position than entity $j$. Further, if another entity $\ell \neq i, j$, is assigned to rank 2 then entity $i$ is preferred for rank 3 ($\sigma_2$) over entity $j$.

Understanding the preference of the entities under the Extended Plackett-Luce model based on values of $\lambda$ and $\sigma$ can be made more straightforward if we first introduce the inverse of the choice order permutation $\sigma^{-1}$. This is defined such that $\sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = I$, the identity permutation, where $\circ$ denotes composition of permutations which, in terms of vectors, implies that if $z = x \circ y$ then $z_i = x_{y_i}$. Here the $j$th element of $\sigma^{-1}$ denotes the stage of the ranking process at which rank $j$ is assigned. We can then obtain directly the modal ordering of the entities under the EPL($\lambda, \sigma$) model, $\hat{x}(\sigma, \lambda)$, and thus obtain a representation of the preference of the entities. Here $\hat{x}(\sigma, \lambda)$ is obtained without enumerating any probabilities by permuting the entries in $\hat{x}$ (the modal ordering under the standard Plackett-Luce model conditional on $\lambda$), by $\sigma^{-1}$, that is $\hat{x}(\sigma, \lambda) = \hat{x} \circ \sigma^{-1}$. In other words, if $\hat{x} = \text{order}_i(\lambda)$ then $\hat{x}_j(\sigma, \lambda) = \hat{x}_{\sigma^{-1} j}$, where $\sigma^{-1} j$ denotes the $j$th element of $\sigma^{-1}$. Let $\hat{x}^{-1}$ represent the ranks assigned to the entities under the standard Plackett-Luce model; this is obtained as the inverse permutation corresponding to $\hat{x}$, that is, the permutation such that $\hat{x}^{-1} \circ \hat{x} = I$, the identity permutation. Now define $\eta(\sigma) = \hat{x}(\sigma, \lambda) \circ \hat{x}^{-1}$; this represents the permutation of the entities ranked under the EPL model at the stage corresponding to the rank assigned to entities 1 to $K$ under the standard Plackett-Luce model. It follows that, if $\lambda(\sigma)$ has $j$th element $\lambda_j(\sigma) = \lambda_{\eta_j(\sigma)}$, where $\eta_j(\sigma)$ is the $j$th element of $\eta(\sigma)$, then $\hat{x}(\sigma, \lambda(\sigma)) \equiv \hat{x}$ for all $\sigma \in S_K$, and the modal preference ordering is preserved.

Some simplification is possible if we first order the entities in terms of preferences. Clearly, if $\hat{x} = I$, the identity permutation, then $\hat{x}(\sigma, \lambda) = \sigma^{-1}$, and so the modal ordering is given by the inverse choice order permutation. Moreover, if $\hat{x} = I$ then $\hat{x}^{-1} = I$ and so $\eta(\sigma) = \sigma^{-1}$. It follows that choosing $\lambda(\sigma)$ such that its $j$th element is $\lambda_j(\sigma) = \lambda_{\sigma^{-1} j}$, then $\hat{x}(\sigma, \lambda(\sigma)) \equiv I$ for all $\sigma \in S_K$. Therefore if the entities are labelled in preference order then permuting the $\lambda$ parameters from the standard Plackett-Luce model by the inverse of the choice order permutation will preserve the modal permutation to be in the same preference order. This suggests a simple strategy for specifying prior distributions for the entity parameters which preserves modal preferences under different choice orders; we revisit this in Section 3.1.2.
3 Bayesian modelling

Suppose we have data consisting of \( n \) independent rank orderings, denoted \( \mathcal{D} = \{ x_1, x_2, \ldots, x_n \} \). The likelihood of \( \lambda, \sigma \) is

\[
\pi(D|\lambda, \sigma) = \prod_{i=1}^{n} \Pr(x_i|\lambda, \sigma) = \prod_{i=1}^{n} \prod_{j=1}^{K} \frac{\lambda_{x_i\sigma_j}}{\sum_{m=j}^{K} \lambda_{x_i\sigma_m}}.
\]

(3)

We wish to make inferences about the unknown quantities in the model \( \sigma, \lambda \) as well as future observable rank orderings \( x \). Specifically we adopt a Bayesian approach to inference in which we quantify our uncertainty about the unknown quantities (before observing the data) through a suitable prior distribution.

3.1 Prior specification

We adopt a joint prior distribution for \( \sigma \) and \( \lambda \) of the form \( \pi(\sigma, \lambda) = \pi(\lambda|\sigma)\pi(\sigma) \) which explicitly emphasizes the dependence of \( \lambda \) on \( \sigma \).

3.1.1 Prior for \( \sigma \)

For the choice ordering \( \sigma \) we need to define a discrete distribution \( P_\sigma \) over the \( K! \) elements of \( \mathcal{S}_K \). If \( K \) is not small, perhaps larger than 4, then this could be a rather daunting task. Given the choice order parameter \( \sigma \) is a permutation, or equivalently a complete rank ordering, one flexible option is to use the Plackett-Luce model to define the prior probabilities for each choice order parameter. More specifically we let \( \sigma|q \sim \text{PL}(q) \) where \( q = (q_1, \ldots, q_K)' \in \mathbb{R}_+^K \) are to be chosen \textit{a priori} and

\[
\Pr(\sigma|q) = \prod_{j=1}^{K} \frac{q_{\sigma_j}}{\sum_{m=j}^{K} q_{\sigma_m}}.
\]

If desired, it is straightforward to assume each choice order is equally likely \textit{a priori} by letting \( q_k = q \) for \( k = 1, \ldots, K \). Furthermore, the inference framework that follows is sufficiently general and does not rely on this prior choice. In particular, if we only wish to consider a
subset of all the possible choice orderings $\mathcal{R}$, for example the restricted space as in Mollica and Tardella (2014), then this can be achieved by making an appropriate choice of prior probabilities for all $\sigma \in \mathcal{R}$ and letting $\Pr(\sigma) = 0$ for all $\sigma \in S_K \setminus \mathcal{R}$. Alternatively, the Plackett-Luce prior is sufficiently flexible that it can mimic the main features of the restricted Mollica and Tardella (2018) prior by suitable choice of $q$ with $q_i = q_{K+1-i}$ and $q_i > q_{i+1}$, for $i < \lceil K/2 \rceil$.

3.1.2 Prior for $\lambda|\sigma$

It is natural to wish to specify prior beliefs in terms of preferences for the entities. However, we have seen in Section 2.2 that the interpretation of the entity parameters $\lambda$ in terms of preferences is dependent on the value of $\sigma$. It follows that specifying an informative prior for the entity parameters is problematic unless the choice order $\sigma$ is assumed to be known. We therefore consider separate prior distributions for $\lambda$ conditional on the value of $\sigma$. Since the entity parameters $\lambda_k > 0$ must be strictly positive, a suitable, relatively tractable, choice of conditional prior distribution is a gamma distribution with mean $a_k^{(\sigma)}/b_k^{(\sigma)}$, that is $\lambda_k|\sigma \overset{\text{indep}}{\sim} \text{Ga}(a_k^{(\sigma)}, b_k^{(\sigma)})$ for $k = 1, \ldots, K$ and $\sigma \in S_K$. Without loss of generality we set $b_k^{(\sigma)} = b = 1$, for all $k$ and $\sigma$ since $b$ is not likelihood identifiable. Our proposed strategy for specifying the hyper-parameters $a_k^{(\sigma)}$ is to first consider the prior distribution for $\lambda$ under the standard Plackett-Luce model with $\sigma = I$. If we specify $a = (a_1, \ldots, a_K)'$ then $\hat{x}$, the modal preference ordering from the prior predictive distribution, is $\hat{x} = \text{order}_I(a)$. Then in order to preserve the beliefs about the modal preference ordering over different values of $\sigma$ we can use the arguments of Section 2.2 to specify $a_k^{(\sigma)} = a_\eta_k^{(\sigma)}$ for $k = 1, \ldots, K$, where $\eta_k^{(\sigma)}$ is as defined in Section 2.2 with $\hat{x}$ now representing the modal preference ordering under the prior predictive distribution conditional on $\sigma = I$ (the standard Plackett-Luce model). The modal entity preferences will therefore be preserved under each value of $\sigma \in S_K$. As in Section 2.2, some simplification of notation is achievable if we first re-order the entities so that $\hat{x} = I$, in which case $a_k^{(\sigma)} = a_{\sigma_k^{-1}}$ for $k = 1, \ldots, K$. Clearly, letting $a_k = a$ for all $k$ induces a uniform prior predictive distribution over all preference orders (irrespective of the choice order $\sigma$). Such a prior represents the situation where we are unwilling to favour any particular preference ordering a priori.
3.2 Bayesian model

The complete Bayesian model is

\[ X_i | \lambda, \sigma \overset{\text{indep}}{\sim} \text{EPL}(\lambda, \sigma), \quad i = 1, \ldots, n, \]
\[ \lambda_k | \sigma, a \overset{\text{indep}}{\sim} \text{Ga}(\alpha_k, 1), \quad k = 1, \ldots, K, \]
\[ \sigma | q \overset{\text{indep}}{\sim} \text{PL}(q), \]

that is, we assume that our observations follow the distribution specified by the Extended Plackett-Luce model (2) and the prior distribution for \((\lambda, \sigma)\) is as described in Section 3.1.

The full joint density of all stochastic quantities in the model (with dependence on fixed hyper-parameters suppressed) is

\[ \pi(\sigma, \lambda, D) = \pi(D | \lambda, \sigma) \pi(\lambda | \sigma) \pi(\sigma). \]

From which we quantify our beliefs about \(\sigma\) and \(\lambda\) through their joint posterior density

\[ \pi(\sigma, \lambda | D) \propto \pi(D | \lambda, \sigma) \pi(\lambda | \sigma) \pi(\sigma) \]

which is obtained via Bayes’ Theorem. The posterior density \(\pi(\sigma, \lambda | D)\) is not available in closed form and so we use simulation-based methods to sample from the posterior distribution as described in the next section.

3.3 Posterior sampling

Due to the complex nature of the posterior distribution we use Markov chain Monte Carlo (MCMC) methods in order to sample realisations from \(\pi(\sigma, \lambda | D)\). The structure of the model lends itself naturally to consider sampling alternately from two blocks of full conditional distributions: \(\pi(\sigma | \lambda, D)\) and \(\pi(\lambda | \sigma, D)\).

3.3.1 Sampling the choice order parameter \(\sigma\) from \(\pi(\sigma | \lambda, D)\)

Given the choice order parameter \(\sigma\) is a member of \(S_K\) it is fairly straightforward to obtain its (discrete) full conditional distribution; specifically this is the discrete distribution with
probabilities

\[ \Pr(\sigma = \sigma_j | \lambda, D) \propto \pi(D | \lambda, \sigma = \sigma_j) \pi(\lambda | \sigma = \sigma_j) \Pr(\sigma = \sigma_j) \]

for \( j = 1, \ldots, K! \). Clearly sampling from this full conditional will require \( K! \) evaluations of the EPL likelihood \( \pi(D | \lambda, \sigma = \sigma_j) \) and so sampling from \( \Pr(\sigma = \sigma_j | \lambda, D) \) for \( j = 1, \ldots, K! \) (a Gibbs update) is probably only plausible if \( K \) is sufficiently small; perhaps not much greater than 5. Of course, the probabilities \( \Pr(\sigma = \sigma_i | \lambda, D) \) and \( \Pr(\sigma = \sigma_j | \lambda, D) \) are conditionally independent for \( i \neq j \) and so could be computed in parallel which may facilitate this approach for slightly larger values of \( K \).

So as to free ourselves from the restriction to the case where \( K \) is small we instead consider a more general sampling strategy by constructing a Metropolis-Hastings proposal mechanism for updating \( \sigma \). Our investigation into the likelihood of the Extended Plackett-Luce model given different choice orders in Section 3 of the supplementary material revealed that \( \pi(D | \lambda, \sigma) \) is likely to be multi-modal. Further, local modes can be separated by large distances within permutation space. In an attempt to effectively explore this large discrete space we consider 5 alternative proposal mechanisms; each of which occurs with probability \( p_\ell \) for \( \ell = 1, \ldots, 5 \). The 5 mechanisms to construct the proposed permutation \( \sigma^\dagger \) are as follows.

1. The random swap: sample two positions \( \phi_1, \phi_2 \in \{1, \ldots, K\} \) uniformly at random and let the proposed choice order \( \sigma^\dagger \) be the current choice order \( \sigma \) where the elements in positions \( \phi_1 \) and \( \phi_2 \) have been swapped.

2. The Poisson swap: sample \( \phi_1 \in \{1, \ldots, K\} \) uniformly at random and let \( \phi_2 = \phi_1 + m \) where \( m = (-1)^\tau f, \tau \sim \text{Bern}(0.5) \) and \( f \sim \text{Po}(t) \). Note that \( t \) is a tuning parameter and \( \phi_2 \rightarrow \{ (\phi_2 - 1) \mod K \} + 1 \) as appropriate. Again the proposed choice order \( \sigma^\dagger \) is formed by swapping the elements in positions \( \phi_1 \) and \( \phi_2 \) of the current choice order \( \sigma \).

3. The random insertion (Bezáková et al., 2006): sample two positions \( \phi_1 \neq \phi_2 \in \{1, \ldots, K\} \) uniformly at random and let the proposed choice order \( \sigma^\dagger \) be formed by taking the value in position \( \phi_1 \) and inserting it back into the permutation so that it is instead in position \( \phi_2 \).
4. The prior proposal: here $\sigma^\dagger$ is simply an independent draw from the prior distribution, that is, $\sigma^\dagger|q \sim \text{PL}(q)$.

5. The reverse proposal: here $\sigma^\dagger$ is defined to be the reverse ordering of the current permutation $\sigma$, that is, $\sigma^\dagger = \sigma_{K:1} = (\sigma_K, \ldots, \sigma_1)$.

Note that performing either of the swap or insertion moves (1–3) above may result in slow exploration of the set of all permutations as the proposal $\sigma^\dagger$ may not differ much from the current value $\sigma$. To alleviate this potential issue we propose to iteratively perform each of these moves $S$ times, where $S$ is to be chosen (and fixed) by the analyst. More formally (when using proposal mechanisms 1–3) we construct intermediate proposals $\sigma^\dagger_s$ from the “current” choice order $\sigma^\dagger_{s-1}$ for $s = 1, \ldots, S$. Here $\sigma^\dagger_0 = \sigma$ and the proposed value for which we evaluate the acceptance probability is $\sigma^\dagger = \sigma^\dagger_S$. Further, for moves 1 and 2 it may seem inefficient to allow for the “null swap” $\phi_1 = \phi_2$, however this is done to avoid only proposing permutations with the same (or opposing) parity as the current value. Put another way, as $S \to \infty$ we would expect $\Pr(\sigma^\dagger|\sigma) > 0$ for all $\sigma^\dagger$, $\sigma \in S_K$ and this only holds if we allow for the possibility that $\phi_1 = \phi_2$. Finally we note that each of these proposal mechanisms is “simple” in the respect that $\Pr(\sigma^\dagger|\sigma) = \Pr(\sigma|\sigma^\dagger)$ and so the proposal ratio cancels in each case. The full acceptance ratio is presented within the algorithm outline in Section 3.4.

3.3.2 Sampling the entity parameters $\lambda$ from $\pi(\lambda|\sigma, D)$

Bayesian inference for variants of standard Plackett-Luce models typically proceeds by first introducing appropriate versions of the latent variables proposed by Caron and Doucet (2012), which in turn facilitate a Gibbs update for each of the entity parameters (assuming independent Gamma prior distributions are chosen). However we found that this strategy does not work well for entity parameter inference under the Extended Plackett-Luce model (not reported here). We therefore propose to use a Metropolis-Hastings step for sampling the entity parameters, specifically we use (independent) log normal random walks for each entity parameter in turn and so the proposed value is $\lambda_k^\dagger \sim \text{LN}(\log\lambda_k, \sigma_k^2)$ for $k = 1, \ldots, K$. We also implement a rescaling step in the MCMC scheme, analogous to that in Caron and Doucet (2012), in order to mitigate the poor mixing that is caused by the invariance of the Extended Plackett-Luce
likelihood to scalar multiplication of the $\lambda$ parameters. Full details are given in Section 3.4.

### 3.3.3 Metropolis coupled Markov chain Monte Carlo

Unfortunately the sampling strategy described above in Sections 3.3.1 and 3.3.2 proves ineffective when $K$ is not small, with the Markov chain suffering from poor mixing, particularly for $\sigma$ where the chain is prone to becoming stuck in local modes (results not reported here). In an attempt to resolve these issues, and therefore aid the exploration of the posterior distribution we appeal to Metropolis coupled Markov chain Monte Carlo, or *parallel tempering*.

Metropolis coupled Markov chain Monte Carlo (Geyer, 1991), is a sampling technique that aims to improve the mixing of Markov chains in comparison to standard MCMC methods particularly when the target distribution is multi-modal (Gilks and Roberts, 1996; Brooks, 1998). The basic premise is to consider $C$ chains evolving simultaneously, each of which targets a tempered posterior distribution $\tilde{\pi}_c(\theta_c|D) \propto \pi(x|\theta_c)^{1/T_c} \pi(\theta_c)$, where $T_c \geq 1$ is the temperature of chain $c$, and $\theta_c = \{\sigma_c, \lambda_c\}$. Note that the posterior of interest is recovered when $T_c = 1$. Further note that we have only considered a tempered likelihood component as we suggest that any prior beliefs should be consistent irrespective of the model choice. Now, as the posteriors $\tilde{\pi}_c(\theta_c|D)$ are conditionally independent given $D$, we can consider them to be targeting the joint posterior

$$\pi(\theta_1, \ldots, \theta_C|D) = \prod_{c=1}^{C} \tilde{\pi}_c(\theta_c|D).$$

(4)

Suppose now we propose to swap $\theta_i$ and $\theta_j$ for some $i \neq j$ within a Markov chain targeting the joint posterior (4). If we let $\theta = (\theta_1, \ldots, \theta_C)$ denote the current state and $\theta^\dagger = (\theta_1^\dagger, \ldots, \theta_C^\dagger)$ the proposed state where $\theta_i^\dagger = \theta_j$, $\theta_j^\dagger = \theta_i$ and $\theta_\ell^\dagger = \theta_\ell$ for $\ell \neq i, j$. Then, assuming a symmetric proposal mechanism, the acceptance probability of the state space swap is $\min(1, A)$ where

$$A = \frac{\pi(D|\theta_j)^{1/T_i} \pi(D|\theta_i)^{1/T_j}}{\pi(D|\theta_i)^{1/T_i} \pi(D|\theta_j)^{1/T_j}}.$$  

Of course, if the proposal mechanism is not symmetric then the probability $A$ must be multiplied by the proposal ratio $q(\theta|\theta^\dagger)/q(\theta^\dagger|\theta)$. Further, it is straightforward to generalise the above acceptance probability to allow the states of more than 2 chains to be swapped. However, this is typically avoided as such a proposal can result in poor acceptance rates. Our specific Metropolis coupled Markov chain Monte Carlo algorithm is outlined in the next section.
3.4 Outline of the posterior sampling algorithm

A parallel Metropolis coupled Markov chain Monte Carlo algorithm to sample from the joint posterior distribution of the skill parameters $\lambda$ and the choice order parameter $\sigma$ is as follows.

1. Tune:
   - choose the number of chains ($C$); let $T_1 = 1$ and choose $T_c > 1$ for $c = 2, \ldots, C$
   - choose appropriate values for the MH proposals outlined in Sections 3.3.1 and 3.3.2

2. Initialise: take a prior draw or alternatively choose $\sigma_c \in S_K$ and $\lambda_c \in \mathbb{R}^K_{>0}$ for $c = 1, \ldots, C$

3. For $c = 1, \ldots, C$ perform (in parallel) the following steps:
   - For $k = 1, \ldots, K$
     - draw $\lambda_{ck}^\dagger | \lambda_{ck} \sim \text{LN}(\log \lambda_{ck}, \sigma_{ck}^2)$
     - let $\lambda_{ck} \rightarrow \lambda_{ck}^\dagger$ with probability $\min(1, A)$ where
       $$A = \left\{ \frac{\pi(D|\lambda_{c,-k}, \lambda_{ck} = \lambda_{ck}^\dagger, \sigma_c)}{\pi(D|\lambda_c, \sigma_c)} \right\}^{1/T_c} \times \left( \frac{\lambda_{ck}^\dagger}{\lambda_{ck}} \right)^{a_k(\sigma)} e^{(\lambda_{ck} - \lambda_{ck}^\dagger)}$$
     - Sample $\ell$ from the discrete distribution with probabilities $\Pr(\ell = i) = p_{i,c}$ for $i = 1, \ldots, 5$
       - propose $\sigma_c^\dagger$ using proposal mechanism $\ell$
       - let $\sigma_c \rightarrow \sigma_c^\dagger$ with probability $\min(1, A)$ where
         $$A = \left\{ \frac{\pi(D|\lambda_c, \sigma_c^\dagger)}{\pi(D|\lambda_c, \sigma_c)} \right\}^{1/T_c} \times \frac{\pi(\lambda_c | \sigma_c^\dagger) \Pr(\sigma_c^\dagger)}{\pi(\lambda_c | \sigma_c) \Pr(\sigma_c)}$$
     - Rescale
       - sample $\Lambda_c^\dagger \sim \text{Ga} \left( \sum_{k=1}^K a_k(\sigma), 1 \right)$
       - calculate $\Sigma_c = \sum_{k=1}^K \lambda_{ck}$.
       - let $\lambda_{ck} \rightarrow \lambda_{ck} \Lambda_c^\dagger/\Sigma_c$ for $k = 1, \ldots, K$. 

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4. Sample a pair of chain labels \((i, j)\) where \(1 \leq i \neq j \leq C\)

- let \((\lambda_i, \sigma_i) \to (\lambda_j, \sigma_j)\) and \((\lambda_j, \sigma_j) \to (\lambda_i, \sigma_i)\) with probability \(\min(1, A)\) where

\[
A = \frac{\pi(D | \lambda_j, \sigma_j)^{1/T_i} \pi(D | \lambda_i, \sigma_i)^{1/T_j}}{\pi(D | \lambda_i, \sigma_i)^{1/T_i} \pi(D | \lambda_j, \sigma_j)^{1/T_j}}
\]

5. Return to Step 3.

3.4.1 Tuning the MC\(^3\) algorithm

The Metropolis coupled Markov chain Monte Carlo scheme targets the joint density (4) by simultaneously evolving \(C\) chains; each of which targets an alternative (tempered) density \(\tilde{\pi}_c(\theta_c | D)\). Given the data \(D\), these chains are conditionally independent and should therefore be individually tuned to target their respective density in a typical fashion. Of course, it may not be possible to obtain near optimal acceptance rates within the posterior chain (and other chains with temperatures \(\approx 1\)) however the analyst should aim to ensure reasonable acceptance rates; even if this results in small moves around the parameter space. Tuning the between chain proposal (Step 4 of the MC\(^3\) algorithm) can be tricky in general. The strategy we suggest, also advocated by Wilkinson (2013), is that where the temperatures are chosen such that they exhibit geometric spacing, that is, \(T_{c+1} / T_c = r\) for some \(r > 1\); this eliminates the burden of specifying \(C - 1\) temperatures and instead only requires a choice of \(r\). We also suggest only considering swaps between adjacent chains as intuitively the target densities are most similar when \(|T_c - T_{c+1}|\) is small. It is generally accepted that between chain acceptance rates of around 20% to 60% provide reasonable mixing (with respect to the joint density of \(\theta_1, \ldots, \theta_C\)); see, for example, Geyer and Thompson (1995); Altekar et al. (2004). A suitable choice of the temperature ratio \(r\) can be guided via pilot runs of the MC\(^3\) scheme and individual temperatures can also be adjusted as appropriate.

4 Simulation study

To investigate the performance of the posterior sampling algorithm outlined in Section 3.4 we apply it on several synthetic datasets. We consider \(K \in \{5, 10, 15, 20\}\) entities and generate 500
rank orderings for each choice of $K$. Further we subset each of these datasets by taking the first $n \in \{20, 50, 200, 500\}$ orderings thus giving rise to 16 (nested) datasets. The parameter values $(\lambda', \sigma')$ from which these data are generated are drawn from the prior distribution outlined in Section 3.1 with $a_k = q_k = 1$ for $k = 1, \ldots, K$. That is, all choice orders and entity preferences (specified by the $(\lambda', \sigma')$ pair) are equally likely. The values of the parameters for each choice of $K$ are given in Section 2 of the supplementary materials. For each dataset, posterior samples were obtained via the algorithm outlined in Section 3.4. We choose to use $C = 5$ chains in each case, with both the temperatures and tuning parameters chosen appropriately. The raw posterior draws are also thinned to obtain (approximately) $10K$ un-autocorrelated draws from the posterior distribution. Note that standard MCMC diagnostics were applied to the (continuous) parameters $\lambda$ and also the (log) observed data likelihood (3). To alleviate potential concerns about the sampling of the discrete choice order parameter ($\sigma$) we checked that the marginal posterior distribution $\pi(\sigma|D)$ was consistent under multiple runs of our algorithm.

Table 1 shows the posterior probability $Pr(\sigma'|D)$ of the choice order parameter used to generate each respective dataset. Perhaps unsurprisingly we see that for each $K \in \{5, 10, 15, 20\}$ the posterior support for the choice order parameter used to generate the data increases with the number of observations (rank orderings) considered, that is, $Pr(\sigma'|D) \rightarrow 1$ as $n \rightarrow \infty$. Interestingly we observe reasonable posterior support for $\sigma'$ when only considering $n = 20$ preference orders of $K = 10$ entities. However for some of the analyses, those where $n$ is

| $K$ | $Pr(\sigma'|D)$ | $\frac{1}{K} \sum_k [E(\log \lambda_k|D) - \log \lambda_k]^2$ |
|-----|----------------|--------------------------------------------------|
| 5   | 0.294*         | 0.045                                           |
| 10  | 0.156*         | 0.276                                           |
| 15  | 0.000          | 0.000                                           |
| 20  | 0.000          | 0.000                                           |

Table 1: Posterior probability $Pr(\sigma'|D)$ of the choice order used to generate each dataset along with mean squared error between the (log) values $\lambda'$ used to generate the data and the posterior expectation of the (log) entity parameters conditional on $\sigma'$. * indicates that $\sigma'$ is also the (posterior) modal observed choice order.
Figure 1: Synthetic data: heat maps of \( \Pr(\sigma_j = k|D) \) for those analyses where \( \sigma' \) was not observed; the crosses highlight \( \Pr(\sigma_j = \sigma'_j|D) \) in each case.

relatively small in comparison to \( K \), the choice order \( \sigma' \) is not observed in any of the 10K posterior draws. Further inspection of the marginal posterior (of \( \sigma \)) for these analyses reveals that there is a large amount of uncertainty on the choice order parameter. That said, the posterior draws of \( \sigma \) are reasonably consistent with the \( \sigma' \) used to generate the respective datasets; this can be seen by considering the marginal posterior distribution for each stage in the ranking process, that is, \( \Pr(\sigma_j = k|D) \) for \( j, k \in \{1, \ldots, K\} \). Figure 1 shows heat maps of \( \Pr(\sigma_j = k|D) \) for those analyses where \( \sigma' \) was not observed; the crosses highlight \( \Pr(\sigma_j = \sigma'_j|D) \) in each case. These figures reveal that, even with limited information, we are able to learn the lower entries in \( \sigma \) fairly well and much of the uncertainty resides within the first few stages of the ranking process. Section 2.2 of the supplementary materials presents the \( \Pr(\sigma_j = k|D) \) from Figure 1 in tabular form along with the image plots for the remaining analyses.

For the Extended Plackett-Luce model we are not only trying to quantify our uncertainty about the choice order parameter but also about the entity parameters. As discussed in Section 2 the entity parameter values \( \lambda \) only have a meaningful interpretation for a given choice order parameter \( \sigma \). That said, the values of the entity parameters are of little interest here and so we instead consider the mean squared error between the (log) values \( \lambda' \) used to generate the data and the posterior expectation of the (log) entity parameters (conditional on the \( \sigma' \) used to generate the data). Table 1 therefore shows \( \frac{1}{K} \sum_k \left[ \mathbb{E}_{\lambda_k, \sigma = \sigma'|D}(\log \lambda_k|D) - \log \lambda'_k \right]^2 \) from which we see that, in general, the inferred posterior means agree with the values \( \lambda' \) used the
generate the data. Of course, there is also uncertainty on these parameters; Section 2 of the supplementary materials contains boxplots of the marginal posterior distributions of $\log \lambda$ and these show that there is reasonable posterior support for $\lambda'$, even when $n$ is small relative to $K$. Naturally we cannot obtain $E_{\lambda, \sigma = \sigma' \mid D}(\log \lambda)$ for those analyses where $\sigma'$ is not observed. However, although prohibitive for $\lambda$ inference, this does not prohibit inferences on observable quantities (rank orders) as this is achieved via the posterior predictive distribution; this is the topic of the next section.

5 Inference and model assessment via the posterior predictive distribution

In this section we consider methods for performing inference for the entities by appealing to the posterior predictive distribution which will also provide us with a mechanism for detecting lack of model fit. We also outline methods for obtaining the mode of the posterior predictive distribution when the number of entities is large. By definition the modal ranking (from the posterior predictive distribution) is that which is most likely given the data and so this can be thought of as the *aggregate* ranking (Johnson et al., 2020) from a rank aggregation perspective if desired.

5.1 Inference for entity preferences

The Extended Plackett-Luce model is only defined for complete rankings and so the posterior predictive distribution is a discrete distribution defined over all possible observations $\tilde{x} \in S_K$. It is straightforward to approximate these probabilities by taking the expectation of the EPL probability (2) over the posterior distribution for $\lambda, \sigma$. Specifically the posterior predictive probability of any observation $\tilde{x}$ is $Pr(\tilde{x} \mid D) \simeq E_{\lambda, \sigma \mid D}[Pr(\tilde{x} \mid \lambda, \sigma)]$ where the approximation is exact in the limit of infinite posterior samples, and where $Pr(\tilde{x} \mid \lambda, \sigma)$ is given in Equation (2). It follows that, in principle, we can obtain the full posterior predictive distribution by simply computing $Pr(\tilde{x} \mid D)$ for each of the $K!$ possible observations $\tilde{x} \in S_K$. We can then use this distribution, for example, to obtain the marginal posterior predictive probability that entity $k$
is ranked in position \( j \), that is, \( \Pr(\tilde{x}_j = k|\mathcal{D}) \) for \( j, k \in \{1, \ldots, K\} \). Further, the modal ordering \( \tilde{x} \) is also straightforward to obtain and is simply that which has largest posterior predictive probability. However, when the number of entities is larger than say 9, this procedure involves enumerating the predictive probabilities for more than \( \mathcal{O}(10^6) \) possible observations. Clearly this becomes computationally infeasible as the number of entities increases; particularly as computing the posterior predictive probability also involves taking the expectation over many thousands of posterior draws. When the number of entities renders full enumeration infeasible we suggest approximating the posterior predictive distribution via a Monte Carlo based approach as in Johnson et al. (2020). In particular we obtain a collection \( P = \{\tilde{x}^{(m)}_\ell\}_{m=1,\ell=1}^{M,L} \) of draws from the posterior predictive distribution by sampling \( L \) rank orderings at each iteration of the \( M \) iterations of the posterior sampling scheme. We can then approximate \( \Pr(\tilde{x}_j = k|\mathcal{D}) \) by the empirical probability computed from the collection of rankings \( P \), that is \( \hat{\Pr}(\tilde{x}_j = k|\mathcal{D}) = \frac{1}{ML} \sum_{m=1}^{M} \sum_{\ell=1}^{L} I(\tilde{x}^{(m)}_{\ell j} = k) \), where \( I(x) \) denotes an indicator function which returns 1 if \( x \) is true and 0 otherwise. Finally, in order to find the mode of the posterior predictive distribution we propose using an efficient optimisation algorithm based on cyclic coordinate ascent; full details are provided in Johnson et al. (2020).

5.2 Model assessment via posterior predictive checks

In the Bayesian framework assessment of model fit to the data can be provided by comparing observed quantities with potential future observations through the posterior predictive distribution; the basic idea being that the observed data \( \mathcal{D} \) should appear to be a plausible realisation from the posterior predictive distribution. This approach to Bayesian goodness of fit dates back at least to Guttman (1967) and is described in detail in Gelman et al. (2013), for example. Several methods for assessing goodness of fit for models of rank ordered data were proposed in Cohen and Mallows (1983) and more recently similar methods have been developed in a Bayesian framework by, amongst others, Yao and Böckenholt (1999), Mollica and Tardella (2017), Johnson et al. (2020) and, specifically for the Extended Plackett Luce model, by Mollica and Tardella (2018). In the illustrative examples on real data in Section 6 we propose a range of diagnostics tailored to the specific examples. For example, one generic method for diagnosing lack of model fit is to monitor the (absolute value of the) discrepancy be-
tween the marginal posterior predictive probabilities of entities taking particular ranks with the corresponding empirical probabilities computed from the observed data. That is, we consider
\[ d_{jk} = | \Pr(\tilde{x}_j = k|D) - \Pr(x_j = k) | \]
where \( \Pr(x_j = k) = \frac{1}{n} \sum_{i=1}^{n} I(x_{ij} = k) \) is computed from those \( x \in D \) and the posterior predictive probabilities \( \Pr(\tilde{x}_j = k|D) \) are computed as described in Section 5.1. These discrepancies \( d_{jk} \) for \( j, k \in \{1, \ldots, K\} \) can then be depicted as a heat map where large values could indicate potential lack of model fit. By focusing on the marginal probabilities \( \Pr(x_j = k) \) we obtain a broad-scale “first-order” check on the model, but, as described in Cohen and Mallows (1983), we could also look at finer-scale features such as pairwise comparisons, triples and so on. Of course, if the full posterior predictive distribution over all \( K! \) possible observations is available (that is, if \( K \) is small) then we could compare the empirical distribution with the posterior predictive distribution directly; this is considered in the example in Section 6.1.

6 Illustrative examples

We now summarise analyses of two real datasets which together highlight how valuable insights can be obtained by considering the Extended Plackett-Luce model as opposed to simpler alternatives. Our conclusions are compared to those obtained under a standard Plackett-Luce analysis; here posterior samples are obtained using the Gibbs sampling scheme of Caron and Doucet (2012).

6.1 Song data

For our first example we consider a dataset with a long standing in the literature that was first presented in Critchlow et al. (1991). The original dataset was formed by asking ninety-eight students to rank \( K = 5 \) words, (1) score, (2) instrument, (3) solo, (4) benediction and (5) suit, according to the association with the target word “song”. However, the available data given in Critchlow et al. (1991) is in grouped format and the ranking of 15 students are unknown and hence discarded. The resulting dataset therefore comprises \( n = 83 \) ranking orderings and is reproduced in the supplementary materials.
Posterior samples are obtained via the algorithm outlined in Section 3.4 where the prior specification is as in Section 3.1 with \( q_k = 1 \) and \( a_k = 1 \) (for \( k = 1, \ldots, K \)) and so all choice and preference orderings are equally likely \textit{a priori}. The following results are based on a typical run of our (appropriately tuned) MC\(^3\) scheme initialised from the prior, with appropriate burn-in and thin to obtain 10K (almost) un-autocorrelated realisations from the posterior distribution.

As in the simulation studies we check that \( \pi(\sigma|D) \) is consistent under multiple runs of our algorithm and also use standard MCMC diagnostics on the \( \lambda \) parameters and the (log) observed data likelihood (3). The algorithm runs fairly quickly, with C code on a five threads of an Intel Core i7-4790S CPU (3.20GHz clock speed) taking around 18 seconds.

Investigation of the posterior distribution reveals there is no support for the standard (or reverse) Plackett-Luce model(s) with \( \Pr(\sigma = (3, 2, 1, 4, 5)|D) = 0.9918 \), \( \Pr(\sigma = (5, 4, 1, 2, 3)|D) = 0.0080 \) and the remaining posterior mass (0.0002) assigned to \( \sigma = (2, 3, 1, 4, 5) \). It is interesting to see that, although it receives relatively little posterior support, the 2nd most likely choice order parameter value is that given by reversing the elements of the posterior modal value. It is also worth noting that the posterior modal choice order \( (\sigma = (3, 2, 1, 4, 5)) \) is not contained within the restricted set considered by Mollica and Tardella (2018); this perhaps explains their conclusion that the (constrained) extended Plackett-Luce model performs poorly for these data. With this in mind we now also question whether the additional complexity of the Extended Plackett-Luce model allows us to better describe the data. Put another way, does the EPL model give rise to improved model fit. To this extent we investigate the (full) posterior predictive distribution; where the predictive probabilities for each possible future observation

![Figure 2: Song data: Full posterior predictive distribution for each of the 5! = 120 possible observations \( \bar{x} \in S_K \) under the EPL (left) and SPL (right) analyses. Crosses (×) highlight the probabilities that correspond to observations within the dataset, that is, those \( \bar{x} \in D \).](image-url)

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\( \bar{x} \in S_K \) are computed from the MCMC draws as described in Section 5. For comparative purposes we also compute the predictive distribution obtained from under a standard Plackett-Luce analysis of these data; Figure 2 shows the posterior predictive distribution under the extended (left) and standard (right) Plackett-Luce analyses. The crosses (×) highlight the probabilities that correspond to observations within the dataset (those \( \bar{x} \in D \)), and visual inspection clearly suggests that the observed data look more plausible under the EPL when compared to the SPL. To further support this conclusion we consider the discrepancies \( d_{jk} = | \Pr(\bar{x}_j = k|D) - \Pr(x_j = k)| \) as described in Section 5.2; Figure 3 shows these values as a heat map for \( j, k \in \{1, \ldots, K\} \). Note that the predictive probabilities \( \Pr(\bar{x}_j = k|D) \) are computed based on synthetic data simulated from the predictive distribution as discussed in Section 5 with \( L = 10 \). Again these figures suggest the EPL model describes the data much better than the standard Plackett-Luce model. In particular, there is a rather large discrepancy (0.34) between the predictive and empirical probabilities that entity \( k = 1 \) (Score) is ranked in position \( j = 3 \) under the SPL analysis.

Turning now to inference (for observable quantities) we again appeal to the posterior predictive distribution. More specifically we can now use the (predictive) probabilities \( \Pr(\bar{x}_j = k|D) \) to deduce the likely positions of entities within rankings. Figure 4 shows these probabilities as a heat map for \( j, k \in \{1, \ldots, K\} \). Focusing on the Extended Plackett-Luce analysis, it is fairly clear that “Suit” (5) is the least preferred entity and “Benediction” (4) is the 4th most

Figure 3: Song data: heat maps showing \( d_{jk} = | \Pr(\bar{x}_j = k|D) - \Pr(x_j = k)| \) for the EPL analysis (left) and the SPL analysis (right). Probabilities based on \( L = 10 \) draws from the predictive distribution per MCMC iteration.
Figure 4: Song data: heat maps showing $\Pr(\tilde{x}_j = k|\mathcal{D})$ for the EPL analysis (left) and the SPL analysis (right). Probabilities based on $L = 10$ draws from the predictive distribution per MCMC iteration.

preferred, with relatively little (predictive) support for any other entities in these positions. There is perhaps more uncertainty on those entities that are ranked within positions $j = 1, 2, 3$, although the figure would suggest that the preference of the entities is (Solo, Instrument, Score, Benediction, Suit). Indeed this is the modal predictive ranking and has predictive probability 0.232. Interestingly there appears to be much more uncertainty, particularly for the top 3 entities, under the SPL analysis; further the modal (predictive) ranking is (Instrument, Solo, Score, Benediction, Suit) and occurs within probability 0.122.

6.2 F1 data

We now analyse a dataset containing the finishing orders of drivers within the 2018/19 Formula 1 (F1) season and so we have $n = 21$ rank orderings of the $K = 20$ drivers. It will be interesting to see whether we are able to gain more valuable insights using the EPL model when compared to the standard PL model. In particular whether we are able to gain any information about the choice order parameter $\sigma$ in this setting as $K$ is fairly large, relative to $n$. The rank orderings considered here were collected from www.espn.co.uk and also reproduced in the supplementary materials.

Numerous variants of the Plackett-Luce model have previously been developed for the analysis of F1 finishing orders; see Henderson and Kirrane (2018) and the discussion therein. In general,
models derived from the reverse Plackett-Luce (RPL) model appear to perform better than the standard Plackett-Luce model in the sense that they give rise to better model fit. We choose to incorporate this prior information by letting \( q = (1, \ldots, K) \) and so \textit{a priori} the modal choice ordering is \( \hat{\sigma} = (K, \ldots, 1) \), that is, the choice ordering corresponding to the reverse Plackett-Luce model. We also take \( a_k = 1 \) (for \( k = 1, \ldots, K \)) and so, although we provide information about the likely choice ordering, each rank ordering remains equally likely under this prior specification. For completeness we also consider an analysis with \( q = a = 1 \) and note that the posterior distribution is not particularly sensitive to this choice. Put another way, these data are rather informative about the choice order parameter \( \sigma \) which is perhaps unsurprising given what we have seen from the simulation studies in Section 4. The following results are based on a typical run of our (appropriately tuned) MC\(^3\) scheme initialised from the prior, with appropriate burn-in and thin to obtain \( 10K \) (almost) un-autocorrelated realisations from the posterior distribution. Again we check that \( \pi(\sigma|D) \) is consistent under multiple runs of our algorithm and also use standard MCMC diagnostics on the \( \lambda \) parameters and the (log) observed data likelihood (3). This analysis takes around 21 minutes using C code on five threads of an Intel Core i7-4790S CPU (3.20GHz clock speed).

Investigation of the posterior distribution reveals that there is a large amount of uncertainty on the choice order parameter \( \sigma \) and also potential bi-modality within certain ranking stages. That

![Heatmaps](image)

Figure 5: F1 data: heat maps of \( \Pr(\sigma_j = k|D) \) for the analysis with \( q = (K, \ldots, 1) \) (left), \( q = 1 \) (middle) and the absolute value of the difference (right) for \( j, k \in \{1, \ldots, K\} \). Crosses (\( \times \)) highlight the probabilities corresponding to \( \sigma = (K, \ldots, 1) \), the reverse Plackett-Luce model.

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| Driver Name (Country) | Wins | Podiums | Points | EPL Wins | EPL Podiums | EPL Points | SPL Wins | SPL Podiums | SPL Points |
|-----------------------|------|---------|--------|----------|------------|------------|----------|------------|------------|
| Lewis Hamilton (GBR)  | 11   | 17      | 20     | 10.37    | 16.49      | 20.19      | 5.20     | 12.59      | 20.63      |
| Sebastian Vettel (GER)| 5    | 12      | 20     | 4.27     | 12.60      | 19.39      | 3.03     | 8.64       | 19.47      |
| Kimi Räikkönen (FIN) | 1    | 12      | 17     | 1.89     | 8.99       | 18.45      | 1.31     | 4.23       | 14.98      |
| Max Verstappen (NED)  | 2    | 11      | 17     | 2.21     | 9.70       | 18.65      | 1.35     | 4.32       | 15.00      |
| Valtteri Bottas (FIN) | 0    | 8       | 19     | 1.64     | 8.36       | 18.30      | 2.29     | 6.89       | 18.33      |
| Daniel Ricciardo (AUS)| 2    | 2       | 13     | 0.59     | 4.90       | 16.79      | 0.76     | 2.57       | 10.86      |

Table 2: F1 data: Observed number of wins, podiums (top 3) and points (top 10) finishes and also the expected numbers under the predictive distributions for the EPL and SPL analyses for the top six drivers in the 2018/19 season.

said, further inspection of the marginal posterior distributions given by $\Pr(\sigma_j = k|D)$ reveals that there is a surprisingly small amount of uncertainty on the ranks allocated in the 13th-20th stages; see Figure 5 (left). Further within these positions ($\sigma_{13}, \ldots, \sigma_{20}$) the ranks allocated are consistent with the choice order parameter corresponding to the reverse Plackett-Luce model which suggests why previous authors may have found the RPL model to be preferable to the SPL model for modelling F1 results. We also note that these marginal posterior distributions seem fairly robust to the choice of $q$; Figure 5 (right) shows the (absolute value of the) discrepancy between the posterior probabilities under each prior choice.

To assess whether the EPL model allows for a good description of these data we again appeal to the posterior predictive distribution. Here complete enumeration of the posterior predictive probabilities for each $\tilde{x} \in S_K$ is computationally infeasible as $K!$ is of $O(10^{18})$. We therefore consider the number of times we would expect each of the top 6 drivers to win a race, feature on the podium (top 3), and also obtain a points (top 10) finish based on the predictive probabilities under the EPL model (with $q = (K, \ldots, 1)$) and under an SPL analysis for comparison. More specifically Table 2 shows $n \times \sum_{k=1}^p \Pr(\tilde{x}_j = k|D)$ for $p = 1, 3, 10$ along with the observed number of times computed from those $x \in D$. Note that the predictive probabilities $\Pr(\tilde{x}_j = k|D)$ are computed based on synthetic data simulated from the predictive distribution as discussed in Section 5 with $L = 10$. It is interesting to see that the expected number of points (top 10) finishes under both the extended and standard Plackett-Luce models are fairly
consistent with the observed data. However, the shortcomings of the more simple standard Plackett-Luce model become clear if we instead consider the expected number of wins/podiums. For example we observed that Hamilton won 11 races and the SPL model would suggest that he would expect to win around 5 races within an F1 season whereas the EPL model suggests 10 wins which is much more consistent with the observed data. Again additional insight into the question of model fit can be obtained via heat maps showing the discrepancies $d_{jk} = |\Pr(\hat{x}_j = k|\mathcal{D}) - \Pr(x_j = k)|$ for $j, k \in \{1,\ldots,K\}$; these are provided in Section 4 of the supplementary materials.

In this setting (large $K$) we do not have access to the full posterior posterior predictive distribution and so we use an efficient optimisation algorithm based on cyclic coordinate ascent (Johnson et al., 2020) to find the (global) mode of this distribution. Table 3 shows these (aggregate) rankings under both the extended and standard Plackett-Luce analyses along with the observed finishing order based on the driver points (also reported). It is pleasing to see that both models are able to predict that Hamilton and Vettel are the two best drivers. There is some disagreement between those drivers ranked 3rd-5th, however this is perhaps not surprising given these drivers obtained a similar number of points in the 18/19 season. One of the more concerning observations is that the mode obtained under the SPL model does not contain Ricciardo in 6th place even though he obtained a much larger number of points (> 100) than

Figure 6: F1 data: heat maps showing the predictive probabilities $\Pr(\hat{x}_j = k|\mathcal{D})$ for the EPL analysis with $q = (1\ldots,K)$ and SPL analysis for $j, k \in \{1,\ldots,K\}$
Table 3: F1 dataset: 2018/19 final Drivers’ Championship standings along with the global mode of the posterior predictive distribution (aggregate ranking) under both the EPL and SPL analyses

| Driver Name (Country)       | Points | Rank | EPL | SPL |
|----------------------------|--------|------|-----|-----|
| Lewis Hamilton (GBR)       | 408    | 1    | 1   | 1   |
| Sebastian Vettel (GER)     | 320    | 2    | 2   | 2   |
| Kimi Räikkönen (FIN)       | 251    | 3    | 4   | 5   |
| Max Verstappen (NED)       | 249    | 4    | 3   | 4   |
| Valtteri Bottas (FIN)      | 247    | 5    | 5   | 3   |
| Daniel Ricciardo (AUS)     | 170    | 6    | 6   | 10  |
| Nico Hülkenberg (GER)      | 69     | 7    | 7   | 6   |
| Sergio Perez (MEX)         | 62     | 8    | 11  | 8   |
| Kevin Magnussen (DEN)      | 56     | 9    | 12  | 16  |
| Carlos Sainz Jr (ESP)      | 53     | 10   | 10  | 9   |
| Fernando Alonso (ESP)      | 50     | 11   | 15  | 12  |
| Esteban Ocon (FRA)         | 49     | 12   | 17  | 14  |
| Charles Leclerc (MON)      | 39     | 13   | 19  | 17  |
| Romain Grosjean (FRA)      | 37     | 14   | 18  | 15  |
| Pierre Gasly (FRA)         | 29     | 15   | 16  | 13  |
| Stoffel Vandoorne (BEL)    | 12     | 16   | 14  | 7   |
| Marcus Ericsson (SWE)      | 90     | 17   | 20  | 11  |
| Lance Stroll (CAN)         | 6      | 18   | 9   | 18  |
| Brendon Hartley (NZL)      | 4      | 19   | 8   | 20  |
| Sergey Sirotkin (RUS)      | 1      | 20   | 13  | 19  |

For those drivers ranked below 7th there is some general agreement between the ranks under both the extended and standard Plackett-Luce models however we note that there is a large amount of uncertainty about which drivers are placed within these positions; see Figure 6 which shows the likely position of drivers within the rank orderings based on the predictive probabilities $\Pr(\tilde{x}_j = k|\mathcal{D})$. 

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7 Conclusion

We have considered the problem of implementing a fully Bayesian analysis of rank ordered data using the Extended Plackett-Luce model. In particular we have considered carefully the problem of prior specification, proposing a Plackett-Luce model as the prior for the choice order parameter $\sigma$ and proposing a prior distribution on the entity parameters that preserves the modal ordering under the prior predictive distribution. We have also tackled the challenging issue of posterior sampling of a potentially highly multi-modal posterior distribution with both discrete and continuous components via a Metropolis coupled Markov chain Monte Carlo scheme. This has enabled efficient posterior sampling which potentially facilitates further analyses based on the Extended Plackett-Luce model and further extensions of the model. Finally, we have focused on predictive inference for observable quantities; this admits a natural solution to the rank aggregation problem and also has facilitated the assessment of model adequacy.

Reproducibility

With reproducibility in mind, the code to run the algorithm outlined in Section 3.4 can be found at the GitHub repository https://github.com/srjresearch/ExtendedPL. This repository also contains each of the datasets considered within the paper along with detailed comments on how to execute the C code should a user wish to perform their own study. C code for performing a standard Plackett-Luce analysis is also provided.

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A copy of the supplementary materials can be obtained by contacting the authors.
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