Path connectivity of line graphs*

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Abstract

Dirac showed that in a $(k - 1)$-connected graph there is a path through each $k$ vertices. The path $k$-connectivity $\pi_k(G)$ of a graph $G$, which is a generalization of Dirac’s notion, was introduced by Hager in 1986. In this paper, we study path connectivity of line graphs.

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1 Introduction

An interpersonal network is represented as a graph, where a node is a processor and an edge is a communication link. Broadcasting is the process of sending a message from the source node to all other nodes in a network. It can be accomplished by message dissemination in such a way that each node repeatedly receives and forwards messages. Some of the nodes and/or links may be faulty. However, multiple copies of messages can be disseminated through disjoint paths. We say that the broadcasting succeeds if all the healthy nodes in the network finally obtain the correct message from the source node within a certain limit of time. A lot of attention has been devoted to fault-tolerant broadcasting in networks [13, 18, 20, 42]. In order to measure the degree of fault-tolerance, the above disjoint path structure connecting two nodes is generalized into some tree structures connecting more than two nodes, see [25, 27, 32]. To show the properties of these generalizations clearly, we hope to state from the connectivity in Graph Theory. We divide our introduction into the following four subsections to state the motivations and our results of this paper.

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1.1 Connectivity and \( k \)-connectivity

All graphs considered in this paper are undirected, finite and simple. We refer to [3] for graph theoretical notation and terminology not described here. For a graph \( G \), let \( V(G) \), \( E(G) \) and \( \delta(G) \) denote the set of vertices, the set of edges and the minimum degree of \( G \), respectively. For \( S \subseteq V(G) \), we denote by \( G - S \) the subgraph obtained by deleting from \( G \) the vertices of \( S \) together with the edges incident with them. Connectivity is one of the most basic concepts in graph theory, both in combinatorial sense and in algorithmic sense. As we know, the classical connectivity has two equivalent definitions. The connectivity of \( G \), written \( \kappa(G) \), is the minimum size of a vertex set \( S \subseteq V(G) \) such that \( G - S \) is disconnected or has only one vertex. We call this definition the ‘cut’ version definition of connectivity. A well-known theorem of Whitney [44] provides an equivalent definition of connectivity, which can be called the ‘path’ version definition. For any two distinct vertices \( x \) and \( y \) in \( G \), the local connectivity \( \kappa_G(x, y) \) is the maximum number of internally disjoint paths connecting \( x \) and \( y \). Then \( \kappa(G) = \min \{ \kappa_G(x, y) \mid x, y \in V(G), x \neq y \} \) is defined to be the connectivity of \( G \). Similarly, the classical edge-connectivity also has two equivalent definitions. The edge-connectivity of \( G \), written \( \lambda(G) \), is the minimum size of an edge set \( M \subseteq E(G) \) such that \( G - M \) is disconnected or has only one vertex. We call this definition the ‘cut’ version definition of edge-connectivity. For any two distinct vertices \( x \) and \( y \) in \( G \), the local edge-connectivity \( \lambda_G(x, y) \) is the maximum number of edge-disjoint paths connecting \( x \) and \( y \). Then \( \lambda(G) = \min \{ \lambda_G(x, y) \mid x, y \in V(G), x \neq y \} \) is defined to be the edge-connectivity of \( G \), which can be called the ‘path’ version definition; see [12]. For connectivity and edge-connectivity, Oellermann published a survey paper; see [38].

Although there are many elegant and powerful results on connectivity in graph theory, the classical connectivity and edge-connectivity cannot be satisfied considerably in practical uses. So people want some generalizations of both connectivity and edge-connectivity. For the ‘cut’ version definition of connectivity, we are looking for a minimum vertex-cut with no consideration about the number of components of \( G - S \). Two graphs with the same connectivity may have different degrees of vulnerability in the sense that the deletion of a vertex cut-set of minimum cardinality from one graph may produce a graph with considerably more components than in the case of the other graph. For example, the star \( K_{1,n} \) and the path \( P_{n+1} \) (\( n \geq 3 \)) are both trees of order \( n + 1 \) and therefore connectivity 1, but the deletion of a cut-vertex from \( K_{1,n} \) produces a graph with \( n \) components while the deletion of a cut-vertex from \( P_{n+1} \) produces only two components. Chartrand et al. [6] generalized the ‘cut’ version definition of connectivity. For an integer \( k \) (\( k \geq 2 \)) and a graph \( G \) of order \( n \) (\( n \geq k \)), the \( k \)-connectivity \( \kappa'_k(G) \) is the smallest number of vertices whose removal from \( G \) produces a graph with at least \( k \) components or a graph with fewer than \( k \) vertices. Thus, for \( k = 2 \), \( \kappa'_2(G) = \kappa(G) \). For more details about \( k \)-connectivity, we refer to [6] [21] [38] [39]. The \( k \)-edge-connectivity, which is a generalization of the ‘cut’ version definition of classical edge-connectivity was initially introduced by Boesch and
Chen [4] and subsequently studied by Goldsmith in [22, 23] and Goldsmith et al. [24]. For more details, we refer to [2, 37].

### 1.2 Generalized connectivity and generalized edge-connectivity

The generalized connectivity of a graph $G$, introduced by Hager [16], is a natural and nice generalization of the ‘path’ version definition of connectivity. For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, an $S$-Steiner tree or a Steiner tree connecting $S$ (or simply, an $S$-tree) is a subgraph $T = (V', E')$ of $G$ that is a tree with $S \subseteq V'$. Two Steiner trees $T$ and $T'$ connecting $S$ are said to be internally disjoint if $E(T) \cap E(T') = \emptyset$ and $V(T) \cap V(T') = S$. For $S \subseteq V(G)$ and $|S| \geq 2$, the generalized local connectivity $\kappa(S)$ is the maximum number of internally disjoint Steiner trees connecting $S$ in $G$. Note that when $|S| = 2$ a minimal Steiner tree connecting $S$ is just a path connecting the two vertices of $S$. For an integer $k$ with $2 \leq k \leq n$, generalized $k$-connectivity (or $k$-tree-connectivity) is defined as $\kappa_k(G) = \min \{ \kappa(S) | S \subseteq V(G), |S| = k \}$. Clearly, when $|S| = 2$, $\kappa_2(G)$ is nothing new but the connectivity $\kappa(G)$ of $G$, that is, $\kappa_2(G) = \kappa(G)$, which is the reason why one addresses $\kappa_k(G)$ as the generalized connectivity of $G$. By convention, for a connected graph $G$ with less than $k$ vertices, we set $\kappa_k(G) = 1$. Set $\kappa_k(G) = 0$ when $G$ is disconnected. This concept appears to have been introduced by Hager in [16]. It is also studied in [7] for example, where the exact value of the generalized $k$-connectivity of complete graphs are obtained. Note that the generalized $k$-connectivity and the $k$-connectivity of a graph are indeed different. Take for example, the graph $H_1$ obtained from a triangle with vertex set $\{v_1, v_2, v_3\}$ by adding three new vertices $u_1, u_2, u_3$ and joining $v_i$ to $u_i$ by an edge for $1 \leq i \leq 3$. Then $\kappa_3(H_1) = 1$ but $\kappa'_3(H_1) = 2$. There are many results on the generalized connectivity or tree-connectivity, we refer to [7, 26, 27, 30, 31, 32, 33, 34, 40].

The following Table 1 shows how the generalization proceeds.

|                  | Classical connectivity | Generalized connectivity |
|------------------|------------------------|--------------------------|
| **Vertex subset** | $S = \{x, y\} \subseteq V(G)$ ($|S| = 2$) | $S \subseteq V(G)$ ($|S| \geq 2$) |
| **Set of Steiner trees** | $\mathcal{P}_{x,y} = \{P_1, P_2, \ldots, P_\ell\}$, $\{x, y\} \subseteq V(R_i)$, $E(P_i) \cap E(P_j) = \emptyset$, $V(P_i) \cap V(P_j) = \{x, y\}$ | $\mathcal{P}_S = \{T_1, T_2, \ldots, T_\ell\}$, $S \subseteq V(T_i)$, $E(T_i) \cap E(T_j) = \emptyset$, $V(T_i) \cap V(T_j) = S$ |
| **Local parameter** | $\kappa(x,y) = \max |\mathcal{P}_{x,y}|$ | $\kappa(S) = \max |\mathcal{P}_S|$ |
| **Global parameter** | $\kappa(G) = \min_{x,y \in V(G)} \kappa(x,y)$ | $\kappa_k(G) = \min_{S \subseteq V(G), |S| = k} \kappa(S)$ |

Table 1. Classical connectivity and generalized connectivity

As a natural counterpart of the generalized connectivity, we introduced in [34] the concept of generalized edge-connectivity, which is a generalization of the ‘path’ version
definition of edge-connectivity. For \( S \subseteq V(G) \) and \(|S| \geq 2\), the \textit{generalized local edge-connectivity} \( \lambda(S) \) is the maximum number of edge-disjoint Steiner trees connecting \( S \) in \( G \). For an integer \( k \) with \( 2 \leq k \leq n \), the \textit{generalized \( k \)-edge-connectivity} \( \lambda_k(G) \) of \( G \) is then defined as \( \lambda_k(G) = \min \{ \lambda(S) | S \subseteq V(G) \text{ and } |S| = k \} \). It is also clear that when \(|S| = 2\), \( \lambda_2(G) \) is nothing new but the standard edge-connectivity \( \lambda(G) \) of \( G \), that is, \( \lambda_2(G) = \lambda(G) \), which is the reason why we address \( \lambda_k(G) \) as the generalized edge-connectivity of \( G \). Also set \( \lambda_k(G) = 0 \) when \( G \) is disconnected. Results on the generalized edge-connectivity can be found in [31, 34, 34].

1.3 Path connectivity and path edge-connectivity

Dirac [10] showed that in a \((k - 1)\)-connected graph there is a path through each \( k \) vertices. Related problems were inquired in [45]. In [17], Hager revised this statement to the question of how many internally disjoint paths \( P_i \) with the exception of a given set \( S \) of \( k \) vertices exist such that \( S \subseteq V(P_i) \). The path connectivity of a graph \( G \), introduced by Hager [17], is a natural specialization of the generalized connectivity and is also a natural generalization of the ‘path’ version definition of connectivity. For a graph \( G = (V, E) \) and a set \( S \subseteq V(G) \) of at least two vertices, a \textit{path connecting} \( S \) (or simply, \textit{an S-path}) is a subgraph \( P = (V', E') \) of \( G \) that is a path with \( S \subseteq V' \). Note that a path connecting \( S \) is also a tree connecting \( S \). Two paths \( P \) and \( P' \) connecting \( S \) are said to be \textit{internally disjoint} if \( E(P) \cap E(P') = \emptyset \) and \( V(P) \cap V(P') = S \). For \( S \subseteq V(G) \) and \(|S| \geq 2\), the \textit{local path connectivity} \( \pi_G(S) \) is the maximum number of internally disjoint paths connecting \( S \) in \( G \), that is, we search for the maximum cardinality of edge-disjoint paths which contain \( S \) and are vertex-disjoint with the exception of the vertices in \( S \). For an integer \( k \) with \( 2 \leq k \leq n \), the \textit{path \( k \)-connectivity} is defined as \( \pi_k(G) = \min \{ \pi_G(S) | S \subseteq V(G), |S| = k \} \), that is, \( \pi_k(G) \) is the minimum value of \( \pi_G(S) \) when \( S \) runs over all \( k \)-subsets of \( V(G) \). Clearly, \( \pi_1(G) = \delta(G) \) and \( \pi_2(G) = \kappa(G) \). For \( k \geq 3 \), \( \pi_k(G) \leq \kappa_k(G) \) holds because each path is also a tree.

The relation between generalized connectivity and path connectivity are shown in the following Table 2.

| Vertex subset | Generalized connectivity | Path-connectivity |
|---------------|-------------------------|------------------|
| \( S \subseteq V(G), |S| \geq 2 \) | \( S \subseteq V(G), |S| \geq 2 \) |
| Set of Steiner trees | \( \mathcal{T}_S = \{ T_1, T_2, \cdots, T_k \} \) \( S \subseteq V(T_i), E(T_i) \cap E(T_j) = \emptyset, \) | \( \mathcal{S}_S = \{ P_1, P_2, \cdots, P_k \} \) \( S \subseteq V(P_i), E(P_i) \cap E(P_j) = \emptyset, \) |
| Local parameter | \( \kappa(S) = \max |\mathcal{T}_S| \) | \( \pi(S) = \max |\mathcal{S}_S| \) |
| Global parameter | \( \kappa_k(G) = \min_{S \subseteq V(G), |S| = k} \kappa(S) \) | \( \pi_k(G) = \min_{S \subseteq V(G), |S| = k} \pi(S) \) |

Table 2. Two kinds of tree-connectivities

As a natural counterpart of path \( k \)-connectivity, we recently introduced the concept of
path $k$-edge-connectivity. Two paths $P$ and $P'$ connecting $S$ are said to be edge-disjoint if $E(P) \cap E(P') = \emptyset$. For $S \subseteq V(G)$ and $|S| \geq 2$, the path local edge-connectivity $\omega_G(S)$ is the maximum number of edge-disjoint paths connecting $S$ in $G$. For an integer $k$ with $2 \leq k \leq n$, the path $k$-edge-connectivity is defined as $\omega_k(G) = \min\{\omega_G(S) \mid S \subseteq V(G), |S| = k\}$, that is, $\omega_k(G)$ is the minimum value of $\omega_G(S)$ when $S$ runs over all $k$-subsets of $V(G)$. Clearly, we have

$$
\begin{cases}
\omega_k(G) = \delta(G), & \text{for } k = 1; \\
\omega_k(G) = \lambda(G), & \text{for } k = 2; \\
\omega_k(G) \leq \lambda_k(G), & \text{for } k \geq 3.
\end{cases}
$$ (1)

The following observation is immediate.

**Observation 1.1**  (1) Let $G$ be a connected graph. Then $\pi_k(G) \leq \omega_k(G) \leq \delta(G)$.

(2) Let $G$ be a connected graph with minimum degree $\delta$. If $G$ has two adjacent vertices of degree $\delta$, then $\pi_k(G) \leq \omega_k(G) \leq \delta - 1$.

**Lemma 1.2**  [16] Let $k, n$ be two integers with $3 \leq k \leq n$, and let $K_n$ be a complete graph of order $n$. Then

$$
\pi_k(K_n) = \left\lceil \frac{2n + k^2 - 3k}{2(k-1)} \right\rceil.
$$

Note that each graph is a spanning subgraph of a complete graph. So the following result is immediate.

**Observation 1.3** Let $k, n$ be two integers with $3 \leq k \leq n$, and let $G$ be a graph of order $n$. Then

$$
0 \leq \pi_k(G) \leq \left\lceil \frac{2n + k^2 - 3k}{2(k-1)} \right\rceil.
$$

**Lemma 1.4** Let $G$ be a graph of order $n$. If $n$ is even, then $\pi_3(G) = \frac{n}{2}$ if and only if $G$ is a complete graph of order $n$.

**Proof.** From Lemma 1.2 we have $\pi_3(K_n) = \left\lceil \frac{n}{2} \right\rceil = \frac{n}{2}$. Actually, the complete graph $K_n$ is the unique graph with this property. We only need to show that $\pi_3(K_n \setminus uv) < \frac{n}{2} - 1$ for any $uv \in E(K_n)$. Pick one vertex $w \in V(K_n) - \{u, v\}$. Choose $S = \{u, v, w\}$. If we choose $uvw$ as a path, then any other $S$-Steiner path occupies at least two vertices of $V(G) - S$, and hence $\pi(S) \leq 1 + \left\lceil \frac{n-3}{2} \right\rceil = \frac{n}{2} - 1$. Suppose that $uv$ and $vw$ belong to different $S$-Steiner path. Then $\pi(S) \leq 2 + \left\lceil \frac{n-3}{2} \right\rceil = \frac{n}{2} - 1$. So $\pi_3(G) \leq \pi(S) \leq \frac{n}{2} - 1$, as desired. □

**Lemma 1.5**  [17] For any connected graph $G$, $\pi_3(G) \geq \frac{1}{2} \kappa(G)$. Moreover, the lower bound is sharp.
In [19], Jain et al. obtained the following result.

**Lemma 1.6** [19] Let $G$ be a graph and $S = \{v_1, v_2, v_3\}$ be a subset of $V(G)$. Assume that $v_1$ and $v_2$ are $\ell$-edge-connected, and $v_1$ and $v_3$ are $\frac{\ell}{2}$-edge-connected in $G$. Then $G$ has $\frac{\ell}{2}$ edge-disjoint $S$-Steiner trees.

In their proof, the $\frac{\ell}{2}$ edge-disjoint $S$-Steiner trees are actually edge-disjoint $S$-Steiner paths. So the following result is immediate.

**Corollary 1.7** For any connected graph $G$, $\omega_3(G) \geq \frac{1}{2}\lambda(G)$. Moreover, the lower bound is sharp.

1.4 Application background of these parameters

In addition to being a natural combinatorial measure, the path $k$-(edge-)connectivity and generalized $k$-(edge-)connectivity can be motivated by their interesting interpretation in practice. For example, suppose that $G$ represents a network. If one considers to connect a pair of vertices of $G$, then a path is used to connect them. However, if one wants to connect a set $S$ of vertices of $G$ with $|S| \geq 3$, then a tree has to be used to connect them. This kind of tree for connecting a set of vertices is usually called a Steiner tree, and popularly used in the physical design of VLSI circuits (see [14, 15, 41]). In this application, a Steiner tree is needed to share an electric signal by a set of terminal nodes. Steiner tree is also used in computer communication networks (see [11]) and optical wireless communication networks (see [9]). Usually, one wants to consider how tough a network can be, for the connection of a set of vertices. Then, the number of totally independent ways to connect them is a measure for this purpose. The path $k$-connectivity and generalized $k$-connectivity can serve for measuring the capability of a network $G$ to connect any $k$ vertices in $G$.

1.5 Main results of this paper

Chartrand and Stewart [8] investigated the relation between the connectivity and edge-connectivity of a graph and its line graph.

**Theorem 1.8** [8] If $G$ is a connected graph, then

1. $\kappa(L(G)) \geq \lambda(G)$ if $\lambda(G) \geq 2$.
2. $\lambda(L(G)) \geq 2\lambda(G) - 2$.
3. $\kappa(L(L(G))) \geq 2\kappa(G) - 2$.

In Section 2, we investigate the relation between the path connectivity and path edge-connectivity of a graph and its line graph.
In their book, Capobianco and Molluzzo [5], using $K_{1,n}$ as their example, note that the difference between the connectivity of a graph and its line graph can be arbitrarily large.

They then proposed an open problem: Whether for any two integers $p, q \ (1 < p < q)$, there exists a graph $G$ such that $\kappa(G) = p$ and $\kappa(L(G)) = q$. In [1], Bauer and Tindell gave a positive answer of this problem, that is, for every pair of integers $p, q \ (1 < p < q)$ there is a graph of connectivity $p$ whose line graph has connectivity $q$.

Note that the difference between the path $k$-connectivity of a graph $G$ and its line graph $L(G)$ can be arbitrarily large. Let $n, k$ be two integers with $2 \leq k \leq n$, and let $G = K_{1,n}$. Then $L(G) = K_n$, $\pi_k(G) = 0$ and $\pi_k(L(G)) = \left\lfloor \frac{2n+k^2-3k}{2(k-1)} \right\rfloor$. In fact, we can consider a similar problem: Whether for any two integers $p, q$, $1 < p < q$, there exists a graph $G$ such that $\pi_k(G) = p$ and $\pi_k(L(G)) = q$. It seem to be not easy to solve this problem for a general $k$. In this paper, we focus our attention on the case $k = 3$ and $q \geq 2p - 1$, and give a positive answer of this problem.

## 2 Path connectivity of line graphs

Hager [16] obtain the following result.

**Lemma 2.1** [16] Let $G$ be a graph of order $n$, and let $k$ be an integer with $3 \leq k \leq n$. If $\kappa(G) \geq 2^{k-2} \ell$, then $\pi_k(G) \geq \ell$.

**Corollary 2.2** Let $G$ be a graph of order $n$, and let $k$ be an integer with $3 \leq k \leq n$. Then

$$2^{2-k}\kappa(G) \leq \pi_k(G) \leq \kappa(G).$$

For general $k$, we can easily derive the following result.

**Proposition 2.3** Let $G$ be a graph of order $n$. If $G$ is 2-edge-connected, then

1. $2^{2-k}\omega_k(G) \leq \pi_k(L(G))$;
2. $\omega_k(L(G)) \geq 2^{2-k}\omega_k(G)$;
3. $\pi_k(L(L(G))) \geq 2^{3-k}(\pi_k(G) - 1)$.

**Proof.** For (1), from Corollary 2.2 and Theorem 1.8, we have

$$\pi_k(L(G)) \geq 2^{2-k}\kappa(L(G)) \geq 2^{2-k}\lambda(G) \geq 2^{2-k}\kappa(G) \geq 2^{2-k}\pi_k(G).$$

For (2), from (1) of this proposition, Corollary 2.2 and Theorem 1.8, we have

$$\omega_k(L(G)) \geq \pi_k(L(G)) \geq 2^{2-k}\kappa(L(G)) \geq 2^{2-k}\lambda(G) \geq 2^{2-k}\omega_k(G).$$

For (3), from (1) of this proposition and Corollary 2.2, we have

$$\pi_k(L(L(G))) \geq 2^{2-k}\kappa(L(L(G))) \geq 2^{2-k}(2\kappa(G) - 2) = 2^{3-k}(\kappa(G) - 1) \geq 2^{3-k}(\pi_k(G) - 1).$$
For $k = 3$, we can improve the above result, and obtain the following theorem.

**Theorem 2.4** Let $G$ be a 2-connected. Then

(1) $\omega_3(G) \leq \pi_3(L(G))$.

(2) $\omega_3(L(G)) \geq \omega_3(G) - 1$.

(3) $\pi_3(L(G)) \geq \pi_3(G) - 1$.

**Proof.** For (1), let $e_1, e_2, e_3$ be three arbitrary distinct vertices of the line graph of $G$ such that $\omega_3(G) = \ell$ with $\ell \geq 1$. Let $e_1 = v_1v_1'$, $e_2 = v_2v_2'$ and $e_3 = v_3v_3'$ be those edges of $G$ corresponding to the vertices $e_1, e_2, e_3$ in $L(G)$, respectively.

Consider three distinct vertices of the six end-vertices of $e_1, e_2, e_3$. Without loss of generality, let $S = \{v_1, v_2, v_3\}$ be three distinct vertices. Since $\omega_3(G) = \ell$, there exist $\ell$ edge-disjoint $S$-Steiner paths in $G$, say $P_1, P_2, \cdots, P_\ell$. We define a minimal $S$-Steiner path $P$ as an $S$-Steiner path whose sub-path obtained by deleting any edge of $P$ does not connect $S$. Choosing any two edge-disjoint minimal $S$-Steiner paths $P_i$ and $P_j$ ($1 \leq i, j \leq \ell$) in $G$,

![Type a](image1)

![Type b](image2)

Figure 1: Three possible types of $T_i \cup T_j$.

we will show that the paths $P_i'$ and $P_j'$ corresponding to $P_i$ and $P_j$ in $L(G)$ are internally disjoint $S$-Steiner paths. It is easy to see that $P_i \cup P_j$ has two possible types, as shown in Figure 1. Since $P_i$ and $P_j$ are edge-disjoint in $G$, we can find internally disjoint Steiner paths $P_i'$ and $P_j'$ connecting $e_1, e_2, e_3$ in $L(G)$. We give an example of Type a, see Figure 2. So $\pi_3(L(G)) \geq \ell$, as desired.

(2) From Corollary 1.7 and Theorem 1.8 we have

$$\omega_3(L(G)) \geq \frac{1}{2} \lambda(L(G)) \geq \frac{1}{2}(2\lambda(G) - 2) = \lambda(G) - 1 \geq \omega_3(G) - 1.$$ 

(3) From Lemma 1.5 and Theorem 1.8 we have

$$\pi_3(L(L(G))) \geq \frac{1}{2} \kappa(L(L(G))) \geq \frac{1}{2}(2\kappa(G) - 2) = \kappa(G) - 1 \geq \pi_3(G) - 1.$$
3 Graphs with prescribed path connectivity and path edge-connectivity

In [16], Hager got the following result.

**Lemma 3.1** [16] Let $K_{a,b}$ be a complete bipartite graph with $a + b$ vertices, and let $k$ be an integer with $2 \leq k \leq a + b$. Then

$$\pi_k(K_{a,b}) = \min \left\{ \frac{a}{k-1}, \frac{b}{k-1} \right\}.$$ 

The following corollary is immediate from the above theorem.

**Corollary 3.2** Let $a, b$ be two integers with $2 \leq a \leq b$, and $K_{a,b}$ denote a complete bipartite graph with a bipartition of sizes $a$ and $b$, respectively. Then

$$\pi_3(K_{a,b}) = \left\lfloor \frac{a}{2} \right\rfloor.$$ 

In this section, we consider the problem mentioned in Subsection 1.5 for the case $q \geq 2p - 1$. Let us put our attention on the complete bipartite graph $G = K_{2p, 2q-2p+2}$. Since $q \geq 2p - 1$, it follows that $2q - 2p + 2 \geq 2p$. From Corollary 3.2, $\pi_3(G) = \pi_3(K_{2p, 2q-2p+2}) = p$. Now we turn to consider the line graph of complete bipartite graph $G = K_{2p, 2q-2p+2}$.

Recall that the Cartesian product (also called the square product) of two graphs $G$ and $H$, written as $G \square H$, is the graph with vertex set $V(G) \times V(H)$, in which two vertices $(u,v)$ and $(u',v')$ are adjacent if and only if $u = u'$ and $(v,v') \in E(H)$, or $v = v'$ and $(u,u') \in E(G)$. Clearly, the Cartesian product is commutative, that is, $G \square H \cong H \square G$.

The following lemma is from [43].
Lemma 3.3 \((43)\) For a complete bipartite graph \(K_{r,s}\), \(L(K_{r,s}) = K_r \square K_s\).

From the above lemma, \(L(G) = L(K_{2p,2q-2p+2}) = K_{2p} \square K_{2q-2p+2}\). In order to obtain the exact value of \(\pi_3(L(G)) = \pi_3(K_{2p} \square K_{2q-2p+2})\), we consider to determine the exact value of the Cartesian product of two complete graphs.

Before proving Lemma 3.3, we define some notation. Let \(G\) and \(H\) be two connected graphs with \(V(G) = \{u_1, u_2, \ldots, u_n\}\) and \(V(H) = \{v_1, v_2, \ldots, v_m\}\), respectively. Then \(V(G \circ H) = \{(u_i, v_j) | 1 \leq i \leq n, 1 \leq j \leq m\}\). For \(v \in V(H)\), we use \(G(v)\) to denote the subgraph of \(G \circ H\) induced by the vertex set \(\{(u_i, v) | 1 \leq i \leq n\}\). Similarly, for \(u \in V(G)\), we use \(H(u)\) to denote the subgraph of \(G \circ H\) induced by the vertex set \(\{(u, v_j) | 1 \leq j \leq m\}\).

Lemma 3.4 Let \(p, q\) be two integers with \(p \geq 2\) and \(q \geq 3\). Then

\[\pi_3(K_{2p} \square K_{2q-2p+2}) = q.\]

Proof. Let \(G = K_{2p}\) and \(H = K_{2q-2p+2}\). Set \(V(G) = \{u_1, u_2, \ldots, u_{2p}\}\) and \(V(H) = \{v_1, v_2, \ldots, v_{2q-2p+2}\}\). From Lemma \(1.4\) we have

\[\pi_3(G \square H) = \pi_3(K_{2p} \square K_{2q-2p+2}) \leq \pi_3(K_{2q-2p+2}) \leq q.\]

It suffices to show that \(\pi_3(G \square H) \geq q\). We need to show that for any \(S = \{x, y, z\} \subseteq V(G \square H)\), there exist \(q\) internally disjoint S-Steiner paths. We complete our proof by the following three cases.

Case 1. \(x, y, z\) belongs to the same \(V(H(u_i)) (1 \leq i \leq r)\).

Without loss of generality, we assume \(x, y, z \in V(H(u_1))\). Since \(\pi_3(H) = \pi_3(K_{2q-2p+2}) = q - p + 1\), there exist \(q - p + 1\) internally disjoint S-Steiner paths \(P_1, P_2, \ldots, P_{q-p+1}\) in \(H(u_1)\). Let \(x_j, y_j, z_j\) be the vertices corresponding to \(x, y, z\) in \(H(u_j) (2 \leq j \leq 2p)\). Then the paths \(Q_i\) induced by the edges in \(\{xx_{2i}, xx_{2i+1}, xy_{2i}, yy_{2i}, yz_{2i}, zz_{2i+1}, zz_{2i+1}, zy_{2i+1}, zy_{2i+2}, zy_{2i+1}, zy_{2i+2}\} (1 \leq i \leq p-1)\) are \(p-1\) internally disjoint S-Steiner paths. These paths together with the paths \(P_1, P_2, \ldots, P_{q-p+1}\) are \(q\) internally disjoint S-Steiner paths, as desired.

Case 2. Only two vertices of \(\{x, y, z\}\) belong to some copy \(H(u_i)\).

We may assume \(x, y \in V(H(u_1)), z \in V(H(u_2))\). Let \(x', y'\) be the vertices corresponding to \(x, y\) in \(H(u_2)\), and let \(z'\) be the vertex corresponding to \(z\) in \(H(u_1)\).

Suppose \(z' \notin \{x, y\}\). Without loss of generality, let \(x = (u_1, v_1), y = (u_1, v_2)\) and \(z' = (u_1, v_3)\). The path \(Q_1\) induced by the edges in \(\{xy, yz', z'y\}\), the path \(Q_2\) induced by the edges in \(\{xx', xz', z'z, z'y\}\) and \(P_i\) induced by the edges in \(\{x(u_1, v_2i), (u_1, v_2i)y, y(u_1, v_2i+1), (u_1, v_2v_2i+1)(u_2, v_2i+1), (u_2, v_2i+1)z, zy_{2i+1}, zy_{2i+2}\} (2 \leq i \leq q-p)\) are \(q-p+1\) internally disjoint S-Steiner paths. Note that all the edges of these paths are between \(H(u_1)\) and \(H(u_2)\). Let \(x_j, y_j, z_j\) be the vertices corresponding to \(x, y, z\) in \(H(u_j) (3 \leq j \leq 2p)\). The the paths \(Q_j\) induced by the edges in \(\{xx_{2j+1}, xx_{2j+2}y_{2j+1}, xx_{2j+2}y_{2j+2}, yy_{2j+2}, yy_{2j+2}+2z_{2j+2}, zz_{2j+2}z\} (1 \leq j \leq
Suppose $z' \in \{x, y\}$. Without loss of generality, let $x = (u_1, v_1)$ and $y = (u_2, v_2)$. Then the path $Q$ induced by the edges in $\{xy, yz\}$ and $P_j$ induced by the edges in $\{x(u_1, v_{2j-1}), (u_1, v_{2j-1})y, y(u_1, v_{2j}), (u_1, v_{2j})(u_2, v_{2j}), (u_2, v_{2j})z\}$ $(2 \leq j \leq q - p + 1)$ are $q - p + 1$ internally disjoint $S$-Steiner paths. Let $x, y, z$ be the vertices corresponding to $x, y, z'$ in $H(u_j)$ $(3 \leq j \leq 2p)$. Then the paths $Q_i$ induced by the edges in $\{xx_{2i-1}, x_{2i-1}y_{2i-1}, y_{2i-1}y_{2i-2}, y_{2i-2}z_{2i-2}, z_{2i-2}z_{2i}\}$ $(1 \leq i \leq p - 1)$ are $p - 1$ internally disjoint $S$-Steiner paths. These paths together with the paths $Q, P_1, P_2, \cdots, P_{q-p}$ are $q$ internally disjoint $S$-Steiner paths, as desired.

**Case 3.** $x, y, z$ are contained in distinct $H(u_i)$s.

We may assume that $x \in V(H(u_1))$, $y \in V(H(u_2))$, $z \in V(H(u_3))$. Let $y', z'$ be the vertices corresponding to $y, z$ in $H(u_1)$, $x', z''$ be the vertices corresponding to $x, z$ in $H(u_2)$ and $x'', y''$ be the vertices corresponding to $x, y$ in $H(u_3)$.

Suppose that $x, x', z'$ are distinct vertices in $H(u_1)$. Without loss of generality, let $x = (u_1, v_1)$, $y = (u_2, v_2)$ and $z = (u_3, v_3)$. Let $x, y, z$ be the vertices corresponding to $x, y, z'$ in $H(u_j)$ $(4 \leq j \leq 2p)$. Then the path $R_1$ induced by the edges in $\{xx', yx', yy'', y'y''z\}$, the path $R_2$ induced by the edges in $\{xy', yx'', y'y''z\}$, the path $R_3$ induced by the edges in $\{xz', zz', yy_2y_2, y_2z_2, z_2z\}$ and $P_j$ induced by the edges in $\{x(u_1, v_{2j-1}), (u_1, v_{2j-1})(u_2, v_{2j-1}), (u_2, v_{2j-1})y, y(u_2, v_{2j}), (u_2, v_{2j}), (u_2, v_{2j})z\}$ $(2 \leq j \leq q - p)$ are $q - p + 2$ internally disjoint $S$-Steiner paths. The paths $Q_i$ induced by the edges in $\{xx_{2i-1}, x_{2i-1}y_{2i-1}, y_{2i-1}y_{2i-2}, y_{2i-2}z_{2i-2}, z_{2i-2}z_{2i}\}$ $(2 \leq i \leq p - 1)$ are $p - 2$ internally disjoint $S$-Steiner paths. These paths together with the trees $R_1, R_2, R_3, P_2, P_3, \cdots, P_{q-p-1}$ are $q$ internally disjoint $S$-Steiner paths, as desired.

Suppose that two of $x, x', z'$ are the same vertex in $H(u_1)$. Without loss of generality, let $x = (u_1, v_1)$, $y = (u_2, v_1)$ and $z = (u_3, v_2)$. Let $x, y, z$ be the vertices corresponding to $x, y, z'$ in $H(u_j)$ $(4 \leq j \leq 2p)$. Then the path $R_1$ induced by the edges in $\{xy, yx''', x''''z\}$, the path $R_2$ induced by the edges in $\{xx_2, x_2y_2, y_2z_2, z_2z\}$ and $P_j$ induced by the edges in $\{x(u_1, v_{2j-1}), (u_1, v_{2j-1})(u_2, v_{2j-1}), (u_2, v_{2j-1})y, y(u_2, v_{2j}), (u_2, v_{2j})(u_3, v_{2j}), (u_3, v_{2j})z\}$ $(2 \leq j \leq q - p + 1)$ are $q - p + 2$ internally disjoint $S$-Steiner paths. The paths $Q_i$ induced by the edges in $\{xx_{2i-1}, x_{2i-1}y_{2i-1}, y_{2i-1}y_{2i-2}, y_{2i-2}z_{2i-2}, z_{2i-2}z_{2i}\}$ $(2 \leq i \leq p - 1)$ are $p - 2$ internally disjoint $S$-Steiner paths. These paths together with the paths $R_1, R_2, P_2, P_3, \cdots, P_{q-p-1}$ are $q$ internally disjoint $S$-Steiner paths, as desired.

Suppose that $x, x', z'$ are the same vertex in $H(u_1)$. Without loss of generality, let $x = (u_1, v_1)$, $y = (u_2, v_1)$ and $z = (u_3, v_1)$. Then the path $R_1$ induced by the edges in $\{xy, yz\}$, the path $R_2$ induced by the edges in $\{xz, y(u_2, v_{2q-2p+2}), (u_2, v_{2q-2p+2})z, (u_2, v_{2q-2p+2})z\}$ and $P_j$ induced by the edges in $\{x(u_1, v_{2j}), (u_1, v_{2j})(u_2, v_{2j}), y(u_2, v_{2j}), y(u_2, v_{2j+1}), y(u_2, v_{2j+1})(u_3, v_{2j+1}), (u_3, v_{2j+1})z\}$ $(1 \leq j \leq q - p)$ are $q - p + 2$ internally disjoint $S$-Steiner paths. Let $x, y, z$ be the vertices corresponding to $x, y, z'$ in
$H(u_j)$ ($4 \leq j \leq 2p$). The paths $Q_i$ induced by the edges in \{\(xx_{2i}, x_{2i}y_{2i}, y_{2i}y_{2i+1}, y_{2i+1}z_{2i+1}, z_{2i+1}z\)$ \((2 \leq i \leq p - 1)\) are $p - 2$ internally disjoint $S$-Steiner paths. These paths together with the paths $R_1, R_2, P_1, P_2, \ldots, P_{q-p}$ are $q$ internally disjoint $S$-Steiner paths, as desired.

From the above argument, we conclude that for any $S = \{x, y, z\} \subseteq V(G \Box H)$, there exist $q$ internally disjoint $S$-Steiner paths, and hence $\pi(S) \geq q$. From the arbitrariness of $S$, we have $\pi_k(G) = q$.

From Lemma 3.4, the following result holds.

**Theorem 3.5** For any two integers $p, q$ with $q \geq 2p - 1$, $p \geq 2$ and $q \geq 3$, there exists a graph $G$ such that $\pi_3(G) = p$ and $\pi_3(L(G)) = q$.

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