On large deviations in the averaging principle for SDE’s with a “full dependence”, correction

A. Yu. Veretennikov
(School of Mathematics, University of Leeds, UK & Institute for Information Transmission Problems, Russia)

We establish the large deviation principle for stochastic differential equations with averaging in the case when all coefficients of the fast component depend on the slow one, including diffusion.

1 Introduction

This is a corrected version of the paper [16]. We consider the SDE system

\[ \begin{align*}
    dX_t &= f(X_t, Y_t)dt, \quad X_0 = x_0, \\
    dY_t &= \varepsilon^{-2}B(X_t, Y_t)dt + \varepsilon^{-1}C(X_t, Y_t)dW_t, \quad Y_0 = y_0.
\end{align*} \]

Here \( X_t \in E^d, Y_t \in M, M \) is a compact manifold of dimension \( \ell \) (e.g. torus \( T^\ell \)), \( f \) is a function with values in \( d \)-dimensional Euclidean space \( E^d \), \( B \) is a function with values in \( TM \), \( C \) is a function with values in \( (TM)^\ell \) (i.e., in local coordinates an \( \ell \times \ell \) matrix), \( (W_t) \) is an \( \ell \)-dimensional Wiener process with respect to some increasing and right continuous filtration \( (\mathcal{F}_t) \) on some probability space \( (\Omega, F, P) \), \( \varepsilon > 0 \) is a small parameter, i.e., \( \varepsilon \to 0 \). Concerning SDE’s on manifolds we refer to [5].

The large deviation principle (LDP) for such systems with a “full dependence”, that is, \( C(X_t, Y_t) \), was not treated before [16]. Only the case \( C(Y_t) \) was considered in the papers by [1, 2, 3] for a compact state space and by [14] for a non-compact one. Also the papers [10], [15] and [11] on similar or close topics for more general systems with small additive diffusions should be mentioned, which, however, all concern only the case \( C(Y_t) \). Concerning most recent developments the reader is referred to [7] and the references therein.

The LDP for systems like (1) is important in averaging and homogenization, in the KPP equation theory, for stochastic approximation algorithms with averaging and so forth. The problem of an LDP for the case \( C(X_t, Y_t) \) has arisen since [1, 2]. Intuitively, the scheme used for \( C(Y_t) \) should work; at least, almost all main steps go well. Indeed, there was only one lacuna; the use of Girsanov’s transformation did not allow freezing of \( X_t \) if \( C \) depended on the

---

1. AMS 1991 subject classifications. 60F10, 60J60.
2. Key words and phrases. Large deviations, averaging, stochastic differential equation.
slow motion, while it worked well and very naturally for the drift \( B(X_t, Y_t) \). Yet the problem remained unresolved for years and the answer was unclear. Notice that this difficulty does not appear in analogous discrete-time systems (see [1, Chapter 11]).

It turned out that the use of Girsanov’s transformation in some sense prevented from resolving the problem. Our approach in this paper is based on a new technical result, Lemma 5 below. The main new idea is to use two different scales of partitions of the interval \([0, T]\), a “first-order partition” by points \( \Delta, 2\Delta, \ldots \), which do not depend on the small parameter \( \varepsilon \) and “second-order partitions” which depend on \( \varepsilon \) in a special way, by points \( \varepsilon^2 t(\varepsilon), 2\varepsilon^2 t(\varepsilon), \ldots \). Then the exponential estimates needed for the proof of the result can be established in two steps. First, the estimates for a “small” partition interval are derived using the uniform bound of Lemma 3 (see below) and the estimates for stochastic integrals. It is important that in the “second” scale the fast motion is still close enough to its frozen version [the bound (14) below]. Second, the bounds for “small” partitions and induction give one the estimate for a “large” partition interval.

The original proof in [17] contained some gap relates to a boundedness of some auxiliary constant \( b \) in the proof: in the original version this constant may depend implicitly on the partition size \( \Delta \), while the choice of \( \Delta \) could depend on \( b \), hence generating a vicious circle. The main aim of this version of the paper is to present the “patch”. A provisional version of this correction may be found in [17]. The present version is simplified further. The correction uses improved approximations that keep this constant \( b \) bounded in the lower and upper bounds, and it uses also a truncated Legendre transformation in the upper bound. The author is deeply indebted to Professor Yuri Kifer for discovering this vicious circle in the original version of the paper. The main technical tool remains the Lemma 5. All standing assumptions are the same as in the original version.

The main result is stated in Section 2. In Section 3 we present auxiliary lemmas, among them the main technical Lemma 5 with its proof and a version of an important lemma from [3] (see Lemma 6) which requires certain comments. Those comments along with other related remarks are given in the Appendix, the latter has been also slightly extended. The proof of the main theorem is presented in Section 4.

## 2 Main result

We make the following assumptions.

\( (A_f) \) The function \( f \) is bounded and satisfies the Lipschitz condition.

\( (A_C) \) The function \( C \) is bounded, uniformly nondegenerate, \( C \) satisfies the Lipschitz condition.

\( (A_B) \) The function \( B \) is bounded and satisfies the Lipschitz condition.
Some conditions may be relaxed; for example, $B$ may be assumed locally bounded, $C$ locally (with respect to $x$) nondegenerate and so on.

The family of processes $X^\varepsilon$ satisfies a large deviation principle in the space $C([0,T]; R^d)$ with a normalizing coefficient $\varepsilon^{-2}$ and a rate function $S(\varphi)$ if the following three conditions are satisfied:

\begin{equation}
\limsup_{\varepsilon \to 0} \varepsilon^2 \log P_x(X^\varepsilon \in F) \leq - \inf_F S(\varphi), \quad \forall F \text{ closed },
\end{equation}

\begin{equation}
\liminf_{\varepsilon \to 0} \varepsilon^2 \log P_x(X^\varepsilon \in G) \geq - \inf_G S(\varphi), \quad \forall G \text{ open },
\end{equation}

and $S$ is a “good” rate function; that is, for any $s \geq 0$, the set

\[ \Phi(s) := \{ \varphi \in C([0,T]; R^d) : S(\varphi) \leq s, \varphi(0) = x \} \]

is compact in $C([0,T]; R^d)$. We will establish the following equivalent set of assertions due to Freidlin and Wentzell, where $\rho(\phi, \psi) = \sup_{0 \leq s \leq T} |\phi_s - \psi_s|$,

\begin{equation}
\limsup_{\delta \to 0} \limsup_{\varepsilon \to 0} \varepsilon^2 \log P_x (\rho(X^\varepsilon, \Phi(s)) \geq \delta) \leq -s, \quad \forall s > 0,
\end{equation}

where $\Phi(s) := \{ \varphi \in C([0,T]; R^d), S(\varphi) \leq s \}$, and

\begin{equation}
\liminf_{\delta \to 0} \liminf_{\varepsilon \to 0} \varepsilon^2 \log P_x (\rho(X^\varepsilon, \varphi) < \delta) \geq -S(\varphi), \quad \forall \varphi,
\end{equation}

where $S$ is a “good” rate function (see above). In what follows, $\dot{\varphi}_t$ is a derivative function for $\varphi_t$ and if it does not exist almost everywhere or if the integral $\int_0^T L(\varphi_t, \dot{\varphi}_t) dt$ diverges, then by definition $\int_0^T L(\varphi_t, \dot{\varphi}_t) dt := +\infty$.

**Theorem 1** Let $(A_f), (A_B), (A_C)$ be satisfied. Then the family $(X^\varepsilon_t = X_t, \ 0 \leq t \leq T)$ satisfies the LDP as $\varepsilon \to 0$ in the space $C([0,T]; R^d)$ with a rate function

\[ S(\varphi) = \int_0^T L(\varphi_t, \dot{\varphi}_t) dt, \]

where

\[ L(x, \alpha) = \sup_{\beta} (\alpha \beta - H(x, \beta)), \]

\[ H(x, \beta) = \lim_{t \to \infty} t^{-1} \log E \exp \left( \beta \int_0^t f(x, y^x_s) ds \right). \]

The limit $H$ exists and is finite for any $\beta$, the functions $H$ and $L$ are convex in their last arguments $\beta$ and $\alpha$ correspondingly, $L \geq 0$ and $H$ is continuously differentiable in $\beta$.

The differentiability of $H$ at any $\beta$ will be provided by the compactness of the state space of the fast component. The constants $C$ in the calculus may change from line to line, unlike $K, C_f, L_f$ and some other.
3 Auxiliary lemmas

Let $\tilde{W}_t = \epsilon^{-1}W_{t^2}, \ y_t = Y_{t^2}, \ x_t = X_{t^2}$, and let $y^x_t$ solve an SDE,

$$dy^x_t = B(x, y^x_t)dt + C(x, y^x_t)d\tilde{W}_t, \quad y^x_0 = y_0. \quad (6)$$

Below $\tilde{F}_t := F_{t^2}, \ \beta \in E^d, \ \beta f$ means a scalar product and the index $y$ in $E_y$ stands for the initial value of $y^x_t$ at $t = 0$. Let us consider the semigroup of operators $T^{\beta}_t, t \geq 0$, on $C(M)$ defined by the formula

$$T^{x',x,\beta}_t g(y) = T^{\beta}_t g(y) = E_y g(y^x_t) \exp \left( \int_0^t \beta f(x', y^x_s)ds \right),$$

where $\beta \in E^d, \ \beta f$ is a scalar product and the index $y$ in $E_y$ means the initial value of $y^x_t$ at $t = 0$. In the case if some inequality is uniform over $y \in M$, this index may be dropped in the calculus.

**Lemma 1** Let assumptions $(A_f), (A_B), (A_C)$ be satisfied. Then for any $\beta$ the operator $T^{\beta}_1$ is compact in the space $C(M)$.

**Lemma 2** Let assumptions $(A_f), (A_B), (A_C)$ be satisfied. Then the spectral radius $r(T^{\beta}_1)$ is a simple eigenvalue of $T^{\beta}_1$ separated from the rest of the spectrum and its eigen–function $e^{\beta}_1$ belongs to the cone $C^+(M)$. Moreover, function $r(T^{\beta}_1)$ is smooth (of $C^\infty$) in $\beta$ and for any $b > 0$ the function $e^{\beta}_1$ is bounded and separated away from zero uniformly in $|\beta| < b$ and all $x', x$.

**Lemma 3** Let $\beta \in E^d$, and let assumptions $(A_f), (A_B), (A_C)$ be satisfied. Then there exists a limit uniformly in $x, x'$,

$$\tilde{H}(x', x, \beta) = \lim_{t \to \infty} t^{-1} \log E_y \exp \left( \beta \int_0^t f(x', y^x_s)ds \right);$$

moreover, $\tilde{H}(x', x, \beta) = \log r(T^{x',x,\beta}_1)$. The function $\tilde{H}(x', x, \beta)$ is of $C^\infty$ in $\beta$ and convex in $\beta$. For any $b > 0$ there exists $C(b)$ such that, for any $y, |\beta| < b$ and for all values of $t > 0$ uniformly in $x, x'$,

$$|t^{-1} \log E_y \exp \left( \beta \int_0^t f(x', y^x_s)ds \right) - \tilde{H}(x', x, \beta)| \leq C(b)t^{-1}. \quad (7)$$

Notice that $|\tilde{H}(x', x, \beta)| \leq \|f\|_C|\beta|$.

In what follows, $\nabla_\beta \tilde{H}$ stands for the gradient of $\tilde{H}$ with respect to $\beta$.

**Lemma 4** Let assumptions $(A_f), (A_B), (A_C)$ be satisfied. Then for any $b > 0$ the functions $\tilde{H}$ and $\nabla_\beta \tilde{H}$ are uniformly continuous in $(x', x, \beta)$, for $|\beta| < b$.

Lemmas [1]–[4] are standard (cf. [14] or [13]). They are based on Frobenius-type theorems for positive compact operators (see [8]) and the theory of perturbations of linear operators (see [6 Chapter 2]).
Lemma 5 Let the assumptions \((A_f), (A_B), (A_C)\) hold true, \(b > 0, t(\varepsilon) \to \infty\) and \(t(\varepsilon) = o(\log \varepsilon^{-1})\) as \(\varepsilon \to 0\). Then for any \(\nu > 0\) there exist \(\delta(\nu) > 0, \varepsilon(\nu) > 0\) such that for \(\varepsilon \leq \varepsilon(\nu)\) uniformly with respect to \(t_0, x', x, x_0, y_0, x_{t_0}, |\beta| \leq b\), the inequality holds on the set \(\{|x_{t_0} - x| < \delta(\nu)\}\),

\[
\left| \log E(\exp(\beta \int_{t_0}^{t_0+\varepsilon} f(x', y_s)ds) - \varepsilon H(x', x, \beta)) \right| \leq \nu t(\varepsilon). \tag{8}
\]

Moreover, if \(\Delta \leq \Delta(\nu) = (1 + \|f\|_C)^{-1}\delta(\nu)/2\) and \(\varepsilon\) is small enough, then uniformly with respect to \(t_0, x', x, x_0, y_0, \delta \leq \delta(\nu), |x_{t_0} - x| < \delta\), and \(|\beta| \leq b\),

\[
\exp(\varepsilon^{-2}\Delta H(x', x, \beta) - \nu \Delta \varepsilon^{-2})
\leq E \left( \exp(\frac{\beta}{t_0+\Delta} \int_{t_0}^{t_0+\Delta} f(x', y_s)ds) \right) \leq \exp(\varepsilon^{-2}\Delta H(x', x, \beta) + \nu \Delta \varepsilon^{-2}). \tag{9}
\]

Remark. Let us emphasize that any couple \((\Delta, \delta)\) satisfying only \(\Delta \leq \Delta(\nu)\) and \(\delta \leq \delta(\nu)\) would do.

Proof. Step 1. It suffices to prove \((8)\) and \((9)\) for \(t_0 = 0\). Moreover, since \(H\) is continuous, it suffices to check both inequalities for \(x = x_0\). Indeed, the bound

\[
\left| \log E \exp \left( \beta \int_{0}^{t(\varepsilon)} f(x', y_s)ds \right) - \varepsilon H(x', x_0, \beta) \right| \leq \nu t(\varepsilon)
\]

implies

\[
\left| \log E \exp \left( \beta \int_{0}^{t(\varepsilon)} f(x', y_s)ds \right) - \varepsilon H(x', x, \beta) \right|
\leq t(\varepsilon)(\nu + |\tilde{H}(x', x, \beta) - H(x', x_0, \beta)|),
\]

and we use the uniform continuity of the function \(H\) on compact sets (remind that \(|\beta| \leq b\)). The same arguments are applicable to the second inequality of the assertion of the lemma. So, in the sequel we consider the case \(x_0 = x\).

Let us show first that

\[
\sup_{x', x_0} |t(\varepsilon)^{-1} \log E \exp \left( \beta \int_{0}^{t(\varepsilon)} f(x', y_s)ds \right) - H(x', x, \beta)| \leq \nu \tag{10}
\]

if \(\varepsilon\) is small enough. Due to Lemma 3 it would be correct if \(y_s\) were replaced by \(y_s^x\) and \(t(\varepsilon) \geq \nu^{-1}C(b)\). We will also use the bounds

\[
\sup_{0 \leq s \leq t} |x_s - x_0| \leq \varepsilon^2 t \|f\|_C, \quad \exp(Ct(\varepsilon))t(\varepsilon)^2 \varepsilon^2 \to 0 (\forall C), \quad \varepsilon \to 0. \tag{11}
\]
Let \(|f(x', y) - f(x', y')| \leq L_f|y - y'|\) for all \(y, y', x', L_f > 0\), \(C_f = \|f\|_C\). We estimate for \(t(\varepsilon) > \nu^{-1}C(b)/4\),

\[
E \exp \left( \beta \int_0^{t(\varepsilon)} f(x', y_s)ds \right)
\]

\[
\times \left\{ I \left( \sup_{0 \leq t \leq t(\varepsilon)} |y_t - y_t^x| \leq \nu/(4L_fb) \right) + I \left( \sup_{0 \leq t \leq t(\varepsilon)} |y_t - y_t^x| > \nu/(4L_fb) \right) \right\}
\]

\[
\leq E \exp \left( \beta \int_0^{t(\varepsilon)} f(x', y_s^x)ds + t(\varepsilon)\nu/4 \right) I \left( \sup_{0 \leq t \leq t(\varepsilon)} |y_t - y_t^x| \leq \nu/(4L_fb) \right)
\]

\[
+ \exp(C_fbt(\varepsilon)\nu)EI \left( \sup_{0 \leq t \leq t(\varepsilon)} |y_t - y_t^x| > \nu/(4L_fb) \right)
\]

\[
\leq E \exp \left( \beta \int_0^{t(\varepsilon)} f(x', y_s^x)ds \right) \exp(t(\varepsilon)\nu/4)
\]

\[
+ 16L_f^2b^2 \exp(C_fbt(\varepsilon)\nu - \nu^2E \sup_{t \leq t(\varepsilon)} |y_t - y_t^x|^2). \quad (12)
\]

By virtue of the Lemma \(\mathbb{3}\) we have

\[
E \exp \left( \beta \int_0^{t(\varepsilon)} f(x', y_s^x)ds \right) \leq \exp(t(\varepsilon)(\bar{H}(x', x, \beta) + \nu/4)), \quad (13)
\]

if \(\varepsilon\) is small enough.

Let us estimate the second term in (12). By virtue of the inequalities for the Itô and Lebesgue integrals, we have

\[
E \sup_{t' \leq t} |y_{t'} - y_0^x|^2 \leq CE \int_0^t |C(x_s, y_s) - C(x, y_s^x)|^2ds
\]

\[
+ C tE \int_0^t |B(x_s, y_s) - B(x, y_s^x)|^2ds
\]

\[
\leq C \int_0^t E|x_s - x|^2ds + C \int_0^t E \sup_{u \leq s} |y_s - y_s^x|^2ds
\]

\[
\leq Ct^2\varepsilon^2 + C \int_0^t E \sup_{u \leq s} |y_u - y_u^x|^2ds.
\]

By virtue of Gronwall's lemma, one gets

\[
E \sup_{t' \leq t} |y_{t'} - y_0^x|^2 \leq Ct^2\varepsilon^2 \exp(Ct).
\]
In particular,
\[ E \sup_{t' \leq t(\varepsilon)} \left| y_{t'} - y_{t'}^x \right|^2 \leq Ct(\varepsilon)^2 \varepsilon^2 \exp(Ct(\varepsilon)). \] (14)

So the second term in (12) does not exceed the value \( o(\exp(Kt(\varepsilon))) \) for any \( K < 0 \). Indeed, for any such \( K \) we have,
\[ \exp(t(\varepsilon)(C_f b t(\varepsilon) - K)) - 2Ct(\varepsilon)^2 \varepsilon^2 \rightarrow 0, \]
because \( \exp(t(\varepsilon)C) \) for any \( C > 0 \)
increases slower than \( \varepsilon^{-2} \) due to the assumption \( t(\varepsilon) = o(\log \varepsilon^{-1}), \varepsilon \rightarrow 0 \).
Hence, we get with any \( K < 0 \) for \( \varepsilon > 0 \) small enough,
\[
E \exp \left( \beta \int_0^{t(\varepsilon)} f(x', y_s) ds \right) \leq E \exp \left( \beta \int_0^{t(\varepsilon)} f(x', y_s^x) ds \right) \exp(t(\varepsilon)\nu/4) + C \exp(Kt(\varepsilon))
\]
\[
\leq \exp(t(\varepsilon)(\tilde{H}(x', x, \beta) + \nu/2)) + C \exp(Kt(\varepsilon)),
\]
by virtue of (13). The upper bound in (10) follows.

The lower bound in (10) may be established similarly. For the convenience of the reader we show the calculus. We estimate for \( t(\varepsilon) > \nu^{-1}C(b)/4 \),
\[
E \exp \left( \beta \int_0^{t(\varepsilon)} f(x', y_s^x) ds \right) \times \left\{ I \left( \sup_{0 \leq t \leq t(\varepsilon)} \left| y_t - y_t^x \right| \leq \nu/(4L_f b) \right) + I \left( \sup_{0 \leq t \leq t(\varepsilon)} \left| y_t - y_t^x \right| > \nu/(4L_f b) \right) \right\}
\]
\[
\leq E \exp \left( \beta \int_0^{t(\varepsilon)} f(x', y_s) ds + t(\varepsilon)\nu/4 \right) I \left( \sup_{0 \leq t \leq t(\varepsilon)} \left| y_t - y_t^x \right| \leq \nu/(4L_f b) \right)
\]
\[
+ \exp(C_f b t(\varepsilon)\nu) EI \left( \sup_{0 \leq t \leq t(\varepsilon)} \left| y_t - y_t^x \right| > \nu/(4L_f b) \right)
\]
\[
\leq E \exp \left( \beta \int_0^{t(\varepsilon)} f(x', y_s) ds \right) \exp(t(\varepsilon)\nu/4)
\]
\[
+ 16L_f^2 b^2 \exp(C_f b t(\varepsilon)\nu) \nu^{-2} E \sup_{t \leq t(\varepsilon)} \left| y_t - y_t^x \right|^2.
\] (15)

Since the second term in (12) is \( o(\exp(Kt(\varepsilon))) \) with any \( K < 0 \), this implies
the bound
\[
E \exp \left( \beta \int_0^{t(\varepsilon)} f(x', y_s^x) ds \right)
\]
\[
\leq E \exp \left( \beta \int_0^{t(\varepsilon)} f(x', y_s) ds \right) \exp(t(\varepsilon)\nu/4) + C \exp(K t(\varepsilon)),
\]

or, equivalently,
\[
E \exp \left( \beta \int_0^{t(\varepsilon)} f(x', y_s) ds \right) \geq E \exp \left( \beta \int_0^{t(\varepsilon)} f(x', y_s) ds \right) \exp(-t(\varepsilon)\nu/4) - C \exp(K t(\varepsilon)).
\]

Now due to (13), we get, with any \( K < 0 \) and \( \varepsilon > 0 \) small enough,
\[
E \exp \left( \beta \int_0^{t(\varepsilon)} f(x', y_s) ds \right) \geq \exp(t(\varepsilon)(\bar{H}(x', x, \beta) - \nu/2)) - C \exp(K t(\varepsilon)),
\]

which implies the lower bound in (11).

Notice that both bounds in (11) are uniform with respect to \( |\beta| \leq b \) and \( x', x, y_0 \). Since the function \( H \) is continuous, we get on the set \( \{|x_{t_0} - x| < \delta(\nu)\} \),
\[
\sup_{x', x_0, t_0, |\beta| \leq b} \left| \log E \left( \exp \left( \beta \int_0^{t_0 + t(\varepsilon)} f(x', y_s^\nu) ds \right) \big| \mathcal{F}_{t_0} \right) - t(\varepsilon)(\bar{H}(x', x, \beta)) \right| \leq \nu t(\varepsilon)(16)
\]
if \( \delta(\nu) \) is small enough.

**Step 2.** Let \( \Delta = (1 + \|f\|_C)^{-1}\delta(\nu)/2 = \Delta(\nu) \) and \( N = \Delta^{-2} t(\varepsilon)^{-1} \). Then \( \sup_{0 \leq s \leq N t(\varepsilon)} |x_s - x_0| \leq \delta(\nu)/2 \). Let \( |x - x_0| < \delta(\nu)/2 \). So, \( \sup_{0 \leq s \leq N t(\varepsilon)} |x_s - x| < \delta(\nu) \). In particular, \( |x_{kt(\varepsilon)} - x| < \delta(\nu) \) for any \( 1 \leq k \leq N \). By induction, we get from (16) for such \( k \),
\[
\exp(kt(\varepsilon)(\bar{H}(x', x, \beta) - \nu kt(\varepsilon)))
\leq E \exp \left( \beta \int_0^{kt(\varepsilon)} f(x', y_s) ds \right) \leq \exp(kt(\varepsilon)(\bar{H}(x', x, \beta) + \nu kt(\varepsilon))),
\]

or, after the time change,
\[
\exp(kt(\varepsilon)(\bar{H}(x', x, \beta) - \nu kt(\varepsilon)))
\leq E \exp \left( \beta \int_0^{kt(\varepsilon)\varepsilon^2} f(x', Y_s) ds \right) \leq \exp(kt(\varepsilon)(\bar{H}(x', x, \beta) + \nu kt(\varepsilon))).
\]
Since $H$ is continuous then we obtain for $k = N$,

$$
\exp(\varepsilon^{-2} \Delta \tilde{H}(x', x, \beta) - \nu \Delta \varepsilon^{-2})
\leq E \exp \left( \beta \varepsilon^{-2} \int_{0}^{\Delta} f(x', Y_s) ds \right)
\leq \exp(\varepsilon^{-2} \Delta \tilde{H}(x', x_0, \beta) + \nu \Delta \varepsilon^{-2}). \tag{17}
$$

The Lemma is proved. QED

The next Lemma is an improved version of the Lemma 7.5.2 from [3]. Although we will not use it explicitly, its technique is essential.

**Lemma 6** ([2, 3]). Let $S(\varphi) < \infty$. If $\psi^n$ is a sequence of step functions tending uniformly to $\varphi$ in $C([0, T]; \mathbb{R}^d)$ as $n \to \infty$, then there exists a sequence of piecewise linear functions $\chi^n$ (with the same partitions) which also tend uniformly to $\varphi$ and such that

$$
\limsup_{n \to \infty} \int_{0}^{T} L(\psi^n_s, \chi^n_s) ds \leq S(\varphi).
$$

Moreover, one may assume without loss of generality that for any $s$ there exists a value

$$
\beta_s = \arg\max_{\beta} (\beta \dot{\chi}^n_{s+} - \tilde{H}(\psi^n_s, \dot{\psi}^n_s, \beta))
$$

and

$$
L(\psi^n_s, \alpha) > L(\psi^n_s, \dot{\chi}^n_{s+}) + (\alpha - \dot{\chi}^n_{s+}) \beta_s \quad \forall \alpha \neq \dot{\chi}^n_s.
$$

If $\hat{\psi}$ is close enough to $\psi^n_s$ then there exists a value

$$
\hat{\beta}_s = \arg\max_{\beta} (\beta \dot{\chi}^n_{s+} - \tilde{H}(\psi^n_s, \hat{\psi}, \beta)),
$$

$$
L(\psi^n_s, \hat{\psi}, \alpha) > L(\psi^n_s, \hat{\chi}^n_{s+}) + (\alpha - \dot{\chi}^n_{s+}) \hat{\beta}_s \quad \forall \alpha \neq \dot{\chi}^n_s
$$

and

$$
L(\psi^n_s, \hat{\psi}, \dot{\chi}^n_{s+}) \to L(\psi^n_s, \psi^n_s, \dot{\chi}^n_{s+}), \quad \hat{\psi} \to \psi^n_s.
$$

We added to the original assertion the property that $\chi^n_t$ may be chosen piecewise linear. Indeed, such functions are used in the proof; see [3, Section 7.5]. The existence of $\beta_s$ asserted in the lemma also follows from the proof; see [2] or [3]. Assertions about $\hat{\psi}$ and $\hat{\beta}_s$ also added to the original assertion can be deduced from the proof using similar arguments.

In fact, there is a little gap in the original proof, namely, an additional assumption was used which was not formulated explicitly. This is why we present a precise statement and give necessary comments in the Appendix.
4 Proof of theorem 1

1. First part of the proof: the lower bound. Let $S(\varphi) < \infty$, and $\nu > 0$. To establish the lower bound, we will show the inequality: given any $\nu > 0$, and any $\delta > 0$, we have for $\varepsilon > 0$ small enough,

$$\varepsilon^2 \log P_x(\rho(X^{\varepsilon}, \varphi) < \delta) \geq -S(\varphi) - \nu.$$ 

Denote $H(x, \beta) = \tilde{H}(x, x', \cdot)$. The existence of the limit $\tilde{H}(x, x', \cdot)$ for any $x, x'$, and its differentiability and continuity are asserted in Lemmas 3 and 4.

Throughout the proof, we may and will assume that for any $s$, $L(\varphi_s, \dot{\varphi}_s) < \infty$. Indeed, this may be violated only on a set of Lebesgue measure zero.

Notice that due to the boundedness of the function $f$, this inequality implies $\sup_s |\dot{\varphi}_s| \leq \|f\|_C$, since for any $|\alpha| > \|f\|_C$, we have $L(x, \alpha) = +\infty$. Unlike in the previous section, in the sequel both $X_0 = x_0$ and $Y_0 = y_0$ are fixed, hence, the symbols $P$ and $E$ will be used without indices.

2. We are going to reduce the problem of estimation from below the probability $P(\rho(X, \varphi) < \delta)$ to that for the probability $P(\rho(X^\varphi, \varphi) < \delta')$, where $X_t^\psi := x_0 + \int_0^t f(\psi_s, Y_s)ds$, $\forall \psi$, and further to $P(\rho(X^\psi, \chi) < \delta')$, where both $\psi, \chi$ approximate $\varphi$. The rough idea is eventually to choose a step function as $\psi$ and piecewise linear one as $\chi$, however we are going to perform these approximations gradually. A step function is needed because we only have a technical tool – the Lemma 5 – established for this very case. A piecewise linear $\psi$ is not necessary, but convenient. Eventually we will consider a finite-dimensional “discretized” subset of the set $\{\rho(X, \varphi) < \delta\}$ with appropriately chosen $\Delta$, $X^\psi$, deterministic curves $\psi, \chi$, and constants $\delta'_s$: in particular, we will choose $\delta'_1 << \delta'_2 << \ldots \ll \delta'_{T/\Delta} \ll \delta$. While performing all these approximations, we need to establish simultaneously a special property: at any point $s$, the Fenchel-Legendre adjoint to the $\chi_s$ variable $\beta_s = \beta_s[\psi_s, \chi_s]$ (see below) can be chosen uniformly bounded.

3. For any nonrandom curve $\psi \in C([0, T]; E^d)$ – although we will apply this firstly to $\varphi$, but other functions are also necessary for the analysis below – we have, due to the Lipschitz condition on $f$,

$$\{\rho(X, \varphi) < \delta\} \supset \{\rho(X^\psi, \chi) < \delta'\} \quad (18)$$

if $\delta'$ and $\lambda := \rho_{0,T}(\varphi, \psi)$ are small enough with respect to $\delta$. (A small constant $\lambda > 0$ is used just within this step.) E.g., $\delta' < \delta(e^{CT} CT + 1)^{-1}/2$, $\lambda < \delta(e^{CT} CT + 1)^{-1}/2$ suffice, see below. Indeed,

$$X_t = x + \int_0^t f(X_s, Y_s)ds, \quad X_t^\psi = x + \int_0^t f(\psi_s, Y_s)ds,$$
thence,
\[ |X_t - X_t^\psi| \leq \int_0^t |f(X_s, Y_s) - f(\psi_s, Y_s)| ds \leq C \int_0^t |X_s - \psi_s| ds \]
\[ \leq C \int_0^t |X_s - X_s^\psi| ds + C \int_0^t |X_s^\psi - \chi_s| ds + C \int_0^t |\chi_s - \psi_s| ds; \]
so on the set \( \{\rho(X^\psi, \chi) < \delta'\} \),
\[ |X_t - X_t^\psi| \leq C \int_0^t |X_s - X_s^\psi| ds + C\delta't + C\lambda t, \]
and, moreover, for every \( \omega \in \{\rho(X^\psi, \chi) < \delta'\} \) and \( 0 \leq t \leq T \),
\[ \sup_{0 \leq s' \leq s''} |X_{t'} - X_{t''}^\psi|(\omega) \leq C \int_0^t \sup_{0 \leq s' \leq s} |X_s - X_s^\psi|(\omega) ds + C(\delta' + \lambda)t. \]
Since all SDE solutions \( X_t, X_t^\psi \) are continuous, \( \sup_{0 \leq s' \leq T} |X_{t'} - X_{t''}^\psi| < \infty \) for each \( \omega \in \Omega \). By the standard “non-random” Gronwall inequality this implies that on the same set \( \{\rho(X^\psi, \chi) < \delta'\} \),
\[ \rho(X, X^\psi)(\omega) \leq e^{CT}(\delta' + \lambda)T. \]
Now, still for any \( \omega \in \{\rho(X^\psi, \chi) < \delta'\} \),
\[ \rho(X, \varphi)(\omega) \leq \rho(X, X^\psi)(\omega) + \rho(X^\psi, \chi)(\omega) + \rho(\chi, \varphi) \]
\[ \leq e^{CT}(\delta' + \lambda)T + \delta' + \lambda = (\delta' + \lambda)(e^{CT}T + 1). \]
Therefore, (18) holds true. For example,
\[ \delta' < \delta(e^{CT}T + 1)^{-1}/2, \quad \lambda < \delta(e^{CT}T + 1)^{-1}/2 \]
suffice. In particular, it is true that
\[ \{\rho(X, \varphi) < \delta\} \supset \{\rho(X^\psi, \chi) < \delta'\}, \]
if \( \delta' \) and \( \lambda \) are small enough with respect to \( \delta \). This bound will be used while establishing a lower bound.

4. While establishing an upper bound, an opposite inclusion will be useful,
\[ \{\rho(X, \varphi) < \delta\} \subset \{\rho(X^\psi, \chi) < 2\delta(KT + 1)\}, \quad (19) \]
if \( \lambda := \max(\rho(\varphi, \psi), \rho(\varphi, \chi)) \leq \delta \). Indeed,
\[ |X_t - X_t^\psi| \leq \int_0^t |f(X_s, Y_s) - f(\psi_s, Y_s)| ds \leq K \int_0^t |X_s - \psi_s| ds \]
\[ \leq K \int_0^t |X_s - \varphi_s| ds + K \int_0^t |\psi_s - \varphi_s| ds; \]
so on the set \( \{ \rho(X, \varphi) < \delta \} \),
\[
|X_t - X_t^\psi| \leq K\delta t + K\lambda t,
\]
and, moreover, on the same set,
\[
\rho(X, X^\psi) \leq K(\delta + \lambda)T.
\]
Now, (19) follows from the inequalities,
\[
\rho(X^\psi, \chi) \leq \rho(X, X^\psi) + \rho(X, \varphi) + \rho(\chi, \varphi)
\]
\[
\leq K(\delta + \lambda)T + \delta + \lambda.
\]
5. Our next goal is the choice of appropriate functions \( \chi \) and \( \psi \). It is essential to keep the integral \( \int_0^T L(\varphi_s, \dot{\chi}_s) \, ds \) close to \( S(\varphi) \). Also, by technical reasons we want some discretization. Hence, we will use a trick well-known in the definition of stochastic integrals based on the following Lemma.

**Lemma 7** Suppose \( g \in L_1([0, T]; R^d) \) and let \( \kappa_m(a) := [2^m a]2^{-m} \). Then there exists a sequence \( m' \to \infty \) such that for almost every \( a \in [0, 1] \),
\[
\int_0^T |g(s) - g(\kappa_{m'}(s + a) - a)| \, ds \to 0, \quad m' \to \infty.
\]
(20)

For the proof for \( g \in L_2([0, T]; R^d) \) see [9, Theorem 2.8.2], however, for \( L_1([0, T]; R^d) \) the proof practically does not change: we approximate \( g \) by continuous functions \( g_n \) which are dense in \( L_1([0, T]; R^d) \) — and integrate with respect to \( a \in [0, 1] \). Then, for each \( g_n \in C([0, T]; R^d) \) the statement follows for every \( a \) and for the limiting function \( g \) the assertion (20) follows for almost every \( \omega \) over some subsequence, as required.

Hence, applying this Lemma we may fix some \( a \in [0, 1] \) for which there exists a sequence \( m' \to \infty \) such that
\[
\int_0^T |L(\varphi_s, \dot{\varphi}_s) - L(\varphi_{\kappa_{m'}(s+a)-a}, \dot{\varphi}_{\kappa_{m'}(s+a)-a})| \, ds \to 0, \quad m' \to \infty.
\]
(21)

Simultaneously for almost every \( a \in [0, 1] \), by virtue of the same Lemma and because \( \varphi \) is absolutely continuous, we also have,
\[
\int_0^T |\dot{\varphi}_s - \dot{\varphi}_{\kappa_{m'}(s+a)-a}| \, ds \to 0, \quad m' \to \infty,
\]
(22)
and
\[
\sup_{0 \leq t \leq T} |\varphi_t - \varphi_{\kappa_{m'}(t+a)-a}| \to 0, \quad m' \to \infty,
\]
(23)
each time over a new subsequence. Yet, to simplify notations, in the sequel \( m' \) will be replaced by \( m \). Denote
\[
\psi_t^m := \varphi_{\kappa_m(t+a)-a}, \quad \dot{\psi}_t^m := \dot{\varphi}_{\kappa_m(t+a)-a}, \quad \chi_t^m := \chi_t := \varphi_0 + \int_0^t \dot{\chi}_s \, ds.
\]
Notice that $\psi$ is piecewise constant (step function) with finitely many values, while $\chi$ is piecewise linear with finitely many values of slopes.

Let

$$S^\rho(\chi) := \int_0^T L(\varphi_s, \dot{\chi}_s) \, ds.$$ 

Notice that $S^\varphi(\varphi) = S(\varphi)$. Then (21) implies

$$|S(\varphi) - S^\psi(\chi)| \to 0, \quad m' \to \infty. \tag{24}$$

At the same time we have,

$$\{\rho(X, \varphi) < \delta\} \supset \{\rho(X, \psi) < \delta/2\}, \tag{25}$$

if $m$ is large enough. Moreover, in addition,

$$\{\rho(X, \psi) < \delta/2\} \supset \{\rho(X^\varphi, \chi) < \tilde{\delta}'\}, \tag{26}$$

if $\tilde{\delta}'$ and $\lambda = \rho(\varphi, \chi)$ are small enough with respect to $\delta$; hence, we can fix the value $\tilde{\delta}'$ here.

So, we can choose the functions $\psi$ and $\chi$ so that, firstly, $2^{-m} \leq \Delta(\nu)$ (a value from the Lemma [5]); secondly,

$$|S^\psi(\chi) - S(\varphi)| \leq \nu; \tag{27}$$

and, finally (see above (18)), if $\tilde{\delta}'$ is small enough then also

$$\{\rho(X^\varphi, \chi) < \tilde{\delta}'\} \supset \{\rho(X^\psi, \chi) < \tilde{\delta}'\}; \tag{28}$$

for the latter we need only $\rho(\varphi, \chi) + \rho(\varphi, \psi)$ to be small enough.

6. Suppose for some $s \in [0, T]$, the set $\{\alpha : L(\psi_s, \alpha) < \infty\}$ has a non-empty interior with respect to its linear hull $L[f, \psi_s]$, that is, to the minimal linear subspace containing $\{\alpha : L(\psi_s, \alpha) < \infty\}$. For this interior – non-empty or empty – we will use notation $L^\circ[f, \psi_s]$. Since $L(\psi_s, \dot{\chi}_s) < \infty$, this value is attained as a lim inf of the values $L(\psi_s, \alpha)$, $\alpha \in L^\circ[f, \psi_s]$, as $\alpha \to \dot{\chi}$ in the case $L^\circ[f, \psi_s] \neq \emptyset$, see [12]. It is a property of any such $\alpha$ that there exists a finite adjoint vector $\beta = \arg\max_{\beta} (\alpha \beta - H(\psi_s, \beta))$ given $\alpha$, although this adjoint may not be necessarily unique which we will discuss shortly. Notice that, in particular, we have

$$H(\psi_s, \beta) = (\alpha \beta - L(\psi_s, \alpha)), \quad \text{as well as} \quad L(\psi_s, \alpha) = (\alpha \beta - H(\psi_s, \beta)).$$

We can choose a vector $\dot{\dot{\chi}}_s := \alpha \in L^\circ[f, \psi_s]$ so that the value $L(\psi_s, \dot{\dot{\chi}}_s)$ is close enough to $L(\psi_s, \dot{\chi}_s)$. Recall that there are finitely many vector-values of $\dot{\chi}_s$ for any given $m$ and $\alpha$; correspondingly, we will choose finitely many approximations satisfying $\dot{\dot{\chi}}_s \in L^\circ[f, \psi_s]$. Let us also choose some adjoint $\beta$ for each $\alpha = \dot{\dot{\chi}}_s$ and denote it by $\beta[\psi_s, \dot{\dot{\chi}}_s]$. \hfill 13
In the case if the set \( \mathcal{L}^\circ[f, \varphi_s] \) is empty, the function \( H(\varphi_s, \beta) \) is linear in \( \beta \) and one can choose \( \hat{\varphi}_s := \varphi_s \) and \( \beta[\varphi_s, \hat{\varphi}_s] = 0 \), see Appendix A.

Notice that whatever is the case – the interior \( \mathcal{L}^\circ[f, \psi_s] \) empty or not – and whatever is the choice of \( \beta \) – if not unique – in all cases there are finitely many of vectors \( \beta[\psi_s, \hat{\varphi}_s] \) chosen. Hence, we may denote

\[
\max_{0 \leq s \leq T} |\beta[\psi_s, \hat{\varphi}_s]| =: b < \infty. \tag{29}
\]

Notice that this value is fixed from now on. Let

\[
\tilde{\chi}_t := x + \int_0^t \dot{\tilde{\chi}}_s \, ds, \quad S^\psi(\tilde{\chi}) := \int_0^T L(\psi_s, \dot{\tilde{\chi}}_s) \, ds.
\]

We may assume that \( \tilde{\chi} \) is as close to \( \varphi \) as we like, say, \( \rho(\tilde{\chi}, \psi) < \nu/3 \) and also

\[
|S^\psi(\tilde{\chi}) - S(\varphi)| \leq \nu/3. \tag{30}
\]

7. In the general case, the discretisations of \( \varphi \) should be read \( \varphi^{\Delta, a} = (\varphi_{\Delta - \hat{a}}, \varphi_{2\Delta - \hat{a}}, \ldots, \varphi_{m\Delta - \hat{a}}, \varphi_T) \), where \( \hat{a} = a - [a/\Delta]\Delta \); if \( a = 0 \) then we may use the approximation \( \varphi^{\Delta} = (\varphi_{\Delta}, \varphi_{2\Delta}, \ldots, \varphi_{m\Delta}) \), \( m\Delta = T \). Notice that ‘almost every value’ of \( a \) does not guarantee any particular value, so that we cannot be sure about taking \( a = 0 \). Hence, let us consider the general case here. Denote \( k\Delta - \hat{a} =: t_k, 1 \leq k \leq m \), and \( t_{m+1} := T \) in the case of \( \hat{a} \neq 0 \) (and no \( t_{m+1} \) in the case of \( \hat{a} = 0 \)).

Since the drift of the diffusion \( X^\psi \) is bounded – \( \|f\|_{C} < \infty \) – we have straight away (however, cf. [3] proof of the Lemma 7.5.1)),

\[
\{\rho(X^\psi, \chi) < \delta'\} \supset \{\rho((X^\psi)^{\Delta, a}, \chi^{\Delta, a}) < \delta''\}, \tag{31}
\]

if \( \delta'' \) and \( \Delta \) are small enough,

\[
\delta'' < \delta''(\delta') \quad \text{and} \quad \Delta \leq \Delta(\delta') \tag{32}
\]

(notice that here \( \Delta \leq \Delta(\delta'') \) is not required), and assuming all our curves start at \( x_0 \) at time zero (hence, we do not include the starting point into the definition of \( \varphi^{\Delta} \)). Here for discretized curves we use the metric,

\[
\rho(\psi^{\Delta, a}, \chi^{\Delta, a}) := \sup_k |\psi_{t_k} - \chi_{t_k}|.
\]

Now, we are going to estimate from below the value in the right hand side of the inequality,

\[
P(\rho((X^\psi)^{\Delta, a}, \chi^{\Delta, a}) < \delta'') \geq E \prod_i I(|X_{t_k}^\psi - \chi_{t_k}| < \delta_i'), \tag{33}
\]

where \( \delta'_1 < \delta'_2 < \ldots < \delta'_{m+1} = \min(\delta(\nu), \delta'') \), \( i = 1, \ldots, m \), and \( \delta(\nu) \) is from the Lemma 5; here all values \( \delta'_i \) and certain auxiliary values \( z_i \) will be chosen in the next two steps as follows:

\[
m_{\nabla_\beta H}(\delta_{k-1} + z_{k-1}) + \frac{\kappa}{2} \delta_{k-1} \leq \frac{\kappa}{2} \delta_k, \quad \& \quad \delta'_{k-1} \leq \frac{\delta'_k}{2}, \quad \& \quad m_H(\delta'_k) \leq \nu.
\]
where $0 < \kappa \leq 1$. Emphasize that $\delta''$ and $\Delta$ may be chosen arbitrarily small at this stage; in particular, we require that they should satisfy the conditions of the Lemma 5, which will be used in the sequel, that is, we do require $\delta'' \leq \delta(\nu)$ and $\Delta \leq \Delta(\nu)$. Hence, both $\delta''$ and $\Delta$ are fixed at this stage.

8. Now everything is prepared for the lower estimate. We start with the estimation of the conditional expectation $E(I(\{|X_{t_{m+1}}^\psi - \chi_{t_{m+1}}| < \delta'_{m+1}\} \mid \mathcal{F}_{t_m})$ on the set $\{|X_{t_{m}}^\psi - \chi_{t_{m}}| < \delta'_m\}$. Let us apply the Cramér transformation of measure. Let $|\beta| \leq b$, we will choose this vector a bit later (as $[\psi_{t_m}, \chi_{t_{m+1}}]$). We get,

$$E(I(\{|X_{t_{m+1}}^\psi - \chi_{t_{m+1}}| < \delta'_{m+1}\} \mid \mathcal{F}_{t_m}) = E^\beta(I(\{|X_{t_{m+1}}^\psi - \chi_{t_{m+1}}| < \delta'_{m+1}\}) \times \exp \left( -\varepsilon^{-2}\beta(X_{t_{m+1}}^\psi - X_{t_{m}}^\psi) - \varepsilon^{-2}\Delta_m \tilde{H}_m^\varepsilon,\psi(X_{t_{m}}^\psi, \beta) \right) \mid \mathcal{F}_{t_m}),$$

where $E^\beta$ is the (conditional) expectation with respect to the measure $P^\beta$ defined on the sigma-field $\mathcal{F}_{t_{m+1}}$ given $\mathcal{F}_{t_m}$, by its density

$$\frac{dP^\beta}{dP}(\omega) = \exp \left( -\varepsilon^{-2}\beta(X_{t_{m+1}}^\psi - X_{t_{m}}^\psi) - \varepsilon^{-2}\Delta_m \tilde{H}_m^\varepsilon,\psi(X_{t_{m}}^\psi, \beta) \right),$$

where $\Delta_m = t_{m+1} - t_m$ (and later on, $\Delta_k = t_{k+1} - t_k$; notice that all $\Delta_k \leq \Delta$)

$$\varepsilon^{-2}\Delta_m \tilde{H}_m^\varepsilon,\psi(X_{t_{m}}^\psi, \beta) := \log E \left( \exp \left( -\varepsilon^{-2}\beta(X_{t_{m+1}}^\psi - X_{t_{m}}^\psi) \right) \mid \mathcal{F}_{t_m} \right).$$

Notice that by virtue of the Lemma 5,

$$\tilde{H}_m^\varepsilon,\psi(X_{t_{m}}^\psi, \beta) \rightarrow \tilde{H}(\psi_{t_m}, X_{t_{m}}^\psi, \beta), \quad \varepsilon \rightarrow 0,$$

uniformly over $|\beta| \leq b$. Indeed, by definition of $X^\psi$,

$$X_{t_{m+1}}^\psi - X_{t_{m}}^\psi = \int_{t_m}^{t_{m+1}} f(\psi_{t_m}, Y_s) \, ds.$$

Thus, the inequality [5] of the Lemma 5 implies,

$$\left| \tilde{H}_m^\varepsilon,\psi(X_{t_{m}}^\psi, \beta) - \tilde{H}(\psi_{t_m}, X_{t_{m}}^\psi, \beta) \right|$$

$$= \left| \varepsilon^2 \Delta_m^{-1} \log E \left( \exp \left( -\varepsilon^{-2}\beta(X_{t_{m+1}}^\psi - X_{t_{m}}^\psi) \right) \mid \mathcal{F}_{t_m} \right) - \tilde{H}(\psi_{t_m}, X_{t_{m}}^\psi, \beta) \right|$$

$$= \left| \varepsilon^2 \Delta_m^{-1} \log E \left( \exp \left( -\varepsilon^{-2}\beta \int_{t_m}^{t_{m+1} + \Delta_m} f(\psi_{t_m}, Y_s) \, ds \right) \mid \mathcal{F}_{t_m} \right) - \tilde{H}(\psi_{t_m}, X_{t_{m}}^\psi, \beta) \right|$$

$$\leq \nu \varepsilon^{-2} \Delta_m.$$  

Also notice that on the set $\{|X_{t_{m}}^\psi - \chi_{t_{m}}| < \delta'_m\}$ we have

$$\varepsilon^{-2} \left| \beta(X_{t_{m}}^\psi - \chi_{t_{m}}) \right| \leq \varepsilon^{-2} b \delta'_m.$$
and on the set the set \( \{|X_{t_{m+1}}^\psi - \chi_{t_{m+1}}| < \delta'_{m+1}\}\),

\[
\varepsilon^{-2} \beta(X_{t_{m+1}}^\psi - \chi_{t_{m+1}}) \leq \varepsilon^{-2} b \delta'_{m+1}.
\]

Hence, for \( \varepsilon > 0 \) small enough on the set \( \{|X_{t_m}^\psi - \chi_{t_m}| < \delta'_m\} \) we estimate,

\[
E \left[ I(|X_{t_{m+1}}^\psi - \chi_{t_{m+1}}| < \delta'_{m+1}) \mid \mathcal{F}_t \right] \\
= E^\beta \left[ I(|X_{t_{m+1}}^\psi - \chi_{t_{m+1}}| < \delta'_{m+1}) \right. \\
\times \exp \left( \varepsilon^{-2} \beta(X_{t_{m+1}}^\psi - X_{t_m}^\psi) - \varepsilon^{-2} \Delta_m \tilde{H}_{m,\psi}(X_{t_{m+1}}^\psi, \beta) \right) \mid \mathcal{F}_t \right] \\
\geq E^\beta \left( I(|X_{t_{m+1}}^\psi - \chi_{t_{m+1}}| < \delta'_{m+1}) \exp \left( -\varepsilon^{-2} \Delta_m \beta \left( (\chi_{t_{m+1}} - \chi_{t_m}) / \Delta_m \right) \right) \\
- \frac{\Delta_m \beta}{\varepsilon^2} (\tilde{H}(\psi_{t_m}, X_{t_m}^\psi, \beta) + \nu) - \frac{b(\delta'_{m+1} + \delta'_m)}{\varepsilon^2} \right) \mid \mathcal{F}_t \right) .
\]

(34)

Now, let us choose \( \beta = \beta(m+1) = \beta[\psi_{t_m}, \hat{\chi}_{t_m}] = \text{argmax}_\beta (\beta \hat{\chi}_{t_m} - H(\psi_{t_m}, \beta)). \) As was explained above, \( |\beta(m+1)| \leq b \) and, moreover,

\[
\beta(m+1) \hat{\chi}_{t_m} - H(\psi_{t_m}, \beta(m+1)) = L(\psi_{t_m}, \hat{\chi}_{t_m}),
\]

and

\[
\hat{\chi}_{t_m} = \nabla_\beta H(\psi_{t_m}, \beta(m+1)).
\]

(35)

So (34) implies (with \( \beta = \beta(m+1) \)),

\[
E \left[ I(|X_{t_{m+1}}^\psi - \chi_{t_{m+1}}| < \delta'_{m+1}) \mid \mathcal{F}_t \right] \\
\geq \exp \left( -\varepsilon^{-2} \Delta_m (L(\psi_{t_m}, \hat{\chi}_{t_m}) + \nu) - b \varepsilon^{-2}(\delta'_{m+1} + \delta'_m) \right) \times \\
\times \exp \left( -\varepsilon^{-2} \Delta_m (\tilde{H}(\psi_{t_m}, X_{t_{m+1}}^\psi, \beta) - \tilde{H}(\psi_{t_m}, \psi_{t_{m+1}})) \right) \\
\times E^\beta(m+1) \left( I(|X_{t_{m+1}}^\psi - \chi_{t_{m+1}}| < \delta'_{m+1}) \mid \mathcal{F}_t \right) \\
\geq \exp \left( -\varepsilon^{-2} \Delta_m (L(\psi_{t_m}, \hat{\chi}_{t_m}) + 2\nu) - b \varepsilon^{-2}(\delta'_{m+1} + \delta'_m) \right) \times \\
\times E^\beta(m+1) \left( I(|X_{t_{m+1}}^\psi - \chi_{t_{m+1}}| < \delta'_{m+1}) \mid \mathcal{F}_t \right) .
\]

(36)

We have used uniform continuity of \( \tilde{H}(x, \cdot, \beta) \) over \( |\beta| \leq b \) and \( x \in \mathbb{R}^d \):

\[
|\tilde{H}(\psi_{t_m}, X_{t_{m+1}}^\psi, \beta) - \tilde{H}(\psi_{t_m}, \psi_{t_{m+1}}, \beta)| \\
\leq m_\tilde{H}(|X_{t_{m+1}}^\psi - \psi_{t_m}|) \leq m_\tilde{H}(\delta'_m) \leq \nu
\]
on the set $|X_{t_m}^\psi - \psi_{t_m}| \leq \delta'_m$ (recall that here $m_{\tilde{H}}$ stands for the modulus of continuity of $\tilde{H}$ for $|\beta| \leq b$), as $\delta'_m$ is small enough.

9. Let us show that given $\delta'_m < 0$, there exists $C_{m+1} > 0$ such that on the set $\{|X_{t_m}^\psi - \chi_{t_m}| < \delta'_m\}$,

$$E^{(m+1)}(I((X_{t_m}^\psi - \chi_{t_m+1}) > \delta'_m) | \mathcal{F}_{t_m}) \geq 1 - \exp(-C_{m+1} \varepsilon^{-2}),$$

(37)

if $\varepsilon$ is small enough. There exists a finite number of vectors $v_1, v_2, \ldots, v_{2d}$ such that $\|v_k\| = 1 \forall k$ (any orthonormal basis would do accomplished by its “symmetric” transformation, i.e. with each coordinate vector $v$ we consider $-v$ as well), and for any (non-random) vector $\xi$ and any positive $c$,

$$|\xi| > c \implies \exists 1 \leq k \leq 2d : \xi v_k > \kappa c,$$

where $\kappa = (1/d)^{1/2}$ (notice that $\kappa \leq 1$). Then,

$$E^{(m+1)}(I((X_{t_m}^\psi - \chi_{t_m+1}) > \delta'_m) | \mathcal{F}_{t_m})$$

$$\leq \sum_{k=1}^{2d} E^{(m+1)}(I((X_{t_m}^\psi - X_{t_m}^\psi - \chi_{t_m+1} + \chi_{t_m})v_k$$

$$> \kappa(\delta'_m - \delta'_m) | \mathcal{F}_{t_m}),$$

given $\{|X_{t_m}^\psi - \chi_{t_m}| < \delta'_m\}$. Let $\nu'_m > 0$ (this is a new constant which has nothing to do with $\nu$ and will be fixed shortly, see (41) below; we need it only while establishing the inequality (37)). By exponential Chebyshev’s inequality we estimate, for any $v := v_k$ and any $0 \leq z \leq 1$ on the set $\{|X_{t_m}^\psi - \chi_{t_m}| < \delta'_m\}$,

$$E^{(m+1)}(I((X_{t_m}^\psi - X_{t_m}^\psi - \chi_{t_m+1} + \chi_{t_m})v > \kappa(\delta'_m - \delta'_m) | \mathcal{F}_{t_m})$$

$$= E^{(m+1)}(I(z \varepsilon^{-2} (X_{t_m}^\psi - X_{t_m}^\psi - \chi_{t_m+1} + \chi_{t_m})v$$

$$> z \varepsilon^{-2} \kappa(\delta'_m - \delta'_m) | \mathcal{F}_{t_m})$$

$$\leq \exp(-(\delta'_m - \delta'_m) z \varepsilon^{-2})$$

$$\times E^{(m+1)} \exp(z \varepsilon^{-2} (X_{t_m}^\psi - X_{t_m}^\psi - \chi_{t_m+1} + \chi_{t_m})v)$$

$$\leq \exp(-(\delta'_m - \delta'_m) z \varepsilon^{-2}) \exp(\varepsilon^{-2}[-z v \chi_{t_m+\Delta m}$$

$$+H^\varepsilon,\psi(X_t^\psi, \beta(m+1) + vz) - \tilde{H}^\varepsilon,\psi(X_t^\psi, \beta(m+1)) + 2\nu'_{m-1}]$$

$$\leq \exp(-(\delta'_m - \delta'_m) z \varepsilon^{-2}) \exp(\varepsilon^{-2}[-z v \chi_{t_m+\Delta m}$$
Recall that a slightly stronger assumption was used in the rule of choosing or, equivalently, and we will need a stronger version in a minute, see (40) below.

\[ + \tilde{H}(\psi_{t_m}, X_{t_m}^\psi, \beta(m + 1) + vz) \]

\[- \tilde{H}(\psi_{t_m}, X_{t_m}^\psi, \beta(m + 1)) + 2\nu'_m \], \hspace{1cm} (38)

if \( \varepsilon \) is small enough. We used here the identity \( \chi_{t_{m+1}} = \Delta_m \chi_{t_m} \pm \). Denote

\[ h(z) := (\delta_{m+1}' - \delta_m')\kappa z + \dot{\chi}_{t_m} + vz\Delta_m \]

\[-[\tilde{H}(\psi_{t_m}, X_{t_m}^\psi, \beta(m + 1) + vz) - \tilde{H}(\psi_{t_m}, X_{t_m}^\psi, \beta(m + 1))] \],

so that the rightmost side of (38) may be represented as

\[ \exp(-\varepsilon^{-2}h(z)) \].

Notice that \( h(0) = 0 \). Moreover, since \( \dot{\chi}_{t_m+} = \nabla_\beta \tilde{H}(\psi_{t_m}, \psi_{t_m}, \beta(m + 1)) \) (see (35)), we have on the set \( \{ |X_{t_m}^\psi - \chi_{t_m}| < \delta_m' \} \),

\[ h'(0) = (\delta_{m+1}' - \delta_m')\kappa + \dot{\chi}_{t_m} + v\Delta_m \]

\[-\nabla_\beta \tilde{H}(\psi_{t_m}, X_{t_m}^\psi, \beta(m + 1))v\Delta_m \]

\[ (\delta_{m+1}' - \delta_m')\kappa \Delta_m + \nabla_\beta \tilde{H}(\psi_{t_m}, X_{t_m}^\psi, \beta(m + 1))v\Delta_m \]

\[ \geq (\delta_{m+1}' - \delta_m')\kappa - m\nabla_\beta \tilde{H}(\delta_m') \Delta =: C_{m+1}' > 0 \]

(recall that \( \Delta_m \leq \Delta \) and that here \( m\nabla_\beta \tilde{H} \) stands for the modulus of continuity of the function \( \nabla_\beta \tilde{H} \) given \( |\beta(m)| \leq b + 1 \) (\( b + 1 \) will be useful in the sequel, although here \( b \) would be enough)). The inequality \( C_{m+1}' = (\delta_{m+1}' - \delta_m')\kappa - m\nabla_\beta \tilde{H}(\delta_m') \Delta > 0 \) holds true provided \( \delta_m' \) is small enough in comparison to \( (\delta_{m+1}' - \delta_m') \), e.g.,

\[ m\nabla_\beta \tilde{H}(\delta_m') \Delta \leq \frac{\kappa}{2}(\delta_{m+1}' - \delta_m') \],

or, equivalently,

\[ m\nabla_\beta \tilde{H}(\delta_m') \Delta + \frac{\kappa}{2} \delta_m' \leq \frac{\kappa}{2} \delta_{m+1}' \]. \hspace{1cm} (39)

Recall that a slightly stronger assumption was used in the rule of choosing \( \delta_m' \)

and we will need a stronger version in a minute, see (40) below.

Moreover, since \( \nabla_\beta \tilde{H} \) is bounded and continuous due to the Lemma 4, then \( h'(z) \geq C_m/2 \) for small \( z \), say, for \( 0 \leq z \leq z_m \) (thus, \( z_m \) is fixed here), on the set \( \{ |X_{t_m}^\psi - \chi_{t_m}| < \delta_m' \} \). Indeed,

\[ h'(z) = (\delta_{m+1}' - \delta_m')\kappa + \dot{\chi}_{t_m} + vz \Delta_m \]

\[-\nabla_\beta \tilde{H}(\psi_{t_m}, X_{t_m}^\psi, \beta(m + 1) + vz) v\Delta_m \]
if we choose (cf. (40)), as well as all auxiliary values \( C \) with \( z \) rather than (39). Hence, under the assumption of (40), the right hand side in

\[
= (\delta_{m+1} - \delta_m') \kappa + \nabla_\beta \tilde{H}(\psi_{tm}, \psi_{tm}, \beta(m + 1))v \Delta_m
\]

\[-\nabla_\beta \tilde{H}(\psi_{tm}, X_{tm}^\psi, \beta(m + 1) + vz) v \Delta_m \]

\[\geq (\delta_{m+1} - \delta_m') \kappa - m_{\nabla \tilde{H}}(\delta_m' + z) \Delta.\]

So, \( h(z_m) \geq C_{m+1}z_m/2 \), provided \( z_m \) along with \( \delta_m' \) are both small in comparison to \( (\delta_{m+1} - \delta_m') \), for example, if

\[m_{\nabla \tilde{H}}(\delta_m' + z_m) \Delta \leq (\delta_{m+1} - \delta_m') \kappa/2, \quad (40)\]

rather than (39). Hence, under the assumption of (40), the right hand side in (38) with \( z = z_m \) on the set \( \{|X_{tm}^\psi - \chi_{tm}| < \delta_m'\} \) does not exceed the value

\[\exp(\varepsilon^{-2}(2\nu' - h(z_m))) \leq \exp(-C_{m+1}z_m\varepsilon^{-2}/4)\]

if we choose

\[\nu' < C_{m+1}z_m/8. \quad (41)\]

Recall that the constant \( \nu' \) should have been fixed in the beginning of this step of the proof; hence, we can do it now, once we have chosen \( z_m \), since the latter does not require any knowledge of \( \nu' \). Given \( \{|X_{tm}^\psi - \chi_{tm}| < \delta_m'\} \), this implies the bound,

\[E^{\beta(m+1)} \left( I(|X_{tm+1}^\psi - \chi_{tm+1}| \geq \delta_{m+1}'|F_{tm}) \right) \leq \exp(-C_{m+1}z_m\varepsilon^{-2}/4),\]

which is equivalent to (37) with \( C_{m+1} := C_{m+1}z_m \). In turn, (37) implies the estimate

\[P(|X_{tm+1}^\psi - \chi_{tm+1}| < \delta_{m+1}'|F_{tm}) \]

\[\geq \exp \left( -\varepsilon^{-2} \Delta_m L(\psi_{tm}, \chi_{tm+1}) + 3\nu \right) - b\varepsilon^{-2}(\delta_{m+1}' + \delta_m') \],

still on \( \{|X_{tm}^\psi - \chi_{tm}| < \delta_m'\} \), if \( \varepsilon \) is small enough. Indeed, \( \nu \), \( C_{m+1} \) and \( \Delta_m \) being fixed, one can choose \( \varepsilon \) so that

\[1 - \exp(-C_{m+1}\varepsilon^{-2}) \geq \exp(-1) \geq \exp(-\nu(\Delta_m\varepsilon^{-2})).\]

10. By “backward” induction from \( k = m \) to \( k = 1 \), choosing at each step \( \delta_{k-1}' \) and \( z_{k-1} \) small enough in comparison to \( \delta_k' - \delta_{k-1}' \),

\[m_{\nabla \tilde{H}}(\delta_{k-1}' + z_{k-1}) \Delta + \frac{\kappa}{2} \delta_{k-1}' \leq \frac{\kappa}{2} \delta_k', \quad \delta_{k-1}' \leq \delta_k'/2, \quad m_{\tilde{H}}(\delta_{k-1}') < \nu \quad (42)\]

(cf. (40)), as well as all auxiliary values \( C_{k-1} \), for \( \varepsilon \) small enough and since \( \sum_{m+1} \delta_k' \leq 2\delta_{m+1}' \), we get the desired lower bound:

\[P(|X_{tm+1}^\psi - \varphi_{tm+1}| < \delta_{m+1}', \ldots, |X_{t}^\psi - \varphi_t| < \delta_1')\]
\[ \geq \exp \left( -\varepsilon^2 \sum_{i=0}^{m} (L(\eta_{m-i}\Delta_i, \dot{x}_{m-i}\Delta_i) + 3\nu)\Delta_i - 2b\varepsilon^{-2} \sum_{k=1}^{m+1} \delta_k' \right) \]

\[ \geq \exp \left( -\varepsilon^2 \left( \int_0^T L(\eta_s, \dot{x}_s) \, ds + 3\nu T \right) - 4b\varepsilon^{-2} \delta_{m+1}' \right) \]

\[ \geq \exp \left( -\varepsilon^2 (S_{0T}(\varphi) + \nu (3T + 2)) \right), \quad \varepsilon \to 0, \]

provided \( 4b\delta_{m+1}' < \nu \). This is equivalent to (5). This bound is uniform in \( x \in \mathbb{R}^d \). \( |y| \leq r \), and \( \varphi \in \Phi_x(s) \) for any \( r, s > 0 \), similar to the Lemma 7.4.1 from [3].

11. The property of the rate function \( S \) to be a “good rate function” can be shown as in [3], using the semi-continuity of the function \( L(x, y) \) with respect to \( y \) and continuity with respect to \( x \) variable (see [3] Lemma 7.4.2).

12. Second part of the proof: the upper bound. Assume that the assertion (1) is not true, that is, there exist \( s \) and \( \nu > 0 \) with the following properties:

\[ \forall \delta > 0, \text{ there exists } \delta_0 < \bar{\delta}, \forall \varepsilon, \text{ there exists } \varepsilon < \bar{\varepsilon} : \]

\[ P(\rho(X, \Phi_x(s)) > \delta_0) > \exp(-\varepsilon^{-2}(s - \nu)). \]

In other words, for some (hence, actually, for any) \( \delta_0 > 0 \) arbitrarily close to zero, there exists a sequence \( \varepsilon_n \to 0 \) such that

\[ P(\rho(X, \Phi_x(s)) > \delta_0) > \exp(-\varepsilon_n^{-2}(s - \nu)). \quad (43) \]

We fix any such \( \delta_0 > 0 \).

13. Since \( f \) is bounded, all possible trajectories of \( X^\psi \) for any \( \psi \) belong to some compact \( F \subset C[0, T; \mathbb{R}^d] \). Due to semi-continuity of the functional \( S^\psi(\varphi) \) with respect to \( \psi \), for any \( \nu > 0 \) there exists a value \( \delta > 0 \) such that \( \rho(\varphi, \psi) < \delta \) and \( S(\varphi) > s \) imply \( S^\psi(\varphi) > s - \nu/2 \). Hence, let us define for each \( \varphi \in C[0, T; \mathbb{R}^d] \) a positive value (notice that this definition differs slightly from that given in [3]; for the latter – without sup – there is no reason to be necessarily semi-continuous)

\[ \delta_\nu(\varphi) := \sup(\delta : \rho(\varphi, \psi) < \delta \text{ and } S(\varphi) > s \implies S^\psi(\varphi) > s - \nu/2). \]

Since \( S^\psi(\varphi) \) is lower semi-continuous with respect to \( \varphi \), too, similarly to \( S(\varphi) \), then it follows that \( \delta_\nu(\varphi) \) is also lower semi-continuous with respect to \( \varphi \). Indeed, let \( \varphi^n \to \varphi, n \to \infty; \) we ought to show that \( \liminf_{n \to \infty} \delta_\nu(\varphi^n) \geq \delta_\nu(\varphi) \). We have,

\[ \delta_\nu(\varphi^n) := \sup(\delta : \rho(\varphi^n, \psi) < \delta \text{ and } S(\varphi^n) > s \implies S^\psi(\varphi^n) > s - \nu/2). \]

Suppose \( 0 < \bar{\delta} < \delta_\nu(\varphi) \) and \( \rho(\varphi^n, \psi) < \bar{\delta} \). We want to show that \( S^\psi(\varphi^n) > s - \nu/2 \). Since \( \rho(\varphi^n, \psi) < \delta_\nu(\varphi) - \bar{\delta} \) for \( n \) large enough, then we also have \( \rho(\varphi, \psi) < \delta_\nu(\varphi) \). Then, by definition of \( \delta_\nu(\varphi) \), \( S^\psi(\varphi) > s - \nu/2 \). Since by
Fatou’s lemma, \( \liminf_{n \to \infty} S^\nu (\varphi^n) > s - \nu / 2 \), this implies \( S^\nu (\varphi^n) > s - \nu / 2 \) for \( n \) large enough. The latter signifies that, indeed, \( \liminf_{n \to \infty} \delta_\nu (\varphi^n) \geq \delta_\nu (\varphi) \), that is, that \( \delta_\nu (\varphi) \) is lower-semicontinuous, as required.

Thus, as every lower semi-continuous function, \( \delta_\nu (\varphi) \) attains its minimum on any compact and, hence, the minimum over any compact must be positive.

Further, consider \( F_1 \), the compact obtained from \( F \) by dropping the \( \delta_\nu /2 \)-neighbourhood of the set \( \Phi_X (s) = \{ \varphi \in C[0, T; R^d] : \varphi_0 = x, S(\varphi) \leq s \} \). Denote \( \delta_\nu = \inf_{\varphi \in F_1} \delta_\nu (\varphi) \), and take any \( \delta' \leq \min \left( \delta_\nu / (4KT + 2), \delta_0 / 2 \right) \) where \( K \) is a Lipschitz constant of \( f \). Choose a finite \( \delta' \)-net for the set \( F_1 \), let \( \varphi^1, \ldots, \varphi^N \) be its elements. All of them do not belong to \( \Phi_X (s) \), hence, \( S(\varphi^i) \geq s' \) with some \( s' > s \). Notice that

\[
\{ \rho (X, \Phi_X (s)) > \delta_0 \} \subset \bigcup_{i=1}^N \{ \rho (X, \varphi^i) < \delta' \}.
\]

Then, for any \( n \) there exists an index \( i \) such that

\[
P(\rho (X, \varphi^i) \leq \delta') > N^{-1} \exp (-\varepsilon_n^{-2} (s - \nu)). \tag{44}
\]

There is a finite number of \( i = 1, \ldots, N \). Thus, there exists at least one \( i \) such that (44) holds true for this \( i \) for some subsequence \( n' \to \infty \) and correspondingly \( \varepsilon_n' \to 0 \); however, we will keep the notation \( n \) for simplicity. We may rewrite (44) as

\[
P(\rho (X, \varphi^i) \leq \delta') > \exp (-\varepsilon_n^{-2} (s - \nu)), \tag{45}
\]

since \( N \) does not depend on \( \varepsilon_n \), strictly speaking with some new \( \nu > 0 \); however, it is again convenient to keep the same notation. Denote \( \varphi (\delta') := \varphi^i \) with this \( i \) (any one if not unique).

14. Consider a sequence \( \delta' \to 0 \) such that a corresponding function \( \varphi (\delta') \) does exist for any \( \delta' \) from this sequence. Recall that \( \delta_0 \) is fixed. All these functions satisfy inequality

\[
S(\varphi (\delta')) \geq s' > s,
\]

since \( \rho (\varphi^i, \Phi_X (s)) \geq \delta_0 / 2 \). Also we have, \( S(\varphi (\delta')) < \infty \), which implies

\[
\sup_t |\dot{\varphi}_t (\delta')| \leq C,
\]

because, due to the boundedness of \( f \), function \( L(x, \alpha) \) equals infinity for every \( |\alpha| > \| f \| \). By virtue of the Arzelà-Ascoli Theorem, it is possible to extract from this set of functions a subsequence which converges in \( C[0, T; R^d] \) to some limit, \( \tilde{\varphi} \). Since \( \rho (\varphi (\delta'), \Phi_X (s)) \geq \delta_0 / 2 \), we have, \( \rho (\tilde{\varphi}, \Phi_X (s)) \geq \delta_0 / 2 \), hence,

\[
S(\tilde{\varphi}) > s,
\]

and, in particular, the lower bound (5) can be applied. However, due to the construction, the function \( \tilde{\varphi} \) satisfies one more lower bound,

\[
\liminf_{\delta' \to 0} \limsup_{\varepsilon \to 0} \varepsilon^2 \ln P(\rho (X, \tilde{\varphi}) < \delta') \geq -s + \nu. \tag{46}
\]
Indeed, the latter follows from (45) because, e.g.,

\[ P(\rho(X, \varphi) \leq \delta' + \rho(\varphi, \varphi(\delta'))) \geq P(\rho(X, \varphi(\delta')) \leq \delta') > \exp(-\varepsilon_n^{-2}(s - \nu)). \]

Due to (46), there exists \( \delta' > 0 \) such that for smaller \( \delta' \)'s (a sequence)

\[ \limsup_{\varepsilon \to 0} \varepsilon^2 \ln P(\rho(X, \varphi) < \delta') \geq -s + \nu/2. \]

In fact, this implies the same inequality for any \( \delta' > 0 \), because with any \( \delta' \) for which the inequality holds true, every greater value would do as well. Therefore, for any \( \delta' \), there exists \( \varepsilon > 0 \) (arbitrarily small) such that

\[ \varepsilon^2 \ln P(\rho(X, \varphi) < \delta') \geq -s + \nu/3 = -(s - \nu/3). \] (47)

We are going to show that this leads to a contradiction.

15. Consider the case \( S(\varphi) < \infty \). Remind that \( S(\varphi) > s \). Denote

\[ L^b(x, y) = \sup_{|\beta| \leq b} (\beta y - H(x, \beta)), \]

\[ \ell^b(x, y) := L(x, y) - L^b(x, y) \equiv \sup_{|\beta| \leq b} (\beta y - H(x, \beta)) - \sup_{|\beta| \leq b} (\beta y - H(x, \beta)). \]

Consider the function \( \ell^b(\hat{\varphi}_t, \hat{\varphi}_t) \). We have,

\[ 0 \leq \ell^b(\hat{\varphi}_t, \hat{\varphi}_t) \leq L(\hat{\varphi}_t, \hat{\varphi}_t). \]

Moreover,

\[ \ell^b(\hat{\varphi}_t, \hat{\varphi}_t) \to 0, \quad b \to \infty, \]

and the function \( \ell \) is decreasing with \( b \to \infty \). Hence, given \( \nu > 0 \), one can choose a \( b > 0 \) such that

\[ \int_0^T \ell^b(\hat{\varphi}_t, \hat{\varphi}_t) dt < \nu/20. \]

Notice that we have chosen \( b \), which is now fixed for the second part of the proof of the Theorem. Moreover, one can also choose a discretisation step \( \Delta \) (see above, step 5 of the proof and, in particular, the Lemma 7) such that for almost every \( a \in [0, 1] \)

\[ \int_0^T \ell^b(\hat{\varphi}_{\kappa_m(t+a)}, \hat{\varphi}_{\kappa_m(t+a)}) dt < \nu/10, \]

and, correspondingly,

\[ \int_0^T L^b(\hat{\varphi}_{\kappa_m(t+a)}, \hat{\varphi}_{\kappa_m(t+a)}) dt > s - \nu/10. \] (48)
In addition, we require $\Delta \leq \Delta(\nu/20)$ (see the Lemma 5). Hence, we have chosen $\Delta$ and $m = T/\Delta$. We also fix any $a \in [0, 1]$ satisfying (48).

16. Further, let

$$\psi_t := \hat{\varphi}_{k_m(t+a) - a}, \quad \chi_t := \hat{\varphi}_{k_m(t+a) - a}, \quad \chi_0 = x.$$  

We have, with a unique constant $C = 2(KT + 1)$ (see (19)) and for any $\delta'$,

$$P(\rho(X, \varphi) < \delta') \leq P(\rho(X, \chi) < C\delta') \leq P(\rho(X^{\psi, \Delta, a}, \chi^{\Delta, a}) < C\delta').$$

Denote $\delta'' = C\delta'$. Let us choose $\delta'' \leq \delta(\nu/20)$ (the notation from the Lemma 5 is used), and consider the following inequality, with the sequence $(\delta'_i, 1 \leq i \leq m), \delta'_m = \delta''$, constructed via the value $\nu/20$ instead of $\nu$ (compare to (42)), where the requirement related to $m\nabla_H$ could be now dropped,

$$P(\rho(X, \varphi) < \delta'_1) \leq E \prod_{i=1}^{m+1} 1(|X^{\psi, \Delta, a}_{t_k} - \chi_{t_k}^{\Delta, a}| < \delta'_i),$$

and $(t_k)$ are chosen as in the step 5. In particular, we require $4\delta'' = 4\delta'_m \leq \nu/20$, and $\sum_{i=1}^{m+1} \delta'_i \leq 2\delta''$. Then, due to the Lemma 5 and using the same calculus as at the step 5, we get on the set $\{|X^{\psi}_{t_m} - \chi_{t_m}| < \delta'_m\}$ and for any $|\beta| \leq b$,

$$E \left( I(|X^{\psi}_{t_{m+1}} - \chi_{t_{m+1}}| < \delta'_{m+1}) \mid \mathcal{F}_{t_{m}} \right)$$

$$\leq E^\beta \left( I(|X^{\psi}_{t_{m+1}} - \chi_{t_{m+1}}| < \delta'_{m+1}) \exp \left( -\varepsilon^{-2}\Delta_m \beta \left( (\chi_{t_{m+1}} - \chi_{t_{m}}) / \Delta_m \right) \right. \right.$$  

$$\left. -\varepsilon^{-2}\Delta_m (H(\psi_{t_m}, \chi_{t_m}, \beta) - \nu/20) + b \frac{\delta'_{m+1} + \delta'_m}{\varepsilon^2} \right) \mid \mathcal{F}_{t_{m}} \right)$$

(compare to (331)). The only change in comparison to the step 5 is that now we want an upper bound, so indicators in the estimation will be just replaced by 1. Thus, we replace here $I(|X^{\psi}_{t_m} - \chi_{t_m}| < \delta'_{m+1})$ by 1 and drop the expectation sign – because there remains nothing random in the expression – then on the set $\{|X^{\psi}_{t_m} - \chi_{t_m}| < \delta'_m\}$ and for any $|\beta| \leq b$ we get,

$$E \left( I(|X^{\psi}_{t_{m+1}} - \chi_{t_{m+1}}| < \delta'_{m+1}) \mid \mathcal{F}_{t_{m}} \right)$$

$$\leq \exp \left( -\varepsilon^{-2}\Delta_m \beta \left( (\chi_{t_{m+1}} - \chi_{t_{m}}) / \Delta_m \right) \right.$$  

$$-\varepsilon^{-2}\Delta_m (H(\psi_{t_m}, \chi_{t_m}, \beta)) + b \frac{\delta'_{m+1} + \delta'_m}{\varepsilon^2} \right),$$

$$\leq \exp \left( -\varepsilon^{-2}\Delta_m \beta \left( (\chi_{t_{m+1}} - \chi_{t_{m}}) / \Delta_m \right) \right.$$  

$$-\varepsilon^{-2}\Delta_m (H(\psi_{t_m}, \chi_{t_m}, \beta) - \nu/20) + b \frac{\delta'_{m+1} + \delta'_m}{\varepsilon^2} \right),$$

$$(50)$$

23
once we have chosen $m_\tilde{H}(\delta''') \leq \nu/20$ (remind that $m_\tilde{H}$ here means the modulus of continuity of the function $\tilde{H}(\cdot, \cdot, \beta)$ with respect to the first two variables on the set $|\beta| \leq b + 1$).

Let $\beta$ satisfy a condition,

$$
\beta(\chi_{t_{m+1}} - \chi_{t_m})/\Delta_m - \tilde{H}(\psi_{t_m}, \psi_{t_m}, \beta)
$$

$$
= \sup_{|\beta| \leq b} \left( \beta(\chi_{t_{m+1}} - \chi_{t_m})/\Delta_m - \tilde{H}(\psi_{t_m}, \psi_{t_m}, \beta) \right)
$$

$$
= L^b(\psi_{t_m}, \dot{\chi}_{t_m}).
$$

Then, on the set $\{|X^\psi_{t_m} - \chi_{t_m}| < \delta'_m\}$,

$$
E \left( I(|X^\psi_{t_{m+1}} - \chi_{t_{m+1}}| < \delta'_{m+1}) \mid \mathcal{F}_{t_m} \right)
$$

$$
\leq \exp \left( -\varepsilon^{-2}\Delta_m (L^b(\psi_{t_m}, \chi_{t_m}) + \varepsilon^{-2}\Delta_m \nu / 20 + b \frac{\delta'_{m+1} + \delta'_m}{\varepsilon^2} \right).
$$

Similarly and by induction and due to (48), we get

$$
P(\rho(X, \chi) < \delta_1')
$$

$$
\leq \exp \left( -\varepsilon^{-2} \int_0^T L^b(\psi_t, \dot{\chi}_t) \, dt + \varepsilon^{-2} \nu / 20 + 4b \varepsilon^{-2} \delta'_m \right)
$$

$$
\leq \exp \left( -\varepsilon^{-2} (s - \nu/5) \right).
$$

This evidently contradicts (47).

17. Consider the case $\bar{\varphi}$ absolute continuous and $S(\bar{\varphi}) = \infty$. In this case, due to monotone convergence $L^b \rightarrow L$, there exist $b > 0$, $m$ and $a \in [0, 1]$ such that

$$
\int_0^T L^b(\bar{\varphi}_t, \dot{\bar{\varphi}}_t) \, dt \geq s - \nu / 20, \quad \int_0^T L^b(\bar{\varphi}_{\kappa_m(t+a)-a}, \dot{\bar{\varphi}}_{\kappa_m(t+a)-a}) \, dt \geq s - \nu / 10.
$$

The rest is similar to the main case, $S(\bar{\varphi}) < \infty$, and leads again to

$$
P(\rho(X, \bar{\varphi}) < \delta_1') \leq \exp \left( -\varepsilon^{-2} (s - \nu/5) \right).
$$

This contradicts (47).

18. Consider the last possible case, $\bar{\varphi}$ not absolute continuous. In this case, for any constant $c$, in particular, for $c = \|f\|_C + 1$, there exist two values $0 \leq t_1 < t_2 < T$, such that $|\bar{\varphi}_{t_2} - \bar{\varphi}_{t_1}| > c(t_2 - t_1)$; indeed, otherwise $\bar{\varphi}$ must be Lipschitz with $|\dot{\varphi}| \leq c$. Therefore, for $\delta < (t_2 - t_1)/2$, probability $P(\rho(X, \bar{\varphi}) < \delta)$ necessarily equals zero, because the event $\{\rho(X, \bar{\varphi}) < \delta\}$ is empty. This evidently contradicts (47). In all possible cases, we got to contradictions. Hence, the assumption is wrong, that is, the upper bound (4) holds true. The Theorem is proved.
APPENDIX

A. Comments on the Lemma[6] To explain that the Lemma[6] is valid without additional assumptions, we have to review very briefly its proof and show those assumptions.

Let 0 = t_0 < t_0 < \ldots < t_m = T be a partition, \gamma_k(\beta) := \int_{t_{k-1}}^{t_k} H(\varphi_s, \beta) ds, \ell_k(\alpha) = \sup_{\beta}(\alpha \beta - \gamma_k(\beta)), A_k = \{\alpha : \ell_k(\alpha) < \infty\}, A^c_k its interior with respect to the linear hull L_{A_k}.

The inequality \( S(\varphi) = \int_0^T L(\varphi_t, \dot{\varphi}_t) dt < \infty \) implies

\[
\sum_{k=1}^{m} \sup_{\beta} ((\varphi_{t_k} - \varphi_{t_{k-1}}) - \gamma_k(\beta)) = \sum_{k=1}^{m} \ell_k(\varphi_{t_k} - \varphi_{t_{k-1}}) \leq S(\varphi).
\]

Under additional assumption \( A^c_k \neq \emptyset \) it is proved in [3] using the arguments from [12] that for any \( \nu > 0 \), there exists a function \( \tilde{\varphi} \) such that \( \rho(\varphi, \tilde{\varphi}) < \nu \) and there exist \( \beta_k \) such that

\[
\ell_k(\tilde{\varphi}_{t_k} - \tilde{\varphi}_{t_{k-1}}) = (\tilde{\varphi}_{t_k} - \tilde{\varphi}_{t_{k-1}}) \beta_k - \gamma_k(\beta_k)
\]

and

\[
\tilde{\varphi}_{t_k} - \tilde{\varphi}_{t_{k-1}} = \nabla \gamma_k(\beta_k).
\]

The proof goes well if \( A^c_k \neq \emptyset \) \( \forall k \).

Let us show that the same is true if \( A^c_k = \emptyset \) for some \( k \)'s. The property \( A^c_k = \emptyset \) is equivalent to \( \dim L_{A_k} = 0 \). In this case, \( \gamma_k(\beta) = c_k \beta \) with some \( c_k \in \mathbb{R}^d \). Hence, \( \ell_k(\alpha_k) < \infty \) means that \( \ell_k(\alpha_k) = 0 \) and for any other \( \alpha \), \( \ell_k(\alpha) = +\infty \) and \( \gamma_k(\beta) = \alpha_k \beta \). So, we have

\[
\ell_k(\varphi_{t_k} - \varphi_{t_{k-1}}) = 0 = (\varphi_{t_k} - \varphi_{t_{k-1}}) \beta - \gamma_k(\beta)
\]

for any \( \beta \). Let \( \beta_k = 0 \). Evidently,

\[
\varphi_{t_k} - \varphi_{t_{k-1}} = \nabla \gamma_k(\beta_k).
\]

Hence, in the case \( A^c_k = \emptyset \), one just should not change the curve \( \varphi_s \) on the interval \((t_{k-1}, t_k)\); that is, (53) and (54) are valid in this case also.

The rest of the proof remains unchanged. For any step function \( \zeta \), one defines a piecewise linear \( \chi \) by the formula

\[
\chi_0 = \varphi_0, \quad \dot{\chi}_s = \nabla \beta H(\zeta_s, \beta_k), \quad t_{k-1} < s < t_k, \quad k = 1, 2, \ldots, m.
\]

Then it is shown that \( \zeta^n \to \varphi \) implies \( \chi^n \to \varphi \) due to the property that the convergence of smooth convex functions to the limit implies the convergence of their gradients. Then there exists a partition such that this construction gives one

\[
\int_0^T L(\zeta_t, \dot{\chi}_t) dt \leq S(\varphi) + \nu.
\]

So, the lemma holds true without additional assumptions. The assertions about \( \dot{\zeta} \) and \( \dot{\beta}_s \) can be shown similarly.

25
B. Comments on the property $A_k^0 \neq \emptyset$, and characterization of the set $L^o[f, x]$. Denote the interior of $A(x) = \{ \alpha : L(x, \alpha) < \infty \}$ with respect to its linear hull $L_{A(x)}$ by $A^o(x)$. Then $A_k^0 = \emptyset \iff A^o(\varphi_{k,-1}) = \emptyset$. In this section we show the following equivalence:

$$\text{card}(f \in R^d : f = f(x, y), y \in M) = 1 \iff \dim L_{A(x)} = 0 \iff A^o(x) = \emptyset.$$ 

Since $A(x)$ is convex, clearly the first two conditions are equivalent.

If $\{f(x, \cdot)\}$ contains only one point then $H(x, \beta)$ is linear with respect to $\beta$; hence, $A(x)$ consists of a unique point and $A^o(x) = \emptyset$.

Now, let $\{f(x, \cdot)\}$ contain at least two different points, say, $f(x, y_1) \neq f(x, y_2)$. Then there exists $1 \leq k \leq d$ such that $(f(x, y_1) - f(x, y_2))_k \neq 0$. Denote $M_k = \sup_y f_k(x, y)$, $m_k = \inf_y f_k(x, y)$. Let $0 < \nu < (f(x, y_1) - f(x, y_2))_k/2$. Take two points $y'$ and $y''$ such that $f_k(x, y') < m_k + \nu/5$ and $f_k(x, y'') > M_k - \nu/5$. There exist two open sets $B' \subset M$ and $B'' \subset M$ such that $\sup_{y \in B'} f_k(x, y) < m_k + \nu/4$ and $\inf_{y \in B''} f_k(x, y) > M_k - \nu/4$.

Since the process $y^x_t$ is a nondegenerate ergodic diffusion, there exists $\lambda > 0$ such that

$$P(y^x_s \in B', 1 \leq s \leq t) \geq \lambda^{t-1}, \quad P(y^x_s \in B'', 1 \leq s \leq t) \geq \lambda^{t-1}, \quad t \to \infty.$$ 

Let $\beta = z\beta_k$ where $\beta_k \in E^d$ is a $k$th unit coordinate vector and $z \in R$. Then for $z > 0$ we have,

$$z^{-1}t^{-1} \log E \exp(z\beta_k \int_0^t f(x, y^x_s) \, ds)$$

$$\geq z^{-1}t^{-1} \log E \exp(z\beta_k \int_0^t f(x, y^x_s) \, ds) I(y^x_s \in B'', 1 \leq s \leq t)$$

$$\geq z^{-1}t^{-1} \log \{ \exp(z(M_k - \nu/2)t)\lambda^{t-1} \}$$

$$= M_k - \nu/4 - \frac{t-1}{t}z^{-1} \log \lambda \geq -\frac{t-1}{t}M_k - \nu/2,$$

if $z$ is large enough. In other words, for large positive $z$ one has $H(x, z\beta_k) \geq z(M_k - 2\nu)$. Similarly, for large negative $z$

$$|z|^{-1}t^{-1} \log E \exp(z\beta_k \int_0^t f(x, y^x_s) \, ds)$$

$$\geq |z|^{-1}t^{-1} \log E \exp(z\beta_k \int_0^t f(x, y^x_s) \, ds) I(y^x_s \in B'', 1 \leq s \leq t)$$

$$\geq |z|^{-1}t^{-1} \log \{ \exp(z(m_k + \nu/4)t)\lambda^{t-1} \}$$

$$= -(m_k + \nu/4) - \frac{t-1}{t}|z|^{-1} \log \lambda \geq -\frac{t-1}{t}m_k - \nu/2,$$

if $|z|$ is large enough. In other words, for negative $z$ with large absolute values one has $H(x, z\beta_k) \geq z(m_k + \nu)$. Therefore, $\{ \alpha : \alpha = \beta_k \theta, m_k + \nu < \theta < M_k - \nu \} \subset A(x)$. 

26
On the other hand, it is obvious that if \( \alpha = \beta_k \theta, \theta \in R^1, \) with \( \theta > M_k \) or \( \theta < m_k \), then \( L(x, \alpha) = \infty \), because \( m_k z \leq H(x, \beta_k z) \leq M_k z \), and, hence (say, if \( \theta > M_k \)), for \( z >> 1 \),

\[
\beta_k \theta \beta_k z - H(x, \beta_k z) \geq (\theta - M_k) z \to +\infty, \quad z \to +\infty.
\]

A similar calculus and similar inequalities are valid for any unit vector \( \beta_0 \). This shows, in particular, that \( \text{dim} L \{ \in \theta < \mu \} \) any \( \text{inf} y \mu \) \( L \) if \( m \) \( L \) the set \( \text{for} \) \( z > \) be shown similarly that for any \( v \) \( \text{proof of} \) the \( \text{theorem} 1 \), \( \text{if} \) \( \text{we} \) show that for any \( \text{direction} \) \( \text{Denote} \) \( A \)

\[
\text{A} \circ \text{L} \left( \alpha \right) = \text{inf} \left( \frac{\beta}{y} f(x,y) \right), \quad M_\beta(x) := \text{sup} \left( \frac{\beta}{y} f(x,y) \right). \quad \text{ Moreover, it can be shown similarly that for any } x, \tilde{x} \text{ (although we do not need it here),}
\]

\[
\mathcal{L}^0[f,x,\tilde{x}]=\mathcal{L}^0[f,x].
\]

**C. About** \( \hat{\alpha}_s \in \mathcal{L}^0[f,\varphi_s] \). Let \( x = \varphi_s, \hat{\alpha} = \hat{\alpha}[x,\hat{x}] \) as described in the proof of the theorem 1. If we show that for any direction \( v \) (a unit vector) satisfying the property \( m_v < M_v \), the strict double inequality holds true

\[
m_v < \partial H(x, zv)/\partial z |_{z=0} < M_v,
\]

\( z \in R^1, \) then it would follow \( \hat{\alpha}_s \in \mathcal{L}^0[f,\varphi_s] \). Let \( \nu > 0 \) and again two open sets \( B' \) and \( B'' \) be chosen such that \( \sup_{y \in B'} v f(x,y) < m_v + \nu/2, \) and \( \inf_{y \in B''} v f(x,y) > M_v - \nu/2. \) Let \( \mu_{\text{inv}}(B'') \) be invariant measure for the event \( \{ y_t \in B'' \} \). We can choose \( \nu \) and correspondingly \( B'' \) so that \( \mu_{\text{inv}}(B'') < 1. \) Then, due to large deviation asymptotics for the process \( y_t \), for any \( \mu_{\text{inv}}(B'') < \zeta < 1 \) there exists \( \lambda > 0 \) such that

\[
P \left( t^{-1} \int_0^t 1(y_s \in B'') ds \geq \zeta \right) \leq \exp(-\lambda t), \quad t \geq t_\zeta.
\]

Denote \( A_{\zeta} = \left\{ t^{-1} \int_0^t 1(y_s \in B'') ds < \zeta \right\}, \) \( A_{\zeta} = \left\{ t^{-1} \int_0^t 1(y_s \in B'') ds \geq \zeta \right\}, \) then for \( z > 0, \)

\[
E \exp(zv \int_0^t f(x,y^v) ds)
\]
\[
\leq E \exp(z \int_0^t (M_v 1(y^x_s \in B'' + (M_v - \nu)1(y^x_s \not\in B''))) \, ds) \, 1(A^c) + E \exp(z \int_0^t (M_v 1(y^x_s \in B'' + (M_v - \nu)1(y^x_s \not\in B''))) \, ds) \, 1(A^c)
\]

\[
\leq E \exp(ztM_v + zt(M_v - \nu)) \, 1(A^c) + E \exp(ztM_v \zeta + zt(M_v - \nu)) \, 1(A^c)
\]

\[
\leq \exp(ztM_v + zt(M_v - \nu) - z\lambda t/z) + \exp(ztM_v \zeta + zt(M_v - \nu))
\]
hence,

\[
\limsup_{z \to 0} \limsup_{t \to \infty} (tz)^{-1} \ln E \exp(zv \int_0^t f(x, y^x) \, ds) < M_v.
\]

Similarly, using \( B' \) one can get

\[
\liminf_{z \to 0} \liminf_{t \to \infty} (tz)^{-1} \ln E \exp(zv \int_0^t f(x, y^x) \, ds) > m_v.
\]

Thus,

\[
m_v < \partial H(x, zv) / \partial z |_{z=0} < M_v.
\]

Therefore, \( \hat{\alpha} \in \mathcal{L}^o[x, f] \).

**Acknowledgements**

The author is grateful to Professor Yuri Kifer who initiated the process of correction and to the unknown referee of this version of the paper for very helpful remarks that resulted in the final simplifications and improvements.

**References**

[1] Freidlin, M. I. (1976) Fluctuations in dynamical systems with averaging. *Dok. Acad. Nauk SSSR* 226 273-276 (in Russian).

[2] Freidlin, M. I. (1978) Averaging principle and large deviations. *Uspekhi Matem. Nauk.* 33 107-160 (in Russian).

[3] Freidlin, M. I. and Wentzell, A. D. (1984) *Random perturbations of dynamical systems.* Springer, New York.

[4] Gulinsky, O. V. and Veretennikov, A. Yu. (1993) *Large Deviations for Discrete–Time Processes with Averaging.* VSP, Utrecht.

[5] Ikeda, N. and Watanabe, S. (1989) *Stochastic differential equations and diffusion processes.* 2nd ed. North-Holland, Amsterdam.

[6] Kato, T. (1976) *Perturbation Theory for Linear Operators.* 2nd ed. Springer, New York.
[7] Kifer, Yu. (2009) Large Deviations and Adiabatic Transitions for Dynamical Systems and Markov Processes in Fully Coupled Averaging, Memoirs of the Amer. Math. Soc. 944, AMS, Providence, RI.

[8] Krasnosel’skii, M. A., Lifshitz, E. A. and Sobolev, A. V. (1989) Positive linear systems. Helderman, Berlin.

[9] Krylov, N. V. (1995), Introduction to the Theory of Random Processes, AMS, Providence, RI.

[10] Liptser, R. S. (1996), Large deviations for two scaled diffusions, *Probability Theory and Related Fields*, 106(1), 71–104; preprint version (2005) at arXiv: math/0510029.

[11] Liptser, R., Spokoiny, V., Veretennikov, A. Yu. (2002), Freidlin–Wentzell type large deviations for smooth processes, *Markov Processes and Related Fields*, 611–636.

[12] Rockafellar, R. T. (1970) *Convex analysis*. Princeton Univ. Press., Princeton, NJ.

[13] Veretennikov, A. Yu. (1992) On large deviations in the averaging principle for stochastic differential equations with periodic coefficients 2. *Math. USSR Izvestiya*, 39 677-701.

[14] Veretennikov, A. Yu. (1994) Large deviations in averaging principle for stochastic differential equation systems (noncompact case). *Stochastics Stochastics Rep.* 48 83-96.

[15] Veretennikov, A. Yu. (1998), On large deviations for stochastic differential equations with a small diffusion and averaging, *Theory Probab. Appl.* 43, 335-337.

[16] Veretennikov, A. Yu. (1999) On large deviations in the averaging principle for SDE’s with a “full dependence”, *Ann. Probab.* 27 no. 1, 284–296.

[17] Veretennikov, A. Yu. (2005) On large deviations in the averaging principle for SDE’s with a “full dependence”, correction. arXiv:math/0502098v1 [math.PR]