ON DISTINGUISHED CURVES IN PARABOLIC GEOMETRIES

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Abstract. All parabolic geometries, i.e., Cartan geometries with homogeneous model a real generalized flag manifold, admit highly interesting classes of distinguished curves. The geodesics of a projective class of connections on a manifold, conformal circles on conformal Riemannian manifolds, and Chern–Moser chains on CR-manifolds of hypersurface type are typical examples. We show that such distinguished curves are always determined by a finite jet in one point, and study the properties of such jets. We also discuss the question when distinguished curves agree up to reparametrization and discuss the distinguished parametrizations in this case. We give a complete description of all distinguished curves for some examples of parabolic geometries.

Elie Cartan’s idea of ‘generalized spaces’ as curved analogs of Felix Klein’s geometries (i.e., homogeneous spaces) is a well understood geometrical concept, which, for a Lie subgroup $P \subset G$, generalizes the Maurer–Cartan form on the total space of the principal $P$-bundle $G \to G/P$ to Cartan connections on principal $P$-bundles, see e.g., the introductory book [17]. The concept of parabolic geometries refers to those cases where $P$ is a parabolic subgroup in a (real or complex) semisimple Lie group $G$. In [9], C. Fefferman initiated a program to exploit the representation theory of parabolic subgroups in semisimple Lie groups in order to understand invariants of geometric structures like CR-
geometries, projective geometries, or conformal Riemannian geometries. This approach has proved to be extremely powerful. First, all parabolic geometries can be described in terms of weaker analogies of classical G-structures on smooth manifolds and, similarly to the examples mentioned above, all such structures give rise to canonical normal Cartan connections, [19, 14, 3]. In fact, these constructions express Cartan’s method of equivalence using the language of the modern representation theory and natural cohomological reasoning. The existence of the Cartan connection provides an effective calculus to deal with invariant objects, see e.g., [5] and the references therein. To a large extent, the understanding of the general (curved) geometries can be reduced to properties of the homogeneous model, and thus to purely algebraic questions.

The goal of this paper is to use this approach in order to understand invariantly defined systems of distinguished curves for parabolic geometries, which we call (generalized) geodesics. After recalling basic concepts of parabolic geometries, geodesics are introduced and discussed along the lines of the classical approach in affine geometry, which uses the development of curves. This approach may be found in a similar context in [17] and [13]. In this way, many aspects of the study of the curves are reduced to the case of the homogeneous model. Thus the original ‘smooth’ question on curved manifolds can be transformed to an ‘algebraic’ problem, which is discussed in Section 2. In particular, we obtain estimates on the order of jets necessary to determine a geodesic, and this approach also leads to an algebraic description of all jets of geodesics in a point. The third section is devoted to the study of possible reparametrizations in the class of geodesics. Specializing the general results to [1]-graded Lie algebras, we obtain generalizations of some well-known results on conformal, projective, and quaternionic geometries (see e.g., [1]). The final section provides further refinements for specific classes of curves, see in particular Theorems 4.2 and 4.3.

Acknowledgments. Part of the work was done during a stay of the second author at the University of Adelaide under an ARC financial support, and his discussions with Michael Eastwood were most helpful and illuminating. The first author supported by project P15747 of the FWF. The second and third authors acknowledge the support from GACR, Grant Nr. 201/02/1390.

1. General concepts

1.1. Parabolic geometries

Let us briefly recall the basic facts, more details can be found in [4] or [17], and the references therein. Let $G$ be a real semisimple Lie group with Lie algebra $\mathfrak{g}$, and $P \subset G$ a parabolic subgroup with Lie algebra $\mathfrak{p}$. A (real) parabolic geometry $(\mathcal{G}, \omega)$ of type $(G, P)$ is a principal bundle $\mathcal{G}$ with structure group $P$ over a manifold $M$, equipped with a smooth one-form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, which satisfies

1. $\omega(\zeta_Z)(u) = Z$ for all $u \in \mathcal{G}$ and fundamental fields $\zeta_Z$, $Z \in \mathfrak{p} \subset \mathfrak{g}$, i.e., $\omega$ reproduces the generators of fundamental vector fields,

2. $(r^b)^*\omega = \text{Ad}(b^{-1}) \circ \omega$ for all $b \in P$, i.e., $\omega$ is $P$-equivariant with respect to the adjoint representation, and

3. $\omega|_{T_u\mathcal{G}} : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$, i.e., $\omega$ is an absolute parallelism on $\mathcal{G}$.

The curvature of a parabolic geometry $(\mathcal{G}, \omega)$ is the horizontal two-form $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$