ON THE HARDY–LITTLEWOOD MAXIMAL FUNCTIONS IN HIGH DIMENSIONS: CONTINUOUS AND DISCRETE PERSPECTIVE

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Dedicated to Fulvio Ricci on the occasion of his 70th birthday.

Abstract. This is a survey article about recent developments in dimension-free estimates for maximal functions corresponding to the Hardy–Littlewood averaging operators associated with convex symmetric bodies in $\mathbb{R}^d$ and $\mathbb{Z}^d$.

1. Introduction

1.1. Statement of the results: continuous perspective. Let $G$ be a convex symmetric body in $\mathbb{R}^d$, which is simply a bounded closed and symmetric convex subset of $\mathbb{R}^d$ with non-empty interior. In the literature it is usually assumed that a symmetric convex body $G \subset \mathbb{R}^d$ is open. In fact, in $\mathbb{R}^d$ there is no difference whether we assume $G$ is closed or open, since the boundary of a convex set has Lebesgue measure zero. However, in the discrete case, if $G \cap \mathbb{Z}^d$ is considered, it matters. Therefore, later on in order to avoid some technicalities, we will assume that a symmetric convex body $G \subset \mathbb{R}^d$ is always closed.

For every $t > 0$ and for every $x \in \mathbb{R}^d$ we define the Hardy–Littlewood averaging operator

$$M_t^G f(x) = \frac{1}{|G_t|} \int_{G_t} f(x-y)dy \quad \text{for } f \in L^1_{\text{loc}}(\mathbb{R}^d),$$

where $G_t = \{ y \in \mathbb{R}^d : t^{-1}y \in G \}$ denotes a dilate of the body $G \subset \mathbb{R}^d$.

For $p \in (1,\infty]$, let $C_p(d,G) > 0$ be the best constant such that the following maximal inequality

$$\sup_{t>0} \|M_t^G f\|_{L^p(\mathbb{R}^d)} \leq C_p(d,G) \|f\|_{L^p(\mathbb{R}^d)}$$

holds for every $f \in L^p(\mathbb{R}^d)$.

The question we shall be concerned with, in this survey, is to decide whether the constant $C_p(d,G)$ can be estimated independently of the dimension $d \in \mathbb{N}$ for every $p \in (1,\infty]$. If $p = \infty$, then (1.2) holds with $C_p(d,G) = 1$, since $M_t^G$ is an averaging operator. By appealing to a covering argument for $p = 1$, and a simple interpolation with $p = \infty$, we can conclude that $C_p(d,G) < \infty$ for every $p \in (1,\infty)$ and for every convex symmetric body $G \subset \mathbb{R}^d$. However, then the implied upper bound for $C_p(d,G)$ depends on the dimension, since the interpolation with a weak type (1,1) estimate does not give anything reasonable in these kind of questions, and generally it is better to work with $p \in (1,\infty)$ to obtain any non-trivial result concerning the behavior of $C_p(d,G)$ as $d \to \infty$.

The problem about estimates of $C_p(d,G)$, as $d \to \infty$, has been extensively studied by several authors for nearly four decades. The starting point was the work of the third author [33], where, in the case of the Euclidean balls $G = B^2$, it was shown that $C_p(d,B^2)$ is bounded independently of the dimension for every $p \in (1,\infty]$. Not long afterwards it was proved by the first author, in [31] for $p = 2$, that $C_p(d,G)$ is bounded by an absolute constant, which is independent of the underlying convex symmetric body $G \subset \mathbb{R}^d$. This result was extended in [11], and independently by Carbery [10], for all $p \in (3/2,\infty]$.

It is conjectured that the inequality in (1.2) holds for all $p \in (1,\infty]$ and for all convex symmetric bodies $G \subset \mathbb{R}^d$ with $C_p(d,G)$ independent of $d \in \mathbb{N}$. It is reasonable to believe that this is true, since it was verified for a large class of convex symmetric bodies.
For $q \in [1, \infty]$, let $B^q$ be a $q$-ball in $\mathbb{R}^d$ defined by

$$B^q = \left\{ x \in \mathbb{R}^d : |x|_q = \left( \sum_{1 \leq k \leq d} |x_k|^q \right)^{1/q} \leq 1 \right\} \quad \text{for } q \in [1, \infty),$$

(1.3)

$$B^\infty = \left\{ x \in \mathbb{R}^d : |x|_\infty = \max_{1 \leq k \leq d} |x_k| \leq 1 \right\}.$$  

For the $q$-balls $G = B^q$ the full range $p \in (1, \infty)$ of dimension-free estimates for $C_p(d, B^q)$ was established by Müller in [20] (for $q \in [1, \infty]$) and in [28] (for cubes $q = \infty$) with constants depending only on $q$. More about the current state of the art and papers [5, 6, 8, 26, 33] will be given in Section 2.

The proof of Theorem 1 will be presented in Section 4 using a new flexible approach, which recently resulted in dimension-free bounds in $r$-variations and jump inequalities corresponding to the operators $M^G_t$ from (1.1), see [9] and [23]. An important feature of this method is that it is also applicable to the discrete settings, see [10] and [23]. The method is described in Section 3, the proof of Theorem 1 is given in Section 4. Our aim is to continue the investigations in the discrete settings as well. Similar types of questions were recently studied by the authors [10] for the discrete analogues of the Hardy–Littlewood maximal functions with symmetric convex bodies $G \subset \mathbb{R}^d$, which independently were the subject of [6] and [13]. We prove the following theorem.

**Theorem 1.** Let $p \in (3/2, \infty)$, then there exists a constant $C_p > 0$ independent of dimension $d \in \mathbb{N}$ and a symmetric convex body $G \subset \mathbb{R}^d$ such that the constant $C_p(d, G)$ defined in (1.2) satisfies

$$C_p(d, G) \leq C_p.$$  

(1.4)

Moreover, a dyadic variant of (1.4) remains true for all $p \in (1, \infty)$. More precisely, for every $p \in (1, \infty)$ there exists a constant $C_p > 0$ independent of dimension $d \in \mathbb{N}$ and a symmetric convex body $G \subset \mathbb{R}^d$ such that for every $f \in L^p(\mathbb{R}^d)$ we have

$$\|\sup_{n \in \mathbb{Z}} |M^G_t f|\|_{L^p(\mathbb{Z}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}.$$  

(1.5)

The proof of Theorem 1 will be presented in Section 3 using a new flexible approach, which recently resulted in dimension-free bounds in $r$-variations and jump inequalities corresponding to the operators $M^G_t$ from (1.1), see [9] and [23]. The method is described in Section 3, the proof of Theorem 1 is given in Section 4. Our aim is to continue the investigations in the discrete settings as well. Similar types of questions were recently studied by the authors [10] for the discrete analogues of the operators $M^G_t$ in $\mathbb{Z}^d$.

1.2. Statement of the results: discrete perspective. For every $t > 0$ and for every $x \in \mathbb{Z}^d$ we define the discrete Hardy–Littlewood averaging operator

$$M^G_t f(x) = \frac{1}{|G_t \cap \mathbb{Z}^d|} \sum_{y \in G_t \cap \mathbb{Z}^d} f(x - y) \quad \text{for } f \in \ell^1(\mathbb{Z}^d).$$  

(1.6)

We note that the operator $M^G_t$ is a discrete analogue of $M^G_t$ from (1.1).

For $p \in (1, \infty]$, let $C_p(d, G) > 0$ be the best constant such that the following maximal inequality

$$\sup_{t > 0} \|M^G_t f\|_{L^p(\mathbb{Z}^d)} \leq C_p(d, G) \|f\|_{\ell^p(\mathbb{Z}^d)}.$$  

(1.7)

holds for every $f \in \ell^p(\mathbb{Z}^d)$.

Arguing in a similar way as in (1.1), we conclude that $C_p(d, G) < \infty$ for every $p \in (1, \infty]$ and for every convex symmetric body $G \subset \mathbb{R}^d$. The question now is to decide whether $C_p(d, G)$ can be bounded independently of the dimension $d$ for every $p \in (1, \infty)$.

In [10] the authors examined this question in the case of the discrete cubes $B^\infty \cap \mathbb{Z}^d$, and showed that for every $p \in (3/2, \infty]$ there is a constant $C_p > 0$ independent of the dimension such that $C_p(d, B^\infty) \leq C_p$. It was also shown in [17] that if the supremum in (1.7) is restricted to the dyadic set $D = \{2^n : n \in \mathbb{N} \cup \{0\}\}$, then (1.7) holds for all $p \in (1, \infty]$ and $C_p(d, G)$ is independent of the dimension.

The general case in much more complicated. However, it is not difficult to show [10] that for every symmetric convex body $G \subset \mathbb{R}^d$ there exists $t_G > 0$ with the property that the norm of the discrete maximal function $\sup_{t > t_G} |M^G_t f|$ is controlled by a constant multiple of the norm of its continuous counterpart, and the implied constant is independent of the dimension. This is a simple comparison argument yielding dimension-free estimates for $\sup_{t > t_G} |M^G_t f|$ as long as the corresponding dimension-free bounds are available for their continuous analogues. As a corollary, for $q$-balls $G = B^q$, if $p \in (1, \infty]$ and $q \in [1, \infty]$, we obtain that there is a constant $C_{p,q} > 0$ independent of the dimension $d \in \mathbb{N}$ such that for all $f \in \ell^p(\mathbb{Z}^d)$ we have

$$\sup_{t \geq d^{1+1/q}} \|M^B_t f\|_{\ell^p(\mathbb{Z}^d)} \leq C_{p,q} \|f\|_{\ell^p(\mathbb{Z}^d)}.$$  

(1.8)
At this stage, the whole difficulty lies in estimating $\sup_{t \geq C_d} |M^B_t f|$, where the things are getting more complicated. Nevertheless, as we shall see below, in some cases improvements are possible.

We show that in the case of $M^B_1$, which together with $M^B_\infty$, is presumably the most natural setting for the discrete Hardy–Littlewood maximal functions, the range in [13] can be improved. Namely, the main discrete result of this paper is, an extension of (1.8) for $G = B^2$, stated below.

**Theorem 2.** For each $p \in (1, \infty]$ there is a constant $C_p > 0$ independent of the dimension $d \in \mathbb{N}$ such that for every $f \in \ell^p(\mathbb{Z}^d)$ we have

$$\| \sup_{t \geq C_d} |M^B_t f| \|_{\ell^p(\mathbb{Z}^d)} \leq C_p \|f\|_{\ell^p(\mathbb{Z}^d)},$$

(1.9)

for an appropriate absolute constant $C > 0$.

The proof of Theorem 2 is based on a delicate refinement of the arguments from [10], which in the end reduce the matters to the comparison of the norm of the sup of $M^B_t f$ with the norm of its continuous analogue, and consequently to the dimension-free estimates of $C_p(d, B^2)$ for all $p \in (1, \infty]$, that are guaranteed by [33]. The proof of Theorem 2 is contained in Section 5.

Surprisingly, as it was shown in [10], the dimension-free estimates in the discrete case are not as broad as in the continuous setup and there is no obvious conjecture to prove. This is due to the fact that there exists a simple example of a convex symmetric body in $\mathbb{Z}^d$ for which maximal estimate (1.7) on $\ell^p(\mathbb{Z}^d)$, for every $p \in (1, \infty)$, involves the smallest constant $C_p(d, G) > 0$ unbounded in $d \in \mathbb{N}$. In order to carry out the construction it suffices to fix a sequence $1 \leq \lambda_1 < \ldots < \lambda_d < \ldots < \sqrt{2}$ and consider, as in [10], the ellipsoid

$$E(d) = \left\{ x \in \mathbb{R}^d : \sum_{k=1}^d \lambda_k^2 x_k^2 \leq 1 \right\}.$$

Then one can prove that for every $p \in (1, \infty)$ there is $C_p > 0$ such that for every $d \in \mathbb{N}$ one has

$$C_p(d, E(d)) \geq C_p(d, G) \geq C_p(d, G) \geq C_p(d, G),$$

(1.10)

Inequality (1.10) shows that the dimension-free phenomenon for the Hardy–Littlewood maximal functions in the discrete setting is much more delicate, and the dimension-free estimates even in the Euclidean case for $C_p(d, B^2)$ may be very difficult. However, there is an evidence, gained recently by the authors in [11], in favor of the general problem, which makes the things not entirely hopeless. Namely, in [11] a dyadic variant of inequality (1.7) for $G = B^2$ was studied and we proved the following result.

**Theorem 3.** For every $p \in [2, \infty]$ there exists a constant $C_p > 0$ independent of $d \in \mathbb{N}$ such that for every $f \in \ell^p(\mathbb{Z}^d)$ we have

$$\| \sup_{t \geq 0} |M^B_t f| \|_{\ell^p(\mathbb{Z}^d)} \leq C_p \|f\|_{\ell^p(\mathbb{Z}^d)}.$$

(1.11)

All the aforementioned results give us strong motivation to understand the situation more generally. In particular, in the case of $q$-balls $G = B^q$ where $q \in (1, \infty)$, which is well understood in the continuous setup. More about the methods available in the discrete setting is in Section 6.

1.3. **Notation.** Here we fix some further notation and terminology.

1. Throughout the whole paper $d \in \mathbb{N}$ denotes the dimension and $C > 0$ denotes a universal constant, which does not depend on the dimension, but it may vary from occurrence to occurrence.

2. We write that $A \lesssim B$ (or $A \gtrsim B$) to say that there is an absolute constant $C_B > 0$ (which possibly depends on $\delta > 0$) such that $A \leq C_B B$ (or $A \geq C_B B$), and we write $A \asymp B$ when $A \lesssim B$ and $A \gtrsim B$ hold simultaneously.

3. Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of positive integers let $N_0 = \mathbb{N} \cup \{0\}$, and let $\mathbb{D} = \{2^n : n \in \mathbb{N}_0\}$ denote the set of all dyadic numbers. We set $\mathbb{N}_N = \{1, 2, \ldots, N\}$ for any $N \in \mathbb{N}$.

4. The Euclidean space $\mathbb{R}^d$ is endowed with the standard inner product

$$x \cdot \xi = \langle x, \xi \rangle = \sum_{k=1}^d x_k \xi_k$$

for every $x = (x_1, \ldots, x_d)$ and $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$. 

For a countable set $Z$ (usually $Z = \mathbb{Z}^d$) endowed with the counting measure we shall write that
\[
L^p(Z) = \{ f : Z \to \mathbb{C} : \|f\|_{L^p(Z)} < \infty \}
\]
for any $p \in [1, \infty]$, where for any $p \in [1, \infty)$ we have
\[
\|f\|_{L^p(Z)} = \left( \sum_{m \in Z} |f(m)|^p \right)^{1/p} \quad \text{and} \quad \|f\|_{L^\infty(Z)} = \sup_{m \in Z} |f(m)|.
\]

We shall abbreviate $\| \cdot \|_{L^p(\mathbb{R}^d)}$ to $\| \cdot \|_{L^p}$, and $\| \cdot \|_{L^p(Z)}$ to $\| \cdot \|_{L^p}$.

Let $(X, \mathcal{B}, \mu)$ be a $\sigma$-finite measure space. Let $p \in [1, \infty]$ and suppose that $(T_t)_{t \in \mathbb{R}}$ is a family of linear operators such that $T_t$ maps $L^p(X)$ to itself for every $t \in \mathbb{R} \subseteq (0, \infty)$. Then the corresponding maximal function will be denoted by
\[
T_* f = \sup_{t \in \mathbb{R}} |T_t f|, \quad \text{for every } f \in L^p(X).
\]
We shall abbreviate $T_* f$ to $T_*$, if $\mathcal{B} = (0, \infty)$.

Let $(B_1, \| \cdot \|_{B_1})$ and $(B_2, \| \cdot \|_{B_2})$ be Banach spaces. For a linear or sub-linear operator $T : B_1 \to B_2$ its norm is defined by
\[
\|T\|_{B_1 \to B_2} = \sup_{\|f\|_{B_1} \leq 1} \|T(f)\|_{B_2}.
\]

Let $\mathcal{F}$ denote the Fourier transform on $\mathbb{R}^d$ defined for any function $f \in L^1(\mathbb{R}^d)$ as
\[
\mathcal{F} f(\xi) = \int_{\mathbb{R}^d} f(x) e^{2\pi i \xi \cdot x} \, dx \quad \text{for any } \xi \in \mathbb{R}^d.
\]
If $f \in L^1(\mathbb{Z}^d)$ we define the discrete Fourier transform by setting
\[
\hat{f}(\xi) = \sum_{x \in \mathbb{Z}^d} f(x) e^{2\pi i \xi \cdot x} \quad \text{for any } \xi \in \mathbb{T}^d,
\]
where $\mathbb{T}^d \equiv [-1/2, 1/2]^d$ is the $d$-dimensional torus. We shall denote by $\mathcal{F}^{-1}$ the inverse Fourier transform on $\mathbb{R}^d$ or the inverse Fourier transform (Fourier coefficient) on the torus $\mathbb{T}^d$. This will cause no confusions and the meaning will be always clear from the context.

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2. A review of the current state of the art

In the 1980s dimension-free estimates for the Hardy–Littlewood maximal functions over convex symmetric bodies had begun to be studied \[33\] \[34\] and went through a period of considerable changes and developments \[11\] \[12\] \[13\] \[14\] \[18\] \[23\] \[30\]. However, the dimension-free phenomenon in harmonic analysis had been apparent much earlier, see for instance \[36\] Chapter 14, §3 in Vol.III, as well as \[32\] and the references given there. We refer also to more recent results \[11\] \[12\] \[13\] \[14\] \[18\] \[23\] \[30\] and the survey article \[16\] for a very careful and detailed exposition of the subject.

2.1. Dimension-free estimates for semigroups. Consider the Poisson semigroup $(P_t)_{t \geq 0}$ defined on the Fourier transform side by
\[
\mathcal{F}(P_t f)(\xi) = p_t(\xi) \mathcal{F}(f)(\xi),
\]
for every $t \geq 0$ and $\xi \in \mathbb{R}^d$, with the symbol
\[
p_t(\xi) = e^{-2\pi |t| L |\xi|},
\]
involving an isotropic constant $L = L(G) > 0$ defined in (2.11). The dilation by the isotropic constant is a technical assumption, which will simplify our further discussion.

For every $x \in \mathbb{R}^d$ we introduce the maximal function
\[
P_* f(x) = \sup_{t > 0} |P_t f(x)|,
\]
and the square function
\[ g(f)(x) = \left( \int_0^\infty t \left| \frac{d}{dt} P_t f(x) \right|^2 \, dt \right)^{1/2}, \]
associated with the Poisson semigroup. From [32] we know that for every \( p \in (1, \infty) \) there exists a constant \( C_p > 0 \), which does not depend on \( d \in \mathbb{N} \), such that for every \( f \in L^p(\mathbb{R}^d) \) we have
\[ \| P_t f \|_{L^p} \leq C_p \| f \|_{L^p}, \tag{2.1} \]
and
\[ \| g(f) \|_{L^p} \leq C_p \| f \|_{L^p}. \tag{2.2} \]

For the proof of (2.1) and (2.2) one has to check that \((P_t)_{t \geq 0}\) is a symmetric diffusion semigroup in the sense of [32] Chapter III. For the convenience of the reader we recall the definition of a symmetric diffusion semigroup from [32] Chapter III, p.65. Let \((X, \mathcal{B}(X), \mu)\) be a \( \sigma\)-finite measure space, and \((T_t)_{t \geq 0}\) be a strongly continuous semigroup on \( L^2(X) \), which maps \( L^1(X) + L^\infty(X) \) to itself for every \( t \geq 0 \). We say that \((T_t)_{t \geq 0}\) is a symmetric diffusion semigroup, if it satisfies for all \( t \geq 0 \) the following conditions:

1. **Contraction property**: for all \( p \in [1, \infty) \) and \( f \in L^p(X) \) we have \( \| T_t f \|_{L^p(X)} \leq \| f \|_{L^p(X)} \).
2. **Symmetry property**: each \( T_t \) is a self-adjoint operator on \( L^2(X) \).
3. **Positivity property**: \( T_t f \geq 0 \), if \( f \geq 0 \).
4. **Conservation property**: \( T_t 1 = 1 \).

One major advantage of using the above-mentioned conditions is that the probabilistic techniques are applicable to understand properties of \( T_t \). This is the reason why, in particular, inequalities (2.1) and (2.2) hold, see [32] Chapter III, for more details, and also [15] for an even more relaxed conditions. The semigroup \( P_t \) is closely linked to the averaging operator \( M_t^d \). Namely, both operators are contractive on \( L^p(\mathbb{R}^d) \) spaces for all \( p \in [1, \infty) \), preserve the class of nonnegative functions, and satisfy \( P_1 = M_1^d = 1 \).

Later on, we shall need a variant of the Littlewood–Paley inequality. For every \( n \in \mathbb{Z} \) we define the Poisson projections \( S_n \) by setting
\[ S_n = P_{2^n-1} - P_{2^n}. \]

Then, the sequence \((S_n)_{n \in \mathbb{Z}}\) is a resolution of the identity on \( L^2(\mathbb{R}^d) \). Namely, we have
\[ f = \sum_{n \in \mathbb{Z}} S_n f, \quad \text{for every} \quad f \in L^2(\mathbb{R}^d). \tag{2.3} \]

Observe that
\[ S_n f(x) = - \int_{2^n-1}^{2^n} \frac{d}{dt} P_t f(x) \, dt. \]

Then by the Cauchy–Schwarz inequality we obtain, for every \( n \in \mathbb{Z} \) and \( x \in \mathbb{R}^d \), the following bound
\[ |S_n f(x)|^2 \leq \left( \int_{2^n-1}^{2^n} \frac{d}{dt} P_t f(x) \right)^2 \leq 2^{n-1} \int_{2^n-1}^{2^n} \left| \frac{d}{dt} P_t f(x) \right|^2 \, dt \leq \frac{1}{2^n} \int_0^\infty \left| \frac{d}{dt} P_t f(x) \right|^2 \, dt. \]

Now summing over \( n \in \mathbb{Z} \) and using (2.2) one shows that for every \( p \in (1, \infty) \), there is a constant \( C_p > 0 \) independent of \( d \in \mathbb{N} \) such that for every \( f \in L^p(\mathbb{R}^d) \) the following Littlewood–Paley inequality holds
\[ \left\| \left( \sum_{n \in \mathbb{Z}} |S_n f|^2 \right)^{1/2} \right\|_{L^p} \leq C_p \| f \|_{L^p}. \tag{2.4} \]

Inequality (2.4) will play an important role in the proof of Theorem 1.

We finish this subsection by showing a simple pointwise inequality between the Poisson semigroup and the Hardy–Littlewood maximal function associated with the Euclidean balls, which motivates, to some extent, the study of dimension-free estimates for the Hardy–Littlewood maximal functions. Namely, let \( K_t \) be the kernel corresponding to \( P_t \), assume that \( f \geq 0 \) and observe that
\[ P_t f(x) = K_t * f(x) = \int_{\mathbb{R}^d} \int_0^{K_t(x-y)} ds f(y) \, dy = \int_0^{\infty} \int_{\{y \in \mathbb{R}^d : K_t(x-y) \geq s\}} f(y) \, dy \, ds. \]

The set \( \{y \in \mathbb{R}^d : K_t(x-y) \geq s\} \) is an Euclidean ball centered at \( x \in \mathbb{R}^d \), since \( K_t \) is radially decreasing. Thus
\[ P_t f(x) = K_t * f(x) \leq \left( \int_0^{\infty} |\{y \in \mathbb{R}^d : K_t(x-y) \geq s\}| \, ds \right) M_x^{B_t^2} f(x) = \| K_t \|_{L^1} M_x^{B_t^2} f(x). \]
Hence we conclude that
\[ P_* f(x) \leq M_* B^2 f(x). \]  
Inequality (2.1) gives us a bound independent of the dimension for \( \|P_*\|_{L^p \rightarrow L^p} \), and in view of (2.2) we obtain \( \|P_*\|_{L^p \rightarrow L^p} \leq C_p(d, B^2) \). Now a natural question arises whether \( C_p(d, B^2) \) can be bounded independently of the dimension. This problem was investigated by the third author in [33].

2.2. The case of the Euclidean balls [33,35]. The third author obtained in [33], see also the joint paper with Str"omberg [35] for more details, that for every \( p \in (1, \infty) \) there is a constant \( C_p > 0 \) independent of the dimension \( d \in \mathbb{N} \) such that
\[ C_p(d, B^2) \leq C_p. \]  
Let us briefly describe the method used in [33] to prove (2.6). In \( \mathbb{R}^d \), as \( d \to \infty \), most of the mass of the unit ball \( B^2 \) concentrates at the unit sphere \( S^{d-1} \) in \( \mathbb{R}^d \). In fact, if \( \varepsilon \in (0, 1) \), we have
\[ \int_0^1 r^{d-1} dr = 1, \quad \text{while} \quad \lim_{d \to \infty} d \int_0^{1-\varepsilon} r^{d-1} dr = 0. \]  
Therefore, the key idea is to use the spherical averaging operator, defined for any \( r > 0 \) and \( x \in \mathbb{R}^d \) by
\[ A^d_r f(x) = \int_{S^{d-1}} f(x - r\theta) d\sigma_{d-1}(\theta), \]  
where \( \sigma_{d-1} \) denotes the normalized surface measure on \( S^{d-1} \). Using polar coordinates one easily sees that
\[ M_* B^2 f(x) = d \int_0^1 r^{d-1} A^d_r f(x) dr, \]  
which immediately implies
\[ |M_* B^2 f(x)| \leq |A^d f(x)|. \]  
By the earlier result of the third author [31], we know that for every \( d \geq 3 \) and for every \( p > \frac{d}{d-1} \) there is a constant \( C_{d,p} > 0 \) such that for every \( f \in L^p(\mathbb{R}^d) \) one has
\[ \|A^d f\|_{L^p} \leq C_{d,p} \|f\|_{L^p}. \]  
Inequality (2.9) is also true when \( d = 2 \), but this turned out to be a more difficult result, obtained by the first author in [3]. Now, the matters are reduced to show that the best constant in (2.9) can be taken to be independent of the dimension. For this purpose, the method of rotations enables one to view high-dimensional spheres as an average of rotated low-dimensional ones, and consequently one can conclude that for every \( d \geq 3 \) and \( p > \frac{d}{d-1} \) we have
\[ \|A^d_{\mathbb{R}^{d+1}} f\|_{L^p(\mathbb{R}^{d+1})} \leq \|A^d_f\|_{L^p(\mathbb{R}^d)}. \]  
Hence the best constant in (2.9) is non-increasing, and in particular bounded, in \( d \in \mathbb{N} \).

In order to prove (2.6) it suffices to take an integer \( d_0 > \frac{d}{p-1} \). If \( d \leq d_0 \), then there is nothing to do. If \( d > d_0 \), taking into account (2.8) and (2.10), we see that
\[ \|M_* f\|_{L^p} \leq \|A^d_{\mathbb{R}^{d_0}} f\|_{L^p(\mathbb{R}^{d_0})} \|f\|_{L^p}, \]  
and we obtain (2.6) as claimed.

The method described above is limited to the Euclidean balls. The case of general convex symmetric bodies will require a different approach.

2.3. The \( L^2 \) result for general symmetric bodies via Fourier transform methods [5]. In [5] the first author proposed a different approach, which is based on the estimates of the averaging operators \( M^G_{\mathbb{R}^d} \) on the Fourier transform side. Before we present the main result from [5] we have to fix some notation and terminology. We begin with the most important definition of this paper.

Definition 2.1. We say that a convex symmetric body \( G \subset \mathbb{R}^d \) is in the isotropic position, if it has Lebesgue measure \( |G| = 1 \), and there exists a constant \( L = L(G) > 0 \) depending only on \( G \) such that
\[ \int_G \langle x, \xi \rangle^2 dx = L(G) |\xi|^2 \quad \text{for any} \quad \xi \in \mathbb{R}^d. \]  
The constant \( L(G) \) in (2.11) is called the isotropic constant of \( G \).
and to obtain the following dimensional-free maximal estimate

\[ L(G)^2 = \frac{1}{d} \int_G |x|^2 \, dx. \]  \hfill (2.12)

**Lemma 2.1.** For every convex symmetric body \( G \subset \mathbb{R}^d \), there exists a linear transformation \( U \) of \( \mathbb{R}^d \) such that \( U(G) \) is in the isotropic position.

**Proof.** Observe that

\[ M(\xi) = \int_G \langle x, \xi \rangle x \, dx = \left( \int_G \langle x, \xi \rangle x_1 \, dx, \ldots, \int_G \langle x, \xi \rangle x_d \, dx \right) \]

is a positive operator on \( \mathbb{R}^d \). Thus one can find a positive operator \( S \) such that \( M = S^2 \). Setting \( U = c(G, S)S^{-1} \), where \( c(G, S) = \| \det S \|^{1/d} \| G \|^{-1/d} \), we see that \( \| U(G) \| = 1 \) and

\[
\int_{U(G)} \langle x, \xi \rangle^2 \, dx = c(G, S)^2 \| G \|^{-1} \int_G \langle S^{-1} x, \xi \rangle^2 \, dx \\
= c(G, S)^2 \| G \|^{-1} \langle M(S^{-1} \xi), S^{-1} \xi \rangle \\
= c(G, S)^2 \| G \|^{-1} |\xi|^2.
\]

Hence \( U(G) \) is in the isotropic position, with the isotropic constant \( L(U(G)) = c(G, S)\| G \|^{-1/2} > 0 \). \( \square \)

Observe that if the body \( G \) in \([1,2]\) is replaced with any other set of the form \( U(G) \), where \( U \) is an invertible linear transformation of \( \mathbb{R}^d \), then the \( L^p(\mathbb{R}^d) \) bounds from \([1,2]\) remain unchanged and we have

\[ C_p(d, G) = C_p(d, U(G)). \]  \hfill (2.13)

Indeed, considering an isometry \( U_p \) of \( L^p(\mathbb{R}^d) \) given by

\[ U_p f = |\det U|^{-1/p} f \circ U^{-1}, \quad \text{for any} \quad p \geq 1, \]

we obtain \([2,13]\), since

\[ U_p \circ M^G_t = M^{U(G)}_t \circ U_p. \]

In view of \([2,13]\) the dimension-free estimates are unaffected by a change of the underlying body to an equivalent one. Therefore, from now on unless otherwise stated, we assume that \( G \subset \mathbb{R}^d \) is in the isotropic position. For a symmetric convex body \( G \subset \mathbb{R}^d \), let

\[ m^G(\xi) = \mathcal{F}(1_G)(\xi) \]

be the multiplier corresponding to the operator \( M^G_t \) from \([1,1]\).

In \([5]\) the first author provided the estimates for \( m^G \) and its derivatives in terms of the isotropic constant \( L(G) \), see Theorem \([4]\) below.

**Theorem 4** \([5\text{ eq. (10),(11),(12)}]\). Let \( G \) be a symmetric convex body \( G \subset \mathbb{R}^d \) which is in the isotropic position. Let \( L = L(G) \) be the isotropic constant of \( G \). Then for every \( \xi \in \mathbb{R}^d \setminus \{0\} \) we have

\[ |m^G(\xi)| \leq 150(L|\xi|)^{-1}, \quad \text{and} \quad |m^G(\xi) - 1| \leq 150(L|\xi|), \]  \hfill (2.14)

and

\[ ||\xi, \nabla m^G(\xi)|| \leq 150. \]  \hfill (2.15)

In Section \([4]\) for the sake of completeness, we provide a detailed proof of Theorem \([4]\). In fact, as we shall see later on, the estimates in \([2,13]\) and \([2,15]\) will be the core of the proof of Theorem \([4]\).

Using Theorem \([4]\) as the main tool, it was proved in \([5]\) that

\[ C_2(d, G) \leq C, \]  \hfill (2.16)

where \( C > 0 \) is a constant that does not depend neither on \( d \in \mathbb{N} \) nor the underlying body \( G \subset \mathbb{R}^d \). In view of the dimensional-free estimates for the Poisson semigroup \([2,1]\) in order to prove \([2,16]\) it suffices to obtain the following dimensional-free maximal estimate

\[ \| \sup_{t > 0} |(M^G_t - P_t)f| \|_{L^2} \leq C \| f \|_{L^2}. \]  \hfill (2.17)
The estimate (2.17) in turn, was reduced, using some square function argument and the Plancherel theorem, to the uniform in \( \xi \in \mathbb{R}^d \) estimate
\[
\sum_{n \in \mathbb{Z}} \min \left\{ (2^n L(G)(\xi)), (2^n L(G)(\xi))^{-1} \right\} \leq C,
\]
where \( C > 0 \) is a universal constant independent of \( d \in \mathbb{N} \) and the body \( G \subset \mathbb{R}^d \). It is easy to see that (2.15) indeed holds. Moreover, it is true regardless of the exact value of the isotropic constant \( L(G) \). Remarkably, we do not need to know whether \( L(G) \) is comparable to a dimension-free constant.

### 2.4. Interlude: the isotropic conjecture.

As we have already underlined, the approach from [3] does not require any information on the size of the isotropic constant \( L(G) \). Recall at this point that \( L(G) \) is known to be bounded from below by an absolute constant.

**Proposition 2.2.** There is a universal constant \( c > 0 \) independent of the dimension such that for all convex symmetric bodies \( G \subset \mathbb{R}^d \) in the isotropic position we have \( L(G) \geq c \).

**Proof.** Let \( r_d \) be such that \( |r_d B^2| = 1 \). Then \( r_d B^2 \) is in the isotropic position and \( r_d = |B^2|^{-1/d} \approx d^{1/2} \).

Using (2.12) and polar coordinates one has
\[
L(r_d B^2)^2 = \frac{1}{d} \int_{r_d B^2} |x|^2 dx = \frac{|B^2|^d}{d+2} = \frac{r_d^2}{d+2} \simeq 1.
\]

Clearly, \( |x| \geq r_d \) on \( G \setminus r_d B^2 \) and \( |x| \leq r_d \) on \( r_d B^2 \setminus G \). Thus, using (2.12) and the observation that \( G \setminus r_d B^2 \) and \( r_d B^2 \setminus G \) have the same volume, we estimate
\[
dL(G)^2 = \int_G |x|^2 dx = \int_{G \cap r_d B^2} |x|^2 dx + \int_{G \setminus r_d B^2} |x|^2 dx \geq \int_{G \cap r_d B^2} |x|^2 dx + r_d^2 |G \setminus r_d B^2| \\
\geq \int_{G \cap r_d B^2} |x|^2 dx + \int_{r_d B^2 \setminus G} |x|^2 dx = dL(r_d B^2)^2.
\]

Therefore, we see that \( L(G) \geq L(r_d B^2) \geq c > 0 \). This completes the proof. \( \square \)

Conversely, it is not difficult to show the following upper bound.

**Proposition 2.3.** There is a universal constant \( C > 0 \) independent of the dimension such that for all convex symmetric bodies \( G \subset \mathbb{R}^d \) in the isotropic position we have \( L(G) \leq C d^{1/2} \).

**Proof.** If \( r(G) \) is the largest radius \( r > 0 \) such that \( r B^2 \subseteq G \) then there is an absolute constant \( c > 0 \) such that
\[
cL(G) \leq r(G),
\]
we refer to [12] Section 3.1, p.108 for more details. It follows that \( cL(G)B^2 \subseteq G \) and
\[
(cL(G))^d |B^2| \leq |G| = 1,
\]
and consequently, using \( |B^2|^{-1/d} \approx d^{1/2} \) we obtain the desired claim. \( \square \)

The estimate from Proposition 2.3 was improved by the first author in [7], where it was shown that \( L(G) = O(d^{1/4} \log d) \). Klartag [19] proved that \( L(G) = O(d^{1/4}) \), and this is the best currently available general estimate for \( L(G) \). However, the uniform bound from above for \( L(G) \) is a well-known open problem with several equivalent formulations. More precisely, we are lead to the following conjecture.

**Conjecture 1.** There is a constant \( C > 0 \) independent of \( d \in \mathbb{N} \) such that for all convex symmetric bodies \( G \subset \mathbb{R}^d \) in the isotropic position we have \( L(G) \leq C \).

This conjecture was verified for various classes of convex symmetric bodies. To give an example we consider the class of 1-unconditional symmetric convex bodies. Let \( \{e_1, \ldots, e_d\} \) denote the canonical basis in \( \mathbb{R}^d \). We say that \( G \subset \mathbb{R}^d \) is such a body, whenever, for every choice of signs \( \varepsilon_1, \ldots, \varepsilon_d \in \{-1, 1\} \), we have
\[
\left\| \sum_{i=1}^d \varepsilon_i x_i e_i \right\|_G = \|x\|_G, \text{ for all } x = (x_1, \ldots, x_d) \in \mathbb{R}^d,
\]
where \( \|x\|_G = \inf \{t > 0 : x \in tG\} \) denotes the Minkowski norm associated with \( G \).
Proposition 2.4. There is a constant $C > 0$ independent of $d \in \mathbb{N}$ such that for all 1-unconditional convex bodies $G \subset \mathbb{R}^d$ in the isotropic position we have $L(G) \leq C$.

For the proof of Proposition 2.4 and a more detailed exposition about the subject of geometry of isotropic convex bodies we refer to the monograph [12]. Interestingly, the issue of the isotropic constant did not impact the proofs of the dimension-free bounds for the Hardy–Littlewood maximal function (1.2) obtained in [5]. This gives us strong motivation to understand the role of the isotropic constant $L(G)$ in the estimates for $C_p(d, G)$ for all $p \in (1, \infty]$ for general convex symmetric bodies $G \subset \mathbb{R}^d$.

2.5. The $L^p$ results for $p \in (3/2, \infty]$ and fractional integration method. The first author [6], and independently Carbery [13], extended the $L^2(\mathbb{R}^d)$ result from [5], and showed that for every $p \in (3/2, \infty]$ there exists a numerical constant $C_p > 0$, which does not depend on the dimension $d \in \mathbb{N}$ such that for every convex symmetric body $G \subset \mathbb{R}^d$ we have

$$C_p(d, G) \leq C_p. \quad (2.19)$$

They also showed that if the supremum in (2.19) is restricted to the set of dyadic numbers $\mathbb{D}$, then inequality (2.19) remains valid for all $p \in (1, \infty]$. The methods used in these papers were completely different. We shall focus our attention merely on Carbery’s paper [13], since it was an important starting point for the papers [20] and [31], which will be discussed in the next subsection.

The first main idea introduced in [13] relied on a fractional derivative/integration method, and it was used to prove (2.20). Let $F_R$ denote the one dimensional Fourier transform. For $\alpha \in (0, 1)$, let $D^\alpha$ be the fractional derivative

$$D^\alpha F(t) = D^\alpha_p F(t) = D^\alpha_u F(u)|_{u=t} = F_R((2i\pi)^\alpha F_R^{-1}(F)\xi)(t), \quad \text{for} \quad t \in \mathbb{R}.$$ 

This formula gives a well defined tempered distribution on $\mathbb{R}$. Simple computations show that

$$D^\alpha_m G(t \xi) = \int_G (2\pi i x \cdot \xi)^\alpha e^{-2\pi i x \xi} dx, \quad \text{for} \quad t > 0,$$

where $m^G(t \xi) = F(k_G(t \xi))$. Moreover, [13, Lemma 6.6] guarantees that

$$D^\alpha_m G(t \xi) = \frac{1}{\Gamma(1-\alpha)} \int_t^\infty (u-t)^{-\alpha-1} \frac{d}{du} m^G(u \xi) du, \quad \text{for} \quad \xi \in \mathbb{R}^d.$$ 

If $P_\alpha^u$ is the operator associated with the multiplier

$$p_\alpha^u(\xi) = u^{\alpha+1} D^\alpha \left( \frac{m^G(u \xi)}{v} \right) \bigg|_{v=u}, \quad \text{for} \quad \xi \in \mathbb{R}^d,$$

then one can see that

$$M^G_t f(x) = F^{-1}(m^G(t \xi) F f)(x) = \frac{1}{\Gamma(\alpha)} \int_t^\infty \left( 1 - \frac{t}{u} \right)^{\alpha-1} P_\alpha^u f(x) \frac{du}{u}. \quad (2.21)$$

It was shown [13] that for general symmetric convex bodies one has

$$\|P_\alpha^u f\|_{L^p} \lesssim \|f\|_{L^p} + \|T_{(\xi \nabla)^\alpha m^G} f\|_{L^p}, \quad (2.22)$$

where $T_{(\xi \nabla)^\alpha m^G} f$ is the multiplier operator associated with the symbol

$$(\xi \cdot \nabla)^\alpha m^G(\xi) = D^\alpha m^G(t \xi)|_{t=1}.$$ 

The estimate from (2.22) immediately implies that

$$\sup_{u>0} \|P_\alpha^u\|_{L^p \rightarrow L^p} \lesssim 1 + \|T_{(\xi \nabla)^\alpha m^G}\|_{L^p \rightarrow L^p}, \quad (2.23)$$
since the multipliers \( p_\alpha \) are dilations of \( p_0 \). Using (2.21) and (2.23) one controls
\[
\left\| \sup_{\xi \in \mathbb{S}^{d-1}} |M_{\xi}^G(f)| \right\|_{L^p} \leq C_p \left(1 + \|T_{(\xi \cdot \nabla)^p}\|_{L^p \to L^p}\right) \|f\|_{L^p},
\]
whenever \( \alpha > 1/p \). Now, since \( T_{(\xi \cdot \nabla)^p} \) is associated with the symbol \( (\xi \cdot \nabla)m^G = \xi \cdot \nabla m^G(\xi) \), by Plancherel’s theorem and (2.15) we have \( \|T_{(\xi \cdot \nabla)^p}\|_{L^2 \to L^2} \leq C \). Clearly, \( T_{(\xi \cdot \nabla)^p} = M_1 \) is a contraction on \( L^1(\mathbb{R}^d) \). Then by complex interpolation, as in [13], we get \( \|T_{(\xi \cdot \nabla)^p}\|_{L^p \to L^p} \leq C_\alpha \), whenever \( \alpha < 2/p' \). In view of the restriction for \( \alpha > 1/p \) in (2.23) we obtain (2.22) for \( p \in (3/2, 2] \).

The above-mentioned method of fractional integration was exploited in [26] and [8].

2.6. The \( L^p \) result for \( p \in (1, \infty) \), the case of \( q \)-balls. Müller [26] proved, for all \( p \in [1, \infty] \) and for every symmetric convex body \( G \subset \mathbb{R}^d \), a remarkable upper bound for \( C_p(d, G) \) in terms of certain geometric invariants. To be more precise, assuming that the body \( G \) is in the isotropic position, we define two constants, geometric invariants, by setting
\[
\sigma(G)^{-1} = \max \{ \varphi_G^G(0) : \xi \in \mathbb{S}^{d-1} \},
\]
and
\[
Q(G) = \max \{ \text{Vol}_{d-1}(\pi_\xi(G)) : \xi \in \mathbb{S}^{d-1} \},
\]
where \( \varphi_G^G(0) = \text{Vol}_{d-1}(\{ x \in G : x \cdot \xi = 0 \}) \), while \( \pi_\xi : \mathbb{R}^d \to \mathbb{R}^\perp_\xi \) denotes the orthogonal projection of \( \mathbb{R}^d \) onto the hyperplane perpendicular to \( \xi \). It follows from (1.4) that
\[
\frac{1}{d}L(\xi) \leq \sigma(G) \leq 8L(G).
\]

Using these two linear invariants \( \sigma(G) \) and \( Q(G) \) it was proved in [26] that for every \( p \in (1, \infty] \) and for every symmetric convex body \( G \subset \mathbb{R}^d \) there is a constant \( C(p, \sigma(G), Q(G)) \) independent of the dimension \( d \in \mathbb{N} \) such that
\[
C_p(d, G) \leq C(p, \sigma(G), Q(G)).
\]
In other words \( C_p(d, G) \) may depend on \( \sigma(G) \) and \( Q(G) \), but not explicitly on the dimension \( d \in \mathbb{N} \). For \( p \in (3/2, \infty] \) inequality (2.23) is weaker than the estimates from [6] and [13], which show that \( C_p(d, G) \) can be even chosen independently of \( d \) and \( G \). However, using (2.23) it was proved that \( C_p(d, B^p) \) is independent of the dimension for all \( q \in (1, \infty] \), since \( \sigma(B^q) \) and \( Q(B^q) \) can be explicitly computed and they are independent of the dimension, but they depend on \( q \). For the cubes \( G = B^\infty \) it turned out that \( \sigma(B^\infty) \) is independent of the dimension, but \( Q(B^\infty) = d^{1/2} \), and at that time the cubes were thought of as candidates for a counterexample. However, the first author refined Müller’s approach, and provided the dimensional-free bounds for \( C_p(d, B^\infty) \) for all \( p \in (1, \infty] \) as well. We shall now give a description of Müller’s methods, which resulted in inequality (2.22).

As in [13], the proof of (2.23) in [26] is reduced to estimates of the \( L^p(\mathbb{R}^d) \) norm of the operator \( T_{(\xi \cdot \nabla)^p} \). Recall, that the complex interpolation allowed Carbery to prove dimension-free \( L^p(\mathbb{R}^d) \) bounds for \( T_{(\xi \cdot \nabla)^p} \) only in the restricted range of \( \alpha < 2/p' \). Müller, by considering a suitable admissible family of Fourier multiplier operators, was able to prove that, for all \( p \in (1, \infty) \) and for all \( \alpha \in (1/2, 1) \), one has
\[
\|T_{(\xi \cdot \nabla)^p}\|_{L^p \to L^p} \leq C_p(\alpha, \sigma(G), Q(G)).
\]
More precisely, by using complex interpolation it was shown in [26] that
\[
\|T_{(\xi \cdot \nabla)^p}\|_{L^p \to L^p} \leq C_p \left(1 + \|T_{-2\pi |\xi| m^G(\xi)}\|_{L^p \to L^p}\right),
\]
for \( \alpha \in (1/2, 1) \), where \( T_{-2\pi |\xi| m^G(\xi)} \) is the multiplier operator associated with the symbol \( -2\pi |\xi| m^G(\xi) \).

Finally, (2.26) reduced the task to justifying
\[
\|T_{-2\pi |\xi| m^G(\xi)}\|_{L^p \to L^p} \leq C_p(\sigma(G), Q(G)),
\]
for all \( p \in (1, \infty) \). Since \( T_{-2\pi |\xi| m^G(\xi)} \) is self-adjoint while proving (2.24) we can assume that \( p \in [2, \infty] \). The key part of the proof of (2.24) in [26] is based on the following identity
\[
-2\pi |\xi| m^G(\xi) = \sum_{j=1}^d \left( -i\frac{\xi_j}{|\xi|} \right) (-2\pi i\xi_j m^G(\xi)).
\]
Thus, defining the measures \( \mu_j = \frac{d}{dx_j} 1_G(x) \) we see that
\[
T_{-2\pi |\xi| m^G(\xi)} = \sum_{j=1}^d R_j(\mu_j * f),
\]
where \( R_j \) is the Riesz transform, corresponding to the multiplier \( -i\xi_j/|\xi| \) for \( j \in \{1, \ldots, d\} \).
We now are at the stage, where the dimension-free estimates for the vector of Riesz transforms enter into the game. The third author [34] proved that for every $p \in (1, \infty)$ there is a constant $C_p > 0$ independent of the dimension $d \in \mathbb{N}$ such that the following estimate
\[
\left\| \left( \sum_{j=1}^{d} |R_j f|^2 \right)^{1/2} \right\|_{L^p} \leq C_p \|f\|_{L^p},
\]
holds for every $f \in L^p(\mathbb{R}^d)$.

Then, dimension-free estimates for the vector of Riesz transforms (2.28) on $L^p(\mathbb{R}^d)$, together with a duality argument reduce the problem to the following square function estimate
\[
\left\| \left( \sum_{j=1}^{d} |R_j f|^2 \right)^{1/2} \right\|_{L^p} \leq C_p(\sigma(G),Q(G))\|f\|_{L^p},
\]
for $p \in [2, \infty)$, which was achieved by interpolating between its $p = 2$ and $p = \infty$ endpoints.

As it has been mentioned above this approach resulted in dimension-free estimates for $C_p(d,B^q)$ for all $p \in (1, \infty]$ and $q \in [1, \infty)$, since in these cases the geometric invariants $\sigma(B^q)$ and $Q(B^q)$ turned out to be independent of $d \in \mathbb{N}$. For $q = \infty$ one obtains $Q(B^\infty) = d^{1/2}$, which resulted in no further progress for the Hardy–Littlewood maximal function for the cube.

However, for $q = \infty$ the first author observed [8] by a careful inspection of Müller’s proof, that (2.25) for $p = 2$ can be estimated by a constant, which depends only on $\sigma(B^\infty)$, and the dependence on $Q(B^\infty)$ enters in (2.25) only for $p = \infty$. Therefore, instead of interpolating between $p = 2$ and $p = \infty$ in (2.25) it was natural to try, loosely speaking, to bound (2.25) for $p = q$ with large $q \geq 2$, and then interpolate with the improved estimate for $p = 2$, to obtain (2.25) with the implied constant depending only on $p$ and $\sigma(B^\infty)$.

In [8], in the proof of (2.25) for $p = q$ with large $q \geq 2$ the explicit formula for the multiplier
\[
m^{B^\infty}(\xi) = \prod_{j=1}^{d} \frac{\sin(\pi \xi_j)}{\pi \xi_j},
\]
for $\xi \in \mathbb{R}^d$ was essential. From Theorem 4 we have seen that $|m^{B^\infty}(\xi)| \leq C|\xi|^{-1}$. However, $m^{B^\infty}(\xi)$, for most of $\xi$, decays much faster than $|\xi|^{-1}$ and the worst case happens only for $\xi$ in narrow conical regions along the coordinate axes. This observation was implemented by making suitable localizations on the frequency space. An important ingredient, necessary to make these arguments rigorous in [8], was Pisier’s holomorphic semigroup theorem [29]. The arguments presented in [8] are based on a very explicit analysis which does not immediately carry over to other convex symmetric bodies. Therefore, new methods will need to be invented to understand the growth of $C_p(d,G)$, as $d \to \infty$, in inequality (1.2) for general symmetric convex bodies $G \subset \mathbb{R}^d$ when $p \in (1,3/2]$.

2.7. Weak type $(1,1)$ considerations. So far we have only discussed the question of dimension-free estimates on $L^p(\mathbb{R}^d)$ spaces for $p \in (1, \infty]$. However, one may ask about a dimension-free bound for the best constant $C_1(d,G)$ in the weak type $(1,1)$ estimate
\[
\sup_{\lambda > 0} \lambda \left\{ \left\{ x \in \mathbb{R}^d : \sup_{t>0} |M_{t}^{G} f(x)| > \lambda \right\} \right\} \leq C_1(d,G)\|f\|_{L^1},
\]
Appealing to the Vitali covering lemma one can easily show that $C_1(d,G) \leq 3^d$. In [35] the third author and Strömberg proved that for general symmetric convex bodies $G \subset \mathbb{R}^d$ one has
\[
C_1(d,G) \leq C d \log d,
\]
where $C > 0$ is a universal constant independent of $d \in \mathbb{N}$. This is the best known result to date, see also [28] for generalizations of (2.31). The proof of inequality (2.31) is based on a rather complicated variant of the Vitali covering idea. The authors in [35] were also able to sharpen this estimate in the case of the Euclidean balls by proving
\[
C_1(d,B^2) \leq C d,
\]
with a universal constant $C > 0$ independent of the dimension. For justifying (2.32) the authors used a comparison with the heat semigroup together with the Hopf maximal ergodic theorem, see [32].

Now, in view of these results a natural question arises, whether we can take a dimension-free constant in (2.31) and (2.32). This was resolved in the case of the cube $G = B^\infty$ by Aldaz [1] who proved that
\[
C_1(d,B^\infty) \geq C_d,
\]
where $C_d$ is a constant that tends to infinity as $d \to \infty$. The constant $C_d$ was made more explicit by Aubrun [27], who proved with $C_d \approx (\log d)^{1+\varepsilon}$ for every $\varepsilon > 0$, and by Iakovenko and Strömbärg [15], who considerably improved the latter lower bound by showing that $C_d \approx d^{1/4}$. The arguments in the papers [11, 23, 15], were based on careful analysis of a discretized version of the initial problem. The function $f$ realizing the supremum was then chosen as an appropriate sum of Dirac's deltas.

The case of the cube is the only one where we have a definitive answer on the size of $C_1(d, G)$ in [2, 30]. Remarkably even in the case of the Euclidean ball $B^d$ it is unknown whether the weak type $(1,1)$ constant is dimension-free.

3. Overview of the methods of the paper

This section is intended to present a new flexible approach, which recently resulted in dimension-free bounds in $r$-variational and jump inequalities corresponding to the operators $M^G$ from [1, 11], see [9] and [23]. An important feature of this method is that it is also applicable to the discrete settings, see [10] and [23].

For clarity of exposition we shall only be working with maximal functions on $L^p(\mathbb{R}^d)$ or $\ell^p(\mathbb{Z}^d)$. For a more abstract setting we refer to [23], and also [22].

3.1. Continuous perspective. We shall briefly outline the method of the proof of Theorem [1]. The proof of (1.4) is based on the following simple decomposition

$$\sup_{t > 0} |M^G_t f| \leq \sup_{n \in \mathbb{Z}} |M^G_{2^n} f| + \left( \sum_{n \in \mathbb{Z}} \sup_{t \in [2^n,2^{n+1}]} |(M^G_t - M^G_{2^n}) f|^2 \right)^{1/2}. \quad (3.1)$$

In other words, the full maximal function corresponding to the operators $M^G_t$ is controlled by the dyadic maximal function and the square function associated with maximal functions restricted to dyadic blocks.

The estimates, on $L^p(\mathbb{R}^d)$ for $p \in (1, \infty]$, of the dyadic maximal function (in fact inequality [12]) are based, upon comparing $\sup_{n \in \mathbb{Z}} |M^G_{2^n} f|$ with the Poisson semigroup $P_t$, see [11, 11], on a variant of bootstrap argument. The idea of the bootstrap goes back to [27], where the context of differentiation in lacunary directions was studied. Later on, these ideas were used in many other papers [13, 17], including their applications in dimension-free estimates [13]. Recently, it turned out that certain variant of bootstrap arguments may be also used to obtain dimension-free estimates in $r$-variational inequalities [9, 10] and in jump inequalities [23]. In the latter paper applications to the operators of Radon type are discussed as well. The methods of [23], presented as a part of an abstract theory, immediately give the desired conclusion. However, in Section 4 for the sake of clarity, we give a simple direct proof and deduce from inequality [1.12], which immediately leads to a bootstrap inequality in [1.13]. In particular, three tools, with dimension-free estimates, that we now highlight are used to obtain [1.12].

1. The maximal inequality (2.1) for the Poisson semigroup $P_t$.
2. The Littlewood–Paley inequality (2.3) associated with the Poisson projections $S_n$.
3. The estimates of the Fourier multiplies corresponding to $M^G_t$ from Theorem [4].

The details are given in the second part of Section 4. In the third part of Section 4 we estimate, on $L^p(\mathbb{R}^d)$ for $p \in (3/2, 2]$, the square function from (3.1). In order to do so, we shall employ an elementary numerical inequality, as in [3, 10], see also [23], which asserts that for every $n \in \mathbb{Z}$ and for every function $a : [2^n, 2^{n+1}] \to \mathbb{C}$ we have

$$\sup_{t \in [2^n,2^{n+1}]} |a(t) - a(2^n)| \leq \sqrt{2} \sum_{l \in \mathbb{Z}_0} \left( \sum_{m=0}^{2^l-1} |a(2^n + 2^{n-l}(m + 1)) - a(2^n + 2^{n-l}m)|^2 \right)^{1/2}. \quad (3.2)$$

The inequality is the crucial new ingredient, which on the one hand, replaces the fractional integration argument from [13]. This is especially important in the discrete setting as it is not clear, due to the lack of the dilation structure on $\mathbb{Z}^d$, whether the fractional integration argument is available there. On the other hand, (3.2) reduces estimates for a supremum (or even for $r$-variations, see [23]) restricted to a dyadic block to the situation of certain square functions, where the division intervals over which differences are taken (in these square functions) are all of the same size, see inequality (1.18).

A variant of inequality (5.20) was proved by Lewko–Lewko [20, Lemma 13], and it was used to study variational Rademacher–Menšhov type results for orthonormal systems. Inequality (5.20), essentially in this form, was independently obtained in [23, Lemma 1] by the second author and Trojan in the context of $r$-variational estimates for discrete Radon transforms, see also [21, 22].

Upon applying inequality (3.2) to control the square function from (3.1) the problem is reduced to control a new square function like in [4, 19]. The problem now is well suited to an application of the
Fourier transform methods, and the estimates from Theorem 4 combined with the Littlewood–Paley inequality do the job and we obtain the desired claim.

The approach described above does not allow us to improve the range for \( p \in (3/2, \infty] \) in the inequality from [14]. To see this, it suffices to consider the maximal function corresponding to the spherical means in \( \mathbb{R}^3 \), see (2.7). Indeed, adopting the method from Section 4 we obtain that the spherical maximal function is bounded on \( L^p(\mathbb{R}^3) \) for every \( p \in (3/2, \infty] \), but unbounded on \( L^{3/2}(\mathbb{R}^3) \), see [31].

Any extension of the range \( p \in (3/2, \infty] \) in (1.4) will require more refined information besides the positivity of the operators \( M^G_q \) and estimates of the Fourier multipliers \( m^G_q \) from Theorem 4. To be more precise, assume that \( p_0 \in [1,2] \) and let \( \alpha = 1/p_0 < 1 \). Suppose that there is a constant \( C_{p_0} > 0 \) independent of the dimension \( d \in \mathbb{N} \) such that for every \( t > 0 \) and \( h \in (0,1) \) and for every \( f \in L^{p_0}(\mathbb{R}^d) \) the following Hölder continuity condition holds

\[
\| (M^G_{t,h} - M^G_q) f \|_{L^{p_0}} \leq C_{p_0} \left( \frac{h}{t} \right)^\alpha \| f \|_{L^{p_0}}. \tag{3.3}
\]

Then, it was proved in [23] using a certain bootstrap argument, for every \( p \in (p_0,2] \) we have

\[
C_p(d,G) \lesssim_p 1, \tag{3.4}
\]

with the implicit constant independent of the dimension. Therefore, the general problem is reduced to understand (3.3). In the case of \( q \)-balls \( G = B^q \) for \( q \in [1,\infty] \), inequality (3.3), and consequently (3.4), can be verified as it was shown in [4] [23]. The general case is reduced, anyway, to understand the norm \( \| T_\epsilon \|_{L^p \to L^p} \) as in Müller’s proof [26]. But, as we said before, this will need new ideas.

3.2. Discrete perspective. As we have seen in the introduction the dimension-free estimates in the discrete setting for \( C_p(d,G) \) may be very hard, and in general there is no obvious conjecture to prove. However, for the \( q \)-balls \( G = B^q \) as in (1.3), in view of the methods presented above, the problem may be reduced to estimates of the Fourier multipliers. For \( q \in [1,\infty] \), let \( m^B_q \) be the multiplier corresponding to the operator \( \mathcal{M}^B_q \) as in (1.6). Let us define the proportionality factor

\[
\kappa_q(d,N) = Nd^{-1/q},
\]

which can be identified with the isotropic constant corresponding to \( B^q_N \), if the normalization assumption \( |B^q| = 1 \) in definition (2.11) is dropped. If we could prove that there exists a constant \( C_q > 0 \) independent of the dimension \( d \in \mathbb{N} \) such that for every \( N \in \mathbb{N} \) and \( \xi \in \mathbb{T}^d \) we have

\[
| m^B_q(\xi) - 1 | \leq C_q \kappa_q(d,N) |\xi|,
\]

\[
| m^B_q(\xi) | \leq C_q ( \kappa_q(d,N)|\xi| )^{-1},
\]

\[
| m^B_{q+1}(\xi) - m^B_q(\xi) | \leq C_q N^{-1}, \tag{3.5}
\]

where \( |\xi| \) denotes the Euclidean norm restricted to the torus \( \mathbb{T}^d = [-1/2, 1/2]^d \); then, using the methods from the proof of Theorem 1, we would be able to conclude that the best constant \( C_p(d,B^q) \) in inequality (1.4) is bounded independently of the dimension for every \( p \in (3/2, \infty] \).

Therefore, the problem of estimating \( C_p(d,B^q) \) with bounds independent of the dimension is reduced to establishing (3.5). Even though, estimates (3.5) can be thought of as discrete analogues of the estimates for the continuous multipliers \( m^G_q \), from Theorem 4 with \( G = B^q \), the method of the proof of Theorem 4 is not applicable to derive (3.5). For \( q \in [1,\infty) \) the question seems to be very hard due to the lack of reasonable estimates for the number of lattice points in the sets \( B^q_N \).

However, if \( q = \infty \) then \( B^\infty_N = [-N,N]^d \) is a cube. Thus the number of lattice points is not a problem any more, and we easily have \( |B^\infty_N \cap \mathbb{Z}^d| = (2N+1)^d \). This property distinguishes the cubes from the \( q \)-balls for \( q \in [1,\infty) \). Using the product structure of the cubes we were able to analyze the behavior of the multiplier \( m^B_\infty \) associated with the operator \( \mathcal{M}^B_\infty \) and obtain (3.6), see [14] for more details. The multiplier \( m^B_\infty \) is an exponential sum, which is the product of one dimensional Dirichlet’s kernels. The explicit formula for \( m^B_\infty \) in terms of the Dirichlet kernels was essential for our approach and permitted us to establish (3.6) for \( q = \infty \) with \( \kappa_\infty(d,N) = N \). Applying (3.6) we showed in [10] as it was mentioned in the introduction, that for every \( p \in (3/2, \infty] \) there is a constant \( C_p > 0 \) independent of the dimension such that \( C_p(d,B^\infty) \leq C_p \). However, if the supremum in (1.7) is restricted to the dyadic set \( \mathbb{D} \), then (1.7) holds for all \( p \in (1,\infty] \) and \( C_p(d,B^\infty) \) is independent of the dimension as well. The inequalities in (3.5), for \( q = \infty \), are based on elementary estimates, which are interesting in their own right. For this reason our method does not extend to discrete convex bodies other than \( B^\infty \). This is the second
place which sets the operators \( \mathcal{M}_N^B \) over the cubes apart from the operators \( \mathcal{M}_N^B \) over the \( q \)-balls for \( q \in [1, \infty) \).

Now it is desirable to understand whether inequalities (3.5) hold for \( q \in [1, \infty) \). The absence of the product structure for \( q \in [1, \infty) \) makes the estimates incomparably harder. However, using crude estimates for the number of lattice points in the \( q \)-balls \( B_q^0 \), if \( p \in (1, \infty) \) and \( q \in [1, \infty] \), we obtain, as in [10], that there is \( C_{p,q} > 0 \) independent of the dimension \( d \in \mathbb{N} \) such that for all \( f \in L^p(\mathbb{Z}^d) \) we have

\[
\left\| \sup_{N \geq 2^{d+1/q}} |\mathcal{M}_N^B f| \right\|_{L_p} \leq C_{p,q} \|f\|_{L_p}.
\]

(3.6)

Inequality (3.6) follows from a simple comparison argument, which permits us to dominate the \( L^p(\mathbb{Z}^d) \) norm of the maximal function \( \sup_{N \geq 2^{d+1/q}} |\mathcal{M}_N^B f| \) by a constant multiple of \( C_p(d, B^1) \), which we know is independent of the dimension for every \( p \in (1, \infty) \) due to [20] for \( q \in [1, \infty) \), and due to [5] for \( q = \infty \).

In Section 3 for \( q = 2 \), we shall extend the range in the supremum in (3.6) and we show that \( d^{1+1/q} = d^{3/2} \), (for \( q = 2 \)), can be replaced by a constant multiple of \( d \), see Theorem 2. Our argument is a subtle refinement of the arguments from [10]. Even though, we will also use crude estimates for the number of lattice points in the balls \( B_N^0 \), the essential improvement comes from the fact that the Euclidean norm corresponds to the scalar product \( |x|^2 = \langle x, x \rangle \). See Lemma 5.1 and Lemma 5.2, where this observation plays the key role. The rest of the argument reduces the problem to the comparison of the \( L^p(\mathbb{Z}^d) \) norm of \( \sup_{N \geq 2^{d}} |\mathcal{M}_N^B f| \) with \( C_p(d, B^2) \), which is independent of the dimension for all \( p \in (1, \infty) \). Now the matters are reduced to understand \( \sup_{1 \leq N \leq 2^d} |\mathcal{M}_N^B f| \).

In [11] the authors initiated investigations in this direction and the case of the discrete Euclidean balls with dyadic radii was studied. We obtained Theorem 3 which gives us some evidence that inequality (3.5) with dimension-free bounds in not entirely hopeless, at least for \( q = 2 \). The methods of the proof of Theorem 3 shed a new light on the general problem (1.7), but the best what we can do for the full maximal function at this moment is Theorem 2 and new methods will surely need to be invented to attack this case.

The proof of Theorem 3 is based on the the estimates for \( m_N^B \), which in turn are based on delicate combinatorial arguments that differ completely from the methods used to obtain estimates (3.5) for \( m_N^\infty \). In particular, we proved analogues of the first two inequalities from (3.5) for \( m_N^B \). However, the second inequality is perturbed by a negative power of \( \kappa_2(d, N) \), which makes our method limited to the dyadic scales, and nothing reasonable beyond \( L^2(\mathbb{Z}^d) \) theory can be said in (1.7). Our aim now is to understand whether the second estimate can be improved. If we succeeded in doing so, we could extend inequality (4.1) to \( L^p(\mathbb{Z}^d) \) spaces for all \( p \in (1, \infty) \). The second task, which seems to be quite challenging, is to obtain the third inequality in (3.5) for the multiplier \( m_N^B \). This inequality, if proved, would allow us to think about dimension-free estimates of \( C_p(d, B^2) \) for all \( p \in (3/2, \infty) \). We refer to [11] for more details.

4. Continuous perspective: proof of Theorem 1

The purpose of this section is to provide dimension-free estimates on \( L^p(\mathbb{R}^d) \), with \( p \in (3/2, \infty) \), for the Hardy–Littlewood maximal function associated with convex symmetric bodies in \( \mathbb{R}^d \). However, we begin with the proof of Theorem 4 which will allow us to build up the \( L^2(\mathbb{R}^d) \) theory in Theorem 1.

4.1. Fourier transform estimates: proof of Theorem 4

For \( \zeta \in \mathbb{S}^{d-1} \) and \( u \in \mathbb{R} \) we define the set

\[
A_\zeta(u) = \{ x \in G: x \cdot \zeta = u \},
\]

and an even and compactly supported function by setting

\[
\varphi_\zeta^G(u) = \text{Vol}_{d-1}(A_\zeta(u))^rac{1}{d-1},
\]

where \( \text{Vol}_{d-1} \) denotes \((d - 1)\)-dimensional Lebesgue measure. We observe that for all \( \lambda \in [0, 1] \) and for all \( u, v \in \mathbb{R} \) such that \( \varphi_\zeta^G(u) \neq 0 \) and \( \varphi_\zeta^G(v) \neq 0 \) we obtain

\[
\text{Vol}_{d-1}(A_\zeta(\lambda u + (1 - \lambda)v)) = \lambda \text{Vol}_{d-1}(A_\zeta(u)) + (1 - \lambda)\text{Vol}_{d-1}(A_\zeta(v)).
\]

(4.1)

This can be verified using Brunn–Minkowski’s inequality (in dimension \((d - 1)\)), since, by convexity of \( G \), for every \( u, v \in \mathbb{R} \) if \( A_\zeta(u) \neq \emptyset \) and \( A_\zeta(v) \neq \emptyset \) then

\[
\lambda A_\zeta(u) + (1 - \lambda)A_\zeta(v) \subseteq A_\zeta(\lambda u + (1 - \lambda)v).
\]

(4.2)

For \( \zeta \in \mathbb{S}^{d-1} \) define \( S_\zeta = \{ x \in \mathbb{R}: \varphi_\zeta^G(x) = 0 \} \). If \( u_0 \in S_\zeta \) then, using (4.2), it is not difficult to see that for every \( u \in \mathbb{R} \) such that \(|u| > |u_0|\) we have \( \varphi_\zeta^G(u) = 0 \). This ensures that \( S_\zeta \) is a symmetric interval...
contained in \([-u_\zeta, u_\zeta]\), where \(u_\zeta = \sup\{x \geq 0: \varphi^G_\zeta(x) \neq 0\}\). Taking \(v = -u\) in (4.1) we obtain that 
\[
\varphi^G_\zeta((2\lambda - 1)u) \geq \varphi^G_\zeta(u) \quad \text{for all } \lambda \in [0, 1] \text{ and } u \in \mathbb{R}.
\]
This implies that \(\varphi^G_\zeta\) is decreasing on \(S_\zeta \cap [0, \infty)\) as well as on \([0, \infty)\). Inequality (4.1) shows that the function \(\varphi^G_\zeta(x)\) is concave on \(S_\zeta\). In particular, \(\varphi^G_\zeta\) is differentiable almost everywhere in \((-u_\zeta, u_\zeta)\), since it is absolutely continuous on each closed interval contained in \((-u_\zeta, u_\zeta)\). The inequality between the weighted arithmetic and geometric means together with (4.1) also implies the log-concavity of \(\varphi^G_\zeta\). Namely, for \(\lambda \in [0, 1]\) and \(u, v > 0\) we have
\[
\varphi^G_\zeta(\lambda u + (1 - \lambda)v) \geq \varphi^G_\zeta(u)^{\lambda} \varphi^G_\zeta(v)^{1 - \lambda}.
\]
Note that using Fubini’s theorem we have, for \(\xi \in \mathbb{R}^d \setminus \{0\}\), that
\[
m^G(\xi) = \int_{\mathbb{R}} \varphi^G_{\xi/|\xi|}(u)e^{2\pi i \xi u} du.
\]
More generally, for any \(h \in L^\infty(\mathbb{R})\) and \(\xi \in \mathbb{R}^d \setminus \{0\}\), one has
\[
\int_{G} h(x \cdot \xi) dx = \int_{\mathbb{R}} \varphi^G_{\xi/|\xi|}(u) h(|\xi| u) du.
\]
From the above properties of \(\varphi^G_\zeta\) we shall deduce, as in \([3\text{ Lemma 1}]\), that
\[
\varphi^G_\zeta(u) \leq 2\varphi^G_\zeta(0)e^{-\varphi^G_\zeta(0)|u|}, \quad \text{for all } u \in \mathbb{R}, \quad \text{and } \zeta \in S^{d-1}.
\]
For this purpose, we fix \(\zeta \in S^{d-1}\) and let us consider the function \(\psi^G_\zeta(u) = \varphi^G_\zeta(0)e^{-\varphi^G_\zeta(0)|u|}\), whose logarithm is a linear function. We have that \(\varphi^G_\zeta(0) = \psi^G_\zeta(0)\), and suppose that there is a point \(u_0 \in (0, \infty)\) such that \(\varphi^G_\zeta(u_0) = \psi^G_\zeta(u_0)\). By the log-concavity we obtain that
\[
\varphi^G_\zeta(u) \leq \psi^G_\zeta(u), \quad \text{for } u > u_0,
\]
and in this case there is nothing to do. Moreover, the log-concavity also gives
\[
\varphi^G_\zeta(u) \geq \psi^G_\zeta(u), \quad \text{for } 0 \leq u \leq u_0.
\]
In this case, using (4.4) with \(h(u) = 1_{[0,u_0]}(u)\) we obtain
\[
\frac{1}{2} = \int_{\mathbb{R}}^\infty \psi^G_\zeta(0)du = \int_{\mathbb{R}}^{u_0} \varphi^G_\zeta(0) e^{-u\varphi^G_\zeta(0)} du = \int_{\mathbb{R}}^{u_0} e^{-u\varphi^G_\zeta(0)} du = 1 - e^{-u_0\varphi^G_\zeta(0)},
\]
and, consequently, \(e^{-u_0\varphi^G_\zeta(0)} \leq 1/2\), so that \(u_0\varphi^G_\zeta(0) \leq \log 2\). Hence, (4.5) follows, since
\[
\varphi^G_\zeta(u) \leq \varphi^G_\zeta(0)e^{u\varphi^G_\zeta(0)} \leq 2\varphi^G_\zeta(0)e^{-u\varphi^G_\zeta(0)}, \quad \text{for } 0 \leq u \leq u_0.
\]
If \(u = 0\) is the unique point such that \(\varphi^G_\zeta(0) = \psi^G_\zeta(0)\), then \(\varphi^G_\zeta(u) \leq \psi^G_\zeta(u)\) or \(\varphi^G_\zeta(u) \geq \psi^G_\zeta(u)\) for all \(u \in S_\zeta\). If the first inequality holds then we are done, so we may assume that the second inequality is true. Arguing in a similar way as in (4.4) with \(u_\zeta\) in place of \(u_0\) we obtain that \(u_\zeta\varphi^G_\zeta(0) \leq \log 2\), and consequently
\[
\varphi^G_\zeta(u) \leq \varphi^G_\zeta(0)e^{u\varphi^G_\zeta(0)} \leq 2\varphi^G_\zeta(0)e^{-u\varphi^G_\zeta(0)}, \quad \text{for } 0 \leq u \leq u_\zeta.
\]
Hence, (4.5) follows, since \(\varphi^G_\zeta(u) = 0\) for \(u \in S_\zeta^c\).

Since \(G\) is in the isotropic position we can also prove that \(\varphi^G_\zeta(0)\) is of the same order, uniformly in \(\zeta \in S^{d-1}\). More precisely, as in \([3\text{ Lemma 2}]\), we have
\[
\frac{3}{16} \leq L \varphi^G_\zeta(0) \leq 3, \quad \text{for every } \zeta \in S^{d-1},
\]
where \(L\) is the isotropic constant. To prove the right-hand side inequality in (4.7) we show, with the aid of (4.1) (for \(h(u) = u^2\)) and (4.5), that
\[
L^2 = \int_{\mathbb{R}} u^2 \varphi^G_\zeta(u) du \leq 4\varphi^G_\zeta(0) \int_{\mathbb{R}}^\infty u^2 e^{-\varphi^G_\zeta(0)u} du \leq 8\varphi^G_\zeta(0)^{-2}.
\]
For the left-hand side inequality in (4.7) we calculate
\[
1 = \int_{\mathbb{R}} \varphi^G_\zeta(u) du \leq 4L\varphi^G_\zeta(0) + \frac{1}{4L^2} \int_{|u| \geq 2L} u^2 \varphi^G_\zeta(u) du \leq 4L\varphi^G_\zeta(0) + \frac{1}{4},
\]
which implies \(L\varphi^G_\zeta(0) \geq 3/16\, and (4.7) is justified. We now pass to the proof of Theorem 4.
Proof of Theorem 4. We begin with the proof of inequalities in (2.14). For \( \xi \in \mathbb{R}^d \setminus \{0\} \) we set \( \zeta = \xi/|\xi| \), then integration by parts allows us to rewrite (4.4) as

\[
m^G(\xi) = \int_\mathbb{R} \varphi^G(\zeta)(2\pi|\xi|u)du = \lim_{u \to \zeta} \varphi^G(\zeta) \frac{\sin(2\pi|\xi|u)}{\pi|\xi|} - \frac{1}{2\pi|\xi|} \int_{-\zeta}^{\zeta} (\varphi^G)'(u) \sin(2\pi|\xi|u)du.
\]

Then using (4.7) we obtain the first inequality in (2.14), since

\[
|m^G(\xi)| \leq (\pi|\xi|)^{-1} \varphi^G(0) + (2\pi|\xi|)^{-1} \int_{-\zeta}^{\zeta} |(\varphi^G)'(u)|du
\]

\[
= (\pi|\xi|)^{-1} \varphi^G(0) - (\pi|\xi|)^{-1} \int_{0}^{\zeta} (\varphi^G)'(u)du
\]

\[
\leq 6\pi^{-1}(L|\xi|)^{-1}.
\]

To prove the second inequality in (2.14), we use (4.5) and (4.7) to write

\[
|m^G(\xi) - 1| \leq \int_\mathbb{R} \varphi^G(\zeta) \cos(2\pi|\xi|u) - 1|du
\]

\[
\leq 4\pi|\xi| \int_{0}^{\infty} u\varphi^G(\zeta)du
\]

\[
\leq 8\pi|\xi|\varphi^G(0)^{-1}
\]

\[
\leq 45\pi(L|\xi|).
\]

This completes the proof of (2.14). To justify (2.15), we use (4.4) and integrate by parts to get

\[
(\xi, \nabla m^G(\xi)) = \int_{G} 2\pi i \langle x, \xi \rangle e^{2\pi i x \cdot \xi} dx
\]

\[
= \int_{-\zeta}^{\zeta} (2\pi i \langle \xi \rangle e^{2\pi i u \xi} (u\varphi^G(\zeta)))du
\]

\[
= \lim_{u \to \zeta} e^{2\pi i u \xi} (u\varphi^G(\zeta)) - \lim_{u \to -\zeta} e^{2\pi i u \xi} (u\varphi^G(\zeta)) - \int_{-\zeta}^{\zeta} e^{2\pi i u \xi} \frac{d}{du}(u\varphi^G(\zeta))du.
\]

This leads, in view of (1.5), to the estimate

\[
|(\xi, \nabla m^G(\xi))| \leq 4\zeta \varphi^G(0)e^{-\varphi^G(0)\zeta} + \int_{-\zeta}^{\zeta} \varphi^G(u)du + \int_{-\zeta}^{\zeta} |u||\varphi^G(u)|du
\]

\[
\leq 5 - 2\int_{-\zeta}^{\zeta} u(\varphi^G)'(u)du,
\]

where we used the fact that \( \varphi^G(\zeta) \) is decreasing in \( u \). Hence, integrating by parts once again we reach \( |(\xi, \nabla m^G(\xi))| \leq 10 \), which gives (2.15). The proof of Theorem 4 is completed. \( \square \)

The approach we shall use to prove Theorem 1 was presented as a part of an abstract theory in [23]. The method has recently found many applications in \( r \)- variation and jump estimates (including dimension-free estimates) in the continuous and discrete settings, see [9, 10, 22, 23]. However here, for the sake of clarity, we shall only focus our attention on the maximal functions in the continuous setup.

Since we are working with a family of averaging operators only the range for \( p \in (3/2, 2] \) will be interesting in Theorem 1. The range for \( p \in (2, \infty] \) will follow then by a simple interpolation with the obvious \( L^\infty(\mathbb{R}^d) \) bound. For instance, in order to prove dimension-free bounds for the dyadic maximal function, it will suffice to show that for every \( p \in (1, 2] \) and for every \( f \in L^p(\mathbb{R}^d) \) we have

\[
\left\| \sup_{n \in \mathbb{Z}} |M_{2^n}^G f| \right\|_{L^p} \lesssim \|f\|_{L^p}.
\]

In particular, (4.8) proves inequality (1.5) from Theorem 4. Then, in view of (2.1), the proof of inequality (2.1) will be completed, if we show that for \( p \in (3/2, 2] \) and for every \( f \in L^p(\mathbb{R}^d) \) we have

\[
\left\| \left( \sum_{n \in \mathbb{Z}} \sup_{t \in [2^n, 2^{n+1}]} |(M_{2^n}^G - M_{2^{n+1}}^G)f|^2 \right)^{1/2} \right\|_{L^p} \lesssim \|f\|_{L^p}.
\]

In the next two subsections we prove inequalities (4.8) and (4.9) respectively.
4.2. Proof of inequality \(4.8\). We fix \(N \in \mathbb{N}\) and define

\[
B_p(N) = \sup_{\|f\|_{L^p} \leq 1} \left\| \sup_{|n| \leq N} |M^G_{2n}f| \right\|_{L^p}.
\]

We see that \(B_p(N) \leq 2N + 1\) for every \(N \in \mathbb{N}\), since \(M^G_{2n}\) is an averaging operator. Our aim will be to show that for every \(p \in (1, 2]\) there is a constant \(C_p > 0\) independent of the dimension and the underlying body \(G \subset \mathbb{R}^d\) such that

\[
\sup_{N \in \mathbb{N}} B_p(N) \leq C_p.
\]  

Observe that, by \(2.3\), we have

\[
\left\| \sup_{|n| \leq N} |M^G_{2n}f| \right\|_{L^p} \leq \left\| \sup_{t > 0} |P_{t}f| \right\|_{L^p} + \left\| \sup_{|n| \leq N} |(M^G_{2n} - P_{2n})f| \right\|_{L^p}
\]

\[
\lesssim \left\| f \right\|_{L^p} + \sum_{j \in \mathbb{Z}} \left( \sum_{|n| \leq N} |(M^G_{2n} - P_{2n})S_{j+n}f|^2 \right)^{1/2}
\]

where in the last line we have used decomposition from \(2.3\). The proof of \(4.8\) will be completed, if we show that for every \(p \in (1, 2]\) there is \(C_p > 0\) independent of \(d, N\), and the body \(G \subset \mathbb{R}^d\) such that for every \(j \in \mathbb{Z}\) and for every \(f \in L^p(\mathbb{R}^d)\) we have

\[
\left\| \sum_{|n| \leq N} |(M^G_{2n} - P_{2n})S_{j+n}f|^2 \right\|_{L^2}^{1/2} \leq C_p(1 + B_p(N))^2 \left\| f \right\|_{L^p}.
\]

Assume momentarily that \(1.12\) has been proven. Then combining \(1.11\) with \(1.12\) we obtain that

\[
B_p(N) \lesssim_p 1 + (1 + B_p(N))^{2^{1/p}}
\]

with the implicit constant independent of \(d, N\) and the body \(G \subset \mathbb{R}^d\). Thus we conclude, using \(4.13\), that \(4.10\) holds, and the proof of \(4.8\) and consequently \(1.5\) from Theorem 1 is completed.

4.2.1. Proof of inequality \(4.12\) for \(p = 2\). Using Theorem 3 we show that \(4.12\) holds for \(p = 2\). Let \(k(\xi) = m^G(\xi) - p_1(\xi) = m^G(\xi) - e^{-2\pi j L |\xi|}\) be the multiplier associated with the operator \(M^G - P_1\). Observe that by Theorem 3 and the properties of \(p_1(\xi)\) there exists a constant \(C > 0\) independent of the dimension and the body \(G \subset \mathbb{R}^d\) such that

\[
|k(\xi)| \leq C \min \{ L|\xi|, (L|\xi|)^{-1} \}
\]

where \(L = L(G)\) is the isotropic constant as in \(2.11\). Now by \(2.11\) and Plancherel’s theorem we get

\[
\left\| \left( \sum_{n \in \mathbb{Z}} |(M^G_{2n} - P_{2n})S_{j+n}f|^2 \right)^{1/2} \right\|_{L^2} \lesssim \left( \int_{\mathbb{R}^d} \sum_{n \in \mathbb{Z}} \left| k(2^n \xi) \left( e^{-2\pi j L |\xi|} - e^{-2\pi j L |\xi|} \right)^2 |Ff(\xi)|^2 \right| \right)^{1/2}
\]

\[
\lesssim 2^{-|j|/2} \left( \int_{\mathbb{R}^d} \sum_{n \in \mathbb{Z}} \min \{ 2^n |L|, (2^n L |\xi|)^{-1} \} |Ff(\xi)|^2 \right)^{1/2}
\]

\[
\lesssim 2^{-|j|/2} \left\| f \right\|_{L^2},
\]

with the implicit constant independent of \(d, N\) and the body \(G \subset \mathbb{R}^d\). This proves \(1.12\) for \(p = 2\).

4.2.2. Proof of inequality \(1.12\) for \(p \in (1, 2]\). For \(s \in (1, 2]\) and \(r \in [1, \infty]\), let \(A_N(s, r)\) be the smallest constant in the following inequality

\[
\left\| \left( \sum_{|n| \leq N} |(M^G_{2n} - P_{2n})g_n|^r \right)^{1/r} \right\|_{L^s} \leq A_N(s, r) \left\| \left( \sum_{|n| \leq N} |g_n|^r \right)^{1/r} \right\|_{L^s}.
\]

It is easy to see that \(A_N(s, r) < \infty\). Let \(u \in (1, p)\) be such that \(\frac{1}{u} = \frac{1}{s} + \frac{1}{r}\). Now it is not difficult to see that \(A_N(1, 1) \lesssim 1\), since \(\| (M^G_{2n} - P_{2n})f \|_{L^p} \leq 2 \| f \|_{L^p}\). Moreover, by \(2.1\), if \(g = \sup_{|n| \leq N} |g_n|\) then

\[
\left\| \sup_{|n| \leq N} |(M^G_{2n} - P_{2n})g_n| \right\|_{L^p} \lesssim (B_p(N) + 1) \| g \|_{L^p}.
\]

Hence by the complex interpolation we obtain

\[
A_N(u, 2) \leq A_N(1, 1)^{1/2} A_N(p, \infty)^{1/2} \lesssim (B_p(N) + 1)^{1/2}.
\]
We now take $\rho \in (0, 1]$ satisfying $\frac{1}{p} = \frac{1}{u} + \frac{\rho}{2}$, then $\rho = p - 1$ and $1 - \rho = 2 - p$. Interpolation between (4.14) and (4.17) yields (4.12) for $p \in (1, 2)$ as desired.

4.3. Proof of inequality (4.19). To estimate (4.19) we use (2.3) and (3.2) and obtain

$$
\left\| \left( \sum_{n \in \mathbb{Z}} \sup_{t \in [2^n, 2^{n+1}]} |(M_G^G - M_G^L) f|^2 \right)^{1/2} \right\|_{L^p} \leq \left\| \left( \sum_{n \in \mathbb{Z}} \left( \sum_{l \geq 0} \left( \sum_{l \in \mathbb{Z}} |(M_G^G - M_G^L) s_{j+n} f|^2 \right)^{1/2} \right) \right)^{1/2} \right\|_{L^p}.
$$

Our aim now is to show that for every $q \in (1, 2)$ and $\theta \in [0, 1]$ such that $\frac{1}{p} = \frac{\theta}{2} + \frac{1 - \theta}{q}$ we have, for every $f \in L^p(\mathbb{R}^d)$, the following estimate

$$
\left\| \left( \sum_{n \in \mathbb{Z}} \sum_{m=0}^{2^l-1} |(M_G^G - M_G^L) s_{j+n} f|^2 \right)^{1/2} \right\|_{L^p} \leq 2^{-\theta l/2 + (1-\theta)l} \min \left\{ 1, 2^{l \frac{|j|}{2}} \right\} \|f\|_{L^p},
$$

with the implicit constant independent of the dimension and the underlying body $G \subset \mathbb{R}^d$.

Assume momentarily that (4.19) has been proven. Then we combine (4.18) with (4.19) and obtain estimate (4.9), since the double series

$$
\sum_{l \geq 0} \sum_{j \in \mathbb{Z}} 2^{-\theta l/2 + (1-\theta)l} \min \left\{ 1, 2^{l \frac{|j|}{2}} \right\} \lessapprox 1
$$

is summable, whenever $\theta/2 - (1-\theta) > 0$, which forces $p$ to satisfy $\frac{3}{1+1/q} < p \leq 2$, due to $\theta = \frac{2p-q}{p-2q}$. This completes the proof of (4.14) from Theorem 4.

4.3.1. Proof of inequality (4.19) for $p = 2$. Using inequalities (2.14) and arguing in a similar way as in (4.14) we obtain

$$
\left\| \left( \sum_{n \in \mathbb{Z}} \sum_{m=0}^{2^l-1} |(M_G^2 - M_G^G) s_{j+n} f|^2 \right)^{1/2} \right\|_{L^2} \leq 2^{l/2} 2^{-|j|/2} \|f\|_{L^2}.
$$

Note that inequality (2.15) implies

$$
|m^G((2^n + 2^{n-l}(m+1)) \xi)| - m^G((2^n + 2^{n-l}m) \xi)| \leq \int_{2^n + 2^{n-l}m}^{2^n + 2^{n-l}(m+1)} |(t \xi, \nabla m^G(t \xi))| \frac{dt}{t} \leq 2^{-l}.
$$

Therefore, by Plancherel’s theorem

$$
\left\| \left( \sum_{n \in \mathbb{Z}} \sum_{m=0}^{2^l-1} |(M_G^G - M_G^L) s_{j+n} f|^2 \right)^{1/2} \right\|_{L^2} = \left( \sum_{n \in \mathbb{Z}} \sum_{m=0}^{2^l-1} |(M_G^G - M_G^L) s_{j+n} f|^2 \right)^{1/2} \lessapprox \left\{ \sum_{n \in \mathbb{Z}} \left| s_{j+n} f \right|^2 \right\}^{1/2} \lessapprox 2^{-l/2} \|f\|_{L^2}.
$$

Combining (4.20) and (4.21) we obtain

$$
\left\| \left( \sum_{n \in \mathbb{Z}} \sum_{m=0}^{2^l-1} |(M_G^G - M_G^L) s_{j+n} f|^2 \right)^{1/2} \right\|_{L^2} \leq 2^{-l/2} \min \left\{ 1, 2^{l \frac{|j|}{2}} \right\} \|f\|_{L^2},
$$

as desired.
which proves (4.19) for \( p = 2 \).

4.3.2. Proof of inequality (4.19) for \( p \in (3/2, 2) \). We begin with a general remark, a consequence of (4.15), which states that for every \( q \in (1, \infty) \) there is a constant \( C_q > 0 \) independent of the dimension and the underlying body \( G \subset \mathbb{R}^d \) such that for every sequence \((g_n)_{n \in \mathbb{Z}} \in L^q(\ell^2(\mathbb{R}^d)) \) we have

\[
\left\| \left( \sum_{n \in \mathbb{Z}} |M_{n}^G g_n|^2 \right)^{1/2} \right\|_{L^q} \leq C_q \left\| \left( \sum_{n \in \mathbb{Z}} |g_n|^2 \right)^{1/2} \right\|_{L^q}.
\]

(4.23)

Indeed, let \( A(q, r) \) be the best constant in the following inequality

\[
\left\| \left( \sum_{n \in \mathbb{Z}} |M_{n}^G g_n|^r \right)^{1/r} \right\|_{L^q} \leq A(q, r) \left\| \left( \sum_{n \in \mathbb{Z}} |g_n|^r \right)^{1/r} \right\|_{L^q}.
\]

By the complex interpolation and duality \( (A(q, r) = A(q', r')) \) and inequality (1.5) we obtain

\[
A(q, 2) \leq A(q, 1)^{1/2} A(q, \infty)^{1/2} = A(q', \infty)^{1/2} A(q, \infty)^{1/2} \leq C_q^{1/2} C_q^{1/2},
\]

which implies (4.23). Observe that by (4.23) and (2.4), since \( M_{2^n}^G = M_{2^{(1+t)}}^G \), we obtain

\[
\left\| \left( \sum_{n \in \mathbb{Z}} \left| \left( (M_{2^{(1+t)}(n+1)}^G - M_{2^{n+1}}^G) S_{j+n} f \right)^2 \right| \right)^{1/2} \right\|_{L^q} \leq 2^t \sup_{t \in [0, 1]} \left\| \left( \sum_{n \in \mathbb{Z}} |M_{2^{(1+t)}(n+1)}^G S_{j+n} f|^2 \right)^{1/2} \right\|_{L^q}
\]

\[
\lesssim 2^t \left\| \left( \sum_{n \in \mathbb{Z}} |S_{j+n} f|^2 \right)^{1/2} \right\|_{L^q}
\]

\[
\lesssim 2^t \|f\|_{L^q}.
\]

(4.24)

Interpolating (1.22) with (4.24) we obtain (1.19) as desired.

5. Discrete perspective: proof of Theorem 2

The main objective of this section is to provide dimension-free estimates on \( \ell^p(\mathbb{Z}^d) \), for \( p \in (1, \infty) \), of the norm of the maximal function corresponding to the operators \( \mathcal{M}_{\| \cdot \|}^G \) from (4.10) with large scales \( N \geq Cd \) for some \( C > 0 \), where \( N > 0 \) is a real number. The estimate in (4.10) will be deduced by comparison of \( \sup_{N \geq Cd} \| \mathcal{M}_{\| \cdot \|}^G f \| \) with its continuous analogue, for which we have dimension-free bounds provided by the third author in [33]. Namely, we know that for every \( p \in (1, \infty) \) there is \( C_p > 0 \) independent of the dimension such that for every \( f \in L^p(\mathbb{R}^d) \) we have

\[
\| \mathcal{M}_{\| \cdot \|}^G f \|_{L^p} \leq C_p \| f \|_{L^p}.
\]

(5.1)

Throughout this section, unless otherwise stated, \( N > 0 \) is always a real number and \( Q = [-1/2, 1/2]^d \) denotes the unit cube. A fundamental role, in the proofs of this section, will be played by the fact that the Euclidean norm corresponds to the scalar product \( |x|^2 = \langle x, x \rangle \). We begin with crude estimates for the number of lattice points the Euclidean balls \( B_N^2 \).

**Lemma 5.1.** Let \( N > 0 \) and set \( N_1 = (N^2 + d/4)^{1/2} \). Then

\[
|B_N^2 \cap \mathbb{Z}^d| \leq 2|B_N^2|.
\]

(5.2)

Moreover, if \( N \geq Cd \) for some fixed \( C > 0 \), then we have

\[
|B_N^2 \cap \mathbb{Z}^d| \leq 2e^{1/(8C^2)}|B_N^2|.
\]

(5.3)

**Proof.** For \( x \in B_N^2 \) and \( z \in Q \) we have

\[
|x + z|^2 \leq N^2 + \frac{d}{4} + 2(x, z).
\]

Moreover, for all \( x \in B_N^2 \) we have

\[
|\{ z \in Q : \langle x, z \rangle \leq 0 \}| \geq \frac{1}{2}.
\]
Hence
\[ |B_N^2 \cap \mathbb{Z}^d| = \sum_{x \in B_N^2} 1 \leq 2 \sum_{x \in B_N^2} \int_Q \mathbb{1}_{\{z \in Q : \langle x, z \rangle \leq 0\}}(y)dy \]
\[ \leq 2 \sum_{x \in B_N^2} \int_Q \mathbb{1}_{\{z \in Q : |x+z| \leq N\}}(y)dy \]
\[ \leq 2 \sum_{x \in \mathbb{Z}^d} \int_{x + Q} \mathbb{1}_{B_N^2}(y)dy = 2 |B_N^2|, \]

This proves (5.2). For (5.3) note that for \( N \geq C d \) we get
\[ |B_N^2| = \frac{\pi^{d/2} N^d}{\Gamma(d/2 + 1)} = |B_N^2| \left(1 + \frac{d}{4N^2}\right)^{d/2} \leq |B_N^2| \left(1 + \frac{1}{4CN}\right)^{d/2} \leq e^{1/(8C^2)}|B_N^2|, \]
which proves (5.3).

\[ \square \]

**Lemma 5.2.** Assume that \( N \geq C d \) for some fixed \( C > 0 \) and let \( t > 0 \). Then for every \( x \in \mathbb{R}^d \) such that \( |x| \geq N(1 + t/N)^{1/2} \) we have
\[ |Q \cap (B_N^2 - x)| = |\{y \in Q : x + y \in B_N^2\}| \leq 2e^{-ct^2}, \tag{5.4} \]
where \( c = \frac{7}{32} \frac{c^2}{(C+1)^2} \).

**Proof.** Let \( |x| \geq N(1 + t/N)^{1/2} \). Then for \( y \in Q \) and \( x + y \in B_N^2 \), we have
\[ N^2 + Nt \leq |x|^2 = |x + y|^2 \leq N^2 - 2(x, y) - |y|^2 \leq N^2 + 2|\langle x, y \rangle|. \]

Thus for \( \bar{x} = x/|x| \) one has
\[ |\langle \bar{x}, y \rangle| \geq \frac{1}{2} \frac{Nt}{|y|} \geq \frac{1}{2} \frac{Nt}{|x+y| + |y|} \geq \frac{1}{2} \frac{Nt}{N + d^{1/2}} \geq \frac{1}{2} \frac{Ct}{C+1}, \]
and consequently we get
\[ |\{y \in Q : x + y \in B_N^2\}| \leq |\{y \in Q : |\langle \bar{x}, y \rangle| \geq Ct/(2C + 2)\}|. \tag{5.5} \]

We claim that for every unit vector \( z \in \mathbb{R}^d \) and for every \( s > 0 \) we have
\[ |\{y \in Q : \langle z, y \rangle \geq s\}| \leq e^{-\frac{C}{4} s^2}. \tag{5.6} \]

Taking \( s = Ct/(2C + 2) \) in (5.6) and coming back to (5.5) we complete the proof of (5.4) with \( c = \frac{7}{32} \frac{c^2}{(C+1)^2} \).

In the proof of (5.6) we will appeal to the inequality \( e^x + e^{-x} \leq 2e^{\frac{1}{2}x^2} \), which holds for all \( x \geq 0 \). Indeed, for every \( \alpha > 0 \) we get
\[ e^{\alpha s}|\{y \in Q : \langle z, y \rangle \geq s\}| \leq \int_Q e^{\alpha \sum_{j=1}^d z_j y_j}dy \]
\[ = \prod_{j=1}^d \int_0^{1/2} e^{\alpha z_j y_j} + e^{-\alpha z_j y_j}dy_j \]
\[ \leq \prod_{j=1}^d 2 \int_0^{1/2} e^{\frac{1}{2} \alpha^2 (z_j y_j)^2}dy_j \]
\[ \leq \prod_{j=1}^d e^{\frac{1}{2} \alpha^2 z_j^2} \sum_{j=1}^d z_j^2 \]
\[ = e^{\frac{1}{2} \alpha^2 s^2}. \]

Taking \( \alpha = s \) in the inequality above and dividing by \( e^{s^2} \) we obtain (5.6) and the proof is completed. \( \square \)
Lemma 5.3. There are constants $C_1, C_2 > 0$ such that for every $N \geq C_1 d$ we have
\[ |B_N^2| \leq C_2 |B_N^2 \cap \mathbb{Z}^d|. \] (5.7)
Moreover, combined with (5.3) from Lemma 5.1 yields
\[ C_2^{-1} |B_N^2| \leq |B_N^2 \cap \mathbb{Z}^d| \leq 2e^{1/(8C_2)} |B_N^2| \]
for every $N \geq C_1 d$.

Proof. We show that there is $J \in \mathbb{N}$ such that for every $M \geq d$ we have
\[ |B_M^2| \leq 2 |B_M^2 \cap (1 + J/M)^{1/2} \cap \mathbb{Z}^d|. \] (5.8)
Assume momentarily that (5.8) is proven, then (5.7) follows. Indeed, for every $N \geq C_1 d$, where $C_1 = 2(1 + J)$ we find $M \geq d$ such that $N = M(1 + J/M)^{1/2}$, hence, (5.8) implies
\[ |B_M^2| \leq 2 |B_M^2 \cap \mathbb{Z}^d|. \] (5.9)
On the other hand we have
\[ |B_M^2| \leq |B_M^2| \leq (1 + J/M)^{d/2} |B_M^2| \leq e^J |B_M^2|, \]
since $M \geq d$. This estimate combined with (5.3) gives (5.7) with $C_2 = 2e^J$.

Our aim now is to prove (5.8). For this purpose let $J \in \mathbb{N}$ be a large number such that
\[ \sum_{j \geq J} e^{-j^2/32} e^j \leq \frac{1}{8e}. \]
Define $U_j = \{ x \in \mathbb{R}^d : M \left(1 + \frac{1}{M} \right)^{1/2} < |x| \leq M \left(1 + \frac{(j+1)}{M} \right)^{1/2} \}$ and observe that
\[ |B_M^2| = \sum_{x \in \mathbb{Z}^d} \int_{x+Q} 1_{B_M^2(y)} \, dy = \sum_{x \in \mathbb{Z}^d} \int_Q 1_{B_M^2}(x + y) \, dy \]
\[ \leq \sum_{x \in \mathbb{Z}^d} \int_Q 1_{B_M^2}(x + y) \, dy + \sum_{j \geq J} \sum_{x \in U_j \cap \mathbb{Z}^d} \int_Q 1_{B_M^2}(x + y) \, dy \]
\[ \leq |B_M^2 \cap \mathbb{Z}^d| + \sum_{0 \leq j < J} \sum_{x \in U_j \cap \mathbb{Z}^d} |Q \cap (B_M^2 - x)| + \sum_{j \geq J} \sum_{x \in U_j \cap \mathbb{Z}^d} |Q \cap (B_M^2 - x)| \]
\[ \leq |B_M^2 \cap (1 + J/M)^{1/2} \cap \mathbb{Z}^d| + \sum_{j \geq J} \sum_{x \in U_j \cap \mathbb{Z}^d} |Q \cap (B_M^2 - x)|. \]
By (5.3), since $M \geq d$, we get
\[ |B_M^2 \cap (1 + (j+1)/M)^{1/2} \cap \mathbb{Z}^d| \leq 2e^{1/8} |B_M^2 \cap (1 + (j+1)/M)^{1/2} \cap \mathbb{Z}^d| \]
\[ \leq 2e^{1/8} \left(1 + \frac{j+1}{d} \right)^{d/2} |B_M^2| \]
\[ \leq 2e^{1/8} e^{(j+1)/2} |B_M^2|. \]
Using this estimate, the definition of the sets $U_j$ and Lemma 5.2 we obtain for any $M \geq d$ that
\[ \sum_{j \geq J} \sum_{x \in U_j \cap \mathbb{Z}^d} |Q \cap (B_M^2 - x)| \leq 2 \sum_{j \geq J} e^{-j^2/32} |B_M^2 \cap (1 + (j+1)/M)^{1/2} \cap \mathbb{Z}^d| \]
\[ \leq 4e^{5/8} |B_M^2| \sum_{j \geq J} e^{-j^2/32} e^j \]
\[ \leq \frac{1}{2} |B_M^2|. \]
Combining (5.11) with (5.10) we obtain (5.8) as desired. This completes the proof of Lemma 5.3.

□

We now are ready to prove Theorem 2.
Proof of Theorem 2. Let \( f : \mathbb{Z}^d \to \mathbb{C} \) and define its extension \( F : \mathbb{R}^d \to \mathbb{C} \) on \( \mathbb{R}^d \) by setting

\[
F(x) = \sum_{y \in \mathbb{Z}^d} f(y) \mathbbm{1}_{y+Q}(x).
\]

Then, clearly \( \|F\|_{L^p(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{Z}^d)} \) for every \( p \geq 1 \).

From now on we assume that \( f \geq 0 \). For every \( N \geq C_1d \), with \( C_1 \) as in Lemma 5.3 we define \( N_1 = (N^2 + d/4)^{1/2} \). Observe that for \( z \in Q \) and \( y \in B_N^2 \) we have

\[
|y+z|^2 = |y|^2 + |z|^2 + 2(z,y) \leq N_1^2
\]

on the set \( \{ z \in Q : (z,y) \leq 0 \} \), which has measure 1/2. Then by Lemma 5.3 for all \( x \in \mathbb{Z}^d \) we obtain

\[
\mathcal{M}_N^B f(x) = \frac{1}{|B_N^2 \cap \mathbb{Z}^d|} \sum_{y \in B_N^2 \cap \mathbb{Z}^d} f(x+y) \mathbbm{1}_{B_N^2}(y)
\]

\[
\lesssim \frac{1}{|B_N^2|} \sum_{y \in \mathbb{Z}^d} f(x+y) \int_Q \mathbbm{1}_{B_{N_1}^2}(y+z)dz
\]

\[
= \frac{1}{|B_N^2|} \sum_{y \in \mathbb{Z}^d} f(y) \int_{x+B_{N_1}^2} \mathbbm{1}_{y+Q}(z)dz
\]

\[
= \frac{1}{|B_N^2|} \int_{x+B_{N_1}^2} F(z)dz
\]

\[
= \left( \frac{N_1}{N} \right)^d \frac{1}{|B_N^2|} \int_{B_{N_1}^2} F(x+z)dz
\]

\[
\lesssim \frac{1}{|B_{N_1}^2|} \int_{B_{N_1}^2} F(x+z)dz
\]

\[
= M_{N_1}^B F(x).
\]

Finally, take \( N_2 = (N_1^2 + d/4)^{1/2} \). Similarly as above, for \( y \in Q \) and \( z \in B_{N_1}^2 \) we have

\[
|y+z|^2 \leq |y|^2 + |z|^2 + 2(z,y) \leq N_2^2
\]

on the set \( \{ y \in Q : (z,y) \leq 0 \} \), which has Lebesgue measure 1/2. Therefore, Fubini’s theorem leads to

\[
M_{N_1}^B F(x) = \frac{1}{|B_{N_1}^2|} \int_{B_{N_1}^2} F(x+z)dz
\]

\[
\leq \frac{2}{|B_{N_1}^2|} \int_{\mathbb{R}^d} F(x+z) \mathbbm{1}_{B_{N_1}^2}(z) \int_Q \mathbbm{1}_{B_{N_2}^2}(z+y)dydz
\]

\[
\lesssim \frac{1}{|B_{N_2}^2|} \int_Q \int_{\mathbb{R}^d} F(x+z-y) \mathbbm{1}_{B_{N_2}^2}(z)dzdy
\]

\[
= \int_{x+Q} M_{N_2}^B F(y)dy.
\]

Combining (5.12) with (5.13), applying Hölder’s inequality, and invoking (5.1) we arrive at

\[
\sup_{N \geq C_1d} \left\| \mathcal{M}_N^B f \right\|_{L^p(\mathbb{R}^d)} \lesssim \sum_{x \in \mathbb{Z}^d} \int_{x+Q} \sup_{N \geq C_1d} \left\| M_{N}^B F(y) \right\|^p dy
\]

\[
= \sup_{N \geq C_1d} \left\| M_{N}^B F \right\|_{L^p(\mathbb{R}^d)}^p
\]

\[
\lesssim \|f\|_{L^p(\mathbb{R}^d)}^p \lesssim \|f\|_{L^p(\mathbb{Z}^d)}^p.
\]

This proves Theorem 2 with \( C = C_1 \).
[36] A. Zygmund, *Trigonometric series, Vols. I and II*. Third edition. Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2002.

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