On Nash-solvability of $n$-person graphical games under Markov and a-priori realizations

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Abstract
We consider graphical $n$-person games with perfect information that have no Nash equilibria in pure stationary strategies. Solving these games in stationary mixed strategies, we introduce probability distributions in all non-terminal positions. The corresponding plays can be analyzed under two different basic assumptions: the Markov and a-priori realizations. The former one guarantees existence of a uniform best response for each player in every situation. Nevertheless, Nash equilibrium may fail to exist even in mixed strategies. The classical Nash’s theorem is not applicable, since Markov realizations may result in discontinuous limit distributions and expected payoffs. Although a-priori realizations does not share many nice properties of Markov realizations (for example, the existence of uniform best responses) but in return, Nash’s theorem is applicable. We illustrate both realizations in details by two examples with 2 and 3 players. We also survey some general results related to Nash-solvability, in pure and mixed stationary strategies, of stochastic $n$-person games with perfect information and $n$-person graphical games among them.

Keywords: Graphical Games, Stochastic Games, Nash Equilibrium, Uniform Nash Equilibrium, Markov Process

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1. Introduction

1.1. Graphical n-person games with terminal payoffs

Game structures

Let $G = (V, E)$ be a finite directed graph (digraph) whose vertices $v \in V$ and directed edges (arcs) $e \in E$ are interpreted as positions and moves of a game of $n$ players $I = \{1, \ldots, n\}$. Furthermore, let $D$ be a partition of $V$ into $n + 2$ subsets: $D : V = V_I \cup \ldots \cup V_n \cup V_R \cup V_T$, interpreted as follows:

- $V_T$ are terminal positions of $G$, from which there are no moves,
- $V_R$ are positions of chance,
- $V_i$ are positions controlled by the player $i \in I$.

For each position $v \in V_i$ player $i$ chooses a move from position $v$, that is, an arc $(v, v') \in E$.

For each $v \in V_R$ we fix a probability distribution $q(v)$ over the set of all moves $(v, v')$ from $v$. In other words, we define $q = \{q(v) \mid v \in V_R\}$ by assigning real numbers $q(v, v') \geq 0$ to each arc $(v, v') \in E$ such that $\sum_{(v', v') \in E} q(v, v') = 1$ for each $v \in V_R$. Probabilities $q(v, v') = 0$ are allowed. In this case arc $(v, v')$ can be deleted from $E$ while vertices $v$ and $v'$ remain in $V$.

Without loss of generality (WLOG) we assume that for each position $v \in V \setminus V_T$ the number of moves from $v$ is at least 2. Indeed, if $(v, v')$ is a forced move in $v$ then we can contract this edge, that is, we delete it and merge $v$ and $v'$.

The initial position $v_0 \in V$ may be fixed or not. A quadruple $(G, D, q, v_0)$ and triplet $(G, D, q)$ will be referred to as the graphical game structure: initialized and not initialized, respectively. To simplify our notation, we replace $(G, D, q)$ by $G$.

A game structure is called

- deterministic if $V_R = \emptyset$;
- almost deterministic if it is initialized, $V_R = \{v_0\}$, and there is a move $(v_0, v) \in E$ to each $v \in V \setminus \{V_T \cup \{v_0\}\}$;
- play-once if $|V_i| = 1$ for all $i \in I$, that is, if each of $n$ players controls a unique position.
Initializing extensions

Given a non-initialized game structure $G$, let us add to its digraph $G$ a new position of chance $v_0$ and a move $(v_0, v)$ to every $v \in V \setminus V_T$. Then, let us fix an arbitrary probability distribution $q(v_0)$ on these new edges. We denote the obtained initialized game structure by $G' = (G', D', q', v_0)$ and call $G'$ the initializing extension of $G$.

By definition, $G'$ is almost deterministic if and only if $G$ is deterministic.

**Remark 1.** Let us note that initializing game structure $G$ is obviously equivalent with introducing an initial probability distribution in it, instead of fixing an initial position. However, these two approaches differ in the general framework of Nash-solvability. To construct a deterministic Nash equilibrium free graphical game is more difficult than an almost deterministic one; see Subsection 1.3 for the definitions and Section 4 for more details.

Two examples

2- and 3-person non-initialized deterministic play-once game structures $G_2$ and $G_3$ are shown in Figures 1 and 2. Each player $i \in I$ controls a unique position $v_i$ in which (s)he has two possible moves: $(f)$ to follow the cycle and $(t)$ to terminate in $a_i$; for $G_2$ and $G_3$, we have $n = 2$ and $n = 3$, respectively. The initializing extensions $G'_2$ and $G'_3$ are given in the same two figures.

![Figure 1: Game structures $G_2$ and $G'_2$.](image)
Plays, outcomes, payoffs, and games

A play is as a directed walk in $G$ that begins in some position $v \in V$. In the initialized case we assume that $v = v_0$, while in the non-initialized case, $v$ can be any position in $V \setminus V_T$. A play is finite if and only if it ends in $V_T$. In this case it is called terminal, otherwise, it is called infinite.

Every terminal $v \in V_T$ is an outcome; all terminal plays ending in $v$ are treated as equivalent; they form a single outcome $v$; also all infinite plays are treated as equivalent; they form one extra outcome $c$. The set of outcomes will be denoted by $A = \{V_T \cup \{c\}\} = \{a_1, \ldots, a_m, c\}$.

A payoff function is defined as a mapping $u : I \times A \to \mathbb{R}$; the real number $u_i(a)$ is interpreted as the profit of the player $i \in I$ in case the outcome $a \in A$ is realized. A triplet $(G, u, v_0)$ and a pair $(G, u)$ will be called initialized and non-initialized graphical games, respectively. Deterministic graphical games were introduced in [51] for the 2-person zero-sum case. We generalize this model allowing $n$ players and positions of chance.

1.2. Pure stationary strategies and normal forms of deterministic game structures

A pure stationary strategy (or simply a strategy, for short) of a player $i \in I$ is a mapping $s^i : V_i \to E$ that assigns to each position $v \in V_i$ a move $(v, v') \in E$ from $v$. In other words, player $i$ in advance makes a decision, how (s)he will play in each position.

An $n$-tuple $s = \{s^i \mid i \in I\}$ of strategies of all $n$ players is called a situation.

If game structure $(G, D, v_0)$ is deterministic and initialized then each situation $s$ uniquely defines a walk $W = W(s)$ called a play. It begins in the initial position $v_0$. Assume that $v_0 \in V_i$. Then $W$ proceeds from $v_0$ with the move.
\((v_0, v') \in E\) chosen in \(v_0\) by strategy \(s^i \in s\). Assume that \(v' \in V_j\). (Equality \(i = j\) is allowed.) Then \(W\) proceeds from \(v'\) with the move \((v', v'') \in E\) chosen in \(v'\) by strategy \(s^j \in s\), etc. Play \(W\) either ends in a terminal \(a \in V_T\) (in which case each player \(i \in I\) gets a profit \(u_i(a)\)) or \(W\) lasts infinitely. Since digraph \(G\) is finite, in the latter case walk \(W\), sooner or later, will revisit a position, thus making a directed cycle. Let us consider the first such revisiting and the corresponding directed cycle. This cycle is simple (that is, it has no self-intersections) and play \(W(s)\) will repeat this cycle infinitely, because the players are restricted to their pure stationary strategies. Such an infinite play will be called a lasso. It consists of the initial part (before the first revisiting, which is empty if the play returns to \(v_0\)). If \(W\) is a lasso then each player \(i \in I\) gets a profit \(u_i(c)\), because we assume that all infinite plays are equivalent and form a single outcome.

If player \(i\) controls \(t_i\) positions with \(\ell^i_1, \ldots, \ell^i_{t_i}\) outgoing arcs, then \(i\) has \(k_i = \prod_{j=1}^{t_i} \ell^i_j\) pure stationary strategies. In our examples \(G_2\) and \(G_3\) each player has only two such strategies.

Given an initialized deterministic game structure \((G, D, v_0)\), let \(S^i\) denote the set of pure stationary strategies of player \(i \in I\) and let \(S = S^1 \times \ldots \times S^n\) be the direct product of \(n\) these sets. Mapping \(g^v : S \rightarrow A\) that assigns to each situation \(s \in S\) either a terminal outcome \(v \in V_T\), if play \(W(s)\) ends in \(v\), or the special outcome \(c\), if \(W(s)\) is a lasso, is called the normal form of the (deterministic initialized) game structure \(G\).

Given a non-initialized deterministic game structure \((G, D)\), we define its normal form as the mapping \(g : S \rightarrow 2^A\), where \(g(s) = \{g^v(s) \mid v \in V \setminus V_T\}\). Two examples are given in Figure 3.

1.3. Mixed and stationary mixed strategies

A mixed strategy \(x^i\) of a player \(i \in I\) is defined as a probability distribution over the set of his pure strategies. Thus, the dimension of this set is \(k_i - 1\).

A stationary mixed strategy \(y^i\) of a player \(i \in I\) is defined as a set of probability distributions for all \(v \in V_i\): each one over all moves \((v, v') \in E\) from \(v\). The moves are chosen randomly, in accordance with these probability distributions, and independently for all \(v \in V_i\).

The dimension of the set of stationary mixed strategies of a player \(i\) is equal to

\[
k'_i = \sum_{j=1}^{t_i} (\ell^i_j - 1) = \sum_{j=1}^{t_i} \ell^i_j - t_i
\]
Figure 3: Normal forms $g_2$ and $g_3$ of the non-initialized game structures $G_2$ and $G_3$ from Figures 1 and 2, respectively. Also normal forms $g_2^{v_1}$ and $g_2^{v_2}$ of $G_2$ initialized in $v_1$ and $v_2$, and normal forms $g_3^{v_1}$, $g_3^{v_2}$, $g_3^{v_3}$ of $G_3$ initialized in $v_1$, $v_2$, $v_3$, respectively.

$U_2$: $u_1(c) > u_1(a_1) > u_1(a_2); u_2(a_1) > u_2(a_2) > u_2(c)$

$U_3$: $u_1(a_2) > u_1(a_1) > u_1(a_3) > u_1(c)$
$u_2(a_3) > u_2(a_2) > u_2(a_1) > u_2(c)$
$u_3(a_1) > u_3(a_3) > u_3(a_2) > u_3(c)$
Obviously, \( k'_i \leq k_i \) and the equality holds if and only if \( |V_i| = 1 \). Thus, by definition, the set of stationary mixed strategies is a subset of the set of mixed strategies. For the play-once games, and only in this case, the above two sets coincide.

WLOG we assume that there are no forced positions, that is, \( \ell^i_j > 1 \) for all \( i \in I \) and \( j = 1, \ldots, t_i \).

**Remark 2.** Stationary mixed strategies are closely related to the so-called behaviour strategies introduced in [38] for games with imperfect information; see also [39, 2].

### 1.4. Markov and a-priori realizations; expected payoffs

A non-initialized game structure \( G = (G, D, q) \) defines a probability distribution \( q(v) \) for each \( v \in V_R \). Furthermore, each stationary mixed strategy \( y^i \) of a player \( i \in I \) is a set of probability distributions \( p(v) \) for each \( v \in V_i \). Thus, given a situation \( y = (y^i \mid i \in I) \) in stationary mixed strategies, one obtains a probability distribution over the set of moves \( (v, v') \) for all positions \( v \in V_1 \cup \ldots V_n \cup V_R = V \setminus V_T \).

These distributions naturally define a Markov chain on \( G \). For any initial position \( v_0 \in V \setminus V_T \) we can efficiently compute the unique limiting distribution \( P_M = \{ P_M(c, v_0), P_M(a, v_0) \mid a \in V_T \} \) over the set of outcomes \( A = V_T \cup \{c\} = \{a_1, \ldots, a_m, c\} \); see, for example, [37].

The limiting distribution \( P_M \) is defined as a function of probabilities from the distributions \( \{q(v) \mid v \in V_R\} \) and \( \{p(v) \mid v \in V_1 \cup \ldots \cup V_n\} \). It is important to note that this function may be discontinuous already in the deterministic play-once case, \( V_R = \emptyset \) and \( |V_i| = 1 \) for all \( i \in I \).

Consider, for example, game structures \( \mathcal{G}_2 \) and \( \mathcal{G}_3 \). For a positions \( v_i \), denote by \( p_i \) the probability to stay on the cycle; then \( 1-p_i \) is the probability to terminate in \( a_i \); here \( i = 1, 2 \) for \( \mathcal{G}_2 \) and \( i = 1, 2, 3 \) for \( \mathcal{G}_3 \).

For \( \mathcal{G}_2 \), if \( p_1 = p_2 = 1 \), the play will cycle with probability 1 resulting in the limiting distribution \((0, 0, 1)\) on \((a_1, a_2, c)\). Otherwise, if \( p_1 < 1 \) or \( p_2 < 1 \), for the initial positions \( v_1 \) and \( v_2 \), we obtain the following limiting distributions, respectively:

\[
\begin{pmatrix}
\frac{1-p_1}{1-p_1 p_2}, & \frac{p_1(1-p_2)}{1-p_1 p_2}, & 0
\end{pmatrix},
\begin{pmatrix}
\frac{p_2(1-p_1)}{1-p_1 p_2}, & \frac{1-p_2}{1-p_1 p_2}, & 0
\end{pmatrix}
\]

(1)

For \( \mathcal{G}_3 \), if \( p_1 = p_2 = p_3 = 1 \), the play will cycle with probability 1 resulting in the limiting distribution \((0, 0, 0, 1)\) on \((a_1, a_2, a_3, c)\). Otherwise, if \( p_i < 1 \)
for some \( i \in \{1, 2, 3\} \), for the initial positions \( v_1, v_2, \) and \( v_3 \), we obtain the following limiting distributions, respectively:

\[
\begin{pmatrix}
(1 - p_1, p_1(1 - p_2), p_1p_2(1 - p_3), 0), \\
(1 - p_1p_2p_3, 1 - p_1p_2p_3, 1 - p_1p_2p_3, 0), \\
(p_2p_3(1 - p_1), 1 - p_2, p_2(1 - p_3), 0), \\
(p_3(1 - p_1), p_1p_3(1 - p_2), 1 - p_3, 0), \\
(1 - p_1p_2p_3, 1 - p_1p_2p_3, 1 - p_1p_2p_3, 0)
\end{pmatrix},
\]

(2)

It is important to note that for \( \mathcal{G}_n \), the limiting probability \( P_M(c, v_0) \) of the cycle, as a function of \((p_1, \ldots, p_n)\), has a discontinuity at point \((1, \ldots, 1)\); it is 1 when \( p_1 = \ldots = p_n = 1 \) and 0 otherwise, for any \( n \geq 2 \). Cases \( n = 2 \) and \( n = 3 \) were considered above.

Typically, for solving graphical games in stationary mixed strategies the Markov realization is applied; see for example, [40, 42]. In [11, 10] the following alternative approach was suggested. Suppose a play revisits a position \( v \in V \setminus V_T \). Then, the new move in \( v \) must coincide with the previously chosen one.

In other words, before the play begins, in each position \( v \in V \) a move \((v, v') \in E\) is chosen according to \( q(v) \) for \( v \in V_R \) and to \( p(v) \) for \( v \in V_1 \cup \ldots \cup V_n \). After this, the play begins in an initial position and follows these chosen moves until it terminates or cycles. This rule defines the \textit{a-priori realization}, which differs essentially from the Markov one. Under the latter, a move \((v, v') \in E\) from a position \( v \in V \setminus V_T \) is also chosen in accordance with a distribution \( p(v) \) or \( q(v) \), and such random choice with the same distribution is repeated whenever the play returns to \( v \), but the resulting move itself is not necessarily repeated. In contrast for the a-priori realization, the limiting distribution \( P_{apr} = \{P_{apr}(c, v_0), P_{apr}(a, v_0) \mid a \in V_T\} \) over the set of outcomes \( A = V_T \cup \{c\} = \{a_1, \ldots, a_m, c\} \) is well-defined (unique), whenever an initial position \( v_0 \in V \setminus V_T \) is fixed. Furthermore, \( P_{apr} \) is a continuous function of probabilities from the distributions \( q(v), v \in V_R \), and \( p(v), v \in V_1 \cup \ldots \cup V_n \). Indeed, for any play (terminal one or a lasso) beginning in \( v_0 \), its probability equals the product of probabilities of all moves involved in this play. Then, to compute \( P_{apr}(c, v_0) \) and \( P_{apr}(v, v_0) \) for all \( v \in V_T \), we “simply” sum up the probabilities of the corresponding plays: all lassos in the former case and all plays terminating in \( v \) in the latter case.

Let us note, however, that the number of plays may be exponential in the size of digraph \( G \). So, unlike the Markov case, the above algorithm computing
the limiting distribution is not efficient. Whether a polynomial one exists is an open problem. We conjecture that it does not.

**Remark 3.** We acknowledge that the Markov realization has many advantages with respect to (WRT) the a-priori one: the Markov one has many practical applications, the limiting distribution can be efficiently computed, etc. Yet, we will show that Nash-solvability in mixed strategies of an initialized play-once game holds under the a-priori realization. In contrast, under the Markov one, Nash-solvability may fail (already in the play-once case) because of the discontinuity of the Markov expected payoff; see Section 2.1 and also Appendix B for more details.

After all, when a play revisits a position, why should we roll the dice again? We already did it and can reuse the result.

Let us consider $G_2$. Assuming that the initial positions are $v_1$ and $v_2$, we obtain the following limiting a-priori distributions for the outcomes $(a_1, a_2, c)$, respectively:

$$
(1 - p_1, p_1(1 - p_2), p_1 p_2),
(p_2(1 - p_1), 1 - p_2, p_2 p_1).
$$

(3)

For $G_3$, assuming that the initial positions are $v_1, v_2, v_3$, we obtain the following limiting a-priori distributions, for the outcomes $(a_1, a_2, a_3, c)$, respectively:

$$
(1 - p_1, p_1(1 - p_2), p_1 p_2 (1 - p_3), p_1 p_2 p_3),
(p_2 p_3 (1 - p_1), 1 - p_2, p_2 (1 - p_3), p_2 p_3 p_1),
(p_3(1 - p_1), p_3 p_1 (1 - p_2), 1 - p_3, p_3 p_1 p_2).
$$

(4)

The probability of outcome $c$ is $p_1 p_2$ for $G_2$ and $p_1 p_2 p_3$ and $G_3$, and it is strictly positive whenever $p_i > 0$, for all $i \in I$. Indeed, in contrast to the Markov realization, under the a-priori one, the cycle will be repeated infinitely whenever it appears once.

Given a limiting distribution $P = \{P(c, v_0), P(a, v_0) \mid a \in V_T\}$ over the set of outcomes $A$, which can be the Markov or the a-priori one, and a payoff function $u : I \times A \to \mathbb{R}$, the expected payoff is defined as the linear combination:

$$
u_i(P) = P(a_1, v_0)u_i(a_1) + \ldots + P(a_m, v_0)u_i(a_m) + P(c, v_0)u_i(c),
$$

$i \in I, v_0 \in V \setminus V_T$.

When $P$ is defined by a situation $y = (y^i \mid i \in I)$ in mixed stationary strategies, we will use the notation $u = (u_i(y) \mid i \in I)$.
1.5. Nash equilibria and Uniform Nash equilibria in pure, mixed, and stationary mixed strategies

Normal form games

Given a set of players $I = \{1, \ldots, n\}$, a finite set $S^i$ of pure strategies of each player $i \in I$, and a set of outcomes $A$, a game form is defined as a mapping $g : S \to A$ that assigns an outcome $a \in A$ to each situation $s = (s^1, \ldots, s^n) \in S = S^1 \times \ldots \times S^n$.

Given also a payoff function $u : I \times A \to \mathbb{R}$, where $u(i, a) = u_i(a)$ is interpreted as a profit of player $i \in I$ in case of outcome $a \in A$, the pair $(g, u)$ defines a game in normal form.

A situation $s \in S$ is called a Nash equilibrium (NE) in pure strategies in game $(g, u)$ if $u_i(g(s)) \geq u_i(g(s'))$ for any $i \in I$ and for any situation $s' \in S$ that may differ from $s$ only in the components $i$. In other words, $s \in S$ is a NE if no player $i \in I$ can improve $s$ for himself by choosing some strategy, $s^i$ instead of $s^i$, provided other players $(j \in I \setminus \{i\})$ apply their old strategies $s^j$.

In this case, $s^i$ is called a best response to the strategies $(s^j | j \in I \setminus \{i\})$. Thus, situation $s = (s^1, \ldots, s^n)$ is an NE if and only if the strategy of each player is a best response to the strategies of the remaining $n - 1$ players.

Remark 4. Frequently, game $(g, u)$ has no NE in pure strategies. However, some conditions on game form $g$ may guarantee Nash-solvability. For $n = 2$ such necessary and sufficient conditions were obtained in [22, 23]; see also [30]. However, these conditions do not work for $n > 2$ [22]; see also [2, 17].

A mixed strategy $x^i \in X^i$ of a player $i \in I$ is defined as a probability distribution over $S^i$ determined by probabilities $p(s^i | x^i)$ for all $s^i \in S^i$. Then, each situation $x = (x^1, \ldots, x^n) \in X^1 \times \ldots \times X^n = X$ in mixed strategies uniquely determines a probability distribution over $S$ given by probabilities $p(s | x) = \prod_{i \in I} p(s^i | x^i)$ for all $s = (s^1, \ldots, s^n) \in S$, and also expected payoff $u : I \times X \to \mathbb{R}$, where $u(i, x) = u_i(x) = \sum_{s \in S} p(s | x)u_i(g(s))$ is the expected payoff of player $i$ in situation $x$.

A situation $x \in X$ is called a NE in mixed strategies in the normal form game $(g, u)$ if $u_i(x) \geq u_i(x')$ for every $i \in I$ and each situation $x' \in X$ that may differ from $x$ only in the component $i$.

In other words, situation $x = (x^i \mid i \in I = \{1, \ldots, n\})$ is an NE if and only if $x^i$ is a best response of player $i$ to the strategies $(x^j \mid j \in I \setminus \{i\})$ of the remaining $n - 1$ players.

Nash [44, 45] proved that every normal form game $(g, u)$ has an NE in mixed strategies.
Graphical games

We apply the above definitions to the normal form of an initialized graphical game \((\mathcal{G}, v_0, u)\) to obtain the following concepts:

(i) NE in pure strategies;
(ii) NE in stationary mixed strategies under the Markov realization;
(iii) NE in stationary mixed strategies under the a-priori realization.

Note that the concept of an NE in mixed strategies can also be defined by the mixed extension of the normal form, yet, it is not realized by a position-wise independent randomization. However, mixed and stationary mixed strategies coincide for the play-once games.

Uniform Nash equilibria

Given a non-initialized graphical game \((\mathcal{G}, u)\), we define a uniform NE (UNE) as a situation which is an NE in \((\mathcal{G}, v_0, u)\) for any \(v_0 \in V \setminus V_T\). This modification is applicable in all three above cases: (i), (ii), and (iii). Two examples of the UNE-free games are given in Figures 1 and 2.

Remark 5. The name of subgame perfect NE is common in the literature, but we prefer to call such NE uniform, because the concept of a subgame itself loses its meaning in presence of cycles.

2. Main results

2.1. Markov realization

Uniform best responses

Note that a Markov decision process can be viewed as a graphical one-person game under the Markov realization. The main result in this area states that there exists a uniform best pure strategy, which can be found as a solution of a linear program \([35, 42]\). (As usual, “uniform best” means “best WRT any initial position \(v \in V \setminus V_T\”).) For the \(n\)-person case this result can be reformulated as follows.

Proposition 1. Given an \(n\)-person graphical game under the Markov realization, for any set of mixed stationary strategies \((y_j^i \mid j \in I \setminus \{i\})\) of \(n-1\) players there exists a uniform best response of player \(i\) in pure strategies. \(\square\)

Remark 6. This result in a more general setting, namely, for stochastic games with countable state and action spaces and semi-Markov strategies was obtained in \([36]\). The proof is based on the results of \([3, 47]\).
On Nash equilibria in initializing extensions of graphical games

In its turn, the last statement implies the following relation between UNE in stationary mixed strategies in a non-initialized game structure \((G, u)\) and NE in its initializing extension \((G', u)\); see Section 1.1 for definitions and Figures 1 and 2 for examples.

**Proposition 2.** Given a situation \(y\) in stationary mixed strategies in a non-initialized game \((G, u)\),

(i) if \(y\) is an UNE in \((G, u)\) then \(y\) is an NE in \((G', u)\) WRT every distribution \(q(v_0)\).

(ii) if \(y\) is an NE in \((G', u)\) for some strictly positive \(q(v_0)\) then \(y\) is an UNE in \((G, u)\).

This statement appeared in [4] for the case of pure stationary strategies. Here we extend it to the case of stationary mixed strategies.

**Proof.** Implication (i) \(\Rightarrow\) (ii) is obvious. If \(x\) is an NE \((G, u)\) WRT every initial position \(v \in V \setminus V_T\) then \(x\) is an NE in \((G', u)\) WRT \(v_0\). Indeed, \(v\) will follow \(v_0\) with probability \(q(v_0, v)\) and after this play never returns to \(v_0\). Hence, all expected payoffs in \((G', u)\) initialized in \(v_0\) equal linear combinations of the corresponding payoffs in \((G, u)\) initialized in \(v\) with non-negative coefficients \(q(v_0, v)\). This operation respects inequalities.

Implication (ii) \(\Rightarrow\) (i) follows from Proposition 1. Suppose that \(x = (x^i | i \in I)\) is not a UNE in \((G, u)\). Then, there is a position \(v^* \in V \setminus V_T\) and a player \(i \in I\) who can strictly improve situation \(x\) for himself replacing \(x^i\) by \(\tilde{x}^i\), provided the game begins in \(v^*\). But \(i\) has a uniform best response in situation \(x\). WLOG we can assume that it is \(\tilde{x}^i\). Hence, \(\tilde{x}^i\) strictly improves \(x\) for \(i\) when the game begins in \(v^*\) and it gets at least as good result as in \(x\) when the game begins in any position \(v \in V_T\), just because \(x^i\) is also a response in \(x\). Thus, \(x\) is not an NE in \((G', u)\) provided \(q(v_0, v) > 0\). \(\square\)

We will see that the last condition is essential.

On Nash equilibria in games \((G_2, u)\) and \((G_3, u)\)

It was shown in [1] that game \((G_2, u)\) has no NE in pure strategies if and only if \(u \in U_2\), where \(U_2\) is defined by the system of inequalities:

\[
 u_1(c) > u_1(a_1) > u_1(a_2); \ u_2(a_1) > u_2(a_2) > u_2(c). \tag{5}
\]

In Appendix B.2 we will extend this result to the case of mixed strategies as follows.
Proposition 3. Game \((G_2, u)\) has no UNE in mixed strategies when \(u \in U_2\).

Then, by Proposition \[2\] the following statement holds.

Proposition 4. Initializing extension \((G_2', u)\) has no NE in mixed strategies when \(u \in U_2\).

It was shown in \[7\] that game \((G_3, u)\) has no UNE in pure strategies if \(u \in U_3\), where \(U_3\) is defined by the following system of inequalities:

\[
\begin{align*}
    u_1(a_2) &> u_1(a_1) > u_1(a_3) > u_1(c), \\
    u_2(a_3) &> u_2(a_2) > u_2(a_1) > u_2(c), \\
    u_3(a_1) &> u_3(a_3) > u_3(a_2) > u_3(c).
\end{align*}
\]

(6)

In [Appendix B.2] we strengthen this result as follows. Let us set

\[
\mu_1 = \frac{u_1(a_2) - u_1(a_1)}{u_1(a_1) - u_1(a_3)}, \quad \mu_2 = \frac{u_2(a_3) - u_2(a_2)}{u_2(a_2) - u_2(a_1)}, \quad \mu_3 = \frac{u_3(a_1) - u_3(a_3)}{u_3(a_3) - u_3(a_2)}.
\]

(7)

Obviously, \(\mu_i > 0\) for \(i = 1, 2, 3\) when \(u \in U_3\).

Proposition 5. For \(u \in U_3\), game \((G_3, u)\) has no UNE in mixed strategies when \(\mu_1 \mu_2 \mu_3 \geq 1\). Otherwise, if \(\mu_1 \mu_2 \mu_3 < 1\), game \((G_3, u)\) has a unique UNE in mixed strategies determined by probabilities

\[
\begin{align*}
    p_1 &= \frac{\mu_3(1 + \mu_1 + \mu_1 \mu_2)}{1 + \mu_3 + \mu_3 \mu_1}, \\
    p_2 &= \frac{\mu_1(1 + \mu_2 + \mu_2 \mu_3)}{1 + \mu_1 + \mu_1 \mu_2}, \\
    p_3 &= \frac{\mu_2(1 + \mu_3 + \mu_3 \mu_1)}{1 + \mu_2 + \mu_2 \mu_3}.
\end{align*}
\]

(8)

This statement appears in \[11\], yet, no complete proof was given; we will give it in [Appendix B]. Right now let us only note that each of the three equations of \[8\] implies that \(\mu_1 \mu_2 \mu_3 \leq 1\) and this inequality is strict whenever \(p_i < 1, i = 1, 2, 3\).

By Proposition\[2\] we conclude that the initializing extension \((G_3', u)\) of this game has the same NE in mixed strategies, provided \(u \in U_3\) and all probabilities \(q(v_0, v)\) are strictly positive. The last condition is essential. It is not difficult to verify that if \(u \in U_3\) and \(q(v_0) = (q(v_0, v_1), (q(v_0, v_2), q(v_0, v_3)) = (1, 0, 0)\) or \((1/2, 0, 1/2)\) then \(p = (p_1, p_2, p_3) = (0, 1, 1)\) is a pure strategy NE in game \((G_3', u)\), while \((G_3, u)\) has no UNE when \(u \in U_3\).
However, if we restrict the players to their strictly mixed strategies \( p_i > 0 \) for \( i = 1, 2, 3 \), then games \((G_3', u)\) for all \( q(v_0) \) become equivalent to \((G_3, u)\), that is, all these games have the same NE.

**Proposition 6.** For \( u \in U_3 \), game \((G_3', u)\) has a unique mixed strategy NE given by (8) when \( \mu_1\mu_2\mu_3 < 1 \); otherwise, if \( \mu_1\mu_2\mu_3 \geq 1 \), then game \((G_3', u)\) has no NE in strictly mixed strategies.

Propositions 3, 5, and 6 will be proven in Appendix B.3.

Finally, let us recall that mixed and stationary mixed strategies coincide for the play-once game structures \( G_2 \), \( G_3 \) and \( G_2', G_3' \).

**Why does Nash’s theorem fail in case of a Markov realization?**

Indeed, at the first glance, one may decide that an NE in mixed strategies must exist in games \((G_2', u)\) and \((G_3', u)\) (and more generally, in the initializing extension of every play-once game) due to the classical Nash theorem [44, 45]. Yet, it works only in case of the a-priori realization, but not for the Markov one. As we have already mentioned, in the latter case the limiting distribution (and hence the expected payoff as well) may be a discontinuous function of probabilities \( p(v) \) and \( q(v) \).

**Remark 7.** Although there are results that extend Nash-solvability to some discontinuous payoff functions (see, for example, [13, 46]) yet, these results do not cover the Markov realization.

### 2.2. A-priori realization

Nash’s theorem implies existence of an NE in every initialized play-once graphical game under the a-priori realization. One obtains such an NE just solving in mixed strategies the normal form of this game.

Naturally, Nash-solvability in mixed stationary strategies may fail if the game is not play-once; see, e.g., the main example in [10]. This is not a surprise, since the mixed and stationary mixed strategies coincide only for the play-once games, otherwise the former set is a proper subset of the latter.

The uniform Nash-solvability may fail even in the play-once case. In Appendix B.3 we show it for game structures \( G_2 \) and \( G_3 \).

**Proposition 7.** Under the a-priori realization, games \((G_2, u)\) and \((G_3, u)\) have no UNE in mixed strategies whenever \( u \in U_2 \) and \( u \in U_3 \), respectively.
Moreover, a uniform best strategy may fail to exist already for one player, that is, for a Markov decision process. Consider, for example, game structure $G_1$ in Figure 4.

The player controls position $v_1$, while $v_0$ is a position of chance with two equal probabilities: $1/2$ and $1/2$. Consider any payoff $u$ satisfying inequalities

$$\frac{1}{2} (u(a_2) + u(c)) > u(a_1) > u(c).$$

Then, if game begins in $v_1$ the optimal player’s move is to $v_0$, while if the initial position is $v_0$, it is better to terminate in $a_1$, avoiding $c$. Note that this happens only under the a-priori realization, while under the Markov realization, move $(v_1, v_0)$ will be the best for both initial position: $v_0$ or $v_1$.

This argument shows why Proposition 2 cannot be extended to the a-priori realization: compare Proposition 5 and the opening claim of this subsection.

3. Two main examples are UNE-free

Our two main examples are given by two play-once non-initialized 2- and 3-person game structures $G_2$ and $G_3$ given on Figures 1 and 2, respectively. In normal form both are represented in Figure 3. The corresponding games $(G_2, u)$ and $(G_3, u)$ have no UNE in pure strategies whenever payoffs are ordered in accordance with systems of strict inequalities $U_2$ and $U_3$ defined by (5) and (6), respectively. Interpretation of both games are given in Appendix C.

3.1. Game $G_2(u)$ for $u \in U_2$

Game $(G_2, u)$ has no UNE when $u \in U_2$. 

![Diagram of $G_1$](image-url)
The game is play-once. Each player $i \in I = \{1, 2\}$ controls a unique position $v_i$ and has two strategies: either to terminate in $a_i$ or to follow the cycle: $s^i \in \{t, f\}$.

We have to show that none of the four situations $s = (s^1, s^2)$ is a UNE, that is, at least one player can improve it WRT at least one initial position $v_0 = v_i$, $i \in \{1, 2\}$.

Consider $s = (f, f)$: all players follow the cycle. The play results in $c$ for any initial position. Player 2 can improve his result choosing $t$ rather than $f$ and getting $a_2$, WRT any initial position.

Consider $s = (f, t)$. The play results in $a_2$ WRT any initial position. Player 1 can improve her result WRT $v_1$ choosing $t$ rather than $f$ and terminating in $a_1$ instead of $a_2$. Yet, WRT $v_2$ there is no improvement.

Consider $s = (t, t)$. The play results in $a_i$ WRT initial position $v_i$, for $i = 1, 2$. Player 2 can improve his result WRT initial position $v_2$ choosing $f$ rather than $t$ and terminating in $a_1$ instead of $a_2$. Yet, WRT $v_1$ the outcome is $a_1$ for both his strategies.

Consider $s = (t, f)$. The play results in $a_1$ for any initial position. Player 1 can improve her result WRT any initial position choosing $f$ rather than $t$ and getting $c$ instead of $a_1$.

Thus, we obtain $s = (f, f)$ again. All four situations belong to an improvement cycle of length 4. It is shown in Figure 3. Hence, none of them is a UNE in game $(G_2, u)$ when $u \in U_2$.

In contrast, both initialized game structures $G_3^{v_1}$ and $G_3^{v_2}$, with initial positions $v_1$ and $v_2$, as well as the corresponding game forms $g_2^{v_1}$ and $g_2^{v_2}$, are Nash-solvable; see Figures 1 and 3.

In fact, Nash-solvability holds for any deterministic initialized game structure of two players. This result was derived in [7] from a general criterion of Nash-solvability [20, 22]; see also [28, 30]. However, this criterion holds only for $n = |I| = 2$; see Section 4 for more details.

3.2. Game $(G_3, u)$ for $u \in U_3$

Game $(G_3, u)$ has no UNE when $u \in U_3$ [1]. For completeness, we provide here a simplified proof.

The game is play-once. Each player $i \in I = \{1, 2, 3\}$ controls a unique position $v_i$ and has two strategies: either to terminate in $a_i$ or to follow the cycle: $s^i \in \{t, f\}$. We have to show that none of the eight situations $s = (s^1, s^2, s^3)$ is a UNE when $u \in U_3$, that is, at least one player can improve $s$ WRT at least one initial position $v_0 = v_i$, $i \in \{1, 2, 3\}$.
Consider \( s = (f, f, f) \): all 3 players follow the cycle. For any initial position the play results in \( c \) and each player can improve his result choosing \( t \) rather than \( f \), WRT any initial position.

Consider \( s = (t, t, t) \): all 3 players terminate. Each one can improve the situation choosing \( f \) rather than \( t \). Then, the next player will terminate, which is better, according to \( U_3 \). Note, yet, that improvement for player \( i \) is strict only when \( v_0 = v_i \). Otherwise, the outcome will not change.

The remaining six situations form an improvement cycle.

Indeed, in situation \( (f, f, t) \) player 1 is unhappy and will switch from \( f \) to \( t \). Doing so (s)he improves the situation, at least when \( v_0 = v_1 \). In this case \( a_3 \) is replaced by \( a_1 \), which is better to player 1, according to \( U_3 \). Note that the outcome will remain unchanged when \( v_0 = v_2 \) or \( v_0 = v_3 \).

The obtained situation \( (t, f, t) \) can be improved by player 3 by switching from \( t \) to \( f \), at least when \( v_0 = v_2 \) or \( v_0 = v_3 \). In both cases \( a_3 \) is replaced by \( a_1 \), which is better to 3 according to \( U_3 \). However, if \( v_0 = v_1 \), the outcome will not change.

The obtained situation \( (t, f, f) \) is a “shift” of \( (f, f, t) \), which was already considered. Repeating the same arguments two more times, we obtain the improvement cycle of length 6:

\[ (f, f, t), (t, f, t), (t, f, f), (t, f, f), (t, t, f), (f, t, f), (f, t, t) \]

. It is shown in Figure 3. Thus, none of eight situations of game \( (G_3, u) \) is a UNE in pure stationary strategies, when \( u \in U_3 \); see Figures 2 and 3.

In contrast, all three initialized game structures \( G_3^{v_1}, G_3^{v_2}, G_3^{v_3} \) with initial positions \( v_1, v_2, v_3 \), as well as the corresponding game forms \( g_3^{v_1}, g_3^{v_2}, g_3^{v_3} \) are Nash-solvable; see Figures 2 and 3.

Let us note, however, that initialized deterministic \( n \)-person games without NE in pure stationary strategies exist for \( n > 2 \). First such example for \( n = 4 \) was obtained in [34]. Then in [10] a much smaller 3-person game was constructed that has no NE even in stationary mixed strategies. However, these NE-free games are not play-once. It remains an open question, whether a play-once NE-free example exist; see Section 4 for more details.

### 3.3. Generalizations and possible applications

A tedious but routine case analysis allows to verify that a UNE, in pure stationary strategies, exists in games \( (G_2, u) \) and \( (G_3, u) \) whenever \( u \notin U_2 \) and \( u \notin U_3 \), respectively.
For \( n > 3 \) there are very many similar UNE-free examples; some of them will be given in Appendix A.

Using these examples one can try to solve an important open problem: Construct an initialized deterministic \( n \)-person game that has no NE in pure stationary strategies and satisfies the following condition

\[ (C) \quad \text{outcome } c \text{ is worse than each terminal outcome } a_j \in A \text{ for every player } i \in I. \]

Note that \( U_3 \) satisfies \( (C) \) while \( U_2 \) does not.

For more details see Section 4 and [10], in particular, Remark 3 there. As we already mentioned, for \( n = 2 \), by [20, 22, 7], an NE in pure strategies exists, even if condition \( (C) \) is waved.

4. Main results and open problems related to Nash-solvability in pure stationary strategies

4.1. Uniform Nash-solvability in presence of moves of chance

There are two important classes of games that always have a UNE in pure stationary strategies:

- Graphical \( n \)-person games on acyclic digraphs. In this case a special UNE in pure strategy strategies can be found by Backward Induction [16, 39]; see also [24, 25].

- Two-person zero-sum graphical games. In this case, the existence of an UNE follows from basic results of the stochastic game theory [17, 41].

More details can be found in [8]. In both cases, the uniform Nash-solvability holds not only for the terminal effective payoffs, considered in this paper, but also for a wide family of more general types: limiting mean, total, or \( k \)-total effective payoffs; see, for example, [3, 15, 17, 18, 29, 41, 43, 47, 49, 50].

Let us mention also that 2-person deterministic graphical games are Nash-solvable; see Subsection 4.3 below. Yet, as we know, such games may be NE-free, and hence, Nash-solvability may fail for initialized non-deterministic graphical games with only one (initial) position of chance.

4.2. NE-free graphical games with a unique position of chance and a unique directed cycle

Recall our main two examples \( (G_2', u), u \in U_2 \) and \( (G_3', u), u \in U_3 \). Both game structure \( G_2 \) and \( G_3 \) are play-once, contain a unique directed cycle and
a unique position of chance, which is the initial position, and both are not Nash-solvable: have no NE in pure stationary strategies when \( u \in U_2 \) and \( u \in U_3 \), respectively. Furthermore, in both cases, by deleting the initial position, we obtain a UNE-free non-initialized graphical game, \((G_2, u), u \in U_2\) and \((G_3, u), u \in U_3\).

Condition \((C)\) of Subsection 3.3 holds for \( U_3 \) but not for \( U_2 \).

However, 2-person UNE-free graphical games satisfying \((C)\) also exist. An example \((G_6, u)\) with \( u \in U_6 \) was constructed first in [4], where \( U_6 \) is determined. The 2-person deterministic game structure \( G_6 \) still have only one directed cycle \( C_6 \). Yet, this game is not play-once: players 1 and 2 alternate in \( C_6 \), so each of them controls three positions.

It was proven in the present paper that games \((G_2, u)\) and \((G_3, u)\) remain NE-free even in mixed stationary strategies under the Markov realization, for all \( u \in U_2 \) and for some \( u \in U_3 \). Also \((G_6, u)\) remains UNE-free for both Markov and a-priori realizations for all \( u \in U_6 \); a proof was sketched in [11].

Note, however, that corresponding initialized game structures \( G_2' \) and \( G_3' \) are Nash-solvable for any payoff \( u \) under the a-priori realization. This follows from the classic Nash theorem [44, 45], which is applicable in case of the a-priori realization, because both \( G_2 \) and \( G_3 \) are play-once.

Thus, already one (initial) position of chance may destroy Nash-solvability, even for play-once games and for games satisfying \((C)\).

So, for the rest of this section, we restrict ourselves to the so-called deterministic graphical (DG) games, (without positions of chance) and show (or sometimes conjecture) that Nash-solvability of such games, in pure stationary strategies, can be saved by some additional assumptions. By default, we assume that considered DG games are initialized unless it is explicitly said otherwise.

For the beginning, let us note that DG games may be NE-free under the above assumptions. The first example, with \( n = 4 \), was generated by a computer code [34]. Then, a much simpler 3-person DG game was constructed in [10], where it was also shown that this game has no NE not only in pure but also in stationary mixed strategies, under both the Markov and a-priori realizations. Yet, this game is not play-once; there is player who controls two (adjacent) positions.

4.3. Nash-solvable deterministic graphical \( n \)-person games

Two-person case, \( n = 2 \).

Nash-solvability of the 2-person DG games was derived in [7] from Nash-solvability of the so-called tight game forms. The latter result is old. For the
zero-sum case it was obtained by Edmonds and Fulkerson in 1970 [14], see also [19]. Then, it was extended to the general case in [20, 22]. Recently, a much shorter proof was given in [30].

Let us underline that condition (C) is not required for Nash-solvability of the 2-person DG games.

Although the concept of tight game forms can be naturally extended to the case \( n \geq 2 \), yet, for \( n > 2 \) tightness is no longer related to Nash-solvability: it is neither necessary [22], nor sufficient [20, 22]; see also [7]. Several new classes of tight game forms were recently found in [24, 28, 31, 32].

In [24] the class of the DG games is extended to a larger class of the so-called multi-stage DG games. The outcomes of a DG game are formed by all terminals of its digraph \( G \) and one special outcome \( c \) corresponding to all infinite plays of \( G \). In contrast, the outcomes of a multi-stage DG game are formed by the strongly connected components of its digraph; furthermore, some outcomes may be merged. It is shown in [24] that multi-stage DG game forms are tight. This statement is stronger than tightness of the DG game forms shown in [7].

**Play-once DG games and DG games satisfying (C).**

The play-once \( n \)-person DG games satisfying (C) are Nash-solvable. This is the main result of [7]. Moreover, we conjecture that each of these two conditions is sufficient for Nash-solvability. We have no example of an \( n \)-person NE-free DG game that is either play-once or (C) holds.

A stronger version of the second conjecture, (called “Catch 22”) was suggested in [26]: In every NE-free \( n \)-person DG game there exist at least two players for each of which outcome \( c \) is better than at least 2 terminal outcomes. In other words, \( c \) cannot be either the worst or the second worst for all players, and not even for all but one.

**Symmetric digraphs.**

The digraph \( G \) is called symmetric if \((v', v'')\) is its arc whenever \((v'', v')\) is unless \( v' \) or \( v'' \) is a terminal. Recently it was shown in [6] that every \( n \)-person DG game on a symmetric digraph is Nash-solvable. Condition (C) is not needed, although it simplifies the proof.

A wider class of the so-called \( n \)-person shortest path games was also studied in [6]. A local cost \( \ell(i, e) = -r(i, e) \) is defined for each player \( i \in [n] = \{1, \ldots, n\} \) and move \( e \) of such game. Condition (C) holds if all \( \ell(i, e) > 0 \). In this case, given a play \( P \), the effective cost of \( P \) for \( i \) is the sum of the corresponding local costs, \( L(i, P) = \sum_{e \in P} \ell(i, P) \), if \( P \) is a terminal play and
$L(i, P) = +\infty$ if $P$ is an infinite play (a lasso, whenever all players apply their pure stationary strategies).

Nash-solvability of the $n$-person shortest path games satisfying (C) on symmetric digraphs was proven in [6], where it was also conjectured that the last condition (symmetry) can be waved if $n = 2$. This is the so-called bi-shortest path conjecture [27]. However, an NE-free shortest path game exists if $n = 3$ and the digraph is not symmetric [33].

It was also shown in [6] that a (non-initialized) DG game has a UNE whenever (i) its digraph is symmetric, (ii) $n = 2$, and (iii) (C) holds. Conversely, a UNE may fail to exist if at least one of the above three conditions fails.

Somewhat related results were obtained in [9]. For DG games we assume that all lassos form a unique outcome $c$. The case when all cycles and terminals form pairwise distinct outcomes was considered in [9], where a criterion of Nash-solvability was obtained for the 2-person such games on symmetric digraphs.

5. Graphical games and stochastic games with perfect information

Here we will show that graphical games can be viewed as a special subclass of the stochastic games with perfect information and, thus, Nash-solvability of both can be studied, for the $n$-person case, simultaneously.

5.1. On Nash-solvability of mean payoff games

Two-person zero-sum stochastic games were introduced in 1953 by Shapley [47]. In 1957 Gillette [17] considered the subclass of stochastic games with zero stop probability, introduced limiting mean effective rewards for this case, and proved the existence of a UNE in mixed stationary strategies. (For the 2-person zero-sum case an NE is just a saddle point.) The proof was far from simple; Gillette’s approach was based on the Hardy-Littlewood Tauberian Theorem and all conditions of the latter were accurately verified (and thus the proof finalized) only in 1969 by Ligette and Lippman [41].

Also, Gillette outlined the subclass of games with perfect information and showed that they can be solved in uniform optimal and pure stationary strategies.

These games remain of interest even in absence of moves of chance, when two players control all non-terminal position. (Each one is controlled by one
player.) Such games are called deterministic; 2-person zero-sum deterministic
stochastic games with zero stop probability, perfect information, and the limit-
ing average rewards are known as the mean payoff games. They were in-
tensively studied since 1970s [43, 15, 29] mostly because of the algorithmic
complexity of their solution [29]. No polynomial algorithm for the mean pay-
off games is still known. Recently, a quasi-polynomial one was obtained for
the so-called parity games, which form a special subclass of the mean payoff
games [12]. However, in the present paper we study Nash-solvability rather
than polynomial solvability.

All above definitions can be naturally extended from the 2-person zero-sum
case to the $n$-person one. Thus, we can talk about $n$-person mean payoff or
stochastic games, with or without positions of chance.

Already 2-person (but not zero-sum) mean payoff games may have no NE
in pure stationary strategies. The first NE-free example was given in 1988
[21]; see also [29]. It is constructed on the complete bipartite $3 \times 3$ digraph;
each player controls 3 positions, that is, one part of it, and the local rewards
are symmetric, that is, the same for the moves from $u$ to $w$ and from $w$ to
$u$. This game can be interpreted as an ergodic extension of the correspond-
ing $3 \times 3$ bimatrix game [43].

The normal form of this game is of size $3^3 \times 3^3 = 27 \times 27$ and it is an open
question whether it has an NE in stationary mixed strategies.

In [23] it was shown that this example is, in a way, minimal: Every 2-person
mean payoff game on a bipartite $2 \times k$ digraph has an NE in pure stationary
strategies.

In [5] it was shown that the above $3 \times 3$ example disproves Nash-solvability
not only of the mean payoff games, but also of a much larger family of the
so-called $k$-total payoff games for any integer nonnegative $k$. Case $k = 0$
is associated with the mean payoffs, while $k = 1$ is assigned to the so-called total
payoffs introduced in [49, 50].

5.2. Graphical games can be viewed as transition-free mean payoff games

Recall that, by definition, all infinite plays of a graphical game (and in
particular, all lassos in its digraph) are equivalent, that is, form a single out-
come. In contrast, mean payoffs depend on the directed cycle of the lasso
that appears in the game after all $n$ players have chosen their pure stationary
strategies.

Graphical games can be viewed as a special subfamily of mean payoff games
(with or without positions of chance).

Given an $n$-person graphical game $(G, u)$ on a digraph $G$, let us add a
loop $\ell_v$ to each terminal position $v \in V_T$ in $G$ and for each player $i \in [n] =$
\{1, \ldots, n\} set the local reward \( r(i, \ell_v) \) on \( \ell_v \) equal to the terminal payoff of \( i \) in \( v \). Furthermore, set the local reward \( r(i, e) = 0 \) for any other edge of digraph \( G \) and each player \( i \in [n] \).

By this construction, in the obtained game all its infinite plays (more precisely, all plays that do not come to a terminal loop) are equivalent, since the corresponding effective payoff is 0 for each player, while on “finite” plays (that end in terminal loops) \( n \) players may have arbitrary effective payoffs.

Obviously, condition (C) holds if and only if \( r(i, \ell_v) < 0 \) for each terminal \( v \in V_T \); in other words, if the cost of every terminal is positive for each player.

We can naturally call the obtained mean payoff games transition-free, because players do not pay for the moves of the play, they pay (or are payed) only in the terminals. Obviously, the obtained transition-free mean payoff games are equivalent with the original graphical game.

Thus, two main UNE-free examples \((G_2, u)\) with \( u \in U_2 \) and \((G_3, u)\) with \( u \in U_3 \) of the present paper, as well as the 2-person UNE-free example \((G_6, u)\) with \( u \in U_6 \) from \[4\], provide non-initialized UNE-free and transition-free mean payoff games. Furthermore, the initialized 3- and 4-person NE-free examples from \[10\] and \[34\], respectively, provide initialized NE-free and transition-free mean payoff games.

Thus, results of the present paper can be viewed within the framework of further studies of Nash-solvability in pure stationary strategies of stochastic games with perfect information.

Finally, let us note that Markov realization corresponds exactly to solving stochastic games in stationary mixed strategies, which is standard, while the a-priori realization is a different approach, which was not applied to stochastic games yet.

5.3. Nash-solvability in pure history-dependent strategies

In 1997 Thuijsman and Raghavan \[48\] proved Nash-solvability in pure history dependent strategies for the mean payoff stochastic games with perfect information. As we already mentioned, this class of games contains graphical games considered in the present paper. For this reason, we are trying to solve them in stationary strategies.

Let us remark that the result of \[48\] holds, in fact, for many other classes of effective payoffs, in particular, for total \[49\, 50\, 33\] and \( k \)-total \[\square\] ones.

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APPENDIX

Appendix A. A large family of $n$-person deterministic graphical games without UNE in pure stationary strategies

Consider the following $n$-person play-once non-initialized game structure $G_n$. Given a digraph $G_n = (V, E)$, where

$$V = \{v_1, \ldots, v_n; a_1, \ldots, a_n\},$$

$$E = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1); (v_1, a_1), \ldots, (v_n, a_n)\}.$$ 

For any $n > 2$ set $I = \{1, \ldots, n\}$ and let each player $i \in I$ make a move in $v_i$. A set of payoffs $U_n$ is defined by the following properties:

(a) For each player $i \in I$ their own terminal $a_i$ is worse for them than each of the next $\lfloor n/2 \rfloor$ terminals, in cyclical order. (Among themselves these terminals may be ordered arbitrarily and this order may depend on $i$.)

(b) Among the first $\lfloor (n-1)/2 \rfloor$ of them there is at least one, $a_j$, that is worse than $a_i$ for player $j$.

(c) Finally, condition (C) of Section 3.3 holds.

Let us note that for $n = 3$ conditions (a) - (c) uniquely define the family of payoffs $U_3$, while for $n = 2$ they do not define $U_2$.

Proposition 8. The non-initialized play-once $n$-person game $(G_n, u)$ is UNE-free whenever $u \in U_n$.

Proof. Each player $i \in I$ controls a unique position $v_i$ and, thus, has only two pure strategies: to terminate at $a_i$ or to follow the cycle: $s^i \in \{t, f\}$. We have to prove that any situation $s = \{s^i \mid i \in I\}$ is not a UNE. Consider three cases.

Case 0. No player terminates, that is, all choose $f$. Then, the play results in the cycle and, by condition (C), each player can improve choosing $t$ rather than $f$. This holds for any initial position, $v_0 = v_i, i \in I$.
Case 1. One player \(i \in I\) terminates, while all others choose \(f\). Then, by condition (b), there exists a player \(j \in \{i + 1, \ldots, i + \lfloor \frac{n-1}{2} \rfloor\}\) who can improve her result by choosing \(t\) instead of \(f\), at least when \(v_0 = v_j\).

Case 2. At least two players terminate. Obviously, there exist two of them \(i, j \in I\) such that distance from \(v_i\) to \(v_j\) along the cycle is at most \(\lfloor \frac{n}{2} \rfloor\). Then, by (a), player \(i\) can improve her result by switching from \(f\) to \(t\), at least for \(v_0 = v_i\).

Appendix B. Markov and a-priori realizations for two main examples; proofs of Propositions 3, 5-7

Here we study the uniform Nash-solvability of these games and Nash-solvability of their initializing extensions in the mixed strategies under the Markov and a-priori realizations and prove Propositions 3-6.

Appendix B.1. Games \((G_2, u)\) with \(u \in U_2\) and \((G_3, u)\) with \(u \in U_3\) might have UNE only in strictly mixed strategies, under both the Markov or a-priori realizations

As we already know, games \((G_2, u)\) and \((G_3, u)\) have no UNE in pure strategies when \(u \in U_2\) and \(u \in U_3\), respectively. We will strengthen this claim as follows:

**Lemma 1.** For both the Markov or a-priori realizations, games \((G_2, u)\) with \(u \in U_2\) and \((G_3, u)\) with \(u \in U_3\) may have UNE only in strictly mixed strategies; in other words, only when \(0 < p_i < 1\) for \(i \in \{1, 2, 3\}\).

**Proof.** Let \(p = (p_1, p_2)\) be a UNE in \((G_2, u)\) with \(u \in U_2\). We will show that if \(p_i\) equals 0 or 1 then the same property holds for \(p_{3-i}\), where \(i \in \{1, 2\}\). In fact, this was already shown in Section 1.4 for every pure strategy of player \(i\) there exists a unique uniform best response of the opponent, and this response is realized by a pure strategy, while every strictly mixed response, \(0 < p_{3-i} < 1\), can be improved WRT at least one initial position.

Let \(p = (p_1, p_2, p_3)\) be a UNE in \((G_3, u)\) with \(u \in U_3\). If \(p_i\) equals 0 or 1 for a player \(i \in I = \{1, 2, 3\}\) then the same property holds for the two remaining players. In fact, we can just repeat the arguments of Section 1.4. Since this case is cyclically symmetric (unlike the previous one) WLOG we can set \(i = 3\).

Suppose \(p_3 = 0\), that is, player 3 terminates in \(a_3\). Then player 2 has a unique uniform best response: to follow the cycle with move \((v_2, v_3)\). Then, player 1 also has a unique uniform best response: to terminate with move \((v_1, a_1)\).
Suppose $p_3 = 1$, that is, player 3 follows the cycle by move $(v_3, v_1)$. Then player 2 has a unique uniform best response: to terminate by move $(v_2, a_2)$. Then, player 1 also has a unique uniform best response: to follow the cycle with move $(v_1, v_2)$.

In each of the four above cases the best response is unique and it is realized by a pure strategy.

It is important to note that

By definition of an NE $(p_i | i \in I)$, the strategy $p_i$ of each player $i \in I$ is a best response (not necessarily unique) to the set of strategies of the remaining players $I \setminus \{i\}$.

All above claims hold for both the Markov and a-priori realizations. Although in the latter case a uniform best response may fail to exist, in general, but for games, $(G_2, u)$ with $u \in U_2$, and $(G_3, u)$ with $u \in U_3$, it exists in all considered cases.

**Remark 8.** As we know, no UNE in pure strategies exists for both games under both realizations. Yet, a UNE in strictly mixed strategies might exist. This question will be studied in the next two Sections.

In what follows we denote by $J$ the set of indices of non-terminal positions and by $F_{ji}$ the expected payoff of player $i$, provided the play starts at $v_j$. Observe that $F_{ji}$ are continuously differentiable functions of $p_i$ when $0 < p_i < 1$, $i \in I$. Thus, if $p = (p_i | i \in I)$ is a uniform NE in strictly mixed strategies $(0 < p_i < 1, i \in I)$ under either the Markov or a-priori realization, then

$$\frac{\partial F_{ji}}{\partial p_i} = 0, \text{ for all } i \in I, j \in J.$$  \hspace{1cm} (B.1)

**Appendix B.2. Markov realization**

**Proof of Proposition** \(\Box\) Let $p = (p_1, p_2)$ be a uniform NE in game $(G_2, u)$ under the Markov realization. If $p_1 = p_2 = 1$, the probability of cycling is 1. Otherwise, the limiting distributions for initial positions $v_1$ or $v_2$ are given by \(\Box\), and hence, the expected payoffs are

$$F_{11} = \frac{(1 - p_1)u_1(a_1) + p_1(1 - p_2)u_1(a_2)}{1 - p_1 p_2},$$
$$F_{12} = \frac{(1 - p_1)u_2(a_1) + p_1(1 - p_2)u_2(a_2)}{1 - p_1 p_2},$$
$$F_{21} = \frac{(1 - p_2)u_1(a_2) + p_2(1 - p_1)u_1(a_1)}{1 - p_1 p_2},$$
$$F_{22} = \frac{(1 - p_2)u_2(a_2) + p_2(1 - p_1)u_2(a_1)}{1 - p_1 p_2}.$$
In this case, relations (B.1) have the following form:

\[
\begin{align*}
\frac{(u_1(a_1) - u_1(a_2))(1 - p_2)}{(1 - p_1p_2)^2} &= 0, \\
\frac{-p_1(u_2(a_1) - u_2(a_2))(1 - p_1)}{(1 - p_1p_2)^2} &= 0.
\end{align*}
\]

Since \(0 < p_i < 1\) and \(u \in U_2\), this system has no solutions. Thus, game \((G_2, u)\) has no UNE in mixed strategies.

Similar arguments provide an alternative proof for Proposition 4.

Let \(p = (p_1, p_2)\) be a uniform NE in the game \((G'_2, u)\) under the Markov realization. Denote the expected payoff function of player \(i\) by \(F_i\), \(i = 1, 2\). If \(p_1 = p_2 = 1\), the probability of cycling is 1. Otherwise, from (1) we obtain

\[
\begin{align*}
F_1 &= q_1[(1 - p_1)u_1(a_1) + p_1(1 - p_2)u_1(a_2)] + q_2[(1 - p_1)p_2u_1(a_1) + (1 - p_2)u_1(a_2)]
\frac{1}{1 - p_1p_2}, \\
F_2 &= q_1[(1 - p_1)u_2(a_1) + p_1(1 - p_2)u_2(a_2)] + q_2[(1 - p_1)p_2u_2(a_1) + (1 - p_2)u_2(a_2)]
\frac{1}{1 - p_1p_2}.
\end{align*}
\]

Relations (B.1) have the following form in this case:

\[
\begin{align*}
\frac{(q_1 + p_2q_2)(u_1(a_1) - u_1(a_2))(1 - p_2)}{(1 - p_1p_2)^2} &= 0, \\
\frac{(q_2 + p_1q_1)(u_2(a_1) - u_2(a_2))(1 - p_1)}{(1 - p_1p_2)^2} &= 0.
\end{align*}
\]

Since \(u \in U_2\), and for \(i = 1, 2\), both \(q_i\) cannot be 0 and by Lemma 1, \(0 < p_i < 1\), this system has no solutions. Thus, \((G'_2, u)\) has no NE in mixed strategies.

Proof of Proposition 5. Let \(p = (p_1, p_2, p_3)\) be a uniform NE in the game \((G_3, u)\) under the Markov realization. If \(p_1 = p_2 = p_3 = 1\), the probability of cycling is 1. Otherwise, assuming that the initial positions are \(v_1, v_2\) or \(v_3\),
the limiting distributions are given by (2) and

\[ F_{11} = \frac{(1 - p_1)u_1(a_1) + p_1(1 - p_2)u_1(a_2) + p_1p_2(1 - p_3)u_1(a_3)}{1 - p_1p_2p_3}, \]
\[ F_{21} = \frac{(1 - p_2)u_1(a_2) + p_2(1 - p_3)u_1(a_3) + p_2p_3(1 - p_1)u_1(a_1)}{1 - p_1p_2p_3}, \]
\[ F_{31} = \frac{(1 - p_3)u_1(a_3) + p_3(1 - p_1)u_1(a_1) + p_1p_3(1 - p_2)u_1(a_2)}{1 - p_1p_2p_3}, \]
\[ F_{12} = \frac{(1 - p_1)u_2(a_1) + p_1(1 - p_2)u_2(a_2) + p_1p_2(1 - p_3)u_2(a_3)}{1 - p_1p_2p_3}, \]
\[ F_{22} = \frac{(1 - p_2)u_2(a_2) + p_2(1 - p_3)u_2(a_3) + p_2p_3(1 - p_1)u_2(a_1)}{1 - p_1p_2p_3}, \]
\[ F_{32} = \frac{(1 - p_3)u_2(a_3) + p_3(1 - p_1)u_2(a_1) + p_1p_3(1 - p_2)u_2(a_2)}{1 - p_1p_2p_3}, \]
\[ F_{13} = \frac{(1 - p_1)u_3(a_1) + p_1(1 - p_2)u_3(a_2) + p_1p_2(1 - p_3)u_3(a_3)}{1 - p_1p_2p_3}, \]
\[ F_{23} = \frac{(1 - p_2)u_3(a_2) + p_2(1 - p_3)u_3(a_3) + p_2p_3(1 - p_1)u_3(a_1)}{1 - p_1p_2p_3}, \]
\[ F_{33} = \frac{(1 - p_3)u_3(a_3) + p_3(1 - p_1)u_3(a_1) + p_1p_3(1 - p_2)u_3(a_2)}{1 - p_1p_2p_3}. \]
Relations (B.1) have the following form in this case:

\[-(u_1(a_1) - u_1(a_2) + p_2u_1(a_2) - p_2u_1(a_3) - p_2p_3u_1(a_1) + p_2p_3u_1(a_3)) \frac{1}{(1 - p_1p_2p_3)^2} = 0,\]
\[-(p_2p_3(u_1(a_1) - u_1(a_2) + p_2u_1(a_2) - p_2u_1(a_3) - p_2p_3u_1(a_1) + p_2p_3u_1(a_3))) \frac{1}{(1 - p_1p_2p_3)^2} = 0,\]
\[-(p_3(u_1(a_1) - u_1(a_2) + p_2u_1(a_2) - p_2u_1(a_3) - p_2p_3u_1(a_1) + p_2p_3u_1(a_3))) \frac{1}{(1 - p_1p_2p_3)^2} = 0,\]
\[-(u_2(a_2) - u_2(a_3) - p_3u_2(a_1) + p_3u_2(a_3) + p_1p_3u_2(a_1) - p_1p_3u_2(a_2)) \frac{1}{(1 - p_1p_2p_3)^2} = 0,\]
\[-(p_1p_3(u_2(a_2) - u_2(a_3) - p_3u_2(a_1) + p_3u_2(a_3) + p_1p_3u_2(a_1) - p_1p_3u_2(a_2))) \frac{1}{(1 - p_1p_2p_3)^2} = 0,\]
\[-(p_1(u_2(a_2) - u_2(a_3) - p_3u_2(a_1) + p_3u_2(a_3) + p_1p_3u_2(a_1) - p_1p_3u_2(a_2))) \frac{1}{(1 - p_1p_2p_3)^2} = 0,\]
\[-(u_3(a_3) - u_3(a_1) + p_1u_3(a_1) - p_1u_3(a_2) - p_1p_2u_3(a_3) + p_1p_2u_3(a_2)) \frac{1}{(1 - p_1p_2p_3)^2} = 0,\]
\[-(p_1p_2(u_3(a_3) - u_3(a_1) + p_1u_3(a_1) - p_1u_3(a_2) + p_1p_2u_3(a_3) - p_1p_2u_3(a_2))) \frac{1}{(1 - p_1p_2p_3)^2} = 0,\]
\[-(p_2(u_3(a_3) - u_3(a_1) + p_1u_3(a_1) - p_1u_3(a_2) - p_1p_2u_3(a_3) + p_1p_2u_3(a_2))) \frac{1}{(1 - p_1p_2p_3)^2} = 0.\]

Since \(p_i > 0\) and \(q_i\) are not all equal to 0 for \(i = 1, 2, 3\), we have
\[u_1(a_i) - u_1(a_2) + p_2u_1(a_2) - p_2u_1(a_3) - p_2p_3u_1(a_1) + p_2p_3u_1(a_3) = 0,\]
\[u_2(a_2) - u_2(a_3) - p_3u_2(a_1) + p_3u_2(a_3) + p_1p_3u_2(a_1) - p_1p_3u_2(a_2) = 0,\]
\[u_3(a_3) - u_3(a_1) + p_1u_3(a_1) - p_1u_3(a_2) - p_1p_2u_3(a_3) + p_1p_2u_3(a_2) = 0,\]
and using (7) we transform equations (B.2) to
\[\mu_1(1 - p_2) = p_2(1 - p_3),\]
\[\mu_2(1 - p_3) = p_3(1 - p_1),\]
\[\mu_3(1 - p_1) = p_1(1 - p_2).\]

Recall that \(0 < p_i < 1\) for \(i = 1, 2, 3\), by Lemma [1] Solving (B.3) WRT \(p_i\) yields [8], provided \(p_i > 0\), and each equality of (8) implies that \(\mu_1\mu_2\mu_3 < 1\), provided \(p_i < 1\).

Proof of Proposition [6]. Let \(p = (p_1, p_2, p_3)\) be a uniform NE in the game \((G'_3, u)\) under Markov realization. Let the expected payoff of player \(i\) be denoted by \(F_i, i = 1, ..., 3\). If \(p_1 = p_2 = p_3 = 1\), the probability of a cycle is 1.
Otherwise,
\[
F_1 = \frac{q_1[(1 - p_1)u_1(a_1) + p_1(1 - p_2)u_1(a_2) + p_1p_2(1 - p_3)u_1(a_3)]}{1 - p_1p_2p_3} \\
+ \frac{q_2[(1 - p_1)p_2p_3u_1(a_1) + (1 - p_2)u_1(a_2) + p_2(1 - p_3)u_1(a_3)]}{1 - p_1p_2p_3} \\
+ \frac{q_3[p_3(1 - p_1)u_1(a_1) + p_1p_3(1 - p_2)u_1(a_2) + (1 - p_3)u_1(a_3)]}{1 - p_1p_2p_3},
\]
\[
F_2 = \frac{q_1[(1 - p_1)u_1(a_1) + p_1(1 - p_2)u_1(a_2) + p_1p_2(1 - p_3)u_1(a_3)]}{1 - p_1p_2p_3} \\
+ \frac{q_2[(1 - p_1)u_1(a_1)p_2p_3 + (1 - p_2)u_1(a_2) + p_2(1 - p_3)u_1(a_3)]}{1 - p_1p_2p_3} \\
+ \frac{q_3[p_3(1 - p_1)u_1(a_1) + p_1p_3(1 - p_2)u_1(a_2) + (1 - p_3)u_1(a_3)]}{1 - p_1p_2p_3},
\]
\[
F_3 = \frac{q_1[(1 - p_1)u_1(a_1) + p_1(1 - p_2)u_1(a_2) + p_1p_2(1 - p_3)u_1(a_3)]}{1 - p_1p_2p_3} \\
+ \frac{q_2[(1 - p_1)p_2p_3 + (1 - p_2)u_1(a_2) + p_2(1 - p_3)u_1(a_3)]}{1 - p_1p_2p_3} \\
+ \frac{q_3[p_3(1 - p_1)u_1(a_1) + p_1p_3(1 - p_2)u_1(a_2) + (1 - p_3)u_1(a_3)]}{1 - p_1p_2p_3}.
\]

Relations (B.1) have the following form in this case:
\[
-(q_1 + q_2p_2p_3 + q_3p_3)(u_1(a_1) - u_1(a_2) + p_2u_1(a_2) - p_2u_1(a_3) - p_2p_3u_1(a_1) + p_2p_3u_1(a_3)) \\
(1 - p_1p_2p_3)^2 = 0,
\]
\[
-(q_2 + q_3p_1p_3 + q_1p_1)(u_2(a_2) - u_2(a_3) - p_3u_2(a_3) + p_3u_2(a_3) + p_1p_3u_2(a_1) - p_1p_3u_2(a_2)) \\
(1 - p_1p_2p_3)^2 = 0,
\]
\[
-(q_3 + q_1p_1p_2 + q_2p_2)(u_3(a_1) - u_3(a_3) - p_1u_3(a_1) + p_1u_3(a_2) - p_1p_3u_3(a_2) + p_1p_2u_3(a_3)) \\
(1 - p_1p_2p_3)^2 = 0.
\]

(B.4)

Obviously, for \(i = 1, 2, 3\), not all \(q_i\) are 0, since \(q_1 + q_2 + q_3 = 1\), and furthermore, by Lemma 1, \(0 < p_i < 1\). Hence, in the LHS of each equation in (B.4) the denominator and all three first factors are not 0. Therefore, all three second factors are 0, which exactly means (B.2). As before, using (7) we transform (B.2) to (B.3) and obtain (8), and conditions \(p_i < 1\) for \(i = 1, 2, 3\) imply that \(\mu_1\mu_2\mu_3 < 1\). Thus, if \(\mu_1\mu_2\mu_3 < 1\), (8) defines a unique NE, otherwise, if \(\mu_1\mu_2\mu_3 \geq 1\), there is no NE. \(\square\)

Let us note that the above proof works for any distribution \(q(v_0)\), not necessarily strictly positive.
Appendix B.3. The a-priori realization

Proof of Proposition 7. For game \((G_2, u)\) the limiting a-priori distributions for the outcomes \((a_1, a_2, c)\) WRT initial positions \(v_1\) and \(v_2\), are given by \([3]\). In particular, \([3]\) implies that the expected payoffs \(F_{12}\) and \(F_{21}\) are
\[
F_{12} = (1 - p_1)u_2(a_1) + p_1(1 - p_2)u_2(a_2) + p_1p_2u_2(c),
F_{21} = (1 - p_2)u_1(a_2) + p_2(1 - p_1)u_1(a_1) + p_1p_2u_1(c).
\]

Then, the equations \([B.1]\) have the following form:
\[
\begin{align*}
p_1(u_2(c) - u_2(a_2)) &= 0, \\
p_2(u_1(c) - u_1(a_1)) &= 0.
\end{align*}
\]

If \(u \in U_2\), the above system of equations has no solutions, and hence, game \((G_2, u)\) has no UNE in mixed strategies under the a-priori realization.

For game \((G_3, u)\) the limiting a-priori distributions on the outcomes \((a_1, a_2, a_3, c)\), WRT initial positions \(v_1\), \(v_2\), and \(v_3\), are given by \([4]\).

In particular, the expected mean payoffs \(F_{21}\), \(F_{32}\) and \(F_{13}\) are
\[
\begin{align*}
F_{21} &= (1 - p_2)u_1(a_2) + p_2(1 - p_3)u_1(a_3) + p_2p_3(1 - p_1)u_1(a_1) + p_1p_2p_3u_1(c), \\
F_{32} &= (1 - p_3)u_2(a_3) + p_3(1 - p_1)u_2(a_1) + p_1p_3u_2(a_2) + p_1p_2p_3u_2(c), \\
F_{13} &= (1 - p_1)u_3(a_1) + p_1(1 - p_2)u_3(a_2) + p_1p_2(1 - p_3)u_3(a_3) + p_1p_2p_3u_3(c).
\end{align*}
\]

The equations \([B.1]\) for them turn into
\[
\begin{align*}
p_2p_3(u_1(c) - u_1(a_1)) &= 0, \\
p_1p_3(u_2(c) - u_2(a_2)) &= 0, \\
p_1p_2(u_3(c) - u_3(a_3)) &= 0.
\end{align*}
\]

These three equation together contradict Lemma \([1]\) \((0 < p_i < 1)\), when \(u \in U_3\).
Thus, in this case, game \((G_3, u)\) has no UNE in mixed strategies under the a-priori realization. \(\square\)

Appendix C. Interpretation of two main examples

Appendix C.1. Game \((G_2, u)\) with \(u \in U_2\)

Two mechanics \(M_1\) and \(M_2\) may replace a device in their garage. There are two options of such replacement: \(a_1\) and \(a_2\). Both prefer \(a_1\) to \(a_2\), so the solution seems obvious. Yet, there is a third option, \(c\): they do not replace device at all unless they come to consensus. For \(M_1\) outcome \(c\) is the best.
option: better than \(a_1\) (he prefers to save), while for \(M_2\) \(c\) is the worst option: worse than \(a_2\). They negotiate in pure strategies in accordance with the game structure \(G_2\) on Figure 1.

Suppose \(M_1\) makes a move \((v_1, a_1)\), thus, agreeing to buy device \(a_1\). Then, naturally, \(M_2\) supports \(M_1\) by making move \((v_2, v_1)\). Yet, \(M_1\) can improve the obtained situation \(((v_1, a_1), (v_2, v_1))\) for himself rejecting \(a_1\); that is, he switches from \((v_1, a_1)\) to \((v_1, v_2)\) thus getting \(c\), which is best for him. This happens for any initial position: \(v_0 = v_1\) or \(v_0 = v_2\).

Recall that \(c\) is the worst outcome for \(M_2\), so he is unhappy and will improve for himself the current situation \(((v_1, v_2), (v_2, v_1))\) by switching from \((v_2, v_1)\) to \((v_2, a_2)\) and getting \(a_2\) instead of \(c\). Again, this happens for any initial position: \(v_0 = v_1\) or \(v_0 = v_2\).

Recall that \(a_2\) is the worst outcome for \(M_1\), so he is unhappy and will improve for himself the situation \(((v_1, v_2), (v_2, a_2))\) switching from \((v_1, v_2)\) to \((v_1, a_1)\) and getting \(a_1\), at least when the play begins in \(v_1\). If it begins in \(v_2\), outcome \(a_2\) remains. Nevertheless, \(M_1\) makes a strict improvement when \(v_0 = v_1\) and he gets the same result when \(v_0 = v_2\).

Finally, \(M_2\) can improve the obtained situation \(((v_1, a_1), (v_2, a_2))\) for himself, switching from \((v_2, a_2)\) to \((v_2, v_1)\). At least, \(a_2\) is replaced by \(a_1\) when the play begins in \(v_2\), and if it begins in \(v_1\) then outcome \(a_1\) remains. Nevertheless, \(M_2\) makes a strict improvement when \(v_0 = v_2\) and he gets the same result when \(v_0 = v_2\).

Appendix C.2. Game \((G_3, u)\) with \(u \in U_3\)

Behavioral interpretation

Once upon a time there was a family: grandmother (GM), mother (M), and little girl (LG, not too little, yet) corresponding to players 1, 2, and 3. The family has a work to do, say, cleaning, washing, or shopping. Each player can terminate, which means to do the work herself. This is the second best outcome for each. Alternatively, each can follow the 3-cycle, thus, asking the next player to do the work, in the cyclic order: GM, M, LG. The best outcome for each is when the next player does the work. The third best is when the previous will. Finally, \(c\) means that nobody did the work, which is the worst outcome for all. The following psychological motivation can be suggested.

GM prefers M to work, but she pampers LG and would prefer to work herself instead of her.

M prefers LG to work, but has a mercy for GM and would prefer to replace her.
LG, who is already spoiled by GM, prefers her to work, but not M, because in this case M may get angry and punish LG somehow in the future.

Financial interpretation

Two projects are considered:

(i) constructing a bridge across the Raritan river in Middlesex County, NJ; 
(ii) including this bridge into a highway (route 18) in future.

Project (ii) is essentially more expensive than (i). Only (i) is under consideration at the present. Three players are Local (L), State of New Jersey (S), and Federal (F) governments. All are interested in projects (i) and (ii), but also in saving money from their budgets. Part (ii) is too expensive for L, so either S or F pays for it; S could pay for (i) or (ii) but not for both; F could pay for both, but in this case (ii) will be started only in 8-9 years after (i). (You are not alone!) Otherwise, if L or S pays for (i), then (ii) can be started much sooner, say, in 1-2 years. The big delay is OK with S, but not with L. Both are happy to provide convenient transit, but the bridge, not included in a highway will be served only by local roads and result in frequent local traffic jams.

Each player can terminate, which means paying for (i), or refuse to pay, asking the next player to do it; in the cyclic order L, S, F.

The best for L if S pays for (i); then F will pay for (ii) in 1-2 years. Yet, if F pays for (i) then (ii) will be delayed; so L would prefer to pay for (i).

The best for S if F pays for (i), and then, in 8-9 years, for (ii). If L pays for (i) then S will have to pay for (ii) in 1-2 years; so S would rather pay for (i) now, which is much cheaper.

The best for F if L pays for (i) and then, in 1-2 years, S will pay for (ii). If S pays for (i) then F will have to pay for (ii) in 1-2 years; so F would rather pay for (i) now and for (ii) in 8-9 years.

For all three parties c is the worst outcome: if all refuse to pay then projects (i) and (ii) will not be realized.