A new higher order Yang–Mills–Higgs flow on Riemannian 4-manifolds

Hemanth Saratchandran¹, Jiaogen Zhang², Pan Zhang³

1. School of Mathematical Sciences, University of Adelaide, Adelaide 5005, Australia; hemanth.saratchandran@adelaide.edu.au
2. School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, P.R. China; zjgmath@ustc.edu.cn
3. School of Mathematical Sciences, Anhui University, Hefei 230601, P.R. China; panzhang20100@ahu.edu.cn

Abstract: Let \((M, g)\) be a closed Riemannian 4-manifold and let \(E\) be a vector bundle over \(M\) with structure group \(G\), where \(G\) is a compact Lie group. In this paper, we consider a new higher order Yang–Mills–Higgs functional, in which the Higgs field is a section of \(\Omega^0(\text{ad}E)\). We show that, under suitable conditions, solutions to the gradient flow do not hit any finite time singularities. In the case that \(E\) is a line bundle, we are able to use a different blow up procedure and obtain an improvement of the long time result in [18]. The proof is rather relevant to the properties of the Green function, which is very different from the previous techniques in [8, 11, 18].

Keywords: higher order Yang–Mills–Higgs flow; line bundle; long-time existence

2020 Mathematics Subject Classification. 58C99; 58E15; 81T13

1. Introduction

Let \((M, g)\) be a closed Riemannian manifold of real dimension 4 and let \(E\) be a vector bundle over \(M\) with structure group \(G\), where \(G\) is a compact Lie group. The Yang–Mills functional, defined on the space of connections of \(E\), is given by

\[ \mathcal{YM}(\nabla) = \frac{1}{2} \int_M |F_\nabla|^2 \text{dvol}_g, \]

where \(\nabla\) is a metric compatible connection, \(F_\nabla\) denotes the curvature and the pointwise norm \(|\cdot|\) is given by \(g\) and the Killing form of \(\text{Lie}(G)\).

\(\nabla\) is called a Yang–Mills connection of \(E\) if it satisfies the Yang–Mills equation

\[ D_\nabla^* F_\nabla = 0. \]
A solution of the Yang–Mills flow is given by a family of connections $\nabla_t := \nabla(x, t)$ such that

$$\frac{\partial \nabla_t}{\partial t} = -D_{\nabla_t}^* F_{\nabla_t}, \quad \text{in} \quad M \times [0, T).$$

The Yang–Mills flow was initially studied by Atiyah–Bott [3] and was suggested to understand the topology of the space of connections by infinite dimensional Morse theory.

We consider the Yang–Mills–Higgs $k$-functional (or Yang–Mills–Higgs $k$-energy)

$$\mathcal{J}_k(\nabla, u) = \frac{1}{2} \int_M \left[ |\nabla^{(k)} F_{\nabla}|^2 + |\nabla^{(k+1)} u|^2 \right] d\text{vol}_g,$$  \hspace{1cm} (1.1)

where $k \in \mathbb{N} \cup \{0\}$, $\nabla$ is a connection on $E$ and $u$ is a section of $\Omega^0(\text{ad} E)$. In [18], we have considered the case when $u$ is a section of $\Omega^0(E)$. When $k = 0$, (1.1) is the Yang–Mills–Higgs functional studied in [5]. In [5], Hassell proved the global existence of the Yang–Mills–Higgs flow in 3-dimensional Euclidean space. In [6], Hong–Tian studied the global existence of Yang–Mills–Higgs flow in 3-dimensional Hyperbolic space, their results yields non-self dual Yang–Mills connections on $S^4$. In the new century, the study of Yang–Mills–Higgs flow has aroused a lot of attention (see [17,10,12,13,16,17] and references therein).

The Yang–Mills–Higgs $k$-system, i.e. the corresponding Euler–Lagrange equations of (1.1), is

$$\begin{cases}
(-1)^k D_{\nabla}^{(k)} \Delta^{(k)}_{\nabla} F_{\nabla} + \sum_{v=0}^{2k-1} P_1^{(v)}[F_{\nabla}] + P_2^{(2k-1)}[F_{\nabla}]
+ \sum_{i=0}^{k} \nabla^{*(i)}(\nabla^{(k+1)} u \ast \nabla^{(k-i)} u) = 0, \\
\nabla^{*(k+1)} \nabla^{(k+1)} u = 0,
\end{cases}$$  \hspace{1cm} (1.2)

where $\Delta^{(k)}_{\nabla}$ denotes $k$ iterations of the Bochner Laplacian $-\nabla^* \nabla$, and the notation $P$ is defined in (2.1).

A solution of the Yang–Mills–Higgs $k$-flow is given by a family of pairs $(\nabla(x, t), u(x, t)) := (\nabla_t, u_t)$ such that

$$\begin{cases}
\frac{\partial \nabla_t}{\partial t} = (-1)^{(k+1)} D_{\nabla_t}^{(k)} \Delta^{(k)}_{\nabla_t} F_{\nabla_t} + \sum_{v=0}^{2k-1} P_1^{(v)}[F_{\nabla_t}]
+ P_2^{(2k-1)}[F_{\nabla_t}] + \sum_{i=0}^{k} \nabla_{\nabla_t}^{*(i)}(\nabla_{\nabla_t}^{(k+1)} u_t \ast \nabla_{\nabla_t}^{(k-i)} u_t), \\
\frac{\partial u_t}{\partial t} = -\nabla_t^{*(k+1)} \nabla_{\nabla_t}^{(k+1)} u_t, \quad \text{in} \quad M \times [0, T).
\end{cases}$$  \hspace{1cm} (1.3)

When $k = 0$, the flow (1.3) is a Yang–Mills–Higgs flow [6].

From an analytic point of view, the Yang–Mills–Higgs $k$-flow (1.3) admits similar properties to the case in which the Higgs field takes value in $\Omega^0(E)$. In fact, by the
approach in [18], we can prove the following theorem.

**Theorem 1.1.** Let $E$ be a vector bundle over a closed Riemannian 4-manifold $(M, g)$. Assume the integer $k > 1$, for every smooth initial data $(\nabla_0, u_0)$, there exists a unique smooth solution $(\nabla_t, u_t)$ to the Yang–Mills–Higgs $k$-flow (1.3) in $M \times [0, +\infty)$.

Our motivation for considering such flows comes from recent work of A. Waldron who was able to prove long time existence for the Yang–Mills flow [15], thereby settling a long standing conjecture in the area. In the context of the Yang–Mills–Higgs flow it is still unknown whether the flow exists for all times on a Riemannian 4-manifold. The above theorem shows that provided $k > 1$, the Yang-Mills-Higgs $k$ flow does obey long time existence on a 4-manifold. A question that arises at this point is to understand what the optimum value for $k$ is. By assuming our bundle $E$ is a line bundle, we are able to make progress on this question and show that long time existence holds for all positive $k$:

**Theorem 1.2.** Let $E$ be a line bundle over a closed Riemannian 4-manifold $(M, g)$. Assume the integer $k > 0$, for every smooth initial data $(\nabla_0, u_0)$, there exists a unique smooth solution $(\nabla_t, u_t)$ to the Yang–Mills–Higgs $k$-flow (1.3) in $M \times [0, +\infty)$.

We point out that at present we don’t know if the above theorem is optimal. Meaning we cannot rule out the case that long time existence occurs for $k = 0$.

The proof of Theorem 1.1 involves local $L^2$ derivative estimates, energy estimates and blowup analysis. An interesting aspect of this work is that by using a different blow up procedure we are able to obtain a proof of Theorem 1.2 which may be of independent interest. Another interesting aspect is that the proof of long-time existence obstruction (see Theorem 3.2) is rather relevant to the properties of the Green function, which is very different from the previous techniques in [8, 11, 18].

2. Preliminaries

In this section we introduce the basic setup and notation that will be used throughout the paper. Our approach follows the notation of [8, 11, 18].

Let $E$ be a vector bundle over a smooth closed manifold $(M, g)$ of real dimension $n$. The set of all smooth unitary connections on $E$ will be denoted by $A_E$. A given connection $\nabla \in A_E$ can be extended to other tensor bundles by coupling with the corresponding Levi–Civita connection $\nabla_M$ on $(M, g)$.

Let $D\nabla$ be the exterior derivative, or skew symmetrization of $\nabla$. The curvature
tensor of \( E \) is denoted by
\[
F_{\nabla} = D_{\nabla} \circ D_{\nabla}.
\]
We set \( \nabla^*, D^*_{\nabla} \) to be the formal \( L^2 \)-adjoint of \( \nabla, D_{\nabla} \), respectively. The Bochner and Hodge Laplacians are given respectively by
\[
\Delta_{\nabla} = -\nabla^* \nabla, \quad \Delta_{D_{\nabla}} = D_{\nabla} D^*_{\nabla} + D^*_{\nabla} D_{\nabla}.
\]
Let \( \xi, \eta \) be \( p \)-forms valued in \( E \) or \( \text{End}(E) \). Let \( \xi \ast \eta \) denote any multilinear form obtained from a tensor product \( \xi \otimes \eta \) in a universal way. That is to say, \( \xi \ast \eta \) is obtained by starting with \( \xi \otimes \eta \), taking any linear combination of this tensor, taking any number of metric contractions, and switching any number of factors in the product. We then have
\[
|\xi \ast \eta| \leq C|\xi||\eta|.
\]
Denote by
\[
\nabla^{(i)} = \underbrace{\nabla \cdots \nabla}_{i \text{ times}}.
\]
We will also use the \( P \) notation, as introduced in [9]. Given a tensor \( \xi \), we denote by
\[
P^{(k)}_{v}[\xi] := \sum_{w_1 + \cdots + w_v = k} (\nabla^{(w_1)} \xi) \ast \cdots \ast (\nabla^{(w_v)} \xi) \ast T,
\]
where \( k, v \in \mathbb{N} \) and \( T \) is a generic background tensor dependent only on \( g \).

3. Long-time existence obstruction

We can use De Turck’s trick to establish the local existence of the Yang–Mills–Higgs \( k \)-flow. We refer to [11,13] for more details. As the proof is standard, we will omit the details.

**Theorem 3.1.** (local existence) Let \( E \) be a vector bundle over a closed Riemannian manifold \((M,g)\). There exists a unique smooth solution \((\nabla_t, u_t)\) to the Yang–Mills–Higgs \( k \)-flow \((1.3)\) in \( M \times [0, \epsilon) \) with smooth initial value \((\nabla_0, u_0)\).

Following [8,11], we can derive estimates of Bernstein–Bando–Shi type, which is similar to [13 Proposition 4.10].

**Proposition 3.1.** Let \( q \in \mathbb{N} \) and \( \gamma \in C^\infty_c(M) \) \((0 \leq \gamma \leq 1)\). Suppose \((\nabla_t, u_t)\) is a solution to the Yang–Mills–Higgs \( k \)-flow \((1.3)\) defined on \( M \times I \). Suppose
\( Q = \max\{1, \sup_{t \in I} |F_{\nabla_t}|\}, \quad K = \max\{1, \sup_{t \in I} |u_t|\}, \) and choose \( s \geq (k+1)(q+1). \)
Then for \( t \in [0, T) \subset I \) with \( T < \frac{1}{K_k}, \) there exists a positive constant
\( C_q := C_q(\dim(M), \rk(E), G, q, k, s, g, \gamma) \in \mathbb{R}_{>0} \) such that
\[
\|\gamma^s \nabla_t^{(q)} F_{\nabla_t}\|_{L^2}^2 + \|\gamma^s \nabla_t^{(q)} u_t\|_{L^2}^2 \leq C_q t^{-\frac{s}{q+1}} \sup_{t \in [0,T)} \left( \|F_{\nabla_t}\|_{L^2}^2 + \|u_t\|_{L^2}^2 \right). \tag{3.1}
\]

The following corollary is a direct consequence of the above proposition, which will be used in the blow-up analysis. The proof relies on Sobolev embedding \( W^{p,2} \subset C^0 \)
provided \( p > \frac{2}{q}, \) and the Kato’s inequality \( |d|u_t| \leq |\nabla_t u_t|. \) More details can be found in Kelleher’s paper (\[8\] Corollary 3.14).

**Corollary 3.1.** Suppose \((\nabla_t, u_t)\) solves the Yang–Mills–Higgs k-flow (1.3) defined on \( M \times [0, \tau) \). Set \( \tau := \min\{\tau, 1\}. \) Suppose \( Q = \max\{1, \sup_{t \in [0,\tau)} |F_{\nabla_t}|\}, \)
\( K = \max\{1, \sup_{t \in [0,\tau)} |u_t|\}. \) Assume \( \gamma \in C^\infty_c(M) \) \((0 \leq \gamma \leq 1). \) For \( s, l \in \mathbb{N} \)
with \( s \geq (k+1)(l+1) \) there exists \( C_l := C_l(\dim(M), \rk(E), K, Q, G, s, k, l, \gamma) \in \mathbb{R}_{>0} \) such that
\[
\sup_{M} \left( \|\gamma^s \nabla_t^{(l)} F_{\nabla_t}\|_{L^2}^2 + \|\gamma^s \nabla_t^{(l)} u_t\|_{L^2}^2 \right) \leq C_l \sup_{M \times [0,\tau)} \left( \|F_{\nabla_t}\|_{L^2}^2 + \|u_t\|_{L^2}^2 \right).
\]

Using Corollary 3.1 we have the following corollary, which can be used for finding obstructions to long time existence.

**Corollary 3.2.** Suppose \((\nabla_t, u_t)\) solves the Yang–Mills–Higgs k-flow (1.3) defined on \( M \times [0, T) \) for \( T \in [0, +\infty). \) Suppose
\[
Q = \max\{1, \sup_{t \in [0,T)} |F_{\nabla_t}|, \sup_{t \in [0,T)} \|F_{\nabla_t}\|_{L^2} \}
\]
and
\[
K = \max\{1, \sup_{t \in [0,T)} |u_t|, \sup_{t \in [0,T)} \|u_t\|_{L^2} \}
\]
are finite. Assume \( \gamma \in C^\infty_c(M) \) \((0 \leq \gamma \leq 1). \) Then for \( t \in [0, T), \) \( s, l \in \mathbb{N} \) with \( s \geq (k+1)(l+1) \), there exists \( C_l := C_l(\nabla_0, u_0, \dim(M), \rk(E), K, Q, G, s, k, l, \gamma) \in \mathbb{R}_{>0} \)
such that
\[
\sup_{M \times [0,T)} \left( \|\gamma^s \nabla_t^{(l)} F_{\nabla_t}\|_{L^2}^2 + \|\gamma^s \nabla_t^{(l)} u_t\|_{L^2}^2 \right) \leq C_l.
\]

In the following, we will use Corollary 3.2 to show that the only obstruction to long time existence of the Yang–Mills–Higgs k-flow (1.3) is a lack of supremal bound on \( |F_{\nabla_t}| + |\nabla_t u_t|. \) Before doing so, we need following proposition, which is similar to \([8\] Proposition 4.15).

**Proposition 3.2.** Suppose \((\nabla_t, u_t)\) is a solution to the Yang–Mills–Higgs k-flow (1.3) defined on \( M \times [0, T) \) for \( T \in [0, +\infty). \) Suppose that for all \( l \in \mathbb{N} \cup \{0\} \) there
exists $C_t \in \mathbb{R}_{>0}$ such that
\[
\max \left\{ \sup_{M \times [0,T)} |\nabla^{(l)}_t \frac{\partial}{\partial t} l|, \sup_{M \times [0,T)} |\nabla^{(l)}_t \frac{\partial u_t}{\partial t}| \right\} \leq C_t.
\]
Then $\lim_{t \to T} (\nabla_t, u_t) = (\nabla_T, u_T)$ exists and is smooth.

The following proposition is straightforward.

**Proposition 3.3.** Suppose $(\nabla_t, u_t)$ is a solution to the Yang–Mills–Higgs $k$-flow (1.3) defined on $M \times [0,T)$. We have
\[
\sup_{t \in [0,T)} \|u_t\|_{L^2} < +\infty.
\]

Using Propositions 3.2 and 3.3 we are ready to prove the main result in this subsection.

**Theorem 3.2.** Assume $E$ is a line bundle. Suppose $(\nabla_t, u_t)$ is a solution to the Yang–Mills–Higgs $k$-flow (1.3) for some maximal $T < +\infty$. Then
\[
\sup_{M \times [0,T)} (|F\nabla_t| + |\nabla_t u_t|) = +\infty.
\]

**Proof.** Suppose to the contrary that
\[
\sup_{M \times [0,T)} (|F\nabla_t| + |\nabla_t u_t|) < +\infty,
\]
which means
\[
\sup_{M \times [0,T)} |F\nabla_t| < +\infty, \quad \sup_{M \times [0,T)} |\nabla_t u_t| < +\infty.
\]
Denote by $G_t(x,y)$ the Green function associated to the operator $\Delta\nabla_t$, then for any fixed $x \in M$, $\|\nabla_0 G_t(x,\cdot)\|_{L^\infty(M)} \leq C_G$ for a constant $C_G$ [2, Appendix A]. Note that $\nabla_t G_t - \nabla_0 G_t = [\nabla_t - \nabla_0, G_t] = 0$, we conclude that $\|\nabla_t G_t\|_{L^\infty(M)}$ is also uniformly bounded. Therefore, using the properties of the Green function in [2, Appendix A], we have
\[
|u_t(x) - \frac{1}{\text{Vol}(M)} \int_M u_t(y)dy| = |\int_M \Delta\nabla_t G_t(x,y)u_t(y)dy| = |\int_M \nabla_t G_t(x,y)\nabla_t u_t(y)dy| < +\infty,
\]
which together with Proposition 3.3 implies
\[
\sup_{M \times [0,T)} |u_t| < +\infty.
\]
By Corollary 3.2, for all $t \in [0, T)$ and $l \in \mathbb{N} \cup \{0\}$, we have $\sup_M \left( |\nabla_l^l F_{\nabla_t}|^2 + |\nabla_l^l u_t|^2 \right)$ is uniformly bounded and so by Proposition 3.2, $\lim_{t \to T} (\nabla_t, u_t) = (\nabla_T, u_T)$ exists and is smooth. However, by local existence (Theorem 3.1), there exists $\epsilon > 0$ such that $(\nabla_t, u_t)$ exists over the extended domain $[0, T + \epsilon)$, which contradicts the assumption that $T$ was maximal. Thus we prove the theorem. \hfill \Box

4. Blow-up analysis

In this section, we will address the possibility that the Yang–Mills–Higgs $k$-flow admits a singularity given no bound on $|F_{\nabla_t}| + |\nabla_t u_t|$. To begin with, we first establish some preliminary scaling laws for the Yang–Mills–Higgs $k$-flow.

**Proposition 4.1.** Suppose $(\nabla_t, u_t)$ is a solution to the Yang–Mills–Higgs $k$-flow (1.3) defined on $M \times [0, T)$. We define the 1-parameter family $\nabla_t^\rho$ with local coefficient matrices given by

$$\Gamma_t^\rho(x) := \rho \Gamma_{\rho^{2(k+1)}t}(\rho x),$$

where $\Gamma_t(x)$ is the local coefficient matrix of $\nabla_t$. We define the $\rho$-scaled Higgs field $u_t^\rho$ by

$$u_t^\rho(x) := \rho u_{\rho^{2(k+1)}t}(\rho x).$$

Then $(\nabla_t^\rho, u_t^\rho)$ is also a solution to the Yang–Mills–Higgs $k$-flow (1.3) defined on $[0, \frac{1}{\rho^{2(k+1)}T})$.

Next we will show that in the case that the curvature coupled with a Higgs field is blowing up, as one approaches the maximal time, one can extract a blow-up limit. The proof will closely follow the arguments in [8, Proposition 3.25] and [18, Theorem 5.2].

**Theorem 4.1.** Assume $E$ is a line bundle. Suppose $(\nabla_t, u_t)$ is a solution to the Yang–Mills–Higgs $k$-flow (1.3) defined on some maximal time interval $[0, T)$ with $T < +\infty$. Then there exists a blow-up sequence $(\nabla_t, u_t)$ and converges pointwise to a smooth solution $(\nabla_t^\infty, u_t^\infty)$ to the Yang–Mills–Higgs $k$-flow (1.3) defined on the domain $\mathbb{R}^n \times \mathbb{R}_{<0}$.

**Proof.** From Theorem 3.2 we must have

$$\lim_{t \to T} \sup_{M} \left( |F_{\nabla_t}| + |\nabla_t u_t| \right) = +\infty.$$
Therefore, we can choose a sequence of times $t_i \nearrow T$ within $[0, T)$, and a sequence of points $x_i$, such that

$$|F_{\nabla t_i}(x_i)| + |\nabla_t u_{t_i}(x_i)| = \sup_{M \times [0,t_i]} \left( |F_{\nabla t}| + |\nabla_t u_t| \right).$$

Let $\{\rho_i\} \subset \mathbb{R}_{>0}$ be constants to be determined. Define $\nabla^i_t(x)$ by

$$\Gamma^i_t(x) = \rho_i^{-\frac{1}{2(k+1)}} \Gamma_{\rho_i t + t_i}(\rho_i^{-\frac{1}{2(k+1)}} x + x_i)$$

and

$$u^i_t(x) = \rho_i^{-\frac{1}{2(k+1)}} u_{\rho_i t + t_i}(\rho_i^{-\frac{1}{2(k+1)}} x + x_i).$$

By Proposition 4.1 $(\nabla^i_t, u^i_t)$ are also solutions to Yang–Mills–Higgs $k$-flow (1.3) and the domain for each $(\nabla^i_t, u^i_t)$ is $B_0(\rho_i^{-\frac{1}{2(k+1)}}) \times [-\frac{t}{\rho_i}, \frac{T - t}{\rho_i})$. We observe that

$$F^i_t(x) := F_{\nabla^i t}(x) = \rho_i^{-\frac{1}{2(k+1)}} F_{\nabla^i t + t_i}(\rho_i^{-\frac{1}{2(k+1)}} x + x_i),$$

which means

$$\rho_i \sup_{t \in [-\frac{t}{\rho_i}, \frac{T - t}{\rho_i})} \left( |F^i_t(x)| + |\nabla^i_t u^i_t(x)| \right)$$

$$\rho_i \sup_{t \in [0,t_i]} \left( |F_{\nabla t}(x)| + |\nabla_t u_t(x)| \right)$$

$$\rho_i^{-(k+1)} \left( |F_{\nabla t_i}(x_i)| + |\nabla_t u_{t_i}(x_i)| \right).$$

Therefore, setting

$$\rho_i = \left( |F_{\nabla t_i}(x_i)| + |\nabla_t u_{t_i}(x_i)| \right)^{-(k+1)},$$

which gives

$$1 = |F^i_0(0)| + |\nabla^i_0 u^i_0(0)| = \sup_{t \in [-\rho_i, 0]} \left( |F^i_t(x)| + |\nabla^i_t u^i_t(x)| \right). \quad (4.1)$$

Now, we are ready to construct smoothing estimates for the sequence $(\nabla^i_t, u^i_t)$. Let $y \in \mathbb{R}^n$, $\tau \in \mathbb{R}_{\leq 0}$. For any $s \in \mathbb{N}$,

$$\sup_{t \in [\tau - 1, \tau]} \left( |\gamma^s_\tau F^i_t(x)| + |\gamma^s_\tau \nabla^i_t u^i_t(x)| \right) \leq 1.$$ 

Note that $E$ is a line bundle, and similar to the proof of Theorem 3.2 it suffices to use Corollary 3.3. Then for all $q \in \mathbb{N}$, one may choose $s \geq (k+1)(q+1)$ so that
there exists positive constant $C_q$ such that
\[
\sup_{x \in B_y(\frac{1}{2})} \left( |(\nabla^i \tau)^{(q)} F^i_\tau(x)| + |(\nabla^i \tau)^{(q)} u^i_\tau(x)| \right)
\leq \sup_{x \in B_y(1)} \left( |\gamma^i_\rho(\nabla^i \tau)^{(q)} F^i_\tau(x)| + |\gamma^i_\rho(\nabla^i \tau)^{(q)} u^i_\tau(x)| \right)
\leq C_q.
\]
Then by the Coulomb Gauge Theorem of Uhlenbeck [14, Theorem 1.3] (also see [6]) and Gauge Patching Theorem [4, Corollary 4.4.8], passing to a subsequence (without changing notation) and in an appropriate gauge, $(\nabla_t^i, u^i_t) \to (\nabla^\infty_t, u^\infty_t)$ in $C^\infty$. 

5. Proof of Theorem 1.2

The following energy estimates are similar to the ones in [18, Section 6].

**Proposition 5.1.** Suppose $(\nabla_t, u_t)$ is a solution to the Yang–Mills–Higgs k-flow (1.3) defined on $M \times [0, T)$. The Yang–Mills–Higgs k-energy (1.1) is decreasing along the flow (1.3).

**Proposition 5.2.** Suppose $(\nabla_t, u_t)$ is a solution to the Yang–Mills–Higgs k-flow (1.3) defined on $M^4 \times [0, T)$ with $T < +\infty$, then the Yang–Mills–Higgs energy
\[
\mathcal{YMH}(\nabla_t, u_t) = \frac{1}{2} \int_M \left[ |F_{\nabla_t}|^2 + |\nabla_t u_t|^2 \right] d\text{vol}_g
\]
is bounded along the flow (1.3).

Next, we will complete the proof of Theorem 1.2. To accomplish this, we first show that the $L^p$-norm controls the $L^\infty$-norm by blow-up analysis.

**Proposition 5.3.** Assume $E$ is a line bundle. Suppose $(\nabla_t, u_t)$ is a solution to the Yang–Mills–Higgs k-flow (1.3) defined on $M^4 \times [0, T)$ and
\[
\sup_{t \in [0, T)} (\|F_{\nabla_t}\|_{L^p} + \|\nabla_t u_t\|_{L^p}) < +\infty.
\]
If $p > 2$, then
\[
\sup_{t \in [0, T)} (\|F_{\nabla_t}\|_{L^\infty} + \|\nabla_t u_t\|_{L^\infty}) < +\infty.
\]

**Proof.** So as to obtain a contradiction, assume
\[
\sup_{t \in [0, T)} (\|F_{\nabla_t}\|_{L^\infty} + \|\nabla_t u_t\|_{L^\infty}) = +\infty.
\]
As we did in Theorem 4.1, we can construct a blow-up sequence \((\nabla_i t, u_i t)\), with blow-up limit \((\nabla_\infty t, u_\infty t)\). Noting that \(4.1\), by Fatou’s lemma and natural scaling law,
\[
\|F_{\nabla_\infty t}\|_{L^p}^p + \|\nabla_\infty u_\infty \|_{L^p}^p \leq \lim_{i \to +\infty} \inf (\|F_{\nabla_i t}\|_{L^p}^p + \|\nabla_i u_i \|_{L^p}^p)
\]
\[
\leq \lim_{i \to +\infty} \rho_i^{\frac{2p-4}{2}} (\|F_{\nabla_i t}\|_{L^p}^p + \|\nabla_i u_i \|_{L^p}^p).
\]
Since \(\lim_{i \to +\infty} \rho_i^{\frac{2p-4}{2}} = 0\) when \(p > 2\), the right hand side of the above inequality tends to zero, which is a contradiction since the blow-up limit has non-vanishing curvature.

Now we are ready to give the proof of Theorem 1.2.

**Proof of Theorem 1.2** By the Sobolev embedding theorem, we solve for \(p\) such that \(W^{k,2} \subset L^p\), then \(k > 0\). In this case, using the interpolation inequalities \([9, \text{Corollary 5.5}]\) we have
\[
\|F_{\nabla_t}\|_{L^p} + \|\nabla_t u_t \|_{L^p}
\]
\[
\leq C S_{k,p} \sum_{j=0}^{k} (\|\nabla_t^{(j)} F_{\nabla_t} \|_{L^2}^2 + \|\nabla_t^{(j)} u_t \|_{L^2}^2 + 1)
\]
\[
\leq C S_{k,p} (\|\nabla_t^{(k)} F_{\nabla_t} \|_{L^2}^2 + \|F_{\nabla_t} \|_{L^2}^2 + \|\nabla_t^{(k+1)} u_t \|_{L^2}^2 + \|u_t \|_{L^2}^2 + 1)
\]
\[
\leq C S_{k,p} \mathcal{YMH}_k (\nabla_t, u_t) + \mathcal{YMH}_k (\nabla_t, u_t) + \|u_t \|_{L^2}^2 + 1).
\]
Then using Propositions 5.1, 3.3 and 5.2, we conclude that the flow exists smoothly for all time.

**Acknowledgments**

HS was supported by the Australian Research Council via grant FL170100020. PZ was supported by the Natural Science Foundation of Anhui Province [Grant Number 2108085QA17].

**References**

[1] A. Afuni, Local monotonicity for the Yang–Mills–Higgs flow, Calc. Var. PDE 55 (2016) 13.
[2] S. Alesker, E. Shelukhin, On a uniform estimate for the quaternionic Calabi problem, Israel J. Math. 197 (2013) 309-327.
[3] M. Atiyah, R. Bott, The Yang-Mills equations over Riemann surfaces, Philos. Trans. R. Soc. Lond. 308 (1982) 523-615.
[4] S.K. Donaldson, P. Kronheimer, The Geometry of Four-Manifolds, Oxford University Press, 1990.
A new higher order Yang–Mills–Higgs flow in Riemannian 4-manifold

[5] A. Hassell, The Yang–Mills–Higgs heat flow on $\mathbb{R}^3$, J. Funct. Anal. 111 (1993) 431-448.
[6] M.C. Hong, G. Tian, Global existence of the $m$-equivariant Yang–Mills flow in four dimensional spaces, Comm. Anal. Geom. 12 (2004) 183-211.
[7] M.C. Hong, G. Tian, Asymptotical behavior of the Yang–Mills flow and singular Yang–Mills connections, Math. Ann. 330 (2004) 441-472.
[8] C. Kelleher, Higher order Yang–Mills flow, Calc.Var. 100 (2019), Article number: 100.
[9] E. Kuwert, R. Schätzle, Gradient flow for the Willmore functional, Commun. Anal. Geom. 10 (2002) 307-339.
[10] J. Li, X. Zhang, The limit of the Yang–Mills–Higgs flow on Higgs bundles, Inter. Math. Res. Not. 2017 (2016) 232-276.
[11] H. Saratchandran, Higher order Seiberg-Witten functionals and their associated gradient flows, Manuscripta Math. 160 (2019) 411-481.
[12] C. Song, C. Wang, Heat flow of Yang–Mills–Higgs functionals in dimension two, J. Funct. Anal. 272 (2017) 4709-4751.
[13] S. Trautwein, Convergence of the Yang–Mills–Higgs flow on gauged holomorphic maps and applications, Inter. J. Math. 29 (2018) 1850024.
[14] K.K. Uhlenbeck, Connections with $L^p$-bounds on curvature, Commun. Math. Phys. 83 (1982) 31-42.
[15] A. Waldron, Long-time existence for Yang-Mills flow, Inventiones mathematicae 217 (2019) 1069-1147.
[16] G. Wilkin, The reverse Yang–Mills–Higgs flow in a neighbourhood of a critical point, J. Differ. Geom. 115 (2020) 111-174.
[17] C. Zhang, P. Zhang, X. Zhang, Higgs bundles over non-compact Gauduchon manifolds, Trans. Amer. Math. Soc. 374 (2021) 3735-3759.
[18] P. Zhang, Gradient flows of higher order Yang–Mills–Higgs functionals, J. Austr. Math. Soc., 2021.