NEUTRINO-EXCHANGE INTERACTIONS IN 1-, 2-, and 3-DIMENSIONS

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ABSTRACT

We examine several recent calculations of the self-energy of a neutron star arising from neutrino-exchange. It is shown that the results of Abada, et al. in 1+1 dimensions have no bearing on a 3-dimensional neutron star, since the criticality parameter $G_F N/R^2$ is always much smaller than unity in 1+1 dimensions. The calculation of Kiers and Tytgat in 3-dimensions is shown to disagree with the lowest order 2-body contribution, which is known exactly. This discrepancy raises the possibility that the description of a neutron star as a continuous medium may be inappropriate when calculating higher-order many-body effects. We conclude that none of the recent calculations contradict the earlier claims that the neutrino-exchange contributions to the self-energy of a neutron star are unphysically large, when calculated in the standard model. The implication of this result, that neutrinos must have a non-zero mass, $m_\nu > 0.4 \text{ eV}/c^2$, remains intact.
It has been argued that many-body forces arising from the exchange of massless neutrinos can lead to an unphysically large energy density in white dwarfs and neutron stars [1, 2]. Intuitively this comes about because in the absence of any other physical parameters, the $k$-body ($k = 2, 4, 6, ...$) contribution $W^{(k)}$ to the binding energy $W$ of a neutron star of radius $R$ must be of the form \[1, 2\]

\[W^{(k)} \propto \frac{G_F^k}{R^{2k+1}} \binom{N}{k}.\] (1)

Here $G_F$ is the Fermi constant and $\binom{N}{k}$ is the binomial coefficient,

\[\binom{N}{k} = \frac{N!}{k!(N-k)!} \cong \frac{N^k}{k!} \quad \text{for} \quad N \gg k.\] (2)

Combining Eqs. (1) and (2) we find \[1, 2\]

\[W^{(k)} \sim \frac{1}{k!} \frac{1}{R} \left( \frac{G_F N}{R^2} \right)^k.\] (3)

For a typical neutron star $(G_F N/R^2) = \mathcal{O}(10^{13})$, from which it follows that for $k \ll N$ higher-order many-body interactions make increasingly larger contributions to $W^{(k)}$ \[3\]. The energy, $W = \Sigma_k W^{(k)}$ can exceed the mass of the neutron star, as is shown explicitly in Ref. [1], and this eventually leads to the conclusion that neutrinos must have a minimum mass, $m_\nu > 0.4$ eV/c$^2$. The effect of a nonzero $m_\nu$ is to produce a “saturation” of the neutrino-exchange force, and ultimately a physically acceptable value for $W$.

Following the publication of Ref. [1], a number of papers appeared dealing with various questions raised by this calculation [4-10]. Here we focus on attempts to calculate the energy density $w$ non-perturbatively starting from variants of the Schwinger formula [11] used in Ref. [1]:

2
\[ W = \frac{i}{2\pi} \text{Tr} \left\{ \int_{-\infty}^{\infty} dE \, \ell \ln \left[ 1 + \frac{G_F a_n}{\sqrt{2}} N_\mu \gamma_\mu \left( 1 + \gamma_5 \right) S_F^{(0)}(E) \right] \right\}. \]  

(4)

In Eq. (4) \( a_n = -1/2 \) is the neutrino-neutron coupling constant, \( N_\mu \) is the neutrino current, and \( S_F^{(0)}(E) \) is the free neutrino propagator. In the notation of Ref. [1], \( N_\mu = i \rho(\vec{x}) \delta_\mu 4 \), where \( \rho(\vec{x}) \) is the neutron density, with
\[ \int d^3 x \rho(\vec{x}) = N = O(10^{57}). \]

The perturbative method of Ref. [1] treats neutrons as discrete particles, and \( W \) can be calculated for any \( \rho(\vec{x}) \). Our results, to be presented elsewhere show that the result in Ref. [1] is robust: For a variety of symmetric and asymmetric matter distributions the calculation of \( W \) given in Ref. [1] leads to an unphysically large value for massless neutrinos.

By constrast the non-perturbative calculations treat the neutron star as a continuous medium whose density is constant, either throughout all space [4, 8] or in some sub-region [7, 10]. Since various cancellations occur in both approaches, one explanation for the differences among the results may be the different pictures used to describe a neutron star (i.e. discrete neutrons versus a continuous medium), as we discuss in more detail elsewhere. Along with other considerations, treating the neutron star as an object of finite extent is essential.

Two recent papers have attempted to calculate \( W \) for a “neutron star” of finite extent, rather than calculating the energy density of an infinite medium. Abada, et al. [7] have calculated the effect of many-body neutrino exchange in a 1+1 dimensional “neutron star”, and Kiers and Tytgat [10] have carried out a similar calculation in 3+1 dimensions.

Returning to Eqs. (1)-(3) we note from the heuristic derivation of Eq. (3) — or from the formal treatment in Ref. [1] — that \( W^{(k)} \) will have the same functional form in any number of spatial dimensions. The differences among
the 1-, 2-, and 3-dimensional cases arise when \( (G_F N/R^2) \) is re-expressed in terms of the corresponding linear, surface and volume densities:

1-dimension: For a linear array of length \( L \) and density \( \lambda \), \( N = \lambda L \) and \( R = L/2 \) so that

\[
G_F N/R^2 = 4G_F \lambda /L .
\]

(5)

The maximum value of \( \lambda \) corresponds to taking the inter-neutron spacing to be given by the hard core radius \( r_c = 0.5 \times 10^{-13} \) cm, which gives \( \lambda = 2 \times 10^{13} \) cm\(^{-1} \). Taking \( R \equiv R_{10} = 10 \) km then gives

\[
G_F N/R_{10}^2 \big|_{1\text{-dim}} = 2 \times 10^{-25} .
\]

(6)

Hence in 1-dimension the “criticality parameter” \( G_F N/R^2 \) in Eq. (3) is always much less than unity, from which one could conclude at the outset that \( W \) would always remain acceptably small. Physically this comes about because in 1-dimension there are too few neutrons in the neighborhood of a given neutron to allow the combinatoric factor \( \binom{N}{k} \) to provide a sufficiently large enhancement.

From Eq. (6) we recognize that many-body \((k > 2)\) effects are supressed in 1-dimension, and hence the leading contribution to the energy comes from the 2-body neutron-neutron potential [1, 2, 12-16],

\[
V^{(2)}(r) = \frac{G_F^2 a_n^2}{4\pi^3 r^5} .
\]

(7)

The average interaction energy \( U^{(2)}_{1\text{-dim}} \) of a pair of neutrons in a linear array of length \( L \) would be given by

\[
U^{(2)}_{1\text{-dim}} = \int_{r_c}^{L} dr P_1(r) \frac{G_F^2 a_n^2}{4\pi^3 r^5} .
\]

(8)
Where $P_1(r)$ is the 1-dimensional probability density given by [1]

$$P_1(r) = \frac{2}{L} - \frac{2r}{L^2}. \quad (9)$$

The leading term to the average energy is found by combining Eqs. (8) and (9),

$$U_{1-\text{dim}}^{(2)} = \frac{G_F^2 a_n^2 \lambda^2 L}{8\pi^3 r_c^4 L}. \quad (10)$$

We note that this result is unaffected by inclusion of any contributions from a trapped neutrino sea, since the 2-body potential is dominated by small values of $r$ [9]. For a linear array of $N$ discrete neutrons, there are approximately $N^2/2$ pairs that can be formed, hence the energy in 1-dimension is

$$W_{1-\text{dim}} = W_{1-\text{dim}}^{(2)} = \frac{G_F^2 a_n^2 \lambda^2 L}{16\pi^3 r_c^4}. \quad (11)$$

Consequently, the energy density $w_{1-\text{dim}}$ is a constant, given by

$$w_{1-\text{dim}} = \frac{G_F^2 a_n^2 \lambda^2}{16\pi^3 r_c^4}. \quad (12)$$

This result conflicts with the 1-dimensional results of Abada et al. [7] who find $w_{1-\text{dim}}$ to be identically zero. We note that since the Abada et al. results can be expanded in a power series in $G_F$, they must agree with the lowest-order perturbative results given in Eqs. (11) and (12).

2-dimensions: For a planar circular array of neutrons with surface density $\sigma$, we have $N = \sigma \pi R^2$ so that

$$G_F N/R^2 = \pi G_F \sigma = \text{constant}. \quad (13)$$

Assuming, as before, a hard core radius $r_c = 0.5 \times 10^{-13}$ cm gives $\sigma = 1 \times 10^{26}$ cm$^{-2}$, and for $R = R_{10} = 10$ km we find
\[ G_F N/R_{10}^2 \mid_{2-\text{dim}} = 2 \times 10^{-6}. \tag{14} \]

Since this is again smaller than unity, we conclude that the neutrino-exchange energy \( W \) is well-behaved in 2-dimensions and dominated by the 2-body contribution. The energy density in 2-dimensions \( w_{2-\text{dim}} \), can be obtained in a similar manner to the 1-dimensional result of Eq. (12), and we find

\[ w_{2-\text{dim}} = \frac{G_F^2 a_n^2 \sigma^2}{12\pi^2 r_c^3}. \tag{15} \]

**3-dimensions:** For a spherical volume of radius \( R \) and number density \( \rho \) we have in 3-dimensions \( N = \frac{4}{3}\pi R^3 \rho \), which gives

\[ G_F N/R^2 = (4/3)\pi G_F \rho R. \tag{16} \]

Using \( \rho = 4 \times 10^{38} \text{ cm}^{-3} \) from Ref. [1] gives

\[ G_F N/R_{10}^2 \mid_{3-\text{dim}} = 8 \times 10^{12}, \tag{17} \]

and hence it is only in 3 spatial dimensions that the possibility exists for an unphysically large energy density arising from the exchange of massless neutrinos. Thus the recent calculation of Abada, *et al.*, which is done in 1-dimension, has little bearing on the calculation in Ref. [1].

We turn next to the paper of Kiers and Tytgat (KT) [10] which also calculates \( W \) under the assumption that a neutron star can be described as a continuous medium. KT work in a finite 3-dimensional volume, and hence their starting point is closest to that of Ref. [1] amongst the recent papers which use the continuous medium approximation. Since KT do not find a large value for \( W \), in contrast to Ref. [1], it is instructive to compare the discrete and continuum calculations of \( W \).
For present purposes we focus on the 2-body contribution $W^{(2)}$ to the self-energy of a neutron star $W$, which can be calculated \textit{exactly} with no approximations \cite{2}. The 2-body potential is given by Eq. (7). The average interaction energy $U^{(2)}$ of a pair of neutrons having a uniform density distribution in a spherical volume of radius $R$ is then given by \cite{1, 2}

$$U^{(2)} = \int_{r_c}^{2R} dr P(r) \frac{(G_F a_n)^2}{4\pi^3 r_c^5},$$

(18)

where $P(r)$ is the probability density function\footnote{The normalization factor $\eta(r_c, R)$ was approximated by unity in Refs. \cite{1} and \cite{2}.} given by \cite{1, 2}

$$P(r) = \frac{3r^2}{R^3} \left[ 1 - \frac{3}{2} \left( \frac{r}{2R} \right) + \frac{1}{2} \left( \frac{r}{2R} \right)^3 \right] \eta(r_c, R),$$

(19a)

$$\eta(r_c, R) = \frac{1}{1 - 8 \left( \frac{r_c}{2R} \right)^3 + 9 \left( \frac{r_c}{2R} \right)^4 - 2 \left( \frac{r_c}{2R} \right)^6}.$$ 

(19b)

Combining Eqs.(18) and (19) we find that the energy per pair of neutrons, $U^{(2)}$ is \textit{exactly} given by

$$U^{(2)} = \frac{3}{8\pi^3} \frac{(G_F a_n)^2}{\hbar c} \frac{1}{R^3 r_c^2} \left( 1 - \frac{r_c}{2R} \right)^3 \eta(r_c, R),$$

(20)

where we have reinstated the factor $\hbar c$. For a neutron star containing $N$ discrete neutrons, there are $N(N-1)/2$ pairs and hence the 2-body contribution $W^{(2)}$ to $W$ is given by

$$W^{(2)} = \frac{3}{16\pi^3} \frac{(G_F a_n)^2}{\hbar c} \frac{N(N-1)}{R^3 r_c^2} \left( 1 - \frac{r_c}{2R} \right)^3 \eta(r_c, R),$$

(21)

which is also an \textit{exact} result. Since $N \gg 1$ for neutron star, we can replace $N(N-1)$ by $N^2$, and re-express $N$ in terms of the number density $\rho$: 

\[\text{7}\]
\[ W^{(2)} = \frac{1}{3\pi} \frac{(G_F a_n \rho)^2}{\hbar c} \frac{R^3}{r_c^2} \left(1 - \frac{r_c}{2R}\right)^3 \eta(r_c, R). \]  

(22)

Also of interest is the energy density \( w^{(2)} = W^{(2)}/V \),

\[ w^{(2)} = \frac{1}{4\pi^2} \frac{(G_F a_n \rho)^2}{\hbar c r_c^2} \left(1 - \frac{r_c}{2R}\right)^3 \eta(r_c, R). \]  

(23)

It follows from Eq. (23) that the energy density \( w^{(2)} \) is an increasing function of \( R \), and approaches a constant as \( R \to \infty \).

We now compare these results to those obtained by KT for the 2-body contribution. KT find for \( W^{(2)} \)

\[ W^{(2)}_{\text{KT}} \approx \left( \frac{G_F \rho a_n}{\sqrt{2}} \right)^2 R^2 \Lambda, \]  

(24)

where \( \Lambda \) is a cutoff which plays a role similar to \( r_c \) in Eq. (21). KT then use Eq. (24) to conclude that

\[ w^{(2)}_{\text{KT}} \approx \left( \frac{G_F \rho a_n}{\sqrt{2}} \right)^2 \frac{\Lambda}{R} \xrightarrow{R \to \infty} 0. \]  

(25)

Comparing Eqs. (24) and (25) to Eqs. (22) and (23), respectively, we see that the \( R \)-dependence of the KT results disagrees with that of the exact results obtained by assuming discrete neutrons. Since the KT results can be expanded in a power series in \( G_F \), the lowest order KT results must agree with the corresponding perturbative results given in Eqs. (22) and (23). Furthermore the KT results are seen to underestimate \( W^{(2)} \) and the energy density \( w^{(2)} \) as \( R \) increases. This is a particularly troublesome point since it raises the possibility that the many-body contributions may be similarly underestimated in the KT formalism.
It should be emphasized that the discrepancy between the KT results and the exact results is a matter of principle, and cannot be dismissed on the grounds that $W^{(2)}$ is in either case small. Until the origin of this discrepancy is fully understood one cannot be certain that it does not affect other parts of their calculation as well.

CONCLUSIONS

The recent papers by Abada, et al. [7], and by Kiers and Tytgat [10] have been analyzed. We have shown explicitly that the results of Ref. [7] in 1-dimension have no bearing on whether the neutrino-exchange energy density in 3-dimensions is unphysically large in a neutron star. The KT calculation, although carried out in 3-dimensions, fails to reproduce the 2-body result which is known exactly and, moreover, underestimates this contribution. Obtaining the correct 2-body result is a litmus test for the validity of any calculational scheme, and hence the significance of the KT calculation is unclear at present. One question that must be explored is the validity of approximating the neutron star by a continuous medium. Although such a picture is routinely employed in discussing the MSW effect (in which $G_F$ enters only in lowest order), its appropriateness for calculating self-energy interactions involving higher-order weak processes remains to be demonstrated. Another question that remains to be understood is where the medium description breaks down: Clearly an atomic nucleus should be described in terms of discrete neutrons for purposes of calculating $W$. Since a neutron star behaves in some way as a big nucleus, the transition from a description in terms of discrete neutrons to one in terms of a medium must be fully understood. These and other related questions will be explored in more detail elsewhere.
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