Model and Set-Theoretic Aspects of Exotic Smoothness Structures on $\mathbb{R}^4^*$

Jerzy Król
University of Silesia, Institute of Physics,
ul. Uniwersytecka 4, 40-007 Katowice, Poland,
iriking@wp.pl

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Abstract

Model-theoretic aspects of exotic smoothness were studied long ago uncovering unexpected relations to noncommutative spaces and quantum theory. Some of these relations were worked out in detail in later work. An important point in the argumentation was the forcing construction of Cohen but without a direct application to exotic smoothness. In this article we assign the set-theoretic forcing on trees to Casson handles and characterize small exotic smooth $\mathbb{R}^4$ from this point of view. Moreover, we show how models in some Grothendieck toposes can help describing such differential structures in dimension 4. These results can be used to obtain the deformation of the algebra of usual complex functions to the noncommutative algebra of operators on a Hilbert space. We also discuss the results in the context of the Epstein-Glaser renormalization in QFT.

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1 Infinite geometric constructions and set-theoretic forcing

Currently it is a bit of a folklore to say that dimension 4 is exceptional both in physics and mathematics. On the one hand this is the dimension where Einstein theories of relativity were formulated, where the physics of particles and quantum fields found their marvelous realization on (curved) Minkowski spacetimes, and where the cosmological evolution of our world is to be described. On the other hand, many curious mathematical facts, like the existence of exotic $R^4$, or in fact, of a continuum many of them, take place exactly in this dimension.

It was a big effort of many mathematicians in 1980’s like Donaldson, Freedman, Gompf, Taubes and many others whose work on topology and geometry of manifolds in dimension 4 opened our eyes on the unique 4-dimensional topological and ‘smooth’ world and help in its understanding. However, taking seriously advanced and technical mathematical findings as applicable to physics, required much scientific imagination and courage in those days. It was Carl Brans who took the step in a series of papers \cite{8, 9, 10, 11}. Soon after, there appeared the work of Torsten Asselmeyer-Maluga (e.g. \cite{1}) and Jan Sladkowski (e.g. \cite{42, 43}) who approached the role of exotic $R^4$’s in physics from various perspectives. Carl’s Brans ideas and the papers above were an inspiration to me and I have been lucky as a researcher to work together with Torsten and Jan within the recent years. It is a big honor and pleasure to me to contribute to the volume celebrating the work of Carl Brans.

Exotic smoothness structures on $R^4$ are just Riemannian, curved smooth 4-manifolds (exotic $R^4$) which topologically are (homeomorphic to) $R^4$. In this chapter, I will show that the perspective of set theory and Grothendieck toposes, hence foundations of mathematics, is the right one when considering physical applications of exotic, open 4-smoothness. Even though this is neither obvious nor widely accepted approach, the use of model and set-theoretic methods in physics has a firm and vivid tradition arisen from the foundations of mathematics (e.g. \cite{39, 12, 44, 29}). That was developed substantially further in recent years (e.g. \cite{13, 14, 15, 25, 23, 31}).

In physics, set theory is usually considered informally as unchanged eternal background which goes together with the classical 2-valued logic. However, when one allows for variations in such background more formal, axiomatic formulation is needed. That is why set the-
ory is understood as the first order axiomatic Zermelo-Fraenkel (ZF) theory of sets with possible addition of the axiom of choice (AC) – ZFC. Similarly arithmetic is usually described as an axiomatic first order theory - Peano arithmetic (PA) (see the discussion regarding the order of formal theories vs. set theory in [45]). The variations in the theories can be grasped by considering various models of these theories. Classically such models (Tarski) are built in the category Set of sets and functions between them. All models of first order theories undergo usual limitations and benefits which follow the Goedel or Loewenheim-Skolem-like theorems (and much more, see e.g. [26]). We also will be using more general models of (intuitionistic) set theory in other categories like toposes where the logic becomes intuitionistic [35].

The forcing method is known from the independence results in set theory since 1960’s [17] and allows for changing the models. In general, forcing in mathematics is a very rich, technical and advanced subject (see e.g. [26] [7]). For the purpose of this work it is a method for studying the real numbers line. Thus Cohen forcing in a narrow sense used in the chapter can be seen as a mechanism of adding real numbers to the model and thus changing the model of ZF(C) and the real line. This is also a tool for exploring the exotic smooth $R^4$’s (see e.g. [31] [32] [30]).

We start with infinities appearing in some geometric constructions in dimensions 3 and 4 like Casson handles and Alexander’s horned sphere (wild embeddings). These infinities are the inevitable and intrinsic features of the constructions. On the other hand, infinity by itself is a natural and central topic in set theory. The key for understanding this relation is precisely the Cohen forcing. On the algebraic level a forcing is generated by some complete atomless Boolean algebra - in this case the forcing is nontrivial and can eventually add some reals to the ground model $M$ of ZFC. In the case of Cohen forcing the algebra is the unique atomless Boolean algebra with a dense countable subset. In fact it holds true:

**Lemma 1 (Corollary 25.4, p. 189 [27])** Let $A$ be a complete atomless Boolean algebra that contains a countable dense subset. Then $A$ is isomorphic to the algebra $\text{RO}(CS)$ of regular open subsets of the Cantor set $CS$.

Any (signed) tree canonically generates a partial order (partially ordered set). A partial order $(\mathbb{P}, \leq)$ is called separative if for all $p, q \in \mathbb{P}$
such that $p \not< q$ there exists $r \leq p$ with $r \perp q$. Here $r \perp q$ means incompatibility relation i.e. there does not exist $k$ that neither $q \leq k$ nor $r \leq k$ is true. Then, it holds true the important lemma:

**Lemma 2 (Lemma 13.33, [27])** Every separative partial order $\mathbb{P}$ can be completed to a complete Boolean algebra $B$ such that $\mathbb{P}$ is dense in $B \setminus \{0\}$ and the partial order in $\mathbb{P}$ agrees with $\leq_B$. $B$ is unique up to isomorphism.

Next we ask the question: which rooted trees do represent a separative partial order? One easily finds that the full binary tree (the one which has precisely 2 branches at every node) does. Moreover:

**Lemma 3** The full binary tree represents the countable dense subset (partial order) of some complete atomless Boolean algebra.

This is because the full binary tree represents the Cantor set in $(0, 1)$ interval: one assigns to every branch 0 or 2 numbers which appear in the three-mal decompositions $0.x_1x_2x_3...$ of numbers in $(0, 1)$. Then missing numbers correspond precisely to $x_i = 1, i = 1, 2, 3,...$. The nodes of the tree represent the members of the countable partial order which is dense in the partial order of the tree hence in the corresponding Boolean algebra. The algebra is RO($\text{CS}$) which is atomless and generates the nontrivial Cohen forcing. □

Now the point is that the Cantor set generated by the binary tree is frequently realized geometrically by Casson handles construction in dimension 4 and by wild embeddings of spheres in dimension 3 (see e.g. [21]). Casson handles (CH) (see e.g. [20, 22, 2]) appear in the handle-body decompositions of small exotic smooth open 4-manifolds [22] are also represented by the infinite signed rooted trees [22][10]. If the tree was finite and the CH smooth, the Casson handle would be the ordinary smooth 2-handle\[1\].

Let me quote an important and elementary observation by Kato ([28], p. 114) which ensures that given a signed tree we have a Casson handle spanned on that tree:

There are sufficiently many Casson handles. In fact to each infinite signed tree, one can associate a Casson handle.

Let $M$ be a model of ZFC and $M[G]$ its generic extension by Cohen forcing [26][7]. Then we can prove the following:

---

\[1\]Every CH is topologically (as a pair) homeomorphic to the standard 2-handle which was shown by Freedman [20].
Theorem 1: A general Casson handle appearing in the handlebody of a small exotic $\mathbb{R}^4$ determines a nontrivial Cohen forcing adding a Cohen real in some generic model $M[G]$ of ZFC.

proof: First, any Casson handle can be embedded in the simplest CH which is the linear tree with only one, positive or negative, self-intersection at each level. This follows from the fact that every CH with a bigger signed tree than the tree of another CH is embeddable in this ‘smaller-tree-CH’. One should respect the rule that the smaller tree is homeomorphically embedded into the bigger one. Adding self-intersections on any level and killing the generators by gluing kinky handles determines the embedding. Moreover, the resulting embeddings of CH’s preserves the attaching areas of CH (or at least attaching circles and their framings). The last means that whenever the simplest CH were exotic (the attaching circle determines the non-smooth slice) the embedded CH with a bigger tree would be exotic too [16].

Second, instead of attaching an arbitrary CH let us attach the simplest one (see figs. 2 and 1) with the linear signed tree in which we know the bigger one is embeddable. In general we do not know whether the CH with such a tree is exotic although we know it is exotic for the ‘only +’ or ‘only -’ trees.

Next, let us consider the Casson handle determined by the full binary tree (BT) with one infinite branch identical with the linear one above. Such ‘binary-tree-CH’ embeds in the linear CH and let us forget the signs in the binary CH. Then from Lemmas 3 and 1 the algebra $\text{RO}(CS)$ is the unique Cohen forcing algebra generated by BT. □

Note that every CH determines the same (up to isomorphism) Cohen algebra thus the nontrivial Cohen forcing in a generic model $M[G]$. In dimension 3 given wildly embedded 3-sphere, say horned Alexander sphere, a ‘grope’ is assigned naturally to it which is spanned on the infinite binary tree again ([21], pp. 18-19). Thus Cohen forcing can be built also in this case. We do not discuss the meaning of it here but note only that wild embeddings in dimension 4 are second sides of exotic open 4-smoothness and this can be understood physically as a quantum state [4, 5].

Cohen forcing changes the real line substantially, namely the reals in the model $M$ constitute merely measure zero subset of the extended real line in $M[G]$, hence of $\mathbb{R}$. As shown above it is also assigned to replacing the standard smooth 2-handles by an exotic Casson handle, hence to changing the smoothness structures on $\mathbb{R}^4$. If the forcing
acted over \( \mathbb{R} \)-line in \( \mathbb{R}^4 \) and resulted in exotic \( \mathbb{R}^4 \) the following important question would arise: Can an extension of the real line by forcing be a valid tool when exploring exotic smoothness in dimension 4? In some sense this kind of forcing should add reals to the full \( \mathbb{R} \) resulting in the same \( \mathbb{R} \) since \( \mathbb{R}^4 \) is again the Riemannian smooth real manifold. We will analyze this problem in the next and subsequent sections.

2 From the standard to categorical \( \mathbb{R}^4 \)

One needs 'adding' more real numbers to the already full \( \mathbb{R} \). What is the meaning of such procedure? We will show that the modification of logic and set theory is needed.

From the external absolute point of view a set-theoretic forcing adds reals (if any at all) to subsets \( R_M \) of \( \mathbb{R} \) where \( R_M \) is a set of real numbers in some model \( M \) of ZFC. Internally there is no difference between (1st order) properties of real lines \( R_M \) and \( \mathbb{R} \). Suppose that we already have a well defined model of the standard real line \( \mathbb{R} \). Starting with \( \mathbb{R} \) can one add consistently more reals to the line? More precisely: can one construct a bigger real line which would have the same properties as \( \mathbb{R} \) but be different as a set (thus containing more reals)? Our general motivation for considering such questions, as observed in the 1st section, is that we expect such procedure to possibly modify the smoothness of manifolds.

Reducing the properties of the real line to its 1st order properties, and the logic to first order logic, Robinson showed \cite{40} that there are non-standard models of arithmetic \(*N\) and analysis \(*R\). They are end-extensions of the standard \( \mathbb{N} \) and \( \mathbb{R} \) respectively and contain infinite natural and real numbers. Moreover, \(*R\) contains infinitesimal invertible real elements. Now, every true 1st order formula \( \phi \) about natural numbers is fulfilled in \(*N\) iff it is fulfilled in \( \mathbb{N} \), i.e. \(*N \models \phi \equiv \mathbb{N} \models \phi\). We say that \(*N\) and \( \mathbb{N} \) are elementary equivalent and write:

\[
* \mathbb{N} \simeq_1 \mathbb{N} \ (\ * \mathbb{R} \simeq_1 \mathbb{R} ),
\]

meaning, one can not distinguish the two models just by their 1st order properties. We would like to strengthen the indistinguishability

\footnote{A formal theory giving rise to the unique up to isomorphism model of real numbers should use the 2nd order logic. Such theories are called categorical (in \( \aleph_1 \)). The theory of Archimedean complete ordered field is categorical. It is a second order theory.}
as above and consider something like $^*N \simeq_{2,3,...} \mathbb{N}$ ($^*R \simeq_{2,3,...} \mathbb{R}$).\footnote{It would be sufficient to consider $^*N \simeq_2 \mathbb{N}$ since there are theorems reducing the higher order to 2nd order logic (e.g. \cite{24,36}).}

It is seemingly a trivial task, since 2nd order theory of natural or real numbers are categorical and the real line $\mathbb{R}$ is the only (up to isomorphism) model allowed, hence indeed $^*N \simeq_{2,3,...} \mathbb{N}$.

That is why we are rather looking for an environment (the twist) where non-standard models for arithmetic and analysis may exist, are nontrivial, i.e. different, and are valid for higher order theories, i.e. some second order properties of the models become identical after the twisting. Without any twist these particular properties would not coincide. As noted above we can not achieve the nontrivial realization of the full classical indistinguishability $^*N \simeq_2 \mathbb{N}$ ($^*R \simeq_2 \mathbb{R}$) since 2nd order arithmetic has isomorphic models.

To imagine how the twist could work one can introduce three parameters ($w, \alpha, \epsilon$) controlling the twist - $w$ corresponds to the weakening of the arithmetic and/or the logic, and the other two to the fractions (belonging to $(-1, 1)$) of the numbers of all true formulas of the first and second orders correspondingly. $\alpha = 0$ and $w = 0$ mean that all true 1st order formulas of both models, ($^*R$ and $\mathbb{R}$), are determined with respect to the first order (i.e. $\alpha = 0$) classical (i.e. $w = 0$) predicate logic. Similarly, $\epsilon = 0$ and $w = 0$ mean that all second order formulas of the models are determined w.r.t. the classical second order logic. Thus one writes

$$^*N \simeq^{w}_{1-\alpha,1+\epsilon} \mathbb{N} (^*R \simeq^{w}_{1-\alpha,1+\epsilon} \mathbb{R})$$

(2)

when the logic is weakened and the sets of the first order formulas and second order formulas have been modified and especially some 2nd order formulas become identical in both models after the twist. The $+, -$ signs indicate the twist or the rotation in the parameter space. The value of the parameters depends on the degree of how much of weak and nonclassical logic is used. We do not need to determine the relation between the parameters more precisely here. Instead, let us consider the important example. We will weaken the logic and arithmetic considerably and take the models in a constructive set-up, i.e. in toposes.

This weak Peano arithmetic was recognized in detail by Moerdijk and Reyes \cite{37} when they considered the non-standard models of numbers in smooth toposes and build the smooth topos model for synthetic
differential geometry. We present the discussion of the elements of their construction important for us in the Appendix 4.2.

The important point is that the objects of natural numbers (NNO) in smooth toposes like Zariski ($\mathcal{Z}$) and Basel topos ($\mathcal{B}$) determined by the natural embedding of manifolds from Set to the toposes, i.e. the map $s : \mathbb{M} \to \mathcal{Z}$, sends $\mathbb{N}$ to the standard natural numbers $s(\mathbb{N})$ in $\mathcal{Z}$, $\mathcal{B}$
 fails to generate a proper object of real numbers $s(\mathbb{R}) = R_\mathcal{Z}$ (or $R_\mathcal{B}$). For example: $R_\mathcal{Z}$ is nonarchimedean with respect to $s(\mathbb{N})$ so thus (21) does not hold. Besides $[0,1] \subset R_\mathcal{Z}$ is noncompact with respect to $s(\mathbb{N})$. As the consequence this last property devastates the homology theory of manifolds in $\mathcal{Z}$ ([37], pp. 280-284.).

To cure this one should turn to the modified object of natural numbers $N_\mathcal{Z}$ (smooth natural numbers) which is not the canonical standard NNO $s(\mathbb{N})$ in $\mathcal{Z}$. As shown by Moerdijk and Reyes the axioms of the weak logic (17,18,19) are fulfilled in $\mathcal{Z}$ however the type $N$ is interpreted now as $N_\mathcal{Z}$ i.e. it is the smooth NNO. $R_\mathcal{Z}$ is now Archimedean w.r.t. $N_\mathcal{Z}$, $[0,1]$ is compact (smooth compact, or s-compact), the homologies of manifolds are tractable and in particular the internal topologies of manifolds in $\mathcal{Z}$ are well-defined. Internal in $\mathcal{Z}$ constructions and theories are formulated such as the true natural numbers are $N_\mathcal{Z}$ rather than the standard $s(\mathbb{N})$. The shift $s(\mathbb{N}) \to N_\mathcal{B}$ changes some second order properties of real and natural numbers such that now in $\mathcal{Z}$ internal constructions are more like the external ones.

The construction of $N_\mathcal{Z}$ follows the filterproduct construction. Namely, the object $R_\mathcal{Z} \simeq s(\mathbb{R})$:

$$R_\mathcal{Z} = s(\mathbb{R}) = L(-, lC^\infty(\mathbb{R}))$$

is the representable object of $\mathcal{Z}$ [37]. It is non-archimedean with respect to $s(\mathbb{N})$ as said above. Instead one defines the object of smooth natural numbers $N_\mathcal{Z}$ thus allowing for the modification of finiteness. Let $(\sin(\pi x))$ be the ideal in $C^\infty(\mathbb{R})$. The representable object in $\mathcal{Z}$ of smooth integer numbers $Z_\mathcal{Z}$ is now defined as ([37], p. 252)

$$Z_\mathcal{Z} = l(C^\infty(\mathbb{R})/(\sin \pi x)), \quad N_\mathcal{Z} = l(C^\infty(\mathbb{R})/(\sin \pi x, x \geq 0)) \quad (3)$$

As the object in a topos this standard NNO is the constant sheaf of natural numbers.

$l(\cdot)$ is the member of $L$ – the category of loci which is opposite to the category of (finitely generated) smooth rings ([37], p. 58).
Taking the ideal $F$ of functions which are non-zero only on finite initial segments of $\mathbb{N}$, then the quotient $l(C^{\infty}(\mathbb{N})/F)$ represents a non-standard infinite natural number in $\mathcal{Z}$.

To have the standard $s(\mathbb{N}) \simeq \mathbb{N}$ one can define it as the subtype of $N_{\mathcal{Z}}$:

$$N = \{ n \in N_{\mathcal{Z}} : \forall S \in P(N_{\mathcal{Z}})(0 \in S \wedge \forall n \in N_{\mathcal{Z}}(m \in S \rightarrow m+1 \in S) \rightarrow n \in S) \}$$

which means $N$ fulfills the strong induction scheme we know from Peano arithmetic [37]. However, when logic is weakened (in the metatheory) the 'true' natural numbers are defined with respect to the coherent induction scheme (17), in which case one does not distinguish $\mathbb{N}$ and $N_{\mathcal{Z}}$. We do not dwell upon such metatheoretic considerations here (see however [31]).

Even if the subtype $\mathbb{N} \subset N_{\mathcal{Z}}$ can be defined as in (4) still it is undecidable:

$$\mathcal{Z} \models (\mathbb{N} \neq N_{\mathcal{Z}}) \rightarrow (\mathbb{N} \text{ is not decidable in } N_{\mathcal{Z}}).$$

The important question is the extend up to which one can consistently replace $\mathbb{N}$ by $N_{\mathcal{Z}}$. What is crucial here is that the 2nd order property of $R_{\mathcal{Z}}$ of being Archimedean is again retrieved with respect to $N_{\mathcal{Z}}$. Similarly, the interval $[0, 1]$ is compact again with respect to $N_{\mathcal{Z}}$. The twist (2) is realized by the shift:

$$s(\mathbb{N}) \rightarrow N_{\mathcal{Z}}$$

which allows for the retrieving of some internal higher order properties of theories in $\mathcal{Z}$ which were lost when the canonical standard NNO was in use.

We will demonstrate how this intuitionistic model for weak arithmetic and especially the shift (5) is related to both smoothness structures in dimension 4 and the procedure of adding reals by forcing.

### 2.1 Smooth natural numbers in $B$

Weak logic as described in the previous section (and in the Appendix) guarantees that there is a NNO different than $s(\mathbb{N})$, i.e. $N_{\mathcal{Z}}$ which replaces consistently the standard NNO in the intuitionistic set-up.

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6 A subset $A \subset B$ is decidable when $a \in A$ is decidable property, i.e. when $\forall_{a \in B}(a \in A \lor a \notin A)$. 
The crucial point is that \( N_Z \) contains also non-standard natural numbers what indicates that \( N_Z \) is an intuitionistic analogue of \(*N\) known from the non-standard analysis (NA). Internal in the toposes, higher order intuitionistic theories are formulated internally in \( Z \) w.r.t. \( N_Z \) and \( R_Z \) leaving aside their standard counterparts. But such radical departure from standardness modifies finiteness such that infinite big non-standard natural numbers are considered as \( s\)-finite.

In general there are two kinds of infinitesimal elements in \( R_Z \): invertible (\( I \subset R_Z \)) and nilpotent ones. Nilpotent elements are required by the synthetic differential geometry approach and they represent forms like \( dx \) (\( d^2 = 0 \)), while invertible elements are predicted by the non-standard analysis of Abraham Robinson which can be generated by taking inverses of infinite non-standard natural numbers. The smooth topos unifies both kinds of infinitesimals in the one real line \( R \) where they exist as real numbers. Moreover, most internal higher order theories perceive the smooth numbers as true real and natural numbers. The important class of such theories are differentiable manifolds whose category \( M \) is mapped into the smooth toposes via \( s\) transform, and they require \( s\)-numbers to define their topology, compactness, connectedness or homologies.

However, do there really exist ‘non-standard’ and invertible infinitesimal elements of \( R_Z \), i.e. \( I \) in \( Z \)? In fact it holds \[ Z \models \neg\neg\exists x \in R_Z \cap I \] (6)
which is a rather weak version of the existence of invertible infinitesimals (recall that the logic in \( Z \) is intuitionistic and double negation does not cancel in general). To strengthen this result the Authors of \[37\] proposed to modify the topos \( Z \) towards \( B \) such that now one proves:
\[ B \models \exists x \in R_Z \cap I. \] (7)
To obtain this result one has to modify the Grothendieck topology in \( Z \) and then to be sure invertible infinitesimals do exist, one adds them by the forcing on stages (see the Appendix 1 in \[37\]). Thus, indeed in the internal environment of \( B \) the non-standard real numbers are added by forcing. This is the extension of the real line by adding new reals which we discussed in Secs. 1 and 2. Such procedure is not in general possible in higher orders and in the classical \( \{0,1\} \) logic, but it is possible in the weaker logic of the topos \( B \) realizing the twist \[2\] by the shift \[5\].
2.2 The smooth topos \( B \) localized on \( \mathbb{R}^n \)

Here we want to show that smoothness structures on \( \mathbb{R}^4 \) can have their origins at the level of models of the real line. Moreover, continuum many different exotic smoothness structures \( \mathbb{R}^4 \)'s can be understood at that level. Given the real line (higher order, classical) \( \mathbb{R} \) it is Archimedean with respect to \( \mathbb{N} \). To have such a unique model \( \mathbb{R} \) we can think of it as the model for the second order theory of real numbers or the theory of an Archimedean complete ordered field, both having unique (up to isomorphisms) models. On the contrary, reducing the properties of \( \mathbb{N} \) or \( \mathbb{R} \) to the first order we get a plurality of non-standard models \( *\mathbb{N} \) and \( *\mathbb{R} \) in every infinite cardinality. Can one have different non-standard models \( *\mathbb{R} \) all having the cardinality of continuum? The answer is the following:

**Lemma 4** Under the Continuum Hypothesis (or under \( 2^{<\mathfrak{c}} = \mathfrak{c} \)) there are \( 2^\mathfrak{c} \) different non-isomorphic models \( *\mathbb{R} \) all having the cardinality \( \mathfrak{c} \).

The part of the proof important to us is the observation that every non-principal ultrafilter \( \mathcal{U} \) on the set \( \mathbb{N} \) generates a non-standard \( *\mathbb{R}_\mathcal{U} \) of the cardinality continuum as an ultrapower construction, and two such ultrapowers are isomorphic if and only if the ultrafilters generating them are isomorphic w.r.t. a permutation of \( \mathbb{N} \). Finally there are \( 2^\mathfrak{c} \) non-isomorphic ultrafilters on \( \mathbb{N} \). □

Thus starting with the higher order \( \mathbb{R} \) one has up to \( 2^\mathfrak{c} \) possibilities to choose its 1st order continuous reducts \( \mathbb{R} \to *\mathbb{R} \). This extends to the relation basic to us (especially for \( n = 4 \)) with 1 to \( 2^\mathfrak{c} \) possibilities:

\[
\mathbb{R}^n \xrightarrow{2nd \to 1st} *\mathbb{R}^n. \tag{8}
\]

Let us complete this correspondence with another one as follows:

\[
\mathbb{R}^n \xrightarrow{2nd \to 1st} *\mathbb{R}^1 \xrightarrow{n \, sh} *\mathbb{R}^2 \xrightarrow{n \, 2nd \to 1st} \mathbb{R}^n. \tag{9}
\]

We would like to have \( (9) \) realized as smooth correspondence also in the middle arrow, and valid in the higher orders. This is the point where the topos \( B \) and the twist \( (5) \) come into play. We are further extending the correspondence \( (9) \) into the following \( B \)-modified one:

\[
\mathbb{R}^n \xrightarrow{2nd \to 1st} *\mathbb{R}^1 \xrightarrow{n \, \epsilon_1} \mathbb{R}^n_{Bd} \to \mathbb{R}^n_{Be} \xleftarrow{R_2} *\mathbb{R}^2 \xrightarrow{n \, 2nd \to 1st} \mathbb{R}^n. \tag{10}
\]

We are going to determine the internal in \( B \) [d]-continuous and even differentiable map. Let \( F_{\infty} \) be the ideal in \( P(\mathbb{N}) \) of finite subsets of \( \mathbb{N} \).
The algebra $P(\mathbb{N})/\text{Fin} = P(\omega)/\text{Fin}$ is an atomless Boolean algebra. Moreover, all nonprincipal ultrafilters on $\mathbb{N}$ are the members of the Stone space $\beta[\omega]\setminus \omega$ of the algebra $P(\omega)/\text{Fin}$. Recall that the Frechet cofinite filter $\mathcal{F}$ on $\mathbb{N}$ is defined as:

$$\mathcal{F} = \{ F \in P(\mathbb{N}) : \mathbb{N}\setminus F \in \text{Fin} \}$$

(11)

The following obvious but important lemma holds true:

**Lemma 5** Every nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ contains the Frechet cofinite filter $\mathcal{F}$.

Let us consider now the specific relation of non-standard models $^*\mathbb{N}$, $^*\mathbb{R}$ in classical logic (Set) and in toposes (higher order intuitionistic logic).

**Lemma 6** In $\mathcal{B}$ and $\mathcal{Z}$ the non-standard models are built as filter-product constructions based on the Frechet filter $\mathcal{F}$ rather than on ultrafilters.

This follows from the direct construction of smooth natural numbers in $\mathcal{B}$ (see [37], p 252). Moreover, to respect constructivism in toposes one cannot base on the AC especially using ultrafilters strongly depends on AC. In [38] Moerdijk showed explicitly that the constructive non-standard PA in the topos $\text{Sh}(\mathcal{F})$ of sheaves on the category of filters is based on the smooth natural numbers constructed with respect to the Frechet filter $\mathcal{F}$.

**Corollary 1** All non-standard models $^*\mathbb{N}$ ($^*\mathbb{R}$) are mapped by $e_1,e_2$ in (10), into the single intuitionistic non-standard model $\mathbb{N}_B$ ($\mathbb{R}_B$) in $\mathcal{B}$.

This is the consequence of: (1) All ultrafilters are the extensions of the unique Frechet filter (Lemma 5). (2) Different nonstandard models of $\mathbb{R}$ (with the cardinality continuum) are constructed on the base of non-isomorphic nonprincipal ultrafilters on $\mathbb{N}$. (3) Lemma 6 $\square$

Let us consider relations on $\mathbb{N}$ modulo the ideal of finite subsets Fin, e.g. the equality becomes $A =^* B$ meaning $A \Delta B = A \setminus B \cup B \setminus A \in \text{Fin}$. We call a 1 : 1 function $f : D_f \to \text{Im}_f, D_f, \text{Im}_f \subset \mathbb{N}$ an almost permutation of $\mathbb{N}$ whenever domain of $f$, $D_f$, and its image $\text{Im}_f$ are almost $\mathbb{N}$, i.e. $D_f =^* \mathbb{N} =^* \text{Im}_f$. 

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Each such almost permutation \( f \) of \( \mathbb{N} \) gives rise to the automorphism \( d_f \) of the Boolean algebra \( P(\omega) / \text{Fin} \). Namely

\[
d_f([A]) = [f(A \cap D_f)] \text{ for } [A] \in P(\omega) / \text{Fin}.
\]

Even though there can be up to \( 2^c \) nontrivial automorphisms of the algebra \( P(\omega) / \text{Fin} \) [40], it is still valid that:

**Lemma 7** There are \( c \) automorphisms of \( P(\omega) / \text{Fin} \) which give rise to almost permutations of \( \mathbb{N} \).

This is crucial for us to consider such trivial automorphisms since they forbid \( \mathbb{N} \), hence \( \mathbb{R} \), to be constant and give definite transformations of \( \mathbb{N} \). Moreover, as shown by Shelah [41], the statement that there are only \( c \) automorphisms of \( P(\omega) / \text{Fin} \) (only trivial) is consistent with ZF. So in the above sense we restrict our considerations to the trivial automorphisms case. Let us note that:

**Lemma 8** Every trivial automorphism of \( P(\omega) / \text{Fin} \) represented by a permutation \( \sigma : \omega \rightarrow \omega \) corresponds to a mapping (shift) between non-isomorphic non-standard models of \( \mathbb{R} \) of the cardinality \( c \).

This is a direct consequence of the relation of the nonprincipal ultrafilters and non-standard models of \( \mathbb{R} \), and the fact that the Stone space of \( P(\omega) / \text{Fin} \), i.e. \( \beta[\omega] \), contains all nonprincipal ultrafilters on \( \omega \), i.e. \( \beta[\omega] \setminus \omega \). Every permutation of \( \mathbb{N} \) extends to a homeomorphism \( \beta(\sigma) : \beta[\omega] \rightarrow \beta[\omega] \) and to an automorphism of \( \beta[\omega] \setminus \omega \) (e.g. 3.41, p. 88 in [40]). This last defines the shift between the non-standard models.

\( \square \)

In fact we need the following converse relation:

**Corollary 2** For every automorphism of \( P(\omega) / \text{Fin} \) there exists the shift-map between non-isomorphic non-standard \( c \)-models of \( \mathbb{R} \) such that the automorphism realizes this shift between the models.

Now given the shift-map \( \text{sh} : *R_1 \rightarrow *R_2 \) as in [9] we can think of it as determined by some automorphism of \( P(\omega) / \text{Fin} \). Note that this correspondence is obviously non-unique. Taking an internal in \( \mathcal{B} \) extension \([d]\) of the shift as in [10] gives rise to the following:

**Theorem 2** Every external shift \( \text{sh} : *R_1 \rightarrow *R_2 \) determines the internal \( s \)-differentiable maps \([d]_{1,2}, [d]_{2,1} : R_{\mathcal{B}}^n \rightarrow R_{\mathcal{B}}^n, n = 1, 2, 3, \ldots \).
Note that \([d]_{1,2}\) and \([d]_{2,1}\) are generated in \(\text{Set}\) by the ‘inverse’ almost permutations of \(\mathbb{N}\). \textit{proof}: First, any non-standard model \(*R_i\) is obtained via the ultraproduct construction w.r.t. an ultrafilter \(U_i\). \(U\) is the extension of the Frechet filter \(\mathcal{F}\). In \(\mathcal{B}\) the ‘non-standard’ real line is \(R_B\) obtained via the filter construction w.r.t. \(\mathcal{F}\). Hence we have \([d]_{1,2} : R_B \rightarrow R_B\). Second, every internal \([d]\) is continuous in \(\mathcal{B}\) (see Theorem 3.6, p. 270 in [37]). Next, since \(\mathcal{B}\) is the model of synthetic differential geometry (there exist indempotent infinitesimals \(D \subset R_B\)) it holds true the Kock-Lawvere axiom in \(\mathcal{B}\), which gives ([37], p. 302):

\[
\forall f \in R_B \forall x \in R_B \exists ! f'(x) \in R_B \forall h \in D f(x + h) = f(x) + hf'(x)
\]

where \(R\) stands for \(R_B\). Note that \(f'(x)\) is just the symbol for the unique \(y = f'(x)\) such that \(y \in R\). Repeating the procedure we determine subsequently \(f''(x), f'''(x), \ldots\) Thus \(f \in R^R\) is a standardly infinitely many times differentiable internal function. Finally, we apply again the Kock-Lawvere axiom to the ‘inverse’ map \([d]_{2,1}\) which leads to a similar differentiability. \(\Box\)

\textbf{Definition 1} The pair \(([d]_{1,2}^n, [d]_{2,1}^n)\), or \([d]_{1,2}^n\) to shorten, is called an internal diffeomorphism or \(s\)-diffeomorphisms of \(R_B^n, n = 1, 2, 3\).

Note that any internal diffeomorphism as above is generated by the shift between the non-standard \(e\)-models of \(R\). One could wonder whether the \(s\)-diffeomorphisms can be non-identity maps since they all are generated w.r.t. the Frechet filter. However, due to Lemma [7] there is precisely \(e\) shifts which guarantee that \(\mathbb{N}\) hence \(R_B\) are not constant.

Now we are ready to define the central object in this section (cf. [33]):

\textbf{Definition 2} Let \(M^n\) be a smooth \(n\)-dimensional manifold and \(\{U_\alpha : U_\alpha \in \mathcal{O}\\}\) its regular open cover. We call \((\mathcal{B})M^n\) a \(n\)-dimensional manifold \(M^n\) locally modified by the topos \(\mathcal{B}\), or the smooth \(\mathcal{B}\) structure on \(M^n\), whenever it holds:

- For every regular open cover \(\{U_\alpha\}\) of \(M^n\) there exists some \(U_\alpha \in \{U_\alpha\}\) such that \(U_\alpha\) is internal object of the internal in \(\mathcal{B}\) topology of \(s(M^n)\).

- If two such open \(U_\alpha, U_\beta\) are internal in \(\mathcal{B}\) their nonempty internal meet defines the local change of coordinates in \(\mathcal{B}\) which contain the \(s\)-diffeomorphisms: \(\eta_{\alpha\beta} = [d]_{1,2} : U_\alpha \cap U_\beta \rightarrow U_\alpha \cap U_\beta\).
Next we would like to ensure that $s$-diffeomorphisms do not arise from a Set-based diffeomorphism. To this end let the class of trivial automorphisms of $P(\omega)/\text{Fin}$ be suitably limited: one allows only those trivial automorphisms whose almost permutations of $\mathbb{N}$ contain at least one non-identical almost cycle - cyclic almost permutations and we will call them cyclic permutations if it does not cause any confusion.\footnote{An almost cyclic permutation is an almost permutation $p$, i.e. $p : A \xrightarrow{1:1} B$, $A =^* \mathbb{N} =^* B$, which reverses the order of elements of some $C \subset A$ when compared to the order of $p(C)$.} There still exist continuum many such almost permutations and none of them is extendable in Set to any orientation preserving diffeomorphism of $\mathbb{R}$.

Summarizing:

1. $s$-diffeomorphisms are not images of diffeomorphisms from Set, hence the local modification by $\mathcal{B}$ of the smoothness structure of $M^n$ is nontrivial and categorical.
2. $s$-diffeomorphism is generated in Set by a cyclic almost permutation of $\mathbb{N} \subset \mathbb{R}$ so it is not extendable to any orientation-preserving diffeomorphism of $\mathbb{R}$.
3. In $\mathcal{B}$ each permutation of $s(\mathbb{N}) \subset R_{\mathcal{B}}$ gives rise to the $s$-diffeomorphism $([d]_{ij}, [d]_{ji})$.

## 3 From the categorical to exotic $\mathbb{R}^4$

Given the local $\mathcal{B}$-modification of the smooth structure on $M^n$ we are interested in its impact on the actual classical smoothness of manifolds. One obvious classical limit (this which does not depend on $\mathcal{B}$) of the $\mathcal{B}$-modified structure on $M^n$ is just the smooth structure of $M^n$ we started with. In this case all open $U_\alpha \in \mathcal{O}$ become (again) Set based external objects. There is however another, more refined possibility.

**Definition 3** 1. We say that a classical limit of the $\mathcal{B}$-deformed smooth structure on $M^n$ factors through the non-standard models $^*\mathbb{R}_1$ and $^*\mathbb{R}_2$ whenever they are c-models of $\mathbb{R}$ and the $\mathcal{B}$-deformation was performed according to (9) and (10) where now $\mathcal{O} \ni U_\alpha \simeq \mathbb{R}^n$ on the l.h.s. and $\mathcal{O} \ni U_\beta \simeq \mathbb{R}^n$ on the r.h.s. of these relations.
2. A nontrivial classical limit of the $\mathcal{B}$-deformed smooth structure of $M^n$ is a smooth structure on $M^n$ which factors through some non-standard models of $\mathbb{R}$ while reaching the Set and higher order levels.

The point is that even though local $\mathcal{B}$-modifications of $M^n$ take all almost permutations of $\mathbb{N}$ into internal $s$-diffeomorphisms, hence a single $\mathcal{B}$-deformed structure emerges, on the Set level it is not so. Namely we can prove the important result:

**Theorem 3** For different non-isomorphic $c$-models $^*R_1$ and $^*R_2$ with the cyclic automorphism shifting them, the classical nontrivial limit (if it exists!) of the local $\mathcal{B}$-deformed structure of $\mathbb{R}^4$ is some exotic smooth $R^4_{1,2}$.

To fix the result we need the simple but crucial observation:

**Lemma 9** Given a smooth structure on $\mathbb{R}^4$ if there does not exist any open cover of $\mathbb{R}^4$ containing a single coordinate patch $\mathbb{R}^4$ this structure has to be exotic.

If a smooth $\mathbb{R}^4$ has a single coordinate patch $U \simeq \mathbb{R}^4$ it is diffeomorphic to the standard $\mathbb{R}^4$. If none of its open covers contains a single element, such $\mathbb{R}^4$ can not be diffeomorphic to the standard $\mathbb{R}^4$.

□ **proof:** (Theorem 3) We will show that any coordinate patch of the $\mathcal{B}$-modified $\mathbb{R}^4$ can not contain the single chart. On the contrary let there exist a single coordinate patch $\mathbb{R}^4$ for the classical limit of the $\mathcal{B}$-modified $\mathbb{R}^4$ as above. But in this case any open cover can be deformed by diffeomorphisms to a cover whose transition functions are identities. However, the factorization of some $U_\alpha$ through $^*R_1$ and $U_\beta$ through $^*R_2$ and the cyclic condition on the permutations of $\mathbb{N}$ excludes the identities. □

Note that this proof works in the case of $\mathcal{B}$-modified $\mathbb{R}^n$ since in this case for the standard $\mathbb{R}^n$ one can have a single coordinate patch. It is known that exotic $\mathbb{R}^n$'s exist only in dimension $n = 4$ so that means that classical smooth limits of the categorical $\mathcal{B}$-modifications do not exist for $n \neq 4$. What is so special in dimension 4 that enables the existence of the limit as above? Some explanation comes from the special relation of Casson handles and geometric constructions in dimension 4 with the smooth NNO in $\mathcal{B}$. 

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3.1 Casson handles in $\mathcal{B}$

When proving that emerged smooth $\mathbb{R}^4$ is exotic we left aside the case when there are external diffeomorphisms which can be mapped onto the internal ones. The reason is that they could be ‘gauged out’ to the identity on the intersections by some external diffeomorphisms. However, making the additional assumption, which is also partly and implicitly present in so far analysis, we can include some external diffeomorphisms as generating exotic smooth $\mathbb{R}^4$’s. Namely, assume explicitly that natural numbers $\mathbb{N}$ are generated as different objects by non-isomorphic $\mathfrak{c}$-models of $\mathbb{R}$. This means that given the almost permutations of $\mathbb{N}$ generated by different $\mathfrak{c}$-models of $\mathbb{R}$ they can not be ‘gauged out’ to the identity whenever the models of $\mathbb{R}$ are non-isomorphic. This is rather strong low level assumption which reverses our ‘natural thinking’ about the relation of real and natural numbers.

However, in $\mathcal{B}$ we had a similar situation: given the canonical object of real numbers $\mathbb{R}_\mathcal{B}$ which is the image $s(\mathbb{R})$ from Set, we had to modify the NNO $s(\mathbb{N})$ to the smooth $\mathbb{N}_\mathcal{B}$. The real numbers determined the NNO. Now we want to follow this line of reasoning and show that Casson handles are related with the smooth NNO in $\mathcal{B}$.

Let us consider one example. The simplest known exotic $\mathbb{R}^4$ can be represented in the Kirby calculus language as a handle-body with a single Casson handle (fig. 1, [22], p. 363). The simplest possible Casson handle with a single positive intersection at each level (fig. 2, see [22], p. 363). Let us assign a partial non-cyclic permutation $p$ of $\mathbb{N}$ to this CH, namely define it by: the number of level, i.e. $n$, plus 'the number of intersections at each level, i.e. 1. It results in the following

![Figure 1: The simplest small exotic $\mathbb{R}^4$ with the simplest possible Casson handle attached to the Akbulut cork.](image)

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partial permutation:

\[ p : n \to n + 1, n \in \mathbb{N}. \]  

(13)

Such a permutation defines the automorphism of \( P(\omega)/\text{Fin} \) according to \( [2] \) and thus corresponds to the shifts between the \( \epsilon \)-models of \( \mathbb{R} \). Based on the assumption we indeed arrive at the exotic \( \mathbb{R}^4 \).

This simple example justifies the assumption as a basic rule in the context of exotic smooth structures on \( \mathbb{R}^4 \). It also shows that Casson handles are nontrivially related with the object of NN in \( \mathcal{B} \). One can make this relation even more direct by interpreting the trees spanning CH’s as built w.r.t. the smooth rather than standard NN. In this case we say that a Casson handle is spanned by a tree in \( \mathcal{B} \). Let us turn again to the simplest CH represented by its Kirby diagram in the fig. 2 (see [22], p. 363). The tree is just infinite \(+\)-signed linear order of levels. The crucial information is its infiniteness resulting from the geometric construction. More precisely:

**Lemma 10** If the smooth Casson handle construction terminated after finitely many steps it is the standard smooth 2-handle.

This means that any smooth \( \mathbb{R}^4 \) with a handle-body containing all smooth finite CH’s becomes the standard smooth \( \mathbb{R}^4 \). Let us now associate the smooth NN to the levels of the simplest CH:

\[
\# \text{ of level} \to n \in \mathbb{N} \subset \mathcal{N}_\mathcal{B}
\]

just by taking the infinite set of levels as complementary to the finite set \( \{0\} \) thus becoming a member of the Frechet filter \( \mathcal{F} \). But this means that the infinite tree of this CH is just \( s \)-finite in \( \mathcal{B} \). When one performs similar enumerating of infinite number of levels in an arbitrary CH the result is the following:

![Diagram](image_url)

Figure 2: The simplest possible Casson handle which gives rise to an exotic \( R^4 \).
Lemma 11  Infinite Casson handles are spanned in $B$ by $s$-finite trees.

This together with Lemma 10 indicates that indeed the internal arithmetic of $B$ has something to do with exotic smoothness, since one can state:

Corollary 3  Exotic smoothness structures on $\mathbb{R}^4$ (smooth 4-manifolds), while transformed into $B$ by $s$, belong to the class of $s$-standard smooth $\mathbb{R}^4$. They all are internally $s$-diffeomorphic.

This result in fact agrees with our previous observation that external distinct, even discontinuous maps lead to internal $s$-diffeomorphisms. What was crucial in establishing it was the shift (replacement) from the standard $\mathbb{N}$ to the smooth $N_B$. The same shift is crucial in the above seeing CH’s as $s$-finite objects. Observe that turning to the locally modified by $B$ structures of manifolds, allows for the shifting between various exotic $\mathbb{R}^4$’s, not necessarily between exotic and the standard ones. Namely it holds:

Theorem 4  Let $\mathbb{R}^4$ be a small exotic $\mathbb{R}^4$ whose handle-body contains $k$ many CH’s for some $k \in \mathbb{N}$. Let a local $B$-modification of $\mathbb{R}^4$ be performed such that $l < k, l \in \mathbb{N}$ many CH’s belong to the local open neighborhood which is internal in $B$. Then, there exists a classical limit of this modification which is an exotic $\mathbb{R}^4_{k-l}$ (with only $k-l$ nontrivial CH’s).

proof: Observe that internally $l$ CH’s becomes $s$-finite CH’s (those corresponding to the $s$-finite spanning trees). It is enough to define the classical limit as $\mathbb{R}^4_{k-l}$ by requiring that $s$-finite CH’s are sent to the actually finite ones. □

Now, we see that the local modification of the manifold smooth structures by $B$ and taking classical limits, works as an analog of the large diffeomorphism where the actual smooth exotic $\mathbb{R}^4$’s represent a kind of generalized isotopy classes of embeddings (or small, coordinate-like diffeomorphisms). Working entirely in Set one can not realize exotic $\mathbb{R}^4$’s as merely isotopy classes of embeddings since there is no diffeomorphism at all connecting different exotic $\mathbb{R}^4$’s. Moreover, in this generalized set-up, one can study a class of topological and smooth manifolds allowing for the local categorical modifications (and the resulting new concept of equivalence). The local character of the modification leads to generalized manifolds which are partially both in Set and $B$. 
4 Some consequences to Physics

Starting with \( \mathbb{R} \) and \(*\mathbb{R}\) and creating the pairs of such reals for both models we arrive at the isomorphic fields of complex numbers, even though \( \mathbb{R} \) and \(*\mathbb{R}\) are non-isomorphic. This is connected with the fact that in \( \mathbb{C} \) one can not define the NNO \( \mathbb{N} \) (starting from the axioms of the complete ordered algebraically closed field of characteristic zero). But this means that we can use \(*\mathbb{R}\) instead of \( \mathbb{R} \) in the case of the complete ordered algebraically closed field of characteristic zero, i.e. \( \mathbb{C} \).

Given the divergent expression \( 1+2+3+4+\ldots = \sum_{i=1}^{\infty} i \) it is bigger than any \( n \in \mathbb{N} \) so this sum, if existed as the natural number (and in 1st order language), corresponds to a non-standard number of some model \(*\mathbb{N}\) hence \(*\mathbb{R} \). Moreover, such non-standard element exists in any non-standard \( \varepsilon \)-model of \( \mathbb{R} \), since every non-standard model of \( \mathbb{N} \) is the (conservative) end-extension of \( \mathbb{N} \).

Note that we get the same \( \mathbb{C} \) (up to isomorphism) starting from any \(*\mathbb{R}\) by building the space of pairs with the algebraic operations of \( \mathbb{C} \). This is the consequence of categoricity of \( \mathbb{C} \). Thus, possibly the non-standard big values, like the infinite sum above, should correspond (via the isomorphisms of models) to some finite value in \( \mathbb{C} \).

Indeed, suppose such value does not exist, then each pair of the form \( (\sum_{i=1}^{\infty} i, b), b \in *\mathbb{R} \) can not correspond to any standard complex number. But it does since every \( *\mathbb{C} \iso \mathbb{C} \) (\( \mathbb{C} \) is \( \varepsilon \)-categorical). Moreover it has to correspond via the isomorphism to some standard pair \( z \in \mathbb{C}, z = (x, y); x, y \in \mathbb{R} \). The point is the following: \( \mathbb{C} \) allows \( 2^\varepsilon \) non-trivial automorphisms and they give rise to the isomorphisms \( *\mathbb{C} \iso \mathbb{C} \) for every \( *\mathbb{C} \) generated via the ultrafilter constructions. On the other hand there are only 2 automorphisms of \( \mathbb{C} \) that send \( \mathbb{R} \) to \( \mathbb{R} \) - the identity and the complex conjugation. This, together with the fact that fixed points of all automorphisms of \( \mathbb{C} \) are all rational numbers, i.e. \( \forall_{\phi \in \text{Aut}(\mathbb{C})} \forall_{r \in \mathbb{Q}} \phi(r) = r \), give that the image of \( \sum_{i=1}^{\infty} i \) under any isomorphism \( *\mathbb{C} \iso \mathbb{C} \) has to be irrational pair \((x, y) \in \mathbb{C} : x, y \in \mathbb{I} \). This is in fact result of a very discontinuous and wild behavior of the (wild) automorphisms of \( \mathbb{C} \) realizing the above isomorphism. On the other hand if one would like to have a finite value assigned to this iso which would not be dependent on the choice of the non-standard model \(*\mathbb{R}\) it has to be rational number as it is a fixed point of every automorphism. In what follows we would like to consider this
model-theoretic mechanism for assigning finite values to divergent expressions in context of exotic smoothness structures on \( \mathbb{R}^4 \). Then we try to understand this phenomenon in context of renormalization and regularization ever-present in perturbative quantum field theories.

**Lemma 12** For any exotic smooth \( \mathbb{R}^4 \) (which is topologically \( \mathbb{R}^4 \)) any diffeomorphic image of it can not send smooth coordinate line \( \mathbb{R} \) to the smooth \( \mathbb{R} \).

If there were such diffeomorphism the exotic \( \mathbb{R}^4 \) would factorize as \( \mathbb{R} \times \mathbb{R}^3 \) which is necessary standard. □

One can equivalently state the lemma as: *If the topological \( \mathbb{R} \) is smooth line in a smooth \( \mathbb{R}^4 \) this has to be standard \( \mathbb{R}^4 \).* Thus, when a smooth diffeomorphism of \( \mathbb{R}^4 \) preserves \( \mathbb{R} \) as the factor this can happen only for the standard \( \mathbb{R}^4 \). In the case of automorphisms of \( \mathbb{C} \) when \( \mathbb{R} \) is send to \( \mathbb{R} \) then the automorphism can not be wild. Otherwise, any wild automorphism scatters in a very discontinuous way the real line in the complex plane (leaving the rational numbers fixed). For any exotic diffeomorphism of \( \mathbb{R}^4 \) it can not smoothly send the line \( \mathbb{R} \) to itself, though continuously it does. As we explained in the previous sections and in this one, both situations are connected with non-standard \( \mathbb{C} \)-models of \( \mathbb{R} \).

Let us consider the non-standard \( \mathbb{C} \) (though isomorphic to \( \mathbb{C} \)) as generated by pairs of the non-standard reals, i.e. \( \mathbb{C} \simeq \{(a,b) \in \mathbb{R} \times \mathbb{R}\} \). Then, make the product: \( \mathbb{C}^2 \simeq \mathbb{R}^4 \). When turning to the higher orders one gets the unique (up to isomorphisms) standard real field and the equality reads: \( \mathbb{C}^2 \simeq \mathbb{R}^4 \). Instead, one can use an automorphism of \( \mathbb{C} \) to obtain (non-canonical) isomorphism \( \mathbb{C} \simeq \mathbb{C} \) and thus \( \mathbb{C}^2 \simeq \mathbb{R}^4 \). Given different \( \mathbb{R}^4 \)'s one gets different automorphisms of \( \mathbb{C} \) and thus different realizations of the isomorphism above. It follows that one can use different wild automorphisms of \( \mathbb{C} \) to distinguish (index) different non-standard models of \( \mathbb{R} \). Given \( \mathbb{R}^4 \) locally modified by \( \mathcal{B} \) and taking its classical limit which factors through \( \mathbb{R}_1, \mathbb{R}_2 \), this results in the exotic \( \mathbb{R}^4 \) and thus the correspondence follows:

**Corollary 4** Pairs \((\alpha_1, \alpha_2)\) of automorphisms of \( \mathbb{C} \), where at least one automorphism is wild, distinguishes different exotic \( \mathbb{R}^4 \)’s.

\(^8\)This part of the work was performed in the cooperation with Krzysztof Bielas.
This relation can be expressed in terms of eq. (10) which for \( n = 4 \) and by turning to the \( \mathbb{C} \) leads to the fully external description:

\[
\mathbb{C}^2 \simeq \mathbb{R}^4 \xrightarrow{2nd\rightarrow1st} *R_1^4 \rightarrow *C_1 \xrightarrow{(iso,0)} \mathbb{C} \times \mathbb{C} \xleftarrow{(0,iso)} *C_2 \leftarrow *R_2^4 \xleftarrow{2nd\rightarrow1st} \mathbb{R}^4 \simeq \mathbb{C}^2.
\]

(14)

The middle \( \mathbb{C} \times \mathbb{C} \) product emerges from the component-wise automorphisms of \( \mathbb{C} \) giving rise to the isomorphisms \( \alpha_i : *C_i \simeq \mathbb{C}, i = 1, 2 \) and this is the pair \( (\alpha_1, \alpha_2) \) which represents exotic \( R_{1,2}^4 \). The relation is, however, highly non-constructive, similarly to the wild automorphisms of \( \mathbb{C} \) and the ultrafilters constructions.

As we observed the wild automorphisms of \( \mathbb{C} \) should somehow allow for the assignment of finite values to some divergent expressions. We can make this point more tractable by turning to the relation with exotic \( R_{1,2}^4 \) and making use of the very special properties of \( \mathcal{B} \). So we turn again to (10) from (14). The point is that \( \mathcal{B} \) locally modifies \( \mathbb{R}^4 \) and the theory of distributions in \( \mathcal{B} \) looks very special, namely all distributions in \( \mathcal{B} \) are regular (constructive and w.r.t. the smooth real line and natural numbers) and each external distribution is canonically mapped into the internal one. In fact it holds (37, Th. 3.6 p. 324 and Remark on p. 322):

**Theorem 5 (Moerdijk, Reyes, 1991)** In \( \mathcal{B} \) for every distribution \( \mu \) on \( \mathbb{R}^n \) there exists a predistribution (function) \( \mu_0 : \mathbb{R}^n \rightarrow \mathbb{R} \) such that for all \( f \in F_n \):

\[
\mu(f) = \int f(x)\mu_0(x)dx.
\]

Here \( F_n \) denotes the internal space of test functions in dimension \( n \). Also as stated by Theorem 3.15.3 p. 336 in [37], there exists a bijection between the external distributions in Set and the internal in \( \mathcal{B} \) given by the global section functor \( \Gamma : \mathcal{B} \rightarrow \text{Set} \). In particular, the product and the square roots of distributions are thus well-defined in \( \mathcal{B} \) as operations on the representing internal functions.

### 4.1 Renormalization in the coordinate space

Now we can discuss the problem of renormalization in perturbative quantum field theory based on this special representation of distributions in \( \mathcal{B} \). Note also that in \( \mathcal{B} \) the standard NNO, i.e. \( \mathbb{N} \), is replaced with the smooth NNO, \( \mathbb{N}_B \), such that ‘finite’ in \( \mathcal{B} \) is ‘infinite’ externally in Set. Thus indeed \( \mathcal{B} \) is a natural category for addressing
renormalization questions. Given the interaction Lagrangian $\mathcal{L} = \frac{\lambda}{4!} \phi^k$ of the $\phi^k$ neutral scalar massive quantum field theory its $S$-matrix is determined in Dyson series representation, as [15]:

$$S = \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{M^n} T(\mathcal{L}_I(x_1)\mathcal{L}_I(x_2)...\mathcal{L}_I(x_n))dx_1dx_2...dx_n$$

(15)

where $M = \mathbb{R}^{1,3}$, $M^n = M \times ... \times M$ $n$-times and $T$ stays for the time-ordered products of the operator-valued distributions $\mathcal{L}_I$, hence $S$ is the operator-valued distribution either. The time ordering is defined for two operator valued functions $A, B$ on $M$ as (we follow the presentation in [15]):

$$T(A(x_1)B(x_2)) = \Theta(x_1^0 - x_2^0)A(x_1)B(x_2) + \Theta(x_2^0 - x_1^0)B(x_2)A(x_1)$$

(16)

where $\Theta(x)$ is the Heaviside function on $\mathbb{R}$, i.e. $\Theta(x) = 0, x < 0$ and $\Theta(x) = 1, x \geq 0$. Here $x_i^0, i = 1, 2$ are time coordinates of $x_i \in M, i = 1, 2$. However, as noted in [15] for general operator-valued irregular distributions one cannot create the products of them by discontinuous functions like $\Theta$. If one, however, works outside the thick diagonal in $M^n = M \times ... \times M$, $D_n = \{x \in M^n : \exists i \neq j, x_i = x_j\}$, then the $\Theta$ is continuous hence the product (16) well-defined. This is the core of the various problems in perturbative QFT, let us quote the opinion of Authors of [15]:

In fact the mathematical origin for the appearance of short-distance singularities in perturbation theory is the ill-defined notion of time-ordering reviewed above. Epstein and Glaser proposed a way to construct well-defined time ordered products $T_n$, one for each power $n$ of the coupling constant, that satisfy a set of suitable conditions explained below, the most prominent being that of locality or micro-causality. The power series $S$ constructed by [15] using the Epstein-Glaser time-ordered product $T$ is a priori finite in every order, and renormalization corresponds then to stepwise extension of distributions from $M_n \setminus D_n$ to $M_n$. In general, distributions can not be extended uniquely onto diagonals. The resulting degrees of freedom are in one-to-one correspondence with the degrees of freedom (finite renormalizations) in momentum space renormalization programs like BPHZ and dimensional regularization.
Instead of reviewing the Epstein-Glaser construction let us observe that for regular distributions the problem in (16) does not arise since they can be represented by the operator-valued functions, their product is well-defined and they can be multiplied by Θ. True problem arises for irregular distributions like Dirac δ. Moreover, if all distributions were regular the Epstein-Glaser construction would give as the extension over $M_n \setminus D_n$ just the regular distributions we started with.

Now recall that in $\mathcal{B}$: 1. every distribution is regular (Theorem 5), 2. every distribution in Set can be naturally mapped to a distribution in $\mathcal{B}$ (Theorem 3.15.3 p. 336, [37]), and 3. the function $^B\Theta : R \to R$ is continuous. This observations motivate the following procedure:

Given $M^n$ ($n$-product of the Minkowski spacetime, $n = 1, 2, 3...)$ let the diagonal $D_n \subset U_\beta \in \mathcal{O}$ for some regular open cover $\{U_\alpha\}_{\alpha \in I}$ where $U_\beta \in \{U_\alpha\}_{\alpha \in I}$. Then, one locally modifies $M^n$ by $\mathcal{B}$ such that $U_\beta \in \mathcal{B}$ according to Def. 2.2.

Under the procedure above one indeed has well-defined extensions of distributions over the diagonals $D_n$ in a sense of internal logic of $\mathcal{B}$ localized on $M^n$. Observe also that the local modification of $M^n$ implies some local modification of spacetime $M$ itself (if not, all factors in $M^n$ are not modified, hence $M^n$ neither).

**Corollary 5** Varying the underlying geometry of a spacetime manifold by the local modification of its smooth structure by $\mathcal{B}$, i.e. $(B)M$, gives rise to the renormalization of some perturbative QFT when formulated on such modified manifolds.

Let us introduce the following additional suppositions:

All the local deformations of $M$ are generated by the underlying local deformations by $\mathcal{B}$ of $\mathbb{R}^4$, and let the classical limit of them factorize through some $^*R_i$, $i = 1, 2$, thus leading to exotic $R^4_{1,2}$. Then it follows:

**Corollary 6** The renormalization problem of some perturbative QFT can be translated into the geometry of some (Euclidean) exotic $R^4$ background which complements the Minkowski flat spacetime.

One can restate the corollary as: Ultraviolet (UV) divergencies in some perturbative QFT determine exotic smoothness of the Euclidean $\mathbb{R}^4$ background. We expect that ultraviolet divergencies counterterms of some perturbative QFT’s on Minkowski spacetime are expressible in terms of the Riemannian (sectional) curvature of $R^4_{1,2}$. This Euclidean curved 4-background complements the Minkowski’s one. Recall that
exotic $R^4$’s are just Riemannian smooth 4-manifolds which can not be flat. Thus the Corollary 6 indicates that a curvature in spacetime, hence nonzero density of gravitational energy emerges, when renormalization problem is solved geometrically. This connection with gravity is a rather universal, non-perturbative phenomenon of different perturbative QFT’s and it is an important feature of the approach.

4.2 QM on smooth $R^4$ and model theory

The specific model-theoretic approach to exotic smoothness of open 4-manifolds like $R^4$ presented here has also the advantage that one can still think in terms of local differentiation and (global) functions and arrive at the model-theoretic set-up. This is complementary to the approach via Riemannian structures and curvature, which anyway indicates that exotic $R^4$’s are ‘normal’ smooth 4-manifolds and functions are local objects on them. We follow the work [32] and the case of exotic $R^4$’s is again crucial here. We work in the complementary picture and the analysis is based on model-theoretic tools but it is worth mentioning that strong connection of small exotic smooth $R^4$’s with QM formalism, noncommutative spaces, QFT and quantum gravity, was indeed shown and developed by purely geometric and topological methods (see e.g. [4, 6, 34]).

Lemma 13 Let $R^4$ be some exotic smoothness structure on $R^4$. There has to exist a continuous (non-standard smooth) real-valued function on $R^4$ which would be smooth on $R^4$, or a continuous real-valued function smooth on $R^4$ but merely continuous on $R^4$.

If such function did not exist that means that the precisely the same functions would be smooth in both structures and the smoothness structures would be equivalent and manifolds diffeomorphic (being homeomorphic). □

So, let $f : R^4 \to R, f \in C^0(R^4)$ and $f \in C^\infty(R^4)$ so $f$ is exotic smooth. $f$ can not be everywhere standardly differentiable on $R^4$ but, when changing the smoothness structure into $R^4$, it can. Moreover, the differentiation is locally the same as a standard one, since $R^4$ is a Riemannian smooth 4-manifold. What would happen if one tried to differentiate globally any nondifferentiable continuous function? One should follow the pattern of generalized differentiation of functions or distributions. Outside the domains where the function is not continuous, the differentiation agrees with normal local differentiation. We
are looking for the model-theoretic compensation (representation) for such global ‘non-standard’ distributional differentiation. The result is precisely the $\mathbb{R}^4$ locally modified by $B$.

Namely, it is always possible to choose open neighborhoods containing the nonsmooth domains of the function $f$ such that in these domains the functions would be represented by regular distributions. However, iterating differentiation of them leads to irregular distributions as well, like Dirac $\delta$-distribution. Then, we can turn to a $\mathbb{R}^4$ locally modified by $B$ such that the neighborhoods are internal in $B$ and every external distribution, also irregular, is represented internally by regular one (Theorem 3.15.3 p. 336, \textbf{[37]}), i.e. by some internal smooth function. This is the model-theoretic smoothing of continuous functions on $\mathbb{R}^4$. Taking the classical nontrivial limit of this local modification by $B$ the result is some exotic $\mathbb{R}^4$ as in Theorem \textbf{[3]} On the contrary, every local modification by $B$ sends some irregular distributions to the internal smooth functions. Thus the following definition is natural and direct in this context: we call the modification by $B$ the \textit{model-theoretic representation of an exotic smooth structure on $\mathbb{R}^4$} \textbf{[32]} provided it sends some irregular distributions to the internal smooth functions. Let exotic smooth $R^{4}_{1,2}$ be the classical limit of our $^{(B)}\mathbb{R}^4$ which factorizes through $^\ast R_i$, $i = 1, 2$.

**Lemma 14** Let the model-theoretic representation of the exotic smooth $R^{4}_{1,2}$ be $^{(B)}\mathbb{R}^4$. In the classical trivial, i.e. standard $\mathbb{R}^4$ limit, the space of exotic smooth functions on $R^{4}_{1,2}$ contains some irregular external distributions on the standard $\mathbb{R}^4$.

First, in the classical limit we do not have the dependence on $B$ any longer. Next, suppose that classical limit as in the formulation of the lemma does not contain the distribution. Then the global differentiation of every smooth function on $R^{4}_{1,2}$ agrees with the global standard differentiation on $\mathbb{R}^4$. So, the smoothness structure of $R^{4}_{1,2}$ has to be the standard one. □

Next consider the Fourier transform of smooth functions $\text{FT} : C^\infty(\mathbb{R}^4) \rightarrow C^\infty(\mathbb{R}^4)$. Let us represent the discontinuous functions in some open neighborhood by the corresponding irregular distributions as before. FT extends over the space of $L^2$-functions and distributions on $\mathbb{R}^4$ thus over $C^\infty(R^{4}_{1,2})$ in the $\mathbb{R}^4$ representation. The image of such Fourier operator is again $C^\infty(R^{4}_{1,2})$. This is the core of the interpretation of QM formalism on exotic $\mathbb{R}^4$. 

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**Lemma 15** The FT of $\delta$ and $\delta$-distribution itself, they both belong to the model-theoretic representation of an exotic smooth $\mathbb{R}^4$ in the standard $\mathbb{R}^4$ limit.

We would like to interpret this result directly on exotic $\mathbb{R}^4$. Note that the FT of $\delta$ is $\sim 1$ and it is geometrically a straight line, say coordinate axes, in the standard structure. However, this line can not be any smooth coordinate line in any exotic $\mathbb{R}^4$, since this would give the factorization and the collapse of the structure to the standard one. However, the tangent space of every exotic $\mathbb{R}^4$ is trivial, i.e. $T\mathbb{R}^4 \approx T_b\mathbb{R}^4$ ($\mathbb{R}^4$ is contractible) and we consider this $1(x)$ as the coordinate line in the tangent space $T\mathbb{R}^4$ \[32\] This coordinate line is spanned by $\sim \partial_x$ in the generator tangent space. Thus FT mixes the standard tangent space with coordinate space $\mathbb{R}^4$ and thus $\partial_x$ is sent to the multiplication operation in the model-theoretic representation of exotic $\mathbb{R}^4$. Given a large exotic $\mathbb{R}^4$ (which can not be embedded into the standard $\mathbb{R}^4$) its contraction to a ball in $\mathbb{R}^4$ gives rise to:

**Theorem 6 (Corollary 4, \[32\])** One can interpret the noncommutative relations of the position and momentum operators in the, contracted to a 4-ball, classical limit of the model-theoretic representation of a large exotic smooth $\mathbb{R}^4$.

Based on this interpretation the mechanism of decoherence in space-time was proposed where QM effects disappear by taking uncontracted limit of such contracted $\mathbb{R}^4$ \[32\].

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\[9\] One could also think about such structures as having the generalized tangent spaces like e.g. $TR^4 \oplus *TR^4$. Indeed, one can relate \[3\] some small exotic $R^4$'s with deformations of Hitchin structures (gerbes) defined on $TS^3 \oplus *TS^3, S^3 \subset \mathbb{R}^4$. 

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27
Appendix

Weak arithmetic in smooth toposes

In order to work constructively in arbitrary topos the correct logic is intuitionistic - one avoids the axiom of choice (AC) and the law of excluded middle (e.g. [35]). Next, instead of the axiom of choice, and even finite AC, one has the axiom of bounded search [37] as in (19) below, recursion rule is replaced by the finitely presented type recursion (18) and full induction is replaced by the following (17) coherent induction scheme:

\[
\text{Ind} : \phi(0) \land \forall x \in \mathbb{N}(\phi(x) \rightarrow \phi(x+1)) \rightarrow \forall x \in \mathbb{N}\phi(x), \text{ for } \phi \text{ coherent (17)}
\]

\[
\text{Rec} : \forall f \in S \times T \forall a \in ST \exists ! g \in S \times T \forall x \in T(g(0, x)) = a(x) \land \forall n \in \mathbb{N}g(n+1, x) = f(g(n, x))
\]

weak AC : \( \forall A \in P(\mathbb{N} \times \mathbb{N})(\forall n \in \mathbb{N}\exists m \in \mathbb{N}A(n, m) \rightarrow (19) \)

\( \forall n_0 \in \mathbb{N}\exists m_0 \in \mathbb{N}\forall n \leq n_0 \exists m \leq m_0 \land A(n, m) \).

The type S in (18) has to be finitely presented and the formula \( \phi \) in (17) coherent (e.g. [37] pp. 297-298) which results in further weakening of the logic. Given such substantial weakening of the logic and arithmetic one gains the degree of indistinguishability of the standard and certain non-standard models of natural numbers. These weak properties are augmented by the usual subset of PA axioms (still in the intuitionistic logic):

\( N \) is a subtype of \( R; \)

\( R \) is Archimedean : \( \forall x \in R \exists n \in \mathbb{N}z < n; \)

\( 0 \in N \) and \( \forall x \in R(x \in N \rightarrow x + 1 \in N) \) and

\( \forall x \in R(x \in N \land x + 1 = 0 \rightarrow \bot). \)

As shown by Moerdijk and Reyes [37] these properties characterizing weak intuitionistic arithmetic along with coherent formulas and type restrictions as in (17,18) above, are fulfilled in some smooth toposes like smooth Zariski topos \( Z \) or Basel topos \( B \).

\textsuperscript{10}These axioms are written within the varying types formalism of S. Feferman [37, 19]

\textsuperscript{11}P(\mathbb{N} \times \mathbb{N}) is the power set of \( \mathbb{N} \times \mathbb{N} \) and \( A(n, m) \) means \( (n, m) \in A \in P(\mathbb{N} \times \mathbb{N}). \)
References

[1] T. Asselmeyer, “Generation of source terms in general relativity by differential structures”, Gen. Rel. Grav. 34, 597 (1996).

[2] T. Asselmeyer-Maluga, C. H. Brans, Exotic Smoothness and Physics. Differential Topology and Spacetime Models, (World Scientific, Singapore, 2007).

[3] T. Asselmeyer-Maluga and J. Król, “Abelian gerbes, generalized geometries and foliations of small exotic $R^4$”, (2009). [arXiv: 0904.1276v5]

[4] T. Asselmeyer-Maluga and J. Król, “Topological quantum D-branes and wild embeddings from exotic smooth $R^4$”, Int. J. Mod. Phys. A 26, No 20, 3421 (2011). [arXiv: 1105.1557]

[5] T. Asselmeyer-Maluga and J. Król, “Quantum Geometry and wild embeddings as quantum states”, Int.J. Geom. Meth. Mod. Phys. 10, 10 (2013).

[6] T. Asselmeyer-Maluga and J. Król, “Inflation and topological phase transition driven by exotic smoothness”, Adv. High En. Phys. Article ID 867460, http://dx.doi.org/10.1155/2014/867460, special issue ”Experimental Tests of Quantum Gravity and Exotic Quantum Field Theory Effects (QGEQ)” (2014)

[7] T. Bartoszyński, H. Judah, Set Theory: On the Structure of the Real Line, (A.K. Peters, Wellesley, Massachusetts, USA, 1995).

[8] C. H. Brans, “Roles of space-time models”, in A. Marlow (ed.), Quantum Theory and Gravitation (Academic Press, New York, 1980).

[9] C. H. Brans, “Exotic smoothness and physics”, J. Math. Phys. 35, 5494 (1994).

[10] C. H. Brans, “Localized exotic smoothness”, Class. Quant. Grav. 11, 1785 (1994).

[11] C. H. Brans, “Absolute spacetime: the twentieth century ether”, Gen. Rel. Grav. 31, 597 (1999).

[12] P. Benioff, “Models of Zermelo–Frankel set theory as carriers for the mathematics of physics I, II,” J. Math. Phys. 17 No 5, 618 (1976).
[13] P. Benioff, “Language is physical,” Quant. Inf. Proc. 1, 495 (2002).

[14] P. Benioff, “Space and time dependent scaling of numbers in mathematical structures: Effects on physical and geometric quantities”, will appear in Quantum Information Processing, Howard Brandt memorial issue.

[15] C. Bergbauer and D. Kreimer, “The Hopf algebra of rooted trees in Epstein-Glaser renormalization”, Annales Henri Poincaré (2005) 6, 343 (2005). [arXiv: hep-th/0403207]

[16] Ž. Bizaca, “An explicit family of exotic Casson handles”, Proc. AMS 123, No 4 (1995).

[17] P. J. Cohen, Proc. Nat. Acad. Sci. USA 50, 1143 (1963).

[18] A. Doering and C. J. Isham, “A topos foundation for theories of physics: I. Formal languages for physics”, J. Math. Phys. 49, 053515 (2008).

[19] S. Feferman, “A theory of variable types”, in Proc. 5th Lattin American Symp. Math. Log., (Eds.) X. Caicedo, N. C. A. da Costa and R. Chuaqui (1985).

[20] M. H. Freedman, “The topology of four-dimensional manifolds”, J. Diff. Geom. 17, 357 (1982).

[21] M. H. Freedman and Team Freedman, “Bing Topology and Casson Handles” 2013 Santa Barbara/Bonn Lectures (2013).

[22] R. E. Gompf and A. I. Stipsicz, An Introduction to 4-Manifolds and Kirby Calculus (American Mathematical Society, Rhode Island, 1999).

[23] C. Heunen, N. P. Landsman and B. Spitters, “A topos for algebraic quantum theory”, Comm. Math. Phys. 291 No 1, 63 (2009).

[24] K. J. Hintikka, “Reductions in the theory of types,” in Two Papers on Symbolic Logic, Acta Philosophica Fennica, No. 8, Helsinki (1955).

[25] C. J. Isham, Topos Methods in the Foundations of Physics in Deep Beauty, ed. H. Halvorson (Cambridge University Press, 2010).

[26] T. Jech, Set Theory, (Springer, New York, 2003).
[27] W. Just and M. Weese, Discovering Modern Set Theory. II.
Set-Theoretic Tools for Every Mathematician (AMS, Providence, 1997).

[28] T. Kato, “Spectral analysis on tree-like spaces from gauge the-
oretic point of view.” Cont. Math. 347 (2004).

[29] A. Kock, Synthetic Differential Geometry (London Math. Sci.
Lect. Notes 51, Cambridge U. Press, 1981).

[30] J. Król, “Set theoretical forcing in quantum mechanics and
AdS/CFT correspondence”, Int. J. Theor. Phys. 42 No. 5, 921
(2003). [arXiv: quant-ph/0303089]

[31] J. Król, “Background independence in quantum gravity and
forcing constructions”, Found. Phys. 34, No.3, 361 (2004).

[32] J. Król, “Exotic smoothness and noncommutative spaces.
The model-theoretical approach”, Found. Phys. 34 No.5, 843
(2004).

[33] J. Król, “A model for spacetime: the role of interpretation
in some Grothendieck topoi”, Found. Phys. 36, No. 7, 1070
(2006).

[34] J. Król, “(Quantum) gravity effects via exotic $R^4$”, Ann. Phys.
(Berlin) 19, No. 3-5, 355 (2010).

[35] S. MacLane and I. Moerdijk, Sheaves in Geometry and Logic.
A First Introduction to Topos Theory (Springer, New York,
1992).

[36] R. Montague, “Reductions of higher-order logic,” in The Theory
of Models (Eds.: J. W. Addison, L. Henkin, and A.
Tarski, North-Holland Publishing Co., Amsterdam, pp. 251-
264, 1965).

[37] I. Moerdijk and G. E. Reyes, Models for Smooth Infinitesimal
Analysis (Springer, New York, 1991).

[38] I. Moerdijk, “A model for intuitionistic non-standard arith-
metic”, Ann. Pure App. Logic 73, 37 (1995).

[39] F. W. Lawvere, “Continuously varying sets: Algebraic geometry = Geometric logic”, in Logic Coll. 73, 135 (North Holland,
Bristol, 1975).

[40] A. Robinson, Non-Standard Analysis (Studies in Logic and the
Foundations of Mathematics, Amsterdam, 1966).
[41] S. Shelah, *Proper Forcing*, Lec. N. Math., 940, (Springer, Berlin-New York, 1982).

[42] J. Sładowski, “Exotic smoothness, noncommutative geometry and particle physics”, Int. J. Theor. Phys. 35, 2075 (1996).

[43] J. Sładowski, “Gravity on exotic $R^4$ with few symmetries”, Int. J. Mod. Phys. D, 10, 311 (2001).

[44] G. Takeuti, *Two applications of logic to mathematics* (Math. Soc. Japan 13, Kano Memorial Lecture 3, 1978).

[45] J. Vaananen, “Second-order logic and foundations of mathematics,” Bul. Sym. Log. 7, 504 (2001).

[46] R. C. Walker, *The Stone-Cech Compactification*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 83 (Springer, Berlin, Heidelberg, New York, 1974).