Phases of the electronic two-level model under rotating wave approximation

M T Thomaz1, A C Aguiar Pinto2 and M Moutinho2

1 Instituto de Física, Universidade Federal Fluminense, Av. Gal. Milton Tavares de Souza s/nº, CEP 24210-346, Niterói-RJ, Brazil
2 Curso de Engenharia Física, Universidade Estadual de Mato Grosso do Sul, Caixa Postal 351, Cidade Universitária de Dourados, CEP 79804-970, Dourados-MS, Brazil
E-mail: moutinho@uems.br

Received 22 February 2012
Accepted for publication 13 June 2012
Published 12 July 2012
Online at stacks.iop.org/PhysScr/86/025001

Abstract

We present the time evolution of the electronic two-level model in rotating wave approximation and calculate the Aharonov–Anandan phase of the vector states with cyclic evolution either within the period of an external monochromatic electric field or the period corresponding to the generalized Rabi frequency. The Aharonov–Anandan phase in the most general case is shown to be dependent on the initial vector state, unless the system evolves in the adiabatic regime; in the latter case, the Aharonov–Anandan phase recovers Berry’s phase. In the quasi-resonant regime of the electronic two-level model, the different relations between Γ and the frequency ω of the electric field correspond to distinct regimes of the coupling between this external field and matter.

PACS numbers: 03.65.Vf, 03.65.−w, 03.65.Aa

1. Introduction

A quantum system coupled to a classical monochromatic laser in the quasi-resonant regime can be approximated by an \( N \)-level model, \( N \) being finite, in rotating wave approximation (RWA) \([1, 2]\).

The RWA of the two-level model has been used as a candidate for modeling geometric phase gates between ion-qubits in the presence of an external electric or magnetic field \([3]\). These two-level models can be realized through the interaction of an external magnetic field with a spin-1/2 particle \([4, 5]\), or matter coupled to an external electric field, in the electric dipole approximation \([1]\).

Recently, we have studied the adiabatic evolution of a two-level model \([6]\) (i) under a (real) classical monochromatic electric field and (ii) when only the contribution of the positive frequency of the electric field is taken into account (its RWA). In the RWA of the two-level model, we recover the geometric phases acquired by the instantaneous eigenstates of energy already known in the literature \([7]\). On the other hand, we verified the absence of geometric phase in the adiabatic evolution of the instantaneous eigenstates of the two-level model coupled to a real monochromatic electric field.

Let \( ω \) be the angular frequency of the monochromatic external electric field and \( Δε \) the energy difference of the two-level model. The matter–field coupling can be approximated by the two-level model only in the quasi-resonant regime \([1]\), in which \( ω ≈ Δε \) (in natural units, \( c = \hbar = 1 \)). Certainly, this condition does not fulfill the adiabatic condition \([6]\) \( ω ≪ Δε \). In 2002, Li et al \([8]\) proposed a scheme to perform quantum geometric computation in the non-adiabatic regime with trapped ions. For a review on holonomic quantum computation, see \([9]\) and references therein.

Recently, Imai et al \([10]\) and Imai and Morinaga \([11]\) measured the geometric phases in the two-level model coupled to a monochromatic electric field. In \([10, 11]\), it is mentioned that ‘A geometric phase solely depends on the amount of the solid angle enclosed by the evolution path, not on the details . . . , or the initial and final states of the evolution’. This statement is certainly true when the quantum model is under adiabatic evolution \([12]\), but it is not obvious that this comment is valid in the RWA of the two-level model coupled to a monochromatic electric field in the quasi-resonant regime (\( ω = Δε + δ, \frac{Δε}{δ} ≪ 1 \)).

The two-level model under the action of an either weak or strong, either periodic or quasi-periodic external magnetic field has been studied previously \([13, 14]\). In \([15]\), exact solutions of this model were presented for a set of
time-dependent field configurations. The present work aims at studying the dynamics of the two-level model under the RWA and coupled to a linearly polarized monochromatic electric field, in the electric dipole approximation [1], for any value of the ratio \( \frac{\hbar}{\epsilon} \). We want to determine the initial conditions under which a vector state acquires an Aharonov–Anandan phase [16] within different periods and if these phases can appear in any regime of the coupling between matter and the external electric field. Our model is simpler than the one studied in [10, 11], where different laser beams are applied to matter in order that the quantum system acquires the geometric phases. It is important to mention the dependence of the Aharonov–Anandan phase on the vector state, that is presented in the introduction of Xin et al [8].

In section 2, we present the Hamiltonian of the model under study. Its dynamics is derived by mapping it onto a spin-1/2 model coupled to an effective magnetic field. The study of global phases in the non-adiabatic evolution of the spin-1/2 model in the presence of a rotating magnetic field was carried out previously in [17–19]. In section 3, we study the cyclic evolution of the vector states under the action of the electronic two-level model. The Aharonov–Anandan phase acquired by the states is considered for state vectors with periodicity within that of the external electric field or that of the generalized Rabi frequency. Part of our results have not been discussed in [18, 19]. We also verify if the calculated Aharonov–Anandan phases can be implemented, in the quasi-resonant regime, for any coupling between matter and electric field. A summary of our results is presented in section 4. The appendix presents the average energy for any state vector.

2. The dynamics of the RWA of the two-level model

We begin with the Hamiltonian \( H_0 \) of the quantum two-level model, without interaction [20], \( H_0 = \varepsilon_i |1\rangle \langle 1| + \varepsilon_2 |2\rangle \langle 2| \), where \( H_0 |i\rangle = \varepsilon_i |i\rangle \), \( i = 1, 2 \). We take \( \varepsilon_1 < \varepsilon_2 \). We use the natural units, \( \hbar = 1 \).

The Hamiltonian of the two-level model, including the interaction between matter and electric field, in the electric dipole approximation, \( H_{\text{e}}(t) \), represented in the basis of the eigenstates of \( H_0 \), is [21]

\[
H_{\text{e}}(t) = \begin{pmatrix} \varepsilon_1 & \tilde{d}_{12} \cdot \tilde{E}(t) \\ (\tilde{d}_{12} \cdot \tilde{E}(t))^* & \varepsilon_2 \end{pmatrix},
\]

in which \( \tilde{d}_{12} = -\varepsilon (|1\rangle \langle 2|)/\hbar \), and \( \tilde{x} \) is the position operator of the electron. We assume that the electronic states \( |i\rangle \), \( i \in \{1, 2\} \), are spherically symmetric, and hence do not have a permanent electric dipole \( \tilde{d}_{ii} = 0 \), \( i = 1, 2 \). Throughout our calculations, we have chosen the basis of eigenstates of \( H_0 \), such that the components of \( \tilde{d}_{12} \) are real; we have taken \( \varepsilon > 0 \), so that the electronic charge is \( -\varepsilon \).

In the RWA the classical linearly polarized electric field in Hamiltonian (1) becomes \( \tilde{E}(t) \approx \tilde{E}_0 \ e^{-i(\omega t - \phi_0)} \), where \( \tilde{E}_0 \in \mathbb{R}^3 \), \( \omega > 0 \) and the phase \( \phi_0 \) is chosen such that \( \tilde{d}_{12} \cdot \tilde{E}_0 \geq 0 \).

For the complex electric field in the RWA, the interaction term in Hamiltonian (1) is a complex number and it evolves in a closed circle in the parameter space, during the time interval \([0, \frac{\pi}{\omega}]\).

The dynamics of a spin-1/2 coupled to a magnetic field that precesses around a fixed direction has previously been studied, in any dynamic regime, by calculating: (i) the time evolution of state vectors [18, 22, 23]; (ii) the dynamics of the Bloch vector [19].

A shift in Hamiltonian (1) transforms it into the Hamiltonian of a spin-1/2 coupled to an effective magnetic field, that is,

\[
H_\text{e}(t) = \frac{(\varepsilon_1 + \varepsilon_2)}{2} \mathbf{1} + \frac{\mu}{2} \tilde{B}_\text{eff}(t) \cdot \tilde{\sigma},
\]

where \( \mathbf{1} \) is the identity operator of dimension \( 2 \times 2 \), \( \tilde{\sigma} \) are the Pauli matrices, \( \mu = \mu_B g \), in which \( g \) is the Landé factor and \( \mu_B \) is the Bohr magneton. The components of the effective magnetic field on the rhs of equation (2) are,

\[
\begin{align*}
B_{\text{eff}}^x(t) &= \frac{2D_0}{\mu} \cos(\omega t - \phi_0), \\
B_{\text{eff}}^y(t) &= \frac{2D_0}{\mu} \sin(\omega t - \phi_0), \\
and \quad B_{\text{eff}}^z(t) &= -\frac{\Delta \varepsilon}{\mu},
\end{align*}
\]

where \( D_0 = \tilde{d}_{12} \cdot \tilde{E}_0 \geq 0 \) and \( \Delta \varepsilon = \varepsilon_2 - \varepsilon_1 > 0 \).

We should note that \( D_0 \) is known in the literature as the Rabi frequency [5]. Although the term \( \frac{(\varepsilon_1 + \varepsilon_2)}{2} \) in Hamiltonian (2) does not affect its dynamics, we keep it in order to have a direct translation between the results of the magnetic two-level model and Hamiltonian (1).

The components of the effective magnetic field \( \tilde{B}_{\text{eff}}(t) \) can also be written in terms of polar coordinates, \( B_{\text{eff}}(t) = (B \sin(\theta) \cos(\omega t'), B \sin(\theta) \sin(\omega t'), B \cos(\theta)) \), where \( t' = t - \frac{\phi_0}{\omega} \) and

\[
\cos(\theta) = -\frac{1}{2} \sqrt{\frac{\Delta \varepsilon}{D_0^2 + (\frac{\Delta \varepsilon}{2})^2}}, \quad \sin(\theta) = \frac{D_0}{\sqrt{D_0^2 + (\frac{\Delta \varepsilon}{2})^2}},
\]

\( \theta \in [\frac{\pi}{4}, \pi] \). The norm \( B \) of the effective magnetic field is

\[
B = \sqrt{D_0^2 + (\frac{\Delta \varepsilon}{2})^2}.
\]

We choose to describe the dynamics of the two-level model, described by Hamiltonian (2), through the time evolution of state vectors on the basis of the eigenvectors of \( H_{\text{e}}(t) \). The eigenvectors and the eigenvalues of the second term on the rhs of Hamiltonian (2) were calculated in ([17, 18, 22, 23] and references therein). The eigenvectors of the Hamiltonian of a spin-1/2 coupled to a precessing magnetic field calculated in these two references differ by global phases. From now on we follow Pinto et al [22] and for the sake of completeness we rewrite eigenvectors and eigenvalues in terms of the parameters of Hamiltonian (1), allowing us to discuss the presence of phases for different coupling regimes between matter and the classical linearly polarized electric field.

\[
\text{Phys. Scr. 86 (2012) 025001} \quad \text{M T Thomaz et al}
\]
The eigenvalue equation of $\mathbf{H}_t(t)$ is, $\mathbf{H}_t(t)|\phi_i; t\rangle = E_i|\phi_i; t\rangle$, $i \in [1, 2]$, where
\[
E_i = \frac{(\varepsilon_1 + \varepsilon_2)}{2} + \bar{E}_i, \quad i \in [1, 2],
\]
with $\bar{E}_i = (-1)^i \sqrt{D_0^2 + \left(\frac{\Delta \varepsilon t}{2}\right)^2}$, $i \in [1, 2]$.

The eigenvectors of $\mathbf{H}_t(t)$ associated with $E_1(t)$ and $E_2(t)$ are respectively
\[
|\phi_1; t\rangle = -\sin \left(\frac{\theta t}{2}\right) |1\rangle + \cos \left(\frac{\theta t}{2}\right) e^{-i(t-\phi_0)} |2\rangle, \quad (6a)
\]
\[
|\phi_2; t\rangle = \cos \left(\frac{\theta t}{2}\right) |1\rangle + \sin \left(\frac{\theta t}{2}\right) e^{i(t-\phi_0)} |2\rangle. \quad (6b)
\]

From the previous expressions of the eigenvalues and eigenvectors of $\mathbf{H}_t(t)$, we verify that the ratio $\left(\frac{\Delta \varepsilon}{\omega}\right)$ characterizes the coupling strength between matter and external electric field: (i) $\frac{\Delta \varepsilon}{\omega} \ll 1$, means a weak coupling; (ii) $\frac{\Delta \varepsilon}{\omega} \gg 1$, means a strong coupling. We recall that the strength of the coupling depends not only upon the norm of the electric field but also on its orientation relative to the vector $\vec{d}_2$.

Let us consider the initial state vector $|\psi(0)\rangle = |\psi(0)\rangle_0 = c_1(0)|\phi_1; 0\rangle + c_2(0)|\phi_2; 0\rangle$, where $c_1(0)$ and $c_2(0) \in \mathbb{C}$. The states $|\phi_i; 0\rangle$, $i = 1, 2$, are equal to the states in equations (6a) and (6b) at $t = 0$. These coefficients satisfy the normalization condition: $|c_1(0)|^2 + |c_2(0)|^2 = 1$.

The time evolution of the vector $|\psi(t)\rangle$ is set by the Schrödinger equation,
\[
\mathbf{H}_t(t) |\psi(t)\rangle = i\frac{d|\psi(t)\rangle}{dt}, \quad (7)
\]
and it is subjected to its initial condition.

Using the instantaneous eigenstates of energy (6a) and (6b) to decompose the vector $|\psi(t)\rangle$,
\[
|\psi(t)\rangle = \sum_{j=1}^{2} c_j(t) e^{-iE_j t} |\phi_j; t\rangle, \quad (8)
\]
and substituting equation (8) into (7), we obtain the equations satisfied by the coefficients $c_1(t)$ and $c_2(t)$. These equations are identical to equations (12) and (13) of [22] provided that the following replacements are made: $\omega_1 = \frac{E_2 - E_1}{2}$ and $\omega_2 = \omega$.

In order to simplify the expressions of the coefficients, we include the dynamical phases in them and define $\tilde{c}_j(t) = c_j(t) e^{-iE_j t}$, $j \in [1, 2]$. The expressions of the coefficients $\tilde{c}_j(t)$, $j \in [1, 2]$, at any instant $t$, are [22]
\[
\tilde{c}_j(t) = e^{-i\frac{1}{2}(\varepsilon_1 + \varepsilon_2)\omega t} \left\{ c_j(0) \cos(G t) + (-1)^j \frac{\sin(G t)}{G} \right\} \\
\times \left[ \bar{E}_2 - \frac{\omega}{2} \cos(\theta) \right] c_j(0) + \frac{(-1)^j}{2} \frac{\sin(\theta)}{\omega} c_j(0), \quad (9)
\]
where $k = 1, 2$ and $k \neq j$. $\Gamma$ is the generalized Rabi frequency
\[
\Gamma = \sqrt{D_0^2 + \frac{1}{4} (\Delta \varepsilon + \omega)^2}. \quad (10)
\]

Equations (6a) and (6b) give that the eigenvectors $|\phi_i; t\rangle$, $i = 1, 2$, are periodic vector states with period $T = \frac{\omega}{\omega_2}$. Both coefficients $\tilde{c}_j(t)$, $i \in [1, 2]$ (see (9)), have global phases $e^{\frac{\omega}{\omega_2} t j}$ that do not contribute to the mean value of any physical measurement. The only relevant frequency that drives the contribution of $\tilde{c}_1(t)$ and $\tilde{c}_2(t)$ is the generalized Rabi frequency $\Gamma$.

The expressions (9) are valid for any value of the ratio $\left(\frac{\omega}{\omega_2}\right)$, including the quasi-resonant regime, $\frac{\omega}{\omega_2} \approx 1$.

3. The Aharonov–Anandan phase

In 1987, Aharonov and Anandan extended the concept of geometric phase to any time evolution of a quantum system [16]. Suppose the state vector $|\psi(t)\rangle$ evolves under the action of a Hamiltonian $\mathbf{H}(t)$. The cyclic evolution of the quantum system is related to its state vector during the period $\tau$, and
\[
|\psi(\tau)\rangle = e^{i\phi} |\psi(0)\rangle, \quad (11)
\]
in which $\phi$ is a unique-valued real phase which comes from the time evolution of the initial vector state $|\psi(0)\rangle$ under the action of the Hamiltonian $\mathbf{H}(t)$.

In [16], the authors define a shifted state vector $|\tilde{\psi}(t)\rangle = e^{-i\frac{\Delta \varepsilon(t)}{2}} |\psi(t)\rangle$, where the function $f(t)$ satisfies the condition $f(t) - f(0) = \phi$. It is shown in this reference that
\[
\phi = \beta = \int_0^\tau dt \langle \psi(t)|\mathbf{H}(t)|\psi(t)\rangle. \quad (12a)
\]

Subtracting the dynamical phase from the global phase $\phi$ of the state vector $|\psi(\tau)\rangle$, we obtain that the expression of the $\beta$ phase (the Aharonov–Anandan phase), is [16]
\[
\beta = \int_0^\tau dt \langle \tilde{\psi}(t)|\frac{d}{dt}(|\tilde{\psi}(t)\rangle)\rangle. \quad (12b)
\]
The $\beta$ phase is independent of the function $f(t)$. It is the Aharonov–Anandan phase that is formally identical to the phase acquired by the instantaneous eigenstates of energy when they evolve adiabatically [12]. We point out that, in contrast to Berry’s phase, the state vector in equation (11) is any state that satisfies this cyclic condition.

The time evolution of any initial state vector of the two-level model, at any instant $t \geq 0$, can be written as
\[
|\tilde{\psi}(t)\rangle = \sum_{j=1}^{2} \tilde{c}_j(t) |\phi_j; t\rangle. \quad (13)
\]
The expressions of the coefficients $\tilde{c}_i(t)$, $i \in [1, 2]$, are given by equations (9), and $|\phi_j; t\rangle$, $j = 1, 2$, are the instantaneous eigenstates of $\mathbf{H}_t(t)$ (see equations (6a) and (6b)).

Along an adiabatic evolution, the coefficient $\tilde{c}_j(t)$, $j = 1, 2$, differs from its respective initial coefficient $c_j(0)$, $j \in [1, 2]$, at most a phase. That is not the general case.
equations (9), where for arbitrary instant $t$ we have $|\tilde{c}_i(t)| \neq |c_j(0)|$, $j \in \{1, 2\}$.

Let $T$ be the period of the external electric field ($\omega T = 2\pi$). Due to the global phase $e^{-\frac{1}{\hbar}i(\varepsilon_1 + \varepsilon_2 + \omega)t}$ we redefine the coefficients (9) as

$$\tilde{c}_i(t) = e^{-\frac{1}{\hbar}i(\varepsilon_1 + \varepsilon_2 + \omega)t} g_i(t), \quad i \in \{1, 2\}. \tag{14}$$

Let $T_\Gamma$ be the period associated with the generalized Rabi frequency $\Gamma$ ($\Gamma T_\Gamma = 2\pi$). From the definitions of $g_j(t)$, $j \in \{1, 2\}$, we verify that $g_i(0) = g_i(nT_\Gamma)$, $i \in \{1, 2\}$ where $n \in \{0, 1, 2, \ldots\}$. We call attention to the fact that, differently from the eigen vectors (6a) and (6b), the coefficients $g_i(t)$ ($i \in \{1, 2\}$) differ from their initial values by a phase of $\pi$ in the half-period $g_i(\Delta\Gamma + \frac{\pi}{2}) = -g_i(0)$ ($m \in \{3, 5, 7, \ldots\}$ and $i \in \{1, 2\}$).

3.1. Cyclic evolution at $t = T$

The state vector (8) at $t = T$ is

$$|\psi(T)\rangle = -e^{-\frac{1}{\hbar}i(\varepsilon_1 + \varepsilon_2)T} \left[ g_1(T)|\psi_1; 0\rangle + g_2(T)|\psi_2; 0\rangle \right], \tag{15}$$

where

$$g_j(T) = c_j(0) \cos(\Gamma T) - (1)^j \frac{\sin(\Gamma T)}{\Gamma} \left[ \tilde{\mathcal{E}}_2 - \frac{\omega}{2} \cos(\theta) \right] c_j(0) + (1)^j \frac{\omega}{2} \sin(\theta) c_k(0). \tag{16}$$

with $j, k \in \{1, 2\}$ and $k \neq j$.

We verify from equations (15) that for the state vector $|\psi(T)\rangle$ to differ by a global phase from $|\psi(0)\rangle$ (see equation (11)), particular values should be chosen for the ratio ($\frac{\omega}{\Gamma}$) or we have to choose special values for the initial coefficients $c_1(0)$ and $c_2(0)$. We begin the discussion of $t = T$ with two special cases [18, 19].

1. $\Gamma = \hbar \omega$, $n \in \{1, 2, 3, \ldots\}$.

For any initial state vector, the state vector at $t = T$ is written as equation (11) with

$$\phi = -\pi - \frac{\pi}{2} (\varepsilon_1 + \varepsilon_2) T, \quad \text{mod}(2\pi). \tag{17}$$

In order to derive the Aharonov–Anandan phase defined in equation (12a), we use equation (A.2) to calculate the integral of the expectation value of the energy in the state $|\psi(t)\rangle$ during the interval $t \in [0, T]$. The coefficient $c_1(0) \in \mathbb{C}$ and we write it as $c_1(0) = |c_1(0)| e^{i\delta_1}, \delta_1 \in \mathbb{R}$ from now on.

Subtracting the dynamical phase from the global phase (17), we obtain

$$\beta_1 = -\pi - \frac{\pi}{\cos(\theta)} \frac{\Delta \varepsilon}{\omega} \left[ (|c_2(0)|^2 - |c_1(0)|^2) \left(1 - \frac{\sin^2(\theta)}{4n^2}\right) + \frac{\sin(\theta)}{2n} \sqrt{1 - \frac{\sin^2(\theta)}{4n^2}} |c_1(0)| \left[ c_2(0) e^{-i\delta_1} + c_2^*(0) e^{i\delta_1} \right] \right] \equiv \beta(c_1(0), c_2(0); n). \tag{18}$$

In the adiabatic limit ($\Delta \varepsilon \gg \omega$), with $c_2(0) = 0$, we recover from equation (18) Berry’s phase acquired by the instantaneous eigenstate of energy $|\psi_1; t\rangle$ (see equation (26) of [22]). In the same limit, the Aharonov–Anandan phase with $c_1(0) = 0$ is equal to Berry’s phase associated with the state $|\psi_2; t\rangle$ (see equation (28) of [22]). In each case, Berry’s phase, or equivalently the Aharonov–Anandan phase in the adiabatic limit, is independent of the initial coefficients $c_1(0)$ and $c_2(0)$.

The $\beta_1$ phase (18) is acquired by any state vector $|\psi(t)\rangle$ at $t = \Gamma$ when $\Gamma = \hbar \omega$, $n \in \{1, 2, 3, \ldots\}$, but its value depends on the initial coefficients $c_1(0)$ and $c_2(0)$, rather than what has been stated in [10] and [11], unless the initial state vector is an eigenstate of $\mathbf{H}_0(0)$ ($c_2(0) = 0$ or $c_2(0) = 0$).

In equation (18), we write the $\beta_1$ phase as a function of $\cos(\theta)$ and $\sin(\theta)$ in order to show that in the quasi-resonant regime ($\frac{\Delta \varepsilon}{\omega} \approx 1$) this phase is not equal to the solid angle enclosed by the evolution of the effective magnetic field around the $z$-axis.

The generalized Rabi frequency $\Gamma$ depends on the frequency $\omega$ of the electric field and also on the strength $D_0$ of its coupling with matter. We want to check if it is physically possible to implement the condition $\Gamma = \hbar \omega$, $n \in \{1, 2, 3, \ldots\}$. Equation (10) permits rewriting this condition as

$$D_0^2 = n^2 \omega^2 - \frac{1}{4} (\Delta \varepsilon + \omega)^2, \quad n \in \{1, 2, 3, \ldots\} \tag{19}$$

We assume $D_0 > 0$; for each value of $n$ we have only one value of $D_0$ that satisfies equation (19).

At this point, we restrict the discussion to the quasi-resonant regime ($\omega \approx \Delta \varepsilon + \delta, \frac{\omega}{\Delta \varepsilon} \ll 1$). We recall that in this regime the RWA of the two-level models has been considered for modeling geometric phase gates. In this regime, equation (19) becomes, up to the first order in $\frac{\delta}{\Delta \varepsilon}$,

$$\left( \frac{D_0}{\Delta \varepsilon} \right)^2 \approx (n^2 - 1) + (2n^2 - 1) \left( \frac{\delta}{\Delta \varepsilon} \right), \quad n \in \{1, 2, 3, \ldots\} \tag{20}$$

For $n = 1$, equation (20) is $\left( \frac{D_0}{\Delta \varepsilon} \right)^2 \approx \frac{\delta}{\Delta \varepsilon}$. Since $\Delta \varepsilon > 0$, the previous equation has solution only for $\delta > 0$ and the condition (20) with $n = 1$ is satisfied by the limit of weak coupling between matter and electric field ($D_0 \ll \frac{\Delta \varepsilon}{\omega}$). On the other hand, for $n \geq 2$, the equality (20) cannot be accomplished by a weak coupling.

2. $\Gamma = \hbar \omega$, $m \in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\}$.

In this case, we have $g_j(\frac{2\pi n}{\Gamma}) = -g_i(0), i = 1, 2, m = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, and for any initial state vector and it acquires at $t = T$ the global phase

$$\phi = -\frac{\pi}{2} (\varepsilon_1 + \varepsilon_2) T, \quad \text{mod}(2\pi). \tag{21}$$

The Aharonov–Anandan phase in this case is

$$\beta_2 = \pi + \beta(c_1(0), c_2(0); m), \tag{22}$$

with $m \in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\}$ and the function $\beta(c_1(0), c_2(0); m)$ is defined in equation (18).

Again, we verify that the $\beta_2$ phase depends on the initial state vector, unless the initial vector state is an eigenstate of $\mathbf{H}_0(0)$. 


In order to satisfy the condition \( \Gamma = m \omega \), \( m \in \{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \} \), in the quasi-resonant regime, the coupling \( D_0 \) has to be

\[
\left( \frac{D_0}{\Delta \epsilon} \right)^2 \approx (m^2 - 1) + (2m^2 - 1) \left( \frac{\delta}{\Delta \epsilon} \right), \quad m = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots
\]

(23)

We verify that equation (23) does not have solutions for \( m = \frac{1}{2} \). For \( m \in \{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \} \), we once more conclude that condition (23) cannot be fulfilled by a weak coupling between matter and electric field.

We now consider a third condition among the frequencies \( \Gamma \) and \( \omega \), which was not previously discussed in [18, 19].

(3) \( \Gamma \neq \omega l_0, l \in \{ \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots \} \).

Although we are still looking at the vector states at \( t = T \), when the Hamiltonian completes a period, in the present case we have \( g_i(T) \neq g_i(0) \), \( i = 1, 2 \), which means that not all initial vector states satisfy the periodic condition (11).

For the state vector to satisfy the cyclic condition (11) at \( \tau = T \), the initial coefficients \( c_1(0) \) and \( c_2(0) \) have to satisfy the coupled linear equations,

\[
M \begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

(24)

The entries of the matrix \( M \) are

\[
M_{jj} = \Gamma \tilde{E}_2 (\cos(\Gamma T) + e^{i \phi}) - (\omega \sin(\Gamma T)) \left( \tilde{E}_2 + \frac{\omega \Delta \epsilon}{4} \right),
\]

\( j \in \{ 1, 2 \} \),

(25a)

\[
M_{12} = M_{21} = -i \frac{\omega D_0}{2} \sin(\Gamma T).
\]

(25b)

The phase \( \gamma \) is defined as \( \gamma \equiv \phi + \frac{\omega (1 + \epsilon_2)}{2} T \).

Equation (24) has non-trivial solutions only if \( \det(M) = 0 \). From the entries of the matrix \( M \), we obtain

\[
det(M) = \Gamma^2 \tilde{E}_2^2 \left( 1 + 2 \cos(\Gamma T) e^{i \phi} + e^{2i \phi} \right).
\]

(26)

When \( \gamma = \pi \pm \Gamma T \) we have \( \det(M) = 0 \), and consequently we obtain the global phase

\[
\phi = \pi \pm \Gamma T - \frac{(\epsilon_1 + \epsilon_2) T}{2}.
\]

(27)

Solving equation (24) at \( \phi = \pi \pm \Gamma T - \frac{(\epsilon_1 + \epsilon_2) T}{2} \) and imposing that the coefficients \( c_1(0) \) and \( c_2(0) \) satisfy the normalization condition, we obtain

\[
c_1(0) = \frac{\omega D_0 e^{i \delta_1}}{\omega^2 D_0^2 + 4 \left[ \tilde{E}_2 (\tilde{E}_2 \mp \Gamma) + \frac{\omega \Delta \epsilon}{4} \right]^2},
\]

(28a)

\[
c_2(0) = \frac{2 \left[ \tilde{E}_2 (\tilde{E}_2 \mp \Gamma) + \frac{\omega \Delta \epsilon}{4} \right] e^{i \delta_1}}{\omega^2 D_0^2 + 4 \left[ \tilde{E}_2 (\tilde{E}_2 \mp \Gamma) + \frac{\omega \Delta \epsilon}{4} \right]^2};
\]

(28b)

with \( \delta_1 \in \mathbb{R} \). From the previous equations, we verify that only for particular values of the coefficients \( c_1(0) \) and \( c_2(0) \) the state vector satisfies the condition (11) at \( \tau = T \) when \( \Gamma \neq \omega l_0 \), \( l \in \{ \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots \} \). The global phase of \( c_1(0) \) and \( c_2(0) \) must be the same. Otherwise, the vector state would not satisfy the periodic condition (11).

The Aharonov–Anandan phase acquired by the state vector at \( \tau = T \) when \( \phi = -\frac{(\epsilon_1 + \epsilon_2) T}{2} \) and \( \Gamma \pm \Gamma T \) is

\[
\beta_3 = \pi \pm \Gamma T - \frac{\pi}{\cos(\theta)} \left( \frac{\Delta \epsilon}{\omega} \right) \left( |c_2(0)|^2 - |c_1(0)|^2 \right)^{\frac{1}{2}} \times \left[ 1 - \frac{\omega^2}{4 \Gamma^2} \sin^2(\theta) \left( 1 - \frac{\omega}{4 \Gamma} \sin(2 \Gamma T) \right) \right] + \left[ 1 - \frac{\omega}{8 \pi T} \sin(2 \Gamma T) \right] H(c_1(0), c_2(0))
\]

(29)

with

\[
H(c_1(0), c_2(0)) \equiv \omega \sin(\theta) \sqrt{1 - \frac{\omega^2}{4 \Gamma^2} \sin^2(\theta)} \times |c_1(0)| (c_2(0) e^{i \delta_1} + c_1(0) e^{i \delta_1}).
\]

(30)

The coefficients \( c_1(0) \) and \( c_2(0) \) in the previous equation are given by the expressions (28a) and (28b), respectively. In this case, the \( \beta_3 \) phase is independent of the value of \( \delta_1 \in \mathbb{R} \).

As a final comment on the cyclic evolution of the state vector during the period of the external electric field, we should note that if the ratio

\[
\frac{\Gamma}{\omega} = \frac{m}{2n}, \quad m, n \in \{ 1, 2, 3, \ldots \},
\]

(31)

then the state vector at \( \tau = nT \), \( |\psi(nT)\rangle \), satisfies the condition (11) for any initial state vector, and its global phase is \( \phi = -\frac{\pi}{2}(\epsilon_1 + \epsilon_2) nT + (m - nT) \), with \( m, n \in \{ 1, 2, 3, \ldots \} \). For each value of the ratio (31) one must determine the value of \( D_0 \) that permits this ratio to be valid.

3.2. Cyclic evolution at \( \tau \neq nT \), \( n = 1, 2, 3, \ldots \)

When the cyclic condition (11) is verified at \( \tau \neq nT \), \( n \in \{ \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots \} \), we have \( |\phi_j, \tau \rangle \neq |\phi_j, 0 \rangle \), \( j \in [1, 2] \). Moreover, we note that for \( \tau \neq nT \), \( n \in \{ 1, 2, 3, \ldots \} \), the classical electric field does not perform a closed path in its parameter space when the initial state vector acquires a phase, and the Hamiltonian (1) does not complete a period.

In order to discuss the condition (11) at \( \tau \neq nT \), it is better to write the states \( |\psi(0)\rangle \) and \( |\psi(\tau)\rangle \) on the basis of the eigenstates of the Hamiltonian \( H_0 \). In order to fulfill the condition (11), the coefficients in the initial state vector must satisfy the coupled linear equations analogous to equation (24) and the entries of the new matrix \( \tilde{M} \) are

\[
\tilde{M}_{11} = \sin \left( \frac{\theta}{2} \right) [e^{i \phi} - \cos(\Gamma \tau)] + q(\theta, \Gamma),
\]

(32a)

\[
\tilde{M}_{12} = -\cos \left( \frac{\theta}{2} \right) [e^{i \phi} - \cos(\Gamma \tau)] - p(\theta, \Gamma),
\]

(32b)
\[ \dot{M}_{21} = \cos \left( \frac{\theta}{2} \right) [ -e^{i(\gamma - \omega \tau)} + \cos(\Gamma \tau) ] + p(\theta, \Gamma), \quad (32c) \]
\[ \dot{M}_{22} = \sin \left( \frac{\theta}{2} \right) [ -e^{i(\gamma - \omega \tau)} + \cos(\Gamma \tau) ] + q(\theta, \Gamma), \quad (32d) \]

where

\[
q(\theta, \Gamma) = i \frac{\sin(\Gamma \tau)}{\Gamma} \left[ -\sin \left( \frac{\theta}{2} \right) \left( E_2 - \frac{\omega \cos(\theta)}{2} \right) - \frac{\omega \sin(\theta)}{2} \cos \left( \frac{\theta}{2} \right) \right],
\]

and

\[
p(\theta, \Gamma) = i \frac{\sin(\Gamma \tau)}{\Gamma} \left[ \cos \left( \frac{\theta}{2} \right) \left( E_2 - \frac{\omega \cos(\theta)}{2} \right) - \frac{\omega \sin(\theta)}{2} \sin \left( \frac{\theta}{2} \right) \right].
\]

The phase \( \tilde{\gamma} \) is defined as \( \tilde{\gamma}(\tau) = \phi + (\epsilon_1 + \epsilon_2 + \omega \tau) \).

We have non-trivial solutions only if \( \det(\dot{M}) = 0 \). By direct calculation, we obtain

\[
\det(\dot{M}) = -1 + [\cos(\Gamma \tau) \left( e^{i\tilde{\gamma}} + e^{i(\gamma - \omega \tau)} \right) - e^{i(\gamma - \omega \tau)}] - i \frac{\sin(\Gamma \tau)}{2 \Gamma} (\omega + \Delta \varepsilon) \left( e^{i\tilde{\gamma}} - e^{i(\gamma - \omega \tau)} \right).
\]  

(33)

All the cases discussed in section 3.1 satisfy identically the condition \( \det(\dot{M}) = 0 \) when the relations between \( \Gamma \) and \( \omega \) are substituted on the rhs of equation (33).

We will not discuss the general case when (33) is null for arbitrary value of \( \tau \).

From this point on to the end of this subsection, we let \( \tau = T \Gamma \), in which \( T \Gamma \) is the period associated with the generalized Rabi frequency.

The state vector at \( \tau = T \Gamma \) is

\[
|\psi(T \Gamma)\rangle = e^{-i(\epsilon_1 + \epsilon_2 + \omega \tau) T \Gamma} \times \left[ \left\{ -\sin \left( \frac{\theta}{2} \right) c_1(0) + \cos \left( \frac{\theta}{2} \right) c_2(0) \right\} |1\rangle + e^{-i\phi} e^{i\omega \tau} \left\{ \cos \left( \frac{\theta}{2} \right) c_1(0) + \sin \left( \frac{\theta}{2} \right) c_2(0) \right\} |2\rangle \right].
\]

(34)

The state vector (34) is equal to the initial state up to a global phase, independently of the values of \( c_1(0) \) and \( c_2(0) \), if \( \Gamma = \frac{\omega}{n}, \) \( n \in \{1, 2, 3, \ldots\} \). This relation is identical to \( \Gamma = n T \). The case \( \tau = T \) was already discussed in section 3.1.

As our final case we still consider \( \tau = T \Gamma \) but for \( T \Gamma \neq n T \), with \( n \in \{1, 2, 3, \ldots\} \). The state vector \( |\psi(T \Gamma)\rangle \) is still given by the expression (34). Since \( T \Gamma \neq n T \), \( n = 1, 2, \ldots \), we verify that is not possible for all normalized coefficients \( c_1(0) \) and \( c_2(0) \in \mathbb{C} \) that \( |\psi(T \Gamma)\rangle \) to satisfy condition (11).

At \( \tau = T \Gamma \), we decompose \( \det(\dot{M}) \) into its real and imaginary parts,

\[
\det(\dot{M}) = 2 \sin \left( \frac{\gamma'}{2} \right) \left\{ -\sin \left( \frac{\gamma'}{2} \right) + \sin \left( \frac{3\gamma'}{2} - \omega T \Gamma \right) \right\} + 2i \sin \left( \frac{\gamma' - \omega T \Gamma}{2} \right) \sin \left( \gamma' - \omega T \Gamma \right)
\]

(35)

where \( \gamma' = \tilde{\gamma}(T \Gamma) \).

For arbitrary values of \( \omega T \Gamma \), one solution to \( \det(\dot{M}) = 0 \) is

\[
\gamma' \sim 0, \quad \text{mod}(2\pi).
\]

(36)

In order to calculate the values of the initial coefficients \( c_1(0) \) and \( c_2(0) \) such that the vector \( |\psi(T \Gamma)\rangle \) acquires the global phase

\[
\phi = - \frac{(\epsilon_1 + \epsilon_2)}{2} - \frac{\omega T \Gamma}{2},
\]

(37)

and the Aharonov–Anandan phase

\[
\beta_4 = - \frac{\pi \omega}{\Gamma} - \frac{\pi}{\cos(\theta)} \left( \frac{\omega}{\Gamma} \right) \left( \frac{\Delta \varepsilon}{\omega} \right) \times \left\{ \left[ |c_2(0)|^2 - |c_1(0)|^2 \right] \left( 1 - \frac{\omega^2}{4 \Gamma^2} \sin^2(\theta) \right) + \frac{1}{2} \left( H(c_1(0), c_2(0)) \right) \right\}
\]

(38)

the solution (36) must be replaced into the coupled equations satisfied by these coefficients. The expression of the function \( H(c_1(0), c_2(0)) \) is given by equation (30).

Besides solution (36), the real and the imaginary parts on the rhs of (35) have the common root \( \gamma' \sim \omega T \Gamma, \text{mod}(2\pi) \), valid for arbitrary values of \( \omega T \Gamma \). The state vector at \( \tau = T \Gamma \), for particular values of \( c_1(0) \) and \( c_2(0) \), acquires the global phase

\[
\phi \sim - \frac{1}{2} (\epsilon_1 + \epsilon_2) T \Gamma + \frac{\omega T \Gamma}{2}, \quad \text{mod}(2\pi).
\]

(39)

The Aharonov–Anandan phase (the \( \beta \) phase) in this case is equal to

\[
\beta_5 = \frac{2\pi \omega}{\Gamma} + \beta_4.
\]

(40)

4. Conclusions

We study the dynamical evolution of a two-level model coupled to a classical linearly polarized monochromatic electric field in the RWA. We map this model onto a spin-1/2 model coupled to an effective magnetic field. The dynamics of the latter model has already been studied in the literature [18, 19, 22, 23], for any regime. In particular, its Aharonov–Anandan phases were studied in [18, 19] under the condition \( \Gamma = \omega n, \) \( n \in \{1, 1, \frac{1}{2}, 2, \ldots \} \).

Our results on the dynamics of the RWA of the electronic two-level model (see Hamiltonian (1)) are valid for any value
of the ratio $\frac{\omega}{\tau}$. In particular, this model in the quasi-resonant regime is a good description of the coupling between matter and an electric field in the electric dipole approximation, being a candidate for modeling gates in quantum computation [3, 8].

We show that when a cycle of the state vector occurs within a cycle of the external electric field, the Aharonov–Anandan phase depends on the initial state vector, unless: (i) the quantum system evolves adiabatically; or (ii) the quantum system evolves non-adiabatically from an initial vector state which is an eigenstate of the Hamiltonian, at $T = 0$. Our results show that caution must be taken in considering the general statements on the Aharonov–Anandan phases discussed in [10, 11].

We agree with Ni et al [18] and Zhu et al [19] that any state vector driven by Hamiltonian (1) satisfies the periodic condition (11) when $\Gamma = \frac{\omega}{2} n \in \{\frac{3}{2}, 1, \frac{3}{2}, \ldots\}$ with $\tau = T$. What is special about the electronic two-level model is that in the quasi-resonant regime, under which it is a candidate of modeling a quantum computation device, the condition $\Gamma = \omega$ is only implemented in the weak coupling regime $(\frac{\omega}{2\tau} \ll 1)$ and the previous condition with $n = \frac{1}{2}$ cannot be implemented in this regime of the model. On the other hand, in order to obtain the ratio $\Gamma = \frac{\omega}{2} n \in \{\frac{3}{2}, 2, \frac{5}{2}, \ldots\}$, the coupling between matter and electric field cannot be weak. For the case of $\tau = T$ we also show that for $\Gamma \neq \frac{\omega}{2}$, with $n \in \{\frac{1}{2}, 1, \frac{3}{2}, \ldots\}$ only very particular state vectors acquire an Aharonov–Anandan phase. This very distinct behavior of the state vectors with respect to their periodicity with $\Gamma = \frac{\omega}{2}$ and $\Gamma \neq \frac{\omega}{2}$ makes very important having a fine-tuning of the orientation of the external electric field with respect to the direction of the electric dipole of matter.

We also show that the periodic behavior of the state vector (see equation (17)) can also happen within the period $T_\Gamma$ associated with the generalized Rabi frequency (10). Again, the Aharonov–Anandan phase depends on the initial coefficients of the initial state vector when this vector is not an eigenstate of the Hamiltonian. In this case, some special state vectors satisfy condition (11). For $\tau = T_\Gamma$, we verify that there is no coupling between matter and electric field where the condition $\Gamma = \frac{\omega}{2} n \in \{2, 3, 4, \ldots\}$ can be implemented in the electronic two-level model in the quasi-resonant regime.

In both cases, we verified that in the non-adiabatic regime the Aharonov–Anandan phase in the most general case depends on the initial state and on the interval of time for the state vector to return to its initial state, up to a global phase. Although our results are valid for any value of the $\frac{\omega}{2\tau}$, part of our results are discussed in the quasi-resonant regime, where we expect our results to be applicable to the modeling of a quantum phase gate in an electronic device.

**Acknowledgments**

The authors thank E V Corrêa Silva for careful reading of part of this manuscript. M T Thomaz (Fellowship CNPq, Brazil, Proc. No.: 30.0549/83-FA) thanks CNPq for partial financial support.

**Appendix. The average energy in any time-dependent state vector**

Aharonov and Anandan [16] write the periodic condition of the state vector $|\psi(t)\rangle$ at $t = \tau$ as

$$|\psi(t)\rangle = e^{i\phi} |\psi(0)\rangle = e^{i\phi} e^{-i\int_0^\tau \langle \psi(\tau)|H_\tau|\psi(\tau)\rangle d\tau} |\psi(0)\rangle,$$

(A.1)

where the dynamical phase is subtracted from the global phase $\phi$.

We assume the initial state vector $|\psi(0)\rangle = c_1|0\rangle + c_2|\phi\rangle$, with $c_1(0)$ and $c_2(0) \in \mathbb{C}$. The coefficient $c_1(0)$ is written as $c_1(0) = |c_1(0)| e^{i\delta_1}$, with $\delta_1 \in \mathbb{R}$.

The expectation value of the energy for the state vector (8), at any instant $t$, is

$$\langle \psi(t)|H_\tau|\psi(t)\rangle = \frac{(\epsilon_1 + \epsilon_2)}{2} + \bar{E}_2 \left( \left| c_2(0) \right|^2 - \left| c_1(0) \right|^2 \right)$$

+ $\frac{i}{2} \omega \sin(\theta) \left| c_2(0) \right| \sin(2\Gamma t) \left| c_2(0) e^{-i\delta_1} - c_2(0) e^{i\delta_1} \right|$

+ $\frac{\omega^2}{4T^2} \left| c_2(0) \right|^2 \sin^2(\theta)

+ $\frac{\omega}{2} \sin(\theta) \left( 1 - \frac{\omega^2}{4T^2} \sin^2(\theta) \right) \left| c_1(0) \right|$

$$\left| c_2(0) e^{-i\delta_1} + c_2(0) e^{i\delta_1} \right|.$$  

(A.2)

**References**

[1] Meystre P and Sargent III 1998 *Elements of Quantum Optics* 3rd edn (Berlin: Springer) section 3.3

[2] Vedral V 2006 *Modern Foundations of Quantum Optics* (London: Imperial College Press) section 8.2

[3] Nielsen M A and Chuang I L 2007 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press) chapter 7

[4] Rabi I I 1936 *Phys. Rev.* 49 324–8

[5] Rabi I I 1937 *Phys. Rev.* 51 652–4

[6] Aguiar Pinto A C, Moutinho M and Thomaz M T 2009 *Braz. J. Phys.* 39 326–30

[7] Garrison J C and Wright E M 1988 *Phys. Lett. A* 128 177–81

[8] Li X-Q et al 2002 *Phys. Rev. A* 66 042320

[9] Sjöqvist E 2008 *Physics* 1 35

[10] Imai H, Otsubo Y and Morinaga A 2007 *Phys. Rev. A* 76 012116

[11] Imai H and Morinaga A 2007 *Phys. Rev. A* 76 062111

[12] Berry M V 1984 *Proc. R. Soc. Lond. A* 392 45–57

[13] Barata J C A 2000 *Rev. Math. Phys.* 12 25–64

[14] Barata J C A and Cortez D A 2002 *Phys. Lett. A* 301 350–60

[15] Bagrov V G, Gitman D M, Baldiotti M C and Levin A D 2006 *Ann. Phys.* 174 764–89

[16] Aharonov Y and Anandan J 1987 *Phys. Rev. Lett.* 58 1593–6

[17] Bulgac A 1988 *Phys. Rev. A* 37 4084–9

[18] Ni G-J, Chen S-q and Shen Y-I 1995 *Phys. Lett. A* 197 100–6

[19] Zhu S-L, Wang Z D and Zhang Y-D 2000 *Phys. Rev. B* 61 1142–8

[20] Cohen-Tannoudji C, Diu B and Laloe F 1977 *Quantum Mechanics* vol 1 (Paris: Wiley) p 405

[21] Lamb W E Jr, Schicher R R and Scully M O 1987 *Phys. Rev. A* 36 2763–72

[22] Aguiar Pinto A C, Nemes M C, Peixoto de Faria J G and Thomaz M T 2000 *Ann. J. Phys.* 68 955–8

[23] García de Polavieja G and Sjöqvist E 1998 *Ann. J. Phys.* 66 431–8