BOUNDEDNESS IN PREY-TAXIS SYSTEM WITH ROTATIONAL
FLUX TERMS

HENGLING WANG AND YUXIANG LI*

School of Mathematics
Southeast University, Nanjing 210096, P.R. China

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Abstract. This paper investigates prey-taxis system with rotational flux terms
\[
\begin{align*}
  u_t &= \Delta u - \nabla \cdot (uS(x,u,v)\nabla v) + uv - \rho u, \quad x \in \Omega, \quad t > 0, \\
  v_t &= \Delta v - \xi uv + \mu v(1 - \alpha v), \quad x \in \Omega, \quad t > 0, \\
  (\nabla u - uS(x,u,v)\nabla v) \cdot \nu &= 0, \quad \nabla v \cdot \nu = 0, \quad x \in \partial \Omega, \quad t > 0, \\
  u(x,0) &= u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega,
\end{align*}
\]
under no-flux boundary conditions in a bounded domain \( \Omega \subset \mathbb{R}^n \) \((n \geq 1)\) with smooth boundary. Here the matrix-valued function \( S \in C^2(\overline{\Omega} \times [0, \infty))^2; \mathbb{R}^{n \times n} \) fulfills \( |S(x,u,v)| \leq \frac{S_0(m)}{(1 + u)^\theta} \) for all \((x,u,v) \in \bar{\Omega} \times [0, \infty)\) with some nondecreasing function \( S_0 \). It is proved that for nonnegative initial data \( u_0 \in C_0(\Omega) \) and \( v_0 \in W^{1,q}(\Omega) \) with some \( q > \max\{n, 2\} \), if one of the following assumptions holds: (i) \( n = 1 \), (ii) \( n \geq 2, \theta = 0 \) and \( S_0(m) m < \frac{2}{\sqrt{3n(n+2)}} \), (iii) \( \theta > 0 \), then the model possesses a global classical solution that is uniformly bounded. Where \( m := \max\{\|v_0\|_{L^\infty(\Omega)}, \frac{1}{\alpha}\} \).

1. Introduction. In this paper, we consider the global existence and boundedness of classical solutions to prey-taxis system with rotational flux terms

\[
\begin{align*}
  u_t &= \Delta u - \nabla \cdot (uS(x,u,v)\nabla v) + uv - \rho u, \quad x \in \Omega, \quad t > 0, \\
  v_t &= \Delta v - \xi uv + \mu v(1 - \alpha v), \quad x \in \Omega, \quad t > 0, \\
  (\nabla u - uS(x,u,v)\nabla v) \cdot \nu &= 0, \quad \nabla v \cdot \nu = 0, \quad x \in \partial \Omega, \quad t > 0, \\
  u(x,0) &= u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega,
\end{align*}
\]
in a bounded domain \( \Omega \subset \mathbb{R}^n \) \((n \geq 1)\) with smooth boundary, where \( \nu \) is the outward unit normal vector field on \( \partial \Omega \), the scalar functions \( u = u(x,t) \) and \( v = v(x,t) \) denote the predator density and the prey population density, respectively. \( S(x,u,v) \) is a general chemotactic sensitivity function. When external forces are considered in prey-taxis system, the movement of predators may be affected. Comparing with bacterial chemotaxis, we assume that chemotactic sensitivity \( S \) is a tensor. We suppose that \( \rho, \xi, \mu \) and \( \alpha \) are positive constants, \( u_0 \) and \( v_0 \) are given nonnegative functions, and there exists nondecreasing function \( S_0 : [0, \infty) \rightarrow \mathbb{R} \) such that \( S \) satisfies

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* Corresponding author.
with some $\theta \geq 0$.

Prey-taxis, the movement of predators towards the area with higher-density of prey population, plays an extremely important part in biological control and ecological balance such as maintaining the pest population below some economic threshold or leading to outbreaks of pest density [9, 11]. Karevia and Odell first derived a PDE prey-taxis model to illustrate a population model of spatially heterogeneous predator-prey interactions [5]. The prototypical prey-taxis model can be written as

$$
\begin{cases}
  u_t = \Delta u - \nabla \cdot (u\phi(u, v)\nabla v) + \gamma uF(v) - uh(u), \\
  v_t = D\Delta v - uF(v) + f(v),
\end{cases}
$$

(1.3)

where the term $\nabla \cdot (u\phi(u, v)\nabla v)$ represents prey-taxis with a coefficient $\phi(u, v)$ and $D$ is prey diffusion rate. $uF(v)$ stands for the inter-specific interaction, $uh(u)$ and $f(v)$ accounts for the intra-specific interaction. $F(v)$ accounts for the intake rate of predators, $h(u)$ is the predator mortality rate function and $f(v)$ is the prey growth function. The parameter $\gamma > 0$ denotes the intrinsic predation rate.

Before introducing some results on (1.3), we interpret chemotaxis processes in which the cells are subject to external forces. Based on recent experiments, the movement of cells is not directed toward the concentration of chemical signal, but with a rotational motion. Consequently, the chemotactic sensitivity is a tensor, see [8] for more details. This can be formulated as

$$
\begin{cases}
  u_t = \Delta u - \nabla \cdot (uS(x, u, v)\nabla v), \\
  v_t = \Delta v - f(x, u, v),
\end{cases}
$$

(1.4)

where $S$ is a matrix-valued function. Li et al. [8] proved that the initial-boundary value problem of (1.4) possesses a global bounded classical solution for $n = 2$ under the assumption that $\|v_0\|_{L^\infty(\Omega)}$ is sufficiently small. In [15], a global generalized solution was constructed for the initial-boundary value problem of (1.4) without any restriction on $\|v_0\|_{L^\infty(\Omega)}$. For the case of $n = 2$, Winkler [17] showed that these solutions eventually becomes smooth and satisfies $(u(\cdot, t), v(\cdot, t)) \to (\bar{u}_0, 0)$ as $t \to \infty$ uniformly with respect to $x \in \Omega$. Zhang [18] generalized the result of [8] to general bounded domain. It was shown in [18] that if one of the following assumptions holds:

(i) $n = 1$,

(ii) $n \geq 2, \theta = 0, S_0(\|v_0\|_{L^\infty(\Omega)})\|v_0\|_{L^\infty(\Omega)} < \frac{2}{\sqrt{3n(11n + 2)}}$,

(iii) $\theta > 0$,

then the initial-boundary value problem of (1.4) admits a unique global classical solution that is uniformly bounded.

(1.3) has also been studied by many authors, beginning the work of Lee et al.[6, 7] and Ainseba et al.[1]. Recently, Jin and Wang [4] investigated the predator-prey system with prey-taxis in a two-dimensional bounded domain with Neumann boundary condition. They obtained the global boundedness of a unique classical solution. In [13], global classical solution was constructed for two-predator and one-prey models with nonlinear prey-taxis.
Winkler [16] considered a taxis system of the form
\[
\begin{cases}
u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi uv - \rho u, & x \in \Omega, \ t > 0, \\
v_t = D \Delta v - \xi uv + \mu v(1 - \alpha v), & x \in \Omega, \ t > 0, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, \ t > 0, \\
u(x, 0) = u_0(x), \ v(x, 0) = v_0(x), & x \in \Omega,
\end{cases}
\] (1.5)
in a bounded domain \(\Omega \subset \mathbb{R}^2\). They obtained the global existence and uniform boundedness of weak solution to (1.6) for any \(m > \frac{11}{4} - \sqrt{3}\). Moreover, they discussed the large time behavior of these global bounded solutions.

Inspired by Li et al. [8], we investigate the prey-taxis system with rotational flux terms (1.1). To the best of our knowledge, whether any kind of solution to (1.1) exists globally is an open problem. In this paper, we shall study the global existence and boundedness of classical solutions to (1.1).

Our main result reads as follows. Throughout the sequel, we shall suppose that \(S\) satisfies the conditions of Theorem 1.1.

**Theorem 1.1.** Let \(S \in C^2(\overline{\Omega} \times [0, \infty)^2; \mathbb{R}^{n \times n})\) satisfies (1.2). Suppose that \(u_0 \in C^0(\overline{\Omega})\) and \(v_0 \in W^{1,n}(\Omega)\) with some \(q > \max\{n, 2\}\) are nonnegative functions. If one of the following assumptions is satisfied:
(i) \(n = 1\); (ii) \(n \geq 2, \theta = 0\) and \(S_0(m)m < \frac{2}{\sqrt{3n(11n+2)}}\), (iii) \(\theta > 0\),
where
\[
m := \max\{\|v_0\|_{L^\infty(\Omega)}, \frac{1}{\alpha}\},
\] (1.7)
then (1.1) possesses a global classical solution that is bounded in \(\Omega \times (0, \infty)\).

**Remark 1.1.** In contrast to (1.4), we obtain similar results to Zhang [18]. The difference between (1.1) and (1.4) is that (1.1) has \(uv - \rho u, \mu v(1 - \alpha v)\). \(uv\) indicates the intake of predators, \(-\rho u\) is the linear degradation of the predator, \(\mu v(1 - \alpha v)\) represents the growth of prey. For reader’s convenience, we point out the problems induced by these terms. In contrast to [18], we need to deal with \(K_4\) and \(K_5\) in the proof of Lemma 4.1. According to standard parabolic regularity results ([10, Theorem 1.3]), whether \(uv + \mu v\) satisfies the condition is crucial to the Hölder continuity of \(u\).

**Remark 1.2.** As already observed by Zhang [18], the important part (iii) strongly relies on the fact that unlike in the situation of classical Keller-Segel frameworks, the “signal” evolution in (1.1) is in its essence governed by an absorption process, rather than a production mechanism. The regularization effect of the signal absorption term is already clarified by Tao and Winkler [12], where they considered the oxygen
consumption system
\[
\begin{aligned}
    u_t &= \Delta u - \nabla \cdot (u \nabla v), \quad x \in \Omega, \ t > 0, \\
    v_t &= \Delta v - uv, \quad x \in \Omega, \ t > 0.
\end{aligned}
\]

The authors showed that for arbitrarily large initial data, this problem possesses a global weak solution for which there exists \( T > 0 \) such that \((u, v)\) is bounded and smooth in \( \Omega \times (T, \infty) \).

The rest of this paper is organized as follows. In Section 2, we introduce a family of regularized problems and obtain the local existence of the regularized problems. In addition, we establish some preliminary estimates. In Section 3, we prove the global existence and boundedness of classical solutions to regularized problem in the high-dimensional case. Finally, we give the proof of the Theorem 1.1 in Section 5.

2. Regularized problems. According to the idea from \([8]\), we consider the regularized problems

\[
\begin{aligned}
    u_{\epsilon t} &= \Delta u_{\epsilon} - \nabla \cdot (u_{\epsilon} S_{\epsilon}(x, u_{\epsilon}, v_{\epsilon}) \nabla v_{\epsilon}) + u_{\epsilon} v_{\epsilon} - \rho u_{\epsilon}, \quad x \in \Omega, \ t > 0, \\
    v_{\epsilon t} &= \Delta v_{\epsilon} - \xi u_{\epsilon} v_{\epsilon} + \mu v_{\epsilon}(1 - \alpha v_{\epsilon}), \quad x \in \Omega, \ t > 0, \\
    \frac{\partial u_{\epsilon}}{\partial \nu} &= \frac{\partial v_{\epsilon}}{\partial \nu} = 0, \quad x \in \partial \Omega, \ t > 0, \\
    u_{\epsilon}(x, 0) &= u_0(x), \quad v_{\epsilon}(x, 0) = v_0(x), \quad x \in \Omega,
\end{aligned}
\]

for all \( \epsilon \in (0, 1) \), where

\[
S_{\epsilon}(x, u_{\epsilon}, v_{\epsilon}) := \rho_{\epsilon}(x) S(x, u, v) \quad \text{for all} \quad (x, u_{\epsilon}, v_{\epsilon}) \in \bar{\Omega} \times [0, \infty)^2.
\]

with \( \rho_{\epsilon} \in C_0^\infty(\Omega) \) which satisfy \( 0 \leq \rho_{\epsilon} \leq 1 \) and \( \rho_{\epsilon} \not\to 1 \) as \( \epsilon \searrow 0 \) in \( \Omega \).

All the above approximate problems admit local-in-time smooth solutions. The proof is based on a well-established contraction mapping argument (see \([2]\) for details).

**Lemma 2.1.** Let \( S_{\epsilon} \) satisfies (1.2) and (2.2). Suppose that \( u_0 \in C^0(\bar{\Omega}) \) and \( v_0 \in W^{1,q}(\Omega) \) with some \( q > \max\{n, 2\} \) are nonnegative functions. Then for any \( \epsilon \in (0, 1) \), there exist \( T_{\max, \epsilon} \in (0, \infty) \) and a uniquely determined pair \((u_{\epsilon}, v_{\epsilon})\) of nonnegative functions

\[
\begin{aligned}
    u_{\epsilon} &\in C^0(\Omega \times [0, T_{\max, \epsilon})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max, \epsilon})), \\
    v_{\epsilon} &\in C^0(\Omega \times [0, T_{\max, \epsilon})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max, \epsilon})) \cap L^\infty_{\text{loc}}([0, T_{\max, \epsilon}); W^{1,q}(\Omega)),
\end{aligned}
\]

such that \((u_{\epsilon}, v_{\epsilon})\) is a classical solution of (2.1) in \( \Omega \times (0, T_{\max, \epsilon}) \). Moreover, if \( T_{\max, \epsilon} < \infty \), then

\[
\|u_{\epsilon}(\cdot, t)\|_{L^\infty(\Omega)} + \|v_{\epsilon}(\cdot, t)\|_{W^{1,q}(\Omega)} \to \infty, \quad \text{as} \quad t \searrow T_{\max, \epsilon}.
\]

The following estimates of \( u_{\epsilon} \) and \( v_{\epsilon} \) are basic but important in the proof of our result.

**Lemma 2.2.** Let \( S_{\epsilon} \) satisfies (1.2) and (2.2). Suppose that \( u_0 \in C^0(\bar{\Omega}) \) and \( v_0 \in W^{1,q}(\Omega) \) with some \( q > \max\{n, 2\} \) are nonnegative functions. For each \( \epsilon \in (0, 1) \), there exists some constant \( C > 0 \) such that

\[
\int_{\Omega} u_{\epsilon}(\cdot, t) \leq C \quad \text{for all} \ t \in (0, T_{\max, \epsilon})
\]
and
\[ \|v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq m \quad \text{in } \Omega \times (0, T_{\max, \varepsilon}). \] (2.5)
where \( m \) is defined by (1.7).

**Proof.** Set \( \tilde{v}(x, t) := y(t) \) for \( x \in \overline{\Omega} \) and \( t \geq 0 \), where \( y(t) \in C^1([0, \infty)) \) denotes the solution of
\[
\begin{cases}
y'(t) = \mu y(t) - \mu \alpha y^2(t), & t > 0, \\
y(0) = y_0 := \|v_0\|_{L^\infty(\Omega)}. 
\end{cases}
\]
Obviously, \( y(t) \leq m := \max\{\|v_0\|_{L^\infty(\Omega)}, \frac{1}{2}\} \). It follows from the comparison principle that \( v_\varepsilon \leq \tilde{v} \) in \( \Omega \times (0, T_{\max, \varepsilon}) \). So we have (2.5).

Integrating the first and second equation in (2.1), we obtain
\[
\frac{d}{dt} \int_{\Omega} u_\varepsilon = \int_{\Omega} u_\varepsilon v_\varepsilon - \rho \int_{\Omega} u_\varepsilon
\]
and
\[
\frac{d}{dt} \int_{\Omega} v_\varepsilon = -\xi \int_{\Omega} u_\varepsilon v_\varepsilon + \mu \int_{\Omega} v_\varepsilon - \mu \alpha \int_{\Omega} v_\varepsilon^2 \leq -\xi \int_{\Omega} u_\varepsilon v_\varepsilon + \mu \int_{\Omega} v_\varepsilon.
\]
Therefore,
\[
\frac{d}{dt} \left( \xi \int_{\Omega} u_\varepsilon + \int_{\Omega} v_\varepsilon \right) + \xi \rho \int_{\Omega} u_\varepsilon \leq \mu \int_{\Omega} v_\varepsilon.
\]
Combining with (2.5), we have
\[
\frac{d}{dt} \left( \xi \int_{\Omega} u_\varepsilon + \int_{\Omega} v_\varepsilon \right) + \rho \left( \xi \int_{\Omega} u_\varepsilon + \int_{\Omega} v_\varepsilon \right) \leq (\mu + \rho) \int_{\Omega} v_\varepsilon \leq (\mu + \rho)m_\Omega |\Omega|.
\]
This yields (2.4). \( \square \)

3. **A refined extensibility criterion.** This section is devoted to establishing an improved extensibility criterion which will be convenient for proving the global existence of classical solutions to (2.1). The proof of Lemma 3.1 and Lemma 3.2 is similar to [18], so we omit the details.

**Lemma 3.1.** Suppose that \( S_\varepsilon \) satisfies (1.2) and (2.2). Moreover, assume that \( u_0 \in C^0(\Omega) \) and \( v_0 \in W^{1,q}(\Omega) \) with some \( q > \max\{n, 2\} \) are nonnegative functions. Let \( p \geq 1 \) fulfill \( q < \frac{np}{(n-p)+} \). Then there exists \( C > 0 \) such that
\[
\|\nabla v_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \leq C \left( 1 + \sup_{t \in (0, T_{\max, \varepsilon})} \|u_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \right) \] (3.1)
for all \( t \in (0, T_{\max, \varepsilon}) \) and all \( \varepsilon \in (0, 1) \).

Next we can use the above lemma to derive an improved extensibility criterion.

**Lemma 3.2.** Assume that the initial data \( u_0 \in C^0(\Omega) \) and \( v_0 \in W^{1,q}(\Omega) \) with some \( q > \max\{n, 2\} \) are nonnegative, and that \( S_\varepsilon \) satisfies (1.2) and (2.2). Let \( p \geq 1 \) be such that \( p > \frac{n}{2} \). If there exists \( C(p) > 0 \) such that for each \( \varepsilon \in (0, 1) \),
\[
\sup_{t \in (0, T_{\max, \varepsilon})} \|u_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \leq C(p),
\] (3.2)
then \( T_{\max, \varepsilon} = \infty \). Moreover, there exists \( C > 0 \) (independent of \( \varepsilon \)) such that
\[
\sup_{t \in (0, \infty)} \left( \|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(\cdot, t)\|_{W^{1,q}(\Omega)} \right) \leq C.
\]
In view of (2.4), the above lemma immediately implies the global existence and boundedness of classical solutions to (2.1) in the one-dimensional case.

**Corollary 3.3.** Let $n = 1$. Assume that the initial data $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,q}(\Omega)$ with some $q > 2$ are nonnegative, and that $S_\varepsilon$ satisfies (1.2) and (2.2). Then for all $\varepsilon \in (0,1)$, (2.1) possesses a global classical solution which is bounded in $\Omega \times (0, \infty)$.

4. **Global boundedness in the approximate problem.** In this section, we turn to global boundedness of solutions to (2.1) in the dimension $n \geq 2$. To this end, we need to establish a uniform bound for $u_\varepsilon$ in $L^p(\Omega)$ with $p \geq 1$. The approach pursued in Lemma 4.1 can go back to [14].

**Lemma 4.1.** Let $p > 1$ and $\varepsilon \in (0,1)$. Suppose that $S_\varepsilon$ satisfies (1.2) and (2.2), and that the initial data $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,q}(\Omega)$ with some $q > \max\{n, 2\}$ are nonnegative functions. Recall that $m$ is defined by (1.7). If either

$$\theta = 0 \quad \text{and} \quad S_0(m)m \leq \frac{1}{\sqrt{3p(11p + 1)}}$$

or $\theta > 0$, then there exists $C(p) > 0$ such that for each $\varepsilon \in (0,1)$

$$\|u_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \leq C(p) \quad \text{for all } t \in (0, T_{\max, \varepsilon}).$$

**Proof.** Given $p > 1$, we define $\psi(s) = e^{\beta s^2}$, $s \in [0, m]$, where in the case $\theta = 0$, we pick $\beta$ satisfying

$$\frac{3p(p-1)S_0^2(m)}{2} \leq \beta \leq \frac{p-1}{2(11p + 1)m^2},$$

in the case $\theta > 0$, we choose $\beta$ such that

$$0 < \beta \leq \frac{p-1}{2(11p + 1)m^2}.$$
where we use $\psi'(v_\varepsilon) \geq 0$ for all $v_\varepsilon \geq 0$. Rearranging some terms, we get
\[
\frac{1}{p} \frac{d}{dt} \int_\Omega (u_\varepsilon + 1)^p \psi(v_\varepsilon) + (p - 1) \int_\Omega (u_\varepsilon + 1)^{p-2} \psi(v_\varepsilon)|\nabla u_\varepsilon|^2 \\
+ \frac{1}{p} \int_\Omega (u_\varepsilon + 1)^p \psi''(v_\varepsilon)|\nabla v_\varepsilon|^2 \\
\leq -2 \int_\Omega (u_\varepsilon + 1)^{p-1} \psi'(v_\varepsilon)\nabla u_\varepsilon \cdot \nabla v_\varepsilon \\
+ (p - 1) \int_\Omega u_\varepsilon (u_\varepsilon + 1)^{p-2} \psi(v_\varepsilon)(S_\varepsilon(x, u_\varepsilon, v_\varepsilon)\nabla v_\varepsilon) \cdot \nabla u_\varepsilon \\
+ \int_\Omega u_\varepsilon (u_\varepsilon + 1)^{p-1} \psi'(v_\varepsilon)(S_\varepsilon(x, u_\varepsilon, v_\varepsilon)\nabla v_\varepsilon) \cdot \nabla v_\varepsilon + \int_\Omega u_\varepsilon (u_\varepsilon + 1)^{p-1} \psi'(v_\varepsilon)v_\varepsilon \\
\quad + \frac{1}{p} \int_\Omega (u_\varepsilon + 1)^p \psi''(v_\varepsilon)v_\varepsilon \\
:= K_1 + K_2 + K_3 + K_4 + K_5 \text{ for all } t \in (0, T_{max, \varepsilon}). \tag{4.5}
\]

Using Young’s inequality, (1.2), (2.2) and (2.5), we find that
\[
K_1 \leq \frac{p - 1}{4} \int_\Omega (u_\varepsilon + 1)^{p-2} \psi(v_\varepsilon)|\nabla u_\varepsilon|^2 + \frac{4}{p - 1} \int_\Omega (u_\varepsilon + 1)^{p-2} \psi(v_\varepsilon)^2 |\nabla v_\varepsilon|^2, \tag{4.6}
\]
\[
K_2 \leq \frac{p - 1}{4} \int_\Omega (u_\varepsilon + 1)^{p-2} \psi(v_\varepsilon)|\nabla u_\varepsilon|^2 \\
+ (p - 1) S_0^2(m) \int_\Omega (u_\varepsilon + 1)^{p-20} \psi(v_\varepsilon)|\nabla v_\varepsilon|^2, \tag{4.7}
\]
\[
K_3 \leq S_0(m) \int_\Omega (u_\varepsilon + 1)^{p-\theta} \psi'(v_\varepsilon)|\nabla v_\varepsilon|^2 \text{ for all } t \in (0, T_{max, \varepsilon}). \tag{4.8}
\]

Substituting (4.6)-(4.8) into (4.5), we obtain
\[
\frac{1}{p} \frac{d}{dt} \int_\Omega (u_\varepsilon + 1)^p \psi(v_\varepsilon) + \frac{p - 1}{2} \int_\Omega (u_\varepsilon + 1)^{p-2} \psi(v_\varepsilon)|\nabla u_\varepsilon|^2 \\
+ \frac{1}{p} \int_\Omega (u_\varepsilon + 1)^p \psi''(v_\varepsilon)|\nabla v_\varepsilon|^2 \\
\leq \frac{4}{p - 1} \int_\Omega (u_\varepsilon + 1)^{p-\theta} \frac{\psi''(v_\varepsilon)}{\psi(v_\varepsilon)} |\nabla v_\varepsilon|^2 + (p - 1) S_0^2(m) \int_\Omega (u_\varepsilon + 1)^{p-20} \psi(v_\varepsilon)|\nabla v_\varepsilon|^2 \\
\quad + S_0(m) \int_\Omega (u_\varepsilon + 1)^{p-\theta} \psi'(v_\varepsilon)|\nabla v_\varepsilon|^2 + K_4 + K_5 \text{ for all } t \in (0, T_{max, \varepsilon}). \tag{4.9}
\]

Now we divide the proof into two cases. In the case $\theta = 0$, we set
\[
F_0(s) := \frac{1}{p} \psi''(s), \quad F_1(s) := \frac{4}{p - 1} \psi^2(s), \\
F_2(s) := (p - 1) S_0^2(m) \psi(s), \quad F_3(s) := S_0(m) \psi'(s).
\]
Recalling the choice of $\beta$, we show that the first three terms on the right side of (4.9) are dominated by $\frac{1}{p} \int_\Omega (u_\varepsilon + 1)^p \psi''(v_\varepsilon)|\nabla v_\varepsilon|^2$.

A simple computation shows that
\[
\frac{F_0(s)}{3F_1(s)} = \frac{(p - 1) \psi(s) \psi''(s)}{\psi^2(s)} = \frac{(p - 1)}{12p} \frac{(2\beta + 4\beta^2 s^2)}{4\beta^2 s^2}
\]
for all \( t \theta \)

This completes the proof of (4.2) for the case \( \theta = 0 \).

In the case \( \theta > 0 \), it follows from Young’s inequality that

\[
(p - 1) S_0^2(m) \int \Omega (u_\epsilon + 1)^{p-2} \psi(v_\epsilon) |\nabla v_\epsilon|^2 \\
\leq \frac{1}{3p} \int \Omega (u_\epsilon + 1)^{p} \psi''(v_\epsilon) |\nabla v_\epsilon|^2 + C_5 \int \Omega \psi^{\frac{p}{2}} (\psi'')^2 |\nabla v_\epsilon|^2
\]
\[
\begin{align*}
\frac{1}{3p} \int_\Omega (u_\varepsilon + 1)^p \psi'(v_\varepsilon) |\nabla v_\varepsilon|^2 + C_5 B_1 \int_\Omega |\nabla v_\varepsilon|^2 & \\
& \leq \frac{1}{3p} \int_\Omega (u_\varepsilon + 1)^p \psi''(v_\varepsilon) |\nabla v_\varepsilon|^2 + C_6 \int_\Omega (\psi')^{\frac{p+2}{p+\varepsilon}} |\nabla v_\varepsilon|^2 \\
& \leq \frac{1}{3p} \int_\Omega (u_\varepsilon + 1)^p \psi''(v_\varepsilon) |\nabla v_\varepsilon|^2 + C_6 B_2 \int_\Omega |\nabla v_\varepsilon|^2,
\end{align*}
\]  

where \(C_5, C_6 > 0\), \(B_1 := \max_{s \in [0,m]} \psi_\varepsilon^\rho(\psi''(s))^{\frac{p+2}{p+\varepsilon}}\) and \(B_2 := \max_{s \in [0,m]} (\psi'(s))^{\frac{p+2}{p+\varepsilon}}\). Next we prove that the first term on the right side of (4.9) is dominated by \(\frac{1}{\beta^2} \int_\Omega (u_\varepsilon + 1)^p \psi''(v_\varepsilon) |\nabla v_\varepsilon|^2\). It can be easily obtained by the above computation \(F_1(s) \leq \frac{1}{\beta} F_0(s)\). By virtue of (4.9), (4.11), (4.12) and (4.13), there exists \(C_7\) such that

\[
\frac{1}{p} \frac{d}{dt} \int_\Omega (u_\varepsilon + 1)^p \psi(v_\varepsilon) + C_1 \int_\Omega (u_\varepsilon + 1)^p \psi(v_\varepsilon) \leq C_7 \int_\Omega |\nabla v_\varepsilon|^2 + 2C_3 \tag{4.14}
\]

for all \(t \in (0, T_{\max, \varepsilon})\). Multiplying the second equation in (2.1) by \(v_\varepsilon\) and then integrating by parts, we have

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega v_\varepsilon^2 + \int_\Omega |\nabla v_\varepsilon|^2 = -\xi \int_\Omega u_\varepsilon v_\varepsilon^2 + \mu \int_\Omega \varepsilon^2 (1 - \alpha v_\varepsilon) \leq \mu \int_\Omega v_\varepsilon^2 \leq \mu\mu^{-2}|\Omega|, \tag{4.15}
\]

where we use (2.5). A combination of (4.14) and (4.15) shows that

\[
\frac{d}{dt} \left\{ \frac{1}{p} \int_\Omega (u_\varepsilon + 1)^p \psi(v_\varepsilon) + \frac{C_7}{2} \int_\Omega v_\varepsilon^2 \right\} + C_1 \int_\Omega (u_\varepsilon + 1)^p \psi(v_\varepsilon) \leq 2C_3 + C_7 \mu\mu^{-2}|\Omega|.
\]

Denote \(\bar{g}(t) := \frac{1}{p} \int_\Omega (u_\varepsilon + 1)^p \psi(v_\varepsilon) + \frac{C_7}{2} \int_\Omega v_\varepsilon^2\), then \(\bar{g}(t)\) satisfies

\[
\bar{g}'(t) + C_1 \bar{g}(t) \leq C_8
\]

with \(C_8 := 2C_3 + C_7 \mu\mu^{-2}|\Omega| + \frac{1}{\beta^2} C_1 C_7 \mu m^{-2}|\Omega|\). Integrating the above inequality, we complete the proof of Lemma 4.1. \(\square\)

Combining Lemma 4.1 with Lemma 3.2, we have the following corollary.

**Corollary 4.2.** Let \(n \geq 2\) and \(\varepsilon \in (0,1)\). Suppose that \(S_\varepsilon\) satisfies (1.2) and (2.2), and that the initial data \(u_0 \in C^0(\Omega)\) and \(v_0 \in W^{1,q}(\Omega)\) with some \(q > n\) are nonnegative functions. If either

\[
\theta = 0 \quad \text{and} \quad S_0(m) m < \frac{2}{\sqrt{3n(11n + 2)}}, \tag{4.16}
\]

or \(\theta > 0\), then the solution of (2.1) is global in time.

5. **Global classical solution for (1.1).** In this section, we turn to the proof of global existence of classical solutions to the original equations (1.1).

**Lemma 5.1.** Assume (1.2) and (2.2). We further suppose that (4.16) holds when \(n \geq 2\) and \(\theta = 0\). For any \(T > 0\), there exists \(C(T) > 0\) such that for all \(\varepsilon > 0\),

\[
\int_0^T \int_\Omega |\nabla u_\varepsilon|^2 \leq C(T), \tag{5.1}
\]
In conjunction with (5.1) and (5.2), there exists

$$\int_0^T \int_{\Omega} |\nabla v_\varepsilon|^2 \leq C(T), \quad \tag{5.2}$$

$$||\nabla v_\varepsilon(\cdot, t)||_{L^2(\Omega)} \leq C \quad \text{for all } t > 0. \tag{5.3}$$

**Proof.** Testing the second equation of (2.1) by $v_\varepsilon$ and integrating by parts, we have

$$\frac{1}{2} \frac{d}{dt} \int_\Omega v_\varepsilon^2 + \int_\Omega (|\nabla v_\varepsilon|^2 = -\xi \int_\Omega u_\varepsilon v_\varepsilon^2 + \mu \int_\Omega v_\varepsilon^2 (1 - \alpha v_\varepsilon) \leq \mu \int_\Omega v_\varepsilon^2.$$ Integrating the above inequality over $(0, T)$ gives (5.2). Multiplying the first equation of (2.1) by $u_\varepsilon$ and integrating by parts, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_\Omega u_\varepsilon^2 + \int_\Omega (|\nabla u_\varepsilon|^2 = \int_\Omega u_\varepsilon S_\varepsilon(x, u_\varepsilon, v_\varepsilon) \nabla v_\varepsilon \cdot \nabla u_\varepsilon + \int_\Omega u_\varepsilon^2 v_\varepsilon - \rho \int_\Omega u_\varepsilon^2.$$ It follows from the Young’s inequality and (2.5) that

$$\frac{1}{2} \frac{d}{dt} \int_\Omega u_\varepsilon^2 + \int_\Omega (|\nabla u_\varepsilon|^2 \leq \frac{1}{2} \int_\Omega (|\nabla u_\varepsilon|^2 + \frac{1}{2} \|u_\varepsilon\|_{L^2(\Omega \times (0, \infty))}^2 \int_\Omega |\nabla v_\varepsilon|^2 + m \int_\Omega u_\varepsilon^2.$$ An integration over $(0, T)$ yields (5.1). (3.1) implies (5.3). \qed

Before giving the Proof of Theorem 1.1, we can derive strong compactness properties from the estimates obtained in Lemma 5.1.

**Lemma 5.2.** Let $S_\varepsilon$ satisfies (1.2) and (2.2). Moreover, (4.16) holds when $n \geq 2$ and $\theta = 0$. For any $T > 0$, there exists $C(T) > 0$ such that

$$\|u_{\varepsilon t}\|_{L^2((0, T); (W^{1, 2}(\Omega))^*)} \leq C(T), \quad \|v_{\varepsilon t}\|_{L^2((0, T); (W^{1, 2}(\Omega))^*)} \leq C(T).$$

**Proof.** Multiplying the first equation in (2.1) by $\varphi \in C^1(\Omega)$ and integrating by parts, we have

$$\int_\Omega \int_\Omega (|\nabla u_\varepsilon|^2 + \int_\Omega u_\varepsilon S_\varepsilon(x, u_\varepsilon, v_\varepsilon) \nabla v_\varepsilon \cdot \nabla \varphi | + \int_\Omega u_\varepsilon v_\varepsilon |\nabla \varphi | + \rho \int_\Omega u_\varepsilon |\varphi | \leq \left( |\nabla u_\varepsilon|_{L^2(\Omega)} + S_0(m) \right) \|u_\varepsilon\| \|\nabla v_\varepsilon\|_{L^2(\Omega)} + \|u_\varepsilon v_\varepsilon\|_{L^2(\Omega)} + \|u_\varepsilon |\nabla \varphi |_{W^{1, 2}(\Omega)}$$

by H"older inequality. Therefore,

$$\|u_{\varepsilon t}\|_{(W^{1, 2}(\Omega))^*} \leq |\nabla u_\varepsilon|_{L^2(\Omega)} + S_0(m) \|u_\varepsilon\|_{L^\infty(\Omega \times (0, \infty))} \|\nabla v_\varepsilon\|_{L^2(\Omega)} + \|u_\varepsilon v_\varepsilon\|_{L^2(\Omega)} + \rho \|u_\varepsilon |\nabla \varphi |_{W^{1, 2}(\Omega)}.$$ In conjunction with (5.1) and (5.2), there exists $C(T) > 0$ such that

$$\int_0^T \|u_{\varepsilon t}\|^2_{(W^{1, 2}(\Omega))^*} \leq C\left( \int_0^T \int_\Omega (|\nabla u_\varepsilon|^2 + S_0(m) \|u_\varepsilon\|_{L^\infty(\Omega \times (0, \infty))} \int_0^T \|\nabla v_\varepsilon|^2 \right.\int_0^T \int_\Omega u_\varepsilon^2 + \rho \int_0^T \int_\Omega u_\varepsilon^2 \left. \leq C(T). \right.$$

Let $h(x, u_\varepsilon, v_\varepsilon) := \xi u_\varepsilon v_\varepsilon - \mu v_\varepsilon (1 - \alpha v_\varepsilon)$. Considering (2.5), we obtain

$$|h(x, u_\varepsilon, v_\varepsilon)| \leq C(u_\varepsilon + 1). \tag{5.4}$$

Therefore,

$$\|v_{\varepsilon t}\|_{(W^{1, 2}(\Omega))^*} \leq \|\nabla v_\varepsilon\|_{L^2(\Omega)} + \|h(x, u_\varepsilon, v_\varepsilon)\|_{L^2(\Omega)}.$$
In view of (5.2) and (5.4), we find $C(T) > 0$ such that
\[
\int_0^T \|v_{εt}\|_{W^{1,2}(Ω)}^2 \leq 2 \int_0^T \int_Ω |∇v_ε|^2 + 2 \int_0^T \int_Ω |h(x, u_ε, v_ε)|^2 \leq C(T). \]

Finally we can prove Theorem 1.1.

**Proof of Theorem 1.1.** Based on (2.5), we can obtain from Lemma 3.2 and Lemma 5.1 upon a standard extraction procedure that along a subsequence $(ε_j)_{j ∈ N} ⊆ (0, 1)$ satisfying $ε_j ↘ 0$ as $j → ∞$, we have
\[
u_ε \rightarrow u \quad \text{and} \quad v_ε \rightarrow v \quad \text{in} \quad L^∞(Ω × (0, ∞)).
\]
(5.5)

In view of $H^1(Ω) \hookrightarrow L^2(Ω) \hookrightarrow (W^{1,2}(Ω))^*$, the boundedness of $u_ε, v_ε$ in $L^2([0, T]; H^1(Ω))$ and the boundedness of $u_{εt}, v_{εt}$ in $L^2([0, T]; (W^{1,2}(Ω))^*)$, we can infer from the Aubin-Lions Lemma that
\[
u_ε \rightarrow u \quad \text{and} \quad v_ε \rightarrow v \quad \text{a. e. in} \quad Ω × (0, ∞)
\]
(5.7)
as $ε = ε_j ↘ 0$. This implies,
\[
u_ε S_ε(x, u_ε, v_ε) → uS(x, u, v) \quad \text{a. e. in} \quad Ω × (0, ∞).
\]
According to (2.5), Lemma 3.2 and the dominated convergence theorem, we obtain
\[
u_ε S_ε(x, u_ε, v_ε) → uS(x, u, v) \quad \text{in} \quad L^2_{loc}(Ω × [0, ∞)),
\]
(5.8)
as $ε = ε_j ↘ 0$.

Let $ϕ ∈ C^∞_0(Ω × [0, ∞))$. Testing the first equation of (2.1) by $ϕ$ and integrating by parts, we have
\[
− \int_Ω u_0ϕ(·, 0) − \int_0^∞ \int_Ω u_εϕ_t = − \int_0^∞ \int_Ω ϕ · ∇u_ε + \int_0^∞ \int_Ω u_ε(S_ε(x, u_ε, v_ε) · ∇v_ε) · ∇ϕ
\]
\[
+ \int_0^∞ \int_Ω u_εv_εϕ - ρ \int_0^∞ \int_Ω u_εϕ.
\]
(5.5) yields that
\[
− \int_Ω \int_0^∞ \int_Ω u_εϕ_t → − \int_Ω \int_u^∞ \int_Ω ϕ_t, \quad \int_Ω \int_0^∞ \int_Ω u_εϕ → \int_Ω \int_u^∞ \int_Ω uϕ.
\]
By (5.6) and (5.8),
\[
− \int_0^∞ \int_Ω ∇u_ε · ∇ϕ → − \int_0^∞ \int_Ω u · ∇ϕ, \quad \int_Ω \int_0^∞ \int_Ω u_ε(S_ε(x, u_ε, v_ε) · ∇v_ε) · ∇ϕ
\]
\[
− \int_Ω \int_0^∞ \int_Ω u(S(x, u, v) · ∇v) · ∇ϕ.
\]
(5.7) together with (2.5), Lemma 3.2 and the dominated convergence theorem imply that
\[
\int_0^∞ \int_Ω u_εv_εϕ \rightarrow \int_Ω \int_Ω uvϕ.
\]
Similarly, we conclude that
\[
− \int_Ω u_0ϕ(·, 0) − \int_0^∞ \int_Ω ϕ_t = − \int_0^∞ \int_Ω u · ∇ϕ + \int_0^∞ \int_Ω u(S(x, u, v) · ∇v) · ∇ϕ
\]
\[
+ \int_0^∞ \int_Ω uvϕ - ρ \int_0^∞ \int_Ω uϕ,
\]
(5.9)
and
\[
- \int_\Omega v_0 \varphi(\cdot, 0) - \int_0^\infty \int_\Omega v \varphi_t = - \int_0^\infty \int_\Omega \nabla v \cdot \nabla \varphi - \int_0^\infty \int_\Omega h(x, u, v) \varphi
\]
where \( h(x, u, v) = \xi uv - \mu v(1 - \alpha v) \). Therefore \( v \) is a weak solution of the Neumann problem for \( v_t = \Delta v - h(x, u, v) \) in \( \Omega \times (0, \infty) \) with \( v(x, 0) = v_0(x) \). By standard parabolic Hölder estimates ([10, Remark 1.4]), \( v \) satisfies
\[
\|v\|_{C^{\alpha_1, \frac{2}{\alpha_1}}(\Omega \times [t, t+1])} \leq c_1 \text{ for all } t \geq 0
\] (5.10)
with \( \alpha_1 \in (0, 1) \).

Likewise, \( u \) is a weak solution of the initial-boundary value problem for
\[
\begin{cases}
  u_t = \nabla \cdot a(x, t, \nabla u) + uv - \rho u, & x \in \Omega, \ t > 0, \\
  a(x, t, \nabla u) \cdot \nu = 0, & x \in \partial \Omega, \ t > 0, \\
  u(x, 0) = u_0(x), & x \in \Omega,
\end{cases}
\]
where \( a(x, t, p) := p - uS(x, u, v) \cdot \nabla v, (x, t, p) \in \Omega \times (0, \infty) \times \mathbb{R}^2 \). Obviously,
\[
a(x, t, p) \cdot p = p^2 - uS(x, u, v) \nabla v \cdot p \geq \frac{p^2}{2} - c_2 |\nabla v|^2 \text{ in } \Omega \times (0, \infty) \times \mathbb{R}^2,
\]
and
\[
|a(x, t, p)| \leq |p| + c_3 |\nabla v| \text{ in } \Omega \times (0, \infty) \times \mathbb{R}^2,
\]
with \( c_2, c_3 > 0 \). According to (5.3), \( |\nabla v|, |\nabla v|^2 \) and \( uv + \rho u \) belong to \( L^\infty((0, \infty), L^{1+\eta}(\Omega)) \) with \( \eta := \frac{q-2}{2} > 0 \). As a consequence of standard parabolic regularity results ([10, Theorem 1.3]), for each \( \tau > 0 \), there exist \( c_4(\tau) > 0 \) and \( \alpha_2(\tau) \in (0, 1) \) such that
\[
\|u\|_{C^{\alpha_2, \frac{2}{\alpha_2}}(\Omega \times [t, t+1])} \leq c_4(\tau) \text{ for all } t \geq \tau.
\] (5.11)
The proof of higher-order regularity information is similar to [8, Lemma 4.3], so we omit the details.

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REFERENCES

[1] B. Ainseba, M. Bendahmane and A. Noussair, A reaction-diffusion system modeling predator-prey with prey-taxis, Nonlinear Anal. Real World Appl., 9 (2008), 2086–2105.
[2] D. Horstmann and M. Winkler, Boundedness vs. blow-up in a chemotaxis system, J. Differ. Equ., 215 (2005), 52–107.
[3] C. Jin, Y. Wang and J. Yin, Global solvability and stability to a nutrient-taxis model with porous medium slow diffusion, preprint, arXiv:1804.03964.
[4] H. Y. Jin and Z. A. Wang, Global stability of prey-taxis systems, J. Differ. Equ., 262 (2017), 1257–1290.
[5] P. Kareiva and G. Odell, Swarms of predators exhibit "preytaxis" if individual predators use area-restricted search, Amer. Nat., 130 (1987), 233–270.
[6] J. M. Lee, T. Hillen and M. A. Lewis, Continuous traveling waves for prey-taxis, Bull. Math. Biol., 70 (2008), 654–676.
[7] J. M. Lee, T. Hillen and M. A. Lewis, Pattern formation in prey-taxis systems, J. Biol. Dyn., 3 (2009), 551–573.
[8] T. Li, A. Suen, M. Winkler and C. Xue, Global small-data solutions of a two-dimensional chemotaxis system with rotational flux terms, Math. Models Meth. Appl. Sci., 25 (2015), 721–746.
[9] W. W. Murdoch, J. Chesson and P. L. Chesson, Biological control in theory and practice, Amer. Nat., 125 (1985), 344–366.
[10] M. M. Porzio and V. Vespri, Hölder estimates for local solutions of some doubly nonlinear degenerate parabolic equations, *J. Differ. Equ.*, 103 (1993), 146–178.

[11] N. Sapoukhina, Y. Tyutyunov and R. Arditi, The role of prey taxis in biological control: A spatial theoretical model, *Amer. Nat.*, 162 (2003), 61–76.

[12] Y. Tao and M. Winkler, Eventual smoothness and stabilization of large-data solutions in a three-dimensional chemotaxis system with consumption of chemoattractant, *J. Differ. Equ.*, 252 (2012), 2520–2543.

[13] J. Wang and M. Wang, Boundedness and global stability of the two-predator and one-prey models with nonlinear prey-taxis, *Z. Angew. Math. Phys.*, 69 (2018), Art. 63, 24 pp.

[14] M. Winkler, Absence of collapse in a parabolic chemotaxis system with signal-dependent sensitivity, *Math. Nachr.*, 283 (2010), 1664–1673.

[15] M. Winkler, Large-data global generalized solutions in a chemotaxis system with tensor-valued sensitivities, *SIAM J. Math. Anal.*, 47 (2015), 3092–3115.

[16] M. Winkler, Asymptotic homogenization in a three-dimensional nutrient taxis system involving food-supported proliferation, *J. Differ. Equ.*, 263 (2017), 4826–4869.

[17] M. Winkler, Can rotational fluxes impede the tendency toward spatial homogeneity in nutrient taxis-(Stokes) systems? *Int. Math. Res. Not.*, 2019.

[18] Q. Zhang, Boundedness in chemotaxis systems with rotational flux terms, *Math. Nachr.*, 289 (2016), 2323–2334.

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*E-mail address*: hlwang@seu.edu.cn

*E-mail address*: lieyx@seu.edu.cn