The Casimir effect for mass dimension one fields in a Hořava-Lifshitz-like theory

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Abstract

In this work, we obtain the Casimir energy for the real scalar field and the Elko (a mass dimension one spinor fields) in an Hořava-Lifshitz (HL) spacetime. We analyze both the massless case as the massive case for both fields using dimensional regularization. We obtain a result of Casimir energy in terms of dimensional parameter and of the HL parameter. Particularizing our result, we can recover the usual results without HL parameter and in (3+1)dimensions present in the literature. Moreover, we compute the effects of the HL parameter in the thermal corrections for the massless scalar field.

Keywords: Casimir effect; Hořava-Lifshitz spacetime; Elko fields; Finite temperature

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I. INTRODUCTION

The Casimir Effect is characterized by force between two neutral parallel conducting plates, separated by a very short distance, in the vacuum \[1\]. Such effect was conceived by Hendrik Brugt Gerhard Casimir, in 1948. The Casimir Effect arises by the difference into vacuum expectation value of the energy due to the quantization of the electromagnetic field \[2, 3\]. Experimentally, the Casimir effect was verified at micrometer scale by Al plates in 1958 \[4\], and layers of Cu and Au in 1997 \[5\]. A modern review of experimental methods, issues, precisions and realistic measures is present in \[6, 7\]. Once that the zero energy can be modified by several factors, the Casimir force depends on many parameters, as the geometry and the boundary \[8–10\], the type of field studied \[10, 11, 13\], and the presence of extra-dimensions \[14\]. Moreover, some results of Casimir force were applied to black holes \[8\], cosmic strings \[9\] and the Hořava-Lifshitz(HL) theory \[15–17\]. In particular, the scalar fields was studied in the context of Casimir force in several works, as \[15–23\] which will be detailed along all the paper.

On another hand, the Elko fields (dual-helicity eigenspinors of the charge conjugation operator) \[24, 25\] are neutral fermion 1/2 fields with mass dimension one in (3 + 1) dimensions \[24, 25, 27, 28\]. Because this, the interactions of Elko spinor fields are restricted to only the gravity and the Higgs field. Hence these fields are dark matter candidates \[29, 30\]. Besides, Elko fields have several recent application into the cosmological inflation \[31–33\], Very Special Relativity (VSR) \[34\], Hawking radiation \[35\] and braneworlds \[36, 37\]. Moreover, some phenomenological constraints and attempts to detect Elko at the LHC have been proposed \[38, 39\]. The influence of Elko in the Casimir Effect on (3 + 1) spacetime was studied in Ref. \[13\], where the Casimir force differ both those found for the scalar fields as the Dirac fields. The Casimir force is repulsive for the Elko fields, due to the fact of the Elko fields be a spinor with anti-commutation relation \[13\].

In another way, in the point of view of modified gravity, the Hořava-Lifshitz theory allows the gravity to be power-counting renormalizable \[40\]. The HL theory introduces a space-time anisotropy by the scaling of coordinates such as \(x^i \rightarrow bx^i\) and \(t \rightarrow b^\xi t\), being \(b\) a length constant and \(\xi\) the critical exponent \[40\]. This proposal induces a Lorentz violation at high energies, however, the usual Lorentz symmetry is preserved at low energies \[15, 40\]. The HL theory has several applications in cosmology \[41–43\], black holes \[44, 45\], gravitational waves.
and black body radiation [48]. The Casimir force was studied in the influence of the HL theory in the Ref. [15, 16], where the z critical exponent can result in an attractive, repulsive and even a null Casimir force.

In this work, we deal with the Casimir force due to a massive real scalar field and the spinor Elko fields with mass dimension one in a modified gravity (3 + 1) Hořava-Lifshitz (HL) scenario. We will note that the Elko differ the Casimir energy for the scalar fields only by a multiplicative factor. In order to regularize the Casimir energy, we apply the dimensional regularization. Hence the Casimir force will be obtained in terms of main parameters: dimension $d$, mass $m$ and the HL critical exponent $\xi$. Due to the complexity of Casimir Energy, we will detail our results in term of particular choices of the mass and $\xi$. Moreover, we consider the influence of the HL parameter in the Casimir effect thermal corrections.

The papers follow the structure: in Sec. II we introduce the massive scalar field and the mass dimension fermion Elko into a Hořava-Lifshitz space-time. In Sec. III we present our results: In subsection III A we will obtain the results for the Casimir energy for the massless Elko in HL. We will recover both the results for the usual massless scalar field in a HL scenarios [15], such as the particular massless case of Elko present in Ref. [13]. In subsection III B we study the massive cases. The result for the massive scalar and Elko fields will be recovered for $\xi = 1$. Moreover, the first non-trivial result to HL with $\xi = 2$ for the massive Elko fields will be studied, and also the massive perturbative case for all $\xi$. In Sec. IV the role of the Hořava-Lifshitz in the temperature of massless scalar field is discussed. The paper finish in section V where the main results are summarized.

II. CASIMIR ENERGY IN HORAVA-LIFSHITZ-LIKE THEORY

Let us start computing the Casimir energy for a free massive real scalar fields $\phi$ in Hořava-Lifshitz-like theory (HL) through the following action in Natural units $\hbar = c = 1$:

$$S = \frac{1}{2} \int dt d^{D-1}x \left( \partial_0 \phi \partial_0 \phi - \ell^{2(\xi-1)} \partial_1 \partial_2 \cdots \partial_{\xi} \phi \partial_1 \partial_2 \cdots \partial_{\xi} \phi - m^2 \phi^2 \right),$$

(1)

with repeated Latin indices summed from one to $D - 1$, and $\xi$ is the above mentioned critical exponent (which will be considered in this work as a positive integer $\xi = 1, 2, 3, \ldots$). The usual gravity is recovered when $\xi = 1$. As required by consistency, the constant $\ell$ has
the dimension of length.

The corresponding equation of motion is

$$\left( \partial_0^2 + \ell^{2(\xi-1)}(-1)^\xi \partial_1 \cdots \partial_\xi \partial_1 \cdots \partial_\xi + m^2 \right) \phi(x) = 0, \quad (2)$$

In (3 + 1)-dimension, the operator $\partial_1 \cdots \partial_\xi \partial_1 \cdots \partial_\xi$ takes the following form:

$$\partial_1 \cdots \partial_\xi \partial_1 \cdots \partial_\xi = (\partial_x^2 + \partial_y^2 + \partial_z^2)^\xi, \quad (3)$$

which reads,

$$\left[ \partial_0^2 + \ell^{2(\xi-1)}(-1)^\xi \left( \partial_x^2 + \partial_y^2 + \partial_z^2 \right)^\xi + m^2 \right] \phi(x) = 0. \quad (4)$$

For two large parallel plates with area $L^2$ and separated into orthogonal $z$ axis by a small distance $a$ ($a \ll L$), the Dirichlet boundary conditions reads

$$\phi(x)_{z=0} = \phi(x)_{z=a} = 0. \quad (5)$$

Adopting the standard procedure [12], the quantum field can be written as

$$\hat{\phi}(x) = \sum_{n=1}^{\infty} \sqrt{\frac{2}{a}} \int \frac{d^2k}{(2\pi)^2} \frac{1}{2k_0} \sin \left( \frac{n\pi z}{a} \right) \left[ a_n(k)e^{-ikx} + a_n^\dagger(k)e^{ikx} \right], \quad (6)$$

$n$ is an integer and we have defined $kx \equiv k_0x_0 - k_xx - k_yy$, being

$$k_0 \equiv \omega_{n,\xi}(k) = \ell^{\xi-1} \sqrt{\mu^{2\xi} + \left( \frac{k_x^2 + k_y^2 + \left( \frac{n\pi}{a} \right)^2}{a} \right)^\xi}, \quad (7)$$

with $\mu^\xi \equiv m\ell^{1-\xi}$. The annihilation and creation operators $a_n(k)$ and $a_n^\dagger(k)$ obey the following commutation relations

$$\left[ a_n(k), a_{n'}^\dagger(k') \right] = (2\pi)^2 2\omega_{n,\xi}(k) \delta^{(2)}(k-k') \delta_{nn'},$$

$$[a_n(k), a_{n'}(k')] = \left[ a_n^\dagger(k), a_{n'}^\dagger(k') \right] = 0. \quad (8)$$

Hence, the Hamiltonian operator, $\hat{H}$, resulting from canonical quantization reads

$$\hat{H} = \sum_{n=1}^{\infty} \int \frac{d^2k}{(2\pi)^2} \frac{1}{2} \left( a_n^\dagger(k)a_n(k) + L^2 \omega_{n,\xi}(k) \right). \quad (9)$$

This expression leads to the vacuum energy for the massive real scalar field at the HL scenario given by

$$E_0^{(\text{scalar})} = \langle 0|\hat{H}|0 \rangle = \frac{L^2}{(2\pi)^2} \int d^2k \sum_{n=1}^{\infty} \frac{1}{2} \omega_{n,\xi}(k), \quad (10)$$
or more explicitly as

$$E_0^{(\text{scalar})} = \frac{\ell^{\xi-1} L^2}{(2\pi)^2} \int d^2 k \sum_{n=1}^{\infty} \frac{1}{2} \sqrt{\mu^{2\xi} + \left( k_x^2 + k_y^2 + \left( \frac{n\pi}{a} \right)^2 \right)^{\xi}}. \quad (11)$$

It is easy to see that the vacuum energy (11) is infinite and thus, some renormalization procedure must be applied to remove the divergences. Before studying the regularization of the Casimir energy (11), let us obtain the Casimir energy for the so-called Elko (an acronym for “eigenspinors of the charge conjugation operator”). The Casimir effect for the massive Elko fields in the usual flat spacetime was already be made in Ref. [13], in this work we perform a comparative of the massive Elko spinor and the massive scalar field in an HL theory.

The action for the free massive Elko field $\eta(x)$ in Hořava-Lifshitz-like theory is

$$S = \int dt d^{D-1} x \left( \partial_0 \tilde{\eta} \partial_0 \eta - \ell^{2(\xi-1)} \partial_1 \partial_{i_1} \cdots \partial_{i_2} \tilde{\eta} \partial_1 \partial_{i_1} \cdots \partial_{i_2} \eta - m^2 \tilde{\eta} \eta \right), \quad (12)$$

where the $\eta(x)$ represents the Elko (and its dual $\tilde{\eta}(x)$).

Again, the Elko equation of motion in $(3 + 1)$-dimension is similar to equation (4)

$$\left[ \partial_0^2 + \ell^{2(\xi-1)} (-1)^{\xi} \left( \partial_x^2 + \partial_y^2 + \partial_z^2 \right)^{\xi} + m^2 \right] \eta(x) = 0. \quad (13)$$

Furthermore, with the Dirichlet boundary conditions

$$\eta(x)_{z=0} = \eta(x)_{z=a} = 0, \quad \tilde{\eta}(x)_{z=0} = \tilde{\eta}(x)_{z=a} = 0, \quad (14)$$

the Elko quantum field operator stands for

$$\eta(x) = \sum_{n=1}^{\infty} \sqrt{\frac{2}{a}} \int \frac{d^2 k}{(2\pi)^2} \frac{\sin(n\pi z/a)}{\sqrt{2mk_0}} \sum_{\beta} \left[ a_{\beta,n}(k) \lambda_{\beta}^S(k)e^{-ikx} + a_{\beta,n}^\dagger(k) \lambda_{\beta}^A(k)e^{ikx} \right], \quad (15)$$

and

$$\tilde{\eta}(x) = \sum_{n=1}^{\infty} \sqrt{\frac{2}{a}} \int \frac{d^2 k}{(2\pi)^2} \frac{\sin(n\pi z/a)}{\sqrt{2mk_0}} \sum_{\beta} \left[ a_{\beta,n}^\dagger(k) \lambda_{\beta}^S(k)e^{ikx} + a_{\beta,n}(k) \lambda_{\beta}^A(k)e^{-ikx} \right], \quad (16)$$

with the same frequency $\omega_{n,\xi}(k)$ defined in Eq. (7) associated.

The spin one-half eigenspinors, $\lambda_{\beta}^{S/A}(k)$, satisfy the eigenvalue equation $C\lambda_{\beta}^{S/A}(k) = \pm \lambda_{\beta}^{S/A}(k)$, being $C$ the charge conjugation operator and $\lambda^S$ stands for the self-conjugate spinor (positive), while $\lambda^A$ stands for its anti self-conjugate (negative). Moreover, the helicity is represented by $\beta = (\{+, -\}, \{-, +\}) [13, 25, 26]$. 


The spinor and its dual satisfy the following orthonormality relations [25, 26]:

\[
\begin{align*}
\lambda_{\beta'}^S(k)\lambda_{\beta}^S(k) &= 2m\delta_{\beta\beta'}, \\
\lambda_{\beta'}^A(k)\lambda_{\beta}^A(k) &= -2m\delta_{\beta\beta'}, \\
\lambda_{\beta'}^S(k)\lambda_{\beta}^A(k) &= \lambda_{\beta'}^A(k)\lambda_{\beta}^S(k) = 0.
\end{align*}
\] (17)

In contrast to the bosonic scalar field, the creation and annihilation operators for the Elko fields satisfy anti-commutation relations like fermions [25, 26]:

\[
\{a_{\beta,n}(k), a_{\beta',n'}(k')\} = (2\pi)^2\delta^{(2)}(k - k')\delta_{\beta\beta'}\delta_{nn'},
\]

(18)

\[
\{a_{\beta,n}(k), a_{\beta',n'}(k')\} = \left\{a_{\beta,n}^\dagger(k), a_{\beta',n'}^\dagger(k')\right\} = 0.
\]

(19)

The Hamiltonian operator \(\hat{H}\) for the Elko fields is

\[
\hat{H} = \sum_{n=1}^{\infty} \int \frac{d^2k}{(2\pi)^2} 2\omega_{n,\xi}(k) \sum_{\beta} \left(a_{\beta,n}^\dagger(k) a_{\beta,n}(k) - a_{\beta,n}(k) a_{\beta,n}^\dagger(k)\right),
\]

(20)

and the corresponding vacuum energy for the Elko fields may be written as

\[
E_0^{(Elko)} = -4 E_0^{(scalar)} = -4 \frac{\ell^{\xi-1} L^2}{(2\pi)^2} \int d^2k \sum_{n=1}^{\infty} \frac{1}{2} \sqrt{\mu^2 \xi + \left(k_x^2 + k_y^2 + \left(\frac{n\pi}{a}\right)^2\right)^2}.
\]

(21)

As can be seen from equation (21) and (11), the difference between the vacuum energies of the Elko fields and the real scalar field lies on the -4 factor in front of the zero-point energy. The minus sign reminds us of the fermionic character of the Elko fields, and the factor of 4 refers to the fact that all the four degrees of freedom associated with \(\eta(x)\) carry a vacuum energy = \(-(1/2)\omega(k)\) [25, 26].

III. THE VACUUM ENERGY IN HORA-LIFSHITZ-LIKE THEORY

The vacuum energy represented by the integral (11) is infinite, and so some regularization scheme must be employed to remove this divergence. Several regularization methods can be implemented to calculate the Casimir energy; for example, the adoption of a well-behaved cutoff was used in Ref. [15] to determine the Casimir energy for the massless scalar field in the Hořava-Lifshitz scenario. In Ref. [16], the authors use the Abel-Plana formula in order to obtain the Casimir energy in this same context. In Ref. [13] the Casimir energy for the Elko fields was determined using the Poisson sum formula. A technique widely used in the
general context of quantum field theories is the so-called dimensional regularization, based on the analytical continuation in the spatial dimension number \( n \). This regularization scheme allows us to observe the behavior of the Casimir force concerning the dimension of the transverse space and leads to finite energies without any explicit subtraction [20, 21]. In what follows, we will use the dimensional regularization to evaluate the integral \((11)\) in different cases involving the mass \( m \) and the HL critical exponent \( \xi \).

**A. Casimir energy and Horava-Lifshitz modifications: massless case**

As the most straightforward case for the calculation of the Casimir energy modified by the HL critical exponent let us consider a real massless scalar field. In this case, the dimensionally regularized integral \((11)\) takes the form

\[
E_0^{(reg)}(\ell) = \ell^{\xi-1} L^d \sum_{n=1}^{\infty} \frac{d^d k}{(2\pi)^d} \left( k^2 + \left( \frac{n\pi}{a} \right)^2 \right)^{\xi/2},
\]

where \( d \) is the transverse dimension assumed as a continuous, complex variable. The term in brackets under the integral can be expressed conveniently through the Schwinger proper-time representation:

\[
\frac{1}{a^z} = \frac{1}{\Gamma(z)} \int_0^\infty dt t^{z-1} e^{-at}.
\]

After performing the moment integration, the equation \((22)\) becomes

\[
E_0^{(reg)}(\ell) = \frac{\ell^{\xi-1}}{2(4\pi)^{d/2} \Gamma(-\xi/2)} \sum_{n=1}^{\infty} \int_0^\infty dt t^{-\frac{(d+\xi+2)}{2}} e^{-\left(\frac{n\pi}{a}\right)^2 t},
\]

where \( E_0 \) is the energy density between the plates. The remaining integral can be made using the Euler Representation for gamma function, and the summation in \( n \) is carried out employing the definition of the Riemann zeta function. Therefore, the energy density is expressed by:

\[
E_0^{(reg)}(\ell) = \frac{\ell^{\xi-1}}{2(4\pi)^{d/2} \Gamma(-\xi/2)} \left( \frac{\pi}{a} \right)^{d+\xi} \Gamma\left( -\frac{(d+\xi)}{2} \right) \zeta(-d-\xi).
\]

We note that this expression is indeterminate for \( d + \xi \) positive integer even. Nevertheless, we can apply the reflection formula [19, 21, 49]

\[
\Gamma\left( \frac{d}{2} \right) \zeta(z) \pi^{-\frac{d}{2}} = \Gamma\left( \frac{1-z}{2} \right) \zeta(1-z) \pi^{z-1},
\]

\[
\Gamma\left( \frac{\xi}{2} \right) \zeta(z) \pi^{-\frac{\xi}{2}} = \Gamma\left( \frac{1-z}{2} \right) \zeta(1-z) \pi^{\frac{z-1}{2}},
\]

\[
\Gamma\left( \frac{1}{2} \right) \zeta(z) \pi^{-\frac{1}{2}} = \Gamma\left( \frac{1-z}{2} \right) \zeta(1-z) \pi^{\frac{z-1}{2}},
\]

\[
\Gamma\left( \frac{d+\xi}{2} \right) \zeta(z) \pi^{-\frac{d+\xi}{2}} = \Gamma\left( \frac{1-z}{2} \right) \zeta(1-z) \pi^{\frac{z-1}{2}},
\]
and rewrite the energy density (25) as

$$\mathcal{E}_0^{(\text{reg})} = \frac{\ell^{d-1}}{2^{d+1} a^{d+\xi} \pi^{\frac{d+1}{2}}} \Gamma\left(-\frac{\xi}{2}\right) \Gamma\left(d + \xi + 1\right) \zeta(d + \xi + 1),$$

which is finite for every $d$ and $\xi$ positive integer.

A case of particular interest is when $d = 2$, which implies a Casimir energy per unit area

$$\mathcal{E}_0^{(\text{cas})} \equiv \frac{E_0^{(\text{cas})}}{L^2} = -\frac{\ell^{d-1}}{2^\xi a^{d+\xi + 2} \pi^2} \sin\left(\frac{\pi \xi}{2}\right) \Gamma(\xi + 2) \zeta(\xi + 3),$$

which corresponds exactly to the result found in Ref. [16] obtained via Abel-Plana formula. In particular, we note that the Casimir energy is always zero for $\xi$ even and it is changing the signal to $\xi$ odd.

Once that the Casimir force per unit of area (Casimir pressure) is obtained by $\mathcal{F} = -\frac{\partial \mathcal{E}_0}{\partial a}$, let us explicit some values $\xi$ for the Casimir energy and force with for $d = 2$ in Table 1.

| Field   | $\xi = 1$ | $\xi = 2$ | $\xi = 3$ | $\xi = 4$ | $\xi = 5$ |
|---------|-----------|-----------|-----------|-----------|-----------|
| Scalar  | - $\frac{\pi^2}{1440 a^3}$ | 0         | $\frac{\pi^2}{5040 a^6}$ | 0         | $-\frac{\pi^2}{6720 a^7}$ |
| $\mathcal{F}$ | $-\frac{\pi^2}{480 a^4}$ | 0         | $\frac{\pi^2}{1008 a^6}$ | 0         | $-\frac{\pi^2}{960 a^7}$ |
| Elko    | $\frac{\pi^2}{360 a^3}$ | 0         | $-\frac{\pi^2}{1260 a^6}$ | 0         | $\frac{\pi^2}{1080 a^7}$ |
| $\mathcal{F}$ | $+\frac{\pi^2}{1200 a^4}$ | 0         | $-\frac{\pi^2}{2700 a^6}$ | 0         | $+\frac{\pi^2}{2400 a^7}$ |

Table I: The regularized Casimir energy per unit of area and the Casimir force per unit of area for the massless real scalar fields and the massless Elko with some values of critical exponent $\xi$ and $d = 2$.

From Table I we note that the result are in agreement with the massless scalar field in HL theory present in Ref. [15, 16]. In particular, for $\xi = 1$ we have the well-know case for the massless real scalar field in $(3 + 1)$ dimensions ($\mathcal{F}^{(\text{scalar})} = -\frac{\pi^2}{480 a^4}$) [21]. Moreover, the Elko fields with $\xi = 1$ and small masses agrees with the result found in Ref. [13] where ($\mathcal{E}_0^{(\text{reg})} = +\frac{\pi^2}{360 a^3} - \frac{m^2}{2 a}$).

**B. Casimir energy and Horava-Lifshitz modifications: massive case**

For the case of a massive real scalar field, the Casimir energy modified by the HL critical exponent $\xi$ is given by

$$\mathcal{E}_0^{(\text{reg})} = \frac{\ell^{d-1}}{\Gamma\left(-\frac{1}{2}\right) \Gamma\left(\frac{d}{2}\right) (4\pi)^{d/2}} \sum_{n=1}^{\infty} \int_0^\infty dt t^{-3/2} e^{-(\mu^2 + \xi^2) t} \int_0^\infty dk k^{d-1} e^{-\left(k^2 + \left(\frac{\mu}{2}\right)^2\right)^{\frac{d}{2}}},$$

where $\mu$ is the chemical potential.
where $\mu^\xi = m \ell^{1-\xi}$. In this general case, the integration over $k$ does not have a closed form in terms of known functions. Therefore, we will restrict ourselves to evaluating the exact form of (29) for some particular values of $\xi$.

Assuming $\xi = 1$ and proceeding in a similar way to the massless case, the equation (29) takes the form

$$E_{0}^{\text{(reg)}}(\xi = 1) = -\frac{1}{4\sqrt{\pi}(4\pi)^{\frac{d}{2}}} \Gamma\left(-\frac{d+1}{2}\right) \sum_{n=1}^{\infty} \left(m^2 + \frac{n^2 \pi^2}{a^2}\right)^{\frac{d+1}{2}}. \tag{30}$$

We can perform the summation using the functional relation (an Epstein-Hurwitz Zeta function type) [19, 22]

$$\sum_{n=-\infty}^{\infty} \left(n^2 + \mu^2\right)^{-s} = \frac{\sqrt{\pi}}{\sqrt{b}} \Gamma\left(s - \frac{1}{2}\right) \mu^{1-2s} + \frac{\pi^s}{\sqrt{b} \Gamma(s)} \sum_{n=-\infty}^{\infty} \mu^{\frac{1}{2}-s} \left(n \sqrt{b}\right)^{s-\frac{1}{2}} K_{\frac{s}{2}}\left(2\mu \sqrt{n} \sqrt{b}\right), \tag{31}$$

where $K_{\nu}(z)$ is the modified Bessel function and the prime means that the term $n = 0$ has to be excluded in the sum. After some algebra, it can be shown that the expression (30) results in

$$E_{0}^{\text{(reg)}}(\xi = 1) = -\frac{1}{2(4\pi)^{\frac{d+2}{2}}} \left\{-\sqrt{\pi} \Gamma\left(-\frac{d+1}{2}\right) m^{d+1} + am^{d+2} \Gamma\left(-\frac{d+2}{2}\right) + \frac{4m \frac{d+2}{4}}{a^{\frac{d+2}{2}}} \sum_{n=1}^{\infty} n^{-\frac{d+2}{2}} K_{\frac{d+2}{2}}(2amn)\right\}. \tag{32}$$

The first term in brackets does not depend on $a$, so it does not contribute to the Casimir force. The second term in brackets depends linearly on the distance between the plates and produces a constant Casimir force. It is in fact related to the Casimir energy of the vacuum in the absence of the plates and therefore can be discarded. The third term in brackets is the relevant part for the Casimir energy. Thus, we find the following Casimir energy density for $\xi = 1$:

$$E_{0}^{\text{(cas)}}(\xi = 1) = -2 \left(\frac{am}{4\pi}\right)^{\frac{d+2}{4}} \frac{1}{a^{d+1}} \sum_{n=1}^{\infty} n^{-\frac{d+2}{2}} K_{\frac{d+2}{2}}(2amn), \tag{33}$$

which recovers the classical result for the Casimir energy for Dirichlet condition [21]. For completeness, let us evaluate the general result (33) at $d = 2$. This expression implies a Casimir energy

$$E_{0}^{\text{(cas)}}(d = 2, \xi = 1) = -\frac{L^2}{8\pi^2} \frac{m^2}{a} \sum_{n=1}^{\infty} \frac{1}{n^2} K_2(2amn), \tag{34}$$
whose asymptotic behavior for \( m \ll a^{-1} \) gives
\[
E_{0}^{(\text{cas})}(d = 2, \xi = 1) = -\frac{L^2 \pi^2}{1440} \frac{1}{a^3} + \frac{L^2 m^2}{96} \frac{1}{a} + \cdots ,
\tag{35}
\]
so that the first term corresponds to the Casimir energy for the usual massless scalar field.

At the other extreme, i.e., for \( m \gg a^{-1} \), the Casimir energy (34) decays exponentially with the mass of the particle.

\[
E_{0}^{(\text{cas})}(d = 2, \xi = 1) = -\frac{L^2}{16} \frac{m^2}{\pi^2} \frac{1}{a} \left( \frac{\pi}{ma} \right)^{1/2} e^{-2ma},
\tag{36}
\]
leading to a small force at the non-relativistic limit \([21, 23]\).

Now, we consider the massive case when the HL critical exponent assumes the value of \( \xi = 2 \). In this case, the integrations over \( k \) and on the \( t \) parameter can still be performed analytically. We can show that for this case the Eq. (29) results in
\[
\mathcal{E}_{0}^{(\text{reg})} = -\frac{L^2 \mu^{2\xi}}{2d+2^\frac{d+1}{2}} \left\{ \frac{\Gamma \left( \frac{d+1}{2} \right) \Gamma \left( \frac{d-2}{4} - 1 \right)}{\Gamma \left( \frac{d}{4} + 1 \right)} \sum_{n=1}^{\infty} \left( \frac{n\pi}{a} \right)^{d+2} _2F_1 \left( -\frac{d-2}{4}, -\frac{d-1}{4}; 1; \left( \frac{a\mu}{n\pi} \right)^4 \right) + \Gamma \left( \frac{d-1}{2} \right) \sum_{n=1}^{\infty} \left( \frac{n\pi}{a} \right)^d \left( \frac{a^4 + \left( \frac{n\pi}{a} \right)^4}{d+1} \right)^{\frac{d+1}{2}} _2F_1 \left( \frac{d}{4}; \frac{d+1}{4}; \frac{d+3}{2}; \left( \frac{a\mu}{n\pi} \right)^4 + 1 \right) \right\} ,
\tag{37}
\]
where \(_2F_1(a, b; c; z)\) is the hypergeometric function. This expression is convergent only to \( d < -2 \) and we have not found any proprieties (such as reflection/duplication) for the \( \Gamma \) or \( \zeta \) functions that converge the energy for \( d > 0 \).

**Perturbative case with arbitrary \( \xi \)**

Now we will consider a perturbative approach to the massive case in which we assume \( \mu^\xi \equiv m^\xi a^\xi \ll a^{-1} \), such that we can write
\[
\left[ \mu^{2\xi} + \left( k^2 + \left( \frac{n\pi}{a} \right)^2 \right)^\xi \right]^{\frac{1}{2}} = \left[ k^2 + \left( \frac{n\pi}{a} \right)^2 \right]^{\frac{\xi}{2}} + \mu^{2\xi} \left[ k^2 + \left( \frac{n\pi}{a} \right)^2 \right]^{-\frac{\xi}{2}} + \mathcal{O}(\mu^{2\xi}).
\tag{38}
\]
Substituting the above expression in (11), the first term in (38) results in the Casimir energy for the massless scalar field identical to that obtained in (27). The second term gives rise to correction due to mass and can be put into the form
\[
\Delta \mathcal{E}_{0}^{(\text{reg})}(d, \xi, m) = \frac{m^2 \ell^{1-\xi}}{2d+2^\frac{d+1}{2} a^{d-\xi}} \Gamma \left( \frac{d-\xi+1}{2} \right) \zeta \left( d - \xi + 1 \right).
\tag{39}
\]

To $d = 2$

$$
\Delta \xi_0^{(\text{reg})}(d = 2, \xi, m) = \frac{m^2 \ell^{1-\xi}}{2^{5-\xi} \pi^2 a^{2-\xi}} \sin \left( \frac{\pi \xi}{2} \right) \Gamma (2 - \xi) \zeta (3 - \xi)
$$

$$
= \frac{m^2 \pi^{1-\xi} a^{\xi-2} \ell^{1-\xi}}{8(\xi - 2)} \zeta (\xi - 2),
$$

where we use the reflection (26) and the Legendre duplication formula $\Gamma(s) = 2^{s-1} \pi^{-s/2} \Gamma(s/2) \Gamma(s + 1/2)$. Considering $\xi > 0$, the equation (40) diverge only for $\xi = 2$ and $\xi = 3$, as can be noted in Table II

| $\xi$ | $E_0^{(\text{reg},1)}$ | $E_0^{(\text{reg},1)}$ | $E_0^{(\text{reg},1)}$ | $E_0^{(\text{reg},1)}$ | $E_0^{(\text{reg},1)}$ | $E_0^{(\text{reg},1)}$ |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $\xi = 1$ | $\frac{m^2}{96a^2}$ | $\infty$ | $\infty$ | $\frac{a^2 m^2}{90 \pi \ell}$ | $\frac{a^3 m^2 \zeta (3)}{24 \pi \ell^3}$ | $\frac{a^4 m^2}{2880 \pi \ell^5}$ |
| $\xi = 2$ | | $\infty$ | | $\frac{a^3 m^2 \zeta (3)}{24 \pi \ell^3}$ | | $\frac{a^5 m^2 \zeta (5)}{40 \pi^4 \ell^7}$ |
| $\xi = 3$ | | | $\frac{a^4 m^2 \zeta (5)}{40 \pi^4 \ell^7}$ | | | |

Table II: Pertubative case of regularized Casimir energy per unit of area for the massive scalar fields with some values of critical exponent $\xi$ and $d = 2$.

### IV. THE ROLE OF THE HORA-LIFSHITZ PARAMETER IN THE TEMPERATURE

In this section, we study the thermal corrections to the Casimir energy for the massless case of the scalar field in a Hořava-Lifshitz theory. We obtain the dependence of temperature effect in term of parameters $d$ and $\xi$. Following the approach described in Refs. [3, 21, 23], the Casimir free energy $F$ can be decomposed in the form

$$
F = F_0^{(\text{cas})} + \Delta_T F,
$$

where the $F_0^{(\text{cas})}$ is the zero temperature energy contribution (previously computed in Eq. (27)) and the $\Delta_T F$ the finite temperature energy contribution which has the form [3, 21]:

$$
\Delta_T F = \kappa_B TV \int \frac{d^d k}{(2\pi)^{d+1}} \frac{\pi}{a} \sum_{n=-\infty}^{+\infty} \ln \left( 1 - e^{-\beta \omega_{n,\xi}(k)} \right),
$$

where $\omega_{n,\xi}(k) = \ell^{\xi-1} \left[ k^2 + \left( \frac{an}{\ell} \right)^2 \right]^{\xi/2}$, $\beta = (\kappa_B T)^{-1}$, being $T$ the temperature and $\kappa_B$ the Boltzmann constant, and $V = L^d a$ is the $(d + 1)$-dimensional volume between the plates.

Now we study the high and the low limits for the temperatures. For high temperature ($\beta \ll 1$), we can expand the exponential and keep the first order term in $\beta$. So, we can write
Eq. (42) as
\[
\Delta_{T\gg 1} F = \frac{k_B T V \xi}{2a (4\pi)^{\frac{d}{2}} \Gamma (\frac{d}{2})} \frac{d}{ds} \left\{ \left( \beta \ell^{-1} \right)^{\frac{2}{\xi}} \sum_{n=-\infty}^{+\infty} \int_{0}^{\infty} dk \, k^{d-1} \left[ k^2 + \left( \frac{n\pi}{a} \right)^2 \right]^s \right\}_{s=0},
\]
where the following identity has been used \( \ln x = \left( \frac{d}{ds} x^s \right)_{s=0} \). Integrating over \( k \), we get
\[
\Delta_{T\gg 1} F = \frac{k_B T V \xi}{2a (4\pi)^{\frac{d}{2}}} \left( \frac{\pi}{a} \right)^{\frac{d}{2}} \Gamma \left( \frac{d}{2} \right) \left( \frac{d}{2} + s \right) \zeta (-d - 2s) \right\}_{s=0}. \tag{44}
\]
Using the reflection property, after deriving in \( s \) and taking the limit \( s \to 0 \), we find
\[
\Delta_{T\gg 1} F = -\xi \frac{k_B T V}{(2a \sqrt{\pi})^{d+1}} \Gamma \left( \frac{d+1}{2} \right) \zeta (d+1). \tag{45}
\]
Note that the difference with the usual case \[21\] is the presence of the multiplicative factor \( \xi \). Moreover, as a classical limit \[3, 21\], the dependence of \( \ell^{-1} \) is gone.

To \( d = 2 \), we have \( \Delta_{T\gg 1} F(d = 2) = -\xi (L^2 k_B T / 16 \pi a^2) \zeta (3) \), which results in a thermal correction to the Casimir pressure \( F = -\frac{\partial}{\partial a} \Delta_{T\gg 1} F \) in the form
\[
\Delta_{T\gg 1} F(d = 2) = -\xi \frac{k_B T V}{8 \pi a^3} \zeta (3). \tag{46}
\]
Again, we note that the usual thermal correction to the Casimir pressure \[2, 3, 19, 21\] is modified by a simple multiplicative HL factor \( \xi \). This implies that, in the high temperature limit, the temperature is linearly increased by the HL parameter.

For low temperature \( (\beta \gg 1) \), we start from the formula for Helmholtz free energy:
\[
\Delta_{T\ll 1} F = \frac{k_B T V}{a (4\pi)^{\frac{d}{2}} \Gamma (\frac{d}{2})} \int_{0}^{\infty} dk \, k^{d-1} \left\{ \sum_{n=-\infty}^{+\infty} \ln \left[ 1 - \exp \left( -\beta \ell^{-1} \left( k^2 + \frac{n^2 \pi^2}{a^2} \right)^{\xi/2} \right) \right] \right\}. \tag{47}
\]
Now, let us first consider only the gapless case \( (n = 0) \):
\[
\Delta_{T\ll 1} F_{n=0} = -\frac{k_B T V}{a (4\pi)^{\frac{d}{2}} \Gamma (\frac{d}{2})} \int_{0}^{\infty} dk \, k^{d-1} \ln \left[ 1 - e^{-\beta \ell^{-1} k^2} \right]. \tag{48}
\]
Using the expansion
\[
\ln (1 - x) = -\sum_{m=1}^{\infty} \frac{x^m}{m}, \quad \text{(convergent for } |x| < 1),
\]
and after integration in \( k \), we obtain
\[
\Delta_{T\ll 1} F_{n=0} = -\frac{V \ell^{d-2} \left( k_B T \right)^{d+1} \Gamma \left( \frac{d}{\xi} \right) \zeta \left( \frac{d+\xi}{\xi} \right)}{a (4\pi)^{\frac{d}{2}} \xi \Gamma \left( \frac{d}{2} \right)}. \tag{50}
\]
The above equation shows that the power-like behavior of the Casimir thermal corrections is modified by the presence of the HL parameter $\xi$. In the particular case $d = 2, \xi = 1$, we recover the usual result [3, 23]:

$$
\Delta T_{\ll 1} F_{n=0}(d = 2, \xi = 1) = -L^2 \frac{(\kappa_B T)^3}{4\pi} \xi. \quad (51)
$$

Next, we consider the case of a nonzero $n$. It is clear that the dominating contribution comes from small $k$ and the smallest $n \neq 0$. We substitute $\ln(1-x)$ for $-x$ and expand the exponent in equation (47) for small $k$. Then, after integration we obtain

$$
\Delta T_{\ll 1} F_{n \neq 0} = -\frac{V \kappa_B T}{2^{\frac{d}{2}} \pi^\frac{d}{2}(\xi-1)} \frac{d}{a} \sum_{n=1}^{\infty} e^{-\beta \xi^{-1} \left(\frac{a}{n}\right)^\xi \left(\frac{d}{2} - \frac{d}{2}\right)} \exp \left[ -\frac{n}{\kappa_B T} \left(\frac{\pi}{\xi}\right)^\xi \right], \quad (52)
$$

The dominant contribution is obtained by taking $n = 1$. Thus,

$$
\Delta T_{\ll 1} F_{n=1} = -\frac{V (\kappa_B T)^\frac{d+1}{2}}{2\pi^\frac{d}{2}(\xi-1)^2} \frac{d}{a} \exp \left[ -\frac{\xi^{-1}}{\kappa_B T} \left(\frac{\pi}{\xi}\right)^\xi \right], \quad (53)
$$

showing an exponentially suppressed contribution.

Usual case for the low temperature limit is obtained by substituting $d = 2, \xi = 1$ [3, 23]:

$$
\Delta T_{\ll 1} F_{n=1}(d = 2, \xi = 1) = -\frac{L^2 (\kappa_B T)^2}{2a} \exp \left( -\frac{\pi}{\alpha \kappa_B T} \right). \quad (54)
$$

In $d = 2$, the total dominant contribution for low temperature is the sum $\Delta T_{\ll 1} F_{n=0} + \Delta T_{\ll 1} F_{n=1}$, which gives the $\Delta T_{\ll 1} F$ as

$$
\Delta T_{\ll 1} F = -\frac{L^2 \kappa_B T}{4\pi \xi} \left[ \frac{\kappa_B T}{\ell^{\xi-1}} \frac{\ell}{\xi} \Gamma \left(\frac{2}{\xi}\right) \left(\frac{2 + \xi}{\xi}\right) + \frac{2\pi \kappa_B T}{(\pi \ell)^{\xi-1} a^{2-\xi}} e^{-\frac{\xi-1}{\pi B^2} (\pi a)^{\xi}} \right]. \quad (55)
$$

For $\xi = 1$, the equation (56) coincides with its non-HL version present in Ref. [23] (Less than a 1/2 factor due to degrees of freedom). In contrast to the high-temperature case, the thermal correction is proportional to $1/\xi$ and decreases when $\xi$ increase.

Finally, the Casimir pressure from (55), discarding second order terms in the temperature, reads

$$
\Delta T_{\ll 1} F (d = 2) = \frac{\pi}{2a^3} \kappa_B T e^{-\frac{\xi-1}{\pi B^2} (\pi a)^{\xi}}, \quad (56)
$$

and this expression for $\xi = 1$ is identical those in Ref. [23]. The Casimir pressure for the low temperature limit is only modified by the HL parameter $\xi$ in the exponential decay factor.
V. CONCLUSIONS

In this work, we perform the calculation of Casimir energy and force for two fields: the real scalar field and the mass dimension one Elko fields in an Hořava-Lifshitz (HL) spacetime. We study the motion equation for both fields and remark its differences. Using the dimensional regularization, we obtain the expression for the Casimir effect in term of dimensional parameters $d$ and the HL parameter $\xi$. Our results generalize those obtained in Refs. [15, 16] for the massless scalar field and also for the Elko fields in Ref. [13]. For the massive case, we observe that the Casimir energy modified by the HL critical exponent becomes singular when $\xi = 2$ and $\xi = 3$. Besides, we study the thermal correction to the Casimir effect for the high-temperature and the low-temperature limits. At both the limits, the HL parameter modifies the usual results. It increases the thermal corrections linearly for high-temperature and decreases for low-temperature limit.

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