HIGHER ORDER SYMMETRIC DUALITY FOR MULTIOBJECTIVE FRACTIONAL PROGRAMMING PROBLEMS OVER CONES

Arshpreet KAUR
School of Mathematics, Thapar Institute of Engineering and Technology (Deemed University), Patiala 147004, India
arshpreet.kaur@thapar.edu

Mahesh Kumar SHARMA
School of Mathematics, Thapar Institute of Engineering and Technology (Deemed University), Patiala 147004, India
mksharma@thapar.edu

Received: June 2020 / Accepted: February 2021

Abstract: This article studies a pair of higher order nondifferentiable symmetric fractional programming problem over cones. First, higher order cone convex function is introduced. Then using the properties of this function, duality results are set up, which give the legitimacy of the pair of primal dual symmetric model.

Keywords: Higher order symmetric duality, Higher order \((\Phi, \rho)\)-convexity, Fractional Programs, Nondifferentiable programs, Generalized convexity.

MSC: 90C29, 90C30, 90C32, 90C46.

1. INTRODUCTION

In mathematical programming, symmetric programs are those programs in which primal is the dual of the dual. In other words, the programming problems in which the dual of the dual is primal again, are the symmetric programming problems. Linear programs naturally fall into this category of programs. But for nonlinear programs, such occurrence is quite exceptional. Dorn [7] first studied symmetric quadratic programs, and Dantzig [6] formulated symmetric nonlinear programs and established weak and strong duality theorems.
Bazaraa and Goode [1] generalized the formulation of symmetric duality introduced in [6] to include the case where the inequality constraints are defined via convex cones and their polar sets. Mond [19] studied nonsmooth functions called support functions of a compact convex set thus introducing non-differentiable symmetric primal dual pairs. Gulati et al. [10] formulated multiobjective symmetric type programs and gave duality results for Wolfe and Mond-Weir type symmetric dual multiobjective programming problems. The symmetric programs in which the objective function is a ratio of two functions, namely fractional programs, were given by Chandra et al. [4]. The notion was further extended to a multiobjective fractional symmetric program by Weir [24]. Another class of fractional symmetric programs are studied by Jayswal and Jha [13]. Kim et al. [17] studied multiobjective symmetric program with cone constraints, which was later extended to a non-differentiable multiobjective program involving cones in [16].

As it is known that dual gives a bound on the value of the primal program, the second and higher order duals give further tighter bounds due to the addition of parameters. So they help in finding better approximation to the value of the primal problem. Bector and Chandra [2] introduced second order symmetric dual program for pseudobonvex and pseudoboncave functions. The multiobjective counterpart was studied by Yang and Hou [25], which was further extended to a multiobjective program over cone constraints for second order cone convex functions in [18]. Gulati and Mehndiratta [11] considered a non-differentiable multiobjective symmetric dual pair involving arbitrary cones, thus generalizing the existing classes.

Talking about higher order duality, Gulati and Gupta [9] first studied higher order duality for a symmetric program. Then, Chen [5] discussed about higher order multiobjective non-differentiable symmetric program. Gupta et al. [12] introduced higher order \((F, \alpha, \rho, d)\)-convex functions and studied Wolfe and Mond-Weir type dual symmetric models, whereas Suneja and Louhan [21] studied higher order symmetric programs with cone invexity and cone constraints. Recently, higher order multiobjective non-differentiable fractional symmetric programs with cone constraints are studied in [8, 23]. Some higher order programs are also discussed in [15].

In this paper, motivated by the work of Dubey and Gupta [8], we study fractional vector optimization problems in which constraints are defined over cones and the ordering of the objectives is described with respect to some closed convex cones. This aspect of symmetric programs is not studied so far. The class of functions used in this direction is higher order \((\Phi, \rho)\)-cone convex function. Lastly, we formulate and prove weak, strong, and converse duality theorems.

2. PRELIMINARIES AND DEFINITIONS

The preference among the alternatives must conform to the decision maker’s inclinations. So, a suitable domination structure is defined to find an optimized solution of a mathematical program. This leads to the study of mathematical programming problems over arbitrary cones. Let \(\mathbb{R}^k\) be a \(k\)-dimensional Euclidean
space and $\mathbb{R}^k_+$ denote its nonnegative orthant. Let $K$ be a closed convex pointed cone in $\mathbb{R}^k$ with non-empty interior. Consider a general vector optimization problem in which ordering is defined with respect to the convex cone $K$:

$$
(VP) \quad \text{K - Minimize} \quad f(x)
$$

where $S_0$ is the set of feasible solutions and $S_0 \subseteq X \subseteq \mathbb{R}^n$ and $f : X \to \mathbb{R}^k$.

**Definition 1.** A point $u \in S_0$ is a weak efficient solution of $(VP)$ if $\exists \ x \in S_0$ such that $f(u) - f(x) \in \text{int} \ K$. A point $u \in S_0$ is an efficient solution of $(VP)$ if $\exists \ x \in S_0$ such that $f(u) - f(x) \in K \setminus \{0\}$.

Now we give definition of generalized convex functions named as $K - (\Phi, \rho)$ convex functions. First we give a brief overview of $(\Phi, \rho)$ convexity and its generalizations. The concept of $(\Phi, \rho)$ convexity was set forth by Caristi et al. [3] to extend the class of $(F, \rho)$ convex functions. The following definitions have made grounds for the definition introduced in this paper.

Consider a vector valued function $f = (f_1, f_2, \ldots, f_k) : X \to \mathbb{R}^k$ differentiable on $X$, so we have component functions $f_i$ given by $f_i : X \to \mathbb{R}$ for $i = 1, 2, \ldots, k$ and a vector $\rho = (\rho_1, \rho_2, \ldots, \rho_k) \in \mathbb{R}^k$.

For a natural number $n$ and a set $X \subseteq \mathbb{R}^n$ consider a functional $\Phi : X \times X \times \mathbb{R}^{n+1} \to \mathbb{R}$. Any element of $\mathbb{R}^{n+1}$ takes the form $(a, b)$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

**Definition 2.** $\Phi$ is convex on $\mathbb{R}^{n+1}$ if, for fixed $x, u \in X$, the following holds:

$$
\Phi(x, u; (\lambda(\xi_1, \rho_1) + (1 - \lambda)(\xi_2, \rho_2))) \leq \lambda \Phi(x, u; (\xi_1, \rho_1)) + (1 - \lambda) \Phi(x, u; (\xi_2, \rho_2)),
$$

for all $\xi_1, \xi_2 \in \mathbb{R}^n$, $\rho_1, \rho_2 \in \mathbb{R}$, and $\lambda \in [0, 1]$. Throughout this paper, we assume that $\Phi(x, u; (0, r)) \geq 0$ for $x, u \in X$, and $r \in \mathbb{R}_+$.

The $(\Phi, \rho)$ convex functions introduced in [3] are defined as follows.

**Definition 3.** The scalar valued functions $f_i : X \to \mathbb{R}$, is $(\Phi, \rho_i)$ convex on $X$ if

$$
f_i(x) - f_i(u) \geq \Phi(x, u; (\nabla f_i(u), \rho_i)), \quad \forall \ x \in X
$$

and some $\rho_i \in \mathbb{R}$.

This class of functions were extended to cone $(\Phi, \rho)$ convex functions by Kapoor [14] who gave the following definition.

**Definition 4.** [14] $f : X \to \mathbb{R}^k$ is $K - (\Phi, \rho)$ convex at $u$ on $X$ if for every $x \in X$, the following holds:

$$
f(x) - f(u) - \Phi(x, u; (\nabla f(u), \rho)) \in K.
$$

In this, vector $\Phi$ is given by

$$
\Phi(x, u; (\nabla f(u), \rho)) = (\Phi(x, u; (\nabla f_1(u), \rho_1)), \Phi(x, u; (\nabla f_2(u), \rho_2)), \ldots, \Phi(x, u; (\nabla f_k(u), \rho_k)).
$$
The higher-order convex functions were defined by Tripathy and Devi [22] in the next definition.

**Definition 5.** A scalar function $f_1$ given by $f_1 : X \to \mathbb{R}$ is higher-order $(\Phi, \rho_1)$-invex at $u \in X$ with respect to $h_1 : X \times \mathbb{R}^n \to \mathbb{R}$ if there exists real functional $\Phi$ and scalar $\rho_1 \in \mathbb{R}$ such that for all $x \in X$ we have

$$f_1(x) - f_1(u) - h_1(u, p) + p^T \nabla \rho_1 h_1(u, p) \geq \Phi(x, u; (\nabla f_1(u) + \nabla \rho_1 h_1(u, p), \rho_1)).$$

Now we define higher-order $K - (\Phi, \rho)$-convex functions. For this, assume $f, \Phi, \rho$ as defined above and $F : X \times \mathbb{R}^n \to \mathbb{R}^k$ be a differentiable function.

**Definition 6.** A function $f$ is higher-order $K - (\Phi, \rho)$-convex at $u \in X$ with respect to $F$ and $\rho$ where $p = (p_1, p_2, ..., p_k)$ and each $p_i \in \mathbb{R}^n$, if there exists real functional $\Phi$ and $\rho$ such that for all $x \in X$ we have

$$\begin{bmatrix}
f_1(x) - f_1(u) - F_1(u, p_1) + p_1^T \nabla_{p_1} F_1(u, p_1) - \Phi(x, u; (\nabla f_1(u) + \nabla_{p_1} F_1(u, p_1), \rho_1)) \\
f_2(x) - f_2(u) - F_2(u, p_2) + p_2^T \nabla_{p_2} F_2(u, p_2) - \Phi(x, u; (\nabla f_2(u) + \nabla_{p_2} F_2(u, p_2), \rho_2)) \\
\vdots \\
f_k(x) - f_k(u) - F_k(u, p_k) + p_k^T \nabla_{p_k} F_k(u, p_k) - \Phi(x, u; (\nabla f_k(u) + \nabla_{p_k} F_k(u, p_k), \rho_k))
\end{bmatrix} \in K$$

**Special Cases:**

1. In this definition if $k = 1$ and $K = \mathbb{R}_+$ then we have the higher-order $(\Phi, \rho)$-convex functions defined by Tripathy and Devi [22].

2. If $p = 0$ or no approximation functions are used, then we have $K - (\Phi, \rho)$-convex functions defined by Kapoor [14]. In addition to this, if $k = 1$ and $K = \mathbb{R}_+$ then the function reduce to $(\Phi, \rho)$-convex functions defined by Caristi et al. [3].

A class of higher-order $(\Phi, \rho)$ convex functions is also discussed in [20].

**Definition 7.** For a cone $K$, the positive polar cone(or dual cone) of $K$, denoted by $K^*$, is defined as

$$K^* = \{y \in \mathbb{R}^n : x^T y \geq 0 \ \forall \ x \in K\}$$

We consider a non-differentiable problem in this paper. The non-differentiable part is due to support function and we briefly discuss the related notions below.

**Definition 8.** If $C$ is a compact and convex subset of $\mathbb{R}^n$, then support function of $C$ at $x$ is given by $S(x|C) := \max \{x^T y : y \in C\}$. This function being convex and finite has subdifferential

$$\partial S(x|C) := \{z \in C : x^T z = S(x|C)\}.$$
Definition 9. If $D \subseteq \mathbb{R}^n$ is convex, then the normal cone at a point $z$ in $D$ is defined as:

$$ND(z) := \{y \in \mathbb{R}^n : y^T(x - z) \leq 0, \forall x \in D\}$$

If $D$ is compact convex set, then taking into consideration both the definitions, one can infer that $y \in ND(z) \iff z^T y = S(y|D)$ or say $z \in \partial S(y|D)$.

Consider $F(\cdot, \cdot)$ to be continuously differentiable such that $F(x, y) : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, then

- $\nabla_x F, \nabla_y F$ denote gradient vectors with respect to $x, y$, respectively.
- $\nabla_{xx} F, \nabla_{yy} F$ are $n_1 \times n_1$ and $n_2 \times n_2$ symmetric Hessian matrices respectively.

3. HIGHER ORDER SYMMETRIC PROGRAMS

In this section we introduce multiobjective fractional symmetric primal dual pair. Let us suppose that $S_1$ and $S_2$ are two non-empty open sets in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively. Further, suppose that $A_1$ and $A_2$ are closed and convex cones in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively, such that $A_1 \times A_2 \subset S_1 \times S_2$. Consider the following non-differentiable multiobjective fractional symmetric programs (MFNSP) and (MFNSD):

**Primal (MFNSP)**

$$K - \text{Min} \quad L(x, y, p) = (L_1(x, y, p), L_2(x, y, p), \ldots, L_k(x, y, p))$$

where $L_i(x, y, p) = \frac{f_i(x, y) + S(x|B_i) - y^T z_i + F_i(x, y, p_i) - p_i^T \nabla_{p_i} F_i(x, y, p_i)}{g_i(x, y) - S(x|E_i) + y^T r_i + G_i(x, y, p_i) - p_i^T \nabla_{p_i} G_i(x, y, p_i)}$

subject to

$$-\sum_{i=1}^k \lambda_i (\nabla_y f_i(x, y) - z_i + \nabla_{p_i} F_i(x, y, p_i)$$

$$- L_i(x, y, p_i)(\nabla_y g_i(x, y) + r_i + \nabla_{p_i} G_i(x, y, p_i)) \in A_2^\ast,$$

$$-L_i(x, y, p_i)(\nabla_y g_i(x, y) + r_i + \nabla_{p_i} G_i(x, y, p_i)) \geq 0,$$

$\lambda \in \text{int} K^\ast, \; x \in A_1, \; z_i \in D_i, \; r_i \in F_i, \; i = 1, 2, \ldots, k, \; \lambda^T e = 1.$

**Dual (MFNSD)**

$$K - \text{Max} \quad M(u, v, q) = (M_1(u, v, q), M_2(u, v, q), \ldots, M_k(u, v, q))$$

where $M_i(u, v, q) = \frac{f_i(u, v) - S(v|D_i) + u^T w_i + F_i(u, v, q_i) - q_i^T \nabla_{q_i} F_i(u, v, q_i)}{g_i(u, v) + S(v|E_i) - u^T t_i + G_i(u, v, q_i) - q_i^T \nabla_{q_i} G_i(u, v, q_i)}$
subject to
\[ \sum_{i=1}^{k} \lambda_i (\nabla_x f_i(u,v) + w_i + \nabla_q \tilde{F}_i(u,v,q_i)) - M_i(u,v,q_i)(\nabla_x g_i(u,v) - t_i + \nabla_q \tilde{G}_i(u,v,q_i)) \in A_1^*, \]
\[ u^T \sum_{i=1}^{k} \lambda_i (\nabla_x f_i(u,v) + w_i + \nabla_q \tilde{F}_i(u,v,q_i)) - M_i(u,v,q_i)(\nabla_x g_i(u,v) - t_i + \nabla_q \tilde{G}_i(u,v,q_i)) \leq 0, \]
\[ \lambda \in \text{int} K^*, v \in A_2, w_i \in B_i, t_i \in E_i, \quad i = 1, 2, \ldots, k, \lambda^T e = 1. \]

For \( i = 1, 2, \ldots, k \) the following assumptions have been used while constructing the above programs pair:

1. \( f_i, g_i : S_1 \times S_2 \to \mathbb{R} \) are differentiable functions,

2. The differentiable functions \( F_i, G_i, \tilde{F}_i, \tilde{G}_i \) are such that \( F_i, G_i : S_1 \times S_2 \times \mathbb{R}^{n_2} \to \mathbb{R} \), are higher-order approximation functions of \( f_i, g_i \), respectively, with respect to second variable. \( \tilde{F}_i, \tilde{G}_i : S_1 \times S_2 \times \mathbb{R}^{n_1} \to \mathbb{R} \) are higher-order approximation functions of \( f_i, g_i \), respectively, with respect to first variable.

3. \( B_i, E_i \) are compact convex sets in \( \mathbb{R}^{n_1} \) and \( D_i, F_i \) are compact convex sets in \( \mathbb{R}^{n_2} \),

4. \( p_i \in \mathbb{R}^{n_2}, q_i \in \mathbb{R}^{n_1}, \lambda \in \mathbb{R}^k \),

5. \( A_1^*, A_2^* \) are positive polar cones of \( A_1, A_2 \), respectively,

6. in the feasible region, the numerators and denominators are assumed to be nonnegative and positive, respectively.

**Special Cases:**

1. If \( K = \mathbb{R}^k_+ \), \( F_i = H_i, \phi_i = \tilde{F}_i, \xi_i = \tilde{G}_i \) and \( C_1 = A_1, C_2 = A_2 \), then this above discussed model becomes the one discussed by Dubey and Gupta [8].

2. If \( k = 1, A_1 = \mathbb{R}^{n_1}_+, A_2 = \mathbb{R}^{n_2}_+ \), with all the higher-order approximations are taken to be zero, or \( p_i, q_i = 0 \) and the sets \( B_i = E_i = F_i = \{0\} \). Then the symmetric programs (MFNSP) and (MFNSD) reduce to the symmetric fractional program discussed by Chandra et al. [4].

To make the theorems easier, the following parametric program has been formulated. We take two parameters \( l = (l_1, l_2, \ldots, l_k) \), and \( m = (m_1, m_2, \ldots, m_k) \) and express the programs (MFNSP) and (MFNSD) equivalently as multiobjective
non-differentiable symmetric programs (EMNSP) and (EMNSD), respectively.

Primal (EMNSP)

\[ K - \text{Min} \quad l = (l_1, l_2, \ldots, l_k) \]

subject to

\[
\begin{align*}
& f_i(x, y) + S(x) B_i - y^T z_i + F_i(x, y, p_i) - p_i^T \nabla_p F_i(x, y, p_i) \\
& - l_i(g_i(x, y) - S(x) E_i) + y^T r_i + G_i(x, y, p_i) - p_i^T \nabla_p G_i(x, y, p_i) = 0, \\
& - \sum_{i=1}^{k} \lambda_i (\nabla_y f_i(x, y) - z_i + \nabla_p F_i(x, y, p_i)) \\
& - l_i(\nabla_y g_i(x, y) + r_i + \nabla_p G_i(x, y, p_i)) \in A^*_2, \\
& g^T y \sum_{i=1}^{k} \lambda_i (\nabla_y f_i(x, y) - z_i + \nabla_p F_i(x, y, p_i)) \\
& - l_i(\nabla_y g_i(x, y) + r_i + \nabla_p G_i(x, y, p_i)) \geq 0, \quad \lambda \in \text{int} K^*, \ x \in A_1, \ z_i \in D_i, \ r_i \in F_i, \ i = 1, 2, \ldots, k, \ l^T e = 1.
\end{align*}
\]

Dual (EMNSD)

\[ K - \text{Max} \quad m = (m_1, m_2, \ldots, m_k) \]

subject to

\[
\begin{align*}
& f_i(u, v) - S(v) D_i + u^T w_i + F_i(u, v, q_i) - q_i^T \nabla_q F_i(u, v, q_i) \\
& - m_i(g_i(u, v) + S(v) E_i) - u^T t_i + G_i(u, v, q_i) - q_i^T \nabla_q G_i(u, v, q_i), \\
& \sum_{i=1}^{k} \lambda_i (\nabla_x f_i(u, v) + w_i + \nabla_q F_i(u, v, q_i)) \\
& - m_i(\nabla_x g_i(u, v) - t_i + \nabla_q G_i(u, v, q_i)) \in A^*_1, \\
& u^T \sum_{i=1}^{k} \lambda_i (\nabla_x f_i(u, v) + w_i + \nabla_q F_i(u, v, q_i)) \\
& - m_i(\nabla_x g_i(u, v) - t_i + \nabla_q G_i(u, v, q_i)) \leq 0, \quad \lambda \in \text{int} K^*, \ v \in A_2, \ w_i \in B_i, \ t_i \in E_i, \ i = 1, 2, \ldots, k, \ l^T e = 1.
\end{align*}
\]

4. DUALITY RESULTS

In this Section, we validate the duality relations between the equivalent symmetric programs (EMNSP) and (EMNSD) under generalized convexity assumptions.

Theorem 10 (Weak Duality Theorem). Let \((x, y, l, z, r, \lambda, p)\) be a feasible solution of (EMNSP) and \((u, v, m, w, l, \lambda, q)\) be a feasible of (EMNSD). Assume that:

(i) \(f(\cdot, v) + (\cdot)^T w - m(g(\cdot, v) - (\cdot)^T t)\) is higher-order \(K - (\Phi^1, \rho^1)\) convex, in first variable, at \(u\) for fixed \(v\), with respect to \(F = mG\) and \(q\) where \(\Phi^1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_1+1} \rightarrow \mathbb{R}\) and \(\rho^1 = (\rho^1_1, \ldots, \rho^1_k)\).
(ii) \((-f(x,\cdot) + (\cdot)^T z) + l(g(x,\cdot) - (\cdot)^T r)\) is higher-order \(K - (\Phi^2, \rho^2)\) convex, in second variable, at \(y\) for fixed \(x\), with respect to \(-H + lG\) and \(p\) where 
\[
\Phi^2 : \mathbb{R}^{n_2} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_2+1} \to \mathbb{R} \quad \text{and} \quad \rho^2 = (\rho_1^2, \ldots, \rho_k^2),
\]

(iii) \(\Phi^3(x, u, (a, \bar{\rho})) + a^T u \geq 0, \forall a \in A_1^i, \rho^3 \geq 0\) 
and \(\Phi^2(v, y, (b, \bar{\rho})) + b^T y \geq 0, \forall b \in A_2^i\) and \(\rho^2 \geq 0\).

(iv) \(\mathbb{R}_+^k \subseteq K, g_i(x, v) + v^T t_i - x^T t_i > 0, \ i = 1, 2, \ldots, k, \) and for some \(\omega \in \text{int}\mathbb{R}_+^k, \kappa \in K \setminus \{0\}, \omega \kappa \in K \setminus \{0\}\).

Then \((m - l) \not\in K \setminus \{0\}\).

Proof. We validate the theorem by contradiction. Suppose that the weak duality theorem does not hold, which means that \((m - l) \in K \setminus \{0\}\). Now for some \(\lambda \in \text{int}K^*\) and using (iv) we have

\[
\sum_{i=1}^{k} \lambda_i (l_i - m_i)(g_i(x, v) + v^T t_i - x^T t_i) < 0. \tag{9}
\]

(i) gives that

\[
\begin{bmatrix}
\mathbf{f}_1(x, v) + x^T w_1 - m_1(g_1(x, v) - x^T t_1) - (\mathbf{f}_1(u, v) + u^T w_1 - m_1(g_1(u, v) - u^T t_1)) \\
\mathbf{f}_2(x, v, q_1) - m_2(g_1(u, v, q_1)) + q_1^T \nabla g_1(\mathbf{f}_1(u, v, q_1) - m_1 G_1(u, v, q_1)) \\
- \Phi^3(x, u, (\nabla_x \mathbf{f}_1(u, v) + w_1 - m_1(\nabla_x g_1(u, v) - t_1) + \nabla q_1(\mathbf{f}_1(u, v, q_1) - m_1 G_1(u, v, q_1)), \rho_i^3)) \\
\end{bmatrix} \in K
\]

Since \(\lambda \in \text{int}K^*\) we get the following

\[
\sum_{i=1}^{k} \lambda_i \left[ \mathbf{f}_i(x, v) + x^T w_i - m_i(g_i(x, v) - x^T t_i) - (\mathbf{f}_i(u, v) + u^T w_i - m_i(g_i(u, v) - u^T t_i)) \right. \\
\left. - (\mathbf{f}_i(u, v, q_i) - m_i G_i(u, v, q_i)) + q_i^T \nabla g_i(\mathbf{f}_i(u, v, q_i) - m_i G_i(u, v, q_i)) \right. \\
- \Phi^3(x, u; (\nabla_x \mathbf{f}_i(u, v) + w_i - m_i(\nabla_x g_i(u, v) - t_i) + \nabla q_i(\mathbf{f}_i(u, v, q_i) - m_i G_i(u, v, q_i)), \rho_i^3)) \right] \geq 0. \tag{10}
\]

Divide this equation by \(\sum_{i=1}^{k} \lambda_i = \tau\) (it is clear that \(\sum_{i=1}^{k} \frac{\lambda_i}{\tau} = 1\) and using
convexity of $\Phi^1$, the following is deduced

$$
\sum_{i=1}^{k} \frac{\lambda_i}{\tau} [f_i(x, v) + x^T w_i - m_i(g_i(x, v) - x^T t_i) - (f_i(u, v) + u^T w_i - m_i(g_i(u, v) - u^T t_i))
- (\bar{F}_i(u, v, q_i) - m_i\bar{G}_i(u, v, q_i)) + q_i^T \nabla_{q_i}(\bar{F}_i(u, v, q_i) - m_i\bar{G}_i(u, v, q_i))]
\geq \Phi^1(x, u, \frac{1}{\tau} \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u, v) + w_i - m_i(\nabla_x g_i(u, v) - t_i)
+ \nabla_{q_i}(\bar{F}_i(u, v, q_i) - m_i\bar{G}_i(u, v, q_i)), \rho_i^1)).
$$

(11)

As $a_1 = \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u, v) + w_i - m_i(\nabla_x g_i(u, v) - t_i) + \nabla_{q_i}(\bar{F}_i(u, v, q_i) - m_i\bar{G}_i(u, v, q_i))) \in A^*_1$ due to (6), multiply this with $\frac{1}{\tau} > 0$ to get $a = \frac{a_1}{\tau}$ and since $A^*_1$ is a convex cone, we have $a \in A^*_1$. Use this and $\bar{\rho} = \sum_{i=1}^{k} \frac{\lambda_i}{\tau} \rho_i^1 \geq 0$ (by hypothesis (iii)) to get the following inequality

$$
\frac{1}{\tau} \sum_{i=1}^{k} \lambda_i[f_i(x, v) + x^T w_i - m_i(g_i(x, v) - x^T t_i) - (f_i(u, v) + u^T w_i - m_i(g_i(u, v) - u^T t_i))
- (\bar{F}_i(u, v, q_i) - m_i\bar{G}_i(u, v, q_i)) + q_i^T \nabla_{q_i}(\bar{F}_i(u, v, q_i) - m_i\bar{G}_i(u, v, q_i))]
\geq \Phi^1(x, u, \frac{1}{\tau} \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u, v) + w_i - m_i(\nabla_x g_i(u, v) - t_i)
+ \nabla_{q_i}(\bar{F}_i(u, v, q_i) - m_i\bar{G}_i(u, v, q_i)), \rho_i^1))
= \Phi^1(x, u; (a, \bar{\rho}))
\geq -a^T u
\geq 0, \quad \text{(due to (6)).}
$$

(12)

On rearranging the terms in above equation and adding (5), then using feasibility conditions (4) and (8), we obtain the following

$$
\sum_{i=1}^{k} \lambda_i(f_i(x, v) + x^T w_i - S(v|D_i) - m_i(g_i(x, v) - x^T t_i + v^T r_i) \geq 0.
$$

(13)

On the same lines, using (ii) the following can be obtained

$$
\sum_{i=1}^{k} \lambda_i(-f_i(x, v) + v^T z_i - S(x|B_i) + l_i(g_i(x, v) + v^T r_i - x^T t_i) \geq 0.
$$

(14)

Adding equations (13) and (14) we get

$$
\sum_{i=1}^{k} \lambda_i(l_i - m_i)(g_i(x, v) + v^T r_i - x^T t_i) \geq 0.
$$
This is a contradiction to equation (9). So it is concluded that the supposition was wrong and hence, weak duality holds under the given set of assumptions. □

**Theorem 11 (Strong Duality Theorem).** Let \((\bar{x}, \bar{y}, \bar{t}, \bar{r}, \bar{\lambda}, \bar{\rho})\) be an efficient solution of (EMNSP) and fix \(\lambda = \bar{\lambda}\) in (EMNSD). Further, if the following assumptions hold:

(i) For \(i = 1, 2, \ldots, k\),
\[
F_i(\bar{x}, \bar{y}, 0) = \nabla_{\bar{p}_i} F_i(\bar{x}, \bar{y}, 0) = \nabla_{\bar{z}} F_i(\bar{x}, \bar{y}, 0) = \nabla_{\bar{y}} F_i(\bar{x}, \bar{y}, 0) = 0,
\]
\[
G_i(\bar{x}, \bar{y}, 0) = \nabla_{\bar{p}_i} G_i(\bar{x}, \bar{y}, 0) = \nabla_{\bar{z}} G_i(\bar{x}, \bar{y}, 0) = \nabla_{\bar{y}} G_i(\bar{x}, \bar{y}, 0) = 0,
\]
\[
F_i(\bar{x}, \bar{y}, 0) = \nabla_{\bar{q}_i} F_i(\bar{x}, \bar{y}, 0) = \nabla_{\bar{z}} F_i(\bar{x}, \bar{y}, 0) = \nabla_{\bar{y}} F_i(\bar{x}, \bar{y}, 0) = 0
\]

(ii) for any \(i = 1, 2, \ldots, k\), the Hessian matrix \(\nabla_{\bar{p}_i \bar{p}_i}(F_i(\bar{x}, \bar{y}, \bar{p}_i) - l_i G_i(\bar{x}, \bar{y}, \bar{p}_i))\) is positive or negative definite,

(iii) The set of vectors \(\{\nabla_{\bar{y}} F_i(\bar{x}, \bar{y}) - z_i + \nabla_{\bar{y}} G_i(\bar{x}, \bar{y}) + r_i + \nabla_{\bar{y}} G_i(\bar{x}, \bar{y}) - z_i + \nabla_{\bar{y}} F_i(\bar{x}, \bar{y}, \bar{p}_i) - l_i \nabla_{\bar{y}} G_i(\bar{x}, \bar{y}) + r_i + \nabla_{\bar{y}} G_i(\bar{x}, \bar{y}, \bar{p}_i), i = 1, 2, \ldots, k\}\) is linearly independent.

(iv) if for \(\bar{p}_i \in \mathbb{R}^{n_2}\) such that \(\bar{p}_i \neq 0\) implies \(\sum_{i=1}^{k} \bar{p}_i (\nabla_{\bar{y}} F_i(\bar{x}, \bar{y}) - z_i + \nabla_{\bar{y}} G_i(\bar{x}, \bar{y}) - l_i \nabla_{\bar{y}} G_i(\bar{x}, \bar{y}, \bar{p}_i)) \neq 0\)

(v) \(\bar{l}_i > 0, i = 1, 2, \ldots, k\)

then the point \((\bar{x}, \bar{y}, \bar{t}, \bar{w}, \bar{r}, \bar{\lambda}, \bar{q} = 0)\) is a feasible solution of the dual (EMNSD). Furthermore, if hypotheses of weak duality theorem hold for every feasible solution of dual, then \((\bar{x}, \bar{y}, \bar{t}, \bar{w}, \bar{r}, \bar{\lambda}, \bar{q} = 0)\) is an efficient solution of (EMNSD) and objective function values of (EMNSP) and (EMNSD) are equal.

**Proof.** Since \((\bar{x}, \bar{y}, \bar{t}, \bar{w}, \bar{r}, \bar{\lambda}, \bar{q} = 0)\) is given to be an efficient solution of the primal (EMNSP), then by necessary optimality conditions \([21]\), there exist \(\alpha \in K^*, \beta \in \mathbb{R}_+^k, \gamma \in \mathbb{A}_2, \delta \in \mathbb{R}_+, \eta \in \mathbb{R}, \bar{w}_i\) and \(l_i \in \mathbb{R}^{n_1}\) such that the following hold

\[
\begin{bmatrix}
\sum_{i=1}^{k} \beta_i (\nabla_x f_i(\bar{x}, \bar{y}) + w_i + \nabla_x F_i(\bar{x}, \bar{y}, \bar{p}_i) - l_i (\nabla_x g_i(\bar{x}, \bar{y}) - l_i + \nabla_x G_i(\bar{x}, \bar{y}, \bar{p}_i))) \\
+ (\gamma - \delta \bar{y})^T \sum_{i=1}^{k} \lambda_i (\nabla_{yx} f_i(\bar{x}, \bar{y}) - l_i \nabla_{yx} g_i(\bar{x}, \bar{y})) \\
+ \sum_{i=1}^{k} \nabla_{p_i x} (F_i(\bar{x}, \bar{y}, \bar{p}_i) - l_i G_i(\bar{x}, \bar{y}, \bar{p}_i))^T ((\gamma - \delta \bar{y}) \lambda_i - \beta_i \bar{p}_i) \end{bmatrix}^T (x - \bar{x}) \geq 0, \forall x \in \mathbb{A}_1.
\]

(15)
\[
\sum_{i=1}^{k} \beta_i (\nabla_y f_i(x, \bar{y}) - z_i + \nabla_y F_i(x, \bar{y}, \bar{\mu}_i) - l_i (\nabla_y g_i(x, \bar{y}) + \bar{r}_i + \nabla_y G_i(x, \bar{y}, \bar{\mu}_i)))
\]
\[+ (\gamma - \delta \bar{y})^T \sum_{i=1}^{k} \lambda_i \nabla_{yy} f_i(x, \bar{y}) - l_i g_i(x, \bar{y}) + \sum_{i=1}^{k} (\nabla p_i y G_i(x, \bar{y}, \bar{\mu}_i)) - \bar{l}_i (\nabla_y g_i(x, \bar{y}) + \bar{r}_i + \nabla_p G_i(x, \bar{y}, \bar{\mu}_i)) = 0.\]

(16)

\[
\alpha_i - \beta_i ((g_i(x, \bar{y}) - S(x|E_i) + \bar{y}^T r_i + G_i(x, \bar{y}, \bar{\mu}_i)) - \bar{p}_i^T \nabla_p G_i(x, \bar{y}, \bar{\mu}_i))
\]
\[-(\gamma - \delta \bar{y})^T (\nabla_y g_i(x, \bar{y}) + r_i + \nabla_p G_i(x, \bar{y}, \bar{\mu}_i)) = 0, \quad i = 1, 2, ..., k.\]

(17)

\[
0 \leq \lambda_i \leq \lambda^{\text{bar}}, \lambda_i \in N_D_i(z_i),
\]
\[0 \leq \beta_i \bar{y} + (\gamma - \delta \bar{y}) l_i \lambda_i \in N_{F_i}(r_i),\]

(18)

\[
(\gamma - \delta \bar{y})^T (\nabla_y f_i(x, \bar{y}) - z_i + \nabla_p F_i(x, \bar{y}, \bar{\mu}_i))
\]
\[-l_i (\nabla_y g_i(x, \bar{y}) + r_i + \nabla_p G_i(x, \bar{y}, \bar{\mu}_i)) = 0,\]

(21)

\[
\gamma^T \sum_{i=1}^{k} \lambda_i (\nabla_y f_i(x, \bar{y}) - z_i + \nabla_p F_i(x, \bar{y}, \bar{\mu}_i))
\]
\[-l_i (\nabla_y g_i(x, \bar{y}) + r_i + \nabla_p G_i(x, \bar{y}, \bar{\mu}_i)) = 0,\]

(22)

\[
\delta \bar{y}^T \sum_{i=1}^{k} \lambda_i (\nabla_y f_i(x, \bar{y}) - z_i + \nabla_p F_i(x, \bar{y}, \bar{\mu}_i))
\]
\[-l_i (\nabla_y g_i(x, \bar{y}) + r_i + \nabla_p G_i(x, \bar{y}, \bar{\mu}_i)) = 0,\]

(23)

(\alpha, \beta, \gamma, \delta, \xi, \eta) \neq 0.

(24)

In (24) we have \(\xi^T \bar{\lambda} = 0\). As \(\mathbb{R}^+_k \subseteq K \Rightarrow K^* \subseteq \mathbb{R}^+_k\) we have \(\text{int}K^* \subseteq \text{int} \mathbb{R}^+_k\) implies \(\bar{\lambda} > 0\). So we have \(\xi = 0\). From hypothesis (ii) and equation (18) we get

\[
(\gamma - \delta \bar{y}) \bar{\lambda}_i = \beta_i \bar{\mu}_i, \quad i = 1, 2, ..., k.
\]

(27)

**CLAIM:** \(\beta_i \neq 0\) for any \(i = 1, 2, ..., k\).

Because if \(\beta_{i_0} = 0\), for some \(i_0 \in 1, 2, ..., k\), then we have

\[
(\gamma - \delta \bar{y}) \bar{\lambda}_{i_0} = 0 \xrightarrow{\lambda_{i_0} > 0} \gamma = \delta \bar{y}
\]

(28)

Put this in equation (27), we get \(\beta_i \bar{\mu}_i = 0\) for each \(i \in 1, ..., k\). Using these values
in (16) to get

\[
\sum_{i=1}^{k} \beta_i (\nabla_y f_i(x, y) - z_i + \nabla_x F_i(x, \bar{y}, \bar{p}_i) - l_i(\nabla_y g_i(x, \bar{y}) + r_i + \nabla_x G_i(x, \bar{y}, \bar{p}_i)))
\]

\[= \sum_{i=1}^{k} \delta \lambda_i (\nabla_y f_i(x, y) - z_i + \nabla_p F_i(x, \bar{y}, \bar{p}_i) - l_i(\nabla_y g_i(x, \bar{y}) + r_i + \nabla_p G_i(x, \bar{y}, \bar{p}_i))) = 0.
\]

(29)

This due to (iii) gives \(\beta_i = 0\) and \(\delta \lambda_i = 0\), for all \(i = 1, 2, \ldots, k\) \(\Rightarrow \delta = 0\), (\(\therefore \lambda > 0\)).

Now, due to (17) \(\alpha_i = 0\), \(\forall i = 1, 2, \ldots, k\). Equation (28) gives \(\gamma = 0\) and equation (21) \(\Rightarrow \eta = 0\) respectively. \(\xi\) is already 0. \(\Rightarrow (\alpha, \beta, \gamma, \delta, \xi, \eta) = 0\), which is a contradiction to necessary optimality conditions constructed above. So we have proved our claim that \(\beta_i \neq 0\) for any \(i = 1, 2, \ldots, k\).

Now multiply equation (21) with \(\lambda_i\) and take its sum over the range of \(i\) to get

\[(\gamma - \delta \eta) \sum_{i=1}^{k} \lambda_i (\nabla_y f_i(x, y) - z_i + \nabla_p F_i(x, \bar{y}, \bar{p}_i) - l_i(\nabla_y g_i(x, \bar{y}) + r_i + \nabla_p G_i(x, \bar{y}, \bar{p}_i))) + \eta^T(\lambda e) = 0
\]

and (22)-(23) give

\[(\gamma - \delta \eta) \sum_{i=1}^{k} \lambda_i (\nabla_y f_i(x, y) - z_i + \nabla_p F_i(x, \bar{y}, \bar{p}_i) - l_i(\nabla_y g_i(x, \bar{y}) + r_i + \nabla_p G_i(x, \bar{y}, \bar{p}_i))) = 0.
\]

(31)

Now (31)-(30) give

\[\eta^T \lambda e = 0
\]

\[\Rightarrow \eta = 0\] (\(\therefore \lambda \neq 0\), \(e \neq 0\)).

(32)

Putting this in (21), we get

\[(\gamma - \delta \eta)^T(\nabla_y f_i(x, y) - z_i + \nabla_p F_i(x, \bar{y}, \bar{p}_i) - l_i(\nabla_y g_i(x, \bar{y}) + r_i + \nabla_p G_i(x, \bar{y}, \bar{p}_i))) = 0.
\]

(33)

So due to (27)

\[\beta_i \bar{p}_i (\nabla_y f_i(x, y) - z_i + \nabla_p F_i(x, \bar{y}, \bar{p}_i) - l_i(\nabla_y g_i(x, \bar{y}) + r_i + \nabla_p G_i(x, \bar{y}, \bar{p}_i))) = 0.
\]

(34)

As \(\beta_i \neq 0\),

\[\bar{p}_i (\nabla_y f_i(x, y) - z_i + \nabla_p F_i(x, \bar{y}, \bar{p}_i) - l_i(\nabla_y g_i(x, \bar{y}) + r_i + \nabla_p G_i(x, \bar{y}, \bar{p}_i))) = 0.
\]

(35)
Or \( \sum_{i=1}^{k} \bar{p}_i(\nabla_y f_i(\bar{x}, \bar{y}) - z_i + \nabla_p F_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{l}_i(\nabla_y g_i(\bar{x}, \bar{y}) + r_i + \nabla_p G_i(\bar{x}, \bar{y}, \bar{p}_i))) = 0 \),

(36)

and by (iv), we have each of \( \bar{p}_i = 0 \). So from (27), we get

\[ \gamma = \delta \bar{y}. \quad (37) \]

By putting the obtained values in (15) and (16), we get

\[ \left[ \sum_{i=1}^{k} \beta_i(\nabla_x f_i(\bar{x}, \bar{y}) + \bar{w}_i - \bar{l}_i(\nabla_x g_i(\bar{x}, \bar{y}) - \bar{r}_i)) \right]^T (x - \bar{x}) \geq 0, \forall x \in A_1, \quad (38) \]

and

\[ \sum_{i=1}^{k} (\beta_i - \delta \lambda_i)(\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i - \bar{l}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i)) \right]^T = 0. \quad (39) \]

Again, using (iii), we have \( \beta_i - \delta \lambda_i = 0 \Rightarrow \beta_i = \delta \lambda_i \). As \( \lambda_i > 0, \beta_i \in \mathbb{R}_+^k \) and \( \beta \neq 0 \), we have \( \delta > 0 \). Use this in (38) to get

\[ \left[ \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(\bar{x}, \bar{y}) + \bar{w}_i - \bar{l}_i(\nabla_x g_i(\bar{x}, \bar{y}) - \bar{r}_i)) \right]^T (x - \bar{x}) \geq 0, \forall x \in A_1. \quad (40) \]

Put \( x = 0 \) (as (40) holds for any \( x \in A_1 \)) we have

\[ - \left[ \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(\bar{x}, \bar{y}) + \bar{w}_i - \bar{l}_i(\nabla_x g_i(\bar{x}, \bar{y}) - \bar{r}_i)) \right]^T \bar{x} \geq 0. \quad (41) \]

\[ \Rightarrow \left[ \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(\bar{x}, \bar{y}) + \bar{w}_i - \bar{l}_i(\nabla_x g_i(\bar{x}, \bar{y}) - \bar{r}_i)) \right]^T \bar{x} \leq 0. \quad (42) \]

For convex cone \( A_1 \), \( \bar{x} \in A_1 \Rightarrow x + \bar{x} \in A_1, \forall x \in A_1 \). So (40) becomes

\[ \left[ \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(\bar{x}, \bar{y}) + \bar{w}_i - \bar{l}_i(\nabla_x g_i(\bar{x}, \bar{y}) - \bar{r}_i)) \right]^T x \geq 0, \forall x \in A_1. \quad (43) \]

The above equation holds for every \( x \in A_1 \) so we can say that

\[ \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(\bar{x}, \bar{y}) + \bar{w}_i - \bar{l}_i(\nabla_x g_i(\bar{x}, \bar{y}) - \bar{r}_i)) \in A_1^*, \quad (44) \]

which is dual feasibility condition (6). Since (43) holds for every \( x \in A_1 \) and \( \bar{x} \) is a feasible solution of primal, so at \( x = \bar{x} \) the following is true. By putting \( x = \bar{x} \) in (43), we have

\[ \left[ \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(\bar{x}, \bar{y}) + \bar{w}_i - \bar{l}_i(\nabla_x g_i(\bar{x}, \bar{y}) - \bar{r}_i)) \right]^T \bar{x} \geq 0. \quad (45) \]
As $\bar{x} \in A_1$ \Rightarrow (41) and (45) give
\[
\sum_{i=1}^{k} \bar{\lambda}_i (\nabla_x f_i(\bar{x}, \bar{y}) + \bar{w}_i - \bar{\ell}_i (\nabla_x g_i(\bar{x}, \bar{y}) - \bar{\ell}_i)) = 0.
\] (46)

Hence, the third feasibility condition of the dual, given by equation (7), is obtained. Now in (19), as $\gamma = \delta \bar{y}$, $\beta > 0 \Rightarrow \bar{y} \in N_{D_i}(\bar{z}_i)$ which means, $\bar{y}^T \bar{z}_i = S(\bar{y}|D_i)$. Similarly, due to (20), (25) and hypothesis (v) we have $\bar{y}^T \bar{r}_i = S(\bar{y}|F_i)$. Moreover $p_i = 0$ as obtained above, use them in equation (1) to get
\[
\begin{align*}
\sum_{i=1}^{k} \bar{\lambda}_i (\nabla_x f_i(\bar{x}, \bar{y}) + \bar{w}_i - \bar{\ell}_i (\nabla_x g_i(\bar{x}, \bar{y}) - \bar{\ell}_i)) & = 0, \\
\end{align*}
\] (47)

Equation (37) gives that $\bar{y} = \bar{q} \in A_2$ From equations (44), (46), and (47) we can conclude that $(\bar{x}, \bar{y}, \bar{\ell}, \bar{\ell}, \bar{v}, \bar{\lambda}, \bar{q} = 0)$ is a feasible solution of dual (EMNSD) and the objective values of (EMNSP) and (EMNSD) are equal, i.e., $\bar{l}_i = m_i$. Now, for the second part of the theorem, if $(\bar{x}, \bar{y}, \bar{\ell}, \bar{\ell}, \bar{v}, \bar{\lambda}, \bar{q} = 0)$ is not an efficient solution for dual (EMNSD), then there exists another feasible solution $(\bar{u}, \bar{v}, \bar{m}, \bar{w}, \bar{\ell}, \bar{\lambda}, \bar{q})$ of dual such that $\bar{m} - \bar{l} \in K \setminus \{0\}$. But this contradicts weak duality theorem. Hence, $(\bar{x}, \bar{y}, \bar{\ell}, \bar{\ell}, \bar{v}, \bar{\lambda}, \bar{q} = 0)$ is efficient for (EMNSD). Hence, the result. \qed

**Theorem 12 (Converse Duality Theorem).** Let $(\bar{u}, \bar{v}, \bar{m}, \bar{w}, \bar{\ell}, \bar{\lambda}, \bar{q})$ be an efficient solution of (EMNSD) and fix $\lambda = \lambda$ in (EMNSP). Then under the following set of assumptions:

(i) For $i = 1, 2, \ldots, k$,
\[
F_i(\bar{u}, \bar{v}, 0) = \nabla_{\bar{u}} F_i(\bar{u}, \bar{v}, 0) = \nabla_{\bar{v}} G_i(\bar{u}, \bar{v}, 0) = 0,
\]
\[
F_i(\bar{x}, \bar{y}, 0) = \nabla_{\bar{x}} F_i(\bar{u}, \bar{v}, 0) = \nabla_{\bar{y}} F_i(\bar{u}, \bar{v}, 0) = 0,
\]
\[
G_i(\bar{x}, \bar{y}, 0) = \nabla_{\bar{x}} G_i(\bar{u}, \bar{v}, 0) = \nabla_{\bar{y}} G_i(\bar{u}, \bar{v}, 0) = 0
\]

(ii) for any $i = 1, 2, \ldots, k$, the Hessian matrix $\nabla_{\bar{x}, \bar{v}}(\bar{F}_i(\bar{u}, \bar{v}, \bar{q}_i) - m_i G_i(\bar{u}, \bar{v}, \bar{q}_i))$ is positive or negative definite,

(iii) the set of vectors $\{\nabla_x f_i(\bar{u}, \bar{v}) + w_i + \nabla_x F_i(\bar{u}, \bar{v}, \bar{q}_i) - m_i \nabla_x g_i(\bar{u}, \bar{v}) - t_i + \nabla_x G_i(\bar{u}, \bar{v}, \bar{q}_i)\}$, $\nabla_x f_i(\bar{u}, \bar{v}) + w_i + \nabla_x F_i(\bar{u}, \bar{v}, \bar{q}_i) - m_i \nabla_x g_i(\bar{u}, \bar{v}) - t_i + \nabla_x G_i(\bar{u}, \bar{v}, \bar{q}_i), i = 1, 2, \ldots, k$ are linearly independent.

(iv) if for $\bar{q}_i \in \mathbb{R}^{n_i}$ such that $\bar{q}_i \neq 0$, implies $\sum_{i=1}^{k} \bar{q}_i (\nabla_x f_i(\bar{u}, \bar{v}) + w_i + \nabla_x F_i(\bar{u}, \bar{v}, \bar{q}_i) - m_i \nabla_x g_i(\bar{u}, \bar{v}) - t_i + \nabla_x G_i(\bar{u}, \bar{v}, \bar{q}_i)) \neq 0$

(v) $\bar{m}_i > 0$, $i = 1, 2, \ldots, k$.

the point $(\bar{u}, \bar{v}, \bar{m}, \bar{w}, \bar{\ell}, \bar{\lambda}, \bar{q})$ is a feasible solution of (EMFNSP). Furthermore if hypotheses of weak duality theorem hold for every feasible solution of (EMFNSP), then $(\bar{u}, \bar{v}, \bar{m}, \bar{w}, \bar{\ell}, \bar{\lambda}, \bar{q})$ is an efficient solution of (EMFNSP).

**Proof.** The proof follows on the lines of Theorem 11. \qed
REFERENCES

[1] Bazaraa, M. S. and Goode, J. J., “On symmetric duality in nonlinear programming”, Operations Research, 21 (1973) 1–9.
[2] Bector, C.R. and Chandra, S., “Second order symmetric and self dual programs”, OPSEARCH, 23 (1986) 89–95.
[3] Caristi, G., Ferrara, M. and Stefanescu, A., “Mathematical programming with \((\phi, \rho)-\)invexity”, In Generalized Convexity and Related Topics, pages 167–176. Springer, 2007.
[4] Chandra, S., Craven, B. D., and Mond, B., “Symmetric dual fractional programming”, Mathematical Methods of Operations Research, 29 (1985) 59–64.
[5] Chen, X., “Higher-order symmetric duality in nondifferentiable multiobjective programming problems”, Journal of Mathematical Analysis and Applications, 290 (2004) 423–435.
[6] Dantzig, G., Eisenberg, E., and Cottle, R., “Symmetric dual nonlinear programs”, Pacific Journal of Mathematics, 15 (1965) 809–812.
[7] Dorn, W., “A symmetric dual theorem for quadratic programs”, Journal of Operations Research Society of Japan, 2 (1960) 93–97.
[8] Dubey, R. and Gupta, S. K., “Duality for a nondifferentiable multiobjective higher-order symmetric fractional programming problems with cone constraints”, Journal of Nonlinear Analysis and Optimization: Theory & Applications, 7 (2016) 1–15.
[9] Gulati, T. R. and Gupta, S. K., “Higher-order symmetric duality with cone constraints”, Bulletin of the Australian Mathematical Society, 56 (1997) 25–36.
[10] Gulati, T. R. and Mehdiratta, G., “Nondifferentiable multiobjective Mond–Weir type second-order symmetric duality over cones”, Optimization Letters, 4 (2010) 293–309.
[11] Gupta, S. K., Kailey, N., and Sharma, M. K., “Higher-order (\(F, \alpha, \rho, d\))-convexity and symmetric duality in multiobjective programming”, Computers & Mathematics with Applications, 60 (2010) 2373–2381.
[12] Jayswal, A. and Jha, S., “Second order symmetric duality in fractional variational problems over cone constraints”, Yugoslav Journal of Operations Research, 28 (2018) 39–57.
[13] Kapoor, M., “Vector optimization over cones involving support functions using generalized \((\phi, \rho)-\)convexity”, OPSEARCH, 54 (2017) 351–364.
[14] Verma, K., Verma, J. P. and Ahmad, I., “A new approach on multiobjective higher-order symmetric duality under cone-invexity and other related concepts”, Journal of Computational and Applied Mathematics, 255 (2014) 825–836.
[24] Weir, T., “Symmetric dual multiobjective fractional programming”, *Journal of the Australian Mathematical Society*, 50 (1991) 67–74.
[25] Yang, X.-M. and Hou, S.-H., “Second-order symmetric duality in multiobjective programming”, *Applied Mathematics Letters*, 14 (2001) 587–592.