STRICHARTZ ESTIMATES FOR HIGHER-ORDER SCHRÖDINGER EQUATIONS AND THEIR APPLICATIONS

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ABSTRACT. We consider the higher-order linear Schrödinger equations which are formal finite Taylor expansions of the linear pseudo-relativistic Schrödinger equation. In this paper, we establish global-in-time Strichartz estimates for these higher-order equations which hold uniformly in the speed of light. As nonlinear applications, we show that the higher-order Hartree(-Fock) equations approximate the corresponding pseudo-relativistic equation on an arbitrarily long time interval, with higher accuracy than the non-relativistic model. We also prove small data scattering for the higher-order nonlinear Schrödinger equations.

1. INTRODUCTION

1.1. Background. We consider the pseudo-relativistic linear Schrödinger equation

\[ i\hbar \partial_t \psi = \mathcal{H}(c) \psi, \quad (\text{pLS}) \]

where \( \psi = \psi(t, x) : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C} \) is the wave function and the operator

\[ \mathcal{H}(c) = \sqrt{m^2 c^4 - c^2 \hbar^2 \Delta - mc^2} \]

is the Fourier multiplier of symbol \( \sqrt{m^2 c^4 + c^2 \hbar^2 |\xi|^2 - mc^2} \). Here, \( m > 0 \) represents the particle mass, \( \hbar \) is the reduced Plank constant, and \( c \gg 1 \) denotes the speed of light. The operator \( \mathcal{H}(c) \) is called pseudo-relativistic or semi-relativistic in that it behaves both relativistically and non-relativistically depending on frequencies. Indeed, in the non-relativistic regime \( |\xi| \ll \frac{mc}{\hbar^2} \), the Taylor series expansion yields

\[ \sqrt{h^2 c^2 |\xi|^2 + m^2 c^4 - mc^2} = mc^2 \left( 1 + \frac{h^2 |\xi|^2}{m^2 c^2} - 1 \right) = \frac{h^2 |\xi|^2}{2m} - \frac{h^2 |\xi|^4}{8m^3 c^2} + \cdots \approx \frac{h^2 |\xi|^2}{2m}. \quad (1.1) \]

The pseudo-relativistic operator and associated linear and nonlinear models arise in various physics literature. For instance, the mean-field dynamics of relativistic fermion particles is described by the pseudo-relativistic Hartree-Fock equation

\[ i\hbar \partial_t \psi_k = \mathcal{H}(c) \psi_k + H \psi_k - F_k(\psi_k), \quad k = 1, 2, \ldots, N, \quad (\text{pHF}) \]

where \( \psi_k = \psi_k(t, x) : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C} \),

\[ H \psi_k = \sum_{\ell=1}^{N} \left( \frac{\kappa}{|x|} * |\psi_{\ell}|^2 \right) \psi_k \]
is the Hartree nonlinearity, and

\[ F_k(\psi_k) = \sum_{\ell=1, \ell \neq k}^N \left( \frac{\kappa}{|x|} * (\overline{\psi_\ell} \psi_k) \right) \psi_\ell \]

is the Fock exchange term. A real constant \( \kappa \) determines the strength of self-interaction among quantum particles; it is repulsive if \( \kappa > 0 \), and attractive if \( \kappa < 0 \). By the Pauli exclusion principle, \( \psi_k \)'s are assumed to be mutually \( L^2 \)-orthogonal. Dropping the exchange term, the Hartree-Fock equation is reduced to the pseudo-relativistic Hartree equation

\[ i\hbar \partial_t \psi_k = H^{(c)} \psi_k + H \psi_k, \quad k = 1, 2, \ldots, N. \]  

(pH)

Since the exchange term has lower-order in mean-field approximation, this Hartree model is considered as a simplified Hartree-Fock model. Without the orthogonal condition, the equation with \( \kappa < -1 \) describes the mean field dynamics of boson stars, so it is called the boson star equation. For the rigorous derivation of the semi-relativistic models, we refer to [21,22,13]. For the dynamics of the system, the Hartree(-Fock) equation is globally well-posed in the energy space \( H^1(\mathbb{R}^3) \) [14,18]. It is locally well-posed below the energy space [9,15]. For dynamical properties, we refer to [18,19,20].

For the relativistic models, an important question is to prove their non-relativistic limits \( c \to \infty \). Note that by the convergence (1.1) in low frequencies, it is expected that the pseudo-relativistic operator \( H^{(c)} \) in the above models ((pLS), (pHF) and (pH)) can be replaced by the non-relativistic one \( -\frac{\hbar^2}{2m} \Delta \) in the non-relativistic regime, and that the non-relativistic linear Schrödinger equation and the Hartree(-Fock) equation are derived respectively. Indeed, proving the non-relativistic limits justifies consistency of the relativistic modification. On the other hand, it shows that non-relativistic models are good approximations to the pseudo-relativistic models, which are computationally extremely expensive due to the presence of the non-local operator \( H^{(c)} \).

Similar non-relativistic limits can be formulated for other types of relativistic models, and there have been numerous results on this subject. For instance, in the work of Machihara, Nakanishi and Ozawa [23,24], the nonlinear Schrödinger equations are derived from the Klein-Gordon and the Dirac equations as non-relativistic limits. In [2,3,4], Bechouche, Mauser and Selberg established the convergence from the Dirac-Maxwell (resp., Klein-Gordon-Maxwell) system to the Vlasov-Poisson (resp., Schrödinger-Poisson) system. For the non-relativistic limits of the Dirac-Maxwell, Klein-Gordon-Maxwell and Klein-Gordon-Zakharov systems, we refer to the work of Masmoudi and Nakanishi [26,27,28].

For better accuracy to the non-local pseudo-relativistic equations, Carles and Moulay [7] introduced higher-order linear Schrödinger equations of the form

\[ i\hbar \partial_t \psi = H^{(c)}_J \psi, \]  

(hLS)
where the operator $H^{(c)}_J$ is a formal finite Taylor expansion of the pseudo-relativistic operator, that is,

$$H^{(c)}_J = -\sum_{j=1}^{J} \frac{\alpha(j)\hbar^{2j}}{m^{2j-1}c^{2j-1}} \Delta^j$$

with

$$\alpha(j) = \frac{(2j - 2)!}{j!(j - 1)!2^{2j-1}} \quad (j \geq 1).$$

Note that such higher-order models include the non-relativistic Schrödinger equation $ih\partial_t \psi = -\frac{\hbar^2}{2m} \Delta \psi$ and the fourth-order equation $ih\partial_t \psi = (-\frac{\hbar^2}{2m} \Delta - \frac{\hbar^4}{8mc^2} \Delta^2) \psi$. In [6], Carles, Lucha and Moulay showed that the higher-order linear flow provides a more accurate approximation as $c \to \infty$ [6, Theorem A.1], precisely,

$$\|e^{-i\hbar H^{(c)}_J} \psi_0 - e^{-i\hbar J} \psi_0\|_{L^2(\mathbb{R}^d)} \leq \frac{2T}{\hbar} \frac{\alpha(J + 1)}{m^{2J+1}c^{2J}} \|\psi_0\|_{H^{2J+2}(\mathbb{R}^d)}.$$  (1.2)

Moreover, employing the local-in-time Strichartz estimates [6, Lemma 4.3], the authors developed a well-posedness theory for higher-order Hartree(-Fock) equations [6, Theorem 4.9].

1.2. Strichartz estimates. In this paper, we establish “global-in-time” Strichartz estimates for higher-order equations (hLS) which hold uniformly in the speed of light $c \geq 1$. It turns out that such estimates are distinguished by odd and even orders of the Taylor expansion. For the statement, we call $(q,r)$ odd-admissible if

$$2 \leq q, r \leq \infty, \quad (q,r,d) \neq (2,\infty,2), \quad \frac{2}{q} + \frac{d}{r} = \frac{d}{2}. \quad (1.3)$$

Our first main result asserts that for odd expansions, the uniform global-in-time Strichartz estimates hold for odd-admissible pairs.

**Theorem 1.1** (Strichartz estimates for (hLS): odd case). Let $J \in 2\mathbb{N} - 1$. Then, there exists a constant $A$, independent of $c \geq 1$, such that for any odd-admissible pairs $(q,r)$ and $(\tilde{q},\tilde{r})$, we have

$$\|e^{-i\hbar H^{(c)}_J} \psi_0\|_{L^q_x(L^r_t(\mathbb{R}^d))} \leq A \left(\frac{m}{\hbar}\right)^{\frac{d}{q}} \|\psi_0\|_{L^2(\mathbb{R}^d)} \quad (1.4)$$

and

$$\left\| \int_0^t e^{-i(t-s)H^{(c)}_J} F(s)ds \right\|_{L^q_x(L^r_t(\mathbb{R}^d))} \leq A^2 \left(\frac{m}{\hbar}\right)^{\frac{d}{q}} \|F\|_{L^q_t(L^r_x(\mathbb{R}^d))}. \quad (1.5)$$

**Remark 1.2.**

1. The odd-admissible pairs are exactly the same as those for the non-relativistic model. Indeed, the non-relativistic case $J = 1$ is included.

2. The estimates (1.4) and (1.5) are independent of $c \geq 1$, and they hold globally in time. In this sense, it improves the previous local-in-time bound [6, Lemma 4.3]. Moreover, they can be a proper analysis tool to analyze both the non-relativistic limit ($c \to \infty$) and the large-time asymptotics ($t \to \infty$) for nonlinear problems (see Theorem 1.6 and 1.8).
Remark 1.3. (1) By simple but careful analysis on the symbol of the higher-order operator $H^J(c)$, we show that if $J \in \mathbb{N}$ is odd, the Hessian of the symbol has a strict lower bound, independent of $c \geq 1$ (see Proposition 3.1). Once it is proved, Strichartz estimates follow immediately from the standard oscillatory integral theory (see Lemma 2.2).

(2) In general, the symbols of higher-order Schrödinger operators may have degenerate Hessian. Such examples include the bi-harmonic operator $\Delta^2$ and the even-order expansion $H^J(2J_0)$. The associate linear Schrödinger equations necessarily obey only weaker dispersive estimates.

(3) The proof of Theorem 1.1 heavily relies on the algebraic structure of symbols, originated from the pseudo-relativistic operator. Indeed, the signs of ordered terms in the Hessian are alternating. Nevertheless, fortunately in odd case, negative terms can be controlled by neighboring positive terms. These are not true in general.

Next, we consider higher-order equations (hLS) with even Taylor expansions. In this case, the situation becomes much more complicated, because the Hessian of the symbol can be degenerate. Precisely, there are two spheres in the frequency space where the Hessian is degenerate (see Lemma 4.2). Our second main result provides weaker Strichartz estimates for even expansions. We now call $(q, r)$ even-admissible if

$$
\begin{align*}
2 \leq q & \leq \infty, \quad 2 \leq r < \infty, \quad \begin{cases} 
\frac{3}{q} + \frac{1}{r} = \frac{1}{2} & \text{for } d = 1, \\
\frac{2}{q} + \frac{1}{r} = \frac{1}{2} & \text{for } d \geq 2.
\end{cases}
\end{align*}
$$

(1.6)

Theorem 1.4 (Strichartz estimates for (hLS): even case). Let $J \in 2\mathbb{N}$. Then, there exists $A > 0$, independent of $c \geq 1$, such that for an even-admissible pair $(q, r)$,

$$
\| e^{-itH^J(c)} \psi_0 \|_{L^q_t(L^r_x(\mathbb{R}^d))} \leq \begin{cases} 
A \left( \frac{m}{\hbar} \right)^{\frac{1}{q}} \| \psi_0 \|_{H^{\frac{1}{2}}(\mathbb{R}^d)} & \text{for } d = 1, \\
A \left( \frac{m}{\hbar} \right)^{\frac{1}{q}} \| \psi_0 \|_{H^{\frac{2(d-1)}{q}}(\mathbb{R}^d)} & \text{for } d \geq 2.
\end{cases}
$$

(1.7)

Remark 1.5. (1) In even case, we have weaker Strichartz estimates. Note that in odd case, the Strichartz estimates (1.4) and the Sobolev inequality yield the estimates of the form (1.7).

(2) As mentioned above, in even case, the Hessian of the symbol has two degenerate spheres. It causes additional technical difficulties. To handle this, we need to employ the Littlewood-Paley decomposition.

(3) We do not claim optimality of the estimates (1.7). The additional derivative on the right hand side of (1.7) could be reduced, for instance, by applying a more delicate theory of the resolution of singularity [1].

1.3. Applications to nonlinear problems. Strichartz estimates are one of the fundamental tools for linear and nonlinear dispersive PDEs (see [8]). In this article, we discuss two nonlinear problems which can be solved by taking the full advantages of uniform global-in-time Strichartz estimates (Theorem 1.1); one is higher-order approximation estimate for...
the pseudo-relativistic Hartree(-Fock) equation via its higher-order expansions. The other
is the small data scattering for the higher-order nonlinear Schrödinger equations.

1.3.1. Higher-order approximation. Let \( J \in 2\mathbb{N} - 1 \) be an odd number. Then, as formal finite
expansions of the pseudo-relativistic Hartree(-Fock) model mentioned earlier, we introduce
the corresponding higher-order Hartree-Fock equation

\[
i\hbar \partial_t \phi_k = \mathcal{H}_j^{(c)} \phi_k + H \phi_k - F_k(\phi_k), \quad k = 1, 2, ..., N, \tag{hHF}\]

and the higher-order Hartree equation

\[
i\hbar \partial_t \phi_k = \mathcal{H}_j^{(c)} \phi_k + H \phi_k, \quad k = 1, 2, ..., N. \tag{hH}\]

By Strichartz estimates (Theorem 1.1) and the mass conservation law, both (hHF) and
(hH) are globally well-posed in \( L^2(\mathbb{R}^3; \mathbb{C}^N) \) (see Proposition 5.3).

The following theorem provides precise global-in-time error bounds for the higher-order
approximations of the pseudo-relativistic models.

**Theorem 1.6** (Higher-order approximation). Let \( J \in 2\mathbb{N} - 1 \) and \( c \geq 1 \). Suppose that \( \Psi_0 = \{\psi_{k,0}\}_{k=1}^N \in H^{1/2}(\mathbb{R}^3; \mathbb{C}^N) \). If \( \kappa < 0 \), we further assume that \( \|\Psi_0\|_{H^{1/2}(\mathbb{R}^3; \mathbb{C}^N)} \) is sufficiently
small. Let \( \Psi^{(c)}(t) = \{\psi^{(c)}(t)\}_{k=1}^N \in C(\mathbb{R}; H^{1/2}(\mathbb{R}^3; \mathbb{C}^N)) \) be the global solution to \( (pHF) \)
(resp., \( (hHF) \)) with initial data \( \Psi_0 \), and let \( \Phi^{(c)}(t) = \{\phi^{(c)}(t)\}_{k=1}^N \in C(\mathbb{R}; H^{1/2}(\mathbb{R}^3; \mathbb{C}^N)) \) be
the global solution to \( (pH) \) (resp., \( (hH) \)) with the same initial data. Then, there exist
\( A, B > 0 \), depending on \( \|\Psi_0\|_{H^{1/2}(\mathbb{R}^3; \mathbb{C}^N)} \) but independent of \( c \geq 1 \), such that

\[
\|\Phi^{(c)}(t) - \Psi^{(c)}(t)\|_{L^2(\mathbb{R}^3; \mathbb{C}^N)} \leq A e^{-\frac{t}{1+c}} e^{Bt}. \tag{1.8}
\]

**Remark 1.7.**

1. The limit behavior of stationary states to the pseudo-relativistic equations
have been studied. In [19][12][10], the authors established the non-relativistic
limits of stationary states, which corresponds to the \( J = 1 \) case. Then, the higher-
order approximation estimates are proved [11].

2. The convergence rate in (1.8) is getting better as \( J \) increases, but we do not claim
its optimality.

3. The proof is quite standard. We write the solutions in Duhamel formulae and
measure the difference in \( L^2 \). The convergence rate comes from the Taylor expansion
of symbol and the regularity gap from \( H^{1/2} \) where the initial data is given.

4. The smallness condition on initial data when \( \lambda < 0 \) is imposed just for the global
well-posedness of (pHF). Even \( J \)'s are not included, but the same result can be
proved once the loss of regularity in Strichartz estimates is reduced as for odd \( J \)'s.
The argument can be easily applied to higher dimensional case \( d \geq 3 \).

1.3.2. Small data scattering. Let \( J \in 2\mathbb{N} - 1 \). We now consider the higher-order nonlinear
Schrödinger equation,

\[
i\partial_t \psi = \mathcal{H}_j^{(c)} \psi + \kappa |\psi|^{\nu-1} \psi, \tag{hNLS}\]

where \( \psi = \psi(t, x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}, \kappa \in \mathbb{R} \) and \( \nu > 1 \). This equation could be considered
as the \( J \) th-order approximation to the pseudo-relativistic NLS \( i\partial_t \psi = \mathcal{H}^{(c)} \psi + \kappa |\psi|^{\nu-1} \psi \). Using the Strichartz estimates (Theorem 1.1), we prove small data scattering.
Theorem 1.8 (Small data scattering in $H^1(\mathbb{R}^d)$ for \[hNLS\]). Let $J \in 2\mathbb{N} - 1$. Suppose that $\nu$ satisfies
\[
\begin{cases}
1 + \frac{4}{d} < \nu < \frac{d+2}{d-2}, & \text{if } d \geq 3 \\
1 + \frac{4}{d} < \nu < \infty, & \text{if } d = 1, 2.
\end{cases}
\] (1.9)

Then, there exists $\epsilon_0 > 0$ (independent of $c \geq 1$) such that if $\|\psi_{c,0}\|_{H^1(\mathbb{R}^d)} \leq \epsilon_0$, then there exists a unique global solution $\psi_c(t) \in C(\mathbb{R}; H^1(\mathbb{R}^d))$ to \[hNLS\] with initial data $\psi_{c,0}$, and it moreover scatters in $H^1(\mathbb{R}^d)$, i.e., there exist $\psi^{(c)}_{J,\pm} \in H^1(\mathbb{R}^d)$ such that
\[
\lim_{t \to \pm \infty} \|\psi_{c}(t) - e^{-itH_{c}^{(c)}} \psi^{(c)}_{J,\pm}\|_{H^1(\mathbb{R}^d)} = 0.
\]

Remark 1.9. (1) The proof of (1.8) is identical to that in the non-relativistic case (see [8]), because Strichartz estimates (1.4) are exactly the same.

(2) An interesting question would be to show how accurately the scattering state $\psi^{(c)}_{J,\pm}$ for the higher-order model approximates the relativistic scattering state. The non-relativistic limit ($J = 1$) of scattering states is proved for the nonlinear Klein-Gordon equation [29].

1.4. Organization. In Section 2, we provide basic lemmas for oscillatory integrals, which will be repeatedly used to estimate the kernel of linear flows. In Section 3, we prove the global Strichartz estimates which are uniform in $c$ when $J$ is odd. In Section 4, the Strichartz estimates with derivative loss are established when $J$ is even, which requires additional technical issues. In Section 5, we arrange well-posedness results for \[pHF\] and prove the global well-posedness of \[hHF\] as applications of uniform Strichartz estimates. In addition, we show that the nonlinear solutions are uniformly bounded with respect to $c$. Finally, we prove the convergence of solutions to \[hHF\] towards solutions to \[pHF\] as $c$ goes to infinity. In Section 6, as an application of global Strichartz estimates, we consider the power type nonlinearity and prove the small data scattering result.

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2. Preliminaries

2.1. Notations. Let $\xi = (\xi_1, \cdots, \xi_d) \in \mathbb{R}^d$.

- $\{e_j\}_{j=1}^d$ denotes the standard basis of $\mathbb{R}^d$. 

• We identify a vector in \( \mathbb{R}^d \) as a \( d \times 1 \) column matrix via the standard isomorphism

\[
(\xi_1, \cdots, \xi_d) \leftrightarrow \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_d \end{bmatrix}
\]

• For \( \eta \in \mathbb{R}^d \), we denote the inner product by \( \xi \cdot \eta = \sum_{i=1}^{d} \xi_i \eta_i \).

• Fix \( j \in \{1, \cdots, d\} \). We denote \( \xi_j = (\xi_1, \cdots, \xi_{j-1}, \xi_{j+1}, \cdots, \xi_d) \in \mathbb{R}^{d-1} \).

• For an \( n \)-tuple \( \alpha = (\alpha_1, \cdots, \alpha_d) \) of nonnegative integers and a function \( f \) on \( \mathbb{R}^d \), we define

\[
\partial^{\alpha} f := \partial^{\alpha_1} \partial^{\alpha_2} \cdots \partial^{\alpha_d} f
\]

Let \( \Omega : \mathbb{R}^d \to \mathbb{R} \) be a \( C^2 \) function. We denote the Hessian of \( \Omega \), \( d \times d \) matrix, by \( \mathbf{H}(\Omega) \) whose \((i, j)\) component is given by

\[
[\mathbf{H}(\Omega)]_{ij} = \partial_{\xi_i} \partial_{\xi_j} \Omega \quad \text{for } i, j = 1, 2, \cdots, d.
\]

We say that \( \Omega \) is \textit{degenerate} at \( x \in \mathbb{R}^d \) if \( \det \mathbf{H}(\Omega)(x) = 0 \).

We generalize the notion of Hessian. Let \( E \) be a set of \( k \) orthonormal vectors in \( \mathbb{R}^d \), say, \( E = \{ u_1, \cdots, u_k \} \). We define a \textit{Hessian of \( \Omega \) with respect to} \( E \) by \( k \times k \) matrix \( \mathbf{H}_{\{u_1, \cdots, u_k\}}(\Omega) \) whose \((m, n)\) component is given by

\[
[\mathbf{H}_{\{u_1, \cdots, u_k\}}(\Omega)]_{mn} = \partial_{u_m} \partial_{u_n} \Omega \quad \text{for } m, n = 1, 2, \cdots, k,
\]

where \( \partial_{u_m} \) denotes the directional derivative along \( u_m \).

The rank of a matrix \( A \) is the dimension of the vector space generated by its columns. If \( A \) has a rank \( k \), we denote \( \text{rank} A = k \).

Let \( \chi \in C^\infty_c(\mathbb{R}) \) be a radial non increasing function such that \( \chi(x) = 1 \) for \( |x| \leq 1 \) and \( \chi(x) = 0 \) for \( |x| \geq 2 \), and

\[
\sum_{N \in \mathbb{Z}} \chi_N(|x|) = 1, \quad \text{for } x \in \mathbb{R}^d \setminus \{0\},
\]

where \( \chi_N = \chi\left(\frac{x-i}{2^N}\right) - \chi\left(\frac{x}{2^N}\right) \) for \( N \in \mathbb{Z} \). We define the projection operator \( P_N \) by the fourier multiplier such that

\[
\widehat{P_N f}(\xi) = \chi_N(|\xi|) \hat{f}(\xi).
\]

Then, \( f = \sum_{N \in \mathbb{Z}} P_N f \).

Let us define \( \Theta_j = \{ \xi \in \mathbb{R}^d : |\xi_j| \geq \frac{1}{\sqrt{2^d}}|\xi| \} \) with \( j = 1, 2, \cdots, d \). Then, \( \mathbb{R}^d \setminus \{0\} = \cup_{j=1}^{d} \Theta_j \) and there is a partition of unity \( \{ \theta^j \} \) subordinate to the covering \( \Theta_j \) (the \( \theta^j \) can be defined on the sphere and extended such that they are homogeneous of order zero) satisfying that

\[
\sum_{j=1}^{d} \theta^j(\xi) = 1 \quad \text{and} \quad |\xi_j| \geq \frac{1}{\sqrt{2^d}}|\xi| \text{ on the support of } \theta^j.
\]
2.2. Stationary phase method. By taking the Fourier transform, the linear solution to (2.1) is given by

\[ e^{-itH^{(c)}_j} \psi_0(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi - it\Omega_j^{(c)}(\xi)} \hat{\psi}_0(\xi) d\xi, \]  

where the dispersion relation is given by

\[ \Omega_j^{(c)}(\xi) := \sum_{j=1}^{j} (-1)^{j+1} \frac{(2j-2)! h^{2j-1}}{(j-1)! j! (2m)^{2j-1} c^{2j-2}} |\xi|^{2j}. \]

We observe that \( \Omega_j^{(c)} \) is a radial function, so we have

\[ \Omega_j^{(c)}(\xi) = \omega_j^{(c)}(|\xi|) \text{ with } \omega_j^{(c)} : [0, \infty) \to \mathbb{R}, \]

explicitly,

\[ \omega_j^{(c)}(r) = \sum_{j=1}^{j} (-1)^{j+1} \frac{(2j-2)! h^{2j-1}}{(j-1)! j! (2m)^{2j-1} c^{2j-2}} r^{2j}. \]

The linear solution can be written as kernel form

\[ e^{-itH^{(c)}_j} \psi_0(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi - it\Omega_j^{(c)}(\xi)} d\xi \right) \psi_0(y) d\xi = I_j^{(c)}(t, \cdot) * \psi_0(x), \]

where we introduced the oscillatory integral

\[ I_j^{(c)}(t, v) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{itv\cdot\xi - t\Omega_j^{(c)}(\xi)} d\xi. \]

In this subsection, we list basic analysis tools to estimate the oscillatory integrals. For one dimensional case, the decay rate of oscillatory integral is totally determined from the nondegeneracy of the derivatives of the phase function (see [30, Chapter VIII]).

**Lemma 2.1** (Van der Corput Lemma). Let \( k \in \mathbb{Z}^+ \) and \( |\phi^{(k)}(x)| \geq 1 \) for all \( x \in [a, b] \) with \( \phi'(x) \) monotonic in the case \( k = 1 \). Then,

\[ \left| \int_a^b e^{i\lambda \phi(x)} dx \right| \leq C_k \lambda^{-\frac{1}{k}} \]  

and

\[ \left| \int_a^b e^{i\lambda \phi(x)} \eta(x) dx \right| \leq C_k \left( \int_a^b |\eta'(x)| dx + |\eta(b)| \right) \lambda^{-\frac{1}{k}}, \]

where the constant \( C_k \) is independent of \( a, b \) and \( \phi \).

For multi dimensional case, the rank of Hessian matrix of the phase function plays a crucial role (see [31, Chapter 8]).

**Lemma 2.2** (Stationary phase method). For given a real-valued phase function \( \Omega \in C^\infty(\mathbb{R}^d) \) and amplitude function \( \eta \in C^\infty_0(\mathbb{R}^d) \), assume that

\[ \text{rank} H(\Omega) = m \text{ on the support of } \eta, \]

(2.6)
for $0 < m \leq d$, in other words, there exist a coordinate $x = (x', x'') \in \mathbb{R}^m \times \mathbb{R}^{d-m}$ with a basis $\{ u_j \}_{j=1}^d$ such that
\[
| \det H_{\{ u_1, \ldots, u_m \}} \Omega | \geq 1 \text{ on the support of } \eta.
\] (2.7)

Then,
\[
\left| \int_{\mathbb{R}^d} e^{i\Omega(x)} \eta(x) dx \right| \leq C_m |\text{supp } \eta| C_{\Omega, \eta} \lambda^{-m/2},
\]
where
\[
(C_{\Omega, \eta})^2 \leq 1 + \| \eta \|_{C^{m+1}} \max_{2 \leq |a| \leq m+2} \{ C_\alpha : \sup_{x \in \text{supp } \eta} |\partial^\alpha \Omega(x)| \leq C_\alpha, \}
\]
and $C_m$ depends only on $m$.

2.3. **Hessian of radial function.** In the previous subsection, we found that the Hessian matrix of phase function played an essential role in analysis of oscillatory integrals. Recall that $\Omega^{(c)}_j$ in the phase function in (2.3) is radial, so the formula for Hessian is quite simple.

**Lemma 2.3 (Hessian of radial function).** Let $\omega : [0, \infty) \to \mathbb{R}$ and $\Omega : \mathbb{R}^d \to \mathbb{R}$ be given by $\Omega(\xi) = \omega(|\xi|)$. Then, for $\xi \in \mathbb{R}^d \setminus \{0\}$
\[
\det (H \Omega)(\xi) = \omega''(|\xi|) \{ \omega'(|\xi|) |\xi|^{-1} \}^{d-1}.
\] (2.8)

**Proof.** Without loss of generality, we assume $\xi_1 \neq 0$. We compute
\[
\partial_{\xi_i} \partial_{\xi_j} \Omega(\xi) = \omega''(|\xi|) \frac{\xi_i \xi_j}{|\xi|^2} + \omega'(|\xi|) \left( \frac{\delta_{ij}}{|\xi|} - \frac{\xi_i \xi_j}{|\xi|^3} \right) = \frac{\omega'(|\xi|)}{|\xi|} \delta_{ij} + \left( \omega''(|\xi|) - \frac{\omega'(|\xi|)}{|\xi|} \right) \frac{\xi_i \xi_j}{|\xi|^2}.
\]

For notational convenience, we denote
\[
A = \frac{\omega'(|\xi|)}{|\xi|}, \quad B = \omega''(|\xi|) - \frac{\omega'(|\xi|)}{|\xi|}.
\] (2.9)

Then, we have $\nabla \partial_{\xi_j} \Omega(\xi) = Ae_j + (B \frac{\xi_j}{|\xi|}) \xi$. Thus, by Gaussian elimination (or the row reduction), we obtain
\[
\det (H \Omega)(\xi) = \det \begin{bmatrix}
Ae_1 + (B \frac{\xi_1}{|\xi|}) \xi \\
Ae_2 + (B \frac{\xi_2}{|\xi|}) \xi \\
\vdots \\
Ae_d + (B \frac{\xi_d}{|\xi|}) \xi
\end{bmatrix} = \det \begin{bmatrix}
Ae_1 + (B \frac{\xi_1}{|\xi|}) \xi \\
Ae_2 - B \frac{\xi_1}{|\xi|} Ae_1 \\
\vdots \\
Ae_d - B \frac{\xi_1}{|\xi|} Ae_1
\end{bmatrix} = \det \begin{bmatrix}
(A + B)e_1 \\
Ae_2 - B \frac{\xi_1}{|\xi|} Ae_1 \\
\vdots \\
Ae_d - B \frac{\xi_1}{|\xi|} Ae_1
\end{bmatrix} = (A + B)^{d-1}. \quad (2.10)
\]

Since $A + B = \omega''(|\xi|)$, we prove the desired formula. \qed
3. Strichartz estimates: Odd case

Throughout the paper, we are interested in asymptotic phenomena of higher-order equation in terms of $c$, thus, in what follows, we fix $h = m = 1$ via rescaling, for simplicity of calculation.

When $J = 1$ corresponding to the Schrödinger equations, we can easily show from (2.8) that

$$\det H (v \cdot \xi - \frac{1}{2} |\xi|^2) = 1 \quad \text{for all } \xi \in \mathbb{R}^d,$$

which gives by Lemma 2.2 the dispersive estimates

$$\sup_{v \in \mathbb{R}^d} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{it(v \cdot \xi - \frac{1}{2} |\xi|^2)} d\xi \leq At^{-\frac{d}{2}},$$

In this section, we show that the argument can be extended to all odd cases $J \geq 3$.

**Proposition 3.1.** Let $J \geq 3$ be an odd integer. Then,

$$\det H(\Omega^{(c)}_J)(\xi) \gtrsim (1 + \frac{|\xi|}{c})^{(2J - 2)},$$

and as a consequence we have

$$\sup_{v \in \mathbb{R}^d} |I^{(c)}_J(t, v)| \lesssim t^{-\frac{d}{2}},$$

where the implicit constants are independent of $c$.

**Proof.** First, we show that (3.2) follows from (3.1). When $d = 1$, it is direct application of Lemma 2.1. Also, when $d \geq 2$, it can be shown by applying Lemma 2.2. Indeed, we write

$$I^{(c)}_J(t, v) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{it(v \cdot \xi - \Omega^{(c)}_J(\xi))} \chi(\xi) d\xi + \sum_{N \in \mathbb{Z}^d, 2^N > c} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{it(v \cdot \xi - \Omega^{(c)}_J(\xi))} \chi_N(\xi) d\xi,$$

and change variables to obtain

$$I^{(c)}_J(t, v) = \frac{c^d}{(2\pi)^d} \int_{\mathbb{R}^d} e^{iR^2t(R^{-1}v \cdot \xi - R^{-2}\Omega^{(c)}_J(R\xi))} \chi(\xi) d\xi + \sum_{N \in \mathbb{Z}^d, 2^N > c} \frac{c^{Nd}}{(2\pi)^d} \int_{\mathbb{R}^d} e^{iR^2t(2^{-N}v \cdot \xi - 2^{-2N}\Omega^{(c)}_J(2^N\xi))} \chi_0(\xi) d\xi =: O_1 + O_2.$$

Once proving that for $R > 0$

$$R^d \left| \int_{\mathbb{R}^d} e^{iR^2t(R^{-1}v \cdot \xi - R^{-2}\Omega^{(c)}_J(R\xi))} \eta(\xi) d\xi \right| \lesssim \begin{cases} (1 + \frac{R}{c})^{(J-1)} t^{-\frac{d}{2}} & \text{for } \eta = \chi, \\ (1 + \frac{R}{c})^{(J-1)(1-d)} t^{-\frac{d}{2}} & \text{for } \eta = \chi_0, \end{cases} (3.3)$$

we immediately have

$$O_1 \lesssim t^{-\frac{d}{2}},$$

and

$$O_2 \lesssim \sum_{N \in \mathbb{Z}^d, 2^N > c} (1 + \frac{2N}{c})^{(J-1)(1-d)} t^{-\frac{d}{2}} \lesssim t^{-\frac{d}{2}}.$$
due to $J \geq 3$ and $d \geq 2$. From (3.1), we see that
\[
\det \mathbf{H} \left( R^{-1} v \cdot \xi - R^{-2} \Omega_j^c (R \xi) \right) = \left[ \det \mathbf{H} (\Omega_j^c) \right] (R \xi) \gtrsim (1 + \frac{R(\xi)}{c})^{(2J-2)d},
\] (3.4)
and a direct computation gives
\[
\sup_{2 \leq |\alpha| \leq d+2} \left\{ C_\alpha : \sup_{x_0 \in \text{supp} \eta} |\partial_x^\alpha \eta| \left\{ (R^{-1} v \cdot \xi - R^{-2} \Omega_j^c (R \xi)) \right\} \leq C_\alpha, \right\} \lesssim (1 + \frac{R(\xi)}{c})^{(2J-2)}.
\] (3.5)
Note that the right-hand side of (3.4) is further bounded below by 1 for $\eta = \chi$, and $(\frac{R(\xi)}{c})^{(2J-2)d}$ for $\eta = \chi_0$. Note also that the implicit constants in (3.4) and (3.5) are independent of both $v \in \mathbb{R}^d$ and $c$, but dependent on $J$. Now, we apply Lemma 2.2 to obtain
\[
\text{LHS of (3.3)} \lesssim \begin{cases} 
R^d (1 + \frac{R(\xi)}{c})^{\frac{1}{2}(2J-2)} (R^2 t)^{-\frac{d}{2}} & \text{for } \eta = \chi, \\
R^d (1 + \frac{R(\xi)}{c})^{\frac{1}{2}(2J-2)} \left( 1 + \frac{R(\xi)}{c} \right)^{(2J-2)} R^2 t)^{-\frac{d}{2}} & \text{for } \eta = \chi_0,
\end{cases}
\]
which proves (3.3).

Next, we prove (3.1). Recall from (2.8) that
\[
\det \mathbf{H} (\Omega_j^c (\xi)) = (\omega_j^c)''(|\xi|) \left\{ (\omega_j^c)'(|\xi|)|\xi|^{-1} \right\}^{d-1}.
\]
Let $J = 2J_0 - 1$ with $J_0 \in \mathbb{N}$. For $\omega_j^c(r)$, collecting the positive and negative terms, we write
\[
r^{-1}(\omega_j^c)'(r) = \sum_{j=0}^{J_0-1} \frac{(4j)!}{((2j)!)^2 2^{2j} e^{4j} r^{4j}} - \sum_{j=1}^{J_0-1} \frac{(4j-2)!}{((2j-1)!)^2 2^{2j-2} e^{4j-2} r^{4j-2}}.
\]
For the second term, by the Cauchy-Schwarz inequality, we have
\[
\frac{(4j-2)!}{((2j-1)!)^2 2^{2j-2} e^{4j-2} r^{4j-2}} \leq \frac{1}{2} \frac{(4j-4)!}{((2j-2)!)^2 2^{2j-4} e^{4j-4} r^{4j-4}} + \frac{1}{2} \frac{(4j)!}{((2j)!)^2 2^{2j} e^{4j} r^{4j}}
\]
which proves
\[
|r^{-1}(\omega_j^c)'(r)| \geq \frac{1}{2} + \frac{(2J-2)!}{((J-1)!)^2 2^{2J-2}} \left( \frac{r}{c} \right)^{2J-2}.
\]
Similarly for $\omega_j^c(r)$, we write
\[
(\omega_j^c)'(r) = \sum_{j=1}^{2J_0-1} \frac{(-1)^{j+1}(2j-1)!}{(j-1)! (j-1)! 2^{2j-2} e^{2j-2} r^{2j-2}}
\]
\[
= \sum_{j=0}^{J_0-1} \frac{(4j+1)!}{((2j)!)^2 2^{2j} e^{4j}} r^{4j} - \sum_{j=1}^{J_0-1} \frac{(4j-1)!}{((2j-1)!)^2 2^{2j-2} e^{4j-2} r^{4j-2}}
\]
\[
= 1 + \sum_{j=1}^{J_0-1} \frac{(4j+1)!}{((2j)!)^2 2^{2j} e^{4j}} r^{4j} \left( 1 - \frac{c^2}{r^2} \frac{4j}{4j+1} \right).
\]
When $r \geq c$, one can see that the summand is positive for all $1 \leq j \leq J_0 - 1$, thus we have
\[
(\omega_j^c)'(r) \geq 1 + \frac{(2J-2)!}{((J-1)!)^2 2^{2J-2} e^{2J-2} r^{2J-2}}.
\]
On the other hand, for \(0 \leq r < c\), since \(\omega_J^{(c)}\) is Taylor expansion of
\[
c^2 \left\{ \sqrt{1 + \left(\frac{r}{c}\right)^2} - 1 \right\},
\]
by the term-by-term differentiation and Taylor’s theorem, we obtain
\[
\left(1 + \left(\frac{r}{c}\right)^2\right)^{-\frac{3}{2}} = (\omega_J^{(c)})''(r) + \frac{(-1)^J (2J + 1)!}{(J)! (J)! 2^{2J} c^{2J}} (r_*)^{2J}
\]
for some \(r_* \in [0, r)\). Since \(J\) is odd, we conclude that for \(0 \leq r < c\)
\[
(\omega_J^{(c)})''(r) \geq \left(1 + \left(\frac{r}{c}\right)^2\right)^{-\frac{3}{2}} \geq 2^{-\frac{3}{4}}.
\]
\[\square\]

**Remark 3.2.** For one dimensional case, we applied Lemma 2.1 in the above proof, where the constant in (2.5) only depends on the lower bound of second derivative of phase function. Thus, one sees that if \(d = 1\), the implicit constant in (3.2) is also independent of \(J\).

Recall that
\[
e^{-it\mathcal{H}_J^{(c)}} \psi_0(x) = \mathcal{I}_J^{(c)}(t, \mathbf{r}) * \psi_0(x).
\]
So, by Young’s inequality and (3.2) we have
\[
\left\| e^{-it\mathcal{H}_J^{(c)}} \psi_0 \right\|_{L^\infty(\mathbb{R}^d)} \leq \sup_{v \in \mathbb{R}^d} \left| \mathcal{I}_J^{(c)}(t, v) \right| \| \psi_0 \|_{L^1} \leq At^{-\frac{d}{2}} \| f \|_{L^1(\mathbb{R}^d)}.
\]
It is well-known that (hLS) enjoys mass conservation law
\[
\left\| e^{-it\mathcal{H}_J^{(c)}} \psi_0 \right\|_{L^2(\mathbb{R}^d)} = \| \psi_0 \|_{L^2(\mathbb{R}^d)} \text{ for all } t \in \mathbb{R}.
\]
Interpolating two estimates, we obtain for \(2 \leq p \leq \infty\)
\[
\left\| e^{-it\mathcal{H}_J^{(c)}} \psi_0 \right\|_{L^p(\mathbb{R}^d)} \lesssim t^{-\frac{d}{2}(1 - \frac{2}{p})} \| f \|_{L^p(\mathbb{R}^d)}.
\]
Now, the Strichartz estimates (1.4) follows from the well-known \(TT^*\) argument in [17]. We omit the details.

4. Strichartz estimates: Even case

As we did in the odd case, we will apply Lemma 2.2 to estimate the kernel of linear solution. So, we begin with describing the properties of the dispersion function, \(\Omega_J^{(c)}\).
4.1. Hessian of the phase function. We examine the behavior of the radial function \( \omega_j^{(c)} \) satisfying that \( \Omega_j^{(c)}(\xi) = \omega_j^{(c)}(|\xi|) \) for even case, \( J \in 2\mathbb{N} \) (see (2.1)).

**Lemma 4.1.** Let \( J \in 2\mathbb{N} \). Then, there exists \( A > 0 \) only depending on \( J \) satisfying the followings:

1. \( (\omega_j^{(c)})'''(r) \leq -3 \cdot 2^{-\frac{5}{2}}c^{-2}r \).
2. \( (\omega_j^{(c)})'' \) has a unique zero \( r_2 \) in \( (\frac{c}{2}, c) \). Furthermore,
   \[ (\omega_j^{(c)})''(r) > \frac{5}{2} \quad \text{for} \quad r \leq \frac{c}{2} \quad \text{and} \quad (\omega_j^{(c)})''(r) < -A \left( \frac{c}{J} \right)^2J^{-2} \quad \text{for} \quad r > c. \]
3. \( (\omega_j^{(c)})' \) has a unique zero \( r_1 \in (c, 2c) \). Furthermore,
   \[ r^{-1}(\omega_j^{(c)})'(r) > \frac{1}{c} \quad \text{for} \quad r < c \quad \text{and} \quad r^{-1}(\omega_j^{(c)})'(r) < -A \left( \frac{r}{c} \right)^2J^{-2} \quad \text{for} \quad r > 2c. \]

**Proof.** Let \( J = 2J_0 \) with \( J_0 \in \mathbb{N} \). We begin with verifying estimates (I) for the third derivative. For \( r \geq c \), we have

\[
(\omega_j^{(c)})'''(r) = \frac{2J_0}{r^2} \sum_{j=2}^{J_0} \frac{(-1)^{j+1}(2j-1)!}{(j-1)!2^{2j-3}c^2j^2-2}r^{2j-3}.
\]

On the interval \((0, c)\), \( (\omega_j^{(c)})''' \) is the \((J - 2)\)-th degree Taylor polynomial of \(-3c^{-1}r(1 + \left( \frac{r}{c} \right)^2)^{-\frac{5}{2}}\). Hence, for \( 0 < r < c \), there exists \( 0 < r_0 < r \) such that

\[
-3c^{-1}r \left( 1 + \left( \frac{r}{c} \right)^2 \right)^{-\frac{5}{2}} = (\omega_j^{(c)})'''(r) + \frac{4J_0 + 1}{(2J_0)(2J_0 - 1)!2^{4J_0 - 1}c^{4J_0 - 5}r^{4J_0 - 1},}
\]

which implies that

\[
(\omega_j^{(c)})'''(r) \leq -3c^{-1}r \left( 1 + \left( \frac{r}{c} \right)^2 \right)^{-\frac{5}{2}} \leq -3 \cdot 2^{-\frac{5}{2}}c^{-2}r.
\]

Secondly, a direct computation gives that

\[
(\omega_j^{(c)})''(r) = \sum_{j=1}^{J_0} \frac{4j - 3)!r^{4j-2}}{(2j-2)!2^{4j-4}c^{4j-2}}(c^2 - \frac{4j - 1}{4j - 2}r^2).
\]

Since \( (\omega_j^{(c)})'' \) is decreasing on \([0, \infty)\), we have

\[
(\omega_j^{(c)})''(r) > (\omega_j^{(c)})'(c/2) > \frac{5}{2}, \quad \text{for} \quad r < \frac{c}{2}.
\]

Also, since \( 1 < \frac{4j - 1}{4j - 2} \leq \frac{3}{2} \) for all \( j \in \mathbb{N} \), all terms in the summation are negative for \( r > c \), which gives that

\[
(\omega_j^{(c)})''(r) < -\frac{(2J - 2)!}{(J - 2)!2^{2J-5}c^{2J-2}} \left( \frac{r}{c} \right)^{2J-2}, \quad \text{for} \quad r > c.
\]
We conclude that \((\omega^c_J)^\prime\) has a unique zero \(r_2 \in (c/2, c)\).

Similarly, one can easily check that
\[
(\omega^c_J)(0) = 0 \text{ and } (\omega^c_J)(c) > 0 \text{ and } (\omega^c_J)^\prime(2c) < 0.
\]

Then, since \((\omega^c_J)^\prime(0)\) equals 1 at 0, is decreasing, and has the unique zero at \(r_2\), \((\omega^c_J)^\prime\) is increasing on \((0, r_2)\) and decreasing on \((r_2, \infty)\). Thus, we conclude that \((\omega^c_J)^\prime\) has a unique zero \(r_1 \in (c, 2c)\). Next, we compute that
\[
r^{-1}(\omega^c_J)(r) = \sum_{j=1}^{J_0} \frac{(4j-4)!}{{(2j-2)!}{(2j-2)!}} \frac{r^{4j-2}}{c^{2j-2}} \left( \frac{c^2}{r^2} - \frac{4j-3}{4j-2} \right).
\]

Since \(1 \leq \frac{4j-3}{4j-2} < 1\) for all \(j \in \mathbb{N}\), if \(r > 2c\), all terms in the summation at the last line are negative, so we have
\[
r^{-1}(\omega^c_J)^\prime(r) < -\frac{(2J-4)!}{{(J-2)!}{(J-2)!}} \frac{r^{2J-2}}{c^{2J-2}}.
\]

On the other hand, for \(r < c\), all terms are positive, so
\[
r^{-1}(\omega^c_J)^\prime(r) > \left( \frac{r}{c} \right)^2 \left( \frac{c^2}{r^2} - \frac{1}{2} \right) > \frac{1}{2}.
\]

We recall from Lemma 2.33 that
\[
\text{det} H(\Omega^c_J)(\xi) = (\omega^c_J)^\prime(|\xi|) \{ (\omega^c_J)^\prime(|\xi|)|\xi|^{-1} \}^{d-1}.
\]

Thus, we check from the above lemma that the determinant of Hessian of \(\Omega^c_J\) is zero if and only if \(|\xi| = r_1\) or \(r_2\). Since \(c/2 < r_1, r_2 < 2c\), heuristically speaking, the phase function \(\Omega^c_J\) for an even number \(J\) could be degenerate provided that the norm of momentum is close to the speed of light. Next, we characterize the rank of the Hessian matrix on each degenerate sphere.

**Lemma 4.2.** Let \(J \in 2\mathbb{N}\).

\[
\text{rank} H(\Omega^c_J)(\xi) = \begin{cases} 
   d - 1 & \text{for } |\xi| = r_2, \\
   1 & \text{for } |\xi| = r_1.
\end{cases}
\]

**Proof.** We show that \(H(\Omega^c_J)(\xi) = \text{span}\{\xi\} \text{ for } |\xi| = r_1\) and \(H(\Omega^c_J)(\xi) = \text{span}\{\xi^\perp\} \text{ for } |\xi| = r_2\), where \(\xi^\perp\) denotes a vector orthogonal to \(\xi\). Given any nonzero \(\eta \in \mathbb{R}^d\), we decompose \(\eta = k\xi + \xi^\perp\) for some vector \(\xi^\perp\) orthogonal to \(\xi\) and \(k \in \mathbb{R}\). Then, using the notations in (2.2), we calculate
\[
\left( H(\Omega^c_J)(\xi) \right)_n = \sum_{\ell=1}^{d} \left( A\delta_{n\ell} + B\frac{\xi_n\xi_{\ell}}{|\xi|^2} \right) \eta_{\ell} = A\eta_n + B\frac{\xi \cdot \eta}{|\xi|^2} \xi_n
\]
\[
= k(A + B)\xi_n + A(\xi^\perp)_n = k(\omega^c_J)^\prime(|\xi|)\xi_n + (\omega^c_J)^\prime(|\xi|)|\xi|^{-1}(\xi^\perp)_n.
\]
Thus, on the smaller sphere $|\xi| = r_2$, \[ \det(\Omega_j^c)(\xi) \eta = 0 \] if and only if $\xi^\perp = 0$ (or $\eta = k\xi$), whereas on the larger sphere $|\xi| = r_1$, \[ \det(\Omega_j^c)(\xi) \eta = 0 \] if and only if $k\xi = 0$ (or $\eta = \xi^\perp$). □

From the viewpoint of Section 2.2, the above lemma says that we can expect a time decay of solutions to (SLS) to be at most $t^{-\frac{d+1}{2}}$ on the larger sphere $\{ |\xi| = r_1 \}$ and $t^{-\frac{d-1}{2}}$ on the smaller sphere $\{ |\xi| = r_2 \}$. Also, we infer from the proof of lemma that the Hessian of the phase function $\Omega_j^c$ restricted to radial (resp. perpendicular) direction is nondegenerate on larger (resp. smaller) sphere. First, let us verify this at a point on intersection of the sphere and axis along $e_j$, the simplest case, where the differentiation with radial direction or perpendicular direction is easy to find
\[ \det H_{\{e_j\}}(\Omega_j^c)(r_1 e_j) = (\omega_j^c)'(r_1), \quad \det H_{\{e_1,\ldots,e_{j-1},e_{j+1},\ldots,e_d\}}(\Omega_j^c)(r_2 e_j) = r_2^{-1}(\omega_j^c)'(r_2). \]

Then, we check from Lemma 4.3 that
\[ |\det H_{\{e_j\}}(\Omega_j^c)(r_1 e_j)| \geq \frac{1}{2}, \quad |\det H_{\{e_1,\ldots,e_{j-1},e_{j+1},\ldots,e_d\}}(\Omega_j^c)(r_2 e_j)| \geq \frac{1}{2}, \]
where both lower bounds are independent of $c$. In the next lemma, we prove that the determinants of the above submatrices of Hessian at points near the each degenerate sphere and around $e_j$ direction also have the uniform-in-$c$ lower bound. In addition, we prove that outside the degenerate spheres the determinant of Hessian matrix has the uniform-in-$c$ lower bound.

Lemma 4.3. Let $J \in 2N$ and $\xi \in \mathbb{R}^d$. There exist $0 < \delta \ll 1$ independent of $c$ such that
\[ \min(c - r_2, r_1 - c) \geq \delta. \] (4.3)

Then, we have
\[ |\det H(\Omega_j^c)(\xi)| \geq \left( 1 + \frac{|\xi|}{c} \right)^{2J-2} \] if $||\xi| - r_1| > c\delta$ or $||\xi| - r_2| > c\delta$, (4.4)
where the implicit constant is independent of $c$. Furthermore, if $\xi \notin \Theta_j$ for some $1 \leq j \leq d$, one has
\[ |\det H_{\{e_j\}}(\Omega_j^c)(\xi)| \geq \frac{1}{4d}(|\omega_j^c)'(\xi)| \geq \frac{1}{8d}, \quad \text{if } ||\xi| - r_1| \leq 2c\delta, \] (4.5)
\[ |\det H_{\{e_1,\ldots,e_{j-1},e_{j+1},\ldots,e_d\}}(\Omega_j^c)(\xi)| \geq \frac{1}{4d} \left( \frac{|(\omega_j^c)'(\xi)|}{|\xi|} \right)^{d-1} \geq \frac{1}{8d}, \quad \text{if } ||\xi| - r_2| \leq 2c\delta. \] (4.6)

Proof. (1). Proof of (4.3). By applying Mean value theorem to $(\omega_j^c)'$, we have
\[ (\omega_j^c)'(c) = (\omega_j^c)'(c) - (\omega_j^c)'(r_2) = (\omega_j^c)'(r_*) (c - r_2) \] for some $r_* \in (r_2, c)$.

Then, from (4.1),
\[ \frac{1}{2} \leq |(\omega_j^c)'(c)| = |(\omega_j^c)'(r_*)||c - r_2| \leq 3 \cdot 2^{-\frac{d}{2}} c^{-1}|c - r_2|, \]
which gives that
\[ c - r_2 \geq 3^{-1/2}2^{3/2}c. \]

Similarly, we have
\[ \frac{c}{2} \leq |(\omega_j^{(c)})''(c)| = |(\omega_j^{(c)})''(r_*)||r_1 - c| \leq |(\omega_j^{(c)})''(2c)||r_1 - c|, \]
for some \( r_* \in (c, r_1) \), which gives that
\[ r_1 - c \geq \frac{1}{2}|(\omega_j^{(c)})''(2c)|^{-1}c. \]

Here, we observe that \( |(\omega_j^{(c)})''(2c)| \) is independent of \( c \).

(2). Proof of (4.4). Recall from (2.8) that
\[ \det H(\Omega_j^{(c)})(\xi) = (\omega_j^{(c)})''(|\xi|)\{ (\omega_j^{(c)})'(|\xi|)|\xi|^{-1}\}^{d-1}, \]
so, it can be easily shown from Lemma 4.1 that
\[ |\det H(\Omega_j^{(c)})(\xi)| \geq \left( 1 + \frac{|\xi|^2}{r^2} \right)^{(2J-2)d} \text{ for } |\xi| \leq \frac{c}{2} \text{ or } |\xi| \geq 2c. \] (4.7)

Next, consider \( \frac{c}{2} < |\xi| \leq 2c \). If \( r \in (\frac{c}{2}, 2c) \) and \( |r - r_2| \geq c\delta \), by Mean value theorem, we have
\[ |(\omega_j^{(c)})''(r)| = |(\omega_j^{(c)})''(r_2) + (\omega_j^{(c)})''(r_*)(r - r_2)| \text{ for some } r_* \text{ between } r \text{ and } r_2 \]
\[ = |(\omega_j^{(c)})''(r_*)||r - r_2| \geq 3 \cdot 2^{-\frac{7}{2}}\delta, \]
where we used (1) in Lemma 4.1.

Also, if \( r \in (\frac{c}{2}, 2c) \) and \( |r - r_1| \geq c\delta \),
\[ |(\omega_j^{(c)})'(r)| = |(\omega_j^{(c)})'(r_1) + (\omega_j^{(c)})''(r_*)(r - r_1)| \text{ for some } r_* \text{ between } r \text{ and } r_1 \]
\[ = |(\omega_j^{(c)})''(r_*)||r - r_1| \geq \frac{1}{2}c\delta, \]
which gives that
\[ |r^{-1}(\omega_j^{(c)})'(r)| \geq \frac{1}{2}\delta. \]

By applying two inequalities to the above formula for determinant with \( |\xi| = r \), we obtain for \( \frac{c}{2} < |\xi| \leq 2c \) with \( ||\xi| - r_1| \geq c\delta \) or \( ||\xi| - r_2| \geq c\delta \) that
\[ |\det H(\Omega_j^{(c)})(\xi)| \geq 3 \cdot 2^{-\frac{7}{2}}2^{2(d-1)}\delta^d. \]

(3). Proof of (4.5). A direct computation gives that
\[ \partial^2_{\xi_j} \Omega_j^{(c)}(\xi) = (\omega_j^{(c)})''(|\xi|)\frac{\xi_j^2}{|\xi|^2} + (\omega_j^{(c)})'(|\xi|)|\xi|^{-1}\frac{|\xi_j|^2}{|\xi|^2}. \]

We claim that the latter term on the right-hand side is dominated by the former if \( \delta \) is sufficiently small. Indeed, by Mean value theorem, we have
\[ (\omega_j^{(c)})'(r) = (\omega_j^{(c)})'(r) - (\omega_j^{(c)})'(r_1) = (\omega_j^{(c)})''(r_*)(r - r_1), \text{ for some } r_1 < r_* < r. \]
we have \( \xi \) from which we estimate the second term of the right-hand side that for \( r \)

Thus, by making \( \delta \) sufficiently small if necessary, we can obtain (4.6).

Using the notations in (2.9), we compute

\[
\mathbf{H}_{\{e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_d\}}(\Omega^{(e)}_j)(\xi) = \begin{bmatrix}
Ae_1 + (B \frac{\xi_j}{|\xi|}) \xi_j \\
\vdots \\
Ae_{i-1} + (B \frac{\xi_j}{|\xi|}) \xi_j \\
Ae_i + (B \frac{\xi_j}{|\xi|}) \xi_j \\
\vdots \\
Ae_d + (B \frac{\xi_j}{|\xi|}) \xi_j
\end{bmatrix}.
\]

By performing Gaussian elimination as in Lemma 2.3, we obtain

\[
\det \mathbf{H}_{\{e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_d\}}(\Omega^{(e)}_j)(\xi) = \left( \frac{(\omega_j^{(e)})'(\xi_j)}{|\xi_j|} \right)^{d-2} \left\{ \left( \frac{(\omega_j^{(e)})'(\xi_j)}{|\xi_j|} \right) \frac{\xi_j^2}{|\xi|^2} + (\omega_j^{(e)})''(\xi_j) \frac{\xi_j^2}{|\xi|^2} \right\}.
\]

We show that the first term of the right-hand side is dominant. By Mean value theorem and (1) we have for \( r \in (c/2, c) \),

\[
|\omega_j^{(e)}(r)| = |\omega_j^{(e)}(r_*)||r - r_2| \leq 3 \cdot 2^{-\frac{5}{6}} c^{-2} r_* |r - r_2| \leq 3 \cdot 2^{\frac{3}{2}} \delta.
\]

from which we estimate the second term of the right-hand side that for \( \xi \in \Theta_i \)

\[
\left| (\omega_j^{(e)})''(\xi_j) \frac{\xi_j^2}{|\xi|^2} \right| \leq \frac{2d}{2d - 1} \left| (\omega_j^{(e)})''(\xi_j) \right| \leq \frac{2d}{2d - 1} 3 \cdot 2^{\frac{3}{2}} \delta.
\]

On the other hand, for \( \xi \in \Theta_j \) and \( |\xi| \leq c \)

\[
\left| (\omega_j^{(e)})'(\xi_j) \frac{\xi_j^2}{|\xi|^2} \right| \geq \frac{1}{4d} \left| (\omega_j^{(e)})'(\xi_j) \right| \geq \frac{1}{16d} \left| (\omega_j^{(e)})'(\xi_j) \right| + \frac{1}{8d}.
\]

Thus, by making \( \delta \) sufficiently small if necessary, we can obtain (4.6).

4.2. Dispersive estimates. Using the projection operator (2.1), we write the linear solution to (4.1) as

\[
e^{-it\mathcal{H}_j} \psi_0(x) = \sum_{N \in \mathbb{Z}} e^{-it\mathcal{H}_j^{(N)}} P_N \psi_0(x) = \sum_{N \in \mathbb{Z}} \mathcal{I}_{j,N}(t, \xi) \ast \psi_0(x),
\]

where the kernel is given by

\[
\mathcal{I}_{j,N}(t, \xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{it \cdot (\xi - \Omega_j^{(e)}(\xi))} \chi_N(\xi) d\xi.
\]
We investigate the time decay of $T^c_{J,N}(t,v)$ by using the results in the previous subsection.

4.2.1. Multi dimensional case $d \geq 2$.

**Theorem 4.4.** Let $J \in 2\mathbb{N}$ and $N \in \mathbb{Z}$. For $d \geq 2$, we have

$$\sup_{v \in \mathbb{R}^d} |I^{(c)}_{J,N}(t,v)| \lesssim \begin{cases} t^{-\frac{d}{2}} & \text{for } 2^N < c/4 \text{ or } 2^N > 4c, \\ c^{d-1} t^{-\frac{d}{2}} & \text{for } c/4 \leq 2^N \leq 4c, \end{cases} \quad (4.10)$$

where the implicit constants are independent of $c \geq 1$.

**Remark 4.5.** We do not claim the sharpness of time decay when $c/4 \leq 2^N \leq 4c$.

**Proof.** For $2^N < c/4$ or $2^N > 4c$ when the degenerate points are excluded, $\det(\Omega^{(c)}_J)$ has the same lower bound (4.4) as in the odd case (see Proposition 3.1), so by applying Lemma 2.2 we can show that

$$|I^{(c)}_{J,N}(t,v)| \lesssim t^{-\frac{d}{2}}.$$

Next, we consider the case when $c/4 \leq 2^N \leq 4c$. Let us decompose the support of integral $I^{(c)}_{J,N}$ into

$$I^{(c)}_{J,N} = K_1 + K_2 + K_3,$$

$$K_1 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{it(v \cdot \xi - \Omega^{(c)}_J(\xi))} \chi \left( \frac{|\xi| - r_1}{c} \right) \chi_N(\xi) d\xi,$$

$$K_2 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{it(v \cdot \xi - \Omega^{(c)}_J(\xi))} \left(1 - \chi \left( \frac{|\xi| - r_1}{c} \right) \right) \chi \left( \frac{|\xi| - r_2}{c} \right) \chi_N(\xi) d\xi,$$

$$K_3 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{it(v \cdot \xi - \Omega^{(c)}_J(\xi))} \left(1 - \chi \left( \frac{|\xi| - r_1}{c} \right) \right) \left(1 - \chi \left( \frac{|\xi| - r_2}{c} \right) \right) \chi_N(\xi) d\xi,$$

for $\delta \ll 1$ chosen in the Lemma 4.3. We observe that $K_1$ is supported near the larger sphere $\{ |\xi| = r_1 \}$, $K_2$ is near the smaller sphere $\{ |\xi| = r_2 \}$ and $K_3$ is outside two spheres. Since $\delta$ in Lemma 4.3 was chosen independent of $c$, even though all constants in the estimates below might be dependent on $\delta$, we do not write them explicitly and focus on $c$.

$K_3$ can be estimated similarly as the above case when $2^N < c/4$ or $2^N > 4c$ thanks to (4.4)

$$|K_3| \lesssim t^{-\frac{d}{4}}.$$

Interpolating this with the trivial bound $|K_3| \lesssim c^d$, we can obtain the desired result.

Now, we consider $K_1$ and $K_2$ the supports of which contain the degenerate points. Applying the partition of unity (2.2), we write $K_j = \sum_{\ell=1}^d K_{j,\ell}$ for $j = 1, 2$ where

$$K_{1,\ell} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{it(v \cdot \xi - \Omega^{(c)}_J(\xi))} \chi \left( \frac{|\xi| - r_1}{c} \right) \chi_N(\xi) \theta^\ell(\xi) d\xi,$$

$$K_{2,\ell} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{it(v \cdot \xi - \Omega^{(c)}_J(\xi))} \left(1 - \chi \left( \frac{|\xi| - r_1}{c} \right) \right) \chi \left( \frac{|\xi| - r_2}{c} \right) \chi_N(\xi) \theta^\ell(\xi) d\xi.$$
For $K_2^\ell$, we change variables as
\[
K_2^\ell = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{it(v \cdot \xi - \Omega_j^{(c)}(\xi))} \left( 1 - \chi \left( \frac{|\xi| - r_1}{c_0} \right) \right) \chi \left( \frac{|\xi| - r_2}{c_0} \right) \chi_N(\xi) \theta^\ell(\xi) d\xi
\]
\[
= \frac{2N^d}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i2N_t \left( 2^{-N} v \cdot \xi - 2^{-N} \Omega_j^{(c)}(2^N \xi) \right)} \left( 1 - \chi \left( \frac{|2^N \xi| - r_1}{c_0} \right) \right) \chi \left( \frac{|2^N \xi| - r_2}{c_0} \right) \chi_0(\xi) \theta^\ell(\xi) d\xi.
\]
We claim that the integral with $\hat{\xi}_t$ variables is bounded as follows:
\[
\left| \int_{\mathbb{R}^{d-1}} e^{i2N_t \left( 2^{-N} v \cdot \xi - 2^{-N} \Omega_j^{(c)}(2^N \xi) \right)} \left( 1 - \chi \left( \frac{|2^N \xi| - r_1}{c_0} \right) \right) \chi \left( \frac{|2^N \xi| - r_2}{c_0} \right) \chi_0(\xi) \theta^\ell(\xi) d\hat{\xi}_t \right| \leq (2^N t)^{-d-1},
\]
(4.11)
where the implicit constant is independent of $c$. For $c/4 \leq 2N \leq 4c$, we have from (4.6)
\[
\left| \det H_{\xi_t} \left[ (2^{-2N} \Omega_j^{(c)})(2^N \xi) \right] \right| \geq 1,
\]
and we can show by direct computations that
\[
\sup_{2 \leq |\alpha| \leq d+1} \left\{ C_\alpha : \sup_{\xi \in \text{supp} h} \left| \partial_\xi^\alpha \left( 2^{-N} v \cdot \xi - 2^{-2N} \Omega_j^{(c)}(2^N \xi) \right) \right| \leq C_\alpha \right\} \lesssim 1,
\]
\[
\sup_{|\alpha| \leq d} \left\{ C_\alpha : \sup_{\xi} \left| \partial_\xi^\alpha \left( 1 - \chi \left( \frac{|2^N \xi| - r_1}{c_0} \right) \right) \chi \left( \frac{|2^N \xi| - r_2}{c_0} \right) \chi_0(\xi) \theta^\ell(\xi) \right| \right\} \leq C_\alpha, \right\} \lesssim 1,
\]
where the implicit constants are independent of $v \in \mathbb{R}^d$, $N$, and $c$. Then, (4.11) can be derived by applying Lemma 2.2 with the help of above estimates.Since $\xi_t$ belongs to the interval of length at most $\frac{c_0}{2^N}$, we conclude that
\[
K_2^\ell \lesssim 2^N \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} e^{i2N_t \left( 2^{-N} v \cdot \xi - 2^{-2N} \Omega_j^{(c)}(2^N \xi) \right)} \left( 1 - \chi \left( \frac{|2^N \xi| - r_1}{c_0} \right) \right) \chi \left( \frac{|2^N \xi| - r_2}{c_0} \right) \chi_1(\xi) \theta^\ell(\xi) d\xi_t d\hat{\xi}_t
\]
\[
\lesssim 2^N \cdot (2^N t)^{-d-1} \cdot \frac{c}{2^N} \lesssim t^{-d-1} c.
\]
Finally, by interpolating this with the trivial bound $|K_2^\ell| \lesssim c^d$, we obtain the desired result.

For $K_1^\ell$, the strategy is similar. We first use Lemma 2.2 in terms of $\xi_t$, and measure the set of $\hat{\xi}_t$ variables. To be specific, since
\[
\xi \in \Theta_t \text{ and } |\xi| - r_1 \leq c_0 \text{ on the support of integral } K_1^\ell,
\]
we have the uniform-in-$c$ lower bound of the Hessian in terms of $\xi_t$ variable, i.e., (4.5), which implies by Lemma 2.1 that
\[
\left| \int_{\mathbb{R}} e^{it(v \cdot \xi - \Omega_j^{(c)}(\xi))} \chi \left( \frac{|\xi| - r_1}{c_0} \right) \chi_N(\xi) \theta^\ell(\xi) d\xi_t \right| \lesssim t^{-\frac{1}{2}},
\]
where the implicit constant is independent of $c$. Since $\hat{\xi}_t$ belongs to the $(d-1)$ dimensional sphere of volume at most $c^{d-1}$, we conclude that
\[
|K_1^\ell| \lesssim \int_{\mathbb{R}^{d-1}} \left| \int_{\mathbb{R}} e^{it(v \cdot \xi - \Omega_j^{(c)}(\xi))} \chi \left( \frac{|\xi| - r_1}{c_0} \right) \chi_N(\xi) \theta^\ell(\xi) d\xi_t \right| d\hat{\xi}_t
\]
\[
\lesssim t^{-\frac{1}{2}} c^{d-1}.
\]
4.2.2. One dimensional case \( d = 1 \).

**Proposition 4.6.** Let \( J \in 2\mathbb{N} \) and \( N \in \mathbb{Z} \). For \( d = 1 \), we have

\[
\sup_{v \in \mathbb{R}} \left| T_{J,N}^{(c)}(t,v) \right| \lesssim \begin{cases} t^{-\frac{d}{2}} & \text{for } 2^N < c/4 \text{ or } 2^N > 4c, \\ c^3 t^{-\frac{d}{2}} & \text{for } c/4 \leq 2^N \leq 2c, \end{cases}
\]  

(4.12)

where the implicit constants are independent of \( c > 1 \).

**Proof.** We observe that one of the second or third derivative of the phase function has a lower bound over each dyadic pieces. More precisely, we have from Lemma 4.1 that

\[ |(\omega^{(c)}_J)^{''}(r)| \geq b \text{ for } r \in \text{supp} \chi_N, \quad 2^N < c/4 \text{ or } 2^N > 4c, \]

which implies by Lemma 2.1 that

\[ |T_{J,N}^{(c)}(t,v)| \lesssim t^{-\frac{d}{2}} \text{ for } 2^N < c/4 \text{ or } 2^N > 4c, \]

where the implicit constant is independent of \( c \). Next, we have from (1) in Lemma 4.1 that

\[ |(\omega^{(c)}_J)^{''}(r)| \geq 3c^{-1} \cdot 2^{-\frac{d}{2}}, \text{ for } r \in \text{supp} \chi_N, \quad c/4 \leq 2^N \leq 2c \]

which implies again by Lemma 2.1 that

\[ |T_{J,N}^{(c)}(t,v)| \lesssim c^3 t^{-\frac{d}{2}} \text{ for } c/4 \leq 2^N \leq 2c. \]

\[ \square \]

4.3. **Proof of Theorem 1.4.** We only consider multi dimensional case \( d \geq 2 \). A slight modification would give the proof for one dimensional case. From (4.10), we can obtain, by applying the \( TT^* \) argument in Keel-Tao [17], the frequency localized Strichartz estimates are given by

\[
\left\| e^{-it\mathcal{H}^{(c)}_J} P_N \psi_0 \right\|_{L^3_t L^6_x} \leq Ac^{\frac{2(d-1)}{q}} \| \psi_0 \|_{L^2(\mathbb{R}^d)} \quad \text{for } c/4 \leq 2^N \leq 4c, \\
\left\| e^{-it\mathcal{H}^{(c)}_J} P_N \psi_0 \right\|_{L^3_t L^6_x} \leq A \| \psi_0 \|_{L^2(\mathbb{R}^d)} \quad \text{for } 2^N < c/4 \text{ or } 2^N > 4c,
\]

which holds for all pair \((q, r), (\tilde{q}, \tilde{r})\) satisfying \( 2 \leq q, r, \tilde{q}, \tilde{r} \leq \infty \) and \( \frac{2}{q} + \frac{1}{r} = \frac{1}{2} \) and \( \frac{2}{\tilde{q}} + \frac{d}{\tilde{r}} = \frac{d}{2} \) with \( r \neq \infty \) and \((\tilde{q}, \tilde{r}, d) \neq (2, \infty, 2)\). Then, for \( c/4 \leq 2^N \leq 4c \) we have

\[
\left\| e^{-it\mathcal{H}^{(c)}_J} P_N \psi_0 \right\|_{L^3_t L^6_x} \leq A c^{\frac{2(d-1)}{q}} \| P_N \psi_0 \|_{L^2(\mathbb{R}^d)} \leq A \| \nabla \|^{\frac{2(d-1)}{q}} P_N \psi_0 \|_{L^2(\mathbb{R}^d)}.
\]

If \( 2^N < c/4 \text{ or } 2^N > 4c \), we apply the Sobolev embedding to obtain

\[
\left\| e^{-it\mathcal{H}^{(c)}_J} P_N \psi_0 \right\|_{L^3_t L^6_x} \leq \| \nabla \|^{\frac{2(d-1)}{q}} e^{-it\mathcal{H}^{(c)}_J (-\Delta)} P_N \psi_0 \|_{L^3_t L^6_x} \leq A \| \nabla \|^{\frac{2(d-1)}{q}} P_N \psi_0 \|_{L^2(\mathbb{R}^d)},
\]
where $\frac{q}{2} + \frac{d}{r} = \frac{4}{q}$. Then, by the Littlewood-Paley inequality and Minkowski inequality with $q, r \geq 2$, we obtain that

$$\|e^{-itH_j^{(c)}} P_N \psi_0\|_{L_t^q L_x^r} \approx \left\{ \sum_{N \in \mathbb{Z}} \left| e^{-itH_j^{(c)}(-\Delta)} P_N \psi_0 \right|^2 \right\}^{\frac{1}{2}} \lesssim \left\{ \sum_{N \in \mathbb{Z}} \|\nabla \|^\frac{2(d-1)}{q} P_N \psi_0 \|_{L_x^2}^2 \right\}^{\frac{1}{2}} \approx \|\psi_0\|^2 \|H\|^\frac{2(d-1)}{q}.$$

5. **Application 1: Non-relativistic approximation of Hartree-Fock or Hartree equations**

As applications of Strichartz estimates for higher-order linear Schrödinger equation, we study the non-relativistic limit problems for nonlinear equations with Hartree-Fock or Hartree nonlinearities when $J$ is odd. Here, we only deal with Hartree-Fock nonlinearity, because analogous argument can be directly applied to Hartree nonlinearity. Recall the pseudo-relativistic Hartree-Fock equation (with $\hbar, m = 1$)

$$i\partial_t \psi^{(c)}_k = H^{(c)} \psi^{(c)}_k + H(\psi^{(c)}_k) - F_k(\psi^{(c)}_k), \quad k = 1, 2, \ldots, N, \quad (5.1)$$

where

$$H(\psi^{(c)}_k) = \sum_{\ell=1}^N \left( \frac{\kappa}{|x|} * |\psi^{(c)}_{\ell}|^2 \right) \psi^{(c)}_k \quad \text{and} \quad F_k(\psi^{(c)}_k) = \sum_{\ell=1, \ell \neq k}^N \left( \frac{\kappa}{|x|} * (\psi^{(c)}_{\ell} \psi^{(c)}_k) \right) \psi^{(c)}_k.$$

5.1. **Uniform bound for nonlinear solutions.**

5.1.1. **Results on pseudo-relativistic equations.** The well-posedness results for $[5, 11]$ on the energy space $H^{\frac{1}{2}}(\mathbb{R}^3)$ have been established in $[9, 14, 18]$. The proof of the local well-posedness follows from the standard contraction mapping principle with Sobolev embedding, and this local solution can be extended to the global one via conservation laws of the energy and mass. We emphasize that global bounds of the solutions are independent of $c$. The statement of the result (without proof) is given below:

**Proposition 5.1.** Let $c \geq 1$ and $\Psi_0 \in H^{\frac{1}{2}}(\mathbb{R}^3; \mathbb{C}^N)$ be given initial data. Suppose that $\kappa > 0$, or $\kappa < 0$ and $\|\Psi_0\|_{L_x^2(\mathbb{R}^3; \mathbb{C}^N)}$ is sufficiently small. Then, there exists a global solution $\Psi^{(c)}(t) \in C(\mathbb{R} : H^{\frac{1}{2}}(\mathbb{R}^3; \mathbb{C}^N))$ to $(5.1)$ such that

$$\sup_{t \in \mathbb{R}} \|\Psi^{(c)}(t)\|_{H^{\frac{1}{2}}(\mathbb{R}^3; \mathbb{C}^N)} \leq C(M, E), \quad (5.2)$$
where the bound $C(M, E)$ depends on mass and energy given by

\[ M = M(\Psi_0) = \sum_{k=1}^{N} \| \psi_{k,0} \|_{L^2(\mathbb{R}^3)}^2 \quad \text{and} \]

\[ E = E(\Psi_0) = \sum_{k=1}^{N} \left\{ \frac{1}{2} \left\langle \psi_{k,0}, \left\{ \sqrt{c^4 - c^2 \Delta} - c^2 m \right\} \psi_{k,0} \right\rangle \\
+ \frac{1}{4} \left\langle \sum_{\ell=1}^{N} \left( \frac{\kappa}{|x|} * |\psi_{\ell,0}|^2 \right) \psi_{k,0} - \sum_{\ell=1}^{N} \left( \frac{\kappa}{|x|} * \psi_{k,0} \psi_{\ell,0} \right) \psi_{\ell,0}, \psi_{k,0} \right\rangle \right\}. \]

Remark 5.2. A key estimate to handle Hartree-type nonlinearities is as follows (see [12 Lemma 3.2] for the details of the proof): for $f_j \in H^s(\mathbb{R}^3)$, $j = 1, 2, 3$,

\[ \left\| \left( \frac{1}{|x|} * (f_1 f_2) \right) f_3 \right\|_{H^s(\mathbb{R}^3)} \lesssim \prod_{j=1}^{3} \| f_j \|_{H^s(\mathbb{R}^3)} \quad \text{for } s \geq \frac{1}{2}, \tag{5.3} \]

which follows from endpoint Sobolev inequality

\[ \left\| \frac{1}{|x|} * (f_1 f_2) \right\|_{L^\infty(\mathbb{R}^3)} \lesssim \| f_1 \|_{H^s(\mathbb{R}^3)} \| f_2 \|_{H^s(\mathbb{R}^3)} \quad \text{for } s \geq \frac{1}{2}, \tag{5.4} \]

and the fractional Leibniz rule [16]. Moreover, the particular case when $s = 0$ can be obtained by using the Sobolev inequality

\[ \left\| \left( \frac{1}{|x|} * (f_1 f_2) \right) f_3 \right\|_{L^2(\mathbb{R}^3)} \lesssim \| f_1 \|_{L^4(\mathbb{R}^3)} \| f_2 \|_{L^4(\mathbb{R}^3)} \| f_3 \|_{L^4(\mathbb{R}^3)}. \tag{5.5} \]

We are now focus on the higher-order equation with the Hartree-Fock nonlinearity:

\[ i\partial_t \phi_k^{(c)} = \mathcal{H}_j^{(c)} \phi_k^{(c)} + H(\phi_k^{(c)}) - F_k(\phi_k^{(c)}), \quad k = 1, 2, ..., N. \tag{5.6} \]

The local well-posedness of (5.6) in $H^s(\mathbb{R}^3)$, $s \geq \frac{1}{2}$ follows from the same argument in the proof of Proposition 5.1. When $J$ is odd, this local solution can be extended to the global one in $H^1(\mathbb{R}^3)$ thanks to the conservation of the mass and energy:

\[ \mathcal{M}(\Phi_0) = \| \Phi_0 \|_{L^2(\mathbb{R}^3; C^N)}^2 \]

\[ \mathcal{E}(\Phi_0) = \sum_{k=1}^{N} \left\{ \left\langle \phi_{k,0}, \sum_{j=0}^{J} \frac{(-1)^j (2j)!}{(j+1)!2^j+1} (-\Delta)^{j+1} \phi_{k,0} \right\rangle \\
+ \frac{1}{4} \left\langle \sum_{\ell=1}^{N} \left( \frac{\kappa}{|x|} * |\phi_{\ell,0}|^2 \right) \phi_{k,0} - \sum_{\ell=1}^{N} \left( \frac{\kappa}{|x|} * \phi_{k,0} \phi_{\ell,0} \right) \phi_{\ell,0}, \phi_{k,0} \right\rangle \right\}. \]

Indeed, the kinetic part in the energy controls $\dot{H}^1(\mathbb{R}^3)$ norm of solutions

\[ \left\langle \phi_{k,0}, \sum_{j=0}^{J} \frac{(-1)^j (2j)!}{(j+1)!2^j+1} (-\Delta)^{j+1} \phi_{k,0} \right\rangle = \int \omega_j^\prime(|\xi|) |\hat{\phi}_{k,0}(\xi)|^2 d\xi \]

\[ \geq \frac{1}{4} \int |\xi|^2 |\hat{\phi}_{k,0}(\xi)|^2 d\xi \gtrsim \| \phi_{k,0} \|_{\dot{H}^1}^2. \]
5.1.2. Uniform bounds of solutions to higher order equations. In order to prove the convergence of (5.6) globally in time for initial data given in $H^2(\mathbb{R}^3; \mathbb{C}^N)$, the existence of global solutions to (5.6) in $H^2(\mathbb{R}^3; \mathbb{C}^N)$ is required. The following proposition guarantees that the global solutions to (5.6) exist in $L^2(\mathbb{R}^3; \mathbb{C}^N)$, and it satisfies uniform-in-$c$ boundedness.

**Proposition 5.3** (Global $L^2$ solutions with uniform Strichartz bounds). Let $c \geq 1$ and $J \in 2\mathbb{N} - 1$. For given $\Phi_0 \in L^2(\mathbb{R}^3; \mathbb{C}^N)$, there exists a unique global solution $\Phi^{(c)}$ to (5.6) in the class

$$
\Phi^{(c)} \in C(\mathbb{R}; L^2(\mathbb{R}^3; \mathbb{C}^N)) \cap L^4_t(\mathbb{R}; L^3(\mathbb{R}^3; \mathbb{C}^N)).
$$

Moreover, the global solution satisfies

$$
\|\Phi^{(c)}\|_{L^4_t([0,T]; L^3(\mathbb{R}^3; \mathbb{C}^N))} \lesssim (1 + T)^{1/4},
$$

where the implicit constant depends only on $\mathcal{M}(\Phi_0)$, but independent of $c$.

**Proof.** First, we prove the local well-posedness. Let $I = [0, T_0]$ be a sufficiently small interval to be chosen later. Duhamel’s principle ensures that the solution to (5.6) is equivalent to the following integral formula:

$$
\Gamma_k(\Phi^{(c)}) = e^{-it\mathcal{H}_J(-\Delta)}\phi_{k,0} - i\int_0^t e^{-i(t-s)\mathcal{H}_J(-\Delta)}\left(\mathcal{H}(\phi^{(c)}_k) - F_k(\phi^{(c)}_k)\right)(s)\, ds.
$$

Let $B$ be a constant given by

$$
B = 2A,
$$

for $A$ as in (1.4). Then, the standard argument shows the map $\Gamma$ is contractive in the ball

$$
X_{1,B} = \left\{ \Phi^{(c)} \in C(I; L^2(\mathbb{R}^3; \mathbb{C}^N)) \cap L^4_t(I; L^3(\mathbb{R}^3; \mathbb{C}^N)) : \right. \left. \|\Phi^{(c)}\|_{X_I} := \|\Phi^{(c)}\|_{C(I; L^2(\mathbb{R}^3; \mathbb{C}^N))} + \|\Phi^{(c)}\|_{L^4_t(I; L^3(\mathbb{R}^3; \mathbb{C}^N))} \leq 4B\|\Phi_0\|_{L^2(\mathbb{R}^3; \mathbb{C}^N)} \right\}.
$$

Indeed, by unitarity and Strichartz estimates (1.4), we have

$$
\|\Gamma(\Phi^{(c)})\|_{C(I; L^2(\mathbb{R}^3; \mathbb{C}^N))} + \|\Gamma(\Phi^{(c)})\|_{L^4_t(I; L^3(\mathbb{R}^3; \mathbb{C}^N))} \\
\leq 2B\|\Phi_0\|_{L^2(\mathbb{R}^3; \mathbb{C}^N)} \\
+ 2|\lambda| \sum_{k,\ell=1}^N \left\{ \|(x^{-1} \ast \phi^{(c)}_{\ell})\phi^{(c)}_k\|_{L^4_t(I; L^2(\mathbb{R}^3))} + \|(|x|^{-1} \ast \langle \phi^{(c)}_k \rangle \phi^{(c)}_\ell)\|_{L^4_t(I; L^2(\mathbb{R}^3))} \right\}.
$$
Then, by using (5.5) and Hölder inequality in time, we conclude that
\[
\| \Gamma(\Phi^{(c)}) \|_{X_t} \leq 2B\| \Phi_0 \|_{L^2(\mathbb{R}^3; \mathbb{C}^N)} + CT \sum_{k,\ell=1}^N \| \phi_k^{(c)} \|_{L^\infty_t(I; L^2(\mathbb{R}^3))} \| \phi_\ell^{(c)} \|_{L^4_t(I; L^4(\mathbb{R}^3))} \|
\] (5.8)
\[
\leq 2B\| \Phi_0 \|_{L^2(\mathbb{R}^3; \mathbb{C}^N)} + CT \frac{1}{\alpha} \| \Phi^{(c)} \|_{X_t}^3.
\]

Analogously, we obtain such an a-priori bound for difference of two solutions \( \Phi^{(c)}_1, \Phi^{(c)}_2 \in X_{I,B} \)
\[
\| \Gamma(\Phi^{(c)}_1) - \Gamma(\Phi^{(c)}_2) \|_{X_t} \lesssim 2CT \frac{1}{\alpha} (\| \Phi^{(c)}_1 \|_{X_t} + \| \Phi^{(c)}_2 \|_{X_t})^2 \| \Phi^{(c)}_1 - \Phi^{(c)}_2 \|_{X_t}.
\] (5.9)

By taking an appropriate small \( T_0 \) depending on \( \| \Phi_0 \|_{L^2(\mathbb{R}^3; \mathbb{C}^N)} \), the map \( \Gamma \) is a contraction map in \( X_{I,B} \), thus we can find the solution \( \Phi^{(c)} \) satisfying
\[
\| \Phi^{(c)} \|_{X_t} \leq 4B\| \Phi_0 \|_{L^2(\mathbb{R}^3; \mathbb{C}^N)}.
\] (5.10)

For any given \( T > 0 \), using \( \mathcal{M}(\Phi_0) \) and (5.10) inductively, we show
\[
\| \Phi^{(c)} \|_{L^4_t(I; L^3(\mathbb{R}^3; \mathbb{C}^N))} \lesssim \sum_{j=1}^{[T/T_0]+1} \| \Phi^{(c)} \|_{L^4_t(I_j; L^3(\mathbb{R}^3; \mathbb{C}^N))} \lesssim T,
\] (5.11)
where \( I_j = [(j-1)T_0, jT_0] \) and the implicit constant depends only on \( \| \Phi_0 \|_{L^2(\mathbb{R}^3; \mathbb{C}^N)} \), but not \( c \).

5.2. Approximation : Proof of Theorem 1.6 Before proving Theorem 1.6 we employ the following lemma.

Lemma 5.4. Let \( c > 1 \) and \( t \in \mathbb{R} \). For any \( f \in H^{\frac{1}{2}}(\mathbb{R}^3) \), we have
\[
\| (e^{it(\sqrt{c^4+c^2\Delta-c^2})} - e^{it\mathcal{H}_j(-\Delta)}) f \|_{L^2(\mathbb{R}^3)} \lesssim c^{-\frac{j}{2(j+1)}} \langle t \rangle \| f \|_{H^{\frac{1}{2}}(\mathbb{R}^3)}.
\]

Proof. We write
\[
(e^{it(\sqrt{c^4+c^2\Delta-c^2})} - e^{it\mathcal{H}_j(-\Delta)}) f
\]
\[
= (e^{it(\sqrt{c^4+c^2\Delta-c^2})} - e^{it\mathcal{H}_j(-\Delta)}) P_{low} f + e^{it(\sqrt{c^4+c^2\Delta-c^2})} P_{high} f + e^{it\mathcal{H}_j(-\Delta)} P_{high} f,
\]
where \( \hat{P}_{low} f = \mathbf{1}_{|\xi| \leq \frac{t}{c(j+1)}} \hat{f} \) and \( P_{high} = 1 - P_{low} \). By Taylor remainder theorem, we have
\[
\left| \sqrt{c^4 + c^2|\xi|^2 - c^2} - \sum_{j=1}^J \frac{(-1)^{j+1}(2j-2)!}{(j-1)!2^{j-1}(2j-2)} |\xi|^{2j} \right| \leq \alpha \frac{c^{-2J} |\xi|^2J + 2}{|\xi|^2J + 2}.
\]
for some $0 < |\xi^*| < |\xi|$ and constant $\alpha_J$, which enables us to control the low frequency part as

$$\left\| (e^{it\sqrt{c^2 - c^2\Delta - c^2}} - e^{it\mathcal{H}_J(\Delta)}) P_{\text{low}} f \right\|_{L^2(\mathbb{R}^3)} \lesssim |t| \left\| \{\sqrt{c^2 + c^2|\xi|^2} - c^2 - \mathcal{H}_J(\langle \xi \rangle) \} \hat{f} \right\|_{L^2(\langle \xi \rangle \leq c^{-\frac{1}{2}})} \lesssim \alpha_J c^{-2J} \| \| \phi \|_{H^{\frac{1}{2}}} \|
$$

On the other hand, it is not difficult to control high frequency parts:

$$\left\| e^{it\sqrt{c^2 - c^2\Delta - c^2}} P_{\text{high}} f \right\|_{L^2} \lesssim \| |\xi|^{-\frac{1}{2}} |\xi|^{\frac{1}{2}} \hat{f} \|_{L^2(\langle \xi \rangle \geq c^{-\frac{1}{2}})} \lesssim c^{-\frac{\pi J + 1}{2}} \| f \|_{H^\frac{1}{2}}.$$

The same argument can be applied to $e^{it\mathcal{H}_J(\Delta)} P_{\text{high}} f$. Collecting all, we complete the proof.

Proof of Theorem 1.6. We write the difference of $[\mathcal{H}_c \mathcal{H}_f]$ and $[\mathcal{P}_c \mathcal{P}_f]$ with $\Psi_0 = \Phi_0$ as

$$\psi_k^{(c)} - \phi_k^{(c)} = (e^{-it\sqrt{c^2 - c^2\Delta - c^2}} - e^{-it\mathcal{H}_J(\Delta)}) \psi_{k,0}$$

$$- i \int_0^t \left( e^{-i(t-s)\sqrt{c^2 - c^2\Delta - c^2}} - e^{-i(t-s)\mathcal{H}_J(\Delta)} \right) \left( H(\psi_k^{(c)} - F_k(\psi_k^{(c)})) (s) \right) ds \, ds$$

$$+ i \kappa \int_0^t e^{-i(t-s)\mathcal{H}_J(\Delta)} \mathcal{N}_1(\psi_k^{(c)}, \phi_k^{(c)})(s) \, ds$$

$$+ i \kappa \int_0^t e^{-i(t-s)\mathcal{H}_J(\Delta)} \mathcal{N}_2(\psi_k^{(c)}, \phi_k^{(c)})(s) \, ds$$

$$=: \mathcal{I} + \mathcal{II} + \mathcal{III} + \mathcal{IV},$$

for $k = 1, 2, \cdots, N$, where

$$\mathcal{N}_1(\psi_k^{(c)}, \phi_k^{(c)}) = \sum_{\ell=1}^N \left( |x|^{-1} \ast (|\psi_{\ell}^{(c)}|^2 - |\phi_{\ell}^{(c)}|^2) \right) \phi_k^{(c)} - \sum_{\ell=1}^N \left( |x|^{-1} \ast (|\psi_{\ell}^{(c)}| \psi_k^{(c)} - |\phi_{\ell}^{(c)}| \phi_k^{(c)}) \right) \phi_{\ell}^{(c)},$$

$$\mathcal{N}_2(\psi_k^{(c)}, \phi_k^{(c)}) = \sum_{\ell=1}^N \left( |x|^{-1} \ast (|\psi_{\ell}^{(c)}|^2) \right) (\psi_k^{(c)} - \phi_k^{(c)}) - \sum_{\ell=1}^N \left( |x|^{-1} \ast (|\psi_{\ell}^{(c)}| \psi_k^{(c)}) \right) (\psi_k^{(c)} - \phi_k^{(c)}).$$

By Lemma 5.4 we immediately obtain

$$\| \mathcal{I} \|_{L^2(\mathbb{R}^3)} \lesssim c^{-\frac{\pi J + 1}{2}} \| \psi_{k,0} \|_{H^\frac{1}{2}(\mathbb{R}^3)}.$$ (5.12)

Moreover, by using Lemma 5.4, (5.3) and Proposition 5.1, we have

$$\| \mathcal{II} \|_{L^2(\mathbb{R}^3)} \lesssim c^{-\frac{\pi J + 1}{2}} \int_0^t (t-s) \left\| H(\psi_k^{(c)})(s) - F_k(\psi_k^{(c)})(s) \right\|_{H^\frac{1}{2}(\mathbb{R}^3)} ds$$

$$\lesssim c^{-\frac{\pi J + 1}{2}} (t)^2 \sup_{s \in [0,t]} \| \psi_k^{(c)}(s) \|^3_{H^\frac{1}{2}(\mathbb{R}^3)} \lesssim c^{-\frac{\pi J + 1}{2}} (t)^2 \| \psi_{k,0} \|^3_{H^\frac{1}{2}(\mathbb{R}^3)}.$$ (5.13)
Furthermore, by (5.5) and Sobolev embedding $H^{2/3}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$, and by (5.4), we have

$$\|\mathcal{II}\|_{L^2(\mathbb{R}^3)} \lesssim \int_0^t \left\| N_1(\psi_k^{(c)}, \phi_k^{(c)})(s) \right\|_{L^2(\mathbb{R}^3)} \, ds$$

$$\lesssim \int_0^t \left( \| \Psi^{(c)}(s) \|_{H^{2/3}(\mathbb{R}^3)} + \| \Phi^{(c)}(s) \|_{L^3(\mathbb{R}^3)} \right) \left( \| \Psi^{(c)} - \Phi^{(c)}(s) \|_{L^2(\mathbb{R}^3)} \right) \, ds$$

$$+ \int_0^t \left( \| \Psi^{(c)}(s) \|_{H^{2/3}(\mathbb{R}^3)}^2 + \| \Phi^{(c)}(s) \|_{L^3(\mathbb{R}^3)}^2 \right) \left( \| \psi_k^{(c)} - \phi_k^{(c)}(s) \|_{L^2(\mathbb{R}^3)} \right) \, ds$$

and

$$\|\mathcal{IV}\|_{L^2(\mathbb{R}^3)} \lesssim \int_0^t \left\| N_2(\psi_k^{(c)}, \phi_k^{(c)})(s) \right\| \, ds$$

$$\lesssim \int_0^t \| \Psi^{(c)}(s) \|^2_{H^{1/3}(\mathbb{R}^3)} \| \psi_k^{(c)} - \phi_k^{(c)}(s) \|_{L^2(\mathbb{R}^3)} \, ds$$

$$+ \int_0^t \| \Psi^{(c)}(s) \|^2_{H^{1/3}(\mathbb{R}^3)} \| \Psi^{(c)} - \Phi^{(c)}(s) \|_{L^2(\mathbb{R}^3)} \| \psi_k^{(c)} \|_{H^{2/3}(\mathbb{R}^3)} \, ds$$

respectively. Thus, by collecting (5.12)–(5.15), and by applying Proposition 5.1, we conclude that

$$\| \Psi^{(c)}(t) - \Phi^{(c)}(t) \|_{L^2(\mathbb{R}^3)}$$

$$\lesssim c^{-\frac{1}{2(J+1)}} \left( t^2 \| \Psi_0 \|_{H^{2/3}(\mathbb{R}^3)} + \| \Psi_0 \|_{H^{1/3}(\mathbb{R}^3)} \right)$$

$$+ \int_0^t \left( \| \Psi_0(s) \|^2_{H^{1/3}(\mathbb{R}^3)} + \| \Phi^{(c)}(s) \|^2_{L^3(\mathbb{R}^3)} \right) \| \Psi^{(c)} - \Phi^{(c)}(s) \|_{L^2(\mathbb{R}^3)} \, ds.$$

By Gronwall’s inequality, in addition to (5.7), we have

$$\| \Psi^{(c)}(t) - \Phi^{(c)}(t) \|_{L^2(\mathbb{R}^3)} \lesssim e^{-\frac{1}{2(J+1)} A t} B t,$$

where the constants $A, B$ depend on $\| \Psi_0 \|_{H^{2/3}(\mathbb{R}^3)}$, but not $c$. 

6. Application 2: Small data scattering results

Let $J \in 2\mathbb{N} - 1$. Recall the higher-order nonlinear Schrödinger equations (Hils):

$$i\partial_t \psi_c = \mathcal{H}^{(c)}_J \psi_c + \kappa |\psi_c|^{\nu-1} \psi_c.$$  

(6.1)

We first show the global well-posedness of (6.1) in $H^1(\mathbb{R}^d)$. In what follows, we only focus on $d \geq 2$, since one-dimensional case can be easily proved by only using Sobolev embedding ($H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$). Let $r := \nu + 1$, and let fix $(q, r)$ as the admissible pair in (1.3). Let $0 < \epsilon \leq \delta$ be sufficiently small to be chosen later. Then, for $\psi_{c,0} \in H^1(\mathbb{R}^d)$ with $\| \psi_{c,0} \|_{H^1(\mathbb{R}^d)} \leq \epsilon$, the map

$$\Gamma(\psi) = e^{-it\mathcal{H}^{(c)}_J(-\Delta)} \psi_{c,0} - i\kappa \int_0^t e^{-i(t-s)\mathcal{H}^{(c)}_J(-\Delta)} |\psi|^{\nu-1} \psi(s) \, ds$$

(6.2)
is contractive in the ball
\[ X_\delta := \{ \psi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C} : \|\psi\|_{X^1} := \|\psi\|_{L^\infty(\mathbb{R}; H^1(\mathbb{R}^d))} + \|\psi\|_{L^q_t(\mathbb{R}; W^{1,r}(\mathbb{R}^d))} \leq \delta \} \]
with the topology induced by the metric \[ d(u, v) = \|u - v\|_{L^\infty(\mathbb{R}; L^2(\mathbb{R}^d))} + \|u - v\|_{L^q_t(\mathbb{R}; L^r(\mathbb{R}^d))}. \]

By Strichartz estimates \((1.4)\) and \((1.5)\), we have that
\[ \|\Gamma(\psi)\|_{X^1} \leq A\|\psi_{c,0}\|_{H^1(\mathbb{R}^d)} + C\|\psi^{\nu-1}\|_{L^{\tilde{r}'}(\mathbb{R}; W^{1,r'}(\mathbb{R}^d))}, \]
for \(A\) as in \((1.3)\) and some constant \(C > 0\), independent of \(c\) and \(J\), where \((q', r')\) is the H"older conjugate of \((q, r)\). Note that
\[
\nabla(|\psi|^{\nu-1}\psi) = \left(1 + \frac{\nu - 1}{2}\right) |\psi|^{\nu-1}\nabla\psi + \frac{\nu - 1}{2} |\psi|^{\nu-3}\psi^2 \nabla\bar{\psi}. \tag{6.3}
\]

Using H"older inequality, we immediately obtain
\[ \||\psi|^{\nu-1}\psi\|_{L^{\tilde{r}'}(\mathbb{R}^d)} \leq \||\psi|^{\nu-1}\|_{L^{\nu'}(\mathbb{R}^d)} \|\psi\|_{L^{\tilde{r}'}(\mathbb{R}^d)} \leq \|\psi\|_{L^{\tilde{r}'}(\mathbb{R}^d)}, \tag{6.4}\]
where
\[ \frac{1}{\tilde{r}} = \frac{1}{r'} - \frac{1}{r} = \frac{\nu - 1}{\nu} \quad \text{which gives } \tilde{r}(\nu - 1) = \nu + 1 = r. \tag{6.5}\]

Analogously, we have from \((6.3)\) that
\[ \|\nabla(|\psi|^{\nu-1}\psi)\|_{L^{\tilde{r}'}(\mathbb{R}^d)} \lesssim \|\psi\|_{L^{\tilde{r}'}(\mathbb{R}^d)} \nabla\psi\|_{L^{\tilde{r}'}(\mathbb{R}^d)}. \]
Collecting all, we have
\[ \|\psi\|_{L^{\tilde{q}'}(\mathbb{R}; W^{1,r'}(\mathbb{R}^d))} \lesssim \|\psi\|_{L^{\tilde{q}'}(\mathbb{R}; W^{1,r'}(\mathbb{R}^d))} \|\psi\|_{L^{\tilde{r}'}(\mathbb{R}; W^{1,r'}(\mathbb{R}^d))}, \tag{6.6}\]
where \(\tilde{q}\) satisfies the condition \((1.9)\) that
\[
\frac{1}{\tilde{q}} = \frac{1}{q'} - \frac{1}{q} = \frac{2 - d}{2(\nu + 1)} > 0 \quad \text{which gives } q < \tilde{q}(\nu - 1). \tag{6.7}\]

Under the condition \(q < \tilde{q}(\nu - 1)\), a direct computation, in addition to Sobolev embedding \((H^1(\mathbb{R}^d) \hookrightarrow L^r(\mathbb{R}^d))\), yields
\[
\|\psi\|_{L^{\tilde{q}'}(\mathbb{R}; H^1(\mathbb{R}^d))} \lesssim \left( \int_{\mathbb{R}} \|\psi(t)\|_{\tilde{q}'(\nu - 1)}^{-1} \|\psi(t)\|_{L^\tilde{q}'(\mathbb{R}^d)} \frac{dt}{\tilde{q}'(\nu - 1)} \right)^{\frac{1}{\tilde{q}'}} \lesssim \|\psi\|_{L^{\tilde{q}'}(\mathbb{R}; H^1(\mathbb{R}^d))} \|\psi\|_{L^{\tilde{r}'}(\mathbb{R}; L^r(\mathbb{R}^d))} \lesssim \|\psi\|_{X^1}, \tag{6.8}\]
thus we conclude that
\[ \|\Gamma(\psi)\|_{X^1} \leq A\|\psi_{c,0}\|_{H^1(\mathbb{R}^d)} + C_1\|\psi\|_{X^1}, \]
for some constant \(C_1 > 0\). By taking \(\epsilon, \delta > 0\) satisfying
\[ A\epsilon + C_1\delta^{\nu} < \delta, \tag{6.9}\]
1We refer to \([\mathbf{3}]\) (in the proof of Theorem 4.4.1) for verifying that \((L^\infty H^1 \cap L^d W^{1,r}, d)\) is a complete metric space.
we claim that the map $\Gamma$ is well-defined from $X_\delta$ to itself.

Now, we show the map $\Gamma$ is contractive with respect to the metric $d$. Let $\psi_1$ and $\psi_2$ satisfy (6.2) with $\psi_1(0, x) = \psi_2(0, x)$. A direct computation gives

$$|\psi_1|^{\nu-1}\psi_1 - |\psi_2|^{\nu-1}\psi_2 = |\psi_1|^{\nu-1}(\psi_1 - \psi_2) + (|\psi_1|^{\nu-1} - |\psi_2|^{\nu-1})\psi_2,$$

which, in addition to the Mean Value theorem for complex-valued functions, implies

$$\|\psi_1|^{\nu-1}\psi_1 - |\psi_2|^{\nu-1}\psi_2\| \leq (|\psi_1|^{\nu-1} + |\psi_2|^{\nu-1})|\psi_1 - \psi_2|.$$

Then, similarly as in (6.4) and (6.8), we have

$$d(\Gamma(\psi_1), \Gamma(\psi_2)) = \|\Gamma(\psi_1) - \Gamma(\psi_2)\|_{L^\infty_t(L^2_x)} + \|\Gamma(\psi_1) - \Gamma(\psi_2)\|_{L^3_t(L^{\infty}_x)} \leq C_2(|\psi_1|^{\nu-1} + |\psi_2|^{\nu-1})\|\psi_1 - \psi_2\|_{L^2_t(L^\infty_x)}$$

By taking $\delta > 0$ satisfying $2C_2\delta^{\nu-1} < \frac{1}{2}$ and (6.9), we complete the proof of the global well-posedness of (6.1) in $H^1(\mathbb{R}^d)$.

Now, we prove that the solution $\psi_c(t) \in C(\mathbb{R}; H^1(\mathbb{R}^d))$ to (6.2), that we found above, scatters to a free solution as $t \to \pm \infty$. Let us define scattering state $\psi^{(c)}_t$ by the formula

$$\psi^{(c)}_t := \psi_{c,0} - ik \int_0^\infty e^{-i\mathcal{H}^{(c)}_j(\Delta)} (|\psi_c|^{\nu-1}\psi_c)(s) \, ds.$$

Then, an analogous argument as in the proof of (6.6) with (6.8), we show

$$\left\| \int_{t_1}^{t_2} e^{-i\mathcal{H}^{(c)}_j(\Delta)} (|\psi_c|^{\nu-1}\psi_c)(s) \, ds \right\|_{H^1(\mathbb{R}^d)} \lesssim \|\psi_c|^{\nu-1}\psi_c\|_{L^2_t(L^{\infty}_x); W^{1,\nu}(\mathbb{R}^d)} \lesssim \|\psi_c\|_{L^{\infty}_t(t_1, t_2); H^{1+\frac{\nu}{2}}(\mathbb{R}^d)}.$$ 

which says $\psi^{(c)}_t \in H^1(\mathbb{R}^d)$, and the left-hand side tends to 0 as $t_1, t_2 \to \infty$. This immediately shows

$$\left\| \psi_c(t) - e^{-i\mathcal{H}^{(c)}_j(\Delta)} \psi^{(c)}_t \right\|_{H^1(\mathbb{R}^d)} = \left\| \int_t^\infty e^{-i(t-s)\mathcal{H}^{(c)}_j(\Delta)} (|\psi_c|^{\nu-1}\psi_c)(s) \, ds \right\|_{H^1(\mathbb{R}^d)} \to 0$$

as $t \to \infty$, which completes the proof.

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Footnotes:

2The continuous dependence on initial data follows from the analogous way.

3An analogous way also allows the same conclusion when $t \to -\infty$. 
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