The inverse nodal problem and the Ambarzumyan problem for the $p$-Laplacian

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We study the issues of the reconstruction and stability of the inverse nodal problem for the one-dimensional $p$-Laplacian eigenvalue problem. A key step is the application of a modified Prüfer substitution to derive a detailed asymptotic expansion for the eigenvalues and nodal lengths. Two associated Ambarzumyan problems are also solved.

1. Introduction

There has recently been a lot of interest in the study of the $p$-Laplacian eigenvalue problem

$$-\Delta_p y + q |y|^{p-2} y = \lambda |y|^{p-2} y,$$

$$y|_{\partial \Omega} = 0,$$

where $p > 1$ and $q \in L^2(\Omega)$. This is a quasilinear partial differential equation but many of its properties are analogous to the linear case, when $p = 2$. For example, the variational eigenvalues are associated with the $p$-energy functional

$$\int_{\Omega} |\nabla u|^p \, ds / \int_{\Omega} |u|^p \, dx,$$

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and a version of the Courant nodal domain theorem also holds [11]. However, not all properties are analogues of the 2-Laplacian case. For example, for a periodic boundary-value problem, there exist some non-variational eigenvalues [3]. Also, Fredholm alternatives may not hold, even for the one-dimensional case [10].

For the one-dimensional case the problem becomes, after scaling,

\[-(u^{(p-1)})' = (p-1)(\lambda - q(x))u^{(p-1)},\]

\[u(0) = u(1) = 0,\]

where \(p > 1\), \(f^{(p-1)} := |f|^{p-1} \text{sgn} f\) and \(q \in L^2(0,1)\). A generalized Prüfer substitution helps to establish the classical Sturm–Liouville properties [2,19]: the existence of countably infinite real and simple eigenvalues whose associated eigenfunctions \(u_n\) have exactly \(n-1\) zeros in \((0,1)\). On the interval \([0,\infty)\), the limit-point theory seems to be valid. In particular, when \(q(x) \to \infty\) as \(x \to \infty\), there is a sequence \(\{\lambda_n\}\) tending to infinity such that the associated solution, \(u_n\), lies in \(L^p(0,\infty)\) and \(u_n\) has exactly \(n-1\) zeros [4,5].

In this paper we investigate the solvability of some inverse problems in the one-dimensional case, namely the inverse nodal problem and the Ambarzumyan problem.

The inverse nodal problem for the classical Sturm–Liouville operator is now quite well understood. Using the nodal set as data, some issues of uniqueness, reconstruction and stability of any potential function in \(L^1(0,1)\) have been resolved [13, 14, 16,18,20]. Let \(\{x_i^{(n)}\}_{i=1}^{n-1}\) be the zeros of \(u_n(x)\), and define the nodal set

\[X_n = \{x_i^{(n)}\}_{i=1}^{n-1}.\]

Define the nodal length

\[\ell_i^{(n)} = x_i^{(n+1)} - x_i^{(n)}\]

for \(i = 1, \ldots, n-1\). The following results were proved in [1] and [16]. (See also [7,9].)

**Theorem 1.1** (The case when \(p = 2\)).

(i) Given the nodal set \(X_n\), the potential function \(q\) in \(L^1\) for the Dirichlet problem (1.1), (1.2) can be reconstructed using the following formula:

\[q(x) = \lim_{n \to \infty} 2n^2 \pi^2 (n\ell_j^{(n)} - 1) + n\ell_j^{(n)} \int_0^1 q(x) \, dx,\]

where \(j = j_n(x) := \max\{k : x_k^{(n)} \leq x\}\).

(ii) If the Neumann eigenvalues of (1.1) are given by \(\mu_n = (n-1)^2 \pi^2\), \(n \in \mathbb{N}\), then \(q = 0\) in \(L^1\). If the Dirichlet eigenvalues are \(\lambda_n = (n\pi)^2\) and

\[\int_0^1 q(x) \cos(2x) \, dx = 0,\]

then \(q = 0\).

Part (ii) above is called the Ambarzumyan theorem for the classical Sturm–Liouville operator. Recently, Chakravarty and Acharyya [6] generalized it to a \(2 \times 2\)
vectorial Sturm–Liouville system. In 1997, Chern and Shen [8] extended it to any $n$-dimensional vectorial Sturm–Liouville system. Chern et al. [9] managed to solve the Ambarzumyan problem for the Sturm–Liouville operator (scalar and vectorial) for more general separated boundary conditions, in particular Dirichlet boundary conditions, with one additional assumption on $q$.

We are looking for a $p$-analogue of the above theorem. For this, we need to introduce a generalized sine function $S_p$ satisfying

\[ -(S_p^{(p-1)})' = (p-1)S_p^{(p-1)}, \]
\[ S_p(0) = 0, \quad S'_p(0) = 1. \] (1.3)

The functions $S_p$ and $S'_p$ are in fact periodic functions [12,17] satisfying the identity

\[ |S_p(x)|^p + |S'_p(x)|^p = 1 \] (1.4)

for any $x \in \mathbb{R}$. The functions $S_p$ and $S'_p$ are in fact $p$-analogues of sine and cosine functions in the classical case. It is well known that

\[ \hat{\pi} = 2\pi/p \sin(\pi/p) \]

is the first zero of $S_p$. We shall develop some further properties of $S_p$ in order to derive more detailed eigenvalue asymptotics. This asymptotic behaviour is crucial to the solution of our problems. Our main theorems are as follows.

**Theorem 1.2.** For the Dirichlet problem (1.1) and (1.2) with $q \in C^1([0,1])$, $F_n$ converges to $q$ pointwisely and in $L^1(0,1)$, where

\[ F_n(x) := p(n\hat{\pi})^p(n\ell_j^{(n)} - 1) + n\ell_j^{(n)} \int_0^1 q(t) \, dt. \]

Define the space $\Omega$ and the space of all nodal sequences $\Sigma$ by

\[ \Omega = \left\{ q \in C^1([0,1]) : \int_0^1 q(t) \, dt = 0 \right\}, \]
\[ \Sigma = \left\{ \mathbf{X} = \{x_k^{(n)}\} : \mathbf{X} \text{ is the nodal set associated with some } q \in \Omega \right\}. \]

We shall see that, when equipped with some suitable metrics, $\Omega$ and $\Sigma$ are homeomorphic to each other (see theorem 4.5). Hence, when $\mathbf{X}$ is the nodal set associated with $\bar{q}$ and $\bar{X}$ is close to $\mathbf{X}$ in $\Sigma$, then $\bar{q}$ is close to $q$ in $\Omega$. That is, the inverse nodal problem is stable. Note that here we use $L^r(0,1)$, $r \geq 1$, for $\Omega$, which is more general than previous metrics (see §4).

Finally, we study the Ambarzumyan problem.

**Theorem 1.3.** Let $q \in C^1([0,1])$. If the Neumann eigenvalues for (1.1) are $\mu_n = (n - 1)p\hat{\pi}^p$, $n \in \mathbb{N}$, then $q = 0$ on $(0,1)$.

In §2, we derive the eigenvalue asymptotics with the help of a modified Prüfer substitution. In §3, we prove theorem 1.2, solving the inverse nodal problem, and in §4 we define the metrics and prove theorem 4.5. Finally, in §5, we prove theorem 1.3. We also solve the Ambarzumyan problem for the Dirichlet boundary condition.
Throughout the paper we assume that $q \in C^1([0,1])$. It would be desirable to extend them to work for more general potentials.

2. Eigenvalue asymptotics

To start with, we study the properties of $S_p$.

**Lemma 2.1.**

(i) Whenever $S_p' \neq 0$,

$$(S_p')' = -\frac{S_p}{S_p'} \frac{|S_p|^{p-2}}{S_p};$$

(ii) $(S_pS_p'^{(p-1)})' = |S_p'|^p - (p-1)|S_p|^p = 1 - p|S_p|^p = (1 - p) + p|S_p'|^p$.

**Proof.** Part (i) follows easily from (1.4). For (ii), by (1.3),

$$(S_pS_p'^{(p-1)})' = S_p' S_p'^{(p-1)} + S_p (S_p'^{(p-1)})' = |S_p'|^p - (p-1)|S_p|^p.$$  

The last two equalities in (ii) follow from (1.4). The proof is complete. □

Note that when $p = 2$, (ii) becomes

$$\cos^2 x - \sin^2 x = \frac{1}{2}(\sin 2x)' = \cos 2x,$$

which is a familiar double-angle formula.

Next we define a modified Prüfer substitution.

$$u(x) = r(x)S_p(\lambda^{1/p}\theta(x)), \quad u'(x) = \lambda^{1/p}r(x)S_p'(\lambda^{1/p}\theta(x)),$$

i.e.

$$\frac{u'(x)}{u(x)} = \frac{\lambda^{1/p}S_p'(\lambda^{1/p}\theta(x))}{S_p(\lambda^{1/p}\theta(x))}.$$  

Differentiating the above equation with respect to $x$ and applying lemma 2.1, we obtain

$${\theta'} = 1 - \frac{q}{\lambda} |S_p(\lambda^{1/p}\theta(x))|^p.$$  

That is, for sufficiently large $\lambda$,

$${\theta'} = 1 - O\left(\frac{1}{\lambda}\right), \quad {\theta''} = O\left(\frac{1}{\lambda^{1-1/p}}\right).$$  

Now we are ready to establish the basic asymptotics for eigenvalues.

**Theorem 2.2.** The eigenvalues $\lambda_n$ of the Dirichlet problem (1.1), (1.2) satisfy

$$\lambda_n^{1/p} = n \hat{\pi} + \frac{1}{p \lambda_n^{1-1/p}} \int_0^1 q(t) \, dt + O\left(\frac{1}{\lambda_n}\right) = n \hat{\pi} + \frac{1}{p(n \hat{\pi})^{p-1}} \int_0^1 q(t) \, dt + O\left(\frac{1}{n^p}\right)$$  

as $n \to \infty$. 
Remark 2.3. In [2], various estimates were given for more general problems, but they do not include (2.4).

Proof. For this problem, let $\lambda = \lambda_n$, $\theta(0) = 0$. Then

$$\theta(1) = \frac{n\pi}{\lambda_n^{1/p}}.$$ Integrating both sides of (2.2) over $[0, 1]$ and applying the identity

$$\frac{d}{dt}[S_p(\lambda_n^{1/p}\theta(t))S'_p(\lambda_n^{1/p}\theta(t))^{(p-1)}] = (1 - p|S_p(\lambda_n^{1/p}\theta(t))|^{p})\lambda_n^{1/p}\theta'(t)$$

we obtain, from lemma 2.1(ii),

$$\frac{n\pi}{\lambda_n^{1/p}} = 1 - \frac{1}{p\lambda_n} \int_0^1 q(t) \, dt + \frac{1}{p\lambda_n} \int_0^1 q(t) \, \frac{d}{dt} \left( \frac{1}{\lambda_n^{1/p}\theta'} \right) S_p(\lambda_n^{1/p}\theta(t)) S'_p(\lambda_n^{1/p}\theta(t))^{(p-1)} \, dt.$$ (2.5)

Write

$$F(\lambda_n^{1/p}\theta(x)) = S_p(\lambda_n^{1/p}\theta(x))S'_p(\lambda_n^{1/p}\theta(x))^{(p-1)}.$$ Note that $F(\lambda_n^{1/p}\theta(x)) = 0$ when $x = 0, 1$. Hence, from (2.5) we have, by (2.3),

$$\int_0^1 \frac{q(t)}{\lambda_n^{1/p}\theta'} \, d\left(\frac{1}{\lambda_n^{1/p}\theta'} \right) F(\lambda_n^{1/p}\theta(t)) \, dt$$

$$= \left[ \frac{F(\lambda_n^{1/p}\theta(x))}{\lambda_n^{1/p}\theta'} \right]_0^1 - \int_0^1 \frac{d}{dt} \left( \frac{q(t)}{\lambda_n^{1/p}\theta'} \right) F(\lambda_n^{1/p}\theta(t)) \, dt$$

$$= -\lambda_n^{-1/p} \int_0^1 \left( \frac{q(t)}{\theta'} \right)' \, F(\lambda_n^{1/p}\theta(t)) \, dt$$

$$= O(\lambda_n^{-1/p}).$$ (2.6)

Thus, by (2.5), as $n \to \infty$,

$$\lambda_n^{1/p} = n\pi \left( 1 - \frac{1}{p\lambda_n} \int_0^1 q(t) \, dt + O(\lambda_n^{-1+1/p}) \right)^{-1}.$$ Thus, (2.4) follows.

Remark 2.4. Hence, the asymptotics for $\lambda_n$ is

$$\lambda_n = (n\pi)^p + \int_0^1 q(t) \, dt + O\left(\frac{1}{n}\right).$$

3. The inverse nodal problem

In this section we derive the asymptotics for nodal lengths and deduce a reconstruction formula for the function $q$. Define $\delta_j^{(n)} = x_j^{(n)} - x_{j+1}^{(n)}$, $1 \leq j \leq n - 1$, as the nodal lengths of $u(x; \lambda_n)$, where $x_n^{(n)} = 1$. 
Theorem 3.1.

\[ \ell_j^{(n)} = \frac{n}{\lambda n^{1/p}} + \frac{1}{p\lambda n} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(t) \, dt + O\left(\frac{1}{\lambda^{1+2/p}}\right) \]
\[ = \frac{1}{n} + \frac{1}{p(n\pi)^p} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(t) \, dt - \frac{1}{p(n\pi)^p} \int_0^1 q + O\left(\frac{1}{n^{p+2}}\right) \tag{3.1} \]
as \( n \to \infty \).

Proof. For sufficiently large \( n \in \mathbb{N} \), we integrate (2.2) over \([x_j^{(n)}, x_{j+1}^{(n)}]\) and then

\[ \frac{n}{\lambda n^{1/p}} = \ell_j^{(n)} - \frac{1}{\lambda n} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(t)|S_p(\lambda n^{1/p}\theta(t))|^p \, dt \]
\[ = \ell_j^{(n)} - \frac{1}{p\lambda n} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(t) \, dt + \frac{1}{p\lambda n} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(t) \lambda n^{1/p}\theta(t) \, dt \]
by lemma 2.1. Let \( F(\tau) = S_p(\tau)S_p'(\tau)^{(p-1)} \), where \( \tau = \lambda n^{1/p}\theta(x) \). Similarly to (2.6), we have

\[ \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(t)\lambda n^{1/p}\theta(t) \, dt = - \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} \left( \frac{q(t)}{\lambda n^{1/p}\theta(t)} \right) F(\lambda n^{1/p}\theta(t)) \, dt \]
\[ = - \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} \left( \frac{q(t)}{\lambda n^{1/p}\theta(t)} \right) F(\tau) \, d\tau \]
\[ = O\left(\frac{1}{\lambda^{2/p}}\right). \]

Now we can go back to (3.2) and easily obtain the desired result. \( \square \)

The above theorem shows that the nodal set \( X \) is dense in \((0,1)\). Furthermore,

\[ \frac{\lambda n^{1/p}\ell_j^{(n)}}{\pi} = 1 + O\left(\frac{1}{n^\theta}\right) \tag{3.3} \]
uniformly for \( j = 1, \ldots, n-1 \).

Theorem 3.2. The function \( q \) is given by

\[ q(x) = \lim_{n \to \infty} p\lambda n \left( \frac{\lambda n^{1/p}\ell_j^{(n)}}{\pi} - 1 \right) \]
for \( x \in (0,1) \), where \( j = j_n(x) := \max\{k : x_k^{(n)} \leq x\} \).

Proof. By the mean-value theorem for integrals in (3.1), with fixed \( n \), for each \( j \), there exists \( \xi \in (x_j^{(n)}, x_{j+1}^{(n)}) \) such that

\[ \ell_j^{(n)} = \frac{n}{\lambda n^{1/p}} + \frac{q(\xi)}{p\lambda n} \ell_j^{(n)} + O\left(\frac{1}{\lambda^{1+2/p}}\right) \]

for sufficiently large \( n \).
as }n \to \infty\text{. Hence, }
\begin{align*}
q(\xi) = p\lambda_n \left( \frac{\hat{\pi}}{\lambda_n^{1/p} \ell_j^{(n)}} \right) \left( \frac{\lambda_n^{1/p} \ell_j^{(n)}}{\pi} - 1 \right) + O \left( \frac{1}{\lambda_n^{1/p}} \right).
\end{align*}

Let }n \to \infty\text{. The proof is complete.}

**Remark 3.3.**

(i) One can also apply the Sturm–Liouville comparison theorem (see [2] for a survey) to prove this theorem. The proof is analogous to that for the Sturm–Liouville operator in [15].

(ii) With the asymptotic expression up to the order }\lambda_n^{1/p}\text{, theorem 2.2 implies that }
q(x) = \lim_{n \to \infty} F_n(x),\text{ where }F_n\text{ is determined only by the nodal data and the constant }
\int_0^1 q:
F_n(x) := p(n\hat{\pi})^p(n\ell_j^{(n)} - 1) + n\ell_j^{(n)} \int_0^1 q(t) \, dt.

So we have proved the first part of theorem 1.2. To show that the convergence is }L^1\text{, we need the following lemma.

**Lemma 3.4.** As }n \to \infty\text{, with }j = j_n(x),
\begin{align*}
\left\| \frac{\lambda_n^{1/p}}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q - q(x) \right\|_1 \to 0.
\end{align*}

**Proof.** By the mean-value theorem, with }x_0 \in [x, y]\text{, there exists }\xi \in (x, y)\text{ such that }
\begin{align*}
\left| \frac{1}{y - x} \int_x^y q - q(x_0) \right| = |q(\xi) - q(x_0)|.
\end{align*}

Due to the uniform continuity of }q\text{, there is a }\delta > 0\text{ such that the above difference is small whenever }|y - x| < \delta.\text{ Hence, we have }
\begin{align*}
\left| \frac{\lambda_n^{1/p}}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q - q(x) \right| &\leq \left| \frac{\lambda_n^{1/p} \ell_j^{(n)}}{\pi} \left[ \frac{1}{x_{j+1}^{(n)} - x_j^{(n)}} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q - q(x) \right] \right| + |q(x)| \left| \frac{\lambda_n^{1/p} \ell_j^{(n)}}{\pi} - 1 \right|.
\end{align*}

By (3.3), we have
\begin{align*}
\frac{\lambda_n^{1/p} \ell_j^{(n)}}{\pi} = O(1)\text{ and }\left| \frac{\lambda_n^{1/p} \ell_j^{(n)}}{\pi} - 1 \right| = O \left( \frac{1}{n^p} \right)
\end{align*}
for sufficiently large }n\text{. Hence, given }\varepsilon > 0\text{, when }n\text{ is large enough such that }\ell_j^{(n)} < \delta\text{ with }j = j_n(x),\text{ we have }
\begin{align*}
\left| \frac{\lambda_n^{1/p}}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q - q(x) \right| \leq 2\varepsilon.
\end{align*}
Therefore, if \( q \in C([0, 1]) \), then
\[
\left\| \frac{\lambda_n^{1/p}}{\pi p} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q - q(x) \right\|_1
\]
can be arbitrarily small as \( n \to \infty \).

**Proof of theorem 1.2.** It suffices to show that the convergence is \( L^1 \). Since
\[
\left| F_n(x) - p\lambda_n \left( \frac{\lambda_n^{1/p} \ell_j^{(n)}}{\pi} - 1 \right) \right| = O \left( \frac{1}{n} \right)
\]
by the asymptotic estimate of \( \lambda_n^{1/p} \), it suffices to show that as \( n \to \infty \),
\[
\left\| p\lambda_n \left( \frac{\lambda_n^{1/p} \ell_j^{(n)}}{\pi} - 1 \right) - q \right\|_1 \to 0.
\]
By (3.1), we have
\[
p\lambda_n \left( \frac{\lambda_n^{1/p} \ell_j^{(n)}}{\pi} - 1 \right) = \frac{\lambda_n^{1/p}}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(t) \, dt + O \left( \frac{1}{\lambda_n} \right),
\]
and so it converges to \( q \) in \( L^1(0, 1) \) by lemma 3.4.

4. Stability

In this section we define the metrics on \( \Omega \) and \( \Sigma \) and study the stability of the inverse nodal problem. For \( r \geq 1 \), let
\[
S_r^\Sigma(X, \bar{X}) = \pi^p n^{p-1/r} \left( \sum_{k=0}^{n-1} |\ell_k^{(n)} - \bar{\ell}_k^{(n)}|^r \right)^{1/r}.
\]
Define the metric on \( \Sigma \) as
\[
d_0^\Sigma(X, \bar{X}) = \lim_{n \to \infty} S_r^\Sigma(X, \bar{X})
\]
and
\[
d_0^\Sigma(X, \bar{X}) = \lim_{n \to \infty} \frac{S_r^\Sigma(X, \bar{X})}{1 + S_r^\Sigma(X, \bar{X})}.
\]

**Proposition 4.1.** The function \( d_0^\Sigma(\cdot, \cdot) \) is a pseudometric on \( \Sigma \).

**Remark 4.2.** Note that our definition of \( d_0^\Sigma \) is similar to that found in [14]. Thus, if \( d_0^\Sigma(X, \bar{X}) < \infty \), then
\[
d_0^\Sigma(X, \bar{X}) \leq d_0^\Sigma(X, \bar{X}) < \infty.
\]
Thus \( d_0^\Sigma(X, \bar{X}) \) is close to 0 if and only if \( d_0^\Sigma(X, \bar{X}) \) is close to 0. In particular, \( d_0^\Sigma(X, \bar{X}) = 0 \) if and only if \( d_0^\Sigma(X, \bar{X}) = 0 \).
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Proof. It is obvious that \( d^r_{\Sigma}(\cdot, \cdot) \) is finite and symmetric. To prove the triangle inequality, it suffices to show that

\[
\frac{S^r_n(X, \bar{X})}{1 + S^r_n(X, \bar{X})} \leq \frac{S^r_n(X, Y)}{1 + S^r_n(X, Y)} + \frac{S^r_n(Y, \bar{X})}{1 + S^r_n(Y, \bar{X})},
\]
or, equivalently,

\[
S^r_n(X, Y)S^r_n(Y, \bar{X})S^r_n(X, \bar{X}) + 2S^r_n(X, Y)S^r_n(Y, \bar{X})
+ S^r_n(X, Y) + S^r_n(Y, \bar{X}) - S^r_n(X, \bar{X}) \geq 0.
\]
But this is valid as from the triangle inequality:

\[
S^r_n(X, Y) + S^r_n(Y, \bar{X}) - S^r_n(X, \bar{X}) \geq 0.
\]

\[\square\]

Remark 4.3. In fact, \( d^r_{\Sigma} \) is a metric on \( \Sigma \). It follows from the proof of theorem 4.5 that \( d^r_{\Sigma}(X, \bar{X}) = 0 \) if and only if \( q = \bar{q} \). That means \( X = \bar{X} \).

Lemma 4.4. Suppose that \( X, \bar{X} \in \Sigma \). Then, when \( n \) is sufficiently large,

(i) the interval \( I^k_r \) between \( x^{(n)}_k \) and \( \bar{x}^{(n)}_k \) has length \( O(1/n^p) \),

(ii) for all \( x \in (0, 1) \), \( |j_n(x) - \bar{j}_n(x)| \leq 1 \).

Proof. (i) It follows from (3.1) that, assuming \( \int_0^1 q = 0 \), the asymptotic expansion of a nodal point \( x^{(n)}_k \) is given by

\[
x^{(n)}_k = \frac{k}{n} + \frac{1}{p(n\pi)^p} \int_0^{x^{(n)}_k} q(t) \, dt + O\left(\frac{1}{n^{p+1}}\right).
\]

(ii) Fix \( x \in (0, 1) \). Let \( j = j_n(x) \), \( \bar{j} = \bar{j}_n(x) \). Since

\[
\frac{j}{n} + O\left(\frac{1}{n^p}\right) = x^{(n)}_j \leq x \leq x^{(n)}_{j+1} = \frac{j + 1}{n} + O\left(\frac{1}{n^p}\right)
\]
and

\[
\frac{\bar{j}}{n} + O\left(\frac{1}{n^p}\right) = \bar{x}^{(n)}_{\bar{j}} \leq \bar{x} \leq \bar{x}^{(n)}_{\bar{j}+1} = \frac{\bar{j} + 1}{n} + O\left(\frac{1}{n^p}\right),
\]
when \( n \) is large enough, \( \bar{j} + 1 \geq j \) and \( j + 1 \geq \bar{j} \). Hence, \( -1 \leq \bar{j} - j \leq 1 \). \[\square\]

Theorem 4.5. For any \( r \geq 1 \), \( d^r_{\Sigma} \) is a metric on \( \Sigma \). Furthermore, the metric space \((\Sigma, d^r_{\Sigma})\) is homeomorphic to the space \( \Omega \) with the metric induced by \( \| \cdot \|_r \).

Proof. It suffices to show that

\[
\|q - \bar{q}\|_r = pd^r_{\Sigma}(X, \bar{X}).
\]

For \( x \in (0, 1) \), by theorem 3.2, we have

\[
q(x) - \bar{q}(x) = \lim_{n \to \infty} p(n\pi)^p n(\ell^{(n)}_{j_n(x)} - \ell^{(n)}_{\bar{j}_n(x)}).
\]
Hence, by Fatou’s lemma,

\[ \|q - \bar{q}\|_r \leq \lim_{n \to \infty} pn^{p+1} \hat{\pi}^p \|\ell_{j_n(x)}^{(n)} - \tilde{\ell}_{j_n(x)}^{(n)}\|_r \]

\[ \leq p\hat{\pi}^p \lim_{n \to \infty} (n^{p+1} \|\ell_{j_n(x)}^{(n)} - \tilde{\ell}_{j_n(x)}^{(n)}\|_r + n^{p+1} \|\tilde{\ell}_{j_n(x)}^{(n)} - \tilde{\ell}_{j_n(x)}^{(n)}\|_r). \]

Here, by lemma 4.4, the second term becomes

\[ n^{p+1} \|\tilde{\ell}_{j_n(x)}^{(n)} - \tilde{\ell}_{j_n(x)}^{(n)}\|_r = n^{p+1} \left( \sum_{k=0}^{n-1} |\ell_{k+1}^{(n)} - \tilde{\ell}_{k}^{(n)}| \|I_k^{(n)}\|_r \right)^{1/r} \]

\[ = O(n^{(1-p)/r}) \]

\[ = o(1) \]

because, by (3.1) and continuity of \( q \),

\[ |\ell_{k}^{(n)} - \tilde{\ell}_{k}^{(n)}| = \frac{1}{p(n\hat{\pi})} \left| \int_{x_{k}^{(n)}}^{x_{k+1}^{(n)}} q(t) \, dt - \int_{x_{k}^{(n)}}^{x_{k+1}^{(n)}} \bar{q}(t) \, dt \right| + O \left( \frac{1}{n^{p+2}} \right) \]

\[ = O \left( \frac{1}{n^{p+1}} \right). \]

Hence, for any \( r \geq 1 \),

\[ \|q - \bar{q}\|_r \leq p\hat{\pi}^p \lim_{n \to \infty} n^{p+1} \left( \sum_{k=0}^{n-1} |\ell_{k+1}^{(n)} - \tilde{\ell}_{k}^{(n)}| \|I_k^{(n)}\|_r \right)^{1/r} \]

\[ = p\hat{\pi}^p \lim_{n \to \infty} n^{p+1-1/r} \left( \sum_{k=0}^{n-1} \left[ 1 + O \left( \frac{1}{n^p} \right) \right] |\ell_{k}^{(n)} - \tilde{\ell}_{k}^{(n)}| \right)^{1/r} \]

\[ = pd_0^r(X, \hat{X}). \]

Conversely, using the above derivations, for sufficiently large \( n \),

\[ \|q - \bar{q}\|_r + o(1) = pn^{p+1} \hat{\pi}^p \|\ell_{j_n(x)}^{(n)} - \tilde{\ell}_{j_n(x)}^{(n)}\|_r \]

\[ \geq pn^{p+1} \hat{\pi}^p \|\ell_{j_n(x)}^{(n)} - \tilde{\ell}_{j_n(x)}^{(n)}\|_r - O(n^{(1-p)/r}) \]

\[ = pn^{p+1} \hat{\pi}^p \left( \sum_{k=0}^{n-1} |\ell_{k+1}^{(n)} - \tilde{\ell}_{k}^{(n)}| \|I_k^{(n)}\|_r \right)^{1/r} - O(n^{(1-p)/r}) \]

\[ = pn^{p+1-(1/r)} \hat{\pi}^p \left( \sum_{k=0}^{n-1} |\ell_{k}^{(n)} - \tilde{\ell}_{k}^{(n)}| \right)^{1/r} - O(n^{(1-p)/r}). \]

Hence, as \( n \to \infty \),

\[ pd_0^r(X, \hat{X}) \leq \|q - \bar{q}\|_r. \]

The proof is complete. □
5. Ambarzumyan problems

We first consider the Ambarzumyan problem for Dirichlet boundary conditions.

**Theorem 5.1.** If the eigenvalues of problem (1.1), (1.2) are \( \lambda_n = (n\pi)^p \), \( n \geq 1 \), and the function \( q \) satisfies

\[
\int_0^1 q(x)(S_p(\pi x)S_p'(\pi x)(p-1)')\,dx = 0, \tag{5.1}
\]

then \( q = 0 \) on \([0, 1]\).

**Proof.** By (2.4),

\[
\int_0^1 q(x) = 0.
\]

Next we show that \( S_p(\pi x) \) is the first eigenfunction. By the variational principle (see [2]),

\[
\lambda_1 = \inf_{u \neq 0} \frac{\int_0^1 |u'|^p + (p-1)\int_0^1 q|u|^p}{(p-1)\int_0^1 |u|^p},
\]

where \( u \in C^2[0, 1] \), satisfying the boundary condition (1.2). Now \( S_p(\pi x) \) satisfies (1.2), and

\[
\hat{\lambda}_1 = \inf \frac{\int_0^1 |S_p'(\pi t)|^p \, dt + (p-1)\int_0^1 q(t)|S_p(\pi t)|^p \, dt}{(p-1)\int_0^1 |S_p(\pi t)|^p \, dt}, \tag{5.2}
\]

By lemma 2.1(ii) and (5.1), we have

\[
\int_0^1 |S_p'(\pi t)|^p \, dt = \int_0^1 \frac{p-1}{p} \, dt + \frac{1}{p} \int_0^1 (S_p(\pi t)S_p'(\pi t)(p-1)') \, dt = \frac{p-1}{p},
\]

\[
\int_0^1 q(t)|S_p(\pi t)|^p \, dt = \frac{1}{p} \int_0^1 q(t) \, dt - \frac{1}{p} \int_0^1 q(t)(S_p(\pi t)S_p'(\pi t)(p-1)') \, dt = 0.
\]

So in (5.2), both numerator and denominator are \((p-1)/p\). Hence \( S_p(\pi x) \) achieves the minimum value and is thus the first eigenfunction. Substituting this into (1.1), we obtain \( q \equiv 0 \) on \((0, 1)\). \( \square \)

Finally, we study the Ambarzumyan problem for Neumann boundary conditions:

\[
u'(0) = u'(1) = 0. \tag{5.3}
\]

We need the following lemma on eigenvalue asymptotics.

**Lemma 5.2.** The eigenvalues \( \lambda_n \) of problems (1.1) and (5.3) satisfy

\[
\lambda_n^{1/p} = (n-1)\hat{\pi} + \frac{1}{p((n-1)\hat{\pi})^{p-1}} \int_0^1 q(x) \, dx + O\left(\frac{1}{n^p}\right) \tag{5.4}
\]

as \( n \to \infty \).
Proof. The proof is similar to that of theorem 2.2. From the phase equation (2.2), we let $\lambda = \lambda_n$ and $\theta(0) = \hat{\pi}/2$. Then $\theta(1) = (n - \frac{1}{2})\hat{\pi}$. Integrate both sides of (2.2) over $[0, 1]$. Thus, we obtain

$$\left(\frac{n - 1}{\lambda_n^{1/p}}\right)\hat{\pi} = 1 - \frac{1}{p\lambda_n} \int_0^1 q(t)\,dt + \frac{1}{p\lambda_n} \int_0^1 \frac{q(t)}{\lambda_n^{1/p}} \frac{d}{dt} \left(S_p(\lambda_n^{1/p}(\theta(t)))S'_p(\lambda_n^{1/p}(\theta(t))^{(p-1)})\right)\,dt$$

$$= 1 - \frac{1}{p\lambda_n} \int_0^1 q(t)\,dt + O(\lambda_n^{-1-(1/p)}).$$

Hence,

$$\lambda_n^{1/p} = (n - 1)\hat{\pi} + \frac{1}{p\lambda_n^{1-1/p}} \int_0^1 q(t)\,dt + O\left(\frac{1}{\lambda_n}\right)$$

as $n \to \infty$, and the lemma holds.

The proof of theorem 1.3 is now ready as an analogue to that of theorem 5.1. However, we give an alternative proof using Yurko’s argument [21].

Proof of theorem 1.3. By lemma 5.2,

$$\int_0^1 q(x)\,dx = 0.$$

Let $u_1(x)$ be an eigenfunction associated with $\lambda_1 = 0$. According to Binding and Drábek [2], the eigenfunction $u_1(x)$ has no zeros in the interval $[0, \hat{\pi}]$. Taking into account the relation

$$\left(\left(\frac{u'_1}{u_1}\right)^{(p-1)}\right)' = \left(\frac{u'^{(p-1)}_1}{u_1^p}\right)' - (p - 1) \left|\frac{u'_1}{u_1}\right|^p,$$

we obtain (cf. [21])

$$0 = \int_0^1 q(x)\,dx = \frac{1}{p - 1} \int_0^1 \left(\frac{u'^{(p-1)}_1}{u_1^p}\right)\,dx = \int_0^1 \left|\frac{u'_1}{u_1}\right|^p\,dx.$$

Thus, $u'_1(x) = 0$, and $u_1$ is constant. So $q(x) = 0$. 

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