The converse of Weyl’s eigenvalue inequality

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Abstract

We establish the converse of Weyl’s eigenvalue inequality for additive Hermitian perturbations of a Hermitian matrix.

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1. Introduction

Let $A = (a_{ij})_{m \times n}$ be a complex matrix. The conjugate transpose $A^* = (b_{ij})_{n \times m}$ of $A$ is defined by $b_{ij} = \overline{a_{ji}}$. An $n \times n$ complex square matrix $A$ is called Hermitian if $A^* = A$. It is well known that the eigenvalues of a Hermitian matrix are all real. Throughout this paper we adopt the convention that they always arranged in non-increasing order:

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A).$$

We also set $\lambda_i(A) = +\infty$ for $i < 1$ and $\lambda_i(A) = -\infty$ for $i > n$ for convenience.

In 1912 Hermann Weyl [12] posed the following problem: given the eigenvalues of two $n \times n$ Hermitian matrices $A$ and $B$, how does one determine all possible sets of eigenvalues of the sum $A + B$? He gave partial answers:

$$\lambda_{i+j-1}(A + B) \leq \lambda_i(A) + \lambda_j(B).$$

Note that $\lambda_i(-C) = -\lambda_{n+1-i}(C)$ for any $n \times n$ Hermitian matrix $C$. Hence (1.1) is equivalent to

$$\lambda_{i+j-n}(A + B) \geq \lambda_i(A) + \lambda_j(B).$$

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Although Weyl’s problem has been completely solved (we refer the reader to Allen Knutson and Terence Tao’s survey article [5] on this topic), Weyl’s inequality (1.1) is still the source of a great many eigenvalue inequalities (see [4, §4.3] for instance). Let \( n_+(B) \) (\( n_-(B) \), resp.) denote the positive (negative, resp.) index of inertia of \( B \). Then \( n_+(B) \) (\( n_-(B) \), resp.) is the number of positive (negative, resp.) eigenvalues of \( B \) by Sylvester’s law of inertia. Denote \( n_+(B) = p \) and \( n_-(B) = q \). Then by (1.1) and (1.2),

\[
\lambda_{i+q}(A) \leq \lambda_i(A + B) \leq \lambda_{i-p}(A). \tag{1.3}
\]

We also call (1.3) Weyl’s inequality, since it is equivalent to (1.1). Indeed, assume that (1.3) holds for any \( A \) and \( B \). Noting \( n_+ (B - \lambda_j(B)I) \leq j - 1 \), we have

\[
\lambda_{i+j-1}(A + B) = \lambda_{i+j-1}(A + B - \lambda_j(B)I) + \lambda_j(B) \leq \lambda_i(A) + \lambda_j(B)
\]

by (1.3). Such a form (1.3) of Weyl’s inequality is often more convenient to use. For example, if \( B \) is positive semi-definite, i.e., \( n_-(B) = 0 \), then \( \lambda_i(A) \leq \lambda_i(A + B) \) by (1.3), which is the monotonicity theorem. Further, if \( B = \alpha \alpha^* \) for some column vector \( \alpha \in \mathbb{C}^n \), then \( n_+(B) \leq 1 \) and \( n_-(B) = 0 \), and so

\[
\lambda_1(A) \leq \lambda_n(A + B) \leq \lambda_{n-1}(A) \leq \cdots \leq \lambda_2(A + B) \leq \lambda_1(A) \leq \lambda_1(A + B),
\]

which is the interlacing theorem for a rank-one Hermitian perturbation of a Hermitian matrix. It is well known that the converse of this interlacing theorem is true (see [4, Theorems 4.3.26] for instance or Lemma 2.7). In this paper we consider a more general problem: the converse of Weyl’s inequality. We first introduce some definitions and notations.

Let \( \mathcal{R} \) (\( \mathcal{R}_n \), resp.) denote the set of real polynomials (of degree \( n \), resp.) with only real roots and with positive leading coefficients. In particular, let \( \mathcal{R}^{(1)} \) denote the set of monic polynomials in \( \mathcal{R} \). For \( g \in \mathcal{R} \), we use \( r_i(g) \) denote its roots and arrange them in non-increasing order: \( r_1(g) \geq r_2(g) \geq \cdots \geq r_n(g) \). For convenience, set \( r_i(g) = +\infty \) for \( i < 1 \) and \( r_i(g) = -\infty \) for \( i > \deg g \).

**Definition 1.1.** Let \( f, g \in \mathcal{R} \) and \( p, q \in \mathbb{N} \). The polynomial \( f \) is said to \( (p,q) \)-interlace the polynomial \( g \), denoted by \( f \overset{p}{\sim}_q g \), if

\[
r_{i+p}(g) \leq r_i(f) \leq r_{i-q}(g)
\]

for all \( i \in \mathbb{Z} \).
The following properties are immediate from the definition.

**Proposition 1.2.** (a) $f_p \sim_q f$ for any $p$ and $q$.
(b) $f_p \sim_q g$ is equivalent to $g_q \sim_p f$.
(c) $f_p \sim_q h$ and $h_s \sim_t g$ imply that $f_{p+s} \sim_q g_{q+t}$.
(d) $f_p \sim_q g$ implies that $f_s \sim_t g$ for any $s \geq p$ and $t \geq q$.
(e) $f_p \sim_q g$ implies that $-p \leq \deg f - \deg g \leq q$.

There are two particular interesting special cases in the definition. When $f_1 \sim_0 g$, we say that $f$ *interlaces* $g$ and denote by $f \preceq \text{int} g$. It is well known that $f(x)$ and $g(x)$ are interlacing ($f \preceq \text{int} g$ or $g \preceq \text{int} f$) if and only if for any $a, b \in \mathbb{R}$, all roots of the polynomial $af(x) + bg(x)$ are real. (see Obreschkoff [9, Satz 5.2] for instance). When $f_1 \sim_1 g$, we say that $f$ and $g$ are *compatible* and denote simply by $f \sim g$. Chudnovsky and Seymour [1, Theorem 3.4] showed that $f$ and $g$ are compatible if and only if for any $a, b \geq 0$, all roots of the polynomial $af(x) + bg(x)$ are real. Compatible and interlacing properties of polynomials are often encountered in combinatorics [1, 3, 7, 8, 10, 11, 13].

Let $A$ and $B$ be two Hermitian matrices. Denote $A_p \sim_q B$ if their characteristic polynomials $\det(\lambda I - A)_p \sim_q \det(\lambda I - B)$. Using this notation, Weyl’s inequality (1.3) can be restated as follows.

**Weyl’s Inequality.** Let $A$ and $B$ be two Hermitian matrices of the same order. Assume that $n_+(B - A) \leq p$ and $n_-(B - A) \leq q$. Then $A_p \sim_q B$.

For $f \in \mathcal{R}^{(1)}$, denote by $\mathcal{H}(f)$ the set of Hermitian matrices with characteristic polynomial $f$. The objective of this note is to establish the converse of Weyl’s inequality.

**Theorem 1.3.** Let $f, g \in \mathcal{R}^{(1)}$ have the same degree and $f_p \sim_q g$. Then there exist $A \in \mathcal{H}(f)$ and $B \in \mathcal{H}(g)$ such that $n_+(B - A) \leq p$ and $n_-(B - A) \leq q$.

In the next section we investigate the $(p, q)$-interlacing property. Some known results about the interlacing and compatible polynomials will be extended to $(p, q)$-interlacing polynomials. As an application we prove Theorem 1.3. In §3 we discuss some results closely related to Theorem 1.3. These results can also be obtained from the same approach used in §2.
2. \((p, q)\)-interlacing property and proof of the theorem

Let \(n(f, r)\) be the number of real roots of \(f(x)\) in the interval \([r, +\infty)\). It is well known that \(f\) interlaces \(g\) if and only if \(n(f, r) \leq n(g, r) \leq n(f, r) + 1\) for any \(r \in \mathbb{R}\). Chudnovsky and Seymour [1, Theorem 3.4] showed that \(f\) and \(g\) are compatible if and only if \(|n(g, r) - n(f, r)| \leq 1\) for any \(r \in \mathbb{R}\). A common generalization of these two results is the following.

**Lemma 2.1.** Suppose that \(f, g \in \mathcal{R}\) and \(p, q \in \mathbb{N}\). Then \(f \sim_q g\) if and only if \(-p \leq n(f, r) - n(g, r) \leq q\) for any \(r \in \mathbb{R}\).

**Proof.** Assume that \(f \sim_q g\). Let \(r \in \mathbb{R}\) and \(n(f, r) = i\). Then \(r_i - q \geq r_i - q = n(f, r) - q\). Similarly, \(g \sim_p f\) implies that \(n(f, r) \geq n(g, r) - p\). We conclude that \(-p \leq n(f, r) - n(g, r) \leq q\).

Conversely, assume that \(-p \leq n(f, r) - n(g, r) \leq q\) for any \(r \in \mathbb{R}\). Then \(n(g, r_i(f)) \geq n(f, r_i(f)) - q \geq i - q\), and so \(r_i - q \geq r_i\). Similarly, \(r_i - p \geq r_i\). Thus \(r_i + p \leq r_i \leq r_i - q\), i.e., \(f \sim_q g\).

**Corollary 2.2.** Let \(f, g, h \in \mathcal{R}\). Then \(f \sim_q g\) if and only if \((fh) \sim_q (gh)\).

We next show that the converse of Proposition 1.2 (c) is also true.

**Lemma 2.3.** Suppose that \(f \sim_q g\). For \(0 \leq s \leq p\) and \(0 \leq t \leq q\), denote \(k = \max\{\deg f - t, \deg g - p + s\}\) and \(m = \min\{\deg f + s, \deg g + q - t\}\). Then for each integer \(d \in [k, m]\), there exists a real-rooted polynomial \(h\) of degree \(d\) such that \(f \sim_t h\) and \(h \sim_{p-s} q-t g\).

**Proof.** By the definition of \(f \sim_q g\), we have
\[
    r_{i+t}(f) \leq r_{i-q+t}(g), \quad r_{i+p-s}(g) \leq r_{i-s}(f),
\]
and so
\[
    \max\{r_{i+t}(f), r_{i+p-s}(g)\} \leq \min\{r_{i-s}(f), r_{i-q+t}(g)\}.
\]
For \(i = 1, 2, \ldots, m\), denote
\[
    a_i = \max\{r_{i+t}(f), r_{i+p-s}(g)\}, \quad b_i = \min\{r_{i-s}(f), r_{i-q+t}(g)\}
\]
and \(I_i = [a_i, b_i]\) (\(I_i = (a_i, b_i]\) if \(a_i = -\infty\) or \(I_i = [a_i, b_i)\) if \(b_i = +\infty\)). Clearly, \((a_i)\) is non-increasing. Hence we may choose \(r_i \in I_i\) such that \((r_i)\) is non-increasing. Now for each \(d \in [k, m]\), define a polynomial \(h(x) = \prod_{i=1}^d (x - r_i)\). Then \(r_i(h) = r_i \in I_i\) for \(i = 1, 2, \ldots, d\), and so
\[
    r_{i+t}(f) \leq r_i(h) \leq r_{i-s}(f), \quad r_{i+p-s}(g) \leq r_i(h) \leq r_{i-q+t}(g).
\]
Thus \(f \sim_t h\) and \(h \sim_{p-s} q-t g\). \(\square\)
Corollary 2.4 ([1, Theorem 3.5]). Suppose that \( f, g \in \mathbb{R} \). Then \( f \sim g \) if and only if there exists \( h \in \mathbb{R} \) such that \( h \preceq_{\text{int}} f \) and \( h \preceq_{\text{int}} g \).

It is easy to see that if \( f \preceq_{\text{int}} g \), then \( f' \preceq_{\text{int}} g' \). Also, Chudnovsky and Seymour [1, Theorem 3.1] showed that if \( f \sim g \), then \( f' \sim g' \). We have the following more general result.

Corollary 2.5. If \( f \sim_q g \), then the derivative \( f' \sim_q g' \).

Proof. We proceed by induction on \( p + q \). The statement for \( p + q = 1 \) is just the interlacing case. Suppose that \( p + q > 1 \) and \( f \sim_q g \). We may assume, without loss of generality, that \( p \geq 1 \). Then by Lemma 2.3, there exists \( h \) such that \( f_{p-1} \sim_q h \) and \( h_{1+0} = g \). By the inductive hypothesis, \( f'_{p-1} \sim_q h' \) and \( h'_{1+0} = g' \). Thus \( f' \sim_q g' \) by Proposition 1.2 (c).

For convenience, we introduce an abbreviative notation. If a Hermitian matrix \( A = \sum_{i=1}^{p} \alpha_i \alpha_i^* \), where \( \alpha_i \) are \( p \) (zero or nonzero) complex vectors, then we denote simply by \( A = \sum_{p} \). Also, denote \( \sum_{0} = 0 \). Obviously, \( \sum_{p} \) is positive semi-definite of rank at most \( p \). It is also clear that \( \sum_{p} + \sum_{q} = \sum_{p+q} \) and \( U \left( \sum_{p} \right) U^* = \sum_{p} \) for any matrix \( U \).

Lemma 2.6. Let \( H \) be a Hermitian matrix. Then \( n_+(H) \leq p \) and \( n_-(H) \leq q \) if and only if \( H = \sum_{p} - \sum_{q} \).

Proof. Clearly,

\[
n_+ \left( \sum_{p} - \sum_{q} \right) \leq n_+ \left( \sum_{p} \right) \leq \text{rk} \left( \sum_{p} \right) \leq p
\]

and

\[
n_- \left( \sum_{p} - \sum_{q} \right) = n_+ \left( \sum_{q} - \sum_{p} \right) \leq q.
\]

Conversely, if \( n_+(H) \leq p \) and \( n_-(H) \leq q \), then by the spectral decomposition and Sylvester’s law of inertia for Hermitian matrices,

\[
H = \sum_i \lambda_i(H) P_i P_i^* = \sum_{p} - \sum_{q},
\]

where \( P_i \) are the corresponding orthonormal eigenvectors to \( \lambda_i(H) \).
The following folklore result is the converse of the interlacing theorem for a rank-one Hermitian perturbation of a Hermitian matrix (see [4, Theorems 4.3.26] for instance). We give a direct proof of it for completeness.

**Lemma 2.7.** Let $f, g \in \mathcal{R}_n^{(1)}$ and $f \preceq_{\text{int}} g$. Then there exist a Hermitian matrix $A$ and a complex vector $\alpha$ such that $A \in \mathcal{H}(f)$ and $A + \alpha\alpha^* \in \mathcal{H}(g)$.

**Proof.** Consider first the special case that $f$ and $g$ are coprime. Then $f$ has only simple roots by $f \preceq_{\text{int}} g$. Denote $r_i = r_i(f)$ and $f_i(x) = \frac{f(x)}{x-r_i}$. Then $f(x), f_1(x), \ldots, f_n(x)$ form a basis of the vector space $\mathbb{R}[x]_n$ of real polynomials with degree less than $n+1$. Let $g(x) = f(x) + \sum_{i=1}^n c_if_i(x)$. Then $g(r_i) = c_if_i(r_i)$. Note that $\text{sign}[f_i(r_i)] = (-1)^{i-1}$ and $\text{sign}[g(r_i)] = (-1)^i$. Hence $c_i < 0$. Define $A = \text{diag}(r_1, \ldots, r_n)$ and $\alpha = (a_1, \ldots, a_n)^T$, where $a_i = \sqrt{-c_i}$. Then $\det(xI - A) = f(x)$. On the other hand, we have

$$
\det(xI - A - \alpha\alpha^*) = \det(xI - A) - \sum_{i=1}^n a_i^2 \prod_{j \neq i} (x - r_j) = g(x).
$$

Thus $A \in \mathcal{H}(f)$ and $A + \alpha\alpha^* \in \mathcal{H}(g)$.

Consider next the general case. Let $f = (f, g)f_1$ and $g = (f, g)g_1$. Then $f_1 \preceq_{\text{int}} g_1$ and $(f_1, g_1) = 1$. Thus there are two Hermitian matrices $A_1 \in \mathcal{H}(f_1)$ and $A_1 + \alpha_1\alpha_1^* \in \mathcal{H}(g_1)$. Assume that $(f, g) = \prod_{j=1}^m(x - s_j)$ and define

$$
A = \begin{bmatrix}
A_1 & s_1 & \cdots \\
& \ddots & \iddots \\
& & \ddots & s_m
\end{bmatrix}, \quad \alpha = \begin{bmatrix}
\alpha_1 \\
0 \\
\vdots \\
0
\end{bmatrix}.
$$

Then $A \in \mathcal{H}(f)$ and $A + \alpha\alpha^* \in \mathcal{H}(g)$. This completes the proof. 

**Lemma 2.8.** Let $f, g \in \mathcal{R}_n^{(1)}$ and $f \preceq_{\text{int}} g$. Then there exist two Hermitian matrices $A \in \mathcal{H}(f)$ and $B \in \mathcal{H}(g)$ such that $B - A = \sum p$.

**Proof.** We proceed by induction on $p$. The statement for case $p = 1$ follows from Lemma 2.7. Suppose now that $p > 1$ and $f \preceq_{\text{int}} g$. Then by Lemma 2.3, these exists $h \in \mathcal{R}_n^{(1)}$ such that $f \preceq_{\text{int}} h$ and $h \preceq_{\text{int}} g$. By $f \preceq_{\text{int}} h$ and the induction hypothesis, these exist $A \in \mathcal{H}(f)$ and $C \in \mathcal{H}(h)$ such that $C - A = \sum_{p-1}$. By $h \preceq_{\text{int}} g$ and Lemma 2.7, there exist $B_1 \in \mathcal{H}(g)$ and $C_1 \in \mathcal{H}(h)$ such that $B_1 - C_1 = \sum_1$. Since two Hermitian matrices $C$ and
C_1 have the same characteristic polynomial, there is a unitary matrix U such that UC_1U^* = C. Define B = UB_1U^*. Then B ∈ \mathcal{H}(g) and

\[ B - C = U(B_1 - C_1)U^* = U \left( \sum_1 \right) U^* = \sum_1. \]

It follows that

\[ B - A = (B - C) + (C - A) = \sum_1 + \sum_{p-1} = \sum_p, \]

as required. Thus the proof is complete by induction.

We are now in a position to prove the theorem.

Proof of Theorem 1.3. By Lemma 2.6, it suffices to show that if \( f \sim g \), then there exist two Hermitian matrices \( A \in \mathcal{H}(f) \) and \( B \in \mathcal{H}(g) \) such that \( B - A = \sum_p - \sum_q \).

If \( p = 0 \) or \( q = 0 \), then the statement follows from Lemma 2.8. Assume next that \( p, q > 0 \). Then by Lemma 2.3, there exists \( h \in \mathfrak{H}_h^{(1)} \) such that \( f \sim h \) and \( g \sim h \). By \( f \sim h \) and Lemma 2.8, there exist \( A \in \mathcal{H}(f) \) and \( C \in \mathcal{H}(h) \) such that \( C - A = \sum_p \). By \( g \sim h \) and Lemma 2.8, there exist \( B_1 \in \mathcal{H}(g) \) and \( C_1 \in \mathcal{H}(h) \) such that \( C_1 - B_1 = \sum_q \). Since two Hermitian matrices \( C \) and \( C_1 \) have the same characteristic polynomial, there is a unitary matrix \( U \) such that \( UC_1U^* = C \). Define \( B = UB_1U^* \). Then \( B \in \mathcal{H}(g) \) and

\[ C - B = U(C_1 - B_1)U^* = U \left( \sum_q \right) U^* = \sum_q. \]

It follows that

\[ B - A = (C - A) - (C - B) = \sum_p - \sum_q, \]

as required. This completes the proof of the theorem.

Remark 2.9. We can also prove Theorem 1.3 directly from Lemmas 2.3 and 2.7 by induction on \( p + q \).
3. Remarks

Let \( \alpha = (a_1, \ldots, a_k) \) and \( \beta = (b_1, \ldots, b_k) \) be two real vectors whose entries arranged in non-increasing order. Denote \( \alpha \leq \beta \) if \( \beta - \alpha \) is non-negative, and further denote \( \alpha \leq_{r} \beta \) if \( \beta - \alpha \) has at least \( r \) positive entries. Let \( f \) and \( g \) be two monic real-rooted polynomials of degree \( n \). By the definition, \( f \sim g \) is equivalent to \( (r_1(f), \ldots , r_n(f)) \leq (r_1(g), \ldots , r_n(g)) \) and \( (r_{p+1}(f), \ldots , r_n(f)) \leq (r_1(f), \ldots , r_{n-p}(f)) \). Denote \( f \approx g \) if

\[
(r_{q+1}(f), \ldots , r_n(f)) \leq_p (r_1(g), \ldots , r_{n-q}(g))
\]

and

\[
(r_{p+1}(g), \ldots , r_n(g)) \leq_q (r_1(f), \ldots , r_{n-p}(f))
\]

The notation \( \approx \) enjoys some properties similar to \( \sim \). For example,

(i) If \( f \approx g \), then there exists \( h \) such that \( f \approx_0 h \) and \( g \approx_0 h \).

(ii) If \( f \approx_0 g \), then there exists \( h \) such that \( f \approx_{p-1} h \) and \( h \preceq \text{int} g \).

To prove them, it suffices to choose \( r_i(h) \) in the proof of Lemma 2.3 as far as possible within the interval \( I_i \) unless \( I_i \) consists of only one point.

Theorem 1.3 states that if \( f \sim g \), then there exist two Hermitian matrices \( A \in \mathcal{H}(f) \) and \( B \in \mathcal{H}(g) \) such that \( n_+(B - A) \leq p \) and \( n_-(B - A) \leq q \). Given \( 0 \leq s \leq p \) and \( 0 \leq t \leq q \), a natural problem is when there exist two Hermitian matrices \( A \in \mathcal{H}(f) \) and \( B \in \mathcal{H}(g) \) such that \( n_+(B - A) = s \) and \( n_-(B - A) = t \). Li and Poon [6, Theorem 2.1] gave the characterization for the case \( s = p \) and \( t = q \), which can also be proved by a similar technique used in the previous section. We omit the details for the sake of simplicity.

**Theorem 3.1** ([6, Theorem 2.1]). Let \( f, g \in \mathcal{X}_n^{(1)} \) and \( f \sim g \). Suppose that \( p + q \leq n \) and \( f \approx g \). Then there exist \( A \in \mathcal{H}(f) \) and \( B \in \mathcal{H}(g) \) such that \( n_+(B - A) = p \) and \( n_-(B - A) = q \).

Another result closely related to Theorem 1.3 is Cauchy’s interlacing theorem, which states that a Hermitian matrix \( A \) interlaces its bordered Hermitian matrix \( \begin{bmatrix} A & a \\ \alpha^* & a \end{bmatrix} \) (see [4, Theorem 4.3.17] for instance). More generally, the inclusion principle states that two Hermitian matrices \( A \sim B \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \), where \( C \) is a \( p \times p \) Hermitian matrix (see [4, Theorem 4.3.28] for instance).
Fan and Pall [2, Theorem 1] established the following converses of the inclusion principle.

**Theorem 3.2** ([2, Theorem 1]). *Let $f$ and $g$ be two monic real-rooted polynomials satisfying $\deg g = \deg f + p$ and $f, g \sim_0 g$. Then there are two Hermitian matrices $A$ and $B$ such that their characteristic polynomials are $f$ and $g$ respectively.*

We can prove the converses of Cauchy’s interlacing theorem by the same technique used in the proof of Lemma 2.7, and then prove the converses of the inclusion principle by induction on $p$.

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