MOTIVIC INTERPRETATION OF ALBANESE VARIETIES OF
SMOOTH VARIETIES

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Abstract. For any noetherian scheme $X$ smooth and separated over an algebraically closed field $k$, we describe the Albanese variety $\text{Alb} X$ of $X$ in the world of $\text{DM}^{\text{eff}}(k)$. As an application, we explain the structure of the Picard functor $\text{Pic}^0_{X/k}$.

1. Introduction

1.1. Throughout this paper, let $k$ be an algebraically closed field, and let $p$ be its exponential characteristic. See [1-10] for the reason why we should restrict to this case.

1.2. There are some objects constructed schematically first that are considered as a “motivic object” later in an appropriate sense. For example, Chow groups $CH^n(X)$ were first defined by cycles in a scheme $X$, and after Voevodsky introduced the triangulated category of motives $\text{DM}^{\text{eff}}(k)$, $CH^n(X)$ was also considered as a motivic cohomology group $\text{Hom}_{\text{DM}^{\text{eff}}(k)}(M(X), \mathbb{Z}(n)[2n])$.

Following this phenomenon, we can try to give a motivic structure to an object already constructed schematically. In this paper, we give a motivic structure to Albanese varieties, which are constructed schematically as follows.

Let $X$ be an integral noetherian scheme over $k$, and let $x$ be its closed point. The Albanese morphism (with respect to $(X, x)$) is a morphism $X \to \text{Alb} X$ of schemes over $k$ to a semi-abelian variety $\text{Alb} X$ mapping $x$ to 0 that is universal among morphisms from $X$ to semi-abelian varieties mapping $x$ to 0. Then $\text{Alb} X$ is called the Albanese variety of $X$. The existence of $\text{Alb} X$ is proven by Serre [20].

When $X$ is smooth over $k$, the motivic interpretation of $\text{Alb} X$ is that this is the “1-motivic part” of the motive $M(X)$ in $\text{DM}^{\text{eff}}(k)$. To realize this interpretation, we introduce the following construction. The 0-motivic part of $M(X)$ is $\mathbb{Z}^r$ where $r$ is the number of connected components of $X$. To remove $\mathbb{Z}^r$ from $M(X)$, we take the cocone

$$M_{\geq 1}(X) := \text{ccone}(M(X) \to \mathbb{Z}^r).$$

Then to remove the $m$-motivic parts of $M_{\geq 1}(X)$ for $m \geq 2$, we use the duality $\text{Hom}_{\text{DM}^{\text{eff}}(k)}(-, \mathbb{Z}(1)[2])$ since this operation should make the $m$-motivic part of $M(X)$ into the $(2 - m)$-motivic part of $\text{Hom}_{\text{DM}^{\text{eff}}(k)}(M(X), \mathbb{Z}(1)[2])$, and it is easier to remove the $(2 - m)$-motivic part for $m \geq 2$. The 0-motivic part of $\text{Hom}_{\text{DM}^{\text{eff}}(k)}(M_{\geq 1}(X), \mathbb{Z}(1)[2])$ is the

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Néron-Severi group $\text{NS}(X)$. To remove $\text{NS}(X)$ from $\text{Hom}_{\text{DM}^\text{eff}(k)}(M_{\geq 1}(X), \mathbb{Z}(1)[2])$, we take the cocone

$$M_1(X)^\vee := \text{cocone}(\text{Hom}_{\text{DM}^\text{eff}(k)}(M_{\geq 1}(X), \mathbb{Z}(1)[2]) \rightarrow \text{NS}(X)).$$

Then we can consider $M_1(X)^\vee$ as the dual of the motivic Albanese object of $X$. The comparison of $M_1(X)^\vee$ and the Albanese variety $\text{Alb}(X)$ is as follows, which is our main theorem.

**Theorem 1.3** (Theorem 6.21). Let $X$ be a noetherian scheme smooth and separated over $k$. Then $M_1(X)^\vee[-1]$ is isomorphic to the Cartier dual of the semi-abelian variety $\text{Alb}(X)$.

1.4. As an application of this theorem, we explain the structure of Picard functors as follows.

**Corollary 1.5** (Corollary 6.22). Let $X$ be a noetherian scheme smooth and separated over $k$, and express the structure of $\text{Alb}(X)$ as an exact sequence

$$0 \rightarrow T_{\text{Alb}}(X) \otimes \mathbb{G}_m \rightarrow \text{Alb}(X) \rightarrow \overline{\text{Alb}}(X) \rightarrow 0$$

of semi-abelian varieties where $T_{\text{Alb}}(X)$ is a lattice and $\overline{\text{Alb}}(X)$ is an abelian variety over $k$. Then there is an exact sequence

$$T_{\text{Alb}}(X)^{\vee} \rightarrow \overline{\text{Alb}}(X)^{\vee} \rightarrow \text{Pic}^0_{X/k} \rightarrow 0$$

of Nisnevich sheaves with transfers where $T_{\text{Alb}}^{\vee}$ (resp. $\overline{\text{Alb}}(X)^{\vee}$) denotes the dual lattice (resp. dual abelian variety) of $T_{\text{Alb}}$ (resp. $\overline{\text{Alb}}(X)$).

1.6. Within the category $\text{DM}^\text{eff}(k, \mathbb{Z}[1/p])$, Theorem 1.3 is proved in [6, The main theorem 9.2]. Our version is more refined than this since $p$ is not inverted. Not inverting $p$ is useful since we can consider crystalline realizations of $M_1(X)^\vee$ within $\text{DM}^\text{eff}(k)$, which is not clear within $\text{DM}^\text{eff}(k, \mathbb{Z}[1/p])$.

The usage of Nisnevich topology is also an improvement. By Corollary 1.5, we have an exact sequence

$$(1.6.1) \quad L \rightarrow A \rightarrow \text{Pic}^0_{X/k} \rightarrow 0$$

of Nisnevich sheaves with transfers for some $L$ and $A$. An étale version is that there is an exact sequence $1.6.1$ of étale sheaves with transfer.

The étale version is proved in the argument of [6, Proposition 3.5.1]. A surjective morphism $A \rightarrow \text{Pic}^0_{X/k}$ of étale sheaves with transfer may not be a surjective morphism of Nisnevich sheaves with transfer. Thus more careful choices are needed to establish the Nisnevich version. This means that the above corollary contains more data. Note also that the exactness of 1.6.1 as étale sheaves does not insure that $A \cong \overline{\text{Alb}}(X)$ since the choice of $A$ is not canonical.

1.7. Let us explain what kinds of extra burden appear if we work with $\text{DM}^\text{eff}(k)$ instead of $\text{DM}^\text{eff}(k, \mathbb{Z}[1/p])$. An isogeny $A \rightarrow B$ of abelian varieties over $k$ is not an epimorphism of Nisnevich sheaves with transfer in general. Thus many proofs in the literature cannot be used for the Nisnevich topology. For example, with the étale topology, lots of arguments can be reduced to the case of curves since any abelian variety over $k$ is isogenous to a direct summand of $\text{Alb}(C)$ for some curve $C$ smooth and proper over $k$. Such a technique is not available with the Nisnevich topology.
Note also that Theorem 1.3 and Corollary 1.5 can be easily deduced from Serre’s result ([19]) if we assume resolution of singularities. Since we are working with the Nisnevich topology and \( \mathbb{Z} \)-coefficient, the de Jong alteration is not a perfect replacement for resolution of singularities. Hence we need more arguments.

This paper provides a way to weaken the burden.

1.8. Albanese varieties appear naturally in the theory of 1-motives. Therefore our results may form a basic tool for developing a theory of integral 1-motives relating \( \text{DM}^{\text{eff}}(k) \).

1.9. Organization of the paper. In Section 2, we study Picard functors and their basic properties. In Section 3, we use a kind of Hilbert’s Theorem 90 to use alterations to \( \text{DM}^{\text{eff}}(k) \). In Section 4, we recall the notion of 0-motivic sheaves and prove some vanishing results. In Section 5, the structure of \( M_1(X)^{\vee} \) is described using the previous sections. In Section 6, we study the motivic dual of semi-abelian varieties, and we deduce that \( M_1(X)^{\vee}[-1] \) is isomorphic to the Cartier dual of \( \text{Alb}_X \).

1.10. Condition that \( k \) is algebraically closed. We cannot weaken the condition that \( k \) is algebraically closed if we work within \( \text{DM}(k) \). By the following two related reasons, \( k \) should be separably closed.

(i) The triangulated category \( \text{DM}(k) \) does not admit a reasonable motivic t-structure by [21] Proposition 4.3.8 if there is a conic over \( k \) without rational points.

(ii) Let \( X \) be a noetherian scheme smooth and proper over \( k \). The Picard functor \( \text{Pic}_{X/k} \) is not an étale sheaf (so not representable) in general if \( X \) has no rational points.

We should assume that \( k \) is perfect since [15] Theorem 13.8] is wrong without the condition.

1.11. Notations and convention.

(1) \( \text{Sm}/k \) denotes the category of noetherian schemes smooth and separated over \( k \).

(2) \( \text{Sh}^{\text{tr}}(k) \) denotes the category of Nisnevich sheaves with transfers on \( \text{Sm}/k \).

(3) A lattice is a constant sheaf on \( \text{Sm}/k \) associated with a finitely generated free abelian group \( \mathbb{Z}^d \).

(4) For any abelian group \( F \), by abuse of notation, we write \( F \) for the associated constant Nisnevich sheaf with transfers.

(5) For any semi-abelian variety \( G \), by abuse of notation, we write \( G \) for the associated Nisnevich sheaf with transfers on \( \text{Sm}/k \) given in [17].

(6) For any closed monoidal triangulated category \( \mathcal{T} \), \( \text{Hom}_{\mathcal{T}} \) denotes the internal Hom.

(7) For any morphism \( F \to G \) in an abelian category, let \( [F \to G] \) denote the complex where \( F \) sits in degree 0.

2. Picard functors

2.1. In this section, we study the Picard functor for a noetherian scheme \( X \) smooth and separated over \( k \). Let \( \text{Pic}_{X/k} \) denote the presheaf of abelian groups on \( \text{Sm}/k \) given by

\[
\text{Pic}_{X/k}(T) := \text{Pic}(X \times T)/\text{Pic}(T).
\]
It is the restriction of the usual Picard functor to \( Sm/k \), and it has a transfer structure since the functors \( T \mapsto \text{Pic}(X \times T) \) and \( T \mapsto \text{Pic}(T) \) are presheaves with transfers on \( Sm/k \).

When \( T \) is integral, consider the composition \( \text{Pic}(X \times T) \xrightarrow{i^*} \text{Pic}(X) \rightarrow \text{NS}(X) \) where \( i \) is the pullback of a closed immersion \( \{x\} \rightarrow T \) from a rational point \( x \) of \( X \). The composition is independent of the choice of \( i \), and any element of \( \text{Pic}(T) \) in \( \text{Pic}(X \times T) \) maps to 0 in \( \text{NS}(X) \). This induces a morphism

\[
\text{Pic}_{X/k} \rightarrow \text{NS}(X)
\]
of presheaves with transfers. We denote by \( \text{Pic}^0_{X/k} \) its kernel.

2.2. Let \( X \) be an integral noetherian scheme smooth and separated over \( k \). Recall that \( \pi_0(X) \) denotes the disjoint union of \( r \) copies of \( \text{Spec} k \) if \( X \) has \( r \) connected components. Note that we have the structure morphism \( X \rightarrow \pi_0(X) \). Set \( \pi_0(\mathbb{Z}^{tr}(X)) := \mathbb{Z}^r \). We denote by \( \mathbb{Z}^{tr}_{\geq 1}(X) \) the kernel of the induced morphism

\[
\mathbb{Z}^{tr}(X) \rightarrow \pi_0(\mathbb{Z}^{tr}(X))
\]
of Nisnevich sheaves with transfers. Since each connected component of \( X \) has a rational point, we have a decomposition

\[
\mathbb{Z}^{tr}(X) \cong \mathbb{Z}^{tr}_{\geq 1}(X) \oplus \pi_0(\mathbb{Z}^{tr}(X)).
\]

Set

\[
M_{\geq 1}(X) := C^*_s \mathbb{Z}^{tr}_{\geq 1}(X),
\]
and consider it as an object of \( \text{DM}^{eff}(k) \).

**Proposition 2.3.** Let \( X \) be a scheme smooth and separated over \( k \). Then \( \text{Pic}_{X/k} \) is a Nisnevich sheaf with transfers on \( Sm/k \).

**Proof.** Since we have checked that \( \text{Pic}_{X/k} \) is a presheaf with transfers in 2.1, it remains to show that \( \text{Pic}_{X/k} \) is a Nisnevich sheaf. If \( T \) is a disjoint union of \( T_1 \) and \( T_2 \), then

\[
\text{Pic}(X \times_k T) \cong \text{Pic}(X \times T_1) \oplus \text{Pic}(X \times T_2), \quad \text{Pic}(T) \cong \text{Pic}(T_1) \oplus \text{Pic}(T_2).
\]

Thus

(2.3.1) \[
\text{Pic}_{X/k}(T) \cong \text{Pic}_{X/k}(T_1) \oplus \text{Pic}_{X/k}(T_2).
\]

Let

\[
\begin{array}{ccc}
U & \xrightarrow{g} & T' \\
\downarrow f' & & \downarrow f \\
U & \xrightarrow{g} & T
\end{array}
\]

be a Nisnevich distinguished square where \( f \) is étale, \( g \) is an open immersion, and the induced morphism \( f^{-1}(T - g(U)) \rightarrow T - g(U) \) is an isomorphism. Here, we consider \( T - g(U) \) having the reduced scheme structure. To check that \( \text{Pic}_{X/k} \) is a Nisnevich sheaf, it suffices to show that the induced sequence

\[
0 \rightarrow \text{Pic}_{X/k}(T) \xrightarrow{T'f} \text{Pic}_{X/k}(T') \oplus \text{Pic}_{X/k}(U) \xrightarrow{U'g} \text{Pic}_{X/k}(U') \rightarrow 0
\]
is exact by \( \text{[23 Corollary 2.17]} \). By (2.3.1) we reduce to the case when \( T \) is connected. Then \( T \) is irreducible since \( T \) is smooth over \( k \).
Let $d$ be the number of irreducible components of $T - g(U)$ whose dimensions are $(\dim T - 1)$. Then by the localization sequence for higher Chow groups, we have the commutative diagrams

$$
\begin{array}{ccccccc}
Z^d & \xrightarrow{u} & \Pic(T) & \xrightarrow{g^*} & \Pic(U) & \xrightarrow{f} & 0 \\
\downarrow{\text{id}} & & \downarrow{f^*} & & \downarrow{f^*} & & \\
Z^d & \xrightarrow{v} & \Pic(X \times T) & \xrightarrow{g'^*} & \Pic(U') & \xrightarrow{f'} & 0 \\
\downarrow{\text{id}} & & \downarrow{g'^*} & & \downarrow{g'^*} & & \\
Z^d & \xrightarrow{v'} & \Pic(X \times T') & \xrightarrow{g'^{''*}} & \Pic(U') & \xrightarrow{f''} & 0
\end{array}
$$

of abelian groups such that each row is exact.

Consider the induced commutative diagram

$$
\begin{array}{ccccccc}
\Pic(T) & \xrightarrow{p} & \Pic(T') \oplus \Pic(U) & \xrightarrow{q} & \Pic(U') & \xrightarrow{f} & 0 \\
\downarrow{r} & & \downarrow{r'} & & \downarrow{f'} & & \\
\Pic(X \times T) & \xrightarrow{g'} & \Pic(X \times T') \oplus \Pic(X \times U) & \xrightarrow{g''} & \Pic(X \times U') & \xrightarrow{f''} & 0 \\
\downarrow{r} & & \downarrow{r'} & & \downarrow{r''} & & \\
\Pic_{X/k}(T) & \xrightarrow{p''} & \Pic_{X/k}(T') \oplus \Pic_{X/k}(U) & \xrightarrow{q''} & \Pic_{X/k}(U') & \xrightarrow{f''} & 0
\end{array}
$$

(2.3.2)

of abelian groups where $p = (f^*, -g^*)$ and $q$ is the summation of $f'^*$ and $g'^*$. Note that columns are exact by definition. Taking $\text{Hom}_{\text{DM}^{\text{eff}}(k)}(-, \mathbb{Z}(1)[2])$ to the distinguished triangle

$$M(U') \to M(T') \oplus M(U) \to M(T) \to M(U')[1]$$

in $\text{DM}^{\text{eff}}(k)$, we get the exactness of the top row in (2.3.2). Similarly, the middle row in (2.3.2) is also exact.

Let us show that the bottom row in (2.3.2) is exact. Consider an element $b'' \in \Pic_{X/k}(T') \oplus \Pic_{X/k}(U)$ such that $q''(b'') = 0$. Choose $b' \in \Pic(X \times T') \oplus \Pic(X \times U)$ such that $s''(b') = b''$. Then $s''(q'(b')) = q''(s'(b')) = 0$, so $q'(b') = r''(c)$ for some $c \in \Pic(U')$. Choose $b \in \Pic(T') \oplus \Pic(U)$ such that $q(b) = c$. Then $q'(b - r'(b)) = q'(b') - r''(c) = 0$, so $b' = r'(b) = p'(a')$ for some $a' \in \Pic(X \times T)$. Thus $b'' = s'(b') = s'(b' - r'(b)) = s'(p'(a')) = p''(s(a'))$, so $b''$ is in the image of $p''$. Since $q'$ and $s''$ are surjective, $q''$ is also surjective. We have shown that the bottom row in (2.3.2) is exact.

The remaining is to show that $p''$ is injective. Let $a'' \in \Pic_{X/k}(T)$ be an element such that $p''(a'') = 0$. Choose $a' \in \Pic(X \times T)$ such that $s(a') = a''$. Then $s'(p'(a')) = p'(a'') = 0$, so $p'(a') = r'(b)$ for some $b \in \Pic(T') \oplus \Pic(U)$. Thus $r''(q(b)) = q'(r''(b)) = q'(p'(a')) = 0$. Since $k$ is algebraically closed, the projection $X \times U' \to U'$ has a section. Thus $r''$ is injective. Then $q(b) = 0$ since $r''(q(b)) = 0$. Thus $b = p(a)$ for some $a \in \Pic(T)$. Now $p'(a' - r(a)) = p'(a') - r'(b) = 0$. Thus $v(t) = a' - r(a)$ for some $t \in Z^d$. Then $a' - r(a) = r(u(t))$, so $a'$ is in the image of $r$. Thus $a'' = 0$. This proves that $p''$ is injective. \qed
Definition 2.4. Recall that an object $F$ of a triangulated category $\mathcal{T}$ is **compact** if $\text{Hom}_\mathcal{T}(F, -)$ commutes with small sums.

Definition 2.5. Let $\mathcal{T}$ be triangulated category having small sums, and let $\mathcal{F}$ be a essentially small class of compact objects in $\mathcal{T}$. Recall from [3 Proposition 2.1.70] that there is a $t$-structure such that the category of $t$-positive objects is the smallest full subcategory of $\mathcal{T}$ containing $\mathcal{F}$ and stable under small sums, suspensions, and extensions. This $t$-structure is called the $t$-structure on $\mathcal{T}$ generated by $\mathcal{F}$.

For $i \in \mathbb{Z}$, we denote by $h_i$ the homology functor, and we denote by $\tau_{\leq i}$ and $\tau_{\geq i}$ the homological truncation functors.

According to the definition and properties of $t$-structures, we have the following.

(i) For any $M \in \mathcal{T}$, $\tau_{\geq 0}M$ is $t$-positive, and $\tau_{\leq 0}M$ is $t$-negative.

(ii) For any $t$-positive object $M$ and $t$-negative object $N$, $\text{Hom}_\mathcal{T}(M, N[-1]) = 0$.

(iii) For any $M \in \mathcal{T}$ and $i \in \mathbb{Z}$, there is a canonical distinguished triangle

$$\tau_{\geq i}M \to M \to \tau_{\leq i-1}M \to \tau_{\geq i}M[1].$$

(iv) For any $M \in \mathcal{T}$ and $i \in \mathbb{Z}$,

$$h_i(M) = \tau_{\geq i}\tau_{\leq i}M[-i] \cong \tau_{\leq i}\tau_{\geq i}M[-i],$$

and this is in the heart.

Proposition 2.6. Let $X$ be a noetherian scheme smooth and separated over $k$. Then $M(X)$ and $M_{\geq 1}(X)$ are compact in $\text{DM}^{tr}(k)$.

Proof. By Proposition [3 Proposition 5.1.32, Example 5.1.29(2)], $M(X)$ is compact in $\text{DM}^{tr}(k)$. Then $\mathbb{Z} \cong M(k)$ is compact in $\text{DM}^{tr}(k)$, so $M_{\geq 1}(X)$ is compact in $\text{DM}^{tr}(k)$ since we have the distinguished triangle

$$M_{\geq 1}(X) \to M(X) \to \mathbb{Z}^r \to M_{\geq 1}(X)[1]$$

in $\text{DM}^{tr}(k)$ where $r$ is the number of connected components of $X$. □

Definition 2.7 ([3 Proposition 3.3]). The $0$-motivic $t$-structure (or homotopy $t$-structure) on $\text{DM}^{tr}(k)$ is the $t$-structure generated by objects of the form

$$M(X)$$

where $X$ is a noetherian scheme smooth and separated over $k$. Note that the heart of this $t$-structure is equivalent to the category of homotopy invariant Nisnevich sheaves with transfers on $\text{Sm}/k$ by the following paragraph of [3 Definition 3.1].

Proposition 2.8. Let $X$ be a noetherian scheme smooth and separated over $k$. Then

$$h_0(\text{Hom}_{\text{DM}^{tr}(k)}(M(X), \mathbb{Z}(1)[2])) \cong \text{Pic}_{X/k}.$$

Proof. For any noetherian scheme $T$ smooth and separated over $k$,

$$\text{Hom}_{\text{DM}^{tr}(k)}(M(T), \text{Hom}_{\text{DM}^{tr}(k)}(M(X), \mathbb{Z}(1)[2])) \cong \text{Pic}(T \times X).$$

Thus $h_0(\text{Hom}_{\text{DM}^{tr}(k)}(M(X), \mathbb{Z}(1)[2]))$ is the Nisnevich sheaf associated with the presheaf

$$T \mapsto \text{Pic}(T \times X).$$

By Proposition 2.3, $\text{Pic}_{X/k}$ is a Nisnevich sheaf with transfers, and we have the morphism

$$p : h_0(\text{Hom}_{\text{DM}^{tr}(k)}(M(X), \mathbb{Z}(1)[2])) \to \text{Pic}_{X/k}.$$
of Nisnevich sheaves given by taking the quotient $q : \text{Pic}(T \times X) \to \text{Pic}(T \times X)/\text{Pic}(T)$. Let us show $p$ is an isomorphism. It suffices to check that $q$ is an isomorphism when $T$ is a henselian local scheme. In this case, we are done since $\text{Pic}(T) = 0$.

**Proposition 2.9.** Let $X$ be a noetherian scheme smooth and proper over $k$. Then

$$\text{Hom}_{\text{DM}^{\text{eff}}(k)}(M_{\geq 1}(X), \mathbb{Z}(1)[2]) \cong \text{Pic}_{X/k}.$$ 

**Proof.** We only need to consider the case when $X$ is connected. By Proposition 2.8, for any $i \in \mathbb{Z} - \{0\}$, it suffices to show that the morphism

$$h_!(\text{Hom}_{\text{DM}^{\text{eff}}(k)}(M(k), \mathbb{Z}(1)[2])) \to h_!(\text{Hom}_{\text{DM}^{\text{eff}}(k)}(M(X), \mathbb{Z}(1)[2]))$$

of homotopy invariant Nisnevich sheaves induced by the structure morphism $X \to k$ is an isomorphism. These are the Nisnevich sheaves associated with the presheaves

$$T \mapsto H^1_{\text{Nis}}(X \times T, \mathbb{G}_m), \quad T \mapsto H_{\text{Nis}}^1(T, \mathbb{G}_m)$$

respectively. By [15] Vanishing Theorem 19.3, these are 0 if $i \neq 0, 1$. Thus we only need to show that the induced homomorphism

$$(2.9.1) \quad H^0_{\text{Nis}}(T, \mathbb{G}_m) \to H^0_{\text{Nis}}(X \times T, \mathbb{G}_m)$$

is an isomorphism for any noetherian scheme $T$ smooth and separated over $k$. We also only need to consider the case when $T$ is connected.

Let us argue as in [14] Lemma 2.12. Since $k$ is algebraically closed, the projection $p : X \times T \to T$ has a section $i : T \to X \times T$. Hence it suffices to show that $(2.9.1)$ is surjective. By [13] III.7.7.6, the induced homomorphism

$$p^* : H^0(T, \mathcal{O}_T) \to H^0(X \times T, \mathcal{O}_{X \times T})$$

is an isomorphism. Let $a \in H^0_{\text{Nis}}(X \times T, \mathbb{G}_m)$ be an element. Then $a = p^* b$ for some $b \in H^0(T, \mathcal{O}_T)$. Since $i^* a = i^* p^* b = b$, $b$ is in $H^0_{\text{Nis}}(T, \mathbb{G}_m)$. This shows that $(2.9.1)$ is surjective. \hfill ∎

**2.10.** Let $X$ be a noetherian scheme smooth and separated over $k$. Consider the composition

$$\text{Hom}_{\text{DM}^{\text{eff}}(k)}(M_{\geq 1}(X), \mathbb{Z}(1)[2]) \to h_!(\text{Hom}_{\text{DM}^{\text{eff}}(k)}(M_{\geq 1}(X), \mathbb{Z}(1)[2])) \cong \text{Pic}_{X/k} \to \text{NS}(X)$$

in $\text{DM}^{\text{eff}}(k)$ where the second arrow is given in Proposition 2.8. Let $M_1(X)^\vee$ denote its cocone. Then we have the distinguished triangle

$$(2.10.1) \quad M_1(X)^\vee \to \text{Hom}_{\text{DM}^{\text{eff}}(k)}(M_{\geq 1}(X), \mathbb{Z}(1)[2]) \to \text{NS}(X) \to M_1(X)^\vee[1]$$

in $\text{DM}^{\text{eff}}(k)$.

**2.11.** In [11], $M_1(X)^\vee$ is constructed using an $h$-hypercover $X_\bullet \to X$, and it is shown that it is independent of the choice of $X_\bullet \to X$. Our approach is different. We have given a canonical definition of $M_1(X)^\vee$ without using any $h$-hypercover, and we will prove that $M_1(X)^\vee[-1]$ is isomorphic to a complex $[L \to A]$ where $L$ is a lattice and $A$ is an abelian variety. In the proof, we use an $h$-hypercover.

**Proposition 2.12.** Let $G$ be a semi-abelian variety over $k$, and let $L$ be a lattice. Consider them as Nisnevich sheaves on $\text{Sm}/k$. Then $G \oplus L$ has a unique transfer structure.
Proof. Set $F := G \oplus L$. By [17], $G$ has a transfer structure. Thus $F$ has a transfer structure. The remaining is the uniqueness. Let $X$ and $Y$ be noetherian schemes smooth and separated over $k$, and let $Z \in \text{Cor}(X,Y)$ be a correspondence. If $F$ has another transfer structure, the two transfer structures and $Z$ induce homomorphisms

$$p : F(Y) \to F(X), \quad q : F(Y) \to F(X)$$

of abelian groups. We need to show that $p = q$.

Morphisms in $Sm/k$ are determined by closed points, so the homomorphism

$$F(X) \to \coprod_{x \in \text{cl}X} F(\{x\})$$

of abelian groups induced by the closed immersions $\{x\} \to X$ is injective where $\text{cl}X$ denotes the set of closed points of $X$. Hence we reduce to the case when $X = \text{Spec} k$. Then $\text{Cor}(\text{Spec} k, Y) \cong \mathbb{Z}[Y(k)]$, so we are done since the two transfer structures agree on the level of morphisms. □

Definition 2.13. Let $G$ be a group scheme over $k$. We say that $G$ is an abelian (resp. semi-abelian) group scheme over $k$ if $\pi_0(G)$ is a finitely generated abelian group and the connected component of the identity with the reduced scheme structure is an abelian (resp. semi-abelian) variety.

Proposition 2.14. Let $X$ be a noetherian scheme smooth and proper over $k$. Then $\text{Pic}^0_{X/k}$ is a Nisnevich sheaf with transfers representable by an abelian variety over $k$.

Proof. Let $A$ be the presheaf of abelian groups on $\text{Sch}/k$ given by

$$A(T) = \text{Pic}(X \times T)/\text{Pic}(T)$$

for scheme $T$ over $k$. The restriction of $A$ to $Sm/k$ is $\text{Pic}_{X/k}^0$. It is known that $A$ is representable by an abelian group scheme over $k$. Let $B$ be the connected component of $A$ containing the identity of $A$. Then the restriction of $B$ to $Sm/k$ is $\text{Pic}_{X/k}^0$, and $B_{\text{red}}$ is representable by an abelian variety over $k$. Let $T$ be a scheme smooth and separated over $k$. For any morphism $f : T \to B$ of schemes over $k$, $f$ factors through $B_{\text{red}}$ since $T$ is reduced. Thus $B_{\text{red}}$ and $B$ represent the same presheaf on $Sm/k$. Then the presheaf $\text{Pic}_{X/k}^0$ is representable by an abelian variety over $k$. It has the usual transfer structure given in [15] Example 2.5, and this agrees with the transfer structure in [17] by Proposition 2.12. □

3. Hilbert’s Theorem 90

3.1. If $X$ is not proper over $k$, then $\text{Pic}_{X/k}^0$ is not representable by an abelian variety. Instead, a little improved version of [6 Proposition 3.5.1] is that there is an exact sequence

$$L \to A \to \text{Pic}_{X/k}^0 \to 0$$

of étale sheaves where $L$ is a lattice and $A$ is an abelian variety. We need its Nisnevich version, which asserts that there is an exact sequence (3.1.1) of Nisnevich sheaves. Note that the Nisnevich version is not a consequence of the étale version. Indeed, for any isogeny $B \to A$ of abelian varieties, there is an exact sequence

$$L' \to B \to \text{Pic}_{X/k}^0 \to 0$$
of étale sheaves such that $L'$ is a lattice. Thus there are lots of possible choices of $A$, and we need to choose one of them fitting in (3.1.1). We do this by explicitly constructing $L$ and $A$. For this purpose, we use de Jong alterations to reduce to the case when $X$ is smooth and projective over $k$.

3.2. Let us begin with recalling Hilbert’s theorem 90 in étale cohomology theory. It asserts that the induced homomorphism

$$H^1_{\text{Zar}}(X, G_m) \to H^1_{\text{ét}}(X, G_m)$$

of abelian groups is an isomorphism for any scheme $X$ over $k$. The same proof works if we replace $\text{Zar}$ to $\text{Nis}$, so the induced homomorphism

$$H^1_{\text{Nis}}(X, G_m) \to H^1_{\text{ét}}(X, G_m)$$

of abelian groups is an isomorphism.

There is also a simplicial version. In [7, Proposition 4.4.1], it is shown that the induced homomorphism

$$H^1_{\text{Zar}}(X_\bullet, G_m) \to H^1_{\text{ét}}(X_\bullet, G_m)$$

of abelian groups is an isomorphism for any simplicial scheme $X_\bullet$ over $k$. As above, the induced homomorphism

$$H^1_{\text{Nis}}(X_\bullet, G_m) \to H^1_{\text{ét}}(X_\bullet, G_m)$$

of abelian groups is an isomorphism by the same proof.

Now we apply it to study the structure of $\text{Hom}_{\text{DM}^{eff}(k)}(M(X), \mathbb{Z}(1)[2])$ as follows.

**Proposition 3.3.** Let $X$ be an integral scheme smooth and separated over $k$, and let $p : X_\bullet \to X$ be an $h$-hypercover. Then the induced morphisms

$$\tau_{\leq 1}\text{Hom}_{\text{DM}^{eff}(k)}(M(X), \mathbb{Z}(1)[2]) \to \text{Hom}_{\text{DM}^{eff}(k)}(M(X), \mathbb{Z}(1)[2]),$$

$$\tau_{\leq 1}\text{Hom}_{\text{DM}^{eff}(k)}(M(X), \mathbb{Z}(1)[2]) \to \tau_{\leq 1}\text{Hom}_{\text{DM}^{eff}(k)}(M(X_\bullet), \mathbb{Z}(1)[2])$$

in $\text{DM}^{eff}(k)$ are isomorphisms.

**Proof.** Let $f : X \to k$ be the structure morphism. By [21] Proposition 3.2.8, we need to show that the induced morphisms

$$\tau_{\leq 1}Rf_*f^*G_m \to Rf_*f^*G_m,$$

$$\tau_{\leq 1}Rf_*f^*G_m \to \tau_{\leq 1}Rf_*Rp_*p^*f^*G_m$$

in $D(k_{\text{Nis}}, \mathbb{Z})$ are isomorphisms. In other words, we need to show that

1. $R^if_*f^*G_m = 0$ for $i \neq 0, 1$,
2. the induced morphism $R^if_*f^*G_m \to R^i(fp)_*(fp)^*G_m$ is an isomorphism for $i = 0, 1$.

Then it suffices to show that for any local henselian scheme $Y$ over $k$,

1. $H^i_{\text{Nis}}(Y \times X, G_m) = 0$ for $i \neq 0, 1$,
2. the induced homomorphism $H^i_{\text{Nis}}(Y \times X, G_m) \to H^i_{\text{Nis}}(Y \times X_\bullet, G_m)$ is an isomorphism for $i = 0, 1$.

By [15] Vanishing Theorem 19.3, $H^i_{\text{Nis}}(T, G_m) = 0$ for any $i \neq 0, 1$ and noetherian scheme $T$ smooth and separated over $k$. Then we get (1) since $H^i_{\text{Nis}}(-, G_m)$ commutes with filtered limits by [2, VI.8.7.7]. Hence the remaining is (2').

Using Hilbert’s Theorem 90, it suffices to show that the induced homomorphism

$$H^i_{\text{ét}}(Y \times X, G_m) \to H^i_{\text{ét}}(Y \times X_\bullet, G_m)$$
is an isomorphism for any \( i \geq 0 \). This follows from applying [2] VI.8.7.7 to Proposition 3.4 below.

**Proposition 3.4.** Let \( X \) be an integral scheme smooth and separated over \( k \), and let \( p : X_\bullet \to X \) be an \( h \)-hypercover. Then the induced homomorphism

\[
H^i_{\text{ét}}(X, G_m) \to H^i_{\text{ét}}(X_\bullet, G_m)
\]

of abelian groups is an isomorphism for any \( i \).

**Proof.** Let \( f : X \to k \) be the structure morphism, and let \( K \) be a cone of the induced morphism

\[
R_{\text{ét}}fp_*f^*G_m \to R_{\text{ét}}fp_\ast R_{\text{ét}}p_*p^*f^*G_m
\]

in the derived category \( D(k_{\text{ét}}, Z) \) of étale sheaves of abelian groups on \( k_{\text{ét}} \). We only need to show that

\[
\text{Hom}_{D(k_{\text{ét}}, Z)}(Z, K[i]) = 0
\]

for any \( i \in Z \) since the category of étale sheaves of abelian groups on \( k_{\text{ét}} \) is equivalent to the category of \( k \)-modules.

By [9] Theorems 16.1.3, 16.2.18, Corollary 16.2.22, any object of \( DM_{\text{ét}}(k, Q) \) satisfies \( h \)-descent. Thus by [10] Corollary 1.8.5,

\[
H^i_{\text{ét}}(X, G_m \otimes Q) \cong \text{Hom}_{DM_{\text{ét}}(k, Q)}(M(X), G_m[i])
\]

\[
\cong \text{Hom}_{DM_{\text{ét}}(k, Q)}(M(X_\bullet), G_m[i])
\]

\[
\cong H^i_{\text{ét}}(X_\bullet, G_m \otimes Q).
\]

Then we get

\[
\text{Hom}_{D(k_{\text{ét}}, Z)}(Z, K \otimes Q[i]) = 0
\]

for any \( i \in Z \). By [2] VI.5.3,

\[
\text{Hom}_{D(k_{\text{ét}}, Z)}(Z, K[i]) \otimes Q = 0
\]

for any \( i \in Z \). This means that \( \text{Hom}_{D(k_{\text{ét}}, Z)}(Z, K[i]) \) is a torsion abelian group. Then it remains to show that

\[
\text{Hom}_{D(k_{\text{ét}}, Z)}(Z, K[i]) \otimes Z/n = 0
\]

for any \( i \in Z \) and prime power \( n \).

We have the exact sequences

\[
0 \to G_m^\pm \to G_m \to G_m/G_m^n \to 0 \quad \text{if } \text{char } k \mid n,
\]

\[
0 \to \mu_n \to G_m^\pm \to G_m \to 0 \quad \text{if } \text{char } k \nmid n
\]

of étale sheaves. Set \( F := G_m/G_m^n \) if \( \text{char } k \mid n \) and \( F := \mu_n \) if \( \text{char } k \nmid n \). Then \( F \) is an étale sheaf of \( Z/n \)-modules. In the proof of [10] Proposition 5.3.3, it is shown that any \( h \)-cover is universally of cohomological descent with respect to the fibered category of étale sheaves of \( Z/n \)-modules. By [11] Theorem 7.10, any \( h \)-hypercover is universally of cohomological descent. This implies that the induced homomorphism

\[
H^i_{\text{ét}}(X, F) \to H^i_{\text{ét}}(X_\bullet, F)
\]

of abelian groups is an isomorphism for any \( i \in Z \). Thus we have that

\[
\text{Hom}_{D(k_{\text{ét}}, Z)}(Z, K \otimes Z/n[i]) = 0
\]

for any \( i \in Z \), which implies that \( \text{Hom}_{D(k_{\text{ét}}, Z)}(Z, K[i]) \otimes Z/n = 0 \) for any \( i \in Z \). □
4. 0-motivic sheaves

Definition 4.1. A constant sheaf on $\text{Sm}/k$ is called a 0-motivic sheaf. We denote by $\text{Sh}^{0-\text{mot}}(k)$ the category of 0-motivic sheaves.

Remark 4.2. The definition of 0-motivic sheaves is simpler than that in [5] since we assume that $k$ is algebraically closed.

4.3. Recall from [5, Corollary 1.2.5] that the inclusion functor $\text{Sh}^{0-\text{mot}}(k) \rightarrow \text{Sh}^{\text{tr}}(k)$ admits a left adjoint denote by $\pi_0: \text{Sh}^{\text{tr}}(k) \rightarrow \text{Sh}^{0-\text{mot}}(k)$.

Note that $\pi_0(\mathbb{Z}^{\text{tr}}(X)) \cong \mathbb{Z}$ if $X$ has $r$ connected components. Thus $\pi_0(\mathbb{Z}^{\text{tr}}_{\geq 1}(X)) = 0$.

Proposition 4.4. Let $X$ be a noetherian scheme smooth and separated over $k$, and let $F$ be a constant sheaf on $\text{Sm}/k$. Then

$$\text{Hom}_{\text{DM}}(M_{\geq 1}(X), F) = 0.$$ 

Proof. We may assume that $X$ is integral. For any $i \in \mathbb{Z}$ and noetherian scheme $T$ smooth and separated over $k$, it suffices to show that the induced homomorphism

$$\text{Hom}_{\text{DM}}(M(T), F[i]) \rightarrow \text{Hom}_{\text{DM}}(M(T \times X), F[i])$$

is an isomorphism. Hence we just need to prove that the induced homomorphism

$$(4.4.1) \quad H^i_{\text{Nis}}(k, F) \rightarrow H^i_{\text{Nis}}(X, F)$$

is an isomorphism.

Assume that $F = \mathbb{Z}$. Then by [15, Vanishing Theorem 19.3], $H^i_{\text{Nis}}(X, F) = 0$ for $i > 0$. For $i = 0$, (4.4.1) is an isomorphism since $\mathbb{Z}$ is constant. Thus we are done for this case. We also have that (4.4.1) is an isomorphism if $F$ is a lattice.

Now we treat the general case. Since $F$ is a colimit of lattices, there is an exact sequences

$$0 \rightarrow F'' \rightarrow F' \rightarrow F \rightarrow 0$$

of constant sheaves on $\text{Sm}/k$ such that $F'$ and $F''$ are direct sums of lattices. Then $F'$ and $F''$ are filtered colimit of lattices. Since $H^i_{\text{Nis}}(X, -)$ commutes with filtered colimits by [2, VI.5.3], the induced homomorphisms

$$H^i_{\text{Nis}}(k, F') \rightarrow H^i_{\text{Nis}}(X, F'), \quad H^i_{\text{Nis}}(k, F'') \rightarrow H^i_{\text{Nis}}(X, F'')$$

are isomorphisms by the above paragraph. Then (4.4.1) is an isomorphism by the five lemma. □

Proposition 4.5. Let $X$ be a noetherian scheme smooth and separated over $k$. Then

$$\pi_0(\text{Pic}_{X/k}) \cong \text{NS}(X).$$

Proof. We have the exact sequence

$$0 \rightarrow \text{Pic}_{X/k} \rightarrow \text{Pic}_{X/k} \rightarrow \text{NS}(X) \rightarrow 0$$

of Nisnevich sheaves with transfers. Since $\pi_0$ is the left adjoint of the inclusion functor, $\pi_0$ is right exact. Thus we have the induced exact sequence

$$\pi_0(\text{Pic}_{X/k}^0) \rightarrow \pi_0(\text{Pic}_{X/k}) \rightarrow \pi_0(\text{NS}(X)) \rightarrow 0.$$
Since NS(X) is constant, \( \pi_0(\text{NS}(X)) \cong \text{NS}(X) \). Hence it suffices to show that \( \pi_0(\text{Pic}^0_{X/k}) = 0 \).

Since \( \pi_0(\text{Pic}^0_{X/k}) \) is constant, we only need to show that \( \pi_0(\text{Pic}^0_{X/k})(k) = 0 \). Let \( a \) be its element, and let \( Z \in \text{Pic}^0_{X/k}(k) = \text{Pic}^0(X) \) be an element whose image in \( \pi_0(\text{Pic}^0_{X/k})(k) \) is \( a \). Then by definition, \( Z \) is algebraically equivalent to 0. Thus there is a curve \( C \) smooth and separated over \( k \) and an element \( W \) of \( \text{Pic}^0_{X/k}(k) \) such that the pullback of \( W \) to closed points \( x_1 \) and \( x_2 \) is \( Z \) and 0 respectively. Then \( W \) gives a morphism \( Z^{tr}(C) \rightarrow \text{Pic}^0_{X/k} \) of Nisnevich sheaves with transfers. The composition \( Z \rightarrow Z^{tr}(C) \rightarrow \text{Pic}^0_{X/k} \) is zero where the first arrow is induced by \( x_2 \), so this induces a morphism

\[
Z^{tr}_{\geq 1}(C) \rightarrow \text{Pic}^0_{X/k}
\]

of Nisnevich sheaves with transfers. Note that \( Z \) is in its image of \( Z^{tr}_{\geq 1}(C)(k) \rightarrow \text{Pic}^0_{X/k}(k) \). Taking \( \pi_0 \), we get the homomorphism

\[
\pi_0(Z^{tr}_{\geq 1}(C))(k) \rightarrow \pi_0(\text{Pic}^0_{X/k})(k).
\]

Then \( a \) is in its image. By Proposition 4.3, \( \pi_0(Z^{tr}_{\geq 1}(C)) = 0 \). Thus \( a = 0 \).

**Proposition 4.6.** Let \( G \) be a semi-abelian variety, and consider the functor \( \pi_0 \) in Proposition 4.3. Then \( \pi_0(G) = 0 \).

**Proof.** Consider an exact sequence

\[
0 \rightarrow L \otimes G_m \rightarrow G \rightarrow A \rightarrow 0
\]

of Nisnevich sheaves with transfers where \( L \) is a lattice and \( A \) is an abelian variety. Since \( \pi_0 \) is the left adjoint of the inclusion functor, it is right exact. Thus we have the exact sequence

\[
\pi_0(L \otimes G_m) \rightarrow \pi_0(G) \rightarrow \pi_0(A) \rightarrow 0
\]

of Nisnevich sheaves with transfers. Since \( G_m \cong Z^{tr}_{\geq 1}(G_m) \), \( \pi_0(G_m) = 0 \) by 4.3. By Proposition 4.3,

\[
\pi_0(A) \cong \pi_0(\text{Pic}^0_{A^\vee/k}) = 0.
\]

Thus we have that \( \pi_0(G) = 0 \) from the above exact sequence.

**Proposition 4.7.** Let \( G \) be a semi-abelian variety over \( k \). Then

\[
\text{Hom}_{\text{DM}^{rig}(k)}(G, Z[i]) = 0
\]

for \( i \leq 1 \).

**Proof.** Since \( G \) and \( Z \) are in the heart of the 0-motivic \( t \)-structure, we are done when \( i < 0 \). For \( i = 0 \), we are done by Proposition 4.6. Hence the remaining case is when \( i = 1 \).

By [8, Theorem 2] (and see also the computation \( H_{n+1}(G, n; Z) = 0 \) in [8, p. 26]), we have the exact sequence

\[
Z[G \times G \times G] \oplus Z[G \times G] \xrightarrow{u} Z[G \times G] \xrightarrow{v} Z[G] \rightarrow G \rightarrow 0
\]

of Nisnevich sheaves (without transfers). Here, \( u \) and \( v \) are given by

\[
u([x, y]) = [x] + [y] - [x + y],
\]

\[
v([x, y, z], [p, q]) = [y, z] - [x + y, z] + [x, y + z] - [x, y] + [p, q] - [q, p].
\]
Let Sh$_{Nis}(Sm/k)$ denote the category of Nisnevich sheaves on $Sm/k$. The induced sequence

\[
\begin{align*}
\text{Hom}_{Sh_{Nis}(Sm/k)}(Z[G], Z) &\to \text{Hom}_{Sh_{Nis}(Sm/k)}(Z[G \times G], Z) \\
&\to \text{Hom}_{Sh_{Nis}(Sm/k)}(Z[G \times G \times G] \oplus Z[G \times G], Z)
\end{align*}
\]

(4.7.2)

is isomorphic to the induced sequence

\[
\begin{align*}
\text{Z}[\text{Hom}_{Sm/k}(G, \text{Spec } k)] &\to \text{Z}[\text{Hom}_{Sm/k}(G \times G, \text{Spec } k)] \\
&\to \text{Z}[\text{Hom}_{Sm/k}(G \times G \times G, \text{Spec } k) \oplus \text{Z}[\text{Hom}_{Sm/k}(G \times G, \text{Spec } k)],
\end{align*}
\]

which is isomorphic to

\[
\begin{align*}
\text{Z} &\xrightarrow{id} \text{Z} \\
&\to \text{Z} \oplus \text{Z}.
\end{align*}
\]

Note that this is an exact sequence. For any scheme $X$ over $k$ and $i > 0$,

\[
\text{Ext}^i_{Sh_{Nis}(Sm/k)}(Z[X], Z) \cong H^i_{Nis}(X, Z) = 0.
\]

Thus by (4.7.1) and the exactness of (4.7.2), we have that

\[
\text{Ext}^1_{Sh_{Nis}(Sm/k)}(G, Z) = 0.
\]

Let

\[
0 \to Z \to F \to G \to 0
\]

be an exact sequence of Nisnevich sheaves with transfers. The above paragraph shows that $F \cong Z \oplus G$ in the category of Nisnevich sheaves. Then by Proposition 2.12, $F \cong Z \oplus G$ in the category of Nisnevich sheaves with transfers. Thus

\[
\text{Ext}^1_{Sh^{tr}(k)}(G, Z) = 0.
\]

Then we are done since $Z$ is $A^1$-local. \qed

5. Structure of $M_1(X)^{\vee}$

5.1. Now we begin the proof that $M_1(X)^{\vee}[-1]$ is isomorphic to a complex $[L \to A]$ where $L$ is a lattice and $A$ is an abelian variety over $k$. The strategy is that we will use de Jong alterations in Proposition 5.4 to find the structure of $\text{Hom}_{DM^{eff}(k)}(M_{\geq 1}(X), Z(1)[2])$. Here, we will need Proposition 3.3 since $h$-hypercovers appear in de Jong alterations. Combining with Proposition 4.5, we will get Theorem 5.6.

Definition 5.2. Let $X$ be a noetherian scheme smooth and separated over $k$, and let $U$ be an open subscheme of $X$. Set

\[
M(X/U) := C_*(Z^{tr}(X)/Z^{tr}(U)),
\]

and consider it as an object of $DM^{eff}(k)$.

Proposition 5.3. Let $X$ be a noetherian scheme smooth and separated over $k$, and let $U$ be an open subscheme of $X$ whose complement $Z$ is a strict normal crossing divisor. Then

\[
\text{Hom}_{DM^{eff}(k)}(M(X/U), Z(1)[2]) \cong \mathbb{Z}^d
\]

where $d$ is the number of irreducible components of $Z$. 
Proof. We can choose open immersions
\[ U = U_0 \to U_1 \to \cdots \to U_d = X \]
such that for each \( i \), \( U_i \) is the complement of a divisor \( Z_i \) in \( U_{i+1} \) such that \( Z_i \) is smooth and separated over \( k \). Then by [15, Theorem 15.15],
\[ M(U_{i+1}/U_i) \cong M(Z_i)(1)[2]. \]
Thus
\[
\text{Hom} \_{\text{DM}^\text{eff}(k)}(M(U_{i+1}/U_i), \mathbb{Z}(1)[2]) \cong \text{Hom} \_{\text{DM}^\text{eff}(k)}(M(Z_i)(1)[2], \mathbb{Z}(1)[2]) \\
\cong \text{Hom} \_{\text{DM}^\text{eff}(k)}(M(Z_i), \mathbb{Z}) \\
\cong \mathbb{Z}.
\]
Here, the second isomorphism is given by the cancellation theorem ([22]), and the third isomorphism is given by Proposition [4.4].

Let us show that
\[(5.3.1) \quad \text{Hom} \_{\text{DM}^\text{eff}(k)}(M(U_i/U), \mathbb{Z}(1)[2]) \cong \mathbb{Z}^i \]
by induction on \( i \). It is trivial when \( i = 0 \). Consider the distinguished triangle
\[ M(U_i/U) \to M(U_{i+1}/U) \to M(U_{i+1}/U_i) \to M(U_i/U)[1] \]
in \( \text{DM}^\text{eff}(k) \). We have shown that \( \text{Hom} \_{\text{DM}^\text{eff}(k)}(M(U_{i+1}/U_i), \mathbb{Z}(1)[2]) \cong \mathbb{Z} \). Assume (5.3.1) for \( i \). Since \( \text{Hom} \_{\text{DM}^\text{eff}(k)}(\mathbb{Z}, \mathbb{Z}')(1)[1]) = 0 \), the distinguished triangle split. Then we get
\[ \text{Hom} \_{\text{DM}^\text{eff}(k)}(M(U_{i+1}/U_i), \mathbb{Z}(1)[2]) \cong \mathbb{Z}^{i+1}.
\]
This completes the induction argument. Then we are done if we set \( i = d \). □

Proposition 5.4. Let \( X \) be an integral scheme smooth and separated over \( k \). Then in \( \text{DM}^\text{eff}(k) \),
\[
\text{Hom} \_{\text{DM}^\text{eff}(k)}(M_{\geq 1}(X), \mathbb{Z}(1)[2])[-1]
\]
is isomorphic to a complex \([L \to B]\) where
(i) \( L \) is a lattice,
(ii) \( \pi_0(B) \) is finitely generated,
(iii) \( \ker(B \to \pi_0(B)) \) is an abelian variety.

Proof. By [12 §1], we can choose a diagram
\[
\begin{array}{ccc}
X & \overset{j}{\to} & \overline{X} \\
\downarrow p & & \downarrow a \\
& \leftarrow D & \\
\end{array}
\]
of simplicial schemes over \( k \) where
(i) each \( \overline{X}_i \) is proper and smooth and separated over \( k \),
(ii) \( p \) is an \( h \)-hypercover,
(iii) each \( D_i \) is a strict normal crossing divisor in \( \overline{X}_i \) with complement \( X_i \).

Consider the distinguished triangle
\[ M(X_i) \to M(\overline{X}_i) \to M(\overline{X}_i/X_i) \to M(X_i)[1] \]
in \( \text{DM}^\text{eff}(k) \). Taking out \( M_0(X_i) \cong M_0(\overline{X}_i) \cong \mathbb{Z} \), we get the distinguished triangle
\[ M_{\geq 1}(X_i) \to M_{\geq 1}(\overline{X}_i) \to M(\overline{X}_i/X_i) \to M(X_i)[1] \]
in $\text{DM}^{\text{eff}}(k)$. By Propositions 5.4 and 5.3

$$\text{Hom}_{\text{DM}^{\text{eff}}(k)}(M_{\geq 1}(\varpi_i), \mathbf{Z}(1)[2]) \cong \text{Pic} \varpi_i/k,$$

$$\text{Hom}_{\text{DM}^{\text{eff}}(k)}(M(\varpi_i/X_i), \mathbf{Z}(1)[2]) \cong \mathbf{Z}^{d_i}$$

where $d_i$ is the number of irreducible components of $D_i$. Thus in $\text{DM}^{\text{eff}}(k)$,

$$\text{Hom}_{\text{DM}^{\text{eff}}(k)}(M_{\geq 1}(X), \mathbf{Z}(1)[2])[-1]$$

is isomorphic to the total complex of the double complex

$$\mathbf{Z}^{d_i} \to \text{Pic} \varpi_i/k$$

where $\mathbf{Z}^{d_i}$ sits in degree 0.

By Proposition 3.3

$$\text{Hom}_{\text{DM}^{\text{eff}}(k)}(M_{\geq 1}(X), \mathbf{Z}(1)[2])[-1] \cong \langle \tau_{\leq 1} \text{Hom}_{\text{DM}^{\text{eff}}(k)}(M_{\geq 1}(X_*), \mathbf{Z}(1)[2])[-1].$$

From the description of the above double complex, we see that the latter is isomorphic to

$$[\mathbf{Z}^{d_0} \to \ker((\mathbf{Z}^{d_1} \oplus \text{Pic} \varpi_0/k) \to (\mathbf{Z}^{d_2} \oplus \text{Pic} \varpi_1/k)].$$

Set $B = \ker((\mathbf{Z}^{d_1} \oplus \text{Pic} \varpi_0/k) \to (\mathbf{Z}^{d_2} \oplus \text{Pic} \varpi_1/k))$. It remains to show (ii) and (iii).

We can consider $\mathbf{Z}^{d_i} \oplus \text{Pic} \varpi_0/k$ as a $d_i$-copies of the abelian variety $\text{Pic} \varpi_0/k$. Thus $\mathbf{Z}^{d_i} \oplus \text{Pic} \varpi_0/k$ is an abelian group scheme over $k$. The same is true for $\mathbf{Z}^{d_2} \oplus \text{Pic} \varpi_1/k$.

Then by Lemma 5.5 below, the kernel $B$ is also an abelian group scheme over $k$. Then we get (iii). By Proposition 1.5 $\pi_0(\mathbf{Z}^{d_1} \oplus \text{Pic} \varpi_0/k) \cong \mathbf{Z}^{d_1} \oplus \text{NS}(X)$. Then we get (ii) since $\pi_0(B)$ is a subsheaf of a finitely generated constant sheaf $\mathbf{Z}^{d_1} \oplus \text{NS}(X)$. \hfill $\square$

**Lemma 5.5.** Let $0 \to F \to A \xrightarrow{f} B$ be an exact sequence of Nisnevich sheaves with transfers, and assume that $A$ and $B$ are abelian group schemes over $k$. Then $F$ is an abelian group scheme over $k$.

**Proof.** Let $T$ be a noetherian scheme smooth and separated over $k$, and let $i : C \to A$ be the kernel of $A \to B$ computed in the category of commutative group schemes over $k$. It suffices to show that the induced homomorphism $\alpha : C(T) \to F(T)$ of abelian groups is an isomorphism. Let $g : T \to A$ be a morphism of schemes over $k$ such that $fg = 0$. Then $f$ factors through $C$, so $\alpha$ is surjective.

Let $h : T \to C$ be a morphism of schemes over $k$ such that $\alpha(h) = 0$. Then $ih = 0$. This means that $ih$ factors through the zero in $A$, so $h$ factors through the zero in $C$. Thus $h = 0$, so $\alpha$ is injective. \hfill $\square$

**Theorem 5.6.** Let $X$ be a scheme smooth and separated over $k$. Then there is an abelian variety $A$ over $k$ and a lattice $L$ such that $M_1(X)^{\vee}[-1]$ is isomorphic to the complex

$$[L \to A]$$

in $\text{DM}^{\text{eff}}(k)$.

**Proof.** By Proposition 5.4 $\text{Hom}_{\text{DM}^{\text{eff}}(k)}(M_{\geq 1}(X), \mathbf{Z}(1)[2])$ is isomorphic to a complex $[L \to B]$ in $\text{DM}^{\text{eff}}(k)$ satisfying the conditions (i)-(iii) in Proposition 5.4. Since

$$h_0(\text{Hom}_{\text{DM}^{\text{eff}}(k)}(M_{\geq 1}(X), \mathbf{Z}(1)[2])) \cong \text{Pic} X/k$$
by Proposition 2.8, we have the exact sequence

$$L \to B \to \text{Pic}_{X/k} \to 0$$

of Nisnevich sheaves with transfers. The functor $\pi_0$ is right exact since $\pi_0$ is left adjoint to the inclusion functor. Thus by Proposition 4.5, we have the exact sequence

$$(5.6.1) \quad L \to \pi_0(B) \to \text{NS}(X) \to 0$$

of Nisnevich sheaves with transfers. By (2.10.1), $M_1(X)[\cdot - 1]$ is isomorphic to the total complex of the double complex $L B$ in $\text{DM}^\text{eff}(k)$, which is isomorphic to the complex $[\ker(L \to \pi_0(B)) \to \ker(B \to \pi_0(B))]$ in $\text{DM}^\text{eff}(k)$ since the sequence (5.6.1) is exact. By the condition (iii) in Proposition 5.4, $\ker(B \to \pi_0(B))$ is an abelian variety. Since $\ker(L \to \pi_0(B))$ is a subsheaf of $L$, $\ker(L \to \pi_0(B))$ is a lattice. Thus we are done. □

6. Cartier duality

6.1. One of the purposes of this section is to show that

$$h_0(\text{Hom}_{\text{DM}^\text{eff}(k)}(A, Z(1)[2])) \cong A^\vee$$

for any abelian variety $A$ over $k$. Here, $A^\vee$ denotes the dual abelian variety of $A$. In [6], for the étale topology with $Z[1/p]$-coefficient, this is reduced to showing that

$$\text{Hom}_{\text{DM}^\text{eff}(k, Z[1/p])}(M(C), Z[1/p](1)[2]) \cong M(C)$$

for any curve $C$ smooth and projective over $k$. This is possible since $A$ is a direct summand of $\text{Alb}(C)$ up to isogeny for some curve $C$, and the étale topology can deal with such a situation well. However, for the Nisnevich topology, it is not clear how to get $A$ from $M(C)$. Hence we need to take an alternative way.

Our strategy is to show directly that

$$\text{Hom}_{\text{DM}^\text{eff}(k)}(M(X), \text{Hom}_{\text{DM}^\text{eff}(k)}(A, Z(1)[2])) \cong \text{Hom}_{\text{DM}^\text{eff}(k)}(M(X), A^\vee)$$

for any noetherian scheme $X$ smooth and separated over $k$. In the proof, the results in the previous sections are used.

We first collect several results on homomorphisms and extensions of semi-abelian group schemes.

**Proposition 6.2.** Let $G$ be a semi-abelian variety over $k$. Then $G$ is a homotopy invariant Nisnevich sheaf with transfers.

**Proof.** By [17] Lemme 3.2.1, $G$ has a transfer structure. Hence the remaining is to show that $G$ is homotopy invariant. By [17] Lemme 3.3.1, it is proven when $G$ is an abelian variety or $G_m$. Then we are done by the five lemma. □
Proposition 6.3. Let \( A \) and \( B \) be semi-abelian varieties over \( k \). Then
\[
\hom_{sAV/k}(A, B) \cong \hom_{DM^{eff}(k)}(A, B).
\]
Here, \( sAV/k \) denotes the category of semi-abelian varieties over \( k \) where morphisms are homomorphisms of semi-abelian varieties over \( k \).

Proof. By Proposition 6.2
\[
\hom_{DM^{eff}(k)}(A, B) \cong \hom_{Sh^{tr}(k)}(A, B).
\]
Let \( f : A \to B \) a morphism of Nisnevich sheaves with transfers. We only need to show that \( f \) is a homomorphism of semi-abelian varieties. Since \( f \) is a morphism of abelian sheaves, the diagram
\[
\begin{array}{ccc}
A(T) \times A(T) & \longrightarrow & A(T) \\
\downarrow & & \downarrow \\
B(T) \times B(T) & \longrightarrow & B(T)
\end{array}
\]
of sets commutes for any noetherian scheme \( T \) smooth and separated over \( k \) where the horizontal arrows are the multiplication functions. This means that the diagram
\[
\begin{array}{ccc}
A \times A & \longrightarrow & A \\
\downarrow & & \downarrow \\
B \times B & \longrightarrow & B
\end{array}
\]
of schemes over \( k \) commutes where the horizontal arrows are the multiplication morphisms. Thus \( f \) preserves the multiplication structure. Similarly, \( f \) preserves the identity. Thus \( f \) is a homomorphism of semi-abelian varieties. \( \square \)

Proposition 6.4. Let \( A \) be an abelian variety. Then
\[
\text{Ext}^1_{sAV/k}(A, \mathbb{G}_m) \cong \text{Hom}_{DM^{eff}(k)}(A, \mathbb{Z}(1)[2])
\]

Proof. An element of \( \text{Ext}^1_{sAV/k}(A, \mathbb{G}_m) \) is given by an exact sequence
\[
0 \to \mathbb{G}_m \to F \xrightarrow{p} A \to 0
\]
of semi-abelian varieties the modulo equivalence relation. Note that \( G \) is a Nisnevich sheaf with transfers.

Since \( G \) is a \( \mathbb{G}_m \)-torsor on \( A \), it comes from an element of \( H^1_{\text{pro}}(A, \mathbb{G}_m) \). By Hilbert’s Theorem 90, \( H^1_{\text{pro}}(A, \mathbb{G}_m) \cong \text{Pic}(A) \). From the description of this isomorphism, we see that there is a line bundle \( V \) on \( A \) such that \( G \cong V - V_0 \) where \( V_0 \) is the zero section of \( V \). Since \( V \) is Zariski locally a trivial line bundle, \( G \) is Zariski locally a trivial \( \mathbb{G}_m \)-torsor. This means that \( p \) is surjective in the category of Zariski sheaves. In particular, the sequence \( (6.4.1) \) is exact in the category of Nisnevich sheaves with transfers. Thus we can consider \( \text{Ext}^1_{sAV/k}(A, \mathbb{G}_m) \) as a subset of \( \text{Ext}^1_{Sh^{tr}(k)}(A, \mathbb{G}_m) \).

Let
\[
0 \to \mathbb{G}_m \to F \to A \to 0
\]
be an exact sequence of Nisnevich sheaves with transfers. Then \( F \) is representable by \( [16\text{ Proposition 17.4}] \), so \( F \) is isomorphic to a semi-abelian variety over \( k \) in the category of Nisnevich sheaves. Then \( F \) is isomorphic to a semi-abelian variety over
$k$ in the category of Nisnevich sheaves with transfers by Proposition 2.12. This establishes that
\[
\text{Ext}^1_{sAV/k}(A, G_m) \cong \text{Ext}^1_{\text{Sh}^{tr}(k)}(A, G_m).
\]
Since $G_m$ is $\mathbb{A}^1$-local,
\[
\text{Hom}_{\text{DM}^{eff}(k)}(A, Z(1)[2]) \cong \text{Hom}_{\text{DM}^{eff}(k)}(A, G_m[1])
\cong \text{Hom}_{\text{Sh}^{tr}(k)}(A, G_m)[1] \cong \text{Ext}^1_{\text{Sh}^{tr}(k)}(A, G_m).
\]
Then we get the conclusion. \(\square\)

**Proposition 6.5.** Let $G$ be a semi-abelian variety over $k$. Then
\[
\text{Hom}_{\text{DM}^{eff}(k)}(G_m, G)
\]
is constant.

**Proof.** Consider a distinguished triangle
\[
L \otimes G_m \to G \to A \to L \otimes G_m[1]
\]
in $\text{DM}^{eff}(k)$ where $L$ is a lattice and $A$ is an abelian variety over $k$. By the cancellation theorem ([22]),
\[
\text{Hom}_{\text{DM}^{eff}(k)}(G_m, L \otimes G_m) \cong L.
\]
Thus it suffices to show that $\text{Hom}_{\text{DM}^{eff}(k)}(G_m, A) = 0$. It is the contraction $A_{-1}$ of $A$ given in [13 §23]. Thus it suffices to show that any morphism $f : X \times G_m \to A$ factors through the open immersion $j : X \times G_m = X \times (\mathbb{A}^1 - \{0\}) \to X \times \mathbb{A}^1$.

Let $x$ be a rational point of $X$. Then the composition $\{x\} \times G_m \to X \times G_m \to A$ is constant since $H^0_{\text{Nis}}(A, G_m) = k^*$. This means that the composition
\[
X \times G_m \xrightarrow{p} X \xrightarrow{i_1} X \times G_m \xrightarrow{f} A
\]
is equal to $f$. Here, $p$ denotes the projection, and $i_1$ denotes the morphism induced by the 1-section $\text{Spec} k \to G_m$. In particular, $f$ factors through $p$. Then $f$ factors through $j$. \(\square\)

**Proposition 6.6.** Let $A$ be an abelian variety over $k$. Then
\[
\text{Hom}_{\text{DM}^{eff}(k)}(A[i], Z(1)[2]) = 0
\]
for any $i > 0$.

**Proof.** If $i > 1$, then $A[i-2]$ is $t$-positive for the 0-motivic structure, and $Z(1)[1] \cong G_m$ is $t$-negative for the 0-motivic structure. Thus we are done for this case.

If $i = 1$, then
\[
\text{Hom}_{\text{DM}^{eff}(k)}(A[1], Z(1)[2]) \cong \text{Hom}_{sAV/k}(A, G_m)
\]
by Proposition 6.3. Since $A$ is projective,
\[
\text{Hom}_{\text{Sch}/k}(A, G_m) \cong H^0_{\text{Nis}}(A, G_m) \cong k^*.
\]
Thus any morphism $f : A \to G_m$ of schemes is a constant morphism. If $f$ is a homomorphism of semi-abelian varieties over $k$, $f$ should be the zero morphism. This shows that $\text{Hom}_{sAV/k}(A, G_m) = 0$. \(\square\)

**Lemma 6.7.** Let $A$ be an abelian variety over $k$. Then $\text{Hom}_{\text{DM}^{eff}(k)}(A, Z(1)[2])$ is $t$-negative for the 0-motivic $t$-structure.
Proof. We need to show that
\[ \text{Hom}_{DM^{sf}(k)}(M(X)[i], \text{Hom}_{DM^{sf}(k)}(A, Z(1)[2])) = 0 \]
for any \( i > 0 \) and integral scheme \( X \) smooth and separated over \( k \). Using the distinguished triangle
\[ M_{\geq 1}(X) \to M(X) \to Z \to M_{\geq 1}(X)[1] \]
in \( DM^{sf}(k) \), we only need to show that
\[ \text{Hom}_{DM^{sf}(k)}(Z[i], \text{Hom}_{DM^{sf}(k)}(A, Z(1)[2])) = 0, \]
(6.7.1) \[ \text{Hom}_{DM^{sf}(k)}(M_{\geq 1}(X)[i], \text{Hom}_{DM^{sf}(k)}(A, Z(1)[2])) = 0 \]
for any \( i > 0 \). The first one holds by Proposition 6.6 so the remaining is the second.

We have that
\[ \text{Hom}_{DM^{sf}(k)}(M_{\geq 1}(X)[i], \text{Hom}_{DM^{sf}(k)}(A, Z(1)[2])) \]
\[ \cong \text{Hom}_{DM^{sf}(k)}(A[i], \text{Hom}_{DM^{sf}(k)}(M_{\geq 1}(X), Z(1)[2])). \]

Consider the distinguished triangle
\[ M_1(X)^{\vee} \to \text{Hom}_{DM^{sf}(k)}(M_{\geq 1}(X), Z(1)[2]) \to \text{NS}(X) \to M_1(X)^{\vee}[1] \]
in \( DM^{sf}(k) \). Since \( \text{NS}(X) \) is constant,
\[ \text{Hom}_{DM^{sf}(k)}(A[i], \text{NS}(X)) = 0, \]
for any \( i > 0 \) by Proposition 1.7. Hence it suffices to show that
\[ \text{Hom}_{DM^{sf}(k)}(A[i], M_1(X)^{\vee}) = 0 \]
for any \( i > 0 \). Then by Theorem 5.6 it suffices to show that
\[ \text{Hom}_{DM^{sf}(k)}(A[i], Z(1)) = 0, \quad \text{Hom}_{DM^{sf}(k)}(A[i], B) = 0 \]
for any \( i > 0 \) and abelian variety \( B \) over \( k \). By Proposition 4.7 the first one holds. The second one holds since \( A \) and \( B \) are in the heart of the 0-motivic \( t \)-structure. \( \square \)

6.8. Let \( X \) be a noetherian scheme smooth and separated over \( k \). If we fix a point \( x_i \) for each connected component of \( X \), then we have the Albanese morphism
\[ \text{Alb} : X \to \text{Alb}(X) \]
mapping each \( x_i \) to 0. It is universal among morphisms \( X \to G \) to semi-abelian varieties over \( k \) such that each \( x_i \) maps to 0.

The base point free version is the Albanese morphism \( \text{Alb} : Z_{\geq 1}(X) \to \text{Alb}(X) \).
It is universal among morphisms \( Z_{\geq 1}(X) \to G \) to semi-abelian varieties over \( k \).

6.9. Let \( A \) be an abelian variety over \( k \). Consider the Albanese morphism \( \text{Alb} : M_{\geq 1}(A) \to \text{Alb}(A) \cong A \), and consider its dual
\[ q : \text{Hom}_{DM^{sf}(k)}(A, Z(1)[2]) \to \text{Hom}_{DM^{sf}(k)}(M_{\geq 1}(A), Z(1)[2]). \]
Since \( \text{Hom}_{DM^{sf}(k)}(M_{\geq 1}(A), Z(1)[2]) \cong \text{Pic}_{A/k} \) by Proposition 2.4 we have the distinguished triangle
(6.9.1) \[ A^{\vee} \to \text{Hom}_{DM^{sf}(k)}(M_{\geq 1}(A), Z(1)[2]) \to \text{NS}(A) \to A^{\vee}[1] \]
in \( DM^{sf}(k) \).
Let us show that the composition $\Hom_{DM^{eff}(k)}(A, \mathbb{Z}(1)[2]) \to \NS(A)$ is 0. It suffices to show that the homomorphism $\Hom_{DM^{eff}(k)}(M(X)[i], \Hom_{DM^{eff}(k)}(A, \mathbb{Z}(1)[2])) \to \Hom_{DM^{eff}(k)}(M(X)[i], \NS(A))$ of abelian groups is 0 for any $i \in \mathbb{Z}$ and noetherian scheme $X$ smooth and separated over $k$. We only need to consider the case when $X = \Spec k$ and $i = 0$ since $\NS(A)$ is a constant sheaf. Thus by Proposition 6.4, it suffices to show that the homomorphism

$$\Ext^1_{AV/k}(A, \mathbb{G}_m) \to \NS(A)$$

of abelian groups is 0. This follows from [17 Proposition 17.6]. Thus from (6.9.1), we see that $q$ has a factorization

$$\Hom_{DM^{eff}(k)}(A, \mathbb{Z}(1)[2]) \xrightarrow{\eta_{AV}} A^\vee \to \Hom_{DM^{eff}(k)}(M_{\geq 1}(A), \mathbb{Z}(1)[2]).$$

**Proposition 6.10.** Let $A$ be an abelian variety over $k$, and let $X$ be a noetherian scheme smooth and separated over $k$. Then the homomorphism

$$\Hom_{DM^{eff}(k)}(M_{\geq 1}(X), \Hom_{DM^{eff}(k)}(A, \mathbb{Z}(1)[2])) \to \Hom_{DM^{eff}(k)}(M_{\geq 1}(X), A^\vee)$$

of abelian groups induced by $\eta_{AV}$ is an isomorphism.

**Proof.** By Proposition 6.4

$$\Hom_{DM^{eff}(k)}(M_{\geq 1}(X)[i], \NS(A)) = 0$$

for any $i \in \mathbb{Z}$. Then using the distinguished triangle (6.9.1), we have that

$$\Hom_{DM^{eff}(k)}(M_{\geq 1}(X), \Hom_{DM^{eff}(k)}(M_{\geq 1}(A), \mathbb{Z}(1)[2])) \cong \Hom_{DM^{eff}(k)}(M_{\geq 1}(X), A^\vee).$$

Thus it suffices to show that the induced homomorphism

$$\Hom_{DM^{eff}(k)}(M_{\geq 1}(X), \Hom_{DM^{eff}(k)}(A, \mathbb{Z}(1)[2])) \to \Hom_{DM^{eff}(k)}(M_{\geq 1}(X), \Hom_{DM^{eff}(k)}(M_{\geq 1}(A), \mathbb{Z}(1)[2]))$$

of abelian groups is an isomorphism. Then it suffices to show that the induced homomorphism

$$\Hom_{DM^{eff}(k)}(A, \Hom_{DM^{eff}(k)}(M_{\geq 1}(X), \mathbb{Z}(1)[2])) \to \Hom_{DM^{eff}(k)}(M_{\geq 1}(A), \Hom_{DM^{eff}(k)}(M_{\geq 1}(X), \mathbb{Z}(1)[2]))$$

of abelian groups is an isomorphism. By Propositions 6.4 and 6.7, we have that

$$\Hom_{DM^{eff}(k)}(M_{\geq 1}(A), \NS(X)[i]) = 0, \quad \Hom_{DM^{eff}(k)}(A, \NS(X)[i]) = 0$$

for any $i \leq 1$. Hence using the distinguished triangle (2.10.1), it suffices to show that the induced homomorphism

$$\Hom_{DM^{eff}(k)}(A, M_1(X)^\vee) \to \Hom_{DM^{eff}(k)}(M_{\geq 1}(A), M_1(X)^\vee)$$

of abelian groups is an isomorphism. By Theorem 5.6, there is a distinguished triangle

$$B \to M_1(X)^\vee \to L[1] \to B[1]$$

in $DM^{eff}(k)$ where $B$ is an abelian variety over $k$ and $L$ is a lattice. We have that

$$\Hom_{DM^{eff}(k)}(M_{\geq 1}(A), L[i]) = 0, \quad \Hom_{DM^{eff}(k)}(A, L[i]) = 0$$
for any \( i \leq 1 \) by Propositions [4.3] and [4.7]. Hence it suffices to show that the induced homomorphism
\[
\text{Hom}_{\text{DM}^{\text{eff}}(k)}(A, B) \to \text{Hom}_{\text{DM}^{\text{eff}}(k)}(M_{\geq 1}(A), B)
\]
of abelian groups is an isomorphism. This follows from the universality of the Albanese morphism \( M_{\geq 1}(A) \to A \). □

**Proposition 6.11.** Let \( A \) be an abelian variety over \( k \). Then the homomorphism
\[
\text{Hom}_{\text{DM}^{\text{eff}}(k)}(A, \mathbb{Z}(1)[2]) \to \text{Hom}_{\text{DM}^{\text{eff}}(k)}(\mathbb{Z}, A')
\]
of abelian groups induced by \( \eta_{A'} \) is an isomorphism.

**Proof.** It is equivalent to showing that the induced homomorphism
\[
\text{Hom}_{\text{DM}^{\text{eff}}(k)}(A, \mathbb{Z}(1)[2]) \to \text{Hom}_{\text{DM}^{\text{eff}}(k)}(M_{\geq 1}(A), \mathbb{Z}(1)[2]) \cong \text{Pic}(A)
\]
of abelian groups is injective and has image \( \text{Pic}^0(A) \). Here, the right isomorphism comes from Proposition [2.9]. This follows from Proposition [6.4] and [16, Proposition 17.6]. □

**Theorem 6.12.** Let \( A \) be an abelian variety over \( k \). Then the morphism
\[
h_0(\eta_{A'}) : h_0(\text{Hom}_{\text{DM}^{\text{eff}}(k)}(A, \mathbb{Z}(1)[2])) \to h_0(A') \cong A'
\]
of Nisnevich sheaves with transfers is an isomorphism.

**Proof.** It suffices to show that the induced homomorphism
\[
\text{Hom}_{\text{DM}^{\text{eff}}(k)}(M(X), \text{Hom}_{\text{DM}^{\text{eff}}(k)}(A, \mathbb{Z}(1)[2])) \to \text{Hom}_{\text{DM}^{\text{eff}}(k)}(M(X), A')
\]
of abelian groups is an isomorphism for any noetherian scheme \( X \) smooth and separated over \( k \). Since \( M(X) \cong \mathbb{Z}^d \oplus M_{\geq 1}(X) \) where \( d \) is the number of connected components of \( X \), we are done by Propositions [6.10] and [6.11]. □

**Proposition 6.13.** Let \( f : A \to A' \) be a homomorphism of abelian varieties over \( k \). Then the diagram
\[
\begin{array}{ccc}
\text{Hom}_{\text{DM}^{\text{eff}}(k)}(A', \mathbb{Z}(1)[2]) & \xrightarrow{h_0(\eta_{A'})} & A' \\
h_0(\text{Hom}_{\text{DM}^{\text{eff}}(k)}(A, \mathbb{Z}(1)[2])) & \xrightarrow{f} & A' \\
\end{array}
\]
in \( \text{DM}^{\text{eff}}(k) \) commutes where \( f' \) is induced by the dual homomorphism \( f' \) of \( f \).

**Proof.** Since the dual homomorphism \( f' : A' \to A' \) is a correspondence, there is a commutative diagram
\[
\begin{array}{ccc}
M_{\geq 1}(A') & \xrightarrow{u} & A' \\
g \downarrow & & \downarrow f' \\
M_{\geq 1}(A') & \xrightarrow{u'} & A'
\end{array}
\]
in \( \text{DM}^{\text{eff}}(k) \) where \( u \) and \( u' \) are the Albanese morphisms. By Lemma [6.7] we have the morphisms
\[
h_0(\text{Hom}_{\text{DM}^{\text{eff}}(k)}(A', \mathbb{Z}(1)[2])) \to \text{Hom}_{\text{DM}^{\text{eff}}(k)}(A', \mathbb{Z}(1)[2]),
\]
\[
h_0(\text{Hom}_{\text{DM}^{\text{eff}}(k)}(A', \mathbb{Z}(1)[2])) \to \text{Hom}_{\text{DM}^{\text{eff}}(k)}(A', \mathbb{Z}(1)[2]).
\]
in \(\text{DM}^{\text{eff}}(k)\). Consider the diagram

\[
\begin{array}{c}
h_0(\text{Hom}_{\text{DM}^{\text{eff}}(k)}(A^\vee, Z(1)[2])) \xrightarrow{v} \text{Hom}_{\text{DM}^{\text{eff}}(k)}(A^\vee, Z(1)[2]) \\
\downarrow h_0(f') \\
h_0(\text{Hom}_{\text{DM}^{\text{eff}}(k)}(A'^\vee, Z(1)[2])) \xrightarrow{v'} \text{Hom}_{\text{DM}^{\text{eff}}(k)}(A'^\vee, Z(1)[2]) \\
\end{array}
\]

in \(\text{DM}^{\text{eff}}(k)\) where \(i\) and \(i'\) come from (6.9.1) and Proposition 2.9. By taking \(\text{Hom}_{\text{DM}^{\text{eff}}(k)}(−, Z(1)[2])\) to (6.13.1), we see that the right front square commutes.

Then

\[i'\eta_A v = g^* i\eta_A v = i'\eta_A' v' = i'\eta_A' v' h_0(f').\]

Since we have the distinguished triangle

\[A' \rightarrow \text{Pic}_{A'/k} \rightarrow \text{NS}(A') \rightarrow A'[1]\]

in \(\text{DM}^{\text{eff}}(k)\), to show that \(f \eta_A v = \eta_A' v' h_0(f)\), it suffices to show that

\[\text{Hom}_{\text{DM}^{\text{eff}}(k)}(h_0(\text{Hom}_{\text{DM}^{\text{eff}}(k)}(A^\vee, Z(1)[2])), \text{NS}(A')[i]) = 0\]

for \(i = −1, 0\).

By Theorem 6.12 it suffices to show that

\[\text{Hom}_{\text{DM}^{\text{eff}}(k)}(A, \text{NS}(A')[i]) = 0\]

for \(i = −1, 0\). This follows from Proposition 4.7. \(\square\)

6.14. Let \(G\) be a semi-abelian variety over \(k\) with an exact sequence

(6.14.1)

\[0 \rightarrow L \otimes \text{G}_m \rightarrow G \rightarrow A \rightarrow 0\]

of group schemes over \(k\) where \(L\) is a lattice and \(A\) is an abelian variety over \(k\). Then the Cartier dual of \(G\) is \([L^\vee \rightarrow A^\vee]\) where \(L^\vee\) denotes the dual lattice of \(L\).

Set

\[G^\vee := [L^\vee \rightarrow A^\vee][1].\]

Consider the commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_{\text{DM}^{\text{eff}}(k)}(L^\vee[1], Z(1)[2]) & \xrightarrow{\sim} & L \otimes \text{G}_m \\
\downarrow & & \downarrow \\
\text{Hom}_{\text{DM}^{\text{eff}}(k)}(G^\vee, Z(1)[2]) & \xrightarrow{\eta_A} & G \\
\downarrow & & \downarrow \\
\text{Hom}_{\text{DM}^{\text{eff}}(k)}(A^\vee, Z(1)[2]) & \xrightarrow{\eta_A} & A \\
\downarrow & & \downarrow \\
\text{Hom}_{\text{DM}^{\text{eff}}(k)}(L^\vee, Z(1)[2]) & \xrightarrow{\sim} & L \otimes \text{G}_m[1]
\end{array}
\]

in \(\text{DM}^{\text{eff}}(k)\). Then there is a morphism

\[\eta_G : \text{Hom}_{\text{DM}^{\text{eff}}(k)}(G^\vee, Z(1)[2]) \rightarrow G\]

in \(\text{DM}^{\text{eff}}(k)\) such that the above diagram still commutes after adding this.

Lemma 6.15. Let \(G\) be a semi-abelian variety over \(k\). Then \(\text{Hom}_{\text{DM}^{\text{eff}}(k)}(G^\vee, Z(1)[2])\) is \(t\)-negative for the 0-motivic \(t\)-structure.
Proof. Since $\text{Hom}_{\text{DM-eff}}(k)(L^\vee[1], \mathbb{Z}(1)[2]) \cong L \otimes \mathbb{G}_m$ is in the heart of the 0-motivic $t$-structure, we are done by Lemma 6.7.

Proposition 6.16. Let $G$ be a semi-abelian variety over $k$. Then the morphism

$$h_0(\eta_G) : h_0(\text{Hom}_{\text{DM-eff}}(k)(G^\vee, \mathbb{Z}(1)[2])) \to h_0(G) \cong G$$

of Nisnevich sheaves with transfers is an isomorphism.

Proof. Consider the exact sequence (6.14.1). Then we have the commutative diagram

\[
\begin{array}{c}
0 \\
\downarrow \sim \\
L \otimes \mathbb{G}_m \\
\downarrow \sim \\
G \\
\downarrow \\
A \\
\downarrow \\
0
\end{array}
\]

of Nisnevich sheaves with transfers. By Theorem 6.12, $h_0(\eta_A)$ is an isomorphism. Since $h_0(\eta_G) \circ r$ is injective, $r$ is injective. Thus $h_0(\eta_G)$ is an isomorphism by the five lemma.

Proposition 6.17. The morphism

$$h_0(\eta_G) : h_0(\text{Hom}_{\text{DM-eff}}(k)(G^\vee, \mathbb{Z}(1)[2])) \to h_0(G) \cong G$$

is functorial on $G$.

Proof. Let $f : G \to G'$ be a homomorphism of semi-abelian varieties over $k$ with a commutative diagram

\[
\begin{array}{c}
0 \\
\downarrow h \\
L \otimes \mathbb{G}_m \\
\downarrow f \\
G \\
\downarrow g \\
A \\
\downarrow g \\
0
\end{array}
\]

of groups schemes over $k$ where

(i) each row is exact,
(ii) $A$ and $A'$ are abelian varieties over $k$,
(iii) $L$ and $L'$ are lattices.

Consider the induced diagram

\[
\begin{array}{c}
L \otimes \mathbb{G}_m \\
\downarrow f' \\
G' \\
\downarrow f \\
A \\
\downarrow g \\
A'
\end{array}
\]

of Nisnevich sheaves with transfers. By Proposition 6.13, the right square commutes. We have to show that the left square commutes.
We have the commutative diagram
\[
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
h_0(L^\vee[1], Z(1)[2]) & \longrightarrow & L' \otimes G_m \\
\downarrow & & \downarrow \\
h_0(\text{Hom}_{\text{DM}^{\text{eff}}(k)}(G^\vee, Z(1)[2])) & \longrightarrow & G' \\
\downarrow & & \downarrow \\
h_0(\text{Hom}_{\text{DM}^{\text{eff}}(k)}(A^\vee, Z(1)[2])) & \longrightarrow & A' \\
\downarrow & & \downarrow \\
0 & & 0 \\
\end{array}
\]

of Nisnevich sheaves with transfers where each column is exact. The above diagram still commutes after adding \(fu : h_0(\text{Hom}_{\text{DM}^{\text{eff}}(k)}(G^\vee, Z(1)[2])) \to G'\) to the above diagram. Since \(gv = u'g'\), the same holds if we add \(f'\). Hence to show that \(f'\) is an isomorphism, it suffices to check the conditions of [18 Lemma 3.3], which are as follows:

\[
\begin{align*}
\text{Hom}_{\text{DM}^{\text{eff}}(k)}(L \otimes G_m, A') &= 0, \\
\text{Hom}_{\text{DM}^{\text{eff}}(k)}(L \otimes G_m, A'[-1]) &= 0, \\
\text{Hom}_{\text{DM}^{\text{eff}}(k)}(A, L' \otimes G_m) &= 0
\end{align*}
\]

The first and third ones follow from Proposition 6.3. The second one holds since \(L \otimes G_m\) and \(A'\) are in the heart of the 0-motivic \(t\)-structure. \(\square\)

**Proposition 6.18.** Let \(G\) be a semi-abelian variety over \(k\), and let \(X\) be a noetherian scheme smooth and separated over \(k\). Then the homomorphism

\[
\text{Hom}_{\text{DM}^{\text{eff}}(k)}(M_{\geq 1}(X), \text{Hom}_{\text{DM}^{\text{eff}}(k)}(G, Z(1)[2])) \to \text{Hom}_{\text{DM}^{\text{eff}}(k)}(M_{\geq 1}(X), A^\vee)
\]

of abelian groups induced by \(\eta_{A^\vee}\) is an isomorphism.

**Proof.** There is an exact sequence

\[
0 \to L \otimes G_m \to G \to A \to 0
\]

of group schemes over \(k\) where \(L\) is a lattice and \(A\) is an abelian variety over \(k\). Then we have the induced commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_{\text{DM}^{\text{eff}}(k)}(M_{\geq 1}(X), \text{Hom}_{\text{DM}^{\text{eff}}(k)}(A^\vee, Z(1)[2])[-1]) & \longrightarrow & \text{Hom}_{\text{DM}^{\text{eff}}(k)}(M_{\geq 1}(X), A[-1]) \\
\downarrow & & \downarrow \\
\text{Hom}_{\text{DM}^{\text{eff}}(k)}(M_{\geq 1}(X), \text{Hom}_{\text{DM}^{\text{eff}}(k)}(L^\vee, Z(1)[2])[-1]) & \supseteq & \text{Hom}_{\text{DM}^{\text{eff}}(k)}(M_{\geq 1}(X), L \otimes G_m[-1]) \\
\downarrow & & \downarrow \\
\text{Hom}_{\text{DM}^{\text{eff}}(k)}(M_{\geq 1}(X), \text{Hom}_{\text{DM}^{\text{eff}}(k)}(G^\vee, Z(1)[2])) & \longrightarrow & \text{Hom}_{\text{DM}^{\text{eff}}(k)}(M_{\geq 1}(X), G) \\
\downarrow & & \downarrow \\
\text{Hom}_{\text{DM}^{\text{eff}}(k)}(M_{\geq 1}(X), \text{Hom}_{\text{DM}^{\text{eff}}(k)}(A^\vee, Z(1)[2])) & \longrightarrow & \text{Hom}_{\text{DM}^{\text{eff}}(k)}(M_{\geq 1}(X), A) \\
\downarrow & & \downarrow \\
\text{Hom}_{\text{DM}^{\text{eff}}(k)}(M_{\geq 1}(X), \text{Hom}_{\text{DM}^{\text{eff}}(k)}(L^\vee, Z(1)[2])) & \longrightarrow & \text{Hom}_{\text{DM}^{\text{eff}}(k)}(M_{\geq 1}(X), L \otimes G_m)
\end{array}
\]

in \(\text{DM}^{\text{eff}}(k)\). By Proposition 6.10 the fourth horizontal arrow is an isomorphism. Hence by the five lemma, to show that the third horizontal arrow is an isomorphism, it suffices to show that the first horizontal arrow is an isomorphism.
By Lemma 6.17, \( \text{Hom}_{DM^eff(k)}(A^\vee, \mathbb{Z}(1)[2]) \) is \( t \)-negative for the 0-motivic \( t \)-structure. Since \( M_{\geq 1}(X) \) is \( t \)-positive for the 0-motivic \( t \)-structure,

\[
\text{Hom}_{DM^eff(k)}(M_{\geq 1}(X), \text{Hom}_{DM^eff(k)}(A^\vee, \mathbb{Z}(1)[2])[-1]) = 0.
\]

We also have that \( \text{Hom}_{DM^eff(k)}(M_{\geq 1}(X), A[-1]) = 0 \) since \( A \) is in the heart of the 0-motivic \( t \)-structure. Thus the third horizontal arrow is an isomorphism. \( \square \)

6.19. Now we will construct an isomorphism

\[
\text{Hom}_{DM^eff(k)}(M_{\geq 1}(X), G) \xrightarrow{\sim} \text{Hom}_{DM^eff(k)}(G^\vee, M_1(X)^\vee)
\]

of abelian groups which is functorial on \( G \) where \( X \) is a noetherian scheme smooth and separated over \( k \) and \( G \) is a semi-abelian variety over \( k \).

Let \( f : G \to G' \) be a homomorphism of semi-abelian varieties over \( k \). Consider the diagram

\[
\begin{array}{ccc}
\text{Hom}_{DM^eff(k)}(M_{\geq 1}(X), G) & \to & \text{Hom}_{DM^eff(k)}(M_{\geq 1}(X), G') \\
\uparrow & & \uparrow \\
\text{Hom}_{DM^eff(k)}(M_{\geq 1}(X), h_0(\text{Hom}_{DM^eff(k)}(G^\vee, \mathbb{Z}(1)[2]))) & \to & \text{Hom}_{DM^eff(k)}(M_{\geq 1}(X), h_0(\text{Hom}_{DM^eff(k)}(G'^{\vee}, \mathbb{Z}(1)[2])))} \\
\downarrow & & \downarrow \\
\text{Hom}_{DM^eff(k)}(G^\vee, \text{Hom}_{DM^eff(k)}(M_{\geq 1}(X), \mathbb{Z}(1)[2])) & \to & \text{Hom}_{DM^eff(k)}(G^{\vee}, \text{Hom}_{DM^eff(k)}(M_{\geq 1}(X), \mathbb{Z}(1)[2]))} \\
\uparrow & & \uparrow \\
\text{Hom}_{DM^eff(k)}(G^{\vee}, M_1(X)^{\vee}) & \to & \text{Hom}_{DM^eff(k)}(G'^{\vee}, M_1(X)^{\vee})
\end{array}
\]

in \( DM^{eff}(k) \). Here,

(i) \( u \) (resp. \( u' \)) is induced by \( h_0(\eta_G) \) (resp. \( h_0(\eta_{G'}) \)),
(ii) \( v \) and \( v' \) are obtained by Lemma 6.15
(iii) \( w \) and \( w' \) comes from \( \text{(2.10.1)} \).

The above diagram commutes by Proposition 6.17 and \( u \) and \( u' \) are isomorphisms by Proposition 6.19. Since \( M_{\geq 1}(X) \) is \( t \)-positive for the 0-motivic \( t \)-structure, \( v \) and \( v' \) are isomorphisms by Lemma 6.15.

Let us show that \( w \) is an isomorphism. By \( \text{(2.10.1)} \), it suffices to show that

\[
\text{Hom}_{DM^eff(k)}(G^\vee, \text{NS}(X)[i]) = 0
\]

for \( i = -1, 0 \). Since we have the distinguished triangle

\[
L^\vee \to A^\vee \to G^\vee \to L^\vee[1]
\]

in \( DM^{eff}(k) \), it suffices to show that

\[
\text{Hom}_{DM^eff(k)}(L^\vee[1], \text{NS}(X)[i]) = 0, \quad \text{Hom}_{DM^eff(k)}(A^\vee, \text{NS}(X)[i]) = 0
\]

for \( i = -1, 0 \). The first one holds since \( L^\vee \) and \( \text{NS}(X) \) are in the heart of the 0-motivic \( t \)-structure. Since there is a distinguished triangle

\[
\mathbb{Z}^r \to \mathbb{Z}^r \to \text{NS}(X) \to \mathbb{Z}^r[1]
\]

in \( DM^{eff}(k) \) for some \( r \) and \( s \), it suffices to show that

\[
\text{Hom}_{DM^eff(k)}(A^\vee, \mathbb{Z}[i]) = 0
\]

for any \( i \leq 1 \). This follows from Proposition 6.7.

Therefore, we have proven the following theorem...
Theorem 6.20. Let $X$ be a noetherian scheme smooth and separated over $k$, and let $G$ be a semi-abelian variety over $k$. Then there is an isomorphism
\[ \text{Hom}_{\text{DM}^{eff}(k)}(M_{\geq 1}(X), G) \cong \text{Hom}_{\text{DM}^{eff}(k)}(G^\vee, M_1(X)^\vee) \]
of abelian groups which is functorial on $G$.

Theorem 6.21. Let $X$ be a noetherian scheme smooth and separated over $k$. Then $M_1(X)^\vee[-1]$ is isomorphic to the Cartier dual of $\text{Alb}(X)$.

Proof. By Theorem 6.20 a morphism $M_{\geq 1}(X) \to G$ in $\text{DM}^{eff}(k)$ is universal among morphisms from $M_{\geq 1}(X)$ to semi-abelian varieties over $k$ if and only if the corresponding morphism $G^\vee[-1] \to M_1(X)^\vee[-1]$ is universal among morphisms from the Cartier duals of semi-abelian varieties over $k$ to $M_1(X)^\vee[-1]$. The identity morphism $M_1(X)^\vee[-1] \to M_1(X)^\vee[-1]$ is the solution for the universal problem since $M_1(X)^\vee[-1]$ is the Cartier dual of a semi-abelian variety by Theorem 5.6. This means that $M_1(X)^\vee[-1]$ is isomorphic to the Cartier dual of $\text{Alb}(X)$.

Remark 6.22. Until Theorem 6.20, the existence of Albanese varieties is only used for abelian varieties, and this can be proved by the duality theory for abelian varieties. Thus Theorem 6.21 gives a new proof of the existence of Albanese varieties for noetherian schemes smooth and separated over $k$.

Corollary 6.23. Let $X$ be a noetherian scheme smooth and separated over $k$, and express the structure of $\text{Alb}(X)$ as an exact sequence
\[ 0 \to T_{\text{Alb}}(X) \otimes \mathbb{G}_m \to \text{Alb}(X) \to \overline{\text{Alb}}(X) \to 0 \]
of semi-abelian varieties where $T_{\text{Alb}}(X)$ is a lattice and $\overline{\text{Alb}}(X)$ is an abelian variety over $k$. Then there is an exact sequence
\[ T_{\text{Alb}}(X)^\vee \to \overline{\text{Alb}}(X)^\vee \to \text{Pic}_X^0(k) \to 0 \]
of Nisnevich sheaves with transfers where $T_{\text{Alb}}^\vee$ (resp. $\overline{\text{Alb}}(X)^\vee$) denotes the dual lattice (resp. dual abelian variety) of $T_{\text{Alb}}$ (resp. $\overline{\text{Alb}}(X)$).

Proof. Since $h_1(\text{NS}(X)) = 0$, by Proposition 2.8 and the distinguished triangle $2.10.1$, $h_0(M_1(X)^\vee) \cong \text{Pic}_X^0(k)$. Then we are done by Theorem 6.21.

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