A UNIVERSAL CHARACTERIZATION OF
HIGHER ALGEBRAIC $K$-THEORY

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ABSTRACT. In this paper we establish a universal characterization of higher algebraic $K$-theory in the setting of stable $\infty$-categories. Specifically, we prove that connective algebraic $K$-theory is the universal additive invariant, i.e., the universal functor with values in a stable presentable $\infty$-category which inverts Morita equivalences, preserves filtered colimits, and satisfies Waldhausen’s additivity theorem. Similarly, we prove that non-connective algebraic $K$-theory is the universal localizing invariant, i.e., the universal functor that moreover satisfies the “Thomason-Trobaugh-Neeman” localization theorem.

To show the latter result, we extend Schlichting’s axiomatization of non-connective algebraic $K$-theory to produce a functor of structured ring spectra (and more generally stable $\infty$-categories). In addition, by adapting the standard cosimplicial affine space to the setting of stable categories, we generalize the classical notion of algebraic homotopy invariance and prove that Karoubi-Villamayor’s algebraic $K$-theory is the universal additive homotopy invariant and that Weibel’s homotopy $K$-theory is the universal localizing homotopy invariant.

In order to prove these results, we construct and study various stable symmetric monoidal $\infty$-categories of “non-commutative motives”; one associated to additivity, another to localization and the remaining two to the respective homotopy invariant versions. In these $\infty$-categories, Waldhausen’s $S^\bullet$ construction corresponds to the suspension functor and the various algebraic $K$-theory spectra becomes corepresentable by the unit object in the corresponding $\infty$-category. Moreover, these $\infty$-categories are enriched over Waldhausen’s $A$-theory of a point and the homotopy $K$-theory of the sphere spectrum, respectively.

To work with these categories, we establish comparison theorems between the category of spectral categories localized at the “Morita equivalences” and the category of small idempotent-complete stable $\infty$-categories. We also explain in detail the comparison between the $\infty$-categorical version of Waldhausen $K$-theory and the classical definition.

We give several applications of our theory. We obtain a complete classification of all natural transformations from higher algebraic $K$-theory to topological Hochschild homology ($THH$) and topological cyclic homology ($TC$). Notably, we obtain a canonical construction and universal description of the cyclotomic trace map. We also exhibit a lax symmetric monoidal structure on the different algebraic $K$-theory functors, implying in particular that $E_n$ ring spectra give rise to $E_{n-1}$ $K$-theory spectra.

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1. Introduction

Algebraic K-theory. Algebraic K-theory is a fundamental algebro-geometric invariant, capturing information in arithmetic, algebraic geometry, and topology. The algebraic K-theory of a ring encodes many of its classical number-theoretic invariants, such as its Picard and Brauer groups. More generally, the algebraic K-theory of a scheme encodes arithmetic information as well as information about its singularities. The extension of algebraic K-theory from classical algebraic objects to ring spectra and to derived schemes provides a connection to geometric topology; notably, Waldhausen’s A-theory (i.e., the K-theory of the sphere spectrum) is essentially equivalent to stable pseudo-isotopy theory [64].

The subject originated with Grothendieck’s definition of $K_0$ (the “Grothendieck group”) in the course of his work on the Riemann-Roch theorem. By construction, $K_0$ is the universal receptacle for Euler characteristics, i.e., functors $\phi$ from categories, equipped with a suitable notion of equivalence and extension, to abelian groups which send an equivalence $X \simeq Y$ to the equality $\phi(X) = \phi(Y)$ and an extension

$$X \to Y \to Z$$

to the equality

$$\phi(X) - \phi(Y) + \phi(Z) = 0.$$

During the 70’s and early 80’s, Quillen [44] and then Waldhausen [62] extended Grothendieck’s work making use of tools from algebraic topology. They defined the connective algebraic K-theory spectrum ($K$) of a suitable category; the homotopy groups of this spectrum are the higher algebraic K-groups $K_i$, $i \geq 0$. Subsequently, Thomason and Trobaugh [58] generalized Bass’ work [3] on negative algebraic K-groups and introduced the non-connective algebraic K-theory spectrum ($\mathcal{K}$) in order to properly capture Mayer-Vietoris phenomena for schemes.

However, and in contrast with the Grothendieck group, the construction of these algebraic K-theory spectra does not provide a universal characterization. This raises the following natural question:

*What universal property characterizes the K-theory spectra?*
The most basic theorem concerning connective algebraic $K$-theory, the additivity theorem [62], essentially says that the $S\bullet$ construction forces the chosen cofiber sequences to split. McCarthy’s simplicial proof of the additivity theorem [39] for any theory satisfying a few simple axioms suggests that the $S\bullet$ construction is a universal construction for imposing additivity on a functor from categories to spectra. Moreover, the construction of the connective algebraic $K$-theory spectrum in terms of iterating the $S\bullet$ construction (along with the additivity theorem applied to a simplicial path fibration) implies that the $S\bullet$ construction functions as a kind of delooping functor.

However, although these shadows of a universal description have been known for a long time, a precise characterization of the universal property of algebraic $K$-theory has proved elusive. A major technical impediment has been the absence of a framework in which to systematically express homotopical constructions in the category of categories. Recent developments in the foundations of “higher category theory” (i.e., derivators, $\infty$-categories) now permit the formulation of the necessary universal characterization.

In this paper we answer the above question in the setting of Lurie’s theory of stable $\infty$-categories. We believe that the $\infty$-category of stable $\infty$-categories is the correct domain for the algebraic $K$-theory functor since it provides a natural home for the main examples of interest coming from algebraic geometry and algebraic topology: the $\infty$-category of perfect complexes associated to a scheme (or a stack), the $\infty$-category of compact module spectra associated to a ring spectrum, and the $\infty$-category of stable retractive spaces are all examples of stable $\infty$-categories.

**Universal characterization.** Let $\text{Cat}_{\infty}^{\text{ex}}$ be the $\infty$-category of small stable $\infty$-categories and $\text{Cat}_{\infty}^{\text{perf}}$ the full subcategory of idempotent-complete small stable $\infty$-categories; see §2.2. An exact functor $F : A \to B$ in $\text{Cat}_{\infty}^{\text{ex}}$ is called a Morita equivalence when its idempotent completion $\text{Idem}(F) : \text{Idem}(A) \to \text{Idem}(B)$ is an equivalence of $\infty$-categories; see definition 3.18. This is equivalent to the condition that the induced map $F_! : \text{Mod}(A) \to \text{Mod}(B)$ on module categories is an equivalence of $\infty$-categories; see proposition 2.9.

A sequence $A \to B \to C$ in $\text{Cat}_{\infty}^{\text{perf}}$ is called exact if $A \to B$ is fully-faithful and the map $B/A \to C$, from the cofiber of the inclusion of $A$ into $B$ to $C$, is an equivalence; see proposition 4.12. The sequence is called split-exact if there exist adjoint splitting maps such that the relevant composites are the respective identities; see definition 4.14. More generally, a sequence $A \to B \to C$ in $\text{Cat}_{\infty}^{\text{ex}}$ is (split-) exact if $\text{Idem}(A) \to \text{Idem}(B) \to \text{Idem}(C)$ is (split-) exact. Now, let

$$E : \text{Cat}_{\infty}^{\text{ex}} \to D$$

be a functor with values in a stable presentable $\infty$-category. We say that $E$ is an additive invariant if it inverts Morita equivalences, preserves filtered colimits, and sends split-exact sequences to (split) cofiber sequences; see definition 5.1. This last condition is the stable $\infty$-categorical analogue of Waldhausen’s additivity theorem. If $E$ in fact sends all exact sequences to cofiber sequences we say that it is a localizing invariant; this is Neeman’s generalization of the Thomason-Trobaugh localization theorem [46, 58]. Examples of localizing invariants include non-connective algebraic $K$-theory, topological Hochschild homology, and topological cyclic homology; see sections 6 and 10. Connective algebraic $K$-theory is an example of an additive invariant which is not a localizing one. Our first main results are the following.
Theorem 1.1. (see theorems 5.9 and 7.7) There are stable presentable $\infty$-categories $\mathcal{M}_{\text{add}}$ and $\mathcal{M}_{\text{loc}}$ and universal additive and localizing invariants
\begin{equation}
\mathcal{U}_{\text{add}} : \text{Cat}^\text{ex}_{\infty} \longrightarrow \mathcal{M}_{\text{add}} \quad \mathcal{U}_{\text{loc}} : \text{Cat}^\text{ex}_{\infty} \longrightarrow \mathcal{M}_{\text{loc}}.
\end{equation}
That is, given any stable presentable $\infty$-category $\mathcal{D}$, we have equivalences of $\infty$-categories
\begin{align*}
(\mathcal{U}_{\text{add}})^* : \text{Fun}^L(\mathcal{M}_{\text{add}}, \mathcal{D}) & \sim \text{Fun}^\text{add}(\text{Cat}^\text{ex}_{\infty}, \mathcal{D}) \\
(\mathcal{U}_{\text{loc}})^* : \text{Fun}^L(\mathcal{M}_{\text{loc}}, \mathcal{D}) & \sim \text{Fun}^\text{loc}(\text{Cat}^\text{ex}_{\infty}, \mathcal{D}),
\end{align*}
where the left-hand sides denote the $\infty$-categories of colimit-preserving functors and the right-hand sides the $\infty$-categories of additive and localizing invariants.

From a motivic perspective, the $\infty$-categories $\mathcal{M}_{\text{add}}$ and $\mathcal{M}_{\text{loc}}$ should be considered as candidate categories of non-commutative motives. In fact, theorem 1.1 shows us that every additive (respectively localizing) invariant factors uniquely through $\mathcal{M}_{\text{add}}$ (through $\mathcal{M}_{\text{loc}}$). That is, all the information concerning additive (localizing) invariants is encoded in $\mathcal{M}_{\text{add}}$ (in $\mathcal{M}_{\text{loc}}$). Further support for this viewpoint is provided by corollary 1.13 below, which gives a natural enrichment of $\mathcal{M}_{\text{add}}$ and $\mathcal{M}_{\text{loc}}$ over Waldhausen’s $A$-theory of a point.

Our second main result is the following characterization of higher algebraic $K$-theory. Recall that any stable $\infty$-category (and in particular $\mathcal{M}_{\text{add}}$ and $\mathcal{M}_{\text{loc}}$) admits natural mapping spectra; that is, a stable $\infty$-category is naturally enriched over the $\infty$-category $S^\infty$ of spectra (see section 3).

Theorem 1.3. (see theorems 6.9 and 8.9) Let $\mathcal{A}$ be an idempotent-complete small stable $\infty$-category. Then, there are natural equivalences of spectra
\begin{align*}
\text{Map}(\mathcal{U}_{\text{add}}(S^\infty_{\infty}), \mathcal{U}_{\text{add}}(\mathcal{A})) & \simeq K(\mathcal{A}) \\
\text{Map}(\mathcal{U}_{\text{loc}}(S^\infty_{\infty}), \mathcal{U}_{\text{loc}}(\mathcal{A})) & \simeq \mathbb{K}(\mathcal{A}),
\end{align*}
where $S^\infty_{\infty}$ is the small stable $\infty$-category of compact spectra. In particular, we have isomorphisms of abelian groups
\begin{align*}
\text{Hom}(\mathcal{U}_{\text{add}}(S^\infty_{\infty}), \Sigma^{-n}\mathcal{U}_{\text{add}}(\mathcal{A})) & \simeq K_n(\mathcal{A}) \\
\text{Hom}(\mathcal{U}_{\text{loc}}(S^\infty_{\infty}), \Sigma^{-n}\mathcal{U}_{\text{loc}}(\mathcal{A})) & \simeq \mathbb{K}_n(\mathcal{A}) \quad n \in \mathbb{Z}
\end{align*}
in the triangulated categories $\text{Ho}(\mathcal{M}_{\text{add}})$ and $\text{Ho}(\mathcal{M}_{\text{loc}})$.

Remark 1.4. In fact, a stronger result is true; namely, in the equivalences above, $S^\infty_{\infty}$ can be replaced by any compact idempotent-complete small stable $\infty$-category $\mathcal{B}$ and the right-hand side by the $K$-theory spectrum of $\text{Fun}^\text{ex}(\mathcal{B}, \mathcal{A})$; see theorem 6.9 and theorem 11.16.

In particular, when $\mathcal{A}$ is the $\infty$-category of perfect complexes over a suitable scheme (or stack), we obtain the $K$-spectra of the scheme (stack); see example 9.18. Taking $\mathcal{A}$ to be the $\infty$-category of compact modules over a ring spectrum $R$, we obtain the $K$-theory of $R$. When $R$ is a connective ring spectrum, the non-connective $K$-theory is determined by the non-connective $K$-theory of $\pi_0(R)$: we prove in theorem 8.7 that when $R$ is connective, $\pi_{-n}(R) \cong \pi_{-n}(\pi_0 R)$ for $n \geq 0$. On the other hand, we expect the non-connective $K$-theory of a non-connective ring spectrum to be an interesting new invariant.

We would like to emphasize that the left-hand sides of the natural equivalences (and isomorphisms) of theorem 1.3 are defined solely in terms of universal constructions on presheaf categories; algebraic $K$-theory is not used in their construction.
Therefore, theorem 1.3 provides a new and intrinsic characterization of algebraic $K$-theory, answering the question we posed above:

Connective (respectively non-connective) algebraic $K$-theory is the universal functor from the category of small stable categories to spectra that inverts Morita equivalences, preserves filtered colimits, and satisfies additivity (respectively localization).

**Homotopy invariance.** Roughly at the same time that Quillen discovered the higher algebraic $K$-theory spectrum, geometric considerations motivated efforts to construct versions of algebraic $K$-theory which satisfied algebraic homotopy invariance; i.e., covariant functors of rings such that the natural map $R \to R[t]$ induces an equivalence. Karoubi-Villamayor gave a construction of a homotopy invariant connective $K$-theory spectrum ($KV$) in the 70’s [30, 31], and then at the end of the 80’s Weibel extended this definition to produce the non-connective homotopy $K$-theory spectrum ($KH$) [65]. Following an insight of Rector [47, 23], these theories are best described as the realization of the simplicial spectrum obtained by applying $K$-theory levelwise to the standard cosimplicial affine scheme (simplicial ring)

$$[n] \mapsto \mathbb{Z}[t_0, \ldots, t_n]/(1 - \sum t_i).$$

By extending this construction to the setting of stable $\infty$-categories, in section 9 we introduce and study a generalized notion of homotopy invariance, culminating in the following theorem:

**Theorem 1.5.** (see theorem 9.21) There are stable presentable $\infty$-categories $\mathcal{M}_{add}^\Lambda$ and $\mathcal{M}_{loc}^\Lambda$ and localization functors $L_{add}^\Lambda: \mathcal{M}_{add} \to \mathcal{M}_{add}^\Lambda$ and $L_{loc}^\Lambda: \mathcal{M}_{loc} \to \mathcal{M}_{loc}^\Lambda$ such that the compositions

$$U_{add}^\Lambda: \text{Cat}^{ex}\mathcal{M}_{add} \xrightarrow{U_{add}} \mathcal{M}_{add} \xrightarrow{L_{add}^\Lambda} \mathcal{M}_{add}^\Lambda \quad U_{loc}^\Lambda: \text{Cat}^{ex}\mathcal{M}_{loc} \xrightarrow{U_{loc}} \mathcal{M}_{loc} \xrightarrow{L_{loc}^\Lambda} \mathcal{M}_{loc}^\Lambda$$

are the universal additive and localizing homotopy invariant invariants.

From a motivic perspective, the $\infty$-categories $\mathcal{M}_{add}^\Lambda$ and $\mathcal{M}_{loc}^\Lambda$ should be regarded as the homotopy invariant categories of non-commutative motives. Graphically, we have the following commutative diagram summarizing the situation:

\[\begin{array}{ccc}
\text{Cat}^{ex}\mathcal{M}_{add} & \xrightarrow{U_{add}} & \mathcal{M}_{add} \\
\downarrow & & \downarrow \quad \downarrow \\
\mathcal{M}_{add} & \xrightarrow{L_{add}^\Lambda} & \mathcal{M}_{add}^\Lambda \\
\uparrow & & \downarrow \\
\mathcal{M}_{loc} & \xrightarrow{L_{loc}^\Lambda} & \mathcal{M}_{loc}^\Lambda
\end{array}\]

**Theorem 1.7.** (see theorem 9.23) Let $A$ be a idempotent-complete small stable $\infty$-category. Then there are natural equivalences of spectra

$$\text{Map}(U_{add}^\Lambda(S^\infty_{\omega}), U_{add}^\Lambda(A)) \simeq KV(A) \quad \text{Map}(U_{loc}^\Lambda(S^\infty_{\omega}), U_{loc}^\Lambda(A)) \simeq KH(A).$$

**Remark 1.8.** Once again, a stronger result is true; namely, in the equivalences above, $S^\infty_{\omega}$ can be replaced by any compact idempotent-complete small stable $\infty$-category $B$ and the right-hand side by the $KV$-theory or $KH$-theory spectrum of $\text{Fun}^{ex}(B, A)$; see theorem 11.16.
Theorem 1.7 complements the intrinsic characterization of algebraic \( K \)-theory given in theorem 1.3. Intuitively, when we impose the homotopy invariance property we pass respectively from connective algebraic \( K \)-theory to Karoubi-Villamayor \( K \)-theory and from non-connective \( K \)-theory to homotopy \( K \)-theory. In particular, when \( A \) is the \( \infty \)-category of perfect complexes over a suitable scheme (or stack), we obtain the Karoubi-Villamayor and homotopy \( K \)-theory spectra of the scheme (stack); see example 9.18.

**Morita theory.** The main technical device in our proofs of theorems 1.1 and 1.3 is a comparison result between the theory of small spectral categories (see §2.1) and the theory of small stable \( \infty \)-categories. The category \( \text{Cat}_S \) of small spectral categories carries a Quillen model category structure in which the weak equivalences are the \( DK \)-equivalences, i.e., the functors which are fully-faithful and essentially surjective up to weak homotopy equivalence; see theorem 2.2. As a consequence, we can form the associated \( \infty \)-category \( (\text{Cat}_S)_\infty \) of small spectral categories.

A spectral functor \( F : A \to B \) is called a **triangulated equivalence** if it induces a weak equivalence on the triangulated closures of \( A \) and \( B \), and it is called a **Morita equivalence** if it induces a weak equivalence on the thick closures of \( A \) and \( B \); see definition 2.7. Our comparison result, which can be regarded as an \( \infty \)-categorical version of the Morita theory of [51], is the following.

**Theorem 1.9.** (see theorems 3.19 and 3.20) The localization of \( (\text{Cat}_S)_\infty \) along the triangulated equivalences is equivalent to \( \text{Cat}_{\text{ex}}^\infty \), and the (further) localization of \( (\text{Cat}_S)_\infty \) along the Morita equivalences is equivalent to \( \text{Cat}_{\text{perf}}^\infty \).

We use this comparison result to deduce structural properties of the categories \( \text{Cat}_{\text{ex}}^\infty \) and \( \text{Cat}_{\text{perf}}^\infty \), notably that they are compactly generated, complete, and cocomplete; see corollary 3.22. Furthermore, we prove theorem 1.3 by using theorem 1.9 to lift split-exact sequences of small \( \infty \)-categories to split-exact sequences of small spectral categories so we can apply Waldhausen’s \( K \)-theory construction in the setting of Waldhausen categories.

We believe that theorem 1.9 is of independent interest and expect it will find various useful applications in the near future. For instance, this theorem provides concise and conceptual proofs of the main theorems of Toën’s work on internal hom objects in the category of DG-categories [59], as well as the (previously unknown) extension to the context of spectral categories.

**Cyclotomic trace map.** One of the major revolutions in the calculational study of algebraic \( K \)-theory of rings and schemes in the past two decades has been the development of trace methods, following the ideas of Goodwillie and Bokstedt-Hsiang-Madsen [9]. The cyclotomic trace from \( K \)-theory to topological cyclic homology \( TC \) and topological Hochschild homology \( THH \) (stable homotopy theory generalizations of negative cyclic homology and Hochschild homology) has allowed major calculational advances. The fiber of this map is well understood by work of Goodwillie, McCarthy, and Dundas [40, 13], and the target is relatively computable using the methods of equivariant stable homotopy theory (e.g., see the extensive body of work by Hesselholt and Madsen on the Quillen-Lichtenbaum conjecture [26]). One application of the co-representability of algebraic \( K \)-theory (theorems 1.3 and 1.7) is the complete classification of all natural transformations with source the algebraic \( K \)-theory functor.
Theorem 1.10. (see theorem 10.2) Given an additive invariant
\[ E : \text{Cat}_\infty^{\text{perf}} \to S_\infty \]
with values in the stable \( \infty \)-category of spectra, we have a natural equivalence of spectra
\[ \text{Map}(K, E) \simeq E(S_\infty), \]
where \( \text{Map}(K, E) \) denotes the spectrum of natural transformations of additive invariants; i.e., those natural transformations which preserve filtered colimits. The analogous result for localizing (and homotopy invariant) invariants holds. In the particular case where \( E \) is topological Hochschild homology, we obtain an isomorphism
\[ \pi_0 \text{Map}(K, \text{THH}) \simeq \pi_0 \text{THH}(S_\infty) \simeq \pi_0 \text{THH}(S) \simeq \mathbb{Z}. \]
A calculation then provides a canonical construction and universal description of the cyclotomic trace map.

Corollary 1.11. (see section 10) The ring of homotopy classes of natural transformations of additive invariants from connective algebraic \( K \)-theory to \( \text{THH} \) is isomorphic to \( \mathbb{Z} \). Similarly, the ring of homotopy classes of natural transformations of localizing invariants from non-connective algebraic \( K \)-theory to \( \text{THH} \) is isomorphic to \( \mathbb{Z} \). In particular, the cyclotomic trace is characterized up to homotopy as the natural transformation \( K \to \text{THH} \) corresponding to the unit \( 1 \in \mathbb{Z} \).

That is, up to suspension, the cyclotomic trace is the only natural transformation of additive invariants between connective algebraic \( K \)-theory and \( \text{THH} \). This provides a direct proof that all known constructions of the cyclotomic trace map agree up to homotopy.

Working directly with topological cyclic homology \( (\text{TC}) \) is somewhat more complicated: \( \text{TC} \) does not preserve filtered colimits in general, and is therefore not an additive or localizing functor. However, for precisely this reason it is standard (e.g., see [21]) to regard \( \text{TC} \) as giving rise to a functor to pro-spectra — this variant of \( \text{TC} \) then does preserve filtered colimits. As the category of pro-spectra is a cocomplete stable category, regarding \( K \)-theory as giving rise to the constant pro-spectrum and applying theorem 1.3 provides an identification of the cyclotomic trace as determined by the unit map.

Finally, we note that in the localizing setting our results provide an extension of the cyclotomic trace from non-connective algebraic \( K \)-theory to the non-connective versions of \( \text{TC} \) and \( \text{THH} \) (the former regarded as a functor to pro-spectra, as above). This generalizes and extends the non-connective traces constructed in [22] and [4] for rings and schemes.

Monoidal structure. The \( \infty \)-category \( \text{Cat}_{\infty}^{\text{perf}} \) has a symmetric monoidal structure in which the tensor product \( \otimes^\vee \) is characterized by the property that maps out of \( A \otimes^\vee B \) are correspond to maps out of the product \( A \times A \) which preserve colimits in each variable; see section 2.3 for a discussion of this structure, following the work of Ben-Zvi, Francis, and Nadler [2]. We show that the \( \infty \)-categories \( \mathcal{M}_{\text{add}} \) and \( \mathcal{M}_{\text{loc}} \) (and their homotopy invariant versions) inherit a symmetric monoidal structure from \( \text{Cat}_{\infty}^{\text{perf}} \) and furthermore that the universal invariants (1.2) and (1.6) are lax monoidal. We prove the following theorem in section 11.
Theorem 1.12. The ∞-categories $\mathcal{M}_{\text{add}}, \mathcal{M}_{\text{loc}}, \mathcal{M}^{h}_{\text{add}},$ and $\mathcal{M}^{h}_{\text{loc}}$ are symmetric monoidal stable presentable ∞-categories, with units given by the images of $S^0_\infty$ under the various localization functors. The associated ∞-functors $U_{\text{add}}, U_{\text{loc}}, U^{h}_{\text{add}}$ and $U^{h}_{\text{loc}}$ are lax symmetric monoidal.

This theorem has several interesting applications. Combined with theorems 1.3 and 1.7, it implies that connective $K$-theory, non-connective $K$-theory, Karoubi-Villamayor $K$-theory, and homotopy $K$-theory are all lax monoidal functors; see theorem 11.13. That is, these functors preserve multiplicative structures on the input category, taking $E_n$ monoidal ∞-categories (in the sense of [37]) to $E_n$ ring spectra. Since an $E_{n+1}$ ring spectrum has an $E_n$ monoidal module category, this implies that the $K$-theory of an $E_n$ ring is an $E_{n-1}$ ring. We note that this result has previously been announced by Barwick using different technology.

Furthermore, from theorems 1.12 and 1.3 we can deduce that both $\mathcal{M}_{\text{add}}$ and $\mathcal{M}_{\text{loc}}$ are enriched in modules over Waldhausen’s $A$-theory of a point (as the connective and non-connective $K$-theory of the sphere spectrum $K(S)$ agree, see theorem 8.7). Similarly, both $\mathcal{M}^{h}_{\text{add}}$ and $\mathcal{M}^{h}_{\text{loc}}$ are enriched over $KV(S)$.

Corollary 1.13. Let $A$ and $B$ be small stable ∞-categories. The mapping spectrum $\text{Map}(U_{\text{add}}(B), U_{\text{add}}(A))$ is a module over

$$\text{Map}(U_{\text{add}}(S^0_\infty), U_{\text{add}}(S^0_\infty)) \simeq K(S) = A(*).$$

Similarly, the mapping spectrum $\text{Map}(U_{\text{loc}}(B), U_{\text{loc}}(A))$ is a module over

$$\text{Map}(U_{\text{loc}}(S^0_\infty), U_{\text{loc}}(S^0_\infty)) \simeq IK(S) \simeq A(*).$$

We have the expected analogues of these results in the homotopy invariant settings; see proposition 11.15.

Related works. The “motivic” idea of constructing universal invariants is not new and appears in several different subjects: for example, Cortiñas-Thom’s work [11] on bivariant algebraic $K$-theory, Higson’s work [27] on Kasparov’s bivariant $K$-theory, Meyer-Nest’s work [43] on $C^*$-algebras, Morel-Voevodsky’s work [41] on $K$-homotopy theory of schemes, and Voevodsky’s work [61] on (mixed) motives.

In this vein, over the past few years the third author of this paper has carried out a program [54] of providing universal characterizations of algebraic $K$-theory via the formalism of derivators. Our work in this paper completes this program by extending these results to stable ∞-categories, producing definitions of non-connective $K$-theory and homotopy $K$-theory to this setting, and by solving a key open question left open in the previous work of the third author, namely identifying the cyclotomic trace in terms of the co-representability results.

Finally, we would like to mention that Barwick and Rognes also have recent work on a universal characterization of higher algebraic $K$-theory, in the context of a detailed study of the algebraic $K$-theory of ∞-categories.

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2. Spectral categories and stable $\infty$-categories

The purpose of this section is to collect and recall the results about spectral categories and stable $\infty$-categories we will require for our constructions.

2.1. Review of spectral categories. We write $\mathcal{S}$ for the symmetric monoidal simplicial model category of symmetric spectra [29], and $\mathcal{T}$ for simplicial sets. Recall that a small spectral category $\mathcal{A}$ is a small category enriched in the category of symmetric spectra. Specifically, a small spectral category is given by:

- A set of objects $\text{obj}(\mathcal{A})$ (usually denoted by $\mathcal{A}$ itself),
- for each pair of objects $(x, y)$ of $\mathcal{A}$, a symmetric spectrum $\mathcal{A}(x, y)$,
- for each triple of objects $(x, y, z)$ of $\mathcal{A}$, a composition morphism in $\text{Sp}$
  $$\mathcal{A}(y, z) \wedge \mathcal{A}(x, y) \to \mathcal{A}(x, z),$$
  satisfying the usual associativity condition, and
- for any object $x$ of $\mathcal{A}$, a morphism $\mathcal{S} \to \mathcal{A}(x, x)$ in $\text{Sp}$, satisfying the usual unit condition with respect to the above composition.

We denote by $\text{Cat}_{\mathcal{S}}$ the category of small spectral categories and spectral (enriched) functors. References on spectral categories are [4, §2], [51, Appendix A] and [55, §2].

We now briefly recall the Quillen model structure on spectral categories we work with in this paper. Given a spectral category $\mathcal{A}$, we can form a genuine category $[\mathcal{A}]$ by keeping the same set of objects and defining the set of morphisms between $x$ and $y$ in $[\mathcal{A}]$ to be the set of morphisms in the homotopy category $\text{Ho}(\text{Sp})$ from $\mathcal{S}$ to $\mathcal{A}(x, y)$. We obtain in this way a functor

$$[-] : \text{Cat}_{\mathcal{S}} \to \text{Cat},$$

with values in the category of small categories. Equivalently, we can think of $[-]$ as computed by passing to $\pi_0$ on the morphism spectra, and so we will also refer to $[\mathcal{A}]$ as the homotopy category $\text{Ho}(\mathcal{A})$.

**Definition 2.1.** A spectral functor $F : \mathcal{A} \to \mathcal{B}$ is a DK-equivalence, if:

- for all objects $x, y \in \mathcal{A}$, the morphism in $\mathcal{S}$
  $$F(x, y) : \mathcal{A}(x, y) \to \mathcal{B}(Fx, Fy)$$
  is a stable equivalence and
- the induced functor
  $$[F] : [\mathcal{A}] \to [\mathcal{B}]$$
  is an equivalence of categories.

**Theorem 2.2.** ([55, Thm. 5.10]) The category $\text{Cat}_{\mathcal{S}}$ carries a right proper Quillen model structure whose weak equivalences are the DK-equivalences.

For technical control, we require the following corollary:

**Corollary 2.3.** The category $\text{Cat}_{\mathcal{S}}$ endowed with the model structure of theorem 2.2 is Quillen equivalent (via a natural zig-zag) to a simplicial category with a left proper combinatorial simplicial model structure. There are simplicial cofibrant and fibrant replacement functors.
**Proof.** Theorem 2.2 is proved by localizing a cofibrantly-generated model structure on \( \text{Cat}_S \) in which the weak equivalences are the levelwise equivalences \([55, \S 4]\), i.e., the spectral functors \( F : A \to B \) such that for all objects \( x, y \in A \), the morphism \( A(x, y) \to B(Fx, Fy) \) is a levelwise equivalence, and the induced simplicial functor \( \Omega^\infty(A) \to \Omega^\infty(B) \), see (2.4), is a DK-equivalence. It is straightforward to verify that the category \( \text{Cat}_S \) is locally presentable; a set of small generators is given by applying the functor \( U \) (see [55, A.1]) to a set of small generators for the category of symmetric spectra. Therefore the levelwise model structure on \( \text{Cat}_S \) is combinatorial. The machinery of Dugger [12] then permits us to replace \( \text{Cat}_S \) with a Quillen equivalent simplicial model category (the simplicial objects over \( \text{Cat}_S \)) which is combinatorial and left proper. Using the localization arguments of [55] we obtain a generating set of DK-equivalences, and then the general theory of Bousfield localization for combinatorial model categories allows us to conclude the result. □

Recall from [55, \S 2] the natural adjunction

\[
\text{Cat}_S \xrightarrow{\Omega^\infty} \text{Cat}_\Delta, \quad \Sigma^\infty, \Omega^\infty
\]

between spectral and simplicial categories, where \( \Omega^\infty \) (also denoted \((-)_0 \)) is the space of maps from the unit (equivalently, restriction to the 0-th space of the spectrum). We will use this adjunction to pass between spectral categories and \( \infty \)-categories. Using the model structure on simplicial categories of [1], the pair \((\Sigma^\infty, \Omega^\infty)\) is a Quillen adjunction. By applying the techniques of [12, 48] (as in the proof of corollary 2.3), we can promote this adjunction to a simplicial Quillen adjunction. Specifically, both the categories of simplicial categories and spectral categories are Quillen equivalent to suitable model structures on their simplicial objects, and the simplicial prolongation of the adjunction forms a Quillen pair on the categories of simplicial objects [48, 6.1].

Next, we discuss modules over spectral categories. Let \( A \) be a (fixed) small spectral category, and let \( A^{\text{op}} \) denote the opposite spectral category, defined by \( A^{\text{op}}(x, y) = A(y, x) \).

**Definition 2.5.** A \( A \)-module is a spectral functor from \( A^{\text{op}} \) to the spectral category \( S \) of symmetric spectra. We denote by \( \hat{A} \) the spectral category of \( A \)-modules.

By [51, Thm. A.1.1], \( \hat{A} \) can be given a cofibrantly generated spectral model structure in which the weak equivalences are given by the pointwise stable equivalences (referred to as the projective model structure). We will denote by \( \hat{A}^p \) the full spectral subcategory of \( \hat{A} \) on the cofibrant and fibrant \( A \)-modules, and by \( D(A) \) the derived category of \( A \), i.e., the homotopy category \( \text{Ho} \hat{A} \) associated to the model structure. Clearly we have an equivalence \( [\hat{A}^p] \simeq D(A) \).

Notice that we have a (fully-faithful) spectral Yoneda embedding \( A \to \hat{A} \) which sends the object \( z \) to the functor \( A(-, z) : A^{\text{op}} \to S \) represented by \( z \). Note that when \( A \) is fibrant, the Yoneda embedding lands in \( \hat{A}^p \). By [51, \S A.1], a spectral functor \( F : A \to B \) gives rise to a restriction/extension Quillen adjunction

\[
\hat{B} \xleftarrow{F_*} \hat{A}
\]

and therefore a total left-derived functor \( LF_* : D(A) \to D(B) \).
We will be most interested in spectral categories $\mathcal{A}$ for which the homotopy category $\text{Ho} \mathcal{A}$ has a triangulated structure compatible with the mapping spectra; we refer to \cite[4.4]{4} for the definition of a pretriangulated spectral category, and highlight the essential consequence \cite[4.6]{4} that the homotopy category of a pretriangulated spectral category is triangulated (with distinguished triangles given by the Puppe sequences). This is the stable homotopy theory analogue of the notion of a pretriangulated DG-category. Using the Yoneda embedding, we can construct minimal pretriangulated categories “containing” the spectral category $\mathcal{A}$.

Given a spectral category $\mathcal{A}$, the proof of \cite[4.5]{4} constructs a functorial “triangulated closure” $\hat{\mathcal{A}}_{\text{tri}}$ which is a pretriangulated spectral category. Briefly, $\hat{\mathcal{A}}_{\text{tri}}$ consists of the subcategory of cofibrant-fibrant objects in $\hat{\mathcal{A}}$ which have the homotopy type of finite cell objects (in the projective model structure). A slight modification of this construction of \cite[4.5]{4} produces a functorial “thick closure” $\hat{\mathcal{A}}_{\text{perf}}$, which is an idempotent-complete pretriangulated spectral category.

**Remark 2.6.** In order for the preceding definitions to produce small spectral categories, we need to restrict the sizes of the sets in the spaces of the mapping spectra. A careful discussion of this issue appears in \cite[§4]{4}; see also \cite{6}.

Given a spectral functor $f: \mathcal{A} \to \mathcal{B}$, denote by $F_f^\circ: \hat{\mathcal{A}}^\circ \to \hat{\mathcal{B}}^\circ$ the composite of $F_f$ with a fibrant replacement (note that we do not need a cofibrant replacement here since $F_f$ preserves cofibrant objects). This is a model of the derived functor of $F_f$ as a Quillen functor.

**Definition 2.7.** A spectral functor $F: \mathcal{A} \to \mathcal{B}$ is called
- a **triangulated equivalence** if the induced functor
  $$F_f^\circ: \hat{\mathcal{A}}_{\text{tri}} \to \hat{\mathcal{B}}_{\text{tri}}$$
  is a DK-equivalence of spectral categories.
- a **Morita equivalence** if the induced functor
  $$F_f^\circ: \hat{\mathcal{A}}_{\text{perf}} \to \hat{\mathcal{B}}_{\text{perf}}$$
  is a DK-equivalence of spectral categories.

**Remark 2.8.** Given a spectral functor $F: \mathcal{A} \to \mathcal{B}$, the induced functor $F_f^\circ: \hat{\mathcal{A}}^\circ \to \hat{\mathcal{B}}^\circ$ sends perfect $\mathcal{A}$-modules to perfect $\mathcal{B}$-modules. Since $\hat{\mathcal{A}}^\circ$ is generated by $\hat{\mathcal{A}}_{\text{perf}}$ under filtered homotopy colimits and Morita equivalences are stable under filtered homotopy colimits, it follows that $F$ is a Morita equivalence if and only if $F_f^\circ: \hat{\mathcal{A}}^\circ \to \hat{\mathcal{B}}^\circ$ is a DK-equivalence.

We can relate these notions to definitions purely on the level of triangulated categories (the relationship between triangulated constructions and enriched constructions is discussed further in Section 4). For a spectral category $\mathcal{A}$, let $D_{\text{tri}}(\mathcal{A})$ denote the smallest triangulated subcategory of $D(\mathcal{A})$ containing the image of the $\mathcal{A}$ under the Yoneda embedding, and $D_{\text{perf}}(\mathcal{A})$ denote the smallest thick subcategory of $D(\mathcal{A})$ containing the image of $\mathcal{A}$ under the Yoneda embedding. Observe that $D_{\text{tri}}(\mathcal{A}) \simeq \text{Ho}(\hat{\mathcal{A}}_{\text{tri}})$ and $D_{\text{perf}}(\mathcal{A}) \simeq \text{Ho}(\hat{\mathcal{A}}_{\text{perf}})$. As a consequence, we immediately obtain the following proposition.

**Proposition 2.9.** A spectral functor $F: \mathcal{A} \to \mathcal{B}$ is...
• a triangulated equivalence if and only if the derived induced functor
  \[ LF: D_{\text{tri}}(A) \to D_{\text{tri}}(B) \]
  is an equivalence of (triangulated) categories,

• a Morita equivalence if and only if the derived induced functor
  \[ LF: D_{\text{perf}}(A) \to D_{\text{perf}}(B) \]
  is an equivalence of (triangulated) categories.

Finally, note that we can use \( \Omega^\infty \) to obtain simplicial models of the triangulated and thick closures. Define \( \text{Mod}(A) \) to be the simplicial category \( \Omega^\infty \hat{A} \), \( \text{Mod}^{\text{tri}}(A) \) to be the simplicial category \( \Omega^\infty \hat{A}_{\text{tri}} \), and \( \text{Mod}^{\text{perf}}(A) \) to be the simplicial category \( \Omega^\infty \hat{A}_{\text{perf}} \). Of course, it is also possible to give intrinsic definitions of the latter two categories in terms of \( \text{Mod}(A) \).

Summarizing the relationships between the various categories, when \( A \) is fibrant we have the following commutative diagram (with horizontal arrows induced by the Yoneda embedding and subsequent inclusions):

\[
\begin{array}{cccc}
A & \to & \hat{A}_{\text{tri}} & \to & \hat{A}_{\text{perf}} & \to & \hat{A}^p \\
\Omega^\infty & \downarrow & \Omega^\infty & \downarrow & \Omega^\infty & \downarrow & \Omega^\infty \\
\Omega^\infty A & \to & \text{Mod}(A)_{\text{tri}} & \to & \text{Mod}(A)_{\text{perf}} & \to & \text{Mod}(A) \\
\text{Ho}(A) & \downarrow & \text{Ho} & \downarrow & \text{Ho} & \downarrow & \text{Ho} \\
D_{\text{tri}}(A) & \to & D_{\text{perf}}(A) & \to & D(A) \\
\end{array}
\]

2.2. The \( \infty \)-categories \( \text{Cat}^{\text{ex}}_{\infty} \) and \( \text{Cat}^{\text{perf}}_{\infty} \). In this section we give a rapid review of the relevant background on the theory of quasicategories, as developed by Joyal and Lurie (and which, following Lurie, we refer to as \( \infty \)-categories). The best reference for this material remains Lurie’s book [33] and various papers on derived algebraic geometry (e.g., [34, 35, 36]).

Let \( \text{Cat}_{\infty} \) denote the \( \infty \)-category of small \( \infty \)-categories and functors. Given an \( \infty \)-category \( C \), we can form its homotopy category \( \text{Ho}(C) \), which is an ordinary category [33, §1.2.3]. There is a simplicial nerve functor \( N \) from simplicial categories to simplicial sets which is the right Quillen functor of a Quillen equivalence [33, §1.1.5.5, 1.1.5.13, 2.2.5.1]

\[ N: \text{Cat}_{\Delta} \rightleftarrows \text{Set}_{\Delta} : \mathcal{C}. \]

Here the model structure on the left is Bergner-Dwyer-Kan’s model structure on simplicial categories [1] and the model structure on the right is Joyal’s model structure on simplicial sets. Given a simplicial model category \( C \), we produce an \( \infty \)-category by restricting to the cofibrant-fibrant objects and applying \( N \); we denote this composite by \( N(C^\circ) \). More generally, given a category \( C \) with a subcategory of weak equivalences \( wC \), the Dwyer-Kan simplicial localization \( LC \) [15] provides a corresponding simplicial category and then applying the simplicial nerve yields an associated \( \infty \)-category.

Although a simplicial left Quillen functor \( C \to D \) does not typically induce a functor between the subcategories of cofibrant-fibrant objects in \( C \) and \( D \) respectively, as in Definition 2.7, composing with a fibrant replacement functor does
yield an induced functor on the simplicial nerves. Furthermore, given a simplicial Quillen adjunction \((F,G)\), there is an induced adjunction of functors on the level of \(\infty\)-categories by [33, 5.2.4.6].

Given an \(\infty\)-category \(C\), there is a maximal \(\infty\)-groupoid (Kan complex) \(C_{\text{Iso}}\) inside of \(C\), obtained by restricting to the subcategory of \(C\) consisting of those arrows which become isomorphisms in the homotopy category \(Ho(C)\). The functor which associates to the \(\infty\)-category \(C\) its maximal subgroupoid \(C_{\text{Iso}}\) is right adjoint to the inclusion of \(\infty\)-groupoids into \(\infty\)-categories. We have the following proposition relating this to other, possibly more familiar, notions (see also [60, 2.3]).

**Proposition 2.10.** Let \(C\) be a category with a subcategory \(wC\) of weak equivalences which satisfies a homotopy calculus of two-sided fractions (in the sense of Dwyer and Kan [16, 6.1]). Then there is a weak equivalence of simplicial sets between \(NwC\) and the \(\infty\)-groupoid \((N(L^H C))_{\text{Iso}}\), where here \(N(-)\) denotes the usual nerve simplicial set and \(L^H C\) denotes the hammock version of the simplicial localization [16].

**Proof.** There is an “inclusion” functor \(C \to L^H C\). Restricting to the weak equivalences and passing to simplicial nerves via \(N\), we obtain a map of simplicial sets

\[(2.11) \quad NwC \to N(L^H wC)\]

(using the fact that \(NwC \cong N(wC)\), regarding \(wC\) as a constant simplicial category [33, 1.1.5.8]). Since \((N(L^H wC))_{\text{Iso}}\) is isomorphic to \(N(L^H wC)\), the inclusion \(L^H wC \to L^H C\) induces a natural map \(N(L^H wC) \to (N(L^H C))_{\text{Iso}}\); under the hypothesis that \(C\) satisfies a homotopy calculus of left fractions, this map is a weak equivalence [16, 6.4]. Therefore, it suffices to show that the map of equation 2.11 is a weak equivalence. We consider the map on components; for each homotopy equivalence class \([x]\), both sides are equivalent to \(BH\)aut\((x)\) and the map clearly induces an equivalence. \(\square\)

The \(\infty\)-category of functors between two \(\infty\)-categories \(C\) and \(D\) is denoted \(\text{Fun}(C,D)\). Note that the space of functors from \(C\) to \(D\) is precisely the maximal subgroupoid \(\text{Fun}(C,D)_{\text{Iso}}\) of the exponential \(\text{Fun}(C,D)\) [33, 1.2.5.3.0.0.1].

An \(\infty\)-category is stable [34, 2.9] if it has finite limits and colimits and pushout and pullback squares coincide [34, 4.4]. Let \(\text{Cat}^\infty_{\text{s}}\) denote the category of small stable \(\infty\)-categories and exact functors (i.e., functors which preserve finite limits and colimits) [34, §5]. The \(\infty\)-category of exact functors between \(A\) and \(B\) is denoted by \(\text{Fun}^\text{ex}(A,B)\). Note that the \(\infty\)-category \(\text{Cat}_{\text{ex}}^\infty\) is pointed. For a small stable \(\infty\)-category \(C\), the homotopy category \(Ho(C)\) is triangulated, with the exact triangles determined by the cofiber sequences in \(C\) [34, 3.11]. Small stable \(\infty\)-category corresponds to the notion of a pretriangulated spectral category, and the weak equivalences are given by exact functors which induce triangulated equivalences on passage to the homotopy category. We will make this correspondence precise in Section 3, but for now observe that given a pretriangulated spectral category \(C\), the \(\infty\)-category \(N((C\text{-Mod})^\text{op})\) is stable. This follows from the observation that given a stable model category \(C\), the \(\infty\)-category \(N(C^\text{op})\) is stable. Given an \(\infty\)-category \(C\) with finite limits, we can form the stabilization \(\text{Stab}(C)\) [34, §8, §15].

Recall that an \(\infty\)-category \(C\) is idempotent-complete if the image of \(C\) under the Yoneda embedding \(C \to \text{Pre}(C)\) is closed under retracts (see also [33, §4.4.5]); here \(\text{Pre}(C)\) denotes the \(\infty\)-category \(\text{Fun}(C^\text{op}, N(T^\circ))\) of presheaves of simplicial
sets on $\mathcal{C}$. Let $\text{Cat}^\text{perf}_\infty$ denote the $\infty$-category of small idempotent-complete stable $\infty$-categories. There is an idempotent completion functor given as the left adjoint to the inclusion $\text{Cat}^\text{perf}_\infty \to \text{Cat}^\text{ex}_\infty$ [33, 5.1.4.2], which we denote by $\text{Idem}$.

**Definition 2.12.** Let $\mathcal{A}$ and $\mathcal{B}$ be small stable $\infty$-categories. Then we will say that $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent if $\text{Idem}(\mathcal{A})$ and $\text{Idem}(\mathcal{B})$ are equivalent.

We will verify shortly that this notion of Morita equivalence is compatible with the definition given in terms of spectral categories in definition 2.7.

2.3. Compact objects and compactly-generated $\infty$-categories. The categorical data which serves as the input to algebraic $K$-theory is typically obtained as the objects in a larger ambient category (with weak equivalences and extension sequences) that satisfy some sort of “smallness" condition; e.g., the perfect complexes as a subcategory of all complexes. A key insight initially codified by Thomason-Trobaugh [58] and subsequently elaborated upon by Neeman [45] is that this example is generic, and the typical situation involves working with the compact objects in some model of a triangulated category, which is generated under homotopy colimits by those compact objects. Thus, we systematically regard the small stable idempotent-complete $\infty$-categories that are the domain of the algebraic $K$-theory functor as arising as the compact objects in a larger category.

This notion of looking at large categories which are in some sense determined by the compact objects forms the basis for Lurie’s theory of presentable $\infty$-categories, which is the analogue in the $\infty$-category setting of the homotopy theories encoded by presentable combinatorial model category structures (see also Simpson’s related work in the context of Segal spaces [52]). In addition to Lurie’s work, the paper of Ben-Zvi, Francis, and Nadler [2] provides an excellent exposé of this theory in the context of the study of geometric function theory from a perspective with its origin in Thomason-Trobaugh; we refer the interested reader to sections 2 and 4.1 of that paper.

Roughly speaking, presentable $\infty$-categories are large $\infty$-categories that are generated under sufficiently large filtered colimits by some small $\infty$-category. To make this precise, we need to discuss the notion of the Ind category. Given any small $\infty$-category $\mathcal{C}$, we can form the $\infty$-category $\text{Pre}(\mathcal{C})$ of presheaves of simplicial sets on $\mathcal{C}$, which is the formal closure of $\mathcal{C}$ under colimits; that is, there is a fully-faithful Yoneda embedding $\mathcal{C} \to \text{Pre}(\mathcal{C})$, and $\text{Pre}(\mathcal{C})$ is generated by the image of $\mathcal{C}$ under small colimits [33, §5.1.5.8]. For any $\infty$-category $\mathcal{C}$ and a regular cardinal $\kappa$, we can form the $\text{Ind}$-category $\text{Ind}_\kappa(\mathcal{C})$ which is the formal closure under $\kappa$-filtered colimits of $\mathcal{C}$ [33, §5.3.5]. The $\infty$-category $\text{Ind}_\kappa(\mathcal{C})$ is a full subcategory of $\text{Pre}(\mathcal{C})$, and the Yoneda embedding $\mathcal{C} \to \text{Pre}(\mathcal{C})$ factors as $\mathcal{C} \to \text{Ind}_\kappa(\mathcal{C}) \to \text{Pre}(\mathcal{C})$. We record here the following useful properties of the construction of the $\text{Ind}$-category.

**Proposition 2.13.** Let $\mathcal{C}$ be a small stable $\infty$-category and $\kappa$ a regular cardinal.

- The $\infty$-category $\text{Ind}_\kappa(\mathcal{C})$ admits all $\kappa$-small colimits [33, 5.3.5.14, 5.5.1.1].
- The functor $\mathcal{C} \to \text{Ind}_\kappa(\mathcal{C})$ preserves $\kappa$-filtered colimits [33, 5.3.5.2, 5.3.5.3].
- $\text{Ind}_\kappa(\mathcal{C})$ is a stable $\infty$-category [34, 4.5].
- The image of $\mathcal{C}$ in $\text{Ind}_\kappa(\mathcal{C})$ provides a set of compact objects which generates $\text{Ind}(\mathcal{C})$ under $\kappa$-filtered colimits [33, 5.3.5.5, 5.3.5.11].
An ∞-category \( \mathcal{C} \) is presentable if it arises as \( \text{Ind}_\kappa(\mathcal{D}) \) for a small ∞-category \( \mathcal{D} \) which admits \( \kappa \)-small colimits [33, 5.5.1.1]. The correct notion of functors between presentable ∞-categories are the colimit-preserving functors. We let \( \text{LPr} \) denote the ∞-category of presentable ∞-categories and colimit-preserving functors; the ∞-category of colimit-preserving functors is denoted by \( \text{Fun}^L(\, - \,,\, - \,) \). In fact, \( \text{Fun}^L(\, - \,,\, - \,) \) is in fact itself a presentable ∞-category [33, 5.5.3.8], and so provides an internal hom object for \( \text{LPr} \).

We now restrict attention to the situation in which \( \kappa = \omega \). Recall that an object \( x \) of an ∞-category \( \mathcal{C} \) is compact if the functor \( \mathcal{C}^{\text{op}} \to \mathcal{T} \) represented by \( x \) commutes with filtered colimits [33, §5.3.4]. Given an an ∞-category \( \mathcal{C} \), let \( \mathcal{C}^\omega \) denote the full subcategory of \( \mathcal{C} \) consisting of the compact objects of \( \mathcal{C} \). A presentable ∞-category \( \mathcal{C} \) is compactly generated if the natural functor \( \text{Ind}(\mathcal{C}^\omega) \to \mathcal{C} \), which sends a filtered diagram in \( \mathcal{C}^\omega \) to its colimit in \( \mathcal{C} \), is an equivalence. There is a correspondence between small idempotent-complete ∞-categories and compactly generated ∞-categories given by the construction of the Ind-category [33, §5.5.7].

More generally, the construction of the Ind-category sets up a correspondence between the ∞-category of compactly-generated presentable ∞-categories with morphisms colimit-preserving functors that preserve compact objects and \( \text{Cat}^{\text{perf}}_\infty \); the other direction is given by passage to compact objects [33, 5.5.7.10].

The preceding discussion carries over when we restrict attention to stable categories. The ∞-category of stable presentable ∞-categories \( \mathcal{C}^\omega_{\text{Pr},\sigma} \) is a full subcategory of \( \text{LPr} \), and the Ind-category sets up a correspondence between \( \mathcal{C}^\omega_{\text{Pr},\sigma} \) and compactly generated stable ∞-categories. We may also apply Ind to non-idempotent-complete stable ∞-categories to obtain a correspondence between \( \mathcal{C}^\omega_{\infty} \) and compactly generated stable ∞-categories; however, these two ∞-categories are rather less closely related, as the full subcategory of compact objects is always idempotent-complete.

**Lemma 2.14.** \( \mathcal{C}^\omega_{\text{Pr},\sigma} \) is a reflective subcategory of \( \mathcal{C}^\omega_{\infty} \), and the localization functor \( \text{Idem}: \mathcal{C}^\omega_{\infty} \to \mathcal{C}^\omega_{\text{Pr},\sigma} \) is given by the formula \( \text{Idem}(\mathcal{C}) \simeq \text{Ind}(\mathcal{C})^\omega \).

**Proof.** Certainly the compact objects \( \text{Ind}(\mathcal{C})^\omega \) of \( \text{Ind}(\mathcal{C}) \) is an idempotent-complete stable ∞-category, so that \( \text{Idem} \) is indeed a functor \( \mathcal{C}^\omega_{\infty} \to \mathcal{C}^\omega_{\text{Pr},\sigma} \). Now for small stable ∞-categories \( \mathcal{C} \) and \( \mathcal{D} \) with \( \mathcal{D} \) idempotent-complete, we have a commuting square

\[
\begin{array}{ccc}
\text{Fun}^\omega(\text{Idem}(\mathcal{C}), \mathcal{D}) & \longrightarrow & \text{Fun}^L(\text{Ind}(\text{Idem}(\mathcal{C})), \text{Ind}(\mathcal{D})) \\
\downarrow & & \downarrow \\
\text{Fun}^\omega(\mathcal{C}, \mathcal{D}) & \longrightarrow & \text{Fun}^L(\text{Ind}(\mathcal{C}), \text{Ind}(\mathcal{D}))
\end{array}
\]

in which the horizontal maps are the inclusions of the full subcategories of functors which preserve compact objects, and the right vertical map is an equivalence as the natural map \( \text{Ind}(\mathcal{C}) \to \text{Ind}(\text{Idem}(\mathcal{C})) \) is an equivalence. Hence \( \text{Ind}(\mathcal{C})^\omega \to \text{Ind}(\text{Idem}(\mathcal{C}))^\omega \) is an equivalence, and thus the left vertical map is as well. \(\square\)

Recall that the category of presentable stable ∞-categories is closed symmetric monoidal with product \( \otimes \) and internal hom given by the presentable stable ∞-category \( \text{Fun}^L(\mathcal{A}, \mathcal{B}) \) of colimit-preserving functors [35, §4.1]. Following [2, §4.1.2],
we can then define the tensor product on small idempotent-complete stable ∞-categories as
\[ C \otimes^\vee D = (\text{Ind}(C) \otimes \text{Ind}(D))^\omega. \]

The tensor product of small stable idempotent-complete ∞-categories is characterized by the universal property that maps out of \( A \otimes B \) correspond to maps out of the product \( A \times B \) which preserve finite colimits in each variable \([2, 4.4]\). As will see in Section 3, this symmetric monoidal product corresponds to the derived smash product of spectral categories with respect to the Morita equivalences. We have the following theorem:

**Theorem 2.15.** The category of small idempotent-complete stable ∞-categories is a closed symmetric monoidal category with respect to \( \otimes^\vee \). The unit is the category \( S^\omega_\infty \) of compact spectra and the internal mapping object is given for small idempotent-complete stable ∞-categories \( A \) and \( B \) by \( \text{Fun}^\text{ex}(A, B) \).

For a small stable idempotent-complete ∞-category \( A \) and a presentable ∞-category \( B \), \( \text{Fun}^\text{ex} \) and \( \text{Fun}^L \) are related by the following universal property:

\[ \text{Fun}^\text{ex}(A, B) \simeq \text{Fun}^L(\text{Ind}(A), B). \]

Given a small stable idempotent-complete ∞-category \( A \), we have the category of \( A \)-modules given by the compactly-generated stable ∞-category \( \text{Fun}^\text{ex}(A^{\text{op}}, S) \). There is an equivalence \( \text{Fun}^\text{ex}(A^{\text{op}}, S) \simeq \text{Ind}(A) \), and the Yoneda embedding provides an exact functor

\[ A \to \text{Fun}^\text{ex}(A^{\text{op}}, S). \]

The essential image of the Yoneda embedding is precisely the compact objects in \( A \)-modules:

\[ A \simeq \text{Fun}^\text{ex}(A^{\text{op}}, S)^\omega. \]

If \( A \) is a small stable ∞-category (but not necessarily idempotent-complete), the natural Yoneda map \( A \to \text{Fun}^\text{ex}(A^{\text{op}}, S)^\omega \) models the idempotent-completion of \( A \).

We use the preceding results to characterize \( \text{Fun}^\text{ex}(A, B) \) in terms of a certain subcategory of \( \text{Fun}^L(A \otimes^\vee B^{\text{op}}, S) \) (the category of \( A \)-\( B \) bimodules). Specifically, the Yoneda embedding \( B \to \text{Fun}^\text{ex}(B^{\text{op}}, S) \) provides the following composite

\[ \text{Fun}^\text{ex}(A, B) \to \text{Fun}^\text{ex}(A, \text{Fun}^\text{ex}(B^{\text{op}}, S)) \]
\[ \to \text{Fun}^L(\text{Ind}(A), \text{Fun}^L(\text{Ind}(B^{\text{op}}), S)) \]
\[ \to \text{Fun}^L(\text{Ind}(A) \otimes \text{Ind}(B^{\text{op}}), S) \]
\[ \to \text{Fun}^\text{ex}(A \otimes^\vee B^{\text{op}}, S), \]

which exhibits \( \text{Fun}^\text{ex}(A, B) \) as a full subcategory of \( A \otimes^\vee B^{\text{op}} \)-modules.

We have the following useful corollary, which is the analogue of a characterization originally written down by Toën \[59\]. For each object \( a \in A \), the unit map \( S \to A(a,a) \) composed with the inclusion \( A(a,a) \to A \) and the Yoneda embedding \( A \to \text{Ind}(A) \) induces a map

\[ \nu_a : \text{Fun}^\text{ex}(A \otimes^\vee B^{\text{op}}, S) \to \text{Fun}^\text{ex}(B^{\text{op}}, S). \]

If the image of an element of \( \text{Fun}^\text{ex}(A \otimes^\vee B^{\text{op}}, S) \) under \( \nu_a \) is compact for every \( a \in A \), we will say that the element is right-compact.
Corollary 2.16. Let $A$ and $B$ be small stable idempotent-complete $\infty$-categories. There is an equivalence of small stable idempotent-complete $\infty$-categories between $\text{Fun}^\text{ex}(A,B)$ and $\infty$-category of right-compact $A \otimes^\vee B^{\text{op}}$-modules.

Proof. Since the Yoneda embedding is fully-faithful, it suffices to look at the essential image of the composite. The image of $\text{Fun}^\text{ex}(A,B)$ inside $\text{Fun}^\text{ex}(A \otimes^\vee B^{\text{op}}, S)$ under $\nu_a$ is identified with the image of $\text{Fun}^\text{ex}(A,B)$ inside $\text{Fun}^\text{ex}(A, \text{Fun}^\text{ex}(B^{\text{op}}, S))$ under the unit $S \to A(a,a) \to A$. Since this lies inside the image of $B$ inside $\text{Fun}^\text{ex}(B^{\text{op}}, S)$ under the Yoneda embedding, the result follows. $\square$

3. Morita theory

There is a close connection between stable $\infty$-categories and spectral categories. On the one hand, for every pair of objects in a stable $\infty$-category we can extract a mapping spectrum. On the other hand, given a category $A$ enriched in spectra, the category of (right) $A$-modules has a standard projective model structure and the associated $\infty$-category is stable.

The purpose of this section is to provide a precise account of the relationship between small spectral categories and small stable $\infty$-categories. The moral of this section is that the homotopy theory of small spectral categories localized at the Morita equivalences is the same as the homotopy theory of small stable idempotent-complete $\infty$-categories. Specifically, we prove theorem 1.9. The proof of the theorem establishes an $\infty$-categorical version of the Morita theory of [51]; that is, small stable idempotent-complete $\infty$-categories are categories of modules, and the $\infty$-category of exact functors between two such is a stable subcategory of the category of bimodules.

Establishing this correspondence serves several purposes for us. For one thing, having models of $\text{Cat}_{\text{ex}}^\infty$ and $\text{Cat}_{\text{perf}}^\infty$ as accessible localizations of an $\infty$-category which arises as the nerve of a model category provides technical control on $\text{Cat}_{\text{ex}}^\infty$ and $\text{Cat}_{\text{perf}}^\infty$; we use this to show that $\text{Cat}_{\text{ex}}^\infty$ and $\text{Cat}_{\text{perf}}^\infty$ are compactly generated in corollary 3.22. For another, it permits us to rectify diagrams of small stable $\infty$-categories to strict diagrams in $\text{Cat}_S$. We exploit this to pass to rigid models for the purposes of using Waldhausen’s $K$-theory machinery in Section 6.

We have two options for producing an $\infty$-category of spectral categories (with respect to the DK-equivalences): we can use the combinatorial simplicial model structure of Corollary 2.3 and take $N((\text{Cat}_S)^\circ)$ or we can construct $NL^H(\text{Cat}_S)$, where here $L^H(\text{Cat}_S)$ denotes the Dwyer-Kan simplicial localization of $\text{Cat}_S$ at the DK-equivalences. The resulting $\infty$-categories are equivalent, and we will refer interchangably to them as “the” $\infty$-category of small spectral categories.

Given any spectral category $C$, we can produce an associated $\infty$-category by passing to the associated simplicial category $\Omega^\infty C$ and applying the simplicial nerve: This yields a functor

$$\text{Cat}_S \to \text{Cat}_{\infty}^\Delta,$$

from the category $\text{Cat}_S$ of small spectral categories to the category $\text{Cat}_{\infty}^\Delta$ of small $\infty$-categories, which sends $C$ to the simplicial nerve $N(\Omega^\infty C)$. Precomposing with the functors $(\_)^{\text{perf}}$ and $(\_)^{\text{tri}}$, we obtain functors

$$\psi_{\text{tri}}, \psi_{\text{perf}}: \text{Cat}_S \to \text{Cat}_{\infty}^\Delta$$
and a natural transformation \( \psi_{\text{tri}} \rightarrow \psi_{\text{perf}} \). First, we observe that these functors are compatible with the weak equivalences of Theorem 2.2.

**Lemma 3.1.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be small spectral categories, and let \( f : \mathcal{A} \rightarrow \mathcal{B} \) be a DK-equivalence. Then the induced maps \( \psi_{\text{tri}}(f) \) and \( \psi_{\text{perf}}(f) \) are categorical equivalences of simplicial sets.

**Proof.** If \( f : \mathcal{A} \rightarrow \mathcal{B} \) is a DK-equivalence, then one can check that \((f_! , f^*)\) gives a Quillen equivalence between the spectral model categories \( \mathcal{A} \)-modules and the spectral model category \( \mathcal{B} \)-modules. Passing to underlying simplicial categories of cofibrant and fibrant objects, we see that \( \Omega^\infty(\mathcal{A})^o = \text{Mod}(\mathcal{A})^o \) and \( \Omega^\infty(\mathcal{B}) = \text{Mod}(\mathcal{B})^o \) are DK-equivalent simplicial categories. Finally, applying the simplicial nerve yields categorically equivalent simplicial sets. Restricting to various full subcategories yields the result for \( \psi_{\text{tri}}(f) \) and \( \psi_{\text{perf}}(f) \).

Lemma 3.1 and [16, 3.4] now imply that \( \psi_{\text{tri}} \) and \( \psi_{\text{perf}} \) induce simplicial functors between the Dwyer-Kan simplicial localization \( L^H(\text{Cat}_S) \) at the DK-equivalences and the Dwyer-Kan simplicial localization \( L^H(\text{Cat}_\Delta^\infty) \) at the categorical equivalences.

**Lemma 3.2.** There exist simplicial functors \( \tilde{\psi}_{\text{tri}} , \tilde{\psi}_{\text{perf}} \) lifting \( \psi_{\text{tri}} \) and \( \psi_{\text{perf}} \).

Therefore, applying the simplicial nerve produces functors connecting the \( \infty \)-category of spectral categories and the \( \infty \)-category of \( \infty \)-categories, equipped with a natural transformation

\[
\Psi_{\text{tri}} \longrightarrow \Psi_{\text{perf}} : N(L^H(\text{Cat}_S)) \longrightarrow N(L^H(\text{Cat}_\Delta^\infty)) \simeq \text{Cat}_\infty.
\]

In fact, by construction these functors preserve triangulated and Morita equivalences respectively. Furthermore, \( \Psi_{\text{tri}} \) lands in small stable \( \infty \)-categories and \( \Psi_{\text{perf}} \) lands in idempotent-complete stable \( \infty \)-categories respectively.

**Lemma 3.3.** Let \( \mathcal{C} \) be a small spectral category. Then \( \Psi_{\text{tri}} \mathcal{C} \) is an element of \( \text{Cat}_\infty^{\text{ex}} \) and \( \Psi_{\text{perf}} \mathcal{C} \) is an element of \( \text{Cat}_\infty^{\text{perf}} \).

Therefore, \( \Psi_{\text{tri}} \) descends to a functor from the localization of \( N(L^H(\text{Cat}_S)) \) at the image of the triangulated equivalences to the \( \infty \)-category \( \text{Cat}_\infty^{\text{ex}} \subset \text{Cat}_\infty \) and \( \Psi_{\text{perf}} \) descends to a functor from the localization of \( N(L^H(\text{Cat}_S)) \) at the image of the Morita equivalences to the \( \infty \)-category \( \text{Cat}_\infty^{\text{perf}} \subset \text{Cat}_\infty \).

**Remark 3.4.** Using the machinery of combinatorial simplicial model categories, we can also localize the combinatorial model structure of corollary 2.3 on \( \text{Cat}_S \) at the triangulated or Morita equivalences directly to obtain “triangulated” or “Morita” simplicial model categories on small spectral categories and then pass to simplicial nerves; this is equivalent to localizing the \( \infty \)-category \( N((\text{Cat}_S)^o) \).

The content of theorem 1.9 is that these functors are equivalences. We prove this theorem by producing an “inverse” to \( \Psi_{\text{tri}} \) and \( \Psi_{\text{perf}} \) such that the composite is a localization functor on \( N(L^H(\text{Cat}_S)) \).

**Definition 3.5.** A simplicial category \( \mathcal{A} \) is **stable** if the simplicial nerve of a fibrant replacement of \( \mathcal{A} \) is a stable \( \infty \)-category. A spectral category \( \mathcal{A} \) is **stable** if its underlying simplicial category \( \Omega^\infty \mathcal{A} \) is stable.
We write $\text{Cat}^\text{ex}_\Delta$ for the simplicial category of small stable simplicial categories. This is the subcategory of the simplicial category $L^H(\text{Cat}_\Delta)$ of small simplicial categories where the objects are the stable simplicial categories and the mapping spaces are computed by restriction to the vertices of the mapping space of simplicial functors which represent exact functors upon passage to the simplicial nerve.

**Proposition 3.6.** The $\infty$-category obtained by applying the simplicial nerve to $\text{Cat}^\text{ex}_\Delta$ is equivalent to the $\infty$-category $\text{Cat}^\text{ex}_\infty$.

**Proof.** It suffices to show that the mapping spaces in $\text{Cat}^\text{ex}_\Delta$ have the correct homotopy type, and this follows from the comparison between the mapping spaces of $\text{Cat}_\Delta$ and $\text{Cat}_\infty$ [33, 2.2.0.1] and the fact that on both sides we define the mapping spaces by (the same) restriction of vertices. □

That is, the equivalence (induced by the simplicial nerve [33, 2.2.0.1])

$$N(L^H(\text{Cat}_\Delta)) \longrightarrow \text{Cat}_\infty$$

restricts to an equivalence

$$N(L^H(\text{Cat}^\text{ex}_\Delta)) \longrightarrow \text{Cat}^\text{ex}_\infty.$$  

Recall that a spectrum object of a pointed $\infty$-category $\mathcal{C}$ consists of a bi-indexed family of objects $A(i,j)$ of $\mathcal{C}$ and maps $A(i,j) \rightarrow A(i+1,j), A(i,j) \rightarrow A(i,j+1)$ such that $A(i,j)$ is zero object whenever $i \neq j$ and the square

$$\begin{array}{ccc} A(i,i) & \longrightarrow & A(i,i+1) \\ \downarrow & & \downarrow \\ A(i+1,i) & \longrightarrow & A(i+1,i+1) \end{array}$$

is cartesian for all $i$ [34, §8]. Since the restriction of $A$ to the diagonal carries all the relevant information of $A$, we set $A_i = A(i,i)$ and often refer to $A$ simply by the collection of pointed objects $\{A_i\}$. We write $\text{Sp}(\mathcal{C})$ for the $\infty$-category of spectrum objects in $\mathcal{C}$ and $\Omega^\infty : \text{Sp}(\mathcal{C}) \rightarrow \mathcal{C}$ for the functor which associates to the spectrum object $A$ its zero space $A_0 = A(0,0)$. This is an explicit model for the stabilization $\text{Stab}(\mathcal{C})$ discussed previously. We will usually just write $S_\infty$ or $\text{Sp} := \text{Sp}(N^T)$ for the $\infty$-category $N\Sigma^\infty$ of spectra.

If $\mathcal{C}$ is a stable $\infty$-category, then $\Omega^\infty : \text{Sp}(\mathcal{C}) \rightarrow \mathcal{C}$ is an equivalence with inverse $\Sigma^\infty : \mathcal{C} \rightarrow \text{Sp}(\mathcal{C})$ given by $(\Sigma^\infty a)_i = \Sigma^i a$. In this case, the pointed Yoneda embedding

$$\mathcal{C} \longrightarrow \text{Fun}(\mathcal{C}^{\text{op}}, N(T^0)_*)$$

defined by the assignment $a \mapsto \text{map}(-,a)/\text{map}_0(-,a)$ (the cofiber of the inclusion of the zero maps $\text{map}_0(-,a) \rightarrow \text{map}(-,a)$) gives rise to a spectral Yoneda embedding

$$\mathcal{C} \simeq \text{Sp}(\mathcal{C}) \longrightarrow \text{Sp}(\text{Fun}(\mathcal{C}^{\text{op}}, N(T^0)_*)) \simeq \text{Fun}(\mathcal{C}^{\text{op}}, S_\infty),$$

where the last equivalence follows from the fact that limits in functor categories are computed pointwise. The spectral Yoneda embedding is adjoint to the mapping spectrum functor $\text{Map} : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow S_\infty$, which is described by the formula $\text{Map}(a,b) = \text{map}(a,\Sigma^\infty b)$; that is, $\text{Map}(a,b)_i \simeq \text{map}(a,\Sigma^i b)$. We wish to characterize the image of $\mathcal{C}$ under the spectral Yoneda embedding:
Definition 3.7. Let $\mathcal{C}$ be a pointed $\infty$-category. Then we will say that a functor $X : \mathcal{C}^{\text{op}} \to S_\infty$ is stably representable if there exists a spectrum object $A \in \text{Sp}(\mathcal{C})$ and an equivalence between $\text{Map}(-, A)$ and $X$.

Proposition 3.8. Let $\mathcal{C}$ be a stable $\infty$-category. Then a functor $X : \mathcal{C}^{\text{op}} \to S_\infty$ is stably representable if and only if it is representable.

Proof. This follows from the spectral Yoneda embedding $\mathcal{C} \simeq \text{Sp}(\mathcal{C}) \to \text{Fun}(\mathcal{C}^{\text{op}}, S_\infty)$.

We now produce a lifting of the ($\infty$-categorical) spectral Yoneda embedding to stable simplicial categories. Given a stable simplicial category $\mathcal{C}$, the simplicial category of simplicial functors $\text{Fun}(\Delta, \text{Sp}(\mathcal{C}))$ is a spectral model category (under the projective model structure) with "underlying" $\infty$-category $\text{Fun}(\text{NC}^{\text{op}}, S_\infty)$. Since $\text{NC}$ is stable, the $\infty$-categorical spectral Yoneda embedding $\text{NC} \to \text{Fun}(\text{NC}^{\text{op}}, S_\infty)$ allows us to make the following definition.

Definition 3.9. Let $\text{Spec}(\mathcal{C})$ be the full spectral subcategory of $\text{Fun}(\Delta, \text{Sp}(\mathcal{C}))$ consisting of those objects $X$ of $\text{Fun}(\Delta, \text{Sp}(\mathcal{C}))$ for which $\text{NC} : \text{NC}^{\text{op}} \to \text{NS}_\infty$ is stably representable.

Here we are making use of the equivalence of $\infty$-categories $\text{Fun}(\mathcal{C}^{\text{op}}, S_\infty) \to \text{Fun}(\text{NC}^{\text{op}}, S_\infty)$.

Observe that if $\mathcal{C}$ is a stable simplicial category, then $\Omega^\infty \text{Spec}(\mathcal{C}) \simeq \mathcal{C}$ since $\mathcal{C}$ is stable, and so in this case $\Omega^\infty \text{Spec}(\mathcal{C}) \simeq \mathcal{C}$.

Proposition 3.10. The assignment which associates to the stable simplicial category $\mathcal{C}$ the spectral category $\text{Spec}(\mathcal{C})$ defines a simplicial functor $\text{Spec} : L^H(\text{Cat}^{\text{ex}}_\Delta) \to L^H(\text{Cat}_S)$.

Proof. We first check that the construction of $\text{Spec}$ actually induces a functor $\text{Cat}^{\text{ex}}_\Delta \to \text{Cat}_S$. Let $f : \mathcal{C} \to \mathcal{D}$ be a map of stable simplicial categories and write $f^\circ : \text{Fun}_{\Delta}(\mathcal{C}, S)^\circ \to \text{Fun}_{\Delta}(\mathcal{D}, S)^\circ$ for the induced spectral functor. Suppose that $X : \mathcal{C} \to \text{Sp}$ is projectively cofibrant and fibrant and $\text{NX} : \text{NC}^{\text{op}} \to \text{NS}_\infty$ is stably representable via the spectrum object $\{A_i\}$ in $\mathcal{C}$. Since the diagram

\[
\begin{array}{ccc}
\text{NC} & \to & \text{ND} \\
\downarrow & & \downarrow \\
\text{Fun}(\text{NC}^{\text{op}}, S_\infty) & \longrightarrow & \text{NFun}(\mathcal{D}^{\text{op}}, S_\infty)
\end{array}
\]

commutes (where the vertical maps are the stable Yoneda embeddings), we see that $f^\circ$ restricts to a spectral functor $\text{Sp}(\mathcal{C}) \to \text{Sp}(\mathcal{D})$.

To verify that $\text{Spec}$ induces a simplicial functor between the Dywer-Kan simplicial localizations, we must check that it preserves equivalences of stable simplicial categories. So suppose that $f : \mathcal{C} \to \mathcal{D}$ is an equivalence of stable simplicial categories. Then it follows immediately that $f^\circ$ is a DK-equivalence of spectral categories, as is its restriction to the stably representable objects. \qed
Let $M : \text{Cat}_S \to \text{Cat}_S$ denote the composite functor

$$M : \text{Cat}_S^{\psi_{\text{tri}}} \xrightarrow{\psi_{\text{tri}}} \text{Cat}_S^{\text{ex}} \xrightarrow{\text{Spec}} \text{Cat}_S$$

The work above shows that $M$ induces a simplicial functor on the Dwyer-Kan simplicial localizations and therefore descends to the level of $\infty$-categories to produce a functor

$$NM : NL^H(\text{Cat}_S) \to NL^H(\text{Cat}_S).$$

There is a natural transformation $\eta : \text{id} \to M$ which sends the object $a$ to the object

$$\Sigma^\infty \Omega^\infty \text{map}_{A_{\infty}}(-, \widehat{a}),$$

where $\widehat{a} = \text{map}_A(-, a)$.

**Proposition 3.12.** For any spectral category $A$, the spectral functor $\eta_A : A \to MA$ is fully-faithful.

**Proof.** Given two objects $a$ and $b$ of $A$, we have equivalences

$$\text{Map}_{MA}(\eta_A(a), \eta_A(b)) \simeq \text{map}_{A_{\infty}}(\widehat{a}, \Sigma^i \widehat{b}) \simeq \Omega^\infty \Sigma^i \text{Map}_A(a, b) \simeq \text{Map}_A(a, b),$$

and a diagram chase shows that these equivalences are compatible with the natural map induced by $\eta_A$. □

**Proposition 3.13.** The spectral functor $\eta_A : A \to MA$ is essentially surjective if and only if $A$ is stable.

**Proof.** Indeed, $A \to MA$ is essentially surjective if and only if $\Omega^{\infty} A \to \Omega^{\infty} MA$ is essentially surjective, which is the case if and only if $A$ is already stable. □

Combining propositions 3.12 and 3.13, we obtain the following corollary.

**Corollary 3.14.** The spectral functors $\eta_{MA}$ and $M \eta_A$ induce equivalent functors on passage to simplicial nerves. In particular, both $\eta_{MA}$ and $M \eta_A$ are equivalences.

Next, we want to verify that $M$ is suitable to compute a localization.

**Proposition 3.15.** The pair of natural transformations $\eta_{MA}, M \eta_A : MA \to M^2A$ induce a square

$$\begin{array}{c}
A \xrightarrow{\eta_A} MA \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
MA \xrightarrow{M \eta_A} M^2A
\end{array}$$

which commutes up to homotopy after passage to simplicial nerves.

**Proof.** First note that $M \eta_A : MA \to M^2A$ sends $x : \widehat{A} \to S_{\infty}$ to the functor $\widehat{\eta}_{A_{\infty}} : \widehat{MA} \to S_{\infty}$ induced by homotopy left Kan extension along $\widehat{\eta}_A : \widehat{A} \to \widehat{MA}$. If $x = \text{Map}_A(-, a)$ is represented by the object $a$ of $A$, then the universal properties of representable functors and homotopy left Kan extensions force an equivalence $\widehat{\eta}_{A_{\infty}} x \simeq \text{Map}(-, \eta_A(a))$, so that $\widehat{\eta}_{A_{\infty}} x$ is represented by $\eta_A(a)$. It follows that the restrictions of $\eta_{MA}$ and $M \eta_A$ to $A$ are equivalent. □

**Corollary 3.16.** The spectral functors $\eta_{MA}$ and $M \eta_A$ induce equivalent functors on passage to simplicial nerves. In particular, both $\eta_{MA}$ and $M \eta_A$ are equivalences.
Proof. Since $\mathcal{A}$ generates $M\mathcal{A}$ under finite homotopy colimits and desuspensions, it suffices to show that $M\eta_\mathcal{A}$ preserves finite homotopy colimits and desuspensions. The fact that $M\eta_\mathcal{A}$ preserves finite homotopy colimits follows from the fact that $M\eta_\mathcal{A}$ is homotopy left Kan extension along $\hat{\eta}_\mathcal{A}$. But suspension is an example of a finite homotopy colimit, so we have that $M\eta_\mathcal{A}(\Sigma x) \simeq \Sigma M\eta_\mathcal{A}(x)$. Hence $M\eta_\mathcal{A}(x) \simeq \Sigma^{-1} M\eta_\mathcal{A}(x)$. The final statement is a consequence of corollary 3.14 and the fact that $M\mathcal{A}$ is a stable spectral category. □

Corollary 3.17. The functor $NM : N(L^H(Cats)) \to N(L^H(Cats))$ defines a localization of $N(L^H(Cats))$ with essential image the simplicial category of stable spectral categories.

Proof. Passing to nerves, $M$ induces an endofunctor $NM$ of $N(Cats)$, equipped with a natural transformation $\eta_N : \text{id} \to NM$, such that for each (cofibrant and fibrant) spectral category $\mathcal{A}$, the maps $\eta_{M\mathcal{A}}, M\eta_\mathcal{A} : M\mathcal{A} \to M^2\mathcal{A}$ are equivalences. The result is now an immediate consequence of [33, Prop. 5.2.7.4]. □

Following Definition 2.7, we make the following definitions.

Definition 3.18. A map of small spectral $\infty$-categories $f : \mathcal{A} \to \mathcal{B}$ is:

- A triangulated equivalence if $\Psi_{\tri f} : \Psi_{\tri A} \to \Psi_{\tri B}$ is an equivalence of (stable) $\infty$-categories, and
- A Morita equivalence if $\Psi_{\perf f} : \Psi_{\perf A} \to \Psi_{\perf B}$ is an equivalence of (idempotent-complete) stable $\infty$-categories.

Now, assembling the work of this section we obtain the following two localization results.

Theorem 3.19. The functor $\Psi_{\tri} : N(L^H(Cats)) \to \text{Cat}_\infty^{\text{ex}}$ admits a fully-faithful right adjoint

$$\text{Spec} : \text{Cat}_\infty^{\text{ex}} \to N(L^H(Cats)).$$

That is, the $\infty$-category of stable $\infty$-categories is the localization of the $\infty$-category of spectral categories obtained by inverting the triangulated equivalences.

Proof. The follows from the factorization $M : \text{Cat}_\mathcal{S} \to \text{Cat}_\Delta^{\text{ex}} \to \text{Cat}_\mathcal{S}$ (recall equation 3.11) and the fact that $\text{Cat}_\mathcal{S}^{\text{ex}} \simeq N\text{Cat}_\Delta^{\text{ex}}$. □

Recall that we have a stable idempotent completion functor $\text{Idem} : \text{Cat}_\mathcal{S}^{\text{ex}} \to \text{Cat}_\mathcal{S}^{\text{perf}}$. Since $\text{Idem}$ is left adjoint to the (fully-faithful) inclusion $\text{Cat}_\mathcal{S}^{\text{perf}} \to \text{Cat}_\mathcal{S}^{\text{ex}}$, $\text{Cat}_\mathcal{S}^{\text{perf}}$ is the localization of $\text{Cat}_\mathcal{S}^{\text{ex}}$ obtain by inverting idempotent completion maps. Further, recall that there is an equivalence $\text{Idem} \circ \Psi_{\tri} \simeq \Psi_{\perf}$. □

Theorem 3.20. The functor $\Psi_{\perf} : N(L^H(Cats)) \to \text{Cat}_\mathcal{S}^{\text{perf}}$ admits a fully-faithful right adjoint

$$\text{Spec} : \text{Cat}_\mathcal{S}^{\text{perf}} \to N\text{Cat}_\mathcal{S}^{\text{S}}.$$ 

That is, the $\infty$-category of idempotent-complete stable $\infty$-categories is the localization of the $\infty$-category of spectral categories obtained by inverting the Morita equivalences.
Proof. The ∞-category of idempotent-complete stable ∞-categories is a localizing subcategory of the ∞-category of stable ∞-categories. □

Remark 3.21. Theorems 3.19 and 3.20 imply that computing the localizations of the model category structure on spectral categories from Corollary 2.3 at the triangulated and Morita equivalences (as discussed in Remark 3.4) and passing to the simplicial nerve also yields the ∞-categories Cat^ex_∞ and Cat^perf_∞ respectively.

We conclude the section with the promised applications of the theory. First, we immediately obtain the following corollary about the structure of Cat^ex_∞ and Cat^perf_∞.

Corollary 3.22. The ∞-categories Cat^ex_∞ and Cat^perf_∞ are compactly generated, complete, and cocomplete.

A further consequence of these results is the following compatibility result about the symmetric monoidal structure. Here we refer to the symmetric monoidal product on spectral categories studied in [55], generalizing the DG-category definition of [59].

Corollary 3.23. Let A and B be idempotent-complete pretriangulated spectral categories. Then there is a Morita equivalence

\[ \Psi_{\text{perf}}(A \wedge L B) \simeq \Psi_{\text{perf}}(A) \otimes^\vee \Psi_{\text{perf}}(B). \]

Similarly, we make the following definition.

Definition 3.24. Let A and B be small idempotent-complete stable ∞-categories. We write \text{rep}(B, A) = \text{Spec}(\text{Fun}^ex(B, A)) for the small pretriangulated spectral category associated to the small stable ∞-category of exact functors from B to A.

Corollary 3.25. Let A and B be small idempotent-complete stable ∞-categories and let A and B be spectral categories lifting A and B. Then \text{Nrep}(A, B) \simeq \text{Fun}^ex(A, B) is equivalent to the ∞-category of right-compact A^L \otimes B^{op}-modules.

Proof. This follows from theorem 3.20 and corollary 2.16. □

Finally, we record an important technical lifting result which is a consequence of the work in this section.

Proposition 3.26. Let I be a small category. Given a diagram D of small stable ∞-categories indexed by N(I), there exists an I-diagram of pretriangulated spectral categories D lifting D.

Proof. This is a consequence of [33, 4.2.4.4]. Given a diagram of small stable ∞-categories, the equivalence proved above gives rise to a diagram in the localization of N(L^H(Cat_S)). Including the localization into N(L^H(Cat_S)), we now obtain a diagram in N(L^H(Cat_S)) and we can use [33, 4.2.4.4] to lift this to a rigid diagram in Cat_S. □

4. Exact sequences

In this section we discuss the various definitions of exact sequence, relating notions for triangulated categories, spectral categories, and stable ∞-categories. Arguably the most fundamental definition is that of an exact sequence of triangulated...
categories, as it turns out that exact sequences in both spectral and stable \(\infty\)-categories can be detected on the level of the homotopy category.

Recall that a sequence of triangulated categories

\[
\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}
\]

is called exact if the composite is zero, the functor \(\mathcal{A} \rightarrow \mathcal{B}\) is fully-faithful and the induced functor from the Verdier quotient \(\mathcal{B}/\mathcal{A}\) to \(\mathcal{C}\) is cofinal, i.e., it becomes an equivalence after idempotent completion. Said differently, a triangulated functor \(\mathcal{C}' \rightarrow \mathcal{C}\) is cofinal if every object of \(\mathcal{C}\) is a summand of an object of \(\mathcal{C}'\).

There are analogues of these notions for presentable stable \(\infty\)-categories. Recall that the Ind category participates in an equivalence between \(\text{Cat}_{\text{perf}}^{\infty}\) and the \(\infty\)-category of compactly generated stable \(\infty\)-categories. As a consequence, corollary 3.22 implies that the category of compactly generated stable \(\infty\)-categories is cocomplete.

Throughout this section, we let \(\kappa\) denote an infinite regular cardinal.

**Definition 4.1.** Let \(\mathcal{A}\) and \(\mathcal{B}\) be \(\infty\)-categories and let \(f : \mathcal{A} \rightarrow \mathcal{B}\) be a functor. We say that \(\mathcal{A}\) is \(\kappa\)-closed if \(\mathcal{A}\) is \(\kappa\)-cocomplete, that is, if \(\mathcal{A}\) admits all \(\kappa\)-small colimits, and we say that \(f\) is \(\kappa\)-continuous if \(f\) preserves \(\kappa\)-filtered colimits.

We write \(\text{Cat}_{\infty}^{\text{ex}(\kappa)}\) for the \(\infty\)-category of small \(\kappa\)-closed stable \(\infty\)-categories and \(\kappa\)-small colimit-preserving functors thereof; note that if \(\kappa > \omega\), any small \(\kappa\)-closed stable \(\infty\)-category \(\mathcal{A}\) is necessarily idempotent complete. Given a small \(\kappa\)-closed stable \(\infty\)-category \(\mathcal{A}\), the \(\infty\)-category \(\text{Ind}_{\kappa}(\mathcal{A})\) is a \(\kappa\)-compactly generated stable \(\infty\)-category such that \(\mathcal{A} \simeq \text{Ind}_{\kappa}(\mathcal{A})^\kappa\) [33, Proposition 5.5.7.10]. In fact, provided \(\kappa > \omega\), restriction to subcategories of \(\kappa\)-compact objects determines an equivalence between the \(\infty\)-category of \(\kappa\)-compactly generated stable \(\infty\)-categories \(LPr_{\kappa}\) and the \(\infty\)-category \(\text{Cat}_{\infty}^{\text{ex}(\kappa)}\) of small \(\kappa\)-closed stable \(\infty\)-categories, with inverse \(\text{Ind}_{\kappa}\).

**Lemma 4.2.** Let \(\mathcal{A}\) be a small \(\infty\)-category and let \(S\) be a set of arrows of \(\mathcal{A}\). Then the natural map \(\text{Ind}_{\kappa}(\mathcal{A})[S^{-1}] \rightarrow \text{Ind}_{\kappa}(\mathcal{A})^\kappa\) is an equivalence.

**Proof.** Let \(\mathcal{B}\) be an \(\infty\)-category with \(\kappa\)-filtered colimits. Then, by the universal properties of \(\text{Ind}_{\kappa}\) and localization, the square

\[
\begin{array}{ccc}
\text{Fun}_{\kappa}(\text{Ind}_{\kappa}(\mathcal{A})[S^{-1}], \mathcal{B}) & \rightarrow & \text{Fun}_{\kappa}(\text{Ind}_{\kappa}(\mathcal{A}), \mathcal{B}) \\
\downarrow & & \downarrow \\
\text{Fun}(\mathcal{A}[S^{-1}], \mathcal{B}) & \rightarrow & \text{Fun}(\mathcal{A}, \mathcal{B})
\end{array}
\]

is cartesian. Hence the left vertical map is an equivalence. But the inclusion \(\mathcal{A}[S^{-1}] \rightarrow \text{Ind}_{\kappa}(\mathcal{A})[S^{-1}]\) induces an equivalence

\[
\text{Fun}_{\kappa}(\text{Ind}_{\kappa}(\mathcal{A}[S^{-1}], \mathcal{B}) \rightarrow \text{Fun}(\mathcal{A}[S^{-1}], \mathcal{B}),
\]

so it follows that \(\text{Ind}_{\kappa}(\mathcal{A})[S^{-1}] \rightarrow \text{Ind}_{\kappa}(\mathcal{A}[S^{-1}])\) is also an equivalence. \(\Box\)

**Lemma 4.3.** Let \(\mathcal{B}\) be a small \(\infty\)-category and \(S\) a set of arrows of \(\mathcal{B}\). Then the natural map \(\text{Ind}(\mathcal{B})^\kappa[S^{-1}] \rightarrow \text{Ind}(\mathcal{B}[S^{-1}])^\kappa\) is an equivalence.
Proof. The fact that the map $\Ind(B)^\kappa[S^{-1}] \to \Ind(B[S^{-1}])^\kappa$ is fully faithful follows from the commuting square

$$
\begin{array}{ccc}
\Ind(B)^\kappa[S^{-1}] & \longrightarrow & \Ind(B[S^{-1}])^\kappa \\
\downarrow & & \downarrow \\
\Ind_\kappa(\Ind(B)^\kappa[S^{-1}]) & \longrightarrow & \Ind_\kappa(\Ind(B[S^{-1}])^\kappa)
\end{array}
$$

since by lemma 4.2, the bottom horizontal map is an equivalence. For essential surjectivity, note that the map $\Ind(B) \to \Ind(B[S^{-1}])$ is essentially surjective, and that any preimage of a $\kappa$-compact object in $\Ind(B[S^{-1}])$ is $\kappa$-compact, as the right adjoint $\Ind(B[S^{-1}]) \to \Ind(B)$ preserves $\omega$-filtered colimits and thus also $\kappa$-filtered colimits. Hence any object in $\Ind(B[S^{-1}])^\kappa$ has a preimage in $\Ind(B)^\kappa$, so the result follows from the fact that the map $\Ind(B)^\kappa \to \Ind(B[S^{-1}])^\kappa$ factors through $\Ind(B)^\kappa[S^{-1}]$. \qed

Lemma 4.4. Let $\mathcal{A}$ be a small $\infty$-category, let $S$ be a set of arrows of $\mathcal{A}$, and let $T$ be a set of arrows of $\Ind_\kappa(\mathcal{A})$ which contains $S$ (viewed as a set of arrows of $\Ind_\kappa(\mathcal{A})$ via the inclusion $\mathcal{A} \to \Ind_\kappa(\mathcal{A})$) such that the elements of $T$ become invertible in $\Ind_\kappa(\mathcal{A})[S^{-1}]$. Then $\Ind_\kappa(\mathcal{A})[T^{-1}] \simeq \Ind_\kappa(\mathcal{A})[S^{-1}]$.

Proof. By assumption, the identity map $\Ind_\kappa(\mathcal{A})[S^{-1}] \to \Ind_\kappa(\mathcal{A})[S^{-1}]$ factors as the composite

$$\Ind_\kappa(\mathcal{A})[S^{-1}] \to \Ind_\kappa(\mathcal{A})[T^{-1}] \to \Ind_\kappa(\mathcal{A})[S^{-1}].$$

Hence $\Ind_\kappa(\mathcal{A})[S^{-1}] \to \Ind_\kappa(\mathcal{A})[T^{-1}]$ is fully faithful. But, since $T$ contains $S$, $\Ind_\kappa(\mathcal{A})[T^{-1}]$ is a localization of $\Ind_\kappa(\mathcal{A})[S^{-1}]$, and any localization map is essentially surjective. \qed

Proposition 4.5. Let $A \to B$ be a fully-faithful inclusion of presentable stable $\infty$-categories, and write $B/A$ for the localization of $B$ with respect to the collection of maps $f$ for which $\Cone(f)$ is in the essential image of $A$. Then $B/A$ is a presentable stable $\infty$-category, and the localization map $B \to B/A$ admits a fully-faithful right adjoint $B/A \to B$.

Proof. We must show that the localization $B \to B/A$ is accessible, which implies that $B/A$ is presentable, and also that $B/A$ is a stable subcategory of $B$ (via the fully faithful right adjoint $B/A \to B$).

By assumption, there exists a cardinal $\kappa$ such that $A \to B$ preserves $\kappa$-compact objects and $A \simeq \Ind_\kappa(\mathcal{A}^\kappa)$, $B \simeq \Ind_\kappa(\mathcal{B}^\kappa)$. Let $S$ denote the set of arrows in $\mathcal{B}^\kappa$ whose cones lie in (the essential image of) $\mathcal{A}^\kappa$, and let $T$ denote the (not necessarily small) collection of arrows in $\mathcal{B}$ whose cones lie in $\mathcal{A}$. Clearly we have a map $B[S^{-1}] \to B[T^{-1}] \simeq B/A$, and this map is a localization, so it is enough to show that the arrows in $T$ become invertible in $B[S^{-1}]$.

To see this, let $X \to Y$ be an element of $T$ cofiber $Z$, i.e. $Z$ is in the essential image of $A$. Then $Z = \colim_\alpha Z_\alpha$ is a $\kappa$-filtered colimit of $\kappa$-compact objects $Z_\alpha$, and $Y = \colim_\alpha Y_\alpha$ is a $\kappa$-filtered colimit of objects $Y_\alpha = Y \times Z_\alpha$. Of course, $Y_\alpha$ may not be $\kappa$-compact, so write $Y_\alpha = \colim_\beta Y_{\alpha\beta}$ for $\kappa$-compact objects $Y_{\alpha\beta}$, and set $Z_{\alpha\beta} = Z_\alpha$. Then $Y \to Z$ is a $\kappa$-filtered colimit of maps $Y_{\alpha\beta} \to Z_{\alpha\beta}$. Repeating this procedure with $Y_{\alpha\beta}$ and $X$, we obtain cofiber sequences $X_{\alpha\beta\gamma} \to Y_{\alpha\beta\gamma} \to Z_{\alpha\beta\gamma}$ between $\kappa$-compact objects such that $Z_{\alpha\beta\gamma}$ is in the essential image of $\mathcal{A}^\kappa$. Hence
\[ X_{\alpha \beta \gamma} \to Y_{\alpha \beta \gamma} \] is invertible in \( B[S^{-1}] \), so its colimit \( X \to Y \) is as well. It follows that \( B/A \) is an accessible localization of \( B \) and therefore that \( B/A \) is a presentable \( \infty \)-category by [33, Prop. 5.5.4.15].

An object \( b \) of \( B \) lies in the full subcategory \( B/A \) if and only if \( \text{map}(a, b) \simeq * \) for each object \( a \) of \( A \), as these are the objects which see maps in \( B \) with cofibers in \( A \) as equivalences. Evidently, this is a stable subcategory of \( B \), for it contains a zero object and is closed under formation of limits and therefore finite colimits as well. \( \square \)

The localization \( B/A \) can be characterized in terms of the cofiber of the inclusion in the \( \infty \)-category of stable \( \infty \)-categories.

**Proposition 4.6.** Let \( i : A \to B \) be a fully-faithful inclusion of presentable stable \( \infty \)-categories. Then the quotient \( B/A \) is equivalent to the cofiber of \( i \).

**Proof.** We have a natural map from the cofiber \( C \) of \( i \) to \( B/A \), since the composite \( A \to B \to C \) is trivial. Thus, the Yoneda lemma implies that it suffices to check that \( \text{Fun}(B/A, D) \simeq \text{Fun}(C, D) \) for any presentable stable \( \infty \)-category \( D \). Equivalently, we must check that we have a fiber sequence

\[ \text{Fun}(B/A, D) \to \text{Fun}(B, D) \to \text{Fun}(A, D), \]

which is to say that \( \text{Fun}(B/A, D) \) is the full subcategory of \( \text{Fun}(B, D) \) on those functors \( \varphi \) whose restriction to \( A \) is trivial. But by the universal property of \( B/A \), \( \text{Fun}(B/A, D) \) is the full subcategory of \( \text{Fun}(B, D) \) of those \( \varphi \) such that

\[ \varphi(\text{Cone}(f)) \simeq \text{Cone}(\varphi(f)) \simeq 0 \]

for those \( f \) in \( B \) with cofibers in \( A \); that is, \( \varphi(a) \simeq 0 \) for all objects \( a \) of \( A \). \( \square \)

This suggests the following definition.

**Definition 4.7.** A sequence of presentable stable \( \infty \)-categories \( A \to B \to C \) is exact if the composite is trivial, \( A \to B \) is fully-faithful, and the map \( B/A \to C \) is an equivalence.

Somewhat surprisingly, we can detect exact sequences on the level of homotopy categories, despite the fact that functors which are fully-faithful on homotopy categories are not typically fully-faithful as functors of \( \infty \)-categories.

**Proposition 4.8.** Let \( A \to B \) be a fully-faithful inclusion of finitely presentable stable \( \infty \)-categories. Then the natural map \( \text{Ho}(B)/\text{Ho}(A) \to \text{Ho}(B/A) \) is an equivalence. In other words, \( \text{Ho} \) preserves cofibers of fully-faithful functors.

**Proof.** The localization on the level of stable \( \infty \)-categories descends to a localization on the level of triangulated categories. Thus it is enough to show that \( \text{Ho}(B/A) \) is the full subcategory of \( \text{Ho}(B) \) on those objects \( b \) with \( \pi_0 \text{map}(a, b) \simeq * \) for each object \( a \) of \( A \). In other words, if \( \pi_0 \text{map}(a, b) \simeq * \) for all objects \( a \) of \( A \) then \( \text{map}(a, b) \simeq * \) for all objects \( a \) of \( A \). This follows from the fact that \( A \) is a stable subcategory of \( B \), for then \( \pi_n \text{map}(a, b) \simeq \pi_0 \text{map}(\Sigma^a, b) \simeq * \), so \( \text{map}(a, b) \simeq * \). \( \square \)

The argument for the previous proposition also implies the following characterization of fully-faithful maps.

**Proposition 4.9.** A map of stable \( \infty \)-categories \( A \to B \) is fully-faithful if and only if \( \text{Ho}A \to \text{Ho}B \) is fully-faithful.
**Corollary 4.10.** A map of stable $\infty$-categories $A \to B$ is an equivalence if and only if $\Ho A \to \Ho B$ is an equivalence.

We now extend this definition to the $\infty$-category $\Cat_{\infty}^{\kappa(x)}$ of small $\kappa$-cocomplete stable $\infty$-categories and $\kappa$-cocontinuous functors.

**Definition 4.11.** A sequence of small $\kappa$-cocomplete stable $\infty$-categories $A \to B \to C$ is exact if $\Ind_\kappa(A) \to \Ind_\kappa(B) \to \Ind_\kappa(C)$ is an exact sequence of presentable stable $\infty$-categories.

Proposition 4.8 now implies the expected characterization of exact sequences of small $\kappa$-cocomplete stable $\infty$-categories.

**Proposition 4.12.** A sequence of small $\kappa$-cocomplete stable $\infty$-categories $A \to B \to C$ is exact if and only if the associated sequence $\Ho A \to \Ho B \to \Ho C$ of triangulated categories is exact.

**Proof.** The “if” direction follows from proposition 4.8 and the fact that $\Ho A \to \Ho B$ is full whenever $A \to B$ is full. The “only if” direction follows from proposition 4.9 and the fact that, given a sequence $A \to B \to C$, then $C \equiv B/A$ if $\Ho(C) \equiv \Ho(B)/\Ho(A)$. To see the latter, note that $C \equiv B/A$ is the full subcategory of $B$ on those objects $b$ with $\text{map}(a,b) \equiv \ast$ for all objects $a$ of $A$. By assumption, $\Ho(B/A) \equiv \Ho(B)/\Ho(A) \equiv \Ho(C)$, so that $\Ho(B/A)$ is the full subcategory of $\Ho(B)$ on those objects $b$ with $\pi_0 \text{map}(a,b) \equiv \ast$ for all objects $a$ of $A$. Since $A$ is stable, it follows that $\pi_0 \text{map}(a,b) \equiv 0$ if and only if $\pi_n \text{map}(a,b) \equiv 0$ for all natural numbers $n$. □

**Proposition 4.13.** Let $A \to B \to C$ be an exact sequence of small stable $\infty$-categories. Then for any regular cardinal $\kappa \geq \omega$, $\Ind_\kappa(A)^\kappa \to \Ind_\kappa(B)^\kappa \to \Ind_\kappa(C)^\kappa$ is an exact sequence of (idempotent-complete) small stable $\infty$-categories.

**Proof.** Without loss of generality we may assume that $A$, $B$, and $C$ are idempotent-complete, in which case the exactness condition means precisely that $C \equiv B[S^{-1}]$ is the localization of $B$ with respect to the set of maps $S$ in $B$ whose cones lie in full subcategory of $B$ determined by the fully faithful inclusion $A \to B$. Since $\Ind(-)$ preserves fully faithful functors, we must verify that $\Ind(B[S^{-1}])^\kappa$ is equivalent to the localization of $\Ind(B)^\kappa$ with respect to the set of maps $T$ in $\Ind(B)^\kappa$ whose cones lie in (the essential image of) $\Ind(A)^\kappa$.

To see this, we show that the natural map $\Ind(B)^\kappa[S^{-1}] \to \Ind(B)^\kappa[T^{-1}]$ is an equivalence; the result then follows from the two-out-of-three property by lemma 4.3. Since this map is a localization, it is automatically essentially surjective. To see that the map is fully faithful, consider the commuting square

$$
\begin{array}{ccc}
\Ind(B)^\kappa[S^{-1}] & \longrightarrow & \Ind(B)^\kappa[T^{-1}] \\
\downarrow & & \downarrow \\
\Ind_\kappa(\Ind(B)^\kappa[S^{-1}]) & \longrightarrow & \Ind_\kappa(\Ind(B)^\kappa[T^{-1}])
\end{array}
$$

in which (by lemmas 4.2 and 4.3) the vertical maps are inclusions of subcategories of $\kappa$-compact objects. Clearly, the arrows of $\Ind(B)$ which lie in $T$ (viewed as a set of arrows in $\Ind(B)$ via the inclusion of the $\kappa$-compact objects $\Ind(aB)^\kappa \to \Ind(B)$)
become invertible in $\text{Ind}(\mathcal{B}[S^{-1}])$, so according to lemmas 4.2 and 4.4, we have equivalences

$$\text{Ind}_\kappa(\text{Ind}(\mathcal{B}^\kappa[S^{-1}])) \simeq \text{Ind}(\mathcal{B})[S^{-1}] \simeq \text{Ind}(\mathcal{B})[T^{-1}] \simeq \text{Ind}_\kappa(\text{Ind}(\mathcal{B}^\kappa[T^{-1}])).$$

Since the other maps in this square are fully faithful, the top horizontal map must be fully faithful as well. \qed

**Definition 4.14.** An exact sequence of small $\kappa$-cocomplete stable $\infty$-categories

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is called split-exact if there exist exact functors $i: \mathcal{B} \to A$ and $j: \mathcal{C} \to \mathcal{B}$, right adjoint to $f$ and $g$, respectively, such that $i \circ f \simeq \text{Id}$ and $g \circ j \simeq \text{Id}$ via the adjunction morphisms.

In order to localize with respect to the (split-) exact sequences, we need to be able to choose a set of representatives that generate them under filtered colimits.

**Lemma 4.15.** The full subcategory $(\text{Cat}_{\text{perf}}^\kappa)^\omega \subset \text{Cat}_{\text{perf}}^\infty$ of compact small stable idempotent-complete $\infty$-categories is essentially small.

**Proof.** The result follows from the fact that $\text{Cat}_{\text{perf}}^\infty$ is an accessible localization of $\text{Cat}_{\text{ex}}^\infty$, and $\text{Cat}_{\text{ex}}^\infty$ itself is an accessible localization of the finitely presentable $\infty$-category of small spectral categories via theorem 1.9. \qed

This has the following immediate and essential corollary:

**Corollary 4.16.** For any regular cardinal $\kappa$, there exists a set $\mathcal{E}$ of representatives of split-exact sequences of $\kappa$-compact small idempotent-complete stable $\infty$-categories.

It is easy to see that filtered colimits of exact sequences of such $\infty$-categories are exact, since filtered colimits of fully-faithful functors are fully-faithful, and colimits commute.

**Lemma 4.17.** Given a filtered diagram of exact sequences $A_\alpha \to B_\alpha \to C_\alpha$ of compact small stable idempotent-complete $\infty$-categories, the colimit $A \to B \to C$ is an exact sequence of idempotent-complete small stable $\infty$-categories; that is, $A \to B$ is fully-faithful with cofiber $C$.

We have the following approximation result:

**Proposition 4.18.** For any regular cardinal $\kappa$, any split-exact sequence $A \to B \to C$ of idempotent-complete small stable $\infty$-categories is a filtered colimit of split-exact sequences $A_\alpha \to B_\alpha \to C_\alpha$ of $\kappa$-compact small idempotent-complete stable $\infty$-categories.

**Proof.** By the discussion above, we can write $C \simeq \text{colim} C_\alpha$ for compact small stable idempotent-complete $\infty$-categories $C_\alpha$. Let $\theta: C_\alpha \to C$ to denote the inclusion into the colimit. Observe that $\theta$ induces an equivalence of mapping spaces $\text{Hom}_{C_\alpha}(x, y) \to \text{Hom}_C(\theta x, \theta y)$. Let $B_\alpha$ denote the image of the composite $C_\alpha \to C \to B$, and define $A_\alpha$ to be the pullback

$$\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
A_\alpha & \longrightarrow & B_\alpha
\end{array}$$
First, we verify that $C_\alpha$ is the image of the composite $B_\alpha \to B \to C$. By definition, $b$ is an object of $B_\alpha$ if it is equivalent to an object of the form $\theta c_\alpha$ for some object $c_\alpha$ of $C_\alpha$. Hence an object is in the image of $B_\alpha \to C$ if it is equivalent to an object of the form $g_j(\theta c_\alpha) \simeq \theta c_\alpha$ for some object $c_\alpha$ of $C_\alpha$. Similarly, we can check that $A_\alpha$ is the image of the composite $B_\alpha \to B \to A$.

That is, we have the following commutative diagram:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{f_\alpha} & & \downarrow{g_\alpha} \\
A_\alpha & \xrightarrow{i_\alpha} & B_\alpha \\
\end{array}
\begin{array}{ccc}
 & & C \\
\downarrow{j} & & \downarrow{j_\alpha} \\
 & & C_\alpha
\end{array}
$$

By construction, $A_\alpha \to B_\alpha$ is fully-faithful, and the discussion above implies that $C_\alpha$ is equivalent to the cofiber of $A_\alpha \to B_\alpha$. To check the adjunctions, observe that by [33, 5.2.2.8] it suffices to show that there are “unit transformation” $\text{id}_B : j_\alpha \circ g_\alpha$ and $\text{id}_A : i_\alpha \circ f_\alpha$. These are induced from the unit transformations for $(f, i)$ and $(g, j)$. Finally, it is clear that the filtered colimit of the sequences $A_\alpha \to B_\alpha \to C_\alpha$ is equivalent to the sequence $A \to B \to C$. \hfill \square

In the context of spectral categories, we have the following definition and proposition.

**Definition 4.19.** A sequence $A \to B \to C$ of spectral categories is exact if the induced sequence of stable presentable $\infty$-categories

$$
\text{N Mod}(A) \to \text{N Mod}(B) \to \text{N Mod}(C)
$$

is exact.

**Proposition 4.20.** A sequence $A \to B \to C$ of spectral categories is exact if the induced sequence of triangulated categories

$$
\text{D}(A) \to \text{D}(B) \to \text{D}(C)
$$

is exact.

Next, observe that we can relate these notions as follows (the proof of which is immediate):

**Proposition 4.21.** Let $A \to B \to C$ be a (split-) exact sequence of small spectral categories. Then $\Psi_{\text{perf}}(A) \to \Psi_{\text{perf}}(B) \to \Psi_{\text{perf}}(C)$ is a (split-) exact sequence of small stable $\infty$-categories.

We also have an essential converse statement.

**Proposition 4.22.** Let $A \to B \to C$ be a (split-) exact sequence of small stable $\infty$-categories. Then there exists a (split-) exact sequence of small stable spectral categories

$$
\tilde{A} \to \tilde{B} \to \tilde{C}
$$

such that $\Psi_{\text{perf}}(\tilde{A} \to \tilde{B} \to \tilde{C})$ is naturally equivalent to the original sequence $A \to B \to C$.

**Proof.** This follows from proposition 3.26. \hfill \square
4.1. Strict-exact sequences.

**Definition 4.23.** An exact sequence (see §4) of the form

\[ A \to B \to B/A \]

is called *strict-exact* if \( A \to B \) is the inclusion of a full subcategory and any object of \( B \) which is a summand of an object of \( A \) is also in \( A \). In particular, every split-exact sequence (see definition 4.14) is equivalent to a strict-exact exact sequence.

We denote by \( \mathcal{E}^e_{\kappa} \) a set of representatives of strict-exact sequences \( A \to B \to B/A \) with \( B \) in \((\text{Cat}_{\infty}^e)^\kappa\).

**Proposition 4.25.** Any strict-exact sequence \( A \to B \to B/A \) is a \( \kappa \)-filtered colimit of strict-exact sequences \( A_\alpha \to B_\alpha \to B_\alpha/A_\alpha \) in \( \mathcal{E}^e_{\kappa} \).

**Proof.** Write \( B \simeq \text{colim}_\alpha B_\alpha \) as a \( \kappa \)-filtered colimit of \( \kappa \)-compact stable \( \infty \)-categories \( B_\alpha \), and define \( A_\alpha = A \times_B B_\alpha \) to be the full subcategory of \( A \) consisting of those objects of \( A \) which lie in the image of \( B_\alpha \). Evidently, \( A \to B \to B/A \) is the \( \kappa \)-filtered colimit of the exact sequences \( A_\alpha \to B_\alpha \to B_\alpha/A_\alpha \), and \( A_\alpha \to B_\alpha \to B_\alpha/A_\alpha \) is strict-exact because if \( Y \in B_\alpha \) is a summand of \( X \in A_\alpha \) then \( Y \in A_\alpha \) because the image of \( Y \) in \( B \) lies in \( A \).

\( \Box \)

We denote by \( \mathcal{E}^e \) a set of representatives of maps of the form \( A \to \text{Idem}(A) \) with \( A \) in \((\text{Cat}_{\infty}^e)^\kappa\).

**Proposition 4.26.** Any map of the form \( A \to \text{Idem}(A) \) is a \( \kappa \)-filtered colimit of elements of \( \mathcal{E}^e \).

**Proof.** Write \( A \simeq \text{colim}_\alpha A_\alpha \) as a \( \kappa \)-filtered colimit of \( \kappa \)-compact small stable \( \infty \)-categories \( A_\alpha \). Then \( \text{Idem}(A) \simeq \text{colim}_\alpha \text{Idem}(A_\alpha) \), since \( \text{Idem} \) (viewed as an endo-functor of \( \text{Cat}_{\infty}^e \)) commutes with \( \kappa \)-filtered colimits.

\( \Box \)

5. Additivity

In this section we construct the universal additive invariant of small stable \( \infty \)-categories; see theorem 5.9. Its construction is divided in two steps: first, using proposition 4.18, we construct the unstable version; see theorem 5.6. Then, using Lurie’s theory of stabilization of \( \infty \)-categories, we obtain the universal additive invariant.

**Definition 5.1.** Let \( D \) be a stable presentable \( \infty \)-category. A functor

\[ E : \text{Cat}_{\infty}^e \to D \]

is called an *additive invariant* of small stable \( \infty \)-categories if it inverts Morita equivalences (see definition 2.12), preserves filtered colimits, and satisfies *additivity*, i.e., given a split-exact sequence

\[ \begin{array}{ccc}
A & \xrightarrow{f} & C \\
\downarrow{i} & & \downarrow{g} \\
B & \xleftarrow{j} & \end{array} \]

the functors \( i \) and \( g \) induce an equivalence in \( D \)

\[ E(A) \vee E(B) \sim E(C). \]

We denote by \( \text{Fun}_{\text{add}}(\text{Cat}_{\infty}^e, D) \) the \( \infty \)-category of additive invariants with values in \( D \).
Example 5.3. As we discuss in Sections 6 and 10, appropriate versions of algebraic
$K$-theory, topological Hochschild homology (THH), and topological cyclic homology
(TC) provide additive invariants of small stable $\infty$-categories (where we regard the
latter as taking values in pro-spectra).

5.1. Unstable version. Let us denote by $\text{Pre}((\text{Cat}_{\text{perf}}^\infty)_{\omega}; N(T^\circ),\ast)$ the $\infty$-category
of pointed presheaves of pointed simplicial sets on the essentially small $\infty$-category
$(\text{Cat}_{\text{perf}}^\infty)_{\omega}$ of compact small idempotent-complete stable $\infty$-categories.

Lemma 5.4. Let $D$ be a stable presentable $\infty$-category. Then, we have an equiva-
rence of $\infty$-categories $\text{Fun}_{\text{L}}(\text{Pre}((\text{Cat}_{\text{perf}}^\infty)_{\omega}; N(T^\circ),\ast), D) \simeq \text{Fun}_{\text{R}}(\text{Ind}((\text{Cat}_{\text{perf}}^\infty)_{\omega}), D)$,
where the right-hand side denotes the $\infty$-category of morphisms of $\infty$-categories
which preserve filtered colimits.

Proof. The proof is a consequence of the following equivalences
$\text{Fun}_{\text{L}}(\text{Pre}((\text{Cat}_{\text{perf}}^\infty)_{\omega}; N(T^\circ),\ast), D) \simeq \text{Fun}(\text{Cat}_{\text{perf}}^\infty, D) \simeq \text{Fun}_{\text{R}}(\text{Ind}((\text{Cat}_{\text{perf}}^\infty)_{\omega}), D) \simeq \text{Fun}_{\text{R}}(\text{Cat}_{\text{perf}}^\infty, D)$,
where the last one follows from corollary 3.22. □

Let $\phi: \text{Cat}_{\text{perf}}^\infty \rightarrow \text{Pre}((\text{Cat}_{\text{perf}}^\infty)_{\omega}; N(T^\circ),\ast)$
be the functor obtained by first taking the Yoneda embedding and then restrict-
ing the presheaves to the category $(\text{Cat}_{\text{perf}}^\infty)_{\omega}$. Recall from corollary 4.16 that we can choose a fixed set $E$ of representatives of split-exact sequences in $(\text{Cat}_{\text{perf}}^\infty)_{\omega}$. We denote by $M_{\text{add}}^\infty$ the localization of $\text{Pre}((\text{Cat}_{\text{perf}}^\infty)_{\omega}; N(T^\circ),\ast)$ [33, 5.5.4.15] with respect to the set of maps
$\text{Cone}(\phi(A) \rightarrow \phi(C)) \rightarrow \phi(B)$,
where $A \rightarrow C \rightarrow B$ is a split-exact sequence in $E$. Finally, let $U_{\text{add}}^\infty$ be the following
composite
$\text{Cat}_{\text{perf}}^\infty \xrightarrow{\text{Idem}(-)} \text{Cat}_{\text{perf}}^\infty \xrightarrow{\phi} \text{Pre}((\text{Cat}_{\text{perf}}^\infty)_{\omega}; N(T^\circ),\ast) \xrightarrow{\gamma} M_{\text{add}}^\infty$,
where $\gamma$ is the localization functor.

Theorem 5.6. The functor $U_{\text{add}}^\infty$ inverts Morita equivalences, preserves filtered
colimits, and sends split-exact sequences to cofiber sequences
$A \xleftarrow{f} B \rightarrow C \xleftarrow{g} \rightarrow \xrightarrow{j} U_{\text{add}}^\infty(A) \rightarrow U_{\text{add}}^\infty(C) \rightarrow U_{\text{add}}^\infty(B)$.

Moreover, $U_{\text{add}}^\infty$ is universal with respect to these properties, i.e., given any stable
presentable $\infty$-category $D$, we have an equivalence of $\infty$-categories
$(U_{\text{add}}^\infty)^*: \text{Fun}_{\text{L}}(M_{\text{add}}^\infty, D) \xrightarrow{\sim} \text{Fun}_{\text{R}}(\text{Cat}_{\text{perf}}^\infty, D)$,
where the right-hand denotes the full subcategory of $\text{Fun}(\text{Cat}_{\text{perf}}^\infty, D)$ of morphisms of $\infty$-categories which satisfy the above conditions.
Proof. The result follows from definition 2.12, lemma 5.4 and from the general theory of localization (see [33, §5.2.7, §5.5.4]): The functor $\phi$ preserves filtered colimits and by proposition 4.18 any split-exact sequence can be approximated by a filtered colimit of split-exact sequences in $E$. □

5.2. Universal additive invariant. Let $\mathcal{M}_{\text{add}}$ be the stabilization $\text{Stab}(\mathcal{M}_{\text{add}}^{\text{un}})$ [34, §8, §15] of $\mathcal{M}_{\text{add}}^{\text{un}}$; by construction, this is a stable $\infty$-category. Denote by $\mathcal{U}_{\text{add}}$ the following composite:

$$\text{Cat}_{\infty}^\text{ex} \xrightarrow{\mathcal{U}_{\text{add}}^{\text{un}}} \mathcal{M}_{\text{add}}^{\text{un}} \xrightarrow{\mathcal{U}_{\text{add}}} \text{Stab}(\mathcal{M}_{\text{add}}^{\text{un}}).$$

Remark 5.7. Note that we have the following equivalences

$$\text{Stab}(\text{Pre}((\text{Cat}_{\infty}^\text{perf})^\omega; N(T)^*)) = \text{Stab}(\text{Fun}(((\text{Cat}_{\infty}^\text{perf})^\omega)^{\text{op}}, N(T)^*))$$

$$\simeq \text{Fun}(((\text{Cat}_{\infty}^\text{perf})^\omega)^{\text{op}}, \text{Stab}(N(T)^*))$$

where the last one follows from [34, 10.13]. Therefore, defining

$$\text{Pre}((\text{Cat}_{\infty}^\text{perf})^\omega; S_{\infty}) = \text{Fun}(((\text{Cat}_{\infty}^\text{perf})^\omega)^{\text{op}}, S_{\infty})$$

and writing

$$\psi : \text{Cat}_{\infty}^\text{perf} \longrightarrow \text{Pre}((\text{Cat}_{\infty}^\text{perf})^\omega; S_{\infty})$$

for the natural functor, we see that $\mathcal{M}_{\text{add}}$ can alternately be described as the localization of $\text{Pre}((\text{Cat}_{\infty}^\text{perf})^\omega; S_{\infty})$ with respect to the set of maps

$$\text{Cone}(\psi(A) \longrightarrow \psi(C)) \longrightarrow \psi(B),$$

where $A \to C \to B$ is a split-exact sequence in $E$.

Theorem 5.9. The functor $\mathcal{U}_{\text{add}}$ is the universal additive invariant, i.e., given any stable presentable $\infty$-category $D$, we have an equivalence of $\infty$-categories

$$(\mathcal{U}_{\text{add}})^* : \text{Fun}^L(\mathcal{M}_{\text{add}}, D) \xrightarrow{\sim} \text{Fun}_{\text{add}}(\text{Cat}_{\infty}^\text{ex}, D).$$

Proof. The result follows from theorem 5.6 and from the general properties of stabilization [34, §§8, §15]. Note that since the stabilization morphism preserves colimits and $\mathcal{U}_{\text{add}}^{\text{un}}$ sends split-exact sequences to cofiber sequences, the split-exact sequence (5.2) is sent to a cofiber sequence

$$\mathcal{U}_{\text{add}}(A) \longrightarrow \mathcal{U}_{\text{add}}(C) \longrightarrow \mathcal{U}_{\text{add}}(B) \longrightarrow \mathcal{U}_{\text{add}}(A)[1]$$

in $\mathcal{M}_{\text{add}}$. The stability of $\mathcal{M}_{\text{add}}$ then implies that the spectral functors $i$ and $g$ induce a splitting of this triangle. □

6. Connective $K$-theory

In this section, we verify that higher algebraic $K$-theory provides an additive invariant of small stable $\infty$-categories; theorem 5.9 then applies to show that this invariant descends to $\mathcal{M}_{\text{add}}$. Furthermore, we prove the essential fact that algebraic $K$-theory in fact becomes co-representable in $\mathcal{M}_{\text{add}}$ (see theorem 6.9). This allows us to understand transformations between additive theories from algebraic $K$-theory via the Yoneda lemma; we use this in Section 10 to characterize the cyclotomic trace map. We begin by developing the necessary background on the construction of algebraic $K$-theory for small $\infty$-categories with finite colimits.
6.1 Background on $K$-theory of $\infty$-categories. Waldhausen’s algebraic $K$-theory functor takes as input a category with cofibrations and weak equivalences. It is now well understood that the $K$-theory spectrum is determined by the Dwyer-Kan localization $L^H C$ of the Waldhausen category $C$ [60, 7, 10]. Since $N(L^H C)$ yields the $\infty$-category associated to $C$, these results can be interpreted as saying that the algebraic $K$-theory of a Waldhausen category is an invariant of the underlying $\infty$-category. In this subsection we study a direct construction of the algebraic $K$-theory of $\infty$-categories due to Lurie [34, 11.1] (see also [60, §7] for a sketch of such a definition in the context of Segal categories). We prove that the algebraic $K$-theory of a Waldhausen category is equivalent as a spectrum to the $K$-theory of the associated $\infty$-category.

Let $C$ be a small pointed $\infty$-category with finite colimits. The following definition is the $\infty$-categorical analogue of Waldhausen’s $S_\bullet$ construction.

**Definition 6.1.** Let $\text{Ar}[n]$ denote the category of arrows in $[n]$: $\text{Ar}[n]$ has objects $(i, j)$ for $0 \leq i \leq j \leq n$ and a unique map $(i, j) \to (i', j')$ for $i \leq i'$ and $j \leq j'$. Denote by $\text{Gap}([I], C)$ the full subcategory of $\text{Fun}(N(\text{Ar}[n]), C)$ spanned by the functors $N(\text{Ar}[n]) \to C$ such that, for each $i \in I$, $F(i, i)$ is a zero object of $C$, and for each $i < j < k$, the square

$$
\begin{array}{ccc}
F(i, j) & \longrightarrow & F(i, k) \\
\downarrow & & \downarrow \\
F(j, j) & \longrightarrow & F(j, k)
\end{array}
$$

is cocartesian.

Following [34, 11.4], we define a simplicial $\infty$-category $S^\infty C$ by the rule $iS_n C = \text{Gap}([n], C)$. Applying passage to the largest Kan complex levelwise, we obtain a simplicial space $(S^\infty C)_\text{iso}$. This is the $\infty$-categorical version of Waldhausen’s $K$-theory space. Furthermore, for each $n$, $\text{Gap}([n], C)$ is itself a small pointed $\infty$-category with finite colimits: we can can iterate this procedure. Since $\text{Gap}([0], C)$ is contractible (with preferred basepoint given by the point in $C$) and $\text{Gap}([1], C)$ is equivalent to $C$, there is a natural map

$$
S^1 \wedge (C)_\text{iso} \longrightarrow ((S^\infty C)_\text{iso})
$$

given by the inclusion into the 1-skeleton. Therefore, the spaces $|(S^\infty C)^n(C)_\text{iso}|$ assemble to form a spectrum $K(C)$; this is the $\infty$-categorical version of Waldhausen’s $K$-theory spectrum. It is clear from the construction that this construction is functorial in exact functors which preserve the point.

In practice, we find it more convenient to use a reformulation of the definition of the iterated $S_\bullet$ construction (e.g., see [7, A.5.4], [8, 2.2], the appendix to [21], and also [49, §2]).

**Definition 6.2.** Write $\text{Ar}[n_1, \ldots, n_q]$ for $\text{Ar}[n_1] \times \cdots \times \text{Ar}[n_q]$. For a functor

$$
A: N(\text{Ar}[n_1, \ldots, n_q]) = N(\text{Ar}[n_1] \times \cdots \times \text{Ar}[n_q]) \longrightarrow C,
$$

we write $A_{i_1, j_1; \ldots; i_q, j_q}$ for the value of $A$ on the object $((i_1, j_1), \ldots, (i_q, j_q))$. Let $\text{Gap}([n_1], \ldots, [n_q], C)$ be the full subcategory of $\text{Fun}(N(\text{Ar}[n_1, \ldots, n_q]), C)$ spanned by the functors such that

- $A_{i_1, j_1; \ldots; i_q, j_q} \simeq *$ whenever $i_k = j_k$ for some $k$. 

• For every object \((i_1, j_1; \ldots ; i_q, j_q)\) in \(\text{Ar}[n_1] \times \cdots \times \text{Ar}[n_q]\), every \(1 \leq r \leq q\), and every \(j_r \leq k \leq n_r\), the square

\[
\begin{array}{ccc}
A_{i_1, j_1; \ldots ; i_r, j_r; \ldots ; i_q, j_q} & \longrightarrow & A_{i_1, j_1; \ldots ; i_r, k; \ldots ; i_q, j_q} \\
\downarrow & & \downarrow \\
A_{i_1, j_1; \ldots ; i_r, j_r; \ldots ; i_q, i_q} & \longrightarrow & A_{i_1, j_1; \ldots ; i_r, j_r; \ldots ; i_q, i_q}
\end{array}
\]

is a cocartesian square.

Now we define the multisimplicial \(\infty\)-category

\[
(S^\infty)^{(q)}_{n_1, \ldots , n_q} C = \text{Gap}([(n_1, \ldots , [n_q]), C]).
\]

We regard \((S^\infty)^{(0)}\) as \(C\) and it is clear that \((S^\infty)^{(1)}\) is \(\text{Gap}([n], C)\). Now we directly define the \(K\)-theory spectrum of a Waldhausen category \(C\) to be the spectrum with \(q\)-th space

\[
K\mathcal{C}(q) = ((S^\infty)^{(q)}_{\bullet, \ldots , \bullet})_{\text{iso}},
\]

The suspension maps \(\Sigma K\mathcal{C}(q) \to K(q+1)\) are induced on diagrams by the projection map

\[
\text{Ar}[n_1] \times \cdots \times \text{Ar}[n_q] \times \text{Ar}[n_{q+1}] \to \text{Ar}[n_1] \times \cdots \times \text{Ar}[n_q].
\]

Defining an action of \(\Sigma_q\) on \(K\mathcal{C}(q)\) by permuting the simplicial directions, we see from the explicit description of \(S^\infty_{\bullet, \ldots , \bullet}\) above, that \(K\mathcal{C}\) forms a symmetric spectrum.

We now establish a comparison between Waldhausen’s algebraic \(K\)-theory of a Waldhausen category \(C\) and the \(\infty\)-categorical version of the algebraic \(K\)-theory of the associated simplicial category \(L^H C\). The comparison is essentially a consequence of the rigification of homotopy coherent diagrams to strict diagrams in a model category [17], which allows us to pass between \(\infty\)-categorical diagrams and point-set diagrams, and the “homotopical” \(S^n\) construction of [6], which allows us to replace the use of pushouts by homotopy pushouts for suitable Waldhausen categories.

The version of the comparison of homotopy coherent diagrams to strict diagrams is originally due to Simpson [53] in the context of Segal spaces. Since we use \(\infty\)-categories in this paper, we work with the version proved by Lurie in that setting [33, 4.2.4.4]. Let \(S\) be a small simplicial set, \(\mathcal{D}\) a small simplicial category, and \(u : C[S] \to D\) an equivalence. Let \(\mathcal{A}\) be a combinatorial simplicial model category, and let \(\mathcal{U}\) be a \(\mathcal{D}\)-chunk of \(\mathcal{A}\) (see [33, A.3.4.9] for a discussion of \(D\)-chunks). Then the induced map

\[
N((\mathcal{U}^D)^{\circ}) \to \text{Fun}(S, N(\mathcal{U}^D))
\]

is a categorical equivalence of simplicial sets. Here the notation \((\mathcal{U}^D)^{\circ}\) indicates the full subcategory of \(\mathcal{A}^D\) consisting of cofibrant-fibrant objects (in the projective model structure) landing in \(\mathcal{U}\).

Specializing to our situation, assume that \(S\) is the (ordinary) nerve \(N(J)\) of a diagram (small category) \(J\); that is, \(J\) is regarded as a discrete simplicial category. Then the counit map \(C[N(J)] \to J\) is an equivalence and so we have that the induced map

\[
N((\mathcal{U}^J)^{\circ}) \to \text{Fun}(N(J), N(\mathcal{U}^J))
\]

is a categorical equivalence of simplicial sets.

**Lemma 6.3.** Let \(\mathcal{A}\) be a combinatorial simplicial model category and \(\mathcal{C} \subset \mathcal{A}\) a full subcategory. Then for each \(n\) the induced map

\[
N((\mathcal{C}^{[n]})^{\circ}) \to \text{Fun}(N([n]), N(\mathcal{C}^{\circ}))
\]
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is a categorical equivalence of simplicial sets.

Proof. By [33, A.3.4.15], we can choose a small subcategory \( V \subset A \) which contains \( C \) and such that \( V \) is an \((|n|^{[1]}))\)-chunk for each \( n \) and moreover \( N((C)^\circ) \) is equivalent to \( N((V)^\circ) \). Then as discussed above, [33, 4.2.4.4] implies that for each \( n \) the natural map

\[ N((|n|^{[1]}))^{\circ} \to \text{Fun}(N([n]^{[1]}), N((V)^\circ)) \]

is a categorical equivalence of simplicial sets.

Now, the \( S'_n \) construction [6, 2.7] is a variant of Waldhausen’s \( S_n \) construction defined by replacing the cocartesian squares in the definition of \( S_n \) with homotopy cocartesian squares. Essentially by construction, the \( S'_n \) construction is functorial in weakly exact functors, i.e., functors that preserve weak equivalences and homotopy cocartesian squares. Since a square is homotopy cocartesian in \( C \) if and only if it is a pushout square in \( N((C)^\circ) \), in this setting the equivalence of Lemma 6.3 restricts to give an equivalence

\[ N((S'_n C)^\circ) \to \text{Gap}(|n|, N((C)^\circ)). \]

Similar considerations for the iterated \( S'_n \) construction [7, A.5.4] (as modeled in Definition 6.2) yield the equivalence

\[ N((S^{(q)}_{n_1, \ldots, n_q} C)^\circ) \to \text{Gap}((|n_1|, \ldots, |n_q|), C). \]

Applying proposition 2.10, we obtain the following comparison of algebraic \( K \)-theory spaces and spectra.

**Corollary 6.4.** Let \( A \) be a simplicial model category and \( C \subset A \) a small full subcategory which has all finite homotopy colimits. Then for each \( n \) there is a weak equivalence of simplicial sets

\[ |N_w S'_n C| \simeq |(S_n^\infty N((C)^\circ))_{iso}|. \]

and for each \((n_1, \ldots, n_q)\) there is a weak equivalence of simplicial sets

\[ |N_w S^{(q)}_{n_1, \ldots, n_q} C| \simeq |((S_q^\infty)_{n_1, \ldots, n_q} N((C)^\circ))_{iso}|. \]

In particular, this yields the following theorem:

**Theorem 6.5.** Let \( A \) be a simplicial model category and \( C \subset A \) a small full subcategory of the cofibrants which admits all homotopy pushouts and is a Waldhausen category via the model structure on \( A \). Then there is an equivalence of spectra

\[ K(C) \simeq K(N((C)^\circ)). \]

The comparison is natural in weakly exact functors.

Finally, specializing to our case, we find the following result.

**Corollary 6.6.** Let \( C \) be a small pretriangulated spectral category and let \( M_C \) denote the category of perfect \( C \)-modules with its Waldhausen structure induced by the model structure on \( C \)-modules. Then there is an isomorphism in the stable category

\[ K(M_C) \simeq K(\Psi_{perf}C). \]

As a consequence, Waldhausen’s additivity theorem applies to prove the following proposition.
Proposition 6.7. The algebraic \( K \)-theory functor from the \( \infty \)-category \( \text{Cat}_{\infty}^{\leq} \) of small stable \( \infty \)-categories to the \( \infty \)-category \( \mathcal{S}_{\infty} \) of spectra is an additive invariant.

Proof. The only property for which verification is not immediate is additivity. Corollary 6.6 allows us to reduce to consideration of split-exact sequences of spectral categories

\[
\text{perf} \, \mathcal{A} \longrightarrow \text{perf} \, \mathcal{C} \longrightarrow \text{perf} \, \mathcal{B}.
\]

As in [57], we observe that this sequence is Morita equivalent to the sequence

\[
\text{perf} \, \mathcal{A} \longrightarrow E(\text{perf} \, \mathcal{A}, \text{perf} \, \mathcal{C}, \text{perf} \, \mathcal{B}) \longrightarrow \text{perf} \, \mathcal{B}
\]

(where \( E \) denotes Waldhausen’s category of cofiber sequences in \( \mathcal{C} \) with first term in the image of \( \mathcal{A} \) and cofiber in the image of \( \mathcal{B} \)). Now Waldhausen’s additivity theorem implies the desired splitting on \( K \)-theory. \( \square \)

So far, all of our comparison results assume that the Waldhausen category we are working with arises as a subcategory of a model category. In fact, we can extend our comparison and functoriality results to Waldhausen categories \( \mathcal{C} \) such that cofibrations admit (not necessarily functorial) factorizations and which are \( \text{DKHS-saturated} \) (i.e., such that a map \( f \) is a weak equivalence in \( \mathcal{C} \) if and only if its image in the homotopy category is an isomorphism). We do this as follows, using a construction due to Cisinski [10, §4].

Given a Waldhausen category \( \mathcal{C} \), let \( \mathcal{P}(\mathcal{C}) \) denote here the pointed simplicial presheaves on \( \mathcal{C} \) with the projective model structure (i.e., weak equivalences and fibrations are determined pointwise). We can localize \( \mathcal{P}(\mathcal{C}) \) at the set of presheaves which are pointwise Kan complexes, preserve weak equivalences, and take homotopy cocartesian squares in \( \mathcal{C} \) to homotopy pullback squares in \( \mathcal{P}(\mathcal{C}) \); denote this category by \( \mathcal{P}_{\text{ex}}(\mathcal{C}) \). Let \( \mathcal{M}(\mathcal{C}) \) denote the full subcategory of the localized category consisting of the objects which are cofibrant and weakly equivalent to representable presheaves; this can be regarded as a Waldhausen category, inheriting structure from the model structure on \( \mathcal{P}_{\text{ex}}(\mathcal{C}) \).

The Yoneda embedding induces a DK-equivalence \( \mathcal{C} \to \mathcal{M}(\mathcal{C}) \), and so an equivalence \( K(\mathcal{C}) \to K(\mathcal{M}(\mathcal{C})) \) [7, 10, 60]. Moreover, a weakly exact functor \( \mathcal{C} \to \mathcal{C}' \) induces a left Quillen functor \( \mathcal{P}_{\text{ex}}(\mathcal{C}) \to \mathcal{P}_{\text{ex}}(\mathcal{C}') \) by left Kan extension, and hence an exact functor \( \mathcal{M}(\mathcal{C}) \to \mathcal{M}(\mathcal{C}') \) by restriction. Since the category \( \mathcal{M}(\mathcal{C}) \) satisfies the hypothesis of Theorem 6.5, we obtain a comparison of the \( K \)-theory of Waldhausen categories that are saturated and admit factorization to the associated \( K \)-theory of \( \infty \)-categories, natural in weakly exact functors.

Remark 6.8. On the other hand, given a (homotopically) pointed simplicial category with finite homotopy colimits, we can use the essentially the same construction to produce a DK-equivalent Waldhausen category. In this case, one considers the pointed simplicial presheaves and localizes at the simplicial functors which take finite homotopy colimits to finite limits. This construction is functorial in simplicial categories and the composite now gives us a functor from the category of pointed simplicial categories with finite colimits and exact functors to Waldhausen categories and exact functors. Finally, we can deduce from Theorem 6.5 that the constructions of algebraic \( K \)-theory on the two sides agree.
6.2. Co-representability. This subsection is entirely devoted to the proof of theorem 6.9. The proof will follow from propositions 6.13 and 6.14.

**Theorem 6.9.** Let $\mathcal{A}$ be a small stable $\infty$-category and $\mathcal{B}$ be a compact idempotent-complete small stable $\infty$-category. Then there is a natural equivalence of spectra

$$\text{Map}(\mathcal{U}_{\text{add}}(\mathcal{B}), \mathcal{U}_{\text{add}}(\mathcal{A})) \simeq K(\text{Fun}^{\text{ex}}(\mathcal{B}, \text{Idem}(\mathcal{A}))).$$

When $\mathcal{B}$ is the small stable $\infty$-category $\mathcal{S}_\infty^n$ of compact spectra [35, 4.2.8], there is a natural equivalence of spectra

$$\text{Map}(\mathcal{U}_{\text{add}}(\mathcal{S}_\infty^n), \mathcal{U}_{\text{add}}(\mathcal{A})) \simeq K(\text{Idem}(\mathcal{A})).$$

In particular, we have isomorphisms of abelian groups

$$\text{Hom}(\mathcal{U}_{\text{add}}(\mathcal{S}_\infty^n), \mathcal{U}_{\text{add}}(\mathcal{A})[-n]) \simeq K_n(\text{Idem}(\mathcal{A})) \quad n \geq 0,$$

in the triangulated category $\text{Ho}(\mathcal{M}_{\text{add}})$.

**Notation 6.10.** Given a small stable $\infty$-category $\mathcal{A}$, we denote by $K^w_\mathcal{A}$ the object $\mathcal{B} \mapsto (\text{Fun}^{\text{ex}}(\mathcal{B}, \text{Idem}(\mathcal{A})))_{\text{iso}}$ in $\text{Pre}((\text{Cat}^\perp(\mathcal{B}))_\omega; \mathbb{N}(\mathcal{T}^\perp)_{\omega})$ and by $K_\mathcal{A}$ the object $\mathcal{B} \mapsto K(\text{Fun}^{\text{ex}}(\mathcal{B}, \text{Idem}(\mathcal{A})))$ in $\text{Pre}((\text{Cat}^\perp(\mathcal{B}))_\omega; \mathcal{S}_\infty)$.

**Remark 6.11.** Recall that corollary 3.25 allow us to model the small $\infty$-category of exact functors $\text{Fun}^{\text{ex}}(\mathcal{B}, \text{Idem}(\mathcal{A}))$ as the pretriangulated spectral category $\text{rep}(\mathcal{B}, \mathcal{A})$ of right-compact $\mathcal{A} \wedge \mathcal{B}^\perp$-modules. Combined with proposition 2.10, this implies that the associated mapping space $(\text{Fun}^{\text{ex}}(\mathcal{B}, \text{Idem}(\mathcal{A})))_{\text{iso}}$ can be calculated as $|N.w\text{rep}(\mathcal{B}, \text{Idem}(\mathcal{A}))|$. Moreover, $\text{rep}(\mathcal{B}, \text{Idem}(\mathcal{A}))$ has a natural Waldhausen structure: we can also consider the algebraic $K$-theory space $|N.w\text{rep}(\mathcal{B}, \text{Idem}(\mathcal{A}))|$ and associated spectrum.

In the following results, we will use the observation that Waldhausen’s $S_\bullet$ construction applied to a spectral category which is a Waldhausen category with the cofibrations inherited from a spectral model structure produces a spectral category (where the mapping spectra are given by an appropriate end). We also need the following lemma which allows us to bring the $S_\bullet$ construction inside:

**Lemma 6.12.** Let $\mathcal{A}$ and $\mathcal{B}$ be small stable $\infty$-categories. Then we have an equivalence of simplicial $\infty$-categories

$$S_\infty^\infty\text{Fun}^{\text{ex}}(\mathcal{B}, \mathcal{A}) \simeq \text{Fun}^{\text{ex}}(\mathcal{B}, S_\infty^\infty \mathcal{A})$$

and correspondingly an equivalence of spaces

$$|(S_\infty^\infty\text{Fun}^{\text{ex}}(\mathcal{B}, \mathcal{A})))_{\text{iso}}| \simeq |(\text{Fun}^{\text{ex}}(\mathcal{B}, S_\infty \mathcal{A})))_{\text{iso}}|.$$

We can now relate $\mathcal{M}_{\text{add}}$ to the algebraic $K$-theory presheaf.

**Proposition 6.13.** Let $\mathcal{A}$ be a small stable $\infty$-category. Then, we have a natural equivalence $\Sigma(\mathcal{U}_{\text{add}}^\un(\mathcal{A})) \simeq K^w_\mathcal{A}$ in $\mathcal{M}_{\text{add}}^\un$ and a natural equivalence $\mathcal{U}_{\text{add}}(\mathcal{A})[1] \simeq K_{\mathcal{A}}[1]$ in $\mathcal{M}_{\text{add}}$.

**Proof.** We begin by handling the unstable case. Theorem 3.20 implies that we can model $\mathcal{A}$ by a small spectral category (which we still denote by $\mathcal{A}$). Following [38, 3.3], we consider the following sequence of simplicial spectral categories

$$S_\mathcal{A} \xrightarrow{I} PS_\mathcal{A} \xrightarrow{Q} S_\mathcal{A},$$
where \( A_* \) is a constant simplicial object and \( PS_\bullet A \) is the simplicial path object of \( S_\bullet A \). By applying the functor \( \mathcal{U}_{\text{add}}^\text{un} \) to this sequence, we obtain an induced morphism

\[
\Theta: \text{Cone}[\mathcal{U}_{\text{add}}^\text{un}(A_*) \to \mathcal{U}_{\text{add}}^\text{un}(PS_\bullet A)] \to \mathcal{U}_{\text{add}}^\text{un}(S_\bullet A)
\]

of simplicial objects in \( \mathcal{M}_{\text{add}}^\text{un} \). We now show that each component \( \Theta_n \) of \( \Theta \) is an equivalence. For each \( n \geq 0 \), we have a split-exact sequence:

\[
A \xrightarrow{I_n} PS_n A = S_{n+1} A \xrightarrow{S_n} S_n A,
\]

where \( I_n \) maps \( A \mapsto \cdots \xrightarrow{\text{Id}} A \), \( Q_n \) maps \( (\ast \to A_0 \to A_1 \to \cdots) \mapsto (A_1/A_0 \to \cdots A_{n-1}/A_0) \), \( S_n \) maps \( (\ast \to A_0 \to A_1 \to \cdots A_{n-1}) \mapsto (\ast \to \ast \to A_0 \to \cdots A_{n-1}) \), and \( R_n \) maps \( (\ast \to A_0 \to A_1 \to \cdots A_{n-1}) \mapsto A_0 \).

By construction of \( \mathcal{M}_{\text{add}}^\text{un} \) (and of \( \mathcal{U}_{\text{add}}^\text{un} \)), we conclude that the induced morphisms

\[
\Theta_n: \text{Cone}[\mathcal{U}_{\text{add}}^\text{un}(A) \to \mathcal{U}_{\text{add}}^\text{un}(PS_n A)] \to \mathcal{U}_{\text{add}}^\text{un}(S_n A) \quad n \geq 0,
\]

are equivalences in \( \mathcal{M}_{\text{add}}^\text{un} \). This allows us to obtain the following cocartesian square

\[
\begin{array}{ccc}
\mathcal{U}_{\text{add}}^\text{un}(A) & \xrightarrow{\text{cone}} & \mathcal{U}_{\text{add}}^\text{un}(PS_n A) \\
\downarrow \text{cone} & & \downarrow \text{cone} \\
\ast & \xrightarrow{\text{cone}} & \mathcal{U}_{\text{add}}^\text{un}(S_n A)
\end{array}
\]

and so a natural equivalence

\[
\Sigma(\mathcal{U}_{\text{add}}^\text{un}(A)) \xrightarrow{\sim} \mathcal{U}_{\text{add}}^\text{un}(S_\bullet A)
\]

in \( \mathcal{M}_{\text{add}}^\text{un} \). Finally, since we have a natural equivalence

\[
\mathcal{U}_{\text{add}}^\text{un}(S_\bullet A) \simeq K^w_A
\]

by Lemma 6.12, the assertion is proved. The identification in the stable setting now follows from the unstable considerations and the usual passage from results on the \( K \)-theory space to the \( K \)-theory spectrum.

\[ \square \]

**Proposition 6.14.** Let \( \mathcal{A} \) be a small stable \( \infty \)-category. Then, the presheaves \( K^w_{\mathcal{A}} \) and \( K_{\mathcal{A}} \) (see notation 6.10) are local, i.e., given any split-exact sequence \( B \to C \to D \) in \( \mathcal{E} \), the induced maps of spectra (see (5.5) and (5.8)):

\[
\begin{align*}
\text{map}(\phi(D), K^w_{\mathcal{A}}) & \xrightarrow{\sim} \text{Map}(\text{Cone}(\phi(A) \to \phi(C)), K^w_{\mathcal{A}}) \\
\text{map}(\psi(D), K_{\mathcal{A}}) & \xrightarrow{\sim} \text{Map}(\text{Cone}(\psi(A) \to \psi(C)), K_{\mathcal{A}})
\end{align*}
\]

are weak equivalences.
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Proof. The argument is exactly the same in both cases. Therefore, we discuss only the stable $K_A$. Since $B$, $C$ and $D$ belong to $(\text{Cat}_{\infty}^{\text{perf}})^{\omega}$, the enriched Yoneda lemma shows us that we need to prove that the induced sequence of spectra

$$K(\text{Fun}^{\text{ex}}(D, \text{Idem}(A))) \rightarrow K(\text{Fun}^{\text{ex}}(C, \text{Idem}(A))) \rightarrow K(\text{Fun}^{\text{ex}}(B, \text{Idem}(A)))$$

is a cofiber sequence. Proposition 4.22 implies we can lift $B \rightarrow C \rightarrow D$ to a split-exact sequence of small spectral categories $\tilde{B} \rightarrow \tilde{C} \rightarrow \tilde{D}$. Furthermore, using corollary 3.25 it suffices to consider the split-exact sequence of small spectral categories

$$\text{rep}(\tilde{D}, A) \rightarrow \text{rep}(\tilde{C}, A) \rightarrow \text{rep}(\tilde{B}, A).$$

Since $A$ is a pretriangulated spectral category, all of the above spectral categories carry a natural Waldhausen structure. We will apply Waldhausen’s fibration theorem [62, 1.6.4]. We have the Waldhausen category $v\text{rep}(C, A)$, whose weak equivalences are the morphisms $f$ such that $\text{Cone}(f)$ is contractible, and also the Waldhausen category $w\text{rep}(C, A)$, with the same cofibrations as $v\text{rep}(C, A)$ but whose weak equivalences are the morphisms $f$ such that $\text{Cone}(f)$ belongs to $\text{rep}(D, A)$. Note that we have a natural inclusion $v\text{rep}(C, A) \subseteq w\text{rep}(C, A)$ and an equivalence $w\text{rep}(C, A) \simeq \text{rep}(C, A)$; see [62, §1.6]. The conditions of [62, Thm. 1.6.4] are satisfied and so we obtain a cofiber sequence of spectra

$$K(\text{rep}(D, A)) \rightarrow K(\text{rep}(C, A)) \rightarrow K(\text{rep}(B, A)).$$

Propositions 6.13 and 6.14 allow us to prove theorem 6.9 as follows: let $A$ be stable $\infty$-category and $B$ a compact small idempotent-complete stable $\infty$-category. By proposition 6.13 we have an equivalence $U_{\text{add}}(A) \simeq K_A$ and by proposition 6.14 $K_A$ is local. Therefore, we have the following natural equivalence

$$\text{Map}(U_{\text{add}}(B), U_{\text{add}}(A)) \simeq \text{Map}(\psi(B), K_A),$$

where the right-hand side is calculated in $\text{Pre}((\text{Cat}_{\infty}^{\text{perf}})^{\omega}; S_\infty)$. Since $B$ belongs to $(\text{Cat}_{\infty}^{\text{perf}})^{\omega}$, the presheaf $\psi(B)$ is representable and so by the enriched Yoneda lemma we have $\text{Map}(\psi(B), K_A) \simeq K_A(B)$. Finally, since by definition of $K_A$ we have $K_A(B) = K(\text{Fun}^{\text{ex}}(B, \text{Idem}(A)))$ the proof is finished.

□

7. Localization

The definition of additivity we study in this paper is given in terms of the condition that algebraic $K$-theory takes models of split-exact sequences of triangulated categories to (homotopy) cofiber sequences of spectra. This perspective is motivated in part by Neeman’s reformulation of the Thomason-Trobaugh localization theorem [46]. Neeman observed that following Thomason-Trobaugh and using the construction of Bousfield localization, one could regard algebraic $K$-theory as in fact taking exact sequences of triangulated categories to cofiber sequences of spectra, provided one worked with non-connective $K$-theory. We will refer to such a theory as satisfying localization. In this section we construct the universal localizing invariant of small stable $\infty$-categories; see theorem 7.7.

Definition 7.1. Let $\mathcal{D}$ be a stable presentable $\infty$-category. A functor

$$E : \text{Cat}_{\infty}^{\text{ex}} \rightarrow \mathcal{D}$$

is called a universal localizing invariant of $\mathcal{D}$ if it satisfies the following properties:

1. **Additivity:** $E$ is a monoidal functor, $E$ is additive, and $E$ preserves cofiber sequences of spectra.
2. **Localization:** For any $\mathcal{B}$ in $\text{Cat}_{\infty}^{\text{ex}}$, the induced map $E(\mathcal{B}) : K_A(\mathcal{B}) \rightarrow K_A(E(\mathcal{B}))$ is an equivalence of spectra.
3. **Universality:** For any other localizing invariant $F : \text{Cat}_{\infty}^{\text{ex}} \rightarrow \mathcal{D}$, there exists a natural transformation $\eta : E \Rightarrow F$.

Theorem 7.7. There exists a universal localizing invariant $E : \text{Cat}_{\infty}^{\text{ex}} \rightarrow \mathcal{D}$.
is called a *localizing invariant* of small stable $\infty$-categories if it inverts Morita equivalences (see definition 2.12), preserves filtered colimits, and satisfies *localization*, i.e., sends exact sequences (see definition 4.11)

$$A \to B \to C$$

to cofiber sequences in $D$:

$$E(A) \to E(B) \to E(C) \to E(A)[1].$$

We denote by $\text{Fun}_{\text{loc}}(\text{Cat}_{\infty}^\text{ex}, D)$ the $\infty$-category of localizing invariants with values in $D$.

Every localizing invariant is an additive invariant (see definition 5.1), since a split-exact sequence is exact. The converse does not hold, however: the impetus for the definition of non-connective $K$-theory was precisely the fact that the connective algebraic $K$-theory functor does not satisfy localization. As we discuss in sections 8 and 10, non-connective algebraic $K$-theory ($IK$), topological Hochschild homology ($THH$), and topological cyclic homology ($TC$) provide localizing invariants of small stable $\infty$-categories.

Although universal localizing invariant can be constructed by a direct localization, as in section 5 for the universal additive invariant, we use a more complicated procedure:

(i) First, we construct a variant of the universal additive invariant; see proposition 7.3. We work with a general infinite regular cardinal $\kappa$, and we do not factor through $\text{Cat}_{\infty}^\text{perf}$; that is, Morita equivalences are not inverted. This produces the functor

$$U_{\text{add}}^\kappa: \text{Cat}_{\infty}^\text{ex} \to \mathcal{M}_{\text{add}}^\kappa.$$

(ii) We localize $\mathcal{M}_{\text{add}}^\kappa$ so that the exact sequences

$$A \to B \to B/A,$$

with $\text{Ho} A$ a thick triangulated subcategory of $\text{Ho} B$, are sent to cofiber sequences; see proposition 7.5. We then obtain the functor

$$U_{\text{wloc}}^\kappa: \text{Cat}_{\infty}^\text{ex} \to \mathcal{M}_{\text{wloc}}^\kappa.$$

(iii) We perform a localization of $\mathcal{M}_{\text{wloc}}^\kappa$ to force Morita equivalences to be sent to isomorphisms; see proposition 7.6. We obtain then the functor

$$U_{\text{loc}}^\kappa: \text{Cat}_{\infty}^\text{ex} \to \mathcal{M}_{\text{loc}}^\kappa.$$

(iv) Finally, we localize $\mathcal{M}_{\text{loc}}^\kappa$ so that the functor $U_{\text{loc}}^\kappa$ preserves filtered colimits; see theorem 7.7. We end up with the universal localizing invariant

$$U_{\text{loc}}^\kappa: \text{Cat}_{\infty}^\text{ex} \to \mathcal{M}_{\text{loc}}^\kappa.$$

The point of the seemingly circuitous process above is that it enables a clear and conceptual proof of the co-representability of non-connective $K$-theory in $\mathcal{M}_{\text{loc}}^\kappa$; see section 8.

**Notation 7.2.** From now on and until the end of section 8 we will work with a fixed infinite regular cardinal $\kappa$ larger than $\omega$. We will denote by $(\text{Cat}_{\infty}^\text{ex})^\kappa$ the category of $\kappa$-compact small stable $\infty$-categories (see §2.3).
7.1. Additive $\kappa$-variant. Let
\[
\psi : \text{Cat}^{\infty}_{\kappa} \to \text{Pre}((\text{Cat}^{\infty}_{\kappa})^{\kappa}; S_{\infty})
\]
be the functor obtained by first taking the Yoneda embedding and then restricting
the presheaves to the category $(\text{Cat}^{\infty}_{\kappa})^{\kappa}$. Corollary 4.16 allow us to choose a fixed
set $E_{A}^{\kappa}$ of representatives of split-exact sequences in $(\text{Cat}^{\infty}_{\kappa})^{\kappa}$. We denote by $M_{\text{add}}^{\kappa}$
the localization of $\text{Pre}((\text{Cat}^{\infty}_{\kappa})^{\kappa}; S_{\infty})$ with respect to the set of maps
\[
\text{Cone}(\psi(A) \to \psi(C)) \to \psi(B),
\]
where $A \to C \to B$ is a split-exact sequence in $E_{A}^{\kappa}$. Let $U_{\text{add}}^{\kappa}$ be the following
composite
\[
\text{Cat}^{\infty}_{\kappa} \xrightarrow{\psi} \text{Pre}((\text{Cat}^{\infty}_{\kappa})^{\kappa}; S_{\infty}) \xrightarrow{\gamma} M_{\text{add}}^{\kappa},
\]
where $\gamma$ is the localization functor.

**Proposition 7.3.** The functor $U_{\text{add}}^{\kappa}$ preserves $\kappa$-filtered colimits and sends split-
exact sequences to equivalences in $M_{\text{add}}^{\kappa}$.

Moreover, $U_{\text{add}}^{\kappa}$ is universal with respect to these two properties, i.e., given any
stable presentable $\infty$-category $D$, we have an equivalence of $\infty$-categories
\[
(U_{\text{add}}^{\kappa})^*: \text{Fun}^L(M_{\text{add}}^{\kappa}, D) \xrightarrow{\sim} \text{Fun}^L_{\text{add}}(\text{Cat}^{\infty}_{\kappa}, D),
\]
where the right-hand side denotes the full subcategory of $\text{Fun}(\text{Cat}^{\infty}_{\kappa}, D)$ of morphisms of $\infty$-categories which satisfy the above two conditions.

**Proof.** The result follows from the analogue of the argument for lemma 5.4 in the
context of $\kappa$-compact objects and Ind$_{\kappa}$, and from the general theory of localization
(see [33, §5.2.7, §5.5.4]): note that the functor $\psi$ preserves $\kappa$-filtered colimits and
proposition 4.18 shows that any split-exact sequence can be approximated by a
$\kappa$-filtered colimit of split-exact sequences in $E_{A}^{\kappa}$. □

Next, we localize $M_{\text{add}}^{\kappa}$ with respect to the set of maps
\[
(7.4) \quad \text{Cone} \left( U_{\text{add}}^{\kappa}(A) \to U_{\text{add}}^{\kappa}(B) \right) \to U_{\text{add}}^{\kappa}(B/A),
\]
where $A \to B \to B/A$ is a strict-exact sequence in $E_{A}^{\kappa}$. Let $U_{\text{loc}}^{\kappa}$ be the following composite
\[
\text{Cat}^{\infty}_{\kappa} \xrightarrow{U_{\text{add}}^{\kappa}} M_{\text{add}}^{\kappa} \xrightarrow{\gamma} M_{\text{loc}}^{\kappa},
\]
where $\gamma$ is the localization functor.

**Proposition 7.5.** The functor $U_{\text{loc}}^{\kappa}$ preserves $\kappa$-filtered colimits and sends strict-
exact sequences to cofiber sequences in $M_{\text{loc}}^{\kappa}$
\[
A \to B \to B/A \to U_{\text{loc}}^{\kappa}(A) \to U_{\text{loc}}^{\kappa}(B) \to U_{\text{loc}}^{\kappa}(B/A) \to U_{\text{loc}}^{\kappa}(A)[1].
\]
Moreover, $U_{\text{loc}}^{\kappa}$ is universal with respect to these two properties, i.e., given any
stable presentable $\infty$-category $D$, we have an equivalence of $\infty$-categories
\[
(U_{\text{loc}}^{\kappa})^*: \text{Fun}^L(M_{\text{loc}}^{\kappa}, D) \xrightarrow{\sim} \text{Fun}_{\text{loc}}^{\kappa}(\text{Cat}^{\infty}_{\kappa}, D),
\]
where the right-hand side denotes the full subcategory of $\text{Fun} (\text{Cat}^\infty_{\omega}, D)$ of morphisms of $\infty$-categories which satisfy the above two conditions.

Proof. The result follows from propositions 7.3 and 4.25, and from the general theory of localization. □

7.2. Morita equivalences. Next, we localize $M^c_{\omega_{\text{loc}}}$ with respect to the a set of maps

$$U^c_{\omega_{\text{loc}}} (A \rightarrow \text{Idem}(A)),$$

where $A \rightarrow \text{Idem}(A)$ belongs to $E^c_{\omega_{\text{loc}}}$. Let $U^c_{\text{loc}}$ be the following composition

$$\text{Cat}^\infty \xrightarrow{U^c_{\omega_{\text{loc}}}} M^c_{\omega_{\text{loc}}} \xrightarrow{\gamma} M^c_{\text{loc}},$$

where $\gamma$ is the localization functor.

**Proposition 7.6.** The functor $U^c_{\text{loc}}$ inverts Morita equivalences, preserves $\kappa$-filtered colimits, and sends exact sequences to cofiber sequences in $M^c_{\text{loc}}$

$$A \rightarrow B \rightarrow C \mapsto U^c_{\text{loc}} (A) \rightarrow U^c_{\text{loc}} (B) \rightarrow U^c_{\text{loc}} (C) \rightarrow U^c_{\text{loc}} (A)[1].$$

Moreover, $U^c_{\text{loc}}$ is universal with respect to these two properties, i.e., given any stable presentable $\infty$-category $D$, we have an equivalence of $\infty$-categories

$$(U^c_{\text{loc}})^* : \text{Fun}^1 (M^c_{\text{loc}}, D) \xrightarrow{\sim} \text{Fun}^c_{\text{loc}} (\text{Cat}^\infty_{\omega}, D),$$

where the right-hand side denotes the full subcategory of $\text{Fun} (\text{Cat}^\infty_{\omega}, D)$ of morphisms of $\infty$-categories which satisfy the above three conditions.

Proof. The fact that $U^c_{\text{loc}}$ preserves $\kappa$-filtered colimits is clear. Since an functor $A \rightarrow B$ is a Morita equivalence if and only if the induced functor $\text{Idem}(A) \rightarrow \text{Idem}(B)$ is an equivalence, proposition 4.26 allow us to conclude that $U^c_{\text{loc}}$ inverts Morita equivalences. We now show that $U^c_{\text{loc}}$ sends exact sequences to cofiber sequences. Let

$$A \rightarrow B \rightarrow C$$

be an exact sequence. Since we have an induced Morita equivalence $B/A \rightarrow C$, it suffices to show that $U^c_{\text{loc}}$ sends exact sequences of shape

$$A \rightarrow B \rightarrow B/A$$

to cofiber sequences in $M^c_{\text{loc}}$. Consider the following commutative diagram

$$\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
\text{Idem}(A) & \rightarrow & \text{Idem}(B) \\
\end{array} \quad \begin{array}{ccc} \\
& \rightarrow & \\
\downarrow & & \\
& \rightarrow & \\
\end{array} \quad \begin{array}{ccc} \\
& \rightarrow & \\
\downarrow & & \\
& \rightarrow & \\
\text{Idem}(B)/\text{Idem}(A), & & \\
& & \\
\end{array}$$

where the right-hand vertical map is the induced one. Since the first two vertical maps are Morita equivalences so is the right vertical one. The bottom line is a strict-exact sequence, and so we conclude that $U^c_{\text{loc}}$ sends exact sequences to cofiber sequences.

The universality of $U^c_{\text{loc}}$ follows from propositions 7.3 and 7.5, and from the general theory of localization. □
7.3. **Universal localizing invariant.** We denote by $E_L$ the set of maps in $\mathcal{M}_\text{loc}^\kappa$ of shape

$$\colim_{\alpha} U^\kappa_{\text{loc}}(A_{\alpha}) \to U^\kappa_{\text{loc}}(A),$$

where $\{A_{\alpha}\}$ is a filtered diagram in $\text{Cat}^\infty_\kappa$ whose colimit is a compact small stable $\infty$-category $A$. Localize $\mathcal{M}_\text{loc}^\kappa$ with respect to the set $E_L$. Let $U_{\text{loc}}$ be the following composition

$$\text{Cat}^\infty \xrightarrow{U^\kappa_{\text{loc}}} \mathcal{M}_\text{loc}^\kappa \xrightarrow{\gamma} \mathcal{M}_\text{loc},$$

where $\gamma$ is the localization functor.

**Theorem 7.7.** The functor $U_{\text{loc}}$ is the universal localizing invariant, i.e., given any stable presentable $\infty$-category $D$, we have an equivalence of $\infty$-categories

$$(U_{\text{loc}})^*: \text{Fun}^\kappa(\mathcal{M}_\text{loc}, D) \xrightarrow{\sim} \text{Fun}_\text{loc}(\text{Cat}^\infty_\kappa, D).$$

**Proof.** The result follows from propositions 7.3, 7.5 and 7.6, and from the general theory of localization. $\Box$

8. **Non-connective $K$-theory**

Bass introduced the negative $K$-groups in order to measure the failure of $K_0$ and $K_1$ to satisfy localization; this perspective was studied in detail in Thomason-Trobaugh and led to the definition of the Bass-Thomason non-connective $K$-theory spectrum for rings and schemes. In fact, any theory which is “like $K$-theory” and satisfies localization must be non-connective; there is a nice discussion of this in [32]. In this section we introduce the non-connective algebraic $K$-theory of $\infty$-categories and show that it becomes co-representable in $\mathcal{M}_\text{loc}^\kappa$; see theorem 8.9. Our work depends critically on the multistage construction of Section 7.

8.1. **Non-connective $K$-theory of $\infty$-categories.** In order to construct the non-connective $K$-theory spectrum associated to a small stable $\infty$-category, we use a generalization of the axiomatic framework due to Schlichting [42]. For an uncountable regular cardinal $\kappa$, we will produce endofunctors $F_\kappa$ and $\Sigma_\kappa$ on $\text{Cat}^\infty_\kappa$ such that for any small stable $\infty$ category $A$:

(i) $K(F_\kappa A)$ is contractible,

(ii) there are natural transformations

$$\text{Id} \to F_\kappa \to \Sigma_\kappa$$

such that $A \to F_\kappa A \to \Sigma_\kappa A$ is exact,

(iii) the functors $F_\kappa$ and $\Sigma_\kappa$ preserve exact sequences,

(iv) and $F_\kappa$ and $\Sigma_\kappa$ preserve $\kappa$-filtered colimits in $\text{Cat}^\infty_\kappa$.

The idea is that $F_\kappa A$ is a “$K$-theoretic cone” and so $\Sigma_\kappa A$ is a “suspension” of $A$. Fix an uncountable regular cardinal $\kappa$, and for a stable $\infty$-category $C$ recall from Section 2.3 that $C^\kappa$ denotes the $\kappa$-compact objects in $C$.

**Definition 8.1.** Using proposition 2.13, we define $F_\kappa A = (\text{Ind}_\kappa(A))^\kappa$ and $\Sigma_\kappa A$ to be the cofiber $((\text{Ind}_\kappa(A))^\kappa)/A$.

**Remark 8.2.** One might wish to simply use $\text{Ind} A$ as the cone construction; however, this will rarely turn out to be a small $\infty$-category, whereas passing to the $\kappa$-compact objects clearly yields an (essentially) small $\infty$-category.
By construction and Propositions 4.6 and 4.8, we have an exact sequence
\[ A \longrightarrow \mathcal{F}_\kappa A \longrightarrow \Sigma \kappa A, \]
natural in small stable \( \infty \)-categories \( A \). Next, we check that \( \mathcal{F}_\kappa \) satisfies property (i).

**Lemma 8.3.** Let \( A \) be a small stable \( \infty \)-category. Then \( K(\mathcal{F}_\kappa A) \) is trivial.

**Proof.** Since \( \kappa \) is uncountable, \( \mathcal{F}_\kappa A \) has countable coproducts, and so the usual Eilenberg swindle argument implies that its \( K \)-theory vanishes. \( \square \)

Next, we need to check that \( \mathcal{F}_\kappa \) and \( \Sigma \kappa \) preserve exact sequences of small stable \( \infty \)-categories.

**Proposition 8.4.** Let \( A \to B \to C \) be an exact sequence of small stable \( \infty \)-categories. Then the induced sequences
\[
\mathcal{F}_\kappa A \longrightarrow \mathcal{F}_\kappa B \longrightarrow \mathcal{F}_\kappa C \quad \Sigma \kappa A \longrightarrow \Sigma \kappa B \longrightarrow \Sigma \kappa C
\]
are exact.

**Proof.** It suffices to show the result for \( \mathcal{F}_\kappa \), as the statement for \( \Sigma \kappa \) follows because colimits commute. Thus, we need to verify that
\[(\text{Ind}_\omega(A))^\kappa \longrightarrow (\text{Ind}_\omega(B))^\kappa \longrightarrow (\text{Ind}_\omega(C))^\kappa\]
is exact. The sequence
\[
\text{Ind}_\omega A \longrightarrow \text{Ind}_\omega B \longrightarrow \text{Ind}_\omega C
\]
is exact by Definition 4.11 and Proposition 4.12. Now the result follows from Proposition 4.13. \( \square \)

Passing to the triangulated homotopy category by composing with the functor \( Ho \), we get a series of functors which satisfies Schlichting’s setup of [42, §2.2] and so produces negative \( K \)-groups. Furthermore, we can define the non-connective \( K \)-theory spectrum as follows, following [42, §12].

**Definition 8.5.** Let \( A \) be a small stable \( \infty \)-category. Its non-connective \( K \)-theory spectrum \( IK(A) \) is given by
\[
IK(A) := \text{colim}_n \Omega^n K(\Sigma^{(n)} \kappa A).
\]
Here, \( K \) stands for the \( K \)-theory spectrum of §6.1 and the structure maps are induced from the exact sequences
\[
\Sigma^{(n)} \kappa A \longrightarrow \mathcal{F}_\kappa \Sigma^{(n)} \kappa A \longrightarrow \Sigma^{(n+1)} \kappa A \quad n \geq 0.
\]

**Remark 8.6.** In fact, this stabilization procedure takes any additive invariant to a localizing invariant. This process will be studied in more detail in the forthcoming thesis of P. Szuta.

Schlichting’s axiomatic framework implies that this construction agrees with his when both are defined, and therefore we deduce from his comparison results [42, §8] that the non-connective \( K \)-theory spectrum of Definition 8.5 agrees with the various classical constructions of non-connective \( K \)-theory spectra. Furthermore, we observe that for connective ring spectra \( R \), the non-connective \( K \)-theory spectrum has negative homotopy groups determined by the non-connective \( K \)-theory spectrum of \( \pi_0 R \).
Theorem 8.7. Let $R$ be a connective $S$-algebra. Then for $i \leq 0$, the natural map $R \to H\pi_0 R$ induces isomorphisms $\pi_i K(R) \to \pi_i K(H\pi_0 R) \cong \pi_i K(R)$.

Proof. For a connective ring spectrum $R$, the algebraic $K$-theory spectrum constructed from the associated Waldhausen category of perfect $R$-modules agrees with the description using the the extension of the plus construction introduced in [64] to structured ring spectra [19, §VI.7]. Without loss of generality, we assume that $R$ is an EKMM $S$-algebra. Let $C_R$ denote the full subcategory of finite cell $R$-modules, and denote by $M_n R$ the topological mapping space $C_R(\vee^n S_R, \vee^n S_R)$, where $S_R$ denotes a cofibrant replacement of $R$ as an $R$-module. Now we define $GL_n(R)$ as the pullback

\[
\begin{array}{ccc}
GL_n(R) & \longrightarrow & M_n(R) \\
\downarrow & & \downarrow \\
GL_n(\pi_0 R) & \longrightarrow & M_n(\pi_0 R) \cong \pi_0 M_n(R).
\end{array}
\]

Since $GL_n(R)$ is a monoid, we can form its classifying space $BGL_n(R)$. Moreover, there are natural inclusions $GL_n(R) \to GL_{n+1}(R)$ which induce maps $BGL_n(R) \to BGL_{n+1}(R)$. Let $BGL(R)$ denote the colimit colim $GL_n(R)$, we can form the plus construction, and we define the $K$-theory space to be the infinite loop space $K_0(\pi_0 R) \times BGL(R)^+$.

It now follows from classical work on the $K$-theory of $A_\infty$ ring spaces that the Postnikov 0-section map $R \to H\pi_0 R$ induces an isomorphism on $K_i$, for $i \leq 0$ [20]. Specifically, one can check that the definition of $K$-theory of $R$ and $H\pi_0 R$ in terms of the plus construction above is equivalent to a definition in terms of a plus construction on the $A_\infty$ ring spaces $\Omega^\infty R$ and $\Omega^\infty H\pi_0 R$, respectively. The desired comparison then follows immediately from the discussion following [20, 1.1].

Finally, we establish the final technical condition; this will be needed in the following sections.

Lemma 8.8. The functors $F_\kappa$ and $\Sigma_\kappa$ preserve $\kappa$-filtered colimits.

Proof. For $F_\kappa$, this is a consequence of Proposition 2.13 and the fact that the passage to $\kappa$-compact objects preserves $\kappa$-filtered colimits. Once again, the result follows for $\Sigma_\kappa$ since colimits commute.

8.2. Co-representability. This subsection is entirely devoted to the proof of the following co-representability result.

Theorem 8.9. Let $A$ be a small stable $\infty$-category. Then there is a natural equivalence of spectra

\[
\text{Map} (U_{\text{loc}}(S^\infty_\kappa), U_{\text{loc}}(A)) \simeq K(A).
\]

In particular, for each integer $n$, we have isomorphisms of abelian groups

\[
\text{Hom} (U_{\text{loc}}(S^\infty_\kappa), \Sigma^{-n} U_{\text{loc}}(A)) \simeq K_n(A)
\]

in the triangulated category $\text{Ho}(M_{\text{loc}})$.

The proof of theorem 8.9 will follow from theorems 8.10 and 8.11, and from propositions 8.18 through 8.27.
Theorem 8.10. Let $\mathcal{A}$ and $\mathcal{B}$ be small stable $\infty$-categories such that $\mathcal{B}$ is $\kappa$-compact. Then there is a natural equivalence of spectra

$$\text{Map}(U^\kappa_{\text{add}}(\mathcal{B}), U^\kappa_{\text{add}}(\mathcal{A})) \simeq K(\text{Fun}^\kappa(\mathcal{B}, \mathcal{A})).$$

If $\mathcal{B} = S^\infty_\omega$ is the $\infty$-category of compact spectra \[35, 4.2.8\], this reduces to an equivalence

$$\text{Map}(U^\kappa_{\text{add}}(S^\infty_\omega), U^\kappa_{\text{add}}(\mathcal{A})) \simeq K(A).$$

Proof. The proof is analogous to the proof of theorem 6.9; instead of the idempotent-complete stable $\infty$-category $\text{Fun}^\kappa(\mathcal{B}, \text{Idem}(\mathcal{A}))$ we consider the small stable $\infty$-category $\text{Fun}^\kappa(\mathcal{B}, \mathcal{A})$. Note that since $\kappa > \omega$, $S^\infty_\omega$ belongs to $(\text{Cat}^\infty_{\text{ex}})^\kappa$. □

Theorem 8.11. Let $\mathcal{A}$ be a small stable $\infty$-category. Then there is a natural equivalence of spectra

$$\text{Map}(U^\kappa_{\text{wloc}}(S^\infty_\omega), U^\kappa_{\text{wloc}}(\mathcal{A})) \simeq K(A).$$

Proof. By construction, the object $U^\kappa_{\text{add}}(S^\infty_\omega)$ is compact in $M^\kappa_{\text{add}}$. Let $S$ denote the set of maps in (7.4), $\overline{S}$ the strongly saturated collection of arrows generated by $S$ \[33, Definition 5.5.4.5\], and let $X$ be an $S$-local object such that the map $U^\kappa_{\text{add}}(A) \to X$ is an $S$-local equivalence (i.e., $U^\kappa_{\text{add}}(A) \to X$ is in $\overline{S}$). Then by definition,

$$\text{Map}(U^\kappa_{\text{wloc}}(S^\infty_\omega), U^\kappa_{\text{wloc}}(A)) \simeq \text{Map}(U^\kappa_{\text{add}}(S^\infty_\omega), X),$$

so it suffices to show that the functor

$$R := \text{Map}(U^\kappa_{\text{add}}(S^\infty_\omega), -) : M^\kappa_{\text{add}} \to S^\infty_\omega$$

sends the maps in $\overline{S}$ to equivalences of spectra. Since $M^\kappa_{\text{add}}$ is a stable $\infty$-category and $U^\kappa_{\text{add}}(S^\infty_\omega)$ is compact, $R$ preserves small colimits, so (by the two-out-of-three property) we immediately reduce to checking that $R$ sends the elements of $S$ to equivalences.

Consider the following diagram

$$\begin{array}{ccc}
U^\kappa_{\text{add}}(A) & \longrightarrow & U^\kappa_{\text{add}}(B) \\
\downarrow & & \downarrow \\
U^\kappa_{\text{add}}(A) & \longrightarrow & U^\kappa_{\text{add}}(B) \\
\downarrow & & \downarrow \\
\text{Cone}(U^\kappa_{\text{add}}(A) \to U^\kappa_{\text{add}}(B)) & \longrightarrow & U^\kappa_{\text{add}}(B/A).
\end{array}$$

By applying the functor (8.12) to the above diagram (8.13) we obtain by theorem 8.10 a diagram in $S$

$$\begin{array}{ccc}
K(A) & \longrightarrow & K(B) \\
\downarrow & & \downarrow \\
K(A) & \longrightarrow & K(B) \\
\downarrow & & \downarrow \\
\text{Cone}(K(A) \to K(B)) & \longrightarrow & K(B/A),
\end{array}$$

where the upper row is a homotopy cofiber sequence. Now, an argument analogous to the one used in the proof of proposition 6.14 (where we make use of Waldhausen’s fibration theorem) allow us to conclude that the lower row in the above diagram (8.14) is also a homotopy cofiber sequence. This completes the argument. □
Let $V$ be the partially ordered set $\{(i, j) : |i - j| \leq 1, i, j \geq 0\} \subset \mathbb{N} \times \mathbb{N}$. Given a small stable $\infty$-category $\mathcal{A}$, we denote by $\text{Dia}(\mathcal{A})$ the $V$-diagram

\begin{equation}
\cdots \rightarrow \mathcal{F}_n \Sigma^{(n)}(\mathcal{A}) \rightarrow \Sigma^{(n+1)}(\mathcal{A}) \rightarrow \cdots
\end{equation}

where $\mathcal{F}_n$ and $\Sigma_n$ are as in Definition 8.1.

**Lemma 8.16.** Let $\mathcal{A}$ be a small stable $\infty$-category. Then $\Sigma^{(n)}(\mathcal{A})/\Sigma^{(n)}(\mathcal{A})$ and $\mathcal{F}_n \Sigma^{(n)}(\mathcal{A})$ become trivial after application of $\mathcal{U}^n_{\text{wloc}}$. 

**Proof.** The object $\Sigma^{(n)}(\mathcal{A})/\Sigma^{(n)}(\mathcal{A})$ is already trivial in $\text{Cat}_{\infty}^n$. Since Proposition 2.13 implies that $\mathcal{F}_n \Sigma^{(n)}(\mathcal{A})$ admits all $\kappa$-small colimits, for any $\mathcal{B}$ in $(\text{Cat}_{\infty}^n)_{\kappa}$ the small stable $\infty$-category $\text{Fun}(\mathcal{B}, \mathcal{F}_n \Sigma^{(n)}(\mathcal{A}))$ also admits all $\kappa$-small colimits. Thus, the connective $K$-theory spectrum $K(\text{Fun}(\mathcal{B}, \mathcal{F}_n \Sigma^{(n)}(\mathcal{A})))$ is trivial. Finally, theorem 8.10 and the fact that the objects $\mathcal{U}^n_{\text{add}}(\mathcal{B})$, with $\mathcal{B}$ in $(\text{Cat}_{\infty}^n)_{\kappa}$ generate the category $\mathcal{M}^\kappa_{\text{add}}$ allow us to conclude that $\mathcal{F}_n \Sigma^{(n)}(\mathcal{A})$ becomes trivial after application of $\mathcal{U}^n_{\text{add}}$, and thus after application of $\mathcal{U}^n_{\text{wloc}}$. \qed

Let $\mathcal{A}$ be a small stable $\infty$-category. We denote by $V(\mathcal{A})$ the object

\[ V(\mathcal{A}) = \text{colim}_n \Sigma^{-n} \mathcal{U}^n_{\text{wloc}}(\Sigma^{(n)}(\mathcal{A})) \]

in $\mathcal{M}^\kappa_{\text{wloc}}$ whose indexing maps are induced from the above diagram (8.15). Note that $V(\mathcal{A})$ is functorial in $\mathcal{A}$ and that we have a natural map $\mathcal{U}^n_{\text{wloc}}(\mathcal{A}) \rightarrow V(\mathcal{A})$. We obtain then a well-defined functor $V$ along with a natural transformation:

\begin{equation}
(8.17) \quad V(-) : \text{Cat}_{\infty}^\kappa \rightarrow \mathcal{M}^\kappa_{\text{wloc}} \quad \mathcal{U}^n_{\text{wloc}} \Rightarrow V(-).
\end{equation}

**Proposition 8.18.** Let $\mathcal{A}$ be a small stable $\infty$-category. Then, there is a natural isomorphism in the stable homotopy category of spectra

\[ \text{Map}(\mathcal{U}^n_{\text{wloc}}(\mathcal{S}_\infty^\kappa), V(\mathcal{A})) \simeq \mathcal{K}(\mathcal{A}). \]

**Proof.** This follows from the following equivalences

\begin{align}
\mathcal{K}(\mathcal{A}) &= \text{hocolim}_{n \geq 0} \Omega^n K(\Sigma^{(n)}(\mathcal{A})) \\
(8.19) &\simeq \text{hocolim}_{n \geq 0} \Omega^n \text{Map}(\mathcal{U}^n_{\text{wloc}}(\mathcal{S}_\infty^\kappa), \mathcal{U}^n_{\text{wloc}}(\Sigma^{(n)}(\mathcal{A}))) \\
(8.20) &\simeq \text{Map}(\mathcal{U}^n_{\text{wloc}}(\mathcal{S}_\infty^\kappa), \text{colim}_{n \geq 0} \Sigma^{-n} \mathcal{U}^n_{\text{wloc}}(\Sigma^{(n)}(\mathcal{A}))) \\
&\simeq \text{Map}(\mathcal{U}^n_{\text{wloc}}(\mathcal{S}_\infty^\kappa), V(\mathcal{A})).
\end{align}
Equivalence (8.19) comes from theorem 8.11 and equivalence (8.20) comes from the compactness of $\mathcal{U}_{\text{wloc}}^\kappa(S^\omega_\infty)$ in $\mathcal{M}^\kappa_{\text{wloc}}$. □

**Proposition 8.21.** The functor $V$ (8.17) inverts Morita equivalences.

**Proof.** It suffices to show that $V(-)$ sends maps of shape $A \to \text{Idem}(A)$ to isomorphisms. Consider the following diagram

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\mathcal{F}_\kappa(A)} & \Sigma_\kappa(A) \\
\downarrow P & & \downarrow \Sigma_\kappa(P) \\
\text{Idem}(A) & \xrightarrow{\mathcal{F}_\kappa(\text{Idem}(A))} & \Sigma_\kappa(\text{Idem}(A)).
\end{array}
\]

Proposition 2.13 implies that $\mathcal{F}_\kappa(P)$ is an equivalence. Therefore, since both rows are strict-exact sequences and $\text{Ho}(\mathcal{A})$ and $\text{Ho}(\text{Idem}(\mathcal{A}))$ differ by direct summands, we conclude that $S(P)$ is an equivalence. The definition of the functor $V(-)$ allow us to conclude the proof. □

**Proposition 8.22.** Assume that $\kappa > \omega$. Then, the functor $V(-) : \text{Cat}^{\omega}_{\text{ex}} \to \mathcal{M}^\kappa_{\text{wloc}}$ inverts Morita equivalences, preserves $\kappa$-filtered colimits, and sends exact sequences to cofiber sequences.

**Proof.** Proposition 8.21 implies that $V(-)$ inverts Morita equivalences. Furthermore, by Lemma 8.8, $\mathcal{F}_\kappa$ and $\Sigma_\kappa$ preserve $\kappa$-filtered colimits for $\kappa > \omega$, and so $V(-)$ does as well. Now, let

\[A \to B \to C\]

be an exact sequence. Proposition 8.21 implies that we can assume that $\text{Ho}(\mathcal{A})$ is a thick triangulated subcategory of $\text{Ho}(\mathcal{B})$. Consider the following diagram

\[
\begin{array}{ccc}
\text{Dia}(\mathcal{A}) & \xrightarrow{\text{Dia}(\mathcal{B})} & \text{Dia}(\mathcal{A}, \mathcal{B}) := \text{Dia}(\mathcal{A})/\text{Dia}(\mathcal{B}) \\
\downarrow & & \downarrow \partial \\
\text{Dia}(\mathcal{A}) & \xrightarrow{\text{Dia}(\mathcal{B})} & \text{Dia}(\mathcal{C}),
\end{array}
\]

where $\text{Dia}(\mathcal{A}, \mathcal{B})$ is obtained by passage to the cofiber objectwise. Note that since in the above diagram (8.23) the upper row is objectwise a strict-exact sequence, we obtain a cofiber sequence

\[V(\mathcal{A}) \to V(\mathcal{B}) \to V(\mathcal{B}, \mathcal{A}) \to \Sigma V(\mathcal{A})\]

in $\mathcal{M}^\kappa_{\text{wloc}}$, where

\[V(\mathcal{B}, \mathcal{A}) := \text{colim}_n \Sigma^{-n}\mathcal{U}_{\text{wloc}}^\kappa(\Sigma^{(n)}_\kappa(\mathcal{A})/\Sigma^{(n)}_\kappa(\mathcal{B})).\]

We now show that the induced map

\[V(\mathcal{B}, \mathcal{A}) \to V(\mathcal{C})\]
is an equivalence. For this, consider the following commutative diagram

\[
\begin{array}{ccc}
\Sigma^{(n)}_\kappa(A) & \to & \mathcal{F}_\kappa \Sigma^{(n)}_\kappa(A) \\
\downarrow & & \downarrow \\
\Sigma^{(n)}_\kappa(B) & \to & \mathcal{F}_\kappa \Sigma^{(n)}_\kappa(B) \\
\downarrow & & \downarrow \\
\Sigma^{(n)}_\kappa(B)/\Sigma^{(n)}_\kappa(A) & \to & \mathcal{F}_\kappa \Sigma^{(n)}_\kappa(B)/\mathcal{F}_\kappa \Sigma^{(n)}_\kappa(A) \\
\downarrow \theta_n & & \downarrow D_n \\
\Sigma^{(n)}_\kappa(C) & \to & \mathcal{F}_\kappa \Sigma^{(n)}_\kappa(C) \\
\end{array}
\]

Since the induced triangulated functor

\[ \text{Ho}(\mathcal{F}_\kappa \Sigma^{(n)}_\kappa(A)) \to \text{Ho}(\mathcal{F}_\kappa \Sigma^{(n)}_\kappa(B)) \]

preserves \(\kappa\)-small colimits, [42, §3.1, Thm. 2] implies that the triangulated category

\[ \text{Ho}(\mathcal{F}_\kappa \Sigma^{(n)}_\kappa(B)/\mathcal{F}_\kappa \Sigma^{(n)}_\kappa(A)) \]

is idempotent complete. Therefore, \(\theta_n\) is an equivalence, and we obtain maps

\[ \psi_n : \Sigma^{(n)}_\kappa(C) \to \mathcal{F}_\kappa \Sigma^{(n)}_\kappa(B)/\mathcal{F}_\kappa \Sigma^{(n)}_\kappa(A) \]

which induce maps

\[ \Psi_n : \Sigma^{-n}U^\kappa_{wloc}(\Sigma^{(n)}_\kappa(C)) \to \Sigma^{-n}U^\kappa_{wloc}(\Sigma^{(n+1)}_\kappa(B)/\Sigma^{(n+1)}_\kappa(A)) \]

It follows that the natural map

\[ \text{colim}_n \Sigma^{-n}U^\kappa_{wloc}(\Sigma^{(n)}_\kappa(B)/\Sigma^{(n)}_\kappa(A)) \to \text{colim}_n \Sigma^{-n}U^\kappa_{wloc}(\Sigma^{(n)}_\kappa(C)) \]

is an equivalence, which implies that the map (8.24) is an equivalence. \(\square\)

**Corollary 8.25** (of proposition 8.22). If \(\kappa > \omega\), there is an functor

\[ \text{Loc} : \mathcal{M}^\kappa_{loc} \to \mathcal{M}^\kappa_{wloc} \]

such that \(\text{Loc}(U^\kappa_{loc}(A)) \simeq V(A)\), for every small stable \(\infty\)-category \(A\).

**Proof.** This follows from propositions 8.22 and 7.6. \(\square\)

**Proposition 8.26.** If \(\kappa > \omega\), the two functors

\[ \text{Loc}, \gamma^* : \mathcal{M}^\kappa_{loc} \to \mathcal{M}^\kappa_{wloc} \]

are canonically isomorphic, where \(\gamma^*\) is the right adjoint of the localization functor.

**Proof.** Let us denote by \(L\) the \(\infty\)-endofunctor \(\text{Loc} \circ \gamma\) of \(\mathcal{M}^\kappa_{wloc}\). Note that we have a natural transformation \(\text{Id} \Rightarrow L\) of functors. Making use of the definition of \(V(-)\) and of the Fubini rule (for colimits), we observe that \(L\) is a localization functor on \(\mathcal{M}^\kappa_{wloc}\). Therefore, it suffices to show that a map in \(\mathcal{M}^\kappa_{wloc}\) becomes an isomorphism in \(\mathcal{M}^\kappa_{loc}\) if and only if it becomes an isomorphism after application of \(L\). This follows from the fact that for every small stable \(\infty\)-category \(A\), we have an equivalence \(\gamma(V(A)) \simeq U^\kappa_{loc}(A)\): note that we have cofiber sequences in \(\mathcal{M}^\kappa_{loc}\)

\[ U^\kappa_{loc}(\Sigma^{(n)}_\kappa(A)) \to U^\kappa_{loc}(\mathcal{F}_\kappa \Sigma^{(n)}_\kappa(A)) \to U^\kappa_{loc}(\Sigma^{(n+1)}_\kappa(A)) \to U^\kappa_{loc}(\Sigma^{(n)}_\kappa(A))[1]. \]
Proposition 8.27. Let $\mathcal{A}$ be small stable $\infty$-category. If $\kappa > \omega$, we have a natural isomorphism in the stable homotopy category of spectra

$$\text{Map}(\mathcal{U}_\kappa^c(S^\omega_\infty), \mathcal{U}_\kappa^c(\mathcal{A})) \simeq K(\mathcal{A}).$$

Proof. This follows from the following equivalences

$$\text{Map}(\mathcal{U}_\kappa^c(S^\omega_\infty), \mathcal{U}_\kappa^c(\mathcal{A})) \simeq \text{Map}(\mathcal{U}_\kappa^c(S^\omega_\infty), \gamma^* (\mathcal{U}_\kappa^c(\mathcal{A})))$$
$$\simeq \text{Map}(\mathcal{U}_\kappa^c(S^\omega_\infty), \text{Loc}(\mathcal{U}_\kappa^c(\mathcal{A})))$$
$$\simeq K(\mathcal{A}).$$

Equivalence (8.28) comes from proposition 8.26, equivalence (8.29) comes from corollary 8.25, and equivalence (8.30) is proposition 8.18.

Proof of theorem 8.9. Recall from subsection 7.3 that $\mathcal{M}_\text{loc}$ is obtained by localizing $\mathcal{M}_\kappa^c$ with respect to the set $\mathcal{E}_L$. Since $\mathcal{U}_\kappa^c(S^\omega_\infty)$ is compact in $\mathcal{M}_\kappa^c$, it is sufficient by proposition 8.27 and the general theory of localization to show that the functor

$$\text{Map}(\mathcal{U}_\kappa^c(S^\omega_\infty), -) : \mathcal{M}_\text{loc}^c \rightarrow S_\infty$$

sends the elements of $\mathcal{E}_L$ to equivalences. This follows from the fact that the non-connective $K$-theory construction preserves filtered colimits (see [42, §7, Lemma 6]), and so the proof is finished.

9. Homotopy invariance

Classically, a functor $F$ from rings to spectra satisfies algebraic homotopy invariance if for any ring $R$ the natural map $R \rightarrow R[t]$ induces an equivalence $F(R) \simeq F(R[t])$. Geometric considerations motivated much early work on developing a homotopy invariant version of algebraic $K$-theory, culminating in the Karoubi-Villamayor $K$-theory [30, 31]. This is a connective theory, and following Thomason’s construction of the non-connective $K$-theory spectrum, Weibel defined homotopy $K$-theory as a non-connective generalization [65].

Following an observation of Rector [47, 23], both Karoubi-Villamayor’s $K$-theory and Weibel’s homotopy $K$-theory of a ring $R$ can be defined in terms of the “standard cosimplicial affine space”. More precisely, passing to functions the standard cosimplicial affine space yields a simplicial ring which in degree $n$ is the hyperplane $\mathbb{Z}[t_0, \ldots, t_n]/(1 - \sum t_i)$ in the $(n + 1)$-dimensional affine space $\mathbb{Z}[t_0, \ldots, t_n]$. Tensoring with the ring $R$ yields a simplicial ring, and the Karoubi-Villamayor and homotopy $K$-theories of $R$ are defined to be the geometric realizations of the simplicial spectra obtained by applying connective and non-connective $K$-theory levelwise, respectively.

In this section we construct the universal additive (respectively, localizing) invariant from stable $\infty$-categories to spectra which is homotopy invariant; see theorem 9.21. Moreover, we establish co-representability results for Karoubi-Villamayor (respectively, Weibel’s homotopy $K$-theory) in this context; see theorem 9.23. Most of the work of this section goes into suitably generalizing the notion of homotopy invariance to the setting of stable $\infty$-categories and developing a version of the standard (co)simplicial affine space over $S$. 
Remark 9.1. Note that in the following subsection we will always be working in the $\infty$-categorical setting, unless otherwise noted; e.g., all discussion of ring spectra and module categories should be interpreted in this way.

9.1. Cosimplicial affine space over the sphere. For each $n \in \mathbb{N}$, let $A^n = \text{Perf}(S[\mathbb{N}^n])$ denote the $\infty$-category of perfect $S[\mathbb{N}^n]$-modules, where $S[\mathbb{N}^n] \simeq \Sigma_+^\infty \mathbb{N}^n$ is the $E_\infty$ spectrum freely associated to the $E_\infty$ space $\mathbb{N}$. In mild abuse of notation, we will usually write $S[t_1, \ldots, t_n]$ in place of $S[\mathbb{N}^n]$. Note that $\pi_0 S[t_1, \ldots, t_n] \cong \mathbb{Z}[t_1, \ldots, t_n]$ is the ring of functions on $A^n_\mathbb{Z} \cong A^n \otimes_\mathbb{Z} \mathbb{Z}$, the $n$-dimensional affine space over the integers.

Observe that the stable $\infty$-categories $A^n$ are naturally commutative algebra objects; the multiplicative structure is induced from the commutative algebra structure on $S[t_1, \ldots, t_n]$. In addition, there is an algebra endomorphism $e : S[t] \to S[t]$ given by the composition of the “evaluation at zero” map $\epsilon : S[t] \to S$ with the unique algebra map $S \to S[t]$.

Given a ring spectrum $R$ and an $S$-module $M$, we write $\text{Mod}_R(M)$ for the space of $R$-module structures on $M$ and $\text{Alg}_S(R, \text{End}(M))$ for the space of $S$-algebra maps from $R$ to $\text{End}(M)$.

Lemma 9.2. There is a canonical equivalence $\text{Mod}_R(M) \simeq \text{Alg}_S(R, \text{End}(M))$. In particular, taking $R = S[t]$ yields the following equivalence:

$$\text{Mod}_S(S[t]) \simeq \text{Alg}_S(S[t], \text{End}(M)) \simeq \text{End}(M).$$

That is, the space of $S[t]$-module structures on $M$ is canonically equivalent to the space of $S$-module endomorphisms of $M$.

Proof. The first statement follows from corollary 2.7.9 of [35]. The second one follows from the fact that the $S$-algebra $S[t]$ is the free $S$-algebra generated by the free $S$-module $S$. \qed

Proposition 9.3. Let $M = S[t_1, \ldots, t_n]$. Then we have isomorphisms

$$\pi_0 \text{Mod}_S(M) \cong \pi_0 \text{End}(M) \cong \text{End}(\pi_0 M) \cong \text{Mod}_S(\pi_0 M),$$

which is to say that a $\mathbb{Z}[t]$-module structure on $\pi_0 M$ specifies a unique homotopy class of $S[t]$-module structure on $M$.

Proof. The result follows from the natural isomorphisms

$$\pi_0 \text{End}(M) \cong \pi_0 \text{map}(\mathbb{N}^n, S[\mathbb{N}^n]) \cong \text{map}(\mathbb{N}^n, \mathbb{Z}[\mathbb{N}^n]) \cong \text{End}(\pi_0 M)$$

together with the description of $\text{Mod}_S(M)$ of lemma 9.2 above. \qed

For each natural number $n$, multiplication by $t_0 + \cdots + t_n \in \pi_0 S[t_0, \ldots, t_n]$ endows the commutative $S$-algebra $S[t_0, \ldots, t_n]$ with the structure of an $S[t]$-module. In particular, $S[u, v]$ is an $S[t]$-module such that the $S$-algebra map $S[t] \to S[u, v]$ which sends $t$ to $u + v$ is an $S[t]$-module map.

For the next result, we need to use the notion of flatness discussed in [35, §4.6] in order to retain calculational control. Recall that an $R$-module $M$ is flat in this sense if $\pi_0 M$ is flat as a $\pi_0 R$ module and for each $n \in \mathbb{Z}$, the canonical map $\pi_n R \otimes_{\pi_0 R} \pi_0 M \to \pi_n M$ is an isomorphism [35, 4.6.9].

Lemma 9.4. For each pair of natural numbers $n$, there is a natural $S$-module equivalence

$$S[u_0, \ldots, u_m] \otimes_S S[t_0, \ldots, t_n] \to S[t_0, \ldots, t_{m+n}].$$
Proof. Define an $\mathcal{S}$-module map

$$\mathcal{S}[u_0, \ldots, u_m] \wedge \mathcal{S}[v_0, \ldots, v_n] \rightarrow \mathcal{S}[t_0, \ldots, t_{m+n}]$$

by tensoring (via the multiplication on the commutative $\mathcal{S}$-algebra $\mathcal{S}[t_0, \ldots, t_{m+n}]$) the $\mathcal{S}$-algebra maps

$$\mathcal{S}[u_i] \rightarrow \mathcal{S}[t_0, \ldots, t_{m+n}] \quad \mathcal{S}[v_j] \rightarrow \mathcal{S}[t_0, \ldots, t_{m+n}]$$

which send $u_i$ to $t_i$ for $i < m$, $u_m$ to $t_m + \cdots + t_{m+n}$, $v_0$ to $t_0 + \cdots + t_m$, and $v_j$ to $t_{m+j}$ for $j > 0$. We must show that this map induces an equivalence between $\mathcal{S}[t_0, \ldots, t_{m+n}]$ and the realization of the simplicial $\mathcal{S}$-module which in degree $k$ is

$$\mathcal{S}[u_0, \ldots, u_m] \wedge \mathcal{S}[t_0^\wedge k] \wedge \mathcal{S}[v_0, \ldots, v_n].$$

To see this, we use the Tor spectral sequence

$$\text{Tor}_{p,q}^\mathcal{S}[t](\mathcal{S}[u_0, \ldots, u_m], \mathcal{S}[v_0, \ldots, v_n]) \Rightarrow \pi_{p+q}(\mathcal{S}[u_0, \ldots, u_m] \wedge \mathcal{S}[v_0, \ldots, v_n])$$

for the homotopy groups of the $\mathcal{S}[t]$-module tensor product [35, 4.6.13]. Since each of the $\mathcal{S}$-modules $\mathcal{S}[u_0, \ldots, u_m]$, $\mathcal{S}[v_0, \ldots, v_n]$, and $\mathcal{S}[t_0, \ldots, t_{m+n}]$ are flat, the spectral sequence degenerates and we have that the map

$$\mathcal{S}[u_0, \ldots, u_m] \wedge \mathcal{S}[v_0, \ldots, v_n] \rightarrow \mathcal{S}[t_0, \ldots, t_{m+n}]$$

is an equivalence if it is an equivalence on $\pi_0$ [35, 4.6.21]. But this follows from the fact that the coproduct of the commutative $\mathbb{Z}[t]$-algebras $\mathbb{Z}[u_0, \ldots, u_m]$ and $\mathbb{Z}[v_0, \ldots, v_n]$ is isomorphic to $\mathbb{Z}[t_0, \ldots, t_{m+n}]$.

Proposition 9.5. The $\mathcal{S}[t]$-modules $\mathcal{S}[t_0, \ldots, t_n]$ assemble to form a simplicial $\mathcal{S}[t]$-module in which the simplicial structure maps are given by the formulae

$$d_r(t_j) = \begin{cases} t_j & \text{if } j < r \\ 0 & \text{if } j = r \\ t_{j-1} & \text{if } j > r \end{cases} \quad s_r(t_j) = \begin{cases} t_j & \text{if } j < r \\ t_j + t_{j+1} & \text{if } j = r \\ t_{j+1} & \text{if } j > r \end{cases}.$$

Moreover, for each natural number $n$, we have an $\mathcal{S}[t]$-module equivalence

$$\mathcal{S}[u, v]^\wedge [t_0, \ldots, t_n] \rightarrow \mathcal{S}[t_0, \ldots, t_n],$$

where the $\mathcal{S}[t]$-module on the left is the $n$-fold $\mathcal{S}[t]$-module tensor power of the $\mathcal{S}[t]$-module $\mathcal{S}[u, v]$.

Proof. The face and degeneracy maps defined above are $\mathcal{S}[t]$-module maps because (at least on the level of homotopy) $d_r(\sum t_j) = \sum t_j$ and $s_r(\sum t_j) = \sum t_j$. The equivalences $\mathcal{S}[u, v]^\wedge [t_0, \ldots, t_n]$ follow from lemma 9.4 above; the naturality of these equivalences is a simple computation.

Corollary 9.6. For each $n$, $\mathcal{S}[t_0, \ldots, t_n]$ is canonically an algebra object in the $\infty$-category of $\mathcal{S}[t]$-modules, that is, an $\mathcal{S}[t]$-algebra. Moreover, for each ordinal map $[m] \rightarrow [n]$, the induced map $\mathcal{S}[t_0, \ldots, t_n] \rightarrow \mathcal{S}[t_0, \ldots, t_m]$ is an $\mathcal{S}[t]$-algebra map.

Proof. According to the previous proposition, there is a simplicial $\mathcal{S}[t]$-module which in degree $n$ is $\mathcal{S}[u, v]^\wedge [t_0, \ldots, t_n] \simeq \mathcal{S}[t_0, \ldots, t_n]$. This gives (by forgetting the two augmentations $\mathcal{S}[u, v] \rightarrow \mathcal{S}[t]$) $\mathcal{S}[u, v]$ the structure of an $\mathcal{S}[t]$-algebra, from which it follows that $\mathcal{S}[u, v]^\wedge [t_0, \ldots, t_n] \simeq \mathcal{S}[t_0, \ldots, t_n]$ is also an $\mathcal{S}[t]$-algebra. The simplicial structure maps are all induced by the $\mathcal{S}[t]$-algebra structure map $\mathcal{S}[t] \rightarrow \mathcal{S}[u, v]$ together with the $\mathcal{S}[t]$-algebra maps $\text{id} \otimes \epsilon, \epsilon \otimes \text{id} : \mathcal{S}[u, v] \simeq \mathcal{S}[t] \otimes \mathcal{S}[t] \rightarrow \mathcal{S}[t]$, and are therefore $\mathcal{S}[t]$-algebra maps.
Proposition 9.8. The simplicial algebra \( \Delta_S \) is equivalent to the free \( S[t] \)-algebra \( S[t_0, \ldots, t_n] \) on the \( S \)-algebra \( S[t] \). In particular, the canonical equivalences

\[
S[t_0, \ldots, t_n] \wedge_S S[t] \wedge_S S[t] \rightarrow S[t_1, \ldots, t_n]
\]

yield equivalences

\[
S[t_0, \ldots, t_n] \wedge_S S[t] \rightarrow S[t_1, \ldots, t_n].
\]

Proof. The \( S[t] \)-algebra map \( S[t_1, \ldots, t_n] \wedge_S S[t] \rightarrow S[t_0, \ldots, t_n] \), adjoint to the obvious \( S \)-algebra map \( S[t_1, \ldots, t_n] \rightarrow S[t_0, \ldots, t_n] \), sends \( t \) to \( \sum_{i \geq 0} t_i \) and \( t_i \) to \( t_i \) for \( i > 0 \). It has an inverse which sends \( t_0 \) to \( -\sum_{i \geq 0} t_i \) and \( t_i \) to \( t_i \) for \( i > 0 \).

Proposition 9.9. The simplicial \( S \)-algebra \( \Delta_S \) is flat, and moreover

\[
\pi_0(\Delta_S) \cong \mathbb{Z}[t_0, \ldots, t_n] \otimes_{\mathbb{Z}[t]} \mathbb{Z} \cong \mathbb{Z}[t_0, \ldots, t_n]/(1 - \sum_{i=0}^n t_i).
\]

Proof. First, note that the isomorphism \( \pi_* S[t] \cong \pi_0 S[t] \otimes_{\pi_0 S} \pi_* S \) yields isomorphisms

\[
\pi_* S[t_0, \ldots, t_n] \cong \pi_0 S[t_0, \ldots, t_n] \otimes_{\pi_0 S} \pi_* S \cong \pi_0 S[t_0, \ldots, t_n] \otimes_{\pi_0 S[t]} \pi_* S[t].
\]

Now, it suffices to show that the map \( \delta : \mathbb{Z}[t] \rightarrow \mathbb{Z}[u, v] \) which sends \( t \) to \( u + v \) is flat; this implies that the map \( \mathbb{Z}[t] \rightarrow \mathbb{Z}[t_0, \ldots, t_n] \) factors through the flat map

\[
\mathbb{Z}[t_0, \ldots, t_{n-1}] \cong \mathbb{Z}[t_0, \ldots, t_{n-2}] \otimes \mathbb{Z}[t] \xrightarrow{id \otimes \delta} \mathbb{Z}[t_0, \ldots, t_{n-2}] \otimes \mathbb{Z}[u, v] \cong \mathbb{Z}[t_0, \ldots, t_n],
\]

and inductively we may assume that \( \mathbb{Z}[t_0, \ldots, t_{n-1}] \) is flat over \( \mathbb{Z}[t] \). Since the ring map \( \rho : \mathbb{Z}[t] \rightarrow \mathbb{Z}[u, v] \) which sends \( t \) to \( u \) is evidently flat, it is enough to define an algebra automorphism \( \alpha \) of \( \mathbb{Z}[u, v] \) such that \( \delta = \alpha \circ \rho \). Taking \( \alpha(u) := u + v \) and \( \alpha(v) := v \), we obtain an algebra endomorphism of \( \mathbb{Z}[u, v] \) with \( \delta = \alpha \circ \rho \). Finally, since \( \alpha \) has an inverse \( \beta \) given by \( \beta(u) = u - v \) and \( \beta(v) = v \), the proof is finished.

Corollary 9.9. There are natural equivalences Perf((\( \Delta_S \))_n) \cong \mathbb{A}^n.

Given a small stable \( \infty \)-category \( \mathcal{C} \) and an \( S \)-algebra \( R \), we will typically write \( \mathcal{C} \otimes \mathbb{V} R \) for the small stable idempotent-complete \( \infty \)-category \( \mathcal{C} \otimes \mathbb{V} \text{Perf}(R) \). Thus, for a small stable \( \infty \)-category \( \mathcal{C} \), we have a simplicial stable \( \infty \)-category \( \mathcal{C} \otimes \mathbb{V} \Delta_S \) with \( (\mathcal{C} \otimes \mathbb{V} \Delta_S)_n = \mathcal{C} \otimes \mathbb{V} (\Delta_S)_n \cong \mathcal{C} \otimes \mathbb{V} A^n \). The map \( A^0 \rightarrow \Delta_S \) (regarding \( A^0 \) as a constant simplicial stable \( \infty \)-category) then induces a map \( \mathcal{C} \rightarrow \mathcal{C} \otimes \mathbb{V} \Delta_S \), and hence maps \( \mathcal{C} \rightarrow \mathcal{C} \otimes \mathbb{V} A^n \) for each natural number \( n \).
Remark 9.10. By taking $C = \text{Perf}(H\mathbb{Z})$ to be the $\infty$-category of perfect $H\mathbb{Z}$-modules, we find that $C \otimes^\vee \Delta_S$ is the simplicial stable $\infty$-category associated to the simplicial commutative $\mathbb{Z}$-algebra $\Delta_S \simeq \Delta_S \wedge \mathbb{Z}$ which in degree $n \geq 0$ is the ring $\mathbb{Z}[t_0, \ldots, t_n]/(1 - \sum_{i=0}^{n} t_i)$ of functions on the standard (co)simplicial affine scheme $A^n_S$, with simplicial structure given by the formulae

\[
d_r(t_j) = \begin{cases} 
t_j & \text{if } j < r \\
0 & \text{if } j = r \\
t_j - 1 & \text{if } j > r 
\end{cases}
\]

\[
s_r(t_j) = \begin{cases} 
t_j & \text{if } j < r \\
t_j + t_{j+1} & \text{if } j = r \\
t_{j+1} & \text{if } j > r 
\end{cases}
\]

This shows that $\Delta_S$ should be regarded as a version of $\Delta_S$ which is defined over the sphere. In other words, $\Delta_S$ is a (non-commutative) version of a “standard simplicial affine space” over the sphere. Note that we describe $\Delta_S$ simplicially because we are working algebraically; it may be more intuitive to think geometrically, in which case we would obtain a (non-commutative) cosimplicial affine scheme.

9.2. Homotopy invariance. We can now extend the notion of algebraic homotopy invariance to the setting of small stable $\infty$-categories.

Definition 9.11. A functor $E : \text{Cat}^\text{ex}_\infty \to \mathcal{D}$ is homotopy invariant if, for each small stable $\infty$-category $C$, $E$ inverts the map $C \to C \otimes^\vee \mathbb{A}^1$.

Note that if $E : \text{Cat}^\text{ex}_\infty \to \mathcal{D}$ is a homotopy invariant, then $E$ inverts any map of the form $C \otimes^\vee \mathbb{A}^m \to C \otimes^\vee \mathbb{A}^n$. In particular, $E(C \otimes^\vee \Delta_S)$ is a constant (up to equivalence) simplicial object in $\mathcal{D}$.

Definition 9.12. Let $\mathcal{D}$ be a cocomplete stable $\infty$-category. An additive (respectively, localizing) and homotopy invariant functor $\text{Cat}^\text{perf}_\infty \to \mathcal{D}$ will be called an additive homotopy invariant (respectively, a localizing homotopy invariant).

Let $E : \text{Cat}^\text{ex}_\infty \to \mathcal{D}$ be an additive (respectively, localizing) invariant and consider the functor $L^A E : \text{Cat}^\text{ex}_\infty \to \mathcal{D}$ defined by

\[
L^A E(C) = |E(C \otimes^\vee \Delta_S)|,
\]

the geometric realization of the simplicial object which in degree $n$ is $E$ applied to the stable $\infty$-category $C \otimes \mathbb{A}^n$. Note that $L^A$ comes equipped with a natural transformation from the identity $\eta : \text{id} \to L^A$, induced from the map $\mathbb{A}^0 \to \Delta_S$ (regarding $\mathbb{A}^0$ as a constant simplicial object).

Proposition 9.13. $L^A$ preserves additive and localizing invariants.

Proof. We must show that for any additive (respectively, localizing) invariant $E : \text{Cat}^\text{ex}_\infty \to \mathcal{D}$, $L^A E$ is again an additive (respectively, localizing) invariant. The fact that $L^A E$ inverts Morita equivalences and preserves filtered homotopy colimits is clear. Let $A \to B \to C$ be a (split) exact sequence of small stable $\infty$-categories, so that $E(A) \to E(B) \to E(C)$ is a (split) cofiber sequence in $\mathcal{D}$. Then,

\[
A \otimes^\vee \Delta_S \to B \otimes^\vee \Delta_S \to C \otimes^\vee \Delta_S
\]
is a simplicial (split) exact sequence of small stable ∞-categories,
\[ E(A \otimes^\vee \Delta_3) \longrightarrow E(B \otimes^\vee \Delta_3) \longrightarrow E(C \otimes^\vee \Delta_3) \]
is a simplicial (split) cofiber sequence in \( D \), and finally
\[ |E(A \otimes^\vee \Delta_3)| \longrightarrow |E(B \otimes^\vee \Delta_3)| \longrightarrow |E(C \otimes^\vee \Delta_3)| \]
is a (split) cofiber sequence in \( D \). \( \square \)

Next, we show that \( L^A E \) is a homotopy invariant.

**Proposition 9.14.** For any functor \( E : \text{Cat}^e_{\infty} \rightarrow D \), \( L^A E \) is homotopy invariant.

**Proof.** It is enough to show that the composite
\[
(\mathbb{A}^1 \xrightarrow{\epsilon} \mathbb{A}^0 \xrightarrow{\eta} \mathbb{A}^1) \otimes^\vee \Delta_3
\]
is simplicially homotopic to the identity of \( \mathbb{A}^1 \otimes^\vee \Delta_3 \). For each \( 0 \leq i \leq n + 1 \), we have a composite map
\[
h^i_n : \mathbb{A}^{n+1} \longrightarrow \mathbb{A}^i \otimes^\vee \mathbb{A}^{n+1-i} \longrightarrow \mathbb{A}^i \otimes^\vee \mathbb{A}^0 \longrightarrow \mathbb{A}^i \otimes^\vee \mathbb{A}^{n+1-i} \longrightarrow \mathbb{A}^{n+1}
\]
in which the first and last maps are equivalences, and the middle maps are the identity on the first factor and the map \( \mathbb{A}^{n+1-i} \rightarrow \mathbb{A}^0 \rightarrow \mathbb{A}^{n+1-i} \) induced by (tensor powers of) \( \epsilon : \mathbb{A}^1 \rightarrow \mathbb{A}^0 \) on the second factor. These maps specify a simplicial homotopy \( h : \mathbb{A}^1 \otimes^\vee \Delta_3 \otimes^\vee \Delta^1 \rightarrow \mathbb{A}^1 \otimes^\vee \Delta_3 \) between (9.15) and the identity. \( \square \)

Lastly, we observe that the functor \( L^A \) is a localization.

**Proposition 9.16.** Let \( D \) be a presentable stable ∞-category. Then \( L^A \) defines accessible localizations of the presentable stable ∞-categories \( \text{Fun}_{\text{add}}(\text{Cat}^e_{\infty}, D) \) and \( \text{Fun}_{\text{loc}}(\text{Cat}^e_{\infty}, D) \). That is, \( L^A \) preserves filtered colimits and the composite
\[
\eta \circ L^A : L^A \longrightarrow L^A \circ L^A
\]
is an equivalence.

**Proof.** The fact that \( L^A \) preserves filtered colimits is an immediate consequence of the fact that geometric realization preserve filtered colimits. For the second statement, let \( E : \text{Cat}^e_{\infty} \rightarrow D \) be an additive or localizing invariant. Then \( L^A E \) is homotopy invariant, so for any \( C \) in \( (\text{Cat}^e_{\text{perf}})^e \) the simplicial object \( L^A E(C \otimes^\vee \Delta_3) \) is constant (up to homotopy), and consequently \( L^A E(C) \simeq |L^A E(C \otimes^\vee \Delta_3)| \simeq L^A(L^A E)(C) \). \( \square \)

This leads us to give the following definition.

**Definition 9.17.** The Karoubi-Villamayor \( K \)-theory \( KV \) is the homotopy invariant version \( L^A K \) of connective \( K \)-theory; similarly, the homotopy \( K \)-theory \( KH \) is the homotopy invariant version \( L^A KH \) of non-connective \( K \)-theory. In other words, for any small stable ∞-category \( \mathcal{C} \), we have
\[
KV(C) = |K(C \otimes^\vee \Delta_3)| \quad \text{and} \quad KH(C) = |KH(C \otimes^\vee \Delta_3)|.
\]

The following discussion justifies the use of these names; we show that Karoubi-Villamayor and homotopy \( K \)-theories as we have defined them and the classical definitions for schemes agree.
modules, promote it to a spectral category, and pass to the associated
∞
Example 9.18. Let \((X, \mathcal{O}_X)\) be a scheme (or stack) defined over \(\mathbb{Z}\) and consider the stable \(\infty\)-category \(\text{Perf}(H\mathcal{O}_X)\) of perfect \(H\mathcal{O}_X\)-modules (the compact objects in the stable \(\infty\)-category of quasi-coherent \(H\mathcal{O}_X\)-modules). Recall here that \(H\mathcal{O}_X\) denotes the derived structure sheaf obtained from the sheaf \(\mathcal{O}_X\) by applying the Eilenberg-Mac Lane functor pointwise and sheafifying. Equivalently, we could take the \(DG\)-category of compact objects in the \(DG\)-category of quasi-coherent \(\mathcal{O}_X\)-modules, promote it to a spectral category, and pass to the associated \(\infty\)-category.

We wish to show that Karoubi-Villamayor and homotopy \(K\)-theory of the stable \(\infty\)-category \(\text{Perf}(H\mathcal{O}_X)\) agrees with Karoubi-Villamayor and homotopy \(K\)-theory of the scheme (or stack) \(X\). For this, it suffices to check that \(\text{Perf}(H\mathcal{O}_X) \otimes^\vee \Delta_S \simeq \text{Perf}(H(\mathcal{O}_X \otimes \Delta_S))\). Here \(\Delta_S\) stands for the simplicial commutative ring of functions on the standard cosimplicial affine scheme. Since \(\Delta_S\) is flat over \(\mathbb{Z}\), we observe that \(H(\mathcal{O}_X \otimes \Delta_S) \simeq H\mathcal{O}_X \wedge_{\mathcal{H}_2} H\Delta_S\). Finally, this implies the following natural equivalences:

\[
\text{Perf}(H(\mathcal{O}_X \otimes \Delta_S)) \simeq \text{Perf}(H\mathcal{O}_X \otimes^\vee \text{Perf}(H\mathcal{Z})) \text{Perf}(H\Delta_S) \simeq \text{Perf}(H\mathcal{O}_X) \otimes^\vee \Delta_S,
\]

since \(\text{Perf}(\Delta_S) \simeq \text{Perf}(H\mathcal{Z}) \otimes^\vee \Delta_S\).

Further, note that just as the connective and non-connective \(K\)-theory of \(S\) agree by theorem 8.7, an elaboration of this argument shows that \(KV\) and \(KH\) of \(S\) coincide.

Although Goodwillie showed in [24] that over \(\mathbb{Q}\) periodic cyclic homology is \(\mathbb{A}^1\)-homotopy invariant, the homotopy invariant versions of \(THH\) and \(TC\) are trivial; it is an interesting open problem to determine the correct analogues of \(TC\) and \(THH\) in this setting.

**Proposition 9.19.** For any small stable \(\infty\)-category \(\mathcal{C}\), the spectra \(|THH(\mathcal{C} \otimes^\vee \Delta_S)|\) and (at any prime \(p\)) \(|TC(\mathcal{C} \otimes^\vee \Delta_S)|\) are trivial.

**Proof.** Unlike \(K\)-theory, \(THH\) commutes with geometric realization; this is clear from consideration of the description of \(THH\) in terms of the cyclic bar construction. Now, the geometric realization of the simplicial \(\infty\)-category \(\mathcal{C} \otimes^\vee \Delta_S\) is contractible: just as in the classical situation, the two degeneracies on the one simplices of \(\mathcal{C} \otimes^\vee \Delta_S\) provide a path between the zero object and the unit. This implies that \(|THH(\mathcal{C} \otimes^\vee \Delta_S)|\) is contractible. The result for \(TC\) now follows by induction over the fundamental cofibration sequence (e.g., [25, 2.1.4])

\[
THH(\mathcal{C})_{C^p_{n-1}} \longrightarrow THH(\mathcal{C})_{C^p_{n-2}} \longrightarrow \cdots,
\]

(where the left-hand term denotes the homotopy orbit space) since the homotopy orbit space also commutes with geometric realization. \(\square\)

9.3. The universal additive/localizing homotopy invariant. In this section we use the definition of the previous subsection to produce universal homotopy invariant functors via our localization machinery. Let \(\mathcal{E}_{\text{add}}\) and \(\mathcal{E}_{\text{loc}}\) be the sets

\[
\{\mathcal{U}_{\text{add}}(\mathcal{C} \longrightarrow \mathcal{C} \otimes^\vee \mathbb{A}^1)\} \quad \{\mathcal{U}_{\text{loc}}(\mathcal{C} \longrightarrow \mathcal{C} \otimes^\vee \mathbb{A}^1)\}
\]

of maps in \(\mathcal{M}_{\text{add}}\) (respectively, in \(\mathcal{M}_{\text{loc}}\)), where \(\mathcal{C}\) ranges over the (equivalence classes of) compact objects of \(\text{Cat}_{\text{perf}}^\infty\). We denote by \(\mathcal{M}_{\text{add}}^\wedge\) (respectively, \(\mathcal{M}_{\text{loc}}^\wedge\)) the localization of \(\mathcal{M}_{\text{add}}\) (respectively, \(\mathcal{M}_{\text{loc}}\)) with respect to the set \(\mathcal{E}_{\text{add}}\) (respectively, \(\mathcal{E}_{\text{loc}}\)), and we write \(L^\wedge_{\text{add}} : \mathcal{M}_{\text{add}} \rightarrow \mathcal{M}_{\text{add}}^\wedge\) (respectively, \(L^\wedge_{\text{loc}} : \mathcal{M}_{\text{loc}} \rightarrow \mathcal{M}_{\text{loc}}^\wedge\)) for the localization functors. The \(\infty\)-category of additive homotopy invariants (with values}
in the cocomplete stable \( \infty \)-category \( \mathcal{D} \) will be denoted \( \text{Fun}^{h}_{\text{add}}(\text{Cat}_{\infty}^{\text{ex}}, \mathcal{D}) \), and the \( \infty \)-category of localizing homotopy invariants will be denoted \( \text{Fun}^{h}_{\text{loc}}(\text{Cat}_{\infty}^{\text{ex}}, \mathcal{D}) \).

**Proposition 9.20.** Let \( \mathcal{D} \) be a cocomplete stable \( \infty \)-category. Then the \( \infty \)-category \( \text{Fun}^{L}(\mathcal{M}_{\text{add}}^{h}, \mathcal{D}) \) fits into a pullback square

\[
\begin{array}{ccc}
\text{Fun}^{L}(\mathcal{M}_{\text{add}}^{h}, \mathcal{D}) & \longrightarrow & \text{Fun}^{h}_{\text{add}}(\text{Cat}_{\infty}^{\text{perf}}, \mathcal{D}) \\
L_{\text{add}}^{h} & \downarrow & \downarrow i \\
\text{Fun}^{L}(\mathcal{M}_{\text{add}}^{h}, \mathcal{D}) & \sim & \text{Fun}_{\text{add}}(\text{Cat}_{\infty}^{\text{perf}}, \mathcal{D})
\end{array}
\]

and the \( \infty \)-category \( \text{Fun}^{L}(\mathcal{M}_{\text{loc}}^{h}, \mathcal{D}) \) fits into a pullback square

\[
\begin{array}{ccc}
\text{Fun}^{L}(\mathcal{M}_{\text{loc}}^{h}, \mathcal{D}) & \longrightarrow & \text{Fun}^{h}_{\text{loc}}(\text{Cat}_{\infty}^{\text{perf}}, \mathcal{D}) \\
L_{\text{loc}}^{h} & \downarrow & \downarrow i \\
\text{Fun}^{L}(\mathcal{M}_{\text{loc}}^{h}, \mathcal{D}) & \sim & \text{Fun}_{\text{add}}(\text{Cat}_{\infty}^{\text{perf}}, \mathcal{D})
\end{array}
\]

Here \( i \) denotes the right adjoint of the localization functor \( L^{h} \).

**Proof.** The proofs are similar, so we will treat only the additive case. We begin with the observation that, since \( L_{\text{add}}^{h} \) is a localization, the left-hand vertical map is fully faithful with essential image those colimit-preserving functors \( F: \mathcal{M}_{\text{add}} \rightarrow \mathcal{D} \) which are homotopy invariant in the sense that, for any compact idempotent-complete small stable \( \infty \)-category \( \mathcal{C} \), the map \( F(U^{\text{add}}_{\text{add}}(\mathcal{C})) \rightarrow F(U^{\text{add}}_{\text{add}}(\mathcal{C} \otimes \wedge_{A}^{1})) \) is an equivalence. Since the right-hand vertical map is by definition fully faithful and the bottom map is an equivalence, the homotopy invariance condition implies directly that the composite

\[
\text{Fun}^{L}(\mathcal{M}_{\text{add}}^{h}, \mathcal{D}) \rightarrow \text{Fun}^{L}(\mathcal{M}_{\text{add}}^{h}, \mathcal{D}) \simeq \text{Fun}_{\text{add}}(\text{Cat}_{\infty}^{\text{perf}}, \mathcal{D})
\]

factors through the full subcategory \( \text{Fun}^{h}_{\text{add}}(\text{Cat}_{\infty}^{\text{perf}}, \mathcal{D}) \). Moreover, we also conclude that the top map is fully faithful.

It remains to show that the top map is essentially surjective. Suppose we are given an additive homotopy invariant \( E: \text{Cat}_{\infty}^{\text{perf}} \rightarrow \mathcal{D} \). Then \( E \simeq F \circ U_{\text{add}} \) for some colimit-preserving functor \( F: \mathcal{M}_{\text{add}} \rightarrow \mathcal{D} \), and the homotopy invariance of \( E \) implies by the discussion above that \( F \) factors as a composite \( \mathcal{M}_{\text{add}} \rightarrow \mathcal{M}_{\text{add}}^{h} \rightarrow \mathcal{D} \).

We denote by \( U^{h}_{\text{add}} \) and \( U^{h}_{\text{loc}} \) the composites

\[
\text{Cat}_{\infty}^{\text{ex}} \xrightarrow{U_{\text{add}}} \mathcal{M}_{\text{add}} \xrightarrow{L_{\text{add}}^{h}} \mathcal{M}_{\text{add}}^{h} \quad \text{and} \quad \text{Cat}_{\infty}^{\text{ex}} \xrightarrow{U_{\text{loc}}} \mathcal{M}_{\text{loc}} \xrightarrow{L_{\text{loc}}^{h}} \mathcal{M}_{\text{loc}}^{h},
\]

respectively.

Now proposition 9.20 implies the following universal characterization:

**Theorem 9.21.** The morphism \( U^{h}_{\text{add}} \) is the universal additive homotopy invariant; that is, given a cocomplete stable \( \infty \)-category \( \mathcal{D} \), we have an equivalence of \( \infty \)-categories

\[
(U^{h}_{\text{add}})^{*}: \text{Fun}^{L}(\mathcal{M}_{\text{add}}^{h}, \mathcal{D}) \xrightarrow{\sim} \text{Fun}^{h}_{\text{add}}(\text{Cat}_{\infty}^{\text{ex}}, \mathcal{D}).
\]
Similarly, the morphism $U^k_{\text{loc}}$ is the universal localizing homotopy invariant; that is, given a cocomplete stable $\infty$-category $D$, we have an equivalence of $\infty$-categories

$$(U^k_{\text{loc}})^*: \text{Fun}^1(M^k_{\text{loc}}, D) \xrightarrow{\sim} \text{Fun}^k_{\text{loc}}(\text{Cat}^\omega_\infty, D).$$

Furthermore, we show in proposition 11.15 that these universal homotopy invariant functors are in fact lax symmetric monoidal.

9.4. Co-representability. In this section, we establish the co-representability results providing universal characterizations of Karoubi-Villamayor and homotopy $K$-theory. To do this, we need to prove the following technical fact about the unit object in $M_{\text{add}}$ and $M_{\text{loc}}$.

**Proposition 9.22.** $U_{\text{add}}(S^\omega_\infty)$ is a compact object of $M_{\text{add}}$, and $U_{\text{loc}}(S^\omega_\infty)$ is a compact object of $M_{\text{loc}}$.

**Proof.** First, observe that the image of $S^\omega_\infty$ under the Yoneda embedding is compact in the presheaf category $\text{Pre}((\text{Cat}^{\text{perf}}_\omega)^\omega, S_\infty)$. Since $M_{\text{add}}$ is formed by localizing this $\infty$-category at a set of maps with source and target compact, it is straightforward to check that the subcategory of local objects in $\text{Pre}((\text{Cat}^{\text{perf}}_\omega)^\omega, S_\infty)$ is stable under filtered colimits. By [33, 5.5.7.3], we conclude that $M_{\text{add}}$ is compactly generated and that $U_{\text{add}}(S^\omega_\infty)$ is a compact object.

The situation with $M_{\text{loc}}$ is slightly more complicated. We use the following observation: Suppose we are given a stable $\infty$-category $C$ and a set of maps $S$ to localize at. If $X$ is compact in $C$ and the functor $\text{Map}_C(X, -)$ (to spectra) takes all maps in $S$ to weak equivalences, then the image of $X$ is compact in the localized category $C_S$. We can see this via an easy computation. Denote the localization from $C$ to $C_S$ by $L$ and $L^*$ the inclusion back into $C$. We can describe $C_S$ as the localization of $C$ by the smallest strongly saturated collection of maps containing $S$, which we write $\tilde{S}$ [33, 5.5.4.5]. Since $X$ is compact, $\text{Map}_C(X, -)$ in fact takes all maps in $\tilde{S}$ to equivalences. Now, since the unit map $Y \to L^*LY$ is a map in $\tilde{S}$, we have the following equivalences:

$$\text{Map}_{C_S}(LX, LY) \cong \text{Map}_C(X, L^*LY) \cong \text{Map}_C(X, Y).$$

The claim now follows, using the preceding comparison and the observation that $LL^*Y \simeq Y$ for $Y \in C_S$ to obtain the relations

$$\text{Map}_{C_S}(LX, \colimi Y_i) \cong \text{Map}_{C_S}(LX, \colimi L^*Y_i) \cong \text{Map}_{C_S}(LX, L(\colimi L^*Y_i)) \cong \text{Map}_C(X, \colimi L^*Y_i) \cong \colimi \text{Map}_C(X, L^*Y_i) \cong \colimi \text{Map}_{C_S}(LX, Y_i) \cong \colimi \text{Map}_{C_S}(LX, Y_i).$$

Finally, tracing through the localizations of subsection 9.4, recall that we have already verified this condition in the proofs of theorem 8.9 and 8.11 for each of the relevant functors and so we see that $U_{\text{loc}}(S^\omega_\infty)$ is a compact object of $M_{\text{loc}}$. $\Box$

Using proposition 9.22, it is now straightforward to extend theorems 5.9 and 8.9 to the homotopy invariant setting.
Theorem 9.23. Let \( A \) be a small stable \( \infty \)-category. Then there are natural equivalences of spectra

\[
\text{Map}(U^\Delta_{\text{add}}(S^\infty_\omega), U^\Delta_{\text{loc}}(A)) \simeq K V(A) \quad \text{Map}(U^\Delta_{\text{loc}}(S^\infty_\omega), U^\Delta_{\text{loc}}(A)) \simeq K H(A),
\]

where \( S^\infty_\omega \) is the small stable \( \infty \)-category of compact spectra.

Proof. The proof is analogous in both situations and so we will treat only the localizing case. Consider the following equivalences

\[
\begin{align*}
(9.24) & \quad \text{Map}(U^\Delta_{\text{loc}}(S^\infty_\omega), U^\Delta_{\text{loc}}(A)) \simeq \text{Map}(U^\Delta_{\text{loc}}(S^\infty_\omega), U^\Delta_{\text{loc}}(A)) \\
& \quad \simeq \text{Map}(U^\Delta_{\text{loc}}(S^\infty_\omega), |U^\Delta_{\text{loc}}(A \otimes^\forall \Delta_Z)|) \\
(9.25) & \quad \simeq |\text{Map}(U^\Delta_{\text{loc}}(S^\infty_\omega), U^\Delta_{\text{loc}}(A \otimes^\forall \Delta_Z))| \\
(9.26) & \quad \simeq |K(A \otimes^\forall \Delta_Z)|.
\end{align*}
\]

Equivalence (9.24) follows from the fact that \( L^\Delta_{\text{loc}} \) is a localization, equivalence (9.25) follows from the compactness of \( U^\Delta_{\text{loc}}(S^\infty_\omega) \) in \( \mathcal{M}_{\text{loc}} \) by proposition 9.22, and equivalence (9.26) follows from definition 9.17.

\[\square\]

9.5. The \( E_\infty \) affine space. We close this section with a few remarks about the \( E_\infty \) analogue of \( \Delta_Z \). Although the \( A_\infty \) ring spectra \( S[\mathbb{N}] \) are canonically \( E_\infty \) ring spectra, the simplicial structure maps seem to only be \( A_\infty \) maps. This is why we regard the affine spaces \( \mathbb{A}^n \) as noncommutative versions, defined over \( S \), of the usual affine spaces \( \mathbb{A}^n_\mathbb{Z} \), defined over \( \mathbb{Z} \). There are, however, commutative analogues of \( \mathbb{A}^n \), obtained from the free \( E_\infty \) ring spectra \( S[B\Sigma^n] \), where \( B\Sigma = \coprod_{n \in \mathbb{N}} B\Sigma_n \) is the free \( E_\infty \) space on one generator.

However, \( S[B\Sigma^n] \) is no longer flat over \( S \). Writing \( \Delta'_S \) for the resulting simplicial stable \( \infty \)-category (that is, \((\Delta'_S)_n \simeq \text{Perf}(S[B\Sigma^n])\)), we have a map

\[
\text{Perf}(H\mathcal{O}_X) \otimes^\forall \Delta'_S \rightarrow \text{Perf}(H\mathcal{O}_X) \otimes^\forall \text{Perf}(H\Delta_Z) \simeq \text{Perf}(H(\mathcal{O}_X \otimes \Delta_Z))
\]

induced from the canonical map \( \Delta'_S \rightarrow \text{Perf}(\Delta_Z) \) of simplicial stable \( \infty \)-categories, which is an equivalence in general only if \( \mathcal{O}_X \) is rational (since \( \Delta'_S \rightarrow \text{Perf}(\Delta_Z) \) only becomes an equivalence after rationalization). In particular, in this setting the homotopy invariant versions of connective and non-connective \( K \)-theory agree with Karoubi-Villamayor and homotopy \( K \)-theory only if \( \mathcal{O}_X \) is rational.

10. Cyclotomic trace map

In this section we apply the work of the preceding sections to give a universal characterization of the cyclotomic trace map [9]

\[
K(-) \rightarrow TC(-) \rightarrow THH(-).
\]

More generally, we identify all natural transformations of additive functors from \( K \)-theory to \( THH \): they are suspensions of the cyclotomic trace map. This identification provides a very satisfying conceptual construction of the cyclotomic trace map, and of course makes it clear that all known definitions are consistent.

We begin by observing that \( THH \) provides an additive (in fact, localizing) invariants of small stable \( \infty \)-categories. Although it is possible to do this directly in the setting of \( \infty \)-categories (i.e., for \( THH \) see the more general discussion of topological chiral homology in [37, 3.5.7] or the constructions outlined in [2, 5.1.1]), in order to skirt the complexities of the equivariant structure of \( TC \) we instead use the
expedient definitions offered by theorem 1.9. Then the following proposition follows from the analogous properties of $THH$ in the setting of small spectral categories [57, 4].

**Proposition 10.1.** $THH$ is a localizing invariant of small stable $\infty$-categories.

Next, we briefly review the construction of the cyclotomic trace map as a natural transformation of additive invariants from $K$-theory to $THH$. The work of section 3 implies that we can model the trace on small stable $\infty$-categories by considering the trace for the spectral category $\hat{A}_{perf}$ of perfect modules over a small spectral category $A$. This category has a Waldhausen structure inherited from the spectral model structure on $\hat{A}$; as a consequence, the weak equivalences of the Waldhausen structure (used in the construction of $K$-theory) are compatible with the spectral enrichment (used in the construction of $THH$). This allows us to construct a trace using the perspective of [39] by “mixing” a cyclic bar construction and Waldhausen’s $S_\bullet$ construction.

In the case of a Waldhausen category with a compatible DG-enrichment, Appendix B of [4] constructs a cyclotomic trace in this fashion. The adaptation of this definition to the spectral situation is relatively straightforward and is written out in detail in [5] (which handles the more general setting of construction a trace for a version of $THH$ defined using only the input data of a Waldhausen category). We refer the interested reader to these sources for the details of the construction.

The force of the co-representability result for algebraic $K$-theory (theorem 6.9) is that it easily implies, using the enriched Yoneda lemma, the following identification of the spectrum of natural transformations of additive functors:

**Theorem 10.2.** Given an additive invariant

\[ E : \text{Cat}^{ex}_{\infty} \rightarrow \mathcal{S}_{\infty} \]

with values in the stable $\infty$-category of spectra, we have a natural equivalence of spectra

\[ \text{Nat}(K, E) \simeq E(\mathcal{S}_\omega), \]

where $\text{Nat}(K, E)$ denotes the spectrum of natural transformations from $K$ to $E$ as additive invariants from small stable $\infty$-categories to spectra; i.e., the natural transformations which preserve filtered colimits.

In particular, for $THH$, theorem 10.2 implies that we have an equivalence of spectra

\[ \text{Nat}(K(-), THH(-)) \cong THH(\mathcal{S}_\omega) \cong THH(\mathcal{S}) \simeq \mathcal{S}. \]

Passing to $\pi_0$ on both sides we obtain an isomorphism between homotopy classes of natural transformations and $\mathbb{Z}$.

Now, given a point $\phi$ in $\text{Nat}(K(-), THH(-))$ (a specific natural transformation, that is), we can describe the corresponding element in $\pi_0\mathcal{S}$ as the homotopy class represented by the composite

\[ \mathcal{S} \xrightarrow{\phi} K(\mathcal{S}) \xrightarrow{\text{unit}} THH(\mathcal{S}) \simeq \mathcal{S}, \]

where the first map is the unit. But when $\phi$ is the cyclotomic trace, it’s a classical result of Waldhausen [64, 5.2] that the cyclotomic trace splits the unit; this composite is (homotopic to) the identity map. That is, the usual cyclotomic trace
is (up to homotopy) the natural transformation given by the identity element in $\pi_0(S) = \mathbb{Z}$.

When attempting to apply our co-representability results to $TC$, we must study a somewhat more refined invariant. Although $TC$ is Morita invariant and satisfies localization [4], it does not preserve filtered colimits and is therefore not a localizing (or additive) invariant. Instead (as in e.g., [21, 22]), we study the functor $K \mapsto \{TC^n\}$, where the latter is a certain pro-spectrum (whose limit is the usual $TC$ construction).

We begin by recalling the definition of $TC^n$. Our review is brief; we refer the interested reader to [26] for an authoritative treatment. Fix a prime $p$. For a spectral category $C$, $THH(C)$ is a cyclotomic $S^1$-spectrum. As such, we can consider the associated non-equivariant spectra

$$TR^n(C) = (THH(C)^{C_{p^n-1}}),$$

the fixed points with respect to the induced $C_{p^n-1}$ action. The inclusion of fixed points and the cyclotomic structure give rise to maps $F$ and $R$ respectively

$$F, R: TR^n \longrightarrow TR^{n-1}.$$

Define $TC^n(C)$ to be the homotopy equalizer

$$\text{holim}_{F,R} TR^n(C) \longrightarrow TR^{n-1}(C).$$

Clearly, we have

$$TC(C) = \text{holim}_n TC^n(C),$$

where we form the homotopy limit over the maps induced by the restriction. But if we regard $\{TC^n(C)\}$ as a pro-spectrum, we then obtain a functor from spectral categories to pro-spectra which preserves filtered colimits. The category of pro-spectra is itself a stable category. However, the $\infty$-category of pro-spectra is not presentable; pro-categories are very rarely accessible. Nonetheless, the $\infty$-category of pro-spectra is cocomplete, and this is sufficient for the proofs of our main theorems to go through. Thus, the next proposition will allow us to apply our main results to this functor.

**Proposition 10.3.** The functor $TC^n$ is a localizing invariant of stable $\infty$-categories with values in the stable $\infty$-category of pro-spectra.

**Proof.** First, we verify that $TC^n(C)$ preserves filtered colimits. Since filtered colimits and finite homotopy limits commute, it suffices to show that $TR^n(C)$ preserve filtered colimits. This now follows inductively from consideration of the fundamental cofibration sequence (e.g., [25, 2.1.4])

$$(THH(C)^{C_{p^{n-1}}}) \longrightarrow TR^n(C) \longrightarrow TR^{n-1}(C),$$

(where the left-hand term denotes the homotopy orbit space) and the fact that homotopy orbits commute with filtered colimits. Next, it follows from the results of [4] that $\{TC^n(C)\}$ is Morita invariant and satisfies localization. □

By letting $K$-theory take $C$ to the constant pro-spectrum $\{K(C)\}$, we can again apply theorem 6.9 to produce a model of the refined cyclotomic trace. Specifically, the set of homotopy classes of localizing natural transformations from $\{K(-)\}$ to $\{TC^n(-)\}$ is given by $\text{holim}_p TC^n(S) = TC(S)$. Completing at any prime $p$, recall that $TC(S) \simeq S \vee \Sigma CP_{p-1}^{\infty}$ [50]. Furthermore, since $\pi_0(\Sigma CP_{p-1}^{\infty}) = 0$ and we know
that the unit $S \to TC(*)$ induces the splitting (from the trace $TC(*) \to S$), once again we see that the cyclotomic trace is represented by the unit.

11. Multiplicative structure

Our goal in this section is to prove that the universal invariants $U_{\text{add}}$ and $U_{\text{loc}}$ (and its homotopy invariant versions) are lax symmetric monoidal, regarded as functors of small stable (idempotent-complete, in the additive case) $\infty$-categories. As a consequence, for all $1 \leq n \leq \infty$, algebraic $K$-theory (connective, non-connective and homotopy invariant) of an $E_n$ monoidal stable $\infty$-category is canonically an $E_n$ ring spectrum (where here $E_n$ is meant in the sense of [37]). In particular, we recover the classical results that the $K$-theory of a symmetric monoidal stable $\infty$-category is an $E_\infty$ ring spectrum and the $K$-theory of a monoidal stable $\infty$-category is an $A_\infty$ ring spectrum. As further applications, we use the symmetric monoidal structures to extend the co-representability results and study the localizing subcategory generated by the unit object in $M_{\text{add}}$, $M_{\text{loc}}$, $M_{\text{add}}^b$, and $M_{\text{loc}}^b$.

Our results in this section are analogous to the multiplicative structure theory of Elmendorf and Mandell [18] (see [8] for a discussion of this in the context of Waldhausen categories). We do not provide a comparison herein between these theories and our multiplicative structures; verification of consistency and further exploration will appear in a forthcoming paper (with Barwick).

Recall that the $\infty$-category $\hat{\text{Cat}}_{\infty,\sigma}$ of stable presentable $\infty$-categories admits a symmetric monoidal structure with unit the $\infty$-category $S_\infty$ of spectra, and the category $\text{Cat}_{\infty,\sigma}$ inherits a symmetric monoidal structure from this. Now, $U_{\text{add}}$ is the composite of the functor

$$\phi: \text{Cat}_{\infty,\sigma} \longrightarrow \text{Pre}((\text{Cat}_{\infty,\sigma})^\omega; N(T_\sigma^\omega))$$

(given by the Yoneda embedding and the inclusion $(\text{Cat}_{\infty,\sigma})^\omega \to \text{Cat}_{\infty,\sigma}$), localization at the set $E$ of split-exact sequences of corollary 4.16, and stabilization.

We begin with a few technical results concerning the behavior of compact small stable idempotent-complete $\infty$-categories with respect to the closed symmetric monoidal structure on $\text{Cat}_{\infty,\sigma}$.

**Proposition 11.1.** Let $A$ be a compact object of $\text{Cat}_{\infty,\sigma}$ and $\{B_i\}_{i \in I}$ a filtered diagram in $\text{Cat}_{\infty,\sigma}$. Then the map

$$\colim_{i \in I} \text{Fun}^\text{ex}(A, B_i) \longrightarrow \text{Fun}^\text{ex}(A, \colim_{i \in I} B_i)$$

is an equivalence of small stable idempotent-complete $\infty$-categories.

**Proof.** The inclusion $\text{Fun}^\text{ex}(A, B) \to \text{Fun}^\text{ex}(A \otimes^\text{v} B_{\text{op}}, S_\infty)$ is the full subcategory on those $f: A \otimes^\text{v} B_{\text{op}} \to S_\infty$ which restrict to representable functors for each $a$ in $A$. This gives a commuting square

$$\begin{array}{ccc}
\colim \text{Fun}^\text{ex}(A, B_i) & \longrightarrow & \colim \text{Fun}^\text{ex}(A \otimes^\text{v} B_{\text{op}}^i, S_\infty) \\
\downarrow & & \downarrow \\
\text{Fun}^\text{ex}(A, \colim B_i) & \longrightarrow & \text{Fun}^\text{ex}(A \otimes^\text{v} \colim B_{\text{op}}^i, S_\infty)
\end{array}$$

where we use the fact that $(\colim B_i)_{\text{op}} \simeq \colim B_{\text{op}}^i$. The horizontal maps are fully faithful inclusions and the vertical map on the right is an equivalence since $\text{Fun}^\text{ex}(A \otimes^\text{v} (-), S_\infty)$ preserves filtered colimits. It follows that the vertical map...
Proposition 11.2. The ∞-category of compact objects \((\mathbf{Cat}_\infty^{perf})^\omega\) in \(\mathbf{Cat}_\infty^{perf}\) is a symmetric monoidal subcategory.

Proof. Let \(\mathcal{A}\) and \(\mathcal{B}\) be compact small stable idempotent-complete ∞-categories. We must show that \(\mathcal{A} \otimes^\mathbb{V} \mathcal{B}\) is compact. To this end, let \(\mathcal{C}\) be a filtered system of small stable idempotent-complete ∞-categories. Then by the previous proposition 11.1,

\[
\text{map}(\mathcal{A} \otimes^\mathbb{V} \mathcal{B}, \colim_{i \in I} C_i) \simeq \text{map}(\mathcal{A}, \text{Fun}^\omega(\mathcal{B}, \colim_{i \in I} C_i)) \\
\simeq \text{map}(\mathcal{A}, \colim_{i \in I} \text{Fun}^\omega(\mathcal{B}, C_i)) \\
\simeq \colim_{i \in I} \text{map}(\mathcal{A}, \text{Fun}^\omega(\mathcal{B}, C_i)) \simeq \text{map}(\mathcal{A} \otimes^\mathbb{V} \mathcal{B}, C_i),
\]

so that \(\text{map}(\mathcal{A} \otimes^\mathbb{V} \mathcal{B}, -)\) commutes with filtered colimits.

Proposition 11.3. Let \(\mathcal{C}\) be a symmetric monoidal ∞-category. Then \(\text{Ind}(\mathcal{C})\) is a symmetric monoidal subcategory of \(\text{Pre}(\mathcal{C})\), where the latter is symmetric monoidal with respect to the convolution product.

Proof. Let \(X\) and \(Y\) be presheaves on \(\mathcal{C}\) and suppose that \(X \simeq \colim_{i \in I} X_i\) and \(Y \simeq \colim_{j \in J} Y_j\) are filtered colimits of representable presheaves \(X_i \simeq \text{map}(-, A_i)\) and \(Y_j \simeq \text{map}(-, B_j)\). Because the tensor on presheaves commutes with colimits, we have \(X \otimes Y \simeq \colim_{(i,j) \in I \times J} X_i \otimes Y_j\) and \(X_i \otimes Y_j \simeq \text{map}(-, A_i \otimes B_j)\) since the Yoneda embedding \(\mathcal{C} \to \text{Pre}(\mathcal{C})\) is symmetric monoidal. Now \(I \times J\) is a filtered ∞-category, so we see that \(X \otimes Y\) is a filtered colimit of representable presheaves.

Proposition 11.4. The stabilization functor

\[
(-) \otimes S_\infty : \widehat{\mathbf{Cat}_\infty^{LPr}} \longrightarrow \widehat{\mathbf{Cat}_\infty^{LPr,\sigma}}
\]

is symmetric monoidal. In particular, if \(\mathcal{R}\) is a commutative algebra object in \(\mathbf{LPr}_\infty\), then \(\text{Stab}(\mathcal{R}) \simeq \mathcal{R} \otimes S_\infty\) is a commutative algebra object in \(\mathbf{Cat}_\infty^{LPr,\sigma}\).

Proof. Since \(S_\infty\) is a commutative algebra object in \(\mathbf{Cat}_\infty^{LPr}\), it follows that stabilization is lax symmetric monoidal. It is symmetric monoidal because the multiplication map \(S_\infty \otimes S_\infty \simeq S_\infty\) is an equivalence.

Definition 11.5. Let \(\mathcal{R}\) be a commutative algebra object in the ∞-category of stable presentable ∞-categories. An ideal of \(\mathcal{R}\) is an \(\mathcal{R}\)-module \(I\) equipped with a fully-faithful \(\mathcal{R}\)-module map \(I \to \mathcal{R}\).

If \(\mathcal{R}\) is compactly generated, then an ideal \(I\) of \(\mathcal{R}\) is compactly generated if \(I \simeq \text{Ind}(I^\omega) \to \text{Ind}(\mathcal{R}^\omega) \simeq \mathcal{R}\) for some full subcategory \(I^\omega\) of \(\mathcal{R}^\omega\). In this case, the quotient \(\mathcal{R}/I\) exists, and \(\mathcal{R}/I\) is the localization of \(\mathcal{R}\) with respect to those maps \(f\) whose cofibers lie in \(I\). We say that an object \(Z\) is \(I\)-local if it is equivalent to an object in the full subcategory \(\mathcal{R}/I\) of \(\mathcal{R}\); in other words, \(Z\) is \(I\)-local if \(\text{map}(I, Z) \simeq *\) for all objects \(I\) of \(\mathcal{I}\).

Theorem 11.6. Let \(\mathcal{R}\) be a commutative algebra object in the ∞-category of stable presentable ∞-categories and let \(I\) be a compactly generated ideal of \(\mathcal{R}\). Write \(\mathcal{R}/I\) for the full subcategory of \(\mathcal{R}\) on the \(I\)-local objects and \(L : \mathcal{R} \to \mathcal{R}/I\) for the localization functor. Then \(\mathcal{R}/I\) is a naturally a commutative \(\mathcal{R}\)-algebra.
Proof. If $Z$ is $I$-local, then the function object $Z^Y$ is also $I$-local; indeed, given any object $I$ of $I$, we have that $\text{map}(I, Z^Y) \simeq \text{map}(I \otimes Y, Z) \simeq *$ since $I$ is an ideal. Hence

$$\text{map}(LX \otimes Y, Z) \simeq \text{map}(LX, Z^Y) \simeq \text{map}(X, Z^Y) \simeq \text{map}(X \otimes Y, Z),$$

and we see that $X \otimes Z \to LX \otimes Y$ is a local equivalence. It follows that $L : \mathcal{R} \to \mathcal{R}/I$ is a symmetric monoidal functor.

Proposition 11.7. Let $I$ be the ideal of $\mathcal{R} = \text{Fun}(((\text{Cat}^\text{perf})^\omega)^\text{op}, S_\infty)$ generated by those objects of the form $\tilde{C}/(\tilde{B}/\tilde{A})$ for each split-exact sequence $A \to B \to C$ in the set $E$ of corollary 4.16. Then $\mathcal{R}/I \simeq M_{\text{add}}$.

Proof. A priori, $\mathcal{R}/I$ is only a full subcategory of

$$M_{\text{add}} \simeq \text{Fun}_{\text{add}}(((\text{Cat}^\text{perf})^\omega)^\text{op}, S_{\infty}),$$

so we must show that any additive functor $f : ((\text{Cat}^\text{perf})^\omega)^\text{op} \to S_{\infty}$ is $I$-local. Now $I$ is the smallest full subcategory of $\mathcal{R}$ which contains the objects $\tilde{C}/(\tilde{B}/\tilde{A})$ and is closed under colimits and tensors with arbitrary objects of $\mathcal{R}$; since these tensors commute with colimits and $\mathcal{R}$ is generated by representables, it is enough to check that $\text{map}(\tilde{D} \otimes (\tilde{C}/(\tilde{B}/\tilde{A})), f) \simeq *$ for all $\mathcal{D}$ and split-exact sequences $A \to B \to C$ in $E$. But

$$\tilde{D} \otimes (\tilde{C}/(\tilde{B}/\tilde{A})) \simeq \tilde{D} \otimes C/(\tilde{D} \otimes B/\tilde{D} \otimes A),$$

and $\mathcal{D} \otimes A \to \mathcal{D} \otimes B \to \mathcal{D} \otimes C$ is split-exact since $A \to B \to C$ is split-exact.

Corollary 11.8. The additive localization $\text{Fun}(C^{\text{op}}, S_{\infty}) \to \text{Fun}_{\text{add}}(C^{\text{op}}, S_{\infty})$ is a symmetric monoidal functor.

Proof. The symmetric monoidal structure on

$$\text{Fun}(C^{\text{op}}, S_{\infty}) \simeq \text{Fun}(C^{\text{op}}, \text{Spaces}) \otimes S_{\infty}$$

is defined so that the Yoneda embedding $C \to \text{Fun}(C^{\text{op}}, \text{Spaces})$ is symmetric monoidal. The result then follows from theorem 11.6 and proposition 11.7.

Proposition 11.9. Let $J$ be the ideal of $\mathcal{R} = \text{Fun}(((\text{Cat}^\text{perf})^\omega)^\text{op}, S_{\infty})$ generated by those objects of the form $\tilde{C}/(\tilde{B}/\tilde{A})$ for each (equivalence class of) exact sequence $A \to B \to C$ with $B$ $\kappa$-compact. Then $\mathcal{R}/J \simeq M_{\text{loc}}$.

Proof. We must show that any localizing functor $f : ((\text{Cat}^\text{perf})^\omega)^\text{op} \to S_{\infty}$ is $J$-local. As in the proof of 11.7, it is enough to check that $\text{map}(\tilde{D} \otimes (\tilde{C}/(\tilde{B}/\tilde{A})), f) \simeq *$ for all $\mathcal{D}$ and exact sequences $A \to B \to C$ in with $B$ $\kappa$-compact. The result follows because

$$\tilde{D} \otimes (\tilde{C}/(\tilde{B}/\tilde{A})) \simeq \tilde{D} \otimes C/(\tilde{D} \otimes B/\tilde{D} \otimes A),$$

and $\mathcal{D} \otimes A \to \mathcal{D} \otimes B \to \mathcal{D} \otimes C$ is exact since $A \to B \to C$ is exact.

Corollary 11.10. The localization

$$\text{Fun}(C^{\text{op}}, S_{\infty}) \to \text{Fun}_{\text{loc}}(C^{\text{op}}, S_{\infty})$$

is a symmetric monoidal functor.
Proof. The symmetric monoidal structure on
\[ \text{Fun}(C^{\text{op}}, S_\infty) \simeq \text{Fun}(C^{\text{op}}, \text{Spaces}) \otimes S_\infty \]
is defined so that the Yoneda embedding \( C \to \text{Fun}(C^{\text{op}}, \text{Spaces}) \) is symmetric monoidal. The result then follows from theorem 11.6 and proposition 11.9. □

**Proposition 11.11.** If \( R \) is a commutative algebra object in \( \hat{\text{Cat}}_{\text{Pr}, \sigma}^{\text{Pr}} \), then the map \( R \to S_\infty \), right adjoint to the unit map \( S_\infty \to R \), is a lax symmetric monoidal functor.

Proof. The unit map \( S_\infty \to R \) is necessarily symmetric monoidal, so it follows from lemma 11.12 below that the right adjoint \( R \to S_\infty \) is lax symmetric monoidal. □

**Lemma 11.12.** Let \( f^\otimes : D^\otimes \to C^\otimes \) be a symmetric monoidal functor whose restriction \( f : D \to C \) admits a right adjoint \( g : C \to D \). Then \( g \) extends canonically to a lax symmetric monoidal functor \( g^\otimes : C^\otimes \to D^\otimes \).

**Theorem 11.13.** Let \( A \) be an \( E_n \)-object in the \( \infty \)-category of perfect \( \infty \)-categories, \( 1 \leq n \leq \infty \). Then \( K(A) \) and \( IK(A) \) are \( E_n \)-ring spectra. In particular, both \( K \) and \( IK \) take \( E_{n+1} \)-rings to \( E_n \)-ring spectra.

Proof. As both \( \text{Map}_{\text{Madd}}(S_\infty, -) \) and \( \text{Map}_{\text{Mloc}}(S_\infty, -) \) are lax symmetric monoidal functors by proposition 11.11, they preserve \( E_n \) structures. Since we have the equivalences
\[ K(A) \simeq \text{Map}_{\text{Madd}}(S_\infty, A) \quad \text{and} \quad IK(A) \simeq \text{Map}_{\text{Mloc}}(S_\infty, A), \]
the result follows. □

Finally, we want to extend our analysis to the homotopy invariant categories \( \text{Madd}_{\text{loc}} \) and \( \text{Mloc}_{\text{loc}} \). For this, we need the following easy technical lemma, which follows from lemma 4.15 and the properties of \( \otimes^\vee \).

**Lemma 11.14.** Let \( A \) be a small stable \( \infty \)-category. There exists a filtered direct system \( \{ C_j \to C_j \otimes^\vee A^1 \}_{j \in J} \), with each \( C_j \) a compact object object in \( \text{Cat}_{\infty}^{\text{Perf}} \), such that
\[ \text{colim}_{j \in J} \{ C_j \to C_j \otimes^\vee A^1 \} \xrightarrow{\sim} (A \to A \otimes^\vee A^1). \]

We can now prove that the symmetric monoidal structures on \( \text{Madd} \) and \( \text{Mloc} \) induce corresponding structures on \( \text{Madd}_{\text{loc}} \) and \( \text{Mloc}_{\text{loc}} \).

**Proposition 11.15.** The symmetric monoidal structures on \( \text{Madd} \) and \( \text{Mloc} \) induce symmetric monoidal structures on \( \text{Madd}_{\text{loc}} \) and \( \text{Mloc}_{\text{loc}} \) in such a way that the localization functors \( L_{\text{add}} \) and \( L_{\text{loc}} \) are lax symmetric monoidal.

Proof. Since \( \text{Madd} \) and \( \text{Mloc} \) are generated by objects of the form \( \text{U}_{\text{loc}}(B) \), with \( B \) a small stable \( \infty \)-category, it is enough to show that the maps
\[ \text{U}_{\text{add}}(B) \otimes (\text{U}_{\text{add}}(A) \to \text{U}_{\text{add}}(A \otimes^\vee A^1)) \]
and
\[ \text{U}_{\text{loc}}(B) \otimes (\text{U}_{\text{loc}}(A) \to \text{U}_{\text{loc}}(A \otimes^\vee A^1)) \]
are respectively \( \mathcal{E}_{\text{add}} \)-local and \( \mathcal{E}_{\text{loc}} \)-local equivalences. These equivalences follow from lemma 11.14 and the fact that \( \text{U}_{\text{add}} \) and \( \text{U}_{\text{loc}} \) are symmetric monoidal and preserve filtered colimits. □
In contrast to theorem 6.9, theorems 8.9 and 9.23 identify the mapping spectra in \( \mathcal{M}_{\text{loc}} \), \( \mathcal{M}_{\text{add}}^\infty \), and \( \mathcal{M}_{\text{loc}}^\infty \) only when the domain is \( U_{\text{loc}}(S^\omega_\infty) \), \( U_{\text{mot}}(S^\omega_\infty) \), or \( U_{\text{loc}}^h(S^\omega_\infty) \). In this subsection, we use the symmetric monoidal structure to prove the following extensions of these results:

**Theorem 11.16.** Let \( \mathcal{A} \) be a small stable \( \infty \)-category and let \( \mathcal{B} \) be a compact idempotent-complete small stable \( \infty \)-category. Then, there are natural equivalences of spectra

\[
\text{Map}(U_{\text{loc}}(\mathcal{B}), U_{\text{loc}}(\mathcal{A})) \simeq K(\text{Fun}^{ex}(\mathcal{B}, \mathcal{A}))
\]

and

\[
\text{Map}(U_{\text{loc}}^h(\mathcal{B}), U_{\text{loc}}^h(\mathcal{A})) \simeq KH(\text{Fun}^{ex}(\mathcal{B}, \mathcal{A})).
\]

Furthermore, there is a natural equivalence of spectra

\[
\text{Map}(U_{\text{add}}^h(\mathcal{B}), U_{\text{add}}^h(\mathcal{A})) \simeq KV(\text{Fun}^{ex}(\mathcal{B}, \mathcal{A})).
\]

**Proof.** Since compact idempotent-complete small stable \( \infty \)-categories are dualizable with respect to \( \otimes^\omega \), \( U_{\text{loc}}(\mathcal{B}) \) and \( U_{\text{loc}}^h(\mathcal{B}) \) are dualizable objects of \( \mathcal{M}_{\text{loc}} \) and \( \mathcal{M}_{\text{loc}}^h \) respectively. Next, since there is an equivalence

\[
\text{Map}(U_{\text{loc}}(\mathcal{B}), U_{\text{loc}}(\mathcal{A})) \simeq \text{Map}(U_{\text{loc}}(S^\omega_\infty), D(U_{\text{loc}}(\mathcal{B})) \otimes U_{\text{loc}}(\mathcal{A})),
\]

we deduce the first part of the theorem from theorem 6.9 and the fact that \( U_{\text{loc}} \) is (lax) symmetric monoidal. The second statement follows from the first part by an analogous argument, and the last statement is an immediate consequence of theorem 6.9 as well. \( \square \)

11.1. **The localizing subcategory generated by the unit.** In any symmetric monoidal stable \( \infty \)-category \( \mathcal{C} \), we can consider the smallest stable subcategory \( \text{Loc}_{\text{CC}}(1) \) generated by the unit object \( 1 \) which is closed under (not necessarily finite) direct sum; this is a lift of the localizing subcategory of the homotopy category generated by the image of the unit. If \( \mathcal{C} \) is generated by the unit, then this subcategory is actually all of \( \mathcal{C} \); in general, it is smaller.

Let \( F_{\mathcal{C}}(\cdot, \cdot) \) denote the mapping spectrum in \( \mathcal{C} \). The endomorphism spectrum \( \text{End}_{\mathcal{C}}(1) = F_{\mathcal{C}}(1, 1) \) is a commutative ring spectrum, and there is a functor

\[
F_{\mathcal{C}}(1, -) : \mathcal{C} \longrightarrow \text{End}_{\mathcal{C}}(1)-\text{Mod}.
\]

When \( 1 \) is a compact object in \( \mathcal{C} \), this functor induces an equivalence between the category of modules over \( \text{End}_{\mathcal{C}}(1) \) and \( \text{Loc}_{\mathcal{C}}(1) \) (see [14] for a nice discussion of this kind of “generalized Morita theory”). Moreover, there is an induced equivalence

\[
F_{\mathcal{C}}(X, Y) \longrightarrow F_{\text{End}_{\mathcal{C}}(1)}(F_{\mathcal{C}}(1, X), F_{\mathcal{C}}(1, Y)).
\]

for every \( X \in \text{Loc}_{\mathcal{C}}(1) \) and \( Y \in \mathcal{C} \). Once consequence of this is the following Ext spectral sequence (e.g., see [19, 4.1]):

**Corollary 11.17.** Given objects \( X_1 \) and \( X_2 \) in \( \text{Loc}_{\mathcal{C}}(1) \), we have a (convergent) spectral sequence

\[
E_2^{p,q} = \text{Ext}^{p,q}_{\text{End}_{\mathcal{C}}(1)}(\pi_{-p}F_{\mathcal{C}}(1, X_1), \pi_{-q}F_{\mathcal{C}}(1, X_2)) \Rightarrow \pi_{-p-q}F_{\text{End}_{\mathcal{C}}(1)}(F_{\mathcal{C}}(1, X_1), F_{\mathcal{C}}(1, X_2))
\]

and we can interpret both the \( E_2 \) term and the target in terms of maps in \( \mathcal{C} \).

In our setting these categories are fairly interesting:

(i) If \( \mathcal{C} = \mathcal{M}_{\text{add}} \) then \( \text{Loc}_{\mathcal{M}_{\text{add}}}(1) = A(*)-\text{Mod} \).
(ii) If $\mathcal{C} = \mathcal{M}_{\text{loc}}$ then $\text{Loc}_{\mathcal{M}_{\text{loc}}}^\text{(1)} = \mathcal{K}(S)\text{-Mod}$.

(iii) If $\mathcal{C} = \mathcal{M}_{\text{add}}^\text{A}$ then $\text{Loc}_{\mathcal{M}_{\text{add}}^\text{A}}^\text{(1)} = \mathcal{K}V(S)\text{-Mod}$.

(iv) If $\mathcal{C} = \mathcal{M}_{\text{loc}}^\text{A}$ then $\text{Loc}_{\mathcal{M}_{\text{loc}}^\text{A}}^\text{(1)} = \mathcal{K}H(S)\text{-Mod}$.

As such, the spectral sequence of the preceding corollary might be useful for computation. Furthermore, the following diagram of functors between these categories:

\[
\begin{array}{ccc}
\text{Loc}_{\mathcal{M}_{\text{add}}}^\text{(1)} & \longrightarrow & \text{Loc}_{\mathcal{M}_{\text{add}}^\text{A}}^\text{(1)} \\
\downarrow & & \downarrow \\
\text{Loc}_{\mathcal{M}_{\text{loc}}}^\text{(1)} & \longrightarrow & \text{Loc}_{\mathcal{M}_{\text{loc}}^\text{A}}^\text{(1)}
\end{array}
\]

can be obtained by applying the extension of scalars functors induced from the following diagram of commutative symmetric ring spectra

\[
\begin{array}{ccc}
A(\ast) & \longrightarrow & \mathcal{K}V(S) \\
\downarrow & & \downarrow \\
\mathcal{K}(S) & \longrightarrow & \mathcal{K}H(S).
\end{array}
\]

It is an interesting question to characterize these subcategories.

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