Toeplitz Density Operators and their Separability Properties

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Abstract

Toeplitz operators (also called localization operators) are a generalization of the well-known anti-Wick pseudodifferential operators studied by Berezin and Shubin. When a Toeplitz operator is positive semi-definite and has trace one we call it a density Toeplitz operator. Such operators represent physical states in quantum mechanics. In the present paper we study several aspects of Toeplitz operators when their symbols belong to some well-known functional spaces (e.g. the Feichtinger algebra) and discuss (tentatively) their separability properties.

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1 Introduction

There is a vast mathematical literature on Toeplitz operators and their variants (generalized anti-Wick operators), but these operators are much less known and used in quantum mechanics. This is a kind of paradox since Toeplitz operators were advertised and developed under the influence of Berezin and Shubin [3, 4] in the context of quantization. Certain particular cases are however known to most quantum physicists under the name of

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“anti-Wick quantization” or “Husimi function”. Still, the theory of Toeplitz operators is much better known and more often used in the related discipline of time-frequency analysis; among many references the reader might want to consult the following papers [1, 5, 6, 7, 8, 9, 10, 26, 27, 32, 35, 38, 39] to get an idea of what is going on in the field (I refrain from opposing quantum mechanics and time-frequency analysis, one supposedly being a physical theory and the other a mathematical theory, since both concern themselves with physical objects. It is just their aims and “philosophy” which differ). The aim of this paper is not to review quite generally the theory of Toeplitz operators, but more modestly to focus on the case where such operators are of trace class, more precisely density operators: a density operator on a complex separable Hilbert space $\mathcal{H}$ is a positive semidefinite trace class operator $\hat{\rho}$ with trace $\text{Tr}(\hat{\rho}) = 1$. We assume in this paper that $\mathcal{H} = L^2(\mathbb{R}^n)$. The following properties of density operators are well-known: (i) $\hat{\rho}$ is self-adjoint; (ii) $\hat{\rho}$ is the product of two Hilbert–Schmidt operators (and hence compact); (iii) $\hat{\rho}$ is positive semidefinite: $\hat{\rho} \geq 0$. By the spectral theorem, there exists an orthonormal basis $(\phi_j)_j$ of $L^2(\mathbb{R}^n)$ and coefficients satisfying $\lambda_j \geq 0$ and $\sum_j \lambda_j = 1$ such that $\hat{\rho}$ can be written as a convex sum $\sum_j \lambda_j \hat{\Pi}_{\phi_j}$ of orthogonal projections $\hat{\Pi}_{\phi_j} : L^2(\mathbb{R}^n) \rightarrow \mathbb{C}\phi_j$ converging in the strong operator topology. The importance of density operators in quantum mechanics comes from the fact that they represent (and are identified with) “mixed quantum states”; these are mixtures of $L^2$-normalized “pure states” $(\psi_j)_j$ in $L^2(\mathbb{R}^n)$ weighted by probabilities $\mu_j \geq 0$ summing up to one; the corresponding mixed state is then by definition the operator $\hat{\rho} = \sum_j \mu_j \hat{\Pi}_{\psi_j}$ and represents the maximal knowledge one has about the system under consideration. It is not difficult [18, 22] to check that the operator $\hat{\rho}$ thus defined indeed is a density operator; note that the decomposition $\sum_j \mu_j \hat{\Pi}_{\psi_j}$ of $\hat{\rho}$ has no reason to be unique (Jayne’s theorem, see however [23] where we compare different expansions of pure states). A density operator is de facto a Weyl operator in view of Schwartz’s kernel theorem; its Weyl symbol is $(2\pi \hbar)^n \rho$ where $\rho$ is the “Wigner distribution of $\hat{\rho}$” defined by

$$\rho = \sum_j \mu_j W\psi_j$$  \hspace{1cm} (1)

the series being convergent in the $L^2$-norm. Here $W\psi_j$ is the usual Wigner transform of $\psi_j$. Consider now, as we did in [19], a family of functions

$$\phi_{x\lambda}(x, p) = e^{i p\lambda(x - x\lambda)} \phi(x - x\lambda)$$
the \( z_\lambda = (x_\lambda, p_\lambda) \) belonging to some lattice \( \Lambda \subset \mathbb{R}^{2n} \) and \( \phi \in L^2(\mathbb{R}^n) \) (hereafter called “window”) is a fixed function with unit \( L^2 \)-norm. We can define a corresponding density operator by

\[
\hat{\rho} = \sum_{z_\lambda \in \Lambda} \mu_{z_\lambda} \hat{\Pi}_{\phi z_\lambda}
\]

and its Wigner distribution is given by

\[
\rho(z) = \sum_{z_\lambda \in \Lambda} \mu_{z_\lambda} W \phi(z - z_\lambda)
\]

in view of the translational properties of the Wigner transform. We can rewrite the formula above as the convolution product

\[
\rho = \mu * W \phi , \quad \mu = \sum_{z_\lambda \in \Lambda} \mu_{z_\lambda} \delta_{z_\lambda}
\]

where \( \delta_{z_\lambda} \) is the Dirac measure on \( \mathbb{R}^{2n} \) centered at \( z_\lambda \). This suggests to consider more general operators with Weyl symbol \((2\pi \hbar)^n \rho \) with \( \rho = \mu * W \phi \) where \( \mu \) is a Borel measure on \( \mathbb{R}^{2n} \). The aim of this paper is to study the properties of such operators; when they are density operators we will call them \emph{Toeplitz density operators}. Such operators were first considered by Berezin [3, 4], and have been developed since then by various authors, sometimes under the name of \emph{localization operators}, in time-frequency analysis [5, 6, 10, 26, 32, 35]

\textbf{Notation} \quad We write \( x = (x_1, ..., x_n) \) and \( p = (p_1, ..., p_n) \); we will use the notation \( px \) for the inner product \( p_1 x_1 + \cdots + p_n x_n \). The \textbf{scalar product on} \( L^2(\mathbb{R}^n) \) \textbf{is defined by}

\[
(\psi|\phi)_{L^2} = \int_{\mathbb{R}^n} \psi(x)\overline{\phi(x)}dx.
\]

\textbf{The space} \( T^*\mathbb{R}^n \equiv \mathbb{R}^{2n} \) \textbf{will be equipped with the canonical symplectic structure} \( \sigma = \sum_{j=1}^n dp_j \wedge dx_j \), \textbf{given in matrix notation by} \( \sigma(z, z') = Jz \cdot z' \) where \( J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \) \textbf{is the standard symplectic matrix}.

\section{Weyl and Anti-Wick Operators}

We are using here the notation in [18] which the Reader can consult for details and proofs.
2.1 Weyl pseudodifferential operators

Recall that the cross-Wigner transform of \((\psi, \phi) \in L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\) is defined by the absolutely convergent integral

\[
W(\psi, \phi)(x, p) = \left( \frac{1}{2\pi \hbar} \right)^n \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} \hat{p} y} \psi(x + \frac{1}{2} y) \overline{\phi(x - \frac{1}{2} y)} dy ;
\]

in particular \(W(\psi, \psi) = W\psi\) is the usual Wigner transform. We have the important relation:([18], §9.2)

\[
\int_{\mathbb{R}^{2n}} W(\psi, \phi)(z) dz = (\psi | \phi)_{L^2}.
\]

Let \(a \in S'(\mathbb{R}^{2n})\); the Weyl operator \(\hat{A} = \text{Op}_W(a)\) with symbol \(a\) is (by definition) the unique operator \(\hat{A} : S(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n)\) such that

\[
\langle \hat{A} \psi, \varphi \rangle = \langle \langle a, W(\psi, \phi) \rangle \rangle
\]

for all \((\psi, \phi) \in S(\mathbb{R}^n) \times S(\mathbb{R}^n)\) where \(\langle \cdot, \cdot \rangle\) (resp. \(\langle \langle \cdot, \cdot \rangle \rangle\)) is the distributional bracket on \(\mathbb{R}^n\) (resp. \(\mathbb{R}^{2n}\)). Using the Heisenberg displacement operator

\[
\hat{T}(z_0)\psi(x) = e^{\frac{i}{\hbar} \hat{p}_0 (x - \frac{1}{2} p_0 x_0)} \psi(x - x_0)
\]

we have the harmonic decomposition

\[
\hat{A} = \left( \frac{1}{2\pi \hbar} \right)^n \int_{\mathbb{R}^{2n}} a_\sigma(z) \hat{T}(z) dz
\]

where \(a_\sigma\) is the symplectic Fourier transform of \(a\):

\[
a_\sigma(z) = \left( \frac{1}{2\pi \hbar} \right)^n \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar} \sigma(z, z')} a(z) dz.
\]

When \(a \in S(\mathbb{R}^{2n})\) we get the familiar textbook definition

\[
\hat{A}\psi(x) = \left( \frac{1}{2\pi \hbar} \right)^n \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{-\frac{i}{\hbar} \hat{p}(x-y)} a(\frac{1}{2}(x+y), p) \psi(y) dp dy
\]

valid for \(\psi \in S(\mathbb{R}^n)\). In particular, the distributional kernel of \(\hat{A}\) is

\[
K(x, y) = \left( \frac{1}{2\pi \hbar} \right)^n \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} \hat{p}(x-y)} a(\frac{1}{2}(x+y), p) dp
\]
hence, using the Fourier inversion formula,

\[ a(x,p) = \int_{\mathbb{R}^n} e^{-i\frac{\hbar}{2} y} K(x + \frac{1}{2} y, x - \frac{1}{2} y) dy. \]  

(13)

We will also need the notion of transpose of a Weyl operator. Let \( \hat{A} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \) be a linear operator; the transpose \( \hat{A}^T \) of \( \hat{A} \) is the unique operator \( \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \) such that \( \langle \hat{A}\psi, \phi \rangle = \langle \psi, \hat{A}^T \phi \rangle \) for all \( (\psi, \phi) \in \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \). If \( \hat{A} \) has kernel \( (x,y) \mapsto -K(x,y) \) then \( \hat{A}^T \) has the kernel \( (x,y) \mapsto K(y,x) \) hence, in view of (13),

\[ \hat{A}^T = \text{Op}_W(a \circ I) , \quad I(x,p) = (x,-p). \]  

(14)

Weyl pseudo-differential operators enjoy the property of symplectic covariance: let \( \text{Sp}(n) \) be the standard symplectic group of \( \mathbb{R}^{2n} \). It is the group of all linear automorphisms of \( \mathbb{R}^{2n} \) such that \( S^*\sigma = \sigma \) where \( \sigma(z,z') = p \cdot x' - p' \cdot x \) if \( z = (x,p), z' = (x',p') \). We denote by \( \text{Mp}(n) \) the unitary representation in \( L^2(\mathbb{R}^n) \) of the double cover of \( \text{Sp}(n) \); \( \text{Mp}(n) \) is the metaplectic group \( [18] \); every \( S \in \text{Sp}(n) \) is thus the projection of two elements \( \pm \hat{S} \) of \( \text{Mp}(n) \). We have (\[18\], §10.3.1):

\[ \text{Op}_W(a \circ S^{-1}) = \hat{S} \text{Op}_W(a) \hat{S}^{-1}. \]  

(15)

Here is a general result for the calculation of the trace of a Weyl operator:

**Proposition 1** Let \( \hat{A} = \text{Op}_W(a) \) be a trace class operator. If \( a \in L^1(\mathbb{R}^{2n}) \) then

\[ \text{Tr}(\hat{A}) = \left( \frac{1}{2\pi\hbar} \right)^n \int_{\mathbb{R}^{2n}} a(z) dz. \]  

(16)

We will give refinements of this statement in Propositions 2 and 5 using the so-called Feichtinger algebra.

Notice that Proposition 1 requires that we know from the beginning that \( \hat{A} \) is of trace class. We can get stronger statement if we assume that the symbol \( a \) belongs to some appropriate Shubin class \( \Gamma^m_0(\mathbb{R}^{2n}) \). A function \( a \in C^\infty(\mathbb{R}^{2n}) \) belongs to the Shubin class \( \Gamma^m_0(\mathbb{R}^{2n}) \) if for every \( \alpha \in \mathbb{N}^{2n} \) there exists a constant \( C_\alpha \geq 0 \) such that

\[ |\partial^\alpha a(z)| \leq C_\alpha (1 + |z|)^{m-\delta|\alpha|} \text{ for } z \in \mathbb{R}^{2n}. \]  

(17)

**Proposition 2** Let \( \hat{A} = \text{Op}_W(a) \) with \( a \in \Gamma^m_0(\mathbb{R}^{2n}) \). If \( m < -2n \) then \( \hat{A} \) is of trace class and we have

\[ \text{Tr}(\hat{A}) = \left( \frac{1}{2\pi\hbar} \right)^n \int_{\mathbb{R}^{2n}} a(z) dz = a_\sigma(0) \]  

(18)

where \( a_\sigma = F_\sigma a \) is the symplectic Fourier transform \( [10] \) of the symbol \( a \).
See [18], §1.2.3, for a discussion of various trace formulas occurring in the literature.

Gaussian functions of the type

\[ \rho(z) = \left( \frac{1}{2\pi} \right)^n (\det \Sigma)^{-1/2} e^{-\frac{1}{2} \Sigma^{-1} z \cdot z} \]

(19)

where \( \Sigma \) is a real positive definite \( 2n \times 2n \) matrix clearly satisfy the conditions in Proposition 2 and we have

\[ \int_{\mathbb{R}^{2n}} \rho(z) dz = 1. \]

(20)

However, the operator \( \hat{\rho} = (2\pi \hbar)^n \text{Op}_W(\rho) \) does not qualify as a density operator unless the matrix \( \Sigma \) satisfies the condition

\[ \Sigma + \frac{i\hbar}{2} J \geq 0 \]

(21)

where \( \geq 0 \) stands for “positive semi-definite”; this condition ensures the positivity of \( \hat{\rho} \) [18, 33]. It can be shown [18] that condition (21) is a symplectically invariant reformulation of the uncertainty principle of quantum mechanics [18, 22].

### 2.2 Anti-Wick operators

There are several ways to define anti-Wick operators; in [36] Shubin uses the following definition: given a symbol \( a \in S(\mathbb{R}^n) \) the associated anti-Wick operator \( A_{AW} = \text{Op}_{AW}(a) \) is, by definition,

\[ \hat{A}_{AW} = \int_{\mathbb{R}^{2n}} a(z) \hat{\Pi}_0(z) dz \]

(22)

where \( \hat{\Pi}_0(z) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n) \) is the orthogonal projection onto the ray generated by \( \hat{T}(z)\phi_0 \) where \( \phi_0 \) the standard Gaussian

\[ \phi_0(x) = (\pi \hbar)^{-n/4} e^{-|x|^2/2\hbar}. \]

(23)

This action of this projection is explicitly given by

\[ \hat{\Pi}_0(z)\psi = (\psi|\hat{T}(z)\phi_0)_{L^2} \hat{T}(z)\phi_0 \]

(24)

and hence the operator \( \hat{A}_{AW} \) is given by

\[ \hat{A}_{AW}\psi = \int_{\mathbb{R}^{2n}} a(z)(\psi|\hat{T}(z)\phi_0)_{L^2} \hat{T}(z)\phi_0. \]

(25)
We observe that ([18], §11.4.1) \((\psi|\hat{T}(z)\phi_0)_{L^2}\) is, up to a factor, the radar ambiguity transform [24] of the pair \((\psi,\phi_0)\); in fact
\[
(\psi|\hat{T}(z)\phi_0)_{L^2} = (2\pi\hbar)^n \text{Amb}(\psi,\phi_0)
\] (26)
where, by definition,
\[
\text{Amb}(\psi,\phi_0)(z) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} -\frac{i}{\hbar} p_y \psi(y + \frac{1}{2} x)\phi_0(y - \frac{1}{2} x) dy.
\] (27)

Formula (25) can thus be rewritten
\[
\hat{A}_{AW}\psi = (2\pi\hbar)^n \int_{\mathbb{R}^n} a(z) \text{Amb}(\psi,\phi_0)(z) \hat{T}(z)\phi_0.
\] (28)

Recall ([18], §9.3) the following simple relation between \text{Amb}(\psi,\phi) and the cross-Wigner transform:
\[
\text{Amb}(\psi,\phi)(z) = 2^{-n} W(\psi,\phi^{\vee})(\frac{1}{2} z)
\] (29)
where \(\phi^{\vee}(x) = \phi(-x)\). It follows that (26) can be rewritten, since \(\phi_0^{\vee} = \phi_0\),
\[
(\psi|\hat{T}(z)\phi_0)_{L^2} = (\pi\hbar)^n W(\psi,\phi_0)(\frac{1}{2} z)
\] (30)
so that we also have
\[
\hat{A}_{AW}\psi = (\pi\hbar)^n \int_{\mathbb{R}^n} a(z) W(\psi,\phi_0)(\frac{1}{2} z) \hat{T}(z)\phi_0.
\] (31)

The following characterization in terms of Weyl operators is often taken as a definition of anti-Wick quantization:

**Proposition 3** Let \(a \in L^1(\mathbb{R}^n)\). The Weyl symbol \(b\) of the operator \(\hat{A} = \text{Op}_{AW}(a)\) is
\[
b = (2\pi\hbar)^n W\phi_0 \ast a
\] (32)
that is
\[
b(z) = 2^n \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar} |z-z'|^2} a(z') dz'.
\] (33)

**Proof.** Let \(\pi(z_0)\) be the Weyl symbol of the orthogonal projection [24]; we thus have, in view of definition (8),
\[
(\Pi_0(z)|\psi|\phi)_{L^2} = \int_{\mathbb{R}^{2n}} \pi(z_0)(z') W(\psi,\phi)(z') dz'
\]
for all $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$ (see e.g. [18], §10.1.2); the translational covariance of
the Wigner transform ([18], §9.2.2) that is, we have,

$$\pi(z_0)(z') = (2\pi\hbar)^n W(\hat{T}(z)\phi_0)(z') = (2\pi\hbar)^n W\phi_0(z - z')$$

and hence

$$(\hat{\Pi}_0(z)|\psi\phi)_{L^2} = (2\pi\hbar)^n \int_{\mathbb{R}^{2n}} W\phi_0(z - z')W(\psi, \phi)(z')dz'.$$

Using Fubini’s theorem we get for $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$

$$(\hat{A}\psi|\phi)_{L^2} = \int_{\mathbb{R}^{2n}} a(z)(\hat{\Pi}_0(z)|\psi\phi)_{L^2}dz$$

$$= (2\pi\hbar)^n \int_{\mathbb{R}^{2n}} a(z) \left[ \int_{\mathbb{R}^{2n}} W\phi_0(z - z')W(\psi, \phi)(z')dz' \right]dz$$

$$= (2\pi\hbar)^n \int_{\mathbb{R}^{2n}} \left[ \int_{\mathbb{R}^{2n}} a(z)W\phi_0(z - z')dz \right]W(\psi, \phi)(z')dz'.$$

hence the Weyl symbol of $\hat{A}$ is

$$b(z) = (2\pi\hbar)^n \int_{\mathbb{R}^{2n}} a(z)W\phi_0(z - z')dz' = (2\pi\hbar)^n (a \ast W\phi_0)(z).$$

Formula (33) follows in view of the identity [2, 18]

$$W\phi_0(z) = (\pi\hbar)^{-\frac{n}{2}} e^{-\frac{1}{2}|z|^2}.$$  \hspace{1cm} (34)

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\textbf{Remark 4} Note that it follows from formula (33) that the Weyl symbol $b$ is
a real analytic function, hence we cannot expect an arbitrary Weyl operator
to be an anti-Wick operator [36].

\subsection{2.3 The Feichtinger algebra and its dual}

The Feichtinger algebra and its dual are the simplest examples of modulation
spaces. They were introduced in the early 1980’s by H. Feichtinger [14, 15],
and have since played an increasingly important role in time-frequency
analysis and in Gabor theory; for a full textbook treatment see Gröchenig’s
treatise [24]; in [29] Jakobsen gives an up-to-date review of the Feichtinger
algebra. Modulation spaces were originally defined in terms of the short-
time Fourier transform (or Gabor transform) widely used in time-frequency
analysis; we have redefined them in [18] in terms of the cross-Wigner transform, which is more flexible, and has the indisputable advantage that the metaplectic invariance of modulation spaces becomes immediately obvious. We are following here our treatment in [18], Chapter 16 and 17.

By definition the Feichtinger’s algebra $M^1(\mathbb{R}^n)$ (sometimes denoted $S_0(\mathbb{R}^n)$) consists of all distributions $\psi \in S'(\mathbb{R}^n)$ such that $W(\psi, \phi) \in L^1(\mathbb{R}^{2n})$ for some window $\phi \in \mathcal{S}(\mathbb{R}^n)$; when this is the case we have $W(\psi, \phi) \in L^1(\mathbb{R}^{2n})$ for all windows $\phi \in \mathcal{S}(\mathbb{R}^n)$ and the formula

$$
||\psi||_\phi = \int_{\mathbb{R}^{2n}} |W(\psi, \phi)(z)|dz = ||W(\psi, \phi)||_{L^1}
$$

defines a norm on the vector space $M^1(\mathbb{R}^n)$; another choice of window $\phi'$ the leads to an equivalent norm and one shows that $M^1(\mathbb{R}^n)$ is a Banach space for the topology thus defined. We have the following continuous inclusions:

$$
M^1(\mathbb{R}^n) \subset C^0(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \cap FL^1(\mathbb{R}^n)
$$

and $\mathcal{S}(\mathbb{R}^n)$ is dense in $M^1(\mathbb{R}^n)$. Moreover, $\psi \in L^2(\mathbb{R}^n)$ is in $M^1(\mathbb{R}^n)$ if and only if $||W\psi||_{L^1} < \infty$.

Let $\tilde{S} \in Mp(n)$ (the metaplectic group) cover $S \in Sp(n)$; then $W(\tilde{S}\psi, \tilde{S}\phi) = W(\psi, \phi) \circ S^{-1}$; it follows from this covariance formula and the fact that the choice of window $\phi$ is irrelevant, that $\tilde{S}\psi \in M^1(\mathbb{R}^n)$ if and only if $S\psi \in M^1(\mathbb{R}^n)$ (metaplectic invariance $M^1(\mathbb{R}^n)$). It follows in particular that $M^1(\mathbb{R}^n)$ is invariant by Fourier transform, so it follows from the second inclusion (36) that we have

$$
M^1(\mathbb{R}^n) \subset FL^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n).
$$

As a consequence, using the Riemann–Lebesgue lemma, every $\psi \in M^1(\mathbb{R}^n)$ is bounded and vanishes at infinity. It also follows from the metaplectic invariance property that $M^1(\mathbb{R}^n)$ is stable under linear changes of variables: suppose $L \in GL(n, \mathbb{R})$; then the operator $\tilde{M}_{L,m}$ defined by $\tilde{M}_{L,m}\psi(x) = i^m \sqrt{\det L} \psi(Lx)$ for a choice of $m \mod 4$ corresponding to the argument of $\det L$ is in $Mp(n)$; if $\psi \in M^1(\mathbb{R}^n)$ we have $\tilde{M}_{L,m}\psi \in M^1(\mathbb{R}^n)$.

It turns out that $M^1(\mathbb{R}^n)$ is in addition an algebra for both pointwise product and convolution; in fact if $\psi, \psi' \in M^1(\mathbb{R}^n)$ then $||\psi * \psi'||_\phi \leq ||\psi||_{L^1}||\psi'||_\phi$ so we also have

$$
M^1(\mathbb{R}^n) * M^1(\mathbb{R}^n) \subset M^1(\mathbb{R}^n).
$$

Taking Fourier transforms we conclude that $M^1(\mathbb{R}^n)$ is also closed under pointwise product.
Replacing $\mathbb{R}^n$ with $\mathbb{R}^{2n}$ elements of the Feichtinger algebra can be viewed as pseudodifferential symbols; the following result was proven by Gröchenig in [25] (Theorem 3); also see Gröchenig and Heil [27] or Cordero and Gröchenig [8]; it is the announced refinement of Proposition 1:

Proposition 5 Let $a \in M^1(\mathbb{R}^{2n})$, then $\hat{A} = \text{Op}_W(a)$ is of trace class and we have

$$\text{Tr} \hat{A} = (\frac{1}{2\pi\hbar})^n \int_{\mathbb{R}^{2n}} a(z) dz.$$ 

The dual Banach space of the Feichtinger algebra $M^1(\mathbb{R}^n)$ is denoted by $M^\infty(\mathbb{R}^n)$; it is the space of tempered distributions consisting of all $\psi \in S'(\mathbb{R}^n)$ such that $W(\psi, \phi) \in L^\infty(\mathbb{R}^{2n})$ for one (and hence all) windows $\phi \in M^1(\mathbb{R}^n)$. The duality bracket is given by the pairing

$$(\psi, \psi') = \int_{\mathbb{R}^{2n}} W(\psi, \phi)(z) W(\psi', \phi)(z) dz. \quad \text{(39)}$$

(It follows from the fact that $L^\infty(\mathbb{R}^{2n})$ is the dual space of $L^1(\mathbb{R}^{2n})$; see [24], §11.3.) The formula

$$||\psi||'_\phi = \sup_{z \in \mathbb{R}^{2n}} |W(\psi, \phi)(z)| = ||W(\psi, \phi)||_{L^\infty} < \infty$$

defines a norm on $M^\infty(\mathbb{R}^n)$. Since $M^1(\mathbb{R}^n)$ is the smallest Banach space isometrically invariant under the action of the metaplectic group its dual $M^\infty(\mathbb{R}^n)$ is essentially the largest space of distributions with this property.

3 Toeplitz States

3.1 Toeplitz operators and their Weyl symbols

We defined in Section 2.2 the anti-Wick operator $\hat{A}_{\text{AW}} = \text{Op}_{\text{AW}}(a)$ by

$$\hat{A}_{\text{AW}} = \int_{\mathbb{R}^{2n}} a(z) \hat{\Pi}_0(z) dz \quad \text{(40)}$$

where $\hat{\Pi}_0(z)$ is the orthogonal projection on the ray $\mathbb{C}(\hat{T}(z)\phi_0)$ and $\phi_0$ is the standard Gaussian [23]. The notion of Toeplitz operator generalizes this definition to arbitrary windows $\phi \in L^1(\mathbb{R}^n)$:
Definition 6 Let $\phi \in M^1(\mathbb{R}^n)$ and $a \in L^1(\mathbb{R}^{2n})$. The Toeplitz operator $
abla_{\phi} = \text{Op}_\phi(a)$ with window $\phi$ and symbol $a$ is

$$
\nabla_{\phi} = \int_{\mathbb{R}^{2n}} a(z) \hat{\Pi}_\phi(z) dz
$$

(41)

where $\hat{\Pi}_\phi : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is the orthogonal projection onto the ray $\mathbb{C}(\hat{T}(z)\phi)$.

Explicitly, for $\psi \in \mathcal{S}(\mathbb{R}^n)$.

$$
\hat{A}_{\phi} \psi = \int_{\mathbb{R}^{2n}} a(z)(\psi|\hat{T}(z_0)\phi)_{L^2} \hat{T}(z_0)\phi dz_0.
$$

(42)

We will discuss the convergence of the integral (42) in a moment, but we first note that in view of the formulas (27) and (29) we can rewrite the definition (41) in the two equivalent forms

$$
\hat{A}_{\phi} \psi = (2\pi \hbar)^n \int_{\mathbb{R}^{2n}} a(z) \text{Amb}(\psi, \phi)(z) \hat{T}(z)\phi dz
$$

(43)

$$
\hat{A}_{\phi} \psi = (\pi \hbar)^n \int_{\mathbb{R}^{2n}} a(z) W(\psi, \phi^\vee)\left(\frac{1}{2}z\right) \hat{T}(z)\phi dz;
$$

(44)

the second relation is essentially the definition of the single-windowed Toeplitz (or localization) operators given in the time-frequency analysis literature (see e.g. [7, 10, 8, 38, 39]).

The following statement is the analogue of Proposition 3 in the framework of Toeplitz operators:

**Proposition 7** Let $\phi \in M^1(\mathbb{R}^n)$ and $a \in L^1(\mathbb{R}^{2n})$. The Toeplitz operator

$$
\hat{A}_{\phi} = \text{Op}_\phi(a) = \int_{\mathbb{R}^{2n}} a(z_0) \hat{\Pi}_{\phi}(z_0) dz_0
$$

has Weyl symbol

$$
a_{\phi} = (2\pi \hbar)^n (a \ast W\phi),
$$

(45)

that is,

$$
\hat{A}_{\phi} = (2\pi \hbar)^n \text{Op}_W(a \ast W\phi).
$$

(46)

**Proof.** Let us determine the Weyl symbol $\pi_{\phi}(z_0)$ of the orthogonal projection $\hat{\Pi}_{\phi}(z_0)$. It is easily seen, using (42), that the kernel of $\hat{\Pi}_{\phi}(z_0)$ is the function

$$
K_{\phi}(x, y) = \hat{T}(z_0)\phi(x) \overline{\hat{T}(z_0)\phi(y)}
$$

$$
= e^{-\frac{i}{\hbar}p_0(y-x)\phi(x-x_0)\phi(y-x_0)}
$$

$$
= e^{-\frac{i}{\hbar}p_0(y-x)\phi(x-x_0)\phi(y-x_0)}
$$

(47)

11
hence, by formula (13),
\[
\pi_\phi(z_0)(z) = \int_{\mathbb{R}^n} \frac{1}{\sqrt{\pi}} \phi(x - x_0 + \frac{1}{2}x) \phi_0(y - x_0 - \frac{1}{2}x) dy
\]
that is
\[
\pi_\phi(z_0)(z) = (2\pi\hbar)^n W\phi(z - z_0).
\]
It follows, using (8), that
\[
(\hat{\Pi}_\phi(z) \psi|\chi)_{L^2} = (2\pi\hbar)^n \int_{\mathbb{R}^{2n}} W\phi(z - z_0) W(\psi, \chi)(z_0) dz_0
\]
for all \( \psi, \chi \in \mathcal{S}(\mathbb{R}^n) \) and hence
\[
(\hat{A}_\phi \psi|\chi)_{L^2} = \int_{\mathbb{R}^{2n}} a(z) (\hat{\Pi}_\phi(z) \psi|\chi)_{L^2} dz
\]
\[
= (2\pi\hbar)^n \int_{\mathbb{R}^{2n}} a(z) \left[ \int_{\mathbb{R}^{2n}} W\phi(z - z_0) W(\psi, \chi)(z_0) dz_0 \right] dz.
\]
Using the Fubini–Tonnelli theorem we get
\[
(\hat{A}_\phi \psi|\chi)_{L^2} = \int_{\mathbb{R}^{2n}} a(z) (\hat{\Pi}_\phi(z) \psi|\chi)_{L^2} dz
\]
\[
= (2\pi\hbar)^n \int_{\mathbb{R}^{2n}} a(z) \left[ \int_{\mathbb{R}^{2n}} W\phi(z - z_0) W(\psi, \chi)(z_0) dz_0 \right] dz
\]
\[
= (2\pi\hbar)^n \int_{\mathbb{R}^{2n}} \left[ \int_{\mathbb{R}^{2n}} a(z) W\phi(z - z_0) dz \right] W(\psi, \chi)(z_0) dz_0
\]
hence the Weyl symbol of \( \hat{A}_\phi \) is \( a_\phi = (2\pi\hbar)^n W\phi \ast \mu \), as claimed. 

3.2 Toeplitz operators as density operators

The following result characterizes Toeplitz density operators. It is an extension to the continuous case of the formula (14).

**Proposition 8** Let \( \mu \in M^1(\mathbb{R}^{2n}) \) be a probability density:
\[
\mu \geq 0 \text{ and } \int_{\mathbb{R}^{2n}} \mu(z) dz = 1.
\]
For every \( \phi \in M^1(\mathbb{R}^n) \) such that \( ||\phi||_{L^2} = 1 \), the Toeplitz operator
\[
\hat{\rho} = (2\pi\hbar)^n \text{Op}_\phi(\mu) = (2\pi\hbar)^n \text{Op}_W(\mu \ast W\phi)
\]
(47)
is a density operator on \( L^2(\mathbb{R}^n) \).
Proof. The operator $\hat{\rho}$ is positive semidefinite: by definition (42) we have

$$(\text{Op}_\phi(\mu)\psi|\psi)_{L^2} = \int_{\mathbb{R}^{2n}} \mu(z)|\psi| T(z)\psi|_{L^2}^2 \, dz$$

hence $(\text{Op}_\phi(\mu)\psi|\psi)_{L^2} \geq 0$ because $\mu \geq 0$ being a probability density. Let us prove that $\hat{\rho}$ is of trace class. In view of Proposition 5 it is sufficient to show that the Weyl symbol $a = (2\pi \hbar)^n (\mu \ast W\phi)$ is in $M^1(\mathbb{R}^{2n})$. In view of the convolution algebra property (38) of $M^1(\mathbb{R}^{2n})$ it is sufficient for this to show that $M^1(\mathbb{R}^n)$ implies that $W\phi \in M^1(\mathbb{R}^{2n})$; this property is in fact a consequence of a more general result (Prop.2.5 in [8]), but we give here a direct independent proof. Let $\Phi \in \mathcal{S}(\mathbb{R}^{2n})$; denoting by $W^{2n}$ the cross-Wigner transform on $\mathbb{R}^{2n}$ we have, in view of formula (7),

$$|\int_{\mathbb{R}^{4n}} W^{2n}(W\phi,\Phi)(z,\zeta) \, dz \, d\zeta| = |(W\phi|\Phi)_{L^2(\mathbb{R}^{2n})}| < \infty$$

and hence $W\phi \in M^1(\mathbb{R}^{2n})$ as claimed so that $\hat{\rho}$ is trace class. In view of the convolution property (38) of the Feichtinger algebra we have $\mu \ast W\phi \in M^1(\mathbb{R}^{2n})$ as desired. Let us finally prove that $\text{Tr}(\hat{\rho}) = 1$. We have $a \in M^1(\mathbb{R}^{2n}) \subset L^1(\mathbb{R}^{2n})$ hence, by Proposition 2

$$\text{Tr}(\hat{\rho}) = \left(\frac{1}{2\pi \hbar}\right)^n \int_{\mathbb{R}^{2n}} a(z) \, dz = a_\sigma(0).$$

Denoting by $F_\sigma a$ the symplectic Fourier transform $a_\sigma$ we have

$$a_\sigma = (2\pi \hbar)^n F_\sigma (\mu \ast W\phi) = (2\pi \hbar)^{2n} (F_\sigma \mu)(F_\sigma W\phi)$$

so that

$$\text{Tr}(\hat{\rho}) = (2\pi \hbar)^{2n} (F_\sigma \mu)(0)(F_\sigma W\phi)(0).$$

Since $||\phi||_{L^2} = 1$, we have

$$F_\sigma W\phi(0) = \left(\frac{1}{2\pi \hbar}\right)^n \int_{\mathbb{R}^{2n}} W\phi(z) \, dz = \left(\frac{1}{2\pi \hbar}\right)^n$$

and hence

$$\text{Tr}(\hat{\rho}) = (2\pi \hbar)^n F_\sigma \mu(0) = \int_{\mathbb{R}^{2n}} \mu(z) \, dz = 1.$$
3.3 Example: Gaussian Toeplitz operators

Let us return to the Gaussian Wigner distribution (19), which we write here
\[
\rho_{\Sigma}(z) = \left(\frac{1}{2\pi}\right)^{n} (\det \Sigma)^{-1/2} e^{-\frac{1}{2} \Sigma^{-1}z \cdot z};
\] (48)
the real positive definite $2n \times 2n$ matrix $\Sigma$ (the “covariance matrix”) satisfies
the condition
\[
\Sigma + \frac{i}{\hbar} J \geq 0 \tag{49}
\]
which ensures the positivity of the corresponding density operator $\rho_{\Sigma} = (2\pi\hbar)^n \text{Op}_W(\rho_{\Sigma})$. We are going to see that the corresponding Weyl operators $\hat{\rho}_{\Sigma} = (2\pi\hbar)^n \text{Op}_W(\rho_{\Sigma})$ are Toeplitz operators (in fact generalized anti-Wick operators) if we assume that $\Sigma$ satisfies a certain condition.

Recall that the symplectic eigenvalues $\lambda^\sigma_j$ [18] of $\Sigma$ are the moduli of the eigenvalues of $J \Sigma$ ($J$ the standard symplectic matrix); since $J \Sigma$ has the same eigenvalues as the antisymmetric matrix $\frac{1}{2} J \Sigma \frac{1}{2}$ these are of the type $\pm i \lambda^\sigma_j$ with $\lambda^\sigma_j > 0$. One proves that [16] [18]:

**Lemma 9** The condition (49) is equivalent to $\lambda^\sigma_j \geq \frac{1}{2} \hbar$ for $1 \leq j \leq n$.

We call $\Lambda = (\lambda^\sigma_1, ..., \lambda^\sigma_n)$ the symplectic spectrum of $\Sigma$ (the $\lambda^\sigma_j$ are usually ranked in decreasing order: $\lambda^\sigma_{j+1} \geq \lambda^\sigma_j$). Associated to $\Lambda$ is the Williamson diagonalization of $\Sigma$: there exists $S \in \text{Sp}(n)$ such that
\[
\Sigma = SDS^T, \quad D = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}.
\] (50)
In particular, if all the symplectic eigenvalues are equal to one then $\Lambda = \frac{1}{2} \hbar I_{n \times n}$, and $\Sigma$ becomes $\Sigma_0 = \frac{1}{2} \hbar SS^T$ hence, by formula (34)
\[
\rho_{\Sigma_0}(Sz) = (\pi\hbar)^{-n} e^{-\frac{\hbar}{\pi} |z|^2} = W\phi_0(z)
\]
where $\phi_0$ is the standard Gaussian (23). Thus, by the symplectic covariance of the Wigner transform,
\[
\rho_{\Sigma_0}(z) = W\phi_0(S^{-1}z) = W(\hat{S}\phi_0)(z)
\] (51)
where $\hat{S} \in \text{Mp}(n)$ is anyone of the two metaplectic operators covering $S \in \text{Sp}(n)$.

**Proposition 10** The Weyl operator $\hat{\rho}_{\Sigma} = (2\pi\hbar)^n \text{Op}_W(\rho_{\Sigma})$ is a Toeplitz density operator if the symplectic eigenvalues $\lambda^\sigma_j$ of $\Sigma$ are all larger than $\frac{1}{2} \hbar$. 

14
Proof. Notice that the conditions $\lambda_j^\sigma > \frac{1}{2}h$ ensure us that the condition (49) is satisfied (Lemma 9). We know that $\hat{\rho}_{\Sigma}$ is a density operator so there remains to show that $\hat{\rho}_{\Sigma}$ is Toeplitz, i.e. that there exist $\mu$ and $\phi$ such that $\rho_{\Sigma} = \mu * W \phi$. We begin by remarking that a well-known formula from elementary probability theory says that if $\Sigma'$ and $\Sigma''$ are two symmetric real positive definite $2n \times 2n$ matrices then $\rho_{\Sigma' + \Sigma''} = \rho_{\Sigma'} * \rho_{\Sigma''}$. Choose in particular for $\Sigma'$ the matrix $\Sigma_0 = \frac{1}{2}hSS^T$ defined above, and $\Sigma'' = \Sigma - \Sigma_0$; then $\Sigma' + \Sigma'' = \Sigma$. In view of the diagonalization result (50) and the assumption $\lambda_j^\sigma > \frac{1}{2}h$ for $1 \leq j \leq n$ we have

$$\Sigma'' = S (D - \frac{1}{2}hI_{2n\times 2n}) S^T > 0$$

(52)

hence $\rho_{\Sigma''} \in S(\mathbb{R}^{2n})$. On the other hand $\Sigma' = \Sigma_0$ implies that $\rho_{\Sigma'} = W(\hat{S}_0\phi_0)$ in view of formula (51). The proposition follows taking $\phi = \hat{S}_0\phi_0$ and $\mu = \rho_{\Sigma''}$.

4 Separability Properties of Toeplitz Operators

We will now use the following notation. We introduce the splitting $\mathbb{R}^{2n} = \mathbb{R}^{2n_A} \oplus \mathbb{R}^{2n_B}$ and write: $z = (z_A, z_B) = z_A \oplus z_B$ with $z_A = (x_1, p_1, ..., x_{n_A}, p_{n_A})$ and $z_B = (x_{n_A+1}, p_{n_A+1}, ..., x_n, p_n)$. We equip the symplectic spaces $\mathbb{R}^{2n_A}$ and $\mathbb{R}^{2n_B}$ with their canonical bases. The symplectic structure on $\mathbb{R}^{2n}$ is then $\sigma(z, z') = Jz \cdot z'$ with $J = J_A \oplus J_B$ where

$$J_A = \bigoplus_{k=1}^{n_A} J_k , \quad J_k = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and likewise for $J_B$. Thus $J_A$ (resp. $J_B$) determines the symplectic structure on the partial phase space $\mathbb{R}^{2n_A}$ (resp. $\mathbb{R}^{2n_B}$). We denote by $I_A$ the identity $(x_A, p_A) \mapsto (x_A, p_A)$ and by $I_B$ the involution $(x_B, p_B) \mapsto (x_B, -p_B)$ ("partial reflection").

4.1 The notion of separability

A density operator $\hat{\rho}$ on $L^2(\mathbb{R}^n)$ is AB-separable if there exist sequences of density operators $\hat{\rho}_j^A$ on $L^2(\mathbb{R}^{n_A})$ and $\hat{\rho}_j^B$ on $L^2(\mathbb{R}^{n_B})$ ($n_A + n_B = n$) and real numbers $\alpha_j \geq 0$, $\sum_j \alpha_j = 1$ such that

$$\hat{\rho} = \sum_{j \in \mathbb{I}} \alpha_j \hat{\rho}_j^A \otimes \hat{\rho}_j^B$$

(53)
where the convergence is for the trace class norm. When \( \hat{\rho} \) is not separable, it represents an entangled state in quantum mechanics \([22]\). Here is a well-known necessary condition for a density operator to be AB-separable. We recall that the transpose of the Weyl operator \( \hat{A} = \text{Op}_W(a) \) is \( \hat{A}^T = \text{Op}_W(a \circ \mathcal{T}) \) where \( \mathcal{T} \) is the involution \( (x, p) \mapsto (x, -p) \) (formula \([14]\)). Similarly, one defines the partial transpose \( \hat{A}^{TB} \) with respect to the \( B \) variables by

\[
\hat{A}^{TB} = \text{Op}_W(a \circ (I_A \oplus I_B)).
\]

(54)

For notational simplicity we will write \( a \circ I_B, \rho_j \circ I_B, \) etc. instead of \( a \circ (I_A \oplus I_B), \rho_j \circ (I_A \oplus I_B), \) etc. The following result can be found in many physics texts, we give a rigorous proof thereof below:

**Proposition 11** Let \( \hat{\rho} \) be a density operator on \( L^2(\mathbb{R}^n) \). Suppose that the AB-separability condition \([53]\) holds. Then the partial transpose

\[
\hat{\rho}^{TB} = \sum_j \alpha_j \hat{\rho}_j^A \otimes (\hat{\rho}_j^B)^T
\]

is also a density operator.

**Proof.** (We are following the argument in de Gosson \([22], \S 16.2.2\)). In view of \([54]\) the transpose \( (\hat{\rho}_j^B)^T \) is explicitly given by

\[
(\hat{\rho}_j^B)^T = (2\pi \hbar)^n \text{Op}_W(\rho_j \circ \mathcal{T}_B).
\]

(56)

Suppose that the separability condition \([53]\) holds; then the Wigner distribution of \( \hat{\rho} \) is \( \rho = \sum_j \lambda_j \rho_j^A \otimes \rho_j^B \) with

\[
\rho_j^A = \sum_{\ell} \alpha_{j,\ell} W^A \psi_{j,\ell}^A, \quad \rho_j^B = \sum_m \beta_{j,m} W^B \psi_{j,m}^B
\]

where \( (\psi_{j,\ell}^A, \psi_{j,m}^B) \in L^2(\mathbb{R}^{n_A}) \times L^2(\mathbb{R}^{n_B}) \) and \( \alpha_{j,\ell}, \beta_{j,m} \geq 0 \) are such that \( \sum_{\ell} \alpha_{j,\ell} = 1 \) and \( \sum_m \beta_{j,m} = 1 \) (\( W^A \) and \( W^B \) are the Wigner transforms in the \( z_A \) and \( z_B \) variables, respectively). Thus

\[
\rho = \sum_{j,\ell,m} \gamma_{j,\ell,m} W^A \psi_{j,\ell}^A \otimes W^B \psi_{j,m}^B
\]

where \( \gamma_{j,\ell,m} = \lambda_j \alpha_{j,\ell} \beta_{j,m} \geq 0 \). We have

\[
\rho(\mathcal{T}_B z) = \sum_{j \in \mathcal{I}} \lambda_j \rho_j^A(z_A) \rho_j^B(\mathcal{T}_B z_B)
\]
and thus
\[
\rho \circ T_B = \sum_{j,\ell,m} \gamma_{j,\ell,m} W(\psi_{j,\ell}^A \otimes \overline{\psi}_{j,m}^B)
\]
hence $\text{Op}_W(\rho \circ I_B)$ is also a positive semidefinite trace class operator. That we have $\text{Tr}(\hat{\rho}^T_B) = 1$ is obvious. ■

The result above is sometimes called in physics the “PPT criterion” for “positive partial transpose”. It is known that while the PPT criterion gives a necessary condition for separability [34] it is also sufficient in the case $n_A n_B \leq 6$ [28].

The problem of finding a general sufficient condition for separability of density operators is still unsolved.

### 4.2 Separability: the Toeplitz case

We apply Proposition 11 to Toeplitz density operators. Let us begin by calculating the partial transpose of the density operator
\[
\hat{\rho} = (2\pi \hbar)^n \text{Op}_W(\mu \ast W\phi), \quad \mu \in M^1(\mathbb{R}^{2n}), \quad \phi \in M^1(\mathbb{R}^n).
\]
We have, by formula (54),
\[
\tilde{\rho}^T_B = (2\pi \hbar)^n \text{Op}_W((\mu \ast W\phi) \circ I_B).
\] (57)
A simple calculation shows that
\[
(\mu \ast W\phi) \circ I_B = (\mu \circ I_B) \ast (W\phi \circ I_B).
\]
Obviously $\mu \circ I_B \in M^1(\mathbb{R}^{2n})$ (the Feichtinger algebra is closed under linear changes of variables). Let us examine whether $W\phi \circ I_B$ is the Wigner transform of some $\phi' \in M^1(\mathbb{R}^n)$. It follows from a general (non-) covariance result (Theorem 1 in Dias et al. [11]) that we cannot expect in general $W\phi \circ I_B$ to be a Wigner transform. There are however two exception. Assume that $\phi = \phi_A \otimes \phi_B$ with $\phi_A \in M^1(\mathbb{R}^{n_A})$ and $\phi_B \in M^1(\mathbb{R}^{n_B})$. Then
\[
W\phi \circ I_B = W_A\phi_A \otimes W_B(\phi_B \circ I_B) = W_A\phi_A \otimes W_B\overline{\phi_B}
\]
so that we have in this case
\[
\tilde{\rho}^T_B = (2\pi \hbar)^n \text{Op}_W(\mu \circ I_B) \ast (W_A\phi_A \otimes W_B\overline{\phi_B}).
\] (58)
It follows that:
Proposition 12 Assume that $\phi = \phi_A \otimes \phi_B$ with $\phi_A \in M^1(\mathbb{R}^{n_A})$ and $\phi_B \in M^1(\mathbb{R}^{n_B})$. The partial transpose $\rho^{T_B}$ of the Toeplitz density operator $\rho = (2\pi \hbar)^n \text{Op}_W(\mu * W\phi)$ is also a Toeplitz density operator, given by formula (58), that is

$$\rho^{T_B} = (2\pi \hbar)^n \text{Op}_W(\mu \circ I_B) * W(\phi_A \otimes \overline{\phi_B})$$

(59)

and we have $\mu \circ I_B \in M^1(\mathbb{R}^{2n})$ and $\phi_A \otimes \overline{\phi_B} \in M^1(\mathbb{R}^n)$.

The second case where $W\phi \circ I_B$ is a Wigner transform is when the window $\phi$ is a generalized Gaussian

$$\phi_{X,Y}(x) = \left(\frac{1}{\pi \hbar}\right)^{n/4} \left(\det X\right)^{1/4} e^{-\frac{1}{2\hbar}(X+iY)x\cdot x}$$

(60)

where $X$ and $A$ are symmetric and $X$ positive definite. The Wigner transform of this function is well-known [2, 17, 31] and given by

$$W\phi_{X,Y}(z) = (\pi \hbar)^{-n} e^{-\frac{i}{\hbar}Gz\cdot z}$$

(61)

where $G$ is the symmetric positive definite symplectic matrix

$$G = \begin{pmatrix} X + YX^{-1}Y & XY^{-1} \\ YX^{-1} & X^{-1} \end{pmatrix}$$

(62)

in the $z = (x, p)$ ordering. One verifies that $G = SS^T$ where

$$S = \begin{pmatrix} X^{1/2} & YX^{-1/2} \\ 0 & X^{-1/2} \end{pmatrix} \in \text{Sp}(n).$$

(63)

Setting $\Sigma^{-1} = \frac{2}{\hbar}G$ the function $W\phi_{X,Y}$ is the Wigner distribution of the Gaussian $\rho_\Sigma$ which is a pure state. Now,

$$W\phi_{X,Y}(I_Bz) = (\pi \hbar)^{-n} e^{-\frac{i}{\hbar}G^T_B z\cdot z}$$

and we have

$$I_B G^T_B \equiv (I_A \oplus I_B) G (I_A \oplus I_B) \in \text{Sp}(n).$$

After a few calculations and a convenient reordering of the coordinates we arrive at the fact that $T_B G^T_B$ is obtained from $G$ by replacing the matrix $Y$ with the matrix $Y'$ defined by

$$Y'(x_A, x_B) \cdot (x_A, x_B) = Y(x_A, -x_B) \cdot (x_A, -x_B).$$

More intuitively, this amounts to saying that $W\phi_{X,Y} \circ I_B = W\phi_{X,Y'}$ where $\phi_{X,Y'}$ is obtained from $\phi_{X,Y}$ by taking the partial complex conjugate with respect to the variables $x_B = (x_1, ..., x_{n_B})$.  

18
4.3 Application: separability of Gaussian density operators

Werner and Wolf have proven ([40], Prop.1) the following necessary and sufficient condition for separability: a Gaussian density operator \( \hat{\rho}_\Sigma \) is separable if and only if there exist two partial covariance matrices \( \Sigma_A \) and \( \Sigma_B \) of dimensions \( 2n_A \times 2n_A \) and \( 2n_B \times 2n_B \) satisfying the conditions

\[
\Sigma_A + \frac{i\hbar}{2} J_A \geq 0 \quad \text{and} \quad \Sigma_B + \frac{i\hbar}{2} J_B \geq 0
\]  

and such that

\[
\Sigma \geq \Sigma_A \oplus \Sigma_B .
\]  

(65)

In [12] we have proven with Dias and Prata that this condition is equivalent to the existence of two symplectic matrices \( S_A \in \text{Sp}(n_A) \) and \( S_B \in \text{Sp}(n_B) \) such that

\[
\Sigma \geq \frac{\hbar}{2} (S_A S_A^T \oplus S_B S_B^T).
\]  

(66)

In [20,21] we have shown that every Gaussian density operator can be “disentangled” by a symplectic rotations. More precisely, we showed that for every \( \hat{\rho}_\Sigma = (2\pi \hbar)^n \text{Op}^W(\rho_\Sigma) \) there exists \( U \in U(n) \) such that \( \hat{\rho}_U = (2\pi \hbar)^n \text{Op}^W(\rho_\Sigma \circ U) \) is separable. It turns out that the use of the Toeplitz formalism considerably simplifies the proof:

**Proposition 13** Let \( \hat{\rho}_\Sigma \) be the density operator with Weyl symbol \( (2\pi \hbar)^n \rho_\Sigma \) where

\[
\rho_\Sigma(z) = \left( \frac{1}{2\pi} \right)^n (\det \Sigma)^{-1/2} e^{-\frac{1}{2} \Sigma^{-1} z \cdot z}.
\]  

(67)

There exists a symplectic rotation \( U \in U(n) \) such that

\[
\hat{\rho}_U = (2\pi \hbar)^n \text{Op}^W(\rho_\Sigma \circ U^{-1})
\]  

is a separable Toeplitz density operator and we have \( \hat{\rho}_U = \hat{U} \rho_\Sigma \hat{U}^{-1} \) where \( \hat{U} \in \text{Mp}(n) \) is anyone of the two metaplectic operators covering \( U \).

**Proof.** Let us write as above \( \rho_\Sigma = \rho_{\Sigma - \Sigma_0} \ast \rho_{\Sigma_0} \) where \( \Sigma_0 = \frac{1}{2} \hbar SS^T \), \( S \in \text{Sp}(n) \) as in the Williamson diagonalization ([50]). Since \( SS^T \) is positive definite and symplectic there exists \( U \in U(n) \) such that \( SS^T = UT \Delta U \) where \( \Delta \) is a diagonal matrix whose diagonal elements are the eigenvalues \( \lambda_1, ..., \lambda_{2n} \) of \( SS^T \) ([17], Prop. 2.13). We thus have

\[
\Sigma_0^U = U \Sigma_0 U^T = \frac{1}{2} \hbar \Delta
\]  

(67)
and the positive definite symmetric matrix
\[ \Sigma^U = U \Sigma U^T = U (\Sigma - \Sigma_0) U^T + \Sigma_0^U \geq \Sigma_0^U \]
is the covariance matrix of the Gaussian function \( \rho^U = \rho \circ U^{-1} \). To the latter corresponds the operator \( \hat{\rho}^U \) in view of the symplectic covariance of Weyl operators [17]. Let us prove that \( \rho^U \) is separable. The eigenvalues of \( SS^T > 0 \) are all positive and appear in pairs \((\lambda, 1/\lambda)\). In the AB-ordering \( \Delta \) has the form \( \Delta = \Delta_A \oplus \Delta_B \) with
\[
\Delta_A = \bigoplus_{k=1}^{n_A} \Delta_k \ , \ \Delta_B = \bigoplus_{k=n_A+1}^{n_A+n_B} \Delta_k
\]
where \( \Delta_k = \begin{pmatrix} \lambda_k & 0 \\ 0 & \lambda_k^{-1} \end{pmatrix} \) for \( k = 1, \ldots, n \). Clearly \( \Delta_A \in \text{Sp}(n_A) \) and \( \Delta_B \in \text{Sp}(n_B) \). Since \( \Sigma^U \geq \Sigma_0^U \) it follows that
\[
\Sigma^U \geq \frac{1}{2} \hbar (\Delta_A \oplus \Delta_B)
\]
hence \( \hat{\rho}^U \) is separable as claimed in view of (66) setting \( S_A = \Delta_A^{1/2} \) and \( S_B = \Delta_B^{1/2} \).

**Remark 14** In the physical literature symplectic rotations are called “passive linear transformations” [41]. The result above can thus be restated by saying that every Gaussian state can be made separable by a passive linear transformation.

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