BICONFORMAL CHANGES OF METRIC AND
PSEUDO-HARMONIC MORPHISMS

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Abstract. Pseudo-harmonic morphisms give rise on the domain space to a
distribution which admits an almost complex structure compatible with the
given Riemannian metric. We shall show that this property, together with the
harmonicity, are preserved by a biconformal change of the domain metric. The
special case of the pseudo-horizontally homothetic harmonic morphisms is also
treated.

1. Introduction

Pseudo-harmonic morphisms are a special class of harmonic maps into a Hermit-
ian manifold with the aditional property called Pseudo Horizontal Weak Confor-
mality (PHWC), cf. [6], [7]. This property states that the following relation holds:

\[ [d\varphi \circ d\varphi^*, J] = 0, \]

where \( \varphi : (M, g) \rightarrow (N, J, h) \) and \( d\varphi^* \) denotes the adjoint map.

The geometric meaning of (1.1) will become more transparent if we remark
that the differential of a submersion \( \varphi \) from a Riemannian manifold \( (M, g) \) into a
Hermitian manifold \( (N, J, h) \) induces an almost complex structure on the horizontal
bundle, defined by \( J_H = d\varphi|^{-1}_H \circ J \circ d\varphi|_H \). One can prove that PHWC condition is
equivalent to the compatibility of \( J_H \) with domain metric \( g \).

Moreover, if the codomain is Kähler, pseudo-harmonic morphisms have a nice
description similar to harmonic morphisms. According to [7], pseudo-harmonic
morphisms are those maps that pull back local complex-valued holomorphic func-
tions on \( N \) to local harmonic functions on \( M \).

Then, the analogue of horizontal homothety in this context was introduced by
M.A. Aprodu, M. Aprodu and V. Brinzanescu in [3]. A Pseudo-Horizontally Ho-
mothetic (PHH) map is a PHWC map \( \varphi \) which satisfies:

\[ [d\varphi \circ \nabla_X^M \circ d\varphi^*, J] = 0, \forall X \in \Gamma(H) \]

In turn, this condition means that \( J_H \) is parallel (with respect to \( \nabla^H \)) in hori-
zontal directions, so satisfies a transversal Kähler condition. Any PHH harmonic
submersion onto a Kähler manifold exhibits a particularly nice geometric property:
it pulls back complex submanifolds into minimal submanifolds, cf. [3]. Moreover,
in [11] it is shown that PHH harmonic submersions are (weakly) stable (in partic-
ular such maps minimise the energy-functional in their homotopy class). Further
properties and examples of PHH harmonic submersions can be found in [2], [4].

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In this paper we shall explore the invariance of the above two geometric conditions (PHWC and PHH) together with harmonicity under certain changes of metric. These results are particularly interesting in finding new examples of pseudo-harmonic morphisms and also they point out a nice analogy with the case of harmonic morphisms.

2. Changes of metric which preserve pseudo-harmonic morphisms

According to [8], for a PHWC map \( \varphi : (M^m, g) \rightarrow (N^{2n}, J, h) \) from a Riemannian manifold to a Kähler one, the tension field is given by:

\[
\tau(\varphi) = -d\varphi(F^\varphi \delta F^\varphi),
\]

where \( F^\varphi \) is the \( f \)-structure on \( M \), naturally induced by the PHWC map \( \varphi \) (in fact, \( F^\varphi \) extends \( J_H \) with zero on \( \mathcal{V} \)) and \( \delta F^\varphi = \text{trace} \nabla F^\varphi \) is the divergence of \( F^\varphi \).

If we consider an adapted frame \( \{e_i, F^\varphi e_i, e_\alpha\} \) (i.e. an orthonormal frame such that \( e_\alpha \in \text{Ker} \; F^\varphi \)), then the local form of the above formula is:

\[
(2.1) \quad \tau(\varphi) = -d\varphi \left( \sum_{i=1}^{n} F^\varphi \left[ (\nabla e_i F^\varphi)(e_i) + (\nabla F^\varphi e_i)(F^\varphi e_i) \right] + (m - 2n) \mu \right).
\]

**Definition 2.1.** Let \((M, g)\) be a Riemannian manifold endowed with two complementary distributions \( \mathcal{V} \) and \( \mathcal{H} \) which induce a natural decomposition of the metric: \( g = g^\mathcal{V} + g^\mathcal{H} \). Then one call biconformal change of metric \( g \), the association of a new metric on \( M \) defined by:

\[
\tilde{g} = \sigma^{-2} g^\mathcal{H} + \rho^{-2} g^\mathcal{V},
\]

where \( \sigma, \rho : M \rightarrow (0, \infty) \) are smooth functions.

The next result for PHWC submersions is similar to that one for horizontally conformal submersions (see [6], Lemma 4.6.6, p. 126).

**Proposition 2.1.** Let \( \varphi : (M^m, g) \rightarrow (N^{2n}, J, h) \) be a PHWC submersion. With respect to the vertical and horizontal distributions, one take a biconformal change of the metric \( g \), denoted by \( \tilde{g} \). Then, regarded as a map from \((M, \tilde{g})\) to \((N^{2n}, J, h)\), \( \varphi \) has tension field:

\[
(2.2) \quad \tilde{\tau}(\varphi) = \sigma^2 \left[ \tau(\varphi) + d\varphi(\text{grad}(\ln \rho^{2n-m} \sigma^{2-2n})) \right]
\]

**Proof.** Consider an adapted frame \( \{e_i, F^\varphi e_i, e_\alpha\} \) for \((M, g)\). Then \( \{\sigma e_i, \sigma F^\varphi e_i, \rho e_\alpha\} \) will be an adapted frame with respect to the perturbed metric \( \tilde{g} \). Let \( f_i \) denote \( e_i \) when \( i = 1, n \) and \( F^\varphi e_{1-n} \) when \( i = n + 1, 2n \). Then, using Koszul formula, one obtain for any \( X, Y \in \Gamma(\mathcal{H}) \):

\[
(2.3) \quad \mathcal{H}(\nabla_X Y) = \mathcal{H}(\nabla_X Y) + \sum_{i=1}^{n} \left[ -X(\ln \sigma)g(Y, f_i) - Y(\ln \sigma)g(X, f_i) + f_i(\ln \sigma)g(X, Y) \right] f_i.
\]

In the same way, but for \( V \in \Gamma(\mathcal{V}) \) one get:

\[
(2.4) \quad \mathcal{H}(\nabla_V V) = \frac{\sigma^2}{2} \left[ 2 \rho^{-2} \mathcal{H}(\nabla_V V) + \sum_{i=1}^{n} \left[ -f_i(\rho^{-2})g(V, V) \right] f_i \right].
\]

Now, applying (2.4) for \( V = \rho e_\alpha, \alpha = 2n + 1, m \) and summing, one get the following result which has the same form as in the harmonic morphism case (see [5], Remark 4.6.8, p. 126):
Lemma 2.1. At a biconformal change of the metric, the mean curvature of the fibers of a pseudo-harmonic morphism changes in this way:

\begin{equation}
\nabla^\nu = \sigma^2 \left[ \mu^\nu + \mathcal{H}(\text{grad}(\ln \rho)) \right]
\end{equation}

Now, in order to get the formula stated by the theorem, one has to see how changes the first term in equation (1.1), that is:

\[
F^\nu \text{div}_H F^\nu = F^\nu \left[ (\nabla^\nu, F^\nu)(e_i) + (\nabla_{F^\nu e_i} F^\nu)(F^\nu e_i) \right] = \\
\mathcal{H}(\nabla^\nu e_i) + \mathcal{H}(\nabla_{F^\nu e_i} F^\nu e_i) + F^\nu(\nabla^\nu e_i) = F^\nu(\nabla_{F^\nu e_i} e_i).
\]

So we have to apply formula (2.2) in four particular cases:

\[
\begin{align*}
\mathcal{H}(\nabla^\nu e_i) &= \sigma^2 \mathcal{H}(\nabla^\nu e_i) - \sigma^2 e_i(\ln \sigma) e_i + \sigma^2 \text{grad}_H(\ln \sigma), \\
\mathcal{H}(\nabla_{F^\nu e_i} F^\nu e_i) &= \sigma^2 \mathcal{H}(\nabla_{F^\nu e_i} F^\nu e_i) - \sigma^2 F^\nu e_i(\ln \sigma) F^\nu e_i + \sigma^2 \text{grad}_H(\ln \sigma), \\
\mathcal{H}(\nabla_{F^\nu e_i} e_i) &= \sigma^2 \mathcal{H}(\nabla_{F^\nu e_i} e_i) - \sigma^2 e_i(\ln \sigma) F^\nu e_i.
\end{align*}
\]

Now, summing these four terms as they appear in \( F^\nu \text{div}_H F^\nu \) we get:

\begin{equation}
F^\nu \text{div}_H F^\nu = \sigma^2 (F^\nu \text{div}_H F^\nu + (2n - 2)\text{grad}(\ln \sigma))
\end{equation}

Now, from (2.4) and (2.6), the stated result follows. \( \square \)

The following result is analogous with the one proved in [3].

Corollary 2.1. Let \( \varphi : (M^m, g) \rightarrow (N^{2n}, J, h) \) be a PHWC submersion (with \( m > 2n \)). For any smooth function \( \sigma : M \rightarrow (0, \infty) \), set

\begin{equation}
g_\sigma = \sigma^{-2} g^H + \sigma^{\frac{4n-4}{m-2n}} g^Y.
\end{equation}

Then \( \varphi \) is a pseudo-harmonic morphism with respect to \( g \) if and only if \( \sigma \) is a pseudo-harmonic morphism with respect to \( g_\sigma \).

Now, suppose that \( \varphi \) is a pseudo-horizontally homothetic map. The condition \( (1.2) \) translates simply into \( d\varphi \left( (\nabla_X F^\nu)(Y) \right) = 0, \forall X, Y \in \Gamma(H) \), so:

\[
[(\nabla_X F^\nu)Y]^H = 0, \forall X, Y \in \Gamma(H).
\]

In particular, \( F^\nu \text{div}_H F^\nu = 0 \), so the formula (2.1) reduces to:

\[
\tau(\varphi) = -(m - 2n) \text{div}_\nu \left( \mu^\nu \right).
\]

From the above relation, one can see that a pseudo-horizontally homothetic map is harmonic if and only if \( \sigma \) is constant. In particular, for any any strictly positive constant \( c \), set

\[
g_c = c^{-2} g^H + c^{\frac{4n-4}{m-2n}} g^Y.
\]

Then a PHH submersion \( \varphi \) is a PHH harmonic morphism with respect to \( g \) if and only if \( \sigma \) is a PHH harmonic morphism with respect to \( g_c \).
Proof. In particular, $\varphi$ is a PHWC submersion. Using (2.3), for any $X$ tangent to $H$ we derive:
\[
\mathcal{H} \left( (\nabla_X F^\varphi) Y \right) = \mathcal{H} \left( (\nabla_X F^\varphi) Y \right) + \text{grad}_H (\ln \sigma) g(X, F^\varphi) Y + g(X, Y) \sum_{i=1}^{n} \left[ F^\varphi e_i (\ln \sigma) e_i + e_i (\ln \sigma) F^\varphi e_i \right] - F^\varphi Y (\ln \sigma) X + Y (\ln \sigma) F^\varphi X.
\]
As $\varphi$ is a PHH harmonic morphism with respect to $g$, if $\sigma$ is constant, then it is obvious from the above formula that $\varphi$ is PHH harmonic morphism with respect to $g_\sigma$, too.

Conversely, if $\varphi$ is PHH harmonic morphism with respect to $g_\sigma$, then $F^\varphi \text{div}_H F^\varphi = 0$. But the hypothesis implies also $F^\varphi \text{div}_H F^\varphi = 0$ and the relation (2.6) tells us precisely that $\sigma$ must be a constant. □

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