A Cutoff Procedure and Counterterms for Differential Renormalization

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A Cutoff Procedure and Counterterms for Differential Renormalization

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Abstract

Explicit divergences and counterterms do not appear in the differential renormalization method, but they are concealed in the neglected surface terms in the formal partial integration procedure used. A systematic real space cutoff procedure for massless $\phi^4$ theory is therefore studied in order to test the method and its compatibility with unitarity. Through 3-loop order, it is found that cutoff bare amplitudes are equal to the renormalized amplitudes previously obtained using the formal procedure plus singular terms which can be consistently cancelled by adding conventional counterterms to the Lagrangian. Renormalization group functions $\beta(g)$ and $\gamma(g)$ obtained in the cutoff theory also agree with previous results.

1. Introduction

The purpose of this paper is to explore another aspect of the recently proposed differential renormalization procedure [1]. That method relies on the observation that (essentially all) primitively divergent Feynman graphs are well defined in real space for non-coincident points, but too singular at short distance to allow a Fourier transform. A regularization procedure must supply a prescription for the real space amplitudes which defines the short distance singularities such that integrals over them are well defined. This was done in [1] by a method which simultaneously regularizes and renormalizes amplitudes. The ideas involved are quite simple and best stated in terms of the 1-loop 4-point bubble graph of massless $\phi^4$ theory in 4 dimensions. This involves the singular function $1/x^4$, where $x^4 = (x_\mu x_\mu)^2$, and is regulated as follows:

1. Express such singular functions as derivatives of other functions which have well defined Fourier transforms. For example,

\[
\frac{1}{x^4} = -\frac{1}{4} \ln \frac{M^2 x^2}{x^2}
\]

is an identity for $x \neq 0$, and the function $\ln \left( \frac{M^2 x^2}{x^2} \right)$ has Fourier transform $-4\pi^2 \ln \left( \frac{p^2}{M^2} \right) / p^2$ where $M = 2M/\gamma$ and $\gamma = 1.781 \ldots$ is Euler’s constant.

2. Use formal partial integration of the derivatives in (1.1) to compute integrals such as the Fourier transform. Thus the regulated Fourier transform of $1/x^4$ is defined as $-\pi^2 \ln \left( \frac{p^2}{M^2} \right)$.

In Ref. [1] it was shown in a very explicit study of massless $\phi^4$ theory through 3-loop order, that these ideas can be extended to renormalize all 1PI vertex functions, including both primitively divergent graphs and those with divergent subgraphs. It was also shown that the resulting amplitudes satisfy the renormalization group equations in which $M$ appears as the expected scale variable. Further applications of differential renormalization to gauge theories, supersymmetry, and amplitudes with massive particles have recently appeared [2].

Explicit divergences and the counterterms which cancel them never occur in differential renormalization. The usual ultraviolet divergences of field theory are hidden in the short distance surface terms which are dropped in step 2 above. It was an implicit article-of-faith in [1], justified only in 1-loop order, that these surface terms could be cancelled by counterterms for wave function, mass, and coupling renormalizations. Since this is crucial to the consistency of the procedure, we undertake to demonstrate it here up to 3-loop order in $\phi^4$ theory. For
this purpose we wish to repeat the calculations of [1] using an explicit cutoff, implemented by modifying the
Euclidean massless scalar propagator as follows,

\[ \Delta(x) = \frac{1}{4\pi^2 \ x^2} \rightarrow \Delta(x, \varepsilon) = \frac{1}{4\pi^2 \ x^2 + \varepsilon^2}. \] (1.2)

Real space calculations with this propagator are modelled as closely as possible on the differential methods of [1],
leading to bare amplitudes \( \Gamma^b(x, \varepsilon) \). In the limit of small \( \varepsilon \), we show that the bare amplitudes for each diagram
can be expressed as the renormalized amplitudes of [1] \( \Gamma^r(x, M) \) plus additional singular terms involving \( 1/\varepsilon^2 \)
or \( (\ln \varepsilon^2 M^2)^n \). The latter are cancelled by adding local counterterms to the Lagrangian and including graphs
generated by counterterm vertices. The scale \( M \) is required for dimensional reasons in the separation of regular
and singular terms as \( \varepsilon \rightarrow 0 \). It will be clear from our calculations that the singular terms are related to the
surface terms neglected in [1], and the consistency of Step 2 above is thereby demonstrated.

This investigation was primarily motivated by skeptics of the methods of [1] who were not convinced that
overlap divergences were treated correctly and suspected an attendant violation of unitarity. We believe that the
present investigation resolves such doubts at the concrete level of current calculations. Specifically, the fact that
the singular cutoff dependence of bare amplitudes is cancelled by a Hermitean modification of the Lagrangian
effectively proves that the results of [1] are unitary, provided that any non-unitarity in the cutoff chosen vanishes
as the cutoff is removed. In the present case the Fourier transform of the cutoff propagator is

\[ \Delta(x, \varepsilon) = \int \frac{d^4 p}{(2\pi)^4} e^{ip \cdot x} \frac{\varepsilon}{p} K_1(\varepsilon p) \] (1.3)

and contains a logarithmic branch point at \( p = 0 \), with the cut corresponding to time-like Lorentzian momentum.
This behavior is displayed in the limiting form for small \( \varepsilon \) and fixed \( p \),

\[ \frac{\varepsilon}{p} K_1(\varepsilon p) \sim \frac{1}{p^2} + \frac{1}{4} \varepsilon^2 \left[ \ln \left( \frac{\varepsilon^2 p^2 \gamma^2}{4} \right) - 1 \right] + \mathcal{O}(\varepsilon^4). \] (1.4)

Unitarity is satisfied as \( \varepsilon \rightarrow 0 \) since effects of the logarithm vanish quadratically and the quadratic divergences in
cutoff amplitudes are directly cancelled by additive mass counterterms, so there are no \( 1/\varepsilon^2 \) terms which multiply
(1.4).

One should note that \( \frac{\varepsilon}{p} K_1(\varepsilon p) \) falls exponentially as \( p \rightarrow \infty \). Our real space computations are therefore
equivalent to a momentum space approach with a damped propagator similar to that used in [3] to study the
renormalization group in quantum field theory.

Renormalization group equations are usually derived [4] by studying the relation between bare and renor-
malized amplitudes. Since we have now systematically defined cutoff bare amplitudes within the differential
renormalization method, we can repeat this derivation and obtain \( \beta(g) \) and \( \gamma(g) \) as conventional scale derivatives
of the renormalization constants. Our results agree with those found using the “experimental” approach to the
renormalization group equations taken in [1].

2. The Cutoff Method in 1-Loop Order

The Lagrangian of massive Euclidean signature \( \phi^4 \) theory is

\[ \mathcal{L} = \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{24} \lambda \phi^4. \] (2.1)

We will be concerned only with the massless case in this paper. The bare amplitudes for all Feynman diagrams
will be computed using the Feynman rules of (2.1) with the cutoff propagator (1.2). These Feynman diagrams
are shown in Figs 1–2, in which the same designation as in [1] is used.
See figure 1 of Reference [1]

Fig. 1: Diagrams which contribute to $\Gamma^{(2)}(x-y)$ in $\phi^4$ theory. The divergence associated to the tadpole diagram $b$ is immediately cancelled by an appropriate counterterm in our renormalization scheme, so 2- and 3-loop graphs which include tadpoles need not be considered.

The classical contribution to the 4-point function, diagram $e$, and the bare amplitude for the 1-loop bubble graph $f$, are given by

$$\Gamma_e(x_i) = -16\pi^2 g\delta_{12}\delta_{13}\delta_{14}$$ \hspace{1cm} (2.2)

and

$$\Gamma_f^b(x_i, \varepsilon) = 8g^2\delta_{12}\delta_{34}\frac{1}{[x_{13} + \varepsilon^2]^2} + 2-\text{perms}$$ \hspace{1cm} (2.3)

where

$$g = \frac{\lambda}{16\pi^2},$$ \hspace{1cm} (2.4)

and the notation

$$x_{ij} = x_i - x_j$$ \hspace{1cm} (2.5)

will be used throughout.

As in Section I of [1], we express $1/(x^2 + \varepsilon^2)^2$ as $G(x^2, \varepsilon^2)$, leading to the ordinary linear differential equation

$$\frac{1}{(z + \varepsilon^2)^2} = \frac{4}{z} \frac{d}{dz} \left( z^2 \frac{dG}{dz} \right)$$ \hspace{1cm} (2.6)

in the variable $z = x^2$. The general solution is

$$G(x^2, \varepsilon^2) = -\frac{1}{4} \ln \left( \frac{(x^2 + \varepsilon^2)/\varepsilon^2}{x^2} \right) + \frac{a}{x^2} + b.$$ \hspace{1cm} (2.7)

The additive constant $b$ is irrelevant and can be dropped. The basic differential identity then becomes

$$\frac{1}{(x^2 + \varepsilon^2)^2} = -\frac{1}{4} \ln \left( \frac{(x^2 + \varepsilon^2)/\varepsilon^2}{x^2} \right).$$ \hspace{1cm} (2.8)

We have chosen $a = 0$ because the right hand side would otherwise have a $\delta(x)$ singularity not present on the left. One also sees that the behavior of $G(x^2, \varepsilon^2)$ is sufficiently soft as $x \to 0$, so that the derivatives in (2.8) can be freely integrated by parts without generating a short distance surface term.

We now introduce the dimensional constant $M^2$ and separate (2.8) into the two terms

$$\frac{1}{(x^2 + \varepsilon^2)^2} = -\frac{1}{4} \ln \left( \frac{(x^2 + \varepsilon^2)/\varepsilon^2}{x^2} \right) M^2 \boxed{\frac{1}{x^2}} + \frac{1}{4} \ln \varepsilon^2 M^2 \boxed{\frac{1}{x^2}}.$$ \hspace{1cm} (2.9)

Here $\boxed{\cdot}$ can be interpreted as operating to the right or left. Note that

$$\boxed{\frac{1}{x^2}} = -4\pi^2 \delta(x).$$ \hspace{1cm} (2.10)

As $\varepsilon \to 0$ we obtain

$$\frac{1}{(x^2 + \varepsilon^2)^2} \to \frac{1}{4} \ln \frac{x^2 M^2}{x^2} - \pi^2 \ln \varepsilon^2 M^2 \delta(x),$$ \hspace{1cm} (2.11)

in which we have introduced the notation $\boxed{\cdot}$ to indicate that the derivative must now be interpreted as acting to the left in integrals, since ill defined singularities would otherwise be obtained.
To see the consistency of this interpretation, and its compatibility with Step 2 of the differential renormalization method let us compute the Fourier transform of (2.9) and compare with that of (2.11). The transform of the left side of (2.9) is easily evaluated using parametric differentiation,

$$\int d^4x e^{ip \cdot x} \frac{1}{(x^2 + \varepsilon^2)^2} = -\frac{1}{2\varepsilon} \frac{\partial}{\partial \varepsilon} \int d^4x e^{ip \cdot x} \frac{1}{(x^2 + \varepsilon^2)}$$

$$= -\frac{2\pi^2}{\varepsilon} \frac{\partial}{\partial \varepsilon} \left( \frac{\varepsilon}{p} K_1(\varepsilon p) \right)$$

$$= 2\pi^2 K_0(\varepsilon p)$$

$$= \pi^2 p^2 \left\{ \frac{2}{p^2} K_0(\varepsilon p) + \ln \frac{\varepsilon^2 M^2}{p^2} \right\} .$$

We have taken the simple exact result in the third line, and rewritten it with a mass scale $M$ introduced, so that the final form corresponds to the Fourier transform of the two terms on the right side of (2.9) with $\delta_{ij}$ interpreted as the factor $-p^2$. The limiting form of this result as $\varepsilon \to 0$, namely,

$$-\pi^2 \ln \frac{p^2}{M^2} - \pi^2 \ln \varepsilon^2 M^2$$

(2.13)

coincides with the Fourier transform of the right side of (2.11).

The physical interpretation of these manipulations can be seen by substitution of (2.11) in (2.3) which then reads,

$$\Gamma^b_{ij}(x_i, \varepsilon) \longrightarrow -2g^2 \delta_{ij} \frac{\ln x_{13}^2 M^2}{x_{13}^2} + 2\text{perms}$$

$$-24\pi^2 g^2 \delta_{12 \delta_{13}} \ln \varepsilon^2 M^2$$

$$= \Gamma^b_f(x_i, M) - 24\pi^2 g^2 \delta_{12 \delta_{13}} \ln \varepsilon^2 M^2 .$$

This is the sum of the renormalized amplitude of [1] plus a local term with singular coefficient $\ln \varepsilon^2 M^2$ which is removed by adding the counterterm

$$-\pi^2 g^2 \ln \varepsilon^2 M^2 \phi^4$$

(2.15)

to the Lagrangian (2.1). One can also use (2.13) to obtain the same interpretation in momentum space, and the renormalized amplitude, obtained from the first term of (2.13), agrees with the formal partial integration rule of Step 2.

The other 1-loop diagram is the tadpole contribution $b$ to the 2-point function. Using the damped propagator (1.2) one finds the bare contribution to the 1PI 2-point function

$$\Gamma^b_b(x - y, \varepsilon) = \frac{2g}{\varepsilon^2} \delta(x - y) .$$

(2.16)

This can be cancelled immediately by adding the mass counterterm

$$-\frac{g}{\varepsilon^2} \phi^2$$

(2.17)

to (2.1). Insertions of the tadpole in other diagrams are also cancelled by the counterterm graph from (2.17), and higher order tadpoles can be cancelled similarly. For this reason we do not consider tadpole diagrams further in this paper.
The main purpose of this section was to show in 1-loop order using the cutoff procedure that the bare amplitude can be expressed for small \( \varepsilon \) as the renormalized amplitude of \([1]\) plus a singular term in \( \varepsilon \) which can be absorbed by coupling renormalization, and that this term corresponds closely to the singular surface term neglected in the formal partial integration prescription of differential renormalization.

3. 2-Loop Amplitudes

We now apply the cutoff method of calculation to 2-loop diagrams, where we shall encounter some new features.

Graph \( c \) is the first non-trivial contribution to the 2-point function. Its bare amplitude is

\[
\Gamma^b_c(x, \varepsilon) = -\frac{2g^2}{3\pi^2} \frac{1}{(x^2 + \varepsilon^2)^3}.
\]  
(3.1)

To handle this we simply differentiate the identity (2.8), obtaining

\[
\frac{1}{(x^2 + \varepsilon^2)^3} = \frac{3}{32} \ln \frac{x^2 + \varepsilon^2}{x^2} + \frac{3\varepsilon^2}{(x^2 + \varepsilon^2)^4}.
\]  
(3.2)

The second term in (3.2) is a function whose limit as \( \varepsilon \to 0 \) vanishes for \( x \neq 0 \), but is singular for \( \varepsilon = 0 \). One suspects that its limiting form is a distribution, and this may be confirmed by studying the integral

\[
\int d^4x f(x) \frac{3\varepsilon^2}{(x^2 + \varepsilon^2)^4} = \frac{3}{8} \int d^4y f(\varepsilon y) \frac{1}{(y^2 + 1)^4},
\]  
(3.3)

where \( f(x) \) is a smooth function which is damped at large distances. After the change of variables \( x = \varepsilon y \), one can evaluate the integral for small \( \varepsilon \) by expanding \( f(\varepsilon y) \) in a Taylor series through second order, and noticing that the contribution of the remainder is a convergent integral at large distance which vanishes as \( \varepsilon \to 0 \). After explicit evaluation of two elementary integrals one finds the result

\[
\int d^4x f(x) \frac{3\varepsilon^2}{(x^2 + \varepsilon^2)^4} \to \frac{\pi^2}{2\varepsilon^2} f(0) + \frac{\pi^2}{8} \delta(0) \delta(x),
\]  
(3.4)

which is equivalent to the statement

\[
\lim_{\varepsilon \to 0} \frac{3\varepsilon^2}{(x^2 + \varepsilon^2)^4} \to \frac{\pi^2}{2\varepsilon^2} \delta(x) + \frac{\pi^2}{8} \delta(x).
\]  
(3.5)

Such “representations of distributions” appear frequently in our work, and they are collected systematically in the Appendix.

The final step in the treatment of (3.2) is to introduce the scale \( M \) in the first term, obtaining

\[
\ln \frac{x^2 + \varepsilon^2}{x^2} \to \ln \frac{x^2 M^2}{x^2} + 4\pi^2 \ln \varepsilon^2 M^2 \delta(x)
\]  
(3.6)

by steps similar to those leading from (2.8) to (2.11). We now combine (3.4) and (3.6) and insert the result in (3.1) obtaining

\[
\Gamma^b_c(x, \varepsilon) \to \frac{g^2}{48\pi^2} \ln \frac{x^2 M^2}{x^2} + \frac{g^2}{12} (\ln \varepsilon^2 M^2 - 1) \delta(x) - \frac{g^2}{3\varepsilon^2} \delta(x).
\]  
(3.7)

The first term is the renormalized amplitude \( \Gamma^r_c(x, M) \) of \([1]\) while the last two terms can be cancelled by wave function and mass counterterms.

We now turn to the 4-point function, for which graphs \( g \) and \( h \) contribute in 2-loop order. The bare amplitude for graph \( g \) is

\[
\Gamma^b_g(x_i, \varepsilon) = -\frac{4g^3}{\pi^2} \delta_{12} \delta_{34} I_\varepsilon(x_{13}) + 2\text{--perms},
\]  
(3.8)
where $I_\varepsilon(x)$ is the convolution integral evaluated exactly in (A.2). The limiting form for small $\varepsilon$ is given by (A.4), and we rewrite it as

$$I_\varepsilon(x) \to -\frac{\pi^2}{4} \ln^2 \frac{x^2 M^2}{x^2} + \frac{\pi^2}{2} \ln \varepsilon^2 M^2 \left[ \ln \frac{x^2 M^2}{x^2} + 4 \pi^2 \ln \varepsilon^2 M^2 \delta(x) \right] - \pi^4 \ln^2 \varepsilon^2 M^2 \delta(x). \quad (3.9)$$

When inserted in (3.8) the first term gives the renormalized amplitude $\Gamma^r$, and the second term is proportional to the limiting form of $\Gamma^\prime(x_1, \varepsilon)$ in (2.12) times the singular coefficient $\ln \varepsilon^2 M^2$. This non-local divergence will be cancelled, as we show systematically in Section 5, by the bubble graph generated by the coupling counterterm (2.15). The last term in (3.9) can be cancelled by an order $g^4 \ln^2 \varepsilon^2 M^2 \delta^4$ counterterm in the Lagrangian. The amplitude obtained after insertion of (3.9) in (3.8) is presented in Table 2.

Graph $h$ requires a longer discussion because it is the first diagram whose basic form is triangular. The bare amplitude is

$$\Gamma^h_{h}(x_1, \varepsilon) = -\frac{8g^3}{\pi^2} \delta_{12} \frac{1}{x_{13} + \varepsilon^2} - \frac{1}{x_{14} + \varepsilon^2} + \frac{1}{(x_{34} + \varepsilon^2)^2} + 5\mathrm{-perms}. \quad (3.10)$$

We let $x_{14} = x$, $x_{34} = y$. Using (2.9) or (A.10), and (2.10) together with the antisymmetric derivative identity

$$A \square B = \partial_{\mu} \left( A \partial_{\mu} B \right) + B \square A, \quad (3.11)$$

one sees that

$$\begin{aligned}
\frac{1}{(x-y)^2 + \varepsilon^2} \frac{1}{(x^2 + \varepsilon^2)} \frac{1}{(y^2 + \varepsilon^2)} &= -\frac{1}{4} \frac{\partial}{\partial y_\mu} \left[\frac{1}{(x-y)^2 + \varepsilon^2} \frac{1}{x^2 + \varepsilon^2} \frac{1}{y^2} \right] \\
& - \frac{1}{4} \frac{1}{(x-y)^2 + \varepsilon^2} \frac{1}{(x^2 + \varepsilon^2)} \ln(y^2 + \varepsilon^2) M^2 y^2 \\
& + \pi^2 \delta(y) \ln M^2 \varepsilon^2 \left[ \frac{1}{4} \ln M^2 (x^2 + \varepsilon^2) + \pi^2 \ln M^2 \varepsilon^2 \delta(x) \right].
\end{aligned} \quad (3.12)$$

Note that (2.9) was used both to replace $1/(y^2 + \varepsilon^2)^2$ on the left side of (3.12) and to replace $1/(x^2 + \varepsilon^2)^2$ in the last term. The first term is regular as $\varepsilon \to 0$ provided we understand that $\frac{\partial}{\partial y_\mu}$ must be integrated by parts. A similar remark applies to $\ln M^2 (x^2 + \varepsilon^2)$ term in the last line.

The second term in (3.12) is an example of something we call a triangular structure. To study its limit as $\varepsilon \to 0$, we use the simple identities

$$\frac{1}{(x-y)^2 + \varepsilon^2} = \frac{1}{(x-y)^2} - \frac{\varepsilon^2}{(x-y)^2 [(x-y)^2 + \varepsilon^2]} \quad (3.13)$$

$$\square \frac{1}{(x-y)^2 + \varepsilon^2} = -4 \pi^2 \delta(x-y) - \frac{\varepsilon^2}{(x-y)^2 [(x-y)^2 + \varepsilon^2]} \quad (3.14)$$

Using (3.14) the second term in (3.12) can be written as

$$\pi^2 \delta(x-y) \ln(x^2 + \varepsilon^2) M^2 x^2 + \frac{1}{4} \frac{\varepsilon^2}{(x-y)^2 [(x-y)^2 + \varepsilon^2]} \ln \left( \frac{y^2 + \varepsilon^2}{x^2 + \varepsilon^2} \right) M^2 y^2. \quad (3.15)$$

The limiting form of the first term is obtained from (A.11a) which is essentially a differential identity.

We claim that the limiting form of the second term is that of an eight dimensional delta function $-\frac{1}{4} C \delta(x) \delta(y)$ corresponding, after insertion in (3.10), to another two-loop order coupling constant counter-term. This claim can be verified by studying the integral of the term in question with a test function $f(x, y)$.
After scaling variables, \( x \to \epsilon x, y \to \epsilon y \), one sees that the limiting contribution involves only \( f(0,0) \), and that the constant \( C \) is given by

\[
C = \int \frac{d^4x \, d^4y}{(x^2 + 1) \, y^2} \left( \frac{1}{(x - y)^2 ((x - y)^2 + 1)} \right) \left( \ln \epsilon^2 M^2 + \ln(y^2 + 1) \right) = -4\pi^4 \left( \ln \epsilon^2 M^2 - B \right)
\]  

(3.16)

where the first integral becomes trivial after partial integration of \( \Box \). The evaluation of the second integral is discussed briefly in the Appendix, see (A.13), and the result of the argument beginning with (3.13) is given in (A.12b). The identities (3.13) and (3.14) are useful because the product terms fall off fast enough in the infrared so that their contribution can be obtained by scaling arguments.

It is well worth noting that when scaling arguments are used in the study of integrals involving test functions, the question of the limit as \( \epsilon \to 0 \) is effectively transferred to the question of the behavior of the large \( x, y \) behavior of integrals over the scaled variables. It is a correct rule of thumb, which can be verified by more careful limiting arguments, that the limiting contribution of a term in the bare amplitude is a \( \delta \)-function or product of \( \delta \)-functions, if the integral determining the naive coefficient of the \( \delta \)-functions is infrared convergent as is the case for \( B \) and \( C \) in (3.16).

The results (A.11a) and (A.12b) are now combined with the simple limits of the first and third terms of (3.12) to obtain the limiting form of \( \Gamma^b_i \) given in Table 2. Again one finds the renormalized amplitude of [1] plus singular terms to be cancelled by counterterms.

4. 3-Loop Diagrams

We now continue the program of the last two sections and study the limit as \( \epsilon \to 0 \) of cutoff amplitudes for the 3-loop graphs shown in Figs. 1 and 2. It is worth emphasizing that bare amplitudes are independent of the mass scale \( M \),

\[
M \frac{\partial}{\partial M} \Gamma^b_i(x_1, \epsilon) = 0 ,
\]

because \( M \) is introduced only to separate \( \Gamma^b_i(x_1, \epsilon) \) into regular and singular terms. Thus one can use the property (4.1) as a check on the intermediate steps of the calculation of a complicated amplitude. The same mass scale is used in all diagrams in order to agree with the renormalization scheme of [1].

Graph \( d \) is the only contribution to the 2-point function at 3-loop order. Its bare amplitude is

\[
\Gamma^b_d(x, \epsilon) = \frac{g^3}{\pi^4} \frac{1}{x^2 + \epsilon^2} \int d^4u \frac{1}{(u^2 + \epsilon^2)^2} \left( \frac{1}{((u - x)^2 + \epsilon^2)^2} \right) .
\]

(4.2)

We now use the result (A.2) for the convolution, except that we compute \( \Box \) acting on (A.2). We obtain,

\[
\Gamma^b_d(x, \epsilon) = -\frac{2g^3}{\pi^2} \frac{1}{x^2 + \epsilon^2} \left\{ \frac{(x^2 + 2\epsilon^2) \ln (x^2 + 2\epsilon^2 - |x|\sqrt{x^2 + 4\epsilon^2}) / 2\epsilon^2}{|x|^3 \sqrt{x^2 + 4\epsilon^2}} + \frac{1}{x^2(x^2 + 4\epsilon^2)} \right\} .
\]

(4.3)

Inspired by the form of the renormalized amplitude, we compare (4.3) and

\[
- \frac{g^3}{32\pi^2} \frac{\ln^2(x^2 + \epsilon^2)/\epsilon^2 + 3\ln(x^2 + \epsilon^2)/\epsilon^2}{x^2} = - \frac{2g^3}{\pi^2(x^2 + \epsilon^2)} \left\{ \frac{(x^2 - 2\epsilon^2) \ln(x^2 + \epsilon^2)/\epsilon^2}{(x^2 + \epsilon^2)^3} + \frac{x^2}{(x^2 + \epsilon^2)^3} \right\} .
\]

(4.4)

The difference between (4.3) and (4.4) again involves a representation of the distributions \( \delta(x) \) and \( \Box \delta(x) \), as one can verify by integration with a smooth \( f(x) \) as in (3.3–4.4). One can then write
\[ \Gamma_b(x,\varepsilon) = -\frac{g^3}{32\pi^2} \ln^2(x^2 + \varepsilon^2)/\varepsilon^2 + 3 \ln(x^2 + \varepsilon^2)/\varepsilon^2 \cdot \frac{2g^3}{\pi^2\varepsilon^2} D_1 \delta(x) - \frac{g^3}{4\pi^2} D_2 \delta(x) + O(\varepsilon), \quad (4.5) \]

where \( D_1 \) and \( D_2 \) are the purely numerical values of the following integrals,

\[
D_1 = \int \frac{d^4x}{x^2+1} \left\{ \frac{(x^2+2) \ln \left( x^2 + 2 - |x|\sqrt{x^2 + 4} \right) / 2}{|x|^3\sqrt{x^2 + 4}} + \frac{1}{x^2(x^2 + 4)} \right. \\
- \frac{(x^2-2) \ln(x^2+1)}{(x^2+1)^3} - \frac{x^2}{(x^2+1)^3} \right\},
\]

\[
D_2 = \int \frac{d^4x}{x^2+1} \left\{ \frac{(x^2+2) \ln \left( x^2 + 2 - |x|\sqrt{x^2 + 4} \right) / 2}{|x|^3\sqrt{x^2 + 4}} + \frac{1}{x^2(x^2 + 4)} \right. \\
- \frac{(x^2-2) \ln(x^2+1)}{(x^2+1)^3} - \frac{x^2}{(x^2+1)^3} \right\}. \quad (4.6)
\]

The logarithmic terms in (4.5) may now be separated into regular and singular terms in \( \varepsilon \) after introduction of the mass scale \( M \). One then finds the limiting form given in Table 1. In this expression \( D_1 \) and \( D_2 \) are the coefficients of finite counterterms at 3-loop order. These terms become relevant to the cancellation of divergences only at the 4-loop level, so the integrals (4.6) need not be evaluated.

**Table 1.** Bare cutoff amplitudes for graphs contributing to the 1PI 2-point function. The subscripts denote the graphs shown in Fig. 1, and the tadpole graph \( b \) is omitted for reasons discussed in Sec. 2. The first term in each entry is the renormalized amplitude obtained in [1], and this is followed by cutoff dependent terms in the limit of small \( \varepsilon \). The numerical constants \( D_1 \) and \( D_2 \) are the values of the integrals in (4.6).

\[
\Gamma_a(x) = -\delta^4(x)
\]

\[
\Gamma_b(x,\varepsilon) \rightarrow \frac{g^2}{48\pi^2} \ln x^2 M^2 \frac{x}{x^2} - \frac{g^2}{12} (1 - \ln x^2 M^2) \delta^4(x) - \frac{g^2}{3} \frac{1}{\varepsilon^2} \delta^4(x)
\]

\[
\Gamma_c(x,\varepsilon) \rightarrow -\frac{g^3}{32\pi^2} \ln^2 x^2 M^2 + 3 \ln x^2 M^2 \\
+ 3g \ln x^2 M^2 \Gamma_c(x,\varepsilon) - \frac{g^3}{8} \left( \ln^2 x^2 M^2 + \ln x^2 M^2 + \frac{2D_2}{\pi^2} \right) \delta^4(x) \\
+ g^3 \frac{\ln x^2 M^2}{\varepsilon^2} \left( \frac{2D_1}{\pi^2} - \frac{\ln x^2 M^2 - 2D_1/\pi^2}{\varepsilon^2} \right) \delta^4(x)
\]

We now begin our treatment of the 8 graphs which contribute to the 4-point function. We shall be rather brief in our discussion of the easier graphs and concentrate on the more difficult ones, namely, \( j, l, n, \) and \( o \).

Graph \( i \) is particularly easy since the bare amplitude

\[
\Gamma_i^{b}(x,\varepsilon) = \frac{2g^4}{\pi^4} \delta_{1234} \int d^4u d^4v \frac{1}{[(x_1 - u)^2 + \varepsilon^2][[(u - v)^2 + \varepsilon^2][[(v - x_3)^2 + \varepsilon^2]^{2} + 2-\text{perms}, \quad (4.7)}
\]
is a double convolution of factors for which the identity (2.8) may be used. As discussed in the Appendix, the limiting form of the convolution is correctly given by the convolution of the limiting form (2.11) of each factor. This leads to the result given in Table 2.

Graph j is somewhat more involved. We start by writing its bare amplitude,

$$\Gamma_j(x, \varepsilon) = -\frac{4g^4}{\pi^2} \delta_{12} \delta_{34} \frac{1}{x_{13} + \varepsilon^2} \frac{1}{x_{14} + \varepsilon^2} \int d^4u \frac{1}{(x_3 - u)^2 + \varepsilon^2} \frac{1}{(x_4 - u)^2 + \varepsilon^2} \ln^2 \left( \frac{x_{34}^2 + 2\varepsilon^2 - |x_{34}|\sqrt{x_{34}^2 + 4\varepsilon^2}}{x_{34}^2} \right) / 2\varepsilon^2 + 5\text{-perms}.$$  (4.8)

We substitute the result of the integral (A.2) and then use (3.11), which splits the amplitude into two parts,

$$\Gamma_j(x, \varepsilon) = -\frac{4g^4}{\pi^2} \delta_{12} \partial_{3\mu} \left( \frac{1}{x_{13} + \varepsilon^2} \frac{1}{x_{14} + \varepsilon^2} \partial_{3\mu} \ln^2 \left( \frac{x_{34}^2 + 2\varepsilon^2 - |x_{34}|\sqrt{x_{34}^2 + 4\varepsilon^2}}{x_{34}^2} \right) / 2\varepsilon^2 \right) + 5\text{-perms}.$$  (4.9)

We take the limit $\varepsilon \to 0$ in the first one, and introduce $M$. This yields

$$-\frac{4g^4}{\pi^2} \delta_{12} \partial_{3\mu} \left( \frac{1}{x_{13} + \varepsilon^2} \frac{1}{x_{14} + \varepsilon^2} \partial_{3\mu} \ln^2 \left( \frac{x_{34}^2 + 2\varepsilon^2 - |x_{34}|\sqrt{x_{34}^2 + 4\varepsilon^2}}{x_{34}^2} \right) / 2\varepsilon^2 \right) \rightarrow$$

$$-\frac{4g^4}{\pi^2} \delta_{12} \partial_{3\mu} \left( \frac{1}{x_{13} + \varepsilon^2} \frac{1}{x_{14} + \varepsilon^2} \partial_{3\mu} \ln^2 \left( \frac{x_{34}^2 + 2\varepsilon^2 - |x_{34}|\sqrt{x_{34}^2 + 4\varepsilon^2}}{x_{34}^2} \right) / 2\varepsilon^2 \right) \ln x^2 M^2.$$  (4.10)

The first term of the R.H.S. is a piece of the renormalized amplitude of Ref. [1]. The second is a piece of $g \ln x^2 M^2 \Gamma_h^b(x, \varepsilon)$.

The third term requires a little more work. We undo relation (3.11) recognizing a delta term plus the triangular structure (A.12a),

$$-\frac{4g^4}{\pi^2} \delta_{12} \partial_{3\mu} \left( \frac{1}{x_{13} + \varepsilon^2} \frac{1}{x_{14} + \varepsilon^2} \partial_{3\mu} \frac{1}{x_{34}^2} \right) \ln^2 \varepsilon^2 M^2 =$$

$$4g^4 \delta_{12} \delta_{34} \frac{1}{(x_{13} + \varepsilon^2)^2} \ln \varepsilon^2 M^2 - 4g^4 \frac{\ln \varepsilon^2 M^2}{x_{34}^2 (x_{34} + \varepsilon^2)} \delta_{12} \delta_{13} + 4g^4 \frac{\varepsilon^2}{x_{34}^2} \delta_{12} \delta_{13} \delta_{14} \ln \varepsilon^2 M^2.$$  (4.11)

Let us now turn our attention to the second term of (4.9), which is a triangular structure independent of scale $M$. It will have a representation of the following form in the limit as $\varepsilon \to 0$,

$$-\frac{4g^4}{\pi^2} \delta_{12} \partial_{3\mu} \ln^2 \left( \frac{x_{34}^2 + 2\varepsilon^2 - |x_{34}|\sqrt{x_{34}^2 + 4\varepsilon^2}}{x_{34}^2} \right) / 2\varepsilon^2 \left( \frac{1}{x_{34}^2 + \varepsilon^2} \partial_{3\mu} \frac{1}{x_{34}^2} \right) \left( \frac{1}{x_{34}^2 + \varepsilon^2} \partial_{3\mu} \frac{1}{x_{34}^2} \right)$$

$$g^4 \delta_{12} \left( \delta_{13} \frac{f(x_{34}^2)}{x_{34}^2 (x_{34} + \varepsilon^2)} + \pi^2 B_3 \delta_{13} \delta_{14} \right),$$  (4.12)

where $B_3$ is a numerical constant. We shall determine the function $f(x_{34}^2)$ by the trick of comparing the left side of (4.12) with a similar expression in which the logarithm with complicated argument is replaced by $\ln^2(x^2 + \varepsilon^2)/\varepsilon^2$.

It can be checked that the difference between the two expressions amounts, in the small $\varepsilon$ limit, to a delta function whose constant coefficient $C_j$ is given by

$$\pi^2 C_j = \int d^4 x \frac{\ln^2(x^2 + 2\varepsilon^2 - |x|\sqrt{x^2 + 4\varepsilon^2}) / 2\varepsilon^2 - \ln^2(x^2 + \varepsilon^2)/\varepsilon^2}{x^2(x^2 + \varepsilon^2)}.$$  (4.13)
Thus, we have

\[
-\frac{g^4}{\pi^2} \delta_{12} \frac{1}{x_{14}^2 + \varepsilon^2} \left( \frac{1}{x_{13}^2 + \varepsilon^2} \right) \ln^2 \left( \frac{x_{34}^2 + 2\varepsilon^2 - |x_{34}| \sqrt{x_{34}^2 + 4\varepsilon^2}}{x_{34}^2} \right) / 2\varepsilon^2 =
\]

\[
4g^4 \delta_{12} \delta_{13} \frac{\ln^2(x_{34}^2 + \varepsilon^2)/\varepsilon^2}{x_{34}^2(x_{34}^2 + \varepsilon^2)} + 4g^4 \pi^2 (B_j + C_j) \delta_{12} \delta_{13} \delta_{14}.
\]  

(4.14)

We introduce \( M \) inside the logarithm,

\[
4g^4 \delta_{12} \delta_{13} \frac{\ln^2(x_{34}^2 + \varepsilon^2)/\varepsilon^2}{x_{34}^2(x_{34}^2 + \varepsilon^2)} =
\]

\[
4g^4 \delta_{12} \delta_{13} \frac{\ln^2(x_{34}^2 + \varepsilon^2) M^2 - 2 \ln(x_{34}^2 + \varepsilon^2) M^2 \ln \varepsilon^2 M^2 + \ln^2 \varepsilon^2 M^2}{x_{34}^2(x_{34}^2 + \varepsilon^2)}. \quad (4.15)
\]

Notice that the second term of the R.H.S. is another piece of \( g \ln \varepsilon^2 M^2 \Gamma_h^b(x, \varepsilon) \). The third term cancels with the second term of (4.11). Finally, we use (A.11b) to regularize the first term. These results are collected in the amplitude of Table 2, where \( b_j = C_j + B_j \).

We now turn to graph \( k \), whose bare amplitude reads

\[
\Gamma_h^b(x, \varepsilon) =
\]

\[
\frac{4g^4}{\pi^2} \delta_{12} \int d^4u \frac{1}{[(x_1 - u)^2 + \varepsilon^2]^2 [x_{34}^2 + \varepsilon^2] [(x_4 - u)^2 + \varepsilon^2]} \cdot \frac{1}{[x_{34}^2 + \varepsilon^2]} + 5 - \text{perms}. \quad (4.16)
\]

This is a convolution of bare amplitudes for the bubble and ice-cream cone subgraphs. However, the renormalized amplitude \([1]\) for this graph is also a convolution of the corresponding renormalized amplitudes interpreted using formal partial integration. Since our earlier results show that the limiting forms of the bare amplitudes for graphs \( f \) and \( h \) are equal to the renormalized amplitudes with \( \bullet \) or \( \ast \) derivatives plus singular terms, these expressions from Table 2 can simply be inserted in (4.16). One finds the renormalized convolution amplitude plus terms containing \( \delta^4(u - x_i) \) which render the \( d^4u \) integrals trivial, and this leads immediately to the result in Table 2.

We now study graph \( \ell \) starting from the bare amplitude

\[
\Gamma_h^b(x, \varepsilon) = \frac{8}{3\pi^4} g^4 \delta_{12} \delta_{13} L(x_{13}) + 2 - \text{perms}
\]  

(4.17)

where

\[
L(x) = \frac{1}{x^2 + \varepsilon^2} I(x)
\]

\[
I(x) = \int d^4u d^4v \frac{1}{(x - u)^2 + \varepsilon^2} \frac{1}{v^2 + \varepsilon^2} \left[ \frac{1}{(u - v)^2 + \varepsilon^2} \right]^3.
\]  

(4.18)

The new problem that arises here is that the integral \( I(x) \) is infrared divergent, as may be seen by fixing \( u - v \) and considering the integral over \( u + v \). The physical reason for this is that the subgraph \( c \) contains a mass shift which causes the subsequent integration over the massless propagators to diverge. One thus expects an infrared convergent result only when the mass shift counterterm insertion from (3.5) or (3.7) is subtracted. Our treatment will make this clear.

We insert the identity (3.2) in (4.18), and study separately the two integrals

\[
I_1(x) = -\frac{1}{32} \int d^4u d^4v \frac{1}{(x - u)^2 + \varepsilon^2} \frac{1}{v^2 + \varepsilon^2} \frac{\ln \left[ (u - v)^2 + \varepsilon^2 \right]/\varepsilon^2}{(u - v)^2}.
\]  

(4.19)
\[ I_2(x) = 3 \int d^4u \, d^4v \frac{1}{(x-u)^2 + \varepsilon^2} \frac{1}{v^2 + \varepsilon^2} \frac{\varepsilon^2}{[u-v]^2 + \varepsilon^2} \]  

(4.20)

The first convolution integral is infrared finite, essentially because the Fourier transform of the \( \Box \) log term contains a factor of \( p^2 \) which amply compensates for the \( 1/p^4 \) factors in the transforms of the other two propagators. Therefore we treat \( I_2(x) \) first.

It is not difficult to verify the following differential identity

\[ \frac{3\varepsilon^2}{(\varepsilon^2 + 4\varepsilon^2)^2} = \frac{\varepsilon^2}{8\varepsilon^2 (\varepsilon^2 + 4\varepsilon^2)^2} \pi^2 \delta(z) \]  

(4.21)

in which the \( \delta(z) \) singularity cancels between the two terms. We insert (4.21) with argument \( z \to u - v \) in (4.20). The contribution of the first term, called \( I_{21}(x) \), is infrared finite, again because there is a factor \( p^2 \) in momentum space from the \( \Box \) in (4.21). The infrared divergence is now isolated in the contribution \( I_{22}(x) \) of the last \( \delta(u - v) \) term in (4.21), and it is clear that this integral will be cancelled completely when the mass counterterm for subgraph \( c \) is inserted. See (3.5).

After partial integration of \( \Box \), and use of (A.6), we find that \( I_{21}(x) \) can be written as

\[ I_{21}(x) = -\int \frac{d^4u \, d^4v}{(x-u)^2 + \varepsilon^2} \frac{\varepsilon^4}{(v^2 + \varepsilon^2)^3} \frac{1}{((u-v)^2 + \varepsilon^2)^2} \]  

= \[ -\int \frac{d^4u \, d^4v}{(x-u)^2 + \varepsilon^2} \frac{1}{(v^2 + \varepsilon^2)^3} \frac{1}{((u-v)^2 + \varepsilon^2)^2} \]  

(4.22)

where we have scaled \( u \to \varepsilon u \) and \( v \to \varepsilon v \) in the last line. It is legitimate to take the \( \varepsilon \to 0 \) limit inside the integral because the residual integral is finite. This gives the simple result

\[ I_{21}(x) = -\frac{C}{x^2} \]  

(4.23)

with

\[ C = \int \frac{d^4u \, d^4v}{(v^2 + \varepsilon^2)^3(u-v)^2 ((u-v)^2 + 1)^2} \]  

= \[ \int \frac{d^4v}{(v^2 + \varepsilon^2)^3} \int \frac{d^4z}{x^2(z^2 + 1)^2} \]  

= \[ \frac{1}{2} \pi^4 \]  

(4.24)

We are really interested in the contribution of \( I_{21}(x) \) to \( L(x) \) in (4.18) which is given by the product

\[ L_{21}(x) = \frac{1}{x^2 + \varepsilon^2} I_{21}(x) \]  

(4.25)

and it is not correct to say that the limiting form of this product is obtained simply by inserting the limiting form of (4.23) of \( I_{21}(x) \). Instead we note the following general structure of \( I_{21}(x) \), namely

\[ I_{21}(x) = -\frac{1}{2} \pi^4 \frac{1}{x^2} F \left( \varepsilon^2 / x^2 \right) \]  

(4.26)

which follows simply from (4.22). Because of the result (4.23), we know that

\[ \lim_{x^2 \to \infty} F \left( \varepsilon^2 / x^2 \right) = 1 \]  

(4.27)

We can therefore write

\[ L_{21}(x) = \frac{1}{2} \pi^4 \frac{1}{x^2 + \varepsilon^2} \frac{1}{x^2} \frac{1}{F \left( \varepsilon^2 / x^2 \right)} - \frac{1}{2} \pi^4 \frac{1}{x^2 + \varepsilon^2} \frac{1}{x^2} \left[ F \left( \varepsilon^2 / x^2 \right) - 1 \right] \]  

(4.28)
which is an exact representation. Using (A.9.a) one can see that the limiting form of the first term is that of the bare amplitude for the bubble graph \( f \) plus a \( \delta(x) \) term. Using again a test function and scaling argument, one can show that the limiting contribution of the second term in (4.20) is also of the local form \( C' \delta(x) \) where \( C' \) is a numerical constant defined by the infrared convergent integral

\[
C' = -\frac{1}{2} \pi^4 \int \frac{d^4x}{x^2(x^2 + 1)} \left[ F(1/x^2) - 1 \right].
\]  

(4.29)

A more explicit form can be found using (4.22), but is not necessary.

The integral \( I_1(x) \) remains to be studied. We use the identity (3.13) for each propagator factor obtaining a representation with four terms

\[
I_1(x) = \int d^4 u d^4 v \left[ \frac{1}{(x-u)^2 v^2} - \frac{\varepsilon^2}{(x-u)^2 [\varepsilon^2 + (x-u)^2] v^2} - \frac{\varepsilon^2}{(x-u)^2 v^2 (v^2 + \varepsilon^2)} + \frac{\varepsilon^4}{(x-u)^2 [\varepsilon^2 + (x-u)^2] v^2 (v^2 + \varepsilon^2)} \right] \ln \left[ \frac{(u-v)^2 + \varepsilon^2}{(u-v)^2} \right].
\]

(4.30)

After partial integration of \( \square \), the first term trivially becomes

\[
I_{11}(x) = 16 \pi^4 \ln \left( \frac{x^2 + \varepsilon^2}{x^2} \right)/x^2.
\]

(4.31)

Each of the last 3 terms in (4.30) has the structure \( \frac{1}{x^2} F (\varepsilon^2/x^2) \) because of dimensional considerations. If we can show that \( F (\varepsilon^2/x^2) \) vanishes as \( x^2 \to \infty \), then the contribution of these terms to \( L(x) \) of (4.18) can be shown to be purely local by a scaling argument similar to that used for the second term of (4.28). We next discuss how to establish that \( F (\varepsilon^2/x^2) \) vanishes for each of the last 3 terms.

After partial integration of \( \square \), the second term can be written as

\[
I_{12}(x) = 4 \pi^2 \varepsilon^2 \int \frac{d^4 u}{(x-u)^2 [(x-u)^2 + \varepsilon^2]} \ln \left[ \frac{(u^2 + \varepsilon^2)}{\varepsilon^2} \right] u^2
\]

(4.32)

\[
= -16 \pi^2 \varepsilon^2 \int \frac{d^4 u}{(x-u)^2 [(x-u)^2 + \varepsilon^2] (u^2 + \varepsilon^2)^2}
\]

where (2.8) has been used. A crude estimate of the asymptotic behavior gives

\[
I_{12}(x) \sim \frac{\varepsilon^2}{x^4} \int \frac{d^4 u}{(u^2 + \varepsilon^2)^2}.
\]

(4.33)

This is not quite correct since the residual \( u \)-integral is logarithmically infrared divergent, but it means that the correct falloff is

\[
I_{12}(x) \sim \frac{\varepsilon^2}{x^4} \int \ln x^2/\varepsilon^2.
\]

The function \( F (\varepsilon^2/x^2) \) thus falls nearly a full power of \( \varepsilon^2/x^2 \) faster than necessary, so we are content with the heuristic argument above. The third integral in (4.30), namely \( I_{13}(x) \), can be shown to be equal to \( I_{12}(x) \) after partial integration of \( \square \) and change of variables, so \( I_{13}(x) \) also satisfies (4.33).

The fourth integral \( I_{14}(x) \) is more complicated, but it is not difficult to show that it falls rapidly as \( x \to \infty \). If we crudely extract the factor \( 1/x^4 \) and use (2.8) we find

\[
I_{14}(x) \sim \frac{\varepsilon^4}{x^4} \int d^4 u d^4 v \square, \quad \frac{1}{[u-v]^2 + \varepsilon^2} \frac{1}{v^2 (v^2 + \varepsilon^2)}
\]

(4.34)

\[
\sim \frac{\varepsilon^4}{x^4} \int d^4 u d^4 v \nabla_u \frac{1}{[u-v]^2 + \varepsilon^2} \cdot \nabla_v \left( \frac{1}{v^2 (v^2 + \varepsilon^2)} \right)
\]

\[
\sim \frac{\varepsilon^4}{x^4} \int d^4 u \nabla \left( \frac{1}{u^2 + \varepsilon^2} \right) \cdot d^4 v \nabla \left( \frac{1}{v^2 (v^2 + \varepsilon^2)} \right).
\]
The translation of variables $u' = u - v$ is permitted because the eight dimensional integral is convergent. Each of the two four-dimensional convergent integrals in the last line vanishes by symmetry, and this means that $I_{14}(x)$ actually falls faster than $1/x^4$. This is more than enough to conclude that its contribution to $L(x)$ is purely local as $\varepsilon \to 0$.

The total contribution of $I_1(x)$ to $L(x)$ is therefore

$$L_1(x) = 16\pi^4 \frac{\ln \left( x^2 + \varepsilon^2 \right) / \varepsilon^2}{x^4 \left( x^2 + \varepsilon^2 \right)} + C'' \delta(x)$$

(4.35)

where the first term comes from (4.31). We introduce the scale $M$ in the logarithm and apply the differential identity (A.11a). Similarly (A.9a) and (A.10a) are applied to the first term of (4.28). The limiting form of the bare amplitude $\Gamma_b^b(x, \varepsilon)$ given in Table 2 is the contribution of these two terms plus the infrared divergent mass counterterm integral $I_{22}(x)$ discussed below (4.21) and a local triple $\delta$-term.

The bare amplitude of graph $m$ is

$$\Gamma^b_m(x_i, \varepsilon) = \frac{4g^4}{\pi^4} \frac{1}{(x^2 + \varepsilon^2)^2} \frac{1}{(x^2 + \varepsilon^2)^2} \frac{1}{x_{13}^2 + \varepsilon^2} \frac{1}{x_{24}^2 + \varepsilon^2} + 5\text{-perms}.$$  

(4.36)

We now use (2.9) for the bubble subgraph amplitudes which leads to

$$\Gamma^b_m(x_i) = \frac{9}{4\pi^4} \ln \left( x_{12}^2 + \varepsilon^2 \right) M^2 \ln \left( x_{34}^2 + \varepsilon^2 \right) M^2 \frac{1}{x_{13}^2 + \varepsilon^2} \frac{1}{x_{24}^2 + \varepsilon^2} + 5\text{-perms} \nonumber + g \ln \varepsilon^2 M^2 \Gamma_b^b(x_i, \varepsilon) - g^2 \ln \varepsilon^2 M^2 \Gamma_b^b(x_i, \varepsilon).$$

(4.37)

Using (3.11) we obtain the antisymmetric derivative terms in Table 2, plus the expression

$$F = \frac{9}{4\pi^4} \left( \frac{1}{x_{13}^2 + \varepsilon^2} \right) \ln \left( x_{12}^2 + \varepsilon^2 \right) M^2 \frac{1}{x_{12}^2 + \varepsilon^2} \ln \left( x_{34}^2 + \varepsilon^2 \right) M^2 \frac{1}{x_{34}^2 + \varepsilon^2}.$$  

(4.38)

The limiting behavior of $F$ can be obtained by a procedure involving the use of the identity (3.14) in two factors of (4.38), namely those involving $\square$ of the cutoff propagators with arguments $x_{13}$ and $x_{14}$. A detailed discussion would be lengthy, and since there are no essentially new techniques involved, we give only a brief description. It is convenient to split the product of logarithms in (4.38) into terms proportional to $\ln^2 \varepsilon^2 M^2$, $\ln \varepsilon^2 M^2$, and $M$-independent terms, noting that factors such as $(\ln \left( x^2 + \varepsilon^2 \right) / \varepsilon^2)$ are non-singular as $x \to 0$. Each of the three terms above gives rise to four terms coming from the product of the two identities based on (3.14), and each term can be studied separately in a straightforward way. A minor difficulty occurs in the $\ln^2 \varepsilon^2 M^2$ term, because the product $1/ \left( x_{12}^2 x_{34}^2 \right)$ becomes ultraviolet singular when the arguments are identified in the $\delta(x_{13}) \delta(x_{24})$ term of the identities (3.14). To handle this one uses essentially (3.13), namely

$$\frac{1}{x_{12}^2} = \frac{1}{x_{12}^2 + \varepsilon^2} + \frac{\varepsilon^2}{x_{12}^2 \left( x_{12}^2 + \varepsilon^2 \right)}.$$  

The first term cuts off the ultraviolet singularity of the product $1/ \left( x_{12}^2 x_{34}^2 \right)$. The contribution of the second term to the $\ln^2 \varepsilon^2 M^2$ term of (4.38) is then studied, without use of (3.14), and can be shown to be of the local form $\delta_{12} \delta_{13} \delta_{14}$ as $\varepsilon \to 0$.

The result of the analysis above is the following formula for the $\varepsilon \to 0$ limit of $F$:

$$F \to 4g^4 \delta_{13} \delta_{24} \frac{\ln^2 \left( x_{12}^2 + \varepsilon^2 \right) M^2}{x_{12}^2 \left( x_{12}^2 + \varepsilon^2 \right)} - 4g^4 \pi^2 \delta_{13} \delta_{12} \delta_{14} \left( \ln \varepsilon^2 M^2 - 2B \ln \varepsilon^2 M^2 + B_m \right).$$

(4.39)

All triple $\delta$-terms were obtained by studying the limiting behavior of integrals with a test function $h \left( x_{12}, x_{13}, x_{14} \right)$ of the three independent variables in (4.38). The purely numerical coefficient $B_m$ comes from a combination of
integrals from various terms in the analysis above. One would expect that the singularity associated with the bubble subgraphs of $m$ should involve the amplitude of the ice-cream cone subgraph $h$. It is therefore important to verify, as we have done in detail, that the coefficient $B$ in (4.39) is given by the same integral, see (3.16) or (A.13), that appeared in the original analysis of graph $h$. Only then we will have a consistent cancellation of divergences as $\varepsilon \to 0$ by counterterms. A further check on (4.39) can be obtained by verifying that the limiting form of $\Gamma^b_m$ given in Table 2 satisfies (4.1). The net coefficient of $B \ln \varepsilon^2 M^2$ is easily seen to cancel.

Graph $n$ is moderately complicated. Its bare amplitude is

$$\Gamma_n^b(x_i, \varepsilon) = \frac{8g^4}{\pi^4} \delta_{12} \frac{1}{x_{14}^2 + \varepsilon^2} \frac{1}{\varepsilon} \cdot \int d^4 u \frac{1}{[(x_1 - u)^2 + \varepsilon^2][(x_4 - u)^2 + \varepsilon^2][(x_3 - u)^2 + \varepsilon^2]^2} + 11 - \text{perms}. \quad (4.40)$$

We focus attention on the integral, use (2.8), and follow the spirit of the corresponding steps in [1] to obtain the two terms

$$\begin{align*}
\Gamma_n^b(x_i, \varepsilon) &= -\frac{2g^4}{\pi^4} \delta_{12} \frac{1}{x_{14}^2 + \varepsilon^2} \frac{1}{\varepsilon} \cdot \int d^4 u \frac{1}{(u^2 + \varepsilon^2)((x_{14} - u)^2 + \varepsilon^2)(x_{13} - u)} \\
&\quad \ln \left(\frac{(x_{13} - u)^2 + \varepsilon^2}{(x_{14} - u)^2 + \varepsilon^2}\right) + \frac{8g^4}{\pi^2} \delta_{12} \frac{1}{x_{13}^2 + \varepsilon^2} \frac{1}{x_{14}^2 + \varepsilon^2} \ln\left(\frac{x_{34}^2 + \varepsilon^2}{x_{34}^2 + \varepsilon^2}\right) + 11 - \text{perms}
\end{align*} \quad (4.41)$$

as an exact result. We now use (3.11) to split the first term above into two parts. The term with the anti-symmetric derivative is already finite as $\varepsilon \to 0$, so we can use the techniques of [1] to obtain the corresponding term in the renormalized amplitude.

The second term resulting from use of (3.11) in (4.41) contains the difference of the two triangular structures (A.12c) for $n = 1$ and (A.12d). The $u$-integral cancels and one finds that the total contribution of the first term in (4.41) has the limiting form

$$-\frac{g^4}{\pi^4} \delta_{12} \partial_\mu \left(\frac{1}{x_{14}^2 x_{34}^2} \partial_\mu \ln \frac{x_{13}^2}{x_{14}^2} K(x_{13}, x_{14}) \right) + 11 - \text{perms} + g^2(4B + 4)\Gamma_j^b(x_i, \varepsilon) \quad (4.42)$$

where

$$K(x, y) = \int du (x - u)^2 (y - u)^2 \quad (4.43)$$

is the same function introduced in [1]. $K(x, y)$ has an ultraviolet finite Fourier transform.

The remaining task is to study the second term in (4.41). This is straightforward if we use (A.10c) and introduce the scale $M$ to obtain the exact result

$$-\frac{g^4}{\pi^4} \delta_{12} \frac{1}{x_{13}^2 + \varepsilon^2} \frac{1}{x_{14}^2 + \varepsilon^2} \cdot \frac{\ln(x_{34}^2 + \varepsilon^2)M^2 + 2 \ln(x_{34}^2 + \varepsilon^2)M^2}{M^2} + 11 - \text{perms}$$

$$+ 2g^2 \ln \varepsilon^2 M^2 \Gamma_j^b(x_i, \varepsilon) - 2g^2 \left(\ln \varepsilon^2 M^2 + 2 \ln \varepsilon^2 M^2\right) \Gamma_j^b(x_i, \varepsilon). \quad (4.44)$$

We use (3.11) again, obtaining an ultraviolet finite antisymmetric derivative term and a triangular structure, which is a combination of (A.12a – A.12c). We then use (A.11a) and (A.11b) and assemble our results to complete the amplitude given in Table 2.

We shall use a different technique to analyze graph $o$ in order to avoid an intractable interplay of internal integrals and $\varepsilon \to 0$ limits. Namely, we will obtain the renormalized amplitudes plus non-local divergent terms directly, but we will use (4.1) to obtain the coefficients of scale dependent local triple $\delta$ terms. This is a mathematically correct shortcut because the bare amplitude is independent of $M$. These considerations determine
the limiting form of amplitude except for a purely numerical multiple of \(\delta_{12}\delta_{13}\delta_{14}\) which is simply a change in the renormalization scheme of \([1]\).

The bare amplitude of graph \(o\) is

\[
\Gamma^b_o(x_i, \epsilon) = \frac{4g^4}{\pi^4} \delta_{12}\delta_{13}\delta_{14} f^b_o(x_{13})
\]

\[
f^b_o(x) = \int \frac{d^4u d^4v}{((u-v)^2 + \epsilon^2)^2} \frac{1}{u^2 + \epsilon^2} \frac{1}{v^2 + \epsilon^2} \frac{1}{(x-u)^2 + \epsilon^2} \frac{1}{(x-v)^2 + \epsilon^2}.
\]

We regulate the bubble subgraph factor using (2.9) and integrate by parts obtaining the three integrals

\[
f^b_o(x) = -\frac{1}{4} \int d^4u d^4v \left\{ \frac{1}{u^2 + \epsilon^2} \frac{1}{v^2 + \epsilon^2} \frac{1}{(x-u)^2 + \epsilon^2} \frac{1}{(x-v)^2 + \epsilon^2} \right. \\
+ \frac{1}{u^2 + \epsilon^2} \frac{1}{v^2 + \epsilon^2} \frac{1}{(x-u)^2 + \epsilon^2} \frac{1}{(x-v)^2 + \epsilon^2} \\
\left. + 2\partial_\mu u \frac{1}{u^2 + \epsilon^2} \frac{1}{v^2 + \epsilon^2} \partial_\mu v \frac{1}{(x-u)^2 + \epsilon^2} \frac{1}{(x-v)^2 + \epsilon^2} \right\} \ln \left(\frac{(u-v)^2 + \epsilon^2}{(u-v)^2}\right) \frac{M^2}{(u-v)^2}.
\]

plus a contribution to \(\Gamma^b_o(x_i, \epsilon)\) proportional to the bare amplitude of graph \(g\), namely \(g \ln \epsilon^2 M^2 \Gamma^b_g(x_i, \epsilon)\).

The first two terms of (4.46) have the same form, containing the triangular structure (A.12b). We are entitled to use formula (A.12b) inside the integrand provided that \(x \neq 0\). To account for a possible singularity in this limit, we include a \(\delta(x)\) term, with unknown coefficient \(F^b_0(\epsilon M)\). Such coefficient will be fixed at the end of the computation, requiring the amplitude to satisfy equation (4.1). So we have

\[
4\pi^2 \int d^4v \frac{1}{x^2 + \epsilon^2} \frac{1}{(x-v)^2 + \epsilon^2} \frac{1}{v^2 + \epsilon^2} \frac{\ln(v^2 + \epsilon^2)M^2}{v^2 + \epsilon^2} + 4\pi^4(\ln \epsilon^2 M^2 - B) \frac{1}{(x^2 + \epsilon^2)^2} + F^b_0(\epsilon M) \delta(x).
\]

We regulate the divergent term in the integral using (A.11a), integrate by parts and regulate again using (A.11a) and (A.11b). One thus obtains the following limiting form of the first two integrals in (4.46)

\[
-\frac{\pi^4}{12} \ln^3 x^2 M^2 + 6 \ln^2 x^2 M^2 + 12 \ln x^2 M^2 + \\
+ \pi^4 \left( - \ln^2 \epsilon^2 M^2 - 2 \ln \epsilon^2 M^2 + 2 B + 2 \right) \frac{1}{(x^2 + \epsilon^2)^2} + F^b_0(\epsilon M) \delta(x).
\]

We now study the last integral in (4.46). If \(x \neq 0\), the integral actually converges if the limit \(\epsilon \to 0\) is taken in the integrand, and the result

\[
4\pi^4 \left[ \frac{\ln x^2 M^2}{x^4} + 2 \frac{1}{x^4} \right]
\]

was obtained for this limit in \([1]\) by mathematically correct steps not requiring formal partial integration. The role of the cutoff \(\epsilon\) is therefore to determine the \(\delta(x)\) term in the result of the integral. We are therefore entitled to assume that the integral takes the form

\[
4\pi^4 \left[ \frac{\ln(x^2 + \epsilon^2)M^2}{x^2(x^2 + \epsilon^2)} + 2 \frac{1}{(x^2 + \epsilon^2)^2} \right] + F^b_0(\epsilon M) \delta(x)
\]

as \(\epsilon \to 0\). Any changes in the way the \(x^{-4}\) singularities are cut off results only in a change in the function \(F^b_0(\epsilon M)\). See, for example, (A.9a–A.9b). The first two terms are then regulated in the standard fashion, using
We now insert the results (4.48) and (4.50) in (4.46) and use (4.1) to determine the scale dependent part of $F_3^0(\varepsilon M) + F_3^0(\varepsilon M)$. In this way we obtain the complete limiting form of the cutoff amplitude, except for a $\delta_{12} \delta_{13} \delta_{14}$ term whose unknown numerical constant is called $b_0$.

The last graph needed, namely $p$, is primitively divergent. A special device was used to regulate it in Ref. [1] (see below) and it is particularly useful to see that the same renormalized amplitude can be obtained from the cutoff procedure. In this discussion below we will use some arguments from the treatment of Ref. [1] which do not involve the assumption of formal partial integration.

The bare amplitude for graph $p$ is

$$\Gamma_p^0(x_i, \varepsilon) = \frac{16g^4}{\pi^4} f(x_{12}, x_{13}, x_{14}, \varepsilon)$$

Because the graph is primitively divergent, it is sufficient to cut off only one of the six propagators to obtain an amplitude with a well-defined Fourier transform. We therefore add and subtract the product of the last five propagators without $\varepsilon$ and write

$$f(x, y, z, \varepsilon) = \frac{1}{x^2 + \varepsilon^2} \frac{1}{y^2 + \varepsilon^2} \frac{1}{z^2 + \varepsilon^2} \frac{1}{(x-y)^2 + \varepsilon^2} \frac{1}{(y-z)^2 + \varepsilon^2} \frac{1}{(z-x)^2 + \varepsilon^2} + r(x, y, z, \varepsilon).$$

It is easy to show by scaling that the limiting form of the remainder term $r(x, y, z, \varepsilon)$ is $C \delta(x) \delta(y) \delta(z)$ where $C$ is given by an integral which is infrared convergent because the integrand is a difference of two terms with the same leading infrared behavior. We drop this term henceforth.

We now write

$$f(x, y, z, \varepsilon) = \frac{1}{(x^2 + \varepsilon^2)^2} \frac{1}{y^2 z^2 (x-y)^2 (y-z)^2 (z-x)^2}$$

where we have imitated the special device of Ref. [1] in which the degree of singularity of the first propagator was artificially increased. We drop the explicit $\varepsilon^2$ term in the numerator of (4.53) because it is also shown easily to contribute a finite triple $\delta$-term as $\varepsilon \to 0$. We now use (2.8) and study

$$f(x, y, z, \varepsilon) = -\frac{1}{4} \left[ \ln \left( \frac{x^2 + \varepsilon^2}{x^2} \right) \right] \frac{x^2}{y^2 z^2 (x-y)^2 (y-z)^2 (z-x)^2}.$$  

The term in brackets is regular as $x \to 0$, so that the box operator can be integrated by parts without surface terms in integrals of (4.54) with smooth test functions such as the Fourier transform studied in Ref. [1]. We indicate this partial integration with $\hat{\Box}$, and split the argument of the log to obtain

$$f(x, y, z, \varepsilon) = -\frac{1}{4} \hat{\Box} \left[ \ln \left( \frac{x^2 + \varepsilon^2}{x^2} \right) M^2 - \ln \varepsilon^2 M^2 \right] \frac{x^2}{y^2 z^2 (z-y)^2 (y-z)^2 (z-x)^2}.$$  

If we replace $\ln \left( \frac{x^2 + \varepsilon^2}{x^2} \right) M^2 \to \ln x^2 M^2$ in the first term, the result is just the renormalized amplitude of Ref. [1], which was shown there to give ultraviolet convergent integrals with test functions. Note that in these integrals the derivatives in $\hat{\Box}$ are applied both to the test function and the second factor in (4.55). The replacement made above can be justified by studying integrals in which the difference

$$\frac{\ln \left( \frac{x^2 + \varepsilon^2}{x^2} \right) M^2 - \ln x^2 M^2}{x^2} = \frac{\ln \left( 1 + \varepsilon^2/x^2 \right)}{x^2}$$  

is inserted in (4.55). Scaling arguments show that such integrals vanish as $\varepsilon \to 0$, because of the structure found in Ref. [1] for the $y-z$ subintegrals as $x \to 0$ and because the integrand vanishes faster as $x \to \infty$ than that of (4.55) itself.
In the second term of (4.55), we reverse the partial integration and pick up a
surface term which is exactly that obtained in the rigorous derivation of (2.10) from
Green’s identity. The second term is then
\[
-\pi^2 \delta(x) \lim_{x \to 0} \frac{x^2}{g^2 z^2 (y - z)^2 (z - x)^2} = -6\pi^4 \zeta(3) \delta(x) \delta(y) \delta(z)
\]
(4.57)
where \(\zeta(s)\) is the Riemann \(\zeta\)-function as found from a study of the limit as \(x \to 0\) in Ref. [1].

The result of this analysis is given in Table 2 in which we have added \(-96\pi^2 g^4 b_p \delta(x) \delta(y) \delta(z)\) to account for the finite triple \(\delta\)-terms dropped above.

**Table 2.** The limiting form of bare amplitudes for graphs contributing to the 1PI 4-point function. Subscripts denote the graphs shown in Fig. 2. The first term in each entry is the renormalized amplitude obtained in [1], including the number of permutations of external points \(x_i\) required to obtain the full contribution of a given graph. Cutoff dependent terms in the limit of small \(\varepsilon\) are then given. The constant \(B\) is given in (A.13). Numerical constants \(b_t, b_m, \ldots\), in 3-loop graphs can be expressed as similar integrals, but their specific form is not required.

\[
\Gamma_{\epsilon}(x_i) = -16\pi^2 g^2 \delta_{12} \delta_{13} \delta_{14}
\]

\[
\Gamma^b_j(x_i, \varepsilon) \to -2g^2 \delta_{12} \delta_{34} \bullet \frac{\ln x_{13}^2 M^2}{x_{13}^2} + 2\text{--perms}
\]

\[
-24\pi^2 g^2 \ln \varepsilon^2 M^2 \delta_{12} \delta_{13} \delta_{14}
\]

\[
\Gamma^b_g(x_i, \varepsilon) \to g^3 \delta_{12} \delta_{4}(x_{34}) \bullet \frac{\ln x_{13}^2 M^2}{x_{13}^2} + 2\text{--perms}
\]

\[
+g \ln \varepsilon^2 M^2 \Gamma_j^b(x_i, \varepsilon)
\]

\[
+12\pi^2 g^3 \ln \varepsilon^2 M^2 \delta_{12} \delta_{13} \delta_{14}
\]

\[
\Gamma^b_h(x_i, \varepsilon) \to \frac{2g^3}{\pi^2} \left[ \frac{\pi^2}{2} \delta_{12} \delta_{13} \bullet \frac{\ln x_{34}^2 M^2 + 2 \ln x_{34}^2 M^2}{x_{34}^2} \right]
\]

\[
+\delta_{12} \frac{\partial}{\partial x_{3}^2} \left( \frac{1}{x_{14}^2 x_{34}^2} \delta_{12} \frac{\partial}{\partial x_{3}^2} \ln x_{34}^2 M^2 \right) \right] + 5\text{--perms}
\]

\[
+2g \ln \varepsilon^2 M^2 \Gamma_j^b(x_i, \varepsilon)
\]

\[
+48\pi^2 g^3 \left( \frac{1}{2} \ln \varepsilon^2 M^2 + \ln \varepsilon^2 M^2 - 1 - B \right) \delta_{12} \delta_{13} \delta_{14}
\]

\[
\Gamma^b_i(x_i, \varepsilon) \to -g^4 \delta_{12} \delta_{34} \bullet \frac{\ln x_{13}^2 M^2}{x_{13}^2} - 16\pi^2 \left( 1 - \zeta(3) \right) \delta_{13} + 2\text{--perms.})
\]

\[
+\frac{3g}{2} \ln \varepsilon^2 M^2 \Gamma_j^b(x_i, \varepsilon) - \frac{3g^2}{4} \ln \varepsilon^2 M^2 \Gamma_f^b(x_i, \varepsilon)
\]
\[ -6\pi^2 g^4 \ln^3 \varepsilon^2 M^2 \delta_{12} \delta_{13} \delta_{14} \]

\[ \Gamma_j^b(x_i, \varepsilon) \to \frac{g^4}{\pi^2} \delta_{12} \left[ \frac{\partial}{\partial x_3^\mu} \left( \frac{1}{x_{13}^2 x_{14}^2} \frac{\partial}{\partial x_3^\mu} \ln x_{34}^2 M^2 \right) \right] + 5\text{−perms} \]

\[ + g \ln \varepsilon^2 M^2 \Gamma_h^b(x_i, \varepsilon) - g^2 \ln \varepsilon^2 M^2 \Gamma_j^b(x_i, \varepsilon) \]

\[ - \pi^2 g^4 \left( 8 \ln^3 \varepsilon^2 M^2 + 24 \ln^2 \varepsilon^2 M^2 - 48 B \ln \varepsilon^2 M^2 - 12 - 6 b_j \right) \delta_{12} \delta_{13} \delta_{14} \]

\[ \Gamma_k^b(x_i, \varepsilon) \to \frac{g^4}{4\pi^2} \delta_{12} \left[ \int d^4 u \frac{\ln(u - x_1)^2 M^2}{(u - x_1)^2} \right] \]

\[ + \frac{\pi^2}{2} \delta^4(u - x_3) \ln^2(u - x_4)^2 M^2 + 2 \ln(u - x_4)^2 M^2 \]

\[ + g \ln \varepsilon^2 M^2 \Gamma_h^b(x_i, \varepsilon) + 2 g \ln \varepsilon^2 M^2 \Gamma_j^b(x_i, \varepsilon) \]

\[ - g^2 \left( \frac{3}{2} \ln \varepsilon^2 M^2 + \ln \varepsilon^2 M^2 - 1 - B \right) \Gamma_f^b(x_i, \varepsilon) \]

\[ - 24 \pi^2 g^4 \left( \frac{\ln \varepsilon^2 M^2}{2} + \ln \varepsilon^2 M^2 - \ln \varepsilon^2 M^2 - B \ln \varepsilon^2 M^2 \right) \delta_{12} \delta_{13} \delta_{14} \]

\[ \Gamma_l^b(x_i, \varepsilon) \to \frac{g^4}{6} \delta_{12} \delta_{34} \frac{\ln^2 x_{13}^2 M^2 + 2 \ln x_{13}^2 M^2}{x_{13}^2} + 2\text{−perms} \]

\[ + \frac{g^2}{6} (\ln \varepsilon^2 M^2 - 1) \Gamma_j^b(x_i, \varepsilon) \]

\[ - \frac{4 g^4}{3 \pi^2 \varepsilon^2} \delta_{12} \delta_{34} \int d^4 u \frac{1}{x_1^2 - u^2(x_4 - u)^2} + 2\text{−perms} \]

\[ + 2 \pi^2 g^4 \left( \ln \varepsilon^2 M^2 + 2 \ln \varepsilon^2 M^2 - 2 + b_j \right) \delta_{12} \delta_{13} \delta_{14} \]

\[ \Gamma_m^b(x_i, \varepsilon) \to \frac{g^4}{4\pi^2} \left[ \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_3} \left( \frac{1}{x_{24}^2} \frac{\partial}{\partial x_2} \ln x_{12}^2 M^2 \right) \right] \left( \frac{1}{x_{13}^2} \frac{\partial}{\partial x_3} \ln x_{34}^2 M^2 \right) \]

\[ - 4 \pi^2 \delta_{24} \frac{\ln x_{14}^2 M^2}{x_{14}^2} \frac{\partial}{\partial x_3} \left( \frac{1}{x_{13}^2} \frac{\partial}{\partial x_3} \ln x_{34}^2 M^2 \right) \]

\[ - 4 \pi^2 \delta_{13} \frac{\ln x_{34}^2 M^2}{x_{34}^2} \frac{\partial}{\partial x_2} \left( \frac{1}{x_{24}^2} \frac{\partial}{\partial x_2} \ln x_{12}^2 M^2 \right) \]
unitarity. After it is finished we can all go to the beach.

cancellation is the nontrivial check of the consistency of the entire procedure and the keystone of the proof of
generate non-local graphs which must cancel the non-local divergent terms in the bare amplitudes. This last
functions, directly determine counterterms which are added to the Lagrangian. The counterterm vertices in turn
much the same way as in any other regularization method, we now show that the divergent terms can be cancelled

5. Counterterms and Unitarity

In the previous section, we computed the bare amplitudes for 2- and 4-point functions. These amplitudes consist of the renormalized amplitudes of [1] plus terms which diverge as the ultraviolet cutoff $\varepsilon$ goes to 0. In much the same way as in any other regularization method, we now show that the divergent terms can be cancelled by local counterterms in the Lagrangian. Specifically, the purely local divergent terms, those involving only delta-functions, directly determine counterterms which are added to the Lagrangian. The counterterm vertices in turn generate non-local graphs which must cancel the non-local divergent terms in the bare amplitudes. This last cancellation is the nontrivial check of the consistency of the entire procedure and the keystone of the proof of unitarity. After it is finished we can all go to the beach.

Of course renormalization conditions are necessarily involved in the process of determining counterterms, and we implicitly use the conditions of [1]. The essential role of these conditions is to fix the finite parts of the divergent terms, and the consistent cancellation of these by the finite parts of the counterterms below is crucial to unitarity. We remind the reader that in the standard form of the renormalization procedure used in this section, the coupling $g$ which appears in the Tables is the “physical” coupling within the renormalization scheme of [1].
The simplest illustration of this program is the 1-loop renormalization of the coupling already discussed in Section 2. There we saw how the local divergent term in the bare amplitude (2.14) of graph \( \Gamma \) is cancelled by a counterterm in the Lagrangian obtained from the coupling shift (2.15).

Let us now study the case of the 2-point function. Its total bare amplitude is found adding the contributions of graphs a, c, and d from Table 1. The local divergences are cancelled by the following counterterms:

1. A wave function renormalization,
   \[ -\square \delta(x) (Z_\phi - 1) = -\square \delta(x) \left[ -\frac{g^2}{12} \left( 1 - \ln \frac{\varepsilon^2 M^2}{\Lambda^2} \right) \right. \]
   \[ \left. + \frac{g^3}{8} \left( \ln^2 \varepsilon^2 M^2 + \ln \varepsilon^2 M^2 + \frac{2D_2}{\pi^2} \right) \right], \quad (5.1) \]

2. And a perturbative mass counterterm,
   \[ \delta m^2 Z_\phi \delta(x) = \delta(x) \frac{1}{\varepsilon^2} \left[ \frac{1}{3} \frac{g^2}{\varepsilon^2} - \frac{1}{\pi^2 \varepsilon^2} \left( \pi^2 \ln \varepsilon^2 M^2 - 2D_1 \right) \right]. \quad (5.2) \]

Finally, the non-local divergence of graph d,
   \[ 3g \ln \varepsilon^2 M^2 \Gamma_b(x, \varepsilon) \]
   cancels when we take into account in the computation of c, the one loop modification of the coupling (2.15). We have, thus, shown how counterterms can eliminate all divergences, local as well as non-local, in the two point function. We recover the renormalized amplitude of [1].

The same can be done for the four point function at higher loops. The local triple-delta terms are easily absorbed into a redefinition of the vertex. One includes, then, all the new counterterm vertices, that is coupling, kinetic and mass corrections, in the computation of the amplitude and checks that the non-local divergences cancel. Let us stress the non-triviality of such cancellation, which is particularly dramatic for the \( \Gamma_f \) terms at the three loop level. Here, the sum of the 8 3-loop graphs \((i-p)\) contains the term
   \[ -g^2 \left( \frac{27}{4} \ln^2 \varepsilon^2 M^2 + \frac{35}{6} \ln \varepsilon^2 M^2 - \frac{35}{6} - 6B \right) \Gamma_f(x, \varepsilon) \]
   and this is neatly cancelled by the counterterm vertices from both 1- and 2-loop coupling renormalization inserted in graph \( \Gamma \). In the end, we recover the result of [1].

These results can be summarized by adding the counterterms to the Lagrangian (2.1). The result is the so-called bare Lagrangian,

\[ L_{bare} = \frac{1}{2} Z_\phi (\partial_\mu \phi)^2 + \frac{1}{2} Z_\phi \delta m^2 \phi^2 + \frac{16\pi^2}{4!} g Z g Z_\phi^2 \phi^4, \quad (5.5) \]

with
\[ Z_g Z_\phi^2 = 1 - \frac{3}{2} \frac{g^2 \ln \varepsilon^2 M^2 + g^2 \left( \frac{9}{4} \ln^2 \varepsilon^2 M^2 + 3 \ln \varepsilon^2 M^2 - 3 - 3B \right)}{16} \]
\[ + \frac{1}{16} g^3 \left( -54 \ln^3 \varepsilon^2 M^2 - 178 \ln^2 \varepsilon^2 M^2 \right. \]
\[ \left. + (216B - 44 - 96\zeta(3)) \ln \varepsilon^2 M^2 + c \right) + O(g^4), \quad (5.6) \]

\[ Z_\phi = 1 - \frac{g^2}{12} \left( 1 - \ln \varepsilon^2 M^2 \right) - \frac{g^3}{8} \left( \ln^2 \varepsilon^2 M^2 + \ln \varepsilon^2 M^2 + \frac{2D_2}{\pi^2} \right) + O(g^4), \quad (5.7) \]
\[
\delta m^2 Z_\phi = \frac{1}{\varepsilon^2} \left[ \frac{1}{3} g^2 - \frac{1}{\pi^2} g^3 \left( \ln \varepsilon^2 M^2 - 2D_1 \right) \right] + O(g^4).
\] (5.8)

We can rewrite 5.5 defining the bare field \( \phi_0 = Z_\phi^{1/2} \phi \), the bare coupling \( g_0 = gZ_g \) and the bare mass \( m_0^2 = \delta m^2 \).

Then, the bare Lagrangian becomes
\[
\mathcal{L}_{\text{bare}} = \frac{1}{2} \left( \partial_\mu \phi_0 \right)^2 + \frac{1}{2} m_0 \phi_0^2 + \frac{16\pi^2}{4!} g_0 \phi_0^4.
\] (5.9)

The constant \( c \) in (5.6) is the sum of the various numerical constants in the \( g^4 \delta_{12} \delta_{13} \delta_{14} \) terms of the 3-loop graphs of Table 2. Its value would become relevant only if the calculations of this paper are extended to 4-loop order.

6. Renormalization Group Equations

The renormalizability of massive \( \phi^4 \) theory means that, in any correct regularization procedure with short-distance cutoff \( \varepsilon \) and any renormalization scheme with scale parameter \( M \), renormalized and bare 1PI \( n \)-point functions are related by
\[
\Gamma^{(n)}(x_i, g, m, M) = Z_{\phi}^\frac{n}{2} Z_{\phi}(g_0, M, \varepsilon) \Gamma^{(n)}_{\text{bare}}(x_i, g_0, m_0, \varepsilon).
\] (6.1)

The right-hand side has a finite limit as \( \varepsilon \to 0 \) with physical coupling \( g \) and mass \( m \) held fixed.

The fact that
\[
M \frac{\partial}{\partial M} \Gamma^{(n)}_{\text{bare}}(x_i, g_0, m_0, \varepsilon) \bigg|_{g_0, m_0, \varepsilon \text{ fixed}} = 0
\] (6.2)

leads directly [4] to the Callan-Symanzik equation
\[
\left[ M \frac{\partial}{\partial M} + \beta(g) \frac{\partial}{\partial g} - n \gamma(g) - \delta(g) m \frac{\partial}{\partial m} \right] \Gamma^{(n)}_{\text{ren}}(x_i, g, m, M) = 0
\] (6.3)

with
\[
\beta(g) = M \frac{\partial}{\partial M} g(g_0, M, \varepsilon) \bigg|_{g_0, \varepsilon \text{ fixed}}
\] (6.4)
\[
\gamma(g) = \frac{1}{2} M \frac{\partial}{\partial M} \ln Z_{\phi}(g_0, M, \varepsilon) \bigg|_{g_0, \varepsilon \text{ fixed}}
\] (6.5)
\[
\delta(g) = -\frac{M}{2m^2} \frac{\partial}{\partial M} m^2(g_0, m_0, M, \varepsilon).
\] (6.6)

The relations between bare and physical parameters, such as \( g_0 = gZ_g \), must be inverted to compute the functions \( \beta(g), \gamma(g), \delta(g) \).

The cancellation of divergent terms with counterterms in Section 5 establishes that the differential renormalization procedure is correct through three-loop order for \( m = 0 \), so that (6.1) holds up to three-loop order in the massless theory. One can think of the bare amplitudes as defined via (6.1) by substituting \( g(g_0) \), computed from (5.6)–(5.7), in the renormalized amplitudes of Ref. [1] and multiplying by \( Z_{\phi}^{-n/2} \). However, our computational procedure also provides quite directly the explicit form of the bare amplitudes. Specifically, the entries in Tables 1 and 2 are the result of a systematically cutoff computation with Lagrangian coupling \( g \), zero Lagrangian mass and unit normalization of propagators. Thus if we simply replace \( g \to g_0 \) in Table 1, we can interpret the entries there as \( \Gamma^{(n)}_{\text{bare}}(x_i, g_0, 0, \varepsilon) \), and one can verify directly from the table that the apparent \( M \) dependence of these
amplitudes cancels. To obtain \( \Gamma^{(n)}_{\text{bare}}(x_i, g_0, m, \varepsilon) \), one simply adds the mass insertion (5.2) or (5.8) rewritten in terms of the bare coupling as

\[
m_0^2 = \delta m^2 = \frac{1}{\varepsilon^2} \left( \frac{1}{3} g_0^2 - \frac{2 D_1}{\pi^2} g_0^3 \right) + \mathcal{O}(g_0^4) \,.
\]  

(6.7)

The sole effect of this is to cancel all quadratically divergent terms in the entries of the Tables, leaving amplitudes which clearly satisfy (6.2) because \( M \frac{\partial}{\partial M} \delta m^2 = 0 \).

The previous arguments establish the validity of the standard formulae (6.4)–(6.5) for the renormalization group functions through three-loop order for \( m = 0 \). It is a straightforward matter to use (5.6)–(5.7), with proper attention to the inversion of \( g \leftrightarrow g(0) \), and obtain

\[
\beta(g) = 3 g^2 - \frac{17}{3} g^3 + (31 + 12 \zeta(3)) g^4 + \mathcal{O}(g^5)
\]  

(6.8)

\[
\gamma(g) = \frac{1}{12} g^2 - \frac{3}{8} g^3 + \mathcal{O}(g^4) \,.
\]  

(6.9)

Our calculations have probed only the massless theory, and it is clear that the \( m \frac{\partial}{\partial m} \) term in (6.3) vanishes as \( m \to 0 \) because there are no infrared divergences. Therefore, we do not discuss \( \delta(g) \) here because it requires information about \( m \neq 0 \). The results for \( \beta(g) \), \( \gamma(g) \) found here in the standard framework of the renormalization group equations agree with those of the “experimental” approach taken in Ref. [1]. This provides another check that the differential renormalization procedure is correct.

7. An Alternate Cutoff Method

The cutoff method used in Sections 1–4, which is based on the damped propagator (1.2), is actually the second method we have applied to this problem. In our first approach, which was described briefly in the first article of Ref. [1], regularization of bare amplitudes was achieved by the exclusion of small balls of radius \( \varepsilon \) about short distance singularities. Integrals involving such cutoff bare amplitudes then converge, and the singular contributions as \( \varepsilon \to 0 \) are quite clearly related to the surface terms dropped in the partial integration rule of differential renormalization.

The systematic rules used in this regularization method were the following:

1) Each propagator connecting vertices \( x \) and \( y \) of a diagram is replaced by the cutoff propagator

\[
\Delta(x - y) = \frac{1}{4 \pi^2} \frac{1}{(x - y)^2} \to \Delta(x - y, \varepsilon) = \frac{1}{4 \pi^2} \frac{\Theta(|x - y| - \varepsilon)}{(x - y)^2}
\]  

(7.1)

where \( \Theta(z) \) is the step function.

2) For each pair of (internal or external) vertices \( x_i, x_j \) not connected by a propagator, the bare amplitude is multiplied by an additional cutoff factor \( \Theta(|x_i - x_j| - \varepsilon) \).

Calculations using this approach were generally simpler than in the current method because the differential identities of the Appendix of [1] could be used directly. Further, after partial integration, one finds \( \delta(|x_i - x_j| - \varepsilon) \) terms when derivatives act on the step function cutoff factors, and these effectively reproduce the surface terms which are the crucial issue. Complete results through 3-loop order were obtained, and we found that the cutoff amplitudes for each graph could be expressed as the renormalized amplitudes of [1], plus singular terms which could be consistently compensated by counterterms in the Lagrangian. The renormalization group functions \( \beta(g) \) and \( \gamma(g) \) were calculated from the cutoff dependence of \( Z_g \) and \( Z_\phi \), as in (6.4) and (6.5), again with results identical to those of [1].
Despite the successful result and relative ease of calculation, we now believe that this method does not support the conclusion that the renormalized amplitudes of [1] satisfy perturbative unitarity. To discuss this we first note that the cutoff propagator (7.1) has Fourier transform

\[ \Delta(p, \varepsilon) = \frac{1}{p^2 J_0(p\varepsilon)} \]  

(7.2)

where \( J_0(z) \) is the Bessel function which is analytic in its argument. Since the only singularity of \( \Delta(p, \varepsilon) \) is the standard \( 1/p^2 \) pole, the method would give a plausible argument for unitarity provided that calculations could be done using only the propagator cutoff of Rule 1) above.

In principle, the propagator cutoff is sufficient to make all required integrals converge, but it was technically too difficult to do many integrals in this way. Instead we adopted the procedure of performing subintegrals using Rule 1), but then substituting the limiting form of this result before studying further singularities of a graph which were cutoff by factors from Rule 2). These Rule 2) cutoff factors cannot be described as a modification of the Lagrangian which is Hermitean below some cutoff energy scale, and this raises more questions about unitarity.

In view of the above, one may wonder whether the result of the consistent counterterms mechanism found in this method was an accident or whether it encapsulates some truth. We think that the latter is correct, because our procedures, albeit somewhat sloppy, were used consistently. Subgraphs of a given graph were handled by the same steps as in their initial appearance in lower order.

Very recently, it has been shown that \( x \)-space dimensional regularization can be combined with differential identities so as to reproduce several of the lower order amplitudes of [1] plus local counterterms [5]. In higher order, this method could lead to a useful relation between the amplitudes of the differential renormalization and dimensional regularization procedures.

8. Concluding Remarks

We believe that the calculations of Secs. 2 – 4 have fulfilled their intended goals. Namely, a systematic real space cutoff method for \( \phi^4 \) theory has been used to show that through 3-loop order, the bare 2- and 4-point correlation functions can be expressed as the sum of the renormalized amplitudes of [1] plus a combination of singular and finite terms. This combination can be compensated by adding the traditional counterterms to the Lagrangian. Indirectly this demonstrates that the major heuristic rule of the differential renormalization procedure, namely formal partial integration, is consistent. Since the cutoff method is based on a damped propagator whose Fourier transform (1.4) consists of the usual pole plus a cut whose effects vanish as \( \varepsilon^2 \), our results also imply that differential renormalization obeys perturbative unitarity. Finally we have shown that the same renormalization group functions \( \beta(g) \) and \( \gamma(g) \) are obtained in the cutoff theory and in the method of [1], and this is an additional consistency check.

An important subsidiary purpose of our work was to convince skeptics that overlap divergences are correctly treated in differential renormalization. The results above do demonstrate this since all non-local divergences are exactly cancelled by the local counterterms added to the lagrangian. However, it may be useful to restate and amplify upon the common belief that overlap divergences are not a problem in real space calculations, because subdivergent regions remain distinct and can be regulated before the overall divergence of a graph is studied. Let us illustrate this in the case of the most conspicuous overlap graph in our work, the 3-loop cateye graph \( o \). We note that the treatment of this graph in [1] started with the expression

\[ \Gamma_o(x_i) = -\frac{4g^4}{\pi^4} \delta_{12} \delta_{34} f_o(x_{13}) + 2\text{-perms} \]

(8.1)

\[ f_o(x) = -\frac{1}{4} \int \frac{d^4u d^4v}{u^2 v^2 (x-u)^2 (x-v)^2} \ln(u-v)^2 M^2 \cdot \frac{\Box}{(u-v)^2} . \]
This expression can also be obtained from (4.45) by setting $\varepsilon = 0$ and using (1.1) to regulate the central bubble subgraph.

There are three subdivergent regions in this graph; namely the 4-dimensional region $u \sim v$, and the eight dimensional regions $u \sim v \sim 0$ and $u \sim v \sim x$. The first of these is already regulated by the use of (1.1) which is an identity for $u - v \neq 0$, and is defined at the singularity by the partial integration rule. This means that the divergent 1-bubble subgraph is treated exactly in the same way as the renormalized amplitude for graph $f$. After partial integration of $\Box_u$ in (8.1) one finds [1] two integrals with $\delta(u)$ and $\delta(x - u)$ factors together with a cross term, as one can see from (4.46) at $\varepsilon = 0$. The cross term is complicated but contains no subdivergences. Indeed the $\delta(u)$ and $\delta(x - u)$ factors give a partial localization of the 8-dimensional singular regions, and it is not difficult to see that the integrand in these regions is treated in [1] exactly as the product of the renormalized amplitude of the ice cream cone subgraph (specifically, the first term in the entry for $\Gamma_b^0$ in Table 2 with correct designation of variables) times the remaining non-singular propagator factors. The internal integrals $d^4 u d^4 v$ are then performed leading to the explicit form of $f_\circ(x)$ containing $(\ln x^2 M^2)^n/x^4$ overall singularities which are easily regulated. Independent of a systematic cutoff procedure, our confidence that overlap divergences are properly treated in differential renormalization is based on the property that the amplitude in subdivergent regions is regulated in the same way as the appropriate subgraph.

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Appendix

We include in this Appendix some technical results. Most of them have been used in the calculations of several of the graphs discussed in Sections 2 – 4 of the main text.

1. The convolution integral

\[ I_\varepsilon(x) = \int d^4u \frac{1}{[u^2 + \varepsilon^2]^2} \left( (x - u)^2 + \varepsilon^2 \right) \]

is required to evaluate graphs d, g, 1, and j. The integral can be done using the fact (see (2.12)) that the Fourier transform of \( \frac{1}{u^2 + \varepsilon^2} \) is

\[ \frac{2\pi}{2} K_0(p\varepsilon) \]

We can write

\[ I_\varepsilon(x) = \int d^4p \left( \frac{2\pi}{2} \right)^4 e^{-ip \cdot x} 4\pi^4 K_0^2(p\varepsilon) \]

This is an exact result. As \( \varepsilon \to 0 \) we obtain

\[ I_\varepsilon(x) \to -\frac{\pi^2}{4} \ln^2 \left( \frac{x^2 + \varepsilon^2}{\varepsilon^2} \right) \]

If we introduce the mass scale \( M \), this can be rewritten as

\[ I_\varepsilon(x) \to -\frac{\pi^2}{4} M^2 \ln^2 \left( \frac{x^2}{\varepsilon^2} \right) + \frac{\pi^2}{2} \ln \varepsilon^2 M^2 \frac{\ln x^2 M^2}{x^2} + \pi^4 \delta(x) \ln^2 \varepsilon^2 M^2 \]

This is the same result we would have obtained if the limiting form of the regulated bubble

\[ \frac{1}{[u^2 + \varepsilon^2]^2} \to \frac{1}{4} \ln x^2 M^2 \frac{\ln x^2 M^2}{x^2} - \pi^2 \ln \varepsilon^2 M^2 \delta(x) \]

were inserted in (A.1), and the integral computed using formal partial integration as in [1]. In other words, it is justified in this case to take the limit in the integrand. The reason for this appears to lie in (2.8) in which the cutoff bubble amplitude is expressed as \( \Box \) of a function which has a soft singularity as \( \varepsilon \to 0 \). The \( \Box \) operator can be transferred to the external variables in (A.1), leaving a function which is smooth enough that the \( \varepsilon \to 0 \) limit can be taken in the integrand. This is sufficient for the evaluation of graphs g and i. However, in graphs
d and j, where other singular factors multiply \( I_\epsilon(x) \), the more accurate form (A.2) is required to study the limit of the bare amplitudes.

2. Representations of the distributions \( \delta(x) \) and \( \Box \delta(x) \) appear throughout our work. The simplest example is the basic equation for the cutoff propagator,

\[
\Box \Delta(x, \epsilon) = \frac{1}{4\pi^2} \frac{1}{(x^2 + \epsilon^2)} = -\frac{2\epsilon^2}{\pi^2(x^2 + \epsilon^2)^3} .
\] (A.6)

Of course we expect the limiting relation

\[
\frac{2\epsilon^2}{\pi^2(x^2 + \epsilon^2)^3} \to \delta(x) .
\] (A.7)

To prove this we integrate this candidate \( \delta \)-function with a smooth test function \( f(x) \) which is damped at long distances. We write

\[
\int d^4x f(x) \frac{2\epsilon^2}{\pi^2(x^2 + \epsilon^2)^2} = \frac{2}{\pi^2} \int d^4 y f(\epsilon y) \frac{1}{(y^2 + 1)^3}
\]

\[
= \frac{2}{\pi^2} f(0) \int \frac{d^4 y}{(y^2 + 1)^3} + \frac{2}{\pi^2} \int d^4 y \frac{f(\epsilon y) - f(0)}{(y^2 + 1)^3} .
\]

It follows from Taylor’s theorem that \( f(\epsilon y) - f(0) \sim O(\epsilon) \) as \( \epsilon \to 0 \) so the limit of the second infrared convergent integral vanishes. The first integral is easy to evaluate,

\[
\int \frac{d^4 y}{(y^2 + 1)^3} = \int d^4 y \int_0^\infty \frac{y^3 dy}{(y^2 + 1)^3}
\]

\[
= 2\pi^2 \int_0^\infty \frac{y^3 dy}{(y^2 + 1)^3} = \frac{\pi^2}{2} .
\]

Other representations of distributions are

\[
\frac{3\epsilon^2}{(x^2 + \epsilon^2)^4} \to \frac{\pi^2}{2\epsilon^2} \delta(x) + \frac{\pi^2}{8} \Box \delta(x) \] (A.8a)

\[
\frac{\epsilon^2 \ln(x^2 + \epsilon^2) M^2}{x^2(x^2 + \epsilon^2)^2} \to \pi^2 (\ln \epsilon M^2 + 1) \delta(x) .
\] (A.8b)

When \( \Box \delta(x) \) is involved, it is necessary to expand the test function in a Taylor series through second order in order to extract the limiting form. Similar relations which hold as \( \epsilon \to 0 \) are

\[
\frac{1}{x^2(x^2 + \epsilon^2)} = \frac{1}{x^2 + \epsilon^2} - \pi^2 \delta(x)
\] (A.9a)

\[
\frac{\ln(x^2 + \epsilon^2)/\epsilon^2}{(x^2 + \epsilon^2)^2} = \frac{\ln(x^2 + \epsilon^2)/\epsilon^2}{x^2(x^2 + \epsilon^2)} - \pi^2 \delta(x)
\] (A.9b)

\[
\frac{\ln^n(x^2 + \epsilon^2)/\epsilon^2}{(x^2 + \epsilon^2)^2} = \frac{\ln^n(x^2 + \epsilon^2)/\epsilon^2}{x^2(x^2 + \epsilon^2)} - \pi^2 n! \delta(x)
\] (A.9c)

3. Differential Regularization identities for cutoff singular functions are useful throughout our calculations. These include the following identities,

\[
\frac{1}{(x^2 + \epsilon^2)^2} = -\frac{1}{4} \Box \ln(x^2 + \epsilon^2)/\epsilon^2
\] (A.10a)
\[
\frac{1}{(x^2 + \varepsilon^2)^3} = \frac{1}{32} \ln(x^2 + \varepsilon^2)/x^2 + \frac{3\varepsilon^2}{(x^2 + \varepsilon^2)^4} \quad \text{(A.10b)}
\]
\[
\frac{\ln(x^2 + \varepsilon^2)/x^2}{[x^2 + \varepsilon^2]^2} = \frac{1}{8} \frac{\ln^2(x^2 + \varepsilon^2)/x^2 + 2\ln(x^2 + \varepsilon^2)/x^2}{x^2} \quad \text{(A.10c)}
\]

As \( \varepsilon \to 0 \), we also have

\[
\frac{\ln(x^2 + \varepsilon^2)M^2}{x^2(x^2 + \varepsilon^2)} \to -\frac{1}{8} \frac{\ln^2(x^2 + \varepsilon^2)M^2 + 2\ln(x^2 + \varepsilon^2)M^2}{x^2} - \frac{\pi^2}{2} \frac{\ln^2 \varepsilon^2 M^2}{x^2} \text{ } \delta(x) \quad \text{(A.11a)}
\]
\[
\frac{\ln^2(x^2 + \varepsilon^2)M^2}{x^2(x^2 + \varepsilon^2)} \to -\frac{1}{12} \frac{\ln^3(x^2 + \varepsilon^2)M^2 + 3\ln(x^2 + \varepsilon^2)M^2 + 6\ln(x^2 + \varepsilon^2)M^2}{x^2} - \pi^2 \left( \frac{1}{3} \ln^2 \varepsilon^2 M^2 - \frac{1}{2} \right) \delta(x) . \quad \text{(A.11b)}
\]

4. Triangular structures have the schematic form,

\[
\left[ \text{a representation of } \delta(x - y) \right] \times \begin{bmatrix} \text{smooth } f(y) \end{bmatrix} \times \begin{bmatrix} \text{smooth } g(x) \end{bmatrix}
\]

Such structures first appear in the 2-loop graph \( \gamma \), and we also encounter them in many 3-loop graphs. The basic strategy to obtain their limiting form, which involves integration with a test function \( f(x, y) \) of two variables was discussed in connection with graph \( \gamma \). Using this strategy one obtains the limiting forms

\[
\left[ \frac{1}{(x - y)^2 + \varepsilon^2} \right] \frac{1}{x^2 + \varepsilon^2} \frac{1}{y^2} \to -4\pi^2 \delta(x - y) \frac{1}{y^2(y^2 + \varepsilon^2)} + 4\pi^4 \delta(x) \delta(y) \quad \text{(A.12a)}
\]
\[
\left[ \frac{1}{(x - y)^2 + \varepsilon^2} \right] \frac{1}{y^2 + \varepsilon^2} \frac{1}{y^2} \left( \frac{\ln(y^2 + \varepsilon^2)}{y^2 + \varepsilon^2} \right) \to -4\pi^2 \delta(x - y) \frac{\ln(y^2 + \varepsilon^2)}{y^2(y^2 + \varepsilon^2)} + 4\pi^4 (\ln \varepsilon^2 M^2 - B) \delta(x) \delta(y) \quad \text{(A.12b)}
\]
\[
\left[ \frac{1}{(x - y)^2 + \varepsilon^2} \right] \frac{1}{x^2 + \varepsilon^2} \frac{1}{y^2} \ln \left( \frac{y^2 + \varepsilon^2}{x^2 + \varepsilon^2} \right) \to -4\pi^2 \delta(x - y) \ln \left( \frac{y^2 + \varepsilon^2}{x^2 + \varepsilon^2} \right) \frac{1}{y^2(y^2 + \varepsilon^2)} - 4\pi^4 B_\varepsilon \delta(x) \delta(y) \quad \text{(A.12c)}
\]
\[
\left[ \frac{1}{(x - y)^2 + \varepsilon^2} \right] \frac{1}{x^2 + \varepsilon^2} \frac{1}{y^2} \ln \left( \frac{y^2 + \varepsilon^2}{x^2 + \varepsilon^2} \right) \to -4\pi^2 \delta(x - y) \ln \left( \frac{y^2 + \varepsilon^2}{x^2 + \varepsilon^2} \right) \frac{1}{y^2(y^2 + \varepsilon^2)} + 4\pi^4 \delta(x) \delta(y) . \quad \text{(A.12d)}
\]

The coefficient of \( \delta(x) \delta(y) \) in (A.12b) is simply the double integral of the last term in (3.16). This leads to the following expression for \( B \),

\[
B = \frac{1}{4\pi^4} \int d^4x d^4y \left[ \frac{1}{(x - y)^2 + 1} \right] \frac{1}{x^2(x^2 + 1)} \frac{\ln(y^2 + 1)}{y^2} \quad \text{(A.13)}
\]

\[
= -\frac{1}{\pi^4} \int d^4x d^4y \frac{1}{(x - y)^2 + 1} \frac{1}{x^2(x^2 + 1)} \left[ \frac{1}{y^2 + 1} \right]^2 \quad \text{(A.13)}
\]

\[
= \frac{2\pi^2}{9} - \frac{1}{3} \psi \left( \frac{1}{3} \right) \approx -1.171953 . \quad \text{(A.13)}
\]

One can go from the first to the second line of (A.13) by integrating \( \square \) by parts back onto the log. Finally, this last integral is computed using the standard Feynman parametrisation. Similar expressions can be found for the numerical constants \( B_\varepsilon \). For example, the coefficients of the triple delta terms can be read off by integrating the difference of both sides of the equalities (A.12).