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Inverse cascade and magnetic vortices in kinetic Alfvén-wave turbulence

G. Miloshevich, D. Laveder, T. Passot, P.L. Sulem

Université Côte d’Azur, CNRS, Observatoire de la Côte d’Azur, Laboratoire J.L. Lagrange, Boulevard de l’Observatoire, CS 34229, 06304 Nice Cedex 4, France

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A Hamiltonian two-field gyrofluid model for kinetic Alfvén waves (KAWs) in a magnetized electron-proton plasma, retaining ion finite-Larmor-radius corrections and parallel magnetic field fluctuations, is used to study the inverse cascades that develop when turbulence is randomly driven at sub-ion scales. In the directions perpendicular to the ambient field, the dynamics of the cascade turns out to be nonlocal and sensitive to the ratio $\chi_f$ of the wave period to the characteristic nonlinear time at the driving scale. When $\chi_f$ is not too large, decay instability can develop, enhancing for a while inverse transfers. The balanced state, obtained at early time when the two counter-propagating waves are equally driven, also becomes unstable for small $\chi_f$, leading to an inverse cascade restricted to a limited spectral range. For $\beta_e$ smaller than a few units, the cascade slows down when reaching the low-dispersion spectral range. For higher $\beta_e$, the ratio of the KAW to the Alfvén frequencies displays a local minimum. At this transverse wavenumber, a condensate is formed, associated with the development of ion-scale magnetic vortices. The cascade towards larger scales is then inhibited. In the parallel direction, depending on the parameters, a local inverse cascade can develop, leading to elongated vortices.

1. Introduction

Large-scale magnetic structures are commonly observed in astrophysical media and have also been shown to stabilize the H mode in tokamaks [Solano et al. 2010]. The question then arises as to what are the physical processes contributing to their formation and prescribing their characteristic size. In astrophysics, among the potential mechanisms, turbulent inverse cascades have been suggested. In this context, special attention was paid in the literature to the cascade of magnetic helicity in the framework of incompressible magnetohydrodynamics (MHD) in the absence of an ambient field [Frisch et al. 1975; Pouquet et al. 1976; Meneguzzi et al. 1981; Alexakis et al. 2006; Müller et al. 2012; Linkmann & Dallas 2016, 2017; Pouquet et al. 2019]. Extension to supersonic MHD with Mach number that goes up to unity [Balsara & Pouquet 1999; Brandenburg 2001], as well as the case of incompressible Hall-MHD [Pouquet et al. 2020] were also studied. The cascade is however prevented by the presence of an ambient field, due to the non-conservation of the magnetic helicity based on the fluctuations, and to the lack of gauge-invariance of the conserved generalized magnetic helicity that can be defined in this regime [Matthaeus & Goldstein 1982; Stribling et al. 1994; Brandenburg & Matthaeus 2004]. Conservation of the magnetic helicity together with the existence of an inverse cascade [Kim & Cho 2015] are nevertheless recovered in electron magnetohydrodynamics (EMHD) that describes whistler waves at electron scales where the ion motion is negligible and the dispersion associated with the Hall effect is significant. One is thus naturally led to wonder about the possibility of the existence of an inverse cascade for kinetic Alfvén waves (KAWs) at sub-ion scales.
A question arises concerning the possible drivers for such inverse cascades. By bypassing the classical local direct cascade, energy could be directly injected at small scales via nonlocal interactions mediated by magnetic reconnection occurring in thin ion-scale current sheets. It has recently been proposed that these reconnection events could generate an inverse flux toward larger scales (Franci et al. 2017). Another example of possible mechanism is discussed in the context of Alfvénic turbulence in the distant ion foreshock region where observation of an inverse cascade was reported, related to the existence of nonlinear parametric instabilities generated by upstream accelerated protons reflected on the bow shock (He et al. 2019).

In addition to providing a mechanism for the formation of large-scale structures in fluids and plasmas, inverse cascades reveal various interesting phenomena, like critical transitions that are observed in split-cascade configurations when the relevant dimensionless parameter is varied (Alexakis & Biferale 2018). This includes transitions from inverse to forward cascade of energy in thin-layer turbulence (Benavides & Alexakis 2017) and transitions from MHD to fluid turbulence, when the relative strength of the magnetic forcing parameter is varied (Seshasayanan et al. 2014). Furthermore, there are examples of such criticality in rotating and stratified flows where helicity conservation can be broken when dynamical parameters such as Rossby and Froude numbers are varied (Marino et al. 2013).

The dynamics of a strongly magnetized plasma characterized by small perturbations of a homogeneous equilibrium is appropriately described by the gyrokinetic formalism, from which reduced gyrofluid models of various complexity can be derived. Such models can provide a uniform description covering a spectral range extending from MHD to electron scales. They capture the transition from Alfvén waves to KAWs, which are known to play a dominant role in the solar wind at MHD (Belcher & Davis 1971) and sub-ion (Alexandrova et al. 2009; Sahraoui et al. 2010; Salem et al. 2012; Podesta 2013) scales, respectively. Their nonlinear dynamics is isolated in the description provided by a reduced two-field gyrofluid model discussed in Passot et al. (2018); Passot & Sulem (2019); Miloshevich et al. (2019). In particular, the model enables the study of imbalanced turbulence, characterized by an excess of the energy carried by one of the two types of counter-propagating waves. Imbalanced Alfvénic turbulence is ubiquitous in the solar wind (Tu et al. 1989; Lucek & Balogh 1998; Wicks et al. 2013), with the degree of imbalance dependent on the type of wind (Tu et al. 1990; Bruno et al. 2014; Bruno et al. 2017; D’Amicis et al. 2019) and of the distance from the Sun (Roberts et al. 1987; Marsch & Tu 1990; Chen et al. 2020). In the framework of the two-field model, imbalance is easily characterized by the generalized cross-helicity (GCH) which is an ideal quadratic invariant that reduces to the negative cross-helicity at the MHD scales and to the magnetic helicity at the sub-ion scales. When turbulence is driven by injection of energy and of GCH, these quantities are expected to cascade forward to the smaller scales and/or backward to the larger ones, depending in particular on the injection scale compared to the ion Larmor radius (or the sonic Larmor radius). Indeed, no inverse cascade can take place at the MHD scales in the presence of a strong ambient field, while an inverse cascade of magnetic helicity was predicted in the (dispersive) sub-ion range, by analogy with EMHD (Schekochihin et al. 2009), and also on the basis of absolute equilibrium arguments (Passot et al. 2018). Thus, based on these observations, one expects that as the inverse cascade approaches the MHD scales, its properties will be significantly affected.

The present paper addresses the existence and the properties of the inverse cascades
which can develop when energy and GCH are injected and small-scale dissipation is supplemented to the two-field gyrofluid model. We will mainly vary two control parameters, namely the ratio $\beta_e$ of electron thermal to the magnetic pressures and the driving scale, which will be shown to affect the properties of the inverse cascade. The case where the forcing drives the two counter-propagating waves in a balanced way (zero GCH injection) is also addressed. Section 2 provides a description of the model and of its various limits, with a brief discussion of the limitations of the Fjørtoft argument often used for predicting the existence of an inverse cascade. Section 3 specifies the numerical set up and the conditions of the simulations. Section 4 discusses the cascade dynamics together with the effect of the decay instability and the stability of balanced turbulence. Section 5 is concerned with the arrest of the cascade and the generation of a finite-scale condensate. The coherent structures, in the form of magnetic vortices that are generated in physical space as consequences of the inverse cascades and of their arrest, are described in Section 6. The question of the relative importance of the nonlocal interactions is considered in Section 7. Section 8 is the Conclusion. The Appendices include a brief description of the decay instability in the context of the present model and a discussion of toy models in the form of few-mode shell models, aimed at heuristically illustrating main features of the spectral dynamics.

2. The two-field gyrofluid model

2.1. Equations and conservation laws

A description of the Alfvén wave dynamics from the MHD to the electron scales is provided by the two-field gyrofluid model which involves the gyrokinetic scaling corresponding to a strong spectral anisotropy and weak nonlinearity. In the absence of dissipation and driving, it involves two equations for the electron gyrocenter number density $N_e$ and the parallel component of the magnetic potential $A_\parallel$, in the form

$$\partial_t N_e + [\varphi, N_e] - [B_z, N_e] + \frac{2}{\beta_e} \nabla_\perp \Delta A_\parallel = 0$$

(2.1)

$$\partial_t L_e A_\parallel - \left[ \varphi, \frac{2\delta^2}{\beta_e} \Delta A_\parallel \right] + \left[ B_z, \frac{2\delta^2}{\beta_e} \Delta A_\parallel \right] + \nabla_\perp (\varphi - N_e - B_z) = 0.$$  (2.2)

In the above equations, the parallel magnetic fluctuations $B_z$ and the electron gyrocenter number density $N_e$ are related to the electric potential $\varphi$ by $B_z = M_1 \varphi$ and $N_e = -M_2 \varphi$, where $M_1$ and $M_2$ are Fourier multiplier operators written as $M_1 = L_1^{-1} L_2$ and $M_2 = L_3 + L_4 L_1^{-1} L_2$ with

$$L_1 = \frac{2}{\beta_e} + (1 + 2\tau)(\Gamma_0 - \Gamma_1)$$

(2.3)

$$L_2 = 1 + \frac{1 - \Gamma_0}{\tau} - \Gamma_0 + \Gamma_1$$

(2.4)

$$L_3 = \frac{1 - \Gamma_0}{\tau} - \delta^2 \Delta_\perp$$

(2.5)

$$L_4 = 1 - \Gamma_0 + \Gamma_1$$

(2.6)

$$L_e = 1 - \frac{2\delta^2}{\beta_e} \Delta_\perp.$$  (2.7)

Here, $\Delta_\perp = \partial_{xx} + \partial_{yy}$ is the Laplacian in the plane transverse to the ambient field and $[f, g] = \partial_x f \partial_y g - \partial_y f \partial_x g$ the canonical bracket of two scalar functions $f$ and $g$. Furthermore, $\Gamma_n$ denotes the (nonlocal) operator $\Gamma_n(-\tau \Delta_\perp)$ associated with the Fourier
Figure 1: Normalized parallel-phase velocity $v_{ph}\sqrt{\frac{\beta_e}{2}}$ (or equivalently, ratio of the KAW frequency $v_{ph}k\parallel$ to the Alfvén frequency $k\parallel v_A$), versus the transverse wavenumber $k\perp$ for various values of $\beta_e$.

multiplier $\Gamma_n(\tau k^2 \parallel)$, defined by $\Gamma_n(x) = I_n(x)e^{-x}$ where $I_n$ is the modified Bessel function of first type of order n. For a scalar function $f$, the parallel gradient operator $\nabla_\parallel$ is defined by

$$\nabla_\parallel f = -[A_\parallel, f] + \frac{\delta f}{\delta z}. \quad (2.8)$$

One recovers the fluctuating magnetic field from the expression of $B_z$ above and $B_\perp = \nabla_\perp \times (A_\parallel \hat{z})$. The model is written in a non-dimensional form, using the following units: time is normalized to the inverse ion gyrofrequency $\Omega^{-1}_i = (eB_0/(m_i c))^1/2$, and spatial scales to the sonic Larmor radius $\rho_s = c_s/\Omega_i$ where $c_s = \sqrt{T_{i0}/m_i}$ is the sound speed. Thus wavenumbers and wavevector components are measured in units of $\rho_s^{-1}$. The other normalization conventions include the ambient magnetic field $B_0$ for the parallel magnetic fluctuations $B_z$, the equilibrium number density $n_0$ for the electron gyrocenter density $N_e$, $T_e/e$ for the electric potential $\varphi$ and $B_0\rho_s$ for the parallel magnetic potential $A_\parallel$. In the formulas above, $e$ and $m_i$ denote the charge and the mass of the proton respectively and $c$ is the light velocity. We also introduce the ion and electron temperatures $T_i$ and $T_e$, assumed isotropic. Furthermore, the parameter $\beta_e = 8\pi n_0 T_{e0}/B_0^2$ is the ratio of the equilibrium electron pressure to the magnetic pressure due to the ambient field and $\tau = T_{i0}/T_{e0}$ the ratio of the equilibrium ion and electron temperatures.

At the level of the linear approximation, the KAW parallel-phase velocity $v_{ph} = \omega/k_z$ is given by

$$v_{ph}^2 \equiv \left(\frac{\omega}{k_z}\right)^2 = s^2 \left(\frac{k_\perp^2}{1 + \frac{2k_\perp^2}{\beta_e}}\right) \frac{1 - \tilde{M}_1 + \tilde{M}_2}{\tilde{M}_2}, \quad (2.9)$$

where the hat refers to the Fourier symbol of the operator and $s = (2/\beta_e)^{1/2}$ is the Alfvén velocity measured in sound speed units ($s = v_A/c_s$). We plot in Fig. 1 $v_{ph}$, normalized for presentation purposes by $s$, versus the transverse wavenumber $k\perp$, when electron inertia is neglected. For all values of $\beta_e$, the ratio $v_{ph}/s$, that can also be viewed as the wave frequency normalized by the Alfvén wave frequency $v_Ak\parallel$, is asymptotically equal to unity at large scale (dispersionless limit). On the other hand, while for small ($\beta_e = 0.2$) or moderate ($\beta_e = 2$) values of the electron beta parameter, $v_{ph}/s$ involves a monotonous
transition to a $k_-\text{-scaling at sub-ion scales, for larger } \beta_e \text{ (e.g. 10 or 50), it displays a local minimum at a wavenumber that (in inverse sonic Larmor radius units) increases with } \beta_e \text{ at fixed } \tau \text{ and only depends on } \beta_i \text{ when measured in units of the ion Larmor radius } \rho_i$. A similar behaviour is observed in Fig. 3 ($\tau = 100$) and 5 ($\tau = 1$) of Howes et al. (2006) where this quantity, is computed from the gyrokinetic theory and also in the framework of the full linear kinetic theory taken in the quasi-transverse limit. As discussed below, it turns out that the sensitivity of the dispersion to the transverse wavenumber, as measured by the parallel-phase velocity $v_{ph} = \omega / k_1$, has an important effect on the nonlinear dynamics of the cascade. In particular, it has an impact on the generation of a finite-size condensate and on the formation of large-scale coherent structures.

In physical space, the eigenmodes, which can be referred to as Elsasser potentials, are given by

$$\mu^\pm = \Lambda \varphi \pm sA_\parallel,$$

where

$$\Lambda = D_e^{-1}(1 + M_2 - M_1)^{1/2}M_2^{1/2},$$

with $D_e^2 = (-\Delta_\perp)L_e$. They obey

$$\partial_t \mu^\pm \pm V_{ph} \partial_z \mu^\pm + \frac{1}{4} \Lambda^{-1} D_e^{-2} M_3 \left\{ [\Lambda^{-1} M_5 (\mu^+ \pm \mu^-), \Lambda^{-1} M_2 (\mu^+ \pm \mu^-)] + [(\mu^+ - \mu^-), \Delta_\perp (\mu^+ - \mu^-)] \right\} + \frac{1}{4} D_e^{-2} \Delta_\perp \left\{ [(\mu^+ - \mu^-), \Lambda^{-1} M_2 (\mu^+ \pm \mu^-)] + [L_e (\mu^+ - \mu^-), \Lambda^{-1} (1 - M_1) (\mu^+ \pm \mu^-)] \right\} = 0.$$  \tag{2.12}

The systems (2.1)-(2.2) or (2.12) preserve the energy $E$ and the generalized cross-helicity $C$, defined as

$$E = \frac{1}{\beta_e} \int \left( \frac{2}{\beta_e} |\Delta_\perp A|^2 + \frac{4\delta^2}{\beta_e^2} |\Delta_\perp A||^2 - N_e (\varphi - N_e - B_z) \right) d^3x,$$

$$C = - \int N_e \left( 1 - \frac{2\delta^2}{\beta_e} \Delta_\perp \right) A_\parallel d^3x.$$

where $V_{ph}$ is the operator in physical space, which in Fourier space corresponds to the multiplication by $v_{ph}$. Note that $C$ was defined with an opposite sign in Passot et al. (2018).

2.2. Normal-field formulation

As is common with noncanonical Hamiltonian systems, the associated Poisson bracket possesses Casimir invariants, corresponding to $C_\pm = \int G^\pm d^3x$, where $G^\pm = L_e A_\parallel \pm \delta N_e$ are referred to as normal fields. In terms of these fields, the two-field gyrofluid model can be rewritten in the form

$$\partial_t G^\pm + [\varphi^\pm, G^\pm] + \partial_z \left( \varphi^\pm \mp \frac{1}{\delta} G^\pm \right) = 0,$$  \tag{2.15}
where $\varphi^\pm = \varphi - B_z \pm \frac{1}{\delta} A_1$. In terms of these variables, GCH simply reads

$$C = \frac{-1}{4\delta} \int \{ G^+)^2 - (G^-)^2 \} \, d^3x. \quad (2.16)$$

Similarly,

$$\mathcal{E} = \frac{1}{2} \int \left\{ \frac{1}{2\beta_e} L_c^{-1}|\nabla_\perp (G^+ + G^-)|^2 + \frac{1}{4\delta^2} M_3 M_2^{-1}(G^+ - G^-)^2 \right\} \, d^3x. \quad (2.17)$$

Note that the above description degenerates in the limit $\delta \to 0$, as the leading order of Eq. (2.15) reproduce the equation for $A_1$ only, while the equation for $N_e$ corresponds to the first order in a development in $\delta^2$. As seen below, this formulation nevertheless has a major interest for estimating the energy and GCH spectral transfers discussed in Section 7.

### 2.3. Asymptotic regimes

The two-field model simplifies in two asymptotic regimes arising when electron inertia is neglected ($\delta = 0$).

- **Two-field Hall reduced magnetohydrodynamics (HRMHD)** for dispersive Alfvén waves, under the conditions $\tau k^2_\perp \ll 1$, $\tau \ll 1$ and $\beta_e \ll 1$:

  $$\partial_t A_1 + \nabla_\parallel \left( \varphi - \frac{2}{\beta_e} \frac{1}{1 + \frac{2}{\beta_e}} \Delta_\perp \varphi \right) = 0 \quad (2.18)$$

  $$\partial_t \Delta_\perp \varphi + [\varphi, \Delta_\perp \varphi] + \frac{2}{\beta_e} \nabla_\parallel \Delta_\perp A_1 = 0. \quad (2.19)$$

  In this regime, $N_e = \Delta_\perp \varphi$ which also corresponds to the ion vorticity (with a single component along the $z$ direction). The invariant $C$ thus rewrites $C = -\int u_\perp \cdot B_\perp \, d^3x$, where $u_\perp = -\nabla \times (\varphi \hat{z})$ denotes the transverse ion velocity. In this limit, $C$ thus appears as the opposite of the usual cross-helicity. Alternatively, in terms of magnetic helicity, one has, for small $\tau$, $C = (2/\beta_e + 1) \int B_z A_1 \, d^3x$.

  Reduced magnetohydrodynamics (RMHD) equations are obtained after dropping the second term in the parenthesis of Eq. (2.18), which accounts for the Hall term.

  - **Electron reduced magnetohydrodynamics (ERMHD)** (Schekochihin et al. 2009; Boldyrev et al. 2013), under the conditions $\tau k^2_\perp \gg 1$, $\tau \sim 1$ and $\beta_e \ll 1$:

    $$\partial_t A_\parallel + \left( 1 + \frac{1}{\tau} \right) \nabla_\parallel \varphi = 0 \quad (2.20)$$

    $$\partial_t \varphi - \frac{2\tau}{1 + \frac{2}{\beta_e} \left( 1 + \frac{1}{\tau} \right)} \nabla_\parallel \Delta_\perp A_1 = 0. \quad (2.21)$$

    In this regime, (leaving out the electron inertia and FLR contributions), $B_z$ and $N_e$ are proportional to $\varphi$, and thus the generalized cross-helicity is proportional to the magnetic helicity, in the form

    $$C = \left( 1 + \frac{2}{\beta_e} \frac{1}{1 + \tau} \right) \int A_\parallel B_z \, d^3x. \quad (2.22)$$

    It is interesting to note that in the incompressible limit where the beta parameter tends to infinity, $C = \int B_z A_1 \, d^3x$, thus recovering, in the quasi-transverse limit, the generalized magnetic helicities of EMHD (Biskamp et al. 1999), or of extended MHD (XMHD) when the ion velocity and electron inertia are taken to zero (Eq. (35) of Abdelhamid et al. 2016 and Eq. (29) of Miloshevich et al. 2017).
When formally considering the limit of large $\beta_e$, it is convenient to write the ERMHD equations with the ion inertial length used as length unit (instead of the ion sonic Larmor radius $\rho_s$). One then gets

$$
\partial_t A_1 + \frac{\beta_e}{2} \left( 1 + \frac{1}{\tau} \right) \nabla_\parallel \varphi = 0 \quad (2.23)
$$

$$
\partial_t \varphi - \frac{2}{\beta_e} \left( \frac{1}{1 + \frac{1}{\tau}} \right) \nabla_\parallel \Delta A_1 = 0, \quad (2.24)
$$
or in terms of $B_z = \frac{\beta_e}{2} \left( 1 + \frac{1}{\tau} \right) \varphi$,

$$
\partial_t A_\parallel + \nabla_\parallel B_z = 0 \quad (2.25)
$$
$$
\partial_t B_z - \nabla_\parallel \Delta A_\parallel = 0, \quad (2.26)
$$

which identifies with the quasi-transverse limit of EMHD (Galtier & Meyrand 2015).

Equations (2.25)-(2.26) admit two invariants: the energy $\frac{1}{\beta_e} \int |B|^2 d^3x$ where $B = \nabla_\perp \times (A_\parallel \hat{z}) + B_z \hat{z}$ and the magnetic helicity $\int A_\parallel B_z d^3x$ which identifies with the GCH. Introducing $A_\perp$ so that $B_z = \nabla_\perp \times A_\perp$, one has $\int A_\parallel B_z d^3x = (1/2) \int A \cdot B d^3x$ with $A = A_\perp + B_z \hat{z}$ (see e.g. Eq. (F6) of Schekochihin et al. (2009)), together with $\int |B|^2 d^3x = \int k_3^2 |A|^2 d^3x$. Like in incompressible MHD without ambient field (see e.g. Pouquet et al. (1999) for a recent review), assuming that the helicity is of a given sign and that it is maximal, one gets $E_c(k_\perp) = (1/k_\perp)E(k_\perp)$, which leads to conjecture the existence of an inverse cascade of magnetic helicity, by generalizing the argument developed by Fjørtoft-type argument does not directly extend to the case of finite $\beta_e$ where, in the units of Eqs. (2.20)-(2.21), the energy $E = \frac{1}{\beta_e} \int \left( |B|^2 + \frac{2}{\beta_e} \frac{1}{1 + \frac{1}{\tau}} B^2 \right) d^3x$ includes a term corresponding to the internal energy originating from compressibility. This led us to address the existence of a GCH inverse cascade, numerically.

2.4. The strong imbalance limit

We first rewrite the invariants in the form

$$
\mathcal{E} = \frac{1}{2} \int \left( \frac{2}{\beta_e} |\nabla_\perp A_\parallel|^2 + \varphi M_2 (1 - M_2 + M_1) \varphi \right) d^3x
$$

$$
= \frac{1}{2} \int \left( \frac{2}{\beta_e} |\nabla_\perp A_\parallel|^2 + |DAe|^2 \right) d^3x \quad (2.27)
$$

$$
C = \int M_2 A e d^3x. \quad (2.28)
$$

In the strong imbalance limit, where only one type of waves contributes significantly to the invariants, one assumes the relation $\varphi = \pm s A^{-1} A_1$. The invariants then become

$$
\mathcal{E} = \frac{2}{\beta_e} \int |B_\perp|^2 d^3x \quad (2.29)
$$

$$
C = \pm \sqrt{\frac{2}{\beta_e}} \int |M_2^{1/2} A^{-1/2} A_\parallel|^2 d^3x, \quad (2.30)
$$

where $\pm$ refers to the considered type of waves.

In the ERMHD regime (see Table 1 of (Passot & Sulem 2019)), $\hat{M}_2$ is a constant and $\hat{A}$ scales like $1/k_\perp$. Consequently, $C \propto \int (1/k_\perp) |B_\perp|^2 d^3k$. As in the previous subsection, a Fjørtoft-type argument then suggests the existence of a direct energy cascade and an inverse cascade of magnetic helicity.
each direction, depending on the simulations, and the time stepping is performed using

\[ \chi = \text{supplemented with driving and dissipation, in a periodic domain using a Fourier} \]

\[ \text{maximal wavenumber. Resolutions of} \]

\[ \begin{align*}
R_{1,3}^0 & : 0.2 \quad 1.3 \quad 30.1 \quad 27.5 \quad 360 \quad 7.8 \times 10^{-5} \quad 1.0 \times 10^{-5} \quad 8.0 \times 10^{-3} \quad 0.179 \\
R_{1,3}^2 & : 2 \quad 1.3 \quad 6.34 \quad 27.5 \quad 360 \quad 7.8 \times 10^{-5} \quad 1.0 \times 10^{-5} \quad 1.0 \times 10^{-2} \quad 0.884 \\
R_{6,5}^2 & : 2 \quad 6.5 \quad 27.2 \quad 5.5 \quad 240 \quad 4.0 \times 10^{-8} \quad 1.0 \times 10^{-6} \quad 4.0 \times 10^{-3} \quad 0.350 \\
R_{13}^2 & : 2 \quad 13 \quad 54.2 \quad 2.75 \quad 240 \quad 2.5 \times 10^{-11} \quad 1.0 \times 10^{-6} \quad 5.0 \times 10^{-4} \quad 0.221 \\
R_{135}^2 & : 2 \quad 13 \quad \infty \quad 2.75 \quad 240 \quad 2.5 \times 10^{-11} \quad 1.0 \times 10^{-6} \quad 5.0 \times 10^{-4} \quad 0.221 \\
R_{36}^2 & : 2 \quad 36.0 \quad 149 \quad 1.0 \quad 240 \quad 5.0 \times 10^{-14} \quad 1.0 \times 10^{-6} \quad 5.0 \times 10^{-4} \quad 0.112 \\
R_{56}^2 & : 2 \quad 36.0 \quad \infty \quad 1.0 \quad 240 \quad 5.0 \times 10^{-14} \quad 1.0 \times 10^{-6} \quad 5.0 \times 10^{-4} \quad 0.112 \\
R_{0.87}^{10} & : 10 \quad 0.87 \quad 2.05 \quad 20.3 \quad 360 \quad 1.0 \times 10^{-5} \quad 1.0 \times 10^{-5} \quad 4.0 \times 10^{-3} \quad 2.548 \\
R_{1.3}^{10} & : 10 \quad 1.3 \quad 1.94 \quad 13.5 \quad 360 \quad 4.0 \times 10^{-7} \quad 1.0 \times 10^{-5} \quad 3.125 \times 10^{-3} \quad 3.245 \\
R_{2}^{10} & : 10 \quad 2.0 \quad 2.20 \quad 9.05 \quad 360 \quad 1.5 \times 10^{-8} \quad 1.0 \times 10^{-5} \quad 2.0 \times 10^{-3} \quad 3.679 \\
R_{6.5}^{10} & : 10 \quad 6.5 \quad 6.34 \quad 5.5 \quad 240 \quad 4.0 \times 10^{-8} \quad 1.0 \times 10^{-6} \quad 2.0 \times 10^{-2} \quad 2.014 \\
R_{13}^{10} & : 10 \quad 13 \quad 12.6 \quad 2.75 \quad 240 \quad 2.5 \times 10^{-11} \quad 1.0 \times 10^{-6} \quad 5.0 \times 10^{-4} \quad 1.275 \\
R_{13}^{20} & : 50 \quad 13 \quad 2.63 \quad 2.75 \quad 240 \quad 2.5 \times 10^{-11} \quad 1.0 \times 10^{-6} \quad 5.0 \times 10^{-4} \quad 9.757 \\
\end{align*} \]

Table 1: Parameters of the runs, together with the nonlinear parameter \( \chi_f \) at the driving scale. In all the simulations, \( \epsilon_E = 0.372 \), and \( \epsilon_{E^+}/\epsilon_{E^-} = 1.5 \), except in run \( R_{56}^2 \) where \( \epsilon_{E^+}/\epsilon_{E^-} = 1 \) (balanced driving).

In contrast, at scales small compared to the sonic Larmor radius, \( \hat{M}_2 \propto k_\perp^2 \) while \( \hat{\Lambda} \) becomes asymptotically constant. It follows that \( \mathcal{C} \) behaves like \( \int |B_\perp|^2 d^3 x \), proportional to \( \mathcal{E} \), making the Fjortoft argument not applicable, even in the special case of the strong imbalance limit.

3. Numerical setup and conditions of the simulations

We performed numerical simulations of the two-field gyrofluid, in the form given by Eq. 2.15, supplemented with driving and dissipation, in a periodic domain using a Fourier pseudo-spectral method where aliasing is suppressed by spectral truncation at 2/3 of the maximal wavenumber. Resolutions of \( N = 240 \) or \( N = 360 \) grid points were used in each direction, depending on the simulations, and the time stepping is performed using
a third-order Runge-Kutta method, with a time step $\Delta t$ that can be reduced, as the simulation proceeds.

In all the simulations, the driving corresponds to a white-noise in time random forcing in the equations for $D_{\epsilon \chi}^\pm$ (which reduce to the usual Elsasser variables $z^\pm$ in the MHD limit). In Fourier space, this forcing has an amplitude given by a shifted Gaussian function $f_k^\pm = \frac{1}{\sqrt{2\pi}} A^\pm \exp \left[-\frac{(k_1 - k_0)^2}{\sigma^2_{\parallel}}\right] \exp \left[-\frac{(k_2 - k_2^0)^2}{\sigma^2_{\perp}}\right]$, truncated in a way that only wavenumbers $k_{\parallel}$ and $k_{\perp}$ such that $|k_{\parallel} - k_0| < 3\sigma$ are driven (in practice, only $3$ Fourier modes are forced in each direction), and is further multiplied by a factor $ae^{2i\pi b}$ where $a$ and $b$ are Gaussian random variables of zero mean value and variance unity, drawn at each time step. The injection rates associated with the forcing terms $f_k^\pm$ are denoted $\epsilon_{E \pm}$. From the functions $f_k^\pm$, one easily derives the corresponding driving terms in the equations for $G^\pm$. Here, we choose $k_0 = k_0 = k_f$ and $\sigma_0 = \sigma_0 = \sigma_f$. Due to the ordering underlying the derivation of the gyrofluid model (where longitudinal gradient balances transverse nonlinearity), this choice corresponds to a quasi-transverse driving in the primitive physical variables. The purpose is to prevent the development of a small-scale inertial range. The absence of such a range is always smaller than the dissipation one). In order to make the effects associated with electron inertia negligible, we prescribed an electron-to-proton mass ratio $\delta^2$ smaller than the physical one. The specific parameters of the different simulations reported in this paper are summarized in Table 1.

The various simulations are conveniently characterized by the nonlinear parameter at the driving scale that estimates the ratio of the nonlinear to the wave frequencies and thus the relative importance of the nonlinear and linear effects at this scale. On a purely dimensional basis, and not discriminating between the two kinds of waves, one can estimate, using Eqs. (67)-(68) of Passot et al. (2018) with $\delta = 0$, that the characteristic eddy-turnover time of turbulence $\tau_{NL}$, resulting from energy injection with a rate $\epsilon_E$ at a wavenumber $k_f$, scales like

$$\tau_{NL} \sim \epsilon_E^{-1/3} (1 + M_2(k_f)/M_2(k_f))^{-2/3} k_f^{-2} (v_{ph}(k_f)/s)^{2/3}. $$

The period of the wave at this scale $(k_f v_{ph})^{-1}$ writes $(k_f v_{ph})^{-1}$ since the driving wavevector is such that $k_{\parallel} = k_{\perp}$. The nonlinearity parameter thus reads

$$\chi_f = \epsilon_E^{1/3} (k_f/s)(v_{ph}(k_f)/s)^{-5/3} (1 + M_2(k_f)/M_2(k_f))^{2/3}. $$

(3.2)

Assuming that $\tau$ is of order unity, we find, using Table 1 in Passot & Sulem (2019), that at large scales $v_{ph} \sim s$, $M_2(k_f) \sim k_f^2$ and at small scales $(1 + M_2(k_f)/M_2(k_f))$ is always a constant of order unity, while $v_{ph} \sim s k_f$ when $\beta_e \ll 1$ and $v_{ph} \sim s^2 k_f$ when $\beta_e$ is
large. For a driving acting at sub-ion scales, $\chi_f \sim E_f^{1/3}k_f^{-2/3}\beta_e^{1/2}$ for small or moderate $\beta_e$ and $\chi_f \sim E_f^{1/3}k_f^{-2/3}\beta_e^{4/3}$ for large $\beta_e$. The values of $\chi_f$, as defined by Eq. (3.2), for the different runs considered in this paper are given in Table 1.

4. Inverse cascades

As previously discussed, the two-field gyrofluid reduces to HRMHD, in the cold-ion limit, assuming large or finite transverse scales. When neglecting the Hall contribution (last term in Eq. 2.18) by also taking the large-scale limit, one further gets the RMHD equations, for which only direct cascades of both energy and cross helicity can develop. Differently, for $\tau$ of order unity, numerical integration of the two-fluid model with a forcing at a wavenumber $k_f \ll 1$, shows enhancement of the energy and GCH at increasing scales and on a longer time scale, as $k_f$ decreases (not shown). In the small-scale limit, the model reduces to ERMHD. When $\beta_e \rightarrow \infty$, this latter model can be derived from EMHD for which direct numerical simulations in the case of imbalanced driving displays an inverse cascade of magnetic helicity (Kim & Cho 2015). Therefore, it is of interest to study the dynamics when the driving takes place at scales comparable to or moderately smaller than the sonic Larmor radius (taken as the length unit), for finite values of $\beta_e$.

4.1. Spectra and fluxes

In this section, we concentrate on the spectral density in the transverse plane $E(k_{\perp})$ and $E_C(k_{\perp})$ (hereafter, transverse spectra) and in the parallel direction $E(k_{\parallel})$ and $E_C(k_{\parallel})$ (hereafter, parallel spectra) of the energy and GCH, respectively. Integration of these spectra with respect to the transverse ($k_{\perp}$) or parallel ($k_{\parallel}$) wavenumber respectively, reproduces the respective ideal quadratic invariants. It is also of interest to consider the Elsasser energy and GCH transverse spectra $E_{\pm}(k_{\perp})$ and $E_{C\pm}(k_{\perp})$ related to the energy and GCH transverse spectra by the relations $E(k_{\perp}) = \frac{1}{2}(E_{\pm}(k_{\perp}) + E^{\pm}(k_{\perp}))$ and $E_C(k_{\perp}) = \frac{1}{2}(E_{C\pm}(k_{\perp}) - E_{C}^{\pm}(k_{\perp}))$ with $E_{C\pm}(k_{\perp}) = E^{\pm}(k_{\perp})/v_{ph}(k_{\perp})$. Here, $E^{\pm}(k_{\perp})$ can be viewed as the spectrum of the field $D_e\mu^{\pm}$ and $E_{C\pm}(k_{\perp})$ of the field $V_{ph}^{-1/2}D_e\mu^{\pm}$. Furthermore, the perpendicular and parallel fluxes of the energy and GCH are respectively defined as the negative nonlinear contributions to $\partial_t \int_0^{k_{\perp}} E(k_{\perp})dk_{\perp}$ or $\partial_t \int_0^{k_{\perp}} E_C(k_{\perp})dk_{\perp}$ (for the perpendicular fluxes) or to $\partial_t \int_0^{k_{\parallel}} E(k_{\parallel})dk_{\parallel}$ or $\partial_t \int_0^{k_{\parallel}} E_C(k_{\parallel})dk_{\parallel}$ (for the parallel fluxes).

We first consider the case where $\beta_e = 2$ and the driving is centered at $k_f = 1.3$ (run $R_{1.3}^2$). The top panels of Fig. 2 respectively display the transverse spectra $E(k_{\perp})$ and $E_C(k_{\perp})$ (left) and the parallel spectra $E(k_{\parallel})$ and $E_C(k_{\parallel})$ (right) at increasing times. Inspection of these spectra indicates the existence of an inverse cascade. At early time, both $E(k_{\perp})$ and $E_C(k_{\perp})$ develop a self-similar range displaying a $k_{\perp}^{-2}$ power-law, the similarity between the two spectra resulting from the very weak variation of the parallel phase velocity $v_{ph}$ in this spectral range for $\beta_e = 2$ (see Fig. 1). This self-similar dynamics does not however proceed to longer times. After the cascade reaches scales where the variation of $v_{ph}(k_{\perp})$ becomes very small, the transfer to larger scales significantly slows down and the spectra tend to bend, flattening close to the driving wavenumber and developing a spectral bump at the minimal excited wavenumber. This effect can be viewed as the onset of a condensate, associated with the formation of coherent structures in physical space. As time elapses, the bump moves more and more slowly to the largest scales, while its amplitude still increases under the effect of the persistent driving. Interestingly, this cascade depletion is the result of the inner turbulence dynamics, due
Figure 2: Perpendicular (top left) and parallel (top right) energy and GCH spectra for run $R_{1,3}$, together with the perpendicular (middle) or parallel (bottom) energy (left) and GCH (right) fluxes. The integration time that extends up to $t = 3000$ (in $Ω_i^{-1}$ units) is indicated by the color bars, where $+GCH$ refers to positive value and $-GCH$ to the modulus of negative values (dashed lines visible at the smallest scales) of GCH spectra. For clarity, the axes of the energy (left) and GCH (right) spectra have been shifted. Same convention on forthcoming graphs.
to a strong decrease of the dispersion, with no need for an external effect such as a hypo-viscosity, as required for example in simulations of the inverse magnetic helicity cascade in incompressible isotropic MHD (Linkmann & Dallas 2017) or in EMHD (Kim & Cho 2015). In contrast, in the parallel direction, no inverse cascade is observed. Both the energy and GCH spectra become flat, corresponding to an absolute equilibrium regime.

In order to analyze more precisely the spectral dynamics, we also show in Fig. 2, the perpendicular (middle panels) and parallel (bottom panels) fluxes of the energy (left) and GCH (right) for this simulation. The existence of an inverse cascade of GCH in the transverse spectral plane is conspicuous when considering the transverse GCH flux which displays a flat range between the injection wavenumber and the maximum of the perpendicular spectra. The perpendicular energy flux is also flat in this spectral range, but it is significantly stronger at scales smaller than the driving, indicating a split cascade with most of the energy transferred to the small scales where, if the computation grid were much more refined, power-law spectra would develop. In contrast, in the parallel direction, the energy flux decreases progressively from the injection to the largest scales, while a flux towards the small scales dominates. The parallel GCH flux to the large scales strongly oscillates in time. Note that in the range of wavenumbers where the parallel (and to some extent the perpendicular) GCH fluxes are negative, the GCH spectrum is also negative, as a result of an imperfect pinning, possibly related to the use of hyperviscous dissipation. Thus there is no indication of inverse transfer in this range.

A qualitatively different dynamics from the early-time self-similar cascade is observed when, preserving all the other parameters, $\beta_e$ is taken to be smaller. As seen in Fig. 1, decreasing $\beta_e$ results in an increase of $v_{ph}$ and thus of the wave frequency at a fixed $k_\perp$. Consequently, the relative strength of the linear effects are enhanced. This regime is illustrated by run $R_{1.3}^{0.2}$, where we prescribe $\beta_e = 0.2$. The corresponding energy and GCH spectra are shown in Fig. 3 (left). In this case, no early self-similar cascade is observed. Instead, the amplitude of the modes successively excited by the nonlinear coupling does not saturate at their maximal values (as in the case of a standard self-similar cascade), but decreases when modes at slightly larger scales get amplified. This results in the propagation towards the large scales of a spectral bump formed at early time near the driving wavenumber, a dynamics qualitatively similar to that obtained in numerical simulations of EMHD (Kim & Cho 2015) and also of incompressible MHD in
the absence of an ambient field (Müller et al. 2012; Linkmann & Dallas 2017). This type of temporal evolution of the spectral modes is also observed in simulations of the shell model of Appendix B when nonlocal interactions are important. In run $R_{1.3}^{0.2}$, the transverse inverse cascade also slows down, as the spectral bump reaches quasi-dispersionless scales. The right panel of Fig. 3 displays the Elsasser transverse spectra in the late phase of the simulation. The inverse cascade of transverse GCH is here associated to an inverse transfer of $E^+(k_\perp)$, while the sub-dominant wave energy $E^-(k_\perp)$ reaches an absolute equilibrium characterized by a $k_\perp$ scaling, at scales larger than the injection. This behaviour is specific to cases where the ratio $\epsilon^+/\epsilon^-$ is small or moderate, as also seen in EMHD simulations by Kim & Cho (2015).

When, keeping $\beta_e = 2$, the driving acts at smaller scales ($k_f = 13$), the typical dynamics that establishes already at early time is similar to that of run $R_{1.3}^{0.2}$, with no self-similar regime and a spectral bump propagating towards large transverse scales. This is illustrated in Fig. 4 (top, left). In this case, the inverse cascade extends over more than one decade before reaching the weakly dispersive scales. At the final time of the simulation, the energy has not started to accumulate at the lowest modes, as in Run $R_{1.3}^{2}$. Note that the slope of the envelope of the spectral bumps is here shallower than that of run $R_{1.3}^{0.2}$, indicating that it is not universal. The corresponding GCH flux displayed in the top right panel shows at late times a remarkably constant range, associated with an inverse cascade, in the spectral region between the peak and the forcing. Interestingly, a self-similar inverse cascade develops in the parallel direction as well, as shown in Fig. 4 (middle, left), associated with a negative and constant GCH flux (middle right panel). The existence of this parallel inverse cascade has important consequences on the types of structures that form in physical space, as discussed in Section 6. In the bottom panels of Fig. 4 we show the Elsasser spectra $E^\pm(k_\perp)$ (left) and the time evolution of the total Elsasser energies $E^\pm = \sum_{k_\perp} E^\pm(k_\perp)$ and of the positive/negative GCHs $E^\pm_C = \sum_{k_\perp} E^\pm_C(k_\perp)$ (right).

The inverse GCH cascade is, as for run $R_{1.3}^{0.2}$, only associated with an inverse transfer of $E^+$, although a non-negligible transfer of $E^-$ is also visible which might develop into a different type of inverse cascade involving longer spatial and temporal scales. Simulations with a spectral range extending to much larger scales and spanning a longer time interval would be necessary to characterize this effect more precisely. Both $E^+$ and $E^+_C$ grow at long time as power laws with exponents similar but not identical to those reported in Kim & Cho (2015) in the case of EMHD. Furthermore, we verified that, with the parameters of run $R_{1.3}^{2}$, but when only one type of wave is injected (infinitely imbalanced driving), the other type of wave is driven by interactions of injected modes. While the energy of the dominant mode is transferred to the large scales through the propagation of a spectral bump whose envelope obeys a $k_\perp^{-3/2}$ scaling law, the subdominant wave undergoes a self-similar cascade with a $k_\perp^{-1}$ spectrum (not shown), indicating a behavior of both spectra similar to that observed in EMHD simulations with maximal helicity injection (Kim & Cho 2015).

4.2. Effect of the decay instability

In contrast with Alfvén waves in ideal MHD, three KAWs can interact resonantly, producing a decay instability (Voitenko 1998a) (see Appendix A for a brief description of this linear instability in the context of the two-field model, in the absence of injection and dissipation). This instability is however competing with the usual turbulence dynamics. Relevance of the decay instability thus requires the turbulence driving to be not too strong (see Shi et al. (2017); Fu et al. (2018) for a study in the MHD context). This is achieved
Figure 4: Perpendicular (top) and parallel (middle) energy and GCH spectra (left) and GCH fluxes (right) for Run $R_{13}$, together with (bottom panels) perpendicular Elsasser spectra (left) and time evolution of the total Elsasser energies $E^\pm$ and GCHs $E^\pm_C$ (right).

when the nonlinearity parameter at the driving scale $\chi_f$ is small enough. As seen using its estimates in Section 3, $\chi_f$ decreases with the injection scale and $\beta_e$. On this basis, we performed a simulation (run $R_{36}$) with $\beta_e = 2$ but with the driving wavenumber increased to $k_f = 36$. Figure 5 (left) clearly shows that, after the development of self-similar spectra with exponents -2 for energy and -3 for GCH, there is a transient regime corresponding to
Figure 5: Transverse energy and GCH spectra (left) and time variation of the total energy and GCH (right) for run $R_{36}^2$.

Figure 6: Color scale plot of $\ln|\hat{\mu}|^2$ for run $R_{36}^2$ showing decay instability. Left: Cross-section in a $(k_x,k_z)$-plane. Because the forcing is statistically isotropic in the $(k_x,k_y)$-plane, this plot is sufficient to characterize the full $(k_x,k_y,k_z)$ space (see text). Among the visible peaks, some of them, such as $(50,-16)$, $(25,20)$ and $(36,36)$ (which corresponds to the forcing) are associated with resonant triads. Right: Three-dimensional view. The dominant peaks visible as dots in the $(k_x,k_y)$-plane appear as circles symmetric around the $k_z$ axis. Some circles are not visible since a transparency mask was applied to demonstrate the most intensive peaks clearly.

the onset of unstable modes, an effect that does not persist after turbulence has become sufficiently developed. Such a competition between resonant coupling and regular triad interactions typical of strong turbulence is supported by the observation that reducing the injection rates (or increasing the hyperviscosity which, in the absence of small-scale cascade, indirectly produces the same effect) favors the development of decay instability. This was verified numerically by decreasing the energy injection (keeping the same percentage of GCH injection) in run $R_{13}^2$. When decreasing the injection rate by a factor of 64 (although a factor 16 seems to be close to the critical value), a transient decay
event appears after a very slow self-similar inverse transfer characterized by the same spectral exponent as in run $R_{36}^2$ (not shown). This suggests that, if the decay instability can be a leading mechanism in the regime of weak turbulence (Voitenko 1998b), its effect cannot be more than a transient in the strong turbulence regime. When present, the decay instability nevertheless significantly affects the global dynamics. As seen in Fig. 5 (right), a sharp increase of the total energy and GCH takes place when the decay instability is acting. The instability can be identified at a time close to $t = 7$. Later on, energy and GCH are still growing, but at lower rates, consistent with the existence of an inverse cascade. As expected, increasing $\beta_e$ for example to $\beta_e = 50$ (not shown), the other parameters being fixed, suppresses the decay instability.

In order to highlight the presence of resonant interactions in run $R_{36}^2$, we display in Fig. 6 (left) the color scale plot of the two-dimensional spectrum $|\tilde{\mu}(k)|^2$ in the plane $(k_\perp, k_z)$, showing decay instability at work (at $t \approx 8$). Since the forcing is statistically isotropic in the $(k_x, k_y)$-plane, this plot is sufficient to demonstrate the features of the full $k_x, k_y$ and $k_z$ dependence. We see numerous peaks, some of them associated with resonant triads. In fact, due to axisymmetry, each peak is essentially a circle penetrating the plane (see Fig. 6 (right)). This is important when evaluating the resonance condition since it provides more freedom in choosing the corresponding $k_x$ and $k_y$ wavevector components. Of particular interest are triads with coordinates in the $(k_x, k_z)$-plane given by $(50, -16)$, $(25, 20)$ and $(36, 36)$ (which corresponds to the forcing). We indeed see that, for this triad, both $k_{1z} + k_{2z} = k_{fz}$ and $\omega_1 + \omega_2 = \omega_f$. In this context, what appears as $k_x$ should be viewed as $k_{\perp}$. Resonance conditions in the transverse plane then consist of 5 equations (3 norms of transverse wavevectors and 2 resonance conditions) for 6 unknowns. Choosing arbitrarily one of them, we can easily construct a resonant triad. There is thus an infinite number of such resonant triads which can be viewed as defining a resonant manifold.

4.3. Instability of the balanced state

Decay instability plays an important dynamical role when the $\mu^+$ and $\mu^-$ waves are driven in a balanced way, i.e. with a zero GCH injection rate. This regime is exemplified in Fig. 7 by run $R_{36}^2$. The left panel shows the perpendicular energy spectrum in the transverse plane. One observes the onset of the decay instability, followed by the formation of a spectral bump propagating towards the large scales. Nevertheless, after a
The balanced state is unstable and evolves to a stationary solution where $\mathcal{E}^-/\mathcal{E}^+ \approx 2$. sufficient number of modes has been excited, non-resonant interactions become dominant, making the later-time evolution qualitatively similar to that resulting from an imbalanced driving. The right panel displays the Elsasser energy spectra separately. Associated with the absence of GCH injection, these two spectra identify at early times, but they separate after the onset of the decay instability (between times 9 and 17, see the colormap of the right panel of Fig. 7) and their difference increases in time, up to the moment where, due to the absence of GCH injection, the backward cascade cannot proceed further. The case of a balanced driving is also illustrated with run $R_{13}^2$ in Fig. 8, which shows the Elsasser spectra $E^\pm$ (left) and the time evolution of the ratio $E^+/E^-$ for both runs $R_{13}^2$ and $R_{13}^b$. As in run $R_{36}^2$, we observe that the balanced regime is unstable, but in the present case no decay instability develops. While positive GCH is transferred towards small scales and dissipated, negative GCH is transferred towards large scales, creating a finite global imbalance. After some time, the inverse cascade stops and the global imbalance of run $R_{13}^2$, as measured by the ratio $E^+/E^-$, saturates to a value close to 0.5, in contrast with the simulation with imbalanced driving for which this ratio keeps increasing with time.

It turns out that, like for the decay instability, the stability of the balanced state is controlled by the nonlinearity parameter $\chi_f$. In EMHD, associated with the large $\beta_e$ limit, the balanced regime is known to be stable (Kim & Cho 2015). This suggests that instability requires $\chi_f$ to be small enough. In order to support this statement, we performed additional simulations with balanced driving, for different values of $\beta_e$, $k_f$ and $\epsilon_E$. As previously mentioned, for $\beta_e = 2$, when turbulence is driven at a relatively small scales ($k_f = 36$ or 13), the balanced state is unstable. When for the same $\beta_e$ (or even larger, e.g. $\beta_e = 10$), the wavenumber $k_f$ is reduced to 1.3, the instability is suppressed. When, keeping $k_f = 1.3$, while $\beta_e$ is reduced to 0.2, the instability of the balanced state is recovered. Since $\chi_f$ is proportional to the driving amplitude, one expects that the balanced state will be restored, provided the driving is increased beyond some threshold value. Indeed, this is what is observed (not shown) when, still considering the case $k_f = 1.3$, $\beta_e = 0.2$, the injection rate is increased by a factor 16, confirming the prediction that it is a single parameter $\chi_f$ that governs the apparition of the instability of...
Figure 9: Time evolution of the GCH spectral density (normalized by $\epsilon_E/\epsilon_C$) in the shell $k_\perp = k_f/2$, for three values of $\beta_e$ and two forcing wavenumbers. Left: $k_\perp = 0.65$ with $\beta_e = 0.2$ (1), $\beta_e = 2$ (2) and $\beta_e = 10$ (3) corresponding respectively to runs $R_{0.2}^{1.3}$, $R_{1.3}^{2}$ and $R_{1.3}^{10}$. Right: $k_\perp = 6.5$ with $\beta_e = 2$ (1) , $\beta_e = 10$ (2) and $\beta_e = 50$ (3), corresponding respectively to runs $R_{13}^{2}$, $R_{13}^{10}$ and $R_{13}^{50}$.

the balanced state. A more quantitative study would be required to determine a precise threshold.

4.4. Nature of the cascades

4.4.1. Inverse transverse cascade

The inverse cascades displayed by the simulations previously discussed may suggest the existence of qualitatively different kinds of cascades. In fact, a continuous transition between a self-similar cascade and a spectral bump propagating to large scales can be perceived. Inspection of the time evolution of individual energy and GCH modes indicates that each transverse Fourier mode grows when the inverse cascade reaches the corresponding wavenumber and then decays before saturating. While for given forcing wavenumbers and injection rates, the characteristic growth time is similar for various $\beta_e$ parameters, the decay time is shorter when $\beta_e$ is decreased. This is illustrated in Fig. 9 which displays the time evolution of the GCH spectral density (normalized by $\epsilon_E/\epsilon_C$) in the shell at $k_\perp = k_f/2$, for three values of $\beta_e$, with $k_f = 1.3$ (left) and $k_f = 13$ (right). The cascade property thus continuously depends on $\beta_e$. It can be qualified as self-similar, at least at early times, when $\beta_e$ is large, i.e. when the decay time is very large. Differently, it displays a clear spectral bump that propagates to larger scales as this decay time is shorter, i.e. for $\beta_e$ smaller. The different types of time evolution observed for the spectra of the various runs we performed turn out to be consistently classified by the parameter $\chi_f$, which seems to quantify the deviation from a self-similar behavior. We conjecture that the pure self-similar cascades observed in the absence of waves, as e.g. in incompressible 2D Navier-Stokes flows, are gradually transformed into "propagating spectral bump" cascades as the influence of waves becomes more important, i.e. when $\chi_f$ decreases. In particular, the long time spectra that result from the small $\chi_f$ evolution do not display power laws. Another example supporting this claim is provided by simulations for which the forcing wavevectors are taken quasi-perpendicular to the ambient magnetic field, making $\chi_f$ very large. In this case, the dynamics is quasi-2D, waves have very long time scales, and the dynamics is very much influenced by the quasi-conservation of the squared magnetic potential, which is observed to undergo a self-similar inverse cascade (not shown). A more detailed investigation is nevertheless necessary to determine the
precise conditions under which the inverse cascade is self-similar or displays a propagating bump.

If we analyze the location of the spectral bump as time elapses, we get a power law, at least for time long enough. In the dispersive range, the bump wavenumber scales like \( t^{-2/3} \). When the maximum approaches the non-dispersive scales, as it is the case for run \( R_{1.3}^2 \), the dynamics gets slower with a wavenumber scaling like \( t^{-1/2} \). Longer integrations are needed to check whether this behaviour is persistent.

As suggested before, the fact that the transfer to large transverse scales proceeds through the formation of a propagating bump can originate from the presence of waves. As it will be shown in Section 7, this transfer is mostly non local. In Appendix B, a shell model restricted to the transverse plane, thus ignoring the wave component is used to study the role of nonlocal interactions. When the latter are strong enough, a propagating spectral bump is observed. The relation between wave and nonlocality is, to our knowledge, an open question that deserves a detailed study, using for example shell models including explicitly the parallel wave dynamics, in the spirit of the hybrid-shell model of Nigro et al. (2004); Verdini & Grappin (2012). Interestingly, this latter reference includes a control parameter which identifies with \( \chi_f \) and governs the strong or weak character of the turbulence.

4.4.2. **Inverse parallel cascade**

When comparing parallel spectra and the signs of the fluxes in Figures 2 and 4, it becomes evident that there is a transition from the split cascade with a significant inverse parallel GCH component for run \( R_{13}^2 \) at large \( k_f \), to a predominantly forward cascade for run \( R_{1.3}^2 \) at intermediate \( k_f \). In the latter case, the beating of \( \pm k_z \) modes can feed the zero mode in the parallel direction and further interactions can then drive the intermediate modes to an absolute equilibrium. While the parallel inverse cascade disappears, the transverse inverse GCH cascade survives as \( k_f \) is decreased. The latter can in fact be arrested due to the depression in the phase velocity of KAW provided \( \beta_e \) is large enough. This effect, together with the possible existence of a parallel inverse cascade at large values of \( \beta_e \) will be discussed in the next section.

5. **Impediment to the transverse cascade and formation of a finite-scale condensate**

As mentioned previously, the shape of the dispersion curve has an important effect on the cascade dynamics. This point is exemplified in Fig. 10, which shows the energy and GCH perpendicular (left) and parallel (right) spectra for run \( R_{6.5}^{10} \). As seen in Fig. 11, for \( \beta_e = 10 \), the parallel-phase velocity \( v_{ph} \) displays a local minimum at \( k_\perp = 1.34 \). We observe that after the inverse cascade has reached the largest scales in the simulation, energy and GCH perpendicular spectra develop a condensate at this specific wavenumber. The parallel spectra display a significant transfer towards the largest available scales that are rapidly reached, leading to a GCH spectrum close to a power law with an exponent of approximately -1.8. Later on, the energy and GCH spectra increase continuously in time, preserving their shape. In order to confirm that the spectral location of this condensate corresponds to the minimum of the parallel phase velocity, we show in Fig. 11 two simulations with a driving at \( k_f = 13 \), where \( \beta_e = 10 \) (top) and \( \beta_e = 50 \) (bottom). In the case where \( \beta_e = 10 \), the cascade clearly stops at a perpendicular wavenumber slightly larger than unity, where the energy and GCH accumulate at late times. Note that the largest scale of the simulation is slightly smaller here than in the run \( R_{6.5}^{10} \), so that the spectral range is not large enough to allow for a continuation of the inverse
Figure 10: Energy and GCH perpendicular (left) and parallel (right) spectra for run $R_{6.5}^{10}$.

Figure 11: Energy and GCH spectra (left) and perpendicular GCH flux (right) for Run $R_{13}^{10}$ (top) and $R_{13}^{50}$ (bottom). In the former simulation, the maximum of the spectral bump is located at $k_\perp = 1.3$ and in the latter at $k_\perp = 2$, corresponding to the respective local minima of the parallel-phase velocity $v_{\phi h}$ for the respective $\beta_e$, displayed in Fig. [?].
Figure 12: Perpendicular GCH spectra (left) and perpendicular GCH fluxes (right) for runs $R_{10.87}^0$ ((1)), $R_{2}^{10}$ ((2)) and $R_{1.3}^{10}$ ((3)). The perpendicular GCH flux of Run $R_{2}^{10}$ (2) (middle right) displays a nearly constant negative zone between the forcing and the minimum-dispersion wavenumber. $E_{\pm} (k_{\perp})$ spectra for run $R_{2}^{10}$ ((2)) (middle left) and $R_{0.87}^{10}$ ((1)) (bottom left).
cascade to scales larger than the peak. A similar behavior is observed in the left bottom panel for $\beta_e = 50$, but in this case the peak is centered at a slightly larger wavenumber, at a value close to $k_\perp = 2$. It turns out that the wavenumbers where the cascade is arrested precisely correspond to the minimum of the parallel phase velocity, as displayed in Fig. 1, and only depends on the parameter $\beta_e$. It is remarkable that such a small variation in the curvature of this quantity (e.g. between the cases $\beta_e = 2$ and $\beta_e = 10$) can lead to such an important dynamical effect.

In both cases, the arrest of the cascade leads to the development of a flat GCH spectrum, together with an energy spectrum growing like $k K^q$ at scales slightly larger than the injection scales. Such an effect was also observed in simulations of the inverse cascade of magnetic helicity in MHD after the cascade has been arrested by the effect of an hypo-diffusivity (Linkmann & Dallas 2017). These spectra correspond to those predicted by absolute equilibrium arguments (Linkmann & Dallas 2016). In the case of the present model, the GCH spectrum $E_C(k_\perp)$ and the energy spectrum $E(k_\perp)$, are related, in the absolute equilibrium regime, by $E_C(k_\perp) = E(k_\perp)/v_{ph}(k_\perp)$, which leads us to predict that they will have a similar slope for scales larger than the ion-scale but that $E_C(k_\perp)$ will be steeper than $E(k_\perp)$ by a factor of $k_\perp$ at sub-ion scales. Energy equipartition between the modes corresponds to transverse and parallel energy spectra scaling like $k_\perp$ and $k_\perp^0$ respectively.

The above observation suggests to perform simulations, keeping $\beta_e = 10$ but where the injection wavenumber $k_f$ is taken slightly larger than, equal to or slightly smaller than the wavenumber $k_m = 1.3$ where $v_{ph}$ has a minimum. The top left panel of Fig. 12 displays the transverse GCH spectra corresponding to these three runs $R_{10}^{1.2}$, $R_{1.3}^{10}$ and $R_{0.87}^{10}$. For $k_f = 2$, thus larger than $k_m$, the cascade is arrested at this latter wavenumber. A zone of negative GCH transverse flux develops in the small spectral range between the driving and the spectral peak (middle right panel). In contrast, for $k_f = 1.3$ or 0.87 (thus $k_f \leq k_m$), a weak non self-similar inverse transfer of energy and GCH develops at large scales, with a GCH flux that is essentially zero (top and bottom right panels). A main observation concerning the former simulation is that the condensate forms on the dominant wave only. Transfer to the large scales visible on run $R_{0.87}^{10}$ takes place in a similar way for both waves, consistent with the fact that at these large scales and for this relatively large value of $\beta_e$, the degree of imbalance prescribed by the driving tends to remain constant with no ”balance instability” (see discussion in the previous Section). Concerning the small scales, we note that pinning of $E^\pm$ takes place when the driving wavenumber $k_f$ is smaller than the minimum of $v_{ph}$, but not when it is larger.

A drastically different behavior is observed on both sides of the condensate upon inspection of the perpendicular GCH flux (Fig. 12 middle right panel). Whereas the GCH flux is constant and mostly negative (with some temporal fluctuations) in the spectral range between the condensate and the forcing, it decreases and reaches values at least 100 times smaller at scales larger than the condensate. This large-scale part of the cascade, more clearly developed for run $R_{0.87}^{10}$, is thus of a different nature and much
Figure 13: Magnetic-fluctuation field lines for Run $R_{1.3}^2$, showing a layer of looping filaments at a time corresponding to the late stage of the simulation. Field lines are differentiated by colors. Arrows indicate the direction of the field. Color map corresponds to the magnitude of the magnetic field both on the arrows and on the horizontal and the vertical cuts. The typical size of the 3D structures is indicated using green rulers.

slower, possibly due to the negative curvature of the parallel phase velocity in this range. This point deserves a more detailed investigation.

6. Structures in physical space

The inverse cascade of GCH discussed previously in Fourier space, has an interesting counterpart in physical space. As in two-dimensional hydrodynamics, we can expect the formation of large-scale vortices, but the situation is more complex due to the three-dimensional nature of the dynamics and the different types of behavior in the parallel direction. To give a general overview, the magnetic structures we observe in $B_\perp$ can be seen to form various inter-connected two-dimensional vortices, with a typical diameter that can be appreciated from the wavenumber corresponding to the maximum in the energy or helicity spectrum. In other words, as the perpendicular inverse cascade proceeds, the structures, which originate from initial fluctuations at a scale comparable

† In the context of weak turbulence, resonant three-wave interactions are impossible for negative dispersion (i.e. negative curvature) ([Zakharov et al. 1992](#)) and four-wave processes then have to be taken into account.
Figure 14: Elsasser potential $\mu^+$ for run $R^2_{1.3}$ in selected $(x, y)$ (left) and $(x, z)$ (right) planes. The $z$-scale is much finer than the perpendicular one, hence the term "pancake" to qualify the observed structures.

Figure 15: Magnetic-perturbation field lines for Run $R^2_{0.5}$ showing three magnetic vortices with a vertical extension much greater than in Run $R^2_{1.3}$, hence the term vortex. The field has been filtered to remove the imprint of the small scales such as forcing and dissipation. Same graphical conventions were used as in Fig. 13.

to the forcing one, grow over time. This process continues either until we reach the finite-$k_\perp$ condensate ($\beta_\perp > 3.5$), or (below that value) the dynamics slows down so much so that it becomes impractical to integrate the equations numerically. The simulations are stopped before the cascade reaches the smallest wavenumber and therefore typically several large-scale vortices exist simultaneously in the full domain, even at late times.
Figure 16: Magnetic-perturbation field lines for run $R_{13}^2$ at a late time of the simulation, showing the formation of an elongated vortex. The field has been filtered to remove the imprint of the small scales such as forcing and dissipation. Same graphical conventions were used as in Fig. 13.

Another aspect of the dynamics is the parallel inverse cascade which controls the parallel correlation length of the fluctuating magnetic field of vortices. In run $R_{1,3}^2$, no inverse cascade takes place in the parallel direction and, not surprisingly, the structures in physical space are found to be flattened in horizontal layers, as shown in Fig. 13, which displays a few field line filaments associated with the magnetic perturbations calculated based on $B_\perp = \nabla_\perp \times (A_\parallel \mathbf{z})$ and $B_z$ defined in Section 2. The aspect ratio of the magnetic loops is more easily appreciated when looking at color scale plots in the ($x, y$)– and ($x, z$)-planes for the Elsasser potential $\mu^+$ of the dominant wave, as displayed in Fig. 14. The cross-section containing the longitudinal $z$-axis (the right panel) gives a clear indication that the structures are shaped as pancakes in the gyrofluid variables. Figures 15 and 16 display typical magnetic vortices found in runs $R_{6,5}^2$ and $R_{13}^2$, together with the color plots of the magnetic field magnitude in three orthogonal planes. The aspect ratio of the vortices, equal to $9.69/7.92 \approx 1.2$ and $9.19/3.17 \approx 2.9$ respectively, increases with the driving wavenumber. The vortex of run $R_{13}^2$ is particularly coherent in the parallel direction because this run develops a significant inverse cascade in this direction (see Fig. 4, middle panels).

An even more elongated magnetic vortex (with an aspect ratio of $13/3.75 \approx 3.47$)
Figure 17: Run $R_{6.5}^{10}$ vortices. They appear to be more correlated in $z$ and with a smaller perpendicular radius than for the corresponding Run $R_{6.5}^2$. The field has been filtered to remove the imprint of the small scales such as forcing and dissipation. Same graphical conventions were used as in Fig. 13.

Figure 18: Elsasser potential $\mu^+$ for run $R_{6.5}^{10}$ seen in in selected $(x, y)$ (left) and $(x, z)$ (right) planes. The vortices have a tendency to group themselves into a large-scale structure.
Figure 19: Phase space diagram in the \((k_f, \beta_e)\)-plane, where the locations of the various types of cascades and structures in physical space are indicated as shaded areas. The relevant runs are also added at their corresponding place together with the bottleneck locus associated with the values of \(k_\perp\) where the parallel phase velocity has a minimum.

is found in run \(R_{10}^{10}\) (see Fig. 17). This is the case when a condensate has formed at \(k_\perp = 1.3\), leading to a specific ion-scale transverse size for the magnetic vortex, together with a continuation of the inverse cascade towards larger scales, responsible for the piling up of several vortices, as also seen on the \(\mu^+\) color plots in the \((x, y)\) and \((x, z)\) planes.

Alfvén vortices were found analytically in Petviashvili & Pokhotelov (1992) and also directly observed in the solar wind (Perrone et al. 2017) or in the terrestrial magnetosheath (Wang et al. 2019). The simulations we have discussed above provide a natural way to form vortices, and to demonstrate their dynamical stability, but the question arises whether these vortices are related to the Alfvén vortices mentioned above. Further development is needed to analyze the nature of these structures and the process of their formation, i.e. whether it proceeds in a conventional two-dimensional hydrodynamics sense of vortex merger/nucleation, and if one can observe instances of reconnection, as discussed by Zhou et al. (2020) in the context of RMHD.

Figure 19 provides a graphical summary of the performed simulations, together with the phenomenology observed in different regimes, in a phase diagram whose axes correspond to the driving wavenumber \(k_f\) and the electron beta parameter \(\beta_e\). We can see from the figure that there are two main intersecting areas at sub-ion scales: the green part of the phase space, which corresponds to the elongated vortices, and the red part which corresponds to the runs with \(\beta_e > 3.5\) that, due to the existence of a local minimum in the phase velocity curve, produce a finite-\(k_\perp\) condensate. The “bottleneck” curve is the locus of points on the diagram for which \(v_{ph}\) displays such a minimum.

For moderate values of \(\beta_e\), a transition is also observed close to the ion scale between situations without a parallel inverse cascade (at smaller values of \(k_f\)), where pancake structures are observed, and those at larger \(k_f\) where a clear inverse cascade develops in the parallel direction, associated with strongly elongated vortices. As we cross the bottleneck curve we encounter a critical behavior described in Section 5 associated with
runs at \( k_f \sim 1 \). Beyond, for scales much larger than the ion sonic Larmor radius, the dynamics is dominated by the forward cascade of GCH, that we do not address in this paper. Below the red region, we have runs that do not produce finite-\( k_\perp \) condensate, since the minimum in phase velocity disappears. It must be born in mind that only a selected choice of runs is available for drawing general conclusions, and therefore, this diagram has to be understood to only carry a schematic meaning.

7. Nonlocality of energy and GCH shell-to-shell transfers

Useful information on the cascade dynamics is provided by analyzing the energy and GCH shell-to-shell transfers in Fourier space, when proceeding as in [Alexakis et al. (2007)] and [Mininni et al. (2007)]. As in the calculation of the transverse and parallel energy and GCH spectra, we perform a partition of the Fourier space either in cylindrical shells \( S_{k_\perp} \) including all the wavevectors whose transverse component obeys \( K_\perp < |k_\perp| \leq K_\perp + dk_\perp \), or in slabs \( S_{k_\parallel} \) including all the wavevectors whose parallel component \( k_\parallel \) obeys \( K_\parallel < |k_\parallel| \leq K_\parallel + dk_\parallel \), with the goal to study the perpendicular and parallel transfer respectively. Here, \( dk_\perp \) and \( dk_\parallel \) refer to the spectral mesh along the transverse axes and along the parallel axis, respectively.

Our aim is to estimate the energy and GCH transfer rates \( T_E(K, P) \) and \( T_C(K, P) \), respectively, between a shell \( K \) that receives and a shell \( P \) that gives, such that it is possible to write \( \hat{\partial}_t E_K = \sum_P T_E(K, P) \) and \( \hat{\partial}_t C_K = \sum_P T_C(K, P) \), where \( E_K \) and \( C_K \) are the contributions of a cylindrical or of a slab-shaped shell \( S_K \) to the energy \( \mathcal{E} \) and to the GCH \( \mathcal{C} \) respectively (\( \mathcal{E} = \sum_K E_K \) and \( \mathcal{C} = \sum_K C_K \)). Due its physical meaning, we expect that \( T_E(P, K) = -T_E(K, P) \) and \( T_C(P, K) = -T_C(K, P) \) are antisymmetric, since the amount of the invariant that the \( K \)-shell gives to \( P \) equals the amount of the same invariant that the \( P \)-shell receives. Finally, the total fluxes of energy \( \Pi_E(K) \) and GCH \( \Pi_C(K) \) through a wavenumber \( K \)

\[
\Pi_E(K) := -\sum_{K'=0}^{K} \sum_{P=0}^{\infty} T_E(K', P), \quad \Pi_C(K) := -\sum_{K'=0}^{K} \sum_{P=0}^{\infty} T_C(K', P) \quad (7.1)
\]

are obtained from the shell-to-shell transfer \( T_E(K', P) \) or \( T_C(K', P) \) by the summing over \( P \) and over \( K' \leq K \).

A scalar field \( A \) restricted to a shell \( S_K \) (here and in the following \( K \) holds for either \( K_\perp \) or \( K_\parallel \)) can be represented in terms of its Fourier modes \( \hat{A}_k \) by

\[
A_K(x) = \sum_{k \in S_K} \hat{A}_k e^{i k \cdot x}, \quad (7.2)
\]

so that

\[
\sum_{K} A_K(x) = A(x). \quad (7.3)
\]

When considering two scalar fields \( A \) and \( B \), one has,

\[
\int \, d^3 x A_K(x) B(x) = \sum_{k \in S_K^+} \int \, d^3 x \hat{A}_k e^{i k \cdot x} B(x) = \frac{1}{(2\pi)^3} \sum_{k \in S_K^+} \hat{A}_k \hat{B}_k^* \quad (7.4)
\]

and

\[
\int \, d^3 x A_K(x) B_K(x) = \sum_{k \in S_K^+} \sum_{k' \in S_K^+} \int \, d^3 x \hat{A}_k e^{i (k+k') \cdot x} \hat{B}_k' = \frac{1}{(2\pi)^3} \sum_{k \in S_K^+} \hat{A}_k \hat{B}_k^*, \quad (7.5)
\]
where again $K$ holds for either $K_\perp$ or $K_\parallel$, depending if perpendicular or parallel transfers are considered. In practice, the latter expression is used to evaluate various integrals arising in the expression of the transfer $T_E(K, P)$ and $T_C(K, P)$.

For the two-fluid gyrofluid, the two-fluid Lie-dragging formulation (2.15) can be conveniently used when electron inertia is retained. The total GCH reads

$$C = \int C d^3 x,$$

where we define the GCH density as

$$\frac{\partial C}{\partial t} = \frac{1}{4\delta} \left\{ (G^+)^2 - (G^-)^2 \right\}, \tag{7.6}$$

It follows that

$$\frac{\partial C}{\partial t} = \frac{1}{2\delta} \left\{ G^+[\varphi^+, G^+] - G^-[\varphi^-, G^-] - \frac{1}{2\delta} \delta_z ((G^+)^2 + (G^-)^2) ight\} + \frac{2}{\delta} L_e A_\parallel \delta_z (1 - M_1) \varphi \right\}. \tag{7.7}$$

To simplify the notation, we will use

$$\sum_{\pm} G^\pm [\varphi, G^\pm] = G^\pm \varphi + G^\pm_\parallel - G^- [\varphi, -G^-].$$

In this case, the contribution $C_K = \int_{S_K} C d^3 x$ of shell $K$ to GCH ($C = \sum_K C_K$) obeys

$$\dot{C}_K = \frac{1}{2\delta} \int d^3 x \sum_{\pm} G^\pm_K [\varphi, G^\pm_\parallel], \tag{7.8}$$

where the contribution of the terms involving a $z$-derivative cancel out by space integration, when using that $N_e = -M_e \varphi$ and the hermiticity of the operators $M_1, M_2$ and $-\Delta_\perp$. The formulation given by Eq. (7.8) is interesting, since it is as though the transfer is achieved via Lie-drag over the flow corresponding to $\varphi^\pm$. In terms of primitive fields, the expression evaluates to

$$\dot{C}_K = \frac{1}{2\delta} \int d^3 x \sum_{\pm} (L_e A_\parallel)_K \left[ \left( \frac{A_\parallel}{\delta} \right)_K - \left( L_e A_\parallel \right)_K \right] - (M_3 - M_2) \varphi; \tag{7.9}$$

$$\delta(M_2 \varphi)_p - \delta(M_2 \varphi)_K \left[ (M_3 - M_2) \varphi, (L_e A_\parallel)_p \right] + \delta(M_2 \varphi)_K \left[ \frac{A_\parallel}{\delta}, \delta(M_2 \varphi)_p \right].$$

The first summand, which involves a term $A_{\parallel K}[A_\parallel, A_{\parallel p}] / \delta$ has a singular behavior as $\delta \to 0$, but it does not contribute to the flux and is thus physically irrelevant. We will ignore it in the estimate of the shell-to-shell transfer.

Turning to the energy, we write $E = \sum_K E_K$ where $E_K$, given by

$$E_K = \frac{1}{2} \int \left\{ (M_2 \varphi)_K ((1 - M_1) \varphi) + (M_2 \varphi)_K^2 - \frac{2}{\beta_e} (L_e A_K) (\Delta_{\parallel K} A_K) \right\} d^3 x, \tag{7.10}$$

is the contribution of shell $K$ to the energy. The operator arising in the above equation being hermitian, we can write

$$\frac{\partial E_K}{\partial t} = \int \left\{ (1 - M_1) \varphi_K \frac{\partial}{\partial t} (M_2 \varphi) + (M_2 \varphi_K) \frac{\partial}{\partial t} (M_2 \varphi) - \frac{2}{\beta_e} (\Delta_{\parallel K} A_K) \frac{\partial}{\partial t} (L_e A_K) \right\} d^3 x \tag{7.11}$$

or

$$\frac{\partial E_K}{\partial t} = \int \left\{ (1 - M_1) \varphi_K \frac{\partial}{\partial t} (M_2 \varphi) + (M_2 \varphi_K) \frac{\partial}{\partial t} (M_2 \varphi) - \frac{2}{\beta_e} (\Delta_{\parallel K} A_K) \frac{\partial}{\partial t} (L_e A) \right\} d^3 x,$$
\[
\partial_t E_K = \frac{1}{4} \left\{ - (\varphi_K^+ + \varphi_K^-) \partial_t \frac{G^+ - G^-}{\delta} + \frac{G^+_K - G^-_K}{\delta} \partial_t \frac{G^+ - G^-}{\delta} + \frac{G^+_K + G^-_K}{\delta} \partial_t \frac{G^+ + G^-}{\delta} \right. \\
\left. - (\varphi_K^+ - \varphi_K^-) \partial_t \frac{G^+ + G^-}{\delta} \right\} d^3 x
\]

(7.12)

where we have replaced \( L_e \) by its definition. Equivalently, after a simple algebraic rearrangement,

\[
\partial_t E_K = \frac{1}{2\delta} \int \left\{ \left( \frac{G^+_K - \varphi_K^+}{\delta} \right) \partial_t G^+ + \left( \frac{G^-_K + \varphi_K^-}{\delta} \right) \partial_t G^- \right\} d^3 x,
\]

(7.13)

that we rewrite in the more compact form

\[
\partial_t E_K = \frac{1}{2\delta} \sum_{\pm} \left( \frac{G^+_K \mp \varphi_K^+}{\delta} \right) \partial_t G^\pm d^3 x.
\]

(7.14)

We then use the model equations in the form

\[
\partial_t G^\pm + [\varphi^\pm, G^\pm \mp \delta \varphi^\pm] + \partial_z (\varphi^\pm \mp \frac{1}{\delta} G^\pm) = 0
\]

(7.15)

where we include a additional term in the second argument of the bracket which does not contribute, and get

\[
\partial_t E_K = -\frac{1}{2} \int \sum_{\pm} \left( \frac{G^+_K \mp \varphi_K^+}{\delta} \right) \left[ \varphi^\pm, \frac{G^\pm}{\delta} \mp \varphi^\pm \right] d^3 x.
\]

(7.16)

This enables defining the shell-to-shell energy transfer \( T(K, P) \) such that

\[
\partial_t E_K = \sum_P T(K, P)
\]

(7.17)

via

\[
T(K, P) = \frac{1}{2} \int \sum_{\pm} \left( \frac{G^+_K \mp \varphi_K^+}{\delta} \right) \left[ \varphi^\pm, \frac{G^\pm}{\delta} \mp \varphi^\pm \right] d^3 x,
\]

(7.18)

which is antisymmetric in \( K \) and \( P \).

In terms of the primitive variables \( (M_3 = 1 + M_2 - M_1) \),

\[
\left( \frac{G^+_K \mp \varphi_K^+}{\delta} \right) \left[ \varphi^\pm, \frac{G^\pm}{\delta} \mp \varphi^\pm \right] = \\
\left( \mp (M_3 \varphi)_K - \frac{2\delta}{\beta_e} A_\perp A_K \right) \left\{ M_3 \varphi \pm \frac{1}{\delta} A_\parallel, \mp (M_3 \varphi)_P - \frac{2\delta}{\beta_e} A_\perp A_P \right\}
\]

(7.19)

includes a term \( (M_3 \varphi)_K [(M_3 \varphi), (M_3 \varphi)_P] \) that does not contribute to the flux, and that we thus suppress. As a result, the renormalized transfers we report are calculated according to the formulas

\[
T_C(K, P) = \int d^3 x \left( \frac{1}{\delta} A_\parallel K [A_\parallel, A_\parallel P] - \frac{1}{2\delta} \sum_{\pm} \pm G^+_K \left[ \varphi^\pm, \frac{G^\pm}{\delta} \right] \right)
\]

(7.20)

and

\[
T_E(K, P) = \int d^3 x \left( - (M_3 \varphi)_K [(M_3 \varphi), (M_3 \varphi)_P] - \frac{1}{2} \sum_{\pm} \left( \frac{G^+_K \mp \varphi_K^+}{\delta} \right) \left[ \varphi^\pm, \frac{G^\pm}{\delta} \mp \varphi^\pm \right] \right).
\]

(7.21)
Figure 20: Early-time shell-to-shell perpendicular (left) and parallel (right) transfers of GCH for run $R_{1.3}^2$. Strong nonlocality is visible in both directions, in spite of the self-similar character of the transverse cascade at this time. The colormap simultaneously shows relatively low transfer via light colors (like cyan and orange), and contrasts them with stronger one (displayed via dark red and dark blue), typically involving the forcing scale.

Figure 21: Perpendicular (left) and parallel (right) GCH transfers of Run $R_{1.3}^2$ at a late time of the simulation. Local transfer develops in the perpendicular direction, the spectral bump producing a forward cascade of energy and GCH. Parallel transfer looks very different and lacking coherence, consistent with the pancake structures seen in Fig. [14]
Figure 22: Perpendicular (left) and parallel (right) transfers of GCH for run $R_{13}^2$ at a late time of the simulation, when the run displays an extended inverse cascade in the transverse direction. In the parallel direction, the inverse cascade appears to be much more local.

In the following, we analyze the transfer in simulations $R_{13}^2$ and $R_{13}^2$ for which the cascades display significantly different features. For run $R_{13}^2$, Figs. 20 and 21 show the GCH shell-to-shell transfers both in the transverse plane and in the longitudinal direction, respectively at an early time (when the cascade is self-similar) and at a time close to the end of the simulation. Transfers in the transverse directions are computed from Eqs. (7.20) and (7.21). Similar formula are used for the longitudinal transfers, cylindrical shells being replaced by slabs.

At early time (Fig. 20), both the transverse and parallel inverse transfers of GCH appear to be strongly nonlocal, despite the fact that the cascade displays self-similar features. Like in most of the simulations, we observe that the transfers on the modes located on the upper vertical and left horizontal axes that meet at the forcing change sign, being associated to an inverse transfer close to the forcing and to a direct one for some distant modes (a characteristic cross signature, demonstrating strong interactions between the forcing and other scales). Such features are also found in the simulation of the shell model discussed in Appendix [B], as illustrated in Fig. 26. On the contrary, the nonlocal transfers on these horizontal and vertical axes involving wavenumbers larger than the forcing, are consistently associated with direct transfers. In the parallel direction, there is no significant inverse transfer but interacting triads involving the forcing and its first harmonic are clearly visible (see also the parallel spectrum at the first output in the top right panel of Fig. 2). The oblique lines ($k_{||} + p_{||} = k_f$ and circular permutations) and those parallel to the axes ($k_{||} = k_f$ or $p_{||} = k_f$) correspond to the interactions between triads including the driving mode. No other interaction is significant at this early time.

At late times, displayed in Fig. 21, no effective parallel transfer is identified at scales larger than the forcing, as confirmed by the fact that the parallel spectra do not show a significant increase at these scales. Note that the quasi-local interactions near the diagonal at the top left corner of the perpendicular GCH transfer panel, are indicative of a transfer from the largest excited scales towards smaller scales. This corresponds
to the forward cascade originating from the propagating bump. This transfer is in fact
subdominant compared to the somewhat less local inverse transfers in surrounding shells
and to the more diffuse inverse transfer involving wavenumbers located above (orange)
and on the left (cyan) of the forcing (thus between pair of cylindrical shells for which one
of the elements has an index smaller than that of the driving). We also clearly see local
backward transfer at scales close to the forcing.

In contrast, the transfers of run $R_{13}^2$, plotted in Fig. 22 at late times display more local
features, both in the parallel and perpendicular directions. The perpendicular inverse
transfer of GCH still involves nonlocal interactions between the spectral bump and the
forcing, while energy (not shown) has predominantly local features. It is noteworthy to
mention that early bump propagation is always dominated by the nonlocal cross-type
interaction between the forcing and the bump.

The transfers displayed for these two runs are typical of most of the simulations
discussed in this paper. The transfers involving scales larger than the forcing are usually
quite complex, but clearly indicate that the inverse cascade of GCH is intrinsically
nonlocal. The nonlocal shell model discussed in Appendix B also indicates that the
transfers which persist on long times result from nonlocal interactions involving the
forcing and the energy peak.

8. Conclusion

Direct numerical simulations of a two-field gyrofluid model for KAW turbulence
randomly driven at the sub-ion scales have been presented. A main observation is the
development of an inverse cascade of GCH in the transverse direction and, to a lesser
degree, of energy. This cascade is essentially nonlocal and its properties are strongly
sensitive to the nonlinearity parameter $\chi_f$ (ratio of the period of the waves to the
characteristic nonlinear time at the driving scale), and to the profile of the dispersion
relation. When $\chi_f$ is large enough, a self-similar cascade develops while, for smaller
values of $\chi_f$, a spectral bump is observed to propagate towards small wavenumbers.
In the latter case, transient parametric decay instability events are also possible, which
enhance the development of the cascade. At late times, when the cascade reaches non-
dispersive scales, it slows down and energy and GCH accumulate through the formation
of a spectral bump with, in some cases, the establishment of an absolute equilibrium
between this bump and the forcing wavenumber. For the moderate values of the GCH
injection rate we use throughout the paper, one type of wave (Elsasser component) is
dominant, while the other develops an equilibrium spectrum, except when the driving
takes place at a wavenumber smaller than the minimum of the parallel-phase velocity
$v_{ph} = \omega/k_\parallel$ (which exists when $\beta_e \gtrsim 3.5$). When the injection wavenumber is large or
$\beta_e$ is small enough (i.e. when $\chi_f$ is small), such a cascade is even possible when the
driving is balanced. This result suggests that fast magnetic reconnection events could
not only drive small-scale turbulence (Cerri & Califano 2017), but also induce a cascade
to larger scales, a dynamics somewhat similar to the island merging proposed in Franci
et al. (2017). In the parallel direction, depending on the parameters, an inverse cascade
resulting mostly from local interactions can also develop.

Another result concerns the capability of the transverse GCH inverse cascade to
lead to the formation of coherent vortices with a transverse size comparable to the
ion scale, possibly related to the structures observed in the solar wind (Perrone et al.
2017). Depending on the plasma parameters, their longitudinal correlation length (in
the rescaled variables involved by the model) can be either comparable to the injection
scale or much larger when an inverse cascade develops in the parallel direction. Note that
when transforming the (rescaled) coordinates to the physical ones, the structures are to be stretched along this direction by a quantity given by the inverse expansion parameter, for example the degree of spectral anisotropy. Estimating this parameter depends on the situation that is supposed to be modeled. An interesting development would consist of comparing the signatures of the vortices obtained in our simulations with satellite data.

Further developments of the present study includes the coupling to slow magnetosonic waves, using a four-field gyrofluid model (Tassi et al. 2020). This description appears especially relevant in the regions near the Sun, explored by Parker Solar Probe and Solar Orbiter, where large-scale compressibility effects are important. Such a model allows for the development of the Alfvén wave parametric decay instability, an effect that is viewed as an efficient process for generating counter-propagating waves, requested for the existence of turbulence at the MHD scales (Vias & Goldstein 1991, Del Zanna et al. 2001, Shoda & Yokoyama 2018).

Going back to the description of the Alfvén vortices, it would be of great interest to develop more precise diagnostics for comparing the vortices produced in this work to those observed in the solar wind and associated by Alexandrova (2008) to the $k'^4$ transition range. Another, more theoretical avenue, could involve understanding the impediment of the inverse cascade observed in the simulations, and in particular the role of a small depression in the KAW parallel-phase velocity in the formation of a finite-$k_\perp$ condensate.

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**Appendix A. Decay instability**

Using the same notation for the operators and their Fourier symbols, we define the Fourier components $c_{k}^{\sigma_k} = D(k_\perp)\mu_k^{\sigma_k}$ of the fields $D\mu^\pm$ (where $\sigma_k = \pm$) and, in the interaction representation,

$$a_{k}^{\sigma_k} = e^{i\omega_k^{\sigma_k} t} c_{k},$$

with $\omega_k^{\sigma_k} = \sigma_k v_{ph}(k_\perp)k_z$. The gyrofluid equations rewrite

$$\dot{c}_t a_{k}^{\sigma_k} - \int_{\sigma_p,\sigma_q} e^{i\Omega_{k,pq}^{\sigma_k\sigma_p\sigma_q}} V_{kpq}^{\sigma_k\sigma_p\sigma_q} a_{p}^{\sigma_p} a_{q}^{\sigma_q} \delta(p + q - k) dp dq = 0,$$

where we define

$$\Omega_{k,pq}^{\sigma_k\sigma_p\sigma_q} = \omega_k^{\sigma_k} - \omega_p^{\sigma_p} - \omega_q^{\sigma_q} = \sigma_k v_{ph}(k_\perp)k_\parallel - \sigma_p v_{ph}(p_\perp) p_\parallel - \sigma_q v_{ph}(q_\perp) q_\parallel.$$

When neglecting electron inertia,

$$V_{kpq}^{\sigma_k\sigma_p\sigma_q} = \frac{1}{8} \frac{\sigma_p \sigma_q (\hat{x} \cdot (p \times q))}{k_\perp p_\perp q_\perp} \left( \frac{\sigma_p}{\xi(p_\perp)} - \frac{\sigma_q}{\xi(q_\perp)} \right) (\sigma_k k_\parallel^2 \xi(k_\perp) + \sigma_p p_\parallel^2 \xi(p_\perp) + \sigma_q q_\parallel^2 \xi(q_\perp))$$

with $\xi = s/v_{ph}$, consistent with the vertex given in Eq. (6.4) of Voitenko (1998a), directly derived from the Vlasov-Maxwell equations.

At this point it is easy to estimate the growth rate of the decay parametric instability of a KAW into two other KAWs. For this purpose, one considers a pump of type $\sigma_k$, with
wavenumber $k$, frequency $\omega_k$ and complex amplitude $a_k^{\sigma_k} = D_e (A \varphi(k, t) + \sigma_k A_k'(k, t))$ (see Eq. (2.10)). It can interact with KAWs of wavevectors $p$ and $q$ that satisfy the resonance conditions

$$k = p + q,$$

$$\omega_k = \omega_p + \omega_q,$$

in the way that

$$\dot{a}_p a_p^{*} = V_{p-k-q} a_k^{\sigma_k} a_q^{\sigma_q}$$

$$\dot{a}_q a_q^{*} = V_{-q-kp} a_k^{\sigma_k} a_p^{\sigma_p}.$$  

This results in a growth rate $\gamma$ for the modes $a_p^{\sigma_p}$ and $a_q^{\sigma_q}$ given by

$$\gamma^2 = V_{p-k-q} V_{-q-kp} a_k^{\sigma_k} a_p^{\sigma_p} a_q^{\sigma_q}$$

$$- \frac{1}{64} \frac{(z \cdot (p \times q))^2}{\xi(p_{\perp}) \xi(q_{\perp})} \left( \frac{\sigma_k}{\xi(q_{\perp})} - \frac{\sigma_p}{\xi(k_{\perp})} \right) \left( \frac{\sigma_p}{\xi(k_{\perp})} - \frac{\sigma_k}{\xi(p_{\perp})} \right)$$

$$\times \left( \sigma_k q_{\perp}^2 \xi(k_{\perp}) + \sigma_p p_{\perp}^2 \xi(p_{\perp}) + \sigma_q q_{\perp}^2 \xi(q_{\perp}) \right)^2 |a_k^{\sigma_k}|^2,$$

where the eigenmode $a_k^{\sigma_k}$ satisfies $|a_k^{\sigma_k}|^2 = (8/\beta_e) |B_k|^2$ (Passot & Sulem 2019). Instability thus requires

$$\left( \frac{\sigma_k}{\xi(q_{\perp})} - \frac{\sigma_p}{\xi(k_{\perp})} \right) \left( \frac{\sigma_p}{\xi(k_{\perp})} - \frac{\sigma_k}{\xi(p_{\perp})} \right) > 0.$$  

Equation (7.5) of Voitenko (1998a) (see also Zhao et al. 2010b) is reproduced when noting that in the latter equation the length unit is not $\rho_s$ but $\rho_i$ defined as $\sqrt{\tau/\rho_s}$ and that $\xi$, defined as the ratio of the Alfvén to the KAW frequency, is independent of the length unit. In the framework of the present paper, the growth rate of the parametric decay instability scales, in the ERMHD regime, as $\gamma \sim k^2 \beta_e^{-1/2} |B_k|$. In the ERMHD regime with the additional condition $\beta_e > 1$, it follows from Table 1 of Passot & Sulem (2019) that

$$\xi(k_{\perp}) = \frac{1}{k_{\perp}} \sqrt{\frac{\beta_e}{2}}.$$  

Noting that a wavenumber $k_{\perp}$ in $\rho_s^{-1}$ units rewrites

$$k_{\perp} = (\overline{k}_{\perp} d_i) \sqrt{\frac{\beta_e}{2}},$$  

where $\overline{k}_{\perp}$ denotes the corresponding dimensional wavenumber and thus $\overline{k}_{\perp} d_i$ the corresponding value in units of the inverse ion inertial length $d_i^{-1}$, we get $\xi = 1/(\overline{k}_{\perp} d_i)$. We conclude that, in this regime, the growth rate $\gamma$ does not scale with $\beta_e$ when $d_i$ is used as length unit, as it is usually the case in EMHD for whistler waves (Zhao et al. 2010a).

**Appendix B. A shell model with nonlocal interactions**

Let us consider a shell model for $(N + 1)$ real field amplitudes $A_i(t)$ obeying the following equations

$$\frac{d}{dt} A_i(t) = c_{i,1}^1 A_{i+1}(t) A_{i+2}(t) + c_{i,1}^2 A_{i-1}(t) A_{i+1}(t) + c_{i,1}^3 A_{i-2}(t) A_{i-1}(t)$$

where $c_{i,1}^j$ are the nonlocal interaction coefficients.
The discrete Heaviside function is here defined such that $H[n] = 0$ for $n < 0$ and $H[n] = 1$ for $n \geq 0$. Terms involving indices $i < 0$ or $i > N$ are not included. What we call a mode $A_i$ is a representative (or some kind of average) of all the modes contained in a shell delimited by the perpendicular wavenumbers $k_{i,i}$ and $k_{i,(i+1)}$. These shells are assumed to be logarithmically spaced ($k_{i,i} = \lambda^i$), the 0-index corresponding to the smallest wavenumber. Equations (B.1)-(B.2) do not attempt to give an accurate modelling of Eqs. (2.1)-(2.2), but rather aim at capturing some essential and generic behaviors. They constitute a very simplified shell model that does not distinguish between the eigenmodes $a_k$ (defined in Eq. (A.1)) with different values of $\sigma$ (related to the propagation direction). More elaborated systems were derived for MHD (Gloaguen et al. 1985, Frick & Sokoloff 1998, Giuliani & Carbone 1998, Giuliani 1999, Lessinnes et al. 2009) or Hall-MHD (Galtier & Buchlin 2007), without nonlocal interactions. These models all originate from the GOY model (Gledzer 1973, Ohkitani & Yamada 1989), which identifies with the Sabra model (L'vov et al. 1998) when the phases of the (in principle complex) modes are not taken into account. For the simple considerations such as the ones presented in this Appendix, the restriction to real amplitudes gives essentially the same results as the Sabra model (see also Ditlevsen & Mogensen 1996). The present model is inspired from the more sophisticated one given in Plunian & Stepanov (2007). In addition to the local interactions of a given shell $i$ with the two neighbouring shells of index $j$ satisfying $i - 2 \leq j \leq i + 2$, it also includes nonlocal interactions with more distant shells (whose coupling coefficients have an index $m > 1$). Here we restrict the nonlocal interactions to those involving the forced mode $A_N$ since, at least at the beginning of the simulations, its amplitude dominates over the other ones. After proper normalization, we can take for the forcing $F = 1$. All the nonlinearities involve separated triads. Our choice of coupling coefficients ensures, in the absence of forcing and damping, a detailed conservation (among all triads taken in isolation) of the total energy, which for the whole system reads $E = \sum_{i=0}^{N} A_i^2(t)$. Another quadratic invariant is $H = \sum_{i=0}^{N} (-\alpha)^i A_i^2$, which, depending on the parameters, is not necessarily positive definite. This invariant is similar to the cross-helicity defined in a shell model of MHD turbulence (Frick & Sokoloff 1998).

As discussed in Ditlevsen & Mogensen (1996), $\alpha > 0$ corresponds to 3D fluid models (usually $\alpha = 2$ is chosen), whereas 2D fluid turbulence with its notorious inverse cascade corresponds to $\alpha < 0$. The case $0 < \alpha < 1$ corresponds to a situation where the last mode of a given triad can decay into the first two. We shall here focus on this case since we expect an inverse energy transfer originating from the injection acting at the smallest Taking a hyperviscous form for the dissipation coefficients, we set $\nu_i = \nu \lambda^{-2\gamma i}$. Here we shall assume $\gamma = 1$, $\alpha = 0.5$ and $\lambda = 2$, so that there remain three free parameters: the damping coefficient $\nu$, the parameter $\mu$ and the overall ratio $r$ between the nonlocal and the local interaction coefficients. All the modes are initialized with $A_i(0) = 10^{-6}$.

B.1. A five-mode system to illustrate the parametric decay process

We first consider the system (B.1)-(B.2) for $N = 4$ with only three interacting triads in order to illustrate, with the least amount of complexity, the decay events observed
For example in the early-time dynamics of run $R^2_{36}$. In this case, the modes $A_i$ can also be interpreted as individual Fourier modes. The three-wave equations for an isolated parametric decay process are similar to the equations for 3 shells in local interaction. They involve complex amplitudes, but we verified that restricting to real amplitudes gives similar results. Taking $\alpha = 0.5$, a given mode of index $i$ is unstable to two other modes with indices $i-1$ and $i-2$. For KAWs, this corresponds to the case of counter-propagating wave interactions \cite{Voitenko1998a}. The situation where the decay leads to the growth of modes with indices $i-1$ and $i+1$ (co-propagating waves) would require to take $\alpha < 0$. We do not consider this situation here.

In this restricted system, the mode which is forced, here $A_4$, undergoes a parametric decay, resulting in the growth of two other modes, $A_3$ and $A_2$, the latter being later subject to a secondary decay process leading to the growth of $A_1$ and $A_0$. The two successive decays can be observed within the system composed of the two triads $T_1 = (A_2, A_3, A_4)$ and $T_2 = (A_0, A_1, A_2)$ with the couplings associated with the indices $(i = 2, m = 1)$ and $(i = 0, m = 1)$ of the coefficients $c_{i,m}^k$. A more interesting behavior is however observed if we also include the nonlocal interaction with the triad $T_3 = (A_0, A_3, A_4)$ associated with the coupling $(i = 0, m = 3)$. For convenience, we here write the resulting system

$$\frac{d}{dt} A_0(t) = c_{0,1}^1 A_1(t) A_2(t) + c_{0,3}^1 A_3(t) A_4(t) - \nu_0 A_0(t)$$  \hspace{1cm} (B3)

$$\frac{d}{dt} A_1(t) = c_{0,1}^2 A_0(t) A_2(t) - \nu_1 A_1(t)$$  \hspace{1cm} (B4)

$$\frac{d}{dt} A_2(t) = c_{0,1}^3 A_0(t) A_1(t) + c_{2,1}^3 A_3(t) A_4(t) - \nu_2 A_2(t)$$  \hspace{1cm} (B5)

$$\frac{d}{dt} A_3(t) = c_{2,1}^2 A_2(t) A_4(t) + c_{0,3}^2 A_0(t) A_4(t) - \nu_3 A_3(t)$$  \hspace{1cm} (B6)

$$\frac{d}{dt} A_4(t) = c_{2,1}^3 A_2(t) A_3(t) + c_{0,3}^3 A_0(t) A_3(t) + F - \nu_4 A_4(t).$$  \hspace{1cm} (B7)

For this system, the value of $\mu$ is not important since there is only one nonlocal triad; we take it equal to $-1$.

In Fig. [23] we display the typical behavior observed with the triads $T_1$ and $T_2$ only. The nonlocal interaction associated with $T_3$ is suppressed by taking $r = 0$. In this case, we clearly observe that the modes $A_2$ and $A_3$ grow simultaneously as $A_4$ decreases. Some
time after saturation of the decay products, another decay starts with the growth of $A_0$ and $A_1$ together with a sharp decrease of $A_2$. It turns out however that immediately after, the whole process starts again and repeats quasi-periodically, for as long as the duration of the simulation (see the right panel of Fig. 23).

When including some amount of nonlocal interactions, the oscillations get damped, more rapidly as $r$ increases. In Fig. 24 we display the resulting dynamics for the cases $r = 0.7$ and $r = 1$. In addition to the damping of the oscillations, a larger value of $r$ gives rise to another phenomenon: The amplitude of some modes undergo a significant decrease before saturating at late times, while the mode $A_0$ grows to a large amplitude. This new behavior will now be studied within the context of the full model.

### B.2. Simulations of the full shell model

We now consider the full shell model with $N = 8$ with the aim to identify the role of the nonlocal interactions. We display in Fig. 25 the energy spectra $A_i^2 / k_{-i}$ for several times before reaching the stationary regime and for different cases. In the top-left panel, the system corresponds to the local model ($r = 0$) with $N = 8$. An interesting feature emerges, associated with a clear inverse transfer of energy. The decay instability discussed in the previous section is claimed to be at the origin of such a dynamics either with Alfvén waves (Del Zanna et al. 2001; Réville et al. 2020) or KAWs (Voitenko 1998b). We should point out that within the context of this shell model, there is no "true" parametric decay instability because frequency resonance conditions are not required. But as argued in Waleffe (1992), and also in Hasegawa (1985), the values of the mode coupling coefficients...
directly affect the direction of the cascade. We also see in this simulation that late time spectra (in magenta) have a maximum amplitude at almost all scales. The general behavior is thus consistent with a regular, self-similar inverse cascade. A very different behavior takes place when \( r \neq 0 \) (all other panels). We choose first to fix \( r = 1 \) and vary the relative strength of the nonlocal coupling by changing the value of \( \mu \). For very large negative values of \( \mu \), there is very little influence of nonlocal interactions. We choose to display in the top right panel the case \( \mu = -1 \) for which the local and nonlocal interactions are of the same order. The spectra still display an inverse transfer but the time dynamics is not monotonic. This inverse transfer is however more efficient, the shell 0 accumulating much more energy at the end of the simulation. The other extreme case is \( \mu = 0 \), for which all the nonlocal interactions are of the same strength and dominate over
Figure 26: Nonlocal (left panels) and local (right panels) transfers as a function of early (top) or later (bottom) times are displayed for the shell model with $N = 8$, $\nu = 5 \times 10^{-4}$, $r = 1$ and $\mu = -0.2$. The various transfers are ordered as explained in the text, with associated colors red, green, brown, black, turquoise, blue, and magenta.

the local ones. We choose to display in the two bottom panels of Fig. 25 the case $\mu = -0.2$ for early (left) and longer (right) times. At early times, one can clearly identify, as in the direct simulations, the propagation to large scales of a “spectral bump”: at a fixed intermediate wavenumber, energy starts to grow and then decreases, the same dynamics taking place at a smaller wavenumber and at a later time. When the stationary regime is reached, the efficiency of the inverse transfer is comparable to the case $\mu = -1$, but now the final spectrum reaches a minimum for all the wavenumbers $k_i$ with $i > 0$. This is again consistent with the simulations of the gyrofluid model where a relatively low-amplitude flat spectrum is often observed to develop behind the propagating bump. In this inverse cascade where nonlocal interactions play a dominant role, no power-law spectrum emerges at late times.

Let us now turn to a more detailed analysis of the energy transfers between wavenumbers. The quantities $c_{i,1}A_i(t)A_{i+1}(t)A_{i+2}(t)$ (respectively $c_{i,N-i-1}A_i(t)A_{N-1}(t)A_N(t)$)
refer to the rate of energy transfer from wavenumber $k_{\perp(i+2)}$ to $k_{\perp i}$ resulting from local interactions (respectively from $k_{\perp N}$ to $k_{\perp i}$ for nonlocal interactions). A negative value indicates that the energy transfer takes place in the opposite direction, i.e. from large to small scales. From Fig. 26, it can be seen that, at early times, the local transfers (top right) start as an inverse transfer from the small to the large scales and, at times $t > 20$, reverse to a direct transfer from large to small scales, except for $i = 6$ which still receives energy from the forcing wavenumber. The situation is similar at early times for the nonlocal transfers (top left) but differs at later times with a large inverse transfer observed for the large-scale modes ($i = 0, 1, 3$). The modes $i = 2, 5$ give energy to the forcing one at $t > 20$, this transfer becoming negligible at long times. The bottom panels of Fig. 26 show the nonlocal (left) and local (right) transfers for the long times. It is clearly seen that the local transfers are dominantly direct for $20 < t < 100$ and all become negligible at large times. The nonlocal transfers on the other hand remain finite for $i = 0, 3$, the transfer from the forcing wavenumber to $k_{\perp 0}$ remaining large for asymptotically long times.

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