THREE-DIMENSIONAL MAGNETOHYDRODYNAMICS SYSTEM
FORCED BY SPACE-TIME WHITE NOISE

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Abstract. We consider the three-dimensional magnetohydrodynamics system forced by noise that is white in both time and space. Its complexity due to four non-linear terms makes its analysis very intricate. Nevertheless, taking advantage of its structure and adapting the theory of paracalculated distributions from [25], we prove its local well-posedness. A first challenge is to find an appropriate paracalculated ansatz which must consist of both the velocity and the magnetic fields. Second challenge is that for some non-linear terms, renormalizations cannot be achieved individually; we overcome this obstacle by employing a technique which may be appropriately called a coupled renormalization. This technique of coupled renormalizations seems to be new and is expected to be crucial for any other systems of non-linearly coupled differential equations such as the Boussinesq system. Our proof is also inspired by the work of [56]. To the best of the author’s knowledge, this is the first result of a well-posedness of a non-linearly coupled system of equations forced by a space-time white noise.

Keywords: Feynman diagrams; Gaussian hypercontractivity; magnetohydrodynamics system; paracalculated distributions; renormalization; Wick products.

1. Introduction
2. Preliminaries
3. Proof
3.1. Group 1: \( u_1^{\epsilon, i} \odot u_2^{\epsilon, j}, b_1^{\epsilon, i} \odot b_2^{\epsilon, j}, u_1^{\epsilon, i} \odot b_2^{\epsilon, j}, b_1^{\epsilon, i} \odot u_2^{\epsilon, j} \)
3.2. Group 2: \( u_2^{\epsilon, i} \odot u_2^{\epsilon, j}, b_2^{\epsilon, i} \odot b_2^{\epsilon, j}, b_2^{\epsilon, i} \odot u_2^{\epsilon, j} \)
3.3. Group 3: \( \pi_0, (u_3^{\epsilon, i}, u_1^{\epsilon, j}), \pi_0, (b_3^{\epsilon, i}, b_1^{\epsilon, j}), \pi_0, (u_3^{\epsilon, i}, b_1^{\epsilon, j}), \pi_0, (b_3^{\epsilon, i}, u_1^{\epsilon, j}) \)
3.4. Group 4: \( \pi_0, (P^{\epsilon, i}, \partial_x K^{\epsilon, j}, u_1^{\epsilon, j}), \pi_0, (P^{\epsilon, i}, \partial_x K^{\epsilon, j}, u_1^{\epsilon, j}), \pi_0, (P^{\epsilon, i}, \partial_x K^{\epsilon, j}, b_1^{\epsilon, j}), \pi_0, (P^{\epsilon, i}, \partial_x K^{\epsilon, j}, b_1^{\epsilon, j}) \)

Acknowledgments
4. Appendix
References

2010MSC : 35B65; 35Q85; 35R60
1. Introduction

When solutions to a system of partial differential equations (PDE) lack sufficient regularity, a common remedy is to multiply by a sufficiently smooth function, integrate by parts to rid of any derivative on the solution, and only ask that its integral formulation is well-defined; this is the standard definition of a weak solution (e.g. [16, 20]). However, if the PDE are non-linear, then the lack of regularity creates difficulty in understanding any product of the solution with itself because there is no universal agreement on the definition of a product of distributions. Some physically meaningful models which have found rich applications in the real world were forced by a term that is white in both space and time, the so-called space-time white noise (STWN), ever since its first derivation. A prominent example is the Kardar-Parisi-Zhang (KPZ) equation (4) ([37, Equation (1)]); we also refer to [2, 23, 33, 48] concerning the Boussinesq system forced by STWN. While considering the mild solution formulation typically solved the issue in case the noise is white only in time, the STWN leads to a lack of spatial regularity of the solution, and the construction of a solution has created a significant obstacle because the non-linear term seemed to be ill-defined in the classical sense. Let us briefly describe the very recent remarkable developments that ultimately led to the two novel approaches of the theory of regularity structures by Hairer [30] and the theory of paracontrolled distributions by Gubinelli, et al. [25].

Following the notations of Young [54, pg. 258], given a function $f(x)$ over $[x', x'']$, let us denote

$$V_p(f) \triangleq \sup_{[x', x'']=[x_{r-1}, x_{r-1}], |x_r - x_{r-1}| < \delta} \left( \sum_{r} |f(x_r) - f(x_{r-1})|^p \right)^{\frac{1}{p}}, \quad p > 0,$$

where $\delta = |x'' - x'|$ and call it the $p$-variation of $f$. We also write $f \in W_p$ if $V_p(f) < \infty$, and point out that the space of functions of bounded variations (BV) corresponds to $W_p$ in case $p = 1$. It follows (see e.g. [22, Proposition 5.3]) that if $f$ is continuous over $[x', x'']$ and $1 \leq p \leq p' < \infty$, then $V_{p'}(f) \leq V_p(f)$.

It would be instructive here to recall that $f$ is an element of $C_\alpha$, specifically of Hölder continuous with exponent $\alpha \geq 0$ if the following norms are finite:

$$|f|_{\alpha-\text{Höл};[x', x'']} \triangleq \left\{ \begin{array}{ll}
\|f\|_{C([x', x''])} + \sup_{x' \leq x_r < x' \leq x''} \frac{|f(x_r) - f(x_{r-1})|}{|x_r - x_{r-1}|^\alpha} & \text{if } \alpha \leq 1,
\sup_{k \leq |\alpha|} \left( \|\partial_k^\alpha f\|_{L^\infty([x', x''])} + \sup_{x' \leq x_r < x' \leq x''} \frac{|\partial_k^\alpha f(x_r) - \partial_k^\alpha f(x_{r-1})|}{|x_r - x_{r-1}|^{|\alpha|}} \right) & \text{if } \alpha > 1
\end{array} \right.$$

where $\partial_k \triangleq \frac{\partial}{\partial x_k}$ and $\partial_k^\alpha \triangleq \frac{\partial^k}{(\partial x_k)^\alpha}$ (e.g. [3, Definition 1.49]). It is immediate that $f \in C^\frac{1}{2}$ for $p \in (0, \infty)$ is continuous and of finite $p$-variation, but a function of finite $p$-variation need not be continuous (consider a step function) (see [22, pg. 78] for this discussion). Let us recall in particular that Brownian motion is locally Hölder continuous with exponent $\alpha$ for every $\alpha \in (0, \frac{1}{2})$ ([37, Chapter 2, Remark 2.12]) but nowhere locally so for $\alpha > \frac{1}{2}$, and in fact it is not locally Hölder continuous with $\alpha = \frac{1}{2}$ everywhere on $[0, \infty)$ ([37, pg. 113–114]). Therefore, Brownian motion has finite $p$-variation for every $p > 2$ but does not have finite 2-variation.

Now a well known sufficient condition for a Lebesgue-Stieltjes integral $\int g(x) \, df(x)$ to be well-defined requires that $f, g \in BV$ and at least one of them is continuous
we recall the Fourier transform $A$ of a field and the pressure scalar field, respectively. Additionally, by denoting by
\[ \partial \] the viscous diffusivity and
\[ \nu \] the common discontinuities, then their Stieltjes integral still exists. In other words, Young’s contribution states that instead of requiring both $f$ and $g$ to be in $W_1$, each has to satisfy the weaker conditions of $W_p$ or $W_q$ where $\frac{1}{p} + \frac{1}{q} > 1$ and instead of one function being continuous at all $x$, each must be continuous wherever the other is not (see also [40]).

In order to understand the implication of Young’s theory of integration better, let us now introduce the Navier-Stokes equations (NSE). Let us denote by $u : \mathbb{T}^N \times \mathbb{R}_+ \to \mathbb{R}^N$ and $\pi : \mathbb{T}^N \times \mathbb{R}_+ \to \mathbb{R}$ the $N$-dimensional (N-d) velocity vector field and the pressure scalar field, respectively. Additionally, by denoting by $\nu \geq 0$ the viscous diffusivity and $\partial_t \triangleq \frac{\partial}{\partial t}$, we are able to write down the NSE as
\[ \partial_t u + (u \cdot \nabla) u + \nabla p - \nu \Delta u = \xi_u, \quad \nabla \cdot u = 0, \] with initial data $u_0(x) \triangleq u(x,0)$, where $\xi_u$ is the Gaussian field that is white in both time and space; i.e.
\[ \mathbb{E}[\xi_u(x,t)\xi_u(y,s)] = \delta(x-y)\delta(t-s). \]

We will also need the definition of the Hölder space with negative exponent; for this purpose, let us recall the basic background of Besov spaces ([25] and also [34] on how the Littlewood-Paley theory on $\mathbb{R}^3$ may be transferred to $\mathbb{T}^3$). Let us use the notation of $A_{a,b} \lesssim_{a,b} B$ in case there exists a non-negative constant $C = C(a,b)$ that depends on $a, b$ such that $A \leq CB$; similarly let us write $A \approx_{a,b} B$ in case $A \approx CB$. Moreover, unless elaborated in detail, we denote $\sum_{k \in \mathbb{Z}^3}$ by $\sum_k$. Firstly we recall the Fourier transform
\[ \hat{f}(k) \triangleq \mathcal{F}_{\mathbb{T}^3}(f)(k) \triangleq \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} f(x)e^{i x \cdot k} dx \]
with its inverse denoted by $\mathcal{F}_{\mathbb{T}^3}^{-1}$, let $\mathcal{D}$ be the set of all smooth functions with compact support on $\mathbb{T}^3$, $\mathcal{D}'$ its dual and thus the set of all distributions on $\mathbb{T}^3$. We let $\chi, \rho \in \mathcal{D}$ be non-negative, radial such that the support of $\chi$ is contained in a ball while that of $\rho$ in an annulus and satisfy
\[ \chi(\xi) + \sum_{j \geq 0} \rho(2^j \xi) = 1 \quad \forall \xi, \quad \text{supp}(\chi) \cap \text{supp}(\rho(2^{-j} \cdot)) = \emptyset \quad \forall \quad j \geq 1, \]
\[ \text{supp}(\rho(2^{-i} \cdot)) \cap \text{supp}(\rho(2^{-j} \cdot)) = \emptyset \quad \text{for} \quad |i - j| > 1. \]

We realize that $\chi(\cdot) = \rho(2^{-1} \cdot)$ and $\rho_j \triangleq \rho(2^{-j} \cdot)$, and define Littlewood-Paley operator as $\Delta_j f(x) \triangleq (2\pi)^{-3} \sum_k e^{i x \cdot k} \rho_j(k) \mathcal{F}_{\mathbb{T}^3}(f)(k) = \mathcal{F}_{\mathbb{T}^3}^{-1}(\rho_j \mathcal{F}_{\mathbb{T}^3}(f))(x)$. We also write $S_j f \triangleq \sum_{j \leq -1} \Delta_j f$. Now for $\alpha \in \mathbb{R}, p, q \in [1, \infty]$, we may define the inhomogeneous Besov space
\[ B^\alpha_{p,q}(\mathbb{T}^3) \triangleq \{ f \in \mathcal{D}'(\mathbb{T}^3) : \| f \|_{B^\alpha_{p,q}(\mathbb{T}^3)} \triangleq \| 2^j \| \Delta_j f \|_{L^p(\mathbb{T}^3)} \|_{l^q(\{ 1 \geq -1 \})} < \infty \}. \]
The Hölder-Besov space $C^\alpha(\mathbb{T}^3)$ is the special case when $p = q = \infty$; i.e. $C^\alpha(\mathbb{T}^3) = B^\alpha_{\infty,\infty}(\mathbb{T}^3)$. For $\alpha \in (0, \infty) \setminus \mathbb{N}$, $C^\alpha(\mathbb{T}^3) = C^\alpha(\mathbb{T}^3)$; however, for $k \in \mathbb{N}$, $C^k(\mathbb{T}^3)$ is strictly larger than $C^k(\mathbb{T}^3)$ (\cite{33} pg. 99). We point out that
\[ ||| \cdot |||_{C^\beta} \lesssim ||| \cdot |||_{L^\infty} \lesssim ||| \cdot |||_{C^\alpha} \quad \text{if} \quad \beta \leq 0 \leq \alpha \quad \text{and} \quad ||| S_j \cdot |||_{L^\infty} \lesssim 2^{-j\beta} ||| \cdot |||_{C^\alpha} \quad \forall \alpha < 0. \]
Now for simplicity let us consider the 1-d analogue of \((u \cdot \nabla)u\) in the NSE (1), specifically \(u \partial_x u\) corresponding to the non-linear term of the Burgers’ equation which was studied by Da Prato, et al. [12]. Following the discussion of [27, pg. 1548], assuming that its solution \(u \in C^\alpha\) for \(\alpha > \frac{1}{2}\), we may multiply this non-linear term by a smooth periodic function \(\psi\) and understand it as

\[
\int_T \psi(x)u(x)\,du(x)
\]

which is well-defined as a Young’s integral because \(\psi u \in C^\alpha\) for \(\alpha > \frac{1}{2}\). Of course, we can also write \(u \partial_x u = \frac{1}{2} \partial_x u^2\) and integrate by parts. However, when one considers a generalized Burgers’ equation with non-linear term of the form \(g(u)\partial_x u\) where \(g \neq \partial_x G\) for some function \(G : \mathbb{R} \to \mathbb{R}\), as considered by Hairer in [27], integration by parts becomes out of reach and one may only turn to Young’s theory of integration. Unfortunately the assumption of \(u \in C^\alpha\) for \(\alpha > \frac{1}{2}\) turns out to be a wishful thinking. In fact, in the general case when the spatial dimension is \(N\), considering that the space-time dimension is \(N+1\) so that the scaling \(S \in \mathbb{N}^N\) is

\[
S = (S_1, \ldots, S_{N+1}) = (2, 1, \ldots, 1) \quad \text{N-many}
\]

with the first entry informally representing the dimension of time due to \(\partial_t\) and \(\Delta\), we actually know that \(\xi \in C^\alpha(\mathbb{T}^N)\) in space for \(\alpha < -\frac{|S|}{2}\) where \(|S| = N + 2\) by [30, Lemma 10.2] (see also [30, Lemma 3.20]). This leads to \(u \in C^\alpha(\mathbb{T}^N)\) for \(\alpha < 2 - \frac{N+2}{2}\) due to regularization from the diffusion (see [30, pg. 417, 481]). Therefore, the Young’s integral (3) is ill-defined even in case \(N = 1\) as \(2 - \frac{N+2}{2} = \frac{1}{2}\) if \(N = 1\).

At this point one may turn to the theory of stochastic integrals such as the Ito’s integral in hope for some help; such integrals have been known to be extendable to a wider class of semi-martingales and discovered remarkable applications in the real world. However, its limitations have also been noticed over decades (e.g. [24, pg. 6], [27, pg. 1548]): Ito’s integral requires an “arrow of time,” specifically a filtration and adapted integrands, a probability measure because it is defined as the \(L^2\)-limit of appropriate approximations, and the integrand must have the \(L^2\)-orthogonal increments. In order to complement the theory of Ito’s integrals, Lyons developed a theory of rough path ([41, 42]). As stated on [42, pg. 28], rough path is informally a continuous path on which a sequence of iterated path integrals may be constructed. Recall (e.g. [22, pg. 405]) that a fractional Brownian motion (fBm) \(\beta^H\) with Hurst parameter \(H \in (0, 1)\) is a zero-mean Gaussian process with covariance of

\[
\mathbb{E}[\beta^H(t)\beta^H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).
\]

Because fBm for \(H \neq \frac{1}{2}\) is not a semi-martingale ([10, pg. 12–13]), the Ito’s integral cannot be extended to fBm. Nevertheless, using rough path theory, we may still construct a path-wise integral with respect to fBm (see [22, Proposition 15.5]). Subsequently, Gubinelli in [24] extended the Lyon’s rough path theory furthermore; we refer to [22, 21, 20, 27, 32] for further study and applications of rough path theory. As one of the most prominent examples of a result inspired from the rough path theory, let us briefly discuss recent developments of the KPZ equation [11]; this discussion will be relevant anyhow because we will subsequently need the notions
of renormalizations. The KPZ equation as an interface model of flame propagation was first derived in [38, Equation (1)] as
\[ \partial_t h = \partial_x^2 h + \lambda (\partial_x h)^2 + \xi_h \] (4)
where \( h(x,t) \) represents the interface height, \( \lambda > 0 \) is the coupling strength, \( x \in \mathbb{S}^1 \) and \( \xi_h \) is the STWN. Following [29, pg. 562], inspired by the Cole-Hopf transform, let us consider a multiplicative stochastic heat equation
\[ dZ = \partial_x^2 Z dt + \lambda Z dW \]
where \( \partial_t W = \xi_h \). We denote by \( Z^\epsilon \) the solution to the same equation with \( W \) replaced by a mollified noise \( W^\epsilon \), which is obtained from multiplying the \( k \)-th Fourier component of \( W \) by \( f(k \epsilon) \) for a smooth cut-off function \( f \) with compact support such that \( f(0) = 1 \). Then Ito’s formula shows that \( h^\epsilon(x,t) \equiv \frac{1}{\lambda} \ln Z^\epsilon(x,t) \) solves
\[ \partial_t h^\epsilon = \partial_x^2 h^\epsilon + \lambda (\partial_x h^\epsilon)^2 - \lambda \sum_{k \in \mathbb{Z}} f^2(k \epsilon) + \xi_h^\epsilon \] (5)
where \( \sum_{k \in \mathbb{Z}} f^2(k \epsilon) \approx \frac{1}{\epsilon} \int_{\mathbb{R}} f^2(x) dx \to \infty. \) Therefore, informally the limiting process as \( \epsilon \to 0 \) actually solves
\[ \partial_t h = \partial_x^2 h + \lambda (\partial_x h)^2 - \infty + \xi_h \]
(see [28] for further discussion of such a phenomenon). This simple computation displays the necessity to rely on techniques from quantum field theory (e.g. [43] and [40, Section 4 on pg. 295]) such as renormalization, which amounts to strategically subtracting off a large constant from a regularized equation, and replacing a standard product by Wick product (e.g. [25, pg. 23], also [14, 15]) which informally guarantees mean zero condition. Let us point out that these techniques actually have long history of its utility in stochastic quantization. In particular Da Prato and Debussche [11] proved the existence of a unique strong solution to the 2-d stochastic quantization equation for almost all initial data with respect to the invariant measure using such techniques. We also refer to [3] for works on the KPZ equation using these techniques. Without delving into further details, we mention that Hairer [29] in particular discovered two additional logarithmically divergent constants beside the \( \frac{1}{\epsilon} \) in (5) and successfully introduced a completely new concept of a solution to the KPZ equation (4) using rough path theory.

Let us now discuss this direction of research in the case of the NSE (1). To the best of the author’s knowledge, Flandoli and Gozzi [18] were the first to consider the 2-d NSE in \( \mathbb{T}^2 \) with the forcing that is not regular; they proved in [18, Theorem 4.3] that the Kolmogorov equation associated to the NSE with covariance operator that is an identity has a weak solution. However, due to the spatial roughness of the noise, the authors in [18] were not able to make the connection to the original equation. Da Prato and Debussche [10] overcame this difficulty using techniques of renormalization and Wick products.

At this point let us introduce the magnetohydrodynamics (MHD) system of our main concern because the failure to apply the proofs of [10, 18], which we will explain shortly, displays clearly the complexity of the MHD system in contrast to the NSE. Let us denote by \( b : \mathbb{T}^N \times \mathbb{R}^+ \to \mathbb{R}^N \) the magnetic \( N \)-d vector field and \( \eta \geq 0 \) the magnetic diffusivity. Then the MHD system reads as
\[ \partial_t u + (u \cdot \nabla) u + \nabla \pi = \nu \Delta u + (b \cdot \nabla)b + \xi_u, \quad \nabla \cdot u = 0, \] (6a)
\[ \partial_t b + (u \cdot \nabla)b = \eta \Delta b + (b \cdot \nabla)u + \xi_t, \quad \nabla \cdot b = 0, \quad \text{(6b)} \]

for which we write the solution as \( y \triangleq (y^1, \ldots, y^6) \triangleq (u, b) \triangleq (u^1, u^2, u^3, b^1, b^2, b^3) \), with initial data \( y_0(x) \triangleq (u_0(x), b_0(x)) = (u(b), x, 0) \), and \( \xi \triangleq (\xi_u, \xi_b) \) where \( \xi_u \triangleq (\xi^1_u, \xi^2_u, \xi^3_u) = (\xi^1, \xi^2, \xi^3) \) and \( \xi_b \triangleq (\xi^1_b, \xi^2_b, \xi^3_b) = (\xi^1, \xi^2, \xi^3) \), is a Gaussian field which is white in both space and time. For simplicity of computation, let us assume that \( \nu = \eta = 1 \) as well as that \( \int_{T^3} \xi_u dx = \int_{T^3} \xi_b dx = 0 \) which in turn allows us to assume that \( (u, b) \) are also mean zero; this may be justified via a standard scaling argument of the solution to the MHD system.

**Remark 1.1.** As a STWN, the correlation of \( \xi_u \) and that of \( \xi_b \) are both products of a delta function in \( x \) with another delta function in \( t \). In the literature on Boussinesq system such as [23] Equation (3), the authors make an assumption corresponding to the MHD system that the correlation of \( \xi_u \) and \( \xi_b \) vanish; i.e. \( \mathbb{E}[\xi^i_u, \xi^j_b] = 0 \) for all \( i, j \in \{1, 2, 3\} \). Considering that there is no physical reason why \( \xi_u \) and \( \xi_b \) should have any independence, in this manuscript we shall assume that the correlation of \( \xi_u \) and \( \xi_b \) is also a product of a delta function in \( x \) with another delta function in \( t \) (see (11) which is a corollary of this assumption). Our computations are thus more general. Indeed, it is easy to recover the case \( \mathbb{E}[\xi^i_u, \xi^j_b] = 0 \) for all \( i, j \in \{1, 2, 3\} \) because many terms within our proof vanish due to the mixed non-linear terms such as \( (u \cdot \nabla)b \) and \( (b \cdot \nabla)u \). This is actually an interesting difference from the case of the NSE; the computations of the mixed non-linear terms can be actually much simpler than the case of the NSE under the assumption of the zero correlation among \( \xi_u \) and \( \xi_b \).

It is well known that if we take the \( L^2(\mathbb{T}^N) \)-inner products of (4) with \( u \), then the non-linear term, as well as the pressure term, both vanish by divergence-free property; e.g. \( \int_{T^3} (u \cdot \nabla)u \cdot udx = \frac{1}{2} \int_{T^3} |u|^2 dx = 0 \). An analogous attempt of taking \( L^2 \)-inner products on (6a) with \( u \) actually fails because

\[ \int_{T^3} (b \cdot \nabla)b \cdot udx \neq 0 \quad \text{(7)} \]

in general. Yet, if we take \( L^2(\mathbb{T}^N) \)-inner products on (6b) with \( b \) simultaneously and add the two resulting equations, then all the non-linear terms and the pressure term in (6a)–(6b) do indeed vanish because \( \int_{T^3} (u \cdot \nabla)b \cdot bdx = \frac{1}{4} \int_{T^3} (u \cdot \nabla)|b|^2 dx = 0 \) and

\[ \int_{T^3} (b \cdot \nabla)b \cdot u + (b \cdot \nabla)u \cdot bdx = 0. \quad \text{(8)} \]

Even though there exist some extensions of techniques on the NSE to the MHD system such as this, attempts to modify the proofs of [10] [18] on the 2-d NSE to the 2-d MHD system face a non-trivial difficulty. In both works of [10] [18], the authors rely on the following key identity:

\[ \int_{T^2} (u \cdot \nabla)u \cdot \Delta u dx = 0. \quad \text{(9)} \]

This follows immediately from the vector calculus identity of \( \nabla \times (\nabla \times f) = \nabla(\nabla \cdot f) - \Delta f \), integration by parts and divergence-free property of \( u \); in fact, one of the reasons why the authors admit that extending to other boundary conditions beside \( T^2 \) is not easy (e.g. [18] pg. 312) is exactly this identity (9). The identity (9) was used in [18] pg. 328 and [10] pg. 190, and this identity actually fails in the case
of the MHD system because \( \int_{T^3} [(u \cdot \nabla)u - (b \cdot \nabla)b] \cdot \Delta u \, dx \neq 0 \) and even if we add similarly to (8),
\[
\int_{T^3} [(u \cdot \nabla)u - (b \cdot \nabla)b] \cdot \Delta u + [(u \cdot \nabla)b - (b \cdot \nabla)u] \cdot \Delta b \, dx \neq 0
\] (10)
in general. In fact, the identity (9), which is equivalent to
\[
\int_{T^2} (u \cdot \nabla)u \cdot \Delta u = - \int_{T^2} (u \cdot \nabla)(\nabla \times u) \cdot (\nabla \times u) \, dx
\] has also been used crucially in various other works on the NSE (e.g. [31]), many of which have not been extended to the MHD system with (10) being one of the sources of the technical issues.

Zhu and Zhu [56, pg. 4444–4445] gave a very nice discussion of how the proof within [10] cannot be extended to the 3-d NSE and thus most certainly has no chance of being extended to the 3-d MHD system; let us recollect it here. Da Prato and Debussche [10] considered (1) in \( T^2 \), \( z \) to be the solution to the linear Stokes equation forced by the fixed STWN \( \xi u \) and the equation solved by \( v \triangleq u - z, q \triangleq \pi - p \), specifically
\[
\partial_t z = \Delta z - \nabla p + \xi u, \quad \nabla \cdot z = 0,
\]
\[
\partial_t v = \Delta v - \nabla q - \frac{1}{2} \text{div}[(v + z) \otimes (v + z)], \quad \nabla \cdot v = 0.
\]
Similarly to the discussion of the Burgers’ equation in [31], due to [30] Lemma 10.2 (see also [30] Lemma 3.20) the solution \( z \) is very rough, and only in \( C^\alpha(T^N) \) for \( \alpha < 1 - \frac{N}{2} \). Thus, if \( N = 2 \), then \( z \in C^\alpha(T^2) \) for \( \alpha < 0 \) and considering \( \text{div}(z \otimes z) \in C^\alpha(T^2) \) for \( \alpha < -1 \), the diffusion leads to \( v \in C^\alpha(T^2) \) for \( \alpha < 1 \). This implies that according to Bony’s estimates (see Lemma 1.1 (4)) the product \( v \otimes v \) and even \( v \otimes z \) can be well-defined, leaving only \( z \otimes z \) for which one can turn to Wick products, to be described in more detail subsequently. However, in the case \( N = 3 \) we would have \( z \in C^\alpha(T^3) \) for \( \alpha < - \frac{1}{2} \) and thus \( \text{div}(z \otimes z) \in C^\alpha(T^3) \) for \( \alpha < -2 \) so that the diffusion leads to \( v \in C^\alpha(T^3) \) for \( \alpha < 0 \). This implies that not only \( z \otimes z \) but even \( z \otimes v \) is ill-defined.

Two novel approaches have been developed to bring about a resolution to such an issue, specifically the theory of regularity structures due to Hairer [30] and that of paracontrolled distributions due to Gubinelli, et al. [25]. Both of these theories were strongly inspired by the rough path theory due to Lyons [41]: in fact, it is described in [30] Section 4.4 that rough path may be considered as an example of a regularity structure, and it is also acknowledged on [25] pg. 1 that their work is inspired from the theory of controlled rough path [24].

Without delving into the deep theory of the regularity structures, the heuristic behind it is the key observation that even though a function is typically said to be smooth if it may approximated by a Taylor polynomial, it is already somewhat misguided to apply this notion to a solution of an equation forced by STWN because locally it does not behave similarly to a polynomial but more similarly to the STWN convoluted with the Green function from diffusion. The work of Hairer [30] allows one to construct a regularity structure endowed with a whole set of calculus operations such as multiplication, integration and differentiation, so that one can recover a fixed point theory, and finally rely on the reconstruction theorem to conclude the existence and uniqueness of a solution to the original problem.
On the other hand, the theory of paracontrolled distributions relies heavily on the Bony’s decomposition (e.g. [3], g. 86]) beside the rough path theory, which we now describe briefly. The purpose of the Bony’s decomposition is to split \( fg \) in parts where the frequency of \( f \) and \( g \) are low and high, specifically
\[
fg = \sum_{i,j \geq -1} \Delta_i f \Delta_j g = \pi_<(f,g) + \pi_>(f,g) + \pi_0(f,g)
\]

where
\[
\pi_<(f,g) = \sum_{j \geq -1} S_j f \Delta_j g, \quad \pi_>(f,g) = \sum_{j \geq -1} \Delta_j f \Delta_j g, \quad \pi_0(f,g) = \sum_{j \geq -1; |i-j| \leq 1} \Delta_j f \Delta_i g.
\]

The terms \( \pi_<(f,g) \) and \( \pi_>(f,g) \) are called paraproducts while \( \pi_0(f,g) \) is the remainder. The key observation by Bony was that \( \pi_<(f,g) \) and similarly \( \pi_>(f,g) \) are well-defined distributions such that the mapping \( (f,g) \mapsto \pi_<(f,g) \) is a bounded bi-linear operator from \( C^\alpha(\mathbb{T}^N) \times C^\beta(\mathbb{T}^N) \) to \( C^\beta(\mathbb{T}^N) \) if \( \alpha > 0, \beta \in \mathbb{R} \). Heuristically \( \pi_<(f,g) \) behaves at large frequencies similarly to \( g \), and \( f \) provides only a modulation of \( g \) at large scales. The key lemma on which we will rely heavily is the following:

**Lemma 1.1.** ([25, Lemma 2.1], [8, Proposition 2.3]) Let \( \alpha, \beta \in \mathbb{R} \). Then

1. \( \|\pi_<(f,g)\|_{C^\beta} \lesssim \|f\|_{L^\infty} \|g\|_{C^\beta} \) for \( f \in L^\infty(\mathbb{T}^N), g \in C^\beta(\mathbb{T}^N) \),
2. \( \|\pi_>(f,g)\|_{C^{\alpha+\beta}} \lesssim \|f\|_{C^{\alpha}} \|g\|_{C^\beta} \) for \( \beta < 0, f \in C^\alpha(\mathbb{T}^3), g \in C^\beta(\mathbb{T}^3) \),
3. \( \|\pi_0(f,g)\|_{C^{\alpha+\beta}} \lesssim \|f\|_{C^{\alpha}} \|g\|_{C^\beta} \) for \( \alpha + \beta > 0, f \in C^{\alpha}(\mathbb{T}^3), g \in C^\beta(\mathbb{T}^3) \),
4. Consequently, \( fg \) is well-defined for \( f \in C^{\alpha}(\mathbb{T}^3), g \in C^\beta(\mathbb{T}^3) \) if \( \alpha + \beta > 0 \) and \( \|fg\|_{C^{\min(\alpha,\beta)}} \lesssim \|f\|_{C^{\alpha}} \|g\|_{C^\beta} \).

By our discussion, only difficulty in defining the product \( fg \) boils down to \( \pi_0(f,g) \), and for this purpose, Gubinelli, et al. in [25] relied on a paracontrolled ansatz (see [29] and [32]) and a commutator lemma (see Lemma 4.2).

Beside the work of Zhu and Zhu in [56], we wish to mention the work of Catellier and Chouk [8], by which our work was inspired directly. The purpose of this manuscript is to prove the local existence of a unique solution to the MHD system forced by the STWN (6a)-(6b); the specific statement will be stated in Theorems 1.2 and 3.2. To the best of the author’s knowledge, this is a first attempt on the local well-posedness of a system of non-linearly coupled equations forced by STWN. It is worth noting that one should be able to provide another proof of our main result, or its analogue, by relying on the theory of regularity structures. Indeed, Hairer [30] introduced the notion of local subcriticality ([30, Assumption 8.3]), and showed that his theory may be applied particularly to parabolic Anderson model and \( \Phi^4_3 \) model (see [39] pg. 417–418 and Sections 9.1, 9.2) because they are locally subcritical. Instead of going through lengthy proof, let us convince ourselves to believe so by observing that the 3-d MHD system, similarly to the 3-d NSE, is locally subcritical satisfying the [30, Assumption 8.3]. Let us denote \( L \triangleq \partial_t - \Delta \) and apply the Leray projection \( \mathcal{P} \) on the MHD system (6a)-(6b) to deduce
\[
Ly = \left( -\mathcal{P}(u \cdot \nabla)u + \mathcal{P}(b \cdot \nabla)b \right) + \left( \mathcal{P} \xi_a \mathcal{P}_b \right) = F(u, b, \nabla u, \nabla b, \xi).
\]
Then we may let $\beta = 2$ considering $\Delta$ (see \[30\] pg. 417) and we already know that $|S| = 2 + N$ and $\xi$ belongs to $C^\alpha(T^N)$ for $\alpha < -\frac{|S|}{2} = -1 - \frac{N}{2}$ so that

$$\beta + \alpha < 1 - \frac{N}{2} \leq S_i \quad \forall i = 1, \ldots, N + 1,$$

and $\beta + \alpha < 1 - \frac{N}{2} \leq 0$ assuming $N \geq 2$. Thus, we replace $F(u, b, \nabla u, \nabla b, \xi)$ by

$$\left( -\mathcal{P}(U_uP_u) + \mathcal{P}(U_bP_b) \right) + \left( \frac{\mathcal{P}u}{\mathcal{P}b} \right).$$

By \[30\] Assumption 8.3, e.g. we know $U_bP_b$ has homogeneity $(\beta + \alpha) + (\beta + \alpha - 1) = 2\beta + 2\alpha - 1$ and the definition of local subcriticality requires $2\beta + 2\alpha - 1 > \alpha$ which boils down to $4 > N$. Therefore, the 3-d MHD system is indeed locally subcritical.

**Remark 1.2.** We wish to point out here that by definition of local subcriticality in \[30\] Assumption 8.3, the 4-d NSE is actually not locally subcritical, and perhaps may be considered as locally critical if anything. Curiously in \[49\], there is a lengthy discussion of how fourth dimension is indeed the critical dimension in the study of Serrin regularity criteria, as well as partial regularity theory, for the NSE. This connection seems to be no coincidence and yet remains unclear to the author at the time of writing this manuscript.

Without further ado, let us state our main result; the precise statement is deferred until Theorem 3.2.

**Theorem 1.2.** Let $\delta_0 \in (0, \frac{1}{2})$, $z \in (\frac{1}{2}, \frac{1}{2} + \delta_0)$ and $u_0, b_0 \in C^{-z}(\mathbb{T}^3)$. Then there exists a unique local solution to

$$\partial_t u^i - \Delta u^i = \sum_{i_1=1}^{3} \mathcal{P}^{i_11} \xi^i_{u} - \frac{1}{2} \sum_{i_1,j=1}^{3} \mathcal{P}^{i_11} \partial_{x^j}(u^i u^j) + \frac{1}{2} \sum_{i_1,j=1}^{3} \mathcal{P}^{i_11} \partial_{x^j}(b^i b^j), \quad (11a)$$

$$\partial_t b^i - \Delta b^i = \sum_{i_1=1}^{3} \mathcal{P}^{i_11} \xi^i_{b} - \frac{1}{2} \sum_{i_1,j=1}^{3} \mathcal{P}^{i_11} \partial_{x^j}(b^i u^j) + \frac{1}{2} \sum_{i_1,j=1}^{3} \mathcal{P}^{i_11} \partial_{x^j}(b^i b^j), \quad (11b)$$

$$u(x, 0) = \mathcal{P}u_0(\cdot), \quad b(x, 0) = \mathcal{P}b_0(\cdot), \quad (11c)$$

for $i \in \{1, 2, 3\}$, where $\mathcal{P}^{im}(k) = \delta(l - m) - \frac{k^l}{|k|^2}$.

**Remark 1.3.** The proof of an analogue of the Theorem 1.2 for the 2-d MHD system goes through verbatim as in our case of the dimension being three. However, considering our discussion concerning local subcriticality before Remark 1.2, we suspect significant difficulty will arise in an attempt to extend Theorem 1.2 for the 4-d case.

Let us also emphasize that it is completely inaccurate and actually misleading to believe that any result on the NSE may be generalized to the MHD system via more computations. As already mentioned, the work of Hairer and Mattingly \[51\] on the ergodicity of the 2-d NSE seems difficult to be extended to the 2-d MHD system. In the deterministic case, there exist also abundance of results for which an extension from the case of the NSE to the MHD system is a challenging open problem. For example, although Yudovich \[55\] over 55 years ago proved the global regularity of the solution to the 2-d NSE with zero viscous diffusion, which is the Euler equations, its extension to the 2-d MHD system with zero viscous diffusion remains open despite extensive interest from many mathematicians (e.g. \[7\] \[17\] \[38\] \[51\]).
Remark 1.4. We point out an interesting open problem of extending our result to the Hall-MHD system:

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u + \nabla \pi &= \Delta u + (b \cdot \nabla) b + \xi_u, \\
\nabla \cdot u &= 0, \\
\partial_t b + (u \cdot \nabla)b &= \Delta b + (b \cdot \nabla) u - \epsilon (\nabla \times ((\nabla \times b) \times b) + \xi_b), \\
\nabla \cdot b &= 0,
\end{align*}
\]

where \( \epsilon \geq 0 \) is the Hall parameter. We note that the case \( \epsilon = 0 \) reduces (12a)-(12b) to the MHD system (6a)-(6b). Since this system was introduced by Lighthill [39] over 75 years ago, it has found rich applications in astrophysics, geophysics and plasma physics; we refer to [1, 9] for its study in the deterministic case and [50, 53] in the stochastic case. By the same computation of how we showed that the 3-d MHD system is locally subcritical, it can be shown that the \( N \)-d Hall-MHD system is not locally subcritical for any \( N \geq 2 \). Indeed, \((\nabla \times b) \cdot \nabla b\) from Hall term would become \( P_b P_b \) by the notations of [30, Assumption 8.3] which would have homogeneity of \((\beta + \alpha - 1) + (\beta + \alpha - 1)\) with \( \beta = 2 \) and in order for the Hall-MHD system with spatial dimension being \( N \) to be locally subcritical, it requires \((\beta + \alpha - 1) + (\beta + \alpha - 1) > \alpha \) which boils down to \( 2 > N \). In fact, Hairer [30] (e.g. [30, Abstract]) clear states that his theory works for a semi-linear PDE while the Hall-MHD system is quasi-linear. The author believes extending Theorem 1.2 to the Hall-MHD system is a mathematically challenging and physically meaningful open problem.

Remark 1.5. To the best of the author’s knowledge, this is the first well-posedness result for a system of PDE which are non-linearly coupled and forced by STWN. All the previous work on the MHD and related systems forced by random force have been devoted to the case the noise is white in only time and not space (e.g. [4, 45, 47, 52]). With this accomplished, it has become clearer how to establish similar results for other systems such as the Boussinesq system for which its study with STWN has been suggested by physicists for decades [12, 23, 33, 48] but shied away by mathematicians due to technical difficulty. Moreover, it will be interesting to study a system of PDE forced partially by STWN. For example, the Boussinesq system with only the equation of the temperature forced by STWN has been studied in the case the noise is white only in time (e.g. [19]).

We employ the theory of paracontrolled distributions from [25] and follow the work of [8, 56]; some interesting non-trivial modifications must be made within our proof. In particular, renormalizations must be done very carefully considering the coupling (see Remark 3.4). Moreover, it is crucial to make appropriate paracontrolled ansatz (see [20] and [32]), and this is difficult due to the complex structure of the MHD system (see Remark 3.2).

2. Preliminaries

We review basic facts and computations on Wick products which will be used repeatedly throughout. We first need the following definition of a Feynman diagram:

Definition 2.1. ([33, Definition 1.35]; we also refer to [43]) A Feynman diagram of order \( n \geq 0 \) and rank \( r \geq 0 \) is a graph consisting of a set of \( n \) vertices and a set of \( r \) edges without common endpoints. There are, thus, \( r \) disjoint pairs of vertices, each joined by an edge, and \( n - 2r \) unpaired vertices. The Feynman diagram is complete if \( r = \frac{n}{2} \), so that all vertices are paired off and incomplete if \( r < \frac{n}{2} \), so that some vertices are unpaired. A Feynman diagram labelled by \( n \)
random variables $\xi_1, \ldots, \xi_n$ is a Feynman diagram of order $n$ with vertices $1, \ldots, n$, where we think of $\xi_i$ as attached to vertex $i$. The value of such a labelled Feynman diagram $\gamma$ with edges $(i_k, j_k), k = 1, \ldots, r$, and unpaired vertices $\{i : i \in A\}$ is $v(\gamma) \triangleq \prod_{k=1}^r \mathbb{E}[\xi_{i_k} j_{j_k}] \prod_{i \in A} \xi_i$. Similarly we denote the order and rank of a Feynman diagram $\gamma$ by $n(\gamma)$ and $\nu(\gamma)$, respectively.

The formula which is most useful in order to compute Wick products of the form $: \xi_1 \ldots \xi_n :$ is the following:

**Lemma 2.1.** ([35] Theorem 3.4) The Wick product is given by

$$: \xi_1 \ldots \xi_n : = \sum_{\gamma} (-1)^{r(\gamma)} v(\gamma),$$

where we sum over all Feynman diagrams $\gamma$ labeled by $\{\xi_i\}_{i=1}^n$.

**Example 2.1.** The following examples are consequences of Lemma 2.1:

1. $: \xi_1 : = \xi_1,$
2. $: \xi_1 \xi_2 : = \xi_1 \xi_2 - \mathbb{E}[\xi_1 \xi_2],$
3. $: \xi_1 \xi_2 \xi_3 : = \xi_1 \xi_2 \xi_3 - \mathbb{E}[\xi_2 \xi_3] \xi_1 - \mathbb{E}[\xi_1 \xi_3] \xi_2 - \mathbb{E}[\xi_1 \xi_2] \xi_3,$
4. $: \xi_1 \xi_2 \xi_3 \xi_4 : = \xi_1 \xi_2 \xi_3 \xi_4 - \mathbb{E}[\xi_1 \xi_2] \xi_3 \xi_4 - \mathbb{E}[\xi_1 \xi_3] \xi_2 \xi_4 - \mathbb{E}[\xi_1 \xi_4] \xi_2 \xi_3$
   $- \mathbb{E}[\xi_2 \xi_3] \xi_1 \xi_4 - \mathbb{E}[\xi_2 \xi_4] \xi_1 \xi_3 - \mathbb{E}[\xi_3 \xi_4] \xi_1 \xi_2$
   $+ \mathbb{E}[\xi_1 \xi_2] \mathbb{E}[\xi_3 \xi_4] + \mathbb{E}[\xi_1 \xi_3] \mathbb{E}[\xi_2 \xi_4] + \mathbb{E}[\xi_1 \xi_4] \mathbb{E}[\xi_2 \xi_3].$

The following lemma allows us to compute an expectation of products of Wick products.

**Lemma 2.2.** ([35] Theorem 3.12) Let $Y_i = : \xi_{i_1} \ldots \xi_{i_l} :$, where $\{\xi_{i_j}\}_{1 \leq j \leq l}$ are (real or complex) centered jointly normal variables, with $k \geq 0$ and $l_1, \ldots, l_k \geq 0$. Then

$$\mathbb{E}[Y_1 \ldots Y_k] = \sum_{\gamma} v(\gamma)$$

where the summation is over all complete Feynman diagrams $\gamma$ labeled by $\{\xi_{i_j}\}_{i,j}$ such that no edge joins two variables with $\xi_{i_1,j_1}$ and $\xi_{i_2,j_2}$ with $i_1 = i_2$.

**Example 2.2.** The following examples are consequences of Lemma 2.2:

$$\mathbb{E}[\xi_{i_1} \xi_{i_2} : \mathbb{E}[\xi_{i_3} \xi_{i_2}]] = \mathbb{E}[\xi_{i_1} \xi_{i_2}] \mathbb{E}[\xi_{i_3} \xi_{i_2}] + \mathbb{E}[\xi_{i_1} \xi_{i_2}] \mathbb{E}[\xi_{i_3} \xi_{i_2}]$$

$$\mathbb{E}[\xi_{i_1} \xi_{i_2} \xi_{i_3} : \mathbb{E}[\xi_{i_4} \xi_{i_2}]] = \mathbb{E}[\xi_{i_1} \xi_{i_2} \xi_{i_3}] \mathbb{E}[\xi_{i_4} \xi_{i_2}] + \mathbb{E}[\xi_{i_1} \xi_{i_2} \xi_{i_3}] \mathbb{E}[\xi_{i_4} \xi_{i_2}] + \mathbb{E}[\xi_{i_1} \xi_{i_2} \xi_{i_3}] \mathbb{E}[\xi_{i_4} \xi_{i_2}]$$

$$\mathbb{E}[\xi_{i_1} \xi_{i_2} \xi_{i_3} \xi_{i_4} : \mathbb{E}[\xi_{i_5} \xi_{i_2}]] = \mathbb{E}[\xi_{i_1} \xi_{i_2} \xi_{i_3} \xi_{i_4}] \mathbb{E}[\xi_{i_5} \xi_{i_2}] + \mathbb{E}[\xi_{i_1} \xi_{i_2} \xi_{i_3} \xi_{i_4}] \mathbb{E}[\xi_{i_5} \xi_{i_2}] + \mathbb{E}[\xi_{i_1} \xi_{i_2} \xi_{i_3} \xi_{i_4}] \mathbb{E}[\xi_{i_5} \xi_{i_2}].$$
Finally, the following inequality is standard and will be used many times:

\[ \sup_{a \in \mathbb{R}} |a|^r e^{-a^2} \leq e \quad \text{for all } r \geq 0. \]  

(14)

We leave the rest of the preliminaries results in the Appendix.

3. Proof

Hereafter, we denote \( C^\alpha (T^3) \) by simply \( C^\alpha \). We consider \( \{\xi^\epsilon\}_{\epsilon > 0} \), a family of smooth approximations of \( \xi = (\xi_u, \xi_b) \), to be specified subsequently, and study the MHD system corresponding to \( \xi^\epsilon \); we should formally denote its solution as \( y^\epsilon \triangleq (u^\epsilon, b^\epsilon) \) but for brevity omit it until \( (111) \) when it is clear. We define \( L \triangleq \partial_t - \Delta \) and study the following system:

\[
Lu^i = \sum_{i_1=1}^{3} \mathcal{P}^{ii_1} \xi_u^{i_1}, \quad \frac{1}{2} \sum_{i_1,j=1}^{3} \mathcal{P}^{ii_1, \partial_x j} (u^i u^j) + \frac{1}{2} \sum_{i_1,j=1}^{3} \mathcal{P}^{ii_1, \partial_x j} (b^i b^j),
\]

\[
Lb^i = \sum_{i_1=1}^{3} \mathcal{P}^{ii_1} \xi_b^{i_1} - \frac{1}{2} \sum_{i_1,j=1}^{3} \mathcal{P}^{ii_1, \partial_x j} (b^i u^j) + \frac{1}{2} \sum_{i_1,j=1}^{3} \mathcal{P}^{ii_1, \partial_x j} (u^i b^j),
\]

(15)

where \( \xi \triangleq (\xi_u, \xi_b) \) are periodic, independent STWN. Let us now approximate \( (11a) - (111) \) as follows:

\[
Lu^i_1 = \sum_{i_1=1}^{3} \mathcal{P}^{ii_1} \xi_u^{i_1}, \quad Lb^i_1 = \sum_{i_1=1}^{3} \mathcal{P}^{ii_1} \xi_b^{i_1},
\]

(16)

\[
Lu^i_2 = -\frac{1}{2} \sum_{i_1,j=1}^{3} \mathcal{P}^{ii_1, \partial_x j} (u^i_1 \circ u^j_1 - b^i_1 \circ b^j_1), \quad u_2(\cdot ,0) = 0,
\]

(17)

\[
Lb^i_2 = -\frac{1}{2} \sum_{i_1,j=1}^{3} \mathcal{P}^{ii_1, \partial_x j} (b^i_1 \circ u^j_1 - u^i_1 \circ b^j_1), \quad b_2(\cdot ,0) = 0,
\]

\[
Lu^i_3 = -\frac{1}{2} \sum_{i_1,j=1}^{3} \mathcal{P}^{ii_1, \partial_x j} (u^i_1 \circ u^j_1 + u^j_1 \circ u^i_1 - b^i_1 \circ b^j_1 - b^j_1 \circ b^i_1),
\]

(18)

\[
Lb^i_3 = -\frac{1}{2} \sum_{i_1,j=1}^{3} \mathcal{P}^{ii_1, \partial_x j} (b^i_1 \circ u^j_1 + b^j_1 \circ u^i_1 - u^i_1 \circ b^j_1 - u^j_1 \circ b^i_1),
\]

(19)

\[
y_3(\cdot ,0) = 0,
\]

and finally with initial data of

\[
y_4(\cdot ,0) = \mathcal{P} y_0(\cdot) - y_1(\cdot ,0),
\]
indeed; this will be crucially used in (10).

Remark 3.1. The first system (10) immediately deduces

$$u_1^i(x,t) = \int_{-\infty}^{t} \sum_{i_1=1}^{3} P^{i_1} P_{-s} \xi_{i_1}^{i_1}(x,s) ds,$$

$$b_1^i(x,t) = \int_{-\infty}^{t} \sum_{i_1=1}^{3} P^{i_1} P_{-s} \xi_{i_1}^{i_1}(x,s) ds,$$

and consequently we see that $(u_2^i, b_2^i)$ may be solved in (7) using that $(u_1^i, b_1^i)$ is known, $(u_3^i, b_3^i)$ may be solved in (8) using that $(u_1^i, b_1^i), (u_3^i, b_3^i)$ are known, but $(u_4^i, b_4^i)$ in (10) - (24) are the unknown. We also point out that another important feature of this construction is that $y_2(\cdot, 0) = 0, y_3(\cdot, 0) = 0$ but $y_4(\cdot, 0) = P y_0(\cdot) - y_1(\cdot, 0)$ so that $\sum_{j=1}^{4} y_j(\cdot, 0) = P y_0(\cdot)$. Finally, let us observe that

$$||y_4(\cdot, 0)||_{C^{-\epsilon}} \lesssim ||y_0(\cdot)||_{C^{-\epsilon}} + ||y_1(\cdot, 0)||_{C^{-\epsilon}} \lesssim 1$$

by the hypothesis of Theorem 1.3 that $y_0 \in C^{-\varepsilon}, z \in (\frac{1}{4}, \frac{1}{4} + \delta_0), \delta_0 \in (0, \frac{1}{4})$ and $y_1 \in C^\alpha$ for $\alpha < -\frac{1}{4}$ due to (10) being a linear heat equation so that $y_1 \in C^{-\varepsilon}$ indeed; this will be crucially used in (10).

We now specify that

$$u_4^i \circ u_2^i = \pi_<(u_4^i, u_1^i) + \pi_>(u_2^i, u_1^i),$$

(23a)

$$u_1^i \circ u_4^i = \pi_<(u_2^i, u_1^i) + \pi_>(u_4^i, u_1^i),$$

(23b)

$$b_1^i \circ b_4^i = \pi_<(b_4^i, b_1^i) + \pi_>(b_1^i, b_1^i) + \pi_{0,0}(b_2^i, b_1^i),$$

(23c)

$$b_4^i \circ b_1^i = \pi_<(b_1^i, b_2^i) + \pi_>(b_1^i, b_1^i),$$

(23d)

$$b_1^i \circ u_3^i = \pi_<(b_3^i, b_1^i) + \pi_>(u_3^i, b_1^i),$$

(23e)

$$b_1^i \circ u_1^i = \pi_<(u_3^i, b_1^i) + \pi_>(u_1^i, b_1^i),$$

(23f)

$$u_4^i \circ b_3^i = \pi_<(b_3^i, u_1^i) + \pi_>(b_3^i, u_1^i),$$

(23g)

$$u_4^i \circ b_2^i = \pi_<(b_2^i, u_1^i) + \pi_>(b_2^i, u_1^i),$$

(23h)

$$u_1^i \circ u_1^i = u_1^i u_1^i, b_1^i \circ b_2^i = b_1^i b_2^i, b_1^i \circ u_2^i = b_1^i u_2^i, u_1^i \circ b_3^i = u_1^i b_3^i, u_1^i \circ b_4^i = u_1^i b_4^i.$$

(24a)
we postpone specific description of the constants; e.g. in (117), (162) and (195), respectively. Now we consider the following equations and define
\[ \pi_0, (u^i_3, u^j_4) = \pi_0(u^i_3, u^j_4) - C_{1,1}^{ij}, \] (25a)
\[ \pi_0, (b^i_3, b^j_4) = \pi_0(b^i_3, b^j_4) - C_{1,2}^{ij}, \] (25b)
\[ \pi_0, (u^i_4, b^j_4) = \pi_0(u^i_4, b^j_4) - C_{1,3}^{ij}, \] (25c)
\[ \pi_0, (b^i_3, u^j_4) = \pi_0(b^i_3, u^j_4) - C_{1,4}^{ij}. \] (25d)

we postpone specific description of the constants; e.g. \[ C_{0,1}^{ij}, C_{2,3}^{ij} \] and \[ C_{1,3}^{ij} \] are given in [117], [162] and [195], respectively. Now we consider the following equations
\[ LK^i_u = u^i_1, \quad K^i_u(0) = 0 \quad \text{and} \quad LK^i_b = b^i_1, \quad K^i_b(0) = 0 \] (26)

and define \[ \pi_0, (u^i_2, u^j_1) \] of (23b) as follows:
\[ \pi_0, (u^i_2, u^j_1) = - \frac{1}{2} \pi_0, (\sum_{i_1, j_1=1}^{3} P^{ii_1} \pi < (u^i_3 + u^i_4, \partial_{x_1} K_u^{j_1}), u^j_1) \]
\[ + \pi_0, (\sum_{i_1, j_1=1}^{3} P^{ij_1} \pi < (u^j_3 + u^j_4, \partial_{x_1} K_u^{i_1}), u^i_1) \]
\[ + \sum_{i_1, j_1=1}^{3} \pi_0(P^{ii_1} \pi < (\partial_{x_1} (u^i_3 + u^i_4), K_u^{j_1}), u^j_1) \]
\[ + \sum_{i_1, j_1=1}^{3} \pi_0(P^{ij_1} \pi < (\partial_{x_1} (u^j_3 + u^j_4), K_u^{i_1}), u^i_1) \]
\[ - \pi_0, (\sum_{i_1, j_1=1}^{3} P^{ii_1} \pi < (b^i_3 + b^i_4, \partial_{x_1} K_b^{j_1}), u^j_1) \]
\[ - \pi_0, (\sum_{i_1, j_1=1}^{3} P^{ij_1} \pi < (b^j_3 + b^j_4, \partial_{x_1} K_b^{i_1}), u^i_1) \]
\[ - \sum_{i_1, j_1=1}^{3} \pi_0(P^{ii_1} \pi < (\partial_{x_1} (b^i_3 + b^i_4), K_b^{j_1}), u^j_1) \]
\[ - \sum_{i_1, j_1=1}^{3} \pi_0(P^{ij_1} \pi < (\partial_{x_1} (b^j_3 + b^j_4), K_b^{i_1}), u^i_1) \]
where

\[
\begin{align*}
\pi_{0,\phi}(P^{ii}, \pi_< (u_3^i + u_4^i, \partial_{x^{j,i}}, K_u^j), u_1^j) & = \pi_0(P^{ii}, \pi_< (u_3^i + u_4^i, \partial_{x^{j,i}}, K_u^j), u_1^j) - \pi_0(\pi_< (u_3^i + u_4^i, P^{ii}, \partial_{x^{j,i}}, K_u^j), u_1^j) \\
& \quad + \pi_0(\pi_< (u_3^i + u_4^i, P^{ii}, \partial_{x^{j,i}}, K_u^j), u_1^j) - (u_3^i + u_4^i)\pi_0(\pi_{ii}, \partial_{x^{j,i}}, K_u^j, u_1^j) \\
& \quad + (u_3^i + u_4^i)\pi_{0,\phi}(P^{ii}, \partial_{x^{j,i}}, K_u^j, u_1^j), \\
\pi_{0,\phi}(P^{ii}, \pi_< (b_3^i + b_4^i, \partial_{x^{j,i}}, K_b^j), u_1^j) & = \pi_0(P^{ii}, \pi_< (b_3^i + b_4^i, \partial_{x^{j,i}}, K_b^j), u_1^j) - \pi_0(\pi_< (b_3^i + b_4^i, P^{ii}, \partial_{x^{j,i}}, K_b^j), u_1^j) \\
& \quad + \pi_0(\pi_< (b_3^i + b_4^i, P^{ii}, \partial_{x^{j,i}}, K_b^j), u_1^j) - (b_3^i + b_4^i)\pi_0(\pi_{ii}, \partial_{x^{j,i}}, K_b^j, u_1^j) \\
& \quad + (b_3^i + b_4^i)\pi_{0,\phi}(P^{ii}, \partial_{x^{j,i}}, K_b^j, u_1^j).
\end{align*}
\] (28a)

\[
\begin{align*}
\pi_{0,\phi}(P^{ii}, \partial_{x^{j,i}}, K_u^j, u_1^j) & = -\frac{1}{2} \sum_{i_1,j_1=1}^3 P^{ii} \partial_{x^{j,i_1}} [\pi_< (u_3^i + u_4^i, K_u^j) + \pi_< (u_3^j + u_4^j, K_u^i) - \pi_< (b_3^i + b_4^i, K_b^j) - \pi_< (b_3^j + b_4^j, K_b^i)] + u_1^j.
\end{align*}
\] (29)

We also define a paracontrolled ansatz of

\[
\begin{align*}
u_4^j = -\frac{1}{2} \sum_{i_1,j_1=1}^3 P^{ii} \partial_{x^{j,i_1}} [\pi_< (u_3^i + u_4^i, K_u^j) + \pi_< (u_3^j + u_4^j, K_u^i) - \pi_< (b_3^i + b_4^i, K_b^j) - \pi_< (b_3^j + b_4^j, K_b^i)] + u_1^j.
\end{align*}
\] (30)

Additionally we define

\[
\begin{align*}
\pi_{0,\phi}(P^{ii}, \partial_{x^{j,i}}, K_u^j, u_1^j) & \triangleq \pi_0(P^{ii}, \partial_{x^{j,i}}, K_u^j, u_1^j), \\
\pi_{0,\phi}(P^{ii}, \partial_{x^{j,i}}, K_u^j, u_1^j) & \triangleq \pi_0(P^{ii}, \partial_{x^{j,i}}, K_u^j, u_1^j), \\
\pi_{0,\phi}(P^{ii}, \partial_{x^{j,i}}, K_u^j, u_1^j) & \triangleq \pi_0(P^{ii}, \partial_{x^{j,i}}, K_u^j, u_1^j), \\
\pi_{0,\phi}(P^{ii}, \partial_{x^{j,i}}, K_u^j, u_1^j) & \triangleq \pi_0(P^{ii}, \partial_{x^{j,i}}, K_u^j, u_1^j).
\end{align*}
\] (30)
Similarly we may define $\pi_0, \phi(b^4_i, b'_1)$ of (28a) as follows:

$$
\pi_0, \phi(b^4_i, b'_1) = -\frac{1}{2} \pi_0, \phi(\sum_{i_1, i_2 = 1}^3 P^{i_1 i_2} \pi_0 (u^i_{34} + u^i_{34}, \partial_{x^i}, K^i_{b^4_1}, b'_1))^
+ \pi_0, \phi(\sum_{i_1, i_2 = 1}^3 P^{i_1 i_2} \pi_0 (u^i_{34} + u^i_{34}, \partial_{x^i}, K^i_{b^4_1}, b'_1))
- \pi_0 (P^{i_1 i_2} \pi_0 (u^i_{34} + u^i_{34}, K^i_{b^4_1}, b'_1))
+ \pi_0 (P^{i_1 i_2} \pi_0 (u^i_{34} + u^i_{34}, K^i_{b^4_1}, b'_1))
+ \sum_{i_1, i_2 = 1}^3 \pi_0 (P^{i_1 i_2} \pi_0 (u^i_{34} + u^i_{34}, K^i_{b^4_1}, b'_1))
+ \sum_{i_1, i_2 = 1}^3 \pi_0 (P^{i_1 i_2} \pi_0 (u^i_{34} + u^i_{34}, K^i_{b^4_1}, b'_1))
- \sum_{i_1, i_2 = 1}^3 \pi_0 (P^{i_1 i_2} \pi_0 (u^i_{34} + u^i_{34}, K^i_{b^4_1}, b'_1)) + \pi_0 (b^4_i, b'_1)
$$

(31)

where $\pi_0, \phi(P^{i_1 i_2} \pi_0 (u^i_{34} + u^i_{34}, \partial_{x^i}, K^i_{b^4_1}, b'_1))$ is defined identically as $\pi_0, \phi(P^{i_1 i_2} \pi_0 (u^i_{34} + u^i_{34}, \partial_{x^i}, K^i_{b^4_1}, u^i_1))$ in (28a) with $u^i_1$ replaced by $b^i_1$ and $K^i_{b^4_1}$ replaced by $K^i_{b^4_1}$ while $\pi_0, \phi(P^{i_1 i_2} \pi_0 (u^i_{34} + u^i_{34}, \partial_{x^i}, K^i_{b^4_1}, b'_1))$ is defined as $\pi_0, \phi(P^{i_1 i_2} \pi_0 (b^i_{34} + b^i_{34}, \partial_{x^i}, K^i_{b^4_1}, u^i_1))$ in (28b) with $u^i_1$ replaced by $b^i_1$ and $K^i_{b^4_1}$ replaced by $K^i_{b^4_1}$. We also define a para-controlled ansatz of

$$
b^i_4 = -\frac{1}{2} \sum_{i_1, i_2 = 1}^3 P^{i_1 i_2} \partial_{x^i} [-\pi_0 (u^i_{34} + u^i_{34}, K^i_{b^4_1}) + \pi_0 (u^i_{34} + u^i_{34}, K^i_{b^4_1})]
+ \pi_0 (b^i_{34} + b^i_{34}, K^i_{b^4_1}) - \pi_0 (b^i_{34} + b^i_{34}, K^i_{b^4_1}) + b^i_{34};
$$

(32)

aditionally we define

$$
\phi_0, \phi(P^{i_1 i_2} \partial_{x^i} K^i_{b^4_1}, b^i_{34}) = \phi_0 (P^{i_1 i_2} \partial_{x^i} K^i_{b^4_1}, b^i_{34}),
\phi_0, \phi(P^{i_1 i_2} \partial_{x^i}, K^i_{b^4_1}, b^i_{34}) = \phi_0 (P^{i_1 i_2} \partial_{x^i}, K^i_{b^4_1}, b^i_{34}),
\phi_0, \phi(P^{i_1 i_2} \partial_{x^i}, K^i_{b^4_1}, b^i_{34}) = \phi_0 (P^{i_1 i_2} \partial_{x^i}, K^i_{b^4_1}, b^i_{34}),
\phi_0, \phi(P^{i_1 i_2} \partial_{x^i}, K^i_{b^4_1}, b^i_{34}) = \phi_0 (P^{i_1 i_2} \partial_{x^i}, K^i_{b^4_1}, b^i_{34}).
$$

Remark 3.2. This step is absolutely crucial and it took a few trials and errors to finally see what it should be, even following the case of the NSE in [50], particularly the signs of the four terms within (32) were not clear at first. We chose (32) in order to make the proof work, particularly bearing in mind the crucial steps at (24), (51) and (71).
For $\pi_{0,\circ}(u_4^i, b_1^j)$ of (23), it is essentially identical to $\pi_{0,\circ}(u_4^i, u_4^j)$ in (27) with $u_4^j$ replaced by $b_1^j$ because $u_4^j$ has already been defined in (29). We leave details here:

\[
\pi_{0,\circ}(u_4^i, b_1^j) = -\frac{1}{2}(\pi_{0,\circ}(\sum_{i_1,j_1=1}^{3} \mathcal{P}^{i_1i_1} \pi_{\circ}(u_3^{i_1} + u_4^{i_1}, \partial_{x^{i_1}} K^{j_1}_{\mu}), b_1^j)) \\
+ \pi_{0,\circ}(\sum_{i_1,j_1=1}^{3} \mathcal{P}^{i_1i_1} \pi_{\circ}(u_3^{i_1} + u_4^{i_1}, \partial_{x^{i_1}} K^{j_1}_{\mu}), b_1^j)) \\
+ \frac{3}{3} \pi_{0}(\mathcal{P}^{i_1i_1} \pi_{\circ}(\partial_{x^{j_1}} (u_3^{i_1} + u_4^{i_1}), K^{j_1}_{\mu})), b_1^j)) \\
+ \sum_{i_1,j_1=1}^{3} \pi_{0}(\mathcal{P}^{i_1i_1} \pi_{\circ}(\partial_{x^{j_1}} (u_3^{i_1} + u_4^{i_1}), K^{j_1}_{\mu})), b_1^j)) \\
- \frac{3}{3} \pi_{0}(\mathcal{P}^{i_1i_1} \pi_{\circ}(b_3^{i_1} + b_4^{i_1}, \partial_{x^{j_1}} K^{j_1}_{\mu})), b_1^j)) \\
- \sum_{i_1,j_1=1}^{3} \pi_{0}(\mathcal{P}^{i_1i_1} \pi_{\circ}(b_3^{i_1} + b_4^{i_1}, \partial_{x^{j_1}} K^{j_1}_{\mu})), b_1^j)) \\
- \sum_{i_1,j_1=1}^{3} \pi_{0}(\mathcal{P}^{i_1i_1} \pi_{\circ}(\partial_{x^{j_1}} (b_3^{i_1} + b_4^{i_1}), K^{j_1}_{\mu})), b_1^j)) \\
- \sum_{i_1,j_1=1}^{3} \pi_{0}(\mathcal{P}^{i_1i_1} \pi_{\circ}(\partial_{x^{j_1}} (b_3^{i_1} + b_4^{i_1}), K^{j_1}_{\mu})), b_1^j)) \\
+ \pi_{0}(\mathcal{P}^{i_1i_1} \pi_{\circ}(u_3^{i_1} + u_4^{i_1}, \partial_{x^{j_1}} K^{j_1}_{\mu})), b_1^j))
\]

where $\pi_{0,\circ}(\mathcal{P}^{i_1i_1} \pi_{\circ}(u_3^{i_1} + u_4^{i_1}, \partial_{x^{j_1}} K^{j_1}_{\mu})), b_1^j)$ is defined identically to $\pi_{0,\circ}(\mathcal{P}^{i_1i_1} \pi_{\circ}(u_3^{i_1} + u_4^{i_1}, \partial_{x^{j_1}} K^{j_1}_{\mu})), u_4^j)$ in (28) with $u_4^j$ replaced by $b_1^j$, and similarly $\pi_{0,\circ}(\mathcal{P}^{i_1i_1} \pi_{\circ}(b_3^{i_1} + b_4^{i_1}, \partial_{x^{j_1}} K^{j_1}_{\mu})), b_1^j)$ is defined as $\pi_{0,\circ}(\mathcal{P}^{i_1i_1} \pi_{\circ}(b_3^{i_1} + b_4^{i_1}, \partial_{x^{j_1}} K^{j_1}_{\mu})), u_4^j)$ in (28) with $u_4^j$ replaced by $b_1^j$. For $\pi_{0,\circ}(b_4^i, u_4^j)$ of (23), it is also identical to $\pi_{0,\circ}(b_4^i, b_1^j)$ with $b_1^j$ replaced by $u_4^j$, which is automatic because we already defined $b_1^j$ in (29). In detail,

\[
\pi_{0,\circ}(b_4^i, u_4^j) = -\frac{1}{2}(\pi_{0,\circ}(\sum_{i_1,j_1=1}^{3} \mathcal{P}^{i_1i_1} \pi_{\circ}(u_3^{i_1} + u_4^{i_1}, \partial_{x^{j_1}} K^{j_1}_{\mu})), u_4^j)) \\
+ \pi_{0,\circ}(\sum_{i_1,j_1=1}^{3} \mathcal{P}^{i_1i_1} \pi_{\circ}(u_3^{i_1} + u_4^{i_1}, \partial_{x^{j_1}} K^{j_1}_{\mu})), u_4^j)) \\
- \sum_{i_1,j_1=1}^{3} \pi_{0}(\mathcal{P}^{i_1i_1} \pi_{\circ}(u_3^{i_1} + u_4^{i_1}, K^{j_1}_{\mu})), u_4^j)) \\
+ \sum_{i_1,j_1=1}^{3} \pi_{0}(\mathcal{P}^{i_1i_1} \pi_{\circ}(u_3^{i_1} + u_4^{i_1}, K^{j_1}_{\mu})), u_4^j)) \\
+ \pi_{0,\circ}(\sum_{i_1,j_1=1}^{3} \mathcal{P}^{i_1i_1} \pi_{\circ}(b_3^{i_1} + b_4^{i_1}, \partial_{x^{j_1}} K^{j_1}_{\mu})), u_4^j)) \\
- \sum_{i_1,j_1=1}^{3} \pi_{0}(\mathcal{P}^{i_1i_1} \pi_{\circ}(b_3^{i_1} + b_4^{i_1}, K^{j_1}_{\mu})), u_4^j)) \\
+ \pi_{0,\circ}(\sum_{i_1,j_1=1}^{3} \mathcal{P}^{i_1i_1} \pi_{\circ}(b_3^{i_1} + b_4^{i_1}, \partial_{x^{j_1}} K^{j_1}_{\mu})), u_4^j))
\]
\[-\pi_{0,\varnothing}(\sum_{i_1,j_1=1}^{3} \mathcal{P}^{i_1i_1} \pi_{<}(b_{i_3}^{j_1} + b_{i_4}^{j_1}, \partial_{x,i_1} K_{u}^{j_1}), u_{i_1}^{j_1})
\] 
\[+ \sum_{i_1,j_1=1}^{3} \pi_{0}(\mathcal{P}^{i_1i_1} \pi_{<}(\partial_{x,i_1} (b_{i_3}^{j_1} + b_{i_4}^{j_1}), K_{u}^{j_1}), u_{i_1}^{j_1})
\] 
\[+ \sum_{i_1,j_1=1}^{3} \pi_{0}(\mathcal{P}^{i_1i_1} \pi_{<}(\partial_{x,i_1} (b_{i_3}^{j_1} + b_{i_4}^{j_1}), K_{u}^{j_1}), u_{i_1}^{j_1})\]

where \(\pi_{0,\varnothing}(\mathcal{P}^{i_1i_1} \pi_{<}(u_{i_3}^{j_1} + u_{i_4}^{j_1}, \partial_{x,i_1} K_{b}^{j_1}), u_{i_1}^{j_1})\) is defined identically to \(\pi_{0,\varnothing}(\mathcal{P}^{i_1i_1} \pi_{<}(u_{i_3}^{j_1} + u_{i_4}^{j_1}, \partial_{x,i_1} K_{b}^{j_1}), u_{i_1}^{j_1})\) in (28a) with \(K_{b}^{j_1}\) replaced by \(K_{b}^{j_1}\) and similarly \(\pi_{0,\varnothing}(\mathcal{P}^{i_1i_1} \pi_{<}(b_{i_3}^{j_1} + b_{i_4}^{j_1}, \partial_{x,i_1} K_{u}^{j_1}), u_{i_1}^{j_1})\) is defined identically to \(\pi_{0,\varnothing}(\mathcal{P}^{i_1i_1} \pi_{<}(b_{i_3}^{j_1} + b_{i_4}^{j_1}, \partial_{x,i_1} K_{b}^{j_1}), u_{i_1}^{j_1})\) in (28a) with \(K_{b}^{j_1}\) replaced by \(K_{u}^{j_1}\). Now from (20), for all \(\delta \in [0, 1]\) we may compute

\[\|K_{u}^{j_1}(t)\|_{C^{\frac{d}{2} - \delta}} \lesssim \sup_{s \in [0,t]} \|b_{i_1}^{j_1}(s)\|_{C_{-\frac{n}{4} - \frac{d}{2}}} \approx \sup_{s \in [0,t]} \|b_{i_1}^{j_1}(s)\|_{C_{-\frac{n}{4} + \frac{d}{2}}} (33)\]

because e.g.

\[K_{b}^{j_1}(t)\|_{C^{\frac{d}{2} - \delta}} \lesssim \int_{0}^{t} (t - s)^{-\frac{d}{2} - \frac{d}{4}} \|b_{i_1}^{j_1}(s)\|_{C_{-\frac{n}{4} - \frac{d}{2}}} ds \lesssim \sup_{s \in [0,t]} \|b_{i_1}^{j_1}(s)\|_{C_{-\frac{n}{4} - \frac{d}{2}}} (34)\]

by (20) and Lemma 4.4. We fix

\[0 < \delta < \delta_0 \wedge \frac{1 - 2\delta_0}{3} \wedge \frac{1 - z}{4} \wedge (2 - \frac{1}{z}). (35)\]

Let us assume that

\[u_{i_1}^{j_1}, b_{i_1}^{j_1} \in C([0, T]; C^{-\frac{d}{2} - \frac{d}{4}}), (36a)\]
\[u_{i_1}^{j_1} \circ u_{i_1}^{j_1}, b_{i_1}^{j_1} \circ b_{i_1}^{j_1}, u_{i_1}^{j_1} \circ u_{i_1}^{j_1}, b_{i_1}^{j_1} \circ u_{i_1}^{j_1} \in C([0, T]; C^{-1 - \frac{d}{4}}), (36b)\]
\[u_{i_1}^{j_1} \circ u_{i_1}^{j_1}, b_{i_1}^{j_1} \circ b_{i_1}^{j_1}, b_{i_1}^{j_1} \circ u_{i_1}^{j_1}, b_{i_1}^{j_1} \circ u_{i_1}^{j_1} \in C([0, T]; C^{-\frac{d}{2} - \frac{d}{4}}), (36c)\]
\[u_{i_1} \circ u_{i_1}^{j_1}, b_{i_1} \circ b_{i_1}^{j_1}, b_{i_1} \circ u_{i_1}^{j_1}, b_{i_1} \circ u_{i_1}^{j_1} \in C([0, T]; C^{-d}), (36d)\]
\[\pi_{0,\varnothing}(u_{i_1}^{j_1}), \pi_{0,\varnothing}(u_{i_1}^{j_1}), \pi_{0,\varnothing}(b_{i_1}^{j_1}), \pi_{0,\varnothing}(b_{i_1}^{j_1}), \pi_{0,\varnothing}(b_{i_1}^{j_1}, u_{i_1}^{j_1}) \in C([0, T]; C^{-d}), (36e)\]
\[\pi_{0,\varnothing}(\mathcal{P}^{i_1i_1} \partial_{x}, K_{i_1}^{j_1}), \pi_{0,\varnothing}(\mathcal{P}^{i_1i_1} \partial_{x}, K_{i_1}^{j_1}), \pi_{0,\varnothing}(\mathcal{P}^{i_1i_1} \partial_{x}, K_{u}^{j_1}), \pi_{0,\varnothing}(\mathcal{P}^{i_1i_1} \partial_{x}, K_{b}^{j_1}), (36f)\]
for all $i,j,i_1,j_1 \in \{1, 2, 3\}$ so that we may define a finite number of

\[
C_{\xi}' \triangleq \sup_{t \in [0,T]} \left\{ \sum_{i=1}^{3} \left\| (u_i^1, b_i^1) (t) \right\|_{C^{-\frac{1}{2} - \frac{\epsilon}{2}}} + 3 \sum_{i,j=1}^{3} \left\| (u_i^1 \circ u_j^1, b_i^1 \circ b_j^1, u_i^2 \circ b_j^1, b_i^1 \circ u_j^2, b_i^2 \circ u_j^1) (t) \right\|_{C^{-\frac{1}{2} - \frac{\epsilon}{2}}} + 3 \sum_{i,j=1}^{3} \left\| (u_i^2 \circ u_j^1, b_i^1 \circ b_j^2, b_i^2 \circ u_j^1) \right\|_{C^{-\delta}} + 3 \sum_{i,j=1}^{3} \left\| (\pi_{0,0}(u_i^1, u_j^1), \pi_{0,0}(b_i^1), \pi_{0,0}(u_i^2, b_i^1), \pi_{0,0}(b_i^2, u_i^1)) \right\|_{C^{-\delta}} + 3 \sum_{i,i_1,j,j_1=1}^{3} \left\| (\pi_{0,0}(\mathcal{P}^{i_{i_1}i,i} \partial_x \partial_x \partial_x K_u^{i_1}, u_i^1), \pi_{0,0}(\mathcal{P}^{i_{i_1}i,i} \partial_x \partial_x \partial_x K_u^{i_1}, u_i^1), \pi_{0,0}(\mathcal{P}^{i_{i_1}i,i} \partial_x \partial_x \partial_x K_u^{i_1}, b_i^1), \pi_{0,0}(\mathcal{P}^{i_{i_1}i,i} \partial_x \partial_x \partial_x K_u^{i_1}, b_i^1)) \right\|_{C^{-\delta}} \right\} 
\]

(37)

let us write $C_{\xi}'$ in case $\epsilon = 0$. We mention in particular the inclusion of the last two summations in (37) will be crucial in (70) and (76). Now from (17) we see that

\[
\sup_{t \in [0,T]} \| y_2 (t) \|_{C^{-\delta}} \lesssim \sum_{i,j=1}^{3} \sup_{t \in [0,T]} \int_0^t (t-s)^{-\frac{3-\epsilon}{2}} \left\| (u_i^1 \circ u_j^1, b_i^1 \circ b_j^1, b_i^1 \circ u_j^1, u_i^1 \circ b_j^1) \right\|_{C^{-\frac{1}{2} - \frac{\epsilon}{2}}} ds \lesssim C_{\xi} t^{\frac{1}{2}} 
\]

(38)

by Lemmas 4.3 and 4.4 [36] and [37]. Similarly from [18], we may compute

\[
\sup_{t \in [0,T]} \| y_2 (t) \|_{C^{-\frac{1}{2} - \frac{\epsilon}{2}}} \lesssim \sum_{i,i_1,j,j_1=1}^{3} \left\| (u_i^{i_1} \circ u_j^1, u_i^{i_1} \circ u_j^1, b_i^{i_1} \circ b_j^1, b_i^{i_1} \circ b_j^1, b_i^{i_1} \circ u_j^1, b_i^{i_1} \circ u_j^1) \right\|_{C([0,T],C^{-\frac{1}{2} - \frac{\epsilon}{2}})} \times \int_0^t (t-s)^{-\frac{3-\epsilon}{2}} ds \lesssim C_{\xi} t^{\frac{1}{2}} 
\]

by Lemma 4.3 [35] and [37], and therefore

\[
\| y_2 \|_{C([0,T],C^{-\delta})} + \| y_3 \|_{C([0,T],C^{-\frac{1}{2} - \frac{\epsilon}{2}})} \lesssim C_{\xi} t^{\frac{1}{2}}. 
\]

(39)
Next, from (19)–(21), we may compute
\[
\sup_{t \in [0,T]} t^{\frac{\epsilon - \delta_0 + z}{2}} \| y_1^2(t) \|_{C_t^{\frac{\epsilon - \delta_0}{2}}} \\
\lesssim \sup_{t \in [0,T]} t^{\frac{\epsilon - \delta_0 + z}{2}} \int_0^t \| P_{t-s} (\mathcal{P} y_0^1 - y_1^1(0)) ds \|_{C_t^{\frac{\epsilon - \delta_0}{2}}} \\
+ \sup_{t \in [0,T]} t^{\frac{\epsilon - \delta_0 + z}{2}} \sum_{i,j=1}^3 \int_0^t \| P_{t-s} (u_2^i + u_1^i + u_4^i + u_1^i) \|_{C_t^{\frac{\epsilon - \delta_0}{2}}} \]
\]
\[= u_2^i + u_1^i + u_4^i (u_3^i + u_4^i) + u_2^i (u_3^i + u_4^i) \\
+ (u_1^i + u_4^i) (u_3^i + u_4^i) - b_3^i + b_4^i) - (b_3^i + b_4^i) \circ b_1^i \\
- b_1^i \circ b_3^i - b_2^i (b_3^i + b_4^i) - b_2 (b_3^i + b_4^i) - (b_3^i + b_4^i)(b_3 + b_4), \\
\]
\[b_1^i \circ (u_3^i + u_4^i) + (b_3^i + b_4^i) \circ u_1^i + b_2^i \circ u_2^i \\
+ b_2^i (u_3^i + u_4^i) + u_2^i (b_3^i + b_4^i) + (b_3^i + b_4^i)(u_3^i + u_4^i) \\
- u_1^i \circ (b_3^i + b_4^i) - (u_3^i + u_4^i) \circ b_1^i - u_2^i \circ b_2^i \\
\]
\[= \| P_{t-s} (u_2^i + u_1^i + u_4^i)(b_3^i + b_4^i)(u_3^i + u_4^i) \|_{C_t^{\frac{\epsilon - \delta_0}{2}}} ds \triangleq I_1 + I_2^1 
\]
by Lemma 4.3 where it is immediate that we may estimate for \( \epsilon \in (0, 1) \) fixed,
\[I_1^1 = \sup_{t \in [0,T]} t^{\frac{\epsilon - \delta_0 + z}{2}} \int_0^t \| P_{t-s} (\mathcal{P} y_0^1 - y_1^1(0)) ds \|_{C_t^{\frac{\epsilon - \delta_0}{2}}} \\
\lesssim \sup_{t \in [0,T]} t^{\frac{\epsilon - \delta_0 + z}{2}} \int_0^t (t-s)^{-\frac{(\epsilon - \delta_0 + z)}{2}} (\| \mathcal{P} y_0^1 \|_{C^{\epsilon - z}} + \| y_1^1(0) \|_{C^{\epsilon - z}}) ds \lesssim 1 
\]
due to Lemma 4.4, (35) and Remark 3.1. Thus, we now focus on \( I_2^1 \). Firstly, we may estimate also for \( \epsilon \in (0, 1) \) fixed,
\[\sup_{t \in [0,T]} t^{\frac{\epsilon - \delta_0 + z}{2}} \int_0^t \| P_{t-s} (b_3^i \circ u_2^i) \|_{C_t^{\frac{\epsilon - \delta_0}{2}}} ds \leq \sup_{t \in [0,T]} t^{\frac{\epsilon - \delta_0 + z}{2}} \int_0^t (t-s)^{-\frac{\epsilon - \delta_0 + z}{2}} \| b_3^i \circ u_2^i \|_{C^{\epsilon - z}} ds \lesssim 1 
\]
by Lemma 4.3 (36) and (36d). Secondly, e.g. we may also estimate
\[\sup_{t \in [0,T]} t^{\frac{\epsilon - \delta_0 + z}{2}} \int_0^t \| P_{t-s} (u_4^i b_4^i) \|_{C_t^{\frac{\epsilon - \delta_0}{2}}} ds \leq \sup_{t \in [0,T]} t^{\frac{\epsilon - \delta_0 + z}{2}} \int_0^t (t-s)^{-\frac{\epsilon}{2}} \| u_4^i \|_{C_t^{\frac{\epsilon - \delta_0}{2}}} \| b_4^i \|_{C_t^{\frac{\epsilon - \delta_0}{2}}} ds 
\]
\[\lesssim (\sup_{t \in [0,T]} t^{\frac{\epsilon - \delta_0 + z}{2}} \| y_4(t) \|_{C_t^{\frac{\epsilon - \delta_0}{2}}}^2 T_\epsilon^{\frac{\epsilon - \delta_0 + z}{2}} \lesssim 1 
\]
by Lemma 4.3 and Lemma 11 (4). Similar computations on other terms in \( I_2^1 \) of (40) show that for all \( \epsilon \in (0, 1) \) fixed, there exists a maximal existence time \( T_\epsilon > 0 \) and \( y_4 \in C([0,T_\epsilon); C^{\frac{\epsilon - \delta_0}{2}}) \) such that \( y_4 \) satisfies (19)–(21) and
\[\sup_{t \in [0,T_\epsilon]} t^{\frac{\epsilon - \delta_0 + z}{2}} \| y_4(t) \|_{C_t^{\frac{\epsilon - \delta_0}{2}}} = +\infty. 
\]
Now we set
\[
\frac{\delta}{2} < \beta < z + 2\delta - \frac{1}{2} < \frac{1}{2} - 2\delta
\] (44)
and realize that in the computation of (42), we could have instead estimated
\[
t^{\frac{1}{2} + \frac{\delta + z}{2}} \int_0^t \| P_{t-s}(u_1^{i_1} b_2^{j_1}) \|_{C^{\frac{1}{2} + \delta}} ds
\leq t^{\frac{1}{2} + \frac{\delta + z}{2}} \int_0^t (t-s)^{-\frac{1}{2} + \frac{\delta - \delta_0}{2}} \| u_i^{i_1} \|_{C^{\frac{1}{2} + \delta_0}} \| b_j^{j_1} \|_{C^{\frac{1}{2} - \delta_0}} ds
\leq t^{\frac{1}{2} + \frac{\delta - \delta_0}{2}} \left( \sup_{s \in [0, t]} s^{\frac{1}{2} - \frac{\delta_0 + z}{2}} \| y_4(s) \|_{C^{\frac{1}{2} - \delta_0}} \right)^2
\] (45)
by Lemma 4.4 (44), (35) and Lemma 1.1 (4). Thus, similar computations on other terms in $I_1^t$ and $I_2^t$ of (40) lead to
\[
t^{\frac{1}{2} + \frac{\delta + z}{2}} \| y_4(t) \|_{C^{\frac{1}{2} + \delta}} \leq C(\varepsilon, \| y_0 \|_{C^{\frac{1}{2} - \delta}}, y_1, y_2, y_3)
+ t^{\frac{1}{2} + \frac{\delta - \delta_0}{2}} \left( \sup_{s \in [0, t]} s^{\frac{1}{2} - \frac{\delta_0 + z}{2}} \| y_4(s) \|_{C^{\frac{1}{2} - \delta_0}} \right)^2
\] (46)
for all $t \in (0, T_\varepsilon)$. This shows that $(u^{\delta, i}, b^{\delta, j})(t) \in C^{\frac{1}{2} + \beta}$ for all $t \in (0, T_\varepsilon)$ due to (13). This leads us to the next estimate of
\[
\| u_1^{i_1} \|_{C^{\frac{1}{2} + \delta}} + \| b_2^{j_1} \|_{C^{\frac{1}{2} - \delta}}
\leq \sum_{i_1, j_1 = 1}^3 \| (P^{i_1} \partial_{x_1} [\pi < (u_1^{i_1} + u_4^{i_1}, K_1^{i_1}) + \pi < (u_1^{i_1} + u_4^{i_1}, K_1^{i_1})]) \|_{C^{\frac{1}{2} + \delta}}
+ \| (P^{i_1} \partial_{x_1} [\pi < (b_3^{i_1} + b_4^{i_1}, K_1^{i_1}) + \pi < (b_3^{i_1} + b_4^{i_1}, K_1^{i_1})]) \|_{C^{\frac{1}{2} - \delta}}
+ \| (P^{i_1} \partial_{x_1} [\pi < (u_1^{i_1} + u_4^{i_1}, K_1^{i_1}) + \pi < (u_1^{i_1} + u_4^{i_1}, K_1^{i_1})]) \|_{C^{\frac{1}{2} - \delta}}
\] (47)
by the paracontrolled ansatz (51) and (32). Firstly, we may estimate
\[
\| (P^{i_1} \partial_{x_1} [\pi < (b_3^{i_1} + b_4^{i_1}, K_1^{i_1}) + \pi < (b_3^{i_1} + b_4^{i_1}, K_1^{i_1})]) \|_{C^{\frac{1}{2} + \delta}}
\leq \| b_3^{i_1} + b_4^{i_1} \|_{C^{\frac{1}{2} - \delta}} \| K_1^{i_1} \|_{C^{\frac{1}{2} - \delta}} + \| b_3^{i_1} + b_4^{i_1} \|_{C^{\frac{1}{2} - \delta}} \| K_1^{i_1} \|_{C^{\frac{1}{2} - \delta}}
\] (48)
by Lemma 4.5 Lemma 1.1 (1), and (2). Similar estimates may be deduced for
\[
\| (P^{i_1} \partial_{x_1} [\pi < (u_1^{i_1} + u_4^{i_1}, K_1^{i_1}) + \pi < (u_1^{i_1} + u_4^{i_1}, K_1^{i_1})]) \|_{C^{\frac{1}{2} - \delta}},
\] and
\[
\| (P^{i_1} \partial_{x_1} [\pi < (u_1^{i_1} + u_4^{i_1}, K_1^{i_1}) + \pi < (u_1^{i_1} + u_4^{i_1}, K_1^{i_1})]) \|_{C^{\frac{1}{2} + \delta}},
\] as well as \( \| (P^{i_1} \partial_{x_1} [\pi < (b_3^{i_1} + b_4^{i_1}, K_1^{i_1}) + \pi < (b_3^{i_1} + b_4^{i_1}, K_1^{i_1})]) \|_{C^{\frac{1}{2} - \delta}} \). Moreover, it is immediate that \( C^{\frac{1}{2} + \beta} \hookrightarrow C^{\frac{1}{2} - \delta} \) by (14). Therefore, we obtain
\[
\| u_1^{i_1} \|_{C^{\frac{1}{2} + \delta}} + \| b_2^{j_1} \|_{C^{\frac{1}{2} - \delta}} \leq \| (u^{\delta, i}, b^{\delta, j}) \|_{C^{\frac{1}{2} + \beta}}
\sum_{i_1, j_1 = 1}^3 \| (u_1^{i_1} + u_4^{i_1}, b_3^{i_1} + b_4^{i_1}, u_3^{i_1} + u_4^{i_1}, b_3^{i_1} + b_4^{i_1}) \|_{C^{\frac{1}{2} - \delta}} \| (K_1^{i_1}, K_1^{i_1}) \|_{C^{\frac{1}{2} - \delta}}.
\] (49)
Now we obtain from (29)

\[
Lu^{i,i} = -\frac{1}{2} \sum_{i,j=1}^{3} \mathcal{P}^{i,i,j,j} \partial_{x^i} \left[ \pi_<(u_3^i + u_4^i, u_1^i) + \pi_<(u_3^i + u_4^i, u_1^i) \right]
\]

\[
+ \pi_{0,0}(u_3^i + u_1^i) + \pi_{0,0}(u_3^i, u_1^i) \\
+ \pi_<(u_3^i + u_4^i, u_1^i) + \pi_<(u_3^i + u_4^i, u_1^i) + \pi_{0,0}(u_3^i, u_1^i) + \pi_{0,0}(u_4^i, u_1^i) \\
u_2^i + \pi_<(u_3^i + u_4^i, u_1^i) + \pi_<(u_3^i + u_4^i, u_1^i) + \pi_{0,0}(u_3^i, u_1^i) + \pi_{0,0}(u_4^i, u_1^i) \\
u_2^i \cdot \left( u_2^i + u_3^i + u_4^i + u_3^i + u_4^i \right) + \pi_{0,0}(u_3^i + u_4^i, u_1^i) + \pi_{0,0}(u_4^i, u_1^i) \\
- \pi_<(u_3^i + u_4^i, u_1^i) + \pi_<(u_3^i + u_4^i, u_1^i) + \pi_{0,0}(u_3^i, u_1^i) + \pi_{0,0}(u_4^i, u_1^i) \\
- \pi_<(u_3^i + u_4^i, u_1^i) + \pi_<(u_3^i + u_4^i, u_1^i) + \pi_{0,0}(u_3^i, u_1^i) + \pi_{0,0}(u_4^i, u_1^i) \\
- \pi_<(u_3^i + u_4^i, u_1^i) + \pi_<(u_3^i + u_4^i, u_1^i) + \pi_{0,0}(u_3^i, u_1^i) + \pi_{0,0}(u_4^i, u_1^i) \\
- \pi_<(u_3^i + u_4^i, u_1^i) + \pi_<(u_3^i + u_4^i, u_1^i) + \pi_{0,0}(u_3^i, u_1^i) + \pi_{0,0}(u_4^i, u_1^i) \\
+ \frac{1}{2} \sum_{i,j=1}^{3} \mathcal{P}^{i,i,j,j} \partial_{x^i} \left[ \pi_<(L(u_3^i + u_4^i), K^i_u) + \pi_<(u_3^i + u_4^i, u_1^i) \right]
\]

where we used (20), that \( L = \partial_t - \Delta \) (20). (23a)-(23d). We make a crucial observation that we can cancel out \( \pi_<(u_3^i + u_4^i, u_1^i) \), \( \pi_<(u_3^i + u_4^i, u_1^i) \), \( \pi_<(u_3^i + u_4^i, u_1^i) \), \( \pi_<(u_3^i + u_4^i, u_1^i) \) and \( \pi_<(u_3^i + u_4^i, u_1^i) \) to deduce

\[
Lu^{i,i} = -\frac{1}{2} \sum_{i,j=1}^{3} \mathcal{P}^{i,i,j,j} \partial_{x^i} \left[ \pi_<(u_3^i + u_4^i, u_1^i) + \pi_{0,0}(u_3^i, u_1^i) + \pi_{0,0}(u_4^i, u_1^i) \right]
\]

\[
+ \pi_<(u_3^i + u_4^i, u_1^i) + \pi_{0,0}(u_3^i, u_1^i) + \pi_{0,0}(u_4^i, u_1^i) \\
+ \pi_<(u_3^i + u_4^i, u_1^i) + \pi_<(u_3^i + u_4^i, u_1^i) + \pi_{0,0}(u_3^i, u_1^i) + \pi_{0,0}(u_4^i, u_1^i) \\
u_2^i + \pi_<(u_3^i + u_4^i, u_1^i) + \pi_<(u_3^i + u_4^i, u_1^i) + \pi_{0,0}(u_3^i, u_1^i) + \pi_{0,0}(u_4^i, u_1^i) \\
u_2^i \cdot \left( u_2^i + u_3^i + u_4^i + u_3^i + u_4^i \right) + \pi_{0,0}(u_3^i + u_4^i, u_1^i) + \pi_{0,0}(u_4^i, u_1^i) \\
- \pi_<(u_3^i + u_4^i, u_1^i) + \pi_<(u_3^i + u_4^i, u_1^i) + \pi_{0,0}(u_3^i, u_1^i) + \pi_{0,0}(u_4^i, u_1^i) \\
- \pi_<(u_3^i + u_4^i, u_1^i) + \pi_<(u_3^i + u_4^i, u_1^i) + \pi_{0,0}(u_3^i, u_1^i) + \pi_{0,0}(u_4^i, u_1^i) \\
- \pi_<(u_3^i + u_4^i, u_1^i) + \pi_<(u_3^i + u_4^i, u_1^i) + \pi_{0,0}(u_3^i, u_1^i) + \pi_{0,0}(u_4^i, u_1^i) \\
- \pi_<(u_3^i + u_4^i, u_1^i) + \pi_<(u_3^i + u_4^i, u_1^i) + \pi_{0,0}(u_3^i, u_1^i) + \pi_{0,0}(u_4^i, u_1^i) \\
+ \frac{1}{2} \sum_{i,j=1}^{3} \mathcal{P}^{i,i,j,j} \partial_{x^i} \left[ \pi_<(L(u_3^i + u_4^i), K^i_u) + \pi_<(u_3^i + u_4^i, u_1^i) \right]
\]

Similarly we can compute

\[
Lu^{i,i} = -\frac{1}{2} \sum_{i,j=1}^{3} \mathcal{P}^{i,i,j,j} \partial_{x^i} \left[ \pi_<(u_3^i + u_4^i, b_1^i) + \pi_<(u_3^i + u_4^i, b_1^i) \right]
\]

\[
+ \pi_{0,0}(u_3^i, b_1^i) + \pi_{0,0}(u_4^i, b_1^i) \\
+ \pi_<(u_3^i + u_4^i, b_1^i) + \pi_<(u_3^i + u_4^i, b_1^i) + \pi_{0,0}(u_3^i, b_1^i) + \pi_{0,0}(u_4^i, b_1^i) \\
u_2^i \cdot \left( u_2^i + u_3^i + u_4^i + u_3^i + u_4^i \right) + \pi_{0,0}(u_3^i + u_4^i, b_1^i) + \pi_{0,0}(u_4^i, b_1^i) \\
- \pi_<(u_3^i + u_4^i, b_1^i) + \pi_<(u_3^i + u_4^i, b_1^i) + \pi_{0,0}(u_3^i, b_1^i) + \pi_{0,0}(u_4^i, b_1^i) \\
- \pi_<(u_3^i + u_4^i, b_1^i) + \pi_<(u_3^i + u_4^i, b_1^i) + \pi_{0,0}(u_3^i, b_1^i) + \pi_{0,0}(u_4^i, b_1^i) \\
- \pi_<(u_3^i + u_4^i, b_1^i) + \pi_<(u_3^i + u_4^i, b_1^i) + \pi_{0,0}(u_3^i, b_1^i) + \pi_{0,0}(u_4^i, b_1^i) \\
+ \frac{1}{2} \sum_{i,j=1}^{3} \mathcal{P}^{i,i,j,j} \partial_{x^i} \left[ \pi_<(L(u_3^i + u_4^i), K^i_u) + \pi_<(u_3^i + u_4^i, b_1^i) \right]
\]

(50)
\[ + \pi_<(b_3^i + b_4^i, u_4^i) + \pi_>(b_3^i + b_4^i, u_4^i) + \pi_0(0.9(b_3^i + u_4^i) + \pi_0(0.9(b_3^i + u_4^i) + \\
+ b_3^i \circ u_4^i + b_4^i, u_4^i) + u_2^i(b_3^i + b_4^i) + (b_3^i + b_4^i)(u_3^i + u_4^i) \\
- \pi_<(b_3^i + b_4^i, u_4^i) - \pi_>(b_3^i + b_4^i, u_4^i) - \pi_0(0.9(b_3^i + u_4^i) - \pi_0(0.9(b_3^i + u_4^i) \\
- \pi_<(u_3^i + u_4^i, b_4^i) - \pi_0(0.9(u_3^i + b_4^i) - \pi_0(0.9(u_3^i + b_4^i) \\
- u_2^i \circ b_4^i - u_3^i(b_3^i + b_4^i) - b_2^i(u_3^i + u_4^i) - (u_3^i + u_4^i)(b_3^i + b_4^i) \\
+ \frac{1}{2} \sum_{i,j=1}^3 \mathcal{P}^{ii} \partial_{x^j} [\pi_<(L(u_3^i + u_4^i), K_4^i)] - \pi_<(u_3^i + u_4^i, b_4^i) \\
+ \pi_<(L(u_3^i + u_4^i), K_4^i) + \pi_<(u_3^i + u_4^i, b_4^i) \\
+ \pi_<(L(b_3^i + b_4^i), K_4^i) + \pi_<(b_3^i + b_4^i, u_4^i) \\
- \pi_<(L(b_3^i + b_4^i), K_4^i) - \pi_<(b_3^i + b_4^i, u_4^i) \\
+ 2\pi_<(\nabla(u_3^i + u_4^i), \nabla K_4^i) - 2\pi_<(\nabla(u_3^i + u_4^i), \nabla K_4^i) \\
- 2\pi_<(\nabla(b_3^i + b_4^i), \nabla K_4^i) + 2\pi_<(\nabla(b_3^i + b_4^i), \nabla K_4^i)] \]

by (32), that \( L = \partial_t - \Delta \), (26), (21), (23a)-(23h). Again we cancel out \( \pi_<(u_3^i + u_4^i, b_4^i) \), \( \pi_<(b_3^i + b_4^i, u_4^i) \), \( \pi_<(b_3^i + b_4^i, u_4^i) \) and \( \pi_<(u_3^i + u_4^i, b_4^i) \) and obtain

\[ Lb_4^{z,i} = - \frac{1}{2} \sum_{i,j=1}^3 \mathcal{P}^{ii} \partial_{x^j} [\pi_<(L(u_3^i + u_4^i, b_4^i)] + \pi_0(0.9(u_3^i + u_4^i) + \pi_0(0.9(u_3^i + u_4^i) + \\
+ \pi_>(b_3^i + b_4^i, u_4^i) + \pi_0(0.9(b_3^i + u_4^i) + \pi_0(0.9(b_3^i + u_4^i) \\
+ b_3^i \circ u_4^i + b_4^i, u_4^i) + u_2^i(b_3^i + b_4^i) + (b_3^i + b_4^i)(u_3^i + u_4^i) \\
- \pi_>(b_3^i + b_4^i, u_4^i) - \pi_0(0.9(b_3^i + u_4^i) - \pi_0(0.9(b_3^i + u_4^i) \\
- u_2^i \circ b_4^i - u_3^i(b_3^i + b_4^i) - b_2^i(u_3^i + u_4^i) - (u_3^i + u_4^i)(b_3^i + b_4^i) \\
+ \pi_<(L(u_3^i + u_4^i), K_4^i) - \pi_<(L(u_3^i + u_4^i), K_4^i) \\
- \pi_<(L(b_3^i + b_4^i), K_4^i) + \pi_<(L(b_3^i + b_4^i), K_4^i) \\
- 2\pi_<(\nabla(u_3^i + u_4^i), \nabla K_4^i) + 2\pi_<(\nabla(u_3^i + u_4^i), \nabla K_4^i) \\
+ 2\pi_<(\nabla(b_3^i + b_4^i), \nabla K_4^i) - 2\pi_<(\nabla(b_3^i + b_4^i), \nabla K_4^i)] \triangleq \phi_4^{z,i} \]
In contrast to the NSE, we not only have to define \( \pi_0(u^i_4, u^i_1) \) but also \( \pi_0(b^i_4, b^i_1) \), \( \pi_0(u^i_4, b^i_1) \) and \( \pi_0(b^i_4, u^i_1) \). Firstly,

\[
\begin{align*}
\pi_0(u^i_4, u^i_1) &= -\frac{1}{2} \sum_{i_1, j_1=1}^3 \pi_0(\mathcal{P}^{i_{11}} \pi_<(u^i_3 + u^i_4, \partial_{x_{j_1}} K^{j_1}_u), u^i_1) \\
&\quad - \frac{1}{2} \sum_{i_1, j_1=1}^3 \pi_0(\mathcal{P}^{i_{11}} \pi_<(u^i_4 + u^i_3, \partial_{x_{j_1}} K^{j_1}_u), u^i_1) \\
&\quad + \frac{1}{2} \sum_{i_1, j_1=1}^3 \pi_0(\mathcal{P}^{i_{11}} \pi_<(b^i_3 + b^i_4, \partial_{x_{j_1}} K^{j_1}_b), u^i_1) \\
&\quad + \frac{1}{2} \sum_{i_1, j_1=1}^3 \pi_0(\mathcal{P}^{i_{11}} \pi_<(b^i_4 + b^i_3, \partial_{x_{j_1}} K^{j_1}_b), u^i_1) \\
&\quad - \frac{1}{2} \sum_{i_1, j_1=1}^3 \pi_0(\mathcal{P}^{i_{11}} \pi_<(\partial_{x_{j_1}} (u^i_3 + u^i_4), K^{j_1}_u), u^i_1) \\
&\quad - \frac{1}{2} \sum_{i_1, j_1=1}^3 \pi_0(\mathcal{P}^{i_{11}} \pi_<(\partial_{x_{j_1}} (u^i_4 + u^i_3), K^{j_1}_u), u^i_1) \\
&\quad + \frac{1}{2} \sum_{i_1, j_1=1}^3 \pi_0(\mathcal{P}^{i_{11}} \pi_<(\partial_{x_{j_1}} (b^i_3 + b^i_4), K^{j_1}_b), u^i_1) \\
&\quad + \frac{1}{2} \sum_{i_1, j_1=1}^3 \pi_0(\mathcal{P}^{i_{11}} \pi_<(\partial_{x_{j_1}} (b^i_4 + b^i_3), K^{j_1}_b), u^i_1) \\
&\quad + \pi_0(u^i_4, u^i_1)
\end{align*}
\]
by (29) and Leibniz rule. We can define \( \pi \) only consider the first four terms in (52) and Leibniz rule. Similarly, for the second term in (52) we write

\[
\pi_0(b_4^i, b_1^j) = \frac{1}{2} \sum_{i_1, j_1=1}^3 \pi_0(\mathcal{P}^{i_1} \pi_{<} (u_3^{i_1} + u_4^{i_1}, \partial_x^{j_1} K_b), b_1^j)
\]

\[
- \frac{1}{2} \sum_{i_1, j_1=1}^3 \pi_0(\mathcal{P}^{i_1} \pi_{<} (u_3^{i_1} + u_4^{i_1}, \partial_x^{j_1} K_b^{j_1}), b_1^j)
\]

\[
- \frac{1}{2} \sum_{i_1, j_1=1}^3 \pi_0(\mathcal{P}^{i_1} \pi_{<} (b_3^{i_1} + b_4^{i_1}, \partial_x^{j_1} K_u^{j_1}), b_1^j)
\]

\[
+ \frac{1}{2} \sum_{i_1, j_1=1}^3 \pi_0(\mathcal{P}^{i_1} \pi_{<} (b_3^{i_1} + b_4^{i_1}, \partial_x^{j_1} K_u^{j_1}), b_1^j)
\]

(53)

by (52) and Leibniz rule. We can define \( \pi_0(u_4^i, b_1^j) \) and \( \pi_0(b_4^i, u_1^j) \) similarly. We only consider the first four terms in \( \pi_0(u_4^i, u_1^j) \) of (52) and \( \pi_0(b_4^i, b_1^j) \) of (53) as other terms are similar. For the first term in \( \pi_0(u_4^i, u_1^j) \) of (52) we write

\[
\pi_0(\mathcal{P}^{i_1} \pi_{<} (u_3^{i_1} + u_4^{i_1}, \partial_x^{j_1} K_u), u_1^j)
\]

\[
= \pi_0(\mathcal{P}^{i_1} \pi_{<} (u_3^{i_1} + u_4^{i_1}, \partial_x^{j_1} K_u^{j_1}), u_1^j) - \pi_0(\pi_{<} (u_3^{i_1} + u_4^{i_1}, \mathcal{P}^{i_1} \partial_x^{j_1} K_u^{j_1}), u_1^j)
\]

\[
+ \pi_0(\pi_{<} (u_3^{i_1} + u_4^{i_1}, \mathcal{P}^{i_1} \partial_x^{j_1} K_u^{j_1}), u_1^j) - (u_3^{i_1} + u_4^{i_1})\pi_0(\mathcal{P}^{i_1} \partial_x^{j_1} K_u^{j_1}, u_1^j)
\]

\[
+ (u_3^{i_1} + u_4^{i_1})\pi_0(\mathcal{P}^{i_1} \partial_x^{j_1} K_u^{j_1}, u_1^j),
\]

(54)

for the second term in \( \pi_0(u_4^i, u_1^j) \) of (52) we write

\[
\pi_0(\mathcal{P}^{i_1} \pi_{<} (u_3^{i_1} + u_4^{i_1}, \partial_x^{j_1} K_u), u_1^j)
\]

\[
= \pi_0(\mathcal{P}^{i_1} \pi_{<} (u_3^{i_1} + u_4^{i_1}, \partial_x^{j_1} K_u^{j_1}), u_1^j) - \pi_0(\pi_{<} (u_3^{i_1} + u_4^{i_1}, \mathcal{P}^{i_1} \partial_x^{j_1} K_u^{j_1}), u_1^j)
\]

\[
+ \pi_0(\pi_{<} (u_3^{i_1} + u_4^{i_1}, \mathcal{P}^{i_1} \partial_x^{j_1} K_u^{j_1}), u_1^j) - (u_3^{i_1} + u_4^{i_1})\pi_0(\mathcal{P}^{i_1} \partial_x^{j_1} K_u^{j_1}, u_1^j)
\]

\[
+ (u_3^{i_1} + u_4^{i_1})\pi_0(\mathcal{P}^{i_1} \partial_x^{j_1} K_u^{j_1}, u_1^j),
\]

(55)
for the third term in $\pi_0(u_1^j, u_1^1)$ of (52) we write

$$\pi_0(\calP^{i_1} \pi_<(b_3^{i_1} + b_4^{i_1}, \partial_{x_1} K_b^{j_1}), u_1^1)$$

\begin{align*}
&= \pi_0(\calP^{i_1} \pi_<(b_3^{i_1} + b_4^{i_1}, \partial_{x_1} K_b^{j_1}), u_1^1) - \pi_0(\pi_<(b_3^{i_1} + b_4^{i_1}, \calP^{i_1} \partial_{x_1} K_b^{j_1}), u_1^1) \\
&\quad + \pi_0(\pi_<(b_3^{i_1} + b_4^{i_1}, \calP^{i_1} \partial_{x_1} K_b^{j_1}), u_1^1) - (b_3^{i_1} + b_4^{i_1}) \pi_0(\calP^{i_1} \partial_{x_1} K_b^{j_1}, u_1^1) \\
&\quad + (b_3^{i_1} + b_4^{i_1}) \pi_0(\calP^{i_1} \partial_{x_1} K_b^{j_1}, u_1^1),
\end{align*}

(56)

and for the fourth term in $\pi_0(u_1^j, u_1^1)$ of (52) we write

$$\pi_0(\calP^{i_1} \pi_<(b_3^{i_1} + b_4^{i_1}, \partial_{x_1} K_b^{j_1}), u_1^1)$$

\begin{align*}
&= \pi_0(\calP^{i_1} \pi_<(b_3^{i_1} + b_4^{i_1}, \partial_{x_1} K_b^{j_1}), u_1^1) - \pi_0(\pi_<(b_3^{i_1} + b_4^{i_1}, \calP^{i_1} \partial_{x_1} K_b^{j_1}), u_1^1) \\
&\quad + \pi_0(\pi_<(b_3^{i_1} + b_4^{i_1}, \calP^{i_1} \partial_{x_1} K_b^{j_1}), u_1^1) - (b_3^{i_1} + b_4^{i_1}) \pi_0(\calP^{i_1} \partial_{x_1} K_b^{j_1}, u_1^1) \\
&\quad + (b_3^{i_1} + b_4^{i_1}) \pi_0(\calP^{i_1} \partial_{x_1} K_b^{j_1}, u_1^1).
\end{align*}

(57)

Similarly we can write the first four terms of $\pi_0(b_3^{i_1}, b_4^{i_1})$. For the convergence of $\pi_0(\calP^{i_1} \partial_{x_1} K_b^{j_1}, u_1^1)$, $\pi_0(\calP^{i_1} \partial_{x_1} K_b^{j_1}, u_1^1)$, $\pi_0(\calP^{i_1} \partial_{x_1} K_b^{j_1}, u_1^1)$, $\pi_0(\calP^{i_1} \partial_{x_1} K_b^{j_1}, u_1^1)$ as $\epsilon \to 0$, we need to do renormalization. We now estimate

$$\| \pi_{0, \epsilon}(\calP^{i_1} \pi_<(u_3^{i_1} + u_4^{i_1}, \partial_{x_1} K_b^{j_1}), u_1^1) \|_{C^{-\delta}}$$

\begin{align*}
&\lesssim \| \pi_0(\calP^{i_1} \pi_<(u_3^{i_1} + u_4^{i_1}, \partial_{x_1} K_b^{j_1}), u_1^1) - \pi_0(\pi_<(u_3^{i_1} + u_4^{i_1}, \calP^{i_1} \partial_{x_1} K_b^{j_1}), u_1^1) \|_{C^{-\delta}} \\
&\quad + \| \pi_0(\pi_<(u_3^{i_1} + u_4^{i_1}, \calP^{i_1} \partial_{x_1} K_b^{j_1}), u_1^1) - (u_3^{i_1} + u_4^{i_1}) \pi_0(\calP^{i_1} \partial_{x_1} K_b^{j_1}, u_1^1) \|_{C^{-\delta}} \\
&\quad + \| (u_3^{i_1} + u_4^{i_1}) \pi_0(\calP^{i_1} \partial_{x_1} K_b^{j_1}, u_1^1) \|_{C^{-\delta}}
\end{align*}

(58)

by (28b). For

$$\delta \leq \delta_0 < \frac{1}{2} - \frac{3\delta}{2},$$

(59)

we may firstly estimate

$$\| \pi_0(\calP^{i_1} \pi_<(u_3^{i_1} + u_4^{i_1}, \partial_{x_1} K_b^{j_1}), u_1^1) - \pi_0(\pi_<(u_3^{i_1} + u_4^{i_1}, \calP^{i_1} \partial_{x_1} K_b^{j_1}), u_1^1) \|_{C^{-\delta}}$$

\begin{align*}
&\lesssim \| \pi^{i_1} \pi_<(u_3^{i_1} + u_4^{i_1}, \partial_{x_1} K_b^{j_1}) - \pi_<(u_3^{i_1} + u_4^{i_1}, \calP^{i_1} \partial_{x_1} K_b^{j_1}) \|_{C^{1-\delta}} \| u_1^1 \|_{C^{-\frac{3\delta}{2} - \delta_0}} \\
&\lesssim \| u_3^{i_1} + u_4^{i_1} \|_{C^{\frac{1}{2} - \delta_0}} \| K^{j_1}_b \|_{C^{\frac{3\delta}{2} - \delta_0}} \| u_1^1 \|_{C^{-\frac{3\delta}{2} - \delta_0}}
\end{align*}

(60)

by linearity of $\pi_0(\cdot, \cdot)$, that $-\delta \leq \frac{1}{2} - \frac{3\delta}{2} - \delta_0$ due to (59), (2), Lemma 1.1 (3) and Lemma 4.3. Secondly we may estimate

$$\| \pi_0(\pi_<(u_3^{i_1} + u_4^{i_1}, \calP^{i_1} \partial_{x_1} K_b^{j_1}), u_1^1) - (u_3^{i_1} + u_4^{i_1}) \pi_0(\calP^{i_1} \partial_{x_1} K_b^{j_1}, u_1^1) \|_{C^{-\delta}}$$

\begin{align*}
&\lesssim \| u_3^{i_1} + u_4^{i_1} \|_{C^{\frac{1}{2} - \delta_0}} \| \calP^{i_1} \partial_{x_1} K_b^{j_1} \|_{C^{\frac{3\delta}{2} - \delta_0}} \| u_1^1 \|_{C^{-\frac{3\delta}{2} - \delta_0}} \\
&\lesssim \| u_3^{i_1} + u_4^{i_1} \|_{C^{\frac{1}{2} - \delta_0}} \| K_b^{j_1} \|_{C^{\frac{3\delta}{2} - \delta_0}} \| u_1^1 \|_{C^{-\frac{3\delta}{2} - \delta_0}}
\end{align*}

(61)

where we used that $-\delta \leq \frac{1}{2} - \frac{3\delta}{2} - \delta_0$ due to (59), Lemmas 4.2 and 4.3.

**Remark 3.3.** Let us emphasize strongly that this estimate (61) seems very difficult, if not impossible, without relying on the commutator estimate Lemma 4.2, e.g. by utilizing only Lemma 4.4.
Similarly we can deduce
\[
\|(u_3^{i_3} + u_4^{i_4})\pi_{0,\circ}(P^{i_1 i_2 i_3} \partial_{x^{i_1 i_2}} K_u^{j_1}, u_1^i)\|_{c^{-\delta}} \leq \|u_3^{i_3} + u_4^{i_4}\|_{c^{\frac{1}{2} - \delta}} \|\pi_{0,\circ}(P^{i_1 i_2 i_3} \partial_{x^{i_1 i_2}} K_u^{j_1}, u_1^i)\|_{c^{-\delta}}
\] (62)
by Lemma 111 (4), 125 and 126. Applying (60)-(62) to (138) implies
\[
\|\pi_{0,\circ}(P^{i_1 i_2 i_3} \pi_{<}(u_3^{i_3} + u_4^{i_4}, \partial_{x^{i_1 i_2}} K_u^{j_1}), u_1^i)\|_{c^{-\delta}} \leq \|u_3^{i_3} + u_4^{i_4}\|_{c^{\frac{1}{2} - \delta}} \|K_u^{j_1}\|_{c^{\frac{1}{2} - \delta}} \|u_1^i\|_{c^{-\frac{1}{2} - \frac{\delta}{2}}} \|\pi_{0,\circ}(P^{i_1 i_2 i_3} \partial_{x^{i_1 i_2}} K_u^{j_1}, u_1^i)\|_{c^{-\delta},\theta}.
\] (63)
Similarly we can deduce
\[
\|\pi_{0,\circ}(P^{i_1 i_2 i_3} \pi_{<}(b_3^{i_3} + b_4^{i_4}, \partial_{x^{i_1 i_2}} K_b^{j_1}), u_1^i)\|_{c^{-\delta}} \leq \|b_3^{i_3} + b_4^{i_4}\|_{c^{\frac{1}{2} - \delta}} \|K_b^{j_1}\|_{c^{\frac{1}{2} - \delta}} \|u_1^i\|_{c^{-\frac{1}{2} - \frac{\delta}{2}}} \|\pi_{0,\circ}(P^{i_1 i_2 i_3} \partial_{x^{i_1 i_2}} K_b^{j_1}, u_1^i)\|_{c^{-\delta},\theta}
\] (64)
as well as
\[
\|\pi_{0,\circ}(P^{i_1 i_2 i_3} \pi_{<}(u_3^{i_3} + u_4^{i_4}, \partial_{x^{i_1 i_2}} K_b^{j_1}), b_1^i)\|_{c^{-\delta}} \leq \|u_3^{i_3} + u_4^{i_4}\|_{c^{\frac{1}{2} - \delta}} \|K_b^{j_1}\|_{c^{\frac{1}{2} - \delta}} \|b_1^i\|_{c^{-\frac{1}{2} - \frac{\delta}{2}}} \|\pi_{0,\circ}(P^{i_1 i_2 i_3} \partial_{x^{i_1 i_2}} K_b^{j_1}, b_1^i)\|_{c^{-\delta},\theta}
\] (65a)
\[
\|\pi_{0,\circ}(P^{i_1 i_2 i_3} \pi_{<}(b_3^{i_3} + b_4^{i_4}, \partial_{x^{i_1 i_2}} K_u^{j_1}), b_1^i)\|_{c^{-\delta}} \leq \|b_3^{i_3} + b_4^{i_4}\|_{c^{\frac{1}{2} - \delta}} \|K_u^{j_1}\|_{c^{\frac{1}{2} - \delta}} \|b_1^i\|_{c^{-\frac{1}{2} - \frac{\delta}{2}}} \|\pi_{0,\circ}(P^{i_1 i_2 i_3} \partial_{x^{i_1 i_2}} K_u^{j_1}, b_1^i)\|_{c^{-\delta},\theta}
\] (65b)
This leads to
\[
\|\pi_{0,\circ}(u_1^i, u_1^i)\|_{c^{-\delta}} \leq \sum_{i_1, j_1 = 1}^{3} \left( \|u_3^{i_3} + u_4^{i_4}\|_{c^{\frac{1}{2} - \delta}} \|K_u^{j_1}\|_{c^{\frac{1}{2} - \delta}} \|u_1^i\|_{c^{-\frac{1}{2} - \frac{\delta}{2}}} \right)
+ \|u_3^{i_3} + u_4^{i_4}\|_{c^{\frac{1}{2} - \delta}} \|\pi_{0,\circ}(P^{i_1 i_2 i_3} \partial_{x^{i_1 i_2}} K_u^{j_1}, u_1^i)\|_{c^{-\delta}}
+ \|u_3^{i_3} + u_4^{i_4}\|_{c^{\frac{1}{2} - \delta}} \|\pi_{0,\circ}(P^{i_1 i_2 i_3} \partial_{x^{i_1 i_2}} K_u^{j_1}, u_1^i)\|_{c^{-\delta},\theta}
+ \|b_3^{i_3} + b_4^{i_4}\|_{c^{\frac{1}{2} - \delta}} \|K_b^{j_1}\|_{c^{\frac{1}{2} - \delta}} \|u_1^i\|_{c^{-\frac{1}{2} - \frac{\delta}{2}}} \|\pi_{0,\circ}(P^{i_1 i_2 i_3} \partial_{x^{i_1 i_2}} K_u^{j_1}, u_1^i)\|_{c^{-\delta}}
+ \|b_3^{i_3} + b_4^{i_4}\|_{c^{\frac{1}{2} - \delta}} \|\pi_{0,\circ}(P^{i_1 i_2 i_3} \partial_{x^{i_1 i_2}} K_b^{j_1}, u_1^i)\|_{c^{-\delta},\theta}
+ \sum_{i_1, j_1 = 1}^{3} \|\pi_{0}(P^{i_1 i_2 i_3} \pi_{<}(\partial_{x^{i_1 i_2}} (u_3^{i_3} + u_4^{i_4}), K_u^{j_1}), u_1^i)\|_{c^{-\delta}}
+ \|\pi_{0}(P^{i_1 i_2 i_3} \pi_{<}(\partial_{x^{i_1 i_2}} (u_3^{i_3} + u_4^{i_4}), K_b^{j_1}), u_1^i)\|_{c^{-\delta}}
+ \|\pi_{0}(P^{i_1 i_2 i_3} \pi_{<}(\partial_{x^{i_1 i_2}} (b_3^{i_3} + b_4^{i_4}), K_u^{j_1}), u_1^i)\|_{c^{-\delta}}
+ \pi_{0}(u_3^{i_3}, u_1^i)\|_{c^{-\delta}}
by (27), (33) and (64). We may further estimate firstly within (66),

$$\|\pi(\mathcal{P}^{\alpha} \pi_1 (u_{3j} + u_{4j}), K_{2j}^{\frac{1}{2}}, u_{4j})\|_{C^{-\frac{1}{2}}}$$

$$+ \|\pi(\mathcal{P}^{\alpha} \pi_1 (u_{3j} + u_{4j}), K_{2j}^{\frac{1}{2}}, u_{4j})\|_{C^{-\frac{1}{2}}}$$

$$+ \|\pi(\mathcal{P}^{\alpha} \pi_1 (b_{3j} + b_{4j}), K_{2j}^{\frac{1}{2}}, u_{4j})\|_{C^{-\frac{1}{2}}}$$

$$+ \|\pi(\mathcal{P}^{\alpha} \pi_1 (b_{3j} + b_{4j}), K_{2j}^{\frac{1}{2}}, u_{4j})\|_{C^{-\frac{1}{2}}}$$

$$\lesssim (\|\mathcal{P}^{\alpha} \pi_1 (u_{3j} + u_{4j}), K_{2j}^{\frac{1}{2}}\|_{C^{1-\delta}}$$

$$+ \|\mathcal{P}^{\alpha} \pi_1 (u_{3j} + u_{4j}), K_{2j}^{\frac{1}{2}}\|_{C^{1-\delta}}$$

$$+ \|\mathcal{P}^{\alpha} \pi_1 (b_{3j} + b_{4j}), K_{2j}^{\frac{1}{2}}\|_{C^{1-\delta}}$$

$$+ \|\mathcal{P}^{\alpha} \pi_1 (b_{3j} + b_{4j}), K_{2j}^{\frac{1}{2}}\|_{C^{1-\delta}}$$

$$\lesssim (\|\mathcal{P}^{\alpha} \pi_1 (u_{3j} + u_{4j})\|_{C^{1-\delta}}$$

$$+ \|\mathcal{P}^{\alpha} \pi_1 (u_{3j} + u_{4j})\|_{C^{1-\delta}}$$

$$+ \|\mathcal{P}^{\alpha} \pi_1 (b_{3j} + b_{4j})\|_{C^{1-\delta}}$$

$$+ \|\mathcal{P}^{\alpha} \pi_1 (b_{3j} + b_{4j})\|_{C^{1-\delta}}$$

$$\lesssim 2 C_{\frac{1}{2}} + (\|u_4\|_{C^{1-\delta}} + \|b_4\|_{C^{1-\delta}}) C_{\xi}$$

by Lemma 1.1 (3) as $\frac{1}{2} - \delta_0 - \frac{3}{2} \delta > 0$ due to (69), Lemma 4.9, Lemma 1.1 (2), (35), (2), (35), (37) and (39). Secondly, within (66) we may estimate

$$\|\pi_0 (u_{3j}, u_{4j})\|_{C^{-\frac{1}{2}}} \lesssim \|u_{3j}\|_{C^{1-\delta}} + \|u_{4j}\|_{C^{-\frac{1}{2}}} \lesssim \|u_{3j}\|_{C^{1-\delta}} C_{\xi}$$

as $\beta > \frac{1}{2}$ due to (43), Lemma 1.1 (3) and (37). Thirdly, within (66) we may estimate

$$\|u_{3j} + u_{4j}\|_{C^{0-\delta}} \|K_{2j}^{\frac{1}{2}}\|_{C^{0-\delta}} \|u_{4j}\|_{C^{-\frac{1}{2}}} \lesssim \|u_{3j}\|_{C^{0-\delta}} C_{\xi} + \|u_4, b_4\|_{C^{0-\delta}} C_{\xi}$$

by (39), (37), (37) and (39). Fourthly, within (66) we estimate

$$\|u_{3j} + u_{4j}\|_{C^{0-\delta}} \|\pi_0 (\mathcal{P}^{\alpha} \pi_1 (u_{3j}, u_{4j})\|_{C^{-\frac{1}{2}}}$$

$$+ \|u_{3j} + u_{4j}\|_{C^{0-\delta}} \|\pi_0 (\mathcal{P}^{\alpha} \pi_1 (u_{3j}, u_{4j})\|_{C^{-\frac{1}{2}}}$$

$$+ \|b_{3j} + b_{4j}\|_{C^{0-\delta}} \|\pi_0 (\mathcal{P}^{\alpha} \pi_1 (b_{3j}, b_{4j})\|_{C^{-\frac{1}{2}}}$$

$$+ \|b_{3j} + b_{4j}\|_{C^{0-\delta}} \|\pi_0 (\mathcal{P}^{\alpha} \pi_1 (b_{3j}, b_{4j})\|_{C^{-\frac{1}{2}}}$$

$$\lesssim 3 \sum_{j=1}^3 (\|u_{3j}\|_{C^{0-\delta}} + \|u_{4j}\|_{C^{0-\delta}} + \|b_{3j}\|_{C^{0-\delta}} + \|b_{4j}\|_{C^{0-\delta}}) C_{\xi}$$

$$\lesssim 2 C_{\frac{1}{2}} + (\|u_4\|_{C^{0-\delta}} + \|b_4\|_{C^{0-\delta}}) (C_{\xi} + 1)$$

by (39), (37) and (39). Therefore, by applying (67)-(71) in (66) we obtain

$$\|\pi_0 (u_{3j}, u_{4j})\|_{C^{-\frac{1}{2}}} \lesssim C_{\xi} + (\|u_4\|_{C^{0-\delta}} + \|b_4\|_{C^{0-\delta}}) (C_{\xi} + 1) + \|u_4\|_{C^{0-\delta}} C_{\xi} + 1.$$
Similarly,
\[
\|\pi_0,\beta(b^i_4, b^j_4)\|_{c^{-s}} \\
\lesssim \sum_{i_1, j_1=1}^3 (\|u_3^{i_1} + u_4^{i_1}\|_{c^{\frac{1}{2}} - s_0} \|K_j^{i_1}\|_{c^{\frac{1}{2}} - s} \|b_j^i\|_{c^{-\frac{1}{2}} - \frac{3}{2}} \\
+ \|u_3^{i_1} + u_4^{i_1}\|_{c^{\frac{1}{2}} - s_0} \|\pi_0,\beta(P^{i_1, i_2} \partial_{x^{j_1}}, K_j^{i_1})\|_{c^{-s}} \\
+ \|b_3^{i_1} + b_4^{i_1}\|_{c^{\frac{1}{2}} - s_0} \|K_j^{i_1}\|_{c^{\frac{1}{2}} - s} \|b_j^i\|_{c^{-\frac{1}{2}} - \frac{3}{2}} \\
+ \|b_3^{i_1} + b_4^{i_1}\|_{c^{\frac{1}{2}} - s_0} \|\pi_0,\beta(P^{i_1, i_2} \partial_{x^{j_1}}, K_j^{i_1})\|_{c^{-s}} \\
+ \|b_3^{i_1} + b_4^{i_1}\|_{c^{\frac{1}{2}} - s_0} \|\pi_0,\beta(P^{i_1, i_2} \partial_{x^{j_1}}, K_j^{i_1})\|_{c^{-s}} + \|\pi_0(b_3^{i_1}, b_4^{i_1})\|_{c^{-s}}) \\
\]  
(72)
\]
by (31), (35a) and (35b), where tracing previous inequalities (67)-(70), we see that
\[
\|\pi_0(P^{i_1, i_2} \xi_{x^{i_1}}(u_3^i + u_4^i), K_j^{i_1} b_j^i)\|_{c^{-s}} \\
+ \|\pi_0(P^{i_1, i_2} \xi_{x^{i_1}}(u_3^i + u_4^i), K_j^{i_1} b_j^i)\|_{c^{-s}} \\
+ \|\pi_0(P^{i_1, i_2} \xi_{x^{i_1}}(b_3^i + b_4^i), K_j^{i_1} b_j^i)\|_{c^{-s}} \\
+ \|\pi_0(P^{i_1, i_2} \xi_{x^{i_1}}(b_3^i + b_4^i), K_j^{i_1} b_j^i)\|_{c^{-s}} \lesssim C_\xi^3 + \|u_4, b_4\|_{c^{\frac{1}{2}} - s_0} C_\xi^2, \\
\|\pi_0(b^{i_1, i_2}, b_j^i)\|_{c^{-s}} \lesssim \|b^{i_1, i_2}\|_{c^\frac{1}{2} + s} C_\xi, \\
\|b_3^i + b_4^i\|_{c^{\frac{1}{2}} - s_0} \|K_j^{i_1}\|_{c^{\frac{1}{2}} - s} \|b_j^i\|_{c^{-\frac{1}{2}} - \frac{3}{2}} \lesssim C_\xi^3 + \|u_4, b_4\|_{c^{\frac{1}{2}} - s_0} C_\xi^2, \\
\]  
(74)
\[
\|u_3^i + u_4^i\|_{c^{\frac{1}{2}} - s_0} \|K_j^{i_1}\|_{c^{\frac{1}{2}} - s} \|b_j^i\|_{c^{-\frac{1}{2}} - \frac{3}{2}} \\
+ \|b_3^i + b_4^i\|_{c^{\frac{1}{2}} - s_0} \|K_j^{i_1}\|_{c^{\frac{1}{2}} - s} \|b_j^i\|_{c^{-\frac{1}{2}} - \frac{3}{2}} \lesssim C_\xi^3 + \|u_4, b_4\|_{c^{\frac{1}{2}} - s_0} C_\xi^2, \\
\]  
(75)
and
\[
\|u_3^i + u_4^i\|_{c^{\frac{1}{2}} - s_0} \|\pi_0,\beta(P^{i_1, i_2} \xi_{x^{i_1}}(K_j^{i_1} b_j^i))\|_{c^{-s}} \\
+ \|u_3^i + u_4^i\|_{c^{\frac{1}{2}} - s_0} \|\pi_0,\beta(P^{i_1, i_2} \xi_{x^{i_1}}(K_j^{i_1} b_j^i))\|_{c^{-s}} \\
+ \|b_3^i + b_4^i\|_{c^{\frac{1}{2}} - s_0} \|\pi_0,\beta(P^{i_1, i_2} \xi_{x^{i_1}}(K_j^{i_1} b_j^i))\|_{c^{-s}} \\
+ \|b_3^i + b_4^i\|_{c^{\frac{1}{2}} - s_0} \|\pi_0,\beta(P^{i_1, i_2} \xi_{x^{i_1}}(K_j^{i_1} b_j^i))\|_{c^{-s}} \\
\lesssim C_\xi^3 + 1 + \|u_4, b_4\|_{c^{\frac{1}{2}} - s_0} (C_\xi^2 + 1). \\
\]  
(76)
Thus, by applying (72)-(75) to (72) we obtain
\[
\|\pi_0,\beta(b_4^i, b_4^j)\|_{c^{-s}} \lesssim C_\xi^3 + \|u_4, b_4\|_{c^{\frac{1}{2}} - s_0} (C_\xi^2 + 1) + \|b^i\|_{c^{\frac{1}{2} + s}} C_\xi + 1 \\
\]  
(77)
and similar estimates for \(\|\pi_0,\beta(u_4^i, b_4^j)\|_{c^{-s}}\) and \(\|\pi_0,\beta(b_4^i, u_4^j)\|_{c^{-s}}\) follow.
Firstly, within (78)-(79) we may estimate

\[ \| L(u_3^j + u_4^j) \|_{C^{-\frac{3}{2}-\frac{1}{4}}} \]

\[ = \left\| -\frac{1}{2} \sum_{i,j=1}^{3} P^{i \overline{i}} \partial_{x^i} [u_3^j \circ u_2^i + u_4^j \circ u_2^i - b_2^i \circ b_4^j - b_2^j \circ b_4^i] \right\|_{C^{-\frac{3}{2}-\frac{1}{4}}} \]

\[ + \pi < (u_3^j + u_4^j, u_3^i) + \pi_0, 0 (u_3^j, u_3^i) + \pi > (u_3^j + u_4^j, u_3^i) + \pi_0, 0 (u_3^j, u_3^i) \]

\[ + \pi < (u_3^j + u_4^j, u_3^i) + \pi_0, 0 (u_3^j, u_3^i) + \pi > (u_3^j + u_4^j, u_3^i) + \pi_0, 0 (u_3^j, u_3^i) \]

\[ + u_3^j \circ u_2^j + u_4^j (u_3^j + u_2^j) + u_2^i (u_3^j + u_2^j) + (u_3^j + u_4^j)(u_3^j + u_4^j) \]

\[ - \pi < (b_3^j + b_4^j, b_3^i) - \pi_0, 0 (b_3^j, b_3^i) - \pi > (b_3^j + b_4^j, b_3^i) - \pi_0, 0 (b_3^j, b_3^i) \]

\[ - b_3^j \circ b_2^j - b_3^j (b_3^j + b_4^j) - b_4^j (b_3^j + b_4^j) - (b_3^j + b_4^j)(b_3^j + b_4^j) \right\|_{C^{-\frac{3}{2}-\frac{1}{4}}} \]

and

\[ \| L(b_3^j + b_4^j) \|_{C^{-\frac{3}{2}-\frac{1}{4}}} \]

\[ = \left\| -\frac{1}{2} \sum_{i,j=1}^{3} P^{i \overline{i}} \partial_{x^i} [b_3^j \circ u_2^i + b_4^j \circ u_2^i - u_2^i \circ b_4^j - u_2^j \circ b_4^i] \right\|_{C^{-\frac{3}{2}-\frac{1}{4}}} \]

\[ + \pi < (b_3^j + b_4^j, b_3^i) + \pi_0, 0 (b_3^j, b_3^i) + \pi > (b_3^j + b_4^j, b_3^i) + \pi_0, 0 (b_3^j, b_3^i) \]

\[ + \pi < (b_3^j + b_4^j, b_3^i) + \pi_0, 0 (b_3^j, b_3^i) + \pi > (b_3^j + b_4^j, b_3^i) + \pi_0, 0 (b_3^j, b_3^i) \]

\[ + b_3^j \circ b_2^j + b_4^j (u_3^j + u_2^j) + u_2^i (u_3^j + u_2^j) + (b_3^j + b_4^j)(u_3^j + u_4^j) \]

\[ - \pi < (b_3^j + b_4^j, u_3^i) - \pi_0, 0 (b_3^j, u_3^i) - \pi > (b_3^j + b_4^j, u_3^i) - \pi_0, 0 (b_3^j, u_3^i) \]

\[ - u_3^j \circ u_2^j - u_3^j (b_3^j + b_4^j) - b_3^j (u_3^j + u_4^j) - (u_3^j + u_4^j)(b_3^j + b_4^j) \right\|_{C^{-\frac{3}{2}-\frac{1}{4}}} \]

Firstly, within (78)-(79) we may estimate

\[ \| P^{i \overline{i}} \partial_{x^i} (u_1^j \circ u_2^i + u_2^j \circ u_1^i, b_1^j \circ b_2^i, b_2^j \circ b_1^i) \|_{C^{-\frac{5}{2}-\frac{1}{2}}} \leq C \xi \]
by Lemma 4.5 and (37). Secondly, within (78)-(79) we may estimate

\[
\|\mathcal{P}^{\text{ini}} \partial_x [\pi_+ (u_3' + u_4', u_1') + \pi_- (u_3' + u_4', u_1')] \\
+ \pi_+ (u_3' + u_4', u_1') + \pi_- (u_3' + u_4', u_1') \\
- \pi_- (b_3' + b_4, b_1') - \pi_+ (b_3' + b_4, b_1') \\
- \pi_- (b_3' + b_4, b_1') - \pi_+ (b_3' + b_4, b_1') \|_{C^{-\frac{3}{2} - \frac{\delta}{4}}}
\]

(81)

due to Lemma 4.5 that \(-\frac{1}{2} - \frac{\delta}{4} \leq -\frac{\delta}{4} - \delta_0\), Lemma 4.1 (1), Lemma 4.1 (2), (37), (38) and (39). Thirdly, within (78)-(79) we may estimate

\[
\|\mathcal{P}^{\text{ini}} \partial_x [\pi_0,0 (u_3', u_1') + \pi_0,0 (u_3', u_1') + \pi_0,0 (u_3', u_1')] \\
- \pi_0,0 (b_3', b_1') - \pi_0,0 (b_3', b_1') - \pi_0,0 (b_3', b_1') - \pi_0,0 (b_3', b_1') \|_{C^{-\frac{3}{2} - \frac{\delta}{4}}}
\]

(82)

by Lemma 4.5 that \(-\frac{1}{2} - \frac{\delta}{4} \leq -\delta\), (38), (37), (39) and (37). Fourthly, within (78)-(79) we may estimate

\[
\|\mathcal{P}^{\text{ini}} \partial_x (u_2' \circ u_2', b_1' \circ b_2', b_1' \circ b_2' \circ u_2', u_2' \circ b_2') \|_{C^{-\frac{3}{2} - \frac{\delta}{4}}}
\]

(83)

\[
\lesssim \|(u_2' \circ u_2', b_1' \circ b_2', b_1' \circ b_2' \circ u_2', u_2' \circ b_2') \|_{C^{-\delta}} \lesssim C_\xi
\]
by Lemma 4.5 that \(-\frac{1}{2} - \frac{\delta}{2} \leq -\delta\), (39), and (37). Fifthly, within (78)-(79) we may estimate

\[
\|P^{111}\partial_x(u_{11}^{11} + u_{11}^{4}) + \sum_{i,j=1}^\infty \|C\|_\delta \int_{-1}^{1} (a_{i,j}^{11} + a_{i,j}^{4}) (u_{11}^{11} + u_{11}^{4}) (u_{11}^{11} + u_{11}^{4}),
\]

\[
b_{11}^{11} (b_{11}^{11} + b_{11}^{4}), b_{12} (b_{12}^{11} + b_{12}^{4}), (b_{13}^{11} + b_{13}^{4}) (b_{13}^{11} + b_{13}^{4}),
\]

\[
b_{12} (u_{12}^{11} + u_{12}^{4}), (b_{13}^{11} + b_{13}^{4}) (b_{13}^{11} + b_{13}^{4}),
\]

\[
u_{12}^{11} (b_{12}^{11} + b_{12}^{4}), b_{13} (b_{13}^{11} + b_{13}^{4}), (b_{14}^{11} + b_{14}^{4}) (b_{14}^{11} + b_{14}^{4}),
\]

\[
\left\langle \int (u_{11}^{11} + u_{11}^{4}, (b_{11}^{11} + b_{11}^{4}), (b_{12}^{11} + b_{12}^{4}) (b_{12}^{11} + b_{12}^{4}), (b_{13}^{11} + b_{13}^{4}) (b_{13}^{11} + b_{13}^{4}), (b_{14}^{11} + b_{14}^{4}) (b_{14}^{11} + b_{14}^{4}) \right\rangle_{C^{-\frac{1}{2} - \frac{\delta}{4}}}
\]

\[
\lesssim \| (u_{11}^{11} + u_{11}^{4}, u_{12}^{11} + u_{12}^{4}) \|_{C^{-\frac{1}{2} - \frac{\delta}{4}}} + \| \sum_{i,j=1}^\infty \|C\|_\delta \int_{-1}^{1} (a_{i,j}^{11} + a_{i,j}^{4}) (u_{11}^{11} + u_{11}^{4}) (u_{11}^{11} + u_{11}^{4}),
\]

\[
\lesssim C_{11}^3 + 1 + (1 + C_{11}^3) \|(u_{11}, b_{11})\|_{C^{-\frac{1}{2} - \frac{\delta}{4}}} + C_{11} \|(u_{11}, b_{11})\|_{C^{-\frac{1}{2} - \frac{\delta}{4}}} + \|(u_{11}, b_{11})\|_{C^{-\frac{1}{2} - \frac{\delta}{4}}} \|
\]

Therefore,

\[
\| \pi_{c_1} (u_{11}^{11} + u_{11}^{4}), \pi_{c_1} (u_{12}^{11} + u_{12}^{4}), \pi_{c_1} (u_{13}^{11} + u_{13}^{4}), \pi_{c_1} (u_{14}^{11} + u_{14}^{4})\|_{C^{-\frac{1}{2} - \frac{\delta}{4}}}
\]

\[
\lesssim \| (u_{11}^{11} + u_{11}^{4}, u_{12}^{11} + u_{12}^{4}) \|_{C^{-\frac{1}{2} - \frac{\delta}{4}}} + \| \sum_{i,j=1}^\infty \|C\|_\delta \int_{-1}^{1} (a_{i,j}^{11} + a_{i,j}^{4}) (u_{11}^{11} + u_{11}^{4}) (u_{11}^{11} + u_{11}^{4}),
\]

\[
\lesssim C_{11}^3 + 1 + (1 + C_{11}^3) \|(u_{11}, b_{11})\|_{C^{-\frac{1}{2} - \frac{\delta}{4}}} + C_{11} \|(u_{11}, b_{11})\|_{C^{-\frac{1}{2} - \frac{\delta}{4}}} + \|(u_{11}, b_{11})\|_{C^{-\frac{1}{2} - \frac{\delta}{4}}} \|
\]

by Lemma 4.1 (2) and (85). Next, we estimate

\[
\| (\pi_{c_1} (\nabla (u_{11}^{11} + u_{11}^{4}), \nabla K_{11}^{11}), \pi_{c_1} (\nabla (u_{12}^{11} + u_{12}^{4}), \nabla K_{12}^{11}),
\]

\[
\pi_{c_1} (\nabla (b_{11}^{11} + b_{11}^{4}), \nabla K_{11}^{11}), \pi_{c_1} (\nabla (b_{12}^{11} + b_{12}^{4}), \nabla K_{12}^{11})\|_{C^{-\frac{1}{2} - \frac{\delta}{4}}}
\]

\[
+ \| \pi_{c_1} (u_{11}^{11} + u_{11}^{4}, u_{12}^{11} + u_{12}^{4}), \pi_{c_1} (u_{11}^{11} + u_{11}^{4}, u_{12}^{11} + u_{12}^{4}),
\]

\[
\pi_{c_1} (b_{11}^{11} + b_{11}^{4}, b_{11}^{4}), \pi_{c_1} (b_{12}^{11} + b_{12}^{4}, b_{12}^{4}), \pi_{c_1} (b_{13}^{11} + b_{13}^{4}, b_{13}^{4}),
\]

\[
+ \| \pi_{c_1} (u_{11}^{11} + u_{11}^{4}, u_{12}^{11} + u_{12}^{4}), \pi_{c_1} (u_{11}^{11} + u_{11}^{4}, u_{12}^{11} + u_{12}^{4}),
\]

\[
\pi_{c_1} (b_{11}^{11} + b_{11}^{4}, b_{11}^{4}), \pi_{c_1} (b_{12}^{11} + b_{12}^{4}, b_{12}^{4}), \pi_{c_1} (b_{13}^{11} + b_{13}^{4}, b_{13}^{4}),
\]

\[
\lesssim \| (\nabla (u_{11}^{11} + u_{11}^{4}), \nabla (u_{12}^{11} + u_{12}^{4}), \nabla (b_{11}^{11} + b_{11}^{4}), \nabla (b_{12}^{11} + b_{12}^{4}))\|_{C^{-\frac{1}{2} - \frac{\delta}{4}}}
\]

\[
+ \| \sum_{i,j=1}^\infty \|C\|_\delta \int_{-1}^{1} \| (a_{i,j}^{11} + a_{i,j}^{4}) (u_{11}^{11} + u_{11}^{4}) (u_{11}^{11} + u_{11}^{4}),
\]

\[
\lesssim C_{11} \|(u_{11}, b_{11})\|_{C^{-\frac{1}{2} - \frac{\delta}{4}}} + \|(u_{11}, b_{11})\|_{C^{-\frac{1}{2} - \frac{\delta}{4}}} \|
\]

\[
+ \sum_{i_1, j_1=1}^{3} \| (u_{i_1 j_1}^{11} + u_{i_1 j_1}^{4}, b_{i_1 j_1}^{11} + b_{i_1 j_1}^{4}, u_{i_1 j_1}^{11} + u_{i_1 j_1}^{4}, b_{i_1 j_1}^{11} + b_{i_1 j_1}^{4})\|_{C^{-\frac{1}{2} - \frac{\delta}{4}}} \|
\]
by Lemma (2), (33), (37) and (49). Now we have

$$
\| \phi_{b,i}^3 \|_{C^{1-2s}} = \left\| -\frac{1}{2} \sum_{i_1,i_2=1}^{3} \mathcal{P}^{i_1 i_2} \partial_{\xi^j} [\pi > (u_{d3}^j + u_{d4}^j, u_{d1}^j) + \pi_{0,0}(u_{d3}^j, u_{d1}^j) + \pi_{0,0}(u_{d4}^j, u_{d1}^j)]
\right. \\
+ \pi_{b_i}(u_{d3}^j + u_{d4}^j, u_{d1}^j) + \pi_{0,0}(u_{d3}^j, u_{d1}^j) + \pi_{0,0}(u_{d4}^j, u_{d1}^j)
+ u_{d3}^j \circ u_{d2}^j (u_{d3}^j + u_{d4}^j) + u_{d2}^j (u_{d3}^j + u_{d4}^j) + (u_{d3}^j + u_{d4}^j)(u_{d3}^j + u_{d4}^j)
- \pi_{b_i}(b_{d3}^j + b_{d4}^j, b_{d1}^j) - \pi_{0,0}(b_{d3}^j, b_{d1}^j)
- \pi_{b_i}(b_{d4}^j + b_{d1}^j, b_{d1}^j) - \pi_{0,0}(b_{d4}^j, b_{d1}^j)
- b_{d3}^j \circ b_{d2}^j (b_{d3}^j + b_{d4}^j) - b_{d2}^j (b_{d3}^j + b_{d4}^j) - (b_{d3}^j + b_{d4}^j)(b_{d3}^j + b_{d4}^j)
- \pi_{<}(L(u_{d3}^j + u_{d4}^j), K_{b_i}^j) - \pi_{<}(L(u_{d3}^j + u_{d4}^j), K_{b_i}^j)
+ \pi_{<}(L(b_{d3}^j + b_{d4}^j), K_{b_i}^j) + \pi_{<}(L(b_{d3}^j + b_{d4}^j), K_{b_i}^j)
+ 2\pi_{<}(\nabla (u_{d3}^j + u_{d4}^j), \nabla K_{b_i}^j) + 2\pi_{<}(\nabla (u_{d3}^j + u_{d4}^j), \nabla K_{b_i}^j)
- 2\pi_{<}(\nabla (b_{d3}^j + b_{d4}^j), \nabla K_{b_i}^j) - 2\pi_{<}(\nabla (b_{d3}^j + b_{d4}^j), \nabla K_{b_i}^j)) \|_{C^{1-2s}}
$$

by (50) and

$$
\| \phi_{b,i}^3 \|_{C^{1-2s}} = \left\| -\frac{1}{2} \sum_{i_1,i_2=1}^{3} \mathcal{P}^{i_1 i_2} \partial_{\xi^j} [\pi > (u_{d3}^j + u_{d4}^j, b_{d1}^j) + \pi_{0,0}(u_{d3}^j, b_{d1}^j) + \pi_{0,0}(u_{d4}^j, b_{d1}^j)]
\right. \\
+ \pi_{b_i}(b_{d3}^j + b_{d4}^j, u_{d1}^j) + \pi_{0,0}(b_{d3}^j, u_{d1}^j) + \pi_{0,0}(b_{d4}^j, u_{d1}^j)
+ b_{d3}^j \circ u_{d2}^j (u_{d3}^j + u_{d4}^j) + u_{d2}^j (b_{d3}^j + b_{d4}^j) + (b_{d3}^j + b_{d4}^j)(u_{d3}^j + u_{d4}^j)
- \pi_{b_i}(u_{d3}^j + u_{d4}^j, u_{d1}^j) - \pi_{0,0}(u_{d3}^j, u_{d1}^j) - \pi_{0,0}(u_{d4}^j, u_{d1}^j)
- \pi_{b_i}(u_{d3}^j + u_{d4}^j, b_{d1}^j) - \pi_{0,0}(u_{d3}^j, b_{d1}^j) - \pi_{0,0}(u_{d4}^j, b_{d1}^j)
- u_{d3}^j \circ b_{d2}^j (b_{d3}^j + b_{d4}^j) - b_{d2}^j (u_{d3}^j + u_{d4}^j) - (u_{d3}^j + u_{d4}^j)(b_{d3}^j + b_{d4}^j)
+ \pi_{<}(L(u_{d3}^j + u_{d4}^j), K_{b_i}^j) - \pi_{<}(L(u_{d3}^j + u_{d4}^j), K_{b_i}^j)
- \pi_{<}(L(b_{d3}^j + b_{d4}^j), K_{b_i}^j) + \pi_{<}(L(b_{d3}^j + b_{d4}^j), K_{b_i}^j)
- 2\pi_{<}(\nabla (u_{d3}^j + u_{d4}^j), \nabla K_{b_i}^j) + 2\pi_{<}(\nabla (u_{d3}^j + u_{d4}^j), \nabla K_{b_i}^j)
+ -2\pi_{<}(\nabla (b_{d3}^j + b_{d4}^j), \nabla K_{b_i}^j) - 2\pi_{<}(\nabla (b_{d3}^j + b_{d4}^j), \nabla K_{b_i}^j)) \|_{C^{1-2s}}
$$

by (51). Firstly, we may bound within (88) - (89):

$$
\| \mathcal{P}^{i_1 i_2} \partial_{\xi^j} [u_{d3}^j (u_{d3}^j + u_{d4}^j) + u_{d2}^j (u_{d3}^j + u_{d4}^j) + (u_{d3}^j + u_{d4}^j)(u_{d3}^j + u_{d4}^j)]
\right. \\
- b_{d2}^j (b_{d3}^j + b_{d4}^j) - b_{d2}^j (b_{d3}^j + b_{d4}^j) - (b_{d3}^j + b_{d4}^j)(b_{d3}^j + b_{d4}^j)) \|_{C^{1-2s}}
\leq \| \phi_{b,i}^3 \|_{C^{1-s}} \| (u_{d3}^j + u_{d4}^j, u_{d1}^j, u_{d3}^j + u_{d4}^j, b_{d3}^j + b_{d4}^j, b_{d1}^j) \|_{C^{1-2s}}
+ \| (u_{d3}^j + u_{d4}^j, b_{d1}^j, b_{d1}^j) \|_{C^{s}} \| (u_{d3}^j + u_{d4}^j, b_{d3}^j + b_{d4}^j) \|_{C^{s}}
\leq (1 + C_4^j) \| \phi_{b,i}^3 \|_{C^{1-s}} + \| \phi_{b,i}^3 \|_{C^{2s}}
$$
by Lemma 1.3 that \(-2\delta \leq -\delta,\) Lemma 1.1 (4), (59) and (39). Similar computations show that

\[
\|\mathcal{P}^{ij_1} \partial_x [(b^i_j (u^i_3 + u^i_4) + u^i_3 (b^i_j + b^{j_1}_i) + (b^i_j + b^{j_1}_i)(u^i_3 + u^i_4) \\
- u^i_3 (b^i_j + b^{j_1}_i) - b^i_3 (u^i_3 + u^i_4) - (u^i_3 + u^i_4)(b^j_i + b^{j_1}_i)]\|_{C^{-1-2\delta}} \lesssim (1 + C^j_\xi)^3 [1 + \|(u_4, b_2)\|_{C^{\frac{1}{2}-\delta_0}} + \|(u_4, b_2)\|_{C^{2}}^2].
\]

Secondly, we bound within (88)-(89)

\[
\|\mathcal{P}^{ij_1} \partial_x (u^i_3 \circ u^j_2, b^{j_1}_i \circ b^j_1)\|_{C^{-1-2\delta}} \lesssim \|(u^i_3 \circ u^j_2, b^{j_1}_i \circ b^j_1)\|_{C^{-2\delta}} \lesssim C_\xi
\]

by Lemma 1.3 and (37). Similarly,

\[
\|\mathcal{P}^{ij_1} \partial_x (b^{j_1}_i \circ u^j_2, u^i_3 \circ b^j_1)\|_{C^{-2\delta}} \lesssim C_\xi.
\]

Thirdly, we bound within (88)-(89)

\[
\|\mathcal{P}^{ij_1} \partial_x [(\pi_\geq (u^i_3 + u^j_2, u^i_4), \pi_\leq (u^i_3 + u^j_2, b^{j_1}_i), \pi_\leq (b^{j_1}_i + b^j_1), \pi_\leq (L(u^i_3 + u^j_2, K^j_\mu), \pi_\leq (L(u^i_3 + u^j_2, K^{j_1}_\mu), \pi_\leq (\nabla(u^i_3 + u^j_2), \nabla K^j_\mu), \pi_\leq (\nabla(u^i_3 + u^j_2), \nabla K^{j_1}_\mu), \pi_\leq (\nabla(b^{j_1}_i + b^j_1), \nabla K^j_\mu), \pi_\leq (\nabla(b^{j_1}_i + b^j_1), \nabla K^{j_1}_\mu))\|_{C^{-1-2\delta}}
\lesssim C_\xi (\|(u^i_3, b^{j_1}_i)\|_{C^{\frac{1}{2}+\beta}} + \|(u^i_3, b^{j_1}_i, b^j_1, b^{j_1}_i)\|_{C^{\frac{1}{2}-\delta}})
\]

\[
+ \sum_{i_1, i_2=1}^3 \|(u^i_3 + u^i_4, b^{i_1}_i + b^j_1, u^i_3 + u^i_4, b^{j_1}_i + b^j_1)\|_{C^{\frac{1}{2}-\delta_0}} (\|K_\mu, K^{j_1}_\mu\|_{C^{\frac{1}{2}+\beta}})
\]

\[
+ C^j_\xi [\|(u^2, b^2)\|_{C^{\frac{1}{2}+\beta}} + \|(u_4, b_4)\|_{C^{\frac{1}{2}-\delta_0}} + \|(u_4, b_4)\|_{C^{2}}^2] \times \|K_\mu, K^{j_1}_\mu\|_{C^{\frac{1}{2}-\delta}}
\lesssim (1 + C^j_\xi)^2 [1 + \|(u^2, b^2)\|_{C^{\frac{1}{2}+\beta}} + \|(u_4, b_4)\|_{C^{\frac{1}{2}-\delta_0}} + \|(u_4, b_4)\|_{C^{2}}^2]
\]

by Lemma 1.3, (21), (37), (36), (38), (37) and (39). Similarly we bound

\[
\|\mathcal{P}^{ij_1} \partial_x [(\pi_\geq (u^i_3 + u^j_2, b^{j_1}_i), \pi_\leq (b^{j_1}_i + b^j_1), \pi_\leq (L(u^i_3 + u^j_2, K^j_\mu), \pi_\leq (L(u^i_3 + u^j_2, K^{j_1}_\mu), \pi_\leq (\nabla(u^i_3 + u^j_2), \nabla K^j_\mu), \pi_\leq (\nabla(u^i_3 + u^j_2), \nabla K^{j_1}_\mu), \pi_\leq (\nabla(b^{j_1}_i + b^j_1), \nabla K^j_\mu), \pi_\leq (\nabla(b^{j_1}_i + b^j_1), \nabla K^{j_1}_\mu))\|_{C^{-1-2\delta}}
\lesssim (1 + C^j_\xi)^2 [1 + \|(u^2, b^2)\|_{C^{\frac{1}{2}+\beta}} + \|(u_4, b_4)\|_{C^{\frac{1}{2}-\delta_0}} + \|(u_4, b_4)\|_{C^{2}}^2].
\]
Fourthly, we bound within (88) - (89)

\[ \| \mathcal{P}^{\pi,i} \partial_{\tau_{i,j}} (\mathcal{P}_{i,j} C_{i,j}^s) \|_{\ell_2} \leq \sum_{i,j} \left[ \| u_{i,j}^s \|_{\ell_2} + \| b_{i,j}^s \|_{\ell_2} \right] \| \mathcal{P}_{i,j} \|_{\ell_2} \| C_{i,j}^s \|_{\ell_2} \]

(96)

by Lemma 4.4, 4.7, 4.7, 4.7. Therefore, inserting (90)-(96) in (88) and (89) gives

\[ \| (\phi_{a,i}^s, \phi_b^s) (t) \|_{\ell_2} \leq (1 + C_4^s) [1 + \| (u^s, b^s) \|_{\ell_2} + \| y_4 \|_{\ell_2} + \| y_4 \|_{\ell_2}^2]. \]

(97)

Now from the paracontrolled ansatz (29) and (32), for any \( t \in [0, T], T > 0 \) depending only on \( C_4 \), we can obtain

\[ \| (u^s_{i,j}, b^s_{i,j}) (t) \|_{\ell_2} \leq \sum_{i,j} \left[ \| u^s_{i,j} \|_{\ell_2} + \| b^s_{i,j} \|_{\ell_2} \right] \| (\mathcal{P}_{i,j} \partial_{\tau_{i,j}}) \|_{\ell_2} \| C_{i,j}^s \|_{\ell_2} \]

(98)

for some \( C \geq 0 \) by Lemma 1.1. (1), (59), (83) and (87). Therefore, this gives

\[ \sum_{i,j} \| (u^s_{i,j}, b^s_{i,j}) (t) \|_{\ell_2} \leq C_2^s + \sum_{i,j} \left[ \| u^{s,i}_{i,j} \|_{\ell_2} + \| b^{s,i}_{i,j} \|_{\ell_2} \right] \]

(99)

for \( t \in [0, \frac{1}{C_4^s}] \) due to (89). Similarly for any \( t \in [0, T], T > 0 \) depending only on \( C_4 \),

\[ \| (u^s_{i,j}, b^s_{i,j}) (t) \|_{\ell_2} \]

\[ \leq \sum_{i,j} \left[ \| u^s_{i,j} \|_{\ell_2} + \| b^s_{i,j} \|_{\ell_2} \right] \| \mathcal{P}_{i,j} \|_{\ell_2} \| C_{i,j}^s \|_{\ell_2} \]

(100)

by (29), Lemma 1.1 (2), (59), (83) and (87). This gives

\[ \sum_{i,j} \| (u^s_{i,j}, b^s_{i,j}) (t) \|_{\ell_2} \leq C_2^s + \sum_{i,j} \left[ \| u^{s,i}_{i,j} \|_{\ell_2} + \| b^{s,i}_{i,j} \|_{\ell_2} \right] \]

(101)
for $t \in [0, (\frac{1}{C \xi})^\delta]$ due to (39). Now due to (51), (51), (29), (32), (19) and (20) we see that
\[
u^{3,i}(t) = P_t(\sum_{i_1=1}^{3} \mathcal{P}^{i_1} u_{0}^{i_1} - u_1^{i_1}(0)) + \int_0^t P_{t-s} \phi_u^{3,i}(s) ds, \tag{102a}
\]
\[
\nu^{3,i}(t) = P_t(\sum_{i_1=1}^{3} \mathcal{P}^{i_1} b_{0}^{i_1} - b_1^{i_1}(0)) + \int_0^t P_{t-s} \phi_b^{3,i}(s) ds. \tag{102b}
\]

Then we obtain
\[
\begin{align*}
t^{\delta+z} \|& (u^\delta, b^\delta)(t) \|_{C^{\frac{\delta}{2+\delta}}} \\
\lesssim & \| P y_0 - y_1(0) \|_{C^{-\delta}} + t^{\delta+z} \int_0^t (t-s)^{-\frac{\delta}{2+\delta}-\delta} \| (\phi_u^\delta, \phi_b^\delta)(s) \|_{C^{-1-2\delta}} ds \tag{103}
\end{align*}
\]
by (102a), (102b), Lemma 4.4 and (34). We are also able to estimate
\[
\begin{align*}
t^{\delta+z} \|& (u^\delta, b^\delta)(t) \|_{C^{-1-2\delta}}^2 \\
\lesssim & t^{\delta+z}[\| P_t( P y_0 - y_1(0)) \|^2_{C^{-1-2\delta}} + \left( \int_0^t \| P_{t-s} (\phi_u^\delta, \phi_b^\delta)(s) \|_{C^{-1-2\delta}} ds \right)^2 ] \\
\lesssim & \| P y_0 - y_1(0) \|^2_{C^{-1-2\delta}} \\
& + t^{\frac{\delta}{2+\delta}} \int_0^t (t-s)^{-\frac{(3+\delta)}{2+\delta}} s^{-(\delta+z)} (s^{\delta+z} \| (\phi_u^\delta, \phi_b^\delta)(s) \|_{C^{-1-2\delta}})^2 ds \tag{104}
\end{align*}
\]
by (102a), (102b), Lemma 4.4, Hölder’s inequality, (59) and (44). Thus,
\[
\begin{align*}
t^{\delta+z} \| & (\phi_u^\delta, \phi_b^\delta)(t) \|_{C^{-1-2\delta}} \\
\lesssim & t^{\delta+z} \left( (1 + C_\xi^4) [1 + \| (u^\delta, b^\delta)(t) \|_{C^{\frac{\delta}{2+\delta}}} + C_\xi^4 + \| (u^\delta, b^\delta)(t) \|_{C^{-1-2\delta}}] \\
\lesssim & (1 + C_\xi^4) [1 + (1 + C_\xi^4) \| P y_0 - y_1(0) \|^2_{C^{-1-2\delta}} \tag{105} \\
& + t^{\delta+z} \int_0^t (t-s)^{-\frac{\delta}{2+\delta} \delta} s^{-(\delta+z)} (s^{\delta+z} \| (\phi_u^\delta, \phi_b^\delta)(s) \|_{C^{-1-2\delta}}) ds \\
& + t^{\frac{\delta}{2+\delta}} \int_0^t (t-s)^{-\frac{(3+\delta)}{2+\delta}} s^{-(\delta+z)} (s^{\delta+z} \| (\phi_u^\delta, \phi_b^\delta)(s) \|_{C^{-1-2\delta}})^2 ds \right]
\end{align*}
\]
by (97), (99), (101), (103) and (104). By Lemma 4.8 and Remark 5.1 this implies that for $\delta < \frac{1}{2-C_\xi}$ there exists some $T_0 \in (0, \mathcal{T})$ which is independent of $\epsilon \in (0, 1)$ such that
\[
\sup_{t \in [0, T_0]} t^{\delta+z} \| (\phi_u^\delta, \phi_b^\delta)(t) \|_{C^{-1-2\delta}} \lesssim C(T_0, C_\xi, \| y_0 \|_{C^{-\delta}}, \| y_1(0) \|_{C^{-\delta}}) \tag{106}
\]
Thus, if $C_\xi$ is uniformly bounded over $\epsilon \in (0, 1)$, then (106) holds for all $\epsilon \in (0, 1)$. Next, we estimate
\[
\begin{align*}
t^{\frac{\delta}{2+\delta}} \| (u^\delta, b^\delta)(t) \|_{C^{\frac{\delta}{2+\delta}-\delta}} \\
\lesssim & t^{\frac{\delta}{2+\delta}-\delta} \| ( P_t( P y_0 - y_1(0)) \|_{C^{\frac{\delta}{2+\delta}-\delta}} + \int_0^t \| P_{t-s} (\phi_u^\delta, \phi_b^\delta)(s) \|_{C^{\frac{\delta}{2+\delta}-\delta}} ds \tag{107} \\
\lesssim & \| P y_0 - y_1(0) \|_{C^{-\delta}} + t^{\frac{\delta}{2+\delta}-\frac{3}{2}} \left( \sup_{s \in [0, t]} s^{\delta+z} \| (\phi_u^\delta, \phi_b^\delta)(s) \|_{C^{-1-2\delta}} \right)
\end{align*}
\]
Thus, for \( t \in [0, T_0] \) for some constant \( a > 0 \), equipped with product topology. Then we may show via similar arguments that for \( T \) by (99), (107), (106), (44) and Remark 3.1. By (43) and (108) we conclude that by (109), (102a), (102b), Lemma 4.4 and (106). Based on (37) we now define

\[
\|u(t)\|_{C^{-\frac{1}{2}}} \leq C(T, C_\xi, \|y_0\|_{C^{-\frac{1}{2}}}, \|y_1(0)\|_{C^{-\frac{1}{2}}})
\]

by (109), (102a), (102b), Lemma 4.4 and Remark 3.1. By (103) and (108) we conclude that \( T_1 \geq T_0 \). Finally,

\[
\|u(t)\|_{C^{-\frac{1}{2}}} \lesssim \|\pi_\leq(u_3 + u_4, K_u), \pi_\leq(b_3 + b_4, K_u)\|_{C^{-1-\frac{1}{2}}}
+ \|\pi_\leq(u_3 + u_4, K_b), \pi_\leq(b_3 + b_4, K_b)\|_{C^{-1-\frac{1}{2}}} + \|(u^2, b^2)\|_{C^{-1}}
\leq C[\|u(t)\|_{C^{-\frac{1}{2}}} + \|u(t)\|_{C^{-\frac{1}{2}}} + \|(u^2, b^2)\|_{C^{-1}}]
\]

for some constant \( C \geq 0 \) by (29), (32), Lemma 1.1 (2), (35), (39), (35) and (37).

Thus, for \( t \in [0, (\frac{1}{C C_\xi})^\frac{1}{2}) \) we have

\[
\|u(t)\|_{C^{-\frac{1}{2}}} \leq \frac{C}{1 - C C_\xi t^2} [C^2 t^2 + \|(u^2, b^2)\|_{C^{-\frac{1}{2}}}]
\]

by (109), (102a), (102b), Lemma 4.4 and (106). Based on (37) we now define

\[
Z(\xi') \triangleq \{u_1', b_1', u_1' \otimes u_1', b_1' \otimes b_1', u_1' \otimes b_1', b_1' \otimes u_1', u_1' \otimes u_2', b_1' \otimes b_2', b_2' \otimes u_1', w_2' \otimes b_2', u_2' \otimes b_2', b_2' \otimes u_2', w_2' \otimes b_2', \pi_{0,0}(u_3', u_1'),
\pi_{0,0}(b_3', b_1'), \pi_{0,0}(u_3', b_1'), \pi_{0,0}(b_3', u_1'), \pi_{0,0}(PDK', u_1'),
\pi_{0,0}(PDK', b_1'), \pi_{0,0}(PDK', b_1')\}
\]

equipped with product topology. Then we may show via similar arguments that for all \( a > 0 \), there exists \( T_0 > 0 \) sufficiently small such that the mapping \((y_0, Z(\xi')) \to y_4\) is Lipschitz in a norm of \( C([0, T_0]; C^{-2})\) on the set \( \{(y_0, Z(\xi')) : \max\{\|y_0\|_{C^{-2}}, C_\xi\} \leq a\} \). This implies the following result.

**Proposition 3.1.** Let \( \delta_0 \in (0, \frac{1}{2}) \), \( z \in (\frac{1}{2}, \frac{1}{2} + \delta_0) \) and \( (\xi')_{\epsilon > 0} \) be a family of smooth functions converging to \( \xi \) as \( \epsilon \to 0 \). Suppose that for any \( \epsilon > 0, y_0 \in C^{-2} \) given, \( y^e\)
is the unique maximal solution to

\[
Ly^i = \begin{cases} 
\sum_{i_1=1}^{3} P^{i_1} \xi^i_{u} - \frac{1}{2} \sum_{i_1,j=1}^{3} P^{i_1} \partial_{x^j} (u^i u^j) \\
+ \frac{1}{2} \sum_{i_1,j=1}^{3} P^{i_1} \partial_{x^j} (b^i b^j) & \text{if } i \in \{1, 2, 3\}, \\
\sum_{i_1=1}^{3} P^{i_1} \xi^i_{v} - \frac{1}{2} \sum_{i_1,j=1}^{3} P^{i_1} \partial_{x^j} (b^i u^j) \\
+ \frac{1}{2} \sum_{i_1,j=1}^{3} P^{i_1} \partial_{x^j} (u^i b^j) & \text{if } i \in \{4, 5, 6\}, 
\end{cases}
\]  

(112)

\[ y^i(\cdot, 0) = P y_0(\cdot), \]

such that \( y^i = (u^i_{k}, b^i_{k}) \in (C((0, T); \mathbb{C}^{\frac{1}{2} - \delta}))^2 \). Suppose \( Z(\xi^e) \) converges in \( \mathcal{X} \) so that for \( i, i_1, j, j_1 \in \{1, 2, 3\} \), there exist \( (v^i_{k})_{k \in \{1, 2, 3\}, i \in \{1, 2, 3\} \}, (v^i_{k})_{k \in \{3, \ldots, 17\}, i, j \in \{1, 2, 3\}}, (v^i_{k1})_{k \in \{18, \ldots, 21\}, i, i_1, j, j_1 \in \{1, 2, 3\}} \) satisfying

\[
\begin{align*}
&v^i_{1,1} \rightarrow v^i_{1}, b^i_{1,1} \rightarrow v^i_{2}, b^i_{1,1} \rightarrow v^i_{3} \text{ in } C([0, T]; C^{-\frac{1}{2} - \delta}), \\
&v^i_{1,1} \circ v^i_{1,1} \rightarrow v^i_{1}, b^i_{1,1} \circ b^i_{1,1} \rightarrow v^i_{1}, \\
&v^i_{1,1} \circ b^i_{1,1} \rightarrow v^i_{5}, b^i_{1,1} \circ u^i_{1} \rightarrow v^i_{6} \text{ in } C([0, T]; C^{-1 - \delta}), \\
&v^i_{1,1} \circ u^i_{2} \rightarrow v^i_{7}, b^i_{1,1} \circ b^i_{2} \rightarrow v^i_{8}, \\
&b^i_{1,1} \circ u^i_{2} \rightarrow v^i_{9}, b^i_{2} \circ u^i_{2} \rightarrow v^i_{10} \text{ in } C([0, T]; C^{-\frac{1}{2} - \delta}), \\
&v^i_{2,1} \circ u^i_{2} \rightarrow v^i_{11}, b^i_{2} \circ b^i_{2} \rightarrow v^i_{12}, b^i_{1,1} \circ u^i_{2} \rightarrow v^i_{13} \text{ in } C([0, T]; C^{-\delta}), \\
&\pi_{0,0}(u^i_{3}, u^i_{1}) \rightarrow v^i_{14}, \pi_{0,0}(b^i_{3}, b^i_{1}) \rightarrow v^i_{15}, \\
&\pi_{0,0}(u^i_{3}, b^i_{1}) \rightarrow v^i_{16}, \pi_{0,0}(b^i_{3}, u^i_{1}) \rightarrow v^i_{17} \text{ in } C([0, T]; C^{-\delta}), \\
&\pi_{0,0}(P^{i_1} \partial_{x^j} K^e_{u^j}, u^i_{1}) \rightarrow v^i_{18} \text{ in } C([0, T]; C^{\delta}), \\
&\pi_{0,0}(P^{i_1} \partial_{x^j} K^e_{b^j}, u^i_{1}) \rightarrow v^i_{19} \text{ in } C([0, T]; C^{-\delta}), \\
&\pi_{0,0}(P^{i_1} \partial_{x^j} K^e_{b^j}, b^i_{1}) \rightarrow v^i_{20} \text{ in } C([0, T]; C^{-\delta}), \\
&\pi_{0,0}(P^{i_1} \partial_{x^j} K^e_{b^j}, b^i_{1}) \rightarrow v^i_{21} \text{ in } C([0, T]; C^{-\delta}). 
\end{align*}
\]  

(113)
as $\epsilon \to 0$, where

\[
\begin{align*}
    u_1^{\epsilon,i} \circ u_1^{\epsilon,j} &= u_1^{\epsilon,i} u_1^{\epsilon,j} - C^{\epsilon,ij}_{0,1}, b_1^{\epsilon,i} \circ b_1^{\epsilon,j} = b_1^{\epsilon,i} b_1^{\epsilon,j} - C^{\epsilon,ij}_{0,2}, \\
    u_1^{\epsilon,i} \circ b_1^{\epsilon,j} &= u_1^{\epsilon,i} b_1^{\epsilon,j} - C^{\epsilon,ij}_{0,3}, b_1^{\epsilon,i} \circ u_1^{\epsilon,j} = b_1^{\epsilon,i} u_1^{\epsilon,j} - C^{\epsilon,ij}_{0,4}, \\
    u_1^{\epsilon,i} \circ u_2^{\epsilon,j} &= u_1^{\epsilon,i} u_2^{\epsilon,j}, b_1^{\epsilon,i} \circ b_2^{\epsilon,j} = b_1^{\epsilon,i} b_2^{\epsilon,j}, \\
    u_1^{\epsilon,i} \circ b_2^{\epsilon,j} &= u_1^{\epsilon,i} b_2^{\epsilon,j}, b_1^{\epsilon,i} \circ u_2^{\epsilon,j} = b_1^{\epsilon,i} u_2^{\epsilon,j}, \\
    u_2^{\epsilon,i} \circ u_2^{\epsilon,j} &= u_2^{\epsilon,i} u_2^{\epsilon,j} - C^{\epsilon,ij}_{2,1}, b_2^{\epsilon,i} \circ b_2^{\epsilon,j} = b_2^{\epsilon,i} b_2^{\epsilon,j} - C^{\epsilon,ij}_{2,2}, \\
    b_2^{\epsilon,i} \circ u_2^{\epsilon,j} &= b_2^{\epsilon,i} u_2^{\epsilon,j} - C^{\epsilon,ij}_{2,3}, \\
    \pi_0,\circ(u^{\epsilon,i}, u^{\epsilon,j}) &= \pi_0(u^{\epsilon,i}, u^{\epsilon,j}) - C^{\epsilon,ij}_{1,1}, \\
    \pi_0,\circ(b^{\epsilon,i}, b^{\epsilon,j}) &= \pi_0(b^{\epsilon,i}, b^{\epsilon,j}) - C^{\epsilon,ij}_{1,2}, \\
    \pi_0,\circ(u^{\epsilon,i}, b^{\epsilon,j}) &= \pi_0(u^{\epsilon,i}, b^{\epsilon,j}) - C^{\epsilon,ij}_{1,3}, \\
    \pi_0,\circ(b^{\epsilon,i}, u^{\epsilon,j}) &= \pi_0(b^{\epsilon,i}, u^{\epsilon,j}) - C^{\epsilon,ij}_{1,4}, \\
    \pi_0,\circ(P^{ii,j}\partial_x K^{\epsilon,j}, u^{\epsilon,j}_1) &= \pi_0(P^{ii,j}\partial_x K^{\epsilon,j}, u^{\epsilon,j}_1), \\
    \pi_0,\circ(P^{ii,j}\partial_x K^{\epsilon,j}, b^{\epsilon,j}_1) &= \pi_0(P^{ii,j}\partial_x K^{\epsilon,j}, b^{\epsilon,j}_1), \\
    \pi_0,\circ(P^{ii,j}\partial_x K^{\epsilon,j}, b^{\epsilon,j}_1) &= \pi_0(P^{ii,j}\partial_x K^{\epsilon,j}, b^{\epsilon,j}_1), \\
    \pi_0,\circ(P^{ii,j}\partial_x K^{\epsilon,j}, b^{\epsilon,j}_1) &= \pi_0(P^{ii,j}\partial_x K^{\epsilon,j}, b^{\epsilon,j}_1), \\
    \pi_0,\circ(P^{ii,j}\partial_x K^{\epsilon,j}, b^{\epsilon,j}_1) &= \pi_0(P^{ii,j}\partial_x K^{\epsilon,j}, b^{\epsilon,j}_1),
\end{align*}
\]

(114)

with \{C^{\epsilon,ij}_{0,k}\}_{\epsilon>0}, \{C^{\epsilon,ij}_{2,k}\}_{\epsilon>0}, \{C^{\epsilon,ij}_{1,k}\}_{\epsilon>0} \subset \mathbb{R}$ for $k \in \{1, 2, 3, 4\}$ to be specified subsequently, e.g. $C^{\epsilon,ij}_{0,1}$, $C^{\epsilon,ij}_{2,1}$ and $C^{\epsilon,ij}_{1,3}$ in \[17\], \[16\] and \[19\] respectively. Then there exists a unique $y \in C([0,\tau]; C^{-\frac{2}{3}})$ where $\tau = \tau(y_0, v_1, \ldots, v_{20}) > 0$ such that $\lim_{\epsilon \to 0} \|y' - y\|_C([0,\tau]; C^{-\frac{2}{3}}) = 0$, and $y$ depends only on $(y_0, \{v_k\}_{k=1,\ldots,21})$, and not on the approximating family.

We refer to \[56\] Remark 3.9 and \[8\] Theorem 3.1, Proposition 3.2, Corollary 3.3 for further discussions. Hereafter let us write $X_{i,u} \triangleq u_1(t), X_{i,b} \triangleq b_1(t)$ where $y = (u^1, u^2, u^3, b^1, b^2)$ and following \[8\] Notation 4.1, for $k_1, \ldots, k_6 \in \mathbb{Z}^3$, we also write $k_{1,\ldots,6} \triangleq \sum_{i=1}^n k_i$. Since $X_{i,u} = u_1(t), X_{i,b} = b_1(t)$, we have

\[
X_{i,u} = \sum_{k \neq 0} \hat{X}_{i,u}^k(k) e_k, \quad X_{i,b} = \sum_{k \neq 0} \hat{X}_{i,b}^k(k) e_k, \quad e_k \triangleq (2\pi)^{-\frac{2}{3}} e^{i x \cdot k}
\]

(115)

where $\hat{X}_{i,u}(0) = 0, \hat{X}_{i,b}(0) = 0$ due to mean-zero property of $\xi_u$ and $\xi_b$ and

\[
\begin{align*}
    E[\hat{X}_{i,u}^k(k) \hat{X}_{j,u}^k(k')] &= 1_{k+k' = 0} \sum_{i_1=1}^3 \frac{e^{-|k|^2|t-s|}}{2|k|^2} \tilde{\rho}^{ii}(k) \tilde{\rho}^{i\bar{j}}(k), \\
    E[\hat{X}_{i,b}^k(k) \hat{X}_{j,b}^k(k')] &= 1_{k+k' = 0} \sum_{i_1=1}^3 \frac{e^{-|k|^2|t-s|}}{2|k|^2} \tilde{\rho}^{ii}(k) \tilde{\rho}^{i\bar{j}}(k), \\
    E[\hat{X}_{i,u}^k(k) \hat{X}_{j,b}^k(k')] &= 1_{k+k' = 0} \sum_{i_1=1}^3 \frac{e^{-|k|^2|t-s|}}{2|k|^2} \tilde{\rho}^{ii}(k) \tilde{\rho}^{i\bar{j}}(k), \\
    E[\hat{X}_{i,b}^k(k) \hat{X}_{j,u}^k(k')] &= 1_{k+k' = 0} \sum_{i_1=1}^3 \frac{e^{-|k|^2|t-s|}}{2|k|^2} \tilde{\rho}^{ii}(k) \tilde{\rho}^{i\bar{j}}(k), \\
    E[\hat{X}_{i,b}^k(k) \hat{X}_{j,b}^k(k')] &= 1_{k+k' = 0} \sum_{i_1=1}^3 \frac{e^{-|k|^2|t-s|}}{2|k|^2} \tilde{\rho}^{ii}(k) \tilde{\rho}^{i\bar{j}}(k),
\end{align*}
\]

(116)
for $k \in \mathbb{Z}^3 \setminus \{0\}$ due to $[10]$. Now we regularize the noise $\xi$ by $\xi^\epsilon = \sum_k f(\epsilon k) \hat{\xi}(k) e_k$ where $f$ is a smooth radial cut-off function with compact support such that $f(0) = 1$ so that

$$X^{\epsilon, i}_{t, u} = \int_t^\infty \sum_{i_1=1}^3 \mathcal{P}^{i_1 i} P_{t-s} \sum_{k \neq 0} f(\epsilon k) \hat{\xi}^{\epsilon, i_1}(k, s) ds,$$

$$X^{\epsilon, b}_{t, b} = \int_t^\infty \sum_{i_1=1}^3 \mathcal{P}^{i_1 i} P_{t-s} \sum_{k \neq 0} f(\epsilon k) \hat{\xi}^{\epsilon, i_1}(k, s) ds,$$

and the covariance of $X^{\epsilon, i}_{t, u}, X^{\epsilon, b}_{t, b}$ follow from $[110]$, only multiplied by $f(\epsilon k)^2$.

We now devote ourselves to convergence and renormalizations. Firstly, the existence of $v_1, v_2$ such that $u^i_1 \to v_1, b^i_1 \to v_2$ in $L^p(\Omega; C([0, T]; C^{-\frac{3}{2}}))$ for all $p \geq 1$ as $\epsilon \to 0$ is immediate from $[10]$. Secondly, the convergence issues of $u^{\epsilon, i} \circ u^{\epsilon, j} = u^{\epsilon, i} \circ b^{\epsilon, j} = u^{\epsilon, i} \circ b^{\epsilon, j} = b^{\epsilon, i} \circ b^{\epsilon, j} = u^{\epsilon, i} \circ b^{\epsilon, j} = u^{\epsilon, i} \circ b^{\epsilon, j} = v^{ij}_3, b^{\epsilon, i} \circ u^{\epsilon, j} = b^{\epsilon, i} \circ u^{\epsilon, j} = b^{\epsilon, i} \circ b^{\epsilon, j} \to v^{ij}_5$ by $[113]$ in $L^p(\Omega; C([0, T]; C^{-\frac{3}{2}}))$ for all $p \geq 1$ as $\epsilon \to 0$ are clear because $\xi_1 \xi_2 := \xi_1 \xi_2 - \mathbb{E} [\xi_1 \xi_2]$ by $[131]$ so that e.g.

$$C^{\epsilon, ij}_{0, 1} = \mathbb{E} [u^{\epsilon, i}(t) u^{\epsilon, j}(t)] = (2\pi)^{-\frac{3}{2}} \sum_{k_1 \neq 0} \sum_{i_1=1}^3 f(\epsilon k_1)^2 \frac{1}{2|k_1|^2} \hat{P}^{i_1 i}(k_1) \hat{P}^{j_1 i}(k_1)$$

by $[110]$ and $[111]$. It follows that $C^{\epsilon, ij} \to \infty$ as $\epsilon \to 0$.

We need to perform renormalizations on the following groups in $[114]$: a first group of $u^{\epsilon, i} \circ u^{\epsilon, j}, b^{\epsilon, i} \circ b^{\epsilon, j}, u^{\epsilon, i} \circ b^{\epsilon, j}, b^{\epsilon, i} \circ u^{\epsilon, j}$, a second group of $u^{\epsilon, i} \circ u^{\epsilon, j}, b^{\epsilon, i} \circ b^{\epsilon, j}$, and $b^{\epsilon, i} \circ u^{\epsilon, j}$, a third group of $\pi_{0,0}(u^{\epsilon, i}_3, u^{\epsilon, j}_1), \pi_{0,0}(b^{\epsilon, i}_3, b^{\epsilon, j}_1), \pi_{0,0}(u^{\epsilon, i}_3, b^{\epsilon, j}_1)$ and $\pi_{0,0}(b^{\epsilon, i}_3, u^{\epsilon, j}_1)$, a fourth group of $\pi_{0,0}(P^{i_1 i, i_2, j}_1, u^{\epsilon, j}_1), \pi_{0,0}(P^{i_1 i, j, k}_1, u^{\epsilon, j}_1), \pi_{0,0}(P^{i_1 i, i_2, j}_1, u^{\epsilon, j}_1)$, and finally $\pi_{0,0}(P^{i_1 i, j, k}_1, b^{\epsilon, j}_1)$.

3.1. **Group 1**: $u^{\epsilon, i} \circ u^{\epsilon, j}, b^{\epsilon, i} \circ b^{\epsilon, j}, u^{\epsilon, i} \circ b^{\epsilon, j}, b^{\epsilon, i} \circ u^{\epsilon, j}$. Within the group 1 of $[114]$, we focus on $b^{\epsilon, i} \circ u^{\epsilon, j}$ and prove that $b^{\epsilon, i} \circ u^{\epsilon, j} \to v^{ij}_9$ in $C([0, T]; C^{-\frac{3}{2}})$, and for simplicity of notations we write $b^{\epsilon, j} u^{\epsilon, j}$. Firstly, from $[17]$, $[144]$ and $[145]$, we obtain

$$b^{\epsilon, j}(t) u^{\epsilon, i}(t)$$

$$= - \frac{1}{2(2\pi)^3} \sum_k \sum_{i_1, i_2=1}^3 \sum_{k_1, k_2, k_3; k_{123} = k} \int_0^t e^{-|k_{12}|^2 |t-s|} \hat{P}^{i_1 i}(k_{12}) ik_{12}^2$$

$$\times [\hat{X}^{\epsilon, j}_{t, b}(k_3) \hat{X}^{\epsilon, i}(k_1) \hat{X}^{\epsilon, i_2}(k_2) - \hat{X}^{\epsilon, j}_{t, b}(k_3) \hat{X}^{\epsilon, i_1}(k_1) \hat{X}^{\epsilon, i_2}(k_2)] ds e_k.$$
We rely on (13c) and (116) to deduce

\[
\hat{X}_{i,j}^\epsilon,t,b(k_3) \hat{X}_{s,u}^\epsilon,i_1(k_1) \hat{X}_{s,b}^\epsilon,i_2(k_2) - \hat{X}_{i,j}^\epsilon,t,b(k_3) \hat{X}_{s,u}^\epsilon,i_1(k_1) \hat{X}_{s,b}^\epsilon,i_2(k_2)
\]

\[
= \hat{X}_{i,j}^\epsilon,t,b(k_3) \hat{X}_{s,u}^\epsilon,i_1(k_1) \hat{X}_{s,b}^\epsilon,i_2(k_2) : 
\]

\[
+ 1_{k_2 = 0, k_1 \neq 0} \sum_{i_3 = 1}^3 e^{-|k_2|^2 |t-s|} \frac{f(\epsilon k_2)^2 \hat{P}_{j,k}^{i_3} (k_2) \hat{P}_{j,k}^{i_3} (k_2) \hat{X}_{s,u}^\epsilon,i_1 (k_1)}{2|k_2|^2} 
\]

\[
+ 1_{k_1 = 0, k_2 \neq 0} \sum_{i_3 = 1}^3 e^{-|k_1|^2 |t-s|} \frac{f(\epsilon k_1)^2 \hat{P}_{j,k}^{i_3} (k_1) \hat{P}_{j,k}^{i_3} (k_1) \hat{X}_{s,b}^\epsilon,i_2 (k_2)}{2|k_1|^2} 
\]

\[
- \hat{X}_{i,j}^\epsilon,t,b(k_3) \hat{X}_{s,u}^\epsilon,i_1(k_1) \hat{X}_{s,b}^\epsilon,i_2(k_2) : 
\]

\[
- 1_{k_2 = 0, k_1 \neq 0} \sum_{i_3 = 1}^3 e^{-|k_2|^2 |t-s|} \frac{f(\epsilon k_2)^2 \hat{P}_{j,k}^{i_3} (k_2) \hat{P}_{j,k}^{i_3} (k_2) \hat{X}_{s,b}^\epsilon,i_1 (k_1)}{2|k_2|^2} 
\]

\[
- 1_{k_1 = 0, k_2 \neq 0} \sum_{i_3 = 1}^3 e^{-|k_1|^2 |t-s|} \frac{f(\epsilon k_1)^2 \hat{P}_{j,k}^{i_3} (k_1) \hat{P}_{j,k}^{i_3} (k_1) \hat{X}_{s,b}^\epsilon,i_2 (k_2)}{2|k_1|^2}. 
\]
42 KAZUO YAMAZAKI

Applying (119) to (118) finally gives

\[ \frac{1}{2\pi} \sum_{k} \sum_{i_1, i_2 = 1}^{3} \sum_{k_1, k_2, k_3, k_{123} = k} \int_0^t e^{-|k_{12}|^2|t-s|} \times \hat{P}^{i_{13}}(k_{12})k_{12}^i \hat{X}_{i_1}^i(k_1) \hat{X}_{i_2}^i(k_2) : dse_k \]

\[ = \frac{1}{2\pi} \sum_{k} \sum_{i_1, i_2 = 1}^{3} \sum_{k_1, k_2, k_3, k_{123} = k} \int_0^t e^{-|k_{12}|^2|t-s|} \times \hat{P}^{i_{13}}(k_{12})k_{12}^i \hat{X}_{i_1}^i(k_1) \hat{X}_{i_2}^i(k_2) : dse_k \]

\[ = \frac{1}{2\pi} \sum_{k} \sum_{i_1, i_2 = 1}^{3} \sum_{k_1, k_2, k_3, k_{123} = k} \int_0^t e^{-|k_{12}|^2|t-s|} \times \hat{P}^{i_{13}}(k_{12})k_{12}^i \hat{X}_{i_1}^i(k_1) \hat{X}_{i_2}^i(k_2) : dse_k \]

\[ \triangleq \sum_{l=1}^{9} I_l^{1,1} \]

where \( I_l^{1,1} \) are the terms in the third chaos while \( I_l^{1,2}, I_l^{1,3}, I_l^{1,4}, I_l^{1,5} \) are in the first chaos.

3.1.1. Terms in the first chaos. Let us work on \( I_l^{1,5} \) of (120). We first rewrite

\[ I_l^{1,5} = \frac{1}{2\pi} \sum_{k} \sum_{i_1, i_2 = 1}^{3} \sum_{k_1, k_2, k_3, k_{123} = k} \int_0^t e^{-|k_{12}|^2|t-s|} \times ik_{12}^i \hat{X}_{i_1}^i(k_1) e^{-|k_{12}|^2|t-s|} \times f(ek_{12})^2 \hat{P}^{i_{13}}(k_{12}) \hat{p}_{i_2}^{i_{13}}(k_{12}) \hat{p}_{i_3}^{i_{13}}(k_{12}) dse_k \]

and write

\[ I_l^{1,5} = I_l^{1,5} - \bar{I}_l^{1,5} + \tilde{I}_l^{1,5} - \sum_{i_1=1}^{3} \hat{X}_{i_1}^i C_l^{i_{13}} \]

(122)
MAGNETOHYDRODYNAMICS SYSTEM

\begin{equation}
\tilde{\mathcal{I}}_{t,e}^5 \triangleq \frac{1}{2(2\pi)^3} \sum_{i_3,i_3=1}^{3} \sum_{k_1} \int_0^t e^{-|k_1|^2|t-s|} \times ik_1^{i_2} \hat{X}_{s,b}^{i_1}(k_1) \frac{e^{-|k_2|^2|t-s|} f(ek_2)^2}{2|k_2|^2} \hat{\rho}^{i_3 i_1}(k_2) \hat{\rho}^{i_2 i_3}(k_2) \hat{\rho}^{i_3 i_3}(k_2) \, ds \, k_1, \end{equation}

We compute within \((122)\),

\begin{align}
E[\Delta_q(\mathcal{I}_{t,e}^5 - \tilde{\mathcal{I}}_{t,e}^5)]^2 &= E[\sum_{k_1} F_{k_1}^3 \Delta_q(\mathcal{I}_{t,e}^5 - \tilde{\mathcal{I}}_{t,e}^5)] (k_1)^2 ] \\
&= E[\sum_{k_1} \theta(2^{-q}k_1) F_{k_1}^3 (\mathcal{I}_{t,e}^5 - \tilde{\mathcal{I}}_{t,e}^5) (k_1) (k_1) e_{k_1}^2 ] \\
&\approx E[\sum_{k_1 \neq 0} \theta(2^{-q}k_1) \sum_{i_3,i_3=1}^{3} \sum_{k_1} \int_0^t e^{-|k_1|^2|t-s|} \times ik_1^{i_2} \hat{X}_{s,b}^{i_1}(k_1) \hat{X}_{t,b}^{i_1}(k_1) \frac{e^{-|k_2|^2|t-s|} f(ek_2)^2}{2|k_2|^2} \hat{\rho}^{i_3 i_1}(k_2) \hat{\rho}^{i_2 i_3}(k_2) \hat{\rho}^{i_3 i_3}(k_2) \, ds \, k_1] ] \\
&\lesssim E[\sum_{i_1,i_2,i_3,i_1,i_2,i_3=1}^{3} \int_0^t \sum_{k_1 \neq 0} \theta(2^{-q}k_1) e_{k_1} \sum_{k_2 \neq 0} e^{-|k_2|^2|t-s|} \times ik_1^{i_2} \hat{X}_{s,b}^{i_1}(k_1) \hat{X}_{t,b}^{i_1}(k_1) \frac{e^{-|k_2|^2|t-s|} f(ek_2)^2}{2|k_2|^2} \hat{\rho}^{i_3 i_1}(k_2) \hat{\rho}^{i_2 i_3}(k_2) \hat{\rho}^{i_3 i_3}(k_2) \hat{\rho}^{i_3 i_3}(k_2) \hat{X}_{t,b}^{i_1}(k_1) \hat{X}_{s,b}^{i_1}(k_1) \, ds ] ] \\
&\lesssim E[\sum_{i_1,i_2,i_3,i_1,i_2,i_3=1}^{3} \int_0^t \sum_{k_1 \neq 0} \theta(2^{-q}k_1) \theta(2^{-q}k_1') a_{k_1}^{i_2 i_3 i_3}(t-s) a_{k_1'}^{i_1 i_1 i_1}(t-s') ] \\
&\times E[\sum_{i_1,i_2,i_3} (\hat{X}_{s,b}^{i_1}(k_1) - \hat{X}_{t,b}^{i_1}(k_1)) (\hat{X}_{s,b}^{i_1}(k_1) - \hat{X}_{t,b}^{i_1}(k_1)) ] \, ds \, ds' \\
&\ \\
&\end{align}

where we denoted

\begin{equation}
a_{k_1}^{i_2 i_3 i_3}(t-s) \triangleq \sum_{k_2 \neq 0} e^{-|k_1|^2|t-s|} e^{-|k_2|^2|t-s|} f(ek_2)^2 \\
\times \hat{\rho}^{i_3 i_1}(k_2) \hat{\rho}^{i_2 i_3}(k_2) \hat{\rho}^{i_3 i_3}(k_2). 
\end{equation}
Thus, applying (128) to (127) gives
\[
\mathbb{E}[|\hat{\Delta}^{\epsilon}_{q}(\hat{I}_{5,\epsilon}^{t}-\hat{I}_{5,\epsilon}^{t})|^{2}] \lesssim \sum_{i_1,i_2,i_3,i_4,i'_1,i'_2=1}^{3} \int_{[0,t]^{2}} \sum_{k_{1} \neq 0} \theta(2^{-q}k)^{2} \times |a_{k_{1}}^{i_1,i_2,i_3}(t-s)a_{k_{1}}^{i_4,i'_2,i'_3}(t-\bar{s})| \frac{f(ek_{1})^{2}}{|k_{1}|^{2}} |k_{1}|^{2n} |t-s|^{\frac{n}{2}} |t-\bar{s}|^{\frac{n}{2}}.
\]
Moreover,
\[
|a_{k_{1}}^{i_1,i_2,i_3}(t-s)| \lesssim \sum_{k_{2} \neq 0} e^{-|k_{2}|^{2}(t-s)} |k_{2}|^{2} \lesssim \frac{1}{(t-s)^{1+\frac{n}{2}}}
\]
by (126) and (16). This gives
\[
\sum_{i_1,i_2,i_3,i_4,i'_1,i'_2=1}^{3} \int_{[0,t]^{2}} |a_{k_{1}}^{i_1,i_2,i_3}(t-s)a_{k_{1}}^{i_4,i'_2,i'_3}(t-\bar{s})| |t-s|^{\frac{n}{2}} |t-\bar{s}|^{\frac{n}{2}} ds d\bar{s} \lesssim \int_{[0,t]^{2}} (t-s)^{-\frac{n}{2}-\frac{n}{4}} (t-\bar{s})^{-\frac{n}{2}-\frac{n}{4}} ds d\bar{s} \lesssim t^{n-\epsilon}.
\]
Thus, applying (128) to (127) gives
\[
\mathbb{E}[|\hat{\Delta}^{\epsilon}_{q}(\hat{I}_{5,\epsilon}^{t}-\hat{I}_{5,\epsilon}^{t})|^{2}] \lesssim \sum_{k_{1} \neq 0} \theta(2^{-q}k)^{2} t^{n-\epsilon} |k_{1}|^{2n-2} \approx t^{n-2q(1+2n)}.
\]
Next, for any \(\eta \in (0, 1)\), we estimate within (122),
\[
\mathbb{E}[|\hat{\Delta}^{\epsilon}_{q}(\hat{I}_{5,\epsilon}^{t}-\sum_{i=1}^{3} X_{t,b}^{i} C_{t}^{i})|^{2}]
\]
\[
\lesssim \sum_{k_{1}} \mathbb{E}[|\hat{\Delta}^{\epsilon}_{q}(k_{1})|^{2}] \theta(2^{-q}k_{1})^{2} \sum_{i_1,i_2,i_3=1}^{3} \sum_{k_{2} \neq 0} \int_{0}^{t} e^{-|k_{2}|^{2}(t-s)} \frac{f(ek_{2})^{2}}{|k_{2}|^{2}} \times (e^{-|k_{12}|^{2}(t-s)} k_{12}^{12} \hat{\Delta}_{i_1,i_1}(k_{12}) - e^{-|k_{12}|^{2}(t-s)} k_{12}^{12} \hat{\Delta}_{i_2,i_2}(k_{12}) \hat{\Delta}_{i_3,i_3}(k_{12}) ds)^{2}
\]
\[
\lesssim \sum_{k_{1} \neq 0} \frac{f(ek_{1})^{2}}{|k_{1}|^{2}} \theta(2^{-q}k_{1})^{2} \sum_{k_{2} \neq 0} \int_{0}^{t} e^{-|k_{2}|^{2}(t-s)} \frac{f(ek_{2})^{2}}{|k_{2}|^{2}} |k_{1}|^{2} (t-s)^{-(1-n)} ds^{2}
\]
by (123), Lemma 3.6 and (116). We furthermore estimate for \( \varepsilon \in (0, \eta) \),
\[
\left( \sum_{k_2 \neq 0} \int_0^t \frac{e^{-|k_2|^2(t-s)} f(k_2)^2}{|k_2|^2} (t-s)^{-\frac{3-2\eta}{2}} ds \right)^2 \lesssim t^{\eta-\varepsilon} \quad (131)
\]
by (14). We also estimate
\[
\sum_{k_1 \neq 0} \frac{f(k_1)^2}{|k_1|^{2-2\eta}} \theta(2^{-q}k_1)^2 \lesssim \sum_{k_1 \neq 0} \frac{1}{|k_1|^{3-2\eta}} \theta(2^{-q}k_1)^2 \lesssim 2^{q(1+2\eta)},
\]
applying this and (131) to (130) leads to, together with (129),
\[
\mathbb{E}[|\Delta_q I_{\varepsilon}^1|] \lesssim t^{\eta-\varepsilon} 2^{q(1+2\eta)}, \quad (132)
\]
Similarly we can show \( \sum_{k=2,3,6} \mathbb{E}[|\Delta_q I_{\varepsilon}^1|] \lesssim t^{\eta-\varepsilon} 2^{q(1+2\eta)} \).

3.1.2. Terms in the third chaos. We work on \( I_{\varepsilon}^2 \) of (120):
\[
\mathbb{E}[|\Delta_q I_{\varepsilon}^2|] \approx \sum_{k_1} \theta(2^{-q}k_1)^2 \quad (133)
\]
\[
\times \int_{[0,t]^2} \mathbb{E}[\hat{X}^{e,i_1}_{s,u}(k_1)\hat{X}^{e,i_2}_{s,u}(k_2)\hat{X}^{e,j}_{t,b}(k_3) ;] \\
\times : \hat{X}^{e,i_1'}_{s,u}(k_1)\hat{X}^{e,i_2'}_{s,u}(k_2)\hat{X}^{e,j'}_{t,b}(k_3) ;] b^{i_1'i_2'}_{k_1k_2}(t-s) b^{i_1'i_2'}_{k_1k_2}(t-\tilde{s}) ds d\tilde{s}
\]
where we used (120), (135) and denoted \( b^{i_1'i_2'}_{k_1k_2}(t-s) \equiv e^{-|k_{12}|^2(t-s)} k_{12} \hat{P}^{i_1i_1'}(k_{12}) \). We now rely on Example 2.2 and (110) to write for \( k_1, k_2, k_3 \neq 0 \),
\[
\mathbb{E}[\hat{X}^{e,i_1}_{s,u}(k_1)\hat{X}^{e,i_2}_{s,u}(k_2)\hat{X}^{e,j}_{t,b}(k_3) ;] = [k_{1}+k'=0, k_1 \neq 0] \sum_{l=1}^3 \frac{e^{-|k_{1}|^2s-\tilde{s}}}{2|k_1|^2} f(k_1)^2 \hat{P}^{i_1i_1'}(k_1) \hat{P}^{i_1'i_1'}(k_1)
\]
\[
\times [k_{2}+k''=0, k_2 \neq 0] \sum_{l=1}^3 \frac{e^{-|k_{2}|^2s-\tilde{s}}}{2|k_2|^2} f(k_2)^2 \hat{P}^{i_2i_2'}(k_2) \hat{P}^{i_2'i_2'}(k_2)
\]
\[
\times [k_{3}+k'_3=0, k_3 \neq 0] \sum_{l=1}^3 \frac{e^{-|k_{3}|^2s-\tilde{s}}}{2|k_3|^2} |\hat{P}^{j_3j_3'}(k_3)|^2
\]
\[
+ [k_{1}+k'=0, k_1 \neq 0] \sum_{l=1}^3 \frac{e^{-|k_{1}|^2s-\tilde{s}}}{2|k_1|^2} f(k_1)^2 \hat{P}^{i_1i_1'}(k_1) \hat{P}^{i_1'i_1'}(k_1)
\]
\[
\times [k_{2}+k''=0, k_2 \neq 0] \sum_{l=1}^3 \frac{e^{-|k_{2}|^2s-\tilde{s}}}{2|k_2|^2} f(k_2)^2 \hat{P}^{i_2i_2'}(k_2) \hat{P}^{i_2'i_2'}(k_2)
\]
\[
\times [k_{3}+k'_3=0, k_3 \neq 0] \sum_{l=1}^3 \frac{e^{-|k_{3}|^2s-\tilde{s}}}{2|k_3|^2} f(k_3)^2 \hat{P}^{j_3j_3'}(k_3) \hat{P}^{j_3'j_3'}(k_3)
\]
\[
+ [k_{1}+k'=0, k_1 \neq 0] \sum_{l=1}^3 \frac{e^{-|k_{1}|^2s-\tilde{s}}}{2|k_1|^2} f(k_1)^2 \hat{P}^{i_1i_1'}(k_1) \hat{P}^{i_1'i_1'}(k_1)
\]
\[ \times \left[ 1_{k_2 + k'_2 = 0, k_2 \neq 0} \sum_{l=1}^{3} e^{-|k_2|^2|s - t|} \frac{f(\epsilon k_2)^2}{2|k_2|^2} \hat{P}_{i_2 l} (k_2) \hat{P}_{i_2 l} (k_2) \right] \]

\[ \times \left[ 1_{k_3 + k'_3 = 0, k_3 \neq 0} \sum_{l=1}^{3} \frac{f(\epsilon k_3)^2}{2|k_3|^2} |\hat{P}_{j l} (k_3)|^2 \right] \]

\[ + \left[ 1_{k_2 + k'_2 = 0, k_2 \neq 0} \sum_{l=1}^{3} e^{-|k_2|^2|s - t|} \frac{f(\epsilon k_1)^2}{2|k_1|^2} \hat{P}_{i_1 l} (k_1) \hat{P}_{i_1 l} (k_1) \right] \]

We see that \( III^1 \) and \( III^3 \) may be bounded by a constant multiple of

\[ \prod_{i=1}^{3} \frac{f(\epsilon k_i)^2}{|k_i|^2} e^{-\frac{|k_i|^2 + |k_2|^2|s - t|}{|s - t|}}, \]

while \( III^2 \) by a constant multiple of

\[ \prod_{i=1}^{3} \frac{f(\epsilon k_i)^2}{|k_i|^2} e^{-\frac{|k_i|^2 + |k_2|^2|s - t| - |k_3|^2|t - s|}{|s - t|}}. \]
by switching variables $k_1$ and $k_2$. Similarly switching variables in $III^4$ lead us to an estimate of

$$
\mathbb{E}[|\Delta q I_{t^1, t^2}|^2] \lesssim \sum_{k} \sum_{i_1, i_2} \sum_{i_1', i_2'} \frac{\theta(2^{-q}k)^2}{|k|_t^2} F(k) \int_{[0, t]^2} \prod_{i=1}^{3} \frac{f(\epsilon k_i)^2}{|k_i|} |e^{-(k_1^2 + k_2^2)|s-\tau|} b_{k_{12}}^{i_1, i_2}(t-s) b_{k_{12}}^{i_1', i_2'}(t-\tau) | ds d\tau
$$

(134)

$$
\leq \Pi_{t^1, t^2}^{1, 1} + \Pi_{t^1, t^2}^{1, 2}
$$

We may further estimate for any $\eta \in (0, 1)$,

$$
|b_{k_{12}}^{i_1, i_2}(t-s)| \lesssim \frac{1}{|k_{12}|^{1-\eta}(t-s)^{1-2\eta}}
$$

(135)

by (14). Applying (135) to (134) shows that

$$
\Pi_{t^1, t^2}^{1, 1} \approx \sum_{k} \sum_{i_1, i_2} \sum_{i_1', i_2'} \frac{\theta(2^{-q}k)^2}{|k|_t^2} F(k) \int_{[0, t]^2} \prod_{i=1}^{3} \frac{f(\epsilon k_i)^2}{|k_i|} |e^{-(k_1^2 + k_2^2)|s-\tau|} b_{k_{12}}^{i_1, i_2}(t-s) b_{k_{12}}^{i_1', i_2'}(t-\tau) | ds d\tau
$$

(136)

$$
\lesssim \sum_{k} \theta(2^{-q}k) \sum_{k_1, k_2, k_3 \neq 0; k_12 = k} \frac{1}{|k_i|_t^2 |k_{12}|^{2-2\eta}} t^{\eta}
$$

$$
\lesssim \sum_{k} \theta(2^{-q}k) \frac{t^{\eta}}{|k|_t^{2-2\eta}} \lesssim t^{\eta} 2^{q(1+2\eta)}
$$

(137)

where we used Lemma 1.7. Next,

$$
\Pi_{t^1, t^2}^{1, 2} \approx \sum_{k} \sum_{i_1, i_2} \sum_{i_1', i_2'} \frac{\theta(2^{-q}k)^2}{|k|_t^2} F(k) \int_{[0, t]^2} \prod_{i=1}^{3} \frac{f(\epsilon k_i)^2}{|k_i|} |e^{-(k_1^2 + k_2^2)|s-\tau|} b_{k_{12}}^{i_1, i_2}(t-s) b_{k_{12}}^{i_1', i_2'}(t-\tau) | ds d\tau
$$

(138)

due to (134) and (135). At this point, this is identical to the estimate of $II_{t^1, t^2}$ in (136); thus, it may be bounded by the same bound on $II_{t^1, t^2}$ in (136). Therefore, we now conclude from (134), (132) and (120) that

$$
\mathbb{E}[|\Delta q b_t^{i_1, j}(t_2) u_t^{e_1, i}(t)|^2] \lesssim t^{\eta-\gamma} 2^{q(1+2\eta)}
$$

(139)

for any $t \in (0, 1)$.

Let us now first assume that for $t_1 < t_2$,

$$
\mathbb{E}[|\Delta q (b_t^{i_1, j} u_t^{e_1, i} (t_1) - b_t^{i_1, j} u_t^{e_1, i} (t_2)) - b_t^{i_2, j} u_t^{e_2, i} (t_1) + b_t^{i_2, j} u_t^{e_2, i} (t_2))|^2] \lesssim (\epsilon_1^\gamma + \epsilon_2^\gamma) (t_1 - t_2)^{\eta} 2^{q(1+2\eta)(1+\beta_0)}
$$

(140)

for $\epsilon_1, \epsilon_2 \in (0, \eta)$, $\gamma > 0$ and $\beta_0 \in (0, \frac{1}{2})$ sufficiently small. Now it is clear that

$$
\|f\|_{C^{-\eta(1+\beta_0)-\epsilon}} \lesssim \|f\|_{B_{p,p}^{\eta(1+\beta_0)-\epsilon}} \lesssim \|f\|_{B_{p,p}^{-\eta(1+\beta_0)-\epsilon}}
$$

(141)
by Lemma 4.1. Therefore,

$$E[|b_{1}^{e_{1},i}u_{1}^{e_{1},i}(t_{1}) - b_{1}^{e_{1},j}u_{2}^{e_{1},j}(t_{2}) - b_{1}^{e_{1},j}u_{2}^{e_{1},j}(t_{1}) + b_{1}^{e_{1},j}u_{2}^{e_{1},j}(t_{2})|^{p}]\leq E[\sum_{q\geq 1}2^{q(p-\frac{1}{2}-\eta(1+\beta_{0})-\epsilon)}\|\Delta_{q}(b_{1}^{e_{1},j}u_{2}^{e_{1},i}(t_{1}) - b_{1}^{e_{1},j}u_{2}^{e_{1},i}(t_{2})\|_{L^{p}}]$$

(141)

by (140), Gaussian hypercontractivity [35 Theorem 3.50] and (139). Thus, for every $\eta, \epsilon, \beta$ and $l$ sufficiently small, we may assume that there exists $\nu^{j}_{1,2}$ such that $b_{1}^{e_{1},j}u_{2}^{e_{1},j} \rightarrow \nu^{j}_{1,2}$ as $\epsilon \rightarrow 0$ in $C([0, T]; C^{-\frac{1}{2}, \frac{1}{2}})$ as desired in (133) if $\eta(1+\beta_{0})+\epsilon + \frac{4}{p} \leq \frac{\delta}{2}$; therefore, by taking $p$ sufficiently large and $\eta, \epsilon, \beta > 0$ sufficiently small, we may assume that $\beta > 0$ is arbitrary small. Now to prove (139), we may use that $b_{1}^{e_{1},j}(t_{1})u_{2}^{e_{1},j}(t_{1}) = \sum_{l=1}^{6} I_{l,e}^{j}$ from (120) so that

$$b_{1}^{e_{1},j}u_{1}^{e_{1},i}(t_{1}) - b_{1}^{e_{1},j}u_{2}^{e_{1},j}(t_{1}) - b_{1}^{e_{1},j}u_{2}^{e_{1},j}(t_{1}) + b_{1}^{e_{1},j}u_{2}^{e_{1},j}(t_{2})$$

$$= \left(\sum_{l=1}^{6} II_{l,e}^{1}\right) - \left(\sum_{l=1}^{6} III_{l,e}^{1}\right) - \left(\sum_{l=1}^{6} IV_{l,e}^{1}\right) + \left(\sum_{l=1}^{6} IV_{l,e}^{1}\right).$$

(142)

For brevity we only consider when $l = 5$, and rewrite

$$II_{5,e}^{1} - III_{5,e}^{1} = \sum_{i=1}^{3} x_{t_{1},i}^{1,1} c_{t_{1},i}^{1,1}$$

(143)

$$= \sum_{i=1}^{3} x_{t_{2},i}^{2,1} c_{t_{2},i}^{2,1}$$

$$+ \sum_{i=1}^{3} x_{t_{1},i}^{1,1} c_{t_{1},i}^{1,1}$$

$$- [II_{5,e}^{1}]_{1,2} - [II_{5,e}^{1}]_{2,1} + \sum_{i=1}^{3} x_{t_{1},i}^{1,1} c_{t_{1},i}^{1,1}$$

$$- [II_{5,e}^{1}]_{1,2} - [II_{5,e}^{1}]_{2,1} + \sum_{i=1}^{3} x_{t_{1},i}^{1,1} c_{t_{1},i}^{1,1}$$

$$+ [II_{5,e}^{1}]_{1,2} - [II_{5,e}^{1}]_{2,1} + \sum_{i=1}^{3} x_{t_{1},i}^{1,1} c_{t_{1},i}^{1,1}$$

$$= \sum_{i=1}^{16} IV^{i}$$
as we did in \((12.2)\) and \((12.3)\). For brevity we only consider \(IV^3 + IV^4 + IV^7 + IV^8\); i.e. \((\tilde{I}_{t_1}^\delta - \sum_{i_1=1}^{3} X^{e_{t_1},i_1} C^{e_{t_1},i_1} - (\tilde{I}_{t_2}^\delta - \sum_{i_1=1}^{3} X^{e_{t_2},i_1} C^{e_{t_2},i_1})\). We first compute

\[
\mathbb{E}[\| \sum_{i_1,i_2,i_3=1}^{3} X^{e_{t_1},i_1} (k_1) \theta(2^{-q}k_1) e_{k_1} \| \sum_{i_1,i_2,i_3=1}^{3} X^{e_{t_2},i_1} (k_2) \theta(2^{-q}k_2) e_{k_2} ]
\]

\[
\leq \mathbb{E}[\| \sum_{i_1,i_2,i_3=1}^{3} X^{e_{t_1},i_1} (k_1) \theta(2^{-q}k_1) e_{k_1} \| \sum_{i_1,i_2,i_3=1}^{3} X^{e_{t_2},i_1} (k_2) \theta(2^{-q}k_2) e_{k_2} ]
\]

\[
\times \left[ \sum_{k_2 \neq 0} \int_{0}^{t_1} e^{-|k_2|^2|t_1-s|} f(\epsilon_1 k_2)^2 \hat{P}^{i_2 i_3}(k_2) \hat{P}^{j i_3}(k_2) \right.
\]

\[
\times \left( e^{-|k_2|^2|t_1-s|} k_2^{i_2} \hat{P}^{j i_3}(k_12) - e^{-|k_2|^2|t_1-s|} k_2^{i_2} \hat{P}^{j i_3}(k_12) \right) ds
\]

\[
- \sum_{k_2 \neq 0} \int_{0}^{t_2} e^{-|k_2|^2|t_2-s|} f(\epsilon_1 k_2)^2 \hat{P}^{i_2 i_3}(k_2) \hat{P}^{j i_3}(k_2) \right.
\]

\[
\times \left( e^{-|k_2|^2|t_2-s|} k_2^{i_2} \hat{P}^{j i_3}(k_12) - e^{-|k_2|^2|t_2-s|} k_2^{i_2} \hat{P}^{j i_3}(k_12) \right) ds \right]^{(144)}
\]

\[
+ \mathbb{E}[\| \sum_{i_1,i_2,i_3=1}^{3} X^{e_{t_1},i_1} (k_1) - X^{e_{t_2},i_1} (k_1) \| \sum_{i_1,i_2,i_3=1}^{3} X^{e_{t_2},i_1} (k_2) - X^{e_{t_2},i_1} (k_2) ]
\]

\[
\times \theta(2^{-q}k_1) e_{k_1} \hat{P}^{i_2 i_3}(k_2) \hat{P}^{j i_3}(k_2) \right| \sum_{k_2 \neq 0} \int_{0}^{t_2} e^{-|k_2|^2|t_2-s|} f(\epsilon_1 k_2)^2 \right.
\]

\[
\times \left( e^{-|k_2|^2|t_2-s|} k_2^{i_2} \hat{P}^{j i_3}(k_12) - e^{-|k_2|^2|t_2-s|} k_2^{i_2} \hat{P}^{j i_3}(k_12) \right) ds \right]^{(144)}
\]

by \((12.3)\). Now we have two expectations in \((144)\). For the first expectation in \((144)\), we can simply rewrite it for \(0 \leq t_1 < t_2 \leq T\) as

\[
\mathbb{E}[\| \sum_{i_1,i_2,i_3=1}^{3} X^{e_{t_1},i_1} (k_1) \theta(2^{-q}k_1) e_{k_1} \| \sum_{i_1,i_2,i_3=1}^{3} X^{e_{t_2},i_1} (k_2) \theta(2^{-q}k_2) e_{k_2} ]
\]

\[
\times \left[ \sum_{k_2 \neq 0} \int_{0}^{t_1} e^{-|k_2|^2|t_1-s|} f(\epsilon_1 k_2)^2 \hat{P}^{i_2 i_3}(k_2) \hat{P}^{j i_3}(k_2) \right.
\]

\[
\times \left( e^{-|k_2|^2|t_1-s|} k_2^{i_2} \hat{P}^{j i_3}(k_12) - e^{-|k_2|^2|t_1-s|} k_2^{i_2} \hat{P}^{j i_3}(k_12) \right) ds
\]

\[
- \sum_{k_2 \neq 0} \int_{0}^{t_2} e^{-|k_2|^2|t_2-s|} f(\epsilon_1 k_2)^2 \hat{P}^{i_2 i_3}(k_2) \hat{P}^{j i_3}(k_2) \right.
\]

\[
\times \left( e^{-|k_2|^2|t_2-s|} k_2^{i_2} \hat{P}^{j i_3}(k_12) - e^{-|k_2|^2|t_2-s|} k_2^{i_2} \hat{P}^{j i_3}(k_12) \right) ds \right]^{(145)}
\]

\[
\leq V_{t_1} + V_{t_1}^2 + V_{t_2}^3
\]

where

\[
V_{t_1} \equiv \sum_{k_1 \neq 0} \sum_{i_1,i_2=1}^{3} \frac{1}{|k_1|^2} \theta(2^{-q}k_1)^2 \left( \sum_{k_2 \neq 0} \int_{0}^{t_1} e^{-|k_2|^2|t_1-s|} \frac{1 - e^{-|k_2|^2|t_2-t_1|}}{|k_2|^2} \right)
\]

\[
\times \left( e^{-|k_2|^2|t_1-s|} k_2^{i_2} \hat{P}^{j i_3}(k_12) - e^{-|k_2|^2|t_1-s|} k_2^{i_2} \hat{P}^{j i_3}(k_12) \right) ds \right]^{(146a)}
\]
\[ V_{t_1}^2 \triangleq \sum_{k_1 \neq 0} \sum_{i_1, i_2} \frac{1}{|k_1|^2} \theta(2^{-q}k_1^2) \left( \sum_{k_2 \neq 0} \int_{0}^{t_1} e^{-|k_2|^2(t_2-s)} \right) \]
\[ \times \left( e^{-|k_{12}|^2(t_1-s)} k_{12}^2 \hat{P}_{\theta i i_1}(k_{12}) - e^{-|k_{12}|^2(t_1-s)} k_{12}^2 \hat{P}_{\theta i i_1}(k_2) \right. \]
\[ \left. - e^{-|k_{12}|^2(t_2-s)} k_{12}^2 \hat{P}_{\theta i i_1}(k_{12}) + e^{-|k_2|^2(t_2-s)} k_{12}^2 \hat{P}_{\theta i i_1}(k_2) \right) \right\}^2 \] (146b)

\[ V_{t_1,t_2}^3 \triangleq \sum_{k_1 \neq 0} \sum_{i_1, i_2} \frac{1}{|k_1|^2} \theta(2^{-q}k_1^2) \left( \sum_{k_2 \neq 0} \int_{t_1}^{t_2} e^{-|k_2|^2(t_2-s)} \right) \]
\[ \times \left( e^{-|k_{12}|^2(t_2-s)} k_{12}^2 \hat{P}_{\theta i i_1}(k_{12}) - e^{-|k_{12}|^2(t_2-s)} k_{12}^2 \hat{P}_{\theta i i_1}(k_2) \right) \right\} \right\}^2 \] (146c)

due to (118). On the other hand, the second expectation in (144) may be bounded clearly as follows:

\[ E \left[ \left( \sum_{i_1, i_2=1}^{3} \sum_{k_1} \left( \hat{X}_{t_1,b}^{i_1,i_1}(k_1) - \hat{X}_{t_2,b}^{i_1,i_1}(k_1) \right) \theta(2^{-q}k_1^2) e_{k_1} \hat{P}_{\theta i i_1}(k_2) \hat{P}_{\theta i i_1}(k_2) \right) \times \left( \sum_{k_2 \neq 0} \int_{0}^{t_2} e^{-|k_2|^2(t_2-s)} f(k_2^2) \right) \] (147)
\[ \times \left( e^{-|k_{12}|^2(t_2-s)} k_{12}^2 \hat{P}_{\theta i i_1}(k_{12}) - e^{-|k_{12}|^2(t_2-s)} k_{12}^2 \hat{P}_{\theta i i_1}(k_2) \right) \right\} \right\}^2 \] (147)

where we used that \( \hat{X}_{t_1,b}^{i_1,i_1}(0) - \hat{X}_{t_2,b}^{i_1,i_1}(0) = 0 \). Now on \( V_{t_1}^2 \), we may bound

\[ |e^{-|k_{12}|^2(t_1-s)} k_{12}^2 \hat{P}_{\theta i i_1}(k_{12}) - e^{-|k_{12}|^2(t_1-s)} k_{12}^2 \hat{P}_{\theta i i_1}(k_2)| \]
\[ \leq |e^{-|k_{12}|^2(t_1-s)} k_{12}^2 \hat{P}_{\theta i i_1}(k_{12}) - e^{-|k_{12}|^2(t_1-s)} k_{12}^2 \hat{P}_{\theta i i_1}(k_2)| \]
\[ \leq |e^{-|k_{12}|^2(t_2-s)} k_{12}^2 \hat{P}_{\theta i i_1}(k_{12}) - e^{-|k_{12}|^2(t_2-s)} k_{12}^2 \hat{P}_{\theta i i_1}(k_2)| \] (148)

or we may bound it instead by

\[ |e^{-|k_{12}|^2(t_1-s)} k_{12}^2 \hat{P}_{\theta i i_1}(k_{12}) - e^{-|k_{12}|^2(t_2-s)} k_{12}^2 \hat{P}_{\theta i i_1}(k_{12})| \]
\[ + |e^{-|k_{12}|^2(t_2-s)} k_{12}^2 \hat{P}_{\theta i i_1}(k_{12}) - e^{-|k_{12}|^2(t_2-s)} k_{12}^2 \hat{P}_{\theta i i_1}(k_2)| \] (149)

In the first case of (148) we may bound by

\[ |k|^2 |t_1 - s|^{-\frac{n}{2}} + |k|^2 |t_2 - s|^{-\frac{n}{2}} \lesssim |k|^2 |t_1 - s|^{-\frac{n}{2}} \] (150)

for \( \eta \in (0, 1) \) due to Lemma 1.8. In the second case of (149) we may bound by

\[ |k_{12}| |e^{-|k_{12}|^2(t_1-s)} \hat{P}_{\theta i i_1}(k_{12})| \]
\[ + |k_2| |e^{-|k_{12}|^2(t_2-s)} \hat{P}_{\theta i i_1}(k_2)| \] (151)
by mean value theorem and (14). Applying (148) - (151) to (146b) gives for any $\beta_0 \in (0, 1)$,

$$V_{t_1}^2 \lesssim \sum_{k_1 \neq 0} |k_1|^{2\eta(1-\beta_0)} \frac{\theta(2-\eta k_1)^2}{|k_1|^2} |t_2 - t_1|^2 \beta_0$$

$$\times \left( \sum_{k_2 \neq 0} \frac{1}{|k_2|^2} (|k_2|^{2\eta \beta_0} + |k_2|^{2\eta \beta_0}) \int_0^{t_1} e^{-|k_2|^2(t_s-s)}(t_1 - s)^{\frac{(1-\beta_0)}{2}} ds \right)^2$$

(152)

Furthermore, we can compute

$$\int_0^{t_1} e^{-|k_2|^2(t_2-s)}(t_1-s)^{-\frac{(1-\beta_0)}{2}} ds \lesssim \int_0^{t_1} e^{-|k_2|^2(t_1-s)}(t_1-s)^{-\frac{(1-\beta_0)}{2}} ds \lesssim |k_2|^{-(1+\frac{\beta_0}{2})}$$

by (14). Therefore, we may estimate from (152)

$$V_{t_1}^2 \lesssim |t_2 - t_1|^2 \beta_0 2^g(1+2\eta(1+\beta_0)) \sum_{k_1 \neq 0} \frac{\theta(2-\eta k_1)^2}{|k_1|^2} \lesssim |t_2 - t_1|^2 \beta_0 2^g(1+2\eta(1+\beta_0))$$

(153)

if we choose $\beta_0 < \frac{1}{4}$. Similar estimates may be obtained for $V_{t_1}^1, V_{t_1}^3$ and $V_{t_1}^4$ so that applying these estimates in (142) and (143) lead to

$$\mathbb{E}[\Delta_q (IV^3 + IV^4 + IV^7 + IV^8)]^2 \lesssim |t_2 - t_1|^2 \beta_0 2^g(1+2\eta(1+\beta_0))$$

Through (143) and (142), this finally leads to (159).

3.2. **Group 2:** $u_2^{\epsilon,i} \circ u_2^{\epsilon,j}, b_2^{\epsilon,i} \circ b_2^{\epsilon,j}, b_2^{\epsilon,i} \circ u_2^{\epsilon,j}$. Within the Group 2 of (114), we focus on $b_2^{\epsilon,i} \circ u_2^{\epsilon,j}$ and prove that $b_2^{\epsilon,i} \circ u_2^{\epsilon,j} \rightarrow v_{13}^{\epsilon,j}$ as $\epsilon \rightarrow 0$ in $C([0, T]; C^{\beta})$. Due to (17), (24c), (24d) and relying on the representation of $u_2^{\epsilon,j}(t)$ in (118), we may compute

$$b_2^{\epsilon,i} u_2^{\epsilon,j}(t) = \frac{1}{4(2\pi)^2} \sum_{i_1, i_2, j_1, j_2 = 1} \sum_{k_1, k_2, k_3, k_4, k_{234} = k} \int_{[0,t]^2} e^{-|k_1|^2(t-s)} e^{-|k_2|^2(t-\tau)} \tilde{P}_{i_1 j_1}(k_{12}) \tilde{P}_{j_2 i_2}(k_{234})$$

$$\times [\tilde{X}_{s,b}^{i_1}(k_1) \tilde{X}_{s,u}^{j_2}(k_2) \tilde{X}_{s,u}^{j_1}(k_3) \tilde{X}_{s,b}^{i_2}(k_4)]$$

$$+ [\tilde{X}_{s,b}^{i_1}(k_1) \tilde{X}_{s,b}^{j_2}(k_2) \tilde{X}_{s,u}^{j_1}(k_3) \tilde{X}_{s,u}^{i_2}(k_4)]$$

$$- [\tilde{X}_{s,b}^{i_1}(k_1) \tilde{X}_{s,u}^{j_2}(k_2) \tilde{X}_{s,u}^{j_1}(k_3) \tilde{X}_{s,b}^{i_2}(k_4)]$$

$$- [\tilde{X}_{s,u}^{i_1}(k_1) \tilde{X}_{s,b}^{j_2}(k_2) \tilde{X}_{s,u}^{j_1}(k_3) \tilde{X}_{s,u}^{i_2}(k_4)]$$

(154)

Now we rely on (13d) and (13e) to rewrite (154) as

$$b_2^{\epsilon,i} u_2^{\epsilon,j}(t) = V I_{t}^1 + V I_{t}^2 + V I_{t}^3$$

(155)
where

$$V I_1^1 \triangleq \frac{1}{4(2\pi)^2} \sum_{i_1, i_2, j_1, j_2 = 1}^{3} \sum_{k_1, k_2, k_3, k_4 = 1}^{3} \sum_{k = 1}^{4} e^{-|k_{12}|^2(t-s)-|k_{34}|^2(t-s)} \hat{P}^{i_1i_2} (k_{12}) \hat{P}^{j_1j_2} (k_{34}) i k_{12}^2 i k_{34}^2 \times \int_{[0,t]^2} e^{-|k_{12}|^2(t-s)-|k_{34}|^2(t-s)} \hat{P}^{i_1i_2} (k_{12}) \hat{P}^{j_1j_2} (k_{34}) i k_{12}^2 i k_{34}^2 \times [ \hat{X}^{\epsilon, i_1}_{s,b} (k_1) \hat{X}^{\epsilon, i_2}_{s,u} (k_2) \hat{X}^{\epsilon, j_1}_{\Sigma} (k_3) \hat{X}^{\epsilon, j_2}_{\Sigma} (k_4) ] : ds ds \hat{e}_k, \quad (156)$$
$$V I^2_j \triangleq \frac{1}{4(2\pi)^{\frac{3}{2}}} \sum_{i_1, i_2, j_1, j_2 = 1}^{3} \sum_{k_1, k_2, k_3, k_4 : k_1 k_2 k_3 k_4 = k} \int_{[0, t]^2} e^{-|k_1 z|^2(t - s) - |k_3 z|^2(t - \tau)} \hat{\mathbf{p}}^{i_1 i_2}(k_{12}) \hat{\mathbf{p}}^{j_1 j_2}(k_{34}) i k_{12}^{i_1} i k_{34}^{j_2} \times \left[ E \left[ \hat{X}^{i_1}_{s, b}(k_1) \hat{X}^{i_2}_{s, u}(k_2) \right] : \hat{X}^{j_1}_{s, u}(k_3) \hat{X}^{j_2}_{s, b}(k_4) \right] + E \left[ \hat{X}^{i_1}_{s, b}(k_1) \hat{X}^{i_2}_{s, u}(k_3) \right] : \hat{X}^{j_1}_{s, u}(k_2) \hat{X}^{j_2}_{s, b}(k_4) + E \left[ \hat{X}^{i_1}_{s, u}(k_1) \hat{X}^{i_2}_{s, b}(k_2) \right] : \hat{X}^{j_1}_{s, u}(k_3) \hat{X}^{j_2}_{s, b}(k_4) + E \left[ \hat{X}^{i_1}_{s, u}(k_1) \hat{X}^{i_2}_{s, b}(k_2) \right] : \hat{X}^{j_1}_{s, u}(k_3) \hat{X}^{j_2}_{s, b}(k_4)$$

(157)
\[ V I_3^2 = \frac{1}{4(2\pi)^2} \sum_{i_{12}, i_{34} = 1} 3 \sum_k \sum_{k_{12}, k_{34}}^{k_{12}, k_{34}} e^{-|k_{12}|^2(t-s) - |k_{34}|^2(t-\tau)} \hat{P}_{i_{12}j_{12}}(k_{12}) \hat{P}_{j_{12}j_{12}}(k_{34}) i_{12} k_{12} i_{34} k_{34} \]

\[ \times \int_{[0, t]^2} e^{-(|k_{12}|^2(t-s) - |k_{34}|^2(t-\tau))} \hat{P}_{i_{12}j_{12}}(k_{12}) \hat{P}_{j_{12}j_{12}}(k_{34}) i_{12} k_{12} i_{34} k_{34} \]

\[ \times \left[ E[\tilde{X}_{s, t}^{i_{12}}(k_{2}) \tilde{X}_{s, t}^{j_{12}}(k_{3})] E[\tilde{X}_{s, t}^{i_{12}}(k_{1}) \tilde{X}_{s, t}^{j_{12}}(k_{4})] + E[\tilde{X}_{s, t}^{i_{12}}(k_{2}) \tilde{X}_{s, t}^{j_{12}}(k_{4})] E[\tilde{X}_{s, t}^{i_{12}}(k_{1}) \tilde{X}_{s, t}^{j_{12}}(k_{3})] + E[\tilde{X}_{s, t}^{i_{12}}(k_{3}) \tilde{X}_{s, t}^{j_{12}}(k_{4})] E[\tilde{X}_{s, t}^{i_{12}}(k_{1}) \tilde{X}_{s, t}^{j_{12}}(k_{2})] - E[\tilde{X}_{s, t}^{i_{12}}(k_{2}) \tilde{X}_{s, t}^{j_{12}}(k_{3})] E[\tilde{X}_{s, t}^{i_{12}}(k_{1}) \tilde{X}_{s, t}^{j_{12}}(k_{4})] - E[\tilde{X}_{s, t}^{i_{12}}(k_{2}) \tilde{X}_{s, t}^{j_{12}}(k_{4})] E[\tilde{X}_{s, t}^{i_{12}}(k_{1}) \tilde{X}_{s, t}^{j_{12}}(k_{3})] - E[\tilde{X}_{s, t}^{i_{12}}(k_{3}) \tilde{X}_{s, t}^{j_{12}}(k_{4})] E[\tilde{X}_{s, t}^{i_{12}}(k_{1}) \tilde{X}_{s, t}^{j_{12}}(k_{2})] - E[\tilde{X}_{s, t}^{i_{12}}(k_{2}) \tilde{X}_{s, t}^{j_{12}}(k_{3})] E[\tilde{X}_{s, t}^{i_{12}}(k_{1}) \tilde{X}_{s, t}^{j_{12}}(k_{4})] + E[\tilde{X}_{s, t}^{i_{12}}(k_{2}) \tilde{X}_{s, t}^{j_{12}}(k_{4})] E[\tilde{X}_{s, t}^{i_{12}}(k_{1}) \tilde{X}_{s, t}^{j_{12}}(k_{3})] + E[\tilde{X}_{s, t}^{i_{12}}(k_{3}) \tilde{X}_{s, t}^{j_{12}}(k_{4})] E[\tilde{X}_{s, t}^{i_{12}}(k_{1}) \tilde{X}_{s, t}^{j_{12}}(k_{2})] + E[\tilde{X}_{s, t}^{i_{12}}(k_{2}) \tilde{X}_{s, t}^{j_{12}}(k_{3})] E[\tilde{X}_{s, t}^{i_{12}}(k_{1}) \tilde{X}_{s, t}^{j_{12}}(k_{4})] + E[\tilde{X}_{s, t}^{i_{12}}(k_{3}) \tilde{X}_{s, t}^{j_{12}}(k_{2})] E[\tilde{X}_{s, t}^{i_{12}}(k_{1}) \tilde{X}_{s, t}^{j_{12}}(k_{4})] + E[\tilde{X}_{s, t}^{i_{12}}(k_{2}) \tilde{X}_{s, t}^{j_{12}}(k_{3})] E[\tilde{X}_{s, t}^{i_{12}}(k_{1}) \tilde{X}_{s, t}^{j_{12}}(k_{2})] + E[\tilde{X}_{s, t}^{i_{12}}(k_{3}) \tilde{X}_{s, t}^{j_{12}}(k_{2})] E[\tilde{X}_{s, t}^{i_{12}}(k_{1}) \tilde{X}_{s, t}^{j_{12}}(k_{2})] + E[\tilde{X}_{s, t}^{i_{12}}(k_{2}) \tilde{X}_{s, t}^{j_{12}}(k_{3})] E[\tilde{X}_{s, t}^{i_{12}}(k_{1}) \tilde{X}_{s, t}^{j_{12}}(k_{2})] + E[\tilde{X}_{s, t}^{i_{12}}(k_{3}) \tilde{X}_{s, t}^{j_{12}}(k_{2})] E[\tilde{X}_{s, t}^{i_{12}}(k_{1}) \tilde{X}_{s, t}^{j_{12}}(k_{2})] + E[\tilde{X}_{s, t}^{i_{12}}(k_{2}) \tilde{X}_{s, t}^{j_{12}}(k_{3})] E[\tilde{X}_{s, t}^{i_{12}}(k_{1}) \tilde{X}_{s, t}^{j_{12}}(k_{2})] + E[\tilde{X}_{s, t}^{i_{12}}(k_{3}) \tilde{X}_{s, t}^{j_{12}}(k_{2})] E[\tilde{X}_{s, t}^{i_{12}}(k_{1}) \tilde{X}_{s, t}^{j_{12}}(k_{2})] + E[\tilde{X}_{s, t}^{i_{12}}(k_{2}) \tilde{X}_{s, t}^{j_{12}}(k_{3})] E[\tilde{X}_{s, t}^{i_{12}}(k_{1}) \tilde{X}_{s, t}^{j_{12}}(k_{2})] + E[\tilde{X}_{s, t}^{i_{12}}(k_{3}) \tilde{X}_{s, t}^{j_{12}}(k_{2})] E[\tilde{X}_{s, t}^{i_{12}}(k_{1}) \tilde{X}_{s, t}^{j_{12}}(k_{2})]) ds dx \]

For \( V I_3^2 \) in (168), there are 12 terms; however, 3rd, 6th, 9th and 12th terms vanish due to \( k_{12} = 0, k_{34} = 0 \) and \( i_{12} k_{12} i_{34} k_{34} \) within the integrand. Therefore,

\[ V I_3^2 = \frac{1}{4(2\pi)^2} \sum_{i_{12}, i_{34} = 1} 3 \sum_k \sum_{k_{12}, k_{34}}^{k_{12}, k_{34}} e^{-|k_{12}|^2(t-s) - |k_{34}|^2(t-\tau)} \hat{P}_{i_{12}j_{12}}(k_{12}) \hat{P}_{j_{12}j_{12}}(k_{34}) i_{12} k_{12} i_{34} k_{34} \]

\[ \times \int_{[0, t]^2} e^{-(|k_{12}|^2(t-s) - |k_{34}|^2(t-\tau))} \hat{P}_{i_{12}j_{12}}(k_{12}) \hat{P}_{j_{12}j_{12}}(k_{34}) i_{12} k_{12} i_{34} k_{34} \]

\[ \times \left[ \hat{P}_{i_{12}j_{12}}(k_{2}) \hat{P}_{j_{12}j_{12}}(k_{3}) \hat{P}_{i_{12}j_{12}}(k_{1}) \hat{P}_{j_{12}j_{12}}(k_{4}) \right] ds dx \]

\[ \times \sum_{j_{13}, j_{13} = 1} 3 \left[ \hat{P}_{i_{12}j_{13}}(k_{2}) \hat{P}_{j_{12}j_{13}}(k_{3}) \hat{P}_{i_{12}j_{13}}(k_{1}) \hat{P}_{j_{12}j_{13}}(k_{4}) \right] ds dx \]

by (168). We need to also work on \( V I_2^2 \) of (157). There are 24 terms; 1st, 6th, 7th, 12th, 13th, 18th, 19th, 24th terms all vanish due to \( k_{12} = 0, k_{34} = 0 \) and \( i_{12} k_{12} i_{34} k_{34} \).
within the integrand. Therefore,

\begin{align*}
V T_i^2 &= \frac{1}{4(2\pi)^2} \sum_{i_1, i_2; j_1, j_2 = 1}^3 \sum_k \sum_{k_1, k_2, k_3, k_4, k_{1234} = k} \\
&\times \int_{[0, t]^2} e^{-|k_{12}|^2(t-s) - |k_{34}|^2(t-\tau)} \hat{p}^{i_1 j_1}(k_{12}) \hat{p}^{i_2 j_2}(k_{34}) i_{12} i_{34} k_{1234}^2 \\
&\times \left[ (1_{k_{12}=0, k_1 \neq 0} \sum_{j_5 = 1}^3 e^{-|k_1|^2 |s-\tau|} f(\epsilon k_1)^2 \hat{p}^{i_1 j_5}(k_1) \hat{p}^{j_5 s_2}(k_1) : \hat{X}^{i_1 j_2}(k_2) \hat{X}^{j_2 s_2}(k_4) : \\
+ (1_{k_{12}=0, k_2 \neq 0} \sum_{j_5 = 1}^3 e^{-|k_2|^2 |s-\tau|} f(\epsilon k_2)^2 \hat{p}^{i_2 j_5}(k_2) \hat{p}^{j_5 s_2}(k_2) : \hat{X}^{i_1 j_1}(k_1) \hat{X}^{j_1 s_2}(k_3) : \\
+ (1_{k_{12}=0, k_1 \neq 0} \sum_{j_5 = 1}^3 e^{-|k_2|^2 |s-\tau|} f(\epsilon k_1)^2 \hat{p}^{i_1 j_5}(k_1) \hat{p}^{j_5 s_2}(k_1) : \hat{X}^{i_1 j_1}(k_1) \hat{X}^{j_1 s_2}(k_3) : \\
+ (1_{k_{12}=0, k_2 \neq 0} \sum_{j_5 = 1}^3 e^{-|k_2|^2 |s-\tau|} f(\epsilon k_1)^2 \hat{p}^{i_1 j_5}(k_1) \hat{p}^{j_5 s_2}(k_1) : \hat{X}^{i_1 j_1}(k_1) \hat{X}^{j_1 s_2}(k_3) : \\
- (1_{k_{12}=0, k_1 \neq 0} \sum_{j_5 = 1}^3 e^{-|k_1|^2 |s-\tau|} f(\epsilon k_1)^2 \hat{p}^{i_1 j_5}(k_1) \hat{p}^{j_5 s_2}(k_1) : \hat{X}^{i_1 j_1}(k_1) \hat{X}^{j_1 s_2}(k_3) : \\
+ (1_{k_{12}=0, k_2 \neq 0} \sum_{j_5 = 1}^3 e^{-|k_2|^2 |s-\tau|} f(\epsilon k_1)^2 \hat{p}^{i_1 j_5}(k_1) \hat{p}^{j_5 s_2}(k_1) : \hat{X}^{i_1 j_1}(k_1) \hat{X}^{j_1 s_2}(k_3) : \\
+ (1_{k_{12}=0, k_1 \neq 0} \sum_{j_5 = 1}^3 e^{-|k_2|^2 |s-\tau|} f(\epsilon k_1)^2 \hat{p}^{i_1 j_5}(k_1) \hat{p}^{j_5 s_2}(k_1) : \hat{X}^{i_1 j_1}(k_1) \hat{X}^{j_1 s_2}(k_3) : \\
+ (1_{k_{12}=0, k_2 \neq 0} \sum_{j_5 = 1}^3 e^{-|k_2|^2 |s-\tau|} f(\epsilon k_1)^2 \hat{p}^{i_1 j_5}(k_1) \hat{p}^{j_5 s_2}(k_1) : \hat{X}^{i_1 j_1}(k_1) \hat{X}^{j_1 s_2}(k_3) : \\
+ (1_{k_{12}=0, k_1 \neq 0} \sum_{j_5 = 1}^3 e^{-|k_1|^2 |s-\tau|} f(\epsilon k_1)^2 \hat{p}^{i_1 j_5}(k_1) \hat{p}^{j_5 s_2}(k_1) : \hat{X}^{i_1 j_1}(k_1) \hat{X}^{j_1 s_2}(k_3) : \\
+ (1_{k_{12}=0, k_2 \neq 0} \sum_{j_5 = 1}^3 e^{-|k_2|^2 |s-\tau|} f(\epsilon k_1)^2 \hat{p}^{i_1 j_5}(k_1) \hat{p}^{j_5 s_2}(k_1) : \hat{X}^{i_1 j_1}(k_1) \hat{X}^{j_1 s_2}(k_3) : \\
+ (1_{k_{12}=0, k_1 \neq 0} \sum_{j_5 = 1}^3 e^{-|k_1|^2 |s-\tau|} f(\epsilon k_1)^2 \hat{p}^{i_1 j_5}(k_1) \hat{p}^{j_5 s_2}(k_1) : \hat{X}^{i_1 j_1}(k_1) \hat{X}^{j_1 s_2}(k_3) : \\
+ (1_{k_{12}=0, k_2 \neq 0} \sum_{j_5 = 1}^3 e^{-|k_2|^2 |s-\tau|} f(\epsilon k_1)^2 \hat{p}^{i_1 j_5}(k_1) \hat{p}^{j_5 s_2}(k_1) : \hat{X}^{i_1 j_1}(k_1) \hat{X}^{j_1 s_2}(k_3) : \\
+ (1_{k_{12}=0, k_1 \neq 0} \sum_{j_5 = 1}^3 e^{-|k_1|^2 |s-\tau|} f(\epsilon k_1)^2 \hat{p}^{i_1 j_5}(k_1) \hat{p}^{j_5 s_2}(k_1) : \hat{X}^{i_1 j_1}(k_1) \hat{X}^{j_1 s_2}(k_3) : \\
+ (1_{k_{12}=0, k_2 \neq 0} \sum_{j_5 = 1}^3 e^{-|k_2|^2 |s-\tau|} f(\epsilon k_1)^2 \hat{p}^{i_1 j_5}(k_1) \hat{p}^{j_5 s_2}(k_1) : \hat{X}^{i_1 j_1}(k_1) \hat{X}^{j_1 s_2}(k_3) : \\
\right] \\
\times \text{d} s d e_k \overset{\Delta}{=} \sum_{k = 1}^{16} V T_i^{2, k}.
\end{align*}
We may furthermore write

\[ VI_{t}^{2,1} = \frac{1}{4(2\pi)^{2}} \sum_{i_1, i_2, j_1, j_2 = 1}^{3} \sum_{k, k_2, k_4 = k, k_1 \neq 0}^{3} \sum_{k_1, k_4 - k_1} \int_{[0,t]^2} e^{-|k_1|^2(t-s) - |k_4 - k_1|^2(t-\tau)} \hat{\rho}_{i_1 i_2}^{j_1} (k_2) \hat{\rho}_{j_1 j_2}^{i_1} (k_4 - k_1) \hat{\rho}_{i_1 i_2}^{j_1} (k_4 - k_1) ik_1^{j_1} (k_4 - k_1) ik_1^{j_2} (k_4 - k_1) \]

by (100) and

\[ VI_{t}^{2,2} = \frac{1}{4(2\pi)^{2}} \sum_{i_1, i_2, j_1, j_2 = 1}^{3} \sum_{k, k_2, k_4 = k, k_1 \neq 0}^{3} \sum_{k_1, k_4 - k_1} \int_{[0,t]^2} e^{-|k_1|^2(t-s) - |k_4 - k_1|^2(t-\tau)} \hat{\rho}_{i_1 i_2}^{j_1} (k_2) \hat{\rho}_{j_1 j_2}^{i_1} (k_4 - k_1) \hat{\rho}_{i_1 i_2}^{j_1} (k_4 - k_1) ik_1^{j_1} (k_4 - k_1) ik_1^{j_2} (k_4 - k_1) \]

\[ \times 1_{i_5 = i_2, i_4 = i_1, j_5 = j_2, j_4 = j_1} \]
by (160) and a change of variable $k_3$ to $k_4$. We may repeat a similar procedure for $V_{1}^{2,3}, \ldots, V_{1}^{2,16}$ in (160) to deduce

\[
VI_t^2 = \frac{1}{4(2\pi)^2} \sum_{i_1, i_2, j_1, j_2=1}^{3} \sum_{k_2, k_4, k_24 = k, k_1 \neq 0} \int_{[0,1]^2} e^{-|k_{12}|^2 (t-s) - |k_4-k_{12}|^2 (t-s)} 
\times \hat{\mathcal{P}}^{i_{12}}(k_{12}) \hat{\mathcal{P}}^{j_{12}}(k_4 - k_1) i k_{12}^2 i (k_{12}^2 - k_{12}^2) 
\times \sum_{j_3=1}^{3} e^{-|k_1|^2 (t-s)} f(k_1)^2 \hat{\mathcal{P}}^{i_{12} j_3}(k_1) \hat{\mathcal{P}}^{j_{12} j_3}(k_4) : \hat{X}^{i_{12} j_3}(k_2) \hat{X}^{j_{12} j_3}(k_4) : dsd\xi_k
\]
\[
\times \left[ 1_{i_3=i_2, j_3=j_2, j_4=j_1} + 1_{i_3=i_2, i_4=i_1, j_3=j_1, j_4=j_2} - 1_{i_3=i_1, i_4=i_2, j_3=j_2, j_4=j_1} - 1_{i_3=i_1, i_4=i_2, j_3=j_1, j_4=j_2} \right]
\]
\[
+ \frac{1}{4(2\pi)^2} \sum_{i_1, i_2, j_1, j_2=1}^{3} \sum_{k_2, k_4, k_24 = k, k_1 \neq 0} \int_{[0,1]^2} e^{-|k_{12}|^2 (t-s) - |k_4-k_{12}|^2 (t-s)} 
\times \hat{\mathcal{P}}^{i_{12}}(k_{12}) \hat{\mathcal{P}}^{j_{12}}(k_4 - k_1) i k_{12}^2 i (k_{12}^2 - k_{12}^2) 
\times \sum_{j_3=1}^{3} e^{-|k_1|^2 (t-s)} f(k_1)^2 \hat{\mathcal{P}}^{i_{12} j_3}(k_1) \hat{\mathcal{P}}^{j_{12} j_3}(k_4) : \hat{X}^{i_{12} j_3}(k_2) \hat{X}^{j_{12} j_3}(k_4) : dsd\xi_k
\]
\[
\times \left[ 1_{i_3=i_2, i_4=i_1, j_3=j_2, j_4=j_1} + 1_{i_3=i_2, i_4=i_1, j_3=j_1, j_4=j_2} - 1_{i_3=i_1, i_4=i_2, j_3=j_2, j_4=j_1} - 1_{i_3=i_1, i_4=i_2, j_3=j_1, j_4=j_2} \right]
\]
\[
+ \frac{1}{4(2\pi)^2} \sum_{i_1, i_2, j_1, j_2=1}^{3} \sum_{k_2, k_4, k_24 = k, k_1 \neq 0} \int_{[0,1]^2} e^{-|k_{12}|^2 (t-s) - |k_4-k_{12}|^2 (t-s)} 
\times \hat{\mathcal{P}}^{i_{12}}(k_{12}) \hat{\mathcal{P}}^{j_{12}}(k_4 - k_1) i k_{12}^2 i (k_{12}^2 - k_{12}^2) 
\times \sum_{j_3=1}^{3} e^{-|k_1|^2 (t-s)} f(k_1)^2 \hat{\mathcal{P}}^{i_{12} j_3}(k_1) \hat{\mathcal{P}}^{j_{12} j_3}(k_4) : \hat{X}^{i_{12} j_3}(k_2) \hat{X}^{j_{12} j_3}(k_4) : dsd\xi_k
\]
\[
\times \left[ -1_{i_3=i_2, i_4=i_1, j_3=j_2, j_4=j_1} - 1_{i_3=i_2, i_4=i_1, j_3=j_1, j_4=j_2} + 1_{i_3=i_1, i_4=i_2, j_3=j_2, j_4=j_1} + 1_{i_3=i_1, i_4=i_2, j_3=j_1, j_4=j_2} \right]
\]
\[
+ \frac{1}{4(2\pi)^2} \sum_{i_1, i_2, j_1, j_2=1}^{3} \sum_{k_2, k_4, k_24 = k, k_1 \neq 0} \int_{[0,1]^2} e^{-|k_{12}|^2 (t-s) - |k_4-k_{12}|^2 (t-s)} 
\times \hat{\mathcal{P}}^{i_{12}}(k_{12}) \hat{\mathcal{P}}^{j_{12}}(k_4 - k_1) i k_{12}^2 i (k_{12}^2 - k_{12}^2) 
\times \sum_{j_3=1}^{3} e^{-|k_1|^2 (t-s)} f(k_1)^2 \hat{\mathcal{P}}^{i_{12} j_3}(k_1) \hat{\mathcal{P}}^{j_{12} j_3}(k_4) : \hat{X}^{i_{12} j_3}(k_2) \hat{X}^{j_{12} j_3}(k_4) : dsd\xi_k
\]
\[
\times \left[ -1_{i_3=i_2, i_4=i_1, j_3=j_2, j_4=j_1} - 1_{i_3=i_2, i_4=i_1, j_3=j_1, j_4=j_2} + 1_{i_3=i_1, i_4=i_2, j_3=j_2, j_4=j_1} + 1_{i_3=i_1, i_4=i_2, j_3=j_1, j_4=j_2} \right]
\]
\[
+ 1_{i_3=i_2, i_4=i_1, j_3=j_2, j_4=j_1} + 1_{i_3=i_2, i_4=i_1, j_3=j_1, j_4=j_2} \triangleq \sum_{i=1}^{16} VI_t^2.
\]

Finally, from (169) we define

\[
VI_t^2 \triangleq C_{2,3}^{i,j}.
\]

3.2.1. Terms in the second chaos. In order to estimate $E[|\Delta_q V_{1}^2|^2]$, we consider only $VI_t^{15}$ within (161) as others are similarly estimated. We may rely on Example
to write

$$
E[: \hat{X}_{\sigma,b}(k_2) \hat{X}_{\pi,b}(k_4) :] = 1_{k_2 + k'_2 = 0, k_4 + k'_4 = 0, k_2, k_4 \neq 0} 
\sum_{i_3,i'_3 = 1} 3 \frac{e^{-|k_2|^2 |s-\sigma|}}{2|k_2|^2} \times \hat{P}_{i_3 i'_3} (k_2) \hat{P}_{i_3 i'_3} (k_4) e^{-|k_4|^2 |s-\sigma|} f(\epsilon k_2) \frac{f(\epsilon k_4)}{2|k_4|^2} \hat{P}_{i_3 i'_3} (k_4) \hat{P}_{i_3 i'_3} (k_4)
$$

(163)

so that from (161) we obtain

$$
E[|\Delta_4 VI l_{15}^t|^2] \leq \sum_k \theta(2^{-q} k)^2 \sum_{k_2,k_4 \neq 0, k_2 = k, k_1 = k_2 \neq k, k_4 \neq k \neq 0} |e^{-|k_2|^2 (t-s) - |k_4 - k_1|^2 (t-\pi)} - |k'_2 - k_2|^2 (t-\sigma) - |k'_4 - k'_3|^2 (t-\pi)|
\sum_{i_3,i'_3 = 1} 3 \frac{e^{-|k_2|^2 |s-\sigma|}}{2|k_2|^2} \frac{e^{-|k_4|^2 |s-\sigma|}}{2|k_4|^2} \frac{1}{|k_1|^2 |k_2|^2 |k_3|^2 |k_4|^2} dsd\sigma d\delta d\sigma
$$

(164)

where we denoted $k'_1 = k'_2 + k'_3$. Considering the term that is multiplied by the first characteristic function $1_{k_2 + k'_2 = 0, k_4 + k'_4 = 0}$, we see that it may be estimated by

$$
\sum_k \theta(2^{-q} k)^2 \sum_{k_2,k_4 \neq 0, k_2 = k, k_1 = k_2 \neq k, k_4 \neq k \neq 0} |e^{-|k_2|^2 (t-s) - |k_4 - k_1|^2 (t-\pi)} - |k'_2 - k_2|^2 (t-\sigma) - |k'_4 - k'_3|^2 (t-\pi)|
\sum_{i_3,i'_3 = 1} 3 \frac{e^{-|k_2|^2 |s-\sigma|}}{2|k_2|^2} \frac{e^{-|k_4|^2 |s-\sigma|}}{2|k_4|^2} \frac{1}{|k_1|^2 |k_2|^2 |k_3|^2 |k_4|^2} dsd\sigma d\delta d\sigma
\sum_{k_2,k_4 \neq 0, k_2 = k, k_1 = k_2 \neq k, k_4 \neq k \neq 0} \theta(2^{-q} k)^2 \frac{1 - e^{-|k_2|^2 t}}{|k_2|^2} \frac{1 - e^{-|k_4 - k_1|^2 t}}{|k_4 - k_1|^2}
\sum_{|k_2|^2 |k_2|^2 |k_3|^2 |k_4|^2} 1_{k_2 \neq 0, k_4 - k_1 \neq 0, k_2 \neq 0, k_4 - k_3 \neq 0}
\sum_{|k_2|^2 |k_2|^2 |k_3|^2 |k_4|^2} \theta(2^{-q} k)^2 \prod_{j=1}^4 \frac{1}{|k_j|^2 |k_4 - k_1|^2 - \epsilon |k_4 - k_3|^2 - \epsilon}
\leq \epsilon^2 2^{2q} \sum_{k \neq 0} \theta(2^{-q} k)^2 \frac{1}{|k|^3} \leq \epsilon^2 2^{2q}
$$

(165)
where we used a change of variable $k'_1$ with $-k_3$, mean value theorem and Lemma 4.7.

3.2.2. Terms in the fourth chaos. We wish to estimate

$$\mathbb{E}[|\Delta \nabla I_t^1|^2] = \mathbb{E}[(\sum_k \theta(2^{-q}k)\tilde{V}I_t^1(k)e_k)^2]$$ (166)

where $\nabla I_t^1$ is that of (156) of which it suffices to estimate for example a mix term such as second and third terms multiplied; i.e.

$$\mathbb{E}[|\sum_k \theta(2^{-q}k) \sum_{i_1,i_2,j_1,j_2=1}^3 k_1,k_2,k_3,k_4,k_{1234}=k \int [0,t]^2 e^{-|k_{12}|^2(t-s)-|k_{34}|^2(t-\sigma)}$$
$$\times :X^e_{\sigma,b} (k_1) X^e_{\sigma,b} (k_2) \tilde{X}^e_{\tau,u} (k_3) \tilde{X}^e_{\tau,u} (k_4) : dsd\sigma e_k$$
$$\times : \hat{P}^{i_1i_2} (k_{12}) \hat{P}^{j_1j_2} (k_{34}) i_k j_l k_{1234} |$$

$$\times \int |k_{12}|^2(t-s)-|k_{34}|^2(t-\sigma)$$

$$\mathbb{E}[|\sum_k \theta(2^{-q}k') \sum_{i_1',i_2',j_1',j_2'=1}^3 k_{1}'=k_{12}',k_{2}'=k_{34}',k_{1234}'=k' \int [0,t]^2 e^{-|k_{12}'|^2(t-s)-|k_{34}'|^2(t-\sigma)}$$
$$\times :X^e_{\sigma,b} (k_{1}') X^e_{\sigma,b} (k_{2}') \tilde{X}^e_{\tau,u} (k_{3}') \tilde{X}^e_{\tau,u} (k_{4}') : dsd\sigma e_k'$$
$$\times : \hat{P}^{i_1'i_2'} (k_{12}') \hat{P}^{j_1'j_2'} (k_{34}') i'(k_{12}') j_l' k_{1234}' |$$

$$\times |k_{12}'|^2(t-s)-|k_{34}'|^2(t-\sigma)$$

$$\times |k_{12}'|^2(t-s)-|k_{34}'|^2(t-\sigma)$$

$$\times |k_{12}|^2(t-s)-|k_{34}|^2(t-\sigma)$$

$$(167)$$
We can compute by Example 2.2 that

\[
E[\tilde{X}_{s,u}^{c,i,s}(k_1)\tilde{X}_{s,u}^{c,i,s}(k_2)\tilde{X}_{s,u}^{c,j,u}(k_3)\tilde{X}_{s,u}^{c,j,u}(k_4)]:
\]

\[
\times: \tilde{X}_{s,b}^{c,i,b}(k_1)\tilde{X}_{s,b}^{c,i,b}(k_2)\tilde{X}_{s,b}^{c,j,b}(k_3)\tilde{X}_{s,b}^{c,j,b}(k_4)
\]

\[
= E[\tilde{X}_{s,u}^{c,i,s}(k_1)\tilde{X}_{s,u}^{c,i,s}(k_2)]
\]

\[
\times (E[\tilde{X}_{s,b}^{c,i,b}(k_2)\tilde{X}_{s,b}^{c,i,b}(k_3)]E[\tilde{X}_{s,u}^{c,j,u}(k_3)\tilde{X}_{s,u}^{c,j,u}(k_4)]E[\tilde{X}_{s,u}^{c,j,u}(k_4)\tilde{X}_{s,b}^{c,j,b}(k_4)]
\]

\[
+ E[\tilde{X}_{s,b}^{c,i,b}(k_2)\tilde{X}_{s,b}^{c,i,b}(k_4)]E[\tilde{X}_{s,u}^{c,j,u}(k_3)\tilde{X}_{s,u}^{c,j,u}(k_4)]E[\tilde{X}_{s,u}^{c,j,u}(k_4)\tilde{X}_{s,b}^{c,j,b}(k_4)]
\]

\[
+ E[\tilde{X}_{s,b}^{c,i,b}(k_2)\tilde{X}_{s,b}^{c,i,b}(k_3)]E[\tilde{X}_{s,u}^{c,j,u}(k_3)\tilde{X}_{s,u}^{c,j,u}(k_4)]E[\tilde{X}_{s,u}^{c,j,u}(k_4)\tilde{X}_{s,b}^{c,j,b}(k_4)]
\]

\[
+ E[\tilde{X}_{s,b}^{c,i,b}(k_2)\tilde{X}_{s,b}^{c,i,b}(k_3)]E[\tilde{X}_{s,u}^{c,j,u}(k_3)\tilde{X}_{s,u}^{c,j,u}(k_4)]E[\tilde{X}_{s,u}^{c,j,u}(k_4)\tilde{X}_{s,b}^{c,j,b}(k_4)]
\]

\[
+ E[\tilde{X}_{s,b}^{c,i,b}(k_2)\tilde{X}_{s,b}^{c,i,b}(k_3)]E[\tilde{X}_{s,u}^{c,j,u}(k_3)\tilde{X}_{s,u}^{c,j,u}(k_4)]E[\tilde{X}_{s,u}^{c,j,u}(k_4)\tilde{X}_{s,b}^{c,j,b}(k_4)]
\]

\[
+ E[\tilde{X}_{s,b}^{c,i,b}(k_2)\tilde{X}_{s,b}^{c,i,b}(k_3)]E[\tilde{X}_{s,u}^{c,j,u}(k_3)\tilde{X}_{s,u}^{c,j,u}(k_4)]E[\tilde{X}_{s,u}^{c,j,u}(k_4)\tilde{X}_{s,b}^{c,j,b}(k_4)]
\]

\[
+ E[\tilde{X}_{s,b}^{c,i,b}(k_2)\tilde{X}_{s,b}^{c,i,b}(k_3)]E[\tilde{X}_{s,u}^{c,j,u}(k_3)\tilde{X}_{s,u}^{c,j,u}(k_4)]E[\tilde{X}_{s,u}^{c,j,u}(k_4)\tilde{X}_{s,b}^{c,j,b}(k_4)]
\]

\[
+ E[\tilde{X}_{s,b}^{c,i,b}(k_2)\tilde{X}_{s,b}^{c,i,b}(k_3)]E[\tilde{X}_{s,u}^{c,j,u}(k_3)\tilde{X}_{s,u}^{c,j,u}(k_4)]E[\tilde{X}_{s,u}^{c,j,u}(k_4)\tilde{X}_{s,b}^{c,j,b}(k_4)]
\]

\[
+ E[\tilde{X}_{s,b}^{c,i,b}(k_2)\tilde{X}_{s,b}^{c,i,b}(k_3)]E[\tilde{X}_{s,u}^{c,j,u}(k_3)\tilde{X}_{s,u}^{c,j,u}(k_4)]E[\tilde{X}_{s,u}^{c,j,u}(k_4)\tilde{X}_{s,b}^{c,j,b}(k_4)]
\]

\[
+ E[\tilde{X}_{s,b}^{c,i,b}(k_2)\tilde{X}_{s,b}^{c,i,b}(k_3)]E[\tilde{X}_{s,u}^{c,j,u}(k_3)\tilde{X}_{s,u}^{c,j,u}(k_4)]E[\tilde{X}_{s,u}^{c,j,u}(k_4)\tilde{X}_{s,b}^{c,j,b}(k_4)]
\]

\[
+ E[\tilde{X}_{s,b}^{c,i,b}(k_2)\tilde{X}_{s,b}^{c,i,b}(k_3)]E[\tilde{X}_{s,u}^{c,j,u}(k_3)\tilde{X}_{s,u}^{c,j,u}(k_4)]E[\tilde{X}_{s,u}^{c,j,u}(k_4)\tilde{X}_{s,b}^{c,j,b}(k_4)]
\]

\[
+ E[\tilde{X}_{s,b}^{c,i,b}(k_2)\tilde{X}_{s,b}^{c,i,b}(k_3)]E[\tilde{X}_{s,u}^{c,j,u}(k_3)\tilde{X}_{s,u}^{c,j,u}(k_4)]E[\tilde{X}_{s,u}^{c,j,u}(k_4)\tilde{X}_{s,b}^{c,j,b}(k_4)]
\]

\[
(168)
\]

\[
\Delta \sum_{i=1}^{24} VI_{i}^{1,i}
\]
where

\[
\sum_{i=1}^{6} V F_{t}^{i,j} = 1_{k_{1}+k'_{1}=0,k_{2}+k'_{2}=0,k_{3}+k'_{3}=0,k_{4}+k'_{4}=0} \sum_{i_{3,4,5,6}=1} 1_{k_{1},k_{2},k_{3},k_{4}\neq 0} \\
\times \frac{e^{-|k_{1}|^{2}[x-z]} f(\epsilon k_{1})^{2}}{2|k_{1}|^{2}} \hat{p}_{i_{1}i_{3}}(k_{1}) \hat{p}_{i_{1}i_{3}}(k_{1}) \frac{e^{-|k_{2}|^{2}[x-z]} f(\epsilon k_{2})^{2}}{2|k_{2}|^{2}} \hat{p}_{i_{2}i_{4}}(k_{2}) \hat{p}_{i_{2}i_{4}}(k_{2}) \\
\times \frac{e^{-|k_{3}|^{2}[\sigma+\tau]} f(\epsilon k_{3})^{2}}{2|k_{3}|^{2}} \hat{p}_{i_{1}i_{5}}(k_{3}) \hat{p}_{i_{1}i_{5}}(k_{3}) \frac{e^{-|k_{4}|^{2}[\sigma+\tau]} f(\epsilon k_{4})^{2}}{2|k_{4}|^{2}} \hat{p}_{i_{2}i_{6}}(k_{4}) \hat{p}_{i_{2}i_{6}}(k_{4}) \\
+ 1_{k_{1}+k'_{1}=0,k_{2}+k'_{2}=0,k_{3}+k'_{3}=0,k_{4}+k'_{4}=0} \sum_{i_{3,4,5,6}=1} 1_{k_{1},k_{2},k_{3},k_{4}\neq 0} \\
\times \frac{e^{-|k_{1}|^{2}[x-z]} f(\epsilon k_{1})^{2}}{2|k_{1}|^{2}} \hat{p}_{i_{1}i_{3}}(k_{1}) \hat{p}_{i_{1}i_{3}}(k_{1}) \frac{e^{-|k_{2}|^{2}[x-z]} f(\epsilon k_{2})^{2}}{2|k_{2}|^{2}} \hat{p}_{i_{2}i_{4}}(k_{2}) \hat{p}_{i_{2}i_{4}}(k_{2}) \\
\times \frac{e^{-|k_{3}|^{2}[\sigma+\tau]} f(\epsilon k_{3})^{2}}{2|k_{3}|^{2}} \hat{p}_{i_{1}i_{5}}(k_{3}) \hat{p}_{i_{1}i_{5}}(k_{3}) \frac{e^{-|k_{4}|^{2}[\sigma+\tau]} f(\epsilon k_{4})^{2}}{2|k_{4}|^{2}} \hat{p}_{i_{2}i_{6}}(k_{4}) \hat{p}_{i_{2}i_{6}}(k_{4})
\]
\[ \sum_{i=7}^{12} \mathcal{V} \mathcal{I}^1_i \]

\[ = 1_{k_1+k'_2=0,k_2+k'_3=0,k_3+k'_4=0,k_4+k'_4=0} \sum_{i_3,i_4,i_5,i_6=1}^3 1_{k_1,k_2,k_3,k_4 \neq 0} \]

\[ \times e^{-|k_1|^2 [\sigma_e - \sigma_e]} f(k_1)^2 \tilde{\mathcal{P}}^{i_1 i_3} (k_1) \tilde{\mathcal{P}}^{i_4 i_6} (k_1) e^{-|k_2|^2 [\sigma_e - \sigma_e]} f(k_2)^2 \tilde{\mathcal{P}}^{i_1 i_6} (k_2) \tilde{\mathcal{P}}^{i_4 i_3} (k_2) \]

\[ \times e^{-|k_3|^2 [\sigma_e - \sigma_\pi]} f(k_3)^2 \tilde{\mathcal{P}}^{j_1 j_5} (k_3) \tilde{\mathcal{P}}^{j_4 j_6} (k_3) e^{-|k_4|^2 [\sigma_e - \sigma_\pi]} f(k_4)^2 \tilde{\mathcal{P}}^{j_1 j_6} (k_4) \tilde{\mathcal{P}}^{j_4 j_5} (k_4) \]

\[ + 1_{k_1+k'_2=0,k_2+k'_3=0,k_3+k'_4=0,k_4+k'_3=0} \sum_{i_3,i_4,i_5,i_6=1}^3 1_{k_1,k_2,k_3,k_4 \neq 0} \]

\[ \times e^{-|k_1|^2 [\sigma_e - \sigma_e]} f(k_1)^2 \tilde{\mathcal{P}}^{i_1 i_3} (k_1) \tilde{\mathcal{P}}^{i_4 i_6} (k_1) e^{-|k_2|^2 [\sigma_e - \sigma_e]} f(k_2)^2 \tilde{\mathcal{P}}^{i_1 i_6} (k_2) \tilde{\mathcal{P}}^{i_4 i_3} (k_2) \]

\[ \times e^{-|k_3|^2 [\sigma_e - \sigma_\pi]} f(k_3)^2 \tilde{\mathcal{P}}^{j_1 j_5} (k_3) \tilde{\mathcal{P}}^{j_4 j_6} (k_3) e^{-|k_4|^2 [\sigma_e - \sigma_\pi]} f(k_4)^2 \tilde{\mathcal{P}}^{j_1 j_6} (k_4) \tilde{\mathcal{P}}^{j_4 j_5} (k_4) \]

\[ + 1_{k_1+k'_2=0,k_2+k'_3=0,k_3+k'_4=0,k_4+k'_3=0} \sum_{i_3,i_4,i_5,i_6=1}^3 1_{k_1,k_2,k_3,k_4 \neq 0} \]

\[ \times e^{-|k_1|^2 [\sigma_e - \sigma_e]} f(k_1)^2 \tilde{\mathcal{P}}^{i_1 i_3} (k_1) \tilde{\mathcal{P}}^{i_4 i_6} (k_1) e^{-|k_2|^2 [\sigma_e - \sigma_e]} f(k_2)^2 \tilde{\mathcal{P}}^{i_1 i_6} (k_2) \tilde{\mathcal{P}}^{i_4 i_3} (k_2) \]

\[ \times e^{-|k_3|^2 [\sigma_e - \sigma_\pi]} f(k_3)^2 \tilde{\mathcal{P}}^{j_1 j_5} (k_3) \tilde{\mathcal{P}}^{j_4 j_6} (k_3) e^{-|k_4|^2 [\sigma_e - \sigma_\pi]} f(k_4)^2 \tilde{\mathcal{P}}^{j_1 j_6} (k_4) \tilde{\mathcal{P}}^{j_4 j_5} (k_4) \]

\[ (170) \]
\[
\sum_{i=1}^{18} V f_{i,j}^{1,i} = 1_{k_1 + k_2' = 0, k_2 + k_1' = 0, k_3 + k_4' = 0, k_4 + k_3' = 0} \sum_{k_1, k_2, k_3, k_4} \sum_{k_1, k_2, k_3, k_4} \frac{e^{-|k_1|^2 |\sigma|} f(k_1)^2}{2|k_1|^2} \hat{P}_{i_1 i_3}^{*} (k_1) \hat{P}_{i_3 i_5}^{*} (k_1) \frac{e^{-|k_2|^2 |\sigma|} f(k_2)^2}{2|k_2|^2} \hat{P}_{i_2 i_4}^{*} (k_2) \hat{P}_{i_4 i_6}^{*} (k_2) \\
+ \frac{e^{-|k_3|^2 |\sigma|} f(k_3)^2}{2|k_3|^2} \hat{P}_{i_1 i_3}^{*} (k_3) \hat{P}_{i_3 i_5}^{*} (k_3) \frac{e^{-|k_4|^2 |\sigma|} f(k_4)^2}{2|k_4|^2} \hat{P}_{i_2 i_4}^{*} (k_4) \hat{P}_{i_4 i_6}^{*} (k_4)
\]

(171)
and

\[
\sum_{i=19}^{24} V_{ij}^{1,j} = \sum_{i_1,i_2,i_3,i_4,i_5,i_6=1}^3 1_{k_1,k_2,k_3,k_4 \neq 0}
\]

\[
\times \frac{e^{-|k_1|^2 |\sigma|}}{2|k_1|^2} f(k_1)^2 \bar{\mathcal{P}}_{j_1,i_3}^1 (k_1) \bar{\mathcal{P}}_{j_2,i_3}^1 (k_1) \frac{e^{-|k_2|^2 |\sigma|}}{2|k_2|^2} \bar{\mathcal{P}}_{j_3,i_4}^1 (k_2) \bar{\mathcal{P}}_{j_4,i_4}^1 (k_2)
\]

\[
\times \frac{e^{-|k_3|^2 |\sigma|}}{2|k_3|^2} f(k_3)^2 \bar{\mathcal{P}}_{j_5,i_5}^1 (k_3) \bar{\mathcal{P}}_{j_6,i_5}^1 (k_3) \frac{e^{-|k_4|^2 |\sigma|}}{2|k_4|^2} \bar{\mathcal{P}}_{j_7,i_6}^1 (k_4) \bar{\mathcal{P}}_{j_8,i_6}^1 (k_4)
\]

\[
+ 1_{k_1,k_2,k_3,k_4 \neq 0} \sum_{i_1,i_2,i_3,i_4,i_5,i_6=1}^3 1_{k_1,k_2,k_3,k_4 \neq 0}
\]

\[
\times \frac{e^{-|k_1|^2 |\sigma|}}{2|k_1|^2} f(k_1)^2 \bar{\mathcal{P}}_{j_1,i_3}^1 (k_1) \bar{\mathcal{P}}_{j_2,i_3}^1 (k_1) \frac{e^{-|k_2|^2 |\sigma|}}{2|k_2|^2} \bar{\mathcal{P}}_{j_3,i_4}^1 (k_2) \bar{\mathcal{P}}_{j_4,i_4}^1 (k_2)
\]

\[
\times \frac{e^{-|k_3|^2 |\sigma|}}{2|k_3|^2} f(k_3)^2 \bar{\mathcal{P}}_{j_5,i_5}^1 (k_3) \bar{\mathcal{P}}_{j_6,i_5}^1 (k_3) \frac{e^{-|k_4|^2 |\sigma|}}{2|k_4|^2} \bar{\mathcal{P}}_{j_7,i_6}^1 (k_4) \bar{\mathcal{P}}_{j_8,i_6}^1 (k_4)
\]

(172)
Thus, applying \(108\) to \(167\) leads to

\[
\begin{align*}
\mathbb{E}\left[ \sum_k \theta(2^{-q}k) \sum_{i_1,i_2,j_1,j_2=1,k_1,k_2,k_3,k_4,k_1234=k}^3 \int_{[0,t]^4} e^{-|k_{12}|^2(t-s)-|k_{34}|^2(t-\tau)} \\
\times \left( \hat{X}_{k_{12}}^{r_1,s_1}(k_1) \hat{X}_{k_{12}}^{r_2,s_2}(k_2) \hat{X}_{k_{12}}^{r_3,s_3}(k_3) \hat{X}_{k_{12}}^{r_4,s_4}(k_4) : d\sigma d\sigma_k \hat{P}_{ij}(k_1) \hat{P}_{ij}(k_4) i_{k_{12}}^2 i_{k_{12}}^2 \right) \right] \\
\times \left( \sum_k \theta(2^{-q}k') \sum_{i_1',i_2',j_1',j_2'=1,k_1',k_2',k_3',k_4',k_1'2344=k'}^3 \int_{[0,t]^4} e^{-|k_{12}'|^2(t-s)-|k_{34}'|^2(t-\tau)} \\
\times \left( \hat{X}_{k_{12}'}^{r_1,s_1}(k_1') \hat{X}_{k_{12}'}^{r_2,s_2}(k_2') \hat{X}_{k_{12}'}^{r_3,s_3}(k_3') \hat{X}_{k_{12}'}^{r_4,s_4}(k_4') : d\sigma d\sigma_k' \hat{P}_{ij}(k_1') \hat{P}_{ij}(k_4') i_{k_{12}'}^2 i_{k_{12}'}^2 \right) \right)
\end{align*}
\]

with each \(\{VI_t^{1,i}\}_{i=1}^{24}\) elaborated in \(169\)–\(172\). Now let us observe that in every product of indicator functions in \(169\)–\(172\) such as \(1_{k_1,k_2,k_3,k_4,k_1234=1}\), we always have \(k_1+k_2=0\), \(k_2+k_3=0\), \(k_3+k_4=0\), \(k_4+k_3=0\), and therefore it implies \(k=k_1234=-k_1\). Thus, e.g. we may bound the case of \(VI_t^{1,1}\) in \(169\) by

\[
\begin{align}
\sum_k \theta(2^{-q}k) \sum_{k_1,k_2,k_3,k_4,k_1234=k}^3 \int_{[0,t]^4} e^{-|k_{12}|^2(t-s)-|k_{34}|^2(t-\tau)} |k_{12}|^2 |k_{34}|^2 |k_{12}||k_{14}||k_{14}||k_{23}| |k_{12}|^2 |k_{23}|^2 |k_{34}|^2 d\sigma d\sigma_k.
\end{align}
\]

(173)

For the case of \(VI_t^{1,2}\) in \(169\), we see that we have the same bound as \(VI_t^{1,1}\). For the case of \(VI_t^{1,3}\) in \(169\), we may bound by

\[
\begin{align}
\sum_k \theta(2^{-q}k) \sum_{k_1,k_2,k_3,k_4,k_1234=k}^3 \int_{[0,t]^4} e^{-|k_{12}|^2(t-s)-|k_{34}|^2(t-\tau)-|k_{14}|^2(t-\sigma)} |k_{12}| |k_{34}| |k_{14}| |k_{23}| |k_{12}|^2 |k_{23}|^2 |k_{34}|^2 |k_{12}|^2 |k_{23}|^2 |k_{34}|^2 d\sigma d\sigma_k d\sigma d\sigma_k.
\end{align}
\]

(174)

where the equality is due to switching variables \(k_3\) and \(k_4\), as well as \(\sigma\) and \(\tau\). Other terms \(VI_t^{1,k}\) for \(k \in \{4,\ldots,24\}\) may be bounded similarly so that we deduce

\[
\begin{align}
\mathbb{E}[|\Delta_q VI_t^{1,2}|^2] \leq \sum_k \theta(2^{-q}k) \sum_{k_1,k_2,k_3,k_4,k_1234=k}^3 \int_{[0,t]^4} e^{-|k_{12}|^2(t-s)-|k_{34}|^2(t-\tau)-|k_{14}|^2(t-\sigma)} |k_{12}| |k_{34}| |k_{14}| |k_{23}| |k_{12}|^2 |k_{23}|^2 |k_{34}|^2 |k_{12}|^2 |k_{23}|^2 |k_{34}|^2 d\sigma d\sigma_k d\sigma d\sigma_k.
\end{align}
\]

(175)
Within (175) we may further estimate for $k_1, k_2, k_3, k_4 \neq 0$,
\[
\int_{[0,t]^4} e^{-|k_1|^2(2t-s-\tau)-|k_3|^2(2t-\tau-\sigma)} \frac{|k_{12}|^2|k_{34}|^2}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2} ds d\sigma d\tau d\sigma \lesssim 1_{k_{12},k_{34}\neq 0} \frac{1}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2|k_{12}|^{2-\epsilon}|k_{34}|^{2-\epsilon}}
\] (176)
where we used mean value theorem, while for $k_1, k_2, k_3, k_4 \neq 0$,
\[
\int_{[0,t]^4} e^{-|k_1|^2(t-s)-|k_3|^2(t-\tau)-|k_4|^2(t-\sigma)} \frac{|k_{12}|^2|k_{34}|^2}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2} ds d\sigma d\tau d\sigma \lesssim 1_{k_{12},k_{34},k_{14},k_{23}\neq 0} \frac{1}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2|k_{12}|^{1-\frac{\theta}{2}}|k_{34}|^{1-\frac{\theta}{2}}|k_{14}|^{1-\frac{\theta}{2}}|k_{23}|^{1-\frac{\theta}{2}}}
\] (177)
by mean value theorem. Therefore, applying (176) and (177) to (175) gives
\[
\mathbb{E}(|\Delta_q V ||_1^2) \lesssim t^\epsilon \sum_k \theta(2^{-q}k)^2 |VII^1 + \sqrt{VII^1} \sqrt{VII^2}|
\] (178)
where
\[
VII^1 \triangleq \sum_{k_1, k_2, k_3, k_4 \neq 0; k_{12} = k} \frac{1}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2|k_{12}|^{2-\epsilon}|k_{34}|^{2-\epsilon}},
\]
\[
VII^2 \triangleq \sum_{k_1, k_2, k_3, k_4 \neq 0; k_{12} = k} \frac{1}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2|k_{14}|^{2-\epsilon}|k_{23}|^{2-\epsilon}},
\]
due to Hölder’s inequality. We may estimate
\[
t^\epsilon \sum_k \theta(2^{-q}k)^2 \sqrt{VII^1} \sqrt{VII^2} \lesssim 2^{2q} t^\epsilon \sum_k \theta(2^{-q}k)^2 \frac{1}{|k|^{2\epsilon}} \left( \frac{1}{|k|^{12-2\epsilon}} \right)^{\frac{\theta}{2}} \left( \frac{1}{|k|^{12-2\epsilon}} \right)^{\frac{\theta}{2}} \lesssim 2^{2q} t^\epsilon
\] (179)
by Lemma 4.7. We may apply identical estimates to $\sum_k \theta(2^{-q}k)^2 VII^1$ in (178) to deduce
\[
\mathbb{E}(|\Delta_q V ||_1^2) \lesssim t^{2q}.
\] (180)
Similarly to how we deduced (139) from (138), we can obtain an analogous Lipschitz bound on
\[
\mathbb{E}(|\Delta_q (b^{1,i}_2 \circ u^{1,j}_2(t_1) - b^{1,i}_2 \circ u^{1,j}_2(t_2) - b^{2,i}_2 \circ u^{2,j}_2(t_1) + b^{2,i}_2 \circ u^{2,j}_2(t_2))|^2),
\]
with which similar arguments using Besov embedding, Gaussian hypercontractivity [29, Theorem 3.50], as we did in (139)-(141), imply that there exists $v^{ij}_{13} \in C([0,T];C^{-\gamma})$ for $i,j \in \{1,2,3\}$ such that for all $p \in (1,\infty)$, $b^{1,i}_2 \circ u^{1,j}_2 \rightarrow v^{ij}_{13}$ in $L^p(\Omega;C([0,T];C^{-\delta}))$ as desired in (113).
3.3. Group 3: \( \pi_{0,0}(u_{3}^{\epsilon,i}, u_{1}^{\epsilon,j}), \pi_{0,0}(b_{3}^{\epsilon,i}, b_{1}^{\epsilon,j}), \pi_{0,0}(u_{3}^{\epsilon,i}, b_{1}^{\epsilon,j}), \pi_{0,0}(b_{3}^{\epsilon,i}, u_{1}^{\epsilon,j}) \). Within the Group 3 of (114), we focus on \( \pi_{0,0}(u_{3}^{\epsilon,i}, b_{1}^{\epsilon,j}) \). Considering (18), (24a)-(24d) and (17) we see that we may rewrite \( L_{u_{3}^{\epsilon},i}^{(0)} = \sum_{i=1}^{8} L_{u_{3}^{\epsilon},i}^{(0)} \) where

\[
L_{u_{31}^{\epsilon}}^{(0)} \triangleq \frac{1}{4} \sum_{i_{3},j_{3}=1}^{3} \mathcal{P}^{i_{3}i_{1}} \partial_{x_{1}} (u_{1}^{i_{1}} [\int_{0}^{t} P_{t-s} \sum_{i_{2},i_{3}=1}^{3} \mathcal{P}^{j_{3}i_{2}} \partial_{x_{3}} (u_{1}^{i_{2}} u_{1}^{i_{3}})(s) ds]),
\]

\[
L_{u_{32}^{\epsilon}}^{(0)} \triangleq -\frac{1}{4} \sum_{i_{3},j_{3}=1}^{3} \mathcal{P}^{i_{3}i_{1}} \partial_{x_{1}} (u_{1}^{i_{1}} [\int_{0}^{t} P_{t-s} \sum_{i_{2},i_{3}=1}^{3} \mathcal{P}^{j_{3}i_{2}} \partial_{x_{3}} (b_{1}^{i_{2}} b_{1}^{i_{3}})(s) ds]),
\]

\[
L_{u_{33}^{\epsilon}}^{(0)} \triangleq \frac{1}{4} \sum_{i_{3},j_{3}=1}^{3} \mathcal{P}^{i_{3}i_{1}} \partial_{x_{1}} ([\int_{0}^{t} P_{t-s} \sum_{i_{2},i_{3}=1}^{3} \mathcal{P}^{j_{3}i_{2}} \partial_{x_{3}} (u_{1}^{i_{2}} u_{1}^{i_{3}})(s) ds] u_{1}^{i_{1}}),
\]

\[
L_{u_{34}^{\epsilon}}^{(0)} \triangleq -\frac{1}{4} \sum_{i_{3},j_{3}=1}^{3} \mathcal{P}^{i_{3}i_{1}} \partial_{x_{1}} ([\int_{0}^{t} P_{t-s} \sum_{i_{2},i_{3}=1}^{3} \mathcal{P}^{j_{3}i_{2}} \partial_{x_{3}} (b_{1}^{i_{2}} b_{1}^{i_{3}})(s) ds] u_{1}^{i_{1}}),
\]

\[
L_{u_{35}^{\epsilon}}^{(0)} \triangleq -\frac{1}{4} \sum_{i_{3},j_{3}=1}^{3} \mathcal{P}^{i_{3}i_{1}} \partial_{x_{1}} ([\int_{0}^{t} P_{t-s} \sum_{i_{2},i_{3}=1}^{3} \mathcal{P}^{j_{3}i_{2}} \partial_{x_{3}} (b_{1}^{i_{2}} u_{1}^{i_{3}})(s) ds] b_{1}^{i_{1}}),
\]

\[
L_{u_{36}^{\epsilon}}^{(0)} \triangleq \frac{1}{4} \sum_{i_{3},j_{3}=1}^{3} \mathcal{P}^{i_{3}i_{1}} \partial_{x_{1}} ([\int_{0}^{t} P_{t-s} \sum_{i_{2},i_{3}=1}^{3} \mathcal{P}^{j_{3}i_{2}} \partial_{x_{3}} (u_{1}^{i_{2}} b_{1}^{i_{3}})(s) ds] b_{1}^{i_{1}}),
\]

\[
L_{u_{37}^{\epsilon}}^{(0)} \triangleq -\frac{1}{4} \sum_{i_{3},j_{3}=1}^{3} \mathcal{P}^{i_{3}i_{1}} \partial_{x_{1}} ([\int_{0}^{t} P_{t-s} \sum_{i_{2},i_{3}=1}^{3} \mathcal{P}^{j_{3}i_{2}} \partial_{x_{3}} (u_{1}^{i_{2}} b_{1}^{i_{3}})(s) ds] b_{1}^{i_{1}}). \tag{181}
\]

By (25e) we have \( \pi_{0,0}(u_{3}^{\epsilon,i}, b_{1}^{\epsilon,j}) = \pi_{0}(u_{3}^{\epsilon,i}, b_{1}^{\epsilon,j}) - C_{1,3}^{\epsilon,i,j} \) where

\[
\pi_{0}(u_{3}^{\epsilon,i}, b_{1}^{\epsilon,j}) = \sum_{k=1}^{8} \pi_{0}(u_{3k}^{\epsilon,i}, b_{1}^{\epsilon,j}) \tag{182}
\]
due to linearity. Now by necessity, as we will see, we shall actually work on \( \pi_{0}(u_{31}^{\epsilon,j} + u_{32}^{\epsilon,j}, b_{1}^{\epsilon,j}), \pi_{0}(u_{33}^{\epsilon,j} + u_{34}^{\epsilon,j}, b_{1}^{\epsilon,j}), \pi_{0}(u_{35}^{\epsilon,j} + u_{36}^{\epsilon,j}, b_{1}^{\epsilon,j}), \pi_{0}(u_{37}^{\epsilon,j} + u_{38}^{\epsilon,j}, b_{1}^{\epsilon,j}) \). Without loss of generality we work on the last one, elaborating on the computations of \( u_{38}^{\epsilon,j} \) first. Firstly, we see from (181) that

\[
\pi_{0}(u_{38}^{\epsilon,i}, b_{1}^{\epsilon,j})(t) = -\frac{1}{4(2\pi)^{2}} \sum_{k=1}^{8} \sum_{k_{1},k_{2},k_{3},k_{4}=1}^{3} \sum_{k_{1},i_{2},i_{3},j_{1}=1}^{3} \theta(-i_{1} k_{123}) \theta(-j_{1} k_{1})
\]

\[
\times \int_{0}^{t} e^{-|k_{123}|^{2}(t-s)} \int_{0}^{s} X_{u,i,s}(k_{1}) X_{b,j,s}(k_{2}) X_{b,i,s}(k_{3}) X_{b,j,s}(k_{4}) X_{b,j,t}(k_{4})
\]

\[
\times e^{-|k_{123}|^{2}(s-\sigma)} d\sigma d\tilde{k}_{123}^{(s)} \tilde{P}_{i_{12}}^{(s)}(k_{12}) \tilde{P}_{i_{3}}^{(s)}(k_{3}) \tilde{P}_{k_{4}}^{(s)}(k_{4}) \tag{183}
\]
Now by (134) we may rewrite

\[ \pi_0(t_{\text{in}}, b_1^{j_0})(t) = \frac{-1}{4(2\pi)^2} \sum_k \sum_{|i-j|\leq 1} \sum_{k_1, k_2, k_3, k_4: k_1 + k_2 = i \neq j, i_1, i_2, i_3, i_4, j_1 = 1} \theta(2^{-i}k_1) \theta(2^{-j}k_4) \]

\[ \times \int_0^t e^{-|k_{12}|^2(t-s)} \int_0^s \sum_{a, \sigma} \dot{X}_{u, \sigma}^e(k_1) \dot{X}_{b, \sigma}^e(k_2) \dot{X}_{b, a}^e(k_3) \dot{X}_{b, t}^e(k_4) : \]

\[ + \mathbb{E} \left[ \dot{X}_{u, \sigma}^e(k_1) \dot{X}_{b, a}^e(k_2) : \right] \dot{X}_{b, \sigma}^e(k_3) \dot{X}_{b, t}^e(k_4) \]

(184)

by switching variables \( k_1 \) and \( k_2 \). Next, we similarly compute using (116).

Next, we similarly compute using (116),

\[ \text{VIII}_i^{8, 3} = -\frac{1}{4(2\pi)^2} \sum_k \sum_{|i-j|\leq 1} \sum_{k_1, k_2, k_3, k_4: k_1 + k_2 = i \neq j, i_1, i_2, i_3, i_4, j_1 = 1} \theta(2^{-i}k_1) \theta(2^{-j}k_4) \]

\[ \times \int_0^t e^{-|k_{12}|^2(t-s)} \int_0^s \sum_{a, \sigma} \dot{X}_{u, \sigma}^e(k_1) \dot{X}_{b, \sigma}^e(k_2) : \]

\[ \times \mathbb{P}^{i_{12}}(k_2) \mathbb{P}^{i_{12}}(k_2) e^{-|k_{12}|^2(s-\sigma)} d\sigma ds e^{-|k_{12}|^2(s-\sigma)} \]

(185)

\[ \text{VIII}_i^{8, 3} = \frac{-1}{4(2\pi)^2} \sum_k \sum_{|i-j|\leq 1} \sum_{k_1, k_2, k_3, k_4: k_1 + k_2 = i \neq j, i_1, i_2, i_3, i_4, j_1 = 1} \theta(2^{-i}k_1) \theta(2^{-j}k_4) \]

\[ \times \int_0^t e^{-|k_{12}|^2(t-s)} \int_0^s \sum_{a, \sigma} \dot{X}_{u, \sigma}^e(k_1) \dot{X}_{b, \sigma}^e(k_2) : \]

\[ \times k_{12}^{j_1, j_2} \mathbb{P}^{i_{12}}(k_2) \mathbb{P}^{i_{12}}(k_1) e^{-|k_{12}|^2(s-\sigma)} ds \]

(186)
\[ \mathcal{VIII}_t^{8.4} = - \frac{1}{4(2\pi)^2} \sum_{k} \sum_{|l-j| \leq 1} \sum_{k_1,k_2,k_{14}=k,k_2 \neq 0,i_1,i_2,i_3,i_4,j_1=1}^3 \theta(2^{-i}k_1) \times \theta(2^{-j}k_4) \int_{0}^{t} e^{-|k_1|^2(t-s)} \int_{0}^{t} e^{-|k_2|^2(s-\sigma)} f(ek_2)^2 \frac{\hat{\phi}_{i_1i_4}(k_2)}{2|k_2|^2} \mathcal{MAGNETOHYDRODYNAMICS SYSTEM 69} \]

\[ \times \hat{\mathcal{P}}_{i_1i_4}(k_2) : \hat{X}_{u,\sigma}(k_1) \hat{X}_{b,\sigma}(k_4) : e^{-|k_1|^2(s-\sigma)} d\sigma ds \]

\[ \times k_{12} k_{12} k_{12} k_{13} \hat{\phi}_{i_2i_3}(k_12) \hat{\phi}_{i_0i_1}(k_23) e_{i_5=i_2,i_6=i_3} \triangleq \mathcal{I}X_t^{8.5} , \]  

\[ \mathcal{VIII}_t^{8.5} = - \frac{1}{4(2\pi)^2} \sum_{k} \sum_{|l-j| \leq 1} \sum_{k_1,k_2,k_{13}=k,k_2 \neq 0,i_1,i_2,i_3,i_4,j_1=1}^3 \theta(2^{-i}k_1) \times \theta(2^{-j}k_3) \int_{0}^{t} e^{-|k_1|^2(t-s)} \int_{0}^{s} e^{-|k_3|^2(s-\sigma)} f(ek_1)^2 \frac{\hat{\phi}_{i_0i_4}(k_1) \hat{\phi}_{i_0i_4}(k_1)}{2|k_1|^2} \]

\[ \times \hat{\mathcal{P}}_{i_1i_4}(k_3) : \hat{X}_{u,\sigma}(k_1) \hat{X}_{b,\sigma}(k_3) : e^{-|k_1|^2(s-\sigma)} d\sigma ds \]

\[ \times k_{12} k_{12} k_{13} \hat{\phi}_{i_2i_3}(k_12) \hat{\phi}_{i_2i_3}(k_13) e_{i_5=i_2,i_6=i_3} \triangleq \mathcal{I}X_t^{8.6} , \]  

\[ \mathcal{VIII}_t^{8.6} = - \frac{1}{4(2\pi)^2} \sum_{k} \sum_{|l-j| \leq 1} \sum_{k_1,k_2,k_{14}=k,k_2 \neq 0,i_1,i_2,i_3,i_4,j_1=1}^3 \theta(2^{-i}k_1) \times \theta(2^{-j}k_2) \int_{0}^{t} e^{-|k_1|^2(t-s)} \int_{0}^{s} f(ek_1)^2 f(ek_2)^2 \frac{\hat{\phi}_{i_2i_3}(k_1) \hat{\phi}_{i_1i_4}(k_1)}{4|k_1|^2|k_2|^2} \]

\[ \times \hat{\mathcal{P}}_{i_1i_4}(k_2) \hat{\phi}_{i_2i_3}(k_2) e^{-|k_1|^2(s-\sigma)-|k_1|^2(s-\sigma)-|k_2|^2(t-\sigma)} d\sigma ds \]

\[ \times k_{12} k_{12} k_{13} \hat{\phi}_{i_2i_3}(k_12) \hat{\phi}_{i_2i_3}(k_13) e_{i_5=i_2,i_6=i_3} \triangleq \mathcal{I}X_t^{8.7} , \]  

\[ \text{and} \]

\[ \mathcal{VIII}_t^{8.7} = - \frac{1}{4(2\pi)^2} \sum_{k} \sum_{|l-j| \leq 1} \sum_{k_1,k_2,k_{14}=k,k_2 \neq 0,i_1,i_2,i_3,i_4,j_1=1}^3 \theta(2^{-i}k_1) \theta(2^{-j}k_2) \int_{0}^{t} e^{-|k_2|^2(t-s)} \int_{0}^{s} e^{-|k_2|^2(t-\sigma)} e^{-|k_1|^2(s-\sigma)} f(ek_1)^2 f(ek_2)^2 \frac{\hat{\phi}_{i_2i_3}(k_1) \hat{\phi}_{i_2i_3}(k_2)}{4|k_1|^2|k_2|^2} \]

\[ \times \hat{\mathcal{P}}_{i_2i_3}(k_1) \hat{\phi}_{i_2i_3}(k_1) \hat{\phi}_{i_2i_3}(k_2) \hat{\phi}_{i_2i_3}(k_2) e^{-|k_1|^2(s-\sigma)} d\sigma ds \]

\[ \times k_{12} k_{12} k_{13} \hat{\phi}_{i_2i_3}(k_12) \hat{\phi}_{i_2i_3}(k_13) , \]  

\[ \mathcal{VIII}_t^{8.8} = - \frac{1}{4(2\pi)^2} \sum_{k} \sum_{|l-j| \leq 1} \sum_{k_1,k_2,k_{14}=k,k_2 \neq 0,i_1,i_2,i_3,i_4,j_1=1}^3 \theta(2^{-i}k_1) \theta(2^{-j}k_2) \int_{0}^{t} e^{-|k_2|^2(t-s)} \int_{0}^{s} e^{-|k_2|^2(t-\sigma)} e^{-|k_1|^2(s-\sigma)} f(ek_1)^2 f(ek_2)^2 \frac{\hat{\phi}_{i_2i_3}(k_1) \hat{\phi}_{i_2i_3}(k_2)}{4|k_1|^2|k_2|^2} \]

\[ \times \hat{\mathcal{P}}_{i_2i_3}(k_1) \hat{\phi}_{i_2i_3}(k_1) \hat{\phi}_{i_2i_3}(k_2) \hat{\phi}_{i_2i_3}(k_2) e^{-|k_1|^2(s-\sigma)} d\sigma ds \]

\[ \times k_{12} k_{12} k_{13} \hat{\phi}_{i_2i_3}(k_12) \hat{\phi}_{i_2i_3}(k_13) \triangleq \mathcal{I}X_t^{8.8} . \]
We define the sum of right hand side of $VIII_t^{8,7},VIII_t^{8,8}$ in (190)-(191) to be $IX_t^{8,7}$; i.e.

$$IX_t^{8,7}$$

$$\triangleq \frac{1}{4(2\pi)^2} \sum_{|i-j|\leq 1} \sum_{k_1,k_2 \neq 0,i_1,i_2,i_3,i_4,i_5,j_1=1}^3 \theta(2^{-i}k_2)\theta(2^{-j}k_2)$$

$$\times \int_0^t e^{-|k_2|^2(t-s)} \int_0^s f(ek_1)^2 f(ek_2)^2 e^{-|k_1|^2|s-\tau|-|k_1|^2(s-\sigma)-|k_2|^2(t-\sigma)} e^{-|k_{12}|^2|s-\tau|-|k_1|^2(s-\sigma)-|k_2|^2(t-\sigma)}$$

$$\times \{ \tilde{P}^{i_1,i_4}(k_1)\tilde{P}^{j_1,i_4}(k_1)\tilde{P}^{j_2,i_4}(k_2)\tilde{P}^{j_3,i_4}(k_2)\} e^{-|k_{12}|^2|s-\tau|-|k_1|^2(s-\sigma)-|k_2|^2(t-\sigma)}$$

(192)

and formally

$$IX_t^{8,7} \triangleq C_{1,3,8}^{e,i_0,j_0}$$

where we observe that $\lim_{t \to 0} C_{1,3,8}^{e,i_0,j_0} = \infty$. Due to (185)-(192) applied to (184), we see that

$$\pi_0(u_{38}^{\epsilon,i_0,j_0}(b_1^{j_0}))(t) = \sum_{k=1}^{7} IX_t^{8,k} = \sum_{k=1}^{6} IX_t^{8,k} + C_{1,3,8}^{e,i_0,j_0}.$$  

Repeating similar procedure for $\pi_0(u_{38}^{\epsilon,i_0,j_0}(b_1^{j_0}))(t)$ for $k \in \{1, \ldots, 7\}$ within (182), we can similarly define $C_{1,3,k}^{e,i_0,j_0}$ for $k \in \{1, \ldots, 7\}$. Thereafter we shall define

$$C_{1,3}^{e,i_0,j_0} = \sum_{k=1}^{8} C_{1,3,k}^{e,i_0,j_0}.$$  

(195)

3.3.1. Terms in the second chaos. Within (194) we see that $IX_t^{8,1}$ is a term in the fourth chaos while $IX_t^{8,k}$ for $k \in \{2, \ldots, 6\}$ are in the second chaos. Let us first work on $IX_t^{8,2}$ as follows:

$$E[|\Delta_\theta IX_t^{8,2}|^2]$$

$$\approx \sum_{k,k'} \sum_{|i-j|\leq 1} \sum_{i'j'\leq 1} \sum_{k_1,k_2,k_3,k_4,k_5} \sum_{k_1'k_2'k_3'k_4'k_5'=0}^3 \theta(2^{-i}k_2)\theta(2^{-j}k_1)$$

$$\times \theta(2^{-i'}k_2)\theta(2^{-j'}k_1)$$

$$\times \int_0^t e^{-|k_2|^2(t-s)} e^{-|k_1'|^2(t-s)}$$

$$\times \int_0^s \mathbb{E}[\tilde{X}_{b_1}^{i_1,i_4}(k_2)\tilde{X}_{b_2}^{i_1,i_4}(k_3) : \tilde{X}_{b_3}^{i_1,i_4}(k_2)\tilde{X}_{b_4}^{i_1,i_4}(k_3) : \tilde{X}_{b_5}^{i_1,i_4}(k_2)\tilde{X}_{b_6}^{i_1,i_4}(k_3) : ]$$

$$\times e^{-|k_1|^2(t-s)} e^{-|k_1'|^2(t-s)} f(ek_1)^2 f(ek_1')^2$$

$$\times 2|k_1|^2 2|k_1'|^2$$

(196)

$$\times \tilde{P}^{i_1,i_4}(k_1)\tilde{P}^{j_1,i_4}(k_1')\tilde{P}^{j_2,i_4}(k_1)\tilde{P}^{j_3,i_4}(k_1') e^{-|k_{12}|^2|s-\tau|-|k_1|^2(s-\sigma)-|k_2|^2(t-\sigma)} e^{-|k_{12}|^2|s-\tau|-|k_1|^2(s-\sigma)-|k_2|^2(t-\sigma)}$$

$$\times \tilde{P}^{i_1,i_4}(k_1)\tilde{P}^{j_1,i_4}(k_1')\tilde{P}^{j_2,i_4}(k_1)\tilde{P}^{j_3,i_4}(k_1') e^{-|k_{12}|^2|s-\tau|-|k_1|^2(s-\sigma)-|k_2|^2(t-\sigma)} e^{-|k_{12}|^2|s-\tau|-|k_1|^2(s-\sigma)-|k_2|^2(t-\sigma)}$$

$$\times \tilde{P}^{i_1,i_4}(k_1)\tilde{P}^{j_1,i_4}(k_1')\tilde{P}^{j_2,i_4}(k_1)\tilde{P}^{j_3,i_4}(k_1') e^{-|k_{12}|^2|s-\tau|-|k_1|^2(s-\sigma)-|k_2|^2(t-\sigma)} e^{-|k_{12}|^2|s-\tau|-|k_1|^2(s-\sigma)-|k_2|^2(t-\sigma)}$$

$$\times \tilde{P}^{i_1,i_4}(k_1)\tilde{P}^{j_1,i_4}(k_1')\tilde{P}^{j_2,i_4}(k_1)\tilde{P}^{j_3,i_4}(k_1') e^{-|k_{12}|^2|s-\tau|-|k_1|^2(s-\sigma)-|k_2|^2(t-\sigma)} e^{-|k_{12}|^2|s-\tau|-|k_1|^2(s-\sigma)-|k_2|^2(t-\sigma)}$$

$$\times \tilde{P}^{i_1,i_4}(k_1)\tilde{P}^{j_1,i_4}(k_1')\tilde{P}^{j_2,i_4}(k_1)\tilde{P}^{j_3,i_4}(k_1') e^{-|k_{12}|^2|s-\tau|-|k_1|^2(s-\sigma)-|k_2|^2(t-\sigma)} e^{-|k_{12}|^2|s-\tau|-|k_1|^2(s-\sigma)-|k_2|^2(t-\sigma)}$$
due to (180). By Example (22) we can compute
\[\begin{align*}
E[: X_{b, \alpha}^{i_1 j_1}(k_2) \hat{X}_{b, \alpha}^{i_2 j_2}(k_3) \cdot \hat{X}_{b, \alpha}^{i_3 j_3}(k') \cdot :]
&= \sum_{i_5, i_5' = 1}^{3} e^{-|k_2|^2 |\sigma| \mathcal{I}_f (e k_2)^2} \frac{1}{2 |k_2|^2} \\
&\quad \times \hat{\mathcal{P}}^{i_5 i_5}(k_2) \hat{\mathcal{P}}^{i_5 i_5}(k_2) \hat{\mathcal{P}}^{j_1 j_2}(k_3) \hat{\mathcal{P}}^{i_1 i_5}(k_3) \hat{\mathcal{P}}^{i_1 i_5}(k_3) \\
&+ \sum_{i_5, i_5' = 1}^{3} e^{-|k_2|^2 |\sigma| \mathcal{I}_f (e k_2)^2} \frac{1}{2 |k_2|^2} \\
&\quad \times \hat{\mathcal{P}}^{i_5 i_5}(k_2) \hat{\mathcal{P}}^{i_5 i_5}(k_2) \hat{\mathcal{P}}^{j_1 j_2}(k_3) \hat{\mathcal{P}}^{i_1 i_5}(k_3) \hat{\mathcal{P}}^{i_1 i_5}(k_3)
\end{align*}\]
(197)

due to (116). We apply (197) to (196) to deduce
\[\begin{align*}
E[|\Delta q X_{i}^{8,2}|^2]
&\leq \sum_{k} \sum_{[i-j] \leq 1} \sum_{|i'| \leq 1} \sum_{k_2, k_3: k_2+k_3 \neq k, k_1, k_4 \neq 0} \theta(2^{-1} k_{123}) \theta(2^{-1} k_{234}) \theta(2^{-j} k_1) \\
&\quad \times \theta(2^{-j} k_4) \theta(2^{-q} k_2) \prod_{i=1}^{4} \frac{f(e k_i)}{|k_i|^2} \int_{[0, t]^{2}} e^{-|k_{123}|^2 (s-\tau)} - |k_{234}|^2 (t-\tau) \\
&\quad \times \int_{0}^{s} \int_{0}^{\tau} e^{-|k_{12}|^2 (s-\sigma)} - |k_{24}|^2 (\sigma-\tau) |k_{12}||k_{24}| |k_{123}||k_{234}| \\
&\quad + e^{-|k_{12}|^2 (s-\sigma)} - |k_{34}|^2 (\sigma-\tau) |k_{12}||k_{34}| |k_{123}||k_{234}| \\
&\quad \times e^{-|k_{1}^2 (s-\tau)} - |k_{4}|^2 (t-\tau) d\sigma d\tau ds d\tau
\end{align*}\]
(198)
by a change of variable of $k_1'$ to $-k_4$. Within (198), we may estimate furthermore for $k_1, k_2, k_3, k_4 \neq 0$,
\[\begin{align*}
&\prod_{i=1}^{4} \frac{f(e k_i)}{|k_i|^2} \int_{[0, t]^{2}} e^{-|k_{123}|^2 (s-\tau)} - |k_{234}|^2 (t-\tau) \\
&\quad \times \int_{0}^{s} \int_{0}^{\tau} e^{-|k_{12}|^2 (s-\sigma)} - |k_{24}|^2 (\sigma-\tau) |k_{12}||k_{24}| |k_{123}||k_{234}| \\
&\quad + e^{-|k_{12}|^2 (s-\sigma)} - |k_{34}|^2 (\sigma-\tau) |k_{12}||k_{34}| |k_{123}||k_{234}| \\
&\quad \times e^{-|k_{1}^2 (s-\tau)} - |k_{4}|^2 (t-\tau) d\sigma d\tau ds d\tau
\end{align*}\]
(199)
\[\leq \prod_{i=1}^{4} \frac{1}{|k_i|^2} e^{-|k_{123}|^2 - |k_{234}|^2 - |k_{1}^2 - |k_{4}^2|^2 s} \int_{[0, t]^{2}} e^{k_{123}^2 s + |k_{234}|^2 \tau} |k_{12}||k_{123}||k_{234}| \\
&\quad \times \frac{1}{|k_{24}|^2} e^{-|k_{12}|^2 + |k_{4}^2| |k_{12}|^2 + |k_{1}^2|^2} e^{-|k_{24}|^2 + |k_{4}^2| |k_{1}^2|^2} 1_{k_{12}, k_{24} \neq 0} \\
&\quad + \frac{1}{|k_{34}|^2} e^{-|k_{12}|^2 + |k_{4}^2| |k_{12}|^2 + |k_{1}^2|^2} e^{-|k_{34}|^2 + |k_{4}^2| |k_{1}^2|^2} 1_{k_{12}, k_{34} \neq 0} ds d\tau
\end{align*}\]
\[\leq \prod_{i=1}^{4} \frac{1}{|k_i|^2} \frac{e^{-|k_{123}|^2 - |k_{234}|^2 - |k_{1}^2 - |k_{4}^2|^2 s}}{|k_{24}|^2 + |k_{1}|^2} e^{-|k_{12}|^2 (s-\tau) - |k_{24}|^2 (\tau-\sigma) - |k_{34}|^2 (\tau-\sigma) - |k_{4}|^2 (t-\tau)} d\sigma d\tau ds d\tau
\]
by mean value theorem. Thus, applying (199) to (198) leads to
\[
\mathbb{E}(|\Delta_q I X_t^{8,2}|) \lesssim \sum_k \sum_{|i-j|\leq 1, |i'-j'|\leq 1} \sum_{k_2, k_3 \neq 0: k_2 = k, k_3 = k} \sum_{k_4 \neq 0} \theta(2^{-i} k_{123}) \theta(2^{-j} k_3) \theta(2^{-i'} k_{234}) \theta(2^{-j'} k_4) \theta(2^{-q} k)^2 \tag{200}
\]
Now \(2^q \approx |k| = |k_2 + k_3| \leq |k_{123}| + |k_3| \approx 2^q\) as \(|i - j| \leq 1\) so that \(q \lesssim i\). Similarly \(2^q \lesssim 2^{i'}\) as \(|i' - j'| \leq 1\) so that \(q \lesssim i'\). Thus for \(\epsilon \in (0, 1 - \eta)\) sufficiently small we estimate from (200),
\[
\mathbb{E}(|\Delta_q I X_t^{8,2}|) \lesssim \sum_k \sum_{k_2, k_3 \neq 0: k_2 = k, k_3 = k} \sum_{k_4 \neq 0} \frac{2^q \theta(2^{-q} k)^2}{|k_2|^2 |k_3|^2 |k_4|^{4 - \eta}} \lesssim 2^q \theta(2^{-q} k) \tag{201}
\]
by Lemma 4.7. The estimate of \(I X_t^{8,3}\) may be achieved very similarly to \(I X_t^{8,2}\).
We now consider \(I X_t^{8,4}\) of (194). Let us make an important remark here.

**Remark 3.4.** In particular, this is the renormalization on which we must diverge from the previous study of a single equation (stochastic quantization [8] or NSE [56]) instead of a system of coupled non-linear PDE such as the MHD system. For example, if we write
\[
I X_t^{8,4} = I X_t^{8,4} - \tilde{I} X_t^{8,4} + \tilde{I} X_t^{8,4} - \sum_{i_1=1}^{3} u_{i_2}^{i_1}(t) C_{3}^{i_1 i_1}(t) \tag{202}
\]
where
\[
\tilde{I} X_t^{8,4} \triangleq (2\pi)^{-\frac{7}{2}} \sum_{k \neq 0} \sum_{|i-j|\leq 1, k_{12} = k, k_3 \neq 0, i_1, i_2, i_3, j_1, j_2} \sum_{j_1, j_2} \theta(2^{-i} k_{123}) \theta(2^{-j} k_3)
\times \int_0^t \int_{\mathbb{R}^d} e^{-|k_{12}|^2 (t-s)} f(\partial_k) \frac{(e(\partial_k))^2}{|k_2|^2} \mathcal{P}^{11_{j_1} i_1}(k_3) \mathcal{P}^{j_{i_2} i_2}(k_3) \mathcal{P}^{j_{i_3} i_3}(k_{123}) \mathcal{P}^{i_{i_1} i_1}(k_{123}) dx \tag{203}
\]
and
\[
C_{3}^{i_1 i_1}(t) \triangleq (2\pi)^{-\frac{7}{2}} \sum_{|i-j|\leq 1} \sum_{k_{12} = k, k_3 \neq 0} \sum_{j_1, j_2} \theta(2^{-i} k_3) \theta(2^{-j} k_3) \int_0^t \frac{e^{-2|k_2|^2 (t-s)} f(\partial_k)}{|k_2|^2} \mathcal{P}^{11_{j_1} i_1}(k_3) \mathcal{P}^{j_{i_2} i_2}(k_3) \mathcal{P}^{j_{i_3} i_3}(k_{123}) \mathcal{P}^{i_{i_1} i_1}(k_{123}) dx \tag{204}
\]

as Zhu and Zhu did for the NSE (see [56] pg. 4489), then the necessary estimate of \(\tilde{I} X_t^{3,4} - \sum_{i_1=1}^{3} u_{i_2}^{i_1}(t) C_{3}^{i_1 i_1}(t)\) on [56] pg. 4491] works well because
\[
Lu_{i_2}^{i_1} = -\frac{1}{2} \sum_{i_1=1}^{2} \mathcal{P}^{i_1} \left( \sum_{j=1}^{3} \partial_{x_1} (u_{i_1}^{i_1} \circ u_{i_1}^{j}) \right)
\]
The system does not work due to the additional term of \( b_1^i \circ b_1^i \) in (7).

This creates a huge obstacle.

We can actually overcome this difficulty remarkably by considering the sum of (189) and (205). This technique of coupled renormalizations is very reminiscent of the basic energy identity (7) and (8) actually. We emphasize that it must be \( u_{00} \) that we couple with \( u \), not any other \( u_{0k} \) for \( k \in \{1, \ldots, 6\} \).

Now recalling (189), we see that the only differences between \( Lu_{00}^i \) and \( Lu_{00}^i \) in (181) consist of the sign and \( b_1^i u_1^i \) replaced by \( u_1^i b_1^i \) so that we have

\[
IX_t^{7,4} = \frac{1}{4(2\pi)^2} \sum_k \sum_{[i-j] \leq 1} \sum_{k, k_1, k_2, k_{12} = k, k_3 \neq 0 \ i_1, i_2, i_3, i_4, j_1 = 1} \sum_{2} \theta(2^{-i} k_{123}) \theta(2^{-j} k_3)
\]

\[
\times \int_0^t e^{-|k_{123}|^2 (t-s)} \int_0^t e^{-|k_3|^2 (t-s)} f(ek_3)^2 \frac{\hat{P}_{i1i2}(k_3)}{2|k_3|^2} \hat{P}_{j0i3}(k_3) \theta(2^{-i} k_{123}) \theta(2^{-j} k_3)
\]

\[
\times : \hat{X}_{b,\sigma}^\epsilon i_2(k_1) \hat{X}_{u,\sigma}^\epsilon i_3(k_2) : e^{-|k_{123}|^2 (t-s)} d\sigma ds
\]

\[
\times k_{12}^i k_{123}^i \hat{P}_{i1i2}(k_1) \hat{P}_{j0i3}(k_3) e_k.
\]

In sum of (189) and (205) we obtain

\[
IX_t^{7,4} + IX_t^{8,4} = \frac{1}{4(2\pi)^2} \sum_k \sum_{[i-j] \leq 1} \sum_{k, k_1, k_2, k_{12} = k, k_3 \neq 0 \ i_1, i_2, i_3, i_4, j_1 = 1} \sum_{2} \theta(2^{-i} k_{123}) \theta(2^{-j} k_3)
\]

\[
\times \int_0^t e^{-|k_{123}|^2 (t-s)} \int_0^t e^{-|k_3|^2 (t-s)} f(ek_3)^2 \frac{\hat{P}_{i1i2}(k_3)}{2|k_3|^2} \hat{P}_{j0i3}(k_3)
\]

\[
\times : \hat{X}_{b,\sigma}^\epsilon i_2(k_1) \hat{X}_{u,\sigma}^\epsilon i_3(k_2) : e^{-|k_{123}|^2 (t-s)} d\sigma ds k_{12}^i k_{123}^i \hat{P}_{i1i2}(k_1) \hat{P}_{j0i3}(k_3) e_k.
\]

We define now

\[
IX_t^{7,8,4} = \frac{1}{4(2\pi)^2} \sum_k \sum_{[i-j] \leq 1} \sum_{k, k_1, k_2, k_{12} = k, k_3 \neq 0 \ i_1, i_2, i_3, i_4, j_1 = 1} \sum_{2} \theta(2^{-i} k_{123}) \theta(2^{-j} k_3)
\]

\[
\times \int_0^t e^{-|k_{123}|^2 (t-s)} \int_0^t e^{-|k_3|^2 (t-s)} f(ek_3)^2 \frac{\hat{P}_{i1i2}(k_3)}{2|k_3|^2} \hat{P}_{j0i3}(k_3) d\sigma
\]

\[
\times \int_0^t e^{-|k_{123}|^2 (t-s)} e^{-|k_3|^2 (t-s)} f(ek_3)^2 \frac{\hat{P}_{i1i2}(k_3)}{2|k_3|^2} \hat{P}_{j0i3}(k_3) k_{123}^i dse_k,
\]
Within (210) we first focus on
\[ \int_0^t e^{-(|k_3|)^2(s-t)} e^{-|k_3|^2(s-t)} f(\varepsilon k_3)^2 2|k_3|^2 ds \]
where it can be readily confirmed that \( C_3^{7,8,e,i} (t) = 0 \). Now we split
\[ IX_t^{7,4} + IX_t^{8,4} = (IX_t^{7,4} + IX_t^{8,4}) - \tilde{IX}_t^{7,4} + \tilde{IX}_t^{7,4} - \sum_{i=1}^3 f_{7,8} (t) C_3^{7,8,e,i} (t). \] (209)
Within (209) we first work on
\[ (IX_t^{7,4} + IX_t^{8,4}) - \tilde{IX}_t^{7,4} \]
\[ \frac{1}{4(2\pi)^2} \sum_k \sum_{|i-j|\leq 1} \sum_{k_1,k_2,k_3 \neq 0} \sum_{t_{1,2,3},j_1=1} \theta(2^{-i} k_3) \theta(2^{-j} k_3) \]
\[ \times \int_0^t e^{-|k_3|^2(s-t)} e^{-|k_3|^2(s-t)} f(\varepsilon k_3)^2 2|k_3|^2 ds \]
\[ \int_0^s : \hat{X}^{e,i} (k_1) \hat{X}^{e,i} (k_2) : - : \hat{X}^{e,i} (k_1) \hat{X}^{e,i} (k_2) : e^{-|k_3|^2(s-t)} d\sigma \]
\[ - \int_0^s : \hat{X}^{e,i} (k_1) \hat{X}^{e,i} (k_2) : - : \hat{X}^{e,i} (k_1) \hat{X}^{e,i} (k_2) : e^{-|k_3|^2(s-t)} d\sigma \]
\[ \int_0^s : \hat{X}^{e,i} (k_1) \hat{X}^{e,i} (k_2) : - : \hat{X}^{e,i} (k_1) \hat{X}^{e,i} (k_2) : \]
\[ \times (e^{-|k_3|^2(s-t)} - e^{-|k_3|^2(t-s)}) d\sigma \]
\[ \int_0^s : \hat{X}^{e,i} (k_1) \hat{X}^{e,i} (k_2) : - : \hat{X}^{e,i} (k_1) \hat{X}^{e,i} (k_2) : e^{-|k_3|^2(t-s)} d\sigma. \]
\[ (211) \]
where we relied on (206) and (207). Within (210) we first focus on
\[ \int_0^s : \hat{X}^{e,i} (k_1) \hat{X}^{e,i} (k_2) : - : \hat{X}^{e,i} (k_1) \hat{X}^{e,i} (k_2) : e^{-|k_3|^2(s-t)} d\sigma \]
\[ - \int_0^s : \hat{X}^{e,i} (k_1) \hat{X}^{e,i} (k_2) : - : \hat{X}^{e,i} (k_1) \hat{X}^{e,i} (k_2) : e^{-|k_3|^2(s-t)} d\sigma \]
\[ \int_0^s : \hat{X}^{e,i} (k_1) \hat{X}^{e,i} (k_2) : - : \hat{X}^{e,i} (k_1) \hat{X}^{e,i} (k_2) : \]
\[ \times (e^{-|k_3|^2(s-t)} - e^{-|k_3|^2(t-s)}) d\sigma \]
\[ \int_0^s : \hat{X}^{e,i} (k_1) \hat{X}^{e,i} (k_2) : - : \hat{X}^{e,i} (k_1) \hat{X}^{e,i} (k_2) : e^{-|k_3|^2(t-s)} d\sigma. \]
\[ (212) \]
We also define for \( k_3 \neq 0 \),
\[ C_{k_1,k_2,k_3}^{j_1} (t-s) \triangleq \sum_{i=1}^3 e^{-|k_3|^2(s-t)} e^{-|k_3|^2(s-t)} f(\varepsilon k_3)^2 2|k_3|^2 |k_3| \tilde{\P}^{j_1} (k_3) | \tilde{\P}^{j_1} (k_3) | \]
\[ (212) \]
so that we can now estimate

\[
E[\Delta_t((IX_t^{7,4} + IX_t^{8,4}) - \hat{I}X_t^{7,8,4})^2] \\
\leq \sum_{k,k'} \theta(2^{-g}k)\theta(2^{-g}k') \sum_{|i-j|\leq 1, |i'-j'|\leq 1} \sum_{k_1,k_2,k_3,k_4} \theta(2^{-i}k_1)\theta(2^{-j}k_2)\theta(2^{-i'}k_1')\theta(2^{-j'}k_2') \int_{[0,4]^2} \\
\times C_{k_1,k_2,k_3,k_4}^{|t-s|} d\sigma d\sigma \\
\times \int_0^4 \int_0^4 \mathbb{E}[\hat{X}_{b,\sigma}^{i_1,j_2}(k_1)\hat{X}_{u,\sigma}^{i_3,j_4}(k_2) : \hat{X}_{b,\sigma}^{i_1',j_2'}(k_1')\hat{X}_{u,\sigma}^{i_3',j_4'}(k_2')] \\
\times (e^{-|k_{12}|^2|s-\sigma|} - e^{-|k_{12}|^2(t-\sigma)}) e^{-|k_{12}'|^2|\sigma-\tau|} - e^{-|k_{12}'|^2(t-\tau)} d\sigma d\tau \tag{213} \\
+ \int_0^s \int_0^s \mathbb{E}[\hat{X}_{b,\sigma}^{i_1,j_2}(k_1)\hat{X}_{u,\sigma}^{i_3,j_4}(k_2) : \hat{X}_{b,\sigma}^{i_1',j_2'}(k_1')\hat{X}_{u,\sigma}^{i_3',j_4'}(k_2')] \\
\times e^{-|k_{12}|^2|(t-\sigma)+|t-\tau|) d\sigma d\tau \\
+ \int_0^s \int_0^s \mathbb{E}[\hat{X}_{u,\sigma}^{i_1,j_2}(k_1)\hat{X}_{b,\sigma}^{i_3,j_4}(k_2) : \hat{X}_{b,\sigma}^{i_1',j_2'}(k_1')\hat{X}_{u,\sigma}^{i_3',j_4'}(k_2')] \\
\times (e^{-|k_{12}|^2(s-\sigma) - e^{-|k_{12}|^2(t-\sigma)}) e^{-|k_{12}'|^2|\sigma-\tau|} - e^{-|k_{12}'|^2(t-\tau)} d\sigma d\tau \\
+ \int_0^s \int_0^s \mathbb{E}[\hat{X}_{u,\sigma}^{i_1,j_2}(k_1)\hat{X}_{b,\sigma}^{i_3,j_4}(k_2) : \hat{X}_{u,\sigma}^{i_1',j_2'}(k_1')\hat{X}_{b,\sigma}^{i_3',j_4'}(k_2')] \\
\times e^{-|k_{12}|^2|(t-\sigma)+|t-\tau|) d\sigma d\tau \\
\triangleq \sum_{i=1}^4 \Delta_i \\
\]
due to (110). Applying this to (213) leads to

\[ X^1 + X^2 \lesssim \sum_k \theta(2^{-q}k)^2 \sum_{|i-j| \leq 1, |i'-j'| \leq 1} \sum_{k_1, k_2 \neq 0, k_1 = k, k_3, k_4 \neq 0} \sum_{j_1, j_2=1}^{3} |k_{12}|^2 \times \int_{[0, t]^2} \theta(2^{-i}k_{123})\theta(2^{-i'}k_{124})\theta(2^{-j}k_3)\theta(2^{-j'}k_4) \times C_{k_{123}, k_3}^{j_1}(t-s)C_{k_{124}, k_4}^{j_2}(t-\sigma) \int_0^s \frac{e^{-(|k_1|^2 + |k_2|^2)|\sigma-\tau|}}{|k_1|^2|k_2|^2} \times (e^{-|k_{12}|^2(t-\sigma)} - e^{-|k_{12}|^2(t-\sigma)}) \times (e^{-|k_{12}|^2(t-\sigma)} - e^{-|k_{12}|^2(t-\tau)}) d\sigma d\bar{\sigma} \]

(215)

where we used a change of variable of \( k_3 \) with \(-k_4 \). Within (215) we may further estimate for \( k_{12} \neq 0 \),

\[ \int_0^s \frac{e^{-(|k_1|^2 + |k_2|^2)|\sigma-\tau|}}{|k_1|^2|k_2|^2} \times (e^{-|k_{12}|^2(t-\sigma)} - e^{-|k_{12}|^2(t-\sigma)}) \times (e^{-|k_{12}|^2(t-\sigma)} - e^{-|k_{12}|^2(t-\tau)}) d\sigma d\bar{\sigma} \]

(216)

due to mean value theorem and (14). Therefore, applying (216) to (215) gives

\[ X^1 + X^2 \lesssim \sum_k \theta(2^{-q}k)^2 \sum_{|i-j| \leq 1, |i'-j'| \leq 1} \sum_{k_1, k_2 \neq 0, k_1 = k, k_3, k_4 \neq 0} \sum_{j_1, j_2=1}^{3} |k_{12}|^2 \times \theta(2^{-i}k_{123})\theta(2^{-i'}k_{124})\theta(2^{-j}k_3)\theta(2^{-j'}k_4) \int_{[0, t]^2} C_{k_{123}, k_3}^{j_1}(t-s) \times C_{k_{124}, k_4}^{j_2}(t-\sigma) \frac{1}{|k_{123}|^2|k_3|^2|k_4|^2} (t-s)^{3} (t-\sigma)^{\frac{1}{2}} dsd\bar{\sigma}. \]

(217)

Moreover, for \( k_3, k_4 \neq 0 \),

\[ \int_{[0, t]^2} (t-s)^{\frac{1}{2}}(t-\sigma)^{\frac{1}{2}} C_{k_{123}, k_3}^{j_1}(t-s)C_{k_{124}, k_4}^{j_2}(t-\sigma) dsd\bar{\sigma} \]

\[ \lesssim |k_{123}|^2|k_{124}|^2, \frac{1-e^{-\frac{1}{2}(|k_{123}|^2 + |k_3|^2)t}}{|k_3|^2|k_4|^2}, \frac{1-e^{-\frac{1}{2}(|k_{124}|^2 + |k_4|^2)t}}{|k_{124}|^2 + |k_4|^2} \]

\[ \lesssim |k_3|^2|k_4|^2(|k_{123}|^2 + |k_3|^2)^{\frac{1}{2}} - (\frac{1}{2} + \frac{1}{2}) \]

(218)
by (212) and (14). Applying (218) to (217) leads to

\[
X^1 + X^2 \lesssim \sum_{k \neq 0} \theta(2^{-q}k)^2 \sum_{k_1, k_2 \neq 0; k_1 = k} \frac{t^2(t + \theta)}{|k_{12}|^2 |k_1|^2 |k_2|^2} \times \sum_{q \leq k \leq 2q} \frac{1}{2^{q}(|k - \hat{k} + \theta|)} \frac{1}{2^{q}(|k - \hat{k} + \theta|)} \lesssim t^2(t + \theta)2^{q(3(t + \theta))} \sum_{2^{q-1} \leq |k| \leq 2^{q+1}} \frac{1}{|k|^3} \lesssim t^2(t + \theta)2^{q(3(t + \theta))}
\]  

(219)

where we used that \(2^q \lesssim 2^i, 2^q \lesssim 2^i\) and Lemma 4.7. Similar estimates may be obtained for \(X^3\) and \(X^4\). Therefore, we conclude by applying (219) to (218) that

\[
E[\|\Delta_q((IX_i^7,4 + IX_i^8,4) - \tilde{I}_X^7,8,4)]^2] \lesssim t^2(t + \theta)2^{q(3(t + \theta))}. 
\]

(220)

Next, within (209) we now work on \(E[\|\Delta_q(\tilde{I}_X^7,8,4 - \sum_{i_1=1}^3 b_{2,i_1}^e(t)C_{3}^{7,8,e,i_1}(t))|^2]\) where we may write

\[
\sum_{i_1=1}^3 b_{2,i_1}^e(t)C_{3}^{7,8,e,i_1}(t) = \frac{1}{4(2\pi)^2} \sum_{k \neq 0} \sum_{|\sigma| \leq 1} \sum_{k_1,k_2,k_3 \neq 0} \sum_{i_1,i_2,i_3,i_4,j_1=1}^3 \int_0^t e^{-|k_{12}|^2(t-\sigma)} \tilde{P}^{j_1,i_1}(k_1) \tilde{P}^{j_2,i_2}(k_2) - \tilde{P}^{j_1,i_1}(k_1) \tilde{P}^{j_2,i_2}(k_2) d\sigma \times \theta(2^{-i}k_3) \theta(2^{-j}k_3) \int_0^t \frac{e^{-2|k_3|^2(t-s)} f(k_3)^2}{2|k_3|^2} \times \tilde{P}^{j_3,i_3}(k_3) \tilde{P}^{j_4,i_4}(k_3) \tilde{P}^{j_5,i_5}(k_3) dse_k 
\]

(221)

by (17), (24d) and (24e). Thus, by (207) and (221) we obtain

\[
\tilde{I}_X^7,8,4 - \sum_{i_1=1}^3 b_{2,i_1}^e(t)C_{3}^{7,8,e,i_1}(t) 
\]

\[
= \frac{1}{4(2\pi)^2} \sum_{k \neq 0} \sum_{|\sigma| \leq 1} \sum_{k_1,k_2,k_3 \neq 0} \sum_{i_1,i_2,i_3,i_4,j_1=1}^3 \int_0^t e^{-|k_{12}|^2(t-\sigma)} \tilde{P}^{j_1,i_1}(k_1) \tilde{P}^{j_2,i_2}(k_2) \times \frac{e^{-|k_{12}|^2(t-s)} \theta(2^{-i}k_3) \theta(2^{-j}k_3) \tilde{P}^{j_3,i_3}(k_3) \tilde{P}^{j_4,i_4}(k_3) \tilde{P}^{j_5,i_5}(k_3) dse_k}{2|k_3|^2} 
\]

(222)
Due to similarity, let us work only on $XI_1^i$, to which we use (13b) to deduce

$$
XI_1^i = \frac{1}{4(2\pi)^2} \sum_{k \neq 0} \sum_{|i-j| \leq 1} \sum_{k_1, k_2 = k, k \neq 0} \sum_{i_1, i_2, j_1 = 1} \sum_{j_1 = 1}^{3} e^{-|k_1|^2 |t-s|} \hat{\theta}(2^{-j} k_3) \hat{X}_{\sigma}^{\epsilon, i_3} (k_1) \hat{X}_{\sigma}^{\epsilon, i_3} (k_2) \\
\times \int_{[0,t]}^2 e^{-|k_1|^2 |t-s|} \hat{P}_{i_1 i_3} (k_2) \hat{P}_{i_2 i_3} (k_2) \hat{P}_{j_1 i_3} (k_3) \hat{P}_{j_2 i_3} (k_3) d\sigma ds.
$$

Now upon computing $E[\Delta_q (\hat{X}_t^{7,8,4} - \sum_{i=1}^3 b_{i}^{\epsilon, i_1} (t) C_{i}^{7,8,\epsilon, i_3} (t))]$, we need to compute $E[|\Delta_q X I_1^i|^2]$. In its endeavor, we rely on (13d), Lemma 2.2 and (116) to deduce

$$
E[\hat{X}_{\sigma}^{\epsilon, i_1} (k_1) \hat{X}_{\sigma}^{\epsilon, i_3} (k_2) \hat{X}_{\sigma}^{\epsilon, i_1} (k_1) \hat{X}_{\sigma}^{\epsilon, i_3} (k_2)] \\
= 1_{k_2 + k_2' = 0, k_1 + k_2' = 0, k_1, k_2 \neq 0} \sum_{i_1, i_2 = 1}^{3} e^{-|k_2|^2 |\sigma-\bar{\sigma}|} f(k_2)^2 \\
\times \hat{P}_{i_1 i_2} (k_2) \hat{P}_{i_3 i_2} (k_2) \hat{P}_{i_1 i_3} (k_1) \hat{P}_{i_2 i_3} (k_1) \\
+ 1_{k_2 + k_2' = 0, k_1 + k_2' = 0, k_1, k_2 \neq 0} \sum_{i_1, i_2 = 1}^{3} e^{-|k_2|^2 |\sigma-\bar{\sigma}|} f(k_2)^2 \\
\times \hat{P}_{i_1 i_2} (k_2) \hat{P}_{i_3 i_2} (k_2) \hat{P}_{i_1 i_3} (k_1) \hat{P}_{i_2 i_3} (k_1) \\
+ 1_{k_1 + k_1' = 0, k_1 + k_1' = 0, k_1, k_1' \neq 0} \sum_{i_1, i_2 = 1}^{3} f(k_1)^2 \\
\times \hat{P}_{i_1 i_2} (k_1) \hat{P}_{i_3 i_2} (k_1) \hat{P}_{i_1 i_3} (k_1) \hat{P}_{i_2 i_3} (k_1)
$$

where the third term will vanish due to $1_{k_1 = 0}$ and $k_1^3$ in the integrand of (223). Thus

$$
E[|\Delta_q X I_1^i|^2] \\
\leq \sum_{k, k \neq 0} \sum_{|i-j| \leq 1} \sum_{k_1, k_1' = k, k \neq 0} \sum_{i_1, i_2 = 1}^{3} e^{-|k_1|^2 |t-s|} - k_1^3 |t-s|} \hat{\theta}(2^{-j} k_3) \hat{\theta}(2^{-j} k) d\sigma d\bar{\sigma}
\times \left[ \sum_{i-j = 1}^3 \sum_{i_1, j_1 = 1}^{3} \sum_{k_3 \neq 0} \theta(2^{-j} k_3) \int_0^t e^{-|k_3|^2 |t-s|} f(k_3)^2 \right]
$$
\( \times \left( e^{-\|k_{123}\|^2(t-s)} \theta(2^{-i}k_{123}) \hat{P}^{ioi_1}(k_{123})k_{123}^{j_1} - e^{-\|k\|^2(t-s)} \theta(2^{-i}k) \hat{P}^{ioi_1}(k)k_{12}^{j_1} \right) ds \),

where we may observe that \(|k_{12}| = |k_{12}| \) due to a straight-forward extension of Lemma 4.6, Lemma 4.7 and (14). We obtain

\[
\int_{[0,t]^2} e^{-|k_{12}|^2(t-\sigma)}(t-\sigma)^{-\frac{5}{2}} |k_{12}| |k_{12}| e^{-\|k_{12}\|^2+|k_{12}|^2} e^{-\|k\|^2} ds d\sigma \lesssim \frac{1}{|k_{12}|^2}
\]

for \( k_{12} \neq 0 \). Therefore, (210) gives for any \( \eta \in (0,1) \),

\[
\mathbb{E}[|\Delta_t X_{1t}|^2] \lesssim \sum_{k \neq 0} \sum_{k_1,k_2 
eq 0:k_{12}=k} \theta(2^{-q}k)^2
\times \left[ \sum_{|j|=1} \sum_{k_3 \neq 0} \theta(2^{-j}k_3) \int_0^t e^{-|k_3|^2(t-s)} f(ck_3)^2 \right]
\times (e^{-|k_{12}|^2(t-s)} \theta(2^{-i}k_{12}) \hat{P}^{ioi_1}(k_{123})k_{123}^{j_1},

\]

or (225).

due to a straight-forward extension of Lemma 4.6, Lemma 4.7 and (14). We obtain similar estimates for \( \mathbb{E}[|\Delta_t X_{2t}|^2] \) in (222). Together with (225), this concludes our estimate of

\[
\mathbb{E}[|\Delta_t (IX_{t}^{7,4} + IX_{t}^{8,4})|^2] \lesssim 2^{2q(\eta+\tilde{\eta})} t^{\tilde{\eta}(3\tilde{\eta}+\tilde{\eta})}
\]

if we choose \( \epsilon, \eta > 0 \) such that \( \epsilon \leq \frac{4}{3} \).

For \( IX_{t}^{k}, k \in \{1,\ldots,6\} \), in (194), we obtained estimates of \( IX_{t}^{8,2} \) in (201) and \( IX_{t}^{7,4} + IX_{t}^{8,4} \) in (222). Next, within (194), let us work on

\[
IX_{t}^{8,5} = IX_{t}^{8,5} - \tilde{IX}_{t}^{8,5} + \tilde{IX}_{t}^{8,5} - \tilde{IX}_{t}^{8,5}
\]

where

\[
\tilde{IX}_{t}^{8,5} \triangleq \frac{1}{4(2\pi)^2} \sum_{k} \sum_{|j|=1} \sum_{k_1,k_2 \neq 0} \sum_{k_3,k_4} \sum_{i_1,i_2,i_3,i_4,j_1=1} \theta(2^{-i}k_1) \theta(2^{-j}k_4)
\times \int_0^t \tilde{X}_{u}^{i_1,i_2}(k_1) \tilde{X}_{u}^{i_3,i_4}(k_4) e^{-|k_i|^2(t-s)} k_1^{i_1} \hat{P}^{ioi_1}(k_1) \int_0^s e^{-|k_{12}|^2(s-\sigma)} \frac{e^{-|k_{12}|^2(t-\sigma)} f(ck_3)^2}{|k_2|^2} \hat{P}^{ioi_2}(k_{12}) \hat{P}^{ioi_3}(k_2) \hat{P}^{ioi_4}(k_2) d\sigma ds e_k
\]

(228)
and
\[
\mathcal{T}X_t^{3,5} = -\frac{1}{4(2\pi)^2} \sum_k \sum_{|i-j|\leq 1} \sum_{k_1 k_2 k_3 k_4 = k, k_2 \neq 0} \sum_{i_1 i_2 i_3 j_1 = 1}^3 \theta(2^{-i} k_1) \theta(2^{-j} k_4) \\
\times \int_0^t \hat{X}_{u,i}^{t,12}(k_1) \hat{X}_{b,t}^{t,0}(k_4) : e^{-|k_1|^2 (t-s) k_1^4} \mathcal{P}_{\gamma i i i}(k_1) \int_0^s e^{-|k_2|^2 (s-\sigma)} \\
\times \frac{f(\epsilon k_2)^2}{|k_2|^2} t_2 \mathcal{P}_{\gamma i i 2}(k_2) \mathcal{P}_{\gamma i i 4}(k_2) d\sigma \epsilon_k
\]
so that \( \mathcal{T}X_t^{3,5} = 0 \). We define for \( k_2 \neq 0 \),
\[
d_{k_1 k_2}(s-\sigma) \triangleq \sum_{i_1 i_2 i_3 = 1}^3 e^{-|k_1|^2 (t-s)} e^{-|k_2|^2 (s-\sigma)} f(\epsilon k_2)^2 |k_2|^2 \mathcal{P}_{\gamma i i 12}(k_2).
\]
Then we see that
\[
\mathbb{E} [\hat{X}_{u,i}^{t,12}(k_1) \hat{X}_{b,t}^{t,0}(k_4) : - : \hat{X}_{u,j}^{t,12}(k_1) \hat{X}_{b,t}^{t,0}(k_4) ] = 0
\]
by Example 222. Now if we group within (229), then for \( k_1, k_4 \neq 0 \) we may estimate
\[
(XII^3 + XII^6) + (XII^5 + XII^7) + (XII^2 + XII^4) + (XII^6 + XII^8)
\leq (1_{k_1 + k_2 = 0, k_4 + k_3 = 0} |k_1|^n |k_4|^n + 1_{k_1 + k_2 = 0, k_4 + k_3 = 0} |k_4|^n |k_1|^n)
\times \frac{1}{|k_1||k_2||k_4|} |s - \sigma|^n
\]
by (116) and mean value theorem while instead, if we regroup to
\[
(XII^1 + XII^5) + (XII^3 + XII^7) + (XII^2 + XII^6) + (XII^4 + XII^8),
\]
we get a same bound in (232) except with $|s - \sigma|^\eta$ instead of $|\sigma - \sigma'|^\eta$. Hence, we obtain by applying (232) to (231) and interpolation that

$$
\mathbb{E} : \hat{X}_{t,0}^{\epsilon,\sigma}(k_1) \hat{X}_{t,0}^{\epsilon,\omega}(k_4) : - : \hat{X}_{t,0}^{\epsilon,\sigma}(k_1) \hat{X}_{t,0}^{\epsilon,\omega}(k_4) :
\times : \hat{X}_{t,0}^{\epsilon,\sigma}(k_1') \hat{X}_{t,0}^{\epsilon,\omega}(k_4') :
\lesssim (1_{k_1+k_1'=0,k_4+k_4'=0} |k_1|^\eta |k_4'|^\eta + 1_{k_1+k_4=0,k_1+k_4'=0} |k_4|^\eta |k_1'|^\eta)
\times \frac{1}{|k_1||k_1'||k_4||k_4'|} |s - \sigma|^\frac{2}{2+\eta} |s - \sigma'|^\frac{2}{2+\eta}.
$$

(233)

We use (117), (228), (230) and (233) to estimate now

$$
\mathbb{E} [\|\Delta_y (IX_t^{8.5} - IX_t^{8.5})\|^2]
\lesssim \sum_k \theta (2^{-\eta} k)^2 \sum_{|t-j| \leq 1, |t'-j'| \leq 1} \sum_{k_1+k_1'=0:k_4+k_4'=k,k_2,k_3 \neq 0}
\times \theta (2^{-i} k_1) \theta (2^{-i'} k_1') \theta (2^{-j} k_4) \theta (2^{-j'} k_4')
\times \int_{[0,t]^2} \int_{[0,s] \times [0,\mathbb{R}]} e^{-|k_1|^2 (t-s-t-\sigma)} \frac{|k_1|^{2+\eta}}{|k_1|^2 |k_4|^2} |s - \sigma|^\frac{2}{2+\eta} |s - \sigma'|^\frac{2}{2+\eta}
\times d_{k_1,k_2}(s - \sigma) d_{k_3,k_4} (\sigma - \sigma') |k_1|^2 |k_4|^2 |s - \sigma|^\frac{2}{2+\eta} |s - \sigma'|^\frac{2}{2+\eta}
\times d_{k_1,k_2}(s - \sigma) d_{k_3,k_4} (\sigma - \sigma') |k_1| |k_4| d\sigma d\bar{\sigma} ds d\bar{s}
$$

(234)

where we used a change of variable of $k_2'$ with $-k_3$. We can estimate for $k_1, k_4 \neq 0$,

$$
\sum_{k_2,k_3 \neq 0} \int_{[0,t]^2} \int_{[0,s] \times [0,\mathbb{R}]} e^{-|k_1|^2 (t-s-t-\sigma)} \frac{|k_1|^{2+\eta}}{|k_1|^2 |k_4|^2} |s - \sigma|^\frac{2}{2+\eta} d_{k_1,k_2}(s - \sigma) d_{k_3,k_4} (\sigma - \sigma') d\sigma d\bar{\sigma} ds d\bar{s}
\lesssim \frac{1}{|k_1|^{2+\eta} |k_4|^{2+\eta}} \sum_{k_2,k_3 \neq 0} \frac{|k_1|^{2+\eta} |k_3|^{2+\eta}}{|k_2|^{2+\eta} |k_3|^{2+\eta}}
\times \int_{[0,t]^2} \int_{[0,s] \times [0,\mathbb{R}]} e^{-|k_1|^2 s} e_{\frac{1}{2}}^2 |k_1|^2 |k_2|^2 |k_3|^2 |k_4|^2 |s - \sigma|^\frac{2}{2+\eta} |s - \sigma'|^\frac{2}{2+\eta}
\times \frac{1}{2} |k_1|^2 |k_3|^2 |k_1|^2 |k_2|^2 |k_3|^2 |k_4|^2 |s - \sigma|^\frac{2}{2+\eta} |s - \sigma'|^\frac{2}{2+\eta}
\times \frac{1}{2} \frac{(1 - e^{-|k_1|^2 t})^2}{|k_1|^4} \lesssim \frac{t^{2+\eta}}{|k_1|^{3+\eta} |k_3|^{3+\eta} |k_4|^{2+\eta}}
$$

(235)
by (230) and (14). Similarly for \( k_1, k_4 \neq 0 \),

\[
\sum_{k_2, k_3 \neq 0} \int_0^t \int_{[0, t]^2} \int_{[0, \pi] \times [0, \pi]} e^{-\frac{1}{2} |k_1|^2(t-s) - |k_4|^2(t-s)} \left| k_1 \right| \left| k_4 \right|^{\frac{1}{2}} \sum_{\eta = 0}^{\infty} \eta ! \left| \delta_s - \delta_f \right|^\eta \nonumber
\]

\[
\times d k_1, k_2 (s-\sigma) d k_3, k_3 (\delta_s - \delta_f) d s d \sigma d \bar{\sigma} d s \nonumber
\]

\[
\lesssim \frac{1}{\left| k_1 \right| \left| k_4 \right|^{1/2}} \sum_{k_2, k_3 \neq 0} \frac{1}{\left| k_2 \right|^{3+\eta} \left| k_3 \right|^{3+\eta}} \nonumber
\]

\[
\times \int_{[0, t]^2} e^{-\frac{1}{2} |k_1|^2 + |k_4|^2} e^{\frac{1}{2} |k_1|^2} \left| k_1 \right| \left| k_4 \right|^{\frac{1}{2}} ds d s d \sigma \lesssim \frac{t^{2(\frac{s}{4} + \frac{t}{4})}}{\left| k_1 \right|^{|1/2 - 2(\frac{s}{4} + \frac{t}{4})| |k_4|^{3/2 - \frac{3}{4}}}} \nonumber
\]

by (230) and (14). Therefore, applying (233) and (230) to (234) gives

\[
\mathbb{E} \left[ \left| \Delta_4 (I X_t^{8.5} - \overline{I X_t^{8.5}}) \right|^2 \right] \nonumber
\]

\[
\lesssim t^{2(\frac{s}{4} + \frac{t}{4})} \sum_k \theta(2^{-q} k)^2 \sum_{k_1, k_4 \neq 0; k_1 = k_3 = 0} \nonumber
\]

\[
\times \left( \sum_{i \leq 2} \frac{2^{-i}}{\left| k_1 \right|^{3-\frac{3}{4} - \frac{3}{4}} \left| k_4 \right|^2} + \frac{2^{-i-2} \left( \frac{s}{4} + \frac{t}{4} \right)}{\left| k_1 \right|^{3-4 \left( \frac{s}{4} + \frac{t}{4} \right)} \left| k_4 \right|^{3-\frac{3}{4} - \frac{3}{4}}} \right) \nonumber
\]

\[
\lesssim t^{2(\frac{s}{4} + \frac{t}{4})} 2 \theta(2^{-q} k)^2 \sum_{k \neq 0} \frac{1}{\left| k \right|^3} \lesssim t^{2(\frac{s}{4} + \frac{t}{4})} 2 \theta(2^{-q} k) \nonumber
\]

where we used that \( 2^q \lesssim 2^1 \) and \( 2^q \lesssim 2^1 \) and Lemma 4.7. Next, within (227) we estimate

\[
\mathbb{E} \left[ \left| \Delta_4 (I X_t^{8.5} - \overline{I X_t^{8.5}}) \right|^2 \right] \nonumber
\]

\[
\approx \mathbb{E} \left[ \sum_k \theta(2^{-q} k) \sum_{i = 1}^{3} \sum_{k \neq 0} \theta(2^{-i} k) \right] \nonumber
\]

\[
\times \int_0^t : X_{u, i_k}^{e, i_{j_k}} (k_1) \tilde{X}_{b, d^{(i)}}^{e, i_{j_d}} (k_2) : e^{-\frac{1}{2} |k_1|^2 (t-s)} k_1^{i_k} \tilde{\mathcal{P}}^{i_k i_{j_k}} (k_1) \nonumber
\]

\[
\times \int_0^s \left[ e^{-|k_2|^2 (s-\sigma)} k_{12} \tilde{\mathcal{P}}^{i_1 i_2} (k_2) - e^{-|k_2|^2 (s-\sigma)} k_{12} \tilde{\mathcal{P}}^{i_1 i_2} (k_2) \right] \nonumber
\]

\[
\times e^{-|k_2|^2 (s-\sigma)} f(e \tilde{k}_2)^2 \frac{\hat{P}_{i_k i_{j_k}} (k_2) \hat{P}_{i_1 i_{j_1}} (k_2) e_k ds d s}{\left| k_2 \right|^2} \nonumber
\]

due to (228) and (229) where we rewrite for \( k_1, k_4 \neq 0 \),

\[
\mathbb{E} \left[ : \tilde{X}_{u, i_k}^{e, i_{j_k}} (k_1) \tilde{X}_{b, d^{(i)}}^{e, i_{j_d}} (k_2) : \tilde{X}_{u, i_k}^{e, i_{j_k}} (k_1) \tilde{X}_{b, d^{(i)}}^{e, i_{j_d}} (k_1) \right] \nonumber
\]

\[
= 1_{k_1 + k_4 = 0, k_1 + k_4 = 0} \sum_{i_k, i_{j_k} = 1} e^{-|k_1|^2 |s-\sigma| f(e \tilde{k}_1)^2} \frac{f(e \tilde{k}_1)^2}{2 |k_1|^2} \nonumber
\]

\[
\times \tilde{P}_{i_k i_{j_k}} (k_1) \tilde{P}_{i_1 i_{j_1}} (k_1) \tilde{P}_{i_k i_{j_k}} (k_1) \nonumber
\]

\[
= 1_{k_1 + k_4 = 0, k_1 + k_4 = 0} \sum_{i_k, i_{j_k} = 1} e^{-|k_1|^2 |s-\sigma| f(e \tilde{k}_1)^2} \frac{f(e \tilde{k}_1)^2}{2 |k_1|^2} \nonumber
\]

\[
\times \tilde{P}_{i_k i_{j_k}} (k_1) \tilde{P}_{i_1 i_{j_1}} (k_1) \tilde{P}_{i_k i_{j_k}} (k_1) \nonumber
\]
by Example 2.2 and (116). Therefore,

\[
\mathbb{E}[|\Delta_q(\bar{X}_t^{8.5} - \bar{X}_t^{4.5})|^2] \\
\leq \sum_{k,k'} \theta(2^{-q} k) \theta(2^{-q} k') \sum_{|i-j| \leq 1, |i'-j'| \leq 1} \sum_{k_1, k_4 \neq 0} \theta(2^{-i} k_1) \theta(2^{-j} k_4) \int_{[0,t]^2} |k_1|^2 |k_4|^2 |k_1| |k_4|
\times \sum_{i_1, i_2, i_3, i_4} \sum_{k_i, k_{i'}} \theta(2^{-i} k_1) \theta(2^{-i'} k_{i'}) \theta(2^{-i} k_4) \theta(2^{-j} k_{i'}) \int_{[0,t]^2} |k_1|^2 |k_{i'}|^2 |k_1| |k_{i'}|
\times \left( \int_0^s e^{-|k_{i'}|^2 (s-\tau)} k_{i'}^4 \tilde{P}_{i'3} (k_{i'}) e^{-|k_4|^2 (s-\tau)} k_4^3 \tilde{P}_{43} (k_4) \right)
\times \left( \int_0^s e^{-|k_1|^2 (s-\tau)} k_1^4 \tilde{P}_{13} (k_1) e^{-|k_{i'}|^2 (s-\tau)} k_{i'}^3 \tilde{P}_{i'3} (k_{i'}) \right)
\times \frac{e^{-|k_1|^2 (s-\tau)} e^{-|k_{i'}|^2 (s-\tau)}}{|k_1|^2 |k_{i'}|^2} d\sigma ds d\sigma ds = \sum_{i=1}^{2} X_{III}^i
\]

where \(X_{III}^1\) involves \(k_1 + k_4 = 0, k_4 + k_{i'} = 0\) and \(X_{III}^2\) involves \(k_1 + k_4 = 0, k_4 + k_{i'} = 0\). We bound

\[
X_{III}^1 \leq \sum_k \theta(2^{-q} k)^2 \sum_{|i-j| \leq 1, |i'-j'| \leq 1} \sum_{k_1, k_4 \neq 0} \theta(2^{-i} k_1) \theta(2^{-j} k_4) \int_{[0,t]^2} e^{-|k_1|^2 (s-\tau)} |k_1|^2 |k_4|^2 |k_1|^2 |k_4|^2
\times \int_{[0,s] \times [0,\pi]} e^{-|k_1|^2 (s-\tau)} |k_1|^2 |k_4|^2 |k_1|^2 |k_4|^2 d\sigma ds d\sigma ds (241)
\]

for any \( \eta \in (0, 1) \) due to a change of variable of \( k_3' \) with \( k_3 \) and Lemma 4.6. On the other hand, we may bound

\[
X_{III}^2 \lesssim \sum_{k \neq 0} \theta(2^{-q} k)^2 \sum_{|i-j| \leq 1, |i'-j'| \leq 1} \sum_{k_1, k_4 \neq 0; k_1 = k, k_2, k_3 \neq 0} \theta(2^{-i} k_1) \theta(2^{-i'} k_4) \\
\quad \times \theta(2^{-j} k_4) \theta(2^{-j'} k_1) \int_{[0,t]^2} \frac{e^{-2 |k_1|^2 (t-s)} e^{-2 |k_4|^2 (t-\sigma)}}{|k_1||k_4|} \\
\quad \times \int_{[0,s] \times [0,\sigma]} \frac{|k_1|^\eta |s-\sigma|^{-\frac{q-1}{2}} |k_4|^\eta |\sigma-\sigma'|^{-\frac{q-1}{2}}}{|k_2|^2 |k_3|^2} \\
\quad \times e^{-|k_2|^2 (s-\sigma) - |k_3|^2 (\sigma-\sigma')} d\sigma d\sigma' ds d\sigma}
\]

(242)
due to a change of variable of $k'_3$ with $k_3$ and Lemma 1.6. Therefore, applying (241) and (242) to (240) gives

\[
E[|\Delta_q(\mathbb{I}X^8_{t,5} - \mathcal{T}X^8_{t,5})|^2] 
\leq \sum_k \theta(2^{-q}k)^2 \sum_{|i-j| \leq 1, |i' - j'| \leq 1, k_1, k_4 \neq 0, k_1 = k_2, k_3 \neq 0} \theta(2^{-i}k_1) \theta(2^{-i'}k_1) 
\times \theta(2^{-j}k_4) \theta(2^{-j'}k_4) \int_{[0,t]^2} e^{-|k_1|^2(|s - \tau| + 2(t - s - \tau))} \frac{|k_1|^{2\eta}}{|k_2^2| |k_3|^2 |k_4|^2} 
\times \left( \int_0^s |s - \sigma|^{-(1 - \eta)} d\sigma \right)^{\frac{1}{2}} \left( \int_0^s e^{-2|k_2|^2(s - \sigma)} d\sigma \right)^{\frac{1}{2}} 
\times \left( \int_0^{\tau} \frac{\tau - \sigma}{1 - \eta} d\sigma \right)^{\frac{1}{2}} \left( \int_0^{\tau} e^{-2|k_4|^2(\tau - \sigma)} d\sigma \right)^{\frac{1}{2}} ds d\tau 
\times \theta(2^{-q}k)^2 \sum_{|i-j| \leq 1, |i' - j'| \leq 1, k_1, k_4 \neq 0, k_1 = k_2, k_3 \neq 0} \theta(2^{-i}k_1) \theta(2^{-i'}k_1) 
\times \theta(2^{-j}k_4) \theta(2^{-j'}k_4) \int_{[0,t]^2} e^{-2|k_1|^2(t - s) - 2|k_4|^2(t - \tau)} \frac{|k_1|^{\eta}}{|k_2^2| |k_3|^2 |k_4|^2} 
\times \left( \int_0^s |s - \sigma|^{-(1 - \eta)} d\sigma \right)^{\frac{1}{2}} \left( \int_0^s e^{-2|k_2|^2(s - \sigma)} d\sigma \right)^{\frac{1}{2}} 
\times \left( \int_0^{\tau} \frac{\tau - \sigma}{1 - \eta} d\sigma \right)^{\frac{1}{2}} \left( \int_0^{\tau} e^{-2|k_4|^2(\tau - \sigma)} d\sigma \right)^{\frac{1}{2}} ds d\tau \quad (243)
\]

due to Hölder's inequality. We continue to bound this by

\[
E[|\Delta_q(\mathbb{I}X^8_{t,5} - \mathcal{T}X^8_{t,5})|^2] 
\leq \sum_k \theta(2^{-q}k)^2 \sum_{|i-j| \leq 1, |i' - j'| \leq 1, k_1, k_2, k_4 \neq 0, k_1 = k_2, k_3 \neq 0} \theta(2^{-i}k_1) \theta(2^{-i'}k_1) 
\times \theta(2^{-j}k_4) \theta(2^{-j'}k_4) \left( \frac{1 - e^{-|k_1|^2t}}{|k_1|^{2\eta}} \right)^2 \frac{|k_1|^{2\eta}}{|k_2^2|^{\eta} |k_3^3 + |k_4|^2}
\]
\[
+ \sum_k \theta(2^{-q}k)^2 \sum_{|i-j| \leq 1, |i’-j’| \leq 1 \text{ and } k_1, k_2, k_3, k_4 \neq 0} \sum_{k_4 = k} \theta(2^{-i}k_1) \theta(2^{-i'}k_4) \\
\times \theta(2^{-j}k_4) \theta(2^{-j'}k_1) \left( \frac{1 - e^{-2|k_1|^2}}{|k_1|^2} \right) \left( \frac{1 - e^{-2|k_4|^2}}{|k_4|^2} \right) \frac{|k_1|^q|k_4|^q}{|k_1||k_2|^3|k_3|^3|k_4|^3} \\
\lesssim t^{2(\frac{q}{2} + \frac{q}{4})} \sum_{k \neq 0} \theta(2^{-q}k)^2 \sum_{q \leq j} \sum_{k_1, k_4 \neq 0} \theta(2^{-i}k_1) \theta(2^{-i'}k_4) \\
\times \left[ \frac{1}{|k_1|^{3-\frac{5n}{2}-\frac{5}{2}}} \frac{1}{|k_4|^{3-\frac{5n}{2}-\frac{5}{2}}} \right] \\
\lesssim t^{2(\frac{q}{2} + \frac{q}{4})(\frac{10n}{2} + \frac{5}{2})}
\]

due to the mean value theorem, that \( 2^q \lesssim 2^i \) and Lemma 4.7. Combining this with (237) in (227) gives

\[
\mathbb{E}(|\Delta_q I X_{t}^{8,5}|^2) \lesssim t^{2(\frac{q}{2} + \frac{q}{4})(\frac{10n}{2} + \frac{5}{2})}.
\]

(244)

Similar estimates for \( I X_{t}^{8,6} \) may be deduced as well.

### 3.3.2. Terms in the fourth chaos.

We finally work on \( I X_{t}^{8,1} \) of (194), specifically the first term of (184) where

\[
I X_{t}^{8,1} = -\frac{1}{4(2\pi)^2} \sum_k \sum_{|i-j| \leq 1} \sum_{k_1, k_2, k_3, k_4 = 1} \sum_{i_1, i_2, i_3, i_4 = 1} \theta(2^{-i}k_1) \theta(2^{-i}k_4) \\
\times \int_0^t e^{-|k_1|^2(t-s)} \int_0^s \hat{X}_{u,\sigma}^{i_1, i_2}(k_1) \hat{X}_{u,\sigma}^{i_3, i_4}(k_2) \hat{X}_{u,\sigma}^{i_5, i_6}(k_3) \hat{X}_{u,\sigma}^{i_7, i_8}(k_4) : \\
\times e^{-|k_2|^2(s-\sigma)} d\sigma ds k_{123}^{i_1} k_{123}^{i_2} \hat{P}_{i_1}^{i_2}(k_{12}) \hat{P}_{i_1}^{i_2}(k_{12}) e_k.
\]

(245)
We need to compute using Example 2.2 that
\[
\mathbb{E}[\hat{X}_{b,s,\sigma}^{\epsilon,i} (k_1) \hat{X}_{u,\sigma}^{\epsilon,i} (k_2) \hat{X}_{b,t}^{\epsilon,j_1} (k_3) \hat{X}_{b,t}^{\epsilon,j_0} (k_4) ] \\
\times : \hat{X}_{u,\sigma}^{\epsilon,i} (k_1') \hat{X}_{b,t}^{\epsilon,j_1} (k_2') \hat{X}_{b,t}^{\epsilon,j_0} (k_3') \hat{X}_{b,t}^{\epsilon,j_0} (k_4') ] \\
= \mathbb{E}[\hat{X}_{a,\sigma}^{\epsilon,i} (k_1) \hat{X}_{a,\sigma}^{\epsilon,i} (k_1') ] \\
\times (\mathbb{E}[\hat{X}_{b,\sigma}^{\epsilon,i} (k_2) \hat{X}_{b,\sigma}^{\epsilon,i} (k_2') ] \mathbb{E}[\hat{X}_{b,s,\sigma}^{\epsilon,j_1} (k_3) \hat{X}_{b,s,\sigma}^{\epsilon,j_1} (k_3') ] \mathbb{E}[\hat{X}_{b,t}^{\epsilon,j_0} (k_4) \hat{X}_{b,t}^{\epsilon,j_0} (k_4') ] \\
+ \mathbb{E}[\hat{X}_{b,\sigma}^{\epsilon,i} (k_2) \hat{X}_{b,\sigma}^{\epsilon,i} (k_2') ] \mathbb{E}[\hat{X}_{b,s,\sigma}^{\epsilon,j_1} (k_3) \hat{X}_{b,s,\sigma}^{\epsilon,j_1} (k_3') ] \mathbb{E}[\hat{X}_{b,t}^{\epsilon,j_0} (k_4) \hat{X}_{b,t}^{\epsilon,j_0} (k_4') ] \\
+ \mathbb{E}[\hat{X}_{b,\sigma}^{\epsilon,i} (k_2) \hat{X}_{b,\sigma}^{\epsilon,i} (k_2') ] \mathbb{E}[\hat{X}_{b,s,\sigma}^{\epsilon,j_1} (k_3) \hat{X}_{b,s,\sigma}^{\epsilon,j_1} (k_3') ] \mathbb{E}[\hat{X}_{b,t}^{\epsilon,j_0} (k_4) \hat{X}_{b,t}^{\epsilon,j_0} (k_4') ] \\
+ \mathbb{E}[\hat{X}_{b,\sigma}^{\epsilon,i} (k_2) \hat{X}_{b,\sigma}^{\epsilon,i} (k_2') ] \mathbb{E}[\hat{X}_{b,s,\sigma}^{\epsilon,j_1} (k_3) \hat{X}_{b,s,\sigma}^{\epsilon,j_1} (k_3') ] \mathbb{E}[\hat{X}_{b,t}^{\epsilon,j_0} (k_4) \hat{X}_{b,t}^{\epsilon,j_0} (k_4') ] \\
+ \mathbb{E}[\hat{X}_{b,\sigma}^{\epsilon,i} (k_2) \hat{X}_{b,\sigma}^{\epsilon,i} (k_2') ] \mathbb{E}[\hat{X}_{b,s,\sigma}^{\epsilon,j_1} (k_3) \hat{X}_{b,s,\sigma}^{\epsilon,j_1} (k_3') ] \mathbb{E}[\hat{X}_{b,t}^{\epsilon,j_0} (k_4) \hat{X}_{b,t}^{\epsilon,j_0} (k_4') ] \\
+ \mathbb{E}[\hat{X}_{b,\sigma}^{\epsilon,i} (k_2) \hat{X}_{b,\sigma}^{\epsilon,i} (k_2') ] \mathbb{E}[\hat{X}_{b,s,\sigma}^{\epsilon,j_1} (k_3) \hat{X}_{b,s,\sigma}^{\epsilon,j_1} (k_3') ] \mathbb{E}[\hat{X}_{b,t}^{\epsilon,j_0} (k_4) \hat{X}_{b,t}^{\epsilon,j_0} (k_4') ] \\
+ \mathbb{E}[\hat{X}_{b,\sigma}^{\epsilon,i} (k_2) \hat{X}_{b,\sigma}^{\epsilon,i} (k_2') ] \mathbb{E}[\hat{X}_{b,s,\sigma}^{\epsilon,j_1} (k_3) \hat{X}_{b,s,\sigma}^{\epsilon,j_1} (k_3') ] \mathbb{E}[\hat{X}_{b,t}^{\epsilon,j_0} (k_4) \hat{X}_{b,t}^{\epsilon,j_0} (k_4') ] \\
+ \mathbb{E}[\hat{X}_{b,\sigma}^{\epsilon,i} (k_2) \hat{X}_{b,\sigma}^{\epsilon,i} (k_2') ] \mathbb{E}[\hat{X}_{b,s,\sigma}^{\epsilon,j_1} (k_3) \hat{X}_{b,s,\sigma}^{\epsilon,j_1} (k_3') ] \mathbb{E}[\hat{X}_{b,t}^{\epsilon,j_0} (k_4) \hat{X}_{b,t}^{\epsilon,j_0} (k_4') ] \\
+ \mathbb{E}[\hat{X}_{b,\sigma}^{\epsilon,i} (k_2) \hat{X}_{b,\sigma}^{\epsilon,i} (k_2') ] \mathbb{E}[\hat{X}_{b,s,\sigma}^{\epsilon,j_1} (k_3) \hat{X}_{b,s,\sigma}^{\epsilon,j_1} (k_3') ] \mathbb{E}[\hat{X}_{b,t}^{\epsilon,j_0} (k_4) \hat{X}_{b,t}^{\epsilon,j_0} (k_4') ] \\
+ \mathbb{E}[\hat{X}_{b,\sigma}^{\epsilon,i} (k_2) \hat{X}_{b,\sigma}^{\epsilon,i} (k_2') ] \mathbb{E}[\hat{X}_{b,s,\sigma}^{\epsilon,j_1} (k_3) \hat{X}_{b,s,\sigma}^{\epsilon,j_1} (k_3') ] \mathbb{E}[\hat{X}_{b,t}^{\epsilon,j_0} (k_4) \hat{X}_{b,t}^{\epsilon,j_0} (k_4') ] \\
+ \mathbb{E}[\hat{X}_{b,\sigma}^{\epsilon,i} (k_2) \hat{X}_{b,\sigma}^{\epsilon,i} (k_2') ] \mathbb{E}[\hat{X}_{b,s,\sigma}^{\epsilon,j_1} (k_3) \hat{X}_{b,s,\sigma}^{\epsilon,j_1} (k_3') ] \mathbb{E}[\hat{X}_{b,t}^{\epsilon,j_0} (k_4) \hat{X}_{b,t}^{\epsilon,j_0} (k_4') ] \\
(246)
We bound this by a constant multiples of
\[
\sum_{i=1}^{24} XIV_i \]
when \(k_1, k_2, k_3, k_4 \neq 0\). Thus, applying (247) and (246) to (245) gives

\[
\mathbb{E}[(\Delta_q I X_i^{k_1,k_2})^2] 
\lesssim \sum_{k,k'} \sum_{|i-j| \leq 1, |i'-j'| \leq 1} \theta(2^{-q}k) \theta(2^{-q}k') \sum_{k_1,k_2,k_3,k_4 = k,k',k_1',k_2',k_3',k_4'} e^{-|k_{1234}|^2(t-s)} e^{-|k_{1234}'|^2(t-\tau)} \int_{[0,t]^2} e^{-|k_{1234}|^2(t-s)} e^{-|k_{1234}'|^2(t-\tau)} d\sigma d\sigma' ds d\sigma e_k e_{k'}.
\]

We realize that in every term of \(XIV_i\) for \(i \in \{1, \ldots, 24\}\) we always have the characteristic function \(1_{k_1+k'_1=0,k_2+k'_2=0,k_3+k'_3=0,k_4+k'_4=0}\) for \(i,j,l,m\) which are distinct and elements of \(\{1,2,3,4\}\). Therefore, \(|k'_{1234}| = |k_{1234}|\) allowing us to reduce \(\sum_{k,k' \neq 0}\) to \(\sum_{k \neq 0}\). Thus, we can bound the term corresponding to \(XIV_i\) by

\[
\sum_{k} \theta(2^{-q}k)^2 \sum_{|i-j| \leq 1, |i'-j'| \leq 1} \sum_{k_1,k_2,k_3,k_4 = k} e^{-|k_{1234}|^2(t-s)} e^{-|k_{1234}'|^2(t-\tau)} \int_{[0,t]^2} e^{-|k_{1234}|^2(t-s)} e^{-|k_{1234}'|^2(t-\tau)} d\sigma d\sigma' \sum_{k_1,k_2,k_3,k_4 = k} e^{-|k_{1234}|^2(t-s)} e^{-|k_{1234}'|^2(t-\tau)} d\sigma d\sigma' ds d\sigma e_k e_{k'}.
\]

(249)
For the term corresponding to $XIV^2_t$, we bound by the same as $XV^1_t$ except $1_{k_1+k'_1=0,k_2+k'_2=0,k_3+k'_3=0, k_4+k'_4=0}$ is replaced by $1_{k_1+k'_1=0,k_2+k'_2=0,k_3+k'_3=0, k_4+k'_4=0}$ and we denote it by $XV^3_t$.

For the term corresponding to $XIV^3_t$, we bound by the same as $XV^1_t$ except $1_{k_1+k'_1=0,k_2+k'_2=0,k_3+k'_3=0, k_4+k'_4=0}$ is replaced by $1_{k_1+k'_1=0,k_2+k'_2=0,k_3+k'_3=0, k_4+k'_4=0}$ and we denote it by $XV^5_t$.

For the term corresponding to $XIV^4_t$ we change variable $k_1$ with $k_2$, as well as $k'_1$ with $k'_2$ so that we may bound by $XV^1_t$ but $1_{k_1+k'_1=0,k_2+k'_2=0,k_3+k'_3=0, k_4+k'_4=0}$ replaced by $1_{k'_2+k_2=0,k_1+k'_1=0,k_3+k'_3=0, k_4+k'_4=0}$ and we denote it by $XV^6_t$.

For the term corresponding to $XIV^5_t$ we replace $k_1$ by $k_2$, as well as $k'_1$ with $k'_2$ so that we may bound by $XV^1_t$ but with $1_{k_1+k'_1=0,k_2+k'_2=0,k_3+k'_3=0, k_4+k'_4=0}$ replaced by $1_{k_1+k'_1=0,k_2+k'_2=0,k_3+k'_3=0, k_4+k'_4=0}$ which we denote by $XV^7_t$.

For the term corresponding to $XIV^6_t$ we switch $k_1$ with $k_2$, as well as $k'_1$ with $k'_2$ so that we may bound this by $XV^1_t$ but with $1_{k_1+k'_1=0,k_2+k'_2=0,k_3+k'_3=0, k_4+k'_4=0}$ replaced by $1_{k_1+k'_1=0,k_2+k'_2=0,k_3+k'_3=0, k_4+k'_4=0}$, which we denote by $XV^2_t$.

For the term corresponding to $XIV^7_t$, we switch $k_1$ with $k_2$ so that we can actually bound by $XV^1_t$.

For the term corresponding to $XIV^8_t$, we switch $k_1$ with $k_2$ so that we can actually bound by $XV^3_t$.

For the term corresponding to $XIV^9_t$, we switch $k'_1$ with $k'_2$ so that we can bound by $XV^5_t$.

For the term corresponding to $XIV^{10}_t$, we switch $k_1$ with $k_2$ so that we can bound by $XV^6_t$.

For the term corresponding to $XIV^{11}_t$, we switch $k_1$ with $k_2$ so that we can bound by $XV^7_t$.

For the term corresponding to $XIV^{12}_t$, we switch $k_1$ with $k_2$ so that we can bound by $XV^2_t$.

For the term corresponding to $XIV^{13}_t$, we switch $k_1$ with $k_2$ so that we can bound by $XV^5_t$.

For the term corresponding to $XIV^{14}_t$, we switch $k'_1$ with $k'_2$ so that we can bound by $XV^6_t$.

For the term corresponding to $XIV^{15}_t$, we switch $k_1$ with $k_2$, as well as $k'_1$ with $k'_2$ so that we can bound by $XV^7_t$.

For the term corresponding to $XIV^{16}_t$, we bound by $XV^6_t$.

For the term corresponding to $XIV^{17}_t$, we bound by $XV^2_t$ but by substituting $1_{k_1+k'_1=0,k_2+k'_2=0,k_3+k'_3=0, k_4+k'_4=0}$ for $1_{k_1+k'_1=0,k_2+k'_2=0,k_3+k'_3=0, k_4+k'_4=0}$ and denote it by $XV^4_t$.

For the term corresponding to $XIV^{18}_t$, we switch $k'_1$ with $k'_2$ so that we can bound it by $XV^4_t$.

For the term corresponding to $XIV^{19}_t$, we switch $k'_1$ with $k'_2$ so that we could bound it by $XV^7_t$.

For the term corresponding to $XIV^{20}_t$ we switch $k'_1$ with $k'_2$ so that we could bound by $XV^2_t$.

For the term corresponding to $XIV^{21}_t$ we bound by $XV^7_t$.

For the term corresponding to $XIV^{22}_t$ we bound by $XV^2_t$.

For the term corresponding to $XIV^{23}_t$, we switch $k_1$ with $k_2$ so that we could bound by $XV^4_t$. 


For the term corresponding to $XV_t^{24}$ we switch $k_1$ with $k_2$, as well as $k'_1$ with $k'_2$ so that we could bound by $XV_t^{4}$.

**Remark 3.5.** Actually for the terms which may be bounded by $XV_t^{7}$, we may take advantage of the symmetry of $(k_1', k_2', k_3', k_4')$, switch $k_1$ with $k_1'$, $k_2$ with $k_2'$, $k_3$ with $k_3'$, $k_4$ with $k_4'$, $s$ with $\sigma$, $\sigma$ with $\bar{\sigma}$, $i$ with $i'$, as well as $j$ with $j'$ and actually bound by $XV_t^{6}$.

Therefore we have shown

$$E[|\Delta_q IX_t^{8,1}|^2] \leq \sum_{k=1}^{6} XV_t^k. \quad (250)$$

It suffices to see the estimate on $XV_t^1$ as others are similar. We estimate within $XV_t^1$,

$$\int_{[0,1]^3} e^{-|k_{123}|^2(t-s-t-\bar{t})} \int_{[0,1]} e^{-|k_{12}|^2(s-\sigma-t-\bar{\sigma})} d\sigma d\bar{\sigma} ds \bar{d}\sigma \leq e^{-2\bar{t}|k_{123}|^2} \left(\frac{|k_{12}|^2 t}{|k_{123}|^2 t - 1}\right)^2 1_{k_{12},k_{123} \neq 0} \leq \frac{1}{|k_{12}|^4 |k_{123}|^2} \frac{t^n}{|k_{12}|^4 |k_{123}|^2-2\eta} 1_{k_{12},k_{123} \neq 0}$$

by mean value theorem so that

$$XV_t^1 \leq t^n \sum_k \sum_{|i-j| \leq 1} \sum_{1 \leq k_1,k_2,k_3,k_4 \neq 0,k_1234=0} \theta(2^{-q}k)2\theta(2^{-i}k_{123})\theta(2^{-j}k_{13}) \times \theta(2^{-q}k)2\theta(2^{-i}k_{123})\theta(2^{-j}k_{13})$$

$$\leq \frac{1}{|k_{12}|^4 |k_{123}|^2} \frac{t^n}{|k_{12}|^4 |k_{123}|^2-2\eta} 1_{k_{12},k_{123} \neq 0}$$

$$\leq t^n 2\eta(2q+\varepsilon) \sum_{k \neq 0} \theta(2^{-q}k)^2 \frac{1}{|k|^3} \leq t^n 2\eta(2q+\varepsilon)$$

by Lemma [17] and that $2\eta \leq 2^i$. Therefore, by applying (251) to (250) we deduce

$$E[|\Delta_q IX_t^{8,1}|^2] \leq t^n 2\eta(2q+\varepsilon). \quad (252)$$

By applying (211), (220), (241) and (252) to (194) we have shown that

$$E[|\Delta_q \pi_{0,\sigma}(u_{3,i}^1,b_{1,j}^{1,0})|^2] \leq t\eta(2^{q+\varepsilon} + \varepsilon)$$

due to (181). Similarly to how we deduced (159) from (138), we can also prove

$$E[|\Delta_q (\pi_{0,\sigma}(u_{3,i}^1,b_{1,j}^{1,0})(t_1) - \pi_{0,\sigma}(u_{3,i}^1,b_{1,j}^{1,0})(t_2) - \pi_{0,\sigma}(u_{3,i}^1,b_{1,j}^{1,0})(t_1) + \pi_{0,\sigma}(u_{3,i}^1,b_{1,j}^{1,0})(t_2))|^2]$$

$$\leq (c_\gamma^2 + c_\gamma^2) |t_1 - t_2|^{2(\frac{\delta}{2} + \frac{\varepsilon}{2})} 2q(\frac{10\eta}{3} + \delta). \quad (253)$$

Recalling again that $B_{p,p}^{-\frac{2}{p}-\varepsilon} \to C^{-\frac{2}{p}-\varepsilon-\frac{\varepsilon}{p}}$ as in (140), we deduce

$$E[|\pi_{0,\sigma}(u_{3,i}^1,b_{1,j}^{1,0})(t_1) - \pi_{0,\sigma}(u_{3,i}^1,b_{1,j}^{1,0})(t_2) - \pi_{0,\sigma}(u_{3,i}^1,b_{1,j}^{1,0})(t_1) + \pi_{0,\sigma}(u_{3,i}^1,b_{1,j}^{1,0})(t_2)|^p] \leq t\eta \left(\frac{2^q+\varepsilon}{2}\right)^2 2q(\frac{10\eta}{3} + \delta).$$
by the Gaussian hypercontractivity theorem [35, Theorem 3.50] and (20) as we did in (141). If we choose \( \eta, \epsilon, p > 0 \) such that \( \frac{5p}{3} + \epsilon + \frac{3}{p} \leq \delta \), we have proven that there exists \( v_{16}^{t_0,j_0} \in C([0,T]; C^{-\delta}) \) for \( i_0, j_0 \in \{1,2,3\} \) such that \( \pi_{0,0}(v_{3}^{t_0,j_0}, v_{1}^{t_0,j_0}) \rightarrow v_{16}^{t_0,j_0} \) as \( \epsilon \rightarrow 0 \) in \( L^p(\Omega; C([0,T]; C^{-\delta})) \) as desired in (143).

3.4. **Group 4**: \( \pi_{0,0}(\mathcal{P}^{ii}; \partial_{x_i} K_{u}^{e,j}, u_1^{e,j_i}), \pi_{0,0}(\mathcal{P}^{ii}; \partial_{x_i} K_{b}^{e,j}, u_1^{e,j_i}), \pi_{0,0}(\mathcal{P}^{ii}; \partial_{x_i} K_{u}^{e,j}, b_1^{e,j_i}), \pi_{0,0}(\mathcal{P}^{ii}; \partial_{x_i} K_{b}^{e,j}, b_1^{e,j_i}) \). Within Group 4 of (114), we work on \( \pi_{0,0}(\mathcal{P}^{ii}; \partial_{x_i} K_{u}^{e,j}, b_1^{e,j_1}) \) and show the existence of \( v_{20}^{t_0,j_0} \in C([0,T]: C^{-\gamma}) \) such that \( \pi_{0,0}(\mathcal{P}^{ii}; \partial_{x_i} K_{u}^{e,j}, b_1^{e,j_1}) \rightarrow v_{20}^{t_0,j_0} \) as \( \epsilon \rightarrow 0 \). Due to (20) we can write down

\[
\pi_0(\mathcal{P}^{ii} \partial_{x_0} K_{u}^{e,j_0}, b_1^{e,j_1})(t) = \frac{1}{(2\pi)^2} \sum_k \sum_{|i-j| \leq 1} \sum_{k_1,k_2,k_{12}} \theta(2^{-i}k_1)\theta(2^{-j}k_2) \int_0^t e^{-|k_1|^2(t-s)}i k_1 j_0 \times \hat{X}_{u,s}^{e,j_0}(k_1) \hat{X}_{b,t}^{e,j_1}(k_2) \cdot d s e_k \hat{\mathcal{P}}^{ii} \hat{i} \theta(2^{-i}k_1) \hat{\mathcal{P}}^{ii} \hat{j} \theta(2^{-j}k_2) e_k \tag{255}
\]

by (13b) where the second term can be shown to be actually zero. Thus,

\[
\mathbb{E}[|\Delta_{q} \pi_0(\mathcal{P}^{ii} \partial_{x_0} K_{u}^{e,j_0}, b_1^{e,j_1})(t)|^2] \\
\approx \sum_{k,k'} \sum_{|i-j| \leq 1} \sum_{k_1,k_2,k_{12}} \theta(2^{-i}k_1)\theta(2^{-i'}k_1')\theta(2^{-j}k_2) \times \theta(2^{-j'}k_2') \theta(2^{-q}k) \int_{[0,t]^2} e^{-|k|^2(t-s)}|k_1^2(t-s)| |k_1| \tag{256}
\]

\[
\mathbb{E}[: \hat{X}_{u,s}^{e,j_0}(k_1) \hat{X}_{b,t}^{e,j_1}(k_2) :: \hat{X}_{u,s}^{e,j_0}(k_1') \hat{X}_{b,t}^{e,j_1}(k_2') :] \leq 1_{k_1+k_1'=0,k_2+k_2'=0} \frac{e^{-|k_1|^2(t-s)}e^{-|k_2|^2(t-s)}}{|k_1|^2|k_2|^2} + 1_{k_1+k_1'=0,k_2+k_2'=0} \frac{e^{-|k_1|^2(t-s)}e^{-|k_2|^2(t-s)}}{|k_1|^2|k_2|^2} \tag{257}
\]

where we may estimate for \( k_1, k_2 \neq 0 \),
due to Example 2.2 and (110). We obtain by applying (257) to (256)

\[
\mathbb{E}[|\Delta_{\eta}\pi_0(P^{i_1i_2}_x \partial_{x_{i_0}} \mathcal{K}^{\epsilon,j_0}_{\eta}, b_3^{\epsilon,j_1})(t)|^2] 
\leq \sum_{k} \sum_{|i-j| \leq 1, |i'-j'| \leq 1} \sum_{k_1, k_2 \neq 0: k_1 = k}
\times [\theta(2^{-j}k_1)\theta(2^{-j'}k_1)\theta(2^{-j}k_2)\theta(2^{-j'}k_2)\theta(2^{-q}k_2)^2 \frac{1}{|k_2|^2} \left(1 - \frac{|2^{-j}k_1|^2}{|k_1|^2}\right)^2 + \theta(2^{-j}k_1)\theta(2^{-j'}k_2)\theta(2^{-j}k_2)\theta(2^{-j'}k_1)\theta(2^{-q}k_2)^2 \frac{1}{|k_1|^2} \left(1 - \frac{|2^{-j}k_2|^2}{|k_2|^2}\right)^2]
\leq t^{\eta} \sum_{k} [\theta(2^{-q}k_2)^2 \sum_{q \leq i} \frac{1}{|k_2|^2} |2^{-i}k_1|^{-j} \leq t^{\eta} 2^{q} \sum_{k \neq 0} [\theta(2^{-q}k_2)^2 \frac{1}{|k_2|^2} \leq t^{\eta} 2^{q}]
\]

where we used mean value theorem, Lemma 4.7 and that \(2^q \lesssim 2^i\). Similarly to how we deduced (139) from (138) we can also show

\[
\mathbb{E}[|\Delta_{\eta}(\pi_0,\alpha(P^{i_1i_2}_x \partial_{x_{j_0}} \mathcal{K}^{\epsilon,j_0}_{\eta}, b_3^{\epsilon,j_1})(t_1) - \pi_0,\alpha(P^{i_1i_2}_x \partial_{x_{j_0}} \mathcal{K}^{\epsilon,j_0}_{\eta}, b_3^{\epsilon,j_1})(t_2))|^2] 
\leq (\epsilon_i^2 + \epsilon_j^2)|t_1 - t_2|^2 \sum_{k} [\theta(2^{-q}k_2)^2 \frac{1}{|k_2|^2} \leq t^{\eta} 2^{q}]
\]

so that applications of Besov embedding and Gaussian hypercontractivity theorem [35] Theorem 3.50] as we did in (139) - (141) implies that there exists \(v_{20}^{i_1i_2,j_0,j_1}\) in \(C([0,T]; C^{-\delta})\) for \(i_1, i_2, j_0, j_1 \in \{1, 2, 3\}\) such that for all \(p \in [1, \infty)\), we have \(\pi_0,\alpha(P^{i_1i_2}_x \partial_{x_{j_0}} \mathcal{K}^{\epsilon,j_0}_{\eta}, b_3^{\epsilon,j_1})(t) \to v_{20}^{i_1i_2,j_0,j_1}\) as \(\epsilon \to 0\) in \(L^p(\Omega; C([0,T]; C^{-\delta}))\).

With all these convergence results, we may now state and prove the main result.

**Theorem 3.2.** Let \(\delta_0 \in (0, \frac{1}{7})\) and then \(z \in (\frac{1}{7}, \frac{1}{7} + \delta_0)\), as well as \(y_0 = (u_0, b_0) \in C^{-z}\). Then there exists a unique local solution to

\[
Ly' = \left(\sum_{i,j=1}^{3} P^{i_1i_2}_x \partial^{\epsilon}_{x_i} (u'^{i_1}_{j_0} f (u')^{i_2}) + \frac{1}{2} \sum_{i,j=1}^{3} P^{i_1i_2}_x \partial^{\epsilon}_{x_i} (u'^{i_1}_{j_0} b_3^{i_2}) \right) y, (\cdot, 0) = P(u_0, b_0)(\cdot).
\]

Specifically, suppose \(\epsilon = \sum_{k} f(\epsilon(k)) \xi(k) e_k\) for \(\epsilon > 0\) and \(f\) that is smooth radial cut-off function with compact support such that \(f(0) = 1\), and \(y' = (u', b')\) is the maximal unique solution to

\[
Ly'^{\epsilon} = \left(\sum_{i,j=1}^{3} P^{i_1i_2}_x \partial^{\epsilon}_{x_i} (u'^{i_1}_{j_0} f (u'^{i_2})) + \frac{1}{2} \sum_{i,j=1}^{3} P^{i_1i_2}_x \partial^{\epsilon}_{x_i} (b_3^{i_2}) \right) y'^{\epsilon}, (\cdot, 0) = P(u_0, b_0)(\cdot).
\]

such that \(u'^{\epsilon}_{j}, b'_3\) belong to \(C([0,T^*]; C^{\frac{1}{7} + \delta_0})\). Then there exists \(y \in C([0,T]; C^{-\frac{1}{7}})^2\) and \(\{\tau_L\}_L\) such that \(\tau_L\) converges to the explosion time \(\tau = (u, b)\) such that \(\sup_{t \in [0, \tau_L]} \|y'^{-} - y\|_{C^{-z}} \to 0 \) as \(\epsilon \to 0\) in probability.

**Proof.** By a similar argument that we showed already, and in particular (36a) and (39), we can prove the existence of \(\gamma > 0\), \((u_1, b_1) \in C([0,T]; C^{-\frac{1}{7} - \frac{1}{7}})^2\), \((u_2, b_2) \in C([0,T]; C^{-\frac{1}{7}})^2\), and then conclude using a similar strategy as in the proof of Theorem 3.50.
C([0, T]; C^{−δ})^2, (u_3, b_3) \in C([0, T]; C^{\frac{1}{2}−δ})^2 such that for all p > 0,
\[ \mathbb{E}[\|(u'_1, b'_1) - (u_1, b_1)\|_{C([0,T]; C^{-\frac{1}{2}-\frac{1}{4}})}^p] \lesssim \epsilon^\gamma, \]
\[ \mathbb{E}[\|(u'_2, b'_2) - (u_2, b_2)\|_{C([0,T]; C^{-\delta})}^p] \lesssim \epsilon^\gamma, \quad (262) \]
\[ \mathbb{E}[\|(u'_3, b'_3) - (u_3, b_3)\|_{C([0,T]; C^{\frac{1}{2}-\gamma})}^p] \lesssim \epsilon^\gamma. \]

Letting \( \epsilon_k \triangleq 2^{-k} \) and \( \epsilon > 0 \), proving
\[ \sum_{k=1}^{\infty} \mathbb{P}(\|u^k - (u_1, b_1)\|_{C([0,T]; C^{-\frac{1}{2}-\frac{1}{4}})} > \epsilon) \lesssim \sum_{k=1}^{\infty} \frac{1}{\epsilon}(\epsilon_k) \lesssim 1 \] 
by Chebychev’s inequality, Hölder’s inequality and (262) is standard. By Borel-Cantelli lemma, this implies \((u^{\epsilon \kappa_i}, b^{\epsilon \kappa_i}) \to (u^1, b_1)\) in \(C([0,T]; C^{-\frac{1}{2}-\frac{1}{4}})\) \(\mathbb{P}\)-a.s. as \( k \to \infty \) and analogous conclusions hold for \((u^{3\kappa_i}, b^{3\kappa_i})\) and \((u^{2\kappa_i}, b^{2\kappa_i})\). Hence, we have shown that sup_{\epsilon \kappa_i \to 0} \epsilon \kappa_i \to C_\epsilon^\kappa < \infty \mathbb{P}-a.s. where \( C_\epsilon^\kappa \) is that of (27), \((u_4, b_3) = \lim_{k \to \infty} (u^{4k}, b^{4k})\) in \([0, T_0]\), \( y = (u, b) = (u_1 + u_2 + u_3 + u_4, b_1 + b_2 + b_3 + b_4) \) as the solution to (15) on \([0, T_0]\) where \( T_0 \) is independent of \( \epsilon \) and
\[ \sup_{t \in [0, T_0]} \|u^{\epsilon \kappa} - (u, b)\|_{C^{-\gamma}} \to 0 \] 
(264)
as \( k \to \infty \) \(\mathbb{P}\)-a.s. due to Proposition 5.1. The proof that we can extend the solution to the maximal solution that satisfies sup_{t \in [0, T]} \|y(t)\|_{C^{-\gamma}} = +\infty is relatively standard and may be achieved identically to [56]. We omit further details here. □

Acknowledgments

The author expresses deep gratitude to Prof. Carl Mueller and Prof. Marco Romito for valuable discussions. He also thanks Prof. Jared Whitehead for suggesting references [2, 23, 33, 48] on the Boussinesq system.

4. Appendix

Under the same notations introduced in Section 1, let us also recall the Besov embeddings:

Lemma 4.1. ([25 Lemma A.2], [56 Lemma 3.1]) Let 1 \leq p_1 \leq p_2 \leq \infty, N \in \mathbb{N} and 1 \leq q_1 \leq q_2 \leq \infty, and \( \alpha \in \mathbb{R} \). Then \( B^\alpha_{p_1, q_1}(\mathbb{T}^N) \) is continuously embedded in \( B^{-N(\frac{1}{p_2} - \frac{1}{p_1})}_{p_2, q_2}(\mathbb{T}^N) \).

We recall several more useful lemmas here:

Lemma 4.2. ([25 Lemma 2.4], [56 Lemma 3.3]) Suppose \( \alpha \in (0, 1), \beta, \gamma \in \mathbb{R} \) satisfy \( \alpha + \beta + \gamma > 0 \) and \( \beta + \gamma < 0 \). Then for smooth \( f, g, h \), the tri-linear operator
\[ C(f, g, h) \triangleq \pi_0(\pi_\alpha(f, g, h)) - f \pi_0(g, h) \]
satisfies
\[ \|C(f, g, h)\|_{C^{\alpha+\beta+\gamma}} \lesssim \|f\|_{C^\alpha}\|g\|_{C^\beta}\|h\|_{C^\gamma}, \]
and thus \( C \) can be uniquely extended to a bounded tri-linear operator in \( L^3(C^\alpha(\mathbb{T}^3) \times C^\beta(\mathbb{T}^3) \times C^\gamma(\mathbb{T}^3)) \).
Lemma 4.3. ([56, Lemma 3.4]) Let $P$ be the Leray projection, $f \in C^\alpha(T^3)$, $g \in C^\beta(T^3)$ for $\alpha < 1$ and $\beta \in \mathbb{R}$. Then for every $k, l \in \{1, 2, 3\}$,
\[ \| P^{kl} \pi_c (f, g) - \pi_c (f, P^{kl} g) \|_{C^{\alpha+\beta}} \lesssim \| f \|_{C^\alpha} \| g \|_{C^\beta}. \]

Lemma 4.4. ([25, Lemma A.7], [56, Lemma 3.5]) Let $P_t$ be the heat semigroup on $\mathbb{T}^N$. Then for $f \in C^\alpha(T^3)$, $\alpha \in \mathbb{R}$ and $\delta \geq 0$, $P_t f$ satisfies
\[ \| P_t f \|_{C^{\alpha+\delta}} \lesssim t^{-\frac{\delta}{2}} \| f \|_{C^\alpha}. \]

Lemma 4.5. ([56, Lemma 3.6]) Let $P$ be the Leray projection and $f \in C^\alpha(T^N)$ for $\alpha \in \mathbb{R}$. Then for every $k, l \in \{1, 2, 3\}$,
\[ \| P^{kl} f \|_{C^\alpha} \lesssim \| f \|_{C^\alpha}. \]

Lemma 4.6. ([56, Lemma 3.11]) Let $P$ be the Leray projection. Then for any $\eta \in (0, 1), i, j, l \in \{1, 2, 3\}$ and $t > 0$,\[ |e^{-|k_1|^2 t} P^{ij}(k_2) - e^{-|k_2|^2 t} P^{ji}(k_2)| \lesssim |k_1|^\eta |t|^{-\frac{1}{1-\eta}}. \]

Lemma 4.7. ([56, Lemma 3.10]) For any $l, m \in (0, N)$ such that $l + m - N > 0$,
\[ \sum_{k_1, k_2 \in \mathbb{Z}^N \setminus \{0\}: k_1 + k_2 = k} \frac{1}{|k_1|^l |k_2|^m} \lesssim \frac{1}{|k|^l+m-N}. \]

Lemma 4.8. Let $u, f \geq 0$ be continuous on $[0, \infty)$. $w$ be continuous, non-decreasing on $[0, \infty)$ and satisfy $w(u) > 0$ on $(0, \infty)$. If
\[ u(t) \leq \alpha + \int_0^t f(s) w(u(s)) ds \]
where $\alpha \geq 0$ is a constant, then
\[ u(t) \leq G^{-1}(G(\alpha) + \int_0^t f(s) ds) \]
where $G(x) = \int_{x_0}^x \frac{du}{w(u)}$, $x, x_0 > 0$.

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