On the minimizers of energy forms with completely monotone kernel

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This paper is dedicated to Jim Gatheral on the occasion of his 60th birthday.

Abstract

Motivated by the problem of optimal portfolio liquidation under transient price impact, we study the minimization of energy functionals with completely monotone displacement kernel under an integral constraint. The corresponding minimizers can be characterized by Fredholm integral equations of the second type with constant free term. Our main result states that minimizers are analytic and have a power series development in terms of even powers of the distance to the midpoint of the domain of definition and with nonnegative coefficients. In particular, our minimization problem is equivalent to the minimization of the energy functional under a nonnegativity constraint.

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1 Introduction and problem formulation

In this paper, we study the minimization of energy functionals of the form

\[ J_\gamma[\varphi] = \frac{\gamma}{2} \int_0^T \varphi(t)^2 \, dt + \frac{1}{2} \int_0^T \int_0^T G(|t-s|)\varphi(s)\varphi(t) \, ds \, dt, \quad \varphi \in L^2[0, T], \quad (1) \]

where \( \gamma \geq 0 \), \( T > 0 \), and \( G : (0, \infty) \to [0, \infty) \) is a continuous and nonconstant function satisfying

\[ \int_0^T G(t) \, dt < \infty \quad \text{for all } T > 0. \quad (2) \]

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Problems of this type have a long history. An early reference is Hilbert (1904), where the minimization and maximization of $J_0[\varphi]$ is studied under the constraint $\int_0^T \varphi(t)^2 \, dt = 1$ if $G$ is of positive type in the sense that

$$\frac{1}{2} \int_0^T \int_0^T G(|t-s|) \varphi(t) \varphi(s) \, dt \, ds \geq 0 \quad \text{for all } \varphi \in L^2[0,T] \text{ and all } T > 0.$$  

In potential theory, one usually takes $\gamma = 0$ and considers the minimization of

$$J_0[\mu] = \frac{1}{2} \int \int G(|t-s|) \mu(ds) \mu(dt),$$

over Borel probability measures $\mu$ supported on a given compact set $K \subset [0,T]$. If a minimizing measure $\mu^*$ exists, it is the capacitary measure for $K$, and $1/J_0[\mu^*]$ is the capacity of $K$; see, e.g., Choquet (1954). Note that the requirement that $\mu$ is a probability measure corresponds to the infinitely many convex constraints $\mu(K) = 1$ and $\mu(A) \geq 0$ for every Borel set $A \subset K$. It was proved in Gatheral et al. (2012) that, for convex and nonincreasing $G$, the latter constrained minimization problem can be replaced by the much simpler minimization of $J_0[\mu]$ over all finite signed Borel measures $\mu$ on $K$ that have finite total variation and satisfy the single linear constraint $\mu(K) = 1$. This observation enabled in particular an approach to compute $\mu^*$ for $K = [0,T]$ by means of singular control (Alfonsi and Schied 2013). Here, we will instead exploit the fact that, for $\gamma > 0$, minimizers of $J_\gamma[\varphi]$ under the constraint $\int_0^T \varphi(t) \, dt = 1$ can be characterized as the solution of the following Fredholm integral equation of second kind,

$$\gamma \varphi(t) + \int_0^T G(|t-s|) \varphi(s) \, ds = \sigma \quad \text{for a.e. } t \in [0,T],$$  

where the constant $\sigma$ is equal to the minimal energy (see Proposition 2).

In this paper, we focus on the qualitative properties of minimizers. For instance, explicit computations or numerical simulations reveal that minimizers of $J_\gamma$ are often convex functions of $t \in [0,T]$ with a minimum at $T/2$. In addition, it is easy to see that every solution $\varphi$ must be symmetric around $T/2$, i.e., $\varphi(t) = \varphi(T-t)$. These two facts are reminiscent of the celebrated Riesz rearrangement inequality, which states that for decreasing $G$,

$$\int_0^T \int_0^T G(|t-s|) f(s) g(t) \, ds \, dt \leq \int_0^T \int_0^T G(|t-s|) f^*(s) g^*(t) \, ds \, dt,$$

where $f^*$ and $g^*$ are the symmetric decreasing rearrangements of the nonnegative functions $f$ and $g$; see Riesz (1930). Although a lower bound in (5) is generally not available, it would be tempting to conjecture that minimizers of $J_\gamma$ are equal to their symmetric increasing rearrangements. This conjecture, however, cannot be true in general since the choice $G(t) = (1-t)^+$ provides a counterexample; see Example 6 and Figure 1. So the following question arises:

*For which kernels $G$ is the minimizer $\varphi$, respectively the solution of (4), convex with a minimum at $T/2$?*

Our main result shows that this is the case whenever $G$ is completely monotone. As a matter of fact, we will actually prove a much stronger result: If $G$ is completely monotone, then $\varphi$ is *symmetrically totally monotone* in the sense that it is analytic in $(0,T)$ and its power series development in $T/2$ is of the form $\varphi(t) = \sum_{n=0}^\infty a_{2n} (t-T/2)^{2n}$ for coefficients $a_{2n} \geq 0$. 

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Problems such as the minimization of $J_\gamma$ or the solution of Fredholm integral equations (4) have a large number of applications, for instance in machine learning; see, e.g., Chen and Haykin (2002), Gatheral et al. (2012) and Alfonsi and Schied (2013), on the other hand, were motivated by the problem of optimal portfolio liquidation in financial markets. There, the solution $\varphi$ corresponds to an optimal trading rate for liquidating a large initial position of shares during the time interval $[0, T]$. Since the position is large, its liquidation affects asset prices in an unfavorable way, which creates additional execution costs. The temporal evolution of this price impact can be described by means of a kernel $G$, for which some empirical studies suggest a behavior of the form $G(t) \sim t^{-\alpha}$ for some $\alpha \in (0, 1)$; see, e.g., Gatheral (2010). Assumption (3) is reasonable in this context: it excludes the existence of market liquidity. That is, if the answer to the practical significance that it matches the empirically observed U-shape of the daily distribution asked by J. Gatheral, and the possible convexity of the optimal portfolio liquidation strategy $\varphi$ involves fast trading toward the beginning and end of the trading day when liquidity is high and slower trading when liquidity is low.

This paper is organized as follows. In Section 2, we present our main results and a few explicit examples. All proofs are given in Section 3.

2 Main results

Let $G: (0, \infty) \to [0, \infty)$ be continuous and nonconstant satisfying (2) and (3). For $\gamma > 0$, we consider the following variational problem,

$$\text{minimize } J_\gamma[\varphi] = \gamma \int_0^T \varphi(t)^2 \, dt + \frac{1}{2} \int_0^T \int_0^T G(|t-s|) \varphi(t) \varphi(s) \, dt \, ds \quad \text{over } \varphi \in \Phi_1,$$

where $\Phi_1$ consists of all functions $\varphi \in L^2[0, T]$ that satisfy the linear constraint $\int_0^T \varphi(t) \, dt = 1$ and for which the double integral on the right is well-defined and finite. For $\gamma = 0$ we consider the following problem,

$$\text{minimize } J_0[\mu] = \frac{1}{2} \int_{[0,T]} \int_{[0,T]} G(|t-s|) \mu(dt) \mu(ds) \quad \text{over } \mu \in \Phi_0,$$

where we put $G(0) := G(0+) \in (0, \infty]$ and where $\Phi_0$ consists of all signed Borel measures $\mu$ on $[0, T]$ that satisfy $\mu([0,T]) = 1$ and whose total variation measure $|\mu|$ is finite and such that

$$\int_{[0,T]} \int_{[0,T]} G(|t-s|) |\mu|(dt) |\mu|(ds) < \infty.$$

For $\gamma > 0$, standard Hilbert space arguments easily yield the existence and uniqueness of minimizers to (6). For $\gamma = 0$, however, the existence of a minimizer for (7) is nontrivial even if $G$ is bounded. Indeed, it was shown in Gatheral et al. (2012) that minimizers do not exist for a large class of kernels for which $G(|\cdot|)$ is analytic, such as for $G(t) = e^{-t^2}$ or $G(t) = 1/(1 + t^2)$. But it was shown in Theorem 2.24 of Gatheral et al. (2012) that (7) admits a unique minimizer $\mu^* \in \Phi_0$ provided that $G(0, \infty) \rightarrow [0, \infty)$ is convex, nonincreasing, nonconstant, and satisfies (2). It was shown moreover that $\mu^*$ is a probability measure. The following proposition extends this latter result to the case $\gamma > 0$. 

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Its proof also provides an alternative proof for the existence of minimizers of (2). Note that every convex, nonincreasing, and nonnegative function $G$ satisfies (3) due to Equation (14) below.

**Proposition 1.** If $G : (0, \infty) \to [0, \infty)$ is convex, nonincreasing, nonconstant, and satisfies (2), the unique minimizer of (1) is a probability density.

The nonnegativity of minimizers to (6) and (7), which only involve a one-dimensional linear constraint, yield the solutions to the minimization of the functional $J_\gamma$ over probability measures or probability densities. The latter problem is of interest in many applications (see, e.g., Gatheral et al. (2012) and Alfonsi and Schied (2013)).

The following proposition links the minimizer of $J_\gamma$ for $\gamma > 0$ to the solution of a Fredholm integral equation of second kind with constant free term.

**Proposition 2.** For $\gamma > 0$ and a function $\varphi \in \Phi_1$, the following conditions are equivalent.

(a) $\varphi$ solves (6).

(b) There exists a constant $\sigma$ such that $\varphi$ solves (4).

In this case, the constant $\sigma$ from (b) is equal to $2J_\gamma[\varphi] = 2\inf_{\psi \in \Phi_1} J_\gamma[\psi]$ and strictly positive.

Now we prepare for the statement of our main result. Let $\tau \in (0, \infty]$. Recall that a function $f : (0, \tau) \to \mathbb{R}$ is called completely monotone on $(0, \tau)$ if $f$ admits derivatives of all orders throughout $(0, \tau)$ and if $(-1)^n f^{(n)}(x) \geq 0$ for all $x \in (0, \tau)$ and $n = 0, 1, \ldots$. Completely monotone functions on $(0, \infty)$ are special, as they can be represented as the Laplace transforms of positive Radon measures on $[0, \infty)$. This representation may fail if $\tau < \infty$. A simple example is the function $f(t) = e^t + e^{T-t}$ for $T > 0$, which is completely monotone on $(0, T/2)$ but not on $(0, T)$. This function, however, belongs to the following class.

**Definition 3.** A function $f : (0, T) \to \mathbb{R}$ is called symmetrically totally monotone if it is analytic in $(0, T)$ and its power series development in $T/2$ is of the form

$$f(x) = \sum_{n=0}^{\infty} a_{2n}(x - T/2)^{2n}$$

for coefficients $a_{2n} \geq 0$.

This terminology is motivated by the fact that any symmetrically totally monotone function $f$ on $(0, T)$ is symmetric in the sense that $f(x) = f(T - x)$, completely monotone on $(0, T/2)$, and absolutely monotone on $(T/2, T)$ (i.e., $f^{(n)}(x) \geq 0$ for $x \in (T/2, T)$ and $n = 0, 1, \ldots$).

**Theorem 4.** Suppose that that $T > 0$ and that $G : (0, \infty) \to \mathbb{R}_+$ is completely monotone, nonconstant, and satisfies (2).

(a) For $\gamma > 0$, the unique minimizer of (6) is symmetrically totally monotone.

(b) For $\gamma = 0$, the restriction to $(0, T)$ of the unique minimizer $\mu^*$ of (7) admits a symmetrically totally monotone Lebesgue density.

\[ \text{1This fact relies on the well-known result that } G(|\cdot|) \text{ is positive definite in the sense of Bochner for every bounded, convex,} \]

\[ \text{and nonincreasing function } G : [0, \infty) \to [0, \infty). \text{ This latter result is often attributed to Polya (1949), although it is also an easy consequence of Young (1913).} \]
For $\gamma = 0$, the unique minimizer $\mu^*$ has strictly positive point masses in 0 and $T$ as soon as both $G(0)$ and $G'(0+)$ are finite (Gatheral et al. [2012] Theorem 2.23). If, however, $G(0+) = \infty$, then we must have $\mu^*(\{0\}) = \mu^*(\{T\}) = 0$, and so $\mu^*$ will be absolutely continuous with respect to the Lebesgue measure on all of $[0, T]$.

**Example 5** (Exponential kernel). Consider a completely monotone kernel of the form

$$G(t) = \sum_{k=1}^{n} a_k e^{-\sqrt{b_k}t}$$

for coefficients $a_1, a_2, \ldots, a_n > 0$ and $b_n > b_{n-1} > \cdots > b_1 > 0$. We will show in Section 3.4.1 that the unique solution of (4) is of the form

$$\varphi(t) = z_0 + \sum_{i=1}^{n} z_i (e^{\sqrt{c_i}t} + e^{\sqrt{c_i}(T-t)})$$

where $z_i \geq 0$ and the coefficients $c_i$ are equal to the eigenvalues of the matrix $M$ from (21) and satisfy $c_n > b_n > c_{n-1} > b_{n-1} > \cdots > c_1 > b_1 > 0$. This function $\varphi$ is clearly symmetrically totally monotone. In the special case $n = 1$ with $G(t) = e^{-\sqrt{b}t}$, we have $c = b + \frac{2}{\gamma} \sqrt{b}$ and a direct calculation yields that

$$\varphi(t) = \frac{\gamma b}{\gamma c - 2\sqrt{b}} \left(1 + \frac{2(e^{\sqrt{c}t} + e^{\sqrt{c}(T-t)})}{\gamma(e^{\sqrt{c}t}(\sqrt{b} + \sqrt{c}) + \sqrt{b} - \sqrt{c})}\right), \quad t \in [0, T],$$

where the constant $\sigma > 0$ is as in (4). For solving (6), it can be determined through the condition $\int_0^T \varphi(t) \, dt = 1$.

The following two examples illustrate that the assertions of Proposition 1 and Theorem 4 need no longer be true if the corresponding hypotheses are not satisfied. More precisely, the following Example 6 shows that the minimizer $\varphi$ need not be convex even if $G$ is convex and nonincreasing, and Example 7 illustrates that $\varphi$ can become negative if $G$ is merely of positive type and not convex.

**Example 6** (Capped linear kernel). Consider the convex nonincreasing kernel $G(t) = (1 - t)^+$ and the equation

$$\gamma \varphi(t) + \int_0^n (1 - |t - s|)^+ \varphi(s) \, ds = \sigma, \quad (8)$$

where we assume for simplicity that $n \in \mathbb{N}$. For $i = 1, \ldots, n$, define $\lambda_i := 2(1 - \cos\left(\frac{i\pi}{n+1}\right))$ and $b_i := \sqrt{\lambda_i/\gamma}$. Let $B := \text{diag}(b_1, \ldots, b_n)$,

$$Q := \left(\sin\left(\frac{ij\pi}{n+1}\right)\right)_{i,j=1,\ldots,n}$$

and $E(t) := \text{diag}(e^{b_1t}, \ldots, e^{b_nt})$. Furthermore, define $\sigma := (\sigma, \ldots, \sigma) \in \mathbb{R}^n$, denote by $I$ the $n$-dimensional identity matrix, let $J := \text{diag}(1, -1, 1, \ldots, \pm 1) \in \mathbb{R}^{n \times n}$, and put $K := I + (\delta_{j,n-i})_{i,j=1,\ldots,n}$. For the solution $\varphi$ of (8), as provided by Propositions 1 and 2, define $\varphi_i(t) := \varphi(t + i - 1)$ for $t \in [0, 1], i = 1, \ldots, n$. We will prove in Section 3.5 that the functions $\varphi_1, \ldots, \varphi_n$ are given by

$$\left(\varphi_1(t), \ldots, \varphi_n(t)\right)^\top = Q(E(t) + E(1-t)J)a, \quad t \in [0, 1], \quad (9)$$

where

$$a := \left(\gamma Q(E(1) + J) + KQ((E(1) - I)(J - I) + B(E(1) - J))B^{-2}\right)^{-1} \sigma.$$

See Figure 1 for an illustration.
Example 7 (Trigonometric kernel). Let $G(t) = \cos(\rho t)$ for a constant $\rho > 0$. It is well known that $G$ is positive definite and hence satisfies (3), but it is of course not convex. One easily verifies that the solution $\varphi$ of (4) is given by

$$\varphi(t) = \frac{\sigma}{\gamma} \left[ 1 - \frac{2\tan(\rho T/2)(\cos(\rho t) + \cos(\rho(T-t)))}{\rho(2\gamma + T) + \sin(\rho T)} \right].$$

This function can clearly become negative; see Figure 2.

Figure 1: Solution $\varphi$ of (8) for $G(t) = (1 - t)^+$, $\gamma = 0.01$, and $n = 11$. Although $\varphi$ is positive, it is not convex.

Figure 2: Solution $\varphi$ of (4) for $G(t) = \cos(t/2)$, $\gamma = 0.001$, and $T = 1$.

3 Proofs

3.1 Preliminaries and proof of Proposition 2

Let us first recall some facts from Gatheral et al. (2012) on convex, nonincreasing, and nonconstant kernels $G : (0, \infty) \to [0, \infty)$ satisfying the condition (2). By Lemma 4.1 in Gatheral et al. (2012), there exists a positive Radon measure $\eta$ on $(0, \infty)$ such that

$$\int_{(0,\infty)} y \wedge y^2 \eta(dy) < \infty$$

and

$$G(x) = G(\infty-) + \int_{(0,\infty)} (y - x)^+ \eta(dy) \quad \text{for } x > 0. \quad (11)$$

The Fourier transform of a Radon measure $\mu$ on $\mathbb{R}$ for which $\mu([-x, x])$ grows at most polynomially in $x$ can be defined through

$$\hat{\mu}(f) = \int \hat{f} \, d\mu, \quad f \in \mathcal{S}(\mathbb{R}),$$

where $\mathcal{S}(\mathbb{R})$ is the usual Schwartz space of rapidly decreasing $C^\infty$-functions and $\hat{f}(z) = \int e^{izx} f(x) \, dx$ is the Fourier transform of $f$ (in the convention of Gatheral et al. (2012)). With this definition, it was shown in Lemma 4.2 of Gatheral et al. (2012) that $G(|\cdot|)$ can be represented as the Fourier transform of the positive Radon measure

$$\nu(dx) = G(\infty-) \delta_0(dx) + g(x) \, dx,$$
on \( \mathbb{R} \), where the density \( g \) is given by
\[
g(x) = \frac{1}{\pi} \int_{(0, \infty)} \frac{1 - \cos xy}{x^2} \eta(dy)
\] (12)
and the measure \( \eta \) is in (11). Now let \( \mu \) be any signed Borel measure on \([0, T] \) whose total variation measure \( |\mu| \) is finite and such that \( \int \int G(|t - s|) |\mu|(dt) |\mu|(ds) < \infty \). Proposition 4.5 in Gatheral et al. (2012) then shows that
\[
J_0[\mu] = \frac{1}{2} \int |\hat{\mu}(z)|^2 \nu(dz).
\] (13)
It therefore follows from Plancherel’s theorem that for \( \gamma > 0 \),
\[
J_\gamma[\varphi] = \frac{\gamma}{2} \int |\hat{\varphi}(z)|^2 dz + \frac{1}{2} \int |\hat{\varphi}(z)|^2 \nu(dz), \quad \varphi \in \Phi_1.
\] (14)
As a matter of fact, the preceding identity extends to the space \( L^2_G[0, T] \) of all functions \( \varphi \in L^2[0, T] \) for which \( J_\gamma[\varphi] \) is finite. It is clear from (14) and Minkowski’s inequality that \( L^2_G[0, T] \) is a vector space.

**Proof of Proposition** [3]

For \( f, g \in L^2_G[0, T] \), we can define the symmetric bilinear form
\[
J_\gamma[f, g] := \frac{1}{2} (J_\gamma[f + g] - J_\gamma[f] - J_\gamma[g]).
\]
Now suppose \( \varphi \) solves (6). We take a nonzero function \( \psi \in L^2_G[0, T] \) such that \( \int_0^T \psi(t) \, dt = 0 \) and \( \alpha \in \mathbb{R} \). Then \( \varphi + \alpha \psi \in \Phi_1 \) and
\[
J_\gamma[\varphi + \alpha \psi] = J_\gamma[\varphi] + \alpha^2 J_\gamma[\psi] + 2\alpha J_\gamma[\varphi, \psi].
\]
The optimality of \( \varphi \) implies that the right-hand side is minimized at \( \alpha = 0 \), which implies that \( J_\gamma[\varphi, \psi] = 0 \). Thus, \( \gamma \varphi(t) + \int_0^T G(|t - s|) \varphi(s) \, ds \) must be orthogonal to every \( \psi \in L^2_G[0, T] \) with \( \int_0^T \psi(t) \, dt = 0 \), which gives (14).

Conversely, (14) implies that \( J_\gamma[\varphi, \psi] = 0 \) for every \( \psi \in L^2_G[0, T] \) with \( \int_0^T \psi(t) \, dt = 0 \). For every \( \tilde{\varphi} \in \Phi_1 \) and \( \psi := \tilde{\varphi} - \varphi \),
\[
J_\gamma[\tilde{\varphi}] = J_\gamma[\varphi] + J_\gamma[\psi] + 2J_\gamma[\varphi, \psi] \geq J_\gamma[\varphi],
\]
and so \( \varphi \) solves (6). Finally, it is clear that \( J_\gamma[\varphi] = \sigma/2 \) if \( \varphi \) solves (14).

\[ \square \]

### 3.2 Proof of Proposition [1]

The uniqueness of minimizers follows immediately from the fact that \( J_\gamma \) is strictly convex by (14). To show the existence of a nonnegative minimizer, we consider first the case in which \( G(0+) < \infty \). When letting \( G(0) := G(0+) \), the function \( G(|\cdot|) \) is a bounded and continuous function on \( \mathbb{R} \).

For \( n \in \mathbb{N} \), we let \( \Phi_1^{(n)} \) denote the set of all \( \varphi \in L^2[0, T] \) that satisfy \( \int_0^T \varphi(t) \, dt = 1 \) and that are constant on all intervals of the form \([t_k, t_{k+1}]\), where \( t_k = k2^{-n}T \) for \( k = 0, \ldots, 2^n \). Any such \( \varphi \) is thus of the form
\[
\varphi = \sum_{k=0}^{2^n-1} \varphi_k \mathbbm{1}_{[t_k, t_{k+1}]}
\] (15)
for certain real coefficients \( \varphi_k \) that sum up to \( 2^n/T \). In particular, \( \varphi \) belongs to \( L^\infty[0, T] \) and hence to \( \Phi_1 \). We need the following simple lemma.
Lemma 8. For \( \varphi \in \Phi_1^{(n)} \) of the form \([15]\), we have

\[
J_\gamma[\varphi] = \sum_{i,j=0}^{2^n} \varphi_i \varphi_j G_n(|t_i - t_j|)
\]

where

\[
G_n(0) = \gamma 2^{-(n+1)}T + 2^{-2n+1} T^2 \int_0^1 G(2^{-n}Ts) (1 - s) \, ds,
\]

\[
G_n(t) = 2^{-2n} T^2 \int_{-1}^1 G(t + 2^{-n} Ts) (1 - |s|) \, ds \quad \text{for } t \geq 2^{-n},
\]

and \( G_n(t) \) linearly interpolated between \( G_n(0) \) and \( G_n(2^{-n}) \) for \( t \in (0, 2^{-n}) \).

Proof. We clearly have

\[
\frac{\gamma}{2} \int_0^T \varphi(t)^2 \, dt = \frac{\gamma}{2} \sum_{i=0}^{2^n-1} \varphi_i^2 (t_{i+1} - t_i) = \gamma 2^{-(n+1)} T \sum_{i=0}^{2^n-1} \varphi_i^2.
\]

Moreover,

\[
\frac{1}{2} \int_0^T \int_0^T G(|t - s|) \varphi(t) \varphi(s) \, ds \, dt = \frac{1}{2} \sum_{i,j=0}^{2^n-1} \varphi_i \varphi_j \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} G(|t - s|) \, ds \, dt.
\]

For \( i < j \), we have

\[
\int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} G(|t - s|) \, ds \, dt = 2^{-2n} T^2 \int_0^1 \int_0^1 G(t_j - t_i + 2^{-n} T (s - r)) \, ds \, dr = G_n(t_j - t_i).
\]

For \( i = j \), we have

\[
\int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} G(|t - s|) \, ds \, dt = 2^{-2n+1} T^2 \int_0^1 G(2^{-n} Ts)(1 - s) \, ds.
\]

This determines the values of \( G \) in the points \( t_k \) for \( k = 0, \ldots, 2^n \). The values of \( G_n(t) \) for all other \( t \) do actually not matter for the representation of \( J_\gamma[\varphi] \), and hence can be chosen arbitrarily, for instance, as in the statement of the lemma. \( \square \)

On \([0, 2^{-n}]\), the function \( G_n \) is linear. On \([2^{-n}, \infty)\), it is a mixture of the convex, nonincreasing, and nonnegative functions \( G(\cdot + 2^{-n} s) \) for \(-1 \leq s \leq 1\). Hence, \( G_n \) also has these properties on \([2^{-n}, \infty)\). We conclude that \( G_n \) is convex if and only if the left-hand derivative \( G'_{n,-}(2^{-n} T) \) of \( G_n \) in \( 2^{-n} T \) is smaller than or equal to the right-hand derivative \( G'_{n,+}(2^{-n} T) \). Let \( G'_+ \) denote the right derivative of \( G \). Recall that \( G(0) \) is assumed to be finite. The value \( G'_+ (0) \) is \( (\infty, 0) \) provides a lower bound on \( G'_+ \), hence \( G'_{n,+}(2^{-n} T) \geq 2^{-2n} T^2 G^+(0) \). Define \( G(\infty-) := \lim_{t \to \infty} G(t) \). Notice that \( 0 \leq G(\infty-) \leq G(t) \leq G(0) < \infty \) for all \( t \in \mathbb{R}_+ \). Plugging into the definition of \( G_n \) shows that

\[
G'_{n,-}(2^{-n} T) = \frac{G_n(2^{-n} T) - G_n(0)}{2^{-n} T} \leq 2^{-n} T (G(0) - G(\infty-)) - \frac{\gamma}{2}.
\]

If \( n \) is sufficiently large, then the factor \( \gamma/2 \) becomes dominating and ensures the convexity of \( G_n \) on all of \( \mathbb{R}_+ \). Thus, let \( n_0 \) be such that \( G_n \) is nonconstant and convex on \( \mathbb{R}_+ \) for all \( n \geq n_0 \).
Now consider the problem of minimizing $J_\gamma[\varphi]$ over $\varphi \in \Phi_1^{(n)}$. By Lemma 8, this problem is equivalent to the minimization of the quadratic form $\sum_{i,j=0}^{2^n} \varphi_i \varphi_j G_n(|t_i - t_j|)$ over $\varphi_0, \ldots, \varphi_{2^n} \in \mathbb{R}$ that sum up to one. The fact that $G_n$ is convex, nonincreasing, nonnegative, and nonconstant implies that the matrix with entries $G_n(|t_i - t_j|)$ is positive definite due to (13). Thus, our minimization problem has a unique minimizer as soon as $n \geq n_0$. Moreover, by Theorem 1 in Alfaoni et al. (2012), all components $\varphi_k$ of this minimizer will be nonnegative. Thus, also the problem of minimizing $J_\gamma[\varphi]$ over $\varphi \in \Phi_1^{(n)}$ has a unique minimizer $\varphi(\cdot)$, which is nonnegative, as soon as $n \geq n_0$.

Next, since $\Phi_1^{(n)} \subset \Phi_1^{(n+1)}$, we have $J_\gamma[\varphi(n)] \geq J_\gamma[\varphi(n)]$ for all $n \geq n_0$. Since moreover $J_\gamma[\varphi] \geq \frac{\gamma}{2} \|\varphi\|^2$ due to (13), we get that the $L^2$-norms $\|\varphi(n)\|$ are uniformly bounded for all $n \geq n_0$. By passing to a subsequence if necessary, we may therefore assume that the sequence $(\varphi(n))_{n \geq n_0}$ converges weakly in $L^2(0, T)$ to some nonnegative limit $\varphi^*$. We claim that $\varphi^*$ is the minimizer of $J_\gamma[\varphi]$ over $\varphi \in \Phi_1$. To see this, let us assume by way of contradiction that $\varphi^*$ is not the minimizer. Then there exists another function $\hat{\varphi} \in \Phi_1$ with $J_\gamma[\hat{\varphi}] < J_\gamma[\varphi^*]$. By $\hat{\varphi}_n$ we denote the conditional expectation of $\hat{\varphi}$ with respect to the $\sigma$-field on $[0, T]$ generated by the intervals $[t_0, t_1), \ldots, [t_{2^n-1}, t_{2^n})$ and under the (normalized) Lebesgue measure. Then $\hat{\varphi}_n$ belongs to $\Phi_1^{(n)}$, and so $J_\gamma[\hat{\varphi}_n] \geq J_\gamma[\varphi(n)]$. By martingale convergence $\hat{\varphi}_n \to \hat{\varphi}$ in $L^2(0, T)$, the fact that $G(| \cdot |)$ is bounded and continuous now gives that $J_\gamma[\hat{\varphi}_n] \to J_\gamma[\hat{\varphi}]$. On the other hand, the map $\varphi \mapsto J_\gamma[\varphi]$ is weakly lower semicontinuous, and so

$$J_\gamma[\varphi^*] \leq \liminf_{n \uparrow \infty} J_\gamma[\varphi(n)] \leq \liminf_{n \uparrow \infty} J_\gamma[\hat{\varphi}_n] = J_\gamma[\hat{\varphi}],$$

which is a contraction. Therefore, the nonnegative function $\varphi^*$ is indeed the minimizer.

Let us now consider the case $G(0+) = \infty$. As in the proof of Theorem 2.24 of Gatheral et al. (2012), we can consider approximations $G_n$ of $G$ defined through the measures

$$\eta_n(dy) = \mathbb{1}_{(1/n, \infty)}(y) \eta(dy) \quad \quad (16)$$

in (11). These functions $G_n$ are then continuous, nonincreasing and nonnegative and they satisfy $G_n(0+) < \infty$. They correspond to functions $g_n$ defined as in (12) and energy functionals $J_\gamma^{(n)}$ satisfying (14) for $\nu_n(dx) = G(\infty -) \delta_0(dx) + g_n(x) dx$. Then let $(\psi_n)_{n=1,2,\ldots}$ be a minimizing sequence for $J_\gamma$ in $\Phi_1$. Since $g(x) \geq g_n(x)$, we have $J_\gamma[\psi_n] \geq J_\gamma^{(n)}[\psi_n]$. For each $n$, we take moreover a minimizer $\varphi_n$ of $J_\gamma^{(n)}$ in $\Phi_1$. Since $G_n(0+) < \infty$, we already know that $\varphi_n \geq 0$. Moreover, we have $J_\gamma[\psi_n] \geq J_\gamma^{(n)}[\psi_n] \geq J_\gamma^{(n)}[\varphi_n]$. This implies that $\gamma \int_0^T \varphi_n(t)^2 dt$ is uniformly bounded in $n$, and so, after passing to a subsequence if necessary, we may assume that the sequence $(\varphi_n)$ converges weakly in $L^2(0, T)$ to a function $\varphi \in L^2(0, T)$, which must also be nonnegative. Due to the compactness of $[0, T]$, it follows that the Fourier transforms $\hat{\varphi}_n$ converge pointwise to $\hat{\varphi}$. Since moreover $g_n$ increases pointwise to the function $g$ from (12), we get

$$\inf_{\varphi \in \Phi_1} J_\gamma[\varphi] = \lim_{n \uparrow \infty} J_\gamma[\varphi_n] \geq \liminf_{n \uparrow \infty} J_\gamma^{(n)}[\varphi_n]$$

$$= \liminf_{n \uparrow \infty} \left( \frac{\gamma}{2} \int |\hat{\varphi}_n(z)|^2 dz + G_n(\infty -)|\hat{\varphi}_n(0)|^2 + \frac{1}{2} \int |\hat{\varphi}_n(z)|^2 g_n(z) dz \right)$$

$$\geq J_\gamma[\varphi],$$

where we have used Fatou’s lemma in the final step. This shows that the function $\varphi$ is the desired nonnegative minimizer. \hfill \Box
3.3 On symmetrically totally monotone functions

Let us introduce the notation $\Delta_h f(x) := f(x + h) - f(x)$ for a function $f$. We will say that $f$ is symmetric around $T/2$ if $f(x) = f(T - x)$.

Lemma 9. For an analytic function $f : (0, T) \to \mathbb{R}$, the following conditions are equivalent.

(a) $f$ is symmetrically totally monotone.

(b) $f$ is symmetric around $T/2$, completely monotone on $(0,T/2)$, and absolutely monotone on $(T/2,T)$.

(c) $f$ is symmetric around $T/2$ and

$$\Delta^n_h f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x + kh) \geq 0$$

for $x > T/2$, $n = 1, 2, \ldots$, and $h > 0$ with $x + nh < T$.

Proof. The implication (a) $\Rightarrow$ (c) is straightforward. The proof of (c) $\Rightarrow$ (b) relies on results by Bernstein (1914). It was proved there (p. 451) that a function $f$ satisfying condition (c) (without necessarily being analytic a priori) is absolutely monotone on $(T/2, T)$ and admits an analytic continuation $\tilde{f}$ to all of $(0, T)$. By analyticity, $\tilde{f}$ must coincide with $f$ on $(0, T)$. The symmetry of $f$ now implies that $f$ is completely monotone on $(0,T/2)$. To show (b) $\Rightarrow$ (a), note that complete monotonicity in $(0, T/2)$ together with absolute monotonicity in $(T/2, T)$ implies that $f^{(2n)}(T/2) \geq 0$ and $f^{(2n+1)}(T/2) = 0$ for $n = 0, 1, \ldots$. Thus, developing $f$ into a power series at $T/2$ gives (a).

The function $\arcsin(|1 - x|)$ shows that the condition of analyticity in Lemma 9 cannot simply be dropped. The following lemma shows in particular that the class of symmetrically totally monotone functions is closed under pointwise convergence.

Lemma 10. Suppose that $(f_n)$ is a sequence of symmetrically totally monotone functions on $(0,T)$ and $D$ is a dense subset of $(0,T)$ such that for each $x \in D$ the limit $\lim_n f_n(x)$ exists and is finite. Then the limit of $f_n(x)$ exists for every $x \in (0,T)$ and the function $f(x) := \lim_n f_n(x)$ is symmetrically totally monotone. Moreover, $f_n \to f$ uniformly on compact subsets of $(0,T)$ and the coefficients in the power series development $f_n(x) = \sum_{k=0}^{\infty} a_k^{(n)} (x - T/2)^k$ converge to those in the corresponding development of $f$.

Proof. Clearly, every function $f_n$ is convex, so Theorem 10.8 from Rockafellar (1970) yields the existence of the limit $f(x) := \lim_n f_n(x)$ for every $x \in (0,T)$, the uniform convergence $f_n \to f$ on compact subsets of $(0,T)$, and the fact that $f$ is convex and, hence, continuous on $(0,T)$. Moreover, we clearly have $\Delta^n_h f_n(x) \to \Delta^n_h f(x)$ for every $x \in (T/2,T)$, $k = 0, 1, 2, \ldots$, and $h > 0$ with $x + kh < T$. The result from Bernstein (1914) quoted in the proof of Lemma 9 implies that $f$ is analytic on $(T/2,T)$ and can be extended to an analytic function $\tilde{f}$ on $(0,T)$. We will show below that $\tilde{f}$ is symmetrically totally monotone. Then the symmetry of the functions $f_n$, $f$, and $\tilde{f}$ and the continuity of $f$ and $\tilde{f}$ will imply that $f = \tilde{f}$ on all of $(0,T)$, and the result will be proved.

Now we prove that $\tilde{f}$ is symmetrically totally monotone. To this end, let $f_n(x) = \sum_{k=0}^{\infty} a^{(n)}_k (x - T/2)^k$ and $\tilde{f}(x) = \sum_{k=0}^{\infty} \tilde{a}_k (x - T/2)^k$ denote the power series developments of $f_n$ and $\tilde{f}$ around $T/2$. In a first step, we note that $a_0^{(n)} = f_n(T/2) \to \tilde{f}(T/2) = \tilde{a}_0$, according to the convergence established in
the preceding paragraph. Next, consider the functions $f_{n,1}(x) = \sum_{k=0}^{\infty} a_{k+1}^{(n)} |x - T/2|^k$. Since $a_k^{(n)} \geq 0$ and $a_{2k+1}^{(n)} = 0$, these functions are convex and we have

$$f_{n,1}(x) = \text{sgn}(x - T/2) \sum_{k=0}^{\infty} a_{k+1}^{(n)} (x - T/2)^k = \frac{f_n(x) - a_0^{(n)}}{|x - T/2|}.$$ 

Therefore, these functions converge pointwise on $(0, T/2) (T/2, T)$ to $\tilde{f}_1(x) = (\tilde{f}(x) - \tilde{a}_0)/|x - T/2|$. Using once again Theorem 10.8 from Rockafellar (1970), we conclude that $\tilde{f}_1$ has a continuous and convex extension to all of $(0, T)$ and that $f_{n,1} \rightarrow \tilde{f}_1$ locally uniformly. It follows that $a_1^{(n)} = f_{n,1}(T/2) \rightarrow \tilde{f}_1(T/2) = \tilde{a}_1$. Next, by considering the convex functions $f_{n,2}(x) = \sum_{k=0}^{\infty} a_{k+2}^{(n)} (x - T/2)^k$, we conclude in the same way that $a_2^{(n)} \rightarrow \tilde{a}_2$. Iterating this argument further yields that $a_k^{(n)} \rightarrow \tilde{a}_k$ for all $k$, and hence that $\tilde{a}_k \geq 0$ and $\tilde{a}_{2k} = 0$ for all $k$. Therefore, $\tilde{f}$ is indeed symmetrically totally monotone.

**Lemma 11.** Let $\mathcal{M}$ denote the class of all finite measures on $[0, T]$ whose restrictions to $(0, T)$ admit a symmetrically totally monotone Lebesgue density. Then $\mathcal{M}$ is closed with respect to weak convergence of measures.

**Proof.** Let $(\mu_n)_{n=1,2,\ldots}$ be a sequence of measures in $\mathcal{M}$ such that $\mu_n$ converges weakly to a finite measure $\mu_0$ on $[0, T]$ and denote by $F_n(x) = \mu_n([0, x])$ the corresponding distribution functions. Then $F_n(x) \rightarrow F_0(x)$ for all continuity points of $F_0$ and hence on a dense subset of $(0, T)$. By assumption, $F_n$ is the integral of an absolutely monotone function on $(T/2, T)$ and thus absolutely monotone there itself if $n \geq 1$. In particular, $F_n$ is convex on $[T/2, T]$. Since, moreover, $F_n(x) = F_n(T) - F_n(T - x)$ for $x \in [0, T/2]$, it is concave on $[0, T/2]$. By arguing as in the proof of Lemma 10, we thus conclude that $F_n(x) \rightarrow F_0(x)$ for all $x \in (0, T)$, that $F_0$ is absolutely monotone on $(T/2, T)$, and that $F_0$ is analytic on $(0, T)$ with a symmetrically totally monotone derivative there.

**Lemma 12.** The class of all symmetrically totally monotone functions in $L^2[0, T]$ is weakly closed in $L^2[0, T]$.

**Proof.** Let $\mathcal{S}$ denote the class of all symmetrically totally monotone functions in $L^2[0, T]$. If a sequence $(f_n)$ in $\mathcal{S}$ converges in $L^2[0, T]$ to a function $f$, then $f \geq 0$ and the finite measures $f_n(x) \, dx$ converge weakly to the finite measure $f(x) \, dx$. Hence, $f \in \mathcal{S}$ by Lemma 11. Hence, the convex cone $\mathcal{S}$ is closed in $L^2[0, T]$ and thus also weakly closed.

### 3.4 Proof of Theorem 4

#### 3.4.1 Proof of Theorem 4 for exponential kernels

Assume first that $G$ is an exponential kernel (of order $n$), i.e. there are $a_1, a_2, \ldots, a_n > 0$ and $b_n > b_{n-1} > \cdots > b_1 > 0$ such that

$$G(t) = \sum_{k=1}^{n} a_k e^{-\sqrt{b_k}t}. \quad (17)$$

Clearly, any such $G$ is completely monotone and satisfies (2). Let $\varphi$ be the unique minimizer of (6). By Proposition 2 there is a $\sigma > 0$ such that $\varphi$ solves (4). By scaling $\varphi$ and $G$, we may assume without loss of generality that $\sigma = \gamma$.

All matrices considered in this proof are $n$-dimensional square matrices, and all vectors $n$-dimensional column vectors. We denote the diagonal matrix with $x_1, \ldots, x_n$ on its main diagonal as $\text{diag}(x_i)_{i=1,\ldots,n}$, and say that a matrix is a positive diagonal matrix if it is diagonal and all diagonal entries are positive.
Let \( A := \text{diag}(a_i)_{i=1,\ldots,n} \) and \( B := \text{diag}(b_i)_{i=1,\ldots,n} \). Define the function \( \psi = (\psi_1, \psi_2, \ldots, \psi_n) \) via

\[
\psi_k(t) := a_k \int_0^T e^{-\sqrt{b_k}|t-s|} \varphi(s) \, ds, \quad t \in [0,T], \quad k = 1, 2, \ldots, n.
\]

Then

\[
\varphi = 1 - \lambda \sum_k \psi_k = 1 - \lambda 1^T \psi,
\]

where \( \lambda := 1/\gamma \) and \( 1 := (1, 1, \ldots, 1) \in \mathbb{R}^n \).

Let us first give an outline of the proof:

1. Show that \( \psi \) solves a system of \( n \) ordinary differential equations \( \psi'' = M \psi - 2AB^{1/2} 1 \) with boundary conditions \( \psi(0) = \psi(T) \) and \( \psi'(0) = B^{1/2} \psi(0) \). Here, \( M \) is a nonsingular matrix.

2. Show that \( M \) has \( n \) distinct, real eigenvalues \( c_n > c_{n-1} > \cdots > c_1 > 0 \). Let \( C := \text{diag}(c_i)_{i=1,\ldots,n} \).

Obtain an eigendecomposition \( M = QCQ^{-1} \), where \( Q \) is a nonsingular matrix.

3. Conclude with 1. that

\[
\varphi(t) = d \left( 1 + 2 \lambda 1^T \left( e^{M^{1/2}t} + e^{M^{1/2}(T-t)} \right) N^{-1} 1 \right),
\]

where \( d > 0 \) and \( N \) is a nonsingular matrix.

4. Use the eigendecomposition of \( M \) to rewrite (19) as

\[
\varphi(t) = d \left( 1 + 1^T E(t) \tilde{N}^{-1} 1 \right).
\]

Here, \( \tilde{N} \) is a nonsingular matrix. The matrices \( E(t) \) are positive diagonal matrices, and each diagonal entry of the mapping \( t \mapsto E(t) \) is symmetrically totally monotone.

5. Decompose \( \tilde{N}^{-1} = \tilde{N}_1 (\tilde{N}_2 + \tilde{N}_3)^{-1} \tilde{N}_4 \) such that \( \tilde{N}_1 \) and \( \tilde{N}_3 \) are positive diagonal matrices, \( \tilde{N}_2 \) is positive definite, and all off-diagonal entries of \( (\tilde{N}_2 + \tilde{N}_3)^{-1} \) are nonpositive.

6. Show that all entries of \( \tilde{N}_2^{-1} \tilde{N}_4 1 \) are nonnegative. Show that this implies that all entries of \( \tilde{N}^{-1} 1 = \tilde{N}_1 (\tilde{N}_2 + \tilde{N}_3)^{-1} \tilde{N}_4 1 \) are nonnegative.

7. Conclude with (20) and Step 6 that \( \varphi \) is symmetrically totally monotone.

1. Recall that \( A = \text{diag}(a_i)_{i=1,\ldots,n} \) and \( B = \text{diag}(b_i)_{i=1,\ldots,n} \) are positive diagonal matrices, and that \( b_n > b_{n-1} > \cdots > b_1 \). Notice that \( 11^T \) is the matrix containing all ones. Define

\[
M := B + 2\lambda AB^{1/2} 11^T = \begin{pmatrix}
 b_1 + 2\lambda a_1 \sqrt{b_1} & 2\lambda a_1 \sqrt{b_1} & \cdots & 2\lambda a_1 \sqrt{b_1} \\
 2\lambda a_2 \sqrt{b_2} & b_2 + 2\lambda a_2 \sqrt{b_2} & \cdots & 2\lambda a_2 \sqrt{b_2} \\
 \vdots & \vdots & \ddots & \vdots \\
 2\lambda a_n \sqrt{b_n} & 2\lambda a_n \sqrt{b_n} & \cdots & b_n + 2\lambda a_n \sqrt{b_n}
\end{pmatrix}.
\]

1.1 \( \psi \) solves the system of \( n \) ordinary differential equations \( \psi'' = M \psi - 2AB^{1/2} 1 \).

Let \( t \in [0,T] \) and \( k = 1, 2, \ldots, n \). Differentiating and plugging in from (18) shows

\[
\psi_k''(t) = a_k \sqrt{b_k} \frac{d}{dt} \left[ - \int_0^t e^{-\sqrt{b_k}|t-s|} \varphi(s) \, ds + \int_t^T e^{-\sqrt{b_k}(s-t)} \varphi(s) \, ds \right]
\]

\[
= a_k \sqrt{b_k} \left( \sqrt{b_k} \int_0^T e^{-\sqrt{b_k}(t-s)} \varphi(s) \, ds - 2\varphi(t) - b_k \psi_k(t) + 2a_k \sqrt{b_k} \varphi(t) \right)
\]

\[
= b_k \psi_k(t) - 2a_k \sqrt{b_k} \psi_k(t) - b_k \lambda \sum_{i} \psi_i(t).
\]

We conclude \( \psi'' = (B + 2\lambda AB^{1/2} 11^T) \psi - 2AB^{1/2} 1 = M \psi - 2AB^{1/2} 1 \).
1.2 $\psi(0) = \psi(T)$. 
Recall that $\varphi(t) = \varphi(T-t)$ for all $t \in [0,T]$. Let $t \in [0,T]$ and $k = 1, 2, \ldots, n$. Integration by substitution shows

$$
\psi_k(t) = a_k \int_0^T e^{-\sqrt{b_k} s} \varphi(s) \, ds = a_k \int_0^T e^{-\sqrt{b_k} |(T-t)-(T-s)|} \varphi(T-s) \, ds = a_k \int_0^T e^{-\sqrt{b_k} (T-t)} \varphi(s) \, ds = \psi_k(T-t).
$$

In particular, $\psi_k(0) = \psi_k(T)$.

1.3 $\psi'(0) = B^{1/2} \psi(0)$. 
Let $k = 1, 2, \ldots, n$. Then

$$
\psi_k'(0) = a_k \sqrt{b_k} \left[ -\int_0^t e^{-\sqrt{b_k} (t-s)} \varphi(s) \, ds + \int_t^T e^{-\sqrt{b_k} (s-t)} \varphi(s) \, ds \right]_{t=0} = a_k \sqrt{b_k} \int_0^T e^{-\sqrt{b_k} s} \varphi(s) \, ds = \sqrt{b_k} \psi_k(0).
$$

2.1 $M$ has $n$ distinct, real eigenvalues $c_1, c_2, \ldots, c_n$ that satisfy $c_n > b_n > c_{n-1} > b_{n-1} > \cdots > c_1 > b_1 > 0$. 
Let $v := 2\lambda(a_1 \sqrt{b_1}, a_2 \sqrt{b_2}, \ldots, a_n \sqrt{b_n}) \in \mathbb{R}^n$ and $x \in \mathbb{R}^n \setminus \{b_1, b_2, \ldots, b_n\}$. The matrix $xI - M$ is the sum of a diagonal matrix and the outer product $v1^T$. Hence

$$
det(xI - M) = det(xI - B - v1^T) = (1 - v^T (xI - B)^{-1}1) det(xI - B) = \left(1 - 2\lambda \sum_k \frac{a_k \sqrt{b_k}}{x - b_k}\right) \prod_k (x - b_k).
$$

The following argument is due to [Terrell (2017)]. Define $f: \mathbb{R}^n \setminus \{b_1, b_2, \ldots, b_n\} \to \mathbb{R}$ via

$$
f(x) := 1 - 2\lambda \sum_k \frac{a_k \sqrt{b_k}}{x - b_k}.
$$

Let $k = 1, 2, \ldots, n - 1$. Then $f$ is continuous on $(b_k, b_{k+1})$, with

$$
\lim_{x \searrow b_k} f(x) = -\infty \quad \text{and} \quad \lim_{x \nearrow b_{k+1}} f(x) = +\infty.
$$

We conclude that $f$ has a root $c_k \in (b_k, b_{k+1})$. Furthermore,

$$
\lim_{x \searrow b_n} f(x) = -\infty \quad \text{and} \quad \lim_{x \nearrow +\infty} f(x) = 1,
$$

showing that $f$ has another root $c_n \in (b_n, +\infty)$. Since $\det(c_k I - M) = 0$ for $k = 1, 2, \ldots, n$, each $c_k$ is an eigenvalue of $M$. 

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2.2 If \( c \) is an eigenvalue of \( M \), then

\[
\begin{pmatrix}
\frac{a_1}{c-b_1}, & \frac{a_2}{c-b_2}, & \cdots, & \frac{a_n}{c-b_n}
\end{pmatrix}
\]

is a corresponding eigenvector.

Let \( c \in \mathbb{R}_+ \setminus \{b_1, \ldots, b_n\} \) be an eigenvalue of \( M \), and \( v = (v_1, v_2, \ldots, v_n) \) a corresponding eigenvector. The definition \( Mv = cv \) translates into the following system of equations:

\[
b_k v_k + 2\lambda_k \sqrt{b_k} \sum_{l} v_l = cv_k, \quad k = 1, 2, \ldots, n.
\]

It must be true that \( \sum_l v_l \neq 0 \). Otherwise, \( b_k v_k = cv_k \) for all \( k = 1, 2, \ldots, n \). Since \( c \not\in \{b_1, b_2, \ldots, b_n\} \) (see Step 2.1), this implies \( v = 0 \), which contradicts the definition of an eigenvector. Hence we may set \( 1^\top v = \sum_l v_l = \frac{1}{2\lambda} \) without loss of generality. We obtain

\[
v = \begin{pmatrix}
\frac{a_1}{c-b_1}, & \frac{a_2}{c-b_2}, & \cdots, & \frac{a_n}{c-b_n}
\end{pmatrix}.
\]

Let \( c_n > c_{n-1} > \cdots > c_1 > 0 \) be the eigenvalues of \( M \). Define \( C := \text{diag}(c_i)_{i=1,\ldots,n} \),

\[
\hat{Q} := \begin{pmatrix}
\frac{1}{c_j - b_i}
\end{pmatrix}_{i,j=1,2,\ldots,n}
\quad \text{and} \quad
Q := AB^{1/2}\hat{Q}.
\]

2.3 \( M = QCQ^{-1} \).

By Step 2.2 the columns of \( Q \) are eigenvectors corresponding to the eigenvalues \( c_1, c_2, \ldots, c_n \). Eigenvectors corresponding to different eigenvalues are linearly independent, hence \( Q \) is nonsingular. We obtain the eigendecomposition \( M = QCQ^{-1} \).

2.4 \( 1^\top Q = \frac{1}{2\lambda} 1^\top \).

This follows from Step 2.2 where we assumed that each eigenvector contained in \( Q \) sums to \( 1/2\lambda \).

3. Define

\[
d := \left(1 + 2\lambda \sum_k \frac{a_k}{\sqrt{b_k}} \right)^{-1} > 0.
\]

We let \( M^{1/2} := Q \text{diag}(\sqrt{c_i})_{i=1,\ldots,n} Q^{-1} \) and denote by \( e^{M^{1/2}T} = Q \text{diag}(e^{\sqrt{c_i}T})_{i=1,\ldots,n} Q^{-1} \) the matrix exponential of \( M^{1/2}T \). Define

\[
N := A^{-1}\left(M^{1/2}(e^{M^{1/2}T} - I) + B^{1/2}(e^{M^{1/2}T} + I)\right),
\]

where \( I \) denotes the identity matrix.

The general solution to the system of \( n \) ordinary differential equations \( f'' = Mf - 2AB^{1/2}1 \) is

\[
f(t) = e^{M^{1/2}T}x_0 + e^{M^{1/2}(T-t)}x_1 + 2dAB^{-1/2}1, \quad t \in [0, T],
\]

for \( x_0, x_1 \in \mathbb{R}^n \). To see this, let \( t \in [0, T] \) and \( x_0, x_1 \in \mathbb{R}^n \). Writing \( d = 1/(1 + 2\lambda 1^\top AB^{-1/2}1) \) shows

\[
d MAB^{-1/2}1 = d\left(AB^{1/2}1 + 2\lambda AB^{1/2}1 1^\top AB^{-1/2}1\right)
= d\left(1 + \frac{1}{d} - 1\right)AB^{1/2}1
= AB^{1/2}1.
\]
Therefore,
\[
\begin{align*}
f''(t) &= M(e^{M^{1/2}}x_0 + e^{M^{1/2}(T-t)}x_1) \\
&= Mf(t) - 2dMAB^{-1/2}1 \\
&= Mf(t) - 2AB^{1/2}1.
\end{align*}
\]

It remains to choose \(x_0\) and \(x_1\) in such a way that the boundary conditions from Steps 1.2 and 1.3 are satisfied. First, \(f(0) - f(T) = (e^{M^{1/2}T} - I)(x_1 - x_0)\). By Step 2.3
\[
e^{M^{1/2}T} - I = Q \text{ diag } (e^{\sqrt{c_i}T})_{i=1,\ldots,n}Q^{-1} - I = Q \text{ diag } (e^{\sqrt{c_i}T} - 1)_{i=1,\ldots,n}Q^{-1}
\]
is nonsingular. Hence \(f(0) = f(T)\) if and only if \(x_0 = x_1\). Second,
\[
f'(0) - B^{1/2}f(0) = (M^{1/2}(I - e^{M^{1/2}T}) - B^{1/2}(I + e^{M^{1/2}T}))x_0 - 2dA1
\]
\[
= -A(Nx_0 + 2d1).
\]

We show in Step 5.5 that \(N\) is nonsingular. Hence, \(f'(0) = B^{1/2}f(0)\) if and only if \(x_0 = -2dN^{-1}1\). We conclude
\[
\psi(t) = e^{M^{1/2}t}x_0 + e^{M^{1/2}(T-t)}x_1 + 2dAB^{-1/2}1
\]
\[
= (e^{M^{1/2}t} + e^{M^{1/2}(T-t)})x_0 + 2dAB^{-1/2}1
\]
\[
= 2d(AB^{-1/2} - (e^{M^{1/2}t} + e^{M^{1/2}(T-t)})N^{-1})1
\]
for all \(t \in [0, T]\). Notice that
\[
1 - 2d\lambda1^TAB^{-1/2}1 = 1 - d\left(\frac{1}{d} - 1\right) = d,
\]
so
\[
\varphi(t) = 1 - \lambda1^T\psi(t)
\]
\[
= 1 - 2d\lambda1^TAB^{-1/2}1 + 2d\lambda1^T(e^{M^{1/2}t} + e^{M^{1/2}(T-t)})N^{-1}1
\]
\[
= d\left(1 + 2\lambda1^T(e^{M^{1/2}t} + e^{M^{1/2}(T-t)})N^{-1}1\right)
\]
for all \(t \in [0, T]\).

4. Define
\[
E(t) := \text{ diag } \left(\frac{e^{\sqrt{c_i}t} + e^{\sqrt{c_i}(T-t)}}{e^{\sqrt{c_i}T} - 1}\right)_{i=1,\ldots,n}, \quad t \in [0, T],
\]
and
\[
\tilde{N} := A^{-1}(QC^{1/2} + B^{1/2}QE(T)).
\]
\(E(t)\) is a positive diagonal matrix for all \(t \in [0, T]\) and thus nonsingular. The diagonal entries of the mapping \(t \mapsto E(t)\) are symmetrically totally monotone. Using Step 2.3 we obtain
\[
N = A^{-1}(QC^{1/2} \text{ diag } (e^{\sqrt{c_i}T} - 1)_{i=1,\ldots,n}Q^{-1} + B^{1/2}Q \text{ diag } (e^{\sqrt{c_i}T} + 1)_{i=1,\ldots,n}Q^{-1})
\]
\[
= A^{-1}(QC^{1/2} + B^{1/2}QE(T)) \text{ diag } (e^{\sqrt{c_i}T} - 1)_{i=1,\ldots,n}Q^{-1}
\]
\[
= \tilde{N} \text{ diag } (e^{\sqrt{c_i}T} - 1)_{i=1,\ldots,n}Q^{-1}.
\]
Hence \( N \) is nonsingular if and only if \( \tilde{N} \) is nonsingular. This, in combination with Steps 2.3 2.4 and 3, shows

\[
\varphi(t) = d \left( 1 + 2\lambda 1^\top \left( e^{M^{1/2}t} + e^{M^{1/2}(T-t)} \right) N^{-1} 1 \right) \\
= d \left( 1 + 2\lambda 1^\top Q \operatorname{diag} \left( e^{\sqrt{\gamma}t} + e^{\sqrt{\gamma}(T-t)} \right)_{i=1,...,n} Q^{-1} N^{-1} 1 \right) \\
= d \left( 1 + 1^\top E(t) \tilde{N}^{-1} 1 \right)
\]

for all \( t \in [0, T] \).

5. Define the real-valued functions

\[
\beta(x) := \prod_i (x - b_i), \quad \gamma(x) := \prod_i (x - c_i).
\]

Let

\[
D_1 := \operatorname{diag} \left( \beta(c_i) \right)_{i=1,...,n} \quad \text{and} \quad D_2 := \operatorname{diag} \left( \frac{-\gamma(b_i)}{\beta'(b_i)} \right)_{i=1,...,n}.
\]

We show in Step 5.2 that \( D_1 \) and \( D_2 \) are positive diagonal matrices. In particular, they are nonsingular.

5.1 \( \tilde{Q}^{-1} = D_1 \tilde{Q}^\top D_2 \) and \( \tilde{Q}^{-1} 1 = D_1 1 \).

The matrix \(-Q\) as defined in (22) is known as a Cauchy matrix. Both results are due to Schechter (1959).

5.2 \( D_1 \) and \( D_2 \) are positive diagonal matrices.

Let \( k = 1, 2, \ldots, n \). Then

\[
\frac{\beta(c_i)}{\gamma'(c_i)} = \frac{\prod_i (c_i - b_i)}{\sum_m \prod_{l \neq m} (c_i - c_l)} = \frac{\prod_i (c_i - b_i)}{\prod_{l \neq i} (c_i - c_l)} = \left( c_i - b_i \right) \prod_{l \neq i} \frac{c_i - b_l}{c_i - c_l}.
\]

Recall from Step 2.1 that \( c_i > b_i \), and that \( c_i > b_j \) if and only if \( c_i > c_l \) for all \( l = 1, 2, \ldots, n \). Similarly,

\[
-\frac{\gamma(b_i)}{\beta'(c_i)} = \left( c_i - b_i \right) \prod_{l \neq i} \frac{b_l - c_l}{b_i - b_l} > 0.
\]

Define

\[
\tilde{N}_1 := C^{-1/2}, \quad \tilde{N}_2 := \tilde{Q}^\top D_2 B^{-1/2} \tilde{Q}, \quad \tilde{N}_3 := D_1^{-1} E(T) C^{-1/2}, \quad \tilde{N}_4 := \tilde{Q}^\top D_2 B^{-1}.
\]

All four matrices are nonsingular (see Steps 5.1 and 5.2 in particular). We show in Step 5.5 that \( \tilde{N}_2 + \tilde{N}_3 \) is nonsingular.

5.3 \( \tilde{N}^{-1} = \tilde{N}_1 (\tilde{N}_2 + \tilde{N}_3)^{-1} \tilde{N}_4 \).

By definition, \( \tilde{Q} = A^{-1} B^{-1/2} Q \). Using Step 5.1

\[
\tilde{N}_4^{-1} (\tilde{N}_2 + \tilde{N}_3) \tilde{N}_1^{-1} = BD_2^{-1} \tilde{Q}^{-T} \left( \tilde{Q}^\top D_2 B^{-1/2} \tilde{Q} + D_1^{-1} E(T) C^{-1/2} \right) C^{1/2} \\
= \left( B^{1/2} Q C^{1/2} + B (D_1 \tilde{Q}^\top D_2)^{-1} E(T) \right) \\
= \left( A^{-1} Q C^{1/2} + B \tilde{Q} E(T) \right) \\
= A^{-1} (QC^{1/2} + B^{1/2} Q E(T)) \\
= \tilde{N}.
\]
5.4 There exists a positive diagonal matrix $U_k$. By Step 5.2, the same is true for $D_1$ and $D_2$. Hence $D_2B^{-1/2}$ is positive definite. Since $\tilde{Q}$ is nonsingular (see Step 5.1), $\tilde{N}_2 = \tilde{Q}^T D_2 B^{-1/2} \tilde{Q}$ is also positive definite.

5.5 $\tilde{N}_2 + \tilde{N}_3, \tilde{N}$ and $N$ are nonsingular.
It follows from Step 5.4 that $\tilde{N}_2 + \tilde{N}_3$ is positive definite and hence nonsingular. Since $\tilde{N}_1$ and $\tilde{N}_3$ are nonsingular, $\tilde{N}$ is nonsingular. We have shown in Step 4 that $N$ is nonsingular if (and only if) $\tilde{N}$ is nonsingular.

A square matrix is called a Z-matrix if all its off-diagonal entries are nonpositive. Given that some matrix $U$ is a nonsingular Z-matrix, the following two conditions are equivalent:

(M1) There exists a positive diagonal matrix $V$ such that $UV + VU^T$ is positive definite.
(M2) $U$ is nonsingular and all entries of $U^{-1}$ are nonnegative.

In this case, $U$ is called an M-matrix. In particular, condition (M1) implies that every positive definite Z-matrix is an M-matrix. See Theorem 2.3 in Berman and Plemmons (1994) for proofs and further equivalent characterizations of M-matrices.

5.6 $\tilde{N}_2^{-1}$ is a Z-matrix.

With Step 5.1 we obtain

$$\tilde{N}_2^{-1} = \tilde{Q}^{-1} B_2^{1/2} D_2^{-1/2} \tilde{Q}^{-T} = D_1 \tilde{Q}^T D_2 B_2^{1/2} \tilde{Q} D_1.$$

$D_1$ is a positive diagonal matrix, so it suffices to show that all off-diagonal entries of $\tilde{Q}^T D_2 B_2^{1/2} \tilde{Q}$ are nonpositive. Fix $i, j \in \{1, 2, \ldots, n\}$ such that $i \neq j$. Define

$$\alpha := (\tilde{Q}^T D_2 B_2^{1/2} \tilde{Q})_{ij} = -\sum_k \frac{\sqrt{b_k} \gamma(b_k)}{(b_k - c_i)(b_k - c_j)\beta'(b_k)} = -\sum_k \frac{\sqrt{b_k} \prod_{l \neq i, j} (b_k - c_l)}{\prod_{l \neq k} (b_k - b_l)}.$$

The following argument is due to Petrov (2017). Define $f : \mathbb{R}_+ \to \mathbb{R},$

$$f(x) := -\sqrt{x} \prod_{l \neq i, j} (x - c_l).$$

There are positive constants $z_0, z_1, \ldots, z_{n-2}$ such that

$$f(x) = -\sum_{k=0}^{n-2} (-1)^{n-2-k} z_k x^{k+1/2} = \sum_{k=0}^{n-2} (-1)^{n-1-k} z_k x^{k+1/2}.$$

Differentiating $n - 1$ times yields

$$f^{(n-1)}(x) = \sum_{k=0}^{n-2} (-1)^{n-1-k} z_k x^{k-n+3/2} \prod_{l=0}^{n-2} (k + 1/2 - l).$$

For $k = 0, 1, \ldots, n - 2$, the factor $k + 1/2 - l$ is positive if $l = 0, 1, \ldots, k$ and negative if $l = k + 1, k + 2, \ldots, n - 2$. Hence

$$(-1)^{n-1-k} \prod_{l=0}^{n-2} (k + 1/2 - l) = (-1)^{n-1-k} (-1)^{n-2-(k+1)+1} \prod_{l=0}^{n-2} |k + 1/2 - l|$$

$$= -\prod_{l=0}^{n-2} |k + 1/2 - l| < 0.$$
We conclude that \( f^{(n-1)}(x) < 0 \) for all \( x > 0 \).

The Lagrange polynomial interpolation \( p \) of \( f \) in the points \( b_1, b_2, \ldots, b_n \) is

\[
p(x) = \sum_k f(b_k) \prod_{l \neq k} \frac{x - b_l}{b_k - b_l} = \left( -\sum_k \frac{\sqrt{b_k} \prod_{l \neq k} (b_k - c_l)}{\prod_{l \neq k} (b_k - b_l)} \right) x^{n-1} + q(x) = \alpha x^{n-1} + q(x)
\]

for some polynomial \( q \) of degree at most \( n - 2 \). The interpolation is exact for \( x = b_1, b_2, \ldots, b_n \).

By Rolle’s theorem, there is an \( x_0 > 0 \) such that \( f^{(n-1)}(x_0) = p^{(n-1)}(x_0) \) (Milne-Thomson 2000, Chapter 1). Hence

\[
0 > f^{(n-1)}(x_0) = p^{(n-1)}(x_0) = (n-1)! \alpha,
\]

showing that \( (\bar{Q}^\top D_2 B^{1/2} \bar{Q})_{ij} \) is nonpositive if \( i \neq j \).

**5.7** \( \bar{N}_2^{-1} \) is a nonsingular \( M \)-matrix.

We have shown in Step 5.6 that \( \bar{N}_2^{-1} \) is a \( Z \)-matrix. Since \( \bar{N}_2 \) is positive definite by Step 5.4, \( \bar{N}_2^{-1} \) is positive definite as well. Hence \( \bar{N}_2^{-1} \) is a nonsingular \( M \)-matrix by condition (M1).

**5.8** All entries of \( (\bar{N}_2^{-1} + \bar{N}_3^{-1})^{-1} \) are nonnegative.

As a positive diagonal matrix, \( \bar{N}_3^{-1} \) is positive definite and a nonsingular \( M \)-matrix. The sum of positive definite \( Z \)-matrices is again a positive definite \( Z \)-matrix. Hence \( \bar{N}_2^{-1} + \bar{N}_3^{-1} \) is a positive definite \( Z \)-matrix (see Step 5.7); and therefore an \( M \)-matrix. By condition (M2), all entries of \( (\bar{N}_2^{-1} + \bar{N}_3^{-1})^{-1} \) are nonnegative.

**5.9** All off-diagonal entries of \( (\bar{N}_2 + \bar{N}_3)^{-1} \) are nonpositive.

By the Woodbury matrix identity,

\[
(\bar{N}_2 + \bar{N}_3)^{-1} = \bar{N}_3^{-1} - \bar{N}_3^{-1} (\bar{N}_2^{-1} + \bar{N}_3^{-1})^{-1} \bar{N}_3^{-1}.
\]

Recall that \( \bar{N}_3^{-1} \) is a positive diagonal matrix. It follows from Step 5.8 that all off-diagonal entries of \( \bar{N}_3^{-1} (\bar{N}_2^{-1} + \bar{N}_3^{-1})^{-1} \bar{N}_3^{-1} \) are nonnegative.

**6.1** All entries of \( \bar{N}_2^{-1} \bar{N}_4 \) are nonnegative.

Using Step 5.1, we obtain

\[
\bar{N}_2^{-1} \bar{N}_4 = \bar{Q}^{-1} B^{1/2} D_2^{-1} \bar{Q}^{-T} \bar{Q}^\top D_2 B^{-1} \bar{1} \quad \text{or}\quad \bar{N}_2^{-1} \bar{N}_4 = D_1 \bar{Q}^\top D_2 B^{-1/2} \bar{1} = D_1 \bar{Q}^\top D_2 B^{-1/2} \bar{Q} \bar{Q}^{-1} \bar{1} = D_1 \bar{N}_2 D_1 \bar{1}.
\]

We have shown in Step 5.7 that \( \bar{N}_2^{-1} \) is a nonsingular \( M \)-matrix. Hence all entries of \( \bar{N}_2 \) are nonnegative by condition (M2). The same is true for \( D_1 \) by Step 5.2.

**6.2** All entries of \( (\bar{N}_2 + \bar{N}_3)^{-1} \bar{N}_2 \) are nonnegative.

Define \( U := (\bar{N}_2 + \bar{N}_3)^{-1} \bar{N}_2 \). Writing

\[
U = I - (\bar{N}_2 + \bar{N}_3)^{-1} \bar{N}_3
\]

shows that all off-diagonal entries of \( U \) are nonnegative (see Steps 5.4 and 5.9).
This function is clearly symmetrically totally monotone.

3.4.2 Proof of Theorem 4 for arbitrary \( G \)

We now use the following result about positive definite matrices: If two matrices \( U \) and \( V \) are positive definite, then \( U - V \) is positive definite if and only if \( V^{-1} - U^{-1} \) is positive definite (Horn and Johnson 2013, Corollary 7.7.4). The matrices \((\tilde{N}_2 + \tilde{N}_3), \tilde{N}_3 \) and \((\tilde{N}_2 + \tilde{N}_3) - \tilde{N}_3 = \tilde{N}_2 \) are positive definite (see Step 6.1). Hence

\[
\tilde{N}_3^{-1} - (\tilde{N}_2 + \tilde{N}_3)^{-1} = U \tilde{N}_3^{-1}
\]

is positive definite. All entries on the main diagonal of a positive definite matrix are nonnegative. Therefore, all entries on \( U \)'s main diagonal are nonnegative.

6.3 All entries of \( \tilde{N}^{-1} \) are nonnegative.

All entries of \( \tilde{N}_1, (\tilde{N}_2 + \tilde{N}_3)^{-1}\tilde{N}_2 \) and \( \tilde{N}_2^{-1}\tilde{N}_4 \) are nonnegative (see Steps 6.1 and 6.2). Hence all entries of the product

\[
\tilde{N}_1(\tilde{N}_2 + \tilde{N}_3)^{-1}\tilde{N}_2\tilde{N}_2^{-1}\tilde{N}_4 = \tilde{N}_1(\tilde{N}_2 + \tilde{N}_3)^{-1}\tilde{N}_4 = \tilde{N}^{-1}
\]

are nonnegative.

7. Conclude with Steps 4 and 6.3 that there are \( z_1, z_2, \ldots, z_n \geq 0 \) such that

\[
\varphi(t) = d \left( 1 + (1 + 1^T E(t) \tilde{N}^{-1}) \right) = d \left( 1 + \sum_{i=1}^{n} z_i (e^{\sqrt{\pi} t} + e^{\sqrt{\pi} (T-t)}) \right), \quad t \in [0, T].
\]

This function is clearly symmetrically totally monotone.

3.4.2 Proof of Theorem 4 for arbitrary \( G \) and \( \gamma > 0 \)

Let \( G : (0, \infty) \to \mathbb{R} \) be a nonconstant and completely monotone kernel. We assume first that \( G(0+) < \infty \). Then we may assume without loss of generality that \( G(0) := G(0+) = 1 \). By Bernstein’s theorem, there exists a Borel probability measure \( \mu \) on \([0, \infty)\) such that \( G \) is equal to the Laplace transform of \( \mu \). Since the set of finite convex combinations of Dirac measures is dense in the set of all Borel probability measures on \([0, \infty)\) with respect to weak convergence, there exists a corresponding sequence \( (\mu_n)_{n=1,2,\ldots} \) that converges weakly to \( \mu \). Clearly, the corresponding Laplace transforms,

\[
G_n(t) = \int_{[0,\infty)} e^{-tx} \mu_n(dx) \quad \text{for } t \geq 0 \text{ and } n = 1, 2, \ldots
\]

are all exponential kernels of type \( 17 \). The weak convergence \( \mu_n \to \mu \) implies that \( G_n(t) \to G(t) \) for all \( t \geq 0 \). By slight abuse of notation, let us write \( J^{(n)}_\gamma [\varphi] := \gamma \int_0^T \varphi(t)^2 \, dt + \int_0^T \int_0^T G_n(|t - s|) \varphi(s) \varphi(t) \, ds \, dt \) for every \( \gamma \geq 0 \) and \( \varphi \in L^2[0,T] \). Then

\[
|J_0[\varphi] - J^{(n)}_0[\varphi]| \leq \|\varphi\|_{L^2[0,T]} \left( \int_0^T \int_0^T (G(|t - s|) - G_n(|t - s|))^2 \, ds \, dt \right)^{1/2} |\varphi(t)| \, dt
\]

\[
\leq 2\sqrt{T} \|\varphi\|_{L^2[0,T]}^2 \|G - G_n\|_{L^2[0,T]}.
\]

Since \( \|G - G_n\|_{L^2[0,T]} \to 0 \) by dominated convergence, we conclude that \( J^{(n)}_\gamma [\varphi] \to J_\gamma [\varphi] \) uniformly in functions \( \varphi \) from any bounded subset of \( L^2[0,T] \).

For each \( n \), let \( \varphi_n \) be the minimizer in \( \Phi_1 \) of the energy functional \( J^{(n)}_\gamma \). By Section 3.4.1 each function \( \varphi_n \) is symmetrically totally monotone. Since the function \( f \equiv 1/T \) belongs to \( \Phi_1 \), one sees that there exists a constant \( C \) such that \( \|\varphi_n\|_{L^2[0,T]} \leq C \). By passing to a subsequence if necessary, we may
To prove the representation (9), note first that

therefore assume without loss of generality that the sequence \((\varphi_n)_{n \in \mathbb{N}}\) converges weakly in \(L^2[0, T]\) to a limiting function \(\tilde{\varphi}\), which by Lemma 12 admits a symmetrically totally monotone version. Let \(\varphi\) be the minimizer of \(J_\gamma\). Then \(J_\gamma^{(n)}[\varphi] \geq J_\gamma^{(n)}[\varphi_n]\) for each \(n\). Hence, the uniform convergence of \(J_\gamma^{(n)}\) yields that

\[
J_\gamma[\varphi] = \lim_{n \uparrow \infty} J_\gamma^{(n)}[\varphi] \geq \lim \inf_{n \uparrow \infty} J_\gamma^{(n)}[\varphi_n] = \lim \inf_{n \uparrow \infty} J_\gamma[\varphi_n] \geq J_\gamma[\tilde{\varphi}],
\]

where the latter inequality follows from the weak lower semicontinuity of \(J_\gamma\). This shows that \(\varphi = \tilde{\varphi}\) and concludes the proof for \(G(0+) < \infty\).

If \(G\) is weakly singular and satisfies \(G(0+) = \infty\), we use its approximation as in (16) by kernels \(G_n\) with \(G_n(0+) < \infty\). As in the final part of the proof of Proposition 1, one sees that the symmetrically totally monotone minimizers for \(G_n\) converge weakly in \(L^2[0, T]\) to the minimizer for \(G\). Thus, this latter minimizer is also symmetrically totally monotone by Lemma 12.

### 3.4.3 Proof of Theorem 4 for \(\gamma = 0\)

Let \(\mu^*\) be the minimizer of \(J_0\) as provided by Theorem 2.24 of Gatheral et al. (2012). We approximate \(\mu^*\) in the weak topology by probability measures of the form \(\mu_n^*(dx) = \psi_n(x) \, dx\), where each \(\psi_n\) is a bounded nonnegative function on \([0, T]\) satisfying \(\int_0^T \psi_n(x) \, dx = 1\). Then we choose a sequence \(\gamma_n \downarrow 0\) that is such that \(\gamma_n \int_0^T \psi_n(x)^2 \, dx \to 0\). Then it follows from (14) that \(J_{\gamma_n}[\psi_n] \to J_0[\mu^*]\).

Next, we let \(\varphi_n\) be the minimizer of \(J_{\gamma_n}\) in \(\Phi_1\). By passing to a subsequence if necessary, we may assume that the probability measures \(\mu_n(dx) = \varphi_n(x) \, dx\) on \([0, T]\) converge weakly to a probability measure \(\mu\) on \([0, T]\). By Lemma 11, the restriction of \(\mu\) to \((0, T)\) is absolutely continuous with respect to the Lebesgue measure and admits a symmetrically totally monotone density. Finally, we claim that \(\mu = \mu^*\). Indeed,

\[
J_0[\mu] = \lim_{n \uparrow \infty} J_0[\mu_n] \leq \lim \inf_{n \uparrow \infty} J_{\gamma_n}[\varphi_n] \leq \lim \inf_{n \uparrow \infty} J_{\gamma_n}[\psi_n] = J_0[\mu^*],
\]

and so the uniqueness of the minimizer yields \(\mu = \mu^*\).

### 3.5 Proof of the formula from Example 6

To prove the representation (6), note first that

\[
\int_0^n (1 - |t - s|)^+ \varphi(s) \, ds = \begin{cases} 
\int_0^t (1 - t + s) \varphi(s) \, ds + \int_t^{t+1} (1 + t - s) \varphi(s) \, ds, & t \in [0, 1], \\
\int_{t-1}^t (1 - t + s) \varphi(s) \, ds + \int_t^{t+1} (1 + t - s) \varphi(s) \, ds, & t \in [1, n-1], \\
\int_{t-1}^n (1 - t + s) \varphi(s) \, ds + \int_t^n (1 + t - s) \varphi(s) \, ds, & t \in [n-1, n].
\end{cases}
\]

Differentiating this identity twice and replacing \(\varphi\) with \(\varphi_1, \ldots, \varphi_n\) yields

\[
\begin{align*}
\gamma \varphi''(t) &= 2 \varphi_1(t) - \varphi_2(t), \\
\gamma \varphi''(t) &= 2 \varphi_i(t) - \varphi_{i-1}(t) - \varphi_{i+1}(t), & i = 2, \ldots, n-1, \\
\gamma \varphi''(t) &= 2 \varphi_n(t) - \varphi_{n-1}(t).
\end{align*}
\]

Hence \(f := (\varphi_1, \ldots, \varphi_n)\) solves the following \(n\)-dimensional system of ordinary differential equations
on $[0, 1]$:

$$f'' = \frac{1}{\gamma} \begin{pmatrix} 2 & -1 & \ldots & 0 & 0 \\ -1 & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 2 & -1 \\ 0 & 0 & \ldots & -1 & 2 \end{pmatrix} f.$$

Let $M_n$ denote the preceding triangular matrix, denote by $\lambda_1, \ldots, \lambda_n$ its eigenvalues, and let $Q$ contain the corresponding eigenvectors as columns. Then

$$f(t) = Q(E(t)x_0 + E(1-t)x_1)$$

for some vectors $x_0, x_1 \in \mathbb{R}^n$.

Define $m := \lfloor n/2 \rfloor$. Let $I_m, J_m, 0_m$ denote the $m$-dimensional identity matrix, reverse identity matrix, and zero matrix, respectively. The symmetry of $\varphi$ implies that $\varphi_i(t) = \varphi_{n+1-i}(1-t)$, and so

$$\left[ I_m \ 0_m \right] Q(E(t)x_0 + E(1-t)x_1) = \left[ 0_m \ J_m \right] Q(E(1-t)x_0 + E(t)x_1), \ t \in [0, T]. \quad (23)$$

Since

$$\sin \left( \frac{(n + 1 - i) j \pi}{n + 1} \right) = \sin(j \pi) \cos \left( \frac{i j \pi}{n + 1} \right) - \cos(j \pi) \sin \left( \frac{i j \pi}{n + 1} \right) = (-1)^{j+1} \sin \left( \frac{i j \pi}{n + 1} \right)$$

for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, it holds that $\left[ 0_m \ J_m \right] Q = \left[ I_m \ 0_m \right] Q J$. Notice that $J^{-1} = J$. Hence $[23]$ is satisfied if and only if $x_1 = Jx_0$.

Let $t = i \in \{1, \ldots, n-1\}$. Then the symmetry of $\varphi$ shows that

$$\sigma = \gamma \varphi(i) + \int_{i-1}^{i} (1 - i + s) \varphi(s) \, ds + \int_{i}^{i+1} (1 + i - s) \varphi(s) \, ds$$

$$= \gamma \varphi(i) + \int_{i-1}^{i} (1 - i + s) \varphi(i - s + i) \, ds + \int_{i}^{i+1} (1 + i - s) \varphi_{n-i}(1 + i - s) \, ds$$

$$= \gamma \varphi(i) + \int_{0}^{1} s (\varphi(s) + \varphi_{n-i}(s)) \, ds.$$ 

Similar arguments yield $\sigma = \gamma \varphi_{n}(1) + \int_{0}^{1} s \varphi_{n}(s) \, ds$.

A straightforward calculation shows $\int_{0}^{1} sf(s) \, ds = Q(\gamma (E(1) - I)(J - I) + B(E(1) - J))B^{-2}$. Hence

$$\sigma = \left( \gamma Q(E(1) + J) + KQ((E(1) - I)(J - I) + B(E(1) - J))B^{-2} \right)x_0. \quad (24)$$

Existence and uniqueness of a solution to (8) imply that $[24]$ can be uniquely solved for $x_0$.

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