On symmetric and Hermitian rank distance codes

Antonio Cossidente Giuseppe Marino Francesco Pavese

Abstract

Let \( M \) denote the set \( S_{n,q} \) of \( n \times n \) symmetric matrices with entries in \( \text{GF}(q) \) or the set \( H_{n,q^2} \) of \( n \times n \) Hermitian matrices whose elements are in \( \text{GF}(q^2) \). Then \( M \) equipped with the rank distance \( d_r \) is a metric space. We investigate \( d \)-codes in \((M, d_r)\) and construct \( d \)-codes whose sizes are larger than the corresponding additive bounds. In the Hermitian case, we show the existence of an \( n \)-code of \( M \), \( n \) even and \( n/2 \) odd, of size \((3q^n - q^{n/2})/2\), and of a \( 2 \)-code of size \( q^6 + q(q-1)(q^4 + q^2 + 1)/2 \), for \( n = 3 \). In the symmetric case, if \( n \) is odd or if \( n \) and \( q \) are both even, we provide better upper bound on the size of a \( 2 \)-code. In the case when \( n = 3 \) and \( q > 2 \), a \( 2 \)-code of size \( q^4 + q^3 + 1 \) is exhibited. This provides the first infinite family of \( 2 \)-codes of symmetric matrices whose size is larger than the largest possible additive \( 2 \)-code and an answer to a question posed in [25, Section 7], see also [23, p. 176].

Keywords: symmetric rank distance codes; Hermitian rank distance codes; symplectic polar spaces; Hermitian polar spaces; Segre variety.

1 Introduction

Let \( q \) be a power of a prime and let \( \text{GF}(q) \) be the finite field with \( q \) elements. Denote by \( S_{n,q} \) the set of \( n \times n \) symmetric matrices with entries in \( \text{GF}(q) \) and by \( H_{n,q^2} \) the set of \( n \times n \) Hermitian matrices whose elements are in \( \text{GF}(q^2) \). Let \( M \) be \( S_{n,q} \) or \( H_{n,q^2} \). For two matrices \( A, B \in M \), define their rank distance to be

\[
d_r(A, B) = \text{rk}(A - B).
\]

Thus \( d_r \) is a metric on \( M \) and \((M, d_r)\) is a metric space. A rank metric code \( C \) is a non–empty subset of \((M, d_r)\). The minimum distance of \( C \) is

\[
d_r(C) = \min \{ d_r(c_1, c_2) \mid c_1, c_2 \in C, c_1 \neq c_2 \}.
\]

We will refer to a code in \((M, d_r)\) with minimum distance \( d \) as a \( d \)-code. A \( d \)-code is said to be maximal if it maximal with respect to set theoretic inclusion, whereas it is called maximum
if it has the largest possible size. If a $d$–code $C \subset \mathcal{M}$ forms a subgroup of $(\mathcal{M}, +)$, then $C$ is called additive. Upper bounds on the size of a $d$–code of $\mathcal{M}$ were provided in [20, Corollary 7], [21, Lemma 3.5, Proposition 3.7], [24, Proposition 3.4] and [22, Theorems 1 and 2]. In the case when $C$ is additive much better bounds can be obtained. Indeed in [21, Lemmas 3.5 and 3.6], [23, Theorem 4.3], the author proved that the largest additive $d$–codes of $\mathcal{S}_{n,q}$ have size at most either $q^{n(n-d+2)/2}$ or $q^{(n+1)(n-d+1)/2}$, according as $n - d$ is even or odd, respectively, whereas the size of the largest additive $d$–codes of $\mathcal{H}_{n,q^2}$ cannot exceed $q^{n(n-d+1)}$, see [22, Theorem 1]. Moreover there exist additive $d$–codes whose sizes meet the upper bounds for all possible value of $n$ and $d$, except when $\mathcal{M} = \mathcal{H}_{n,q^2}$, $n$, $d$ are both even and $3 < d < n$, see [20, Theorems 12 and 16], [21, Theorem 4.4], [23, Theorem 5.3], [22, Theorems 4 and 5], [24, Theorem 6.1], [6], [7], [8]. If $d$ is odd, a $d$–code attaining the corresponding additive bound is maximum. This is not always true if $d$ is even. However not much is known about $d$–codes whose size is larger than the corresponding additive bound. In the Hermitian case, if $n$ is even, there is an $n$–code of $\mathcal{H}_{n,q^2}$ of size $q^n + 1$ [22, Theorem 6], [11, Theorem 18]. In the symmetric case only sporadic examples of non–additive $d$–codes that are larger than the largest possible additive $d$–code are known [24, Tables 2 and 9].

Let $\mathcal{W}(2n - 1, q)$ be a non–degenerate symplectic polar space and $\mathcal{H}(2n - 1, q^2)$ be a non–degenerate Hermitian polar space. Let $\Pi_1$ be a generator of $\mathcal{W}(2n - 1, q)$ and let $\Lambda_1$ be a generator of $\mathcal{H}(2n - 1, q^2)$. It is known that there exists a bijection $\tau$ between the matrices of $\mathcal{S}_{n,q}$ or $\mathcal{H}_{n,q^2}$ and the generators of $\mathcal{W}(2n - 1, q)$ or $\mathcal{H}(2n - 1, q^2)$ disjoint from $\Pi_1$ or $\Lambda_1$, respectively, see [2, Proposition 9.5.10], [11]. Here an upper bound on the maximum number of generators of $\mathcal{W}(2n - 1, q)$ or $\mathcal{H}(2n - 1, q^2)$ pairwise intersecting in at most an $(n - 3)$–dimensional projective space is derived, see Theorems 5.2 and 5.4. As a by product, by means of $\tau$, the following upper bound on the size of a $2$–code $C$ of $\mathcal{S}_{n,q}$ is obtained:

$$|C| \leq \begin{cases} \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \frac{q^j}{q^n - q^{j+1} + 1} \prod_{i=1}^{j} \frac{q^{2(n-i+1)} - 1}{(q^2 - 1)(q^{i+1} + 1)} & \text{if } n \text{ is odd,} \\ \prod_{i=2}^{n} (q^i + 1) & \text{if } n \text{ is even.} \end{cases} \quad (1.1)$$

Since the previous known upper bound for the size of a $2$–code of $\mathcal{S}_{n,q}$ was $q^{n(n-1)/2+1}(q^{n-1} + 1)/(q + 1)$ for $q$ odd [21, Proposition 3.7], and $q^{n(n+1)/2} - q^n + 1$ for $q$ even [24, Proposition 3.4], it follows that (1.1) provides better upper bounds if $n$ is odd or if $n$ and $q$ are both even.

By using $\tau$, it can be seen that an $n$–code $C$ of $\mathcal{S}_{n,q}$ or $\mathcal{H}_{n,q^2}$ exists if and only if there exists a partial spread of $\mathcal{W}(2n - 1, q)$ or $\mathcal{H}(2n - 1, q^2)$ of size $|C| + 1$, see Lemmas 3.1 and 4.1 [11]. It is well–known that the points of $\mathcal{W}(2n - 1, q)$ can be partitioned into $q^n + 1$ pairwise disjoint generators of $\mathcal{W}(2n - 1, q)$, that is, $\mathcal{W}(2n - 1, q)$ admits a spread. On the other hand, $\mathcal{H}(2n - 1, q^2)$ has no spread. If $n$ is odd, an upper bound for the largest partial spreads of $\mathcal{H}(2n - 1, q^2)$ is $q^n + 1$ [29] and there are examples of partial spreads of that size [18]. If $n$ is even the situation is less clear: upper bounds can be found in [15], as for lower bounds there is a partial spread of $\mathcal{H}(2n - 1, q^2)$ of size $(3q^2 - q)/2 + 1$ for $n = 2, q > 13$, [11, p. 32] and of size $q^n + 2$ for $n \geq 4$ [11]. Here, generalizing the partial spread of $\mathcal{H}(3, q^2)$, we show the existence of a partial spread of
Consider the map defined by

\[ \xi : \text{PG}(1, q) \times \text{PG}(2, q) \rightarrow \text{PG}(5, q), \]

taking a pair of points \( x = (x_1, x_2) \) of \( \text{PG}(1, q) \), \( y = (y_1, y_2, y_3) \) of \( \text{PG}(2, q) \) to their product \( (x_1y_1, \ldots, x_2y_3) \). This is a special case of a wider class of maps called Segre maps [13]. The image of \( \xi \) is an algebraic variety called the Segre variety and denoted by \( \Sigma_{1,2} \). The Segre variety

\( \mathcal{H}(2n - 1, q^2) \), in the case when \( n \) is even and \( n/2 \) is odd, of size \( (3q^n - q^{n/2})/2 + 1 \) (cf. Theorem 6.4) and hence, if \( n \) is even and \( n/2 \) is odd, of an \( n \)-code of \( \mathcal{H}_{n,q^2} \) of size \( (3q^n - q^{n/2})/2 \).

In the remaining part of the paper, we focus on the case \( n = 3 \). First, a further improvement on the size of a 2–code \( C \) of \( S_{3,q} \) is obtained (cf. Corollary 6.7):

\[ |C| \leq \frac{q(q^2 - 1)(q^2 + q + 1)}{2} + 1. \]

Then we construct 2–codes of \( S_{3,q} \) and \( \mathcal{H}_{3,q^2} \) of size \( q^4 + q^3 + 1 \) and \( q^6 + q(q-1)(q^4 + q^2 + 1)/2 \), respectively. This provides the first infinite family of 2–codes of \( S_{3,q} \) whose size is larger than the largest possible additive 2–code and an answer to a question posed in [25, Section 7], see also [23, p. 176].

\section{2 Preliminaries}

\subsection{2.1 Projective and polar spaces}

Let \( \text{PG}(r - 1, q) \) be the projective space of projective dimension \( r - 1 \) over \( \text{GF}(q) \) equipped with homogeneous projective coordinates \( X_1, \ldots, X_r \). We will use the term \( n \)-space of \( \text{PG}(r - 1, q) \) to denote an \( n \)-dimensional projective subspace of \( \text{PG}(r - 1, q) \). We shall find it helpful to represent projectivities of \( \text{PG}(r - 1, q) \) by invertible \( r \times r \) matrices over \( \text{GF}(q) \) and to consider the points of \( \text{PG}(r - 1, q) \) as column vectors, with matrices acting on the left. Let \( U_i \) be the points having 1 in the \( i \)-th position and 0 elsewhere. Furthermore, we denote by \( 0_n \) and \( I_n \) the \( n \times n \) zero matrix and identity matrix, respectively; if \( M \) is an \( n \times n \) matrix over \( \text{GF}(q) \), we denote by \( L(M) \) the \( (n - 1) \)-space of \( \text{PG}(2n - 1, q) \) whose underlying vector space is the vector space spanned by the rows of the \( n \times 2n \) matrix \( (I_n \ M) \); we also use the notation \( L(M) = \langle (I_n \ M) \rangle \). If \( m \) divides \( r \), an \( (m-1) \)-spread of \( \text{PG}(r - 1, q) \) is a set of pairwise disjoint \( (m-1) \)-spaces of \( \text{PG}(r - 1, q) \) which partition the point set of \( \text{PG}(r - 1, q) \).

A \textit{finite classical polar space} \( \mathbf{P} \) arises from a vector space of finite dimension over a finite field equipped with a non–degenerate reflexive sesquilinear form. In this paper we will be mainly concerned with symplectic polar spaces and Hermitian polar spaces. A projective subspace of maximal dimension contained in \( \mathbf{P} \) is called a \textit{generator} of \( \mathbf{P} \). For further details on finite classical polar spaces we refer the readers to [13]. A \textit{partial spread} \( \mathbf{S} \) of \( \mathbf{P} \) is a set of pairwise disjoint \( m \)-spaces of \( \mathbf{P} \) which partition the point set of \( \mathbf{P} \).

\subsection{2.1.1 Segre varieties}

Consider the map defined by

\[ \xi : \text{PG}(1, q) \times \text{PG}(2, q) \rightarrow \text{PG}(5, q), \]

taking a pair of points \( x = (x_1, x_2) \) of \( \text{PG}(1, q) \), \( y = (y_1, y_2, y_3) \) of \( \text{PG}(2, q) \) to their product \( (x_1y_1, \ldots, x_2y_3) \). This is a special case of a wider class of maps called Segre maps [13]. The image of \( \xi \) is an algebraic variety called the \textit{Segre variety} and denoted by \( \Sigma_{1,2} \). The Segre variety
Σ_1,2 has two rulings, say \( R_1 \) and \( R_2 \), containing \( q^2 + q + 1 \) lines and \( q + 1 \) planes, respectively, satisfying the following properties: two subspaces in the same ruling are disjoint, elements of different ruling intersect in exactly one point and each point of \( \Sigma_1,2 \) is contained in exactly one member of each ruling.

Notice that the set \( R_1 \) consists of all the lines of \( \text{PG}(5,q) \) incident with three distinct members of \( R_2 \) and, from [13, Theorem 25.6.1], three mutually disjoint planes of \( \text{PG}(5,q) \) define a unique Segre variety \( \Sigma_1,2 \). A line of \( \text{PG}(5,q) \) shares 0, 1, 2 or \( q + 1 \) points with \( \Sigma_1,2 \). Also, the automorphism group of \( \Sigma_1,2 \) in \( \text{PGL}(6,q) \) is a group isomorphic to \( \text{PGL}(2,q) \times \text{PGL}(3,q) \) [13, Theorem 25.5.13]. For more details on Segre varieties, see [13].

2.2 Graphs

Recall some definitions and results from [3, 10]. Suppose \( \Gamma \) is a (simple, undirected) graph having \( V \) as set of vertices. The adjacency matrix \( A \) of \( \Gamma \) is a symmetric real matrix whose rows and columns are indexed by 1, \ldots, |V|. The eigenvalues of \( \Gamma \) are those of its adjacency matrix \( A \). A graph \( \Gamma \) is called regular of valency \( k \) or \( k \)-regular when every vertex has precisely \( k \) neighbors. If \( \Gamma \) is regular of valency \( k \), then \( A1 = k1 \), where \( 1 \) denotes the all one column vector. Hence \( k \) is an eigenvalue of \( \Gamma \) and for every eigenvalue \( \lambda \) of \( \Gamma \), we have that \( |\lambda| \leq k \). Furthermore the multiplicity of \( k \) equals the number of connected components of \( \Gamma \).

Let \( \Gamma \) be a \( k \)-regular graph and let \( \{V_1, \ldots, V_m\} \) be a partition of \( V \). Let \( A \) be partitioned according to \( \{V_1, \ldots, V_m\} \), that is,

\[
A = \begin{pmatrix}
A_{1,1} & \cdots & A_{1,m} \\
\vdots & & \vdots \\
A_{m,1} & \cdots & A_{m,m}
\end{pmatrix},
\]

such that \( A_{i,j} \) is a square matrix for all 1 \( \leq i \leq m \). The quotient matrix \( B \) is the \( m \times m \) matrix with entries the average row sum of the blocks of \( A \). More precisely,

\[
B = (b_{i,j}), \quad b_{i,j} = \frac{1}{v_i}1^t A_{i,j} 1,
\]

where \( v_i \) is the number of rows of \( A_{i,j} \). If the row sum of each block \( A_{i,j} \) is constant then the partition is called equitable or regular and we have \( A_{i,j} 1 = b_{i,j} 1 \) for 1 \( \leq i, j \leq m \). One important class of equitable partitions arises from automorphisms of \( \Gamma \), indeed the orbits of any group of automorphisms of \( \Gamma \) form an equitable partition. The following result is well known and useful.

**Lemma 2.1** (Lemma 2.3.1, [3], Theorem 9.4.1, [10]). Let \( B \) be the quotient matrix of an equitable partition. If \( \lambda \) is an eigenvalue of \( B \), then \( \lambda \) is an eigenvalue of \( A \).

Let \( \Gamma \) be a vertex-transitive graph and let \( B \) be the quotient matrix of an equitable partition arising from the orbits of some subgroup of \( \text{Aut}(\Gamma) \). If \( |V_i| = 1 \) for some \( i \), then every eigenvalue of \( A \) is an eigenvalue of \( B \).

A coclique of \( \Gamma \) is a set of pairwise nonadjacent vertices. The independence number \( \alpha(\Gamma) \) is the size of the largest coclique of \( \Gamma \). Let \( \lambda_1 \geq \cdots \geq \lambda_{|V|} \) be the eigenvalues of \( \Gamma \). The following
3 Symmetric matrices and symplectic polar spaces

Let $W(2n−1,q)$ be the non–degenerate symplectic polar space of $PG(2n−1,q)$ associated with the following alternating bilinear form

$$
\begin{pmatrix}
X_1, \ldots, X_{2n}
\end{pmatrix}
\begin{pmatrix}
0_n & I_n \\
-I_n & 0_n
\end{pmatrix}
\begin{pmatrix}
Y_1 \\
\vdots \\
Y_{2n}
\end{pmatrix}.
$$

Let $\perp$ denote the symplectic polarity of $PG(2n−1,q)$ defining $W(2n−1,q)$ and let $PSp(2n,q) = Sp(2n,q)/\langle −I \rangle$, where $Sp(2n,q)$ is the group of isometries of the alternating bilinear form previously defined. Hence $PSp(2n,q)$ consists of projectivities of $PG(2n−1,q)$ fixing $W(2n−1,q)$. It acts transitively on the generators of $W(2n−1,q)$. Denote by $Π_1$ the $(n−1)$–space of $PG(2n−1,q)$ spanned by $U_{n+1}, \ldots, U_{2n}$. Then $Π_1$ is a generator of $W(2n−1,q)$. Let $G$ be the stabilizer of $Π_1$ in $PSp(2n,q)$. Then it is readily seen that an element of $G$ is represented by the matrix

$$
\begin{pmatrix}
T^{−t} \\
0_n & T
\end{pmatrix},
$$

where $T ∈ GL(n,q)$ and $S_0 ∈ S_{n,q}$. Hence $G ∼ S_{n,q} × (GL(n,q)/\langle −I_{2n} \rangle)$ has order

$$
\frac{q^{n(n+1)/2} \prod_{i=0}^{n−1} (q^n − q^i)}{gcd(2, q − 1)}
$$

and it acts transitively on the set $G$ of generators of $W(2n−1,q)$ disjoint from $Π_1$.

Define an action of $S_{n,q} × GL(n,q)$ on $S_{n,q}$ as follows

$$
((S_0,T), S) ∈ (S_{n,q} × GL(n,q)) × S_{n,q} \rightarrow TST^t + S_0 ∈ S_{n,q}.
$$

Its orbitals are the relations of an association scheme, the so called association scheme of symmetric matrices [14, 11]. The following result enlightens a correspondence between $S_{n,q}$ and $G$, see also [2, Proposition 9.5.10].

**Lemma 3.1.** There is a bijection between $S_{n,q}$ and $G$ such that $S_{n,q} × GL(n,q)$ acts on $S_{n,q}$ as $G$ acts on $G$. In particular, a $d$–code of $S_{n,q}$ corresponds to a set of generators of $W(2n−1,q)$ disjoint from $Π_1$ pairwise intersecting in at most an $(n − d − 1)$–space, and conversely.

**Proof.** Let $S ∈ S_{n,q}$. Since the rank of the matrix $\begin{pmatrix} I_n & S \\ 0_n & I_n \end{pmatrix}$ is $2n$, it follows that $L(S)$ is disjoint from $Π_1$. The map $S \mapsto L(S)$ is injective. Moreover $|S_{n,q}| = |G|$ and $L(S)$ is a generator of $W(2n−1,q)$, indeed

$$
(I_n \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} \begin{pmatrix} I_n \\ S \end{pmatrix} = 0.
$$
Finally, let \( g \in G \) be represented by the matrix (3.1). Then \( L(S)^g = \langle (T^{-1} ST^t + T^{-1}S_0) \rangle = \langle (I_n \ TST^t + S_0) \rangle = L(TST^t + S_0) \). This completes the proof of the first part of the statement. Let \( \mathcal{C} \) be a \( d \)-code of \( S_{n, q} \) and let \( S_1 \) and \( S_2 \) be two different elements of \( \mathcal{C} \). Since \( \text{rk}(S_1 - S_2) \geq d \), it follows that \( L(S_1) \cap L(S_2) \) is at most an \((n - d - 1)\)-space.

Let \( \Pi_2 = L(0_n) \). The previous lemma implies that the number of orbits of \( G_{\Pi_2} \) on \( G \) equals the number of relations of the association scheme on symmetric matrices. Since \( |G| = q^{n(n+1)/2} \) and \( G \) acts transitively on \( G \), it follows that the stabilizer of \( \Pi_2 \) in \( G \), namely \( G_{\Pi_2} \), has order

\[
\prod_{i=0}^{n-1} \frac{q^n - q^i}{\text{gcd}(2, q - 1)}.
\]

More precisely an element of \( G_{\Pi_2} \) is represented by the matrix

\[
\begin{pmatrix}
T^{-t} & 0_n \\
0_n & T
\end{pmatrix},
\]

(3.2)

where \( T \in \text{GL}(n, q) \). The group \( G_{\Pi_2} \) acts transitively on points and hyperplanes of both \( \Pi_1 \) and \( \Pi_2 \). The action of the group \( G_{\Pi_2} \) on points of \( W(2n - 1, q) \) has been studied in [16, p. 347]. For the sake of completeness a direct proof is given below.

**Lemma 3.2.** The orbits of \( G_{\Pi_2} \) on points of \( W(2n - 1, q) \setminus (\Pi_1 \cup \Pi_2) \) are

- \( \mathcal{P}_0 \) of size \((q^n - 1)(q^{n-1} - 1)/(q - 1)\),

- \( \mathcal{P}_1 \) of size \(q^{n-1}(q^n - 1)\),

if \( q \) is even and

- \( \mathcal{P}_0 \) of size \((q^n - 1)(q^{n-1} - 1)/(q - 1)\),

- \( \mathcal{P}_1, \mathcal{P}_2 \), both of size \(q^{n-1}(q^n - 1)/2\),

if \( q \) is odd.

**Proof.** Let \( g \) be the projectivity of \( G_{\Pi_2} \) associated with the matrix (3.2), for some \( T \in \text{GL}(n, q) \). Then \( g \) stabilizes \( U_{n+1} \) if and only if the first column of \( T \) is \((z, 0, \ldots, 0)^t\), for some \( z \in \text{GF}(q) \setminus \{0\} \), which is equivalent to the requirement that the first row of \( T^{-t} \) is \((z^{-1}, 0, \ldots, 0)\), for some \( z \in \text{GF}(q) \setminus \{0\} \). It follows that \( \text{Stab}_{G_{\Pi_2}}(U_{n+1}) \) has two orbits on points of \( \Pi_2 \), namely \( U_{n+1}^\perp \cap \Pi_2 \) and \( \Pi_2 \setminus U_{n+1}^\perp \). Hence \( G_{\Pi_2} \) permutes in a single orbit the lines of \( W(2n - 1, q) \) meeting both \( \Pi_1 \), \( \Pi_2 \) in a point. Similarly the lines not of \( W(2n - 1, q) \) meeting both \( \Pi_1 \), \( \Pi_2 \) in a point form a unique \( G_{\Pi_2} \)-orbit.

Let \( P = U_2 + U_{n+1} \). A projectivity of \( G_{\Pi_2} \) stabilizing \( P \) has to fix both \( U_2 \) and \( U_{n+1} \). Straightforward calculations show that a member of \( G_{\Pi_2} \) fixes \( P \) if and only if it is associated with the matrix (3.2), where

\[
T = \begin{pmatrix}
x & * & * & \ldots & * \\
0 & x^{-1} & 0 & \ldots & 0 \\
0 & * & \vdots & \ddots & T' \\
0 & * & \vdots & \ddots & \vdots \\
\end{pmatrix},
\]
for some $T' \in \text{GL}(n-2,q)$ and $x \in \text{GF}(q) \setminus \{0\}$. Therefore $|\text{Stab}_{G_{\Pi_2}}(P)| = (q-1)(q^n-1)(q^{n-2}-1)|\text{GL}(n-2,q)|/\gcd(2,q-1)$. Hence $|P^G_{\Pi_2}| = (q^n-1)(q^{n-1}-1)/(q-1)$, which equals the number of points of $W(2n-1,q) \setminus (\Pi_1 \cup \Pi_2)$ that lie on the lines of $W(2n-1,q)$ meeting both $\Pi_1$, $\Pi_2$ in one point.

Let $z \in \text{GF}(q) \setminus \{0\}$ and let $P = U_1 + zU_{n+1}$. As before, a projectivity of $G_{\Pi_2}$ stabilizing $P_z$ has to fix both $U_1$ and $U_{n+1}$. Straightforward calculations show that the projectivity $g \in G_{\Pi_2}$ fixes the line $U_1U_{n+1}$ if and only if it is associated with the matrix $\begin{pmatrix} y & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & T'' & \\ 0 & & & \end{pmatrix}$, where

$$T = \begin{pmatrix} y & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & T'' & \\ 0 & & & \end{pmatrix},$$

for some $T'' \in \text{GL}(n-1,q)$ and $y \in \text{GF}(q) \setminus \{0\}$. Moreover $g$ stabilizes $P_z$ if and only if $y = \pm 1$. In this case we have that $|\text{Stab}_{G_{\Pi_2}}(P)| = |\text{GL}(n-1,q)|$. Hence if $q$ is even, $|P_z^G_{\Pi_2}| = q^n(q^n-1)$, which equals the number of points of $W(2n-1,q) \setminus (\Pi_1 \cup \Pi_2)$ that lie on the lines not belonging to $W(2n-1,q)$ and meeting both $\Pi_1$, $\Pi_2$ in one point. If $q$ is odd, then $|P_z^G_{\Pi_2}| = q^n(q^n-1)/2$.

Representatives for these two orbits are $P_{z_1}$ and $P_{z_2}$, where $z_1$ is a non–zero square in $\text{GF}(q)$ and $z_2$ is a non–square of $\text{GF}(q)$. Indeed there is no element of $G_{\Pi_2}$ sending $P_{z_1}$ to $P_{z_2}$. To see this fact assume on the contrary that there is a projectivity of $G_{\Pi_2}$ mapping $P_{z_1}$ to $P_{z_2}$. Then it has to fix the line $U_1U_{n+1}$. On the other hand such a projectivity sends the point $P_{z_1}$ to $P_{y^2z_1}$. Hence $z_2 = y^2z_1$, a contradiction. \hfill $\Box$

**Remark 3.3.** Note that if $q$ is odd and $\ell$ is a line such that $\ell \cap \Pi_2 = P_2 = (x_1, \ldots, x_n, 0, \ldots, 0)$, $\ell \cap \Pi_1 = P_1 = (0, \ldots, 0, y_{n+1}, \ldots, y_{2n})$ and $\ell$ is not a line of $W(2n-1,q)$, then the point $P = P_1 + zP_2 \in \ell$ belongs to $P_1$ or to $P_2$, according as $z(x_1, \ldots, x_n)(y_{n+1}, \ldots, y_{2n})^t$ is a non–zero square or a non–square in $\text{GF}(q)$. Therefore $|\ell \cap P_1| = |\ell \cap P_2| = (q-1)/2$. Moreover, it can be checked that there are projectivities of $\text{Stab}_{\text{PSp}(2n-1,q)}(\{\Pi_1, \Pi_2\}) \setminus G_{\Pi_2}$ interchanging the two orbits and hence $\text{Stab}_{\text{PSp}(2n-1,q)}(\{\Pi_1, \Pi_2\})$ acts transitively on points of $W(2n-1,q) \setminus (\Pi_1 \cup \Pi_2)$.

From Lemma 3.1 \(\Pi_3\) is a generator of $W(2n-1,q)$ disjoint from both $\Pi_1$ and $\Pi_2$ if and only if $\Pi_3 = L(A)$, where $A$ is an invertible matrix of $S_{n,q}$. Hence there are

$$q^{\frac{n(n+1)}{2}} \sum_{i=1}^{\left[\frac{n}{2}\right]} \left(\prod_{i=1}^{\left[\frac{n}{2}\right]} (q^{2i-1} - 1)\right)$$

generators of $W(2n-1,q)$ disjoint from both $\Pi_1$ and $\Pi_2$, see [10] Theorem 2, [17] Corollary 19. The following result is well known. For the convenience of the reader a direct proof is provided.

**Lemma 3.4** (Theorem 21, [17]). Let $\Pi_3$ be a generator of $W(2n-1,q)$ disjoint from $\Pi_1$ and $\Pi_2$. The points $P \in \Pi_3$ such that there exists a line of $W(2n-1,q)$ through $P$ intersecting $\Pi_1$ and $\Pi_2$ are the absolute points of a non–degenerate polarity which is

- pseudo–symplectic if $q$ is even and $n$ is odd,
orthogonal if $q$ and $n$ are odd,

symplectic or pseudo–symplectic if $q$ and $n$ are even,

elliptic orthogonal or hyperbolic orthogonal if $q$ is odd and $n$ is even.

Proof. Let $\Pi_3 = L(A)$ be a generator of $W(2n - 1, q)$ such that $|\Pi_1 \cap \Pi_3| = |\Pi_2 \cap \Pi_3| = 0$. We show that there is a non–degenerate polarity $\rho$ of $\Pi_3$ associated with the matrix $A$. Observe that the $(n - 1)$–space $\Pi_3$ has equations:

$$
\begin{pmatrix}
X_{n+1} \\
\vdots \\
X_{2n}
\end{pmatrix} = A
\begin{pmatrix}
X_1 \\
\vdots \\
X_n
\end{pmatrix}
$$

Hence the point $P$ belongs to $\Pi_3$ if and only if $P = (x_1, \ldots, x_n, 0, \ldots, 0) + (0, \ldots, 0, x_1, \ldots, x_n) (0_n \quad A)^t$ and $P$ lies on the line $\ell$ joining the points $(0, \ldots, 0, x_1, \ldots, x_n) (0_n \quad A)^t \in \Pi_1$ and $(x_1, \ldots, x_n, 0, \ldots, 0) \in \Pi_2$. Thus $\ell^\perp$ is represented by the equations: $x_1 X_{n+1} + \ldots + x_n X_{2n} = (x_1, \ldots, x_n) A (X_1, \ldots, X_n)^t = 0$ and a point $P' = (y_1, \ldots, y_n, 0, \ldots, 0) + (0, \ldots, 0, y_1, \ldots, y_n) (0_n \quad A)^t \in \Pi_3$ belongs to $\ell^\perp$ if and only if

$$(y_1, \ldots, y_n) A \begin{pmatrix}
x_1 \\
\vdots \\
x_n
\end{pmatrix} = 0.$$ 

This concludes the proof. \qed

Lemma 3.5. The group $G_{\Pi_2}$ has the following orbits on generators of $W(2n - 1, q)$ disjoint from both $\Pi_1$ and $\Pi_2$:

- one orbit if $q$ is even and $n$ is odd,

- two equally sized orbits if $q$ and $n$ are odd,

- two orbits having size

$$q^{-\frac{n(n-2)}{4}} \prod_{i=1}^{\frac{n}{2}} (q^{2i-1} - 1) \quad \text{and} \quad q^{-\frac{n(n-2)}{4}} (q^n - 1) \prod_{i=1}^{\frac{n}{2}} (q^{2i-1} - 1)$$

if $q$ and $n$ are even,

- two orbits having size

$$q^{\frac{n^2}{2}} \left( q^{\frac{n}{2}} + 1 \right) \prod_{i=1}^{\frac{n}{2}} (q^{2i-1} - 1) \quad \text{and} \quad q^{\frac{n^2}{2}} \left( q^{\frac{n}{2}} - 1 \right) \prod_{i=1}^{\frac{n}{2}} (q^{2i-1} - 1)$$

if $q$ is odd and $n$ is even.
Proof. Let $\Pi_3 = L(A)$ be a generator of $W(2n - 1, q)$ such that $|\Pi_1 \cap \Pi_3| = |\Pi_2 \cap \Pi_3| = 0$ and let $g$ be the projectivity of $G_{\Pi_2}$ associated with the matrix (3.2), for some $T \in \mathrm{GL}(n, q)$. From the proof of Lemma 3.2, $\Pi_3^3 = L(TAT^t)$. Therefore $g$ stabilizes $\Pi_3$ if and only if $TAT^t = A$. It follows that $|Stab_{G_{\Pi_2}}(\Pi_3)| = 12$ Appendix I)

$$|Stab_{G_{\Pi_2}}(\Pi_3)| = \begin{cases} 
\frac{(n-1)^2}{4} \prod_{i=1}^{n-1} (q^{2i} - 1) & \text{for } n \text{ odd}, \\
\frac{n^2}{4} \prod_{i=1}^{n-1} (q^{2i} - 1) & \text{for } q, n \text{ even, } a_{ij} = a_{ji}, a_{ii} = 0, \\
q^{n(n-2)} (\frac{n}{2} - 1) \prod_{i=1}^{n-2} (q^{2i} - 1) & \text{for } q, n \text{ even, } a_{ij} = a_{ji}, a_{ii} \neq 0, \text{ for some } i, \\
q^{n(n-2)} (\frac{n}{2} + 1) \prod_{i=1}^{n-2} (q^{2i} - 1) & \text{for } q \text{ odd, } n \text{ even, } \det(A) \text{ square of } GF(q) \setminus \{0\}, \\
q^{n(n-2)} (\frac{n}{2} - 1) \prod_{i=1}^{n-2} (q^{2i} - 1) & \text{for } q \text{ odd, } n \text{ even, } \det(A) \text{ non–square of } GF(q), 
\end{cases}$$

where $A = (a_{ij})$. The result follows. $\Box$

Remark 3.6. We remark that if $q$ and $n$ are odd, then the generator $\Pi_3 = L(A), A \in S_{n,q}$, $\mathrm{rk}(A) = n$, belongs to the first or the second $G_{\Pi_2}$–orbit on generators of $W(2n - 1, q)$ skew to $\Pi_1, \Pi_2$, according as $\det(A)$ is a square or a non–square in $GF(q)$. It can be easily seen that there are projectivities of $Stab_{PSp(2n-1,q)}(\Pi_1, \Pi_2)$ acts transitively on generators of $W(2n - 1, q)$ disjoint from $\Pi_1$ and $\Pi_2$.

3.1 $W(5,q)$

Set $n = 3$. Let $W(5,q)$ be the symplectic polar space of $PG(5, q)$ as described above. Recall that $G$ is the set of $q^6$ planes of $W(5,q)$ that are disjoint from $\Pi_1$, the group $G$ is the stabilizer of $\Pi_1$ in $PSp(6,q)$, $\Pi_2 = L(0_3)$ and $G_{\Pi_2}$ is the stabilizer of $\Pi_2$ in $G$.

Following Lemma 3.2 let $P_0$ be the set of points $R$ of $W(5,q) \setminus (\Pi_1 \cup \Pi_2)$ such that the line through $R$ intersecting $\Pi_1$ and $\Pi_2$ is a line of $W(5,q)$ and let $P$ be its complement in $W(5,q) \setminus (\Pi_1 \cup \Pi_2)$. Note that $P$ coincides with $P_1$ or $P_1 \cup P_2$, according as $q$ is even or odd. Then $|P_0| = (q^2 - 1)(q^2 + q + 1)$ and $|P| = q^5 - q^2$. Let $\ell$ be a line of $W(5,q)$ disjoint from $\Pi_1 \cup \Pi_2$. The hyperplane $(\Pi_2, \ell)$ meets $\Pi_1$ in a line, say $r_\ell$, and the three–space $(\ell, r_\ell)$ meets $\Pi_2$ in a line, say $t_\ell$. Hence the line $\ell$ defines a unique three–space $T_\ell = (r_\ell, t_\ell)$ meeting both $\Pi_1, \Pi_2$ in a line.

Lemma 3.7. Let $\ell$ be a line of $W(5,q)$ disjoint from $\Pi_1 \cup \Pi_2$, then $|\ell \cap P_0|$ belongs to $\{1, q+1\}$, if $q$ is even, and to $\{0, 1, 2\}$, if $q$ is odd.

Proof. There are two possibilities, either $T_\ell^\perp$ is a line of $W(5,q)$ or it is not. In the former case, among the $q + 1$ lines meeting $\ell$, $r_\ell, t_\ell$ in one point, there is exactly one line of $W(5,q)$. If the latter case occurs, then $T_\ell \cap W(5,q)$ is a $W(3,q)$ and the regulus $R$ determined by $\ell, r_\ell, t_\ell$ consists of lines of $W(3,q)$. Thus its opposite regulus consists of either 1 or $q + 1$ lines of $W(3,q)$ if $q$ is even and of 0 or 2 lines of $W(3,q)$ if $q$ is odd. $\Box$
Let us partition the set of lines of $W(5, q)$ disjoint from $\Pi_1 \cup \Pi_2$. Let

- $\mathcal{L}_0 = \{ \ell \text{ line of } W(5, q) : |\ell \cap (\Pi_1 \cup \Pi_2)| = 0, |\ell \cap \mathcal{P}_0| = q + 1 \}$,
- $\mathcal{L}_1 = \{ \ell \text{ line of } W(5, q) : |\ell \cap (\Pi_1 \cup \Pi_2)| = 0, T^\perp_\ell \text{ is a line of } W(5, q) \}$,
- $\mathcal{L}_2 = \{ \ell \text{ line of } W(5, q) : |\ell \cap (\Pi_1 \cup \Pi_2)| = 0, |\ell \cap \mathcal{P}_0| = 1, T^\perp_\ell \text{ is not a line of } W(5, q) \}$,

if $q$ is even, or

- $\mathcal{L}_0 = \{ \ell \text{ line of } W(5, q) : |\ell \cap (\Pi_1 \cup \Pi_2)| = 0, |\ell \cap \mathcal{P}_0| = 0, T^\perp_\ell \text{ is not a line of } W(5, q) \}$,
- $\mathcal{L}_1 = \{ \ell \text{ line of } W(5, q) : |\ell \cap (\Pi_1 \cup \Pi_2)| = 0, T^\perp_\ell \text{ is a line of } W(5, q) \}$,
- $\mathcal{L}_2 = \{ \ell \text{ line of } W(5, q) : |\ell \cap (\Pi_1 \cup \Pi_2)| = 0, |\ell \cap \mathcal{P}_0| = 1, T^\perp_\ell \text{ is not a line of } W(5, q) \}$,

if $q$ is odd. Note that in both cases if $\ell \in \mathcal{L}_1$, then $|\ell \cap \mathcal{P}_0| = 1$, whereas if $q$ is even and $\ell \in \mathcal{L}_0$, then $T^\perp_\ell$ is not a line of $W(5, q)$.

Lemma 3.8. If $q$ is even, then

$$|\mathcal{L}_0| = q^2(q^3 - 1), \quad |\mathcal{L}_1| = q(q^2 - 1)(q^3 - 1), \quad |\mathcal{L}_2| = q^2(q^2 - 1)(q^3 - 1).$$

If $q$ is odd, then

$$|\mathcal{L}_0| = \frac{q^2(q - 1)(q^3 - 1)}{2}, \quad |\mathcal{L}_1| = q(q^2 - 1)(q^3 - 1), \quad |\mathcal{L}_2| = \frac{q^2(q + 1)(q^3 - 1)}{2}.$$

Proof. The line $T^\perp_\ell$ meets both $\Pi_1$, $\Pi_2$ in one point. If $T^\perp_\ell$ is a line of $W(5, q)$, then $T^\perp_\ell \cap W(5, q)$ consists of $q + 1$ generators of $W(5, q)$ through $T^\perp_\ell$ and hence there are $q(q - 1)^2$ lines of $W(5, q)$ contained in $T_\ell$ and disjoint from $\Pi_1 \cup \Pi_2$. Since there are $(q + 1)(q^2 + q + 1)$ lines of $W(5, q)$ meeting both $\Pi_1$, $\Pi_2$ in one point, we get $|\mathcal{L}_1| = q(q^2 - 1)(q^3 - 1)$. If $T^\perp_\ell$ is not a line of $W(5, q)$, then $T^\perp_\ell \cap W(5, q)$ is a non-degenerate symplectic polar space $W(3, q)$ and the regulus $\mathcal{R}$ determined by $\ell, r_\ell, t_\ell$ is a regulus of $W(3, q)$. The point line dual of $W(3, q)$ is a parabolic quadric $Q(4, q)$ and the lines $r_\ell$ and $t_\ell$ correspond to two points $R, T$ such that the line $RT$ meets $Q(4, q)$ only in $R$ and $T$. Moreover, the regulus $\mathcal{R}$ corresponds to a conic $C$ of $Q(4, q)$, where $R, T \in C$.

Assume that $q$ is even, then $\ell$ belongs either to $\mathcal{L}_0$ or to $\mathcal{L}_2$, according as the opposite regulus of $\mathcal{R}$ has $q + 1$ or one line of $W(3, q)$. In this case the parabolic quadric $Q(4, q)$ has a nucleus, say $N$. Moreover, the opposite regulus of $\mathcal{R}$ has $q + 1$ or one line of $W(3, q)$ according as $N$ belongs to the plane $(C)$ or does not. Therefore, in $W(3, q)$, $\ell$ can be chosen in $q - 1$ ways such that it belongs to $\mathcal{L}_0$ and in $(q^2 - 1)(q - 1)$ ways such that it belongs to $\mathcal{L}_2$. Since there are $q^2(q^2 + q + 1)$ lines not of $W(5, q)$ meeting both $\Pi_1$, $\Pi_2$ in one point, we get $|\mathcal{L}_0| = q^2(q^3 - 1)$ and $|\mathcal{L}_2| = q^2(q^2 - 1)(q^3 - 1)$.

If $q$ is odd, then $\ell$ belongs either to $\mathcal{L}_0$ or to $\mathcal{L}_2$, according the opposite regulus of $\mathcal{R}$ has $0$ or $2$ lines of $W(3, q)$. In this case the opposite regulus of $\mathcal{R}$ has $0$ or $2$ lines of $W(3, q)$ according as the polar of $(C)$ with respect to the orthogonal polarity of $Q(4, q)$ is a line external or secant to $Q(4, q)$. Therefore, in $W(3, q)$, $\ell$ can be chosen in $q(q - 1)^2/2$ ways such that it
belongs to $\mathcal{L}_0$ and in $(q^3 - q)/2$ ways such that it belongs to $\mathcal{L}_2$. Since there are $q^2(q^2 + q + 1)$ lines not of $\mathcal{W}(5, q)$ meeting both $\Pi_1$, $\Pi_2$ in one point, we get $|\mathcal{L}_0| = q^3(q - 1)(q^3 - 1)/2$ and $|\mathcal{L}_2| = q^3(q + 1)(q^3 - 1)/2$. 

Lemma 3.9. Let $\Pi_3$ be a plane of $\mathcal{W}(5, q)$ skew to $\Pi_1$ and $\Pi_2$. The $q + 1$ planes of the Segre variety $\Sigma_{1, 2}$ of $\text{PG}(5, q)$ determined by $\Pi_1$, $\Pi_2$, $\Pi_3$ are generators of $\mathcal{W}(5, q)$.

Proof. Let $A$ be an invertible matrix of $S_{3, q}$ and consider the symplectic Segre variety $\Sigma_{1, 2}$ determined by the planes $\Pi_1$, $\Pi_2$ and $L(A)$. Direct computations show that the remaining $q - 2$ planes of $\Sigma_{1, 2}$ are the planes $L(\lambda A)$, where $\lambda \in \text{GF}(q) \setminus \{0, 1\}$. 

We will refer to a Segre variety $\Sigma_{1, 2}$ of $\text{PG}(5, q)$ whose $q + 1$ planes are generators of $\mathcal{W}(5, q)$ as a symplectic Segre variety of $\mathcal{W}(5, q)$. As a consequence of Lemma 3.4, the following corollary arises.

Corollary 3.10. Let $\Sigma_{1, 2}$ be a symplectic Segre variety containing $\Pi_1$, $\Pi_2$ and let $\Pi_3$ be a plane of $\Sigma_{1, 2}$, $\Pi_3 \neq \Pi_1$, $\Pi_3 \neq \Pi_2$. Then $\Pi_3$ contains one line of $\mathcal{L}_0$, $q + 1$ lines of $\mathcal{L}_1$ and $q^2 - 1$ lines of $\mathcal{L}_2$, if $q$ is even, and $q(q - 1)/2$ lines of $\mathcal{L}_0$, $q + 1$ lines of $\mathcal{L}_1$ and $q(q + 1)/2$ lines of $\mathcal{L}_2$, if $q$ is odd.

Proof. From Lemma 3.4 there is a non-degenerate polarity $\rho$ of $\Pi_3$. Note that if $\ell$ is a line of $\Pi_3$, then $\ell^\rho = T_\ell^\perp \cap \Pi_3$. If $q$ is even, $\rho$ is a pseudo–polarity and the unique line of $\Pi_3$ belonging to $\mathcal{L}_0$ is the line $\ell_0$ consisting of its absolute points. The other lines of $\Pi_3$ belong to $\mathcal{L}_1$ if they pass through $\ell_0^\rho$ and to $\mathcal{L}_2$ otherwise. If $q$ is odd, $\rho$ is an orthogonal polarity and its absolute points form a conic, say $C$. A line of $\Pi_3$ belongs either to $\mathcal{L}_1$, or to $\mathcal{L}_0$ or to $\mathcal{L}_2$ according as it is tangent, external or secant to $C$, respectively.

For a point $P \in \Pi_2$, let $\Sigma_P$ denote a 3–space contained in $P^\perp$ and not containing $P$. When restricted to $\Sigma_P$, the polarity $\perp$ defines a non–degenerate symplectic polar space of $\Sigma_P$, say $\mathcal{W}_P$. Moreover $r_P = \Sigma_P \cap \Pi_1$ and $t_P = \Sigma_P \cap \Pi_2$ are lines of $\mathcal{W}_P$. In what follows we investigate the action of the group $G_{\Pi_2}$ on $G$.

Lemma 3.11. The group $G_{\Pi_2}$ has the following orbits on $G$:

- the plane $\Pi_2$;
- $\mathcal{G}_1$ of size $q^3 - 1$ consisting of the planes of $G$ meeting $\Pi_2$ in a line,
- $\mathcal{G}_2$ of size $q^3 - 1$ consisting of the planes of $G$ meeting $\Pi_2$ in a point and no plane of $\mathcal{G}_1$ in a line,
- $\mathcal{G}_3$ of size $(q^2 - 1)(q^3 - 1)$ consisting of the planes of $G$ meeting $\Pi_2$ in a point and $q$ planes of $\mathcal{G}_1$ in a line,
- $\mathcal{G}_4$ of size $q^2(q^3 - 1)(q - 1)$ consisting of the planes of $G$ disjoint from $\Pi_2$,

if $q$ is even.
• the plane $\Pi_2$;
• $G_1$ of size $(q^3 - 1)/2$ consisting of the planes of $G$ meeting $\Pi_2$ in a line and having $q^2$ points of $\mathcal{P}_1$,
• $G_2$ of size $(q^3 - 1)/2$ consisting of the planes of $G$ meeting $\Pi_2$ in a line and having $q^2$ points of $\mathcal{P}_2$,
• $G_3$ of size $q(q - 1)(q^3 - 1)/2$ consisting of the planes of $G$ meeting $\Pi_2$ in a point and $q + 1$ planes of $G_1 \cup G_2$ in a line,
• $G_4$ of size $q(q + 1)(q^3 - 1)/2$ consisting of the planes of $G$ meeting $\Pi_2$ in a point and $q - 1$ planes of $G_1 \cup G_2$ in a line,
• two orbits, say $G_5$ and $G_6$, both of size $q^2(q^3 - 1)(q - 1)/2$, consisting of planes of $G$ disjoint from $\Pi_2$,

if $q$ is odd.

Proof. There are $q^3 - 1$ members of $G$ intersecting $\Pi_2$ in a line. The number of generators of $\mathcal{W}(5, q)$ through $P$ disjoint from $\Pi_1$ and intersecting $\Pi_2$ exactly in $P$ equals the number of lines of $\mathcal{W}_P$ disjoint from both $r_P$ and $t_P$, and they are $q^2(q - 1)$. As the point $P$ varies on $\Pi_2$ we get $q^2(q^3 - 1)$ generators of $\mathcal{W}(5, q)$ disjoint from $\Pi_1$ and intersecting $\Pi_2$ at exactly one point. Hence there are $q^2(q - 1)(q^3 - 1)$ generators of $\mathcal{W}(5, q)$ disjoint from both $\Pi_1$ and $\Pi_2$. Alternatively, if $A \in S_{3,q}$, then $L(A) \cap \Pi_2$ is a $(3 - \text{rk}(A) - 1)$–space of $\Pi_1$.

Let $\pi$ be a plane of $G$ intersecting $\Pi_2$ in a line. Since $G_{\Pi_2}$ is transitive on lines of $\Pi_2$, we may assume without loss of generality that $\pi \cap \Pi_2$ is the line $U_2U_3$. Then $\pi : X_5 = X_6 = 0, X_4 = zX_1$, for some $z \in GF(q) \setminus \{0\}$. Let $g$ be the projectivity of $G_{\Pi_2}$ associated with the matrix $[3, 2]$, for some $T \in GL(3, q)$. Then $g$ stabilizes $\pi$ if and only if

$$T = \begin{pmatrix} \pm 1 & * & * \\ 0 & T' & 0 \end{pmatrix},$$

where $T' \in GL(2, q)$. Hence $|Stab_{G_{\Pi_2}}(\pi)| = \text{gcd}(2, q - 1)q^2/|GL(2, q)|/\text{gcd}(2, q - 1)$ and $|\pi^{G_{\Pi_2}}|$ equals $q^3 - 1$ if $q$ is even or $(q^3 - 1)/2$ if $q$ is odd. In the even characteristic case the planes of $G$ meeting $\Pi_2$ in a line are permuted in a unique orbit, say $G_1$. In the odd characteristic case, there are two orbits, say $G_1$ and $G_2$; it can be seen that representatives for $G_1$ and $G_2$ are $\pi_1 : X_5 = X_6 = X_4 - z_1X_1 = 0$ and $\pi_2 : X_5 = X_6 = X_4 - z_2X_1 = 0$, respectively, where $z_1$ is a non–zero square in $GF(q)$ and $z_2$ is a non–square of $GF(q)$. Moreover $Stab_{G_{\Pi_2}}(\pi)$ acts transitively on the $q^2$ points of $\pi \setminus \Pi_2$.

Let $\pi$ be a plane of $G$ and intersecting $\Pi_2$ in the point $P$ and let $\ell$ be the line of $\mathcal{W}_P$ obtained by intersecting $\pi$ with $\Sigma_P$. Let $\mathcal{R}$ be the regulus determined by $r_P, t_P, \ell$ and $\mathcal{R}^o$ be its opposite regulus.

Assume that $q$ is even. There are $q - 1$ possibilities for the line $\ell$ such that the regulus $\mathcal{R}^o$ contains $q + 1$ lines of $W_P$, and $(q^2 - 1)(q - 1)$ possibilities for $\ell$ such that $\mathcal{R}^o$ contains exactly one
line of $\mathcal{W}_P$. Varying $P$ in $\Pi_2$ we get two sets, namely $\mathcal{G}_2$ and $\mathcal{G}_3$, of size $q^3-1$ and $(q^2-1)(q^3-1)$, respectively. Observe that there are 0 or $q$ planes of $\mathcal{G}_1$ meeting $\pi$ in a line, according as $\pi$ belongs to $\mathcal{G}_2$ or $\mathcal{G}_3$. We claim that $\mathcal{G}_2$ and $\mathcal{G}_3$ are two $G_{\Pi_2}$-orbits. Let $\pi$ be the plane with equations $X_2 + X_6 = X_3 + X_5 = X_4 = 0$. Direct computations show that $\pi \in \mathcal{G}_2$. The projectivity $g$ of $G_{\Pi_2}$ associated with the matrix \((3.2)\), $T \in \text{GL}(3, q)$, stabilizes $\pi$ if and only if

$$
T = \begin{pmatrix}
  x & 0 & 0 \\
  * & T' & \\
  * & & 
\end{pmatrix},
$$

where $x \in \text{GF}(q) \setminus \{0\}$ and $T' \in \text{SL}(2, q)$. Hence $|\text{Stab}_{G_{\Pi_2}}(\pi)| = q^2(q-1)|\text{SL}(2, q)|$ and $|\pi_{G_{\Pi_2}}| = q^3 - 1 = |\mathcal{G}_2|$. Let $\pi$ be the plane having equations $X_2 + X_5 = X_3 + X_6 = X_4 = 0$. Direct computations show that $\pi \in \mathcal{G}_3$. The projectivity $g$ of $G_{\Pi_2}$ associated with the matrix \((3.2)\), $T \in \text{GL}(3, q)$, stabilizes $\pi$ if and only if

$$
T = \begin{pmatrix}
  x & 0 & 0 \\
  * & y & y' \\
  * & y' & y
\end{pmatrix},
$$

where $x, y, y' \in \text{GF}(q)$, $x \neq 0$ and $y^2 + y'^2 = 1$. Hence $|\text{Stab}_{G_{\Pi_2}}(\pi)| = q^3(q - 1)$ and $|\pi_{G_{\Pi_2}}| = (q^2 - 1)(q^3 - 1) = |\mathcal{G}_3|$.

Assume that $q$ is odd. There are $q(q-1)^2/2$ possibilities for the line $\ell$ such that the regulus $\mathcal{R}^0$ contains no line of $\mathcal{W}_P$, and $q(q^2 - 1)/2$ possibilities for $\ell$ such that $\mathcal{R}^0$ contains exactly two lines of $\mathcal{W}_P$. Varying $P$ in $\Pi_2$ we get two sets, say $\mathcal{G}_3$ and $\mathcal{G}_4$ of size $q(q - 1)(q^3 - 1)/2$ and $q(q + 1)(q^3 - 1)/2$ respectively. Observe that there are $q + 1$ or $q - 1$ planes of $\mathcal{G}_1 \cup \mathcal{G}_2$ meeting $\pi$ in a line, according as $\pi$ belongs to $\mathcal{G}_3$ or $\mathcal{G}_4$. Again we want to show that $\mathcal{G}_3$ and $\mathcal{G}_4$ are two $G_{\Pi_2}$-orbits. Let $\alpha$ be a fixed non–square in $\text{GF}(q)$ and let $\pi$ be the plane with equations $X_5 - \alpha^2 X_2 = X_6 + \alpha X_3 = X_4 = 0$. Direct computations show that $\pi \in \mathcal{G}_3$. The projectivity $g$ of $G_{\Pi_2}$ associated with the matrix \((3.2)\), $T \in \text{GL}(3, q)$, stabilizes $\pi$ if and only if

$$
T = \begin{pmatrix}
  x & 0 & 0 \\
  * & y & -\alpha y' \\
  * & y' & -y
\end{pmatrix} \quad \text{or} \quad T = \begin{pmatrix}
  x & 0 & 0 \\
  * & y & \alpha y' \\
  * & y' & y
\end{pmatrix},
$$

where $x, y, y' \in \text{GF}(q)$, $x \neq 0$ and $y^2 - \alpha y'^2 = 1$. Note that there are $q + 1$ couple $(y, y') \in \text{GF}(q) \times \text{GF}(q)$ such that $y^2 - \alpha y'^2 = 1$. Therefore $|\text{Stab}_{G_{\Pi_2}}(\pi)| = q^2(q - 1)(q + 1)$ and $|\pi_{G_{\Pi_2}}| = q(q - 1)(q^3 - 1)/2 = |\mathcal{G}_3|$. Let $\pi$ be the plane having equations $X_2 - X_6 = X_3 - X_5 = X_4 = 0$. Direct computations show that $\pi \in \mathcal{G}_4$. The projectivity $g$ of $G_{\Pi_2}$ associated with the matrix \((3.2)\), $T \in \text{GL}(3, q)$, stabilizes $\pi$ if and only if

$$
T = \begin{pmatrix}
  x & 0 & 0 \\
  * & y & 0 \\
  * & 0 & y^{-1}
\end{pmatrix} \quad \text{or} \quad T = \begin{pmatrix}
  x & 0 & 0 \\
  * & 0 & y \\
  * & y^{-1} & 0
\end{pmatrix},
$$

where $x, y \in \text{GF}(q) \setminus \{0\}$. In this case $|\text{Stab}_{G_{\Pi_2}}(\pi)| = q^2(q-1)^2$ and $|\pi_{G_{\Pi_2}}| = q(q+1)(q^3-1)/2 = |\mathcal{G}_4|$.

From Lemma \((3.5)\) the group $G_{\Pi_2}$ has one or two orbits on generators of $\mathcal{W}(5, q)$ skew to $\Pi_1$ and $\Pi_2$. \[\square\]
Lemma 3.12. Assume that $q$ is even. Let $\Pi \in \mathcal{G}_2$ and $\Pi' \in \mathcal{G}_3$. Then the number of planes of \mathcal{G}_2 meeting $\Pi$ or $\Pi'$ in a line is zero or one, whereas the number of planes of $\mathcal{G}_3$ meeting $\Pi$ or $\Pi'$ in a line equals $q^2 - 1$ or $q^2 - q - 2$.

Proof. Let $P = \Pi \cap \Pi_2$, $P' = \Pi' \cap \Pi_2$, $\ell = \Pi \cap \Sigma_P$, $\ell' = \Pi' \cap \Sigma_{P'}$, $\mathcal{R}$ be the regulus determined by $t_P, t_P, \ell$ and $\mathcal{R}'$ be the regulus determined by $t'_P, t'_P, \ell'$. From the proof of Lemma 3.11, the opposite regulus of $\mathcal{R}$, say $\mathcal{R}^o$, consists of lines of $\mathcal{W}_P$, whereas the opposite regulus of $\mathcal{R}'$, say $\mathcal{R}'^o$, has exactly one line of $\mathcal{W}_P$.

If a plane $\gamma \in \mathcal{G}_2 \cup \mathcal{G}_3$ intersects $\Pi$ in a line, then $\gamma = \langle P, s \rangle$, where $s$ is a line of $\mathcal{W}_P$ intersecting $\ell$ and skew to $r_P$ and $t_P$. Moreover, $\gamma$ belongs to $\mathcal{G}_2$ or $\mathcal{G}_3$ according as there are $q + 1$ or one line of $\mathcal{W}_P$ meeting $s, r_P, t_P$. Since the number of lines of $\mathcal{W}_P$ intersecting $\ell$ at a point and skew to $r_P$ and $t_P$ equals $q^2 - 1$ and, if $s$ is one of these lines, there is exactly one line of $\mathcal{W}_P$ meeting $s, r_P, t_P$, the statement holds true in this case.

Similarly, if a plane $\gamma \in \mathcal{G}_2 \cup \mathcal{G}_3$ intersects $\Pi'$ in a line, then $\gamma = \langle P', s \rangle$, where $s$ is a line of $\mathcal{W}_{P'}$ intersecting $\ell'$ and skew to $r'_P$ and $t'_P$. Moreover, $\gamma$ belongs to $\mathcal{G}_2$ or $\mathcal{G}_3$ according as there are $q + 1$ or one line of $\mathcal{W}_{P'}$ meeting $s, r'_P, t'_P$. The point line dual of $\mathcal{W}_{P'}$ is a parabolic quadric $\mathcal{Q}(4, q)$, the lines $r'_P, t'_P$ and $\ell'$ correspond to three points, say $R, T, L$, such that the line $RT$ meets $\mathcal{Q}(4, q)$ only in $R$ and $T$. Moreover, the regulus $\mathcal{R}'$ corresponds to a conic $C$ of $\mathcal{Q}(4, q)$, where $R, T, L \in C$. Let $N$ be the nucleus of $\mathcal{Q}(4, q)$. Then $N$ does not belong to the plane $\langle C \rangle$ and the points $R, T, N$ span a plane meeting $\mathcal{Q}(4, q)$ in a conic, say $C'$. Hence there is a plane $\gamma \in \mathcal{G}_2$ meeting $\Pi'$ in a line if and only if there is a point $U$ of $C'$ such that the line $UL$ is a line of $\mathcal{Q}(4, q)$. There exists only one such a point: the intersection point between the three–space containing the lines of $\mathcal{Q}(4, q)$ through $L$ and the conic $C'$. Analogously, there is a plane $\gamma \in \mathcal{G}_3$ meeting $\Pi'$ in a line if and only if there is a point $U \in \mathcal{Q}(4, q)$ not belonging to $C'$ such that the line $UL$ is a line of $\mathcal{Q}(4, q)$ and the lines $UR$ and $UT$ are not lines of $\mathcal{Q}(4, q)$. There exist exactly $q^2 - q - 2$ points having these properties. \[\square\]

Lemma 3.13. Assume that $q$ is odd. Let $\Pi \in \mathcal{G}_5$ and $\Pi' \in \mathcal{G}_6$, then either $|\Pi \cap \mathcal{P}_1| = |\Pi' \cap \mathcal{P}_2| = q(q - 1)/2$ and $|\Pi \cap \mathcal{P}_2| = |\Pi' \cap \mathcal{P}_1| = q(q + 1)/2$ or $|\Pi \cap \mathcal{P}_1| = |\Pi' \cap \mathcal{P}_2| = q(q + 1)/2$ and $|\Pi \cap \mathcal{P}_2| = |\Pi' \cap \mathcal{P}_1| = q(q - 1)/2$. Moreover through a line of $\mathcal{L}_0 \cup \mathcal{L}_3$, there pass $(q - 1)/2$ planes of $\mathcal{G}_5$ and $(q - 1)/2$ planes of $\mathcal{G}_6$, whereas the $q$ generators passing through a line of $\mathcal{L}_1$ and skew to $\Pi_1, \Pi_2$ are planes either of $\mathcal{G}_5$ or of $\mathcal{G}_6$.

Proof. Let $A$ be an invertible matrix of $S_{3, q}$ and consider the symplectic Segre variety $\Sigma_{1, 2}$ determined by the $\Pi_1, \Pi_2$ and $L(A)$. The planes of $\Sigma_{1, 2}$ distinct from $\Pi_1$ and $\Pi_2$ are $L(\lambda A)$, where $\lambda \in GF(q) \setminus \{0\}$. Taking into account Remark 3.16 we have that $(q - 1)/2$ members of $\Sigma_{1, 2}$ belong to $\mathcal{G}_5$ and $(q - 1)/2$ members of $\Sigma_{1, 2}$ belong to $\mathcal{G}_6$. Moreover, taking into account Remark 3.13 if $\lambda$ is a non–zero square of $GF(q)$, then the point of $L(A)$ given by $(x, y, z, 0, 0, 0) + (0, 0, 0, x, y, z)(0_3 A)^t$ belongs to $\mathcal{P}_1$ if and only if the point of $L(\lambda A)$ given by $(x, y, z, 0, 0, 0) + (0, 0, 0, x, y, z)\lambda (0_3 A)^t$ belongs to $\mathcal{P}_1$, whereas if $\lambda$ is a non–square of $GF(q)$, then the point of $L(A)$ given by $(x, y, z, 0, 0, 0) + (0, 0, 0, x, y, z)(0_3 A)^t$ belongs to $\mathcal{P}_1$ if and only if the point of $L(\lambda A)$ given by $(x, y, z, 0, 0, 0) + (0, 0, 0, x, y, z)\lambda (0_3 A)^t$ belongs to $\mathcal{P}_2$. Let $\Pi = L(\lambda A) \in \mathcal{G}_5,$
\( \Pi' = L(\lambda' A) \in \mathcal{G}_6 \). From Lemma 3.3, there is a non-degenerate conic \( C \) (resp. \( C' \)) of \( \Pi \) (resp. \( \Pi' \)). Observe that exactly one of the two following possibilities occurs: either \( \Pi \cap \mathcal{P}_1 \) are the points of \( \Pi \) internal to \( C \), \( \Pi \cap \mathcal{P}_2 \) are the points of \( \Pi \) external to \( C \), \( \Pi' \cap \mathcal{P}_1 \) are the points of \( \Pi' \) internal to \( C' \), \( \Pi' \cap \mathcal{P}_2 \) are the points of \( \Pi' \) external to \( C' \), \( \Pi \cap \mathcal{P}_2 \) are the points of \( \Pi \) internal to \( C \), \( \Pi' \cap \mathcal{P}_1 \) are the points of \( \Pi' \) external to \( C' \), \( \Pi' \cap \mathcal{P}_1 \) are the points of \( \Pi' \) internal to \( C' \), \( \Pi' \cap \mathcal{P}_2 \) are the points of \( \Pi' \) external to \( C' \).

Let \( \ell \) be a line of \( \mathcal{W}(5, q) \) and let \( \Pi_3 \) be a generator of \( \mathcal{W}(5, q) \) skew to \( \Pi_1 \), \( \Pi_2 \) such that \( \ell \subset \Pi_3 \). Denote by \( \rho \) the non-degenerate polarity of \( \Pi_3 \) arising from Lemma 3.3 and let \( C_3 \) be the corresponding non-degenerate conic. If \( \ell \in \mathcal{L}_0 \cup \mathcal{L}_2 \), then \( T^\perp_\ell \) is a line meeting both \( \Pi_1 \), \( \Pi_2 \) in a point and it is not a line of \( \mathcal{W}(5, q) \). Hence, by Remark 3.3, \( T^\perp_\ell \) contains \( (q-1)/2 \) points of \( \mathcal{P}_1 \) and \( (q-1)/2 \) points of \( \mathcal{P}_2 \). Since \( \ell^\rho \) is the point \( \Pi_3 \cap T^\perp_\ell \), we have that through \( \ell \) there pass \( (q-1)/2 \) planes of \( \mathcal{G}_5 \) and \( (q-1)/2 \) planes of \( \mathcal{G}_6 \). If \( \ell \in \mathcal{L}_1 \), then \( T^\perp_\ell \) is a line of \( \mathcal{W}(5, q) \) meeting both \( \Pi_1 \), \( \Pi_2 \) in a point. In this case \( \ell \cap C_3 \) consists of one point, say \( Q \). The \( q \) points of \( \ell \) distinct from \( Q \) are external to \( C_3 \) and hence they all lie in a unique point–orbit, that is either \( \mathcal{P}_1 \) or \( \mathcal{P}_2 \). Therefore the \( q \) generators of \( \mathcal{W}(5, q) \) passing through \( \ell \) and skew to \( \Pi_1 \), \( \Pi_2 \) are such that the external points of their corresponding conics are all points belonging to the same point orbit, that is either \( \mathcal{P}_1 \) or \( \mathcal{P}_2 \). Hence these \( q \) generators lie in the same \( G_{\Pi_2} \)–orbit which contains \( \Pi_3 \).

4 Hermitian matrices and Hermitian polar spaces

Let \( \omega \in \text{GF}(q^2) \setminus \{0\} \) such that \( \omega^q = -\omega \) and let \( \mathcal{H}(2n-1, q^2) \) be the non-degenerate Hermitian polar space of \( \text{PG}(2n-1, q^2) \) associated with the following Hermitian form

\[
(X_1, \ldots, X_{2n}) \begin{pmatrix}
0_n & \omega I_n \\
\omega^t I_n & 0_n
\end{pmatrix}
\begin{pmatrix}
Y_1^q \\
\vdots \\
Y_{2n}^q
\end{pmatrix},
\]

Let \( \text{PGU}(2n, q^2) \) be the group of projectivities of \( \text{PG}(2n-1, q^2) \) stabilizing \( \mathcal{H}(2n-1, q^2) \). Denote by \( \Lambda_1 \) the \((n-1)\)–space of \( \text{PG}(2n-1, q^2) \) spanned by \( U_{n+1}, \ldots, U_{2n} \). Then \( \Lambda_1 \) is a generator of \( \mathcal{H}(2n-1, q^2) \). Denote by \( \tilde{G} \) the stabilizer of \( \Lambda_1 \) in \( \text{PGU}(2n, q^2) \). In this case an element of \( \tilde{G} \) is represented by the matrix

\[
\begin{pmatrix}
T^{-t} & 0_n \\
H_0^t T^{-t}
\end{pmatrix},
\]

where \( T \in \text{GL}(n, q^2) \) and \( H_0 \in \mathcal{H}_{n,q^2} \). Hence \( \tilde{G} \cong \mathcal{H}_{n,q^2} \rtimes (\text{GL}(n, q^2)/\{aI_{2n}\}) \), where \( a^{q^2+1} = 1 \), and

\[
|\tilde{G}| = q^{n^2} \prod_{i=0}^{q-1} (q^{2n} - q^{2i}) / q + 1.
\]

Define an action of \( \mathcal{H}_{n,q^2} \rtimes \text{GL}(n, q^2) \) on \( \mathcal{H}_{n,q^2} \) as follows

\[
((H_0, T), H) \in (\mathcal{H}_{n,q^2} \rtimes \text{GL}(n, q^2)) \times \mathcal{H}_{n,q^2} \mapsto TH (T^q)^t + H_0 \in \mathcal{H}_{n,q^2}.
\]

Its orbitals are the relations of an association scheme, the so called association scheme of Hermitian matrices \[30\] [23]. As in the symmetric case there is a correspondence between \( \mathcal{H}_{n,q^2} \) and
the set $\tilde{G}$ of generators of $H(2n-1, q^2)$ disjoint from $\Lambda_1$ (see also [2 Proposition 9.5.10]). The proof is similar to that of the symmetric case and hence it is omitted.

**Lemma 4.1.** There is a bijection between $H_{n,q^2}$ and $\tilde{G}$ such that $H_{n,q^2} \rtimes GL(n, q^2)$ acts on $H_{n,q^2}$ as $G$ acts on $\tilde{G}$. In particular, a $d$-code of $H_{n,q^2}$ corresponds to a set of generators of $H(2n-1, q^2)$ disjoint from $\Lambda_1$ pairwise intersecting in at most an $(n-d-1)$-space, and conversely.

As before, if $\Lambda_2 = L(0_n)$, the previous lemma implies that the number of orbits of $G_{\Lambda_2}$ on $\tilde{G}$ equals the number of relations of the association scheme on Hermitian matrices.

**Lemma 4.2 ([27]).** Let $\Pi_3$ be a generator of $H(2n-1, q^2)$ disjoint from $\Pi_1$ and $\Pi_2$. The points $P \in \Pi_3$ such that there exists a line of $H(2n-1, q^2)$ through $P$ intersecting $\Pi_1$ and $\Pi_2$ are the absolute points of a non-degenerate unitary polarity of $\Pi_3$.

## 5 2-codes of $S_{n,q}$ or $H_{n,q^2}$

Let $\Gamma_W$ or $\Gamma_H$ be the graph whose vertices are the generators of $W(2n-1, q)$ or $H(2n-1, q^2)$ and two vertices are adjacent whenever they meet in an $(n-2)$-space. Then $\Gamma_W$ or $\Gamma_H$ is a distance regular graph having diameter $n$, see [2] Section 9.4. A coclique of $\Gamma_W$ or $\Gamma_H$ is a set of generators of $W(2n-1, q)$ or $H(2n-1, q^2)$ pairwise intersecting in at most an $(n-3)$-space.

**Lemma 5.1 (Theorem 9.4.3, [2]).** The eigenvalues $\theta_j$, $0 \leq j \leq n$, of $\Gamma_W$ are:

$$\theta_j = \frac{q^j(q^{n-2j+1}+1)}{q-1} - 1, \text{ with multiplicity } f_j = \frac{q^j(q^{n-2j+1}+1)}{q^n-j+1} \prod_{i=1}^{j} \frac{q^{2(n-i+1)}-1}{(q^i-1)(q^{i-1}+1)}.$$  

The eigenvalues $\lambda_j$, $0 \leq j \leq n$, of $\Gamma_H$ are:

$$\lambda_j = \frac{q^{2j}(q^{2n-4j+1}+1)}{q^2-1} - \frac{1}{q+1}, \text{ with multiplicity } g_j = \frac{q^{2j}(q^{2n-4j+1}+1)}{q^{2n-2j+1}+1} \prod_{i=1}^{j} \frac{(q^{2n-2i+2}-1)(q^{2n-2i+1}+1)}{(q^{2i}-1)(q^{2i-1}+1)}.$$  

The eigenvalue $\theta_j$ is positive or negative, according as $0 \leq j \leq \left\lceil \frac{n-1}{2} \right\rceil$ or $\left\lceil \frac{n-1}{2} \right\rceil + 1 \leq j \leq n$, respectively, and

$$\deg(f_j) = \begin{cases} nj + j(n-2j+1) & \text{if } 0 \leq j \leq \left\lceil \frac{n-1}{2} \right\rceil, \\ nj + (j-1)(n-2j+1) & \text{if } \left\lceil \frac{n-1}{2} \right\rceil + 1 \leq j \leq n. \end{cases}$$  

Moreover $\deg(f_i) < \deg(f_j)$, if $0 \leq i < j \leq \left\lceil \frac{n-1}{2} \right\rceil$, and $\deg(f_i) > \deg(f_j)$, if $\left\lceil \frac{n-1}{2} \right\rceil + 1 \leq i < j \leq n$. From the Cvetković bound (Lemma 2.2), it follows that $\alpha(\Gamma_W) \leq \sum_{j=0}^{\frac{n-1}{2}} f_j$, if $n$ is odd. Note that the Hoffman bound gives a better upper bound for $\alpha(\Gamma_W)$ than the Cvetković bound in the cases $n$ is even. Hence, the following result arises.

**Theorem 5.2.** Let $X$ be a set of generators of $W(2n-1, q)$ pairwise intersecting in at most an $(n-3)$-space. Then

$$|X| \leq \begin{cases} \sum_{j=0}^{\frac{n-1}{2}} \frac{q^j(q^{n-2j+1}+1)}{q^n-j+1} \prod_{i=1}^{j} \frac{q^{2(n-i+1)}-1}{(q^i-1)(q^{i-1}+1)} & \text{if } n \text{ is odd}, \\ \prod_{i=2}^{n} (q^i+1) & \text{if } n \text{ is even}. \end{cases}$$
9.5.10. In particular, \( \Gamma' \) is also known as the last subconstituent of \( \Gamma_W \) [2, Corollary 8.4.4]:

\[
\| \mathcal{C} \| \leq \begin{cases} 
\sum_{j=0}^{n-1} \frac{q^j(q^n-2j+1)}{q^n-1} \prod_{i=1}^{j} \frac{2^{2(j+1)}}{(q^i-1)(q^{i-1}+1)} & \text{if } n \text{ is odd}, \\
\prod_{i=2}^{n} (q^i + 1) & \text{if } n \text{ is even}.
\end{cases}
\]

The term of highest degree in Corollary 5.3 is \( q^{(n+1)/2} - q^n + 1 \) if \( n \) is odd or \( q^{(n+1)/2} - q^n + 1 \) if \( n \) is even. The previous known upper bound for the size of a 2-code of \( S_{n,q} \) is \( q^{(n-1)/2} + 1 \) for odd \( q \) [21, Proposition 3.7], and \( q^{(n+1)/2} - q^n + 1 \) for even \( q \) [24, Proposition 3.4]. Therefore Corollary 5.3 provides better upper bounds if \( n \) is odd or if \( n \) and \( q \) are both even.

Regarding 2-codes of \( S_{3,q} \), a further improvement will be obtained in Section 6.

In the Hermitian case, \( \lambda_j \) is positive or negative, according to as \( 0 \leq j \leq \left\lfloor \frac{n-1}{2} \right\rfloor \) or \( \left\lceil \frac{n-1}{2} \right\rceil + 1 \leq j \leq n \), respectively, and

\[
\deg(g_j) = \begin{cases} 
4j(n-j) & \text{if } 0 \leq j \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \\
4j(n-j) - (2n - 4j + 1) & \text{if } \left\lceil \frac{n-1}{2} \right\rceil + 1 \leq j \leq n.
\end{cases}
\]

Moreover \( \deg(g_i) < \deg(g_j) \), if \( 0 \leq i < j \leq \left\lfloor \frac{n-1}{2} \right\rfloor \), and \( \deg(g_i) > \deg(g_j) \), if \( \left\lceil \frac{n-1}{2} \right\rceil + 1 \leq i < j \leq n \). From the Cvetković bound, it follows that \( \alpha(\Gamma_{H}) \leq \sum_{j=0}^{n-1} g_j \), if \( n \) is odd and \( \alpha(\Gamma_{H}) \leq n (2^n + 1) g_j \), if \( n \) is even.

**Theorem 5.4.** Let \( \mathcal{X} \) be a set of generators of \( H(2n-1,q^2) \) pairwise intersecting in at most an \( (n-3) \)-space. Then

\[
|\mathcal{X}| \leq \begin{cases} 
\sum_{j=0}^{n-1} \frac{q^{2j}(q^{2n-2j+1}+1)}{q^{2n-1}+1} \prod_{i=1}^{j} \frac{(q^{2n-2i+2}-1)(q^{2n-2i+1}+1)}{(q^{2i-1}+1)(q^{2i-2}+1)} & \text{if } n \text{ is odd}, \\
\sum_{j=2}^{n} \frac{q^{2j}(q^{2n-2j+1}+1)}{q^{2n-1}+1} \prod_{i=1}^{j} \frac{(q^{2n-2i+2}-1)(q^{2n-2i+1}+1)}{(q^{2i-1}+1)(q^{2i-2}+1)} & \text{if } n \text{ is even}.
\end{cases}
\]

From Lemma 3.1 the size of the largest 2-codes of \( H_{n,q^2} \) coincides with the maximum number of members of \( \tilde{G} \), the set of generators of \( H(2n-1,q^2) \) disjoint from \( \Lambda_1 \), such that they pairwise intersect in at most an \( (n-3) \)-space. Let \( \Gamma'_{H} \) be the induced subgraph of \( \Gamma_{H} \) on \( \tilde{G} \). The graph \( \Gamma'_{W} \) is also known as the last subconstituent of \( \Gamma_{W} \) or the Hermitian forms graph, see [2] Proposition 9.5.10. In particular \( \Gamma'_{W} \) is a distance-regular graph of diameter \( n \). The eigenvalues of \( \Gamma'_{H} \) are [2] Corollary 8.4.4:

\[
\frac{(-q)^{2n-j} - 1}{q+1}, \text{ with multiplicity } \prod_{i=1}^{j} \frac{(-q)^{n+i-1} - 1}{(-q)^i - 1} \prod_{i=0}^{j-1} \frac{(-(-q)^n - (-q)^i)}{(-(-q)^n - (-q)^i)},
\]

respectively, with \( 0 \leq j \leq n \).

In this case the Hoffman bound gives a better upper bound for \( \alpha(\Gamma'_{H}) \) than the Cvetković bound, that is \( q^{(n-1)^2}(q^{2n-1} + 1)/(q + 1) \). However this was already known [22, Theorem 2].
6 Large non–additive rank distance codes

In the case when $C$ is additive much better bounds can be provided. Indeed in [23, Theorem 4.3], [22, Theorem 1] the author proved that the largest additive $d$–codes of $S_{n,q}$ have size at most either $q^{n(d+2)/2}$ or $q^{(n+1)(n-d+1)/2}$, according as $n - d$ is even or odd, respectively, whereas the size of the largest additive $d$–codes of $H_{n,q^2}$ cannot exceed $q^{n(n-d+1)}$. As far as regard codes whose size is larger than the additive bound not much is known. In the Hermitian case, if $n$ is even, there is an $n$–code of $H_{n,q^2}$ of size $q^n + 1$ [22, Theorem 6], [11, Theorem 18]. Observe that from Lemma 3.1 and Lemma 4.1, an $n$–code $C$ of $S_{n,q}$ or $H_{n,q^2}$ exists if and only if there exists a partial spread of $W(2n - 1, q)$ or $H(2n - 1, q^2)$ of size $|C| + 1$. It is well–known that the points of $W(2n - 1, q)$ can be partitioned into $q^n + 1$ pairwise disjoint generators of $W(2n - 1, q)$, that is, $W(2n - 1, q)$ admits a spread. On the other hand, $H(2n - 1, q^2)$ has no spread. If $n$ is odd, an upper bound for the largest partial spreads of $H(2n - 1, q^2)$ is $q^{n+1}$ [29] and there are examples of partial spreads of that size [18]. If $n$ is even the situation is less clear: upper bounds can be found in [15], as for lower bounds there is a partial spread of $H(2n - 1, q^2)$ of size $(3q^2 - q)/2 + 1$ for $n = 2, q > 13$, [11, p. 32] and of size $q^{n+2}$ for $n \geq 4$ [11]. Here, generalizing the partial spread of $H(3, q^2)$, we show the existence of a partial spread of $H(2n - 1, q^2)$, in the case when $n$ is even and $n/2$ is odd, of size $(3q^n - q^{n/2})/2 + 1$, see Theorem 6.3. Hence the following result holds true.

**Theorem 6.1.** If $n$ is even and $n/2$ is odd, then there exists an $n$–code of $H_{n,q^2}$ of size $\frac{3q^n - q^{n/2}}{2}$.

For small values of $d$, $q$ and $n$, in [24] there are several $d$–codes of $S_{n,q}$ and $H_{n,q^2}$ whose sizes are larger than the corresponding additive bounds, namely a 2–code of $S_{3,2}$ of size 22, a 2–code of $S_{3,3}$ of size 135, a 2–code of $S_{3,4}$ of size 428, a 2–code of $S_{3,5}$ of size 934, a 2–code of $S_{3,7}$ of size 3100, a 2–code of $S_{4,2}$ of size 320, a 4–code of $S_{5,2}$ of size 96, a 2–code of $H_{3,4}$ of size 120 and a 4–code of $H_{4,4}$ of size 37. Besides these few examples, no $d$–codes whose sizes are larger than the largest possible additive $d$–codes are known.

In the remaining part of this section we focus on the case $n = 3$. From Corollary 5.3 a 2–code $C$ of $S_{3,q}$ has size at most

$$\frac{q(q^2 + 1)(q^2 + q + 1)}{2} + 1.$$ 

First, we improve on the upper bound of the size of $C$. Then we construct 2–codes of $S_{3,q}$ and $H_{3,q^2}$ that are larger than the largest possible additive 2–codes. This provides an answer to a question posed in [25, Section 7], see also [23, p. 176]. The main results are summarized in the following theorem.

**Theorem 6.2.** Let $C$ be a maximum 2–code of $S_{3,q}$, $q > 2$, then

$$q^4 + q^3 + 1 \leq |C| < q(q^2 - 1)(q^2 + q + 1) + 1.$$ 

Let $C$ be a maximum 2–code of $H_{3,q^2}$, then

$$q^6 + q(q - 1)(q^4 + q^2 + 1) \leq |C| \leq q^4(q^4 - q^3 + q^2 - q + 1).$$
6.1 Partial spread of \( \mathcal{H}(8m - 5, q^2) \)

Let us consider the projective line \( \text{PG}(1, q^{4m-2}) \) whose underlying vector space is \( V(2, q^{4m-2}) \), and let \( \mathcal{H}(1, q^{4m-2}) \) be a non-degenerate Hermitian polar space of \( \text{PG}(1, q^{4m-2}) \) associated with \( h \), where \( h \) is a sesquilinear form on \( V(2, q^{4m-2}) \). The vector space \( V(2, q^{4m-2}) \) can be regarded as a \((4m-2)\)-dimensional vector space over \( \text{GF}(q^2) \), say \( \tilde{V} \). More precisely

\[
\tilde{V} = \left\{ \left( x, x^q, \ldots, x^{q^{4m-4}}, y, y^q, \ldots, y^{q^{4m-4}} \right) \mid (x, y) \in V(2, q^{4m-2}) \right\}.
\]

Let \( \text{PG}(4m - 3, q^2) \) be the projective space whose underlying vector space is \( \tilde{V} \) and let

\[
Tr_{q^{4m-2}|q^2} : x \in \text{GF}(q^{4m-2}) \mapsto \sum_{i=0}^{2m-2} x^{q^{2i}} \in \text{GF}(q^2)
\]

denote the usual trace function. Note that

\[
\tilde{h} = Tr_{q^{4m-2}|q^2} \circ h : \tilde{V} \times \tilde{V} \longrightarrow \text{GF}(q^2)
\]

is a non-degenerate sesquilinear form on \( \tilde{V} \) and hence there is a non-degenerate polar space \( \mathcal{H}(4m - 3, q^2) \) of \( \text{PG}(4m - 3, q^2) \) associated with \( \tilde{h} \). See [2] for more details. Let \( \rho \) be the unitary polarity of \( \text{PG}(4m - 3, q^2) \) defining \( \mathcal{H}(4m - 3, q^2) \).

**Lemma 6.3.** There exists a \((2m - 2)\)-spread \( S \) of \( \text{PG}(4m - 3, q^2) \), such that \( q^{2m-1} + 1 \) members of \( S \) are generators of \( \mathcal{H}(4m - 3, q^2) \) and the remaining \( q^{4m-2} - q^{2m-1} \) are such that they occur in \( (q^{4m-2} - q^{2m-1})/2 \) pairs of type \( \{\Delta, \Delta^\rho\} \), where \( |\Delta \cap \Delta^\rho| = 0 \).

**Proof.** With the notation introduced above, if \( W \) is a vector subspace of \((V(2, q^{4m-2})\) of dimension one, then

\[
\left\{ \left( x, x^q, \ldots, x^{q^{4m-4}}, y, y^q, \ldots, y^{q^{4m-4}} \right) \mid (x, y) \in W \right\}
\]

is a \((2m - 1)\)-dimensional vector subspace of \( \tilde{V} \). Hence a point of \( \text{PG}(1, q^{4m-2}) \) is sent to a \((2m - 2)\)-space of \( \text{PG}(4m - 3, q^2) \) and two distinct \((2m - 2)\)-spaces of \( \text{PG}(4m - 3, q^2) \) so obtained are pairwise skew. Let \( S \) be the set of \((2m - 2)\)-spaces of \( \text{PG}(4m - 3, q^2) \) constructed in this way. Then \( S \) is a \((2m - 2)\)-spread of \( \text{PG}(4m - 3, q^2) \) and \( |S| = q^{4m-2} + 1 \).

Note that \( \mathcal{H}(1, q^{4m-2}) \) consists of \( q^{2m-1} + 1 \) points. The polarity of \( \text{PG}(1, q^{4m-2}) \) defining \( \mathcal{H}(1, q^{4m-2}) \) fixes each of these \( q^{2m-1} + 1 \) points and interchanges in pairs the remaining \( q^{4m-2} - q^{2m-1} \) points of \( \text{PG}(1, q^{4m-2}) \). Therefore an element \( \Delta \) of \( S \) is a generator of \( \mathcal{H}(4m - 3, q^2) \) if \( \Delta \) corresponds to a point of \( \mathcal{H}(1, q^{4m-2}) \); otherwise \( |\Delta \cap \Delta^\rho| = 0 \).

Let \( \mathcal{H}(8m - 5, q^2) \) be a non-degenerate Hermitian polar space of \( \text{PG}(8m - 5, q^2) \) and let \( \perp \) be the unitary polarity of \( \text{PG}(8m - 5, q^2) \) defining \( \mathcal{H}(8m - 5, q^2) \).

**Theorem 6.4.** \( \mathcal{H}(8m - 5, q^2) \) has a partial spread of size \( \frac{3q^{4m-2} - q^{2m-1}}{2} + 1 \).

**Proof.** Let \( \Pi_1, \Pi_2, \Pi_3 \) be three pairwise disjoint generators of \( \mathcal{H}(8m - 5, q^2) \). Then \( \Pi_i \simeq \text{PG}(4m - 3, q^2) \). Moreover, there is a non-degenerate unitary polarity \( \rho_i \) of \( \Pi_i \), see Lemma[4,7]. Let \( \mathcal{H}_i \) be the
Hermitian polar space of $\Pi_1$, defined by $\rho_i$, $i = 1, 2, 3$. From Lemma 6.13 there exists a $(2m-2)$–spread $S$ of $\Pi_1$, such that $q^{2m-1} + 1$ members of $S$ are generators of $H_1$ and the remaining $q^{4m-2} - q^{2m-1}$ are such that they occur in $(q^{4m-2} - q^{2m-1})/2$ pairs of type $\{\Delta, \Delta^\rho\}$, where $|\Delta \cap \Delta^\rho| = 0$. For an element $\Delta_1$ of $S$, let $\Delta_2 = \langle \Pi_3, \Delta_1 \rangle \cap \Pi_2$ and $\Delta_3 = \langle \Pi_2, \Delta_1 \rangle \cap \Pi_3$. Then $\Delta_2^\rho = \Delta_2^\perp \cap \Pi_3 = \Delta_2^\perp \cap \Pi_2$ and $\Delta_3^\rho = \Delta_3^\perp \cap \Pi_3 = \Delta_3^\perp \cap \Pi_2$. Similarly, $\Delta_1^\rho = \Delta_1^\perp \cap \Pi_1 = \Delta_1^\perp \cap \Pi_1$.

If $\Delta_1$ is a generator of $H_1$, then $\Delta_1^\rho = \Delta_i$, $i = 1, 2, 3$, and $\langle \Delta_1, \Delta_2 \rangle$ is a generator of $H(8m - 5, q^2)$. Varying $\Delta_1$ among the $q^{2m-1} + 1$ members of $S$ that are generators of $H_1$, one obtains a set $Z_1$ of $q^{2m-1} + 1$ generators of $H(8m - 5, q^2)$ that are pairwise disjoint.

If $\Delta_1$ is such that $|\Delta_1 \cap \Delta_2^\rho| = 0$, then consider the following generators of $H(8m - 5, q^2)$:

$$\langle \Delta_1, \Delta_2^\rho \rangle, \langle \Delta_2^\rho, \Delta_3 \rangle, \langle \Delta_3, \Delta_1^\rho \rangle, \langle \Delta_1^\rho, \Delta_2 \rangle, \langle \Delta_2, \Delta_3^\rho \rangle, \langle \Delta_3^\rho, \Delta_1 \rangle \rangle.$$

Among these six generators, we can always choose three of them such that they are pairwise disjoint. For instance

$$\langle \Delta_1, \Delta_2^\rho \rangle, \langle \Delta_3, \Delta_1^\rho \rangle, \langle \Delta_2, \Delta_3^\rho \rangle$$

are pairwise skew since any two of them span the whole $PG(8m - 5, q^2)$. Repeating this process for each of the $(q^{4m-2} - q^{2m-1})/2$ couples $\{\Delta_1, \Delta_2^\rho\}$ such that $|\Delta_1 \cap \Delta_2^\rho| = 0$, a set $Z_2$ of $3(q^{4m-2} - q^{2m-1})/2$ pairwise disjoint generators of $H(8m - 5, q^2)$ is obtained. Again two members of $Z_1 \cup Z_2$ span the whole $PG(8m - 5, q^2)$. Therefore $Z_1 \cup Z_2$ is a partial spread of $H(8m - 5, q^2)$ of size $3(q^{4m-2} - q^{2m-1})/2 + q^{2m-1} + 1 = (3q^{4m-2} - q^{2m-1})/2 + 1$. 

### 6.2 2–codes of $S_{3,q}$

Let $\perp$ be the symplectic polarity of $PG(5, q)$ defining $W(5, q)$. Recall that $G$ is the set of $q^6$ planes of $W(5, q)$ that are disjoint from $\Pi_1$, the group $G$ is the stabilizer of $\Pi_1$ in $PSp(6, q)$, $\Pi_2 = L(03)$, and $G_{\Pi_2}$ is the stabilizer of $\Pi_2$ in $G$. For a point $P$ in $\Pi_2$, let $\Sigma_P$ denote a 3–space contained in $P^\perp$ and not containing $P$. When restricted to $\Sigma_P$, the polarity $\perp$ defines a non–degenerate symplectic polar space of $\Sigma_P$, say $W_P$. Moreover $r_P = \Sigma_P \cap \Pi_1$ and $t_P = \Sigma_P \cap \Pi_2$ are lines of $W_P$.

#### 6.2.1 The upper bound

The graph $\Gamma_W$ has valency $q(q^2 + q + 1)$. Let $\Gamma_W'$ be the induced subgraph of $\Gamma_W$ on $G$. The graph $\Gamma_W'$ is also known as the second subconstituent of $\Gamma_W$, see [1]. Then $\Gamma_W'$ is connected, has valency $q^3 - 1$ and it is vertex–transitive, since $G \leq Aut(\Gamma_W')$ is transitive on $G$. A 2–code of $S_{3,q}$ is a coclique of $\Gamma_W'$. We want to apply the Cvetković bound (Lemma 2.2) to the graph $\Gamma_W'$. In order to do that we need to compute the spectrum of $\Gamma_W'$. Consider the equitable partition arising from the action of the group $G_{\Pi_2}$ on $G$. Then, according to Lemma 3.11, the set $G$ is partitioned into $\{\Pi_2\}, G_1, G_2, G_3, G_4\}$ if $q$ is even or into $\{\Pi_2\}, G_1, G_2, G_3, G_4, G_5, G_6\}$ if $q$ is odd. Let $B = (b_{ij})$ denote the quotient matrix of this equitable partition. In other words, $b_{ij}$ is the number of planes of $G_j$ intersecting a given plane of $G_i$ in a line.
Lemma 6.5. If \( q \) is even, then
\[
B = \begin{pmatrix}
0 & q^3 - 1 & 0 & 0 & 0 \\
1 & q - 2 & q^3 - q & 0 & 0 \\
0 & 0 & 0 & q^2 - 1 & q^2(q - 1) \\
0 & q & 1 & q^2 - q - 2 & q^2(q - 1) \\
0 & 0 & 1 & q^2 - 1 & q^3 - q^2 - 1
\end{pmatrix}.
\]

If \( q \) is odd, then
\[
B = \begin{pmatrix}
0 & \frac{q^3 - 1}{2} & \frac{q^3 - 1}{2} & 0 & 0 & 0 \\
1 & \frac{q - 3}{2} & \frac{q^3 - q}{2} & \frac{q^3 - q}{2} & 0 & 0 \\
1 & \frac{q - 3}{2} & \frac{q^3 - q}{2} & \frac{q^3 - q}{2} & 0 & 0 \\
0 & \frac{q + 1}{2} & \frac{q^3 - q}{2} & \frac{q^2(q - 1)}{2} & \frac{q^2(q - 1)}{2} & \frac{q^2(q - 1)}{2} \\
0 & \frac{q - 1}{2} & \frac{q + 1}{2} & \frac{(q - 3)(q + 1)}{2} & \frac{q^2(q - 1)}{2} & \frac{q^2(q - 1)}{2} \\
0 & 0 & 0 & \frac{q(q - 1)}{2} & \frac{q(q + 1)}{2} & \frac{q^2(q - 1)}{2}
\end{pmatrix}.
\]

Proof. Let \( q \) be even. Every plane of \( G_1 \) meets \( \Pi_2 \) in a line, no plane of \( \cup_{i=2}^{4} G_i \) meets \( \Pi_2 \) in a line and no plane of \( G_1 \) meets a plane of \( G_4 \) in a line. Hence \( b_{12} = q^3 - 1 \), \( b_{21} = 1 \), \( b_{25} = b_{52} = b_{1j} = b_1 = 0 \), \( j \neq 2 \), \( i \neq 2 \).

Let \( \sigma \in G_1 \) and let \( \gamma \in \cup_{i=1}^{3} G_i \) such that \( \gamma \cap \sigma \) is a line. If \( \gamma \in G_1 \), then \( \sigma \cap \Pi_2 = \gamma \cap \Pi_2 \). Hence \( b_{22} = q - 2 \). From Lemma 3.11 \( \gamma \notin G_2 \) and hence \( b_{23} = 0 \). If \( \gamma \in G_3 \), let \( P \) be the point \( \gamma \cap \Pi_2 \). Then \( P \in \sigma \cap \Pi_2 \). Let \( \ell = \sigma \cap \Sigma P \) and \( s = \gamma \cap \Sigma P \). Then \( |\ell \cap t_P| = 1 \) and \( |\ell \cap r_P| = 0 \). On the other hand the line \( s \) is skew to \( r_P \) and \( t_P \) and meets \( \ell \) in a point. Since \( s \) can be chosen in \( q^2 - q \) ways, we have that there are \( q^2 - q \) planes of \( G_3 \) in \( P \) meeting \( \sigma \) in a line. Varying \( P \) in \( \sigma \cap \Pi_2 \), we obtain \( b_{24} = q^3 - q \).

Let \( \sigma \in G_2 \cup G_3 \), \( P = \sigma \cap \Pi_2 \), and let \( \gamma \in \cup_{i=1}^{4} G_i \) such that \( \gamma \cap \sigma \) is a line. Note that necessarily \( P \in \gamma \). By Lemma 3.11 there is no plane of \( G_2 \) meeting a plane of \( G_1 \) in a line and if \( \sigma \in G_3 \), then there are \( q \) planes of \( G_1 \) meeting \( \sigma \) in a line. Hence \( b_{32} = 0 \) and \( b_{34} = q \). From Lemma 3.12 it follows that \( b_{33} = 0 \), \( b_{34} = q^2 - 1 \), \( b_{43} = 1 \) and \( b_{44} = q^2 - q - 2 \). Through a line of \( \sigma \) not containing \( P \), there pass exactly \( q - 1 \) planes of \( G_4 \). Therefore \( b_{45} = q^2(q - 1) \).

Let \( \sigma \in G_4 \) and let \( \gamma \in \cup_{i=1}^{4} G_i \) such that \( \gamma \cap \sigma \) is a line. If \( \gamma \in G_2 \) or \( G_3 \), then \( \sigma \cap \gamma \in \Sigma_0 \) or \( L_0 \). Also, through a line of \( L_0 \) or \( L_2 \) there pass one plane of \( G_2 \) or \( G_3 \) and \( q - 1 \) planes of \( G_4 \), whereas through a line of \( L_1 \) there are \( q \) planes of \( G_4 \). Since \( \sigma \) contains one line of \( L_0 \), \( q + 1 \) lines of \( L_1 \) and \( q^2 - 1 \) lines of \( L_2 \), we have \( b_{53} = 1 \), \( b_{54} = q^2 - 1 \) and \( b_{55} = q^3 - q^2 - 1 \).

Let \( q \) be odd. Every plane of \( G_1 \cup G_2 \) meets \( \Pi_2 \) in a line, no plane of \( \cup_{i=3}^{6} G_i \) meets \( \Pi_2 \) in a line and no plane of \( G_1 \cup G_2 \) meets a plane of \( G_5 \cup G_6 \) in a line. Hence \( b_{12} = b_{13} = (q^3 - 1)/2 \), \( b_{21} = b_{31} = 1 \), \( b_{26} = b_{27} = b_{36} = b_{37} = b_{62} = b_{72} = b_{63} = b_{73} = b_{1j} = b_1 = 0 \), \( j \neq 2, 3 \), \( i \neq 2, 3 \).

Let \( \sigma \in G_1 \cup G_2 \) and let \( \gamma \in \cup_{i=1}^{4} G_i \) such that \( \gamma \cap \sigma \) is a line. If \( \gamma \in G_1 \cup G_2 \), then \( \sigma \cap \Pi_2 = \gamma \cap \Pi_2 \). Hence \( b_{22} = b_{33} = (q - 3)/2 \) and \( b_{23} = b_{32} = (q - 1)/2 \). If \( \gamma \in G_3 \cup G_4 \), let \( P \) be the point \( \gamma \cap \Pi_2 \). Then \( P \in \sigma \cap \Pi_2 \). The lines \( r_P \), \( t_P \), \( s = \Sigma P \cap \gamma \) are three pairwise disjoint lines of \( W_P \). Let \( R \) be the regulus consisting of the lines of \( W_P \) intersecting both \( r_P \) and \( t_P \). The points covered by the lines of \( R \) form a hyperbolic quadric \( Q^+(3,q) \). The line \( s \) is external or secant to \( Q^+(3,q) \).
according as $\gamma \in G_3$ or $G_4$. Moreover the line $\ell = \sigma \cap \Sigma_P$ is tangent to $Q^+(3, q)$ at the point $\ell \cap t_P$. For a point $L \in \ell \setminus t_P$, the plane $L^+ \cap \Sigma_P$ meets $Q^+(3, q)$ in a non-degenerate conic $C$; in this plane, through the point $L$ there are $(q - 1)/2$ lines of $W_P$ external to $C$ and skew to both $r_P$, $t_P$ and $(q - 1)/2$ lines of $W_P$ secant to $C$ and skew to both $r_P$, $t_P$. By varying the point $P$ over the line $\sigma \cap \Pi_2$, we get $b_{24} = b_{25} = b_{34} = b_{35} = (q^3 - q)/2$.

Let $\sigma \in G_3 \cup G_4$, $P = \sigma \cap \Pi_2$, and let $\gamma \in \cup_{i=1}^6 G_i$ such that $\gamma \cap \sigma$ is a line. Note that necessarily $P \in \gamma$. As before, let $R$ be the regulus consisting of the lines of $W_P$ intersecting both $r_P$ and $t_P$ and denote by $Q^+(3, q)$ the corresponding hyperbolic quadric. Let $\ell$ be the line $\sigma \cap \Sigma_P$ and $s = \gamma \cap \Sigma_P$. The line $\ell$ is skew to $r_P, t_P$ and it is external or secant to $Q^+(3, q)$ according as $\sigma \in G_3$ or $G_4$. If $\gamma \in G_1 \cup G_2$, then $s$ is a line of $W_P$ meeting both $t_P$ and $\ell$, and it is disjoint from $r_P$. Also $s \cap \ell$ belongs to $P_1$ or $P_2$, according as $\gamma \in G_1$ or $G_2$, respectively. From the proof of Lemma 3.13 we have that $|\ell \cap P_1| = |\ell \cap P_2| = (q - 1)/2$ if $\ell$ is secant and $|\ell \cap P_1| = |\ell \cap P_2| = (q + 1)/2$ if $\ell$ is external. Hence $b_{42} = b_{43} = (q + 1)/2$ and $b_{52} = b_{53} = (q - 1)/2$. If $\gamma \in G_3 \cup G_4$, then $s$ is a line of $W_P$ intersecting $\ell$ and disjoint from $r_P$ and $t_P$. Also $|s \cap Q^+(3, q)|$ equals 0 or 2 (i.e., $\ell$ belongs to $L_0$ or $L_2$, respectively), according as $\gamma \in G_1$ or $G_3$. If $\ell$ is external, through a point of $\ell$, there are $(q - 1)/2$ lines of $W_P$ secant to $Q^+(3, q)$ and skew to $r_P$ and $t_P$ and $(q - 3)/2$ lines of $W_P$ external to $Q^+(3, q)$ distinct from $\ell$ and skew to $r_P$ and $t_P$. Hence $b_{44} = (q - 3)(q + 1)/2$ and $b_{45} = (q^2 - 1)/2$. If $\ell$ is secant, through a point of $\ell$ not on $Q^+(3, q)$, there are $(q - 3)/2$ lines of $W_P$ secant to $Q^+(3, q)$ distinct from $\ell$ and skew to $r_P$ and $t_P$ and $(q - 1)/2$ lines of $W_P$ external to $Q^+(3, q)$ and skew to $r_P$ and $t_P$; through a point of $\ell \cap Q^+(3, q)$, there are $q - 1$ lines of $W_P$ secant to $Q^+(3, q)$ distinct from $\ell$ and skew to $r_P$ and $t_P$. Hence $b_{54} = (q - 1)^2/2$ and $b_{55} = (q^2 - 1)/2$. A line of $\sigma$ not containing $P$ belongs to $L_0$ or $L_2$ and, by Lemma 3.13 through such a line there pass exactly $(q - 1)/2$ planes of $G_5$ and $(q - 1)/2$ planes of $G_6$. Therefore $b_{46} = b_{47} = b_{56} = b_{57} = q^2(q - 1)/2$.

Let $\sigma \in G_5 \cup G_6$ and let $\gamma \in \cup_{i=3}^6 G_i$ such that $\gamma \cap \sigma$ is a line. If $\gamma \in G_3$ or $G_4$, then $\sigma \cap \gamma \in L_0$ or $L_2$, respectively. Also, through a line of $L_0$ or $L_2$ there pass one plane of $G_3$ or $G_4$, $(q - 1)/2$ planes of $G_5$ and $(q - 1)/2$ planes of $G_6$, see Lemma 3.13. Moreover, by Lemma 3.13 through a line of $L_1$ there are $q$ planes disjoint from $\Pi_1$ and $\Pi_2$ and they belong either to $G_5$ or to $G_6$ according as $\sigma \in G_5$ or $\sigma \in G_6$, respectively. Since, by Corollary 3.1, $\sigma$ contains $q(q - 1)/2$ lines of $L_0$, $q + 1$ lines of $L_1$ and $q(q + 1)/2$ lines of $L_2$, it follows that $b_{64} = b_{74} = q(q - 1)/2$, $b_{65} = b_{75} = q(q + 1)/2$, $b_{66} = b_{77} = q^2(q - 1)/2 - 1$ and $b_{67} = b_{76} = q^2(q - 1)/2$.

**Theorem 6.6.** The spectrum of the graph $\Gamma_W'$ is

$$(q^3 - 1)^1, (q^2 - 1)^{(2(q+1)(q^3-1))/2}, (-1)^{(q^3 - q^2 + 1)(q^3-1)}, (-q^2 - 1)^{(q(q-1)(q^3-1))/2}.$$  

**Proof.** The matrix $B$ described in Lemma 6.5 has four distinct eigenvalues, three of them are simple: $q^3 - 1$, $q^2 - 1$ and $-q^2 - 1$, whereas the multiplicity of the eigenvalue $-1$ is two or four, according as $q$ is even or odd. From Lemma 2.1 the graph $\Gamma_W'$ has four distinct eigenvalues: $q^3 - 1$, $q^2 - 1$, $-1$, $-q^2 - 1$ with multiplicities $m_0, m_1, m_2, m_3$, respectively. Note that $m_0 = 1$, since $\Gamma_W'$ is connected. Moreover, the following equations have to be satisfied (see for instance
[28] p. 142));

\[ 1 + m_1 + m_2 + m_3 = q^6, \]
\[ q^3 - 1 + (q^2 - 1)m_1 - m_2 - (q^2 + 1)m_3 = 0, \]
\[ (q^3 - 1)^2 + (q^2 - 1)^2m_1 + m_2 + (q^2 + 1)^2m_3 = q^6(q^3 - 1). \]

It follows that \( m_1 = q(q + 1)(q^3 - 1)/2, m_2 = (q^3 - 1)(q^3 - q^2 + 1), m_3 = q(q - 1)(q^3 - 1)/2. \)

By applying the Cvetković bound (Lemma 2.2), we get \( \alpha(\Gamma'_{W}) \leq \frac{q(q^2-1)(q^2+q+1)}{2} + 1. \)

**Corollary 6.7.** Let \( \mathcal{C} \) be a 2–code of \( S_{3,q} \), then \( |\mathcal{C}| \leq \frac{q(q^2-1)(q^2+q+1)}{2} + 1. \)

**Problem 6.8.** We obtained a better upper bound for a 2–code of \( S_{n,q} \) in the case \( n = 3 \), by applying the Cvetković bound to the graph \( \Gamma'_{W} \), the last subconstituent of \( \Gamma_{W} \). Determine whether or not this holds true for \( n > 3 \).

### 6.2.2 The lower bound

Here we provide the first infinite family of 2–codes of \( S_{3,q} \) whose size is larger than the largest possible additive 2–code.

**Construction 6.9.** Let \( \mathcal{F}_P \) be a line–spread of \( W_P \) containing \( r_P \) and \( t_P \) and let \( \mathcal{X}_P \) be the set of \( q^2 - 1 \) generators of \( W(5, q) \) passing through \( P \) and meeting \( \Sigma_P \) in a line of \( \mathcal{F}_P \setminus \{r_P, t_P\} \).

Define the set \( \mathcal{X} \) as follows

\[ \bigcup_{P \in \Pi_2} \mathcal{X}_P \cup \{\Pi_2\}. \]

**Theorem 6.10.** The set \( \mathcal{X} \) consists of \((q + 1)(q^3 - 1) + 1 \) planes of \( W(5, q) \) disjoint from \( \Pi_1 \) and pairwise intersecting in at most one point.

**Proof.** By construction every member of \( \mathcal{X} \) distinct from \( \Pi_2 \) meets \( \Pi_2 \) in exactly one point. Let \( \sigma_1, \sigma_2 \in \mathcal{X} \setminus \{\Pi_2\} \). If \( \sigma_1 \cap \Pi_2 = \sigma_2 \cap \Pi_2 \), then \( |\sigma_1 \cap \Pi_2| = 1 \) and there is nothing to prove. Hence let \( P_i = \sigma_i \cap \Pi_2, i = 1, 2 \), with \( P_1 \neq P_2 \). Assume by contradiction that \( \sigma_1 \cap \Pi_2 \) is a line, say \( \ell \). If \( \ell \cap \Pi_2 \) is a point, say \( R \), then \( R \in \sigma_1 \cap \Pi_2 \). If \( R \neq P_1 \) then the line \( RP_1 \) would be contained in \( \sigma_1 \cap \Pi_2 \), contradicting the fact that \( |\sigma_1 \cap \Pi_2| = 1 \). Similarly if \( R \neq P_2 \), then \( RP_2 \subseteq \sigma_2 \cap \Pi_2 \), a contradiction. Therefore \( R = P_1 = P_2 \), contradicting the fact that \( P_1 \neq P_2 \). Hence \( |\ell \cap \Pi_2| = 0 \) and both \( \sigma_1, \sigma_2 \) are contained in \( \ell^\perp \). However in this case the line \( P_1P_2 \) is a line of \( W(5, q) \) contained in \( \ell^\perp \) and disjoint from \( \ell \); a contradiction. \( \square \)

**Corollary 6.11.** There exists a 2–code of \( S_{3,q} \) of size \((q^2 - 1)(q^3 + q + 1) + 1 \).

From Corollary 6.7 a 2–code of \( S_{3,2} \) has at most 22 elements and hence the 2–code of \( S_{3,2} \) obtained from Construction 6.9 is maximal; an alternative proof of its maximality will be exhibited (Corollary 6.15). Moreover, from [24], this code is the unique largest 2–code of \( S_{3,2} \) of size 22. Our next step is to show that, if \( q > 2 \), the 2–code of \( S_{3,q} \) provided in Construction 6.9 can be further enlarged. In order to do that some preliminary results are required.
Lemma 6.12. Let \( \sigma \not\in \mathcal{X} \) be a plane of \( \mathcal{W}(5,q) \) disjoint from \( \Pi_1 \) and meeting \( \Pi_2 \) in at least one point. Then there exists a plane of \( \mathcal{X} \) meeting \( \sigma \) in a line.

Proof. Let \( \sigma \not\in \mathcal{X} \) be a plane disjoint from \( \Pi_1 \) and meeting \( \Pi_2 \) in one point, say \( P \). Then \( \sigma \) meets \( \Sigma_P \) in a line, say \( s \), and there are \( q+1 \) lines of \( \mathcal{F}_P \setminus \{ r_P, t_P \} \) meeting \( s \) in one point. It follows that there are \( q+1 \) planes of \( \mathcal{X}_P \) meeting \( \sigma \) in a line. \( \square \)

Lemma 6.13. Through a point \( R \) of \( \mathcal{W}(5,q) \setminus (\Pi_1 \cup \Pi_2) \) there pass either \( q \) or \( q+1 \) planes of \( \mathcal{X} \), according as the line through \( R \) intersecting \( \Pi_1 \) and \( \Pi_2 \) is a line of \( \mathcal{W}(5,q) \) or it is not.

Proof. Let \( R \) be a point of \( \mathcal{W}(5,q) \setminus (\Pi_1 \cup \Pi_2) \), let \( \ell_R \) be the unique line through \( R \) intersecting both \( \Pi_1 \) and \( \Pi_2 \) and let \( R_i = \ell_R \cap \Pi_i, i = 1,2 \). Let \( s \) denote the line \( R_i \cap \Pi_2 \). If a plane of \( \mathcal{X}_P \) contains the point \( R \), then \( P \in s \). On the other hand for a fixed \( P \in s \) there is at most one plane of \( \mathcal{X}_P \) containing \( R \). Hence there are at most \( q+1 \) planes of \( \mathcal{X} \) through \( R \). If \( P \in s \) and \( P \neq R_2 \), then both \( R_2 \) and \( R \) are in \( P \). Hence \( \ell_R \subseteq P \), the line \( r_P = P \cap \Pi_i \) contains \( R_1 \) and \( \langle P, r_P \rangle \cap \ell_R = \{ R_1 \} \). Therefore \( R \not\in \langle P, r_P \rangle \) and there exists a plane of \( \mathcal{X}_P \) containing \( R \). On the other hand, if \( P = R_2 \), then \( R \in \langle P, r_P \rangle \). Finally note that \( R_2 \in s \) if and only if \( \ell_R \) is a line of \( \mathcal{W}(5,q) \).

Let \( \mathcal{L} \) be the set of lines of \( \mathcal{W}(5,q) \) disjoint from \( \Pi_1 \cup \Pi_2 \) contained in a plane of \( \mathcal{X} \). Then \( |\mathcal{L}| = q^2(q+1)(q^3-1) \).

Lemma 6.14. If \( q \) is even, then \( \mathcal{L} \subseteq \mathcal{L}_2 \). If \( q \) is odd, then \( |\mathcal{L} \cap \mathcal{L}_0| = |\mathcal{L} \cap \mathcal{L}_2| \).

Proof. Let \( \ell \) be a line of \( \mathcal{L} \). Thus there is a point \( P \in \Pi_2 \) and a plane \( \sigma \) of \( \mathcal{X}_P \) containing \( \ell \). The three–space \( T_\ell \) is contained in \( P \) and does not contain \( P \), otherwise \( |\sigma \cap \Pi_1| \neq 0 \). Hence \( T_\ell \cap \mathcal{W}(5,q) \) is a non–degenerate symplectic polar space \( \mathcal{W}(3,q) \) and \( |\mathcal{L} \cap \mathcal{L}_1| = 0 \). Note that \( \mathcal{D} = \{ T_\ell \cap \gamma : \gamma \in \mathcal{X}_P \} \cup \{ r_\ell, t_\ell \} \) is a line–spread of \( \mathcal{W}(3,q) \). In the point–line dual of \( \mathcal{W}(3,q) \), the line–spread \( \mathcal{D} \) is an ovoid \( \mathcal{O} \) of the parabolic quadric \( \mathcal{Q}(4,q) \) and the regulus determined by \( r_\ell, t_\ell, \ell \), would correspond to three points \( P_1, P_2, P_3 \) of a conic \( \mathcal{C} \) of \( \mathcal{Q}(4,q) \) such that \( P_1, P_2, P_3 \in \mathcal{O} \).

Assume that \( q \) is even. Then the parabolic quadric \( \mathcal{Q}(4,q) \) has a nucleus \( N \). If \( \ell \) were in \( \mathcal{L}_0 \), then \( N \not\in \langle \mathcal{C} \rangle \). Consider a three–space \( Z \) of the ambient projective space of \( \mathcal{Q}(4,q) \) such that \( N \not\in Z \). By projecting points and lines of \( \mathcal{Q}(4,q) \) from \( N \) to \( Z \), we obtain the points and lines of a non–degenerate symplectic polar space \( \mathcal{W} \) of \( Z \). In particular \( \mathcal{C}' = \{ NP \cap Z : P \in \mathcal{C} \} \) is a line of \( Z \) and \( \mathcal{O}' = \{ NP \cap Z : P \in \mathcal{O} \} \) is an ovoid of \( W \). Then we would have \( |\mathcal{C}' \cap \mathcal{O}'| \geq 3 \), a contradiction, see [20].

Assume that \( q \) is odd. The line \( \ell \) belongs to \( \mathcal{L}_2 \) if and only if the line polar to the plane \( \langle P_1, P_2, P_3 \rangle \) with respect to the polarity of \( \mathcal{Q}(4,q) \) is secant to \( \mathcal{Q}(4,q) \). Let \( \mathcal{A} \) be the conic obtained by intersecting \( \mathcal{Q}(4,q) \) with the plane polar to the line \( \langle P_1, P_2 \rangle \) with respect to the orthogonal polarity of \( \mathcal{W}(4,q) \) associated with \( \mathcal{Q}(4,q) \). Let us count the triple \( (R, S, P_3) \), where \( R, S \in \mathcal{A}, R \neq S, P_3 \in \mathcal{O} \setminus \{P_1, P_2\} \) and both \( RP_3, SP_3 \) are lines of \( \mathcal{Q}(4,q) \). The point \( R \) can be chosen in \( q+1 \) ways and for a fixed \( R \), the point \( P_3 \) can be chosen in \( q-1 \) ways. Finally once \( R \) and \( P_3 \) are fixed, the point \( S \) is uniquely determined. Hence there are \( q^2-1 \) such triples. It
turns out that there are \((q^2 - 1)/2\) points \(P_3 \in O \setminus \{P_1, P_2\}\) such that the line polar to the plane \(\langle P_1, P_2, P_3 \rangle\) with respect to the polarity of \(Q(4, q)\) is secant to \(Q(4, q)\) and \((q^2 - 1)/2\) points \(P_3 \in O \setminus \{P_1, P_2\}\) such that the line polar to the plane \(\langle P_1, P_2, P_3 \rangle\) with respect to the polarity of \(Q(4, q)\) is external to \(Q(4, q)\).

\[\square\]

**Corollary 6.15.** The 2–code of \(\mathcal{S}_{3,2}\) obtained from Construction 6.9 is maximal.

**Proof.** From Lemma 6.12 if there exists a plane \(\sigma\) disjoint from \(\Pi_1\) such that it meets every plane of \(\mathcal{X}\) in at most one point, then \(\sigma\) must be disjoint from \(\Pi_2\). From Lemma 3.4 \(\sigma\) contains exactly one line of \(\mathcal{L}_0\), 3 lines of \(\mathcal{L}_1\) and 3 lines of \(\mathcal{L}_2\). Since \(|\mathcal{L}_2| = |\mathcal{L}|\), the result follows.

Let \(\Pi_3\) be a generator of \(\mathcal{W}(5, q)\) disjoint from both \(\Pi_1\) and \(\Pi_2\). Let us denote by \(\Pi_i, 1 \leq i \leq q + 1\), the \(q + 1\) planes of the unique symplectic Segre variety of \(\mathcal{W}(5, q)\) containing \(\Pi_1, \Pi_2, \Pi_3\). In what follows we want to prove that it is possible to construct \(\mathcal{X}\) in such a way that the \(q - 1\) planes \(\Pi_i, 3 \leq i \leq q + 1\) can be added to it.

### 6.2.2.1 The even characteristic case

Assume that \(q > 2\) is even. Since the planes \(\Pi_1, \ldots, \Pi_{q+1}\) are pairwise disjoint generators of \(\mathcal{W}(5, q)\), from Lemma 3.4 there is a non–degenerate pseudo–polarity \(\rho_i\) of \(\Pi_i\). The set of absolute points of \(\rho_i\) are those of a line \(v_i\) of \(\Pi_i\). Let \(V_i = v_i^\rho_i\). Note that the unique line of \(\mathcal{L}_0\) contained in \(\Pi_i\) is \(v_i, 3 \leq i \leq q + 1\), while the \(q + 1\) lines of \(\mathcal{L}_1\) contained in \(\Pi_i\) are those through \(V_i, 3 \leq i \leq q + 1\).

Let \(Q\) be a point of \(\Pi_2\) not on \(v_2\) and distinct from \(V_2\). Let \(\Sigma_Q\) be a 3–space contained in \(Q^\perp\) and not containing \(Q\). In particular we choose \(\Sigma_Q\) spanned by the lines \(Q^- \cap \Pi_1\) and \(Q'^2\). Note that \(\Sigma_Q \cap \Pi_i = Q^\perp \cap \Pi_i\). Indeed, if \(s\) is the unique line (not of \(\mathcal{W}(5, q)\)) containing \(Q\) and meeting each of the planes \(\Pi_i, 1 \leq i \leq q + 1\), in one point, then \((s \cap \Pi_i)^\rho_i = \Sigma_Q \cap \Pi_i\). When restricted on \(\Sigma_Q\), the polarity \(\perp\) defines a non–degenerate symplectic polar space of \(\Sigma_Q\), say \(\mathcal{W}_Q\). As before, let \(r_Q = \Sigma_Q \cap \Pi_1\) and \(Q'^2 = t_Q = \Sigma_Q \cap \Pi_2\). Let \(R_Q\) be the set of \(q + 1\) lines of \(\mathcal{W}_Q\) defined as follows

\[\{\Sigma_Q \cap \Pi_i : 1 \leq i \leq q + 1\}\]

Then \(R_Q\) is a regulus of \(\mathcal{W}_Q\) containing both \(r_Q\) and \(t_Q\); the opposite regulus of \(R_Q\) contains exactly one line of \(\mathcal{W}_Q\) which is the line consisting of the points \(v_i \cap (s \cap \Pi_i)^\rho_i\).

**Lemma 6.16.** There exists a Desarguesian line–spread of \(\mathcal{W}_P\) having in common with \(R_Q\) exactly the lines \(r_P\) and \(t_P\).

**Proof.** Let \(Q(4, q)\) be the point line dual of \(\mathcal{W}_Q\) and let \(N\) be the nucleus of \(Q(4, q)\). The regulus \(R_Q\) corresponds to a conic \(C\) of \(Q(4, q)\) such that \(N \notin (C)\) and the lines \(r_P\) and \(t_P\) correspond to two points of \(C\), say \(R\) and \(T\). The result follows, since there are \(q^2/2 - q\) elliptic quadrics of \(Q(4, q)\) meeting \(C\) exactly in the points \(R, T\).

\[\square\]

For any point \(Q\) of \(\Pi_2\) different from \(V_2\) and not on \(v_2\), let \(F_Q\) be a Desarguesian line–spread of \(\mathcal{W}_Q\) having in common with \(R_Q\) exactly the lines \(r_Q\) and \(t_Q\) and let \(Y_Q\) be the set of \(q^2 - 1\)
generators of \( \mathcal{W}(5, q) \) passing through \( Q \) and meeting \( \Sigma_Q \) in a line of \( \mathcal{F}_Q \setminus \{r_Q, t_Q\} \). For any point \( P \in v_2 \cup \{V_2\} \), let \( \mathcal{X}_P \) be a set of \( q^2 - 1 \) generators of \( \mathcal{W}(5, q) \) passing through \( P \) defined as in Construction 6.9.

Define the set \( \bar{X} \) as follows
\[
\left( \bigcup_{P \in v_2 \cup \{V_2\}} \mathcal{X}_P \right) \cup \left( \bigcup_{Q \in \Pi_2 \setminus (v_2 \cup \{V_2\})} \mathcal{Y}_Q \right) \cup \left( \bigcup_{i=2}^{q+1} \Pi_i \right).
\]

**Theorem 6.17.** The set \( \bar{X} \) consists of \( q^3 + q^2 + 1 \) planes of \( \mathcal{W}(5, q) \) disjoint from \( \Pi_1 \) and pairwise intersecting in at most one point.

**Proof.** It is enough to show that a plane \( \sigma \) of
\[
\left( \bigcup_{P \in v_2 \cup \{V_2\}} \mathcal{X}_P \right) \cup \left( \bigcup_{Q \in \Pi_2 \setminus (v_2 \cup \{V_2\})} \mathcal{Y}_Q \right)
\]
meets \( \Pi_i, 3 \leq i \leq q+1 \), in at most one point. If \( \sigma \) intersects \( \Pi_i \) in a line, say \( \ell \), from Lemma 6.14 \( \ell \in L \subseteq L_2 \). Hence \( \ell \neq v_i \) and \( V_i \neq \ell \). Let \( s \) be the unique line through the point \( \ell \) meeting both \( \Pi_1 \) and \( \Pi_2 \) in one point. Let \( Q = s \cap \Pi_2 \). Then \( Q \) coincides with \( \ell^\perp \cap \Pi_2 \) and \( Q \notin v_2 \cup \{V_2\} \). Moreover \( \sigma = \langle Q, \ell \rangle \in \mathcal{Y}_Q \). But in this case the Desarguesian line–spread \( \mathcal{F}_Q \) would have the three lines \( \ell, r_Q, t_Q \) in common with \( \mathcal{R}_Q \), contradicting the fact that \( |\mathcal{F}_Q \cap \mathcal{R}_Q| = 2 \). \( \square \)

6.2.2.2 The odd characteristic case

Assume that \( q \) is odd. Since the planes \( \Pi_1, \ldots, \Pi_{q+1} \) are pairwise disjoint generators of \( \mathcal{W}(5, q) \), from Lemma 3.4 there is a non–degenerate orthogonal polarity \( \rho_i \) of \( \Pi_i \). The set of absolute points of \( \rho_i \) are those of a conic \( \alpha_i \) of \( \Pi_i \). Note that a line \( \ell \) of \( \Pi_i \) belongs to \( L_j \), according as \( |\ell \cap \alpha_i| = j, 0 \leq j \leq 2, 3 \leq i \leq q+1 \).

Let \( Q \) be a point of \( \Pi_2 \) not on \( \alpha_2 \). Let \( \Sigma_Q \) be a 3–space contained in \( Q^\perp \) and not containing \( Q \). In particular we choose \( \Sigma_Q \) spanned by the lines \( Q^\perp \cap \Pi_1 \) and \( Q^\perp \cap \Pi_i \). Indeed, if \( s \) is the unique line containing \( Q \) and meeting each of the planes \( \Pi_i, 1 \leq i \leq q + 1 \), in one point, then \( (s \cap \Pi_i)^\rho_i = \Sigma_Q \cap \Pi_i \). When restricted on \( \Sigma_Q \), the polarity \( \perp \) defines a non–degenerate symplectic polar space of \( \Sigma_Q \), say \( \mathcal{W}_Q \). As before, let \( r_Q = \Sigma_Q \cap \Pi_1 \) and \( t_Q = \Sigma_Q \cap \Pi_2 \). Let \( \mathcal{R}_Q \) be the set of \( q + 1 \) lines of \( \mathcal{W}_Q \) defined as follows
\[
\{\Sigma_Q \cap \Pi_i : 1 \leq i \leq q + 1\}.
\]

Then \( \mathcal{R}_Q \) is a regulus of \( \mathcal{W}_Q \) containing both \( r_Q \) and \( t_Q \) and the opposite regulus of \( \mathcal{R}_Q \) contains exactly 0 or 2 lines of \( \mathcal{W}_Q \). The proof of the next result is left to the reader.

**Lemma 6.18.** There exists a Desarguesian line–spread of \( \mathcal{W}_P \) having in common with \( \mathcal{R}_Q \) exactly the lines \( r_P \) and \( t_P \).

For any point \( Q \) of \( \Pi_2 \) not on \( \alpha_2 \), let \( \mathcal{F}_Q \) be a Desarguesian line–spread of \( \mathcal{W}_Q \) having in common with \( \mathcal{R}_Q \) exactly the lines \( r_Q \) and \( t_Q \) and let \( \mathcal{Y}_Q \) be the set of \( q^2 - 1 \) generators of
\( \mathcal{W}(5, q) \) passing through \( Q \) and meeting \( \Sigma_Q \) in a line of \( \mathcal{F}_Q \setminus \{r_Q, t_Q\} \). For any point \( P \in \alpha_2 \), let \( \mathcal{X}_P \) be a set of \( q^2 - 1 \) generators of \( \mathcal{W}(5, q) \) passing through \( P \) defined as in Construction 6.9.

Define the set \( \bar{\mathcal{X}} \) as follows
\[
\left( \bigcup_{P \in \alpha_2} \mathcal{X}_P \right) \cup \left( \bigcup_{Q \in \Pi_2 \setminus \alpha_2} \mathcal{Y}_Q \right) \cup \left( \bigcup_{i=2}^{q+1} \Pi_i \right).
\]

A proof similar to that given in the even characteristic case yields the following result.

**Theorem 6.19.** The set \( \bar{\mathcal{X}} \) consists of \( q^4 + q^3 + 1 \) planes of \( \mathcal{W}(5, q) \) disjoint from \( \Pi_1 \) and pairwise intersecting in at most one point.

### 6.3 2–codes of \( \mathcal{H}_{3, q^2} \)

Let \( \perp \) be the Hermitian polarity of \( \text{PG}(5, q^2) \) defining \( \mathcal{H}(5, q^2) \). Recall that \( \mathcal{G} \) is the set of \( q^9 \) planes of \( \mathcal{H}(5, q^2) \) that are disjoint from \( \Lambda_1 \), the group \( \overline{G} \) is the stabilizer of \( \Lambda_1 \) in \( \text{PGU}(6, q^2) \), \( \Lambda_2 = L(0_3) \), and \( G_{\Lambda_2} \) is the stabilizer of \( \Lambda_2 \) in \( \overline{G} \). For a point \( P \) in \( \Lambda_2 \), let \( \overline{\Sigma}_P \) denote a 3–space contained in \( P^{\perp} \) and not containing \( P \). When restricted to \( \overline{\Sigma}_P \), the polarity \( \perp \) defines a non–degenerate Hermitian polar space of \( \overline{\Sigma}_P \), say \( \mathcal{H}_P \). Moreover \( \overline{r}_P = \overline{\Sigma}_P \cap \Lambda_1 \) and \( \overline{t}_P = \overline{\Sigma}_P \cap \Lambda_2 \) are lines of \( \mathcal{H}_P \).

**Construction 6.20.** Let \( \overline{\mathcal{F}}_P \) be a partial spread of \( \mathcal{H}_P \) containing \( \overline{r}_P \) and \( \overline{t}_P \) and let \( \mathcal{Y}_P \) be the set of \( |\overline{\mathcal{F}}_P| - 2 \) generators of \( \mathcal{H}(5, q^2) \) passing through \( P \) and meeting \( \overline{\Sigma}_P \) in a line of \( \overline{\mathcal{F}}_P \setminus \{\overline{r}_P, \overline{t}_P\} \). Define the set \( \mathcal{Y} \) as follows
\[
\bigcup_{P \in \Lambda_2} \mathcal{Y}_P \cup \{\Lambda_2\}.
\]

A proof similar to that given in the symmetric case gives:

**Theorem 6.21.** The set \( \mathcal{Y} \) consists of \((q^4 + q^3 + 1)(|\mathcal{F}| - 2)\) planes of \( \mathcal{H}(5, q^2) \) disjoint from \( \Lambda_1 \) and pairwise intersecting in at most one point.

By selecting \( \mathcal{F} \) as a partial spread of \( \mathcal{H}(3, q^2) \) of size \( (3q^2 - q + 2)/2 \), see [1, p. 32], the following arises.

**Corollary 6.22.** There exists a 2–code of \( \mathcal{H}_{3, q^2} \) of size \( q^6 + \frac{q(q-1)(q^4+q^2+1)}{2} \).

**Acknowledgments.** This work was supported by the Italian National Group for Algebraic and Geometric Structures and their Applications (GNSAGA–INdAM).

**References**

[1] A. Aguglia, A. Cossidente, G.L. Ebert, On pairs of permutable Hermitian surfaces, *Discrete Math.*, 301 (2005), 28–33.

[2] A.E. Brouwer, A.M Cohen, A. Neumaier, *Distance–regular graphs*, Springer-Verlag, Berlin, 1989.
[3] A.E. Brouwer, W.H. Haemers, *Spectra of graphs*, Universitext. Springer, New York, 2012.

[4] S.M. Cioabă, J.H. Koolen, On the connectedness of the complement of a ball in distance-regular graphs, *J. Algebraic Combin.*, 38 (2013), 191–195.

[5] D.M. Cvetković, Graphs and their spectra, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, 354-356 (1971), 1–50.

[6] J.-G. Dumas, R. Gow, J. Sheekey, Rank properties of subspaces of symmetric and Hermitian matrices over finite fields, *Finite Fields Appl.*, 17 (2011), 504–520.

[7] É.M. Gabidulin, N.I. Pilipchuk, Symmetric rank codes, *Probl. Inf. Transm.*, 40 (2004), 103–117.

[8] É.M. Gabidulin, N.I. Pilipchuk, Symmetric matrices and codes correcting rank errors beyond the $\lfloor (d - 1)/2 \rfloor$ bound, *Discret. Appl. Math.*, 154 (2006), 305–312.

[9] N. Gill, Polar spaces and embeddings of classical groups, *New Zealand J. Math.*, 36 (2007), 175–184.

[10] C. Godsil, G. Royle, *Algebraic graph theory*, Graduate Texts in Mathematics, 207. Springer-Verlag, New York, 2001.

[11] R. Gow, M. Lavrauw, J. Sheekey, F. Vanhove, Constant rank–distance sets of Hermitian matrices and partial spreads in Hermitian polar spaces, *Electron. J. Combin.*, 21(1), (Paper 1.26) (2014).

[12] J.W.P. Hirschfeld, *Projective Geometries over Finite Fields*, Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1998.

[13] J.W.P. Hirschfeld, J.A. Thas, *General Galois geometries*, Springer Monographs in Mathematics, Springer, London, 2016.

[14] Y. Huo, Z. Wan, Non–symmetric association schemes of symmetric matrices, *Acta Mathematicae Applicatae Sinica*, 9 (1993), 236–255.

[15] F. Ihringer, P. Sin, Q. Xiang, New bounds for partial spreads of $\mathcal{H}(2d - 1, q^2)$ and partial ovoids of the Ree–Tits octagon, *J. Combin. Theory Ser. A*, 153 (2018), 46–53.

[16] O.H. King, Imprimitive maximal subgroups of the symplectic, orthogonal and unitary groups, *Geom. Dedicata*, 15 (1984), 339–353.

[17] A. Klein, K. Metsch, L. Storme, Small maximal partial spreads in classical finite polar spaces, *Adv. Geom.*, 10 (2010), 379–402.

[18] D. Luyckx, On maximal partial spreads of $\mathcal{H}(2n + 1, q^2)$, *Discrete Math.*, 308 (2008), no. 2-3, 375–379.
[19] J. MacWilliams, Orthogonal matrices over finite fields, *Amer. Math. Monthly*, 76 (1969), no. 2, 152–164.

[20] K.-U. Schmidt, Symmetric bilinear forms over finite fields of even characteristic, *J. Combin. Theory Ser. A*, 117 (2010), no. 8, 1011–1026.

[21] K.-U. Schmidt, Symmetric bilinear forms over finite fields with applications to coding theory, *J. Algebraic Combin.*, 42 (2015), no. 2, 635–670.

[22] K.-U. Schmidt, Hermitian rank distance codes, *Des. Codes Cryptogr.*, 86 (2018), 1469–1481.

[23] K.-U. Schmidt, Quadratic and symmetric bilinear forms over finite fields and their association schemes, *Algebraic Combinatorics*, 3 (2020), 161–189.

[24] M. Schmidt, Rank metric codes. *Masters thesis*, University of Bayreuth, Germany, 2016.

[25] J. Sheekey, MRD Codes: Constructions and Connections, *Combinatorics and Finite Fields: Difference Sets, Polynomials, Pseudorandomness and Applications*, Ed. by Schmidt, K.-U. and Winterhof, A., Series: Radon Series on Computational and Applied Mathematics 23, De Gruyter 2019.

[26] J.A. Thas, Ovoidal translation planes, *Arch. Math. (Basel)*, 23 (1972), 110–112.

[27] J.A. Thas, Old and new results on spreads and ovoids of finite classical polar spaces, *Combinatorics ’90* (Gaeta, 1990), 529–544, *Ann. Discrete Math.*, 52, North-Holland, Amsterdam, 1992.

[28] E.R. van Dam, Regular graphs with four eigenvalues, *Linear Algebra Appl.*, 226/228 (1995), 139–162.

[29] F. Vanhove, The maximum size of a partial spread in $\mathcal{H}(4n + 1, q^2)$ is $q^{2n+1} + 1$, *Electron. J. Combin.*, 16 (1) (2009), Note 13, 6.

[30] Z.-X. Wan, *Geometry of matrices*, World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[31] Y. Wang, M. Jianmin, Association schemes of symmetric matrices over a finite field of characteristic two, *Journal of Statistical Planning and Inference*, 51 (1996), no. 3, 351–371.