Partial Domination in Prisms of Graphs

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Abstract

For any graph $G = (V, E)$ and proportion $p \in (0, 1]$, a set $S \subseteq V$ is a $p$-dominating set if $|N[S]| \geq p|V|$. The $p$-domination number $\gamma_p(G)$ equals the minimum cardinality of a $p$-dominating set in $G$. For a permutation $\pi$ of the vertex set of $G$, the graph $\pi G$ is obtained from two disjoint copies $G_1$ and $G_2$ of $G$ by joining each $v$ in $G_1$ to $\pi(v)$ in $G_2$, i.e., $V(\pi G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{|v, \pi(v)| : v \in V(G_1), \pi(v) \in V(G_2)\}$. The graph $\pi G$ is called the prism of $G$ with respect to $\pi$. In this paper, we find some relations between the domination and the $p$-domination numbers in the context of graph and its prism graph for particular values of $p$.

Keywords/Phrases: Permutation graph, algebraic graph theory, prism graph

1 Introduction

The concept of prisms of graphs was first introduced by Chartrand and Harary \cite{1} in 1967. They used the term permutation graphs to define such graphs; but their definition was different from the one we have for permutation graphs as defined in \cite{2}. Later those graphs were named as prisms of graphs with respect to a permutation. Prisms of graphs play a great role in designing computer networks.

Partial domination \cite{3,4} in graphs is a variation of domination introduced in 2017. In \cite{5}, we see some algebraic properties of the partial dominating sets of a graph. Here, in this paper we study prism graphs in the context of partial domination.

2 Basic Terminologies

Let $G = (V(G), E(G))$ be a finite, simple and undirected graph with $V(G)$ as its vertex set and $E(G)$ as its edge set. For any $v \in V(G)$, $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$ denote the open and the closed neighborhoods of $v$ respectively. A set $S \subseteq V(G)$ is called a dominating set if every vertex in $V - S$ is adjacent to at least one vertex in $S$. The minimum cardinality of a dominating set is called the domination number and is denoted by $\gamma(G)$.

A subset $S \subseteq V$ is called a $p$-dominating set for $p \in (0, 1]$ if $\frac{|N[S]|}{|V|} \geq p$. The $p$-domination number, denoted by $\gamma_p(G)$ is the cardinality of the minimum $p$-dominating set.

In Figure \cite{I} (a), the white vertices dominate all the vertices of the graph. In Figure \cite{I} (b), the white vertices do not dominate the vertices of the graph. However, they dominate exactly 10 vertices of the graph. Hence, we say, the white vertices $\frac{2}{3}$-dominate the graph.
Let \( \pi \) be any permutation on \( V(G) \). The prism \( \pi G \) of \( G \) with respect to \( \pi \) is obtained by taking two disjoint copies of \( G \) and joining each vertex \( v \) in one copy of \( G \) with \( \pi(v) \) in the other copy by means of an edge. The study of domination in prisms started in 2004 [6]. Since then many works related to various domination parameters were studied [7, 8].

In [9], it has been proved that for any graph \( G \), \( \gamma(G) \leq \gamma(\pi G) \leq 2\gamma(G) \). Also, it has been defined in [6] that, \( G \) is called universal \( \gamma \)-fixer, if \( \gamma(G) = \gamma(\pi G) \) for all permutations \( \pi \) of \( V(G) \) and \( G \) is called universal doubler if \( \gamma(\pi G) = 2\gamma(G) \) for all permutations \( \pi \) of \( V(G) \). Analogous to this, in the context of partial domination we give the following definition:

**Definition 2.1.** Let \( p \in [0, 1] \). \( G \) is called universal \( \gamma_p \)-fixer, if \( \gamma_p(G) = \gamma_p(\pi G) \) for all permutations \( \pi \) of \( V(G) \) and \( G \) is called universal \( \gamma_p \)-doubler if \( \gamma_p(\pi G) = 2\gamma_p(G) \) for all permutations \( \pi \) of \( V(G) \).

Figure 2 is an example of a graph \( G \) and its prism with respect to the permutation \( \pi=(234) \).

### 3 Results

**Proposition 3.1.** Let \( G \) be any \( n \)-vertex connected graph without isolated vertices and with \( \gamma=1 \). Then for any permutation \( \pi \) of \( V(G) \) and for any \( p \in (0, 1] \),

\[
\gamma_p(\pi G) = \begin{cases} 
1, & \text{for } p \in (0, \frac{n+1}{2n}] \\
2, & \text{for } p \in (\frac{n+1}{2n}, 1]
\end{cases}
\]

**Proof.** Let \( \pi \) be any permutation of \( V(G) \). Consider \( \pi G \).

Since \( \gamma(G)=1 \), \( \exists v \in V(G) \) such that \( \text{deg}(v)=n-1 \). This \( v \) dominates \( n+1 \) vertices in \( \pi G \).
In this case $S$ dominates at most

\[
\frac{|N_{\pi G}(v)|}{|V(\pi G)|} = \frac{n+1}{2n}.
\]

Thus $\gamma_p(\pi G) = 1$, for $p \in (0, \frac{n+1}{2n}]$.

Now consider $S = \{v, \pi(v)\}$ in $\pi G$.

This $S$ dominates $\pi G$ and is minimum.

Thus $\gamma_p(\pi G) = 2$, for $p \in (\frac{n+1}{2n}, 1]$.

**Corollary 3.1.** Let $G$ be any $n$-vertex connected graph without isolated vertices and with $\gamma=1$. Then $G$ is a universal $\gamma_p$-fixer for $p \in (0, \frac{n+1}{2n}]$ and is a universal $\gamma_p$-doubler for $p \in (\frac{n+1}{2n}, 1]$.

**Proposition 3.2.** Let $G$ be any $n$-vertex graph. Then for any permutation $\pi$ of $V(G)$ and for any $p \in (0, \frac{n+\gamma(G)}{2n}]$, $\gamma_p(\pi G) \leq \gamma(G)$.

*Proof.* Let $\pi$ be any permutation of $V(G)$. Consider $\pi G$.

Let $S$ be a $\gamma$-set of $G$. Then by the definition of $\pi G$, $S$ is a $n + \gamma(G)$ dominating set in $\pi G$.

Also if $p \leq q$ then $\gamma_p \leq \gamma_q$.

Hence for any $p \in (0, \frac{n+\gamma(G)}{2n}]$, $\gamma_p(\pi G) \leq \gamma(G)$.

\[
\gamma_p = \frac{n+\gamma(G)}{2n}.
\]

The following result shows that the above bound is sharp.

**Proposition 3.3.** Let $G$ be $P_n$ or $C_n$ for $n \geq 2$. Then for any permutation $\pi$ on $V(G)$, $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) = \gamma(G)$.

*Proof.* Let $\pi$ be any permutation on $V(G)$. Then by the previous proposition $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) \leq \gamma(G)$.

Hence it is enough if we prove that $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) \geq \gamma(G)$ for any permutation $\pi$ on $V(G)$. Let us assume the contradiction that $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) < \gamma(G)$ for some permutation $\pi$.

WLG let $S \subseteq V(G)$ be a $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G)$-set with $\gamma(G) - 1$ vertices. Let $G_1$ and $G_2$ denote the two copies of $G$ in $\pi G$. Then two cases may arise.

**Case(i):** All the vertices of $S$ are from either $V(G_1)$ or $V(G_2)$.

In this case $S$ dominates at most $(n-1) + (\gamma(G) - 1)$ vertices in $\pi G$.

\[
\Rightarrow \frac{|N[S]|}{2n} \leq \frac{n-1+\gamma(G)-1}{2n} = \frac{n+\gamma(G)-2}{2n} < \frac{n+\gamma(G)}{2n}.
\]

This is a contradiction to our assumption that $S \subseteq V(G)$ is a $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G)$-set. Hence the proof in this case.

**Case(ii):** Vertices of $S$ are from both $G_1$ and $G_2$.

WLG let there be $l$ and $m$ vertices from $G_1$ and $G_2$ respectively, where $l + m = \gamma(G) - 1$ by our assumption. Then we have the following:
Now, $\gamma(G) = \left\lceil \frac{n}{3} \right\rceil < n + 1$

$\implies 3\gamma(G) < n + 3$

$\implies < n + 4$  \hspace{1cm} (2)

Hence from equations (1) and (2) we will get a contradiction to our assumption. Hence the proof.

**Remark 3.1.** For any graph $G$, $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) = 1$ if and only if $\gamma(G) = 1$.

**Proposition 3.4.** Let $G$ be any graph with $u_\Delta$ as a vertex having the maximum degree $\Delta(G)$ and $u'_\Delta$ as its mirror image in the second copy of $G$ in $\pi G$. Then $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) = 2$ if and only if one of the following two conditions holds for $G$:

(i) $\gamma(G) = 2$

(ii) $\gamma(G) \geq 3$ and $\Delta(G) \geq \frac{n+\gamma(G)-4+i}{2}$ where $|N[u_\Delta] \cap N[u'_\Delta]| = i$ for $0 \leq i \leq 2$.

**Proof.** Let us assume that $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) = 2$. Then by the above remark $\gamma(G) \geq 2$. If $\gamma(G) = 2$, then the result is true. Hence we assume that $\gamma(G) \geq 3$. Let $|N[u_\Delta] \cap N[u'_\Delta]| = i$ for $0 \leq i \leq 2$.

We prove by the method of contradiction. Suppose $\Delta(G) < \frac{n+\gamma(G)-4+i}{2}$.

Then $|N_{\pi G}[u_\Delta] \cup N_{\pi G}[u'_\Delta]| < n + \gamma(G)$ which implies that $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) > 2$ which is a contradiction.

Hence the condition is necessary.

For the proof of the sufficient part, let us assume that $\gamma(G) = 2$. Let $S$ be a $\gamma(G)$-set of $G$. Then $S$ is also a $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G)$-set of $\pi G$. Hence $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) = 2$.

Now, let us assume the condition(ii).

Consider $T = \{u_\Delta, u'_\Delta\}$. Then

$N_{\pi G}[T] \geq 2 \left[ \frac{n + \gamma(G) - 4 + i + 4 - i}{2} \right]$

$= n + \gamma(G)$

Also $T$ is minimum in this context. Hence $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) = 2$. \hfill \square

**Proposition 3.5.** Suppose $p \in (0, 1]$ and $\pi$ is any permutation on $V(G)$, where $G$ is any $n$-vertex graph. Then $\gamma_p(G) \leq \gamma_p(\pi G) \leq 2\gamma_p(G)$.

**Proof.** Let us first prove the lower bound part.

Let $p \in (0, 1]$ and $S$ be a $\gamma_p$-set in $\pi G$.

$\implies \frac{|N[S]|}{2n} \geq p$

$\implies |N[S]| \geq 2np$ \hspace{1cm} (3)
Let $G_1$ and $G_2$ denote the two copies of $G$ in $\pi G$.
Now two cases may arise.

**Case(i):** All the vertices of $S$ are from either $V(G_1)$ or $V(G_2)$

Then from equation (3), $|N[S]| \geq 2np > np$

Hence $S$ is a $p$-dominating set in $G_1$ or $G_2$.

Thus $\gamma_p(G) \leq \gamma_p(\pi G)$ in this case.

**Case(ii):** Vertices of $S$ are from both $G_1$ and $G_2$.
Let $S = X \cup Y$ where $X \subseteq V(G_1)$ and $Y \subseteq V(G_2)$.
By our assumption,
$$\frac{|N[X \cup Y]|}{2n} \geq p \implies |N[X]| + |N[Y]| \geq 2np \tag{4}$$

Now two cases may arise:

**Subcase(i)** $|N[X]| \geq np$ and $|N[Y]| \geq np$.
Let $X^* = \{w \in V(G_2)/\pi(v) = w \forall v \in X\}$ and $Y^* = \{w \in V(G_1)/\pi(w) = v \forall v \in Y\}$. Then $X \cup Y^*$ and $X^* \cup Y$ are $p$-dominating sets of $G$.

Thus,
$$\gamma_p(G) \leq |X \cup Y^*| = |X \cup Y| = \gamma_p(\pi G)$$

**Subcase(ii)** WLG let $|N[X]| < np$. Then $|N[Y]| \geq np + (np - |N[X]|)$ by our assumption found in equation (4).

In this case $|N[Y]| > np$.
Hence $X^* \cup Y$ is a $p$-dominating set of $G$, where $X^*$ is defined as in the above subcase. Thus

$$\gamma_p(G) \leq |X^* \cup Y| = |X \cup Y| = \gamma_p(\pi G)$$

Hence $\gamma_p(G) \leq \gamma_p(\pi G)$ in both the cases.

Now let us prove the upper bound part.
Let $S$ be a $\gamma_p$ set of $G$. Let $S^* = \{v^* \in G_2/\pi(v) = v^* \forall v \in S\}$.
Then $\frac{|N[S \cup S^*]|}{2n} \geq p$.
Thus $S \cup S^*$ is a $p$-dominating set of $\pi G$ and $|S \cup S^*| = 2\gamma_p(G)$.

Hence $\gamma_p(\pi G) \leq 2\gamma_p(G)$.

**Proposition 3.6.** Let $G$ be a graph having an independent set $M = \{v_1, v_2, \ldots, v_k\}$ of $k$-vertices ($k \geq 1$) each having the maximum degree $\Delta(G)$. If $N(v_i) \cap N(v_j) = \emptyset \forall v_i, v_j \in M$ then for $i = 1, 2, \ldots, k$,
$$\gamma_p(\pi G) = i$$
for $p \in \left(\frac{(i-1)(\Delta(G)+2)}{2n}, \frac{i(\Delta(G)+2)}{2n}\right)$ for any permutation $\pi$ on $V(G)$.
Proof. Let \( N(v_i) \cap N(v_j) = \emptyset \forall v_i, v_j \in M \).
In this case, for \( 1 \leq i \leq k \), \( S = \{v_1, v_2, \ldots, v_i\} \subseteq M \) dominates \( i(\Delta(G) + 2) \) vertices in \( \pi G \). Also since each vertex in \( M \) is of maximum degree, ‘i’ is the minimum number of vertices that are required to dominate \( i(\Delta(G) + 2) \) vertices in \( \pi G \).
Hence \( \gamma_{i(\Delta(G) + 2)} = i \).
By the same argument we can say that \( \gamma_{\frac{(i-1)(\Delta(G)+2)}{2n}} = i - 1 \).
Also, \( \frac{(i-1)(\Delta(G)+2)}{2n} \) is the best proportion of domination possible with \( i-1 \) vertices. Hence the result is true in this case.

Proposition 3.7. Let \( G \) be a \( n \)-vertex graph having a set of \( k \)-mutually non-adjacent vertices \( (k \geq 1) \) say \( M = \{v_1, v_2, \ldots, v_k\} \) each having the maximum degree \( \Delta(G) \). Let \( G \) and \( G' \) be the two copies of \( G \) in \( \pi G \) for any permutation \( \pi \) on \( V(G) \). Let \( M' = \{v'_1, v'_2, \ldots, v'_k\} \) be the copy of \( M \) in \( G' \). If for each \( v_r \in M \), there exists exactly one \( v_s \in M \) such that \( |N(v_r) \cap N(v_s)| = 1 \), then for \( i = 1, 2, \ldots, k \), \( \gamma_p(\pi G) = i \) for \( p \in \left( \frac{(i-1)(\Delta(G)+2)}{2n}, \frac{i(\Delta(G)+2)}{2n} \right) \) for the following permutations \( \pi \) on \( V(G) \):
(i) \( \pi = 1 \)
(ii) \( \pi(v_r) = v'_s \) and \( \pi(v_s) = v'_r \) for each \( v_r \) and \( v_s \) as defined above.
(iii) \( \pi \neq 1 \) and \( \pi(v_r) \neq v'_s \) or \( \pi(v_s) \neq v'_r \) or both and \( \pi(v_i) \in M' \forall v_i \in M \).

Proof. We shall prove the theorem in three cases. For \( 1 \leq i \leq k \), WLG assume that \( i \) is an even number. Let \( S = \{v_1, v_2, \ldots, v_i\} \subseteq M \) be such that there exists \( \frac{i}{2} \) pairs of \( v_r, v_s \) as defined above.
Case(i): \( \pi = 1 \)
In this case \( \{v_r, v'_s\} \) for each pair of \( v_r, v_s \) will dominate \( \frac{2(\Delta(G)+2)}{2n} \) vertices in \( \pi G \). Thus there exists \( \frac{i}{2} \) such pairs which dominate \( \frac{i(\Delta(G)+2)}{2n} \) vertices in \( \pi G \). Hence by an argument similar to the previous proposition, the result is proved in this case.
Case(ii): \( \pi(v_r) = v'_s \) and \( \pi(v_s) = v'_r \) for each \( v_r \) and \( v_s \) as defined above.
In this case \( \{v_r, v'_s\} \) or \( \{v_s, v'_r\} \) for each pair of \( v_r, v_s \) will serve the purpose for the required result.
Case(iii): \( \pi \neq 1 \) and \( \pi(v_r) \neq v'_s \) or \( \pi(v_s) \neq v'_r \) or both and \( \pi(v_i) \in M' \forall v_i \in M \)
For this case we give an algorithm which returns a set \( T \) of \( k \)-vertices from \( M \cup M' \) whose members are mutually non-adjacent to each other.
Let \( \pi(v_i) = v'_i \forall v_i \in M \) under the above permutations and \( m = k \).
Procedure 1 Algorithm to find T

Input: \( m, \forall i = 1 \) to \( m \) \( v_i, v_i^*, N(v_i), N(v_i^*) \)

Output: \( T \)

1. \( T = \{\}, i = 1, j = 1, k = 1 \)
2. \( \text{while } i \leq m \) \( \text{do} \)
3. \( \text{if } \{v_i, v_i^*\} \cap T = \phi \) \( \text{then} \)
4. \( T = T \cup \{v_i\} \)
5. \( \text{break} \)
6. \( \text{else} \)
7. \( \text{if } |T| = m \) \( \text{then} \)
8. \( \text{return } T \)
9. \( \text{else} \)
10. \( i = i + 1 \) and go to 2
11. \( \text{end if} \)
12. \( \text{end if} \)
13. \( \text{end while} \)
14. \( \text{for all } j \neq i \) and \( j = 1 \) to \( m \) \( \text{do} \)
15. \( \text{if } |N(v_i) \cap N(v_j)| = 1 \) \( \text{then} \)
16. \( T = T \cup v_j^* \)
17. \( \text{break} \)
18. \( \text{end if} \)
19. \( \text{end for} \)
20. \( \text{for all } k \neq j \) and \( k = 1 \) to \( m \) \( \text{do} \)
21. \( \text{if } |N(v_j^*) \cap N(v_k^*)| = 1 \) \( \text{then} \)
22. \( i = k \) and go to 2
23. \( \text{end if} \)
24. \( \text{end for} \)

Each vertex in \( T \) thus got from the algorithm will dominate \( \frac{\Delta(G)+2}{2n} \) vertices in \( \pi G \). And for \( 1 \leq i \leq k \), \( i \) vertices from \( T \) will dominate \( \frac{i(\Delta(G)+2)}{2n} \) vertices in \( \pi G \).

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