AN EXTENSION OF REES’ THEOREM AND TWO INTERPRETATIONS OF A VECTOR IN THE JOINT REDUCTION LATTICE

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Abstract. In [18] Rees gave a characterization for the normal joint reduction number zero of two \textit{m}-primary ideals in an analytically unramified Cohen-Macaulay local ring of dimension two. Rees’ result is a generalization of Zariski’s product theorem for complete ideals in a regular local ring of dimension two. The aim of this paper is to extend Rees’ theorem for the ordinary powers of \textit{m}-primary ideals \textit{I} and \textit{J} in a Cohen-Macaulay local ring of dimension two. Following Rees’ approach, we define the modified Koszul homology modules \( M^1_{r,s}(a,k) \) for a joint reduction \((a, b)\) of \textit{I} and \textit{J}. Under the additional assumption that the associated graded rings of \textit{I} and \textit{J} have positive depth, we obtain a characterization of the joint reduction number zero of \textit{I} and \textit{J} in terms of the vanishing of the module \( M^1_{0,0}(a,b) \), as well as in terms of the Hilbert coefficients and the bigraded Hilbert coefficients. More generally, we introduce the joint reduction lattice and study the vanishing of \( M^1_{r,s}(a,b) \) for any \( r, s \geq 0 \). This gives a characterization for a vector \((r, s)\) to be in the joint reduction lattice of \textit{I} and \textit{J}. We also give a cohomological interpretation of these theorems by investigating the local cohomology modules of the bigraded extended Rees algebra. This gives another characterization for a vector \((r, s)\) to be in the joint reduction lattice and also extends a recent result of Masuti and Verma in [12] for ordinary powers of ideals.

1. Introduction

Throughout this paper \((R, \mathfrak{m})\) is a Noetherian local ring with infinite residue field. Recall that an ideal \textit{I} is complete if \( I = \mathcal{T} := \{ x \in R | x^n + a_1 x^{n-1} + \cdots + a_n, a_i \in I \text{ for } i = 1, \ldots, n \} \). In 1960, Zariski showed that if \((R, \mathfrak{m})\) is a regular local ring of dimension two, then product of complete ideals is complete ([24]). In order to generalize Zariski’s result, in 1981, Rees studied complete ideals in an analytically unramified Cohen-Macaulay local ring of dimension two. To state Rees’ result we need to introduce some notation. It is well-known that if \textit{R} is an analytically unramified local ring of dimension \( d \) and \textit{I} an \textit{m}-primary ideal in \textit{R}, then there exists a polynomial \( \overline{P}_I(x) \in \mathbb{Q}[x] \) such that \( \overline{P}_I(n) = \overline{P}_I := \ell(R/I^n) \) for \( n \gg 0 \) ([16, Theorem 1.4], [17, Theorem 1.1]). Here \( \ell(M) \)
denotes the length of the $R$-module $M$. One can write $P_I(n) := \sum_{i=0}^d (-1)^i e_i(I) \binom{n+d-i-1}{n+d-1}$. Rees proved the following interesting result:

**Theorem 1.** (Rees’ theorem) [18, Theorem 2.5] Let $(R, \mathfrak{m})$ be an analytically unramified Cohen-Macaulay local ring of dimension two and $I, J$ be $\mathfrak{m}$-primary ideals in $R$. Then the following statements are equivalent:

(a) $\varpi_2(IJ) = \varpi_2(I) + \varpi_2(J)$;
(b) $IJ^{r+1}J^{s+1} = aIJ^{r+1}J^s + bIJ^{r+1}J^s$ for all $r, s \geq 0$.

In [18], Rees proved that $\varpi_2(I) = 0$ for every $\mathfrak{m}$-primary ideal $I$ in a regular local ring of dimension two, thus generalizing Zariski’s product theorem for complete ideals. One of the main aims of this paper is to extend Theorem 1 for the ordinary powers of ideals $I$ and $J$. We refer to Theorem 1 as Rees’ theorem throughout this paper.

Consider the filtrations $\mathcal{F} := \{I^r J^s\}_{r,s \in \mathbb{Z}}$ and $\overline{\mathcal{F}} = \{\overline{I^r J^s}\}_{r,s \in \mathbb{Z}}$, where $I$ and $J$ are $\mathfrak{m}$-primary ideals in $R$. An important tool which was used by Rees in [18] in generalizing Zariski’s product theorem was the bigraded normal Hilbert function $H_\mathcal{F}(r, s) := \ell(R/I^r J^s)$, where $r, s \in \mathbb{Z}$. In the same paper Rees showed that in an analytically unramified local ring of dimension $d$, there exists a polynomial $P_\mathcal{F}(x, y) \in \mathbb{Q}[x, y]$ such that $P_\mathcal{F}(r, s) = H_\mathcal{F}(r, s)$ for all $r, s \gg 0$ and can be written as

$$P_\mathcal{F}(r, s) = \sum_{i+j \leq d} (-1)^i j e_{(i, j)}(\mathcal{F}) \binom{r+i-1}{i} \binom{s+j-1}{j}.$$  

For our purpose we consider the bigraded Hilbert function $H_\mathcal{F}(r, s) := \ell(R/I^r J^s)$ for all $r, s \in \mathbb{Z}$. In [1] P. B. Bhattacharya proved that there exists a polynomial $P_\mathcal{F}(x, y) \in \mathbb{Q}[x, y]$ such that $P_\mathcal{F}(r, s) = H_\mathcal{F}(r, s)$ for all $r, s \gg 0$. We call this polynomial the **bigraded Hilbert polynomial**. This polynomial can be written as

$$P_\mathcal{F}(r, s) = \sum_{i+j \leq d} (-1)^i j e_{(i, j)}(\mathcal{F}) \binom{r+i-1}{i} \binom{s+j-1}{j}. \quad (1.1)$$

We call the coefficients $e_{(i, j)}(\mathcal{F})$ the **bigraded Hilbert coefficients**. If the ideals $I$ and $J$ are clear from the context, then for simplicity we set $e_{(i, j)} := e_{(i, j)}(\mathcal{F})$. Let $P_I(x) \in \mathbb{Q}[x]$ be the Hilbert-Samuel polynomial of $I$, i.e., $P_I(x)$ is the polynomial such that $P_I(n) = H_I(n) := \ell(R/I^n)$ for $n \gg 0$. This polynomial can be written as

$$P_I(n) = e_0(I) \binom{n+d-1}{d} - e_1(I) \binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(I)$$

for some integers $e_i(I)$, for $i = 0, \ldots, d$, known as the **Hilbert coefficients of $I$**.

In [18] Rees introduced the modified Koszul homology modules $M_{r, s}^1$ for the filtration $\overline{\mathcal{F}}$ and gave an explicit formula for the normal Hilbert coefficients in terms of these modules. In order to generalize Rees’ theorem we define the modified bigraded Koszul complex $C_*((a^k, b^k), r, s)$ (2.1) for
a \in I \text{ and } b \in J. \text{ Using this complex, we define the modified Koszul homology modules } M_{r,s}^1(a^k, b^k) \text{ for all } r, s \geq 0 \text{ and } k \geq 1, \text{ and elements } a \in I \text{ and } b \in J \text{ (Definition 2.4). We set } M_{r,s}^1 := M_{r,s}^1(a,b). \text{ Let } (R, \mathfrak{m}) \text{ be a Cohen-Macaulay local ring, } I \text{ and } J \text{ be } \mathfrak{m}-\text{primary ideals in } R \text{ and } (a,b) \text{ a joint reduction of } I \text{ and } J. \text{ If either } k \gg 0 \text{ or } r, s \gg 0 \text{ then the asymptotic behaviour of the } M_{r,s}^1(a^k, b^k) \text{ plays an important role in understanding the coefficients of } P_F(r, s). \text{ Recall that } (a,b) \text{ is a joint reduction of } I \text{ and } J \text{ if } a \in I, b \in J \text{ and }

I^{r+1}J^{s+1} = aI^rJ^{s+1} + bI^{r+1}J^s \text{ for some and hence for all } r, s \gg 0 \quad (1.2)

([19]). \text{ We first prove that if } r, s \geq 0, \text{ then } \ell(M_{r,s}^1(a^k, b^k)) \text{ is a polynomial of degree at most one in } k \text{ for all } k \gg 0 \text{ (Proposition 3.9(a)). We explicitly describe this polynomial in terms of the Hilbert coefficients and the bigraded Hilbert coefficients. In addition, if we choose } a \text{ and } b \text{ to be Rees superficial elements (see Definition 2.9), then we study the asymptotic behaviour of } \ell(M_{r,s}^1(a^k, b^k)) \text{ for all } k \geq 1 \text{ and if either } r \text{ or } s \text{ is large ((Proposition 3.9(b)). Using this result, we show that the differences } e_{(1,0)} - e_1(I) \text{ and } e_{(0,1)} - e_1(J) \text{ can be expressed in terms of the length of the modules } M_{r,s}^1 \text{ for } r, s \gg 0, \text{ (Proposition 3.9(c)). Hence, the length of the modules } M_{r,s}^1 \text{ help us to measure for the difference between } e_{(1,0)} \text{ and } e_1(I). \text{ A similar expression gives the difference between } e_{(0,1)} \text{ and } e_1(J).

For the filtration } \mathcal{F}, \text{ Rees showed that } e_{(1,0)}(\mathcal{F}) = \overline{e}_1(I) \text{ and } e_{(0,1)}(\mathcal{F}) = \overline{e}_1(J) [18, \text{ Theorem 1.2}. \text{ This was an important result used in Rees’ proof of Theorem 1}. \text{ For the filtration } \mathcal{F}, \text{ in general, } e_{(1,0)} \text{ (resp. } e_{(0,1)}) \text{ need not be equal to } e_1(I) \text{ (resp. } e_1(J)). \text{ In fact, in [3] the first author and A. Guerrieri proved that } e_{(1,0)} = e_1(I) \text{ and } e_{(0,1)} = e_1(J) \text{ in any Noetherian local ring of dimension } d. \text{ We give a counter-example to their result (Example 5.3). In fact, } e_{(1,0)} - e_1(I) \text{ (resp. } e_{(0,1)} - e_1(J)) \text{ can be as large as possible (Example 5.3). This was an obstruction in generalizing Rees’ theorem for the filtration } \mathcal{F}. \text{ We conclude that } e_{(1,0)} \geq e_1(I) \text{ and } e_{(0,1)} \geq e_1(J), \text{ and give a criteria for the equality in terms of the vanishing of the modules } M_{r,s}^1 \text{ (Corollary 3.16). As a consequence we generalize Rees’ theorem in Theorem 3.25.}

**Definition 1.3.** \text{ For ideals } I \text{ and } J \text{ we define the joint reduction lattice of } I \text{ and } J, \text{ denoted by } \Lambda(I|J), \text{ as }

\Lambda(I|J) := \{(r, s) \in \mathbb{N}^2 : I^{r+1}J^{s+1} = aI^rJ^{s+1} + bI^{r+1}J^s \text{ for some joint reduction } (a,b) \text{ of } I \text{ and } J\}.

\text{ We remark that if } K = \{a, c, d, b\} \text{ is a complete reduction of } I \text{ and } J, \text{ and } r_{K}^{1,1}(I, J) = n \text{ where } r_{K}^{1,1}(I, J) \text{ is the joint reduction number of type } (1,1) \text{ with respect to } K \text{ introduced by Hyry in [6, Definition 3.2], then } (n, n) \in \Lambda(I|J). \text{ Moreover, joint reduction vectors (with respect to a joint reduction of type } (1,1) \text{) introduced in [21] are also in the joint reduction lattice. Recall that the ideals } I \text{ and } J \text{ are said to have joint reduction number zero, denoted by } r(I|J) = 0, \text{ if there exists a joint reduction } (a, b) \text{ of } I \text{ and } J \text{ such that } IJ = aJ + bI \text{ (c.f. [23]). It is clear that } r(I|J) = 0 \text{ if and only if } \Lambda(I|J) = \mathbb{N}^2.\}
In Theorem 3.25, for \( k \gg 0 \), we characterize joint reduction number zero for the ideals \( I^k \) and \( J^k \) in terms of the Hilbert and bigraded Hilbert coefficients, as well as in terms of the vanishing of the modules \( M^1_{i,j}(a^k, b^k) \) for \( k \gg 0 \). If \( G(I) := \bigoplus_{n \geq 0} I^n/I^{n+1} \) and \( G(J) := \bigoplus_{n \geq 0} J^n/J^{n+1} \) have positive depth, then Theorem 3.25 holds true for all \( k \geq 1 \) (Theorem 3.28). However, if depth \( G(I) = 0 \) or depth \( G(J) = 0 \), then the joint reduction number of \( I \) and \( J \) need not be zero even if \( I \) and \( J \) satisfy the equivalent conditions of Theorem 3.25. We demonstrate this in Example 3.27.

More generally, we study the vanishing of the modules \( M^1_{i,j}(a^k, b^k) \) for \( k \gg 0 \) and \( r, s \geq 0 \) (Theorem 3.30). As a consequence, we obtain a sufficient conditions in terms of the Hilbert and bigraded Hilbert coefficients for a vector \((r, s)\) to be in the joint reduction lattice of \( I^k \) and \( J^k \) for \( k \gg 0 \) (Corollary 3.31). Under the additional assumptions (3.34) and (3.35), we obtain criteria for a vector \((r, s)\) to be in the joint reduction lattice of \( I \) and \( J \) (Theorem 3.33).

We now describe the cohomological approach for joint reduction number zero. Let \( \mathcal{R}'(F) := \bigoplus_{r, s \in \mathbb{Z}} I^{r} J^{s} t_1 t_2 \) (resp. \( \mathcal{R}'(F) := \bigoplus_{r, s \in \mathbb{Z}} I^{r} J^{s} t_1 t_2 \)) be the extended bigraded Rees algebra of \( F \) (resp. \( F \)) where \( t_1 \) and \( t_2 \) are indeterminate. In [12] the second author and Verma gave a new approach to Rees’ theorem. They showed that if \((a, b)\) is a good joint reduction of \( F \), then \( \ell([H^2_{I(a, b)}(\mathcal{R}'(F))]_{(0,0)}) = -\overline{e}_2(I) + \overline{e}_2(J) \) (12, Theorem 3.7). Moreover, in [12, Theorem 4.1] they showed that the vanishing of \([H^2_{I(a, b)}(\mathcal{R}'(F))]_{(0,0)}\) is equivalent to the equivalent conditions of Rees’ theorem. This gave a cohomological interpretation of Rees’ theorem. In Section 4 we extend these results for the filtration \( F = \{I^r J^s\}_{r,s \geq 0} \).

One of the important consequences of the equalities \( e_{(1,0)}(F) = \overline{e}_1(I) \) and \( e_{(0,1)}(F) = \overline{e}_1(J) \) is that the local cohomology modules \([H^2_{I(a, b)}(\mathcal{R}'(F))]_{(r,s)}\) have finite length for all \( r, s \geq 0 \), see [12, Theorem 3.7]. However, for the filtration \( F \) the module \([H^2_{I(a, b)}(\mathcal{R}'(F))]_{(0,0)}\) need not have finite length (Example 5.3). In Theorem 4.3 we give necessary and sufficient conditions for \( \ell_R([H^2_{I(a, b)}(\mathcal{R}'(F))]_{(r,s)}) \) to be finite in terms of the Hilbert and bigraded Hilbert coefficients. In particular, we show that the module \([H^2_{I(a, b)}(\mathcal{R}'(F))]_{(0,0)}\) has finite length if and only if \( e_{(1,0)} = e_1(I) \) and \( e_{(0,1)} = e_1(J) \) which in turn is equivalent to the vanishing of the modules \( M^1_{i,k} \) and \( M^1_{k,j} \) for all \( i \geq 0 \) and \( j \geq 0 \) and \( k \gg 0 \) (Corollary 4.4).

We also give necessary and sufficient conditions for the vanishing of the cohomology modules \([H^2_{I(a, b)}(\mathcal{R}'(F))]_{(r,s)}\) (Theorem 4.7). In fact, we give a cohomological interpretation of Theorems 3.30 and 3.33 in Theorems 4.7 and 4.9 respectively. In Theorem 4.9 we give a characterization for \((r_0, s_0) \in \Lambda(I, J)\) in terms of the vanishing of \([H^2_{I(a, b)}(\mathcal{R}'(F))]_{(r_0, s_0)}\). Putting \( r_0 = s_0 = 0 \) in Theorems 4.7 and 4.9 we obtain a cohomological interpretation of Theorems 3.25 and 3.28 in Corollaries 4.8 and 4.10, respectively. These results extend Rees’ theorem and [12, Theorem 3.8] for the filtration \( F \). We remark that Rees’ theorem for joint reduction number zero has been extended for arbitrary filtration in [10, Theorem 6.6] under certain additional assumptions. We do not need these additional assumptions for our results. In fact, we recover [10, Theorem 6.6] for the filtration \( F \) (see Corollary 4.10).
In Section 5, we give an explicit example for which \( e(0,1) \neq e_1(J) \) and hence \( H^2_{(a_1,b_2)}(\mathcal{R}'(F))_{(0,0)} \) is not finite (Example 5.3). We also give an example where both \( e_{(1,0)} \neq e_1(I) \) and \( e_{(0,1)} \neq e_1(J) \) (Example 5.4).

We conclude that the lengths of the modified Koszul homology modules as well as the local cohomology modules are a measure for a vector \((r,s)\) to be in the joint reduction lattice in dimension two Cohen-Macaulay local rings. We hope that these approaches will be useful to find a characterization for a vector to be in the joint reduction lattice in terms of the Hilbert and bigraded Hilbert coefficients in higher dimension (see [11] for results in dimension 3 for the normal filtration).

We refer [13] for all undefined terms.

Acknowledgement: The second author thanks the Institute of Mathematical Sciences (IMSc) and Chennai Mathematical Institute (CMI) for financial support during her post doctoral studies at IMSc and CMI, respectively, during which this work has started. Both the authors thank J. K. Verma for many fruitful discussions.

2. Modified bigraded Koszul Complex

In [18, Lemma 2.2] Rees introduced the modules \( M^1_{r,s} \) which played an important role in relating a vector in the joint reduction lattice and the Hilbert coefficients of the normal filtrations \( \{I^n\}_{n \in \mathbb{Z}} \) and \( \{J^n\}_{n \in \mathbb{Z}} \) [18, Theorem 2.4]. In this section, we construct a modified bigraded Koszul complex \( C_\bullet((a^k,b^k),r,s) \) for all \( r,s \geq 0 \) and \( k \geq 1 \) (c.f. [9] for the modified Koszul complex of \( \mathbb{Z}\)-graded filtrations). We study certain properties of the homology modules of this complex. Using this complex, for all \( k \geq 1 \) and \( r,s \geq 0 \), we define the modified Koszul homology module \( M^1_{r,s}(a^k,b^k) \) (Definition 2.4). The modules \( M^1_{r,s}(a^k,b^k) \) are used to relate the Hilbert coefficients and the bigraded Hilbert coefficients in Section 3, thus extending Rees’ theorem for the filtration \( F \). In this section, we also give a generalization of Huneke’s fundamental Lemma (Lemma 2.8) which relates the modules \( M^1_{r,s}(a^k,b^k) \) and \( H_2(C_\bullet((a^k,b^k),r,s)) \) with the bigraded Hilbert function.

Let \( r,s \geq 0, k \geq 1, a \in I \) and \( b \in J \). We have the complexes \( C_\bullet((a^k),r,s) \) and \( C_\bullet((b^k),r,s) \) given by

\[
C_\bullet((a^k),r,s) : 0 \rightarrow \frac{R}{I^r J^s} a^k \rightarrow \frac{R}{I^{r+k} J^s} \rightarrow 0
\]

\[
C_\bullet((b^k),r,s) : 0 \rightarrow \frac{R}{I^r J^s} b^k \rightarrow \frac{R}{I^r J^{s+k}} \rightarrow 0
\]

where the maps are induced by the Koszul complex \( K_\bullet(a^k;R) \) and \( K_\bullet(b^k;R) \), respectively. We also have the chain map of complexes:

\[ b^k : C_\bullet((a^k),r,s) \rightarrow C_\bullet((a^k),r,s+k) \]
We call the mapping cylinder of this chain map as the modified bigraded Koszul complex which we denote by \( C_\bullet((a^k, b^k), r, s) \) (see [20, page 175] for mapping cylinder). More precisely this complex is

\[
C_\bullet((a^k, b^k), r, s) : 0 \rightarrow \frac{R}{I^r J^s} \phi_1 \rightarrow \frac{R}{I^{r+k} J^{s+k}} \bigoplus \frac{R}{I^r J^{s+k}} \phi_0 \rightarrow \frac{R}{I^{r+k} J^{s+k}} \rightarrow 0
\]

(2.1)

where the maps \( \phi_0 \) and \( \phi_1 \) are induced by the Koszul complex \( K_\bullet(a^k, b^k; R) \). Let \( H_i(C_\bullet((a^k, b^k), r, s)) \) denote the \( i \)-th homology of the complex \( C_\bullet((a^k, b^k), r, s) \).

**Theorem 2.2.** Let \((R, m)\) be a Noetherian local ring of dimension two and \( I, J \) be \( m \)-primary ideals in \( R \). Let \( a \in I \) and \( b \in J \). Then for all \( k \geq 1 \) and \( r, s \geq 0 \),

(a) \( H_0(C_\bullet((a^k, b^k), r, s)) = \frac{R}{I^{r+k} J^{s+k} + (a^k, b^k)} \);

(b) \( H_2(C_\bullet((a^k, b^k), r, s)) = \frac{(I^{r+k} J^{s} : (a^k)) \cap (I^{r} J^{s+k} : (b^k))}{I^{r} J^{s}} \);

(c) If \( a, b \) is a regular sequence, then \( H_1(C_\bullet((a^k, b^k), r, s)) = \frac{(a^k, b^k) \cap I^{r+k} J^{s+k}}{a^k I^{r} J^{s+k} + b^k I^{r+k} J^{s}} \).

**Proof.** (a) and (b) are easy to verify.

(c) Consider the complex

\[
C'_\bullet((a^k, b^k), r, s) : 0 \rightarrow \frac{R}{I^r J^s} \psi_1 \rightarrow \frac{R}{I^{r+k} J^{s+k}} \bigoplus \frac{R}{I^r J^{s+k}} \psi_0 \rightarrow \frac{(a^k, b^k)}{a^k I^{r} J^{s+k} + b^k I^{r+k} J^{s}} \rightarrow 0,
\]

(2.3)

where the maps are induced by the Koszul complex \( K_\bullet(b^k, a^k; R) \). We claim that \( \ker(\psi_0) = \text{Im}(\psi_1) \).

We write \( (\cdot) \) for the image of an element in respective quotients. If \( (\overline{a}, \overline{b}) \in \ker(\psi_0) \), then \( xb^k - ya^k = a^k c + b^k d \) for some \( c \in I^r J^{s+k} \) and \( d \in I^{r+k} J^{s} \). Since \( a, b \) is a regular sequence, there exists \( r \in R \) such that \( x = d + ra^k \) and \( y = -c + rb^k \). Hence \( (\overline{a}, \overline{b}) = \psi_1(\overline{r}) \in \text{Im}(\psi_1) \). This proves the claim.

Thus we have the following commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \ker \psi_0 & \rightarrow & \frac{R}{I^{r+k} J^{s}} & \bigoplus & \frac{R}{I^r J^{s+k}} & \psi_0 & \rightarrow & \frac{(a^k, b^k)}{a^k I^{r} J^{s+k} + b^k I^{r+k} J^{s}} & \rightarrow & 0 \\
0 & \rightarrow & \ker \phi_0 & \rightarrow & \frac{R}{I^{r+k} J^{s}} & \bigoplus & \frac{R}{I^r J^{s+k}} & \phi_0 & \rightarrow & \frac{(a^k, b^k) \cap I^{r+k} J^{s+k}}{I^{r+k} J^{s+k}} & \rightarrow & 0.
\end{array}
\]

By Snake lemma we get

\[
H_1(C_\bullet((a^k, b^k), r, s)) = \frac{\ker \phi_0}{\text{Im} \phi_1} = \frac{\ker \phi_0}{\text{Im} \psi_1} = \frac{\ker \phi_0}{\ker \psi_0} = \ker \eta = \frac{(a^k, b^k) \cap I^{r+k} J^{s+k}}{a^k I^{r} J^{s+k} + b^k I^{r+k} J^{s}}.
\]

Here the second equality is true because \( \phi_1 = \psi_1 \).

Motivated by [18, Lemma 2.2] we introduce the modules \( M^1_{r,s}(a^k, b^k) \) which will be useful to detect whether a vector is in the joint reduction lattice.
**Definition 2.4.** Let \( a \in I, \ b \in J, \ r, s \geq 0 \) and \( k \geq 1 \). We define the first modified homology module to be

\[
M^1_{r,s}(a^k, b^k) := \frac{I^{r+k}J^{s+k}}{a^k I^r J^s + b^k I^{r+k} J^s}
\]  

(2.5)

and we set \( M^1_{r,s} := M^1_{r,s}(a, b) \).

The modules \( M^1_{r,s}(a^k, b^k) \) and \( H_1(C_\bullet((a^k, b^k), r, s)) \) are related as follows:

**Lemma 2.6.** Let \((R, \mathfrak{m})\) be a Noetherian local ring of dimension two and \( a, b \) a regular sequence where \( a \in I \) and \( b \in J \). Then for all \( r, s \geq 0 \) and \( k \geq 1 \),

\[
\ell \left( M^1_{r,s}(a^k, b^k) \right) = \ell \left( \frac{(a^k, b^k) + I^{r+k}J^{s+k}}{(a^k, b^k)} \right) + \ell \left( H_1(C_\bullet((a^k, b^k), r, s)) \right).
\]

**Proof.** By Theorem 2.2 we have the short exact sequence

\[
0 \longrightarrow H_1(C_\bullet((a^k, b^k), r, s)) \longrightarrow M^1_{r,s}(a^k, b^k) \longrightarrow \frac{(a^k, b^k) + I^{r+k}J^{s+k}}{(a^k, b^k)} \longrightarrow 0.
\]  

(2.7)

As all the modules in (2.7) are Artinian we get the result. \( \square \)

As a consequence of Theorem 2.2 we obtain a bigraded version of Huneke’s fundamental lemma (c.f. [5, Lemma 2.4]).

**Lemma 2.8.** [Huneke’s fundamental lemma for two ideals] Let \((R, \mathfrak{m})\) be a Cohen-Macaulay local ring of dimension two. Let \((a, b)\) be a joint reduction of \( I \) and \( J \). Then for all \( r, s \geq 0 \) and \( k \geq 1 \),

\[
\ell \left( M^1_{r,s}(a^k, b^k) \right) - \ell \left( H_2(C_\bullet((a^k, b^k), r, s)) \right) = k^2 e_{(1,1)} - H_\mathcal{F}(r + k, s + k) + H_\mathcal{F}(r, s + k) + H_\mathcal{F}(r + k, s) - H_\mathcal{F}(r, s).
\]

**Proof.** By [19, Theorem 2.4] \( e_{(1,1)} = \ell(R/\langle a, b \rangle) \). Hence using Theorem 2.2(a) and Lemma 2.6 we get

\[
\ell \left( M^1_{r,s}(a^k, b^k) \right) - \ell \left( H_2(C_\bullet((a^k, b^k), r, s)) \right) = \ell \left( \frac{(a^k, b^k) + I^{r+k}J^{s+k}}{(a^k, b^k)} \right) + \ell \left( H_1(C_\bullet((a^k, b^k), r, s)) \right) - \ell \left( H_2(C_\bullet((a^k, b^k), r, s)) \right)
\]

\[
= \ell \left( \frac{R}{(a^k, b^k)} \right) - \ell \left( \frac{R}{(a^k, b^k) + I^{r+k}J^{s+k}} \right) + \ell \left( H_1(C_\bullet((a^k, b^k), r, s)) \right) - \ell \left( H_2(C_\bullet((a^k, b^k), r, s)) \right)
\]

\[
= \ell \left( \frac{R}{(a^k, b^k)} \right) - \ell \left( H_0(C_\bullet((a^k, b^k), r, s)) \right) + \ell \left( H_1(C_\bullet((a^k, b^k), r, s)) \right) - \ell \left( H_2(C_\bullet((a^k, b^k), r, s)) \right)
\]

\[
= \ell \left( \frac{R}{(a^k, b^k)} \right) - \left[ \ell \left( \frac{R}{I^{r+k}J^{s+k}} \right) - \ell \left( \frac{R}{I^{r+k}J^s} \right) - \ell \left( \frac{R}{I^r J^{s+k}} \right) + \ell \left( \frac{R}{I^r J^s} \right) \right]
\]

\[
= k^2 e_{(1,1)} - H_\mathcal{F}(r + k, s + k) + H_\mathcal{F}(r, s + k) + H_\mathcal{F}(r + k, s) - H_\mathcal{F}(r, s).
\]  

\( \square \)
We now study the properties of $H_2(C_\bullet((a^k, b^k), r, s))$. We need a few definitions for this.

**Definition 2.9.** [7, Definition 2.4] (1) We say $a \in I$ (resp. $b \in J$) is a *Rees-superficial element* for the ideals $I$ (resp. $J$) if

\[
(a) \cap I^r J^s = a I^{r-1} J^s \quad \text{for } r \gg 0 \text{ and all } s \geq 0 \tag{2.10}
\]

(resp. $(b) \cap I^r J^s = b I^{r} J^{s-1} \quad \text{for } s \gg 0 \text{ and all } r \geq 0$). \tag{2.11}

(2) We call a joint reduction $(a, b)$ of $I$ and $J$ as a *Rees-joint reduction* of $I$ and $J$ if $a$ and $b$ are Rees-superficial elements for $I$ and $J$, respectively.

Let $(R, m)$ be a Noetherian local ring of dimension two with infinite residue field and $I, J$ be $m$-primary ideals in $R$. Then by [19] there exists a Rees-joint reduction $(a, b)$ of $I$ and $J$.

The Ratliff-Rush closure of an ideal was first introduced in [15] and played an important role in studying the Hilbert coefficients. The Ratliff-Rush closure for the product of ideals was considered in [8]. For our purpose, we introduce a refinement of the Ratliff-Rush closure for two ideals, i.e., the notion of Ratliff-Rush closure of $\mathcal{F}$ with respect to a joint reduction (Definition 2.12). This notion of the Ratliff-Rush closure is important in the study the Koszul homology $H_2(C_\bullet((a^k, b^k), r, s))$.

**Definition 2.12.** Let $I, J$ be $m$-primary ideals and $(a, b)$ a joint reduction of $I$ and $J$. We define the Ratliff-Rush closure of $\mathcal{F}$ with respect to $(a, b)$ to be $\bar{\mathcal{F}}_{(a, b)} := \{ \bar{\mathcal{F}}_{(a, b)}(I^r, J^s) \}_{r, s \in \mathbb{Z}}$, where

\[
\bar{\mathcal{F}}_{(a, b)}(I^r, J^s) := \bigcup_{k \geq 1} (I^{r+k} J^s : a^k) \cap (I^r J^{s+k} : b^k).
\]

**Remark 2.13.** One can verify that $I^r J^s \subseteq \bar{\mathcal{F}}_{(a, b)}(I^r, J^s) \subseteq \bar{I} \bar{J}^s$, where $\bar{I}$ denotes the Ratliff-Rush closure of $I$ introduced in [15] (see [8] for the Ratliff-Rush closure of product of ideals).

**Remark 2.14.** For all $k \geq 1$, $(I^{r+k} J^s : a^k) \cap (I^r J^{s+k} : b^k) \subseteq (I^{r+k+1} J^s : a^{k+1}) \cap (I^r J^{s+k+1} : b^{k+1})$. As $R$ is Noetherian, $\bar{\mathcal{F}}_{(a, b)}(I^r, J^s) = (I^{r+k} J^s : a^k) \cap (I^r J^{s+k} : b^k)$ for some and hence for all $k \gg 0$.

**Lemma 2.15.** Let $(R, m)$ be a Cohen-Macaulay local ring of dimension two and $I, J$ be $m$-primary ideals in $R$. Let $(a, b)$ be a joint reduction of $I$ and $J$.

(a) Fix $r, s \geq 0$. Then there exists $k \gg 0$ (which depends on $r, s$), such that $H_2(C_\bullet((a^k, b^k), r, s)) = \bar{\mathcal{F}}_{(a, b)}(I^r, J^s)$ and hence is independent of $k$ for $k \gg 0$.

(b) Suppose $a$ and $b$ are Rees-superficial elements for $I$ and $J$. If either $r \gg 0$ or $s \gg 0$, then for all $k \geq 1$, $H_2(C_\bullet((a^k, b^k), r, s)) = 0$.

**Proof.** (a): By Remark 2.14 $\bar{\mathcal{F}}_{(a, b)}(I^r, J^s) = (I^{r+k} J^s : a^k) \cap (I^r J^{s+k} : b^k)$ for all $k \gg 0$. Hence, by Theorem 2.2, for $k \gg 0$,

\[
H_2(C_\bullet((a^k, b^k), r, s)) = \frac{\bar{\mathcal{F}}_{(a, b)}(I^r, J^s)}{I^r J^s}.
\]
(b): Since $R$ is Cohen-Macaulay and $a, b$ is a system of parameters, it is a regular sequence. Hence for all $k \geq 1$, either $I^{r+k}J^s : a^k = I^rJ^s$ or $I^rJ^{s+k} : b^k = I^rJ^s$ by (2.10) and (2.11). This implies that $H_2(C_{\bullet}((a^k, b^k), r, s)) = 0$.

3. Characterization of a vector in the joint reduction lattice in terms of modified Koszul homology

Let $I$ and $J$ be $m$-primary ideals in a Cohen-Macaulay local ring $(R, m)$ of dimension two. In [18] Rees used the modules $M_{r,s}^1$ for the filtration $\mathcal{F}$ to characterize the joint reduction number zero of $\overline{\mathcal{F}}$. The aim of this section is to extend Rees’ theorem for the filtration $\mathcal{F} = \{I^rJ^s\}_{r,s \geq 0}$ and to characterize the joint reduction number zero in terms of the vanishing of the modules $M_{0,0}^1(a^k, b^k)$ for $k \gg 0$. We also investigate the relationship between $e_{(1,0)} - e_1(I)$ and $e_{(0,1)} - e_1(J)$ using the modules $M_{r,s}^1$. More generally, we characterize the vector $(r, s) \in \Lambda(I, J)$ in terms of the vanishing of the modules $M_{r,s}^1(a, b)$, and also in terms of the bigraded and the Hilbert coefficients under certain additional assumptions.

Fix $r \geq 0$. Then $\ell(J^s/J^sI^r)$ is a polynomial of degree one for all large $s$, i.e., there exist integers $f_0(I^r)$ and $f_1(I^r)$ such that for all large $s$

$$\ell(J^s/J^sI^r) = f_0(I^r)s - f_1(I^r).$$

Therefore, for all $s \gg 0$ and $r \geq 0$

$$\ell\left(\frac{R}{I^rJ^s}\right) = \ell\left(\frac{R}{J^sI^r}\right) + \ell\left(\frac{J^s}{I^rJ^s}\right) = e_0(J)\left(\frac{s+1}{2}\right) - e_1(J)s + e_2(J) + f_0(I^r)s - f_1(I^r)$$

$$= e_0(J)\left(\frac{s+1}{2}\right) - g_1(r)s + g_2(r), \quad (3.1)$$

where

$$g_1(r) = e_1(J) - f_0(I^r), \quad g_2(r) = e_2(J) - f_1(I^r) \quad \text{for all } r \geq 0. \quad (3.2)$$

In particular,

$$g_1(0) = e_1(J) \text{ and } g_2(0) = e_2(J). \quad (3.3)$$

Comparing the equations (1.1) and (3.1), for $r \gg 0$ we get

$$g_1(r) = e_{(0,1)} - e_{(1,1)}r, \quad g_2(r) = e(I)^{r+1} - e_{(1,0)}r + e_2(IJ) \quad \text{for all } r \gg 0. \quad (3.4)$$

Similarly, fix $s \geq 0$ and for all $r \gg 0$, there exist integers $f_0'(J^s)$ and $f_1'(J^s)$ such that

$$\ell(I^r/J^s) = f_0'(J^s)r - f_1'(J^s)$$
and hence for all \( r \gg 0 \) and \( s \geq 0 \)
\[
\ell \left( \frac{R}{I^r J^s} \right) = \ell \left( \frac{R}{I^r} \right) + \ell \left( \frac{I^r}{I^r J^s} \right) = e_0(I) \left( \frac{r+1}{2} \right) - e_1(I)r + e_2(I) + f_0'(J^s)r - f_1'(J^s)
\]
\[
= e(I) \left( \frac{r+1}{2} \right) - h_1(s)r + h_2(s)
\] (3.5)
where
\[
h_1(s) = e_1(I) - f_0'(J^s), \quad h_2(s) = e_2(I) - f_1'(J^s) \text{ for all } s \geq 0.
\] (3.6)
In particular,
\[
h_1(0) = e_1(I) \text{ and } h_2(0) = e_2(I).
\] (3.7)
Comparing the equations (1.1) and (3.5), for \( s \gg 0 \) we get
\[
h_1(s) = e_{(1,0)} - e_{(1,1)}s, \quad h_2(s) = e(J)(s+1) - e_{(0,1)}s + e_2(IJ) \text{ for all } s \gg 0.
\] (3.8)
In the next proposition we study asymptotic behaviour of the module \( M_{r,s}^1(a^k, b^k) \).

**Proposition 3.9.** Let \((R, m)\) be a Cohen-Macaulay local ring of dimension two and \( I, J \) be \( m \)-primary ideals in \( R \). Let \((a, b)\) be a joint reduction of \( I \) and \( J \). In addition, we assume that \( a \) and \( b \) are Rees-superficial elements for part (b) and (c). Then the following hold true.

(a) Let \( r, s \geq 0 \). Then for \( k \gg 0 \),
\[
\ell(M_{r,s}^1(a^k, b^k)) = \left[ e_{(0,1)} - g_1(r) - e_{(1,1)}r + e_{(1,0)} - h_1(s) - e_{(1,1)}s \right] k
\]
\[- e_{(1,1)}rs + \left[ e_{(0,1)} - g_1(r) \right] s + \left[ e_{(1,0)} - h_1(s) \right] r
\]
\[- e_2(IJ) + g_2(r) + h_2(s) - \ell(R/I^r J^s) + \frac{\ell \left( \frac{I^r}{I^r J^s} \right) \ell \left( \frac{R}{I^r J^s} \right)}{\ell(I^r J^s)}.
\]
In particular, \( \ell(M_{r,s}^1(a^k, b^k)) \) is a polynomial of degree at most one in \( k \).

(b) Let \( k \geq 1 \). Then
\[
\ell(M_{r,s}^1(a^k, b^k)) = k^2 e_{(1,1)} + k [g_1(r + k) - g_1(r)] \text{ for } r \geq 0 \text{ and } s \gg 0,
\] (3.10)
\[
\ell(M_{r,s}^1(a^k, b^k)) = k e_{(1,1)} + k [h_1(s + k) - h_1(s)] \text{ for } s \geq 0 \text{ and } r \gg 0.
\] (3.11)
In particular, if \( r \geq 0 \) and \( s \gg 0 \) (resp. \( s \geq 0 \) and \( r \gg 0 \)), then \( \ell(M_{r,s}^1(a^k, b^k)) \) is independent of \( s \) (resp. \( r \)) and the joint reduction \((a, b)\).

(c) Let \( k \geq 1 \). Then for all \( r, s \gg 0 \)
\[
\sum_{i=0}^{r-1} \ell(M_{r,s}^1(a^k, b^k)) = k \left[ e_{(0,1)} - e_1(J) \right] \text{ and}
\] (3.12)
\[
\sum_{i=0}^{s-1} \ell(M_{r,s}^1(a^k, b^k)) = k \left[ e_{(1,0)} - e_1(I) \right].
\] (3.13)
Proof. (a): From Lemma 2.8 and Lemma 2.15(a), for $k \gg 0$, we have

$$
\ell(M_{r,s}^1(a^k, b^k))
= k^2 \ell_{(1,1)} - \ell \left( \frac{R}{I^{r+k}J^s} \right) - \ell \left( \frac{R}{I^rJ^{s+k}} \right) - \ell \left( \frac{R}{I^rJ^s} \right) + \ell \left( \frac{\bar{F}_{a,b}(I^r, J^s)}{I^rJ^s} \right)
$$

$$
= k^2 \ell_{(1,1)} - \left[ \ell_{(1,1)}(r+k)(s+k) - \ell_{(1,0)}(r+k) - \ell_{(0,1)}(s+k) + \ell_{2}(IJ) \right]
- \left[ h_1(s)(r+k) - h_2(s) + g_1(r)(s+k) - g_2(r) + \ell \left( \frac{R}{I^rJ^s} \right) \right] + \ell \left( \frac{\bar{F}_{a,b}(I^r, J^s)}{I^rJ^s} \right)
$$
from (1.1), (3.1), (3.5)

$$
= \left[ \ell_{(0,1)} - \ell_{(1,1)}r + \ell_{(1,0)} - \ell_{(1,1)}s \right] - [e_{(0,1)} - g_1(r)]s + [e_{(1,0)} - h_1(s)]r
- e_2(IJ) + g_2(r) + h_2(s) - \ell(R/I^rJ^s) + \ell \left( \frac{\bar{F}_{a,b}(I^r, J^s)}{I^rJ^s} \right).
$$

(b): It is enough to prove (3.10) as the proof of (3.11) is similar. Let $r \geq 0$ and $s \gg 0$. As $a$ and $b$ are Rees-superficial elements, $H_2((a^k, b^k), r, s)) = 0$ by Lemma 2.15(b). Hence from Lemma 2.8 and (3.1) we have

$$
\ell(M_{r,s}^1(a^k, b^k)) = k^2 \ell_{(1,1)} + k \left[ g_1(r+k) - g_1(r) \right].
$$

(c): Applying (3.10), for all $s \gg 0$, we get

$$
\sum_{i=0}^{r-1} \ell(M_{ik,s}^1(a^k, b^k)) = r k^2 \ell_{(1,1)} + k \sum_{i=0}^{r-1} \left[ g_1(ik+k) - g_1(ik) \right]
= r k^2 \ell_{(1,1)} + k \left[ g_1(rk) - g_1(0) \right]
= r k^2 \ell_{(1,1)} + k \left[ g_1(rk) - e_1(J) \right] \quad \text{[from (3.3)]}
$$

Since $g_1(r) = e_{(0,1)} - e_{(1,1)}r$ for $r \gg 0$ by (3.4), we get

$$
\sum_{i=0}^{r-1} \ell(M_{ik,s}^1(a^k, b^k)) = k \left[ e_{(0,1)} - e_1(J) \right].
$$

Replacing $g_1(r)$ by $h_1(s), e_{(1,0)}$ by $e_{(1,0)}$ in the proof of (3.12) we get (3.13). \qed

As a corollary we express $\ell(M_{0,0}^1(a^k, b^k))$ in terms of the bigraded Hilbert coefficients and the Hilbert coefficients.

**Corollary 3.15.** Let $I, J$ be $m$-primary ideals in a Cohen-Macaulay local ring $(R, m)$ of dimension two and $(a, b)$ a joint reduction of $I$ and $J$. Then for all $k \gg 0$, $\ell(M_{0,0}^1(a^k, b^k))$ is a polynomial of degree at most one in $k$ and this polynomial can be written as

$$
\ell(M_{0,0}^1(a^k, b^k)) = \left[ e_{(0,1)} - e_1(J) + e_{(1,0)} - e_1(I) \right]k - e_2(IJ) + e_2(J) + e_2(I).
$$

In particular, $\ell(M_{0,0}^1(a^k, b^k))$ is independent of the joint reduction chosen.
Proof. For \( r = s = 0, \ell \left( \frac{\mathfrak{F}(a,b)(P',Q')}{P',Q'} \right) = 0 \). We have \( g_1(0) = e_1(J), g_2(0) = e_2(J) \) by (3.3), and \( h_1(0) = e_1(I) \) and \( h_2(0) = e_2(I) \) by (3.7). Hence substituting \( r = s = 0 \) in Proposition 3.9(a) we get the result. \( \square \)

In Corollary 3.16, we give a formula for the difference \( e_{(1,0)} - e_1(I) \) and \( e_{(0,1)} - e_1(J) \) and a criterion for the equality to hold. This gives a generalization to [18, Theorem 1.2].

**Corollary 3.16.** Let \((R, m)\) be a Cohen-Macaulay local ring of dimension two, \( I, J \) be \( m \)-primary ideals in \( R \) and \((a,b)\) a Rees-joint reduction of \( I \) and \( J \). Then

\[(a)\]
\[
e_{(0,1)} - e_1(J) = \sum_{i=0}^{r-1} \ell(M_{i,s}^1) \text{ for } r \gg 0 \text{ and } s \gg 0.
\]

\[(b)\]
\[
e_{(1,0)} - e_1(I) = \sum_{i=0}^{s-1} \ell(M_{r,i}^1) \text{ for } r \gg 0.
\]

\[(b)\]
\[
e_{(0,1)} \geq e_1(J) \text{ (resp. } e_{(1,0)} \geq e_1(I)) \text{ and the equality holds if and only if for all } i \geq 0 \text{ and } s \gg 0, M_{i,s}^1 = 0 \text{ (resp. for all } i \geq 0 \text{ and } r \gg 0, M_{r,i}^1 = 0). \]

\[(c)\]
\[
\text{Let } i \geq 0. \text{ Then for all } s \gg 0 \text{ (resp. } r \gg 0), \ell(M_{i,s}^1) \text{ (resp. } \ell(M_{r,i}^1)) \text{ is independent of } s \text{ (resp. } r). \]

Proof. Put \( k = 1 \) in Proposition 3.9(c) to get (3.17) and (3.18). (b) is immediate from (3.17) and (3.18). By putting \( k = 1 \) in (3.10) and (3.11) we get 3.16(c). \( \square \)

The inequalities in Corollary 3.16(b) can be strict (see Examples 5.3 and 5.4). In fact, Example 5.3 shows that the difference \( e_{(0,1)} - e_1(J) \) can be as large as possible.

**Notation 3.19.** Let \( i, j \geq 0 \) and \((a,b)\) a Rees-joint reduction of \( I \) and \( J \). Since \( \ell(M_{i,s}^1) \) (resp. \( \ell(M_{r,i}^1) \)) is independent of \( s \) (resp \( r \)) for all \( s \gg 0 \) (resp \( r \gg 0 \)) by Corollary 3.16(c), we set

\[
M_{i,s}^1 := \ell(M_{i,s}^1) \text{ for } s \gg 0
\]

\[
M_{r,i}^1 := \ell(M_{r,i}^1) \text{ for } r \gg 0
\]

In Proposition 3.24 we generalize Corollary 3.16(b) and prove that \( e_{(0,1)} - g_1(r) - re_{(1,1)} \geq 0 \) and \( e_{(1,0)} - h_1(s) - se_{(1,1)} \geq 0 \). We also give a criteria for the equality to hold. We need the following lemma for this purpose.

**Lemma 3.20.** For \( i, j \geq 0 \) and \((a,b)\) a Rees-joint reduction of \( I \) and \( J \),

\[
M_{i,s}^1 = g_1(i + 1) - g_1(i) + e_{(1,1)}, \tag{3.21}
\]

\[
M_{r,j}^1 = h_1(j + 1) - h_1(j) + e_{(1,1)}. \tag{3.22}
\]

In particular, \( g_1(i + 1) - g_1(i) + e_{(1,1)} \geq 0 \) and \( h_1(j + 1) - h_1(j) + e_{(1,1)} \geq 0 \).
Proof. Put \( k = 1 \) in (3.10) and (3.11) to get the result. \( \square \)

**Remark 3.23.** By Lemma 3.20 and (3.4) it follows that for \( i \gg 0, M_{i,*}^1 = 0 \). Similarly, for \( j \gg 0, M_{*,j}^1 = 0 \).

**Proposition 3.24.** Let \((R, m)\) be a Cohen-Macaulay local ring of dimension two, \( I, J \) be \( m \)-primary ideals in \( R \) and \((a, b)\) a Rees-joint reduction of \( I \) and \( J \). Then

(a) For all \( r \geq 0 \), \( e_{(0,1)} - g_1(r) - re_{(1,1)} \geq 0 \) and equality holds if and only if \( M_{i,*}^1 = 0 \) for all \( i \geq r \).

(b) For all \( s \geq 0 \), \( e_{(1,0)} - h_1(s) - se_{(1,1)} \geq 0 \) and equality holds if and only if \( M_{*,j}^1 = 0 \) for all \( j \geq s \).

**Proof.** (a) Since \( M_{i,*}^1 = g_1(i + 1) - g_1(i) + e_{(1,1)} \) by (3.21) and \( g_1(0) = e_1(J) \) by (3.3),

\[
\sum_{i=0}^{r-1} M_{i,*}^1 = g_1(r) - g_1(0) + re_{(1,1)} = g_1(r) - e_1(J) + re_{(1,1)}.
\]

Hence

\[
e_{(0,1)} - g_1(r) - re_{(1,1)} = e_{(0,1)} - e_1(J) - \sum_{i=0}^{r-1} M_{i,*}^1 \geq e_{(0,1)} - e_1(J) - \sum_{i=0}^{\infty} M_{i,*}^1 = 0 \quad \text{ (by (3.17))}.
\]

Hence it follows that \( e_{(0,1)} - g_1(r) - re_{(1,1)} = 0 \) if and only if \( M_{i,*}^1 = 0 \) for all \( i \geq r \). The proof of (b) is similar. \( \square \)

We observe that if we put \( r = 0 \) and \( s = 0 \) in Proposition 3.24 we obtain Corollary 3.16(b).

We are now ready to generalize Rees’ theorem for the filtration \( \mathcal{F} \). This result characterizes joint reduction number zero of \( I^k \) and \( J^k \) for \( k \gg 0 \) in terms of the Hilbert coefficients and the bigraded Hilbert coefficients. We need to make an additional assumption that \( e_{(1,0)} = e_1(I) \) and \( e_{(0,1)} = e_1(J) \). These assumptions hold true for the filtration \( \mathcal{F} \) (c.f. [18, Theorem 1.2]).

**Theorem 3.25.** Let \((R, m)\) be a Cohen-Macaulay local ring of dimension two and \( I, J \) be \( m \)-primary ideals in \( R \). Then the following statements are equivalent:

(a) \( r(I^k|J^k) = 0 \) for all \( k \gg 0 \);

(b) \( r(I^k|J^k) = 0 \) for some \( k \gg 0 \);

(c) \( e_{(1,0)} = e_1(I), e_{(0,1)} = e_1(J) \), and \( e_2(IJ) = e_2(I) + e_2(J) \);

(d) there exists a joint reduction \((a, b)\) of \( I \) and \( J \) such that \( M_{0,0}^1(a^k, b^k) = 0 \) for all \( k \gg 0 \).

**Proof.** (a) \( \implies \) (b) is clear.
(b) \implies (c): Fix \( k \geq 1 \) such that \( r(I^k|J^k) = 0 \). Then \( r(I^{nk}|J^{nk}) = 0 \) for all \( n \geq 1 \). Hence there exists a joint reduction \((a, b)\) of \( I \) and \( J \) such that for all \( n \gg 0 \)

\[
0 = M_{0,0}^1(a^{nk}, b^{nk}) = [e_{(0,1)} - e_1(J) + e_{(1,0)} - e_1(I)]nk - e_2(IJ) + e_2(J) + e_2(I) \quad \text{(by Corollary 3.15)}.
\]

This implies that \( e_2(IJ) = e_2(J) + e_2(I) \) and \( e_{(0,1)} - e_1(J) + e_{(1,0)} - e_1(I) = 0 \). Since \( e_{(0,1)} - e_1(J) \geq 0 \) and \( e_{(1,0)} - e_1(I) \geq 0 \) by Corollary 3.16, \( e_{(0,1)} = e_1(J) \) and \( e_{(1,0)} = e_1(I) \). This proves (c).

(c) \implies (d): Let \((a, b)\) be a joint reduction of \( I \) and \( J \). Then the result follows from Corollary 3.15.

(d) \implies (a): This is clear. \( \square \)

We remark that the equivalent conditions of Theorem 3.25 need not imply that the joint reduction number of \( I \) and \( J \) is zero. Example 3.27 illustrates this. Before that we make the following remark.

**Remark 3.26.** From [22, Theorem 3.2] it follows that if \( r(I,J) = 0 \) for \( m \)-primary ideals \( I \) and \( J \) in a Cohen-Macaulay local ring of dimension two with infinite residue field, then the condition \( IJ = aJ + bI \) holds for every joint reduction of \( I \) and \( J \).

**Example 3.27.** Let \( R = \mathbb{k}[x, y], I = (x^4, x^3y, xy^3, y^4) \) and \( m = (x, y) \). Then \((x^4, y)\) is a joint reduction of \( I \) and \( m \), and \( I^2m^2 = x^8m^2 + y^2I^2 \). Hence \( r(I^2|m^2) = 0 \). Therefore the equivalent conditions of Theorem 3.25 hold true. But \( r(I|m) \neq 0 \), since \( x^2y^3 \in mI \setminus xy^3m + yI \) (c.f. Remark 3.26).

In the next theorem we show that if we make an additional assumption that \( \text{depth} G(I) \geq 1 \) and \( \text{depth} G(J) \geq 1 \), then Theorem 3.25 holds true for \( k = 1 \).

**Theorem 3.28.** Let \((R, m)\) be a Cohen-Macaulay local ring of dimension two and \( I, J \) be \( m \)-primary ideals in \( R \). Assume that \( \text{depth} G(I), \text{depth} G(J) \geq 1 \). Then following statements are equivalent:

(a) \( r(I|J) = 0; \)

(b) \( e_{(0,1)} = e_1(I), e_{(0,1)} = e_1(J) \) and \( e_2(IJ) = e_2(I) + e_2(J); \)

(c) there exists a joint reduction \((a, b)\) of \( I \) and \( J \) such that \( M_{0,0}^1(a^k, b^k) = 0 \) for all \( k \geq 1 \).

**Proof.** (a) \implies (b): Follows from Theorem 3.25.

(b) \implies (c): By [19, Lemma 1.2] there exists a Rees-joint reduction \((a, b)\) of \( I \) and \( J \) such that \( a \in I \setminus mI \) and \( b \in J \setminus mJ \). Let \( a^* \) (resp. \( b^* \)) denotes the image of \( a \) (resp. \( b \)) in \([G(I)]_1 \) (resp. \([G(J)]_1 \)). Since \( \text{depth} G(I) \) (resp. \( \text{depth} G(J) \)) \( \geq 1 \), \( a^* \) (resp. \( b^* \)) is a nonzero-divisor in \( G(I) \) (resp. \( G(J) \)). Hence by [4, Lemma 2.1],

\[
(a) \cap I^n = aI^{n-1} \quad \text{and} \quad (b) \cap J^n = bJ^{n-1} \quad \text{for all} \ n > 0.
\]
To complete the proof we need to show that $I^k J^k = a^k J^k + b^k I^k$ for all $k \geq 1$. By Theorem 3.25 there exists $N$ so that $I^k J^k = a^k J^k + b^k I^k$ for all $k \geq N$. We use decreasing induction on $k$ to show that $I^k J^k = a^k J^k + b^k I^k$ for all $k \geq 1$.

We claim that $I^{N-1} J^{N-1} = a^{N-1} J^{N-1} + b^{N-1} I^{N-1}$. We first show that $I^{N-1} J^N = a^{N-1} J^N + b^N I^{N-1}$. Let $x \in I^{N-1} J^N$. Then $ax \in I^N J^n$. Hence there exist $p \in J^N$ and $q \in I^N$ such that

$$ax = a^N p + b^N q. \quad (3.29)$$

This implies that $q \in (a) \cap I^N = a I^{N-1}$. Hence there exists $q' \in I^{N-1}$ such that $q = aq'$. Plugging in (3.29) we get $ax = a^N p + ab^N q'$. As $a$ is a regular element

$$x = a^{N-1} p + b^N q' \in a^{N-1} J^N + b^N I^{N-1}.$$ 

Hence $I^{N-1} J^N = a^{N-1} J^N + b^k I^{N-1}$. Repeating the above argument we get that $I^{N-1} J^{N-1} = a^{N-1} J^{N-1} + b^{N-1} I^{N-1}$. Thus by decreasing induction on $k$ we get that $I^k J^k = a^k J^k + b^k I^k$ for all $k \geq 1$. Hence $M^1_{0,0}(a^k, b^k) = 0$ for all $k \geq 1$.

(c) $\implies$ (a): This follows from the definition of $M^1_{0,0}(a^k, b^k)$. 

In the following theorem we generalize Theorem 3.25. As a consequence we give a sufficient condition for the vector $(r_0, s_0) \in \Lambda(I^k | J^k)$ for $k \gg 0$ in Corollary 3.31.

**Theorem 3.30.** Let $(R, m)$ be a Cohen-Macaulay local ring of dimension two and $I, J$ be $m$-primary ideals in $R$. Let $(a, b)$ be a joint reduction of $I$ and $J$. Let $r_0, s_0 \geq 0$. Then the following statements are equivalent:

(a) for $k \gg 0$

$$I^{r_0+k} J^{s_0+k} = a^k I^{r_0} J^{s_0+k} + b^k I^{r_0+k} J^{s_0},$$

(b) $e_{(0, 1)} = g_1(r_0) + r_0 e_{(1, 1)}, e_{(1, 0)} = h_1(s_0) + s_0 e_{(1, 1)}$ and $e_2(IJ) = g_2(r_0) + h_2(s_0) - \ell(R/I^{r_0} J^{s_0}) + r_0 s_0 e_{(1, 1)} + \ell \left( \frac{\overline{F}_{(a, b)}(I^{r_0}, J^{s_0})}{I^{r_0} J^{s_0}} \right)$;

(c) $M^1_{r_0, s_0}(a^k, b^k) = 0$ for $k \gg 0$.

**Proof.** (a) $\implies$ (b): Since $I^{r_0+k} J^{s_0+k} = a^k I^{r_0} J^{s_0+k} + b^k I^{r_0+k} J^{s_0}$ for $k \gg 0$, by definition $M^1_{r_0, s_0}(a^k, b^k) = 0$ for $k \gg 0$. Hence by Proposition 3.9(a)

$$[e_{(0, 1)} - g_1(r_0) - r_0 e_{(1, 1)}] + [e_{(1, 0)} - h_1(s_0) - s_0 e_{(1, 1)}] = 0.$$ 

Since $e_{(0, 1)} - g_1(r_0) - r_0 e_{(1, 1)} \geq 0$ and $e_{(1, 0)} - h_1(s_0) - s_0 e_{(1, 1)} \geq 0$ by Proposition 3.24, we get that $e_{(0, 1)} = g_1(r_0) + r_0 e_{(1, 1)}, e_{(1, 0)} = h_1(s_0) + s_0 e_{(1, 1)}$. Now using Proposition 3.9(a) we get

$$e_2(IJ) = g_2(r_0) + h_2(s_0) - \ell(R/I^{r_0} J^{s_0}) + r_0 s_0 e_{(1, 1)} + \ell \left( \frac{\overline{F}_{(a, b)}(I^{r_0}, J^{s_0})}{I^{r_0} J^{s_0}} \right).$$
(b) \implies (c): By Proposition 3.9(a) for every joint reduction \((a, b)\) of \(I\) and \(J\) \(M_{r_0, s_0}^1(a^k, b^k) = 0\) for \(k \gg 0\).

(c) \implies (a): This is clear by definition of \(M_{r_0, s_0}^1(a^k, b^k)\). \(\square\)

**Corollary 3.31.** Let the assumptions be as in Theorem 3.30. If any of the equivalent conditions of Theorem 3.30 are satisfied, then \((r_0, s_0) \in \Lambda(I^k|J^k)\) for all \(k \gg 0\).

**Proof.** By assumption there exists a joint reduction \((a, b)\) of \(I\) and \(J\) such that for \(k \gg 0\)

\[
I^{r_0+k}J^{s_0+k} = a^kI^{r_0}J^{s_0+k} + b^kI^{r_0+k}J^{s_0}. \quad (3.32)
\]

Then multiplying Equation 3.32 by \(I^{(k-1)r_0}J^{(k-1)s_0}\) we get

\[
I^{kr_0+k}J^{ks_0+k} = a^kI^{kr_0}J^{ks_0+k} + b^kI^{kr_0+k}J^{ks_0}.
\]

Hence \((r_0, s_0) \in \Lambda(I^k|J^k)\) for \(k \gg 0\). \(\square\)

We now give a criterion for the a vector \((r_0, s_0)\) to be in \(\Lambda(I|J)\) under the additional assumptions (3.34) and (3.35).

**Theorem 3.33.** Let \((R, m)\) be a Cohen-Macaulay local ring of dimension two and \(I, J\) be \(m\)-primary ideals in \(R\). Let \(r_0, s_0 \geq 0\). Assume that there exists a joint reduction \((a, b)\) of \(I\) and \(J\) such that

\[
(a) \cap I^{r_0+k}J^{s_0} = aI^{r_0+k-1}J^{s_0} \text{ for } k \geq 1 \quad (3.34)
\]

\[
(b) \cap I^{r_0}J^{s_0+k} = bI^{r_0}J^{s_0+k-1} \text{ for } k \geq 1. \quad (3.35)
\]

Then the following statements are equivalent:

(a) \((r_0, s_0) \in \Lambda(I|J)\);

(b) \(e_{(0,1)} = g_1(r_0) + r_0e_{(1,1)}\), \(e_{(1,0)} = h_1(s_0) + s_0e_{(1,1)}\) and \(e_2(IJ) = g_2(r_0) + h_2(s_0) - \ell(R/I^{r_0}J^{s_0}) + r_0s_0\ell e_{(1,1)} + \ell \left( \frac{r_0s_0 e_{(1,1)} + \ell (r_0s_0 e_{(1,1)})}{I^{r_0}J^{s_0}} \right)\);

(c) \(M^1_{r_0, s_0}(a^k, b^k) = 0\) for all \(k \geq 1\).

**Proof.** (a) \implies (b): Let \((a_1, b_1)\) be a joint reduction of \(I\) and \(J\) such that for all \(r \geq r_0\) and \(s \geq s_0\),

\[
I^{r+1}J^{s+1} = a_1I^rJ^{s+1} + b_1I^{r+1}J^s.
\]

Then by induction on \(k\) it follows that for all \(k \geq 1\), and all \(r \geq r_0\) and \(s \geq s_0\)

\[
I^{r+k}J^{s+k} = a_1^kI^rJ^{s+k} + b_1^kI^{r+k}J^s.
\]

Hence the result follows from Theorem 3.30.

(b) \implies (c): By Proposition 3.9(a) \(M^1_{r_0, s_0}(a^k, b^k) = 0\) for \(k \gg 0\). Therefore

\[
I^{r_0+k}J^{s_0+k} = a^kI^{r_0}J^{s_0+k} + b^kI^{r_0+k}J^{s_0}, \text{ say for } k \geq N.
\]
To complete the proof we need to show $I^{r_0+k}J^{s_0+k} = a^kI^{r_0}J^{s_0+k} + b^kI^{r_0+k}J^{s_0}$ for $k \geq 1$. We first show that

$$I^{r_0+N-1}J^{s_0+N} = a^{N-1}I^{r_0}J^{s_0+N} + b^N I^{r_0+N-1}J^{s_0}.$$  

Let $x \in I^{r_0+N-1}J^{s_0+N}$. Then $ax \in I^{r_0+N}J^{s_0+N}$. Let $ax = a^Np + b^Nq$ for some $p \in I^{r_0}J^{s_0+N}$ and $q \in I^{r_0+N}J^{s_0}$. Then $q \in (a) \cap I^{r_0+N}J^{s_0} = aI^{r_0+N-1}J^{s_0}$ by (3.34). Let $q = az$ for some $z \in I^{r_0+N-1}J^{s_0}$. Then $x = a^{N-1}p + b^Nz' \in a^{N-1}I^{r_0}J^{s_0+N} + b^N I^{r_0+N-1}J^{s_0}$.

Similar argument shows that

$$I^{r_0+N}J^{s_0+N-1} = a^N I^{r_0}J^{s_0+N-1} + b^{N-1} I^{r_0+N}J^{s_0}.$$  

Continuing as above we get that for all $k \geq 1$

$$I^{r_0+k}J^{s_0+k} = a^kI^{r_0}J^{s_0+k} + b^k I^{r_0+k}J^{s_0}.$$  

Hence $M^{1}_{r_0,s_0}(a^k,b^k) = 0$ for all $k \geq 1$.

(e) $\implies$ (a): This follows from the definition of $M^{1}_{r_0,s_0}(a^k,b^k)$.  

4. Characterization of a vector in the joint reduction lattice in terms of Local Cohomology

The aim of this section is to characterize a vector in the joint reduction lattice in terms of the vanishing of the local cohomology modules. This is motivated by the work of the second author with Verma in [12] for the filtration $\overline{F}$. In order to extend their result for the filtration $F$, first we derive a formula for $[H^2_{(at_1,bt_2)}(R' F)]_{(r,s)}$ as a direct limit of $M^1_{r,s}(a^k,b^k)$ (Theorem 4.2). In [12] the authors prove that for the filtration $F$ and a good joint reduction $(a,b)$ of $F$

$$[H^2_{(at_1, bt_2)}(R' F)]_{(r,s)} \cong M^1_{r,s}(a^k,b^k;F) := \frac{I^{r+k}J^{s+k}}{a^{k+r}J^s + b^k I^{r+k}J^s},$$

which in particular shows that the length of $[H^2_{(at_1, bt_2)}(R' F)]_{(r,s)}$ is finite. However, for the filtration $F$, $[H^2_{(at_1, bt_2)}(R' F)]_{(r,s)}$ need not be equal to $M^1_{r,s}(a^k,b^k)$ for large $k$. In fact, the length of $[H^2_{(at_1, bt_2)}(R' F)]_{(r,s)}$ can be infinite (see Example 5.3). In this section we investigate the finiteness of the length of the local cohomology modules $[H^2_{(at_1, bt_2)}(R' F)]_{(r,s)}$ and their vanishing in terms of the Hilbert and bigraded Hilbert coefficients, as well as in terms of the vanishing of the modules $M^1_{i,j}$ for all $i \geq r$ and $M^1_{i,j}$ for all $j \geq s$.

For the sake of simplicity we set $R' := R'(F)$. Let $a \in I$ and $b \in J$. Consider the Koszul co-complex

$$K^\bullet((at_1)^k,(bt_2)^k;R') : 0 \longrightarrow R' \overset{\alpha_k}{\longrightarrow} R'(k,0) \oplus R'(0,k) \overset{\beta_k}{\longrightarrow} R'(k,k) \longrightarrow 0,$$

where the maps are defined as,

$$\alpha_k(1) = ((at_1)^k,(bt_2)^k) \quad \text{and} \quad \beta_k(u,v) = -(bt_2)^ku + (at_1)^kv.$$
Then for $i = 0, 1, 2$,
\[
H_{(at_1, bt_2)}^i(\mathcal{R}') = \lim_{k} H^i(K^*((at_1)^k, (bt_2)^k; \mathcal{R}')) \quad [2, \text{Theorem 5.2.9}]. \tag{4.1}
\]

In the following theorem we recover some results from $[12, \text{Theorem 3.6}]$ for $[H^2_{(at_1, bt_2)}(\mathcal{R}')]_{(r,s)}$, where $(a,b)$ is a Rees-joint reduction of $I$ and $J$.

**Theorem 4.2.** Let $(R, m)$ be a Cohen-Macaulay local ring of dimension two. Let $I$ and $J$ be $m$-primary ideals in $R$ and $(a,b)$ a joint reduction of $I$ and $J$. Then for all $r, s \geq 0$,

(a) $[H^2_{(at_1, bt_2)}(\mathcal{R}')]_{(r,s)} \cong \lim_{k} M^1_{r,s}(a^k, b^k)$.

(b) If in addition $a$ and $b$ are Rees-superficial elements for $I$ and $J$, respectively, then for $k \gg 0$ the maps

\[
\mu_k : M^1_{r,s}(a^k, b^k) \ni (ab) \mapsto M^1_{r,s}(a^{k+1}, b^{k+1})
\]

are injective.

**Proof.** (a) Follows from (4.1).

(b) For $x \in I^{r+k}J^{s+k}$, we write $\bar{x}$ for the image of $x$ in $M^1_{r,s}(a^k, b^k)$. Let $x \in I^{r+k}J^{s+k}$ be such that $\mu_k(\bar{x}) = 0$. Then $xab = a^{k+1}p + b^{k+1}q$ for some $p \in I^{r}J^{s+k+1}$ and $q \in I^{r+k+1}J^{s}$. Hence $q \in (a) \cap I^{r+k+1}J^{s} = aI^{r+k}J^{s}$ for $k \gg 0$. Therefore $q = aq'$ for some $q' \in I^{r+k}J^{s}$. Similarly for $k \gg 0$, $p = bp'$ for some $p' \in I^{r}J^{s+k}$. Hence

\[
x = a^{k}p' + b^{k}q' \in a^{k}I^{r}J^{s+k} + b^{k}I^{r+k}J^{s}.
\]

Thus $\bar{x} = 0$ and hence $\mu_k$ is injective for all $k \gg 0$. □

The map $\mu_k$ defined in Theorem 4.2 need not be surjective for $k \gg 0$. In fact, if the map $\mu_k$ in Theorem 4.2 is surjective for $k \gg 0$, then $\mu_k$ is an isomorphism for $k \gg 0$ by Theorem 4.2 and this implies that $[H^2_{(at_1, bt_2)}(\mathcal{R}')]_{(r,s)}$ has finite length which need not be true (see Example 5.3). The non-finiteness of the length of $[H^2_{(at_1, bt_2)}(\mathcal{R}')]_{(r,s)}$ is one of the obstructions in extending Rees’ theorem for the ordinary powers of ideals. In the following theorem we give equivalent conditions for $\ell_R([H^2_{(at_1, bt_2)}(\mathcal{R}')]_{(r,s)})$ to be finite in terms of the Hilbert and the bigraded Hilbert coefficients, and also in terms of the vanishing of the modules $M^1_{i,s}$ and $M^1_{s,j}$. We remark that in $[12, \text{Theorem 3.7}]$ the authors derived a formula for $\ell([H^2_{(at_1, bt_2)}(\mathcal{R}'(\mathcal{F}))]_{(r,s)})$, which in particular shows that $[H^2_{(at_1, bt_2)}(\mathcal{R}'(\mathcal{F}))]_{(r,s)}$ has finite length.

**Theorem 4.3.** Let $(R, m)$ be a Cohen-Macaulay local ring of dimension two, $I, J$ be $m$-primary ideals in $R$ and let $(a,b)$ a Rees-joint reduction of $I$ and $J$. Let $r_0, s_0 \geq 0$ be fixed. Then the following statements are equivalent:

(a) $\ell_R([H^2_{(at_1, bt_2)}(\mathcal{R}')]_{(r_0,s_0)}) < \infty;$
(b) \( e_{(0,1)} = g_1(r_0) + r_0 e_{(1,1)} \) and \( e_{(1,0)} = h_1(s_0) + s_0 e_{(1,1)} \);
(c) \( M^1_{i,*} = 0 \) for \( i \geq r_0 \) and \( M^1_{*,j} = 0 \) for all \( j \geq s_0 \).

If any of the above equivalent conditions hold true, then

\[
\ell_R([H^2_{(at_1, bt_2)}(\mathcal{R}')])_{(r_0, s_0)} = -e_2(IJ) + g_2(r_0) + h_2(s_0) - \ell(R/I^{r_0}J^{s_0}) + r_0 s_0 e_{(1,1)} + \ell\left( \frac{\tilde{F}_{(a,b)}(I^{r_0}, J^{s_0})}{I^{r_0}J^{s_0}} \right).
\]

**Proof.** (a) \( \implies \) (b): By Proposition 3.9(a) for \( k \gg 0 \), \( \ell(M^1_{r_0,s_0}(a^k,b^k)) \) is a polynomial in \( k \) of degree at most 1. By Theorem 4.2 for all \( k \gg 0 \),

\[
\ell\left( M^1_{r_0,s_0}(a^k,b^k) \right) \leq \ell([H^2_{(at_1, bt_2)}(\mathcal{R}')])_{(r_0, s_0)} < \infty.
\]

Hence \( \ell(M^1_{r_0,s_0}(a^k,b^k)) \) is a constant for \( k \gg 0 \). This implies that \( [e_{(0,1)} - g_1(r_0) - r_0 e_{(1,1)}] + [e_{(1,0)} - h_1(s_0) - s_0 e_{(1,1)}] = 0 \) by Proposition 3.9(a). Since \( e_{(0,1)} - g_1(r_0) - r_0 e_{(1,1)} \) and \( e_{(1,0)} - h_1(s_0) - s_0 e_{(1,1)} \) are non-negative by Proposition 3.24,

\[
e_{(0,1)} - g_1(r_0) - r_0 e_{(1,1)} = e_{(1,0)} - h_1(s_0) - s_0 e_{(1,1)} = 0
\]

which gives the result.

(b) \( \implies \) (c): Follows from Proposition 3.24.

(c) \( \implies \) (a): Since \( M^1_{i,*} = 0 \) for all \( i \geq r_0 \) and \( M^1_{*,j} = 0 \) for all \( j \geq s_0 \), by Proposition 3.24

\[
e_{(0,1)} = g_1(r_0) + r_0 e_{(1,1)} \) and \( e_{(1,0)} = h_1(s_0) + s_0 e_{(1,1)} \).

Substituting for \( e_{(0,1)} \) and \( e_{(1,0)} \) in Proposition 3.9(a), we get \( \ell(M^1_{r_0,s_0}(a^k,b^k)) \) is a constant for \( k \gg 0 \). Since \( \mu_k \) is injective for \( k \gg 0 \), we conclude that \( \mu_k \) is also surjective for \( k \gg 0 \) and hence is an isomorphism for \( k \gg 0 \). Thus for \( k \gg 0 \),

\[
[H^2_{(at_1, bt_2)}(\mathcal{R}')]_{(r_0, s_0)} \cong M^1_{r_0,s_0}(a^k,b^k)
\]

and hence \( \ell([H^2_{(at_1, bt_2)}(\mathcal{R}')])_{(r_0, s_0)} < \infty \). \( \square \)

As a corollary we give equivalent conditions for \( \ell_R([H^2_{(at_1, bt_2)}(\mathcal{R}')])_{(0,0)} < \infty \).

**Corollary 4.4.** With the assumptions as in Theorem 4.3, the following statements are equivalent:

(a) \( \ell_R([H^2_{(at_1, bt_2)}(\mathcal{R}')])_{(0,0)} < \infty \);
(b) \( e_{(1,0)} = e_1(I) \) and \( e_{(0,1)} = e_1(J) \);
(c) \( M^1_{i,*} = 0 \) for \( i \geq 0 \) and \( M^1_{*,j} = 0 \) for all \( j \geq 0 \).

If any of the above equivalent conditions hold true, then

\[
\ell_R([H^2_{(at_1, bt_2)}(\mathcal{R}')])_{(0,0)} = -e_2(IJ) + e_2(I) + e_2(J).
\]

**Proof.** Put \( r_0 = s_0 = 0 \) in Theorem 4.3. \( \square \)
Theorem 4.5. With the assumptions as in Theorem 4.3, if $\ell([H^2_{(at_1, bt_2)}(R')](r_0, s_0)) < \infty$ for some $r_0, s_0 \geq 0$, then $\ell([H^2_{(at_1, bt_2)}(R')](r, s)) < \infty$ for all $r \geq r_0$ and $s \geq s_0$.

Proof. Suppose $\ell([H^2_{(at_1, bt_2)}(R')](r_0, s_0)) < \infty$ for some $r_0, s_0 \geq 0$. Then by Theorem 4.3(c) $M_i^s = 0$ for all $i \geq r_0$ and $M_j^s = 0$ for all $j \geq s_0$. As $r \geq r_0$ and $s \geq s_0$, $M_i^s = 0$ for all $i \geq r$ and $M_j^s = 0$ for all $j \geq s$. Therefore using Theorem 4.3 once again we get $\ell([H^2_{(at_1, bt_2)}(R')](r, s)) < \infty$ for all $r \geq r_0$ and $s \geq s_0$.

In Section 5 we will give an example to show that $e_{(1,0)} \neq e_1(I)$ and hence $\ell([H^2_{(at_1, bt_2)}(R')](0,0))$ is not finite. Here, we give an example for which $\ell_R([H^2_{(at_1, bt_2)}(R')](r, s))$ is finite, but need not be zero. Recall that an ideal $K \subseteq I$ is called a reduction of $I$ if $KI^n = I^{n+1}$ for some $n$.

Example 4.6. Let $(R, m)$ be a Cohen-Macaulay local ring of dimension two and $I$ be a $m$-primary ideal of $R$. Let $J = I$. Then $e_{(1,0)} = e_{(0,1)} = e_1(I)$. Hence, for any reduction $(a, b)$ of $I$ such that $a$ and $b$ are superficial elements, $\ell([H^2_{(at_1, bt_2)}(R')](0,0)) = e_2(I) < \infty$ by Corollary 4.4.

In the rest of this section we give necessary and sufficient conditions for the vanishing of $[H^2_{(at_1, bt_2)}(R')](r, s)$. The following theorem gives a cohomological interpretation of Theorem 3.30.

Theorem 4.7. Let $(R, m)$ be a Cohen-Macaulay local ring of dimension two, $I, J$ be $m$-primary ideals in $R$ and let $(a, b)$ a joint reduction of $I$ and $J$. Let $r_0, s_0 \geq 0$. Then the following statements are equivalent:

(a) For all $k \gg 0$

\[ I^{r_0+k}J^{s_0+k} = a^k I^{r_0}J^{s_0+k} + b^k I^{r_0+k}J^{s_0}; \]

(b) $e_{(0,1)} = g_1(r_0)+r_0e_{(1,1)}, e_{(1,0)} = h_1(s_0)+s_0e_{(1,1)}$ and $e_2(IJ) = g_2(r_0)+h_2(s_0) - \ell(R/I^{r_0}J^{s_0}) + r_0s_0 e_{(1,1)} + \ell\left(\frac{\mathcal{F}_{(a,b)}(I^{r_0}, J^{s_0})}{I^{r_0}J^{s_0}}\right)$;

(c) $[H^2_{(at_1, bt_2)}(R')](r_0, s_0) = 0$.

Proof. The equivalence of (a) and (b) follows from Theorem 3.30, while the equivalence of (b) and (c) follows from Theorem 4.3.

As a corollary we obtain a criteria for the vanishing of $[H^2_{(at_1, bt_2)}(R')](0,0) = 0$. This gives a cohomological interpretation of Theorem 3.25.

Corollary 4.8. Let $(R, m)$ be a Cohen-Macaulay local ring of dimension two and $I, J$ be $m$-primary ideals in $R$. Then the following statements are equivalent:

(a) $r(I^k|J^k) = 0$ for $k \gg 0$;

(b) $e_{(1,0)} = e_1(I), e_{(0,1)} = e_1(J)$ and $e_2(IJ) = e_2(I) + e_2(J)$.

(c) for every joint reduction $(a, b)$ of $I$ and $J$, $[H^2_{(at_1, bt_2)}(R')](0,0) = 0$;
Proof. Put $r_0 = s_0 = 0$ in Theorem 4.7.
\[\square\]

In the next theorem we give a criteria for a vector to be in the joint reduction lattice of $I$ and $J$ in terms of the vanishing of \([H^2_{(a_1,b_2)}(\mathcal{R}')]_{(r_0,s_0)}\). This gives a cohomological interpretation of Theorem 3.33.

**Theorem 4.9.** Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension two and $I, J$ be $\mathfrak{m}$-primary ideals in $R$. Let $r_0, s_0 \geq 0$. Assume that there exists a joint reduction $(a, b)$ of $I$ and $J$ such that

\[
(a) \cap I^{r_0+k}s_0 = aI^{r_0+k-1}s_0 \quad \text{for } k \geq 1
\]

\[
(b) \cap J^{s_0+k} = bJ^{s_0+k-1} \quad \text{for } k \geq 1.
\]

Then the following statements are equivalent:

(a) $(r_0, s_0) \in \Lambda(I,J)$;

(b) $e_{(0,1)} = g_1(r_0) + r_0 e_{(1,1)}$, $e_{(1,0)} = h_1(s_0) + s_0 e_{(1,1)}$ and $e_2(IJ) = g_2(r_0) + h_2(s_0) - \ell(R/I^{r_0}s_0) + r_0 s_0 e_{(1,1)} + \ell\left(\frac{\mathcal{F}_{(a,b)}(I^{r_0}s_0)}{I^{r_0}s_0}\right)$;

(c) \([H^2_{(a_1,b_2)}(\mathcal{R}')]_{(r_0,s_0)} = 0\).

Proof. The equivalence of (a) and (b) follows from Theorem 3.33, while the equivalence of (b) and (c) follows from Theorem 4.7.
\[\square\]

As a consequence we obtain the following corollary which gives a cohomological interpretation of Theorem 3.28.

**Corollary 4.10.** Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension two and $I, J$ be $\mathfrak{m}$-primary ideals in $R$. Assume that depth $G(I), \text{depth } G(J) \geq 1$. Then the following statements are equivalent.

(a) $r(I|J) = 0$;

(b) $e_{(1,0)} = e_1(I), e_{(0,1)} = e_1(J)$ and $e_2(IJ) = e_2(I) + e_2(J);

(c) for every joint reduction $(a, b)$ of $I$ and $J$, \([H^2_{(a_1,b_2)}(\mathcal{R}')]_{(0,0)} = 0\).

Proof. By [19, Lemma 1.2] there exists a Rees-joint reduction $(a, b)$ of $I$ and $J$ such that $a \in I \setminus \mathfrak{m}I$ and $b \in J \setminus \mathfrak{m}J$. Let $a^*$ (resp. $b^*$) denotes the image of $a$ (resp. $b$) in $[G(I)]_1$ (resp. $[G(J)]_1$). Since depth $G(I)$ (resp. depth $G(J)$) $\geq 1$, $a^*$ (resp. $b^*$) is a nonzero-divisor in $G(I)$ (resp. $G(J)$) by [4, Lemma 2.1]. Hence

\[
(a) \cap I^n = aI^{n-1} \quad \text{and} \quad (b) \cap J^n = bJ^{n-1} \quad \text{for all } n > 0.
\]

Therefore the equivalence of (a) and (b) follows from Theorem 4.9 by taking $r_0 = s_0 = 0$. The equivalence of (b) and (c) follows from Corollary 4.8.
In the following example we verify Theorem 3.28 and Corollary 4.10 for complete ideals in a regular local ring.

**Example 4.11.** Let \((R, \mathfrak{m})\) be a regular local ring of dimension two and let \(I, J\) be complete ideals (i.e. \(\mathfrak{T} = I\) and \(\mathfrak{T} = J\)) in \(R\). Then \(IJ = aJ + bI\) for any joint reduction \((a, b)\) of \(I\) and \(J\) [22, Theorem 2.1]. Hence \(M^{a,b}_{0,0}(a^k, b^k) = 0\) for all \(k \geq 1\) which implies that \([H^{2\{a_1, a_2\}(R')}_{0,0}] = 0\) by Theorem 4.2(a). Moreover, by [22, Theorem 3.2] \(e_{(0,1)} = e_1(I)\), \(e_{(0,1)} = e_1(J)\) and \(e_2(IJ) = e_2(I) + e_2(J)\). This verifies Theorem 3.28 and Corollary 4.10.

5. Examples

In this section we give an explicit example for which \(e_{(0,1)} \neq e_1(I)\) and \([H^{2\{b_1, a_2\}(R')}_{0,0}] = 0\) is not finite (Example 5.3). We also give an example where \(e_{(1,0)} \neq e_1(I)\) and \(e_{(0,1)} \neq e_1(J)\) (Example 5.4).

Recall that a reduction \(K\) is called a minimal reduction of \(I\) if whenever \(K' \subseteq K\) and \(K'\) is a reduction of \(I\), then \(K' = K\) [14]. The reduction number of \(I\) with respect to a minimal reduction \(K\) of \(I\) is defined as

\[
r_K(I) := \min\{n \geq 0 \mid KI^n = I^{n+1}\}.
\]

The reduction number of \(I\) denoted by \(r(I)\) is defined to be the minimum of \(r_K(I)\) where \(K\) varies over all minimal reductions of \(I\).

In order to obtain Example 5.3 we need the following proposition.

**Proposition 5.1.** Let \((R, \mathfrak{m})\) be a Cohen-Macaulay local ring of dimension two and \(I\) be an \(m\)-primary ideal in \(R\) with \(r(I) \geq 1\). Let \(J = (a, b)\) be a minimal reduction of \(I\) such that \(b \in I\) (resp. \(a \in J\)) is a Rees-superficial element for \(I\) (resp. \(J\)). Then

(a) \(IJ^{s+1} \neq bJ^{s+1} + aIJ^s\) for any \(s \geq 0\).
(b) \(e_{(0,1)} \neq e_1(J)\).

**Proof.** (a) Let \(I = (a, b, z_1, \ldots, z_{t-2})\) be a generating set of \(I\) such that \(z_i \notin (a, b)\) for every \(i = 1, \ldots, t - 2\). Note that \(t > 2\) by [9, Theorem 3.21]. To prove the lemma it is enough to show that for all \(s \geq 0\)

\[
b^{s+1}z_i \notin bJ^{s+1} + aIJ^s \text{ for all } i = 1, \ldots, t - 2.
\]

Suppose \(b^{s+1}z_i \in bJ^{s+1} + aIJ^s\) for some \(i\). Inductively, for all \(s \geq 0\), we have

\[
IJ^{s+1} = (a, b)^{s+2} + (a, b)^{s+1}(z_1, \ldots, z_{t-2}) \quad \text{and} \quad bJ^{s+1} + aIJ^s = (a, b)^{s+2} + a(a, b)^s(z_1, \ldots, z_{t-2}).
\] (5.2)
Hence from (5.2),
\[ b^{s+1}z_i = \sum_{k=0}^{s+2} x_k a^k b^{s+2-k} + a \sum_{j=1}^{s} \left( \sum_{k=0}^{t-2} y_{kj} a^k b^{s-k} z_j \right) \]
where \( x_k, y_{kj} \in R \). This implies that
\[ b^{s+1}(z_i - bx_0) \in (a). \]
As \( a, b \) is a regular sequence in \( R \),
\[ z_i - bx_0 \in (a). \]
Therefore \( z_i \in (a, b) \) which contradicts that \( z_i \notin (a, b) \).

(b) Suppose \( e_{(0,1)} = e_1(J) \). As \( (b, a) \) is a Rees-joint reduction of \( I \) and \( J \), by Corollary 3.16(b) \( M_{i,s}^1(b, a) \) for all \( i \geq 0 \) and \( s \gg 0 \). In particular,
\[ 0 = M_{0,s}^1(b, a) = \frac{IJ^{s+1}}{bJ^{s+1} + aIJ^s}. \]
This contradicts (a).

Now we give an explicit example for which \( e_{(0,1)} \neq e_1(J) \). In fact, the following example shows that the difference \( e_{(0,1)} - e_1(J) \) can be as large as possible.

**Example 5.3.** Let \( R = k[x, y] \), \( m = (x, y) \), \( I = m^t \), \( J = (x^t, y^t) \), \( t \geq 2 \). Put \( a = x^t \) and \( b = y^t \). Then \( b \in I \) (resp. \( a \in J \)) is a Rees-superficial element for \( I \) and \( J \). Therefore by Proposition 5.1 \( e_{(0,1)} \neq e_1(J) \). We explicitly calculate \( e_{(0,1)} - e_1(J) \). For all \( r, s \geq 1 \),
\[ \ell \left( \frac{R}{I^rJ^s} \right) = \ell \left( \frac{R}{m^{t(r+s)}} \right) = \binom{t(r+s)+1}{2} = t^2 \left( \frac{r+1}{2} \right) + t^2rs + t^2 \left( \frac{s+1}{2} \right) - \left( \frac{t}{2} \right)^2 r - \left( \frac{t}{2} \right)^2 s. \]
As \( J \) is a parameter ideal \( e_1(J) = 0 \). Hence \( e_{(0,1)} - e_1(J) = \binom{t}{2} \). Therefore by Corollary 4.4 the length of \( [H_{(b_1, a_2)}^2(R')]_{(0,0)} \) is not finite. \( \square \)

Notice that in Example 5.3 \( e_{(1,0)} = e_1(I) \). Next we give an example for which \( e_{(1,0)} \neq e_1(I) \) as well as \( e_{(0,1)} \neq e_1(J) \).

**Example 5.4.** Let \( R = k[x, y] \) where \( m = (x, y) \) and let \( a \geq 1 \). Put \( I = (x^2, y^2) \) and \( J = (x^3, y^3) \). We claim that

(a) \( e_{(1,0)} \neq e_1(I), e_{(0,1)} \neq e_1(J), e_2(IJ) \neq e_2(I) + e_2(J). \)
(b) \( r(I|J) \neq 0 \) even though \( G(I) \) and \( G(J) \) are Cohen-Macaulay.
(c) $[H^2_{((x^2)t_1,(y^2)t_2)}(\mathcal{R}^q)]_{(0,0)} \neq 0$.

(a) One can verify that $(IJ)^3 = (x^2 J + y^2 J)^3 \subseteq x^2 I^2 J^3 + y^3 I^3 J^2 \subseteq (IJ)^3$. Hence $I^3 J^3 = x^2 I^2 J^3 + y^3 I^3 J^2$ and for all $r, s \geq 3$,

$$I^r J^s = x^2 I^{r-1} J^s + y^3 I^r J^{s-1} = x^{2(r-2)} I^2 J^s + y^{3(s-2)} I^r J^2.$$  \hspace{1cm} (5.5)

For the rest of the proof we will assume that $r, s \geq 3$. Recall the complex $C'_*(((x^{2(r-2)}, y^{3(s-2)}), 2, 2)$ from (2.3)

$$0 \rightarrow \frac{R}{I^2 J^2} \xrightarrow{\psi_1} \frac{R}{I^r J^2} \oplus \frac{R}{I^2 J^s} \xrightarrow{\psi_0} \frac{(x^{2(r-2)}, y^{3(s-2)})}{x^{2(r-2)} I^2 J^s + y^{3(s-2)} I^r J^2} \rightarrow 0$$  \hspace{1cm} (5.6)

where $\psi_1 : \begin{pmatrix} x^{2(r-2)} \\ y^{3(s-2)} \end{pmatrix}$ and $\psi_0 : \begin{pmatrix} y^{3(s-2)} \\ -x^{2(r-2)} \end{pmatrix}$. We claim that the complex (5.6) is exact. Clearly $\psi_0$ is surjective. From the proof of Theorem 2.2(c), $\ker(\psi_0) = \text{Im}(\psi_1)$. We show that $\psi_1$ is injective. From the complex (5.6), $\ker(\psi_1) := (I^2 J^s : y^{3(s-2)}) \cap (I^r J^2 : x^{2(r-2)}) / I^2 J^2$. Therefore for $s \geq 3$

$$(x^9 y) + I^2 J^2 \subseteq (I^2 J^s : y^{3(s-2)})$$

$$= (x^9 I^2 J^{s-3} + y^{3(s-2)} I^2 J^2 : y^{3(s-2)})$$

$$= (x^9 (x^2 I + (y^4) J^{s-3}) + y^{3(s-2)} I^2 J^2 : y^{3(s-2)})$$

$$= \begin{cases} (x^{11}) + (x^9 y) + I^2 J^2 & \text{if } s = 3 \\ (x^{11}) + (x^{12}, x^9 y) + I^2 J^2 & \text{if } s > 3 \end{cases}$$

$$\subseteq (x^9 y) + I^2 J^2.$$  \hspace{1cm} (5.7)

Hence the equality holds in (5.7). Let $r \geq 3$. Then

$$(xy^9) + I^2 J^2 \subseteq (I^r J^2 : x^{2(r-2)})$$

$$= (x^{2(r-2)} I^2 J^2 + y^6 I^{2(r-3)} J^2 : x^{2(r-2)})$$

$$= \begin{cases} I^2 J^2 + y^6 (x^4, xy^3, y^6) & \text{if } r = 3 \\ I^2 J^2 + y^6 (x^4, x^2 y^2, xy^3, y^6) & \text{if } r = 4 \\ I^2 J^2 + y^6 (x^4, x^2 y^2, xy^3, y^4) & \text{if } r > 4 \end{cases}$$

$$\subseteq (xy^9) + I^2 J^2.$$  \hspace{1cm} (5.8)

Hence the equality holds in (5.8). From (5.7) and (5.8) we get

$$(I^2 J^s : y^{3(s-2)}) \cap (I^r J^2 : x^{2(r-2)}) = ((xy^9) + I^2 J^2) \cap ((x^9 y) + I^2 J^2) = (x^9 y^9) + I^2 J^2 = I^2 J^2.$$  \hspace{1cm} (5.9)

This implies that $\psi_1$ is injective. From (5.5) and (5.6), for all $r, s \geq 3$,

$$\ell \left( \frac{R}{I^r J^s} \right) = \ell \left( \frac{R}{x^{2(r-2)} y^{3(s-2)}} \right) + \ell \left( \frac{R}{I^2 J^2} \right) + \ell \left( \frac{R}{I^r J^2} \right) - \ell \left( \frac{R}{I^2 J^2} \right)$$
By induction we can show that for all \( r, s \geq 2 \),
\[
I^2 J^s = (x^{4+3s}, y^{4+3s}) + x^2 y^2 (x, y)^{3s} \quad \text{and} \quad I^r J^2 = (x^{2r+6}, y^{2r+6}) + x^2 y^2 (x, y)^{2r+2}.
\]

Therefore
\[
\ell \left( \frac{R}{I^2 J^s} \right) = \ell \left( \frac{R}{m^{4+3s}} \right) + 2 = \left( \frac{4 + 3s + 1}{2} \right) + 2 \quad \text{and} \quad \ell \left( \frac{R}{I^r J^2} \right) = \ell \left( \frac{R}{m^{2r+6}} \right) + 2 = \left( \frac{2r + 6 + 1}{2} \right) + 2.
\]

Hence for all \( r, s \geq 3 \),
\[
\ell \left( \frac{R}{I^r J^s} \right) = 6(r - 2)(s - 2) + \left( \frac{4 + 3s + 1}{2} \right) + 2 + \left( \frac{2r + 6 + 1}{2} \right) + 2 - \left( \frac{11}{2} \right) - 2 = 17 - 4 - 9 = 4 \neq 6 = e(1,0)(I,J).
\]

Hence \( e_{(1,0)} = 1, e_{(0,1)} = 3 \). As \( I \) and \( J \) are parameter ideals \( e_1(I) = e_1(J) = e_2(I) = e_2(J) = 0 \). Therefore \( e(1,0) - e_1(I) = 1, e_{(0,1)} - e_1(J) = 3 \) and \( e_2(IJ) - e_2(I) - e_2(J) = 2 \).

(b) By Theorem 3.28 \( r(I,J) \neq 0 \), even though \( G(I) \) and \( G(J) \) are Cohen-Macaulay. We verify this directly here. Suppose joint reduction number of \( I \) and \( J \) is zero. Then by [22, Theorem 3.2(d)],
\[
e_{(1,1)}(I,J) = \ell(R/IJ) - \ell(R/I) - \ell(R/J).
\]
But here
\[
\ell \left( \frac{R}{IJ} \right) - \ell \left( \frac{R}{I} \right) - \ell \left( \frac{R}{J} \right) = 17 - 4 - 9 = 4 \neq 6 = e_{(1,1)}(I,J).
\]
(c) Applying (a) and (b) to Theorem 4.10 we conclude that \( [H^2_{(x^2)t_1, (y^3)t_2}(\mathcal{R}^t)]_{(0,0)} \neq 0 \). We verify this directly. By Theorem 4.2(a)
\[
[H^2_{(x^2)t_1, (y^3)t_2}(\mathcal{R}^t)]_{(0,0)} \cong \lim_{k \to \infty} M_{0,0}^1 ((x^2)^k, (y^3)^k).
\]
For all \( k \geq 3 \),
\[
\ell \left( M_{0,0}^1 ((x^2)^k, (y^3)^k) \right) - \ell \left( \frac{R}{I^k J^k} \right) = \ell \left( \frac{R}{(x^2)^k J^k + (y^3)^k} \right) - \ell \left( \frac{R}{I^k J^k} \right) = \ell \left( \frac{R}{(x^2)^k J^k} \right) + \ell \left( \frac{R}{I^k} \right) + \ell \left( \frac{R}{J^k} \right) - \ell \left( \frac{R}{I^k J^k} \right) \quad [\text{by [22, Lemma 3.1]}]
\]
\[
= 6k^2 + 4 \left( \frac{k}{2} + 1 \right) + 9 \left( \frac{k}{2} + 1 \right) - \left[ 4 \left( \frac{k}{2} + 1 \right) + 6k^2 + 9 \left( \frac{k}{2} + 1 \right) - 4k + 2 \right] \quad [\text{From (5.10)}]
\]
\[
= 4k - 2.
\]
As \( k \geq 3 \), \( 4k - 2 \neq 0 \). Hence \([H^2_{((x^2),t_1),(y^3),t_2}((R'))}_{(0,0)} \neq 0\).

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