Connection between Dispersive Transport and Statistics of Extreme Events

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Abstract

A length dependence of the effective mobility in the form of a power law, \( B \sim L^{1-\frac{1}{\alpha}} \) is observed in dispersive transport in amorphous substances, with \( 0 < \alpha < 1 \). We deduce this behavior as a simple consequence of the statistical theory of extreme events. We derive various quantities related to the largest value in samples of \( n \) trials, for the exponential and power-law probability densities of the individual events.

I. INTRODUCTION

Dispersive transport in amorphous materials has been studied for almost three decades now, and yet it continues to attract considerable attention. In these studies, charge carriers are created at one side of a slab of the material and transported across to the other side. It has been realised that the transport phenomena in amorphous media cannot be described by standard concepts like uniform drift and diffusional spreading, see [1–5] for exhaustive reviews on this subject and [6] for a popular account. An important and striking observation is the (apparent) dependence of the mobility on the thickness of the material. The general features of the mechanism of dispersive transport in amorphous materials are well understood; the process is subdiffusive and the delay in the transport occurs mainly due to trapping events in localized centers of various energetic depths. A fairly large body of theory has been developed to describe several aspects of the transport phenomena; already the early theoretical work of Shlesinger [7] and of Scher and Montroll [8] could explain many of the experimentally observed phenomena, in particular the length-dependent mobility. In this note we shall show that this length dependence of the mobility can be easily understood as a simple consequence of the statistics of extreme events.

The statistical theory of extreme events was formulated for discrete random variables more than half a century ago by Fisher and Tippett [9] and Gnedenko [10]; it was extended and popularized by Gumbel [11,12]. Gumbel described various applications that include statistics of extreme floods, droughts and fracture of materials. In the field of condensed matter, however, the statistical theory of extreme events has found fewer applications. The
principal aim of the theory of extreme events is to obtain the statistics of the largest value in a sample of \( n \) independent realizations of a random variable. In particular the aim is to determine the asymptotic (\( n \to \infty \)) dependence of the extreme values on the sample size \( n \). The random variable in the case of dispersive transport is the residence time in the trapping centers. The question then can be posed as follows. Given the distribution of trap depths, how does the largest trapping time increase with the number of trapping centers; the latter quantity is determined by the thickness of the material. (we shall use length synonymously with thickness). Together with the assumption that the extreme events determine the behavior of the sum of the residence times, the statistical theory of extreme events makes then a prediction concerning the dependence of the largest residence time on sample size or the thickness of the material. Since mobility is deduced from the sum of the transit times and this quantity is given by the sum of the residence times, the length dependence of the mobility would follow in a natural fashion, as we shall show in this paper.

The paper is organized as follows. In section II the elements of the statistical theory of extreme events that are necessary for the later derivations are briefly described. The application to dispersive transport in disordered materials is discussed in section III. Detailed calculations of various quantities that are relevant for the characterization of the statistics of extreme events are made in section IV. The paper closes with concluding remarks in section V.

II. ELEMENTS OF STATISTICS OF EXTREME EVENTS

This section reviews standard material of the statistics of extreme events \([11,12]\) which is needed later on. Let \( f(x) \) be the probability density function (PDF) of a random variable \( X \) and \( F(x) = \int_{-\infty}^{x} f(x') \, dx' \) its cumulative distribution function (CDF). \( F(x) \) is the probability that a particular realization of \( X \) has a value \( \leq x \). Let \( \Omega_n = \{x_1, x_2, \ldots, x_n\} \) be a set of \( n \) independent realizations of \( X \) sampled from the density \( f(x) \). Let \( x_{\text{nl}} \) denote the largest value of \( X \) in the set \( \Omega_n \). The CDF of \( x_{\text{nl}} \) is the same as the probability that all the values of \( X \) in the set \( \Omega_n \) are less than \( x \), and hence is given by,

\[
\Phi_n(x) = F^n(x).
\] (1)

The probability density of the largest value of \( X \) in the set \( \Omega_n \) is found by differentiation,

\[
\phi_n(x) = n F^{n-1}(x) f(x).
\] (2)

Given a value of \( u \), let \( g_u(\nu) \) denote the probability that the first \( \nu - 1 \) values of \( X \) are less than \( u \) and the \( \nu \)-th value is greater than \( u \). An expression for \( g_u(\nu) \) can be readily written down as

\[
g_u(\nu) = F^{\nu-1}(u)[1 - F(u)].
\] (3)

The mean value of \( \nu \) is denoted by \( n \) and is given by

\[
n = \langle \nu \rangle = \sum_{\nu=1}^{\infty} \nu g_u(\nu) = \frac{1}{1 - F(u)}.
\] (4)
The quantity \( n \) is the mean number of steps required to exceed a given value of \( u \). The above relation can be ‘inverted’ as follows. For a given \( n \), considered now as a parameter, let \( u_{nl} \) denote the value of \( u \) that obeys Eq. (4). Thus we get an implicit equation for \( u_{nl} \) as,

\[
F(u_{nl}) = 1 - \frac{1}{n}.
\]

(5)

Gumbel [11] calls \( u_{nl} \) as the ‘expected’ largest value of \( X \) in a sample of \( n \) independent realizations. Notice that \( u_{nl} \) is not the mean of the extreme value in a sample of size \( n \). Hence \( u_{nl} \) should strictly be viewed as a quantity defined by Eq. (5). This simple expression for the ‘expected’ largest value helps gain insight into the asymptotic behaviour of the extreme values; we shall use this definition of ‘expected’ largest value to derive the power law dependence of the mobility in the next section.

A simple example is provided by the exponential PDF \( f(x) = \exp(-x) \), \( x \geq 0 \). Solution of eq.(5) with respect to \( u_{nl} \) yields

\[
u_{nl} = \ln(n),
\]

i.e., a logarithmic increase of the expected largest value with sample size.

There are essentially three classes of behavior of the largest value \( u_{nl} \) for large \( n \), leading to different asymptotic forms of \( \Phi_n(x) \) for large \( n \) [11][12]. The first class (I) is formed by probability densities \( f(x) \) which decay at least exponentially for large \( n \). The second class (II) results from probability densities whose moments diverge beyond a certain order. The third class (III) comprises probability densities where the values of \( x \) are bounded. We will encounter class I and class II behavior later, depending on the physical quantity that is considered.

### III. MOBILITY IN DISPERSIVE TRANSPORT

In a typical experiment on dispersive transport, a slab of a finite thickness \( L \) (usually thin films of an amorphous substance) is coated with semitransparent metal electrodes. A constant voltage is maintained across the slab. Charge carriers are created at one surface at time \( t = 0 \) by a laser pulse; the charge carriers are drawn through the slab by the electric field. The current \( I(t) \) exhibits different behavior at short and at long times, which is most clearly seen when plotted on a log-log graph. Shlesinger [7] and Scher and Montroll [8] predicted

\[
I(t) \sim \begin{cases} t^{-1+\alpha} & t \leq t_r \\ t^{-1-\alpha} & t \gg t_r \end{cases},
\]

(7)

where the parameter \( 0 < \alpha < 1 \). Deviations from the ideal behavior Eq.(7) are still a topic of current research (see, e.g. [3][13] and the references therein). The physical explanation of the behavior of \( I(t) \) is by trapping of the charge carriers in trapping centers with widely differing depths. In the short-time regime, most of the charges are within the slab, while at longer times the charges are extracted from the slab.
A transit time $t_{tr}$ can be deduced from the crossover between the short-time and long-time behavior. An effective mobility is then defined by the ratio of effective velocity and applied field $F$,

$$ B = \frac{L}{t_{tr}F}. \tag{8} $$

In the multiple-trapping model, which is employed here, the transit time $t_{tr}$ is the free transit time $t_{free}$ plus the sum of all dwell times $\tau_i$ in the trapping centers. Similar arguments could be applied to trap-controlled hopping models. The free transit time is given by the mobility $B_0$, if there are no trapping events: $t_{free} = L/(B_0F)$; it is usually short compared to the sum of all dwell times. Hence to a good approximation

$$ t_{tr} \approx \sum_{i=1}^{n} \tau_i \tag{9} $$

where $n$ is the number of trapping events. For a constant trapping rate the number of trapping events is proportional to the thickness $L$ of the slab.

For broad distributions of dwell times $\tau_i$, the sum in Eq.(9) should be dominated by the largest dwell time. With this assumption

$$ t_{tr} \sim \tau_{\text{largest}}. \tag{10} $$

The Arrhenius law is assumed for thermally activated processes,

$$ \tau_i = \tau_0 \exp\left(\frac{E_i}{k_BT}\right) \tag{11} $$

where $E_i$ is the energy required for release from the trapping center $i$. The largest dwell time is then determined by the largest trapping energy. The probability density of the depths of the trapping centers, i.e., of the energies necessary for release is assumed to be exponential,

$$ f(E) = E_c^{-1} \exp\left(-\frac{E}{E_c}\right) \quad E \geq 0. \tag{12} $$

Note that $f(E)$ is normalized to unity in the interval $(0, \infty)$. It is easy to convert the probability density of the trapping energies into the probability density of the dwell times $\rho(\tau)$, using the Arrhenius law Eq.(11). We have

$$ \rho(\tau) = \frac{\alpha}{\tau_0} \left(\frac{\tau}{\tau_0}\right)^{-(1+\alpha)} \quad \tau_0 \leq \tau \ll \infty \tag{13} $$

with the parameter $\alpha = k_BT/E_c$. The probability density \((13)\) is normalized, but already its first moment does not exist for $0 < \alpha < 1$, which is the parameter range of interest for dispersive transport. The cumulative distribution function of $\tau$ is given by

$$ P(\tau) = 1 - \left(\frac{\tau}{\tau_0}\right)^{-\alpha}. \tag{14} $$

Equation \((14)\) for the expected largest value in a sample of $n$ trials yields
\[ \tau_{nl} = \tau_0 n^{\frac{1}{\alpha}}. \]  

As already stated, the average number of trapping events \( n \) is proportional to the thickness of the experimental sample \( L \). Hence we expect, using the assumption that the transit time is dominated by the largest dwell time

\[
t_{tr} \sim L^{\frac{1}{\alpha}} \quad B \sim L^{1-\frac{1}{\alpha}} .
\]

This is precisely the behavior of the mobility that has been observed in experiments on dispersive transport, see for instance [8]. Here we have derived this behavior from the statistical theory of extreme events.

**IV. EXPONENTIAL AND POWER-LAW PROBABILITY DENSITIES**

**A. Motivation**

Various questions arise with regard to the validity and the significance of the above result for the length dependence of the mobility. For instance, the meaning of the “expected largest value in a sample of \( n \) trials” is partially intuitive; i.e. the precise meaning of the quantity that follows from Eq.(5) is different from what is suggested by this notion. Precisely defined quantities are the moments of the probability density function of the largest value in a sample of \( n \) trials (if they exist) and the most probable value. Of course, complete information is contained in the PDF itself. Fortunately, all quantities of interest can be derived exactly, if the underlying PDF for one event is exponential, or of power-law form. This section will describe the results of these calculations, including a numerical determination of the PDF of the dwell time.

**B. Exponential probability density for single events**

The basic quantity which determines the dwell times of particles in the multiple-trapping model for dispersive transport is the trapping energy \( E \). It is a random quantity and the simplest, experimentally relevant, assumption is the exponential PDF, see Eq.(12). The dimensionless form of this PDF is \( f(x) = \exp(-x) \) with the variable \( x = E/E_C \) and the restriction \( 0 \leq x \leq \infty \). The cumulative distribution function is then \( F(x) = 1 - \exp(-x) \). The expected largest value in samples of \( n \) trials has already been given in eq.(3). The PDF for the largest value \( x_{nl} \) in a sample of \( n \) trials follows from Eq.(1) as

\[
\varphi^{(1)}_n(x) = n(1 - e^{-x})^{n-1}e^{-x}. \quad (17)
\]

The index (1) shall indicate that this PDF is of type I in the classification of Gumbel [11,12].

The moments of \( \varphi^{(1)}_n(x) \) can be calculated exactly. The first moment, or mean value of \( x_{nl} \) is defined as

\[
\langle x_{nl} \rangle = n \int_0^\infty dx x(1 - e^{-x})^{n-1}e^{-x}. \quad (18)
\]
The evaluation of the integral is made in the Appendix. The result is

\[ \langle x_n \rangle = \psi(n + 1) - \psi(1). \] (19)

The digamma function \( \psi(z) \) is defined as \[1\]

\[ \psi(z) = \frac{d}{dz} \ln(\Gamma(z)) \quad \text{and} \quad \xi(1) = -\gamma_E \] (20)

where \( \gamma_E \) is the Euler constant. For integer \( n \)

\[ \psi(n + 1) = -\gamma_E + \sum_{j=1}^{n} \frac{1}{j}. \] (21)

Thus

\[ \langle x_{nl} \rangle = \sum_{j=1}^{n} \frac{1}{j}. \] (22)

Euler’s constant is asymptotically given by

\[ \gamma_E = \lim_{n \to \infty} \left[ \sum_{j=1}^{n} \frac{1}{j} - \ln(n) \right] \] (23)

Hence the first moment is given for large \( n \) by

\[ \langle x_{nl} \rangle = \gamma_E + \ln(n). \] (24)

Note that the first moment differs from the expected largest value of \( x_{nl} \) by Euler’s constant, cf. Eq.(6). Figure 1 contains the results on the first moment Eq.(22) and the expected largest value Eq.(6). Results on a numerical determination of the mean value \( \langle x_{nl} \rangle \) have been included, to demonstrate that this quantity can also be accurately determined by numerical simulation.

The second moment \( \langle x_{nl}^2 \rangle \) can be calculated by the same technique as employed for the first moment, cf. the Appendix. The result for the second moment is

\[ \langle x_{nl}^2 \rangle = [\psi(n + 1) - \psi(1)]^2 + \psi^{(1)}(1) - \psi^{(1)}(n + 1) \] (25)

where \( \psi^{(1)}(z) \) is the threegamma function defined by \[14\]

\[ \psi^{(1)}(z) = \frac{d^2}{dz^2} \ln(\Gamma(z)). \] (26)

The variance of the extreme value is then given by

\[ \langle x_{nl}^2 \rangle - \langle x_{nl} \rangle^2 = \psi^{(1)}(1) - \psi^{(1)}(n + 1). \] (27)

Therefore, the variance of the extreme value is

\[ \sigma^2 = \sum_{k=1}^{n} \frac{1}{k^2} = \xi(2) - \sum_{k=n+1}^{\infty} \frac{1}{k^2}. \] (28)
Note that $\xi(2) = \pi^2/6$. Hence the variance of the extreme value approaches asymptotically this value, i.e., it is asymptotically independent of $n$. The significance of this fact has been stressed by Gumbel [11]. It is easy to derive the following bounds for the variance:

$$\frac{\pi^2}{6} - \frac{1}{n} < \sigma^2 < \frac{\pi^2}{6} - \frac{1}{n+1}.$$  \hspace{1cm} (29)

The variance of the dimensionless extreme energy as a function of $n$ has been included in Fig.1.

We return to the PDF for the largest value $x_{nl}$ in a sample of $n$ trials, which is generally given by Eq.(2) and for an exponential PDF of single events by Eq.(17). We define the most probable value of the random variable $x_{nl}$, i.e., the most probable extreme value, as the value of $x$ where the PDF $\varphi_n(x)$ has its maximum. Differentiation of Eq.(2) yields the extreme value condition

$$\varphi_n'(x) = n \left\{ F^{n-1}(x)f'(x) + (n-1)F^{n-2}(x)f(x)f(x) \right\} = 0.$$  \hspace{1cm} (30)

This equation can be cast into a simple form,

$$\frac{n-1}{F(x)f(x)} = -\frac{f'(x)}{f(x)}.$$  \hspace{1cm} (31)

For an exponential PDF $f(x) = \exp(-x)$ the solution of this equation yields the most probable extreme value $x_{m.p.}$ as a function of the sample size, and is given by

$$x_{m.p.} = \ln(n).$$  \hspace{1cm} (32)

Thus the most probable extreme value agrees with the expected extreme value in the case of the exponential PDF for the single events, cf. Eq.(5).

\[ \text{C. Power-law probability density for single events} \]

The quantity that determines the effective mobility is the sum of all dwell times, which is assumed to be dominated by the largest dwell time. In this subsection we direct the attention to the statistics of the largest dwell time. The dwell time in a trapping center follows from the trapping energy via the Arrhenius relation Eq.(11). The PDF and the CDF for the single events were already given in Eq.(13) and Eq.(14), respectively. We will use dimensionless variables $y = \tau/\tau_0$ henceforth. The PDF for the extreme value $y$ in a sample of $n$ events is then given by

$$\varphi_n^{(2)}(y) = n\alpha (1 - y^{-\alpha})^{n-1}y^{-(1+\alpha)} \hspace{1cm} 1 \leq y < \infty$$  \hspace{1cm} (33)

where the parameter $\alpha = k_BT/E_c$ and the relevant range is $0 < \alpha < 1$. The index (2) shall also indicate that the distribution of $y$ is of type II in the sense of Gumbel [11][12].

The moments of $\varphi_n^{(2)}(y)$ can be calculated. Let $\langle y_{nl}^k \rangle$ denote the $k$th moment of the $y_{nl}$. It is easily shown,

$$\langle y_{nl}^k \rangle = n\beta \left( 1 - \frac{k}{\alpha}, n \right)$$  \hspace{1cm} (34)
\[ \beta(a,b) \] is the usual beta function.

From the properties of the beta function it is clear that the \( k \)th moment of the extreme value exists only if \( k \) is less than \( \alpha \). Since in our problem, \( \alpha \) is between zero and one, no integer moment \( k \geq 1 \) of the extreme value exists. Nevertheless, the expected extreme value (as defined by Gumbel) exists, from which we have derived the length dependent mobility.

It is instructive to determine the most probable extreme value from the PDF Eq. (33) for the largest value \( y_{nl} \) in samples of \( n \) trials. The condition for an extremum has been given in Eq. (30); evaluation with the PDF for single events Eq. (13) and the CDF Eq. (14) gives

\[
y_{m.p.} = \left( \frac{1 + \alpha n}{1 + \alpha} \right)^{1/\alpha}.
\] (35)

For large \( n \), we can replace \( 1 + \alpha n \) by \( \alpha n \), and we get,

\[
y_{m.p.} = \left( \frac{\alpha}{1 + \alpha} \right)^{1/\alpha} n^{1/\alpha}.
\] (36)

Another quantity, which is often used to characterize broad probability densities, is the typical value. It is defined by

\[
\ln(x_{typ}) = \langle \ln(x) \rangle.
\] (37)

where the brackets indicate a mean value taken with a general PDF. The typical value of the PDF Eq. (33) for the extreme value of \( \tau \) is given by

\[
\ln(y_{typ}) = \int_1^\infty dy \ln(y) n\alpha (1 - y^{-\alpha})^{n-1} y^{-\alpha-1}.
\] (38)

By the substitution \( z = y^{-\alpha} \) this integral is transformed into

\[
\ln(y_{typ}) = -\frac{n}{\alpha} \int_0^1 dz \ln(z) (1 - z)^{n-1}
\] (39)

The above integral can be evaluated exactly employing essentially the technique described in the appendix (for the exponential PDF) and the result is

\[
\ln(y_{typ}) = -\frac{1}{\alpha} [\psi(n + 1) - \psi(1)].
\] (40)

Asymptotically, the typical extreme value behaves as

\[
y_{typ} \longrightarrow n^{\frac{\alpha}{\alpha - 1}} \exp(\frac{\tau E}{\tau_0} \alpha).
\] (41)

That is, \( y_{typ} \) is related to \( < x_{nl} > \) by \( y_{typ} = \exp(\frac{1}{\alpha} < x_{nl} >) \). We point out that the connection of the typical extreme value of \( y \) to the mean extreme value of \( x \) is generally valid, if these two variables \( x \) and \( y \) are related by the exponential transformation \( y = \exp(x) \). Notice that the energy \( E \) and the dwell time \( \tau \) are related to each other by the Arrhenius law (11), which is an exponential transformation.

In Fig.2 we have plotted the expected largest value \( \tau_{nl}/\tau_0 \) of the dimensionless dwell time, the most probable value according to eq. (40), and the typical value following from Eq. (40),
for $\alpha = 0.5$. In the case of the dwell time, the expected, the most probable, and the typical largest value are all proportional to $n^{\frac{1}{\alpha}}$, with different $n$-independent factors for large $n$. It seems to be a general fact that the quantities which characterize the largest value in samples of $n$ trials, show similar behavior with respect to $n$, if they exist.

Although the PDF for the extreme value of $y$ in samples of $n$ trials is exactly given by Eq.(33), the evaluation of this function is not practical for large $n$. Therefore, we have determined this PDF for fixed values of $n$ by numerical simulations. We have generated random energies according to the distribution Eq.(12) with $E_c = 1$ and calculated dwell times using the Arrhenius law (11) with $\tau_0 = 1$ and $\alpha = 0.5$. The normalized dwell times were sorted into bins containing $L$ values. The inverse of the lengths of the bins gives the PDF, when properly normalized. The result for $n = 1024$ is given in Fig.3. The maximum of the distribution agrees with the prediction Eq.(35). The typical value is found right of the maximum, close but not identical to the median value. Similar observations have been made for broad distributions previously, for instance in Ref. [15].

Note that the PDF of the extreme value can have a different form, if it is determined by different techniques. If the dwell times are binned into intervals of logarithmically increasing intervals, the maximum of the resulting PDF is found at a different location. This is due to the fact that by this procedure another PDF is estimated, which is related to the one we used by the usual transformation with a Jacobian. The maximum of the latter distribution is at $n^{\frac{1}{\alpha}}$ which is identical to the expected largest value as introduced by Gumbel. The message is that the precise characterization of an extreme value depends on the PDF used for this purpose.

A final problem is the justification of the replacement of the dwell times, cf. Eq.(10) by the largest dwell time, as was done in Eq.(10). In the subsequent derivation, the largest dwell time was identified with the expected largest time, however, any quantity that characterizes an extreme value can as well be used in the argument. The replacement can be justified by extending the derivations given above to the most probable $K^{th}$ extreme value. Let $\varphi_{n,K}(x)$ denote the probability density function of the $K^{th}$ extreme value. A formal expression for $\varphi_{n,K}(x)$ can be easily derived and is given below:

$$\varphi_{n,K}(x) = \frac{n!}{(n - K)!(K - 1)!} f(x) F^{n-K}(x) [1 - F(x)]^{K-1}$$

Note that if we set $K = 1$, we recover the probability density of the first extreme, which we have considered so far. The condition for an extremal value is

$$\varphi'_{n,K}(x) = 0.$$  \hspace{1cm} (43)

Solution of this equation with respect to $x$ gives the most probable $K^{th}$ extreme value.

The extremum condition Eq.(43) can be solved for a power-law PDF as given in Eq.(13). The result for the dimensionless variable $y \equiv \tau / \tau_0$ is

$$y^{(K)}_{m,p.} = \left( \frac{1 + \alpha n}{1 + \alpha K} \right)^{\frac{1}{\alpha}}.$$  \hspace{1cm} (44)

Let $r_K$ denote the ratio of the most probable $K^{th}$ extreme to that of the most probable (first) extreme. It is given by
For $\alpha = 0.5$ and $K = 2$, we have $r = 9/16$. The ratio becomes rapidly smaller for larger $K$; it is already small for $K = 2$ when $\alpha$ is small. The consequence is that the inclusion of the second, third, etc. extreme value in the estimate of the dwell time would not change the asymptotic dependence on the sample size $n$; only numerical factors would be modified.

V. CONCLUDING REMARKS

The apparent anomalies in the transport properties of charges in amorphous substances, which are summarized as 'dispersive transport', are due to broad distributions of trapping times of the charge carriers. It is satisfying that features like the length dependence of the effective mobility can already be derived from the statistics of extreme events, the extreme events being the occurrence of particularly long trapping times.

Although the length dependence of the effective mobility could already be deduced from the notion of the ‘expected largest value’ of the trapping time, we investigated in detail various quantities in the context of the statistics of extreme events. Explicit expressions could be derived for the mean, the most probable, and the typical extreme values in the case of exponential or power-law probability densities for single events. We could also justify the use of the largest dwell time to estimate the summary transit time by considering the second, third, etc. extreme values. Their effect is to modify numerical factors, but they do not alter the asymptotic dependence on the sample size.

All derivations were made here for the ideal case of an exponential density of energy levels. In practice deviations from the exponential density of states are of interest. It turned out that many features of dispersive transport are also present for a Gaussian density of states \[5\]; if the temperature is sufficiently low. Hence it is of interest to extend the present derivations also to the case of other densities of states, for instance to the Gaussian one. In the case of more complicated probability densities of the single events, analytical derivations are either difficult or impossible. Hence one has to resort to numerical simulation to study the statistics of extreme events in those cases.

Finally we point out that the observation of different behaviors of distributions, depending on which of two exponentially related variables are used, is rather general. For instance, de Arcangelis, Redner, and Coniglio \[16\] found broad distributions of voltage drops in random resistor networks, whose moments are characterized by an infinite set of exponents. To the contrary, the distribution of the logarithm of the voltage drops behaves normally, in that the moments exhibit constant-gap scaling. Similar behavior of the probability distributions of the cluster numbers in the percolation problem was discussed by Stauffer and Coniglio \[17\].

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VI. APPENDIX: EVALUATION OF FIRST MOMENT OF $\varphi^{(1)}_N(X)$

In the expression for the first moment Eq.(18) the substitution $y = \exp(-x)$ is made,

$$\langle u_n \rangle = - \int_0^1 dy \ln y(1-y)^{n-1}.$$  \hspace{1cm} (46)

The following trick [18] is applied:

$$\ln(y) = \lim_{\epsilon \to 0} \frac{y^\epsilon - 1}{\epsilon}$$  \hspace{1cm} (47)

We have

$$\langle u_n \rangle = \lim_{\epsilon \to 0} - n \int_0^1 dy \frac{y^\epsilon - 1}{\epsilon}(1-y)^{n-1}.$$  \hspace{1cm} (48)

The integral can now be performed with the result

$$\langle u_n \rangle = -n \lim_{\epsilon \to 0} \frac{\beta(1+\epsilon,n) - \beta(1,n)}{\epsilon}$$  \hspace{1cm} (49)

where $\beta(a,b) = \int_0^1 dx x^{a-1}(1-x)^{b-1}$. Hence we have

$$\langle u_n \rangle = -n \frac{d}{dz} \beta(z,n)|_{z=1}.$$  \hspace{1cm} (50)

The beta function is related to the gamma function (see [14]),

$$\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$  \hspace{1cm} (51)

Differentiating and observing that $\Gamma(n+1) = n\Gamma(n)$ and $\Gamma(1) = 1$ we obtain

$$\langle u_n \rangle = \frac{\Gamma'(n+1)}{\Gamma(n+1)} - \frac{\Gamma'(1)}{\Gamma(1)}.$$  \hspace{1cm} (52)

Using the definition of the digamma function [14] this can be written as

$$\langle u_n \rangle = \xi(n+1) - \xi(1).$$  \hspace{1cm} (53)
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Figure Captions

Fig. 1: Mean and variance of the dimensionless largest energy in samples of size $n$, taken from an exponential PDF, as functions of $n$. Full line: expected largest value $x_{nl}$, dotted line: mean of largest value, dash-dotted line: variance of largest value. Symbols: ◊ : mean largest value by simulations, + : variance by simulations.

Fig. 2: Normalized extreme dwell times as functions of the sample size $n$. Full line: most probable largest dwell time, short dashes: expected largest dwell time, long dashes: typical largest dwell time.

Fig. 3: Probability distribution function of the largest value of the normalized dwell time in samples of size $n = 1024$. The PDF was determined from $10^5$ realizations. Indicated are: most probable largest value (full line), expected largest value (short dashes), median (points), and typical largest value (long dashes).
fig.1.
fig. 2.
fig. 3