Characterization of Lipschitz Functions via Commutators of Multilinear Singular Integral Operators in Variable Lebesgue Spaces

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Abstract Let \( \vec{b} = (b_1, b_2, \ldots, b_m) \) be a collection of locally integrable functions and \( T_{\vec{b}} \) the commutator of multilinear singular integral operator \( T \). Denote by \( L(\delta) \) and \( L(\delta(\cdot)) \) the Lipschitz spaces and the variable Lipschitz spaces, respectively. The main purpose of this paper is to establish some new characterizations of the (variable) Lipschitz spaces in terms of the boundedness of multilinear commutator \( T_{\vec{b}} \) in the context of the variable exponent Lebesgue spaces, that is, the authors give the necessary and sufficient conditions for \( b_j \) \( (j = 1, 2, \ldots, m) \) to be \( L(\delta) \) or \( L(\delta(\cdot)) \) via the boundedness of multilinear commutator from products of variable exponent Lebesgue spaces to variable exponent Lebesgue spaces. The authors do so by applying the Fourier series technique and some pointwise estimate for the commutators. The key tool in obtaining such pointwise estimate is a certain generalization of the classical sharp maximal operator.

Keywords Multilinear commutator, singular integral operator, Lipschitz function, variable exponent

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1 Introduction and Main Results

Let \( T \) be the classical singular integral operator. The commutator \( [b, T] \) generated by \( T \) and a suitable function \( b \) is defined by

\[
[b, T]f = bT(f) - T(bf).
\]

It is well known that the commutators are intimately related to the regularity properties of the solutions of certain partial differential equations (PDE, see for example [3, 19, 49]). The continuity properties of such commutators, studied in several literatures, have contributed to the development of the PDE’s (such as [2, 12, 42, 43]).

The first result for the commutator \( [b, T] \) was established by Coifman, Rochberg and Weiss in [10], and the authors proved that the BMO is characterized by the boundedness of the singular integral operators’ commutator \( [b, T] \). In 1978, Janson [32] generalized the results in [10] to functions belonging to a Lipschitz functional space and gave a characterization in terms of the boundedness of the commutators of singular integral operators with Lipschitz functions. In
1982, Chanillo [7] proved that BMO can be characterized by mean of the boundedness between Lebesgue spaces of the commutators of fractional integral operators with BMO functions. In 1995, Paluszyński [41] gives some results in the spirit of [7] for the functions belonging to Lipschitz function spaces.

The multilinear Calderón–Zygmund theory was first studied by Coifman and Meyer in [8, 9]. This theory was then further investigated by many authors in the last few decades, see for example [25, 26, 35], for the theory of multilinear Calderón–Zygmund operators with kernels satisfying the standard estimates.

In 2003, Diening and Růžička [22] studied the Calderón–Zygmund operators on variable exponent Lebesgue spaces and gave some applications to problems related to fluid dynamics. In past twenty years since some elementary properties were established by Kováčik and Rákosník in [34]. In 2003, Diening and Růžička [22] studied the Calderón–Zygmund operators on variable exponent Lebesgue spaces and gave some applications to problems related to fluid dynamics.

The multilinear Calderón–Zygmund theory was first studied by Coifman and Meyer in 1982, and there it was further investigated by many authors in the last few decades, see for example [25, 26, 35], for the theory of multilinear Calderón–Zygmund operators with kernels satisfying the standard estimates.

In 2009, Lerner et al. [35] developed a multiple-weight theory that adapts to the multilinear Calderón–Zygmund operators. They established the multiple-weighted norm inequalities for the multilinear Calderón–Zygmund operators. They established the multiple-weighted norm inequalities for the multilinear Calderón–Zygmund operators and their commutators.

The theory of function spaces with variable exponent has been intensely investigated in the past twenty years since some elementary properties were established by Kováčik and Rákosník in [34], Paluszyński [41] gives some results in the spirit of [7] for the functions belonging to Lipschitz function spaces.

Let $\mathbb{R}^n$ be an $n$-dimensional Euclidean space and $(\mathbb{R}^n)^m = \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ be an $m$-fold product space ($m \in \mathbb{N}$). We denote by $\mathcal{S}(\mathbb{R}^n)$ the space of all Schwartz functions on $\mathbb{R}^n$ and by $\mathcal{S}'(\mathbb{R}^n)$ its dual space, the set of all tempered distributions on $\mathbb{R}^n$.

**Definition 1.1** A locally integrable function $K(x, y_1, \ldots, y_m)$, defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, is called an $m$-linear Calderón–Zygmund kernel, if there exists a constant $A > 0$ such that the following conditions are satisfied.

1. **Size estimate**: for all $(x, y_1, \ldots, y_m) \in (\mathbb{R}^n)^{m+1}$ with $x \neq y_j$ for some $j \in \{1, 2, \ldots, m\}$, there is

$$|K(x, y_1, \ldots, y_m)| \leq \frac{A}{(|x - y_1| + \cdots + |x - y_m|)^{mn}}.$$  

2. **Smoothness estimate**: assume that for some $\epsilon > 0$, and for each $j \in \{1, 2, \ldots, m\}$, there are regularity conditions

$$|K(x, y_1, \ldots, y_j, \ldots, y_m) - K(x', y_1, \ldots, y_j, \ldots, y_m)| \leq \frac{A|x - x'|^\epsilon}{(\sum_{j=1}^{m} |x - y_j|)^{mn+\epsilon}}, \quad (1.1)$$

whenever $|x - x'| \leq \frac{1}{2} \max_{1 \leq j \leq m} |x - y_j|$, and for each fixed $j$ with $1 \leq j \leq m$,

$$|K(x, y_1, \ldots, y_j, \ldots, y_m) - K(x, y_1, \ldots, y_j', \ldots, y_m)| \leq \frac{A|y_j - y_j'|^\epsilon}{(\sum_{j=1}^{m} |x - y_j|)^{mn+\epsilon}}$$

whenever $|y_j - y_j'| \leq \frac{1}{2} \max_{1 \leq j \leq m} |x - y_j|.$

We say $T : \mathcal{S}'(\mathbb{R}^n)^m \times \cdots \times \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ is an $m$-linear singular integral operator with an $m$-linear Calderón–Zygmund kernel, $K(x, y_1, \ldots, y_m)$, if

$$T(f_1, \ldots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m$$
whenever $x \notin \bigcap_{j=1}^{m} \text{supp } f_j$ and each $f_j \in C_c^\infty(\mathbb{R}^n)$, $j = 1, \ldots, m$.

If $T$ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ with $1 < p_1, \ldots, p_m < \infty$ and $1/p = 1/p_1 + 1/p_2 + \cdots + 1/p_m$, then we say that $T$ is an $m$-linear Calderón–Zygmund operator (see [26, 30, 38] for more details). If $K(x, y_1, \ldots, y_m)$ is of form $K(x - y_1, \ldots, x - y_m)$, then $T$ is called an operator of convolution type.

Let $\vec{b} = (b_1, b_2, \ldots, b_m)$ be a collection of locally integrable functions, the $m$-linear commutator of $T$ with $\vec{b}$ is defined by

$$T_{\Sigma^b}(\vec{f})(x) = T_{\Sigma^b}(f_1, \ldots, f_m)(x) = \sum_{j=1}^{m} T_{b_j}(\vec{f})(x),$$

where each term is the commutator of $b_j$ and $T$ in the $j$-th entry of $T$, that is,

$$T_{b_j}(\vec{f})(x) = [b_j, T](\vec{f})(x) = b_j(x)T(f_1, \ldots, f_j, \ldots, f_m)(x) - T(f_1, \ldots, b_j f_j, \ldots, f_m)(x)$$

for every $j = 1, 2, \ldots, m$. This definition coincides with the linear commutator $[b, T]$ when $m = 1$. And the iterated commutator $T_{\Pi^b}(\vec{f})$ is defined via [43]

$$T_{\Pi^b}(\vec{f})(x) = [b_1, [b_2, \ldots [b_{m-1}, [b_m, T]_{m-1} \ldots]_2]_1(\vec{f})(x).$$

To clarify the notation, the commutators can be written formally as

$$T_{\Sigma^b}(\vec{f})(x) = \sum_{j=1}^{m} \int_{\mathbb{R}^n} (b_j(x) - b_j(y_j))K(x, y_1, \ldots, y_m)f_1(y_1) \cdots f_m(y_m)dy_1 \cdots dy_m$$

$$= \sum_{j=1}^{m} \int_{\mathbb{R}^n} (b_j(x) - b_j(y_j))K(x, \vec{y}) \prod_{i=1}^{m} f_i(y_i)dy;$$

$$T_{\Pi^b}(\vec{f})(x) = \int_{\mathbb{R}^n} \left( \prod_{j=1}^{m} (b_j(x) - b_j(y_j)) \right) K(x, y_1, \ldots, y_m)f_1(y_1) \cdots f_m(y_m)dy_1 \cdots dy_m$$

$$= \int_{\mathbb{R}^n} \left( \prod_{j=1}^{m} (b_j(x) - b_j(y_j)) \right) K(x, \vec{y}) \prod_{i=1}^{m} f_i(y_i)dy.$$

When $m = 1$, $T_{\Sigma^b}(\vec{f}) = T_{\Pi^b}(\vec{f}) = [b, T]f = bT(f) - T(bf)$, which is the well known classical commutator studied in [10]. These multilinear commutators are early appeared in [53].

Denote by $\mathbb{L}(\delta)$ and $\mathbb{L}(\delta(\cdot))$ the Lipschitz spaces and the variable Lipschitz spaces (see Definition 2.16), respectively. Motivated by the works mentioned above, the main aim of this paper is to obtain some new characterizations of the (variable) Lipschitz spaces via the boundedness of multilinear commutator $T_{\Sigma^b}$ in the context of the variable exponent Lebesgue spaces. The necessary and sufficient conditions for $b_j$ ($j = 1, 2, \ldots, m$) belonging to $\mathbb{L}(\delta)$ or $\mathbb{L}(\delta(\cdot))$ are given by the aid of the boundedness of multilinear commutator from products of variable exponent Lebesgue spaces to variable exponent Lebesgue spaces. The key tools in obtaining the results are Fourier series applied by Janson [32] and certain generalizations of the classical sharp maximal operator, which was introduced by Calderón and Scott in [5] and was extended to the variable context by Cabral, Pradolini and Ramos in [4].

Our main results can be stated as follows. And some notations can refer to Section 2, such as, $p_-, p_+, \mathcal{C}_{\log}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ etc.
Theorem 1.2  Let $T$ be an $m$-linear Calderón–Zygmund operator given in Definition 1.1. Suppose that $0 < \delta \leq 1$, $p_1(\cdot), p_2(\cdot), \ldots, p_m(\cdot) \in C_0^\infty(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ satisfy $\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)} + \cdots + \frac{1}{p_m(x)}$, and $\frac{\min}{\max} < (p_j)_- < (p_j)_+ < \frac{\max}{\delta}$ $(j = 1, 2, \ldots, m)$. Define the variable exponent $q(\cdot)$ by

$$
\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\delta}{n}.
$$

(1) Assume that the associated kernel $K$ of $T$ is a homogeneous of degree $(-mn)$ and the Fourier series of $\frac{1}{K}$ is absolutely convergent on some ball $B \subset \mathbb{R}^{mn}$. Let $\vec{b} = (b_1, b_2, \ldots, b_m)$ be a collection of locally integrable functions. If $T_{b_j}$ $(j = 1, 2, \ldots, m)$ is bounded from $L^{p_1(\cdot)}(\mathbb{R}^n) \times \cdots \times L^{p_m(\cdot)}(\mathbb{R}^n)$ to $L^{q^1(\cdot)}(\mathbb{R}^n)$, then $\vec{b} = (b_1, b_2, \ldots, b_m) \in L(\delta) \times L(\delta) \times \cdots \times L(\delta)$.

(2) If $\vec{b} = (b_1, b_2, \ldots, b_m) \in L(\delta) \times L(\delta) \times \cdots \times L(\delta)$, then $T_{b_j} : L^{p_1(\cdot)}(\mathbb{R}^n) \times L^{p_2(\cdot)}(\mathbb{R}^n) \times \cdots \times L^{p_m(\cdot)}(\mathbb{R}^n) \to L^{q(\cdot)}(\mathbb{R}^n)$ $(j = 1, 2, \ldots, m)$.

Remark 1.3 (i) The above result gives a characterization of the Lipschitz spaces $L(\delta)$ in terms of the boundedness of $T_{\Sigma \vec{b}}$ between variable Lebesgue spaces.

(ii) In order to prove the first part of Theorem 1.2, the authors use the techniques and ideas of Fourier series applied by Janson [32], and modify it to adapt to the multilinear setting.

(iii) In the second part of Theorem 1.2, some tools and techniques similar to [51] are used to prove the result. When $L^{p_j(\cdot)}(\mathbb{R}^n)$ $(j = 1, 2, \ldots, m)$ is the classical Lebesgue spaces, the proof was obtained in [52]. For the bilinear case, the proof was given in [51] when $\beta = \frac{\delta}{2}$ $(i = 1, 2)$.

Theorem 1.4  Let $T$ be an $m$-linear Calderón–Zygmund operator given in Definition 1.1, and let $\epsilon \in (\epsilon_0, 1]$ be given as in Definition 1.1, where $\epsilon_0$ is closely related to $k_0$ in (2.5). Suppose that $0 < \delta(\cdot) < 1$, $0 < \gamma < \eta < 1/m$, $r(\cdot), p_1(\cdot), p_2(\cdot), \ldots, p_m(\cdot) \in C_0^\infty(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ satisfy $\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)} + \cdots + \frac{1}{p_m(x)}$, $r(x) \geq r_\infty$ for almost every $x \in \mathbb{R}^n$ and $1 < \beta \leq r_-$ such that $\frac{1}{p(x) - 1} = \frac{\delta(x)}{n} = \frac{1}{\beta} - \frac{1}{r(x)}$ with $\sup_{x \in \mathbb{R}^n} p_j(x) \delta(x) < n$ $(j = 1, 2, \ldots, m)$.

(1) Assume that the associated kernel $K$ of $T$ is a homogeneous of degree $(-mn)$ and the Fourier series of $\frac{1}{K}$ is absolutely convergent on some ball $B \subset \mathbb{R}^{mn}$. Let $\vec{b} = (b_1, b_2, \ldots, b_m)$ be a collection of locally integrable functions. If $T_{b_j} : L^{p_1(\cdot)}(\mathbb{R}^n) \times L^{p_2(\cdot)}(\mathbb{R}^n) \times \cdots \times L^{p_m(\cdot)}(\mathbb{R}^n) \to L^{q(\cdot)}(\mathbb{R}^n)$ $(j = 1, 2, \ldots, m)$, then $\vec{b} = (b_1, b_2, \ldots, b_m) \in L(\delta(\cdot)) \times L(\delta(\cdot)) \times \cdots \times L(\delta(\cdot))$.

(2) If $\vec{b} = (b_1, b_2, \ldots, b_m) \in L(\delta(\cdot)) \times L(\delta(\cdot)) \times \cdots \times L(\delta(\cdot))$, then $T_{b_j} : L^{p_1(\cdot)}(\mathbb{R}^n) \times L^{p_2(\cdot)}(\mathbb{R}^n) \times \cdots \times L^{p_m(\cdot)}(\mathbb{R}^n) \to L^{q(\cdot)}(\mathbb{R}^n)$ $(j = 1, 2, \ldots, m)$.

Remark 1.5 (i) The above result characterizes the variable Lipschitz spaces $L(\delta(\cdot))$ in terms of the boundedness of $T_{\Sigma \vec{b}}$ between variable Lebesgue spaces. The authors do so by applying the Fourier series technique and certain generalizations of the classical sharp maximal operator with variable exponent.

(ii) In addition, some authors also consider the boundedness of commutators of singular integral operators in the classical (product) Lebesgue spaces. For example, Chaffee [6] characterizes bounded mean oscillation in terms of the boundedness of commutators of various bilinear singular integral operators. However, there are also some other weaker cases than the assumptions in [6] are considered recently, especially the operators do not need to be convolution type. Such as, Guo et al. [27] discussed certain non-degenerate condition of the kernel in linear and
multilinear settings, and in the linear case, it is improved in [31], while the related multilinear case was handled in [36].

Throughout this paper, the letter $C$ always stands for a constant independent of the main parameters involved and whose value may differ from line to line. A cube $Q \subset \mathbb{R}^n$ always means a cube whose sides are parallel to the coordinate axes and denote its side length by $l(Q)$. For some $t > 0$, the notation $tQ$ stands for the cube with the same center as $Q$ and with side length $tl(Q)$. Denote by $|S|$ the Lebesgue measure and by $\chi_S$ the characteristic function for a measurable set $S \subset \mathbb{R}^n$. $B(x, r)$ means the ball centered at $x$ and of radius $r$, and $B_0 = B(0, 1)$. For any index $1 < q(x) < \infty$, we denote by $q'(x)$ its conjugate index, namely, $q'(x) = \frac{q(x)}{q(x) - 1}$. And we will occasionally use the notational $\vec{f} = (f_1, \ldots, f_m)$, $T(\vec{f}) = T(f_1, \ldots, f_m)$, $d\vec{y} = dy_1 \cdots dy_m$ and $(x, \vec{y}) = (x, y_1, \ldots, y_m)$ for convenience. For a set $E$ and a positive integer $m$, we will use the notation $(E)^m = E \times \cdots \times E$.

2 Preliminaries

Over last three decades, the study of variable exponent function spaces has attracted many authors’ attention (see [13, 14, 21, 22] et al.). In fact, many classical operators are discussed in variable exponent function spaces (see [13, 14, 21]).

In this section, we give some definitions and lemmas we need.

2.1 Function Spaces with Variable Exponent

Let $\Omega$ be a measurable set in $\mathbb{R}^n$ with $|\Omega| > 0$. We first define variable exponent Lebesgue spaces.

**Definition 2.1** Let $q(\cdot) : \Omega \to [1, \infty)$ be a measurable function.

(i) The Lebesgue spaces with variable exponent $L^{q(\cdot)}(\Omega)$ is defined by

$$L^{q(\cdot)}(\Omega) = \{ f \text{ is measurable function : } F_q(f/\eta) < \infty \text{ for some constant } \eta > 0 \},$$

where $F_q(f) := \int_{\Omega} |f(x)|^{q(x)}dx$. The Lebesgue space $L^{q(\cdot)}(\Omega)$ is a Banach function space with respect to the norm

$$\|f\|_{L^{q(\cdot)}(\Omega)} = \inf \left\{ \eta > 0 : F_q(f/\eta) = \int_{\Omega} \left( \frac{|f(x)|}{\eta} \right)^{q(x)} dx \leq 1 \right\}.$$

(ii) The space $L_{\text{loc}}^{q(\cdot)}(\Omega)$ is defined by

$$L_{\text{loc}}^{q(\cdot)}(\Omega) = \{ f \text{ is measurable : } f \in L^{q(\cdot)}(\Omega_0) \text{ for all compact subsets } \Omega_0 \subset \Omega \}.$$

The weighted Lebesgue space $L_{\omega}^{q(\cdot)}(\Omega)$ is defined by as the set of all measurable functions for which

$$\|f\|_{L_{\omega}^{q(\cdot)}(\Omega)} = \|\omega f\|_{L^{q(\cdot)}(\Omega)} < \infty.$$

Next we define some classes of variable exponent functions. Given a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the Hardy–Littlewood maximal operator $M$ is defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|dy.$$
Definition 2.2 Given a measurable function \(q(\cdot)\) defined on \(\mathbb{R}^n\). For \(E \subset \mathbb{R}^n\), we write
\[
q_-(E) := \text{ess inf}_{x \in E} q(x), \quad q_+(E) := \text{ess sup}_{x \in E} q(x),
\]
and write \(q_- (\mathbb{R}^n) = q_-\) and \(q_+ (\mathbb{R}^n) = q_+\) simply.

(i) \(q'_- = \text{ess inf}_{x \in \mathbb{R}^n} q'(x) = \frac{q_-}{q_+ - 1}, \quad q'_+ = \text{ess sup}_{x \in \mathbb{R}^n} q'(x) = \frac{q_-}{q_- - 1}.\)

(ii) Denote by \(\mathcal{P}_0(\mathbb{R}^n)\) the set of all measurable functions \(q(\cdot): \mathbb{R}^n \to (0, \infty)\) such that
\[
0 < q_- \leq q(x) \leq q_+ < \infty, \quad x \in \mathbb{R}^n.
\]

(iii) Denote by \(\mathcal{P}_1(\mathbb{R}^n)\) the set of all measurable functions \(q(\cdot): \mathbb{R}^n \to [1, \infty)\) such that
\[
1 \leq q_- \leq q(x) \leq q_+ < \infty, \quad x \in \mathbb{R}^n.
\]

(iv) Denote by \(\mathcal{P}(\mathbb{R}^n)\) the set of all measurable functions \(q(\cdot): \mathbb{R}^n \to (1, \infty)\) such that
\[
1 < q_- \leq q(x) \leq q_+ < \infty, \quad x \in \mathbb{R}^n.
\]

(v) The set \(\mathcal{B}(\mathbb{R}^n)\) consists of all measurable functions \(q(\cdot) \in \mathcal{P}(\mathbb{R}^n)\) satisfying that the Hardy–Littlewood maximal operator \(M\) is bounded on \(L^{q(\cdot)}(\mathbb{R}^n)\).

Definition 2.3 (log-Hölder continuity) Let \(q(\cdot)\) be a real-valued function on \(\mathbb{R}^n\).

(i) Denote by \(\mathcal{E}_0^\text{log}(\mathbb{R}^n)\) the set of all local log-Hölder continuous functions \(q(\cdot)\) which satisfies
\[
|q(x) - q(y)| \leq \frac{-C}{\ln(|x - y|)}, \quad |x - y| \leq 1/2, \quad x, y \in \mathbb{R}^n,
\]
where \(C\) denotes a universal positive constant that may differ from line to line, and \(C\) does not depend on \(x, y\).

(ii) The set \(\mathcal{E}_\infty^\text{log}(\mathbb{R}^n)\) consists of all log-Hölder continuous functions \(q(\cdot)\) at infinity satisfies
\[
|q(x) - q_\infty| \leq \frac{C_\infty}{\ln(e + |x|)}, \quad x \in \mathbb{R}^n,
\]
where \(q_\infty = \lim_{|x| \to \infty} q(x)\).

(iii) Denote by \(\mathcal{E}_\infty^\text{log}(\mathbb{R}^n) := \mathcal{E}_0^\text{log}(\mathbb{R}^n) \cap \mathcal{E}_\infty^\text{log}(\mathbb{R}^n)\) the set of all global log-Hölder continuous functions \(q(\cdot)\).

Remark 2.4 The \(\mathcal{E}_\infty^\text{log}(\mathbb{R}^n)\) condition is equivalent to the uniform continuity condition
\[
|q(x) - q(y)| \leq \frac{C}{\ln(e + |x|)}, \quad |y| \geq |x|, \quad x, y \in \mathbb{R}^n.
\]
The \(\mathcal{E}_\infty^\text{log}(\mathbb{R}^n)\) condition was originally defined in this form in [15].

2.2 Auxiliary Propositions and Lemmas
In this part we state some auxiliary propositions and lemmas which will be needed for proving our main theorems. And we only describe partial results we need.

Lemma 2.5 Let \(p(\cdot) \in \mathcal{P}(\mathbb{R}^n)\).

(1) If \(p(\cdot) \in \mathcal{E}_\infty^\text{log}(\mathbb{R}^n)\), then we have \(p(\cdot) \in \mathcal{B}(\mathbb{R}^n)\).

(2) The following conditions are equivalent:
(i) \(p(\cdot) \in \mathcal{B}(\mathbb{R}^n)\).
(ii) \(p'(\cdot) \in \mathcal{B}(\mathbb{R}^n)\).
(iii) \(p(\cdot)/p_0 \in \mathcal{B}(\mathbb{R}^n)\) for some \(1 < p_0 < p_-\).
(iv) \((p(\cdot)/p_0)' \in \mathcal{B}(\mathbb{R}^n)\) for some \(1 < p_0 < p_-\).
The first part in Lemma 2.5 is independently due to Cruz-Uribe et al. [15] and to Nekvinda [40] respectively. The second of Lemma 2.5 belongs to Diening [20] (see Theorem 8.1 or Theorem 1.2 in [14]).

**Remark 2.6** (a) Since
\[
|q'(x) - q'(y)| \leq \frac{|q(x) - q(y)|}{(q_+ - 1)^2},
\]
it follows at once that if \( q(\cdot) \in C^\log(\mathbb{R}^n) \), then so does \( q'(\cdot) \), that is, if the condition holds, then \( M \) is bounded on \( L^{q(\cdot)}(\mathbb{R}^n) \) and \( L^{q'(\cdot)}(\mathbb{R}^n) \). Furthermore, Diening has proved general results on Musielak–Orlicz spaces.

(b) When \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \), the assumption that \( p(\cdot) \in C^\log(\mathbb{R}^n) \) is equivalent to assuming \( 1/p(\cdot) \in C^\log(\mathbb{R}^n) \), since
\[
\left| \frac{p(x) - p(y)}{(p_+)^2} \right| \leq \frac{1}{p(x)} - \frac{1}{p(y)} = \frac{|p(x) - p(y)|}{p(x)p(y)} \leq \frac{|p(x) - p(y)|}{(p_-)^2}.
\]

As the classical Lebesgue norm, the (quasi-)norm of variable exponent Lebesgue space is also homogeneous in the exponent. Precisely, we have the following result (see Lemma 2.3 in [18]).

**Lemma 2.7** Given \( p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \), then for all \( s > 0 \), we have
\[
\|f|^{s}_{p(\cdot)} = \|f|^{s}_{p(\cdot)}.
\]

The next lemma is known as the generalized Hölder’s inequality on Lebesgue spaces with variable exponent, and the proof can also be found in [34] or [13] (see [13, pp. 27–30]).

**Lemma 2.8** (generalized Hölder’s inequality) (1) (see [21, pp. 81–82, Lemma 3.2.20]) Let \( p(\cdot), q(\cdot), r(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \) satisfy the condition
\[
\frac{1}{r(x)} = \frac{1}{p(x)} + \frac{1}{q(x)} \quad \text{for a.e. } x \in \mathbb{R}^n.
\]

(i) Then, for all \( f \in L^{p(\cdot)}(\mathbb{R}^n) \) and \( g \in L^{q(\cdot)}(\mathbb{R}^n) \), one has
\[
\|fg\|_{r(\cdot)} \leq C\|f\|_{p(\cdot)}\|g\|_{q(\cdot)}. \tag{2.1}
\]

(ii) When \( r = 1 \), then \( p'(\cdot) = q(\cdot) \), hence, for all \( f \in L^{p(\cdot)}(\mathbb{R}^n) \) and \( g \in L^{p'(\cdot)}(\mathbb{R}^n) \), one has
\[
\int_{\mathbb{R}^n} |fg| \leq C\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}. \tag{2.2}
\]

(2) The generalized Hölder’s inequality in Orlicz space (for details and the more general cases see [35, 42, 44]).

(i) Let \( r_1, \ldots, r_m \geq 1 \) with \( \frac{1}{r} = \frac{1}{r_1} + \cdots + \frac{1}{r_m} \) and \( Q \) be a cube in \( \mathbb{R}^n \). Then
\[
\frac{1}{|Q|} \int_{Q} |f_1(x) \cdots f_m(x)g(x)|dx \leq C\|f_1\|_{\exp L^{r_1}, Q} \cdots \|f_m\|_{\exp L^{r_m}, Q} \|g\|_{L(\log L)^{1/r}, Q}.
\]

(ii) Let \( t \geq 1 \), then
\[
\frac{1}{|Q|} \int_{Q} |f(x)g(x)|dx \leq C\|f\|_{\exp L^t, Q} \|g\|_{L(\log L)^{1/t}, Q}. \tag{2.3}
\]
(3) (see [38, Lemma 9.2]) Let \( q(\cdot), q_1(\cdot), \ldots, q_m(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) satisfy the condition

\[
\frac{1}{q(x)} = \frac{1}{q_1(x)} + \cdots + \frac{1}{q_m(x)} \quad \text{for a.e. } x \in \mathbb{R}^n.
\]

Then, for any \( f_j \in L^{q_j(\cdot)}(\mathbb{R}^n), j = 1, \ldots, m, \) one has

\[ \| f_1 \cdots f_m \|_{q(\cdot)} \leq C \| f_1 \|_{q_1(\cdot)} \cdots \| f_m \|_{q_m(\cdot)}. \]

We need the boundedness of bilinear Calderón–Zygmund operators in variable Lebesgue spaces.

**Lemma 2.9** (see in [17, Corollary 4.1]) Let \( p(\cdot), q(\cdot), r(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \) satisfy \( \frac{1}{r(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} \) with \( p_- > 1, q_- > 1 \) and \( r_- > \frac{1}{2} \). Suppose further that \( p(\cdot), q(\cdot) \in \mathcal{B}(\mathbb{R}^n) \). If \( T \) is a bilinear Calderón–Zygmund operator, then there exists a positive constant \( C \) such that

\[ \| T(f, g) \|_{L^{r(\cdot)}} \leq C \| f \|_{L^{p(\cdot)}} \| g \|_{L^{q(\cdot)}} \]

holds for all \( f \in L^{p(\cdot)}(\mathbb{R}^n), g \in L^{q(\cdot)}(\mathbb{R}^n) \).

**Remark 2.10** The boundedness of bilinear Calderón–Zygmund operators in variable Lebesgue spaces was first proved in [30] (see Corollary 2.1) and [38] (see Theorem 3.1). Cruz-Uribe and Naibo [17] improved upon both of these results by Lemma 2.9: the former requires the additional hypothesis \( r(\cdot)/r_\ast \in \mathcal{B} \) for some \( r_\ast \in (0, r_-) \) while the latter assumes \( r(\cdot) \in \mathcal{B} \). And in both cases, the proofs use linear extrapolation in the scale of variable Lebesgue spaces.

The following results are also needed.

**Lemma 2.11** (Norms of Characteristic Functions) (1) Let \( q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n) \) and \( q(x) \leq q_\infty \) for a.e. \( x \in \mathbb{R}^n \). Then there exists a positive constant \( C \) such that the inequality

\[ \| \chi_Q \|_{q(\cdot)} \leq C |Q|^{1/q(x)} \]

holds for every cube \( Q \subset \mathbb{R}^n \) and a.e. \( x \in Q \) (see in [47, Lemma 4.4], or [21, p. 126, Corollary 4.5.9]).

(2) Let \( q(\cdot) \in \mathcal{P}(\mathbb{R}^n) \). \( \frac{1}{Q Q} = \frac{1}{|Q|} \int_Q \frac{1}{q(y)} \, dy \) is the harmonic mean of \( q(\cdot) \). Then the following conditions are equivalent (see in [21, Theorem 4.5.7] or in [13, Proposition 4.66]).

(i) \( \| \chi_Q \|_{q(\cdot)} \| \chi_Q \|_{q(\cdot)} \sim |Q|^{-\frac{1}{q(\cdot)}} \) uniformly for all cubes \( Q \subset \mathbb{R}^n \).

(ii) \( \| \chi_Q \|_{q(\cdot)} \approx |Q|^{-\frac{1}{q(\cdot)}} \) and \( \| \chi_Q \|_{q(\cdot)} \approx |Q|^{-\frac{1}{q(\cdot)}} \) uniformly for all cubes \( Q \subset \mathbb{R}^n \).

(3) Let \( q_+ < \infty \). Then the following conditions are equivalent (see in [21, p. 101, Lemma 4.1.6], or in [47, Lemma 4.2], or [13, Corollary 3.24], or [16, 29]).

(a) The function \( q(\cdot) \in C^{\log}(\mathbb{R}^n) \).

(b) For every cube \( Q \subset \mathbb{R}^n \), there exists a positive constant \( C \) such that

\[ |Q|^{q_-(Q)-q_+(Q)} \leq C. \]

(c) For all cube \( Q \subset \mathbb{R}^n \) and all \( x \in Q \), there exists a positive constant \( C \) such that

\[ |Q|^{q_-(Q)-q(x)} \leq C. \]

(d) For all cube \( Q \subset \mathbb{R}^n \) and all \( x \in Q \), there exists a positive constant \( C \) such that

\[ |Q|^{q(x)-q_+(Q)} \leq C. \]
(4) Let \( q(\cdot) \in C_\infty^\log(\mathbb{R}^n) \). Then \( \|\chi_Q\|_{q(\cdot)} \approx |Q|^{1/q_\infty} \) for every cube \( Q \subset \mathbb{R}^n \) with diameter \( r_Q \geq 1/4 \) (a particular case of [16, Lemma 3.6], see also [47]).

(5) Let \( q(\cdot) \in C_\infty^\log(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n) \). Then \( \|\chi_Q\|_{q(\cdot)} \approx |Q|^{1/q_\infty} \) for every cube (or ball) \( Q \subset \mathbb{R}^n \).

More concretely,
\[
\|\chi_Q\|_{q(\cdot)} \approx \begin{cases} |Q|^{1/q_\infty} & \text{if } |Q| \leq 2^n \text{ and } x \in Q, \\ |Q|^{1/q_\infty} & \text{if } |Q| \geq 1 \end{cases}
\]
for every cube (or ball) \( Q \subset \mathbb{R}^n \) (see [21, Corollary 4.5.9 and Lemma 7.3.19]).

(6) Given a cube \( Q = Q(x_0, r) \), with center in \( x_0 \) and diameter \( r \).

(a) If \( r < 1 \), there exist two positive constants \( a_1 \) and \( a_2 \) such that
\[
a_1 |Q|^{1/q_-(Q)} \leq \|\chi_Q\|_{q(\cdot)} \leq a_2 |Q|^{1/q_+(Q)}.
\]

(b) If \( r > 1 \), there exist positive constants \( c_1 \) and \( c_2 \) such that
\[
c_1 |Q|^{1/q_+(Q)} \leq \|\chi_Q\|_{q(\cdot)} \leq c_2 |Q|^{1/q_-(Q)}.
\]

Therefore, \( \|\chi_Q\|_{q(\cdot)} \leq \max\{\|Q|^{1/q_+(Q)}, |Q|^{1/q_-(Q)}\} \) (see [13, pp. 25–26, Corollary 2.23], or [21, 23, 47]).

(7) If \( p(\cdot) \in C_\infty^\log(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n) \), \( \beta > 1 \) and \( p_+ < \frac{n \beta}{(n-\beta)} \), then there exists a number \( a > 1 \) such that
\[
\|\chi_{Q(x, r)}\|_{p'(\cdot)} \leq \frac{a^{n-n/\beta+1}}{2} \|\chi_{Q(x, r)}\|_{p(\cdot)} \tag{2.4}
\]
for every \( r > 0 \) and \( x \in \mathbb{R}^n \), where \( Q(x, r) \) denotes a cube centered at \( x \) and with diameter \( r \) (see [48, Lemma 2.17] or (2.9) in [47]).

Let \( k_0 \in \mathbb{N} \) satisfy that
\[
a^{k_0-1} < 2 \leq a^{k_0}, \tag{2.5}
\]
where \( a \) is given in condition (2.4), and let \( \epsilon_0 = 1/k_0 \in (\frac{1}{1+\ln 2/\ln a}, \frac{\ln 2}{\ln a}] \) (which will be used in the following sections).

Note that, if \( p(\cdot) \in C_\infty^\log(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n) \), the estimates (2.4) and (2.5) imply the doubling condition for the functional \( a(Q) := \|\chi_Q\|_{p(\cdot)} \), that is
\[
\|\chi_{2Q}\|_{p(\cdot)} \leq C\|\chi_Q\|_{p(\cdot)}
\]
for every cube \( Q \subset \mathbb{R}^n \).

Set \( 0 < \gamma < n \) and \( p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) such that \( 1/q(x) = 1/p(x) - \gamma/n \) with \( p_+ < n/\gamma \). We say a weight \( \omega \in A_{p(\cdot), q(\cdot)}^\gamma(\mathbb{R}^n) \) if there exists a positive constant \( C \) such that for every cube \( Q \), the inequality
\[
\|\omega\chi_Q\|_{q(\cdot)} \omega^{-1}\|\chi_Q\|_{p'(\cdot)} \leq C |Q|^{1-\gamma/n} \tag{2.6}
\]
holds.

When \( \gamma = 0 \), the inequality above is the \( A_{p(\cdot)}(\mathbb{R}^n) \) class given by Cruz-Uribe, Diening and Hästö in [11], that characterizes the boundedness of the Hardy–Littlewood maximal operator on \( L_p^\infty(\mathbb{R}^n) \), that is, the measurable functions \( f \) such that \( f\omega \in L_p^\infty(\mathbb{R}^n) \).

The following result was proved in [1] and gives a relation between the \( A_{p(\cdot)}(\mathbb{R}^n) \) and the \( A_{p(\cdot), q(\cdot)}^\gamma(\mathbb{R}^n) \) classes (see also [47, Lemma 4.14]).
Lemma 2.12 (see [1, Lemma 4.1]) Let $0 < \gamma < n$ and $\omega$ be a weight. Set $p(\cdot), q(\cdot), s(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $1/q(x) = 1/p(x) - \gamma/n$ and $s(x) = (1 - \gamma/n)q(x)$ with $p_+ < n/\gamma$. Then $\omega \in A_{p(\cdot), q(\cdot)}^{\gamma}(\mathbb{R}^n)$ if and only if $\omega^{1/n} \in A_{s(\cdot)}(\mathbb{R}^n)$.

Note that if $q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, then $s(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$. Since $M$ is continuous on $L^{s(\cdot)}(\mathbb{R}^n)$, thus Pradolini and Ramos obtained the following lemma [47].

Lemma 2.13 ([47, Lemma 4.15]) Let $0 \leq \gamma < n$, $p(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ and $1/q(x) = 1/p(x) - \gamma/n$. Then $1 \in A_{p(\cdot), q(\cdot)}^{\gamma}(\mathbb{R}^n)$.

2.3 A Pointwise Estimate

The following notations can be founded in refs. [4, 47].

Definition 2.14 Let $f$ be a locally integrable function defined on $\mathbb{R}^n$.

(1) Set $0 \leq \delta < 1$. The $\delta$-sharp maximal operator is defined by

$$f^\sharp_\delta(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1+\delta/n}} \int_Q |f(y) - f_Q|\,dy,$$

where the supremum is taken over all cube $Q \subset \mathbb{R}^n$ containing $x$, and $f_Q = |Q|^{-1} \int_Q f(z)\,dz$ denotes the average of $f$ over the cube $Q \subset \mathbb{R}^n$.

(2) Let $0 \leq \delta(\cdot) < 1$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $\delta(\cdot)/n = 1/\beta - 1/p(\cdot)$. The $\delta(\cdot)$-sharp maximal operator is defined by

$$f^\sharp_{\delta(\cdot)}(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1/\beta-1} \|\chi_Q\|_{p(\cdot)}} \left( \frac{1}{|Q|} \int_Q |f(y) - f_Q|\,dy \right).$$

And for any $\gamma > 0$, the following generalization of the operator above is

$$f^\sharp_{\delta(\cdot), \gamma}(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1/\beta-1} \|\chi_Q\|_{p(\cdot)}} \left( \frac{1}{|Q|} \int_Q \left| |f(y)|^\gamma - (|f|^\gamma)_Q \right|\,dy \right)^{1/\gamma}.$$

(3) Let $0 \leq \alpha < n$ and $\epsilon > 0$, define the following operators

$$M_\epsilon f(x) = [M(\|f\|^\alpha)(x)]^{1/\epsilon} = \left( \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|^\epsilon\,dy \right)^{1/\epsilon}$$

and

$$M_{\alpha, L(\log L)} f(x) = \sup_{Q \ni x} \|f\|_{L(\log L), Q},$$

where $\|\cdot\|_{L(\log L), Q}$ is the Luxemburg type average defined by

$$\|f\|_{L(\log L), Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q |f(x)| \log(e + |f|/\lambda)\,dx \leq 1 \right\}.$$
and
\[ M_r(\tilde{f})(x) = \sup_{Q \ni x} \prod_{j=1}^{m} \left( \frac{1}{|Q|} \int_{Q} |f_j(y_j)|^{r} \, dy_j \right)^{1/r}, \quad \text{for } r > 1, \]

(2) the maximal functions related to Young function \( \Phi(t) = t(1 + \log^{+} t) \) are defined by
\[ M_{L(\log L)}^{i}(\tilde{f})(x) = \sup_{Q \ni x} \|f_i\|_{L(\log L),Q} \prod_{j=1}^{m} \frac{1}{|Q|} \int_{Q} |f_j(y_j)| \, dy_j \]
and
\[ M_{L(\log L)}(\tilde{f})(x) = \sup_{Q \ni x} \prod_{j=1}^{m} \|f_j\|_{L(\log L),Q}, \]
where the supremum is taken over all the cubes \( Q \) containing \( x \).

Obviously, if \( r > 1 \), then the following pointwise estimates hold
\[ M(\tilde{f})(x) \leq C M_{L(\log L)}^{i}(\tilde{f})(x) \leq C_1 M_{L(\log L)}(\tilde{f})(x) \leq C_2 M_r(\tilde{f})(x). \quad (2.7) \]
The first two inequalities in (2.7) follows from (2.3) with \( t = 1 \), that is
\[ \frac{1}{|Q|} \int_{Q} |f_j(y_j)| \, dy_j \leq \|f_j\|_{L(\log L),Q}, \]
and the last one follows from the generalized Jensen’s inequality (see [38, Lemma 4.2]).

In addition, one can see that \( M_{L(\log L)}(\tilde{f}) \) is pointwise controlled by a multiple of \( \prod_{j=1}^{m} M^{2}(f_j)(x) \), where \( M^{2} = M \circ M \).

**Definition 2.16** (Lipschitz-type spaces)  (1) Let \( 0 < \delta < 1 \). The space \( \Lambda_{\delta} \) of the Lipschitz continuous functions with order \( \delta \) is defined by
\[ \Lambda_{\delta}(\mathbb{R}^{n}) = \{ f : |f(x) - f(y)| \leq C|x - y|^\delta \text{ for a.e. } x, y \in \mathbb{R}^{n} \}, \]
where \( f \) is the locally integrable function on \( \mathbb{R}^{n} \), and the smallest constant \( C > 0 \) will be denoted Lipschitz norm by \( \|f\|_{\Lambda_{\delta}} \).

(2) Let \( 0 \leq \delta < 1 \). The space \( \mathbb{L}(\delta) \) is defined to be the set of all locally integrable functions \( f \), i.e., there exists a positive constant \( C > 0 \), such that
\[ \sup_{Q} \frac{1}{|Q|^{1+\delta/n}} \int_{Q} |f(y) - f_{Q}| \, dy < C, \]
where the supremum is taken over every cube \( Q \subset \mathbb{R}^{n} \) and \( f_{Q} = \frac{1}{|Q|} \int_{Q} f(z) \, dz \). The least constant \( C \) will be denoted by \( \|f\|_{\mathbb{L}(\delta)} \).

(3) [48] Let \( 0 < \alpha < n \) and an exponent function \( p(\cdot) \in \mathcal{P}_{1}(\mathbb{R}^{n}) = \{ 1 \} \cup \mathcal{P}(\mathbb{R}^{n}) \). We say that a locally integrable function \( f \) belongs to \( \mathbb{L}_{\alpha,p(\cdot)}(\mathbb{R}^{n}) \) if there exists a constant \( C \) such that
\[ \frac{1}{|B|^{\alpha/n} \chi_{B} p'(\cdot)} \int_{B} |f(y) - f_{B}| \, dy < C \]
for every ball \( B \subset \mathbb{R}^{n} \), with \( f_{B} = \frac{1}{|B|} \int_{B} f(z) \, dz \). The least constant \( C \) will be denoted by \( \|f\|_{\mathbb{L}_{\alpha,p(\cdot)}} \).
Let \( r(\cdot) \in \mathcal{C}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n) \) such that \( 1 < \beta \leq r_- \leq r(x) \leq r_+ < \frac{n\beta}{(n-\beta)} \), and set \( \frac{\delta(x)}{n} = \frac{1}{\beta} - \frac{1}{r(x)} \). The space \( \mathbb{L}(\delta(\cdot)) \) is defined by the set of the measurable functions \( f \) such that (see [48] for more details)

\[
\|f\|_{\mathbb{L}(\delta(\cdot))} = \sup_B \frac{1}{|B|^{1/\beta} \|\chi_B\|_r^{\cdot}(\cdot)} \int_B |f(y) - f_B| \, dy < \infty.
\]

(5) (weighted Lipschitz integral spaces \( \mathbb{L}_w(\delta) \), see [45] or [46] Let \( w \) be a weight and \( 0 \leq \delta < 1 \), we say that a locally integrable function \( f \) belongs to \( \mathbb{L}_w(\delta) \) if there exists a positive constant \( C \) such that the inequality

\[
\frac{\|w\chi_B\|_\infty}{|B|^{1+\delta/n}} \int_B |f(y) - f_B| \, dy < C
\]

holds for every ball \( B \subset \mathbb{R}^n \). The least constant \( C \) will be denoted by \( \|f\|_{\mathbb{L}_w(\delta)} \).

**Remark 2.17**

(i) In (1) of Definition 2.16, it is well known that the space \( \Lambda_\delta \) coincides with the space \( \mathbb{L}(\delta) \) (see [28, 47]).

(ii) In (2) of Definition 2.16, it is not difficult to see that, for \( \delta = 0 \), the space \( \mathbb{L}(\delta) \) coincides with the space of bounded mean oscillation functions \( \text{BMO} \) (see [33]).

(iii) In (4) of Definition 2.16, denote \( z^+ \) by (see [4])

\[
z^+ = \begin{cases} 
z, & \text{if } z > 0, \\
0, & \text{if } z \leq 0.
\end{cases}
\]

In addition, when \( r(x) \) is equal to a constant \( r \), this space coincides with the space \( \mathbb{L}(n/\beta - n/r) \).

(iv) In (5) of Definition 2.16, it is not difficult to see that, for \( \delta = 0 \), the space \( \mathbb{L}_w(\delta) \) coincides with one of the versions of weighted bounded mean oscillation spaces (see [39]). Moreover, for the case \( w \equiv 1 \), the space \( \mathbb{L}_w(\delta) \) is the known Lipschitz integral space for \( 0 < \delta < 1 \).

**Lemma 2.18** ([47]) Let \( p(\cdot) \in \mathcal{C}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n) \) and \( 1 < \beta \leq p_- \). Then the functional

\[
a(Q) = |Q|^{1/\beta - 1} \|\chi_Q\|_{p^{\cdot}(\cdot)}
\]

satisfies the \( T_\infty \) condition, that is, there exists a positive constant \( C \) such that \( a(Q') \leq Ca(Q) \) for each cube \( Q \) and each cube \( Q' \subset Q \).

**Lemma 2.19** ([37]) Let \( 1 \leq r < \infty \) and \( a \in T_\infty \). Then

\[
\sup_Q \frac{1}{a(Q)} \left( \frac{1}{|Q|} \int_Q |f(x) - (f)_Q|^r \, dx \right)^{1/r} \approx \sup_Q \frac{1}{a(Q)} \frac{1}{|Q|} \int_Q |f(x) - (f)_Q| \, dx.
\]

The following inequalities are also necessary (see [35, (2.16)] or [38, Lemma 4.6] or [24, p. 485]).

**Lemma 2.20** (Kolmogorov’s inequality) Let \( 0 < p < q < \infty \). Then there is a positive constant \( C \) such that

\[
|Q|^{-1/p} \|f\|_{L_p(Q)} \leq C |Q|^{-1/q} \|f\|_{L_q,\infty(Q)}
\]

holds for any measurable function \( f \), where \( L^{q,\infty}(Q) \) denotes the weak space with norm

\[
\|f\|_{L^{q,\infty}(Q)} = \sup_{t>0} t^{1/q} |\{x \in Q : |f(x)| > t\}|^{1/q}.
\]
Lemma 2.21 ([47]) Let $p(\cdot) \in \mathcal{C}^{\text{log}}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, $1 < \beta \leq p_-$ such that $0 \leq \delta(\cdot)/n = 1/\beta - 1/p(\cdot) \leq 1$ and $b \in \mathbb{L}(\delta(\cdot))$.

1. Then there is a positive constant $C$ such that
\[
\sup_Q \frac{\|b - b_Q\|_{\exp L,Q}}{|Q|^{1/\beta - 1}} \chi_{Q'}(\cdot) \leq C\|b\|_{\mathbb{L}(\delta(\cdot))}.
\]

2. Then there is a positive constant $C$ such that
\[
|b_{a^{k_0}(j+1)Q} - b_{a^{k_0}Q}| \leq Cj\|b\|_{\mathbb{L}(\delta(\cdot))}|a^{k_0(j+1)Q}|^{1/\beta - 1} \chi_{a^{k_0}(j+1)Q}_{Q'}(\cdot)
\]
holds for every $j \in \mathbb{N}$.

Lemma 2.22 (see [47, Lemma 4.11]) Let $0 < \gamma < 1$, $r(\cdot), p(\cdot) \in \mathcal{C}^{\text{log}}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, and let $q(\cdot), \beta, \delta(\cdot)$ such that $0 \leq \delta(\cdot)/n = 1/p(\cdot) - 1/q(\cdot) = 1/\beta - 1/r(\cdot) \leq 1/n$. If $\|f\|_{q(\cdot)} < \infty$, then there is a positive constant $C$ such that
\[
\|f\|_{q(\cdot)} \leq C\|f^2_{\delta(\cdot),\gamma}\|_{p(\cdot)}.
\]

The following result is a generalization to the variable context of a pointwise estimate of commutators.

Lemma 2.23 Let $m \geq 2$, $0 < \gamma < \eta < 1/m$, $0 < \epsilon \leq 1$, $p(\cdot) \in \mathcal{C}^{\text{log}}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, $1 < \beta \leq p_-$ such that $0 \leq \delta(\cdot)/n = 1/\beta - 1/p(\cdot) \leq 1$ and $b = (b_1, b_2, \ldots, b_m) \in \mathbb{L}(\delta(\cdot)) \times \mathbb{L}(\delta(\cdot)) \times \cdots \times \mathbb{L}(\delta(\cdot))$.

Then there exists a positive constant $C$ such that
\[
(T_{b_j}(\tilde{f})_{\delta(\cdot),\gamma}^2)(x) \leq C\|b_j\|_{\mathbb{L}(\delta(\cdot))}(M_\eta(T\tilde{f})(x) + M_{L(\log L)}(\tilde{f})(x)) \quad (j = 1, 2, \ldots, m).
\]

Furthermore,
\[
(T_{\Sigma_{b_j}}(\tilde{f})_{\delta(\cdot),\gamma}^2)(x) \leq C\sum_{j=1}^m \|b_j\|_{\mathbb{L}(\delta(\cdot))}(M_\eta(T\tilde{f})(x) + M_{L(\log L)}(\tilde{f})(x)).
\]

Proof Let $Q \subset \mathbb{R}^n$ and $x \in Q$. Due to the fact $|a| - |b| \leq |a - c|\gamma$ for $0 < \gamma < 1$, it is enough to show that, for some constant $C_Q$, there exists a positive constant $C$ such that
\[
\left(\frac{1}{|Q|} \int_Q |T_{b_j}(\tilde{f})(z) - C_Q|^\gamma dz\right)^{1/\gamma} \leq C\|b_j\|_{\mathbb{L}(\delta(\cdot))}(M_\eta(T\tilde{f})(x) + M_{L(\log L)}(\tilde{f})(x)).
\]

For each $j$, we decompose $f_j = f_j^0 + f_j^\infty$ with $f_j^0 = f_j \chi_{a^{k_0}Q}$, where $a$ and $k_0$ are defined as in conditions (2.4) and (2.5) respectively. Then
\[
\prod_{j=1}^m f_j(y_j) = \prod_{j=1}^m (f_j^0(y_j) + f_j^\infty(y_j)) = \sum_{\alpha_1,\ldots,\alpha_m \in \{0,\infty\}} f_1^{\alpha_1}(y_1) \cdots f_m^{\alpha_m}(y_m)
\]
\[
= \prod_{j=1}^m f_j^0(y_j) + \sum_{(\alpha_1,\ldots,\alpha_m) \in \ell} f_1^{\alpha_1}(y_1) \cdots f_m^{\alpha_m}(y_m),
\]
where $\ell = \{(\alpha_1, \ldots, \alpha_m) : \text{there is at least one } \alpha_j \neq 0\}$. Let $\lambda$ be some positive constant to be chosen. It is easy to see that
\[
T_{b_j}(\tilde{f})(x) = (b_j(x) - \lambda)T(\tilde{f})(x) - T(f_j^0, \ldots, (b_j - \lambda)f_j^0, \ldots, f_m^0)(x)
\]
\[
- \sum_{(\alpha_1,\ldots,\alpha_m) \in \ell} T(f_j^{\alpha_1}, \ldots, (b_j - \lambda)f_j^{\alpha_1}, \ldots, f_m^{\alpha_m})(x).
\]
By taking \( \lambda = (b_j)_{a^0Q} \) and \( C_Q = \sum_{(\alpha_1, \ldots, \alpha_m) \in \ell} (T(f_1^{\alpha_1}, \ldots, (b_j - \lambda)f_j^{\alpha_j}, \ldots, f_m^{\alpha_m}))Q \), we obtain
\[
\frac{1}{|Q|} \int_Q |T_{b_j}(\bar{f})(z) - C_Q|^{\gamma} dz \leq C \frac{|Q|^{1/\beta - 1} \|\chi_Q\|_{p'(\cdot)}}{|Q|^{1/\beta - 1} \|\chi_Q\|_{p'(\cdot)}} (I + II + III),
\]
where
\[
I = \left( \frac{1}{|Q|} \int_Q |(b_j(z) - \lambda)T(\bar{f})(z)|^{\gamma} dz \right)^{1/\gamma}
\]
\[
II = \left( \frac{1}{|Q|} \int_Q |T(f_1^0, \ldots, (b_j - \lambda)f_j^0, \ldots, f_m^0)(z)|^{\gamma} dz \right)^{1/\gamma}
\]
\[
III = \sum_{(\alpha_1, \ldots, \alpha_m) \in \ell} \left( \frac{1}{|Q|} \int_Q |T(f_1^{\alpha_1}, \ldots, (b_j - \lambda)f_j^{\alpha_j}, \ldots, f_m^{\alpha_m})(z) - (T(f_1^{\alpha_1}, \ldots, (b_j - \lambda)f_j^{\alpha_j}, \ldots, f_m^{\alpha_m}))Q|^{\gamma} dz \right)^{1/\gamma}
\]
\[
:= \sum_{(\alpha_1, \ldots, \alpha_m) \in \ell} III_{\alpha_1, \ldots, \alpha_m}.
\]

Let us first estimate \( I \). By taking \( 1 < r < \eta/\gamma \) such that \( r'\gamma > 1 \) and using Hölder’s inequality, we obtain
\[
I \leq C \left( \frac{1}{|Q|} \int_Q |(b_j(z) - (b_j)_{a^0Q})|^{r' \gamma} dz \right)^{1/(r' \gamma)} \left( \frac{1}{|Q|} \int_Q |T(\bar{f})(z)|^{\gamma} dz \right)^{1/(r' \gamma)}.
\]
It is known from Lemma 2.18 that the functional \( a(Q) = |Q|^{1/\beta - 1} \|\chi_Q\|_{p'(\cdot)} \) satisfies \( T_\infty \) condition. Then, by Lemma 2.19, we can obtain
\[
\frac{I}{|Q|^{1/\beta - 1} \|\chi_Q\|_{p'(\cdot)}} \leq C \frac{|Q|^{1/\beta - 1} \|\chi_Q\|_{p'(\cdot)}}{|Q|^{1/\beta - 1} \|\chi_Q\|_{p'(\cdot)}} \left( \frac{1}{|Q|} \int_Q |T(\bar{f})(z)|^{\gamma} dz \right)^{1/(r' \gamma)}
\]
\[
\leq C \|b_j\|_{L_\infty(\delta(\cdot))} M_{r' \gamma}(T(\bar{f}))(x)
\]
\[
\leq C \|b_j\|_{L_\infty(\delta(\cdot))} M_{\eta}(T(\bar{f}))(x).
\]

To estimate \( II \), note that \( 0 < \gamma < 1/m \). By using \( L^1 \times L^1 \times \cdots \times L^1 \) to \( L^{1/m, \infty} \) boundedness of \( T \), Kolmogorov’s inequality (Lemma 2.20), Hölder’s inequality (2.3) with \( t = 1 \) and condition (2.7), we have
\[
II = \left( \frac{1}{|Q|} \int_Q |T(f_1^0, \ldots, (b_j - \lambda)f_j^0, \ldots, f_m^0)(z)|^{\gamma} dz \right)^{1/\gamma}
\]
\[
\leq C \frac{|Q|^{1/m}}{|Q|^{1/m}} \|T(f_1^0, \ldots, (b_j - \lambda)f_j^0, \ldots, f_m^0(L))\|_{L^{1/m, \infty}(Q)}
\]
\[
\leq C \frac{|Q|^{1/m}}{|Q|^{1/m}} \|(b_j - (b_j)_{a^0Q})f_j^0\|_{L^1(Q)} \prod_{k=1}^m \|f_k^0\|_{L^1(Q)}
\]
\[
\leq C \frac{|Q|}{|Q|} \int_Q |(b_j(z) - (b_j)_{a^0Q})f_j^0(z)| dz \prod_{k=1}^m \frac{1}{|Q|} \int_Q |f_k^0(z)| dz
\]
\[
\leq C \|b_j - (b_j)_{a^0Q}\|_{L_{a^0Q}L_{a^0Q}f_jL_{(\log L), a^0Q} \prod_{k=1}^m \frac{1}{a^0Q} \int_{a^0Q} |f_k(z)| dz}
\]
\[\leq C\|b_j - (b_j)_{a^0 Q}\|_{\exp L, a^0 Q} M^j_{L(\log L)}(\tilde{f})(x)\]

Thus, by Lemmas 2.18 and 2.21 and doubling condition implied in conditions (2.4) and (2.5), we get

\[\frac{II}{|Q|^{1/\beta - 1} \|\chi_Q\|_{p',(\cdot)}} \leq C\frac{\|b_j - (b_j)_{a^0 Q}\|_{\exp L, a^0 Q} M_{L(\log L)}(\tilde{f})(x)}{|Q|^{1/\beta - 1} \|\chi_Q\|_{p',(\cdot)}}\]

To estimate III, we consider first the case when \(\alpha_1 = \cdots = \alpha_m = \infty\), i.e., we consider the term \(III_{\infty, \ldots, \infty}\) first. For any \(z \in Q\), one has

\[|T(f_1^\infty, \ldots, (b_j - (b_j)_{a^0 Q})f_j^\infty, \ldots, f_m^\infty)(z) - (T(f_1^\infty, \ldots, (b_j - (b_j)_{a^0 Q})f_j^\infty, \ldots, f_m^\infty))|Q| \leq \frac{1}{|Q|} \int_Q \left( \int_{\mathbb{R}^n \setminus a^0 Q}^m |b_j(y_j) - (b_j)_{a^0 Q}| K(z, \bar{y}) - K(w, \bar{y}) \prod_{i=1}^m |f_i^\infty(y_i)|d\bar{y} \right) dw \]

where \(Q_k = (a^{k_0(k+1)} Q) \setminus (a^{k_0} Q)\) for \(k = 1, 2, \ldots\). Noting that, for \(w, z \in Q\) and any \((y_1, \ldots, y_m) \in (Q_k)^m\), one has

\[a^{k_0 l}(Q) \leq |z - y_i| < a^{k_0(k+1) l}(Q)\quad \text{and} \quad |z - w| \leq a^{k_0 l}(Q),\]

and applying (1.1), we have

\[|K(z, \bar{y}) - K(w, \bar{y})| \leq \frac{A|z - w|^\epsilon}{(|z - y_1| + \cdots + |z - y_m|)^{mn+\epsilon}} \leq \frac{Ca^{-k_0 k \epsilon}}{|a^{k_0} Q|^m}.\]  

Then

\[|T(f_1^\infty, \ldots, (b_j - (b_j)_{a^0 Q})f_j^\infty, \ldots, f_m^\infty)(z) - (T(f_1^\infty, \ldots, (b_j - (b_j)_{a^0 Q})f_j^\infty, \ldots, f_m^\infty))|Q| \leq \frac{1}{|Q|} \sum_{k=1}^\infty \int_Q \left( \int_{(Q_k)^m} |b_j(y_j) - (b_j)_{a^0 Q}| K(z, \bar{y}) - K(w, \bar{y}) \prod_{i=1}^m |f_i^\infty(y_i)|d\bar{y} \right) dw \]

\[\leq C \sum_{k=1}^\infty a^{-k_0 k \epsilon} \int_Q \left( \int_{(Q_k)^m} |b_j(y_j) - (b_j)_{a^0 Q}| \frac{1}{|a^{k_0} Q|^m} \prod_{i=1}^m |f_i^\infty(y_i)|d\bar{y} \right) dw \]

\[\leq C \sum_{k=1}^\infty a^{-k_0 k \epsilon} \int_{(Q_k)^m} |b_j(y_j) - (b_j)_{a^0 Q}| \frac{1}{|a^{k_0} Q|^m} \prod_{i=1}^m |f_i^\infty(y_i)|d\bar{y} \]

\[\leq C \sum_{k=1}^\infty a^{-k_0 k \epsilon} \left( \frac{1}{|a^{k_0} Q|^m} \int_{a^{k_0} (k+1) Q}^m |b_j(y_j) - (b_j)_{a^0 Q}| f_j(y_j)|dy_j| \right) \]

\[\times \prod_{i=1}^m \left( \frac{1}{|a^{k_0} (k+1) Q|} \int_{a^{k_0} (k+1) Q}^m f_i(y_i)|dy_i| \right).\]
By Lemma 2.21 and the generalized Hölder’s inequality in Orlicz space (see (2.3) with \(t = 1\)), we have

\[
\frac{1}{|a^{k_0(k+1)}Q|} \int_{a^{k_0(k+1)}Q} |b_j(y_j) - (b_j)_{a^{k_0}Q}| |f_j(y_j)| dy_j \\
\leq \frac{1}{|a^{k_0(k+1)}Q|} \int_{a^{k_0(k+1)}Q} |b_j(y_j) - (b_j)_{a^{k_0}Q}| |f_j(y_j)| dy_j \\
+ \frac{|(b_j)_{a^{k_0}Q} - (b_j)_{a^{k_0}Q}|}{|a^{k_0(k+1)}Q|} \int_{a^{k_0(k+1)}Q} |f_j(y_j)| dy_j \\
\leq C \|b_j - (b_j)_{a^{k_0}Q}\|_{\exp L,a^{k_0}Q} \|f_j\|_{L(\log L),a^{k_0}Q} \\
+ \|b_j\|_{L(\log L),a^{k_0}Q} \|f_j\|_{L(\log L),a^{k_0}Q} \\
\leq C \left( \frac{\|b_j - (b_j)_{a^{k_0}Q}\|_{\exp L,a^{k_0}Q}}{|a^{k_0(k+1)}Q|^{1/\beta-1}} \|X_{a^{k_0}Q}\|_{\beta(p')} + k \|b_j\|_{L(\delta(\cdot))} \right) \\
\times |a^{k_0(k+1)}Q|^{1/\beta-1} \|X_{a^{k_0}Q}\|_{\beta(p')} \|f_j\|_{L(\log L),a^{k_0}Q} \\
\leq C(k+1) \|b_j\|_{L(\delta(\cdot))} a^{k_0(k+1)}Q|^{1/\beta-1} \|X_{a^{k_0}Q}\|_{\beta(p')} \|f_j\|_{L(\log L),a^{k_0}Q}.
\] (2.10)

Therefore, we can apply (2.9) and (2.10) to obtain

III.\(\infty,\ldots,\infty\)

\[
\leq \frac{C}{|Q|} \int_{Q} |\sum_{k=1}^{\infty} a^{-k_0 k} (k+1) a^{k_0(k+1)}Q|^{1/\beta} \|X_{a^{k_0}Q}\|_{\beta(p')} \\
\times \|f_j\|_{L(\log L),a^{k_0}Q} \left( \prod_{i \neq j} \frac{1}{|a^{k_0(k+1)}Q|} \int_{a^{k_0(k+1)}Q} |f_i(y_i)| dy_i \right) \\
\leq C \|b_j\|_{L(\delta(\cdot))} M_{L(\log L)}(\tilde{f})(x) \sum_{k=1}^{\infty} a^{-k_0 k} (k+1) a^{k_0(k+1)}Q|^{1/\beta} \|X_{a^{k_0}Q}\|_{\beta(p')}.
\] (2.11)

Since \(0 < \gamma < 1\), by Hölder’s inequality, Lemma 2.21, (2.4), (2.5), (2.7) and (2.11), and the fact that \(1/k_0 = \epsilon_0 < \epsilon \leq 1\), we get

\[
\frac{\text{III.}\(\infty,\ldots,\infty\)}{|Q|^{1/\beta-1} \|X_{Q}\|_{\beta(p')}} \leq \frac{C \|b_j\|_{L(\delta(\cdot))}}{|Q|^{1/\beta-1} \|X_{Q}\|_{\beta(p')}} M_{L(\log L)}(\tilde{f})(x) \\
\times \left( \sum_{k=1}^{\infty} a^{-k_0 k} (k+1) a^{k_0(k+1)}Q|^{1/\beta} \|X_{a^{k_0}Q}\|_{\beta(p')} \right) \\
\leq C \|b_j\|_{L(\delta(\cdot))} M_{L(\log L)}(\tilde{f})(x) \sum_{k=1}^{\infty} a^{-k_0 k} (k+1) a^{k_0(k+1)\beta-1} \|X_{a^{k_0}Q}\|_{\beta(p')} \\
\leq C \|b_j\|_{L(\delta(\cdot))} M_{L(\log L)}(\tilde{f})(x) \left( \sum_{k=1}^{\infty} a^{-k_0 k} (k+1) a^{k_0(k+1)\beta-1} \right) \\
\leq C \|b_j\|_{L(\delta(\cdot))} M_{L(\log L)}(\tilde{f})(x) \sum_{k=1}^{\infty} \left( \frac{a^{k_0(1-\epsilon)}}{2} \right)^k
\]
\[
\begin{align*}
\leq C\|b_j\|_{L^q(\mathbb{R}^n)} \mathcal{M}_{\log L}(\tilde{f})(x).
\end{align*}
\] (2.12)

From estimates (2.9), (2.10), (2.11) and (2.12), there holds the following inequality, which will be used later,

\[
\begin{align*}
\frac{1}{|Q|^{1/\beta-1}} \|\chi_{Q}\|_{p'(-)} \left( \sum_{k=1}^{\infty} a^{-k\delta_k e} \left( \int_{(a_{k_0}(k+1)Q)^m} \frac{|b_j(y_j) - (b_j)_{a_{k_0}Q}| |f_j(y_j)|}{|a_{k_0}(k+1)Q|^m} \prod_{i=1\atop i \neq j}^m |f_i(y_i)| dy_i \right) \right) \leq C\|b_j\|_{L^q(\mathbb{R}^n)} \mathcal{M}_{\log L}(\tilde{f})(x).
\end{align*}
\] (2.13)

Now, for \((\alpha_1, \ldots, \alpha_m) \in \ell\), let us consider the terms \(III_{\alpha_1, \ldots, \alpha_m}\) such that at least one \(\alpha_j = 0\) and one \(\alpha_i = \infty\). Without loss of generality, we assume that \(\alpha_1 = \cdots = \alpha_l = 0\) and \(\alpha_{l+1} = \cdots = \alpha_m = \infty\) with \(1 \leq l \leq m\). For any \(z \in Q\), set \(Q_k = (a_{k_0(k+1)}Q) \setminus (a_{k_0(k+1)}Q)\) as above, when \(l + 1 \leq j \leq m\), applying (2.8), we obtain

\[
|T(f_1^{\alpha_1}, \ldots, (b_j - (b_j)_{a_{k_0}Q})f_j^{\alpha_j}, \ldots, f_m^{\alpha_m})(z) - (T(f_1^{\alpha_1}, \ldots, (b_j - (b_j)_{a_{k_0}Q})f_j^{\alpha_j}, \ldots, f_m^{\alpha_m}))(z)|
\leq \frac{1}{|Q|} \int_Q \left( \int_{(a_{k_0}Q)^l \setminus (\mathbb{R}^n \setminus (a_{k_0}Q)^m)} |b_j(y_j) - (b_j)_{a_{k_0}Q}| |K(z, \bar{y}) - K(w, \bar{y})| \right.
\times \prod_{i=1}^l |f_i^0(y_i)| \prod_{i=l+1}^m |f_i^\infty(y_i)| dy \right) dw
\leq \frac{C}{|Q|} \int_Q \left( \int_{(a_{k_0}Q)^l \setminus (\mathbb{R}^n \setminus (a_{k_0}Q)^m)} \left( \sum_{k=1}^{\infty} a^{-k\delta_k e} \frac{|b_j(y_j) - (b_j)_{a_{k_0}Q}|}{|a_{k_0}(k+1)Q|^m} \prod_{i=1\atop i \neq j}^m |f_i(y_i)| dy_i \right) \right) dw
\leq \sum_{k=1}^{\infty} a^{-k\delta_k e} \left( \int_{(a_{k_0}Q)^m \setminus (a_{k_0}(k+1)Q)^m} \frac{|b_j(y_j) - (b_j)_{a_{k_0}Q}| |f_j(y_j)|}{|a_{k_0}(k+1)Q|^m} \prod_{i=1\atop i \neq j}^m |f_i(y_i)| dy_i \right).
\]
Let $M^\sharp$ be the standard sharp maximal function, and $M_\gamma^\sharp f(x) = [M^\sharp(|f|^\gamma)(x)]^{1/\gamma}$ for $\gamma > 0$. For $0 < r < \infty$ and $\rho \in [0, n/r)$, defining maximal operator $M_{\rho,r}$ by

$$M_{\rho,r}(f)(x) = \sup_{Q \ni x} \left( \frac{1}{|Q|^{1 - \frac{\rho}{n}}} \int_Q |f(y)|^\gamma \, dy \right)^{1/r},$$

and $M_{\rho,r}(f)(x) = [M_{\rho,1,1}(|f|^\gamma)(x)]^{1/r}$. When $\rho = 0$, we denote $M_{\rho,r}$ simply by $M_r$, and $M_1$ is the standard Hardy--Littlewood maximal function $M$.

Similar to [52, Lemma 2.2] (also see [50, 51]), by elementary calculations and derivations we have the following estimation.

**Lemma 2.2.4** Let $m \geq 2$, $0 < \gamma < \eta < 1/m$, $0 < \delta \leq 1$, $p_i(\cdot) \in \mathcal{C}(\log \mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, $1 < r_i < (p_i)_-$ with $i = 1, 2, \ldots, m$. If $\vec{b} = (b_1, b_2, \ldots, b_m) \in \mathbb{L}(\delta) \times \mathbb{L}(\delta) \times \cdots \times \mathbb{L}(\delta)$, then there exists a positive constant $C$ such that

$$M_\gamma^\sharp(T_{b_j}(\vec{f}))(x) \leq C \|b_j\|_{L(\delta)} \left( M_{\delta,\eta}^{\star}(T\vec{f})(x) + M_{\delta,\rho_j}(f_j)(x) \prod_{i \neq j} M_{\delta,\rho_i}(f_i)(x) \right) \quad (j = 1, 2, \ldots, m)$$

for all $m$-tuples $\vec{f} = (f_1, f_2, \ldots, f_m)$ of bounded measurable functions with compact support.

## 3 Proofs of the Main Results

We now give the proofs of the theorems given in Section 1. First, we prove Theorem 1.2, which uses some techniques and ideas applied by Janson [32] (also see [6, 47, 54]).

**Proof of Theorem 1.2** Without loss of generality, we only consider the case that $m = 2$. Actually, similar procedure work for all $m \in \mathbb{N}$.

(1) We first prove $\vec{b} = (b_1, b_2) \in \mathbb{L}(\delta) \times \mathbb{L}(\delta)$. Assume that $T_{b_j}$ $(j = 1, 2)$ maps $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$. Let $\vec{z} = (z_1, z_2) \in \mathbb{R}^n \times \mathbb{R}^n$, and by the homogeneity of $K$, we may assume without loss of generality that $|z_j - z_{j_0}| < \epsilon \sqrt{n}$ $(j = 1, 2)$ with $\vec{z}_0 = (z_{j_0}, z_{j_0} \in \mathbb{R}^n \setminus \{0\} \times \mathbb{R}^n \setminus \{0\}$ and $\epsilon > 0$, which implies that $|\vec{z} - \vec{z}_0| < \epsilon \sqrt{2n}$. Then $1/K(\vec{z}_0, \epsilon \sqrt{2n}) \subset \mathbb{R}^{2n}$, can be expanded as an absolutely convergent Fourier series

$$\frac{1}{K(\vec{z})} = \frac{1}{K(z_1, z_2)} = \sum_k a_k e^{i \vec{v}_k \cdot \vec{z}} = \sum_k a_k e^{i (v_{1k}, v_{2k}) (z_1, z_2)},$$

where the individual vectors $\vec{v}_k = (v_{1k}, v_{2k}) \in \mathbb{R}^n \times \mathbb{R}^n$ do not play any significant role in the proof.

Set $\vec{z}' = (z_1^*, z_2^*)$. If $z_j^* = \epsilon^{-1} z_{j_0}$, then $|z_j - z_j^*| < \sqrt{n}$ implies that $|\epsilon z_j - z_{j_0}| < \epsilon \sqrt{n}$ $(j = 1, 2)$. Thus, by the homogeneity of $K$, for all $\vec{z} \in B(\vec{z}', \sqrt{2n}) \subset \mathbb{R}^{2n}$, we can obtain

$$\frac{1}{K(\vec{z})} = \frac{1}{K(\vec{z}')} = \epsilon^{-2n} \sum_k a_k e^{i \epsilon \vec{v}_k \cdot \vec{z}}.$$

Let $Q = Q(x_0, l)$ be an arbitrary cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, diameter $l$ and center $x_0$, and set $y_{j_0} = x_0 - l z_j^*$ and $Q_j^* = Q(y_{j_0}, l) \subset \mathbb{R}^n$ $(j = 1, 2)$. By taking $x \in Q, y_j \in Q_j^*$ $(j = 1, 2)$, we have
\[ \frac{|x - y_j|}{l} - z_j^* = \frac{|x - x_0 + x_0 - y_j - y_{j0} - y_j|}{l} \leq \frac{|x - x_0|}{l} + \frac{|y_j - y_{j0}|}{l} \leq \sqrt{n}. \]

So we have \(|(x-\frac{y_1}{l}, x-\frac{y_2}{l}) - z^*| = |(x-\frac{y_1}{l}, x-\frac{y_2}{l}) - (z_1^*, z_2^*)| \leq \sqrt{2n} \), that is, \((x-\frac{y_1}{l}, x-\frac{y_2}{l}) \in B(z^*, \sqrt{2n})\), which means that \((x - y_1, x - y_2)\) is bounded away from the singularity of \(K\).

Without loss of generality, let \(s_1(x) = \text{sgn}(b_1(x) - (b_1)_Q)\), then

\[
\int_Q |b_1(x) - (b_1)_Q|^2 dx = \int_Q (b_1(x) - (b_1)_Q)s_1(x)dx = \int_Q \left( \frac{1}{|Q_1^2|} \int_{Q_1^2} (b_1(x) - b_1(y_1))dy_1 \right) \left( \frac{1}{|Q_2^2|} \int_{Q_2^2} \chi_{Q_2^2}(y_2)dy_2 \right) s_1(x)dx \\
= l^{-2n} \int_{R^n} \left( \int_{R^n} (b_1(x) - b_1(y_1)) \chi_{Q_1^2}(y_1) \chi_{Q_2^2}(y_2) \chi_Q(x) s_1(x) dy_1 dy_2 dx \\
= l^{-2n} \int_{R^n} \left( \int_{R^n} \int_{R^n} \chi_{Q_1^2}(y_1) \chi_{Q_2^2}(y_2) \chi_Q(x) s_1(x) dy_1 dy_2 dx \\
= \epsilon^{-2n} \sum_k a_k \int_{R^n} \int_{R^n} \int_{R^n} (b_1(x) - b_1(y_1))K(x - y_1, x - y_2) e^{-\frac{k\pi}{l}v_1 y_1} \chi_{Q_1^2}(y_1) \\
\times \chi_{Q_2^2}(y_2) e^{\frac{k\pi}{l}v_2 y_2} \chi_Q(x) s_1(x) dy_1 dy_2 dx. \]

Set \(f_{1k}(y_1) = e^{-\frac{k\pi}{l}v_1 y_1} \chi_{Q_1^2}(y_1), f_{2k}(y_2) = e^{-\frac{k\pi}{l}v_2 y_2} \chi_{Q_2^2}(y_2), h_k(x) = e^{\frac{k\pi}{l}(v_1 + v_2) x/|l|} \chi_Q(x)\), then

\[
\int_Q |b_1(x) - (b_1)_Q|^2 dx = \epsilon^{-2n} \sum_k a_k \int_{R^n} \int_{R^n} \int_{R^n} (b_1(x) - b_1(y_1))K(x - y_1, x - y_2) \\
\times f_{1k}(y_1) f_{2k}(y_2) h_k(x) s_1(x) dy_1 dy_2 dx \\
= \epsilon^{-2n} \sum_k a_k \int_{R^n} \left( \int_{R^n} \int_{R^n} (b_1(x) - b_1(y_1))K(x - y_1, x - y_2) f_{1k}(y_1) f_{2k}(y_2) dy_1 dy_2 \right) \\
\times h_k(x) s_1(x) dx \\
= \epsilon^{-2n} \sum_k a_k \int_{R^n} T_{b_1}(f_{1k}, f_{2k})(x) h_k(x) s_1(x) dx \\
\leq C \epsilon^{-2n} \sum_k |a_k| \int_{R^n} |T_{b_1}(f_{1k}, f_{2k})(x)||h_k(x)||s_1(x)| dx \\
\leq C \epsilon^{-2n} \sum |a_k| \int_{R^n} |T_{b_1}(f_{1k}, f_{2k})(x)||h_k(x)||s_1(x)| dx.
\]
\[
\leq C e^{-2n} \sum_k |a_k| \int_Q |T_{b_1}(f_{1k}, f_{2k})(x)|dx. 
\]

(3.1)

Note that \(|f_{jk}(y_j)| \leq c_{Q_j^*}(y_j)| for every \(y_j \in Q_j^*\), which implies that \(f_{jk} \in L^{p_j^*}(\mathbb{R}^n)\) (\(j = 1, 2\)) for every \(k \in \mathbb{N}\). Then, from the generalized H"{o}lder’s inequality (2.2) and the hypothesis, we obtain

\[
\int_Q |T_{b_1}(f_{1k}, f_{2k})(x)|dx \leq C\|T_{b_1}(f_{1k}, f_{2k})\|_{q^*} \|\chi_Q\|_{q^*} 
\leq C\|f_{1k}\|_{p_{1j}} \|f_{2k}\|_{p_{2j}} \|\chi_Q\|_{q^*}. 
\]

Hence, since \(Q, Q_j^* \subset Q_j = Q(x_0, (|z_j^*| + 1)l)\) (\(j = 1, 2\), we have

\[
\int_Q |b_1(x) - (b_1)_{Q_1^*}|dx \leq C e^{-2n} \sum_k |a_k| \|f_{1k}\|_{p_{1j}} \|f_{2k}\|_{p_{2j}} \|\chi_Q\|_{q^*} \leq C e^{-2n} \sum_k |a_k| \|\chi_{Q_{10}}\|_{p_{1j}} \|\chi_{Q_{20}}\|_{p_{2j}} \|\chi_Q\|_{q^*}. 
\]

(2) We now prove the second part. Similar to the techniques used in the proof of [51, Theorem 1], since \(L^\infty(\mathbb{R}^n)\), which denotes all \(L^\infty(\mathbb{R}^n)\) functions with compact support, is dense in \(L^{r}(\mathbb{R}^n)\) for \(r(\cdot) \in \mathcal{P}(\mathbb{R}^n)\), and every Lipschitz function can be approximated by
bounded functions. Therefore we only need to prove that there exists a positive constant C independent of $f_1, f_2 \in L^\infty(\mathbb{R}^n)$ such that

$$\|T_{b_j}(\vec{f})\|_{q(\cdot)} \leq C\|b_j\|_{L(\delta)}\|f_1\|_{p_1(\cdot)}\|f_2\|_{p_2(\cdot)} \quad (j = 1, 2).$$

Modifying the argument in [51], using Lemma 2.24, for $j = 1, 2$, one can obtain

$$\|T_{b_j}(\vec{f})\|_{q(\cdot)} \leq C\|M_{f_j}(T_{b_j}(\vec{f}))\|_{p(\cdot)}$$

$$\begin{align*}
&\leq C\|b_j\|_{L(\delta)} \left(\|M_{\delta, n}(T\vec{f})\|_{q(\cdot)} + \left\|M_{\delta, r_j}(f_j)\prod_{i=1}^2 M_{r_i}(f_i)\right\|_{q(\cdot)} \right).
\end{align*}$$

Furthermore, using similar tools and techniques in [51], we can get the desired result.

Thus, combining the above estimates we finish the proof of Theorem 1.2. \qed

Now, we prove Theorem 1.4.

**Proof of Theorem 1.4** Without loss of generality, we only consider the case that $m = 2$. Actually, similar procedure work for all $m \in \mathbb{N}$.

(1) We first prove $\vec{b} = (b_1, b_2) \in L(\delta(\cdot)) \times L(\delta(\cdot))$. Assume that $T_{b_k}$ maps $L^{p_1(\cdot)}(\mathbb{R}^n) \times L^{p_2(\cdot)}(\mathbb{R}^n)$ into $L^{q_1(\cdot)}(\mathbb{R}^n)$. By proceeding as in (3.1) with $Q = Q(x_0, \ell)$ and $Q_j = Q(y_j, \ell) \subset \mathbb{R}^n (j = 1, 2)$, we have

$$\int_{Q} |b_1(x) - (b_1)_{Q_j^*}|dx \leq C e^{-2n} \sum_k |a_k| \int_{Q} |T_{b_1}(f_{1k}, f_{2k})(x)|dx,$$

where $f_{jk} \in L^{p_j(\cdot)}(\mathbb{R}^n)$ and $\|f_{jk}\|_{p_j(\cdot)} \leq \|\chi_{Q_j^*}\|_{p_j(\cdot)} (j = 1, 2)$ for every $k \in \mathbb{N}$. Then, from the generalized Hölder’s inequality (2.2) and the hypothesis, we obtain

$$\int_{Q} |T_{b_1}(f_{1k}, f_{2k})(x)|dx \leq C \|T_{b_1}(f_{1k}, f_{2k})\|_{q_j(\cdot)} \|\chi_Q\|_{q_j(\cdot)}$$

$$\leq C \|f_{1k}\|_{p_1(\cdot)} \|f_{2k}\|_{p_2(\cdot)} \|\chi_Q\|_{q_j(\cdot)},$$

and then

$$\int_{Q} |b_1(x) - (b_1)_{Q_j^*}|dx \leq C e^{-2n} \sum_k |a_k| \|\chi_{Q_j^*}\|_{p_1(\cdot)} \|\chi_{Q_j^*}\|_{p_2(\cdot)} \|\chi_Q\|_{q_j(\cdot)}.$$

Set $Q, Q_j^* \subset Q_{j0} = Q(x_0, (|z_j^*| + 1)\ell)$ $(j = 1, 2)$. Since $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{\delta(\cdot)}{n} = \frac{1}{\beta} - \frac{1}{\gamma(\cdot)}$, then $1/q(\cdot) = 1/r(\cdot) + (1/\beta - 1/p(\cdot)) = \frac{1}{r(\cdot)} + (\frac{1}{2\beta} - \frac{1}{p_1(\cdot)}) + (\frac{1}{2\beta} - \frac{1}{p_2(\cdot)})$. Hence, by applying the generalized Hölder’s inequality (2.1), we get

$$\|\chi_Q\|_{q(\cdot)} \leq \|\chi_Q\|_{r(\cdot)} \|\chi_Q\|_{(\frac{1}{2\beta} - \frac{1}{p_1(\cdot)})^{-1}} \|\chi_Q\|_{(\frac{1}{2\beta} - \frac{1}{p_2(\cdot)})^{-1}}$$

$$\leq \|\chi_{Q_1}\|_{(\frac{1}{2\beta} - \frac{1}{p_1(\cdot)})^{-1}} \|\chi_{Q_2}\|_{(\frac{1}{2\beta} - \frac{1}{p_2(\cdot)})^{-1}} \|\chi_Q\|_{r(\cdot)}.$$

For any $j = 1, 2$, let $h_j^* = (\frac{1}{2\beta} - \frac{1}{p_j(\cdot)})^{-1}$. We get that $\frac{1}{h_j^*} = \frac{1}{2\beta} - \frac{1}{p_j(\cdot)}$, that is, $\frac{1}{p_j(\cdot)} = \frac{1}{h_j^*} - (1 - \frac{1}{2\beta}) = \frac{1}{h_j^*} - (n - \frac{2\beta}{n})/n$. Thus, since $r(\cdot), p_1(\cdot), p_2(\cdot) \in C^{\text{loc}}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, using Lemma 2.13, we obtain that $1 \in A_{\left(\frac{1}{h_j^*} - \frac{n}{n - \frac{2\beta}{n}}\right), p_j(\cdot)}(\mathbb{R}^n) (j = 1, 2)$. Then, from (2.6), Lemma 2.18 and doubling condition implied in conditions (2.4) and (2.5), we have

$$\|\chi_{Q_j^*}\|_{p_1(\cdot)} \|\chi_{Q_j^*}\|_{p_2(\cdot)} \|\chi_Q\|_{q(\cdot)} \leq \|\chi_{Q_{j0}}\|_{p_1(\cdot)} \|\chi_{Q_{j0}}\|_{p_2(\cdot)} \|\chi_Q\|_{q(\cdot)}$$
Recall the pointwise equivalence for any given $\mathbf{z} = z^*_1, z^*_2$ is fixed, by taking supremum over every $Q \subset \mathbb{R}^n$, we obtain that $b_1 \in \mathbb{L}(\delta(\cdot))$. Similar argument as above, we can get that $b_2 \in \mathbb{L}(\delta(\cdot))$.

(2) We now prove the second part. Assume that $\mathbf{b} = (b_1, b_2) \in \mathbb{L}(\delta(\cdot)) \times \mathbb{L}(\delta(\cdot))$ and $\mathbf{f} = (f_1, f_2) \in L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ such that $\| T_{b_j}(\mathbf{f})\|_{q(\cdot)} < \infty \ (j = 1, 2)$. From Lemma 2.22, for any given $\gamma \in (0, 1)$, it follows that

$$\| T_{b_j}(\mathbf{f})\|_{q(\cdot)} \leq C \| (T_{b_j}(\mathbf{f}))_{d(j, \gamma)}\|_{p(\cdot)}.$$  

Since $\tilde{z}^* = (z^*_1, z^*_2)$ is fixed, by taking supremum over every $Q \subset \mathbb{R}^n$, we obtain that $b_1 \in \mathbb{L}(\delta(\cdot))$. Similar argument as above, we can get that $b_2 \in \mathbb{L}(\delta(\cdot))$.

Recall the pointwise equivalence $M_{L(\log L)}(g)(x) \approx M^2(g)(x)$ for any locally integrable function $g$ (see [42, (21)]) and

$$M_{L(\log L)}(\mathbf{f})(x) \leq 2 \prod_{i=1}^2 \left( \sup_{Q \ni x} \| f_i \|_{L(\log L)}(Q) \right) = \prod_{i=1}^2 M_{L(\log L)}(f_i)(x).$$

Then, by the generalized Hölder’s inequality (Lemma 2.8) and the hypothesis on $p(\cdot)$, one has

$$\| M_{L(\log L)}(\mathbf{f})\|_{p(\cdot)} \leq C \| \prod_{i=1}^2 M_{L(\log L)}(f_i)\|_{p(\cdot)} \leq C \| \prod_{i=1}^2 M^2(f_i)\|_{p(\cdot)} \leq C \prod_{i=1}^2 \| f_i \|_{p_i(\cdot)},$$

where in the last inequality, we make use of the $L^{p(\cdot)}(\mathbb{R}^n)$ boundedness of $M$ twice.

In addition, applying Lemmas 2.5, 2.7, 2.8 and 2.9, it is easy to see that $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and that $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies $(p(\cdot)/p_0)^\prime \in \mathcal{P}(\mathbb{R}^n)$ with some $p_0 \in (0, p_\cdot)$, then we have

$$\| M_q(T\mathbf{f})\|_{p(\cdot)} \leq C \| T(\mathbf{f})\|_{p(\cdot)/\eta} \leq C \| T(\mathbf{f})\|_{p(\cdot)} \leq C \prod_{i=1}^2 \| f_i \|_{p_i(\cdot)}.$$  

Thus, we obtain

$$\| T_{b_j}(\mathbf{f})\|_{q(\cdot)} \leq C \| b_j \|_{L(\delta(\cdot))} \| f_1 \|_{p_1(\cdot)} \| f_2 \|_{p_2(\cdot)}.$$  

Combining these estimates, the proof is completed. \[\square\]

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