Black hole solution for scale-dependent gravitational couplings and the corresponding coupling flow

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Received 28 March 2013, in final form 26 June 2013
Published 8 August 2013
Online at stacks.iop.org/CQG/30/175009

Abstract
We study a particular solution for the generalized Einstein Hilbert action with scale-dependent couplings $G(r)$ and $\Lambda_1(r)$. The form of the couplings is not imposed, but rather deduced from the existence of a non-trivial symmetrical solution. A classical-like choice of the integration constants is found. Finally, the induced flow of the couplings is derived and compared to the flow that is obtained in the context of the exact renormalization group approach.

PACS numbers: 04.60., 04.70.

(Some figures may appear in colour only in the online journal)

1. Introduction

One of the many achievements of quantum field theories like the standard model is the confirmed prediction of a scale dependence of the physical couplings $\alpha \rightarrow \alpha_k$. In order to know in which way the couplings $\alpha$ of a given quantum field theory depend on the energy scale $k$, one usually has to regularize and renormalize the theory. However, when it comes to gravity, a consistent and predictive renormalization and therefore a predictive quantum field theoretical description is still to be found. Independently of how this theory of quantum gravity will look like, in most approaches, it is expected to introduce a non-trivial running to the couplings of classical gravity which are Newton’s ‘constant’ $G \rightarrow G(k)$ and the cosmological constant $\Lambda \rightarrow \Lambda(k)$. In an effective description, those couplings are expected to be present in an improved action and the corresponding solutions.

Since, on the one hand, black holes are key objects in every classical or quantum gravitational theory and their understanding is crucial for the understanding of the whole model, and on the other hand, the scale dependence of gravitational couplings has been extensively studied in the context of exact renormalization group (ERG) equations [1–15], it is natural to study black holes in the context of ERG results. This has been done mostly
by improving the classical solutions [16–26]. This procedure has, however, two weaknesses which basically motivated this study. The first problem is that in order to study black holes in the ERG context, one has to relate the radial scale of a black hole solution \( r \) to the energy scale \( k \) of the ERG calculation, this procedure is, however, not uniquely defined. The second problem is that the improved solution does not (at least not at all scales [27]) resolve the improved equations of motion nor does it minimize the ERG improved action.

As complementary contribution to this programme, we will follow a philosophy that is somewhat inverse to the existing studies on ERG improved black hole solutions. As starting point, we will take the improved equations of motions which contain scale-depending couplings \( G(r) \) and \( \Lambda(r) \) which are \textit{a priori} undetermined. The working hypothesis will then be to ask that for which functional form of those couplings it is possible to solve those equations of motion with the most symmetrically possible metrical ansatz.

The paper is organized as follows. In section 2, it is shown how solving the resulting system of equations determines the resulting black hole metric and the functional form of \( G(r) \) and \( \Lambda(r) \) up to the existence of four integration constants. General properties of this solution such as differences to the studies in the literature, singularities and the existence of classical-like parameter choices are then discussed.

Since, both couplings of the present solution are functions of the radial scale \( r \), the corresponding adimensional couplings \( g(r) \) and \( \lambda(r) \) can be combined in a coupling flow. In section 3, this induced flow with an ultraviolet (UV) fixed point for \( g(r) \) and \( \lambda(r) \) is derived, the flow is compared to the flow for \( g_k \) and \( \lambda_k \) known from ERG calculations. Finally, the anomalous dimensions of the induced couplings and the product of couplings are discussed and compared to the findings in the ERG approach.

After summarizing remarks in section 4, we give complementary discussions in the appendix.

2. Exact solution with cosmological term

2.1. A solution with spherical symmetry

Up to now, black holes were studied in the context of ERG equations by taking the classical solutions (for constant \( G \) and \( \Lambda \)) and then replacing \( G \rightarrow G(k) \) and \( \Lambda \rightarrow \Lambda(k) \) in those solutions. The physical interpretation of those ERG improved solutions depends on how the ERG scale \( k \) is related to the physical scale \( r \). This procedure however was only partially successful, since the improved solutions were actually no more solutions of any form of Einstein or modified Einstein equations. Yet, the improved solution with a cosmological term seems at least to solve the improved equations of motion asymptotically in the UV [27].

In order to avoid the danger which is involved with the choice of \( k \rightarrow k(r) \), we will directly consider the fact that all scales are in the end functions of the physical scale \( G = G(r) \) and \( \Lambda = \Lambda(r) \). When doing so, we pretend to find a solution which is still an exact solution of the equations motion. The corresponding improved equation of motion is [52, 62]

\[
G_{\mu\nu} = -g_{\mu\nu}\Lambda(r) + 8\pi G(r)T_{\mu\nu} - \Delta t_{\mu\nu},
\]

with

\[
\Delta t_{\mu\nu} = G(r)(g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) \frac{1}{G(r)}.
\]

For a discussion of the conservation laws for those equations, see section A.1 in the appendix. At first instance, we are interested in spherically symmetric solutions in regions where the classical matter contribution \( T_{\mu\nu} = 0 \). As ansatz for the metric tensor, we use

\[
d\tilde{s}^2 = -f(r)\, dt^2 + \frac{1}{f(r)}\, dr^2 + r^2\, d\theta^2 + r^2\, \sin(\theta)\, d\phi^2
\]
where
\[ f(r) = \left( 1 - \frac{\Sigma G(r)}{r} - \frac{l(r)}{3} r^2 \right). \] (4)

With this ansatz the Einstein equations (1) reduce to three independent differential equations for the a priori unknown functions \( G(r), \Lambda(r) \) and \( l(r) \). Please note that the constant \( \Sigma \) would only be the mass of the black hole, if \( G \) and \( \Lambda = l \) would be constants. In the context of variable constants, however, \( \Sigma \) is only an arbitrary constant with units of mass, which can take arbitrary (even negative) values.

It was found that, apart from the well-known solutions, which imply constant couplings, there exists a solution with a non-trivial \( r \) dependence
\[ G(r) = -\frac{16\pi c_2}{r - 2c_1}. \] (5)

\[ \Lambda(r) = \frac{-1}{24(r - 2c_1)c_1^2 c_4^4} \left\{ -2c_1 (c_1^2(12c_1^2 + 384\Sigma\pi c_2 + c_3) + 24r^3c_1^4c_4 + 3\sqrt{2} (384\Sigma\pi c_2 + c_3 - 24c_1^4c_4) + 6c_1( -c_1^2 - 384\Sigma\pi c_2 - c_3 + 8c_1^4c_4)) + 3r \left( r^2 - 3rc_1 + 2c_1^2 \right) (384\Sigma\pi c_2 + c_3) \ln[r] - 3r \left( r^2 - 3rc_1 + 2c_1^2 \right) \right\} \times (384\Sigma\pi c_2 + c_3) \ln[r - 2c_1]. \] (6)

\[ l(r) = c_4 + \frac{1}{48c_1^4} \left\{ \frac{576\Sigma\pi c_1 c_2}{r - 2c_1} + \frac{8c_1^3(12c_1^2 + 96\Sigma\pi c_2 + c_3)}{r^3} + \frac{6c_1^2(12c_1^2 + 384\Sigma\pi c_2 + c_3)}{r^2} \right\} + \frac{6c_1(288\Sigma\pi c_2 + c_3)}{r} - 3(384\Sigma\pi c_2 + c_3) \ln[r] + 3(384\Sigma\pi c_2 + c_3) \ln[r - 2c_1]. \] (7)

Those solutions contain four constants of integration \( c_i \). The role and the interpretation of those constants in terms of different physical perspectives will be discussed throughout this paper.

One of the first questions one might ask for this new kind of solution is whether it has the same singularity problem at the origin as the standard solution or whether there exist parameter configurations for which the singularity can be avoided. This can be checked by calculating the invariant tensor density which gives to lowest order in \( 1/r \)
\[ R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{144c_1^2 + 9216\Sigma\pi c_1 c_2 + 147456\Sigma^2\pi^2 c_2^2 + 24c_1^2c_3 + 768\Sigma\pi c_2 c_3 + c_3^2}{27c_1^4 r^6} \] + \( O(r^{-5}) \). (8)

The divergence to this order can be avoided by choosing \( c_2 = \hat{c}_2 \) with
\[ \hat{c}_2 = -\frac{12c_1^2 + c_3}{384\Sigma\pi}. \] (9)

This choice removes a number of divergent terms such that the remaining next-to-next to next-to-leading singularity reads
\[ R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} |_{\hat{c}_2} = \frac{2}{c_1^2 r^2} + O(r^{-1}). \] (10)

If one further tries to remove this singularity, one is forced to take \( \hat{c}_1 = c_1 \rightarrow \infty \), where the value for \( c_2 \) has to be chosen, before one takes the limit in \( c_1 \). This indeed leads to a finite tensor density:
\[ R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} |_{\hat{c}_2, \hat{c}_1^2} = \frac{8}{3} c_4. \] (11)
but this choice of the integration constants is actually almost trivial because it corresponds to flat (anti) de Sitter space:

\[ f(r)|_{\hat{c}_2, \hat{c}_1} = 1 - \frac{c_1}{3r^2}. \]  

When one studies other invariants that can be formed of the Ricci tensor, the result is basically the same. It is however interesting that for the scalar formed from the Weyl tensor

\[ W = \frac{1}{3r^4} \left( r^2 \frac{d^2 f(r)}{dr^2} - 2 + 2f(r) - 2f(r) \frac{df(r)}{dr} \right)^2, \]

the choice (9) renders this scalar finite at the origin

\[ W|_{\hat{c}_2} = \frac{1}{3(r - 2c_1)^4}. \]  

Since the Weyl tensor is constructed such that it is invariant under conformal transformations and since other invariants are not finite in this case, one can conclude that the singularity at the origin after the special choice of the integration constant \( \hat{c}_2 \) (9) is due to a conformal factor of the metric solution.

2.2. Solutions in the literature

Since the equation of motion (1) can be interpreted as a special case of an \( f(R) \) theory \([28–30]\) (see \([31]\) for a review), one has to check whether the above solution has already been discussed in the context of the more general \( f(R) \) theories. A class of solutions with constant curvature \( R = R_0 \) and perturbative expansions around the ‘classical’ solutions has been discussed in \([32, 33]\), and special solutions with conformal anomaly have been discussed in \([34]\). Other perturbative solutions (assuming \( g_{\mu\nu} \approx \hat{g}_{\mu\nu} + h_{\mu\nu} \)) to \( f(R) \) theories in their transition limit to general relativity can be found in \([35]\). Further variations of the classical solution have also been studied in the context of Kerr black holes \([36–38]\). For an exact solution that assumes a finite \( R/(R - R_\partial f(R)) \) and constant curvature \( R_0 \), which generalizes to charged, rotating black holes, see \([39]\). None of the above solutions contains \((5)–(7) \) since here, \( R/(R - R_\partial f(R)) = 0 \), \( R \) is not constant, and the solution is not a perturbation of a classical solution. Thus, to our current knowledge, \((5)–(7) \) have not been discussed in the context of \( f(R) \) theories.

An other very similar approach is Brans–Dicke (BD) theory \([40]\) where one allows us for a varying gravitational constant \( \Phi(x) = 1/G(x) \). In difference to this approach, the BD action contains a kinetic term \( \sim \omega\Phi_{,\mu}\Phi^{,\mu} \). In the limit \( \omega \to \infty \), this theory corresponds to constant coupling and standard general relativity. Solutions of BD with the limit \( \omega \to 0 \) correspond to our ERG-inspired approach. Black hole solutions in pure BD theory have been discussed. First black hole solutions for this theory have been found by Brans \([41]\). In \([42]\), it has been shown that all BD black holes that have constant \( \Phi \) in the outside region and that are static solutions correspond to the known Schwarzschild, Kerr or Kerr–Newman solutions. If one relaxes the condition of constant fields further non-trivial solutions have been found \([43, 44]\). This solution does not apply here, since it has \( \Lambda = 0 \). In \([44]\), BD black holes have been studied in the Lifshitz context. BD black hole solutions with a specific cosmological term \( V(\Phi) \) have been found and discussed in \([45–48]\). Based on those studies, \([49, 50]\) discussed further generalizations of BD AdS BHs. By using dilaton black hole solutions with a cosmological term \( V(\Phi) \), they construct via a conformal transformation \( \Phi = (D - 4)/(4\alpha) \ln \Phi \) the BD solution with the cosmological term \( V(\Phi) \). This method, however, works only for larger space-time dimensions \( D > 4 \), since this conformal transformation is ill defined for \( D = 4 \). This reflects also in the fact that the metric coefficients of those BD solutions diverge for four space-time dimensions.
It is also instructive to do a comparison with an approximated black hole solution of BD theory given by Weinberg [51] (pp 183 and 247) for the case that gives the Schwarzschild solution plus corrections in \( \frac{1}{r} \)

\[
d s^2 = \left( 1 - 2 \frac{M G}{r} + \frac{M^2 G^2}{r^2} + \cdots \right) \, dt^2 - \left( 1 + \frac{M G}{r} + \cdots \right) \, dr^2 - r^2 \, d\theta^2 - r^2 \sin^2 \theta \, d\phi^2. \tag{15}
\]

Comparing this approximation with \( c_4 = 0 \) to the series expansion in \( \frac{1}{r} \) of the exact solution, one finds that one can at best fix the constants \( c_1 \) and \( c_2 \) in combination with a time rescaling \((t \to k t)\), such that

\[
d s^2 = \left( 1 - 2 \frac{M G}{r} + \frac{M^2 G^2}{r^2} + \cdots \right) \, dt^2 - \left( 2 + 2 \frac{M G}{r} + \cdots \right) \, dr^2 - r^2 \, d\theta^2 - r^2 \sin^2 \theta \, d\phi^2. \tag{16}
\]

Both expansions (15) and (16) do not agree in the constant coefficient of the \( rr \) component. If one wants to demand a regime where the exact solution has a constant factor 1 for both \( g_{tt} \) and \( g_{rr} \), one has to fix the constants for an expansion around \( r \ll G_0 \Sigma_1 \), as it is discussed in the following subsection. This also cannot be fixed by just varying the initial metric ansatz by \( g_{rr} \to k^2 g_{rr} \). In terms of physical viability, this is a severe problem, because the above expansion for \( c_4 = 0 \) basically predicts that this new solution would contradict all gravitational lensing effects with relativistic trajectories. In order to address this concern, we will first ask that there are special parameter choices (with \( c_4 \neq 0 \)) for which the new solution approximates to the (for wide ranges of \( r \)) well-confirmed classical solution.

### 2.3. A classical-like choice of parameters

Given the problems for the \( \frac{1}{r} \) expansion for \( c_4 = 0 \), we will now study the case \( c_4 \neq 0 \) and try to approximate the new solution to the standard solution where

\[
f_s(r) = 1 - 2 \frac{G_0 M_0}{r} - r^2 \frac{\Lambda_0}{3}. \tag{17}
\]

As it can be seen from equation (16), not even reproducing the standard result without cosmological constant is trivially possible. Thus, one has to ask the question that whether there is any configuration of the parameters \((c_1, c_2, c_3, c_4)\) which is in agreement with current experimental results, which basically confirm (17). By constructing an approximation to the standard metric (17), it will now be shown that such parameter choices exist.

An important property of the standard metric in the de Sitter case is that it approximates to a plane flat space for large \( r \), before it runs into an other horizon at even larger distances. Thus, we try to fix the constant \( c_4 \) such that there exists a \( r_m \) with

\[
f(r_m) = 1 \quad \text{and} \quad f'|_{r_m} = 0. \tag{18}
\]

An analytic solution of those two combined conditions is possible, it gives

\[
r_m = \sqrt{\frac{\tilde{c}_3 + 12 \tilde{c}_1^2}{3}}. \tag{19}
\]

\[
c_{4,s} = \frac{4\sqrt{\tilde{c}_3} + \frac{16\sqrt{\tilde{c}_3}}{\sqrt{\tilde{c}_3} + 12\tilde{c}_1} - 3\tilde{c}_3 \ln[3] - \tilde{c}_3 \ln[\tilde{c}_3 + 12\tilde{c}_1] + 2\tilde{c}_3 \ln[-6\tilde{c}_1 + \sqrt{3\tilde{c}_3} + 12\tilde{c}_1]}{32\tilde{c}_1^2}, \tag{20}
\]
where \( \tilde{c}_3 = c_3 + 382\pi \Sigma c_2 \). The next step is to choose the constants \( c_1 \) and \( c_3 \) such that the two horizons of \( f(r) \) approximate to the horizons of \( f_1(r) \)

\[
r_s \approx 2G_0 M_0 \quad \text{and} \quad r_{ds} \approx \frac{3}{\Lambda_0},
\]

where we are interested in the cases \( r_1 \gg r_0 \). A numerical optimization to those conditions gives

\[
c_{1,r} = \frac{1}{(12G_0 M_0 \Lambda_0^2)^{1/3}},
\]

\[
c_{3,r} = \frac{27/3 G_0 (3G_0 M_0)^{2/3} (-4\Sigma + 3M_0) \Lambda_0^{4/3} - 18(3G_0 M_0 \Lambda_0^2/2)^{1/3}}{27G_0 M_0 \Lambda_0^3}.
\]

By demanding that \( G(r) \approx G_0 \) for \( r_{ds} \gg r \), one can also fix the missing constant

\[
c_{2,r} = \frac{G_0}{24\pi (4G_0 M_0 \Lambda_0^2/9)^{1/3}}.
\]

Using the definitions (20), (22)–(24), the solution (4)–(6) can be written. \( G(r) \) is given by

\[
G(r) = \frac{(2/3)^{1/3} G_0}{(2/3)^{1/3} - r(G_0 M_0 \Lambda_0^2)^{1/3}}.
\]

In order to reduce this expression, let us introduce the following additional variables:

\[
r_G = \frac{(3/2)^{1/3}}{(G_0 M_0 \Lambda_0^2)^{1/3}} = \left( \frac{2^2}{3} \cdot \frac{r_{ds}}{r_s} \right)^{1/3} r_{ds},
\]

so we can rewrite the last expressions for \( r < r_G \) as

\[
G(r) = \frac{G_0}{1 - r/r_G},
\]

and in terms of those variables, it is straightforward to find \( f(r) \) and \( \Lambda(r) \) :

\[
f(r) = 1 - \frac{r_s}{r} - \frac{9}{2^2} \left( \frac{r_s}{r_{ds}} \right)^{3/4} + \frac{r}{r_{ds}} \left[ 3 \left( \frac{1}{2^2} \cdot \frac{r_s}{r_{ds}} \right)^{3/4} - 27 \left( \frac{1}{2^7} \right)^{3/4} \right] - \frac{3}{2^5} \left( \frac{r_s}{r_{ds}} \right)^{3/4} - 6 \left( \frac{3}{2} \right)^{3/4} \left( \frac{r_s}{r_{ds}} \right)^{3/4} \ln \left[ \frac{27}{2} \left( \frac{3}{2^5} \right)^{3/4} \left( \frac{r_s}{r_{ds}} \right)^{3/4} \right] \left( \frac{r_{ds}}{r} \right) \right]
\]

\[
\left[ 1 - \frac{9}{2^2} \left( \frac{r_s}{r_{ds}} \right)^{3/4} \right] \cdot \ln \left[ \frac{3 \left( 1 - (2^2)^{3/4} \left( \frac{r_s}{r_{ds}} \right)^{3/4} \right) \left( \frac{r_s}{r_{ds}} \right)^{3/4}}{1 - r/r_G} \right]
\]

\[
\left( 2^{11/2} \right) \left( \frac{r_s}{r_{ds}} \right)^{3/4} \left( \frac{r}{r_{ds}} \right)^{3/4} \right] \cdot \ln \left[ \frac{3 \left( 1 - (2^2)^{3/4} \left( \frac{r_s}{r_{ds}} \right)^{3/4} \right) \left( \frac{r_s}{r_{ds}} \right)^{3/4}}{1 - r/r_G} \right]
\]

\[
\left( 2^{11/2} \right) \left( \frac{r_s}{r_{ds}} \right)^{3/4} \left( \frac{r}{r_{ds}} \right)^{3/4} \right] \cdot \ln \left[ \frac{3 \left( 1 - (2^2)^{3/4} \left( \frac{r_s}{r_{ds}} \right)^{3/4} \right) \left( \frac{r_s}{r_{ds}} \right)^{3/4}}{1 - r/r_G} \right]
\]

\[
\left( 2^{11/2} \right) \left( \frac{r_s}{r_{ds}} \right)^{3/4} \left( \frac{r}{r_{ds}} \right)^{3/4} \right] \cdot \ln \left[ \frac{3 \left( 1 - (2^2)^{3/4} \left( \frac{r_s}{r_{ds}} \right)^{3/4} \right) \left( \frac{r_s}{r_{ds}} \right)^{3/4}}{1 - r/r_G} \right]
\]

\[
\left( 2^{11/2} \right) \left( \frac{r_s}{r_{ds}} \right)^{3/4} \left( \frac{r}{r_{ds}} \right)^{3/4} \right] \cdot \ln \left[ \frac{3 \left( 1 - (2^2)^{3/4} \left( \frac{r_s}{r_{ds}} \right)^{3/4} \right) \left( \frac{r_s}{r_{ds}} \right)^{3/4}}{1 - r/r_G} \right]
\]

and

\[
\Lambda(r) = \frac{-42 \frac{3}{2} \Lambda_0}{3^2 \cdot 6^2 (1 - r/r_G)^2} \left\{ -9 \left( \frac{3}{2^5} \right)^{3/4} \left( \frac{r_s}{r_{ds}} \right)^{5/3} \frac{r_{ds}}{r} - 2 \sqrt{6} - 24 \left( \frac{3}{2^5} \right)^{3/4} \left( \frac{r_s}{r_{ds}} \right)^{3/4} \right\}
\]

\[
+ 54 \left( \frac{3}{2^5} \right)^{3/4} \left( \frac{r_s}{r_{ds}} \right)^{3/4} + 162 \left( \frac{3}{2^5} \right)^{1/2} \left( \frac{r_s}{r_{ds}} \right)^{3/4} + \frac{r}{r_{ds}} \left( \frac{3}{2} \right)^{3/4}
\]
scales, and since the special choice of \( c \) will be addressed in future work.

the parameters (integration constants) is, however, beyond the scope of this study, but this point
for the gravitational coupling. A more systematic scan of physically viable values for the four
discussion of physically viable choices of parameters will be subject of future investigation.

that physically viable parameter choices are perfectly possible. An other possible restriction
most likely not the only one that achieves this goal, but the point is that it shows by construction
observational data such as dark matter effects can be incorporated in the current solution (in
agreement with current experimental limits and observations. The question to which extend

can be concluded that the solution (4) also allows us for parameter choices which are also in
The special choice of the parameters \((c_1, c_2, c_3, c_4)\) is not exactly the most compact but it
shows that the classical result \( f_c(r) \) with the parameters \( (\Lambda_0, M_0, G_0) \) can be approximated very
well by the exact solution \( f(r) \) with variable \( G(r) \) and \( \Lambda(r) \). As it can be seen, from figure 1,
\( f(r) \) and \( f_c(r) \) are practically indistinguishable for very small values of \( \Lambda_0 \). A more general
discussion of physically viable choices of parameters will be subject of future investigation.
Since the classical form of (17) has been reproduced by various experiments at different
scales, and since the special choice of \( c_4 \) allows us to approximate very well to this result, it

can be concluded that the solution (4) also allows us for parameter choices which are also in
agreement with current experimental limits and observations. The question to which extend
observational data such as dark matter effects can be incorporated in the current solution (in
the spirit of [52–56]) will be subject of future studies [57]. The above choice of parameters is
most likely not the only one that achieves this goal, but the point is that it shows by construction
that physically viable parameter choices are perfectly possible. An other possible restriction
on the parameter \( c_1 \) could be obtained by demanding positive values for the logarithm and
for the gravitational coupling. A more systematic scan of physically viable values for the four
parameters (integration constants) is, however, beyond the scope of this study, but this point
will be addressed in future work.

\[
\begin{align*}
\times & \left[ 12 \left( \frac{3^2}{27} \right)^{\frac{2}{3}} \left( \frac{r_s}{r_{\text{as}}} \right)^{\frac{1}{3}} + 6 \left( \frac{3^7}{27} \right)^{\frac{2}{3}} \left( \frac{r_s}{r_{\text{as}}} \right) - 81 \left( \frac{3^9}{27} \right)^{\frac{2}{3}} \left( \frac{r_s}{r_{\text{as}}} \right)^{5/3} \\
- & 162 \left( \frac{3^9}{27} \right)^{\frac{2}{3}} \left( \frac{r_s}{r_{\text{as}}} \right)^{\frac{2}{3}} + \frac{r_s^2}{r_{\text{as}}} \left( \frac{3^3}{2} \right)^{1/2} \left[ -12 \left( \frac{1}{2} \right)^{\frac{5}{2}} \left( \frac{r_s}{r_{\text{as}}} \right)^{\frac{2}{3}} \\
+ & 9 \left( \frac{3}{2^2} \right)^{1/2} \left( \frac{r_s}{r_{\text{as}}} \right)^{\frac{1}{3}} + 81 \left( \frac{r_s}{r_{\text{as}}} \right)^{\frac{1}{2}} \right] + 6 \left[ -9 \left( \frac{3^8}{27} \right)^{\frac{2}{3}} \left( \frac{r_s}{r_{\text{as}}} \right)^{\frac{2}{3}} \\
+ & 27 \left( \frac{3}{2^3} \right)^{1/2} \left( \frac{r_s}{r_{\text{as}}} \right) + \frac{r_s}{r_{\text{as}}} \left( \frac{3^3}{2^3} \right)^{\frac{1}{3}} \left( \frac{18}{3^3} \right)^{\frac{2}{3}} \left( \frac{r_s}{r_{\text{as}}} \right)^{\frac{2}{3}} \\
- & 81 \left( \frac{3^9}{27} \right)^{\frac{2}{3}} \left( \frac{r_s}{r_{\text{as}}} \right)^{\frac{2}{3}} \\
+ & \left( \frac{r_s}{r_{\text{as}}} \right)^{2} \left( \frac{3^3}{2^5} \right)^{1/2} \cdot \left[ 8 \left( \frac{3^9}{27} \right)^{\frac{2}{3}} \left( \frac{r_s}{r_{\text{as}}} \right)^{\frac{1}{3}} - 9 \left( \frac{1}{2^2} \right)^{\frac{5}{2}} \left( \frac{r_s}{r_{\text{as}}} \right)^{8/3} \right] \right] \\
\ln \left[ \frac{3 \left( 1 - 3 \left( \frac{r_s}{r_{\text{as}}} \right)^{\frac{1}{3}} \right) \left( \frac{r_s}{r_{\text{as}}} \right)^{\frac{1}{3}}}{3^{1/2}(1 - r/r_{G})} \right] \left[ 3 \cdot 2^4 \left( \frac{r_s}{r_{\text{as}}} \right)^{\frac{8}{3}} \left( \frac{3}{2^2} \right)^{r_{\text{as}}}/r^2 \right]^{-1/2} \right]. \tag{29}\end{align*}
\]

The physical meaning of this special solution can be verified by expanding (28) for small \( \Lambda_0 \)
or \( r < r_{\text{as}} \)

\[
f(r) = 1 - \frac{r_s}{r} + \mathcal{O} \left( \frac{r_s}{r} \right)^{4/3} = 1 - 2 \frac{G_0M_0}{r} + \mathcal{O} \left( \frac{\Lambda_0^{2/3}}{r} \right). \tag{30}\]

Similarly, one can verify the meaning of the cosmological constant parameter, since for large
radii \( r > r_s \), the classical cosmological horizon \( r_{\text{as}} = \sqrt{3/\Lambda_{\text{class}}} \) is recovered:

\[
f(r) = 1 - \left( \frac{r_s}{r_{\text{as}}} \right)^{2} + \mathcal{O} \left( \frac{r_s}{r} \right)^{1/3}. \tag{31}\]

The special choice of the parameters \((c_1, c_2, c_3, c_4)\) is not exactly the most compact but it
shows that the classical result \( f_c(r) \) with the parameters \( (\Lambda_0, M_0, G_0) \) can be approximated very
well by the exact solution \( f(r) \) with variable \( G(r) \) and \( \Lambda(r) \). As it can be seen, from figure 1,
\( f(r) \) and \( f_c(r) \) are practically indistinguishable for very small values of \( \Lambda_0 \). A more general
discussion of physically viable choices of parameters will be subject of future investigation.
Since the classical form of (17) has been reproduced by various experiments at different
scales, and since the special choice of \( c_4 \) allows us to approximate very well to this result, it

can be concluded that the solution (4) also allows us for parameter choices which are also in
agreement with current experimental limits and observations. The question to which extend
observational data such as dark matter effects can be incorporated in the current solution (in
the spirit of [52–56]) will be subject of future studies [57]. The above choice of parameters is
most likely not the only one that achieves this goal, but the point is that it shows by construction
that physically viable parameter choices are perfectly possible. An other possible restriction
on the parameter \( c_1 \) could be obtained by demanding positive values for the logarithm and
for the gravitational coupling. A more systematic scan of physically viable values for the four
parameters (integration constants) is, however, beyond the scope of this study, but this point
will be addressed in future work.
Figure 1. Comparison of the classical metric coefficient (17) with the solution (28) as dashed curves. The numerical values were chosen to be $M_0 = 0.01$, $G_0 = 1$ and from left to right $\Lambda_0 = \{10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}\}$.

3. Coupling flow

3.1. The induced coupling flow

When discussing the flow of scale-dependent couplings like $(G, \Lambda)$, one usually does so for the dimensionless couplings $(\tilde{g}, \tilde{\lambda})$. Therefore, the free parameters that describe this flow should also be adimensional. Hence, one defines a set of four dimensionless parameters $\{\tilde{\lambda}_U, \tilde{l}_I, \tilde{g}_U, \tilde{g}_I\}$ instead of the dimensionfull parameters $\{c_1, c_2, c_3, c_4\}$:

\[
\begin{align*}
    c_1 &= -\frac{g_I g^*_U}{2g_i^2 \Sigma} \\
    c_2 &= -\frac{g_I^2}{16\Sigma^2\pi} \\
    c_3 &= \frac{3g_I(8g_i^2 - g_i g^*_U + 2g_i^3 \lambda^*_U)}{g_i^2 \Sigma^3} \\
    c_4 &= -\frac{\Sigma^2 l_i}{2}
\end{align*}
\]

or inversely

\[
\begin{align*}
    \tilde{\lambda}_U &= -\frac{12c_1^2 + 384c_1 \Sigma \pi}{48c_1^{3/2}} \\
    \tilde{l}_I &= -\frac{2c_2}{\Sigma^2 \pi} \\
    \tilde{g}_U &= \frac{8c_3 \Sigma^2 \pi}{c_3} \\
    \tilde{g}_I &= -16c_4^{3/2}
\end{align*}
\]

(32)

With this reparametrization, the metric solution reads

\[
f(r) = \frac{1}{6g_i g_i^2 \Sigma r} \left\{ g_i \left( -6g_i^3 \Sigma^2 r^2 + 4g_i^3 \lambda^*_U - 6g_i^2 g_i^* \Sigma r \lambda^*_U \\
+ g_i g_i^* \Sigma r (6 + \Sigma^2 r^2 l_i + 12 \Sigma r \lambda^*_U) \right) \right.
\]

\[
+ 6g_i^3 \Sigma^3 r^3 (g_i^* - 2g_i^3 \lambda^*_U) \log \left[ \frac{g_i}{g_i^* \Sigma r} + 1 \right] \right\} .
\]

(33)

One observes that the remaining dimensionfull parameters are $r$ and $\Sigma$, which only appear in dimensionless pairs $r \Sigma$ in (28). Now that convenient parameters have been defined, one can define the dimensionless couplings. In order to make the couplings dimensionless, one has to multiply them with dimensionfull quantities that describe the physical system. In our case, the two dimensionfull quantities that can vary and that describe the physical system are $\Sigma$ and $r$. In principle, every adequate power of those two quantities can do the job, thus one can write

\[
g(r) = G(r) \frac{\Sigma^2}{(\Sigma r)^c}
\]

(34)

\[
\lambda(r) = -\Lambda(r) \frac{(\Sigma r)^c}{\Sigma^2},
\]

(35)
where $a$ and $c$ are numbers that determine the respective impact of $r$ and $\Sigma$ 'adimensionalization!' and the minus sign in (35) is pure convention. The constants $a$ and $c$ are crucial for the expected fixed point behavior of the adimensional coupling flow. Only for certain values, there exists a well-behaved, non-trivial fixed point. Motivated from the results in the ERG approach, which will be introduced in the following section, one demands the existence of a non-trivial UV fixed point. This UV fixed point exists for both couplings for the values $a = 0$ and $c = +1$:

$$g_U(r) = G(r) \Sigma^2$$

$$\lambda_U(r) = - \Lambda(r) \frac{r}{\Sigma}.$$  \hspace{1cm} (36)

The values of the UV fixed points are

$$g_U(r \to 0) = g_U^*$$

$$\lambda_U(r \to 0) = \lambda_U^*.$$  \hspace{1cm} (37)

The limits in (37) show that two of the new dimensionless parameters are such that they represent the numerical value of the corresponding UV fixed points. Therefore, part of the possible physical results are already encoded in the numerical value of this fixed point. The choice (36) is further interesting in the sense that in the adimensional couplings, only the adimensional quantities $\lambda_U^*$, $l_I$, $g_U^*$, $g_I$ and $(r \cdot \Sigma)$ appear. With those parameters the adimensional gravitational couplings read

$$g_U(r) = \frac{g_U^*}{1 + \frac{g_I}{g_U^*} \Sigma r}$$  \hspace{1cm} (38)

and

$$\lambda_U(r) = \frac{1}{2g_I^2(g_I + g_U^*)^2} \left\{ g_I \left( g_I^2 \left( \Sigma r l_I + 2 \lambda_U^* \right) - 12 g_U^* \Sigma^2 r^2 + 3 g_I^2 g_U^* \Sigma r (\Sigma r l_I + 8 \lambda_U^*) \right) \right. \\
+ g_I^2 g_U^* \Sigma r (2 \Sigma^2 r l_I - 11 + 24 \Sigma r \lambda_U^*) + 6 g_U^* \Sigma r (g_I^2 + 3 g_I g_U^* \Sigma r + 2 g_U^2 \Sigma^2 r^2) \right\} \times \left( g_U^* - 2 g_I \lambda_U^* \right) \ln \left( \frac{g_I}{g_U^*} \Sigma r + 1 \right).$$  \hspace{1cm} (39)

It is straightforward to depict the flow of this UV fixed point scenario $a = 0$ and $c = 1$. The corresponding flow is shown in figure 2, where the numerical values of $g_U^*$ and $\lambda_U^*$ were taken from [11].

3.2. Comparing to the ERG flow

In the context of ERG-induced coupling flows, the dimensionless couplings are defined by the use of the energy scale $k$:

$$g(k) = k^2 G(k), \quad \lambda(k) = \frac{\Lambda(k)}{k^2}.$$  \hspace{1cm} (40)

Those running couplings and the corresponding fixed points have been repeatedly calculated numerically [3–7, 9–11, 58–60]. In order to obtain a tractable analytic solution for the running couplings (40), we will use a similar approximation procedure as it was used in [61]. According to [6], the flow equations are given by

$$\partial_k g(k) = \beta_g(\lambda_k, g_k) = [d - 2 + \eta(k)] g(k)$$

$$\partial_k \lambda(k) = \beta_\lambda(\lambda_k, g_k),$$  \hspace{1cm} (41)

where $t = \ln k / \Lambda$ and $\eta$ is the anomalous dimension and the beta functions are for $g$

$$\partial_k g = \beta_g = \frac{(-2 + d) P g k}{P_2 + 4(2 + d) g k}.$$  \hspace{1cm} (42)
and for $\lambda$

$$\partial_t \lambda = \beta_\lambda = \frac{P_1}{P_2 + 4(2 + d)g(k)},$$  \tag{43}$$

with

$$P_1 = d(2 + d)g(k)(-3 + d - 16g(k) + 8dg(k)) + 4(-1 + 10dg(k) + d^2g(k) - d^3g(k))$$

$$\lambda(k) + 4(4 - 10dg(k) - 3d^2g(k) + d^3g(k))\lambda(k)^2 - 16\lambda(k)^3, \tag{44}$$

and

$$P_2 = 2 + 8(-dg(k) - \lambda(k) + \lambda(k)^2). \tag{45}$$

Expanding those beta functions for small values of the couplings ($g, \lambda \ll 1$) and for four space-time dimensions, one obtains

$$\beta_g = g(k)(2 - 24g(k)) \tag{46}$$

and

$$\beta_\lambda = 12g(k) - 2\lambda(k). \tag{47}$$

The approximated beta function (46) can be integrated to

$$g_{\text{ERG}}(k) = \frac{g^*_U}{1 + \frac{g^*_U}{g(k)}} \tag{48}$$

Using (48), one can also integrate (47) giving

$$\lambda_{\text{ERG}}(k) = \lambda^*_U + \frac{1}{k^2}A_0 - \frac{g^*_U \lambda^*_U}{G_0 k^2} \log \left[1 + \frac{k^2}{g^*_U} \right]. \tag{49}$$

Note that the fixed points for this flow equations are the Gaussian fixed point ($\lambda^*_U, g^*_U = (0, 0)$) and in this approximation, the UV non-Gaussian fixed point with $\lambda^*_U = 1/2$ and $g^*_U = 1/12$. The values of the non-Gaussian fixed point were replaced in the solution by their symbols $g^*_U$ and $\lambda^*_U$, which will subsequently be treated as free parameters. The relation between $g$ and $k^2$ (48) can be inverted in order to express equation (49) in terms of $g_{\text{ERG}}$ giving

$$\lambda_{\text{ERG}}(g) = \lambda^*_U + \frac{1}{g} \left(A_0 G_0 (1 - g/g^*_U) - \lambda^*_U g^*_U (1 - g/g^*_U) \ln \left[\frac{1}{1 - g/g^*_U} \right] \right). \tag{50}$$
Figure 3. Flow of the scale-dependent couplings due to the ERG result from equation (50) which are depicted as dashed lines, and by using the parameters $g^*_{U} = 0.707$, $\lambda^*_{U} = 0.193$ and $G_0 = 1$ with $\Lambda_0 = \{-0.1, -0.01, 0, 0.01, 0.1\}$ (blue, green, black, orange, red). This flow is compared to the solution-induced flow $\lambda_U(r)$ and $g_U(r)$ as solid lines for the same parameters with the identifications $\Lambda_0 = l_I - \Sigma^2$ and $G_0 = g_I/\Sigma^2$ with the additional choice of $g_I = 2.5$ and $\Sigma = 1$.

This result can be plotted and compared to the coupling flow from figure 2, the resulting graphical comparison of the two flows is shown in figure 3.

Since the graphical similarity of the solution-induced flow and of the analytic ERG flow in figure 3 is quite striking, we will now proceed with an analytical comparison.

For the adimensional gravitational constant, one finds that the solution-induced $g_U(38)$ and the ERG result $g_{\text{ERG}}(48)$ are exactly identical if one uses the scale setting

$$ r \equiv \frac{g_I}{k^2 G_0 \Sigma}. $$

This scale setting result is interesting since intuitively one might have expected something like $k \sim 1/r$. Please note that this scale setting definition still leaves $g_I$ arbitrary. The next step is to compare the couplings $\lambda_{\text{ERG}}(49)$ and $\lambda_U(35)$. By using the scale setting (51), one finds

$$ \lambda_U(k) = \frac{1}{2g_I k^2 (g_U^* + G_0 k^2)} \left\{ g_0^2 G^2 I k^4 \left( \frac{g_I}{G_0} + 2k^2 \lambda_U^* \right) - 12g^*_U k^2 

+ 3G_0 g_I g_U^* k^2 \left( \frac{g_I}{G_0} + 8k^2 \lambda_U^* \right) + g_U^2 \left( -11G_0 k^4 + 2g_I \left( \frac{g_I}{G_0} + 12k^2 \lambda_U^* \right) \right) 

+ 6g_I (g_U^2 + 3G_0 g_U^* k^2 + G_0 k^4) (g_U^* - 2g_I \lambda_U^*) \text{Log} \left[ 1 + G_0 k^2 \right] \right\}. $$

The limit $\lim_{k \to \infty} = g_I l_I/G_0$ suggests the identification $l_I \equiv \frac{A g_0 G_0}{G_0}$. Apparently, (52) is not identical to (49), but the question is whether and to which extent both are similar. Since equation (49) is an analytic approximation which is assumed to be best close to a small valued fixed point, it is instructive to compare (49) and (52) in the UV regime for large values of $k^2$. For the comparison, we separate (52) in a logarithmic and a non-logarithmic part and perform a Taylor expansion of the coefficients to lowest order in $(1/k^2)$, $\lambda_U^*$, and $g_U^*$, which is analogous to the expansion that was used when deriving $\lambda_{\text{ERG}}(k)$. This gives

$$ \lambda_U(k)|_{\text{UV}} = \lambda_U^* + \frac{1}{k^2 G_0} \frac{6g_I l_I - 3g^*_U}{g_I} \text{Log} \left[ 1 + G_0 \frac{k^2}{g^*_U} \right]. $$

11
One observes that the approximated solution-induced function (53) has the same functional structure as the approximated ERG function (49). Even more, by choosing the remaining free constants to be $l_1 = 2\lambda_0/\Lambda_0$ and $g_1 = 3\lambda_0/(5\lambda_0^2)$, the matching is exact. The approximated BH-induced cosmological constant (53) is then identical to the (approximated) ERG function (49). Please note that due to the UV approximation, the infrared (IR) limit $k \rightarrow 0$ of (53) is a factor of two different from the IR limit of the complete expression (52).

3.3. Anomalous dimension and product of fixed points

One relevant point in the discussion of the running parameter is the behavior of the anomalous dimensions in the UV or IR region. The anomalous dimension $\eta$ is connected to the previously defined beta function $\beta_g$:

$$\beta_g(k) = \beta_g(\lambda_k, g_k) = [d - 2 + \eta(k)]g(k)$$

(54)

where $t = \ln k/\Lambda$. By using (40), $\eta$ can be written as

$$\eta(k) = -2 + \frac{1}{g(k)} \beta_g(k).$$

(55)

The conditions for the existence of non-trivial fixed points in the UV limit, non-Gaussian fixed points, is that $\beta_\lambda = 0$ and $\beta_g = 0$. With those conditions, the UV behavior of the anomalous dimension is given by $k \rightarrow \infty : \eta_k \rightarrow -(d - 2)$. In order to study the behavior of the anomalous dimension in the IR region, we select one of the trajectories which connects the UV non-Gaussian fixed point and runs to the IR region as it is done in figure 4. One can see from the figure that the trajectories have a classical behavior in the IR where the anomalous dimension goes to zero, while the anomalous dimension smoothly goes to the fixed point value in the UV:

- in the IR limit $k \rightarrow 0; \eta_k \rightarrow 0$
- in the UV limit $k \rightarrow \infty ; \eta_k \rightarrow -(d - 2)$.

Since in the previous discussion, it has been shown that $g_{\text{ERG}}(k)$ and $g_U(r)$ are exactly equivalent due to the scale setting condition (51), it is sufficient to work with one of the two, for example $g_U(r)$ (38). Considering this result, we find that anomalous dimension has the form

$$\eta(r) = -2 - \frac{\partial}{\partial \ln r} \ln G(r) = -2 + \frac{r/g}{\beta_{\Sigma} + r/g}.$$  

(56)
Figure 5. Product of the scale-dependent couplings due to the ERG result from equation (50) which are depicted as dashed lines, and by using the parameters $g^*_U = 0.707$, $\lambda^*_U = 0.193$ and $G_0 = 1$ with $\Lambda_0 = [-0.1, -0.01, 0, 0.01, 0.1]$, (blue, green, black, orange, red). This is compared to the solution-induced product $\lambda_U(r)/g_U(r)$ as solid lines for the same parameters with the identifications $\Lambda_0 = h \cdot \Sigma^2$ and $G_0 = g_I/\Sigma^2$ with the additional choice of $g_I = 2.5$ and $\Sigma = 1$.

One can see nicely that $r$ and $g_I$ only appear in pairs $r/g_I \sim /k^2$ which explains that the different values of $g_I$ in figure 4 actually correspond to a rescaling of $r$ in the same function $\eta(r)$.

In the variety of ERG calculations, it turned out that even though the values of the fixed points $g^*_{\text{ERG}}$ and $\lambda^*_{\text{ERG}}$ are scheme dependent, the product $g^*_{\text{ERG}} \cdot \lambda^*_{\text{ERG}}$ is rather robust throughout the different calculations. Therefore, figure 5 shows the product $g^*_{\text{ERG}} \cdot \lambda^*_{\text{ERG}}$ compared to the solution-induced results as a function of $k$.

One can see from figure 5 that the ERG result and the solution-induced results for $\lambda \cdot g$ are also in good agreement.

4. Summary and conclusion

In this paper, we have studied the possibility of a scale-dependent gravitational coupling $G(r)$ and cosmological coupling $\Lambda(r)$. Such scale-dependent couplings have to be studied in the context of improved equations of motion (1). We asked the question of how the scale-dependent couplings would have to look like in order to permit the most simple spherical symmetric metric solution with $g_{tt} = -1/g_{rr}$. Solving the equations of motion lead to a non-trivial metric $g_{tt}(r)$ with non-trivial functions for $G(r)$ and $\Lambda(r)$. This solution contains four constants of integration. Since, a naive expansion with one of the constants zero led to unphysical predictions (16) for large radii, we showed the existence of parameter choices where this problem does not exist (see section 2.3).

From the functional form of the dimensionfull coupling constants ($G(r)$ and $\Lambda(r)$), we defined the dimensionless coupling constants ($g_U(r)$ and $\lambda_U(r)$) with a fixed point in the regime of small radii. The flow of those dimensionless couplings was then compared to the (approximated) functional form of the running couplings of the ERG approach ($g^*_{\text{ERG}}(k)$ and $\lambda^*_{\text{ERG}}(k)$). After a proper scale setting $k = k(r)$ (51), it was found that there exists an exact equivalence between $g_U(r)$ and $g^*_{\text{ERG}}(k)$ and a structural correspondence between $\lambda_U(r)$ and $\lambda^*_{\text{ERG}}(k)$. By approximating the solution-induced result $\lambda_U(r) \rightarrow \lambda_U(r)|_{UV}$ to the same order as the ERG result $\lambda^*_{\text{ERG}}(k)$, one even finds an exact agreement between $\lambda_U(r)|_{UV}$
and $\lambda_{\text{ERG}}(k)$. Finally, the behavior of the anomalous dimension and of the product of the two couplings was discussed.

Given the good qualitative and quantitative agreement between the actual ERG result and the solution-induced results, one is tempted to believe that the solution (4)–(6) is actually a self-consistent and good approximation to a still unknown complete solution of the ERG improved equations of motion. The solution is defined for all scales, but its correspondence to the ERG black hole is expected to be best for small values of $g^*_U$ and $\lambda^*_U$ and for large energy scales $k$. This similarity (and approximate correspondence) was taken as an unexpected surprise and is the main result of this study.

Acknowledgments

The work of BK was supported by proj. Fondecyt 1120360 and anillo Atlas Andino 10201. The work of CC was supported by proj. Fondecyt 1120360 and DGIP grant 11.11.05.

Appendix. Complementary material

A.1. Improved action and improved equation of motion

Coupling the Einstein–Hilbert action to matter with scale-dependent couplings $\Lambda(k)$ and $G_k$ gives the improved action

$$S[g] = \int_M d^4x \sqrt{-g} \left( R - \frac{2\Lambda_k}{16\pi G_k} + \mathcal{L}_m \right) - \frac{1}{8\pi} \int_M d^3x \sqrt{-h} \frac{K}{G(k)}. \quad (A.1)$$

The Gibbons–Hawking boundary term with the trace of the extrinsic curvature $K$ can become relevant for the consistency of solutions that are not asymptotically flat such as (A) dS. The equations of motion for the metric field in (A.1) are

$$G_{\mu\nu} = -g_{\mu\nu} \Lambda_k + 8\pi G_k T_{\mu\nu} - \Delta t_{\mu\nu}, \quad (A.2)$$

where the possible coordinate dependence of $G(k)$ induces an additional contribution to the stress–energy tensor [52, 62]:

$$\Delta t_{\mu\nu} = G_k (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) \frac{1}{G_k}. \quad (A.3)$$

Please note that the tensor $\Delta t_{\mu\nu}$ alone is not covariantly conserved, but due to the Bianchi identities, the conservation holds for whole right-hand side of equation (A.2). Demanding a self-consistency of those equations of motion and a conserved stress–energy tensor for matter $\nabla^\mu T_{\mu\nu} = 0$, the following condition is found [61]:

$$R \nabla_\mu \left( \frac{1}{G(k)} \right) - 2 \nabla_\mu \left( \frac{\Lambda(k)}{G(k)} \right) = 0. \quad (A.4)$$

Given a certain form of the scale-dependent couplings $G(k)$ and $\Lambda(k)$, for example from the ERG approach, the above relation allows us to relate this scale $k$ to the scalar curvature $R$ of a supposed solution.

A.2. A further solution without cosmological term

It is interesting to look for further solutions where $g_{tt} \neq -g^{rr}$. Just adding an other unknown function to (3) would, however, leave more functions than independent equations. Thus, it is straightforward to study scenarios with $\Lambda(r) = 0$ and

$$ds^2 = -f(r) dr^2 + 1/h(r) dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2. \quad (A.6)$$
With this ansatz, we did not succeed to find general solutions with variable \( G(r) \). A special solution was, however, found:

\[
\begin{align*}
  f(r) &= 1 \\
  h(r) &= 1 + \frac{2}{r c_1} \\
  G(r) &= c_2 \sqrt{\frac{r}{r c_1 + 2}}.
\end{align*}
\]  

(A.7)

It is quite interesting that for this solution, the adimensional coupling is the square root of the adimensional coupling of the previous solution.

A.3. Induced flow for a classical-like parameters

Given the schematic behavior of the induced coupling flow in figure 2, one wonders how the classical-like choice of parameters fits into this picture. The relations (20), (22)–(24) in combination with (32) allow us to express the dimensionless parameters \( l_I, \lambda_U, g_i \) and \( g_U \) in terms of the physical parameters \( G_0, \Lambda_0 \) and \( M_0 \) and the undetermined scale \( \Sigma \) that was introduced in order to define the dimensionless parameters. Assuming a small physical cosmological constant \( \Lambda_0 \) adimensional constants that correspond to the solution (28) and (29) read approximately

\[
\begin{align*}
  l_{I,s} &\approx -8\sqrt{2/3} \frac{\Lambda_0}{3\Sigma^2} + O(\Lambda_0^{4/3}) \\
  \lambda_{U,s}^* &\approx -4(4/3)^{1/3} \frac{(G_0 M_0)^{5/3} \Lambda_0^{4/3}}{M} \\
  g_{I,s} &\approx -\frac{1}{2}(9/2)^{1/3} \frac{G_0 \Sigma^3}{(G_0 M_0 \Lambda_0^{2})^{1/3}} \\
  g_{U,s}^* &\approx G_0 \Sigma^2.
\end{align*}
\]  

(A.8)–(A.11)

Those adimensional constants can be combined in various ways in order to study the fixed point behavior when varying \( \Sigma \) and \( M_0 \). But it also allows us to form combinations which only depend on the general physical parameters \( \Lambda_0 \) and \( G_0 \), like \( g_{U,s}^* l_{I,s} \approx -\frac{8}{3\sqrt{2}} \Lambda_0 G_0 \), which establishes a global relation between the UV fixed point of \( g_{U,s}^* \) and the IR parameter \( l_{I,s} \).

The positivity of the gravitational coupling furthermore suggests a negative value for the IR parameter \( l_I \) for positive \( \Lambda_0 \). One can also ask whether the classical-like choice of parameters can be made compatible with the values of the UV fixed points \( \lambda_{U}^* = 0.193, g_{U}^* = 0.707 \) known from the ERG approach [11]. Imposing those fixed point values on the classical-like parameters (A.9)–(A.11) allows us to fix the parameters \( \Sigma \) and \( \Lambda_0 \) and leaves \( G_0 \) and \( M_0 \) as only free physical parameters. In figure A1, it is shown how the corresponding flow would look like for the classical-like scenario. This procedure, however, gives complex values for \( \lambda(0 < r < r_H) \), which strongly suggests that the classical-like scenario is not compatible with the above fixed point values.

Please note that (A.8)–(A.11) is not the only possible choice for the parameters of this solution. It only means that this choice of parameters reproduces the standard form of the black hole metric (17) despite of the fact that \( G(r) \) and \( \Lambda(r) \) are not constants. This result is new and unexpected, because it means that from solely observing the classical de Sitter black hole metric to high precision in the range between the horizons, one cannot conclude that \( G \) and \( \Lambda \) are actually constants.
A.4. Horizons and temperature

The horizons for this solution are given by the conditions $f(r_H) = 0$ and the black hole temperature is obtained by using the radial derivative at this point. This analysis reproduces only in very special cases such as the classical-like scenario all the features that are known for the classical solution (17). Depending on the choice of the four parameters, the horizon structure can be largely different. In most other cases, the finding of the horizons boils down to solving non-analytic equations which only can be done numerically. This numerical study of the possible horizons and corresponding thermodynamical behavior is postponed to a future study.

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