PROPERTIES OF INTERSTELLAR TURBULENCE FROM GRADIENTS OF LINEAR POLARIZATION MAPS

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ABSTRACT

Faraday rotation of linearly polarized radio signals provides a very sensitive probe of fluctuations in the interstellar magnetic field and ionized gas density resulting from magnetohydrodynamic (MHD) turbulence. We used a set of statistical tools to analyze images of the spatial gradient of linearly polarized radio emission ($\nabla P$) for both observational data from a test image of the Southern Galactic Plane Survey (SGPS) and isothermal three-dimensional simulations of MHD turbulence. Visually, in both observations and simulations, a complex network of filamentary structures is seen. Our analysis shows that the filaments in $\nabla P$ can be produced both by interacting shocks and random fluctuations characterizing the non-differentiable field of MHD turbulence. The latter dominates for subsonic turbulence, while the former is only present in supersonic turbulence. We show that supersonic and subsonic turbulence exhibit different distributions as well as different morphologies in the maps of $\nabla P$. Similarly, filaments produced by shocks show a characteristic “double jump” profile at the sites of shock fronts resulting from delta function-like increases in the density and/or magnetic field, while those produced by subsonic turbulence show a single jump profile. In order to quantitatively characterize these differences, we use the topology tool known as the genus curve as well as the probability distribution function moments of the image distribution. We find that higher values for the moments correspond to cases of $\nabla P$ with larger sonic Mach numbers. The genus analysis of the supersonic simulations of $\nabla P$ reveals a “swiss cheese” topology, while the subsonic cases have characteristics of a “clump” topology. Based on the analysis of the genus and the higher order moments, the SGPS test region data have a distribution and morphology that match subsonic- to transonic-type turbulence, which confirms what is now expected for the warm ionized medium.

Key words: ISM: general – magnetohydrodynamics (MHD) – polarization – shock waves – turbulence

Online-only material: color figures

1. INTRODUCTION

The interstellar medium (ISM) of our Milky Way is host to a variety of physical mechanisms that regulate and govern the structure and evolution of the Galaxy. The current understanding of the ISM shows it to be a multi-phase environment composed of a tenuous plasma, consisting of gas and dust, which is both magnetized and highly turbulent (Ferriere 2001; McKee & Ostriker 2007). In particular, the awareness of turbulence as a dominant physical process in the ISM has only happened in the last decade (Elmegreen & Scalo 2004). Turbulence plays a critical role in the areas of star formation, magnetic reconnection, magnetic field amplification, nearly every transport process, cosmic ray acceleration, magnetic dynamo, and the physics in the intercluster medium, to name just a few (see Lazarian & Vishniac 1999; Vishniac & Cho 2001; Elmegreen & Scalo 2004; Mac Low & Klessen 2004; Lazarian 2006; Ballesteros-Paredes et al. 2007; McKee & Ostriker 2007, and references therein). Additionally, turbulence has the unique ability to transfer energy over scales ranging from kiloparsecs down to the proton gyro-radius. This is critical for the ISM, as it explains how energy is distributed from large to small spatial scales in the Galaxy.

Despite the importance of magnetized turbulence, the situation of understanding ISM physics is no less complicated. In spite of the recent advances in our understanding of incompressible and compressible MHD turbulence (see Goldreich & Sridhar 1995; Cho & Vishniac 2000; Maron & Goldreich 2001; Cho et al. 2002, 2003; Cho & Lazarian 2003; Kritsuk et al. 2007; Schmidt et al. 2008; Kowal & Lazarian 2010; Beresnyak & Lazarian 2010; Beresnyak 2011; Federrath et al. 2011a, 2011b; Galtier & Banerjee 2011), the ISM presents a complex environment with multiple energy injection sources, different phases, and various instabilities acting at different scales. Additionally, obtaining properties of ISM turbulence from observations opens ways of gauging numerical simulations and testing theory. In light of these complexities, the most fruitful way of studying astrophysical MHD turbulence and the processes it affects is to use a synergetic approach, which combines the knowledge of theoretical predictions, numerical studies, and observational efforts.

Observationally, there are several ways of studying MHD turbulence. Many of these techniques hinge on density fluctuations in the ionized or neutral media (see Spangler & Gwinn 1990; Armstrong et al. 1995; Ossenkopf & Mac Low 2002; Padoan et al. 2003; Heyer & Brun 2004; Falgarone et al. 2005; Schneider et al. 2011; Chepurnov & Lazarian 2010; Roman-Duval et al. 2011) and are aimed at finding the density power spectrum of turbulence. More recently, ways to find the magnetization and the Mach number of turbulence have been explored (Kowal et al. 2007; Brunt et al. 2010a, 2010b; Burkhart et al. 2009, 2010; Esquivel & Lazarian 2010, 2011; Tofflemire et al. 2011; Price et al. 2011). While column density images are arguably the most common type of data for these studies, they do not contain the full three-dimensional picture and are only passive tracers of the turbulence velocity field. Various ways to study the interstellar velocity field have been explored and include the Velocity Channel Analysis and Velocity Coordinate Spectrum techniques (Lazarian & Pogosyan 2000, 2004, 2006, 2008). These techniques have been used both for atomic H$^i$ and molecular data (see Stanimirović & Lazarian 2001; Padoan...
et al. 2006, 2009; Chepurnov et al. 2010) to find the spectra of the turbulent velocity fields.

Synergetic studies of turbulence and magnetic fields from synchrotron radiation and dust polarization are also of great importance and value for the ISM. In this case, a number of techniques have been explored, e.g., structure functions of the polarization vectors arising from dust polarized emission (see Falceta-Gonçalves et al. 2009; Houde et al. 2011). The first quantitative study of synchrotron intensity fluctuations can be traced to works by Getmanov (1958), while fluctuations of synchrotron polarization3 were used, for instance, to evaluate the spectra of magnetic turbulence in the Hydra cluster (Enßlin & Vogt 2006; Enßlin et al. 2010).

More recently, several authors have discussed the prospects of the use of radio polarization maps to study turbulence (see Havrkon & Heitsch 2004; Fletcher & Shukurov 2006, 2007; Gaensler et al. 2011). Faraday rotation maps of linearly polarized radio signals are especially promising as they provide very sensitive probes of fluctuations in magnetic field and ionized gas density (see Gray et al. 1998; Gaensler et al. 2001; Landecker et al. 2010). The Faraday rotation measure (with units of rad m$^{-2}$) can be calculated as

$$\text{RM} = K \int_{\text{source}}^{\text{observer}} n_e(l) B(l) \, dl,$$

where $K = 0.81 \text{ rad m}^{-2} \text{ pc}^{-1} \text{ cm}^{-3} \mu \text{G}^{-1}$ and $B$, $l$, and $n_e$ are the magnetic field strength in $\mu \text{G}$, the distance along the line of sight (LOS) in parsecs, and the electron density in $\text{cm}^{-3}$ along the LOS, respectively.

Although many objects seen in polarization/Faraday maps can be matched to objects seen in other wavelengths (such as supernova remnants), an extended diffuse polarization emission network that is rich in structure is also present that cannot be mirrored in other wavebands or in total intensity (Fletcher & Shukurov 2007). The intensity variations seen in maps of stokes $Q$, stokes $U$, and linear polarization $|P|$ (see Figure 1, bottom left for an example of $|P|$ from the simulations) are the result of small-scale angular structure in the Faraday rotation induced by foreground ionized gas and are thus an indirect representation of turbulent fluctuations in the free electron density and magnetic field throughout the ISM.

In this paper, we use polarization gradients to study ISM turbulence. The use of gradients to highlight small rapid fluctuations seen in polarization maps was first discussed by Gaensler et al. (2011). When the spatial gradient is applied to maps of vector $\mathbf{P} = (Q, U)$, a complex web of filamentary structures is revealed. These filaments (see the right column of Figure 1 for an example) were interpreted by Gaensler et al. (2011) as rapid fluctuations in $n_e$ and $B$ along the LOS due to turbulence. In this paper, we will further explain the origins of the filamentary structures as they are related to turbulence and develop quantitative methods that can be applied to these data in order to obtain the Mach numbers.

3 For a theoretical description of synchrotron fluctuations for the arbitrary index of cosmic rays and realistic models of anisotropic turbulence, see Lazarian & Pogosyan (2012).
Taking the gradient of rotation measure or linear polarization maps has its advantages and disadvantages. The primary advantage is that the spatial gradient of $P$ satisfies the property that it has both translational and rotational invariance in the $(Q, U)$ plane. Quantities such as the polarization amplitude and polarization angle are not preserved under arbitrary translations and rotations, which can result from a smooth distribution of intervening polarized emission, a smooth uniform screen of foreground Faraday rotation, the effects of missing large-scale structure in an interferometric data set, or a combination of any of the three. Thus, the magnitude of the gradient is the simplest quantity that is not significantly affected by missing large-scale structure or excess foreground emission/Faraday rotation. Taking the gradient allows one to clearly see jumps and discontinuities, regardless of whether single-dish (total power) measurements are present in the data. In particular, this will highlight areas where a sharp change in $n_e$ or $B$ occurs, which is most likely due to turbulent fluctuations or shock fronts in the ISM. However, one must also keep in mind the disadvantages of using gradients, namely the fact that the gradient may enhance noise since gradients sample only the smallest scales.

In this work, we explore the physical causes of the filaments by taking the gradient of polarization maps of isothermal MHD turbulence. We investigate the dependency of the sonic Mach number on the structures seen in $|\nabla P|$ and calculate measures of probability distribution functions (PDFs) as well as the measure of the topology called the genus function. The PDFs have been studied extensively on MHD turbulence in the past and have shown sensitivity to the sonic Mach number in density, column density, and position–position–velocity (PPV) data (Passot & Vazquez-Semadeni 1998; Padoan et al. 1999; Kowal et al. 2007; Federrath et al. 2008, 2010; Burkhart et al. 2009, 2010; Price et al. 2011; Tofflemire et al. 2011). The genus has been extensively used in cosmology studies (e.g., Gott et al. 1986) and was suggested for ISM studies by Lazarian (1999). Its use for synthetic column density maps and observations has been discussed in the literature (Lazarian et al. 2002; Kowal et al. 2007; Kim & Park 2007; Chepurnov et al. 2008).

The paper is organized as follows. In Section 2, we further describe the data sets used, in particular the Southern Galactic Plane Survey (SGPS) and a set of ideal MHD simulations, and our calculation of the rotation measure and the linear polarization maps and their gradients. In Section 3, we discuss the origins of the observed filaments as they relate to the sonic Mach number. We describe different statistical measures of the sonic Mach number in Section 4; in particular the genus and PDFs. We discuss our results in Section 5, followed by conclusions in Section 6.

2. DATA AND METHOD

2.1. Gradient Technique and Relation of $|\nabla P|$ to $|\nabla RM|$:

An important observational quantity connected to interstellar density and magnetic field fluctuations is the Faraday effect. In the presence of magnetic fields and free electrons, birefringence of circularly polarized orthogonal modes occurs, giving these modes two different propagation velocities. In the case of pure polarized background emission propagation through a magnetized medium, the linearly polarized radiation will emerge with its polarization position angle rotated by the amount given in Equation (1). Thus, the relation between the observed position angle, emitted position angle, and the rotation measure is

$$\Theta - \Theta_0 = \text{RM} \lambda^2,$$

which has units of radians. Here $\lambda$ is the wavelength in meters.

Observational determination of Faraday rotation comes from measurements of the linear polarization vector $P \equiv (Q, U)$ (which depends on Stokes $U$ and $Q$ as $|P| = \sqrt{Q^2 + U^2}$) as a function of $\lambda^2$. To avoid confusion between vector and scalar $P$, we use bold notation to denote the vector quantity of the linear polarization map.

We define the gradient of the polarization vector as

$$|\nabla P| = \sqrt{(\partial Q / \partial x)^2 + (\partial Q / \partial y)^2 + (\partial U / \partial x)^2 + (\partial U / \partial y)^2}.$$

We note that in the case of vector $P$ we have

$$P = |P_0|e^{2i(RM)\lambda^2 + \theta_0}.$$

From this equation, one can derive a relationship between $|\nabla P|$ and $|\nabla \text{RM}|$ for Faraday-thin polarized emission as

$$|\nabla \text{RM}| = \frac{|\nabla P|}{2\lambda^2 |P|}.$$

Equation (5) only holds for data in which the entire signal is measured (i.e., single-dish data included) and for which the background is uniform. When $|P| = 1$ and $\lambda = 1$ (which are the assumptions we use for the simulations), one finds a trivial relation between $|\nabla P|$ and $|\nabla \text{RM}|$ as $|\nabla \text{RM}| = |\nabla P|/2$. However, this relation can only be used to calculate $|\nabla P|$ or $|\nabla \text{RM}|$ in the simulations, since the assumption of $|P| = 1$ is almost always too simplistic for the observations because the data are missing single-dish information and/or the background $|\nabla P|$ is not zero.

We calculate simulated maps of RM, $P$ and their gradients from density and LOS magnetic field maps that are perpendicular and parallel to the mean magnetic field in the simulations. We calculate the RM as per Equation (1) at every point and then take its spatial gradient, e.g., we compute the gradient vector at every pixel of the image using neighbor pixels. We can calculate $|P|$ by calculating the stokes vectors as $Q = \cos(2\theta), U = \sin(2\theta)$. These expressions come from applying Equation (2) with assumed values for $\lambda$ and $\theta_0$.

We show a subsonic and supersonic case of column density ($N$), $\nabla N$, LOS magnetic field ($\text{LOS} B$), $\nabla B$, RM, and $|\nabla P|$ in Figure 2 and Figure 3, respectively. A comparison of the SGPS test data and a subsonic case is given in Figure 1. Inspection of maps of $|\nabla P|$ reveals that filaments are created in both cases and that there is some correlation between gradients of column density, magnetic field, and $|\nabla P|$. We will discuss these further in Section 3.

2.2. Southern Galactic Plane Survey

The SGPS is an H I and 1.4 GHz continuum survey of the fourth quadrant of the Galaxy at arcminute resolution. It is a well-studied data set for investigating turbulence in the warm ionized medium (WIM), and thus ideal for the initial development of new statistical tools for this phase of the ISM. We use a subsection of radio continuum images of an 18 deg$^2$ patch of the Galactic plane, observed with the Australia Telescope Compact Array (ATCA; see McClure-Griffiths et al. 2001 and Gaensler et al. 2001 for more details). We examine
Figure 2. Examples of LOS maps and their respective gradients relevant to this paper for subsonic turbulence (model 1). The first column shows column density, LOS magnetic field, and the rotation measure from top to bottom. The second column shows the gradients of column density, LOS magnetic field, and polarization vector. (A color version of this figure is available in the online journal.)

The 1.4 GHz frequency data averaged over adjoining frequency channels with simultaneously recorded Stokes $I$, Stokes $Q$, and Stokes $U$ as part of the SGPS test region (Gaensler et al. 2001). This field consists of 190 mosaicked pointings of ATCA and covers the range $325.5 < l < 332.5, -0.5 < b < 3.5$. Complicated extended structure is seen in linear polarization throughout the test region, almost all of which has no correlation with total intensity. We select a $512 \times 512$ pixel sub-region from these data to match the resolution of the simulations used in our study and display it in Figure 1 in the top row. The SGPS region we select begins at coordinate $l = 332.3373, b = -0.3138$, is not overly contaminated by bad pixels, and contains significant emission.

2.3. Simulations

We generate a database of three-dimensional numerical simulations of isothermal compressible (MHD) turbulence by using the MHD code of Cho & Lazarian (2003) and varying the input values for the sonic and Alfvénic Mach number. The sonic Mach number is classically defined as $M_s \equiv \langle |v|/C_s \rangle$, where $v$ is the local velocity and $C_s$ is the sound speed. However, in our simulations the rms velocity of the system is maintained to be approximately unity, so the velocity can be viewed as the velocity measured in units of the rms velocity of the system and hence, the case of the simulations, it makes more sense to discuss the rms sonic Mach number: $M_{s, \text{rms}} = \sqrt{\langle v^2/C_s^2 \rangle}$. The averaging is done over the whole box. Similarly, the Alfvénic Mach number is classically defined as $M_A \equiv \langle |v|/\sqrt{\rho} \rangle$, where $v = |B|/\sqrt{\rho}$ is the Alfvénic velocity, $B$ is the magnetic field, and $\rho$ is the density. The rms definitions of velocity discussed in the context of the sonic Mach number equally apply to the Alfvénic Mach number. Our simulations and the values for the average sonic and Alfvénic Mach numbers are listed in Table 1. We create 14 simulations with values of $M_s \equiv 0.5, 1.0, 2.0, 3.0, 4.4, 8.0, 10$ all having sub and super-Alfvénic combinations. We briefly
Figure 3. Examples of LOS maps and their respective gradients relevant to this paper for supersonic turbulence (model 6). The first column shows the column density, the LOS magnetic field, and the rotation measure from top to bottom. The second column shows the gradients of column density, LOS magnetic field, and polarization vector.

(A color version of this figure is available in the online journal.)

Table 1

| Model | $p_{\text{gas}}$ | $B_{\text{ext}}$ | $M_{s}$ | $M_{A}$ | Description                        |
|-------|-----------------|-----------------|--------|--------|-----------------------------------|
| 1     | 2.00            | 1.00            | 0.5    | 0.7    | Subsonic and sub-Alfvénic         |
| 2     | 0.70            | 1.00            | 1.0    | 0.7    | Transonic and sub-Alfvénic        |
| 3     | 0.10            | 1.00            | 2.0    | 0.7    | Transonic and sub-Alfvénic        |
| 4     | 0.05            | 1.00            | 3.0    | 0.7    | Supersonic and sub-Alfvénic       |
| 5     | 0.025           | 1.00            | 4.4    | 0.7    | Supersonic and sub-Alfvénic       |
| 6     | 0.0077          | 1.00            | 8.0    | 0.7    | Supersonic and sub-Alfvénic       |
| 7     | 0.0049          | 1.00            | 10.0   | 0.7    | Supersonic and sub-Alfvénic       |
| 8     | 2.00            | 0.10            | 0.5    | 2.0    | Subsonic and super-Alfvénic       |
| 9     | 0.70            | 0.10            | 1.0    | 2.0    | Transonic and super-Alfvénic      |
| 10    | 0.10            | 0.10            | 2.0    | 2.0    | Transonic and super-Alfvénic      |
| 11    | 0.05            | 0.10            | 3.0    | 2.0    | Supersonic and super-Alfvénic     |
| 12    | 0.025           | 0.10            | 4.4    | 2.0    | Supersonic and super-Alfvénic     |
| 13    | 0.0077          | 0.10            | 8.0    | 2.0    | Supersonic and super-Alfvénic     |
| 14    | 0.0049          | 0.10            | 10.0   | 2.0    | Supersonic and super-Alfvénic     |

The code is a second-order-accurate hybrid essentially nonoscillatory scheme (Cho & Lazarian 2003) which solves the ideal MHD equations in a periodic box:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot \left( \rho \mathbf{v} \mathbf{v} + \left( p + \frac{B^2}{8\pi} \right) \mathbf{I} - \frac{1}{4\pi} \mathbf{BB} \right) = \mathbf{f},$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0,$$

with a zero-divergence condition $\nabla \cdot \mathbf{B} = 0$ and an isothermal equation of state $p = C_s^2 \rho$, where $p$ is the gas pressure. On the right-hand side, the source term $\mathbf{f}$ is a random large-scale solenoidal driving force: $\mathbf{f} = \rho \mathbf{d} \mathbf{v}/dt$. The magnetic field consists of the uniform background field and a fluctuating field: $\mathbf{B} = \mathbf{B}_{\text{ext}} + \mathbf{b}$. Initially $\mathbf{b} = 0$.

We scale the simulations to physical units, adopting typical parameters for warm ionized gas. We assume a pixel size of 0.15 parsecs and density of 0.1 cm$^{-3}$. The simulations are assumed to be fully ionized and we do not include the effects of partial ionization.

To make the maps of $\nabla P$, we first calculate the LOS rotation measure at each pixel then we take the gradient of this rotation outline the major points of the numerical setup (for more details see Cho & Lazarian 2003).
Figure 4. Schematic example of three possible scenarios for enhancements in a generic image “\(n\),” where “\(n\)” could be |\(P\)|, RM, or \(\rho/N/EM\) (density, column density, emission measure). Case one (top row) shows an example of a Hölder continuous function that is not differentiable at the origin (applicable to all turbulent fields). Case two (middle row) shows an example of a jump resulting from strong turbulent fluctuations along the LOS or weak shocks. Case three (bottom row) shows a delta function profile resulting from interactions of strong shocks. In this case, the derivative gives a double jump profile which produces morphology that is distinctly different from the previous cases. In all cases we show examples from |\(\nabla P\)| in our simulations.

(A color version of this figure is available in the online journal.)

In the ISM, fluctuations in density and magnetic field will occur as a result of MHD turbulence, which will be visible in polarimetric maps. In the case of taking gradients of a turbulent field, one would expect to find filamentary structure created by shock fronts, jumps, and discontinuities. Figure 4 shows a schematic illustrating these three separate cases of a possible profile and its respective derivative. The cases are as follows.

1. A Hölder continuous profile\(^5\) that is not differentiable at a given point (e.g., the absolute value function at the origin): common for all types of MHD turbulence.
2. A jump profile: weak shocks, strong fluctuations, or edges (e.g., a cloud in the foreground which suddenly stops).
3. A spike profile (e.g., delta function): strong shock regime.

In respect to case one, it is known that the turbulent velocity field in a Kolmogorov-type inertial range both in hydro and MHD is not differentiable, but only Hölder continuous (Bernard et al. 1998; Eyink 2009). Another example is that of any fractal function that displays self-similarity but is not differentiable everywhere. This profile will naturally create discontinuities when one takes its derivative. Therefore, case one can be found in both subsonic- and supersonic-type turbulence. Case one type filaments can be seen in the right column of Figure 2 for gradients of \(N\) (column density), \(B\) LOS, and \(P\) and in Figure 4. Case two creates a structure in the gradient by a shock jump or a large fluctuation in either \(n\) or \(B\). Here again, this type of enhancement in |\(\nabla P\)| could be found in

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\(^5\) Hölder continuous functions satisfy \(|f(x_1, t) - f(x_2, t)| < C|x_1 - x_2|^h\), where the exponent \(h = 1\), this satisfies the Lipschitz condition. In the case of Kolmogorov turbulent fields \(h = 1/3\).
supersonic- and subsonic-type turbulence and is due either to large random spatial increases or decreases due to turbulent fluctuations along the LOS or weak shocks. We expect weak shock turbulence to show a larger amplitude in $|\nabla P|$ than the subsonic case due to increases in density fluctuations. Case three is unique to supersonic turbulence in that it represents a very sharp spike in $n_e$ and/or $B$ across a shock front. The difference between this case and what might be seen in case two is that here we are dealing with interactions of strong shock fronts, which are known to create delta function-like distributions in density (Kim & Ryu 2005). In this case, the derivative of case three has a distinctly different profile with respect to cases one and two. Case three shows a “double jump” profile across the shock front, which can be seen in Figure 4 in the bottom right panel for $|\nabla P|$. This morphological distinction can be used to determine if one is dealing with turbulence that is in a shock-dominated regime (e.g., supersonic) and provides researchers with a promising new avenue of obtaining the sonic Mach numbers from polarimetric data. In the case of the Alfvénic Mach number, the morphological difference is less clear, however, gradients will tend to align along the field lines in the case of strong field (sub-Alfvénic turbulence).

Also of interest is the question of which quantity is providing the dominant contribution to the structures in $|\nabla P|$ or $|\nabla \mathbf{R M}|$: $\nabla n_e$, LOS, $\nabla B$, LOS, or both equally? Especially in the case of compressible turbulence, the magnetic energy is correlated with density: denser regions contain stronger magnetic fields due to the compressibility of the gas and the potential dynamo amplification of the magnetic field in dense gas (Burkhart et al. 2009; Sur et al. 2010; Federrath et al. 2011a, 2011b). This causes the magnetic field to follow the flow of plasma if the magnetic tension is negligible. The compressed regions are dense enough to distort the magnetic field lines, enhance the magnetic field intensity, and effectively trap the magnetic energy due to the frozen-in condition. Thus, for the supersonic cases, the intensity of the structures seen in $|\nabla P|$ is more pronounced than in the subsonic case, which is observed when comparing Figures 2 and 3. However, in the case of subsonic turbulence, there are no compressive motions. In this case, random fluctuations in density and magnetic field will create structures in $|\nabla P|$ and $|\nabla \mathbf{R M}|$.

Due to these effects, we might expect different trends in the correlation of supersonic and subsonic $|\nabla P|$ with $\nabla N$ or VEM (the gradient of the emission measure) and $\nabla B$. We test this by plotting the pixel-by-pixel correlation coefficient of $|\nabla P|$ with the gradients of EM, $N$, and, LOS $B$ versus sonic Mach number in Figure 5. The left panel shows super-Alfvénic models while the right panel shows sub-Alfvénic models. In the case of subsonic turbulence, $|\nabla P|$ better traces out the fluctuations in $\nabla B$ (blue line), while the supersonic cases are dominated by density fluctuations. This is because density enhancements are dominant due to shock fronts in the case of supersonic turbulence, while in subsonic turbulence density is marginally incompressible. In the subsonic case, the magnetic field will dominate the topology of the rotation measure and $|\nabla P|$. This behavior is analogous to velocity in neutral hydrogen radio PPV cubes of turbulence, where density dominates the power spectrum for the case of supersonic turbulence and velocity dominates the spectrum for subsonic turbulence (see Lazarian & Pogosyan 2006). This difference in correlation provides yet another way of gauging the Mach numbers if one has both the emission measure and the linear polarization map. Correlated spatial gradients between the two could indicate regions of shocks.

In the next section, we will explore the utility of gradients of polarimetric data for the determination of the Mach numbers by investigating two different statistical measures of looking at the distribution and topology of the $|\nabla P|$ maps: PDF moments and the genus function.

4. STATISTICAL DETERMINATION OF THE SONIC MACH NUMBER

The previous section provided some theoretical discussion for why we expect $|\nabla P|$ data to be useful for determining the sonic Mach number. In this section, we will attempt to statistically quantify the differences seen in both the morphology and the distribution of maps of $|\nabla P|$. We again note our assumption for the simulations of $|\mathbf{P}| = 1$, thus giving a trivial scaling relationship between $|\nabla P|$ and $|\nabla \mathbf{R M}|$ as $|\nabla \mathbf{R M}| = |\nabla P|/2\lambda$. We also provide an observational comparison for both statistics with the SGPS test region shown in Figure 1.

4.1. Moments

A PDF is the function describing the frequency of occurrence of values in the distribution of intensities. PDFs and their quantitative descriptors have been used to study turbulence in a variety of astrophysical contexts, including diffuse ISM turbulence (Berkhuijsen & Fletcher 2011), turbulence characterization (Federrath et al. 2010; Esquivel & Lazarian 2010; Audit & Hennebelle 2010), solar wind (Burlaga et al. 2007), and molecular ISM (Padoan et al. 1999). Several authors have discussed the use of PDFs in determining the Mach numbers of ISM turbulence, in particular the sonic Mach number (Vazquez-Semadeni 1994; Passot & Vazquez-Semadeni 1998; Padoan et al. 1999; Kowal et al. 2007; Burkhart et al. 2010; Price et al. 2011). However, this technique is almost always used in the context of the column density. To our knowledge, no one has applied this
technique to the rotation measure, polarization maps, or their gradients.

One method of describing PDFs is by using statistical moments to characterize the mean, variance, and departures from Gaussianity. The first- and second-order statistical moments (mean and variance) used here are defined as follows:

\[ \mu_{\xi} = \frac{1}{N} \sum_{i=1}^{N} (\xi_i) \quad \text{and} \quad \sigma_{\xi} = \frac{1}{N-1} \sum_{i=1}^{N} (\xi_i - \bar{\xi})^2, \]

where \( N \) is the total number of elements, \( \xi_i \) is the distribution of intensities, and \( \sigma_{\xi} \) is the standard deviation.

Past works have focused on the relationship between the moments and the density or column density. As the Mach number increases, so does the mean value of density, as shocks increase density (the sonic Mach number goes as \( M_s \approx \rho^{1/2} \) for isothermal gas). The variance, skewness, and kurtosis are less obvious quantities and their relation to the sonic Mach number has been derived numerically (Padoan et al. 1999; Kowal et al. 2007; Burkhart et al. 2010; Price et al. 2011). Variance has some disadvantages to skewness and kurtosis. One is that variance is scale dependent, and values will change between different data set normalizations. This makes direct comparison between simulations and observations difficult. On the other hand, the higher order moments (skewness and kurtosis) describe deviations from Gaussianity and are unitless numbers, and therefore are scale-free. They are also shown to increase more linearly with the sonic Mach number in the case of column density (Kowal et al. 2007; Burkhart et al. 2010). However, the increase is not so pronounced in the case of subsonic-turbulence, making variance a better indicator in this regime.

While the rotation measure is obviously related to the density, the relationship between the moments of \( |\nabla P| \) and the Mach number has never been studied. The polarization gradient may have an advantage over column density/dispersion measure for investigating the Mach numbers of the WIM, as shocks will cause both an increase in density and magnetic field due to correlations between density and field strength in supersonic-type turbulence (see Burkhart et al. 2009). Furthermore, the gradient will highlight interesting features in the observational data which might be buried by observational effects, such as a DC offset, that otherwise would not be seen in the maps of column density or linear polarization (see Figures 2 and 3).

Figure 6 shows an example PDF of a subsonic and supersonic realization of the \( |\nabla P| \) distribution. The supersonic case (represented by + signs) is highly skewed and kurtotic compared with the subsonic case (represented by * signs).

Figure 7 shows the moments of \( |\nabla P| \) versus the sonic Mach number for our simulations with LOS parallel (left panel) to the average magnetic field. Error bars are created by taking the standard deviation between different time snapshots of the well-developed turbulence. We show sub-Alfvénic cases in red and super-Alfvénic cases in black.

(A color version of this figure is available in the online journal.)
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Figure 8. Left panel: mean (top) and variance (bottom) of the polarization angle (or RM $\lambda^2$) perpendicular to the mean magnetic field vs. sonic Mach number. Right panel: mean (top) and variance (bottom) of the polarization angle (or RM $\lambda^2$) along the direction of the mean magnetic field vs. sonic Mach number. A strong Alfvénic dependency is observed for variance in both cases. In the case of the mean value, the dependency exists only in the case of the LOS parallel to the field and is independent of the sonic Mach number. (A color version of this figure is available in the online journal.)

This is not seen for sight lines perpendicular to the mean field (left panel). The variance of the polarization angle shows strong dependency on both sonic and Alfvénic Mach numbers for both sight lines. Thus, to gauge the Alfvénic Mach number using the distribution, the ordered LOS magnetic field is required.

An additional effect that must be considered is the issue of telescope resolution. For example, Figure 9 shows model 6 without smoothing (left) and with Gaussian smoothing with FWHM = 6 pixels (right). Attention to the differences in the color bars clearly shows that smoothing the maps of $Q$ and $U$ changes the distribution of maps of $|\nabla P|$. We plot the moments versus smoothing FWHM for four models in Figure 10. As the resolution of maps of $Q$ and $U$ decrease, so do the moments. Thus, one needs to take the smoothing of the data into account when comparing PDFs of $|\nabla P|$.

We investigate the PDFs and moments of the SGPS test data set and compare them to our simulations. We find that the SGPS data have a mean skewness and a mean value of 0.004 and $3.38 \times 10^{-6}$, respectively. Comparing the skewness and kurtosis values with our numerical setup shows that these values most closely match subsonic to transonic values of the moments, with telescope smoothing taken into account.

4.1.1. Moment Maps

From Section 4.1 it is clear that, for a given map of $|\nabla P|$, as the sonic Mach number increases the skewness and kurtosis also increase. However, these were globally averaged values of both the moments (e.g., the PDF of the entire image) and the sonic Mach number. If we want to look at smaller scale variations, we can calculate the PDFs of smaller portions of the image. Using a moving kernel (box, circle, etc.), we can create a “moment map” which is essentially a smoothed map that calculates the moments at every point with a given box size. Because we are dealing with gradient quantities, we expect these maps to particularly trace regions where the sonic Mach number is changing.

We make skewness and kurtosis moment maps of the $|\nabla P|$ images, similar to the method of Burkhart et al. (2010). This results in a smoothed moment image, which can be compared to the actual image of the LOS local sonic Mach number (LOS average sonic Mach number for each pixel) with the same smoothing kernel as the $|\nabla P|$ moment map. In this section, we investigate the relationship between the LOS sonic Mach number and the values of skewness and kurtosis on a pixel-by-pixel basis.

The key questions one must consider when making a moment map are what the kernel shape should be, what resolution is appropriate for good statistics, and how to compare the values of the higher order moments to the sonic Mach number? The last point is particularly important. While Burkhart et al. (2010) took a linear fit between the sonic Mach number and the moments in the case of column density, we will test how effectively the moment maps can distinguish between supersonic and subsonic regimes on a pixel-by-pixel basis. That is, what is the success rate that a given pixel value of skewness and kurtosis translates to subsonic or supersonic correctly? We stress that we are not trying to calculate the exact sonic Mach number with this method. We...
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Figure 9. Maps of $|\nabla P|$ for model 6 with smoothing FWHM = 6 pixels (right) and without smoothing (left). Careful attention to the color bar shows that the telescope resolution has a strong effect on the distribution of $|\nabla P|$, which is reflected in the statistical moments. However, the morphology remains similar, suggesting the use of a topological statistical tool in the case of smoothed images.

(A color version of this figure is available in the online journal.)

Figure 10. Moments of $|\nabla P|$ vs. smoothing for four different models. Error bars are created by taking the standard deviation between different time snapshots of the data. Smoothing FWHM is given in pixels.

(A color version of this figure is available in the online journal.)

only seek to determine how well a moment map is able to pick out regimes of supersonic and subsonic turbulence. For our initial test, we choose to use a boxcar kernel for both the $|\nabla P|$ maps and for averaging the sonic Mach number, since a boxcar can handle edges in a square image most effectively.

How well the moment map is able to determine if the gas is supersonic or subsonic depends on three parameters: the threshold level for skewness ($\gamma_T$, above which the gas is considered supersonic and below it is considered subsonic), the threshold level for kurtosis ($\beta_T$, above which the gas is considered supersonic and below it is considered subsonic), and the kernel box size chosen. The values for $\gamma_T$ and $\beta_T$ represent the threshold value between supersonic and subsonic regimes. Any pixel below either $\gamma_T$ or $\beta_T$ is deemed subsonic and any pixel above both is deemed supersonic. Of course, an intermediate threshold value could also be chosen to probe the transonic regime, but we omit this here. Again, we should stress that we are not employing this method to get an exact Mach number; we simply view it as a way of determining whether our $|\nabla P|$ image shows supersonic or subsonic characteristics.

The best-fitting parameter values of box size, $\gamma_T$, and $\beta_T$ are those that provide an accurate translation between the moments and the LOS Mach number. In order to determine these values, we use a genetic algorithm which searches possible combinations of these three parameters in order to determine the best fit. The genetic algorithm can provide a more computationally efficient way of testing for fitness rather than iterating over every possible combination of parameter space. See the Appendix for a detailed description of our algorithm.

Although multiple combinations of parameters showed high confidence levels, we chose $\gamma_T = 1.1$ and $\beta_T = 1.58$ and box size $= 64 \times 64$ pixels. With these values, the supersonic models were able to determine the Mach number regime with 98% accuracy, while the subsonic cases had an accuracy of 67% (see Figure 11). This is not surprising however, since the moments are known to be a more robust measure of highly supersonic turbulence (Kowal et al. 2007). We plot the LOS Mach number map for model number 1 in the top panel of Figure 12 with contours from the kurtosis moment map. In general, the moment map traces areas where the sonic Mach number is changing.

We apply the moving box method to the SGPS data cuts using the same parameters for $\gamma_T$, $\beta_T$, and box size as were used for the simulations. For the SGPS data, we obtained average skewness and kurtosis values for this moment map of 0.3 and 0.9 for skewness and kurtosis, respectively. We overplot the SGPS data with the kurtosis moment map contours in the bottom panel of Figure 12. While these values seem to point in the direction that the SGPS data are subsonic or transonic, we note that this method is less accurate for subsonic-type turbulence. However, the PDFs do seem to confirm that the warm ionized gas in this
Figure 11. Success rate of the moment map. A moment map is constructed with parameters $\gamma_T = 1.1$, $\beta_T = 1.58$, and box size (64 pixel), which were chosen with a genetic algorithm. If a value in the moment map is above both $\gamma_T$ and $\beta_T$, then that pixel is considered supersonic. If the value is below either, then the pixel is considered subsonic. The moment map is compared with the actual LOS sonic Mach number map to determine whether the moment map alone is sufficient for recovering regimes of supersonic or subsonic on a pixel-by-pixel basis. The moment map is very successful (almost 100%) in determining if supersonic turbulence is present. The success rate drops to $\approx 60\%$ in the case of diagnosing subsonic turbulence. This further confirms the higher order moment utility in the presence of supersonic flows.

(A color version of this figure is available in the online journal.)

4.2. Topology: The Genus Statistic

The filaments seen in $|\nabla P|$ show substantially different morphology when comparing maps of subsonic and supersonic turbulence, as discussed in Section 3. Thus, a natural avenue of characterization would be to use topological measures in order to pick out different structures. In this section, we will investigate the utility of the genus statistic in order to characterize the topology of $|\nabla P|$ filaments.

The genus statistic was developed to study the topology and deviations from Gaussianity of the universe and the distribution of galaxies in three dimensions (Gott et al. 1986, 1987). The use of the genus statistic for the study of HI was first discussed in Lazarian (1999), and subsequent studies presented the genus curves for the Small Magellanic Cloud (SMC; Lazarian et al. 2002; Lazarian 2004; Chepurnov et al. 2008) and for MHD simulations (Kowal et al. 2007).

Genus is a quantitative measure of topology. It can characterize both two-dimensional and three-dimensional distributions. Generally speaking, the genus is used to detect departures from Gaussianity. When dealing with the ISM, one cannot expect deviations from symmetry to be small, especially in the presence of supersonic flows. In this case, genus can be used to characterize flows that are supersonic since these show large deviations from Gaussianity.

The two-dimensional genus can be represented as (Coles 1988; Melott et al. 1989)

$$G \equiv \text{(No. of isolated connected regions of high density)} - \text{(No. of isolated connected regions of low density)},$$

where low- and high-density regions are selected with respect to a given contour threshold. For instance, a uniform circle would have a genus of 0 (one connected region of high density, i.e., an “island,” and one connected region of low density), while a ring (a donut, for example) would have a genus of $-1$ (one connected region of high density and two connected regions of low density). Thus, the genus can distinguish between “clump” and “swiss cheese” topologies (Gott et al. 1990).

For a two-dimensional image, the genus is simply a number which corresponds to a given threshold value. What is considered to be high or low is dependent on the threshold value, which acts as a free parameter. As a result, for a given two-dimensional image, the threshold value can be varied to construct a curve with the genus value on the $y$-axis and the threshold value on the $x$-axis. This is known as the genus curve. We show an example genus curve for $|\nabla P|$ in Figure 13. This genus curve is for the subsonic sub-Alfvénic (model 1) simulation. In the case of a density or column density field, the genus of subsonic turbulence is close to a Gaussian field. However, the $|\nabla P|$ distribution

Figure 12. Top: LOS sonic Mach number map for model 1 with overplotted contours of the kurtosis moment map with $\gamma_T = 1.1$, $\beta_T = 1.58$, and smoothing kernel $= 64 \times 64$ pixels. The moment map generally outlines areas where the sonic Mach number is changing, which is expected for a gradient quantity. Bottom: map of $|\nabla P|$ for the SGPS test region used in the paper with the kurtosis moment map contours overplotted.

(A color version of this figure is available in the online journal.)
has longer tails and a more pronounced minimum. The maximum and minimum points of the genus curve correspond to percolation of the distribution (see Colombi et al. 2000). The fact that the observed genus falls more slowly at large thresholds than the Gaussian distribution indicates that the clumps are more discrete and pronounced than for the Gaussian distribution.

We expect the sign of the genus curve at the mean intensity level of $|\nabla P|$ (e.g., where the curve crosses the $x$-axis) describes the field topology. In this context, it is more convenient to work with the zero of the genus curve (genus zero) $\nu_0$, because it can be normalized to the field variance. In other words, consider a map in which the mean value of the map is subtracted off. $\nu_0$ represents the location where the genus curve crosses the $x$-axis (e.g., the genus shift), with the origin being the mean value of the field ($\nu_0 = 0$). In this case, for the intensity with the subtracted mean value, a negative $\nu_0$ corresponds to the clumpy topology, while a positive $\nu_0$ indicates a “swiss cheese” or hole topology.

We make plots of the genus zero versus smoothing of the maps of $U$ and $Q$ of all our simulations in Figure 14. A negative shift indicates a clumpy topology, while a positive shift indicates a hole topology. The error bars are derived by estimating the variance of the genus distribution. For the error bar creation, we follow the method used by Chepurnov et al. (2008) in that we generated a set of images with randomly shifted phases of individual harmonics. This procedure causes the field to take Gaussian statistics, and therefore approximately the genus zero is at the origin. However, slight deviations from Gaussian allow us to effectively estimate its variance. The procedure is as follows: we take a fast Fourier transform (FFT) of the region being studied and assign the phase of each harmonic to a random variable uniformly distributed. After the inverse FFT, we calculate the respective $\nu_0$. After repeating this procedure 10 times, we calculate the variance of the $\nu_0$ values.

Figure 14 plots the genus zero versus smoothing because the scale and mode of driving, and the importance of magnetic field in the dynamics of the media. These characteristics can vary like, showing a negative genus shift. In the case of the transonic turbulence, the topology is more neutral, with genus shifts around zero. Interestingly, there is not a strong dependency when we smooth the maps of $Q$ and $U$. The genus shifts remain consistently positive or negative over a range of smoothing.

We show the genus curve for the SGPS $|\nabla P|$ data in Figure 15. The genus zero is at $-0.07$, which indicates a slightly clumpy topology. This value more closely matches the transonic values accounting also for smoothing effects. We smooth the SGPS maps of $Q$ and $U$ to see if this trend varies for smoothing. We find values ranging from $-0.089$ to $-0.03$. This falls in the range of $M_s \approx 1.0–2.0$, accounting for an unknown Alfvénic Mach number, which again confirms the range we obtained using the PDFs.

5. DISCUSSION

Observational studies of ISM turbulence are extremely important. From the observational data, one would like to obtain the characteristics of turbulence, e.g., intensity of its driving, the scale and mode of driving, and the importance of magnetic field in the dynamics of the media. These characteristics can be evaluated if we know sonic and Alfvén Mach numbers, e.g., $M_s$ and $M_A$. For example, correlations of the Mach number in the media with the temperature can provide additional tests of the nature of the turbulent cascade in the ISM. There have been several attempts to develop the techniques to get these numbers from observations using both column density data (see Kowal et al. 2007; Burkhart et al. 2009; Brunt et al. 2010a, 2010b) as well as spectroscopic data. An example of the successful
Figure 14. Genus zero values of simulated $|\nabla P|$. Negative values imply that the topology is clumpy while positive values imply that the topology is hole dominated (swiss cheese). The genus of $|\nabla P|$ is fairly insensitive to the smoothing of maps of $Q$ and $U$. The supersonic cases show a hole topology while the subsonic case shows a clump topology, even in the case of smoothing. Transonic cases show topology that is a mixture of clumpy and hole dominated.

(A color version of this figure is available in the online journal.)

Figure 15. Genus curve for $|\nabla P|$ of the SGPS data set. The genus zero is at $-0.07$ which indicates a slightly clumpy topology. This is most similar to the subsonic–transonic-type genus curves. Horizontal and vertical solid lines reference the origin. The dotted line references a Gaussian distribution.

application of such techniques to the SMC H\textsc{i} data is provided in Burkhart et al. (2010).

This paper explores the utility of a new observationally motivated technique, namely the gradient of the polarization map, to determine $M_s$ and $M_A$. The observational advantage of using gradients stems from the fact that these gradients are easily available from interferometric observations. Our study shows that $|\nabla P|$ is a very useful measure which allows one to study turbulence. We might also expect gradients to be useful for turbulence studies in other types of data sets where shock morphology will be observed (e.g., column density, see Figures 2 and 3).

Quantitative analysis (via PDF moments and genus) of the polarization gradient of the SGPS test image indicates that turbulence in the warm ionized ISM for this sight line is in the range of subsonic to transonic. Our findings are supported by recent studies of Balmer-$\alpha$ emission measures, which have similarly found a Mach number of 2 or less (see Hill et al. 2008). Furthermore, the “Big Power Law” of electron density fluctuations (see Armstrong et al. 1995; Chepurnov & Lazarian...
also indicates that turbulence in the WIM is not highly compressible, as it provides further proof for a parsec to AU ~5/3 power law for this phase of the ISM. The fact that several different techniques (e.g., polarization gradients, power spectrum) applied to different data sets (e.g., emission measure and polarization) give complementary results is very encouraging.

However, studies of turbulence and the Mach numbers are not limited to warm ionized gas or even the Milky Way. For studies of neutral gas, e.g., 21 cm H i gas, the warm phase of the SMC was shown to have properties of transonic gas (Burkhart et al. 2010). In the case of the cold component of H i, several independent statistical and direct measurements applied to the spectral lines and the column density indicate this component of the SMC is supersonic (Stanimirović & Lazarian 2001; Burkhart et al. 2010). These findings were made using several different techniques including investigations of the column density maps, the spatial power spectrum, and the velocity power spectrum. Again, the use of multiple independent techniques is the key for successfully determining the parameters of turbulence in the ISM.

We view this paper as a first taste of the utility of using polarization data (and their gradients) for studies of turbulence. There are many additional avenues that should be explored from this first step. For instance, what are the effects of multiple screens along the LOS? What are the effects of changing the assumption of the constant background polarization? What are the effects of other equations of state and the inclusion of partial ionization?

One may wonder whether an additional way of studying turbulence is valuable, if we already have a few other ways to study turbulence, e.g., with column densities and PPV spectroscopic data cubes. The answer is a resounding yes. First of all, dealing with as complex media as the ISM, we would like to have as many independent measures as possible. Second, different measures may be more sensitive to different phases of the ISM (see the list of the idealized phases and their magnetizations in Yan et al. 2004). For example, the rotation measure, linear polarization, and their gradients are biased toward ionized parts of ISM.

The Alfvén and sonic Mach numbers are not the only characteristics of ISM turbulence. Both the spectra of turbulence and measures of intermittency (see Schmidt et al. 2008; Kowal & Lazarian 2010; Federrath et al. 2010) provide other diagnostics which should be used to study interstellar turbulence in its complexity. Additionally, we do not advocate gradients of linear polarization to be the only method for obtaining information on the Mach numbers. Rather, a larger set of both statistical and observational tools and various tracers should be used to obtain the most accurate results. This paper highlighted two separate measures (PDFs and the genus) that could be used on one tracer, e.g., the gradients of linear polarization. However, additional methods for gauging the sonic Mach number include the LOS velocity dispersion, column density fluctuations, the density and velocity spectrum (Kowal et al. 2007), and the spin/kinetic temperature for H i gas (Heiles & Troland 2003). Many of these methods also give insight into the gas temperature, turbulence driving scale, and optical depth which further highlight the urgency to use them in a synergetic way.

6. CONCLUSIONS

We created maps of the spatial gradient of the polarization vector of isothermal MHD simulations and compared these with observations from the SGPS test region. We tested two statistical methods on gradient polarization maps, namely the genus and higher order moments of the distribution, to determine if these statistics were sensitive to the sonic Mach number. We found the following.

1. Filamentary structure was created over a range of sonic Mach numbers, including cases where both shocks and subsonic turbulence were present.
2. Filaments showed different morphology for different regimes of sonic Mach number.
3. |\n\n\nP| maps with high sonic Mach number showed filaments with a double jump profile that traced shocks, while subsonic cases showed filaments that were due to random fluctuations in the rotation measure along the LOS.
4. The moments of the |\n\n\nP| distribution were higher for larger values of sonic Mach number but were also sensitive to the telescope resolution.
5. There is a strong Alfvénic dependency in the moments of polarization angle and |\n\n\nP| for sight lines parallel to the ordered magnetic field.
6. The skewness and kurtosis moment maps of |\n\n\nP| were successful at picking out subsonic or supersonic pixels 67% and 99% of the time, respectively.
7. The genus of |\n\n\nP| revealed a “hole” topology for supersonic cases and a “clump” topology for subsonic cases. Transonic cases showed neutral topology.
8. We applied the PDF moments and genus to the SGPS test region and found that this area was statistically similar to models of subsonic- to transonic-type turbulence.

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APPENDIX

THE GENETIC ALGORITHM

We employ a genetic algorithm to search for the optimal three parameters (\gamma_r, \beta_T, and box size) in the creation of the moment map, as discussed in Section 4. The literature on the genetic algorithm is vast, so we refer the curious reader to Eiben & Smith (2003) and references therein. Unlike brute force methods, the genetic algorithm relies on principles of biological evolution, such as reproduction and natural selection, to determine the optimal parameters.

A typical genetic algorithm requires a fitness function to evaluate the solution domain. In our case, the fitness metric is how well the local skewness and kurtosis are able to determine if the local sonic Mach number is either in the supersonic or subsonic regime. In our case, local means on a pixel-by-pixel
basis. The main advantage of us choosing a genetic algorithm over a brute force method is the cost in computation time. Rather than looping over all possible combinations of parameters, we only seek a convergence to a range of the best-fit outcomes. We ran the algorithm several separate times on multiple simulations with varying sonic and Alfvénic Mach numbers in order to assure that the convergence obtained was repeatable.

The algorithm we employ is a simple implementation of a genetic algorithm and works as follows. We start with a population of box sizes and threshold values. The algorithm initially picks a random subset of 10 candidates from this larger population pool; that is, it picks 10 combinations of $\gamma_T$, $\beta_T$, and box size. Then, the moment map is created by calculating the moments in the kernel, then moving the kernel pixel-by-pixel and repeating the moment calculation.

We take this map and compare it to the LOS sonic Mach number map (smoothed by the same degree as the moment map) pixel-by-pixel. If a pixel in the moment map is above $\gamma_T$ and $\beta_T$, then it is considered supersonic, otherwise it is subsonic. If this matches with what is seen in the LOS sonic Mach number map then we give this pixel a value of one, if it does not match then we give this pixel a value of zero. We calculate the percentage of successful pixels to determine the fitness of that particular choice of parameters.

Once we have determined the percent of success for all 10 of these initial candidates, we “clone” the most fit of these and then “breed” the candidates to fill out the other nine models of the next generation. This keeps the total population size constant. Our breeding is done by averaging the parameters from 9 random pairs of these 10 candidates (the parent population). The most successful candidate from the parent population and the averaged candidates now become the “children population.”

The algorithm is repeated from the point of making the moment map. This process continues for 10 iterations of parent and child. Because our algorithm uses small population sizes, after a fairly small number of generations the variety in free parameters will shrink significantly, resulting in an “inbred” population. Therefore, every 10th generation, instead of breeding the next generation, we clone the fittest model and then fill out the rest of the subsequent generation with new models, randomly chosen in the same way as the models used in the first generation of the algorithm. This process ensures that the genetic algorithm always has a large region of parameter space to explore.

We ran the genetic algorithm multiple times to ensure the solution of best fit was converging to roughly the same parameters. While there were many parameter sets in the 90%+ success range in the supersonic case, we choose $\gamma_T = 1.1$, $\beta_T = 1.58$, and box size=64 pixels, as it was the best match we found with the subsonic cases (67%).

REFERENCES

Armstrong, J. W., Rickett, B. J., & Spangler, S. R. 1995, ApJ, 443, 209
Audit, E., & Hennebelle, P. 2010, A&A, 512, 76
Ballesteros-Paredes, J., Klessen, R. S., Mac Low, M.-M., & Vazquez-Semadeni, E. 2007, in Protostars and Planets V, ed. B. Reipurth, D. Jewitt, & K. Keil (Tucson, AZ: Univ. Arizona Press), 63
Berkhuijsen, E., & Fletcher, A. 2011, in Proc. of the Role of Disk-Halo Interaction in Galaxy Evolution: Outflow vs. Infall?, Density PDFs of Diffuse Gas in the Milky Way, ed. M. A. de Avillez (EAS) arXiv:1104.2410
Bouchet, F., Federrath, C., & Price, D. J. 2010a, MNRAS, 403, 1507
Brunthall, C. M., Federrath, C., & Price, D. J. 2010b, MNRAS, 405, 56
Burchall, B., Falceta-Gonçalves, D., Kowal, G., & Lazarian, A. 2009, ApJ, 693, 250
Burkhart, B., Stanimirovic, S., Lazarian, A., & Grzegorz, K. 2010, ApJ, 708, 1204
Falgarone, E., Hily-Blant, P., Pety, J., & Pineau Des Forêts, G. 2009, AIP Conf. Proc. 784, Magnetic Fields in the Universe: From Laboratory and Stars to Primordial Structures, ed. E. M. de Gouveia Dal Pino et al. (Melville, NY: AIP), 299
Federrath, C., Chabrier, G., Schober, J., et al. 2011a, Phys. Rev. Lett., 105, 115004
Federrath, C., Klessen, R. S., & Schmidt, W., Mac Low, M.-M. 2010, A&A, 512, 81
Federrath, C., Roman-Duval, J., Klessen, R. S., Schmidt, W., Mac Low, M.-M. 2010, A&A, 512, 81
Federrath, C., Sur, S., Schleicher, D. R. G., et al. 2011b, ApJ, 731, 62
Fletcher, A., & Shukurov, A. 2006, MNRAS, 371, 21
Fletcher, A., & Shukurov, A. 2007, in EAS Publication Series vol. 23, Depolarization Canals and Interstellar Turbulence (Cambridge: Cambridge Univ. Press), 109
Goldreich, P., & Sridhar, S. 1995, ApJ, 438, 763
Gott, J. R., III, Park, C., Juszkiewicz, R., et al. 1990, ApJ, 352, 1
Gott, J. R., III, Weinberg, D. H., & Melott, A. L. 1986, ApJ, 306, 341
Gott, J. R., III, Park, C., Juszkiewicz, R., et al. 1990, ApJ, 352, 1
Gott, J. R., III, Weinberg, D. H., & Melott, A. L. 1987, ApJ, 319, 1
Gray, A. D., Landecker, T. L., Dewdney, P. E., & Taylor, A. R. 1998, Nature, 393, 660
Haverkorn, M., & Heitsch, F. 2004, A&A, 421, 1011
Heiles, C., & Troland, T. 2003, ApJ, 586, 1067
Hill, A., Benjamin, R. A., Kowal, G., et al. 2008, ApJ, 686, 363
Houde, M., Rao, R., Vaillancourt, J. E., & Hilderbrand, R. H. 2011, ApJ, 733, 109
Kowal, G., & Lazarian, A. 2010, ApJ, 720, 742
Kowal, G., Lazarian, A., & Beresnyak, A. 2007, ApJ, 658, 423
Kritsuk, A. G., Norman, M. L., Padoan, P., & Wagner, R. 2007, ApJ, 665, 416
Landecker, T. L., Reich, W., Reid, R. I., et al. 2010, A&A, 520, A80
Lazarian, A. 1999, in Plasma Turbulence and Energetic Particles in Astrophysics, ed. M. Ostrowski & R. Schlickeiser (Kraków: Osserwatorium Astronomiczne, Uniwersytet Jagielloński), 28
Lazarian, A. 2004, ApJ, 616, 94
Lazarian, A. 2006, Astron. Nach., 327, 609
Lazarian, A., Beresnyak, A., Yan, H., Opheer, M., & Liu, Y. 2009, Space Sci. Rev., 143, 387
Lazarian, A., & Pogosyan, D. 2000, ApJ, 537, 720
Lazarian, A., & Pogosyan, D. 2004, ApJ, 616, 943
Lazarian, A., & Pogosyan, D. 2006, ApJ, 652, 1348
Lazarian, A., & Pogosyan, D. 2008, ApJ, 686, 350
Lazarian, A., & Pogosyan, D. 2012, ApJ, 747, 5
