GROWTH OF THE ASYMPTOTIC DIMENSION FUNCTION FOR GROUPS

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Abstract. It is relatively easy to construct a finitely generated group with infinite asymptotic dimension: the restricted wreath product of \( \mathbb{Z} \) by \( \mathbb{Z} \) provides an example. In light of this, it becomes interesting to consider the rate of growth of the asymptotic dimension function of a group. Loosely speaking, we measure the dimension on \( \lambda \)-scale and let \( \lambda \) increase to infinity to recover the asymptotic dimension. In this paper we consider how the asymptotic dimension function is affected by different constructions involving groups.

1. Introduction

The asymptotic dimension of a metric space was introduced by Gromov [8] in his study of asymptotic invariants of infinite groups. Roughly speaking, the asymptotic dimension of a metric space is the large-scale equivalent of covering dimension of a topological space. If \( \Gamma \) is a finitely generated group, one endows \( \Gamma \) with a word metric associated to a finite, symmetric generating set \( S \). Any choice \( S' \) of finite, symmetric generating set gives rise to a quasi-isometric metric space (in fact, they are Lipschitz equivalent). The asymptotic dimension of a metric space is a quasi-isometry invariant, so we can define the asymptotic dimension of a finitely generated group \( \Gamma \), \( \mathrm{asdim} \Gamma \), as the asymptotic dimension of the metric space corresponding to any finite, symmetric generating set. More generally, one can define \( \mathrm{asdim} \) for coarse spaces; it turns out that \( \mathrm{asdim} \) is also a coarse invariant, see [15].

Yu [17] showed that groups with finite asymptotic dimension satisfy the Novikov higher signature conjecture. Later, Yu [18] generalized this result by proving that groups which admit a uniform embedding into Hilbert space, in particular then, groups with Yu’s “property A” satisfy the coarse Baum-Connes conjecture, and hence the Novikov conjecture.

It is not difficult to construct a finitely generated group with infinite asymptotic dimension. Any group containing isomorphic copies of \( \mathbb{Z}^n \) for each \( n \) will have infinite asymptotic dimension. As pointed out by Roe in

Date: September 17, 2018.

2000 Mathematics Subject Classification. Primary: 20F69, Secondary 20E06, 20E22, 20F65.

Key words and phrases. Asymptotic dimension, growth of dimension, graphs of groups, relatively hyperbolic groups.
two simple examples are Thompson’s group $F$, described in \cite{4} and the reduced wreath product of $\mathbb{Z}$ by $\mathbb{Z}$.

In this paper we consider the asymptotic dimension function of a metric space $X$. This function measures the dimension on the scale $\lambda$ of the metric space $X$. Although this function is obviously not an invariant of quasi-isometry class of the metric space, the growth of this function is, see Proposition 2.2.

Higson and Roe \cite{10} showed that finitely generated groups with bounded asymptotic dimension function have Yu’s property A. (By a theorem of Ozawa \cite{13}, Yu’s property A for a finitely generated group is equivalent to $C^*$-exactness of the group, so this property is also often referred to as exactness of the group.) Later, Dranishnikov \cite{6} showed that bounded geometry metric spaces whose asymptotic dimension function grows to infinity sublinearly have property A. In \cite{7}, Dranishnikov generalized this result, showing that groups whose asymptotic dimension function grows at most polynomially have Yu’s property A, so the coarse Baum-Connes conjecture and Novikov conjectures hold for such groups.

Here, we apply the techniques found in \cite{1, 2, 5, 12} and others to show that the growth rate of the asymptotic dimension function can be recovered from examining neighborhoods of stabilizers of an isometric action of the group on a metric space with finite asdim. We apply this method to prove our main theorems:

\textbf{Corollary 4.4} Let $(G,Y)$ be a finite, connected graph of finitely generated groups. Let $\Gamma$ be the fundamental group of $(F,G)$. Then, the asymptotic dimension function of $\Gamma$ grows no faster than the asymptotic dimension function of the vertex groups $G_P$ in the graph of groups.

Applying Osin’s methods from \cite{12} we are also able to conclude:

\textbf{Theorem 5.3} Let $\Gamma$ be a finitely generated group hyperbolic relative to a collection $\{H_1, \ldots, H_n\}$ of subgroups. Then, the asymptotic dimension function of $\Gamma$ grows no faster than the asymptotic dimension function for each of the $H_i$.

We end the paper with some open questions about the asymptotic dimension function.

\section{The asymptotic dimension function}

Let $(X,d)$ be a metric space. Let $\mathcal{U}$ be a cover of $X$. A \textit{Lebesgue number} for $\mathcal{U}$ is a number $\lambda$ for which every set $A \subset X$ with $\text{diam}(A) \leq \lambda$ is entirely contained within a single element of $\mathcal{U}$. We denote the Lebesgue number of $\mathcal{U}$ by $L(\mathcal{U})$. The multiplicity of a cover $\mathcal{U}$ is $m(\mathcal{U}) = \sup_{x \in X} \{|\{U \in \mathcal{U} \mid x \in U\}|\}$. 

We define the \textit{asymptotic dimension function} of the metric space \( X \) by
\[
\text{ad}_X(\lambda) = \min \{ m(U) \mid L(U) \geq \lambda \} - 1,
\]
where the minimum is taken over all covers of \( X \) by uniformly bounded sets. Note that \( \text{ad} \) is monotonic and that
\[
\lim_{\lambda \to \infty} \text{ad}_X(\lambda) = \text{asdim} \ X.
\]

Let \( f, g : \mathbb{R}_+ \to \mathbb{R}_+ \). We write \( f \preceq g \) if there exists a \( k \in \mathbb{N} \) so that \( f(x) \leq kg(kx + k) + k \) for all \( x \in \mathbb{R}_+ \). We write \( f \approx g \) if \( f \preceq g \) and \( g \preceq f \).

The metric spaces \((X_1, d_1)\) and \((X_2, d_2)\) are \textit{quasi-isometric} if there exist constants \( \lambda \geq 1, \epsilon \geq 0 \), and \( C \geq 0 \) and a map \( f : X \to Y \) so that for all \( x \) and \( y \) in \( X_1 \),
\[
\frac{1}{\lambda} d_1(x, y) - \epsilon \leq d_2(f(x), f(y)) \leq \lambda d_1(x, y) + \epsilon,
\]
and every point of \( X_2 \) lies in the \( C \)-neighborhood of the image of \( f \).

It is easy to see that if \( f : X_1 \to X_2 \) is a \((\lambda, \epsilon)\)-quasi-isometry then there is a \textit{quasi-inverse} \( f' \) for \( f \), i.e. a \((\lambda', \epsilon')\)-quasi-isometry \( f' : X_2 \to X_1 \) such that there exists some \( k \) for which \( d(f(f'(x')), x) \leq k \) and \( d(f'(f(x)), x) \leq k \) for all \( x \in X_1 \) and \( x' \in X_2 \).

We shall need the following simple construction to prove the invariance of growth of \( \text{ad} \) for finitely generated groups.

\textbf{Proposition 2.1.} Let \( \mathcal{U} \) be a cover of the metric space \( X \) by uniformly bounded sets with multiplicity \( m(\mathcal{U}) \) and Lebesgue number \( L(\mathcal{U}) \). Let \( k \ll L(\mathcal{U}) \). Then, there is a cover \( \mathcal{V} \) of \( X \) by uniformly bounded sets with Lebesgue number \( \geq L(U) - 2k \) and \( k \)-multiplicity \( \leq m(\mathcal{U}) \).

\textbf{Proof.} Define \( \mathcal{V} \) to be the \(-k\)-neighborhood of \( \mathcal{U} \), i.e.
\[
V_a = X \setminus (N_k(X \setminus U_a)),
\]
where \( N_k(\cdot) \) denotes the \( k \)-neighborhood. To see that \( \mathcal{V} \) covers \( X \) we take \( x \in X \). Since \( k \ll L(\mathcal{U}) \) there is a \( U \in \mathcal{U} \) so that \( B_{2k}(x) \subset U \). Thus \( x \in X \setminus (N_k(X \setminus U)) \). The sets \( V \) are certainly uniformly bounded, so it remains only to check that \( k \)-multiplicity of \( V \leq m(\mathcal{U}) \). To this end, suppose \( B_k(x) \) meets \( V_1, \ldots, V_p \). Since \( N_k(V_i) \subset U_i \) for each \( i \), we see that \( x \in N_k(V_i) \) implies that \( x \in U_i \). Thus, \( p \leq m(\mathcal{U}) \). Finally, observe that \( \text{diam}(A) < L(U) - 2k \) implies that \( \text{diam}(N_k(A)) < L(U) \). So \( N_k(A) \subset U \) for some \( U \in \mathcal{U} \). Thus \( A \subset X \setminus (N_k(X \setminus U)) \). \( \square \)

Dranishnikov remarks in [7] that the growth of \( \text{ad} \) is a quasi-isometry invariant and that this was known to Gromov. In this paper we are only interested in the growth of the asymptotic dimension function for groups, so we prove a weaker version, showing the growth is an invariant of quasi-isometry when the spaces are discrete with bounded geometry.

\textbf{Proposition 2.2.} Let \( X \) and \( Y \) be discrete metric spaces with bounded geometry. Suppose that \( X \) and \( Y \) are quasi-isometric. Then \( \text{ad}_X \approx \text{ad}_Y \). In
particular, the $\approx$-equivalence class of $\text{ad}_\Gamma$ is well-defined for finitely generated group $\Gamma$.

Proof. Let $f : X \to Y$ be an $(\alpha, \epsilon)$-quasi-isometry with $(\alpha', \epsilon')$-quasi-inverse $g : Y \to X$. We have to show that there is some $k > 0$ so that $\text{ad}_Y(\lambda) \leq k \text{ad}_X(\lambda + k) + k$. Take $C$ so large that $N_C(f(X)) \supseteq Y$ and both $d(fg(y), y) < C$ and $d(gf(x), x) < C$ for all $x \in X$ and all $y \in Y$.

Let $\lambda \gg C$ be given. Suppose that $\mathcal{U}$ is a cover of $X$ by uniformly bounded sets such that $\mathcal{U}$ has Lebesgue number $L(\mathcal{U}) > \alpha'\lambda + (\epsilon' + 2C)$ and multiplicity $1 + \text{ad}_X(\alpha' \lambda + (\epsilon' + 2C))$. Use the previous proposition to define a cover $\bar{\mathcal{U}}$ of $X$ with Lebesgue number $> \alpha' \lambda + \epsilon'$ and $C$-multiplicity $\leq \text{ad}_X(\alpha' \lambda + (\epsilon' + 2C)) + 1$. Put $\mathcal{V} = \{N_C(f(U)) \mid U \in \bar{\mathcal{U}}\}$. The collection $\mathcal{V}$ forms a cover of $Y$ by the choice of $C$. It is easy to see that $\mathcal{V}$ consists of uniformly bounded sets, so it remains to compute its Lebesgue number and its multiplicity.

Suppose that $B \subset Y$ with $\text{diam}(B) \leq \lambda$. Then $\text{diam}(g(B)) \leq \alpha' \lambda + \epsilon'$. So there is some $U \in \mathcal{U}$ so that $g(B) \subset U$ and hence $gf(B) \subset f(U)$. If $b \in B$, then $d(b, fg(b)) < C$, so $b \in N_C(U)$ which is an element of $\mathcal{V}$. Thus, $L(\mathcal{V}) > \lambda$.

Finally, suppose $y \in Y$ meets $V_1, \ldots, V_p$ in $\mathcal{V}$. Then $B_C(y)$ meets each $f(U_i)$, say $d(y, f(x_i)) < C$ for all $i = 1, 2, \ldots, p$ with $x_i \in U_i$. We claim that $p \leq c_Y(C)(\text{ad}_X(\alpha' \lambda + \epsilon' + 2C) + 1)$, where $c_Y(C)$ is the constant from the bounded geometry condition on $Y$. First observe that $B_C(y)$ contains at most $c_Y(C)$ distinct points. If one such point, say $y_0$ were the $f$-image of $x_0^1, \ldots, x_0^t$ where $x_0^i$ are in distinct members $U_i$ of the collection $\bar{\mathcal{U}}$, then we see that $d(g(y_0), x_0^i) \leq C$ for all $i = 1, 2, \ldots, t$. Since the $C$-multiplicity of $\bar{\mathcal{U}}$ is bounded above by $\text{ad}_X(\alpha' \lambda + \epsilon' + 2C) + 1$, we see that $p \leq c_Y(C)(\text{ad}_Y(\alpha' \lambda + \epsilon' + 2C) + 1)$ as desired.

Since $\text{ad}$ is monotonic, by setting $k > \max\{c_Y(C), \alpha', \epsilon' + 2C\}$ we see that $\text{ad}_Y(\lambda) \leq k \text{ad}_X(k\lambda + k) + k$, as required. $\square$

Often we will apply the preceding result to an $R$-neighborhood $N_R(A)$ of a set $A$ in a finitely generated group $\Gamma$ to conclude that $\text{ad}_{N_R(A)} \approx \text{ad}_A$.

3. **Groups acting on finite-dimensional spaces**

We will say that a family $\{X_\alpha\}_\alpha$ has **uniform asymptotic dimension growth** if for every $\lambda$ one can find a constant $B = B(\lambda)$ and a family $\{U^\alpha_i\}$ of covers, $U_\alpha$ covering $X_\alpha$ so that $B$ forms a uniform bound on the diameters of the sets in each cover, so that the Lebesgue number of $U_\alpha$ exceeds $\lambda$ for all $\alpha$, and so that the multiplicity of each family is uniformly bounded in $\alpha$. It is clear that any finite collection of subsets of a metric space satisfies these conditions, as does a collection of subsets belonging to the same isometry class.

**Theorem 3.1.** Let $X = \bigcup_\alpha X_\alpha$ be a metric space such that the collection $\{X_\alpha\}$ has uniform asymptotic dimension growth $\text{ad}(\lambda)$. Moreover, for
each \( \lambda \), assume that there is some set \( Y_\lambda \subset X \) so that \( \{X_\alpha \setminus Y_\lambda\} \) forms a \( 3B(\lambda) \)-disjoint collection and that \( Y_\lambda \) is covered by a family of uniformly bounded sets \( \mathcal{V} \) with \( L(\mathcal{V}) > \lambda \) and multiplicity \( \leq m_Y(\lambda) \). Then \( \text{ad}_X(\lambda) \approx \max\{\text{ad}(\lambda), m_Y(\lambda)\} \).

**Proof.** Let \( \lambda > 0 \) be given. For each \( \alpha \), let \( \mathcal{U}^\lambda_\alpha \) be a cover of \( X_\alpha \) so that the collection \( \{\mathcal{U}^\lambda_\alpha\}_\alpha \) satisfies the uniform condition for asymptotic dimension growth, with \( B(\lambda) \gg \lambda \). Define \( \bar{\mathcal{U}}^\lambda_\alpha \) to be the collection \( \{U \setminus N_{-\lambda}(Y_\lambda) \mid U \in \mathcal{U}^\lambda_\alpha\} \), and put \( \mathcal{W} = \mathcal{W}^\lambda = \bigcup_\alpha \bar{\mathcal{U}}^\lambda_\alpha \cup \mathcal{V} \). We claim that \( \mathcal{W} \) covers \( X \), is uniformly bounded, \( L(\mathcal{W}) > \lambda \) and that \( m(\mathcal{W}) \leq \text{ad}(\lambda) + m_Y(\lambda) \).

It is obvious that \( \mathcal{W} \) covers \( X \).

Next, we have \( \text{diam}(\bar{\mathcal{U}}^\lambda_\alpha) \leq B(\lambda) \). Since \( \text{diam}(\mathcal{W}) \leq \text{diam}(\bar{\mathcal{U}}^\lambda_\alpha) + \text{diam}(\mathcal{V}) \), the cover \( \mathcal{W} \) consists of uniformly bounded sets.

Next, let \( x \in X \). First, we claim that there can be at most one index \( \beta \) with \( x \) in sets from \( \bar{\mathcal{U}}^\lambda_\beta \). Suppose that \( x \in U \) and \( x \in U' \) with \( U \in \bar{\mathcal{U}}^\lambda_\beta \) and \( U' \in \bar{\mathcal{U}}^\lambda_\alpha \). Then \( x \in X_\alpha \setminus N_{-\lambda}(Y_\lambda) \cap X_\alpha' \setminus N_{-\lambda}(Y_\lambda) \), but \( \text{dist}(X_\alpha \setminus N_{-\lambda}(Y_\lambda), X_\alpha' \setminus N_{-\lambda}(Y_\lambda)) \geq 3B(\lambda) - 2\lambda \geq \lambda \). Thus, there can be at most \( m_Y(\lambda) \) sets from \( \mathcal{V} \) containing \( x \) and at most \( \text{ad}(\lambda) \) sets of the form \( \bar{\mathcal{U}}^\lambda_\alpha \) containing \( x \). So \( m(\mathcal{W}) \leq m_Y(\lambda) + \text{diam}(\bar{\mathcal{U}}^\lambda_\alpha) \).

Finally, we check that the Lebesgue number is \( > \lambda \). To this end, let \( A \subset X \) with \( \text{diam}(A) \leq \lambda \). There are only two possibilities for \( A \). Either it lies entirely within \( Y_\lambda \), in which case, it is contained in a set from \( \mathcal{V} \), or it is contained in \( X_\alpha \setminus N_{-\lambda}(Y_\lambda) \). In the second case it is contained in one of the elements of \( \bar{\mathcal{U}}^\lambda_\alpha \).

Suppose \( X = A \cup B \). Setting \( Y_\lambda = B \) the disjointness condition is trivially satisfied so we immediately obtain:

**Corollary 3.2.** (Finite Union Theorem) Let \( X = A \cup B \) be a metric space with asymptotic dimension functions \( \text{ad}_A(\lambda) \) and \( \text{ad}_B(\lambda) \), respectively. Then \( \text{ad}_X(\lambda) \approx \max\{\text{ad}_A(\lambda), \text{ad}_B(\lambda)\} \).

A version of the union theorem we have occasion to use frequently is the following:

**Corollary 3.3.** Let \( X \) be a metric space with bounded geometry (or a finitely generated group with a word metric). Suppose \( X_\alpha \) is a collection of isometric subsets of \( X \) and that \( \bigcup_\alpha X_\alpha = X \). Suppose for each \( r > 0 \), there exists a \( Y_r \subset X \) with \( \{X_\alpha \setminus Y_r\} \) \( r \)-disjoint, where \( Y_r \) is quasi-isometric to \( X_\alpha \). Then \( \text{ad}_X \approx \text{ad}_{X_\alpha} \).

**Proof.** Since the \( X_\alpha \) are isometric, their asymptotic dimension functions grow uniformly as \( \text{ad}_{X_\alpha} \). Let \( \lambda > 0 \) be given and take \( r \geq 3B(\lambda) \), where \( B(\lambda) \) is the constant from the definition of uniform asymptotic dimension function growth. Then \( \{X_\alpha \setminus Y_r\} \) is \( 3B(\lambda) \) disjoint. Since \( Y_r \) is quasi-isometric to any element \( X_\alpha \) in the collection, there is a uniformly bounded cover of \( Y_r \) with Lebesgue number \( > \lambda \) and with multiplicity \( m_Y(\lambda) \leq k \text{ad}_{X_\alpha}(k\lambda + k) + k \) for some \( k \in \mathbb{N} \). So, by the union theorem, \( \text{ad}_X \approx \text{ad}_{X_\alpha} \). \( \square \)
Observe that the same result holds if the set $Y_{r}$ is assumed to be a finite union of spaces each of which is quasi-isometric to an $X_{\alpha}$.

**Theorem 3.4.** Let $X$ be a metric space with finite asymptotic dimension. Suppose that the finitely generated group $\Gamma$ acts by isometries on $X$. Finally, suppose there is some $f : \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$ so that for every $R > 0$, $\text{ad}_{W_{R}} \leq f$. Then, $\text{ad}_{\Gamma} \leq f$.

**Proof.** Let $S = S^{-1}$ be a finite generating set for $\Gamma$. Fix a point $x_{0} \in X$, and define $\pi : \Gamma \rightarrow X$ via the action $\gamma \mapsto \gamma.x_{0}$. Let $\mu = \max \{\text{dist}_{X}(s.x_{0}, x_{0}) \mid s \in S\}$. We claim that $\pi$ is $\mu$-Lipschitz. Since the metric on $\Gamma$ is discrete geodesic, it suffices to verify the Lipschitz condition on pairs $(\gamma, \gamma')$, where $\text{dist}_{s}(\gamma, \gamma') = 1$. Such a pair is necessarily of the form $(\gamma, \gamma s)$, where $s \in S$. Computing, we find $\text{dist}_{X}(\pi(\gamma), \pi(\gamma s)) = \text{dist}_{X}(\gamma.x_{0}, \gamma s.x_{0}) = \text{dist}_{X}(x_{0}, s.x_{0}) \leq \mu$.

Let $R > 0$ and $\lambda > 0$ be given. By assumption $\text{asdim} X < \infty$, so put $\text{asdim} X = k - 1$. There exists an $R/2$-uniformly bounded cover $\mathcal{U}$ of $\Gamma.x_{0}$ with Lebesgue number $\geq \lambda \mu$ and multiplicity $\leq k$.

By assumption there is a uniformly bounded cover $\mathcal{V}$ of $W_{R}(x_{0})$ with Lebesgue number $\geq \lambda$ and multiplicity $\leq n f(n \lambda + n) + n$, for some $n$. Denote $n f(n \lambda + n) + n$ by $f_{n}(\lambda)$. For each $U \in \mathcal{U}$, let $\gamma_{U} \in \Gamma$ be an element such that $\gamma_{U}.x_{0} \in U$. Since left-multiplication is an isometry on $\Gamma$, we see that $\gamma_{U}W_{R}(x_{0})$ is isometric to $W_{R}(x_{0})$. Thus, we can take covers of the form $\gamma_{U}V$ for $\gamma_{U}W_{R}(x_{0})$ that are uniformly bounded, have multiplicity $\leq f_{n}(\lambda)$ and have Lebesgue number $\geq \lambda$.

Define a family $\mathcal{W}$ of subsets of $\Gamma$ by

$$\mathcal{W} = \{ \gamma_{U}V \cap \pi^{-1}(U) \mid U \in \mathcal{U}, V \in \mathcal{V} \}.$$ 

We claim that $\mathcal{W}$ is a uniformly bounded cover of $\Gamma$ with Lebesgue number $\geq \lambda$ and multiplicity $\leq k f_{n}(\lambda)$.

If $\gamma \in \Gamma$ then there is some $U \in \mathcal{U}$ so that $\gamma.x_{0} \in U$. Thus, $\text{dist}_{X}(\gamma_{U}.x_{0}, \gamma.x_{0}) \leq R$. So $\gamma \in \pi^{-1}(B_{R}(\gamma_{U}.x_{0}))$ and we conclude that there is some $V \in \mathcal{V}$ for which $\gamma_{U}V$ contains $\gamma$. Thus, $\gamma \in \pi^{-1}(U)$ and $\gamma \in \gamma_{U}V$, so there is a $W \in \mathcal{W}$ containing $\gamma$.

Next, let $A \subseteq \Gamma$ be given with $\text{diam}_{\Gamma}(A) \leq \lambda$. Then, we have $\text{diam}_{X}(\pi(A)) \leq \mu \lambda$. Since $L(\mathcal{U}) \geq \lambda \mu$, there is a $U \in \mathcal{U}$ so that $\pi^{-1}(U) \supset A$. Next, $\text{diam}_{\Gamma}(\gamma_{U}^{-1}(A)) \leq \lambda$, and $\gamma_{U}^{-1}(A) \subseteq W_{R}(x_{0})$, so there is a $V \in \mathcal{V}$ with $\gamma_{U}^{-1}(A) \subseteq V$. Thus, $\gamma_{U}V$ will contain $A$, and so there is an element of $\mathcal{W}$ containing $A$.

Let $\gamma \in \Gamma$. Then $\pi(\gamma)$ is in at most $k$ of the sets in $\mathcal{U}$. On the other hand, in each $\gamma_{U}\mathcal{V}$, $\gamma$ can belong to at most $f_{n}(\lambda)$ of the $\gamma_{U}V$. Thus, the multiplicity of $\mathcal{W}$ is bounded above by $k f_{n}(\lambda)$.

The family $\mathcal{W}$ is uniformly bounded since $\mathcal{V}$ is and since left multiplication is an isometry in $\Gamma$. □

Using the techniques found in [2] it is easy to prove the following result, (compare [4] Section 4):


Corollary 3.5. Let \( \phi : G \to H \) be a surjective homomorphism of finitely generated groups with \( \ker \phi = K \). If \( \text{asdim} \ H < \infty \), then \( \text{ad}_G \approx \text{ad}_K \).

As an extension theorem for the asymptotic dimension function of groups, this is unsatisfactory due to the requirement that the group \( H \) must have finite \( \text{asdim} \). A much more satisfactory statement would be: \( \text{ad}_G \approx \max\{\text{ad}_H, \text{ad}_K\} \), but our techniques do not apply to this situation.

4. Graphs of groups and complexes of groups

The Bass-Serre theory of graphs of groups Bass-Serre \([16]\) is well known. If \( Y \) is a finite, connected graph we can label the vertices of \( Y \) with groups. An edge joining two vertices should be equipped with a group as well as two injective homomorphisms from the edge group into the two vertex groups connected by the edge. One then forms a group (the fundamental group of the graph of groups) by taking the free product of the vertex groups and the edge labels (with formal inverses) and requiring certain compatibility conditions on the group based on the inclusions of the edge groups into the vertex groups. For more details, see below or \([3, 16]\).

The group constructed in this way will then act on a tree by isometries and the quotient of the tree by this action is the original graph \( Y \). A nice feature of the theory is that to every graph of groups there is an associated group and group action, and for every action by isometries of a group on a tree one can recover the original group via the graph of groups construction.

Three standard examples of fundamental groups of graphs of groups are the following:

- If \( Y \) is any graph, and all vertex groups are taken to be the trivial group, then the fundamental group of the graph of groups is the fundamental group \( \pi_1(Y) \).
- If \( Y \) has two vertices and one edge, then the fundamental group of the graph of groups is the free product of the vertex groups amalgamated over the edge group.
- If \( Y \) has one vertex and one edge, then the fundamental group of the graph of groups is the HNN-extension of the single vertex group over the edge group.

There is a natural generalization of the Bass-Serre theory of graphs of groups (called complexes of groups) due to Haefliger \([9]\). An exposition of this theory can be found in \([3]\); we follow the notation found there. Roughly speaking, one replaces the action on a tree with an action on a suitable higher-dimensional replacement called a small category without loops (called a scwol). One issue that arises with the theory of complexes of groups is that there is not always an associated isometric action of the fundamental group of a complex of groups on scwol so that the quotient is the original complex. When this happens, the complex of groups is said to be developable. Since we are interested in exploiting this action to our
advantage, we only consider developable complexes of groups. In the language of complexes of groups, the Bass-Serre theory states that complexes of groups associated to 1-dimensional scwols (see below) are developable.

The notation we need and pertinent results from the theory of complexes of groups follow.

**Definition.** A small category without loops (abbreviated scwol) is a set $\mathcal{X}$ which is the disjoint union of a vertex set $V(\mathcal{X})$ and an edge set $E(\mathcal{X})$. There are maps

$$i : E(\mathcal{X}) \to V(\mathcal{X}) \quad \text{and} \quad t : E(\mathcal{X}) \to V(\mathcal{X})$$

which assign to each edge $a$ the initial vertex of $a$ and the terminal vertex of $a$, respectively. Let $E^{(k)}(\mathcal{X})$ denote the composable sequences of edges of length $k$, i.e., $E^{(k)}(\mathcal{X}) = \{(a_1, \ldots, a_k) \in (E(\mathcal{X}))^k \mid i(a_i) = t(a_{i+1}), \text{ for } i = 1, \ldots, k - 1\}$. By convention, $E^{(0)}(\mathcal{X}) = V(\mathcal{X})$. There is also a map

$$E^{(2)}(\mathcal{X}) \to E(\mathcal{X})$$

which assigns to each pair $(a, b)$ an edge $(ab)$ called the composition of $a$ and $b$. These maps are required to satisfy:

1. $i(ab) = i(b)$, and $t(ab) = t(a)$ for all $(a, b) \in E^{(2)}(\mathcal{X})$;
2. $a(bc) = (ab)c$ for all edges $a, b$ and $c$ with $i(a) = t(b)$ and $i(b) = t(c)$; and
3. $i(a) \neq t(a)$. (the no loops condition)

We define the dimension of the scwol $\mathcal{X}$ to be the maximum $k$ such that $E^{(k)}(\mathcal{X})$ is not empty.

**Definition.** The geometric realization $|\mathcal{X}|$ is a piecewise Euclidean polyhedral complex, with each $k$-cell isometric to the standard simplex $\Delta^k$. There is one such $k$-simplex $A$ for each $A \in E^{(k)}(\mathcal{X})$. The identifications are the obvious ones, induced by the face relation among simplices.

The geometric realization is a Euclidean complex and is given its intrinsic metric.

**Definition.** A group action on a scwol is a homomorphism $G \to \text{Aut}(\mathcal{X})$ satisfying

1. for every $g \in G$, and for all $a \in E(\mathcal{X})$ $g.i(a) \neq t(a)$.
2. for every $g \in G$, and for all $a \in E(\mathcal{X})$ if $g.i(a) = i(a)$, then $g.a = a$.

Notice that a group action on a scwol induces an isometric action of the group on the geometric realization $|\mathcal{X}|$. Since we are primarily concerned with isometric actions on metric spaces, this is the action that we consider.

One forms the quotient $\mathcal{Y} = G \backslash \mathcal{X}$ of the scwol $\mathcal{X}$ by the action of $G$ by taking $V(\mathcal{Y}) = G \backslash V(\mathcal{X})$, and $E(\mathcal{Y}) = G \backslash E(\mathcal{X})$. One can verify that $\mathcal{Y}$ has the structure of a scwol.

**Definition.** A complex of groups $G(\mathcal{Y})$ over a scwol $\mathcal{Y}$ is a collection $G(\mathcal{Y}) = (G_\sigma, \psi_a, g_{a,b})$ satisfying
(1) to each $\sigma \in V(\mathcal{Y})$, there corresponds a group $G_\sigma$ called the local group at $\sigma$;
(2) for each $a \in E(\mathcal{Y})$ there exists an injective homomorphism $\psi_a : G_{i(a)} \to G_{t(a)}$; and
(3) For each $(a, b) \in E^2(\mathcal{Y})$, there is a $g_{a,b} \in G_{t(a)}$ such that
   (i) $\text{Ad}(g_{a,b})\psi_{ab} = \psi_a \psi_b$, where $\text{Ad}(g_{a,b})$ denotes conjugation by $g_{a,b}$, and
   (ii) $\psi_a (g_{a,b})g_{a,b,c} = g_{a,b} g_{ab,c}$, for all $(a, b, c) \in E^3(\mathcal{Y})$.

When a complex of groups is developable, there is an explicit method of constructing both the scwol $\mathcal{X}$ and the group $G$ which acts on the scwol. The scwol $\mathcal{X}$ on which the group acts is simply connected and has an explicit description in a similar way to the construction of the tree $\hat{X}$ in the theory of graphs of groups (see [16]).

Indeed, if $G(\mathcal{Y})$ is a developable complex of groups, then we can define the development $D(\mathcal{Y})$ to be the scwol whose vertices and edges are given by $V(D(\mathcal{Y})) = \{(gG_\alpha, \sigma) \mid \sigma \in V(\mathcal{Y})\}$, and $E(D(\mathcal{Y})) = \{(gG_{i(a)}, a) \mid a \in E(\mathcal{Y})\}$. Then the group $G$ acts on the development $D(\mathcal{Y})$ by left multiplication. The development is isomorphic to the scwol $\mathcal{X}$, mentioned above. (See [3] for more details.)

We describe the fundamental group of the complex of groups $\pi_1(G(\mathcal{Y}))$ which is the group $G$, up to isomorphism. As in the theory of graphs of groups, there are two equivalent descriptions of the fundamental group, but we only describe one. It relies on the construction of the auxiliary group $FG(\mathcal{Y})$. Let $E^\pm(\mathcal{Y})$ denote the collection of symbols $\{a^+, a^-\}$ where $a \in E(\mathcal{Y})$. The elements of $E^\pm(\mathcal{Y})$ can be thought of as oriented edges. If $e = a^+$, then define $i(e) = t(a)$ and $t(e) = i(a)$. Accordingly, if $e = a^-$, define $t(e) = t(a)$ and $i(e) = i(a)$. Then, define $FG(\mathcal{Y})$ to be the free product of the local groups $G_\sigma$ and the free group generated by the collection $E^\pm(\mathcal{Y})$ subject to the additional relations:

1. $(a^+)^{-1} = a^-$, and $(a^-)^{-1} = a^+$;
2. $a^+ b^+ = g_{a,b}(ab)^+$;
3. $\psi_a(g) = a^+ ga^-$, for all $g \in G_{i(a)}$.

An edge path in $\mathcal{Y}$ is a sequence $(e_1, \ldots, e_k)$ with $t(e_i) = i(e_{i+1})$, for all $i = 1, \ldots, k - 1$. By a $G(\mathcal{Y})$-path issuing from the vertex $\sigma_0$ we mean a sequence $(g_0, e_1, g_1, \ldots, e_k, g_k)$, where $(e_1, \ldots, e_k)$ is an edge path in $\mathcal{Y}$, $g_0 \in G_{\sigma_0}$, $i(e_1) = \sigma_0$, and $g_i \in G_{t(e_i)}$, for $i > 0$. We associate the word $g_0 e_1 \ldots e_k g_k \in FG(\mathcal{Y})$ to the path described above. A $G(\mathcal{Y})$-loop based at $\sigma_0$ is a $G(\mathcal{Y})$ path with $t(e_k) = \sigma_0$. The $G(\mathcal{Y})$-loops $\gamma$ and $\gamma'$ are homotopic if the $FG(\mathcal{Y})$-words they represent are equal. The fundamental group $\pi_1(G(\mathcal{Y}), \sigma_0)$ is the collection of all words associated to $G(\mathcal{Y})$-loops based at $\sigma_0$, up to homotopy equivalence.

Let $G(\mathcal{Y})$ be a developable complex of groups. Fix a vertex $\sigma_0$ in $\mathcal{Y}$ and consider the action of the fundamental group $\pi = \pi_1(G(\mathcal{Y}), \sigma_0)$ on the simply connected scwol $\mathcal{X}_c$ induced by the complex of groups. In [1] Proposition...
Lemma 4.1. Let \( \pi \) denote the fundamental group of a complex of groups \( G(\mathcal{Y}) \) where \( \mathcal{Y} \) is finite and connected, and the local groups are finitely generated. Let \( \sigma_0 \) be a vertex. Suppose there is some function \( f \) so that each stabilizer \( \Gamma_\sigma \) satisfies \( \text{ad}_{\Gamma_\sigma}(\lambda) \leq f(\lambda) \). Then \( \text{ad}_{W_R} \leq f \) for all \( R \).

Proof. The proof of this result is analogous to that of \([1] \) Lemma 2.

Let \( K \subset FG(\mathcal{Y}) \) denote the set of all words in \( FG(\mathcal{Y}) \) with an associated path issuing from \( \sigma_0 \). Observe that \( \pi \subset K \) and the set \( K \) acts on \( \mathcal{X} \) by left multiplication. We consider \( W_R(\sigma_0) \) as a subset of \( K \) and show that it has \( \text{ad}_{W_R} \leq f \). It will follow that as a subset of \( \Gamma \), \( \text{ad}_{W_R} \leq f \).

Put \( K_j \) equal to the subset of \( K \) whose associated paths have length exactly equal to \( j \). Then, in light of the finite union theorem, we need only show that \( \text{ad}_{K_j} \leq f \) for all \( j \). We proceed inductively. Observe that \( K_0 = G_{\sigma_0} \), so this is true by assumption. Next, consider the case \( K_{j+1} \) with \( j \geq 0 \). Observe that \( K_{j+1} \subset \bigcup_{a \in E(\mathcal{Y})} K_j aG_{i(a)} \).

The orientation of the edge \( a \) is an issue since it determines whether the group \( G_{i(a)} \) is a domain or codomain of the function \( \psi_a \). Thus, it is necessary to consider two cases separately.

Suppose first that \( a \) has a negative orientation. So, we are considering \( K_j a^- G_{i(a)} \). For every \( r > 0 \) let \( Y_r = K_j a^- N_r(\psi_a(G_{i(a)})) \), where the \( r \)-neighborhood is taken in the group \( FG(\mathcal{Y}) \). Then \( Y_r \) is quasi-isometric to \( K_j a^- \psi_a(G_{i(a)}) \). We observe that \( K_j a^- \psi_a(G_{i(a)}) = K_j G_{i(a)} a^- \), which is just \( K_j a^- \). Since \( K_j a^- \) is quasi-isometric to \( K_j \), we have \( \text{ad}_{Y_r} \approx \text{ad}_{K_j} \), which by the inductive hypothesis grows no faster than \( f \).

Next, we decompose the set \( K_j a^- G_{i(a)} \) into sets of the form \( \{xa^- G_{i(a)}\} \), where the index runs over all \( x \in K_j \) that do not end with an element \( g \in G_{i(a)} \). One can still obtain these elements through the relations of \( FG(\mathcal{Y}) \).

For example, to obtain \( xga^- g' \), with \( x \) of the required form, \( g \in G_{i(a)} \) and \( g' \in G_{i(a)} \), simply take the word \( x a^- \psi_a(g) g' \), which is of the required form. Next, observe that the map \( G_{i(a)} \to xa^- G_{i(a)} \) is an isometry in the (left-invariant) word metric. So the set \( \{xa^- G_{i(a)}\} \) has \( \text{ad} \leq f \), uniformly.

In order to apply the union theorem to this family, it remains to show only that the family \( \{xa^- G_{i(a)} \setminus Y_r\} \) is \( r \)-disjoint. To this end, let \( xa^- z \) and \( x' a^- z' \) be given in different families. Then we compute \( d(xa^- z, x' a^- z') = \|z^{-1}a^+ x^{-1} x'a^- z'\| \). Since \( z \) and \( z' \) lie outside of \( N_r(\psi_a(G_{i(a)})) \), take \( z = \psi_a(g)s \) and \( z' = \psi_a(g')s' \), where \( \|s\| > r \), \( \|s'\| > r \), and \( \psi_a(g) \) and \( \psi_a(g') \) are in \( \psi_a(G_{i(a)}) \). Then,

\[
\|z^{-1}a^+ x^{-1} x'a^- z'\| = \|s^{-1}a^+ g^{-1} x^{-1} x' g' a^- s'\|
\]

Now, in order for this length to be less than \( r \), a reduction must occur in the middle, so that \( a^+ \) and \( a^- \) annihilate each other. In order for this to occur, we must have \( g^{-1} x^{-1} x' g' \in G_{i(a)} \). Thus, \( x^{-1} x' \in G_{i(a)} \). But, this means that
\(xa^{-G_{t(a)}}\) and \(x'a^{-G_{t(a)}}\) define the same set. Thus, in the case that the edge has negative orientation, we have \(\text{ad}_{K_ja^{-G_{t(a)}}} \preceq f\).

Next, we consider the case where the edge \(a\) has positive orientation. In this case, \(K_ja^{+G_{i(a)}} = K_j\psi_a(G_{i(a)})a^{+}\), which is quasi-isometric to \(K_j\). We conclude that \(\text{ad}_{K_ja^{+G_{i(a)}}} \preceq \text{ad}_{K_j} \preceq f\). □

It is clear that scwols of dimension 1 must have precisely two types of vertices: sources and sinks. A source is an initial vertex of every edge it is contained in and a sink is a terminal vertex of every edge it is contained in. Every one-dimensional simplicial complex (graph) can be given the structure of a one-dimensional scowl by placing a source vertex in the middle of every edge, thus giving the original vertices the structure of sinks. It is easy to verify that the theory of complexes of groups over one-dimensional scwols is precisely the same as the theory of graphs of groups. Phrased in terms of the language of complexes of groups, the Bass-Serre structure theorem for groups acting without inversion on graphs says that if \(\dim(Y) = 1\), then \(G(Y)\) is always developable.

Phrasing this result in the language of graphs of groups, we obtain:

**Corollary 4.2.** Suppose the finitely generated group \(\Gamma\) acts by isometries on a tree \(X\) with compact quotient and finitely generated stabilizers such that there is some function \(f\) so that each stabilizer \(\Gamma_x\) satisfies \(\text{ad}_{\Gamma_x}(\lambda) \leq f(\lambda)\). Then \(\text{ad}_{\Gamma_x} \preceq f\) for all \(R\).

Applying Theorem 3.4 we obtain our main result for developable complexes of groups:

**Theorem 4.3.** Let \(\pi\) be the fundamental group of a finite, developable complex of groups \(G(Y)\) such that the development \(X\) has \(\text{asdim} |X| < \infty\), and such that there is some function \(f\) that is an upper bound for the growth of the asymptotic dimension function of every base group \(G_\sigma\). Then \(\text{ad}_\pi \preceq f\).

In the language of graphs of groups this becomes:

**Theorem 4.4.** Let \(\pi\) be the fundamental group of a finite graph of groups. Suppose that the vertex groups are finitely generated and that the growth of the asymptotic dimension of the vertex groups is bounded above by the function \(f\). Then, \(\text{ad}_\pi \preceq f\).

Based on the examples cited above we immediately obtain:

**Corollary 4.5.** Let \(A\) and \(B\) be finitely generated groups. Then we have the following estimates on growth of \(\text{asdim}\) for the amalgamated free product and HNN-extension: \(\text{ad}_{A \ast C B} \approx \max\{\text{ad}_A, \text{ad}_B\}\) and \(\text{ad}_{A \ast C} \approx \text{ad}_A\).

5. **RELATIVELY HYPERBOLIC GROUPS**

In a recent article, Osin [12] proved that a finitely generated group that is hyperbolic relative to a finite collection \(\{H_1, \ldots, H_m\}\) of subgroups has finite asymptotic dimension if \(\text{asdim} H_i < \infty\) for each subgroup \(H_i\). We refer
Since the set $\text{ad}$ only to show that $\text{ad}$

**Lemma 5.2.**

**Proof.** This argument is essentially the one that appears in [12].

We proceed inductively. Observe that the fact that $|S|$ is finite means that $B(1) = S \cup (\bigcup_{\ell=1}^{m} H_{\ell})$ is a finite union of sets with $\text{ad} \preceq f$. Thus, $\text{ad}_{B(1)} \preceq f$.

For $n > 1$, we can write $B(n) = (\bigcup_{\ell=1}^{m} B(n-1) H_{\ell}) \cup (\bigcup_{s \in S} B(n-1)s)$. Since the set $B(n-1)s$ is quasi-isometric to $B(n-1)$, we have $\text{ad}_{B(n-1)s} \approx \text{ad}_{B(n-1)}$. Since $|S| < \infty$, the finite union theorem means that it remains only to show that $\text{ad}_{B(n-1) H_{\ell}} \preceq f$ for each $1 \leq \ell \leq m$. 

Next, we need an analog of [12 Lemma 12]:

**Proposition 5.1.** For any $n \in \mathbb{N}$ define $B(n) = \{\gamma \in \Gamma : |\gamma|_{S \cup \mathcal{H}} \leq n\}$. View $B(n) \subset \Gamma$ with the metric $d_{S}$. Fix $0 \leq \ell \leq m$. Then,

1. the set $B(n-1) H_{\ell}$ can be written as the disjoint union $B(n-1) H_{\ell} = \bigcup_{g} g H_{\lambda}$, where $g$ ranges over a certain (finite) subset of $B(n-1)$; and

2. for any $r > 0$ there is a subset $Y_{r} \subset \Gamma$ so that $Y_{r} = \bigcup_{x} B(n-1)x$ is a finite union and such that the sets $\gamma H_{\ell} \setminus Y_{r}$ and $\gamma H_{\ell} \setminus Y_{r}$ are $r$-separated whenever $\gamma \neq \gamma'$.

**Lemma 5.2.** Suppose that there is a function $f$ so that $\text{ad}_{H_{\ell}} \preceq f$ for all subgroups $H_{\ell}$. Then for any $n \in \mathbb{N}$ we have $\text{ad}_{B(n)} \preceq f$.

**Proof.** This argument is essentially the one that appears in [12].

Dadarlat and Guentner [5] use similar techniques to prove that the group $\Gamma$ is uniformly embeddable in Hilbert space precisely when each subgroup $H_{i}$ is uniformly embeddable in Hilbert space. Using altogether different techniques, Ozawa [14] was able to show the corresponding result for $C^{*}$-exactness of such a group. Ozawa’s result is recovered in the work of Dadarlat-Guentner.

Our goal in this section is to prove the corresponding result for the growth of the asymptotic dimension function for the group $\Gamma$. Osin gives most of the ingredients and techniques for this result, we simply put them together with our results on the growth of the asymptotic dimension function.

First we fix some notation. Throughout this section $\Gamma$ will be a finitely generated group with generating set $S = S^{-1}$ that is hyperbolic with respect to the collection $H_{1}, \ldots , H_{m}$ of subgroups. Put $\mathcal{H} = \bigcup_{i=1}^{m} (H_{i} \setminus \{e\})$. There are two metrics we wish to consider on $\Gamma$. The first is the word metric $d_{S}$ associated to the generating set $S$. The second will be denoted $d_{S \cup \mathcal{H}}$ and it is the (left-invariant) word metric associated to the set $S \cup \mathcal{H}$.

Observe that there is an action of $\Gamma$ on $(\Gamma \times \mathbb{R})$ by isometries. Osin shows that $\text{asdim}(\Gamma, d_{S \cup \mathcal{H}}) < \infty$. So, in order to apply Theorem [8, 9] it remains to examine the $R$-stabilizers of the action. Observe that in this case, $W_{R}(e) = (\text{in Osin’s notation}) B(R) = \{\gamma \in \Gamma : |\gamma|_{S \cup \mathcal{H}} \leq R\}$.

We extract the following results from the proof of [12 Lemma 12]:
By the previous proposition, \( B(n-1)H_\ell = \bigsqcup \gamma H_\ell \) and the growth of \( \text{ad} \) for \( \{ \gamma H_\ell \} \) is uniform in \( \gamma \) since these cosets are isometric. By the remarks following Corollary 3.3, we obtain \( \text{ad}_{B(n)} \preceq f \) as desired. \( \square \)

We are now in a position to prove the main theorem from this section.

**Theorem 5.3.** Let \( \Gamma \) be a finitely generated group that is hyperbolic relative to a finite family \( \{ H_1, \ldots, H_k \} \) of subgroups. Suppose that there is some upper bound \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) for \( \text{ad}_{H_i} \) for all \( i \). Then, \( \text{ad}_\Gamma \preceq f \).

**Proof.** As observed above, \( \Gamma \) acts by isometries on a metric space with finite asymptotic dimension. By Theorem 3.4, we need only check that for every \( R > 0 \) there is some \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) so that \( \text{ad}_{W_R} \preceq f \). But, this is exactly the content of Lemma 5.2. \( \square \)

6. Open Questions

Dranishnikov [7] posed the following question, which could be called the Milnor-type question for growth of \( \text{ad} \):

**Question 1.** Do there exist groups with intermediate dimension growth?

Also in [7], Dranishnikov showed that restricted wreath products of certain groups with finite asymptotic dimension have polynomial dimension growth. That leads to a few natural questions.

**Question 2.** Is the growth of the asymptotic dimension of a restricted wreath product of groups with finite asymptotic dimension at most polynomial?

A positive answer would lead to:

**Question 3.** Is the class of groups with polynomial asymptotic dimension growth closed under the formation of restricted wreath products?

A negative answer would resolve Question 1 and lead to:

**Question 4.** Is there a “critical rate of growth” so that groups whose dimension grows slower than this rate are exact and those with faster growth are not? What about a critical rate for coarse embeddings in Hilbert space?

**Question 5.** Is it true that \( \text{ad}_G \approx \max \{ \text{ad}_{H}, \text{ad}_{K} \} \) for an exact sequence \( 1 \to K \to G \to H \to 1 \) of finitely generated groups?

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