SCHUBERT STRUCTURE COEFFICIENTS VIA DERIVATIVES

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Abstract. Schubert structure coefficients \( c_{w,v}^u \) describe the multiplicative structure of the cohomology rings of flag varieties. Much work has been done on the problem of giving combinatorial formulas for these coefficients in special cases. In particular, the Littlewood–Richardson rule computes \( c_{w,v}^u \) in the case that \( u, v, \) and \( w \) are all \( p \)-Grassmannian permutations for some common \( p \).

Building on work on Wyser (2013), we introduce backstable clans to prove a “dual” positive combinatorial rule that computes \( c_{w,v}^u \) when \( u^{-1} \) is \( p \)-Grassmannian and \( v^{-1} \) is \( q \)-Grassmannian. We derive new families of linear relations among Schubert structure coefficients, which we then use to give a further positive combinatorial rule for \( c_{w,v}^u \) in the case that \( u^{-1} \) is \( p \)-Grassmannian and \( v^{-1} \) is covered in weak Bruhat order by a \( q \)-Grassmannian permutation.

1. Introduction

The flag variety \( \text{Flags}_n \) is the parameter space of complete nestings

\[
V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n
\]

of vector subspaces of \( \mathbb{C}^n \), where \( \dim V_i = i \). As a complex manifold, \( \text{Flags}_n \) has a cell decomposition induced by the Schubert varieties \( X_w \), where \( w \) is an element of the symmetric group \( S_n \). The Poincaré duals of the homology classes of these subvarieties give an additive basis of the integral cohomology ring \( H^\ast(\text{Flags}_n) \).

An important problem is to understand the structure coefficients of this ring (with respect to the Schubert basis) combinatorially. That is, letting \( \sigma_w \) denote the cohomology class corresponding to \( X_w \), we wish to understand the nonnegative integers \( c_{w,v}^u \) appearing in the expansion

\[
\sigma_u \cdot \sigma_v = \sum_w c_{w,v}^u \sigma_w.
\]

These numbers are called the Schubert structure coefficients. Write \( \iota : S_n \to S_{n+1} \) for the map that sends \( w \) to \( w(1)w(2) \cdots w(n)(n+1) \). Schubert structure coefficients are stable with respect to \( \iota \), i.e., \( c_{w,v}^u = \iota_{\ast(w)}c_{w,v}^{\iota(u),\iota(v)} \). In this sense, we may consider Schubert structure coefficients as being labeled by triples of elements in \( S_+ \), the set of permutations of \( \mathbb{Z}_{\geq 0} \) that fix all but finitely many elements.

Positive combinatorial formulas for the Schubert structure coefficients \( c_{w,v}^u \) are known only in very special cases (see, e.g., Les47, Mon59, Sot96, BS98, KP11, Wys13, BKPT16, KZ17).

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Instead of giving exact formulas, there has also been much work on the problem of establishing vanishing and nonvanishing conditions for Schubert structure coefficients (see, e.g., [Knu01, KTW04, Pur06, Pur07, BV08, MNS12, SY20]).

For a permutation \( w \in S_+ \), say that \( i \) is a descent of \( w \) if \( w(i) > w(i+1) \). A permutation \( w \in S_n \) is \( p \)-Grassmannian if it is the identity or has unique descent \( p \). Most famous of the combinatorial formulas for Schubert structure coefficients \( c_{u,v}^w \) are the Littlewood–Richardson rules [LR34, KTW04]. These rules compute the structure coefficients \( c_{u,v}^w \) in the case that \( u, v, w \) are all \( p \)-Grassmannian for the same \( p \).

We extend work of B. Wyser [Wys13] to give rules for new families of Schubert structure coefficients. These rules are based on the combinatorics of objects called clans. Wyser’s formula computes \( c_{u,v}^w \) when \( u \) and \( v \) are respectively \( p \)-inverse Grassmannian and \( q \)-inverse Grassmannian, \( p + q = n \) and \( u \leq w_0^{(p+q)} v \); here, a permutation is \( p \)-inverse Grassmannian if it is the inverse of a \( p \)-Grassmannian permutation and \( w_0^{(p+q)} = (p+q)(p+q-1)\ldots1 \).

To extend this formula to arbitrary pairs of inverse-Grassmannian permutations, we introduce the notion of backstable clans by analogy with the backstable Schubert calculus of [LLS21]. For inverse Grassmannian permutations \( u, v \), we associate a backstable clan \( \gamma_{u,v} \). We also define a rainbow clan \( \Omega_{p,q} \) associated to any pair of integers \( p, q \). Finally, we need an action of the 0-Hecke algebra on clans, denoted by “\( \cdot \)”; we write \( T_w \) for the element of the 0-Hecke algebra corresponding to the permutation \( w \). All of these notions are defined precisely in Section 2.

With these definitions, we have the following first main theorem, which we derive from Wyser’s formula.

**Theorem 1.1.** Let \( u, v, w \in S_+ \) be permutations, where \( u \) is \( p \)-inverse Grassmannian and \( v \) is \( q \)-inverse Grassmannian. Then \( c_{u,v}^w \in \{0,1\} \). Moreover, we have \( c_{u,v}^w = 1 \) if and only if \( \ell(w) = \ell(u) + \ell(v) \) and

\[
T_w \cdot \gamma_{u,v} = \Omega_{p,q}.
\]

Theorem 1.1 is in some sense dual to the classical Littlewood–Richardson rule, except that it is more general in allowing \( u \) and \( v \) to be \( p \)- and \( q \)-inverse Grassmannian for distinct \( p, q \). Contrarily, the structure coefficients of Theorem 1.1 are somewhat simpler than those from the Littlewood–Richardson rule because they are restricted to \( \{0,1\} \), whereas all nonnegative integers can appear from the Littlewood–Richardson rule. On the other hand, this “multiplicity-freeness” property has powerful geometric and combinatorial consequences (e.g., [Bri03, Knu09, HPPW20, PS22]) and has received significant interest in its own right; for example, [TY10] classifies multiplicity-freeness in the setting of Littlewood–Richardson coefficients. For illustrations of the use of Theorem 1.1 see Examples 4.3 and 4.4.

Say that a permutation is subjacent if it is of the form \( s_p w \) for some \( p \)-inverse Grassmannian permutation \( w \neq \text{id} \). Building on Theorem 1.1 we establish the following positive combinatorial rule for Schubert structure coefficients from multiplying inverse Grassmannian Schubert classes by subjacent Schubert classes.

**Theorem 1.2.** Let \( u, v, w \in S_+ \) be permutations, where \( u \neq \text{id} \) is \( p \)-inverse Grassmannian and \( v \) is \( q \)-inverse Grassmannian. Then \( c_{s_p u, v}^w \in \{0,1\} \). Moreover, we have \( c_{s_p u, v}^w = 1 \) if and only if \( \ell(w) = \ell(u) + \ell(v) - 1 \) and

\[
T_w \cdot \gamma_{u,v} \in \Psi_{p,q},
\]

where \( \Psi_{p,q} \) denotes a set of almost rainbow clans defined precisely in Section 2.
Note that, like those of Theorem 1.1, the Schubert structure coefficients of Theorem 1.2 exhibit multiplicity-freeness. For an illustration of the use of Theorem 1.2, see Example 1.8.

Our main tool for deriving Theorem 1.2 from Theorem 1.1 is the following theorem establishing new linear relations among Schubert structure coefficients.

**Theorem 1.3.** Let $u, v, w \in S_+$. Then

\[
\sum_{s_i < u} i c^w_{s_i u, v} + \sum_{s_j < v} j c^w_{u, s_j v} = \sum_{s_k w > w} k c^w_{u, v}.
\]

Theorem 1.3 enables one to discern properties of an unknown $c^w_{u, v}$ from properties of other Schubert structure coefficients. In particular, our relations yield some new vanishing and nonvanishing conditions, as well as congruence conditions. We do not know how to relate our linear relations to the the nonpositive recurrence of [Knu03]. Replacing the cohomology ring of $\text{Flags}_n$ with its $K$-theory ring of algebraic vector bundles yields more general $K$-theoretic Schubert structure coefficients $K^w_{u, v}$; we also obtain analogous linear relations in this setting.

This paper is organized as follows. In Section 2, we recall necessary background on permutations and Schubert polynomials, and we also introduce and initiate a theory of backstable Schubert structure coefficients. In particular, Proposition 3.10 will be key in our proof of Theorem 1.2. Section 4 is devoted to proofs of Theorems 1.1 and 1.2.

2. Preliminaries

2.1. Permutations. We write $[n]$ for the set $\{1, 2, \ldots, n\}$.

Let $S_Z$ denote the group of permutations of $Z$ that fix all but finitely many elements. For $i \in Z$, the **simple transposition** $s_i \in S_Z$ is the involution that switches $i$ and $i + 1$. Note that $S_Z$ is generated by $\{s_i\}_{i \in Z}$. The **(Coxeter) length** $\ell(w)$ of $w \in S_Z$ is the length of a minimal expression for $w$ as a product of simple transpositions.

We write $S_+$ for the subgroup generated by $\{s_i\}_{i > 0}$ and write $S_n$ for the subgroup generated by $\{s_i\}_{0 < i < n}$. For a permutation $w \in S_n$, we often write $w$ in one-line notation as $w(1)w(2)\ldots w(n)$. If we write $w$ in one-line notation, then $ws_i$ is obtained by swapping the entries in positions $i$ and $i + 1$; $s_i w$ is obtained by swapping the letters $i$ and $i + 1$. The **long element** $w_0^{(n)} \in S_n$ is $n(n-1)\ldots 1$. The inclusion map $ι : S_n \rightarrow S_{n+1}$ sends $w$ to $w(1)w(2)\ldots w(n)(n+1)$.

**Left weak order** on permutations is defined by $u \leq_L w$ if $w = vu$ for some permutation $v$ with $\ell(u) + \ell(v) = \ell(w)$. Similarly, in this case, we write $v \leq_R w$ and call this the **right weak order**. Let $t_{i,j} \in S_Z$ denote the involution swapping $i$ and $j$. **Bruhat order** is the transitive closure of the covering relations $wt_{i,j} < w$ for $\ell(wt_{i,j}) = \ell(w) - 1$. We write Bruhat order comparisons as $u \leq w$, without subscripts. The weak orders are weak in the sense that the corresponding relations are subsets of the Bruhat order relation.

For a permutation $w \in S_Z$, say that $i$ is a (**right** descent) of $w$ if $w(i) > w(i + 1)$, equivalently if $ws_i < w$. Say that $i$ is a (**left** descent) of $w \in S_Z$ if $w^{-1}(i) > w^{-1}(i + 1)$, equivalently if $s iw < w$. The **Lehmer code** of a permutation $w \in S_Z$ is the function $c(w) : Z \rightarrow Z_{\geq 0}$ such that $c(w)(i)$ equals the number of $j > i$ such that $w(j) < w(i)$; as a shorthand, we often write $c_i (w) = c(w)(i)$.

A permutation $w \in S_Z$ is **$k$-Grassmannian** if no $j \neq k$ is a descent of $w$. Note that the identity permutation is $k$-Grassmannian for all $k$. We say $w$ is **Grassmannian** if it is
are defined recursively as follows. If \( S \) is a partial matching of the integers such that 
\[ \gamma \in \text{Sect}(S, \mathbb{Z}) \], a corresponding element \( T_w \) in \( \mathcal{H}_\mathbb{Z} \) is obtained by taking 
any reduced decomposition \( w = s_{i_1} \cdot \cdots \cdot s_{i_k} \) and setting \( T_w = T_{i_1} \cdots T_{i_k} \). The elements \( T_w \) for 
\( w \in S_\mathbb{Z} \) are a linear basis of \( \mathcal{H}_\mathbb{Z} \).

2.2. Schubert polynomials. **Schubert polynomials** are defined recursively as follows. 
For \( w_0^{(n)} \in S_n \), set the Schubert polynomial \( \mathcal{S}_{w_0^{(n)}} = x_1^{n-1} x_2^{n-2} \cdots x_n^0 \). For \( w \) such that 
\( ws_i < w \), set \( \mathcal{S}_{ws_i} = N_i \mathcal{S}_w \), where \( N_i \) is the **(Newton) divided difference operator** that acts on 
\( f \in \mathbb{Z}[x_1, \ldots, x_n] \) by
\[
N_i(f) = \frac{f - s_i \cdot f}{x_i - x_{i+1}}.
\]
Here, \( s_i \) acts on a polynomial by swapping variables \( x_i \) and \( x_{i+1} \).

We have \( \mathcal{S}_w = \mathcal{S}_{\iota(w)} \), so we may treat Schubert polynomials as indexed by the elements 
of \( S_\mathbb{Z} \). The set of Schubert polynomials \( \{ \mathcal{S}_w \}_{w \in S_\mathbb{Z}} \) is a linear basis of the free \( \mathbb{Z} \)-module 
\( \mathbb{Z}[x_1, x_2, x_3, \ldots] \). In particular, there are structure coefficients defined by
\[
\mathcal{S}_u \cdot \mathcal{S}_v = \sum_w d_{u,v}^w \mathcal{S}_w.
\]
For \( u, v, w \in S_n \), these structure coefficients agree with the Schubert structure coefficients defined in 
Section 4 i.e., \( c_{u,v}^w = d_{u,v}^w \). Hence, we can study the structure coefficients \( c_{u,v}^w \) by using Schubert 
polynomials in place of Schubert cohomology classes. References for basic facts about Schubert polynomials include 
[Mac91, Man01].

2.3. Backstable clans. A **backstable clan** is a partial matching of the integers such that 
there exist \( i, j \in \mathbb{Z} \) such that \( i - k \) is paired with \( j + k \) for all \( k > 0 \), together with 
an assignment of labels from \( \{+,-\} \) to the unmatched integers. We say that such a backstable 
clan \( \gamma \) is **supported** on \( [i,j] \) and call \( \gamma \) an **[i,j]-clan**. When we draw diagrams to illustrate 
an **[i,j]-clan** \( \gamma \), we often restrict to the interval \( [i,j] \), since all information about \( \gamma \) can be 
exttracted from this finite region.

In the previous literature, “clans” are restricted to the interval \([1,n]\); we identify these 
objects with \([1,n]-clans\). Clans were introduced in [MO90, Yam97] in the context of \( K \)-orbits. 
For more recent work using clans in a related \( K \)-orbit context, see, e.g., [WY14, WW15]. We believe 
that backstable clans will additionally be amenable to the study of backstabilized \( K \)-orbits, analogous to the 
backstable Schubert calculus of [LLS21]; however, we do not pursue that application here. In this paper, 
backstable clans are a tool for explicating the Schubert calculus of \( \text{Flags}_n \).

For a backstable clan \( \gamma \), let \( k(\gamma) \) denote the number of \( + \) labels minus the number of 
\( - \) labels. If \( \gamma \) is supported on \( [i,j] \) and \( k(\gamma) = k \), we say that \( \gamma \) is a **\( (p,q) \)-clan**, where 
\( p = \frac{1}{2}(i + j + k - 1) \) and \( q = \frac{1}{2}(i + j - k - 1) \). Note that \( k = p - q \) and \( i + j - 1 = p + q \);
moreover, a backstable clan $\gamma$ supported on $[i, j]$ with $k(\gamma) = p - q$ and $i + j - 1 = p + q$ is a $(p, q)$-clan.

For a backstable clan $\gamma$, we write $\gamma(i) = j$ if $i$ is matched with $j$ in $\gamma$. Write $\pm$ to denote an unspecified element of $\{+, -\}$. We write $\gamma(i) = \pm$ if $i$ is unmatched in $\gamma$ and labeled with $\pm \in \{+, -\}$. If $i \in \mathbb{Z}$ is matched, we say that $i$ is initial if $\gamma(i) > i$ and final if $\gamma(i) < i$. A backstable clan $\gamma$ is noncrossing if we never have $a < b < c < d \in \mathbb{Z}$ with $\gamma(a) = c$ and $\gamma(b) = d$.

For each pair of integers $p, q \in \mathbb{Z}$, the rainbow clan $\Omega_{p,q}$ is the $(p, q)$-clan such that

$$
\Omega_{p,q}(i) = \begin{cases} 
+ & \text{if } i \in [q+1, p]; \\
- & \text{if } i \in [p+1, q]; \\
p+q+1-i & \text{otherwise.}
\end{cases}
$$

See Figure 1 for some examples. Note that the rainbow clan is always noncrossing and never has both $+$ and $-$ appearing.

**Figure 1.** The rainbow clans $\Omega_{5,2}$ (left) and $\Omega_{0,1}$ (right).

For each generator $T_i$ of the 0-Hecke algebra $\mathcal{H}_\mathbb{Z}$, we define an action of $T_i$ on $(p, q)$-clans. This action is defined through various cases; however, all have the flavor of acting locally at the numbers $i$ and $i + 1$ and of transforming the $(p, q)$-clan to more closely resemble the rainbow clan $\Omega_{p,q}$. Precisely, for a clan $\gamma$, we have:

- if $\gamma(i) = \pm$ and $i + 1$ is initial, then $(T_i \cdot \gamma)(i) = \gamma(i + 1)$ and $(T_i \cdot \gamma)(i + 1) = \pm$;
- if $i$ is final and $\gamma(i + 1) = \pm$, then $(T_i \cdot \gamma)(i) = \pm$ and $(T_i \cdot \gamma)(i + 1) = \gamma(i)$;
- if $i$ and $i + 1$ are both initial with $\gamma(i) > \gamma(i + 1)$, then $(T_i \cdot \gamma)(i) = \gamma(i + 1)$ and $(T_i \cdot \gamma)(i + 1) = \gamma(i)$;
- if $i$ and $i + 1$ are both final with $\gamma(i) < \gamma(i + 1)$, then $(T_i \cdot \gamma)(i) = \gamma(i + 1)$ and $(T_i \cdot \gamma)(i + 1) = \gamma(i)$;
- if $\gamma(i) = \pm$ and $\gamma(i + 1) = \mp$, then $(T_i \cdot \gamma)(i) = i + 1$ and $(T_i \cdot \gamma)(i + 1) = i$;

in all other cases, $T_i$ acts trivially. Since this action respects the braid relations of Equation (2) by [Wys13, p. 839], we obtain an action of each 0-Hecke element $T_w$. (Wyser only considers $(p, q)$-clans supported on $[1, p + q]$; however, by translating $[i, j]$-clans to be supported on $[1, j - i + 1]$, the general result is immediate.) Examples of the 0-Hecke action on backstable clans are shown in Figure 2.

We say that a $(p, q)$-clan $\gamma$ is an almost rainbow clan if $T_i \cdot \gamma = \Omega_{p,q}$ for at least one $i \in \mathbb{Z}$. Examples of almost rainbow clans are depicted in Figure 3. Write $\omega_p$ for the almost rainbow $(p, p)$-clan with $\omega_p(p) = -$ and $\omega_p(p + 1) = +$. For example, $\omega_3$ is illustrated in the center of Figure 3.

Suppose that $u \in S_\mathbb{Z}$ is $p$-inverse Grassmannian and $v \in S_\mathbb{Z}$ is $q$-inverse Grassmannian. We define a noncrossing backstable clan $\gamma_{u,v}$ associated to the pair $(u, v)$. Write $\hat{p} = p + \frac{1}{2}$ and $\hat{q} = q + \frac{1}{2}$. Then $\gamma_{u,v}$ is the unique noncrossing $(p, q)$-clan such that

$$
\gamma_{u,v}(i) = \begin{cases} 
0 & \text{if } i \in [1, \hat{p}]; \\
1 & \text{if } i \in [\hat{p} + 1, \hat{q}]; \\
\hat{p} + \hat{q} - i & \text{otherwise}.
\end{cases}
$$
Figure 2. Some examples of the Hecke action on clans.

Figure 3. Some representative almost rainbow clans.

(1) if \( u(i) < \hat{p} \) and \( v(i) < \hat{q} \), then \( i \) is initial;
(2) if \( u(i) > \hat{p} \) and \( v(i) > \hat{q} \), then \( i \) is final;
(3) if \( u(i) < \hat{p} \) and \( v(i) > \hat{q} \), then \( \gamma_{u,v}(i) = + \); and
(4) if \( u(i) > \hat{p} \) and \( v(i) < \hat{q} \), then \( \gamma_{u,v}(i) = - \).

Lemma 2.1. The backstable clan \( \gamma_{u,v} \) is well-defined.

Proof. By assumption, there exists \( N > 0 \) such that \( u(i) = i \) and \( v(i) = i \) for \( i \geq N \) and for \( i \leq -N \). Therefore, all \( i < -N \) are initial and all \( i > N \) are final. Since these sets are then both countably infinite, there is a unique noncrossing way to pair them up. Thus, \( \gamma_{u,v} \) is a backstable clan; it remains to show that it is a \((p,q)\)-clan.

Choose an interval \([i,j]\) on which \( \gamma_{u,v} \) is supported. Suppose \( p \geq q \). By expanding the interval \([i,j]\) as necessary, assume that \( i \leq q \) and \( p \leq j \).

On the interval \([i,j]\), define
\[
A = \{ z \in [i,j] : z \text{ is initial} \},
B = \{ z \in [i,j] : z \text{ is final} \},
C = \{ z \in [i,j] : \gamma_{u,v}(z) = + \}, \text{ and }
D = \{ z \in [i,j] : \gamma_{u,v}(z) = - \}
\]
and define \( a = |A|, b = |B|, c = |C|, \) and \( c = |D| \).

On the interval \([i,j]\), both \( u \) and \( v \) take all of the values in \([i,j]\). We see that the \( z \in [i,j] \) with \( u(z) \in [i,p] \) are those \( z \in A \cup C \), while those \( z \) with \( u(z) \in [p+1,j] \) are those \( z \in B \cup D \). Therefore, \( a + c = p - i + 1 \) and \( b + d = j - p \). Similarly, the \( z \in [i,j] \) with \( v(z) \in [i,q] \) are those \( z \in A \cup D \), while those \( z \) with \( v(z) \in [q+1,j] \) are those \( z \in B \cup C \). Therefore, \( a + d = q - i + 1 \) and \( b + c = j - q \).
By the definition of “supported,” the interval \([i, j]\) contains equal numbers of initial and of final elements, so \(a = b\). Thus, \(k(\gamma_{u,v}) = c - d = j - q - (j - p) = p - q\), as desired for a \((p, q)\)-clan. Moreover, we have \(p - i + 1 - (j - q) = 0\), so \(p + q = i + j - 1\), and so \(\gamma_{u,v}\) is a \((p, q)\)-clan.

The case \(p < q\) is entirely analogous; we omit the details.

In the case that \(p\) and \(q\) are positive, \(u, v \in S_{p+q}\), and \(u \leq w_0^{(p+q)} v\), Lemma 2.1 was previously established by Wyser [Wys13, p. 840]. See Figure 4 for an example of the backstable clan \(\gamma_{u,v}\).

**Figure 4.** The clan \(\gamma_{u,v}\) for \(u = 12435\) and \(v = 13425\). Here, \(p = 3, q = 2, \hat{p} = 3.5,\) and \(\hat{q} = 2.5\).

For any positive integers \(p, q \in \mathbb{Z}_{>0}\), we define a set \(\Psi_{p,q}\) of almost rainbow \((p, q)\)-clans. Let

\[
\Psi_{p,q} = \begin{cases} 
\{ \gamma : \gamma \text{ is almost rainbow and } T_i \cdot \gamma = \Omega_{p,q} \text{ for some } i \in \mathbb{Z}_{>0} \text{ with } i \neq q \}, & \text{if } p \neq q; \\
\{ \gamma : \gamma \text{ is almost rainbow and } T_i \cdot \gamma = \Omega_{p,q} \text{ for some } i \in \mathbb{Z}_{>0} \text{ with } i \neq q \} \cup \{ \omega_p \}, & \text{if } p = q.
\end{cases}
\]

Note that in general \(\Psi_{p,q} \neq \Psi_{q,p}\). If we relax the conditions \(i \in \mathbb{Z}_{>0}\) to \(i \in \mathbb{Z}\), then the enlarged set of almost rainbow \((p, q)\)-clans can be used to compute backstable Schubert structure coefficients (introduced in [LLS21]).

### 3. Linear relations among Schubert structure coefficients

In this section, we establish new linear relations among Schubert structure coefficients. In the first subsection, we derive linear relations among cohomological structure coefficients; we will use these relations in Section 4 to prove Theorem 1.2. In the second subsection, we derive analogous linear relations among \(K\)-theoretic structure coefficients; these relations will not be explored further in the later sections of this paper. The third subsection studies stabilization phenomena to obtain additional linear relations among cohomological structure coefficients; these relations will also be important to the proof of Theorem 1.2 in Section 4. The fourth subsection considers relations obtained by iterating the technique of the first subsection; these relations also will not be studied further in this paper.

#### 3.1. Cohomology

We will need the differential operator \(\nabla : \mathbb{Z}[x_1, x_2, \ldots] \to \mathbb{Z}[x_1, x_2, \ldots]\) defined by

\[
\nabla = \sum_{i=1}^{\infty} \frac{\partial}{\partial x_i}.
\]

Our key tool will be the following, developed earlier in joint work with Z. Hamaker and D. Speyer.

**Proposition 3.1 (HPSW20 Proposition 1.1).** For \(w \in S_+\), we have

\[
\nabla \mathcal{S}_w = \sum_{s_k w < w} k \mathcal{S}_{skw}.
\]

From this proposition, we can establish the linear relations of Theorem 1.3.
Proof of Theorem 1.3. Write
\[ S_u \cdot S_v = \sum_p c_{u,v}^p S_p \]
and apply the differential operator \( \nabla \) to both sides to obtain
\[ (3.1) \sum_{s_i u < u} iS_{s_i u} S_v + \sum_{s_j v < v} jS_u S_{s_j v} = \sum_p \sum_{s_k p < p} k c_{u,v}^p S_{s_k p} \]
by Proposition 3.1 and the Leibniz formula. Now extract the coefficient of \( S_w \) from both sides of Equation (3.1) to obtain
\[ \sum_{s_i u < u} ic_{s_i u,v} + \sum_{s_j v < v} jc_{u,s_j v} = \sum_{s_k w > w} k c_{u,v}^{s_k w} , \]
as desired. □

Theorem 1.3 has some surprising corollaries. The following result can be extracted straightforwardly from Monk’s formula [Mon59], but we can alternatively derive it easily from Theorem 1.3.

Corollary 3.2. Let \( v \in S_+ \) and let \( i \in \mathbb{Z}_+ \). Then there is some \( k \in \mathbb{Z}_+ \) such that \( c_{s_i,v}^{s_k} > 0 \).

Proof. Specialize Theorem 1.3 to the case \( u = s_i \) and \( w = v \). Then we get
\[ i + \sum_{s_j v < v} jc_{s_i,s_j v} = \sum_{s_k v > v} k c_{s_i v}^{s_k} . \]
Since the sum on the left is nonnegative and \( i > 0 \), we obtain
\[ 0 < \sum_{s_k v > v} k c_{s_i v}^{s_k} . \]
But then
\[ 0 < \sum_{s_k v > v} c_{s_i v}^{s_k} , \]
so there is some \( k \) with \( s_k v > v \) and \( c_{s_i,v}^{s_k} > 0 \). By dimension counting, the second of these conditions implies the first, so the corollary follows. □

Theorem 1.3 also implies many congruence relations among Schubert structure coefficients.

Corollary 3.3. Suppose all left descents of \( u \) and \( v \) are multiples of \( \alpha \in \mathbb{Z}_+ \). Then, for any \( w \in S_+ \), we have
\[ \sum_{s_k w > w} k c_{u,v}^{s_k w} \equiv 0 \pmod{\alpha} . \]
If moreover \( u = v \), then
\[ \sum_{s_k w > w} k c_{u,u}^{s_k w} \equiv 0 \pmod{2\alpha} . \]

Proof. Under the hypotheses of the corollary, every term of each sum on the left of Equation (1.1) is a multiple of \( \alpha \). Hence, the sum on the right is as well. If also \( u = v \), then the two sums on the left of Equation (1.1) are equal to each other. □
What is remarkable about Corollary 3.3 is that although our sum is congruent to 0 modulo \( \alpha \), the individual terms of the sum generally are not. For this reason, knowing some of the relevant Schubert structure coefficients imposes strong conditions on the remaining ones. Before giving an example of the application of Corollary 3.3 we need the following easy lemma.

**Lemma 3.4.** Suppose \( u, v, a \in S_n \) and let \( m > n \). Then \( c_{m\alpha}^{\alpha} = 0 \).

**Proof.** The Schubert polynomials \( \mathcal{G}_u, \mathcal{G}_v, \mathcal{G}_a \) all lie in \( \mathbb{Z}[x_1, \ldots, x_{n-1}] \). On the other hand, \( m \) is a (right) descent of \( m_\alpha \), so \( \mathcal{G}_{m_\alpha} \) involves the variable \( x_m \), so \( \mathcal{G}_{m_\alpha} \notin \mathbb{Z}[x_1, \ldots, x_{n-1}] \).

Hence, \( c_{m\alpha}^{\alpha} = 0 \).

**Example 3.5.** Suppose \( u = 13254 \) and note that it only has left descents 2 and 4. Let \( w = 231645 \) and suppose we have correctly computed already that \( c_{u,u}^w = 1 \).

Now let \( a = s_1 w = 132645 \). Corollary 3.3 gives that

\[
\sum_{s_h \alpha > a} k c_{u,u}^a \equiv 0 \pmod{4}.
\]

But we can expand this sum as

\[
\sum_{s_h \alpha > a} k c_{u,u}^a = 1 c_{u,u}^{s_1} + 3 c_{u,u}^{s_2} + 4 c_{u,u}^{s_4} + 6 c_{u,u}^{s_6},
\]

using Lemma 3.4 to see that the other potential terms vanish. Since we know the term with coefficient 1 contributes 1, the sum of the other terms must be congruent to 3 modulo 4. In particular, we learn for free that \( c_{u,u}^{s_2} \neq 0 \). Even better, it is immediate without further computation that \( c_{u,u}^{s_6} = 1 \).

If we compute also that \( c_{u,u}^{s_4} = 1 \), we learn then that \( c_{u,u}^{s_6} \) must be even. In fact, it turns out that \( c_{u,u}^{s_6} = 0 \).

\( \diamond \)

### 3.2. \( K \)-theory.

The structure sheaves of Schubert varieties \( X_w \subset \text{Flags}_n \) give classes \( [\mathcal{O}_{X_w}] \) in the Grothendieck ring \( \text{K}^0(\text{Flags}_n) \) of algebraic vector bundles over \( \text{Flags}_n \). These classes form an additive basis and give rise to \( K \)-theoretic Schubert structure coefficients \( K_{u,v}^w \), defined by

\[
[\mathcal{O}_{X_u}] \cdot [\mathcal{O}_{X_v}] = \sum_w K_{u,v}^w [\mathcal{O}_{X_w}].
\]

When \( \ell(w) = \ell(u) + \ell(v) \), we have \( K_{u,v}^w = c_{u,v}^w \), but, unlike \( c_{u,v}^w \), \( K_{u,v}^w \) can be nonzero when \( \ell(w) > \ell(u) + \ell(v) \).

**Grothendieck polynomials** \( \mathcal{G}_w \) represent Schubert structure sheaf classes analogously to how Schubert polynomials represent cohomological Schubert classes. We may also define Grothendieck polynomials recursively. For \( w_0^{(n)} \in S_n \), we set \( \mathcal{G}_w^{(n)} = \mathcal{G}_w^{(n)} = x_1^{n-1} x_2^{n-2} \cdots x_n^0 \).

For \( w \) such that \( ws_i < w \), set

\[
\mathcal{G}_{ws_i} = \mathcal{N}_i \mathcal{G}_w,
\]

where \( \mathcal{N}_i(f) = N_i((1-x_{i+1})f) \). We have \( \mathcal{G}_w = \mathcal{G}_{i(w)} \), so we think of Grothendieck polynomials as also being indexed by elements of \( S_n \). The set of Grothendieck polynomials \( \{ \mathcal{G}_w \}_{w \in S_n} \) is another linear basis of \( \mathbb{Z}[x_1, x_2, x_3, \ldots] \). The structure coefficients defined by

\[
\mathcal{G}_u \cdot \mathcal{G}_v = \sum_w I_{u,v}^w \mathcal{G}_w
\]

agree with the \( K \)-theoretic Schubert structure coefficients \( K_{u,v}^w \) provided \( u, v, w \in S_n \).
Let $\beta$ be an indeterminate. Define the \textbf{$\beta$-Grothendieck polynomial} $G^{(\beta)}_w$ by

$$G^{(\beta)}_w(x_1, \ldots, x_n) = (-\beta)^{-\ell(w)} G_w(-\beta x_1, \ldots, -\beta x_n).$$

The $\beta$-Grothendieck polynomials were introduced in [FK94] and represent classes in the connective $K$-theory of Flags$_n$ [Hud14]. We will find the $\beta$-Grothendieck polynomials slightly easier to work with in our context. Let the structure coefficients for $\beta$-Grothendieck polynomials be $K^{\beta}_w u, v$.

Let Des$(w)$ denote the set of descents of the permutation $w$. The \textbf{major index} of $w$ is

$$\text{maj}(w) = \sum_{i \in \text{Des}(w)} i.$$

We also need the following differential operators related to $\nabla$:

$$\nabla^{\beta} = \nabla + \beta^2 \frac{\partial}{\partial \beta} \quad \text{and} \quad E = \sum_{i=1}^{\infty} x_i \frac{\partial}{\partial x_i}.$$

We can now recall [PSW21, Theorem A.1], as reformulated in [PSW21, Remark A.2], an analogue of Proposition 3.1 for Grothendieck polynomials and our key tool in this subsection.

\textbf{Proposition 3.6 (PSW21, Theorem A.1).} For $w \in S_+$, we have

$$\nabla^{\beta} \mathcal{G}^{(\beta)}_w = \beta(\text{maj}(w^{-1}) - \ell(w)) \mathcal{G}^{(\beta)}_w + \sum_{s_k w < w} k \mathcal{G}^{(\beta)}_{s_k w}$$

and

$$(\text{maj}(w^{-1}) + \nabla - E) \mathcal{G}_w = \sum_{s_k w < w} k \mathcal{G}_{s_k w}.$$

Using essentially the same proof as for Theorem 1.3 but with [PSW21, Theorem A.1] in place of [HPSW20, Proposition 1.1], we obtain the following analogue of Theorem 1.3 giving linear relations among $K$-theoretic Schubert structure coefficients.

\textbf{Theorem 3.7.} Let $u, v, w \in S_+$. Then

$$\beta K^{w}_u v(\beta) \left( \text{maj}(u^{-1}) + \text{maj}(v^{-1}) - \text{maj}(w^{-1}) - \ell(u) - \ell(v) + \ell(w) \right)$$

$$+ \sum_{s_i u < u} i K^{w}_{s_i u v}(\beta) + \sum_{s_j v < v} j K^{w}_{u s_j v}(\beta) = \sum_{s_k w > w} k K^{s_k w}_u v(\beta).$$

\textbf{Proof.} Write

$$\sum_p K^{p}_u v(\beta) \mathcal{G}^{(\beta)}_p = \mathcal{G}^{(\beta)}_u \cdot \mathcal{G}^{(\beta)}_v$$

and apply $\nabla^{\beta}$ to both sides, using the first part of Proposition 3.6. Then on the left we have

$$\nabla^{\beta} \left( \sum_p K^{p}_u v(\beta) \mathcal{G}^{(\beta)}_p \right) = \sum_p K^{p}_u v(\beta) \nabla^{\beta} \mathcal{G}^{(\beta)}_p$$

$$= \sum_p K^{p}_u v(\beta) \left( \beta \mathcal{G}^{(\beta)}_p (\text{maj}(p^{-1}) - \ell(p)) + \sum_{s_k p < p} k \mathcal{G}^{(\beta)}_{s_k p} \right),$$
while on the right we have
\[ \nabla^\beta (\mathcal{G}_u^{(\beta)} \cdot \mathcal{G}_v^{(\beta)}) = \nabla^\beta (\mathcal{G}_u^{(\beta)} \mathcal{G}_v^{(\beta)}) + \mathcal{G}_u^{(\beta)} (\nabla^\beta \mathcal{G}_v^{(\beta)}) \]
\[ = \left( \beta \mathcal{G}_u^{(\beta)} (\text{maj}(w^{-1}) - \ell(u)) + \sum_{s_i u < u} i \mathcal{G}_u^{(\beta)} \mathcal{G}_v^{(\beta)} \right) \]
\[ + \mathcal{G}_u^{(\beta)} \left( \beta \mathcal{G}_v^{(\beta)} (\text{maj}(w^{-1}) - \ell(v)) + \sum_{s_j v < v} j \mathcal{G}_s^{(\beta)} \right). \]

Now we can extract the coefficient of $\mathcal{G}_w^{(\beta)}$ from both of these obtain
\[ \sum_{s_k w > w} k K^{s_k w}_{u,v}(\beta) + \beta (\text{maj}(w^{-1}) - \ell(w)) K^{w}_{u,v}(\beta) = \beta (\text{maj}(w^{-1}) - \ell(v)) K^{w}_{u,v}(\beta) \]
\[ + \beta (\text{maj}(u^{-1}) - \ell(u)) K^{w}_{u,v}(\beta) + \sum_{s_i u < u} i K^{w}_{s_i u,v}(\beta) + \sum_{s_j v < v} j K^{w}_{s_i u v}(\beta). \]

The theorem follows by rearranging and collecting terms. \qed

Let $\mathcal{G}_w(1)$ denote the specialization of the Grothendieck polynomial $\mathcal{G}_w$ obtained by setting all variables equal to 1. It is well known to experts that $\mathcal{G}_w(1) = 1$ (see [ST21] for an explicit proof and [MSS22] for further discussion). We present a new short proof using the second part of Proposition 3.6.

**Corollary 3.8.** Given $w \in S_+$, $\mathcal{G}_w(1) = 1$.

**Proof.** We proceed by induction on Coxeter length. In the base case, $\mathcal{G}_{id} = 1$ and there is nothing to show. Now fix $w \in S_+$ with $\ell(w) \geq 1$ and assume the statement holds for all $v \in S_+$ with $\ell(v) < \ell(w)$.

First note, for any $f \in \mathbb{Z}[x_1, x_2, \ldots]$, we have $(\nabla - E)(f)|_{x=1} = 0$. Thus,
\[ \text{maj}(w^{-1}) \mathcal{G}_w(1) = \sum_{s_k w < w} k \mathcal{G}_{s_k w}(1) \quad \text{(by Proposition 3.6)} \]
\[ = \sum_{s_k w < w} k \quad \text{(by induction)} \]
\[ = \text{maj}(w^{-1}). \]

Since $\ell(w) \geq 1$, $\text{maj}(w^{-1}) \neq 0$ which implies $\mathcal{G}_w(1) = 1$. \qed

### 3.3. Stabilization

For a permutation $w \in S_n$, let $1 \times w \in S_{n+1}$ denote the permutation such that $(1 \times w)(1) = 1$ and $(1 \times w)(i) = w(i - 1) + 1$ for $i > 1$. Iterating this operation gives rise to the notion of backstabilization of Schubert calculus; for further discussion, see [LLS21, Nen20]. We will, however, only need to apply this operation once. The following fact is straightforward; moreover, it is a special case of Lemma 4.6 which we prove later.

**Lemma 3.9.** For any $u, v, w \in S_+$, we have $c^w_{u,v} = c^1_{u,v} \times 1$. \qed

The following is an analogue of Theorem 1.3 we drop the coefficients on the linear relations at the expense of adding one extra term.
Proposition 3.10. Let \( u, v, w \in S_+ \). Then

\[
\sum_{s_i < u} c_{s_i u, v}^w + \sum_{s_j < v} c_{u, s_j v}^w = c_{1 \times u, 1 \times v}^w + \sum_{s_k > w} c_{s_k w}^w.
\]

Proof. It follows easily from the pipe dream formula for Schubert polynomials (e.g., [FK96, BB93, KM05]) that

\[
\mathcal{G}_{1 \times w}(0, x_1, x_2, \ldots) = \mathcal{G}_w(x_1, x_2, \ldots).
\]

(Indeed, we will prove a stronger version of this statement as Lemma 4.5.) By Lemma 3.9

\[c_{u, v}^w = c_{1 \times u, 1 \times v}^w.\]

Apply Theorem 1.3 to \( 1 \times u, 1 \times v, \) and \( 1 \times w \). Then we get

\[
\sum_{s_{i+1}(1 \times u) < 1 \times u} (i+1)c_{s_{i+1}(1 \times u), 1 \times v}^w + \sum_{s_{j+1}(1 \times v) < 1 \times v} (j+1)c_{1 \times u, s_{j+1}(1 \times v)}^w = \sum_{s_{k+1}(1 \times w) > 1 \times w} (k+1)c_{1 \times u, 1 \times v}^w.
\]

Thus,

\[
\sum_{s_i < u} (i+1)c_{s_i u, v}^w + \sum_{s_j < v} (j+1)c_{u, s_j v}^w = c_{1 \times u, 1 \times v}^w + \sum_{s_k > w} c_{s_k w}^w.
\]

Furthermore, from applying Theorem 1.3 to \( u, v, \) and \( w \), we have

\[
\sum_{s_i < u} ic_{s_i u, v}^w + \sum_{s_j < v} jc_{u, s_j v}^w = \sum_{s_k > w} kc_{s_k w}^w.
\]

Therefore, subtracting Equation (3.6) from Equation (3.5) yields

\[
\sum_{s_i < u} c_{s_i u, v}^w + \sum_{s_j < v} c_{u, s_j v}^w = c_{1 \times u, 1 \times v}^w + \sum_{s_k > w} c_{s_k w}^w. \tag*{□}
\]

In many cases, one can see that the extra term on the right side of Proposition 3.10 is in fact 0. In these situations, Proposition 3.10 becomes identical to Theorem 1.3 but with the coefficients dropped, yielding two independent linear relations among the same structure coefficients.

One can also do analogous analysis in \( K \)-theory; we omit the details, since we will not use the \( K \)-theoretic analogue in what follows.

3.4. Iterations of differential operators. Iterating the application of \( \nabla \) allows us to obtain additional linear relations among Schubert structure coefficients. These linear relations are somewhat more complicated to state, but the proof is analogous to that of Theorem 1.3

Let \( \mathcal{G}_w(1) \) denote the specialization of the Schubert polynomial \( \mathcal{G}_w \) obtained by setting all variables equal to 1. For \( w \) with \( \ell(w) = k \), a reduced word for \( w \) is a sequence \( (a_1, a_2, \ldots, a_k) \) of positive integers such that \( w = s_{a_1} s_{a_2} \cdots s_{a_k} \). Let \( \mathring{R}(w) \) denote the set of all reduced words for \( w \). The Macdonald reduced word identity [Mac91, Eq. (6.11)] is the following.

Proposition 3.11 ([Mac91, Eq. (6.11)]). Let \( w \in S_+ \) have \( \ell(w) = k \). Then

\[
k! \mathcal{G}_w(1) = \sum_{a \in \mathring{R}(w)} a_1 a_2 \cdots a_k.
\]

We can use the Macdonald reduced word identity to obtain more linear relations.
Proposition 3.12. Let $u, v \in S_+$ and fix $1 \leq k \leq \ell(u) + \ell(v)$. Let $w \in S_+$ with
\[ \ell(w) = \ell(u) + \ell(v) - k. \]

Then,
\begin{equation}
\sum_{w \geq_{L} w} c_{u,v}^{w} \cdot \mathcal{S}_{w^w-1}(1) = \sum_{i=0}^{k} \sum_{\bar{u} \leq_{L} u} \sum_{\bar{v} \leq_{L} v} c_{\bar{u}, \bar{v}}^{w} \cdot \mathcal{S}_{u\bar{u}-1}(1) \cdot \mathcal{S}_{v\bar{v}-1}(1).
\end{equation}

Proof. First note that
\[ \nabla^{k}(\mathcal{S}_{\pi}) = \sum_{\ell(\hat{\pi})=\ell(\pi)-k} \left( \sum_{a \in R(\pi \hat{\pi}^{-1})} a_1 a_2 \cdots a_k \right) \cdot \mathcal{S}_{\hat{\pi}} \]
by Proposition 3.1. By Proposition 3.11, we can replace the inner summation to obtain
\begin{equation}
\nabla^{k}(\mathcal{S}_{\pi}) = \sum_{\ell(\hat{\pi})=\ell(\pi)-k} k! \mathcal{S}_{\pi \hat{\pi}^{-1}}(1) \cdot \mathcal{S}_{\hat{\pi}}.
\end{equation}

Write
\[ \sum_{\hat{p}} c_{u,v}^{\hat{p}} \mathcal{S}_{\hat{p}} = \mathcal{S}_{u} \cdot \mathcal{S}_{v}. \]

Applying $\nabla^{k}$ to both sides, we obtain
\[ \nabla^{k} \left( \sum_{\hat{p}} c_{u,v}^{\hat{p}} \mathcal{S}_{\hat{p}} \right) = \nabla^{k}(\mathcal{S}_{u} \cdot \mathcal{S}_{v}) = \sum_{i=0}^{k} \binom{k}{i} \nabla^{i}(\mathcal{S}_{u}) \nabla^{k-i}(\mathcal{S}_{v}). \]

Thus, Equation (3.8) yields
\begin{align*}
\sum_{\hat{p}} c_{u,v}^{\hat{p}} 
&= \sum_{\ell(\hat{p})=\ell(\pi)-k} k! \mathcal{S}_{\pi \hat{\pi}^{-1}}(1) \mathcal{S}_{\hat{p}} \\
&= \sum_{i=0}^{k} \binom{k}{i} \left( \sum_{\bar{u} \leq_{L} u} i! \mathcal{S}_{u\bar{u}-1}(1) \mathcal{S}_{\bar{u}} \right) \left( \sum_{\bar{v} \leq_{L} v} (k-i)! \mathcal{S}_{v\bar{v}-1}(1) \mathcal{S}_{\bar{v}} \right) \\
&= \sum_{i=0}^{k} k! \sum_{\bar{u} \leq_{L} u} \sum_{\bar{v} \leq_{L} v} \mathcal{S}_{u\bar{u}-1}(1) \mathcal{S}_{v\bar{v}-1}(1) \sum_{q} c_{\bar{u}, \bar{v}}^{q} \mathcal{S}_{q}.
\end{align*}

Extract the coefficient of $\mathcal{S}_{w}$ from each side to obtain
\[ k! \sum c_{u,v}^{w} \mathcal{S}_{w^w-1}(1) = k! \sum_{i=0}^{k} \sum_{\bar{u} \leq_{L} u} \sum_{\bar{v} \leq_{L} v} \mathcal{S}_{u\bar{u}-1}(1) \mathcal{S}_{v\bar{v}-1}(1). \]

The proposition follows by dividing out $k!$.

□
Theorem 1.3 may alternatively be proved as a corollary to Proposition 3.12 by setting $k = 1$. On the other extreme, we have the following corollary. Define

$$
\delta(w, i) = \begin{cases} 
1, & \text{if } i \in \Des(w); \\
0, & \text{if } i \notin \Des(w).
\end{cases}
$$

Corollary 3.13. Let $u, v \in S_+$ and fix $i \in \mathbb{Z}_{>0}$. Then,

$$
(3.9) \quad \sum_{p : i \in \Des(p)} c_{u,v}^p \mathcal{S}_{p_{u_i}}(1) = \delta(v, i) \mathcal{S}_{u_i}(1) \mathcal{S}_{v_{u_i}}(1) + \delta(u, i) \mathcal{S}_{u_{u_i}}(1) \mathcal{S}_{v}(1).
$$

Proof. In Proposition 3.12 take $w = s_i$ and $k = \ell(u) + \ell(v) - 1$. The left side of Equation (3.7) becomes the left side of Equation (3.9). Most of the terms on the right side of Equation (3.7) vanish by degree considerations, leaving only the terms on the right side of Equation (3.9).

The following was observed as [Knu01, Lemma 1.1], where the phenomenon was referred to as dc-triviality. We obtain another easy proof of this fact.

Corollary 3.14 ([Knu01, Lemma 1.1]). Let $u, v \in S_+$ and suppose $i \notin \Des(u) \cup \Des(v)$. Then, $c_{u,v}^p = 0$ for all $p \in S_+$ with $i \in \Des(p)$.

Proof. In Corollary 3.13 both terms on the right side vanish under these hypotheses. The left side is a sum of nonnegative integers, so all terms on the left side also vanish. Since each specialization $\mathcal{S}_{p_{u_i}}(1)$ is strictly positive, all the relevant $c_{u,v}^p$ equal zero.

4. Littlewood–Richardson Rules

In this section, we prove our main results, Theorem 1.1 and Theorem 1.2.

4.1. Discussion and proof of Theorem 1.1. First, we recall [Wys13, Theorem 3.10], which establishes the case of Theorem 1.1 where $u, v, w \in S_{p+q}$ and $u \leq w_0^{(p+q)} v$. We will use this result to prove Theorem 1.1. Theorem 1.1 will then be a major ingredient in our proof of Theorem 1.2 in the next subsection.

Proposition 4.1 ([Wys13, Theorem 3.10]). Let $u, v, w \in S_{p+q}$ be permutations, where $u$ is $p$-inverse Grassmannian, $v$ is $q$-inverse Grassmannian, and $u \leq w_0^{(p+q)} v$. Then $c_{u,v}^w \in \{0, 1\}$. Moreover, we have $c_{u,v}^w = 1$ if and only if $\ell(w) = \ell(u) + \ell(v)$ and

$$
T_w \cdot \gamma_{u,v} = \Omega_{p,q}.
$$

Remark 4.2. Note that the hypotheses of Proposition 4.1 are somewhat restrictive. For instance, Proposition 4.1 is unable to compute any structure coefficient of the form $c_{231,231}^w$, since 231 is 1-inverse Grassmannian but $231 \notin S_{1+1}$. Example 4.3 demonstrates how we can instead compute these structure coefficients using backstable clans and Theorem 1.1.

Similarly, Proposition 4.1 cannot compute any of the structure coefficients $c_{213,312}^w$ because $213 \notin w_0^{(3)} 312 = 132$. See Example 4.4 for a demonstration of computing these structure coefficients through backstable clans and Theorem 1.1.

We can now establish our first main result.
Proof of Theorem 1.1. Fix an interval \([i, j]\) on which \(\gamma_{u,v}\) is supported. If \(i > 1\), then \(\gamma_{u,v}\) is also supported on the interval \([1, p+q]\). Therefore, \(\gamma_{u,v}\) is a noncrossing \([1, p+q]\)-clan, so by [Wys13] Remark 3.9, we have \(u, v \in S_{p+q}\) and \(u \leq w_{0}^{(p+q)}v\). We are then done in this case by Proposition 4.1.

Suppose instead that \(i < 1\). Then, let \(\tilde{\gamma}\) be the horizontal shift of \(\gamma_{u,v}\) to the right by \(1 - i\). That is, let \(\tilde{\gamma}(z) = \gamma_{u,v}(z + i - 1)\), so that \(\tilde{\gamma}\) is supported on \([1, j - i + 1]\). Also define \(\tilde{u}\) and \(\tilde{v}\) by

\[
\tilde{u}(z) = u(z + i - 1) \quad \text{and} \quad \tilde{v}(z) = v(z + i - 1).
\]

Then, \(\tilde{u}\) is \(\tilde{p}\)-inverse Grassmannian and \(\tilde{v}\) is \(\tilde{q}\)-inverse Grassmannian, where \(\tilde{p} = p + 1 - i\) and \(\tilde{q} = q + 1 - i\). Note that, since \(p + q = i + j - 1\), we have \(\tilde{p} + \tilde{q} = j - i + 1\). Further, observe that \(\gamma_{\tilde{u}, \tilde{v}} = \tilde{\gamma}\) by construction.

The clan \(\gamma_{\tilde{u}, \tilde{v}}\) is a noncrossing clan supported on \([1, \tilde{p} + \tilde{q}]\). Therefore, by [Wys13] Remark 3.9, we have \(\tilde{u}, \tilde{v} \in S_{\tilde{p} + \tilde{q}}\) and \(\tilde{u} \leq w_{0}^{(\tilde{p} + \tilde{q})}\tilde{v}\).

Now, define \(\tilde{w}\) by

\[
\tilde{w}(z) = w(z + i - 1).
\]

Note that

\[
(T_{w} \cdot \gamma_{u,v})(z) = (T_{\tilde{w}} \cdot \gamma_{\tilde{u}, \tilde{v}})(z + i - 1)
\]

for all \(z\). This implies that \(T_{w} \cdot \gamma_{u,v} = \Omega_{p,q}\) if and only if \(T_{\tilde{w}} \cdot \gamma_{\tilde{u}, \tilde{v}} = \Omega_{\tilde{p}, \tilde{q}}\).

By Lemma 3.9 we have that

\[
c_{w}^{w} = c_{\tilde{u}, \tilde{v}}^{\tilde{w}},
\]

since \(\tilde{u} = 1^{1-i} \times u\), \(\tilde{v} = 1^{1-i} \times v\), and \(\tilde{w} = 1^{1-i} \times w\). Thus, by Proposition 4.1, we have \(c_{u,v}^{w} \in \{0, 1\}\). Also note that \(\ell(\tilde{u}) = \ell(u)\), \(\ell(\tilde{v}) = \ell(v)\), and \(\ell(\tilde{w}) = \ell(w)\). Thus, Proposition 4.1 additionally yields that \(c_{u,v}^{w} = 1\) if and only if \(T_{w} \cdot \gamma_{u,v} = \Omega_{p,q}\). This completes the proof of Theorem 1.1. \(\Box\)

We now show how Theorem 1.1 uses backstable clans to compute the Schubert structure coefficients described in Remark 4.2.

Example 4.3. Let \(u = 231 \in S_{3}\). The backstable clan \(\gamma_{u,u}\) looks like

\[
\begin{array}{cccc}
-1 & 0 & 1 & 2 \\
\end{array}
\]

We consider all nontrivial actions of 0-Hecke generators \(T_{i}\) on \(\gamma_{u,u}\), until reaching the rainbow clan \(\Omega_{1,1}\).
There are two paths in this diagram from $\gamma_{u,u}$ to $\Omega_{1,1}$ using only $T_i$ with $i > 0$; these paths are labeled by the sequences $(T_2, T_1, T_3, T_2)$ and $(T_2, T_3, T_1, T_2)$, which both correspond to the permutation $3412 = s_2s_3s_1s_2 = s_2s_1s_3s_2$. Thus by Theorem 1.1 we have that $c_{3412}^{3412} = 1$, while $c_{w}^{u,u} = 0$ for all $w \neq 3412$.

**Example 4.4.** Let $u = 213$ and $v = 312$. Then the backstable clan $\gamma_{u,v}$ looks like

There is a unique path in this diagram from $\gamma_{u,v}$ to $\Omega_{1,2}$ using only $T_i$ with $i > 0$, namely that labeled by the sequence $(T_1, T_2, T_3)$. Note that $s_3s_2s_1 = 4123$. We conclude by Theorem 1.1 that $c_{4123}^{4123} = 1$, while $c_{w}^{u,v} = 0$ for all $w \neq 4123$. \(\diamond\)
4.2. Discussion and proof of Theorem 4.2. For a permutation $w \in S_+$ and positive integer $c$, we define $\tau^{-1}(w)$ to be $s_\epsilon \ldots s_2 s_1(1 \times w)$. For a permutation $w \in S_+$, we define the truncation $\tau(w)$ to be the permutation such that $\tau^{-1}(w) = w$. Note that $\tau(w)$ is the unique element of $S_+$ such that

$$c_i(\tau(w)) = \begin{cases} c_{i+1}(w), & \text{if } i > 0; \\ 0, & \text{otherwise.} \end{cases}$$

The following lemmas are closely related to ideas of [BS98, LRS06] and are known to experts; we include (sketches of) proofs for completeness.

Lemma 4.5. For any $w \in S_n$, we have

$$\mathcal{S}_w(1, x_1, x_2, \ldots, x_{n-1}) = \mathcal{S}_{\tau(w)}(1, x_2, \ldots, x_n) + \text{lower degree terms.}$$

Proof. The Schubert polynomial $\mathcal{S}_w$ can be written as a generating function for diagrams $P$ called pipe dreams (cf. [BB93, KM05]), where each $+$ in row $i$ of $P$ contributes the variable $x_i$ to the weight of $P$. Under the specialization of the lemma, the highest-degree terms of $\mathcal{S}_w(1, x_1, x_2, \ldots, x_{n-1})$ come from pipe dreams with as few $+$’s as possible in the first row.

The ladder moves of [BB93] describe a recursive algorithm to generate all pipe dreams for $w$. From this algorithm, it is straightforward that the pipe dreams for $w$ with a minimum number of $+$’s in the first row are identical to the pipe dreams for $\tau(w)$ after deleting their first row and shifting up. \hfill \Box

Lemma 4.6. Let $u, v, w \in S_n$ such that $c_1(w) = c_1(u) + c_1(v)$. Then we have

$$c^w_{u,v} = c^{\tau(w)}_{\tau(u), \tau(v)}.$$

Proof. Write

$$\mathcal{S}_u \mathcal{S}_v = \sum_a c^a_{u,v} \mathcal{S}_a.$$ 

Choose $m$ sufficiently large so that $a \in S_m$ for all $a$ such that $c^a_{u,v} \neq 0$. We may specialize all of the variables in this equation to obtain

$$\mathcal{S}_u(1, x_1, \ldots, x_{m-1}) \mathcal{S}_v(1, x_1, \ldots, x_{m-1}) = \sum_a c^a_{u,v} \mathcal{S}_a(1, x_1, \ldots, x_{m-1}).$$

Now, by Lemma 4.5 applied to all of these Schubert polynomials, we find that

$$\mathcal{S}_{\tau(u)} + f(\mathcal{S}_{\tau(v)} + g) = \sum_a c^a_{u,v}(\mathcal{S}_{\tau(a)} + h_a),$$

where $\deg f < \ell(\tau(u))$, $\deg g < \ell(\tau(v))$ and $\deg h_a < \ell(\tau(a))$ for each $a$. Now observe that $\ell(\tau(a)) = \ell(\tau(u)) + \ell(\tau(v))$ if and only if we have $c_1(a) = c_1(u) + c_1(v)$. Therefore, by extracting the top-degree terms on both sides of Equation (4.1), we obtain

$$\mathcal{S}_{\tau(u)} \mathcal{S}_{\tau(v)} = \sum_{c_1(b) = c_1(u) + c_1(v)} e^b_{u,v} \mathcal{S}_{\tau(b)}.$$

On the other hand, by definition,

$$\mathcal{S}_{\tau(u)} \mathcal{S}_{\tau(v)} = \sum_{d} c^d_{\tau(u), \tau(v)} \mathcal{S}_d.$$
Now, observe that if \( \tau(b_1) = \tau(b_2) \) and \( c_1(b_1) = c_1(b_2) \), then necessarily \( b_1 = b_2 \). Therefore, comparing Equations (1.2) and (4.3) yields

\[
c_{\tau(u), \tau(v)} = c_{u, v}^w,
\]

as desired.

Note that Lemma 3.9 is a special case of Lemma 4.6. With these lemmas in hand, we are now prepared to finish the proof of Theorem 1.2.

Proof of Theorem 1.2. Recall that \( u, v, w \in S_n \) are permutations, where \( u \) is \( p \)-inverse Grassmannian and \( v \) is \( q \)-inverse Grassmannian. Note that this implies that \( p, q > 0 \). Let \( n = p + q \). We consider a few cases in turn.

We can assume that \( \ell(w) = \ell(u) + \ell(v) - 1 \) because otherwise we certainly have \( c_{s_p u, v}^w = 0 \).

(Case 1: \( p \neq q \)): We first consider a technical special case, which we will then be able to extend.

(Case 1.1: \( c_{1 \times 1, 1 \times 1}^w = 0 \)): In this case, we have by Proposition 3.10 that

\[
c_{s_p u, v}^w + c_{u, s_q v}^w = \sum_{s_k w > w} c_{u, v}^w.
\]

and from Theorem 1.3 that

\[
p c_{u, s_q v}^w + q c_{u, s_q v}^w = \sum_{s_k w > w} k c_{u, v}^w.
\]

Thus, multiplying Equation (4.4) by \( q \) and subtracting from Equation (4.5), we find that

\[
(p - q) c_{s_p u, v}^w = \sum_{s_k w > w} (k - q) c_{u, v}^w.
\]

(Case 1.1.1: \( T_w \cdot \gamma_{u, v} \) is not almost rainbow): By Theorem 1.1, \( c_{u, v}^w = 0 \) for all \( k \) such that \( s_k w > w \). Therefore, the right side of Equation (4.6) is 0. Since \( p \neq q \), this implies that \( c_{s_p u, v}^w = 0 \).

(Case 1.1.2: \( T_w \cdot \gamma_{u, v} \) is almost rainbow): Write \( \delta = T_w \cdot \gamma_{u, v} \). We now break into cases according to what sort of almost rainbow clan \( \delta \) is. Observe that \( \Omega_{p, q} \) has a nonzero number of signed unmatched numbers. If \( p < q \), these signs are all \(-\) and appear in positions \( p + 1, \ldots, q \); if \( p > q \), these signs are all \(+\) and appear in positions \( q + 1, \ldots, p \).

(Case 1.1.2.1: \( T_{s_q} \cdot \delta = \Omega_{p, q} \)): We observe that \( T_{s_q} \) must act on \( \delta \) by moving a sign inside an arc (as in the \( T_3 \) or \( T_6 \) arrow of Figure 2). Therefore, we have \( T_{s_r} \cdot \delta = \delta \) for all \( r \neq q \). So, in this case, Equation (4.6) simplifies to

\[
(p - q) c_{s_p u, v}^w = (q - q) c_{u, v}^w = 0.
\]

Since \( p \neq q \), we therefore have \( c_{s_p u, v}^w = 0 \).

(Case 1.1.2.2: \( T_{s_p} \cdot \delta = \Omega_{p, q} \)): We observe that \( T_{s_p} \) must again act on \( \delta \) by moving a sign inside an arc, as in the previous case. Therefore, we have \( T_{s_r} \cdot \delta = \delta \) for all \( r \neq p \). So, in this case, Equation (4.6) simplifies to

\[
(p - q) c_{s_p u, v}^w = (p - q) c_{s_p u, v}^w = 0.
\]

Since \( p \neq q \), we therefore have \( c_{s_p u, v}^w = c_{s_p u, v}^w = 1 \), where the last equality is by Theorem 1.1.
(Case 1.1.2.3: $T_{s_r} \cdot \delta = \Omega_{p,q}$ for some $r \notin \{p,q\}$): In this case, $T_{s_r}$ must act on $\delta$ by uncrossing a pair of adjacent arcs (as in the $T_2$ arrow of Figure 2). Recall that $n-r$ and $n-r+1$ are the labels on the other ends of the crossing arcs from $r, r+1$.

(Case 1.1.2.3.1: $n-r > 0$): In this case, we also have $T_{s_{n-r}} \cdot \delta = \Omega_{p,q}$. Observe that $r \neq n-r$. Moreover, we have $T_{s_t} \cdot \delta = \delta$ for all $t \notin \{r,n-r\}$. By Theorem 1.1,

$$c_{s_{r}w}^{s_{r}w} = 1 = c_{s_{n-r}w}^{s_{n-r}w}. $$

Now, Equation (4.6) simplifies to

$$(p-q)c_{s_{p}u,v}^{w} = (r-q)c_{s_{r}w}^{w} + (n-r-q)c_{s_{n-r}w}^{s_{n-r}w} = (r-q) + (n-r-q) = n-2q = p-q.$$

Thus, $c_{s_{p}u,v}^{w} = 1$.

(Case 1.1.2.3.2: $n-r \leq 0$): Define $\tilde{u} = 1^{r-n+1} \times u, \tilde{v} = 1^{r-n+1} \times v$, and $\tilde{w} = 1^{r-n+1} \times w$. Then $\tilde{u}$ is $\tilde{p}$-inverse Grassmannian, where $\tilde{p} = p + r - n + 1$. Also let $\tilde{\gamma} = \gamma_{\tilde{u},\tilde{v}}$ and notice that $\tilde{\gamma}$ is a horizontal shift of the backstable clan $\gamma_{u,v}$. Therefore, $T_{\tilde{w}} \cdot \tilde{\gamma}$ is a horizontal shift of $T_{\tilde{w}} \cdot \gamma_{u,v} = \delta$. In particular, $T_{\tilde{w}} \cdot \tilde{\gamma}$ is almost rainbow with a pair of crossing arcs. Observe that $s_{\tilde{p}}\tilde{u} = 1^{r-n+1} \times s_{\tilde{p}}u$. By the previous Case 1.1.2.3.1,

$$c_{s_{\tilde{p}}\tilde{u},\tilde{v}}^{w} = c_{1^{r-n+1} \times s_{\tilde{p}}u,1^{r-n+1} \times v}^{w} = 1.$$

Now, Lemma 3.9 gives that

$$c_{s_{\tilde{p}}u,v}^{1^{r-n+1} \times w} = c_{1^{r-n+1} \times s_{\tilde{p}}u,1^{r-n+1} \times v}^{w} = c_{s_{p}u,v}^{w},$$

so $c_{s_{p}u,v}^{w} = 1$, as desired.

(Case 1.2: $c_{s_{1}^{1} \times w}^{1} \neq 0$): Define $\tilde{u} = 1 \times u, \tilde{v} = 1 \times v$, and $\tilde{w} = 1 \times w$. Then $\tilde{u}$ is $\tilde{p}$-inverse Grassmannian and $\tilde{v}$ is $\tilde{q}$-inverse Grassmannian, where $\tilde{p} = p + 1$ and $\tilde{q} = q + 1$. Also let $\tilde{\gamma} = \gamma_{\tilde{u},\tilde{v}}$ and notice that $\tilde{\gamma}$ is a horizontal shift of $\gamma_{u,v}$.

By Proposition 3.10, we have

$$c_{s_{p}u,v}^{w} + c_{u,s_{q}v}^{w} = c_{s_{1}^{1} \tilde{w}}^{u,v} + \sum_{s_{k}w > w} c_{u,v}^{s_{k}w},$$

and

$$c_{s_{p}u,v}^{\tilde{w}} + c_{u,s_{q}v}^{\tilde{w}} = c_{s_{1}^{1} \tilde{w}}^{1 \times \tilde{u},1 \times \tilde{v}} + \sum_{s_{k}w > \tilde{w}} c_{\tilde{u},\tilde{v}}^{s_{k}w}.$$

By Lemma 3.9, we have

$$c_{s_{p}u,v}^{w} = c_{s_{p}u,v}^{\tilde{w}}, \quad c_{u,s_{q}v}^{w} = c_{u,s_{q}v}^{\tilde{w}}, \quad \text{and} \quad c_{s_{1}^{1} \tilde{w}}^{1 \times \tilde{u},1 \times \tilde{v}} = c_{\tilde{u},\tilde{v}}^{s_{1}^{1} \tilde{w}}.$$

Thus, subtracting Equation (4.7) from Equation (4.8) yields that $c_{s_{1}^{1} \times \tilde{w}}^{w} = 0$.

Now, $T_{\tilde{w}} \cdot \tilde{\gamma}$ is a horizontal shift of $T_{\tilde{w}} \cdot \gamma_{u,v}$. Observe that $s_{\tilde{p}}\tilde{u} = 1 \times s_{\tilde{p}}u$. Since Lemma 3.9 gives that $c_{s_{p}u,v}^{w} = c_{s_{p}u,v}^{\tilde{w}}$ and the latter coefficient falls under Case 1.1, we are done.

(Case 2: $p = q$): We establish this case by reduction to Case 1 via stabilization. Choose an interval $[i, j]$ on which $\gamma_{u,v}$ is supported.

(Case 2.1: $i \geq 1$): If $i > 1$, expand the interval $[i, j]$ until $i = 1$.

Define $\tilde{u} = 1 \times u, \tilde{v} = \tau_{p}^{-1}(v)$, and $\tilde{w} = \tau_{p}^{-1}(w)$. Then $\tilde{u}$ is $\tilde{p}$-inverse Grassmannian, where $\tilde{p} = p + 1$. On the other hand, $\tilde{v}$ is $p$-inverse Grassmannian.
By Lemma [4.6] we have

\[ c_{1 \times (s_p u), \tilde{v}}^\omega = c_{\tau(1 \times (s_p u)), \tau(\tilde{v})}^w = c_{s_p u, v}^w. \]

But also \( 1 \times (s_p u) = s_p \tilde{u} \), so

\[ (4.9) \]

\[ c_{s_p \tilde{u}, \tilde{v}}^\omega = c_{s_p u, v}^w. \]

Since \( \tilde{p} \neq p \), the coefficient \( c_{s_p \tilde{u}, \tilde{v}}^\omega \) falls under Case 1. Note also that \( \ell(\tilde{u}) = \ell(s_p \tilde{u}) + \ell(\tilde{v}) \).

Let \( \tilde{\gamma} = \gamma_{\tilde{u}, \tilde{v}} \). Notice that \( \tilde{\gamma} \) is supported on \([1, j + 1]\) and is obtained from \( \gamma_{u, v} \) by placing a \( + \) in position 1 and shifting the rest of \( \gamma_{u, v} \) horizontally one space to the right. That is,

\[ \tilde{\gamma}(z) = \begin{cases} +, & \text{if } z = 1; \\ \gamma_{u, v}(z - 1) & \text{if } z > 1. \end{cases} \]

See Example [4.7] for an illustration of this construction.

Let \( \delta = T_w \cdot \gamma_{u, v} \) and let \( \hat{\delta} = T_{1 \times w} \cdot \tilde{\gamma} \). Notice that \( \hat{\delta} \) is similarly obtained from \( \delta \) by placing a \( + \) in position 1 and shifting the rest of \( \delta \) horizontally one space to the right.

Now notice that \( \tilde{w} = s_p s_{p-1} \cdots s_1 (1 \times w) \), so

\[ T_{\tilde{w}} \cdot \tilde{\gamma} = T_p T_{p-1} \cdots T_1 T_{1 \times w} \cdot \tilde{\gamma} = T_p T_{p-1} \cdots T_1 \cdot \hat{\delta}. \]

(Case 2.1.1: \( \delta \) is not almost rainbow): Let \( h \) be the least positive integer such that there is a permutation \( \theta \) with \( \ell(\theta) = h \) and \( T_{\hat{p}} \cdot \hat{\delta} = \Omega_{p, p} \). Then, it is easy to see that any permutation \( \sigma \) with \( T_{\sigma} \cdot \hat{\delta} = \Omega_{p, p} \) must have \( \ell(\sigma) \geq h + p \). In particular, \( T_{\tilde{w}} \cdot \tilde{\gamma} = T_p T_{p-1} T_{p-2} \cdots T_1 \cdot \hat{\delta} \) is not almost rainbow. Therefore, by Case 1, we have \( c_{s_p \tilde{u}, \tilde{v}}^\omega = 0 \). Therefore, Equation (4.9) gives that \( c_{s_p u, v}^w = 0 \), as desired.

(Case 2.1.2: \( \delta \) is almost rainbow): We break into cases according to what type of almost rainbow clan \( \delta \) is.

(Case 2.1.2.1: \( T_{s_p} \cdot \delta = \Omega_{p, p} \)): In this case, \( T_{s_p} \) must act on \( \delta \) by joining a \( + \) and a \( - \) into an arc (as in the \( T_4 \) arrow of Figure [2]). The action of \( T_{p-1} T_{p-2} \cdots T_1 \) on \( \hat{\delta} \) is then to move another \( + \) from position 1 past \( p - 1 \) initial nodes to land in position \( p \).

(Case 2.1.2.1.1 \( \delta(p) = + \)): Here, \( T_{p-1} T_{p-2} \cdots T_1 \cdot \hat{\delta}(p) = + \) and \( T_{p-1} T_{p-2} \cdots T_1 \cdot \hat{\delta}(p + 1) = + \). Hence, \( T_p T_{p-1} T_{p-2} \cdots T_1 \cdot \hat{\delta} = T_{p-1} T_{p-2} \cdots T_1 \cdot \hat{\delta} \) and, in particular, \( T_p T_{p-1} T_{p-2} \cdots T_1 \cdot \hat{\delta} \) is not an almost rainbow clan. Thus, by Case 1, we then have \( c_{s_p \tilde{u}, \tilde{v}}^\omega = 0 \). Therefore, Equation (4.9) gives that \( c_{s_p u, v}^w = 0 \), as desired.

(Case 2.1.2.1.2 \( \delta(p) = - \)): Here, \( T_{p-1} T_{p-2} \cdots T_1 \cdot \hat{\delta}(p) = + \) and \( T_{p-1} T_{p-2} \cdots T_1 \cdot \hat{\delta}(p + 1) = - \). Hence, \( T_p \) acts on \( T_{p-1} T_{p-2} \cdots T_1 \cdot \hat{\delta} \) by joining these \( + \) and \( - \) into an arc. So \( T_p T_{p-1} T_{p-2} \cdots T_1 \cdot \hat{\delta} \) is an almost rainbow clan in \( \Psi_{\tilde{p}, p} \). Thus, by Case 1, we then have \( c_{s_p \tilde{u}, \tilde{v}}^\omega = 1 \). Therefore, Equation (4.9) gives that \( c_{s_p u, v}^w = 1 \), as desired.

(Case 2.1.2.2: \( T_{s_r} \cdot \delta = \Omega_{p, p} \) for some \( r \neq p \)): In this case, \( T_{s_r} \) must act on \( \delta \) by uncrossing a pair of adjacent arcs. Recall that \( n - r \) and \( n - r + 1 \) are the labels on the other ends of the crossing arcs from \( r, r + 1 \).

The action of \( T_p T_{p-1} \cdots T_1 \) on \( \hat{\delta} \) is then to move the \( + \) from position 1 past \( p \) initial nodes to land in position \( p + 1 \). Thus, \( T_{\tilde{w}} \cdot \tilde{\gamma} = T_p T_{p-1} \cdots T_1 \cdot \hat{\delta} \) is an almost rainbow clan in \( \Psi_{\tilde{p}, p} \). By Case 1, we then have \( c_{s_p \tilde{u}, \tilde{v}}^\omega = 1 \). Therefore, Equation (4.9) gives that \( c_{s_p u, v}^w = 1 \), as desired.
(Case 2.2: $i < 1$): Define $\tilde{u} = 1^{1-i} \times u$, $\tilde{v} = 1^{1-i} \times v$, and $\tilde{w} = 1^{1-i} \times w$. Then $\tilde{u}$ and $\tilde{v}$ are $p$-inverse Grassmannian, where $\tilde{p} = p - i + 1$.

Also let $\tilde{\gamma} = \gamma_{\tilde{u},\tilde{v}}$ and notice that $\tilde{\gamma}$ is a horizontal shift of $\gamma_{u,v}$. Therefore, $T_{\tilde{w}} \cdot \tilde{\gamma}$ is a horizontal shift of $T_{w} \cdot \gamma_{u,v} = \delta$.

Observe that $s_{\tilde{p}} \tilde{u} = 1^{1-i} \times s_{p} u$. Since Lemma 3.9 gives that $c_{w}^{u,v} = c_{\tilde{w}}^{\tilde{u},\tilde{v}}$ and the latter coefficient falls under Case 2.1, we are done.

$\Box$

Example 4.7. We illustrate part of the construction from Case 2.1. Let $u = 51236748$ and $v = 12354678$. Here, $p = q = 4$. The clan $\gamma_{u,v}$ is

\[
\begin{array}{cccccccc}
\text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} & \text{8} \\
\end{array}
\]

Define $\tilde{u} = 1 \times u$ and $\tilde{v} = \tau_{1}^{-1}(v)$. In this case, $\tilde{u} = 162347859$ and $\tilde{v} = 512364789$. Note that $\tilde{u}$ is 5-inverse Grassmannian, while $\tilde{v}$ is 4-inverse Grassmannian.

Let $\tilde{\gamma} = \gamma_{\tilde{u},\tilde{v}}$, which looks like

\[
\begin{array}{cccccccc}
\text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} & \text{8} \\
\end{array}
\]

Notice that $\tilde{\gamma}$ is obtained from $\gamma_{u,v}$ by placing a $+$ in position 1 and shifting the rest of $\gamma_{u,v}$ to the right, as described in Case 2.1.

$\Diamond$

Example 4.8. Let $u = 3142$. Note that $u$ is 2-inverse Grassmannian and that $s_{2} u = 2143$. We use Theorem 2.2 to compute the Schubert structure coefficients $c_{w}^{u,v} = c_{2143,3142}^{w}$ for all $w \in S_{+}$. We have that $\gamma_{u,u}$ looks like

\[
\begin{array}{cccc}
\text{0} & \text{1} & \text{2} & \text{3} & \text{4} & \text{5} \\
\end{array}
\]

We consider all nontrivial actions of 0-Hecke generators $T_{i}$ on $\gamma_{u,u}$, until reaching an almost rainbow clan:
Here, we have drawn the arrows labeled only by $T_i$ with $i \leq 0$ in purple to distinguish them from those that contribute in Theorem 1.2.

First, observe that there is only two almost rainbow clans that we can reach, specifically the almost rainbow clans $\psi_1$ and $\psi_2$ at the bottom of the diagram above. Note that $\psi_1, \psi_2 \in \Psi_{2,2}$. Using only $T_i$ with $i > 0$, there are exactly two paths from $\gamma_{u,u}$ to $\psi_1$. These paths are labeled by the sequences $(T_1, T_2, T_3, T_2, T_4)$ and $(T_1, T_2, T_3, T_4, T_2)$, both corresponding to the permutation 51324. Thus, by Theorem 1.2, we have computed that $c_{51324}^{2143,3142} = 1$.

On the other hand, there are six paths from $\gamma_{u,u}$ to the almost rainbow clan $\psi_2$. These six paths are labeled by the sequences $\pi_1 = (T_1, T_3, T_2, T_3, T_3), \pi_2 = (T_1, T_3, T_4, T_2, T_3), \pi_3 = (T_3, T_1, T_2, T_4, T_3), \pi_4 = (T_3, T_2, T_1, T_2, T_3), \pi_5 = (T_3, T_2, T_1, T_2, T_3), \pi_6 = (T_3, T_4, T_1, T_2, T_3)$. The sequences $\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6$ all yield the permutation 41523. Thus, by Theorem 1.2, we have computed that $c_{41523}^{2143,3142} = 1$. However, $\pi_5$ yields the permutation 4231, so Theorem 1.2 also gives $c_{4231}^{2143,3142} = 1$. Since these are the only paths from $\gamma_{u,u}$
to almost rainbow clans in $Ψ_{2,2}$. Theorem [L2] finally computes that $c_{w}^{2143,3142} = 0$ for all $w / \in \{51324, 41523, 4231\}$. ♦

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