THE ROLE OF NUCLEON STRUCTURE IN FINITE NUCLEI

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Abstract

The quark-meson coupling model, based on a mean field description of non-overlapping nucleon bags bound by the self-consistent exchange of $\sigma$, $\omega$ and $\rho$ mesons, is extended to investigate the properties of finite nuclei. Using the Born-Oppenheimer approximation to describe the interacting quark-meson system, we derive the effective equation of motion for the nucleon, as well as the self-consistent equations for the meson mean fields. The model is first applied to nuclear matter, after which we show some initial results for finite nuclei.
1 Introductory remarks

The nuclear many-body problem has been the object of enormous theoretical attention for decades. Apart from the non-relativistic treatments based upon realistic two-body forces [1, 2], there are also studies of three-body effects and higher [3]. The importance of relativity has been recognised in a host of treatments under the general heading of Dirac-Brueckner [4, 5, 6]. This approach has also had considerable success in the treatment of scattering processes [7]. At the same time, the simplicity of Quantum Hadrodynamics (QHD) [8, 9] has led to its widespread application, and to attempts to incorporate the density dependence of the couplings [10, 11] that seems to be required empirically.

One of the fundamental, unanswered questions in this field concerns the role of subnucleonic degrees of freedom (quarks and gluons) in determining the equation of state. There is little doubt that, at sufficiently high density (perhaps \(5 - 10 \rho_0\), with \(\rho_0\) the saturation density of symmetric nuclear matter), quarks and gluons must be the correct degrees of freedom and major experimental programs in relativistic heavy ion physics are either planned or underway [12, 13] to look for this transition. In order to calculate the properties of neutron stars [14] one needs an equation of state from very low density to many times \(\rho_0\) at the centre. A truly consistent theory describing the transition from meson and baryon degrees of freedom to quarks and gluons might be expected to incorporate the internal quark and gluon degrees of freedom of the particles themselves. Our investigation may be viewed as a first step in this direction. We shall work with the quark-meson coupling (QMC) model originally proposed by one of us [15] and since developed extensively [16, 17]. Related work has been carried out by a number of groups [18].

Within the QMC model [15] the properties of nuclear matter are determined by the self-consistent coupling of scalar (\(\sigma\)) and vector (\(\omega\)) fields to the quarks within the nucleons, rather than to the nucleons themselves. As a result of the scalar coupling the internal structure of the nucleon is modified with respect to the free case. In particular, the small
mass of the quark means that the lower component of its wave function responds rapidly to the \( \sigma \) field, with a consequent decrease in the scalar density. As the scalar density is itself the source of the \( \sigma \) field this provides a mechanism for the saturation of nuclear matter where the quark structure plays a vital role.

In a simple model where nuclear matter was considered as a collection of static, non-overlapping bags it was shown that a satisfactory description of the bulk properties of nuclear matter can be obtained [15, 16]. Of particular interest is the fact that the extra degrees of freedom, corresponding to the internal structure of the nucleon, result in a lower value of the incompressibility of nuclear matter than obtained in approaches based on point-like nucleons – such as QHD [17]. In fact, the prediction is in agreement with the experimental value once the binding energy and saturation density are fixed. Improvements to the model, including the addition of Fermi motion, have not altered the dominant saturation mechanism. Furthermore, it is possible to give a clear understanding of the relationship between this model and QHD [16] and to study variations of hadron properties in nuclear matter [17]. Surprisingly the model seems to provide a semi-quantitative explanation of the Okamoto-Nolen-Schiffer anomaly [18] when quark mass differences are included [20]. A further application of the model, including quark mass differences, has suggested a previously unknown correction to the extraction of the matrix element, \( V_{ud} \), from super-allowed Fermi beta-decay [21]. Finally the model has been applied to the case where quark degrees of freedom are undisputedly involved – namely the nuclear EMC effect [22].

Because the model has so far been constructed for infinite nuclear matter its application has been limited to situations which either involve bulk properties or where the local density approximation has some validity. Our aim here is to overcome this limitation by extending the model to finite nuclei. In general this is a very complicated problem and our approach will be essentially classical. Our starting point will be exactly as for nuclear matter. That is, we assume that on average the quark bags do not overlap and
that the quarks are coupled locally to average $\sigma$ and $\omega$ fields. The latter will now vary with position, while remaining time independent – as they are mean fields. For deformed or polarised nuclei one should also consider the space components of the vector field. In order to simplify the present discussion we restrict ourselves to spherical, spin-saturated nuclei.

Our approach to the problem will be within the framework of the Born-Oppenheimer approximation. Since the quarks typically move much faster than the nucleons we assume that they always have time to adjust their motion so that they are in the lowest energy state. In order to account for minimal relativistic effects it is convenient to work in the instantaneous rest frame (IRF) of the nucleon. Implicitly one then knows both the position and the momentum of the nucleon, so that the treatment of the motion of the nucleon is classical – at least, as long as the quarks are being considered explicitly. The quantisation of the motion of the nucleon is carried out after the quark degrees of freedom have been eliminated.

In Sect. 2 we show how the Born-Oppenheimer approximation can be used to treat the quark degrees of freedom in a finite nucleus. In Sect. 3 we derive the classical equation of motion for a bag in the meson fields, including the spin-orbit interaction which is treated in first order in the velocity. We then quantize the nucleon motion in a non relativistic way. The self consistent equations for the meson fields are derived in Sect. 4. In Sect. 5 we summarize the model in the form of a self-consistent procedure. The use of this non relativistic formulation is postponed to future work. To allow a clear comparison with QHD we propose a relativistic formulation in Sect. 6, which is then applied to nuclear matter in Sect. 7. Some initial results for finite nuclei are also presented. Sect. 8 summarises our main results and identifies directions for future work.
2 The Born-Oppenheimer approximation

In what follows the coordinates in the rest frame of the nucleus (NRF) will be denoted without primes: \((t, \vec{r})\). In this frame the nucleon follows a classical trajectory, \(\vec{R}(t)\). Denoting the instantaneous velocity of the nucleon as \(\vec{v} = d\vec{R}/dt\), we can define an instantaneous rest frame for the nucleon at each time \(t\). The coordinates in this IRF are \((t', \vec{r}')\):

\[
\begin{align*}
    r_L &= r'_L \cosh \xi + t' \sinh \xi, \\
    \vec{r}_\perp &= \vec{r}'_\perp, \\
    t &= t' \cosh \xi + r'_L \sinh \xi,
\end{align*}
\]

(1)

where \(r_L\) and \(\vec{r}_\perp\) are the components respectively parallel and transverse to the velocity and \(\xi\) is the rapidity defined by \(\tanh \xi = |\vec{v}(t)|\).

Our assumption that the quarks have time to adjust to the local fields in which the nucleon is moving is exact if the fields are constant – i.e. if the motion of the nucleon has no acceleration. It is, of course, very important to examine the validity of the approximation for a typical nuclear environment. For this purpose we take the nucleon motion to be non-relativistic. Assume that at time 0 the nucleon is at \(\vec{R}_0\). After time \(t\), assuming \(t\) small enough, we have

\[
\vec{R}(t) = \vec{R}_0 + \vec{v}_0 t + \frac{1}{2} t^2 \vec{\alpha}_0,
\]

(2)

where \(\vec{\alpha}_0\) \((= F/M_N = -\vec{\nabla}V/M_N)\) is the acceleration and \(M_N\) is the free nucleon mass. We shall take the potential to be a typical Woods-Saxon form with depth \(V_0 \sim -50\) MeV, surface thickness \(a \sim 2\) fm and radius \(R_A \sim 1.2A^{1/3}\) fm. The maximum acceleration occurs at \(R = R_A\) and takes the value

\[
\vec{\alpha}_{\text{max}} = V_0 \frac{\vec{R}}{aM_N}.
\]

(3)

Therefore, in the IRF we have

\[
\vec{R}'(t) \sim \vec{R}_0 + \frac{1}{2} t^2 \frac{V_0}{aM_N} \vec{R}_0,
\]

(4)
and, in the worst case, relative to the size of the nucleon itself \( R_B \), the departure from a fixed position, is

\[
\left| \frac{R'(t) - R'_0}{R_B} \right| \sim \frac{1}{2} t^2 \frac{|V_0|}{aM_N R_B} \sim \frac{t^2}{80},
\]

with \( t \) in fm. Thus, as long as the time taken for the quark motion to change is less than \( \sim 9 \) fm, the nucleon position in the IRF can be considered as unchanged. Since the typical time for an adjustment in the motion of the quarks is given by the inverse of the typical excitation energy, which is of order 0.5 fm, this seems quite safe.

As we have just seen, it is reasonable to describe the internal structure of the nucleon in the IRF. In this frame we shall adopt the static spherical cavity approximation to the MIT bag \[23\] for which the Lagrangian density is

\[
L_0 = \bar{q}'(i\gamma^\mu \partial_\mu - m_q)q' - BV,
\]

for \( |\vec{u}'| \leq R_B \),

with \( B \) the bag constant, \( R_B \) the radius of the bag, \( m_q \) the quark mass and \( \vec{u}' \) the position of the quark from the center of the bag (in the IRF): \( \vec{u}' = \vec{r}' - \vec{R}' \). We shall denote as \( u' \) the 4-vector \( (t', \vec{u}') \). The field \( q'(u'_0, \vec{u}') \) is the quark field in the IRF, which must satisfy the boundary condition

\[
(1 + i\vec{\gamma} \cdot \vec{u}')q' = 0, \quad \text{at } |\vec{u}'| = R_B.
\]

Next we must incorporate the interaction of the quarks with the scalar \((\sigma)\) and vector \((\omega)\) mean fields generated by the other nucleons. In the nuclear rest frame they are self-consistently generated functions of position – \( \sigma(\vec{r}) \) and \( \omega(\vec{r}) \). Using the scalar and vector character of these fields, we know that in the IRF their values are

\[
\sigma_{\text{IRF}}(t', \vec{u}') = \sigma(\vec{r}),
\]

\[
\omega_{\text{IRF}}(t', \vec{u}') = \omega(\vec{r}) \cosh \xi,
\]

\[
\tilde{\omega}_{\text{IRF}}(t', \vec{u}') = -\omega(\vec{r}) \hat{v} \sinh \xi,
\]

and the interaction term is

\[
L_I = g_\sigma \bar{q}' q'(u')\sigma_{\text{IRF}}(u') - g_\omega \bar{q}' \gamma_\mu q'(u')\omega^\mu_{\text{IRF}}(u'),
\]

\[7\]
where $g^\sigma_s$ and $g^\omega_s$ are the quark-meson coupling constants for $\sigma$ and $\omega$, respectively. Apart from isospin considerations, the effect of the $\rho$ meson can be deduced from the effect of the $\omega$. Thus we postpone its introduction to the end of this section.

Since we wish to solve for the structure of the nucleon in the IRF we need the Hamiltonian and its degrees of freedom in this frame. This means that the interaction term should be evaluated at equal time $t'$ for all points $\vec{u}'$ in the bag. Suppose that at time $t'$ the bag is located at $\vec{R}'$ in the IRF. Then, in the NRF it will be located at $\vec{R}$ at time $T$ defined by

$$R_L = R'_L \cosh \xi + t' \sinh \xi,$$
$$\vec{R}_\perp = \vec{R}'_\perp,$$
$$T = t' \cosh \xi + R'_L \sinh \xi.$$ (10)

For an arbitrary point $\vec{r}' (= \vec{u}' + \vec{R}')$ in the bag, at the same time $t'$, we have an analogous relation

$$r_L = r'_L \cosh \xi + t' \sinh \xi,$$
$$\vec{r}_\perp = \vec{r}'_\perp,$$
$$t = t' \cosh \xi + r'_L \sinh \xi,$$ (11)

from which we deduce

$$r_L = R'_L \cosh \xi + t' \sinh \xi + u'_L \cosh \xi,$$
$$= R_L + u'_L \cosh \xi,$$ (12)
$$\vec{r}_\perp = \vec{R}_\perp + \vec{u}'_\perp.$$

This leads to the following expression for the $\sigma$ field

$$\sigma_{IRF}(t', \vec{u}') = \sigma(R_L(T) + u'_L \cosh \xi, \vec{R}_\perp(T) + \vec{u}'_\perp),$$ (13)

with a corresponding equation for the $\omega$ field.
The spirit of the Born-Oppenheimer approximation is to solve the equation of motion for the quarks with the position $\vec{R}(T)$ regarded as a fixed parameter. In order to test the reliability of this approximation we consider a non-relativistic system and neglect finite-size effects. That is, we take

$$\sigma_{IRF}(t', \vec{u}') \sim \sigma(\vec{R}(T)).$$

(14)

As we noted earlier, the typical time scale for a change in the motion of the quark is $\tau \sim 0.5$ fm. During this time the relative change of $\sigma$ due to the motion of the bag is

$$\frac{\Delta \sigma}{\sigma} = \vec{v} \cdot \hat{R} \frac{\sigma'}{\sigma} \tau.$$  

(15)

It is reasonable to assume that $\sigma$ roughly follows the nuclear density, and as long as this is constant, $\Delta \sigma$ vanishes and the approximation should be good. The variation of the density occurs mainly in the surface where it drops to zero from $\rho_0$ (the normal nuclear density) over a distance $d$ of about 2 fm. Therefore we can estimate $|\sigma'/\sigma|$ as approximately $1/d$ in the region where $\rho$ varies. The factor $\vec{v} \cdot \hat{R}$ depends on the actual trajectory, but as a rough estimate we take $\vec{v} \cdot \hat{R} \sim 1/3$. That is, we suppose that the probability for $\vec{v}$ is isotropic. For the magnitude of the velocity we take $k_F/M_N$ with $k_F = 1.7$ fm$^{-1}$ the Fermi momentum. With these estimates we find

$$\frac{\Delta \sigma}{\sigma} \sim \frac{0.36 \cdot 0.5}{3 \cdot 2} \sim 3\%,$$

(16)

which is certainly small enough to justify the use of the Born-Oppenheimer approximation. Clearly this amounts to neglecting terms of order $v$ in the argument of $\sigma$ and $\omega$. In order to be consistent we therefore also neglect terms of order $v^2$ – that is, we replace $u'_L \cosh \xi$ by $u'_L$. 


3 Equation of motion for a bag in the nuclear field

3.1 Leading term in the Hamiltonian

Following the considerations of the previous section, in the IRF the interaction Lagrangian density takes the simple, approximate form:

\[ L_I = g_q^2 q q' (t', \bar{u}') \sigma (\bar{R} + \bar{u}') - g_\omega^2 q q' [\gamma_0 \cosh \xi + \bar{\gamma} \cdot \hat{v} \sinh \xi] q' (t', \bar{u}'). \]  \hspace{1cm} (17)

The corresponding Hamiltonian is

\[ H = \int_{R_0}^{R_B} d\bar{u}' \bar{q}' [-i \bar{\gamma} \cdot \bar{\nabla} + m_q - g_q^2 \sigma (\bar{R} + \bar{u}')] 
+ g_\omega^2 \omega (\bar{R} + \bar{u}')(\gamma_0 \cosh \xi + \bar{\gamma} \cdot \hat{v} \sinh \xi)] q' (t', \bar{u}') + BV, \]  \hspace{1cm} (18)

while the momentum is simply

\[ \bar{P} = \int_{R_0}^{R_B} d\bar{u}' \bar{q}' [-i \bar{\nabla}] q'. \]  \hspace{1cm} (19)

As the \( \sigma \) and \( \omega \) fields only vary appreciably near the nuclear surface, where \( R \gg |\bar{u}'| \) (since \( |\bar{u}'| \) is bounded by the bag radius), it makes sense to split \( H \) into two parts:

\[ H = H_0 + H_1, \]  \hspace{1cm} (20)

\[ H_0 = \int_{R_0}^{R_B} d\bar{u}' q' [-i \bar{\gamma} \cdot \bar{\nabla} + m_q - g_q^2 \sigma (\bar{R})] 
+ g_\omega^2 \omega (\bar{R} + \bar{u}')(\gamma_0 \cosh \xi + \bar{\gamma} \cdot \hat{v} \sinh \xi)] q' (t', \bar{u}') + BV, \]  \hspace{1cm} (21)

\[ H_1 = \int_{R_0}^{R_B} d\bar{u}' q' [-g_q^2 (\sigma (\bar{R} + \bar{u}') - \sigma (\bar{R})) 
+ g_\omega^2 (\omega (\bar{R} + \bar{u}')) (\gamma_0 \cosh \xi + \bar{\gamma} \cdot \hat{v} \sinh \xi)] q' (t', \bar{u}'), \]  \hspace{1cm} (22)

and to consider \( H_1 \) as a perturbation.

Suppose we denote as \( \phi^\alpha \) the complete and orthogonal set of eigenfunctions defined by

\[ h \phi^\alpha (\bar{u}') \equiv (-i \gamma^0 \bar{\gamma} \cdot \bar{\nabla} + m_q^* \gamma^0) \phi^\alpha (\bar{u}'). \]

\(^1\)Only the quark degrees of freedom are active. The nucleon position and velocity are parameters as discussed earlier.
\[
\frac{\Omega_0}{R_B} \phi^\alpha(\vec{u}'),
\]
(23)
\[
(1 + i\vec{\gamma} \cdot \vec{u}') \phi^\alpha(\vec{u}') = 0, \quad \text{at } |\vec{u}'| = R_B,
\]
(24)
\[
\int_0^{R_B} d\vec{u}' \: \phi^{\alpha \dagger} \phi^\beta = \delta^{\alpha \beta},
\]
(25)

with \(\{\alpha\}\) a collective symbol for the quantum numbers and \(m^*_q\) a parameter. Here we recall the expression for the lowest positive energy mode, \(\phi^{0m}\), with \(m\) the spin label:

\[
\phi^{0m}(t', \vec{u}') = \mathcal{N} \left( \frac{j_0(xu'/R_B)}{i\beta_q \vec{\sigma} \cdot \vec{\hat{u}}' j_1(xu'/R_B)} \right) \frac{\chi_m}{\sqrt{4\pi}},
\]
(26)

where

\[
\Omega_0 = \sqrt{x^2 + (m^*_q R_B)^2}, \quad \beta_q = \sqrt{\frac{\Omega_0 - m^*_q R_B}{\Omega_0 + m^*_q R_B}}, \quad \mathcal{N}^{-2} = 2R_B^3 j_0^2(x) |\Omega_0 (\Omega_0 - 1) + m^*_q R_B/2|/x^2.
\]
(27)

For this mode, the boundary condition at the surface amounts to

\[
j_0(x) = \beta_q j_1(x).
\]
(29)

We expand the quark field in the following way

\[
q'(t', \vec{u}') = \sum_\alpha e^{-i\vec{k} \cdot \vec{u}'} \phi^\alpha(\vec{u}') b_\alpha(t'),
\]
(30)

with \(\vec{k}\) chosen as

\[
\vec{k} = g^\omega_q \omega(\vec{R}) \hat{\nu} \sinh \xi,
\]
(31)
in order to guarantee the correct rest frame momentum for a particle in a vector field.

Substituting into the equation for \(H_0\) we find

\[
H_0 = \sum_\alpha \Omega_\alpha \frac{b_\alpha \dagger b_\alpha}{R_B} - \sum_{\alpha \beta} (\alpha|g^\omega_q \sigma(\vec{R}) - m_q + m^*_q |\beta\rangle b^\dagger_\alpha b_\beta + \hat{N}_q g^\omega_q \omega(\vec{R}) \cosh \xi + BV
\]
\[
= \sum_{\alpha \beta} (\alpha| - i\vec{\nabla} |\beta\rangle b^\dagger_\alpha b_\beta - \hat{N}_q \vec{k},
\]
(32)

\[
\vec{P} = \sum_{\alpha \beta} (\alpha| - i\vec{\nabla} |\beta\rangle b^\dagger_\alpha b_\beta - \hat{N}_q \vec{k},
\]
(33)
with the notation
\[
\langle \alpha | A | \beta \rangle = \int_0^{RB} d\vec{u}' \phi^{\alpha \dagger}(\vec{u}') A \phi^\beta(\vec{u}') \quad \text{and} \quad \hat{N}_q = \sum \alpha b^\dagger_\alpha b_\alpha.
\] (34)

Choosing \( m^*_q = m_q - g_\sigma^q \sigma(\vec{R}) \) (in which case the frequency \( \Omega_\alpha \) and the wave function \( \phi^\alpha \) become dependent on \( \vec{R} \) through \( \sigma \)) we get for the leading part of the energy and momentum operator
\[
H^{\text{IRF}}_0 = \sum_\alpha \frac{\Omega_\alpha(\vec{R})}{R_B} b^\dagger_\alpha b_\alpha + BV + \hat{N}_q g^q_\omega(\vec{R}) \cosh \xi,
\] (35)
\[
\vec{P}^{\text{IRF}} = \sum_{\alpha\beta} \langle \alpha | -i\vec{\nabla} | \beta \rangle b^\dagger_\alpha b_\beta - \hat{N}_q g^q_\omega(\vec{R}) \hat{v} \sinh \xi.
\] (36)

If we quantize the \( b_\alpha \) in the usual way, we find that the unperturbed part of \( H \) is diagonalised by states of the form \( |N_\alpha, N_\beta, \cdots \rangle \) with \( N_\alpha \) the eigenvalues of the number operator \( b^\dagger_\alpha b_\alpha \) for the mode \( \{ \alpha \} \). According our working hypothesis, the nucleon should be described by three quarks in the lowest mode \( \alpha = 0 \) and should remain in that configuration as \( \vec{R} \) changes. As a consequence, in the expression for the momentum, the contribution of the gradient acting on \( \phi \) averages to zero by parity and we find that the leading terms in the expression for the energy and momentum of the nucleon in the IRF are:
\[
E^{\text{IRF}}_0 = M^*_N(\vec{R}) + 3g^q_\omega(\vec{R}) \cosh \xi,
\] (37)
\[
\vec{P}^{\text{IRF}} = -3g^q_\omega(\vec{R}) \hat{v} \sinh \xi,
\] (38)

with
\[
M^*_N(\vec{R}) = \frac{3\Omega_0(\vec{R})}{R_B} + BV.
\] (39)

Since we are going to treat the corrections to leading order in the velocity, they will not be affected by the boost back to the NRF. Therefore we can apply the Lorentz transformation to Eqs.(37) and (38) to get the leading terms in the energy and momentum in the NRF. Since there is no possibility of confusion now, we write the NRF variables
without the NRF index. The result is

\[ E_0 = M_N^*(\vec{R}) \cosh \xi + 3g_\omega^2 \omega(\vec{R}), \]  

\[ \vec{P} = M_N^*(\vec{R}) \hat{v} \sinh \xi, \]  

which implies

\[ E_0 = \sqrt{M_N^2(\vec{R}) + \vec{P}^2 + 3g_\omega^2 \omega(\vec{R})}. \]  

At this point we recall that the effective mass of the nucleon, \( M_N^*(\vec{R}) \), defined by Eq.(39) does not take into account the fact that the center of mass of the quarks does not coincide with the center of the bag. By requiring that all of the quarks remain in the same orbit one forces this to be realized in expectation value. However, one knows that the virtual fluctuations to higher orbits would decrease the energy. This c.m. correction is studied in detail in the Appendix, where it is shown that it is only very weakly dependent on the external field strength for the densities of interest. For the zero point energy due to the fluctuations of the gluon field, we assume that it is the same as in free space. Thus we parametrize the sum of the c.m. and gluon fluctuation corrections in the familiar form, \(-z_0/R_B\), where \( z_0 \) is independent of the density. Then the effective mass of the nucleon in the nucleus takes the form

\[ M_N^*(\vec{R}) = \frac{3\Omega_0(\vec{R}) - z_0}{R_B} + BV, \]  

and we assume that the equilibrium condition is

\[ \frac{dM_N^*(\vec{R})}{dR_B} = 0, \]  

which is the usual non-linear boundary condition. This is again justified by the Born-Oppenheimer approximation, according to which the internal structure of the nucleon has enough time to adjust to the varying external field so as to stay in its ground state.

The parameters \( B \) and \( z_0 \) are fixed by the free nucleon mass (\( M_N = 939 \) MeV) using Eqs.(43) and (44) applied to the free case. In the following we keep the free bag radius, \( R_B^0 \), as a free parameter. The results are shown in Table 1.
Table 1: $B^{1/4}$ (MeV) and $z_0$ for some bag radii using $m_q = 5$ MeV.

| $R_B^0$ (fm) | 0.6 | 0.8 | 1.0 |
|--------------|-----|-----|-----|
| $B^{1/4}$    | 210.9 | 170.0 | 143.8 |
| $z_0$        | 4.003 | 3.295 | 2.587 |

### 3.2 Corrections due to $H_1$ and Thomas precession

We estimate the effect of $H_1$ in perturbation theory by expanding $\sigma(\vec{R}+\vec{u}')$ and $\omega(\vec{R}+\vec{u}')$ in powers of $\vec{u}'$ and computing the effect to first order. To this order several terms give zero because of parity and one is left with

$$\langle(0)^3|H_1|(0)^3\rangle = g_q^2 \sum_{\alpha\beta} \langle(0)^3|b_\alpha^* b_\beta|(0)^3\rangle \langle \alpha|\gamma^0 \vec{\gamma} \cdot \hat{v} \vec{u}' \sinh \xi|\beta\rangle \cdot \vec{\nabla} \omega(\vec{R}). \quad (45)$$

Here we need to be more precise about the meaning of the labels $\alpha, \beta$. Suppose we set $\{\alpha\} = \{0, m_\alpha\}$ with $m_\alpha$ the spin projection of the quark in mode $\{0\}$, then we find

$$\langle(0)m_\alpha|\gamma^0 \vec{\gamma} \cdot \hat{v} \vec{u}' \sinh \xi|(0)m_\beta\rangle = - I(\vec{R}) \langle m_\alpha|\frac{\vec{\sigma}}{2}|m_\beta\rangle \times \hat{v} \sinh \xi, \quad (46)$$

with

$$I(\vec{R}) = \frac{4}{3} N^2 \int_0^{R_B} du' u'^3 j_0(xu'/R_B) \beta_q j_1(xu'/R_B), \quad (47)$$

$$= \frac{R_B}{3} \left( \frac{4\Omega_0 + 2m_q^* R_B - 3}{2\Omega_0(\Omega_0 - 1) + m_q^* R_B} \right). \quad (48)$$

The integral, $I(\vec{R})$, depends on $\vec{R}$ through the implicit dependence of $R_B$ and $x$ on the local scalar field. Its value in the free case, $I_0$, can be related to the nucleon isoscalar magnetic moment: $I_0 = 3\mu_s/M_N$ with $\mu_s = \mu_p + \mu_n$ and $\mu_p = 2.79$, $\mu_n = -1.91$ the experimental values. By combining Eqs. (45), (46) and (41) we then find

$$\langle(0)^2|H_1|(0)^3\rangle = \mu_s \frac{I(\vec{R})}{I_0} \frac{1}{M_N R} \left( \frac{d}{dR} 3 g_\omega^a \omega(\vec{R}) \right) \vec{S} \cdot \vec{L} = \mu_s \frac{I(\vec{R}) M_N^* (\vec{R})}{I_0 M_N} \frac{1}{M_N^2 (\vec{R}) R} \left( \frac{d}{dR} 3 g_\omega^a \omega(\vec{R}) \right) \vec{S} \cdot \vec{L}, \quad (49)$$

$$= \mu_s \frac{I(\vec{R}) M_N^* (\vec{R})}{I_0 M_N} \frac{1}{M_N^2 (\vec{R}) R} \left( \frac{d}{dR} 3 g_\omega^a \omega(\vec{R}) \right) \vec{S} \cdot \vec{L}, \quad (50)$$
with \( \vec{S} \) the nucleon spin operator and \( \vec{L} \) its angular momentum.

This spin-orbit interaction is nothing but the interaction between the magnetic moment of the nucleon with the “magnetic” field of the \( \omega \) seen from the rest frame of the nucleon. This induces a rotation of the spin as a function of time. However, even if \( \mu_s \) were equal to zero, the spin would nevertheless rotate because of Thomas precession, which is a relativistic effect independent of the structure. It can be understood as follows.

Let us assume that at time \( t \), the spin vector is \( \vec{S}(t) \) in the IRF\((t)\). Then we expect that, at time \( t + dt \) the spin has the same direction if it is viewed from the frame obtained by boosting the IRF\((t)\) by \( d\vec{v} \) so as to get the right velocity \( \vec{v}(t+dt) \). That is, the spin looks at rest in the frame obtained by first boosting the NRF to \( \vec{v}(t) \) and then boosting by \( d\vec{v} \). This product of Lorentz transformation amounts to a boost to \( \vec{v}(t+dt) \) times a rotation. So, viewed from the IRF\((t+dt)\), the spin appears to rotate. In order that our Hamiltonian be correct it should contain a piece \( H_{\text{prec}} \) which produces this rotation through the Hamilton equations of motion. A detailed derivation can be found in Refs.[24, 25] and the result is

\[
H_{\text{prec}} = -\frac{1}{2} \vec{v} \times \frac{d\vec{v}}{dt} \cdot \vec{S}.
\]  

(51)

The acceleration is obtained from the Hamilton equations of motion applied to the leading order Hamiltonian, Eq.(42). To lowest order in the velocity one finds

\[
\frac{d\vec{v}}{dt} = -\frac{1}{M_N^*(\vec{R})} \vec{\nabla}[M_N^*(\vec{R}) + 3g_\omega^2\omega(\vec{R})].
\]  

(52)

If we put this result into Eq.(51) and add the result to Eq.(50), we get the total spin orbit interaction (to first order in the velocity)

\[
H_{\text{prec}} + H_1 = V_{s.o.}(\vec{R}) \vec{S} \cdot \vec{L},
\]  

(53)

where

\[
V_{s.o.}(\vec{R}) = -\frac{1}{2M_N^2(\vec{R})R} \left[ \left( \frac{d}{dR} M_N^*(\vec{R}) \right) + (1 - 2\mu_s \eta_s(\vec{R})) \left( \frac{d}{dR} \frac{3g_\omega^2\omega(\vec{R})}{3g_\omega^2\omega(\vec{R})} \right) \right],
\]  

(54)

and

\[
\eta_s(\vec{R}) = \frac{I(\vec{R})M_N^*(\vec{R})}{I_0M_N}.
\]  

(55)
3.3 Total Hamiltonian for a bag in the meson mean fields

To complete the derivation we now introduce the effect of the neutral $\rho$ meson. The interaction term that we must add to $L_I$ (see Eq.(9)) is

$$L^\rho_I = -g_q^\rho \bar{q} \gamma_\mu \tau_\alpha q'(u')\rho_{\alpha,IRF}(u'),$$  \hspace{1cm} (56)$$

where $\rho_{\alpha,IRF}$ is the $\rho$ meson field with isospin component $\alpha$ and $\tau^\alpha$ are the Pauli matrices acting on the quarks. In the mean field approximation only $\alpha = 3$ contributes. If we denote by $b(\vec{R})$ the mean value of the time component of the field in the NRF, we can transpose our results for the $\omega$ field. The only difference comes from trivial isospin factors which amount to the substitutions

$$3g_\omega^3 \rightarrow g_\rho^\alpha \frac{\tau_3^N}{2}, \quad \mu_s \rightarrow \mu_v = \mu_p - \mu_n,$$  \hspace{1cm} (57)$$

where $\tau_3^N/2$ (with eigenvalues $\pm 1/2$) is the nucleon isospin operator.

This leads to our final result for the NRF energy-momentum of the nucleon moving in the mean fields, $\sigma(\vec{R}), \omega(\vec{R})$ and $b(\vec{R})$:

$$E = M_N^*(\vec{R}) \cosh \xi + V(\vec{R}),$$  \hspace{1cm} (58)$$
$$\vec{P} = M_N^*(\vec{R}) \hat{v} \sinh \xi,$$  \hspace{1cm} (59)$$

with

$$V(\vec{R}) = V_c(\vec{R}) + V_{s,o.}(\vec{R}) \vec{S} \cdot \vec{L},$$  \hspace{1cm} (60)$$
$$V_c(\vec{R}) = 3g_\omega^3 \omega(\vec{R}) + g_\rho^\alpha \frac{\tau_3^N}{2} b(\vec{R}),$$  \hspace{1cm} (61)$$
$$V_{s,o.}(\vec{R}) = -\frac{1}{2M_N^*(\vec{R})} [\Delta_\sigma + (1 - 2\mu_s \eta_s(\vec{R}))\Delta_\omega + (1 - 2\mu_v \eta_v(\vec{R}))\tau_3^N \Delta_\rho],$$  \hspace{1cm} (62)$$
$$\Delta_\sigma = \frac{d}{dR} M_N^*(\vec{R}), \quad \Delta_\omega = \frac{d}{dR} 3g_\omega^3 \omega(\vec{R}), \quad \Delta_\rho = \frac{d}{dR} g_\rho^\alpha b(\vec{R}).$$  \hspace{1cm} (63)$$

Up to terms of higher order in the velocity, this result for the spin-orbit interaction agrees with that of Achtzehnter and Wilets \[26\] who used an approach based on the non-topological soliton model. As pointed out in \[26\], for a point-like Dirac particle one has
μ_s = 1 while the physical value is μ_s = .88. Thus, in so far as the omega contribution to the spin-orbit force is concerned, the point-like result is almost correct. This is clearly not the case for the rho contribution since we still have μ_v = 1 for the point-like particle while experimentally μ_v = 4.7.

3.4 Quantization of the nuclear Hamiltonian

Until now the motion of the nucleon has been considered as classical, but we must now quantize the model. We do that here in the non-relativistic framework. That is, we consider a theory where the particle number is conserved and we keep only terms up to those quadratic in the velocity. (Here we drop the spin dependent correction since it already involves the velocity. It is reinserted at the end.)

The simplest way to proceed is to realize that the equations of motion

\[ \begin{align*}
E &= M_N^*(\vec{R}) \cosh \xi + V_c(\vec{R}), \\
\vec{P} &= M_N^*(\vec{R}) \dot{\vec{v}} \sinh \xi,
\end{align*} \tag{64, 65} \]

can be derived from the following Lagrangian

\[ L(\vec{R}, \vec{v}) = -M_N^*(\vec{R}) \sqrt{1 - \vec{v}^2} - V_c(\vec{R}). \tag{66} \]

Thus the expansion is simply

\[ L_{nr}(\vec{R}, \vec{v}) = \frac{1}{2} M_N^*(\vec{R}) \vec{v}^2 - M_N^*(\vec{R}) - V_c(\vec{R}) + \mathcal{O}(v^4). \tag{67} \]

Neglecting the \( \mathcal{O}(v^4) \) terms we then go back to the (still classical) Hamilton variables and find (note that the momentum is not the same as before the expansion)

\[ H_{nr}(\vec{R}, \vec{P}) = \frac{\vec{P}^2}{2M_N^*(\vec{R})} + M_N^*(\vec{R}) + V_c(\vec{R}). \tag{68} \]

If we quantize by the substitution \( \vec{P} \to -i\hat{\nabla}_R \) we meet the problem of ordering ambiguities since now \( M_N^*(\vec{R}) \) and \( \vec{P} \) no longer commute. The classical kinetic energy term
\( \vec{P}^2 / 2M_N^*(\vec{R}) \) allows 3 quantum orderings

\[
T_1 = \vec{P} \cdot \hat{A} \vec{P}, \quad T_2 = \hat{A} \vec{P}^2, \quad T_3 = \vec{P}^2 \hat{A},
\]

where \( \hat{A} = 1 / (2M_N^*(\vec{R})) \). However, since we want the Hamiltonian to be hermitian the only possible combination is

\[
T = zT_1 + \frac{1 - z}{2}(T_2 + T_3),
\]

with \( z \) an arbitrary real number. Using the commutation relations one can write

\[
T_2 + T_3 = 2T_1 - \frac{1 - z}{2}(\nabla^2_R \hat{A}),
\]

so that the kinetic energy operator

\[
T = T_1 - \frac{1 - z}{2}(\nabla^2_R \hat{A}),
\]

is only ambiguous through the term containing the second derivative of \( \hat{A} = 1 / (2M_N^*(\vec{R})) \).

Since we have expanded the meson fields only to first order in the derivative to get the classical Hamiltonian it is consistent to neglect such terms. So our quantum Hamiltonian takes the form

\[
H_{nr}(\vec{R}, \vec{P}) = \vec{P} \cdot \frac{1}{2M_N^*(\vec{R})} \vec{P} + M_N^*(\vec{R}) + V(\vec{R}),
\]

where we have reintroduced the spin-orbit interaction in the potential \( V(\vec{R}) \). The nuclear quantum Hamiltonian appropriate to a mean field calculation is then

\[
H_{nr} = \sum_{i=1,A} H_{nr}(\vec{R}_i, \vec{P}_i), \quad \vec{P}_i = -i\nabla_i.
\]

The problem is manifestly self consistent because the meson mean fields, upon which \( H_{nr} \) depends through \( M_N^* \) and \( V \), themselves depend on the eigenstates of \( H_{nr} \).

### 4 Equations for the meson fields

The equation of motion for the meson-field operators \( (\hat{\sigma}, \hat{\omega}^\mu, \hat{\rho}^{\mu,\alpha}) \) are

\[
\partial_\mu \partial^\mu \hat{\sigma} + m_\sigma^2 \hat{\sigma} = g_\sigma \vec{q} \vec{q},
\]
\[ \partial_\mu \partial^\nu \hat{\omega}^\nu + m_\omega^2 \hat{\omega}^\nu = g_\omega^2 \gamma^\nu q, \]  
\[ \partial_\mu \partial^\nu \hat{\rho}_{\nu,\alpha} + m_\rho^2 \hat{\rho}_{\nu,\alpha} = g_\rho^2 \gamma^\nu \tau^\alpha \rho q. \]

The mean fields are defined as the expectation values in the ground state of the nucleus, \(|A\rangle\):

\[ \langle A|\hat{\sigma}(t, \vec{r})|A\rangle = \sigma(\vec{r}), \]  
\[ \langle A|\hat{\omega}^\nu(t, \vec{r})|A\rangle = \delta(\nu, 0)\omega(\vec{r}), \]  
\[ \langle A|\hat{\rho}_{\nu,\alpha}(t, \vec{r})|A\rangle = \delta(\nu, 0)\delta(\alpha, 3)b(\vec{r}). \]

The equations which determine them are the expectation values of Eqs. (75), (76) and (77). First we need the expectation values of the sources

\[ \langle A|\bar{q}q(t, \vec{r})|A\rangle, \langle A|\bar{q}\gamma^\nu q(t, \vec{r})|A\rangle \]  
and \[ \langle A|\bar{q}\gamma^\nu \tau^\alpha q(t, \vec{r})|A\rangle. \]

As before, we shall simplify the presentation by not treating the \(\rho\) meson explicitly until the end. In the mean field approximation the sources are the sums of the sources created by each nucleon – the latter interacting with the meson fields. Thus we write

\[ \bar{q}q(t, \vec{r}) = \sum_{i=1,A} \langle \bar{q}q(t, \vec{r}) \rangle_i, \]  
\[ \bar{q}\gamma^\nu q(t, \vec{r}) = \sum_{i=1,A} \langle \bar{q}\gamma^\nu q(t, \vec{r}) \rangle_i, \]

where \(\langle \cdots \rangle_i\) denotes the matrix element in the nucleon \(i\) located at \(\vec{R}_i\) at time \(t\). According to the Born-Oppenheimer approximation, the nucleon structure is described, in its own IRF, by 3 quarks in the lowest mode. Therefore, in the IRF of the nucleon \(i\), we have

\[ \langle \bar{q}'q'(t', \vec{r}') \rangle_i = 3 \sum_m \bar{\phi}_i^{0,m}(\vec{u}')\phi_i^{0,m}(\vec{u}') = 3s_i(\vec{u}'), \]  
\[ \langle \bar{q}'\gamma^\nu q'(t', \vec{r}') \rangle_i = 3\delta(\nu, 0) \sum_m \bar{\phi}_i^{10,m}(\vec{u}')\phi_i^{0,m}(\vec{u}') = 3\delta(\nu, 0)w_i(\vec{u}'), \]

where the space components of the vector current gives zero because of parity. At the common time \(t\) in the NRF we thus have

\[ \vec{R}_{i,L}' = R_{i,L} \cosh \xi_i - t \sinh \xi_i, \quad \vec{R}_{i,\perp}' = \vec{R}_{i,\perp}, \]
\[ r_{L}' = r_L \cosh \xi_i - t \sinh \xi_i, \quad \vec{r}_{\perp}' = \vec{r}_{\perp}. \]
Therefore

\[ u'_i,L = (r_L - R_i,L) \cosh \xi_i, \quad \vec{u}_{i,\perp}' = \vec{r}_\perp - \vec{R}_{i,\perp}, \]  

(88)

and from the Lorentz transformation properties of the fields we get

\[ \langle \bar{q} q(t, \vec{r}) \rangle_i = 3 s_i (r_L - R_i,L) \cosh \xi_i, \vec{r}_\perp - \vec{R}_{i,\perp} \rangle, \]  

(89)

\[ \langle \bar{q} \gamma^0 q(t, \vec{r}) \rangle_i = 3 w_i (r_L - R_i,L) \cosh \xi_i, \vec{r}_\perp - \vec{R}_{i,\perp} \rangle \cosh \xi_i, \]  

(90)

\[ \langle \bar{q} \gamma q(t, \vec{r}) \rangle_i = 3 w_i ((r_L - R_i,L) \cosh \xi_i, \vec{r}_\perp - \vec{R}_{i,\perp} \rangle \hat{v}_i \sinh \xi_i. \]  

(91)

These equations can be re-written in the form

\[ \langle \bar{q} q(t, \vec{r}) \rangle_i = \frac{3}{(2\pi)^3} (\cosh \xi_i)^{-1} \int \! d\vec{k} e^{i\vec{k} \cdot (\vec{r} - \vec{R}_i)} S(\vec{k}, \vec{R}_i), \]  

(92)

\[ \langle \bar{q} \gamma^0 q(t, \vec{r}) \rangle_i = \frac{3}{(2\pi)^3} \int \! d\vec{k} e^{i\vec{k} \cdot (\vec{r} - \vec{R}_i)} W(\vec{k}, \vec{R}_i), \]  

(93)

\[ \langle \bar{q} \gamma q(t, \vec{r}) \rangle_i = \frac{3}{(2\pi)^3} \hat{v}_i \int \! d\vec{k} e^{i\vec{k} \cdot (\vec{r} - \vec{R}_i)} W(\vec{k}, \vec{R}_i), \]  

(94)

with

\[ S(\vec{k}, \vec{R}_i) = \int \! d\vec{u} e^{-i(\vec{u}_\perp \cdot \vec{R}_i + k_L u_L / \cosh \xi_i)} s_i(\vec{u}), \]  

(95)

\[ W(\vec{k}, \vec{R}_i) = \int \! d\vec{u} e^{-i(\vec{u}_\perp \cdot \vec{R}_i + k_L u_L / \cosh \xi_i)} w_i(\vec{u}). \]  

(96)

Finally, the mean field expressions for the meson sources take the form

\[ \langle A|\bar{q} q(t, \vec{r})|A \rangle = \frac{3}{(2\pi)^3} \int \! d\vec{k} e^{i\vec{k} \cdot \vec{r}} \langle A| \sum_i (\cosh \xi_i)^{-1} e^{-i\vec{k} \cdot \vec{R}_i} S(\vec{k}, \vec{R}_i)|A \rangle, \]  

(97)

\[ \langle A|\bar{q} \gamma^0 q(t, \vec{r})|A \rangle = \frac{3}{(2\pi)^3} \int \! d\vec{k} e^{i\vec{k} \cdot \vec{r}} \langle A| \sum_i e^{-i\vec{k} \cdot \vec{R}_i} W(\vec{k}, \vec{R}_i)|A \rangle, \]  

(98)

\[ \langle A|\bar{q} \gamma q(t, \vec{r})|A \rangle = 0, \]  

(99)

where the last equation follows from the fact that the velocity vector averages to zero.

To simplify further, we remark that a matrix element of the form

\[ \langle A| \sum_i e^{-i\vec{k} \cdot \vec{R}_i} \cdots |A \rangle \]  

(100)
is negligible unless \( k \) is less than, or of the order of, the reciprocal of the nuclear radius. But in Eqs. (95) and (96) \( \vec{k} \) is multiplied by \( \vec{u} \) which is bounded by the nucleon radius. Hence, if we restrict the application of the model to large enough nuclei, we can neglect the argument of the exponential in Eqs. (95) and (96). The evaluation of the correction to this approximation will be postponed to a future work.

We now define the scalar, baryonic and isospin densities of the nucleons in the nucleus by

\[
\rho_s(\vec{r}) = \langle A | \sum_i \frac{M^*_e(\vec{R}_i)}{E_i - V(\vec{R}_i)} \delta(\vec{r} - \vec{R}_i) | A \rangle, \\
\rho_B(\vec{r}) = \langle A | \sum_i \delta(\vec{r} - \vec{R}_i) | A \rangle, \\
\rho_3(\vec{r}) = \langle A | \sum_i \frac{T_3^N}{2} \delta(\vec{r} - \vec{R}_i) | A \rangle,
\]

where we have used Eq. (58) to eliminate the factor \((\cosh \xi_i)^{-1}\). Note that the definition of the scalar density makes sense because, in mean field approximation, each nucleon is moving in an orbital with a given energy. The meson sources then take the simple form

\[
\langle A | \bar{q}q(t, \vec{r}) | A \rangle = 3S(\vec{r})\rho_s(\vec{r}), \\
\langle A | \bar{q}\gamma^\nu q(t, \vec{r}) | A \rangle = 3\delta(\nu, 0)\rho_B(\vec{r}), \\
\langle A | \bar{q}\gamma^\nu \tau^\alpha q(t, \vec{r}) | A \rangle = \delta(\nu, 0)\delta(\alpha, 3)\rho_3(\vec{r}),
\]

where we have deduced the source of the \( \rho \) from that for the \( \omega \). We have used the notation

\[
S(\vec{r}) = S(\vec{0}, \vec{r}) = \int d\vec{u} \, s_F(\vec{u}),
\]

\[
= \frac{\Omega_0/2 + m_s^* R_B(\Omega_0 - 1)}{\Omega_0(\Omega_0 - 1) + m_s^* R_B/2},
\]

where the subscript \( \vec{r} \) reminds us that the function \( s_F(\vec{u}) \) must be evaluated in the scalar field existing at \( \vec{r} \).

Since their sources are time independent and since they do not propagate, the mean meson fields are also time independent. So by combining Eqs. (75) to (80) and Eqs. (104),
We get the desired equations for $\sigma(\vec{r}), \omega(\vec{r})$ and $b(\vec{r})$:

\begin{align*}
(-\nabla_{\vec{r}}^2 + m_{\sigma}^2)\sigma(\vec{r}) &= g_\sigma C(\vec{r})\rho_s(\vec{r}), \quad (109) \\
(-\nabla_{\vec{r}}^2 + m_{\omega}^2)\omega(\vec{r}) &= g_\omega \rho_B(\vec{r}), \quad (110) \\
(-\nabla_{\vec{r}}^2 + m_{\rho}^2)b(\vec{r}) &= g_\rho \rho_3(\vec{r}). \quad (111)
\end{align*}

where the nucleon coupling constants and $C$ are defined by

\begin{align*}
g_\sigma &= 3g_\sigma^0 S(\sigma = 0), \quad g_\omega = 3g_\omega^0, \quad g_\rho = g_\rho^0, \quad C(\vec{r}) = S(\vec{r})/S(\sigma = 0). \quad (112)
\end{align*}

For completeness we recall that the mean fields carry the following energy

\begin{equation}
E^{\text{meson}} = \frac{1}{2} \int d\vec{r}[(\vec{\nabla}\sigma)^2 + m_{\sigma}^2 \sigma^2 - (\vec{\nabla}\omega)^2 - m_{\omega}^2 \omega^2 - (\vec{\nabla}b)^2 - m_{\rho}^2 b^2]. \quad (113)
\end{equation}

## 5 The problem of self-consistency

Our quark model for finite nuclei is now complete. As the main equations are scattered through the text, it is useful to summarize the procedure which we have developed as follows:

1. Choose the bare quark mass, $m_q$, and adjust the bag parameters, $B$ and $z_0$, to fit the free nucleon mass and its bag radius (see Eq. (13) and Table 1).

2. Assume that the coupling constants and the masses of the mesons are known (see Table 2 in Sec. 7.1).

3. Evaluate the nucleon properties, $I(\sigma)$ (Eq. (18)) and $S(\sigma)$ (Eq. (105)), for a range of values of $\sigma$ (see also Eqs. (134) and (135) in Sec. 7.1).

4. Guess an initial form for the densities, $\rho_s(\vec{r}), \rho_B(\vec{r})$ and $\rho_3(\vec{r})$, in Eqs. (101), (102) and (103).

5. For $\rho_s(\vec{r})$ fixed, solve Eq. (109) for the $\sigma$ field.
6. For $\rho_B(\vec{r})$ and $\rho_3(\vec{r})$ fixed, solve Eqs. (110) and (111) for the $\omega$ and $\rho$ fields.

7. Evaluate the effective mass $M_N(\vec{r})$ and the potential $V(\vec{r})$ according to Eqs. (43) and (60). The bag radius at each point in the nucleus is fixed by Eq. (44). For practical purposes it is useful to note that $M_N^*(\vec{r})$ and $C(\vec{r})$ depend only on the value of the $\sigma$ field at $\vec{r}$ and that a simple parametrization, linear in $g_\sigma\sigma$, works extremely well at moderate densities (see Eqs. (134) and (136) in Sec. 7.1). So one does not need to solve the bag equations at each point when solving the self-consistent nuclear problem.

8. Solve the eigenvalue problem defined by the nuclear Hamiltonian Eq. (74) and generate the shell model from which the densities $\rho_s(\vec{r}), \rho_B(\vec{r})$ and $\rho_3(\vec{r})$ can be computed according to Eqs. (101), (102) and (103).

9. Go to 5 and iterate until self-consistency is achieved.

This procedure has to be repeated for each nucleus, which certainly implies considerable numerical work. We plan to study the implications of this non-relativistic formulation in a future work.

6 Relativistic formulation

Here we attempt to formulate the model as a relativistic field theory for the nucleon in order to have a direct comparison with the widely used QHD. We make no attempt to justify the formulation of a local relativistic field theory at a fundamental level because this is not possible for a composite nucleon. Our point is that, in the mean field approximation, QHD has had considerable phenomenological success. We therefore try to express our results in this framework. The idea is to write a relativistic Lagrangian and to check that, in some approximation, it is equivalent to our non-relativistic formulation.
As shown earlier, our basic result is that essentially the nucleon in the meson fields behaves as a point like particle of effective mass $M_N^*(\sigma(\vec{r}))$ moving in a potential $g_\omega(\vec{r})$. To simplify we do not consider the $\rho$ coupling and therefore we shall only apply the model to $N = Z$ nuclei.

As already pointed out the spin-orbit force of the $\omega$ is almost the same as the one of a point like Dirac particle. So a possible Lagrangian density for this system is

$$\mathcal{L} = i\overline{\psi}\gamma \cdot \partial \psi - M_N^*(\hat{\sigma})\overline{\psi}\psi - g_\omega \hat{\omega}^\mu \overline{\psi}\gamma_\mu \psi + \mathcal{L}_{mesons},$$

(114)

where $\psi$, $\hat{\sigma}$ and $\hat{\omega}^\mu$ are respectively the nucleon, $\sigma$ and $\omega$ field operators. The free meson Lagrangian density is

$$\mathcal{L}_{mesons} = \frac{1}{2}(\partial_\mu \hat{\sigma} \partial^\mu \hat{\sigma} - m_\sigma^2 \hat{\sigma}^2) - \frac{1}{2} \partial_\mu \hat{\omega}_\nu(\partial^\mu \hat{\omega}^\nu - \partial^\nu \hat{\omega}^\mu) + \frac{1}{2} m_\omega^2 \hat{\omega}^\mu \hat{\omega}_\mu.$$  

(115)

The comparison with QHD is straightforward. If we define the field dependent coupling constant $g_\sigma(\hat{\sigma})$ by

$$M_N^*(\hat{\sigma}) = M_N - g_\sigma(\hat{\sigma})\hat{\sigma},$$

(116)

it is easy to check that $g_\sigma(\sigma = 0)$ is equal to the coupling constant $g_\sigma$ defined in Eq.(112). Then we write the Lagrangian density as

$$\mathcal{L} = i\overline{\psi}\gamma \cdot \partial \psi - M_N \overline{\psi}\psi + g_\sigma \hat{\sigma}\overline{\psi}\psi - g_\omega \hat{\omega}^\mu \overline{\psi}\gamma_\mu \psi + \mathcal{L}_{mesons},$$

(117)

and clearly the only difference from QHD lies in the fact that the internal structure of the nucleon has forced a known dependence of the scalar meson-nucleon coupling constant on the scalar field itself. Note that this dependence is not the same as the one adopted in the density dependent hadron field theory [11] where the meson-nucleon vertices are assumed to depend directly on the baryonic densities.

In the mean field approximation, the meson field operators are replaced by their time independent expectation values in the ground state of the nucleus:

$$\hat{\sigma}(t, \vec{r}) \rightarrow \sigma(\vec{r}), \quad \hat{\omega}^\mu(t, \vec{r}) \rightarrow \delta(\mu, 0)\omega(\vec{r}),$$

(118)
and variation of the Lagrangian yields the Dirac equation
\[
(i\gamma \cdot \partial - M_N^*(\sigma) - g_\omega \gamma_0 \omega)\psi = 0,
\]
(119)
as well as the equations for the meson mean fields
\[
\begin{align*}
(-\nabla_r^2 + m_\sigma^2)\sigma(\vec{r}) &= -\left(\frac{\partial}{\partial\sigma} M_N^*(\sigma)\right) \langle A|\bar{\psi}\psi(\vec{r})|A\rangle, \\
(-\nabla_r^2 + m_\omega^2)\omega(\vec{r}) &= g_\omega \langle A|\bar{\psi}\psi(\vec{r})|A\rangle.
\end{align*}
\]
(120) (121)

Using the Foldy-Wouthuysen transformation (see Ref.[26] for the details), one can show that the Dirac equation (119) gives back our non-relativistic, quantum Hamiltonian (without the \(\rho\) coupling) under the following conditions:

- only terms of second order in the velocity are kept,
- second derivatives of the meson fields are ignored,
- the fields are small with respect to the nucleon mass,
- the difference between \(\mu_v = 4.7\) and \(\mu_v(\text{point}) = 1\) can be neglected.

The fact that \(\mu_v = 4.7\) is very different from \(\mu_v(\text{point}) = 1\) prevents one from using this simple scheme for the \(\rho\) coupling. The spin-orbit potential would be completely different from the one obtained in Eq.(62). Since the \(\rho\) is not required in the treatment of symmetric nuclei we postpone to future work its introduction in the relativistic framework.

In the same approximations one finds that \(\langle A|\bar{\psi}\psi(\vec{r})|A\rangle\) and \(\langle A|\bar{\psi}\psi(\vec{r})|A\rangle\) are respectively equal to the previously defined scalar source, \(\rho_s\) (Eq.(101)), and vector source, \(\rho_B\) (Eq.(102)). (In nuclear matter the approximation is in fact exact). Therefore the equations (120) and (121) for the meson fields reproduce our previous results given in Eqs.(109) and (110) provided the relation
\[
C(\sigma)g_\sigma(\sigma = 0) = -\frac{\partial}{\partial\sigma} M_N^*(\sigma),
\]
(122)
\[
= \frac{\partial}{\partial\sigma}(g_\sigma(\sigma)\sigma),
\]
(123)
is satisfied, which can be checked explicitly from the definition of $M_N^*(\sigma)$ and $C(\sigma)$.

Within the limits specified above, the Lagrangian density given in Eq.(114) looks acceptable and we shall proceed to study its numerical consequences.

7 Applications

Before turning to finite nuclei, we first explain briefly how the general formalism applies in the case of nuclear matter.

7.1 Infinite Nuclear Matter

In symmetric, infinite nuclear matter the sources of the fields are constant and can be related to the nucleon Fermi momentum $k_F$ according to [1]

$$\langle A|\psi^\dagger \psi(\vec{r})|A\rangle = \frac{4}{(2\pi)^3} \int^{k_F} d\vec{k} = \frac{2k_F^3}{3\pi^2},$$

$$\langle A|\bar{\psi}^\dagger \psi(\vec{r})|A\rangle = \frac{4}{(2\pi)^3} \int^{k_F} d\vec{k} \frac{M_N^*}{\sqrt{M_N^*^2 + \vec{k}^2}},$$

where $M_N^*$ denotes the constant value of the effective nucleon mass defined by Eq.(43), or equivalently, Eq.(116). Obviously these equations can also be deduced from the definitions (Eqs.(101) and (102)) of the scalar and vector densities provided that, for $\rho_s$, one uses Eqs.(58) and (59) to write

$$\frac{M_N^*}{E - V} = \frac{1}{\cosh \xi} = \frac{M_N^*}{\sqrt{M_N^*^2 + \vec{k}^2}}.$$ (126)

Let $(\sigma, \omega)$ be the constant mean-values of the meson fields. From Eqs.(109) and (110) we find

$$\omega = \frac{g_\omega \rho_B}{m_\omega^2},$$ (127)
Figure 1: Mean-field values of the $\sigma$ meson for various bag radii as a function of $\rho_B$. The solid, dotted and dashed curves show $g_\sigma \sigma$ for $R^0_B = 0.6$, 0.8 and 1.0 fm, respectively. The quark mass is chosen to be 5 MeV.

$$\sigma = \frac{g_\sigma C(\sigma)}{m_\sigma^2} \frac{4}{(2\pi)^3} \int^{k_F} d\vec{k} \frac{M_N^*}{\sqrt{M_N^2 + \vec{k}^2}}, \quad (128)$$

where $C(\sigma)$ is now the constant value of $C$ in the scalar field.\(^2\) As emphasised by Saito and Thomas \(^{[16]}\), the self-consistency equation for $\sigma$, Eq.\(^{(128)}\), is the same as that in QHD except that in the latter model one has $C(\sigma) = 1$ (i.e. the quark mass is infinitely heavy).

Once the self-consistency equation for $\sigma$ has been solved, one can evaluate the energy per nucleon. From the Dirac equation \(^{(119)}\), or simply using Eq.\(^{(58)}\), the energy of a nucleon with momentum $\vec{k}$ is

$$E(\vec{k}) = V + M_N^* \cosh \xi = g_\omega \omega + \sqrt{M_N^2 + \vec{k}^2}, \quad (129)$$

\(^2\)Note the change in notation from the earlier papers of Saito and Thomas where $C$ was used for what we now call $S$.\(^3\)
Figure 2: Effective nucleon mass \( m_q = 5 \text{MeV} \). The curves are labelled as in Fig. 1.

which contributes to the energy per nucleon by the amount

\[
E^{\text{nuc.}}/A = \frac{4}{\rho_B (2\pi)^3} \int_{k_F}^{k_F} d\vec{k} E(\vec{k}),
\]

(130)

\[
= \frac{1}{\rho_B} \left[ g_\omega \omega \rho_B + \frac{4}{(2\pi)^3} \int_{k_F}^{k_F} d\vec{k} \sqrt{M_N^* + \vec{k}^2} \right].
\]

(131)

The contribution of the energy stored in the meson fields is, from Eq. (113),

\[
E^{\text{meson}}/A = \frac{1}{2\rho_B} (m_\sigma^2 \sigma^2 - m_\omega^2 \omega^2),
\]

(132)

and using the expression for \( \omega \) we finally obtain the following expression for the energy per nucleon

\[
E^{\text{total}}/A = \frac{1}{\rho_B} \left[ \frac{4}{(2\pi)^3} \int_{k_F}^{k_F} d\vec{k} \sqrt{M_N^* + \vec{k}^2} + \frac{m_\sigma^2 \sigma^2}{2} + \frac{g_\omega^2 \rho_B^2}{2m_\omega^2} \right].
\]

(133)

We determine the coupling constants, \( g_\sigma \) and \( g_\omega \), so as to fit the binding energy \((-15.7 \text{ MeV})\) per nucleon and the saturation density \( (\rho_0 = 0.15 \text{ fm}^{-3}) \) for symmetric nuclear matter at equilibrium. The coupling constants and some calculated properties of nuclear
Figure 3: Scalar density ratio, $C(\bar{\sigma})$, as a function of $g_\sigma\bar{\sigma}$ ($m_q = 5$ MeV). The curves are labelled as in Fig.1.

matter (with $m_q = 5$ MeV, $m_\sigma = 550$ MeV and $m_\omega = 783$ MeV) at the saturation density are listed in Table 2. The most notable fact is that the calculated incompressibility, $K$, is well within the experimental range: $K \approx 200 - 300$ MeV [27]. Also our effective nucleon mass is much larger than in the case of QHD. In the last two columns of Table 2 we show the relative modifications (with respect to their values at zero density) of the bag radius and the lowest eigenvalue, $x$, at saturation density. The changes are not large. In order to show the relative insensitivity to the quark mass (as long as it is small) we note that $M_N^* = (756, 753)$ MeV at saturation density and $K = (278, 281)$ MeV for $R_B^0 = 0.8$ fm and $m_q = (0, 10)$ MeV, respectively.

In Figs. 1 and 2 we show the mean-field values of the $\sigma$ meson and the effective nucleon mass in medium, respectively. In both cases their dependence on the bag radius is rather weak. The scalar density ratio, $C(\bar{\sigma})$, and the ratio of the integral $I(\bar{\sigma})$ to $I_0$ are plotted as a function of $g_\sigma\bar{\sigma}$ in Figs. 3 and 4, respectively. As $\bar{\sigma}$ increases, the scalar density ratio...
Figure 4: The ratio of $I(\sigma)$ to $I_0$ ($m_q = 5$ MeV). The curves are labelled as in Fig.1. decreases linearly, while the ratio, $I/I_0$, gradually increases. Here we note that $S(0) = (0.4819, 0.4827, 0.4834)$ and $I_0 = (0.2421, 0.3226, 0.4028)$ fm for $R_{B}^0 = (0.6, 0.8, 1.0)$ fm, respectively.

It would be very useful to give a simple parametrization for $C$ and $I/I_0$ because they are completely controlled by only the strength of the local $\sigma$ field. We can easily see that $C$ is well approximated by the linear form:

$$C(\sigma) = 1 - a \times (g_\sigma \sigma),$$  \hspace{1cm} (134)

with $g_\sigma \sigma$ in MeV and $a = (6.6, 8.8, 11) \times 10^{-4}$, for $R_{B}^0 = (0.6, 0.8, 1.0)$ fm, respectively. For $I/I_0$ we find a quadratic form:

$$\frac{I(\sigma)}{I_0} = 1 + b_1 \times (g_\sigma \sigma) - b_2 \times (g_\sigma \sigma)^2,$$ \hspace{1cm} (135)

with $b_1 = (3.7, 4.9, 6.1) \times 10^{-4}$ and $b_2 = (3.9, 5.2, 6.5) \times 10^{-7}$. More comments and discussion of the results for nuclear matter can be found in the previous publications \cite{15, 16}. 

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Table 2: Coupling constants and calculated nucleon properties in symmetric nuclear matter at normal nuclear matter density. The effective nucleon mass, $M^*_N$, and the nuclear incompressibility, $K$, are quoted in MeV. The bottom row is for QHD.

| $R_B^0$ (fm) | $\frac{g_\sigma^2}{4\pi}$ | $\frac{g_\omega^2}{4\pi}$ | $M^*_N$ | $K$ | $\frac{\delta R_B}{R_B^0}$ | $\frac{\delta R}{x_0}$ |
|--------------|-----------------|-----------------|--------|-----|-----------------|-----------------|
| 0.6          | 5.86            | 6.34            | 729    | 295 | -0.02           | -0.13           |
| 0.8          | 5.40            | 5.31            | 754    | 280 | -0.02           | -0.16           |
| 1.0          | 5.07            | 4.56            | 773    | 267 | -0.02           | -0.21           |
| QHD          | 7.29            | 10.8            | 522    | 540 | –               | –               |

As a practical matter, we note that Eq. (123) is easily solved for $g_\sigma(\sigma)$ in the case where $C(\sigma)$ is linear in $g_\sigma \bar{\sigma}$ — as we found in Eq. (134). In fact, it is easy to show that

$$M^*_N = M_N - \left[ 1 - \frac{a}{2} (g_\sigma \bar{\sigma}) \right] (g_\sigma \bar{\sigma}),$$

(136)

(recall $g_\sigma \equiv g_\sigma (\sigma = 0)$, Eq. (112)) so that the effective $\sigma$N coupling constant decreases at half the rate of $C(\sigma)$. (Equation (136) is quite accurate up to twice nuclear matter density.) Having explicitly solved the nuclear matter problem by self-consistently solving for the quark wave functions in the bag in the mean scalar field one can solve for the properties of finite nuclei without explicit reference to the internal structure of the nucleon. All one needs is Eqs. (134) and (136) for $C(\sigma)$ and $M^*_N$ as a function of $g_\sigma \bar{\sigma}$. 

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7.2 Initial results for finite nuclei

In order to keep the present work to a reasonable length we propose to present detailed numerical studies of the properties of finite nuclei calculated within the present framework in a later paper [28]. However, to illustrate that the approach has some promise, we have performed some preliminary calculations for the doubly closed shell nucleus $^{16}O$. This requires the self-consistent solution of our equations for 6 Dirac orbitals – the $1s_{1/2}$, $1p_{1/2}$ and $1p_{3/2}$ states of the protons and neutrons. The central Coulomb interaction has been taken into account for the protons.

The numerical calculation was carried out using the techniques described by Walecka and Serot [9]. The resulting charge density for $^{16}O$ is shown in Fig. 5 (dotted curve) in comparison with the experimental data [29] (hatched area) and QHD [9]. In this calculation we used the parameters given in Table 2 for $R^0_B = 0.8$ fm. However, as the central density of $^{16}O$ was a little high we increased the model-dependent slope of the scalar

![Figure 5: The charge density of $^{16}O$ in the present model and QHD, compared with the experimental distribution.](image-url)
Figure 6: Scalar, vector and coulomb potentials for $^{16}\text{O}$ in the present model.

density $C(\sigma)$ and the coupling constant $g_{\sigma}(\sigma)$ (i.e. the parameter $a$ in Eqs. (134) and (136) ) by 10\% to obtain the result shown. The corresponding effect on the saturation energy and density of nuclear matter is very small: $\rho_0 \rightarrow 0.1496$ fm$^{-3}$ and the energy per nucleon becomes $-15.65$ MeV. In Fig. 5, we also show the scalar and vector fields corresponding to the charge density of $^{16}\text{O}$ that was shown in Fig. 5. Using the same parameter set one also finds a very reasonable fit to the charge density of $^{40}\text{Ca}$. It is presented in Fig. 7.

To conclude this initial investigation of finite nuclei we record, in Table 3, the calculated single particle binding energies of the protons and neutrons in $^{40}\text{Ca}$, in comparison with the results of QHD and the experimental data [6]. Because of the smaller scalar and vector field strengths in the present model, compared with QHD, the spin-orbit splitting tends also to be smaller – perhaps only $\frac{2}{3}$ of the experimentally observed splittings. In this context it is interesting to show the sensitivity of the present model to just one feature of the underlying structure of the nucleon, namely the mass of the confined quark. As
Figure 7: The charge density of $^{40}\text{Ca}$ in the present model and QHD, compared with the experimental distribution.

As an example, we consider the case $m_q = 300$ MeV, which is a typical constituent quark mass. For $m_q = 300$ MeV and $R_0 = 0.8$ fm, the coupling constants required to fit the saturation properties of nuclear matter are $g_2^2/4\pi = 6.84$ and $g_\omega^2/4\pi = 8.51$, and the effective nucleon mass (at saturation) and the incompressibility become 674 MeV and 334 MeV, respectively. Using these parameters one finds that the charge densities of $^{16}\text{O}$ and $^{40}\text{Ca}$ are again reproduced very well (without any need to vary the slope parameter “$a$”, i.e., $a = 3.9 \times 10^{-4}$). From the table, one can see that a heavier quark mass gives a spectrum closer to that of QHD – as discussed by Saito and Thomas in Ref. [16] – and in better agreement with the observed spin-orbit splittings. In conclusion we notice one more point: the $2s_{1/2}$ and $1d_{3/2}$ levels are inverted compared with the experimental data (see also Ref. [6]). This may be connected with the effect of rearrangement [1, 30], which is not considered here. Also, the effect of antisymmetrization remains to be investigated.
Table 3: Predicted proton and neutron spectra of $^{40}Ca$ compared with QHD and the experimental data. QMC(5, 300) means the present model with $R_0 = 0.8$ fm and $m_q = 5$ and 300 MeV, respectively (see text). All energies are in MeV.

| Shell | neutron | | | | proton | | | |
|-------|---------|---|---|---|---------|---|---|---|
|       | QHD     | QMC(5) | QMC(300) | Expt. | QHD     | QMC(5) | QMC(300) | Expt. |
| $1s_{1/2}$ | 54.9 | 43.5 | 47.1 | 51.9 | 46.7 | 35.5 | 38.9 | 50±10 |
| $1p_{3/2}$ | 38.6 | 32.5 | 34.5 | 36.6 | 30.8 | 24.7 | 26.7 | 34±6  |
| $1p_{1/2}$ | 33.1 | 31.3 | 32.4 | 34.5 | 25.3 | 23.5 | 24.5 | 34±6  |
| $1d_{5/2}$ | 22.6 | 20.0 | 21.0 | 21.6 | 15.2 | 12.6 | 13.6 | 15.5  |
| $2s_{1/2}$ | 14.6 | 15.0 | 15.3 | 18.9 | 7.4 | 7.7 | 7.9 | 10.9  |
| $1d_{3/2}$ | 14.1 | 17.9 | 17.3 | 18.4 | 6.8 | 10.5 | 9.8 | 8.3   |
8 Summary

Starting with a hybrid model in which quarks confined in nucleon bags interact through the exchange of scalar and vector mesons, we have shown that the Born-Oppenheimer approximation leads naturally to a generalisation of QHD with a density dependent scalar coupling. The physical origin of this density dependence, which provides a new saturation mechanism for nuclear matter, is the relatively rapid increase of the lower Dirac component of the wavefunction of the confined, light quark. We confirm the original discovery of Guichon [15] that once the scalar and vector coupling constants are chosen to fit the observed saturation properties of nuclear matter the extra, internal degrees of freedom lead to an incompressibility that is consistent with experiment.

In the case of finite nuclei we have derived a set of coupled differential equations which must be solved self-consistently but which are not much more difficult to solve than the relativistic Hartree equations of QHD. Initial results for $^{16}O$ (see Fig. 3) are quite promising but a full numerical study will be presented in a later work [28].

The successful generalisation of the quark-meson coupling model to finite nuclei opens a tremendous number of opportunities for further work. For example, earlier results for the Okamoto-Nolen-Schiffer anomaly [20], the nuclear EMC effect [22], the charge-symmetry violating correction to super-allowed Fermi beta-decay [21] and so on, can now be treated in a truly quantitative way.

It will be very interesting to explore the connection between the density dependence of the variation of the effective $\sigma$-nucleon coupling constant, which arises so naturally here, with the variation found empirically in earlier work. We note, in particular, that while our numerical results depend on the particular model chosen here (namely, the MIT bag model), the qualitative features which we find (such as the density dependent decrease of the scalar coupling) will apply in any model in which the nucleon contains light quarks and the attractive $N-N$ force is a Lorentz scalar. Of course, it will be important to investigate...
the degree of variation in the numerical results for other models of nucleon structure.

We could list many other directions for future theoretical work: for the replacement of the MIT bag by a model respecting PCAC (e.g. the cloudy bag model [3]), the replacement of $\sigma$-exchange by two-pion exchange, the replacement of $\omega$ exchange by nucleon overlap at short distance, the inclusion of the density dependence of the meson masses themselves [17, 32] and so on. On the practical side, we stress that the present model can be applied to all the problems for which QHD has proven so attractive, with very little extra effort. It will also be interesting to explore its phenomenological consequences in this way.

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Appendix

Here we want to demonstrate that the center of mass correction to the bag energy is essentially independent of the external scalar field. (Since the vector fields do not alter the quark structure of the nucleon we need not consider them.) To avoid the difficulties associated with the confinement by a sharp boundary, we consider a model where the quark mass grows quadratically with the distance from the center of the bag. This is justified because we do not look for the c.m. correction itself but only for its dependence on the external field. Moreover, after the strength of the confining mass has been adjusted to reproduce the lowest eigenfrequency of the bag, we have found that the corresponding wave functions are rather similar to those for the bag.

To estimate the c.m. correction we also make the assumption that the quark number is a good quantum number, which allows us to formulate the problem in the first quantized form. This amounts to neglecting the effect of quark-antiquark excitation and is therefore not a very strong constraint.

Thus the model is defined by the following first-quantized Hamiltonian:

\[ H_B = \sum_{i=1,N} \gamma_0(i)[\vec{\gamma}(i) \cdot \vec{p}_i + m(\vec{r}_i)], \quad \vec{p}_i = -i\vec{\nabla}_i, \]  

(137)

with

\[ m(\vec{r}) = m^* + Kr^2, \]

(138)

where \( m^* \) is the mass of the quark in the presence of the external scalar field.

By assumption, \( N \) is a number, so we can define intrinsic coordinates \((\vec{\rho}, \vec{\pi})\) by

\[ \vec{\pi}_i = \vec{p}_i - \frac{\vec{P}}{N}, \quad \vec{P} = \sum_i \vec{p}_i, \quad \sum_i \vec{\pi}_i = 0, \]

(139)

\[ \vec{\rho}_i = \vec{r}_i - \vec{R}, \quad \vec{R} = \frac{1}{N} \sum_i \vec{r}_i, \quad \sum_i \vec{\rho}_i = 0. \]

(140)

Then we can write the Hamiltonian in the form:

\[ H_B = H_{intr.} + H_{CM}, \]

(141)
\[ H_{\text{intr.}} = \sum_i \gamma_0(i)[\vec{\gamma}(i) \cdot \vec{r}_i + m(\vec{r}_i)], \]  
\[ H_{CM} = \frac{\vec{P}}{N} \cdot \sum_i \gamma_0(i)\vec{\gamma}(i) + \sum_i \gamma_0(i)[m(\vec{r}_i) - m(\vec{r}_i)]. \]

This separation into an intrinsic and a c.m. Hamiltonian is correct because:

1. \( H_{\text{intr.}} \) commutes with \( \vec{P} \) and \( \vec{R} \),

2. One has \([H_{CM}, \vec{R}] = -i \sum_i \gamma_0(i)\vec{\gamma}(i)\). Since for a Dirac particle \( \gamma_0\vec{\gamma} \) is the velocity, one can identify the RHS of the previous equation with the time derivative of \( \vec{R} \), which is consistent.

The fact that the c.m. Hamiltonian depends on the intrinsic coordinates is not a surprise because the separation is only complete in certain special cases.

We now look for the intrinsic energy of the bag, writing

\[ H_{\text{intr.}} = H_B - H_{CM}. \]  

All that we know are the eigenstates of \( H_B \) but we can consider \( H_{CM} \) as a correction of order \( 1/N \) with respect to the leading term in the bag energy. Therefore we estimate its effect in first order perturbation theory, that is

\[ E_{\text{intr.}} = E_B - \langle B | H_{CM} | B \rangle = E_B - E_{CM}, \]

where \( |B\rangle \) is the eigenstate of \( H_B \) with energy \( E_B \). We must therefore evaluate

\[ E_{CM} = \langle B | \frac{\vec{P}}{N} \cdot \sum_j \gamma_0(j)\vec{\gamma}(j) + \sum_i \gamma_0(i)[m(\vec{r}_i) - m(\vec{r}_i - \vec{R})] | B \rangle 
= \langle B | \frac{\vec{P}}{N} \cdot \sum_j \gamma_0(j)\vec{\gamma}(j) + 2K\vec{R} \cdot \sum_i \gamma_0(i)\vec{r}_i - KR^2 \sum_i \gamma_0(i) | B \rangle. \]  

(Note that there are no ordering problems as long as \( N \) is a number). Let \( |\alpha\rangle \) be the one body solutions, that is (in units such that \( R_B = 1 \))

\[ \gamma_0[\vec{\gamma} \cdot \vec{p} + m(\vec{r})]\phi_\alpha = \Omega_\alpha \phi_\alpha. \]
If we assume that $|B\rangle$ has all the quarks in the lowest mode then elementary techniques for many-body systems lead to the results

\[
\langle B | \vec{P} \cdot \sum_j \gamma_0(j) \vec{r}(j) | B \rangle = \Omega_0 - \langle 0 | \gamma_0 (m^* + Kr^2) | 0 \rangle,
\]

\[
\langle B | \vec{R} \cdot \sum_i \gamma_0(i) \vec{r}_i | B \rangle = \langle 0 | \gamma_0 r^2 | 0 \rangle,
\]

\[
\langle B | R^2 \sum_i \gamma_0(i) | B \rangle = \frac{1}{N} \langle 0 | \gamma_0 r^2 | 0 \rangle + \left( 1 - \frac{1}{N} \right) \langle 0 | \gamma_0 | 0 \rangle \langle 0 | r^2 | 0 \rangle.
\]

(148)

so that we get

\[
E_{CM} = \Omega_0 - m^* \langle 0 | \gamma_0 | 0 \rangle + K \left( 1 - \frac{1}{N} \right) \left( \langle 0 | \gamma_0 r^2 | 0 \rangle - \langle 0 | \gamma_0 | 0 \rangle \langle 0 | r^2 | 0 \rangle \right)
\]

\[
= \Omega_0 - m^* \langle 0 | \gamma_0 | 0 \rangle + K \left( \langle 0 | \gamma_0 r^2 | 0 \rangle - \langle 0 | \gamma_0 | 0 \rangle \langle 0 | r^2 | 0 \rangle \right) + O(1/N).
\]

(149)

Note that we keep only the leading term in $1/N$. This is consistent with our initial approximation according to which $E_{CM}$ is computed as a correction of order $1/N$ with respect to the leading term in the bag energy.

To proceed we need to evaluate the single particle matrix elements which appear in the expression for $E_{CM}$. To determine the wave function we solve Eq.(147) numerically and adjust the constant $K$ to give $\Omega_0 = 2.04$ – i.e. the lowest energy level of the free bag, in units such that $R_B = 1$. We found $K = 1.74$.

Then we compute $E_{CM}$ numerically according to Eq.(149), as a function of $m^*$. The result is shown in Fig.8, where we also plot the value of $\Omega_0$. One can see that in the range $-1.5 < m^* < 0$, which certainly contains the possible values of $m^*$ in the case of finite real nuclei, the value of $E_{CM}$ is almost constant. For instance at $m^* = -1$, $E_{CM}$ differs from its free value by only 6%. Furthermore, its variation is clearly negligible with respect to that of $\Omega_0$. Thus, for practical purposes, it is a very reasonable approximation to ignore the dependence of $E_{CM}$ on the external field.
Figure 8: Dependence of $E_{CM}$ (full line) and $\Omega_0$ (dashed line) on $m^*$. 
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