BRAUER GROUP OF MODULI OF TORSORS UNDER BRUHAT-TITS GROUP SCHEME $\mathcal{G}$ OVER A CURVE

YASHONIDHI PANDEY

Abstract. Let $X$ be a smooth projective curve over the complex numbers. We compute the Brauer group of the moduli stack of Bruhat-Tits group scheme $\mathcal{G}$-torsors. When $g(X) \geq 3$ we compute the Brauer group of the regularly stable locus of the coarse moduli space of semi-stable $\mathcal{G}$-torsors.

1. Introduction

Let $X$ be a smooth projective curve over $\mathbb{C}$ and let $G$ be a semi-simple, simply-connected and almost simple group over $\mathbb{C}$. Pappas and Rapoport have introduced in [30] a global Bruhat-Tits group scheme $\mathcal{G}\to X$. We will work with a slightly less general group scheme following Balaji and Seshadri [3]. Let $R$ denote a finite non-empty set of points on $X$. Let $X^\circ = X \setminus R$. For $x \in X$, let $D_x = \text{Spec}(\hat{\mathcal{O}}_x)$, let $K_x$ be the quotient field of $\hat{\mathcal{O}}_x$ and let $D_x^\circ = \text{Spec}(K_x)$. In this paper, by a Bruhat-Tits group scheme $\mathcal{G}\to X$ we shall mean that $\mathcal{G}$ restricted to $X^\circ$ is isomorphic to $X^\circ \times G$, and for any closed point $x \in X$, $\mathcal{G}$ restricted to $D_x$ is a parahoric group scheme (cf §2.3) such that the gluing functions take values in $\text{Mor}(D_x^\circ, G) = G(K_x)$. Our interest in this paper is to compute the Brauer group of the moduli stack and that of the coarse moduli space of regularly stable torsors under a Bruhat-Tits group scheme $\mathcal{G}\to X$ (cf §7.1). Recall that the Brauer group for a scheme $Y$ is defined to be the group of equivalence classes of Azumaya algebras with group operation induced by tensor product (cf [3]). For a stack $Y$ however, we will work with the cohomological Brauer group defined as the torsion subgroup of $H^2_{\text{ét}}(Y, \mathbb{G}_m)$. Let $\mathcal{M}_X(\mathcal{G})$ denote the moduli stack of $\mathcal{G}$-torsors on $X$. Our first main result computes the Brauer group of $\mathcal{M}_X(\mathcal{G})$.

Let us fix a maximal torus and a Borel subgroup $T \subset B \subset G$ over $\mathbb{C}$. Let $a_0$ be the alcove determined by $B$ in the apartment $\mathcal{A}$ of $T$. This determines a set $S$ of affine simple roots. For $x \in R$, let the restriction of $\mathcal{G}$ to $\text{Spec}(\hat{\mathcal{O}}_x)$ correspond to the facet $\sigma \subset \mathcal{A}_T$. Let us suppose that after translation by the Iwahori-Weyl group, it corresponds to the facet $\sigma_x$ of $a$. Let $Z^x$ denote the set of simple affine roots of $S$ not vanishing on $\sigma_x$. Let $\theta$ denote the highest root of $G/\mathbb{C}$. Let $\alpha_0$ be the affine simple root corresponding to the highest root of $G$. We set $a_0^\vee = 1$ following [38, Tits, last para page 650]. For simple roots $\alpha \in S \setminus \alpha_0$ of $G/\mathbb{C}$, let $a_\alpha^\vee$ be integers defined by the relation:

$$\theta^\vee = \sum a_\alpha^\vee a_\alpha^\vee.$$
where $\alpha^\vee$ denotes the coroot of $\alpha$.

**Theorem 1.0.1.** The cohomological Brauer group of $M_X(G)$ is $\mathbb{Z}^{\oplus \mathcal{R}}$ modulo $(1, \cdots, 1)$ and \{$(0, \cdots, a_{\alpha_i}^\vee, \cdots, 0) | a_x \in \mathbb{Z}^x, x \in \mathcal{R}$\}.

So the Brauer group is always trivial when $\mathcal{R}$ consists of a single point.

(Semi)-stability of $G$-torsors, equipped with weights, have been defined in \[3\]. Let $M_X^\sigma(G)$ denote the moduli space of regularly stable $G$ torsors. Our second main result computes the Brauer group of $M_X^\sigma(G)$ when $g(X) \geq 3$. For simplicity, we first state our theorem in the case of one parabolic point $x$ and a minimal facet $\sigma \subset \mathfrak{a}$. Let $\mathcal{G}^\sigma \rightarrow X$ be the global Bruhat-Tits group scheme so defined. Let $\Omega_x = \{w_\alpha | \alpha \in \mathbb{Z}^x\}$ denote the fundamental weights of $G$ corresponding to $\mathbb{Z}^x$ where for $\alpha_0$ we take the trivial weight. We shall view them as characters on $T$.

**Theorem 1.0.2.** Let $g_X \geq 3$. The Brauer group of $M_X^\sigma(G^\sigma)$ is the quotient of $\text{Hom}(Z_G, \mathbb{G}_m)$ by $Z_G \hookrightarrow T \xrightarrow{\omega} \mathbb{G}_m$ where $\alpha \in S$ is the unique affine simple root not vanishing at $\sigma$.

We now return to the general case. For any facet $\sigma_x$ we define $l_x = \text{GCD}\{a_{\alpha}^\vee | \alpha \in \mathbb{Z}^x\}$. We define $f = \text{LCM}\{l_x | x \in \mathcal{R}\}$.

**Theorem 1.0.3.** Let $g_X \geq 3$. For every $x \in \mathcal{R}$, let $I^x$ denote the set of all possible $|\mathbb{Z}^x|$-uple integral solutions $(\cdots, n_{\alpha_i}^x, \cdots) \in \mathbb{Z}^{I^x}$ to

(1.0.2) $\sum_{\alpha \in \mathbb{Z}^x} n_{\alpha}^x a_\alpha^\vee = f$.

For each $e^x := (\cdots, n_{\alpha_i}^x, \cdots) \in I^x$, consider the weight $\omega(e_x) = \sum_{\alpha \in \mathbb{Z}^x} n_{\alpha}^x \omega_\alpha$. For each $e = (\cdots, e^x, \cdots) \in \prod_{x \in \mathcal{R}} I^x$ consider the composite

(1.0.3) $Z_G \xrightarrow{\text{diag}} \prod_{x \in \mathcal{R}} T \xrightarrow{\prod_{x \in \mathcal{R}} \omega(e_x)} \prod_{x \in \mathcal{R}} \mathbb{G}_m \xrightarrow{\prod} \mathbb{G}_m$.

The kernel of $Br(M_X^\sigma(G)) \rightarrow Br(M_X(G))$ is the quotient of $\text{Hom}(Z_G, \mathbb{G}_m)$ by the subgroup generated by elements as in (1.0.3) as one varies over elements $e$ in $\prod_{x \in \mathcal{R}} I^x$. In particular, when $\mathcal{R} = \{x\}$, then $Br(M_X^\sigma(G))$ is given by this quotient.

In (14) we cross-check our result with [6], [9] Cor 6.5 Thm 4.5 and [4]. In Remark 10.2.1 we explain the difference between the proof strategies of [4] and [9]. The latter is developed in this paper. In (11) we outline another proof strategy following [6]. This is the main body of the paper.

We develop the proof strategy of Biswas-Holla [9]. A key computational input is from Biswas-Hoffman [8]. This gives new proofs of [6] and [4]. In [9], [4] and [6], the map $\text{Pic}(M_X^\sigma(G)) \rightarrow \text{Pic}(M_X(G))$ of (8.2.2) is an isomorphism. These do not hold in the general case. For groups other than those of type $A$ and $C$, the cohomological Brauer group of the stack may be non-trivial. The appearance of equation (1.0.2) is a new feature for moduli space. By [5] though the regularly stable locus may change as we vary parabolic weights, but the Brauer group only changes with the quasi-parabolic structure. The same phenomena holds in Theorem 1.0.3.

Let $L_{X'}(G)$ denote the ind-group of morphisms from $X' \rightarrow G$. Let $\text{BL}_{X'}(G)$ denote its classifying space (cf [6]). If $H^*_f(\text{BL}_{X'}(G, \mathbb{G}_m)$ had torsion, then it would appear in the Brauer group too (cf proof of Theorem 7.6.1). So in (9) we show that this group is torsion-free. We prove this by passing to the analytic site of $L_{X'}(G)$. For this reason, we need to restrict ourselves to the complex numbers.
We have restricted the group scheme $G$ to the context of Balaji-Seshadri for three reasons. Firstly, the restriction allows us to construct a morphism of stacks $\mathcal{M}_X(G) \to \mathcal{M}_X(G)$ (cf. §7.5.2). We could have considered gluing functions to take values in $\text{Aut}(G)(D_{\circ}x)$ by enlarging $G$ to non-connected groups. But for simplicity, we have restricted to this case. Secondly, semi-stability conditions in [3] or [5] are known in these cases. Thirdly, to estimate the codimension of non-regularly stable locus in the stack of $G$-torsors, we make reduction to $G$-bundle theory by using [3]. We have restricted ourselves to the case of semi-simple, almost simple, simply-connected group $G$ rather than the case of a general reductive group to avoid technical complications.

We have relegated to the appendix results known to experts for which we could not find suitable references. It formulates a condition for cohomological descent in case of a morphism from an ind-scheme to an algebraic stack. In Remarks 12.2.1 and 12.2.2, we explain why we have chosen to work with the Big-étale site of ind-schemes and comorphisms between sites of ind-schemes and stacks.

1.1. Conventions and Notations. We shall abbreviate complex numbers $\mathbb{C}$ by $k$ whenever the notions and results we use hold over any algebraically closed field of arbitrary characteristic. Set $X^\circ = X \setminus \mathbb{R}$. We shall fix a maximal split torus $T \subset G/k$. We shall often need to appeal to results of groups defined over local fields $K$ and apply them to $G_K$, the base change of $G$ to $K$. By subscripts such as $Y_T$ and $\sigma_T$ we mean the vanishing conditions and by superscripts such as $Y^\sigma$ and $\sigma^Y$ we mean the non-vanishing conditions.

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2. Local group theoretical data

Our base field will always be $k$. Let $O$ or $A := k[[t]]$ and $K := k((t)) = k[[t]][t^{-1}]$, where $t$ denotes a uniformizing parameter. Let $G$ be a semisimple simply connected affine algebraic group defined over $k$. All the notions in this section hold for a connected reductive group over a local field. This is called the twisted case in [29]. But for simplicity, we will specialize these to the untwisted case namely that of $G_k(K)$ over $K$. We shall fix a maximal torus $T \subset G$ and let $Y(T) = \text{Hom}(\mathbb{G}_m, T)$ denote the group of all one-parameter subgroups of $T$. For each maximal torus $T$ of $G$, the standard affine apartment $\mathcal{A}_T$ is an affine space under $Y(T) \otimes_{\mathbb{R}} \mathbb{R}$. We now want to consider the group $G(K)$.

In general, there is no origin in an apartment $\mathcal{A}_T$ (cf. [12]). But for recalling parahoric groups schemes, we may identify $\mathcal{A}_T$ with $Y(T) \otimes_{\mathbb{R}} \mathbb{R}$ (see [3, §2]) by choosing a point $v_0 \in \mathcal{A}_T$. For a root $r$ of $G$ and an integer $n \in \mathbb{Z}$, we get an affine functional

(2.0.1) $\alpha = r + n : \mathcal{A}_T \to \mathbb{R}, x \mapsto r(x - v_0) + n.$
These are called the affine roots of $G$. For any point $x \in A_T$, let $Y_x$ denote the set of affine roots vanishing on $x$. For an integer $n \geq 0$, define

$$
\mathcal{H}_n = \{ x \in A | |Y_x| = n \}.
$$

A facet $\sigma$ of $A_T$ is defined to be the connected component of $\mathcal{H}_n$ for some $n$. The dimension of a facet is its dimension as a real manifold. Since we fixed an origin $v_0 \in A_T$, so one can take convex combinations of points in $A$ to define new points in $A$. Points in any facet $\sigma$ of $A_T$ can be expressed as a convex combination of zero-dimensional facets in the closure of $\sigma$. We will call the coefficients barycentric coordinates. A point is in the interior of a facet if and only if its barycentric coordinates are strictly positive.

Let $R = R(T, G)$ denote the root system of $G$ (cf. [34, p. 125]). Thus for every $r \in R$, we have the root homomorphism $u_r : \mathcal{G}_\alpha \rightarrow G$ [34, Proposition 8.1.1]. For any non-empty subset $\Theta \subset A_T$, the parahoric subgroup $\mathcal{P}_\Theta \subset G(K)$ is defined as

$$
\mathcal{P}_\Theta = \langle T(A) \ u_r(t^{m_r} A) | m_r = m_r(\Theta) = -[\inf_{\theta \in \Theta} (\theta, r)] > .
$$

Moreover, by [13, Section 1.7] we have an affine flat smooth group scheme $\mathcal{G}_\Theta \rightarrow$ Spec$(A)$ corresponding to $\Theta$. The set of $K$-valued (respectively, $A$-valued) points of $\mathcal{G}_\Theta$ is identified with $G(K)$ (respectively, $\mathcal{P}_\Theta$). The group scheme $\mathcal{G}_\Theta$ is uniquely determined by its $A$-valued points. For simplicity, in this paper $\Theta$ will always be a facet of $A_T$. In this case, we may check (2.0.3) for any $\theta$ in the interior $\Theta^\circ$ of $\Theta$. The pro-unipotent radical $\mathcal{P}_\Theta^u \subset \mathcal{P}_\Theta$ is defined as

$$
\mathcal{P}_\Theta^u = \langle T(1 + t) \ u_r(t^{1-\[\theta, r]} A) | \theta \in \Theta^\circ > .
$$

Let $\mathcal{G}_\Theta^u \rightarrow$ Spec$(A)$ denote the affine flat group scheme corresponding to $\mathcal{P}_\Theta^u$.

We choose a Borel $B$ in $G/k$ containing $T$. For a facet $\sigma \subset A$, we shall denote by $\mathcal{G}_\sigma$ the parahoric group scheme over $O$. Let $a_0$ denote the unique alcove in $A$ whose closure contains $v_0$ and is contained in the finite Weyl chamber determined by our chosen Borel. This determines a set $S$ of simple affine roots $\alpha$. For $Z \subset S$, let $\sigma_Z \subset a$ (resp. $\sigma_Z^\circ \subset a$) be the facet where exactly the roots $\alpha_i \in Z$ vanish i.e such that $\alpha_i(\sigma_Z) = 0$ (resp. where exactly $\alpha_i \in Z$ do not vanish i.e $\alpha_i(\sigma_Z^\circ) \neq 0$) for all $\alpha_i \in Z$. Similarly, for a facet $\sigma$ let

$$
Y_{\sigma} = \{ \alpha | \text{ affine root vanishing at } \sigma \}.
$$

It is possible to identify $Y_{\sigma}$ with a closed sub root system of the root system of $G$ as in [11] and we will do so. Let $Z_{\sigma} \subset S$ be the subset corresponding to $Y_{\sigma}$ and let

$$
Z_{\sigma}^\circ \subset S
$$

denote the complementary set. We refer the reader to [13, 4.6.12] for

**Proposition 2.0.1.** The root system of the reductive quotient $\mathcal{G}_\sigma/\mathcal{G}_\sigma^u$ of the special fiber of $\mathcal{G}_\sigma$ is given by $Y_{\sigma}$.

We refer the reader to [31, Corollary 4.2.2]

**Corollary 2.0.2.** Let $s$ and $b$ be facets of $a$. Suppose that $s$ is in the closure of $b$. Let $G_{s, b} \subset Y_s$ be defined by

$$
G_{s, b} = \{ \alpha \in Y_s | 0 \leq \alpha(b) \}.
$$

The group $\mathcal{G}_b/\mathcal{G}_s^u$ is the parabolic subgroup of $\mathcal{G}_s/\mathcal{G}_s^u$ given by $G_{s, b}$.
2.1. Loop groups and their flag varieties. We need to recall some facts from [29, §2, §8] about affine Weyl group and Schubert varieties. In loc. cit. these are stated for arbitrary reductive groups that split over a tamely ramified extension $\bar{K}/K$. Our interest is in applying these results to $G_K$. Since $\bar{K} = K$ for us, so we shall specialize to this case.

Let $k$ be a field. Let $G$ be a flat affine group scheme of finite type over $k[[t]]$. For a $k$-algebra $R$ let $R[[t]]$ denote the formal power series ring and $R((t)) = R[[t]][t^{-1}]$ the field of Laurent polynomials with coefficients in $R$. Recall that the loop group $L \mathcal{G}$ of $\mathcal{G}$ represents the functor mapping $R$ to $\mathcal{G}(R((t)))$. It is represented by an ind-affine scheme. Recall that the jet group $L^+ \mathcal{G}$ represents the functor mapping $R$ to $\mathcal{G}(R[[t]])$. It is represented by a closed subscheme of $L \mathcal{G}$ which is affine. For a facet $\sigma$, the quotient of fpqc-sheaves

\begin{equation}
\mathcal{F}l_\sigma = L \mathcal{G}_\sigma / L^+ \mathcal{G}_\sigma
\end{equation}

is the flag variety associated to $\sigma$. It is represented by an ind-scheme that is ind-projective over $k$. It represents the functor that to $R$ associates the set of pairs $(\mathcal{E}, \theta)$ where $\mathcal{E}$ is a $\mathcal{G}_\sigma$-torsors on $Spec R[[t]]$ together with a trivialization given by $\theta$ over $Spec R((t))$. We recall

**Theorem 2.1.1.** [29 Theorem 1.4] Let $R$ be a strictly henselian ring over $k$, for any point $Spec(R) \rightarrow \mathcal{F}l_\sigma$, we have $Spec(R) \times_{Spec(k)} L^+ \mathcal{G}_\sigma \simeq Spec(R) \times_{\mathcal{F}l_\sigma} LG$.

**Proposition 2.1.2.** [29 Prop 10.1] There is an isomorphism $Pic(\mathcal{F}l_\mathfrak{a}) \simeq \mathbb{Z}^S$ defined by the degrees of the restrictions to $\mathbb{P}^1_\mathfrak{a} = L^+ \mathcal{G}_\mathfrak{a} / L^+ \mathcal{G} \hookrightarrow \mathcal{F}l_\mathfrak{a}$.

2.2. Weyl groups and Schubert varieties. Recall that the Kottwitz homomorphism $\kappa_H$ is defined for a reductive group $H$ over an arbitrary local field $L$ (cf [29 2.a.2, page 127]). Let us specialize to cases of interest to us. By (loc. cit. Step 1.), when $H = G_m$ then writing $f \in LH(k) = k((t))^\times$ in the form $f = t^j u$ where $u \in k[[t]]^\times$, we have $\kappa_H(f) = j$. For the maximal split torus $T_K \subset G_K$, $\kappa_T$ is just the sum of $\kappa_{G_m}$ in each coordinate. Let

$$T(K)_1 = \ker(\kappa_T).$$

Let $N = N(T)$ be the normaliser of $T$. One defines the **Iwahori-Weyl group associated to $T$** as the quotient

\begin{equation}
\hat{W} = N(K) / T(K)_1.
\end{equation}

Setting $W_0 = N(K)/T(K)$ as the relative Wey group, one has

\begin{equation}
0 \rightarrow X_*(T) \rightarrow \hat{W} \rightarrow W_0 \rightarrow 0.
\end{equation}

The choice of a point $v_0 \in \mathcal{A}$ splits the above sequence. We quote

**Proposition 2.2.1.** [29 Appendix Prop 8] Let $I$ denote an Iwahori subgroup of $G_K(K)$. Then $G_K(K) = IN(K)I$ and the map $InI \rightarrow n \in \hat{W}$ induces a bijection $I \setminus G(K)/I \simeq \hat{W}$. More generally, let $P$ and $P'$ be parahoric subgroups associated to facets $F$ (resp $F'$) contained in the apartment corresponding to $T$. Let $\hat{W}' := (N(K) \cap P)/T(K)_1$, and similarly define $\hat{W}'$. Then $P \setminus G_K(K)/P' \simeq \hat{W}' \backslash \hat{W}/\hat{W}'$.

The **affine Weyl group associated to $T$** is the Iwahori-Weyl group $Wa$ of the simply-connected cover $G_{sc,K}$ of the derived subgroup $G_{der,K}$ of $G_K$. Since we have assumed that $G/k$ is simply-connected and semi-simple, so for us

\begin{equation}
Wa = \hat{W}.
\end{equation}
It is known that $W_a$ is a Coxeter group. Then (cf. loc. cit. page 152) $S$ generate $W_a$. For each $w \in W_a$, its length $l(w)$ is the minimal number of factors in a product of $s_i$'s representing $w$. Any product realizing the minimum number of factors of $w$ is called a reduced decomposition of $w$. Let us recall the Bruhat order (cf. loc. cit. page 152). One says that $w' \leq w$ if $w'$ is obtained by replacing some factors of a reduced decomposition of $w$ by 1. It is a fact that the set of such $w'$ is independent of reduced decomposition of $w$ chosen.

We now recall Schubert cells and varieties (cf. loc. cit. Definition 8.3 and [39 §2]). Let $Y, Y' \subset S$ be two subsets. Then the $L^+G_\sigma$-orbit of $F_{l,\sigma}$ are parametrized by $WY \backslash W/WY'$ where $WY$ is the Weyl group of $G_{\sigma} \otimes k$. The Schubert cell $YC_{w}$ (or $\sigma C_w$) is the reduced orbit $L^+G_{\sigma}n_w \subset F_{l,\sigma}$, where $n_w \in N(K)$ a representative of $w \in WY \backslash W/WY'$ (cf Proposition 2.2.1). The Schubert variety $YS_{w}$ (or $\sigma S_w$) is the reduced scheme with underlying set the Zariski closure of $YC_w$.

When $\sigma Y = a$ then we shall simply write $C_w$ (or $C_w'$) and $S_w$ (or $S_w'$). Further when $\sigma Y = \sigma G = a$, then we shall write $C_w$ and $S_w$.

2.3. The group scheme. Let $R \subset X$ be a non-empty finite set of closed points. For each $x \in R$, we choose a facet $\sigma_x \subset A_T$. Let $G_{\sigma_x} \to \text{Spec}(\mathcal{O}_x)$ be the parahoric group scheme corresponding to $\sigma_x$. Let $X^0 = X \setminus R$. For $x \in X$, let $D_x = \text{Spec}(\mathcal{O}_x)$, let $K_x$ be the quotient field of $\mathcal{O}_x$ and let $D_x^0 = \text{Spec}(K_x)$. In this paper, by a Bruhat-Tits group scheme $G \to X$ we shall mean that $G$ restricted to $X^0$ is isomorphic to $X^0 \times G$, and for any closed point $x \in X$, $G$ restricted to $D_x$ is a parahoric group scheme $G_{\sigma_x}$ such that the gluing functions take values in $\text{Mor}(D_x^0, G) = G(K_x)$. The restriction on gluing functions is useful in the construction in 4.5.2. Further, this is also the setup of [5] Balaji-Seshadri Defn 5.2.1 which is our main reference for (semi)-stability conditions. Let us remark the setup of [20] Heinloth and [39] Zhu are the same. They consider more general group schemes which may not be split over the function field of $X$. The global conditions over $G$ demanded in [20] are satisfied in our case (cf. [20] Introduction).

By [20] Lemma 5 it is always possible to glue $X^0 \times G$ with $\{G_x | x \in R\}$ along the fpqc cover $\{X^0\} \cup \{\text{Spec}(\mathcal{O}_x) | x \in R\}$ of $X$ to get a group scheme $G \to X$. Of course, $G$ depends on the gluing data. For instance, if $E \to X$ is a principal $G$-bundle, then $\text{Aut}(E) \to X$ is such a group scheme. Its restriction to $X^0$ and $\text{Spec}(\mathcal{O}_x)$ is always the trivial group scheme, while it may or may not be trivial over $X$. Similarly, let $x_0 \in X$ be a closed point, and let $M$ denote the stack of vector bundles of rank $r$ on $X$ and determinant $O_X(-dx_0)$ where $0 \leq d < r$. Then for any $V \in M$, the adjoint group scheme $\text{Aut}(V) \to X$ is obtained by gluing $X^0 \times G$ with $G_x$ where $\sigma$ is the unique vertex of the alcove $a$ where only the affine simple root $\alpha_d$ does not vanish.

2.4. Pic($F_{l,\sigma}$) in terms of characters on $T$. Recall (cf [23] page 13 or [33] page 131) that one has a canonical central extension

$$1 \to \mathbb{G}_m \to \tilde{L}G \to LG \to 1.$$ (2.4.1)

Let $\tilde{L}^+G_\sigma$ denote the inverse image of $L^+G_\sigma$ in $\tilde{L}G$. When $\sigma = a$ this is called the Iwahori subgroup of Kac-Moody theory (cf [22] page 491, 13.2.2.] and [29] §10). Then the fpqc-quotient $\tilde{L}G/\tilde{L}^+G_\sigma$ also identifies with $F_{l,\sigma}$. From the $\tilde{L}^+G_\sigma$-fibration
\[ \tilde{L}G \to \mathcal{F}l_\sigma \] using any character \( \chi \) of \( \tilde{L}^+G_\sigma \) we can make the line bundle as follows:

\[ L_\chi := \tilde{L}G \times_\chi \mathbb{G}_m \to \tilde{L}G/\tilde{L}^+G_\sigma = \mathcal{F}l_\sigma. \]

Conversely, one knows that line bundles on \( \mathcal{F}l_\alpha \) are all homogenous line bundles coming from characters of the Iwahori subgroup in Kac-Moody theory (cf [29 §10]). Thus their pull-back to \( \tilde{L}G \) is trivial. So

\[ \text{Pic}(\mathcal{F}l_\sigma) \simeq X^*(\tilde{L}^+G_\sigma). \]

Let us make this more explicit. Then \( \text{Pic}(\mathcal{F}l_\sigma) \) is a free abelian group on \( Z^\sigma \) which is the set of affine simple roots in \( S \) not vanishing on \( \sigma \). Let \( \alpha \in Z^\sigma \) be a simple affine root. Let \( \tilde{\omega}_\alpha \) be the fundamental weight of \( \tilde{L}G \) corresponding to \( \alpha \). Then \( \tilde{\omega}_\alpha \) gives a character on \( \tilde{L}^+G_\sigma \). In this way, the character group in \( 2.4.3 \) is a free abelian group on the set \( \tilde{\omega}_{Z^\sigma} \) of fundamental weights of \( \tilde{L}G \) corresponding to \( Z^\sigma \).

The extension \( 2.4.1 \) splits uniquely over \( L^+(G_A) \) (cf [33, 29 §10]). Since \( L^+G_\alpha \to L^+(G_A) \), so its restriction to \( L^+G_\alpha \) also splits. Let \( \tilde{T} \) denote the inverse image of \( T \) in \( \tilde{L}^+G_\alpha \). Then sequence \( 1 \to \mathbb{G}_m \to \tilde{T} \to T \to 1 \) also splits by the following lemma.

**Lemma 2.4.1.** Recall that in [2] we fixed \( k[T/k \subset B/k \subset G/k \) and \( a \). We have a canonical inclusion of group schemes \( T \to L^+G_\alpha \).

**Proof.** Let \( A = k[[t]] \). So we have an inclusion \( k \to A \) giving \( \text{Spec}(A) \to \text{Spec}(k) \). At the closed fiber, we have an inclusion of \( A \)-group schemes \( T_k \to G_k \).

This gives an inclusion of \( A \)-group schemes \( T_A \to G_A \) by pulling-back to \( \text{Spec}(A) \). Consider the reduction at the special fiber: the sections of \( T_A \) land in \( T_k \) which is contained in the Borel \( B_k \) of \( G_k \) by hypothesis. Since the Iwahori group scheme \( G_\alpha \) is defined as the pull-back of \( B_k \subset G(k) \) by the evaluation at zero map, so we have a factorization \( T_A \twoheadrightarrow G_\alpha \twoheadrightarrow G_A \). Since \( T_A = p^*T_k \), so by adjunction we have \( \text{Mor}_A(p^*T_k, G_\alpha) = \text{Mor}_k(T_k, p_*G_\alpha) \). But \( p_*G_\alpha = L^+G_\alpha \). \( \square \)

So we get in inclusion \( T \to \tilde{T} \) and the following sequence splits

\[ 0 \to X^*(T) \to X^*(\tilde{T}) \to \mathbb{Z} \to 0. \]

By [29 (10.4)], we have

\[ Z^S \simeq \text{Pic}(\mathcal{F}l_\alpha) \simeq X^*(\tilde{L}^+G_\alpha) \simeq X^*(\tilde{T}). \]

Let us make the identification \( Z^S \simeq X^*(\tilde{T}) \) more explicit. Let

\[ S = < \alpha_0 = \delta - \theta, \alpha_1, \cdots, \alpha_n > \]

where \( \delta \) pairs with the central \( \mathbb{G}_m \to \tilde{T} \) and is the trivial character on the image of \( T \) in \( \tilde{T} \) under the splitting \( \delta \). \( \theta \) is the highest root and the remaining are the simple roots of \( G/\mathbb{C} \) viewed as affine roots of \( L \tilde{G} \). Let us take as coroots

\[ S^\vee = < c - \theta^\vee, \alpha_1^\vee, \cdots, \alpha_n^\vee > \]

where \( c \) is the image of the central \( \mathbb{G}_m \) in \( \tilde{T} \). We normalize the Cartan-Killing form \( (, \) on \( X^*(T) \otimes_\mathbb{Z} \mathbb{R} \) so that \( (\theta|\theta) = 2 \). The usual Cartan matrix has \( (i, j) \)-entry \( < \alpha_i^\vee, \alpha_j > \) where \( 1 \leq i, j \leq n \). This is extended by adding a 0-th row and column and defining \( A_{0,0} = 2 \), \( A_{0,j} = -\alpha_j(\theta^\vee) \) and \( A_{j,0} = -\theta(\alpha_j^\vee) \) for \( 1 \leq j \leq l \). The choice of \( S \) and \( S^\vee \) realizes the extended Cartan matrix so defined (cf [22, XIII, page 484]). We will take

\[ \{ \epsilon_\alpha \in X^*(\tilde{T})| \alpha \in S \} \text{ dual to the coroots } S^\vee. \]
This makes the identification $\mathbb{Z}^S = \mathbb{X}^*(\hat{T})$ explicit.

Now we wish to express $\epsilon_\alpha$ in terms of the fundamental weights of $G/\mathbb{C}$ and $\delta$. Under this identification, let $\{\omega_\alpha \in \mathbb{X}^*(T) | \alpha \in S \setminus \alpha_0\}$ be a $\mathbb{Z}$-basis of weights of $G/\mathbb{C}$. We will extend $\omega_\alpha$ as characters on $\hat{T}$ by declaring them to be trivial on the central $G_m$ and denote this extension by $\omega_\alpha|_\hat{T}$. For $\alpha \in S \setminus \alpha_0$, let $a^\vee_\alpha$ be positive integers defined by the relation $\theta^\vee = \sum a^\vee_\alpha \alpha^\vee$.

**Proposition 2.4.2.** If $\alpha \in S \setminus \alpha_0$ then $\epsilon_\alpha|_T = \omega_\alpha$ and $\epsilon_0|_T$ is trivial.

Let us remark in passing that this agrees with [29, (10.5)], in other words we have $(1, \cdots, a^\vee_\alpha, \cdots) A = (\cdots, 0, \cdots)$.

**Proof.** We check that weights set as follows are dual to the coroots (2.4.7)

\[
\epsilon_0 = \delta, \quad \epsilon_\alpha = \omega_\alpha|_T + a^\vee_\alpha \delta.
\]

\[\square\]

2.5. **Central charge of line bundles on flag varieties.** For each flag variety $\mathcal{F}_a$, the obstruction to lifting the action of $LG$ to $\text{Pic}(\mathcal{F}_a)$ defines a central extension $LG$ of $LG$. The weight of the action of central $G_m$ on line bundles defines central charge homomorphism (cf [29, Remark 10.2])

(2.5.1) \[ c_\sigma : \text{Pic}(\mathcal{F}_a) \rightarrow \mathbb{Z}. \]

It satisfies the property that $\ker(c_\sigma) = \mathbb{X}^*(G_\sigma \otimes k)$. By [30, page 11, last paragraph] the central charge on $\text{Pic}(\mathcal{F}_a)$ can be defined after pull-back to $\text{Pic}(\mathcal{F}_a)$. On $\mathcal{F}_a$ it sends the line bundle $L_{c_\alpha}$ to $a^\vee_\alpha$ (cf [29, §10]).

3. **Preliminaries on Global constructions**

We begin with a very general cadre to be able to reconcile the references.

Let $C \rightarrow S$ be a smooth curve. Let $G \rightarrow C$ be a smooth affine group scheme over $C$. Let $\text{Spec}(R) \rightarrow S$ be a $S$-scheme and $y : \text{Spec}(R) \rightarrow C$ be a $R$-point of $C$. Let $C_R := \text{Spec}(R) \times_S C$. Let $\Gamma_y \subset C_R$ denote the graph of $y$ and $\hat{\Gamma}_y$ the completion of $C_R$ along $\Gamma_y$. So we have the closed inclusion $\Gamma_y \hookrightarrow \hat{\Gamma}_y$. Let $s^y_R : \text{Spec}(R) \rightarrow C_R$ denote the section corresponding to $y$.

The jet group $\mathcal{L}^+G$ of $G$ is defined as follows. For any $S$-scheme $R$,

\[ (3.0.1) \quad \mathcal{L}^+G(R) = \{(y, \beta) | y : \text{Spec}R \rightarrow C, \beta \in G(\hat{\Gamma}_y)\}. \]

It is representable by a $S$-scheme.

The loop group $\mathcal{L}G$ of a global group scheme $G \rightarrow C$ is defined as follows. Let $\hat{\Gamma}_y = \hat{\Gamma}_y \setminus \Gamma_y$. For any $S$-scheme $R$,

\[ (3.0.2) \quad \mathcal{L}G(R) = \{(y, \beta) | y : \text{Spec}R \rightarrow C, \beta \in G(\hat{\Gamma}_y)\}. \]

It is representable by an ind-scheme over $S$. It represents the functor that to a $S$-scheme $R$ associates $(y, \mathcal{E}, \alpha, \beta)$ where

1. $y : \text{Spec}(R) \rightarrow C$ is a $S$-point
2. $\mathcal{E} \rightarrow C_R$ is a $G$-torsor
3. $\alpha$ is a trivialization of $\mathcal{E}$ restricted to $C_R \setminus s^y_R$
4. $\beta$ is a trivialization of $\mathcal{E}$ restricted to $\hat{\Gamma}_y$.

Let us recall the twisted affine flag manifold. Let $Gr_G \rightarrow C$ represent the functor that to every $S$-scheme $\text{Spec}(R)$ associates $(y, \mathcal{E}, \alpha)$ where
(1) \( y : \text{Spec}(R) \to C \) is a \( S \)-point,
(2) \( \xi \) is a \( \mathcal{G} \)-torsor on \( C \times_S \text{Spec}(R) \) and,
(3) \( \alpha \) is a trivialization of \( \xi \) restricted to \( C_R \setminus s_R^\beta \).

It is a formally smooth ind-scheme over \( C \). It can be constructed by sheafifying the quotient \( \mathcal{L}\mathcal{G}/\mathcal{L}^+\mathcal{G} \) in the fpqc-topology. The natural map \( \mathcal{L}\mathcal{G} \to \mathcal{G} \) forgets the section \( \beta \). One also has a natural forgetful map \( p : \mathcal{G} \to \text{Bun}_C(\mathcal{G}) \).

Let us specialize to the case \( S = \text{Spec}(k) \), \( \text{Spec}(R) = S \) and \( y : \text{Spec}(S) \to C \) is a point \( x \in C(k) \). We shall denote the fiber at \( x \in C(k) \) of \( \mathcal{G} \to \mathcal{G} \) by

\[
\text{Gr}_{\mathcal{G},x}
\]

in this special situation. Similarly, let \( (\mathcal{L}\mathcal{G})_x \) and \( (\mathcal{L}^+\mathcal{G})_x \) denote the fibers at \( x \) of \( \mathcal{L}\mathcal{G} \to \mathcal{G} \) and \( \mathcal{L}^+\mathcal{G} \to \mathcal{G} \). They are isomorphic to \( L(\mathcal{G}|_{\mathcal{G}_x}) \) and \( L^+(\mathcal{G}|_{\mathcal{G}_x}) \) respectively. Let \( (\mathcal{L}\mathcal{G})_x/(\mathcal{L}^+\mathcal{G})_x \) denote the sheafified quotient in fpqc-topology. Thus we have a natural isomorphism \( \mathcal{G} \mathcal{r}_{\mathcal{G},x} = (\mathcal{L}\mathcal{G})_x/(\mathcal{L}^+\mathcal{G})_x \). These notations agree with [20] [2].

Let \( \sigma_x \) denote the facet defining the parahoric group scheme \( \mathcal{G}_x \) obtained by restricting \( \mathcal{G} \to C \) to \( \Gamma_x \). Thus we have a natural isomorphism

\[
\text{Gr}_{\mathcal{G},x} = (\mathcal{L}\mathcal{G})_x/(\mathcal{L}^+\mathcal{G})_x \cong \mathcal{Fl}_{\sigma_x}.
\]

We quote

**Theorem 3.0.1.** [39] Zhu, Prop 4.1] Let \( C \) be a smooth curve over \( k \). Let \( \mathcal{G} \to \mathcal{G} \) be a Bruhat-Tits group scheme such that over the generic point it is almost simple, absolutely simple, and simply-connected. Let \( L \) be a line bundle on \( \mathcal{G} \mathcal{r}_{\mathcal{G}} \). Then the function \( c_L \) that associates to every \( x \in C(k) \), the central charge (see (2.5.1)) of the restriction of \( L \) to \( \mathcal{G} \mathcal{r}_{\mathcal{G},x} \) is constant.

3.1. Central charge of line bundles on moduli stack of \( \mathcal{G} \)-torsors. By [20] [6, 1st paragraph] the central charge of line bundles on \( \mathcal{M}_X(\mathcal{G}) \) is defined at an arbitrary point \( z \in X \) after pulling them back to \( \mathcal{G} \mathcal{r}_{\mathcal{G},z} \) (cf (3.0.3)) and then applying the central charge homomorphism. We make this more precise as follows. Consider the diagram

\[
\begin{array}{ccc}
\mathcal{G} \mathcal{r}_{\mathcal{G}} & \longrightarrow & \mathcal{M}_X(\mathcal{G}) \\
\downarrow & & \downarrow \\
X & \longrightarrow & \mathcal{M}_X(\mathcal{G})
\end{array}
\]

(1) By (3.0.3), \( \mathcal{G} \mathcal{r}_{\mathcal{G},z} \) is naturally isomorphic to \( \mathcal{Fl}_{\sigma_z} \).
(2) For computing central charge, it suffices to pull-line bundles to \( \mathcal{Fl}_a \).
(3) By [39] Prop 4.1], this definition is independent of \( z \in X \).

4. SCHUBERT VARIETIES \( S_w \)

4.1. Bott-Samelson-Demazure-Hansen desingularization of \( S_w \). Let us fix an alcove \( a \). Let \( w \in W \) and fix a reduced decomposition \( w = s_{i_1} \cdots s_{i_r} \), which we will denote by \( \tilde{w} \), in terms of the simple reflections about the walls of \( a \). For a reflection \( s_{i_j} \), let \( \mathcal{G}_{i_j} \) denote the corresponding parahoric group scheme. The Demazure variety \( D(\tilde{w}) \) is defined as

\[
D(\tilde{w}) = L^+\mathcal{G}_{i_1} \times L^+\mathcal{G}_{a} \cdots \times L^+\mathcal{G}_{i_r} / L^+\mathcal{G}_{a}.
\]

**Proposition 4.1.1.** The Schubert cells \( C_w \) (cf (2.2)) are affine spaces over \( k \).
Proof. Recall (cf [29] Prop 8.8, (8.13)) that the Demazure variety $D(\hat{w})$ is the quotient of $\prod_{j=1,\ldots} L^+ G_j$ by $(L^+ G_a)^r$ by the right action

$(p_1, \ldots, p_r)(b_1, \ldots, b_r) = (p_1 b_1^{-1} p_2 b_2, \ldots, b_{r-1}^{-1} p_r b_r)$.

The variety $D(\hat{w})$ has two open subsets $D^{\circ}(\hat{w})$ and $D^0(\hat{w})$ which are defined by the condition that the last co-ordinate does not belong to $L^+ G_a$ and the condition that $(p_1, \ldots, p_r) \in D^0(\hat{w})$ if no $p_i$ belongs to $L^+ G_a$ respectively. The open subset $D^0(\hat{w})$ maps isomorphically onto the Schubert cell $C_w$ (cf [29] proof of Prop 9.6)]. When $w$ consists of a single reflection say $s_i$, the fpqc-quotient $L^+ G_a / L^+ G_a$ identifies with $\mathbb{P}^1_k$ by [29] Prop 8.7 and thus the open subset $C_w$ equals $\mathcal{A}_k$. More generally, let us write $w = w's_i$. We want to show that $D^0(\hat{w}) \rightarrow D^0(\hat{w}')$ is a line bundle.

To this end, let us consider the projection map $\prod_{j=1}^{r-1} L^+ G_{i_j} \rightarrow \prod_{j=1}^{r-1} L^+ G_{i_j}$ which forgets the last coordinate. Quotienting by $L^+ G_a$, we see that $D(\hat{w}) \rightarrow D^{\circ}(\hat{w}')$ is a $\mathbb{P}^1_k$-bundle. Let us restrict this map to the open subset $D(\hat{w}) \cap D^{\circ}(\hat{w})$. Then $D(\hat{w}) \cap D^{\circ}(\hat{w})$ becomes a line bundle. Since we have $D^0(\hat{w}) \rightarrow D^0(\hat{w}')$, so $D^0(\hat{w}) \rightarrow D^0(\hat{w}')$ is also a line bundle. By induction, we may suppose that $D^0(\hat{w}')$ is an affine space over $k$. So $D^0(\hat{w})$ is the trivial line bundle by the Quillen-Suslin theorem ([32]). Thus $C_w = D^0(\hat{w})$ is also an affine space over $k$.

$$\square$$

Proposition 4.1.2. We have $H^n_{\text{ét}}(D(\hat{w}), \mathcal{O}) = 0$ for $n \geq 1$.

Proof. This follows because $D(\hat{w})$ is an iterated $\mathbb{P}^1$-fibration. Indeed, one can induct on the reduced length of $w$. The base case of length one is clear because $D(\hat{w}) = \mathbb{P}^1$. Suppose that length of $w'$ is one less than that of $w$ and $w = w's$ where $s$ is a reflection. Now consider the natural projection $\pi : D(\hat{w}) \rightarrow D(\hat{w}')$, which is a $\mathbb{P}^1$-fibration by the proof of Prop 4.1.1. We have $\pi_* \mathcal{O}_{D(\hat{w})} = \mathcal{O}_{D(\hat{w}')}$. $R^i(\pi_* \mathcal{O}_{D(\hat{w})}) = 0$ for $i > 0$. Now $H^n(\hat{w}', \mathcal{O}_{D(\hat{w})}) = 0$ by induction hypothesis. Thus by considering the spectral sequence corresponding to $R^n(\pi_* \mathcal{O}_{D(\hat{w})})$, it follows that $H^n(\hat{w}, \mathcal{O}_{D(\hat{w})}) = 0$.

4.2. Results on Schubert varieties.

Proposition 4.2.1. For Schubert varieties $S_w$, we have $H^n_{\text{ét}}(S_w, \mathcal{O}) = 0$ for $n \geq 1$. Let $S := \prod S_w$ denote a finite product of Schubert varieties. We have isomorphisms

$$H^2_{\text{ét}}(S, \mathcal{O}) \simeq H^2_{\text{ét}}(S, \mathcal{O})$$

Proof. Let $\pi_w : D(\hat{w}) \rightarrow S_w$ denote the projection map. Since we are in characteristic zero, by [29] Theorem 8.4, $S_w$ is normal. By [29] (9.16), Prop 9.7 d)] for $i > 0$, we have $R^i(\pi_{w_*}) \mathcal{O}_{D(\hat{w})} = 0$ and $(\pi_w)_* (\mathcal{O}_{D(\hat{w})}) = \mathcal{O}_{S_w}$. Consider the composition of functors $\Gamma_{D(\hat{w})} \circ \pi_* = \Gamma_{D(\hat{w})}$. Let us consider the associated Grothendieck spectral sequence

$$H^n_{\text{ét}}(S_w, R^i(\pi_w)_* (\mathcal{O}_{D(\hat{w})})) \Rightarrow H^{n+i}_{\text{ét}}(D(\hat{w}), \mathcal{O}_{D(\hat{w})})$$

For every $n \geq 1$ we have $E_2^{n-2, 1} \rightarrow E_2^{n, 0} \rightarrow E_2^{n+2, -1}$. Now $E_2^{n-2, 1} = 0$ because $R^1(\pi_w)_* (\mathcal{O}_{D(\hat{w})}) = 0$. Thus $E_2^{n, 0} = E_2^{n, 0}$. Now we have $E_2^{n-3, 2} \rightarrow E_2^{n, 0} \rightarrow E_2^{n+3, -2}$. Since $E_2^{n+3, -2}$ vanishes, this $E_2^{n, 0} = E_2^{n, 0} = \cdots = E_\infty^{n, 0}$. Thus we get the
inclusion $H^n(S_w, R^0(\pi_w)_*(O_{D(\tilde{w})})) = E_2^{n,0} = E_\infty^{n,0} \to H^n(D(\tilde{w}), O)$. Now the right term vanishes by Prop 4.1.2. This shows $H^n_{et}(S_w, O) = 0$ for $n \geq 1$.

Now consider the exponential sequence $0 \to \mathbb{Z} \to O_{an} \to O_{an}^\times \to 0$ on $S_w(\mathbb{C})$. The second isomorphism of (4.2.1) follows because $H^n_{an}(S(\mathbb{C}), \mathcal{O}) = 0$ for $n > 0$. This can be proven by induction on the number of factors. For the case of one factor, this follows by reasoning as before for $H^n_{et}(S_w, O)$ but now in the complex analytic category. The general case follows by inducting on the number of factors and considering the Leray sequence for the fibration that forgets one factor. □

Proposition 4.2.2. Let $S := \prod S_w$ denote a finite product of Schubert varieties. We have $H_3(S, \mathbb{Z}) = 0$, $H_1(S, \mathbb{Z}) = 0$ and $H_2(S, \mathbb{Z})$ is a free abelian group on 2-cells in $S_w$. We have $H^3_{et}(S, G_m)_{tor} = 0$.

Proof. By Prop 4.1.1 the Schubert cell $C_w$ is an affine space over $k$. Since $k = \mathbb{C}$, so $S_w$ has the structure of a CW complex with only even dimensional cells. Now the statements on homology groups follow. We have $H^3_{et}(S, \mathbb{Z}) = 0$. This follows from the universal coefficient theorem, since we have

$$0 \to Ext^1_{et}(H_2(S, \mathbb{Z}), \mathbb{Z}) \to H^3_{et}(S, \mathbb{Z}) \to Hom(H_3(S, \mathbb{Z}), \mathbb{Z}) \to 0.$$ 

By (4.2.1) $H^3_{et}(S, G^*_m)_{tor} = 0$.

Consider the exact sequence $1 \to \mu_n \to G_m \to G_m / \mu_n \to 1$ in etale and analytic topology. It induces multiplication by $n$ on cohomology $H^*(\cdot, G_m) \to H^*(\cdot, G_m / \mu_n)$. Any torsion class in $H^2(S, G_m)$ is represented by a class in $H^2(S, \mu_n)$ for some $n$ in both topologies. Therefore $\lim_n H^2(\cdot, \mu_n) = H^2(\cdot, G_m)_{tor}$ in both topologies. For the constant sheaf defined by $n$-th roots of unity $\mu_n$ by [2] Exposé 16 Thm 4.1] which holds for not necessarily smooth schemes, we have

$$H^2_{et}(S, \mu_n) = H^2_{an}(S, \mu_n).$$

So $H^2_{et}(S, G_m)_{tor} = 0$. □

5. IND-GRASSMANNIAN $Q_G$

We denote the ind-Grassmannian by

$$(5.0.1) \quad Q_G = \prod_{x \in R} Gr_G, x = \prod_{x \in R} Fl_{\sigma_x}.$$

It represents the functor that to a scheme $S$ associates families of $G$-torsors on $X \times S$ together with a section on $X^0 \times S$, where $X^0 = X \setminus R$.

For generalities on ind-schemes (such as big-Étale sites) see §12.1

Proposition 5.0.1. The cohomological Brauer group $H^2_{et}(Q_G, O^*_{Q_G})_{tor} = 0$.

Proof. Let us first handle the case of one parabolic point. So $Q_G$ is simply $Fl_{\sigma}$ for a certain facet $\sigma$ in an alcove $a$.

Since in our case, the Iwahori-Weyl group $\tilde{W}$ equals the affine Weyl group $W_a$, so $\tilde{W}$ is a Coxeter group. Hence it has the Bruhat partial order. This partial ordering induces a partial ordering on $\tilde{W} / W_\sigma$ as well as follows. For $\pi = u mod W_\sigma$ and $\bar{\pi} = v mod W_\sigma$, we have $\pi \leq \bar{\pi}$ if there exists a $w \in W_\sigma$ such that $u \leq v w$.

Recall (29) that the Schubert variety $S'_w$ has the structure of a finite dimensional
defines an abelian sheaf on $RZ$ and its cokernel is free over tive. Thus the inverse system

$$w$$

reflections about the walls of $correspond bijectively to the

$$w$$

arises as $c_{26, Prop III.3.1}$), so we will use them interchangeably for Schubert varieties

Since for an abelian sheaf the cohomology on the big and small ´ etale sites agree (cf

$H$) i.e

$$\text{Hom}_{\text{Func}(w)}(\{G_w\}, \mathcal{F}) = \text{Hom}_{\text{Func}(\check{W}, Ab)}(\{G_w\}, \{\Gamma(S_w, \mathcal{F}|_{S_w})\})$$

Thus we have the Grothendieck spectral sequence

$$R^1\lim_w \{H^0_{\text{\acute{e}t}}(S_w, \mathbb{G}_m)\} \implies H^*_\text{\acute{e}t}(\mathcal{F}_\mathbb{A}, \mathbb{G}_m).$$

Since for an abelian sheaf the cohomology on the big and small ´ etale sites agree (cf

$H$) so we will use them interchangeably for Schubert varieties $S_w$.

Now for all $w \in \check{W}$ we have $H^0_{\text{\acute{e}t}}(S_w, \mathbb{G}_m) = \mathbb{C}^*$. Thus for $w \leq w'$, the restriction maps are identity. So the inverse system of abelian groups $\{H^0_{\text{\acute{e}t}}(S_w, \mathbb{G}_m)\}$ arises as $c : Ab \to \text{Func}(\check{W}, Ab)$ where $c$ is the functor that to a group $A$ associates the constant functor $\check{W} \to Ab$. Since by definition, colimit is left-adjoint to the constant functor, so we have an adjunction $Hom_{Ab}(\text{colim}(\{B_w\}_{w \in \check{W}}), A) = Hom_{\text{Func}}(\{B_w\}, c(A))$. Further since $c$ is the initial object, so $colim(\{B_w\}) = B_c$. Thus taking colimits in our case is an exact functor. Thus $c$ has an exact left-adjoint. So it takes injectives to injectives. Now $\mathbb{C}^*$ is an injective abelian group. Thus $c(\mathbb{C}^*)$ is an injective object in $\text{Func}(\check{W}, Ab)$. Thus $R^2\lim_w \{H^0_{\text{\acute{e}t}}(S_w, \mathbb{G}_m)\}$ vanishes.

We have $H^1_{\text{\acute{e}t}}(S_w, \mathbb{G}_m) = Pic(S_w)$. By GAGA, algebraic line bundles correspond functorially and bijectively to homomorphic line bundles. Now by the exponential exact sequence and the vanishing of $H^1_{\text{\acute{e}t}}(S_w, \mathcal{O}_{an})$ for $i = 1, 2$, we see that $H^1_{\text{\acute{e}t}}(S_w, \mathcal{O}_{an}) = H^2_{\text{\acute{e}t}}(S_w, \mathbb{Z})$. By the universal coefficient theorem it follows that $H^2_{\text{\acute{e}t}}(S_w, \mathbb{Z}) = Hom(H_1(S_w, \mathbb{Z}), \mathbb{Z})$ since $H_1(S_w, \mathbb{Z}) = 0$. Recall by Prop 4.2.2 $H_2(S_w, \mathbb{Z})$ is a free-abelian group on the 2-cells. Let us explicitly describe them. Let $a$ be the alcove defining the Iwahori group scheme. Let $s_i$ be the simple reflections about the walls of $a$. Let $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ be a reduced expression of $w$ in terms of $s_i$. The 2-cells of $S_w$ correspond bijectively to the $\mathbb{P}^1 = L^+\mathbb{G}_a/L^+\mathbb{G}_a$ corresponding to the reflections occurring in the reduced expression of $w$ (cf also Prop 2.1.3). Thus for $w \leq w'$, the map $H_2(S_w, \mathbb{Z}) \to H_2(S_w, \mathbb{Z})$ are injective and its cokernel is free over $\mathbb{Z}$. Thus the restriction of line bundle map is surjective. Thus the inverse system $\{Pic(S_w)\}$ satisfies the Mittag-Leffler condition. So $R^1\lim_w \{H^1_{\text{\acute{e}t}}(S_w, \mathbb{G}_m)\} = 0$. 

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Thus the natural morphism \( H^2_{\text{Et}}(\mathcal{F}_a, \mathbb{G}_m) \to \lim_{w} \{ H^2_{\text{Et}}(S_w, \mathbb{G}_m)_w \} \) is injective since it identifies with the edge morphism \( E^2 \to E_2^{0,2} \) of a spectral sequence. Any torsion class in \( H^2_{\text{Et}}(Q_G, \mathbb{G}_m) \) must map to a torsion class \( s \in H^2_{\text{Et}}(S_w, \mathbb{G}_m)_w \) for \( w \in W \). There it must be trivial by Prop 12.22. Since it is trivial in each group, so it must already be trivial in \( H^2_{\text{Et}}(\mathcal{F}_a, \mathbb{G}_m) \).

This finishes the proof when \( Q_G = \mathcal{F}_a \). In the case of \( a \) replaced by a facet \( \sigma \), the preceding arguments carry word-for-word. In the case of more than one point, we have \( Q_G = \prod_{x \in R} \mathcal{F}_{a_x} \). In this case, instead of \( W \) we consider \( \text{Maps}(R, W) \) as a category with Bruhat-order as follows: say \( w \leq w' \) if for each \( x \in R \), we have \( w_x \leq w'_x \). We again have \( H^2_{\text{Et}}(S_w, \mathbb{G}_m) = \mathbb{C}^* \) and by the see-saw theorem, we have \( H^2_{\text{Et}}(S_w, \mathbb{G}_m) = \text{Pic}(S_w) = \prod_{x \in R} \text{Pic}(S_w) \). We can conclude again that the natural morphism \( \prod_{w} \{ H^2_{\text{Et}}(S_w, \mathbb{G}_m)_w \} \to \lim_{w} \{ H^2_{\text{Et}}(S_w, \mathbb{G}_m)_w \} \) is injective. Now the claim follows as before by Prop 12.22. \( \square \)

6. \( L_{X^\circ}(G) \)

Set \( X^\circ = X \setminus R \). Consider the presheaf of sets on the category of \( \mathbb{C} \)-algebras which to a \( \mathbb{C} \)-algebra \( R \) associates

\[
G(\text{Spec}(R) \times_k X^\circ) = \text{Hom}(\text{Spec}(R) \times_k X^\circ, G).
\]

Let \( L_{X^\circ}(G) \) denote the associated sheaf of sets. It is represented by an ind-scheme (cf [20, Lemma 20], [21, proof of Lemma 2.1]).

The main purpose of this section is to prove Corollary 6.3.5. which states that \( H^2_{\text{Et}}(BL_{X^\circ}(G), \mathbb{G}_m) \) is a free abelian group. We quote

**Theorem 6.0.1.** [30 Prop 2.3, Prop 2.4] For \( k = 0, 1, \ldots , \infty \), let \( C^k(X^\circ, G) \) the space of \( k \)-differentiable maps with compact-open topology. The natural inclusions

\[
L_{X^\circ}(G) \subset \text{Hol}(X^\circ, G) \subset C^\infty(X^\circ, G) \subset \cdots C^k(X^\circ, G) \subset \cdots C^0(X^\circ, G)
\]

are homotopy equivalences.

Let \( G_{an} \) denote the analytic space underlying \( G/\mathbb{C} \). The curve \( X^\circ \) is a complex affine curve (\( \Sigma \) in notation of [36]) smoothly deformable to a bouquet of \( N := 2g + |R| - 1 \) loops. It follows [30] cf Page 12, §II that the homotopy type of the analytic space underlying \( L_{X^\circ}(G) \) equals that of \( G_{an} \times \Omega G_{an}^X N \). Now the following corollary follows immediately.

**Corollary 6.0.2.** The ind-group \( L_{X^\circ}(G)_{an} \) is connected and simply-connected.

6.1. **Bar construction.** We now recall some generalities that we will need in this section and the next. Let \( H \) be a topological group with identity \( e \). Let \( S \) be a topological space with left \( H \)-action. The Bar construction

\[
(6.1.1) \quad E H(S)_\bullet.
\]

gives a simplicial topological space as follows. For \( n \geq 1 \), its zero simplicies are \( s \in S \). Its \( n \)-simplices are written suggestively as \( h_0 h_1 \cdots h_{n-1} s \) where \( h_i \in H \) and \( s \in S \). The \( i \)-th degeneracy removes the \( i \)-th bar to make \( h_i \) act on the successive element. The \( j \)-th face operator inserts "\( e \)" at the \( j \)-th place. Let \( * \) be a point with trivial \( H \)-action. The simplicial group \( E H_\bullet := E H_\bullet(\ast)_\bullet \) is contractible. We have a natural \( H \)-action on \( E H_\bullet \) on the leftmost factor. The Borel construction
$B_{H_\bullet}$ is the quotient $EH_\bullet/H$ with the induced simplicial structure and the degrees are shifted by $-1$. So its $-1$ simplices are $*/H$. Its $n$-simplices are $H^n$ where $H^0 = e$. Explicitly, its face and degeneracy maps are given by

\begin{align}
(6.1.2) \quad & s_i(h_1, \cdots, h_n) = (h_1, \cdots, h_i, h_{i+1}, \cdots, h_n) \\
(6.1.3) \quad & d_0(h_1, \cdots, h_n) = (h_2, \cdots, h_n) \\
(6.1.4) \quad & d_i(h_1, \cdots, h_n) = (h_1, \cdots, h_i h_{i+1}, \cdots, h_n) \\
(6.1.5) \quad & d_n(h_1, \cdots, h_n) = (h_1, \cdots, h_{n-1}).
\end{align}

Let $BL_{X^e_\bullet}(G_\bulle)$ be the Borel construction of $L_{X^e_\bullet}(G)$. We view it as a simplicial ind-scheme. In particular, $e \in L_{X^e}(G)$ is Spec($\mathbb{C}$).

**Proposition 6.1.1.** Let $G$ be semi-simple and simply-connected. Then

1. Let $Z$ be the trivial $L_{X^e}(G)$ module. Then $H^1_{\text{et}}(BL_{X^e}(G)_{\bulle}, Z) = 0.$
2. We have $H^1_{\text{et}}(BL_{X^e}(G), \mathbb{G}_m) = e$. 
3. $H^1_{\text{et}}(BL_{X^e}(G)_{\bulle}, \mathbb{G}_m) = H^1_{\text{et}}(L_{X^e}(G), \mathbb{G}_m)$.

**Proof.** For any $q \geq 0$, applying the functor $H^q_{\text{et}}(?, Z)$ to the simplicial space $BL_{X^e}(G)_{\bulle}$, we get the cosimplicial group

$$H^q_{\text{et}}(BL_{X^e}(G)_{\bulle}, Z).$$

We have the Atiyah-Hirzebruch spectral sequence

$$E_2^{p,q} := \pi_p(H^q_{\text{et}}(BL_{X^e}(G)_{\bulle}, Z)_{\bulle}) \Longrightarrow H^p_{\text{et}}(BL_{X^e}(G)_{\bulle}, Z).$$

Now $H^1_{\text{et}}(e, Z) = H^1_{\text{et}}(e, Z)$ vanishes. So $\pi_0(\cdots H^1_{\text{et}}(L_{X^e}(G), Z) \cdots) \Longrightarrow H^1_{\text{et}}(e, Z)$ vanishes. Now consider

$$\pi_1(\cdots H^0_{\text{et}}(L_{X^e}(G)^2, Z) \cdots) \Longrightarrow H^0_{\text{et}}(L_{X^e}(G), Z) \Longrightarrow H^0_{\text{et}}(e, Z)).$$

Since $L_{X^e}(G)$ is connected, this simplifies to

$$\pi_1(Z \xleftarrow{d_0-d_1} Z \xrightarrow{d_0-d_1} Z).$$

By the face-degeneracy formulas (cf (6.1.2)) each differential above is the identity on $Z$. So the above group vanishes. This proves the first assertion. For the following assertions, consider the spectral sequence

$$E_2^{p,q} := \pi_p(H^q_{\text{et}}(BL_{X^e}(G)_{\bulle}, \mathbb{G}_m)_{\bulle}) \Longrightarrow H^p_{\text{et}}(BL_{X^e}(G)_{\bulle}, \mathbb{G}_m).$$

Since $H^1_{\text{et}}(e, \mathbb{G}_m)$ vanishes, so $\pi_0(H^1_{\text{et}}(BL_{X^e}(G)_{\bulle}, \mathbb{G}_m))$ also vanishes. So $E_2^{0,1} = 0$. Let us abbreviate $L_{X^e}G$ as $L$. Consider

$$H^0_{\text{et}}(L^3, \mathbb{G}_m) \Longrightarrow H^0_{\text{et}}(L^2, \mathbb{G}_m) \Longrightarrow H^0_{\text{et}}(L, \mathbb{G}_m) \Longrightarrow H^0_{\text{et}}(e, \mathbb{G}_m).$$

Since by Corollary 6.0.2 $L_{X^e}(G)_{\text{an}}$ is connected and simply-connected, so by [21 Corollary 2.4], there is no non-constant regular map $\lambda : L_{X^e}(G) \to \mathbb{C}^*$. On the other hand, the subgroup of constant maps $L_{X^e}(G)^n \to \mathbb{G}_m$ equals $\mathbb{C}^*$ since our base field is $\mathbb{C}$. So all the groups above identify with $\mathbb{C}^*$. By face-degeneracy relations (cf (6.1.2)), the differentials identify with the identity on $\mathbb{C}^*$. So $\pi_1$ of the above cosimplicial group vanishes. Thus $E_2^{1,0} = 0$. This shows the second assertion.
We now consider the third assertion. Arguing as before, we see that \( \pi_2 \) of the above cosimplicial group vanishes. So \( E^{2,0}_2 = 0 \) and \( E^{3,0}_3 = 0 \). Since \( H^2_{\text{Et}}(e, \mathbb{G}_m) \) vanishes, so \( \pi_0(H^2_{\text{Et}}(BL_X^\ast(G)\bullet, \mathbb{G}_m)) \) also vanishes. So \( E^{0,2}_0 = 0 \). So \( E^2 = E^{1,1}_3 \).

Now consider

\[
(6.1.10) \quad \pi_1(\cdots \rightarrow H^1_{\text{Et}}(X^\ast(G)^2, \mathbb{G}_m) \rightarrow H^1_{\text{Et}}(L_X^\ast(G), \mathbb{G}_m)) \rightarrow H^1_{\text{Et}}(e, \mathbb{G}_m)
\]

We have \( H^1_{\text{Et}}(e, \mathbb{G}_m) = 0 \). An element \( l \in H^1_{\text{Et}}(L_X^\ast(G), \mathbb{G}_m) \) is mapped to \((l, 0), (l, l)\) and \((0, l)\) in \( H^1_{\text{Et}}(L_X^\ast(G)^2, \mathbb{G}_m) \) under the various face maps. So \( \pi_1(\cdots) = E^{1,1}_2 = H^1_{\text{Et}}(L_X^\ast G, \mathbb{G}_m) \). Now by \( E^{1,2}_2 \rightarrow E^{1,1}_2 \rightarrow E^{3,0}_3 \), we get \( E^{1,1}_3 = E^{1,1}_2 \). By \( E^{2,3}_3 \rightarrow E^{3,1}_3 \rightarrow E^{4,-1}_4 \), we have \( E^{3,1}_3 = E^{4,1}_4 \). Hence, \( E^{2,1}_4 = E^{1,1}_4 \). This shows the claim.

\[\square\]

6.2. **The analytic site of** \( L_X^\ast(G) \).

**Proposition 6.2.1.** We have \( L_X^\ast(G) = \lim_n Y_n \) for \( n \in \mathbb{N} \) where \( Y_n \) are affine schemes on \( \mathbb{C} \) and \( Y_n \rightarrow Y_{n+1} \) is a closed immersion.

**Proof.** Let \( L(d) \) denote the geometric line bundle on \( X \) corresponding to \( \mathcal{O}_X(dR) \). Let us fix once for all a closed immersion \( G \hookrightarrow M(k \times k) \) in matrices for some \( k \geq 1 \). Let \( Z_d \) denote the subfunctor of \( \text{Mor}(X \setminus R, M(k \times k)) \) that to a ring \( R \) associates matrices each of whose entries has a pole of order at most \( d \) along \( R \times \text{Spec}(R) \) and is regular on \( X \setminus R \times \text{Spec}(R) \). Thus \( Z_d \) identifies with the functor that to a ring \( R \) associates \( k^2 \)-many sections \( \text{Hom}_X(X \times \text{Spec}(R), L(d)) \) of \( L(d) \rightarrow X \). Since the morphism \( L(d) \rightarrow X \) is quasi-projective, so \( Z_d \) is representable. Remark that \( Z_0 \) is representable by \( M(k \times k) \). More generally, if \( l(d) = \dim \mathcal{O}_X(dR) \) then \( Z_d \) is representable by \( k^{2 \times l(d)} \). With \( a_i \in k^2 \), the closed inclusion \( Z_d \hookrightarrow Z_{d+1} \) corresponds to the inclusion of affine spaces

\[
(a_1, \cdots, a_{l(d)}) \mapsto (a_1, \cdots, a_{l(d)}, 0_{l(d)+1}, \cdots, 0_{l(d+1)}).
\]

So \( \lim Z_d = \text{Mor}(X \setminus R, M(k \times k)) \). Set

\[
Y_d := \text{Mor}(X \setminus R, G) \times_{\text{Mor}(X \setminus R, M(k \times k))} Z_d.
\]

Notice further that each \( Y_d \) is a closed subscheme of \( Z_d \). So it is affine. Further, we have \( L_X^\ast(G) = \lim Y_d \).

\[\square\]

We define \( L_X^\ast(G)_{\text{an}} \) as a colimit of analytic spaces by setting

\[
L_X^\ast(G)_{\text{an}} = \lim Y_{n,\text{an}}.
\]

By the analytic site of \( L_X^\ast(G)_{\text{an}} \) we shall mean the following

1. objects are morphisms \( u : U \rightarrow L_X^\ast(G)_{\text{an}} \) factoring through an analytic morphism \( u : U \rightarrow Y_{n,\text{an}} \) for some \( n \).
2. a morphism from \( u : U \rightarrow L_X^\ast(G)_{\text{an}} \) to \( u' : U' \rightarrow L_X^\ast(G)_{\text{an}} \) is an analytic morphism \( f : U \rightarrow U' \) of analytic spaces such that \( u' \circ f = u \).
3. a covering of \( u : U \rightarrow L_X^\ast(G)_{\text{an}} \) is just an analytically étale covering \( U' \rightarrow U \) of \( U \).

**Proposition 6.2.2.** We have

1. \( H^0_{\text{an}}(L_X^\ast(G), \mathcal{O}) = 0 \) for all \( n \geq 1 \).
(2) the group $H^1_{an}(L_X^\ast(G), G_m)$ is a finitely generated free $\mathbb{Z}$-module.

(3) $H^1_{an}(L_X^\ast(G), \mu_n) = 0$ for all $n$.

Proof. (1) By (6.2.2), $L_X^\ast(G)_{an} = \lim_{\to} Y_{n, an}$. For any sheaf $\mathcal{F}$ on the analytic site of $L_X^\ast(G)$ we have

$$\Gamma_{an}(L_X^\ast(G), \mathcal{F}) = \lim_{\to} \{\Gamma_{an}(Y_n, \mathcal{F})\}_n.$$ 

By repeating the arguments of Proposition 4.0.1 we can establish the Grothendieck spectral sequence in the analytic topology:

$$(6.2.3) \quad R^p \lim_{\to} \{H^q_{an}(Y_n, \mathcal{F}|_{Y_n})\}_n \implies H^*_{an}(L_X^\ast(G), \mathcal{F}).$$

Let us take $\mathcal{F}$ as the coherent analytic sheaf $\mathcal{O}_{an}$. Then since $Y_n$ are affine schemes over $\mathbb{C}$, so $Y_{n, an}$ are closed analytic subspaces of $\mathbb{C}^n$ for some $n$. So they are Stein spaces. Thus for $q \geq 1$, $H^q(Y_n, \mathcal{O})$ vanishes. So their $R^p \lim_{\to}$ is zero. If $q = 0$, then the inverse system $\cdots \to H^0(Y_n, \mathcal{O}) \to H^0(Y_{n+1}, \mathcal{O}) \to \cdots$ is surjective on each arrow. So it satisfies the Mittag-Leffler condition. So its higher $\lim_{\to}$ is zero.

(2) Consider the exact sequence $0 \to \mathbb{Z} \to \mathcal{O}_{an} \xrightarrow{\exp} \mathcal{O}_{an}^\times \to 0$ of sheaves on the analytic site of $L_X^\ast(G)$. So by (1), $H^1_{an}(L_X^\ast(G), G_m) \to H^2(L_X^\ast(G), \mathbb{Z})$ is an isomorphism. Now since $L_X^\ast(G)$ has the homotopy type of $\mathbb{G}_m \times \Omega \mathbb{G}_m$, so it follows that it is $(2-1)$-connected. So $H_1(L_X^\ast(G), \mathbb{Z}) = 0$. Thus by the universal coefficient theorem $H^2(L_X^\ast(G), \mathbb{Z}) \to \text{Hom}(H_2(L_X^\ast(G), \mathbb{Z}), \mathbb{Z})$ is an isomorphism. This shows that it is finitely generated and free over $\mathbb{Z}$.

(3) Consider the short exact sequence of sheaves $1 \to \mu_n \to \mathbb{G}_m \xrightarrow{z \mapsto z^n} \mathbb{G}_m \to 1$ on the analytic site of $L_X^\ast(G)_{an}$. Taking the long exact sequence, we see that $H^1_{an}(L_X^\ast(G), \mu_n)$ injects into $H^1_{an}(L_X^\ast(G), G_m)$ as follows. Since by Corollary 6.0.2 $L_X^\ast(G)_{an}$ is connected and simply-connected, so by [21 Corollary 2.4], there is no non-constant regular map $\lambda : L_X^\ast(G) \to \mathbb{C}^\ast$. So we have $H^0_{et}(L_X^\ast(G), \mathbb{G}_m) = \mathbb{C}^\ast$. Thus the first three terms reduce to $1 \to \mu_n \to \mathbb{C}^\ast \xrightarrow{z \mapsto z^n} \mathbb{C}^\ast \to \cdots$. This shows the injection. Now $H^1_{an}(L_X^\ast(G), \mu_n)$ is torsion. So it vanishes by the second assertion.

6.3. First cohomology on sites and torsors. Let us begin by recalling the notion of a torsor on a Grothendieck site $\mathcal{C}$. Let $\mathcal{G}$ be a sheaf of groups on $\mathcal{C}$. A $\mathcal{G}$-torsor $\mathcal{F}$ on $\mathcal{C}$ is a sheaf of sets on $\mathcal{C}$ endowed with an action $\mathcal{G} \times \mathcal{F} \to \mathcal{F}$ such that

(1) whenever $\mathcal{F}(U)$ is non-empty, the action $\mathcal{G}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ is simply-transitive.

(2) for every $U \in \text{Ob}(\mathcal{C})$, there exists a covering $\{U_i \to U\}_{i \in I}$ of $U$ such that $\mathcal{F}(U_i)$ is non-empty for all $i \in I$.

A trivial $\mathcal{G}$-torsor is the sheaf $\mathcal{G}$ endowed with the natural left-action. We have a contravariant functor $\mathcal{C} \to \text{Ab}$ given by $U \mapsto \Gamma(U, \mathcal{F})$. By definition,

$$\Gamma(\mathcal{C}, \mathcal{F}) = \lim_{\to} \Gamma(U, \mathcal{F}).$$

A $\mathcal{G}$-torsor is trivial if and only if $\Gamma(\mathcal{C}, \mathcal{F}) \neq \emptyset$. We quote
Lemma 6.3.1. \cite{11} Lemma 5.3] Let $C$ be a site. Let $\mathcal{H}$ be an abelian sheaf on $C$. There is a canonical bijection between the set of isomorphism classes of $\mathcal{H}$-torsors and $H^1(C, \mathcal{H})$.

We need to show a technical result which says that the canonical bijection above behaves well under change of sites. Let $\epsilon : C_1 \to C_2$ be a morphism of sites. Let $\epsilon^s$ denote the sheaf pull-back on sites (cf \cite{35} Chapter 1). So $(\epsilon_s, \epsilon^s)$ form an adjoint-pair and $\epsilon_s$ is exact. Let us recall the definition of $\epsilon_p$. For $U_2 \in C_2$, let $(U_2 \downarrow \epsilon)$ denote the category whose

1. objects are $\{(u, U_1)| u : U_2 \to \epsilon(U_1) \in Mor(C_2)\}$.
2. morphisms from $(u, U_1)$ to $(u', U_1')$ are $\{f : U_1 \to U_1'| u' = \epsilon(f)u\}$.

We have a natural forgetful functor $(U_2 \downarrow \epsilon) \to C_1$. Given a presheaf $\mathcal{G}_1$ on $C_1$, we view it as a presheaf on $(U_2 \downarrow \epsilon)$. Now, we define

\[
(6.3.2) \quad \epsilon_p(\mathcal{G}_1)(U_2) = \lim_{(U_2 \downarrow \epsilon)} \mathcal{G}_1(U_1).
\]

Let $\mathcal{H}$ be an abelian sheaf on $C_2$. Consider the edge morphism $\epsilon : H^1(C_1, \epsilon^s\mathcal{H}) \to H^1(C_2, \mathcal{H})$ of the Leray spectral sequence.

Corollary 6.3.2. Let $\mathcal{F}_1$ be a $\epsilon^s\mathcal{H}$-torsor. Then $\epsilon$ maps $\mathcal{F}_1$ to the $\mathcal{H}$-torsor obtained by extending structure group by $\epsilon_s \epsilon^s \mathcal{H} \to \mathcal{H}$ on $\epsilon_s \mathcal{F}_1$.

Proof. From the proof of \cite{11} Lemma 5.3, let us recall the correspondence from $H^1(C, \mathcal{H})$ to torsors. Let $\xi \in H^1(C, \mathcal{H})$ be given. Choose an embedding of $\mathcal{H}$ into an injective sheaf $\mathcal{I}$ and let $\mathcal{Q} = \mathcal{I}/\mathcal{H}$. Since $H^1(C, \mathcal{I}) = 0$, so $\xi$ lifts to some $q \in H^0(C, \mathcal{Q})$. Let $\mathcal{F} \subset \mathcal{I}$ be the subsheaf of sets defined by the following condition: its local sections over any open $U$ of $C$ map to $q|_U$. Then $\mathcal{F}$ is a $\mathcal{H}$-torsor and the canonical bijection maps $\xi \mapsto \mathcal{F}$.

With notations as above, consider the exact sequence of sheaves $0 \to \mathcal{H} \to \mathcal{I} \to \mathcal{Q} \to 0$ on $C_2$. Now $\epsilon^s$ is left-exact and $\epsilon^s \mathcal{I}$ is an injective sheaf. Let $\mathcal{Q}_1 = \epsilon^s \mathcal{I}/\epsilon^s \mathcal{H}$ denote the sheaf quotient on $C_1$. So we have a natural map $\mathcal{Q}_1 \to \epsilon^s \mathcal{Q}$ or equivalently $\epsilon_s \mathcal{Q}_1 \to \mathcal{Q}$ by adjunction. Now consider the diagram

\[
(6.3.3) \quad \begin{CD}
0 @>>> \epsilon_s \epsilon^s \mathcal{H} @>>> \epsilon_s \epsilon^s \mathcal{I} @>>> \epsilon_s \mathcal{Q}_1 @>>> 0 \\
&& \downarrow \cong \quad \downarrow \cong \quad \downarrow \cong \quad \\
0 @>>> \mathcal{H} @>>> \mathcal{I} @>>> \mathcal{Q} @>>> 0
\end{CD}
\]

The left-square commutes because the left two vertical arrows are of adjunction. So the right square also commutes because of exactness of rows. The following diagram is also commutative

\[
(6.3.4) \quad \begin{CD}
H^0(C_1, \mathcal{Q}_1) @>>> H^0(C_2, \epsilon_s \mathcal{Q}_1) @>>> H^1(C_2, \epsilon_s \epsilon^s \mathcal{H}) \\
\downarrow @>>> \downarrow @>>> \downarrow \\
H^0(C_1, \epsilon^s \mathcal{Q}) @>>> H^0(C_2, \mathcal{Q}) @>>> H^1(C_2, \mathcal{H})
\end{CD}
\]

The rightmost vertical arrow extends the structure group, the leftmost comes from $\mathcal{Q}_1 \to \epsilon^s \mathcal{Q}$ and the middle comes from adjunction of $\epsilon^s$.

Let $q_1 \in H^0(C_1, \mathcal{Q}_1)$ map to $\xi_1 \in H^1(C_1, \epsilon^s \mathcal{H})$. The edge morphism $\epsilon$ is defined by composing the left vertical arrow with the horizontal arrows to get $H^0(C_1, \mathcal{Q}_1) \to H^1(C_2, \mathcal{H})$ and then observing that $H^0(C_1, \epsilon^s \mathcal{Q}) \to H^1(C_2, \mathcal{H})$ factors through
$H^1(\mathcal{C}_1, \epsilon^*\mathcal{H})$. This describes the edge morphism. We may instead chase the diagram under the top horizontal row followed by the last vertical arrow.

Then $q_1$ determines a subsheaf $F_1 \subset \epsilon^*\mathcal{I}$ by the correspondence described above. Now by exactness of $\epsilon_*$, we have the exact sequence of sheaves $0 \rightarrow \epsilon_*\epsilon^*\mathcal{H} \rightarrow \epsilon_*\epsilon^*\mathcal{I} \rightarrow \epsilon_*\mathcal{Q}_1 \rightarrow 0$ on $\mathcal{C}_2$. Consider the subsheaf $\epsilon_*F_1 \subset \epsilon_*\epsilon^*\mathcal{I}$. Under the correspondence between cohomology classes and torsors recalled above, we claim that $\epsilon_*F_1$ corresponds to the image $q'_1$ of $q_1$ under $H^0(\mathcal{C}_1, \mathcal{Q}_1) \rightarrow H^0(\mathcal{C}_2, \epsilon_*\mathcal{Q}_1)$. By the definition of presheaf pushforward (6.3.2), local sections of $\epsilon_*F_1$ via $\epsilon_*F_1 \rightarrow \epsilon_*\mathcal{Q}_1$ map to restrictions of $q'_1$ in $\epsilon_*\mathcal{Q}_1$. Therefore local sections of $\epsilon_*F_1$ via $\epsilon_*F_1 \subset \epsilon_*\epsilon^*\mathcal{I} \rightarrow \epsilon_*\mathcal{Q}_1$ map to $q'_1$. So $\epsilon_*F_1$ is contained in the $\epsilon_*\epsilon^*\mathcal{H}$-torsor corresponding to $q'_1$. But it is itself such a torsor. So the claim follows.

Since $\epsilon_*F_1$ is an $\epsilon_*\epsilon^*\mathcal{H}$-torsor corresponding to the image of $q'_1$ under $H^0(\mathcal{C}_2, \epsilon_*\mathcal{Q}_1) \rightarrow H^1(\mathcal{C}_2, \epsilon_*\epsilon^*\mathcal{H})$, so the claim follows.

\[ \square \]

### 6.4. Change of sites from Big-étale to Analytic of sheaves.

The Leray spectral sequence (cf [35] Theorem I(3.7.6)) is defined for a continuous morphism of sites. For us, $L_{\mathcal{X}^*}G$ is only a sheaf on the big-étale site of $\text{Spec}(\mathbb{C})$ and similarly $(L_{\mathcal{X}^*}G)_{\text{an}}$ is only a sheaf on the analytic site of $\mathbb{C}$. We use the setup in [18, Giraud] to define a continuous morphism $F^*$ from the site of sheaves on $\hat{\text{Et}}(\text{Spec}(\mathbb{C}))$ to the site of sheaves on the analytic site of $\mathbb{C}$. Then we apply the Leray spectral sequence to $F^*, L_{\mathcal{X}^*}G$ and $(L_{\mathcal{X}^*}G)_{\text{an}}$ to deduce the results of this subsection.

Consider the functor

\[(6.4.1) \quad F^{-1} : \hat{\text{Et}}(\text{Spec}(\mathbb{C})) \rightarrow \text{an}(\mathbb{C}_{\text{an}})\]

on underlying categories given by mapping $u : U \rightarrow Y_n$ to the analytic morphism $u_{\text{an}} : U_{\text{an}} \rightarrow Y_{n,\text{an}}$. Recall [18] Definition 0.3.1, if $X$ and $Y$ are two sites, then a functor $f^{-1} : Y \rightarrow X$ is said to be continuous if for every sheaf $G$ on $X$, the presheaf

\[(6.4.2) \quad f_*(G)(y) = G(f^{-1}(y)), y \in \text{Ob}(Y)\]

is a sheaf. Then $F^{-1}$ is a continuous morphism of sites by [35] I(1.2.2)] because it satisfies the following conditions:

1. if $\{U_i \rightarrow U\}_{i \in I}$ is a covering of $U$ in the big-étale site, then $\{U_{i,\text{an}} \rightarrow U_{\text{an}}\}_{i \in I}$ is a covering of $U_{\text{an}}$ in the analytic site.
2. for a covering $U' \rightarrow U$ and for any $V \rightarrow U$, we have $F^{-1}(U' \times_U V) = (U' \times_U V)_{\text{an}} = U'_{\text{an}} \times _{U_{\text{an}}} V_{\text{an}} = F^{-1}(U') \times _{F^{-1}(U)} F^{-1}(V)$.

Let us proceed to define a continuous morphism of sites $f_L^{-1} : \hat{\text{Et}}(L_{\mathcal{X}^*}(G)) \rightarrow \text{an}(L_{\mathcal{X}^*}(G))_{\text{an}}$ which is a restriction of $F^{-1}$.

**Proposition 6.4.1.** [18, J.Giraud Chapitre 0 3.1.4] Let $E$ be a site and $\hat{E}$ be the category of presheaves on $E$. Let $\eta : E \rightarrow \hat{E}$ be given by $\eta(S)(T) = \text{Hom}(T, S)$. Let $P \in \text{Ob}(\hat{E})$. Let $E/P = \hat{E}/P \times _{\hat{E}} E$. We equip $E/P$ with the topology that makes the functor $E/P \rightarrow E$ continuous. Consider an object $S = (S, s : \eta(S) \rightarrow P)$ in the comma category $E/P$. There is a natural isomorphism of comma categories

\[(E/P)_S = E/S,\]

which induces a bijection between the set of refinements of $S$ for the topology induced on $E/P$ and the set of refinements of $S$ for the topology of $E$. 

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By definition of sites $\acute{\text{E}}t(L_X^+(G))$ and $an(L_X^+(G))$, we have an equivalence of the following categories

\begin{align}
(6.4.3) \quad \acute{\text{E}}t(L_X^+(G)) &= \acute{\text{E}}t(\text{Spec}(\mathbb{C}))/L_X^+(G) \\
(6.4.4) \quad an(L_X^+(G)) &= an(\text{Spec}(\mathbb{C}))/L_X^+(G)_{an},
\end{align}

By the above proposition, it is also an equivalence of sites. Further $f_{L_X^+}^{-1}$ is the restriction of $F^{-1}$. We now want to define a continuous morphism $F^*$ from the site of sheaves on $\acute{\text{E}}t(\text{Spec}(\mathbb{C}))$ to the site of sheaves on the analytic site of $\mathbb{C}$.

For a site $E$, let $\tilde{E}$ denote the category of sheaves on $E$ with the canonical topology. We put the canonical topology on $\tilde{E}$ which is the strongest topology so that every object is representable. Let $\epsilon = a\eta$ where $a : E \to \tilde{E}$ is the sheafification functor. Further for any continuous morphism $f^{-1} : Y \to X$ of sites, the functor $f_* : \tilde{X} \to \tilde{Y}$ admits a left-adjoint $f^* : \tilde{Y} \to \tilde{X}$ by [18, Prop 3.2]. Further the diagram (cf [18, 0(3.3.2)]) commutes

\begin{equation}
(6.4.5)
\begin{array}{ccc}
X & \xrightarrow{\epsilon} & \tilde{X} \\
\downarrow f^{-1} & & \downarrow f^* \\
Y & \xrightarrow{\epsilon'} & \tilde{Y}
\end{array}
\end{equation}

We quote

**Proposition 6.4.2.** [18, Proposition 3.6] The functor $\epsilon : E \to \tilde{E}$ defines a morphism of sites $\tilde{E} \to E$ if we equip $\tilde{E}$ with the canonical topology. Consider the direct image functor on sheaves which by definition is induced by composition with $\epsilon$. It induces an equivalence between the category of sheaves of sets on $E$ and $\tilde{E}$.

Therefore $\tilde{\epsilon}$ and $\tilde{\epsilon}'$ are equivalence of categories in the commutative diagram

\begin{equation}
(6.4.6)
\begin{array}{ccc}
X & \xrightarrow{\epsilon} & \tilde{X} \\
\downarrow f^{-1} & & \downarrow (f^*)_* \\
Y & \xrightarrow{\epsilon'} & \tilde{Y}
\end{array}
\end{equation}

Therefore $\tilde{\epsilon} f_* \tilde{\epsilon}^{-1} : \tilde{X} \to \tilde{Y}$ identifies with $(f^*)_*$, because they are both right-adjoint to $(f^*)^*$. So $(f^*)_*$ maps sheaves to sheaves because $f_*$ does. So $f^*$ is continuous morphism of sites. We apply this result setting

\begin{equation}
(6.4.7)
Y = \acute{\text{E}}t(\text{Spec}(\mathbb{C})), \quad X = an(\text{Spec}(\mathbb{C})), \quad f^{-1} = F^{-1}.
\end{equation}

Since $F^{-1}$ is a morphism of sites, so the pair $(F^*, F_*): \tilde{X} \to \tilde{Y}$ define a morphism of topoi. In particular, $F^*$ commutes with finite projective limits. So

\begin{equation}
(6.4.8)
F^* : \tilde{Y} \to \tilde{X}
\end{equation}

is a continuous morphism of sites.

Let us view $L_X^+(G) \in \text{Ob}(\tilde{Y})$ and $L_X^+(G)_{an} \in \text{Ob}(\tilde{X})$. Since $L_X^+(G) = \lim Y_n$ by Prop [6.2.1] and $L_X^+(G)_{an} = \lim Y_{n,an}$ by [6.2.2], since $F^*$ commutes with arbitrary colimits, we have

\begin{equation}
(6.4.9)
F^*(L_X^+(G)) = L_X^+(G)_{an}.
\end{equation}
Let \((F^*)^s(= (F^*)^s)\) denote the sheaf pull-back by \(F^*\) for the sheaf defined by \(\mu_n\) on \(\tilde{X}\). By [35 Theorem I(3.7.6)] we have the Leray spectral sequence
\[
E_2^{p,q} = H^p_{\text{ét}}(\tilde{Y}, L_{X^s}(G), R^q(F^*)^s(\mu_n)) \implies E^{p+q} = H^{p+q}_{\text{an}}(\tilde{X}, F^*(L_{X^s}(G)), \mu_n)
\]
This gives the edge morphism
\[
(6.4.10) \quad e : H^1(\tilde{Y}, L_{X^s}(G), (F^*)^s \mu_n) \to H^1(\tilde{X}, L_{X^s}(G))_{\text{an}, \mu_n}.
\]
Let us relate the above cohomologies on sites \(\tilde{Y}\) and \(\tilde{X}\) with those on \(\text{ét}(L_{X^s}(G))\) and \(\text{an}(L_{X^s}(G))\) to simplify \((6.4.10)\).

**Proposition 6.4.3.** Consider the sheaf defined by the group \(\mu_n\) on \(\tilde{X}\). The pull-back sheaf \((F^*)^s(\mu_n)\) on \(\tilde{Y}\) is the sheaf defined by \(\mu_n\) itself.

**Proof.** Consider the presheaf \(\mu_n\) defined by \(\mu_n\) on the sites \(\tilde{X}\) and \(\tilde{Y}\). By the last proposition, we may assume that our sites are \(X\) and \(Y\) instead of \(\tilde{X}\) and \(\tilde{Y}\) and our functor is \(F^{-1}\) instead of \(F^*\). Let us observe that this presheaf is already a sheaf. Since \(\mu_n\) is discrete, so any local section of \(\mu_n\) is locally constant on any open of these sites. To verify the sheaf condition, we may restrict ourselves to Zariski open covers (or open covers in the analytic topology) and a single étale morphism \(V \to U\). In both cases, we may further assume that \(U\) is connected. Let \(V_i, i \leq n\) be the connected components of \(V\) or of an open cover of \(U\). Consider \(p_r^*: p_r^*: \mu_n(V) \to \mu_n(V \times_U V)\). Let \(s \in \mu_n(V)\) and \(s_i\) denote its restriction to \(V_i\). Let us say that \(V_i\) is related to \(V_j\) if \(V_i \times_U V_j \neq \emptyset\). Observe that if \(V_i\) related to \(V_j\), then the value in the group \(\mu_n\) of \(s_i\) and \(s_j\) are equal. This property holds more generally for a pair \(V_i\) and \(V_j\) which are in the same equivalence class generated by this relation. Now observe that over a connected \(U\) any two \(V_i\) and \(V_j\) are related.

By definition \((F^*)^s = \#(F^*)^p i\), where \(i : \text{Sh} \to \text{Prsh}\) is the inclusion functor of sheaves into presheaves, \((F^*)^p\) is the presheaf pull-back defined by \((F^*)^p(F)(U) = F(f(U)))\), and \(\#\) is the sheafification functor. We have \(i(\mu_n)\) is the presheaf defined by \(\mu_n\). Now it suffices to check that \((F^*)^p(\mu_n)\) is the presheaf defined by \(\mu_n\). This holds. So \((F^*)^s(\mu_n) = \mu_n\).

The equivalence of sheaves of sets on \(E\) and \(\tilde{E}\) in Proposition [6.4.2] restricts to an equivalence of abelian sheaves on \(E\) and \(\tilde{E}\). It preserves exactness. Further injective sheaves restrict to injective sheaves. Finally for any sheaf \(\mathcal{F}\) on \(\tilde{E}\), we have the equality of global sections by definition (cf \((6.3.1)\))
\[
(6.4.11) \quad \Gamma(\tilde{Y}, L_{X^s}(G), \mathcal{F}) = \Gamma_{\tilde{E}}(L_{X^s}(G), \epsilon_*(\mathcal{F})).
\]
Therefore their derived functors are isomorphic. Hence for any \(n\) we have
\[
(6.4.12) \quad H^n(\tilde{Y}, L_{X^s}(G), \mathcal{F}) = H^n_{\text{ét}}(L_{X^s}(G), \epsilon_*(\mathcal{F}))
\]
\[
(6.4.13) \quad H^n(\tilde{X}, L_{X^s}(G)_{\text{an}}, \mathcal{F}) = H^n_{\text{an}}(L_{X^s}(G)_{\text{an}}, \epsilon_*(\mathcal{F}))
\]
So \((6.4.10)\) becomes
\[
(6.4.14) \quad e : H^1_{\text{ét}}(L_{X^s}(G), \mu_n) \to H^1_{\text{an}}(L_{X^s}(G)_{\text{an}}, \mu_n).
\]

**Proposition 6.4.4.** The edge morphism \((6.4.14)\) (or \((6.4.10)\)) is injective.
Proof. Let \( c \in \ker(e) \). Let \( L \) be a \( \mu_n \) torsor on \( \mathcal{E}(L_{X^+}(G)) \) representing \( c \) by Lemma 6.3.1. Let \( L_{an} \) represent \( e(c) \). We have a natural map \((F^*)_aL \to L_{an}\) by Corollary 6.3.2. So for any open \( U \in \mathcal{E}(L_{X^+}(G)) \) we have

\[
H^0(U, L) = H^0(F^*(U), (F^*)_aL) \to H^0(F^*(U), L_{an}).
\]

Since \( e(c) = 0 \), so \( L_{an} \) admits a global section \( s^{an} \) on the analytic site. Since \( L_{X^+}(G)_{an} \) is connected, and \( s^{an} \) takes values in \( \mu_n \), so it must be a constant global section. So its restriction \( s^{an}_u \in L_{an}(U) \) to any open \( u : U_{an} \to L_{X^+}(G)_{an} \) in the analytic site is a constant section. Consider an arbitrary open \( u : U \to L_{X^+}(G) \) of the big-étale site. Since \( s^{an}_u \) is a constant section, so in (6.4.15) it may be viewed as a section \( s_u \) in \( H^0(F^*(U), (F^*)_aL) \), and thereby of \( L(U) \) through \( u \). So the sections \( s_u \) are constant for any \( u \in \mathcal{E}(L_{X^+}G) \). So the collection \( \{s_u\} \) define a global section \( s \) of \( L \). So \( L \) is trivial. \( \square \)

**Corollary 6.4.5.** The group \( H^2_{\text{Et}}(BL_{X^+}G, \mathbb{G}_m) \) is finitely-generated and free abelian.

**Proof.** We have \( H^2_{\text{Et}}(BL_{X^+}G, \mathbb{G}_m) = H^1_{\text{Et}}(L_{X^+}G, \mathbb{G}_m) \) by Proposition 6.1.1. Any torsion-class is represented by an element in \( H^1_{\text{Et}}(L_{X^+}G, \mu_n) \) for some \( n \). The second group vanishes by Propositions 6.4.4 and 6.2.2. \( \square \)

For notational convenience, we will denote \( H^1_{\text{Et}}(L_{X^+}G, \mathbb{G}_m) \) by \( \text{Pic}(L_{X^+}G) \).

7. **Brauer group of the moduli stack:** \( G \) is simply connected

7.1. **Parahoric torsors.** Let \( G \to X \) be a group scheme as in (2.3). A quasi-parahoric torsor \( E \) is a \( G \)-torsor on \( X \). This means that \( E \times_X E \simeq \mathcal{E} \times_X G \) and we have an action map \( \alpha : \mathcal{E} \times_X G \to E \) which satisfies the usual axioms of \( G \)-bundles. By weights we mean elements \( \theta = \{\theta_x|x \in \mathcal{R}\} \in (Y(T) \otimes \mathbb{R})^m \) where \( \theta_x \) lies in the interior of the facet \( \sigma_x \) (cf (2.3)) and \( m = \mathcal{R} \). A parahoric torsor \( E(\mathcal{F}, \theta) \) consists of a quasi-parahoric torsor and weights.

7.2. **Uniformization.** Let \( \mathcal{M}_X(G) \) be the moduli stack of parahoric \( G \)-torsors on \( X \). By [20] it is an algebraic stack. Recall that \( Q_G = \prod_{x \in \mathcal{R}} \mathcal{F}_{\mathcal{L}_x} \) and \( X^e = X \setminus \mathcal{R} \). A \( R \)-point of \( Q_G \) classifies \( G \)-torsors on \( X \times \text{Spec}(R) \) together with a section on \( X^e \times \text{Spec}(R) \). The map \( Q \to \mathcal{M} \) forgets the section and the ind-scheme \( L_{X^+}(G) \) acts on \( Q_G \) by changing the section. By the Uniformization theorem (cf [20], Heinloth]) we have an isomorphism of stacks

\[
\begin{align*}
Q_G/L_{X^+}(G) &= \mathcal{M}_X(G) \\
Q \times L_{X^+}(G) &= Q \times \mathcal{M} Q.
\end{align*}
\]

For a stack \( \mathcal{X} \), \( H^q_{\text{Et}}(\mathcal{X}, \mathcal{G}_m) \) is called the cohomological Brauer group. We wish to compute it when \( \mathcal{X} = \mathcal{M}_X(G) \).

7.3. **Cohomology of sheaves on \( \mathcal{M}_X(G) \).** We begin with a generality. Let \( \mathcal{Y} \) be a simplicial ind-scheme and \( \mathcal{X} \) be an Artin stack. Let \( a : \mathcal{Y} \to \mathcal{X} \) be a morphism universally of cohomological descent (cf [12,5]). Let us denote by \( \mathcal{G}_m \) the abelian sheaf defined by \( \mathcal{G}_m \) on the big étale sites of \( \mathcal{Y} \) and \( \mathcal{X} \). We have \( a^*\mathcal{G}_m = \mathcal{G}_m \) since \( a^* \) is just restriction to \( \mathcal{E}(\mathcal{Y}) \) of \( \mathcal{E}(\mathcal{X}) \). Then by Theorem 12.4.1 we have an equality of abutments

\[
H^{p+q}_{\text{Et}}(\mathcal{Y}, \mathcal{G}_m) = H^{p+q}_{\text{Et}}(\mathcal{X}, \mathcal{G}_m).
\]

For our purposes, we now specialize to the case \( p : Q_G \to \mathcal{M}_X(G) \).
Proposition 7.3.1. The morphism \( p \) is universally of cohomological descent.

Proof. Recall that \( Gr_{G,x} \) parametrizes \( G \)-torsors together with a section on \( X \setminus \{x\} \). By [20] Thm 4, for any \( S \)-family \( \mathcal{P} \in \mathcal{M}_X(G)(S) \), there exists an \( \acute{e} \text{tale} \) covering \( S' \to S \) such that \( \mathcal{P}|_{X \setminus \{x\} \times S'} \) is trivial. Recall by (5.0.1) that \( \mathcal{Q}_G = \prod_{x \in R} Gr_{G,x} = \prod_{x \in R} \mathcal{F}l_{\sigma_x} \). So \( p \) admits \( \acute{e} \text{tale} \) local sections in the sense of Definition 12.5.3. So by Theorem 12.5.4 \( p \) is universally of cohomological descent.

Let us abbreviate \( Q_G \) as \( Y \) and \( \mathcal{M}_X(G) \) as \( Z \). Associated to any such projection \( p \), by the coskeleton construction one has a simplicial ind-scheme augmented by \((a, Z)\). It is isomorphic to the quotient of the Bar construction (see (6.1.1))

\[
\text{(7.3.2)}
\]

\[ EL_{X^*}(G)(Y)/L_{X^*}(G). \]

We will abbreviate it as \( Y_* \). Its set of \( n \)-simplices is the quotient by the left-action of \( L_{X^*}(G) \) on \( L_{X^*}(G)^{x \times Y} \) for \( n \geq 0 \). We have a simplicial map

\[
\text{(7.3.3)}
\]

\[ p_* : Y_* \to BL_{X^*}(G)_*, \]

\( p_* \) is defined by projecting on the \( L_{X^*}(G) \)-part. The Grothendieck spectral sequence for \( \Gamma_Y = \Gamma_{BL_{X^*}(G)} \circ p_* \) with values in a simplicial sheaf \( F^* \) on \( Y_* \) has the form

\[
\text{(7.3.4)}
\]

\[ E_2^{p,q} = H_{\acute{e}t}^p(\text{BL}_{X^*}(G)_*, R^q p_{*,*}(F^*)) \implies H_{\acute{e}t}^*(Y_*, F^*), \]

(cf [38] page 10 (1.9) and page 27 (5.5) for similar result). Combining with (7.3.1) and taking \( F^* \) as the sheaf defined by \( \mathbb{G}_m \), we get

\[
\text{(7.3.5)}
\]

\[ E_2^{p,q} = H_{\acute{e}t}^p(\text{BL}_{X^*}(G)_*, R^q p_{*,*}(\mathbb{G}_m)) \implies H_{\acute{e}t}^*(\mathcal{M}_X(G), \mathbb{G}_m). \]

7.4. Big-\( \acute{e} \text{tale} \) site and line bundles on sites and ind-schemes. For the case of schemes, by [26] Prop III.3.1 for an abelian sheaf the cohomology on the big and small \( \acute{e} \text{tale} \) sites agree. Let \( A \to \mathcal{M} \) be an atlas of an algebraic stack \( \mathcal{M} \). Let \( A^{X,MP} \) denote the \( p \)-fold fiber product of \( A \) over \( \mathcal{M} \). We have spectral sequences

\[
\text{(7.4.1)}
\]

\[ E_1^{p,q} = H_{\acute{e}t}^q(A^{X,MP}, \mathbb{G}_m) \implies H_{\acute{e}t}^p(\mathcal{M}, \mathbb{G}_m) \]

\[ E_1^{p,q} = H_{\acute{e}t}^q(A^{X,MP}, \mathbb{G}_m) \implies H_{\acute{e}t}^p(\mathcal{M}, \mathbb{G}_m) \]

which fit vertically by the natural homomorphism from the small to big \( \acute{e} \text{tale} \) groups. So cohomologies on the big-\( \acute{e} \text{tale} \) and small sites agree for algebraic stacks too. The following proposition proves a similar result for ind-projective varieties like \( \mathcal{F}_l \).

Proposition 7.4.1. Let \( \sigma \) be a facet of \( a \). We have \( H_{\acute{e}t}^1(\mathcal{F}_l, \mathbb{G}_m) = \text{Pic}(\mathcal{F}_l). \)

Proof. For simplicity, we explain the case of \( \sigma = a \). The more general case can be proven by a very similar argument. Recall that \( \mathcal{F}_a = \lim_{\longrightarrow} \mathcal{F}_l \) where \( W \) is the Iwahori-Weyl group. We revisit the setup of the proof of Prop 5.0.3. By the Grothendieck spectral sequence (5.0.3) we get

\[ 0 \to R^1 \lim_{\longrightarrow} H_{\acute{e}t}^0(S_w, \mathbb{G}_m) \to H_{\acute{e}t}^1(\mathcal{F}_a, \mathbb{G}_m) \to \lim_{\longrightarrow} H_{\acute{e}t}^1(S_w, \mathbb{G}_m) \to R^2 \lim_{\longrightarrow} H_{\acute{e}t}^0(S_w, \mathbb{G}_m) \]

on the big-\( \acute{e} \text{tale} \) site. Now since \( H_{\acute{e}t}^0(S_w, \mathbb{G}_m) = H^0(S_w, O_{S_w}^*) = C^* \) so the first and last terms vanish. So the middle arrow is an isomorphism. Now by [35] II Theorem(4.3.1)], there is a canonical isomorphism \( H_{\acute{e}t}^1(S_w, \mathbb{G}_m) \cong \text{Pic}(S_w) \) where \( \text{Pic}(S_w) \) is the usual Picard group \( H_{\text{Zar}}^1(S_w, O_{S_w}^*) \). Since the Iwahori-Weyl group
\[ \tilde{W} \text{ equals the affine Weyl group } W_a \text{ for us, so } \lim \text{Pic}(S_w) = \oplus_{i \in S} \text{Pic}(S_{w_i}) \text{ where } w_i \text{ is the simple reflection corresponding to the affine simple root } \alpha_i \in S. \] So we can conclude by [29 Prop 10.1] according to which \( \text{Pic}(\mathcal{F}_a) = \oplus_{i \in S} \text{Pic}(S_{w_i}). \)

7.5. **The case of one parabolic point and line bundles.** In this subsection, let \( x \) be the unique parabolic point, let \( \sigma^{\alpha} \) for \( \alpha \in S \) be the facet of \( a \) where only \( \alpha \) does not vanish. Thus the case \( \sigma^{\alpha_0} = v_0 \) corresponds to the parahoric group \( G(\mathcal{O}). \) For simplicity we will just write \( \sigma \) for a zero-dimensional facet.

7.5.1. **Construction of** \( \mathcal{G}^a \rightarrow X. \)** For any \( \sigma \), we choose an alcove \( a^x \) such that \( \sigma \) lies in its closure. So we have a natural map \( \mathcal{G}_{a^x} \rightarrow \mathcal{G}_{a^x} \) over \( D_x \). Recall that we have assumed in (2.3) that the gluing functions \( \{f_x\} \) lie in \( \text{Mor}(D_0^\circ, G) = G(K_0) \) where \( K_0 \) is the quotient field of \( \mathcal{O}_x \).

As in the introduction, let us agree to denote by \( \mathcal{G}^x \) the Bruhat-Tits group scheme on \( X \) which restricts to \( G_e \). Let \( \mathcal{G}^a \) denote the group scheme which restricts to \( \mathcal{G}_{a^x} \rightarrow D_x \) and \( X^0 \times G \) respectively and is constructed by gluing through \( f_x \). So the natural map \( \mathcal{G}_{a^x} \rightarrow \mathcal{G}_a \) over \( \text{Spec}(\mathcal{O}_x) \) extends to \( \mathcal{G}^a \rightarrow \mathcal{G}^x \) over \( X \). The group scheme \( \mathcal{G}^a \) depends on our choices of \( f_x \) and \( a^x \), but in this paper we will only need its existence. We have a morphism of stacks \( \mathcal{M}_X(\mathcal{G}^a) \rightarrow \mathcal{M}_X(\mathcal{G}). \)

7.5.2. **Construction of** \( \mathcal{M}_X(\mathcal{G}^a) \rightarrow \mathcal{M}_X(\mathcal{G}). \)** While there is not necessarily a morphism of group schemes \( \mathcal{G}^a \rightarrow X \times G \), let us construct a morphism of algebraic stacks \( \mathcal{M}_X(\mathcal{G}^a) \rightarrow \mathcal{M}_X(\mathcal{G}) \) whose fibers are full flag varieties \( G/B \). We will need only its existence in the rest of the paper.

Let us construct a morphism of algebraic stacks \( \mathcal{M}_X(\mathcal{G}^a) \rightarrow \mathcal{M}_X(\mathcal{G}) \). For each \( x \in R \), let \( w_x \) be an element in the affine Weyl group \( W_a \) which maps \( a_0 \) to \( a^x \). Set \( v_x = w_x v_0 \). Let \( N \) denote the normalizer of \( T \) in \( G \). We choose an element \( n_x \in N(K_x) \) which maps to \( w_x \) where \( K_x \) is the quotient field of \( \mathcal{O}_{X,x} \). We shall view \( n_x \) as an element in \( G(K_x) \).

Let \( E \rightarrow X \) be a principal \( G \)-bundle obtained by gluing the trivial bundles on \( X^0 \) and \( \{D_x\}_{x \in R} \) by \( \{n_x f_x\}_{x \in R} \). Let \( \text{Ad}(E) \rightarrow X \) denote the adjoint group scheme of \( E \). Then \( \text{Ad}(E) \) is obtained by gluing the constant group scheme \( X^0 \times G \) with \( G_{v_0} \) by \( \{n_x f_x\} \). Thus \( \text{Ad}(E) \) is obtained by gluing \( X^0 \times G \) with \( G_{v_0} \) via \( \{f_x\} \). Since the group scheme \( \mathcal{G}^a \) is obtained by gluing \( X^0 \times G \) and \( \mathcal{G}_{a^x} \) via \( \{f_x\} \), and we have natural morphisms \( \mathcal{G}_{a^x} \rightarrow \mathcal{G}_{e^x} \), so we obtain a natural map of group schemes \( \mathcal{G}^a \rightarrow \text{Ad}(E) \). This also furnishes

\[
\phi_f : \mathcal{M}_X(\mathcal{G}^a) \rightarrow \mathcal{M}_X(\text{Aut}(E)) \tag{7.5.1}
\]

By Proposition 7.5.2, its fibers are isomorphic to \( G/B \). The principal bundle \( E \) is a left \( \text{Aut}(E) \)-torsor and right \( G \)-torsor on \( X \). We have an isomorphism of stacks \( \mu_E : \mathcal{M}_X(\text{Aut}(E)) \rightarrow \mathcal{M}_X(G) \) which sends a right \( \text{Aut}(E) \)-torsor \( \mathcal{F} \) to the principal \( G \)-bundle \( \mathcal{F} \times_{\text{Aut}(E)} E \). Here \( \mathcal{F} \times_{\text{Aut}(E)} E \) denotes the space where for local sections \( f, g, e \) of \( \mathcal{F}, \text{Aut}(E) \) and \( E \) respectively we identify \( (eg^{-1}f) \) with \( (e, f) \). Its inverse is given by sending \( F \) to \( F \times G E^{op} \). Here \( E^{op} \) has the same underlying space as \( E \) but for local section \( e \) of \( E^{op} \), \( e \) is defined to be \( eg^{-1} \) after viewing it as a local section of \( E \). Thus we obtain a morphism of stacks with desired properties:

\[
\mu_E \circ \phi_f : \mathcal{M}_X(\mathcal{G}^a) \rightarrow \mathcal{M}_X(\text{Aut}(E)) \rightarrow \mathcal{M}_X(G) \tag{7.5.3}
\]
To prove our main theorem, we begin relating line bundles on $\mathcal{M}_X(G^\sigma)$ with those on $\mathcal{M}_X(G^\sigma)$. To this end, we begin by reconciling our references [16] and [20] on the one hand and [8] on the other because we want to use formulation [8, Theorem 4.2.1] of [16, Theorem 17] and a similar formulation of [20, Thm 7].

Let us recall that the references [16, Faltings] and [20, Heinloth] work with rigidified line bundles i.e line bundles whose restriction to the trivial torsor for moduli stacks and on the trivial coset for affine grassmannians becomes trivial. Let us make this more precise. Let $L$ be a line bundle on a stack $X \to S$ together with a section $s : S \to X$ and $L_s$ denotes the restriction of $L$ to $S$ through $s$. Then a rigidification of $L$ is a choice of a trivialization $\alpha : \mathcal{O}_S \cong L_s$. These references work over a connected noetherian base scheme $S$ and the curve $C \to S$ is smooth projective and absolutely irreducible. In particular, the moduli stack $\mathcal{M}_C \to S(G)$ of principal $G$-bundles is fibered over $S$. Therefore let us note that line bundles on $\mathcal{M}$ are not rigidifiable automatically.

Now we recall the set up of [8]. Recall that for us (cf §8.2) $\text{Pic}(?)$ denotes the group of isomorphism classes of line bundles. Recall if $X/k$ is an algebraic stack, then the Picard functor (cf [8, Definition 2.1.1]) $\text{Pic}(X)$ is the functor that to a scheme $T$ of finite type over a field $k$ associates the group $\text{Pic}(X \times T)/\text{pr}_2^*\text{Pic}(T)$. If $\text{Pic}(X)$ is the constant presheaf given by an abelian group $A$, then following [8] we shall say that $\text{Pic}(X)$ is discrete and simply denote $\text{Pic}(X) \cong A$. These definitions naturally generalize to the case of ind-schemes also. For the various stacks $X$ and ind-schemes $X$ of interest to us, the Picard functor $\text{Pic}(X)$ (or $\text{Pic}(X)$) will be a finitely generated free abelian group hence discrete, the base space $S$ will be $\text{Spec}(\mathcal{O})$ and the section $s : S \to X$ will be given by the trivial torsor or the identity coset. Since $S = \text{Spec}(\mathcal{O})$ so all line bundles on $X/S$ are rigidifiable. So when $\text{Pic}(?)$ is discrete, then the constant presheaf defined by $\text{Pic}(?)$ on the category of schemes and $\text{Pic}(?)$ are isomorphic. There is a natural forgetful map from rigidified line bundles to isomorphism classes of line bundles and further, in this case, once we choose arbitrary rigidifications for any set of generators of $\text{Pic}(X)$, then they determine a compatible choice of rigidifications on every line bundle in $\text{Pic}(X)$. Henceforth we choose once for all arbitrary rigidifications for a set of generators of $\text{Pic}(X)$. Moreover, we will consider line bundles upto rigidifications in the following sense.

Let us choose a uniformizer $z \in \hat{\mathcal{O}}_{X,x}$. Recall that we have a map $\text{glue}_{x,z} : F_{\ell} \to \mathcal{M}_X(G^\sigma)$ that on a coset $fL^s G_\sigma$ glues the trivial $G$-torsor on $X \setminus \{x\}$ with the trivial $G_\sigma$-torsor on $\text{Spec}(\hat{\mathcal{O}}_{X,x})$.

Let $Gr_G$ denote $LG/L^\sigma G$. Let $G$ be simply-connected and simple. We will use [8, Theorem 4.2.1] formulation of [16, Theorem 17]: we have $\text{Pic}(Gr_G) = \mathbb{Z}$ and $\text{glue}_{x,z}^* : \text{Pic}(\mathcal{M}_G) \to \text{Pic}(Gr_G)$ is an isomorphism of functor. Similarly, denoting isomorphism classes of line bundles by $\text{Pic}(?)$, by $c$ the homomorphism given by central charge (cf 2.5.1), by $G_z$ the reduction of the group scheme to the closed fiber at $z$, by [20, Thm 7] we have the exact sequence

(7.5.4) $0 \to \prod_{z \in \mathcal{R}} \mathbb{X}^*(G_z) \to \text{Pic}(\mathcal{M}_X(G)) \xrightarrow{\sim} \mathbb{Z} \to 0$.

Let us emphasize that in the reference [20, Thm 7], the middle term above is the Picard group of rigidifiable bundles.
Proposition 7.5.1. Pull-back under $\mathcal{F}_l a \xrightarrow{\alpha_0} \mathcal{M}_X(G^a)$ establishes an isomorphism

\[(7.5.5)\]
$$q_*: \text{Pic}(\mathcal{M}_X(G^a)) \to \text{Pic}(\mathcal{F}_l a) = \oplus_{\alpha \in S} L_{\epsilon_0}.$$

Proof. For $q_\alpha$ from (7.3.5) the terms $0 \to E^{1,0}_2 \to E^{1,1}_2 \to E^{0,1}_2$ work out to

$$0 \to H^1(BL_X \cdot G, \mathcal{G}_m) \to H^1(\mathcal{M}_X(G^a), \mathcal{G}_m) \to H^0(BL_X \cdot G, H^1(\mathcal{F}_l a, \mathcal{G}_m)),$$

where all cohomology groups are computed in the big-étale topology. By Proposition 6.1.1 we have $H^1(BL_X \cdot G, \mathcal{G}_m) = 0$. Since $L_X \cdot G$ is connected, so the third group is just $H^1(\mathcal{F}_l a, \mathcal{G}_m)$. So we have an injective map $H^1(\mathcal{M}_X(G^a), \mathcal{G}_m) \to H^1(\mathcal{F}_l a, \mathcal{G}_m)$.

The ample generator of Pic($\mathcal{M}_X(G)$) has central charge one (cf [16, Theorem 17]). It may be written as $L_{\epsilon_0}$ in our notations. Let $L_{\epsilon_0} \in \mathcal{M}_X(G)$ denote its pull-back under $\mathcal{M}_X(G^a) \to \mathcal{M}_X(G)$ (7.5.3). The central charge does not change under $\mu_E$ (cf 7.5.2) because it is an isomorphism of stacks. It also does not change under $\phi_f$ (cf 7.5.1) by (3.1). So $L_{\epsilon_0} \in \mathcal{M}_X(G^a)$ also has central charge one. Taking $G = G^a$, the sequence (7.5.4) is split by mapping $1 \in \mathbb{Z}$ to $L_{\epsilon_0}(a)$.

Consider $\mathcal{G}_a \to G_A$. At the closed fiber, the image of $\mathcal{G}_a \otimes k$ in $G$ is $B$. So we may view a weight $\omega$ of $G$ as a character on $\mathcal{G}_a^a$ via

\[7.5.6\]
$$\mathcal{G}_a^a = \mathcal{G}_a \otimes k \to B \to T \to \mathcal{G}_m.$$

Let $L_\omega$ be the corresponding line bundle on $\mathcal{M}_X(G^a)$ via (7.5.4). Let $L_G$ denote the pull-back of $L_\omega$ via $\mathcal{F}_l a \to \mathcal{M}_X(G^a)$. By [8, Theorem 4.2.1] we see that $L_{\sigma_0} \to \mathcal{M}_X(G)$ pulls back to the generator of $G_{\mathcal{F}_l a} = \mathcal{F}_l a(\sigma_0)$. This line bundle pulls back to $L_{\epsilon_0} \to \mathcal{F}_l a$ (cf [39, (2.2.6)]). By (2.4.9) it follows that on $\mathcal{F}_l a$ we have an isomorphism

\[7.5.7\]
$$L_{\epsilon_0} \cong L_G \otimes L_{\epsilon_0}^t.$$

This isomorphism together with the splitting of (7.5.4) imply that an isomorphism is induced in (7.5.5) by the pull-back map that sends: $L_\omega \mapsto L_G$ and $L_{\sigma_0}(a) \mapsto L_{\epsilon_0}$.

\[\square\]

Proposition 7.5.2. Let $\sigma$ be any facet. Let $a$ be an alcove such that $\sigma$ lies in its closure. Consider the diagram

\[7.5.8\]
$$\begin{array}{ccc}
\mathcal{F}_l a & \xrightarrow{p} & \mathcal{F}_l a \\
\downarrow{q_*} & & \downarrow{q_*} \\
\mathcal{M}_X(G^a) & \xrightarrow{\pi} & \mathcal{M}_X(G^\sigma)
\end{array}$$

Then the diagram

\[7.5.9\]
$$\begin{array}{ccc}
\text{Pic}(\mathcal{F}_l a) & \xleftarrow{p^*} & \text{Pic}(\mathcal{F}_l a) \\
\downarrow{q_*^*} & & \downarrow{q_*^*} \\
\text{Pic}(\mathcal{M}_X(G^a)) & \xrightarrow{\pi^*} & \text{Pic}(\mathcal{M}_X(G^\sigma))
\end{array}$$

has all arrows injective and is a pull-back square. Further, we have $p_* \mathcal{G}_m = \mathcal{G}_m$ and $\pi_* \mathcal{G}_m = \mathcal{G}_m$. Lastly, let $G^\sigma$ denote the reductive quotient of $G_{\mathcal{F}_l a} \otimes k$ and $F^\sigma$ its full flag variety. The sheaves $R^1 p_* \mathcal{G}_m$ and $R^1 \pi_* \mathcal{G}_m$ are the trivial local systems with fibers isomorphic to Pic($F^\sigma$).
etale locally, the morphisms \( p_L \) faithfully flat. Let us show that the diagram (7.5.9) is a pull-back square. Take any \( \alpha \).

**Proof.** By arguing exactly as in Proposition 7.5.1, we get that \( q^*_\sigma \) is injective. By the uniformization theorem (cf 7.2.1 [20]), it follows that (7.5.2) is cartesian.

Here the fibers of the horizontal maps are isomorphic to the \( k \)-scheme given by \( G_\sigma \otimes k/\text{Im}(G_n \otimes k \to G_n \otimes k) \). By Corollary 2.0.2 this is a flag variety \( F^\sigma \) of \( G^\sigma := G_\sigma/G^\sigma_n \). One can see that \( F^\sigma \) is the full flag variety by Corollary 2.0.2 and its proof as follows. Firstly, no affine root \( \alpha \) takes an integral value on \( a \); so all inequalities in (2.0.7) are strict because, by definition, for \( \alpha \in \Phi \) we have \( \alpha^* \sigma = 0 \) and no root vanishes on \( a \). Thus viewing \( Y_\sigma \) as the root-system of \( G_\sigma/G^\sigma_n \), the set \( G_{\sigma,n} \) (cf (2.0.4)) has exactly one of \( \alpha \) or \( -\alpha \). So \( G_{\sigma,n}/G^\sigma_n \) is a Borel subgroup of \( G^\sigma_n \).

Now \( LG \to F_{\sigma} = LG/L^+G_\sigma \) has etale local sections by Theorem 2.1.1. So \( G_{\sigma,n}/G^\sigma_n \) is flat. Let us show that the diagram (7.5.9) is a pull-back square. Take any line bundle \( L \in \text{Pic}(F_{\sigma}) \). By Proposition 7.5.1 it suffices to show that the natural morphism \( q^*_\sigma q_{\sigma,*} \to L \) of sheaves on \( F_{\sigma} \) is an isomorphism. This can be checked on any faithfully flat covering. So we will check that

\[
(7.5.10) \quad \theta : p^* q^*_\sigma q_{\sigma,*} \to p^* L
\]

is an isomorphism. We first prove the following lemma.

**Lemma 7.5.3.** Let \( X \) be an ind-scheme, say \( \lim_{\longrightarrow \ n \in I} X_n \). Consider \( a : X \to X \) as in the diagram (12.2.4). Let \( \mathcal{G} \in X_{\acute{E}t} \) be a quasi-coherent sheaves on \( X \). Let \( \mathcal{G}_n \) be the restriction of \( \mathcal{G} \) on \( X_n \). Suppose that there is a finite subcategory \( I' \subset I \) such that \( \mathcal{G} = \lim_{\longrightarrow \ n \in I', \mathcal{G}_{\alpha,n}} \). Then for any flat \( u : U \to X \), consider the cartesian square

\[
(7.5.11) \quad \begin{array}{ccc}
X_u & \xrightarrow{u} & X \\
\downarrow a' & & \downarrow a \\
U & \xrightarrow{u} & X
\end{array}
\]

The canonical map \( u^* a_* \mathcal{G} \to a'_* u^* \mathcal{G} \) is an isomorphism.

**Proof.** Consider the cartesian squares

\[
(7.5.12) \quad \begin{array}{ccc}
X_{n,u} & \xrightarrow{u_n} & X_n \\
\downarrow a'_n & & \downarrow a_n \\
U & \xrightarrow{u} & X
\end{array}
\]

So \( X_u = \lim_{\longrightarrow \ n \in I} X_{n,u} \). By definition of the site of ind-schemes and sheaves on it, \( \mathcal{G} \) is determined by its restrictions \( \{ \mathcal{G}_n | n \in I \} \) on \( X_n \) for \( n \in I \). Further we have

\[
(u^* \mathcal{G})_n = u_n^* (\mathcal{G}_n), \quad a_* \mathcal{G} = \lim_{\longrightarrow \ n} a_n \mathcal{G}_n, \quad (u_* a^* \mathcal{G}) = \lim_{\longrightarrow \ n} (u_n^* (a_n^* \mathcal{G}))_n = \lim_{\longrightarrow \ n} a'_n (u_n^* \mathcal{G}_n).
\]

Since pushforward commutes with flat base-changes, we have the isomorphism

\[
(7.5.13) \quad u^* a_* \mathcal{G}_n \to a'_* u_n^* \mathcal{G}_n
\]

Therefore \( u^* a_* \mathcal{G} \to u^* a_* \lim_{\longrightarrow \ n} \mathcal{G}_n \to \lim_{\longrightarrow \ n} u^* (a_n \mathcal{G}_n) \to \lim_{\longrightarrow \ n} a'_n u_n^* \mathcal{G}_n \to a'_* \lim_{\longrightarrow \ n} u_n^* \mathcal{G}_n \to a'_* \lim_{\longrightarrow \ n} (u^* \mathcal{G})_n \to a'_* u^* \mathcal{G}.
\]

All arrows above are isomorphisms: the first and last by hypothesis, the fourth by (7.5.13) and the remaining because pushforward commutes with arbitrary projective limits and pull-back by finite projective limits. The last two assertions follow since \( (a^*, a_*) : X_{\acute{E}t} \to X_{\acute{E}t} \) (cf (12.5.10)) and \( (u^*, u_*) : U_{\acute{E}t} \to X_{\acute{E}t} \) are morphisms of
topoi. So pushforward commutes with projective limits and pull-backs with finite projective limits.

By the proof of Proposition \textit{7.4.1} the hypothesis of Lemma \textit{7.5.3} hold for line bundles on \(F_{\sigma}\). Indeed, we proved the following isomorphisms

\[
H^1_{\text{ét}}(F_{\sigma}, \mathbb{G}_m) \to \lim H^1_{\text{ét}}(S_w, \mathbb{G}_m) \to \lim \text{Pic}(S_w) = \oplus_{i \in S} \text{Pic}(S_w),
\]

and therefore any line bundle on \(F_{\sigma}\) is determined by its restriction to any Schubert variety that contains all \(S_{w_i}\) for \(w_i \in S\). So since \(\pi\) is flat, we have by Lemma \textit{7.5.3}

\[
(7.5.14) \quad \pi^* q_{\sigma,*} L \cong q_{n,*} p^* L.
\]

an isomorphism of sheaves. Applying \(q_{n,*}\), this gives \(p^* q_{\sigma,*} L = q_{n,*} \pi^* q_{\sigma,*} L \to q_{n,*} p^* L\). The adjunction arrow \(adj : q_{n,*} \to \text{Id}\) is an isomorphism because all line bundles on \(F_{\sigma}\) descend to \(\mathcal{M}_X(\mathbb{G}_a)\) by Proposition \textit{7.5.1}. So we have checked that \(\theta\) in \textit{(7.5.10)} is an isomorphism.

Consider the étale local \(F^\sigma\) fibration \(p : F_{\sigma} \to F_{\sigma}\) and an arbitrary open \(u : U \to F_{\sigma}\). Let \((F_{\sigma})_u := U \times_{F_{\sigma}} F_{\sigma}\). Then \((F_{\sigma})_u \to U\) is a \(F^\sigma\) fibration represented by a scheme. Further it is the final object of the category:

\[
\begin{array}{ccc}
V & \longrightarrow & F_{\sigma} \\
\downarrow & & \downarrow p \\
U & \longrightarrow & F_{\sigma}
\end{array}
\]

of objects over \(u\) together with a morphism to \(F_{\sigma}\) as in the above diagram. By Definition \textit{(12.2.5)} we have \((p_* \mathcal{O})_u = \lim p_* \mathcal{O}_V\). The inverse system reduces to \(p_* \mathcal{O}_{(F_{\sigma})_u}\). So it identifies with \(\mathcal{O}_U\). Thus \(p_* \mathcal{O} = \mathcal{O}\) on the big-étale site. Similarly \(\pi_* \mathcal{O} = \mathcal{O}\) because \(\pi\) is an étale local \(F^\sigma\) fibration.

So \(\pi_* \mathbb{G}_m = \mathbb{G}_m\) and \(p_* \mathbb{G}_m = \mathbb{G}_m\) on the big-étale sites. Thus at closed points the fibers of \(R^1 \pi_* \mathbb{G}_m\) and \(R^1 p_* \mathbb{G}_m\) are isomorphic to \(\text{Pic}(F^\sigma)\). This isomorphism can be made canonical by the following observation. Let \(y : \text{Spec}(\mathbb{C}) \to \mathcal{M}_X(\mathbb{G}_a)\) be an arbitrary closed point. Choose any isomorphism \(\theta\) of the fiber \(\pi^{-1}(y)\) with \(F^\sigma\) as left-\(\mathbb{G}_a\) homogenous spaces. Since \(\mathbb{G}_a\) is connected and \(\text{Pic}(F^\sigma)\) is discrete, so \(\text{Pic}(\pi^{-1}(y)) \to \text{Pic}(F^\sigma)\) is independent of \(\theta\). Since \(p\) and \(\pi\) are étale locally \(F^\sigma\)-fibrations and \(C\)-points are dense, this shows that \(R^1 p_* \mathbb{G}_m\) and \(R^1 \pi_* \mathbb{G}_m\) are the trivial local systems. \hfill \square

**Corollary 7.5.4.** When \(\sigma\) is a vertex of \(\mathfrak{a}\), we have \(\text{Pic}(\mathcal{M}_X(\mathbb{G}_a)) = \mathbb{Z}\). Under \(\pi : \mathcal{M}_X(\mathbb{G}_a) \to \mathcal{M}_X(\mathbb{G}_a)\), the pull-back map corresponds to \(Z L_{\alpha} \mapsto \oplus_{\alpha \in S} Z L_{\alpha}\).

Recall that \(G\) is a simply-connected and semi-simple group.

### 7.6 The case \(R = \{x\}\) and the facet \(\sigma\) is alcove \(\mathfrak{a}\).

**Theorem 7.6.1.** The cohomological Brauer group \(H^2_{\text{ét}}(\mathcal{M}_X(\mathbb{G}_a), \mathbb{G}_m)_{\text{tor}} = 0\).

**Proof.** For \(F_{\sigma} \to \mathcal{M}_X(\mathbb{G}_a)\) from the spectral sequence \textit{(7.3.5)} we deduce

\[
0 \to E_2^{1,0} \to E_1^{1} \to E_2^{0,1} \to E_2^{2,0} \to \ker(E_2^{0,2}) \to E_2^{2,1} \to \cdots
\]
Since the square (7.5.9) is cartesian so the images of \( H^1(BL_X, \mathbb{G}_m, \mathbb{G}_m) \) \( \xrightarrow{\sigma} H^0(BL_X, \mathbb{G}_m, H^1(Fl_n, \mathbb{G}_m)) \)
\( \to H^2(BL_X, \mathbb{G}_m, \mathbb{G}_m) \to \ker[H^2(BL_X, \mathbb{G}_m, \mathbb{G}_m)] \) \( \xrightarrow{\sigma} H^0(BL_X, \mathbb{G}_m, H^2(Fl_n, \mathbb{G}_m)) \)
\( \to H^1(BL_X, \mathbb{G}_m, H^1(Fl_n, \mathbb{G}_m)) \to \ldots \)

where all cohomology groups are computed in the big-étale topology. Let us mention some simplifications. Now \( H^1(BL_X, \mathbb{G}_m, \mathbb{G}_m) = 0 \) by Proposition 6.4.1. Since \( L_X \ast G \) is connected and \( H^i(Fl_n, \mathbb{G}_m) \) is discrete for \( i = 1, 2 \), so \( L_X \ast G \) acts trivially. Also from Proposition 6.4.1 we get \( H^1(BL_X, \mathbb{G}_m, H^1(Fl_n, \mathbb{G}_m)) = 0 \). Using Corollary 6.4.5 we denote \( H^2(BL_X, \mathbb{G}_m, \mathbb{G}_m) \) by Pic(\( L_X \ast G \)). So the sequence simplifies to

\[
0 \to \text{Pic}(\mathcal{M}_X(G^\sigma), \mathbb{G}_m) \xrightarrow{\sigma} \text{Pic}(Fl_n) \to \text{Pic}(L_X \ast G) \to \ker[H^2(q_n)] \to 0
\]

where we denote \( H^2(q_n) : H^2(\mathcal{M}_X(G^\sigma), \mathbb{G}_m) \to H^2(Fl_n, \mathbb{G}_m) \). We have the inclusion \( H^2(\mathcal{M}_X(G^\sigma), \mathbb{G}_m)_{\text{tor}} \subset \ker(H^2(q_n)) \). Using Proposition 6.4.5 we denote \( H^2(\mathcal{M}_X(G^\sigma), \mathbb{G}_m) \) by Pic(\( L_X \ast G \)). So the sequence simplifies to

\[
0 \to \text{Pic}(\mathcal{M}_X(G^\sigma), \mathbb{G}_m) \xrightarrow{\sigma} \text{Pic}(Fl_n) \to \text{Pic}(L_X \ast G) \to \ker[H^2(q_n)] \to 0
\]

We will abbreviate \( \mathcal{M}_X(\mathbb{G}_m) \) as \( M^e \) and \( \mathcal{M}_A(\mathbb{G}_m) \) as \( M^a \). We have a similar sequence for \( \pi \) as well because \( H^2(M^e, \mathbb{G}_m) = 0 \) by Theorem 7.6.1. Therefore the two sequences can be put in exact sequences

\[
\begin{array}{ccc}
\text{Pic}(Fl_\sigma) & \xrightarrow{q^*} & \text{Pic}(Fl_n) \xrightarrow{\alpha^*} H^0(Fl_\sigma, R^1p_*\mathbb{G}_m) \xrightarrow{\ker(H^2(\pi))} 0 \\
\downarrow & & \downarrow \\
\text{Pic}(\mathcal{M}_\pi) & \xrightarrow{q^*} & \text{Pic}(\mathcal{M}_\pi) \xrightarrow{\alpha^*} H^0(\mathcal{M}_\pi, R^1p_*\mathbb{G}_m) \xrightarrow{\ker(H^2(\pi))} 0
\end{array}
\]

Since the square (7.5.9) is cartesian so the images of \( \alpha^\pi \) and \( \alpha^\mathcal{M} \) are naturally isomorphic. Further \( R^1p_*\mathbb{G}_m \) and \( R^1\pi_*\mathbb{G}_m \) are the trivial local systems by Proposition 7.5.2. So the middle arrow identifies with identity on \( \text{Pic}(\mathbb{G}_m) \). Thus \( \ker[H^2(\pi)] \to \ker(H^2(\pi)) \) is an isomorphism. So \( \ker(H^2(\pi)) \) is torsion-free. Now \( H^2(\mathcal{M}_e, \mathbb{G}_m, \mathbb{G}_m) \subset \ker(H^2(\pi)) \) because by Theorem 7.6.1 we have \( H^2(\mathcal{M}_e, \mathbb{G}_m, \mathbb{G}_m) = 0 \). So it follows that the cohomological Brauer group \( H^2_{\text{ét}}(\mathcal{M}_e, \mathbb{G}_m) = 0 \).
7.8. Several points with facet alcove at each of the points.

**Proposition 7.8.1.** For each \( x \in \mathcal{R} \), let us choose alcoves \( \mathbf{a}^x \). Let \( \mathcal{G}^a \) be a group scheme that restricts to \( \mathcal{G}_x \to \mathbb{D}_x \) at each \( x \in \mathcal{R} \). Let \( \mathcal{M} \) denote the moduli stack with facet \( \sigma_x = \mathbf{a}^x \) at each \( x \in \mathcal{R} \). The cohomological Brauer group of \( \mathcal{M} \) vanishes.

**Proof.** By repeating the construction in §7.5.2 for multiple points, we get a morphism \( \mathcal{M}_X(G^a) \to \mathcal{M}_X(G) \). Let \( \sigma_0^a \) be the unique vertex of \( \mathbf{a}^x \) corresponding to the vertex of \( \mathbf{a} \) where only \( \alpha_0 \) does not vanish. We have the cartesian square

\[
\begin{array}{ccc}
\prod_{x \in \mathcal{R}} \mathcal{F}_{\mathbf{a}^x} & \xrightarrow{p} & \prod_{x \in \mathcal{R}} \mathcal{F}_{\sigma_0^a} \\
\downarrow{q} & & \downarrow{q_0} \\
\mathcal{M} & \xrightarrow{\pi} & \mathcal{M}_X(G)
\end{array}
\]

Reasoning exactly as in the proof of Theorem 7.6.1, for \( q \) and \( q_0 \) the sequences analogous to (7.5.1) fit into exact sequences

\[
\begin{array}{cccccc}
\text{Pic}(\mathcal{M}) & \xrightarrow{q^*} & \text{Pic}(\prod_{x \in \mathcal{R}} \mathcal{F}_{\mathbf{a}^x}) & \xrightarrow{\text{Id}} & \text{Pic}(L_X \cdot G) & \xrightarrow{\text{ker}(H^2(q))} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Pic}(\mathcal{M}_X(G)) & \xrightarrow{q_0^*} & \text{Pic}(\prod_{x \in \mathcal{R}} \mathcal{F}_{\sigma_0^a}) & \xrightarrow{\text{Id}} & \text{Pic}(L_X \cdot G) & \xrightarrow{\text{ker}(H^2(q_0))} & 0
\end{array}
\]

The leftmost square is a pull-back square by exactly the same reasoning as (7.6.9). So by definition of pull-back square we have \( \text{coker}(q_0^*) \to \text{coker}(q^*) \). By Theorem 3.0.1, and the existence of line bundle of central charge one on \( \mathcal{M}_X(G) \), both these cokernels are equal to \( \text{coker}(\mathbb{Z} \xrightarrow{\text{diag}} \mathbb{Z}^{\oplus \mathcal{R}}) \). Therefore the two images in \( \text{Pic}(L_X \cdot G) \) are equal. Therefore \( \text{ker}(H^2(q_0)) \to \text{ker}(H^2(q)) \) is an isomorphism. Since \( H^2(\mathcal{M}_X(G), \mathbb{G}_m) \) is torsion-free by the one point case, so \( \text{ker}(H^2(q_0)) \) is also torsion-free. Thus \( \text{ker}(H^2(q)) \) is torsion-free. On the other hand, \( H^2(\mathcal{M}, \mathbb{G}_m)_{\text{tor}} \subset \ker(H^2(q)) \) because \( H^2(\prod_{x \in \mathcal{R}} \mathcal{F}_{\mathbf{a}^x}, \mathbb{G}_m)_{\text{tor}} = 0 \) by Proposition 5.0.1. Therefore \( H^2(\mathcal{M}, \mathbb{G}_m)_{\text{tor}} = 0 \). This proves the result.

7.9. General case. For each \( x \in \mathcal{R} \), for a facet \( \sigma_x \) let us choose an alcove \( \mathbf{a}^x \) such that \( \sigma_x \) lies in its closure. Let \( Z^x \subseteq S \) be the set of affine simple roots corresponding to the affine roots bordering \( \mathbf{a}^x \) but not vanishing at \( \sigma_x \). Let \( \mathcal{G}^a \) be the group scheme obtained from \( \mathcal{G} \) as in §7.6.1 using the \( \{ \mathbf{a}^x \}_{x \in \mathcal{R}} \). So we have a morphism \( \mathcal{G}^a \to \mathcal{G} \). We shall abbreviate \( \mathcal{M}_X(G) \) as \( \mathcal{M} \) and \( \mathcal{M}_X(G^a) \) as \( \mathcal{M}^a \).

**Theorem 7.9.1.** The cohomological Brauer group of \( \mathcal{M}_X(G) \) is \( \mathbb{Z}^{\oplus \mathcal{R}} \) modulo \( (1, \cdots, 1) \) and \( \{(0, \cdots, a^{\vee}_{\alpha}, \cdots, 0) | \alpha_x \in Z^x, x \in \mathcal{R} \} \).

**Proof.** We have the cartesian square

\[
\begin{array}{ccc}
\prod_{x \in \mathcal{R}} \mathcal{F}_{\mathbf{a}^x} & \xrightarrow{p} & \prod_{x \in \mathcal{R}} \mathcal{F}_{\sigma_x} \\
\downarrow{q} & & \downarrow{q_0} \\
\mathcal{M}^a & \xrightarrow{\pi} & \mathcal{M}
\end{array}
\]
For $q$ and $q_0$ the sequences analogous to (7.6.1) fit into exact sequences

\[
\begin{array}{ccccccccc}
\text{Pic}(M^a) & \xrightarrow{q^*} & \text{Pic}(\prod_{x \in \mathcal{R}} F_{l_{a^x}}) & \rightarrow & \text{Pic}(L_XG) & \rightarrow & \text{ker}(H^2(q)) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\text{Pic}(\mathcal{M}) & \xrightarrow{q_0^*} & \text{Pic}(\prod_{x \in \mathcal{R}} F_{l_{\sigma_x}}) & \rightarrow & \text{Pic}(L_XG) & \rightarrow & \text{ker}(H^2(q_0)) & \rightarrow & 0 \\
\end{array}
\]

The leftmost square is a pull-back square by exactly the same reasoning as (7.5.9). So by definition of pull-back square we have $i : \text{coker}(q_0^*) \rightarrow \text{coker}(q^*)$. They are subgroups of $\text{Pic}(L_XG)$ which is torsion-free by Proposition 6.4.2. Further they have the same ranks namely $|\mathcal{R}| - 1$ over $\mathbb{Z}$ by (7.5.4). So $\text{coker}(i)$ is torsion. Consider

\[(7.9.2)\quad 0 \rightarrow \text{coker}(q^*) \rightarrow \text{Pic}(L_XG) \rightarrow \text{ker}(H^2(q)) \rightarrow 0\]

So by snake lemma, we have $\text{coker}(i) = \ker(k)$. Thus $\ker(k)$ is torsion. On the other hand, $\ker(H^2(q))$ is torsion-free because $Br(M^a) = 0$ by Proposition 7.8.1. Considering the $\mathbb{Z}$-ranks above, we find that

$$\ker(k) = \ker(H^2(q_0))_{\text{tor}}.$$  

Now $H^2(\mathcal{M}, \mathcal{G}_m)_{\text{tor}} \subset \ker(H^2(q_0))$ because $H^2(\prod_{x \in \mathcal{R}} F_{l_{\sigma_x}} \otimes \mathcal{G}_m)_{\text{tor}} = 0$ by Proposition 5.0.1. Therefore

$$Br(\mathcal{M}) = H^2(\mathcal{M}, \mathcal{G}_m)_{\text{tor}} = \ker(H^2(q_0))_{\text{tor}} = \ker(k) = \text{coker}(i).$$

We have

\[(7.9.3)\quad \text{coker}(i) = \text{coker}(\text{Pic}(\prod_{x \in \mathcal{R}} F_{l_{\sigma_x}} \otimes \text{Pic}(M^a) \rightarrow \text{Pic}(\prod_{x \in \mathcal{R}} F_{l_{a^x}}))\]

Consider the central charge morphism $\text{Pic}(\prod_{x \in \mathcal{R}} F_{l_{a^x}}) \rightarrow \mathbb{Z}^{\oplus \mathcal{R}}$. It is surjective and its kernel is contained in $\text{Pic}(M^a)$. So $\text{coker}(i)$ is $\mathbb{Z}^{\oplus \mathcal{R}}$ modulo the image of $\text{Pic}(\prod_{x \in \mathcal{R}} F_{l_{\sigma_x}}) \oplus \text{Pic}(M^a)$. This image works out to $(1, \cdots, 1)$ and

$$\{(0, \cdots, a_{\alpha_x}^x, \cdots, 0) | \alpha_x \in \mathbb{Z}^x, x \in \mathcal{R}\}.$$  

\[\square\]

So the Brauer group of the moduli stack is trivial for groups of type $A$ and $C$, but it may not be trivial in general. For instance one can take the case of two points with facets vertices corresponding to affine roots whose highest co-root coefficient is two.

8. **Brauer Group of Moduli Space $M^a_\mathcal{X}(\mathcal{G})$**

From [31 Prop 6.1.1] and [31 Prop 6.3.1] we quote

**Proposition 8.0.1.** Let $G$ be a semi-simple simply-connected group. We have

a) The open substack $M^a_\mathcal{X}(\mathcal{G})$ of regularly stable torsors, in characteristic zero, and when $g_X \geq 3$, has complement of co-dimension at least two.

b) The morphism $M^a_\mathcal{X}(\mathcal{G}) \rightarrow M^a_\mathcal{X}(\mathcal{G})$ is a gerbe banded by $Z_G$.

As a corollary we get
Stacks are smooth noetherian stacks over $\mathbb{C}$ (standard proof for schemes (cf [35, Theorem 9.1.5]).

**Lemma 8.0.4.** We have spectral sequences

$$(8.0.1) \quad E_1^{p,q} = H^q_{et}(A_p, \mathbb{G}_m) \implies H^p_{et}(\mathcal{M}, \mathbb{G}_m)$$

$$E_1^{p,q} = H^q_{et}(A'_p, \mathbb{G}_m) \implies H^p_{et}(\mathcal{M}^\circ, \mathbb{G}_m)$$

whose differentials are compatible with restriction morphisms for every $E_r^{p,q}$ including $r = \infty$.

Here below we will not need the compatibility of the restriction of $E_\infty^{p,q}$ and $E^n$.

**Proof.** Let $K \to I^\bullet$ be an injective resolution in the derived category of bounded below complexes of abelian sheaves on $A_\bullet$. Consider the double complex

$$(8.0.2) \quad \Gamma(A_p, I^q|_{A_p}).$$

Then $E_1^{p,q} = H^q_{et}(A_p, I^\bullet|_{A_p}) \implies H^p_{et}(\mathcal{M}, K)$ is the spectral sequence associated to this double complex (cf the proof of Theorem [24.11 or [13 Thm 6.11]). The restriction morphism $res : Ab(\mathcal{M}_\bullet) \to Ab(\mathcal{M}^\circ_\bullet)$ admits a left adjoint $L$ defined as follows: given $G \in Ab(\mathcal{M}_\bullet)$ and $u : U \to A_p \in \mathcal{M}_\bullet$, define $L(G)_u$ by extending $G$ restricted to $U \times_{A_p} A'_p$ by zero outside of $U'$. This is exact. This shows that $res$ sends injectives to injectives. Thus $K_{A_\bullet^\prime} \to I^\bullet|_{A_p}$ is an injective resolution. Since we have a morphism of double complexes compatible with the differentials

$$(8.0.3) \quad \Gamma(A_p, I^q|_{A_p}) \to \Gamma(A'_p, I^q|_{A_p})$$

so for every $E_r^{p,q}$, including $r = \infty$ we get morphisms compatible with differentials.

**Lemma 8.0.5.** The natural restriction map $H^q_{et}(A_p, \mathbb{G}_m) \to H^q_{et}(A'_p, \mathbb{G}_m)$ is an isomorphism for any $p$ and any $q \in \{0, 1, 2\}$.

**Proof.** For the case $q = 0$, this follows by observing that $\mathcal{O}_{A_p}^\times \to \mathcal{O}_{A'_p}^\times$ is an isomorphism since $A'_p \subset A_p$ is an open subset of codimension at least two. For $q = 1$, this follows by identifying $H^1_{et}(A_p, \mathbb{G}_m)$ with the group of line bundles on $A_p$ and since $A_p$ is smooth.

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For $q = 2$, notice that $A_p$ and $A'_p$ are both smooth and quasi-projective. For a smooth variety $Y$, recall that by a theorem of Grothendieck $H^2_{\text{et}}(Y, \mathbb{G}_m)$ is always torsion (cf [20]). On the other hand, by a theorem due to Gabber (cf [15]), for a quasi-projective variety, $H^2_{\text{et}}(Y, \mathbb{G}_m)_{\text{tor}}$ coincides with the Brauer group of morita equivalence classes of Azumaya algebras. So $H^2_{\text{et}}(A_p, \mathbb{G}_m)$ identifies with the Brauer group $Br(A_p)$. Thus the restriction map for $q = 2$, identifies with the restriction map $Br(A_p) \rightarrow Br(A'_p)$. Let $Z_0 \hookrightarrow A_p$ be the closed subscheme whose complement is $A'_p$. Since we are over $\mathbb{C}$, so if $Z_0$ is not itself smooth then its singular locus $Z_1$ is a closed subvariety of strictly smaller dimension. Inductively define $Z_{n+1}$ as the singular subvariety of $Z_n$ till we reach a $k$ such that $Z_k$ is smooth. Such a $k$ exists, for instance when $\dim(Z_k)$ becomes zero. From $A_p$, we remove $Z_k$ and then successively $Z_{n-1} \setminus Z_n$ for $n$ from $k$ to 1. The restriction map $Br(X) \rightarrow Br(X \setminus Z)$ is an isomorphism by [19 Corollaire 6.2] (cf [20 Theorem VI.5.1]) if $X$ and $Z$ are regular and the codimension of $Z$ is at least two. Applying this result repeatedly, it follows that $Br(A_p) \rightarrow Br(A'_p)$ is an isomorphism.

\textbf{Lemma 8.0.6.} The groups of $H^2_{\text{et}}(\mathcal{M}(\mathcal{G}), \mathbb{G}_m)$ and $H^2_{\text{et}}(\mathcal{M}^{rs}(\mathcal{G}), \mathbb{G}_m)$ have two-step filtrations whose associated graded are isomorphic.

\textbf{Proof.} The differentials $d^p_{p+1,q} : E_1^{p+1,q} \rightarrow E_2^{p+1,q}$ in (8.0.1) are compatible with restriction morphisms which are isomorphisms. So we have an isomorphism of groups $E_2^{p,q}$ of both spectral sequences for all $(p, q)$ where $q \leq 2$. Again the differentials $d^q_{q} : E_2^{p,q} \rightarrow E_2^{p+2,q-1}$ are compatible with restriction morphisms. By analysing case by case, we shall show that the induced morphism between the $E_2^{p,q}$ terms, where $p + q = 2$, is an isomorphism. To this end, we first consider the $E_2^{0,2}$ term. We have the sequences $E_2^{-2, 3} \rightarrow E_2^{0, 2} \rightarrow E_2^{2, 1} \rightarrow E_2^{4, 0}$ and $E_2^{1, 1} \rightarrow E_2^{3, 0} \rightarrow E_2^{5, -1}$.

Since $E_2^{2, -3}$ and $E_2^{5, -1}$ is zero, so we conclude that the restriction morphism is again an isomorphism for the sequence $E_2^{-3, 4} \rightarrow E_2^{0, 2} \rightarrow E_2^{3, -3}$. So $E_2^{0, 2}$ are isomorphic for both spectral sequences. But now $E_2^{0, 2} = E_2^{0, 2}$. Now we consider $E_2^{1, 1}$.

We have the sequence $E_2^{-1, 2} \rightarrow E_2^{1, 1} \rightarrow E_2^{3, 0}$. Reasoning as before, we conclude that $E_2^{-2, 3} \rightarrow E_2^{1, 1} \rightarrow E_2^{3, -1}$ are isomorphic for both spectral sequences. But now $E_3^{1, 1} = E_3^{-1, 1}$. Now we consider $E_2^{2, 0}$. We have the sequence $E_2^{0, 1} \rightarrow E_2^{2, 0} \rightarrow E_2^{4, -1}$ with isomorphic terms for both spectral sequences. Thus $E_2^{0, 1} = E_2^{0, 1}$. $E_2^{1, 2} \rightarrow E_2^{2, 0} \rightarrow E_2^{4, -1}$ are isomorphic for both spectral sequences. Thus $E_2^{2, 0} = E_2^{0, 2}$.

Consider the restriction map $H^2(\mathcal{M}(\mathcal{G}), \mathbb{G}_m)_{\text{tor}} \rightarrow H^2(\mathcal{M}^{rs}, \mathbb{G}_m)_{\text{tor}}$. By the so-called 5-lemma it follows that it is an isomorphism.

8.1. Brauer group of $M^{rs}_X(\mathcal{G})$. Recall that an Azumaya algebra of degree $n$ on a scheme $Y$ is a sheaf of $\mathcal{O}_Y$-algebras that is étale locally isomorphic to $M_n(\mathcal{O}_X)$. We say that two Azumaya algebras are Brauer-equivalent if there are locally free sheaves $V$ and $W$ on $Y$ of strictly positive rank at every point of $Y$ and an isomorphism $A \otimes \text{End}(V) \simeq B \otimes \text{End}(W)$ of $\mathcal{O}_Y$-algebras. The Brauer group of $Y$ is the group of equivalence classes of Azumaya algebras on $Y$ with group operation induced by tensor product.

8.2. Reduction to cokernel of weight homomorphism. Let $M^{rs}_X(\mathcal{G})$ denote the moduli space of regularly stable $\mathcal{G}_X$-torsors on $X$.  

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Let us denote the class of the gerbe \( \mathcal{M}_X^r(G) \to \mathcal{M}_X^r(G) \) by \( \psi \in H^2_\mathcal{X}(\mathcal{M}_X^r(G), \mathbb{G}_m) \).

For a stack \( \mathcal{X} \), following \([8, BHf1]\) and \([13, Giraud]\) by Pic(\( \mathcal{X} \)) we shall denote the abelian group of isomorphism classes of line bundles on \( \mathcal{X} \).

Given any line bundle \( L \) on \( \mathcal{M}_X^r(G) \), the weight of \( L \) is a character \( \chi : Z_G \to \mathbb{G}_m \) defined as follows (cf \([8, BHf1]\)). An object \( E \in \mathcal{M}(S) \), may be viewed as a 1-morphism \( S \to \mathcal{M} \). Let \( L_E \to S \) denote the pull-back line bundle. A line bundle \( L \) on \( \mathcal{M} \) defines, for any \( k \)-scheme \( S \), a functor from \( \mathcal{M}(S) \) to the groupoid of line bundles on \( S \). So we have a group homomorphism \( \text{Aut}(E) \to \text{Aut}_S(L_E) \). Since \( Z_G(S) \to \text{Aut}(E) \), we get \( Z_G(S) \to \text{Aut}_S(L_E) \) for every \( E \). When \( E \) is a closed point, then \( L_E \) is a one-dimensional vector space. Thus we get \( wt(L_E) : Z_G \to \text{Aut}_S(L_E) = \mathbb{G}_m \). We now suppose that \( M \) is connected. Then \( wt(L_E) \) is independent of \( E \). So we get the weight homomorphism

\[ (8.2.1) \quad wt(L) : Z_G \to \mathbb{G}_m \]

**Proposition 8.2.1.** We have the exact sequence

\[ (8.2.2) \quad 1 \to \text{Pic}(\mathcal{M}_X^r(G)) \to \text{Pic}(\mathcal{M}_X^r(G)) \xrightarrow{\psi} \text{Br}(\mathcal{M}_X^r(G)) \to \text{Br}(\mathcal{M}_X^r(G)) \]

**Proof.** We start by recalling a general result for any gerbe \( p : M \to \mathcal{M} \). For a sheaf \( A \in \mathcal{M}_{\text{et}} \), we have the Leray spectral sequence \( E_2^{p,q} = H^p_{\text{et}}(M, R^q p_*(A)) \Rightarrow H^*(M, A) \). This gives the so called Brauer sequence (cf \([13, V.3.1.4.1]\))

\[ 0 \to H^1(M, p_* A) \xrightarrow{\phi} H^1(M, A) \xrightarrow{\gamma} H^0(M, R^1 p_*(A)) \xrightarrow{\delta} H^2(M, p_*(A)) \xrightarrow{\psi} H^2(M, A) \]

Let us recall the explicit descriptions of the above morphisms. By \([13, Prop III.3.1.3]\), \( \phi \) pulls back a \( p_*(A) \) torsor and then extends structure group by \( p^* p A \to A \). For \( \phi \), we temporarily specialize to the following case: \( A \) is the sheaf defined by \( \mathbb{G}_m \) on \( \text{et}(M) \). Evaluating over \( U \) we have \( p_*(\mathbb{G}_m)(U) = \mathbb{G}_m(p^{-1}(U)) \) which equals \( \mathbb{G}_m(U) \) since \( p : M \to \mathcal{M} \) is a gerbe by \([24, Lemme 3.18]\). So \( p_*(\mathbb{G}_m) = \mathbb{G}_m \). Thus in our case, \( \phi \) is just the pull-back of line bundles from \( M \) to \( \mathcal{M} \). By \([13, V.2.1.1]\), \( \gamma \) may be described as follows: consider the presheaf \( P \in \mathcal{M}_{\text{et}} \) which on any object \( Y \in \text{et}(M) \) takes value \( H^1(p^{-1}(Y), A) \). Consider the sheafification functor which to a presheaf associates its sheaf. Applying this functor to \( P \) we have a morphism \( P \to R^1 p_*(A) \). Evaluating this map at the open \( M \) of the site \( \text{et}(M) \) gives \( \gamma \).

We specialize the above situation to \( \mathcal{M}_X^r(G) \to \mathcal{M}_X^r(G) \) which is a gerbe banded by \( Z_G \). Setting \( A = \mathbb{G}_m \), we now want to analyse \( \gamma \) and \( H^0(M, R^1 p_*(\mathbb{G}_m)) \). Let \( \text{Hom}(Z_G, \mathbb{G}_m) \) denote the constant sheaf in \( M_{\text{et}} \). Consider the map \( wt : \mathcal{P} \to \text{Hom}(Z_G, \mathbb{G}_m) \) by the weight homomorphism. So the weight homomorphism factors naturally as \( \mathcal{P} \to R^1 p_*(\mathbb{G}_m) \xrightarrow{\theta} \text{Hom}(Z_G, \mathbb{G}_m) \). Let \( m : \text{Spec}(\mathbb{C}) \to M \) be an arbitrary closed point. We have \( R^1 p_*(\mathbb{G}_m)_m = \lim_{N \to \infty} H^1(M \times M', \mathbb{G}_m) \) where the colimit is taken over \( Y' \) in the dual category of étale neighbourhoods of \( m \in M \) (\([35, II.6.4]\)). Now since \( m \) is separably closed, so it is a final object of this category. Thus the stalk \( R^1 p_*(\mathbb{G}_m)_m \) is simply \( H^1_{\text{et}}(M, \mathbb{G}_m) \). By a theorem of Giraud the set of \( Z_G \)-gerbes on \( m \) are classified by \( H^2(m, Z_G) \) which is trivial. Thus the gerbe \( M \times M m \to m \) must be \( BZ_G \). Thus \( H^1_{\text{et}}(M \times M m, \mathbb{G}_m) = H^1_{\text{et}}(BZ_G, \mathbb{G}_m) = H^1(Z_G, \mathbb{G}_m) = \text{Hom}(Z_G, \mathbb{G}_m) \) which is also the stalk of \( \text{Hom}(Z_G, \mathbb{G}_m) \) at \( m \). Since \( M \) is a scheme of finite type over \( \mathbb{C} \), so by \([35, II.5.6(i)]\), it follows that \( \theta \) is an isomorphism of sheaves. Thus after identifying \( H^0(M, R^1 p_*(\mathbb{G}_m)) \) with \( \text{Hom}(Z_G, \mathbb{G}_m) \), the morphism \( \gamma \) is simply \( wt \).

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Summarizing the above, we get the following 5-term short sequence

\[(8.2.3) \quad 1 \to \text{Pic}(M) \to \text{Pic}(\mathcal{M}) \overset{\psi}{\to} H^2_{\text{ét}}(M, \mathbb{G}_m) \to H^2_{\text{ét}}(\mathcal{M}, \mathbb{G}_m).\]

For a smooth variety \(Y\), recall that by a theorem of Grothendieck \(H^2_{\text{ét}}(Y, \mathbb{G}_m)\) is always torsion (cf \[23\]). On the other hand, by a theorem due to Gabber (cf \[15\] Thm 1.1 de Jong]), for a quasi-compact separated scheme \(Y\) endowed with an ample invertible sheaf, \(H^2_{\text{ét}}(Y, \mathbb{G}_m)_{\text{tor}}\) coincides with the Brauer group of morita equivalence classes of Azumaya algebras. Taking \(Y = M^*_X(G)\), since it is contained in the smooth locus of \(M^*_X(G)\) by Proposition 8.0.1, and since by \[3\] Balaji-Seshadri \(M^*_X(G)\) exists as a quasi-projective variety, so by combining the theorems of Grothendieck and Gabber it follows that the Brauer group \(Br(Y), H^2_{\text{ét}}(Y, \mathbb{G}_m)\) and \(H^2_{\text{ét}}(Y, \mathbb{G}_m)_{\text{tor}}\) all coincide.

For any smooth scheme or stack \(Y\), the inclusion of sites \(i : \acute{\text{e}}t(Y) \to \acute{\text{e}}t(Y)\) is a morphism of topologies (cf \[23\] Definition I.1.2.2)). Further \(i\) is full-faithful and we observe that any covering in \(\acute{\text{e}}t(Y)\) of \(u \in \acute{\text{e}}t(Y)\) is also a covering in \(\acute{\text{e}}t(Y)\). Thus by \[35\] Theorem I.3.9.2), the functor \(i^* : Y_{\acute{\text{e}}t} \to Y_{\acute{\text{e}}t}\) defined for \(F \in Y_{\acute{\text{e}}t}\) by \(i^*(F)(u) = F(i(u)) = F(u)\) for \(u : U \to X \in \acute{\text{e}}t(Y)\) is exact. Thus by the Leray spectral sequence \[35\] I.3.7.6], \(E^{pq}_2 = H^p_{\acute{\text{e}}t}(Y, R^q i^*(F)) \Rightarrow H^{p+q}_{\acute{\text{e}}t}(Y, F)\) it follows that cohomology over the big and small étale sites agree. So we may switch to the big-étale site in the above cohomology groups. Further the image of \(H^2_{\acute{\text{e}}t}(M^r_s, \mathbb{G}_m)\) lands in \(H^2_{\acute{\text{e}}t}(M^r_s, \mathbb{G}_m)_{\text{tor}}\). Now by Proposition 8.0.3 we get the desired result.

\[\square\]

Thus to compute the Brauer group of \(M\), we need to compute the image of \(\psi : \text{Pic}(\mathcal{M}) \to \text{Hom}(Z_G, \mathbb{G}_m)\) homomorphism. This is carried out in the subsections below by relating it to the case where at the parabolic point \(x \in R\) the parahoric group scheme corresponds to the Iwahori subgroup.

8.3. The case of one parabolic point \(x\) with facet a vertex \(\sigma\) of the alcove.

8.3.1. cokernel of weight in terms of cokernel of evaluation map. For the convenience of the reader, we give a summary of \[8\] Biswas-Hoffmann]. Let us place ourselves in the context \(G/C\) is any reductive group. Recall that the connected components of the stack \(\mathcal{M}_X(G)\) are parametrized by \(\pi_1(G)\). Let \(d \in \pi_1(G)\) and let \(\mathcal{M}^{rs, d}_X(G)\) denote the corresponding component of the regularly stable locus. Now \[8\] Prop 7.2] gives an alternate description of the cokernel of

\[(8.3.1) \quad \psi^{\text{ad}} : \text{Pic}(\mathcal{M}^{rs, d}_X(G)) \to \text{Hom}(Z_G, \mathbb{G}_m).\]

To explain the alternate description, we need to first introduce some more notation. Let \(\Lambda_T := \text{Hom}(\mathbb{G}_m, T)\) denote the cocharacter group of \(T\). Let \(\Lambda_{\text{coroots}} \subset \Lambda_T\) denote the subgroup generated by the coroots of \(G\). The Weyl group \(W\) acts trivially on \(\Lambda_T/\Lambda_{\text{coroots}}\). This quotient up to canonical isomorphism is \(\pi_1(G)\). A bilinear form \(b\) on \(\Lambda_{\text{coroots}}\) is said to be even if \(b(\lambda \otimes \lambda) \in 2\mathbb{Z}\) for all \(\lambda \in \Lambda_{\text{coroots}}\). Let \(G^{\text{ad}}\) be the adjoint group of \(G\). Following \[8\] Definition 6.3] let

\[(8.3.2) \quad \Psi(G) \subset \text{Hom}(\pi_1(G^{\text{ad}}) \otimes \pi_1(G^{\text{ad}}), \mathbb{Q}/\mathbb{Z}),\]

denote the abelian group of all bilinear maps that come from even \(W\)-invariant symmetric bilinear forms on \(\Lambda_{\text{coroots}} \times \Lambda_{\text{coroots}} \to \mathbb{Z}\). It is determined by the root system of \(G\). If \(G = G_1 \times G_2\), then \(\Psi(G) = \Psi(G_1) \oplus \Psi(G_2)\). We refer the reader to \[8\] Table 1] for an explicit description of \(\Psi(G)\) and its generator for all simple

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groups. Let $G'$ denote the derived subgroup of $G$ and $Z_G^0$ the connected component of identity in $Z_G$. Following [S] Definition 6.4], let $\Psi'(G) \subset \Psi(G)$ be the subgroup of all elements such that

\[(8.3.3) \quad b(\pi_1(G/Z_G^0) \times \pi_1(G')) = 0.\]

Given an element $d \in \pi_1(G)$, with image $\overline{d} \in \pi_1(G/Z_G^0)$, the evaluation map

\[(8.3.4) \quad ev_G^d : \Psi'(G) \to \text{Hom}(\pi_1(G^{\text{red}}), \mathbb{Q}/\mathbb{Z}),\]

sends a bilinear form $b$ to $b(\overline{d}, ?)$. One can check directly that $\text{Hom}(\pi_1(G^{\text{red}}), \mathbb{Q}/\mathbb{Z})$ equals just $\text{Hom}(Z_G, \mathbb{G}_m)$ (for instance cf [S] Remark 6.5). We recall

**Proposition 8.3.1.** [S] Prop 7.2] $\text{coker}(wt^d) \simeq \text{coker}(ev_G^d)$.

With this preparation, let us prove our main result for the case of one point. Recall that the diagram (7.5.4) is cartesian. Let $L_\sigma \in \text{Pic}(\mathcal{M}_X(\mathcal{G}^\sigma))$ correspond to the positive generator of $\text{Pic}(\mathcal{F}_\sigma)$.

**Theorem 8.3.2.** Let $\sigma = \sigma^\alpha$ for $\alpha \in S$. Let $\omega_0$ be the fundamental weight of $G/\mathbb{C}$ for $\alpha \neq e_0$ and set $\omega_{a_0}$ as the trivial weight. Then $L_\sigma$ under $wt : \text{Pic}(\mathcal{M}_X(\mathcal{G}^\sigma)) \to \text{Hom}(Z_G, \mathbb{G}_m)$ goes to $Z_G \hookrightarrow T \simeq \mathbb{G}_m$.

**Proof.** We first consider the case $\sigma = e_0$. In the following lines we show how to deduce this case from [S]. Thus $\mathcal{G}^\sigma$ is the constant group scheme $X \times G$. Now since our group is semi-simple and simply-connected so we have $\Psi'(G) = \Psi(G)$. Since $\pi_1(G)$ is trivial in our case, so we just have to consider $ev_G^e$ where $e \in G$ is the identity element. By the explicit description of $\Psi(G)$ in [S] Table 1, we see that $b(\overline{e}, ?)$ is the trivial homomorphism in $\text{Hom}(Z_G, \mathbb{G}_m)$. So $\text{coker}(ev_G^e) = \text{Hom}(Z_G, \mathbb{G}_m)$. By Proposition 8.3.1 $wt$ is the trivial morphism.

We now consider the case $\alpha \in S \setminus \{0\}$. Consider the morphism of stacks $\pi : \mathcal{M}_X(\mathcal{G}^\alpha) \to \mathcal{M}_X(\mathcal{G}^\sigma)$ defined by extension of structure group $\mathcal{G}^\alpha \to \mathcal{G}^\sigma$.

**Lemma 8.3.3.** The weight map $wt$ as defined in (8.2) factors as follows:

\[(8.3.5) \quad \text{Pic}(\mathcal{M}_X(\mathcal{G}^\sigma)) \xrightarrow{\text{wt}} \text{Pic}(\mathcal{M}_X(\mathcal{G}^\sigma)) \xrightarrow{\text{wt}} \text{Hom}(Z_G, \mathbb{G}_m).\]

**Proof.** To check the claim, setting $S = \text{Spec}(\mathbb{C})$, we take

1. an arbitrary closed point $S : S \to \mathcal{M}_X(\mathcal{G}^\alpha)$,
2. a line bundle $L$ on $\mathcal{M}_X(\mathcal{G}^\sigma)$.

Let $\mathcal{E} \to \mathcal{E}'$ under $\pi : \mathcal{M}_X(\mathcal{G}^\alpha) \to \mathcal{M}_X(\mathcal{G}^\sigma)$. So we have a group homomorphism $\text{Aut}(\mathcal{E}) \to \text{Aut}(\mathcal{E}')$. Let $L_\sigma \in \text{Pic}(S)$ and $L' \to \mathcal{M}_X(\mathcal{G}^\sigma)$ denote the pull-back line bundles $(\mathcal{E}')^*L$ and $\pi^*L$ respectively. Viewing a line bundle on a stack as a sheaf, we have a natural morphism $\alpha : \mathcal{E}' \to (\mathcal{E})^*L = L_\sigma$. It is an isomorphism of line bundles. Let us check that we have a factorization which makes the diagrams

\[(8.3.6) \quad \text{Aut}(\mathcal{E}) \xrightarrow{\text{wt}(L)} \text{Aut}(\mathcal{E}) \xrightarrow{\text{wt}(L')} \text{Aut}(\mathcal{E}').\]

commute. The triangle commutes because $\pi$ is a morphism of stacks. Considering $S \xrightarrow{\mathcal{E}} \mathcal{M}_X(\mathcal{G}^\alpha) \xrightarrow{\mathcal{E}'} \mathcal{M}_X(\mathcal{G}^\sigma)$, by functoriality properties of sheaves on a stack applied
to $L \to M_X (G^n)$ the square also commutes. We have $\text{Aut}_S((E^c)^+ L) = \text{Aut}_S(L_S) = \mathbb{G}_m(S)$. Since $M_X (G^n)$ is connected, so this show that $wt$ factors in (8.3.5). □

So the pull-back $L_{\omega_0} (a) \to M_X (G^n)$ of $L_{\omega_0} \to M_X (G)$ has trivial weight. From (7.5.7) we observe that under $\pi : M_X (G^n) \to M_X (G^\sigma)$ we have

\begin{equation}
(8.3.7) \quad \pi^* L_G \simeq L_{\omega_0} \otimes L_G^{\sigma^n_0} (a).
\end{equation}

We want to compute the weight of $L_G$. So we are reduced to computing the weight of $L_{\omega_0} \to M_X (G^n)$. To carry out the computation, we introduce some notation.

Let $\omega = \omega_0$. Let $S = \text{Spec}(\mathbb{C})$. For any $E \in M_X (G^n)(S)$ viewed as a 1-morphism $S \to M_X (G^n)$, let $L_{\omega,S}$ denote the pull-back of $L_\omega$, let $E_x$ denote the restriction of $E \to X \to S$ to the closed subscheme $x \to S$ and let $E_x(T)$ denote the $T$-bundle obtained by extension of structure group $G^n_\omega \cong G_n \otimes k \to B \to T$. We have morphisms of stacks $M_X (G^n) \to \mathbb{B}(G^n_\omega) \to \mathbb{B}(?)$ where $\mathbb{B}(?)$ denotes the classifying stack of a group scheme. By construction $L_\omega$ comes from a line bundle in $\text{Pic}(\mathbb{B}(T)) = \mathbb{X}^*(T)$. Since $M_X (G^n)$ is connected, so to compute $wt(L_\omega)$ by definition (cf (8.2.1)) we want to compute the composite

$$Z_G \to \text{Aut}(E) \to \text{Aut}(E_x) \to \text{Aut}(E_x(T)) \to \text{Aut}(L_{\omega,S}) = \mathbb{G}_m(S).$$

To this end, we book-keep some morphisms to show that they are canonical.

**Lemma 8.3.4.** We have a natural isomorphism $G_n/G_n^n \to T$.

**Proof.** By Proposition [2.0.1] it follows that $G_n/G_n^n$ has empty root-system since $Y_n$ is empty. For any $\theta \in a^\vee$, we have $-[(\theta, r)] = 1 - [\theta, r]$ where $r$ is a root of $G$. By [2.0.3] and [2.0.4] it follows that $G_n/G_n^n$ is isomorphic to $T$. Further, this isomorphism can be made canonical since $G_n \otimes k$ surjects naturally onto $B$. □

**Lemma 8.3.5.** Let $\phi : Z_G \to \text{Aut}(E)$ be given by the natural inclusion $Z_G \times X \hookrightarrow G^n$ followed by the right $G^n$-action on $E$. Then $Z_G \to \text{Aut}(E)$ is given by $\phi$.

**Proof.** It is well known that for any principal $G$-bundle $F \to M$, the automorphism group scheme $\text{Aut}_M(F)$ is represented by $F \times_{\text{R-act},M,\text{conj}} G$, which is the quotient of $F \times G$ by the right action of $G$ on $F$ and action through conjugation on itself. If $f$ denotes a local section of $F$ and $g \in G$, then the class of $(f, g)$ in $\text{Aut}_M(F)$ represents the unique (local) $G$-bundle automorphism that sends $f \mapsto fg$ and therefore $fh \mapsto fgh = (fh)(h^{-1}gh)$ for $h \in G$. Considering $F \times_{\text{R-act},M,\text{conj}} Z_G$, through $Z_G \to G$ we get $Z_G \times M \hookrightarrow \text{Aut}_M(F)$. Let $z \in Z_G$. If $f \mapsto fz$, then $fh \mapsto fzh$ which equals $fhz$. This means that $Z_G$ action on $F$ through $G$-bundle automorphism coincides with the restriction of the action map $\alpha : F \times G \to F$ from $G$ to $Z_G$. This describes $Z_G \times M \hookrightarrow \text{Aut}_M(F)$. From now on, let $\text{Aut}(F)$ denote $\Gamma(M, \text{Aut}_M(F))$. So we have a description of $Z_G \to \text{Aut}(F)$, which is an inclusion.

More generally, let us describe $Z_G \to \text{Aut}(E)$ by $\Gamma$-$G$ bundle theory (cf (??)) as follows. Assume $E$ corresponds to a $\Gamma$-$G$ bundle $E \to Y$ on a cover $Y \to X$. The composite $Z_G \xrightarrow{\Delta} \text{Aut}(E) = \text{Aut}_{-G}(E/Y) \to \text{Aut}_Y(E)$ identifies with the natural inclusion $Z_G \hookrightarrow \text{Aut}_Y(E)$ for the $G$-bundle case described above. Therefore we have $Z_G \to \text{Aut}(E)$ is an inclusion and it is given by $\phi$. □

**Lemma 8.3.6.** The composite $Z_G \to \text{Aut}(E) \to \text{Aut}(E_x) \to \text{Aut}(E_x(T)) = T$ identifies with the natural inclusion $Z_G \hookrightarrow T$. 36
Proof. Therefore $Z_G$-action on $E_x$ factors via the natural inclusion $Z_G \to G^a$ and $G^a$-action on $E_x$. Thus the composite $Z_G \to \text{Aut}(E) \to \text{Aut}(E_x)$ remains injective. Since $Z_G$-action on $E_x$ is via $Z_G \to G^a = G^a \otimes k$, and by Lemma 8.3.4, the reductive quotient $G^a/G^a$ is isomorphic to $T$ canonically, so

(1) the homomorphism $Z_G \to \text{Aut}(E_x(T))$ is also an inclusion.

(2) Further, the $Z_G$-action on $E_x(T)$ is via the natural inclusion $Z_G \to T$, coming from $Z_G \to G^a \otimes k \to T$, followed by the right $T$-action on $E_x(T)$. Now, $\text{Aut}(E_x(T)) = T$ because $T$ is abelian. Thus $Z_G \to \text{Aut}(E) \to \text{Aut}(E_x) \to \text{Aut}(E_x(T)) = T$ identifies with the natural inclusion $Z_G \to T$. □

We see that $T = \text{Aut}(E_x(T)) = E_x \times G^a \otimes k T \xrightarrow{(\text{Id}, \omega)} E_x \times G^a \otimes k G_m = \text{Aut}(L_\omega,S) = G_m$ is given by $\omega : T \to G_m$ because the $G^a \otimes k$ action is trivial on the second factor. So $Z_G \to \text{Aut}(E_x(T)) \to \text{Aut}(L_\omega,S)$ identifies with $Z_G \to T \xrightarrow{\omega \circ} G_m$. Thus the weight of $L_\omega \to M_X(G^a)$ is given by $Z_G \to T \xrightarrow{\omega} G_m$. To summarize,

$$wt(L_\sigma) = wt(L_{\omega_{\sigma}}) = \omega_\sigma \mid Z_G : Z_G \to T \xrightarrow{\omega \circ} G_m.$$  

8.4. The General case. Let $\theta$ denote the highest root of $G/C$. For simple roots $\alpha$, let $a_\alpha^\vee$ be integers defined by the relation $\theta^\vee = \sum a_\alpha^\vee \alpha^\vee$. Set $a_0^\vee = 1$. For any facet $\sigma_x$ we define $l_{\sigma_x} = \text{GCD}\{a_\alpha^\vee | \alpha \in \sigma_x^\vee\}$. We define $f = \text{LCM}\{l_{\sigma_x} | x \in \mathcal{R}\}$.

Proof of Main Theorem 1.0.3 By Proposition 8.2.1 $\ker(\text{Br}(M^+_X(G^a)) \to \text{Br}(M_X(G)))$ is the cokernel of $w : \text{Pic}(M^a) \to \text{Hom}(Z,G_m)$. We place ourselves in the setup of the proof of Theorem 1.0.1 Let $Q := \prod_{x \in \mathcal{R}} F\ell_{\sigma_x}$. So the following square

\[
\begin{array}{ccc}
\text{Pic}(M^a) & \xrightarrow{q^*} & \text{Pic}(F\ell^a) \\
\pi^* & & p^* \\
\text{Pic}(M) & \xrightarrow{q_0^*} & \text{Pic}(Q)
\end{array}
\]

is a pull-back square. By Theorem 3.0.1 the image of $q^*$ consists of line bundles with equal central charge in each factor. For $x \in \mathcal{R}$, $p^*$ maps to line bundles whose central charge is a multiple of $l_{\sigma_x}$. Therefore the image of $q_0^*$ consists of line bundles whose central charge is a multiple of $f$ in each factor.

Let $m \in \mathbb{Z}$. For $x \in \mathcal{R}$, let $I^*_m$ denote the set of $|\mathbb{Z}^*|$-uple integral solutions $e^x_m := (\ldots, n^{m,x}_\alpha, \ldots) \in \mathbb{Z}^{\mathbb{Z}^*}$ to

$$\sum_{\alpha \in \mathbb{Z}^*} n^{m,x}_\alpha a_\alpha^\vee = mf.$$  

Then for each solution $e^x_m \in I^*_m$, we have a line bundle $L(e^x_m)$ on $F\ell_{\sigma_x}$ of central charge $mf$ given by the box-tensor product over $\alpha \in \mathbb{Z}^*$ of $L_{\omega_{\sigma_x}}$ raised to the power $n^{m,x}_\alpha$. For $e = (\ldots, e^m_x, \ldots) \in \prod_{x \in \mathcal{R}} I^*_m$, the line bundle

$$L := \boxtimes_{x \in \mathcal{R}} L(e^m_x) \to Q$$  

descends to $M_X(G)$. Further we argued above that all line bundles on $M_X(G)$ are exactly of this form. For each $e^m_m := (\ldots, n^{m,x}_\alpha, \ldots) \in I^*_m$, consider the weight

$$\omega(e^m_x) = \sum_{\alpha \in \mathbb{Z}^*} n^{m,x}_\alpha \omega_{\sigma_x}$$  

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of $G/\mathbb{C}$. The weight of $L$ can be computed by the equation (8.3.8) and it works out to

\begin{equation}
Z_G \hookrightarrow \prod_{x \in \mathcal{R}} T \prod_{\omega(e^m)} \prod_{G_m} \mathbb{G}_{m} \to \mathbb{G}_{m}.
\end{equation}

Now varying over all line bundles on $\mathcal{M}_X(G)$ is equivalent to varying over $e \in \cup_{m \in \mathbb{Z}} I^e_m$. To prove the theorem, it suffices to show that we can restrict ourselves to the case $m = 1$. To this end, we shall show that

\begin{equation}
I^e_m = I^e_1 + \cdots + I^e_1
\end{equation}

where by a sum of sets we mean all possible sums in $\mathbb{Z}^e$ of $m$-many elements from $I^e_1$. By definition of $I^e_1$, it follows that it is non-empty. Consider the functional

\begin{equation}
(\cdots, a^\vee, \cdots) : \mathbb{Z}^e \to \mathbb{Z}
\end{equation}

and let $K$ be its kernel. Let $z_1$ be arbitrary such that $z_1 \mapsto mf$. Pick $z \in I^e_1$. So $z \mapsto f$. Thus $z_1 - mz = k \in K$. Now $z + k \in I^e_1$. Thus $z_1 = (m - 1)z + (z + k)$. □

9. Some Computations

In this section, we will use results from [38, §1.2]. The notation will also be from (loc. cit) and [8, §3]. Let $\Lambda$ (resp. $\Lambda'$) denote the weight lattice (resp. root lattice) of $G/\mathbb{C}$. We have $\text{Hom}(Z_G, \mathbb{G}_m) = \Lambda / \Lambda'$. It is called the cocenter. We will denote as $C^*(\Delta)$. For $\alpha \in S$ let $\omega_{\alpha}$ denote the fundamental weight, where for $\alpha_0$, we take the trivial weight. Let $\omega_{\alpha_i}$ be the canonical image of $\omega_{\alpha}$ in $\text{Hom}(Z_G, \mathbb{G}_m)$ and set $\omega_i = \omega_{\alpha_i}$, for $i \geq 0$. If $\sigma = \sigma^\alpha$, then set $G^i = G^\sigma$. In the third column, we put the fundamental weight corresponding to $\alpha_i$. By $(i, j)$ we mean the GCD of $i$ and $j$. If $\alpha \in S$ is the $i$-th root, then let $G^i := G^\alpha$.

| Type | $C^*(\Delta)$ | weight $\omega_i$ | $Br(M^\chi(G^i))$ |
|------|---------------|------------------|------------------|
| $A_1$ | $\mathbb{Z}_l$ | $\omega_i = l\omega_1$ | $\mathbb{Z}_{(l, l+1)}$ |
| $B_l$ | $\mathbb{Z}_l$ | $\omega_i = l\omega_1, i \neq l$ | $\mathbb{Z}_2$ for $i \neq l$, $\mathbb{Z}_2$ for $i = l$ |
| $C_l$ | $\mathbb{Z}_l$ | $\omega_i = l\omega_1$ | $\mathbb{Z}_2$ for $i$ odd |
| $D_l$ | $\mathbb{Z}_4$ \ l odd | $\omega_i = l\omega_1$ | $\mathbb{Z}_2$ for $i$ odd |
| \ | \ | \ | \ |
| $Z_2 \times \mathbb{Z}_2$ \ l even | $\omega_i = l\omega_1$ | $\mathbb{Z}_2$ for $i$ odd |
| \ | \ | \ | \ |
| $G_2$ | $\mathbb{Z}_2$ | $\omega_i = l\omega_1$ | $\mathbb{Z}_2$ for $i$ odd |
| $F_4$ | $\mathbb{Z}_1$ | $\omega_i = l\omega_1$ | $\mathbb{Z}_2$ for $i$ odd |
| $E_6$ | $\mathbb{Z}_3$ | $\omega_i = l\omega_1$ | $\mathbb{Z}_{(3, 1)}$ |
| $E_7$ | $\mathbb{Z}_2$ | $\omega_i = l\omega_1, i = 4, 6, 7$ | $\mathbb{Z}_2$ |
| $E_8$ | $\mathbb{Z}_2$ | $\omega_i = l\omega_1, i \neq 4, 6, 7$ | $\mathbb{Z}_2$ |

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10. Cross-checks

10.1. With \cite{Biswa-Holla}. The case of \cite{Biswa-Holla} is that of principal $G$-bundles, where $G$ is semi-simple. The set $\mathcal{R}$ is reduced to a single element, say $\{x\} \in X$. The parahoric group scheme is $G_A \to \text{Spec}(A)$ where $A = \mathbb{C}[[t]]$. So the facet $\sigma$ is the origin of the apartment $\mathcal{A}_T$. Only the affine simple root $\alpha_0 := \delta - \theta \in S$ (cf (2.1.9)) does not vanish at $\sigma$. So $\sigma = \sigma^{\alpha_0}$ and also $f = l_{\sigma} = a_{\alpha_0} = 1$. Thus $\omega_{\alpha_0}$ is the trivial weight. Thus by Theorem 8.3.2, it follows that $Br(M) = \text{Hom}(Z_G, \mathbb{G}_m) = Z_G^\vee$. This agrees with \cite{Biswa-Holla}.

10.2. With \cite{BBGN}. The case of \cite{BBGN} is that of the moduli space of vector bundles of rank $n$ and degree $d$ of fixed determinant $\xi$. Here below, we proceed to reinterpret this moduli space in the context of this paper which is that of torsors under Bruhat-Tits group scheme. We refer the reader to \cite{ArX}

§3.4 for a summary of $\Gamma$-G bundle theory of \cite{ArX}.

In the case $d = 0$, by taking a $n$-th root of $\xi$, we may reduce to the previous case of \cite{BBGN}. So without loss of generality, we may suppose that $-n < d < 0$. Let us fix a point $x \in X$. By putting the full flag $F_x = V_x$ at $x$ together with parabolic weight $d/n$, the space $M(n, -d, \xi)$ is equivalent to the moduli space $\text{Par}M(n, -d)$ of parabolic vector bundles of fixed parabolic determinant. Note that the parabolic degree has become zero. By the well-known correspondence between $\Gamma$-$SL_n$ bundles and parabolic vector bundles of fixed parabolic determinant \((\cite{ArX} \S3.4)), Par$M(n, -d)$ is isomorphic to $M^\vee_Y(\Gamma, SL_n)$ which parametrizes $S$-equivalences classes of $\Gamma$-$SL_n$ bundles on a cover $p : Y \to X$ where for $y \in Y$, and denoting $\zeta_n$ the $n$-th root of unity, the isotropy representation $\rho_y : \Gamma_y \to \text{Aut}(E_y)$ is given by $\zeta_n \mapsto c^d nId_{|n \times n}$. Since $c_n^d = \zeta_n^{-d}$, we may rewrite $\rho_y$ equivalently as

\[
(\text{10.2.1}) \quad \rho_y(\zeta_n) = \begin{pmatrix} c^d_n \text{Id}_{(n-d) \times (n-d)} & 0 \\ 0 & \zeta_n^{-d} \text{Id}_{d \times d} \end{pmatrix}.
\]

As in \cite{BBGN} Remark 2.2.9], let $\omega$ (resp. $z$) be a local parameter around $y$ (resp. $x$). So $z = \omega^n$. Then if $L$ is the quotient field of the completion of the local ring at $y$, we set a one parameter subgroup $\Delta : G_{m,L} \to SL_n$ by

\[
(\text{10.2.2}) \quad \Delta(\omega) = \begin{pmatrix} \omega^d \text{Id}_{(n-d) \times (n-d)} & 0 \\ 0 & \omega^{-d} \text{Id}_{d \times d} \end{pmatrix}.
\]

Thus to $\Delta$ we can correspond a rational 1-PM subgroup $\theta = \frac{1}{n} \Delta$ in the apartment $\mathcal{A}_T = Y(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Notice that only the affine simple root $\alpha_d$ of $SL_n$, which is $\epsilon_d - \epsilon_{d+1}$, does not vanish on $\theta$. Thus $\theta$ belongs to the facet $\sigma^\vee$ where $Y = \{\alpha_d\}$.

Set $B = \tilde{O}_{Y,y}$ and $N_y = \text{Spec}(B)$ and $U_y = \text{Aut}_{\Gamma, SL_n}(E|_{N_y}) \to N_y$ the unit group scheme of local automorphisms of a $\Gamma$-$SL_n$-bundle, say $E$. By \cite{ArX} Theorem 2.3.1, Proposition 5.1.2 we have an isomorphism of group schemes

\[
(\text{10.2.3}) \quad U_y = G_{\sigma^\vee}.
\]

Thus in (??), we see that $p^\Gamma_G \to G$ where $G \to X$ is obtained by gluing $G_{\sigma^\vee} \to \text{Spec}(\tilde{O}_{X,x})$ with $SL_n \times X \backslash \{x\}$. Now by \cite{ArX} 3.4, $M^\vee_Y(\Gamma, SL_n)$ may be interpreted as $M(G)$, where we take the weight $\theta$ to check semi-stability conditions of parahoric torsors. We are in the setting of Theorem 8.3.2 since $\sigma = \sigma^\vee$, so we consider $Z_G \to T \to G_{\sigma^\vee}$. Thus if $d = 0$, then $Br(M(G)) = \mathbb{Z}/n\mathbb{Z}$. If $d \neq 0$, then $\omega_d = \epsilon_1 + \cdots + \epsilon_d$ where $\epsilon_i : T \to G_{\sigma_d}$ on $T$, which is split, is $T \to \mathbb{G}_m$ followed by projection onto the $i$-th factor. Further $f = l_{\sigma} = a_{\sigma_d} = 1$. By Theorem 8.3.2.
\[ Br(M^{rs}) = \text{Hom}(Z_G, \mathbb{G}_m)/ < Z_G \rightarrow T \overset{\omega_d}{\rightarrow} \mathbb{G}_m >. \] Now \( Z_G \) is generated by \( \zeta_n Id_{n \times n} \). Thus \( \omega_d: Z_G \rightarrow \mathbb{G}_m \) sends \( \zeta_n \rightarrow \zeta_n^d \). So the order of \( \omega_d \) restricted to \( Z_G \) is \( n/(n,d) \). Now the result \( Br(M(n,d)) = \mathbb{Z}/(n,d)\mathbb{Z} \) in [4] follows.

**Remark 10.2.1.** We want to compare the proof strategy of [4] BBGN with our paper. After Proposition 8.2.1 the Brauer group computation reduces to the computing the weight homomorphism (cf [8,2.1]). As we remarked in [8,2.1] \( Le: Z_G \rightarrow \mathbb{G}_m \) is independent of \( \mathcal{E} \) in a connected moduli stack. In [4], this is computed through [27, Drézet-Narasimhan, Prop 5.1] by taking \( \mathcal{E} \) to be a semi-stable vector bundle of non-constant group scheme \( X \), and our paper are different.

So to compute the weight homomorphism, the proof strategies of [4] fixed determinant say \( \xi \). For simplicity let us consider the case of only one point \( x \). Let \( \mathcal{X}, G, \mathcal{G} \) denote the group schemes on \( X \) obtained by patching the constant group scheme \( G \times X \setminus \{p\} \) with a local Iwahori group scheme, or the constant group scheme or an arbitrary Bruhat-Tits group scheme on \( \text{Spec}(\hat{O}_{X,p}) \) respectively. We have a

### 10.3. With [4] Biswas-Dey.

In [6], one treats the case parabolic vector bundles of fixed determinant say \( \xi \). Let us first consider the case when the parabolic degree is zero, since by taking \( n \)-root of \( \xi \), this case corresponds directly to the case \( G = SL_n \) in our paper. For simplicity, we may suppose that there is only one parabolic point \( x \). At \( x \), suppose that the flags are \( 0 \neq F_n \subset \cdots \subset F_1 = E_x \), where \( \text{dim}(F_i) = d_i \). Then the affine roots would be \( \{a_d,|1 \leq i \leq a_x\} \). Since for \( SL_n \), \( a_x = 1 \) for all \( \alpha \in \mathfrak{s} \), so \( f = f_\alpha = \text{GCD}((a_d,|1 \leq i \leq q)) = 1. \) Thus the weights would be \( \{\omega_d,|d_i \neq 0\} \). For \( SL_n \) the fundamental weight \( \omega_d \) equals \( \epsilon_1 + \cdots + \epsilon_d \). Viewing \( Z_G \) as generated by \( \zeta_n Id_{n \times n} \), we have \( \omega_d(\zeta_n) = \zeta_n^d \). So by Theorem 8.3.2 \( Br = \mathbb{Z}_{(n,d,\alpha,1 \leq i \leq a_x)} \). This equals \( \mathbb{Z}_{(n,d,\alpha,1 \leq i \leq a_x,x \in \mathbb{R})} \).

### 11. A possible proof strategy

Let us review how the proof strategy of [6] Biswas-Dey] may used to prove our result. For simplicity let us consider the case of only one point \( p \). Let \( \mathcal{I}, G, \mathcal{G} \) denote the group schemes on \( X \) obtained by patching the constant group scheme \( G \times X \setminus \{p\} \) with a local Iwahori group scheme, or the constant group scheme or an arbitrary Bruhat-Tits group scheme on \( \text{Spec}(\hat{O}_{X,p}) \) respectively. We have a
The idea is essentially to first deduce results on moduli space of \( I \)-torsors from \( G \)-bundles and then deduce results for general \( G \)-torsors.

For vector bundles with fixed parabolic structure, Thaddeus (cf [37]) proved that moduli spaces corresponding to different choice of weights have isomorphic blowups along closed subschemes of codimension at least two. It follows then that their Brauer groups are isomorphic. So once the quasi-parabolic structure is fixed, it becomes sufficient to prove the results for small weights. Now let us suppose below that this result is generalized also to the case of moduli of parahoric torsors.

For small weights, we have a forgetful map
\[
\pi_0 : \mathcal{P}M_s \to \mathcal{N}
\]
from the moduli \( \mathcal{P}M_s \) of stable parabolic bundles to the moduli of semi-stable vector bundles by associating the underlying vector bundle. Let \( \mathcal{N} \subset \mathcal{N} \) denote the stable locus. It is simply connected (cf [4]) and it is known that the moduli of stable \( G \)-bundles is also simply-connected (cf [10]). By considering the Leray spectral sequence for \( \pi_0 \) of the sheaf defined by \( \mathbb{G}_m \), we deduce the exact sequence
\[
(11.0.2) \quad \text{Pic}(\text{Flag}) \to \text{Br}(\mathcal{N}) \to \text{Br}(\mathcal{P}M_s) \to 0,
\]
from which \( \text{Br}(\mathcal{P}M_s) \) is deduced. Analogous to \( \pi_0 \) a similar forgetful map between the moduli spaces \( \pi : M \to M_{ss}(G) \) may also be established. The role of its generator will be played by the generator of \( \text{Br}(M_{ss}(G)) \) which is the class of the \( \mathbb{Z}_G \)-gerbe \( M_{ss}(G) \to M_{ss}(G) \) in \( H^2(M_{ss}(G), \mathbb{G}_m) \) (cf [9]).

If the complement of \( M \) in \( M_{ss}(G) \) has codimension at least two then one may be able to deduce \( \text{Br}(M_{ss}(G)) \).

By similarly adjusting weights and estimating codimensions, one may establish a map from \( \pi' : M \to M_{ss}(G) \) where \( M \) is an open subset \( M \subset M_{ss}(G) \). By considering the descent spectral sequence of \( \pi' \), it may be possible to deduce \( \text{Br}(M_{ss}(G)) \) from \( \text{Br}(M) \).

12. Appendix: Cohomological descent

12.1. Ind-schemes, morphisms and fiber products. Let \( (I, \leq) \) be a partially ordered set. Thus the relation \( \leq \) is reflexive and transitive. We will view it as a category. For applications in this paper, \( (I, \leq) \) will be the Iwahori Weyl group \( \tilde{W} \) with Bruhat order.

Recall an ind-scheme \( X \) indexed by \( I \) is a collection of schemes \( \{X_i\} \) together with a collection of closed immersions \( i_{j,k} : X_j \to X_k \) for \( j \leq k \) where \( j, k \in I \). We shall assume that each \( X_n \) is a finite dimensional scheme. Let \( X \) and \( Y \) be two ind-schemes indexed by categories \( I \) and \( J \). By a morphism \( f : X \to Y \) between ind-schemes we mean the following (cf [22]): for every \( n \in I \), there exists \( m(n) \in J \)
and \( f_n : X_n \to Y_{m(n)} \) a morphism of schemes such that if \( i_1 \leq i_2 \), then \( f_{i_2} \) restricts to \( f_{i_1} \). Thus for each non-empty \( I \), the category of schemes embeds diagonally into the category of ind-schemes indexed by \( I \). For \( n \in I \), consider the partially ordered set \( I_{\geq n} \) of elements \( j \geq n \). Then \( X_n \) extends to an ind-scheme on \( I_{\geq n} \). Let us denote the ind-scheme again by \( X_n \). So we may define a morphism from ind-scheme indexed by \( I_{\geq n} \) to ind-scheme indexed by \( I \)

\[
(12.1.1) \quad X_n \to X
\]

using the \( i_{j,k} \) for \( j, k \geq n \). Let \( a : X \to \mathcal{X} \) and \( b : \mathcal{Y} \to \mathcal{X} \) be two ind-schemes indexed by \( I \) and \( J \) respectively. Then the product category \( I \times J \) has a partial order defined by: \( (i_1, j_1) \leq (i_2, j_2) \) if \( i_1 \leq i_2 \) and \( j_1 \leq j_2 \). By the fibered product

\[
(12.1.2) \quad \mathcal{X} \times_\mathcal{X} \mathcal{Y}
\]

we mean the ind-scheme \( \{ X_i \times_\mathcal{X} Y_j \}_{(i,j) \in I \times J} \) together with the natural closed immersions. We do have projection morphisms \( \mathcal{X} \times_\mathcal{X} \mathcal{Y} \to \mathcal{X} \) and to \( \mathcal{Y} \) using the projection functors from \( I \times J \) to \( I \) and \( J \). We also have the diagonal morphism

\[
(12.1.3) \quad \mathcal{X} \to \mathcal{X} \times_\mathcal{X} \mathcal{X},
\]

by sending \( X_i \to X_i \times_\mathcal{X} X_i \) for \( i \in I \). So given \( \mathcal{X} \to \mathcal{X} \) we can make a simplicial ind-scheme augmented by a stack. If \( J \) is filtered i.e given \( j, k \in J \) there exists \( i \in J \) such that \( j \leq i \) and \( k \leq i \), then given \( f : \mathcal{X} \to \mathcal{Y} \), by the nerve construction, we can make a simplicial ind-scheme augmented by an ind-scheme.

12.2. Sites of ind-schemes.

Remark 12.2.1. Let us explain why we have chosen to work with the Big-étale sites of ind-schemes \( \mathcal{X} \) and stacks \( \mathcal{X} \) in this section. Further, we use the fact that Big-étale site is functorial [28]. In particular, in our context, it is well-adapted to define a morphism of topoi. More precisely, for \( a : \text{Ét}(\mathcal{X}) \to \text{Ét}(\mathcal{X}) \) the condition ‘\( a^* \) commutes with finite limits’ is easier to check for the big-étale site than it is for other sites. Similarly to determine some results for \( L_{X*}(G) \) it is easier to pass to the analytic site from big-étale than other sites.

Remark 12.2.2. Note that for a morphism \( a : \mathcal{X} \to \mathcal{X} \), and a covering \( u : U \to \mathcal{X} \), the fiber product \( \mathcal{X} \times_\mathcal{X} U \) is only an ind-scheme. Further, the notion of a morphism between ind-schemes following [22] has some ambiguity with respect to the indexing set. So to define a morphism of small étale sites \( \text{ét}(\mathcal{X}) \to \text{ét}(\mathcal{X}) \) (cf [17] Definition 2.1) seems to require some foundational work such as defining étale and smooth morphisms between ind-schemes. This is avoided in this paper by using the notion of comorphism of sites following [17] (1.1) and Definitions 1.1, 1.2, 2.3] recalled below.

Let \( C \) be a category. A seive \( R \) of \( C \) is a subcategory characterised by its set of objects by the relation: if \( f : X \to Y \in \text{Mor}(C) \) then \( Y \in R \implies X \in R \). The set of all seives is denoted \( \emptyset(C) \). A functor \( f : C' \to C \) sends a seive \( R \in \emptyset(C) \) to \( R^f := R \times_C C' \). Suppose that for every \( S \in \text{Ob}(C) \) we are given a non-empty set \( J(S) \) of seives of the comma category \( C/S \) satisfying the condition:

\[
(S.1) \quad \forall f : T \to S, J(S)^f \subset J(T).
\]

One says that a collection \( \{ J(S) \subset \emptyset(C/S) \mid S \in \text{Ob}(C) \} \) define a topology on \( C \) if

\[
(S.2) \quad \forall S \in \text{Ob}(C), \forall C \in \emptyset(C/S), \text{if there exists } R \in J(S) \text{ such that } \forall f : X \to S \in \text{Ob}(R), C^f \in J(X), \text{then } C \in J(S).
\]
(12.2.1) \( \forall S \in Ob(\mathcal{C}) \), if \( C \in \emptyset(\mathcal{C}/S) \) and if there exists \( R \in J(S) \) such that \( R \subset C \), then \( C \in J(S) \).

A site is a category with a topology. The \( R \in J(S) \) are called refinements of \( S \).

**Definition 12.2.3.** A comorphism of sites \( f : \mathcal{E} \to \mathcal{E}' \) is a functor on the underlying categories \( f : E \to E' \) such that for every object \( X \in ob(E) \), and every sieve \( R' \) in \( (\mathcal{E}' \downarrow X') \) where \( X' = f(X) \) in \( \mathcal{E}' \), the sieve \( R' \times_{(E \downarrow X)} (E \downarrow X) \) belongs to \( \mathcal{E} \).

Let us recall how a covering gives rise to a sieve. Let \( \mathcal{R} = \{ r_i : R_i \to S | i \in I \} \) be a family of arrows with the range. They define a sieve \( R \) of \( \mathcal{C}/S \) whose objects are \( f : X \to S \) such that there exists a \( i \in I \) such that \( Hom_S(X, R_i) \neq \emptyset \). So to describe the topology whenever it is convenient, we will specify only the coverings.

Let \( C \) be the category of schemes with étale topology. Let \( \mathcal{X} \) be an ind-scheme.

We define the big étale site \( \hat{\mathcal{E}}(\mathcal{X}) \) of \( \mathcal{X} \) as follows:

1. **objects** are morphism \( u : U \to \mathcal{X} \) factoring through \( X_n \) for some \( n \in I \)
2. **morphisms** from \( u : U \to \mathcal{X} \) to \( u' : U' \to \mathcal{X} \) is a morphism \( f : U \to U' \) of schemes over \( \mathcal{X} \).
3. **covering of** \( u : U \to \mathcal{X} \) is just a covering of \( U \) in \( C \).

Let \( f : \mathcal{X} \to \mathcal{Y} \) be a morphism of ind-schemes. We obtain a comorphism of sites \( \hat{\mathcal{E}}(f) : \hat{\mathcal{E}}(\mathcal{X}) \to \hat{\mathcal{E}}(\mathcal{Y}) \), where the functor \( \hat{\mathcal{E}}(f) \) sends \( u : U \to \mathcal{X} \) to \( f \circ u : U \to \mathcal{Y} \).

Let \( \hat{\mathcal{X}} \) (or sometimes \( \mathcal{X}_{\hat{\mathcal{E}}} \)) denote the category of sheaves of sets on \( \hat{\mathcal{E}}(\mathcal{X}) \). It admits the following alternate description. A sheaf of sets \( \mathcal{F} \) on an ind-scheme \( \mathcal{X} \) is a collection of sheaves of sets \( \mathcal{F}_n \) on \( X_n \), where \( n \in I \), together with morphisms \( \phi_{j,k}^n : i_{j,k}^*(\mathcal{F}_k) \to \mathcal{F}_j \), whenever \( j \leq k \), satisfying an obvious co-cycle condition: let \( j \leq k \leq l \), then

\[
\phi_{j,l} = \phi_{j,k} \circ i_{j,k}^{-1}(\phi_{k,l}).
\]

Let \( Ab(\mathcal{X}) \) denote the subcategory of abelian group objects in \( \hat{\mathcal{X}} \). It is well known that the category of abelian presheaves on any Grothendieck site is an abelian category with enough injectives (cf [17, (12.1.1)(12.1.2)]).

**12.2.1. Pull-back and pushforward of (pre)sheaves.** Let \( \mathcal{F} \in \mathcal{Y} \) and \( f : \mathcal{X} \to \mathcal{Y} \) be a morphism. Let \( u : U \to \mathcal{X} \) be an open of \( \mathcal{X} \). We define *pull-back presheaf* by

\[
(f^*\mathcal{F})(U,u) = \mathcal{F}(f_!(u,u)).
\]

The *pull-back sheaf* is obtained by sheafifying it following [17 Prop 2.6, 29].

Similarly, let \( \mathcal{G} \in \hat{\mathcal{X}} \). We view it as a presheaf. Let \( u : U \to \mathcal{Y} \) be an open of \( \mathcal{Y} \). Consider the comma category \( (\hat{\mathcal{E}}(f), u) \). It consists of commutative diagrams

\[
\begin{array}{ccc}
V & \xrightarrow{v} & \mathcal{X} \\
\downarrow f & & \downarrow f \\
U & \xrightarrow{u} & \mathcal{Y}
\end{array}
\quad
\begin{array}{ccc}
V_2 & \xrightarrow{\theta} & V_1 \\
\downarrow f_2 & & \downarrow f_1 \\
U & \xrightarrow{u} & \mathcal{Y}
\end{array}
\]

where objects are \( v : V \to \mathcal{X} \) lying over \( u : U \to \mathcal{Y} \) and morphisms \( \theta : V_2 \to V_1 \) fit compatibly as above. It will be convenient to denote \( V \to U \) also by \( f \).
For push-forward, one considers \((V_1, v_1)\) to be finer than \((V_2, v_2)\) because \(G_{v_1} \to \theta, G_{u_2}\). Taking limit over \((\text{ét}(f), u)\), one defines the push-forward presheaf

\[
(f_*G)_{(U,u)} = \lim_{\to} f_*G_{(V,v)}.
\]

Actually \(f_*G\) is already a sheaf if \(G\) is a sheaf by [17, Prop 2.6, 2°]. Indeed, this follows because pushforward commutes with arbitrary limits and the sheaf condition is formulated in terms of limits. Indeed, by [18 Chapitre 0, §2] or [17 (1.2)], a presheaf \(F\) is a sheaf on a site \(E\) if for every object \(S \in ob(E)\), and every refining \(R \in J(S)\) the natural map \(F(S) \to \lim(F|_R)\) is bijective. By [17, page 196], the functor \(f^*\) commutes with finite limits. So we have a morphism of topoi of sheaves on ind-schemes

\[
(f^*, f_*) : \tilde{X} \to \tilde{Y}.
\]

12.3. **Comorphism between sites of ind-schemes and stacks.** Let \(\mathcal{X}/S\) be an algebraic stack. Let us recall the big-étale site \(\text{Étale}(\mathcal{X})\) of \(\mathcal{X}\) defined as follows:

1. The objects are representable 1-morphisms \(u : U \to \mathcal{X}\) of \(S\)-algebraic stacks, where \(U\) is a scheme.
2. A morphism from \((U_1, u_1)\) to \((U_2, u_2)\) consists of a pair \((\phi, \alpha)\) where

\[
\begin{array}{ccc}
U_1 & \xrightarrow{\phi} & U_2 \\
\downarrow{\phi_1} & & \downarrow{\phi_2} \\
U_1 & \xrightarrow{\alpha} & \mathcal{X}
\end{array}
\]

and \(\alpha : u_1 \to u_2 \circ \phi\) is a 2-morphism.
3. Coverings are families \(\{(\phi_i, \alpha_i) : (U_i, u_i) \to (U, u), i \in I\}\) such that the 1-morphism of \(S\)-schemes \(\bigcup_i \phi_i : \bigcup_i U_i \to U\) is étale and surjective.

We denote by \(\mathcal{X}_{\text{ét}}\) the big-étale topos of sheaves on \(\mathcal{X}\). We shall work with Artin stacks in this paper. So we will tacitly assume the existence of a smooth atlas. Further the diagonal map \(\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}\) is representable.

By a morphism \(a : \mathcal{X} \to \mathcal{X}\) from a stack to a stack we mean a sequence of morphisms \(a_j : X_j \to X_k\) such that through \(X_j \leftrightarrow X_k\), \(a_k\) restricts to \(a_j\). Given an open \(u : U \to \mathcal{X}\) in \(\text{Ét}(\mathcal{X})\), since it factors through some \(X_n\), so let agree to define \(a \circ u\) as \(a_n \circ u\). We thus obtain a comorphism of sites (cf [17, Définition 2.3])

\[
\text{Ét}(a) : \text{Ét}(\mathcal{X}) \to \text{Ét}(\mathcal{X}),
\]

where the functor \(\text{Ét}(a)\) sends \(u : U \to \mathcal{X}\) to \(a \circ u : U \to \mathcal{X}\).

12.3.1. **Pull-back and push-forward of sheaves and presheaves.** To define pullback of (pre)sheaves it will be notionally convenient below to use \((?)^*\) instead of \((?)^{-1}\) even for abelian sheaves. Let \(\mathcal{F} \in \mathcal{X}_{\text{ét}}\). We view it as a presheaf. Let \(v : V \to \mathcal{X}\) be an open of \(\mathcal{X}\). Let us consider the comma category \(\langle v \downarrow \text{Ét}(a)\rangle\). It consists of commutative diagrams

\[
\begin{array}{ccc}
V & \xrightarrow{v} & \mathcal{X} \\
\downarrow{\alpha} & & \downarrow{a} \\
U & \xrightarrow{u} & \mathcal{X}
\end{array}
\quad
\begin{array}{ccc}
V & \xrightarrow{v} & \mathcal{X} \\
\downarrow{\alpha_2} & & \downarrow{a} \\
U_2 & \xrightarrow{u_2} & \mathcal{X}
\end{array}
\quad
\begin{array}{ccc}
V & \xrightarrow{v} & \mathcal{X} \\
\downarrow{\alpha_1} & & \downarrow{a} \\
U_1 & \xrightarrow{u_1} & \mathcal{X}
\end{array}
\]

of objects \(u : U \to \mathcal{X}\) fitting below \(v : V \to \mathcal{X}\) and morphisms \(\theta : U_1 \to U_2\) fitting compatibly as above. It will be convenient to denote \(V \to U\) also by \(a\).
For pull-back, one considers the open \((U_1, u_1)\) as finer than \((U_2, u_2)\) because 
\[
\theta^*F_{u_2} \to F_{u_1}.
\]
Taking colimit over \((v \downarrow \acute{E}t(a))\), one defines
\[(12.3.4) \quad (a^*F)_{(V,v)} = \operatorname{lim} a^*F_{(U,u)}\]
Let \(\phi : (V_1, v_1) \to (V_2, v_2)\) be a morphism between opens of \(\mathcal{X}\). We have a morphism
\[(12.3.5) \quad \phi^*(a^*F)_{(V_2,v_2)} \to (a^*F)_{(V_1,v_1)}\]
because any open \((U, u)\) under \((V_2, v_2)\) is also under \((V_1, v_1)\). This defines the pull-back presheaf \(a^*F\). We note that pull-back commutes with finite limits, and so is afortiori an exact functor. We remark a simplification in the present case. The category \((v \downarrow \acute{E}t(a))\) has \(\acute{E}t(a)(v) = a \circ v : V \to \mathcal{X}\) as a final object with \(a_1 = \operatorname{Id}_V\).
Thus
\[(12.3.6) \quad (a^*F)_{(V,v)} = F_{(V,av)}\]
For \(F \in \mathcal{X}_{\acute{E}t}\), we define the pull-back sheaf \(F\) as the sheaf associated to this presheaf.

Similarly, let \(G \in \mathcal{X}_{\acute{E}t}\). We view it as a presheaf. Let \(u : U \to \mathcal{X}\) be an open of \(\mathcal{X}\). The comma category \((\acute{E}t(a), u)\) consists of commutative diagrams
\[(12.3.7) \quad \begin{array}{ccc}
V & \xrightarrow{v} & X \\
\downarrow a & & \downarrow a \\
U & \xleftarrow{u} & \mathcal{X}
\end{array} \quad \begin{array}{ccc}
V_1 & \xrightarrow{v_1} & X \\
\downarrow a_1 & & \downarrow a_2 \\
U & \xleftarrow{u} & \mathcal{X}
\end{array}
\]
where objects are \(v : V \to \mathcal{X}\) lying over \(u : U \to \mathcal{X}\) and morphisms \(\theta : V_2 \to V_1\) fit compatibly as above. It will be convenient to denote \(V \to U\) also by \(a\).

For push-forward, one considers \((V_1, v_1)\) to be finer than \((V_2, v_2)\) because \(G_{v_1} \to \theta_*G_{v_2}\). Taking limit over \((\acute{E}t(a), u)\), one defines
\[(12.3.8) \quad (a_*G)_{(U,u)} = \operatorname{lim} a_*G_{(V,v)}\]
Let \((\phi, a) : (U_1, u_1) \to (U_2, u_2)\) be a morphism between opens of \(\mathcal{X}\). We have a morphism
\[(12.3.9) \quad (a_*G)_{(U_2,u_2)} \to \phi_*(a_*G)_{(U_1,u_1)}\]
because any open \((V, v)\) over \((U_1, u_1)\) is also over \((U_2, u_2)\). This defines the pushforward presheaf \(a_*G\). Actually \(a_*G\) is already a sheaf because push-forward commutes with arbitrary limits.

**Remark 12.3.1.** Taking \(\acute{E}t(a) : \acute{E}t(\mathcal{X}) \to \acute{E}t(\mathcal{X})\) as \(u : E \to E'\) in \([17\text{ Lemme }2.5]\), we see that \(a^* = u^*\) and \(a_* = u_*\) (cf also \([17\text{ Proposition }2.6 (2)]\) where we set \(f : E \to E'\) as \(\acute{E}t(a)\)). So the constructions of \(a^*\) and \(a_*\) are just specializations to our case of the constructions in \([17]\) for the case of comorphisms. In particular, one has the following adjunction. We give below a self-contained proof.

**Proposition 12.3.2.** Let \(a : \mathcal{X} \to \mathcal{X}\) be a morphism. Let \(G \in \mathcal{X}_{\acute{E}t}\) and \(F \in \mathcal{X}_{\acute{E}t}\). Then we have
\(\operatorname{Mor}_{\mathcal{X}}(a^*G, F) = \operatorname{Mor}_{\mathcal{X}}(G, a_*F)\).

**Proof.** A morphism in \(\operatorname{Mor}_{\mathcal{X}}(a^*G, F)\) is equivalently a functorial assignment for every \((V, v) \in \acute{E}t(\mathcal{X})\) of an element in \(\operatorname{Mor}_{(V,v)}((a^*G)_{(V,v)}, F_{(V,v)})\). This last set is \(\operatorname{lim} \operatorname{Mor}((a^*G)_{(U,u)}), F_{(V,v)})\) where one varies over \((U, u) \in \acute{E}t(\mathcal{X})\) sitting under \((V, v)\) and \(a\) is the corresponding morphism. Thus a morphism in \(\operatorname{Mor}_{\mathcal{X}}(a^*G, F)\) is a functorial assignment for \(\acute{E}t(\mathcal{X}) \times \acute{E}t(\mathcal{X})\) such that \((V, v)\) sits on \((U, u)\) of an element
abelian presheaves on ind-schemes. Further, various operations like sheafification may be carried out "degree-by-degree". Let $u$ checking epimorphisms and monomorphisms, computing inverse and direct limits relative to this site, formation of images and quotients by equivalence relations, since there are enough injective objects in the category of simplicial ind-schemes. One defines the functors 

denote the category of abelian group objects in $\tilde{\mathcal{X}}$. Let $\tilde{\mathcal{X}}$ be the category whose objects are maps $U \to \mathbb{Y}_n$ for some $n$. (2) morphisms from $U \to \mathbb{Y}_n$ to $U' \to \mathbb{Y}_{n'}$ is a pair $(f, \phi)$ that fit in a diagram 

\[
\begin{array}{ccc}
U & \xrightarrow{f} & U' \\
\downarrow & & \downarrow \\
\mathbb{Y}_n & \xrightarrow{Y(\phi)} & \mathbb{Y}_{n'}
\end{array}
\]

where $\phi : [n'] \to [n]$ is a non-decreasing map and $f : U \to U'$ is any morphism of schemes. (3) a covering \{U_i \to U\} of $U \to \mathbb{Y}_n$ is a collection $(f_i, id)$ such that the images of $f_i$ cover $U$ set-theoretically and each $f_i$ is étale.

Let $\tilde{\mathcal{Y}}_\bullet$ denote the category of sheaves of sets on this site. It may equivalently be defined as follows. A sheaf of sets on $\mathbb{Y}_\bullet$ is a collection $\mathcal{F}_n \in \tilde{\mathbb{Y}}_n$ of sheaves on $\mathbb{Y}_n$ together with morphism $[\phi]$ of sheaves on $\mathbb{Y}_n$, 

\[
[\phi] : Y(\phi)^{-1}(\mathcal{F}_{n'}) \to \mathcal{F}_n
\]

for every $\phi \in Mor(\mathbb{Y}_n, [n])$, satisfying an obvious co-cycle relation. Let $\text{Ab}(\tilde{\mathbb{Y}}_\bullet)$ denote the sub-category of abelian group objects in $\tilde{\mathbb{Y}}_\bullet$.

We have a natural restriction morphism $\text{Res}_n : \mathbb{Y}_\bullet \to \tilde{\mathbb{Y}}_n$. It is exact. It has both left and right adjoint [23, §2.4]. Using its right adjoint, we can show that there are enough injective objects in the category of abelian presheaves on simplicial ind-schemes since there are enough injective objects in the category of abelian presheaves on ind-schemes. Further, various operations like sheafification relative to this site, formation of images and quotients by equivalence relations, checking epimorphisms and monomorphisms, computing inverse and direct limits may be carried out "degree-by-degree". Let $u \bullet : \mathcal{X}_\bullet \to \mathbb{Y}_\bullet$ be a map of simplicial ind-schemes. One defines the functors 

\[
u_\bullet, a_\bullet : \tilde{\mathcal{X}}_\bullet \to \tilde{\mathbb{Y}}_\bullet \quad u_\bullet, a_\bullet : \tilde{\mathbb{Y}}_\bullet \to \tilde{\mathcal{X}}_\bullet,
\]

by term-by-term construction of pushforward and pullback.

Let us consider the special case of augmented simplicial spaces $a : \mathbb{Y}_\bullet \to \mathcal{X}$. Let $\mathcal{X}_\bullet$ denote the constant simplicial space and $a_\bullet : \mathbb{Y}_\bullet \to \mathcal{X}_\bullet$ the associated map. We thus obtain a comorphism of sites (cf [17, Définition 2.3])

\[
\tilde{\text{Et}}(a_\bullet) : \tilde{\text{Et}}(\mathbb{Y}_\bullet) \to \tilde{\text{Et}}(\mathcal{X}_\bullet),
\]

where the functor $\tilde{\text{Et}}(a_\bullet)$ sends $u : U \to \mathbb{Y}_n$ to $a_n \circ u : U \to (\mathcal{X}_\bullet)_n = \mathcal{X}$. 

\[\text{(12.3.10)}\] 

\[(a^*, a_*): \mathcal{X}_{\text{Et}} \to \mathcal{X}'_{\text{Et}}.\]
The category \( \mathcal{X}_n \) is canonically identified with the category of co-simplicial objects in the category of sheaves of sets on \( \mathcal{X} \). For presheaves, the pull-back functor (12.4.4)
\[
a^* : \mathcal{X} \rightarrow \mathcal{Y}_n,
\]
defined as \( a^*(G)^n = a^n_*(G) \) in each degree and with natural face and degeneracy relations. Let \( \mathcal{F} \in \mathcal{X} \) and \( u : U \rightarrow Y_n \), then \( a^*(\mathcal{F})_u \) is simply the restriction \((\mathcal{F})_{a \circ u}\). So pull-back is an exact functor on presheaves. For sheaves, we take pull-back of \( G \) as a presheaf and then sheafify it. It has a right adjoint (12.4.5)
\[
a_* : \mathcal{Y}_n \rightarrow \mathcal{X},
\]
given by defining \( a_*(\mathcal{F}) \) as the kernel equalizer of the two ”face”-morphisms \( a_0, \mathcal{F}^0 \rightarrow a_1, \mathcal{F}^1 \). The proof of the adjunction \((a^*, a_*)\) is word-to-word generalisation of the standard proof in the case \( \mathcal{Y}_n \) is a simplicial scheme. The only non-obvious ingredient was proven in Prop [12.3.2] So we have a morphism of topoi (12.4.6)
\[
(a^*, a_*): (\mathcal{Y}_n)_{\text{Et}} \rightarrow (\mathcal{X})_{\text{Et}}.
\]

12.4.2. A spectral sequence. Let \( \mathcal{Y}_n \) be a simplicial ind-scheme. Let \( \mathcal{F}_n \in \text{Ab}(\mathcal{Y}_n) \).
We define \( \Gamma(\mathcal{Y}_n, \mathcal{F}_n) \) as the kernel equalizer of the two ”face”-morphisms (12.4.7)
\[
\Gamma(\mathcal{Y}_0, \mathcal{F}_0) \rightarrow \Gamma(\mathcal{Y}_1, \mathcal{F}_1).
\]
The following is a mild generalization of cf [14 Thm 6.11].

**Theorem 12.4.1.** Let \( \mathcal{Y}_n \) be a simplicial ind-scheme. Let \( D_+(\mathcal{Y}_n) \) denote the derived category of bounded below complexes of sheaves of abelian groups on \( \mathcal{Y}_n \). For any \( K' \in D_+(\mathcal{Y}_n) \), there is a natural spectral sequence 
\[
E^{p,q}_1 = H^q(\mathcal{Y}_p, K'_{\mathcal{Y}_p}) \implies H^{p+q}(\mathcal{Y}_n, K'),
\]
where \( d_1^{p,q} \) is induced from the differential complex structure along \( \mathcal{Y}_n \).

**Proof.** Let \( K' \rightarrow I^* \) be a quasi-isomorphism to a bounded below complex of injectives in \( \text{Ab}(\mathcal{Y}_n) \). Let us consider the 1-st quadrant double complex (12.4.8)
\[
\Gamma(\mathcal{Y}_p, I^n|_{\mathcal{Y}_p})_{p,q}.
\]
Let us study page one of the spectral sequence that arises by filtering (12.4.8) first by rows. If we filter (12.4.9)
\[
H^0(\Gamma(\mathcal{Y}_0, I^n) \rightarrow \Gamma(\mathcal{Y}_1, I^n) \rightarrow \Gamma(\mathcal{Y}_2, I^n) \rightarrow \cdots) = \Gamma(\mathcal{Y}_n, I^n).
\]
At page one, these groups fit vertically on the ”0-th” column to make a complex 
\[
\cdots \rightarrow \Gamma(\mathcal{Y}_p, I^n) \rightarrow \Gamma(\mathcal{Y}_p, I^{n+1}) \rightarrow \cdots
\]
with differentials given by \( I^n \). So the 0-column on page one is isomorphic to \( R\Gamma(\mathcal{Y}_n, K') \) in \( D_+(\mathcal{Y}_n) \).

**Lemma 12.4.2.** In (12.4.8) all horizontal \( H^0 \) away from degree zero vanish.

**Proof.** An injective object in \( Ch_{\geq 0}(\text{Ab}) \) is necessarily acyclic in positive degrees. It suffices to show that for an injective sheaf \( I \in \text{Ab}(\mathcal{Y}_n) \) the complex (12.4.10)
\[
\Gamma(\mathcal{Y}_0, I) \rightarrow \Gamma(\mathcal{Y}_1, I) \rightarrow \Gamma(\mathcal{Y}_2, I) \rightarrow \cdots
\]
is an injective object in \( Ch_{\geq 0}(\text{Ab}) \). To this end, it suffices to construct an exact left-adjoint to the functor \( \text{Ab}(\mathcal{Y}_n) \rightarrow Ch_{\geq 0}(\text{Ab}) \) which sends \( \mathcal{F} \mapsto \{ \Gamma(\mathcal{Y}_p, \mathcal{F}|_{\mathcal{Y}_p}) \}_{p \geq 0} \) where the differentials are given by alternating sum of face maps. Let \( C_{\text{Ab}} \in Ch_{\geq 0}(\text{Ab}) \). Applying the Dold-Kan functor we get a simplicial abelian group. Let \( C_n \in \text{Ab} \) be the n-simplices. We associate to \( C_n \) the sheaf \( \mathcal{C}_n \) which on every non-empty open \( U \) of \( Y_n \) takes value \( C_n \). These \( \mathcal{C}_n \) define the abelian sheaf \( \mathcal{C}_* \) on \( \mathcal{Y}_n \).
Let us call this functor \( dk : Ch_{\geq 0}(Ab) \to Ab(\mathcal{Y}_*) \). It is exact. Further we have the desired adjunction \( \text{Hom}_{Ab(\mathcal{Y}_*)}(dk(C_*), G^\bullet) = \text{Hom}_{Ch_{\geq 0}(Ab)}(C_*, \{\Gamma(\mathcal{Y}_p, G|_{\mathcal{Y}_p})\}) \).

\[ \square \]

In conclusion, at page one only groups in column zero survive. Thus the horizontal spectral sequence abuts to the cohomology groups of \( R\Gamma(\mathcal{Y}_*, K') \).

Now we study page one of the spectral sequence that arises by filtering (12.4.8) by columns. A column consists of \( \Gamma(\mathcal{Y}_p, I^\bullet|_{\mathcal{Y}_p}) \) for a fixed \( p \geq 0 \). Upon taking homology, we get a spectral sequence \( H^q(\Gamma(\mathcal{Y}_p, I^\bullet|_{\mathcal{Y}_p})) \Rightarrow H^\bullet(\mathcal{Y}_*, K') \). Recall that \( K' \to I^\bullet \) is a quasi-isomorphism of bounded below chain complexes in \( Ab(\mathcal{Y}_*) \).

By exactness of the functor restricting sheaves on \( \mathcal{Y} \) to \( \mathcal{Y}_p \) we have \( K'|_{\mathcal{Y}_p} \to I^\bullet|_{\mathcal{Y}_p} \) is also exact. Further \( I^\bullet|_{\mathcal{Y}_p} \) is an injective sheaf. Indeed, the restriction functor \( \text{res}_n : Ab(\mathcal{Y}_*) \to Ab(\mathcal{Y}_n) \) has an exact left adjoint given by

\[ (l_n(G))_k = \oplus_{p \in \text{Hom}_\Delta([n],[k])} \rho^*G. \]

It is exact because \( \rho^* \) is exact for each morphism \( Ab(\mathcal{Y}_n) \to Ab(\mathcal{Y}_n) \). So \( K'|_{\mathcal{Y}_p} \to I^\bullet|_{\mathcal{Y}_p} \) is a bona-fide qis of \( K'|_{\mathcal{Y}_p} \) to an injective in \( Ch_{\geq 0}(Ab(\mathcal{Y}_p)) \).

Thus

\[ H^q(\Gamma(\mathcal{Y}_p, I^\bullet|_{\mathcal{Y}_p})) = H^q(\mathcal{Y}_p, K'|_{\mathcal{Y}_p}). \]

So \( H^q(\mathcal{Y}_p, K'|_{\mathcal{Y}_p}) \) fit to form the page one of the column spectral sequence. Thus we get the desired spectral sequence.

\[ \square \]

12.5. Cohomological descent for maps from an ind-scheme to a stack.

12.5.1. Morphism of sites. Let \( \epsilon : \mathcal{X} \to \mathcal{X} \) be a simplicial ind-scheme over an Artin stack \( \mathcal{X} \). Let \( \epsilon_n : \mathcal{X}_n \to \mathcal{X} \) denote the projection morphisms. We have a comorphism of sites (cf [12.3.4] Definition 2.3))

\[ (\epsilon_1) : \mathcal{X} \to \mathcal{X} \]

which sends \(([n], u : U \to \mathcal{X}_n) \) to \((U, u \circ \epsilon_n) \). This morphism induces a pair of adjoint functors (cf [12.4.4] [12.4.5] [12.4.4.1])

\[ (\epsilon_1^*, \epsilon_1_* : (\mathcal{X})_{\text{F}} \to \mathcal{X}_{\text{F}}. \]

12.5.2. Cohomological Descent. We say that \( \epsilon_* : \mathcal{X} \to \mathcal{X} \) is a morphism of cohomological descent if the natural transformation

\[ \text{id} \to R\epsilon_* \circ \epsilon^*, \]

is an isomorphism on the derived category of bounded below complexes of abelian sheaves \( D_+(\mathcal{X}) \) on \( \mathcal{X} \). We say \( \epsilon_* \) is universally of cohomological descent if it is remains so under arbitrary base-change. We shall say that a morphism \( \epsilon : \mathcal{X} \to \mathcal{X} \) from an ind-scheme to an algebraic stack is a morphism of cohomological descent if \( \epsilon_* := \text{cosk}_0(\epsilon) \) is so (cf [12.1.2] for fibered products of ind-schemes).

**Theorem 12.5.1.** Suppose that \( a : \mathcal{Y}_* \to \mathcal{X} \) is augmented. Let \( K' \in D_+(\mathcal{Y}_*) \). We have a canonical spectral sequence

\[ E_1^{p,q} = R^q a_{p,*}(K'|_{\mathcal{Y}_p}) \Rightarrow R^{p+q} a_*(K'), \]

which is functorial in \( a \). Suppose that \( a \) is universally of cohomological descent. Let \( K \in D_+(\mathcal{X}) \) be a complex and set \( K' := a^* K \). The spectral sequence of Theorem [12.3.1] abuts to \( H^{p+q}(\mathcal{X}, K) \).

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Definition 12.5.3. Suppose that \( \text{Theorem 12.5.4.} \)

Proof. This spectral sequence is a sheafified version of the one in Theorem 12.4.1. It is constructed by working with \( a_p \) and \( a_s \) instead of \( \Gamma(Y_p, \bullet) \) and \( \Gamma(Y_s, \bullet) \) of 12.4.7 respectively. For \( K \in D_+(X) \) we have the following isomorphism

\[ R\Gamma(X, K) \xrightarrow{\text{coh-des}} R\Gamma(X, Ra_*a^*K) \simeq R(\Gamma(X, \bullet) \circ a_s)(a^*K) \]

which is functorial in \( a \). Thus we get

\[ H^i(X, K) \simeq H^i(Y_s, a_*K), \]

which is also natural in \( a \). This shows the equality of abutment terms.

The following is a mild generalization of [14] Thm 7.2 to the case of a morphism \( \epsilon : X \to X' \) from an ind-scheme to an algebraic stack.

Definition 12.5.2. We say that \( \epsilon \) admits a section \( s : X' \to X \) if there is a morphism \( s : X \to X_n \) for some \( n \in I \) such that \( \epsilon_n \circ s = \text{Id}_{X_n} \).

Definition 12.5.3. We say that \( \epsilon \) admits étale local sections if any neighbourhood \( u : U \to X \), admits an étale surjective map \( \theta : U' \to U \) such that making make fiber product of ind-scheme by \( \epsilon_U \) admits a section

\[ (12.5.5) \]

\[ X_{U'} \xrightarrow{\theta} X_U \]

\[ \epsilon_{U'} \]

\[ U' \xrightarrow{\theta} \]

\[ U \]

Theorem 12.5.4. Suppose that \( \epsilon \) admits sections étale locally on \( X \). Then \( \epsilon \) is universally a morphism of cohomological descent.

Proof. The universality follows simply because our hypothesis is preserved under base-change. To check that \( 1 \to R\epsilon_{U*} \circ \epsilon_U^* \) is an isomorphism of functors on \( D_+(U) \), by successively truncating any complex \( K \in D_+(U) \) from above, it suffices to consider the case \( K = F \) for a \( F \in \text{Ab}(U_{\text{ét}}) \).

Let us take a local neighbourhood \( u : U \to X \) of \( X \). Let \( \theta : U' \to U \) be an étale and surjective. Since \( \theta^* : \text{Ab}(U_{\text{ét}}) \to \text{Ab}(U'_{\text{ét}}) \) is exact, so it preserves kernels and cokernels. Thus it suffices to check \( \theta^* \) is of cohomological descent then so is \( \epsilon_U \). So we may suppose that \( X_U \to U \) itself has a section \( s : U \to X_U \). Let us abbreviate \( X_U \) to \( X \) and \( \epsilon_U \) to \( \epsilon \).

Set \( K' = \epsilon^*F \in D_+(X) \). By Theorem 12.5.1 we have a spectral sequence

\[ E_1^{p,q} = R^p\epsilon_*F(K'|_{\text{p}}) \Rightarrow R^{p+q}\epsilon_*(K'). \]

Here the \( q \)-th row has differential given by the simplicial structure on \( X_\bullet \). Let us abbreviate \( X \) simply by \( X \) from now.

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Note $E_1^{p,q}$ makes sense for $p \geq -1$, where $\epsilon_{-1} = id_X$. Thus $E_1^{-1,q} = 0$ for $q \geq 1$ since push-forward along the identity map has vanishing higher derived functors. For $p \geq 1$, consider

$$(12.5.8) \quad h_p = s \times id_{\mathbb{R}^p} : \mathbb{X}_p = \mathbb{X}^{\times, p+1} \to \mathbb{X}^{\times, p+2} = \mathbb{X}_{p+1},$$

defined by inserting the section $s$ along the 0-th coordinate of $\mathbb{X}_{p+1}$. We have thus

$$(12.5.9) \quad \mathbb{X}(\partial^{p}_p)h_p = id_{\mathbb{X}_p} \quad p \geq 0$$

$$(12.5.10) \quad h_{p-1} \mathbb{X}(\partial^{j}_p) = \mathbb{X}(\partial^{j+1}_p)h_p \quad \forall p \geq 1, 0 \leq j \leq p - 1.$$ 

When $K' = \epsilon_* \mathcal{F}$, by $K'|_{\mathbb{X}_{p+1}} = h_p^*(K'|_{\mathbb{X}_{p+1}})$, we have

$$K'|_{\mathbb{X}_{p+1}} \to R\mathbb{H}_p h^{*} h_{p}^{*} h^{*} h_{p}^{*} h_{p}^{*} h_{p}^{*} r_n^{*} (K'|_{\mathbb{X}_{p+1}}) = R\mathbb{H}_p h^{*} (K'|_{\mathbb{X}_{p}}).$$

Now applying $R\epsilon_{p+1} \circ$ we get

$$R\epsilon_{p+1} \circ (K'|_{\mathbb{X}_{p+1}}) \to R\epsilon_{p+1} \circ R\mathbb{H}_p h^{*} h_{p}^{*} h_{p}^{*} h_{p}^{*} h_{p}^{*} h_{p}^{*} r_n^{*} (K'|_{\mathbb{X}_{p+1}}) = R\epsilon_{p+1} \circ R\mathbb{H}_p h^{*} (K'|_{\mathbb{X}_{p}}) = R\mathbb{H}_p \circ (K'|_{\mathbb{X}_{p}}).$$

So we get maps $E_{p+1}^{p+1,q} \to E_{p}^{p,q}$.

**Lemma 12.5.5.** These maps form a homotopy between the identity and zero maps on the augmented differential complexes $E_{1}^{p,q}$ (in degrees $\geq -1$).

**Proof.** Let $p \geq 0$. Let us compute $h_p^* \partial^p_q + \partial^q_p h_{p-1}^*$

$$(12.5.11) \quad = h_p^* \left( \sum_{j=0}^{p+1} (-1)^j \mathbb{X}(\partial^j_p)^* \right) + \left( \sum_{k=0}^{p-1+1} (-1)^k \mathbb{X}(\partial^k_{p-1})^* \right) h_{p-1}^*$$

$$(12.5.12) \quad = h_p^* \mathbb{X}(\partial^p_p)^* + \sum_{k=0}^{p-1+1} (-1)^k (-h_p^* \mathbb{X}(\partial^k_{p-1})^* + \mathbb{X}(\partial^k_{p-1})^* h_{p-1}^*)$$

$$(12.5.13) \quad = id + 0 = id.$$ 

For $p = -1$, we have $d_{1}^{-1,q} = \mathbb{X}(\partial^0_{-1})^*$. Now $h_{-1}^* d_{1}^{-1,q} = h_{-1}^* \mathbb{X}(\partial^0_{-1})^* = (\mathbb{X}(\partial^0_{-1}) h_{-1})^*$ which by (12.5.9) equals $id^*$. Hence the augmented complex is acyclic.

Now let us consider the terms of the spectral sequence restricted to the 1st quadrant. Since $E_1^{q-1,q} = 0$ for $q \geq 1$, thus the rows for $q \geq 1$ remain exact upon restriction to the 1st quadrant. For $q = 0$, we have $E^{0,0}_2 = \epsilon_{*} \epsilon^*_0 \mathcal{F}$. The kernel of $\epsilon_{0,*} \epsilon^*_0 \mathcal{F} = E^{0,0}_1 \to E^{1,0}_1 = \epsilon_{*} \epsilon^*_1 \mathcal{F}$ identifies with $\mathcal{F}$ via the natural inclusion $\mathcal{F} \to \epsilon_{0,*} \epsilon^*_0 \mathcal{F}$. So at the $E_2$ stage, the spectral sequence collapses to just the $(0,0)$-position. Hence the total complex has vanishing homology in higher degrees and homology in degree zero is given by $\mathcal{F}$. Thus $R^n \epsilon_{*} \epsilon^* \mathcal{F} = 0$ for $n \geq 1$ and the natural map $\mathcal{F} = E_{2}^{0,0} \to R^n \epsilon_{*} \epsilon^* \mathcal{F}$ is an isomorphism. Thus $\mathcal{F} \to R\epsilon_{*} \epsilon^* \mathcal{F}$ is an isomorphism of complexes of abelian sheaves on $X$. Thus $\epsilon$ is a morphism of cohomological descent.

**References**

[1] https://stacks.math.columbia.edu/download/sites-cohomology.pdf.

[2] Jean-Louis Verdier Artin, Michael; Alexandre Grothendieck. Séminaire de Géométrie Algébrique du Bois Marie - 1963-64 - Théorie des topos et cohomologie étale des schémas - (SGA 4) - vol. 3. Lecture notes in mathematics (in French) 305. Berlin; New York: Springer Verlag.
[3] V. Balaji and C. S. Seshadri. Moduli of parahoric G-torsors on a compact Riemann surface. *J. Algebraic Geom.*, 24(1):1–49, 2015.

[4] Vikraman Balaji, Indranil Biswas, Ofer Gabber, and Doni hakkalu S. Nagaraj. Brauer obstruction for a universal vector bundle. *C. R. Math. Acad. Sci. Paris*, 345(5):265–268, 2007.

[5] Vikraman Balaji, Indranil Biswas, and Yashonidhi Pandey. Connections on parahoric torsors on curves. *Publ. Res. Inst. Math. Sci.*, pages 551–585, 2017.

[6] Indranil Biswas and Arijit Dey. Brauer group of a moduli space of parabolic vector bundles over a curve. *Journal of K-theory: K-theory and its Applications to Algebra, Geometry, and Topology*, 8:437–449, 2011.

[7] Indranil Biswas and Norbert Hoffmann. The line bundles on moduli stacks of principal bundles on a curve. *Documenta Mathematica*, 15:35–72, 2010.

[8] Indranil Biswas and Norbert Hoffmann. Poincaré families of G-bundles on a curve. *Math. Ann.*, 352(1):133–154, 2012.

[9] Indranil Biswas and Yogish I. Holla. Brauer group of moduli spaces of principal bundles over a curve. *J. Reine Angew. Math.*, 677:225–249, 2013.

[10] Indranil Biswas, Sean Lawton, and Daniel Ramras. Fundamental groups of character varieties: surfaces and tori. *Math. Zeit.*, 281:415–449, 2011.

[11] Armand Borel and Jean de Siebenthal. Les sous-groupes fermés de rang maximum des groupes de lie clos. *Commentarii mathematici Helvetici*, 23:200–221, 1949.

[12] F. Bruhat and J. Tits. Groupes réductifs sur un corps local. *Inst. Hautes Études Sci. Publ. Math.*, (41):5–251, 1972.

[13] F. Bruhat and J. Tits. Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d’une donnée radicielle valuée. *Inst. Hautes Études Sci. Publ. Math.*, (60):197–376, 1984.

[14] Brian Conrad. Cohomological descent. math.stanford.edu/conrad/papers/hypercover.pdf.

[15] A. J de Jong. A result of gabber. http://www.math.columbia.edu/dejong/papers/2-gabber.pdf.

[16] Gerd Faltings. Algebraic loop groups and moduli spaces of bundles. *Journal of the European Mathematical Society*, 5(1):41–68, Mar 2003.

[17] Jean Giraud. Analysis Situs [reprint of MR0193122]. In *Dix exposés sur la cohomologie des schémas*, volume 3 of *Adv. Stud. Pure Math.*, pages 1–11. North-Holland, Amsterdam, 1968.

[18] Jean Giraud. *Cohomologie non abélienne*. Springer-Verlag, Berlin-New York, 1971. Die Grundlehren der mathematischen Wissenschaften, Band 179.

[19] Alexander Grothendieck. Le groupe de Brauer III: Exemples et Complements. In *Dix exposés sur la cohomologie des schémas*, volume 3 of *Adv. Stud. Pure Math.*, pages 88–188. North-Holland, Amsterdam, 1968.

[20] Jochen Heinloth. Uniformization of G-bundles. *Math. Ann.*, 347(3):499–528, 2010.

[21] Shrawan Kumar. Infinite grassmannians and moduli spaces of G-bundles. *Vector Bundles on Curves- New Directions*, (ed. by M.S. Narasimhan). *Springer Lecture Notes in Math.* vol. 1649, 1-49 (1997).

[22] Shrawan Kumar. *Kac-Moody groups, their flag varieties and representation theory*, volume 204 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2002.

[23] Yves Laszlo and Christoph Sorger. The line bundles on the moduli of parabolic G-bundles over curves and their sections. *Ann. Sci. École Norm. Sup. (4)*, 30(4):499–525, 1997.

[24] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2000.

[25] A. S. Merkur’ev. A seven-term sequence in the Galois theory of schemes. *Mat. Sb. (N.S.),* 109(151)(3):395–409, 479, 1979.

[26] James S. Milne. *Etale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.

[27] Drezet J.-M. Narasimhan, M.S. Groupe de picard des variés de modules de fibrés semi-stables sur les courbes algébriques. *Inventiones mathematicae*, 97(1):53–94, 1989.

[28] Martin Olsson. Sheaves on Artin stacks. *J. Reine Angew. Math.*, 603:55–112, 2007.

[29] G. Pappas and M. Rapoport. Twisted loop groups and their affine flag varieties. *Adv. Math.*, 219(1):118–198, 2008. With an appendix by T. Haines and Rapoport.

[30] Georgios Pappas and Michael Rapoport. Some questions about G-bundles on curves. In *Algebraic and Arithmetic Structures of Moduli Spaces (Sapporo 2007)*, pages 159–171, Tokyo, Japan, 2010. Mathematical Society of Japan.
[31] A.J Parameshwaran and Yashonidhi Pandey. Étale fundamental group of moduli of torsors under bruhat-tits group scheme on a curve. arXiv:1911.02861.

[32] Daniel Quillen. Projective modules over polynomial rings. Invent. Math., 36:167–171, 1976.

[33] Christoph Sorger. On moduli of $G$-bundles on a curve for exceptional $G$. Ann. Sci. cole Norm. Sup., pages (4), 32, 127133, 1999.

[34] T. A. Springer. Linear algebraic groups. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, second edition, 2009.

[35] Günter Tamme. Introduction to étale cohomology. Universitext. Springer-Verlag, Berlin, 1994. Translated from the German by Manfred Kolster.

[36] Constantin Teleman. Borel-Weil-Bott theory on the moduli stack of $G$-bundles over a curve. Invent. Math., 134(1):1–57, 1998.

[37] M Thaddeus. Geometric invariant theory and flips. Jour. Amer. Math. Soc., 9:691–723, 1996.

[38] Jacques Tits. Strongly inner anisotropic forms of simple algebraic groups. J. Algebra, 131(2):648–677, 1990.

[39] Xinwen Zhu. On the coherence conjecture of Pappas and Rapoport. Ann. of Math. (2), 180(1):1–85, 2014.

Indian Institute of Science Education and Research, Mohali Knowledge city, Sector 81, SAS Nagar, Manauli PO 140306, India

E-mail address: ypande@iisermohali.ac.in, yashonidhipandey@yahoo.co.uk