CONSTRUCTING SYMPLECTOMORPHISMS BETWEEN SYMPLECTIC TORUS QUOTIENTS

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Abstract. We identify a family of torus representations such that the corresponding singular symplectic quotients at the 0-level of the moment map are graded regularly symplectomorphic to symplectic quotients associated to representations of the circle. For a subfamily of these torus representations, we give an explicit description of each symplectic quotient as a Poisson differential space with global chart as well as a complete classification of the graded regular diffeomorphism and symplectomorphism classes. Finally, we give explicit examples to indicate that symplectic quotients in this class may have graded isomorphic algebras of real regular functions and graded Poisson isomorphic complex symplectic quotients yet not be graded regularly diffeomorphic nor graded regularly symplectomorphic.

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1. Introduction

Let $G$ be a compact Lie group and $G \to U(V)$ a finite dimensional unitary representation of $G$. Here $U(V)$ stands for the unitary group of $V$, i.e. the group of automorphisms preserving the hermitian inner product $\langle \cdot , \cdot \rangle$. To describe the orbit space $V/G$, i.e. the space of $G$-orbits in $V$, invariant theory is employed as follows. There exists a system of fundamental real homogeneous polynomial invariants $\phi_1, \phi_2, \ldots, \phi_m$; we refer to the system $\phi_1, \phi_2, \ldots, \phi_m$ as a Hilbert basis. This means that any real invariant polynomial $f \in \mathbb{R}[V]^G$ can be written as a polynomial in the $\phi$'s, i.e. there exists a polynomial $g \in \mathbb{R}[x_1, x_2, \ldots, x_m]$ such that $f = g(\phi_1, \phi_2, \ldots, \phi_m)$.

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actually a diffeomorphism onto $X$, i.e. the pullback $\overline{\phi}$ via $\overline{\phi}$ induces an isomorphism of algebras $C^\infty(X) := \{g : X \rightarrow \mathbb{R} \mid \exists G \in C^\infty(\mathbb{R}^m) : g = G_{|X}\}$ and $C^\infty(V/G) := C^\infty(V)^G$. Moreover, the restriction of $\overline{\phi}$ to the subalgebra $\mathbb{R}[X] := \{g : X \rightarrow \mathbb{R} \mid \exists G \in \mathbb{R}[x_1, x_2, \ldots, x_m] : g = G_{|X}\}$ isomorphically to $\mathbb{R}[V/G] := \mathbb{R}[V]^G$ preserving the grading. Here we use the natural grading $\deg(x_i) := \deg(\phi_i)$. We say that $\overline{\phi}$ is a graded regular diffeomorphism. The algebra $\mathbb{R}[X]$ can be understood as the quotient of $\mathbb{R}[x_1, x_2, \ldots, x_m]$ by the kernel of the restriction map, which we refer to as the ideal of off-shell relations. Its generators are assumed to be homogeneous in the natural grading. The real variety underlying $\mathbb{R}[X]$ is the Zariski closure $\overline{X}$ of $X$ inside $\mathbb{R}^m$. The space $X$ itself is not a real variety but a semi-algebraic set. How the inequalities cutting out $X$ inside $\overline{X}$ are obtained has been explained in [12].

The hermitian vector space $V$ is equipped with the symplectic form $\omega = \text{Im}(\langle \cdot, \cdot \rangle)$ obtained by taking the imaginary part of hermitian inner product. Moreover, the action of $G$ on $V$ is Hamiltonian and admits a unique homogeneous quadratic moment map $\phi : V \rightarrow g^*$ where $g^*$ denotes the dual of the Lie algebra $g$ of $G$. The zero fibre $Z := J^{-1}(0)$ of $J$ is referred to as the shell. It is a real subvariety of $V$ with a conical singularity at the origin. Due to the $G$-equivariance of $J$ the group $G$ acts on $Z$. The space $M_0 := Z/G$ of $G$-orbits in $Z$ is called the (linear) symplectic quotient. By the work Sjamaar and Lerman [13] the smooth structure $C^\infty(M_0)$ is given by the quotient $C^\infty(V)^G/I_Z^G$ where $I_Z^G$ is the invariant part of the vanishing ideal $I_Z := \{f \in C^\infty(V) \mid f|_Z = 0\}$. Note that $C^\infty(M_0)$ is in a canonical way a Poisson algebra containing the Poisson subalgebra $\mathbb{R}[M_0] := \mathbb{R}[V]^G/I_Z^G$, where $I_Z^G := I_Z \cap \mathbb{R}[V]^G$. The image $Y := \phi(Z)$ of $Z$ under the Hilbert map is a semi-algebraic subset of $X$. Its Zariski closure $\overline{Y}$ is described by the generators of the kernel in $\mathbb{R}[x_1, x_2, \ldots, x_m]$ of the algebra morphism $x_i \mapsto \phi_i|_Z \in C^\infty(M_0)$. We refer to it as the ideal of on-shell relations. The inequalities that cut out $Y$ from $\overline{Y}$ are the same as those cutting out $X$ from $\overline{X}$.

Let us now assume that we have two symplectic quotients $M_0$ and $M'_0$ constructed from the representations $G \rightarrow U(V)$ and $G' \rightarrow U(V')$, respectively. By a symplectomorphism between $M_0$ and $M'_0$ we mean a homeomorphism $F : M_0 \rightarrow M'_0$ such that the pullback $F^* := F^*: C^\infty(M'_0) \rightarrow C^\infty(M_0)$ is an isomorphism of Poisson algebras $F^*: C^\infty(M'_0) \rightarrow C^\infty(M_0)$. We say that $F$ is regular if $F^*(\mathbb{R}[M'_0]) \subseteq \mathbb{R}[M_0]$. A regular symplectomorphism is called graded regular if the map $(F^*)_{|[M'_0]} : \mathbb{R}[M'_0] \rightarrow \mathbb{R}[M_0]$ preserves the grading. By the Lifting Theorem of [2], an isomorphism $f : \mathbb{R}[M'_0] \rightarrow \mathbb{R}[M_0]$ of Poisson algebras gives rise to a unique symplectomorphism if it compatible with the inequalities.

When $G = \mathbb{T}^d$ is a torus, a representation $V$ of complex dimension $n$ can be described in terms of a weight matrix $A \in \mathbb{Z}^d \times n$; we use $M_0(A)$ to denote the symplectic quotient associated with the representation with weight matrix $A$. In [2] Theorem 7, it is demonstrated that for a weight matrix of the form $A = [D[C]]$ where $D$ is an $\ell \times \ell$ diagonal matrix with strictly negative entries on the diagonal and $C$ is an $\ell \times 1$ matrix with strictly positive entries, the corresponding symplectic quotient by $\mathbb{T}^d$ is graded regularly symplectomorphic to the symplectic orbifold $\mathbb{C}/\mathbb{Z}_n$ where $\eta = \eta(A)$ is a quantity determined by the entries of $A$; see Definition 2.1. However, based on the explicit description of the ring $\mathbb{R}[\mathbb{C}]^{\mathbb{Z}_n}$ of real regular functions on the orbifold $\mathbb{C}/\mathbb{Z}_n$ given in the proof of [2] Theorem 7, it is easy to see that $\mathbb{R}[\mathbb{C}]^{\mathbb{Z}_{n_1}}$ and $\mathbb{R}[\mathbb{C}]^{\mathbb{Z}_{n_2}}$ are isomorphic as algebras over $\mathbb{R}$ if and only if $\eta_1 = \eta_2$. Hence, an immediate corollary of [2] Theorem 7 is the following.

**Corollary 1.1.** For $i = 1, 2$, let $A_i = [D_i[C_i]]$ where each $D_i$ is an $\ell_i \times \ell_i$ diagonal matrix with strictly negative entries on the diagonal and each $C_i$ is an $\ell_i \times 1$ matrix with strictly positive entries. Then the symplectic quotients $M_0(A_1)$ and $M_0(A_2)$ are regularly diffeomorphic if and only if $\eta(A_1) = \eta(A_2)$, in which case they are graded regularly symplectomorphic.

More recently, it was shown in [6] Theorem 1.1 that for general symplectic quotients, symplectomorphisms with symplectic orbifolds are rare, even if the graded regular requirements are
dropped; see also [9]. Hence, one cannot use isomorphisms with quotients by finite groups to approach a more general classification of higher-dimensional symplectic quotients by tori.

In this paper, we give a generalization of Corollary [1,4] as a step towards a general classification of linear symplectic quotients by tori into (graded) regular symplectomorphism classes. While Corollary [1,4] addresses a class of symplectic quotients by tori that can be reduced to quotients by finite groups, we consider here a class of symplectic quotients by tori that are graded regularly symplectomorphic to symplectic quotients by the circle $\mathbb{T}^1$. To state our main result, we say that a weight matrix $A \in \mathbb{Z}^{\ell \times (\ell + k)}$ is Type $\Pi_k$ if it can be expressed in the form $A = [D, c_1n, \ldots, c_kn]$ with $D$ a diagonal matrix with strictly negative diagonal entries, $n$ a column matrix with strictly positive entries, and each $c_r \geq 1$. Our main result is that the symplectic associated to a Type $\Pi_k$ matrix of any size is graded regularly symplectomorphic to a symplectic quotient by $\mathbb{T}^1$. Specifically, we have the following; see Definition [2,3] for the definitions of $\alpha$ and $\beta$.

**Theorem 1.2.** Let $A \in \mathbb{Z}^{\ell \times (\ell + k)}$ be the Type $\Pi_k$ matrix of a faithful $\mathbb{T}^\ell$-representation $V$ of dimension $n = \ell + k$. Then the symplectic quotient $M_0(A)$ is graded regularly symplectomorphic to the $\mathbb{T}^1$-symplectic quotient $M_0(B)$ where $B = (-\alpha(A), c_1\beta(A), \ldots, c_k\beta(A)) \in \mathbb{Z}^{1 \times (k + 1)}$.

Theorem 1.2 can be thought of as a dimension reduction formula, allowing one to describe symplectic quotients by $\mathbb{T}^\ell$ associated to Type $\Pi_k$ weight matrices in terms of much simpler quotients by $\mathbb{T}^1$. In particular, it extends results concerning $\mathbb{T}^1$-symplectic quotients to this family of quotients by tori, e.g. the Hilbert series computations of [8] or the representability results of [16]. The graded regular symplectomorphism given by the theorem preserves several structures, and hence can be thought of as a symplectomorphism of symplectic stratified spaces, a graded isomorphism of the corresponding real algebraic varieties, etc., and it induces a graded Poisson isomorphism of the corresponding complex symplectic quotients, the complexifications treated as complex algebraic varieties with symplectic singularities; see [7]. The proof of Theorem 1.2 is given in Section 3 by indicating a Seshadri section for the action of the torus on the zero fiber of the moment map after complexifying; see [10, Corollary, page 169] and [11, Theorem 3.14]. The first proof we obtained of Theorem 1.2, however, was constructive for a smaller class of weight matrices, so-called Type $I_k$ (see Definition 2.1), and used explicit descriptions of the corresponding symplectic quotients and algebras of real regular functions. Because this description has proven useful and may be of independent interest, we give this description and outline the constructive approach in Section 4.

In the case of symplectic quotients of (real) dimension 2 considered in Corollary [1,4] (corresponding to Type $I_1$ weight matrices), the graded regular symplectomorphism class of $M_0(A)$ depends only on the constant $\eta(A)$, which is given by the sum $\alpha(A) + \beta(A)$ (see Definition 2.1). In the case of Type $I_k$ weight matrices with $k > 1$, this is no longer the case; we show in Section 5 that the graded regular symplectomorphism classes of Type $I_k$ symplectic quotients are classified by $k, \alpha(A),$ and $\beta(A)$. For Type $\Pi_k$ weight matrices, though the graded regular symplectomorphism class of $M_0(A)$ is certainly not determined by $k$ and $\eta(A)$, the situation is more subtle, and such a classification would require very different techniques. In Section 5 we indicate this with examples of symplectic quotients associated to Type $\Pi_k$ weight matrices that fail to be graded regularly symplectomorphic, though the corresponding complex algebraic varieties are graded Poisson isomorphic, and hence the Hilbert series of real regular functions coincide.
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2. Background on torus representations

In this section, we give a brief overview of the structures associated to (real linear) symplectic quotients by tori, specializing the constructions described in the Introduction. We refer the reader to [2, 4] for more details.

Let $G = T^r$ and let $V$ be a unitary $G$-module with $\dim_C V = n$. Choosing a basis with respect to which the action of $G$ is diagonal and letting $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ denote coordinates for $V$ with respect to this basis, the action of $G$ is given by

$$tz := \left( t_1^{a_{11}} t_2^{a_{21}} \cdots t_ℓ^{a_{ℓ1}} z_1, t_1^{a_{12}} t_2^{a_{22}} \cdots t_ℓ^{a_{ℓ2}} z_2, \ldots, t_1^{a_{1n}} t_2^{a_{2n}} \cdots t_ℓ^{a_{ℓn}} z_n \right)$$

where $t = (t_1, t_2, \ldots, t_ℓ) \in G$ and $A = (a_{ij}) \in \mathbb{Z}^{ℓ\times n}$ is the weight matrix of the representation. Given a weight matrix $A \in \mathbb{Z}^{ℓ\times n}$, we let $V_A$ denote the $n$-dimensional representation of $T^ℓ$ with weight matrix $A$ along with the corresponding basis for $V_A$. We let $\langle \cdot, \cdot \rangle$ denote the standard hermitian scalar product on $V_A$ corresponding to this basis.

Letting $a_j$ denote the $j$th column of $A$ so that $A = (a_1, \ldots, a_n)$, it will be convenient to define

$$t^{a_j} := t_1^{a_{1j}} t_2^{a_{2j}} \cdots t_ℓ^{a_{ℓj}}$$

so that the action is given by

$$tz = (t^{a_1} z_1, t^{a_2} z_2, \ldots, t^{a_n} z_n).$$

Row-reducing $A$ over $\mathbb{Z}$ corresponds to changing coordinates $(t_1, \ldots, t_2)$ for $G$, so we may assume that $A$ is in reduced echelon form over $\mathbb{Z}$. Similarly, permuting the columns of $A$ corresponds to reordering the basis for $V_A$.

With respect to the symplectic form given by $\omega(z, z') = \text{Im} \langle z, z' \rangle$, the action of $G$ on $V_A$ is Hamiltonian and admits a unique homogeneous quadratic moment map $J_A : V_A \to \mathfrak{g}^*$; we will write $J = J_A$ when there is no potential for confusion. Identifying the Lie algebra $\mathfrak{t}$ of $T^ℓ$ with $\mathbb{R}^\ell$ using a basis for $\mathfrak{t}^\ell$ corresponding to the coordinates $(t_1, \ldots, t_ℓ)$ for $T^ℓ$ and the dual basis for $(\mathfrak{t}^\ell)^*$, $J = (J_1, \ldots, J_ℓ)$ can be expressed in terms of the component functions

$$J_i : V_A \to \mathbb{R}, \quad J_i(z) := \frac{1}{2} \sum_{j=1}^{n} a_{ij} z_j \bar{z}_j, \quad j = 1, \ldots, ℓ.$$

As the action of $T^ℓ$ on $\mathfrak{t}$ is trivial, each component $J_i$ is $T^ℓ$-invariant. Then the shell $Z = Z_A := J^{-1}(0)$ is the $T^ℓ$-stable real algebraic variety in $V_A$ corresponding to this family of quotients. The (real) symplectic quotient $M_0 = M_0(A) := Z_A/T^ℓ$. The algebra of smooth functions $C^∞(M_0)$ is defined by $C^∞(M_0) := C^∞(V)^G/\mathcal{I}_Z^G$ where $\mathcal{I}_Z$ is the vanishing ideal of $Z$ in $C^∞(V)$ and $\mathcal{I}_Z^G := \mathcal{I}_Z \cap C^∞(V)^G$. The algebra $C^∞(M_0)$ inherits a Poisson structure from $C^∞(V)$, where the Poisson bracket is given on coordinates by $\{ z_i, \bar{z}_j \} = -2\sqrt{-1} \delta_{ij}$, see [1].
Equipped with the algebra \(C^\infty(M_0)\) and its Poisson structure, \(M_0\) is a Poisson differential space, see [2] Definition 5].

The algebra of real regular functions \(\mathbb{R}[M_0]\) on \(M_0\) is defined in terms of the real polynomial invariants \(\mathbb{R}[V]^G\). Specifically, \(\mathbb{R}[M_0] := \mathbb{R}[V]^G/I_2^G\) where \(I_2^G := I_2^G \cap \mathbb{R}[V]^G\). The ideal \(I_2^G\) is homogeneous with respect to the grading of \(\mathbb{R}[V]\) by total degree so that \(\mathbb{R}[M_0]\) is a graded algebra; it is as well a Poisson subalgebra of \(C^\infty(M_0)\). We refer to elements of \(\mathbb{R}[V]^G\) as off-shell invariants and the corresponding classes in \(\mathbb{R}[M_0]\) as on-shell invariants. Note that for \(i = 1, \ldots, n\), the real polynomials \(z_i\) are always invariant. We will take advantage of the complex coordinate system on \(V\) for convenience, often expressing \(\mathbb{R}[V]^G\) in terms of polynomials in the \(z_i\) and \(\overline{z_i}\). By this, we mean that the real and imaginary parts of these polynomials are elements of \(\mathbb{R}[V]^G\). Note that the real invariants \(\mathbb{R}[V]^G\) can be computed in terms of the complexification \(V \otimes_{\mathbb{R}} \mathbb{C}\) of \(V\) by [14] Proposition 5.8(1), and \(V \otimes_{\mathbb{R}} \mathbb{C}\) is isomorphic as a \(\mathbb{T}^\ell\)-module to \(V \oplus V^*\).

In this paper, we are primarily interested in the symplectic quotients \(M_0(A)\) associated to weight matrices of a specific form, which we now define.

**Definition 2.1.** We say that an \(\ell \times (\ell + k)\) weight matrix \(A\) is of **Type I\(_k\)** if it is of the form \(A = [D, n_1, \ldots, n_{\ell}]\) where \(D = \text{diag}(-a_1, -a_2, \ldots, -a_\ell)\) with each \(a_i > 0\) and \(n = (n_1, n_2, \ldots, n_\ell)^T\) with each \(n_i > 0\). We will say that \(A\) is of **Type II\(_k\)** if \(A = [D, c_1 n_1, \ldots, c_k n_k]\) with \(D\) and \(n\) as above and each \(c_i \geq 1\). Note that a Type I\(_k\) weight matrix is Type II\(_k\) with each \(c_i = 1\). For a Type II\(_k\) weight matrix, we define

\[
\alpha(A) := \text{lcm}(a_1, \ldots, a_\ell), \quad m_i(A) := \frac{n_i \alpha(A)}{a_i} \quad \text{for} \quad i = 1, \ldots, \ell, \\
\beta(A) := \sum_{i=1}^{\ell} m_i(A), \quad \text{and} \quad \eta(A) := \alpha(A) + \beta(A).
\]

We will often abbreviate \(\alpha(A)\), \(m_i(A)\), \(\beta(A)\), and \(\eta(A)\) as \(\alpha\), \(m_i\), \(\beta\), and \(\eta\), respectively, when \(A\) is clear from the context.

For a weight matrix \(A\) of full rank, the representation \(V_A\) being faithful is equivalent to the nonzero \(\ell \times \ell\) minors of \(A\) having no common factor, see [2]. If \(A\) is of Type II\(_k\), then these minors are of the form \(a_1 \cdots a_r a_{r+1} \cdots a_\ell\) or \(a_1 \cdots a_{j-1} c_i n_j a_{j+1} \cdots a_\ell\) for some \(r = 1, \ldots, k\), i.e. the product of the \(a_i\) or the same product with one \(a_j\) replaced with \(c_i n_j\). The following is an immediate consequence.

**Lemma 2.2.** Let \(A\) be a Type II\(_k\) weight matrix. Then \(V_A\) is a faithful \(\mathbb{T}^\ell\)-module if and only if \(\gcd(a_i, a_j) = 1\) for each \(1 \leq i < j \leq n\), and for each \(j = 1, \ldots, \ell\), there is an \(r \leq k\) such that \(\gcd(a_j, c_r n_j) = 1\).

For a Type I\(_k\) or Type II\(_k\) weight matrix \(A\), the corresponding representation \(V_A\) of the complexification \(\mathbb{T}^\ell\mathbb{C} = (\mathbb{C}^\times)^\ell\) is stable and hence 1-large, see [5] for this result and the definitions. Then by [5] Corollary 4.3], the ideal \(I_Z\) is generated by the components \(J_i\) of the moment map.

Because the \(J_i\) are \(G\)-invariant in the case under consideration, we have

\[
\mathbb{R}[M_0] = \mathbb{R}[V]^G / (J_1, \ldots, J_\ell).
\]

In particular, given Equation (2.1), the quotient map \(\mathbb{R}[V]^G \to \mathbb{R}[M_0]\) can be understood as defining the invariants \(z_i\) for \(i = 1, \ldots, \ell\) in terms of the \(z_i\) for \(i = \ell + 1, \ldots, \ell + k\).
3. Proof of Theorem 1.2

In this section, we give the proof of our main result, Theorem 1.2 which is divided into several auxiliary results. Throughout this section, we consider a Type II \( k \) weight matrix \( A = [D, c_1 n, \ldots, c_k n] \in \mathbb{Z}^{\ell \times (\ell + k)} \) such that \( V_A \) is a faithful \( T^\ell \)-module of dimension \( n = \ell + k \). In addition, we let \( B = (-\alpha(A), c_1 \beta(A), \ldots, c_k \beta(A)) \in \mathbb{Z}^{1 \times (k + 1)} \). We assume throughout this section that \( \ell > 1 \); when \( \ell = 1 \), \( A = B \) so that Theorem 1.2 is trivial.

Our first result demonstrates that the \( T^1 \)-representation \( V_B \) is faithful.

Lemma 3.1. Let \( A = [D, c_1 n, \ldots, c_k n] \in \mathbb{Z}^{\ell \times (\ell + k)} \) be a Type II \( k \) weight matrix. If \( V_A \) is a faithful \( T^\ell \)-module, then \( \gcd(c_1 \beta(A), \ldots, c_k \beta(A)) = 1 \).

Proof. Suppose \( V_A \) is faithful, and let \( p \) be a prime that divides \( \alpha \) and each \( c_i \beta \) for contradiction. As \( p \) divides \( \alpha \), it divides some \( a_j \); assume \( p \mid a_1 \) without loss of generality. By Lemma 2.2 it is not possible that \( p \mid c_r \) for all \( r \), so it must be that \( p \mid \beta \). Similarly, \( p \mid a_i \) for each \( i \neq 1 \). Then \( p \mid m_i = n_i \alpha/a_1 \) for \( i > 1 \), so the fact that \( p \mid \beta = \sum m_i \) implies that \( p \mid m_1 \). But as \( p \) does not divide any \( a_i \) except \( a_1 \), we have \( \gcd(p, \alpha/a_1) = 1 \). Hence \( p \mid n_1 \). As \( p \mid a_1 \) and \( p \mid n_1 \), \( p \) divides the first row of \( A \), contradicting the fact that \( V_A \) is a faithful \( T^\ell \)-module. \( \square \)

Lemma 3.2. The function \( \phi: V_B \to V_A \) defined by

\[
\phi: (z_1, \ldots, z_{k+1}) \mapsto \left( \frac{m_1}{\beta} z_1, \frac{m_2}{\beta} z_1, \ldots, \frac{m_{k+1}}{\beta} z_1, z_2, z_3, \ldots, z_{k+1} \right)
\]

is a symplectic embedding that maps the shell \( Z_B = J_B^{-1}(0) \) into the shell \( Z_A = J_A^{-1}(0) \).

Proof. Using coordinates \((u_1, \ldots, u_n)\) for \( V_A \), we have

\[
\phi^* \sum_{i=1}^n du_i \wedge du_i = \sum_{i=1}^{\ell} \frac{m_i}{\beta} dz_i \wedge dz_i + \sum_{i=2}^{k+1} dz_i \wedge dz_i = \sum_{i=1}^{k+1} dz_i \wedge dz_i
\]

so that \( \phi \) is a symplectic embedding.

Suppose \( z = (z_1, \ldots, z_{k+1}) \in Z_B \) so that

\[
(3.1) \quad -\alpha z_1 \bar{z}_1 + \beta \sum_{j=1}^{k} c_j z_{j+1} \bar{z}_{j+1} = 0.
\]

Then for each \( i = 1, \ldots, \ell \), we have that

\[
(J_A)_i(\phi(z)) = -\frac{a_i m_i}{2 \beta} z_1 \bar{z}_1 + \frac{n_i}{2} \sum_{j=1}^{k} c_j z_{j+1} \bar{z}_{j+1}
\]

\[
= -\frac{n_i \alpha}{2 \beta} z_1 \bar{z}_1 + \frac{n_i}{2} \sum_{j=1}^{k} c_j z_{j+1} \bar{z}_{j+1}
\]

\[
= \frac{n_i}{2 \beta} \left( -\alpha z_1 \bar{z}_1 + \beta \sum_{j=1}^{k} c_j z_{j+1} \bar{z}_{j+1} \right) = 0.
\]

Hence, \( \phi \) maps \( Z_B \) into \( Z_A \). \( \square \)

Complexifying the underlying real spaces, we consider the \( z_i \) and \( w_i := \bar{z}_i \) as independent complex coordinates for \( V_B \otimes \mathbb{R} \mathbb{C} \) and \( u_i \) and \( v_i := \bar{w}_i \) as independent complex coordinates.
for $V_A \otimes_{\mathbb{R}} \mathbb{C}$. Let $N_B$ denote the complex shell $(J_B \otimes_{\mathbb{R}} \mathbb{C})^{-1}(0) \subset V_B \otimes_{\mathbb{R}} \mathbb{C}$, i.e. the set of $(z_1, \ldots, z_{k+1}, w_1, \ldots, w_k) \in V_B \otimes_{\mathbb{R}} \mathbb{C}$ such that

$$-\alpha z_1w_1 + \beta \sum_{j=1}^{k} c_j z_{j+1}w_{j+1} = 0.$$  \hfill (3.2)

Similarly, the complex shell $N_A = (J_A \otimes_{\mathbb{R}} \mathbb{C})^{-1}(0) \subset V_A \otimes_{\mathbb{R}} \mathbb{C}$ is defined by

$$-a_i u_i v_i + n_i \sum_{j=1}^{k} c_j u_{\ell+j}v_{\ell+j} = 0 \quad \text{for } i = 1, \ldots, \ell.$$  \hfill (3.3)

Recall that if $G$ is a connected algebraic group and $X$ is an irreducible $G$-variety, then a subvariety $Y \subset X$ is a Seshadri section if $GY = X$ for each irreducible component $Y_0$ of $Y$, and $Gy \cap Y = N(Y)y$ for any $y \in Y$, where $N(Y) = \{ g \in G \mid gY = Y \}$. By [11] Corollary, page 169] and [11] Theorem 3.14, if $X$ is normal, and a Seshadri section $Y$ satisfies $\text{codim}_X (X \setminus gY) \geq 2$, then $Y$ is a Chevalley section, i.e. restriction of functions to $Y$ defines an isomorphism $\mathbb{C}[X]^G \rightarrow \mathbb{C}[Y]^{N(Y)}$.

We now demonstrate that these hypotheses are satisfied, i.e. the image of $N_B$ under $\phi_C = \phi \otimes_{\mathbb{R}} \mathbb{C}$ is a Seshadri section for the action of $(\mathbb{C}^*)^\ell$ on $N_A$.

**Lemma 3.3.** The image $S := \phi_C(N_B)$ of the complex shell $N_B$ is a Seshadri section for the action of $(\mathbb{C}^*)^\ell$ on the complex shell $N_A \subset V_A \otimes_{\mathbb{R}} \mathbb{C}$. Moreover, the (complex) codimension of $N_A \setminus (\mathbb{C}^*)^\ell S$ in $N_A$ is 2.

**Proof.** First observe that $S$ is given by the set of points in $V_A \otimes_{\mathbb{R}} \mathbb{C}$ given by

$$\left( \sqrt{\frac{m_1}{\beta}} z_1, \sqrt{\frac{m_2}{\beta}} z_1, \ldots, \sqrt{\frac{m_\ell}{\beta}} z_1, z_2, \ldots, z_{k+1}, \sqrt{\frac{m_1}{\beta}} w_1, \sqrt{\frac{m_2}{\beta}} w_1, \ldots, \sqrt{\frac{m_\ell}{\beta}} w_1, w_2, w_3, \ldots, w_k \right)$$

for some $z_i$ and $w_i$ that satisfy Equation (3.2). As the actions of $\mathbb{C}^\times$ and $(\mathbb{C}^*)^\ell$ on $V_B \otimes_{\mathbb{R}} \mathbb{C}$ and $V_A \otimes_{\mathbb{R}} \mathbb{C}$, respectively, are stable and hence 1-large by [5] Proposition 3.1, both $N_A$ and $N_B$ are reduced and irreducible by [5] Theorem 2.2 (3)].

Fix a point $(u, v) \in N_A$, i.e. satisfying Equation (3.3), and assume that each $u_i \neq 0$ for $i \leq \ell$. For $i = 2, \ldots, \ell$, choose $t_i$ such that

$$t_i^{-a_i} = \sqrt{\frac{m_i}{m_1}} u_1 = \sqrt{\frac{m_i}{m_1}} u_i,$$

i.e. $\sqrt{m_i} t_i^{-a_i} u_1 = \sqrt{m_i} u_i$.

Let $z_1 := u_1 \sqrt{\beta/m_1}$, and then

$$\sqrt{\frac{m_i}{\beta}} z_1 = \sqrt{\frac{m_i}{m_1}} u_1 = t_i^{-a_i} u_i.$$  \hfill (3.4)

Similarly, by Equation (3.3), each $v_i$ with $i = 1, \ldots, \ell$ is given by

$$v_i = \frac{n_i}{a_i u_1} \sum_{j=1}^{k} c_j u_{\ell+j}v_{\ell+j}.$$  \hfill (3.5)

Letting

$$w_1 = \frac{\sqrt{m_1} \beta}{\alpha u_1} \sum_{j=1}^{k} c_j u_{\ell+j}v_{\ell+j},$$
we have
\[ v_1 = \frac{m_1}{\alpha u_1} \sum_{j=1}^{k} c_j \alpha u_1 v_{\ell + j} = \sqrt{\frac{m_1}{\beta}} w_1, \]
and, for \( i = 2, \ldots, \ell, \)
\[ t_i^{a_i} v_i = \sqrt{\frac{m_1 m_i}{\alpha u_1}} \sum_{j=1}^{k} c_j \alpha u_1 v_{\ell + j} = \sqrt{\frac{m_i}{\beta}} w_i. \]
Hence, letting \( t = (1, t_2, \ldots, t_\ell) \in (\mathbb{C}^\times)^\ell \) and defining \( z_{i+1} = t^{c_{i+1} n} u_{i+1} \) and \( w_{i+1} = t^{-c_{i+1} n} v_{i+1} \)
for \( i = 1, \ldots, k, \) we have
\[ t(u, v) = \left( \sqrt{\frac{m_1}{\beta}}, \ldots, \sqrt{\frac{m_2}{\beta}}, \ldots, \sqrt{\frac{m_k}{\beta}}, \sqrt{\frac{m_1}{\beta}} \right). \]
Moreover,
\[ -\alpha z_1 w_1 + \beta \sum_{j=1}^{k} c_j z_{j+1} w_{j+1} = -\beta \sum_{j=1}^{k} c_j \alpha u_1 v_{\ell + j} + \beta \sum_{j=1}^{k} c_j z_{j+1} w_{j+1} = 0, \]
so that \( t(u, v) \in S. \) That is, any point \((u, v) \in N_A \) with each \( u_i \neq 0 \) for \( i \leq \ell \) is in the
\( (\mathbb{C}^\times)^\ell \)-orbit of a point in \( S. \) Note that if each \( v_i \neq 0, \) then we can define
\[ t_i^{a_i} = \sqrt{\frac{m_i}{m_1}} v_i \]
for \( i = 2, \ldots, \ell \) and again obtain \( t(u, v) \in S. \) Taking the closure to account for points with
some \( u_i = 0 \) or \( v_i = 0 \) for \( i \leq \ell, \) we have
\[ \overline{(\mathbb{C}^\times)^\ell S} = N_A. \]
In particular, note that \( N_A \setminus (\mathbb{C}^\times)^\ell S \) consists of those points in \( N_A \) where some \( u_i = 0 \) and
some \( v_j = 0 \) for \( i, j \leq \ell; \) in particular \( N_A \setminus (\mathbb{C}^\times)^\ell S \) is closed and has codimension 2 in \( N_A. \)

Now, recall the definition \( N(S) = \{ t \in (\mathbb{C}^\times)^\ell \mid tS = S \}. \) We claim that \( N(S) = \{ (t^{a_1/\alpha_1}, \ldots, t^{a_{\ell}/\alpha_{\ell}}) \mid t \in \mathbb{C}^\times \}. \) Let
\[ (z, w) = \left( \sqrt{\frac{m_1}{\beta}}, \ldots, \sqrt{\frac{m_k}{\beta}}, \sqrt{\frac{m_1}{\beta}}, \sqrt{\frac{m_1}{\beta}}, \ldots, \sqrt{\frac{m_k}{\beta}} \right) \in S, \]
and suppose \( t \in (\mathbb{C}^\times)^\ell \) such that \( t(z, w) \in S. \) We have
\[ t(z, w) = \left( \sqrt{\frac{m_1}{\beta}} t_{1}^{-a_1} z_1, \ldots, \sqrt{\frac{m_k}{\beta}} t_{\ell}^{-a_\ell} z_\ell, t^{c_1 n} z_2, \ldots, t^{c_{k+1} n} z_{k+1}, \right) \]
\[ \left( \sqrt{\frac{m_1}{\beta}} t_{1}^{a_1} w_1, \ldots, \sqrt{\frac{m_k}{\beta}} t^{a_\ell} w_{\ell}, t^{-c_1 n} w_2, \ldots, t^{-c_{k+1} n} w_{k+1} \right). \]
If \( z_1 \neq 0 \) or \( w_1 \neq 0, \) we have \( t_{1}^{a_1} = t_{i}^{a_i} \) for each \( i. \) Choosing \( t \in \mathbb{C}^\times \) such that \( t^{a_1/\alpha_1} = t_1 \)
and noting that \( \gcd(\alpha_1/\alpha_1, \ldots, \alpha_{\ell}/\alpha_{\ell}) = 1 \) by construction, it follows that \( t \) is of the form
\( (t^{a_1/\alpha_1}, \ldots, t^{a_{\ell}/\alpha_{\ell}}). \) Note that for any such \( t, \) we have \( tS = S \) so that \( N(S) = \{ (t^{a_1/\alpha_1}, \ldots, t^{a_{\ell}/\alpha_{\ell}}) \mid t \in \mathbb{C}^\times \}. \)

If \( z_1 = w_1 = 0, \) we have \( \sum_{j=1}^{k} c_j z_{j+1} w_{j+1} = 0. \) Then
\[ t(z, w) = (0, \ldots, 0, t^{c_1 n} z_2, \ldots, t^{c_{k+1} n} z_{k+1}, 0, \ldots, 0, t^{-c_1 n} w_2, \ldots, t^{-c_{k+1} n} w_{k+1}). \]
Choosing an \( s \in \mathbb{C}^\times \) such that \( s^\beta = t^n \), we have
\[
(s^{a_1}, \ldots, s^{a_{k-1}})(z, w) = (0, 0, 0, s^{c_1}z_2, \ldots, s^{c_{k-1}}z_{k+1}),
\]
so that \( (\mathbb{C}^\times)^\ell(z, w) \subset \mathbb{N}(S)(z, w) \). □

As \( S \) is a Seshadri section for the action of \( (\mathbb{C}^\times)^\ell \) on \( N_A \) such that the codimension of \( \overline{N_A \setminus S} \) in \( N_A \) is 2, we have that the restriction of functions to \( S \) defines an isomorphism \( \mathbb{C}[N_A]((\mathbb{C}^\times)^\ell) \to \mathbb{C}[S]^{\mathbb{N}(S)} \) by [10] Corollary, page 169; see also [11] Theorem 3.14. Note that \( \mathbb{N}(S) \) acts on the subspace of \( V_A \) spanned by \((1, \ldots, 1, 0, \ldots, 0)\) and the standard unit vectors \( e_i \) for \( i > \ell \) with weight vector \(-\alpha, c_1 \beta, \ldots, c_k \beta\). Then as \( S \) is isomorphic to the shell \( N_B \) via the embedding \( \phi_C \), it follows that \( \phi_C \) induces an isomorphism \( \phi_C^*: \mathbb{C}[S]^{\mathbb{N}(S)} \to \mathbb{C}[N_B]((\mathbb{C}^\times)^\ell) \). As \( \phi_C \) is a linear map, \( \phi_C^* \) preserves the grading. Then by Lemma 3.2, as the representations of \( (\mathbb{C}^\times)^\ell \) and \( \mathbb{C}^\times \) corresponding to \( A \) and \( B \), respectively, are 1-large, we have that \( \mathbb{R}[Z_A]^T \otimes \mathbb{C} \simeq \mathbb{C}[N_A]((\mathbb{C}^\times)^\ell) \) and \( \mathbb{R}[Z_B]^T \otimes \mathbb{C} \simeq \mathbb{C}[N_B]((\mathbb{C}^\times)^\ell) \). That is, \( \phi^* \) induces a graded isomorphism of the algebras of real regular functions \( \mathbb{R}[M_0(A)] \to \mathbb{R}[M_0(B)] \). By Lemma 3.2, this isomorphism is Poisson.

Summarizing, we have the following.

**Corollary 3.4.** The restriction of functions to \( S \) and pulling back via \( \phi_C \) are both graded isomorphisms
\[
\mathbb{C}[N_B]((\mathbb{C}^\times)^\ell) \xrightarrow{\phi_C^*} \mathbb{C}[S]^{\mathbb{N}(S)} \xrightarrow{\phi_C^*} \mathbb{C}[N_A]((\mathbb{C}^\times)^\ell),
\]
and the composition of these maps induces a graded Poisson isomorphism of the real algebras
\[
\Psi: \mathbb{R}[M_0(A)] \to \mathbb{R}[M_0(B)].
\]

By Lemmas 3.2 and 3.3 and Corollary 4.4, it follows that \( \phi \) induces an isomorphism between the Zariski closures of the real algebraic varieties defined by \( \mathbb{R}[Z_A]^T \) and \( \mathbb{R}[Z_B]^T \). To complete the proof of Theorem 1.2, it remains only to show that the semialgebraic conditions are preserved, i.e. the map \( \phi \) induces a homeomorphism between the symplectic quotients.

**Lemma 3.5.** The map \( \phi \) induces a homeomorphism \( M_0(B) = Z_B/T^1 \to M_0(A) = Z_A/T^\ell \).

**Proof.** It is clear that \( \phi \) maps \( T^1 \)-orbits into \( T^\ell \)-orbits, as if \( z = (z_1, \ldots, z_{k+1}) \in Z_B \) and \( t \in T^1 \), then
\[
\phi(tz) = \phi(t^{-\alpha}z_1, t^{c_1\beta}z_2, \ldots, t^{c_k\beta}z_{k+1})
\]
\[
= \left( \frac{m_1}{\beta} t^{-\alpha}z_1, \ldots, \frac{m_\ell}{\beta} t^{-\alpha}z_1, t^{\beta}z_2, \ldots, t^{\beta}z_{k+1} \right)
\]
\[
= \left( \frac{m_1}{\beta} (t^{\alpha/a_1})^{-a_1}z_1, \ldots, \frac{m_\ell}{\beta} (t^{\alpha/a_1})^{-a_1}z_1, (t^{\alpha/a_1})^{c_1}z_2, \ldots, (t^{\alpha/a_1})^{c_k}z_{k+1} \right)
\]
\[
= (t^{\alpha/a_1}, \ldots, t^{\alpha/a_1})\phi(z).
\]
As \( \phi(Z_B) \subset (Z_A) \) by Lemma 3.2, it is sufficient to show that each element of \( Z_A \) is in the orbit of an element of \( \phi(Z_B) \). So let \( \mathbf{u} = (u_1, \ldots, u_n) \in Z_A \) so that for \( i = 1, \ldots, \ell \),

\[
-a_i u_i u_i + n_i \sum_{j=1}^{k} c_{j} u_{\ell+j} u_{\ell+j} = 0, \quad \text{i.e.} \quad \frac{a_i}{n_i} u_i u_i = \sum_{j=1}^{k} c_{j} u_{\ell+j} u_{\ell+j}.
\]

As each \( a_i, n_i, c_j > 0 \), it follows that if \( u_i = 0 \) for some \( i \leq \ell \), then \( u_i = 0 \) for each \( i > \ell \), i.e. \( \mathbf{u} = \mathbf{0} = \phi(\mathbf{0}) \). Hence, we may assume each \( u_i \) is nonzero. Then for \( i = 2, \ldots, \ell \), we have

\[
|u_i| = \sqrt{\frac{a_1 n_1}{a_i n_i}} |u_1| = \sqrt{\frac{m_i}{m_1}} |u_1|.
\]

Hence for \( i = 2, \ldots, \ell \), there is a \( t_i \in T^1 \) such that

\[
t_i^{-a_i} u_i = \sqrt{\frac{m_i}{m_1}} u_1.
\]

Then setting \( \mathbf{t} := (1, t_2, \ldots, t_{\ell}) \), \( z_1 := u_1 \sqrt{\beta/m_1} \), and \( z_{i+1} := t_1^{c_i} u_{\ell+i} \) for \( i > 1 \), we have that

\[
\mathbf{t}(u_1, \ldots, u_{n}) = (u_1, t_2^{-a_2} u_2, \ldots, t_{\ell}^{-a_\ell} u_\ell, t_1^{c_1} u_{\ell+1}, \ldots, t_1^{c_\ell} u_{n})
\]

as each \( \mathbf{u} \) is Type II. Hence, we may assume each \( u_i \) is nonzero. Then for \( i = 2, \ldots, \ell \), we have

\[
|u_i| = \sqrt{\frac{a_1 n_1}{a_i n_i}} |u_1| = \sqrt{\frac{m_i}{m_1}} |u_1|.
\]

Then setting \( \mathbf{t} := (1, t_2, \ldots, t_{\ell}) \), \( z_1 := u_1 \sqrt{\beta/m_1} \), and \( z_{i+1} := t_1^{c_i} u_{\ell+i} \) for \( i > 1 \), we have that

\[
\mathbf{t}(u_1, \ldots, u_{n}) = (u_1, t_2^{-a_2} u_2, \ldots, t_{\ell}^{-a_\ell} u_\ell, t_1^{c_1} u_{\ell+1}, \ldots, t_1^{c_\ell} u_{n})
\]

as each \( \mathbf{u} \) is Type II. Hence, we may assume each \( u_i \) is nonzero. Then for \( i = 2, \ldots, \ell \), we have

\[
|u_i| = \sqrt{\frac{a_1 n_1}{a_i n_i}} |u_1| = \sqrt{\frac{m_i}{m_1}} |u_1|.
\]

Finally, we note that \( (z_1, \ldots, z_{k+1}) \) satisfy Equation 3.2, as

\[
-\alpha z_1 z_1 + \beta \sum_{j=1}^{k} c_j z_{j+1} z_{j+1} = -\frac{\beta}{m_1} u_1 u_1 + \beta \sum_{j=1}^{k} c_j u_{\ell+j} u_{\ell+j}
\]

so that \( \mathbf{t}(u_1, \ldots, u_{n}) \) is in the orbit \( \phi(Z_B) \). It follows that each \( T^\ell \)-orbit in \( Z_A \) intersects \( \phi(Z_B) \).

We leave it to the reader to show that the inverse homeomorphism is induced by the linear map

\[
(u_1, u_2, \ldots, u_{k+\ell}) \mapsto (\sqrt{\frac{\beta}{m_1}} u_1, u_{\ell+1}, \ldots, u_{k+\ell})
\]

\[ \square \]

Example 3.6. The weight matrix

\[
A = \begin{pmatrix}
-3 & 0 & 0 & 1 & 2 & 3 & 3 \\
0 & -4 & 0 & 3 & 6 & 9 & 9 \\
0 & 0 & -5 & 2 & 4 & 6 & 6
\end{pmatrix}
\]

is Type II with \( \alpha = 60, n_1 = 1, n_2 = 3, n_3 = 2, c_1 = 1, c_2 = 2, \) and \( c_3 = c_4 = 3 \). Hence, \( m_1 = 20, m_2 = 45, m_3 = 24, \) and \( \beta = 89, \) and the symplectic quotient \( M_0(A) \) is graded regularly symplectomorphic to that associated to \((-60, 89, 178, 267, 267)\).
4. Constructive Approach to Theorem 1.2

We first obtained a proof of Theorem 1.2 for Type I_k matrices by determining an explicit description of the symplectic quotient $M_0$ and algebra $\mathbb{R}[M_0]$ of regular functions. This description may be of independent interest and illustrates the structure of these spaces, so we include it here. The proofs of these results are cumbersome computations and hence only summarized.

**Proposition 4.1.** Let $A = [D, \mathbf{n}_1, \ldots, \mathbf{n}_k] \in \mathbb{Z}^{\ell \times (\ell + k)}$ be a type I_k weight matrix such that $V_A$ is a faithful $\mathbb{T}^\ell$-module. Then a generating set for the algebra $\mathbb{R}[V_A]^{\mathbb{T}^\ell}$ of invariants is given by

1. the $\ell$ quadratic monomials $r_i := z_{\mathbf{n}_i}$ for $i = 1, \ldots, \ell$,
2. the $k^2$ quadratic monomials $p_{i,j} := z_{\mathbf{n}_i \mathbf{n}_j}$ for $1 \leq i, j \leq k$,
3. the $(a^{+k-1})$ degree $\eta$ monomials $q_{\alpha} := \prod_{i=1}^{\ell} z_{\mathbf{n}_i}^{m_i} \prod_{i=1}^{k} z_{\ell+1}^{s_i}$ where $\alpha = (s_1, \ldots, s_k)$ and the $s_i$ are any choice of nonnegative integers such that $\sum_{i=1}^{k} s_i = \alpha$, and
4. the $(a^{+k-1})$ degree $\eta$ monomials $\ell_{\alpha}$ for each choice of $\alpha$.

For a generating set for $\mathbb{R}[M_0(A)]$, the generators in (1) can be omitted using the on-shell relations.

A simple computation demonstrates that each of the monomials listed in Proposition 4.1 is invariant. To prove the proposition, one first establishes the result when $k = 1$ by induction on $\ell$; the base case is simple, and the inductive step is accomplished by comparing the invariants of $A$ to those corresponding to submatrices formed by removing a single row and the resulting column of zeros. For general $k$, consider the map $\phi: \mathbb{R}[z_1, \ldots, z_{\ell+k}, w_1, \ldots, z_{\ell+k+1} \mathbb{T}^\ell] \to \mathbb{R}[w_1, \ldots, w_{\ell+1}, \mathbb{T}^\ell]$, which is of degree $1$. It is easy to see that $\phi$ maps $A$-invariants onto $[D, \mathbf{n}]$-invariants, and then the proof is completed by considering the preimages of the $[D, \mathbf{n}]$-invariants, a case with $k = 1$.

**Proposition 4.2.** Let $A = [D, \mathbf{n}_1, \ldots, \mathbf{n}_k] \in \mathbb{Z}^{\ell \times (\ell + k)}$ be a type I_k weight matrix such that $V_A$ is a faithful $\mathbb{T}^\ell$-module. The (off-shell) relations among the $r_i$, $p_{i,j}$, $q_{\alpha}$, and $\ell_{\alpha}$ are generated by the following.

1. $p_{g,h} p_{i,j} - p_{g,j} p_{i,h}$ for $1 \leq g, h, i, j \leq k$ with $g \neq i$ and $h \neq j$.
2. $p_{g,h} q_{\alpha} - p_{h,i} q_{\beta}$ where $s'_g = s_g + 1$, $s'_i = s_i - 1$, and $s'_j = s_j$, for $j \neq g, i$. Note that we must have $s_i \geq 1$.
3. $p_{g,h} q_{\alpha} - p_{h,i} q_{\beta}$ where $s'_g = s_g + 1$, $s'_i = s_i - 1$, and $s'_j = s_j$, for $j \neq g, i$. Note that we must have $s_i \geq 1$.
4. $q_{\alpha} q_{\beta} - q_{\alpha} q_{\beta}$ where $s + s' = t + t'$ and $s \neq t$.
5. $q_{\alpha} q_{\beta} - q_{\alpha} q_{\beta}$ where $s + s' = t + t'$ and $s \neq t$.
6. $\prod_{i=1}^{\ell} r_{g}^{m_i} \prod_{j=1}^{\alpha} p_{g,h} - q_{\alpha} q_{\beta}$ where the vector $(g_1, \ldots, g_\alpha)$ contains each value $g$ exactly $s_g$ times and the vector $(h_1, \ldots, h_\alpha)$ contains each value $h$ exactly $s'_h$ times.

On-shell, the monomials additionally satisfy the defining relations of the moment map, $-a_i r_i + n_i \sum_{j=1}^{k} p_{i,j}$ for $i = 1, \ldots, \ell$.

One verifies that each of these relations holds by direct computation using the definitions of the monomials given in Proposition 4.1. The proof that all relations are generated by these is by induction on $k$. For the case $k = 1$, there is only one nontrivial relation, $p_{g,1}^{m_i} \prod_{j=1}^{\ell} r_{i}^{m_j} - q_{\alpha}(q_{\beta})$, a simple computation of cases demonstrates that this generates all relations. The induction step is demonstrated by considering the preimages of monomials under the map $C[z_1, \ldots, z_{\ell+k+1}] \to C[z_1, \ldots, z_{\ell+k}]$ given by $(z_1, \ldots, z_{\ell+k+1}) \mapsto (z_1, \ldots, z_{\ell+k} + z_{\ell+k+1})$. One then verifies the following by direct computation.
Proposition 4.4. Let \( A = [D, \mathbf{n}, \ldots, \mathbf{n}] \in \mathbb{Z}^{T \times (\ell + k)} \) be a type \( I_k \) weight matrix such that \( V_A \) is a faithful \( \mathbb{T}^\ell \)-module. The Poisson algebra of regular functions. It remains only to determine the semialgebraic description of the symplectic quotient.

The above results give an explicit description of the Poisson algebra of regular functions. It remains only to determine the semialgebraic description of the symplectic quotient.

Proposition 4.4. Let \( A = [D, \mathbf{n}, \ldots, \mathbf{n}] \in \mathbb{Z}^{T \times (\ell + k)} \) be a type \( I_k \) weight matrix associated such that \( V_A \) is a faithful \( \mathbb{T}^\ell \)-module. Using the real Hilbert basis given by the real and imaginary parts of the monomials listed in Proposition 4.2, the image of the Hilbert embedding is described by the relations given in Proposition 4.2 as well as the inequalities \( r_i \geq 0 \) for \( i = 1, \ldots, \ell \) and \( p_j, q_j \geq 0 \) for \( j = 1, \ldots, k \).

From the definition of the monomials, it is easy to see that these inequalities are satisfied. For the converse, choose values of the \( r_i, p_{j,i} \), and \( q_s \) such that each \( r_i \geq 0 \), each \( p_{j,i} \geq 0 \), and the remaining values are arbitrary elements of \( \mathbb{C} \) such that the each \( p_{j,i} = \overline{p_{j,i}} \) and relations in Proposition 4.2 are satisfied. It is then easy to see that the values \( |r_i|, |p_{j,i}| \) for \( i \neq j \), and \( |q_s| \) are determined by the \( p_{j,i} \). Specifically, using the relations of Proposition 4.2(1), we have

\[ |p_{j,i}| = \sqrt{p_{j,i}p_{j,i}} \]

using the moment map, we have

\[ |r_i| = \frac{n_i}{a_i} \sum_{j=1}^{k} p_{j,i} \]

and using the relations of Proposition 4.2(6), we have

\[ q_s = \frac{\ell}{\prod_{i=1}^{\ell} \left( \frac{n_i}{a_i} \right)^{m_i} \left( \sum_{j=1}^{k} p_{j,i} \right)^{\sum_{i=1}^{\ell} m_i} \left( \prod_{j=1}^{k} \alpha_{i,j}^{p_{j,i}} \right)^{\alpha/2}} \]

Similarly, using the relations of Proposition 4.2(3), one checks that the arguments of the \( q_s \) where \( s \) has only one nonzero coordinate (which must be equal to \( \alpha \)) determine the arguments.
of the $p_{i,j}$ and the other $q_{s'}$. It follows that one can find a point $(z_1, \ldots, z_n)$ mapped via the Hilbert embedding to these values of $r_i, p_{i,j}$, and $q_s$ by choosing the modulus of each $z_{k+i}$ to be $\sqrt{-p_k}$, the modulus of each $z_i$ for $i \leq \ell$ to be determined by the moment map, the argument of each $z_i$ for $i \leq \ell$ to be 0, and the argument of each $z_{k+i}$ to be the argument of $q(0,0,0,0,\ldots,0)$ where $\alpha$ occurs in the $i$th position.

With this explicit description of $M_0(A)$ and $\mathbb{R}[M_0(A)]$ the following can be verified by explicit computation.

**Theorem 4.5.** Let $A \in \mathbb{Z}^{\ell \times (\ell+1)}$ be a Type I\(_k\) matrix such that $V_A$ is a faithful $\mathbb{T}^\ell$-module, and let $B = (-\alpha(A), c_1, \beta(A), \ldots, c_k, \beta(A)) \in \mathbb{Z}^{1 \times (k+1)}$. Using coordinates $(w_1, \ldots, w_{k+1})$ for $V_B$, define the map $\Phi: \mathbb{C}[V_A]^\ell \to \mathbb{C}[V_B]$ by

$$
\begin{align*}
  r_i &\mapsto \frac{m_i(A)}{\beta(A)} w_1 w_1, \quad 1 \leq i \leq \ell, \\
p_{i,j} &\mapsto w_{i+1} w_{i+1}, \quad 1 \leq i, j \leq k, \\
q_s &\mapsto \frac{\beta(A)^{-\beta(A)} m_j(A)^{m_j(A)}}{\prod_{j=1}^\ell m_j(A)} w_1 w_{j+1} w_{j+1}, \\
q_s &\mapsto \frac{\beta(A)^{-\beta(A)} m_j(A)^{m_j(A)}}{\prod_{j=1}^\ell m_j(A)} w_1 w_{j+1} w_{j+1}.
\end{align*}
$$

Then $\Phi$ is a well-defined homomorphism $\Phi: \mathbb{C}[V_A]^\ell \to \mathbb{C}[V_B]^1$ inducing an isomorphism $\mathbb{R}[M_0(A)] \to \mathbb{R}[M_0(B)]$ and a graded regular symplectomorphism between $M_0(A)$ and $M_0(B)$.

5. **Classification for Type I\(_k\) matrices**

In the case $k = 1$, Corollary 4.1 implies that two weight matrices $A_1$ and $A_2$ yield graded regularly symplectomorphic symplectic quotients if and only if $\eta(A_1) = \eta(A_2)$, i.e. if and only if $\alpha(A_1) + \beta(A_1) = \alpha(A_2) + \beta(A_2)$. For $k > 1$, this is no longer the case, as we demonstrate with the following.

**Lemma 5.1.** Let $A = (-\alpha, \beta, \ldots, \beta)$ and $B = (-\alpha', \beta', \ldots, \beta')$ such that $V_A$ and $V_B$ are faithful $\mathbb{T}^1$-modules. If the symplectic quotients $M_0(A)$ and $M_0(B)$ are graded regularly diffeomorphic for $k \geq 2$, then $k = k'$, $\alpha = \alpha'$ and $\beta = \beta'$.

*Proof.* First note that the fact that $V_A$ and $V_B$ are faithful implies that $\gcd(\alpha, \beta) = \gcd(\alpha', \beta') = 1$. The existence of a graded regular diffeomorphism implies that $\mathbb{R}[M_0(A)]$ is graded isomorphic to $\mathbb{R}[M_0(B)]$. As the Krull dimensions of $\mathbb{R}[M_0(A)]$ and $\mathbb{R}[M_0(B)]$ are given by $2k$ and $2k'$, respectively, it follows that $k = k'$.

Let $Q(A)$ denote the subalgebra of $\mathbb{R}[M_0(A)]$ that is generated by the quadratic monomials of the form $z_i z_i$ and $I_{\mathbb{Z}_A}$ for $i = 1, \ldots, k+1$ and $z_i z_{i+j}$ and $I_{\mathbb{Z}_A}$ for $1 \leq i, j \leq k$, and define $Q(B)$ similarly as a subalgebra of $\mathbb{R}[M_0(B)]$. Note that $Q(A)$ and $Q(B)$ are obviously graded isomorphic. The lowest-degree element of $\mathbb{R}[M_0(A)]$ that is not an element of $Q(A)$ has degree $\alpha + \beta$, and similarly for $\mathbb{R}[M_0(B)]$, so we can conclude that $\alpha + \beta = \alpha' + \beta'$.

Finally, the number of monomials in $\mathbb{R}[M_0(A)]$ of degree $\alpha + \beta$ that are not elements of $Q(A)$ is $(\alpha + k - 1)$, and hence $(\alpha + k - 1)$ is $(\alpha' + k - 1)$, i.e. $(\alpha + k - 1)!/\alpha! = (\alpha' + k - 1)!/\alpha'!$. As $k > 1$, it follows that $\alpha = \alpha'$, and hence $\beta = \beta'$. \(\Box\)
Corollary 5.2. The graded regular symplectomorphism classes of symplectic quotients associated to Type $I_k$ weight matrices with $k > 1$ are classified by the triple $(k, \alpha(A), \beta(A))$. Moreover, these graded regular symplectomorphism classes coincide with the graded regular diffeomorphism classes.

It is not clear whether an analog to Lemma 5.1 is true for Type $II_k$ matrices, but a proof using only the grading of $R[M_0]$ as in Lemma 5.1 is not possible. First note that such a generalization would require restricting to specific representatives, e.g. requiring that $\gcd(c_1, \ldots, c_k) = 1$. Otherwise, it is possible that a $1 \times (k+1)$ Type $II_k$ matrix could be written in terms of $\alpha, \beta$, and the $c_i$ in more than one way, e.g. $(-1, 4, 12)$ could correspond to $\alpha = 1, \beta = 2, c_1 = 2$, and $c_2 = 3$ or to $\alpha = 1, \beta = 4, c_1 = 1$, and $c_2 = 3$. However, even with such a restriction, it is possible that $R[M_0(A)]$ and $R[M_0(B)]$ have the same Hilbert series yet fail to be graded regularly symplectomorphic. We will illustrate this in the next section.

6. The Hilbert Series Does Not Classify Symplectic Quotients by Tori

The graded regular symplectomorphisms given by Theorem 1.2 were initially discovered by computing Hilbert series of the algebras of regular functions on symplectic quotients associated to large classes of weight matrices and looking for cases that coincide. While the Hilbert series has been a valuable heuristic to indicate potential graded regular symplectomorphisms and an important tool to distinguish between non-graded regularly symplectomorphic cases, one would likely guess that there are cases with the same Hilbert series that are not graded regularly symplectomorphic. In this section, we give examples to indicate that this is the case: the Hilbert series has been a valuable heuristic to indicate potential graded regular symplectomorphisms associated to large classes of weight matrices and looking for cases that coincide. While the Hilbert series has been a valuable heuristic to indicate potential graded regular symplectomorphisms and an important tool to distinguish between non-graded regularly symplectomorphic cases, one would likely guess that there are cases with the same Hilbert series that are not graded regularly symplectomorphic. In this section, we give examples to indicate that this is the case: the Hilbert series is not a fine enough invariant to distinguish graded regularly symplectomorphic classes of symplectic quotients by tori. These examples further illustrate that two symplectic quotients can have several isomorphic structures yet fail to be graded regularly symplectomorphic.

Let $A = (-2, 3, 6)$ and $B = (-3, 2, 6)$. Note that these are both Type $II_2$ weight matrices; $A$ corresponding to $\alpha = 2, \beta = 3, c_1 = 1$, and $c_2 = 2$; and $B$ corresponding to $\alpha = 3, \beta = 2, c_1 = 1$, and $c_2 = 3$. Because the Hilbert series of symplectic quotients by $T^1$ only depends on the sign of the weights (see [8, page 47]), it is clear that the Hilbert series of $R[M_0(A)]$ and $R[M_0(B)]$ coincide. In particular, they are both given by

$$\frac{1 + t^3 + 2t^4 + t^5 + t^8}{(1 - t^3)(1 - t^5)(1 - t^2)^3}.$$  

The off-shell invariants $R[V_A]^{T^1}$ are generated by

\[
\begin{align*}
p_0 &= z_1 z_2, & p_1 &= z_2 z_3, & p_2 &= z_3 z_3, & p_3 &= z_2 z_2, & p_4 &= z_3 z_2, \\
p_5 &= z_3 z_3, & p_6 &= z_2 z_3, & p_7 &= z_3 z_2, & p_8 &= z_2 z_2,
\end{align*}
\]

and the moment map determines $p_0$ via $2p_0 = 3p_1 + 6p_2$. The off-shell invariants $R[V_B]^{T^1}$ are generated by

\[
\begin{align*}
q_0 &= u_1 u_1, & q_1 &= u_2 u_2, & q_2 &= u_3 u_3, & q_3 &= u_1 u_3, & q_4 &= u_1^2 u_1, & q_5 &= u_2 u_3, & q_6 &= u_3 u_2, & q_7 &= u_2 u_3, & q_8 &= u_1^2 u_2^2,
\end{align*}
\]

and the shell relation is given by $3q_0 = 2q_1 + 6q_2$.

Proposition 6.1. For the weight matrices $A = (-2, 3, 6)$ and $B = (-3, 2, 6)$, the following hold true.

(i.) The algebras $R[M_0(A)] \otimes_R \mathbb{C}$ and $R[M_0(B)] \otimes_R \mathbb{C}$ are graded Poisson isomorphic. Hence, the complex symplectic quotients are isomorphic as Poisson varieties.
(ii.) The algebras \( \mathbb{R}[M_0(A)] \) and \( \mathbb{R}[M_0(B)] \) are graded isomorphic. However, no graded isomorphism \( \mathbb{R}[M_0(A)] \rightarrow \mathbb{R}[M_0(B)] \) preserves the inequalities describing the semialgebraic sets \( M_0(A) \) and \( M_0(B) \).

An immediate consequence of (ii.) is that the symplectic quotients \( M_0(A) \) and \( M_0(B) \) are not graded regularly symplectomorphic.

**Proof of Proposition 6.1(ii.)** As in the proof of Lemma 3.3 we complexify the underlying real vector spaces to consider the \( z_i, w_i := \overline{u}_i, u_i, \) and \( v_i := \overline{w}_i \) as independent complex variables. Then an easy-to-identify isomorphism over \( \mathbb{C} \) is induced by the linear map \( \phi: V_A \otimes \mathbb{C} \rightarrow V_B \otimes \mathbb{C} \) given by

\[
\phi: (z_1, z_2, z_3, w_1, w_2, w_3) \mapsto (\sqrt{-1}w_2, \sqrt{-1}w_1, z_3, \sqrt{-1}z_2, \sqrt{-1}z_1, w_3).
\]

A simple computation demonstrates that \( \phi \) is equivariant with respect to the two \( \mathbb{C}^\times \)-actions, implying that the corresponding map \( \phi^*: \mathbb{C}[V_B \otimes \mathbb{C}]^{\mathbb{C}^\times} \rightarrow \mathbb{C}[V_A \otimes \mathbb{C}]^{\mathbb{C}^\times} \) is an isomorphism. Using coordinates \( (u_1, u_2, u_3, v_1, v_2, v_3) \) for \( V_B \otimes \mathbb{C} \), we have

\[
\phi^*(du_1 \wedge dv_1 + du_2 \wedge dv_2 + du_3 \wedge dv_3) = -dw_2 \wedge dz_2 - dw_1 \wedge dz_1 + dz_3 \wedge dw_3 \\
= dz_1 \wedge dw_1 + dz_2 \wedge dw_2 + dz_3 \wedge dw_3
\]

so that \( \phi \) is a symplectic embedding.

Identifying the real and complex invariants via \( w_i = \overline{u}_i \) and \( v_i = \overline{w}_i \), the map \( \phi^* \) is given on generators by

\[
\begin{align*}
\phi^* q_0 &= -z_2 w_2 = -p_1, \\
\phi^* q_2 &= z_3 w_3 = p_2, \\
\phi^* q_4 &= -z_2^2 w_3 = -p_3, \\
\phi^* q_6 &= -\sqrt{-1}z_1^3 z_3 = -\sqrt{-1}p_5, \\
\phi^* q_8 &= \sqrt{-1}z_1^3 z_2 = \sqrt{-1}p_7,
\end{align*}
\]

so that \( \phi^* J_B = J_A \). Hence \( \phi^* \) induces an isomorphism \( \mathbb{R}[M_0(B)] \otimes \mathbb{C} \rightarrow \mathbb{R}[M_0(A)] \otimes \mathbb{C}, \) completing the proof. \( \square \)

Clearly, the isomorphism \( \phi^* \) does not restrict to a map \( \mathbb{R}[M_0(B)] \rightarrow \mathbb{R}[M_0(A)] \) of the real algebras. Hence, to determine an isomorphism over \( \mathbb{R} \), we need a more explicit description of \( \mathbb{R}[M_0(A)] \) and \( \mathbb{R}[M_0(B)] \).
Proof of Proposition \(\textup{(iii)}\). Using Macaulay2 \([3]\), we compute the relations among the generators \(p_1, p_2, \ldots, p_8\) of \(\mathbb{R}[M_0(A)]\) to be

\[
\begin{align*}
2p_0 - 3p_1 - 6p_2, & \quad p_1^2 p_2 - p_4 p_3, \\
p_4 p_6 - p_2 p_8, & \quad p_3 p_5 - p_2 p_7, \\
p_1^2 p_0 - p_3 p_8, & \quad p_1^2 p_5 - p_4 p_7, \\
27p_1^3 p_4 + 216p_2^3 + 27p_1 p_4 p_3 + 162p_2 p_4 p_3 - 8p_5 p_6, \\
324p_1 p_2^2 p_3 + 216p_2^3 + 162p_4 p_3 - 8p_6 p_7, \\
27p_1^4 p_4 + 216p_2^3 + 162p_2^2 p_3 - 8p_5 p_8, \\
432p_2^5 - 81p_1^2 p_3 p_3 - 432p_1^2 p_2 p_3 p_3 - 648p_2^2 p_4 p_3 + 24p_1 p_5 p_6 - 16p_2 p_5 p_6, \\
27p_1^5 + 162p_1^2 p_3 p_3 + 324p_1 p_2 p_4 p_3 + 216p_2^2 p_3 p_3 - 8p_7 p_8, \\
324p_1 p_2^2 p_5 p_6 + 216p_2^3 p_3 p_6 - 8p_5^2 p_7 + 27p_1^2 p_3 p_8 + 162p_2 p_5 p_8, \\
432p_2^2 p_3 p_5 + 24p_1 p_3 p_5 - 16p_2 p_3 p_5 - 81p_1^2 p_5 p_5 - 432p_1^2 p_2 p_5 p_5 - 648p_2^2 p_4 p_5 + 24p_1 p_5 p_8 - 16p_2 p_5 p_8.
\end{align*}
\]

Similarly, the relations among the generators \(q_1, q_2, \ldots, q_8\) of \(\mathbb{R}[M_0(B)]\) are given by

\[
\begin{align*}
3q_0 - 2q_1 - 6q_2, & \quad 4q_1^2 q_2 + 24q_1 q_2^2 + 36q_2^2 - 9q_3 q_4, \\
q_4 q_6 - q_2 q_8, & \quad q_3 q_9 - q_2 q_7, \\
4q_1^2 q_6 + 24q_1 q_2 q_6 + 36q_2^2 q_6 - 9q_3 q_8, & \quad 4q_1^2 q_5 + 24q_1 q_2 q_5 + 36q_2^2 q_5 - 9q_4 q_7, \\
108q_1 q_3^3 + 216q_2^4 + 9q_1 q_3 q_4 - 54q_2 q_3 q_4 - 4q_5 q_6, & \quad q_1^3 q_4 - q_3 q_5, \\
q_3^3 q_3 - q_6 q_7, & \quad q_1^3 q_3 - q_6 q_7, \\
108q_2^3 - 9q_1^2 q_3 q_4 + 18q_1 q_2 q_3 q_4 - 27q_2^2 q_3 q_4 q_4 + 4q_1 q_5 q_6 + 16q_2 q_5 q_6, \\
4q_5^2 + 24q_1 q_5 q_6 + 36q_2 q_5 q_6 - 9q_7 q_8, \\
108q_1 q_3 q_3 q_4 + 216q_2 q_3 q_4 - 4q_6^2 q_7 + 9q_1 q_3 q_6 - 54q_2^2 q_3 q_6, \\
108q_2^3 q_4 q_5 + 216q_2 q_4 q_5 + 9q_1^2 q_4 q_7 - 54q_2^2 q_4 q_7 - 4q_5^2 q_7, \\
108q_2 q_4 q_5 + 4q_1 q_6^2 q_7 + 16q_2 q_6^2 q_7 - 9q_1^2 q_5 q_8 + 18q_1 q_2 q_5 q_8 - 27q_2^2 q_5 q_8, \\
108q_2^4 q_5 - 9q_1^2 q_4 q_7 + 18q_1 q_3 q_4 q_7 - 27q_2^2 q_4 q_7 + 4q_1 q_5 q_8 + 16q_2 q_5 q_8.
\end{align*}
\]

Define the map \(\Psi: \mathbb{R}[M_0(A)] \to \mathbb{R}[M_0(B)]\) by

\[
\begin{align*}
\Psi(p_1) &= q_1 + 3q_2, & \Psi(p_2) &= -\frac{3}{2}q_2, & \Psi(p_3) &= q_1, \\
\Psi(p_4) &= \frac{27}{8}q_3, & \Psi(p_5) &= q_6, & \Psi(p_6) &= -\frac{81}{16}q_5, \\
\Psi(p_7) &= \frac{2}{3}q_8, & \Psi(p_8) &= -\frac{729}{64}q_7.
\end{align*}
\]

A tedious though elementary computation demonstrates that \(\Psi\) maps the ideal of relations of the \(p_i\) into the ideal of relations of the \(q_i\), and \(\Psi^{-1}\) similarly maps the ideal of relations of the \(q_i\) into the ideal of relations of the \(p_i\). Therefore, \(\Psi: \mathbb{R}[M_0(A)] \to \mathbb{R}[M_0(B)]\) is an isomorphism. Note that \(p_2 = z_4 \geq 0\), while \(\Psi(p_2) = -3q_2/2 \leq 0\) so that \(\Psi\) does not preserve the inequalities.

To show that any graded isomorphism \(\mathbb{R}[M_0(A)] \to \mathbb{R}[M_0(B)]\) fails to preserve the inequalities, suppose for contradiction that \(\Phi: \mathbb{R}[M_0(A)] \to \mathbb{R}[M_0(B)]\) is such a graded isomorphism. Let \(Q(A)\) and \(Q(B)\) denote the subalgebras of \(\mathbb{R}[M_0(A)]\) and \(\mathbb{R}[M_0(B)]\), respectively, that are generated by elements of degree at most four. Then \(\Phi\) restricts to an isomorphism \(\Phi|_Q(A): Q(A) \to Q(B)\).
Again using Macaulay2 [3], the algebra \( \mathcal{Q}(A) \) generated by \( p_1, p_2, \ldots, p_6 \) has relations generated by
\[
R_1 = p_1^2 p_2 - p_3 p_4, \quad R_2 = 27(4 p_2^2(3 p_1 + 2 p_2) + (p_1 + 6 p_2) p_3 p_4) - 8 p_5 p_6, \\
R_3 = -81 p_1^2 p_3 p_4 - 8(-54 p_2^2 + 54 p_1 p_2 p_3 p_4 + 81 p_2^2 p_3 p_4 - 3 p_1 p_5 p_6 + 2 p_2 p_5 p_6), \\
R_4 = 27 p_3 p_4 (p_3^2 + 12 p_1 p_2^2 + 8 p_2^2 + 6 p_3 p_4) - 8 p_2^2 p_5 p_6,
\]
and the algebra \( \mathcal{Q}(B) \) generated by \( q_1, q_2, \ldots, q_6 \) has relations generated by
\[
R'_1 = 4 q_2 (q_1 + 3 q_2)^2 - 9 q_3 q_4, \quad R'_2 = 108 q_3^2 (q_1 + 2 q_2) + 9 (q_1 - 6 q_2) q_3 q_4 - 4 q_5 q_6, \\
R'_3 = 108 q_5^2 - 9 (q_3^2 - 2 q_1 q_2 + 3 q_2^2) q_3 q_4 + 4 (q_1 + 4 q_2) q_5 q_6, \\
R'_4 = 9 q_1^2 q_3 q_4 - 4 (q_1 + 3 q_2)^2 q_5 q_6.
\]
As \( \Phi \) preserves the grading, it must be of the form
\[
\Phi(p_1) = c_{11} q_1 + c_{12} q_2, \quad \Phi(p_2) = c_{21} q_1 + c_{22} q_2, \quad \Phi(p_3) = c_{33} q_3 + c_{34} q_4,
\]
(6.1) \( \Phi(p_4) = c_{43} q_3 + c_{44} q_4, \quad \Phi(p_5) = c_{55} q_5 + c_{56} q_6, \quad \Phi(p_6) = c_{65} q_5 + c_{66} q_6, \)
\[
\Phi(p_7) = c_{77} q_7 + c_{78} q_8, \quad \Phi(p_8) = c_{87} q_7 + c_{88} q_8.
\]
Using the fact that \( \Phi \) preserves the grading and maps the ideal of relations for the \( p_i \) into the ideal of relations for the \( q_i \), we must have
\[
\Phi(R_1) = k_1 R'_1, \quad \text{and} \quad \Phi(R_2) = k_2 R'_2 + k_3 q_1 R'_1 + k_4 q_2 R'_2
\]
for some \( k_1, k_2, k_3, k_4 \in \mathbb{R} \). Computing the \( q_1^3, q_3^3 \), and \( q_1^2 q_2 \) coefficients of each side of the \( q_1^3, q_3^3 \), \( q_1^2 q_2 \), \( q_1 q_3 q_4 \), and \( q_1 q_3 q_4 \) coefficients of each side of the second equation yields the system
\[
\Phi(R_1) : \\
q_1^3 : \quad c_{11}^2 c_{21} = 0, \\
q_3^3 : \quad c_{33} c_{43} = 0, \\
q_1^2 q_2 : \quad c_{11}(2c_{12} c_{21} + c_{11} c_{22}) = 4 k_1,
\]
\[
\Phi(R_2) : \\
q_1^3 q_2^3 : \quad 81 c_{21} c_{22}(3c_{12} c_{21} + 3c_{11} c_{22} + 4c_{21} c_{22}) = k_2(6k_3 + k_4), \\
q_1^2 q_3 q_4 : \quad 9 c_{22}(9c_{12} c_{21} + 3c_{11} c_{22} + 8c_{21} c_{22}) = k_2(9 + 3k_3 + 2k_4), \\
q_2^4 : \quad 3 c_{22}^2(3c_{12} + 2c_{22}) = k_2(6 + k_4), \\
q_2 q_3 q_4 : \quad 3(c_{11} + 6c_{21})(c_{34} c_{43} + c_{33} c_{44}) = k_2(1 - k_3), \\
q_2 q_3 q_4 : \quad 3(c_{12} + 6c_{22})(c_{34} c_{43} + c_{33} c_{44}) = -k_2(6 + k_4),
\]
Every solution of this system not corresponding to \( \Phi(p_i) = 0 \) for some \( i \) satisfies \( c_{11} = -2c_{22} / 3, \) \( c_{12} = -2c_{22}, \) and \( c_{21} = 0 \). Hence, though \( p_1 \geq 0 \) and \( p_2 \geq 0 \), either \( c_{22} > 0 \) so that \( \Phi(p_1) = -2c_{22} (q_1 / 3 + q_2) < 0 \) for any nonzero \( q_1 \) or \( q_2 \), or \( c_{22} < 0 \) so that \( \Phi(p_2) = c_{22} q_2 < 0 \) for any nonzero \( q_2 \). In either case, \( \Phi \) does not preserve the inequalities describing the semialgebraic sets \( M_0(A) \) and \( M_0(B) \).

As another example, let \( A' = (-2, 1, 1) \), Type II \( \mathbb{Z}_2 \) with \( \alpha = 2, \beta = 1, c_1 = 1, \) and \( c_2 = 1; \) and let \( B' = (-1, 2, 1), \) Type II \( \mathbb{Z}_2 \) with \( \alpha = 1, \beta = 1, c_1 = 2, \) and \( c_2 = 1 \). As above, \( \mathbb{R}[M_0(A')] \) and \( \mathbb{R}[M_0(B')] \) have the same Hilbert series, given by
\[
\left( \frac{1 + 2t^2 + 4t^3 + 2t^4 + t^6}{(1 - t^3)^2(1 - t^2)^2} \right)
\]
The quadratic off-shell invariants of the action with weight matrix \( A' \) are spanned by \( z_1 z_1, z_2 z_2, z_3 z_3, z_2 z_3, \) and \( z_2 z_2 \) with relation \( (z_2 z_2)(z_2 z_3) = (z_2 z_2)(z_2 z_2) \), and the moment map determines
For the action with weight matrix $B'$, the quadratic off-shell invariants are generated by $u_1u_1$, $u_2u_2$, $u_3u_3$, $u_1u_3$, and $u_2u_3$ with relation $(u_1u_3)(u_2u_3) = (u_1u_1)(u_2u_2)$, and the moment map expresses $u_1u_1 = 2u_2u_2 + u_3u_3$. Considering only the Poisson brackets of the quadratics, computations similar to those above demonstrate that any graded Poisson isomorphism $\Phi: \mathbb{R}[M_0(B')] \rightarrow \mathbb{R}[M_0(A')]$ must map $u_3u_3 \mapsto c^2z_2^2 + (-c^2 + 1)z_4 + \sqrt{-1}d_{z_2^2}$ where $c \in \{0,1\}$ and $d \neq 0$. For each $z_2, z_3 \in \mathbb{C}$, there is a $z_1 \in \mathbb{C}$ such that $(z_1, z_2, z_3) \in ZA'$ so that $z_4$ is not bounded by inequalities. As $u_3u_3, z_2u_2, z_3u_3 \geq 0$, it follows that $\Phi$ cannot preserve the inequalities.

Finally, we consider a closely related example that is not of Type I$_k$ nor II$_k$ for any $k$. Let

$$A'' = \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad B'' = \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$ 

To see that the Hilbert series of $\mathbb{R}[M_0(A'')]$ and $\mathbb{R}[M_0(B'')]$ coincide note that the cotangent-lifted weight matrix corresponding to $A''$,

$$\begin{pmatrix} -1 & 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 & -1 \end{pmatrix},$$

can be transformed into that of $B''$,

$$\begin{pmatrix} -1 & 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 & -1 \end{pmatrix}$$

by transposing the column pairs $(1, 4), (3, 7), (5, 8)$ and row-reducing over $\mathbb{Z}$. The common Hilbert series is given by

$$\frac{1 + 2t^2 + 2t^3 + 2t^4 + t^6}{(1 - t^3)^2(1 - t^2)^2}.$$ 

The quadratic off-shell invariants associated to $A''$ are $z_1z_1, z_2z_2, z_3z_3, z_4z_4, z_1z_3$, and $z_4z_1$, the moment map expresses $z_1z_1$ and $z_2z_2$ in terms of $z_3z_3$ and $z_4z_4$, and we have the relation $(z_3z_3)(z_4z_4) = (z_1z_3)(z_4z_4)$. Similarly, the quadratic off-shell invariants associated to $B''$ are $z_1z_1, z_2z_2, z_3z_3, z_4z_4, z_1z_3$, and $z_2z_4$, the moment map expresses $z_1z_1$ and $z_2z_2$ in terms of $z_3z_3$ and $z_4z_4$, and we have the relation $(z_1z_3)(z_2z_4) = (z_2z_2 + z_3z_3)(z_4z_4)$. Hence, computations identical to those for $A'$ and $B'$ demonstrate that the only Poisson isomorphisms between the algebras $\mathbb{R}[M_0(A'')]$ and $\mathbb{R}[M_0(B'')]$ do not satisfy the semialgebraic conditions, and hence do not correspond to a graded regular symplectomorphism.

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