Multimass schemes for collisionless $N$-body simulations

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ABSTRACT

We present a general scheme for constructing Monte Carlo realizations of equilibrium, collisionless galaxy models with known distribution function (DF) $f_0$. Our method uses importance sampling to find the sampling DF $f_s$ that minimizes the mean-square formal errors in a given set of projections of the DF $f_0$. The result is a multimass $N$-body realization of the galaxy model in which ‘interesting’ regions of phase space are densely populated by lots of low-mass particles, increasing the effective $N$ there, and less interesting regions by fewer, higher mass particles.

As a simple application, we consider the case of minimizing the shot noise in estimates of the acceleration field for an $N$-body model of a spherical Hernquist model. Models constructed using our scheme easily yield a factor of $\sim 100$ reduction in the variance at the central acceleration field when compared to a traditional equal-mass model with the same number of particles. When evolving both models with a real $N$-body code, the diffusion coefficients in our model are reduced by a similar factor. Therefore, for certain types of problems, our scheme is a practical method for reducing the two-body relaxation effects, thereby bringing the $N$-body simulations closer to the collisionless ideal.

Key words: methods: $N$-body simulations – methods: numerical – galaxies: kinematics and dynamics.

1 INTRODUCTION

There are two types of $N$-body simulations in stellar dynamics. In collisional simulations each of the $N$ particles represents an individual star. This type of simulation is most often used to model the evolution of star clusters in which discreteness effects, such as two-body relaxation, are important.

When it comes to modelling galaxies, however, the number of stars is large enough and the dynamical time is long enough, so these discreteness effects are usually unimportant. In the limit of a very large number, $N$, of bodies, stars and dark matter particles move in a smooth mean-field potential $\Phi(x, t)$ and behave as a collisionless fluid in six-dimensional phase space (Binney & Tremaine 1987, hereafter BT87), the (mass) density at any point $(x, v)$ being given by the distribution function (hereafter DF) $f(x, v; t)$. The time-evolution of the DF is described by the collisionless Boltzmann equation (hereafter CBE). Therefore, in a collisionless $N$-body simulation the $N$ particles do not correspond to real stars; instead they provide a Monte Carlo realization of the smooth underlying DF, from which one can estimate the potential $\Phi(x, t)$. By integrating the orbits of these particles, one is solving the CBE by the method of characteristics (Hernquist & Ostriker 1992, hereafter HO92; Leeuwin, Combes & Binney 1993, hereafter LCB1993).

In reality, no simulation is perfectly collisionless because Poisson noise in the estimates of $\Phi(x, t)$ inevitably leads to numerical diffusion in particles’ orbits. To reduce this noise, it is important to make $N$, the number of particles in the simulation, as large as possible. Unfortunately, the cost of running an $N$-body code scales at least linearly with $N$, so increasing $N$ also makes the simulation more expensive to run. The good news is that alternative, more sophisticated weapons are available for use in the fight against small $N$ limitations. A collisionless $N$-body code is essentially a Monte Carlo method and so should be amenable to well-known variance-reduction methods such as importance sampling (e.g. Press et al 1992).

In this paper we present a generally applicable, essentially model-independent method for constructing $N$-body realizations of isolated model galaxies in equilibrium, suitable for use as initial conditions (hereafter ICs) in collisionless simulations. Our scheme uses importance sampling to find a sampling DF $f_s$ that minimizes the mean-square uncertainty in a chosen set of projections of the DF $f_0$. For example, in modelling bar evolution, one might be most interested in following the detailed evolution of the DF around the strongest resonances. It is natural then to try to increase the sampling density near these regions by populating them with lots of low-mass particles. Outside these interesting regions, however, one must also have

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enough particles to maintain accurate estimations of the force field which governs the evolution of the system as a whole.

The paper is organized as follows. After reviewing the basics of Monte Carlo integration and the connection between CBE and N-body simulations, we explain our multimass formulation in Section 2.4. With the notable exception of some heuristic multimass schemes (e.g. Sigurdsson, Hernquist & Quinlan 1995, hereafter SHQ95; Weinberg & Katz 2007; Sellwood 2008; Zemp et al. 2008), most other IC-generation schemes have used equal-mass particles. In Section 3 we give an example of using our scheme to suppress fluctuations in the monopole component of acceleration in a spherical galaxy model. We calculate formal estimates of the noise in N-body simulations and the CBE borrows heavily from LCB93. We assume that N = 1 and that the mass density of stars in phase space is given by a DF \( f(x, \nu; t) = f(u, t) \), where \( u \equiv (x, \nu) \), normalized so that
\[
\int f(u) \, d^6u = 1.
\]

The evolution of the DF is governed by the CBE,
\[
\frac{\partial f}{\partial t} + \nu \cdot \frac{\partial f}{\partial x} - \frac{\partial \Phi}{\partial x} - \frac{\partial f}{\partial \nu} = 0.
\]

It conserves phase-space density, so that
\[
f(w(t); t) = f(w_0; 0),
\]
where \( w(t) \) is the path traced by an individual particle, with \( w_0 \equiv w(t = 0) \). As HO92 and LCB93 point out, in a collisionless N-body simulation one is solving the CBE for these \( w(t) \) by integrating the characteristic equation,
\[
\frac{dw^i}{dt} = \frac{dv_i}{a},
\]
and using Monte Carlo integration to estimate the acceleration
\[
a(x; t) \equiv -\frac{\partial \Phi}{\partial x} = -G\nabla \int \frac{f(u')}{|x - x'|} \, d^6u'.
\]

From (4) it follows that
\[
an \equiv -\nabla \sum_{i=0}^{N} \frac{m_i}{|x - x_i|}.
\]

These \( m_i \) clearly depend on the choice of sampling DF \( f_i \). The simplest choice is \( f_i(w; t) = f(w; t) = f(w_0; 0) \), in which case all particles have equal masses \( m_i = 1/N \). However, one is free to tailor the choice of \( f_i \) to suit the particular problem under study.

The singularities in (11) at \( x = x_i \) yield estimates of \( a(x) \) that suffer from unacceptably large scatter; in fact, they correspond to the direct accelerations appropriate for a collisional N-body code! So, in practice collisionless simulations do not use (11) directly, but instead obtain \( a(x) \) using techniques (e.g. softened force kernels, grid methods or truncated basis function expansions) that reduce the scatter by removing the singularities. More generally, equation (11) provides only the most simple-minded estimate of the integral (10), and one has some leeway in how one reconstructs \( f(w; t) \) from the discrete realization furnished by the \( N \) particles. Of course, the reliability of any sensible reconstruction will be wholly dependent on how well the DF is sampled.

2.3 Observables

What constitutes a ‘good’ choice of sampling density \( f_i \)? The DF \( f \) is a high-dimensional probability density and itself is not measurable. We are usually only interested in coarse-grained projections of the DF,
\[
\langle Q_i \rangle \equiv \int f(u) Q_i(u) \, d^6u,
\]
where the kernels \( Q_i(u) \) are some functions of the phase-space coordinates \( (x, \nu) \). For the purposes of the present paper, we consider a ‘good’ sampling scheme to be one that minimizes the uncertainty in the estimates of some given set of \( \langle Q_i \rangle \). Apart from some general guidance, we do not address the question of how best to choose...
these \( Q \), which usually requires some experience of the particular problem at hand.

We now give some examples. It is helpful to introduce the indicator function
\[
\mathbf{1}_V (w) \equiv \begin{cases} 
1, & \text{if } w \in V \\
0, & \text{otherwise.}
\end{cases}
\]

Then a particularly simple but important choice of kernel is
\[
Q_i (w) = \mathbf{1}_V (w),
\]
for which \( \langle Q \rangle \) measures the mass inside a volume \( V \). For many problems one might choose some of the \( V_i \) to surround important resonances in phase space, so that \( \langle Q_i \rangle \) measures the phase-space density around the resonances. With appropriate choices of projection kernel \( Q \), the expression (13) includes quantities such as the galaxy's density profile, its velocity moments or even its projected line-of-sight velocity distributions.

More fundamentally, an \( N \)-body model should provide a good estimate of the galaxy’s acceleration field. Therefore we recommend that many of the \( \langle Q_i \rangle \) be used to measure at least the monopole component of the galaxy’s acceleration field at a range of points. This can be achieved using (15) with spherical volumes \( V_i \) centred on \( x = 0 \) for a range of radii \( r_i \), encompassing all velocity space for \( |x| < r_i \). Similarly, one can include higher order multipole components of the galaxy’s acceleration field by choosing a slightly more complicated projection kernel \( Q \) (see equation 35 below).

### 2.4 Optimal sampling scheme

The problem we address in this paper is the following. We wish to construct an equilibrium \( N \)-body realization of a galaxy model with some known \( f_0 \). Specifically, we seek ICs that faithfully represent some projections,
\[
\langle Q_i \rangle = \int f_0 Q_i (w) \, d^6 w,
\]
and correctly normalized. This averaging extremizes (18), the proportionality constant being set by the constraint that \( f_i \) should be normalized, \( \int f_i (w) \, d^6 w = 1 \). This direct solution is flawed, however, since for most interesting choices of \( Q \), the resulting \( f_i \) depends on orbit phase; using this \( f_i \) the masses of particles sampling a given orbit would vary along the orbit! Therefore, in practice we use a slightly less direct approach.

We partition phase space into \( n_q \) cells and write \( \tau_j \) for the phase-space volume enclosed by the \( j \)th cell. We parametrize \( f_i \) as
\[
f_i (w) = \sum_{j=1}^{n_q} \frac{1_{f_i}}{a_j} f_0 (w),
\]
so that within the \( j \)th phase-space cell \( f_i \) is given by \( f_0 (w)/a_j \). For the equilibrium models considered, it is natural to choose \( \tau_j \) to be cells in integral space. Substituting this \( f_i \) into (19) yields
\[
(\delta Q_i)^2 = \frac{1}{N} \left[ \sum_{j=1}^{n_q} a_j H_j - 1 \right],
\]
where
\[
H_j = \int f_0 Q_i^2 (w) \, d^6 w / \langle Q_i \rangle^2.
\]

If we further define
\[
H_i = \sum_{j=1}^{n_q} H_j,
\]
then the mean-square fractional uncertainty (18) becomes
\[
S = \frac{1}{N} \sum_{j=1}^{n_q} \left[ a_j H_j - n_q \right].
\]

Our goal is to find the coefficients \( a_j \) that minimize this \( S \), subject to the constraint that the resulting \( f_i \) be normalized. The normalization constraint is that
\[
\int f_i (w) \, d^6 w = \sum_{j=1}^{n_q} \frac{L_j}{a_j} = 1,
\]
where
\[
L_j = \int f_0 \, d^6 w.
\]

Using the method of Lagrange multipliers, the coefficients of the ‘best’ sampling \( f_i \) obtained by minimizing (25) subject to the constraint (26) are simply
\[
a_j = \sqrt{\frac{L_j}{H_j} \sum_{k=1}^{n_q} \sqrt{H_k}},
\]
which is just the direct solution (20) in disguise, but averaged over the phase-space cells \( \tau_j \) and correctly normalized. This averaging means that the resulting \( f_i \) will be smooth, provided that none of the kernels \( Q \) pick out specific regions of integral space.

Substituting the \( f_i \) given by (21) into (12), we have that particles in phase-space cell \( \tau_j \) have masses \( m_i = a_j/N \). One can therefore easily impose additional, direct constraints on the masses of particles within a subset of the phase-space cells \( \tau_j \); simply repeat the minimization of (25) subject to (26) while holding the relevant subset of cells fixed.

Note that we use \( V \) to denote subvolumes of phase space be used in the calculation of the projections (13) of the DF \( f_0 \), and \( \tau \) for the subvolumes used in the discretization of the sampling DF \( f_i \).

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of the $a_j$ fixed at the desired values. For example, when generating an $N$-body realization of a dark matter halo model inside which one intends to embed a disc of light particles, one might want to ensure that those halo particles passing through the disc have the same mass as the disc particles. Of course, a more pedestrian approach would be to introduce additional kernels $Q_j$ to pick out the relevant parts of integral space. We caution, however, that we have not tested how well such a ‘bumpy’ $f_i$ would work in practice; the tests we present later all involve smoothly varying sampling distributions.

2.5 ICs for N-body model

Together with $f_i$, the coefficients $a_j$ completely determine the sampling DF $f_i$ of the form (21). In particular, it reduces to the conventional equal-mass case when all $a_j = 1$.

We apply the following sequence of steps $N$ times to draw particles from this $f_i$, thereby constructing an $N$-body realization of the galaxy model.

1. Choose one of the $n_j$ cells at random, the probability of choosing the $j$th cell being given by $I_j/a_j$. Let $i$ be the index of the chosen cell.

2. Assign a mass $m_i = f_i(x_i)/N f_i(x_i) = a_i/N$ to the particle.

3. Within the $i$th cell, draw $x_i$ from its density distribution, $\rho_i = \int f_i(x, v) \, dv$. For the special case of a spherical galaxy, one can precompute the cumulative mass distribution $M_i(r)$ for each of the $n_i$ cells and use this to draw a radius $r$, followed by angles $\theta_i$ and $\phi_i$.

4. Use an acceptance–rejection method to draw $n_i$ from $f_i(x, v)$ at this fixed value of $x$.

3 An example

In this section we use a simple galaxy model to demonstrate our scheme. Our galaxy model is spherical and isotropic, with density profile (Hernquist 1990)

$$\rho(r) = \frac{M_*}{2\pi a(r+a)^3},$$

(29)

total mass $M_*$ and scale radius $a$. By Jean’s theorem, the model’s DF $f(x, v)$ depends on $(x, v)$ only through the binding energy per unit mass $E$. Hernquist (1990) gives an expression for $f(E)$.

We want to construct an N-body realization of this model that minimizes the mean-square error in the monopole component of the acceleration averaged over many decades in radius, from $r_{\text{min}} = 10^{-4} a$ up to $r_{\text{max}} = 10^2 a$. To achieve this we choose kernels $Q_j = 1_{ij}(r)$ that measure the mass enclosed within a sequence of 25 spheres centred on the origin, with radii $r_i$ spaced logarithmically between $r_{\text{min}}$ and $r_{\text{max}}$. We use (22) to calculate the formal uncertainty $\delta M_i$ in the enclosed mass for a range of discretized sampling densities of the form (21), including (28).

To implement this, we first of all partition integral space $(E, J^2)$ on to a regular $n_E \times n_X$ grid. The $n_E$ energies $E_j$ are chosen to match the potential $E_j = \Psi(r_j)$ with $r_j$ logarithmically spaced between $10^{-6} a$ and $10^3 a$. At each $E_j$, there are $n_X$ values of $X_{ij}$ running linearly from 0 to 1, where $X_{ij} = J_i(E_j) / J_j(E_j)$ is the orbital angular momentum normalized by the circular angular momentum at energy $E_j$. These choices ensure that our $f_i$ samples well the interesting parts of phase space. For the calculations below we take $n_E \times n_X = 200 \times 100$, although a coarser grid (e.g. $50 \times 25$) would suffice. Having defined our projection kernels $Q_j = 1_{ij}$, we use (16) to calculate the expected values of enclosed mass $\langle M_i \rangle$ and the ancillary quantities $H_{ij}$ (from equation 23), and use these to obtain the formal uncertainties $\delta M_i$ in (22).

3.1 Comparison with other schemes: formal errors

Before applying our method, we study two other schemes: the conventional equal-mass scheme and the multimass scheme of SHQ95.

3.1.1 The conventional equal-mass scheme

The most common (albeit implicit) choice of sampling density is $f_i = f_0$, which corresponds to setting all $a_j = 1$ in our equation (21). All particles then have the same $1/N$ mass. For our example Hernquist model the fraction of particles within radius $r$ is $r^2/(a + r)^2$, so that less than 1 per cent of the particles are within 0.1 $a$. As shown in Fig. 1, for this $f_i$ the formal uncertainty, $\delta M(r)$, rises steeply towards the centre. Although this scheme produces accurate estimates of the galaxy’s potential outside the scale radius $a$, it performs poorly in the interesting $r^{-1}$ central density cusp.

3.1.2 Multimass scheme of Sigurdsson et al.

SHQ95 have used an interesting heuristic scheme to improve the resolution of N-body models near galaxy centres. In effect, they use an anisotropic sampling function of the form (21) with coefficients

$$a(\tau) \equiv B \times \left\{ \frac{\zeta_{\text{peri}}(\tau)}{\tau} \right\}^\nu \text{if } r_{\text{peri}} < a, \quad\text{otherwise,}$$

(30)

where $r_{\text{peri}}(\tau)$ is the smallest pericentre radius of any orbit from the phase-space cell $\tau$, and the constant $B$ is chosen to normalize $f_i$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{Figure 1. Formal relative errors $\delta M \equiv (\text{Var}(M))^{1/2}/\langle M \rangle$ (equation 19) in the monopole component of the potential of a Hernquist model (equation 29) constructed using the same number $N = 10^5$ of particles, but drawn from different sampling DFs. The heavy solid curve plots results for our tailored sampling DF. For comparison, we also show results for the conventional equal-mass scheme (light solid curve) and the heuristic multimass scheme of Sigurdsson et al. (1995) (dashed curve).}
\end{figure}
When the parameter \( \lambda = 0 \), then \( a_i = 1 \) and the sampling DF \( f_i \) is identical to \( f_0 \). Increasing \( \lambda \) improves the sampling of the cusp by increasing the number density of particles having pericentres \( r_{\text{pas}} < a \). Consequently, as \( r \to 0 \) the number density of particles rises more rapidly than the mass density, permitting better resolution in the centre. To balance this increase in number density, each particle is assigned a mass \( f_0/N f_i = a_i/N \) so that the phase-space mass density is still given by the desired \( f_0 \).

The dashed line in Fig. 1 shows the formal error \( \delta M(r) \) in our implementation of their scheme for \( \lambda = 1 \). Their scheme does much better than the conventional equal-mass scheme at small radii \( r \ll a \), at the cost of a slightly noisier monopole at \( r \gtrsim a \).

### 3.1.3 Our scheme

It is encouraging to see that SHQ95’s multimass scheme does, to some extent, improve mass resolution at small radius. However, as shown in Fig. 1, \( \delta M \) at \( r = 10^{-4} a \) is still almost two orders of magnitude larger than at \( r = a \). Can we achieve even better results by carefully designing an \( f_i \) that generates a flat \( \delta M(r) \) across a large range of radii? The \( f_i \) given by our optimal choice of coefficients (28) is plotted in Fig. 2. It is qualitatively similar to SHQ95’s results, in the fact that it samples densely the low angular momentum parts of phase space. The detailed shape of the function is different, however, and the thick solid curve in Fig. 1 shows that our scheme provides much better estimates of the monopole components of the acceleration at small radii; in fact, the formal error \( \delta M(r) \) varies by only a factor of \( \sim 4 \) over six decades in radius.

### 3.2 N-body realizations

Fig. 3 shows the spectrum of masses obtained using the algorithm detailed in Section 2.5 to draw \( N = 10^6 \) particles from this optimal \( f_i \). Unlike the conventional scheme which would give all particles the same \( 10^5 M_\odot \) mass if we assume the Hernquist galaxy has a total mass \( M_\star = 10^{10} M_\odot \), our multimass scheme assigns a range of masses between \( 10^{-2} \) and \( 10^6 M_\odot \) (8 decades), with many low-mass particles in the central region.

As a simple sanity check of our formal estimates of the errors in the monopole, we count the mass of particles within the same spheres \( V_i \) used to calculate \( M_\bullet \). The deviations from the mass profile of the target Hernquist model are consistent (Fig. 4) with the expected values of \( \delta M \) from equation (22).

Ultimately, the purpose of our sampling scheme is to improve the numerical modelling of collisionless galaxies close to equilibrium using full N-body integrations. To test how well our scheme succeeds at this task, we use the particle–multiple-mesh code Grommet (Magorrian 2007) to compare the evolution of our multimass models against equal-mass ones. Below, we adopt N-body units \( G = M = a = 1 \).

#### 3.2.1 How well is the acceleration field reproduced?

An important unsolved problem is how best to estimate the accelerations (10) given a discrete N-particle realization of the underlying DF \( f \). The most sophisticated approaches to this problem (e.g. Dehnen (2001) and references therein) have focused on finding softening kernels that minimize the errors in the acceleration field given a static distribution of \( N \) equal-mass particles. In the present paper we do not investigate how different softening lengths or softening kernels affect our multimass models. We simply adopt a nested series of boxes with boundaries at \( |x| = 100 \times 2^{-i} \) with \( i = 0, \ldots, 20 \), each box covered by a \( 60^3 \) mesh. As one moves to smaller lengthscales the effective softening length decreases, with \( \epsilon_{\min} = 200/60 \times 2^{-20} \approx 10^{-2} \).

Fig. 4 shows the fractional error in the radial component of the acceleration field returned by Grommet, in addition to the fractional error in enclosed mass. There is an approximately constant offset between these two quantities for equal-mass and multimass realization. Since our ICs here have been tailored to minimize the variance in the monopole component of the acceleration field, how important is our neglect of the higher order multipoles?
In terms of multipole moments, the radial component of the acceleration is (e.g. BT87)

\[ a_r(r, \theta, \phi) = 4\pi G \sum_{lm} \frac{Y_l^m}{2l + 1} \left[ \frac{l + 1}{r^{l+2}} \int_0^r \rho_{lm}(r') r'^{\delta+1} dr' \right. \right. \\
\left. + \left. r^{\delta+1} \int_0^\infty \rho_{lm}(r') \frac{dr'}{r'^{\delta+1}} \right] , \tag{31} \]

where

\[ \rho_{lm}(r) = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta Y_l^m(\theta, \phi) \rho(r, \theta, \phi). \tag{32} \]

This can be rewritten as

\[ a_r(r, \theta, \phi) = -\frac{4\pi G}{r^2} \sum_{lm} \langle M_{lm}(r) \rangle Y_l^m(\theta, \phi), \tag{33} \]

where \( \langle M_{lm}(r) \rangle \) are given by

\[ \langle M_{lm}(r) \rangle = \int f_0(w') M_{lm}(r, w') d^3 w' \tag{34} \]

with projection kernels

\[ M_{lm}(r, w') = Y_l^m(\theta', \phi') \left[ \frac{l + 1}{2l + 1} \frac{r^\delta}{r^{l+1}} I_{lV,(r)}(w') \right. \right. \\
\left. - \left. \frac{1}{2l + 1} \frac{r^{\delta+1}}{r^{l+2}} I_{lV,(r)}(w') \right] . \tag{35} \]

where \( V(r) \) encompasses all phase-space points having radii less than the (real-space) radius \( r \), and \( V^\prime(r) \) is its complement. For our spherical galaxy,

\[ \langle M_{lm}(r) \rangle = \int f_0(w') M_{lm}(r, w') d^3 w' = \left\{ \begin{array}{ll} M(r), & \text{if } l = m = 0, \\ 0, & \text{otherwise}. \end{array} \right. \tag{36} \]

The corresponding variance in \( a_r(r) \) for an \( N \)-body realization drawn from some choice of \( f_i \) is

\[ \text{Var}(a_r(r)) = \frac{4\pi G}{r^2} \sum_{lm} \text{Var}(\langle M_{lm}(r) \rangle) Y_l^m(\theta, \phi), \tag{37} \]

where, from (17),

\[ \text{Var}(\langle M_{lm}(r) \rangle) = \frac{1}{N} \left[ \int \frac{f_i^2(w')}{f_i(w')} M_{lm}(r, w')^2 d^3 w' \right. \right. \\
\left. - \left. \langle \langle M_{lm}(r) \rangle \rangle^2 \right] . \tag{39} \]

Similarly, the variance in the tangential component of acceleration field can be achieved by using projection kernels

\[ \text{Var}(a_{\theta,\phi}(r)) = \frac{4\pi G}{r^2} \sum_{lm} \text{Var}(\langle M_{lm}^\prime(r) \rangle) \left[ Y_l^m(\theta, \phi) \right]_{\theta, \phi}, \tag{40} \]

where

\[ M_{lm}^\prime(r, w') = Y_l^m(\theta', \phi') \left[ \frac{1}{2l + 1} \frac{r^\delta}{r^{l+1}} I_{lV,(r)}(w') \right. \right. \\
\left. + \left. \frac{1}{2l + 1} \frac{r^{\delta+1}}{r^{l+2}} I_{lV,(r)}(w') \right] . \tag{41} \]

So, given any choice of \( f_i \), we can use the expressions above to calculate the contribution of the higher order multipole moments to the formal errors in the acceleration. We find that, as we progressively include more terms, our estimate of the formal \( \text{Var}(a_r(r)) \) approaches the actual errors observed in the \( N \)-body realization.

Alternatively, we can find our optimal sampling DF \( f_i \) by minimizing

\[ S = \sum_{i=1}^{\sigma} \sum_{l=0}^{l_{\text{max}}} \sum_{m=-l}^{l} \text{Var}(M_{lm}^\prime) \tag{42} \]

truncated at say \( l_{\text{max}} = 2 \). Note that this new \( S \) reduces to the old one in equation (18) when \( l_{\text{max}} = 0 \), but otherwise includes additional terms with \( l > 0 \), each weighted by the monopole component \( l = 0 \). On increasing \( l_{\text{max}} \) from 0 to 2, the formal \( \text{Var}(a_r(r)) \) increases but the shape of the curve remains approximately unchanged and there are no noticeable differences in the resulting \( f_i \). Therefore, our neglect of higher order multipole moments is justified, at least in the present case, provided one bears in mind that the errors in the full acceleration field are going to be larger by an approximately constant factor than what one would estimate from the monopole component alone.
3.2.2 How well are integrals of motion conserved?

This paragraph describes the details of a full N-body implementation. Using both equal-mass and multimass schemes, we draw 10⁶ particles with radii between 10⁻³ < r < 10⁰. In order to suppress any slight deviation from symmetry (the odd terms of higher order multipoles) and remove any intrinsic transient in linear momentum (see also McMillan & Dehnen 2005), ICs (x, v) are extended to include the mirror distribution by reflecting each of the 10⁶ particles with (x, v) → (-x, -v). The full ICs then have N = 2 x 10⁶ particles. Taking the efficiency of integration into consideration, only a 12-level nested series of boxes each covered by a 60² mesh is used, together with a single time-step of 2 x 10⁻⁴. Therefore, we expect our numerical results to be trustworthy at radii greater than a few times 10⁻³.

Fig. 5 plots the inner density profiles of both realizations after evolving each for 200 time units (or 300 circular orbit periods at r = 0.01). The lack of particles at small radius r ≃ 10⁻² in the equal-mass realization means that the initial model is out of exactly detailed equilibrium and causes the central density profiles to flatten. In contrast, the density profile in the multimass case is always much better behaved there.

It is interesting to examine what is going on at the level of individual orbits. Both realizations begin with spherical symmetry and remain spherical, apart from the effects of Poisson noise. The amount of diffusion in the angular momentum J of each particle's orbit serves as a strong gauge of relaxation effects. This is complicated by the fact that many particles in isotropic models being considered here have J(t = 0) ≃ 0. In such cases, even a small change in J(t > 0) would yield a large fractional change when measured in respect to its initial value. To circumvent this artificial problem, for each particle we measure the change in angular momentum relative to its circular value at t = 0 using

\[ \Delta X_i^2 = \left( \frac{J_i(t) - J_i(0)}{J_i(\infty)} \right)^2 \frac{t}{T}. \]  

(43)

Binning particles by energy and calculating the mean \( \Delta X_i^2 \) within each energy bin gives us the time-averaged diffusion rate \( \delta X^2(\mathcal{E}) \).

As shown in Fig. 6, both models suffer diffusion, but due to the enhancement of particle numbers and hence the smoothness of potential field in the central region, diffusion in the multimass scheme is suppressed by two orders of magnitude across the whole system.

As a further test of the robustness of our multimass scheme, we have evolved our multimass ICs using the tree code FALCON (Dehnen 2000) with a single interparticle softening radius of 10⁻³, comparable to the finest mesh size used in the Grommet runs. The dashed curve in Fig. 6 plots the resulting \( \delta X^2 \); our scheme works just as well for tree codes as it does for mesh codes, although the variable softening in Grommet does slightly decrease the amount of diffusion. This is not surprising, since the only difference between the two runs is the approximations used to estimate the accelerations (10).

In any model with a broad spectrum of particle masses, a natural question is what happens if heavy bodies from the outskirts visit the centre full of light mass elements and vice versa. To address this issue, we have measured the \( \delta X^2(\mathcal{E}) \) of equation (43) but, instead of considering all particles of a given \( \mathcal{E} \), we compare the diffusion of particles on radially biased orbits with \( X^2 < 0.1 \) to those on nearly circular orbits with \( X^2 > 0.9 \). As shown in Fig. 7, there are no systematic differences between them. The reason for this is simply that particles with \( X \simeq 0 \) spend most of their time at apocentre, the apocentre radius being only a factor of ~2 larger than the radius of a circular orbit of the same energy. Nevertheless, a particle with \( X \simeq 0 \) will affect all of the more tightly bound orbits as it plunges through the centre of the galaxy, but our measured diffusion rates account for this.

4 CONCLUSIONS

We have presented a general multimass scheme to construct Monte Carlo realizations of collisionless galaxy models with known steady-state DFs f_0. The scheme uses importance sampling to find the tailored sampling DF f, that minimizes the sum of mean-square uncertainties in a given set of observable quantities of the form (16). Although our method works for any reasonably general
collisionless $N$-body code, we note that there are three conditions that must be satisfied before it can be applied successfully.

(i) The system should be in a steady state, or close to one.

(ii) The DF $f_i$ should be quick and cheap to evaluate, either numerically or analytically. The calculation of $f_i$ is not much more demanding for axisymmetric or triaxial galaxies than for spherical isotropic models. Finding $f_i$ for such systems is, however, non-trivial since one rarely has sufficient knowledge of the underlying potential’s integrals of motion, but suitable flattened DFs do exist, including the standard axisymmetric two-integral $f(E, L_z)$ models and also rotating triaxial models such as those used in, e.g. Berczik et al. (2006). An alternative way of constructing flattened multism mass realizations would be to apply the adiabatic sculpting scheme of Holley-Bockelmann et al. (2002) to a spherical $N$-body model constructed using our scheme.

(iii) Finally, the utility of our multimass scheme depends critically on the selection of the projection kernels $Q(\omega)$.

Point (ii) is a well-known and longstanding problem, but the final condition is new. It is probably best addressed by experimenting with different sets of kernels, especially since it is easy to test the consequences of modifying them. Nevertheless, there are cases in which modest physical insight offers some guidance on choosing the $Q$. Here are some examples.

Galaxies with central massive black holes: It is straightforward to extend our treatment of self-consistent galaxy models to models containing a central black hole (hereafter MBH). By choosing kernels (as in Section 3) to measure the monopole component of galaxy’s force field and choosing $f_i$ to minimize their mean-square fractional uncertainty, one also achieves better spatial and mass resolution within the sphere of influence of the black hole.

Loss-cone problems: The rate of supplying stars into MBH’s loss cone is an important ingredient in galaxy models with central MBHs. A thorough understanding of collisionless loss-cone refilling mechanisms and accurate estimates of the resulting refilling rates are particularly critical for the prediction of astrophysical quantities such as the time-scale of binary MBH merger (Begelman, Blandford & Rees 1980; Yu 2002; Milosavljević & Merritt 2003), the tidal disruption rate of stars (Magorrian & Tremaine 1999; Syer & Ulmer 1999; Wang & Merritt 2004). When using $N$-body simulations to study such loss-cone problems, one is often interested in stars on low angular momentum orbits and can therefore choose kernels to pick out such loss-cone phase space for detailed modeling, while simultaneously maintaining accurate estimates of the galaxy’s acceleration field.

Sinking satellites: Kazantzidis, Magorrian & Moore (2004) demonstrate the significance of using equilibrium $N$-body realizations of satellite models when investigating the effect of tidal stripping of CDM substructure haloes (satellites) orbiting inside a more massive host potential. Besides the shape of the background potential and the amount of tidal heating, the mass-loss history is very sensitive to the detailed density profile of the satellite itself. One can therefore make one step further from equal-mass realizations by designing kernels to pick out orbits that pass through the tidal radius, while again maintaining an accurate estimate of the satellite’s acceleration field.

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