Equation of state for Buchdahl star and black hole

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It turns out that the Buchdahl star, the most compact object without horizon, is defined by the potential \( \Phi(R) = 4/9 \) and black hole by \( \Phi(R) = 1/2 \), where \( \Phi(R) = (MR - Q^2/2R)/(R^2 + a^2) \) for the Kerr-Newman charged and rotating object. Equivalently in terms of gravitational and non-gravitational energy, the former is defined when gravitational energy is half of non-gravitational energy while the latter when the two are equal. In this letter we wish to point out that the latter characterization implies that the Buchdahl star is a Virial star as its equilibrium is maintained by average kinetic energy being half of potential energy, and consequently the equation of state \( p = \frac{1}{2} \rho \) and sound velocity, \( v_s^2 = 1/2 \). Similarly this would predict an equation of state for black hole \( p = \rho \) and \( v_s^2 = 1 \).

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It is a matter of prime importance in astrophysics that how compact a star like object could be? The obvious answer in general relativity is black hole with a horizon which however blocks out all the information of its interior. That is why the notions of membrane paradigm and stretched horizon were invoked [1] in which a non-null surface very close to horizon, but not a horizon, is envisaged for which black hole physics and thermodynamics could be explored. That is to have a timelike surface as close as possible to horizon which could be in active physical communication with outside observer.

There does however exist the most compact object without horizon in the Buchdahl star (BS), defined in general by potential, \( \Phi(R) = 4/9 \) [2]. It is identified with the equality in the Buchdahl compactness bound \( M/R \leq 4/9 \) [3]. This bound was derived under very general conditions of isotropic perfect fluid with density decreasing outwards and the fluid interior at the boundary is matched to the exterior vacuum. For the general case of charged and rotating object \( M/R \) is replaced by \( \Phi(R) \). Similarly black hole (BH) is defined by \( \Phi(R) = 1/2 \). This is a general definition and we write for the Kerr-Newman charged and rotating object, \( \Phi(R) = (MR - Q^2/2R)/(R^2 + a^2) = 4/9, 1/2 \) respectively for BS and BH.

Note that the Buchdahl bound is also given by pressure at the center \( p_c(r = 0) \leq \infty \) for the stiffest equation of state of uniform density — incompressible perfect fluid distribution described by the unique Schwarzschild’s interior solution not only for the Einstein but also for the Lovelock gravity in general [4]. The same limit was also obtained [5, 6] by assuming the strong energy condition, \( p_r + 2p_t \leq \rho \), where \( p_r, p_t, \rho \) are respectively radial and transverse pressure, and matter energy density. In this case the limit saturates not for fluid interior but for an infinitely thin shell with \( 2p_t = \rho \). In the previous case the limit saturates for central pressure tending to infinity while in the latter, it is for infinitely thin shell. In either case, the situation is not entirely physically acceptable, it should anyway be taken as a limiting case.

There exists extensive literature on alternative derivations of the bound involving various situations like inclusion of \( \Lambda \) [7], different conditions than Buchdahl’s [6, 8], brane-world gravity [9, 10], modified gravity theories including Lovelock gravity and higher dimensions [11–14]. The limit on maximum mass has been obtained by appealing to the dominant energy condition and sound velocity being subliminal [15]. The Buchdahl bound defines an overriding state which is obtained under very general conditions while more compact distributions are allowed under specific circumstances and conditions.

Recently it has been rigorously established [16] that Buchdahl bound cannot be pierced for a fluid obeying the energy conditions and radial stability of the distribution as well as sound velocity remaining subluminal. Thus Buchdahl bound is to be always respected by any physically reasonable fluid interior for a compact star. Buchdahl star is the limiting case when this bound saturates, and it is the most compact object without horizon.

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Further there are several works finding the compactness bound for a charged star \[ 6, 8, 17, 26 \]. Unlike the neutral case, here they all do not agree. However there is one \[ 17 \] that accord to the general universal prescription of
\[
\Phi(R) = (M - Q^2 / 2R) / r \leq 4/9
\]
and it reads as follows:
\[
M/R \leq \frac{8/9}{1 + \sqrt{1 - (8/9)\alpha^2}}, \quad \alpha^2 = Q^2 / M^2. \tag{1}
\]
It reduces to the Buchdahl bound for uncharged object, \( M/R \leq 4/9 \) for \( Q = 0 \). Like BH, it also has extremal limit, \( \alpha^2 = 9/8 > 1 \) while \( M/R \leq 8/9 < 1 \). This means BS could indeed have \( \alpha^2 > 1 \); i.e. overcharged relative to BH.

Now we come to the gravitational energy and its role in the present discussion of compactness bound. Energy is a very illdefined and ambiguous entity in general relativity particularly because energy of gravitational field resides in space curvature and hence is non-localizable and hard to be computed unambiguously. However there is the Brown-York prescription \[ 27 \] for quasilocal energy that gives total energy contained inside a sphere of radius \( R \) enclosing a gravitating body. In this prescription it is envisaged that an infinitely dispersed system of bare mass \( M \) begins collapsing under its own gravity and as collapse proceeds it picks up gravitational field energy which lies exterior to the object. Then the total energy contained inside would be given by \( E_{\text{tot}}(R) = M - E_{\text{grav}}(R) \), and thus gravitational energy, \( E_{\text{grav}} \) could be computed. How far the collapse goes on and where it halts would determine bound on compactness as well as on gravitational energy. This would mean that the compactness bound could be obtained as a bound on \( E_{\text{grav}} \), which could be computed by using the exterior metric alone without any reference to what happens inside the object. Further the exterior metric is unique for a stationary object.

By using the Brown-York prescription \[ 27 \] of quasilocal energy for computing \( E_{\text{grav}} \), we had long back \[ 28 \] defined BH horizon by equipartition of total energy into gravitational and non-gravitational energy. That is, horizon is defined when gravitational energy equals non-gravitational energy. This could be easily understood as follows: at horizon timelike particle that feels gravitational attraction produced by \( E_{\text{non-grav}} \) tends to be photon that could only feel space curvature produced by \( E_{\text{grav}} \). That is why horizon would be defined when \( E_{\text{grav}} = E_{\text{non-grav}} \). This was a remarkably insightful and physically enlightening explanation.

Relatively recently we had asked \[ 30 \] the question, when would the Buchdahl bound be similarly defined by writing \( E_{\text{grav}} = \gamma E_{\text{non-grav}} \)? It happens for \( \gamma = 1/2 \). That is, Buchdahl bound is defined by gravitational energy being half of non-gravitational energy. All this could be computed by employing only the unique exterior metric without any reference to the interior. The key question is how do we understand it physically as we do equipartition of gravitational and non-gravitational energy for BH horizon?

Before we address this question, let us compute \( E_{\text{grav}} \) from the Brown-York prescription which would be briefly outlined in the Appendix. The total energy contained inside radius \( R \) is in general given by
\[
E_{\text{tot}}(R) = R(1 - \sqrt{1 - 2\Phi(R)}) \tag{2}
\]
where \( \Phi(R) = E_{\text{non-grav}} / R \) and \( E_{\text{non-grav}} = M, M - Q^2 / 2R, M / (1 + a^2 / R^2) \) respectively for neutral, charged and rotating star \[ 2 \]. For large \( R \) it tends to \( E_{\text{tot}}(R) = M - (\text{constant}) = M + M^2 / 2R \) for a neutral static star. This is how gravitational energy gets added to mass \( M \) to give the total energy contained inside a spherical object. That is how compactness is directly proportional to gravitational energy.

On subtracting non-gravitational energy from the above total energy we obtain the gravitational energy given by
\[
E_{\text{grav}}(R) = E_{\text{tot}}(R) - E_{\text{non-grav}} = R(1 - \sqrt{1 - 2\Phi(R)}) - E_{\text{non-grav}} \tag{3}
\]
Now the Buchdahl bound is given by \( E_{\text{grav}}(R) \leq 4/9 E_{\text{non-grav}}(R) \) which would read as
\[
1 - \sqrt{1 - 2\Phi(R)} \leq \frac{3}{2} \Phi(R) \tag{4}
\]
which solves to give the bound
\[
\Phi(R) \leq \frac{4}{9}. \tag{5}
\]
In terms of the compactness parameter it reads as
\[
M/R \leq \frac{8/9}{1 + \sqrt{1 - \lambda^2}}, \tag{6}
\]
\[ \lambda^2 = \frac{(8/9) Q^2}{M^2}, \frac{(8/9)^2 a^2}{M^2} \] respectively for charged and rotating star. It reduces to Eq (1) for the charged case and to the Buchdhal bound, \( M/R \leq 4/9 \) when \( Q = a = 0 \). Note that in terms of potential it has the universal characterization \( \Phi(R) \leq 4/9 \). Note that Eq (4) for \( E_{\text{grav}} = \gamma E_{\text{non-grav}} \) would in general read as

\[ 1 - \sqrt{1 - 2\Phi(R)} = (1 + \gamma)\Phi(R). \]  

Here \( \gamma \) is the compactness parameter and it is BS when \( \gamma = 1/2 \) and BH when \( \gamma = 1 \)

As we have seen above that energy in the interior of an object increases by the amount equivalent to gravitational energy and that goes into building its kinetic energy (KE) while non-gravitational energy is measure of potential energy (PE). Then the Buchdhal bound condition, \( E_{\text{grav}} \leq \frac{1}{2}E_{\text{non-grav}} \) translates to average kinetic energy being less or equal to half of average potential energy. Buchdhal star is defined when this condition is saturated and thus results the well known Virial equilibrium relation, \( KE = \frac{1}{2}PE \). This is remarkable that the Buchdhal star is characterized by the Virial equilibrium condition. This seems to indicate that as compactness increases fluid distribution in the interior of a star tends to attain the limiting Virial distribution which is entirely maintained by average kinetic energy being equal to half of average potential energy.

Fluid pressure is measure of average kinetic energy and density that of average potential energy, then the Virial equilibrium equation implies for BS the equation of state \( p = \frac{1}{3} \rho \) and the sound speed, \( v_s^2 = 1/2 \). Recently there have been very interesting works on exploring sound speed in neutron star. In particular conformal bound on sound speed in neutron star is \( v_s^2 \leq 1/3 \). While for \( M/M_{\text{TOV}} > 0.7 \) the limiting value turns out to be \( v_s^2 = 1/2 \). It is remarkable that the same bound as obtained above comes independently from altogether a different consideration. Following the same line of reasoning, one would arrive at the remarkable result that the equation of state for a black hole would be \( p = \rho \) and \( v_s^2 = 1 \). This is because BH is defined by \( E_{\text{grav}} = E_{\text{non-grav}} - \).

Our main aim was to infer the equation of state for BS and BH from the consideration of gravitational energy as the driving agent for compactness. Having done that, the next pertinent question that arises is, do there exist exact fluid solutions for the interior with these equations of state? There is a vast literature on fluid solutions for compact object but so far as we know there exists none for \( p = \frac{1}{3} \rho \) and \( p = \rho \). However there exists one as discussed in the Klein-Tolman solution, that has an equation of state \( p = k\rho \) admitting \( k = 1/2, 1 \). Unfortunately the distribution turns out to be unbounded as the pressure cannot vanish for any finite \( R \). It cannot therefore serve as an interior to a stellar object. There is extensive effort being invested in finding exact solutions for stellar interiors, yet the absence of one with these equations of state is perhaps indicative of the fact that they are hard to find. Note that in the Buchdhal bound, \( M/R \leq 4/9 \) is obtained by assuming the strong energy condition, \( p_r + 2p_t \leq \rho \), and it saturates for the thin shell with \( p_r = 0 \) and \( 2p_t = \rho \); i.e., \( p = p_t = \frac{1}{2} \rho \), the Virial condition of equilibrium. Thus BS could indeed be described by a thin shell with the equation of state \( p = \frac{1}{2} \rho \). Perhaps we should try to look for solutions of a fluid interior with \( p_r = 0 \) and \( p_t = \frac{1}{2} \rho \) which should be somewhat simpler to obtain.

Unlike static solutions, Schwarzschild and Reissner - Nordström, that describe static object whether black hole or otherwise, the Kerr solution in contrast describes only a rotating BH and not a non-BH rotating star. This is because a rotating star would suffer flattening at the poles due to rotation and that would give rise to multipole moments. For BH all the moments get evaporated out as horizon is approached, leaving horizon to have perfect spherical topology. We do however continue to employ the Kerr metric for consideration of rotating BS with the justification that it is though not a BH but it is quite close to it. The Kerr metric for BS could therefore be taken as good and reasonable approximation. Also note that Eq (3) is valid in general for static as well as stationary objects and involves only potential which is easily computable. It should be emphasized that Eq (3) has made evaluation of gravitational energy for rotating object trivial which was otherwise formidably difficult and almost intractable.

We would once again like to emphasise the fact that BS is physically accessible real astrophysical object that offers without any apology all that one was asking of the membrane or stretched horizon paradigm. All the BH properties including thermodynamical parameters could indeed be explored for BS. It is known that a non-extremal BH cannot be converted to extremal by adiabatic test particle accretion, the same is true for BS. Second, BH can though be over-extremalized under linear order accretion but the result is always overturned when second order perturbations are included; i.e., the weak cosmic censorship conjecture (WCCC) is always respected, the same is again true for BS. Further it is an astrophysical object which is almost like BH, it could serve as an excellent candidate as BH mimicker without any exotic and unusual properties. Rotating BS should therefore be probed as BH mimicker in various astrophysical situations and phenomena.
Another remarkable property of BS is that its extremal limit is over extremal for BH. That is, a BS could hold much more charge or rotation compared to BH because extremal bound for the former is \( \alpha^2 = 9/8 > 1 \) for the charged case, which is over-extremal for BH. This gives rise to an interesting possibility, begin with a BS with \( 1 < \alpha^2 < 9/8 \) and let neutral matter accrete on it and thereby \( \alpha^2 \) could reduce down to 1. It would be interesting to probe whether it would result into a BS with \( \alpha^2 = 1 \) or an extremal BH. If the latter is the case, that would perhaps be the only way to form extremal BH by gravitational collapse. This is how extremality which cannot be reached from the bottom, could now be attained from the top through BS. That is perhaps the only way by which extremal BH could form in gravitational collapse. BS is the most compact stable object without horizon and it could therefore be an immediate intermediate state which could perhaps turn into black hole on further accretion.

It is true that compactness of a star should be governed by its internal structure, binding energy, equation of state and fluid properties in general. However star is formed from collapse of a self gravitating dispersed system, and so as collapse proceeds gravitational energy which is negative keeps on increasing in the exterior. It is to be balanced by increase in (positive) energy in the interior by the same amount. The gain in energy in the interior appears as kinetic and potential energy. In contrast star interior consists of fluid which is in hydrostatic equilibrium by pressure gradient balancing gravitational attraction. Then what it seems to imply is that in the limiting BS compactness state, fluid may break into free elements giving rise to a Virial distribution. This is a very enlightening perspective. Further it is remarkable that it could be computed by using the unique exterior metric which is unique, without reference to the interior at all. This is precisely what we had wished to convey and share.

All this is however done only when we first know the Buchdahl bound and then translate it in terms of potential or gravitational and non-gravitational energy. Without prior knowledge of the bound, it would have been impossible to derive it other way round. Once we know the bound, it could be understood in a different physically enlightening perspective. Further it is remarkable that it could be computed by using the unique exterior metric alone without reference to interior at all. This is precisely what we had wished to convey and share.

It is remarkable that BS is a Virial star as its equilibrium is maintained by the Virial condition. The application of the Virial condition envisages a gravitating system consisting of isolated bodies being in equilibrium purely through average kinetic and potential energy. In contrast star interior consists of fluid which is in hydrostatic equilibrium by pressure gradient balancing gravitational attraction. Then what it seems to imply is that in the limiting BS compactness state, fluid may break into free elements giving rise to a Virial distribution. This is a very remarkable and novel astrophysical prediction that ensues from our simple minded physically motivated analysis. Does fluid break into free elements when it is highly compressed to the limiting state where the Buchdahl bound saturates, \( \Phi(R) = 4/9? \) Until the saturation is reached fluid character persists with \( p < \frac{1}{2} \rho \) and it breaks into the Virial distribution at saturation.

Finally we wish to say that BS is a real astrophysical object which is very close to BH and hence provides a rich astrophysical avenue which should be probed as BH mimicker and otherwise in various astrophysical settings and phenomena. It is remarkable that it is a Virial star whose equilibrium is maintained by the Virial condition, average kinetic energy being half of potential energy. This raises the amazing question, does fluid interior in the limiting compactness state tends to attain the Virial distribution? On the other hand for fluid it would imply the equation of state, \( p = \frac{1}{2} \rho \) and sound velocity, \( v_s^2 = 1/2 \). Similarly for BH it would be \( p = \rho \) and \( v_s^2 = 1 \).

I. APPENDIX

A. Brown-York quasilocal and gravitational energy

Let us briefly recall the Brown-York prescription in which it is envisioned that a space-time region is bounded in a 3-cylindrical timelike surface bounded at the two ends by a 2-surface. Then Brown-York quasilocal energy is defined by

\[
E_{BY} = \frac{1}{8\pi} \int d^2 x \sqrt{|q|} (k - k_0) = E_{tot}(R)
\]

where \( k \) and \( q \) are respectively trace of extrinsic curvature and determinant of metric, \( q_{ab} \) on 2-surface. The reference extrinsic curvature, \( k_0 \) is of some reference space-time, which for asymptotically flat case would naturally be Minkowski...
flat. This is the measure of total energy contained inside a sphere of some radius $R$ around a static object. The evaluation of the above integral for the Reissner-Nordström metric yields,

$$E_{\text{tot}}(R) = R - \sqrt{R^2 - 2MR + Q^2},$$

which at large $R$ approximates to

$$E_{\text{tot}}(R) = M - (Q^2/2R - M^2/2R) = M - Q^2/2R + M^2/2R. \tag{10}$$

This prescription envisions an infinitely dispersed distribution of bare ADM mass $M$ at infinity, while collapsing under its own gravity it picks up gravitational field energy as well as electrostatic field energy. Then gravitational field energy at $R$ would be given by subtracting from it mass $M$ and electrostatic energy $Q^2/2R$ lying outside $R$. That is to evaluate gravitational field energy, subtract non-gravitational matter energy $E_{\text{non-grav}} = M - Q^2/2R$ from total energy, $E_{\text{tot}}(R)$ contained inside radius $R$. Thus gravitational field energy lying outside $R$ is given by

$$E_{\text{grav}}(R) = E_{\text{tot}} - (M - Q^2/2R) = R - \sqrt{R^2 - 2MR + Q^2} - (M - Q^2/2R). \tag{11}$$

Note that total energy, $E_{\text{tot}}(R) = E_{\text{non-grav}}(R) + E_{\text{grav}}(R)$.

The Brown-York prescription for quasilocal energy is good not only because it gives the expected result for gravitational energy in the first approximation but also it has attracted serious consideration and like mass its positivity has also been proven [41].

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This equation of state for black hole has been proposed in [34] from altogether a different perspective.

As discussed above thin shell with \( p_t = \frac{1}{2} \rho \) provides an example of one such realization.