Spinning particles on $S^2$ in accord with the Bianchi classification

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Abstract
Motivated by recent studies of superconformal mechanics extended by spin degrees of freedom, we construct minimally superintegrable models of spinning particles on $S^2$, the spin degrees of freedom of which are represented by a 3–vector obeying the structure relations of a 3d real Lie algebra. Generalisations involving an external field of the Dirac monopole, or the motion on the group manifold of $SU(2)$, or a scalar potential giving rise to two quadratic constants of the motion are discussed. A procedure how to build similar extensions, which rely upon $d = 4, 5, 6$ real Lie algebras, is elucidated.

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1. Introduction

Over the last few decades, models of superconformal mechanics attracted a considerable amount of attention (for a review see [1]). For one thing, they proved useful for describing a super 0–brane propagating on a near horizon extreme black hole background as well as for a microscopic description of the latter. For another thing, they are relevant for understanding the $AdS_2/CFT_1$—correspondence.

More recently, the focus of research shifted to the study of superconformal mechanics extended by spin degrees of freedom [2]–[12]. Such variables typically arise when gauging $U(n)$ isometry of the matrix superfield systems [2] or supersymmetrizing the Euler–type extension of the Calogero model [5]. Because the new variables entail a richer structure of admissible couplings, one can bypass some long–standing problems. To mention a few, the spin–extended superconformal models can be formulated for an arbitrary number of particles and they can accommodate an arbitrary even number of supersymmetries in a way compatible with non–trivial interactions [5, 6].

In a recent work [7], an alternative approach of introducing spin degrees of freedom was advocated, which promoted a dynamical realization of $su(2)$ associated with the model of a relativistic spinning particle propagating on a spherically symmetric curved background to that of the $D(2, 1; a)$ superconformal mechanics.¹ The spin variables were represented by a symmetric Euler top.

Within the Hamiltonian formalism, the Euler top is usually described by the angular velocity vector $J_i$, $i = 1, 2, 3$, obeying the $su(2)$ structure relations $\{J_i, J_j\} = \epsilon_{ijk} J_k$ under the Poisson bracket. Classification of 3d real Lie algebras was accomplished in [13], where $su(2)$ was identified with the type–IX algebra. One may wonder whether the construction in [7] can be generalised to cover other instances from the Bianchi classification.

The aim of this paper is to construct a minimally superintegrable spinning particle on $S^2$, the spin degrees of freedom of which are represented by a 3–vector obeying the structure relations of a 3d real Lie algebra. An extension to the $D(2, 1; a)$ superconformal mechanics is then straightforward [14], which would result in a supermultiplet of the type $(3, 4, 1)$ accompanied by the spin degrees of freedom.²

The work is organised as follows. In Sec. 2, we briefly remind how a dynamical realization of $su(2)$ on some Poisson manifold can be extended to accommodate the $D(2, 1; a)$ superconformal symmetry. In Sec. 3, minimally superintegrable spinning particles on $S^2$ are constructed and ranked in accord with the Bianchi classification [13]. The construction includes a few steps. First, one chooses a 3d real Lie algebra with generators $J_i$, $i = 1, 2, 3$, and structure constants $c_{ij}^k$ and identifies $J_i$ with the spin degrees of freedom obeying the

¹The exceptional supergroup $D(2, 1; a)$ describes the most general $N = 4$ supersymmetric extension of the conformal group in one dimension $SO(2, 1)$ (see e.g. the discussion in [1]). The structure relations of the corresponding Lie superalgebra involve a real parameter $a$.

²In modern literature, it is customary to associate the non–relativistic spin with the $SU(2)$ group. Other options are usually referred to as "internal degrees of freedom". In this work, we loosely use the term "spin variables" irrespectively of a 3d real Lie algebra at hand.

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(degenerate) Poisson bracket \( \{ J_i, J_j \} = \epsilon_{ijk} J_k \). Then one decomposes the spin vector \( \vec{J} \) on the orthonormal frame attached to a point on \( S^2 \) embedded in \( \mathbb{R}^3 \) and extends the conventional angular momentum vector \( \vec{L} \) of a free particle on \( S^2 \) to include the spin part \( \vec{J} = \vec{L} + \vec{J} \).

Afterwards, the Poisson brackets among the momenta canonically conjugate to the angular variables and the spin degrees of freedom are fixed from the requirement that \( \vec{J} \) obeys the \( su(2) \) structure relations, the condition that the Casimir element of a 3d real Lie algebra be an integral of motion of a dynamical system governed by the Hamiltonian \( H = \frac{1}{2} \vec{J}^2 \), and the fulfilment of the Jacobi identities. In Sec. 4, a qualitative dynamical behaviour of the systems is analysed and another perspective on the material in Sec. 3 is offered. A procedure how to build similar minimally superintegrable extensions, which rely upon \( d = 4, 5, 6 \) real Lie algebras, is elucidated. Generalisations of the models in Sec. 3, which are compatible with the minimal superintegrability, are given in Sec. 5. In the concluding Sec. 6, we summarise our results. Poisson brackets among the momenta canonically conjugate to the angular variables and the spin degrees of freedom are gathered in Appendix.

2. Extending a dynamical realization of \( su(2) \) to that of \( D(2, 1; a) \)

Following Ref. [14], let us briefly remind how a dynamical realization of \( su(2) \) on a Poisson manifold can be extended to accommodate the \( D(2, 1; a) \) superconformal symmetry.

Let a Poisson manifold be parametrized by real variables \( \Gamma^\alpha, \alpha = 1, \ldots, d \), obeying the bracket \( \{ \Gamma^\alpha, \Gamma^\beta \} = \Omega_{\alpha\beta}(\Gamma) \). It is assumed that \( \Omega_{\alpha\beta}(\Gamma) = -\Omega_{\beta\alpha}(\Gamma) \) and the Jacobi identities are satisfied. In general, \( \Omega_{\alpha\beta}(\Gamma) \) is allowed to be a degenerate matrix.

Consider three functions \( J_i(\Gamma) \), \( i = 1, 2, 3 \), which obey the \( su(2) \) structure relations

\[
\{ J_i, J_j \} = \epsilon_{ijk} J_k, \tag{1}
\]

where \( \epsilon_{ijk} \) is the Levi–Civita symbol with \( \epsilon_{123} = 1 \). The Lie superalgebra associated with the exceptional superconformal group \( D(2, 1; a) \) involves two \( su(2) \) subalgebras, the first of which is identified with \( R \)–symmetry, while the second transforms fermions only. Below, \( J_i \) will enter the \( R \)–symmetry generator.

In order to incorporate (1) into a dynamical realization of \( D(2, 1; a) \), it suffices to extend \( \Gamma^\alpha \) by bosonic canonical variables \( (x, p) \) and a pair of complex conjugate \( SU(2) \)–spinors \( \psi_\alpha, \bar{\psi}^\alpha = (\psi_\alpha)^* \), \( \alpha = 1, 2 \), which satisfy the brackets

\[
\{ x, p \} = 1, \quad \{ \psi_\alpha, \bar{\psi}^\beta \} = -i \delta^\alpha_\beta. \tag{2}
\]

On such an extended Poisson supermanifold one then considers the set of functions

\[
H = \frac{p^2}{2} + 2a^2 \frac{x^2}{x^2} J_i J_i + 2a \frac{x^2}{x^2} (\bar{\psi} \sigma_i \psi) J_i - \frac{(1 + 2a)}{4x^2} \psi^2 \bar{\psi}^2, \quad D = tH - \frac{1}{2} xp, \\
K = t^2 H - txp + \frac{1}{2} x^2, \quad \mathcal{L}_i = J_i + \frac{1}{2} (\bar{\psi} \sigma_i \psi), \tag{3}
\]
where $a$ is an arbitrary real parameter, $\psi^2 = \psi^\alpha \psi_\alpha$, $\bar{\psi}^2 = \bar{\psi}_\alpha \bar{\psi}^\alpha$, $\bar{\psi}\psi = \bar{\psi}^\alpha \psi_\alpha$, and $(\sigma_i)_\alpha^\beta$ are the Pauli matrices. These prove to reproduce the structure relations of the Lie superalgebra corresponding to $D(2,1; a)$ under the Poisson bracket chosen [14]. $H$ is the Hamiltonian of the resulting dynamical system. $D$ and $K$ are the generators of dilatations and special conformal transformations. $Q_\alpha$ and $S_\alpha$ are linked to supersymmetry transformations and superconformal boosts, while $L_i$ and $I_\pm, I_3$ generate two $su(2)$ subalgebras.

The simplest realization of (1) is provided by the angular momentum vector of a free particle on $S^2$, in which case (3) describes an on-shell $(3,4,1)$ supermultiplet. Vector fields dual to the conventional left–invariant one–forms on $SU(2)$ group manifold give rise to an off–shell $(4,4,0)$ supermultiplet. By properly adjusting $su(2)$ generators characterising a relativistic spinning particle propagating on a spherically symmetric curved background, one can achieve an extension of such supermultiplets by $SU(2)$–spin variables [7].

In the next section, we construct a spinning particle on $S^2$, the spin degrees of freedom of which are represented by a 3–vector obeying the structure relations of a generic 3d real Lie algebra. Making use of the extended framework (3), one can automatically build the corresponding spinning extension of the $(3,4,1)$ supermultiplet.

3. Spinning particles on $S^2$ in accord with the Bianchi classification

A group–theoretic description of a free particle on $S^2$ identifies the geodesic Hamiltonian $H = \frac{1}{2}g_{ij}p_ip_j$ with the Casimir element of $su(2)$ represented by the angular momentum vector $\vec{L}$

$$H = \frac{1}{2} \vec{L}^2, \quad \vec{L} = \left( \begin{array}{c} -p_\theta \sin \phi - p_\phi \cot \theta \cos \phi \\ p_\theta \cos \phi - p_\phi \cot \theta \sin \phi \\ p_\phi \end{array} \right),$$

where $(\theta, p_\theta)$ and $(\phi, p_\phi)$ are canonical pairs obeying the conventional Poisson brackets

$$\{\theta, p_\theta\} = 1, \quad \{\phi, p_\phi\} = 1.$$

In order to build a spinning extension of (5), let us consider a 3d real Lie algebra with generators $J_i$, $i = 1, 2, 3$, and structure constants $c^k_{ij}$ and identify $J_i$ with the spin degrees of freedom obeying the bracket

$$\{J_i, J_j\} = c^k_{ij} J_k.$$

Classification of 3d real Lie algebras dates back to the work of Bianchi [13]. The available options are displayed below in Table 1 (we follow a modern exposition in [15]), which also contains the Casimir invariant for each case.
In what follows we assume that $J_i$ commute with $(\theta, \phi)$

$$\{\theta, J_i\} = 0, \quad \{\phi, J_i\} = 0,$$  \hspace{1cm} (8)

while the brackets $\{p_\theta, p_\phi\}, \{p_\theta, J_i\}, \{p_\phi, J_i\}$ will be fixed below (see (18)). The abelian type–I case will be disregarded as it is of little physical interest.

Note that it is customary nowadays to use such a formalism for the Hamiltonian description of rigid body dynamics. For example, focusing on $su(2)$ for which $c_{ij}^k = \epsilon_{ijk}$ and choosing the Hamiltonian in the form $H = \frac{1}{2} (g_1 J_1^2 + g_2 J_2^2 + g_3 J_3^2)$, where $(g_1, g_2, g_3)$ are constants (moments of inertia), one can represent the Euler top equations as $\dot{J}_i = \{J_i, H\}$.

Further examples of such a kind can be found in [16].

As the next step, one considers a unit two–sphere embedded in $\mathcal{R}^3$, builds an orthonormal frame at each point\(^3\)

$$\vec{e}_\theta = \begin{pmatrix} \cos \theta \cos \phi \\ \cos \theta \sin \phi \\ - \sin \theta \end{pmatrix}, \quad \vec{e}_\phi = \begin{pmatrix} - \sin \phi \\ \cos \phi \\ 0 \end{pmatrix}, \quad \vec{e}_r = \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix},$$

(9)

such that $\vec{L} = p_\theta \vec{e}_\phi - \frac{p_\phi}{\sin \theta} \vec{e}_\theta$, introduces the spin vector \([7]\)

$$\vec{J} = g_1 J_1 \vec{e}_r + g_2 J_2 \vec{e}_\theta + g_3 J_3 \vec{e}_\phi, \quad \vec{J}^2 = g_1^2 J_1^2 + g_2^2 J_2^2 + g_3^2 J_3^2,$$

(10)

where $(g_1, g_2, g_3)$ are nonzero constants (moments of inertia), and finally extends the orbital angular momentum in (5) to include the spin part

$$\vec{J} = \vec{L} + \vec{J}.$$  \hspace{1cm} (11)

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\(^3\)Given the parametric representation $x = \sin \theta \cos \phi, y = \sin \theta \sin \phi, z = \cos \theta$, one computes two vectors $(x', y', z')$, $(x'_\phi, y'_\phi, z'_\phi)$, which specify the tangent plane at a point $(\theta, \phi)$, normalizes them to have the unit length, and then computes their vector product.
After that, one demands \( \mathcal{J} \) to obey the \( su(2) \) structure relations and identifies the Hamiltonian of a spinning particle on \( S^2 \) with the Casimir element

\[
H = \frac{1}{2} \mathcal{J}_x \mathcal{J}_z = \frac{1}{2} \left( p_\theta^2 + \frac{p_\phi^2}{\sin^2 \theta} \right) + \frac{1}{2} \left( g_1^2 J_1^2 + g_2^2 J_2^2 + g_3^2 J_3^2 \right) - \frac{g_2 J_2 p_\phi}{\sin \theta} + g_3 J_3 p_\theta.
\] (12)

Note that discarding the spin degrees of freedom one reproduces the Hamiltonian of a free particle on \( S^2 \). Omitting the angular variables, one gets the Hamiltonian typical for describing 3d rigid body dynamics \[16\].

From the equations \( \{ \mathcal{J}_i, \mathcal{J}_j \} = \epsilon_{ijk} \mathcal{J}_k \) one finds three Poisson brackets

\[
\{ p_\theta, p_\phi \} = -g_1 J_1 \sin \theta - g_2 J_2 \cos \theta - g_2 g_3 \{ J_2, J_3 \} \sin \theta + g_2 \{ p_\theta, J_2 \} \sin \theta + g_3 \{ p_\phi, J_3 \},
\]

\[
\{ p_\theta, J_1 \} = g_3 \{ J_1, J_3 \}, \quad \{ p_\phi, J_1 \} = -g_2 \{ J_1, J_2 \} \sin \theta,
\] (13)

with \( \{ p_\theta, J_2 \} \) and \( \{ p_\phi, J_3 \} \) to be fixed below. In obtaining Eqs. (13), we used the fact that the triple \( (\bar{e}_r, \bar{e}_\theta, \bar{e}_\phi) \) is closed under the Poisson action of \( (p_\theta, p_\phi) \)

\[
\{ p_\theta, \bar{e}_r \} = \bar{e}_r, \quad \{ p_\theta, \bar{e}_\theta \} = 0, \quad \{ p_\theta, \bar{e}_\phi \} = -\bar{e}_\theta,
\]

\[
\{ p_\phi, \bar{e}_r \} = -\cos \theta \bar{e}_\phi, \quad \{ p_\phi, \bar{e}_\theta \} = \sin \theta \bar{e}_r + \cos \theta \bar{e}_\phi, \quad \{ p_\phi, \bar{e}_\phi \} = -\sin \theta \bar{e}_r,
\] (14)

and took into account the identities

\[
\begin{align*}
(e_r)_i (e_\theta)_j - (e_\theta)_i (e_r)_j &= \epsilon_{ijk} (e_\phi)_k, \\
(e_\theta)_i (e_\phi)_j - (e_\phi)_i (e_\theta)_j &= -\epsilon_{ijk} (e_\theta)_k.
\end{align*}
\] (15)

It is straightforward to verify that the second and third brackets in (13) ensure the relations

\[
\{ J_1, \mathcal{J}_i \} = 0.
\] (16)

Thus, at this stage the system is characterised by three functionally independent integrals of motion in involution \( H, \mathcal{J}_3 \) and \( J_1 \).

It seems reasonable to fix the remaining Poisson brackets from the requirement that the ensuing spinning particle on \( S^2 \) be integrable. Focusing on the spin sector, one has two equations of motion in an unparametrized form and hence one more function commuting with \( (H, \mathcal{J}_3, J_1) \) is needed in order to provide the Liouville integrability. From the group–theoretic standpoint, a natural choice is the Casimir element \( \mathcal{I} \) exposed above in Table 1. Demanding \( \{ \mathcal{I}, \mathcal{J}_3 \} = 0 \) and \( \{ \mathcal{I}, H \} = 0 \), one obtains

\[
\{ p_\phi, \mathcal{I} \} = \{ p_\phi, J_i \} \frac{\partial \mathcal{I}}{\partial J_i} = 0, \quad \{ p_\theta, \mathcal{I} \} = \{ p_\theta, J_i \} \frac{\partial \mathcal{I}}{\partial J_i} = 0.
\] (17)

Depending on a 3d real Lie algebra at hand, these equations allow one to express two brackets in terms of the other. Taking into account (13), one concludes that two brackets among \( (p_\theta, p_\phi) \) and \( J_i \) are still missing.
As the final step, one requires the Jacobi identities to hold, which ultimately give
\[
\{p_\theta, p_\phi\} = -g_1 J_1 \sin \theta - g_2 J_2 \cos \theta - g_1 g_3 \{J_1, J_3\} \cos \theta + g_2 g_3 \{J_2, J_3\} \sin \theta, \\
\{p_\theta, J_1\} = g_3 \{J_1, J_3\}, \\
\{p_\theta, J_2\} = g_3 \{J_2, J_3\}, \\
\{p_\theta, J_3\} = 0, \\
\{p_\phi, J_1\} = -g_2 \{J_1, J_2\} \sin \theta, \\
\{p_\phi, J_2\} = -g_1 \{J_1, J_2\} \cos \theta, \\
\{p_\phi, J_3\} = -g_1 \{J_1, J_3\} \cos \theta + g_2 \{J_2, J_3\} \sin \theta. 
\]  
(18)

This completes our construction of a Hamiltonian formulation for a spinning particle on \(S^2\), the spin degrees of freedom of which are described by a 3–vector obeying the structure relations of a generic 3d real Lie algebra. For the reader’s convenience, we expose the Poisson brackets (18) for each instance in the Bianchi classification as well as the second invariant \(I\) in Appendix.

Note that the resulting system is minimally superintegrable as the Liouville integrals of motion \((H, J_3, J_1, I)\) can be extended to include \(J_2\) (or alternatively \(J_1\)). It is straightforward to verify that \((H, J_3, J_1, I, J_2)\) are functionally independent, the only nonzero bracket being \(\{J_2, J_3\}\).

It might seem odd that the Poisson brackets among the spin degrees of freedom \(J_i\) and the angular variables \((p_\theta, p_\phi)\) are not canonical. Yet, it is worth recalling the Poisson structure underlying a general relativistic spinning particle on a curved background [17]
\[
\{x^\mu, p_\nu\} = \delta^\mu_\nu, \\
\{p_\mu, p_\nu\} = -\frac{1}{2} R^\nu_{\mu\lambda\sigma} S^{\lambda\sigma}, \\
\{S^{\mu\nu}, p_\lambda\} = \Gamma^\mu_{\lambda\sigma} S^{\nu\sigma} - \Gamma^\nu_{\lambda\sigma} S^{\mu\sigma}, \\
\{S^{\mu\nu}, S^{\lambda\sigma}\} = g^{\mu\lambda} S^{\nu\sigma} + g^{\nu\sigma} S^{\mu\lambda} - g^{\mu\sigma} S^{\nu\lambda} - g^{\nu\lambda} S^{\mu\sigma}, 
\]  
(19)
where \((x^\mu, p_\mu)\), \(\mu = 0, 1, 2, 3\), are the canonical variables, \(S^{\mu\nu} = -S^{\nu\mu}\) are the spin degrees of freedom, \(g^{\mu\nu}\) is the inverse metric tensor, \(\Gamma^\mu_{\lambda\sigma}\) are the Christoffel symbols, and \(R^\nu_{\mu\lambda\sigma}\) is the Riemann tensor. Focusing on a spherically symmetric background and properly reducing the corresponding \(su(2)\) Killing vector fields, one arrives at a Bianchi type–IX spinning particle on \(S^2\) [7], which is a particular member in the set of models constructed above.

4. Qualitative dynamics

Let us briefly discuss a qualitative dynamical behaviour of the integrable systems built in the preceding section. Passing to the Cartesian coordinates \(\vec{x} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)\), one gets the relations
\[
(\vec{x}, \vec{\dot{x}}) = g_1 J_1, \\
\dot{x}_i \dot{x}_i = \dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta = 2H - g_1^2 J_1^2, 
\]  
(20)
which imply a uniform motion along a circular orbit on \(S^2\), which is an intersection of the cone and the sphere. The apex semi–angle of the cone depends on the energy of the full

\footnote{The momenta \((p_\theta, p_\phi)\) entering the \(su(2)\) generators in (5) are conventionally defined up to a pure gauge vector potential \(p_\theta \to p_\theta + A_\theta(\theta, \phi), p_\phi \to p_\phi + A_\phi(\theta, \phi), \partial_\theta A_\phi - \partial_\phi A_\theta = 0\). When analysing the Jacobi identities, we discarded a pure gauge vector field contributions.}
system and the conserved spin component $J_1$

$$\cos \alpha = \frac{g_1 J_1}{\sqrt{2H}}.$$  \hspace{1cm} (21)

If $g_1 J_1 = 0$ the cone opens to the plane $x_i J_i = 0$, and the orbit becomes a great circle.

Turning to the spin sector, one reveals a conserved component $J_1 = \text{const}$, while $J_2$ and $J_3$ swing in a way dependent on the angular variables. The analysis becomes more transparent if one changes $(p_\theta, p_\phi)$ so as to partially diagonalize the brackets

$$p'_\theta = p_\theta + g_3 J_3, \quad p'_\phi = p_\phi + g_1 J_1 \cos \theta - g_2 J_2 \sin \theta,$$

$$\{p'_\theta, J_i\} = 0, \quad \{p'_\phi, J_i\} = 0, \quad \{p'_\theta, p'_\phi\} = 0.$$ \hspace{1cm} (22)

Then (12) simplifies to

$$H' = \frac{1}{2} \left( p'_\theta^2 + \frac{(p'_\phi - g_1 J_1 \cos \theta)^2}{\sin^2 \theta} + g_1^2 J_1^2 \right).$$ \hspace{1cm} (23)

Remarkably enough, $H'$ coincides with the Hamiltonian of a particle on $S^2$ in the presence of a magnetic monopole field, the magnetic charge being promoted to the conserved spin component $J_1$ multiplied by a constant $g_1$. In this coordinate system it becomes evident that the evolution of $(\theta, \phi)$ is identical to that of a particle on $S^2$ coupled to an external field of the Dirac monopole, while $J_2$, $J_3$ satisfy the equation

$$\dot{J}_{2,3} = \frac{g_1}{\sin^2 \theta} \left( p'_\phi \cos \theta - g_1 J_1 \right) \{J_1, J_{2,3}\}.$$ \hspace{1cm} (24)

Having solved the equations of motion for the angular variables, one can redefine the temporal parameter

$$\tau(t) = g_1 \int dt \left( \frac{p'_\phi(t) \cos \theta(t) - g_1 J_1}{\sin^2 \theta(t)} \right) + \tau_0,$$ \hspace{1cm} (25)

where $\tau_0$ is a constant, and reduce (24) to the linear equations

$$J'_{2,3} = \{J_1, J_{2,3}\},$$ \hspace{1cm} (26)

the prime indicating the derivative with respect to $\tau$, which can be easily integrated. The results are given below in Table 2, in which $C_1$ and $C_2$ denote constants of integration. Because the motion on $S^2$ is periodic, so is the swinging of the vector with components $(J_2(t), J_3(t))$ in the tangent plane.
Let us choose a 3

{Poisson bracket

\{q\} generators involve the magnetic charge

\(J\) of the Dirac monopole. The system is known to be

\(su(5)\). Generalisations

then follows as described in Sec. 2.

Concluding this section, we note that the Hamiltonian (23) offers another perspective on

the material in the preceding section. Consider a particle on \(S^2\) coupled to an external field

of the Dirac monopole. The system is known to be \(su(2)\) invariant and the corresponding

generators involve the magnetic charge \(q\)

\[\vec{L'} = \begin{pmatrix} -p_\theta \sin \phi - p_\phi \cot \theta \cos \phi + q \cos \phi \sin^{-1} \theta \\ p_\theta \cos \phi - p_\phi \cot \theta \sin \phi + q \sin \phi \sin^{-1} \theta \\ p_\phi \end{pmatrix}, \quad \{L'_i, L'_j\} = \epsilon_{ijk} L'_{k}. \quad (27)\]

Let us choose a 3d real Lie algebra and use its structure constants to specify the (degenerate)

Poisson bracket \(\{J_i, J_j\} = \epsilon^{kj}_i J_k.\) Implementing the oxidation with respect to \(q\)

\[q \rightarrow g_1 J_1, \quad (28)\]

where \(g_1\) is a constant, and taking the Casimir element of \(su(2)\) to be the Hamiltonian of the

extended system \(H = \frac{1}{2} \vec{L}^2\), one obtains a spinning extension for which \(J_1\) is a linear integral

of motion.

In Ref. [18], the Casimir invariants were found for all real algebras of dimension up to five

and for all nilpotent real algebras of dimension six. Implementing the oxidation (28), one

automatically gets similar integrable extensions of a free particle on \(S^2\), the internal degrees

degree of freedom of which satisfy a \(d = 4, 5, 6\) real Lie algebra. \(D(2, 1; a)\) supersymmetrization

then follows as described in Sec. 2.

5. Generalisations

Let us discuss some directions in which the analysis in Sec. 3 can be generalised.

When decomposing the spin vector \(\vec{J}\) on the basis \((\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)\) in Eq. (10) above, we chose

\(J_1\) to be a companion of \(\vec{e}_r\). It was later established that \(J_1\) is a constant of the motion of the

spinning particle on \(S^2\), while the interchange of \(J_2\) and \(J_3\) affects the resulting system

Table 2. Evolution of the spin degrees of freedom with time

| Type | \(J_2(t)\) | \(J_3(t)\) |
|------|-------------|-------------|
| type II | \(C_1\) | \(C_2\) |
| type III | \(C_1 (1 + e^{2\tau(t)}) + C_2 (1 - e^{2\tau(t)})\) | \(C_1 (1 - e^{2\tau(t)}) + C_2 (1 + e^{2\tau(t)})\) |
| type IV | \(C_1 e^{\tau(t)} + C_2 \tau(t) e^{\tau(t)}\) | \(C_2 \tau(t) e^{\tau(t)}\) |
| type V | \(C_1 e^{\tau(t)}\) | \(C_2 e^{\tau(t)}\) |
| type VI | \(C_1 (e^{(a-1)\tau(t)}) + C_2 (e^{(a+1)\tau(t)})\) | \(C_1 (e^{(a-1)\tau(t)} - e^{(a+1)\tau(t)}) + C_2 (e^{(a+1)\tau(t)} + e^{(a-1)\tau(t)})\) |
| type VII | \(C_1 \cos \tau(t) + C_2 \sin \tau(t)\) | \(-C_1 \sin \tau(t) - C_2 \cos \tau(t)\) |
| type VIII | \(C_1 \cosh \tau(t) + C_2 \sinh \tau(t)\) | \(-C_1 \sinh \tau(t) - C_2 \cosh \tau(t)\) |
| type IX | \(C_1 \cos \tau(t) + C_2 \sin \tau(t)\) | \(-C_1 \sin \tau(t) + C_2 \cos \tau(t)\) |
only slightly. In a similar fashion one could build models in which either $J_2$ or $J_3$ would be a linear integral of motion. As follows from Table 1, in the latter two cases the Casimir element would degenerate either to linear or quadratic integral of motion depending on the item in the Bianchi classification.

The models in Sec. 3 can be readily coupled to an external field of the Dirac monopole. It suffices to extend (11)

$$\vec{J} \rightarrow \vec{J}' = \vec{J} + q \vec{B}, \quad \vec{B} = \begin{pmatrix} \cos \phi \sin^{-1} \theta \\ \sin \phi \sin^{-1} \theta \\ 0 \end{pmatrix},$$

(29)

where $q$ is the magnetic charge. The presence of the external field alters the dynamics only slightly. For the orbital motion one finds

$$(\vec{x}, \vec{J}')] = g_1 J_1 + q, \quad \dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta = 2H - (g_1 J_1 + q)^2.$$  

(30)

In particular, at $g_1 J_1 + q = 0$ the impact of the spin degrees of freedom on the orbital motion on $S^2$ is compensated by the external field.

The model (29) can be further generalised by introducing into the consideration an extra canonical pair $(\chi, p_\chi)$, obeying the standard Poisson bracket \{\chi, p_\chi\} = 1, and implementing the oxidation with respect to $q$

$$q \rightarrow p_\chi.$$  

(31)

The corresponding $su(2)$ generators are the building blocks to construct a spinning particle propagating on the group manifold of $SU(2)$. Its Liouville integrability is provided by five functionally independent integrals of motion in involution $(H, J_3, J_1, I, p_\chi)$.

As was mentioned after Eq. (21), for $g_1 = 0$ the orbit on $S^2$ is a great circle. In this case one can add external scalar potential without spoiling the minimal superintegrability. It suffices to consider three functions

$$I_1 = J_1^2 + (\nu_1 \sin^{-1} \phi \cot \theta + \nu_3 \sin \phi \tan \theta)^2,$$

$$I_2 = J_2^2 + (\nu_2 \cos^{-1} \phi \cot \theta + \nu_3 \cos \phi \tan \theta)^2,$$

$$I_3 = J_3^2 + (\nu_1 \cot \phi + \nu_2 \tan \phi)^2,$$

(32)

where $(\nu_1, \nu_2, \nu_3)$ are (coupling) constants and verify that they commute with the Hamiltonian $H = \frac{1}{2}(I_1 + I_2 + I_3)$. Four functionally independent integrals of motion in involution include $(H, I_1, I, J_1)$. Adding $I_2$ (or alternatively $I_3$) renders the model minimally superintegrable.

6. Conclusion

To summarise, in this work we have constructed a minimally superintegrable spinning particle on $S^2$, the spin degrees of freedom of which are represented by a 3–vector obeying the
structure relations of a 3d real Lie algebra in accord with the Bianchi classification. Generalisations involving an external field of the Dirac monopole, or the motion on the group manifold of $SU(2)$, or a scalar potential giving rise to two quadratic constants of the motion were proposed. It was argued that similar integrable extensions, the internal degrees of freedom of which satisfy $d = 4, 5, 6$ real Lie algebra, can be constructed by considering a particle on $S^2$ coupled to an external field of the Dirac monopole and implementing an oxidation with respect to the magnetic charge.

As a possible further development, it would be interesting to study whether integrable spinning extensions of a particle on $S^2$ (or $S^3$) can be constructed beyond the oxidation scheme in Sec. 4, i.e. avoiding a linear integral of motion in the spin sector.

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Appendix

In this Appendix, we display the Poisson brackets among $p_\theta$, $p_\phi$, and $J_i$ for each instance in the Bianchi classification. We omit the abelian type–I case as it is of little physical interest. For the type–II case, $J_1$ coincides with the Casimir invariant $I$. In order to provide integrability, $J_2$ was chosen to be the second integral of motion.

|          | type II | type III | type IV | type V   | type VI  |
|----------|---------|----------|---------|----------|----------|
| $\{p_\theta, J_1\}$ | 0       | $-g_3(J_2 - J_3)$ | $g_3J_3$ | $g_3J_3$ | $-g_3(J_2 - aJ_3)$ |
| $\{p_\theta, J_2\}$ | $g_3J_1$ | 0        | 0       | 0        | 0        |
| $\{p_\theta, J_3\}$ | 0       | 0        | 0       | 0        | 0        |
| $\{p_\phi, J_1\}$ | 0       | $-g_2(J_2 - J_3)\sin\theta$ | $-g_2(J_2 + J_3)\sin\theta$ | $-g_2J_2\sin\theta$ | $-g_2(aJ_2 - J_3)\sin\theta$ |
| $\{p_\phi, J_2\}$ | 0       | $-g_1(J_2 - J_3)\cos\theta$ | $-g_1(J_2 + J_3)\cos\theta$ | $-g_1J_2\cos\theta$ | $-g_1(aJ_2 - J_3)\cos\theta$ |
| $\{p_\phi, J_3\}$ | $g_2J_1\sin\theta$ | $g_1(J_2 - J_3)\cos\theta$ | $-g_1J_3\cos\theta$ | $-g_1J_3\cos\theta$ | $g_1(J_2 - aJ_3)\cos\theta$ |
| $\{p_\phi, p_\phi\}$ | $(g_2g_3 - g_1)\times J_1\sin\theta$ | $(g_1g_3 - g_2)J_2\cos\theta$ | $-g_1J_1\sin\theta$ | $-g_1J_1\sin\theta$ | $-g_1J_1\sin\theta$ |
|          | $-g_2J_2\cos\theta$ | $-g_1g_3J_3\cos\theta$ | $-g_1g_3J_3\cos\theta$ | $-g_1g_3J_3\cos\theta$ | $-g_1g_3J_3\cos\theta$ |

$I$  

$J_2$  

$J_2 + J_3$  

$rac{J_2}{J_3} - \ln J_3$  

$rac{J_2}{J_3} J_3^2 \left(1 + \frac{J_2}{J_3}\right)^{1+a} \times \left(1 - \frac{J_2}{J_3}\right)^{1-a}$
\[ \begin{array}{|c|c|c|c|c|c|}
\hline
& \text{type VI}_0 & \text{type VII} & \text{type VII}_0 & \text{type VIII} & \text{type IX} \\
\hline
\{p_0, J_1\} & g_1 J_2 & -g_3 (J_2 - a J_3) & -g_3 J_2 & -g_3 J_2 & -g_3 J_2 \\
\hline
\{p_0, J_2\} & g_3 J_1 & 0 & g_3 J_1 & g_3 J_1 & g_3 J_1 \\
\hline
\{p_0, J_3\} & 0 & 0 & 0 & 0 & 0 \\
\hline
\{p_0, J_1\} & 0 & -g_2 (a J_2 + J_3) \sin \theta & 0 & g_2 J_3 \sin \theta & -g_2 J_3 \sin \theta \\
\hline
\{p_0, J_2\} & 0 & -g_1 (a J_2 + J_3) \cos \theta & 0 & g_1 J_3 \cos \theta & -g_1 J_3 \cos \theta \\
\hline
\{p_0, J_3\} & g_2 J_1 \sin \theta & (g_2 g_3 - g_1) \times J_1 \sin \theta & g_2 J_1 \sin \theta & (g_2 g_3 - g_1) \times J_1 \sin \theta & (g_2 g_3 - g_1) \times J_1 \sin \theta \\
& -g_1 J_2 \cos \theta & + (g_1 g_3 - g_2) J_2 \cos \theta & + (g_1 g_3 - g_2) J_2 \cos \theta & + (g_1 g_3 - g_2) J_2 \cos \theta & + (g_1 g_3 - g_2) J_2 \cos \theta \\
& + (g_1 g_3 - g_2) J_2 \cos \theta & -a g_1 g_3 J_3 \cos \theta & -a g_1 g_3 J_3 \cos \theta & -a g_1 g_3 J_3 \cos \theta & -a g_1 g_3 J_3 \cos \theta \\
\hline
\{p_0, p_0\} & \times \frac{1}{2} \arctan \frac{J_2}{\sqrt{J_2^2 + J_3^2}} & \times \frac{1}{2} \arctan \frac{J_2}{\sqrt{J_2^2 + J_3^2}} & \times \frac{1}{2} \arctan \frac{J_2}{\sqrt{J_2^2 + J_3^2}} & \times \frac{1}{2} \arctan \frac{J_2}{\sqrt{J_2^2 + J_3^2}} & \times \frac{1}{2} \arctan \frac{J_2}{\sqrt{J_2^2 + J_3^2}} \\
\hline
\end{array} \]

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