SPATIAL PROPAGATION FOR A PARABOLIC SYSTEM WITH MULTIPLE SPECIES COMPETING FOR SINGLE RESOURCE

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ABSTRACT. A model of $m$ species competing for a single growth-limiting resource is considered. We aim to use the dynamics of such a problem to describe the invasion and spread of $m$ species which are introduced localized in space $\mathbb{R}^N$. The existence, uniqueness and uniform boundedness of the Cauchy problem are investigated by semigroup theory and local $L^p$-estimates. The asymptotic speed of spread is achieved by uniform persistence ideas. The existence of traveling wave is obtained by upper-lower solutions and sliding techniques. Our result shows that the asymptotic speed of spread for $m$ species is characterized by the minimum wave speed of the positive traveling wave solutions associated with this system.

1. Introduction. In this paper, we consider the following reaction-diffusion system

$$
\begin{align*}
\frac{\partial S}{\partial t} &= d_0 \Delta S - \sum_{i=1}^{m} \gamma_i^{-1} u_i f_i(S), \\
\frac{\partial u_i}{\partial t} &= d_i \Delta u_i + u_i (f_i(S) - k_i), \quad i = 1, 2, \ldots, m,
\end{align*}
$$

(1.1)

where $S(x,t)$ and $u_i(x,t)$ are the concentrations of resource and individuals at position $x \in \mathbb{R}^N (N \geq 1)$ and time $t \in \mathbb{R}^+$, respectively. The parameter $k_i$ is the death rate for the $i$th individual; $\gamma_i$ is yield rate for the $i$th individual feeding on the resource; $d_i$ is the diffusion rate. All those parameters $k_i, \gamma_i$ and $d_i$ are positive constants. The function $f_i$ describes the specific uptake rate of resource by the $i$th individual as a function of resource concentration. They will be assumed to satisfy the assumptions

(i) $f_i : \mathbb{R}^+ \to \mathbb{R}^+$ is monotone increasing with $f_i(0) = 0$;
(ii) $f_i$ is locally Lipschitz continuous.

An important example is the Monod function $f_i(S) = b_i S / (a_i + S)$ where $a_i, b_i > 0$.

Problem (1.1) can be interpreted as an unmixed chemostat model with $m$ species of consumers $u_i$ that compete for a limiting substrate $S$, or as a spatially heterogeneous epidemic model for the spread of an infectious pathogen that comes in $m$ different strains and converts susceptible hosts $S$ into hosts $u_i$ infected with strain $i$. Aside from chemostats or epidemics, the model represents the indirect competition

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between a collection of agents for a common resource, available in limited quan-
tity, wherein each agent, having no direct interaction with its competitors, merely
consumes the common resource so that it may reproduce and offset losses due to
removal or mortality (see [26]).

Our interest in this paper is to study the spatial propagation phenomenon arising
in invasion of species or outbreak of epidemics. When some inoculum of bacteria
were initially introduced into some location in the bio-reactor or some infection
agents were observed at some local epidemic area, one might expect that bacterial
cell or agents would move into new territory. Using such a framework we are
interested in deriving some information about the invasion of the bacteria or agents
in such an environment. To this end, we assume the resource is initially distributed
in the whole space region $\mathbb{R}^N$. More precisely, we assume that the initial conditions

$$S(x,0) = S_0(x), \quad u_i(x,0) = u_{i0}(x), \quad i = 1, 2, \cdots, m, \quad x \in \mathbb{R}^N\quad (1.2)$$

satisfy the following assumptions ($H$):

(i) the functions $S_0 \geq 0$ and $u_{i0} \geq 0, \neq 0$;

(ii) $S_0 \in C(\mathbb{R}^N)$ and $\lim_{|x| \to \infty} S_0(x) = S^* > 0$;

(iii) $u_{i0} \in C(\mathbb{R}^N)$ is compactly supported.

Coming back to system (1.1), we observe that the equation for the resource $S$ has no
feeding term. For this reason, the assumption $\lim_{|x| \to \infty} S_0(x) = S^*$ is a reasonable
condition to ensure the individuals have an involving external supply resource over
their spreading front.

In bounded habitats, the dynamic behaviors of problem (1.1) have been stud-
ied extensively with suitable boundary conditions. For instance, coexistence was
carried out in [14, 21, 33], uniform persistence was studied in [20, 22], asymptotic
behavior was investigated in [12, 13, 22], and more detail results please refer to the
monographs [25] and the references therein.

The existence and uniqueness of traveling wave solutions to problem (1.1) with
$m = 1$ also have received considerable attention. For instance, the existence of
traveling wave solutions was considered in [10, 11, 15, 19, 27, 28, 29, 31], and the
uniqueness of traveling wave solutions was studied in [16, 30]. For the case of $m = 2$,
the partial results on the existence of traveling waves have been obtained in [17] by
shooting method and Lyapunov function. However, there are very few results on
traveling waves of (1.1) with $m > 2$.

It is worth mentioning that despite the asymptotic speed of spread has been intro-
duced by Aronson et al. [1, 2] in the 1970s for scalar reaction-diffusion equation, only
few results about the asymptotic speed of spread for the diffusive prey-predator sys-
tem have been obtained in the literatures. If neglecting the diffusion of the suscep-
tible population in system (1.1) with $m = 1$, some results on the asymptotic speed
for the epidemic models have been achieved. An estimate of asymptotic speed for
an epidemic model for rabies is established in [18]. The asymptotic speeds of spread
for a nonlocal epidemic model and the Kermack-Mekendrick reaction-diffusion sys-
tem are determined in [35] and [7], respectively. Considering the diffusive effect on
the susceptible population, the asymptotic speed of spread for the system (1.1) is
still open, which has been mentioned by [31].

To describe the spatial propagation phenomenon arising in invasion of species or
outbreak of epidemic, we concentrate on the corresponding Cauchy problem (1.1)-
(1.2), which has been neglected in the past decades due to the lack of comparison
principle for the system under consideration. The purpose of this paper is to study
the asymptotic speed of spread, traveling waves and minimal wave speed for system (1.1). Our results provide a clear, coherent picture of the connection between the asymptotic speed and traveling wave solutions, that is, the asymptotic speed of spread for \( m \) populations is characterized by the minimum wave speed of traveling waves.

The paper is organized as follows. In Section 2, we state main results of this paper. In Section 3, we investigate some dynamical properties of the Cauchy problem (1.1)-(1.2), and study the spreading speed of populations. In Section 4, we consider the traveling wave solutions of (1.1), and determine the minimum speed of traveling waves.

2. The main results. In this section we state the main results of the paper. The existence and uniqueness of solution of (1.1)-(1.2) can be obtained by semigroup theory (see Proposition 3.3). The uniform boundedness of solution of (1.1)-(1.2) can be established by \( L^p \) estimates (see Proposition 3.5). Our aim here is to determine the long-time behavior of (1.1)-(1.2).

The biomass reproduction rate of species \( i \) or disease reproduction rate of strain \( i \) is given by

\[
R_0^i = \frac{f_i(S^*)}{k_i}, \quad i = 1, 2, \cdots, m.
\]

According to the definition of the basic reproduction rate in existing literatures (for example, see Chapter 10 in [24]), we introduce the basic reproduction rate of individuals for (1.1) as follows:

\[
R_0 = \min\{R_0^i, i = 1, 2, \cdots, m\}.
\]

Our first result gives the dynamics of (1.1)-(1.2) when \( R_0 < 1 \). In such a situation, the \( i \)th species dies out provided \( R_0^i < 1 \), as indicated by the following theorem.

**Theorem 2.1.** (Extinction) Assume the functions \( S_0 \) and \( \{u_{0i}\}_{i=1}^m \) satisfy the assumptions (H). Let \( (S, u_1, u_2, \cdots, u_m) \) be the solution of (1.1)-(1.2). If \( R_0^i < 1 \) for some \( i \in \{1, 2, \cdots, m\} \), then \( \lim_{t \to \infty} u_i(x, t) = 0 \) uniformly for \( x \in \mathbb{R}^N \). Particularly, if \( \max\{R_0^i, i = 1, 2, \cdots, m\} < 1 \), then \( \lim_{t \to \infty} S(x, t) = S^* \) and \( \lim_{t \to \infty} u_i(x, t) = 0 \) uniformly for \( x \in \mathbb{R}^N \), \( i = 1, 2, \cdots, m \).

In order to go further into the investigation of the dynamical behavior of (1.1)-(1.2) when \( R_0 > 1 \), let us introduce some notations. If \( R_0^i > 1 \), we denote

\[
c_i^* = 2\sqrt{d_i(f_i(S^*) - k_i)}, \quad i = 1, 2, \cdots, m.
\]

Furthermore, if \( R_0 > 1 \), we denote

\[
c^* = \max\{c_i^*, i = 1, 2, \cdots, m\}.
\]

By relabeling the equations without loss of generality, we may assume the parameters \( c_i^* \) satisfying

\[
c_1^* \leq c_2^* \leq \cdots \leq c_m^*.
\]

Then \( c^* = c_m^* \).

The next theorem deals with the spatial spread phenomena for (1.1)-(1.2) when \( R_0 > 1 \).
Theorem 2.2. (Spreading) Assume $R_0 > 1$, the functions $S_0$ and $\{u_{0i}\}_{i=1}^m$ satisfy the assumptions (H). Let $(S, u_1, u_2, \cdots, u_m)$ be the solution of (1.1)-(1.2). The following results hold

(i) For each $c \in (-c^*, c^*)$, each unit vector $e \in \mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ and any $x \in \mathbb{R}^N$, there exists $\epsilon > 0$ and some $i \in \{1, 2, \cdots, m\}$ such that

$$\limsup_{t \to \infty} u_i(x + cte, t) \geq \epsilon.$$  

If $c_i^* < |c| < c^*$ then

$$\limsup_{t \to \infty} u_j(x + cte, t) = 0 \text{ for } 1 \leq j \leq i. \quad (2.1)$$

Particularly, if $c_{m-1}^* < |c| < c_m^*$ then

$$\liminf_{t \to \infty} u_m(x + cte, t) \geq \epsilon, \quad \limsup_{t \to \infty} u_i(x + cte, t) = 0 \text{ for } 1 \leq i \leq m-1. \quad (2.2)$$

(ii) For each $c > c^*$,

$$\limsup_{t \to \infty} u_i(x, t) = 0, \; i = 1, 2, \cdots, m. \quad (2.3)$$

Theorem 2.2 indicates that at least one species locally survives behind the front, and all species go extinct ahead of the front. Namely, some species survive on the expanding spheres $|x| = ct$ with $c \in [0, c^*)$, and all species go extinct on the habitats $|x| > ct$ with $c > c^*$. Unfortunately, we can not definitively determine which species would locally survive when $t \to \infty$. But we know that species $u_1, u_2, \cdots, u_i$ are extinct ultimately if $c > c_i^*$. Particularly, if $c_{m-1}^* < c < c_m^*$, only the species $u_m$ uniformly survives behind the front when $t \to \infty$. Roughly speaking, if an observer was to move forward at a fixed speed greater than $c^*$, the local species density would eventually look like 0. If an observer was to move forward at a fixed speed less than $c^*$, the population density for some species would be strictly larger than 0. Particularly, if $c \in (c_{m-1}^*, c_m^*)$ or $m = 1$, the population density of $u_m$ would eventually be strictly larger than 0. Hence $c^*$ is called the asymptotic spreading speed for the system (1.1)-(1.2).

In order to give more hints about the asymptotic spreading speed $c^*$, we shall look for one-dimensional traveling wave solutions, that are specific solution of (1.1) with the form

$$S = S(x - cte), \; u_i = u_i(x - cte), \quad (2.3)$$

where the unit vector $e \in \mathbb{S}^{N-1}$, $S$ and $u_i$ are the functions of $s = x - cte$. A straight substitution of (2.3) into (1.1) gives the equations for $S(s)$ and $u_i(s)$ as follows

$$\begin{aligned}
&d_0S'' + cS' = \sum_{i=1}^{m} \gamma_i^{-1} u_i f_i(S) \text{ in } \mathbb{R}, \\
&d_i u''_i + cu'_i = u_i (k_i - f_i(S)) \text{ in } \mathbb{R}, \; i = 1, 2, \cdots, m.
\end{aligned} \quad (2.4)$$

Note that (2.4) has infinitely many equilibria $(S_*, 0, \cdots, 0)$ with arbitrary constant $S_* \geq 0$. Here we’re concentrated on the traveling wave solutions which connect two different equilibria $(S^*, 0, \cdots, 0)$ and $(S_*, 0, \cdots, 0)$. Hence we assume that $S(\infty)$ and $u_i(\infty)$ satisfy the boundary conditions

$$S(\infty) = S^* > S(-\infty) = S_*, \; u_i(\pm \infty) = 0. \quad (2.5)$$
Theorem 2.3. (Traveling waves) For any given $S^*$ satisfying $R_0 > 1$, the following results hold.

(i) If $c > c^*$, then there exists $S_* \geq 0$ satisfying

$$\max \left\{ \frac{f_i(S_*)}{k_i}, i = 1, 2, \cdots, m \right\} < 1,$$

such that (2.4)-(2.5) has a positive traveling wave solution $(S, u_1, \cdots, u_m)$, where $S$ is an increasing wave front, and $u_i(i = 1, 2, \cdots, m)$ are increasing initially and decreasing afterwards pulses satisfying

$$ (S^* - S_*) = \sum_{i=1}^{m} \frac{k_i}{c_i} \int_{-\infty}^{\infty} u_i(s) ds. \quad (2.6) $$

(ii) If $c < c^*$, then (2.4)-(2.5) has no positive traveling wave solutions.

For $R_0 > 1$, Theorem 2.3 indicates that the system (2.4)-(2.5) has a positive traveling wave solution if $c > c^*$, whereas the system (2.4)-(2.5) has no such solution if $c < c^*$. Hence, $c^*$ is the minimum wave speed. It is clear that the asymptotic speed of spreading is exactly the minimal wave speed for traveling waves.

3. The spreading speed in the Cauchy problem.

3.1. The Cauchy problem. For convenience of proceeding, we begin with some functional spaces. Let the set

$$ X := \text{BUC}(\mathbb{R}^N, \mathbb{R}^{m+1}) $$

be all bounded and uniformly continuous functions from $\mathbb{R}^N$ to $\mathbb{R}^{m+1}$. With the usual supremum norm, $X$ is a Banach space. Denote

$$ X_+ := \{ \Phi \in X : \Phi(x) \geq 0, x \in \mathbb{R}^N \}. $$

It is easy to see that $X_+$ is a closed cone of $X$ and $X$ is a Banach lattice under the partial order introduced by $X_+$. Introduce the following functional space:

$$ L^p_u(\mathbb{R}^N) := \left\{ \phi \in L^p_{\text{loc}}(\mathbb{R}^N) : \sup_{x \in \mathbb{R}^N} \| \phi \|_{L^p(B(x, 1))} < \infty \right\}, $$

where $p \in [1, \infty)$ and $B(x, 1) \subset \mathbb{R}^N$ denotes the unit ball of center $x$. It is not difficult to verify that $L^p_u(\mathbb{R}^N)$ is a Banach space when it is endowed with the supremum norm

$$ \| \phi \|_{L^p_u(\mathbb{R}^N)} = \sup_{x \in \mathbb{R}^N} \| \phi \|_{L^p(B(x, 1))}, \quad \forall \phi \in L^p_u(\mathbb{R}^N). $$

Denote the Euclidean norm by $| \cdot |$ and inner product by $\langle \cdot, \cdot \rangle$ in Euclidean space $\mathbb{R}^N$.

It is expedient to simplify notation by setting $u_0 = S, u_{00} = S_0$ and to use vector notation where possible. In particular, we introduce the vectors $U = (u_0, u_1, \cdots, u_m), U_0 = (u_{00}, u_{10}, \cdots, u_{m0})$ and $D = (d_0, d_1, \cdots, d_m)$. Let $U_0 \in X_+$ be the vector of initial conditions in (1.2) with the $i$th component $U_{0i} = u_{0i}$. The nonlinear term $G : X_+ \to X$ in (1.1) is denoted by

$$ G(U) = (G_0(U), G_1(U), \cdots, G_m(U)). $$
where $G_0(U) = -\sum_{i=1}^{m} \gamma_i^{-1} u_i f_i(S)$ and $G_i(U) = u_i (f_i(S) - k_i), \ i = 1, 2, \cdots, m$. Then system (1.1)-(1.2) can now be rewritten simply as
\[
\frac{\partial U}{\partial t} = D \Delta U + G(U), \ x \in \mathbb{R}^N, \ t > 0
\] (3.1) with nonnegative initial condition
\[
U(x,0) = U_0(x), \ x \in \mathbb{R}^N.
\] (3.2)
For $U_0 \in X_+$, it is well known that the solution of the initial value problem
\[
\frac{\partial v_i}{\partial t} = d_i \Delta v_i, \ v_i(x,0) = U_{0i}(x), \ x \in \mathbb{R}^N, \ t > 0, \ i = 0, 1, \cdots, m
\] (3.3) can be expressed in terms of heat kernel as the following Poisson’s formula
\[
v_i(x,t) = \frac{1}{(4\pi d_i t)^{N/2}} \int_{\mathbb{R}^N} \exp \left(-\frac{|x - y|^2}{4d_i t} \right) U_{0i}(y) dy := (B_i(t)U_{0i})(x), \quad \text{(3.4)}
\]
where the operator \( \{B(t) = (B_0(t), B_1(t), \cdots, B_m(t))\}_{t \geq 0} \) is an analytic semigroup on $X$ (see Theorem 1.5 in Daners and Median [4], also see Lemma 8.1.4 in [36]). From (3.4), it is easy to see that $B(t)$ is a positive operator: $B(t)X_+ \subseteq X_+$. Moreover, the following properties hold.

**Lemma 3.1.** (i) There exists some constant $\mathcal{M}_i$ depending only on the dimension $N$, such that for every $\varphi \in L_p^\infty(\mathbb{R}^N)$, the following estimates hold
\[
\|B_i(t)\varphi\|_{L_p^\infty(\mathbb{R}^N)} \leq \mathcal{M}_i \left(1 + t^{-\frac{N}{2(p - \frac{N}{2})}}\right) \|\varphi\|_{L_p^\infty(\mathbb{R}^N)}, \quad 1 \leq p \leq q \leq \infty, \ t > 0.
\]
(ii) Assume $S_0$ and \( \{u_{0i}\}_{i=1}^m \) satisfy the assumptions (H). The unique solution of (3.3) satisfies
\[
\lim_{t \to \infty} v_0(x,t) = S^*, \quad \lim_{t \to \infty} v_i(x,t) = 0 \quad \text{for} \quad i = 1, 2, \cdots, m
\] uniformly in $\mathbb{R}^N$.

**Proof.** The proof of (i) can be found in Proposition 2.1 in [3]. We only need to prove (ii). Since $U_{0i} = u_{0i}$ is compactly supported in $\mathbb{R}^N$ for $i = 1, 2, \cdots, m$, it is easy to see that $\lim_{t \to \infty} v_i(x,t) = 0$. It remains to prove $\lim_{t \to \infty} v_0(x,t) = S^*$. By the Poisson’s formula (3.4),
\[
v_0(x,t) = \frac{1}{(4\pi d_0 t)^{N/2}} \int_{\mathbb{R}^N} \exp \left(-\frac{|x - y|^2}{4d_0 t} \right) S_0(y) dy. \quad \text{(3.5)}
\]
It follows from $\lim_{|x| \to \infty} S_0(x) = S^*$ that, for any given $\epsilon > 0$, there exists $R > 0$ such that $|S_0(x) - S^*| < \epsilon$ for $|x| > R$. Rewrite (3.5) as
\[
v_0(x,t) = \frac{1}{(4\pi d_0 t)^{N/2}} \int_{B(0,R)} \exp \left(-\frac{|x - y|^2}{4d_0 t} \right) S_0(y) dy
\]
\[+ \frac{1}{(4\pi d_0 t)^{N/2}} \int_{\mathbb{R}^N \setminus B(0,R)} \exp \left(-\frac{|x - y|^2}{4d_0 t} \right) S_0(y) dy := J_1 + J_2.
\]
Firstly, note that
\[
|J_1| \leq \frac{|B(0,R)|}{(4\pi d_0 t)^{N/2}} \|S_0\|_\infty \to 0 \quad \text{as} \quad t \to \infty
\] (3.6)
uniformly for \( x \in \mathbb{R}^N \), where \(|B(0,R)|\) represents the volume of the ball \( B(0,R) \).

On one hand,

\[
J_2 \leq \frac{1}{(4\pi d_0t)^{N/2}} \int_{\mathbb{R}^N} \exp\left( -\frac{|x-y|^2}{4d_0t} \right) (S^* + \epsilon)dy = S^* + \epsilon. \tag{3.7}
\]

On the other hand,

\[
J_2 \geq \frac{1}{(4\pi d_0t)^{N/2}} \int_{\mathbb{R}^N} \exp\left( -\frac{|x-y|^2}{4d_0t} \right) (S^* - \epsilon)dy
- \frac{1}{(4\pi d_0t)^{N/2}} \int_{B(0,R)} \exp\left( -\frac{|x-y|^2}{4d_0t} \right) (S^* - \epsilon)dy
= S^* - \epsilon - \frac{1}{(4\pi d_0t)^{N/2}} \int_{B(0,R)} \exp\left( -\frac{|x-y|^2}{4d_0t} \right) (S^* - \epsilon)dy
\]

\[
\rightarrow S^* - \epsilon \quad \text{as} \quad t \to \infty
\]

uniformly for \( x \in \mathbb{R}^N \). Since \( \epsilon \) is arbitrary, the desired result is obtained by (3.6)-(3.8).

The following preliminary result exactly demonstrates that for each initial data \( U_0 \in X_+ \), there is a unique solution \( U(\cdot,t) \in X_+ \) of (3.1)-(3.2) defined for \( 0 \leq t < \tau = \tau(U_0) \). Moreover the map \( U_0 \mapsto U(\cdot,t) \) is continuous and satisfies the semi-group property wherever it's defined.

**Lemma 3.2.** The system (3.1)-(3.2) generates a nonlinear local semi-dynamical system on the space \( X_+ \).

**Proof.** We only give a sketch of the proof on this well-known result. First, rewrite the system (3.1)-(3.2) to an integral equation

\[
U(\cdot,t) = B(t)U_0 + \int_0^t B(t-s)G(U(\cdot,s))ds, \quad U(\cdot,0) = U_0.
\]

As \( G \) is Lipschitz on bounded subsets of \( X_+ \), one can show that for each \( U_0 \in X_+ \) there is a unique mild solution, which remains in \( X_+ \) (see Corollary 1.3 in [32]), of the integral equation on a maximal interval of existence \([0, \tau]\). The smoothness assumptions on \( G_i \), and the fact that \( B(t) \) is an analytic semigroup can be used to show that this solution is a classical solution of (3.1)-(3.2). Furthermore, the map \( U_0 \mapsto U(\cdot,t) \) is continuous, where \( U(\cdot,t) \) is the solution corresponding to the initial data \( U_0 \), and the semi-group property holds where the map is defined (see Corollary 2.5 in [32]).

Next, we state the following global existence result for the Cauchy problem (3.1)-(3.2).

**Proposition 3.3.** *(Global existence)* The system (3.1)-(3.2) generates a strongly continuous and globally defined semi-flow on \( X_+ \) denoted by

\[
\{ (B(t)U_0)(x) = U(x,t;U_0) \}_{t \geq 0}
\]

for each \( U_0 \in X_+ \). Moreover, the solution \( U(x,t;U_0) = U(x,t) \) satisfies the properties

(i) for each \( t \geq 0 \) and \( x \in \mathbb{R}^N \), the components of \( U \) satisfy

\[
S(x,t) \leq \|S_0\|_0, \quad u_i(x,t) \leq \|u_0\|_\infty \exp(f_i(\|S_0\|_\infty)t) \quad \text{for} \quad i = 1, 2, \cdots, m;
\]

(ii) \( U \in C(X_+ \times [0, \infty)) \). Particularly, for each \( 0 < \tau < T, U \in C^2(X_+ \times (\tau, T)) \).
Proof. In view of Lemma 3.2, we only need to prove the results (i) and (ii). Since \( G_0 \leq 0 \), we observe that
\[
\frac{\partial u_0}{\partial t} = d_0 \Delta u_0 + G_0(U) \leq d_0 \Delta u_0.
\]
Hence,
\[
u(x, t) = \mathcal{N} \int_{\mathbb{R}^2} \exp\left( -\frac{|x-y|^2}{4d_0 t} \right) S_0(y) dy
\leq \frac{1}{(4\pi d_0 t)^{N/2}} \int_{\mathbb{R}^2} \exp\left( -\frac{|x-y|^2}{4d_0 t} \right) \| S_0 \|_{\infty} dy = \| S_0 \|_{\infty}.
\] (3.9)
Furthermore, for \( i = 1, 2, \cdots, m \), we have
\[
\frac{\partial u_i}{\partial t} = d_i \Delta u_i + u_i(f_i(u_0) - k_i) \leq d_i \Delta u_i + u_i(f_i(\| S_0 \|_{\infty})).
\]
It follows from the parabolic comparison principle that
\[
u_i(x, t) \leq \| U_{0i} \|_{\infty} \exp(f_i(\| S_0 \|_{\infty}) t), \quad i = 1, 2, \cdots, m
\] (3.10)
for \( t \geq 0 \) and \( x \in \mathbb{R}^N \). Hence, the result (i) holds.

Let \([0, T_{max}]\) be the maximal time interval in which the solution exists. By Lemma 3.2, \( T_{max} > 0 \). We show that \( T_{max} = \infty \). Arguing indirectly, assume that \( T_{max} < \infty \). Fix \( t_0 \in (0, T_{max}) \) and \( M > T_{max} \). Set
\[
\mathcal{C} = \max\{\| S_0 \|_{\infty}, \| U_{0i} \|_{\infty} \exp(f_i(\| S_0 \|_{\infty}) M), i = 1, 2, \cdots, m\}.
\]
By (3.9) and (3.10), we have \( u_i \leq \mathcal{C} \) for \( i = 0, 1, \cdots, m \) and \( t \in (t_0, T_{max}) \). It then follows from Lemma 3.2 that there exists a \( \tau > 0 \) depending only on \( \mathcal{C} \) such that the solution of problem (3.1)-(3.2) with initial time \( T_{max} - \tau/2 \) can be extended uniquely to the time \( T_{max} - \tau/2 + \tau \). But this contradicts the assumption. Finally, since \( G(U) \) is continuously differentiable and the semigroup \( B(t) \) is analytical, one can conclude that (ii) holds.

To investigate the uniform boundedness of solutions of (3.1)-(3.2), we first recall the duality estimates for the parabolic inequality, which is achieved by Ducrot (see Theorem 3.8 in [6]).

Lemma 3.4. (Ducrot [6]) Let \( T > 0 \) be given. Let \( h \in L^\infty(\mathbb{R}^N \times (0, T)), (\theta_1, \theta_2) \in \mathbb{R}^2 \) and \( \nu > 0 \) be given. Let \((\tilde{u}, \tilde{v}) \in (W^{1,2}(\mathbb{R}^N \times (0, T)))^2 \) be given such that \( \tilde{u} \geq 0 \) satisfying
\[
(\partial_t - \Delta + \nu)\tilde{u}(x, t) \leq h(x, t) + (\theta_1 \partial_t + \theta_2 \Delta)\tilde{v}(x, t)
\]
for almost every \((x, t) \in \mathbb{R}^N \times (0, T)\). Then for each \( p \in (1, \infty) \), there exists some constant \( C(p) > 0 \) depending only upon \( p \in (1, \infty) \), \( \nu > 0 \) and \( N \geq 1 \) such that
\[
\left[ \int_0^T \| \tilde{u}(\cdot, t) \|_{L_p(\mathbb{R}^N)}^p dt \right]^{1/p} \leq K(p, T) \left[ 1 + \| \tilde{u}(\cdot, 0) \|_{L_p^\infty(\mathbb{R}^N)} + T^{1/p} \right],
\]
wherein \( K(p, T) > 0 \) is defined by
\[
K(p, T) = C(p) [1 + \| h \|_{L^\infty(\mathbb{R}^N \times (0, T))} + \| \tilde{v} \|_{L^\infty(\mathbb{R}^N \times (0, T))}(\| \theta_1 \| + \| \theta_1 \|)].
\]
Now, we are ready to state the boundedness of the solutions of (3.1)-(3.2).
Proposition 3.5. (Uniform boundedness) Assume \( U(x,t) \) is the solution of (3.1)-(3.2). Then there exists \( C > 0 \) such that

\[
\|U(x,t)\|_X \leq C, \quad \forall t \geq 0, \quad x \in \mathbb{R}^N,
\]

where \( C \) depends on \( \|S_0\|_\infty, \|u_{i0}\|_\infty, N \) and all parameters in (3.1)-(3.2).

Proof. In view of Proposition 3.3 (i), we only need to prove \( \|u_i\|_X \leq C \) for \( i = 1, 2, \cdots, m \). To this end, we finish the proof by three steps.

Step 1. Local \( L^p \) estimates. Note that

\[
\frac{\partial u_i}{\partial t} - d_i \Delta u_i + k_i u_i = u_i f_i(S) \leq \gamma_i \left( d_0 \Delta S - \frac{\partial S}{\partial t} \right), \quad i = 1, \cdots, m.
\]

It follows from Lemma 3.4 that for each \( p \in (1, \infty) \), there exists some constant \( C_i(p) > 0 \) such that for each \( 1 \leq \tau \leq T \),

\[
\left[ \int_\tau^T \|u_i(\cdot, t)\|_{L^p_0(\mathbb{R}^N)}^p dt \right]^{1/p} \leq K_i(p, T, \tau) \left[ 1 + \|u_i(\cdot, \tau)\|_{L^p_0(\mathbb{R}^N)} + (T - \tau)^{1/p} \right]
\]

with

\[
K_i(p, T, \tau) = C_i(p) [1 + \gamma_i(1 + d_i)] \|S\|_{L^\infty(\mathbb{R}^N \times [\tau, T])}.
\]

Using (3.9), we obtain that for each \( p \in (1, \infty) \) and each \( 1 \leq \tau \leq T \)

\[
\left[ \int_\tau^T \|u_i(\cdot, t)\|_{L^p_0(\mathbb{R}^N)}^p dt \right]^{1/p} \leq K_i \left[ 1 + \|u_i(\cdot, \tau)\|_{L^p_0(\mathbb{R}^N)} + (T - \tau)^{1/p} \right] \tag{3.11}
\]

with \( K_i = C_i(p) [1 + \gamma_i(1 + d_i)] \|S_0\|_\infty \).

Step 2. We claim that for each \( p \in (1, \infty) \) and \( \rho = \max\{\|S_0(x)\|_\infty, \|u_{i0}(x)\|_\infty, i = 1, \cdots, m\} \) there exist positive constants \( \Lambda_0(p, \rho), \Gamma_0(p, \rho) \) and a nondecreasing sequence \( \{t_k\}_{k \geq 0} \) with \( t_0 = 1 \) such that for each \( k \geq 0 \),

\[
1 \leq t_{k+1} - t_k \leq \Lambda_0(p, \rho), \quad \|u_i(\cdot, t_k)\|_{L^p_0(\mathbb{R}^N)} \leq K_i + 1 \tag{3.12}
\]

and

\[
\int_{t_k}^{t_{k+1}} \|u_i(\cdot, s)\|_{L^p_0(\mathbb{R}^N)}^p ds \leq \Gamma_0(p, \rho). \tag{3.13}
\]

The arguments are motivated by Hollis et al. in Lemma 7 in [9]. Without loss of generality, we assume that \( K_i + 1 \geq \rho \) and \( K_i \geq 1 \). Set \( T_p = K_i^p(K_i + 2)^p \). We first show that if \( \tau \geq 1 \) and \( \|u_i(\cdot, \tau)\|_{L^p_0(\mathbb{R}^N)} \leq K_i + 1 \), then there is a \( t^* \in (\tau, \tau + T_p) \) such that

\[
\|u_i(\cdot, t^*)\|_{L^p_0(\mathbb{R}^N)} \leq K_i + 1. \tag{3.14}
\]

Otherwise, \( \|u_i(\cdot, t)\|_{L^p_0(\mathbb{R}^N)} > K_i + 1 \) for any \( t \in (\tau, \tau + T_p) \), and so

\[
\int_{\tau}^{\tau + T_p} \|u_i(\cdot, t)\|_{L^p_0(\mathbb{R}^N)}^p dt > T_p(K_i + 1)^p.
\]

Nevertheless, it follows from (3.11) with \( T = \tau + T_p \) that

\[
\int_{\tau}^{\tau + T_p} \|u_i(\cdot, t)\|_{L^p_0(\mathbb{R}^N)}^p dt \leq K_i^p \left[ 1 + K_i + 1 + T_p^{1/p} \right]^p
\]

\[
= K_i^p \left( 1 + K_i + 1 + K_i(K_i + 2) \right)^p
\]

\[
= T_p(K_i + 1)^p.
\]
This contradiction means that \(3.14\) holds. Set \(\Lambda_0(p, \rho) = 2T_p\) and inductively define \(\{t_k\}_{k \geq 0}\) by \(t_0 = 1\) and
\[
t_{k+1} = \sup \{T \geq t_k : T - t_k \leq 2T_p \text{ and } \|u_i(\cdot, T)\|_{L^p_\infty(\mathbb{R}^N)} \leq K_i + 1\}.
\]
By the continuity of \(t \mapsto \|u_i(\cdot, t)\|_{L^p_\infty(\mathbb{R}^N)}\) we have that
\[
\|u_i(\cdot, t_{k+1})\|_{L^p_\infty(\mathbb{R}^N)} \leq K_i + 1
\]
and
\[
\|u_i(\cdot, t_k)\|_{L^p_\infty(\mathbb{R}^N)} = K_i + 1 \text{ if } t_{k+1} < t_k + 2T_p.
\]
Moreover, \(t_{k+1} - t_k \geq T_p \geq 1\). Otherwise, if \(t_{k+1} - t_k < T_p\) then \(3.14\) implies that there is a \(t^* \in (t_{k+1}, t_k + T_p)\) such that \(\|u_i(\cdot, t^*)\|_{L^p_\infty(\mathbb{R}^N)} \leq K_i + 1\). But \(t^* \leq t_{k+1} + T_p < t_k + 2T_p\), which contradicts the definition of \(t_{k+1}\). Thus \(\{t_k\}_{k \geq 0}\) is well defined and satisfies (3.12). Finally, by using (3.11), it follows from
\[
\int_{t_k}^{t_{k+1}} \|u_i(\cdot, t)\|_{L^p_\infty(\mathbb{R}^N)}^p dt \leq K_p^p \left[1 + \|u_i(\cdot, t_k)\|_{L^p_\infty(\mathbb{R}^N)} + (t_{k+1} - t_k)^{1/p}\right]^p
\]
\[
\leq K_p^p \left[2 + K_i + (2T_p)^{1/p}\right]^p = \Gamma_0(p, \rho)
\]
that \(3.13\) holds.

**Step 3.** \(L^\infty\) estimates. Let \(p > 1 + N/2\) and integer \(k \geq 1\) be given. Then for each \(t \in [t_k, t_{k+1}]\), the mild solution \(u_i\) of (3.1)-(3.2) satisfies
\[
u_i(\cdot, t) = e^{-k_i(t-t_k-1)}B_i(t-t_k-1)u_i(t_{k+1}) + \int_{t_k}^{t} e^{-k_i(t-s)}B_i(t-s)u_i(\cdot, s)(f_i(S(\cdot, s))ds.
\]
Taking \(1/q + 1/p = 1\) and using Lemma 3.1 (i) and Proposition 3.3 (i), for each \(t \in [t_k, t_{k+1}]\) we have
\[
\|u_i(\cdot, t)\|_{L^\infty_\infty(\mathbb{R}^N)} \leq e^{-k_i(t-t_k-1)} \overline{M}_i \left[1 + (t-t_k)^{-\frac{N}{p}}\right] \|u_i(\cdot, t_k)\|_{L^p_\infty(\mathbb{R}^N)}
\]
\[
+ \overline{M}_i \int_{t_k}^{t} e^{-k_i(t-s)} \left[1 + (t-s)^{-\frac{N}{p}}\right] f_i(||S_0||_\infty) \|u_i(\cdot, s)||_{L^p_\infty(\mathbb{R}^N)}ds,
\]
where \(\overline{M}_i\) depends only on the dimension \(N\). Applying (3.12) and (3.13), for each \(t \in [t_k, t_{k+1}]\), we have
\[
\|u_i(\cdot, t)\|_{L^\infty_\infty(\mathbb{R}^N)} \leq 2\overline{M}_i(K_i + 1)
\]
\[
+ f_i(||S_0||_\infty) \overline{M}_i \left(\int_{t_k}^{t} \left[1 + (t-s)^{-\frac{N}{p}}\right]^q ds\right)^{1/q} \left(\int_{t_k}^{t} \|u_i(\cdot, s)||_{L^p_\infty(\mathbb{R}^N)}^p ds\right)^{1/p}
\]
\[
\leq 2\overline{M}_i(K_i + 1) + f_i(||S_0||_\infty) \overline{M}_i(\Gamma_0(p, l))^{1/p} \left(\int_0^{2\Lambda_0(p,l)} \left[1 + \tau^{-\frac{N}{p}}\right]^q d\tau\right)^{1/q}.
\]
This completes the proof. \(\Box\)

Finally, we turn to prove Theorem 2.1.

**Proof of Theorem 2.1.** Note that
\[
\frac{\partial S}{\partial t} - d_0 \Delta S = -\sum_{i=1}^{m} \gamma_i^{-1} u_i(S) \leq 0.
\]
The comparison principle implies that \( S(x, t) \leq v_0(x, t) \) for \((x, t) \in \mathbb{R}^N \times (0, \infty)\), where \( v_0(x, t) \) is given by (3.4). In view of Lemma 3.1 (ii), \( \lim_{t \to \infty} v_0(x, t) = S^* \) uniformly for \( x \in \mathbb{R}^N \). Hence, for any given \( \epsilon > 0 \) small, there exists \( T = T(\epsilon) > 0 \) such that

\[
S(x, t) \leq v_0(x, t) < S^* + \epsilon \quad \text{for} \quad (x, t) \in \mathbb{R}^N \times (T, \infty).
\]

Observing that \( R_{0i} < 1 \) for some \( i \in \{1, 2, \ldots, m\} \), we can conclude that there exists \( \epsilon > 0 \) small such that \( \sigma_i := k_i - f_i(S^* + \epsilon) > 0 \) and

\[
\frac{\partial u_i}{\partial t} - d_i \Delta u_i = u_i(f_i(S) - k_i) < -\sigma_i u_i
\]

for \( t > T \) and \( x \in \mathbb{R}^N \). It follows from Proposition 3.3 (i) that

\[
\frac{\partial u_i}{\partial t} - d_i \Delta u_i \leq u_i(f_i(\|S\|_\infty) - k_i) \quad \text{for} \quad t > 0, \quad x \in \mathbb{R}^N.
\]

Hence, we have

\[
0 < u_i(x, t) \leq \hat{u}_i(x, t) := e^{(f_i(\|S\|_\infty)-k_i)t} \frac{1}{(4\pi d_i t)^{N/2}} \int_{\mathbb{R}^N} \exp\left(-\frac{|x-y|^2}{4d_i t}\right) u_{i0}(y) dy
\]

(3.15)

where \( v_i(x, t) \) is given by (3.4). Consider the function \( \tilde{u}_i(t) \) defined by

\[
\frac{d\tilde{u}_i}{dt} = -\sigma_i \tilde{u}_i, \quad \tilde{u}_i(0) = \|\hat{u}_i(x, T)\|_\infty.
\]

Then from the maximum principle and comparison principle, one has

\[
0 < u_i(x, t) \leq \tilde{u}_i(x, t) = \|\hat{u}_i(x, T)\|_\infty e^{-\sigma_i(t-T)} \quad \text{for} \quad x \in \mathbb{R}^N \text{ and } t \geq T.
\]

Hence, if \( R_{0i} < 1 \) for some \( i \), then we have \( \lim_{t \to \infty} u_i(x, t) = 0 \) uniformly for \( x \in \mathbb{R}^N \). Furthermore, if \( \max\{R_{0i}, i = 1, 2, \ldots, m\} < 1 \), then \( \lim_{t \to \infty} u_i(x, t) = 0 \) uniformly for \( x \in \mathbb{R}^N, i = 1, 2, \ldots, m \).

It remains to prove that \( \lim_{t \to \infty} S(x, t) = S^* \) uniformly for \( x \in \mathbb{R}^N \) if \( \max\{R_{0i}, i = 1, 2, \ldots, m\} < 1 \). To this end, we first prove that there exists \( \bar{\beta}_i > 0 \) such that

\[
0 < u_i(x, t) \leq \tilde{u}_i(x, t) := \bar{\beta}_i e^{-\sigma_i t} v_i(x, t) \quad \text{for} \quad (x, t) \in \mathbb{R}^N \times [0, \infty).
\]

(3.16)

In fact, for the case of \( 0 \leq t \leq T \), due to (3.15) and \( v_i(x, t) \to u_{i0} \) as \( t \to 0 \), it is easy to see (3.16) holds by taking

\[
\bar{\beta}_i = e^{(f_i(\|S\|_\infty)-k_i+\sigma_i)T}.
\]

On the other hand, for the case of \( t > T \), a straightforward calculation gives

\[
\frac{d\tilde{u}_i}{dt} - d_i \Delta \tilde{u}_i - (f_i(S_i) - k_i) \tilde{u}_i = (-\sigma_i - f_i(S_i) + k_i) \tilde{u}_i
\]

\[
= (f_i(S^* + \epsilon) - f_i(S_i)) \tilde{u}_i \geq 0, \quad \tilde{u}_i(x, T) \geq u_i(x, T)
\]

The maximum principle and comparison principle imply that

\[
0 < u_i(x, t) \leq \tilde{u}_i(x, t) \quad \text{for} \quad x \in \mathbb{R}^N, t > T.
\]

Hence, the inequality (3.16) holds.
Next, we consider asymptotic behavior of \( v_i(x, t) \) when \( |x| \to \infty \). Since \( u_{i0}(x) \) is compactly supported for \( i = 1, 2, \ldots, m \), there exists \( L > 0 \) such that \( u_{i0}(x) = 0 \) for all \( |x| > L \). By the Poisson’s formula we obtain that

\[
v_i(x, t) = \frac{1}{(4\pi d_i t)^{N/2}} \int_{\mathbb{R}^N} \exp\left(-\frac{|x-y|^2}{4d_i t}\right) u_{i0}(y) dy
\]

\[
= \exp\left(-\frac{|x|^2}{4d_i t}\right) \frac{1}{(4\pi d_i t)^{N/2}} \int_{\mathbb{R}^N} \exp\left(-\frac{|y|^2 - 2x \cdot y}{4d_i t}\right) u_{i0}(y) dy
\]

\[
\leq \exp\left(-\frac{|x|^2}{4d_i t}\right) \frac{1}{(4\pi d_i t)^{N/2}} \int_{\mathbb{R}^N} \exp\left(-\frac{|y|^2 + 2L|x|}{4d_i t}\right) u_{i0}(y) dy
\]

\[
= \exp\left(-\frac{|x|^2 + 2L|x|}{4d_i t}\right) \frac{1}{(4\pi d_i t)^{N/2}} \int_{\mathbb{R}^N} \exp\left(-\frac{|y|^2}{4d_i t}\right) u_{i0}(y) dy.
\]

If \( |x| \geq 4L \), then

\[
-|x|^2 + 2L|x| = -|x|||x| - 2L| \leq -\frac{|x|^2}{2}.
\]

Since \( u_{i0}(x) \) is compactly supported and \( v_i(x, t) \to u_{i0}(x) \) as \( t \to 0 \), there exists \( \tilde{K}_i > 0 \) dependent on \( u_{i0} \) such that

\[
v_i(x, t) \leq \tilde{v}_i(x, t) := \tilde{K}_i \exp\left(-\frac{|x|^2}{8d_i t}\right) \frac{1}{(4\pi d_i t)^{N/2}} \text{ for } |x| > 4L \text{ and } t \geq 0. \tag{3.17}
\]

Furthermore, we claim that there exists \( M_i > 0 \) independent of \( t \) such that

\[
\int_{\mathbb{R}^N} v_i(x, t) dx \leq M_i \text{ for } t > 0. \tag{3.18}
\]

It follows from (3.17) and \( v_i \leq \|u_{i0}\|_{\infty} \) that

\[
\int_{\mathbb{R}^N} v_i(x, t) dx = \int_{\mathbb{R}^N \setminus B(0, 4L)} v_i(x, t) dx + \int_{B(0, 4L)} v_i(x, t) dx
\]

\[
\leq \int_{\mathbb{R}^N \setminus B(0, 4L)} \tilde{v}_i(x, t) dx + \int_{B(0, 4L)} v_i(x, t) dx
\]

\[
\leq \int_{\mathbb{R}^N} \tilde{v}_i(x, t) dx + \int_{B(0, 4L)} v_i(x, t) dx
\]

\[
\leq \int_{\mathbb{R}^N} \tilde{K}_i \exp\left(-\frac{|x|^2}{8d_i t}\right) \frac{1}{(4\pi d_i t)^{N/2}} dx + \int_{B(0, 4L)} \|u_{i0}\|_{\infty} dx
\]

\[
= \tilde{K}_i 2^{N/2} + |B(0, 4L)||u_{i0}|_{\infty} := M_i.
\]

At last, applying the Poisson’s formula yields that

\[
S(x, t) = \frac{1}{(4\pi d_{0} t)^{N/2}} \int_{\mathbb{R}^N} \exp\left(-\frac{|x-y|^2}{4d_{0} t}\right) S_{0}(y) dy
\]

\[
- \int_{0}^{t} d\tau \int_{\mathbb{R}^N} \frac{1}{(4\pi d_{0} (t - \tau))^{N/2}} \exp\left(-\frac{|x-y|^2}{4d_{0} (t - \tau)}\right) \sum_{i=1}^{m} \gamma_i^{-1} u_{i}(y, \tau) f_i(S(y, \tau)) dy
\]

\[
:= J_3 - J_4.
\]
By similar arguments as in the proof of Lemma 3.1 (ii), it is easy to see that 
\[ \lim_{t \to 0^+} J_3 = 0. \]
Furthermore, noting that 
\[ \sum_{i=1}^{m} \gamma_i^{-1} u_i(y) \leq \sum_{i=1}^{m} \gamma_i^{-1} \tilde{u}_i(\|S_0\|_\infty), \]
we can conclude that 
\[
J_4 \leq \sum_{i=1}^{m} \int_{0}^{t} d\tau \int_{\mathbb{R}^N} \frac{1}{(4\pi d_0(t-\tau))^{N/2}} \exp\left( -\frac{|x-y|^2}{4d_0(t-\tau)} \right) \gamma_i^{-1} \tilde{u}_i(y, \tau) f_i(\|S_0\|_\infty) dy
\]
\[ := \sum_{i=1}^{m} J_{5i}. \]
It follows from (3.16) and (3.18) that 
\[
J_{5i} = \lim_{\delta \to 0^+} \int_{\delta}^{t} d\tau \int_{\mathbb{R}^N} \frac{f_i(\|S_0\|_\infty)}{\gamma_i(4\pi d_0(t-\tau))^{N/2}} \exp\left( -\frac{|x-y|^2}{4d_0(t-\tau)} \right) \tilde{u}_i(y, \tau) dy
\]
\[ = \lim_{\delta \to 0^+} \int_{\delta}^{t} d\tau \int_{\mathbb{R}^N} \frac{f_i(\|S_0\|_\infty)}{\gamma_i(4\pi d_0(t-\tau))^{N/2}} \exp\left( -\frac{|x-y|^2}{4d_0(t-\tau)} \right) \tilde{v}_i(y, \tau) dy
\]
\[ \leq \frac{f_i(\|S_0\|_\infty) \beta_i \lim_{\delta \to 0^+} \int_{\delta}^{t} e^{-\sigma_\tau} \int_{\mathbb{R}^N} e^{-\sigma \eta} \frac{\sum_{i=1}^{m} \gamma_i^{-1} \tilde{u}_i(\|S_0\|_\infty)}{\gamma_i(4\pi d_0)^{N/2}} \eta^{N-1} d\eta}{\sigma t} \] (\eta = \sqrt{t-\tau}).

By L'Hospital's rule, we have
\[ \lim_{t \to \infty} e^{-\sigma_\tau} \int_{0}^{\sqrt{t}} \frac{e^{\sigma \eta} \eta^{N-1} d\eta}{\eta^{N-1}} = \lim_{t \to \infty} \frac{1}{2\sigma t^{N/2}} = 0. \]
Hence, \( J_{5i} \to 0 \) as \( t \to \infty \) uniformly on \( x \in \mathbb{R}^N \). As a consequence, we have \( \lim_{t \to \infty} S(x, t) = S^* \) uniformly for \( x \in \mathbb{R}^N \). This completes the proof of Theorem 2.1.

3.2. The asymptotic speed for spread. In this section, we consider the spreading properties of system (1.1)-(1.2). Although the main idea is motivated by the works of Ducrot [6] and Magal and Zhao [23], significant changes are needed in the detailed techniques.

Let \( U(x, t) \) be the solution of system (1.1)-(1.2) and \( Q(x, t) = U(x + \text{ct} e, t) \). Then the component \( q_i \) of \( Q \) satisfies
\[
(\partial_t - \text{ct} \cdot \nabla - d_i \Delta) q_i = G_i(Q), \quad x \in \mathbb{R}^N, \quad t > 0,
\]
\[ q_i(x, 0) = U_{i0}(x), \quad x \in \mathbb{R}^N, \quad i = 0, 1, \ldots, m. \] (3.19)

Rewrite the system (3.19) to an integral equation
\[
q_i(x, t) = \int_{\mathbb{R}^N} g_i(x - y, t) U_{i0}(y) dy
\]
\[ + \int_{0}^{t} \int_{\mathbb{R}^N} g_i(x - y, t - \tau) G_i(Q(y, \tau)) dy d\tau,
\]
where
\[
g_i(x, t) = \frac{1}{(4\pi d_i t)^{N/2}} \exp\left( -\frac{|x + \text{ct} e|^2}{4d_i t} \right).
\]
Lemma 3.6. Assume $R_0 > 1$, the functions $S_0$ and $\{u_{i0}\}_{i=1}^m$ satisfy the assumptions $(H)$, and $(S,u_1,u_2,\cdots,u_m)$ is the solution of (1.1)-(1.2). Let $\theta > 0$ and $c_0 \in [0,c^*)$ be given. For each $x \in \mathbb{R}^N$, each unit vector $e \in S^{N-1}$, each $c \in [-c_0,c_0]$ and any $U_0 \in M_{m+1} = \{\varphi \in \text{BUC}(\mathbb{R}^N,\mathbb{R}^{m+1}): 0 \leq \varphi_i \leq \vartheta, \varphi_i \neq 0, 0 \leq i \leq m\}$, there exists $\epsilon = \epsilon(\vartheta,c_0) > 0$ and some $i \in \{1,2,\cdots,m\}$ such that

$$\limsup_{t \to \infty} u_i(x + cte, t; U_0) \geq \epsilon.$$ 

Proof. Suppose for contradiction that for each $n \geq 0$, there exist $U^n_0 \in M_{m+1}^0$, $x_n \in \mathbb{R}^N$, $c_n \in [-c_0,c_0]$ and $e_n \in S^{N-1}$ such that, the components of the solution of (3.1)-(3.2), denoted by $U^n$, satisfies

$$\limsup_{t \to \infty} u^n_j(x_n + c_n e_n, t; U^n_0) \leq \frac{2}{n+1}, \quad j = 1,2,\cdots,m.$$ 

Hence, there exists a sequence $\{t_n > 0\}_{n \geq 0}$ with $0 < t_{n+1} - t_n < n + 1, t_n \to \infty$ as $n \to \infty$ such that

$$u^n_j(x_n + c_n t_n e_n, t; U^n_0) \leq \frac{2}{n+1}, \quad \forall t \geq t_n, \; j = 1,2,\cdots,m. \quad (3.21)$$

Consider the sequence of shifted functions

$$\omega^n(x,t) = U^n(x + x_n + c_n(t + t_n) e_n, t + t_n; U^n_0). \quad (3.22)$$

In view of (3.21), we have

$$\omega^n_j(0,t) = u^n_j(x_n + c_n(t + t_n) e_n, t + t_n; U^n_0) \leq \frac{2}{n+1}, \quad j = 1,2,\cdots,m \quad (3.23)$$

for any $t \geq 0$. Firstly, we claim that

$$\lim_{n \to \infty} \omega^n_0(x,t) = S^*, \quad \lim_{n \to \infty} \omega^n_j(x,t) = 0, \quad j = 1,2,\cdots,m$$

uniformly with respect to $t \geq 0$ and $x$ in bounded sets. In view of the uniform bound established by Lemma 3.5 and parabolic estimates, by passing to a subsequence if necessary we have $\omega^n \to \omega^\infty$ locally uniformly for $(x,t) \in \mathbb{R}^{N+1}$ as $n \to \infty$. Moreover, one may assume $e_n \to e \in S^{N-1}$ and $c_n \to c \in [-c_0,c_0]$ as $n \to \infty$. By the virtue of (3.22), $\omega^\infty$ satisfies

$$(\partial_t - ce \cdot \nabla - d_0 \Delta)\omega^\infty = G_i(\omega^\infty), \quad i = 0,1,\cdots,m,$$

$$0 \leq \omega^\infty_0 \leq \|S_0\|_{\infty}, \quad \omega^\infty_j \geq 0, \quad j = 1,2,\cdots,m.$$ 

Thanks to (3.23), it follows that $\omega^\infty_j(0,t) = 0, \quad j = 1,2,\cdots,m, \quad \forall t \geq 0$. The strong maximum principle implies that

$$\omega^\infty_j(x,t) \equiv 0 \quad \text{for} \quad x \in \mathbb{R}^N, \; t \geq 0, \; j = 1,2,\cdots,m. \quad (3.24)$$

Therefore, $\omega^\infty_0$ becomes an entire solution of the following equation

$$(\partial_t - ce \cdot \nabla - d_0 \Delta)\omega^\infty_0 = 0, \quad 0 \leq \omega^\infty_0 \leq \|S_0\|_{\infty}.$$
Next, we claim that such entire solution $\omega_0^\infty$ satisfies $\omega_0^\infty = S^*$. In fact, to prove $\omega_0^\infty = S^*$ it suffices to show that $\omega_0^\infty (x, 0) = S^*$. It follows from (3.20) that

$$\omega_0^n (x, 0) = u_0^n (x + x_n + c_n t_n e_n, t_n; U_0^n) = q_0^n(x + x_n, t_n; U_0^n)$$

$$= \int_{\mathbb{R}^N} g_0(x + x_n - y, t_n) S_0^n (y) dy$$

$$- \sum_{i=1}^m \int_0^{t_n} \int_{\mathbb{R}^N} g_0(x + x_n - y, t_n - \tau) \gamma_i^{-1} f_i(q_0^n(y, \tau)) q_i^n(y, \tau) dy d\tau$$

$$: = I^n - \sum_{i=1}^m I_i^n,$$

where $q_i^n(x, t) = u_i^n (x + c_n t e_n, t; U_0^n)$, $i = 0, 1, \cdots, m$. By similar arguments as in the proof of Lemma 3.1 (ii) one can obtain that

$$\lim_{n \to \infty} I^n = S^* \text{ uniformly for } x \in \mathbb{R}^N.$$

Now we determine the limit of $I_i^n$ as $n \to \infty$. In view of (3.24), there exists $T > 0$ large enough such that

$$q_i^n(x, t) \leq \frac{1}{(n + 1)^3} \text{ for } x \in \mathbb{R}^N, \ t \geq T.$$

Hence, by setting $T \geq t_0$ we have

$$\lim_{n \to \infty} I_i^n = \lim_{n \to \infty} \gamma_i^{-1} \int_{t_0}^{t_n} \int_{\mathbb{R}^N} g_0(x + x_n - y, t_n - \tau) f_i(q_0^n(y, \tau)) q_i^n(y, \tau) dy d\tau$$

$$\leq \sum_{n=0}^{\infty} \gamma_i^{-1} \int_{t_n}^{t_{n+1}} \int_{\mathbb{R}^N} g_0(x + x_n - y, t_n - \tau) f_i(q_0^n(y, \tau)) q_i^n(y, \tau) dy d\tau$$

$$+ \gamma_i^{-1} \int_0^T \int_{\mathbb{R}^N} g_0(x + x_n - y, t_n - \tau) f_i(q_0^n(y, \tau)) q_i^n(y, \tau) dy d\tau$$

$$\leq \sum_{n=1}^{\infty} \gamma_i^{-1} \int_{t_{n-1}}^{t_n} \int_{\mathbb{R}^N} g_0(x + x_n - y, t_n - \tau) f_i(\|S_0\|_\infty) \frac{1}{n^3} dy d\tau$$

$$+ \gamma_i^{-1} \int_0^T \int_{\mathbb{R}^N} g_0(x + x_n - y, t_n - \tau) f_i(\|S_0\|_\infty) q_i^n(y, \tau) dy d\tau$$

$$\leq \gamma_i^{-1} f_i(\|S_0\|_\infty) \left( \sum_{n=1}^{\infty} \frac{1}{n^2} + T \|u_i\|_\infty \right)$$

$$= \gamma_i^{-1} f_i(\|S_0\|_\infty) \left( \frac{\pi^2}{6} + T \|u_i\|_\infty \right).$$

Thus, for any given $\epsilon > 0$ small, there exists $N_1 > 0$ such that for $n > N_1$ we have

$$\lim_{n \to \infty} \gamma_i^{-1} \int_{t_N_1}^{t_n} \int_{\mathbb{R}^N} g_0(x + x_n - y, t_n - \tau) f_i(q_0^n(y, \tau)) q_i^n(y, \tau) dy d\tau < \frac{\epsilon}{2m}.$$

(3.25)

Next, we claim that for $\epsilon > 0$ small and $T_0 > 0$, there exists $N_2 > N_1$ such that for $n > N_2$ we have

$$\gamma_i^{-1} \int_0^{T_0} d\tau \int_{\mathbb{R}^N} g_0(x + x_n - y, t_n - \tau) f_i(q_0^n(y, \tau)) q_i^n(y, \tau) dy < \frac{\epsilon}{2m}.$$

(3.26)
In fact, it follows from Proposition 3.3 (i) and (3.19) that
\[(\partial_t - ce \cdot \nabla - d_i \Delta)q_i \leq q_i(f_i(\|S_0\|_\infty) - k_i), \ x \in \mathbb{R}^N, \ t > 0,\]
\[q_i(x, 0) = U_{i0}(x), \ x \in \mathbb{R}^N, \ i = 1, \cdots, m.\]
Hence,
\[0 < q_i(x, t) \leq q_i(x, t) := e^{f_i(\|S_0\|_\infty) - k_i} \int_{\mathbb{R}^N} g_i(x - y, t)U_{i0}(y)dy \text{ for } x \in \mathbb{R}^N, \ t > 0.\]
Repeating the same arguments as for (3.18), one can obtain that there exists \(M_i > 0\), independent of \(t\), such that
\[\int_{\mathbb{R}^N} q_i^n(y, t)dy \leq \int_{\mathbb{R}^N} \tilde{q}_i^n(y, t)dy \leq M_i \text{ for } 0 \leq t \leq T_0.\]
Now, we have
\[\gamma_i^{-1} \int_0^{T_0} d\tau \int_{\mathbb{R}^N} g_0(x + x_n - y, t_n - \tau)f_i(q_0^n(y, \tau))q_i^n(y, \tau)dy \leq \gamma_i^{-1} \int_0^{T_0} d\tau \int_{\mathbb{R}^N} \frac{1}{[4\pi d_0(t_n - \tau)]^{\frac{N}{2}}} f_i(\|S_0\|_\infty)q_i^n(y, \tau)dy \]
\[= f_i(\|S_0\|_\infty) \gamma_i^{-1} \int_0^{T_0} \frac{1}{[4\pi d_0(t_n - \tau)]^{\frac{N}{2}}} d\tau \int_{\mathbb{R}^N} q_i^n(y, \tau)dy \]
\[= \gamma_i^{-1} f_i(\|S_0\|_\infty) M_i \int_0^{T_0} \frac{1}{[4\pi d_0(t_n - \tau)]^{\frac{N}{2}}} d\tau \to 0 \text{ as } n \to \infty,\]
which implies that (3.26) holds. By taking \(T \geq t_{N_2}\), we apply (3.25) and (3.26) to yield that \(\sum_{i=1}^m I_i^n < \epsilon\) for \(n > N_2\). Hence,
\[\omega_0^\infty(x, 0) = \lim_{n \to \infty} q_0^n(x + x_n, t_n; U_0^n) = S^*.\]
Next, we show
\[\lim_{n \to \infty} \omega_0^n(x, t) = S^* \quad (3.27)\]
uniformly with respect to \(t \geq 0\) and locally in \(x \in \mathbb{R}^N\). Let \(L > 0\) be given and assume that there exists \(\epsilon > 0\) and a sequence \((x_n, t_n) \in B(0, L) \times [0, \infty)\) such that \(|\omega_0^n(x_n, t_n) - S^*| \geq \epsilon\). Denote \(x_n \to x_\infty \in \mathbb{R}^N\). Using parabolic estimates, we may assume that \(\omega_0^n(\cdot, t_n + \cdot) \to \omega_0^\infty\) locally uniformly in \(\mathbb{R}^N\). Then one gets that
\[|\omega_0^\infty(x_\infty, 0) - S^*| \geq \epsilon, \quad (3.28)\]
where \(\omega_0^\infty\) is a bounded entire solution of
\[(\partial_t - ce \cdot \nabla - d_0 \Delta)\omega_0^\infty(x, t) = 0, \quad \omega_0^\infty(x, 0) = S^*.\]
Hence, \(\omega_0^\infty(x, t) = S^*\), a contradiction to (3.28).
Now, we consider an eigenvalue problem
\[-d_i \Delta \psi - ce \cdot \nabla \psi + a_i \psi = \lambda \psi, \ x \in B(0, L),\]
\[\psi = 0, \ x \in \partial B(0, L), \quad (3.29)\]
where \(a_i = k_i - f_i(S^*) < 0\). By Lemma 5.2 in [6], the principal eigenvalue of (3.29) satisfies
\[\lim_{L \to \infty} \lambda_1^i = a_i + \frac{c^2}{4d_i}.\]
Note that \( c^* = c^*_m \). It follows from \( c \in [-c_0, c_0] \) that \( \vert c \vert < c^*_m \). Hence, there exists \( L > 0 \) large such that the principal eigenvalue \( \lambda_{1m}^N < 0 \) and its eigenfunction \( \psi_{1m}^N > 0 \) in \( B(0, L) \). Recalling (3.27), we have
\[
\lim_{t \to \infty} \omega_0^m(x, t) = S^* \quad \text{uniformly for } x \in \mathbb{R}^N, \quad t \geq 0.
\]
Hence, for any given \( \epsilon > 0 \) small, there exists \( N_{\epsilon} = N(\epsilon) > 0 \) such that
\[
\vert \omega_0^m(x, t) - S^* \vert < \epsilon, \quad t \geq 0 \quad \text{and} \quad x \in B(0, L), \quad n > N_{\epsilon}.
\]
Note that
\[
(\partial_t - c \mathbf{e} \cdot \nabla - d_m \nabla^2\omega_m^n(x, t) = (f_m(\omega_m^n(x, t)) - k_m)\omega_m^n(x, t) \geq (f_m(S_\ast - c) - k_m)\omega_m^n(x, t).
\]
Now choose \( \omega_m^n(x, 0) \geq \epsilon \psi_{1m}^N(x) \) with \( \epsilon > 0 \) small. The comparison principle implies that
\[
\omega_m^n(x, t) \geq \epsilon e^{-\lambda_{1m}^N t} \psi_{1m}^N(x) \quad \text{for } x \in \mathbb{R}^N, \quad t > 0,
\]
which contradicts (3.23). This completes the proof.

**Lemma 3.7.** Assume \( R_0 > 1 \), \( S_0 \) and \( \{u_{i0}\}_{i=1}^m \) satisfy the assumptions (H). Then for each \( c > c^*_m \),
\[
\lim_{t \to \infty} \sup_{|x| \geq ct} u_i(x, t) = 0. \tag{3.30}
\]
Particularly, for each \( c > c^* \),
\[
\lim_{t \to \infty} \sup_{|x| \geq ct} u_i(x, t) = 0, \quad i = 1, 2, \cdots, m. \tag{3.31}
\]

**Proof.** Note that \( u_0(x, t) \leq v_0(x, t) \) for all \( t \geq 0 \) and \( x \in \mathbb{R}^N \). Moreover, by Lemma 3.1, \( v_0(x, t) \leq \|S_0\|_{\infty} \) and \( \lim_{t \to \infty} v_0(x, t) = S^* \) uniformly for \( x \in \mathbb{R}^N \). Hence, for \( \epsilon > 0 \), there is \( T = T(\epsilon) > 0 \) such that \( v_0(x, t) < S^* + \epsilon \) for \( x \in \mathbb{R}^N \) and \( t > T \). Set
\[
\hat{v}_0(t) = \begin{cases} \|S_0\|_{\infty}, & 0 < t \leq T, \\ S^* + \epsilon, & t > T. \end{cases}
\]
Let \( \hat{u}_i(x, t) \) satisfy
\[
\partial_t \hat{u}_i - d_i \Delta \hat{u}_i = (f_i(\hat{v}_0(t)) - k_i)\hat{u}_i, \quad \hat{u}_i(x, 0) = u_{i0}(x).
\]
Hence, \( u_i(x, t) \leq \hat{u}_i(x, t) \) for all \( t \geq 0 \) and \( x \in \mathbb{R}^N \). It follows from the Poisson’s formula that
\[
\hat{u}_i(x, t) = e^{\int_0^t (f_i(\hat{v}_0(\tau)) - k_i) d\tau} v_i(x, t),
\]
where \( v_i \) is given by (3.4). Meanwhile, if \( e \in S^{N-1} \) and \( c \in \mathbb{R} \), then for each \( t > 0 \) and \( x \in \mathbb{R}^N \) we have
\[
v_i(x + cte, t) = \frac{1}{(4\pi d_it)^{N/2}} \int_{\mathbb{R}^N} \exp \left( -\frac{|y - x + cte - y|^2}{4d_it} \right) u_{i0}(y) dy \
\leq \frac{1}{(4\pi d_it)^{N/2}} e^{-\frac{c^2 t}{4d_i} e^{-\frac{c^2 t}{8d_i (e,x)}} J_i(c),
\]
where
\[
J_i(c) = \int_{\mathbb{R}^N} \exp \left( \frac{c|y|}{2d_i} \right) u_{i0}(y) dy < \infty.
\]
Hence
\[
u_i(x + cte, t) \leq e^{\int_0^t (f_i(\hat{v}_0(\tau)) - k_i) d\tau} (4\pi d_it)^{-N/2} e^{-\frac{c^2 t}{8d_i (e,x)}} J_i(c).
\]
Note that for $t > T$,
\[
\int_0^t (f_i(\hat{\nu}_0(\tau)) - k_i) d\tau = (f_i(S^* + \epsilon) - k_i) t + (f_i(\|S_0\|_{\infty}) - f_i(S^* + \epsilon)) T \\
\leq \frac{(c_i^* + \delta_\epsilon)^2 t}{4d_i} + (f_i(\|S_0\|_{\infty}) - f_i(S^* + \epsilon)) T,
\]
where $\delta_\epsilon = \sqrt{c_i^* + 4d_i(f_i(S^* + \epsilon) - f_i(S^*))} - c_i$. Set $C_i = \exp((f_i(\|S_0\|_{\infty}) - f_i(S^* + \epsilon)) T)$. For $t > T$, we have
\[
u_i(x + c t e, t) \leq C_i e^{\frac{(c_i^* + \delta_\epsilon)^2 t^2}{4d_i}} (4\pi d_i t)^{-N/2} e^{-\frac{\sum_{i}\langle e_i \rangle}{\bar{c}_i}} J_i(c_i).
\]
Suppose $c \geq c_i^* + \delta_\epsilon$. By taking $\alpha_i \in (0, Nd_i/(c_i^* + \delta_\epsilon))$ and $e = x/|x|$, if $|x| \geq (c_i^* + \delta_\epsilon) t - \alpha_i \ln t$, we have
\[
u_i(x, t) \leq C_i (4\pi d_i t)^{-N/2} e^{\frac{\alpha_i(c_i^* + \delta_\epsilon)^2 t^2}{4d_i}} J_i(c_i^*) \to 0 \text{ as } t \to \infty.
\]
Since $\epsilon$ is arbitrary, (3.32) implies that (3.30) holds. Moreover, if $c \geq c_i^* + \delta_\epsilon$, by taking $\alpha_i < Nd_i/(c_i^* + \delta_\epsilon)$ with $\tilde{d} = \min\{d_i, i = 1, 2, \cdots, m\}$, (3.32) implies that (3.31) holds too.

We are now in a position to prove Theorem 2.2.

**Proof of Theorem 2.2.** We only prove (2.1) and (2.2), the rest of conclusion in Theorem 2.2 can be obtained directly by Lemmas 3.6 and 3.7. For any $\epsilon > 0$ and $c_i^* < |c| < c_i^*$, suppose there exists $\bar{x} \in \mathbb{R}^N$ such that $\limsup_{t \to \infty} u_i(\bar{x} + c t e, t) \geq \epsilon$. Note that $\bar{x} + c t e = \bar{x} + (c - c_i^*) t e + c_i^* t e$. Hence $|\bar{x} + c t e| > c_i^* t$ for $t$ large enough since $c_i^* < |c|$. By Lemma 3.7, we have $\limsup_{t \to \infty} u_i(\bar{x} + c t e, t) = 0$. This contradiction implies that $\limsup_{t \to \infty} u_i(x + c t e, t) = 0$ if $c_i < |c| < c_i^*$. Similarly, we can prove $\limsup_{t \to \infty} u_j(x + c t e, t) = 0$ for $1 \leq j < i$ if $c_i < |c| < c_i^*$. Hence, (2.1) holds.

Now we claim that (2.2) holds if $c_i^* > |c| > c_i$. Arguing indirectly, for $0 < \epsilon < (c_i^* - c_i^+)/4$, suppose that for each $n \geq 0$, there exist $x_n \in \mathbb{R}^N$, a sequence of initial data $U^n_0 \in M^N_{m+1}$, $|c| \in [c_i^* - \epsilon, c_i^* + \epsilon]$ and a sequence of unit vectors $e_n \in S^{N-1}$ such that the component of the solution $U^n$ to (3.1)-(3.2) satisfies
\[
\liminf_{n \to \infty} u^n_i(x_n + c t e_n, t) \leq \frac{1}{n+1}, \quad \forall n \geq 0.
\]
As a consequence of the above analysis, for each $x \in \mathbb{R}^N$, $|c| \in [c_i^* - \epsilon, c_i^* + \epsilon]$ and each $e \in S^{N-1}$, one has
\[
\limsup_{n \to \infty} u_i(x + c t e, t) = 0, i = 1, 2, \cdots, m - 1.
\]
Furthermore, it follows from Lemma 3.6 that,
\[
\limsup_{t \to \infty} u_i(x + c t e, t) \geq \epsilon.
\]
Set $\tilde{u}_i^n(x, t) = S^n(x_n + x + c t e_n, t)$ and $\tilde{u}_i^n(x, t) = u_i^n(x_n + x + c t e_n, t)$ for $i = 1, 2, \cdots, m$. Then there exists a sequence $\{t_n\}_{n \geq 0}$ tending to infinity and a sequence $\{\eta_n\}_{n \geq 0} \in \mathbb{R}^+$ such that
\[
\tilde{u}_m^n(0, t_n) = \epsilon, \quad \tilde{u}_m^n(0, t_n) \leq \frac{\epsilon}{2} \text{ for } t \in [t_n, t_n + \eta_n], \quad \tilde{u}_m^n(0, t_n + \eta_n) \leq \frac{1}{n+1}.
\]
Up to a subsequence if necessary, we may assume that $\eta = \liminf_{n \to \infty} \eta_n$, $e = \liminf_{n \to \infty} e_n$, $\tilde{u}_i^n(x, t + t_n) \to \tilde{u}_i^n(x, t)$ locally uniformly in $\mathbb{R}^N$. Then $\tilde{u}_m^n(x, t)$
satisfies
\[ \hat{u}_m^\infty(0,0) = \frac{\epsilon}{2}, \quad \hat{u}_m^\infty(0,t) \leq \frac{\epsilon}{2} \text{ for } t \in [0, \eta). \]

Moreover, \( \hat{u}_0^\infty \) and \( \hat{u}_m^\infty \) satisfy
\[
\begin{align*}
(\partial_t - ce \cdot \nabla - d_0 \Delta)\hat{u}_0^\infty &= -\gamma_1^{-1}\hat{u}_m f_m(\hat{u}_0^\infty), \\
(\partial_t - ce \cdot \nabla - d_1 \Delta)\hat{u}_m^\infty &= \hat{u}_m^\infty(f_m(\hat{u}_0^\infty) - k_m).
\end{align*}
\]

If \( \eta < \infty \), one has \( \hat{u}_m^\infty(0,\eta) = 0 \). It follows from the maximum principle that \( \hat{u}_m^\infty(x,t) \equiv 0 \), a contradiction with \( \hat{u}_m^\infty(0,0) = \epsilon/2 \). Hence, the only possibility is \( \eta = \infty \). This means that
\[ \hat{u}_m^\infty(0,t) \leq \frac{\epsilon}{2}, \quad t \in [0, \infty). \] \tag{3.33}

Recall that \( \hat{u}_0^\infty(x,t) = \hat{u}_0^\infty(x - cte, t) \) and \( \hat{u}_m^\infty(x,t) = \hat{u}_m^\infty(x - cte, t) \) satisfy
\[
\begin{align*}
(\partial_t - d_0 \Delta)\hat{u}_0^\infty &= -\gamma_1^{-1}\hat{u}_m f_m(\hat{u}_0^\infty), \\
(\partial_t - d_1 \Delta)\hat{u}_m^\infty &= \hat{u}_m^\infty(f_m(\hat{u}_0^\infty) - k_m).
\end{align*}
\]

In view of \( \hat{u}_m^\infty(x,0) \neq 0 \) and \( (\hat{u}_0^\infty, \hat{u}_m^\infty) \in \text{BUC}(\mathbb{R}^N, \mathbb{R}^2) \), it follows from Lemma 3.6 that \( \limsup_{t \to \infty} \hat{u}_m^\infty(cte, t) \geq \epsilon \). Noting that \( \hat{u}_m^\infty(cte, t) = \hat{u}_m^\infty(0,t) \), we obtain a contradiction to (3.33). This completes the proof.

4. The minimum wave speed of traveling waves.

4.1. A regularized system on finite intervals. To establish the existence of traveling wave solutions, we assume that \( R_0 > 1 \). Consider the equation
\[ h_i(\lambda) = d_i\lambda^2 + c\lambda + f_i(S^*) - k_i = 0. \] \tag{4.1}

If \( \Delta_i = c^2 - 4d_i[f_i(S^*) - k_i] > 0 \), then (4.1) has two different roots
\[ \lambda_{i1} = \frac{-c + \sqrt{c^2 - 4d_i[f_i(S^*) - k_i]}}{2d_i}, \quad \lambda_{i2} = \frac{-c - \sqrt{c^2 - 4d_i[f_i(S^*) - k_i]}}{2d_i}. \]

Clearly, \( \lambda_{i2} < \lambda_{i1} < 0 \). Set
\[ \mathcal{S}(s) = S^*, \quad \mathcal{S}(s) = \max\{S^*(1 - \alpha e^{-\sigma s}), 0\}, \]
\[ \mathcal{U}_i(s) = e^{\lambda_i^* s}, \quad \mathcal{U}_i(s) = \max\{e^{\lambda_i s}(1 - \beta_i e^{-\tau_i s}), 0\}, \quad i = 1, 2, \cdots, m, \]
where \( \epsilon > 0 \) is small enough such that \( \mathcal{U}_i \leq \mathcal{U}_i \), positive constants \( \sigma, \alpha, \beta_i \) and \( \tau_i \) will be determined later.

Lemma 4.1. Suppose
\[ \sigma < \min \left\{ \frac{c}{d_0}, -\frac{\lambda_{i1}}{2}, \ldots, -\frac{\lambda_{m1}}{2} \right\}, \quad \alpha > \max \left\{ \frac{\sum_{i=1}^{m} \gamma_i^{-1} f_i(S)}{S^* \sigma(c - d_i \sigma)}, 1 \right\}. \]

Then \( \mathcal{S} \) satisfies the following inequality
\[ -d_0 \mathcal{S}'' - c\mathcal{S}' + \sum_{i=1}^{m} \mathcal{U}_i \gamma_i^{-1} f_i(\mathcal{S}) \leq 0 \] \tag{4.2}
for \( s \in \mathbb{R} \setminus \{\sigma^{-1} \ln \alpha\} \).
Proof. Note that $S = S^*(1 - \alpha e^{-\sigma s})$ if $s > \sigma^{-1} \ln \alpha$. Direct computation gives
\[
-d_0\varphi'' - c\varphi' + \sum_{i=1}^m \varpi_i \gamma_i^{-1} f_i(S) 
\]
\[
\leq S^* \alpha (d_0 \sigma^2 - c \sigma) e^{-\sigma s} + \sum_{i=1}^m \gamma_i^{-1} f_i(S^*) e^{\lambda_i s} 
\]
\[
= \left[ S^* \alpha (d_0 \sigma^2 - c \sigma) + \sum_{i=1}^m \gamma_i^{-1} f_i(S^*) e^{\lambda_i s} \right] e^{-\sigma s} 
\]
\[
\leq \left[ S^* \alpha (d_0 \sigma^2 - c \sigma) + \sum_{i=1}^m \gamma_i^{-1} f_i(S^*) e^{(\lambda_i + \sigma) \sigma^{-1} \ln \alpha} \right] e^{-\sigma s} 
\]
\[
= \left[ S^* \alpha (d_0 \sigma^2 - c \sigma) + \sum_{i=1}^m \gamma_i^{-1} f_i(S^*) e^{\lambda_i s} \right] e^{-\sigma s} 
\]
\[
\leq \left[ S^* \alpha (d_0 \sigma^2 - c \sigma) + \sum_{i=1}^m \gamma_i^{-1} f_i(S^*) e^{\lambda_i s} \right] e^{-\sigma s} 
\]
In addition, noting that $S(s) = 0$ if $s < \sigma^{-1} \ln \alpha$, it is easy to see that (4.2) holds.

Lemma 4.2. Suppose $R_0 > 1$ and $c < c^*$. Then $\varpi_i$ satisfies the following inequality
\[
-d_i \varpi''_i - c \varpi'_i + [k_i - f_i(S)] \varpi_i \geq 0 \text{ for } s \in \mathbb{R}, \; i = 1, 2, \ldots, m. 
\]
Proof. By a direct computation, we have
\[
-d_i \varpi''_i - c \varpi'_i + |k_i - f_i(S)| \varpi_i = -d_i \varpi''_i - c \varpi'_i + |k_i - f_i(S)| \varpi_i 
\]
for any $s \in \mathbb{R}$.

Lemma 4.3. Suppose $R_{ij} > 1$, $c < c^*$ and $\sigma, \alpha$ satisfy the hypotheses of Lemma 4.1. Let $\tau_i, \beta_i$ satisfy
\[
0 < \tau_i < \sigma < \lambda_{11} - \lambda_{21}, \quad \beta_i^{\sigma/\tau_i} > \max \left\{ -L_i S^* \alpha / h_i(\lambda_{11} - \tau_i), \alpha \right\} 
\]
where $h_i(\lambda_{11} - \tau_i)$ is given by (4.1) and $L_i$ is the Lipschitz constant of function $f_i(S)$. Then $\varpi_i$ satisfies the following inequality
\[
-d_i \varpi''_i - c \varpi'_i + |k_i - f_i(S)| \varpi_i \leq 0 
\]
for $s \in \mathbb{R} \setminus \{ \tau_i \ln \beta_i \}, i = 1, 2, \ldots, m$.

Proof. Note that $\varpi_i(s) = ce^{\lambda_i s}(1 - \beta_i e^{-\tau_i s})$ if $s > \tau_i^{-1} \ln \beta_i$. By a direct computation, we have
\[
-d_i \varpi''_i - c \varpi'_i + |k_i - f_i(S)| \varpi_i 
\]
\[
= \epsilon \beta_i h_i(\lambda_{11} - \tau_i) e^{(\lambda_{11} - \tau_i) s} + [f_i(S^*) - f_i(S)] \varpi_i 
\]
\[
\leq \epsilon \beta_i h_i(\lambda_{11} - \tau_i) e^{(\lambda_{11} - \tau_i) s} + e^{\lambda_i s}(1 - \beta_i e^{-\tau_i s}) L_i S^* \alpha e^{-\sigma s} 
\]
\[
\leq \epsilon \beta_i h_i(\lambda_{11} - \tau_i) e^{(\lambda_{11} - \tau_i) s} + \epsilon L_i S^* \alpha e^{(\lambda_{11} - \sigma) s} 
\]
\[
= \epsilon e^{(\lambda_{11} - \tau_i) s} \left[ \beta_i h_i(\lambda_{11} - \tau_i) + L_i S^* \alpha e^{(\tau_i - \sigma) s} \right] 
\]
\[
\leq \epsilon e^{(\lambda_{11} - \tau_i) s} \left[ \beta_i h_i(\lambda_{11} - \tau_i) + L_i S^* \alpha e^{(\tau_i - \sigma) \tau_i^{-1} \ln \beta_i} \right]. 
\]
If \( \tau_i \in (0, \sigma) \), then \( h_i(\lambda_{i1} - \tau_i) < 0 \). By the assumption (4.3)
\[-d_i u_i'' + cu_i' + |k_i - f_i(S)|u_i \leq \epsilon \epsilon^{(\lambda_i - \tau_i)^s} \left[ \beta_i h_i(\lambda_{i1} - \tau_i) + L_i S^* \alpha \epsilon^{(\tau_i - \sigma) \tau_i^{-1} \ln \beta_i} \right] \leq 0.
\]
Meanwhile, if \( s < \tau_i^{-1} \ln \beta_i \), then \( u_i(s) = 0 \), which implies that (4.4) holds.

To establish the existence of traveling wave solutions of (2.4)-(2.5), we first study the following regularized elliptic boundary value problem

\[
d_0 S'' + c S' = \sum_{i=1}^{m} \gamma_i^{-1} u_i f_i(S), \quad s \in (-l, l), \tag{4.5}
\]
\[d_i u_i'' + c u_i' = u_i(k_i - f_i(S)), \quad i = 1, 2, \cdots, m, \quad s \in (-l, l)
\]
with the boundary conditions
\[S(\pm l) = S(\pm l), \quad u_i(\pm l) = u_i(\pm l). \tag{4.6}
\]

Next we show the existence of solutions to (4.5)-(4.6) by the Schauder fixed-point theorem [37, 38], and weak upper and lower solution method [5].

**Definition 4.4.** (Du [5]) Consider the boundary-value problem

\[-\ddot{u}'' + c(x)\dot{u}' = f(x, u) \quad \text{for} \; x \in (a, b), \quad \dot{u}(a) = \alpha, \quad \dot{u}(b) = \beta. \tag{4.7}
\]

A function \( \tilde{u} \) is called a weak solution of (4.7) if \( \tilde{u} \in W^{1,p}(a, b), f \in (W_0^{1,p})^* \), which is the conjugate space of Banach space \( W_0^{1,p} \), and for every \( \phi \in W_0^{1,p}(a, b), \phi \geq 0,
\[
\int_a^b \tilde{u}'(\phi' + c(x)\phi)dx = \int_a^b f(x, \tilde{u})\phi dx.
\]

For convenience in the sequel, we set
\[w = (w_0, w_1, \cdots, w_m) = (S, u_1, \cdots, u_m), \quad \tilde{w} = (\tilde{w}_0, \tilde{w}_1, \cdots, \tilde{w}_m) = (\tilde{S}, \tilde{u}_1, \cdots, \tilde{u}_m),
\]
\[w = (\bar{w}_0, \bar{w}_1, \cdots, \bar{w}_m) = (\bar{S}, \bar{u}_1, \cdots, \bar{u}_m), \quad \bar{w} = (\bar{w}_0, \bar{w}_1, \cdots, \bar{w}_m) = (\bar{S}, \bar{\bar{u}}_1, \cdots, \bar{\bar{u}}_m).
\]

Let \( \frac{1}{2} \ln \frac{2}{\varphi} \leq \frac{1}{m} \ln \alpha \leq \frac{2}{\varphi} \), \( i = 1, 2, \cdots, m \). Define
\[\Lambda = \left\{ \tilde{w} \in (C[-l, l])^{m+1} \bigg| \begin{array}{l}
\bar{w} \leq \tilde{w} \leq \bar{\bar{w}} \; \text{in} \; (-l, l), \\
\tilde{w}(\pm l) = \bar{\tilde{w}}(\pm l) \end{array} \right\}.
\]

Denote \( F(\tilde{w}) = (F_0(\tilde{w}), F_1(\tilde{w}), \cdots, F_m(\tilde{w})) \), where
\[F_0(\tilde{w}) = -\sum_{i=1}^{m} \tilde{u}_i C^{-1} f_i(\tilde{S}) + M_0 \tilde{S},
\]
\[F_i(\tilde{w}) = \tilde{u}_i f_i(\tilde{S} - k_i) + M_i \tilde{u}_i, \quad i = 1, 2, \cdots, m,
\]
the constants \( M_i > 0 \) are large enough. Meanwhile, denote \( [w]_0 = [w_1, w_2, \cdots, w_m],
\]
\( [w]_i = [w_1, \cdots, w_{i-1}, w_{i+1}, \cdots, w_m] \) for \( 1 \leq i < m \) and \( [w]_m = [w_1, w_2, \cdots, w_{m-1}] \). Then system (4.5)-(4.6) can be rewritten as
\[-d_i w_i'' - c w_i' + M_i w_i = \begin{cases} F_0(w_0, [w]_i) & \text{in} \; (-l, l), \; i = 0, \\
F_i(w_0, w_i, [w]_i) & \text{in} \; (-l, l), \; i \neq 0,
\end{cases} \tag{4.8}
\]
\[w_i(\pm l) = \bar{w}_i(\pm l), \quad i = 0, 1, \cdots, m.
\]

Since the functions \( F_i \) possess the mixed quasi-monotone properties, the definition of the weak upper and lower solutions is introduced for the purpose of subsequent application.
Definition 4.5. The two vectors \( \overline{w}, \underline{w} \in [W^{1,p}(\Omega)]^{m+1} \) are called a pair of weak upper and lower solutions of (4.8) if the following inequalities are satisfied:

(i) \( w(\pm l) \leq \overline{w}(\pm l) \leq \underline{w}(\pm l) \);

(ii) for all \( \phi \in W_0^{1,p}(\Omega) \), \( \phi \geq 0 \),

\[
\int_{-l}^{l} \left[ (d_0 \overline{w}_0 + c \overline{w}_0)\phi' + M_0 \overline{w}_0 \phi \right] ds \geq \int_{-l}^{l} F_0(\overline{w}_0,[\overline{w}]_i)\phi ds,
\]

\[
\int_{-l}^{l} \left[ (d_i \overline{w}_i + c \overline{w}_i)\phi' + M_i \overline{w}_i \phi \right] ds \geq \int_{-l}^{l} F_i(\overline{w}_i,[\overline{w}]_i)\phi ds,
\]

\[
\int_{-l}^{l} \left[ (d_0 \underline{w}_0 + c \underline{w}_0)\phi' + M_0 \underline{w}_0 \phi \right] ds \leq \int_{-l}^{l} F_0(\underline{w}_0,[\underline{w}]_i)\phi ds,
\]

\[
\int_{-l}^{l} \left[ (d_i \underline{w}_i + c \underline{w}_i)\phi' + M_i \underline{w}_i \phi \right] ds \leq \int_{-l}^{l} F_i(\underline{w}_i,[\underline{w}]_i)\phi dx,
\]

with \( i = 1, 2, \ldots, m \).

It follows from Definition 4.4 that the following conclusion holds. This lemma indicates that it suffices to construct piecewise smooth upper-lower solutions, which makes Definition 4.5 easy to use.

Lemma 4.6. Suppose two vectors \( \overline{w}, \underline{w} \in (C([-l,l])^{m+1} \) are a pair of classical upper and lower solutions of (4.8) on \( (-l,l) \setminus \Xi := \{ \xi_j, j = 1, 2, \ldots, k \} \). Moreover, the derivatives of \( \overline{w} \) and \( \underline{w} \) at \( \xi_j \) satisfy

\[
\overline{w}'(\xi_j-) \leq \overline{w}'(\xi_j+) \quad \text{and} \quad \underline{w}'(\xi_j+) \leq \underline{w}'(\xi_j-) \quad \text{for} \quad j = 1, 2, \ldots, k.
\]

Then \( \overline{w} \) and \( \underline{w} \) are a pair of weak upper and lower solutions of (4.8), respectively.

Remark 4.7. It follows from Lemmas 4.1-4.3 and 4.6 that \( (\overline{S}, \overline{u}_1, \ldots, \overline{u}_m) \) and \( (\underline{S}, \underline{u}_1, \ldots, \underline{u}_m) \) are a pair of weak upper-lower solutions of (4.8).

Proposition 4.8. Suppose \( R_0 > 1 \) and \( c > c^* \). Then (4.5)-(4.6) has at least one classical solution in \( \Lambda \).

Proof. For any \( \tilde{w} \in \Lambda \), we consider the following linear boundary value problem

\[
-d_i w_i'' - cw_i' + M_i w_i = F_i(\tilde{w}) \quad \text{in} \quad (-l,l),
\]

\[
w_i(\pm l) = w_i(\pm l), \quad i = 0, 1, 2, \ldots, m. \tag{4.9}
\]

By the linear theory of the standard elliptic equation [8], (4.9) has a unique solution \( w_i \) satisfying \( w_i \in W^{2,p}(-l,l) \cap C(-l,l) \) for any \( \tilde{w} \in \Lambda \) and \( p > 1 \). It follows from the embedding theorems that \( W^{2,p}(-l,l) \hookrightarrow C^{1+\alpha}[-l,l] \) for some \( \alpha \in (0,1) \). Define an operator \( T: \Lambda \to (C([-l, l])^{m+1} \) by \( T\tilde{w} = w \), where \( w = (w_0, w_1, \ldots, w_m) \) and \( w_i \) is the unique solution of (4.9). Then it is easy to see that (4.5)-(4.6) has a solution if and only if \( T \) has a fixed point in \( \Lambda \).

Clearly, it is easy to see that \( T: \Lambda \to (C([-l,l])^{m+1} \) is compact. Next, we prove the operator \( T \) maps \( \Lambda \) into \( \Lambda \). Note that \( \bar{w} = (\overline{S}, \overline{u}_1, \ldots, \overline{u}_m) \leq (\overline{S}, \overline{u}_1, \ldots, \overline{u}_m) = \overline{w} \) are a pair of ordered weak lower and upper solutions of (4.8). For any \( \tilde{w} \in \Lambda \), set \( \bar{w} = w - \overline{w} \). Since \( \bar{w} \in \Lambda \), we have \( \bar{w} \leq w \leq \overline{w} \) and \( \bar{w} \) satisfies

\[
-d_i \bar{w}_i'' - c \bar{w}_i' + M_i \bar{w}_i \geq \begin{cases} F_0(\overline{w}_0,[\overline{w}]_i) \quad & \text{in} \quad (-l,l), \\ F_i(\overline{w}_0,[\overline{w}]_i) \quad & \text{in} \quad (-l,l), \end{cases}
\]

\[
\bar{w}_i(\pm l) \geq w_i(\pm l) \tag{4.10}
\]
in the weak sense. Subtracting (4.9) from (4.10), we obtain
\[-d_i w_i'' - c c w_i' + M_i w_i \leq F_i(w_i) - F_i(\bar{w}_i) \leq 0 \text{ in } (-l, l), \quad w_i(\pm l) \leq 0,
\]
where \([\bar{w}_i] = [\bar{w}_0, \ldots, \bar{w}_{i-1}, \bar{w}_{i+1}, \ldots, \bar{w}_m]. By weak maximum principle we have \(\bar{w}_i \geq w_i\) a.e. in \((-l, l). Furthermore, it follows from the standard elliptic estimates and the embedding theorems that \(\bar{w}_i \geq w_i. Similarly, we obtain \(w_i' \leq u_i. Thus, the operator \(T\) maps \(\Lambda\) into \(\Lambda\).

Clearly, \(\Lambda\) is closed and convex. The Schauder fixed-point theorem implies that there exists weak solution \((S, u_1, \ldots, u_m) \in \Lambda\) such that \((S, u_1, \ldots, u_m) = T(S, u_1, \ldots, u_m)\) in \((-l, l). Since the functions \(f_i(S)\) are sufficiently regular, by the standard \(L^p\) regularity and the Sobolev embedding theorems, one can show that the weak solution obtained above is classical. 

Next, we derive further estimates on solutions of (4.5)-(4.6) which enable us to extend those solutions to the entire real line. Firstly, let us derive some integral formula for the solution \((S, u_1, \ldots, u_m)\) of system (4.5)-(4.6) that will be used later. Multiplying the first equation of (4.5) by \(e^{cs/d_i}\) and integrating from \(-l\) to \(s\), we obtain
\[S'(s) = e^{-\frac{c}{d_i}(l+s)}S'(-l) + \frac{1}{d_i} \int_{-l}^{s} e^{-\frac{c}{d_i}(t-s)} \sum_{i=1}^{m} \gamma_i^{-1} u_i(t) f_i(S(t)) dt. \]  
(4.11)
Integrating (4.11) from \(-l\) to \(s\) yields
\[S(s) = S(-l) + S'(-l) \int_{-l}^{s} e^{-\frac{c}{d_i}(l+t)} dt + \frac{1}{d_i} \int_{-l}^{s} \int_{-l}^{t} e^{-\frac{c}{d_i}(\tau-s)} \sum_{i=1}^{m} \gamma_i^{-1} u_i(t) f_i(S(t)) dt d\tau
\]
\[= S(-l) + \frac{d_0}{c} (1 - e^{-\frac{c}{d_i}(s+l)}) S'(-l) + \frac{1}{c} \int_{-l}^{s} (1 - e^{-\frac{c}{d_i}(s-t)}) \sum_{i=1}^{m} \gamma_i^{-1} u_i(t) f_i(S(t)) dt.
\]  
(4.12)
Similarly, multiplying the equation of (4.5) for \(u_i\) by \(e^{cs/d_i}\) and integrating from \(-l\) to \(s\), we obtain
\[u_i'(s) = e^{-\frac{c}{d_i}(l+s)} u_i'(-l) + \frac{1}{d_i} \int_{-l}^{s} e^{-\frac{c}{d_i}(t-s)} (k_i - f_i(S(t))) u_i(t) dt. \]  
(4.13)
Integrating (4.13) from \(-l\) to \(s\) yields
\[u_i(s) = u_i(-l) + \frac{d_i}{c} (1 - e^{-\frac{c}{d_i}(l+s)}) u_i'(-l) + \frac{1}{c} \int_{-l}^{s} (1 - e^{-\frac{c}{d_i}(s-t)})(k_i - f_i(S(t))) u_i(t) dt.
\]  
(4.14)

**Lemma 4.9.** Suppose \(S\) is a nonnegative solution of the system (4.5)-(4.6). Then

\[0 \leq S \leq S^*, \quad S' > 0. \]

**Proof.** Applying the strong maximum principle to the first equation of (4.5) yields \(0 \leq S \leq S^*.\) In view of \(S(-l) = 0,\) it is easy to see that \(S'(-l) > 0.\) It follows from (4.11) that \(S'(s) > 0\) in \((-l,l).\) 

Lemma 4.10. Suppose \((S,u_1,\ldots,u_m)\) is a nonnegative solution of system (4.5) – (4.6) in \(\Lambda\). Then there exist positive constants \(M_{i1}\) independent of \(l\) such that

\[0 \leq u_i \leq M_{i1}, \quad i = 1, 2, \ldots, m.\]

Proof. It is easy to check that \(S'(-l) > 0, S'(l) \leq S'(l)\) and \(u'_i(-l) > 0, i = 1, 2, \ldots, m\). Direct computations show that \(u'_i(l) \leq u'_i(l) \leq 0\) if \(l\) is large enough.

For \(s \in [-l,l]\), it follows from (4.14) that

\[u_i(s) = u_i(-l) + \frac{d_i}{c} \left(1 - e^{-\frac{c}{s} (s - l)}\right) u'_i(-l) + \frac{1}{c} \int_{-l}^{s} \left(1 - e^{-\frac{c}{s} (s - t)}\right) (k_i - f_i(S(t))) u_i(t) dt\]

\[\leq u_i(-l) + \frac{d_i}{c} u'_i(-l) + \frac{1}{c} \int_{-l}^{l} (k_i + f_i(S(t))) u_i(t) dt.\]

Integrating the equation of (4.5) for \(u_i\) from \(-l\) to \(s\), we have

\[k_i \int_{-l}^{s} u_i(t) dt = \int_{-l}^{s} (d_i u''_i(t) + c u'_i(t) + f_i(S(t)) u_i(t)) dt.\]

Consequently for \(s \in [-l,l]\),

\[\int_{-l}^{s} (k_i + f_i(S(t))) u_i(t) dt \leq \int_{-l}^{l} (k_i + f_i(S(t))) u_i(t) dt\]

\[= \int_{-l}^{l} (d_i u''_i(t) + c u'_i(t) + 2 f_i(S(t)) u_i(t)) dt\]

\[= d_i (u'_i(l) - u'_i(-l)) + c (u_i(l) - u_i(-l)) + 2 \int_{-l}^{l} f_i(S(t)) u_i(t) dt\]

\[\leq d_i (u'_i(l) - u'_i(-l)) + c (u_i(l) - u_i(-l)) + 2 \gamma_i \int_{-l}^{l} \sum_{i=1}^{n} \gamma_i^{-1} f_i(S(t)) u_i(t) dt\]

\[= d_i (u'_i(l) - u'_i(-l)) + c (u_i(l) - u_i(-l)) + 2 \gamma_i \int_{-l}^{l} (d_0 S''(t) + c S'(t)) dt\]

\[\leq d_i (u'_i(l) - u'_i(-l)) + c (u_i(l) - u_i(-l)) + 2 \gamma_i d_0 (S'(l) - S'(-l)) + 2 \gamma_i c S^*.\]

Combining (4.15)-(4.16), we discover

\[u_i(s) \leq u_i(-l) + \frac{d_i}{c} u'_i(-l) + \frac{1}{c} \int_{-l}^{s} (k_i + f_i(S(t))) u_i(t) dt\]

\[\leq u_i(l) + \frac{d_i}{c} u'_i(l) + 2 \gamma_i \frac{d_0}{c} (S'(l) - S'(-l)) + 2 \gamma_i S^*\]

\[\leq u_i(l) + 2 \gamma_i \frac{d_0}{c} S'(l) + 2 \gamma_i S^*\]

by \(u'_i(l) \leq 0, S'(-l) > 0\) and \(S'(l) \leq S'(l)\). Since \(u_i(l)\) and \(S'(l)\) in (4.17) are bounded independent of \(l\) as long as \(l\) is large enough, there must exist positive constants \(M_{i1}\) independent of \(l\) such that \(0 \leq u_i \leq M_{i1}, i = 1, 2, \ldots, m.\)

Lemma 4.11. Let \((S,u_1,\ldots,u_m)\) be a solution to system (4.5) – (4.6) in \(\Lambda\). Then there exist positive constants \(M_{02}\) and \(M_{12}\) independent of \(l\) such that

\[\|S\|_{C^2(-l,l)} \leq M_{02}, \quad \|u_i\|_{C^2(-l,l)} \leq M_{12}, \quad i = 1, 2, \ldots, m.\]
Proof. In view of Lemma 4.9, it follows from (4.12) that

\[ S^* \geq S(0) = S(-l) + \frac{d_0}{c} (1 - e^{-\frac{c}{\pi^2}}) S'(-l) + \frac{1}{c} \int_{-l}^0 (1 - e^{-\frac{c}{\pi^2} t}) \sum_{i=1}^m \gamma_i^{-1} u_i(t) f_i(S(t)) dt \]

\[ \geq \frac{d_0}{c} (1 - e^{-\frac{c}{\pi^2} l}) S'(-l). \]

Hence, \( S'(-l) < 2cS^*/d_0 \) for any \( l > \frac{d_0}{c} \ln 2 \). Using (4.11), we obtain

\[ S'(s) = e^{-\frac{c}{\pi^2} (l+s)} S'(-l) + \frac{1}{d_0} \int_{-l}^s e^{\frac{c}{\pi^2} (t-s)} \sum_{i=1}^m \gamma_i^{-1} u_i(t) f_i(S(t)) dt \]

\[ \leq e^{-\frac{c}{\pi^2} (l+s)} S'(-l) + \frac{1}{d_0} \sum_{i=1}^m \gamma_i^{-1} M_i f_i(S^*) \int_{-l}^s e^{\frac{c}{\pi^2} (t-s)} dt \]

\[ \leq 2 \frac{cS^*}{d_0} + \frac{1}{c} \sum_{i=1}^m \gamma_i^{-1} M_i f_i(S^*). \]

Since \( |S''| \leq (cS' + \sum_{i=1}^m \gamma_i^{-1} u_i f_i(S))/d_0 \), there exists some positive constant \( M_{02} \) which does not depend on \( l \) such that \( \|S\|_{C^2(-l,l)} \leq M_{02} \). Similarly, applying (4.13) and Lemma 4.10 we can prove \( u_i' \) is bounded and independent of \( l \). Combining the equation of (4.5) for \( u_i \), we can obtain that \( u_i \) are bounded in \( C^2(-l,l) \) and independent of \( l \). Hence, we obtain \( \|u_i\|_{C^2(-l,l)} \leq M_{i2} \) for some positive constants \( M_{i2} \) independent of \( l \). \( \square \)

4.2. The traveling waves. Now, we are ready to establish the existence of traveling wave solutions to system (2.4)-(2.5).

Proposition 4.12. (Existence) For any given \( S^* \) satisfying \( R_0 > 1 \) and \( c > c^* \), there exists \( S_* \geq 0 \) satisfying

\[ \max \left\{ \frac{f_i(S_*)}{k_i}, i = 1, 2, \ldots, m \right\} < 1, \]

such that system (2.4)-(2.5) has a positive traveling wave solution \((S, u_1, \ldots, u_m)\), connecting \((S_0, 0, \ldots, 0)\) and \((S^*, 0, \ldots, 0)\). Moreover, \( S_* < S < S^*, 0 < u_i \leq 2\gamma_i(S^* - S_*) \) in \( \mathbb{R} \) for \( i = 1, 2, \ldots, m \), and

\[ (S^* - S_*) = \sum_{i=1}^m \frac{k_i}{c\gamma_i} \int_{-\infty}^{\infty} u_i(s) ds. \]

Proof. Suppose \((S_n, u_{1n}, \ldots, u_{mn})\) is the solution of (4.5)-(4.6) in \( \Lambda \) with \( l = l_n \), where \( l_n \to \infty \) as \( n \to \infty \). It follows from Lemma 4.11 and Lipschitz continuity of \( f_i(S) \) that \((S_{n}^{(j)}, u_{1n}^{(j)}, \ldots, u_{mn}^{(j)})(j = 0, 1, 2)\) are uniformly bounded and equicontinuous in \((-l_n, l_n)\). Due to elliptic regularity, up to a subsequence, one may assume that there exists \((S, u_1, \ldots, u_m) \in (C^2(\mathbb{R}))^{m+1}\) such that the sequence \((S_n, u_{1n}, \ldots, u_{mn}) \to (S, u_1, \ldots, u_m) \) in \((C^2_{loc}(\mathbb{R}))^{m+1}\). Furthermore, \((S, u_1, \ldots, u_m)\) satisfies (2.4). It is easy to see that \((S, u_1, \ldots, u_m) \to (S^*, 0, \ldots, 0)\) as \( s \to \infty \) by \((S(s), u_{1}(s), \ldots, u_{m}(s)) \to (S^*, 0, \ldots, 0)\) and \((S(s), u_{1}(s), \ldots, u_{m}(s)) \to (S^*, 0, \ldots, 0)\) as \( s \to \infty \). By the Fluctuation Lemma 2.2 in [34], we have

\[ (S', u'_1, \ldots, u'_m) \to (0, 0, \ldots, 0) \] as \( s \to \infty \).
Moreover, by (4.22),

\[ d_0 S'(s) = c[S^* - S(s)] - \sum_{i=1}^{m} \int_{s}^{\infty} \gamma^{-1}_i f_i(S(t))u_i(t)dt. \]

We claim that the integral \( f_s^\infty \gamma^{-1}_i f_i(S(t))u_i(t)dt \) is uniformly bounded since \( S \) is uniformly bounded. Otherwise, \( S'(s) \to -\infty \) as \( s \to -\infty \), which leads to \( S(s) \to \infty \) as \( s \to -\infty \), a contradiction. Thus, \( f_i(S)u_i \) is integrable on \( \mathbb{R} \) and \( S' \) is uniformly bounded on \( \mathbb{R} \). Multiplying the first equation of (2.4) by \( e^{cs/d_0} \), we have

\[ d_0 \left( S'(s)e^{\frac{cs}{d_0}} \right)' = \sum_{i=1}^{m} \gamma^{-1}_i f_i(S(s))u_i(s)e^{\frac{cs}{d_0}}. \]  

(4.19)

Since \( S' \) is uniformly bounded, it follows that \( \lim_{s \to -\infty} S'(s)e^{cs/d_0} = 0 \). Integrating (4.19) from \(-\infty \) to \( s \), we get

\[ S'(s) = \sum_{i=1}^{m} \frac{1}{d_0 \gamma_i} \int_{-\infty}^{s} f_i(S(t))u_i(t)e^{-\frac{cs}{d_0}}(s-t)dt \geq 0. \]  

(4.20)

Then, by the fact that \( f(S,u)u \) is integrable on \( \mathbb{R} \),

\[ S'(s) = \sum_{i=1}^{m} \frac{1}{d_0 \gamma_i} \int_{-\infty}^{s} f_i(S(t))u_i(t)e^{-\frac{cs}{d_0}}(s-t)dt \leq \sum_{i=1}^{m} \frac{1}{d_0 \gamma_i} \int_{-\infty}^{s} f_i(S(t))u_i(t)dt \to 0 \]

as \( s \to -\infty \). Hence, \( S'(s) \to 0 \) as \( s \to -\infty \). It follows from \( S'(s) \geq 0 \) that \( \lim_{s \to -\infty} S(s) = S_\ast \geq 0 \). Hence,

\[ c(S^* - S_\ast) = \sum_{i=1}^{m} \gamma^{-1}_i \int_{-\infty}^{\infty} f_i(S(t))u_i(t)dt. \]  

(4.21)

We are now ready to study the limit of \( u(s) \) as \( s \to -\infty \). By the variation of constants formula to the equation of (2.4) for \( u_i \),

\[ u_i(s) = \frac{1}{d_i(\mu_2i - \mu_1i)} \left[ \int_{-\infty}^{s} e^{\mu_1i(s-\xi)} f_i(S(\xi))u_i(\xi)d\xi + \int_{s}^{\infty} e^{\mu_2i(s-\xi)} f_i(S(\xi))u_i(\xi)d\xi \right], \]  

(4.22)

where

\[ \mu_1i = -c - \sqrt{c^2 + 4d_i k_i} < \mu_2i = -c + \sqrt{c^2 + 4d_i k_i}. \]

Since \( f_i(S)u_i \) is integrable on \( \mathbb{R} \), it follows from (4.22) and Fubini’s Theorem that \( u_i \) is also integrable on \( \mathbb{R} \), and

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_i(s)ds = \frac{1}{k_i} \int_{-\infty}^{\infty} f_i(S(s))u_i(s)ds. \]  

(4.23)

Moreover, by (4.22),

\[ u'_i(s) = \frac{1}{d_i(\mu_2i - \mu_1i)} \left[ \mu_1i \int_{-\infty}^{s} e^{\mu_1i(s-\xi)} f_i(S(\xi))u_i(\xi)d\xi + \mu_2i \int_{s}^{\infty} e^{\mu_2i(s-\xi)} f_i(S(\xi))u_i(\xi)d\xi \right]. \]
Consequently,
\[ |u_i'(s)| \leq \frac{f_i(S^*)}{d_i} \int_{-\infty}^{s} u_i(s) ds. \]

In conclusion, \( u' \) is uniformly bounded and \( u \) is integrable on \( \mathbb{R} \), which implies that \( u(s) \to 0 \) as \( s \to -\infty \).

It follows from (4.20) that \( S_* \leq S(s) \leq S^* \). If \( S_* > 0 \), then \( S'(s) > 0 \) by (4.20), which means \( S_* < S(s) < S^* \). If \( S_* = 0 \), we claim that \( S(s) > 0 \) for \( -\infty < s < \infty \), and hence \( 0 < S(s) < S^* \) by (4.20). Otherwise, there exists a point \( s_1 \in \mathbb{R} \) such that \( S'(s_1) = 0 \). The equation (4.20) implies that \( S \equiv 0 \) on \( (-\infty, s_1) \), which leads to \( S'(s_1) = 0 \). Moreover, from the equation of (2.4) for \( u_i \), we can obtain that \( u_i(s_1) = B e^{\mu_2 s_1}, u_i'(s_1) = B \mu_2 e^{\mu_2 s_1} \) for any given \( B \geq 0 \). By the uniqueness of Cauchy problem of ordinary differential equations, it is easy to see that \( S(s) \equiv 0 \) on \( \mathbb{R} \), a contradiction. Since \( u_i(s) \geq 0, \neq 0 \), the maximum principle implies \( u_i(s) > 0 \) for \( s \in \mathbb{R} \). Multiplying the equation of (2.4) for \( u_i \) by \( e^{\alpha s/d_i} \) and integrating from \( -\infty \) to \( s \) twice, we have
\[ u_i(s) = \frac{1}{c} \int_{-\infty}^{s} \left[ 1 - e^{\frac{\alpha}{d_i} (t-s)} \right] [k_i - f_i(S(t))] u_i(t) dt. \]

It follows from (4.21) and (4.23) that
\[ u_i(s) \leq \frac{1}{c} \int_{-\infty}^{s} [k_i + f_i(S(t))] u_i(t) dt \leq \frac{1}{c} \int_{-\infty}^{\infty} [k_i + f_i(S(t))] u_i(t) dt \]
\[ = \frac{2}{c} \int_{-\infty}^{\infty} f_i(S(t)) u_i(t) dt \leq 2 \gamma_i (S^* - S_*) \]
for \( s \in \mathbb{R} \). Moreover, the equations (4.21) and (4.23) imply that (4.18) holds.

It remains to prove that \( f_i(S_*) < k_i \) for \( i = 1, 2, \ldots, m \). For contradiction, suppose there exists some \( i \in \{1, 2, \ldots, m\} \) such that \( f_i(S_*) \geq k_i \). Let \( (S, u_1, \ldots, u_m) \) be a positive solution of (2.4)-(2.5). Since \( S' > 0 \), \( S(-\infty) = S_* \) and \( f_i(S_*) \geq k_i \), there exists positive constant \( A_i \) large enough such that \( k_i < f_i(S(s)) \) for \( s \in (-\infty, -A_i) \). It follows from (2.4) that
\[ u_i'(s) = \frac{1}{d_i} \int_{-\infty}^{s} e^{\frac{\alpha}{d_i} (t-s)} [k_i - f_i(S(t))] u_i(t) dt \leq 0 \text{ for } s \in (-\infty, -A_i). \]

Due to \( u_i(-\infty) = 0 \) and \( u_i \geq 0 \), it leads to \( u_i(s) \equiv 0 \) on \( (-\infty, -A_i) \), which contradicts \( u_i(s) > 0 \) for any finite \( s \in \mathbb{R} \).

\textbf{Remark 4.13.} Although we can not establish the existence of a positive traveling wave solution for (2.4)-(2.5) when \( c = c^* \) and \( m > 1 \), we can see that system (2.4)-(2.5) has at least one nonconstant traveling wave solution (see Theorem 1.1 in [30]).

\textbf{Proposition 4.14.} (Monotonicity) Suppose \( R_0 > 1 \) and \( c > c^* \). Then the positive traveling wave solutions of (2.4)-(2.5) satisfy (i) \( S'(s) > 0 \) for \( s \in (-\infty, +\infty) \); (ii) there are \( s_i \in \mathbb{R} \) such that \( u_i'(s) > 0 \) if \( s < s_i \) and \( u_i'(s) < 0 \) if \( s > s_i \), where \( i = 1, 2, \ldots, m \).
Proof. According to (2.4)-(2.5), using the Proposition 4.12, we have

\[ S'(s) = \frac{1}{d_i} \int_{-\infty}^{s} e^{\frac{s}{\gamma_i(t-s)}} \sum_{i=1}^{m} \gamma_i^{-1} u_i(t)f_i(S(t))dt, \]

\[ u_i'(s) = \frac{1}{d_i} \int_{-\infty}^{s} e^{\frac{s}{\gamma_i(t-s)}} (k_i - f_i(S(t)))u_i(t)dt. \]

Thus, \( S'(s) > 0 \) for all \( s \in (\infty, \infty) \). It is easy to see that \( u_i'(s) > 0 \) if \( s \) is near \( \infty \). Next, we prove that there must be an \( s_i \) such that \( u_i'(s_i) = 0 \). On the contrary, if \( u_i'(s) > 0 \) for all \( s \), then \( \lim_{s \to \infty} u_i(s) = 0 \), which contradicts \( \lim_{s \to \infty} u_i(s) = 0 \) in Proposition 4.12. Let \( s_i \) be the first zero of \( u_i'(s_i) \) from \( -\infty \) to \( +\infty \). The monotonicity of \( S \) implies \( k_i - f_i(S(s)) < 0 \) for \( s > s_i \). Hence for \( s > s_i \) we obtain

\[ u_i'(s) = \frac{1}{d_i} \int_{-\infty}^{s} e^{\frac{s}{\gamma_i(t-s)}} (k_i - f_i(S(t)))u_i(t)dt \]

\[ = \frac{1}{d_i} \int_{-\infty}^{s_i} e^{\frac{s}{\gamma_i(t-s)}} (k_i - f_i(S(t)))u_i(t)dt + \frac{1}{d_i} \int_{s_i}^{s} e^{\frac{s}{\gamma_i(t-s)}} (k_i - f_i(S(t)))u_i(t)dt \]

\[ = u_i'(s_i) + \frac{1}{d_i} \int_{s_i}^{s} e^{\frac{s}{\gamma_i(t-s)}} (k_i - f_i(S(t)))u_i(t)dt \]

\[ = \frac{1}{d_i} \int_{s_i}^{s} e^{\frac{s}{\gamma_i(t-s)}} (k_i - f_i(S(t)))u_i(t)dt < 0. \]

This completes the proof. \( \square \)

**Proposition 4.15.** *(Non-existence)*

**(i)** If there exists \( i \in \{1, 2, \cdots, m\} \) such that \( R_{0i} \leq 1 \), then system (2.4)-(2.5) has no positive traveling wave solution.

**(ii)** If \( R_0 > 1 \) and \( 0 < c < c^* \), then system (2.4)-(2.5) has no positive traveling wave solution.

**Proof.** (i) Suppose that \((S, u_1, u_2, \cdots, u_m)\) is a positive traveling wave solution of system (2.4)-(2.5). With the use of inequality \( k_i \geq f_i(S^*) \) based on \( R_{0i} \leq 1 \) for some \( i \in \{1, 2, \cdots, m\} \), we deduce from the equations of (2.4) for \( u_i \) that

\[ d_i u_i'' + c u_i' = u_i(k_i - f_i(S)) \geq u_i(f_i(S^*) - f_i(S)) \geq 0. \]

Since \( d_i u_i'' + c u_i' = d_i e^{-c/s/d_i}(c^{s/d_i} u_i'(s))' \geq 0 \), we conclude that \( e^{c/s/d_i} u_i'(s) \geq \lim_{s \to -\infty} e^{c/s/d_i} u_i'(s) = 0 \). Hence \( u_i' \geq 0 \). It follows that \( u_i(+\infty) > 0 \) from \( u_i(-\infty) = 0 \), \( u_i' \geq 0 \) and \( u_i \not\equiv 0 \), which contradicts the boundary condition \( u_i(+\infty) = 0 \).

(ii) Suppose that \((S, u_1, u_2, \cdots, u_m)\) is a positive traveling wave solution of system (2.4)-(2.5). Take a sequence \( \{s_n\}_{n \geq 0} \) such that \( s_n \to \infty \) as \( n \to \infty \). Consider the sequence of shifted functions \( S_n = S(s + s_n) \) and \( u_{mn} = u_m(s + s_n) \). Then \( u_{mn} \) satisfies

\[ d_m u_{mn}'' + c u_{mn}' + (f_m(S_n) - k_m)u_{mn} = 0. \]

By Harnack inequality, for each \( r > 0 \) there exists some constant \( M_r > 0 \) dependent on \( r \), such that

\[ u_m(s + s_n) \leq M_r u_m(s_n) \text{ for } s \in [-r, r], \ n \geq 0. \]

Hence, \( u_m(s_n) > 0 \) for \( n \geq 0 \). Setting \( \tilde{u}_{mn}(s) = u_m(s + s_n)/u_m(s_n) \), we have

\[ d_m \tilde{u}_{mn}'' + c \tilde{u}_{mn}' + (f_m(S_n) - k_m)\tilde{u}_{mn} = 0, \ \tilde{u}_{mn}(0) = 1, \ 0 < \tilde{u}_{mn}(s) \leq M_r. \]
As a consequence, one may assume that the sequence \( \{\tilde{u}_{mn}\}_{n \geq 1} \) (if necessary, taking a subsequence) converges locally uniformly towards some function \( \tilde{u}_{m\infty} \) that satisfies
\[
d_m \tilde{u}_{m\infty}'' + c\tilde{u}_{m\infty}' + (f_m(S^*) - k_m)\tilde{u}_{m\infty} = 0, \quad \tilde{u}_{m\infty}(0) = 1, \quad 0 \leq \tilde{u}_{m\infty}(s).
\] (4.24)

The characteristic equation of (4.24) satisfies
\[
d_m \lambda^2 + c\lambda + f_m(S^*) - k_m = 0.
\]
The conditions \( R_0 > 1 \) and \( 0 < c < c^* = c^*_m \) imply that the nontrivial solution \( \tilde{u}_{m\infty} \) of (4.24) is oscillating, and \( \tilde{u}_{m\infty} \) can not keep nonnegative all the time, which contradicts \( \tilde{u}_{m\infty} \geq 0 \) and \( \tilde{u}_{m\infty} \not\equiv 0 \).

Theorem 2.3 now follows directly from Propositions 4.12, 4.14 and 4.15.

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