Affine symmetry in mechanics of collective and internal modes. Part I. Classical models

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Abstract

Discussed is a model of collective and internal degrees of freedom with kinematics based on affine group and its subgroups. The main novelty in comparison with the previous attempts of this kind is that it is not only kinematics but also dynamics that is affinely-invariant. The relationship with the dynamics of integrable one-dimensional lattices is discussed. It is shown that affinely-invariant geodetic models may encode the dynamics of something like elastic vibrations.

Keywords: collective modes, affine invariance, integrable lattices, nonlinear elasticity.

Introduction

In some of our earlier papers including rather old ones we have discussed the concept of affinely-rigid body, i.e., continuous, discrete, or simply finite system of material points subject to such constraints that all affine relations between its elements are frozen during any admissible motion. For example, all material straight lines remain straight lines in the course of evolution, and their parallelism is also a constant, non-violated property. Unlike this, the metrical features, like distances and angles, need not be preserved. In other words, such a body is restricted in its behaviour to rigid translations, rigid rotations, and homogeneous deformations. Models of this kind may be successfully applied in
a very wide spectrum of physical problems like nuclear dynamics \([5]\) (droplet model of the atomic nuclei), molecular vibrations, macroscopic elasticity \([15, 16, 41-45, 48, 50, 52, 53, 54, 55, 60, 62, 63]\) (in situations when the length of excited waves is comparable with the size of the body), in the theory of microstructured bodies \([53]\) (micromorphic continua), in geophysics \([4, 11]\) (the theory of the shape of Earth), and even in large-scale astrophysics (vibrating stars, vibrating concentrations of the cosmic substratum, like galaxies or concentrations of the interstellar dust).

From the purely mathematical point of view such a model provides an interesting example of a system with the group-theoretical background of the geometry of degrees of freedom \([2, 25, 26, 46, 53, 55, 56]\). It is an affine generalization of the usual rigid top with the orthogonal group replaced by the linear one (isometries replaced by affine transformations). Let us remind also that there is an interesting formulation of the general non-constrained continuum mechanics based on the infinite-dimensional "Lie group" of all diffeomorphisms or volume-preserving diffeomorphisms \([1, 2, 3, 13, 14, 33, 34]\). This theory is rather complicated (although heuristically very fruitful) because of serious mathematical problems with infinite-dimensional groups. The mechanics of an affinely-rigid body is a simple compromise between rigid-body mechanics and such a continuum theory, because admitting deformative degrees of freedom it is simultaneously based on the finite-dimensional framework.

Let us stress, however, that, in spite of its non-questionable physical applicability and formally interesting features, the referred mechanics of affinely-rigid body is in a sense disappointing from the point of view of the mathematical theory of Lie group motivated systems. The point is that in the latter theory it is not only kinematics (finally, geometry of degrees of freedom) but also dynamics that is ruled by the underlying group. Due to the isotropy of the physical space, Lagrangian of a free rigid top, i.e., its kinetic energy, is invariant under all left regular translations (all spatial rotations); the same is valid, of course, for the resulting equations of motion (Euler equations). If the material structure of the top is isotropic (spherical inertial tensor), then the model is also invariant under right regular translations. When formulating the theory of ideal incompressible fluids in terms of the group of all volume-preserving diffeomorphisms, one obtains an infinite-dimensional Hamiltonian system invariant under all right regular translations. This is due to the fact that in the usual Euler description of fluid its Lagrangian coordinates are not very essential, and the fluid particles have a rather limited individuality. Summarizing, in these theories one deals with Lagrangians or Hamiltonians based on left-, right-, or even two-side invariant metric tensors on the Lie group used as a configuration space. It is never the case in the above-quoted model of affinely-rigid body. This brings about the question as to the hypothetic affine counterpart of left- and right-invariant geodetic models on the orthogonal group and their potential perturbations. This interest is at least academically motivated. But at the same time, from the physical point of view, such models look rather esoteric. In any case, the previously mentioned applications of affine collective modes are dynamically well-established, because they are based on the d’Alembert principle.
in theory of constrained systems. There are, however, some indications that physical applicability is not a priori excluded.

This problem has to do with the very philosophy of the origin of collective and internal degrees of freedom. We say that a "large" system of material points (continuous, denumerable, or just finite admitted) has collective modes when there exists a "small" number of parameters $q^1, \ldots, q^n$ that are dynamically relevant, i.e., satisfy an approximately autonomous system of evolution equations, if for our purposes the kinematical information about the system, encoded in them, is sufficient, and (very important!) if they depend on individual particles in a non-local way. The latter means that positions and velocities of all particles enter the $q^i$-variables on essentially equal footing, with the same strength, order of magnitude, so to speak. This is, of course, a rough, qualitative introduction of the term, but there is no place here to develop a rigorous mathematical description. As a mathematical model we can realize some quotient manifolds of multiparticle state spaces or their submanifolds (e.g., representatives of cosets). On the contrary, internal degrees of freedom are described in terms of fibre bundles over the physical space, space-time, or the configuration space. They give an account of phenomena which are either essentially non-extended in space, or perhaps cannot be described in terms of composed systems because their spatial details are unapproachable to our experimental abilities. For example, from the point of view of contemporary science, spin systems seem to be based on essentially internal quantities [30, 71]. In any case, spin media do not look like the Cosserat continuous limit of discrete systems of molecular "gyroscopes". The latter model works successfully in the theory of Van der Waals crystals and granular media.

Apparently, the most natural and intuitive origin of collective modes, e.g., of some microstructure variables, is based on the mechanism of constraints and the d’Alembert principle. Collective motion is then "large", whereas non-collective one is "small" and merely reduced to some vibrations about the appropriate constraint submanifold. The collective kinetic energy, i.e., dynamical metric element, is obtained from the restriction of the total one to the constraints surface (the first fundamental quadratic form). This corresponds to the classical relationship between kinetic energy and inertia [2, 7, 8, 69]. In this case, as a rule, the collective kinetic energy is invariant under a proper subgroup of a group underlying geometry of the constraints submanifold. But one can also realize another mechanism, namely, such one that the hidden non-collective motion is just large, and that the emerging collective modes have to do with the averaged behaviour of hidden modes, i.e., with the time dependence of some relatively slowly-varying mean values. Then it is quite natural to expect that the collective Lagrangian will be based on a kinetic energy whose underlying dynamical metric tensor will be non-interpretable in terms of the restriction of the usual multiparticle metric tensor of the kinetic energy to the constraints manifold (i.e., to the first fundamental form of constraints). Similarly, equations of motion need not be derivable from the usual d’Alembert principle based on the original spatial metric. Therefore, the relationship between kinetic energy and inertia may become rather non-classical, to some extent exotic in comparison with the
usual requirements (cf., e.g., the discussion by Capriz and Trimarco [7,8,71]). In such situations the only reasonable procedure is to postulate the kinetic term of the Lagrangian on the basis of some natural and physically justified postulates. Let us mention two examples from the two completely opposite scales of the physical phenomena, namely, the atomic nuclei and vibrating-rotating stars (by the way, the neutron stars are in a sense exotic and gigantic nuclei with $Z = 0$ and enormous $A$). As objects more close to the Earth one can think, e.g., kinetic bodies as discussed by Capriz, and various non-standard microstructure elements like gas bubbles, voids, and defects in solids [7,8,28]. Though bubbles and voids can be hardly treated as constrained pieces of a substance or systems of material points.

Situation is even much more complicated, when one deals with essentially internal degrees of freedom, like, e.g., spin systems [71]. Then, although we have some guiding hints from the theory of extended systems, any choice of Lagrangian, Hamiltonian, or equations of motion is based on some rather hypothetic postulates, first of all, on certain invariance requirements.

There is also another point worth of mentioning. Namely, usually in variational theories of analytical mechanics, Lagrangian consists of the kinetic and potential parts. The first one has to do with inertia, constraints, metric structure, whereas the other one describes true interactions. But even in traditional problems of analytical mechanics there are approaches where the structure of interactions is encoded in an appropriate metric structure, i.e., in a kind of kinetic term. There is a well-known example, namely, the Jacobi-Maupertuis variational principle. If $ds$ is the usual metric (arc) element of the configuration space, and $V$ is the potential energy, then one uses a modified metric [2]

$$d\sigma_V = \sqrt{E - V} \, ds,$$

where $E$ denotes a fixed energy value. This is so-called isoenergetic dynamics, based on the homogeneous "Lagrangian"

$$\mathcal{L} = \sqrt{E - V} \sqrt{g_{ij} \frac{dq^i}{d\lambda} \frac{dq^j}{d\lambda}},$$

$\lambda$ denoting an arbitrary parameter (not time). This variational principle, based on the metric element $d\sigma_V$, gives trajectories with the energy value $E$, but without the time-dependence. There are also spatiotemporal forms of this principle, where the time variable occurs as one of coordinates $q^i$, and there is no restriction to the fixed energy value.

In a slightly different context, in certain problems we will follow the idea of encoding the interaction structure in an appropriately postulated kinetic energy form, i.e., metric tensor on the configuration space.

As mentioned, we concentrate below on models with kinematics (and dynamics) ruled by the linear group $GL(n, \mathbb{R})$, or, more rigorously, affine group $GAf(n, \mathbb{R})$ (physically $n = 2, 3$). Of course, the usual rigid body in $n$ dimensions is ruled by $SO(n, \mathbb{R})$, or, if translations are taken into account, by the isometry group $E(n, \mathbb{R}) = SO(n, \mathbb{R}) \times \mathbb{R}^n$. But there are also other possibilities of
finite-dimensional collective modes, e.g., \( \text{SL}(n, \mathbb{R}) \) or \( \text{SL}(n, \mathbb{R}) \times \mathbb{R}^n \), i.e., incompressible affinely-rigid body, or, just conversely, the Weyl group \( \mathbb{R}^+ \text{SO}(n, \mathbb{R}) \) generated by rotations and translations (the shape of the body is preserved, but not necessarily its size). In some future we are going to investigate systems ruled by the projective group in \( n \) dimensions, \( \text{Pr}(n, \mathbb{R}) \simeq \text{SL}(n+1, \mathbb{R}) \), cf., e.g., [65].

This is quite a natural extension of affinely-rigid body, when the system of material straight-lines is preserved but their parallelism may be violated. Another interesting model would be given by the Euclidean-conformal group \( \text{CO}(n, \mathbb{R}) \).

Let us mention that there was also some very interesting attempt by unjustly forgotten German physicist Westpfahl [72], who invented the idea of using the unitary group \( \text{U}(3) \) as a basis for collective modes in three dimensions, quite independently of later applications of unitary symmetry in elementary particle physics.

Finally, it is quite often so that the complexification idea leads to physically interesting results. It is not excluded that complexifying the physical space \( \mathbb{R}^n \) to \( \mathbb{C}^n \) and replacing the real groups \( \text{GL}(n, \mathbb{R}), \text{U}(n) \) by \( \text{GL}(n, \mathbb{C}) \) we could obtain some interesting models of collective or internal degrees of freedom [73, 75, 76]. The idea is particularly tempting, because \( \text{GL}(n, \mathbb{R}) \) and \( \text{U}(n) \) are two different (and qualitatively opposite) real forms of the same complex group \( \text{GL}(n, \mathbb{C}) \).

But, of course, such exotic ideas are rather far from realization and they are mentioned here only because of their obvious conceptual relationship with the usual and generalized models of affinely-rigid bodies.

The group space is a particular model of systems with kinematics and dynamics ruled by a Lie group. In general, the microstructure or collective configuration space (the manifold \( \mathcal{M} \) in the sense of Capriz book [7] and related papers) is a homogeneous quotient space \( G/H \). Here \( G \) is a fundamental group of the model, and \( H \) is an appropriate subgroup of \( G \), not necessarily normal one, thus \( G/H \) need not inherit the group structure from \( G \) [7, 8].

1 Dynamical systems based on Lie groups

Dynamical systems based on Lie groups and their homogeneous spaces are widely used as a model of internal and collective degrees of freedom [7, 8, 9, 10, 31, 32]. They present also interest by themselves from the purely mathematical point of view. They are realistic and quite often they possess rigorous analytical solutions in terms of special functions and power series; this is probably due to the analytical structure of Lie groups. The first step of analysis is the theory of left- and right-invariant geodetic systems, when the Lagrangian and total energy are identical with the kinetic energy expression based on an appropriate Riemannian structure of \( G \).

For simplicity let us use the language of linear groups; by the way, nonlinear groups are exceptional in applications, and the most known examples are the universal covering groups \( \text{GL}(n, \mathbb{R}), \text{SL}(n, \mathbb{R}) \) of the indicated linear groups. For any curve \( \mathbb{R} \ni t \mapsto g(t) \in G \) its tangent vectors \( \dot{g}(t) \in T_{g(t)}G \) may be transported to the Lie algebra \( G' = T_eG \) with the help of right or left \( g(t)^{-1} \).
translations, resulting in quantities \( \Omega(t) := \dot{g}(t) g(t)^{-1}, \hat{\Omega}(t) := g(t)^{-1} \dot{g}(t) \). In this way the tangent and cotangent bundles \( TG, T^*G \) may be, in two canonical ways, identified with the Cartesian products: \( TG \simeq G \times G', T^*G \simeq G \times G'' \).

It is clear that the left and right regular translations \( g \mapsto L_k(g) = kg, g \mapsto R_k(g) = gk \) transform quasi-velocities either according to the adjoint rule or trivially:

\[
L_k : \Omega \mapsto \text{Ad}_k \Omega = k \Omega k^{-1}, \quad \hat{\Omega} \mapsto \hat{\Omega},
\]

\[
R_k : \Omega \mapsto \Omega, \quad \hat{\Omega} \mapsto \text{Ad}_{k^{-1}} \hat{\Omega} = k^{-1} \hat{\Omega} k.
\]

Left-invariant geodetic systems on \( G \) are based on kinetic energies, which are quadratic forms of \( \Omega \) with constant coefficients. If \( G \) is non-Abelian, then \( \hat{\Omega} \) is a non-holonomic quasi-velocity and the corresponding Riemannian structure on \( G \) is curved. Similarly, right-invariant kinetic energies are quadratic forms of \( \Omega \) with constant coefficients. As a canonical example of left-invariant systems we can realize the free rigid body in \( n \) dimensions, \( G = \text{SO}(n, \mathbb{R}) \) (if we neglect translational motion). If the rigid body is spherical (its inertial tensor is completely degenerate), then \( T \) is also right-invariant, and the underlying metric tensor on \( G \) is proportional to the Killing tensor. Such a pattern may be followed in all semisimple Lie groups \( [2, 35, 36, 42] \). Quite a different example is provided by the theory of the ideal fluids \( [2] \). The configuration space is identified with \( \text{SDiff} \mathbb{R}^3 \) — the infinite-dimensional group of all volume-preserving diffeomorphisms of \( \mathbb{R}^3 \) (provided that we discuss the physical three-dimensional case). If we admitted the fluid to be compressible, we would have to use the full group \( \text{Diff} \mathbb{R}^3 \) of all diffeomorphisms. The functional of kinetic energy is invariant under right regular translations in \( \text{SDiff} \mathbb{R}^3 \). What concerns left regular translations, it is invariant only under the six-dimensional isometry group of \( \mathbb{R}^3 \). The reason for this relatively poor left-hand-side invariance is that the kinetic energy expression depends in an essential way on the spatial metric tensor. At the same time, from the point of view of the material space, the particles of fluid have a rather limited individuality, and that is why the kinetic energy form of incompressible fluid is invariant under the huge group of sufficiently smooth and volume-preserving "permutations" of particles, i.e., under \( \text{SDiff} \mathbb{R}^3 \).

In some of our earlier papers \([19, 20, 21, 22, 23, 37, 39, 44, 45, 48, 49, 52, 53, 54, 55, 60, 62, 63, 64, 66]\) we discussed the object called "affinely-rigid body", i.e., the system of material points "rigid" in the sense of affine geometry, i.e., all affine relationship between constituents being kept fixed during any admissible motion. Such a model is geometrically interesting in itself and has a wide range of applications in macroscopic elasticity, mechanics of micromorphic continua with internal degrees of freedom, molecular vibrations, nuclear dynamics, vibrations of astrophysical objects, and the theory of the shape of Earth \([4, 11]\). Analytically, the configuration space of \( n \)-dimensional affinely-rigid body may be identified with the semi-direct product \( \text{GL}(n, \mathbb{R}) \times_s \mathbb{R}^n \), or simply \( \text{GL}(n, \mathbb{R}) \) when we neglect translational degrees of freedom.

The kinetic energy of an extended affinely-rigid body in Euclidean space may be calculated in the usual way, by summation of kinetic energies of its
constituents. Velocity vectors are squared with the use of the fixed metric tensor of the physical space. The resulting metric tensor of the configuration space is flat, and it is not invariant either under left or right regular translations, except two subgroups isomorphic with the Euclidean group \( SO(n, \mathbb{R}) \times_s \mathbb{R}^n \). Because of this the resulting geodesic model, although kinematically based on the group manifold, dynamically is incompatible with it. Besides, it is physically non-realistic and useless, because geodetics are straight lines in \( L(n, \mathbb{R}) \times_s \mathbb{R}^n \), therefore, in certain directions the body would suffer a non-limited extension or squeezing. It is impossible to avoid such non-physical catastrophic phenomena without introducing some potential term.

The very taste and mathematical machinery of systems with group-manifold degrees of freedom consist in the invariance of geodetic models under the total group of regular translations. This motivates the search for left- or right-invariant kinetic energies, i.e., Riemannian structures on \( GL(n, \mathbb{R}) \times_s \mathbb{R}^n \) or \( GL(n, \mathbb{R}) \). The first step is purely mathematical: the very construction and some primary analysis. Later on some hypotheses are formulated concerning the physical applicability of such apparently exotic "non-d’Alembertian" models.

2 Kinematics and Poisson brackets

Let us remind briefly the basic ideas concerning the extended affinely-rigid body in a flat Euclidean space \([44, 45, 48, 49, 52, 53, 54, 55, 60, 62]\). It is convenient to use the standard terms of continuum mechanics, although the model applies also to discrete or finite systems of material points (provided there exist at least \( n + 1 \) material points in \( n \)-dimensional space). Two Euclidean spaces are used, namely, the material space \((N, U, \eta)\) and the physical space \((M, V, g)\); the symbols \( N, M \) denote the underlying sets, \( U \) and \( V \) are their linear spaces of translations, and \( \eta \in U^* \otimes U^* \), \( g \in V^* \otimes V^* \) are metric tensors. We put \( \dim N = \dim M = n \). The points of \( N \) are labels of material points. The configuration space \( Q \) of affinely-rigid body in \( M \) is given by \( \text{Aff}(N, M) \), i.e., the manifold of affine isomorphisms of \( N \) onto \( M \). Obviously, it is an open submanifold of \( \text{Aff}(N, M) \) — the affine space of all affine mappings of \( N \) into \( M \) (including non-invertible ones). In some configuration \( \Phi \in Q \) the material point \( a \in N \) occupies the spatial position \( x = \Phi(a) \in M \). The co-moving, i.e., Lagrangian, mass distribution within the body will be described by the constant (time-independent) positive measure \( \mu \) on \( N \); it may be \( \delta \)-like (concentrated at single points), continuous with respect to the Lebesgue measure, or mixed. Cartesian (Lagrange) coordinates \( a^K \) in \( N \) are chosen in such a way that their origin is placed at the centre of mass \( C \), i.e.,

\[
\int a^K d\mu(a) = 0.
\]

The manifold \( \text{Aff}(N, M) \) may be identified with the Cartesian product \( M \times \text{LI}(U, V) \), where \( \text{LI}(U, V) \) denotes the manifold of all linear isomorphism of \( U \).
onto \( V \); it is an open submanifold of the linear space \( L(U, V) \). The first factor refers to translational motion, i.e., to the centre of mass position \( x = \Phi(C) \). The linear part of \( \Phi, \varphi = L[\Phi] = D\Phi \in LI(U,V) \), describes the relative (internal) motion. Analytically, when some Cartesian coordinates in \( M \) are used, motion is described by the dependence of Euler (current) coordinates on Lagrangian (material) ones and on the time variable:

\[
\Phi(t,a)^i = \varphi^i_K(t)a^K + x^i(t).
\]

In practical calculations it is often technically convenient, although may be geometrically misleading, to identify both \( U \) and \( V \) with \( \mathbb{R}^n \) and \( \mathbb{Q} \) with semi-direct product \( G\mathbb{A}f(n,\mathbb{R}) \simeq GL(n,\mathbb{R}) \times_s \mathbb{R}^n \). Another natural model of \( \mathbb{Q} \) is \( M \times F(V) \), where \( F(V) \) denotes the manifold of all linear frames in \( V \). By the way, \( F(V) \) as a model of internal (relative-motion) degrees of freedom is essentially identical with \( LI(U, V) \) if we put \( U = \mathbb{R}^n \) and use the natural isomorphism between linear mappings \( \varphi \in LI(\mathbb{R}^n, V) \) and co-moving frames \( e \in F(V) \) frozen into the body and attached at the centre of mass. This must be done when the body is infinitesimal and the relative motion is replaced by the dynamics of essentially internal degrees of freedom. Then \( \mathbb{R}^n \) becomes the micromaterial space of internal motion.

Inertia of the body is described by two constant quantities, namely, the total mass and the second-order moment of internal inertia \( J \in U \otimes U \), i.e.,

\[
m := \int_N d\mu(a), \quad J^{KL} := \int_N a^K a^L d\mu(a)
\]

(cf., e.g., \([44, 45, 48, 49, 52, 53, 54, 55, 60, 62]\); \( J \) is symmetric and positively-definite.

Summing up the kinetic energies of constituents,

\[
T = \frac{1}{2} g_{ij} \int \frac{\partial \Phi^i}{\partial t}(t,a) \frac{\partial \Phi^j}{\partial t}(t,a) d\mu(a),
\]

one obtains:

\[
T = T_{tr} + T_{int} = \frac{m}{2} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} + \frac{1}{2} g_{ij} \frac{d\varphi^i_A}{dt} \frac{d\varphi^j_B}{dt} J^{AB},
\]

the symbols ”tr” and ”int” refer, obviously, to the translational and internal (relative) terms.

The phase space of our system may be identified with the manifold \( P := M \times LI(U,V) \times V^* \times L(V,U) \). The factor \( V^* \) refers to translational canonical momentum, whereas \( L(V,U) \) to the internal one, in the sense of the obvious pairing between \( \pi \in L(V,U) \) and generalized internal velocity \( \xi \in L(U,V) \):

\[
\langle \pi, \xi \rangle = \text{Tr}(\pi \cdot \xi) = \text{Tr}(\xi \cdot \pi).
\]

Cartesian coordinates in \( M \) generate parametrization \( p_i, p^A_i \) of canonical momenta. For Lagrangians of the form \( L = T - V(x, \varphi) \) Legendre transformation

\[
p_i = mg_{ij} \frac{dx^j}{dt}, \quad p^A_i = g_{ij} \frac{d\varphi^j_B}{dt} J^{BA}
\]

(2)
leads to the following kinetic term of the Hamiltonian:

$$T = \frac{1}{2m} g^{ij} p_i p_j + \frac{1}{2} g^{ij} p^A_i p^B_j \tilde{J}_{AB},$$

where, obviously, $g^{ij}$ are components of the reciprocal contravariant metric of $g$, and $\tilde{J} \in U^* \otimes U^*$ is reciprocal to $J$, $\tilde{J}_{AC} J^{CB} = \delta_A^B$, do not confuse it with $J$ with the $\eta$-lowered indices. This kinetic term (and its underlying flat metric on $Q$) is invariant under Abelian additive translations in $Q = M \times \text{LI}(U, V)$; those in the second term are meant in the sense

$$\text{LI}(U, V) \ni \varphi \mapsto \varphi + \alpha, \quad \alpha \in \text{L}(U, V).$$

Therefore, without the interaction term (for $L = T$), the Hamiltonian generators $p_i, p^A_i$ are constants of motion. However, as mentioned above, such geodesic models for deformable bodies are physically non-interesting, because they predict unlimited expansion, contraction, and passing through singular configurations with $\det \varphi = 0$. The latter, although non-acceptable in continuum mechanics, may be to some extent admissible in mechanics of discrete bodies. If we once decide that the internal configuration space is given by $\text{LI}(U, V)$, then the above transformation group is only local. At the same time, even for purely geodesic systems, as mentioned, there is no invariance under geometrically interesting affine groups of left or right affine regular translations in $Q$. Even if, at the present stage, models with affinely-invariant kinetic energy might seem rather academic, they present some interest at least from the purely mathematical point of view. Besides, some physical applications seem to be possible in hydrodynamics, astrophysics, nuclear dynamics, and in certain elastic problems. It is interesting that even without any genuine interactions, on the purely geodesic level such models may predict bounded and stable elastic vibrations of incompressible bodies. It is so as if the interaction was encoded in the very kinetic energy, i.e., configuration metric, so as it is, e.g., in Jacobi-Maupertuis variational principle. To formulate such models we must introduce and partially remind certain geometric objects.

Affine velocity in laboratory representation, i.e., expressed in terms of space-fixed frames, is defined as

$$\Omega := \frac{d\varphi}{dt} \varphi^{-1} \in \text{L}(V), \quad \Omega^i_j = \frac{d\varphi^i}{dt} K_j^i,$$

The corresponding co-moving object, related to the body-fixed frame, is given by

$$\hat{\Omega} := \varphi^{-1} \frac{d\varphi}{dt} \in \text{L}(U), \quad \hat{\Omega}^A_B = (\varphi^{-1})^A_i \frac{d\varphi^i}{dt} B_j.$$

Obviously, $\Omega = \varphi \hat{\Omega} \varphi^{-1}$, $\Omega^i_j = \varphi^i A \hat{\Omega}^A_B (\varphi^{-1})^B_j$. These are Lie-algebraic objects corresponding to the structure of $Q$ as the group space of a Lie group. They provide an affine counterpart of the rigid-body angular velocities, and in fact reduce to them when $\varphi$ is confined to the manifold of isometries of $(U, \eta)$
onto \((V, g)\); then they become skew-symmetric respectively with respect to \(\eta\) or \(g\).

The object \(\Omega\) may be represented in terms of continua as a gradient of the Euler velocity field, namely, the material point passing the fixed spatial point \(y\) has the translational velocity:

\[
E_v(y)^i = \frac{dx^i}{dt} + \Omega^i_j(y^j - x^j),
\]

e.g., simply \(\Omega^i_jy^j\) in the instantaneous rest frame of the centre of mass, placed also at the instantaneous position of this centre in \(M\). Similarly, \(d\varphi^A_B/dt\) has to do with the gradient of the Lagrange velocity field, because the instantaneous velocity of the \(a\)-th particle (\(a \in \mathbb{N}\)) is given by

\[
L_v(a)^i = \frac{dx^i}{dt} + \frac{d\varphi^i_K}{dt}a^K
\]

(concerning the standard concepts of continuum mechanics consult, e.g., [15, 16, 29]). In certain problems it is also convenient to express the centre of mass translational velocity \(v^i = dx^i/dt\) in co-moving terms, i.e., \(\hat{v}^A = (\varphi^{-1})^A_i v^i\).

It is very convenient to introduce the canonical affine spin, also in two representations, the spatial and co-moving ones \(\Sigma \in L(V)\), \(\hat{\Sigma} \in L(U)\). In terms of coordinates they are given by the following formulas: \(\Sigma^i_j = (\varphi^{-1})^i_A p^A_j\), \(\hat{\Sigma}^A_B = p^A_i \varphi^{-1}_i B\). As previously, \(\Sigma = \varphi \Sigma \varphi^{-1}, \hat{\Sigma} = \varphi^A_i \hat{\Sigma}^B_i (\varphi^{-1}) B_j\).

They are purely Hamiltonian quantities defined on the phase space; without any precisely defined Lagrangian or Hamiltonian we cannot relate them to generalized velocities. It is seen, however, that they are dual objects to affine velocities, i.e., they are non-holonomic canonical momenta conjugate to them in the sense of following pairing:

\[
\langle \Sigma, \Omega \rangle = \langle \hat{\Sigma}, \hat{\Omega} \rangle := \text{Tr}(\Sigma \Omega) = \text{Tr}(\hat{\Sigma} \hat{\Omega}) = p^A_i \hat{v}^i_A,
\]

where \(v^i_A\) are generalized velocities of internal (relative) motion. This canonical isomorphism between Lie algebras \(\text{GL}(V)'=L(V)\), \(\text{GL}(V)'=L(V)\) and their duals simplifies remarkably all formulas and considerations.

It is clear that quantities \(\Sigma^i_j\) are Hamiltonian generators of \(\text{GL}(V)\) acting on \(\text{LI}(U, V)\) through the left translations:

\[
\varphi \mapsto A \varphi, \quad \varphi \in \text{LI}(U, V), \quad A \in \text{GL}(V).
\]

Similarly, \(\hat{\Sigma}^A_B\) generate right regular translations in the internal configuration space:

\[
\varphi \mapsto \varphi B, \quad \varphi \in \text{LI}(U, V), \quad B \in \text{GL}(U).
\]

In continuum mechanics these mappings are referred to, respectively, as spatial and material transformations; in this case they include rotations and homogeneous deformations. Obviously, to use correctly such terms we must be given metric tensors in \(V\) and \(U\). Then the \(g\)-antisymmetric part of \(\Sigma\) and
the $\eta$-antisymmetric part of $\hat{\Sigma}$ generate, respectively, spatial and material rigid rotations; the symmetric parts generate deformations.

The doubled antisymmetric parts are referred to as spin $S$ and vorticity $V$[12],

\[ S^i_j = \Sigma^i_j - g^{ik} g_{jm} \Sigma^m_k, \quad V^A_B = \hat{\Sigma}^A_B - \eta^{AC} \eta_{BD} \hat{\Sigma}^D_C. \]  \hspace{1cm} (6)

**Attention!** There is an easy mistake possibility: if motion is not metrically-rigid, then $V$ is not a co-moving representation of $S$, i.e.,

\[ S^i_j \neq \varphi^i_A V^A_B (\varphi^{-1})^B_j. \]

Just as translational velocity, the canonical linear momentum may be expressed in co-moving terms according to the following rule: $\hat{p}_A = p_i \varphi^i_A$.

The objects $\Omega$ and $\Sigma$ are invariant under material transformations, but the spatial action of $A \in \text{GL}(V)$ transforms them according to the adjoint rule, i.e., $\Omega \mapsto A \Omega A^{-1}, \Sigma \mapsto A \Sigma A^{-1}$. On the contrary, $\hat{\Omega}$ and $\hat{\Sigma}$ are invariant under $\text{GL}(V)$ but experience the inverse adjoint rule under $B \in \text{GL}(U)$, i.e., $\hat{\Omega} \mapsto B^{-1} \hat{\Omega} B, \hat{\Sigma} \mapsto B^{-1} \hat{\Sigma} B$. This formally agrees with formulas for systems with configuration spaces identical with Lie groups, but one must stress that there are some subtle differences due to the fact that $\text{LI}(U, V)$ is not a Lie group (may be identified with it, but there is an infinity mutually equivalent identifications).

The translational or orbital affine momentum with respect to some point $O \in M$ is defined as follows:

\[ \Lambda(O)^i_j := x^i p_j, \]

where $x^i$ are Cartesian coordinates of the $O$-radius vector of the current position of the centre of mass in $M$. The total affine momentum with respect to $O$ is given by

\[ I(O)^i_j := \Lambda(O)^i_j + \Sigma^i_j. \]

$\Lambda(O)$ and $I(O)$ depend explicitly on the choice of $O$. Unlike this, $\Sigma$ is objective (in a fixed Galilean reference frame). There is a complete analogy with the properties of angular momentum, the doubled $g$-antisymmetric part of the above objects. The quantity $I(O)$ is a Hamiltonian generator of the group of affine transformations of $M$ preserving $O$ (O-centred affine subgroup).

Poisson brackets of $\Sigma$-quantities follow directly from the standard ones for $x^i$, $p_i$, $\varphi^i_A$, $p^A_i$. The non-vanishing ones are simply given by the structure constants of linear group,

\[ \{\Sigma^i_j, \Sigma^k_l\} = \delta^i_l \Sigma^k_j - \delta^k_l \Sigma^i_j, \quad \{\Sigma^i_j, \hat{\Sigma}^A_B\} = 0, \]

\[ \{\hat{\Sigma}^A_B, \hat{\Sigma}^C_D\} = \delta^C_B \hat{\Sigma}^A_D - \delta^A_B \hat{\Sigma}^C_D \]

(similarly for $\Lambda$, $I$). There are also non-vanishing Poisson brackets related to the left or right affine groups $\text{GAf}(M)$, $\text{GAf}(N)$. Here belong the above ones and besides, those involving linear momenta,

\[ \{\hat{\Sigma}^A_B, \hat{p}_C\} = \delta^A_C \hat{p}_B, \quad \{I^i_j, p_k\} = \{\Lambda^i_j, p_k\} = \delta^i_k p_j. \]
If $F$ is any function depending only on the configurations variables, then, obviously,
\[
\{ F, \Sigma_{ij} \} = \varphi^i_A \frac{\partial F}{\partial \varphi^j_A}, \quad \{ F, \Lambda_{ij} \} = x^i \frac{\partial F}{\partial x^j}, \quad \{ F, \hat{\Sigma}^{AB} \} = \varphi^i_B \frac{\partial F}{\partial \varphi^i_A}.
\]

Geometric meaning of the last formulas is clear, because the differential operators used on their right-hand sides are identical with vector fields on $Q$ generating the action of one-parameter subgroups of $\text{GAf}(M)$ and $\text{GAf}(N)$.

As mentioned, the above Poisson brackets follow directly from the standard definition \cite{2, 18, 24}
\[
\{ F, G \} := \frac{\partial F}{\partial q^\alpha} \frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha} \frac{\partial G}{\partial q^\alpha},
\]
where $q^\alpha$ are generalized coordinates and $p_\alpha$ are their conjugate canonical momenta. In our model $q^\alpha$ are given by $x^i$, $\varphi^i_A$, and $p_\alpha$ by $p_i$, $p^A_i$. In applications it is sufficient to remember that $\{ q^\alpha, q^\beta \} = 0$, $\{ p_\alpha, p_\beta \} = 0$, $\{ q^\alpha, p_\beta \} = \delta^\alpha_\beta$, that Poisson bracket is bilinear (over constant reals $\mathbb{R}$), skew-symmetric, i.e., $\{ F, G \} = -\{ G, F \}$, satisfies the Jacobi identity $\{ \{ F, G \}, H \} + \{ \{ G, H \}, F \} + \{ \{ H, F \}, G \} = 0$, and finally that
\[
\{ F, H(G_1, \ldots, G_k) \} = \sum_{p=1}^k H_p(G_1, \ldots, G_k) \{ F, G_p \},
\]
where commas before indices denote the partial derivatives. The formerly-quoted Poisson brackets together with the above rules are sufficient for all calculations concerning equations of motion and their analysis.

To define a non-dissipative (Hamiltonian) dynamical model, we must be given some Lagrangian $L(q, \dot{q})$, perform the Legendre transformation, $p_\alpha = \partial L/\partial \dot{q}^\alpha$, invert it, i.e., solve with respect to generalized velocities $\dot{q}^\alpha$, and substitute the result to the energy function $E = \dot{q}^\alpha \partial L/\partial \dot{q}^\alpha - L$. In this way one obtains the Hamilton function $H(q, p)$. Equations of motion may be then expressed in terms of Poisson brackets,
\[
\frac{dF}{dt} = \{ F, H \},
\]
where $F$ runs over some finite family of basic functions, e.g., $(p_i, \Sigma^i_j, x^i, \varphi^i_A)$, $(\dot{p}_A, \Sigma^{AB} B, x^i, \varphi^i_A)$, or something else. The basic dynamical laws are given by the balance equations for the linear momentum and affine spin either in laboratory or co-moving representation (one could use equivalently the linear momentum and the total affine momentum, however, the previous choice is more convenient). The procedure based on Poisson brackets and canonical formalism is very often more easy and computationally less embarrassing than the one directly using the Euler-Lagrange equations.

\textbf{Remark:} Legendre transformation may be also expressed in terms of non-holonomic objects, moreover, this is often more convenient and effective than
the use of generalized velocities. Expressing Lagrangian in terms of \((v^i, \Omega^i_j)\) or \((\dot{v}^A, \dot{\Omega}^A_B)\) instead of \((\dot{x}^i, \dot{\varphi}^i_A)\), we can describe the Legendre transformation as follows:

\[ p_i = \frac{\partial L}{\partial v^i}, \quad \Sigma^i_j = \frac{\partial L}{\partial v^i}, \quad \text{or} \quad \dot{p}_A = \frac{\partial L}{\partial \dot{v}^A}, \quad \Sigma^A_B = \frac{\partial L}{\partial \dot{\Omega}^B_A}. \]

When dealing with the Hamiltonian form of equations of motion, we need often Poisson brackets involving deformation tensors and certain by-products of the inertial tensor, like, e.g., the Eulerian quadrupole of the mass distribution.

Obviously, for systems with affine degrees of freedom the Green and Cauchy deformation tensors \(G \in U^* \otimes U^*, \ C \in V^* \otimes V^*\) are respectively given by the following expressions: \(G = \varphi*g, \ C = (\varphi^{-1})^*\eta,\) i.e., in analytical terms \(G_{AB} = g_{ij}\varphi^i_A\varphi^j_B, \ C_{ij} = \eta_{AB} (\varphi^{-1})^A_i (\varphi^{-1})^B_j\). Their inverses \(\tilde{G} \in U \otimes U, \ \tilde{C} \in V \otimes V\) are defined by \(\tilde{G}^{AC}G_{CB} = \delta^A_B, \ \tilde{C}^{ik}C_{kj} = \delta^{ij},\) and one must be careful to avoid mistaking \(\tilde{G}^{AB}, \ \tilde{C}^{ij}\) with \(\eta^{AC}\eta^{BD}G_{CD}, \ g^{ij}g^{kl}C_{kl}\). Therefore, the usual convention of the upper- and lower-case indices may be misleading. Analytically, \(\tilde{G}^{AB} = (\varphi^{-1})^A_i (\varphi^{-1})^B_j g^{ij}, \ \tilde{C}^{ij} = \varphi_A^i \varphi_B^j \eta^{AB}.\)

When there is no deformation, i.e., \(\varphi \in \text{Id}(U, \eta; V, g),\) then \(G = \eta, \ C = g\). The corresponding deformation measures vanishing in the non-deformed state, i.e., Lagrange and Cauchy deformation tensors \(E \in U^* \otimes U^*, \ e \in V^* \otimes V^*\) are given by (see, e.g., [15, 16]):

\( E := \frac{1}{2}(G - \eta), \quad e := \frac{1}{2}(g - C). \)

One uses also their contravariant versions \(E^{AB}, \ e^{ij};\) unlike \(\tilde{G}^{AB}, \ \tilde{C}^{ij}\) they are defined via the \(\eta-\) and \(g-\)raising of indices. 

**Remark:** \(G\) is independent of \(\eta\) and may be defined even if the material space is purely affine, amorphous. Similarly, \(C\) is independent of \(g\) and is well-defined even if the physical space is metric-free. Therefore, the literally meant term "deformation" is better expressed by \(E, e\) than \(G, C\). However, in many formulas \(G, C\) are more natural and convenient. Deformation tensors behave under the action of isometries in a very peculiar way, namely, for any \(A \in \text{O}(V, g), \ B \in \text{O}(U, \eta),\) we have:

\[ G[A\varphi]_{KL} = G[\varphi]_{KL}, \quad G[\varphi B]_{KL} = G[\varphi]_{CD}B^C_kB^D_L, \]

\[ C[A\varphi]_{ij} = C[\varphi]_{ab}(A^{-1})^a_i(A^{-1})^b_j, \quad C[\varphi B]_{ij} = C[\varphi]_{ij}. \]

By the way, the last two formulas are valid for any \(A \in \text{GL}(V), \ B \in \text{GL}(U).\) The first two equations (invariance rules) imply the Poisson-bracket rules

\[ \{G_{KL}, S^i_j\} = 0, \quad \{C_{ij}, V^A_B\} = 0, \]

and similarly for \(E_{KL}, e_{ij}\).

Deformation invariants are important mechanical quantities. They are scalar measures of deformation, basic stretchings, which do not contain any information concerning the orientation of deformation (its principal axes) in the physical
or material space. They may be chosen in various ways, but in an \( n \)-dimensional space exactly \( n \) of them may be functionally independent. The particular choice of \( n \) basic invariants depends on the considered problem and on the computational details. When non-specified, they will be denoted by \( K_a \), \( a = 1, \ldots, n \). Let us define mixed tensors \( \tilde{G} \in U \otimes U^*, \tilde{C} \in V \otimes V^*, \tilde{E} \in U \otimes U^* \), \( \tilde{\epsilon} \in V \otimes V^* \), namely,

\[
\hat{G}^{A \, B} := \eta^{AC} G_{CB}, \quad \hat{C}^{i \, j} := g^{ik} C_{kj}, \quad \hat{E}^{A \, B} := \eta^{AC} E_{CB}, \quad \hat{\epsilon}^{i \, j} := g^{ik} e_{kj}.
\]

A class of possible and geometrically natural choices of \( K_a \) is given by the following expressions: \( \text{Tr}(\hat{G}^k) \), \( \text{Tr}(\hat{C}^k) \), \( \text{Tr}(\hat{E}^k) \), \( \text{Tr}(\hat{\epsilon}^k) \), \( k = 1, \ldots, n \). In certain problems it is convenient to use the following eigenequations:

\[
\det \left[ \hat{G}^{A \, B} - \lambda \delta^{A \, B} \right] = 0, \quad \det \left[ \hat{C}^{i \, j} - \lambda \delta^{i \, j} \right] = 0, \quad \det \left[ \hat{E}^{A \, B} - \lambda \delta^{A \, B} \right] = 0, \quad \det \left[ \hat{\epsilon}^{i \, j} - \lambda \delta^{i \, j} \right] = 0.
\]

These are \( n \)-th order algebraic (polynomial) equations with respect to \( \lambda \). Their solutions provide one of possible choices of basic invariants. Another, very convenient one is given by coefficients at \( \lambda^p \), \( p = 0, (n - 1) \) \cite{15,16} (the coefficient at \( \lambda^n \) is standard and equals one). Deformation invariants are non-sensitive with respect to spatial and material isometries, i.e., for any \( A \in O(V, g) \), \( B \in O(U, \eta) \) we have \( K_a[A\varphi B] = K_a[\varphi] \). This implies the obvious Poisson brackets: \( \{K_a, S^i_j\} = \{K_a, V^{A \, B}\} = 0 \).

In certain computational problems, but also in theoretical analysis, it is very convenient to use quantities \( Q^a = \sqrt{\lambda_a} \), where \( \lambda_a \) are solutions of the above eigenequations, or \( q^a = \ln Q^a \) (i.e., \( Q^a = \exp(q^a) \)). The eigenvalues of \( \tilde{C} \) equal \( (\lambda_a)^{-1} = (Q^a)^{-2} = \exp(-2q^a) \).

Any function \( F \) on the configuration space which depends on \( \varphi \) only through the deformation invariants is doubly isotropic, i.e., satisfies \( F(A\varphi B) = F(\varphi) \) for any \( A \in O(V, g) \), \( B \in O(U, \eta) \), \( \varphi \in \mathbb{R}(U, \eta; V, g) \). All such functions have vanishing Poisson brackets with spin and vorticity, i.e., \( \{F, S^i\}_J = \{F, V^{A \, B}\} = 0 \). In certain formulas we need the spatial inertial quadrupole, \( J[\varphi]^{ab} = \varphi^a K^{ab}_L J^{KL} \). It is related to \( J^{KL} \) just as \( C \) is to \( \eta \). When the body is inertially isotropic, \( J[\varphi] \) becomes proportional to the inverse Cauchy deformation tensor. Unlike the co-moving internal tensor \( J \in U \otimes U \), \( J[\varphi] \in V \otimes V \) is configuration-dependent, thus variable in time.

3 Traditional d’Alembert model

At least for the comparison with more exotic (although geometrically and perhaps physically interesting) suggestions we must start with a brief reporting and extension of the traditional model based on the d’Alembert principle. As shown in \cite{17,18,19,20,51,52}, Lagrangians of the form \( L = T - V(x, \varphi) \) with \( T \) given
by (1) lead to the following dynamical laws:

\[
\frac{dp_i}{dt} = -\frac{\partial V}{\partial x^i} = Q_i, \quad \frac{d\Sigma^i_j}{dt} = \Omega^i_m\Sigma^m_j - \varphi^i_A \frac{\partial V}{\partial \varphi^j_A} = \Omega^i_m\Sigma^m_j + Q^i_j, \quad (7)
\]

expressed in terms of Cartesian coordinate systems. This is the balance for fundamental Hamiltonian generators. It becomes a closed dynamical system when considered together with the Legendre transformation (2) or its equivalent description

\[
p_i = mg_{ij} \frac{dx^j}{dt}, \quad \Sigma^i_j = g_{jk}\Omega^k_mJ[\varphi]^m_i. \quad (8)
\]

Substituting these expressions to the dynamical balance for \(p_i, \Sigma^i_j\) one obtains some reformulation of the Euler-Lagrange equations. Similarly, some form of canonical Hamilton equations is obtained when the balance (7) is unified with the inverse Legendre transformation, i.e.,

\[
\frac{dx^i}{dt} = \frac{1}{m}g^{ij}p_j, \quad \Omega^i_j = \tilde{J}[\varphi]_{jk}\Sigma^k_mg^{mi},
\]

where, obviously, \(J[\varphi]_{ik}\tilde{J}[\varphi]_{kj} = \delta^i_j\) (do not confuse \(\tilde{J}[\varphi]\) with \(g\)-shift of indices of \(J[\varphi]\)).

Obviously, the general balance form may be used for dissipative non-Lagrangean models. Simply the covariant force \(Q_i\) and the generalized internal force \(Q^i_j\) (affine moment of forces, hyperforce) must involve appropriately defined dissipative forces (in the case of affinely-constrained continuum one can also consider the mutual coupling of mechanical phenomena with discretized thermal effects).

As shown in the mentioned papers, the above equations of motion may be formulated in various equivalent forms adapted to the kind of considered problems. For example, instead of the canonical (Hamilton) form, one can write them down in purely kinematical velocity-based terms, i.e.,

\[
m\frac{d^2x^i}{dt^2} = F^i, \quad \varphi^i_A \frac{d^2\varphi_A^j}{dt^2} J^{AB} = N^{ij}, \quad (9)
\]

where contravariant forces \(F^i\) and hyperforce \(N^{ij}\) (affine dynamical moment) may depend on all possible arguments, i.e., \(t, x^i, \varphi^i_A, dx^i/dt, d\varphi^i_A/dt\). Obviously, for potential models they depend only on generalized coordinates and possibly on the time variable \(t\) itself, and then

\[
F^i = g^{ij}Q_j = -g^{ij}\frac{dV}{dx^j}, \quad N^{ij} = Q^i_kg^{kj} = -\varphi^i_A \frac{\partial V}{\partial \varphi^k_A}g^{kj}. \quad (10)
\]

**Remark:** In spite of the tradition based on Riemannian geometry and relativity theory we shall refrain from the graphical identification of symbols \(F^i, N^{ij}\) respectively with \(Q^i, Q^{ij}\). In our treatment this would be just confusing, because we shall use various prescriptions for shifting the tensorial indices, i.e., various isomorphisms between contravariant and covariant objects.
As mentioned, the above equations of the motion may be derived directly in Newtonian terms, basing merely on the d’Alembert principle and its underlying spatial metric $g$ in $M$. The primary quantities of this approach are the monopole and dipole moments of the distributions of linear kinematical momentum and forces within the body. These quantities, just as all high-order multipoles may be defined for any unconstrained system of material points, does not matter finite, discrete, or continuous. Let $\Phi(t, a)$ denote as previously Cartesian coordinates of the current position of the $a$-th material point at the time instant $t$, $x^i(t)$ be the current position of the centre of mass, and $F^i(t, x, \Phi(t, a), dx/dt, (\partial \Phi/\partial t)(t, a); a)$ be the density of forces per unit mass. As mentioned, the affine constraints are not yet assumed.

The monopoles are simply the total quantities: the total kinematical momentum $k^i$ (do not confuse it at this stage with the canonical one $p_i$) and the total force $F^i$ affecting the centre of mass motion, i.e.,

$$k^i = \int (\Phi^i(t, a) - x^i(t, a)) d\mu(a), \quad F^i = \int F^i(t, x, \Phi(t, a), \frac{dx}{dt}, (\partial \Phi/\partial t)(t, a); a) d\mu(a).$$

The dipole moments with respect to the centre of mass current position are referred to as kinematical affine spin $K^{ij}$ (do not confuse it at this stage with the canonical one $\Sigma^i_j$) and the affine moment of forces $N^{ij}$ (not to be confused with its potential version $Q^{ij}$). They are given respectively by the following expressions:

$$K^{ij} = \int (\Phi^i(t, a) - x^i(t, a)) \left(\frac{\partial \Phi^j}{\partial t}(t, a) - \frac{dx^j}{dt}\right) d\mu(a),$$

$$N^{ij} = \int (\Phi^i(t, a) - x^i(t, a)) F^j(t, x, \Phi(t, a), \frac{dx}{dt}, (\partial \Phi/\partial t)(t, a); a) d\mu(a).$$

The dipoles may be also referred to some space-fixed centre $O \in M$, e.g., the origin of Cartesian coordinates in $M$. The difference is that “the lever arm” $(\Phi^i - x^i)$ is then replaced by $\Phi^i$ itself, and its velocity $(\partial \Phi^i/\partial t - dx^i/\partial t)$ by $(\partial \Phi^i/\partial t)(t, a)$. The resulting dipoles will be denoted respectively by $K(O)^{ij}$, $N(O)^{ij}$. For the sake of uniformity, it may be also convenient to denote the previous dipoles by $K(cm)^{ij}$ and $N(cm)^{ij}$ instead of $K^{ij}$, $N^{ij}$. We shall also use affine moments of the centre of mass characteristics with respect to the origin $O$. Thus, the translational (orbital) affine moment of kinematical linear momentum and translational affine moment of forces are as follows:

$$K_{tr}(O)^{ij} = x^i k^j = mx^i \frac{dx^j}{dt}, \quad N_{tr}(O)^{ij} = x^i F^j.$$ 

The doubled skew-symmetric parts of the above quantities, i.e.,

$$S^{ij} = K^{ij} - K^{ji}, \quad L_{tr}(O)^{ij} = K_{tr}(O)^{ij} - K_{tr}(O)^{ji}, \quad J(O)^{ij} = K(O)^{ij} - K(O)^{ji},$$

$$N^{ij} = N^{ij} - N^{ji}, \quad N_{tr}(O)^{ij} = N_{tr}(O)^{ij} - N_{tr}(O)^{ji}, \quad N(O)^{ij} = N(O)^{ij} - N(O)^{ji}.$$
represent the kinematical angular momentum and the moment of forces (torque). They also occur in three versions concerning, respectively, the internal motion (thus $S$ is spin), motion of the centre of mass with respect to $O$, and the total motion with respect to $O$.

If now we assume that the motion is affine, then the above expressions simplify to

$$K^{ij} = \varphi^i_A \frac{d\varphi^j_B}{dt} J^{AB}, \quad K(O)^{ij} = K_{tr}(O)^{ij} + K^{ij} = m x^i \frac{dx^j}{dt} + \varphi^i_A \frac{d\varphi^j_B}{dt} J^{AB},$$

$$N(O)^{ij} = N_{tr}(O)^{ij} + N^{ij} = x^i F^j + N^{ij}.$$  

Obviously, $F$ and $N$ become now functions of $t, x^i, dx^i/dt, \varphi^i_A, d\varphi^i_A/dt$. Let us stress that, just as it was the case with the kinetic energy, the above additive splitting into translational and internal parts is based on the assumption that the current centre of mass has permanently Lagrangian coordinates $a^K = 0$. This is consistent because barycenters are invariants of affine transformations.

By summation of elementary time rates of work over the body constituents, one can show that in the affine motion the total rate is given by

$$\mathcal{P} = g_{ij} \frac{dx^i}{dt} F^j + g_{ij} \Omega^k \kappa^k N^{kj}.$$ 

Let us remind however that, besides of active generalized forces $F, N$ controlling affine modes of motion, there are also hidden structural forces keeping affine constraints, i.e., reactions. Their density $F_R$ does not vanish, however, their monopole and dipole moments $F_R, N_R$ do because, according to the d’Alembert principle, the reaction time rate of work vanishes for any constraints-compatible virtual velocities, i.e., for any possible $dx^i/dt, \Omega$:

$$\mathcal{P}_R = g_{ij} \frac{dx^i}{dt} F_R^j + g_{ij} \Omega^i \kappa^k N_R^{kj} = 0.$$

Therefore, the effective reaction-free equations of motion are obtained from the primary non-constrained system by calculating the monopole and dipole moments.

The above derivation is quite general and valid for all kinds of forces, including non-potential and dissipative ones. It relies only on the metric structure $g$ in $M$ and on the d’Alembert principle. Obviously, if equations of motion follow from the Lagrangian $L = T - V(x, e)$, $T$ given by (1), then the above analysis implies equations (10).

Similarly, one can easily show that

$$K^{ij} = \Sigma^i m g^{m, j}, \quad k^i = g^{ij} p_j.$$  

But these relationships become false when Lagrangian depends on velocities not only through the kinetic energy $T$ but also through some generalized potential $V$, e.g., when magnetic or gyroscopic external forces are present. This is one of reasons we avoid denoting $K^{ij}$ by $\Sigma^{ij}$ or $K^i_j$ by $\Sigma^i_j$.  

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Kinematical quantities $k^i$, $K^{ij}$ are intuitive because of their direct operational interpretation in terms of positions and velocities. At the same time they are lowest-order multipoles (monopoles and dipoles) of the distribution of kinematical linear momentum within the body, and it is difficult to over-estimate the application of multipoles and all moment quantities in practical problems of mechanics and field theory (cf. e.g., all Galerkin-type procedures [67]). On the other hand, their canonical counterparts $p_i$, $\Sigma^{ij}$ have a very deep geometrical interpretation as Hamiltonian generators of fundamental transformation groups. Because of this, they are very often important constants of motions. In mechanics of affinely-rigid body, equations of motion are equivalent to the balance laws for $p_i$, $\Sigma^{ij}$ or, in a sense equivalently, to the ones for $k^i$, $K^{ij}$, because Lagrangians of non-dissipative models, or at least Lagrangians of non-dissipative background dynamics, establish some link between these concepts. Similarly, in rigid-body mechanics equations of motion are equivalent to the balance for $p_i$, $(\Sigma^{ij} - g^{ik}g_{jm}\Sigma^{mk})$ or for $k^i$, $S^{ij}$.

Equations of motion (9) may be written in several mutually equivalent balance forms. Let us quote some of them based on kinematical quantities like $k^i$, $K^{ij}$ or their co-moving representation $\hat{k}^A$, $\hat{K}^{AB}$, where, obviously, $k^i = \varphi_A^i \hat{k}^A$, $K^{ij} = \varphi_A^i \varphi_B^j \hat{K}^{AB}$. The co-moving components $\hat{F}^A$, $\hat{N}^{AB}$ of generalized forces are given by analogous expressions, thus, $F^i = \varphi_A^i \hat{F}^A$, $N^{ij} = \varphi_A^i \varphi_B^j \hat{N}^{AB}$.

The dynamical balance expressed in terms of kinematical (non-canonical) quantities in spatial (Eulerian) representation reads:

$$
\frac{dk^i}{dt} = F^i, \quad \frac{dK^{ij}}{dt} = \frac{d\varphi_A^i}{dt} \frac{d\varphi_B^j}{dt} J^{AB} + N^{ij}.
$$

For non-dissipative potential systems with Lagrangians $L = T - V(x, \varphi)$, it reduces to (7), because then (10) holds. Let us observe that even in the interaction-free case, when $N = 0$, the balance for $K$ is not a conservation law due to the first non-dynamical term on its right-hand side. One can write that

$$
\frac{dK^{ij}}{dt} = N^{ij} + 2 \frac{\partial T_{\text{int}}}{\partial g_{ij}}.
$$

On the Hamiltonian level, this means that the non-conservation of $K$ even in geodetic motion is due to the fact that the kinetic energy depends explicitly on the spatial metric tensor. Affine symmetry of degrees of freedom is broken and reduced to the Euclidean one.

The system (12) may be written in the following form:

$$
\frac{dk^i}{dt} = F^i, \quad \frac{dK(O)^{ij}}{dt} = m \frac{dx^i}{dt} \frac{dx^j}{dt} + \frac{d\varphi_A^i}{dt} \frac{d\varphi_B^j}{dt} J^{AB} + N(O)^{ij},
$$

as a balance for the kinematical linear momentum and the total affine momentum with respect to some space-fixed origin $O \in M$.

If the body is rigid in the usual metrical sense, i.e., all distances between its constituents are constant, then the d’Alembert principle implies that the second
subsystems in (12), (13) are to be replaced by their skew-symmetric parts, thus,

\[ \frac{dS^{ij}}{dt} = N^{ij}, \quad \frac{dJ(\mathcal{O})^{ij}}{dt} = N(\mathcal{O})^{ij}. \]

To these equations the rigidity condition, i.e., \( \eta_{AB} = g_{ij} \varphi^i_A \varphi^j_B \), may be automatically substituted without paying any attention to reaction forces responsible for the metrical rigidity.

The above balance laws for kinematical angular momenta become conservation laws in the interaction-free case, and even under weaker, realistic conditions that \( N \) or \( N(\mathcal{O}) \) is symmetric. This is very natural on the Hamiltonian level, because \( S \), \( J(\mathcal{O}) \) are then directly related to their canonical counterparts. The latter are Hamiltonian generators of the isometry group of \( (M, g) \), thus, according to the Noether theorem, they are constants of motion in geodetic problems, because gyroscopic kinetic energy is isometry-invariant.

Another very convenient balance form of equations of motion is obtained when generalized velocities \( d\varphi^i_A/dt \) are expressed through the non-holonomic quantities \( \Omega_{ij} \), then

\[ \frac{d\hat{k}^i}{dt} = F^i, \quad \frac{dK^{ij}}{dt} = \Omega^i_m K^{mj} + N^{ij}. \]

(14)

Expressing our balance in co-moving (material) terms we obtain

\[ \frac{d\hat{k}^A}{dt} = -\hat{k}^B J_{BC} K^{CA} + \hat{F}^A, \quad \frac{d\hat{K}^{AB}}{dt} = -\hat{K}^{AC} J_{CD} K^{DB} + \hat{N}^{AB}, \]

(15)

or, using non-holonomic velocities,

\[ m \frac{d\hat{v}^A}{dt} = -m \hat{\Omega}^A_B \hat{v}^B + \hat{F}^A, \quad J^{AC} \frac{d\hat{\Omega}^B_C}{dt} = -\hat{\Omega}^B_D \hat{\Omega}^D_C J^{CA} + \hat{N}^{AB}. \]

It is a nice feature of the co-moving representation that all non-dynamical terms are built only of expressions \( \hat{k}^A, K^{AB} \) or \( \hat{v}^A, \hat{\Omega}^A_B \) without any direct using of mixed quantities like \( \varphi^i_A, d\varphi^i_A/dt \). The second (internal) subsystems are exactly affine counterparts of gyroscopic Euler equations and exactly reduce to them when the rigid-body constraints are imposed. The relationship between two co-moving forms is based on the equation \( \hat{K}^{AB} = \hat{\Omega}^B_C J^{CA} \) following directly from the definition of \( K \). There is some relationship between this formula and Legendre transformation for Lagrangians \( L = T - V(x, \varphi) \). Namely, one can show that the internal part of (8) may be equivalently written in the following form:

\[ \hat{\Sigma}^{AB} = \hat{K}^{AC} G_{CB} = G_{BC} \hat{\Omega}^C_D J^{DA}, \]

(16)

where \( G \in U^* \otimes U^* \) denotes as previously the Green deformation tensor. Therefore, the canonical affine spin is obtained from the kinematical one by the \( G \)-lowering of the second index. As we saw, there was a similar formula (11) in the spatial representation, i.e.,

\[ \Sigma^{ij} = K^{im} g_{mj}. \]

(17)
It is seen that there is an easy possibility of confusion. Namely a superficial analogy with the last formula might suggest us to use the $\eta$-shifting of indices for establishing the Legendre link between $K$ and $\Sigma$. However, for any reasonable Lagrangian $\hat{\Sigma}_{AB} \neq \hat{K}^{AC}\eta_{CB}$ except the special case of metrically-rigid motion. This is an additional reason for avoiding ambiguous symbols like $\hat{\Sigma}_{AB}$ or $\hat{K}^{AB}$. More generally, if some tensor objects in $V$ are related to each other by the $g$-shifting of indices, then the corresponding co-moving objects in $U$ are interrelated by the $G$-shifting. And conversely, if two tensors in $U$ are interrelated by the $\eta$-shift of indices, then their spatial counterparts in $V$ are obtained from each other by the $C$-shifting, where $C \in V^* \otimes V^*$ is the Cauchy deformation tensor. The contravariant inverses of $G$ and $C$ are carefully denoted by $\tilde{G} \in U \otimes U$ and $\tilde{C} \in V \otimes V$, i.e., $\tilde{G}^{AC}G_{CB} = \delta^A_B$, $\tilde{G}^{ik}C_{kj} = \delta^i_j$. The notation $G^{AB}$, $C^{ij}$ would be misleading because of the possible confusion with the objects $\eta^{AC}\eta^{BD}G_{CD}$, $g^{ik}g^{jm}C_{km}$ obtained from $G$, $C$ by the usual $\eta$- or $g$-metrical operations on indices.

For Lagrangian systems $L = T - V(x, \varphi)$ generalized forces $Q$, $N$ are interrelated by (10), thus, just as in (17), $Q^i_j = N^{lm}g_{mj}$. But in the co-moving representation, in analogy to (18), we have $\tilde{Q}^{AB} = \tilde{N}^{AC}G_{CB}, \tilde{Q}^{AB} \neq \tilde{N}^{AC}\eta_{CB}$. This has to do with different $\varphi$-transformation properties of $Q$, $N$, i.e., $Q^j_i = \varphi^A_i\tilde{Q}^{AB}(\varphi^{-1})^B_j$, $N^{ij} = \varphi^A_i\varphi^B_j\tilde{N}^{AB}$; similarly for $\Sigma^{ij}$, $K^{ij}$, $\tilde{\Sigma}^{AB}$, $\tilde{K}^{AB}$.

As seen from equations (14), (15) even in the interaction-free case neither $K^{ij}$ nor $\tilde{K}^{AB}$ are constants of motion. The same concerns their canonical counterparts $\Sigma^{ij}$, $\tilde{\Sigma}^{AB}$. The reason is that the kinetic energy is not invariant under spatial and material affine transformations (except translations, of course). At the same time, purely geodetic Hamiltonian models with $L = T$ are physically useless because, except of rest-states, all their trajectories (straight lines in $M \times \text{LI}(U, V)$) escape to infinity. In particular, the body may expand to infinity and contract in finite time to a point. The metric on $Q = M \times \text{LI}(U, V)$ underlying the kinetic energy (1) is unable to encode realistic interactions and predict elastic vibrations in purely geodetic terms.

## 4 Dynamical affine invariance

Basing on the motivation presented in previous sections, we shall now consider some models which are ruled by affine groups not only on the kinematical but also on the dynamical level. In particular, we shall discuss left- and right-invariant Riemann metrics on linear and affine groups or rather, more precisely, on their free-action homogeneous spaces. We concentrate on geodetic models, when there is no potential term and the structure of interactions is encoded in an appropriately chosen metric tensor on the configuration space.

Let us begin with the internal sector, when translational degrees of freedom are frozen and the configuration space reduces to $Q_{\text{int}} = M \times \text{LI}(U, V)$, or equivalently to $F(V)$ (when for simplicity we put $U = \mathbb{R}^n$). According to the transformation rules for $\Omega$, $\hat{\Omega}$, the most general metric tensor on $Q_{\text{int}}$ invariant under the action of $\text{GL}(V)$ through (14) is that underlying the kinetic energy (1) is that underlying the kinetic energy
form given by
\[ T_{\text{int}} = \frac{1}{2} L^B A^D \hat{\Omega}^A_B \hat{\Omega}^C_D, \]  
(18)
where coefficients \( L \) are constant and symmetric in bi-indices \((B, A),(D, C)\). This quadratic form is also assumed to be non-degenerate, although not necessarily positively definite. As \( \Omega \) is a non-holonomic velocity, i.e., it is not a time derivative of any system of generalized coordinates, the underlying metric on \( Q_{\text{int}} \) is curved.

Quite similarly, the most general kinetic energy invariant under material affine transformations \((5)\) has the following form:
\[ T_{\text{int}} = \frac{1}{2} R^j i l j k \Omega^i_j \Omega^l_k, \]  
(19)
where \( R \) is also constant and symmetric in bi-indices \((j, i),(l, k)\). The underlying metric tensor on \( Q_{\text{int}} \) is also curved, i.e., essentially Riemannian.

In general, \((18)\) is not right, i.e., materially, invariant under \( \text{GL}(U) \) acting through \((5)\), and \((19)\) is not invariant under \( \text{GL}(V) \) acting through \((4)\) in the physical space, i.e., on the left. The exceptional situation of simultaneous spatial and material invariance leads us to
\[ T_{\text{int}} = A^2 \hat{\Omega}^i_j \hat{\Omega}^j_i + B^2 (\Omega^i_i \Omega^j_j) = A^2 \hat{\Omega}^K L \hat{\Omega}^L_K + B^2 \hat{\Omega}^K K \hat{\Omega}^L_L, \]
where \( A, B \) are some constants. Using invariant terms, we can say that such \( T_{\text{int}} \) is a linear combination of two basic second-order Casimir invariants, i.e.,
\[ T_{\text{int}} = \frac{A}{2} \text{Tr}(\Omega^2) + \frac{B}{2} (\text{Tr}\Omega)^2 = \frac{A}{2} \text{Tr}(\hat{\Omega}^2) + \frac{B}{2} (\text{Tr}\hat{\Omega})^2. \]  
(20)
Such \( T_{\text{int}} \) is never positively-definite. The reason is that the maximal semisimple subgroups \( \text{SL}(V), \text{SL}(U) \) (their determinants equal to unity) are non-compact, thus, the quadratic form \( \text{Tr}(\Omega^2) = \text{Tr}(\hat{\Omega}^2) \) has the hyperbolic signature \((n(n+1)/2 +, n(n-1)/2 -)\), where the positive contribution corresponds to the "non-compact" dimensions and the negative one to the "compact" dimensions in \( \text{GL}(V), \text{GL}(U) \).

By the way, the above quadratic forms reduce to the Killing form (Killing scalar products) on \( L(V), L(U) \) when \( A = 2n \), \( B = -2 \). As \( L(V), L(U) \) are non-semisimple, in this special unhappy case the scalar product (kinetic energy) is degenerate, thus, non-applicable in usual mechanical problems. The singularity consists of dilatational Lie algebras \( \mathbb{R} \text{Id}_V, \mathbb{R} \text{Id}_U \). More generally, the same holds when \( A = -Bn \). Paradoxically enough, non-degenerate forms \((20)\) \((A \neq -Bn)\) may be mechanically useful in spite of their non-definiteness.

The usual d’Alembert model \((1)\) invariant under additive translations \((3)\) is the special case of general models of the following form:
\[ T_{\text{int}} = \frac{1}{2} A^K i j \frac{d\varphi^i_j}{dt} \frac{d\varphi^j_i}{dt}, \]  
(21)
where $\mathcal{A}$ is constant and symmetric in bi-indices $(K_i, L_j)$. The peculiarity of (1) within this class is that $\mathcal{A}$ factorizes, i.e., $\mathcal{A}^{K_i L_j} = g_{ij} J^{KL}$, and is invariant under the left action of $\text{SO}(V, g)$ and the right action of $\text{SO}(U, J)$ (in particular, $\text{SO}(U, \eta)$, when the inertia is isotropic, i.e., $J = \mu \eta$). It is clear that the $\mathcal{A}$-based models of $T_{\text{int}}$ are never invariant under $\text{GL}(V)$, $\text{GL}(U)$. The underlying metric on $\text{LI}(U, V)$ is flat.

Let us now consider the translational sector of motion. The only model of translational kinetic energy invariant under $\text{GAf}(M)$ (affine group of $M$) is as follows:

$$T_{\text{tr}} = \frac{m}{2} C_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = \frac{m}{2} \eta_{AB} \hat{v}^A \hat{v}^B,$$  \hspace{1cm} (22)

It looks like the usual kinetic energy, however, there is a very essential difference. Namely, in the above expression the velocity vector is not squared with the help of the constant and absolutely fixed spacial metric $g \in V^* \otimes V^*$. Instead of it, the Cauchy deformation tensor $C$ is used as an instantaneous metric tensor of $M$. Being a function of the internal configuration $\varphi \in \text{LI}(U, V)$, it depends on time. It is so as if the instantaneous internal configuration created a dynamical metric in an essentially amorphous affine space $M$. In this sense the model is an oversimplified toy simulation of general relativity. At the same time it is clear that $T_{\text{tr}}$ is not invariant under $\text{GL}(U)$ because the fixed material metric $\eta$ restricts the symmetry to $\text{O}(U, \eta)$ ($\eta$-rotations of $U$).

If we wish the translational kinetic energy to be $\text{GL}(U)$-invariant, then the only reasonable model is the usual one, based on the fixed $M$-metric $g$, i.e.,

$$T_{\text{tr}} = \frac{m}{2} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = \frac{m}{2} G_{AB} \hat{v}^A \hat{v}^B.$$  \hspace{1cm} (23)

It is impossible to construct any model of $T_{\text{tr}}$ and $T = T_{\text{tr}} + T_{\text{int}}$, which would be purely affine both in $M$ and $N$; in one of these spaces some metric structure must be assumed. Therefore, although $T_{\text{int}}$ alone may be affine simultaneously in $M$ and $N$, there are no reasons to stick to such models, the more so they are never positively-definite. These problems have to do with the non-existence of a doubly (left- and right-) invariant pseudo-Riemannian structure on the affine group $\text{GAf}(n, \mathbb{R}) \simeq \text{GL}(n, \mathbb{R}) \times \mathbb{R}^n$. Any doubly-invariant twice covariant tensor field on this group is degenerate. Therefore, it is reasonable to concentrate on kinetic energies which are affine in $M$ and $g$-metrical in $N$ or, conversely, $g$-metrical in $M$ and affine in $N$. The corresponding geodetic models in $Q$ have a maximal possible symmetry, being at the same time really true geodetic problems (non-singular underlying metric). Such models are special cases of (18), (19), (22), (23), thus, we start with some statements concerning the general case.

For the model (18), (22) affine in space and matrical in the material, Legendre transformation has the form:

$$\hat{\Sigma}^A_B = \mathcal{L}^A_B C^D \hat{\Omega}^D_C, \; \; \; \hat{p}_A = m \eta_{AB} \hat{v}^B,$$  \hspace{1cm} (24)

where, obviously, the second equation may be rewritten as

$$p_i = m C_{ij} v^j.$$  \hspace{1cm} (25)
The corresponding geodetic Hamiltonian is given by \( T = T_{tr} + T_{int} \), where
\[
T_{tr} = \frac{1}{2m} \gamma^{AB} \dot{p}_A \dot{p}_B = \frac{1}{2m} \tilde{C}^{ij} p_i p_j, \quad T_{int} = \frac{1}{2} \tilde{L}^{A}_{\ B} C_D \dot{\Sigma}^B_A \dot{\Sigma}^D_C,
\]
and the symmetric bimatrix \( \tilde{L} \) is reciprocal to \( L \), i.e.,
\[
\tilde{L}^{A}_{\ B} K L^{L}_K C_D = \delta^A_D \delta^C_B.
\]
For the model (19), (23), metrical in space and affine in the material, Legendre transformation may be represented as follows:
\[
\Sigma^i_j = R^i_j k l \Omega^l_k, \quad p_i = mg_{ij} v^j, \tag{26}
\]
where, in analogy to the \( L \)-case, the second subsystem may be rewritten as
\[
\dot{p}_A = mG_{AB} \ddot{v}^B.
\]
Inverting the Legendre transformation, we obtain geodetic Hamiltonian \( T = T_{tr} + T_{int} \), where, dually to the \( L \)-formulas,
\[
T_{tr} = \frac{1}{2m} \gamma^{ij} p_i p_j = \frac{1}{2m} \tilde{G}^{AB} \dot{p}_A \dot{p}_B, \quad T_{int} = \frac{1}{2} \tilde{R}^i_j k l \Sigma^l_k \Sigma^i_j.
\]
Obviously again \( \tilde{R} \) denotes the inverse bimatrix of \( R \), i.e., \( \tilde{R}^{a}_{\ b} k l \hat{R}^{l}_{\ k j} = \delta^a_j \delta^i_b \).
Let us admit non-geodetic models of the form \( \tilde{L} = T - V(x, \varphi), \ H = T + V(x, \varphi) \), where \( V(x, \varphi) \) is a usual potential energy depending only on the indicated configuration variables. Then the balance equations for \( \tilde{L} \)-models (affine in space, metrical in the material) read:
\[
\frac{dp_i}{dt} = Q_i, \quad \frac{d\Sigma^i_j}{dt} = -\frac{1}{m} \tilde{C}^{ik} p_k p_j + Q_{ij}, \tag{27}
\]
with the same meaning of generalized forces as previously, i.e.,
\[
Q_i = -\frac{\partial V}{\partial x^i}, \quad Q_{ij} = -\varphi^j_A \frac{\partial V}{\partial \varphi^i_A}.
\]
When taken together with the Legendre transformation, the above balance laws are equivalent to the Hamilton canonical equations. They may be also generalized so as to include some non-Hamiltonian, e.g., dissipative terms on their right-hand sides. It is seen that in the purely geodetic case (when \( Q_i = 0, Q_{ij} = 0 \)) the canonical linear momentum is conserved, but the affine spin is not so due to the first term on the right-hand side of the balance law for \( \Sigma \). The reason is that \( \Sigma^i_j \) generate linear transformations of internal degrees of freedom; these transformations do not affect translational variables. Therefore, the full affine symmetry is broken, and \( \Sigma \) is not a constant of motion. But one can reformulate the balance laws (27), just as in the d’Alembert model, by introducing the total canonical affine momentum \( I(O) \), related to some fixed origin \( O \in M \),
\[
I(O)^{i}_{\ j} := \Lambda(O)^{i}_{\ j} + \Sigma^i_j = x^i p_j + \Sigma^i_j.
\]
The first term refers to the translational motion, the second one to the relative (internal) motion. Something similar may be done for generalized forces,

\[ Q_{\text{tot}}(\mathcal{O})^i_j := Q_{\text{tr}}(\mathcal{O})^i_j + Q^i_j = x^i Q_j + Q^i_j. \]

Then the system of balance equations \((27)\) may be written in the following equivalent form:

\[
\frac{dp}{dt} = Q_i, \quad \frac{dI(\mathcal{O})^i_j}{dt} = Q_{\text{tot}}(\mathcal{O})^i_j.
\]

\((28)\)

It is seen that in the geodetic case one obtains conservation laws for \(p_i, I(\mathcal{O})^i_j\), i.e., for the system of generators of the spatial affine group \(\text{GAf}(M)\), just as expected.

Let us observe some funny feature of our geodetic equations. Namely, the canonical linear momentum is a constant of motion, but the translational velocity is not. This is because the Legendre transformation \((25)\) implies that the translational motion is influenced by internal phenomena. Except some special solutions even the direction of translational velocity is non-constant and depends on what happens with "internal" degrees of freedom. This is a kind of "drunk missile" effect. Something similar occurs in the dynamics of defects in solids \([28]\). It is also non-excluded that the non-conservation of velocity might be an over-simplified model of certain specially-relativistic phenomena (internal motion results in changes of internal energy, and therefore, in the rest mass pulsations; but the latter ones influence the effective inertia, and therefore, the translational motion). One can show that the time-rate of translational velocity may be expressed as follows:

\[
m \frac{dv^a}{dt} = -\tilde{C}^{aj} \frac{dC_{jb}}{dt} v^b + F^a = mv^b (\Omega^a_b + \tilde{C}^{ad} \Omega^{m}_d C_{mb}) + F^a,
\]

where the contravariant force \(F\) is given by the following expression: \(F^a = C^{ab} Q_b\) (because of the formerly mentioned reasons, we avoid denoting \(F^a\) as \(Q^a\)). It is explicitly seen that \(v\) is variable even in the purely geodetic motion. The \(\mathcal{L}\)-based geodetics in \(M \times \text{LI}(U, V)\) do not project onto straight lines in \(M\).

Let us now consider the \(\mathcal{R}\)-based models \((19), (23)\), metrical in space and affine in the material. As mentioned, they are somehow related to the Arnold-Ebin-Marsden-Binz approach to the dynamics of ideal fluids \([1, 2, 3, 13, 14, 33, 34]\). Our Poisson brackets imply that the balance form of equations of motion may be expressed as follows:

\[
\frac{dp_a}{dt} = Q_a, \quad \frac{d\Sigma_A^B}{dt} = \dot{Q}_B^A,
\]

\((29)\)

again with

\[
Q_a = -\frac{\partial V}{\partial x^a}, \quad \dot{Q}_B^A = -\frac{\partial V}{\partial \varphi^A} \varphi^B = (\varphi^{-1})^A_i Q^i_j \varphi^B_j
\]

in the potential case. In geodetic models \(p_a, \Sigma_A^B\) are conserved quantities as explicitly seen from the balance equations and expected on the basis of invariance properties. Indeed, the \(\mathcal{R}\)-model of \(T\) is invariant under the Abelian group
of translations in $M$. Therefore, $p_a$ are constants of motion as Hamiltonian generators of this group. Similarly, as seen from our Poisson brackets, $\Sigma^A_B$ Poisson-commute with $p_a$ and $\Sigma^i_j$, therefore, with the total geodetic Hamiltonian $T$. This is due to the explicitly obvious invariance of the $R$-based $T$ under the group of material linear transformations $\text{GL}(U)$. Surprisingly enough, the co-moving components of linear momentum, $\hat{p}_A = p_i \varphi^i_A$, are not constants of motion. But this is also clear because the material space $N$ has in our model a distinguished point, i.e., the Lagrangian position of the centre of mass. Because of this, translations in $N$ fail to be symmetries and their Hamiltonian generators $\hat{p}_A$ are non-conserved. According to the structure of Legendre transformation, translational velocity $v^a = dx^a/dt = g^{ab} p_b$ is also a constant of motion, just as $p$ itself. Therefore, in geodetic $R$-models, the geodesic curves in $M \times \text{LI}(U, V)$ project to $M$ onto straight lines swept with constant velocities (uniform motions). This means that there is no "drunk missile effect" and contravariant representation of the translational balance takes on the usual form:

$$m \frac{dv^a}{dt} = m \frac{d^2 x^a}{dt^2} = F^a, \quad F^a = g^{ab} Q_b.$$  

As previously, the balance laws (29) become a closed system of equations of motion when considered jointly with the Legendre transformation (26). Let us observe that the structure of (29) is in a sense less "aesthetical" than that of (27) because it is more non-homogeneous. The point is that in (27) both subsystems are written in terms of spatial objects, whereas in (29) one uses the mixed representation: spatial for the translational motion and material for the internal one. Obviously, (29) may be done symmetric, dual to (27), by substituting $\hat{p}_A = p_i \varphi^i_A$. But this immediately makes the translational equation more complicated.

There is also another problem. The simplicity of our balance laws (conservation laws in the geodetic case) is rather illusory. The point is that, as mentioned above, the total system of equations of motion consists of the balance laws and Legendre transformation. The balance (27), (28) looks simple in Euler (spatial) representation, but the corresponding Legendre transformation is simple in Lagrangian (material) representation (24). And quite conversely, the internal part of (29) is simple in the co-moving terms, but its Legendre transformation is simple when expressed in the spatial (Eulerian) form (26).

One can easily show that the internal parts of Legendre transformations (24), (26) may be respectively expressed as follows:

$$\Sigma^i_j = \hat{\mathcal{L}}^i_{j k} \Omega_k^l, \quad \Sigma^A_B = \hat{\mathcal{R}}^A_B C D \hat{\Omega}^D C,$$

where

$$\hat{\mathcal{L}}^i_{j k} = \varphi^j_A (\varphi^{-1})^B_{j C} \varphi^k_C (\varphi^{-1})^D_{j D} \hat{\mathcal{L}}^A_B C D,$$

$$\hat{\mathcal{R}}^A_B C D = (\varphi^{-1})^A_{j C} \varphi^B_C (\varphi^{-1})^D_{j D} \hat{\mathcal{R}}^i_{j k}.$$  

Obviously, this form is rather complicated because the coefficients at $\Omega$ and $\hat{\Omega}$ are non-constant; they depend on the internal configuration $\varphi$. Simplicity of the balance laws is incompatible with simplicity of Legendre transformations.
As mentioned, when translational degrees of freedom are taken into account, there are no sensible models which would be affine simultaneou sly in space and in the material. The highest symmetry of mathematical interest and at the same time physically reasonable is that affine in space and rotational in the material, and conversely, Euclidean in space and (centro-)affine in the body. The latter model is an over-simplified discretization of dynamical systems on diffeomorphisms group as used in hydrodynamics and elasticity.

In materially isotropic $\mathcal{L}$-models the quantity $\mathcal{L}^A B^C D$ is a linear combination of tensors $\eta^{AC} \eta_{BD}$, $\delta^A D \delta^C B$, $\delta^A B \delta^C D$. Similarly, in spatially isotropic $R^i j^k L$-models the tensor $R^i j^k L$ is a linear combination of terms $g^{ik} g_{jl}$, $\delta^{il} \delta^{kj}$, $\delta^{ij} \delta^{kl}$. Therefore, (18), (19) take on, respectively, the following forms:

$$T_{\text{int}} = \frac{I}{2} \eta_{KL} \hat{\Omega}^K L \hat{\Omega}^M L \eta_{MN} + \frac{A}{2} \hat{\Omega}^K L \hat{\Omega}^L K + \frac{B}{2} \hat{\Omega}^K K \hat{\Omega}^L L,$$  \hspace{1cm} (31)

$$T_{\text{int}} = \frac{I}{2} g_{ik} \Omega^i j g_{jl} + \frac{A}{2} \Omega^i j \Omega^j i + \frac{B}{2} \Omega^i i \Omega^j j,$$  \hspace{1cm} (32)

where the constants $I$, $A$, $B$ are generalized internal inertia scalars. It is clear that if $I = 0$, then these expressions become identical. The $I$-terms break the centro-affine symmetry in $U$ and $V$, and restrict it to the metrical one, respectively, in the sense of metric tensors $\eta$ or $g$. The first term in (31), just as (22), may be expressed in terms of the Cauchy deformation tensor, i.e.,

$$\frac{I}{2} C_{ij} \frac{d\varphi^i A}{dt} \frac{d\varphi^j B}{dt} \eta^{AB}.$$  \hspace{1cm} (33)

Let us observe that the isotropic inertial tensor $I \eta^{AB}$ in (33) might be replaced by the general one,

$$\frac{1}{2} C_{ij} \frac{d\varphi^i A}{dt} \frac{d\varphi^j B}{dt} J^{AB} = \frac{1}{2} \eta_{KL} \hat{\Omega}^K L \hat{\Omega}^L B J^{AB}.$$  \hspace{1cm} (34)

This expression is structurally similar to the d’Alembert formula (1). The difference is that the fixed metric $g$ is replaced by the $\varphi$-dependent Cauchy tensor $C$. There is not only formal similarity but also some asymptotic correspondence between (34) and (1). Obviously, for the general $J$, (34) is not metrically isotropic and its internal symmetry is reduced to $O(U, \eta) \cap O(U, \tilde{J})$. The $I$-terms in (31), (32) are positively definite if $I > 0$. Moreover, the total expressions (31), (32) are positively definite for some open range of $(I, A, B) \in \mathbb{R}^3$. Roughly speaking, the absolute values of $A$, $B$ must be “sufficiently small” in comparison with $I$.

The internal part of Legendre transformation (24) for $\mathcal{L}$-models becomes now (i.e., for (31))

$$\dot{\Sigma}^K L = I \eta^{KM} \eta_{LN} \hat{\Omega}^N M + A \hat{\Omega}^K L + B \delta^K L \hat{\Omega}^M M.$$  \hspace{1cm} (35)

This may be alternatively written as follows: $\Sigma^i j = I \tilde{C}^{ib} C^j a \Omega^a b + A \Omega^j i + B \delta^j i \Omega^m m$, it is the very special case of (30).
The inverse Legendre transformation has the same structure, i.e.,

$$
\hat{Ω}^K_L = \frac{1}{I} \tilde{I}^{KM} \eta_{LM} \hat{Σ}^N_M + \frac{1}{A} \hat{Σ}^K_L + \frac{1}{B} \delta^K_L \hat{Σ}^M_M,
$$

(36)

where $\tilde{I} = (I^2 - A^2) / I$, $\tilde{A} = (A^2 - I^2) / A$, $\tilde{B} = -(I + A) (I + A + nB) / B$.

When written in Eulerian (spatial) terms, this formula becomes

$$
\Omega^i_j = \frac{1}{I} \tilde{C}^{ab} C_{ja} \Sigma^a_b + \frac{1}{A} \Sigma^a_j + \frac{1}{B} \delta^i_j \Sigma^m_m,
$$

(37)

with the same as previously meaning of modified inertial coefficients $\tilde{I}$, $\tilde{A}$, $\tilde{B}$.

Similarly, for $R$-models based on (32) the internal sector of Legendre transformation has the following form

$$
\Sigma^i_j = I g^{im} g_{jn} \Omega^a_m + A \Omega^i_j + B \delta^i_j \Omega^m_m,
$$

(37)

which is inverted as

$$
\Omega^i_j = \frac{1}{I} g^{im} g_{jn} \Sigma^a_m + \frac{1}{A} \Sigma^i_j + \frac{1}{B} \delta^i_j \Sigma^m_m.
$$

(38)

The co-moving representation of these formulas is given by the following expressions:

$$
\hat{Σ}^K_L = \frac{1}{I} \tilde{I}^{KM} G_{LN} \hat{Ω}^N_M + A \hat{Ω}^K_L + B \delta^K_L \hat{Ω}^M_M,
$$

$$
\hat{Ω}^K_L = \frac{1}{I} G^{KM} G_{LN} \hat{Σ}^N_M + \frac{1}{A} \hat{Σ}^K_L + \frac{1}{B} \delta^K_L \hat{Σ}^M_M.
$$

The general balance laws (27), (29) considered jointly with these Legendre transformations (including the obvious translational sector) provide the complete system of equations of motion (naturally, the definitions of $Ω$, $\hat{Ω}$ are to be substituted). These equations, due to the very special structure of (31), (32) are relatively readable and effective. At the same time one can show that for incompressible bodies even in the purely geodetic case ($Q_i = 0$, $Q^j_{ij} = 0$) there exists an open set of solutions which are bounded in the internal configuration space $LI(U, V)$, so the elastic vibrations may be encoded in the very kinetic energy (Riemann structure) without the explicite use of forces.

After substituting the above inverse of Legendre transformations to kinetic energies (31), (32), we obtain the following geodetic Hamiltonians of internal motion:

$$
T_{int} = \frac{1}{2I} \eta_{KL} \hat{Σ}^K_M \hat{Σ}^L_N \eta^{MN} + \frac{1}{2A} \hat{Σ}^K_L \hat{Σ}^L_K + \frac{1}{2B} \hat{Σ}^K_K \hat{Σ}^L_L,
$$

(39)

$$
T_{int} = \frac{1}{2I} g_{kl} \Sigma^i_j \Sigma^k_l g^{jl} + \frac{1}{2A} \Sigma^i_j \Sigma^l_i + \frac{1}{2B} \Sigma^i_i \Sigma^l_j.
$$

(40)

In certain problems it may be convenient to write down the first formula in spatial terms; similarly, the second one may be expressed with the use of co-moving
representation. Therefore, we obtain, respectively, the following expressions:

\[ T_{\text{int}} = \frac{1}{2I} C_{kl} \Sigma^k_m \Sigma^l_n \bar{C}^{mn} + \frac{1}{2A} \Sigma^k_i \Sigma^l_k + \frac{1}{2B} \Sigma^i_k \Sigma^l_i, \]

\[ T_{\text{int}} = \frac{1}{2I} G_{KL} \tilde{\Sigma}^K_M \tilde{\Sigma}^L_N \bar{G}^{MN} + \frac{1}{2A} \tilde{\Sigma}^K_L \tilde{\Sigma}^L_K + \frac{1}{2B} \tilde{\Sigma}^K_K \tilde{\Sigma}^L_L. \]

The corresponding velocity-based formulas (31), (32) for kinetic energy may be written in an analogous way. Simply \( \frac{1}{\tilde{I}}, \frac{1}{\tilde{A}}, \frac{1}{\tilde{B}} \) in the last expressions are to be replaced by \( I, A, B \), and simultaneously one must substitute \( \Omega^k_l, \hat{\Omega}^K_L \) instead of \( \Sigma^k_l, \hat{\Sigma}^K_L \).

For certain purposes it is convenient to rewrite geodetic Hamiltonians (39), (40) in an alternative form:

\[ T_{\text{int}} = \frac{1}{2\alpha} \text{Tr}(\hat{\Sigma}^2) + \frac{1}{2\beta} (\text{Tr} \hat{\Sigma})^2 - \frac{1}{4\mu} \text{Tr}(V^2), \quad (41) \]

\[ T_{\text{int}} = \frac{1}{2\alpha} \text{Tr}(\Sigma^2) + \frac{1}{2\beta} (\text{Tr} \Sigma)^2 - \frac{1}{4\mu} \text{Tr}(S^2), \quad (42) \]

where \( \alpha := I + A, \beta := -(I + A)(I + A + nB)/B, \mu := (I^2 - A^2)/I, \) and \( V, S \) denote, respectively, the vorticity and spin given by (6). It is clear that the only distinction between expressions (41) and (42) is that in their third terms, thus, we can rewrite them concisely like

\[ T_{\text{int}} = \frac{1}{2\alpha} C(2) + \frac{1}{2\beta} C(1)^2 - \frac{1}{2\mu} \|V\|^2, \quad (43) \]

\[ T_{\text{int}} = \frac{1}{2\alpha} C(2) + \frac{1}{2\beta} C(1)^2 - \frac{1}{2\mu} \|S\|^2, \quad (44) \]

where \( C(k) \) denotes the \( k \)-th order Casimir expression built of generators, i.e., \( C(k) := \text{Tr}(\Sigma^k) = \text{Tr}(\hat{\Sigma}^k) \).

By analogy with the physical 3-dimensional case the quantity \( -\text{Tr}(S^2) \) may be interpreted as the doubled squared norm of the internal angular momentum, \( -\text{Tr}(S^2) = 2\|S\|^2 \). Similarly, \( -\text{Tr}(V^2) = 2\|V\|^2 \). Therefore, the formulas (43), (44) may be respectively written in the following intuitive and suggestive way:

\[ T_{\text{int}} = \frac{1}{2\alpha} C(2) + \frac{1}{2\beta} C(1)^2 + \frac{1}{2\mu} \|V\|^2, \quad (45) \]

\[ T_{\text{int}} = \frac{1}{2\alpha} C(2) + \frac{1}{2\beta} C(1)^2 + \frac{1}{2\mu} \|S\|^2. \quad (46) \]

Obviously, for the model (20) invariant under the left and right action of linear groups we have

\[ T_{\text{int}} = \frac{1}{2A} C(2) + \frac{1}{2A(n + A/B)} C(1)^2, \quad (47) \]

When performing computations, it is convenient to use orthogonal coordinates, \( \eta_{KL} = \delta_{KL}, g_{ij} = \delta_{ij} \), and rewrite some of the above formulas in terms
of purely matrix operations. Thus, (31), (32) become, respectively,

\[ T_{\text{int}} = \frac{I}{2} \text{Tr}(\hat{\Omega}^T \hat{\Omega}) + \frac{A}{2} \text{Tr}(\hat{\Omega}^2) + \frac{B}{2} (\text{Tr}\hat{\Omega})^2, \]  

(48)

\[ T_{\text{int}} = \frac{I}{2} \text{Tr}(\Omega^T \Omega) + \frac{A}{2} \text{Tr}(\Omega^2) + \frac{B}{2} (\text{Tr}\Omega)^2. \]  

(49)

Obviously, the second and third terms in these formulas are pairwise equal because \( \text{Tr}(\hat{\Omega}^2) = \text{Tr}(\Omega^2) \) and \( \text{Tr}(\hat{\Omega}) = \text{Tr}(\Omega) \).

Legendre transformations (35), (37) and their inverses (36), (38) are as follows:

\[ \hat{\Sigma} = I \hat{\Omega}^T + A \hat{\Omega} + B (\text{Tr}\hat{\Omega}) I_n, \quad \Sigma = I \Omega^T + A \Omega + B (\text{Tr}\Omega) I_n, \]

\[ \hat{\Omega} = \frac{1}{I} \Sigma^T + \frac{1}{A} \hat{\Sigma} + \frac{1}{B} (\text{Tr}\hat{\Sigma}) I_n, \quad \Omega = \frac{1}{I} \Sigma^T + \frac{1}{A} \Sigma + \frac{1}{B} (\text{Tr}\Sigma) I_n, \]

where \( I_n \) denotes the \( n \)-th order identity matrix.

Similarly, for the kinetic Hamiltonians (39), (40) we have, respectively,

\[ T_{\text{int}} = \frac{1}{2I} \text{Tr}(\hat{\Sigma}^T \hat{\Sigma}) + \frac{1}{2A} \text{Tr}(\hat{\Sigma}^2) + \frac{1}{2B} (\text{Tr}\hat{\Sigma})^2, \]

(50)

\[ T_{\text{int}} = \frac{1}{2I} \text{Tr}(\Sigma^T \Sigma) + \frac{1}{2A} \text{Tr}(\Sigma^2) + \frac{1}{2B} (\text{Tr}\Sigma)^2. \]

(51)

This matrix representation is very lucid and useful in calculations. Nevertheless, in comparison with the systematic tensor language, it may be risky and misleading because it obscures the geometric meaning of symbols and concepts. And this is worse than the lack of aesthetics; when no care is taken, this may lead simply to logical and numerical mistakes.

We finish this section with some geometric remarks.

Kinetic energy \( T \) of a non-relativistic mechanical system is equivalent to some Riemannian structure \( \Gamma \) on its configuration space \( Q \). In terms of generalized coordinates and velocities, we have that

\[ T = \frac{1}{2} \Gamma_{\alpha\beta}(q) \frac{dq^\alpha}{dt} \frac{dq^\beta}{dt}, \quad \Gamma = \Gamma_{\alpha\beta}(q) dq^\alpha \otimes dq^\beta. \]

Although usually somehow related to the metric tensor \( g \) of the physical space \( M \), \( \Gamma \) need not be directly interpretable in terms of geometrical distances in \( M \). As a rule, it depends not only on \( g \) but also on certain parameters characterizing inertial properties of the system, i.e., masses, inertial moments, etc.

It is instructive to describe explicitly in a bit more geometric form the metric tensors \( \Gamma \) on \( Q = M \times L I(U,V) \) underlying the kinetic energies defined above. For this purpose it is convenient to introduce auxiliary geometric objects.

Let \( E_A, e_i \) denote the basic vectors in \( U, V \) underlying our Lagrange and Euler coordinates \( a^K, x^j \). The corresponding dual basic covectors in \( U^*, V^* \) will be denoted as usual by \( E^A, e^i \). Generalized coordinates in \( Q \) will be, as previously, denoted by \( x^i, \varphi^i_A \), and no misunderstandings just simplifications follow from
using the same symbol $x^i$ for coordinates in $M$ and for their pull-backs to $Q$. Now, we introduce two families of Pfaff forms (differential one-forms) on $Q$, i.e.,

\[ \hat{\omega}^A_B := (\varphi^{-1})^A_i \varphi^B_i, \omega^i_j := \varphi^j_i (\varphi^{-1})^A_A. \]

These basic systems depend on the choice of bases $E$, $e$, but this is, so to speak, a covariant dependence. In other words the $L(U)$- and $L(V)$-valued one-forms $\hat{\omega}^A_B = \hat{\omega}^A_B E_A \otimes E_B$, $\omega^i_j = \omega^i_j e_i \otimes e_j$ are base-independent.

In addition, we shall need the following two families of Pfaff forms: $\hat{\theta}^A = (\varphi^{-1})^A_i \text{d}x^i$, $\theta^i = \varphi^i A \hat{\theta}^A_A$. These basic systems depend on the choice of bases $E$, $e$, but this is, so to speak, a covariant dependence. In other words the $L(U)$- and $L(V)$-valued one-forms $\hat{\theta}^A = \hat{\theta}^A_E E_A \otimes E_B$, $\theta^i = \theta^i e_i \otimes e_j$ are base-independent.

In addition, we shall need the following two families of Pfaff forms: $\hat{\omega}^A_B = \hat{\omega}^A_B E_A \otimes E_B$, $\omega^i_j = \omega^i_j e_i \otimes e_j$. Just as previously, they give rise to the objective base-independent $U$- and $V$-valued differential one-forms: $\hat{\theta}^A = \hat{\theta}^A_E E_A \otimes E_B$, $\theta^i = \theta^i e_i \otimes e_j$.

The base-independent objects $\hat{\omega}$, $\hat{\theta}$ and $\omega$, $\theta$ could be defined without any use of bases, however, the above definitions are technically simplest. The above objects are closely related to the concept of affine velocity in co-moving (Lagrange) and spatial (Euler) representations. Namely, for any history $\mathbb{R} \ni t \mapsto (x(t), \varphi(t))$, the quantities $\hat{\Omega}^A_B$, $\Omega^i_j$ are evaluations of $\hat{\omega}^A_B$, $\omega^i_j$ on the tangent vectors (general velocities) given analytically by $\text{d}x^i / \text{d}t, \text{d}\varphi^i_A / \text{d}t$. Roughly speaking, we could say that

\[ \hat{\Omega}^A_B = \hat{\omega}^A_B \frac{\text{d}t}{\text{d}t}, \quad \Omega^i_j = \omega^i_j \frac{\text{d}t}{\text{d}t}. \]

This is obviously a kind of joke, but fully justified on the basis of infinitesimal Leibniz notation. Similarly, the co-moving and spatial components of translational velocity are given by evaluations of $\hat{\theta}^A$ and $\theta^i$ on tangent vectors, and using the same trick we could say that

\[ \hat{v}^A = \frac{\hat{\theta}^A}{\text{d}t}, \quad v^i = \frac{\theta^i}{\text{d}t}. \]

At any point of the configuration space the systems $\theta^i, \omega^k_l$ and $\hat{\theta}^A, \hat{\omega}^K_L$ provide two bases in the corresponding space of covariant vectors. We could ask for the corresponding bases of contravariant vector spaces. It is convenient to use the language of contemporary differential geometry, where vector fields $X$ with components $X^i$ (meant in the sense of some local coordinates $q^i$) are identified with first-order differential operators of directional derivatives $\nabla_X$, i.e., $X = X^i \partial / \partial q^i$. One can easily show that the bases $\hat{H}_K, \hat{E}^A_B$ and $H_k, E^a_b$, dual respectively to $\hat{\theta}^K, \hat{\omega}^A_B$ and $\theta^k, \omega^a_b$, are given by the following differential operators:

\[ \hat{H}_K = \varphi^K_p \frac{\partial}{\partial x^p}, \quad H_k = \frac{\partial}{\partial x^k}, \quad \hat{E}^A_B = \varphi^K_B \frac{\partial}{\partial \varphi^K_A}, \quad E^a_b = \varphi^a_K \frac{\partial}{\partial \varphi^b_K}. \]

The general $L$-models \eqref{18}, \eqref{22} are based on metric tensors of the following form:

\[ \Gamma = m \eta_{AB} \hat{\theta}^A \otimes \hat{\theta}^B + L^B_A D^C \hat{\omega}^A_B \otimes \hat{\omega}^C_D. \]

Similarly, for $R$-models \eqref{19}, \eqref{23} we have

\[ \Gamma = m g_{ij} \theta^i \otimes \theta^j + R^j_i A^k B \omega^j_A \otimes \omega^k_B. \]
The corresponding contravariant metrics underlying the kinetic Hamiltonians are given, respectively, for $\mathcal{L}$-models by
\[
\tilde{\Gamma} = \frac{1}{m} \eta^{AB} \hat{H}_A \otimes \hat{H}_B + \tilde{\xi}^A_C \hat{E}^A_B \otimes \hat{E}^C_D
\]
and for $\mathcal{R}$-models by
\[
\tilde{\Gamma} = \frac{1}{m} g^{ij} H_i \otimes H_j + \tilde{\eta}^l_k E^i_j \otimes E^k_l.
\]

If the kinetic energy of internal motion (20) is invariant simultaneously under $\text{GL}(\mathcal{V})$ and $\text{GL}(\mathcal{U})$, then the corresponding metric tensor on $\text{LI}(\mathcal{U}, \mathcal{V})$ is given by
\[
\Gamma^0_{\text{int}} = A \omega^K_L \otimes \hat{\omega}^L_K + B \omega^K_K \otimes \hat{\omega}^L_L = A \omega^k_i \otimes \omega^i_k + B \omega^k_k \otimes \omega^i_l
\]
and its inverse by
\[
\tilde{\Gamma}^0_{\text{int}} = \frac{1}{A} \hat{E}^K_L \otimes \hat{E}^L_K - \frac{B}{A(A + nB)} E^K_L \otimes E^L_L
\]
\[
= \frac{1}{A} E^k_i \otimes E^i_k - \frac{B}{A(A + nB)} E^k_k \otimes E^i_l.
\]

Here $\Gamma^0_{\text{int}}$ is a linear combination of two Casimir-like objects built of Pfaff forms $\omega$ in a quadratic way. As already mentioned, $\Gamma^0_{\text{int}}$ becomes the group-theoretic Killing tensor when $A = 2n$, $B = -2$. This is just the pathological situation to be excluded because for $A/B = -n$ the tensor $\Gamma^0_{\text{int}}$ is singular.

For our models affine in space and metrical in the body we have that
\[
\Gamma = m \eta^{KL} \hat{\theta}^K_L \otimes \hat{\theta}^L_K + I \eta^{MN} \hat{\omega}^K_M \otimes \hat{\omega}^L_N + \Gamma^0_{\text{int}}.
\]

Similarly, for models metrical in space and affine in the body:
\[
\Gamma = mg_{ij} \theta^i \otimes \theta^j + I g^{kl} \omega^i_j \otimes \omega^k_l + \Gamma^0_{\text{int}}.
\]

The corresponding contravariant (reciprocal) metrics are given by
\[
\tilde{\Gamma} = \frac{1}{m} \eta^{KL} \hat{H}_K \otimes \hat{H}_L + \frac{1}{I} \eta^{MN} \hat{E}^K_M \otimes \hat{E}^L_N + \frac{1}{A} \hat{E}^K_K \otimes \hat{E}^L_L + \frac{1}{B} \hat{E}^L_K \otimes \hat{E}^K_L
\]
for spatially affine models and
\[
\tilde{\Gamma} = \frac{1}{m} g^{ij} H_i \otimes H_j + \frac{1}{I} g^{kl} \omega^i_j \otimes \omega^k_l + \frac{1}{A} E^i_j \otimes E^i_j + \frac{1}{B} E^k_k \otimes E^i_l
\]
for materially affine models. Obviously, the last two terms in both expressions coincide. The kinetic (geodetic) terms of Hamiltonians for geodetic and potential systems are based on $\Gamma$-tensors, namely,
\[
T = \frac{1}{2} \tilde{\Gamma}^{\mu\nu}(q)p_\mu p_\nu.
\]
\[ \tilde{\Gamma}^\mu_{\kappa \nu} = \delta^\mu_{\nu} \] and \( p_\mu = \partial L / \partial \dot{q}^\mu = \partial T / \partial \dot{q}^\mu \) are canonical momenta conjugate to \( q^\mu \) (dual objects to generalized velocities \( \dot{q}^\mu \)).

The above objects \( \omega, \theta \) and \( E, H \) possess natural generalizations to curved manifolds with affine connection. They appear there, respectively, as the connection form, canonical form, fundamental vector fields, and standard horizontal vector fields on \( FM \) (the principle fibre bundle of linear frames in a manifold \( M \)) [27]. Such a formalism is used in mechanics of infinitesimal affinely-rigid bodies, when affine degrees of freedom are considered as internal ones, i.e., attached to material points moving in manifolds with curvature and torsion [61].

5 Without translational motion

It is instructive to consider the simplified situation when the centre of mass is at rest and the covariant translational forces do vanish, i.e., \( Q_i = 0 \). For the "usual" d’Alembert model such a situation, characteristic for practical elastic problems, was discussed detailly in our earlier papers [45, 48, 49, 52, 53, 54, 55].

And from the purely geometric symmetry point of view nothing particularly interesting happened there due to this simplification. In our model, based on dynamical affine symmetries, the translationless situation is an important step of the general analysis.

We have mentioned that in spatially affine \( L \)-models with \( Q_i = 0 \), in particular in geodetic ones, canonical linear momentum is a constant of motion but translational velocity is not (except some very special solutions). This violates our ideas about Galileian symmetry, at least in the form developed in the "usual" d’Alembert mechanics. Nevertheless, the concept of translationless motion is well-defined because in the usual potential models equations \( v^i = 0, \ p_i = 0 \) are equivalent; this is one of exceptional cases when the constancy of velocity does not contradict the constancy of linear momentum. One must only remember that the Galilei transforms (in the usual sense) of such space-resting solutions will not be solutions any longer.

In \( L \)-models without translational motion the evolution is ruled by the second of the balance laws (27) with the simplified right-hand side, i.e.,

\[ \frac{d\Sigma^i_j}{dt} = Q^i_j. \]

Affine invariance in \( M \) implies that in the completely geodetic case this becomes simply the Noether conservation law:

\[ \frac{d\Sigma^i_j}{dt} = 0, \]

i.e., affine spin in spatial representation is a constant of motion. As mentioned, to obtained a closed system of equations, one must consider the above balance (conservation) jointly with the Legendre transformation and the definition of affine velocity. Unfortunately, the nice form (24) with constant coefficients cannot be used because \( \Sigma^A_B \) is not a constant of motion in the geodetic case.
And in general, (30) is too complicated to be effectively used. But it turns out that something may be done for our simplified model (31), affine in $M$ and $\eta$-metrical in $U$.

Something similar may be said about $R$-models, moreover, they are in some respects simpler. The balance equations (29) reduce to their internal parts, i.e.,

$$\frac{d\hat{\Sigma}^A_B}{dt} = \hat{Q}^A_B,$$

and become conservation laws for the co-moving affine spin in the geodetic case, i.e.,

$$\frac{d\hat{\Sigma}^A_B}{dt} = 0.$$

As previously, the simplicity is only seeming one because the laboratory components $\Sigma^i_j$ fail to be constants of motion and Legendre transformation expressed in co-moving terms (30), in general, is rather complicated. Fortunately, for our models (32), metrical in space and centro-affine in the material, also something may be done.

Let us begin with the over-simplified model with $I = 0$, affine both in the spatial and material sense. It is easily seen that the general solution for translation-free geodetic motion is then given by the system of orbits of one-parameter subgroups of $GL(V)$ or, equivalently, one-parameter subgroups of $GL(U)$, i.e.,

$$\varphi(t) = \exp(Et)\varphi_0 = \varphi_0 \exp(\hat{E}t),$$

(52)

where $\varphi_0$ is an arbitrary element of $LI(U,V)$, $E$ is an arbitrary element of $L(V) = GL(V)'$, and $\hat{E} = \varphi_0^{-1}E\varphi_0$ is the corresponding element of $L(U) = GL(U)'$ obtained by the $\varphi_0$-similarity. If we identify formally $U$ and $V$ with $\mathbb{R}^n$ (by the particular choice of bases), then the phase portrait consists of all one-parameter subgroups of $GL(n, \mathbb{R})$ and of all their left cosets or, equivalently, of all their right cosets. One must only remember that although the sets of left and right cosets coincide, they are parameterized in a different way by the corresponding generators and initial shifting elements. The reason is that $GL(n, \mathbb{R})$ is non-Abelian and, in general, its one-parameter subgroups are not normal. If we write the group-theoretical version of (52), i.e.,

$$g(t) = \exp(at)h = h \exp(h^{-1}ah)t,$$

it is seen that the coinciding left and right cosets usually refer to different generators $a$ and $h^{-1}ah$, thus, to different subgroups. If some left and right cosets refer to the same subgroup, i.e., the same generator $a$, and have non-empty intersection, then, as a rule, they are different subsets, i.e.,

$$g_1(t) = \exp(at)h \neq h \exp(at) = g_2(t).$$

Only the dilatational subgroup is exceptional because, consisting of central elements, it is a normal subgroup, and $h^{-1}ah = a$ for any dilatation generator $a$; the previous inequality becomes equality for any $h \in GL(n, \mathbb{R})$. 33
Let us notice that in the solution (52) the pairs $\varphi_0, E$ and $\hat{\varphi}_0, \hat{E}$ play the role of differently represented initial conditions; in this sense they label the general solution. Thus, $\varphi_0 = \varphi(0)$ is an initial configuration, whereas $E = \Omega(0)$, $\hat{E} = \hat{\Omega}(0)$ are initial and at the same time permanently constant values of the laboratory and co-moving affine velocities. Therefore, the initial values of generalized velocities are given by $\dot{\varphi}(0) = E\varphi_0 = \varphi_0\hat{E}$.

It is seen that for $I = 0$ the structure of general solution resembles that of the spherical rigid body. It is so for every geodetic model on a semisimple group or its trivial central extension if the kinetic energy is doubly (left and right) invariant [2]. But we should remember that even in the simple case of a free anisotropic rigid body situation changes drastically. Kinetic energy is invariant under left regular translations on $SO(3, \mathbb{R})$ (identified with the configuration space) but no longer under right translations. As a rule, one-parameter subgroups and their cosets fail to be solutions, i.e., they are not geodetics of left-invariant metric tensors on $SO(3, \mathbb{R})$. There are some exceptions, however, namely stationary rotations [2, 35, 36, 42]. They happen when one of main axes of inertia has a fixed orientation in space and the remaining two perform a uniform rotation about it with a fixed angular velocity. Thus, there is a subset of general solution given by three one-parameter subgroups and all their left cosets (the non-moving axis of inertia may be arbitrarily oriented in space). This is the special case of what is known as relative equilibria [1, 35, 36]. They correspond to critical points of geodetic Hamiltonians restricted to co-adjoint orbits in the dual space of the Lie algebra $SO(3, \mathbb{R})' \cong SO(3, \mathbb{R})$ [1, 35, 36]. Such particular solutions, although do not exhaust the phase portrait, contain an important information about its structure.

Something similar happens in our affine models when $I \neq 0$. The general solution is not any longer given by orbits of one-parameter subgroups. Nevertheless, there exist geometrically interesting orbits which are particular solutions, i.e., generalized equilibria.

Let us begin with geodetic $\mathcal{L}$-models left-invariant under $GL(V)$ and right-invariant under $O(U, \eta)$. One can show after some calculations that there exist solutions of the following form:

$$\varphi(t) = \varphi_0 \exp(\eta t), \quad (53)$$

where the initial configuration $\varphi_0 \in L(U, V)$ is arbitrary just as in (52). But now $F \in L(U) \cong GL(U)'$ is not arbitrary any longer, instead it must be $\eta$-normal in the sense of commuting with its $\eta$-transpose, i.e., $F^A_C \eta^{C D} F^E_D \eta_{E B} - \eta^{A D} F^E_D \eta_{E C} F^C_B = 0$. Introducing the $\eta$-transpose symbol,

$$(F^{\eta T})^A_B := \eta^{A C} F^D_C \eta_{D B}, \quad (54)$$

we can write the above condition in the following concise form:

$$[F, F^{\eta T}] = F F^{\eta T} - F^{\eta T} F = 0. \quad (55)$$

It is obvious that for such solutions affine velocities are constant and given by

$$\Omega = \varphi_0 F \varphi_0^{-1}, \quad \hat{\Omega} = F.$$
The initial data \( \phi_0, F \) are independent of each other. The only restriction is that of \( \eta \)-normality imposed on \( F \) alone. This holds, in particular, in two extremely opposite special cases when \( F \) is \( \eta \)-skew-symmetric or \( \eta \)-symmetric, i.e.,

\[
F^{\eta T} = -F, \quad F^{\eta T} = F,
\]

In the skew-symmetric case the one-parameter group generated by \( F \) consists of \( \eta \)-rotations, i.e.,

\[
\exp(\eta t) \in \SO(U, \eta) \subset \GL(U).
\]

If \( F \) is \( \eta \)-symmetric, then so are transformations \( \exp(\eta t) \); they describe pure deformations in \( U \) in the sense of \( \eta \)-polar decomposition.

In calculations one identifies usually \( U \) and \( V \) with \( \mathbb{R}^n \) and their metrics \( \eta \), \( g \) with the Kronecker delta. Then the solutions (53) become all possible left cosets of one-parameter subgroups of \( \GL(n, \mathbb{R}) \) generated by all possible normal matrices \( F \in \mathcal{L}(n, \mathbb{R}) \), i.e.,

\[
[F, F^T] = F F^T - F^T F = 0
\]

(in this formula we mean the usual matrix transposition).

Following (52) we can try to rewrite (53) in terms of the left-acting one-parameter subgroups. It is easy to see that

\[
\phi(t) = \phi_0 \exp(\eta t) = \exp(\tilde{F} t) \phi_0,
\]

where \( \tilde{F} = \phi_0 F \phi_0^{-1} \in \mathcal{L}(V) = \GL(V)' \).

In this representation \( \phi_0 \) is still arbitrary but \( \tilde{F} \) is subject to some restrictions following from (55) and depending on \( \phi_0 \). Namely, \( \tilde{F} \) is normal in the sense of the Cauchy deformation tensor \( C[\phi_0] \) used as a kind of metric in \( V \), i.e.,

\[
\tilde{F}^i_a C[\phi_0]^{ai} \tilde{F}^j_i C[\phi_0]_{ij} = 0.
\]

Introducing in analogy to (54) the \( C[\phi_0] \)-transpose of \( \tilde{F} \), i.e.,

\[
(\tilde{F} C[\phi_0])^i_j := \tilde{C}[\phi_0]^{ik} \tilde{F}^l_k C[\phi_0]_{lj},
\]

we can write simply

\[
[\tilde{F}, \tilde{F} C[\phi_0]] = 0.
\]

Therefore, in the right-cosets representation the initial configuration \( \phi_0 \) and the generator \( \tilde{F} \) are mutually interrelated. Namely, if \( \phi_0 \) is not subject to any restrictions, then \( \tilde{F} \) satisfies the condition (57) explicitly depending on \( \phi_0 \). And conversely, if \( \tilde{F} \) is arbitrary, then the initial conditions of \( \phi_0 \) must be so suited to any particular choice of \( \tilde{F} \) that the commutator condition (57) is non-violated.

Let us observe that in all \( \mathcal{L} \)-models the spatial metric \( g \) does not occur in expressions for the kinetic energy at all. Thus, as a matter of fact it does not need to exist at all and the physical space \( M \) may be purely affine. Only the material metric \( \eta \) in the body is essential for (51). Let us notice, however, that if both \( g \in V^* \otimes V^* \) and \( \eta \in U^* \otimes U^* \) are fixed, then some family of special solutions may be distinguished, for which the relationship between initial configurations...
and infinitesimal generators is simpler and more symmetric. Namely, we can start from the very beginning with the representation

\[ \varphi(t) = \exp(Et) \varphi_0, \]

where \( E \) and \( \varphi_0 \) are respectively some elements of \( L(V) \) and \( LI(U, V) \). It is easy to see that, when some metric \( g \) is fixed in \( V \), then we have at disposal a very natural family of solutions assuming that \( \varphi_0 \in LI(U, V) \) is an isometry and \( E \) is \( g \)-normal,

\[ [E, E^gT] = EE^gT - E^gT E = 0, \quad (58) \]

where in a full analogy to the previous expression we use the definition

\[ (E^gT)^i_j := g^{ik} E^l_k g_{lj}. \]

Obviously, such solutions form a submanifold of the family \((56), (57)\) because then \( C[\varphi_0] = g \).

Now let us consider geodetic \( \mathcal{R} \)-models \((32)\), which are left-invariant only under \( O(V, g) \) but right-invariant under the total \( GL(U) \). Now, as expected, the situation will be reversed. Let us assume solutions in the right-coset form:

\[ \varphi(t) = \exp(Et) \varphi_0, \]

where \( \varphi_0 \in LI(U, V), E \in L(V) \). It is easy to show that such a curve (right coset) satisfies, in fact, equations of geodetic motion if \( E \) is \( g \)-normal, just as in \((58)\), but now \( \varphi_0 \) may be quite arbitrary isomorphism of \( U \) onto \( V \). And if we write the above curve as a left coset, i.e.,

\[ \varphi(t) = \varphi_0 \exp(\tilde{E}t), \quad \tilde{E} = \varphi_0^{-1} E \varphi_0 \in L(U), \]

then it is easy to see that, with a still arbitrary \( \varphi_0 \), \( \tilde{E} \) will be \( G[\varphi_0] \)-normal in the sense of Green deformation tensor \( G[\varphi_0] \in U^* \otimes U^* \), i.e.,

\[ [\tilde{E}, \tilde{E}^G[\varphi_0]T] = \tilde{E} \tilde{E}^G[\varphi_0]T - \tilde{E}^G[\varphi_0]T \tilde{E} = 0, \]

where the \( G[\varphi_0] \)-transpose is defined in a full analogy to the above \( \eta \) - and \( g \) -transposes,

\[ (\tilde{E}^{G[\varphi_0]T})^A_B := G[\varphi_0]^{AC} \tilde{E}^D_C G[\varphi_0]_{DB}. \]

Just as previously, we can distinguish an interesting submanifold of such solutions when some material metric tensor \( \eta \in U^* \otimes U^* \) is fixed (we know it does not exist in \((52)\)). They are given by curves of the following form:

\[ \varphi(t) = \varphi_0 \exp(Ft), \]

where \( \varphi_0 \in O(U, \eta; V, g) \) is an isometry and \( F \in L(U) \) is \( \eta \)-normal in the sense of \((54), (55)\). For such solutions we have \( G[\varphi_0] = \eta \). Manipulating with \( \eta \) we introduce some kind of parametrization, ordering in our manifold of relative equilibria.
The above particular solutions are very special, nevertheless very important. Their position in our model is analogous to that of stationary rotations in rigid body mechanics. They provide a kind of skeleton for the general solution. Nevertheless, some, at least qualitative, rough knowledge of the phase portrait would be mostly welcome. The crucial question is to what extent the purely geodetic models may predict bounded motions. Obviously, this is impossible for compressible bodies, when the configuration space is identical with the total \( LI(U, V) \). To see this it is sufficient to consider the special case \( n = 1 \), when compressibility is the only degree of freedom of internal motion. There is only one affinely-invariant model of \( T_{\text{int}} \). The resulting trivial geodetic model predicts, depending on the sign of the initial internal velocity, either the infinite expansion or contraction, although in the latter case the object shrinks to a single point after infinite time. The only bounded (and non-stable) solution is the rest state. Something similar occurs in \( n \)-dimensional geodetic problems. Namely, degrees of freedom of the isochoric motion are orthogonal to the pure dilatations and completely independent of them. Some purely geometric comments are necessary here. Namely, if \( N \) and \( M \) are purely amorphous affine spaces, in particular no metrics \( \eta \), \( g \) are fixed in \( U \), \( V \), then their volume measures are defined only up to multiplicative constant factors. They are Lebesgue measures, i.e., special cases of Haar measures invariant under additive Abelian translations in \( U \), \( V \) (in \( N \), \( M \)). Let us denote some particular choices respectively by \( \nu_U \), \( \nu_V \). Obviously, for any measurable domain \( Y \subset U \) and for any configuration \( \varphi \in LI(U, V) \) we have

\[
\nu_V(\varphi(Y)) = \Delta(\varphi)\nu_U(Y).
\]

The scalar multiplicator \( \Delta(\varphi) \) depends on \( \varphi \) and on non-correlated normalizations of \( \nu_V \), \( \nu_U \) but does not depend on \( Y \). Obviously, for any \( A \in GL(U) \), \( B \in GL(V) \) we have

\[
\Delta(A\varphi B) = (\det A)\Delta(\varphi)\det B.
\]

The motion is isochoric if \( \Delta \) is constant in the course of evolution. Obviously, this definition is independent of particular normalizations of \( \nu_V \), \( \nu_U \). The manifold \( LI(U, V) \) becomes then foliated by \( (n^2 - 1) \)-dimensional leaves consisting of mutually non-compressed configurations. Every such leaf establishes holonomic constraints, and the total foliation is what is sometimes referred to as semi-holonomic or quasi-holonomic constraints. If some metric tensors \( \eta \in U^* \otimes U^* \), \( g \in V^* \otimes V^* \) are fixed, then the measures \( \nu_U \), \( \nu_V \) may be fixed respectively as \( \nu_\eta \), \( \nu_g \), and in terms of coordinates

\[
d\nu_\eta = \sqrt{\det[\eta_{KL}]} da^1 \cdots da^n, \quad d\nu_g = \sqrt{\det[g_{ij}]} dx^1 \cdots dx^n.
\]

Using Euclidean coordinates we can simply put \( \Delta(\varphi) = \det[\varphi^i_K] \) but, obviously, this convention fails for general coordinates. For non-Euclidean affine coordinates we have

\[
\Delta(\varphi) = \frac{\sqrt{\det[g_{ij}]} \det[\varphi^i_K]}{\sqrt{\det[\eta_{AB}]}}.
\]
Let us remind that the corresponding curvilinear formula reads

\[
\frac{d\nu_g(x(a))}{d\nu_\eta(a)} = \frac{\sqrt{\det[g_{kl}(x(a))]} \det \left[ \frac{\partial x^i}{\partial a^K} \right]}{\sqrt{\det[\eta_{AB}(a)]}}.
\] (59)

If some volumes are fixed in \(U\) and \(V\), e.g., due to some choices of metrics \(\eta\), \(g\), then the volume extension ratio \(\Delta(\varphi)\) is uniquely fixed. In certain formulas it may be convenient to use the additive parameter \(\alpha(\varphi)\) instead of the multiplicative one, i.e., \(\Delta(\varphi) = \exp[\alpha(\varphi)]\). Another convenient dilatation measures are those describing the linear size extension ratio,

\[D(\varphi) = \sqrt{\Delta(\varphi)} = \exp \left[ \frac{\alpha(\varphi)}{n} \right] = \exp[q(\varphi)].\]

The only possibility of stabilizing dilatations is to include some extra potential preventing the unlimited expansion to the infinite size and asymptotic contraction to the point-like object. There is plenty of such phenomenological modelling potentials, e.g.,

\[V_{\text{dil}} = \frac{\kappa}{8} (D^2 + D^{-2} - 2) = \frac{\kappa}{8} (\text{ch}^2 q - 1), \quad \kappa > 0.\]

Obviously, this potential is positively infinite at \(q = \mp\infty\) \((D = 0, D = +\infty)\) and has the global stable equilibrium at \(q = 0\) \((D = 1)\), where it behaves as the harmonic oscillator: \(V_{\text{dil}}(q) \approx \kappa q^2/2\) for \(q \approx 0\). For strongly extended bodies it also behaves harmonically in the \(D\)-variable sense. Another phenomenological model would be just the global form \(\kappa q^2/2\). One can also try to use some toy models predicting "dissociation" of the body (its unlimited size-expansion), unlimited collapse, or both of them above some threshold of the total dilatational energy, e.g.,

\[V_{\text{dil}}(q) = \frac{\kappa}{2} (\text{th}^2 q - 1).\]

In certain problems it may be reasonable to use phenomenological models preventing contraction but admitting dissociation.

In quantized version of the theory one can stabilize dilatations in an easy way with the use of the \(q\)-variable potential well (perhaps with the infinite walls) concentrated around \(q = 0\) \((D = 1)\).

If we identify analytically \(U\) and \(V\) with \(\mathbb{R}^n\) and \(\text{LI}(U, V)\) with \(\text{GL}(n, \mathbb{R})\), then it is clear that the connected component of unity \(\text{GL}^+(n, \mathbb{R})\) becomes the direct product \(\text{GL}^+(n, \mathbb{R}) \simeq \text{SL}(n, \mathbb{R}) \times \text{exp}(\mathbb{R}) = \text{SL}(n, \mathbb{R}) \times \mathbb{R}^+\); the second group factor is obviously meant in the multiplicative sense, as \(\text{GL}^+(1, \mathbb{R})\). It describes pure dilatations, whereas \(\text{SL}(n, \mathbb{R})\) refers to the isochoric motion. Without this identification, \(\text{LI}(U, V)\) may be represented as the Cartesian product of any of the aforementioned leaves (of mutually non-compressed configurations) and the multiplicative group \(\mathbb{R}\setminus\{0\}\). If some volume-standards \(\nu_U, \nu_V\) (e.g., metric-based ones \(\nu_\eta, \nu_g\)) and orientations are fixed in \(U, V\), then \(\text{LI}^+(U, V)\), i.e., the manifold of orientation-preserving isomorphisms, is identified with the product \(\text{SLI}(U, \nu_U; V, \nu_V) \times \text{exp}(\mathbb{R})\), where, obviously, the first term consists
of transformations \( \varphi \) for which \( \Delta(\varphi) = 1 \), i.e., \( q(\varphi) = 0 \). Such a formulation is more correct from the point of view of geometrical purity, however, for our purposes (qualitative discussion of the general solution), the analytical identification of \( LI^+(U, V) \) with \( GL^+(n, \mathbb{R}) \simeq SL(n, \mathbb{R}) \times \exp(\mathbb{R}) \) is sufficient and, as a matter of fact, more convenient. In any case, qualitative analysis of the general solution (bounded and non-bounded trajectories) is not then obscured by cosmetical aspects of geometry. Thus, from now on the internal configuration space \( Q_{\text{int}} = LI(U, V) \) will be identified with \( GL^+(n, \mathbb{R}) \simeq SL(n, \mathbb{R}) \times \exp(\mathbb{R}) \).

Any matrix \( \varphi \in GL^+(n, \mathbb{R}) \) will be uniquely represented as

\[
\varphi = \Psi = \exp(q)\Psi,
\]

where \( \Psi \in SL(n, \mathbb{R}) \). It is convenient to introduce the following shear velocities:

\[
\omega := \frac{d\Psi}{dt}\Psi^{-1}, \quad \hat{\omega} := \Psi^{-1}\frac{d\Psi}{dt}.
\]

Obviously, \( \omega, \hat{\omega} \in SL(n, \mathbb{R})' \), i.e., they are trace-less. Then affine velocities may be expressed as follows:

\[
\Omega = \omega + dq dt I, \quad \hat{\Omega} = \hat{\omega} + dq dt I,
\]

where, obviously, \( I \) denotes the identity matrix.

Analogously, the affine spin splits as follows:

\[
\Sigma = \sigma + \frac{p}{n}I, \quad \hat{\Sigma} = \hat{\sigma} + \frac{p}{n}I, \quad \sigma, \hat{\sigma} \in SL(n, \mathbb{R})',
\]

where \( p \) denotes the dilatational canonical momentum. The velocity-momentum pairing becomes \( \text{Tr}(\Sigma\Omega) = \text{Tr}(\Sigma\hat{\Omega}) = \text{Tr}(\sigma\omega) + pq = \text{Tr}(\hat{\sigma}\hat{\omega}) + pq \).

Poisson-bracket relations for \( \sigma \)-components are based on the structure constants of \( SL(n, \mathbb{R}) \). The same is obviously true for \( \hat{\sigma} \) with the proviso that the signs are reversed. The mixed \( \{\sigma, \hat{\sigma}\} \) brackets do vanish. Obviously, \( \{q, p\} = 1 \), and the quantities \( q, p \) (dilatation) Poisson-commute with \( \Psi, \sigma, \hat{\sigma} \) (shear).

The doubly-invariant ”kinetic energy” \( T \) is a superposition of the isochoric and dilatational terms,

\[
T = \frac{A}{2} \text{Tr}(\omega^2) + \frac{n(A + nB)}{2} q^2 = T_{\text{sh}} + T_{\text{dil}}.
\]

Performing the Legendre transformation, \( \sigma = A\omega, \quad p = n(A + nB)q \), we obtain the following geodetic Hamiltonian:

\[
T = \frac{1}{2A} \text{Tr}(\sigma^2) + \frac{1}{2n(A + nB)} p^2 = T_{\text{sh}} + T_{\text{dil}}.
\]

In these expressions the quantities \( \omega, \sigma \) may be replaced by their co-moving representations \( \hat{\omega}, \hat{\sigma} \). Lagrangians and Hamiltonians of systems with stabilized (controlled) dilatations have the following forms:

\[
L = L_{\text{sh}} + L_{\text{dil}} = T_{\text{sh}} + T_{\text{dil}} - V(q), \quad H = H_{\text{sh}} + H_{\text{dil}} = T_{\text{sh}} + T_{\text{dil}} + V(q).
\]
There is a complete separability of shear and dilatation degrees of freedom; they are mutually independent. This property would not be violated if we included also a shear potential \( V_{sh}(\Psi) \), i.e., if \( V(\Psi, q) = V_{sh}(\Psi) + V_{dil}(q) \). The question arises as to the structure of general solution for the geodetic \( SL(n, \mathbb{R}) \)-model, i.e., for \( V_{sh} = 0 \). A superficial reasoning based on the analogy with d’Alembert models might suggest that the general solution consists only of unbounded motions (and the rest states), because there is no potential and the configuration space is non-compact. However, it is not the case; there is an open subset consisting of bounded orbits.

Indeed, let us assume that some trace-less matrix \( \alpha \in SL(n, \mathbb{R})' \) is similar to an anti-symmetric matrix \( \lambda \in SO(n, \mathbb{R})' \), i.e., there exists such \( \chi \in SL(n, \mathbb{R}) \) that \( \alpha = \chi \lambda \chi^{-1} \). Then every motion \( \Psi(t) = \exp(\alpha t)\Psi_0 \) is bounded. Indeed, \( \exp(\lambda t) \) is a bounded subgroup of \( SO(n, \mathbb{R}) \subset SL(n, \mathbb{R}) \) and so is \( \exp(\alpha t) = \chi \exp(\lambda t)\chi^{-1} \). But similarities are globally defined continuous mappings, therefore, they transform bounded subsets onto bounded ones. Let us observe that for physical dimensions \( n = 2, 3 \) motions of this type are periodic. For higher dimensions periodicity is not necessary, although obviously possible. To see this, let us consider the simplest situation \( n = 4 \) and represent \( \mathbb{R}^4 \) as the direct sum of two complementary \( \mathbb{R}^2 \)-subspaces. Now, let \( \lambda \in SO(n, \mathbb{R})' \) be a block matrix consisting of two skew-symmetric blocks \( \nu_1 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \) and \( \nu_2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \). Rotations generated by separate blocks are obviously periodic, but the total motion is periodic if and only if \( \nu_1 / \nu_2 \in \mathbb{Q} \), i.e., the ratio of angular velocities is a rational number. If it is irrational, the subgroup obtained by exponentiation of \( t \)-multiples of the above block matrix is not a periodic function of the parameter \( t \). It is not a closed subset at all; its closure is a two-dimensional submanifold of \( SO(4, \mathbb{R}) \subset SL(4, \mathbb{R}) \). Therefore, being an algebraic subgroup of \( SO(4, \mathbb{R}) \subset SL(4, \mathbb{R}) \) it is not its Lie subgroup in the literal sense. The same concerns any subgroup of \( SL(4, \mathbb{R}) \) obtained from the above one by a similarity transformation. For an arbitrary \( n \), solutions of this kind are matrix-valued almost periodic functions of the time variable \( t \).

If \( \alpha \in SL(n, \mathbb{R})' \) is similar to a symmetric matrix \( \kappa \in SO(n, \mathbb{R})' \), \( \alpha = \chi \kappa \chi^{-1} \), then, obviously, the motion given by \( \Psi(t) = \exp(\alpha t)\Psi_0 \) is unbounded. But one can show that the previously described bounded almost periodic solutions are "stable" in such a sense that for any skew-symmetric \( \lambda \) there exists some open range of symmetric \( \kappa \)-s such that for \( \alpha = \lambda + \kappa \), or, more generally, for similar matrices \( \alpha = \chi(\lambda + \kappa)\chi^{-1} \), the corresponding solutions \( \Psi(t) = \exp(\alpha t)\Psi_0 \) are also bounded, although not necessarily almost periodic \([43, 74]\) (and not necessarily periodic in dimensions \( n = 2, 3 \)). The arbitrariness of pairs \((\alpha, \Psi_0)\) is sufficient for the corresponding family of bounded solutions to be open in the general solution manifold, thus, \( 2(n^2 - 1) \)-dimensional (topological and differential concepts like openness and dimension are meant in the sense of the manifold of initial conditions). Let us observe that this statement would be false for solutions with generators \( \alpha \) similar to skew-symmetric matrices. At first look, this might seem strange, because the structure of \( SL(n, \mathbb{R})' \) implies that the family of such \( \alpha \)-s is \((n^2 - 1)\)-dimensional and so is the set of initial configurations.
But these data are not independent and mutually interfere in the formula \( \Psi(t) = \exp(\alpha t)\Psi_0 \). Therefore, the very interesting subfamily of almost-periodic solutions is a proper subset of the manifold of all bounded solutions.

By "anti-analogy", for symmetric matrices \( \lambda = \lambda^T \in \text{SL}(n, \mathbb{R})' \), the corresponding solutions \( \Psi(t) = \exp(\lambda t)\Psi_0 \) are non-bounded (escaping in \( \text{SL}(n, \mathbb{R}) \)) and, obviously, so are the solutions generated by matrices similar to symmetric ones, \( \Psi(t) = \chi \exp(\lambda t)\chi^{-1}\Psi_0 = \exp(\chi \lambda \chi^{-1} t)\Psi_0 \). And again this property is stable with respect to small perturbations of \( \lambda \) by skew-symmetric matrices \( \epsilon = -\epsilon^T \in \text{SO}(n, \mathbb{R})' \). Therefore, the general solution of the geodetic doubly-invariant model contains also an open subset of non-bounded (escaping) solutions.

Roughly speaking, using analogy with the Kepler or attractive Coulomb problem we may tell here about motions below and above dissociation threshold, however, without any potential, just in purely geodetic models on the non-compact manifold \( \text{SL}(n, \mathbb{R}) \).

In a quantized version of this model the family of bounded classical trajectories is replaced by the discrete energy spectrum and the \( L^2 \)-class wave functions of stationary states. And similarly, the manifold of non-bounded orbits is a classical counterpart of the continuous spectrum and non-normalized wave functions (scattering situations).

Obviously, the above description in terms of groups \( \text{GL}(n, \mathbb{R}), \text{SL}(n, \mathbb{R}) \) is an analytical simplification used for computational purposes. To use systematically a more correct geometrical language we should replace the terms "skew-symmetric" and "symmetric" by \( g \)- or \( \eta \)-skew-symmetric and symmetric:

\[ \lambda^i_j = \mp g^i_{km} \lambda^m_{kj} \quad \lambda^A_B = \mp \eta^{AC} \lambda^D_C \eta_{DB}. \]

Finally, we can conclude that if dilatations are stabilized by some potential \( V_{dil}(q) \), then for the model with the kinetic energy invariant under spatial and material affine transformations, there exists a \( 2n^2 \)-dimensional family of bounded solutions even if the shear component of motion is purely geodetic. If the stabilizing dilatational potential has an upper bound, there exists also a \( 2n^2 \)-dimensional family of unbounded, escaping motions. The above arguments are based on properties of one-parameter subgroups and their cosets in \( \text{SL}(n, \mathbb{R}) \). Therefore, they do not apply directly to affine-metrical and metrical-affine models. Indeed, as we have seen, if the spatial or material symmetry of the kinetic energy is restricted to the rotation group, then, except some special solutions (relative equilibria), one-parameter subgroups and their cosets fail to be solutions. Nevertheless, our arguments may be used then in a non-direct way.

In analogy to (60) we can rewrite the kinetic Hamiltonians (41), (42), i.e., (45), (46), as follows:

\[
\mathcal{T}_{\text{int}} = \frac{1}{2(I + A)\text{Tr}(\dot{\sigma}^2)} + \frac{1}{2n(I + A + nB)}p^2 + \frac{I}{2(I^2 - A^2)}\|V\|^2 \quad (61)
\]

\[
= \frac{1}{2(I + A)C_{\text{SL}(n)}(2)} + \frac{1}{2n(I + A + nB)}p^2 + \frac{I}{2(I^2 - A^2)}\|V\|^2,
\]
\[ T_{\text{int}} = \frac{1}{2(I + A)} \text{Tr}(\sigma^2) + \frac{1}{2n(I + A + nB)}p^2 + \frac{I}{2(I^2 - A^2)}\|S\|^2 \] 

(62)

where \( C_{\text{SL}(n)}(2) = \text{Tr}(\sigma^2) = \text{Tr}(\dot{\sigma}^2) \).

The formulas (45), (46) or, equivalently, (51), (52) imply that for dilatation-stabilized models \( H = T_{\text{int}} + V_{\text{dil}}(g) \) with the affine-metrical and metrical-affine kinetic terms \( T_{\text{int}} \), all the above statements concerning bounded and unbounded solutions of affine-affine models (60), (61) remain true. In particular, for the purely geodetic incompressible models with \( T_{\text{int}} \) invariant under \( \text{SO}(U, \eta) \) or under \( \text{O}(V, g) \times \text{SL}(U) \), there exists an open subset of bounded solutions (vibrations) and an open subset of non-bounded ones. What concerns spatially affine and materially metrical models, the very rough argument is that the evolution of quantities \( \Sigma, K \) is exactly the same as it was for Hamiltonians \( H \) with \( T_{\text{int}} \) affinely-invariant both in the physical and in the material spaces, in this case in (60) \( A \) is replaced by \( I + A \). This is a direct consequence of equations of motion written in terms of Poisson brackets,

\[ \frac{dF}{dt} = \{F, H\}. \]

In fact, \( \|V\| \) is a constant of motion for both types of Hamiltonians (affine-affine and affine-metrical). In addition to Lie-algebraic relations of \( \text{GL}(V) \)' \( \simeq \text{L}(V) \) satisfied by \( \Sigma^t, j \), we have the following obvious Poisson rules:

\[ \{\Sigma^t, C(2)\} = \{\Sigma^t, C(1)\} = 0, \quad \{\Sigma^t, ||V||^2\} = 0, \quad \{\mathcal{K}_a, ||V||^2\} = 0. \]

The first equations express an obvious property of \( C(k) \) as Casimir invariants of \( \Sigma^t, j \) and \( \hat{\Sigma}^A_B \). The second formula follows from the obvious relationship \( \{\Sigma^t, \hat{\Sigma}^A_B\} = 0 \), because \( ||V||^2 \) is an algebraic function of \( \hat{\Sigma}^A_B \). And the third equation is due to the fact that the deformation invariants \( \mathcal{K}_a \) are invariant under the group of material isometries generated by \( V^A_B \).

Therefore, the time evolution of variables \( \Sigma^t, j, \mathcal{K}_a \) is identical in both types of models, i.e., (60) and (61); the former with \( A \) replaced by \( I + A \). As a matter of fact, for geodetic models with dilatation-stabilizing potentials \( V(g) \), the deviator \( \sigma^t, j = \Sigma^t, j - (1/n)\Sigma^a, a\delta^t, j \) is a constant of motion and, obviously, it is so for the purely geodetic incompressible models. The only difference occurs in degrees of freedom ruled by \( \text{SO}(V, g), \text{SO}(U, \eta) \), describing the orientation of principal axes of deformation tensors \( C \in V^* \otimes V^*, G \in U^* \otimes U^* \). But, roughly speaking, these degrees of freedom have compact topology and their evolution does not influence the bounded or non-bounded character of the total orbits.

The same reasoning applies to dilatationally stabilized geodetic models invariant under \( \text{O}(V, g) \times \text{GL}(U) \) or purely geodetic incompressible models invariant under \( \text{O}(V, g) \times \text{SL}(U) \) (spatially metrical and materially affine models). Then everything follows from Poisson brackets

\[ \{\hat{\Sigma}^A_B, C(2)\} = \{\hat{\Sigma}^A_B, C(1)\} = 0, \quad \{\hat{\Sigma}^A_B, ||S||^2\} = 0, \quad \{\mathcal{K}_a, ||S||^2\} = 0. \]
Now on the level of state variables $\hat{\Sigma}^{AB}$, $\mathcal{K}_a$ the time evolution is exactly identical with that based on the affine-affine model of $T_{\text{int}}$ (again with $A$ in (60) replaced by $I + A$).

Let us stress an important point that it is the time evolution of deformation invariants that decides whether the total motion is bounded or not. This is a purely geometric fact independent on any particular dynamical model. There is an analogy with the material point motion in $\mathbb{R}^n$. An orbit is bounded if and only if the range of the radial variable $r$ is bounded.

The above point plays an essential role in the qualitative discussion of deformative motion. It suggests one to use analytical descriptions of degrees of freedom based on deformation invariants.

### 6 Analytical description

In section 3 some fundamental facts concerning deformation tensors and deformation invariants were summarized. Below we continue this subject and present some natural descriptions of affine degrees of freedom well-adapted to the study of isotropic problems.

The material and physical spaces are endowed with fixed metric tensors, $\eta \in U^* \otimes U^*$, $g \in V^* \otimes V^*$, and any configuration $\varphi \in \text{LI}(U, V)$ gives rise to the symmetric positively definite tensors $G[\varphi] \in U^* \otimes U^*$, $C[\varphi] \in V^* \otimes V^*$, i.e., Green and Cauchy deformation tensors. Raising their first indices respectively with the help of $\eta$ and $g$, we obtain the mixed tensors $\hat{G}[\varphi] \in U \otimes U^*$, $\hat{C}[\varphi] \in V \otimes V^*$ with eigenvalues $\lambda_a$, $\lambda_a^{-1}$, $a = 1, \ldots, n$. It is also convenient to use the quantities $Q^a$, $q^a$, where $Q^a = \exp(q^a) = \sqrt{\lambda_a}$. The diagonal matrix $D = \text{diag}(Q^1, \ldots, Q^n)$ is identified with the linear mapping $D : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

The configuration $\varphi \in \text{LI}(U, V)$ may be characterized by $D$, i.e., by the system of fundamental stretchings $Q^a = \exp(q^a)$, and by the systems of eigenvectors $R_a \in U$, $L_a \in V$ of $G$, $\hat{C}$ normalized, respectively, in the sense of $\eta$ and $g$.

$$
\hat{G}R_a = \lambda_a R_a = \exp(2q^a)R_a, \quad \hat{C}L_a = \lambda_a^{-1}L_a = \exp(-2q^a)L_a.
$$

Obviously, when the spectrum is non-degenerate, then $R_a$, $L_a$ are uniquely defined (up to re-ordering) and pair-wise orthogonal, $\eta(R_a, R_b) = \eta_C R^C_a R^C_b = \delta_{ab} = g(L_a, L_a) = g_{ij} L^i_a L^j_b$. Such a situation is generic, thus, when at some time instant $t \in \mathbb{R}$ $\varphi(t)$ corresponds to degenerate situation, then $L_a(t)$, $R_a(t)$ may be also uniquely defined due to the continuity demand.

The elements of the corresponding dual bases will be denoted respectively by $R^a \in U^*$, $L^a \in V^*$. When necessary, to avoid misunderstandings, we shall indicate explicitly the dependence of the above quantities on $\varphi \in \text{LI}(U, V)$: $q^a[\varphi]$, $R_a[\varphi]$, $L_a[\varphi]$, etc.

Green and Cauchy deformation tensors may be respectively expressed as follows:

$$
G[\varphi] = \sum_a \lambda_a[\varphi] R^a [\varphi] \otimes R^a [\varphi] = \sum_a \exp(2q^a[\varphi]) R^a [\varphi] \otimes R^a [\varphi],
$$
\[ C[\varphi] = \sum_a \lambda_a^{-1}[\varphi] L^a[\varphi] \otimes L^a[\varphi] = \sum_a \exp(-2q^a[\varphi]) L^a[\varphi] \otimes L^a[\varphi]. \]

In this way \( \varphi \) has been identified with the triple of fictitious objects: two rigid bodies in \( U \) and \( V \) with configurations represented, respectively, by orthonormal frames \( R \in F(U, \eta) \), \( L \in F(V, g) \) and a one-dimensional \( n \)-particle system with coordinates \( q^a \) (or \( Q^a \)). Even for non-degenerate spectra of \( \hat{C}[\varphi] \), \( \hat{C}[\varphi] \) this representation is not unique because the labels \( a \) under the summation signs may be simultaneously permuted without affecting \( \varphi \) itself. For degenerate spectra this representation becomes continuously non-unique in a similar (although much stronger) way as, e.g., spherical coordinates at \( r = 0 \).

Let us observe that the linear frames \( L = (\ldots, L_a, \ldots) \) and \( R = (\ldots, R_a, \ldots) \) may be, as usual, identified with linear isomorphisms \( L : \mathbb{R}^n \to V \) and \( R : \mathbb{R}^n \to U \). Similarly, their dual co-frames \( \bar{L} = (\ldots, L^a, \ldots) \) and \( \bar{R} = (\ldots, R^a, \ldots) \) are equivalent to isomorphisms \( L^{-1} : V \to \mathbb{R}^n \) and \( R^{-1} : U \to \mathbb{R}^n \). Identifying the diagonal matrix \( \text{diag}(\ldots, Q_a, \ldots) \) with a linear isomorphism \( D : \mathbb{R}^n \to \mathbb{R}^n \), we may finally represent
\[ \varphi = LDR^{-1}, \]

this is a geometric description of what is sometimes referred to as the two-polar decomposition [33, 55, 57, 58, 59, 68].

Strictly speaking, in continuum mechanics, when the orientation of the body is constant during any admissible motion (no mirror-reflections), one has to fix some pattern orientations in \( U \), \( V \) and admit only orientation-preserving mappings \( \varphi \). And then the non-connected sets of all orthonormal frames \( F(U, \eta) \), \( F(V, g) \) are to be replaced by their connected submanifolds \( F^+(U, \eta) \), \( F^+(V, g) \) of positively oriented frames.

Obviously, the spatial and material orientation-preserving isometries \( A \in \text{SO}(V, g) \), \( B \in \text{SO}(U, \eta) \) affect only the \( L \)- and \( R \)-gyrosopes on the left. Indeed, \( L \mapsto AL, R \mapsto BR \) result in \( \varphi \mapsto A\varphi B^{-1} \). Their Hamiltonian generators, spin and minus-vorticity (i.e., respectively \( V \)- and \( U \)-spatial canonical spins) have identical Poisson-commutation rules.

For any of the mentioned rigid bodies, one can define in the usual way the angular velocity in two representations. One should stress that both \( V \) and \( U \) are from this point of view interpreted as "physical spaces". The "material" ones are both identified with \( \mathbb{R}^n \). The "co-moving" and "current" representations \( \hat{\chi} \in \text{SO}(n, \mathbb{R})' \), \( \chi \in \text{SO}(V, g)' \) for the \( L \)-top are respectively given by
\[ \hat{\chi}^a_b := \langle L^a, \frac{dL^b}{dt} \rangle = L^a, \frac{dL^b}{dt}, \quad \chi := \hat{\chi}^a_b L_a \otimes L^b, \text{ i.e., } \chi^i_j = \frac{dL^i_a}{dt} L^a_j. \]

The corresponding objects \( \hat{\vartheta} \in \text{SO}(n, \mathbb{R})' \), \( \vartheta \in \text{SO}(U, \eta)' \) for the \( R \)-top are defined by analogous formulas:
\[ \hat{\vartheta}^a_b := \langle R^a, \frac{dR^b}{dt} \rangle = R^a, \frac{dR^b}{dt}, \quad \vartheta := \hat{\vartheta}^a_b R_a \otimes R^b, \text{ i.e., } \vartheta^K_L = \frac{dR^K_a}{dt} R^a_L. \]

In certain problems it is convenient to use non-holonomic velocities \( \hat{\vartheta}^a, \hat{\chi}^a_b, \hat{\vartheta}^a_b \) or \( \hat{q}^a, \chi^i_j, \vartheta^A_B \). Similarly, non-holonomic conjugate momenta \( p_a, \hat{p}^a_b, \hat{p}^a_b \) or \( p_a, \).
\( \rho^i, \tau^A \) are used, where again \( \hat{\rho}, \hat{\tau} \in SO(n, \mathbb{R})' \), \( \rho \in SO(V, g)' \), \( \tau \in SO(U, \eta)' \).

The pairing between non-holonomic momenta and velocities is given by

\[
< (\rho, \tau, p), (\chi, \vartheta, \dot{q}) > = p_a q^a + \frac{1}{2} \text{Tr}(\rho \chi) + \frac{1}{2} \text{Tr}(\tau \vartheta) = \frac{1}{2} \text{Tr}(\hat{\rho} \hat{\chi}) + \frac{1}{2} \text{Tr}(\hat{\tau} \hat{\vartheta}).
\]

**Remark:** Our system of notations is slightly redundant, because \( \rho \) and \( \tau \) coincide, respectively, with spin and negative vorticity, \( \rho = S, \tau = -V \). The reason is that they are Hamiltonian generators of transformations \( \varphi \mapsto A \varphi \), \( \varphi \mapsto \varphi B^{-1} \), \( A \in SO(V, g) \), \( B \in SO(U, \eta) \).

The objects \( \hat{\rho}, \hat{\tau} \) generate transformations

\[
L \mapsto LA, \quad R \mapsto RB, \quad A, B \in SO(n, \mathbb{R})
\]

and express the quantities \( \rho, \tau \) in terms of the reference frames given, respectively, by the principal axes of the Cauchy and Green deformation tensors,

\[
\rho = \hat{\rho}^a b L_a \otimes L^b, \quad \tau = \hat{\tau}^a b R_a \otimes R^b.
\]

**Remark:** In dynamical models based on the d’Alembert principle the quantities \( Q^a \) and their conjugate momenta \( P_a \) are more convenient than \( q^a \) and \( p_a \). The latter ones are useful in models with affinely-invariant kinetic energy.

If \( V \) and \( U \) both are identified with \( \mathbb{R}^n \) and \( LI(U, V) \) with \( GL(n, \mathbb{R}) \), then \( L \) and \( R \) in the two-polar splitting \( \varphi = L D R^{-1} \) become elements of \( SO(n, \mathbb{R}) \) and \( D \), as previously, is a diagonal matrix with positive elements. The two-polar decomposition is a by-product of the polar decomposition of \( GL^+(n, \mathbb{R}) \),

\[
\varphi = U A,
\]

where \( U \in SO(n, \mathbb{R}) \), thus, \( U^T = U^{-1} \), and \( A = A^T \) is a symmetric positively-definite matrix. It is well-known that this decomposition is unique, whereas the two-polar one is charged with some multivaluedness. Green and Cauchy deformation tensors are then represented as follows: \( G = \varphi^T \varphi = A^2, \ C = (\varphi^{-1})^T \varphi^{-1} = U A^{-2} U^{-1} \). One can also use the reversed polar decomposition

\[
\varphi = BU, \quad U \in SO(n, \mathbb{R}), \quad B = U A U^{-1} = B^T.
\]

Then \( G = U^{-1} B^2 U, \ C = B^{-2} \). The two-polar decomposition is achieved by the orthogonal diagonalization of the matrix \( A, \ A = V D V^{-1}, \ V \in SO(n, \mathbb{R}) \). Then \( L = U V, \ R = V \).

The polar splitting was described above in an over-simplified standard way, namely, \( U \) and \( V \) were identified with \( \mathbb{R}^n \) and \( LI(U, V) \) with \( GL(n, \mathbb{R}) \). Let us remind that in continuum mechanics the connected components of \( LI(U, V) \) and \( GL(n, \mathbb{R}) \) are used as configuration spaces, \( LI^+(U, V), \ GL^+(n, \mathbb{R}) \), where the first symbol denotes the manifold of orientation-preserving isomorphisms (it is assumed here that some orientations in \( U, V \) are fixed). It is instructive to
see what the both polar splittings are from the geometric point of view, when $U$ and $V$ are distinct linear spaces, non-identified with $\mathbb{R}^n$.

As mentioned above, when metric tensors $\eta \in U^* \otimes U^*$, $g \in V^* \otimes V^*$ are fixed, then any $\varphi \in \text{LI}(U, V)$ with non-degenerate spectra of deformation tensors gives rise to the pair of orthonormal bases $(L_\alpha[\varphi] \in V, \alpha = \overline{1,n})$, $(R_\alpha[\varphi] \in U, \alpha = \overline{1,n})$. There exists exactly one isometry $U[\varphi] : U \to V$ such that $U[\varphi] \cdot R_\alpha[\varphi] = L_\alpha[\varphi]$. Obviously, the isometry property is meant in the sense that $\eta = U[\varphi]^* \cdot g$, i.e., analytically $\eta_{AB} = g_{ij} U[\varphi]^i A U[\varphi]^j B$. Geometric meaning of the polar decomposition is as follows:

$$\varphi = U[\varphi] A[\varphi] = B[\varphi] U[\varphi],$$

where the automorphisms $A[\varphi] \in \text{GL}(U)$, $B[\varphi] \in \text{GL}(V)$ are symmetric, respectively, in the $\eta$- and $g$-sense, i.e., $\eta(A[\varphi] x, y) = \eta(x, A[\varphi] y)$, $g(B[\varphi] w, z) = g(w, B[\varphi] z)$ for arbitrary $x, y \in U$, $w, z \in V$. They are also positively definite, $\eta(A[\varphi] x, x) > 0$, $g(B[\varphi] w, w) > 0$ for arbitrary non-null $x \in U$, $w \in V$.

In spite of the non-uniqueness contained in $L[\varphi]$, $R[\varphi]$, the mappings $U[\varphi]$, $A[\varphi]$, $B[\varphi]$ are unique. And the symmetric parts are obtained from each other by the $U[\varphi]$-intertwining, $B[\varphi] = U[\varphi] A[\varphi] U[\varphi]^{-1}$.

In mechanics of discrete affine systems we are free to admit orientation-reversing isometries $U$ or symmetric mappings $A$, $B$ not necessarily positively-definite.

The non-uniqueness of the two-polar decomposition mentioned above is important in certain computational and also principal problems, so some comments are necessary here. The problem is technically complicated, thus, only necessary facts will be quoted here, some of them formulated in a rather brief, rough way.

The subgroup of $O(n, \mathbb{R})$ consisting of matrices which have exactly one non-vanishing entry in every row and column will be denoted by $K$. Obviously, $K$ is finite and the mentioned entries are $\pm 1$, reals with absolute value 1. The subgroup of proper $K$-rotations will denoted by $K^+ := K \cap SO(n, \mathbb{R})$. Obviously, the orders (numbers of elements) of $K$, $K^+$ equal respectively $2n \cdot n!$ and $n \cdot n!$. Let $W \in K$ be a corresponding similarity transformation preserving the group of diagonal matrices, $\text{Diag}(\mathbb{R}^n) \ni D \mapsto W^{-1} D W \in \text{Diag}(\mathbb{R}^n)$, and resulting in permutation of the diagonal elements of $D = \text{diag}(Q^1, \ldots, Q^n)$, i.e., we have $(Q^1, \ldots, Q^n) \mapsto (Q^{\pi w(1)}, \ldots, Q^{\pi w(n)})$ or $(q^1, \ldots, q^n) \mapsto (q^{\pi w(1)}, \ldots, q^{\pi w(n)})$, where $Q^w = \exp(q^w)$. Obviously, the mapping $K \ni W \mapsto \pi_W \in S(n)$ is a $2n : 1$ epimorphism of $K$ onto the permutation group $S(n)$. Its restriction to $K^+$ has an $n$-element kernel. The non-uniqueness of representation of $\varphi \in \text{GL}^+(n, \mathbb{R})$ through elements of $SO(n, \mathbb{R}) \times \mathbb{R}^n \times SO(n, \mathbb{R})$ depends strongly on the degeneracy of spectra of deformation tensors. The multi-valuedness is discrete, thus, simplest in the case of simple spectra.

Let $\text{GL}^+(n, \mathbb{R}) \subset \text{GL}^+(n, \mathbb{R})$ be the subset of $\varphi$-s with non-degenerate spectra of $C$, $G$. The corresponding subset $M(n)$ of $SO(n, \mathbb{R}) \times \mathbb{R}^n \times SO(n, \mathbb{R})$ consists of such triplets $(L; q^1, \ldots, q^n; R)$ that all $q^i$-s are pairwise distinct. The group $K^+$ may be faithfully realized by the following transformation group $H(n)$ of $M(n)$:

$$(L; q^1, \ldots, q^n; R) \mapsto (LW; q^{\pi w(1)}, \ldots, q^{\pi w(n)}; RW).$$
Obviously, this transformation does not affect $\varphi = LDR^{-1}$. Therefore, we have a diffeomorphism

$$\text{GL}^+(n, \mathbb{R}) \simeq M^{(n)}/H^{(n)}.$$ 

Non-degenerate spectrum is a generic one, nevertheless the coincidence case must be also taken into account because some new qualities appear then and they are relevant for qualitative analysis of classical phase portraits and for quantum conditions on admissible wave functions.

Let $\text{GL}^{+(k:p_1,\ldots,p_k)} \subset \text{GL}^+(n, \mathbb{R})$ consist of $\varphi$-s for which deformation tensors have $k \leq n$ different principal values, every one of them with the corresponding multiplicity $p_\sigma$. $\sum_{\sigma=1}^k p_\sigma = n$. And similarly, let $M^{(k:p_1,\ldots,p_k)}$ be the set of such triplets $(L; q^1, \ldots, q^n; R) \in \text{SO}(n, \mathbb{R}) \times \mathbb{R}^n \times \text{SO}(n, \mathbb{R})$ that there are only $k$ different $q^i$-s with the same conditions concerning multiplicity. And now let us consider the transformation group $H^{(k:p_1,\ldots,p_k)}$ acting on $M^{(k:p_1,\ldots,p_k)}$ as follows:

$$(L; q^1, \ldots, q^n; R) \mapsto (LW; q^{\pi w(1, \ldots, q^n)}; RW),$$

where $W$ runs over the subgroup of $\text{SO}(n, \mathbb{R})$ that is generated by $K^+$ and the subgroup $H^{(k:p_1,\ldots,p_k)} \subset \text{SO}(n, \mathbb{R})$ composed of $k$ blocks $p_\sigma \times p_\sigma$, every one given by the corresponding $\text{SO}(p_\sigma, \mathbb{R})$. Then we have that

$$\text{GL}^{+(k:p_1,\ldots,p_k)} \simeq M^{(k:p_1,\ldots,p_k)}/H^{(k:p_1,\ldots,p_k)}.$$ 

When $k < n$, then at least one of multiplicities is non-trivial and the resulting group $H^{(k:p_1,\ldots,p_k)}$ is continuous. The resulting quotient is lower-dimensional because of this continuity of the divisor transformation group.

In the physical case $n = 3$, we have obviously only two possibilities of the non-trivial blocks, namely the total $\text{SO}(3, \mathbb{R})$ and $\text{SO}(2, \mathbb{R}) \times \text{SO}(1, \mathbb{R})$ (respectively, all three $q^i$’s equal or two of them); obviously $\text{SO}(1, \mathbb{R}) = \{1\}$.

In the extreme case $k = 1, D$ is proportional to the $n \times n$ identity matrix and it is only the total $LR^{-1}$ that is well-defined; on the other hand, $L, R$ separately are meaningless.

It is very convenient and instructive to express our Hamiltonians, kinetic energies and configuration metrics in terms of the two-polar splitting. The previous statements concerning the phase pictures become then much more lucid. Let us introduce some auxiliary quantities $M := -\dot{\rho} - \dot{\tau}$, $N := \dot{\rho} - \dot{\tau}$. One can easily show that the second-order Casimir invariant $C(2)$ occurring in the main terms of our affine-affine, affine-metrical and metrical-affine kinetic Hamiltonians has the following form:

$$C(2) = \sum_a p_a^2 + \frac{1}{16} \sum_{a,b} \frac{(M^a b)^2}{\text{sh}^2 q^a - q^a} - \frac{1}{16} \sum_{a,b} \frac{(N^a b)^2}{\text{ch}^2 q^a - q^a}. \quad (64)$$

Obviously, $M$ and $N$ are antisymmetric in the Kronecker-delta sense, $M^a b = -M^a b = -g_{bk}^a M^k l$, $N^a b = -N^a b = -g_{bk}^a N^k l$. The first term in (64) may be suggestively decomposed into the ”relative” and the ”over-all” ("centre of mass") parts:

$$\frac{1}{2n} \sum_{a,b} (p_a - p_b)^2 + \frac{p^2}{n}.$$
Obviously, $C(1) = p$.

For geodetic systems and for more general systems with potentials $V$ depending only on deformation invariants, spin $S = \rho$ and vorticity $V = \tau$ are constants of motion and may be used for extracting from equations of motion some information concerning the general solution. Unlike this the quantities $\hat{\rho}$, $\hat{\tau}$, thus, also $M$, $N$, fail to be constants of motion except the special case $n = 2$, when the rotation group is Abelian. However, on the level of qualitative analysis, the expression (64) based on $\hat{\rho}$, $\hat{\tau}$ is more convenient because it does not involve $L$, $R$-variables, i.e., rotational degrees of freedom of deformation tensors. Therefore, our Poisson bracket relations imply that on the level of variables $q^a$, $p_a$, $M^a_b$, $N^a_b$ equations of motion based on (61) (equivalently (45)), (62) (equivalently (46)), and (60) with $A$ replaced by $I + A$ are identical. In particular, for geodetic incompressible models and for compressible models with stabilized dilatations there exists an open family of bounded (vibrating) solutions and an open family of non-bounded (decaying) solutions. The reason is that it is so for (60) with $A$ replaced by $I + A$, and the additional terms proportional to $S^2$ or $V^2$ do not influence anything because they have vanishing Poisson brackets with $q^a$, $p_a$, $M^a_b$, $N^a_b$ and only those variables occur in $H$. The only difference appears when the evolution of $L$- and $R$-variables is taken into account. However, the corresponding configuration spaces $F(V, g)$, $F(U, \eta)$ are compact (they are manifolds of orthonormal frames) and do not influence the boundedness of orbits.

Let us observe that after substituting (64), the first main term of (61) (equivalently (45)), (62) (equivalently (46)), and (60) with $A$ replaced by $I + A$ acquires the characteristic lattice structure,

$$T_{\text{latt}} = \frac{1}{2\alpha} \sum_a p_a^2 + \frac{1}{32\alpha} \sum_{a,b} (M^a_b)^2 - \frac{1}{32\alpha} \sum_{a,b} (N^a_b)^2.$$ 

This expression resembles structurally the hyperbolic Sutherland $n$-body system on the straight line. Positions of the fictitious material points are given by deformation invariants $q^a$. The "particles" have identical masses and are indistinguishable. Unlike the hyperbolic Sutherland system, the coupling amplitudes $M^a_b$, $N^a_b$ are non-equal and non-constant; rather they are dynamical variables on the equal footing with $q^a$, $p_a$. The negative $N$-contribution to $T_{\text{latt}}$ describes the attractive forces between lattice points, whereas the positive $M$-term corresponds to repulsion. Under the appropriate initial conditions we have stable bounded vibrations without any use of the potential energy term. Therefore, the non-definiteness of $T_{\text{latt}}$ is not only non-embarrassing, but just desirable as a tool for describing "elastic" vibrations on the basis of purely geodetic models. Let us observe that the purely affine-affine part of (61), (62) (equivalently (45), (46)), i.e., (60) with $A$ replaced by $I + A$ (composed of its first two Casimir terms), splits in the following suggestive way into the binary $\text{SL}(n, \mathbb{R})$-part and dilatational contribution:

$$T_{\text{aff int}} = \frac{1}{2\alpha} C(2) + \frac{1}{2\beta} C(1)^2 = \frac{1}{4\alpha n} \sum_{a,b} (p_a - p_b)^2.$$ 

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\[
+ \frac{1}{32\alpha} \sum_{a,b} \frac{(M_{ab})^2}{\text{sh}^2 \frac{q_a - q_b}{2}} - \frac{1}{32\alpha} \sum_{a,b} \frac{(N_{ab})^2}{\text{ch}^2 \frac{q_a - q_b}{2}} + \frac{n\alpha + \beta}{2n\alpha\beta} p^2,
\]

or, in a more explicit form,

\[
T_{\text{aff}}^\text{int} = \frac{1}{4(I + A)} \sum_{a,b} (p_a - p_b)^2 + \frac{1}{32(I + A)} \sum_{a,b} \frac{(M_{ab})^2}{\text{sh}^2 \frac{q_a - q_b}{2}}
- \frac{1}{32(I + A)} \sum_{a,b} \frac{(N_{ab})^2}{\text{ch}^2 \frac{q_a - q_b}{2}} + \frac{1}{2n(I + A + nB)} p^2.
\] (65)

Obviously, for (61) (equivalently (45)) and (62) (equivalently (46)) we have, respectively,

\[
T_{\text{aff} - \text{metr}}^\text{int} = T_{\text{aff}}^\text{int} + \frac{I}{2(I^2 - A^2)} \|V\|^2,
\] (66)

\[
T_{\text{metr} - \text{aff}}^\text{int} = T_{\text{aff}}^\text{int} + \frac{I}{2(I^2 - A^2)} \|S\|^2.
\] (67)

Comparing this with (61), (62), we conclude that

\[
C_{\text{SL}(n)}(2) = \text{Tr}(\sigma^2) = \text{Tr}(\hat{\sigma}^2)
= \frac{1}{2n} \sum_{a,b} (p_a - p_b)^2 + \frac{1}{16} \sum_{a,b} \frac{(M_{ab})^2}{\text{sh}^2 \frac{q_a - q_b}{2}} - \frac{1}{16} \sum_{a,b} \frac{(N_{ab})^2}{\text{ch}^2 \frac{q_a - q_b}{2}}.
\]

This expression is very suggestive because it expresses the quantity \(C_{\text{SL}(n)}(2)\) and the corresponding contribution to \(T_{\text{int}}\), i.e., the metric tensor on the manifold of incompressible motions, as the sum of \(n(n - 1)/2\) two-dimensional clusters, i.e., \(\mathbb{R}^2\)-coordinate planes in \(\mathbb{R}^n\). Incompressibility is expressed by the fact that the invariants \(q^a\) and their conjugate momenta \(p_a\) enter the above formula through the shape-describing differences \((q^a - q^b)\) (ratios \(Q^a/Q^b\) and \(p_a - p_b\). This expression may be very convenient when studying invariant geodetic models on the projective group \(\text{Pr}(n, \mathbb{R})\), i.e., when dealing with the mechanics of projectively-rigid bodies (bodies subject to such constraints that all geometric relationships of projective geometry are preserved, in particular, the material straight lines remain straight lines). The point is that \(\text{Pr}(n, \mathbb{R})\) may be identified in a standard way with \(\text{SL}(n + 1, \mathbb{R})\).

For the d’Alembert model the two-polar splitting leads to the following kinetic Hamiltonian term:

\[
T_{\text{int}} = \frac{1}{2I} \sum_a p_a^2 + \frac{1}{8I} \sum_{a,b} \frac{(M_{ab})^2}{(Q^a - Q^b)^2} + \frac{1}{8I} \sum_{a,b} \frac{(N_{ab})^2}{(Q^a + Q^b)^2}.
\] (68)

It is purely repulsive on the level of \(Q\)-variables, thus, without any potential term it is non-realistic as a model of elastic vibrations. It is related to the Calogero-Moser lattices similarly as the previous models show some kinship with the hyperbolic Sutherland lattices [61, 40, 41, 57, 58, 59, 68, 70].
What concerns affine models, we can compactify deformation invariants \( q^a \) by taking them modulo \( 2\pi \) (\( n \)-dimensional torus), i.e., by putting formally \( Q^a = \exp(iq^a) \). This is equivalent to replacing \( \text{GL}(n, \mathbb{R}) \) by \( \text{U}(n) \), i.e., another and completely opposite real form of \( \text{GL}(n, \mathbb{C}) \). The Lie algebra \( \text{U}(n) \) consists of anti-Hermitian matrices, and the positively definite kinetic energy may be postulated in the following form:

\[
T_{\text{int}} = -\frac{A}{2} \text{Tr}(\Omega^2) - \frac{B}{2} (\text{Tr}\Omega)^2 = \frac{A}{2} \text{Tr}(\Omega^+\Omega) + \frac{B}{2} \text{Tr}(\Omega^+)\text{Tr}(\Omega),
\]

where \( \Omega = (d\varphi/dt)\varphi^{-1} \), \( A > 0 \), \( B > 0 \). Obviously, in this expression for \( T \), \( \Omega \) may be as well replaced by \( \hat{\Omega} = \varphi^{-1}(d\varphi/dt) \).

Using again the "two-polar" decomposition \( \varphi = LDR^{-1} \), where \( L, R \in \text{SO}(n, \mathbb{R}), \ D = \text{diag}(\ldots, \exp(iq^a), \ldots) \), one obtains for the geodetic Hamiltonian:

\[
T_{\text{int}} = \frac{1}{2A} \sum_a p_a^2 + \frac{1}{32A} \sum_{a,b} \frac{(M^a_b)^2}{\sin^2 \frac{q^a - q^b}{2}} + \frac{1}{32A} \sum_{a,b} \frac{(N^a_b)^2}{\cos^2 \frac{q^a - q^b}{2}} - \frac{B}{2A(A+nB)}p^2.
\]

(69)

The first three terms, corresponding to the \( C(2) \)-Casimir, resemble the usual Sutherland lattice for \( q \)-particles with the same provisos as previously. Geodetic motion is bounded, because \( \text{U}(n) \) is compact. Just as previously, it may be convenient to use the splitting into \( \text{SU}(n) \)- and \( \text{U}(1) \)-terms,

\[
T_{\text{int}} = \frac{1}{4An} \sum_{a,b} (p_a - p_b)^2 + \frac{1}{2n(A+nB)}p^2
\]

\[
+ \frac{1}{32A} \sum_{a,b} \frac{(M^a_b)^2}{\sin^2 \frac{q^a - q^b}{2}} + \frac{1}{32A} \sum_{a,b} \frac{(N^a_b)^2}{\cos^2 \frac{q^a - q^b}{2}}.
\]

And, in particular,

\[
C_{\text{SU}(n)}(2) = \frac{1}{2n} \sum_{a,b} (p_a - p_b)^2 + \frac{1}{16} \sum_{a,b} \frac{(M^a_b)^2}{\sin^2 \frac{q^a - q^b}{2}} + \frac{1}{16} \sum_{a,b} \frac{(N^a_b)^2}{\cos^2 \frac{q^a - q^b}{2}}.
\]

The binary structures of \( C_{\text{SL}(n,\mathbb{R})}(2) \) and \( C_{\text{SU}(n)}(2) \) and their dependence on the variables \( q^a, p_a \) through their differences \( q^a - q^b, p_a - p_b \) is geometrically interesting in itself. The splitting into \( \text{SL}(2, \mathbb{R}) \)- and \( \text{SU}(2) \)-clusters corresponding to all possible coordinate planes \( \mathbb{R}^2 \) in \( \mathbb{R}^n \) may be also analytically helpful. However, some sophisticated mathematical techniques would be necessary then, like, e.g., the Dirac procedure for degenerate/constrained system. The point is that, in general, different clusters are not analytically independent. And any procedure based on some ordering of variables destroys the explicit binary structure and makes the structure of \( T \) rather obscure.

It is interesting that the general solution of \( C(2) \)-based geodetic models contains as a particular subfamily the general solution of the mentioned Calogero-Moser and Sutherland models. It is obtained by putting \( N^a_b = 0 \), and all \( M^a_b \) with \( b \neq a \) equal to some fixed constant \( M \).
As mentioned, we are particularly interested in geodetic affine models. Nevertheless, it is instructive to admit a wider class of Hamiltonians:

\[ H = T + V(q^1, \ldots, q^n), \]  

(70)

where \( T \) is any of the kinetic energy models described above, and the potential \( V \) depends on \( \varphi \) only through the deformation invariants \( q^a \). This means that it is isotropic both in the physical and material space. The mentioned non-uniqueness of the two-polar decomposition implies that \( V \) as a function on \( \mathbb{R}^n \) must be permutation-invariant to represent a well-defined function on the configuration space. When the extra potential, e.g., elastic one, is admitted, then also the "usual" model of \( T \) based on the d’Alembert principle may be sensibly used for describing bounded elastic vibrations. Therefore, from now on all the above models of \( T \) (65), (66), (67), (68) are admitted, although of course the "non-usual" affine models (65), (66), (67) are still particularly interesting for us.

As mentioned, the most convenient way of discussing and solving equations of motion is that based on Poisson brackets,

\[ \frac{dF}{dt} = \{F, H\}, \]

where \( F \) runs over some maximal system of (functionally) independent functions on the phase space. The most convenient and geometrically distinguished choice is \( q^a, p_a, M^a_b, N^a_b, L, R \) or, more precisely, some coordinates on \( \text{SO}(n, \mathbb{R}) \) parameterizing \( L \) and \( R \). In d’Alembert models \( Q^a, P_a \) are more convenient than \( q^a, p_a \).

An important point is that \( q^a, p_a, M^a_b, N^a_b \) generate some Poisson subalgebra, because their Poisson brackets may be expressed by them alone without any use of \( L, R \)-variables. And Hamiltonians also depend only on \( q^a, p_a, M^a_b, N^a_b \), whereas \( L, R \) are non-holonomically cyclic variables. This enables one to perform a partial reduction of the problem. In fact, the following subsystem of equations is closed:

\[
\begin{align*}
\frac{dq^a}{dt} &= \{q^a, H\} = \frac{\partial H}{\partial p_a}, \\
\frac{dM^a_b}{dt} &= \{M^a_b, H\} = \{M^a_b, M^c_d\} \frac{\partial H}{\partial M^c_d} + \{M^a_b, N^c_d\} \frac{\partial H}{\partial N^c_d}, \\
\frac{dp_a}{dt} &= \{p_a, H\} = -\frac{\partial H}{\partial q^a}, \\
\frac{dN^a_b}{dt} &= \{N^a_b, H\} = \{N^a_b, M^c_d\} \frac{\partial H}{\partial M^c_d} + \{N^a_b, N^c_d\} \frac{\partial H}{\partial N^c_d}.
\end{align*}
\]

Obviously, \( \{q^a, p_b\} = \delta^a_b \), \( \{q^a, M^c_d\} = \{p_a, M^c_d\} = \{q^a, N^c_d\} = \{p_a, N^c_d\} = 0 \). Poisson brackets of \( M, N \)-quantities follow directly from those for \( \hat{\rho}, \hat{\tau} \), and the latter ones correspond exactly to the structure constants of \( \text{SO}(n, \mathbb{R}) \), thus,

\[ \{\hat{\rho}_{ab}, \hat{\rho}_{cd}\} = \hat{\rho}_{ad} \delta_{cb} - \hat{\rho}_{cb} \delta_{ad} + \hat{\rho}_{db} \delta_{ac} - \hat{\rho}_{ac} \delta_{db}. \]
The subsystem for \((q^a, p_a, M_{ab}, N_{ab})\) may be in principle autonomously solvable. When the time dependence of \(\hat{\rho} = (N - M)/2\) and \(\hat{\tau} = -(N + M)/2\) is known, then performing the inverse Legendre transformation we can obtain the time dependence of angular velocities \(\hat{\chi}, \hat{\vartheta}\):

\[
\hat{\chi}^a_b = \frac{\partial H}{\partial \hat{\rho}^a}, \quad \hat{\vartheta}^a_b = \frac{\partial H}{\partial \hat{\tau}^a}.
\]

(some care must be taken when differentiating with respect to skew-symmetric matrices). And finally the evolution of \(L, R\) is given by the following time-dependent systems:

\[
\frac{dL}{dt} = L \hat{\chi}, \quad \frac{dR}{dt} = R \hat{\vartheta}.
\]

There is some very important consequence of this reduction procedure, i.e., in doubly-isotropic models spin \(S\), vorticity \(V\), and their magnitudes \(\|S\|, \|V\|\) are constants of motion. Moreover, \(\|S\|\) and \(\|V\|\) have vanishing Poisson brackets with all quantities \(q^a, p_a, M_{ab}, N_{ab}\). Therefore, on the level of these variables, all Hamiltonian systems with the same doubly isotropic potential \(V\) and with three affine models of the kinetic energy \(65, 66, 67\) are identical. In particular, the solutions for variables \(q^a, p_a, M^a_b, N^a_b\) coincide with those for the affine-affine model \(65\). And this applies, in particular, to the geodetic model (when \(V = 0\)) and to the geodetic shear model with extra imposed dilatations stabilized by \(V_{\text{dil}}(q)\), where \(q = (q^1 + \cdots + q^n)/n\). And then, as mentioned, the argument about one-parameter subgroups and their cosets decides about the existence of open subsets of bounded and non-bounded trajectories. The only difference between various \(T\)-models appears only on the level of \(L, R\)-degrees of freedom. But the compactness of the corresponding configuration spaces \(F(V, g), F(U, \eta)\) implies that this part of motion does not influence the property of the total orbits in \(Q = LI(U, V)\) to be bounded or non-bounded. One should stress that for the affine-metrical and metrical-affine geodetic models \(65\), \(66\), \(67\) only exceptional solutions are given by one-parameter subgroups and their cosets (relative equilibria). Nevertheless, extracting from all possible one-parameter subgroups and their cosets their \((q^a, p_a, M^a_b, N^a_b)\)-content, we obtain true statements concerning all three geodetic models \(65, 66, 67\).

Our affine geodetic models \(65\), \(66\), \(67\) have a nice binary structure with an additional degree of freedom related to the motion of the centre \(q \in T, n\). In practical applications this term in \(T_{\text{int}}\) should be stabilized by some extra introduced dilatational potential. If we perturb geodetic models by admitting more general doubly-isotropic potentials,
then it follows from the mentioned structure of $T_{\text{int}}$ that the most natural and computationally effective potentials will be those somehow adapted to the above splitting into shear and dilatation parts, i.e.,

$$V(q^1, \ldots, q^n) = V_{\text{dil}}(q) + \frac{1}{2} \sum_{i,j} V_{\text{sh}}^{ij}(|q^i - q^j|).$$

Here the additional shear part is not only binary but, just as it should be, it is depending only on the relative positions of deformation invariants $|q^i - q^j|$ on $\mathbb{R}$ or on the circle $U(1)$ when the group $U(n)$ is used. Obviously, the model of $V_{\text{sh}} = (1/2) \sum_{i,j} V_{\text{sh}}^{ij}(|q^i - q^j|)$ will be computationally effective only when the structure of functions $V_{\text{sh}}^{ij}$ will have something to do with $\text{sh} \left(\frac{|q^i - q^j|}{2}\right)$, $\text{ch} \left(\frac{|q^i - q^j|}{2}\right)$, $\text{sin} \left(\frac{|q^i - q^j|}{2}\right)$, $\text{cos} \left(\frac{|q^i - q^j|}{2}\right)$.

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