A new proof of the local criterion of flatness

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Abstract

Let \((A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)\) be a local morphism of local noetherian rings and \(M\) a finitely generated \(B\)-module. Then it follows from \(\text{Tor}^1_A(M, A/\mathfrak{m}_A) = 0\) that \(M\) is a flat \(A\)-module. This is usually called the "local criterion of flatness". We give a proof that proceeds along different lines than the usual textbook proofs, using completions and only elementary properties of flat modules and the \(\text{Tor}\)-functor.

1 A new proof of the local criterion of flatness

1.1 Introduction

It is well known, that for a finite \(A\)-module \(M\) over a noetherian local ring \(A\) the truth of \(\text{Tor}^1_A(M, A/\mathfrak{m}_A) = 0\) implies that \(M\) is \(A\)-flat.

Under some assumptions on \(M\) this can be generalized to the case where \(M\) is no longer finitely–generated over \(A\). The respective theorems are often called "local criterion of flatness" (see for example [4, (20.C) Theorem 49], [2, Theorem 6.8])

The proof here given proceeds along different lines than the proofs cited above, making essential use of completions and only the simplest properties of the \(\text{Tor}\)-functor and of flat modules.

For all results in commutative algebra we refer to [2], [4] and [1].

1.2 An introductory lemma

**Lemma 1.1** Let \((A, \mathfrak{m}_A)\) be an artinian local ring and \(M\) an \(A\)-module. Then the following assertions are equivalent

i) \(M\) is free.

ii) \(M\) is projective.
iii) $M$ is $A$-flat.

iv) $\text{Tor}_1^A(M, k_A) = 0$, where $k_A = A/m_A$.

**Proof.** We prove iv) $\Rightarrow$ i). First by considering the sequence

$$0 \to m^p/m^{p+1} \to A/m^{p+1} \to A/m^p \to 0$$

one concludes that

$$\text{Tor}_1^A(M, A/m^p) = 0$$

for all $p \geq 0$.

Now assume we have found inductively an isomorphism:

$$\gamma : F \otimes_A A/m^p \xrightarrow{\sim} M \otimes_A A/m^p$$

where $F = \bigoplus_{i \in I} A$ is a free $A$–module.

First construct a small commutative diagram of $A$–modules

\[\begin{array}{ccc}
0 & \xrightarrow{\gamma} & M \otimes_A A/m^p \\
\downarrow & & \downarrow \\
F & \otimes_A A/m^p & \xrightarrow{\gamma} M \otimes_A A/m^p \\
\downarrow \zeta & & \downarrow \\
0 & & 0
\end{array}\]

By successive tensoring we can construct a diagram:

\[\begin{array}{ccccccc}
0 & & 0 & & 0 \\
\downarrow & & & & & & \downarrow \\
0 & \xrightarrow{\beta} & F \otimes_A A/m^p & \xrightarrow{\gamma} M \otimes_A A/m^p & \xrightarrow{\alpha} & 0 & \xrightarrow{\phi} 0 \\
\downarrow & & \downarrow \phi & & \downarrow & & \downarrow \\
0 & \xrightarrow{\delta} & F \otimes_A A/m^{p+1} & \xrightarrow{\delta} M \otimes_A A/m^{p+1} & \xrightarrow{\beta} P & \xrightarrow{\psi} 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \xrightarrow{\chi} & F \otimes_A m^p/m^{p+1} & \xrightarrow{\chi} M \otimes_A m^p/m^{p+1} & \xrightarrow{\phi} 0 & \xrightarrow{\psi} 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array}\]

Note, especially, that $\chi = \zeta \otimes_A \text{id}_{k_A} \otimes_{k_A} \text{id}_{m^p/m^{p+1}}$ is an isomorphism, because $\zeta \otimes_A \text{id}_{k_A}$ is one.
Now by the snake lemma, for example, follows \( L = 0 \) and \( P = 0 \). So we can conclude from the fact that \( \gamma \) is an isomorphism, that \( \delta \) is an isomorphism too.

So, by induction, \( M \otimes_A A/m^p \) is a free \( A/m^p \)-module for all \( p \geq 0 \). As \( A \) is artinian and \( m^{p_0} = 0 \) for a certain \( p_0 > 0 \), we have that \( M \) is \( A \)-free.

### 1.3 The main theorem

**Theorem 1.1** Let \( (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B) \) be a local morphism of noetherian local rings. Further let \( Y \) be a finitely generated \( B \)-module. Then the following assertions are equivalent:

i) \( Y \) is a flat \( A \)-module.

ii) \( \text{Tor}^1_A(Y, k_A) = 0 \), where \( k_A = A/\mathfrak{m}_A \).

**Proof.** We prove the nontrivial direction: Consider an exact sequence of \( A \)-modules

\[
0 \to N \to F \to Y \to 0
\]

(3)

where \( F = \bigoplus_{i \in I} A \) is a free \( A \)-module.

Now consider in (3) the filtrations

\[
N_k = N \cap \mathfrak{m}^k_A F, \quad m^k_A F = F_k, \quad Y_k = m^k_A Y
\]

(4)

From them result exact sequences of \( \widehat{A} \)-modules, where \( \widehat{A} \) is the \( \mathfrak{m}_A \)-adic completion of \( A \):

\[
\begin{array}{c}
0 \longrightarrow \widehat{N} \longrightarrow \widehat{F} \longrightarrow \widehat{Y} \longrightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow N/N_k \longrightarrow F/F_k \longrightarrow Y \otimes_A A/m^k_A \longrightarrow 0
\end{array}
\]

(5)

There the lower sequences are exact by definition of \( N_k, F_k \) and \( Y_k \).

**An artinian interlude** We use the following Lemma

**Lemma 1.2** Under the conditions of the theorem it follows from \( \text{Tor}^1_A(Y, k_A) = 0 \) that \( Y/m^k_A Y \) is a projective \( A/m^k_A \)-module.
Call \( A' = A/m_A^k \). Then by lemma [17] it is enough to prove, that

\[
\Tor^1_{A'}(Y/m_A^k Y, A'/m_{A'}) = 0.
\]

Now from the sequences

\[
0 \to m_A^p/m_A^{p+1} \to A/m_A^{p+1} \to A/m_A^p \to 0
\]

and the base assertion \( \Tor^1_A(Y, A/m_A) = 0 \) it follows inductively, that

\[
\Tor^1_A(Y, A/m_A^p) = 0
\]

for all \( p \geq 0 \).

Now consider the exact sequence

\[
0 \to L \to F \to Y \to 0
\]

with free \( A \)-module \( F \). Tensoring with \( A/m_A^k \) gives the exact sequence

\[
0 \to L \otimes_A A/m_A^k \to F \otimes_A A/m_A^k \to Y \otimes_A A/m_A^k \to 0.
\]

As \( F \otimes_A A/m_A^k \) is a free \( A/m_A^k \) module it follows that \( \Tor^1_{A'}(Y/m_A^k Y, k_{A'}) = 0 \) if tensoring (6) by \(- \otimes_A k_A \) results in an exact sequence. But (6) \( \otimes_A k_A \) is nothing else but

\[
0 \to L \otimes_A k_A \to F \otimes_A k_A \to Y \otimes_A k_A \to 0
\]

which is exact by \( \Tor^1_A(Y, k_A) = 0 \).

**Climbing the ladder** We can therefore find in the lower sequences of (5) a projective splitting of \( F/F_k \) that climbs from \( k \) to \( k + 1 \):

\[
\begin{array}{ccc}
0 & \xrightarrow{\phi_{k+1}} & N/N_{k+1} \\
\downarrow{\alpha} & & \downarrow{\beta} \\
0 & \xrightarrow{\phi_k} & N/N_k \\
\end{array}
\quad
\begin{array}{ccc}
F/F_{k+1} & \xrightarrow{\psi_{k+1}} & Y/m_A^{k+1} Y \\
\downarrow{\beta} & & \downarrow{\gamma} \\
F/F_k & \xrightarrow{\psi_k} & Y/m_A^k Y \\
\end{array}
\quad
0
\]

First construct \( s'_{k+1} \) from the condition \( \psi_{k+1}s_{k+1}' = \text{id} \). From this follows \( \psi_k (\beta s_{k+1}' - s_k \gamma) = 0 \). So we have a map \( s''_{k+1} = \beta s_{k+1}' - s_k \gamma : Y/Y_{k+1} \to N/N_k \). We lift it to a map \( s'''_{k+1} : Y/Y_{k+1} \to N/N_{k+1} \). Then \( s_{k+1} = s_{k+1}' - \phi_{k+1}s_{k+1}' \) is a lifting of \( s_k \) that makes the diagram (7) commute.
The splitting diagram  So we get from above a commutative diagram with exact rows:

\[
\begin{array}{cccccccc}
0 & \rightarrow & N/N_k & \rightarrow & F/F_k & \rightarrow & Y/m^kAY & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \hat{N} & \rightarrow & \hat{F} & \rightarrow & \hat{Y} & \rightarrow & 0 \\
\end{array}
\]  (8)

Using the splitting  Let \( M \) be a finitely generated \( A \)-module and \( \hat{M} \) its completion. We write

\[
\hat{M} = \lim_{k} M(k) = \lim_{k} M/m^k_A M.
\]

Now consider the mapping

\[
\left( \lim_{k} F(k) \otimes_A M \right) \xrightarrow{u_M} \lim_{k} \left( F(k) \otimes_A M \right)
\]

where we made use of the abbreviation \( X(k) = X/m^k_A X \).

Note also

\[
\lim_{k} (F \otimes_A M(k)) = \lim_{k} (F \otimes_A M) = \lim_{k} (F(k) \otimes_A M(k) \otimes_A M) = \lim_{k} (F(k) \otimes_A M)
\]  (10)

We will prove that \( u_M \) is an isomorphism for a finitely generated \( A \)-module \( M \).

First we prove this for \( M = E \) free of rank \( r \):

\[
\lim_{k} F(k) \otimes_A A^r = \left( \lim_{k} F(k) \right)^r = \lim_{k} F^r(k) = \lim_{k} (F(k) \otimes_A A^r)
\]  (11)

Now consider a presentation

\[
E' \rightarrow E \rightarrow M \rightarrow 0
\]

with finite rank free \( E, E' \) and the diagram

\[
\begin{array}{cccccccc}
\left( \lim_{k} F(k) \right) \otimes_A E' & \rightarrow & \left( \lim_{k} F(k) \right) \otimes_A E & \rightarrow & \left( \lim_{k} F(k) \right) \otimes_A M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\lim_{k} (F \otimes_A E'(k)) & \rightarrow & \lim_{k} (F \otimes_A E(k)) & \rightarrow & \lim_{k} (F \otimes_A M(k)) & \rightarrow & 0 \\
\end{array}
\]  (12)
It proves that $u_M$ is an isomorphism, if we can show, that the bottom row is exact. We will show this in a moment, but first we will further the main line of argument:

Consider the line

$$
\hat{F} \otimes_A M = (\varprojlim_k F(k)) \otimes_A M = \varprojlim_k (F_k \otimes_A M(k)) = \varprojlim_k \bigoplus_i M(k) \rightarrow
\rightarrow \varprojlim_k \prod_i M(k) = \prod_i \varprojlim_k M(k) = \prod_i \hat{M} \tag{13}
$$

So we have a canonical injection

$$
\hat{F} \otimes_A M \hookrightarrow \prod_i \hat{M}. \tag{14}
$$

In the following we use the so called "Mittag–Leffler property" of inverse systems. See for example [3, Proposition II.9.1.] for definition of and elementary facts about this property.

It remains to prove the exactness of the lower row in diagram (12):

Start with the exact sequences

$$
E'_k \rightarrow E_k \rightarrow M_k \rightarrow 0 \tag{15}
$$

which form an inverse system in $k$. Splice them into short exact sequences

$$
0 \rightarrow P_k \rightarrow E'_k \rightarrow Q_k \rightarrow 0 \tag{16}
$$

$$
0 \rightarrow Q_k \rightarrow E_k \rightarrow M_k \rightarrow 0 \tag{17}
$$

The above two systems of sequences each form an inverse system. We note that $(P_k)_k$ and $(Q_k)_k$ have the Mittag–Leffler property (ML) as they consist of Artin–modules only.

Now tensoring the sequences in (16), (17) with $\otimes_A F$ retains their exactness and the (ML)–property on $(P_k \otimes_A F)_k$ and $(Q_k \otimes A F)_k$ too.

This is because, if $\psi_{k'} : P_{k'} \rightarrow P_k$ and we have $\text{im}(\psi_{k'}) = \text{im}(\psi_{k''})$, then $\text{im}(\psi_{k'} \otimes \text{id}_F) = \text{im}(\psi_{k''} \otimes \text{id}_F)$ too.

Now take a sequence $0 \rightarrow M' \rightarrow M$ of two finitely generated $A$–modules. Then consider the diagram

$$
\begin{align*}
0 & \rightarrow X & \rightarrow & M' \otimes_A \hat{F} & \rightarrow & M \otimes_A \hat{F} & \rightarrow & 0 \\
0 & \rightarrow & \prod_{i \in I} \hat{M}' & \rightarrow & \prod_{i \in I} \hat{M} & \rightarrow & 0 
\end{align*} \tag{18}
$$
From this follows $X = 0$ and therefore the conclusion that tensoring with $\otimes_A \hat{F}$ is exact on injections of finitely generated $A$–modules.

From this follows at once that $\hat{F}$ is a flat $A$–module.

As $\hat{Y}$ and $\hat{N}$ are split-summands of $\hat{F}$ they are $A$–flat too:

**Lemma 1.3** The modules $\hat{F}$, $\hat{Y}$, $\hat{N}$ from above are $A$–flat modules.

Additionally we have

**Lemma 1.4** The module $\hat{B}$ is a faithfully-flat $B$–module. There $\hat{B}$ is the completion of $B$ with respect to the filtration $(m_A^k B)$, that is $\lim_{\leftarrow k} B/m_A^k B$.

First $\hat{B}$ is $B$–flat as the $m_A B$–adic completion of $B$. Furthermore, we have $m_B \supset m_A B$ and therefore $\hat{m}_B = m_B \hat{B}$ is a maximal ideal in $\hat{B}$. It has the property, that under $B \to \hat{B}$ we have $\hat{m}_B \cap B = m_B$.

So, together with the going-down property for flat extensions, we conclude, that $\text{Spec} (\hat{B}) \to \text{Spec} (B)$ is surjective and therefore $\hat{B}$ a faithfully $B$–flat module.

**Lemma 1.5** There is an isomorphism $\hat{Y} = Y \otimes_B \hat{B}$

This is well known ([1 Proposition 10.13]).

**The conclusion** Now consider an injection of two finitely generated $A$–Modules $0 \to N' \to N$. Tensoring with $Y$ gives the exact sequence

$$0 \to P \to N' \otimes_A Y \to N \otimes_A Y$$

(19)

Tensoring with $\otimes_B \hat{B}$ leads to

$$0 \to P \otimes_B \hat{B} \to N' \otimes_A (Y \otimes_B \hat{B}) \to N \otimes_A (Y \otimes_B \hat{B})$$

(20)

As $Y \otimes_B \hat{B}$ is $\hat{Y}$ and $\hat{Y}$ is $A$–flat it follows, that $P \otimes_B \hat{B} = 0$.

Now by lemma 1.4 we conclude $P = 0$. So $Y$ is a flat $A$–module.

**References**

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