Spectrum and rearrangement decays of tetraquark states with four different flavors

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We have systematically investigated the mass spectrum and rearrangement decay properties of the exotic tetraquark states with four different flavors using a color-magnetic interaction model. Their masses are estimated by assuming that the Hamiltonian for decay is a constant. According to the adopted method, we find that the most stable states are probably the isoscalar $bs\bar{d}$ and $cs\bar{d}$ with $J^P = 0^+$ and $1^+$. The width for most unstable tetraquarks is about tens of MeVs, but that for unstable $cs\bar{d}$ and $cs\bar{d}$ can be around 100 MeV. For the $X(5568)$, our method cannot give consistent mass and width if it is a $bs\bar{d}$ tetraquark state. For the $I(J^P) = 0(0^+), 0(1^+)$ double-heavy $T_{bc} = bc\bar{u}$ states, their widths can be several MeVs. We also discuss the relation between the tetraquark structure and the total width with the help of defined $K$ factors. We find that: (1) one cannot uniquely determine the structure of a four-quark state just from the decay ratios between its rearrangement channels; (2) it is possible to judge relative stabilities of tetraquark states just from the calculable $K$ factors for the quark-quark interactions; and (3) the range of total width for a tetraquark state can be estimated just with the masses of the involved states.

I. INTRODUCTION

Since 2003, heavy quark exotic XYZ or $P_c$ states have been observed in almost every year [1–4]. It is impossible to assign all of them as conventional mesons or baryons in the traditional quark model. Their observation is consistent with the fact that QCD does not prohibit the existence of multiquark hadrons. To understand the nature of the exotic states, a large number of theoretical studies in the literature were performed, which include discussions about the hadron structures (molecules, compact multiquarks, kinematic effects, or other nonresonant interpretations), investigations on the production and decay properties, and developments of research methods [5–17].

In these exotic states, the $X(5568)$ claimed by the D0 Collaboration [18, 19] is special in that it is a $bs\bar{d}$ or $bsd\bar{u}$ meson with valence quarks of four different flavors. Unfortunately, its existence was not confirmed by later experimental analyses from LHCb [20], CMS [21], CDF [22], and ATLAS [23]. On the theoretical side, various discussions [24–62] had tried to understand the nature of $X(5568)$. Most of investigations could not give a natural explanation for its low mass. Its production is also difficult to understand [55]. However, the nonconfirmation of $X(5568)$ does not exclude the existence of $bs\bar{d}$ or $bsd\bar{u}$ tetraquark and its partner states. In this article, we are going to explore systematically the spectrum and decay properties of tetraquark mesons with four different flavors. There are totally nine systems we may consider, $bc\bar{s}u$, $bs\bar{u}c$, $bnc\bar{s}$, $bud\bar{c}$ or $bd\bar{c}u$, $bs\bar{d}$ or $bd\bar{s}u$, $c\bar{u}d$ or $cs\bar{d}u$, $b\bar{c}ud$, $bs\bar{d}$, and $(cs\bar{d}$, where $n = u$ or $d$. All of them are heavy quark states. Recent studies of such systems within various approaches can be found in the literature [63–73]. One may also consult Refs. [7, 10, 16] for more investigations.

Previously in Ref. [48], we presented a study of the spectra of $qqQ$ ($Q = c, b, q = u, d, s$) systems in the color-magnetic interaction (CMI) model. It was found that the $X(5568)$ can be assigned as a tetraquark state if small quark masses are adopted. A number of its stable partner tetraquarks may also exist. However, the conclusion based on the original form of model Hamiltonian is sensitive to the quark masses and thus different conclusions may be reached. From the following studies of various systems within the same model [69, 72, 74–79], the multiquark masses estimated with the quark masses $m_u \approx 362$ MeV ($n = u, d$), $m_d \approx 540$ MeV, $m_s \approx 1725$ MeV, and $m_b \approx 5053$ MeV can be treated as upper limits. A more reasonable multiquark mass seems to be close to that determined with a related hadron-hadron channel whose threshold can give higher multiquark masses. For example, the masses of the recently observed $P_c$ states can be reproduced with this estimation method [80, 81]. The present study will include an updated analysis on the spectra of some $qqQ$ systems. Then one may get a deeper understanding of the $X(5568)$ state.

Besides the spectroscopy, the decay properties of hadrons are also important in understanding their internal structures. For the decays of multiquark states, the rearrangement mechanism plays a significant role. In Ref. [81], we tried to understand the decay properties of the newly observed $P_c(4312)$, $P_c(4440)$, and $P_c(4457)$ by the LHCb Collaboration [80] with a very simple model $H_{\text{decay}} = \text{const}$.

This article is organized as follows. In Sec. II, we introduce the formalism to study the spectrum and decay widths in the CMI model. Then in Sec. III, the selection of parameters and numerical results for various systems will be
shown. Next in Sec. IV, we discuss the relation between the tetraquark structure and the total width. The last section is for the summary of the present work.

II. FORMULATION

In the CMI model, we first consider the mass splittings of the S-wave tetraquark states by using the color-spin interaction term of the one-gluon-exchange potential and then we estimate the tetraquark masses by adding effective quark masses. One does not need to solve the bound state problem in the model since the contributions from the spacial part of the total wave function have been encoded in the effective parameters. In this section, what we need is to construct the flavor-spin-color wavefunction bases, diagonalize the CMI Hamiltonian, and choose appropriate estimation method for tetraquark masses. A simple decay model is also presented.

A. Wave functions

In this study, we do not consider the isospin breaking effects and the nine systems we will consider are defined as

\[ F_{1-9} = bc\bar{s}u, \, bsc\bar{n}, \, bn\bar{c}s, \, bu\bar{c}d, \, bc\bar{u}d, \, bs\bar{u}d, \, cs\bar{u}d, \, cs\bar{d}u. \]  

Note that the states containing two quark masses. One does not need to solve the bound state problem in the model since the contributions from the spacial part of the total wave function have been encoded in the effective parameters. In this section, what we need is to construct the flavor-spin-color wavefunction bases, diagonalize the CMI Hamiltonian, and choose appropriate estimation method for tetraquark masses. A simple decay model is also presented.

Here, the superscripts (subscripts) represent the spins (color representations) of the quark-quark state, the antiquark-state, and the tetraquark state. The mixing effects between states with the same total spin will be considered.

When combining the flavor, color, and spin wave functions, one needs to include the constraints from the Pauli principle for the \( F_5 = bc\bar{u}d, \) \( F_7 = bs\bar{u}d, \) and \( F_9 = cs\bar{u}d \) cases, but not for other cases. We tabulate the needed spin-color base states for \( F_5, F_7, \) and \( F_9 \) in various \( (I, J) \) combinations in Table I.

| \( I = 0 \) | \( J = 1 \) | \( J = 2 \) |
|---|---|---|
| \( B_1, B_4 \) | \( B_2, B_3 \) | \( B_5, B_6, B_9, B_{10} \) |
| \( B_{11} \) | \( B_{12} \) |

B. Color-magnetic interaction and mass estimation

With the constructed wave functions, the mass splittings between states with different spins for a given system can be calculated by using the CMI Hamiltonian,

\[ \hat{H}_{CMJ} = \sum_{i<j} C_{ij} \langle \vec{\sigma}_i \cdot \vec{\sigma}_j \rangle \langle \vec{\lambda}_i \cdot \vec{\lambda}_j \rangle. \]

(3)

Here, \( C_{ij} \) is the effective coupling strength between the \( i \)th and \( j \)th quark components, \( \vec{\sigma}_i \) is the Pauli matrix acting on the \( i \)th quark component, and \( \vec{\lambda}_i = \lambda_i \) (\( -\lambda_i^* \)) for quarks (antiquarks) with \( \lambda \) being the Gell-Mann matrix.

For the systems \( F_{1,2,3,4,6,8}, \) the three \( \langle \hat{H}_{CMJ} \rangle \) matrices corresponding to the total spin \( J = 2, 0, \) and 1 are respectively

\[
\begin{bmatrix}
\frac{2(5\tau - 2\nu)}{3} & -2\sqrt{2}\nu \\
-2\sqrt{2}\nu & \frac{4(2\tau + \alpha)}{3}
\end{bmatrix}
\quad (J = 0)
\]

\[
\begin{bmatrix}
4\tau & \frac{-10\nu}{\sqrt{3}} & 0 & 2\sqrt{6}\alpha \\
-\frac{10\nu}{\sqrt{3}} & \frac{-4(7\tau + 5\alpha)}{3} & 2\sqrt{6}\alpha & 4\sqrt{2}\nu \\
0 & 2\sqrt{6}\alpha & -8\tau & -\frac{4\nu}{\sqrt{3}} \\
2\sqrt{6}\alpha & 4\sqrt{2}\nu & -\frac{4\nu}{\sqrt{3}} & \frac{8(\tau - \alpha)}{3}
\end{bmatrix}
\quad (J = 2)
\]

\[
\begin{bmatrix}
\frac{2(5\alpha - 2\nu)}{3} & -2\sqrt{2}\nu \\
-2\sqrt{2}\nu & \frac{4(2\tau + \alpha)}{3}
\end{bmatrix}
\quad (J = 2)
\]

\[
\begin{bmatrix}
4\tau & \frac{-10\nu}{\sqrt{3}} & 0 & 2\sqrt{6}\alpha \\
-\frac{10\nu}{\sqrt{3}} & \frac{-4(7\tau + 5\alpha)}{3} & 2\sqrt{6}\alpha & 4\sqrt{2}\nu \\
0 & 2\sqrt{6}\alpha & -8\tau & -\frac{4\nu}{\sqrt{3}} \\
2\sqrt{6}\alpha & 4\sqrt{2}\nu & -\frac{4\nu}{\sqrt{3}} & \frac{8(\tau - \alpha)}{3}
\end{bmatrix}
\quad (J = 0)
\]
vectors (Table I) are three systems where a reference system is introduced. The formula means that we are calculating the mass splitting between the eigenvalue, the tetraquark masses can be estimated with the mass formula for the case

$$\sum \Delta \text{m}_{ij} \text{ref}$$

in the reference state. A natural choice of the reference system is a hadron-(4140) as a reference state in Ref. [69] by assuming it to be the lowest 1++ cscūū compact tetraquark [76, 90]. Then the uncertainty problem in mass differences between different quark flavors, e.g. uncertainty in \(\Delta_{sn} = m_s - m_n\). If we use \(\Delta_{ij} = m_i - m_j\) to represent the mass difference between two quarks of different flavors \(i\) and \(j\), the choice of \(\Delta_{ij}\) depends on the studied systems, which actually implies the fact that the effective quark mass may be affected by the quark environment. Explicitly, the formula we will adopt in the present study is

$$M = [M_{X(4140)} - (E_{CMI})_{X(4140)}] + \sum_{ij} n_{ij} \Delta_{ij} + E_{CMI},$$

where \(n_{ij}\) is an integer. In Ref. [69], the spectra of \(F_{2,3,4} = |bs\bar{c}n\rangle, |bn\bar{c}s\rangle, |bn\bar{c}n\rangle\) with this formula had been explored. In this article, we extend the investigation to other systems.

C. Rearrangement decays

In each system, one may finally obtain the tetraquark wave functions and corresponding eigenvalues with the above CMI matrices. For each state, its spin⊗color wave function can be written as

$$\Psi^{j=0} = x_1 B_1 + x_2 B_2 + x_3 B_3 + x_4 B_4,$$
$$\Psi^{j=1} = x_1 B_5 + x_2 B_6 + x_3 B_7 + x_4 B_8 + c_5 B_9 + x_6 B_{10},$$
$$\Psi^{j=2} = x_1 B_{11} + x_2 B_{12}.$$
The normalization condition requires \( \sum_{i=1}^{4} |x_i|^2 = 1 \) for each tetraquark state. In Ref. [81], we used a simple model in which the Hamiltonian for rearrangement decays is just taken as a constant to investigate the decay properties of the \( uudc \) pentaquark states. It was found that the measured ratios between decay widths of \( \bar{P}_c(4312) \), \( \bar{P}_c(4440) \), and \( \bar{P}_c(4457) \) can be understood well. Here, we also preliminarily investigate the decay properties of the tetraquarks within this simple model. There are two types of rearrangement decays for tetraquark states, \( q_1q_2\bar{q}_3\bar{q}_4 \to (q_1\bar{q}_3)_{1c} + (q_2\bar{q}_4)_{1c} \) and \( q_1q_2\bar{q}_3\bar{q}_4 \to (q_1\bar{q}_3)_{1c} + (q_2\bar{q}_3)_{1c} \). The final state meson-meson wavefunction will be recoupled to the combination of the \( B_{1-12} \) bases. In doing so, we use the formulas

\[
(q_1\bar{q}_3)_{1c} = \frac{\sqrt{6}}{3} (q_1q_2)_{6c}(\bar{q}_3\bar{q}_4)_{6c} - \frac{1}{\sqrt{3}} (q_1q_2)_{3c}(\bar{q}_3\bar{q}_4)_{3c},
\]

\[
(q_1\bar{q}_4)_{1c} = \frac{\sqrt{6}}{3} (q_1q_2)_{6c}(\bar{q}_3\bar{q}_4)_{6c} + \frac{1}{\sqrt{3}} (q_1q_2)_{3c}(\bar{q}_3\bar{q}_4)_{3c}.
\]

Note that we construct the wavefunctions explicitly with the convention used in Refs. [91, 92] and the relative signs are different from those in Refs. [93, 94]. The amplitude \( \mathcal{M} = \langle \text{initial}|H_{\text{decay}}|\text{final} \rangle = \alpha \langle \text{initial}|\text{final} \rangle \) is then obtained and the two-body decay widths can be calculated with the standard formula. Note that each system has its own constant value of \( \alpha \). If we use \( y_i \) \((i = 1, 2, \ldots)\) to denote the coefficients of tetraquark base states in the final meson-meson state, one has

\[
|\mathcal{M}|^2 = \alpha^2 \sum_i (x_iy_i)^2 \equiv \alpha^2 |\tilde{\mathcal{M}}|^2.
\]

For convenience, we will call \( |\tilde{\mathcal{M}}|^2 \) coupling matrix element (CME) in the following discussions.

### III. NUMERICAL RESULTS

**TABLE II: Effective coupling parameters \( C_{ij} \)'s in units of MeV. The extracted effective quark masses are \( m_n = 361.8 \text{ MeV} \), \( m_s = 542.4 \text{ MeV} \), \( m_c = 1724.1 \text{ MeV} \), and \( m_b = 5054.4 \text{ MeV} \).**

| \( C_{ij} \) | \( C_{ij} \) |
|---|---|---|---|---|---|---|---|
| \( n \) | 18.3 | 12.0 | 4.0 | 1.3 | \( n \) | 29.9 | 18.7 | 6.6 | 2.1 |
| \( s \) | 5.7 | 4.4 | 0.9 | \( s \) | 9.3 | 6.7 | 2.3 |
| \( c \) | 3.2 | 2.0 | \( c \) | 5.3 | 3.3 |
| \( b \) | 1.8 | \( b \) | 2.0 |
TABLE III: Quark mass differences (units: MeV) determined with various hadrons. Those from the extracted effective quark masses are $\Delta_{\text{vac}}=3330.3$ MeV $\Delta_{\text{en}}=1362.3$ MeV, and $\Delta_{\text{en}}=180.6$ MeV.

| Hadron | Hadron | $\Delta_{\text{en}}$ | Hadron | Hadron | $\Delta_{\text{en}}$ | Hadron | Hadron | $\Delta_{\text{en}}$ |
|--------|--------|-----------------|--------|--------|-----------------|--------|--------|-----------------|
| $B_s (B^+_s)$ | $D_s (D^+_s)$ | 3340.2 (3340.1) | $D (D^+)$ | $\pi (\rho, \omega)$ | 1356.4 (1357.6, 1350.2) | $K (K^+)$ | $\pi (\rho, \omega)$ | 178.4 (178.1, 170.7) |
| $B_c$ | $\eta_c$ | 3259.0 | $\eta_c (J/\psi)$ | $D (D^+)$ | 1095.9 (1095.3) | $B_s (B^+_s)$ | $B (B^+)$ | 1014.6 |
| $\eta_b$ | $B_c$ | 3117.7 | $B_c$ | $B$ | 1014.6 | $B_s (B^+_s)$ | $B (B^+)$ | 90.6 (89.7) |

$\Lambda_b, \Lambda_c, \Sigma_c$ | 3333.1, 3318.6 | $\Lambda_c, N$ | 1347.5 | $\Lambda_c, N$ | 176.8 |
$\Sigma_b (\Sigma^+_b)$ | $\Sigma_c (\Sigma^+_c)$ | 3331.1 (3329.9) | $\Sigma_c (\Sigma^+_c) N (\Delta)$ | 1362.1 (1362.4) | $\Sigma (\Sigma^*) N (\Delta)$ | 187.0 (186.2) |
$\Sigma_b$ | $\Xi_c$ | 3345.6 | $\Xi_c (\Xi^*), \Lambda$ | 1328.8 | $\Xi_c, \Lambda$ | 169.0 |
$\Xi_b$ | $\Xi_c, \Xi_c$ | 3235.1, 3299.6 | $\Xi (\Xi^*) N (\Delta)$ | 1318.6 (1319.8) | $\Xi (\Xi^*) N (\Delta)$ | 158.7 (159.3) |
$\Xi_b (\Xi^*_b)$ | $\Xi (\Xi^*)$ | 3232.9 (3232.2) | $\Xi_b, \Lambda, \Sigma$ | 1303.3, 1293.0 | $\Omega, \Xi^*$ | 172.7 |
$\Xi_b$ | $\Xi_c, \Xi_c$ | 3349.4 | $\Omega_c (\Omega^*_c) \Xi (\Xi^*)$ | 1295.9 (1273.0) |

$\eta_b$ | $\Omega_c$ | 3313.6 |

$\eta_c$ | 3188.4 | $\eta_c (J/\psi)$ | $\pi (\rho, \omega)$ | 1226.1 (1226.4, 1222.7) |
$\Xi_{cc}$ | $N$ | 1287.2 |

The value of $K_{ij}$ can be calculated when diagonalizing the CMI matrix. From the sign of $K_{ij}$, one can judge whether the effective CMI between the $i$th and $j$th quark components is attractive ($K_{ij} < 0$) or repulsive ($K_{ij} > 0$). From the amplitude of $K_{ij}$, one can roughly understand how important the effect would be from the change of the corresponding CMI. Their values will be explicitly given. For convenience, we will also call these $K_{ij}$’s $K$ factors.

With the above parameters, one may derive numerical results for various systems. We collect the estimated tetraquark masses and $K_{ij}$’s for effective CMI in tables IV and V. For convenience, we also show the relative positions of the tetraquark states and related thresholds in Fig. 1. In these tables and figure, we do not show the results for $F_2 = bs\bar{c}n$, $F_3 = bu\bar{c}s$, and $F_4 = bu\bar{c}d$ since they have been presented in Ref. [69]. Although the systems $F_1 = bc\bar{s}n$ and $F_5 = bc\bar{c}d$ have been discussed in Ref. [72] and the systems $F_6 = bu\bar{s}d$, $F_7 = bs\bar{d}d$, $F_8 = cu\bar{s}d$, and $F_9 = cs\bar{d}d$ discussed in Ref. [48], the updated estimation method may result in different properties. We will discuss details about these results system by system.

The rearrangement decay widths for the considered $F_1 - F_9$ systems are listed in tables VI, VII, and VIII. Since $\alpha$ in the decay Hamiltonian may be different for each system, without knowing its value, one may only estimate the relative decay ratios for those states with the same $\alpha$. If we know the value of $\alpha$, all the partial decay widths for the considered states may be estimated. For this purpose, we assume that the total width equals to the sum of the partial decay widths for the rearrangement channels, i.e. $\Gamma_{\text{total}} \approx \Gamma_{\text{sum}}$, and that the parameter extracted from the LHCb decay width of $X(4140)$, $\Gamma_X(4140) = 83^{+30}_{-25}$ MeV [89], can be applied to other tetraquark systems. In table IX, the results for the rearrangement decay widths of the $cs\bar{c}s$ states are shown. Before the extraction of $\alpha$, we need to understand whether these results are acceptable. From table IX, $\Gamma_{X(4274)}$ is around $14.3/15.7 \times 83 = 76$ MeV when one treats it as a $cs\bar{c}s$ tetraquark state [76]. This value is slightly larger than the measured $\Gamma_{X(4274)} = 56^{+14}_{-16}$.
MeV [89]. Note that the adopted mass of $X(4274)$ (4309.4 MeV) is larger than the LHCb result (4273.3 MeV). If one adopts its measured mass, we get $\Gamma_{X(4274)} \approx 13.1/15.7 \times 83 = 69$ MeV. This width is now consistent with the measured one and thus the assumptions we adopt are acceptable. However, our results in table IX cannot explain the ratio between the widths of $X(4140)$ and $X(4274)$ from Ref. [1] where $\Gamma_{X(4274)} > \Gamma_{X(4140)}$. The experimental information about the widths is further needed to understand the nature of these two exotic states. In the present study, we use the LHCb data. With the assumption $\Gamma_{sum} = \Gamma_{total} = 83$ MeV for $X(4140)$, one gets $\alpha = 7.27$ GeV, a number not far from that in the hidden-charm pentaquark case [81]. The following discussions about the predictions on the tetraquark widths depend on this number. If there is a $c\bar{s}\bar{c}\bar{s}$ state around 4350 MeV, from table IX, its width may be comparable to that of $X(4274)$.


table : Numerical results for $F_1 = bc\bar{s}n$, $F_6 = bu\bar{d}\bar{s}$, and $F_8 = cu\bar{s}\bar{d}$ in units of MeV: CMI eigenvalues ($E_{CMI}$), upper limits for tetraquark masses ($M_{upper}$), tetraquark masses estimated with relevant thresholds and with the scale related to $X(4140)$, and $K$ factors for effective CMI.

\[
| J^P | E_{CMI} | M_{upper} | B_s D | D_s B | X(4140) | K_{bc} | K_{b\bar{s}} | K_{b\bar{n}} | K_{c\bar{s}} | K_{c\bar{n}} | K_{n_s} |\]

| 2+ | 61.0 | 7744 | 7438 | 7450 | 7542 | 2.7 | 1.4 | 1.3 | 1.3 | 1.4 | 2.7 |
| 40.3 | 7723 | 7417 | 7429 | 7521 | -1.3 | 3.3 | 3.4 | 3.4 | 3.3 | -1.3 |
| 1+ | 61.0 | 7744 | 7438 | 7450 | 7542 | 2.7 | 1.4 | 1.3 | 1.3 | 1.4 | 2.7 |
| 40.3 | 7723 | 7417 | 7429 | 7521 | -1.3 | 3.3 | 3.4 | 3.4 | 3.3 | -1.3 |
| 30.6 | 7713 | 7407 | 7419 | 7511 | -1.1 | -2.5 | -4.6 | 1.9 | 0.6 | 2.6 |
| 21.9 | 7705 | 7398 | 7411 | 7502 | -3.4 | -1.2 | 0.7 | 2.4 | 3.6 | -2.0 |
| -43.5 | 7639 | 7333 | 7345 | 7437 | -2.7 | 2.1 | -0.1 | -4.1 | -7.5 | 2.8 |
| -48.2 | 7635 | 7328 | 7340 | 7432 | 0.2 | -6.9 | -4.4 | 0.1 | 3.3 | -3.8 |
| -150.3 | 7592 | 7226 | 7238 | 7330 | 1.5 | 2.7 | 2.8 | -8.5 | -8.2 | 4.6 |
| 0+ | 115.9 | 7799 | 7492 | 7505 | 7596 | -3.6 | -3.7 | 3.7 | -0.3 | 3.7 | 3.6 |
| -36.8 | 7646 | 7340 | 7352 | 7444 | -5.1 | 2.1 | 1.9 | 2.1 | -5.1 |
| -69.6 | 7613 | 7307 | 7319 | 7411 | 3.1 | -6.4 | -6.4 | -6.4 | 3.1 |
| -212.0 | 7471 | 7165 | 7177 | 7269 | -4.2 | -8.7 | -8.6 | -8.6 | -8.7 | 4.2 |

| 2+ | 174.2 | 6495 | 6194 | 6282 | 6374 | -0.6 | 5.2 | 0.8 | 0.8 | 5.2 | -0.6 |
| 90.9 | 6411 | 6111 | 6199 | 6291 | 1.9 | -0.5 | 3.9 | 3.9 | -0.5 | 1.9 |
| 1+ | 222.1 | 6543 | 6242 | 6330 | 6422 | -1.9 | 0.9 | -0.9 | 3.1 | 4.2 | 3.3 |
| 144.6 | 6465 | 6165 | 6252 | 6344 | 2.1 | -7.4 | 1.2 | 1.1 | 4.9 | -0.8 |
| 69.6 | 6390 | 6090 | 6178 | 6269 | -2.4 | -0.9 | -1.7 | 3.1 | 0.3 | 1.0 |
| 38.3 | 6359 | 6058 | 6146 | 6238 | 1.1 | -2.5 | -8.1 | 2.9 | 2.0 | -4.7 |
| 237.0 | 6083 | 5783 | 5871 | 5963 | -1.0 | 0.6 | 3.6 | -11.2 | -2.0 | 2.1 |
| -502.6 | 5818 | 5518 | 5605 | 5697 | 0.7 | 4.7 | 1.2 | -3.8 | -14.0 | -2.3 |
| 0+ | 247.5 | 6568 | 6268 | 6355 | 6447 | 3.4 | 4.1 | 3.3 | 3.5 | 4.1 | 3.4 |
| 53.0 | 6373 | 6073 | 6161 | 6253 | -5.9 | 2.5 | 2.4 | 2.4 | 2.5 | -5.9 |
| -270.3 | 6050 | 5750 | 5838 | 5930 | 2.3 | -2.3 | -10.9 | -10.9 | -2.3 | 2.3 |
| -560.3 | 5760 | 5460 | 5548 | 5640 | -2.4 | -13.7 | -4.2 | -4.2 | -13.7 | -2.4 |

| 2+ | 198.9 | 3189 | 2891 | 2967 | 3059 | -0.7 | 5.1 | 1.0 | 1.0 | 5.1 | -0.7 |
| 111.3 | 3101 | 2803 | 2879 | 2971 | 2.0 | -0.4 | 3.7 | 3.7 | -0.4 | 2.0 |
| 1+ | 225.2 | 3213 | 2914 | 2990 | 3082 | -2.7 | 3.0 | -0.2 | 2.4 | 4.5 | 3.0 |
| 145.1 | 3135 | 2837 | 2913 | 3005 | 3.2 | -2.1 | 3.8 | 3.0 | 3.1 | -2.3 |
| 43.0 | 3033 | 2735 | 2811 | 2903 | -1.6 | -6.2 | -1.0 | 1.2 | 1.9 | 1.4 |
| -22.0 | 2968 | 2670 | 2746 | 2838 | 0.8 | -4.3 | -11.4 | 3.4 | 1.8 | -3.3 |
| -233.8 | 2766 | 2468 | 2544 | 2636 | -1.7 | 0.1 | 3.1 | -11.2 | -1.8 | 2.0 |
| -475.1 | 2515 | 2217 | 2293 | 2385 | 0.5 | 4.8 | 1.0 | -3.5 | -14.2 | -2.2 |
| 0+ | 290.0 | 3280 | 2982 | 3058 | 3150 | 3.5 | 4.1 | 3.4 | 3.4 | 4.1 | 3.5 |
| 59.0 | 3049 | 2751 | 2827 | 2919 | -6.0 | 2.6 | 2.4 | 2.4 | 2.6 | -6.0 |
| -322.8 | 2667 | 2369 | 2445 | 2537 | 2.5 | -2.7 | -10.4 | -10.4 | -2.7 | 2.5 |
| -646.5 | 2344 | 2046 | 2121 | 2213 | -2.6 | -13.2 | -4.8 | -4.8 | -13.2 | -2.6 |
TABLE V: Numerical results for $M_{\text{upper}}$ for tetraquark masses ($M_{\text{upper}}$), tetraquark masses estimated with relevant thresholds and with the scale related to $X(4140)$, and $K$ factors for effective CMI.

| $J^P$ | $E_{\text{CMI}}$ | $M_{\text{upper}}$ | $DB$ | $X(4140)$ | $K_b n$ | $K_b n$ | $K_{\text{cii}}$ | $K_{\text{cii}}$ | $K_{\text{n}}$ |
|-------|------------------|-------------------|------|---------|---------|---------|-------------|-------------|---------|
| 1(2+) | 77.4             | 7579              | 7363 | 7467    | 2.7     | 2.7     | 2.7         | 2.7         | 2.7     |
| 1(1+) | 110.2            | 7612              | 7396 | 7500    | -2.5    | 1.5     | 7.0         | 3.6         | 3.6     |
|       | 47.9             | 7550              | 7334 | 7438    | -1.4    | -6.9    | 2.4         | 2.7         | 2.7     |
|       | -24.0            | 7478              | 7262 | 7366    | -2.8    | 2.7     | -12.1       | 3.1         | 3.1     |
| 1(0+) | 137.4            | 7393              | 7423 | 7527    | 3.6     | 7.4     | 7.4         | 3.6         | 3.6     |
|       | -48.3            | 7454              | 7238 | 7342    | 3.1     | -12.7   | -12.7       | 3.1         | 3.1     |

$F_\gamma = bc\bar{u}\bar{d}$

| $J^P$ | $E_{\text{CMI}}$ | $M_{\text{upper}}$ | $BK$ | $X(4140)$ | $K_b n$ | $K_n$ | $K_{\text{cii}}$ | $K_{\text{cii}}$ | $K_{\text{n}}$ |
|-------|------------------|-------------------|------|---------|---------|-------|-------------|-------------|---------|
| 1(2+) | 106.7            | 6427              | 6215 | 6307    | 2.7     | 2.7    | 2.7         | 2.7         | 2.7     |
| 1(1+) | 197.9            | 3188              | 2966 | 3058    | -3.1    | 2.7    | 6.9         | 3.5         | 3.5     |
|       | 42.4             | 3033              | 2810 | 2902    | -0.9    | -8.0   | 2.7         | 2.7         | 2.7     |
|       | -166.3           | 2824              | 2601 | 2693    | -2.6    | 2.7    | -12.3       | 3.1         | 3.1     |
| 1(0+) | 269.5            | 3260              | 3037 | 3129    | 3.5     | 7.5    | 7.5         | 3.5         | 3.5     |
|       | -206.4           | 6114              | 5902 | 5993    | 3.1     | -12.8  | -12.8       | 3.1         | 3.1     |

$F_\gamma = bs\bar{u}\bar{d}$

| $J^P$ | $E_{\text{CMI}}$ | $M_{\text{upper}}$ | $DK$ | $X(4140)$ | $K_b n$ | $K_n$ | $K_{\text{cii}}$ | $K_{\text{cii}}$ | $K_{\text{n}}$ |
|-------|------------------|-------------------|------|---------|---------|-------|-------------|-------------|---------|
| 1(2+) | 128.0            | 3118              | 2896 | 2988    | 2.7     | 2.7    | 2.7         | 2.7         | 2.7     |
| 1(1+) | 197.9            | 3188              | 2966 | 3058    | -3.1    | 2.7    | 6.9         | 3.5         | 3.5     |
|       | 42.4             | 3033              | 2810 | 2902    | -0.9    | -8.0   | 2.7         | 2.7         | 2.7     |
|       | -166.3           | 2824              | 2601 | 2693    | -2.6    | 2.7    | -12.3       | 3.1         | 3.1     |
| 1(0+) | 269.5            | 3260              | 3037 | 3129    | 3.5     | 7.5    | 7.5         | 3.5         | 3.5     |
|       | -253.1           | 2737              | 2515 | 2607    | 3.1     | -12.8  | -12.8       | 3.1         | 3.1     |

$F_\gamma = cs\bar{u}\bar{d}$

FIG. 1: Mass spectrum for the considered tetraquark states.
TABLE VI: Rearrangement decays for $F_1 = bc\bar{s}n$, $F_2 = bs\bar{c}n$, and $F_3 = bn\bar{c}s$. The numbers in the parentheses are $(100 |M|^2/\alpha^2$, $10^7 \Gamma/\alpha^2)$. The symbol “−” means that the decay is forbidden. Decay channels from left to right are presented with increasing thresholds.

| $F_1$ states | Channels | $\Gamma_{\text{sum}}$ |
|-------------|----------|-----------------|
| $J^P = 2^+$ | $B_s^0 D^*$ $\bar{B}^* D_s^{*+}$ | 7542 (34.6,1.4) (32.1,1.3) 2.7 |
| $J^P = 1^+$ | $B_s^0 D$ $B_s^* D^*_s$ $B_s^0 D^*$ $\bar{B}^* D_s^{*+}$ $B_s^0 D^*$ $\bar{B}^* D_s^{*+}$ | 7521 (65.4,2.5) (67.9,2.4) 4.9 |
| $J^P = 0^+$ | $B_s^0 D$ $B_s^* D_s^*$ $B_s^0 D^*$ $\bar{B}^* D_s^{*+}$ $B_s^0 D^*$ $\bar{B}^* D_s^{*+}$ | 7569 (0.6,0.0) (0.6,0.0) (8.8,0.5) (9.6,0.5) (46.7,2.1) (45.1,2.0) 5.2 |
| $J^P = 2^+$ | $B_s^0 K$ $B_s^* K_s^*$ $B_s^0 K^*$ $\bar{B}^* D_s^{*+}$ $B_s^0 D^*$ $\bar{B}^* D_s^{*+}$ | 7552 (0.1,0.0) (44.7,2.6) (6.4,0.3) (8.0,0.1) (0.0,0.0) (60.7,2.5) 5.5 |
| $J^P = 1^+$ | $B_s^0 K$ $B_s^* K_s^*$ $B_s^0 K^*$ $\bar{B}^* D_s^{*+}$ $B_s^0 D^*$ $\bar{B}^* D_s^{*+}$ | 7513 (0.7,0.0) (82.9,2.8) 2.9 |
| $J^P = 0^+$ | $B_s^0 K$ $B_s^* K_s^*$ $B_s^0 K^*$ $\bar{B}^* D_s^{*+}$ $B_s^0 D^*$ $\bar{B}^* D_s^{*+}$ | 7600 (79.1,4.5) (0.8,0.1) (30.0,1.5) 6.1 |
| $J^P = 2^+$ | $B_s^0 K$ $B_s^* K_s^*$ $B_s^0 K^*$ $\bar{B}^* D_s^{*+}$ $B_s^0 D^*$ $\bar{B}^* D_s^{*+}$ | 7572 (19.5,1.0) (4.9,0.3) (60.7,2.2) 3.7 |
| $J^P = 1^+$ | $B_s^0 K$ $B_s^* K_s^*$ $B_s^0 K^*$ $\bar{B}^* D_s^{*+}$ $B_s^0 D^*$ $\bar{B}^* D_s^{*+}$ | 7535 (3.3,0.2) (1.4,0.0) (83.5,3.4) (4.6,--) 3.7 |
| $J^P = 0^+$ | $B_s^0 K$ $B_s^* K_s^*$ $B_s^0 K^*$ $\bar{B}^* D_s^{*+}$ $B_s^0 D^*$ $\bar{B}^* D_s^{*+}$ | 7120 (96.6,4.9) (0.0,--) (10.8,--) (4.8,--) 4.9 |

A. Systems composed of $b$, $c$, $s$, and $n$ quarks

In this case, we have three systems, $F_1 = bc\bar{s}n$, $F_2 = bs\bar{c}n$, and $F_3 = bn\bar{c}s$. The mass formula for them reads

$$M = [M_{X(1410)} - (E_{\text{CM1}})_{X(1410)}] + \Delta_{bc} - \Delta_{sn} + E_{\text{CM1}}.$$  

The $F_1$ states are partners of the predicted $T_{cc} = cc\bar{u}d$ tetraquark. They were considered previously in Ref. [72] with the same CMI model. The $F_2$ and $F_3$ states considered in Ref. [69] are partners of the $cs\bar{e}s$ tetraquarks. Now, with the present mass formula, the mass splittings between all these involved states are determined by the color-spin...
TABLE VII: Rearrangement decays for isovector $F_4 = ba\bar{c}d$, $F_6 = ba\bar{s}d$, and $F_8 = cu\bar{s}d$. The numbers in the parentheses are $(100|\mathcal{M}|/\alpha^2, 10^7\alpha^2/\text{MeV})$. The symbol "-" means that the decay is forbidden. Decay channels from left to right are presented with increasing thresholds.

| $F_4$ states | Channels | $\Gamma_{\text{sum}}$ |
|--------------|----------|-----------------|
| $\ell^F = 2^{+}$ | $B_{c}^{-}\rho^{+} \bar{B}^{0}\bar{D}^{*0}$ |
| 7567 (99.8,6.0) (13.9,0.8) | 6.8 |
| 7415 (0.2,0.0) (86.1,3.1) | 3.1 |
| $\ell^F = 1^{+}$ | $B_{c}^{-}\pi^{+} \bar{B}^{0}\rho^{+} \bar{B}^{0}\bar{D}^{0} \bar{B}^{0}\bar{D}^{0}$ |
| 7570 (0.0,0.0) (0.4,0.0) (96.8,5.9) (25.0,2.5) (8.4,0.5) (1.7,0.1) | 6.7 |
| 7509 (0.0,0.0) (80.3,5.0) (1.6,0.1) (0.0,0.0) (1.0,0.0) (31.1,1.6) | 6.7 |
| 7448 (0.0,0.0) (18.9,1.1) (1.3,0.1) (6.9,0.4) (10.4,0.5) (46.7,2.0) | 4.1 |
| 7388 (0.1,0.0) (0.2,0.0) (1.0,0.0) (1.0,0.0) (3.2,0.2) (72.1,2.9) (14.0,0.4) | 3.5 |
| 7296 (0.6,0.0) (0.3,0.0) (0.2,0.0) (81.8,3.3) (6.4,0.1) (2.6,--) (3.9,--) | 3.4 |
| 6926 (99.3,4.5) (0.0,--) (0.0,--) (5.6,--) (2.6,--) (3.9,--) | 4.5 |

| $F_6$ states | Channels | $\Gamma_{\text{sum}}$ |
|--------------|----------|-----------------|
| $\ell^F = 2^{+}$ | $\bar{B}_{c}^{+}\rho^{+} \bar{B}^{0}\bar{K}^{*+}$ |
| 6374 (97.0,4.9) (24.1,1.2) | 6.1 |
| 6291 (3.0,0.1) (75.9,2.6) | 2.7 |
| $\ell^F = 1^{+}$ | $B_{c}^{-}\pi^{+} \bar{B}^{0}\bar{K}^{+} \bar{B}^{0}\rho^{+} \bar{B}^{0}\bar{K}^{*+} \bar{B}^{0}\bar{K}^{*+}$ |
| 6422 (0.0,0.0) (0.4,0.0) (12.1,1.0) (13.6,0.8) (55.9,3.2) (27.7,1.5) | 6.4 |
| 6244 (0.0,0.0) (0.2,0.0) (57.2,3.1) (6.4,0.3) (32.0,1.5) (23.9,1.1) | 6.0 |
| 6269 (0.2,0.0) (0.3,0.0) (13.5,0.6) (23.6,0.9) (6.0,0.0) (37.5,1.1) | 2.6 |
| 6238 (0.3,0.0) (3.1,0.2) (16.2,0.6) (53.4,1.7) (10.2,0.3) (6.9,0.1) | 3.0 |
| 5963 (12.4,0.7) (72.4,3.1) (1.0,--) (0.0,--) (1.2,--) (2.4,--) | 3.8 |
| 5967 (87.0,2.6) (23.7,--) (0.0,--) (4.4,--) | 4.0 |
| 5640 (85.2,2.4) (25.8,--) (0.1,--) (2.5,--) | 2.4 |

| $F_8$ states | Channels | $\Gamma_{\text{sum}}$ |
|--------------|----------|-----------------|
| $\ell^F = 2^{+}$ | $D_{s}^{*}\rho^{+} \bar{D}^{*+}K^{*+}$ |
| 3059 (95.8,18.5) (27.0,5.2) | 23.6 |
| 2971 (4.2,0.6) (73.0,9.7) | 10.3 |
| $\ell^F = 1^{+}$ | $D_{s}^{*}\pi^{+} \bar{D}^{*+}K^{*+} \bar{D}_{s}^{*}\rho^{+} \bar{D}^{*+}K^{*+}$ |
| 3082 (0.0,0.0) (0.8,0.3) (4.5,1.2) (9.3,2.5) (71.4,14.5) (22.2,4.5) | 23.0 |
| 3005 (0.3,0.1) (1.3,0.4) (22.8,5.6) (13.0,3) (26.5,4.3) (62.1,9.9) | 20.7 |
| 2903 (0.7,0.2) (0.0,0.0) (4.4,0.1) (12.0,2.4) (0.0,0.0) (12.7,0.1) | 11.8 |
| 2838 (0.0,0.0) (3.2,0.9) (26.4,4.3) (71.3,10.9) (1.5,--) (0.1,--) | 16.0 |
| 2636 (10.5,2.7) (72.9,14.1) (1.8,--) (4.4,--) (0.8,--) (1.5,--) | 16.8 |
| 2385 (88.4,13.7) (21.8,--) (0.0,--) (1.7,--) (0.0,--) (1.4,--) | 13.7 |
| $\ell^F = 0^{+}$ | $D_{s}^{*}\pi^{+} \bar{D}^{*+}K^{*+} \bar{D}_{s}^{*}\rho^{+} \bar{D}^{*+}K^{*+}$ |
| 3150 (0.0,0.0) (0.2,0.1) (63.4,14.5) (46.2,10.6) | 25.2 |
| 2919 (0.7,0.3) (3.1,1.1) (34.0,3.0) (47.5,3.2) | 7.5 |
| 2537 (17.1,5.2) (67.2,16.1) (2.5,--) (4.1,--) | 21.3 |
| 2213 (82.1,12.7) (29.4,--) (0.1,--) (2.1,--) | 12.7 |

Interactions. Once an exotic tetraquark containing $b$, $c$, $s$, and $n$ quarks could be observed, one may use such a state to correct the tetraquark masses obtained here.

Compared with the $F_1$ results using the $B_s D$ threshold (adopted in Ref. [72]), the updated masses are 100 MeV higher, but are still 200 MeV lower than the theoretical upper limits. If the present results are more realistic, the lowest $0^{+}$ and $1^{+}$ states without strong decays in Ref. [72] are not stable any longer. Here “realistic” means that the masses are closer to the measured ones by future experiments. From the values of $K_{ij}$'s, the mass splittings are not affected significantly by the uncertainties of the coupling parameters. We have argued in Ref. [69] that states with negative $K_{qq}$ and positive $K_{q\bar{q}}$ are probably relatively stable. One will see that parts of them satisfy this feature.
TABLE IX: Rearrangement decays for \(cs\bar{s}\) states. The numbers in the parentheses are \((100|M|^2/\alpha^2, 10^7\Gamma/\alpha^2\text{MeV})\). The symbol “\(\ast\)” means that the decay is forbidden. Decay channels from left to right are presented with increasing thresholds.

| States | Channels | \(\Gamma_{\text{sum}}\) |
|--------|----------|-----------------|
| \(J^{PC} = 2^{++}\) | \(J/\psi\) \(D_s^+D_s^-\) | 14.5 |
| 4313.2 | (51.1,4.8) | |
| 4291.2 | (48.9,4.0) | 6.2 |
| \(J^{PC} = 1^{++}\) | \(D_s D_s^*\) \(J/\psi\) | 14.3 |
| 4309.4 | (99.7,11.9) | 15.7 |
| 4146.5 | (0.3,0.0) | |
| \(J^{PC} = 1^{--}\) | \(\eta_s\) \(D_s D_s^*\) \(D_s^+D_s^-\) | |
| 4304.8 | (0.2,0.1) | |
| 4222.3 | (98.1,8.7) | 10.0 |
| 4164.7 | (16.5,3.9) | 10.6 |
| 4095.7 | (0.1,0.0) | 15.5 |
| \(J^{PC} = 0^{++}\) | \(D_s^+D_s^-\) \(J/\psi\) | |
| 4366.9 | (28.4,2.4) | 2.4 |
| 4248.0 | (0.6,0.0) | 13.7 |
| 4079.7 | (13.7,6.1) | 6.5 |
| 3944.4 | (0.8,0.1) | 6.1 |

TABLE VIII: Rearrangement decays for \(F_5 = bc\bar{u}, F_7 = bs\bar{d},\) and \(F_6 = cs\bar{u}\). The numbers in the parentheses are \((100|M|^2/\alpha^2, 10^7\Gamma/\alpha^2\text{MeV})\). The symbol “\(\ast\)” means that the decay is forbidden. Note that the widths for \(\bar{B}^{(*)} D^{(*)}\) include results for both \(B^{(*)} D^{(*)}\) and \(\bar{B}^{(*)} \bar{D}^{(*)}\) channels, the widths for \(\bar{B}^{(*)} K^{(*)}\) include results for both \(B^{(*)} \bar{K}^{(*)}\) and \(\bar{B}^{(*)}\bar{K}^{(*)}\) channels, and the widths for \(D^{(*)}\bar{K}^{(*)}\) include results for both \(D^{(*)}\bar{K}^{(*)}\) and \(D^{(*)}\bar{K}^{(*)}\) channels. Decay channels from left to right are presented with increasing thresholds.

| States | Channels | \(\Gamma_{\text{sum}}\) |
|--------|----------|-----------------|
| \(F_5\) states | Channels | \(\Gamma_{\text{sum}}\) |
| \(I(J^P) = 1(2^+)\) \(B^* D^*\) | \(I(J^P) = 0(2^+)\) \(B^* D^*\) | 4.9 |
| 7467 | (33.3,3.0) | 3.0 |
| \(I(J^P) = 1(1^+)\) \(B^* D^*\) | \(I(J^P) = 0(1^+)\) \(B^* D^*\) | |
| 7500 | (0.8,0.1) | 5.8 |
| 7438 | (12.9,1.5) | 3.2 |
| 7366 | (39.6,4.1) | 4.3 |
| \(I(J^P) = 1(0^+)\) \(B^* D^*\) | \(I(J^P) = 0(0^+)\) \(B^* D^*\) | |
| 7527 | (0.4,0.1) | 6.1 |
| 7342 | (41.2,4.5) | 4.6 |
| \(F_7\) states | Channels | \(\Gamma_{\text{sum}}\) |
| \(I(J^P) = 1(2^+)\) \(B^* K^*\) | \(I(J^P) = 0(2^+)\) \(B^* K^*\) | |
| 6307 | (33.3,2.5) | 2.5 |
| \(I(J^P) = 1(1^+)\) \(B^* K^*\) | \(I(J^P) = 0(1^+)\) \(B^* K^*\) | |
| 6025 | (41.4,4.2) | 4.2 |
| 6423 | (0.2,0.0) | 6.2 |
| 5994 | (41.5,4.4) | 4.4 |
| \(F_6\) states | Channels | \(\Gamma_{\text{sum}}\) |
| \(I(J^P) = 1(2^+)\) \(D^* K^*\) | \(I(J^P) = 0(2^+)\) \(D^* K^*\) | |
| 2988 | (33.3,9.8) | 9.8 |
| \(I(J^P) = 1(1^+)\) \(D^* K^*\) | \(I(J^P) = 0(1^+)\) \(D^* K^*\) | |
| 3058 | (0.4,0.2) | 21.9 |
| 2902 | (1.1,0.7) | 12.8 |
| 2693 | (40.2,18.1) | 18.1 |
| \(I(J^P) = 1(0^+)\) \(D^* K^*\) | \(I(J^P) = 0(0^+)\) \(D^* K^*\) | |
| 3129 | (0.2,0.1) | 24.6 |
| 2607 | (41.5,22.8) | 22.8 |
From table VI, the $F_1$, $F_2$, and $F_3$ states have comparable widths. If we adopt $\alpha = 7.27$ MeV, the widths of these states are about $10 \sim 33$ MeV. If such tetraquarks do exist, one may conclude that all of them are measurable according to the present model calculation. We now examine the decay properties of states in each system.

1. $F_1 = bcs\bar{n}$ states

The lowest $bcs\bar{n}$ tetraquark is a scalar state and it has two rearrangement decay channels $\bar{B}^0D_s$ and $\bar{B}D_s^+$ with almost equal coupling matrix elements and partial widths. The second lowest $0^+$ $bcs\bar{n}$ shares the same channels with the lowest one. Although the coupling matrix elements for decay are smaller than those of the lowest tetraquark, the larger phase spaces lead to larger partial widths. The second highest $0^+$ $bcs\bar{n}$ has four rearrangement channels with comparable partial widths. The couplings with the channels $\bar{B}^0D_s$ and $\bar{B}D_s^+$ are weak but the large phase spaces ensure nonvanishing partial widths. Although the phase spaces for $B^{*0}D^*$ and $B^*D_s^+$ are small, the large coupling matrix elements for decay result in similar widths to the channels $\bar{B}^0D_s$ and $\bar{B}D_s^+$. For the highest $0^+$ $bcs\bar{n}$, the stronger couplings with $\bar{B}^0D^*$ and $B^*D_s^+$ and the larger phase spaces for these two channels give the largest width. In Ref. [69], we have found that the second highest $0^+$ $Qq\bar{q}\bar{q}$ tetraquark usually has a relatively stable structure based on the signs of $K_{ij}$’s. Now, the second highest state of these four $bcs\bar{n}$ tetraquarks is the narrowest one ($\sim 10$ MeV). From table IV, $K_{bc}$ and $K_{bs}$ for this state are both negative while $K_{bs}$, $K_{bn}$, $K_{cs}$, and $K_{cn}$ are all positive, which indicates that this tetraquark tends to be a $((qq)(q\bar{q}))$ but not $((q\bar{q})(q\bar{q}))$ structure. Therefore, the narrow rearrangement decay width is consistent with the argument from $K_{ij}$ for this state.

There are six $J^P = 1^+$ $bcs\bar{n}$ states. The lowest $1^+$ has similar rearrangement decay property to the lowest scalar state: two channels with almost equal coupling matrix elements and comparable widths. The second and third lowest states are almost degenerate, but their dominant coupling channels are different. The former mainly couples to $B^{0}_sD_s^*$ and $\bar{B}D_s^{*+}$ while the latter to $B^{*0}_sD_s$ and $B^*D_s^+$. These channels have important contributions to the decay widths. For the second lowest state, the contribution from the channel $B^*D_s^+$ is also significant, although the coupling matrix element is not so large. The third and second highest states are also almost degenerate. Main contributions to the decay width of the third highest state are from the $B^{*0}_sD^*$ and $B^*D_s^{*+}$ channels as well as the $B^{0}_sD^*$ channel while those to the second highest state are the $\bar{B}^0_sD^*$ and $\bar{B}D_s^{*+}$ channels. As for the highest state around 7570 MeV, it mainly decays into $\bar{B}^{*0}_sD^*$ and $B^*D_s^{*+}$. From the signs of $K_{ij}$’s, one cannot get a correct conclusion about the width. From table IV, only the highest $1^+$ state is probably stable, but the results in table VI illustrate that the second highest state is relatively stable.

The two $2^+$ states share the same rearrangement decay channels. The lower state has larger width than the higher one does, which cannot be understood just from the signs of $K_{ij}$’s. The reason is that the mass difference between the two states is not large but the lower state has stronger couplings with the two channels.

2. $F_2 = bs\bar{c}n$ and $F_3 = bn\bar{c}s$ states

For the four $0^+$ $bs\bar{c}n$ states, each state has a dominant decay channel. The second highest one is relatively stable, which is consistent with the argument with $K_{ij}$’s [69]. For the three lowest $1^+$ states and the highest $1^+$ state, each one has a dominant decay channel. The second highest $1^+$ $bs\bar{c}n$ gets contributions mainly from $B^*D_s^0$ and $B_s^-K^*$. The third highest state has the dominant decay channel $B_s^-K^*$, but the decay into $B^*D_s^0$ is also significant. One cannot understand the relatively stable nature of the second lowest state just from $K_{ij}$’s. The broad width for the lowest $1^+$ state is due to the strong coupling with the $B_s^-K$ channel and the large phase space. For the two $2^+$ $bs\bar{c}n$ tetraquarks, the order of decay width is nothing special. The higher state is broader although the $cn$ and $bs$ diquarks have slightly attractive color-magnetic interactions.

In the $bn\bar{c}s$ case, the features of decay widths and spectrum are similar to the $bs\bar{c}n$ case.

B. Systems composed of b, c, u, and d quarks

Because of the isospin symmetry, there are two systems in this case, $F_4 = bu\bar{c}d$ and $F_5 = bc\bar{u}d$. The mass formula we use is

$$M = [M_{X(4140)} - (E_{CM})_{X(4140)}] + \Delta_{bc} - 2\Delta_{sn} + E_{CM}.$$  \hspace{1cm} (14)

We have studied the spectrum for the $F_4$ states in Ref. [69] and do not repeat the results here. For the $F_5$ tetraquarks, they are also partner states of $T_{cc}$ and their masses with Eq. (7) were estimated in Ref. [72]. Here, with the updated
method, we get 100 MeV higher results (still more than 100 MeV lower than the theoretical upper limits), which makes the previously stable lowest $0^+$ and $1^+$ states decay. Compared with the result in Ref. [95] where the lowest $bc\bar{u}\bar{d}$ is slightly below the $BD$ threshold, the corresponding state in the present method is slightly above the threshold. If the present CMI model is correct, the $F_4$ and $F_5$ spectra can be related and the observation of one state can be used to predict other states. The effects due to the uncertainties of coupling parameters $C_{ij}$'s on the mass splittings can be understood from the values of $K_{ij}$'s in table XI of Ref. [69] and table V.

Now we move on to the decay properties of the $F_4$ states. Since we are interested in tetraquarks with four different flavors, it is not necessary to consider the isoscalar case and we only give results in table VII for the isosvector decay channels. The feature for the four $0^+$ $bc\bar{u}\bar{d}$ is similar to the $F_2 = bs\bar{c}\bar{n}$ case where each state has a dominant decay channel. Different from the expectation from $K_{ij}$'s that the second highest state is relatively stable, the second lowest one has a smaller width. The reason is that the state couples mainly to a channel with smaller phase space. For the six $1^+ bc\bar{d}$ tetraquarks and the two $2^+$ states, one also finds similar features to the $F_2 = bs\bar{c}\bar{n}$ case.

There are two possible isospins for the $F_5 = bc\bar{u}\bar{d}$ states. Because of the Pauli principle, the two isospin cases have different decay properties which can be seen from table VIII. The isovector tetraquarks are generally broader than the isoscalar states. The lowest $I = 1$ and $I = 0$ $bc\bar{u}\bar{d}$ states both decay dominantly to the $BD$ channels (note $BD$ means both $B^-D^+$ and $B^0\bar{D}^0$). The smaller phase space for the $I = 0$ state decides its small width ($\sim 5$ MeV with $\alpha = 7.27$ GeV). The width of the $I = 1$ state is also not very large, although it is about 200 MeV heavier than the $I = 0$ state. The higher $I = 1$ $0^+$ tetraquark mainly decays to the $B^*D^*$ channels. The coupling of the higher $I = 1$ $0^+$ state with these channels is also strong, but the decay is kinematically forbidden. The dominant decay channels for the $I = 1$ and $I = 0$ $1^+ bc\bar{u}\bar{d}$ tetraquarks are manifest from table VIII. For the $2^+$ tetraquarks, the width of the isoscalar state is larger than the isovector state, although the former state is lighter. This is because of the stronger coupling matrix element. We cannot judge the stability just from the signs of $K_{ij}$'s, although several states satisfy the condition $K_{qq} < 0, K_{\bar{q}\bar{q}} > 0$. If we use $\alpha = 7.27$ GeV, the largest decay width for the $F_2$ states is around 35 MeV. Therefore, these possible tetraquark states can be detectable if they do exist. Of course, whether the value of $\alpha$ is reasonable for the double-heavy tetraquarks needs further studies.

C. Systems composed of $b, s, u, d$ quarks

Replacing the $c$ quark in $F_4$ and $F_5$ systems with the $s$ quark, one gets the $F_6 = bu\bar{s}\bar{d}$ and $F_7 = bs\bar{u}\bar{d}$ systems. Now the mass formula reads

$$M = [M_{X(4140)} - (E_{CMI}X(4140))] + \Delta_{bc} - \Delta_{cn} - \Delta_{sn} + E_{CMI}.$$  \hfill (15)

These systems can be generally denoted as $bq\bar{q}\bar{q}$ where $q = u, d, s$. The $X(5568)$ is a $F_6$ state. In Ref. [48], we performed preliminarily a systematic study of the $QQ\bar{q}\bar{q}$ ($Q = b, c$) states with the CMI model. That study suffered from a large uncertainty due to the quark mass choice and the $SU(3)$ breaking effects. Here, we revisit the systems with the updated scheme. In this subsection, we focus on the $Q = b$ case.

1. $F_6 = bu\bar{s}\bar{d}$ states

The mass difference between the lowest and highest tetraquarks is more than 700 MeV. The reason is the system contains three light quarks and the coupling parameters for light quarks are larger than those for heavy quarks. Compared with the former systems, the estimated tetraquark masses of several states will get stronger effects from the uncertainties of coupling parameters, which can be understood from the $K$ factors in table IV. If the obtained masses are realistic, all the $bu\bar{s}\bar{d}$ tetraquarks have rearrangement channels.

The D0 $X(5568)$ is related to the lowest $0^+$ or $1^+$ $bu\bar{s}\bar{d}$ tetraquark state. Our lowest $0^+$ $bu\bar{s}\bar{d}$ is about 70 MeV higher than the $X(5568)$ mass. It has only one rearrangement decay mode, $B_{s0}\pi^+$, with $\Gamma \sim 15$ MeV. The width is roughly consistent with that of $X(5568)$, $\Gamma = 21.9_{-6.4}^{+8.1}$ MeV, but one will get an inconsistent result ($\Gamma \sim 9$ MeV) if we adjust the mass to 5568 MeV. The second lowest $bu\bar{s}\bar{d}$ couples mainly to another mode $B_{s0}K^+$. Its width is around 20 MeV. For the other two higher $0^+$ states, the dominant decay modes are both $B_{s0}\rho^+$ and $B_{s0}K^{*+}$. Again, one finds that the second highest scalar tetraquark has a relatively stable structure. This is consistent with the argument from $K_{ij}$'s.

What the D0 Collaboration measured is probably also a state around 5616 MeV with $J^P = 1^+$. Our lowest $1^+$ $bu\bar{s}\bar{d}$ with $\Gamma \sim 15$ MeV is 80 MeV higher than the experimental result. If we set its mass to 5616 MeV, the width will become a value around 9 MeV. We still cannot understand the experimental result. For the other $1^+$ states, the third highest one is relatively stable, but one cannot use the $K$ factors to judge its stability. The reason for the relatively
narrow width is that it decays mainly into channels having small phase spaces with weaker couplings. For the two $2^+$ states, each one has a dominant rearrangement channel. The higher state is not relatively stable although it satisfies the condition $K_{qq} < 0, K_{q \bar{q}} > 0$.

2. $F_7 = bs \bar{u}d$ states

The spectrum has some similar features to the $F_5 = bc \bar{u}d$ case since the differences come only from the coupling strengths. Compared with the $bc \bar{u}d$ tetraquarks, the widths of the low-lying $I = 0$ $bs \bar{u}d$ are smaller but those of other states are similar.

In Ref. [65], a possible stable $bs \bar{u}d$ tetraquark state was proposed by noticing the higher threshold for rearrangement decay. The investigation with a chiral quark model in Ref. [68] indicates that no stable diquark-antidiquark $bs \bar{u}d$ exists but a bound molecule-type state with $I(J^P) = 0(0^+)$ is possible. The weak decays of the stable $bs \bar{u}d$ were studied in Ref. [96]. Our lowest isoscalar $0^+$ and $1^+$ tetraquarks are both slightly below respective thresholds for rearrangement decay channels and they are probably stable states. If their masses are underestimated, their widths should not be larger than those of the corresponding $I = 1$ states ($\Gamma \sim 22$ MeV). Therefore, the $bs \bar{u}d$ exotic states are worthwhile to search for.

D. Systems composed of $c$, $s$, $u$, and $d$ quarks

The $F_8 = cu \bar{s}d$ and $F_9 = cs \bar{u}d$ systems are obtained with the replacement $b \rightarrow c$ for the $F_6$ and $F_7$ systems. The mass formula in this case is

$$M = [M_{X(4140)} - (E_{CMI})_{X(4140)}] - \Delta_{sn} - \Delta_{sn} + E_{CMI}.$$  \hspace{1cm} (16)

For the spectrum of $F_8$ ($F_9$), the features are similar to those of $F_6$ ($F_7$). However, the decay widths are larger than the corresponding bottom cases. Since the coupling matrix elements have similar values, the difference in width is because of the phase spaces.

In the $F_8$ case, the rearrangement decay width of the lowest $0^+$ $cu \bar{s}d$ is about 70 MeV if we use $\alpha = 7.27$ GeV. The narrowest tetraquark is the second highest $0^+$ state with $\Gamma \sim 40$ MeV, which is consistent with the argument from the $K$ factors. For the highest and the second lowest $0^+$ tetraquarks, the widths are more than 110 MeV. For the $1^+$ states, the widths range from 62 MeV to 122 MeV. Those for the two $2^+$ states are 54 MeV and 125 MeV. These values are all measurable if our results are realistic.

In the $F_9$ case, the lowest $0^+$ and $1^+$ tetraquarks are both states with narrow widths, decaying strongly with small phase space or decaying weakly. The widths of the isovector tetraquarks (52−130 MeV) are comparable to those of the $F_8$ states. For the isoscalar states, the higher $0^+$ is not broad ($\Gamma \sim 10$ MeV), but the other $1^+$ and $2^+$ states are ($\Gamma = 50 \sim 100$ MeV).

IV. $K$ FACTORS AND DECAY WIDTHS

Previously, we conjectured that a tetraquark state is probably relatively stable if $K_{qq} < 0$ and $K_{q \bar{q}} > 0$. The numerical results for rearrangement decays have shown us that this argument is correct only for some states, in particular the second highest $J = 0$ tetraquarks. If the argument is applicable for a state, it is probably an accidental event because the decay width is determined by the coupling matrix element, the phase space, and the number of decay channels together. In the adopted simple decay model, the $K$ factors are only directly related to the coupling matrix elements. Now we move on to the relation between $K_{ij}$’s and $\Gamma_{sum}$ by treating $y_i$ in Eq. (11) as known coefficients. In the following discussions, for convenience, if the state stability can be judged from the $K$ factor(s), we will call it stability rule.

As shown in Ref. [16], the $K$ factors can also be expressed as (see also Appendix A)

$$K_{mn} = x_k X_{mn}^{kl} x_l,$$  \hspace{1cm} (17)

where $x_i$’s form eigenvectors of the matrix $\langle \hat{H}_{CMI} \rangle = \sum_{m<n} X_{mn} C_{mn}$. Therefore, $K_{mn}$ is not an eigenvalue of the matrix $X_{mn}$ but it encodes the structure information of the initial tetraquark state. When one wants to express the wavefunction ($x_1, x_2, \cdots$) with the $K$ factors, this definition becomes a dimensionless equation. On the other hand, $y_i$’s in Eq. (11) may form eigenvectors of $X_{13} (X_{14})$ and/or $X_{24} (X_{23})$ (see Appendix A). From the form of the
coupling matrix element $|\mathcal{M}|^2$, it is possible to rewrite it with the defined $K$ factors. In this situation, the $K$ factors are linearly related with the decay widths. If they can be determined from the partial widths, one may use their values to judge whether the initial meson is a tetraquark- or molecule-dominant state according to their properties. In principle, the $x_i$'s can be further extracted with these $K$ factors and we may get the wavefunctions of the initial state. Since the number of independent $x_i$'s is known for states with a given $J$, the $K$ factors should not be independent and their relations should be nonlinear. However, what we are interested in are the relations in a linear form. Since the $F_5$, $F_7$, and $F_9$ states are affected by the Pauli principle, we consider them separately.

A. The systems without Pauli constraint

For the $F_{1,2,3,4,6,8}$ states, the coupling matrix elements satisfy the constraint

$$0 \leq |x_i y_i|^2 \leq 1$$

according to the normalization conditions and the Cauchy-Buniakowsky-Schwarz Inequality. When $x_i = y_i$, $|x_i y_i|^2$ has the maximum value 1. That is to say, the strongest coupling occurs when the initial state and the final state have the same structure.

In the case $J = 2$, we have

$$X_{12} = X_{34} = \left( -\frac{4}{3}, \frac{0}{3} \right), \quad X_{13} = X_{24} = \left( \frac{10}{3}, -\sqrt{8} \right), \quad X_{14} = X_{23} = \left( \frac{10}{3}, \sqrt{8} \right),$$

where the base order is the same as before. The eigenvalues of $X_{13}$ and $X_{14}$ are $16/3$ and $-2/3$. They are nothing, but the $K$ factors of the $(q\bar{q})_{J=1}^{J=1}$ and $(q\bar{q})_{J=1}^{J=1}$ states (see Appendix A). With the explicit expressions of $K$ factors from Eq. (17), we get the constraints

$$-\frac{2}{3} \leq K_{13} \leq \frac{16}{3}, \quad -\frac{2}{3} \leq K_{14} \leq \frac{16}{3}, \quad -\frac{4}{3} \leq K_{12} \leq \frac{8}{3}. \quad (20)$$

Solving equations of the $K$ factors, one can express the coupling matrix elements for the $[(q_1\bar{q}_3)^{J=1}(q_2\bar{q}_4)^{J=1}]$ and $[(q_1\bar{q}_3)^{J=1}(q_2\bar{q}_4)^{J=1}]$ mode final states as

$$|\mathcal{M}|^2 = \frac{1}{3}(\sqrt{2}x_1 - x_2)^2 = \frac{3K_{13} + 2}{18}, \quad |\mathcal{M}|^2 = \frac{1}{3}(\sqrt{2}x_1 + x_2)^2 = \frac{3K_{14} + 2}{18}, \quad (21)$$

respectively. Therefore, these $K$ factors can be related to the decay widths directly. If $K_{13}$ approaches to $16/3$, the coupling matrix element tends to be 1 and it means that the initial tetraquark state has a significant $(q_1\bar{q}_3)^{J=1}(q_2\bar{q}_4)^{J=1}$ molecule component. If $K_{13}$ approaches to $-2/3$, $|\mathcal{M}|^2 \to 0$ means that the hidden-color component $(q_1\bar{q}_3)^{J=1}(q_2\bar{q}_4)^{J=1}$ contributes to the initial tetraquark significantly. The value of $|\mathcal{M}|^2$ can also reflect the $(q_1\bar{q}_3)(q_2\bar{q}_4)$ contribution.

When we present the values of $K_{13}$ and $K_{14}$ in tables IV and V, the coupling parameters in table II are extracted from the conventional hadrons. For multiquark states, these coupling parameters and thus the values of $K_{13}$ and $K_{14}$ may be different from those shown in the tables. If $K_{13}$ and $K_{14}$ can be extracted from the partial decay widths, one may reversely extract some information about the internal structures of the initial tetraquark states. Now, we assume that the ratio

$$r = \frac{3K_{13} + 2}{3K_{14} + 2} \quad (22)$$

has been measured experimentally. From the definitions, one finds two solutions (structures) for the initial state,

$$x_1 = \pm \frac{(\sqrt{r} + 1)}{\sqrt{3r - 2\sqrt{r} + 3}}, \quad x_2 = \mp \frac{\sqrt{2}(\sqrt{r} - 1)}{\sqrt{3r - 2\sqrt{r} + 3}}; \quad (23)$$

$$x_1 = \pm \frac{(\sqrt{r} - 1)}{\sqrt{3r + 2\sqrt{r} + 3}}, \quad x_2 = \mp \frac{\sqrt{2}(\sqrt{r} + 1)}{\sqrt{3r + 2\sqrt{r} + 3}}. \quad (24)$$

If $r = 0$ or $\infty$, the initial state is a $8_c8_c$ meson-meson-like tetraquark state. In other cases, however, one cannot distinguish the two initial structures just from a given $r$. For example, if $r = 9$ is obtained, the solution is

$$x_1 = \pm \frac{\sqrt{6}}{3}, \quad x_2 = \pm \frac{1}{3}; \quad or \quad x_1 = \pm \frac{1}{3}, \quad x_2 = \pm \frac{2\sqrt{2}}{3}. \quad (25)$$
The former solution corresponds to the \((q_1 \bar{q}_3)(q_2 \bar{q}_4)\) molecule \((K_{13} = 16/3, K_{14} = 0)\) while the latter corresponds to a tetraquark \((K_{13} = 10/3, K_{14} = -2/9)\) (or mixed \((q_1 \bar{q}_3)(q_2 \bar{q}_4)\)\(\sim\)\((q_1 \bar{q}_4)(q_2 \bar{q}_3)\) structure). To determine the unique structure of the initial state, we should know the values of both \(K_{13}\) and \(K_{14}\).

The factor \(K_{12}\) is not directly related to the decay width, but there is a linear relation between \(K_{12}, K_{13},\) and \(K_{14},\)

\[
K_{12} = \frac{16}{3} - (K_{13} + K_{14}).
\]  

(26)

The constraint for the value of \(K_{12}\) and this equation reflects the fact that \(K_{13}\) and \(K_{14}\) are actually not independent. Their relation is nonlinear.

Now we come back to the decay problem argued with \(K_{qq}\). The total width of a tetraquark state in the adopted rearrangement model can be written as the form

\[
\Gamma_{\text{sum}} = \frac{\phi_a}{18}(3K_{13} + 2) + \frac{\phi_b}{18}(3K_{14} + 2),
\]  

(27)

where \(\phi_a\) \((\phi_b)\) is determined by the phase space for the \((q_1 \bar{q}_3)(q_2 \bar{q}_4)\) \([\(q_1 \bar{q}_4)(q_2 \bar{q}_3)\]) type decay and it is equal to \(\alpha^2|p_1|/(8\pi M^2)\) with \(p_1\) being the three-momentum of a final meson in the rest frame of the initial state of mass \(M\). If the difference between \(\phi_a\) and \(\phi_b\) is not large, we have

\[
\Gamma_{\text{sum}} \approx \frac{\phi_a}{18}(20 - 3K_{12}),
\]  

(28)

Therefore, the larger \(K_{12}\) is, the smaller width the initial tetraquark has, but the width is nonvanishing according to the constraint for \(K_{12}\). In the employed model, the approximation \(\phi_a \approx \phi_b\) is roughly satisfied (see Table X) and one can confirm this feature by checking Tables VI and VII together with the calculated \(K_{12}\)'s. Therefore, the stability rule exists in the \(J = 2\) case.

If one does not consider the approximation \(\phi_a \approx \phi_b\), one may theoretically give a constraint for the total width of a tetraquark state in the rearrangement decay model. From Table X, we have \(\phi_a \geq \phi_b\). Considering the range of \(K_{12}\), one gets

\[
2 \frac{\phi_b}{\phi_a} \leq \Gamma_{\text{sum}} \leq \frac{4}{3} \phi_a.
\]  

(29)

The wavefunctions encode the information of the Hamiltonian. Although the wavefunctions could be determined from the rearrangement decays, information from spectrum is still necessary if one wants to further determine the coupling parameters.

In the case \(J = 0\), the involved matrices read

\[
X_{12} = X_{34} = \begin{pmatrix}
4 & 0 & 0 & 0 \\
0 & -\frac{4}{3} & 0 & 0 \\
0 & 0 & -8 & 0 \\
0 & 0 & 0 & \frac{2}{3}
\end{pmatrix},
\]  

\[
X_{13} = X_{24} = \begin{pmatrix}
0 & -\frac{10}{\sqrt{3}} & 0 & 2\sqrt{6} \\
-\frac{10}{\sqrt{3}} & -\frac{20}{3} & 2\sqrt{6} & 4\sqrt{2} \\
0 & 2\sqrt{6} & 0 & -\frac{4}{\sqrt{3}} \\
2\sqrt{6} & 4\sqrt{2} & -\frac{4}{\sqrt{3}} & -\frac{8}{3}
\end{pmatrix},
\]  

\[
X_{14} = X_{23} = \begin{pmatrix}
0 & \frac{10}{\sqrt{3}} & 0 & 2\sqrt{6} \\
\frac{10}{\sqrt{3}} & -\frac{20}{3} & 2\sqrt{6} & -4\sqrt{2} \\
0 & 2\sqrt{6} & 0 & \frac{4}{\sqrt{3}} \\
2\sqrt{6} & -4\sqrt{2} & \frac{4}{\sqrt{3}} & -\frac{8}{3}
\end{pmatrix}.
\]  

(30)

The eigenvalues of \(X_{13}\) and \(X_{14}\) are \(-16, 16/3, 2,\) and \(-2/3\). They correspond to the \(K\) factors of \((q\bar{q})_{1c}^{I=0}, (q\bar{q})_{1c}^{I=1}, (q\bar{q})_{2c}^{I=0},\) and \((q\bar{q})_{2c}^{I=1}\) states, respectively.

Different from the \(J = 2\) case, the present case is more complicated when we rewrite the coupling matrix elements with the defined \(K\) factors. It is not enough to adopt only \(K_{12}, K_{13},\) and \(K_{14}\), since we have ten variables to be expressed, \(x_i x_j (i, j = 1, \cdots, 4)\). More calculable \(K\) factors need to be used. Here, we define \(K_{mn,kl} = \frac{1}{2} x_i (X_{mn} \cdot X_{kl} + X_{kl} \cdot X_{mn})x_j, K_{mn}^{(2)} = K_{mn,mn},\) and \(K_{mn}^{(3)} = x_i (K_{mn} \cdot K_{mn} \cdot K_{mn})x_j.\) For the rearrangement decays, in
TABLE X: Values of $|\langle \mathbf{p}_1 | M_i^2 | \mathbf{p}_{1,h} \rangle|^2$ for various $F_{1,2,3,4,6,8}$ tetraquark states in the rearrangement decays. Here, $\mathbf{p}_1$ is the three-momentum of a final meson in the rest frame of the initial state of mass $M$ and $\mathbf{p}_{1,h}$ and $M_h$ are for the highest state with the same $J^P$ decaying into the channel with largest phase space. The symbol "\*" means that the decay is forbidden. Decay channels from left to right are presented with the increasing thresholds.

| $F_1 = b\bar{c}e\bar{n}$ | $F_2 = b\bar{s}c\bar{n}$ | $F_3 = b\bar{n}c\bar{s}$ | $F_4 = b\bar{u}c\bar{d}$ | $F_5 = b\bar{u}d\bar{c}$ | $F_6 = b\bar{s}d\bar{c}$ | $F_7 = c\bar{u}d\bar{s}$ |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $J^P = 2^+$ | $B_{c}\bar{D}^{*} D^{*}$ | $B_{c}\bar{D}^{*} D^{*}$ | $B_{c}\bar{D}^{*} D^{*}$ | $B_{c}\bar{D}^{*} D^{*}$ | $B_{c}\bar{D}^{*} D^{*}$ | $B_{c}\bar{D}^{*} D^{*}$ |
| 7542 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| 7521 | 0.9 | 0.9 | 0.9 | 0.9 | 0.9 | 0.9 | 0.9 |
| $J^P = 1^+$ | $\bar{B}_{c}^{0} D^{*} B_{c}^{*} D^{*}$ | $\bar{B}_{c}^{0} D^{*} B_{c}^{*} D^{*}$ | $\bar{B}_{c}^{0} D^{*} B_{c}^{*} D^{*}$ | $\bar{B}_{c}^{0} D^{*} B_{c}^{*} D^{*}$ | $\bar{B}_{c}^{0} D^{*} B_{c}^{*} D^{*}$ | $\bar{B}_{c}^{0} D^{*} B_{c}^{*} D^{*}$ |
| 7569 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| 7511 | 0.9 | 0.9 | 0.9 | 0.9 | 0.9 | 0.9 | 0.9 |
| $J^P = 0^+$ | $\bar{B}_{c}^{0} D^{*} B_{c}^{*} D^{*}$ | $\bar{B}_{c}^{0} D^{*} B_{c}^{*} D^{*}$ | $\bar{B}_{c}^{0} D^{*} B_{c}^{*} D^{*}$ | $\bar{B}_{c}^{0} D^{*} B_{c}^{*} D^{*}$ | $\bar{B}_{c}^{0} D^{*} B_{c}^{*} D^{*}$ | $\bar{B}_{c}^{0} D^{*} B_{c}^{*} D^{*}$ |
| 7596 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| 7444 | 0.8 | 0.8 | 0.8 | 0.8 | 0.8 | 0.8 | 0.8 |
| 7411 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 |
| 7269 | 0.3 | 0.3 | 0.3 | 0.3 | 0.3 | 0.3 | 0.3 |

...addition to $K_{13}$ and $K_{14}$, only $K_{13}^{(2,3)}$ and $K_{14}^{(2,3)}$ appear in the coupling matrix elements. For convenience, we define two combinations,

$$K_{mn} = [-68K_{mn} + 44K_{mn}^{(2)} + 3K_{mn}^{(3)} - 64]/1280,$$

$$\tilde{K}_{mn} = [-52K_{mn} + 60K_{mn}^{(2)} - 9K_{mn}^{(3)} - 64]/52992.$$
Then, one has

\[ |\tilde{\mathcal{M}}|^2 = \begin{cases} 
\frac{1}{17}(\sqrt{6}x_1 - \sqrt{2}x_2 - \sqrt{3}x_3 + x_4)^2 = \tilde{K}_{13}, \\
\frac{1}{17}(\sqrt{2}x_1 + \sqrt{6}x_2 - x_3 - \sqrt{3}x_4)^2 = \tilde{K}_{13}, \\
\frac{1}{17}(\sqrt{6}x_1 + \sqrt{2}x_2 + \sqrt{3}x_3 + x_4)^2 = \tilde{K}_{14}, \\
\frac{1}{17}(\sqrt{2}x_1 - \sqrt{6}x_2 + x_3 - \sqrt{3}x_4)^2 = \tilde{K}_{14}, 
\end{cases} \tag{33} \]

We use \(a, b, c, d\) to denote these four decay modes, respectively. Let us take a look at the \( |\tilde{\mathcal{M}}|^2 = 1 \) solutions when \(x_1/y_1 = x_2/y_2 = x_3/y_3 = x_4/y_4\). For mode \(a\), one has \(K_{13} = 16/3, K_{13}^{(2)} = (K_{13})^2\), and \(K_{13}^{(3)} = (K_{13})^3\). For mode \(b\), \(K_{13} = -16, K_{13}^{(2)} = (K_{13})^2\), and \(K_{13}^{(3)} = (K_{13})^3\). For modes \(c\) and \(d\), one obtains similar results from modes \(a\) and \(b\), respectively, with the replacement \(K_{13} \rightarrow K_{14}\). Obviously, the strongest coupling occurs when the initial state and the final state have the same structure. On the other hand, the solution for \(|\tilde{\mathcal{M}}|^2 = 0\) is not unique and there are more possibilities. That is to say, more structures can be orthogonal to the final meson-meson state in this \(J = 0\) case, which is different from the \(J = 2\) case where only one structure is orthogonal to the final state. When \(|\tilde{\mathcal{M}}|^2 \neq 0\), we may define ratios between CMEs. For a \(J = 0\) state, there are three independent ratios. If one set of the ratios is measured, there are 8 solutions for the wavefunctions. Therefore, one cannot determine uniquely the initial structure just from the rearrangement decay ratios. In fact, even if we know the values of \(K_{13}, K_{14}, K_{13},\) and \(K_{14}\), what we obtain are four combinations for the wavefunction:

\[
x_3^2 + x_4^2 = 2 - \frac{3}{2}(\tilde{K}_{13} + \tilde{K}_{14} + \tilde{K}_{13} + \tilde{K}_{14}), \\
\sqrt{2}(x_1x_3 + x_2x_4) = -\frac{3}{4}(\tilde{K}_{13} - \tilde{K}_{14} + \tilde{K}_{13} - \tilde{K}_{14}), \\
2x_2^2 + x_4^2 - \sqrt{6}(x_1x_4 + x_2x_3) = -\frac{3}{4}(\tilde{K}_{13} + \tilde{K}_{14}) + \frac{9}{4}(\tilde{K}_{13} + \tilde{K}_{14}), \\
\sqrt{3}(2x_1x_2 + x_3x_4) - 2\sqrt{2}x_2x_4 = -\frac{3}{4}(\tilde{K}_{13} - \tilde{K}_{14}) + \frac{9}{4}(\tilde{K}_{13} - \tilde{K}_{14}). \tag{34} \]

One still cannot uniquely determine the tetraquark structure.

Similar to the \(J = 2\) case, the total width in the adopted rearrangement decay model can be written as

\[
\Gamma_{\text{sum}} = \phi_a\tilde{K}_{13} + \phi_b\tilde{K}_{13} + \phi_c\tilde{K}_{14} + \phi_d\tilde{K}_{14}, \tag{35} \]

where \(\phi_{a,b,c,d}\) are determined by phase spaces of the decay modes \(a, b, c, d\), respectively. If \(\phi_a \approx \phi_b \approx \phi_c \approx \phi_d\), we have

\[
\Gamma_{\text{sum}} \approx \phi_a(\tilde{K}_{13} + \tilde{K}_{13} + \tilde{K}_{14} + \tilde{K}_{14}) = 2\phi_a \left( \frac{19}{45} - \frac{7}{3} \tilde{K}_{12} \right) \tag{36} \]

Therefore, the larger \(\tilde{K}_{12}\) is, the smaller width the initial state has. From its explicit expression, the range of \(\tilde{K}_{12}\) is between \(-11/7\) and \(4/7\). The reasonable result \(\Gamma_{\text{sum}} > 0\) is then obtained. Previously, we argued the relative stability of tetraquarks with \(\tilde{K}_{12}\). Now, one finds that \(\tilde{K}_{12}\) is more relevant. To check the stability relation between the total width and the effective interaction in the CMI model, we show the modified \(\tilde{K}_{12}, \tilde{K}_{12}\), in table XI. Comparing this table with tables VI and VII, one does not always observe the stability rule. Main reason for the inconsistency should come from the approximation \(\phi_a \approx \phi_b \approx \phi_c \approx \phi_d\). In table X, this approximation should not be applicable to states having forbidden decay channels or channels with numbers much less than 1. To remedy this problem for some states, one may alternatively use another approximation \(\phi_b \approx \phi_d \gg \phi_a \approx \phi_c\). By defining a second modified \(\tilde{K}_{12}\) appropriately, \(\Gamma_{\text{sum}}\) can be written in the same form as Eq. (36), so that the results in these two approximations can be compared directly. However, the procedure is complicated and it is not so meaningful even if the stability rule exists. Thus, we prefer to conclude that the stability rule does not always exist in the \(J = 0\) case.

If one does not consider the approximations for \(\phi_{a,b,c,d}\), we may give a theoretical constraint on the total width by noting the range of \(\tilde{K}_{12}\),

\[
\frac{2}{3}\phi_{\min} \leq \Gamma_{\text{sum}} \leq \frac{4}{3}\phi_{\max} \tag{37} \]

where \(\phi_{\min}, \phi_{\max}\) represents the minimum (maximum) value in \(\phi_{a,b,c,d}\).
matrix elements with the coupling the strongest coupling occurs when the initial state and the final state have the same structure. For the vanishing modes $K_{13}$, \[ |\bar{m}|^2 = \begin{cases} \frac{1}{16} (\sqrt{2}x_1 + \sqrt{2}x_2 - x_4 - x_5)^2 = K_{13} - K_{24} = K_{24} - K_{13}, & [(q_1 \bar{q}_3)^2 = (q_2 \bar{q}_1)] \text{ mode} \\ \frac{1}{17} (\sqrt{2}x_1 - \sqrt{2}x_2 - 2x_3 - x_4 + x_5 + \sqrt{2}x_6)^2 = K_{13}, & [(\bar{q}_1 q_3)^2 = (q_2 \bar{q}_1)] \text{ mode} \\ \frac{1}{18} (\sqrt{2}x_1 + \sqrt{2}x_2 + 2x_3 - x_4 + x_5 + \sqrt{2}x_6)^2 = K_{24}, & [(\bar{q}_1 q_3)^2 = (q_2 \bar{q}_1)] \text{ mode} \\ \frac{1}{19} (\sqrt{2}x_1 + \sqrt{2}x_2 + x_3 + x_4 - x_5 + \sqrt{2}x_6)^2 = K_{23}, & [(\bar{q}_1 q_3)^2 = (q_2 \bar{q}_1)] \text{ mode} \\ \frac{1}{20} (\sqrt{2}x_1 + \sqrt{2}x_2 + 2x_3 - x_4 + x_5 + \sqrt{2}x_6)^2 = K_{14}, & [(\bar{q}_1 q_3)^2 = (q_2 \bar{q}_1)] \text{ mode} \\ \end{cases} \]

As in the $J = 0$ case, the eigenvalues of $X_{13}, X_{24}, X_{14},$ and $X_{23}$ are $-16, \frac{10}{3}, 2,$ and $-\frac{4}{3}$. To rewrite the coupling matrix elements with the $K$ factors in a linear form, we also need $K^{(2, 3)}_{mn}$ whose explicit expressions are different from those in the $J = 0$ case. After some calculations, we find that

$$X_{12} = \begin{pmatrix} -\frac{4}{3} & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & -\frac{4}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{8}{3} & 0 \\ 0 & 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_{34} = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & -\frac{4}{3} & 0 & 0 & 0 \\ 0 & 0 & -\frac{4}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{8}{3} & 0 \\ 0 & 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_{13} = \begin{pmatrix} 0 & 10 & \frac{10\sqrt{2}}{3} & 0 & -2\sqrt{2} \\ 10 & 0 & \frac{10\sqrt{2}}{3} & -2\sqrt{2} & 0 \\ \frac{10\sqrt{2}}{3} & 0 & -\frac{10\sqrt{2}}{3} & -\frac{4}{3} & 2\sqrt{2} \\ 0 & -2\sqrt{2} & -\frac{4}{3} & 0 & \frac{4\sqrt{2}}{3} \\ -2\sqrt{2} & 0 & 4 & \frac{4\sqrt{2}}{3} & 0 \\ -4 & 4 & 2\sqrt{2} & -\frac{4\sqrt{2}}{3} & -\frac{4}{3} \end{pmatrix}, \quad X_{24} = \begin{pmatrix} 0 & 10 & \frac{10\sqrt{2}}{3} & 0 & -2\sqrt{2} \\ 10 & 0 & \frac{10\sqrt{2}}{3} & -2\sqrt{2} & 0 \\ \frac{10\sqrt{2}}{3} & 0 & -\frac{10\sqrt{2}}{3} & -\frac{4}{3} & 2\sqrt{2} \\ 0 & -2\sqrt{2} & -\frac{4}{3} & 0 & \frac{4\sqrt{2}}{3} \\ -2\sqrt{2} & 0 & 4 & \frac{4\sqrt{2}}{3} & 0 \end{pmatrix}, \quad X_{23} = \begin{pmatrix} 0 & 10 & \frac{10\sqrt{2}}{3} & 0 & -2\sqrt{2} \\ 10 & 0 & \frac{10\sqrt{2}}{3} & -2\sqrt{2} & 0 \\ \frac{10\sqrt{2}}{3} & 0 & -\frac{10\sqrt{2}}{3} & -\frac{4}{3} & 2\sqrt{2} \\ 0 & -2\sqrt{2} & -\frac{4}{3} & 0 & \frac{4\sqrt{2}}{3} \\ -2\sqrt{2} & 0 & 4 & \frac{4\sqrt{2}}{3} & 0 \end{pmatrix}.$$ \[ (38) \]

In the $J = 1$ case, the situation is more complicated. Now, more $X$-matrices are involved:

$$|\bar{m}|^2 = \begin{cases} \frac{1}{16} (\sqrt{2}x_1 + \sqrt{2}x_2 - x_4 - x_5)^2 = K_{13} - K_{24} = K_{24} - K_{13}, & [(q_1 \bar{q}_3)^2 = (q_2 \bar{q}_1)] \text{ mode} \\ \frac{1}{17} (\sqrt{2}x_1 - \sqrt{2}x_2 - 2x_3 - x_4 + x_5 + \sqrt{2}x_6)^2 = K_{13}, & [(\bar{q}_1 q_3)^2 = (q_2 \bar{q}_1)] \text{ mode} \\ \frac{1}{18} (\sqrt{2}x_1 + \sqrt{2}x_2 + 2x_3 - x_4 + x_5 + \sqrt{2}x_6)^2 = K_{24}, & [(\bar{q}_1 q_3)^2 = (q_2 \bar{q}_1)] \text{ mode} \\ \frac{1}{19} (\sqrt{2}x_1 + \sqrt{2}x_2 + x_3 + x_4 - x_5 + \sqrt{2}x_6)^2 = K_{23}, & [(\bar{q}_1 q_3)^2 = (q_2 \bar{q}_1)] \text{ mode} \\ \frac{1}{20} (\sqrt{2}x_1 + \sqrt{2}x_2 - 2x_3 - x_4 + x_5 + \sqrt{2}x_6)^2 = K_{14}, & [(\bar{q}_1 q_3)^2 = (q_2 \bar{q}_1)] \text{ mode} \end{cases} \]

These six modes are denoted as $a, \cdots, f,$ respectively. The solution for $|\bar{m}|^2 = 1$ is always unique. It is easy to check that $K_{13} = K_{24} = 16/3, K_{13}^{(2)} = K_{24}^{(2)} = (K_{13})^2, K_{13}^{(3)} = K_{24}^{(3)} = (K_{13})^3$ for mode $a$, $K_{13} = -16, K_{24} = 16/3, K_{13}^{(2)} = (K_{13})^2, K_{13}^{(3)} = (K_{13})^3, K_{24}^{(2)} = (K_{24})^2, K_{24}^{(3)} = (K_{24})^3$ for mode $b$, and $K_{13} = 16/3, K_{24} = -16, K_{13}^{(2)} = (K_{13})^2, K_{13}^{(3)} = (K_{13})^3, K_{24}^{(2)} = (K_{24})^2, K_{24}^{(3)} = (K_{24})^3$ for mode $c$. For modes $d, e,$ and $f$, similar results can be obtained from modes $a, b,$ and $c$, respectively, by performing the replacements $K_{13} \rightarrow K_{14}, K_{24} \rightarrow K_{13}$. Again, one observes that the strongest coupling occurs when the initial state and the final state have the same structure. For the vanishing coupling $|\bar{m}|^2 = 0$, one cannot identify the initial structure from various possibilities. In the general case, there are
five independent ratios between the coupling matrix elements for a $J = 1$ tetraquark state. If one set of them is measured, there are 32 solutions for the wavefunctions and we cannot uniquely determine the initial structure just from the rearrangement decays, which is a feature similar to the $J = 0$ case, but more complicated. Even if we know the values of $\bar{K}_{13}$, $\bar{K}_{14}$, $\bar{K}_{13}$, $\bar{K}_{14}$, $\bar{K}_{23}$, and $\bar{K}_{24}$, what we obtain are six combinations for the wavefunction:

$$
x_1^2 + x_2^2 + x_3^2 = 1 - (x_2^2 + x_3^2 + x_4^2) = -1 + \frac{3}{2}(\bar{K}_{13} + \bar{K}_{14} + \bar{K}_{13} + \bar{K}_{14}),
$$

$$
2\sqrt{2}(x_1x_5 + x_2x_4) + 2x_3^2 + x_6^2 = \frac{3}{2}(\bar{K}_{13} - \bar{K}_{14} - \bar{K}_{13} - \bar{K}_{14} - 2\bar{K}_{23} - 2\bar{K}_{24}),
$$

$$
2\sqrt{2}x_1x_3 + 2(x_2x_6 + x_3x_5) + \sqrt{2}x_4x_6 = \frac{3}{2}(\bar{K}_{13} - \bar{K}_{14} - \bar{K}_{23} - \bar{K}_{24}),
$$

$$
2\sqrt{2}x_2x_3 + 2(x_1x_6 + x_3x_4) + \sqrt{2}x_5x_6 = \frac{3}{2}(\bar{K}_{13} - \bar{K}_{14} - \bar{K}_{23} - \bar{K}_{24}),
$$

$$
4x_1x_2 + 2\sqrt{2}x_3x_6 + 2x_4x_5 = \frac{3}{2}(\bar{K}_{13} - \bar{K}_{14} - \bar{K}_{23} - \bar{K}_{24}),
$$

$$
2\sqrt{2}x_3x_6 + 2\sqrt{2}(x_1x_4 + x_2x_5) = -\frac{3}{2}(\bar{K}_{13} - \bar{K}_{14} - \bar{K}_{13} - \bar{K}_{14}).
$$

(40)

With the above CMEs, the total width in the adopted rearrangement decay model can be written as

$$
\Gamma_{\text{sum}} = \phi_a(\bar{K}_{13} - \bar{K}_{24}) + \phi_b\bar{K}_{13} + \phi_c\bar{K}_{24} + \phi_d(\bar{K}_{14} - \bar{K}_{23}) + \phi_e\bar{K}_{14} + \phi_f\bar{K}_{23},
$$

(41)

where $\phi_a, \cdots, \phi_f$ correspond to the decay modes $a, \cdots, f$, respectively. If $\phi_a \approx \phi_b \approx \phi_c \approx \phi_d \approx \phi_e \approx \phi_f$, we have

$$
\Gamma_{\text{sum}} \approx \phi_a(\bar{K}_{13} + \bar{K}_{13} + \bar{K}_{14} + \bar{K}_{14}) = \phi_a(\bar{K}_{23} + \bar{K}_{23} + \bar{K}_{24} = \bar{K}_{24})
$$

$$
= \phi_a\left[38 - 7(\bar{K}_{12} + \bar{K}_{34})\right],
$$

(42)

where $\bar{K}_{12} \equiv \bar{K}_{12} - 3/16K_{12}^{(2)} - 9/224K_{12}^{(3)}$ and $\bar{K}_{34} \equiv \bar{K}_{34} - 3/16K_{34}^{(2)} - 9/224K_{34}^{(3)}$. One may check that $\bar{K}_{12}$ is equal to $\bar{K}_{34}$ and thus similar formula to the $J = 0$ case is obtained,

$$
\Gamma_{\text{sum}} \approx \frac{2\phi_a}{45}[19 - 7\bar{K}_{12}],
$$

(43)

where the constraint for the modified $\bar{K}_{12}$ is also $-11/7 \leq \bar{K}_{12} \leq 4/7$. Then one also observes that the larger $\bar{K}_{12}$ is, the smaller width the initial state has. From tables X, XI, VI, and VII, the approximation $\phi_a \approx \phi_b \approx \phi_c \approx \phi_d \approx \phi_e \approx \phi_f$ is not always appropriate and the situation is similar to the $J = 0$ case. Although more approximations than the $J = 0$ case can be adopted, we still prefer to conclude that the stability rule does not always exist in the $J = 1$ case.

If one does not consider any approximation, we may give a theoretical constraint on the total width by noting the range of $\bar{K}_{12}$. The result is similar to the $J = 0$ case

$$
\frac{2}{3}\phi_{\text{min}} \leq \Gamma_{\text{sum}} \leq \frac{4}{3}\phi_{\text{max}}.
$$

(44)

From the above discussions, in the rearrangement decay model, we are actually detecting the initial structure with the final meson-meson states. Since only color-singlet quark-antiquark meson states can be observed, structure information of the initial state may have been lost in the decays. With our studies, one concludes: (1) For a $J = 2$ state, it is possible to identify its unique tetraquark structure from its decay properties while, for a $J = 0$ or $J = 1$ state, it is nearly impossible to uniquely identify its structure just from the decay ratios in the rearrangement decay model. (2) Theoretically, an approximate stability rule for tetraquark states may exist which says that the larger the diquark $K$ factor is, the smaller width the tetraquark has. One may estimate the relative stability of a tetraquark with such a rule if all the rearrangement decays are kinematically allowed. (3) The range for the width of a tetraquark state in the adopted model can be estimated by calculating the phase spaces.

### B. The systems with Pauli constraint

For the $F_{5,7,9}$ systems, the situation is different. For convenience, we use $q_1q_2\bar{u}\bar{d}$ or $q_1q_2\bar{n}\bar{n}$ to denote their structure. Because of the effects from the Pauli principle, $|x_iy_i|^2 \leq \sum (y_i)^2 < 1$. This means that the coupling between final
and initial states should not be very strong. Alternatively, when one detects the initial structure with the meson-meson states through rearrangement mechanism, the coupling may be reduced by the Pauli principle compared to the previous states, which is helpful to the stability of the double-heavy tetraquarks. Now the $K$ factors of the final mesons are not eigenvalues of $X_{1\tilde{n}}$ and/or $X_{2\tilde{n}}$ any longer. Four $K$ factors, $K_{12}$, $K_{1\tilde{n}}$, $K_{2\tilde{n}}$, and $K_{nn}$, are involved in the following discussions.

The $J = 2$ case is simple since there is only one base. One has $|\tilde{M}|^2 = |x_1y_1|^2 = (y_1)^2$. If the isospin of the initial state is 1,

$$K_{12} = K_{nn} = K_{1\tilde{n}} = K_{2\tilde{n}} = \frac{8}{3}, \quad |\tilde{M}|^2 = \frac{1}{3}. \quad (45)$$

If the isospin of the initial state is 0,

$$K_{12} = K_{nn} = -\frac{4}{3}, \quad K_{1\tilde{n}} = K_{2\tilde{n}} = \frac{10}{3}, \quad |\tilde{M}|^2 = \frac{2}{3}. \quad (46)$$

In the $J = 0$ case, both isovector and isoscalar states have two tetraquark bases. The situation is somewhat similar to the $J = 2$ states without Pauli constraint.

- $I = 1$ states

The base corresponding to the CMI Hamiltonian is $(B_1, B_4)^T$. The $X$ matrices read

$$X_{12} = X_{nn} = \begin{pmatrix} 4 & 0 \\ 0 & 4/3 \end{pmatrix}, \quad X_{1\tilde{n}} = X_{2\tilde{n}} = \begin{pmatrix} 0 & 4\sqrt{6} \\ 4\sqrt{6} & -16/3 \end{pmatrix}. \quad (47)$$

It is easy to check that the eigenvalues of $X_{1\tilde{n}}$ are different from the $K$ factors of the final state mesons. With the explicit expressions, one finds

$$\frac{8}{3} \leq K_{nn} \leq 4, \quad \frac{4(-2 - \sqrt{38})}{3} (\approx -12.8) \leq K_{1\tilde{n}} \leq \frac{4(-2 + \sqrt{38})}{3} (\approx 7.5). \quad (48)$$

Expressing the coupling matrix elements with the $K$ factors, we get

$$|\tilde{M}|^2 = \frac{1}{12} (\sqrt{6x_1 + x_2})^2 = \frac{1}{48} K_{1\tilde{n}} + \frac{11}{48} K_{nn} - \frac{5}{12}, \quad |\tilde{M}|^2 = \frac{1}{12} (\sqrt{2x_1 - \sqrt{3}x_2})^2 = \frac{1}{48} K_{1\tilde{n}} + \frac{1}{48} K_{nn} + \frac{1}{12}. \quad (49)$$

for the $(q_1\bar{d})^J = 1 (q_2\bar{d})^J = 1$ or $(q_1\bar{d})^J = 1 (q_2\bar{d})^J = 1$ mode final state (mode-a) and $(q_1\bar{d})^J = 0 (q_2\bar{d})^J = 0$ or $(q_1\bar{d})^J = 0 (q_2\bar{d})^J = 0$ mode final state (mode-b), respectively. From the form with $K$ factors, it is easy to see the effects on the couplings from the Pauli principle. Numerically, the effect is not small for mode-a decay ($0.61 < 1/48 K_{nn} < 0.92$) but not large for mode-b decay ($0.06 < 1/48 K_{nn} < 0.08$).

The minimum value of $|\tilde{M}|^2$ is always 0 while the maximum values of $|\tilde{M}|^2$ are 7/12 and 5/12 for the two modes, respectively. When an initial state does not couple to the model-a final state, we have $K_{1\tilde{n}} = -80/7$. When the coupling is strongest, we have $K_{1\tilde{n}} = 128/21$. They are not the minimum/maximum values of $K_{1\tilde{n}}$. For mode-b decay, the corresponding values of $K_{1\tilde{n}}$ are 112/15 and $-64/5$, respectively.

Since the Pauli principle has been considered in the initial bases, one cannot describe a molecule with them and we do not try to distinguish a molecule from a compact tetraquark. Such a problem can be discussed in a future work. In general case, if the ratio between the two coupling matrix elements,

$$r = \frac{(\sqrt{6}x_1 + x_2)^2}{(\sqrt{2}x_1 - \sqrt{3}x_2)^2}, \quad (50)$$

has been determined, similar to the $J = 2$ case, one obtains two solutions

$$x_1 = \pm \frac{\sqrt{3r} - 1}{\sqrt{5r + 2\sqrt{3r} + 7}}, \quad x_2 = \pm \frac{\sqrt{2r + \sqrt{3}}}{\sqrt{5r + 2\sqrt{3r} + 7}}; \quad (51)$$

$$x_1 = \pm \frac{\sqrt{3r} + 1}{\sqrt{5r - 2\sqrt{3r} + 7}}, \quad x_2 = \pm \frac{\sqrt{2r - \sqrt{3}}}{\sqrt{5r - 2\sqrt{3r} + 7}}. \quad (52)$$
To uniquely determine the initial structure, the values of the two coupling matrix elements need to be known.

The total width in the adopted rearrangement decay model can be written as

\[
\Gamma_{\text{sum}} = 2\phi_a\left(\frac{1}{48}K_{1\bar{n}} + \frac{11}{48}K_{nn} - \frac{5}{12}\right) + 2\phi_b\left(-\frac{1}{48}K_{1\bar{n}} + \frac{1}{48}K_{nn} + \frac{1}{12}\right),
\]

where the factor 2 is from the fact that \((q_1\bar{u})(q_2\bar{d})\) and \((q_1\bar{d})(q_2\bar{u})\) are different final states with the same coupling matrix element. If \(\phi_a \approx \phi_b\), we have

\[
\Gamma_{\text{sum}} = \phi_b\left(\frac{1}{2}K_{nn} - \frac{2}{3}\right).
\]

Therefore, the smaller the \(K_{nn}\) is, the smaller width the initial state has. One may confirm this feature from the numerical results in tables V and VIII although the approximation \(\phi_a \approx \phi_b\) is not always appropriate (see table XII). So the stabilization rule exists in this case. The formula we get is interesting because the width is mainly related with the effective (light or heavy) diquark interaction. If the interaction is weakly repulsive \((K_{nn} \text{ is always positive in this isovector case})\), it is helpful to stabilize the tetraquark, which is a reasonable observation.

If we do not use the approximation \(\phi_a \approx \phi_b\), the constraint we may give reads

\[
\frac{2}{3}\phi_a \leq \Gamma_{\text{sum}} \leq \frac{4}{3}\phi_b.
\]

- **I = 0 states**

  From the CMI Hamiltonian, we get the \(X\) matrices:

\[
X_{12} = X_{nn} = \begin{pmatrix}
-\frac{4}{3} & 0 \\
0 & -8
\end{pmatrix}, \\
X_{1\bar{n}} = X_{2\bar{n}} = \begin{pmatrix}
\frac{40}{3} & 4\sqrt{6} \\
4\sqrt{6} & 0
\end{pmatrix},
\]

whose corresponding base is \((B_2, B_3)^T\). The obtained constraints for relevant \(K\) factors are

\[
-8 \leq K_{nn} \leq -\frac{4}{3}, \\
\frac{4(-5 - \sqrt{79})}{3} (\approx -18.5) \leq K_{1\bar{n}} \leq \frac{4(-5 + \sqrt{79})}{3} (\approx 5.2).
\]

What we get for the coupling matrix elements with the \(K\) factors are

\[
|\tilde{M}|^2 = \frac{1}{12}((\sqrt{2}x_1 + \sqrt{3}x_2)^2 = \frac{1}{48}K_{1\bar{n}} + \frac{7}{240}K_{nn} + \frac{29}{60},
\]

\[
|\tilde{M}|^2 = \frac{1}{12}((\sqrt{6}x_1 - x_2)^2) = -\frac{1}{48}K_{1\bar{n}} + \frac{1}{48}K_{nn} + \frac{1}{4},
\]

for the \((q_1\bar{u})^{J=1}(q_2\bar{d})^{J=1}\) or \((q_1\bar{d})^{J=1}(q_2\bar{u})^{J=1}\) mode final state (mode-\(a\)) and \((q_1\bar{u})^{J=0}(q_2\bar{d})^{J=0}\) or \((q_1\bar{d})^{J=0}(q_2\bar{u})^{J=0}\) mode final state (mode-\(b\)), respectively. The effect on CMEs from the Pauli principle is about \(-0.23 \sim -0.04\) for mode-\(a\) decay and about \(-0.17 \sim -0.03\) for mode-\(b\) decay.

The maximum values of \(|\tilde{M}|^2\) are 5/12 and 7/12 for mode-\(a\) and mode-\(b\) decays, respectively. With the wavefunctions corresponding to \(|\tilde{M}|^2 = 0\) and 5/12 for the mode-\(a\) decay, we get \(-88/5\) and 64/15 for the values of \(K_{1\bar{n}}\), respectively. Similar results for the mode-\(b\) decay are 104/21 and \(-128/7\), respectively. All of these numbers are not the minimum/maximum values of \(K_{1\bar{n}}\).

If the ratio between the above two CMEs, \(r\), is known, one may obtain two solutions for the initial wavefunctions,

\[
x_1 = \pm \frac{\sqrt{7} - \sqrt{3}}{\sqrt{7}r + 2\sqrt{3r} + 5}, \quad x_2 = \pm \frac{\sqrt{6r} + \sqrt{2}}{\sqrt{7}r + 2\sqrt{3r} + 5};
\]

\[
x_1 = \pm \frac{\sqrt{7} + \sqrt{3}}{\sqrt{7}r - 2\sqrt{3r} + 5}, \quad x_2 = \pm \frac{\sqrt{6r} - \sqrt{2}}{\sqrt{7}r - 2\sqrt{3r} + 5}.
\]

To uniquely determine the initial structure, both values of the two coupling matrix elements need to be given.
For the decay of the isoscalar tetraquarks, the total width can be written as

\[ \Gamma_{\text{sum}} = 2\phi_a \left( \frac{1}{48} K_{1\bar{n}} + \frac{7}{240} K_{nn} + \frac{29}{60} \right) + 2\phi_b \left( -\frac{1}{48} K_{1\bar{n}} + \frac{1}{48} K_{nn} + \frac{1}{4} \right). \]  

(61)

If \( \phi_a \approx \phi_b \), one has

\[ \Gamma_{\text{sum}} = \phi_a \left( \frac{1}{10} K_{nn} + \frac{22}{15} \right). \]  

(62)

Therefore, the smaller the \( K_{nn} \) is, the smaller width the initial state has, the feature similar to the isovector tetraquarks. However, we cannot confirm this feature in the present case from tables V and VIII, because the approximation \( \phi_a \approx \phi_b \) does not work (see table XII).

If one does not use the approximation \( \phi_a \approx \phi_b \), the constraint we may give also reads

\[ \frac{2}{3} \phi_a \leq \Gamma_{\text{sum}} \leq \frac{4}{3} \phi_b, \]  

(63)

but the values of \( \phi_a \) and \( \phi_b \) are different from those in the last case.

TABLE XII: Values of \( |p_1| M_1^2 / |p_1h| M_1^2 \) for various \( F_{5,7,9} \) tetraquark states in the rearrangement decays. Here, \( p_1 \) is the three-momentum of a final meson in the rest frame of the initial state of mass \( M \) and \( p_{1h} \) and \( M_h \) are for the highest state with the same \( J^P \) decaying into the channel with largest phase space. The symbol “-” means that the decay is forbidden. Decay channels from left to right are presented with the increasing thresholds.

| \( F_5 = bc\bar{n} \) | \( I(J^P) = 1(2^+) \) | \( B^*D^* \) | \( I(J^P) = 0(2^+) \) | \( B^*D^* \) |
|---|---|---|---|---|
| 7467 | 1.0 | | | |
| 7500 | 1.0 | 0.8 | 0.7 | |
| 7438 | 0.9 | 0.7 | 0.6 | |
| 7366 | 0.8 | 0.5 | 0.3 | |
| 7527 | 1.0 | 0.7 | | |
| 7342 | 0.7 | 0.2 | | |

| \( F_2 = bs\bar{n} \) | \( I(J^P) = 1(2^+) \) | \( B^*K^* \) | | \( I(J^P) = 0(2^+) \) | \( B^*K^* \) |
|---|---|---|---|---|---|
| 6307 | 1.0 | | | | |
| 6400 | 1.0 | 0.7 | 0.6 | | |
| 6283 | 0.9 | 0.5 | 0.4 | | |
| 6025 | 0.6 | – | – | | |
| 6423 | 1.0 | 0.6 | | | |
| 5993 | 0.6 | – | | | |

| \( F_0 = cs\bar{n} \) | \( I(J^P) = 1(2^+) \) | \( D^*K^* \) | | \( I(J^P) = 0(2^+) \) | \( D^*K^* \) |
|---|---|---|---|---|---|
| 2988 | 1.0 | | | | |
| 3058 | 1.0 | 0.8 | 0.6 | | |
| 2902 | 0.9 | 0.6 | – | | |
| 2603 | 0.7 | – | – | | |
| 3129 | 1.0 | 0.6 | | | |
| 2607 | 0.7 | – | | | |

Now we move on to the \( J = 1 \) case. The involved states have three tetraquark bases. One will see some new features.

- \( I = 1 \) states
From the CMI Hamiltonian, the involved $X$ matrices are

$$X_{12} = \begin{pmatrix} -\frac{4}{3} & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & \frac{8}{3} \end{pmatrix}, \quad X_{nn} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & \frac{8}{3} & 0 \\ 0 & 0 & \frac{8}{3} \end{pmatrix}, \quad X_{1n} = \begin{pmatrix} 0 & -4\sqrt{2} & -8 \\ -4\sqrt{2} & 0 & -\frac{8\sqrt{2}}{3} \\ -8 & -\frac{8\sqrt{2}}{3} & -\frac{8}{3} \end{pmatrix}, \quad X_{2n} = \begin{pmatrix} 0 & -4\sqrt{2} & 8 \\ -4\sqrt{2} & 0 & \frac{8\sqrt{2}}{3} \\ 8 & \frac{8\sqrt{2}}{3} & -\frac{8}{3} \end{pmatrix}.\tag{64}$$

Their corresponding base is $(B_5, B_9, B_{10})^T$. The constraints for the relevant $K$ factors, $K_{nn}$ and $K_{1n}$, can be found in Eq. (48). The constraint for $K_{2n}$ is the same as $K_{1n}$. To express the coupling matrix elements with the $K$ factors, we also need $K_{1n}^{(2)}$, $K_{2n}^{(2)}$, and the definition

$$\tilde{K}_{mn} = \frac{1}{2688}[40K_{mn} - 3K_{mn}^{(2)}].\tag{65}$$

The resulting coupling matrix elements are

$$|\tilde{M}|^2 = \begin{cases} \frac{1}{6}(\sqrt{2}x_1 - x_2)^2 = \tilde{K}_{1n} + \tilde{K}_{2n} + \frac{5}{24}K_{nn} - \frac{2}{7}, \\ \frac{1}{12}(\sqrt{2}x_1 + x_2 + \sqrt{2}x_3)^2 = -\tilde{K}_{1n} + \frac{1}{4}K_{nn} - \frac{1}{4}, \end{cases}\tag{66}$$

for the $(q_1\bar{u})^{J=1}(q_2\bar{d})^{J=1}$ or $(q_1\bar{d})^{J=1}(q_2\bar{u})^{J=1}$ mode final state (mode-$a$), $(q_1\bar{u})^{J=0}(q_2\bar{d})^{J=1}$ or $(q_1\bar{d})^{J=0}(q_2\bar{u})^{J=1}$ mode final state (mode-$b$), and $(q_1\bar{u})^{J=1}(q_2d)^{J=0}$ or $(q_1\bar{d})^{J=1}(q_2\bar{u})^{J=0}$ mode final state (mode-$c$), respectively. The contributions from the Pauli principle is not small for mode-$a$ decay ($0.56 \sim 0.83$) and not large for mode-$b, c$ decays ($0.06 \sim 0.08$).

Different from the $J = 0$ case, the solution for minimum coupling ($|\tilde{M}|^2 = 0$) is not unique. The maximum $|\tilde{M}|^2$ for the three decay modes are $1/2$, $5/12$, and $5/12$, respectively. They have unique solutions, but we do not find relations $K_{1n}^{(2)}$ or $K_{2n}^{(2)}$ from the solutions. One may understand it from the fact that the value of $K_{1n}/K_{2n}$ corresponding to the maximum $|\tilde{M}|^2$ is not the eigenvalue of $X_{1n}/X_{2n}$, which is an effect from the Pauli constraint.

For tetraquarks having three decay modes, the number of independent ratios between CMEs is two. They may generally give 4 solutions for the initial wavefunction. In the case that the three CMEs are all known, what we get are three constraint relations:

$$\begin{align*}
 x_1^2 &= \frac{3}{4}K_{nn} - 2, \\
 x_3^2 + 2\sqrt{2}x_1x_2 &= -6(\tilde{K}_{1n} + \tilde{K}_{2n}) - \frac{1}{2}K_{nn} + \frac{5}{7}, \\
 2x_1x_3 + \sqrt{2}x_1x_2 &= -3(\tilde{K}_{1n} - \tilde{K}_{2n}).\tag{67}
\end{align*}$$

The total width in the adopted rearrangement decay model can be written as

$$\Gamma_{\text{sum}} = 2\phi_a(\tilde{K}_{1n} + \tilde{K}_{2n} + \frac{5}{24}K_{nn} - \frac{2}{7}) + 2\phi_b(-\tilde{K}_{1n} + \frac{1}{48}K_{nn} - \frac{1}{42}) + 2\phi_c(-\tilde{K}_{2n} + \frac{1}{48}K_{nn} - \frac{1}{42}).\tag{68}$$

If $\phi_a \approx \phi_b \approx \phi_c$, we have

$$\Gamma_{\text{sum}} = \phi_a\left(\frac{1}{2}K_{nn} - \frac{2}{7}\right).\tag{69}$$

This formula is similar to the one for the $J = 0, I = 1$ case, but $K_{nn}$ is different from that one. Therefore, the smaller the $K_{nn}$ is, the smaller width is the initial state has. In the $J = 0, I = 1$ case, $K_{12} = K_{nn}$. In the present case, $K_{12} \neq K_{nn}$, but $K_{12}$ does not appear in CMEs and $\Gamma_{\text{sum}}$. The width is more relevant with the light diquark interaction. If the interaction is weakly repulsive ($K_{nn}$ is always positive in this isovector case), it is helpful to narrow the width of the tetraquark state, which is a reasonable observation. Like in the $J = 0, I = 1$ case, one can confirm this feature from the numerical results in tables $V$ and $VIII$, although the approximation $A \approx B \approx C$ is not always approximate (see table $XII$).

If we do not use the approximation $\phi_a \approx \phi_b \approx \phi_c$, the constraint for the total width we can give is

$$\frac{2}{3}\phi_{\text{min}} \leq \Gamma_{\text{sum}} \leq \frac{4}{3}\phi_{\text{max}}.\tag{70}$$
• $I = 0$ states

Now, the $X$ matrices with corresponding base $(B_6, B_7, B_8)^T$ are

\[
X_{12} = \begin{pmatrix}
4 & 0 & 0 \\
0 & -\frac{4}{3} & 0 \\
0 & 0 & \frac{8}{3}
\end{pmatrix}, \quad X_{nn} = \begin{pmatrix}
-\frac{4}{3} & 0 & 0 \\
0 & -\frac{4}{3} & 0 \\
0 & 0 & -8
\end{pmatrix},
\]

\[
X_{1\bar{n}} = \begin{pmatrix}
0 & -\frac{20\sqrt{2}}{3} & -4\sqrt{2} \\
-\frac{20\sqrt{2}}{3} & -\frac{20}{3} & -8 \\
-4\sqrt{2} & -8 & 0
\end{pmatrix}, \quad X_{2\bar{n}} = \begin{pmatrix}
0 & \frac{20\sqrt{2}}{3} & 8 \\
\frac{20\sqrt{2}}{3} & \frac{20}{3} & 8 \\
8 & 8 & 0
\end{pmatrix}.
\] (71)

The constraints for relevant $K$ factors are slightly different from the $J = 0, I = 0$ case:

\[-8 \leq K_{nn} \leq -\frac{4}{3}, \quad 4\left(\frac{5 - \sqrt{79}}{3}\right) \leq K_{1\bar{n}} \leq \frac{20}{3}, \quad 4\left(\frac{5 - \sqrt{79}}{3}\right) \leq K_{2\bar{n}} \leq \frac{20}{3}.\] (72)

With the definition

\[
\tilde{K}_{mn} = \frac{1}{1344}[68K_{mn} + 3K_{mn}^{(2)}],
\] (73)

the coupling matrix elements can be written as

\[
|\mathcal{M}|^2 = \begin{cases}
\frac{1}{6} (\sqrt{2} x_1 - x_3)^2 = \tilde{K}_{1\bar{n}} + \tilde{K}_{2\bar{n}} + \frac{41}{210} K_{nn} - \frac{41}{210}, \\
\frac{1}{12} (\sqrt{2} x_1 + 2 x_2 + x_3)^2 = -\tilde{K}_{1\bar{n}} + \frac{1}{48} K_{nn} + \frac{13}{28}, \\
\frac{1}{12} (\sqrt{2} x_1 - 2 x_2 + x_3)^2 = -\tilde{K}_{2\bar{n}} + \frac{1}{48} K_{nn} + \frac{13}{28},
\end{cases}
\] (74)

for the $(q_1 \bar{u})_{J=1}(q_2 \bar{d})_{J=1}$ mode final state (mode-$a$), $(q_1 \bar{u})_{J=0}(q_2 \bar{d})_{J=1}$ or $(q_1 \bar{d})_{J=0}(q_2 \bar{u})_{J=1}$ mode final state (mode-$b$), and $(q_1 \bar{u})_{J=1}(q_2 \bar{d})_{J=0}$ or $(q_1 \bar{d})_{J=1}(q_2 \bar{u})_{J=0}$ mode final state (mode-$c$), respectively. The contributions from the Pauli principle are around $-0.07 \sim -0.01$ for mode-$a$ decay and $-0.17 \sim -0.03$ for mode-$b, c$ decays.

Similar to the last case, the solution for $|\mathcal{M}|^2 = 0$ is not unique and the maximum $|\mathcal{M}|^2$ for the three modes are $1/2$, $7/12$, and $7/12$, respectively. Unique solution exists for the largest $|\mathcal{M}|^2$, but we do not find relations $K_{1\bar{n},2\bar{n}} = (K_{1\bar{n},2\bar{n}})^2$ again.

Once the two independent ratios between the three CMEs are determined, one may get 4 possible solutions to the initial structure. If the three CMEs are all known, we cannot get the unique structure, but three constraint relations:

\[
x_1^2 = -\frac{3}{20} K_{nn} - \frac{1}{5}, \\
x_2^2 + \sqrt{2} x_1 x_3 = -3(\tilde{K}_{1\bar{n}} + \tilde{K}_{2\bar{n}}) + \frac{1}{20} K_{nn} + \frac{59}{35}, \\
2\sqrt{2} x_1 x_2 + 2 x_2 x_3 = -3(\tilde{K}_{1\bar{n}} - \tilde{K}_{2\bar{n}}).
\] (75)

We may write the total width in the adopted rearrangement decay model as

\[
\Gamma_{\text{sum}} = \frac{2}{\Gamma_a}(\tilde{K}_{1\bar{n}} + \tilde{K}_{2\bar{n}} + \frac{1}{120} K_{nn} - \frac{41}{210}) + 2\phi_b(-\tilde{K}_{1\bar{n}} + \frac{1}{48} K_{nn} + \frac{13}{28}) + 2\phi_c(-\tilde{K}_{2\bar{n}} + \frac{1}{48} K_{nn} + \frac{13}{28}).\] (76)

If $\phi_a \approx \phi_b \approx \phi_c$, we have

\[
\Gamma_{\text{sum}} = \phi_a \frac{1}{10} K_{nn} + \frac{22}{15}.
\] (77)

This formula is similar to the one for the $J = 0, I = 0$ case, but $K_{nn}$ is different from that one. Again, from tables $V$ and $VIII$, one cannot confirm the feature that the smaller the $K_{nn}$ is, the smaller width the initial state has.

If we do not consider the approximation $\phi_a \approx \phi_b \approx \phi_c$, the form of the constraint for the total width is the same as before,

\[
\frac{2}{3} \phi_{\text{min}} \leq \Gamma_{\text{sum}} \leq \frac{4}{3} \phi_{\text{max}}.
\] (78)
From the discussions for the $F_{5,7,9}$ systems, one finds: (1) It is possible to uniquely determine the structure of a $J = 0$ state from the decay properties. (2) The stability rule that the width is narrow when $K_{mn}$ is small exists for the isovector states. (3) The range for the width of a tetraquark state can also be estimated by calculating the phase spaces.

V. SUMMARY

In studying the properties of the LHCb $P_c$ states [81], we adopted the same CMI model as the present work. In that case, the pentaquark masses estimated with meson-baryon thresholds are consistent with the measured data. However, in the tetraquark case, the masses estimated with meson-meson thresholds may be underestimated, which was illustrated in the $cs\bar{s}s$ system [76]. Thus, we can correct other tetraquark masses by assuming that the $X(4140)$ is a $cs\bar{s}s$ tetraquark. With the updated estimation method, the masses based on $X(4140)$ are always larger than those based on meson-meson thresholds and smaller than those according to Eq. (6). The results show that only several states, e.g. lowest $bs\bar{u}d$ and $cs\bar{u}d$, are probably stable. For most states, strong decays are allowed. It is necessary to investigate their decay widths using appropriate models.

Based on the decay properties of conventional hadrons, one expects several types of two-body strong decays for tetraquarks: 1) rearrangement decays into meson-meson states, 2) baryon-antibaryon decay modes through the creation of a quark-antiquark pair, and 3) decays into lower four-quark states and conventional mesons with a created quark-antiquark pair. Compared with the rearrangement decay patterns, others may be suppressed. In the present study, we only concentrate on the dominant rearrangement decays with a simple model, i.e. $H_{\text{decay}} = \alpha$ is a constant. This model has been successfully used to explain the decay ratios for LHCb $P_c$ states [81]. The present investigation is also helpful for us to clarify the internal structures of the observed four-quark states through their partial decay widths [97].

To get more information in understanding the nature of exotic states, one should combine the analyses from spectrum and decay properties. For the $X(5568)$ observed by the D0 Collaboration, its flavor structure corresponds to our $0^+$ or $1^+ F_6 = bs\bar{u}d$ states. From the numerical results, one cannot understand its mass and decay width consistently in the tetraquark picture. However, if a narrow $bs\bar{u}d$ state about 70 MeV higher were observed, its tetraquark nature could be understood in the present model. On the other hand, the lowest $0^+$ and $1^+$ isoscalar $F_7 = bs\bar{d}u$ states may be stable although their masses are higher than the lowest $F_5$ state, because the number of decay patterns is reduced for such states which are affected by the Pauli principle.

The lowest $F_5 = cs\bar{u}d$ tetraquark states, similar to the $F_7$ states are also possibly stable. For the $F_8 = cu\bar{d}d$ states, we only show results in the $I = 1$ case. In fact, the $I = 0$ states are degenerate with the $I = 1$ states in the present model. According to our results, the lowest $0^+$ isoscalar tetraquark is below the $D^{*0}_{s0}(2317)$ state. On the experimental side, it is interesting to answer whether a lower exotic $cs$-like state exists or not in order to test the model. On the theoretical side, it is worthwhile to investigate the nature of $D^{*0}_{s0}(2317)$ further by considering contributions from both two-quark and four-quark components. Similar studies of $D_{s1}(2460)$ should also be performed.

Another system affected by the Pauli principle is $F_3 = bc\bar{u}d$. There are controversial results on the masses of the lowest $F_3$ states in the literature. The present method leads to unstable $bc\bar{u}d$ tetraquarks. The lowest $0^+$ and $1^+$ isoscalar states are slightly above the $BD$ and $B^*D$ thresholds, respectively. According to the mass difference, one would have an unstable $1^+ T_{cc} = c\bar{c}u\bar{d}$ tetraquark and a stable $1^+ T_{bb} = b\bar{b}u\bar{d}$ tetraquark. The predicted widths (several MeVs or around 10 MeV) for the $bc\bar{u}d$ states can be tested in future experiments.

Tetraquark states with four different flavors are certainly exotic. The exotic phenomenon is easy to be identified at experiments. However, to distinguish the nature of the observed state, a compact tetraquark or a molecule, is still difficult. Generally speaking, if a stable four-quark state is slightly below a relevant meson-meson threshold, a molecular structure is preferred. If its mass is much lower, a compact tetraquark interpretation may be more natural. To understand its nature further, more studies of decay properties are also essential. If the state does not have strong decay channels, studies of weak decays will be helpful. If it has strong decays, the decay ratios will be helpful.

In the present work, we analyzed the rearrangement decays further with the help of the defined $K$ factors. The decay width of each mode can be expressed in a linear form of the $K$ factors. In the limit that the phase spaces for all the channels are equal, the total width $\Gamma_{\text{sum}}$ can be related to the effective CMI for the diquark, from which we discussed the possible stability rules in various cases. Comparing the numerical results of $\Gamma_{\text{sum}}$ with the obtained stability rule, one cannot always find consistent conclusions for states with $J = 0, 1$ when they are not affected by the Pauli principle and for states with $I = 0$ when they are affected by the Pauli principle. If the stability rule exists for a system, one may judge the relative stabilities of the tetraquarks just by using effective color-magnetic interactions for diquarks inside the tetraquarks. For example, a smaller $K_{nn}$ is helpful to decrease the width of an isovector $bc\bar{i}d$ state. Using the relations between the $K$ factors, one may further estimate the range of the total width $\Gamma_{\text{sum}}$ in the
adopted model with simple calculations. The range can be universally given by

\[
\frac{2}{3} \phi_{\text{min}} \leq \Gamma_{\text{sum}} \leq \frac{4}{3} \phi_{\text{max}},
\]

(79)

where \(\phi_{\text{min}} (\phi_{\text{max}})\) is determined by the minimum (maximum) phase space. For example, one gets 55 MeV \(\leq \Gamma_{\text{sum}} \leq 109\) MeV for the lowest \(F_0 = \bar{c}u\bar{d}\) state if \(\alpha = 7.27\) GeV is adopted. This inequality gives at least the order of the strong decay width in the adopted model. We hope that the predictions can be tested in future experiments. If they could be confirmed, it means that we used a reasonable method and the nature that the \(X(4140)\) is a tetraquark state is indirectly supported.

To describe the effective CMI for quark components in multiquark states, we introduced \(K\) factors in Ref. [75]. To describe the coupling matrix elements in the rearrangement model, we define more \(K\) factors such as \(K^{(2)}\) and \(K^{(3)}\). At present, it is not clear whether they have physical meanings or not. Here, one only treats them as calculable parameters.

We also discussed the possibility to extract the color-spin wave function of the initial state with measured decay ratios or partial widths. However, we find that one cannot uniquely determine the initial structure just from the decay ratios. The number of solutions is equal to \(2^{N-1}\) where \(N\) is the number of bases for the initial tetraquark state or the number of rearrangement decay channels. For the case \(N = 2\), it is possible to give a unique solution with a measured partial width. For other cases, the number of conditions is not enough to uniquely determine the initial structure. This means that we can adopt at least two different structures for the initial tetraquark state to explain the observed ratios between its decay channels. In principle, the mass spectrum may provide additional conditions to constrain the tetraquark wave function.

Another issue one may consider is the production of the tetraquark states. According to the investigation in Ref. [55], the production mechanism of \(X(5568)\) at D0 is very strange if it really exists. Up to now, no confirmed multiquark state has been observed in multiproduction processes. Probably this indicates that multiquark states are difficult to produce in such processes [55, 98]. If this conclusion is correct, maybe one would not observe states studied here in multiproduction processes. However, their production in decays of higher hadrons and in electron-positron annihilation processes could be detectable. The lowest tetraquark states we study are those composed of \(c, s, u,\) and \(d\) quarks. They can even be produced at BEPCII.

To summarize briefly, in the present work, we discuss the spectrum and decay properties of the possible tetraquark states with four different flavors in a color-magnetic interaction model. The tetraquark masses are obtained by estimating the mass difference between the tetraquarks and the \(X(4140)\). It is found that most states have rearrangement decays and only lowest \(J = 0, 1\) isoscalar \(bs\bar{u}\bar{d}\) and \(cs\bar{u}\bar{d}\) are probably stable. From the decay widths calculated with a constant Hamiltonian, the studied tetraquarks should be detectable if they do exist. In particular, the widths of the lowest \(I(J^P) = 0(0^+, 1^+)\) \(bc\bar{u}\bar{d}\) tetraquarks are in the order of 10 in units of MeV. From the consistency between the masses and widths of \(F_0 = bs\bar{u}\bar{d}\) states, it is not natural to interpret the \(X(5568)\) as a tetraquark. From the investigations on the relation between the internal structure of a tetraquark and its decay width \(\Gamma_{\text{sum}}\), we find: (1) one cannot uniquely determine the structure of a four-quark state just from the decay ratios between its rearrangement channels; (2) it is possible to judge relative stabilities of tetraquark states just from the calculable \(K\) factors for the quark-quark interactions; and (3) the range of \(\Gamma_{\text{sum}}\) can be estimated just with the masses of the involved states in the adopted rearrangement model. We hope that the present study can be helpful for the future studies of tetraquark states.

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Appendix A: \(K\) factors and CMI Hamiltonian

There are two forms of \(K_{ij}\). We here give a simple proof for their equivalence. Let us consider the eigenvalue problem of an \(n \times n\) CMI Hamiltonian \(\langle \hat{H}_{\text{CMI}} \rangle = \sum_{i<j} X_{ij} C_{ij}\) and assume that \(|x\rangle\) and \(E_{\text{CMI}}\) are one of its eigenvectors and
eigenvalues, respectively. Then we have

\[ \sum_{m<n} X_{mn}|x\rangle C_{mn} = E_{CMI}|x\rangle, \]

\[ \sum_{m<n} (x|X_{mn}|x\rangle C_{mn} = E_{CMI}. \] (A1)

Since the eigenvector is also affected when \( C_{ij} \) is changed, the originally defined \( K_{ij} \) in Refs. [16, 75]

\[ K_{ij} = \frac{\partial E_{CMI}}{\partial C_{ij}} \] (A2)

can be written as

\[ K_{ij} = \langle x|X_{ij}|x\rangle + \sum_{m<n} \left[ \left( \frac{\partial (x)}{\partial C_{ij}} \right) X_{mn}|x\rangle + \langle x|X_{mn} \left( \frac{\partial (x)}{\partial C_{ij}} \right) \right] C_{mn} \]

\[ = \langle x|X_{ij}|x\rangle + E_{CMI} \left[ \left( \frac{\partial (x)}{\partial C_{ij}} \right) |x\rangle + \langle x| \left( \frac{\partial (x)}{\partial C_{ij}} \right) \right]. \] (A3)

The second term is the derivative form of the normalization condition \( \langle x|x\rangle = 1 \) and vanishes (in fact, \( \langle x|\partial|x\rangle/(\partial C_{ij}) = 0 \)). Therefore, the two forms of the \( K \) factor are equivalent,

\[ K_{ij} = \frac{\partial E_{CMI}}{\partial C_{ij}} = \langle x|X_{ij}|x\rangle. \] (A4)

In this appendix, we also give a proof for the fact that \( y_i \)'s in Eq. (11) form the eigenvector of \( X_{13} \) or \( X_{14} \). We just illustrate the \( J = 2 \) case with \( X_{13} \). Basically, one has

\[ (q_1\bar{q}_3)_L (q_2\bar{q}_4)_L \; = \; y_1 (q_1q_2)_{6c}(\bar{q}_3\bar{q}_4)_{\overline{6c}} + y_2 (q_1q_2)_{\overline{3c}}(\bar{q}_3\bar{q}_4)_{3c}. \] (A5)

Since the meson-meson base is equivalent to the diquark-antidiquark base in describing the multiquark states, we also need the state \((q_1\bar{q}_3)_{8c}(q_2\bar{q}_4)_{8c}\). The transformation matrix is denoted as \( Y \),

\[ \left[ \begin{array}{c} (q_1\bar{q}_3)_{Lc}(q_2\bar{q}_4)_{Lc} \\ (q_1\bar{q}_3)_{S8c}(q_2\bar{q}_4)_{S8c} \end{array} \right] = Y \left[ \begin{array}{c} (q_1q_2)_{6c}(\bar{q}_3\bar{q}_4)_{\overline{6c}} \\ (q_1q_2)_{\overline{3c}}(\bar{q}_3\bar{q}_4)_{3c} \end{array} \right]. \] (A6)

Obviously, the elements in the first row of \( Y \) are \( y_1 \) and \( y_2 \). In the meson-meson base, the CMI Hamiltonian is

\[ \langle \hat{H}'_{CMI} \rangle = Y\langle \hat{H}_{CMI} \rangle Y^{-1} \; = \; \sum_{m<n} X'_{mn} C_{mn}. \] (A7)

One may prove that \( X'_{13} = YX_{13}Y^{-1} \) is a diagonal matrix:

\[ (X'_{13})^{ij} = -\langle (q_1\bar{q}_3)_i(q_2\bar{q}_4)_j|((\bar{\sigma}_1 \cdot \bar{\sigma}_3)(\bar{\lambda}_1 \cdot \bar{\lambda}_3))(q_1\bar{q}_3)_{j}(q_2\bar{q}_4)_j \rangle 
\]

\[ = -\langle (q_1\bar{q}_3)_i((\bar{\sigma}_1 \cdot \bar{\sigma}_3)(\bar{\lambda}_1 \cdot \bar{\lambda}_3))(q_1\bar{q}_3)_{j} \rangle \delta_{ij} \]

\[ = -4 \left[ C_2[SU(3)_c] - \frac{8}{3} \right] \left[ J_{13}(J_{13}+1) - \frac{3}{2} \right] \delta_{ij}, \] (A8)

where \( C_2[SU(3)_c] = 0 \) (3) is the quadratic Casimir operator for the color-singlet (octet) quark-antiquark state. It is obvious that the diagonal elements of \( X'_{13} \) are eigenvalues of \( X_{13} \) and \( Y \) is the matrix composed of the eigenvectors of \( X_{13} \). Therefore, one can directly get \( y_i \)'s from the eigenvectors of \( X_{13} \). For the \( X_{14} \) and the cases \( J = 0, 1 \), similar conclusion is also evidently valid.

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