A brief and personal history of stochastic partial differential equations

Lorenzo Zambotti

Laboratoire de Probabilités, Statistique et Modélisation, Sorbonne Université, Université de Paris, CNRS, 4 Place Jussieu, 75005 Paris, France
E-mail address: zambotti@lpsm.paris

Abstract. We trace the evolution of the theory of stochastic partial differential equations from the foundation to its development, until the recent solution of long-standing problems on well-posedness of the KPZ equation and the stochastic quantization in dimension three.

Keywords: Stochastic partial differential equations
MSC classification: 60H15

1. Introduction

In September 2017 I attended a meeting in Trento in honor of Luciano Tubaro, who was retiring. Mimmo Iannelli gave a humorous and affectionate talk whose title was Abstract stochastic equations: when we used to study in Rome’s traffic jams. He talked about the ’70s, when he and Luciano were the first students of Giuseppe Da Prato’s, who around 1975 proposed them to work on a brand new topic: stochastic partial differential equations. Since I was myself a PhD student of Da Prato’s in the late ’90s, on that day in Trento I was being told the story of the beginning of our scientific family.

Then, a month later, I was at the Fields Institute in Toronto for a conference in honor of Martin Hairer, who had been awarded in 2014 a Fields medal "for his outstanding contributions to the theory of stochastic partial differential equations, and in particular for the creation of a theory of regularity structures for such equations" (the official citation of the International Mathematical Union).

Within a few weeks I was therefore confronted with a vivid representation of the beginning of SPDEs and with a celebration of their culminating point so far. I realised that, because of Hairer’s Fields medal,
the mathematical community was suddenly aware of the existence of SPDEs, although very little was commonly known about them.

For example, during his laudatio which introduced Hairer’s talk at the 2014 International Congress of Mathematics in Korea, Ofer Zeitouni felt the need to say to the audience "I guess that many of you had never heard about stochastic partial differential equations". The other three Fields medals in 2014 were awarded for work in, respectively, dynamical systems, Riemann surfaces and number theory. Certainly there was no need to introduce these topics to the mathematicians attending the ICM. However, after forty years of work, with thousands of published papers and hundreds of contributors, SPDEs were still unknown to a large portion of the mathematical world.

I decided to dedicate my talk in Toronto not just to Hairer’s achievements, but to the whole community that had formed and nurtured him. In the last two years I have given several times this talk in different occasions. This special issue of DCDS gives me the opportunity to write down the few thoughts I have to share about this topic, in the hope that someone else may continue this work and enrich this tale with other points of view. I will make no claim to exhaustivity: the topic is vast and I know only a fraction of the literature. I wish to explain the origin and the development of SPDEs from my personal point of view, and I apologise in advance for the aspects of this story that I will fail to explain properly or even mention. I encourage anyone wishing to see this tale completed or told differently and better to do so and continue the work I am starting.

2. The beginnings

In principle, a Stochastic Partial Differential Equation (SPDE) is a Partial Differential Equations (PDE) which is perturbed by some random external force. This definition is however too general: if we have a PDE with some random coefficients, where the randomness appears as a parameter and the equation can be set and solved with classical analytic arguments, then one speaks rather of a random PDE; this is the case for example of a (deterministic) PDE with a random initial condition.

A SPDE is, more precisely, a PDE which contains some stochastic process (or field) and cannot be defined with standard analytic techniques; typically such equations require some form of stochastic integration. In most of the cases, the equation is a classical PDE perturbed by adding a random external forcing. One of the first examples is the
following *stochastic heat equation with additive noise*

\[
\frac{\partial u}{\partial t} = \Delta u + \xi
\]  

(1)

where \( u = (u(t, x))_{t \geq 0, x \in \mathbb{R}^d} \) is the unknown solution and \( \xi = (\xi(t, x)) \) is the random external force. Then one can add non-linearities and, in some cases, multiply the external force by a coefficient which depends on the unknown solution, for example

\[
\frac{\partial u}{\partial t} = \Delta u + f(u) + \sigma(u) \xi
\]  

(2)

where \( f, \sigma : \mathbb{R} \to \mathbb{R} \) are smooth. The product \( \sigma(u) \xi \) is not always well-defined, since in many cases of interest \( \xi \) is a *generalised function* and \( u \) is not expected to be smooth; in this case one writes the equation in an integral form and uses Itô integration to give a sense to the stochastic term.

The idea of associating PDEs and randomness was already present in the physics literature in the ’50s and ’60s, see for example [87, 64, 18, 42]. In the mathematical literature, several authors extended Itô’s theory of stochastic differential equations (SDE) to a Hilbert space setting, see for example Dalecki˘ı [27] and Gross [45]. In a paper published in 1969 [92], Zakai wrote that the unnormalised conditional density in a filtering problem satisfies a linear SPDE.

However, to my knowledge, the first papers which studied explicitly a SPDE as a problem in its own appeared in the ’70s. In 1970 Cabaña [13] considered a linear wave equation

\[
\frac{\partial^2 u}{\partial t^2} + 2b \frac{\partial u}{\partial t} = \Delta u + \xi
\]

with a *space-time white noise* \( \xi \) and one-dimensional space variable \( x \). This is a very important particular choice for the random external force: it is given by a random generalised function \( \xi \) which is *Gaussian* and has very strong independence properties, namely the "values" at different points in space-time are independent.

In 1972 three papers were published on the topic: two in France (Bensoussan-Temam [5] and Pardoux [77]) and one in Canada (Dawson [28]). The French school was strongly influenced by the PDE methods of the time, championed by Jacques-Louis Lions and his collaborators. Bensoussan and Temam [5] considered an evolution equation driven by a monotone non-linear operator \( A_t \)

\[
\frac{dy}{dt} + A_t(y) = \xi
\]
and with an external forcing $\xi$ which we can call now \textit{white in time and coloured in space}; this means that values of the noise on points with different time-coordinate are independent, but there is a non-trivial correlation in space. In [77] Pardoux considered a similar problem with multiplicative noise

$$\frac{dy}{dt} + A_t(y) = B_t(y) \xi$$

where $B_t$ is a non-linear operator and the stochastic term is treated with the Itô integration theory. In 1975 Pardoux defended his PhD thesis written under the supervision of Bensoussan and Temam, which is considered the first extended work on the topic.

Dawson’s paper [28] has a more probabilistic flavour. It treats the stochastic heat equations (1) and (2) with one-dimensional space variable $x$ and space-time white noise $\xi$; it shows that the solution $u$ to the linear equation (1) is almost-surely continuous in $(t, x)$ (this is false in higher dimension, as we are going to see below); moreover, it introduces the non-linear equation (2) with the coefficient $\sigma(u) = \sqrt{u}$, which will soon become famous as the equation of the Super-Brownian motion (for $f = 0$).

In the following years more and more researchers got interested in SPDEs. In particular, the Italian and Russian schools were founded, respectively, in 1976 with Da Prato’s first paper [22] on the topic (together with his students Iannelli and Tubaro) and between 1974 and 1977 with Rozovskii’s papers [82, 83] and Krylov-Rozovskii’s [59].

3. The physical models

In the ’80s some theoretical physicists published a few very influential papers based on applications of SPDEs to several important physical problems: Parisi-Wu’s [78] and Jona Lasinio-Mitter’s [55] on the \textit{stochastic quantization}, and the Kardar-Parisi-Zhang model for the \textit{dynamical scaling of a growing interface} [56]. All these papers would be, thirty years later, an important motivation for the theory of regularity structures, see below.

3.1. The Stochastic Quantization. The 1981 paper [78] by Parisi and Wu proposed a dynamical approach to the construction of probability measures which arise in Euclidean Quantum Field Theory. The difficulty with such measures is that they are supposed to be supported by spaces of distributions (generalised functions) on $\mathbb{R}^d$, which makes the definition of \textit{non-linear} densities problematic. For example one would
like to consider a measure on the space of distributions $\mathcal{D}'([0,1]^d)$ of the form

$$\mu(d\phi) = \frac{1}{Z} \exp \left( - \int_{[0,1]^d} V(\phi(x)) \, dx \right) \mathcal{N}(0, (1-\Delta)^{-1})(d\phi)$$

where $\mathcal{N}(0, (1-\Delta)^{-1})$ is a Gaussian measure with covariance operator $(1-\Delta)^{-1}$, with $\Delta$ the Laplace operator on $[0,1]^d$ with suitable boundary conditions, and $V : \mathbb{R} \to \mathbb{R}$ is some potential. If $d > 1$ then $\mathcal{N}(0, (1-\Delta)^{-1})$-a.s. $\phi$ is a distribution and not a function, and the non-linearity $V(\phi)$ is therefore ill-defined. Parisi-Wu introduce a stochastic partial differential equation

$$\frac{\partial u}{\partial t} = \Delta u - u - \frac{1}{2} V''(u) + \xi, \quad x \in [0,1]^d$$

(3)

which has $\mu$ as invariant measure, namely if $u(0, \cdot)$ has law $\mu$, then so has $u(t, \cdot)$ for all $t \geq 0$. This is an infinite-dimensional analog of the classical Langevin dynamics. By the ergodic theorem, for a generic initial condition $u(0, \cdot)$, the distribution of $u(t, \cdot)$ converges to $\mu$ as $t \to +\infty$. Therefore one can use the stochastic dynamical system $(u(t, \cdot))_{t \geq 0}$ in order to obtain useful information on $\mu$.

We note however that, for $d > 1$, the solution to (3) is expected to be again a distribution on space-time, at least this is the case for the linear equation with $V' \equiv 0$. Therefore a rigorous study of this equation is also problematic, since $V''(u)$ is again ill-defined.

The first rigorous paper on the Parisi-Wu programme was by Jona Lasinio-Mitter [55], where the authors chose the non-linearity $V(\phi) = \phi^4$ and the space dimension $d = 2$, in order to construct the continuum $\phi^4_2$ model of Euclidean Quantum Field Theory [86, 43], and called this equation the stochastic quantization. Jona Lasinio-Mitter studied a modified version of equation (3) and obtained probabilistically weak solutions via a Girsanov transformation; strong solutions to (3) were obtained in a later paper by Da Prato-Debussche [21], see below. The case of space dimension $d = 3$ remained however open until the inception of regularity structures.

3.2. The KPZ equation. The Kardar-Parisi-Zhang (KPZ) equation [56] is the following SPDE

$$\frac{\partial h}{\partial t} = \nu \Delta h + \lambda |\nabla h|^2 + \xi, \quad x \in \mathbb{R}^d$$

(4)

and describes the fluctuations around a deterministic profile of a randomly growing interface, where $\nabla$ is the gradient with respect to the space variable $x$. 
From an analytic point of view, even if $d = 1$ the KPZ equation is very problematic: if we consider the case $\lambda = 0$ then we are back to the stochastic heat equation with additive white noise \([1]\), for which it is known that the solution $u$ is not better than Hölder-continuous in $(t, x)$ and certainly not differentiable; we expect $h$ in \([4]\) to have at best the same regularity as $u$. In particular the gradient in space $\nabla h$ is defined only as a distribution and the term $(\nabla h)^2$ is ill-defined. We restrict ourselves for simplicity to the case $\nu = \lambda = 1/2$.

In the original KPZ paper \([56]\) it was noticed that one can linearize \([4]\) by means of the Cole-Hopf transformation: if we define $\psi = (\psi(t, x))_{t \geq 0, x \in \mathbb{R}}$ as the unique solution to the equation

$$\frac{\partial \psi}{\partial t} = \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + \psi \xi, \quad x \in \mathbb{R}, \tag{5}$$

which is called the stochastic heat equation with multiplicative noise, then $h := \log \psi$ (formally) solves \([1]\).

In the first mathematical paper on KPZ, Bertini-Cancrini \([7]\) studied in 1995 the stochastic heat equation \([5]\) in the Itô sense for $d = 1$. Since Mueller \([70]\) had proved that a.s. $\psi(t, x) > 0$ for all $t > 0$ and $x \in \mathbb{R}$, then the Cole-Hopf solution $h = \log \psi$ is indeed well-defined. Bertini-Giacomin \([8]\) proved in 1997 that the stationary Cole-Hopf solution is the scaling limit of a particle system, the weakly-asymmetric simple exclusion process (WASEP); this celebrated result was the first example of the KPZ universality class, see below.

Since \([5]\) is to be interpreted in the Itô sense, one can apply the Itô formula to $h = \log \psi$ and the result is, at least formally, that $h$ solves

$$\frac{\partial h}{\partial t} = \frac{1}{2} \frac{\partial^2 h}{\partial x^2} + \frac{1}{2} \left[(\partial_x h)^2 - \infty\right] + \xi, \quad x \in \mathbb{R}, \tag{6}$$

which is almost \([1]\), apart from the appearance of the famous infinite constant which is supposed to renormalize the ill-defined term $(\partial_x h)^2$. Making sense of this renormalization and constructing a well-posedness theory for such equations were however open problems for over 15 years until Hairer’s breakthrough \([52]\), see below.

We note that the KPZ equation, and in particular its universality class, has been one of the most fertile topics in probability theory of the last decade, with connections to particle systems, random matrices, integrable probability, random polymers and much else. See the surveys by Quastel \([80]\) and Corwin \([20]\) for more details.

3.3. Superprocesses. SPDEs have also been applied to biological systems, in particular in the context of the so-called superprocesses introduced by Watanabe and Dawson in the ’70s. Superprocesses are limits
of discrete population models of the following type: particles evolve in a $\mathbb{R}^d$ space following some Markovian dynamic, typically Brownian motion, independently of each other; at random exponential times each particle dies and is replaced by a random number of identical particles, which become new elements of the population and behave as all other particles. We refer to the Saint-Flour lecture notes by Dawson [29] and Perkins [79] for pedagogical introductions to this topic.

The total number of members of the population which are alive at time $t \geq 0$ follows a standard branching process and is independent of the motion of the particles. Therefore there are three situations, depending on the value $m$ of the average number of descendants that a particle has when it dies: if $m > 1$ the population grows at an exponential rate, if $m < 1$ it dies after a finite and integrable time, if $m = 1$ it dies after a finite but non-integrable time. The three situations are called, respectively, supercritical, subcritical and critical.

The critical case, with Brownian spatial motion, has a scaling limit which is a Markov process with values in the space of measures on the state space $\mathbb{R}^d$; this process is called the super-Brownian motion. If $d = 1$, then Konno-Shiga [57] proved in 1988 that a.s. this random measure has a continuous density $X_t(x)$ with respect to the Lebesgue measure $dx$ on $\mathbb{R}$, and $(X_t(x))_{t \geq 0, x \in \mathbb{R}}$ solves the SPDE

$$
\frac{\partial X}{\partial t} = \frac{1}{2} \frac{\partial^2 X}{\partial x^2} + \sqrt{X} \xi.
$$

The diffusion coefficient of this equation, already introduced by Dawson in [28], does not satisfy the usual Lipschitz condition and, indeed, pathwise uniqueness for (7) is still an open problem, see the papers by Mytnik-Perkins [74] and Mueller-Mytnik-Perkins [71]. More precisely, the situation is the following: we consider the SPDE

$$
\frac{\partial X}{\partial t} = \frac{1}{2} \frac{\partial^2 X}{\partial x^2} + \sigma(X) \xi,
$$

with $\sigma : \mathbb{R} \to \mathbb{R}$ a Hölder function with exponent $\gamma \in ]0, 1[$, namely $|\sigma(x) - \sigma(y)| \leq C|x - y|^{\gamma}$ and one looks in general for solutions with values in $\mathbb{R}$, rather than in $\mathbb{R}_+$; in particular, for equation (7) one would have $\sigma(u) = \sqrt{|u|}$. Then:

- if $\gamma > 3/4$ we have pathwise uniqueness, namely if we have two solutions $(X^1, \xi)$ and $(X^2, \xi)$ to (5) driven by the same noise $\xi$ with $X^1(0, \cdot) = X^2(0, \cdot)$ a.s., then $X^1 \equiv X^2$ almost surely
- if $\gamma < 3/4$ then pathwise uniqueness fails in general and there are counterexamples
• if $\sigma(0) = 0$ and one is interested only in the class of non-negative solutions, then it is not known whether pathwise uniqueness holds or fails in this class for $\gamma < 3/4$. This leaves in particular the hope that the equation for super-Brownian motion (7) may satisfy pathwise uniqueness. However, for the related equation of super-Brownian motion with immigration the pathwise non-uniqueness was proved by Chen in [19].

If the state space $\mathbb{R}^d$ has dimension greater or equal to 2, then a.s. the measure $X_t(dx)$ is singular with respect to the Lebesgue measure (see [30]), but the equation (7) is still well-defined as a martingale problem, since the diffusion coefficient $\sigma(x) = \sqrt{x}$ has the special property that $\sigma^2(x) = x$ is linear. Remarkably, this martingale problem is well-posed and one can prove uniqueness in law of these superprocesses using a technique called duality due to Watanabe [91], see the cited paper by Konno-Shiga [57]; duality can also be applied to prove uniqueness for other processes, see the works of Shiga [84, 85] and Mytnik [73].

Finally, we mention that superprocesses are related to Le Gall’s Brownian snake, see [63], which also plays a crucial role in the context of planar random maps, see e.g. Miermont’s lecture notes [68].

4. The theory

During the ’80s and the ’90s, several monographs were published with the aim of presenting a systematic theory of SPDEs.

The first major monograph was Walsh’s Saint-Flour lecture notes [90], which were published in 1986. In this course Walsh proposed a general approach to SPDEs which has been very influential; his point of view has a very probabilistic flavour, since it consists in regarding the solution $u = u(t, x)$ of a (parabolic or hyperbolic) SPDE as a multi-parameter process, or more generally a multi-parameter random field. The stochastic integration with respect to space-time white noise is developed according to this point of view, considering $t \mapsto \xi(t, \cdot)$ as a so-called martingale measure, thus generalizing the Itô theory. We have used Walsh’s notations for the equations numbered from (1) to (7) above, and for others below.

In 1992 the first book by Da Prato-Zabczyk [23] was published. This monograph, also known as the red book among Da Prato’s students, is still the reference text for the so-called semigroup approach to SPDEs. Da Prato-Zabczyk’s point of view is to treat a SPDE as an-infinite dimensional SDE, and the solution $u = u(t, \cdot)$ as a function-valued process with a single parameter, the time $t$. The notations are different from those of Walsh; for example the stochastic heat equation with
additive space-time white noise \((1)\) is written as
\[
dx = AX \, dt + dW
\]
where \(X_t = u(t, \cdot) \in L^2(\mathbb{R}) = H\), \(A : D(A) \subset H \to H\) is the realization of \(\partial_t^2\) in \(H\), \((W_t)_{t \geq 0}\) is a cylindrical Wiener process. The solution to this equation is called the stochastic convolution and is written explicitly as
\[
X_t = e^{tA}X_0 + \int_0^t e^{(t-s)A} \, dW_s, \quad t \geq 0.
\]
The general SPDE \((2)\) with non-linear coefficients is written as
\[
dx = (AX + F(X)) \, dt + \Sigma(X) \, dW
\]
where \(F : D(F) \subseteq H \to H\) is some non-linear function and \(\Sigma\) is a map from \(H\) to the linear operators in \(H\). This approach has a more functional-analytical flavour, and is based mainly on the study of the properties of the semigroup \((e^{tA})_{t \geq 0}\) generated by \(A\) in \(H\), and their interplay with the properties of the cylindrical Wiener process \(W\). This non-linear equation is usually written in its mild formulation
\[
X_t = e^{tA}X_0 + \int_0^t e^{(t-s)A} F(X_s) \, ds + \int_0^t e^{(t-s)A} \Sigma(X_s) \, dW_s.
\]

During the ’90s there was also an important activity on infinite-dimensional analysis, namely on elliptic and parabolic PDEs where the space-variable belongs to a Hilbert space. The connection with SPDEs is given by the notion of infinitesimal generator which is associated with a Markov process with continuous paths. As for finite-dimensional diffusions, the transition semigroup of the solution to a SPDE solves a parabolic equation, known as Kolmogorov equation. One can find a systematic theory of these operators in the third book by Da Prato-Zabczyk \([25]\). Much work was dedicated to existence and uniqueness of invariant measures, see the next section; the second Da Prato-Zabczyk book was entirely dedicated to this topic \([24]\).

It can be recalled that Itô introduced his notion of stochastic differential equations in order to give a probabilistic representation of the solution to Kolmogorov equations. Viceversa, if the Kolmogorov equation is well-posed, then it is possible to construct the law of the associated Markov process. This allows to construct weak (in the probabilistic sense) solutions, especially in the form of martingale solutions, see the 1979 monograph by Stroock-Varadhan \([88]\) on the theory for finite dimensional diffusions.

The construction of the transition semigroup of a Markov process in a locally compact space can be done also with another analytical tool,
a Dirichlet Form, for which a theory was developed in particular by Fukushima, see the monographs \cite{39, 40}. The state space of a SPDE is however always a function space, and therefore infinite-dimensional. The extension of Fukushima’s theory to non locally compact spaces was a project of Albeverio-Høegh-Krohn \cite{1} since the ’70s and was finally obtained by Ma-Röckner \cite{66}. Although Dirichlet forms allow to construct only weak solutions, they are a powerful tool in very singular situations, where pathwise methods are often ineffective.

Another approach to SPDEs is given by Krylov’s $L^p$-theory, see for example \cite{58}.

5. Ergodicity of Navier-Stokes

The Navier-Stokes equation for the flow of an incompressible fluid is one of the most prominent PDEs and it is therefore not surprising that its stochastic version was among the first SPDEs to be studied, starting from the 1973 paper \cite{6} by Bensoussan-Temam. The equation has the form (in Walsh’s notation)

$$\frac{\partial u}{\partial t} + (\nabla u) \cdot u = \nu \Delta u - \nabla p + \xi, \quad \text{div } u = 0,$$

where $u(t, x) \in \mathbb{R}^d$ denotes the value of the velocity of the fluid at time $t \geq 0$ and position $x \in \mathbb{R}^d$, $p(t, x)$ is the pressure, $\nu > 0$ and $\xi$ is an external noise whose structure will be made precise below.

The statistical approach to hydrodynamics is based on the assumption that the fluid has a stationary state (invariant measure) on the phase space; by the ergodic theorem, the time average of an observable computed over the dynamics converges for large time to the average of the observable with respect to the invariant measure. This ergodicity property must however be proved, and in the case of the Stochastic Navier Stokes equation in 2D this has been a very active area of research, at least between the 1995 paper by Flandoli-Maslowski \cite{36} and the 2006 paper by Hairer-Mattingly \cite{54}.

5.1. Ellipticity versus hypoellipticity. For stochastic differential equations in general, the choice of the external noise plays a very important role. In most of the literature on SPDEs, the space-time noise $\xi$ is realised as the following series

$$\xi(t, x) = \sum_{k=1}^{\infty} \lambda_k e_k(x) \dot{B}_k(t), \quad t \geq 0, \quad x \in \mathcal{O} \subseteq \mathbb{R}^d,$$
where \((\lambda_k)_k\) is a sequence of real numbers, \((e_k)_k\) an orthonormal basis of \(L^2(\mathcal{O}, dx)\) and \((B_k)_k\) an independent family of standard Brownian motions. If \(\lambda_k = 1\) for all \(k\) then we have space-time white noise, which has the property that for all \(\varphi \in L^2(\mathcal{O}, dx)\) the random variable

\[
\int_{[0,T] \times \mathcal{O}} \varphi(t, x) \xi(t, x) \, dt \, dx := \sum_{k=1}^{\infty} \langle \varphi, e_k \rangle_{L^2(\mathcal{O}, dx)} B_k(T)
\]

has normal law \(\mathcal{N}\left(0, T \| \varphi \|^2_{L^2(\mathcal{O}, dx)} \right)\).

In analogy with the finite-dimensional case, if \(\lambda^2_k \geq \varepsilon > 0\) for all \(k\), then we are in the elliptic case. In finite dimension, we are in a degenerate case as soon as \(\lambda_k = 0\) for some \(k\); in infinite dimension, however, we can have \(\lambda_k > 0\) for all \(k\) but \(\lambda_k \to 0\) as \(k \to +\infty\). This situation is neither degenerate nor elliptic.

The paper by Flandoli-Maslowski proved for the first time ergodicity for a stochastic Navier-Stokes equation in 2D, under the assumption that \(\lambda_k > 0\) for all \(k\) but \(\lambda_k \to 0\) as \(k \to +\infty\) with two (different) power-law controls from above and from below. This article sparked an intense activity and a heated debate which revolved around the following question: what is the most relevant choice of the noise structure, which allows to prove ergodicity?

If, as in Flandoli-Maslowski \[36\], the noise is sufficiently non-degenerate, namely if \(\lambda_k > 0\) and \(\lambda_k \to 0\) not too fast as \(k \to +\infty\), then it is often possible to proveergodicity using an argument due to Doob and based on two ingredients: the Strong-Feller property and irreducibility; the former means that the transition semigroup of the dynamics maps bounded Borel functions on the state space into continuous functions, the latter that all non-empty open sets of the state space are visited with positive probability at any positive time. The Strong-Feller property is proved with ideas coming from Malliavin calculus, in particular on an integration by parts on the path space which is now known as the Bismut-Elworthy-Li formula, see the paper by Elworthy-Li \[33\] and the monograph by Cerrai \[15\]; irreducibility is based on control theory for PDEs. These techniques were explored and applied to a number of examples in the second Da Prato-Zabczyk book \[24\] of 1996.

However, it soon appeared clear that it was possible to consider a degenerate noise and still obtain uniqueness of the invariant measure. Here by degenerate we mean that \(\lambda_k = 0\) for all \(k > N\), where \(N\) is a deterministic integer. The main idea behind this line of research was that, if the noise acted on a sufficiently large but finite number of modes (i.e. the functions \(e_k\)), then the noise is elliptic on the modes which determine the long-time behavior of the dynamics; we can call this
the essentially elliptic case. These results, together with exponential convergence to equilibrium, were proved independently (for Gaussian or for discrete noise) by three groups of authors during the same years: Mattingly [67] and E-Mattingly-Sinai [32], Kuksin-Shirikyan [60, 61], Bricmont-Kupiainen-Lefevere [9, 10].

However in these works the number $N$ of randomly forced modes is not universal but depends on the parameters $\nu$ and $\sum_k \lambda_k^2$ of the equation. This was dramatically improved in the paper by Hairer-Mattingly [54] published in 2006 in Annals of Mathematics, which proved that it is enough to inject randomness only in four well-chosen modes, then the non-linearity propagated the randomness to the whole system for any $\nu > 0$: the so-called hypoelliptic case, for which it is possible to derive uniqueness of the invariant measure for the 2D stochastic Navier-Stokes. One of the main novelties in this paper was the notion of the asymptotic Strong-Feller property, which could be proved in the hypoelliptic case, while the standard Strong-Feller property requires much stronger non-degeneracy properties of the noise.

Let us mention here that the Malliavin Calculus, see e.g. Nualart’s monograph [75], has played an important role for Navier-Stokes like for many other SPDEs.

6. My SPDEs

The results on the ergodicity of the stochastic Navier-Stokes equation seemed at the time to make SPDEs with degenerate noise particularly prominent. Now that singular SPDEs with space-time white noise and regularity structures have become so famous, this may seem even strange. In fact, since the very first papers that I have mentioned, see Cabaña [13] and Dawson [28], the research activity on SPDEs with genuinely infinite-dimensional noise has always been intensive and most of the problems I have mentioned above concern space-time white noise.

The case of degenerate noise is certainly more difficult if one wants to prove ergodicity, as we have seen. However, if the noise is spatially finite-dimensional, then the solution to the SPDE are typically smooth in space, although still Brownian-like in time. In the case of space-time white noise, on the contrary, the solution are rather Brownian-like in space if the space dimension is $d = 1$, and even less regular in time; if $d > 1$, as we have already seen, solutions are rather distributions.

Therefore, SPDEs driven by space-time white noise are particularly strange objects: even the solutions to the simplest equation, as the stochastic heat equation with additive space-time white noise, are far too irregular for any of the derivatives which appear in (1) to make
any sense as a function. The KPZ equation (6) has almost an explicit solution given by the Cole-Hopf transform $h = \log \psi$, with $\psi$ solution to the stochastic heat equation with multiplicative space-time white noise (5); however the KPZ equation itself makes no sense as it is written in (6)!

It is in this topic that I made my first steps as a researcher. I did my PhD at Scuola Normale in Pisa under the supervision of Giuseppe Da Prato (also known as Beppe) from 1997 to 2001. Like Da Prato himself and many of his students, I started as an analyst but felt increasingly attracted by probability theory, in particular stochastic calculus and SDEs. In the shelves of Beppe's office I found the Revuz-Yor monograph, which became one of my favourite mathematics books. I started to dream of unifying two worlds: the classical Itô theory of stochastic calculus based on martingales, and SPDEs.

Chapter 5 in the book by Revuz-Yor on local time and reflecting Brownian motion was one of the topics which most intrigued me. At that time Da Prato was studying equations of the form

$$\text{d}X \in (AX - \partial U(X)) \text{d}t + \text{d}W$$

with $U : H \to \mathbb{R}$ a convex lower semi-continuous but not necessarily differentiable function. In the deterministic setting, this is a classical problem and the set $\partial U(x)$ is the subdifferential at a point $x \in H$, namely the set of all directions $h \in H$ such that the affine subspace $U(x) + \{z \in H : \langle z, h \rangle = 0\}$ lies below the graph of $U$. For a simple example, think of the function $\mathbb{R} \ni x \mapsto |x| \in \mathbb{R}_+$, which is convex and has as subdifferential the set $\{1\}$ for all $x > 0$, the set $\{-1\}$ for all $x < 0$ and the set $[-1, 1]$ for $x = 0$. Then equation (9) is rather a stochastic differential inclusion, and if $U$ is differentiable at $x$ then $\partial U(x) = \{\nabla U(x)\}$. There is an extensive literature on this problem in the finite-dimensional case, see e.g. Cépa [14], much less so in infinite dimension where many problems remain open.

The case of $U$ being equal to 0 on a closed convex set $K \subseteq H$ and to $+\infty$ on $H \setminus K$ seemed to be outside the scope of Da Prato’s techniques. I convinced myself that this case had to be related with reflection on the boundary of $K$, but I was unable to make this precise. Then Samy Tindel pointed out to me a 1992 paper by Nualart and Pardoux [76] on the following SPDE with reflection at 0

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \xi + \eta, \quad t \geq 0, \ x \in [0, 1],$$

where $\eta$ is a Radon measure on $]0, +\infty[ \times ]0, 1[$, $u$ is continuous and non-negative, and the support of $\eta$ is included in the zero set $\{(t, x) :
$u(t, x) = 0$} of $u$, or equivalently

$$u \geq 0, \quad \eta \geq 0, \quad \int_{0, +\infty \times [0, 1]} u \, d\eta = 0. \quad (11)$$

This is a stochastic obstacle problem, the obstacle being the constant function equal to 0, which can be formulated in the abstract setting of the stochastic differential inclusion [9]. Continuity of $(t, x) \mapsto u(t, x)$ here is essential in order to make sense of the condition (11); in this setting the Walsh approach is clearly necessary, since continuity of $t \mapsto u(t, \cdot)$ in $L^2(0, 1)$ would not be sufficient. In higher space dimension, $u$ is not expected to be continuous and indeed it remains an open problem to define in this case a notion of solution to (10)-(11). We note also that this equation arises as the scaling limit of interesting microscopic models of random interfaces: see Funaki-Olla [41] and Etheridge-Labbé [34].

The Nualart-Pardoux paper was motivated by stochastic analysis but it was an entirely deterministic work, which pushed the PDE techniques to cover a situation of minimal regularity for the solution; a probabilistic interpretation of this result remained elusive. This is what I tried to give with the results of my PhD thesis. First I identified in [93] the unique invariant measure of (10)-(11) as the 3-d Bessel bridge (also known as the normalized Brownian excursion), an important process which plays a key role in the study of Brownian motion and its excursion theory, see [81]. Then I proved in [94] an infinite-dimensional integration by parts with respect to the law of the 3-d Bessel bridge, which gave a powerful probabilistic tool to study the reflection measure $\eta$ (it provides its Revuz measure). Then I set out to study the fine properties of the solution, in particular of the contact set $\{(t, x) : u(t, x) = 0\}$ between the solution $u$ and the obstacle 0, see [95] and the paper [26] in collaboration with Dalang and Mueller.

In these papers I tried to realize my dream, by showing that solutions to SPDEs display very rich and new phenomena with respect to finite-dimensional SDEs, and that it was possible to go much beyond results on existence and uniqueness. I found some interesting link between classical stochastic processes arising in the study of Brownian motion and SPDEs. For a more recent account, see my Saint-Flour lecture notes [96].

However it does not seem that this point of view has been followed by many others. As we are going to see, the SPDE community would soon be heading in a very different direction.
7. Rough paths and regularity structures

In 1998 T. Lyons published a paper [65] on a new approach to stochastic integration. Lyons was an accomplished probabilist and an expert of stochastic analysis. Therefore it may seem puzzling that the aim of his most famous contribution to mathematics, the invention of rough paths, is to give a deterministic theory of stochastic differential equations!

The classical Itô theory of stochastic calculus, see again [81], is a wonderful tool to study stochastic processes (more precisely continuous semimartingales). Not only does it allow to prove existence and uniqueness of solutions to stochastic differential equations, but it also allows to compute the law of a great variety of random variables and stochastic processes. The key tool is that of martingales, which allow explicit computations of expectations and probabilities with often deep and surprising results.

In particular one obtains well-posedness of SDEs in $\mathbb{R}^d$ of the form
\begin{equation}
    dX_t = b(X_t) \, dt + \sigma(X_t) \, dW_t,
\end{equation}
with $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ smooth coefficients and $(W_t)_{t \geq 0}$ a Brownian motion in $\mathbb{R}^d$. However, in general $X$ is not better than a measurable function of $W$. This fact is rarely mentioned in courses of stochastic calculus, and probabilists seem used to it. Nevertheless, a physicist may point out that Brownian motion or its derivative, white noise, are an approximation of a real noise, not the other way round; an analyst may found this lack of continuity disturbing. Therefore a theory which is too sensitive on the structure of the noise is not so satisfactory after all. A robust theory would be more convincing from this point of view. In the late ’70s, the works of Doss [31] and Sussmann [89] gave sufficient conditions on the coefficient $\sigma$ for continuity of the maps $W \mapsto X$ in the sup-norm topology on $C([0,T];\mathbb{R}^d)$. These conditions were however very restrictive for $d > 1$.

Following an early intuition by Föllmer [37], Lyons constructed a deterministic (pathwise) approach to stochastic integration. The main result is the construction of a topology that makes the map $W \mapsto X$ continuous. However, there is a very important twist: the topology is not just on $W$ or $X$, but on a richer object which contains more information. If for example $W : [0,T] \to \mathbb{R}^d$ is a deterministic smooth path, then one needs to consider a finite number of iterated integrals of $W$, which take the form
\[ W_{s,t}^n = \int_{s < u_1 < \cdots < u_n < t} dW_{u_1} \otimes \cdots \otimes dW_{u_n}, \quad n \in \mathbb{N}, \ 0 \leq s \leq t \leq T, \]
where $dW_u = \dot{W}_u \, du$. For a fixed $\gamma \in ]0,1[$, one takes $N \in \mathbb{N}$ such that $N\gamma \leq 1 < (N+1)\gamma$ and for every smooth $W : [0,T] \to \mathbb{R}^d$

$$W_{s,t}^{(N)} := 1 + \sum_{n=1}^N W_n^{s,t}, \quad 0 \leq s \leq t \leq T,$$

which belongs to the truncated tensor algebra $T^{(N)} = \oplus_{n=0}^N (\mathbb{R}^d)^\otimes n$. We note that $W_{s,t}^1 = W_t - W_s$, so that $W_{s,t}^{(N)}$ contains the increments of the original process, plus additional information. We can now define a distance between two such objects $W_{s,t}^{(N)}$ and $V_{s,t}^{(N)}$ for smooth $W,V : [0,T] \to \mathbb{R}^d$

$$d_\gamma (W^{(N)}, V^{(N)}) := \sup_{n=1,\ldots,N} \sup_{s \neq t} \frac{|W_n^{s,t} - V_n^{s,t}|}{|t-s|^{n\gamma}}.$$ 

Then Lyons’ result was that the map $W^{(N)} \mapsto X^{(N)}$, where $W,X : [0,T] \to \mathbb{R}^d$ are smooth processes which satisfy (12), is continuous with respect to the metric $d_\gamma$.

Lyons’ paper [65] was astounding for its novelty: it introduced in stochastic analysis a number of concepts which were unknown to many probabilists, in particular the algebraic language based on the work of Chen [17] on iterated integrals. Moreover it presented a radically different approach to the pillar of modern probability theory, the Itô stochastic calculus. For these reasons, it seems that Lyons’ ideas took some time before being widely accepted by the community and became really famous only fifteen years later, when Hairer proved their power in the context of SPDEs. See the book of Friz-Hairer [38] for a pedagogical introduction.

7.1. Singular SPDEs and regularity structures. As we have seen above, several interesting physical models were described in the ’80s with SPDEs such as the dynamical $\phi^4_d$ model, recall the stochastic quantization [8],

$$\frac{\partial \phi}{\partial t} = \Delta \phi - \phi^3 + \xi, \quad x \in \mathbb{R}^d,$$

for $d=2,3$ and the KPZ equation [6]. In both equations there are ill-defined non-linear functionals of some distribution. Equations of this kind are now commonly known as singular SPDEs.

In 2003 Da Prato-Debussche [21] solved the stochastic quantization in $d=2$ with the following idea: they wrote $\phi = z + \nu$, where $z$ is the solution to the linear stochastic heat equation with additive white
noise

\[ \frac{\partial z}{\partial t} = \Delta z + \xi, \quad x \in \mathbb{R}^2, \]

and they wrote an equation for \( v = \phi - z \)

\[ \frac{\partial v}{\partial t} = \Delta v - z^3 - 3z^2v - 3zv^2 - v^3, \]

which is now random only through the explicit Gaussian process \( z \). We note that \( z \) is still a distribution, so that the terms \( z^2 \) and \( z^3 \) are still ill-defined; however it turns out that it is possible to give a meaning to these terms as distributions with the classical Wick renormalization. Then, the products \( z^2v \) and \( zv^2 \) are defined using Besov spaces. This allows to use a fixed point argument for \( v \) and obtain existence and uniqueness for the original (renormalized) equation. However this technique does not work for \( d = 3 \), since in this case the products \( z^2v \) and \( zv^2 \) are still ill-defined.

Since Lyons’ foundational paper of 1998, rough paths have been based on generalised Taylor expansions, with standard monomials replaced by iterated integrals of the driving noise. In 2004 Gubinelli built on this idea a new approach to rough integration based on the notion of controlled paths \[46\] and started to work on the project of a rough approach to SPDEs, see for example the 2010 paper \[50\] with Tindel.

In 2011 Hairer \[51\] considered the equation

\[ \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + g(u) \frac{\partial u}{\partial x} + \xi, \quad t \geq 0, \quad x \in \mathbb{R} \]

with \( u \) and \( \xi \) taking values in \( \mathbb{R}^d \) with \( d > 1 \), and \( g \) taking values in \( \mathbb{R}^{d \times d} \). Although this is less frightening than KPZ, the product \( g(u) \frac{\partial u}{\partial x} \) is ill-defined for the usual reason: the partial derivative of \( u \) is a distribution, the function \( g(u) \) is not smooth, and therefore the product cannot be defined by an integration by parts or other classical tools (the fact that \( u \) is vector valued prevents in general this product from being written as \( \frac{\partial}{\partial x} G(u) \)). The idea was to treat the solution \( u(t, x) \) as a rough path in space.

In 2013 Hairer managed to apply the same techniques to KPZ \[52\], thus giving a well-posedness theory for this equation first introduced in 1986. The importance of this result was amplified by the explosion of activity around the KPZ universality class following the 2011 papers by Balázs-Quastel-Seppäläinen \[41\] and Amir-Corwin-Quastel \[3\], which proved that the Cole-Hopf solution proposed by Bertini-Cancrini has indeed the scaling computed in the original KPZ paper \[56\] with non-rigorous renormalization group techniques.
In order to solve the stochastic quantization in \( d = 3 \), and many other equations, Hairer [53] expanded the theory of rough paths to cover functions of space-time. Da Prato-Debussche [21] had solved the case \( d = 2 \) with the global expansion \( \phi = z + v \) of the solution, in terms of an explicit term \( z \) and a remainder \( v \). Hairer’s idea was to use rather local expansions at each point \((t, x)\) in space-time, with a far-reaching generalization of the classical notion of Taylor expansion. The theory has been developed and expanded in three subsequent papers: Bruned-Hairer-Zambotti [12], Chandra-Hairer [16], Bruned-Chandra-Chevryrev-Hairer [11].

In the meantime, Gubinelli-Imkeller-Perkowski [47] constructed a different approach to singular SPDEs based on paracontrolled distributions, combining the paradifferential calculus coming from harmonic analysis and the ideas of rough paths. This approach is effective in many situations like KPZ and the stochastic quantization, see also the paper [62] by Mourrat-Weber on the convergence of the two-dimensional dynamic Ising-Kac model to the dynamical \( \phi^4_2 \), see [13], but not in all cases which are covered by regularity structures. In my personal opinion it is Hairer’s theory which transposes in the most faithful way Gubinelli’s ideas on rough paths from SDEs to SPDEs.

Another interesting approach to the KPZ equation is that of energy solutions by Gonçalves-Jara [44] and Gubinelli-Jara [48], which is particularly effective in order to prove convergence under rescaling of a large class of particle systems to a martingale problem formulation of KPZ. Uniqueness for such a martingale problem was proved in [49] by Gubinelli-Perkowski. Other construction of the \( \phi^4_3 \) dynamical model are due to Kupiainen [62], using renormalization group methods, and to Albeverio-Kusuoka [2], using finite-dimensional approximations.

8. Conclusions

In this brief and personal history of SPDEs I have left aside many topics that would deserve more attention, for example

- **regularization by noise**, see Flandoli-Gubinelli-Priola [35]
- the stochastic FKPP equation, see Mueller-Mytnik-Quastel [72]
- stochastic dispersive equations, stochastic conservation laws and viscosity solutions for fully non-linear SPDEs
- numerical analysis of SPDEs.

I hope that I have at least managed to express my enthusiasm for this topic. The last seven years have been particularly exciting: Gubinelli and Hairer have clearly influenced each other in a number of
A brief and personal history of stochastic partial differential equations

occasions, and their work has spurred an exceptional activity in this area. Rough paths and regularity structures tend to make relatively little use of classical probability theory, and my project of combining stochastic calculus and SPDEs went exactly in the opposite direction. However in the years before 2013 I felt somewhat discouraged by the lack of progress of this project, and Hairer’s paper on KPZ came as a revelation to me. What came afterwards was one of those rare situations when reality surpasses our own dreams.

The message that I wished to convey is that the ground for the success of today was prepared by a considerable amount of work by a whole community, in particular on equations driven by space-time white noise. I am convinced that this activity has produced many ideas which could and should be of interest for other communities and there are already encouraging signs in this direction.

References

[1] S. Albeverio and R. Høegh-Krohn, “Dirichlet forms and diffusion processes on rigged Hilbert spaces,” Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 40 no. 1, (1977) 1–57. https://doi.org/10.1007/BF00535706

[2] S. Albeverio and S. Kusuoka, “The invariant measure and the flow associated to the \( \Phi^4_3 \)-quantum field model,” ArXiv e-prints (2017), arXiv:1711.07103

[3] G. Amir, I. Corwin, and J. Quastel, “Probability distribution of the free energy of the continuum directed random polymer in 1 + 1 dimensions,” Comm. Pure Appl. Math. 64 no. 4, (2011) 466–537. https://doi.org/10.1002/cpa.20347

[4] M. Balázs, J. Quastel, and T. Seppäläinen, “Fluctuation exponent of the KPZ/stochastic Burgers equation,” J. Amer. Math. Soc. 24 no. 3, (2011) 683–708 https://doi.org/10.1090/S0894-0347-2011-00692-9

[5] A. Bensoussan and R. Temam, “Équations aux dérivées partielles stochastiques non linéaires. I,” Israel J. Math. 11 (1972) 95–129 https://doi.org/10.1007/BF02761449

[6] A. Bensoussan and R. Temam, “Équations stochastiques du type Navier-Stokes,” J. Functional Analysis 13 (1973) 195–222 https://doi.org/10.1016/0022-1236(73)90045-1

[7] L. Bertini and N. Cancrini, “The stochastic heat equation: Feynman-Kac formula and intermittence,” J. Statist. Phys. 78 no. 5-6, (1995) 1377–1401 https://doi.org/10.1007/BF02180136

[8] L. Bertini and G. Giacomin, “Stochastic Burgers and KPZ equations from particle systems,” Comm. Math. Phys. 183 no. 3, (1997) 571–607 https://doi.org/10.1007/s002200050044

[9] J. Bricmont, A. Kupiainen, and R. Lefevere, “Ergodicity of the 2D Navier-Stokes equations with random forcing,”
[10] J. Bricmont, A. Kupiainen, and R. Lefevere, “Exponential mixing of the 2D stochastic Navier-Stokes dynamics,” *Comm. Math. Phys.* **230** no. 1, (2002) 87–132. https://doi.org/10.1007/s00220-002-0708-1

[11] Y. Bruned, A. Chandra, I. Chevyrev, and M. Hairer, “Renormalising SPDEs in regularity structures,” *to appear in J. Eur. Math. Soc. (JEMS)* (Nov., 2017). [arXiv:1711.10239 [math.AP]]

[12] Y. Bruned, M. Hairer, and L. Zambotti, “Algebraic renormalisation of regularity structures,” *Invent. Math.* **215** no. 3, (2019) 1039–1156. https://doi.org/10.1007/s00222-018-0841-x

[13] E. Cabaña, “The vibrating string forced by white noise,” *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **15** (1970) 111–130. https://doi.org/10.1007/BF00531880

[14] E. Cépa, “Problème de Skorohod multivoque,” *Ann. Probab.* **26** no. 2, (1998) 500–532. https://doi.org/10.1214/aop/1022855642

[15] S. Cerrai, *Second order PDE’s in finite and infinite dimension*, vol. 1762 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001. https://doi.org/10.1007/b80743

[16] A. Chandra and M. Hairer, “An analytic BPHZ theorem for Regularity Structures,” *ArXiv e-prints* (Oct., 2016). [arXiv:1612.08138 [math.AP]]

[17] K.-T. Chen, “Integration of paths, geometric invariants and a generalized Baker-Hausdorff formula,” *Ann. of Math. (2)* **65** (1957) 163–178.

[18] Y. M. Chen, “On scattering of waves by objects imbedded in random media: Stochastic linear partial differential equations and scattering of waves by conducting sphere imbedded in random media,” *J. Mathematical Phys.* **5** (1964) 1541–1546. https://doi.org/10.1063/1.1931186

[19] Y.-T. Chen, “Pathwise nonuniqueness for the SPDEs of some super-Brownian motions with immigration,” *Ann. Probab.* **43** no. 6, (2015) 3359–3467. https://doi.org/10.1214/14-AOP962

[20] I. Corwin, “Kardar-Parisi-Zhang universality,” *Notices Amer. Math. Soc.* **63** no. 3, (2016) 230–239. https://doi.org/10.1090/noti1334

[21] G. Da Prato and A. Debussche, “Strong solutions to the stochastic quantization equations,” *Ann. Probab.* **31** no. 4, (2003) 1900–1916. https://doi.org/10.1214/aop/1068646370

[22] G. Da Prato, M. Iannelli, and L. Tubaro, “Stochastic differential equations in Banach spaces, variational formulation,” *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8)* **61** no. 3-4, (1976) 168–176 (1977).

[23] G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions* vol. 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992. https://doi.org/10.1017/CBO9780511666223

[24] G. Da Prato and J. Zabczyk, *Ergodicity for infinite-dimensional systems* vol. 229 of *London Mathematical Society Lecture Note Series*. Cambridge
A brief and personal history of stochastic partial differential equations

University Press, Cambridge, 1996. https://doi.org/10.1017/CBO9780511666282

[25] G. Da Prato and J. Zabczyk, *Second order partial differential equations in Hilbert spaces*, vol. 293 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2002. https://doi.org/10.1017/CBO9780511543210

[26] R. C. Dalang, C. Mueller, and L. Zambotti, “Hitting properties of parabolic s.p.d.e.’s with reflection,” *Ann. Probab.* 34 no. 4, (2006) 1423–1450. https://doi.org/10.1214/009117905000000792

[27] J. L. Daleckiǐ, “Differential equations with functional derivatives and stochastic equations for generalized random processes,” *Dokl. Akad. Nauk SSSR* 166 (1966) 1035–1038.

[28] D. A. Dawson, “Stochastic evolution equations.” *Math. Biosci.* 15 (1972) 287–310 https://doi.org/10.1016/0025-5564(72)90039-9

[29] D. A. Dawson, “Measure-valued Markov processes,” in *École d’Été de Probabilités de Saint-Flour XXI—1991*, vol. 1541 of *Lecture Notes in Math.*, pp. 1–260. Springer, Berlin, 1993. https://doi.org/10.1007/BFb0084190

[30] D. A. Dawson and K. J. Hochberg, “The carrying dimension of a stochastic measure diffusion,” *Ann. Probab.* 7 no. 4, (1979) 693–703. http://links.jstor.org/sici?sici=0091-1798(197908)7:4<693:TCDOAS>2.0.CO;2-E&origin=MSN

[31] H. Doss, “Liens entre équations différentielles stochastiques et ordinaires,” *Ann. Inst. H. Poincaré Sect. B (N.S.)* 13 no. 2, (1977) 99–125.

[32] W. E, J. C. Mattingly, and Y. Sinai, “Gibbsian dynamics and ergodicity for the stochastically forced Navier-Stokes equation,” *Comm. Math. Phys.* 224 no. 1, (2001) 83–106 https://doi.org/10.1007/s00220124083 Dedicated to Joel L. Lebowitz.

[33] K. D. Elworthy and X.-M. Li, “Formulea for the derivatives of heat semigroups,” *J. Funct. Anal.* 125 no. 1, (1994) 252–286 https://doi.org/10.1006/jfan.1994.1124

[34] A. M. Etheridge and C. Labbé, “Scaling limits of weakly asymmetric interfaces,” *Comm. Math. Phys.* 336 no. 1, (2015) 287–336 https://doi.org/10.1007/s00220-014-2243-2

[35] F. Flandoli, M. Gubinelli, and E. Priola, “Well-posedness of the transport equation by stochastic perturbation,” *Invent. Math.* 180 no. 1, (2010) 1–53. https://doi.org/10.1007/s00222-009-0224-4

[36] F. Flandoli and B. Maslowski, “Ergodicity of the 2-D Navier-Stokes equation under random perturbations,” *Comm. Math. Phys.* 172 no. 1, (1995) 119–141. http://projecteuclid.org/euclid.cmp/1104279631

[37] H. Föllmer, “Calcul d’Itô sans probabilités,” in *Seminar on Probability, XV (Univ. Strasbourg, Strasbourg, 1979/1980) (French)*, vol. 850 of *Lecture Notes in Math.*, pp. 143–150. Springer, Berlin, 1981.

[38] P. K. Friz and M. Hairer, *A course on rough paths*. Universitext. Springer, Cham, 2014. https://doi.org/10.1007/978-3-319-08332-2 With an introduction to regularity structures.
[39] M. Fukushima, *Dirichlet forms and Markov processes*, vol. 23 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam-New York; Kodansha, Ltd., Tokyo, 1980.

[40] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet forms and symmetric Markov processes*, vol. 19 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, extended ed., 2011.

[41] T. Funaki and S. Olla, “Fluctuations for $\nabla \phi$ interface model on a wall,” *Stochastic Process. Appl.* 94 no. 1, (2001) 1–27 [https://doi.org/10.1016/S0304-4149(00)00104-6](https://doi.org/10.1016/S0304-4149(00)00104-6).

[42] W. E. Gibson, “An exact solution for a class of stochastic partial differential equations,” *SIAM J. Appl. Math.* 15 (1967) 1357–1362 [https://doi.org/10.1137/0115118](https://doi.org/10.1137/0115118).

[43] J. Glimm and A. Jaffe, *Quantum physics*. Springer-Verlag, New York, second ed., 1987. [https://doi.org/10.1007/978-1-4612-4728-9](https://doi.org/10.1007/978-1-4612-4728-9). A functional integral point of view.

[44] P. Gonçalves and M. Jara, “Nonlinear fluctuations of weakly asymmetric interacting particle systems,” *Arch. Ration. Mech. Anal.* 212 no. 2, (2014) 597–644 [https://doi.org/10.1007/s00205-013-0693-x](https://doi.org/10.1007/s00205-013-0693-x).

[45] L. Gross, “Potential theory on Hilbert space,” *J. Functional Analysis* 1 (1967) 123–181 [https://doi.org/10.1016/0022-1236(67)90030-4](https://doi.org/10.1016/0022-1236(67)90030-4).

[46] M. Gubinelli, “Controlling rough paths,” *Journal of Functional Analysis* 216 no. 1, (2004) 86 – 140 [http://www.sciencedirect.com/science/article/pii/S0022123604000497](http://www.sciencedirect.com/science/article/pii/S0022123604000497).

[47] M. Gubinelli, P. Imkeller, and N. Perkowski, “Paracontrolled distributions and singular PDEs,” *Forum Math. Pi* 3 (2015) e6, 75 [https://0-dx.doi.org.pugwash.lib.warwick.ac.uk/10.1017/fmp.2015.2](https://0-dx.doi.org.pugwash.lib.warwick.ac.uk/10.1017/fmp.2015.2).

[48] M. Gubinelli and M. Jara, “Regularization by noise and stochastic Burgers equations,” *Stoch. Partial Differ. Equ. Anal. Comput.* 1 no. 2, (2013) 325–350 [https://doi.org/10.1007/s40072-013-0011-5](https://doi.org/10.1007/s40072-013-0011-5).

[49] M. Gubinelli and N. Perkowski, “Energy solutions of KPZ are unique,” *J. Amer. Math. Soc.* 31 no. 2, (2018) 427–471 [https://doi.org/10.1090/jams/889](https://doi.org/10.1090/jams/889).

[50] M. Gubinelli and S. Tindel, “Rough evolution equations,” *Ann. Probab.* 38 no. 1, (2010) 1–75 [https://doi.org/10.1214/08-AOP437](https://doi.org/10.1214/08-AOP437).

[51] M. Hairer, “Rough stochastic PDEs,” *Comm. Pure Appl. Math.* 64 no. 11, (2011) 1547–1585 [https://doi.org/10.1002/cpa.20383](https://doi.org/10.1002/cpa.20383).

[52] M. Hairer, “Solving the KPZ equation,”  *Ann. of Math. (2)* 178 no. 2, (2013) 559–664 [https://doi.org/10.4007/annals.2013.178.2.4](https://doi.org/10.4007/annals.2013.178.2.4).

[53] M. Hairer, “A theory of regularity structures,” *Invent. Math.* 198 no. 2, (2014) 269–504 [1303.5113](https://doi.org/10.4007/annals.2013.178.2.4).

[54] M. Hairer and J. C. Mattingly, “Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing,”
[55] G. Jona-Lasinio and P. K. Mitter, “On the stochastic quantization of field theory,” Comm. Math. Phys. 101 no. 3, (1985) 409–436. http://projecteuclid.org/euclid.cmp/1104114183

[56] M. Kardar, G. Parisi, and Y.-C. Zhang, “Dynamic Scaling of Growing Interfaces,” Phys. Rev. Lett. 56 no. 9, (1986) 899–904. https://doi.org/10.1103/PhysRevLett.56.889

[57] N. Konno and T. Shiga, “Stochastic partial differential equations for some measure-valued diffusions,” Probab. Theory Related Fields 79 no. 2, (1988) 201–225. https://doi.org/10.1007/BF00320919

[58] N. V. Krylov, “A W^2_p-theory of the Dirichlet problem for SPDEs in general smooth domains,” Probab. Theory Related Fields 98 no. 3, (1994) 389–421. https://doi.org/10.1007/BF01192260

[59] N. V. Krylov and B. L. Rozovskii, “The Cauchy problem for linear stochastic partial differential equations,” Izv. Akad. Nauk SSSR Ser. Mat. 41 no. 6, (1977) 1329–1347, 1448.

[60] S. Kuksin and A. Shirikyan, “Stochastic dissipative PDEs and Gibbs measures,” Comm. Math. Phys. 213 no. 2, (2000) 291–330. https://doi.org/10.1007/s002200000237

[61] S. Kuksin and A. Shirikyan, “Ergodicity for the randomly forced 2D Navier-Stokes equations,” Math. Phys. Anal. Geom. 4 no. 2, (2001) 147–195. https://doi.org/10.1017/A:1011989910997

[62] A. Kupiainen, “Renormalization group and stochastic PDEs,” Ann. Henri Poincaré 17 no. 3, (2016) 497–535. https://doi.org/10.1007/s00023-015-0408-y

[63] J.-F. Le Gall, Spatial branching processes, random snakes and partial differential equations Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1999. https://doi.org/10.1007/978-3-0348-8683-3

[64] R. H. Lyon, “Response of a nonlinear string to random excitation,” J. Acoust. Soc. Amer. 32 (1960) 953–960. https://doi.org/10.1121/1.1908341

[65] T. J. Lyons, “Differential equations driven by rough signals,” Rev. Mat. Iberoamericana 14 no. 2, (1998) 215–310. https://doi.org/10.4171/RMI/240

[66] Z. M. Ma and M. Röckner, Introduction to the theory of (nonsymmetric) Dirichlet forms Universitext. Springer-Verlag, Berlin, 1992. https://doi.org/10.1007/978-3-642-77739-4

[67] J. C. Mattingly, “Ergodicity of 2D Navier-Stokes equations with random forcing and large viscosity,” Comm. Math. Phys. 206 no. 2, (1999) 273–288. https://doi.org/10.1007/s002200050706

[68] G. Miermont, “Aspects of random maps,” http://perso.ens-lyon.fr/ gregory.miermont/coursSaint-Flour.pdf Saint-Flour Lecture notes, 2014.
[69] J.-C. Mourrat and H. Weber, “Convergence of the two-dimensional dynamic Ising-Kac model to $\Phi^4_2$,” *Comm. Pure Appl. Math.* **70** no. 4, (2017) 717–812. https://doi.org/10.1002/cpa.21655

[70] C. Mueller, “On the support of solutions to the heat equation with noise,” *Stochastics Stochastics Rep.* **37** no. 4, (1991) 225–245. https://doi.org/10.1080/17442509108833738

[71] C. Mueller, L. Mytnik, and E. Perkins, “Nonuniqueness for a parabolic SPDE with $\frac{1}{2} - \epsilon$-Hölder diffusion coefficients,” *Ann. Probab.* **42** no. 5, (2014) 2032–2112. https://doi.org/10.1214/13-AOP870

[72] C. Mueller, L. Mytnik, and J. Quastel, “Effect of noise on front propagation in reaction-diffusion equations of KPP type,” *Invent. Math.* **184** no. 2, (2011) 405–453. https://doi.org/10.1007/s00222-010-0241-5

[73] L. Mytnik, “Superprocesses in random environments,” *Ann. Probab.* **24** no. 4, (1996) 1953–1978. https://doi.org/10.1214/aop/1041903212

[74] L. Mytnik and E. Perkins, “Pathwise uniqueness for stochastic heat equations with Hölder continuous coefficients: the white noise case,” *Probab. Theory Related Fields* **149** no. 1-2, (2011) 1–96. https://doi.org/10.1007/s00440-009-0241-7

[75] D. Nualart, *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, Berlin, second ed., 2006.

[76] D. Nualart and E. Pardoux, “White noise driven quasilinear SPDEs with reflection,” *Probab. Theory Related Fields* **93** no. 1, (1992) 77–89. https://doi.org/10.1007/BF01195389

[77] E. Pardoux, “Sur des équations aux dérivées partielles stochastiques monotones,” *C. R. Acad. Sci. Paris Sér. A-B* **275** (1972) A101–A103.

[78] G. Parisi and Y. S. Wu, “Perturbation theory without gauge fixing,” *Sci. Sinica* **24** no. 4, (1981) 483–496.

[79] E. Perkins, “Dawson-Watanabe superprocesses and measure-valued diffusions,” in *Lectures on probability theory and statistics (Saint-Flour, 1999)*, vol. 1781 of *Lecture Notes in Math.*, pp. 125–324. Springer, Berlin, 2002.

[80] J. Quastel, “Introduction to KPZ,” in *Current developments in mathematics, 2011*, pp. 125–194. Int. Press, Somerville, MA, 2012.

[81] D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, vol. 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third ed., 1999. https://doi.org/10.1007/978-3-662-06400-9

[82] B. L. Rozovskii, “Stochastic differential equations in infinite-dimensional spaces, and filtering problems,” in *Proceedings of the School and Seminar on the Theory of Random Processes (Druskininkai, 1974), Part II (Russian)*, pp. 147–194. 1975.

[83] B. L. Rozovskii, “Stochastic partial differential equations,” *Mat. Sb. (N.S.)* **96(138)** (1975) 314–341, 344.
[84] T. Shiga, “Diffusion processes in population genetics,” *J. Math. Kyoto Univ.* 21 no. 1, (1981) 133–151. https://doi.org/10.1215/kjm/1250522109

[85] T. Shiga, “Existence and uniqueness of solutions for a class of nonlinear diffusion equations,” *J. Math. Kyoto Univ.* 27 no. 2, (1987) 195–215. https://doi.org/10.1215/kjm/1250520714

[86] B. Simon, *The P(φ)_2 Euclidean (quantum) field theory*. Princeton University Press, Princeton, N.J., 1974. Princeton Series in Physics.

[87] M. R. Spiegel, “The random vibrations of a string,” *Quart. Appl. Math.* 10 (1952) 25–33. https://doi.org/10.1090/qam/45976.

[88] D. W. Stroock and S. R. S. Varadhan, *Multidimensional diffusion processes*. Classics in Mathematics. Springer-Verlag, Berlin, 2006. Reprint of the 1997 edition.

[89] H. J. Sussmann, “On the gap between deterministic and stochastic ordinary differential equations,” *Ann. Probability* 6 no. 1, (1978) 19–41. https://doi.org/10.1016/0166-218x(83)90112-9

[90] J. B. Walsh, “An introduction to stochastic partial differential equations,” in *École d’été de probabilités de Saint-Flour, XIV—1984*, vol. 1180 of *Lecture Notes in Math.*., pp. 265–439. Springer, Berlin, 1986. https://doi.org/10.1007/BFb0075920

[91] S. Watanabe, “A limit theorem of branching processes and continuous state branching processes,” *J. Math. Kyoto Univ.* 8 (1968) 141–167. https://doi.org/10.1215/kjm/1250524180

[92] M. Zakai, “On the optimal filtering of diffusion processes,” *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 11 (1969) 230–243. https://doi.org/10.1007/BF00536382

[93] L. Zambotti, “A reflected stochastic heat equation as symmetric dynamics with respect to the 3-d Bessel bridge,” *J. Funct. Anal.* 180 no. 1, (2001) 195–209. https://doi.org/10.1006/jfan.2000.3686

[94] L. Zambotti, “Integration by parts formulae on convex sets of paths and applications to SPDEs with reflection,” *Probab. Theory Related Fields* 123 no. 4, (2002) 579–600. https://doi.org/10.1007/s004400200203

[95] L. Zambotti, “Occupation densities for SPDEs with reflection,” *Ann. Probab.* 32 no. 1A, (2004) 191–215. https://doi.org/10.1214/aop/1078415833

[96] L. Zambotti, *Random obstacle problems*, vol. 2181 of *Lecture Notes in Mathematics*. Springer, Cham, 2017. https://doi.org/10.1007/978-3-319-52096-4 Lecture notes from the 45th Probability Summer School held in Saint-Flour, 2015.