RIEMANNIAN HILBERT MANIFOLDS

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Abstract. In this article we collect results obtained by the authors jointly with other authors and we discuss old and new ideas. In particular we discuss singularities of the exponential map, completeness and homogeneity for Riemannian Hilbert quotient manifolds. We also extend a Theorem due to Nomizu and Ozeki to infinite dimensional Riemannian Hilbert manifolds.

1. Introduction

Let $\mathbb{H}$ be a Hilbert space. A Riemannian Hilbert manifold $(M, \langle \cdot, \cdot \rangle)$, RH manifold for short, is a smooth manifold modeled on the Hilbert space $\mathbb{H}$, equipped with an inner product $\langle \cdot, \cdot \rangle_p$ on any tangent space $T_pM$ depending smoothly on $p$ and defining on $T_pM \cong \mathbb{H}$ a norm equivalent to the one of $\mathbb{H}$.

The local Riemannian geometry of RH manifolds goes in the same way as in the finite dimensional case. We can prove, just like in the finite dimensional case, the existence and uniqueness of a symmetric connection, compatible with the Riemannian metric, the Levi-Civita connection, characterized by the Koszul formula

$$2\langle \nabla_X Y, Z \rangle = X \langle Y, Z \rangle + Y \langle Z, X \rangle - Z \langle X, Y \rangle + \langle [X, Y], Z \rangle + \langle [Z, X], Y \rangle - \langle [Y, Z], X \rangle.$$

Hence we can define covariant differentiation of a vector field along a smooth curve, parallel translation, geodesics, exponential map, the curvature tensor $R$, its sectional curvature $K$ etc., just like in the finite dimensional case (see [10, 18, 22] for details).

The investigation of global properties in infinite dimensional geometry is harder than in the finite dimensional case essentially because of the lack of
local compactness. For example, there exist complete RH manifolds with points that cannot be connected by minimal geodesics, complete connected RH manifolds for which the exponential map is not surjective etc. (see Section 3). Moreover, on some RH manifolds one can construct finite geodesic segments containing infinitely many conjugate points [13]. A complete description of conjugate points along finite geodesic segment is given in [6] and similar questions have been studied in [4, 16, 17, 25, 26, 27, 28, 29, 30].

The aim of this survey is to describe results obtained by the authors jointly with D. Tausk, R. Exel and P. Piccione and others authors [1, 2, 3, 4, 5, 6, 7, 11, 13, 24]. We have tried to avoid technical results in order to make the paper more readable also by non experts in this field. The interested reader will find details and further results in papers and books quoted in the bibliography.

This paper is organized as follows. In Section 2, we investigate complete Riemannian Hilbert manifolds. We extend a Theorem due to Nomizu and Ozeki [31] to Riemannian Hilbert manifolds. We also investigate Hopf-Rinow manifolds, i.e., Riemannian Hilbert manifolds such that there exists minimal geodesic between any two points of $M$, properly discontinuous actions on Riemannian Hilbert manifolds and homogeneity for Riemannian Hilbert quotient manifolds. We also point out that if $M$ has constant sectional curvature then completeness is equivalent to geodesically completeness and there are not non trivial Clifford translations on a Hadamard manifold. In Section 3, following the point of view used by Karcher [15], we introduce the Jacobi flow, we discuss singularities of the exponential map and the main result proved in [6].

2. Complete Riemannian Hilbert Manifolds

Let $M$ be a RH manifold. If $\gamma : [a, b] \subseteq \mathbb{R} \rightarrow M$ is a piecewise smooth curve, the length of $\gamma$ is defined, as in the finite dimensional case, $L(\gamma) = \int_a^b \|\dot{\gamma}(t)\|dt$. Then, if $M$ is connected, we can define a distance function

$$d(p, q) = \inf\{L(\gamma) : \gamma \text{ is a piecewise smooth curve joining } p \text{ and } q\}.$$

The function $d$ is, in fact, a distance and induces the original topology of $M$. [22, 33].

**Definition 2.1.** We will say that a RH manifold $M$ is complete if it is complete as a metric space.

Let $M$ be a Hilbert manifold. A natural question is if there exists a Riemannian metric $\langle \cdot, \cdot \rangle$ such that $(M, \langle \cdot, \cdot \rangle)$ is a complete RH manifold. McAlpin [24] proved that any separable Hilbert manifold modeled on a separable Hilbert space can be embedded as a closed submanifold of a separable Hilbert space. Hence, if $f : M \rightarrow \mathbb{H}'$ is such an embedding, $M$, with the
induced metric, is a complete RH manifold. The following result is an extension to the infinite dimensional case of a Theorem due to Nomizu and Ozeki [31].

**Theorem 2.1.** Let \((M, \langle \cdot, \cdot \rangle)\) be a separable RH manifold modeled on a separable Hilbert space. Then there exists a positive smooth function \(f : M \to \mathbb{R}\) such that \((M, f\langle \cdot, \cdot \rangle)\) is a complete RH manifold.

**Proof.** Consider the geodesic ball \(B(p, \epsilon) = \{q \in M : d(p, q) < \epsilon\}\). By a result of Ekeland [11] there exists a smooth function \(f : \mathbb{R} \to \mathbb{R}\) such that \(d(f(p), f(q)) \leq \epsilon \). Pick \(0 < \epsilon < \frac{d(p, q)}{2}\). The triangle inequality implies \(B(q, d(p, q) - \epsilon) \subseteq B(p, r(p) - \epsilon)\) and so \(B(q, d(p, q) - r(p) - \epsilon) \subseteq B(p, r(p) - \epsilon)\). Hence \(r(q) \geq r(p) - d(p, q)\) and so \(d(p, q) \leq d(p, q)\). Applying a result of Ekeland [11], see also [39], there exists a smooth function \(f : M \to \mathbb{R}\) such that \(f(x) > \frac{1}{r(x)}\) for any \(x \in M\). Pick \(\langle \cdot, \cdot \rangle'(x) = f^2(x)\langle \cdot, \cdot \rangle(x)\). Then \((M, \langle \cdot, \cdot \rangle')\) is a RH manifold. We denote by \(d'\) the distance defined by \(\langle \cdot, \cdot \rangle'\).

Let \(x, y \in M\) and let \(\gamma : [0, 1] \to M\) be a piecewise smooth curve joining \(x\) and \(y\). We denote by \(L\), respectively \(L'\), be the length of \(\gamma\) with respect to \(\langle \cdot, \cdot \rangle\), respectively the length of \(\gamma\) with respect to \(\langle \cdot, \cdot \rangle'\). Then

\[
L' = \int_0^1 f(\gamma(t)) \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle^{1/2} dt \geq f(\gamma(c))L > \frac{1}{r(\gamma(c))}L
\]

where \(0 \leq c \leq 1\). Since \(r(\gamma(c)) \leq r(x) + d(x, \gamma(c)) \leq r(x) + L\), it follows \(L' > \frac{L}{r(x) + L}\). Therefore, as in [31], for any \(0 < \epsilon < 1\) and for any \(x \in M\), we get \(B^{\langle \cdot, \cdot \rangle'}(x, \frac{1}{3-\epsilon})\) is contained in \(B(x, \frac{r(x)}{2-\epsilon})\). Hence \(B^{\langle \cdot, \cdot \rangle'}(x, \frac{1}{3-\epsilon})\) is a complete metric space, with respect to \(d\). We claim that \(B^{\langle \cdot, \cdot \rangle'}(x, \frac{1}{3-\epsilon})\) is a complete metric space with respect to \(d'\) as well.

Let \(\{x_n\}_{n \in \mathbb{N}}\) be a Cauchy sequence of \(B^{\langle \cdot, \cdot \rangle'}(x, \frac{1}{3})\) with respect to \(d'\). Let \(0 < \epsilon < \frac{2}{3}\). Then there exists \(n_0\) such that for any \(n, m \geq n_0\) we get \(d'(x_n, x_m) \leq \frac{\epsilon}{4}\). We claim that if \(\gamma : [0, 1] \to M\) is a curve joining \(x_n\) and \(x_m\), for any \(n, m \geq n_0\), such that \(L(\gamma) < \frac{\epsilon}{2}\), then \(\gamma([0, 1]) \subseteq B(x, \frac{3r(x)}{4})\).
Indeed, let \( t \in [0, 1] \). Then

\[
d'(\gamma(t), x) \leq d'(\gamma(t), x_n) + d'(x_n, x_m) + d'(x_m, x) < \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{1}{3} < \frac{1}{3} + \epsilon = \frac{1}{3} - \epsilon',
\]

where \( \epsilon' = \frac{9\epsilon}{1 + 3\epsilon} \). Hence \( d'(\gamma(t), x) < \frac{1}{5 - \epsilon} \) and so \( d(\gamma(t), x) < \frac{r(x)}{2 - \epsilon} < \frac{3r(x)}{4} \).

Now, \( L' \geq \frac{1}{r(y(c))} L \), for some \( 0 \leq c \leq 1 \). Since \( d(\gamma(c), x) < \frac{3r(x)}{4} \), it follows \( r(\gamma(c)) \leq r(x) + d(x, \gamma(c)) \leq r(x) + \frac{3r(x)}{4} = K_o \) and so \( \frac{1}{K_o} L' \geq K_o d(x, x_m) \). Hence \( d'(x_n, x_m) \geq \frac{1}{K_o} d(x, x_m) \) and so \( \{x_n\}_{n \geq n_o} \) is a Cauchy sequence of \( B(x, \frac{3r(x)}{4}) \) with respect to \( d \). Therefore it converges proving \( B^{(\mathbb{C} \setminus \gamma)}(x, \frac{1}{3}) \) is complete with respect to \( d' \), for any \( x \in M \).

Let \( \{x_n\}_{n \in \mathbb{N}} \) be a Cauchy sequence with respect to the distance \( d' \). Then there exists \( n_o \) such that \( x_n \in B^{(\mathbb{C} \setminus \gamma)}(x_{n_o}, \frac{1}{3}) \) for \( n \geq n_o \). Hence \( \{x_n\}_{n \geq n_o} \) is a Cauchy sequence of \( B^{(\mathbb{C} \setminus \gamma)}(x_{n_o}, \frac{1}{3}) \) and so it converges.

\( \square \)

**Remark 2.1.** In [31] the authors consider the function \( r(x) \) to be the supremum of positive numbers \( r \) such that the neighborhood \( B(x, r) \) is relative compact. This function does not work if \( M \) has infinite dimension due of the lack of the local compactness. Moreover, in the finite dimensional case \( B^{(\mathbb{C} \setminus \gamma)}(x, \frac{1}{3}) \) is a complete metric space with respect to \( d' \) since it is contained in \( B(x, \frac{r(x)}{2}) \) and so it is compact. In our case, we have to check directly that \( B^{(\mathbb{C} \setminus \gamma)}(x, \frac{1}{3}) \) is complete.

If \( M \) is a connected finite dimensional RH manifold, then \( M \) is complete if and only if it is geodesically complete at some point \( p \in M \), i.e., there exists \( p \in M \) such that the maximal interval of definition of any geodesics starting at \( p \) is all of \( \mathbb{R} \) and so the exponential map \( \exp_p \) is defined on all of \( T_p M \). This also implies that the exponential map \( \exp_q \) is defined in all of \( T_q M \) for any \( q \in M \) and any two points can be joined by a minimal geodesic. These facts are not true, in general, for infinite dimensional RH manifolds. The following example is due to Grossman [13].

**Example 2.1.** Let \( \mathbb{H} \) be a separable Hilbert space with an orthonormal basis \( \{e_i, i \in \mathbb{N}\} \). Consider

\[
M = \{ \sum_{i=1}^{\infty} x_i e_i \in \mathbb{H} : x_i^2 + \sum_{i=2}^{\infty} (1 - \frac{1}{i})^2 x_i^2 = 1 \}.
\]

Then \( M \) is a complete RH manifold such that \( e_1 \) and \(-e_1 \in M \) can be connected by infinitely many geodesics but there are not a minimal geodesics between the two points.
Remark 2.2. Atkin [1] modified the above example to construct a complete RH manifold such that the exponential map at some point fails to be surjective.

On the other hand the following result holds.

Theorem 2.2. Let \( M \) be a complete RH manifold and \( p \in M \). Then the exponential map \( \exp_p \) is defined on all of \( T_pM \). Moreover, the set \( \mathcal{M}_p = \{q \in M : there \ exists \ a \ unique \ minimal \ geodesic \ joining \ p \ and \ q \} \) is dense in \( M \)

The first part of the Theorem can be proved as in the finite dimensional case. The second part is a result due to Ekeland [11]. He proved \( \mathcal{M}_p \) contained a countable intersection of open and dense subsets of \( M \). By the Baire’s Theorem it follows \( \mathcal{M}_p \) is dense.

The next result proves that a RH manifold of constant sectional curvature which is geodesically complete it is also complete.

Proposition 2.1. Let \( M \) be a RH manifold of constant sectional curvature \( K_o \). Then \( M \) is a complete RH manifold if and only there exists \( p \in M \) such that \( \exp_p \) is defined in all of \( T_pM \).

Proof. By Theorem 2.2 completeness implies geodesically completeness. Vice versa, if the sectional curvature is non positive then geodesic completeness is equivalent to completeness. This is a consequence of a version of the Cartan-Hadamard Theorem due to McAlpin [24] and Grossman [13] [22]. Hence we may assume \( K_o > 0 \). Let \( p \in M \) and let \( S_{\sqrt{K_o}}(T_pM \times \mathbb{R}) \) the sphere of \( T_pM \times \mathbb{R} \) of radius \( \frac{1}{\sqrt{K_o}} \). Let \( N = (0, \frac{1}{\sqrt{K_o}}) \in S_{\sqrt{K_o}}(\mathbb{H} \times \mathbb{R}) \) and let \( T : T_NS_{\sqrt{K_o}}(\mathbb{H} \times \mathbb{R}) \rightarrow T_pM \) be an isometry. By Proposition 3.1 and a Theorem of Cartan [18, Theorem 1.12.8], the map

\[
F = \exp_p \circ T \circ \exp_N^{-1} : S_{\sqrt{K_o}}(\mathbb{H} \times \mathbb{R}) \setminus \{-N\} \rightarrow M
\]

is a local isometry. Let \( v \in T_NS_{\sqrt{K_o}}(\mathbb{H} \times \mathbb{R}) \) be a unit vector. Then \( \gamma^v(t) = F(\exp_N(tv)) \) is a geodesic in \( M \). Let \( q(v) = \gamma^v(\pi) \). It is easy to see that \( q(v) = q(w) \) for any unit vector \( w \in TS_{\sqrt{K_o}}(\mathbb{H} \times \mathbb{R}) \). Hence we may extend \( F : S_{\sqrt{K_o}}(\mathbb{H} \times \mathbb{R}) \rightarrow M \) and it is easy to check that it is still an isometry. Since \( S_{\sqrt{K_o}}(\mathbb{H} \times \mathbb{R}) \) is complete, by [22, Theorem 6.9 p. 228] we get \( F \) is a Riemannian covering map, and so \( F \) is surjective, and \( M \) is complete. \( \square \)

Definition 2.2. A Hopf-Rinow manifold is a complete RH manifold such that any two points \( x, y \in M \) can be joined by a minimal geodesic.

The unit sphere \( S(\mathbb{H}) \) is Hopf-Rinow. The Stiefel manifolds of orthonormal \( p \) frames in a Hilbert space \( \mathbb{H} \) and the Grassmann manifolds of \( p \) subspaces of \( \mathbb{H} \) are Hopf-Rinow manifolds [7] [14]. These manifolds are homogeneous,
i.e., the isometry group acts transitively on $M$. It is easy to see that homo-
geneity implies completeness \[7\] but it does not imply the existence of path
of minimal length between two points. We also point out that the isometry
group of a complete RH manifold can be turned in a Banach Lie group and
its Lie algebra is given by the Killing vector fields, i.e., vector fields $X$ such
that $L_X \langle \cdot, \cdot \rangle = 0$. Moreover the natural action of the isometry group on $M$
is smooth (see \[20\]).

In \[3, 7\] properly isometric discontinuous actions on the unit sphere of a
Hilbert space $H$ and on the Stiefel and Grassmannian manifolds are studied.
We recall that a group $\Gamma$ of isometries acts properly discontinuously on $M$
if for any $f \in \Gamma$, the condition $f(x) = x$ for some $x \in M$ implies $f = e$
and the orbit throughout any element $x \in M$ is closed and discrete \[21\].
We completely classify properly discontinuous actions of a finitely generated
abelian group on the unit sphere of a separable Hilbert space and we give new
examples of complete RH manifolds, respectively Kähler RH manifolds, with
non negative and non positive sectional curvature with infinite fundamental
group, respectively with non negative holomorphic sectional curvature with
infinite fundamental group (\[3, 7\]). These new examples of RH manifolds
are Hopf-Rinow manifolds due the following simple fact.

**Proposition 2.2.** Let $M$ be a Hopf-Rinow manifold. Let $\Gamma$ be a group
acting isometrically and properly discontinuously on $M$. Then $M/\Gamma$ is also
Hopf-Rinow.

**Proof.** Since $\Gamma$ acts isometrically and properly discontinuously on $M$, it follows
that $M/\Gamma$ admits a Riemannian metric such that $M/\Gamma$ is complete and
$\pi : M \to M/\Gamma$ is a Riemannian covering map \[3, 22\]. Let $p, q \in M/\Gamma$. Since
$\Gamma$ acts properly discontinuously on $M$, then both $\pi^{-1}(p)$ and $\pi^{-1}(q)$ are $\Gamma$
orbits, and also closed and discrete subsets of $M$ \[21\]. Hence given $z \in \pi^{-1}(p)$,
there exists a unique $w \in \pi^{-1}(q)$ such that $d(z, w) \leq d(r, s)$ for every
$r \in \pi^{-1}(p)$ and $s \in \pi^{-1}(q)$, i.e., $d(z, w) = d(\pi^{-1}(p), \pi^{-1}(q))$.
Let $\gamma$ be a minimal geodesic joining $z$ and $w$. We claim that $\pi \circ \gamma$ is a minimal geodesic. Since
$\pi$ is a Riemannian covering map, then $d(p, q) \leq L(\pi \circ \gamma) = L(\gamma) = d(z, w)$.
On the other hand pick a sequence $\gamma_n : [0, 1] \to M/\Gamma$ joining $p$ and $q$ such
that $\lim_{n \to +\infty} L(\gamma_n) = d(p, q)$. Since $\pi$ is a Riemannian covering map there
exists a lift $\tilde{\gamma}_n$ starting at $z$ satisfying $L(\gamma_n) = L(\tilde{\gamma}_n)$. Therefore

$$L(\gamma) = d(z, w) \leq L(\gamma_n) \Rightarrow d(p, q),$$

and so $L(\pi \circ \gamma) = d(p, q)$.

In \[7\] we prove a homogeneity result for Riemannian Hilbert manifolds of
constant sectional curvature. In finite dimension this result was proved by
Wolf \[35, 38\].


An isometry \( f : M \rightarrow M \) is called a Clifford translation if \( \delta_f(x) = d(x, f(x)) \) is a constant function. As in the finite dimensional case, if \( M \) is a homogeneous Riemannian manifold and \( \Gamma \) a group acting on \( M \) isometrically and properly discontinuously on \( M \), then \( M/\Gamma \) is homogeneous if and only if the centralizer of \( \Gamma \), that we denote by \( Z(\Gamma) \), acts transitively on \( M \) \[7, 38\]. In particular if \( M/\Gamma \) is homogeneous then any element \( g \in \Gamma \) is a Clifford translation. Indeed,
\[
d(x, g(x)) = d(h(x), hg(x)) = d(h(x), g(h(x))),
\]
for any \( h \in Z(\Gamma) \). Hence if \( Z(\Gamma) \) acts transitively on \( M \) we get \( f \) is a Clifford translation.

In the finite dimensional case, the homogeneity conjecture says that if \( M \) is a homogeneous simply connected Riemannian manifold then \( M/\Gamma \) is homogeneous if and only if all the elements of \( \Gamma \) are Clifford translations. We point out that the conjecture is true for locally homogeneous symmetric spaces \[36\] and also for locally homogeneous Finsler symmetric spaces \[8\]. In \[7\] we proved the homogeneity conjecture for complete RH manifolds of constant sectional curvature. We leave the investigation of locally homogeneous symmetric space of infinite dimension for future investigation (see \[9, 19, 22\] for basic references of symmetric space in infinite dimension.) The following result proves there are not non trivial Clifford translations on a Hadamard manifold, i.e., a simply connected Riemannian Hilbert manifold with negative sectional curvature.

**Proposition 2.3.** Let \( M \) be a simply connected RH manifold of negative sectional curvature. If \( f : M \rightarrow M \) is a Clifford translation then \( f = \text{Id} \).

**Proof.** Assume \( f(p) \neq p \) for some \( p \in M \), hence for every \( p \in M \). By Cartan-Hadamard Theorem \( M \) is a Hopf-Rinow manifold and so by Lemma 5.2 p. 448 in \[7\], see also \[32\], \( f \) preserves the minimal geodesic, that we denote by \( \gamma_p \), joining \( p \) and \( f(p) \). Let \( p \in M \) and let \( \theta \) be a geodesic different from \( \gamma_p \).

As in the Proof of Theorem 1 p. 16 in \[37\], one can prove that the union \( \gamma_\theta(t) \) is a flat totally geodesic surface which is a contradiction. \( \Box \)

3. **Jacobi fields and conjugate points**

Let \( M \) be a RH manifold and let \( \gamma : [0, b) \rightarrow M \) be a geodesic with \( \gamma(0) = p \). Without lost of generality we assume that \( \gamma(t) = \exp_p(tv) \), with \( \|v\| = 1 \). A Jacobi field along \( \gamma \) is a smooth vector field \( J \) along \( \gamma \) satisfying
\[
\nabla_{\dot{\gamma}(t)} \nabla_{\dot{\gamma}(t)} J(t) + R(\dot{\gamma}(t), J(t))J(t) = 0.
\]
In the sequel we will denote by $J'(t)$ the covariant derivative $\nabla_{J'(t)}J(t)$. If $J_1$ and $J_2$ are Jacobi fields along $\gamma$, then

$$
\langle J'_1(t), J_2(t) \rangle - \langle J_1(t), J'_2(t) \rangle = \text{Constant}.
$$

This formula is due to Ambrose (see [22]). The Jacobi field along $\gamma$ satisfying $J(0) = 0$ and $J'(0) = \nabla_{J(0)}J(0) = w$ is given by $J(t) = (d\exp_p)_{w(t)}(tw)$. Hence $(d\exp_p)_{w}(w) = 0$ if and only if there exists a Jacobi field $J$ along $\gamma(t)$ such that $J(0) = 0$ and $J(1) = 0$.

In infinite dimension there exist two types of singularities of the exponential map.

**Definition 3.1.** We will say that $q = \gamma(t_o), t_o \in (0,b)$, is

- monoconjugate to $p$ along $\gamma$ if $(d\exp_p)_{t_o}v$ is not injective,
- epiconjugate, to $p$ along $\gamma$ if $(d\exp_p)_{t_o}v$ is not surjective.

We also say $q = \gamma(t_o)$ is conjugate of $p$ along $\gamma$ if $(d\exp_p)_{t_o}v$ is not an isomorphism and $t_o \in (0,b)$ is a conjugate, monoconjugate, respectively epiconjugate instant if $\gamma(t_o)$ is conjugate, monoconjugate, respectively epiconjugate of $p$ along $\gamma$.

Let $\tau^t_\gamma : T_{\gamma(t)}M \rightarrow T_{\gamma(s)}M$ be the isometry between the tangent spaces given by the parallel transport along the geodesic $\gamma$. The following result is easy to check.

**Lemma 3.1.** If $V : [0,b] \rightarrow T_pM$, then $\nabla_{\dot{\gamma}(t)}\tau^{t_0}_\gamma(V(t)) = \tau^{t_0}_\gamma(\dot{V}(t))$.

By the above Lemma, a Jacobi field along $\gamma$ such that $J(0) = 0$ is given by $J(t) = \tau^{t_0}_0(T(t)(V))$, where $V \in T_pM$, and $T(t)$ is a family of endomorphism of $T_pM$ satisfying

$$
\begin{aligned}
T'''(t) + R_t(T(t)) &= 0; \\
T(0) &= 0, \ T'(0) = \text{Id},
\end{aligned}
$$

where $R_t : T_pM \rightarrow T_pM$ is a one parameter family of endomorphism of $T_pM$ defined by $R_t(X) = \tau^{t_0}_0(R_t(X), \dot{\gamma}(t))\ddot{\gamma}(t))$. We call the above differential equation the Jacobi flow of $\gamma$.

**Example 3.1.** Assume that $M$ is a RH manifold with constant sectional curvature $K_o$. Then

$$
T(t)(w) = \begin{cases} 
\frac{\sinh(\sqrt{-K_o}tw)}{\sqrt{-K_o}}w & K_o < 0 \\
\frac{\sin(t\sqrt{K_o})}{\sqrt{K_o}}w & K_o = 0 \\
\frac{\sinh(t\sqrt{-K_o})}{\sqrt{-K_o}}w & K_o > 0
\end{cases}
$$

Karcher used the Jacobi flow to get Jacobi fields estimates [15]. By standard properties of the curvature, it follows $R_t$ is a symmetric endomorphism.
of $T_pM$. Since $\tau_0^t \circ T(t) = t(\text{d} \exp_p)_t$, we may thus equivalently state the definitions of monoconjugate, epiconjugate in terms of injectivity, respectively surjectivity of $T(t)$. Moreover, conjugate instants are also discussed in terms of Lagrangian curves [5]. Indeed, the Hilbert space $T_pM \times T_pM$ has a natural symplectic structure given by $\omega((X, Y), (Z, W)) = \langle X, W \rangle - \langle Y, Z \rangle$. It is easy to check that $\Psi(t) : T_pM \times T_pM \to T_pM \times T_pM$ defined by $\Psi(t)(X, Y) = (\tau_t^0(J(t)), \tau_t^0(J'(t)))$, where $J(t)$ is the Jacobi field along $\gamma$ such that $J(0) = X$ and $J'(0) = Y$, is a symplectomorphism of $(T_pM \times T_pM, \omega)$.

Let $E_t = \Phi_t(\{0\} \times T_pM)$ be a curve of Lagrangian subspaces of $T_pM \times T_pM$. Moreover $t_o \in (0, b)$ is a monoconjugate instant, respectively a epiconjugate instant, if and only if $E_t \cap (\{0\} \times T_pM) \neq \{0\}$, respectively if and only if $E_t + (\{0\} \times T_pM) \neq T_pM \times T_pM$.

Let $t_o \in (0, b)$. We compute the transpose of $T(t_o)$. Let $J_1(t) = \tau_0^t(T(t)(v))$ and let $u \in T_pM$. Let $J_2$ be the Jacobi field along the geodesic $\gamma$ such that $J_2(t_o) = 0$, $\nabla_{\dot{\gamma}(t_o)}J_2(t_o) = \tau_0^t(u)$. By (1), we have $\langle J_1(t_o), J_2(t_o) \rangle = \langle J_1'(0), J_2(0) \rangle$ and so $\langle T(t_o)(v), u \rangle = \langle v, \tau_0^t(J_2(t_o)) \rangle$. Let $\overline{\gamma}(t) = \gamma(t_o - t)$ and let

$$\begin{align*}
\left\{ \begin{array}{l}
\tilde{T}''(t) + R_t(\tilde{T}(t)) = 0; \\
\tilde{T}'(0) = 0, \; \tilde{T}'(0) = id,
\end{array} \right.
\end{align*}$$

be the Jacobi flow along $\overline{\gamma}$. Summing up we have proved that $T^*(t_o) = \tau_0^t \circ \tilde{T}(t_o) \circ \tau_0^t$. As a corollary, keeping in mind Example 3.1, we get the following result.

**Proposition 3.1.** The kernel of $T(t_o)$ and the kernel of $T^*(t_o)$ are isomorphic. Hence a monoconjugeate point is also epiconjugeate. Moreover, if $M$ has constant sectional curvature $K_o$, then $T(t)$ is an isomorphism for any $t > 0$ whether $K_o \leq 0$, and $T(t)$ is an isomorphism for $0 < t < \frac{\pi}{\sqrt{K_o}}$ whether $K_o > 0$.

The above result was proven by McAlpin [24] and Grossmann in [13]. Since both Rauch and Berger Comparison Theorems work for RH manifolds [5] [22], they also work for a weak Riemannian Hilbert manifold [3], the second part of Proposition 3.1 holds for any RH manifold with negative sectional curvature and for any RH manifold with sectional curvature bounded above for a constant $K_o > 0$.

Proposition 3.1 implies that if $\text{Im} T(t_o)$ is closed then monoconjugeate implies epiconjugeate and vice-versa. This holds, for example, if $\exp_p$ is Fredholm. We recall that a smooth map between Hilbert manifolds $f : M \to N$ is called Fredholm if for each $p \in M$ the derivative $(df)_p : T_pM \to T_{f(p)}N$ is a Fredholm operator. If $M$ is connected then the index $(df)_p$ is independent of $p$, and one defines the index of $f$ by setting $\text{ind}(f) = \text{ind}(df)_p$ (see [12] [34]). Misiolek proved that the exponential map of a free loop space with
its natural Riemannian metric is Fredholm \cite{Misiolek97}. Misiolek also pointed out that if the curvature is a compact operator, i.e., for any $X \in T_pM$, the map $Z \mapsto R(Z, X)X$ is a compact operator, then $T(t)$ is Fredholm of index zero and so the exponential map is Fredholm as well \cite{Walter72}. Indeed,

$$T(t) = tId - \int_0^t \left( \int_0^h R_s(T(s))ds \right) dh$$

and so $T(t) = tId + K(t)$ where $K(t)$ is a compact operator. Hence $T(t)$ is Fredholm \cite{Walter92} and so $\exp_p$ is Fredholm.

It is convenient to introduce the notion of \textit{strictly epiconjugate} instant, to denote an instant $t \in [0, b]$ for which the range of $T(t)$ fails to be closed. Unlike finite-dimensional Riemannian geometry, conjugate instants can accumulate. The classical example of this phenomenon is given by an infinite dimensional ellipsoid in $\ell^2$ whose axes form a non discrete subset of the real line given by Grossman \cite{Grossman68}.

Let $M = \{ x \in \ell^2 : x_1^2 + x_2^2 + \sum_{i=3}^\infty (1 - \frac{1}{k})^4 x_i^2 = 1 \}$. $M$ is a closed submanifold of $\ell^2$ and the curve $\gamma(t) = \cos te_1 + \sin te_2$ is a geodesic of $M$ since it is the set of fixed points of the isometry

$$F(\sum_{i=1}^\infty x_i e_i) = x_1 e_1 + x_2 e_2 + \sum_{i=3}^\infty (-x_i)e_i.$$

For any $k \geq 3$, $E_k := \{ x_1^2 + x_2^2 + (1 - \frac{1}{k})^4 x_k^2 = 1 \} \hookrightarrow M$ is a totally geodesic submanifold of $M$ since it is the fixed points set of the isometry $F(\sum_{i=1}^\infty x_i e_i) = x_1 e_1 - x_2 e_2 + x_k e_k + \sum_{i=3, i \neq k}^\infty (-x_i)e_i$. Hence $K(\dot{\gamma}(s), e_k) = (1 - \frac{1}{k})^2$, $J_k(t) = \sin(t(1 - \frac{1}{k}))e_k$ is the Jacobi field along $\gamma$ satisfying $J(0) = 0$ and $J'(0) = e_k$. Consider $q_k = k\frac{\pi}{k}$. Then $q_k$ is a sequence of monoconjugate instant such that $\lim_{k \to \infty} q_k = \pi$. We claim that $-e_1 = \gamma(\pi)$ is a strictly epiconjugate point. Indeed,

$$T(\pi)(e_2 + \sum_{k=3}^\infty b_k e_k) = e_2 + \sum_{k=3}^\infty b_k \sin(\frac{k - 1}{k})\pi) e_k$$

which implies $T(\pi)$ is injective. On the other hand $\sum_{i=3}^\infty \frac{1}{k} e_k$ does not lie in $\text{Im } T(\pi)$ and so $\gamma(\pi)$ is strictly epiconjugate. Indeed if $\sum_{i=3}^\infty \frac{1}{k} e_k \in \text{Im } T(\pi)$ then $\sum_{k=3}^\infty \frac{1}{k} e_k = \sum_{k=3}^\infty b_k \sin((1 - \frac{1}{k})\pi)e_k$ and so $-\sin(\pi k/k)b_k = \frac{1}{k}$. Hence

$$\lim_{k \to +\infty} b_k = - \lim_{k \to +\infty} k \sin(\pi \frac{1}{k}) = -\pi$$

which is a contradiction. Hence $\gamma(\pi)$ is a strictly epiconjugate point along $\gamma$ and it is an accumulation point of sequence of monoconjugate points.

In \cite{Grossman68} the authors give a complete characterization of the conjugate instants along a geodesic. In particular the set of conjugate instants is closed.
and the set of strictly epiconjugate points are limit of conjugate points as before. Hence if there is no strictly epiconjugate instant along \( \gamma \) then the set of conjugate instants along any compact segment of \( \gamma \) is finite. Under these circumstances a Morse Index Theorem for geodesics in RH manifolds holds true.

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