Zero (sub)sets
for spaces of holomorphic functions
and (sub)harmonic minorants

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Abstract

Let $\Lambda = \{\lambda_k\}$ be a sequence of points in a domain $\Omega$ of the complex plane $\mathbb{C}$. We obtain various general conditions in terms of the balayage and Green’s functions under which the sequence $\Lambda$ is the zero (sub)set for weighted spaces of holomorphic functions in $\Omega$. In particular, we consider the classical space $A^{-p}$ with $p > 0$, i.e., the set of functions $f$ holomorphic in the unit disk $\mathbb{D}$ satisfying $\sup_{z \in \mathbb{D}} |f(z)|(1-|z|)^p < +\infty$.

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Introduction

Let $\Omega$ be a domain in the complex plane $\mathbb{C}$ with the boundary $\partial \Omega$. Denote by $\text{Hol}(\Omega)$ the space of all holomorphic functions in $\Omega$. We are concerned with finite or infinite sequences $\Lambda = \{\lambda_k\}, k = 1, 2, \ldots$ of not necessarily distinct points from the domain $\Omega$, without limit points in $\Omega$. Let $n_{\Lambda}$ be an integer-valued counting measure of sequence $\Lambda$, defined by

$$n_{\Lambda}(S) := \sum_{\lambda_k \in S} 1, \quad S \subset \Omega. \quad (0.1)$$

The function

$$n_{\Lambda}(z) := n_{\Lambda}(\{z\}), \quad z \in \Omega, \quad (0.2)$$
is called the divisor of sequence $\Lambda$. In our paper the sequence $\Lambda$ coincides with a sequence $\Gamma = \{\gamma_n\}$ (or is equal to $\Gamma$, or $\Lambda = \Gamma$) if and only if $n_\Lambda(z) = n_\Gamma(z)$ for every $z \in \Omega$; $\Lambda \subset \Gamma$ means $n_\Lambda(z) \leq n_\Gamma(z)$ for all $z \in \Omega$.

Given $f : A \to B$ and $b \in B$, we write $f \equiv b$ on $A'$ if $f$ is identically equal to $b$ on $A' \subset A$; in the opposite case, $f \not\equiv b$ on $A'$. For a subset $A' \subset A$, denote by $f|_{A'}$ the restriction of $f$ to $A'$.

Let $A, B \subset [-\infty, +\infty]$. A function $f : A \to B$ is increasing (decreasing resp.) if, for any $x_1, x_2 \in A$, the inequality $x_1 \leq x_2$ implies the inequality $f(x_1) \leq f(x_2)$ ($f(x_1) \geq f(x_2)$ resp.).

"Positive" ("negative" resp.) means "$\geq 0$" ("$\leq 0$" resp.).

Let $f \in \text{Hol}(\Omega)$, $f \not\equiv 0$ on $\Omega$. We denote by $\text{Zero}_f$ the sequence of zeros, counting multiplicities, of the function $f$. The sequence $\Lambda$ is zero set for a class $H \subset \text{Hol}(\Omega)$ (we write $\Lambda \in \text{Zero}(H)$) if and only if there exists a function $f \in H$ such that $\Lambda = \text{Zero}_f$. A function $f \in \text{Hol}(\Omega)$ vanish on $\Lambda$ if and only if $\Lambda \subset \text{Zero}_f$ (we write $f(\Lambda) = 0$). The sequence $\Lambda$ is zero subset for a class $H \subset \text{Hol}(\Omega)$ if and only if there exists a nonzero function $f \in H$ which vanish on $\Lambda$.

By $\mathbb{D}$ denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$.

By $m$ denote the Lebesgue measure on $\mathbb{C}$.

We write $D \subseteq \Omega$ if the closure $\overline{D}$ of $D \subset \Omega$ is a compact subset in $\Omega$, i.e., the set $D$ is a relatively compact subset of $\Omega$.

For domain $D \subset \mathbb{C}$, denote by $g_D(\cdot, z)$ the extended Green’s function for $D$ with a pole at $z \in D$ [1, 5.7.4], i.e., $g_D(\zeta, z) \equiv 0$ for all $\zeta \in \mathbb{C} \setminus \overline{D}$, and $g(\zeta, z)$ is a subharmonic function of $\zeta \in \mathbb{C} \setminus \{z\}$.

For a real-valued function $M$ on an interval of the real axis $\mathbb{R}$, denote by $M'$ the left-hand derivative of $M$.

Given $M : \Omega \to [-\infty, +\infty]$, we put

$$\text{Hol}(\Omega; M) := \left\{ f \in \text{Hol}(\Omega) : \sup_{z \in \Omega} \exp \frac{|f(z)|}{M(z)} < +\infty \right\} . \quad (0.3)$$

A function $M : \mathbb{D} \to \mathbb{R}$ is called radial if $M(z) = M(|z|)$ for all $z \in \mathbb{D}$. Given $z \in \mathbb{C}$ and $t \in \mathbb{R}$, denote by $D(z, t)$ the open disk of radius $t$, centered at $z$. If $t \leq 0$, then $D(z, t)$ is the empty set $\emptyset$. By definition $D(0, t) := D(0, t)$.

Hereinafter by $\Omega$ denote a domain in $\mathbb{C}$ containing the origin, i.e., $0 \in \Omega$.

By $\mathcal{U}_0^d(\Omega)$ denote the class of all subdomains $D \subseteq \Omega$ such that each domain $D \in \mathcal{U}_0^d(\Omega)$ is the union of finite number of disks $D(z, t) \subseteq \Omega$, and $D$ contains the origin, i.e., $0 \in D$.

Let $\Omega_0 \neq \emptyset$ be a relatively compact subdomain of $\Omega$, and the domain $\Omega_0$ contains the origin. By $\mathcal{U}_0^d(\Omega; \Omega_0)$ denote the subclass of $\mathcal{U}_0^d(\Omega)$ such that each domain $D \in \mathcal{U}_0^d(\Omega; \Omega_0)$ includes the domain $\Omega_0$; $\mathcal{U}_0^d(\Omega; \emptyset) := \mathcal{U}_0^d(\Omega)$.

The following Theorems 0.1 and 0.2 illustrate a some of our results.
Theorem 0.1. Let $M : \mathbb{D} \to [0, +\infty)$ be a radial positive function that is continuous at 0. Suppose that $M \big|_{(0,1)}$ is the increasing convex function of log on $(0,1)$, i.e., the superposition $M \circ \exp$ is convex function on $(-\infty, 0)$. Under the condition
\[
\int_{r}^{1^-} (1-t) \, dM(t) = O(1-r) \quad \text{as} \quad r \to 1^-, \quad t, r \geq 0,
\]
the following three statements hold:

(Z_r) A sequence $\Lambda = \{\lambda_n\}, \ 0 \notin \Lambda$, is zero set for the space $\text{Hol}(\mathbb{D}; M)$ if and only if there are constants $a < 1$ and $C$ such that the inequality
\[
\sum_n g_D(\lambda_n, 0) \leq \int_0^{1^-} \left( \frac{1}{2\pi} \int_0^{2\pi} g_D(te^\theta, 0) \, d\theta \right) \, d(tM'_-(t)) + C
\]
holds for each domain $D \in \mathcal{U}_0(\mathbb{D}; D(a))$;

(S_r) A sequence $\Lambda$ in $\mathbb{D}$ is zero subset for the space $\text{Hol}(\mathbb{D}; M)$ if and only if the sequence $\Lambda$ is a zero set for the space $\text{Hol}(\mathbb{D}; M)$;

(M_r) If $f = g/q$ is a meromorphic function on $\mathbb{D}$, and $g, q \in \text{Hol}(\mathbb{D}; M)$, then there are holomorphic functions $g_0, q_0 \in \text{Hol}(\mathbb{D}; M)$ without common zeros such that $f = g_0/q_0$ on $\mathbb{D}$.

Theorem 0.1 is a very special case of Theorem 1.1 $\iff$ Corollary 1.1 $\iff$ the Main Theorem (see exact wordings of these results in Subsection 1.2). Corollary 1.1 is established for spaces $H(\Omega; M)$ in case that $\Omega$ is a simply connected domain, and $M \in \text{SH}(\Omega)$. In Theorem 1.1 we consider generally speaking a case of non-radial subharmonic function $M$ on $\mathbb{D}$.

In our article a main model class $\text{Hol}(\mathbb{D}; M)$ is the space $A^{-p}, \ p \geq 0$, with
\[
M(z) \equiv M(\|z\|) \equiv p \log \frac{1}{1-|z|}, \quad z \in \mathbb{D},
\]
i.e., the set of functions $f \in \text{Hol}(\mathbb{D})$ satisfying
\[
|f(z)| \leq C_f \left( \frac{1}{1-|z|} \right)^p, \quad \forall z \in \mathbb{D},
\]
where $C_f$ is a constant dependent on $f$. For $p = 0$ this space is the space $H^\infty$ of bounded holomorphic functions in $\mathbb{D}$. The classical Nevanlinna theorem gives precise geometric information about the zero sets for $H^\infty$: a sequence $\Lambda$ is zero set for $H^\infty$ if and only if $\sum_{\lambda_k \in \mathbb{D}} (1 - |\lambda_k|) < +\infty$ (the classical Blaschke condition), and class of zero sets for $H^\infty$ coincides with the class of
of zero subsets for $H^\infty$. B. Korenblum introduced in the work [2] a notion of density, in a certain sense generalizing the classical Blaschke condition, and found a complete geometric description of zero sets for the algebra $A^{-\infty} = \cup_{0 \leq p} A^{-p}$. E. Beller [3] proved that for any $p > 0$, the class of zero sets for $A^{-p}$ coincides with the class of zero subsets for $A^{-p}$. D. Pascuas [4] and J. Bruna and X. Massaneda [5] generalized the result of B. Korenblum to weighted algebras $A^\lambda(D) := \{ f \in \text{Hol}(D) : \log |f(z)| \leq C_f \lambda(|z|) \}$ where $\lambda > 0$ is a “slowly increasing” function, and $C_f$ is a constant.

In [6]–[7] K. Seip evolved the method of Korenblum and obtained a similar complete description of zero set for spaces $A^{-p} = \cap_{p' > p} A^{-p'}$ and $A^{-p} = \cup_{p' < p} A^{-p'}$. The joint book of H. Hedenmalm, B. Korenblum, and K. Zhu [8, Ch. 4] contains a detailed analysis of this results together with their improved interpretations.

D. Luecking gave in [9] a criterion of zero sets for $A^{-p}$ in terms of existence of harmonic majorant for special test function constructed by $\Lambda$. A development of Luecking’s method to weighted spaces on $D$ with “slowly increasing” weight was recently proposed in [10].

Following to [11], we will refer to the spaces $A^{-p}$ as uniform Bergman spaces.

In our article we establish necessary and sufficient conditions of zero sets for weighted classes of holomorphic functions and their corollaries for the uniform Bergman spaces in terms of balayage of measures and functions (see Theorems 7.1 and 7.2 in Section 7), and in terms of Green’s functions.

**Theorem 0.2.** Let $\Lambda = \{\lambda_k\}, k = 1, 2, \ldots, 0 \notin \Lambda$, be a sequence of points in $D$ and $0 \leq p < +\infty$. The following three statements are equivalent:

(i) $\Lambda$ is a zero set for $A^{-p}$;

(ii) There exist constants $a < 1$ and $C$ such that for any $D \in \mathcal{U}_0^d(D; D(a))$,

$$\sum_k g_D(\lambda_k, 0) \leq p \int_0^1 \left( \frac{1}{2\pi} \int_0^{2\pi} g_D(te^{i\theta}, 0) \, d\theta \right) \frac{dt}{(1-t)^2} + C; \quad (0.6)$$

(iii) There are constants $a < 1$ and $C$ such that for any $D \in \mathcal{U}_D^d(D; D(a))$,

$$\sum_k (1 - |\lambda_k|^2)^2 \left( \frac{1}{\pi} \int_D g_D(\zeta, 0) \frac{dm(\zeta)}{|1 - \lambda_k \zeta|^4} \right) \leq p \int_0^1 \left( \frac{1}{2\pi} \int_0^{2\pi} g_D(te^{i\theta}, 0) \, d\theta \right) \frac{dt}{(1-t)^2} + C. \quad (0.7)$$

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1 Main notions and results

Let \( \Omega \) be a domain in \( \mathbb{C} \), \( 0 \in \Omega \), and let \( S \) be a subset in \( \Omega \). We write \( C(S) \) for the space of all continuous real-valued functions on \( S \).

By \( \mathcal{M}(S) \) denote the set of all real-valued Borel (Radon) measures on \( S \) (on \( C(S) \)); by \( \mathcal{M}^+(S) \) denote the subcone of \( \mathcal{M}(S) \) consisting of positive Borel measures; by \( \mathcal{M}^+_ac(S) \) denote the subset of \( \mathcal{M}^+(S) \) consisting of measures that are absolutely continuous with respect to Lebesgue measure \( m \).

Let \( \mu \in \mathcal{M}(\Omega) \). Denote by \( \text{supp} \mu \) the support of \( \mu \). We say that a measure \( \mu \in \mathcal{M}^+(\Omega) \) is concentrated in a subset \( S \subset \Omega \) if \( \mu(\Omega \setminus S) = 0 \). Given Borel set \( B \subset S \), we denote by \( \mu \big|_B \) the restriction of \( \mu \) to \( B \).

Given \( \nu \in \mathcal{M}(\Omega) \), \( z \in \Omega \), \( t \geq 0 \), we write
\[
\nu(z, t) := \nu(D(z, t)) \quad \text{if} \quad D(z, t) \subset \Omega,
\]
\[
\nu^{\text{rad}}(t) := \nu(0, t).
\]

We hope that last notation will not create confusion with notion (0.2) for the divisor of sequence \( \Lambda \), defined by the counting measure (0.1), and with notation \( d\nu(\zeta) \) which means that the variable of integration is \( \zeta \).

By \( \text{Harm}(\Omega) \) denote the space of all harmonic functions on \( \Omega \), and by \( \text{SH}(\Omega) \) denote the cone of all subharmonic functions on \( \Omega \). The function \( \equiv -\infty \) on \( \Omega \) belongs to \( \text{SH}(\Omega) \). Besides, \( -\text{SH}(\Omega) \) is the cone of all superharmonic functions on \( \Omega \). For \( u \in \text{SH}(\Omega) \), \( u \neq -\infty \), we denote by \( ^1\nu_u := \frac{1}{2\pi} \Delta u \) the Riesz measure of \( u \).

1.1 Subharmonic kernels

Let \( B \) be a Borel subset of \( \Omega \), and \( \nu \in \mathcal{M}^+(\Omega) \). By definition, \( L^1(B, d\nu) \) is the set of all functions \( q \colon B \to [-\infty, +\infty] \) that are integrable with respect to the restriction of \( \nu \) to \( B \), i.e., \( \int_B |q| \, d\nu < +\infty \).

**Definition 1.** Let \( B \) be a Borel subset of \( \Omega \). Let
\[
h \colon (\zeta, z) \mapsto \mathbb{R}, \quad (\zeta, z) \in B \times \Omega,
\]
be a Borel-measurable function which is locally bounded. Suppose, for every fixed point \( \zeta \in B \), the function \( h(\zeta, \cdot) \) is harmonic on \( \Omega \); then the function
\[
k(\zeta, z) := \log |\zeta - z| + h(\zeta, z), \quad (\zeta, z) \in B \times \Omega,
\]
is a subharmonic kernel on \( B \times \Omega \) (supported by \( B \) with the harmonic component \( h \)).

\(^1\)Here \( \Delta \) is the Laplace operator which acts in sense of distribution theory.
Suppose a $\nu \in \mathcal{M}^+(\Omega)$ is concentrated in $B$, i.e., $\nu(\Omega \setminus B) = 0$. A subharmonic kernel $k$ on $B \times \Omega$ is suitable for $\nu$ if for any $z \in \Omega$ there are a subdomain $D_z \ni z$ of $\Omega$ and a function $q \in L^1((\Omega \setminus D_z) \cap B, d\nu)$ such that

$$|k(\zeta, w)| \leq q(\zeta), \quad \forall \zeta \in (\Omega \setminus D_z) \cap B, \quad \forall w \in D_z.$$ 

**Examples.** Consider some frequently occurring subharmonic kernels.

0. The function $\log |\zeta - z|$ is a suitable subharmonic kernel on $\Omega \times \Omega$ for all positive Borel measures with compact support in $\Omega$.

1. If the domain $\Omega$ possesses a Green’s function $g_\Omega$ ($\partial \Omega$ is non-polar), then the functions $-g_\Omega(\zeta, z)$ and $-g_\Omega(\zeta, z) + \log |\zeta|$ are subharmonic kernels on $\Omega \times \Omega$ and on $(\Omega \setminus \{0\}) \times \Omega$ respectively. These kernels are suitable for measures $\nu \in \mathcal{M}^+(\Omega)$ satisfying (see [12, Theorem 4.5.4])

$$\int_\Omega g_\Omega(\zeta, 0) \, d\nu(\zeta) < +\infty.$$

2. Let $\Omega = \mathbb{D}$. Here we use the Blaschke factor, a variant of this factor, and the pseudohyperbolic distance for $\mathbb{D}$:

$$B_\zeta(z) := \frac{|\zeta|}{\zeta} \frac{\zeta - z}{1 - \overline{\zeta} z}, \quad \zeta \in \mathbb{D} \setminus \{0\}, \quad z \in \mathbb{D}, \quad \text{but} \quad |B_0(z)| := |z|. \quad (1.3B)$$

$$\overline{B}_\zeta(z) := \frac{\overline{\zeta}(\zeta - z)}{|1 - \overline{\zeta} z|} = |\zeta| B_\zeta(z), \quad \zeta \in \mathbb{D}, \quad z \in \mathbb{D}, \quad (1.3B)$$

$$\rho(\zeta, z) := \left| \frac{\zeta - z}{1 - \overline{\zeta} z} \right| = |B_\zeta(z)| = \frac{1}{|\zeta|} |\overline{B}_\zeta(z)|. \quad (1.3p)$$

The following functions are subharmonic kernels:

(B$_0$) The subharmonic *Blaschke kernel* on $\mathbb{D} \times \mathbb{D}$ is the function

$$b_1(\zeta, z) := \log |B_\zeta(z)| = \log |\rho(\zeta, z)| = -g_\mathbb{D}(\zeta, z)$$

which is suitable for measures $\nu \in \mathcal{M}^+(\mathbb{D})$ satisfying $\int_0^{1^+} (1 - t) \, d\nu^{rad}(t) < +\infty$. Similarly, $\overline{b}_1(\zeta, z) := \log |\overline{B}_\zeta(z)| = -g_\mathbb{D}(z, \zeta) + \log |\zeta|$ is a subharmonic kernel on $(\mathbb{D} \setminus \{0\}) \times \mathbb{D}$ which is suitable for the same measures $\nu \in \mathcal{M}^+(\mathbb{D})$.

(D$_p$) For an integer $p \geq 0$, the subharmonic *Dzhrbashyan’s kernel of*
genus\(^2\) \(p\) supported by \(\mathbb{D} \setminus \{0\}\) is the function (see [13]–[17])

\[
\overline{\psi}_p(\zeta, z) := \log \left| \overline{B}_\zeta(z) \right| + \sum_{k=1}^{p} \frac{1}{k} \operatorname{Re} \left( 1 - \overline{B}_\zeta(z) \right)^k
\]

\[
= \log \left| \frac{\overline{\zeta}(\zeta - z)}{1 - \overline{\zeta}z} \right| + \sum_{k=1}^{p} \frac{1}{k} \operatorname{Re} \left( \frac{1 - |\zeta|^2}{1 - \overline{\zeta}z} \right)^k
\]

which coincides with \(-g_\mathbb{D}(\zeta, z) + \log |\zeta|\) for \(p = 0\). This kernel is suitable for measures \(\nu \in \mathcal{M}(\mathbb{D})\) satisfying

\[
\int_{0}^{1^{-}} (1 - t)^{p+1} d\nu^\text{rad}(t) + \int_{0}^{1/2} \frac{\nu^\text{rad}(t)}{t} dt < +\infty. \tag{1.5}
\]

(H\(_2\)) The subharmonic Horowitz’s kernel (see [18], [17]) on \(\mathbb{D} \times \mathbb{D}\) is the function \(h_2(\zeta, z) := \log |1 - (1 - B_\zeta(z))^2|\). This kernel is suitable for measures \(\nu \in \mathcal{M}(\mathbb{D})\) satisfying \(\int_{0}^{1^{-}} (1 - t)^2 d\nu^\text{rad}(t) < +\infty\).

(B\(_s\)) For \(0 < s \leq 6\), the subharmonic Beller’s kernel on \(\mathbb{D} \times \mathbb{D}\) is the function \(b_s(\zeta, z) := \log |1 - (1 - B_\zeta(z))^s|\) (see [3], [17]) which coincides with the Horowitz’s kernel for \(s = 2\) and with the Blaschke kernel for \(s = 1\). This kernel is suitable for measures \(\nu \in \mathcal{M}(\mathbb{D})\) satisfying \(\int_{0}^{1^{-}} (1 - t)^s d\nu^\text{rad}(t) < +\infty\).

(\(\overline{B}_s\)) For \(s \geq 1\), the subharmonic Bomash’s kernel supported by \(\mathbb{D} \setminus \{0\}\) is the function (see [19], [20])

\[
\overline{b}_s(\zeta, z) := \log |1 - (1 - \overline{B}_\zeta(z))^s|.
\]

This kernel is suitable for measures \(\nu \in \mathcal{M}(\mathbb{D})\) satisfying (1.5) for \(p = s - 1\). Below we will use a special case of subharmonic Bomash’s kernel with \(s = 2\):

\[
\overline{b}_2(\zeta, z) := \log |1 - (1 - \overline{B}_\zeta(z))^2| = \log \left| 1 - \left( \frac{1 - |\zeta|^2}{1 - \overline{\zeta}z} \right)^2 \right| \quad \tag{1.6a}
\]

\[
= \log \left( \left| \overline{B}_\zeta(z) \right| |2 - \overline{B}_\zeta(z)| \right) = \log \frac{|\zeta||\zeta - z||2 - |\zeta|^2 - \overline{\zeta}z|}{|1 - \overline{\zeta}z|^2}. \tag{1.6b}
\]

This kernel is suitable for measures \(\nu \in \mathcal{M}(\mathbb{D})\) satisfying

\[
\int_{0}^{1^{-}} (1 - t)^2 d\nu^\text{rad}(t) + \int_{0}^{1/2} \frac{\nu^\text{rad}(t)}{t} dt < +\infty. \tag{1.7}
\]

\(^2\)For \(q > p\), by definition, \(\sum_{k=q}^{p} = 0\), \(\prod_{k=q}^{p} = 1\). Similarly, \(\sum_{k \in \emptyset} = 0\), \(\prod_{k \in \emptyset} = 1\).
(K1) The subharmonic Korenblum’s kernel on \( (\mathbb{D} \setminus \{0\}) \times \mathbb{D} \) (see \cite{2}, \cite{17}) is the function
\[
k_1(\zeta, z) := \log |B_\zeta(z)| + \log \frac{1}{|\zeta|} \Re \frac{\zeta/|\zeta| + z}{\zeta/|\zeta| - z}.
\]
This kernel is suitable for measures \( \nu \in M^+(\mathbb{D}) \) satisfying (1.7).

Note that for each measure \( \nu \in M(\mathbb{D}) \) we can select subharmonic kernel (see Dzhrbashyan’s factorization theory in \cite{14}, \cite{15}, \cite{17}) which is suitable for \( \nu \) and optimal in a certain sense.

3. Let \( \Omega = \mathbb{C} \), \( r_0 > 0 \). The following functions are subharmonic kernels:

( E_q) For an integer \( q \geq 0 \), the subharmonic Hadamard–Weierstrass kernel of genus \( q \) on \( (\mathbb{C} \setminus \{0\}) \times \mathbb{C} \) is the function
\[
e_q(\zeta, z) := \log \left| 1 - \frac{z}{\zeta} \right| + \sum_{k=0}^{q} \frac{1}{k} \Re \frac{z}{\zeta}.
\]
This kernel is suitable for measures \( \nu \in M^+(\mathbb{C}) \) satisfying
\[
\int_0^1 \frac{\nu^{\text{rad}}(t)}{t^{q+1}} \, dt + \int_1^{+\infty} \frac{\nu^{\text{rad}}(t)}{t^{q+2}} \, dt < +\infty.
\]

( W) Let \( \{q_n\} \) be a sequence of nonnegative integer, and let \( \{r_n\} \) be an increasing sequence of positive numbers, \( n \in \mathbb{N} \), \( r_1 > r_0 \). The subharmonic Weierstrass kernel (with respect to these two sequences) on \( \mathbb{C} \times \mathbb{C} \) is the function \( w(\zeta, z) := e_{q_n}(\zeta, z) \) when \( r_{n-1} \leq |\zeta| < r_n \), \( n \in \mathbb{N} \), and \( w(\zeta, z) := \log |\zeta - z| \) when \( |\zeta| < r_0 \). This kernel is suitable for measures \( \nu \in M^+(\mathbb{C}) \) satisfying (see \cite{1, Theorem 4.1})
\[
\sum_{n=1}^{\infty} \int_{r_{n-1}}^{r_n} \frac{\nu^{\text{rad}}(t)}{t^{q_n+2}} \, dt < +\infty.
\]

Note also that, for each measure \( \nu \in M^+(\mathbb{C}) \), we can select subharmonic kernel (see Dzhrbashyan’s factorization theory in \cite{21} and the survey \cite{22}) which is suitable for \( \nu \) and optimal in a certain sense.

The following Proposition 1.1 is a global version of Riesz Decomposition Theorem for subharmonic functions \cite{1}, \cite{12}.

**Proposition 1.1.** Let \( k \) be a subharmonic kernel on \( B \times \Omega \). Suppose a measure \( \nu \in M^+(\Omega) \) is concentrated in \( B \). If the kernel \( k \) is suitable for \( \nu \), then the function
\[
U^\nu_k(z) := \int_{\Omega} k(\zeta, z) \, d\nu(\zeta)
\]
is subharmonic on $\Omega$ with the Riesz measure $\nu$. In particular, every function $M \in SH(\Omega)$ with the Riesz measure $\nu_M = \nu$ can be decomposed as

$$M = U_k^\nu + H, \quad \text{where } H \in \text{Harm}(\Omega). \quad (1.8)$$

The proof is omitted here. It can be obtained from [23, Theorem 2.6.5].

1.2 Main results

We write $L^1_{\text{loc}}(\Omega)$ for the set of all functions $F: \Omega \to [-\infty, +\infty]$ that are locally integrable with respect to $m$. A sequence $\{w_n\}$ from $L^1_{\text{loc}}(\Omega)$ is convergent in $L^1_{\text{loc}}(\Omega)$ if there exists a function $w \in L^1_{\text{loc}}(\Omega)$ such that $\lim_{n \to \infty} \int_K |w_n - w| \, dm \to 0$ for any compact subset $K \subset \Omega$.

We denote by $\text{dist}(z, \partial \Omega)$ the Euclidean distance from $z \in \Omega$ to $\partial \Omega$.

In what follows the measure $m(r) \in M^+(\mathbb{C})$ is obtained from Lebesgue measure $m$ by restricting it to the disk $D(r)$ and normalizing so that $m(r)(D(r)) = 1$, i.e.,

$$m(r) := \frac{1}{m^{\text{rad}}(r)} m\big|_{D(r)}. \quad (1.9)$$

We have

$$v(z) \leq (v * m(r))(z), \quad v \in SH(\Omega), \quad (1.10)$$

if $D(r) \subset \Omega$ (here and below $*$ stands for the convolution operation).

Let $\sigma: \Omega \to (0, +\infty)$ be a function satisfying

$$0 < \sigma(z) < \text{dist}(z, \partial\Omega), \quad \forall z \in \Omega. \quad (1.11)$$

For $F \in L^1_{\text{loc}}(D)$, we put

$$F^{(\sigma)}(z) := \int_{D(\sigma(z))} F(z + w) \, dm^{(\sigma(z))}(w), \quad z \in \Omega. \quad (1.12)$$

We say that a function $F: \Omega \to [-\infty, +\infty]$ admits a harmonic minorant (a subharmonic minorant resp.) on $\Omega$ if and only if there is a $h \in \text{Harm}(\Omega)$ ($h \in SH(\Omega)$, $h \not\equiv -\infty$ on $\Omega$, resp.) such that $h(z) \leq F(z)$ for all $z \in \Omega$.

For $a \in [-\infty, +\infty]$ (for a function $f: X \to [-\infty, +\infty]$ resp.), as usual, we put $a^+ := \max\{a, 0\}$ ($f^+(x) := \max\{f(x), 0\}$ resp.).

The following theorem is the main general result of our article.

**Main Theorem.** Let $M \not\equiv -\infty$ be a subharmonic function with the Riesz measure $\nu_M$ on a domain $\Omega \subset \mathbb{C}$, $0 \in \Omega$. Suppose also that

i) the function $M$ is bounded on a subdomain $\Omega_1 \subset \Omega$, and $0 \in \Omega_1$;

ii) the measure $\nu_M$ is concentrated in a Borel subset $B \subset \Omega$, and subharmonic kernel $k$ on $B \times \Omega$ is suitable for $\nu_M$;
iii) for the function

\[
Q_k^M(z) := \int_B \left( k(\zeta, 0) - k(\zeta, z) \right)^+ \, d\nu_M(\zeta), \quad z \in \Omega,
\]

there exists a majorizing function \( Q \in L^1_{\text{loc}}(\Omega) \) on \( \Omega \), i.e., \( Q_k^M(z) \leq Q(z) \) at almost all (with respect to Lebesgue measure \( m \)) points \( z \in \Omega \).

Let \( u \) be a subharmonic function with the Riesz measure \( \nu_u \) on \( \Omega \), and \( u(0) \neq -\infty \). Then the following two assertions hold:

(Z) Let \( \Omega_0 \) be a relative compact subdomain of \( \Omega_1 \) containing the origin. If there exists a constant \( C \) such that the inequality

\[
\int g_D(\zeta, 0) \, d\nu_u(\zeta) \leq \int g_D(\zeta, 0) \, d\nu_M(\zeta) + C
\]

holds for each domain \( D \in \mathcal{U}_0^d(\Omega; \Omega_0) \), then, for any \( \sigma \in C(\Omega) \) satisfying (1.11), there exists a harmonic function \( h \) on \( \Omega \) such that

\[
uu \left. u(z) + h(z) \right. M(\sigma)(z) + Q(\sigma)(z), \quad \forall z \in \Omega.
\]

(S) If the difference \( M - u \) admits\(^3\) a subharmonic minorant on \( \Omega \), then, for any \( \sigma \in C(\Omega) \) satisfying (1.11), the function

\[
M(\sigma) + Q(\sigma) - u
\]

admits a harmonic minorant on \( \Omega \).

An important consequence from the Main Theorem is

**Corollary 1.1.** Let \( \Omega \subset \mathbb{C} \) be a simply connected domain containing the origin. Let \( M \equiv -\infty \) be a subharmonic continuous function on \( \Omega \) with the Riesz measure \( \nu_M \). Suppose that the conditions i)–iii) of the Main Theorem are fulfilled. Then the following three assertions hold:

(Z\(_{sc}\)) Let \( \Lambda = \{\lambda_n\} \) be a sequence in \( \Omega \), \( 0 \notin \Lambda \), and let \( \Omega_0 \) be a relative compact subdomain of \( \Omega \) containing the origin. If there exists a constant \( C \) such that the inequality

\[
\sum_n g_D(\lambda_n, 0) \leq \int g_D(\zeta, 0) \, d\nu_M(\zeta) + C
\]

holds for each domain \( D \in \mathcal{U}_0^d(\Omega; \Omega_0) \), then, for any \( \sigma \in C(\Omega) \) satisfying (1.11), the sequence \( \Lambda \) is a zero set for \( \text{Hol}(\Omega; M(\sigma) + Q(\sigma)) \).

\(^3\)If \( M(z) = u(z) = -\infty \), then we put \( (M - u)(z) = +\infty \).
If a sequence $\Lambda$ in $\Omega$ is zero subset for the space $\text{Hol}(\Omega; M)$, then, for any $\sigma \in C(\Omega)$ satisfying (1.11), the sequence $\Lambda$ is a zero set for the space $\text{Hol}(\Omega; M(\sigma) + Q^{(\sigma)})$.

Suppose that $f = g/q$ is a meromorphic function on $\Omega$ where $g$ and $q$ belong to $\text{Hol}(\Omega; M)$; then, for any $\sigma \in C(\Omega)$ satisfying (1.11), there are holomorphic functions $g_0, q_0 \in \text{Hol}(\Omega; M(\sigma) + Q^{(\sigma)})$ without common zeros such that $f = g_0/q_0$ on $\Omega$.

Given $z \in \mathbb{D}$ and $\alpha > 0$, the set
$$\square_{\alpha}(z) := \{\zeta \in \mathbb{D} : |z| - \alpha(1 - |z|) \leq |\zeta|, |\arg \zeta - \arg z| \leq \alpha(1 - |z|)\}.$$ (1.18)

is called the Carleson box of relative size $\alpha$ with center at $z$.

Given $0 < \varepsilon < 1$ and $M : \mathbb{D} \to [-\infty, +\infty)$, we put
$$A_{M,\varepsilon}(z) := \frac{1}{2\pi} \int_0^{2\pi} M(z + \varepsilon(1 - |z|)e^{i\theta}) \, d\theta$$ (1.19)

if this integral there exists.

The following Theorem 1.1 is a special version of previous Corollary 1.1 in case that we use the subharmonic Bomash’s kernel (1.6) from the item $(\overline{B}_s)$ of Examples in the role of subharmonic kernel $k$ on $(\mathbb{D} \setminus \{0\}) \times \mathbb{D}$.

**Theorem 1.1.** Let $M \neq -\infty$ be a subharmonic function $M$ on $\mathbb{D}$ with the Riesz measure $\nu_M$. Suppose that the function $M$ is bounded below on every $D \subseteq \mathbb{D}$. Suppose also that the condition
$$\int_0^1 (1 - t)^2 \, d\nu_M^{\text{rad}}(t) < \infty$$ (1.20)

holds. By definition, put (see (1.18))
$$b_{\alpha,M}(z) := \frac{1}{(1 - |z|)^2} \int_{\square_{\alpha}(z)} (1 - |\zeta|)^2 \, d\nu_M(\zeta).$$ (1.21)

Under these conditions and notations, the following three assertions hold:

$(Z)$ Let $\Lambda = \{\lambda_n\}$ be a sequence in $\mathbb{D}$, and $a < 1$. If there exists a constant $C$ such that the inequality (1.17) holds for each domain $D \subseteq \mathcal{U}_0^d(\mathbb{D}; D(a))$, then, for any $0 < \varepsilon < 1$, the sequence $\Lambda$ is a zero set for the space
$$\text{Hol}(\mathbb{D}; 2A_{M,\varepsilon} - C_{\varepsilon} \cdot b_{\alpha,M}^{|\lambda|}),$$ (1.22)

where $C_{\varepsilon}$ is a positive constant dependent only on $\varepsilon$, and $\alpha$ is an absolute constant.
(S\(\mathbb{D}\)) If a sequence \(\Lambda\) in \(\mathbb{D}\) is zero subset for the space \(\text{Hol}(\mathbb{D}; M)\), then, for any \(0 < \varepsilon < 1\), the sequence \(\Lambda\) is a zero set for the space \((1.22)\).

(M\(\mathbb{D}\)) If \(f = g/q\) is a meromorphic function on \(\mathbb{D}\), and \(g, q \in \text{Hol}(\mathbb{D}; M)\), then, for any \(0 < \varepsilon < 1\), there are holomorphic functions \(g_0, q_0\) without common zeros on \(\mathbb{D}\) such that \(f = g_0/q_0\) on \(\mathbb{D}\), and the functions \(g_0, q_0\) belong to the space \((1.22)\).

2 Bases of general approach

2.1 The balayage: Arens–Singer and Jensen measures and functions

Let \(H\) be a convex subcone in \(SH(\Omega)\), \(z \in \Omega\). Let \(\delta, \mu \in M^+(\Omega)\) are measures with compact support on \(\Omega\). We write \(\delta \prec_H \mu\) and say that \(\mu\) is a balayage of \(\delta\) with respect to \(H\) if

\[
\int h \, d\delta \leq \int h \, d\mu, \quad \forall h \in H. \tag{2.1}
\]

Given \(z \in \mathbb{C}\), we denote by \(\delta_z\) the Dirac measure at the point \(z \in \mathbb{C}\), i.e. \(\text{supp} \delta_z = \{z\}, \delta_z(\mathbb{C}) = 1\).

**Definition 2** ([24]–[33]). A measure \(\mu\) with compact support in \(\Omega \ni z\) is called the Jensen (Arens–Singer, or representing resp.) measure for \(z\) on \(\Omega\) if \(\delta_z \prec_H \mu\) where \(H = SH(\Omega)\) \((H = \text{Harm}(\Omega)\) resp.), i.e., if and only if

\[
h(z) \leq \int h \, d\mu, \quad \forall h \in SH(\Omega) \left( h(z) = \int h \, d\mu, \quad \forall h \in \text{Harm}(\Omega) \text{ resp.} \right). \tag{2.2}
\]

Denote by \(J_z(\Omega)\) \((AS_z(\Omega)\) resp.) the class of all Jensen (Arens–Singer resp.) measures for \(z\). If \(z = 0 \in \Omega\), then \(J(\Omega) := J_0(\Omega), AS(\Omega) := AS_0(\Omega)\).

Evidently, \(J_z(\Omega) \subset AS_z(\Omega)\), and every \(\mu \in AS_z(\Omega)\) is a probability measure, i.e.,

\[
\mu(\Omega) = 1. \tag{2.3}
\]

A subclass of \(J_z(\Omega)\) is the class \(H_z(\Omega)\) \((H_z^{\text{reg}}(\Omega)\) resp.) of all harmonic measures \(\omega_D(z, \cdot)\) for domains \(D \in \Omega\) at the point \(z\). We set \(H(\Omega) := H_0(\Omega), H^{\text{reg}}(\Omega) := H_0^{\text{reg}}(\Omega)\).

The potential of a measure \(\mu \in AS(\Omega)\) is defined to be function

\[
V_\mu(\zeta) := \int \log |z - \zeta| \, d\mu(z) - \log |\zeta| = \int \log \left| 1 - \frac{z}{\zeta} \right| \, d\mu(z), \quad \zeta \in \mathbb{C} \setminus \{0\}. \tag{2.4}
\]
Let \( \mu \in \mathcal{A}\mathcal{S}(\Omega) \). For any function \( u_\nu \in SH(\Omega) \) with the Riesz measure \( \nu_u \) the generalized Poisson–Jensen formula [27, Proposition 1.2] holds:

\[
u_u(0) = \int_\Omega u_\nu \, d\mu - \int_\Omega V_\mu \, d\nu. \tag{2.5}
\]

**Proposition 2.1.** Let \( V_\mu \) be the potential (2.4) of a measure \( \mu \in \mathcal{A}\mathcal{S}(\Omega) \), and let \( k \) be a subharmonic kernel on \( B \times \Omega \). Then, for every \( \zeta \in B \setminus \{0\} \), the potential \( V_\mu \) can be represented in the form

\[
V_\mu(\zeta) = \int_\Omega k(\zeta, z) \, d\mu(z) - k(\zeta, 0) = \int_\Omega (k(\zeta, z) - k(\zeta, 0)) \, d\mu(z). \tag{2.6}
\]

**Proof.** By (1.2), for every \( \zeta \in B \setminus \{0\} \), we have representations

\[
k(\zeta, z) = \log |\zeta - z| + h(\zeta, z), \quad k(\zeta, 0) = \log |\zeta| + h(\zeta, 0), \tag{2.7}
\]

where \( h(\zeta, z) \) is harmonic function of \( z \in \Omega \). Hence, for \( \mu \in \mathcal{A}\mathcal{S}(\Omega) \), we obtain

\[
\int_\Omega (k(\zeta, z) - k(\zeta, 0)) \, d\mu(z)
\]

\[
\overset{(2.7)}{=} \int_\Omega (\log |\zeta - z| + h(\zeta, z) - \log |\zeta| - h(\zeta, 0)) \, d\mu(z)
\]

\[
\overset{(2.3)}{=} \int_\Omega (\log |\zeta - z| - \log |\zeta|) \, d\mu(z) + \int_\Omega h(\zeta, z) \, d\mu(z) - h(\zeta, 0)
\]

\[
\overset{(2.4), (2.2)}{=} V_\mu(\zeta) + h(\zeta, 0) - h(\zeta, 0) = V_\mu(\zeta)
\]

for each \( \zeta \in B \setminus \{0\} \), as desired. \( \bullet \)

**Definition 3** ([24]–[29]). A function \( V \in SH(\Omega \setminus \{0\}) \) will be called a Arens–Singer (or representing) function on \( \Omega \) if this function satisfies the following two conditions:

1. There is a set \( K \subseteq \Omega \) such that \( V(\zeta) \equiv 0 \) for all \( \zeta \in \Omega \setminus K \);

2. \( \limsup_{\zeta \to 0} \frac{V(\zeta)}{-\log |\zeta|} \leq 1. \)

If \( V \) satisfies also the condition

3. \( V(\zeta) \geq 0 \) for all \( \zeta \in \Omega \setminus \{0\} \),

then we call \( V \) a Jensen function on \( \Omega \). Denote by \( \mathcal{P}_{AS}(\Omega) \) and \( \mathcal{P}_J(\Omega) \) the classes of all Arens–Singer and Jensen functions respectively.
Evidently, $\mathcal{P}_{\mathcal{J}}(\Omega) \subset \mathcal{P}_{\mathcal{AS}}(\Omega)$. The class $\mathcal{P}_{\mathcal{H}}(\Omega)$ ($\mathcal{P}^{\text{reg}}_{\mathcal{H}}(\Omega)$ resp.) of all extended Green functions $g_D(\cdot,0)$ for domains $D \subseteq \Omega$ (for regular for the Dirichlet problem domains $D \subseteq \Omega$ resp.) is a subclass of $\mathcal{P}_{\mathcal{J}}(\Omega)$.

**Proposition 2.2** ([27, Duality Theorem]). The map $\mathcal{P}: \mu \rightarrow V_\mu$ is an affine\(^4\) bijection from $\mathcal{AS}(\Omega)$ onto $\mathcal{P}_{\mathcal{AS}}(\Omega)$, and from $\mathcal{J}(\Omega)$ onto $\mathcal{P}_{\mathcal{J}}(\Omega)$.

Note that, according to Proposition 2.2, for $V \in \mathcal{P}_{\mathcal{AS}}(\Omega)$ the condition \((2)\) is equivalent to the condition (see [12, Theorem 3.1.2])

\[(2') \quad V(\zeta) \leq -\log |\zeta| + O(1) \quad \text{as} \quad \zeta \to 0.\]

Let $\Omega_0 \Subset \Omega$ be a domain containing the origin.

Denote by $\mathcal{AS}^{\Omega_0}(\Omega)$ ($\mathcal{J}^{\Omega_0}(\Omega)$ resp.) the set of all Arens–Singer (Jensen resp.) measures $\mu$ on $\Omega$ such that $\Omega_0 \cap \text{supp} \mu = \emptyset$.

Denote by $\mathcal{P}^{\mathcal{AS}}_{\Omega_0}(\Omega)$ ($\mathcal{P}^{\mathcal{J}}_{\Omega_0}(\Omega)$ resp.) the set of all Arens–Singer (Jensen resp.) functions $V$ that is harmonic on $\Omega_0 \setminus \{0\}$ such that (cf. \((2')\))

\[V(\zeta) = -\log |\zeta| + O(1), \quad \zeta \to 0.\]  \hspace{1cm} (2.8)

**Proposition 2.3.** The map $\mathcal{P}: \mu \rightarrow V_\mu$ of Proposition 2.2 is an affine bijection from $\mathcal{AS}^{\Omega_0}(\Omega)$ onto $\mathcal{P}^{\mathcal{AS}}_{\Omega_0}(\Omega)$, and from $\mathcal{J}^{\Omega_0}(\Omega)$ onto $\mathcal{P}^{\mathcal{J}}_{\Omega_0}(\Omega)$.

The proof of Proposition 2.3 follows from Proposition 2.2 and [27, Proposition 1.4, (4)].

We recall the notion of classical balayage [34]. Given a bounded regular for the Dirichlet problem domain $\Omega_1 \Subset \Omega$ and a measure $\mu \in M^+(\Omega)$, the classical balayage of $\mu$ relative to $\Omega_1$ is the measure $\mu^{\Omega_1} \in M^+(\Omega)$ defined by

\[\mu^{\Omega_1}(E) := \mu(E \setminus \Omega_1) + \int_{\Omega_1} \omega_{\Omega_1}(z,E) \, d\mu(z), \quad \text{Borel } E \subseteq \Omega.\]  \hspace{1cm} (2.9)

In particular, $\Omega_1 \cap \text{supp} \mu^{\Omega_1} = \emptyset$, and $\mu \prec_{\mathcal{SH}(\Omega)} \mu^{\Omega_1}$ [32, Lemma 6.4]. Hence

\[\mathcal{AS}^{\Omega_1}(\Omega) = \{ \mu^{\Omega_1} : \mu \in \mathcal{AS}(\Omega) \}, \quad \mathcal{J}^{\Omega_1}(\Omega) = \{ \mu^{\Omega_1} : \mu \in \mathcal{J}(\Omega) \} \]  \hspace{1cm} (2.10)

for regular subdomains $\Omega_1 \Subset \Omega$ containing the origin.

Let $D \Subset \mathbb{C}$ be a regular domain containing the origin. For short, denote by $\omega_D$ the harmonic measure for $D$ at the origin.

As before, $\delta_0$ is the Dirac measure at 0.

**Proposition 2.4** ([34, Ch. IV, §1, and Theorem 4.12]). Let $D \Subset \mathbb{C}$, $\Omega' \Subset \mathbb{C}$, and $D \cup \Omega'$ are regular domains containing the origin. Then

\[\delta_0^{\Omega' \cup D} = \omega_{\Omega' \cup D} = (\omega_D)^{\Omega' \cup D} = \omega_{\Omega'} - (\omega_{\Omega'} |_D) + (\omega_{\Omega'} |_{\Omega})^{\Omega' \cup D}.\]  \hspace{1cm} (2.11)

\(^4\)It means that $\mathcal{P}(\alpha \mu_1 + (1-\alpha)\mu_2) = \alpha \mathcal{P}(\mu_1) + (1-\alpha)\mathcal{P}(\mu_2)$ for $0 \leq \alpha \leq 1.$

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2.2 On the existence of (sub)harmonic minorants

Proposition 2.5 (cf. [26, Proposition 7.1]). Let $F : \Omega \to [-\infty, +\infty]$ be a Borel-measurable function. If $F$ admits a harmonic minorant, then

$$-\infty < \inf \left\{ \int \limits_{\Omega} F \, d\mu : \mu \in \mathcal{AS}(\Omega) \right\}.$$  \hspace{1cm} (2.12)

If $F$ admits a subharmonic minorant, and $F$ is bounded below in a certain neighbourhood of origin, then

$$-\infty < \inf \left\{ \int \limits_{\Omega} F \, d\mu : \mu \in \mathcal{J}(\Omega) \right\}.$$  \hspace{1cm} (2.13)

Evidently, the class $\mathcal{AS}(\Omega)$ in (2.12) can be replaced by $\mathcal{J}(\Omega)$, $\mathcal{H}(\Omega)$, $\mathcal{H}^{reg}(\Omega)$, and we can substitute $\mathcal{H}(\Omega)$ or $\mathcal{H}^{reg}(\Omega)$ for $\mathcal{J}(\Omega)$ in (2.13).

Proof. Let $h$ be a Borel-measurable function on $\Omega$ and $F \geq h$ on $\Omega$. Then $\int F \, d\mu \geq \int h \, d\mu$ for every $\mu \in \mathcal{M}^+(\Omega)$. If the minorant $h$ is harmonic, then $\int h \, d\mu \geq h(0)$ for all $\mu \in \mathcal{AS}(\Omega)$ according to (2.2), and (2.12) is proved.

Let us assume that $F$ is bounded below on $D(2\varepsilon) \subset \Omega$, and the minorant $h$ is subharmonic. There exists a subharmonic function $h_\varepsilon$ which coincides with $h$ on $\Omega \setminus D(\varepsilon)$ and harmonic on $D(\varepsilon)$ [1, Theorem 2.18]. Hence the difference $h_\varepsilon - C$ is a subharmonic minorant for $F$ if a constant $C$ is sufficiently large. Therefore, $\int F \, d\mu \geq \int (h_\varepsilon - C) \, d\mu \geq h_\varepsilon(0) - C$ for all $\mu \in \mathcal{J}(\Omega)$ according to Definition 2, and (2.13) is proved.

We need another treatments of Proposition 2.5 for the cases special functions $F$ representing by the difference

$$F = M - u, \quad u \in SH(\Omega), \quad u(0) \neq -\infty.$$  \hspace{1cm} (2.14)

Here the functions $M$ and $u$ can take on the value $-\infty$ at a point $z \in \Omega$ simultaneously if $M \in SH(\Omega)$ or $M \in L^1_{\text{loc}}(\Omega)$. In this case we will be consider that $F(z) = +\infty$. The following Proposition 2.6 is an easy corollary of Propositions 2.5, 2.2, and of generalized Poisson–Jensen formula (2.5).

Proposition 2.6. Let $M : \Omega \to [-\infty, +\infty]$ be a Borel-measurable function, and $u$ be a subharmonic function on $\Omega$ with the Riesz measure $\nu_u$.

If the function $F := M - u$ from (2.14) admits a harmonic minorant on $\Omega$, then there is a constant $C$ such that

$$\int \limits_{\Omega} V_\mu \, d\nu_u \leq \int \limits_{\Omega} M \, d\mu + C$$  \hspace{1cm} (2.15)
for all $\mu \in A\mathcal{S}(\Omega)$ where $V_\mu$ is the potential of $\mu$ (see definition (2.4)). Besides, if $M \in SH(\Omega)$, then there is a constant $C$ such that

$$\int_{\Omega} V \, d\nu_u \leq \int_{\Omega} V \, d\nu_M + C$$

(2.16)

for all Arens–Singer functions $V$ on $\Omega$.

If $F = M - u$ from (2.14) admits a subharmonic minorant, and $M$ is bounded below in a certain neighbourhood of origin, then there is a constant $C$ such that (2.15) holds for all $\mu \in \mathcal{J}(\Omega)$. Besides, if $M \in SH(\Omega)$, then there is a constant $C$ such that (2.16) holds for all $V \in \mathcal{P}_J(\Omega)$.

Let $\{u_n\}$ be a sequence of functions subharmonic in $\Omega$ and uniformly locally bounded above (that is, uniformly bounded above on every $G \subset \Omega$) in $\Omega$. We denote by $\limsup^* u_n$ the upper semicontinuous regularization of the (pointwise) upper limit of $\{u_n\}$, which is, in turn, a subharmonic function.

The following Theorem is in a sense converse to Proposition 2.5.

**Theorem 2.1** ([26, Theorem 7.1]). Let $H \subset SH(\Omega)$ be a convex cone.

Assume that the cone $H$ contains a negative $(\leq 0)$ function, and let $F : \Omega \to [-\infty, +\infty]$ be a function that belongs to $L^1_{\text{loc}}(\Omega)$.

Let us suppose that for any $K \subset \Omega$ and any constant $C$ there is a $h \in H$ such that $h \leq C$ on $K$, and at least one of the following conditions holds:

(L*) if a sequence $\{h_n\}, h_n \in H$, is locally bounded above, then the function $\limsup^* h_n$ belongs to $H$,

(CL) the cone $H$ is sequentially closed in $L^1_{\text{loc}}(\Omega)$.

If

$$-\infty < \inf \left\{ \int_{\Omega} F \, d\mu : \delta_0 \preceq_H \mu, \, \mu \in M^+_\text{ac}(\Omega) \right\},$$

(2.17)

where $\delta_0$ is the Dirac measure at 0, then for any function $\sigma \in C(\Omega)$ satisfying (1.11), there is a function $h \in H$ such that $h \not\equiv -\infty$ on $\Omega$, and

$$h(z) \leq F^{(\sigma)}(z) := \int_{D(\sigma(z))} F(z + w) \, dm^{(\sigma(z))}(w), \quad \forall z \in \Omega.$$  

(2.18)

The space $H = \text{Harm}(\Omega)$ and the cone $H = SH(\Omega)$ satisfy the conditions of Theorem 2.1 (see [25]–[26] for example).

Results more general than Theorem 2.1 were proved for abstract cones $H$ of vector lattices [25, Theorems 5.1, 6.1], and, for $H = SH(\Omega)$, in [31, 5]

\footnote{In [25], we proposed by definition that the cone $SH(\Omega)$ is not containing the function identically equal to $-\infty$.}
Corollary 1.7. They require a certain preparation, and we do not use their here.

We also need another treatments of theorem 2.1 for the cases \( H = \text{Harm}(\Omega) \) and \( H = \text{SH}(\Omega) \), and for special functions \( F \) representing by the difference (2.14) where \( M \in L^1_{\text{loc}}(\Omega) \) or \( M \in \text{SH}(\Omega) \).

The following Theorem 2.2 is an earlier version of Theorem 2.1 which joins Proposition 2.5 with Theorem 2.1 for special functions (2.14) with \( M \in C(\Omega) \).

**Theorem 2.2** ([24, Main Theorem]). Let \( u \in \text{SH}(\Omega) \) and \( M \in C(\Omega) \). The difference (2.14) admits a harmonic (subharmonic resp.) minorant on \( \Omega \) if and only if there is a constant \( C \) such that \( \int_\Omega u \, d\mu \leq \int_\Omega M \, d\mu + C \) for all \( \mu \in \text{AS}(\Omega) \) (for all \( \mu \in \mathcal{J}(\Omega) \) resp.).

We also need an other version of Theorem 2.2 in the case that the function \( M \) in (2.14) belongs to \( L^1_{\text{loc}}(\Omega) \) or \( \text{SH}(\Omega) \).

**Theorem 2.3.** Let \( u \not\equiv -\infty \) be a subharmonic function with the Riesz measure \( \nu_u \), and \( u(0) \not\equiv -\infty \). Let \( M \in L^1_{\text{loc}}(\Omega) \) be a function that is bounded on a subdomain \( \Omega_1 \subseteq \Omega \) containing the origin.

If there are a subdomain \( \Omega_0 \subseteq \Omega_1 \) and constant \( C \) such that the inequality

\[
\int_\Omega u \, d\mu \leq \int_\Omega M \, d\mu + C \quad \text{or} \quad \int_\Omega V_\mu \, d\nu_u \leq \int_\Omega M \, d\mu + C \tag{2.19}
\]

holds for every \( \mu \in \text{AS}(\Omega_0) \cap M_{\text{ac}}(\Omega) \) (for every \( \mu \in \mathcal{J}(\Omega_0) \cap M_{\text{ac}}(\Omega) \) resp.) where \( V_\mu \) is the potential (2.4) of \( \mu \), then, for any function \( \sigma \in C(\Omega) \) satisfying (1.11), there exists a harmonic (subharmonic resp.) function \( h \) (\( h \not\equiv -\infty \) resp.) on \( \Omega \) such that (see (1.12))

\[
u_u \leq \sigma \tag{2.20}
\]

In addition, if \( M \in \text{SH}(\Omega) \), then the inequality (2.19) can be replace by

\[
\int_\Omega V \, d\nu_u \leq \int_\Omega V \, d\nu_M + C \tag{2.21}
\]

for every function \( V \in \mathcal{P}_{\text{AS}}(\Omega) \cap C(\Omega) \) (function \( V \in \mathcal{P}_{\mathcal{J}}(\Omega) \cap C(\Omega) \) resp.).

**Proof.** Here and later we need the following elementary

**Lemma 2.1.** Let \( \Omega_0 \subseteq \Omega_1 \) are subdomains of \( \Omega \), and \( 0 \in \Omega_0 \). Then there is a domain \( \Omega' \in \mathcal{U}(\Omega_1; \Omega_0) \) such that \( \Omega_0 \subseteq \Omega' \subseteq \Omega_1 \). In particular, the domain \( \Omega' \) is regular for the Dirichlet problem.
By Lemma 2.1 we can assume that \( \Omega_1 \) is regular for the Dirichlet problem, \( m(\partial \Omega_1) = 0 \), and \( M \) is bounded in an open neighbourhood of the closure \( \overline{\Omega}_1 \).

Then we can choose a sufficiently small constant \( \varepsilon > 0 \) and a constant \( c \geq 0 \) such that

\[
|M(z + w)| \leq c, \quad \forall z \in \Omega_1, \quad \forall w \in D(\varepsilon),
\]

\[
\bigcup_{z \in \Omega_1} D(z, \varepsilon) \subseteq \Omega, \quad \Omega_0 \cap \left( \bigcup_{z \in \partial \Omega_1} D(z, \varepsilon) \right) = \emptyset.
\]

We put \( F = M - u \), and \( H = SH(\Omega) \) or \( H = \text{Harm}(\Omega) \). According to generalized Poisson–Jensen formula (2.5) and Proposition 2.3, it follows from (2.19) that the inequality

\[
\int F \, d\mu > -C
\]

holds for every measure \( \mu \in M_{ac}(\Omega) \) such that \( \delta_0 \prec_H \mu \) and \( \Omega_0 \cap \text{supp} \mu = \emptyset \).

Let \( \mu \) be an arbitrary measure that belongs to \( M_{ac}(\Omega) \), and \( \delta_0 \prec_H \mu \). Denote by \( \mu_1 \) the restriction of \( \mu \) to \( \Omega_1 \). We have the representation

\[
\mu = \mu_1 + \mu_\infty, \quad \mu_1, \mu_\infty \in M_{ac}^+(\Omega).
\]

The measure \( \mu' := \mu_1^{\Omega_1} \ast m(\varepsilon) \in M_{ac}^+(\Omega) \) is the balayage of \( \mu_1 \) with respect to \( SH(\Omega) \) (see (2.10) and [26, Lemma 7.1]). Hence \( \delta_0 \prec_H (\mu' + \mu_\infty) \in M_{ac}^+(\Omega) \),

\[
\int u \, d\mu_1 \leq \int u \, d\mu',
\]

and, by constructions (2.9) and (2.22Ω), \( \Omega_0 \cap \text{supp}(\mu' + \mu_\infty) = \emptyset \). The inequality (2.23) holds for all such measures. Thus,

\[
\int F \, d(\mu' + \mu_\infty) \geq -C.
\]

Hence we get

\[
\int F \, d\mu \overset{(2.24)}{=} \int F \, d(\mu' + \mu_\infty) + \int F \, d\mu_1 - \int (M - u) \, d\mu'
\]

\[
\overset{(2.26)}{\geq} -C - \left( \int u \, d\mu_1 - \int u \, d\mu' \right) + \int_{\Omega_1} M \, d\mu_1 - \int_{\Omega_1} M \, d\mu'
\]

\[
\overset{(2.25)}{\geq} -C - \int_{\Omega_1} |M| \, d(\mu_1 + \mu') \overset{(2.22M),(2.3)}{\geq} -C - 2c.
\]

The last inequality implies the condition (2.17) of Theorem 2.1. It follows that there is a function \( h \in H \) such that (2.18) holds. Under our notation, (2.18) considers with (2.20).
For the case (2.21), by Proposition 2.3, we see that the map \( \mathcal{P} \) acts from \( \mathcal{A} \mathcal{S}^\Omega_0(\Omega) \cap \mathcal{M}_{\text{ac}}(\Omega) \) into \( \mathcal{P}_{\mathcal{A} \mathcal{S}}(\Omega) \cap C(\Omega) \) and from \( \mathcal{A} \mathcal{S}^\Omega_0(\Omega) \cap \mathcal{M}_{\text{ac}}(\Omega) \) into \( \mathcal{P}_{\mathcal{A} \mathcal{S}}(\Omega) \cap C(\Omega) \). Hence, (2.19) follows from (2.21) according to generalized Poisson-Jensen formula (2.5).

Remark. In Theorem 2.3 we may assume that the inequalities (2.19) are fulfilled only for all measures \( \mu \in \mathcal{A} \mathcal{S}^\Omega_0(\Omega) \cap \mathcal{M}_{\text{ac}}(\Omega) \) (only for all measures \( \mu \in \mathcal{J}^\Omega_0(\Omega) \cap \mathcal{M}_{\text{ac}}(\Omega) \) resp.) where

\[
\mathcal{A} \mathcal{S}^\Omega_0(\Omega) := \{ \mu \in \mathcal{A} \mathcal{S}(\Omega) : \Omega_0 \cap \text{supp} \mu = \emptyset \},
\]

\[
\mathcal{J}^\Omega_0(\Omega) := \{ \mu \in \mathcal{J}(\Omega) : \Omega_0 \cap \text{supp} \mu = \emptyset \}. \quad (2.27j)
\]

It follows from an arbitrary rule for selection of \( \Omega_0 \subset \Omega_1 \).

3 From Green functions to Jensen measures

The aim of this section is to prove that the conditions of statements (Z) and (S) of the Main Theorem give an united condition in terms of Jensen measures. For the statement (S) this condition is the following evident corollary of Proposition 2.6:

\[(J) \quad \text{There exists a constant } C \text{ such that}
\]

\[
\int_{\Omega} V_\mu \, d\nu_u \leq \int_{\Omega} M \, d\mu + C \quad (3.1)
\]

for all measures \( \mu \in \mathcal{J}^\Omega_0(\Omega) \).

In subsections 3.1 and 3.2 we show that the condition (J) also is fulfilled under the conditions of (Z).

3.1 A special approximation of Jensen measures

First we recall joint results of B. Cole and T. Ransford [32] about the approximation of Jensen measures by harmonic measures.

We write \( \text{conv} A \) (\( \overline{\text{conv}} A \) resp.) for the convex (closed convex resp.) hull of a set \( A \).

Let \( S \) be an open subset or a compact subset of \( \Omega \). The space \( C(S) \) is a Fréchet space with the topology of uniform convergence on compact subsets of \( S \). The dual space \( C(S)^* \) may be identified with the space of finite signed Borel measures on \( \Omega \) of compact support. We use only weak*-topology in the space \( C(S)^* \).
Theorem 3.1 ([31, Theorem 6.6]). \( \mathcal{J}(\Omega) = \overline{\text{conv}(\mathcal{H}(\Omega))} \).

Proposition 3.1 ([31, Proposition 2.1], [27, Proposition 1.1]). Let \( \Omega_1 \) be a subdomain of \( \Omega \), \( 0 \in \Omega_1 \). Then \( \mathcal{J}(\Omega_1) \subset \mathcal{J}(\Omega) \). If \( \mu \in \mathcal{J}(\Omega) \) and \( \text{supp} \mu \subset \Omega_1 \), and if each bounded component of \( \mathbb{C} \setminus \Omega_1 \) meets \( \mathbb{C} \setminus \Omega \), then \( \mu \in \mathcal{J}(\Omega_1) \).

Proposition 3.2. Let \( D \) be a subdomain of \( \Omega \), and \( 0 \in D \in \Omega \). Then there exists an increasing sequence of domains \( D_n \in \mathcal{U}_0^d(D) \) such that the sequence of harmonic measures \( \omega_{D_n}(0, \cdot) \) converges to \( \omega_D(0, \cdot) \) in \( C(\Omega)^* \).

Proof. See [34, Theorems 4.15, 5.14].

Proposition 3.3. Let \( \Omega_0 \) and \( \Omega_1 \) are subdomains of \( \Omega \) such that

\[ 0 \in \Omega_0 \in \Omega_1 \in \Omega. \tag{3.2} \]

Let \( K \) be a compact subset of \( \Omega \) satisfying \( K \subset \Omega_0 \). Then there are domains \( \Omega' \in \mathcal{U}_0^d(\Omega_1; \Omega_0) \) and \( \Omega'' \in \mathcal{U}_0^d(\Omega; \Omega_1) \) such that

\[ \Omega_0 \in \Omega' \in \Omega_1 \in \Omega'', \quad K \subset \Omega_1 \setminus \overline{\Omega'}, \tag{3.3} \]

and each bounded component of \( \mathbb{C} \setminus \Omega'' \) meets \( \mathbb{C} \setminus \Omega \).

The proof of Proposition 3.3 follows from Lemma 2.1.

Given domains \( \Omega' \in \Omega_2 \) containing 0, we write

\[
\mathcal{K}^{ud}(\Omega_2) := \{ \omega_D(0, \cdot) : D \in \mathcal{U}_0^d(\Omega_2) \},
\]

\[
\mathcal{K}^{ad}(\Omega_2; \Omega') := \{ \omega_D(0, \cdot) : D \in \mathcal{U}_0^d(\Omega_2; \Omega') \}.
\]

Proposition 3.4. Let \( \mu \) be a Jensen measure for \( 0 \) on \( \Omega \), and \( K := \text{supp} \mu \).

Under the notations and conditions of Proposition 3.3,

\[ \mu \in \overline{\text{conv}\mathcal{K}^{ad}(\Omega''; \Omega')} \tag{3.4} \]

where the closure is taken with respect to the weak*-topology on \( C(\Omega'')^* \).

Proof. By Proposition 3.3 (for \( \Omega'' \) instead of \( \Omega \) there exists one more domain \( \Omega_2 \in \mathcal{U}_0^d(\Omega''; \Omega_1) \) such that \( \Omega_1 \in \Omega_2 \in \Omega'' \), each bounded component of \( \mathbb{C} \setminus \Omega_2 \) meets \( \mathbb{C} \setminus \Omega'' \), and \( \text{supp} \mu \subset \Omega_2 \setminus \overline{\Omega'} \). By Proposition 3.1 (for \( \Omega_2 \) instead of \( \Omega_1 \)) and Theorem 3.1 (for \( \Omega_2 \) instead of \( \Omega \), the measure \( \mu \) belongs to \( \overline{\text{conv}\mathcal{H}(\Omega_2)} \) where the closure is taken with respect to weak*-topology of \( C(\Omega_2)^* \). According to Proposition 3.2 we have

\[ \overline{\text{conv}\mathcal{H}(\Omega_2)} = \overline{\text{conv}\mathcal{K}^{ad}(\Omega_2)} = \overline{\text{conv}\mathcal{K}^{ad}(\Omega_2)}. \]
Proposition 2.4 with $D$ and domains $\Gamma$ there are a finitely many numbers $c_k(\gamma) > 0$ and domains $D_k(\gamma) \in \mathcal{U}_0^d(\Omega_2)$ such that
\[ \sigma_\gamma = \sum_k c_k(\gamma)\omega_{D_k(\gamma)}, \quad \omega_{D_k(\gamma)} := \omega_{D_k(\gamma)}(0, \cdot), \quad \sum_k c_k(\gamma) = 1. \] (3.5)

There is a domain $\Omega_\gamma$ such that $\Omega' \in \Omega_\gamma \in \Omega_1$ and $K \subset \Omega_2 \setminus \overline{\Omega}_\gamma$. For every $k, \gamma$ we can choose a domain $\Omega_k(\gamma) \in \mathcal{U}_0^d(\Omega_\gamma; \Omega')$ such that the domain $D'_k(\gamma) := D_k(\gamma) \cup \Omega_k(\gamma) \in \mathcal{U}_0^d(\Omega_2; \Omega')$ is a regular domain, i.e., the complement $\mathbb{C} \setminus (D_k(\gamma) \cup \Omega_k(\gamma))$ has not of isolated points. Consider the restrictions
\[ \omega_{D_k(\gamma)} := \omega_{D_k(\gamma)}|_{\Omega_k(\gamma)}, \quad \sigma'_\gamma := \sum_k c_k(\gamma)\omega'_{D_k(\gamma)}. \] (3.6)

Since supp $\mu \subset \Omega_2 \setminus \overline{\Omega}_\gamma$, the net $\{\sigma'_\gamma\}$ converges to the null measure in $C(\Omega')^*$:
\[ \sum_k c_k(\gamma)\omega'_{D_k(\gamma)}(\overline{\Omega}) \to 0. \] (3.7)

Now we reconstruct the net $\{\sigma_\gamma\}$ with the help of balayage.
Denote by $(\omega'_{D_k(\gamma)})^{D'_k(\gamma)}$ the classical balayage of $\omega'_{D_k(\gamma)}$ relative to $D'_k(\gamma)$, and denote by $\omega''_{D_k(\gamma)}$ the classical balayage of $\omega_{D_k(\gamma)}$ relative to $D'_k(\gamma)$. By Proposition 2.4 with $D = D_k(\gamma)$ each measure $\omega''_{D_k(\gamma)}$ is exactly the harmonic measure $\omega_{D'_k(\gamma)}(0, \cdot)$. So, by definition (2.9) and equalities (2.11), we obtain
\[ (\omega'_{D_k(\gamma)})^{D'_k(\gamma)}(\Omega_2) = \omega_{D_k(\gamma)}(\Omega'), \quad \text{supp}(\omega'_{D_k(\gamma)})^{D'_k(\gamma)} \subset \overline{\Omega}_2. \] (3.8)
\[ \omega_{D'_k(\gamma)}(0, \cdot) = \omega''_{D_k(\gamma)} = (\omega'_{D_k(\gamma)})^{D'_k(\gamma)} + (\omega_{D_k(\gamma)} - \omega'_{D_k(\gamma)}). \] (3.8')

It follows from (3.7) and (3.8) that
\[ \sum_k c_k(\gamma)(\omega'_{D_k(\gamma)})^{D'_k(\gamma)}(\Omega_2) \to 0. \]

Hence, in view of (3.8'), (3.6), and (3.7), the nets
\[ \sigma''_\gamma = \sum_k c_k(\gamma)(\omega'_{D_k(\gamma)})^{D'_k(\gamma)}, \quad \sigma'_\gamma = \sum_k c_k(\gamma)\omega'_{D_k(\gamma)}, \quad \sigma'_\gamma = \sum_k c_k(\gamma)\omega'_{D_k(\gamma)} \]

converge to the null measure in $C(\Omega_2)^*$. It follows from the representations (3.8'') and (3.5) that the net
\[ \sigma^*_\gamma := \sum_k c_k(\gamma)\omega'_{D_k(\gamma)}(0, \cdot) = \sigma''_\gamma + \sigma'_\gamma - \sigma'_\gamma, \quad \Omega' \subset D'_k(\gamma) \in \Omega_2, \]

converges to $\mu$ in $C(\Omega_2)^*$, and all the more in $C(\Omega')^*$. So, (3.4) is proved. •
3.2 The property (J) for the statement (Z)

Suppose that a Jensen measure $\mu$ belongs to $\mathcal{J}(\Omega)$ (see definition (2.27j)) for the same subdomain $\Omega_0$ as in (Z). Put $K := \text{supp } \mu$. The function $M$ of Main Theorem is bounded on the subdomain $\Omega_1$ of Main Theorem satisfying (3.2). By Propositions 3.3 and 3.4, there are domains $\Omega', \Omega'' \in \mathcal{U}_d(\Omega_1; \Omega_0)$ such that (3.3) and (3.4) are fulfilled.

Put
\[ d' := \text{dist}(\Omega_0, \partial \Omega'), \quad d'' := \text{dist}(\Omega'', \partial \Omega). \tag{3.9} \]

We can choose a number $\varepsilon$ so that
\[ 0 < \varepsilon < \min\{d', d''\}. \tag{3.10} \]

Let $D \in \mathcal{U}_d(\Omega''; \Omega')$. According to (3.9) and (3.10), for any $w \in D(\varepsilon)$, the domain $D_w = \{ z - w : z \in D \}$ belongs to the class $\mathcal{U}_d(\Omega; \Omega_0)$. Therefore, by the main condition of (Z), there is a constant $C$ such that the inequality
\[ \int_{\Omega} g_{D_w}(\zeta, 0) \, d\nu_u(\zeta) \leq \int_{\Omega} g_{D_w}(\zeta, 0) \, d\nu_M(\zeta) + C \tag{3.11} \]
holds for each domain $D \in \mathcal{U}_d(\Omega''; \Omega')$. Here the constant $C$ is independent of $D$, $w$, and $\mu$.

By the generalized Poisson–Jensen formula (2.5), the inequality (3.11) can be rewritten as
\[ \int_{\Omega} u(z) \, d\omega_{D_w}(0, z) \leq \int_{\Omega} M(z) \, d\omega_{D_w}(0, z) + C' \tag{3.12} \]
where the constant $C' = C - u(0) - M(0)$ is independent of $D \in \mathcal{U}_d(\Omega''; \Omega')$, $w \in D(\varepsilon)$, $\mu$, and $\varepsilon$ under condition (3.10). If we replace $z - w$ by a new variable in (3.12), then
\[ \int_{\Omega} u(z + w) \, d\omega_D(0, z) \leq \int_{\Omega} M(z + w) \, d\omega_D(0, z) + C' \tag{3.13} \]
for all $D \in \mathcal{U}_d(\Omega''; \Omega')$ and $w \in D(\varepsilon)$ where the constant $C'$ is independent of $\mu$ and $\varepsilon$ of (3.10). Integrating the last inequality with respect to the probability measure $m^{(\varepsilon)}$ (see (1.9)), by Fubini’s theorem, we obtain
\[
\int_{\Omega} (u * m^{(\varepsilon)})(z) \, d\omega_D(0, z) = \int_{D(\varepsilon)} \int_{D(\varepsilon)} u(z + w) \, d\omega_D(0, z) \, dm^{(\varepsilon)}(w)
\leq \int_{D(\varepsilon)} \int_{\Omega} M(z + w) \, d\omega_D(0, z) \, dm^{(\varepsilon)}(w) + \int_{D(\varepsilon)} C' \, dm^{(\varepsilon)}(w)
= \int_{\Omega} (M * m^{(\varepsilon)})(z) \, d\omega_D(0, z) + C'.
\]
Hence, under notation (1.12),
\[ \int_{\Omega} u^{(\varepsilon)}(z) \, d\omega_D(0, z) \leq \int_{\Omega} M^{(\varepsilon)}(z) \, d\omega_D(0, z) + C' \]
for all domains \( D \in \mathcal{U}_d^b(\Omega''; \Omega') \) where the functions \( u^{(\varepsilon)} \) and \( M^{(\varepsilon)} \) are well defined and continuous on \( \Omega'' \). According to (3.4) of Proposition 3.4, it implies that
\[ \int_{\Omega} u^{(\varepsilon)}(z) \, d\mu(z) \leq \int_{\Omega} M^{(\varepsilon)}(z) \, d\mu(z) + C'. \]
In view of (1.10), we have
\[ \int_{\Omega} u \, d\mu \leq \int_{\Omega} M^{(\varepsilon)} \, d\mu + C'. \]
If \( \varepsilon \) tend to 0, then the decreasing net \( \{ M^{(\varepsilon)} \} \) of continuous functions tend pointwise to \( M \) on \( \Omega'' \) since \( M \) is subharmonic. Thus,
\[ \int_{\Omega} u \, d\mu \leq \int_{\Omega} M \, d\mu + C'. \quad (3.14) \]
But the measure \( \mu \in \mathcal{D}_0(\Omega) \) was any given, and the constant \( C' \) is independent of \( \mu \). Therefore, the inequality (3.14) is fulfilled for every measure \( \mu \in \mathcal{D}_0(\Omega) \). By the generalized Poisson–Jensen formula (2.5), the inequality (3.14) implies the inequality
\[ \int_{\Omega} V_\mu \, d\nu_a \leq \int_{\Omega} M \, d\mu + (C' - u(0)) \quad (3.15) \]
for every measure \( \mu \in \mathcal{D}_0(\Omega) \) where \( V_\mu \) is potential (2.4) of \( \mu \), and the constant \( C'' := C' - u(0) \) is independent of \( \mu \). But the inequality (3.15) is exactly (3.1) with constant \( C'' \) instead of \( C \). Besides, here we can replace the upper index of \( \mathcal{D}_0(\Omega) \) by \( \Omega_0 \). The last follows from an arbitrary rule for selection of \( \Omega_0 \in \Omega_1 \). The property (J) is proved. •

Remark. For the deduction of (J) from (S) or (Z) we did not use a some representation of \( M \) by a subharmonic kernel. In other words, the conditions ii) and iii) of Main Theorem are unnecessary in order to prove (J).

4 The proofs of the Main Theorem

4.1 From Jensen measures and functions to the existence of harmonic minorant

The aim of this subsection is to prove that the condition (J) at the beginning of Section 3 implies the existence of harmonic minorant for the function (1.16). After Section 3 it gives both (S) and (Z).
Suppose that the condition (J) is fulfilled.

By the generalized Poisson–Jensen formula (2.5) and Proposition 2.3, it follows from the condition (J) that, for the constant \( C' = C + M(0) \),

\[(J') \text{ the inequality} \]
\[
\int_{\Omega} V \, d\nu_u \leq \int_{\Omega} V \, d\nu_M + C' \quad (4.1)
\]

holds for every Jensen function \( V \in \mathcal{P}_{\Omega} \) (see above (2.8)).

Now let \( \mu \in \mathcal{A} \mathcal{S}^{\Omega_0}(\Omega) \cap \mathcal{M}_{ac}^+(\Omega) \) be a Arens–Singer measure with the potential \( V_\mu \) (see (2.4)). By Proposition 2.3 the Arens–Singer function \( V_\mu \) belongs to the class \( \mathcal{P}_{\Omega}^{\Omega_0} \). Besides, by Definition 3, the function

\[
V_\mu^+(\zeta) := \max\{V(\zeta), 0\}, \quad \zeta \in \mathbb{C} \setminus \{0\},
\]

is a Jensen function that belongs to the class \( \mathcal{P}_{\Omega}^{\Omega_0} \). Therefore the condition \((J')\) implies that the inequalities

\[
\int_{\Omega} V_\mu \, d\nu_u \leq \int_{\Omega} V_\mu^+ \, d\nu_u \leq \int_{\Omega} V_\mu^+ \, d\nu_M + C' \quad (4.3)
\]

are fulfilled for all Arens–Singer measures \( \mu \in \mathcal{A} \mathcal{S}^{\Omega_0}(\Omega) \).

We shall need to estimate above the last integral

\[
I_\mu := \int_{\Omega} V_\mu^+ \, d\nu_M = \int_B V_\mu^+ \, d\nu_M. \quad (4.4)
\]

According to the representation (2.6) of Proposition 2.1 for the potential \( V_\mu \) we have

\[
I_\mu = \int_B \left( \int_{\Omega} \left( k(\zeta, z) - k(\zeta, 0) \right) \, d\mu(z) \right)^+ \, d\nu_M(\zeta)
\]

\[
\leq \int_B \int_{\Omega} \left( k(\zeta, z) - k(\zeta, 0) \right)^+ \, d\mu(z) \, d\nu_M(\zeta)
\]

\[
= \int_B \int_{\Omega} \left( (k(\zeta, z) - k(\zeta, 0)) + (k(\zeta, 0) - k(\zeta, z))^+ \right) \, d\mu(z) \, d\nu_M(\zeta)
\]

Hence, by Fubini’s theorem,

\[
I_\mu \leq \int_{\Omega} \left( \int_B (k(\zeta, z) - k(\zeta, 0)) \, d\nu_M(\zeta) \right.
\]

\[
+ \int_B (k(\zeta, 0) - k(\zeta, z))^+ \, d\nu_M(\zeta) \bigg) \, d\mu(z)
\]

\[
\overset{(1.13)}{=} \int_{\Omega} \left( \int_B (k(\zeta, z) - k(\zeta, 0)) \, d\nu_M(\zeta) + Q_k^\nu_M(z) \right) \, d\mu(z). \quad (4.5)
\]
By condition ii) and representation (1.8) of Proposition 1.1 the function $M$ can be represented in the form

$$M(z) = \int_B k(\zeta, z) \, d\nu_M(\zeta) + H(z) = U^{\nu_M}_k(z) + H(z), \quad z \in \Omega,$$

where $H \in \text{Harm}(\Omega)$. From this it follows that

$$(M(z) - M(0)) - (H(z) - H(0)) = \int_B (k(\zeta, z) - k(\zeta, 0)) \, d\nu_M(\zeta). \quad (4.6)$$

Substituting the left-hand side of (4.6) in (4.5), we get

$$I^\text{iii)}_\mu \leq \int_{\Omega} \left( (M(z) - M(0)) - (H(z) - H(0)) + Q(z) \right) \, d\mu(z). \quad (4.7)$$

By Definition 2 of Arens–Singer (probability (2.3)) measures the right-hand side of (4.7) is equal to

$$\int_{\Omega} \left( (M(z) + Q(z)) - H(z) \right) \, d\mu(z) - M(0) + H(0)$$

$$\overset{(2.2)}{=} \int_{\Omega} (M(z) + Q(z)) \, d\mu(z) - M(0)$$

since $H \in \text{Harm}(\Omega)$. Now if we recall (4.3), (4.4), and (4.7), we get

$$\int_{\Omega} V_\mu \, d\nu_\alpha \leq \int_{\Omega} (M(z) + Q(z)) \, d\mu(z) + (C' - M(0))$$

where the constant $C'' := C' - M(0)$ is independent of $\mu \in \text{AS}^{\text{th}}(\Omega) \cap \text{M}_c^+(\Omega)$. It means that second inequality in (2.19) is fulfilled for every Arens–Singer measure $\mu \in \text{AS}^{\text{th}}(\Omega) \cap \text{M}_c^+(\Omega)$ with the function $M + Q \in L^1_{\text{loc}}(\Omega)$ instead of $M$, and with the constant $C''$ instead of $C$. Thus, by Theorem 2.3, for any function $\sigma \in C(\Omega)$ satisfying (1.11), there exists a function $h \in \text{Harm}(\Omega)$ such that

$$u(z) + h(z) \leq (M + Q)^{(\sigma)}(z) = (M^{(\sigma)} + Q^{(\sigma)}) (z), \quad \forall z \in \Omega.$$

In other words, the function $h$ is a harmonic minorant for the function (1.16).

This completes the proof of the Main Theorem. \ bullet

### 4.2 The proof of Corollary 1.1

In this subsection we use a some function $\sigma \in C(\Omega)$ satisfying (1.11).
The proof of \((Z_{sc})\). Let \(f_{\Lambda}\) be a holomorphic function with zero set \(\Lambda = \{\lambda_n\}\), which exists by Weierstrass’ theorem. Then the function \(u = \log|f_{\Lambda}|\) is subharmonic with the Riesz measure \(n_{\Lambda}\) where \(n_{\Lambda}\) is the counting measure of sequence \(\Lambda\) (see (0.1)). At that rate the inequality (1.17) is exactly (1.14) with the Riesz measure \(\nu_u = n_{\Lambda}\) of \(u\). By assertion (Z) of the Main Theorem there exists a function \(h \in \text{Harm}(\Omega)\) such that the inequality (1.15) holds. The domain \(\Omega\) is simply connected. Therefore there exists a holomorphic function \(g\) on \(\Omega\) such that
\[
\log|f_{\Lambda}(z)| + \Re g(z) = u(z) + h(z) \quad \leq \quad M^{(\sigma)}(z) + Q^{(\sigma)}(z), \quad \forall z \in \Omega. \quad (4.8)
\]
In other words the function \(f = f_{\Lambda}\exp g\) with Zero\(_f = \text{Zero}_{f_{\Lambda}} = \Lambda\) satisfies the inequality \(\log|f| \leq M^{(\sigma)} + Q^{(\sigma)}\) on \(\Omega\). This proves that \(\Lambda\) is a zero set for \(\text{Hol}(\Omega; M^{(\sigma)} + Q^{(\sigma)})\).

The proof of \((S_{sc})\). If \(\Lambda\) is zero subset for \(\text{Hol}(\Omega; M)\), then there exists a holomorphic function \(f \neq 0\) on \(\Omega\) such that \(f(\Lambda) = 0\) and \(f \in \text{Hol}(\Omega; M)\). Let \(f_{\Lambda}\) be a holomorphic function with zero set \(\Lambda\). Evidently, we have a representation \(f = f_{\Lambda}g\) where \(g \neq 0\) is a holomorphic function on \(\Omega\). The condition \(f \in \text{Hol}(\Omega; M)\) implies an estimate
\[
\log|f_{\Lambda}(z)| + \log|g(z)| \leq M(z) + C, \quad \forall z \in \Omega, \quad (4.9)
\]
where \(C\) is a constant. We can rewrite (4.9) as
\[
h(z) := \log|g(z)| - C \leq M(z) - \log|f_{\Lambda}(z)|, \quad \forall z \in \Omega.
\]
In particular, it means that the difference \(M - \log|f_{\Lambda}|\) admits a subharmonic minorant on \(\Omega\). Therefore, by statement (S) of the Main Theorem (with \(\log|f_{\Lambda}|\) instead of \(u\)), the function \(M^{(\sigma)} + Q^{(\sigma)} - \log|f_{\Lambda}|\) admits a harmonic minorant \(h\) on \(\Omega\). Since the domain \(\Omega\) is simply connected, there exists a holomorphic function \(g\) on \(\Omega\) such that \(h = \Re g\) on \(\Omega\). Thus we get (4.8). As before, this shows that \(\Lambda\) is a zero set for \(\text{Hol}(\Omega; M^{(\sigma)} + Q^{(\sigma)})\).

The proof of \((M_{sc})\). Let \(f = g/q\) be a meromorphic function on \(\Omega\). If \(g, q \in \text{Hol}(\Omega; M)\), then the subharmonic function \(u := \max\{\log|g|, \log|q|\}\) satisfies
\[
\max\{\log|g(z)|, \log|q(z)|\} = u(z) \leq M(z) + C, \quad \forall z \in \Omega, \quad (4.10)
\]
where \(C\) is a constant. It is clear that there is a representation
\[
f = \frac{g_1}{q_1} = \frac{g_1 l}{q_1 l}, \quad g = g_1 l, \quad q = q_1 l, \quad l \in \text{Hol}(\Omega), \quad l \neq 0, \quad (4.11)
\]
where \( g_1, q_1 \) are holomorphic functions without common zeros on \( \Omega \). Hence, in view of (4.10), we have

\[
\max\{\log|g_1(z)|, \log|q_1(z)|\} + \log|l(z)| = u(z) \leq M(z) + C
\]

for all \( z \in \Omega \). Put

\[
u_1 := \max\{\log|g_1|, \log|q_1|\}.
\]

Under this notation we can rewrite (4.12) as \( \log|l| - C \leq M - u_1 \) on \( \Omega \). It means that the difference \( M - u_1 \) admits the subharmonic minorant \( \log|l| - C \).

Therefore, by assertion (S) of the Main Theorem, the function \( M(\sigma) + Q(\sigma) - u_1 \) admits a harmonic minorant. In other words there exists a holomorphic function \( s \) such that \( u_1 + \Re s \leq M(\sigma) + Q(\sigma) \). Thus according to (4.13),

\[
\max\{\log|g_1 e^s|, \log|q_1 e^s|\} \leq M(\sigma) + Q(\sigma).
\]

Hence if we put \( g_0 = g_1 e^s \) and \( q_0 = q_1 e^s \), then in view of (4.11) \( f = g_0/q_0 \), \( g_0, q_0 \in \text{Hol}(\Omega; M(\sigma) + Q(\sigma)) \), and \( \text{Zero}_{g_0} \cap \text{Zero}_{q_0} = \emptyset \) as desired.

\section{The proof of Theorem 1.1}

\subsection{The reduction to an upper estimate of function (1.13)}

Since \( M \) is bounded below at the origin, we have

\[
\int_0^{1/2} \frac{\nu_M^\text{rad}(t)}{t} \, dt < +\infty, \quad \nu_M(\{0\}) = 0.
\]

It and the condition (1.20) imply that the subharmonic Bomash’s kernel (1.6) from the item (B) of Section 1 is suitable for the Riesz measure \( \nu_M \) of \( M \) (see (1.7)). We use the subharmonic Bomash’s kernel \( \overline{\mathcal{b}}_2(\zeta, z) \) in the role of subharmonic kernel \( k \) supported by \( \mathbb{D} \setminus \{0\} \). In this case three conditions i)–iii) of the Main Theorem are fulfilled. It follows from Corollary 1.1 that all statements \((Z_D), (S_D), \) and \((M_D)\) of Theorem 1.1 are fulfilled for the space \( \text{Hol}(\mathbb{D}; M(\sigma) + Q(\sigma)) \) where

\[
Q(z) := \int_\mathbb{D} (\overline{\mathcal{b}}_2(\zeta, 0) - \overline{\mathcal{b}}_2(\zeta, z))^+ \, d\nu_M(\zeta), \quad z \in \Omega.
\]

By construction this function \( Q \) belongs to \( L^1_{\text{loc}}(\mathbb{D}) \) since the function \( M \) is bounded below on every \( D \in \mathbb{D} \).
Lemma 5.1. There is an absolute constant $a$ such that, for any $0 < \varepsilon < 1$,

$$Q(z) \leq A_{M,\varepsilon}(z) - M(z) + C_\varepsilon \cdot b_M^{[\alpha]}(z), \quad a \leq |z| < 1,$$  \hfill (5.2)

where the functions $A_{M,\varepsilon}$ and $b_M^{[\alpha]}$ is defined by (1.19) and (1.21), $\alpha' > 0$ is an absolute constant, and $C_\varepsilon$ depends only on $\varepsilon$.

We shall prove the important upper estimate (5.2) later by a few steps. Now we choose a function $\sigma \in C(\mathbb{D})$ satisfying (1.11) so that

$$\sigma(z) \leq \varepsilon (1 - |z|), \quad (A_{M,\varepsilon})^{(\sigma)}(z) \leq A_{M,\varepsilon}(z) + 1, \quad \forall z \in \mathbb{D}. \hfill (5.3)$$

It is possible since the function $A_{M,\varepsilon}$ is continuous on $\mathbb{D}$ if $M$ is subharmonic. It follows from (5.3) and (5.2) that

$$M^{(\sigma)}(z) + Q^{(\sigma)}(z) \leq A_{M,\varepsilon}(z) + (A_{M,\varepsilon}(z) - M(z) + C_\varepsilon \cdot b_M^{[\alpha]}(z))^{(\sigma)}$$

$$\leq A_{M,\varepsilon}(z) + (A_{M,\varepsilon})^{(\sigma)}(z) - M^{(\sigma)}(z) + C_\varepsilon \cdot (b_M^{[\alpha']})^{(\sigma)}(z)$$

$$\leq 2A_{M,\varepsilon}(z) - M(z) + C_\varepsilon \cdot b_M^{[\alpha' + 1]}(z) + 1, \quad a \leq |z| < 1.$$

Besides, the left-hand side is bounded below in $D(a)$. Thus the statements $(Z_D), (S_D), (M_D)$ hold for the space (1.22) with absolute constant $\alpha = \alpha' + 1$. \hfill ●

### 5.2 The proof of Lemma 5.1

It is enough to estimate of integral (5.1) in the case $z = x \in [0, 1)$. First we investigate sizes of sets (see Examples, (1.3) and (B), (B$_s$) and (1.6))

$$D_1(x) := \{ \varsigma \in \mathbb{D}: \overline{t}_1(\varsigma, 0) > \overline{t}_1(\varsigma, x) \} = \{ \varsigma \in \mathbb{D}: \rho(\varsigma, x) < |\varsigma| \}, \hfill (5.4.1)$$

$$D_2(x) := \{ \varsigma \in \mathbb{D}: \overline{t}_2(\varsigma, 0) > \overline{t}_2(\varsigma, x) \} \overset{(1.6b)}{=} \{ \varsigma \in \mathbb{D}: |\varsigma|^2 (2 - |\varsigma|^2) > |\overline{B}_\varsigma(x)|^2 - 2|\overline{B}_\varsigma(x)| \}. \hfill (5.4.2)$$

The domain $D_2(x)$ determines the function $Q$ from (5.1) since

$$Q(x) = \int_{D_2(x)} (\overline{t}_2(\varsigma, 0) - \overline{t}_2(\varsigma, x)) \, d\nu_M(\varsigma), \quad x \in [0, 1). \hfill (5.5)$$

Besides, the function $t \rightarrow t^2 (2 - t^2)$ is strictly increasing on $[0, 1)$, and $|\overline{B}_\varsigma(x)| < 1$ for $\varsigma \in \mathbb{D}$. Therefore, if $\varsigma \in D_2(x)$, then, by (5.4.2), we have $|\varsigma|^2 > |\overline{B}_\varsigma(x)|$, i.e., $|\varsigma| > \rho(\varsigma, x)$. Thus,

$$D_2(x) \subset D_1(x), \quad \forall x \in [0, 1). \hfill (5.6)$$
The domain $D_1(x)$. We put
\[ \zeta := te^{i\theta}, \quad 0 \leq t < 1, \quad \theta \in (-\pi, \pi]. \] (5.7)

If $\rho(\zeta, x) \leq |\zeta|$, then
\[ \left| \frac{te^{i\theta} - x}{1 - txe^{-i\theta}} \right|^2 = \frac{t^2 - 2tx \cos \theta + x^2}{1 - 2tx \cos \theta + t^2x^2} < t^2. \]

The last inequality is equivalent to
\[ \cos \theta > \frac{x(1 + t^2)}{2t}. \] (5.8)

Hence $\cos \theta > x$ and
\[ |\theta| < \frac{\pi}{2} \sqrt{1 - x^2}. \] (5.9)

Besides, inequality (5.8) implies $x(1 + t^2) < 2t$. Hence, for $t = |\zeta|$ of (5.7), we obtain
\[ 1 > t > \frac{1 - \sqrt{1 - x^2}}{x} \geq x - \sqrt{1 - x^2}, \quad 0 \leq x < 1. \] (5.10)

The inequalities (5.9) and (5.10) give the inclusion
\[ D_1(x) \subset \{ \zeta = te^{i\theta} : x - \sqrt{1 - x^2} < t < 1, \ |\theta| < \frac{\pi}{2} \sqrt{1 - x^2} \}. \] (5.11)

But this inclusion is insufficient for good estimates of integral (5.5) by (5.6). It is important farther what, according to (5.6) and (5.11), the set $D_2(x)$ lies in the right half-plane for $x > 0$.

The domain $D_2(x)$. Let $\zeta \in D_2(x)$ the same as in (5.7).

**Proposition 5.1.** Under the condition $9/10 \leq x < 1$ the Carleson box of relative size 6 with center at $x$ (see (1.18))
\[ \Box_6(x) = \{ \zeta = te^{i\theta} : x - 6(1 - x) < t < 1, \ |\theta| < 6(1 - x) \} \]
(5.12)
includes the domain $D_2(x)$.

**Proof.** In view of (1.6b), we have
\[ \log(|\zeta|(2 - |\zeta|^2)) > \log \frac{|\zeta - x||2 - |\zeta|^2 - \bar{x}|}{|1 - \bar{x}x|^2} \]
In particular, $t > \zeta = e^{i\theta} \in D_{2}(x)$. Hence, under the notation (5.7),
\[
t^2(2 - t^2)^2 > \frac{(t^2 - 2tx \cos \theta + x^2)((2 - t^2)^2 - 2(2 - t^2)tx \cos \theta + t^2x^2)}{(1 - 2tx \cos \theta + t^2x^2)^2}
\]
\[
= \frac{(t - x)^2 + 4tx \sin^2(\theta/2)((2 - t^2 - tx)^2 + 4(2 - t^2)tx \sin^2(\theta/2))}{((1 - tx)^2 + 4tx \sin^2(\theta/2))^2}.
\] (5.13)

In particular, $t > 0$. For convenience, we put
\[
s := \sin^2 \frac{\theta}{2}.
\] (5.14)

Besides, $(2 - t^2) \geq t^2(2 - t^2)^2$ for $t \in [0, 1)$. Therefore, the inequality (5.13) implies
\[
2 - t^2 > \frac{(t - x)^2 + 4txs}{((1 - tx)^2 + 4txs)^2},
\]
whence a direct calculation gives
\[
s \cdot q(t, x) := s \cdot 4tx((2 - t^2 - tx)^2 + (t - x)^2(2 - t^2) - 2(1 - tx)^2(2 - t^2))
\]
\[
< (1 - tx)^2(2 - t^2) - (1 - tx)^2(2 - t^2)^2 =: p(t, x),
\] (5.15)

where the polynomials $q$ and $p$ are defined by first and second equalities respectively. We have the following factorial expansion for co-factor $q$ after $s$:
\[
q(t, x) = 8x^3t(1 - t^2)^2 > 0, \quad t \in (0, 1), \quad x \in (0, 1).
\]

It permits to get an upper estimate for $s$:
\[
s < \frac{p(t, x)}{q(t, x)} = \frac{(1 - tx)^2(2 - t^2) - (t - x)^2(2 - t^2 - tx)^2}{8tx^3(1 - t^2)^2}
\]
\[
= \frac{(1 - t)^2(1 + t^2)(-t^2 - x^4t^2 + 4x^3t + 2 - 4x^2)}{8x^3t(1 - t^2)^2}
\]
\[
= \frac{1}{8x^3t}(-t^2 - x^4t^2 + 4x^3t + 2 - 4x^2)
\]
\[
= \frac{1}{8x^3t}(-(1 + x^4)t^2 + (4x^3)t + 2(1 - 2x^2)) =: \frac{1}{8x^3t}g_x(t),
\] (5.16)

where the quadratic trinomial $g_x$ is defined by last equality. By definition (5.14), $s \geq 0$. Therefore the quadratic trinomial $g_x(t)$ is strictly positive for $\zeta = e^{i\theta} \in D_{2}(x)$, whence
\[
t > \frac{2x^3 - \sqrt{2}(1 - x^2)}{1 + x^4} = x - \left(x - \frac{2x^3 - \sqrt{2}(1 - x^2)}{1 + x^4}\right)
\]
\[
= x - (1 - x^2)\frac{\sqrt{2} + x(1 - x^2)}{1 + x^4} \geq x - 2(1 - x^2) \geq x - 4(1 - x).\] (5.17)
In particular,

\[ \text{if } x \geq 9/10, \text{ then } t > 1/2. \quad (5.18) \]

Now we must find an upper estimate for \( s \). Let us consider the inequality (5.16) again. The quadratic trinomial \( g_x \) attains its maximum at the point \( t_x = 2x^3/(1 + x^4) \) so that

\[
g_x(t_x) = -(1 + x^4) \left( \frac{2x^3}{1 + x^4} \right)^2 + 4x^3 \frac{2x^3}{1 + x^4} + 2(1 - 2x^2)
\]

\[
= \frac{2 - 4x^2 + 2x^4}{1 + x^4} = \frac{2(1 - x^2)^2}{1 + x^4} \leq 2(1 - x)^2 \max_{0 \leq x < 1} \frac{(1 + x)^2}{1 + x^4} \leq 8(1 - x)^2
\]

for each \( x \in [0, 1) \). Hence, in view of (5.14) and (5.16), we get

\[ \sin^2 \frac{\theta}{2} = s < \frac{1}{8x^3t} \cdot 8(1 - x)^2 = \frac{1}{x^3t} \cdot (1 - x)^2 \]

whence, for \( x \geq 9/10 \), by (5.18) we obtain

\[ |\theta| \leq \pi \sqrt{\frac{10^3}{9.3 \cdot (1/2)}} (1 - x) < 6(1 - x). \quad (5.19) \]

Thus, taking into account (5.17), we have the inclusion

\[ D_2(x) \subset \{ \zeta = te^{i\theta}: x - 4(1 - x) < t < 1, |\theta| < 6(1 - x) \} \quad (5.20) \]

for all \( 9/10 \leq x < 1 \). This completes the proof of Proposition 5.1.

The estimates of kernel of integral (5.5). Let \( 0 < \varepsilon < 1 \). We put

\[ \Delta_\varepsilon(x) := D(x, \varepsilon(1 - x)). \quad (5.21) \]

Denote by

\[ g_{\Delta_\varepsilon(x)}(\zeta, x) := \log \left| \frac{\varepsilon(1 - x)}{\zeta - x} \right| \quad (5.22) \]

the Green’s function for the disk \( \Delta_\varepsilon(x) \) with the pole \( x \).

**Proposition 5.2.** If \( 9/10 \leq x < 1 \), then the inequality

\[ \overline{b}_2(\zeta, 0) - \overline{b}_2(\zeta, x) \leq g_{\Delta_\varepsilon(x)}(\zeta, x) + \log \frac{30}{\varepsilon} \quad (5.23) \]

holds for all \( \zeta \in \Delta_\varepsilon(x) \). Besides,

\[ \frac{(1 - |\zeta|)^2}{(1 - x)^2} \geq 1 - \varepsilon, \quad \forall \zeta \in \Delta_\varepsilon(x). \quad (5.24) \]

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Proof. By (1.6b) and (5.22) we obtain the following representation

\[ \bar{b}_2(\zeta, 0) - \bar{b}_2(\zeta, x) = g_{\Delta_\varepsilon(x)}(\zeta, x) + \log \frac{|\zeta|(2 - |\zeta|^2)|1 - \bar{\zeta}x|}{\varepsilon(1 - x)|2 - |\zeta|^2 - \bar{\zeta}x|}. \] (5.25)

Given \( \zeta \in \mathbb{D} \) and \( 0 \leq x < 1 \), we have \( |\zeta|(2 - |\zeta|^2) \leq 2 \),

\[ |2 - |\zeta|^2 - \bar{\zeta}x| = |2 - \bar{\zeta}(\zeta + x)| \geq 2 - |\bar{\zeta}|(|\zeta| + x) \geq 1 - x. \] (5.26)

Finally, for \( 0 < \varepsilon < 1 \), \( 9/10 \leq x < 1 \), under notation (5.7), we obtain

\[ |1 - \bar{\zeta}x|^2 = (1 - tx)^2 + 4tx \sin^2 \frac{\theta}{2} \leq (1 - (x - \varepsilon(1 - x))x)^2 + 4tx \left( \frac{\varepsilon(1 - x)}{x} \right)^2 \]
\[ \leq (1 - x)^2((1 + x + \varepsilon x)^2 + 4\varepsilon^2/x) \leq (1 - x)^2(9 + 4/x) \leq 15(1 - x)^2. \]

These three estimates along with (5.25) give right away (5.23).

The lower estimate (5.24) is trivial.

**Proposition 5.3.** If \( x \notin \Delta_\varepsilon(x) \), then

\[ (\bar{b}_2(\zeta, 0) - \bar{b}_2(\zeta, x))^+ \leq \frac{12}{\varepsilon} \frac{(1 - |\zeta|)^2}{(1 - x)^2}. \] (5.27)

**Proof.** It is follows from the representation (1.6a) that

\[ (\bar{b}_2(\zeta, 0) - \bar{b}_2(\zeta, x))^+ = \log^+ \left| \frac{1 - (1 - |\zeta|^2)^2}{1 - \left( \frac{1 - |\zeta|^2}{1 - \bar{\zeta}x} \right)^2} \right| \]
\[ = \log^+ \left| 1 + \left| \frac{1 - |\zeta|^2}{1 - \frac{1 - |\zeta|^2}{1 - \bar{\zeta}x}} \right|^2 \right| \]
\[ = (1 - |\zeta|^2)^2 \frac{|1 - (1 - \bar{\zeta}x)^2|}{|1 - (1 - \bar{\zeta}x)^2|} = (1 - |\zeta|)^2 \frac{(1 + |\zeta|^2)2|x - \bar{\zeta}x|}{|\zeta - x||2 - |\zeta|^2 - \bar{\zeta}x|}. \]

Hence, if we use (5.26), and the condition \( |\zeta - x| \geq \varepsilon(1 - x) \), then we obtain

\[ (\bar{b}_2(\zeta, 0) - \bar{b}_2(\zeta, x))^+ \leq (1 - |\zeta|)^2 \frac{2^2 \cdot 3}{\varepsilon(1 - x)(1 - x)}, \]

and the estimate (5.27) is proved.
The upper estimate of (5.5). It is follows from the inclusion (5.12) of Proposition 5.1 that

$$Q(x) \leq \int_{\partial_6(x)} (b_2(\zeta, 0) - \overline{b}_2(\zeta, x))^+ d\nu_M(\zeta)$$

$$(5.21)$$

$$= \left( \int_{\Delta_x(x)} + \int_{\partial_6(x) \setminus \Delta_x(x)} \right) \left( b_2(\zeta, 0) - \overline{b}_2(\zeta, x) \right)^+ d\nu_M(\zeta) =: I_\varepsilon(x) + J(x)$$

for all $9/10 < x < 1$, where the integrals $I_\varepsilon(x)$ and $J(x)$ are defined by the last equality. For $I_\varepsilon(x)$, by estimate (5.23) of Proposition 5.2, we obtain the inequality

$$I_\varepsilon(x) := \int_{\Delta_x(x)} \left( b_2(\zeta, 0) - \overline{b}_2(\zeta, x) \right)^+ d\nu_M(\zeta)$$

$$\leq \int_{\Delta_x(x)} g_{\Delta_x(x)}(\zeta, x) d\nu_M(\zeta) + \int_{\Delta_x(x)} \log \frac{30}{\varepsilon} d\nu_M(\zeta)$$

$$(5.24)$$

$$\leq \int_{\Delta_x(x)} g_{\Delta_x(x)}(\zeta, x) d\nu_M(\zeta) + \frac{1}{1 - \varepsilon} \log \frac{30}{\varepsilon} \int_{\Delta_x(x)} \frac{(1 - |\zeta|)^2}{(1 - x)^2} d\nu_M(\zeta)$$

for all $9/10 < x < 1$. But by the Poisson–Jensen formula for the disk $\Delta_x(x)$ we have

$$\int_{\Delta_x(x)} g_{\Delta_x(x)}(\zeta, x) d\nu_M(\zeta) = A_{M,\varepsilon}(x) - M(x).$$

Therefore,

$$I_\varepsilon(x) \leq A_{M,\varepsilon}(x) - M(x) + C'_\varepsilon \cdot \frac{1}{(1 - x)^2} \int_{\Delta_x(x)} (1 - |\zeta|)^2 d\nu_M(\zeta) \quad (5.28)$$

for all $9/10 < x < 1$ where the constant $\frac{1}{1 - \varepsilon} \log \frac{30}{\varepsilon} =: C'_\varepsilon$ depends only on $\varepsilon$.

For the integral $J(x)$, by estimate (5.27) of Proposition 5.3, we have

$$J(x) := \int_{\partial_6(x) \setminus \Delta_x(x)} \left( b_2(\zeta, 0) - \overline{b}_2(\zeta, x) \right)^+ d\nu_M(\zeta)$$

$$\leq \frac{12}{\varepsilon} \int_{\partial_6(x) \setminus \Delta_x(x)} \frac{(1 - |\zeta|)^2}{(1 - x)^2} d\nu_M(\zeta).$$

$$= C''_\varepsilon \cdot \frac{1}{(1 - x)^2} \int_{\partial_6(x) \setminus \Delta_x(x)} (1 - |\zeta|)^2 d\nu_M(\zeta) \quad (5.29)$$

where the constant $12/\varepsilon =: C''_\varepsilon$ also depends only on $\varepsilon$. The addition of (5.28) and (5.29) gives

$$I_\varepsilon(x) + J(x) \leq A_{M,\varepsilon}(x) - M(x) + C'_\varepsilon b^{[6]}_{M}(x) \quad (5.30)$$

for all $9/10 < x < 1$ where the constant $C := \max\{C'_\varepsilon, C''_\varepsilon\}$ depends only on $\varepsilon$. Thus Lemma 5.1 is proved with the constants $a = 9/10$ and $\alpha' = 6$. •

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Remark. Since $\alpha = \alpha' + 1$ in subsection 5.1, it follows that we can put $\alpha = 7$ in the definition of space (1.22) in Theorem 1.1.

6 The proof of Theorem 0.1

We prove Theorem 0.1 as a corollary of Theorem 1.1.

By condition of Theorem 0.1 the function $M \mid_{[0,1]}$ is the increasing positive convex function of log on $(0, 1)$. Therefore the function $M(z) = M(|z|)$ is continuous subharmonic function on $\mathbb{D}$, and there exists the positive left-hand derivative $M'_-(t)$, $t \in (0, 1)$. Besides, the function $tM'_-(t)$ is increasing on $(0, 1)$. In particular,

$$\int_1^1 (1- t) M(t) \, dt = \int_1^1 (1- t) M'_-(t) \, dt \quad (6.1i)$$
\[
\geq r M'_-(r) \int_r^1 \frac{1-t}{t} \, dt \geq r M'_-(r) \frac{1}{r} \frac{1}{2} (1-r)^2 = \frac{1}{2} M'_-(r)(1-r)^2 \quad (6.1r)
\]

whence, according to the condition (0.4), we obtain

$$\lim_{r \to 1^-} M'_-(r)(1-r)^2 = 0. \quad (6.2)$$

An easy calculation of Laplacian of $M$ gives the expression

$$d\nu_M(z) = \frac{1}{2\pi} \, d\theta \otimes d(tM'_-(t)), \quad z := te^{i\theta}, \quad 0 \leq t < 1, \quad (6.3)$$

for the density of Riesz measure $\nu_M$ of $M \in SH(\mathbb{D})$ (in the sense of distribution theory).

Suppose that $\alpha$ is the absolute constant for the space (1.22) in Theorem 1.1. In our case we have $(z = re^{i\theta})$

$$b_M^{[\alpha]}(z) \quad (1.21) \quad b_M^{[\alpha]}(r) \quad (1.18), (6.3) \quad \frac{1}{(1-r)^2} \frac{1}{2\pi} \int_{-\alpha(1-r)}^\alpha (1-t)^2 \, d(tM'_-(t)) \, d\theta$$
\[
= \frac{\alpha}{\pi} \frac{1}{1-r} \int_{r-\alpha(1-r)}^1 (1-t)^2 \, d(tM'_-(t)).
\]

Hence integration by parts gives (taking into account (6.2) and the positivity
of the function $tM'_+(t)$)

$$b_{M}^{[\alpha]}(z) \leq \frac{\alpha}{\pi} \frac{1}{1-r} 2 \int_{r-\alpha(1-r)}^{1-} tM'_+(t)(1-t) \, dt$$

$$\frac{2\alpha}{\pi} \frac{1}{1-r} \int_{r-\alpha(1-r)}^{1-} M'_+(t)(1-t) \, dt \overset{(6.11)}{=} \frac{2\alpha}{\pi} \frac{1}{1-r} \int_{r-\alpha(1-r)}^{1-} (1-t) \, dM(t)$$

$$\overset{(0.4)}{\leq} \frac{2\alpha}{\pi} \frac{1}{1-r} \cdot C(1-(r-\alpha(1-r)) \leq \frac{2\alpha}{\pi} C_M(1+\alpha),$$

where a constant $C_M$ is dictated by condition (0.4). In particular, the condition (1.20) is fulfilled. Thus, under the conditions of Theorem 0.1 the space $\text{Hol}(\mathbb{D}; 2A_{M,\varepsilon} - M + C_{\varepsilon} \cdot b_{M}^{[\alpha]}$ from (1.22) coincides with the space $\text{Hol}(\mathbb{D}, 2A_{M,\varepsilon} - M)$ for every $0 < \varepsilon < 1$.

Let us fix a number $\varepsilon \in (0, 1)$. The function $M$ is increasing and radial. Therefore,

$$A_{M,\varepsilon}(z) \leq M(r + \varepsilon(1-r)), \quad r = |z|. \quad (6.4)$$

By the mean value theorem there exists a point $r'$ such that $r \leq r' \leq r + \varepsilon(1-r)$ and

$$M(r + \varepsilon(1-r)) - M(r) \leq M'_-(r')(1-r') \cdot \varepsilon(1-r) = M'_-(r')(1-r') \cdot \frac{1-r}{1-r'}$$

$$\leq \frac{\varepsilon}{1-\varepsilon} M'_-(r')(1-r') \overset{(6.1)}{\leq} \frac{\varepsilon}{1-\varepsilon} \int_{r'}^{1-} (1-t) \, dM(t) \overset{(0.4)}{\leq} \frac{2\varepsilon}{1-\varepsilon} \cdot C_M,$$

where $C_M$ is a constant which exists according to the condition (0.4). Hence, in view of (6.4), we obtain

$$2A_{M,\varepsilon}(z) - M(z) \leq 2M(r) + 2 \frac{2\varepsilon}{1-\varepsilon} \cdot C_M - M(r) = M(r) + C, \quad r = |z|,$$

where $C = 4C_M\varepsilon/(1-\varepsilon)$ is a constant. It means that the space $\text{Hol}(\mathbb{D}, 2A_{M,\varepsilon} - M)$ coincides with the space $\text{Hol}(\mathbb{D}, M)$. Hence Theorem 0.1 is a special case of Theorem 1.1. •

7 The case of the uniform Bergman spaces

For $V \in \mathcal{P}_{AS}(\mathbb{D})$, we put

$$\tilde{\kappa}(V) := \int_{0}^{1-} \left( \frac{1}{2\pi} \int_{0}^{2\pi} V(te^{i\theta}) \, d\theta \right) \frac{dt}{(1-t)^2}. \quad (7.1)$$
7.1 Weak analogs of Korenblum–Seip’s conditions

Theorem 7.1. Let \( \Lambda = \{ \lambda_k \}, \ k = 1, 2, \ldots, \ 0 \notin \Lambda, \) be a sequence of points in \( \mathbb{D}, \) and \( 0 \leq p < +\infty. \) The following four statements are equivalent:

[A] the sequence \( \Lambda \) is a zero set for the space \( A^{-p}; \)

[G] there exists a constant \( a < 1 \) such that

\[
\sup_{D \in \mathcal{U}_D(\mathbb{D}, D(\alpha))} \left( \sum_k g_D(\lambda_k, 0) - p \hat{\kappa}(g_D(\cdot, 0)) \right) < \infty;
\]

[J] \( \sup_{V \in \mathcal{P}_J(\mathbb{D})} \left( \sum_k V(\lambda_k) - p \hat{\kappa}(V) \right) < \infty; \)

[AS] \( \sup_{V \in \mathcal{P}_{AS}(\mathbb{D})} \left( \sum_k V(\lambda_k) - p \hat{\kappa}(V) \right) < \infty. \)

Evidently, the equivalence \( [\Lambda] \iff [G] \) coincides with the equivalence (i) \( \iff \) (ii) of Theorem 0.2.

Proof. Here we put

\[ M_p(z) := p \log \frac{1}{1 - |z|}, \quad z \in \mathbb{D}. \]  

(7.2)

This function is radial and positive. Besides, the restriction \( M \Big|_{[0,1)} \) is the increasing continuous convex function of log on \( (0,1), \) and

\[ dM_p(t) = \frac{1}{1 - t} \, dt, \quad d(tM'_p(t)) = \frac{1}{(1 - t)^2} \, dt. \]  

(7.3)

In particular, condition (0.4) holds, and \( M_p \) is subharmonic on \( \mathbb{D} \) with the Riesz measure

\[ d\nu_{M_p}(z) = \frac{p}{2\pi} \, d\theta \otimes \frac{dt}{(1 - t)^2}, \quad z = te^{i\theta} \in \mathbb{D}, \ 0 \leq r < 1. \]  

(7.4)

By definition (0.3) we have \( \text{Hol}(\mathbb{D}, M_p) = A^{-p}. \) In this case condition [G] coincides with condition (0.5) of Theorem 0.1 with a constant \( C \) independent of \( D. \) By item (Z_r) of Theorem 0.1 condition [G] implies [\Lambda] and vice versa.

The evident inclusions of the class of all extended Green functions \( g_D(\cdot, 0), \) \( D \in \mathbb{D}, \) into \( \mathcal{P}_J(\mathbb{D}) \subset \mathcal{P}_{AS}(\mathbb{D}) \) give the implications [AS] \( \implies \) [J] \( \implies \) [G].

For the proof of implication [\Lambda] \( \implies \) [AS] we use Proposition 2.6.

Let \( f_\Lambda \in A^{-p} \) be a function with zero set \( \Lambda. \) It means that the function \( \equiv 0 \) is harmonic minorant for the difference \( M_p - \log |f_\Lambda|. \) Therefore, by Proposition 2.6, the inequalities (2.16) hold for all for all Arens–Singer functions \( V \) on \( \mathbb{D} \) with \( u = \log |f_\Lambda|, \) \( \nu_u = n_\Lambda, \) and \( \nu_M = \nu_{M_p}. \) In view of (7.4), the system of inequalities (2.16) is equivalent to [AS].
7.2 Variants of Luecking’s condition

Theorem 7.2. Let $\Lambda = \{\lambda_k\}$, $k = 1, 2, \ldots$, $0 \notin \Lambda$, be a sequence of points in $\mathbb{D}$ and $0 \leq p < +\infty$. The following four statements are equivalent:

[A] The sequence $\Lambda$ is a zero set for the space $A^{-p}$;

[GL] there exists a constant $a < 1$ such that

$$\sup_{D \in \mathcal{U}_D^0(\mathbb{D}; D(a))} \left( \sum_k (1 - |\lambda_k|^2)^2 \left( \frac{1}{\pi} \int_{\mathbb{D}} \frac{g_D(\zeta, 0) \, d\mu(\zeta)}{|1 - \lambda_k \zeta|^4} \right) - p \hat{\kappa}(g_D(\cdot, 0)) \right) < \infty;$$

[JL] $\sup_{V \in \mathcal{P}_J(\mathbb{D})} \left( \sum_k (1 - |\lambda_k|^2)^2 \left( \frac{1}{\pi} \int_{\mathbb{D}} V(\zeta) \, d\mu(\zeta) \right) - p \hat{\kappa}(V) \right) < \infty$;

[ASL] $\sup_{V \in \mathcal{P}_{AS}(\mathbb{D})} \left( \sum_k (1 - |\lambda_k|^2)^2 \left( \frac{1}{\pi} \int_{\mathbb{D}} V(\zeta) \, d\mu(\zeta) \right) - p \hat{\kappa}(V) \right) < \infty$.

Evidently, the equivalence $[\Lambda] \iff [\text{GL}]$ coincides with the equivalence (i) $\iff$ (iii) of Theorem 0.2.

Scheme of the proof. The evident inclusions of the class of all extended Green functions $g_D(\cdot, 0), D \in \mathbb{D}$, into $\mathcal{P}_J(\mathbb{D}) \subset \mathcal{P}_{AS}(\mathbb{D})$ give the implications $[\text{ASL}] \implies [\text{JL}] \implies [\text{GL}]$.

Put

$$K_\Lambda(z) = \frac{|z|^2}{2} \sum_{\lambda_k \in \Lambda} \frac{(1 - |\lambda_k|^2)^2}{|1 - \lambda_k \bar{z}|^2}.$$  (7.5)

The function $K_\Lambda$ is a subharmonic continuous function on $\mathbb{D}$. Easy calculations give

$$\Delta K_\Lambda(z) = 4 \frac{\partial^2}{\partial z \partial \bar{z}} K_\Lambda(z)$$

$$= 2 \sum_k (1 - |\lambda_k|^2)^2 \frac{\partial^2}{\partial z \partial \bar{z}} \frac{z \bar{z}}{(1 - \lambda_k \bar{z})(1 - \lambda_k z)}$$

$$= 2 \sum_k (1 - |\lambda_k|^2)^2 \frac{1}{|1 - \lambda_k \bar{z}|^4}.$$  

Hence we get

$$d\nu_{K_\Lambda}(z) = \frac{1}{\pi} \sum_k \frac{(1 - |\lambda_k|^2)^2}{|1 - \lambda_k \bar{z}|^4} \, d\mu(z).$$  (7.6)
By Luecking’s criterion [9, Theorem A, (b)] a sequence \( \Lambda = \{\lambda_k\} \) is zero set for \( A^{-p} \) if and only if for \( M_p \) from (7.2), the function \( M_p - K_\Lambda \) admits a harmonic minorant.

Let \( M = M_p \), \( \nu_M = \nu_{M_p} \) (see (7.4)), and \( u = K_\Lambda \). It follows from Proposition 2.6 the same way as in the proof of implication \([\Lambda] \Longrightarrow [AS]\) of previous Theorem that the implication \([\Lambda] \Longrightarrow [ASL]\) holds.

For the proof of implication \([GL] \Longrightarrow [\Lambda]\) we can use subharmonic analogs of Corollary 1.1 (item \(((Z_{sc})\))), Theorem 1.1 (item \(((Z_D)\))), and Theorem 0.1 (item \(((Z_r)\)) with the measure \( \nu_{K_\Lambda} \) in place of measure \( n_\Lambda \). In particular, the last is the statement

\[ [\text{SH}_r] \quad \text{Let the function } M \text{ be the same as in Theorem 0.1, and } K \text{ be a subharmonic continuous function on } \mathbb{D} \text{ with Riesz measure } \nu_K. \text{ The function } M - K \text{ admits a harmonic minorant if and only if there are constants } a < 1 \text{ and } C \text{ such that the inequality} \]

\[ \int_{\mathbb{D}} g_D(\zeta,0) \, d\nu_K(\zeta) \leq \int_0^{1 \pi} \left( \frac{1}{2\pi} \int_0^{2\pi} g_D(te^{i\theta},0) \, d\theta \right) \, d(tM'_-(t)) + C \]

\[ \text{holds for each domain } D \in \mathcal{U}_0(\mathbb{D}; D(a)). \]

The system of inequalities in the statement \([\text{SH}_r]\) is equivalent to the statement \([GL]\) with \( K_\Lambda \) and \( M_p \) from (7.5) and (7.2) in place of \( K \) and \( M \) respectively according to (7.6) and (7.3). It proves the equivalence \([GL] \iff [\Lambda]\).

Remarks. 1. The summands in first sums of \([GL]\), \([JL]\), and \([ASL]\) are values at points \( \lambda_k \) of the Berezin transforms (see [8, Ch. 2]) of Green, Jensen and Arens–Singer functions respectively.

2. In Theorems 7.1 and 7.2 in items \([G]\) and \([GL]\) we can suppose that the domain \( D \subset \mathbb{D} \) has final number of points of contact with circumference \( \partial \mathbb{D} \) since the integral (7.1) with the integrand \( V = g_D(\cdot,0) \) is a finite number.

More detailed statement of this work will be submit in the journal “Matematicheskii sbornik” in Russian (the translation from Russian into English is “Sbornik: Mathematics”).

References

[1] W. K. Hayman, P. Kennedy B., Subharmonic Functions. Vol. I. London Math. Soc. Monogr. 9. Academic Press. London. 1986

[2] B. Korenblum, An extension of the Nevanlinna theory // Acta Math. 1975. V. 135. P. 187–219.
[3] E. Beller, Factorization for non-Nevanlinna classes of analytic functions // Israel J. Math. 1977. V. 27. No. 3–4. P. 320–330.

[4] D. Pascuas, Zeros interpolació en espais de funcions holomorfes del disc unitat. Tesi doctoral, Universitat Autónoma de Barcelona, 1988.

[5] J. Bruna, X. Massaneda, Zero sets of holomorphic functions in the unit ball with slow growth // J. Analyse Math. 1995. V. 66. P. 217–252.

[6] K. Seip, On a theorem of Korenblum // Ark. Math. 1994. V. 32. P. 237–243.

[7] K. Seip, On Korenblum’s density condition for the zero sequences of $A^{-\alpha}$ // J. Analyse Math. 1995. V. 67. P. 307–322.

[8] H. Hedenmalm, B. Korenblum, K. Zhu, Theory of Bergman spaces. Graduate Texts in Mathematics. 199. Springer-Verlag, New York. 2000.

[9] D. Luecking, Zero sequences for Bergman spaces // Complex Variables. 1996. V. 30. P. 345–362.

[10] O. Blasco, A. Kukuryka, M. Nowak, Luecking’s condition for zeros of analytic functions // Preprint. 2003. electronically available from: http://www.uv.es/~oblasco/investigacion/CA/Nowak.pdf

[11] A. Borichev, H. Hedenmalm, Harmonic functions of maximal growth: invertibility and cyclicity in Bergman spaces // J. Amer. Math. Soc. 1997. V. 10. 4. P. 761–796.

[12] T. J. Ransford, Potential Theory in the Complex Plane. Cambridge: Cambridge Univ. Press, 1995.

[13] M. M. Dzhrbashyan, On the canonical form of the representation of functions meromorphic in the unit disk // Dokl. Akad. Nauk Armjan. SSR. 1945. V. 3. P. 3–9.

[14] M. M. Dzhrbashyan, Theory of factorization of functions meromorphic in the disk // Mat. Sb. 1969. V. 79(121). P. 517–615 (Russian); Math. USSR Sb. 1969. V. 8. P. 493–592 (English).

[15] M. M. Dzhrbashyan, The theory of factorization and boundary behavior of functions meromorphic in a disk // uspehi Mat. Nauk. 1973. V. 28. No. 4. P. 3–14 (Russian); Russian Math. Surveys. 1973. V. 28. P. 607–610 (English).

[16] M. Tsuji, Potential theory in modern function theory. Maruzen Co. Tokyo. 1959.
[17] P. Colwell, Blaschke Product. Bounded Analytic Functions // Ann Arbor. The University of Michigan Press. 1985.

[18] C. Horowitz, Zeros of functions in the Bergman spaces // Duke Math. 1974. V. 41. P. 693–710.

[19] G. Bomash, A Blashke-type product and random zero sets for Bergman spaces // Arkiv für Math. 1992. V. 30. P. 45–60.

[20] E. Beller, C. Horowitz, Zero sets and random zero sets in certain function spaces // J. Analyse Math. 1994. V. 64. P. 203–217.

[21] M. M. Dzhrbashyan, The factorization of functions that are meromorphic in the finite plane // Izv. Akad. Nauk Armjan. SSR. Mat. 1970. V. 5, No. 6. P. 453–485.

[22] A. A. Gol’dberg, B. Ya. Levin, I. V. Ostrovskii, Entire and meromorphic functions // In book: Itogi VINITI. Sovremennye problemy matematiki. Fundamental’nye napravleniya. V. 85. Moscow. 1991. P. 5–185.

[23] M. Klimek, Pluripotential Theory. Oxford : Clarendon Press, 1991.

[24] B. N. Khabibullin, Dual approach to certain questions for the weighted spaces of holomorphic functions // Israel Math. Conference Proceedings (“Entire Functions in Modern Analysis”, Tel-Aviv, 1997). 2001. V. 15. P. 207–219.

[25] B. N. Khabibullin, Dual representation of superlinear functionals and its applications in function theory. I // Izv. Ross. Akad. Nauk, Ser. Mat. 65, No.4, 205-224 (2001). English transl. in Izvestiya:Mathematics, 65, No.4, 835-852 (2001).

[26] B. N. Khabibullin, Dual representation of superlinear functionals and its applications in function theory. II // Izv. Ross. Akad. Nauk, Ser. Mat. 65, No.5, 167-190 (2001). English transl. in Izvestiya:Mathematics, 65, No.5, 1017-1039 (2001).

[27] B. N. Khabibullin, Criteria for (sub-)harmonicity and continuation of (sub-)harmonic functions // Sib. Mat. Zh., 44, No.4, 905-925 (2003); english translation in Sib. Math. J., 44, No.4, (2003).

[28] T. W. Gamelin, Uniform Algebras and Jensen Measures. Cambridge: Cambridge Univ. Press, 1978.

[29] C. Sundberg, Measures induced by analytic functions and a problem of Walter Rudin // J. Amer. Math. Soc. 2002. V. 16. No. 1. P. 69–90.
[30] P. Koosis, Leçons sur le théorème de Beurling et Malliavin. Montréal: Les Publications CRM, 1996.

[31] B. J. Cole, T. J. Ransford, Subharmonicity without Upper Semicontinuity // J. Funct. Anal. 1997. V. 147. P. 420–442.

[32] B. J. Cole, T. J. Ransford, Jensen measures and harmonic measures // J. reine und angew. Math. 2001. V. 541. P. 29–53.

[33] T. J. Ransford, Jensen measures // In: Approximation, Complex Analysis and Potential Theory (Montréal, Qc, 2000), P. 221–237. NATO Sci. Sér. II. Math-Phys-Chem 37. Kluwer, Dordrecht, 2001.

[34] N. C. Landkof, Foundations of modern potential theory. Berlin–Heidelberg–New York: Springer-Verlag, 1972.

[35] B. N. Khabibullin, Estimations of the volume of the zero sets of holomorphic functions // Izv. Vyssh. Uchebn. Zaved., Mat. 1992, No.3(358), 58-63 (1992); English translation in Russ. Math. 36, No.3, 56-62 (1992).