Up Quark Masses from Down Quark Masses

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Abstract

The quark and charged lepton masses and the angles and phase of the CKM mixing matrix are nicely reproduced in a model which assumes $SU(3) \otimes SU(3)$ flavour symmetry broken by the v.e.v.'s of fields in its bi-fundamental representation. The relations among the quark mass eigenvalues, $m_u/m_c \approx m_c/m_t \approx m_d^2/m_s^2 \approx m_s^2/m_b^2 \approx \Lambda_{GUT}^2/M_{Pl}^2$, follow from the broken flavour symmetry. Large tan $\beta$ is required which also provides the best fits to data for the obtained textures. Lepton-quark grandunification with a field that breaks both $SU(5)$ and the flavour group correctly extends the predictions to the charged lepton masses. The seesaw extension of the model to the neutrino sector predicts a Majorana mass matrix quadratically hierarchical as compared to the neutrino Dirac mass matrix, naturally yielding large mixings and low mass hierarchy for neutrinos.
In a bottom-up approach to the flavour problem, the first task is to decode the variety of experimental data on fermion mass ratios and mixings. By translating that information into simple mass matrix textures that reproduce the empirical relations one makes progress towards the interpretation of these textures in terms of flavour symmetries and their breakings. In spite of the many interesting textures and models present in the literature, the flavour issue still remains a totally open problem. We still ignore why $m_t \gg m_b$ while $m_u < m_d$.

Since the first suggestion of a relation between the Cabibbo angle and quark mass ratios \cite{1}, many models were built where the angles of the CKM mixing matrix are expressed in terms of the quark mass eigenvalues (in practice angles are better known than light quark masses). Many textures have been designed along these lines, in particular, those with a maximal number of vanishing matrix elements which are now excluded by the experiments \cite{2, 3}. More recent ones \cite{4} improve the fit to the data by enriching the textures, however these are not quite consistent with the latest data \cite{5}. Neutrino oscillation experiments have prompted a lot of work on the analogous approach to the lepton mass matrices including solutions compatible with lepton-quark (grand-)unification \cite{6}. The considerable progress obtained in the last years in measurements of the quark mixings - all quite consistent with the Standard Model (SM) description of flavour changing and CP violation effects \cite{7} - as well as in studies of neutrino oscillations, implies a much stronger selection of allowed textures.

One obvious difficulty in the definition of the mass textures is their dependence on the assumed basis for the fermions since only the family mixings inside the weak doublets of quarks or leptons is observable without physics beyond the SM. This allows for several different patterns of mass textures even after the number of free parameters are reduced by theoretical assumptions or educated guess. This fact notwithstanding, these efforts are an important step in the quest for the symmetries underlying the flavour theory which should naturally ensure these relations. Because of the hierarchical nature of the charged fermion masses and observed mixings, the flavour theory must also contain one or more small parameters whose existence would be natural only if protected by spontaneously broken flavour symmetry.

In models based on abelian symmetries, the different scales present in the mass matrices are associated to powers of the small parameters defined by the choice of the fermion abelian charges, all their $O(1)$ complex pre-factors remaining arbitrary \cite{8}. This intrinsically limits the predictivity of these models and their selection by the experimental progress. Instead, non-abelian flavour symmetries potentially establish exact relations and are more constrained but the overall hierarchies require more than one small parameter and more involved symmetry breaking schemes \cite{9}. In this paper we develop a novel approach to these issues based on the following argumentation.

In a recent publication \cite{5}, we have built fermion mass matrices by the identification of a few characteristic features of the mixing angles and phases and their implementation via simple mechanisms and associated textures. Some ingredients were already present in the literature (as referred to in ref. \cite{5}), but were combined to bring forth new textures for $m_{\text{up}}$ and $m_{\text{down}}$ following some observations outlined below. This resulted in textures with five free parameters: two that implement the double seesaw-like texture of $m_{\text{up}}$, often advocated in the literature, plus a necessary, smaller parameter to improve the fit of $m_u$ to data; and two that define a new texture for $m_{\text{down}}$. CP violation is introduced by requiring the so-called maximal CP violation
Figure 1: Fermion masses at the scale $\mathcal{M}_{\text{GUT}}$ for $\tan \beta = 30$. The neutrino mass scales $\sqrt{\Delta m^2_{\text{sol}}}$ and $\sqrt{\Delta m^2_{\text{atm}}}$ have been rescaled by a factor $10^7$.

between two families (namely a phase $i$) [10]. The fit nicely reproduces the masses and the unitarity triangle within the relatively small experimental uncertainties.

In order to further reduce the number of parameters, so to get more insight into the underlying flavour symmetry, further relations about mass ratios in and between $m_{\text{up}}$ and $m_{\text{down}}$ are needed besides those among mixing angles and mass ratios implicit in these textures. There are indeed other intriguing relations in the hierarchies of the $m_{\text{up}}$ and $m_{\text{down}}$ eigenvalues that are latent guidelines in flavour model building, namely, the following approximate relations:

\[
\frac{m_u}{m_c} \approx \frac{m_c}{m_t} \approx \frac{m_d^2}{m_s^2} \approx \frac{m_s^2}{m_b^2} = O \left( \frac{M^2_{\text{GUT}}}{M^2_{\text{Pl}}} \right).
\]

They are better realized for the values of the quark masses run up to the GUT scale $M_{\text{GUT}}$ in a supersymmetric framework, shown in fig. 1, which also makes the last relation more suggestive.

The general strategy in the quest for hidden symmetries in the fermion masses and mixings assumes that they are defined by appropriate effective operators after the flavour symmetry breaking fields are replaced by their v.e.v’s. Therefore one can write:

\[
m_{\text{down}} = \sum_i X_i v \cos \beta
\]
where the $X_i$ matrices are functions of v.e.v.’s associated to the various scales needed to define the fermion masses in units of the cutoff\(^1\), and $v = 174$ GeV. The relations (1) suggest the following expression: $m_{up} = \sum_{i,j} b_{ij} X_i^T X_j v \sin \beta$, which, for $b_{ij} = 1$, trivially satisfies (1) with vanishing mixings. Then by a choice of $O(1)$ numbers $b_{ij}$ the CKM angles and phase could be introduced. Of course, the program makes sense only if the number of free parameters is low enough. However, in a supersymmetric theory $m_{up}$ have to be holomorphic in the fields $X_i$, suggesting to replace that expression by

$$m_{up} = \sum_{i,j} b_{ij} X_i^T X_j v \sin \beta . \tag{3}$$

Let us define the relation corresponding to (2) and (3) as $m_{up} = m_{down}^T \circ m_{down}$. In the presence of a flavour symmetry, $m_{up}$ correctly transforms only under $O(3)$ transformations of the right-handed down quarks. In spite of the fact that $m_{up} = m_{down}^T \circ m_{down}$ alone does not ensure the relations (1), we present in this letter a realistic model that does\(^2\).

We seek out a flavour model realizing both the textures of [5] and the hierarchies in (1) in the framework of effective supergravity and grandunification. We propose a model with $SU(5) \otimes SO(3) \otimes SU(3)$ symmetry under which fermions transform in one ($\overline{5}, 3, 1$) plus one ($10, 1, 3$) (right handed neutrinos would require another $SU(3)$ factor), and the electroweak symmetry breaking Higgs doublets come from a ($\overline{5} \oplus 5, 1, 1$). This flavour symmetry is broken by the v.e.v.’s of three fields, all transforming as the bifundamental representation of the flavour symmetry. Later on the flavour group is upgraded to $SU(3) \otimes SU(3)$, where one factor is broken to $SO(3)$ by a field in its symmetric representation.

The Yukawa couplings in $m_{up}$ and $m_{down}$ are obtained from $SU(5) \otimes SU(3) \otimes SU(3)$ invariant higher-dimension operators after these fields are substituted with their v.e.v.’s. The latter follow from field equations designed to reproduce the required textures. There are only two small parameters, the two ratios between v.e.v.’s, one actually corresponding to $M_{GUT}/M_{Pl}$. Consequently, the three small parameters in $m_{up}$ are related to the two fitted ones in $m_{down}$, up to $O(1)$ factors associated to couplings in the effective supergravity. The relation $m_{up} = m_{down}^T \circ m_{down}$ is realized.

Therefore, this model explains why the hierarchy in $m_{up}$ is approximately quadratic with respect to that in $m_{down}$; it results from both the chiral-like flavour symmetry and the direction of the v.e.v.’s also required by the successful textures. The charged lepton mass eigenvalues are simply obtained à la Georgi-Jarlskog by promoting one of the flavour breaking fields to be the $75$ or the $24$ that breaks the $SU(5)$ GUT symmetry. Interestingly enough, this is also required to explain one small parameter, namely the ratio $m_s/m_b$, as well as to optimize $m_{up}$.

Paradoxically, the setup also provides a solution for the large mixing and low hierarchy that characterizes the neutrino effective mass matrix even if the seesaw Yukawa couplings are hierarchical like the quark and charged lepton ones. Indeed, with a further flavour $SU(3)$ factor associated to right-handed neutrinos, their Majorana mass matrix will result quadratically

\(^1\)In this paper the cutoff is the Planck scale and the $X_i$ are the most general ones allowed by the hidden symmetries. In supersymmetric models with a cutoff scale much below $M_{Pl}$ it makes sense to select some $X_i$ by a choice of states to be integrated out.

\(^2\)For recent work addressing the relations (1) in a different approach see, e.g., Ref.[11].
hierarchical as compared to the Dirac mass matrix, \( m_M = m_D \circ m_D^T \), analogously to the quark sector mass matrices. By a strong compensation of hierarchies, the seesaw mechanism then yields an effective neutrino mass matrix consistent with the experimental one.

**Mass Matrix Textures**

Let us first summarize the steps that lead to the mass textures in [5]. We first concentrate on the quark sector and write the mass matrices as follows:

\[
\begin{align*}
\mathbf{m}_{up} &= U_{us} \text{diag}(m_u, m_c, m_t) U_{us}^\dagger, \\
\mathbf{m}_{down} &= U_{ub} \text{diag}(m_d, m_s, m_b) U_{ub}^\dagger,
\end{align*}
\]

so that the CKM matrix is \( U^{CKM} = U_{us} U_{ub}^\dagger U_{ub}^\dagger \). The unitary matrices are written as:

\[
U = \exp(i \Phi R(\theta_{23}) \Gamma_\delta R(\theta_{13}) \Gamma_\delta R(\theta_{12}) \exp(i \Phi'))
\]

where \( \Phi = \text{diag}(\phi_1), \ \Phi' = \text{diag}(\phi'_1), \ \Gamma_\delta = \text{diag}(1, 1, e^{i \delta}) \) and \( R(\theta_{ij}) \) is a rotation in the \((i,j)\) plane. Only the phases \( \phi_{ij} = \phi_{ij}^{dL} - \phi_{ij}^{uL} - \phi_{ij}^{dL} + \phi_{ij}^{uL} \) are relevant for the CKM matrix.

Our basic assumptions are based on two simple facts. Firstly, the relation \( |U_{ub}^{CKM}| = O(|U_{us}^{CKM}U_{cb}^{CKM}|) \) suggests that \( \theta_{13} \) in \( U^{CKM} \) results from the commutation between the other rotations. Accordingly, we have assumed in [5] that \( \theta_{13}^{uL} = \theta_{13}^{dL} = 0 \). Secondly, it then follows that the unitarity triangle angle \( \alpha = \phi_{12} \) and experimentally it is consistent with \( \pi/2 \). Instead, \( \phi_{23} \) has to be small. Therefore we assume one and only one phase \( \phi_{12} = \pi/2 \) in \( \mathbf{m}_{down} \) and a real \( \mathbf{m}_{up} \). The complete analysis in ref. [5] leads to the following textures:

\[
\begin{align*}
\mathbf{m}_{up} &= m_t \begin{pmatrix} c\lambda^8 & a\lambda^6 & 0 \\
-a\lambda^6 & c\lambda^8 & -b\lambda^2 \\
0 & b\lambda^2 & 1 \end{pmatrix}, \\
\mathbf{m}_{down} &= m_b \begin{pmatrix} 0 & g\lambda^3 & 0 \\
g\lambda^3 & if\lambda^2 & 0 \\
0 & f\lambda^2 & 1 \end{pmatrix}
\end{align*}
\]

up to unobservable unitary transformations\(^3\). The Wolfenstein approximation for \( U^{CKM} \) suggests to introduce in (6) powers of \( \lambda = \sin \theta_C \) to roughly characterize the magnitude of the mass matrix elements as well.

In (6), \( \mathbf{m}_{up} \) has the double seesaw texture with the additional parameter \( c\lambda^8 \) to shift \( m_u \) down and obtain a good fit to all data\(^4\), while \( \mathbf{m}_{down} \) has only two parameters and one single phase, \( i \), which yields \( \alpha \approx \pi/2 \). The fit to the ten experimental observables is quite satisfactory. The mass matrices are defined at the GUT scale and the couplings are run down to the electroweak scale for comparison with data, which introduces a dependence on tan \( \beta \). The fit is better for larger values of tan \( \beta \). The results for tan \( \beta = 45 \) are as follows:

\[
f = 0.33, \quad g = 0.32, \quad a = 1.77, \quad b = 1.01, \quad c = -3.6.
\]

For more details we refer to [5].

The matrices in (6) are the simplest realization of our assumptions up to unitary transformations that do not modify the measured observables, provided they do not introduce new

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\(^3\)Actually some matrix elements in the textures displayed in [5] have opposite signs with respect to (6). Both choices give the same results for the fitted observables but only the textures (6) are consistent with the model discussed in this paper.

\(^4\)While in [5] \( c\lambda^8 \) appears only in \((\mathbf{m}_{up})_{11}\), in the present model it also appears in \((\mathbf{m}_{up})_{22}\) and negligibly affects the results.
parameters. But we stick to the textures in (6) since it has a nice explanation in terms of a flavour model that we now turn to discuss. In [5] the charged lepton mass matrix, $m_\ell$, was obtained by using the $SU(5)$ relation $m_\ell = m_{\text{left}}^T$, but assuming that the coupling $f$ transforms as an element of a $45$ of $SU(5)$, so that

$$m_\ell = m_b \begin{pmatrix} 0 & g\lambda^3 & 0 \\ g\lambda^3 & -3if\lambda^2 & -3f\lambda^2 \\ 0 & 0 & 1 \end{pmatrix}.$$  

(8)

This reasonably fits the charged lepton masses, realizing the relations: $m_e \approx m_d/3$, $m_\mu \approx 3m_s$, $m_\tau \approx m_b$, at $M_{\text{GUT}}$.

### A Flavour Model: Quarks

Let us begin with the quark sector and subsequently extend the results to the lepton one. The maximum flavour symmetry of the gauge interactions in a $SU(5)$ GUT is a chiral $SU(3) \otimes SU(3)$ where the two factors act on the three $\bar{5}$’s and three $10$’s, respectively. As already stated, we first propose a model with the $SO(3) \otimes SU(3)$ subgroup as the overall (possibly gauged) flavour symmetry. Therefore, $m_{\text{down}}$ belongs to the $(\bar{5} \otimes 10)$ representation of $SU(5)$ and the $(3, \bar{3})$ of the flavour group, while $m_{\text{up}}$ transforms in the $(10 \otimes 10)$ and in the $(1, \bar{3} \otimes \bar{3})$, respectively.

Fermion masses are assumed to derive from the effective Yukawa couplings to the electroweak Higgs doublets in a $\bar{5} \oplus 5$, invariant under the flavour symmetry, with the usual doublet-triplet splitting.

These Yukawa couplings are functions of the flavour symmetry breaking v.e.v.’s defined by the allowed $SU(5) \otimes SO(3) \otimes SU(3)$ operators. To realize the hierarchy and texture of $m_{\text{down}}$ in (6) the corresponding fields must be in the $(3, \bar{3})$ and we need three of them with v.e.v.’s:

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Lambda, \quad F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 1 & 0 \end{pmatrix} f\lambda^2 \Lambda, \quad G = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} g\lambda^3 \Lambda,$$

(9)

where the $O(1)$ parameters $g$ and $f$ and the flavour symmetry breaking scale $\Lambda$ are to be fixed by fitting $m_{\text{down}}$. These v.e.v.’s are the solution (up to complexified SO(3)$\otimes SU(3)$ transformations) of the following analytic field equations,

$$P^T P = \Lambda P, \quad T_r\Lambda = \Lambda, \quad P F^T = 0,$$

$$F^T F = 0, \quad P^T G = 0, \quad F^T G + G^T F = f\lambda^2 \Lambda G .$$

(10)

At the scale $\Lambda$, the flavour symmetry is broken to $SO(2) \otimes U(2)$ by $P$, which acts as a projector of the heavy family. The field equations can be implemented in the superpotential (invariant under the complexified flavour group) by introducing Lagrange multiplier fields, although this method could look awkward. The construction of a more satisfactory superpotential will be presented elsewhere (also because it would depend on some options defined below) and we concentrate in the following on the consequences of (9) for the fermion masses.

Since the natural cutoff of the supersymmetric GUT is the Planck mass $M_{\text{Pl}}$, $m_{\text{down}}$ can be written as

$$m_{\text{down}} = \frac{P + F + G}{M_{\text{Pl}}} \frac{v \cos \beta}{},$$

(11)
where $O(1)$ real coefficients have been absorbed by a redefinition of $f$, $g$ and $\Lambda$. In particular, $m_b = \Lambda v \cos \beta / M_{\text{Pl}}$.

Let us derive its consequences for $m_{\text{up}}$ with the same set of v.e.v.'s. The lowest order $SO(3) \otimes SU(3)$ invariant operators are quadratic in $P$, $F$ and $G$. Combining them into products that transform as $(1, 2 \otimes 2)$ and replacing the solutions (10) yields the general expression:

\[
m_{\text{up}} = \frac{v \sin \beta}{M_{\text{Pl}}^2} \left( p(P^T P) + q(F^T F - F^T P) + q'(P^T F + F^T P) + r(F^T G - G^T F) + r'(F^T G + G^T F) + s(G^T G) \right)
\]

(12)

where the couplings $p$, $q$, $q'$, $r$, $r'$ and $s$, of the corresponding invariant operators in the superpotential are expected to be $O(1)$. With $q' = r' = 0$ (to be discussed below), the expression in (12) nicely matches the texture for $m_{\text{up}}$ in (6). By comparison we get $q/p = b/f = 3.1$, $r/p = i a \lambda / f g = 3.8 i$, and $s/p = c \lambda^2 / g^2 = -1.9$ with the fitted values in (7) for $\tan \beta = 45$. Of course, the fact that these numbers come out $O(1)$ is a crucial check of the model. Also, notice the relevance of the condition $F^T F = 0$ to obtain the right texture.

From (11) and (12), with the masses at the scale $\Lambda$:

\[
\frac{m_t}{v \sin \beta} = \frac{p \Lambda^2}{M_{\text{Pl}}^2}, \quad \frac{m_b}{v \cos \beta} = \frac{\Lambda}{M_{\text{Pl}}} \quad \Rightarrow \quad \Lambda \approx \frac{0.8 M_{\text{Pl}}}{\sqrt{p}}, \quad \sqrt{p} \approx \frac{1.2 m_t}{\tan \beta m_b}.
\]

(13)

Hence $\Lambda = O(M_{\text{Pl}})$, meaning a first breaking of the flavour symmetry close to the cutoff $M_{\text{Pl}}$ and involving supergravity effects. The magnitude of the three sequential flavour symmetry breakings are $O(M_{\text{Pl}})$ in $P$, $O(\lambda^2 M_{\text{Pl}})$ in $F$, $O(\lambda^3 M_{\text{Pl}})$ in $G$. With only two small parameters, the model defines a relationship between $m_{\text{up}}$ and $m_{\text{down}}$ that nicely reproduces the much stronger hierarchy of the eigenvalues of the former as well as the relative mixings and phase in the CKM matrix. Because $\Lambda = O(M_{\text{Pl}})$, operators with higher powers of $P/M_{\text{Pl}}$ must be included in $m_{\text{down}}$ and $m_{\text{up}}$. With the fields above it is not possible to write any new relevant operator that does not vanish with the assumed v.e.v.'s.

We still have to naturally enforce $q' = r' = 0$ or, equivalently, the vanishing of the corresponding symmetric operators in (12). As a matter of fact, this can be obtained from a simple assumption which turns out to be also required to fit the charged lepton spectrum (altogether this means a prediction). Indeed, let us take the fields in $F$ to transform under $SU(5)$ in a $\overline{75}$. Taking into account its product with the Higgs $\overline{5}$, the two symmetric operators involving $F$ in (12) transform as a $\overline{50}$ which has no colour singlet, e.w. doublet, so that they do not contribute to $m_{\text{up}}$ as required. Correspondingly, in (11) the term $F$ transforms as the $\overline{45}$. As an alternative, if $F$ transforms as a $\overline{24}$, requiring that the its product with the Higgs $\overline{5}$ transforms as a $\overline{45}$, leads to the same consequences. In a sense, this assumption for the effective coupling is natural in the effective theory because it can be realized by choosing the states that are integrated out.

5In the previous version of this paper, with flavour group $O(3) \otimes O(3)$, $m_{\text{up}}$ would have a flavour singlet component implying an equal mass contribution for all the up-quarks. This cannot be forbidden by the assumption of discrete symmetries as suggested there. We thank Z. Berezhiani for calling our attention to this problem.

6Actually, the realization of the field equations would presumably require more fields, e.g., a field transforming as the conjugate of $P$. Since $P/\Lambda$ is a projector, any polynomial $\phi(P/M_{\text{Pl}}) = \phi(\Lambda/M_{\text{Pl}}) P/\Lambda$ so that the mass matrix textures would be mildly affected.
Consistently, we take the \( F \) v.e.v. for the \( SU(5) \) breaking, keeping in mind that an \( O(1) \) coefficient has been absorbed in its definition in (11). This defines the GUT scale as:
\[
\Lambda_{\text{GUT}} = O\left(f\lambda^2\right)\Lambda = O\left(10^{-2}\right) \ M_{\text{Pl}}
\]
with \( f \) given in (7), which is quite consistent with the gauge coupling unification scale. Hence one small parameter, \( f\lambda^2 \), is naturally related to the \( \Lambda_{\text{GUT}} \) scale. It remains one parameter \( g\lambda/f \approx \theta_C \) to be explained. The other four parameters are \( O(1) \) unknown coefficients of the higher dimension operators in the effective supergravity theory.

By choosing the field \( F \) to transform as a \((\bar{3}, \bar{3})\) of the flavour groups amounts to have nine \( 24 \)'s or nine \( 75 \)'s of \( SU(5) \). The model framework requires the GUT gauge couplings to remain perturbative at least up to the Planck scale, which allows for five or six \( 24 \)'s or one \( 75 \), at most. Since the \( P \) projections suggest that three \( F \) components get masses of \( O(M_{\text{Pl}}) \), it seems consistent with perturbativity to assume \( F \) to transform as a \( 24 \). In the case of the \( 75 \), one must modify the model by writing \( F \) as the direct product of two matrices of fields: an \( SU(5) \) singlet \( Q \) in a bi-fundamental of the flavour group whose v.e.v. is \( O(\Lambda) \) and analogous to \( F \) in eq. (9), and a flavour singlet \( V \) in a \( 75 \) that breaks \( SU(5) \) at the scale \( f\lambda^2 M_{\text{Pl}} \). A discrete symmetry would constrain them to couple up in the mass matrices just as \( F \).

As already noticed, the \( SO(3) \otimes SU(3) \) flavour symmetry considered up to now can be explained by starting from the more natural one, \( SU(3) \otimes SU(3) \) with the fields \( P, F \) and \( G \) in the \((\bar{3}, \bar{3})\) and adding a field \( S \) in the \((6, 1)\) representation; the operators in (12) now contain \( S \) insertions: \( P^T SP, F^T SP, \ldots \). When \( S \) gets an \( SO(3) \) invariant v.e.v., \( \text{diag}(1, 1, 1) \ O(M_{\text{Pl}}) \), the model discussed before is obtained.

Notice that all coefficients in (12) are real with the exception of \( r \), with the phase \( i \) needed for \( m_{\uparrow} \) to be real as in (8). In this way, the CP violation has been introduced by hand - although it is nicely consistent with a maximal CP violation. Introducing the CKM phase through spontaneous CP symmetry breaking is a difficult problem by itself, specially in the context of a flavour theory with a reduced number of fields. Interestingly, there is a simple mechanism to get spontaneous breaking of the CP symmetry in the context of the \( SU(3) \otimes SU(3) \) flavour symmetry, but in practice it does not work. Indeed, introducing CP phases at the level of the \( SU(3) \to SO(3) \) breaking, \( S = \exp(i \text{diag}(\alpha, \beta, 0)) \), with \( \beta \neq 0 \) the important relation \( F^T SF = 0 \) is lost, while for \( \alpha \neq 0 \) the CP violation comes out wrong.

**Lepton-Quark Unification**

The \( SU(5) \) symmetry relates \( m_\ell \) to \( m_{\downarrow} \) but the precise relation depends on insertions of \( SU(5) \) breaking fields in the effective mass matrices. In (4) a simple generalization of the Georgi-Jarlskog mechanism has been proposed in order to ensure the relations \( m_\mu \approx 3m_\tau \) and \( m_\mu m_\mu \approx m_\mu m_\tau \). It amounts to make the product of \( F \) and the electroweak Higgs \( 5 \) to transform as a \( 45 \). This is just what has been imposed above from the study of the quark sector. Thus the charged lepton mass matrix becomes a prediction of the model that reads:
\[
m_\ell = \frac{P^T - 3F^T + G^T}{M_{\text{Pl}}} v \cos \beta .
\]
It correctly accounts for the charged lepton mass eigenvalues within the precision appropriate to the aim of this paper [5].
The mixing angles come out relatively small but, only by coupling this model to a neutrino mass generation mechanism, the prediction for $U_{\text{MNS}}$ could be tested. Indeed, the observable mixing angles in neutrino oscillations are defined by the transformation $U_{\text{MNS}}$ between the bases where the charged and neutral leptons in the $\tilde{5}$'s are mass eigenstates respectively. Since in the basis chosen here the charged lepton angles are small\(^7\), the large atmospheric angle must come from the neutrino sector.

An effective light neutrino mass matrix would transform as the conjugate of $S$ under $SU(3) \otimes SU(3)$. If the corresponding field is included, with the cutoff at $M_{\text{Pl}}$, the resulting (degenerate) neutrino masses are at most $O(\nu^2/M_{\text{Pl}})$, much smaller than the measured mass differences. One needs a model for neutrino masses and the natural choice in this GUT context is the seesaw mechanism with three $SU(5)$ singlets and their Majorana mass matrix. The obvious extension of the flavour symmetry is $SU(3) \otimes SU(3) \otimes SU(3)$ with $P, F, G$ in $(\bar{1}, \bar{3}, \bar{3})$ and additional breaking through Higgs fields in the bi-fundamental representation $(\bar{3}, \bar{3}, 1)$. The complete building of such a model is beyond the scopes of this letter, but it is worth noticing a nice feature of the seesaw mechanism in the present context. It comprises an $SU(5)$ invariant Majorana mass $M_{\text{R}}$ in the representation $(\bar{3} \otimes \bar{3}, 1, 1)$ of the flavour group and a Dirac mass $m_D$ in the $(\bar{3}, \bar{3}, 1)$. The effective neutrino mass matrix is given (in the flavour basis) by the seesaw expression:

$$m_{\nu} = m_D^T M_{\text{R}}^{-1} m_D.$$  \hspace{1cm} (16)

By analogy with the quark case, we would expect a hierarchical structure in $m_D$ while $M_{\text{R}} = m_D \circ m_D^T$ follows from the flavour symmetry, implying a quadratically stronger hierarchy. The resulting hierarchy in $m_{\nu}$ in general is much milder, although the precise relations depend on the structure of the field matrices. The present model provides a non-abelian explanation for hierarchy compensation in the seesaw mechanism.

For the sake of example, we assume for $m_D$ and $M_{\text{R}}$ textures analogous to those introduced in the quark sector, $m_{\text{down}}^T$ and $m_{\text{up}}$ in (6), respectively, and choose the $O(1)$ parameters to fit the experimental data. Although the textures generically predict a large atmospheric mixing as well as a very mild hierarchy among the neutrino mass eigenvalues, some tuning of these parameters is needed to reproduce the data. We obtain the following set: $a = 1.62, b = 1.01, c = .27, g = 1.43, f = .17.$

**Final Remarks**

The present model, based on bottom-up flavour model building, is successful in describing fermion masses and mixings, in explaining the hierarchies of up and down quarks, in exploiting the GUT breaking. It is natural in the technical sense that the mass matrices are defined by the breaking of flavour symmetries through a set of fields. Their configuration - direct product of $SU(3)$ group factors, matter in fundamental representations, flavour symmetry breaking fields in bi-fundamental ones - reminds several setups in various frameworks. The breaking of one $SU(3)$ into its $SO(3)$ subgroup requires an additional field in the symmetric representation and is a main ingredient in the realization of the basic relation, $m_{\text{up}} = m_{\text{down}}^T \circ m_{\text{down}}$. Supersymmetric $SU(5)$ grand-unification plays a crucial role in the set-up of the model, and provides one of the\(^{\ast}\)

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\(^{\ast}\)The alternative texture presented in Ref.[5] with a maximal $\mu - \tau$ mixing angle, obtained through a non-orthogonal transformation, does not correspond to the present model.
two small parameters defining the flavour symmetry breaking scale. The fact that the t-quark coupling to one Higgs field is $O(1)$ implies that the first flavour symmetry breaking occurs close to the cutoff scale $M_{Pl}$ and that $\tan \beta$ has to be large.

Work is still in progress on the construction of a detailed extension of the model to the seesaw mechanism, when both the Dirac and Majorana mass matrices are controlled by the breaking of the $SU(3)$ flavour symmetry associated to the heavy neutrinos. Crucial issues such as proton decay must also be addressed. Since the flavour and gauge symmetry breakings are fixed (and related) one can tackle with some predictivity the supersymmetric flavour problem so providing further tests of the model. Hopefully the features of the present model that might appear as weaknesses - in particular, CP violation - could be improved through the choice of other assumptions and other frameworks, still guided by the relation $m_{up} = m_{down}^T \circ m_{down}$.

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