The $p$—sphere and the geometric substratum of power law probability distributions

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Abstract

Links between power law probability distributions and marginal distributions of uniform laws on $p$-spheres in $\mathbb{R}^n$ show that a mathematical derivation of the Boltzmann-Gibbs distribution necessarily passes through power law ones. Results are also given that link parameters $p$ and $n$ to the value of the non-extensivity parameter $q$ that characterizes these power laws in the context of non-extensive statistics. PACS: 05.30.-d, 05.30.Jp

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1 Introduction

The probability distribution (PD) deduced by Gibbs for the canonical ensemble [1, 2], usually referred to as the Boltzmann-Gibbs (BG) equilibrium distribution

$$p_G(i) = \frac{\exp(-\beta E_i)}{Z_{BG}},$$

(1)

with $E_i$ the energy of the microstate labelled by $i$, $\beta = 1/k_B T$ the inverse temperature ($T$), $k_B$ Boltzmann’s constant, and $Z_{BG}$ the partition function, can fairly be regarded as statistical mechanics’ most notorious and renowned PD. In the last decade this PD has found a counterpart, in the guise of power-law distributions, with reference to the so-called nonextensive thermostatistics (NEXT). NEXT, or Tsallis’ thermostatistics, is currently a very active field, perhaps a new paradigm for statistical mechanics, with applications to several
scientific disciplines \[4, 5, 6\]. Power-law distributions are certainly ubiquitous in physics (critical phenomena are just a conspicuous example \[7\]). Now, as it is well known, both the BG and power-law distributions arise quite naturally in maximizing Shannon’s (resp., Tsallis’) information measure, the so-called Max-Ent approach, which is one of the most powerful statistics-theoretical techniques devised in the last 60 years.

Our goal here is to show that the above mentioned probability distributions can be also derived via purely geometric arguments, which is sure to be of interest to the immense audience of MaxEnt practitioners. In order to motivate our approach we discuss first of all the 2−sphere.

### 2 Physical motivation of the present work

#### 2.1 The 2−sphere

Let us consider a dilute gas of \(N\) hard spheres in a box with hard walls and give these spheres some arbitrary initial distribution of momenta and positions. Classically, after a few mean free times have passed, we expect that the distribution of momenta \(P_i\) will be given by the Maxwell-Boltzmann (MB) formula,

\[
f_{MB}(P) \propto \exp \left(- \frac{\|P\|^2}{2m k_B T}\right), \tag{2}
\]

where the temperature \(T\) is given in terms of the conserved total energy \(U\) by the ideal-gas relation \(U = \frac{3}{2} N k_B T\), with \(k_B\) the Boltzmann constant. This is so because the hamiltonian is simply

\[
H = \frac{1}{2m} \sum_i P_i^2 = \frac{\|P\|^2}{2m}, \tag{3}
\]

where \(P\) is a vector with \(3N\) components and

\[
\|P\|^2 = \sum_i P_i^2. \tag{4}
\]

Since the Hamiltonian \(H\) takes on the constant value \(U\), the allowed values of \(P\) form a sphere called the 2-sphere.

Suppose we now choose \(P\) at random on the 2-sphere. For this to be a meaningful statement, we need to have a measure which tells us which sets of \(P\)’s are equally likely a priori. The obvious choice is to assign equal a priori probabilities to equal areas on the 2-sphere. Why should this be the choice? Because, according to Sinai \[8\] the hard-spheres gas is a chaotic system \[9\].

Thus, if we choose \(P\) at random with respect to this uniform measure, the probability that our choice makes an angle between \(\nu\) and \(\nu + d\nu\) with respect to any particular axis is simply \[9\]

\[
f(\nu)d\nu \approx (\sin \nu)^{3N-2} d\nu \approx (\sin \nu)^{3N-3} d\cos \nu \approx \left[(1 - \cos^2 \nu)^{\frac{3N-3}{2}}\right] d(\cos \nu). \tag{5}
\]
If we now identify \((2mU)^{1/2}\cos(\nu)\) as, say, the value of \(p_{1z}\) (the \(z\) component of the first particle’s momentum), we find, with \(U = 3Nk_B T/2\)

\[
f(p_{1z})dp_{1z} \propto \left[1 - \frac{p_{1z}^2}{2mU}\right]^{3N-2} dp_{1z},
\]
which is a power law distribution \[1, 10\]. In the large-\(N\) limit this probability becomes

\[
f(p_{1z})dp_{1z} \approx \exp \left(-\frac{p_{1z}^2}{2mk_BT}\right)dp_{1z}.
\]

One recovers thereby the MB distribution for \(p_{1z}\), passing through a power law one. This result reminds one of an entirely similar one advanced years ago by Plastino and Plastino, but from a very different viewpoint that uses the notion of canonical ensemble \[6\].

Now consider the probability distribution for \(p_{1y}\) when \(p_{1z}\) is fixed. It is given by the first line of \[5\], with the \(3N\) in the exponent(s) replaced by \(3N - 1\) (since there is one less coordinate when \(p_{1z}\) is fixed) and \(2mU\) replaced by \(2mU - p_{1z}^2\).

In the large-\(N\) limit we can neglect, of course, \(p_{1z}^2\) compared to \(2mU\), so that we find the MB distribution for \(p_{1y}\). In similar fashion, one obtains, passing first through a Tsallis’ distribution, the MB distribution for any \(k\) components of \(P\) as long as \(k \ll N\).

We will generalize the above intuitive notions below to the case where, in equation \[4\], the power to which the summands are raised is any integer \(p\).

### 2.2 Revisiting the equipartition theorem

In classical statistical mechanics there exists a useful general result concerning the energy \(E\) of a system expressed as a function of \(N\) generalized coordinates \(q_i\) and momenta \(p_i\). The result holds in the case of the following (frequent) occurrence

1. the energy splits additively into the form
   \[
   E = \epsilon_i(p_i) + E'(q_1, \ldots, q_N, p_1, \ldots, p_i-1, p_{i+1}, \ldots, p_N),
   \]
   where \(\epsilon_i(p_i)\) involves only the degree of freedom \(i\) (the variable \(p_i\)) and the remaining part \(E'\) does not depend on \(p_i\).

2. the function \(\epsilon_i(p_i)\) is quadratic in \(p_i\).

Thus, \(\langle \epsilon_i \rangle = k_BT/2\). Any independent quadratic term in the Hamiltonian contributes this amount to the mean energy. This is the equipartition theorem \[2\].

Notice the similarity with the considerations of Section 1. Some light is thus shed on the equipartition meaning. The text-book demonstration assumes \[2\] the thermal equilibrium Boltzmann–Gibbs probability distribution

\[
f = \frac{1}{Z} e^{-\beta E},
\]
where \(\beta = 1/k_BT\) is the (Shannon-Boltzmann-Gibbs) Lagrange multiplier associated with the mean-energy constraint \(\langle E \rangle = \int d\tau f E\) and \(d\tau\) the phase-space
volume element. However, it has been shown in [11] that the equipartition theorem can be generalized i) to a non-extensive statistics and ii) to cases in which the Hamiltonian is an homogeneous function of degree $p$. This last fact motivates the considerations that follow below.

3 Geometric derivation of MaxEnt PDs

3.1 Uniform distribution on the $p-$ sphere and its marginals

We say that a random vector $X$ is orthogonally invariant if, for any deterministic orthogonal matrix $A$, random vector $AX$ is distributed as $X$. This is equivalent to the fact that the probability distribution of $X$ depends on $X$ only through its 2-norm. A typical physical example is that of $\mathbb{R}^3$-rotations, that are represented by orthogonal matrices. Obviously, the physical meaning attached to the $X-$distribution will not change if we rotate the coordinate-system [12]. An extension of this definition is as follows: a vector $X$ is $p$-spherically invariant if its probability distribution depends only on the $p$-norm of $X$. Orthogonal invariance corresponds thus to the case $p = 2$.

A uniform distribution on the $p$-sphere in $\mathbb{R}^n$ can be obtained by normalizing a vector distributed according to any $p$-spherically invariant distribution as follows.

**Theorem 1** An $n$-variate random vector $U$ is uniformly distributed on the $p$-sphere if it writes

$$U = \frac{X}{\|X\|_p}$$

where $X$ is $p$-spherically invariant [13].

Remark that vector $U$ has unit $p$-norm:

$$\|U\|_p = \left(\sum_{i=1}^{n} |u_i|^p\right)^{1/p} = 1,$$

which is to be regarded as a constraint. The marginal distributions of a uniform distribution on the $p$-sphere in $\mathbb{R}^n$ can be easily computed as follows:

**Theorem 2** if $U$ is uniformly distributed on the $p$-sphere in $\mathbb{R}^n$ then the marginal density of $V = [U_1, \ldots, U_k]^T$ is

$$f(u_1, \ldots, u_k) = \frac{p^k \Gamma \left( \frac{n}{p} \right)}{2^k \Gamma^k \left( \frac{1}{p} \right) \Gamma \left( \frac{n-k}{p} \right)} \left( 1 - \sum_{i=1}^{k} |u_i|^p \right)^{\frac{n-k-1}{p}} \text{; with } 1 \leq k \leq n - 1.$$  

(9)
Proof. The proof can be found in [14]: the first step consists in proving the result for \( k = n - 1 \), using the change of variable

\[
y_i = \frac{x_i}{\|X\|_p}, 1 \leq i \leq n - 1
\]

\[
y_n = \|X\|_p
\]

(10)

the Jacobian of which writes

\[
J = r^{n-1} \left( 1 - \sum_{i=1}^{n-1} |u_i|^p \right)^{\frac{1-p}{p}}.
\]

(11)

The second step consists in a proof by backward induction on dimension \( k \): assuming the result is true for all \( l \geq k \), it is proved for \( l = k - 1 \) by integrating over variable \( u_k \) the density

\[
f(u_1, \ldots, u_k) = \frac{p^k \Gamma \left( \frac{n}{p} \right)}{2^k \Gamma \left( \frac{1}{p} \right) \Gamma \left( \frac{n-k}{p} \right)} \left( 1 - \sum_{i=1}^{k} |u_i|^p \right)^{\frac{n-k-1}{p}}.
\]

(12)

3.2 Maximum Tsallis entropy distributions

The so-called Tsallis information measure \( H_q \) (with \( q \) a real parameter called the non-extensivity index) associated with a continuous distribution is defined as follows:

\[
H_q = \frac{1}{1 - q} \left( 1 - \int f^q(x) \right)
\]

(13)

As parameter \( q \to 1 \), this information measure converges to the classical (Shannon’s) measure of information. We are tacitly assuming that mean values are to be computed in their customary linear (in the probabilities) fashion [15]. Other ways of expressing “Tsallis” expectation values do exist, of course [4, 10], but appealing to them here would unnecessarily complicate things and obscure our message. The PD with given order—\( p \) moment that maximizes information measure \( H_q \) can be characterized as follows [16].

**Theorem 3** Given \( q > 1 \), the following problem

\[
\text{arg max}_f \frac{1}{1 - q} \left( 1 - \int f^q(x) \right)
\]

with \( EX_i^p = K_i \)

(15)

has for unique solution

\[
f(u_1, \ldots, u_k) = \left( 1 - \sum_{i=1}^{k} \lambda_i |u_i|^p \right)^{\frac{n-k}{p}-1}
\]

(17)
Since each Lagrange multiplier amounts to stretch any component $u_i$ by a factor $(\lambda_i)^{1/p}$, we conclude that the probability distributions given by (9) coincides with the maximum Tsallis' information measure distributions with given order-$p$ moment. Thus, we reach the following conclusion:

**Conclusion 4** If $[u_1, ..., u_n]$ is uniformly distributed on the $p$-sphere in $\mathbb{R}^n$, then all its $k$-variate marginals maximize Tsallis’ entropy with a non-extensivity parameter $q$ given by

$$q = \frac{n - k}{n - k - p}.$$  \hspace{1cm} (18)

We remark moreover that $q \geq 0$ provided that $(n - k)/p - 1 > 0$ or, equivalently, $1 \leq k \leq n - p$.

For example, in the case of norm-2 ($p = 2$), only the marginals of dimensions $1, 2, ..., n - 3$ are Tsallis maximizers with a positive parameter $q$. Additionally, the marginal of dimension $n - p$ is uniform in (not on!) the $p$-sphere in $\mathbb{R}^{n-p}$, that is, $u_1^p + ... + u_{n-p}^p \leq 1$ (not = 1) and thus maximizes Tsallis’ entropy, but with $q = +\infty$. Note that the “large dimension” remaining marginals, i.e., the ones for which $n - p + 1 \leq k \leq n - 1$, maximize Tsallis entropy with a parameter $q < 0$.

Summing up: as the dimension of the marginal decreases, we go from maximizers of Tsallis entropies with

- (A) $q < 0$ if $n - p + 1 \leq k \leq n - 1$
- (B) $q = +\infty$ if $k = n - p$
- (C) $q > 1$ if $k \leq n - p - 1$
- (D) $q \approx 1$ for $n \rightarrow \infty$.

For macroscopic systems, item (D) applies (classical statistics). Consider then the case $n$–finite: in most cases of physical relevance, $k$ is small, so that item (C) applies. Item (A) corresponds to a situation in which we have a great deal of information, that specifies the more important aspects of the problem. Only small details remain to be determined. A distribution with $q < 0$, precisely, amplifies those small details [4]. Item (B) corresponds to a very peculiar situation, the uniform distribution, as discussed below.

### 3.3 Application

Suppose we observe a $k$-variate random vector $Y$ distributed according to a Tsallis distribution with associated parameter $q > 1$ assumed known (in fact it can be estimated easily). The idea is that $Y$ can be interpreted as a restricted set of components of a larger system with $n > k$ degrees of freedom, this larger system being distributed according to a more natural distribution, namely the
uniform distribution on the $p$-sphere in $\mathbb{R}^n$. In such a case, $n$ and $p$ are related to $q$ and $k$ as prescribed in the preceding Section, namely,

$$\frac{1}{q - 1} = \frac{n - k}{p} - 1 \rightarrow q = \frac{n - k}{n - k - p}.$$ 

This supposes that the $n - k$ remaining variables are hidden or unavailable at the time of the measurement.

Strictly speaking, we recover classical statistics ($q = 1$) only for $n \to \infty$. Otherwise, since $k$ is assumed to be small, $q \geq 1$ and we are within the non-extensivity realm. This $q > 1$ restriction on $q$ agrees with considerations recently made from an entirely unconnected viewpoint that employs escort distributions and Fisher’s information measure [17]. For macroscopic systems, however, $n$ is of the order of Avogadro’s number, and thus $q$ is very close to unity.

In standard statistical mechanics’ text-books (see Section 1) we have $p = 2$ and $n$ the number of particles, with $n \approx 10^{24}$. Thus, the classicality criterium $q = 1$ works quite well indeed. What was called the first particle in Section 1 is assumed to be a test-particle, representative of the remaining degrees of freedom, so that the idea that $Y$ can be regarded as a restricted set of components of a larger system with $n > k$ degrees of freedom can be safely ignored.

Strictly speaking though, this larger system is being distributed according to a more natural distribution than the Boltzmann or the Tsallis ones, namely the uniform distribution on the $p$-sphere in $\mathbb{R}^n$.

4 A quantum analogy

In order to get a better grasp of the changes in the values of $q$ described at the end of Section 3, we make now recourse to the following analogy: consider a physical situation that revolves around a system that can exist in any of a large (discrete) set of (same energy $E_o$)-states labelled by a quantum number $i$: $i = 1, \ldots, n$ [18] and that our interest lies in the probability distribution (PD) $p_i$. In quantum mechanics, these states constitute a basis that spans an $n$-dimensional linear vector subspace. All possible physical states of our system that have energy $E_o$ can be expressed as linear combinations of these basis states with complex coefficients, so that we are speaking here of a subset of $\mathbb{C}^n$, which does not really seriously affect our considerations. Indeed, it has been recently pointed out by Caves, Fuchs, and Rungta [19] that real quantum mechanics (that is, quantum mechanics defined over real vector spaces [20, 21, 22, 23]) provides an interesting foil theory whose study may shed some light on which aspects of quantum entanglement are unique to standard quantum theory, and which ones are more generic over other physical theories endowed with the phenomenon of entanglement.

We assume further that i) we only have access to, say, $k < n$ of these states and ii) the (PD) $p_j$ (of finding the system in the basis-state $j$) is uniform both for $k = n$ and for $k = n - p$. 

7
Now, it is well known that i) the entropy $S$ is a functional of the probability distribution that quantifies the degree of ignorance for a given scenario \[3\] and ii) the uniform distribution always yields the largest possible entropic value \[24\]. In the present circumstances we thus have, for Tsallis’ entropy $S_q$, \[10\]

$$S_q^{(k=\nu)} = k_B \frac{n^{1-q} - 1}{1-q}$$

$$S_q^{(k \leq \nu)} = \frac{k_B}{1-q} \left[ \sum_{i=1}^{k} p_i^q - 1 \right], \quad \text{(19)}$$

so that

$$D_1 \equiv S_q^{(k=\nu)} - S_q^{(k \leq \nu)} = \frac{k_B}{1-q} \left[ \sum_{i=1}^{k} p_i^q - n^{1-q} \right]. \quad \text{(20)}$$

$D_1$ quantifies the information gain (or ignorance loss) that, paradoxically, ensues from the fact that one does not have access to $n-k$ states. For $q = 1$, $D_1$ is actually the Kullback-Leibler cross entropy.

Some refinement is still needed with reference to the above considerations. We have seen in Section 3 that a uniform distribution also ensues for $k = n-p$,

$$S_q^{(k=n-p)} = k_B \frac{\left(n-p\right)^{1-q} - 1}{1-q} \leq S_q^{(k=n)}. \quad \text{(21)}$$

It is clear that, according to \[19\], the entropy of a uniform distribution (all the pertinent $p_i$’s are equal), grows with the corresponding number of states. Notice that the above considerations only make sense within the non extensive framework. For $q = 1$ one has $n = \infty$.

Consider now the particular situation $k = n-p+1$. Clearly, then, the pertinent entropy has to be larger than the uniform one for $k = n-p$

$$S_q^{(k=n-p+1)} = k_B \frac{(n-p+1)^{1-q} - 1}{1-q} \geq S_q^{k=n} \Rightarrow (n-p+1)^{1-q} \geq (n-p)^{1-q}, \quad \text{(22)}$$

which implies

$$q < 0, \quad \text{(23)}$$

and we understand now point (A) at the end of Section 3. Conversely, take now $k = n-p-1$. A similar line of reasoning yields

$$S_q^{(k=n-p-1)} = k_B \frac{(n-p-1)^{1-q} - 1}{1-q} \leq S_q^{k=n} \Rightarrow (n-p-1)^{1-q} \leq (n-p)^{1-q}, \quad \text{(24)}$$
which clearly implies

\[ q > 1, \]  \hspace{1cm} (25)

and we thus understand point (B) at the end of Section 3.

5 Conclusions

Boltzmann-Gibbs’ and Tsallis’ probability distributions can be derived via purely geometric arguments starting from a uniform distribution. In particular, we have shown that such geometric considerations can be employed in order to determine the non-extensivity index \( q \). The way of foxing \( q \) remains still an open problem for non-extensive thermostatistics, which gives our result an additional interest.

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