Alliance free sets in Cartesian product graphs

Ismael G. Yero¹, Juan A. Rodríguez-Velázquez¹, Sergio Bermudo²

¹Departament d’Enginyeria Informàtica i Matemàtiques, Universitat Rovira i Virgili, Av. Països Catalans 26, 43007 Tarragona, Spain.
ismael.gonzalez@urv.cat, juanalberto.rodriguez@urv.cat
²Department of Economy, Quantitative Methods and Economic History
Pablo de Olavide University, Carretera de Utrera Km. 1, 41013-Sevilla, Spain
sbernav@upo.es

December 12, 2011

Abstract

Let $G = (V, E)$ be a graph. For a non-empty subset of vertices $S \subseteq V$, and vertex $v \in V$, let $\delta_S(v) = |\{u \in S : uv \in E\}|$ denote the cardinality of the set of neighbors of $v$ in $S$, and let $\overline{S} = V - S$. Consider the following condition:

$$\delta_S(v) \geq \delta_{\overline{S}}(v) + k,$$

which states that a vertex $v$ has at least $k$ more neighbors in $S$ than it has in $\overline{S}$. A set $S \subseteq V$ that satisfies Condition (1) for every vertex $v \in S$ is called a defensive $k$-alliance; for every vertex $v$ in the neighborhood of $S$ is called an offensive $k$-alliance. A subset of vertices $S \subseteq V$, is a powerful $k$-alliance if it is both a defensive $k$-alliance and an offensive $(k + 2)$-alliance. Moreover, a subset $X \subset V$ is a defensive (an offensive or a powerful) $k$-alliance free set if $X$ does not contain any defensive (offensive or powerful, respectively) $k$-alliance. In this article we study the relationships between defensive (offensive, powerful) $k$-alliance free sets in Cartesian product graphs and defensive (offensive, powerful) $k$-alliance free sets in the factor graphs.
1 Introduction

The study of relationships between invariants of Cartesian product graphs and invariants of its factor graphs appears frequently in researches about graph theory. In this sense, there are important open problems which are being investigated now. For instance, the Vizing’s conjecture [11, 12, 31], which is one of the most known open problems in graph theory, states that the domination number of the Cartesian product of two graphs is at least equal to the product of the domination numbers of these two graphs. Some variations and partial results about this conjecture have been developed in the last years, like those in [3, 4, 7, 29].

Apart from the domination number, there are several invariants which have been studied in Cartesian product graphs. For instance, the geodetic number [2, 4, 10, 16], the metric dimension [6], the partition dimension [33], the Menger number [19], the \(k\)-domination number [15], the offensive \(k\)-alliance number [1], the \(k\)-alliance partition number [5, 34] and the offensive \(k\)-alliance partition number [28].

This article concerns the study of alliance free sets in Cartesian product graphs. Since (defensive, offensive and powerful) alliances in graphs were first introduced by Kristiansen et al. [18], several authors have studied their mathematical properties [1, 5, 14, 21, 22, 23, 24, 25, 26, 27, 28, 32, 34] (the reader is referred to the Ph.D. Thesis [32] for a more complete list of references). Applications of alliances can be found in the Ph. D. Thesis [22] where the author studied problems of partitioning graphs into alliances and its application to data clustering. On the other hand, defensive alliances represent the mathematical model of web communities, by adopting the definition of Web Community proposed by Flake et al. in [9], “a Web Community is a set of web pages having more hyperlinks (in either direction) to members of the set than to non-members”. Other applications of alliances were presented in [13] (where alliances were used in a quantitative analysis of secondary RNA structure), [17] (where alliances were used in the study of predator-prey models on complex networks), [30] (where alliances were used in the study of spatial models of cyclical population interactions) and [20] (where
alliances were used as a model of monopoly).

In this work we continue the previous studies \([22, 23, 25, 26, 21]\) on \(k\)-alliance free sets and \(k\)-alliance cover sets focusing our attention on the particular case of Cartesian product graphs. We study the relationships between defensive (offensive, powerful) \(k\)-alliance free sets in Cartesian product graphs and defensive (offensive, powerful) \(k\)-alliance free sets in the factor graphs. The plan of the article is the following: In Section 2 we present the notation and terminology used and we recall the definitions of Cartesian product graph, defensive (offensive and powerful) \(k\)-alliance, defensive (offensive and powerful) \(k\)-alliance free set and defensive (offensive and powerful) \(k\)-alliance cover set. Section 3 is devoted to the study of defensive \(k\)-alliances. More specifically, we give a sufficient condition for the existence of defensive \(k\)-alliance free sets in cartesian product graphs and we study the relationships between the maximum cardinality of a defensive \(k\)-alliance free set in Cartesian product graphs and several invariants of the factor graphs, including the order and the independence number. Analogously, sections 4 and 5, respectively, are devoted to the study of offensive and powerful \(k\)-alliance free sets. In section 6 we present the conclusions.

\section{Notation and terminology}

In this paper \(G = (V, E)\) denotes a simple graph of order \(n\), minimum degree \(δ\) and maximum degree \(Δ\). The independence number of \(G\) is denoted by \(α(G)\). For a non-empty set \(S \subseteq V\) and a vertex \(v \in V\), \(δ_S(v)\) denotes the number of neighbors \(v\) has in \(S\) and \(δ(v)\) denotes the degree of \(v\). The complement of the set \(S\) in \(V\) is denoted by \(\overline{S}\). The set of vertices of \(\overline{S}\) which are adjacent to at least one vertex in \(S\) is denoted by \(\overline{∂S}\).

A non-empty set of vertices \(S \subseteq V\) is called a \textit{defensive} (respectively, an \textit{offensive}) \(k\)-\textit{alliance} in \(G\) if for every \(v \in S\) (respectively, \(v \in \overline{∂S}\)), \(δ_S(v) \geq δ_{\overline{S}}(v) + k\), where \(k \in \{-Δ, ..., Δ\}\) (respectively, \(k \in \{2 − Δ, ..., Δ\}\)). Also, a non-empty set of vertices \(S \subseteq V\) is called a \textit{powerful} \(k\)-\textit{alliance} in \(G\) if it is both, defensive \(k\)-alliance and offensive \((k+2)\)-alliance, \(k \in \{-Δ, ..., Δ − 2\}\). Notice that, since \(V\) is an offensive \(k\)-alliance for every \(k \in \{2 − Δ, ..., Δ\}\), \(V\) is a powerful \(k\)-alliance if and only if it is a defensive \(k\)-alliance.

A set \(X \subseteq V\) is (defensive, offensive, powerful) \(k\)-\textit{alliance free}, \((k\text{-daf, } k\text{-oaf, } k\text{-paf})\), if for all (defensive, offensive, powerful) \(k\)-alliance \(S\), \(S \setminus X \neq \emptyset\), i.e., \(X\) does not contain any (defensive, offensive, powerful) \(k\)-alliance as a
subset \[[24]\].

Associated with the characteristic sets defined above we have the following invariants:

- \(\phi^d_k(G)\): maximum cardinality of a \(k\)-daf set in \(G\), \(k \in \{-\Delta, \ldots, \Delta\}\).
- \(\phi^o_k(G)\): maximum cardinality of a \(k\)-oaf set in \(G\), \(k \in \{2 - \Delta, \ldots, \Delta\}\).
- \(\phi^p_k(G)\): maximum cardinality of a \(k\)-paf set in \(G\), \(k \in \{-\Delta, \ldots, \Delta - 2\}\).

We now state the following fact on (defensive, offensive and powerful) \(k\)-alliance free sets that will be useful throughout this article.

**Remark 1.** If \(X\) is a \(k\)-alliance free set and \(k < k'\), then \(X\) is a \(k'\)-alliance free set.

A set \(Y \subseteq V\) is a (defensive, offensive, powerful) \(k\)-alliance cover set if for all (defensive, offensive, powerful) \(k\)-alliance \(S\), \(S \cap Y \neq \emptyset\), i.e., \(Y\) contains at least one vertex from each (defensive, offensive, powerful) \(k\)-alliance of \(G\).

The following duality between \(k\)-alliance cover sets and \(k\)-alliance free sets allows us to study \(k\)-alliance cover sets from the results obtained on \(k\)-alliance free sets, so in this article we only consider the study of \(k\)-alliance free sets.

**Remark 2.** \(X\) is a \(k\)-alliance cover set if and only if \(\overline{X}\) is a \(k\)-alliance free set.

We recall that the Cartesian product of two graphs \(G_1 = (V_1, E_1)\) and \(G_2 = (V_2, E_2)\) is the graph \(G_1 \times G_2 = (V, E)\), such that \(V = \{(a, b) : a \in V_1, b \in V_2\}\) and two vertices \((a, b) \in V\) and \((c, d) \in V\) are adjacent in \(G_1 \times G_2\) if and only if, either \(a = c\) and \(bd \in E_2\) or \(b = d\) and \(ac \in E_1\).

For a set \(A \subseteq V_1 \times V_2\) we denote by \(P_{V_i}(A)\) the projection of \(A\) over \(V_i\), \(i \in \{1, 2\}\).

### 3 Defensive \(k\)-alliance free sets in Cartesian product graphs

To begin with the study we present the following straightforward result.
Remark 3. Let $G_i$ be a graph of order $n_i$, minimum degree $\delta_i$ and maximum degree $\Delta_i$, $i \in \{1, 2\}$. Then, for every $k \in \{1 - \delta_1 - \delta_2, ..., \Delta_1 + \Delta_2\}$,

$$
\phi^d_k(G_1 \times G_2) \geq \alpha(G_1)\alpha(G_2) + \min\{n_1 - \alpha(G_1), n_2 - \alpha(G_2)\}.
$$

Proof. For every graph $G$ of minimum degree $\delta$ and maximum degree $\Delta$, any independent set in $G$ is a $k$-daf set for $k \in \{1 - \delta, ..., \Delta\}$. Hence, $\phi^d_k(G_1 \times G_2) \geq \alpha(G_1 \times G_2)$, for every $k \in \{1 - \delta_1 - \delta_2, ..., \Delta_1 + \Delta_2\}$, and by the Vizing’s inequality, $\alpha(G_1 \times G_2) \geq \alpha(G_1)\alpha(G_2) + \min\{n_1 - \alpha(G_1), n_2 - \alpha(G_2)\}$, we obtain the result.

Let $G_1$ be the star graph of order $t + 1$ and let $G_2$ be the path graph of order 3. In this case, $\phi^d_k(G_1 \times G_2) = 2t + 1$ for $k \in \{-1, 0\}$. Therefore, the above bound is tight. Even so, Corollary 5 (ii) improves the above bound for the cases where $\phi^d_k(G_i) > \alpha(G_i)$, for some $i \in \{1, 2\}$.

Theorem 4. Let $G_i = (V_i, E_i)$ be a simple graph of maximum degree $\Delta_i$, $i \in \{1, 2\}$, and let $S \subseteq V_1 \times V_2$. Then the following assertions hold.

(i) If $P_{V_i}(S)$ is a $k_i$-daf set in $G_i$, then $S$ is a $(k_i + \Delta_j)$-daf set in $G_1 \times G_2$, where $j \in \{1, 2\}$, $j \neq i$.

(ii) If for every $i \in \{1, 2\}$, $P_{V_i}(S)$ is a $k_i$-daf set in $G_i$, then $S$ is a $(k_1 + k_2 - 1)$-daf set in $G_1 \times G_2$.

Proof. Let $A \subseteq S$ and we suppose $P_{V_i}(S)$ is a $k_i$-daf set in $G_i$. Since $P_{V_i}(A) \subseteq P_{V_i}(S)$, there exists $a \in P_{V_i}(A)$ such that $\delta_{P_{V_i}(A)}(a) < \delta_{P_{V_i}(A)}(a) + k_i$. If we take $b \in V_2$ such that $(a, b) \in A$, then

$$
\delta_A(a, b) \leq \delta_{P_{V_i}(A)}(a) + \delta_{P_{V_2}(A)}(b) < \delta_{P_{V_i}(A)}(a) + k_i + \delta(b) \leq \delta_{P_{V_2}(A)}(a) + k_1 + \Delta_2.
$$

Thus, $A$ is not a defensive $(k_1 + \Delta_2)$-alliance in $G_1 \times G_2$. Therefore, (i) follows.

In order to prove (ii), let $x \in X = P_{V_i}(A)$ such that $\delta_X(x) < \delta_X(x) + k_1$. Let $A_x \subseteq A$ be the set composed by the elements of $A$ whose first component is $x$. On the other hand, since $P_{V_2}(S)$ is a $k_2$-daf set and $Y = P_{V_2}(A_x) \subseteq P_{V_2}(S)$, there exists $y \in Y$ such that $\delta_Y(y) < \delta_Y(y) + k_2$. Notice that $(x, y) \in A$. Let $A_y \subseteq A$ be the set composed by the elements of $A$ whose second
component is \( y \). Hence,

\[
\delta_A(x, y) = \delta_{A_x}(x, y) + \delta_{A_y}(x, y) \\
\leq \delta_Y(y) + \delta_X(x) \\
< \delta_{\overline{X}}(y) + \delta_{\overline{X}}(x) + k_1 + k_2 - 1 \\
\leq \delta_{\overline{A_x}}(x, y) - \delta(x) + \delta_{\overline{A_y}}(x, y) - \delta(y) + k_1 + k_2 - 1 \\
\leq \delta_{\overline{A_x}}(x, y) + \delta_{\overline{A_y}}(x, y) + k_1 + k_2 - 1 \\
= \delta_{\overline{A}}(x, y) + k_1 + k_2 - 1.
\]

Thus, \( A \) is not a defensive \((k_1 + k_2 - 1)\)-alliance in \( G_1 \times G_2 \) and, as a consequence, (ii) follows. \( \square \)

**Corollary 5.** Let \( G_l \) be a graph of order \( n_l \), maximum degree \( \Delta_l \) and minimum degree \( \delta_l \), with \( l \in \{1, 2\} \). Then the following assertions hold.

(i) For every \( k \in \{\Delta_j - \Delta_i, \ldots, \Delta_i + \Delta_j\} \ (i, j \in \{1, 2\}, i \neq j),

\[
\phi^d_k(G_1 \times G_2) \geq n_j \phi^d_{k - \Delta_j}(G_i).
\]

(ii) For every \( k_i \in \{1 - \delta_i, \ldots, \Delta_i\}, i \in \{1, 2\},

\[
\phi^d_{k_1 + k_2 - 1}(G_1 \times G_2) \geq \phi^d_{k_1}(G_1)\phi^d_{k_2}(G_2) + \min\{n_1 - \phi^d_{k_1}(G_1), n_2 - \phi^d_{k_2}(G_2)\}.
\]

**Proof.** By Theorem 4 (i) we conclude that for every \( k_i\)-daf set \( S_i \) in \( G_i, i \in \{1, 2\} \), the sets \( S_1 \times V_2 \) and \( V_1 \times S_2 \) are \((k_1 + \Delta_2)\)-daf and \((k_2 + \Delta_1)\)-daf, respectively, in \( G_1 \times G_2 \). Therefore, (i) follows.

In order to prove (ii), let \( V_1 = \{u_1, u_2, \ldots, u_{n_1}\} \) and \( V_2 = \{v_1, v_2, \ldots, v_{n_2}\} \). Moreover, let \( S_i \) be a \( k_i\)-daf set of maximum cardinality in \( G_i, i \in \{1, 2\} \). We suppose \( S_1 = \{u_1, \ldots, u_t\} \) and \( S_2 = \{v_1, \ldots, v_s\} \). By Theorem 4 (ii) we deduce that \( S_1 \times S_2 \) is a \((k_1 + k_2 - 1)\)-daf set in \( G_1 \times G_2 \). Now let \( X = \{(u_{r+i}, v_{s+i}), i = 1, \ldots, t\} \), where \( t = \min\{n_1 - r, n_2 - s\} \) and let \( S = X \cup (S_1 \times S_2) \). Since, for every \( x \in X \), \( \delta_S(x) = 0 \) and \( k_i > -\delta_i, i \in \{1, 2\} \), we obtain that \( S \) is a \((k_1 + k_2 - 1)\)-daf set in \( G_1 \times G_2 \). Thus, \( \phi^d_{k_1 + k_2 - 1}(G_1 \times G_2) \geq |S| = \phi^d_{k_1}(G_1)\phi^d_{k_2}(G_2) + \min\{n_1 - \phi^d_{k_1}(G_1), n_2 - \phi^d_{k_2}(G_2)\} \). \( \square \)

Now we state the following fact that will be useful for an easy understanding of several examples in this paper.
Proposition 6. Let $G$ be a graph of order $n$ and maximum degree $\Delta$. Then $\phi^d_k(G) = n$, for each of the following cases:

(i) $G$ is a tree of maximum degree $\Delta \geq 2$ and $k \in \{2, \ldots, \Delta\}$.

(ii) $G$ is a planar graph of maximum degree $\Delta \geq 6$ and $k \in \{6, \ldots, \Delta\}$.

(iii) $G$ is a planar triangle-free graph of maximum degree $\Delta \geq 4$ and $k \in \{4, \ldots, \Delta\}$.

Proof. Suppose $S$ is a defensive $k$-alliance in $G = (V, E)$. That is, for every $v \in S$, it follows

$$2\delta_S(v) \geq \delta(v) + k.$$  \hspace{1cm} (2)

If some vertex $v \in S$ satisfies $\delta(v) < k$, then equation (2) leads to $\delta_S(v) > \delta(v)$, a contradiction. Hence, for every $v \in S$ we have $\delta(v) \geq k$ and, as a consequence, equation (2) leads to $\delta_S(v) \geq k$. Now, let $m_s$ be the size of the subgraph induced by $S$. Then we have

$$2m_s = \sum_{v \in S} \delta_S(v) \geq k|S|.$$  \hspace{1cm} (3)

Case (i). Since $G$ is a tree, we obtain $2(|S| - 1) \geq 2m_s \geq k|S| \geq 2|S|$, a contradiction.

For the cases (ii) and (iii) we have $|S| \geq 3$, due to that if $|S| \leq 2$, then equation (2) leads to $2 \geq \delta(v) + k$, a contradiction. It is well-known that the size of a planar graph of order $n' \geq 3$ is bounded above by $3(n' - 2)$. Moreover, in the case of triangle-free graphs the bound is $2(n' - 2)$. Therefore, in case (ii) we have $m_s \leq 3(|S| - 2)$ and, as a consequence, equation (3) leads to $6(|S| - 2) \geq k|S| \geq 6|S|$, a contradiction. Analogously, in case (iii) we have $m_s \leq 2(|S| - 2)$ and, as a consequence, equation (3) leads to $4(|S| - 2) \geq k|S| \geq 4|S|$, a contradiction. \hfill \Box

We emphasize that Corollary 5 and Proposition 6 lead to infinite families of graphs whose Cartesian product satisfies $\phi^d_k(G_1 \times G_2) = n_1n_2$. For instance, if $G_1$ is a tree of order $n_1$ and maximum degree $\Delta_1 \geq 2$, $G_2$ is a graph of order $n_2$ and maximum degree $\Delta_2$, and $k \in \{2 + \Delta_2, \ldots, \Delta_1 + \Delta_2\}$, we have $\phi^d_k(G_1 \times G_2) = \phi^d_{k-\Delta_2}(G_1)n_2 = n_1n_2$. In particular, if $G_2$ is a cycle graph, then $\phi^d_k(G_1 \times G_2) = n_1n_2$.

Another example of equality in Corollary 5 (ii) is obtained, for instance, taking the Cartesian product of the star graph $St$ of order $t + 1$ and the path
Theorem 7. Let $G_i = (V_i, E_i)$ be a graph and let $S_i \subseteq V_i$, $i \in \{1, 2\}$. If $S_1 \times S_2$ is a $k$-daf set in $G_1 \times G_2$ and $S_2$ is a defensive $k'$-alliance in $G_2$, then $S_1$ is a $(k - k')$-daf set in $G_1$.

Proof. If $S \subseteq S_1$, then $S \times S_2 \subseteq S_1 \times S_2$ is a $k$-daf set in $G_1 \times G_2$. So, there exists $(a, b) \in S \times S_2$ such that $\delta_{S \times S_2}(a, b) < \delta_{S_1 \times S_2}(a, b) + k$. Thus, we have

$$\delta_S(a) + \delta_{S_2}(b) = \delta_{S \times S_2}(a, b) < \delta_{S_1 \times S_2}(a, b) + k = \delta_S(a) + \delta_{S_2}(b) + k. \tag{4}$$

As $S_2$ is a defensive $k'$-alliance in $G_2$, for every $b \in S_2$ we have, $\delta_{S_2}(b) \geq \delta_{S_2}(b) + k'$. Hence, from equation (4) we obtain $\delta_S(a) < \delta_S(a) + k - k'$. Therefore, $S$ is not a defensive $(k - k')$-alliance in $G_1$ and, as a consequence, $S_1$ is a $(k - k')$-daf set. \hfill \Box

Taking into account that $V_2$ is a defensive $\delta_2$-alliance in $G_2$ we obtain the following result.

Figure 1: This graph is the Cartesian product $S_3 \times P_4$ where $S = \{(1, 1), (2, 1), (4, 1), (1, 2), (2, 2), (4, 2), (1, 3), (2, 3), (4, 3), (3, 4)\}$ is a maximum defensive 0-alliance free set.
Corollary 8. Let $G_i = (V_i, E_i)$ be a graph, $i \in \{1, 2\}$. Let $\delta_2$ be the minimum degree of $G_2$ and let $S_1 \subseteq V_1$. If $S_1 \times V_2$ is a $k$-daf set in $G_1 \times G_2$, then $S_1$ is a $(k - \delta_2)$-daf set in $G_1$.

By Theorem 4 (i) and Corollary 8 we obtain the following result.

Proposition 9. Let $G_1$ be a graph of maximum degree $\Delta_1$ and let $G_2$ be a $\delta_2$-regular graph. For every $k \in \{\delta_2 - \Delta_1, ..., \Delta_1 + \delta_2\}$, $S_1 \times V_2$ is a $k$-daf set in $G_1 \times G_2$ if and only if $S_1$ is a $(k - \delta_2)$-daf set in $G_1$.

4 Offensive $k$-alliance free sets in Cartesian product graphs

Theorem 10. Let $G_i = (V_i, E_i)$ be a graph, $i \in \{1, 2\}$, and let $S \subseteq V_1 \times V_2$. If $P_{V_i}(S)$ is a $k$-oaf set in $G_i$, then $S$ is a $(k - \delta_j)$-oaf set in $G_1 \times G_2$, where $\delta_j$ denotes the minimum degree of $G_j$ and $j \in \{1, 2\}, i \neq j$.

Proof. If $P_{V_i}(S)$ is a $k$-oaf set in $G_1$ and $A \subseteq S$, then $P_{V_i(A)} \subseteq P_{V_i}(S)$ is a $k$-oaf set in $G_1$. So, there exists $a \in \partial P_{V_i}(A)$, such that $\delta_{P_{V_i}(A)}(a) < \delta_{P_{V_i(A)}}(a) + k$. Let $a' \in P_{V_i}(A)$ such that $a$ and $a'$ are adjacent, and let $Y_a$ be the set of elements of $A$ whose first component is $a'$. Thus, if $b \in P_{V_2}(Y_a')$, then $(a, b) \in \partial A$, so we have

$$\delta_A(a, b) \leq \delta_{P_{V_i}(A)}(a) < \delta_{P_{V_i(A)}}(a) + k \leq \delta_A(a, b) - \delta(b) + k \leq \delta_A(a, b) + k - \delta_2.$$ 

Therefore, $A$ is not an offensive $(k - \delta_2)$-alliance in $G_1 \times G_2$. The proof of the other case is completely analogous. \hfill \Box

From Theorem 10 we conclude that for every $k_i$-oaf set $S_i$ in $G_i$, $i \in \{1, 2\}$, the sets $S_1 \times V_2$ and $V_1 \times S_2$ are $(k_1 - \delta_2)$-oaf and $(k_2 - \delta_1)$-oaf, respectively, in $G_1 \times G_2$. Therefore, we obtain the following result.

Corollary 11. Let $G_i$ be a graph of order $n_i$, maximum degree $\Delta_i$ and minimum degree $\delta_i$, $i \in \{1, 2\}$. Then, for every $k \in \{2 - \delta_j - \Delta_i, ..., \Delta_i - \delta_j\}$, $\phi_k^l(G_1 \times G_2) \geq n_j \phi_{k+\delta_j}^l(G_i)$, where $i, j \in \{1, 2\}, i \neq j$.

Example of equality in the above result is the following. If we take $G_1 = C_4$, $G_2 = P_3$ and $k_2 = 2$, then $\phi_0^0(C_4 \times P_3) = 8 = 4\phi_2^0(P_3)$. 

9
Theorem 12. Let \( G_i = (V_i, E_i) \) be a graph of minimum degree \( \delta_i \) and maximum degree \( \Delta_i \). If \( S_i \) is a \( k_i \)-oaf set in \( G_i, i \in \{1, 2\} \), then for every \( k \in \{k', \ldots, \Delta_1 + \Delta_2\} \), \((S_1 \times V_2) \cup (V_1 \times S_2)\) is a \( k \)-oaf set in \( G_1 \times G_2 \), where \( k' = \max \{k_1 - \delta_2, k_2 - \delta_1, \min\{k_2 + \Delta_1, k_1 + \Delta_2\}\} \).

Proof. Let \( A \subseteq (S_1 \times V_2) \cup (V_1 \times S_2) \). By Theorem 10 we deduce that, if \( A \subseteq S_1 \times V_2 \), then \( A \) is a \((k_1 - \delta_2)\)-oaf set in \( G_1 \times G_2 \). Analogously, if \( A \subseteq V_1 \times S_2 \), then \( A \) is a \((k_2 - \delta_1)\)-oaf set in \( G_1 \times G_2 \).

Now we suppose \( A \not\subseteq S_1 \times V_2 \) and \( A \not\subseteq V_1 \times S_2 \). Let \( B = A \setminus (S_1 \times V_2) \).

For every \( a \in P_1(B) \), the set \( Y_a \), composed by the elements of \( B \) whose first component is \( a \), satisfies that \( P_2(Y_a) \) is a \( k_2 \)-oaf set in \( G_2 \). Then, there exists \( b \in \partial P_2(Y_a) \) such that \( \delta_{P_2(Y_a)}(b) < \delta_{P_2(Y_a)}(b) + k_2 \). Also, notice that \((a, b) \in \partial A \). Thus,

\[
\delta_A(a, b) < \delta_{P_2(Y_a)}(b) + \delta(a) < \delta_{P_2(Y_a)}(b) + k_2 + \delta(a) \leq \delta_{A}(a, b) + k_2 + \Delta_i.
\]

We conclude that \( A \) is not an offensive \((k_2 + \Delta_i)\)-alliance in \( G_1 \times G_2 \). Analogously, \( A \) is not an offensive \((k_1 + \Delta_2)\)-alliance in \( G_1 \times G_2 \). Therefore, the result follows.

Corollary 13. Let \( G_i \) be a graph of order \( n_i \), minimum degree \( \delta_i \) and maximum degree \( \Delta_i, i \in \{1, 2\} \). Let \( k' = \max \{k_1 - \delta_2, k_2 - \delta_1, \min\{k_2 + \Delta_1, k_1 + \Delta_2\}\} \), where \( k_i \in \{2 - \Delta_1, \ldots, \Delta_i\} \). Then, for every \( k \in \{k', \ldots, \Delta_1 + \Delta_2\} \),

\[
\phi^o_k(G_1 \times G_2) \geq n_1 \phi^o_{k_2}(G_2) + n_2 \phi^o_{k_1}(G_1) - \phi^o_{k_1}(G_1) \phi^o_{k_2}(G_2).
\]

For instance, if we take \( G_1 = C_3, G_2 = P_3, k_1 = 1 \) and \( k_2 = 2 \), then \( \phi_2^o(C_3 \times P_3) = 7 = 3\phi_2^o(P_3) + 3\phi_1^o(C_3) - \phi_1^o(C_3) \phi_2^o(P_3) \).

5 Powerful \( k \)-alliance free sets in Cartesian product graphs

Since for every graph \( G \), \( \phi^o_k(G) \geq \max\{\phi^d_k(G), \phi^o_{k+2}(G)\} \), we have that lower bounds on \( \phi^d_k(G) \) and \( \phi^o_{k+2}(G) \) lead to lower bounds on \( \phi^o_k(G) \). So, by the results obtained in the above sections on \( \phi^d_k(G_1 \times G_2) \) and \( \phi^o_{k+2}(G_1 \times G_2) \) we deduce lower bounds on \( \phi^o_k(G_1 \times G_2) \).

We emphasize that there are graphs where \( \phi^o_k(G) > \max\{\phi^d_k(G), \phi^o_{k+2}(G)\} \).

For instance, the graph of above figure satisfies \( \phi^o_2(G) = 9 \) while \( \phi^o_2(G) = 8 \) and \( \phi^o_4(G) = 7 \).
Figure 2: The graph $G = (V, E)$ is the Cartesian product of the cycle graph $C_3$ by the path graph $P_3$ where $S = V \setminus \{(1, 3), (2, 3)\}$ is a maximum offensive 3-alliance free set.

Theorem 14. Let $G_i = (V_i, E_i)$ be a simple graph of maximum degree $\Delta_i$ and minimum degree $\delta_i$, $i \in \{1, 2\}$, and let $S \subseteq V_1 \times V_2$. Then the following assertions hold.

(i) If $P_{V_i}(S)$ is a $k_i$-paf set in $G_i$, then, for every $k \in \{k_i + \Delta_j, \ldots, \Delta_i + \Delta_j - 2\}$, $S$ is a $k$-paf set in $G_1 \times G_2$, where $j \in \{1, 2\}$, $j \neq i$.

(ii) If for every $i \in \{1, 2\}$, $P_{V_i}(S)$ is a $k_i$-paf set in $G_i$, then, for every $k \in \{k', \ldots, \Delta_1 + \Delta_2 - 2\}$, $S$ is a $k$-paf set in $G_1 \times G_2$, where $k' = \max\{k_1 + k_2 - 1, \min\{k_2 - \delta_1, k_1 - \delta_2\}\}$.

Proof. Let $A \subseteq S$. We suppose $P_{V_i}(S)$ is a $k_i$-paf set in $G_i$ for some $i \in \{1, 2\}$. Since $P_{V_i}(A) \subseteq P_{V_i}(S)$, it follows that $P_{V_i}(A)$ is not a powerful $k_i$-alliance in $G_i$. If $P_{V_i}(A)$ is not a defensive $k_i$-alliance, by analogy to the proof of Theorem 4 (i), we obtain that $A$ is not a defensive $(k_i + \Delta_j)$-alliance in

Figure 3: A graph $G = (V, E)$ where $V$ is a powerful 2-alliance free set, although $\{2, 3, 4, 5, 6, 8\}$ is a defensive 2-alliance and $\{3, 4, 5, 6, 7\}$ is an offensive 4-alliance.

Theorem 14. Let $G_i = (V_i, E_i)$ be a simple graph of maximum degree $\Delta_i$ and minimum degree $\delta_i$, $i \in \{1, 2\}$, and let $S \subseteq V_1 \times V_2$. Then the following assertions hold.

(i) If $P_{V_i}(S)$ is a $k_i$-paf set in $G_i$, then, for every $k \in \{k_i + \Delta_j, \ldots, \Delta_i + \Delta_j - 2\}$, $S$ is a $k$-paf set in $G_1 \times G_2$, where $j \in \{1, 2\}$, $j \neq i$.

(ii) If for every $i \in \{1, 2\}$, $P_{V_i}(S)$ is a $k_i$-paf set in $G_i$, then, for every $k \in \{k', \ldots, \Delta_1 + \Delta_2 - 2\}$, $S$ is a $k$-paf set in $G_1 \times G_2$, where $k' = \max\{k_1 + k_2 - 1, \min\{k_2 - \delta_1, k_1 - \delta_2\}\}$.

Proof. Let $A \subseteq S$. We suppose $P_{V_i}(S)$ is a $k_i$-paf set in $G_i$ for some $i \in \{1, 2\}$. Since $P_{V_i}(A) \subseteq P_{V_i}(S)$, it follows that $P_{V_i}(A)$ is not a powerful $k_i$-alliance in $G_i$. If $P_{V_i}(A)$ is not a defensive $k_i$-alliance, by analogy to the proof of Theorem 4 (i), we obtain that $A$ is not a defensive $(k_i + \Delta_j)$-alliance in

11
If $P_{V_i}(A)$ is not an offensive $(k_i + 2)$-alliance in $G_i$, then by analogy to the proof of Theorem 10, we obtain that $A$ is not an offensive $(k_i - \delta_j + 2)$-alliance in $G_1 \times G_2$, $j \neq i$. Since, $k_i + \Delta_j > k_i - \delta_j$, we obtain that $A$ is not a powerful $(k_i + \Delta_j)$-alliance in $G_1 \times G_2$. Therefore, (i) follows.

If for every $l \in \{1, 2\}$, $P_{V_l}(S)$ is a $k_l$-paf set in $G_l$, then $P_{V_i}(A)$ is not a powerful $k_l$-alliance in $G_l$. Hence, we differentiate two cases.

Case (1): For some $l \in \{1, 2\}$, $P_{V_l}(A)$ is not a defensive $k_l$-alliance. We suppose $P_{V_i}(A)$ is not a defensive $k_1$-alliance. Hence, there exists $x \in P_{V_i}(A)$ such that $\delta_{P_{V_l}(A)}(x) < \delta_{P_{V_i}(A)}(x) + k_1$. Let $A_x \subseteq A$ be the set composed by the elements of $A$ whose first component is $x$. If $P_{V_2}(A_x) \subset P_{V_2}(S)$ is not a defensive $k_2$-alliance, then by analogy to the proof of Theorem 4 (ii), we obtain that $A$ is not a defensive $(k_1 + k_2 - 1)$-alliance in $G_1 \times G_2$. On the other hand, if $P_{V_2}(A_x)$ is a defensive $k_2$-alliance, then it is not an offensive $(k_2 + 2)$-alliance. Thus, there exists $y \in \partial P_{V_2}(A_x)$ such that $\delta_{P_{V_2}(A_x)}(y) < \delta_{P_{V_2}(A_x)}(y) + (k_2 + 2)$. We note that $(x, y) \in \partial A$. Hence,

$$\delta_A(x, y) \leq \delta_{P_{V_i}(A)}(x) + \delta_{P_{V_l}(A_x)}(y)$$

$$< \delta_{P_{V_l}(A)}(x) + \delta_{P_{V_2}(A_x)}(y) + k_1 + k_2 + 1$$

$$\leq \delta_A(x, y) + k_1 + k_2 + 1.$$

As a consequence, $A$ is not an offensive $(k_1 + k_2 + 1)$-alliance in $G_1 \times G_2$. Thus, in this case, $A$ is not a powerful $(k_1 + k_2 - 1)$-alliance in $G_1 \times G_2$.

Case (2): For every $i \in \{1, 2\}$, $P_{V_i}(A)$ is not an offensive $(k_i + 2)$-alliance in $G_i$. In this case, as we have shown in the proof of (i), $A$ is not an offensive $(k_i - \delta_j + 2)$-alliance in $G_1 \times G_2$, $j \in \{1, 2\}$, $j \neq i$.

As a consequence, for $k = \max\{k_1 + k_2 - 1, k_1 - \delta_2, k_2 - \delta_1\}$, $A$ is not a powerful $k$-alliance in $G_1 \times G_2$. Hence, $S$ is a $k$-paf set in $G_1 \times G_2$. Therefore, (ii) follows. □

**Corollary 15.** Let $G_1$ be a graph of order $n_1$, maximum degree $\Delta_1$ and minimum degree $\delta_1$, $l \in \{1, 2\}$. Let $k_l \in \{1 - \delta_l, \ldots, \Delta_l - 2\}$. Then the following assertions hold.

(i) For every $k \in \{\Delta_j - \Delta_i, \ldots, \Delta_i + \Delta_j - 2\}$, $(i, j \in \{1, 2\}$, $i \neq j$)

$$\phi_k^p(G_1 \times G_2) \geq n_j \phi_{k-\Delta_j}^p(G_i).$$

(ii) For every $k \in \{k_1 + k_2 - 1, \ldots, \Delta_1 + \Delta_2 - 2\}$,

$$\phi_k^p(G_1 \times G_2) \geq \phi_{k_1}^p(G_1)\phi_{k_2}^p(G_2) + \min\{n_1 - \phi_{k_1}^p(G_1), n_2 - \phi_{k_2}^p(G_2)\}.$$
Proof. By Theorem 14 (i) we conclude that for every $k_i$-paf set $S_i$ in $G_i$, $i \in \{1, 2\}$, the sets $S_1 \times V_2$ and $V_1 \times S_2$ are, respectively, $(k_1 + \Delta_2)$-paf and $(k_2 + \Delta_1)$-paf in $G_1 \times G_2$. Therefore, (i) follows.

In order to prove (ii), let $V_1 = \{u_1, u_2, ..., u_{n_1}\}$ and $V_2 = \{v_1, v_2, ..., v_{n_2}\}$. Let $S_i$ be a $k_i$-paf set of maximum cardinality in $G_i$, $i \in \{1, 2\}$. We suppose $S_1 = \{u_1, ..., u_r\}$ and $S_2 = \{v_1, ..., v_s\}$. By Theorem 14 (ii) we deduce that, for $k \geq k_1 + k_2 - 1$, $S_1 \times S_2$ is a $k$-paf set in $G_1 \times G_2$. Now let $X = \{(u_{r+i}, v_{s+i}), i = 1, ..., t\}$, where $t = \min\{n_1 - r, n_2 - s\}$ and let $S = X \cup (S_1 \times S_2)$. Since, for every $x \in X$, $\delta_S(x) = 0$ and $k_i > -\delta_i, i \in \{1, 2\}$, we obtain that for every $A \subseteq S$, such that $A \cap X \neq \emptyset$, $A$ is not a defensive $(k_1 + k_2 - 1)$-alliance in $G_1 \times G_2$. Hence, $S$ is a $k$-paf set for $k \geq k_1 + k_2 - 1$. As a consequence, $\phi^p_k(G_1 \times G_2) \geq |S| = \phi^p_{k_1}(G_1)\phi^p_{k_2}(G_2) + \min\{n_1 - \phi^p_{k_1}(G_1), n_2 - \phi^p_{k_2}(G_2)\}$. \[\square\]

If $G_1 = C_{n_1}$ is the cycle graph of order $n_1$ and $G_2$ is the graph in Figure 3, then, by Corollary 15 (i), we deduce $\phi^p_k(G_1 \times G_2) = n_1n_2$, that is, $\phi^p_k(G_1 \times G_2) \geq n_1\phi^p_k(G_2) = n_1n_2$. Moreover, if $G_1 = T_{n_1}$ is a tree of order $n_1$ and maximum degree $\Delta_1 \geq 4$ and $G_2$ is the graph in Figure 3, then $\phi^p_k(G_1) = n_1$ and $\phi^p_k(G_2) = n_2 = 9$. Therefore, by Corollary 15 (ii) we deduce $\phi^p_k(G_1 \times G_2) = 9n_1$.

6 Conclusions

This article is a contribution to the study of alliances in graphs. Particularly, we have dealt with defensive (offensive, powerful) $k$-alliance free sets in Cartesian product graphs. We have shown several relationships between defensive (offensive, powerful) $k$-alliance free sets in Cartesian product graphs and defensive (offensive, powerful) $k$-alliance free sets in the factor graphs. Our principal contributions are summarized below.

Let $G_i = (V_i, E_i)$ be a graph of maximum degree $\Delta_i$ and minimum degree $\delta_i$, $i \in \{1, 2\}$:

- We have shown that if the projection of a set $S \subset V_1 \times V_2$ over $V_i$ is a defensive (offensive, powerful) $k_i$-alliance free set in $G_i$, then $S$ is a defensive (offensive, powerful) $k$-alliance free set in $G_1 \times G_2$, where the values of $k$ depend on the values of $k_i, \delta_j$ and $\Delta_j$, with $j \in \{1, 2\}$.

- We have shown the relationships between the maximum cardinality of a defensive (offensive, powerful) $k_i$-alliance free set in $G_i$ and the
maximum cardinality of a defensive (offensive, powerful) \(k\)-alliance free set in \(G_1 \times G_2\), where the values of \(k\) depend on the values of \(K_i\), \(\delta_j\) and \(\Delta_j\), with \(j \in \{1, 2\}\).

References

[1] S. Bermudo, J. A. Rodríguez-Velázquez, Ismael G. Yero and José M. Sigarreta, On global offensive \(k\)-alliances in graphs. *Applied Mathematics Letters* **23** (12) (2010) 1454–1458.

[2] S. Bermudo, J. A. Rodríguez-Velázquez, J. M. Sigarreta and I. G. Yero, On geodetic and \(k\)-geodetic sets in graphs. *Ars Combinatoria* **96** (2010) 469–478.

[3] B. Brešar, Vizing-like conjecture for the upper domination of Cartesian products of graphs-the proof, *The Electronic Journal of Combinatorics* **12** (2005), no. 12.

[4] B. Brešar, S. Klavžar and A. Tepeh Horvat, On the geodetic number and related metric sets in Cartesian product graphs, *Discrete Mathematics* **308** (2008) 5555–5561.

[5] R. C. Brigham, R. Dutton and S. Hedetniemi, A sharp lower bound on the powerful alliance number of \(C_m \times C_n\), *Congressus Numerantium* **167** (2004) 57–63.

[6] J. Cáceres, C. Hernando, M. Mora, I. Pelayo, M. L. Puertas, C. Seara and D. R. Wood, On the metric dimension of Cartesian products of graphs, *SIAM Journal on Discrete Mathematics* **21** (2) (2007) 423–441.

[7] W. E. Clark and S. Suen, An inequality related to Vizing’s conjecture, *The Electronic Journal of Combinatorics* **7** (2000), no. 1, Note 4, 3 pp.

[8] P. Dickson, K. Weaver, Alliance formation: the relationship between national RD intensity and SME size, *Proceedings of ICSB 50th World Conference D.C.* (2005) 123–154.

[9] G. W. Flake, S. Lawrence, C. L. Giles, Efficient identification of web communities. *Proceedings of the 6th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining (KDD-2000)* (2000) 150–160.
[10] R. Gera and P. Zhang, On $k$-geodomination in Cartesian products, Congressus Numerantium 158 (2002) 163–178.

[11] B. Hartnell and D. F. Rall, Domination in Cartesian products: Vizing’s conjecture (In: T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Editors, Domination in Graphs Advanced Topics, Marcel Dekker, New York (1998)) 163–189.

[12] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, Inc. New York, 1998.

[13] T. Haynes, D. Knisley, E. Seier, Y. Zou, A quantitative analysis of secondary RNA structure using domination based parameters on trees, BMC Bioinformatics 7 (108) (2006) 11 pages.

[14] T. W. Haynes and J. A. Lachniet, The alliance partition number of grid graphs. AKCE International Journal of Graphs and Combinatorics 4 (1) (2007) 51–59.

[15] X. Hou and Y. Lu, On the $k$-domination number of Cartesian products of graphs, Discrete Mathematics 309 (2009) 3413–3419.

[16] T. Jiang, I. Pelayo and D. Pritikin, Geodesic convexity and Cartesian products in graphs. Submitted.

[17] B. J. Kim, J. Liu, Instability of defensive alliances in the predator-prey model on complex networks, Physical Reviews E 72 041906 (2005) 5 pages.

[18] P. Kristiansen, S. M. Hedetniemi and S. T. Hedetniemi, Alliances in graphs, Journal of Combinatorial Mathematics and Combinatorial Computing 48 (2004) 157–177.

[19] Meijie Ma, Jun-Ming Xu and Qiang Zhu, The Menger number of the Cartesian product of graphs, Applied Mathematics Letters 24 (2011) 627–629.

[20] S. Mishra, Jaikumar Radhakrishnan and S. Sivasubramanian, On the Hardness of Approximating Minimum Monopoly Problems, Lecture Notes in Computer Science 2556 (2002) 277–288.
[21] J. A. Rodríguez-Velázquez, J. M. Sigarreta, I. G. Yero and S. Bermudo, Alliance free and alliance cover sets, Acta Mathematica Sinica, English Series 27 (3) (2011) 497–504.

[22] K. H. Shafique, Partitioning a Graph in Alliances and its Application to Data Clustering. Ph. D. Thesis, 2004.

[23] K. H. Shafique and R. D. Dutton, Partitioning a graph into alliance free sets, Discrete Mathematics 309 (2009) 3102–3105.

[24] K. H. Shafique and R. D. Dutton, On satisfactory partitioning of graphs, Congressus Numerantium 154 (2002) 183–194.

[25] K. H. Shafique and R. D. Dutton, Maximum alliance-free and minimum alliance-cover sets, Congressus Numerantium 162 (2003) 139–146.

[26] K. H. Shafique and R. Dutton, A tight bound on the cardinalities of maximum alliance-free and minimum alliance-cover sets, Journal of Combinatorial Mathematics and Combinatorial Computing 56 (2006) 139–145.

[27] J. M. Sigarreta and J. A. Rodríguez, On the global offensive alliance number of a graph Discrete Applied Mathematics 157 (2) (2009) 219–226.

[28] J. M. Sigarreta, I. G. Yero, S. Bermudo and J. A. Rodríguez-Velázquez, Partitioning a graph into offensive k-alliances. Discrete Applied Mathematics 159 (4) (2011) 224–231.

[29] L. Sun, A result on Vizing’s conjecture, Discrete Mathematics 275 (2004) 363–366.

[30] G. Szabó, T. Czárán, Defensive alliances in spatial models of cyclical population interactions, Physical Reviews E 64, 042902 (2001) 11 pages.

[31] V. G. Vizing, Some unsolved problems in graph theory, Uspehi Mat. Nauk 23 (144) (1968) 117–134.

[32] I. G. Yero, Contribution to the study of alliances in graphs. Ph. D. Thesis, 2010.
[33] I. G. Yero and J. A. Rodríguez-Velázquez. A note on the partition dimension of Cartesian product graphs. *Applied Mathematics and Computation* **217** (7) (2010) 3571–3574.

[34] I. G. Yero, S. Bermudo, J. A. Rodríguez-Velázquez and J. M. Sigarreta. Partitioning a graph into defensive $k$-alliances. *Acta Mathematica Sinica, English Series* **27** (1) (2011) 73–82.