EQUIVARIANT MORSE INEQUALITIES AND APPLICATIONS

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Abstract. In this paper, we prove equivariant Morse inequalities via Bismut-Lebeau’s analytic localization techniques. As an application, we obtain Morse inequalities on compact manifold with nonempty boundary by applying equivariant Morse inequalities to the doubling manifold.

1. Introduction

In his influential work [15], Witten sketched analytic proofs of the degenerate Morse inequalities of Bott [4] for Morse functions whose critical submanifolds are nondegenerate in the sense of Bott. Rigorous proofs were given by Bismut [2], by using heat kernel methods, and later by Helffer and Sjöstrand [10], by means of semiclassical analysis. Braverman and Farber [6] provided another proof using the Witten deformation techniques suggested by Bismut [2].

Concerning the standard Morse inequalities (i.e., for Morse functions with isolated critical points), an analytic proof is given by Zhang [17, Chap. 5], in the spirit of the analytic localization techniques developed by Bismut-Lebeau [3 §8-9]. Moreover, [17 Chap. 6] contains a complete proof of the isomorphism between the Thom-Smale complex and the Witten instanton complex. Following the ideas in [17], we give here a proof of degenerate Morse inequalities by similar techniques.

Let us mention the related papers [6, 7, 8]. In [6, 7], Braverman, Farber and Silantyev used Witten deformation techniques to study the Novikov number associated to closed differential 1-forms nondegenerate in the sense of Bott and Kirwan, respectively. In this way, they obtained Novikov-type inequalities associated to a closed differential 1-form. When the closed differential form is exact, these inequalities turn to Morse inequalities.

In [8], Feng and Guo establish Nivokov’s type inequalities associated to vector fields instead of closed differential forms under a natural assumption on the zero-set of the vector field.

In this paper, we work out equivariant Morse-Bott inequalities along the lines of [17] (cf. [3 §8-9]). Compared to [8], where Bismut-Lebeau’s analytic localization techniques are applied along the lines of [17], we can choose the geometrical data near the singular points as simple as possible, due to the equivariant Morse’s Lemma [14]. As an application, we get degenerate Morse inequalities for manifolds with nonempty boundary by passing to the doubling manifold. Thus, we extend the result from [16] to the most general situation.

Let $M$ be a smooth $m$-dimensional closed and connected manifold, and let $G$ be a finite group acting smoothly on $M$. Let $f : M \to \mathbb{R}$ be a smooth $G$-invariant Morse-Bott
function \[4\]. This means that the critical points of \(f\) form a union of disjoint connected submanifolds \(Y_1, \ldots, Y_r\) such that for every \(x \in Y_i\) the Hessian of \(f\) is nondegenerate on all subspaces of \(T_x M\) intersecting \(T_x Y_i\) transversally. One verifies directly that the index of the Hessian of \(f\) is constant on any orbit \(G \cdot Y_i\). Set \(\{B_1, \ldots, B_r\} = \{G \cdot Y_1, \ldots, G \cdot Y_r\}\), \(r \leq r'\), where \(B_1, \ldots, B_r\) are pointwise disjoint orbits. Then \(B_i\) is a \(G\)-invariant submanifold of \(M\). For \(1 \leq i \leq r\), let \(n_i\) be the dimension of the submanifold \(B_i\) and \(n_i^-\) be the index of the Hessian of \(f\) on \(B_i\).

Using the equivariant Morse’s Lemma \[14\], we embed each critical submanifold \(B_i\) in a \(G\)-invariant tubular neighborhood \((h, N_i^- \oplus N_i^+)\) of \(B_i\) such that \(h\) equivariantly embeds \(N_i^- \oplus N_i^+\) into \(M\). Moreover, there is an open \(G\)-invariant neighborhood \(B_i\) of \(B_i\) in \(N_i^- \oplus N_i^+\) such that if \(Z = (Z^-, Z^+) \in B_i\), then

\[
(1.1) \quad f \circ h(Z^-, Z^+) = c - \frac{|Z^-|^2}{2} + \frac{|Z^+|^2}{2},
\]

where \(c\) denotes the value of the constant function \(f|_{B_i}\). The rank of \(N_i^-\) is \(n_i^-\), while that of \(N_i^+\) is \(m - n_i - n_i^-\). Let \(o(N_i^-)\) denote the orientation bundle of \(N_i^-\). We call \(n_i^-\) the index of \(B_i\) in \(M\).

In the sequel, we will often omit the subscript \(i\) in \(B_i, n_i, n_i^-\), i.e., \(n\) denotes the dimension of the critical submanifold \(B\) and \(n^-\) is the index. Denote by \(o(N^-)\) the orientation bundle of \(N^-\) over \(B\).

Let \(W_1, W_2\) be two finite-dimensional \(G\)-representations. A morphism between \(W_1\) and \(W_2\) is a linear map which is \(G\)-equivariant. Let \(\text{Hom}_G(W_1, W_2)\) denote the set of all morphism between \(W_1\) and \(W_2\). If \(E_1, E_2\) are two finite-dimensional representations of \(G\), then we say that

\[
(1.2) \quad E_1 \leq E_2
\]

in the representation ring \(R(G)\) if for any irreducible representation \(V\) of \(G\), the multiplicity of \(V\) in \(E_1\) is smaller than the multiplicity of \(V\) in \(E_2\), equivalently,

\[
(1.3) \quad \dim \text{Hom}_G(V, E_1) \leq \dim \text{Hom}_G(V, E_2).
\]

Denote by \(\Omega^i(B, o(N^-))\) the space of smooth differential \(i\)-forms on \(B\) with values in \(o(N^-)\). Set \(\Omega(B, o(N^-)) = \bigoplus_{i=0}^{n} \Omega^i(B, o(N^-))\). Let \(d^B\) denote the exterior differential on \(\Omega(B, o(N^-))\) induced by the flat connection \(\nabla^{o(N^-)}\) on \(o(N^-)\). Denote by \(H^\bullet(B, o(N^-))\) the cohomology of the de Rham complex \((\Omega(B, o(N^-)), d^B)\). Let \(H^\bullet(M)\) denote the de Rham cohomology groups of \(M\).

The main result of this paper is as follows.

**Theorem 1.1.** Let \(M\) be a smooth \(m\)-dimensional closed and connected manifold, and let \(G\) be a finite group acting smoothly on \(M\). Let \(f : M \to \mathbb{R}\) be a smooth \(G\)-invariant Morse-Bott function. Then we have for \(k = 0, 1, \ldots, m\),

\[
(1.4) \quad \sum_{j=0}^{k} (-1)^{k-j} H^j(M) \leq \sum_{i=1}^{r} \sum_{j=n_i^-}^{k} (-1)^{k-j} H^{j-n_i^-}(B_i, o(N_i^-)).
\]
in the sense of (1.2). When \( k = m \), the equality holds,

\[
\sum_{j=0}^{m} (-1)^{m-j} H^j(M) = \sum_{i=1}^{r} \sum_{j=0}^{m} (-1)^{m-j} H^j(B_i, o(N^-_i)).
\]

(1.5)

Let us explain Theorem 1.1 in more detail. Set

(1.6)

\[
F_j = \bigoplus_{i=1}^{r} H^j(B_i, o(N^-_i)), \quad q_j = \sum_{i=1}^{r} \dim H^j(B_i, o(N^-_i)).
\]

(1.6)

Let \( \{ V^\alpha \}_{\alpha=1}^{l_0} \) be the finite set of irreducible representations of \( G \). As representation spaces of \( G, F_j \) and \( H^j(M) \) have the following decompositions:

(1.7)

\[
F_j = \bigoplus_{\alpha=1}^{l_0} \text{Hom}_G(V^\alpha, F_j) \otimes V^\alpha, \quad H^j(M) = \bigoplus_{\alpha=1}^{l_0} \text{Hom}_G(V^\alpha, H^j(M)) \otimes V^\alpha.
\]

(1.7)

For \( k = 0, 1, \ldots, m, \alpha = 1, \ldots, l_0 \), set

(1.8)

\[
d^\alpha_j = \dim \text{Hom}_G(V^\alpha, F_j), \quad b^\alpha_j = \dim \text{Hom}_G(V^\alpha, H^j(M))
\]

(1.8)

Then (1.4) is equivalent to

(1.9)

\[
\sum_{j=0}^{k} (-1)^{k-j} b^\alpha_j \leq \sum_{j=0}^{k} (-1)^{k-j} d^\alpha_j,
\]

and (1.5) is equivalent to:

(1.10)

\[
\sum_{j=0}^{m} (-1)^{m-j} b^\alpha_j = \sum_{j=0}^{m} (-1)^{m-j} d^\alpha_j.
\]

From the equivariant Morse inequalities (1.9) and (1.10), we will obtain the Morse inequalities for manifolds with nonempty boundary. This goes like follows. Let \( M \) be a smooth \( m \)-dimensional connected orientable manifold with nonempty boundary \( \partial M \). Let \( f : M \to \mathbb{R} \) be a smooth function which is a Morse-Bott function in the interior of \( M \). Let \( f|_{\partial M} \) be restriction of \( f \) to the boundary. We also assume the following condition. Let \( \partial M = N_+ \sqcup N_- \) be a disjoint union of closed manifolds such that \( f(u, y) = \frac{1}{2} u^2 + f_+(y) \) in a collar neighborhood \( N_+ \times [0, \eta) \) of \( N_+ \), while \( f(u, y) = -\frac{1}{2} u^2 + f_-(y) \) in a collar neighborhood \( N_- \times [0, \eta) \) of \( N_- \), where \( f_+ \) (resp. \( f_- \)) is a Morse-Bott function on \( N_+ \) (resp. \( N_- \)). This implies that \( f|_{\partial M} \) is also a Morse-Bott function.

Let \( N_+ = N_{a+} \sqcup N_{r+} \) and \( N_- = N_{a-} \sqcup N_{r-} \) be disjoint union of closed manifolds. The subscripts "a" and "r" refer to absolute and relative boundary conditions, respectively. Set \( N_a = N_{a+} \sqcup N_{a-}, N_r = N_{r+} \sqcup N_{r-} \). The Riemannian metric is assumed to take the product form \( g^{TM} = g^{T\partial M} \oplus d^2u \) in the collar neighborhood \( \partial M \times [0, \eta) \), where \( g^{T\partial M} \) is a Riemannian metric on \( \partial M \).

Let \( \{ B_i \}_{i=1}^{r} \) (resp. \( \{ S_{+,i} \}_{i=1}^{t_+} \), resp. \( \{ S_{-,i} \}_{i=1}^{t_-} \)) be the critical submanifolds of \( f \) in the interior of \( M \) (resp. of \( f_+ \) on \( N_+ \), resp. of \( f_- \) on \( N_- \)). Set

(1.11)

\[
S_{a+,i} = S_{+,i} \cap N_a, \quad S_{r-,i} = S_{-,i} \cap N_r.
\]
Let \( o(N_i^-) \) denote the orientation bundle of \( N_i^- \) over \( B_i \) as before. To simplify our notation, we denote by \( o(S_{a+,i}) \) (resp. \( o(S_{r-,i}) \)) the corresponding bundle over \( S_{a+,i} \) (resp. \( S_{r-,i} \)) and by \( n_{a+,i}^- \) (resp. \( n_{r-,i}^- \)) its index in \( N_a \) (resp. \( N_r \)). Set
\[
F_{a+,j} = \bigoplus_{i=1}^{t_a} H^{j-n_{a+,i}}(S_{a+,i}, o(S_{a+,i})), \quad q_{a+,j} = \dim F_{a+,j};
\]
\[
F_{r-,j} = \bigoplus_{i=1}^{t_r} H^{j-n_{r-,i}}(S_{r-,i}, o(S_{r-,i})), \quad q_{r-,j} = \dim F_{r-,j}.
\]
(1.12)

Denote by \( H^*(M, N_r) \) the relative cohomology of \( M \) with respect to \( N_r \).

**Theorem 1.2.** The following inequalities hold for \( k = 0, 1, \ldots, m \),
\[
\sum_{j=0}^{k} (-1)^{k-j} \beta_j(M, N_r) \leq \sum_{j=0}^{k} (-1)^{k-j} \mu_j,
\]
(1.13)

where
\[
\beta_j(M, N_r) = \dim H^j(M, N_r), \quad \mu_j = q_j + q_{a+,j} + q_{r-,j-1}.
\]
(1.14)

The equality holds for \( k = m \).

When \( f|_{\partial M} = 0 \) and the critical points of \( f \) in the interior of \( M \) are isolated and nondegenerate, Theorem 1.2 reduces to Theorem 1 in [16].

2. **Equivariant Morse Inequalities**

This section is organized as follows. In Section 2.1, we calculate the kernel of the Witten Laplacian on Euclidean space. The results of this section will be applied to the fibres of the normal bundle of critical manifolds in \( M \). In Section 2.2, a special metric on the total space \( N \) is constructed such that the critical manifolds are totally geodesic in \( N \). In Section 2.3, we introduce the Witten deformation, the deformed de Rham operator \( D_T \) and state a crucial result (Proposition 2.3) concerning the lower part of the spectrum of \( D_T \). Section 2.4 is devoted to the Taylor expansion of \( D_T \) near the critical manifolds. In Section 2.5, a decomposition of \( D_T \) is established. Various estimates are also briefly described there. Finally in Section 2.6, we prove Proposition 2.3 and then finish the proof of Theorem 1.1.

2.1. **Some calculations on Euclidian space.** In this section, we calculate the kernel of the Witten Laplacian on Euclidian space. The result of this section will be applied to the fibres of the normal bundle to \( B \) in \( M \).

Let \( V \) be an \( l \)-dimensional real vector space endowed with an Euclidean scalar product. Let \( V^+, V^- \) be two subspaces such that \( V = V^- \oplus V^+ \) and \( \dim V^- = n^- \). Let \( f \in C^\infty(V, \mathbb{R}) \) be defined as:
\[
f(Z) = f(0) - \frac{|Z^-|^2}{2} + \frac{|Z^+|^2}{2},
\]
(2.1)

where \( Z^- = (Z_1, \ldots, Z_{n^-}), Z^+ = (Z_{n^-+1}, \ldots, Z_l), (Z^-, Z^+) \) denote the coordinate functions on \( V \) corresponding to the decomposition \( V = V^- \oplus V^+ \).
Let $Z = \sum_{\alpha=1}^l Z_\alpha e_\alpha$ be the radial vector field on $V$. There is a natural Euclidean scalar product on $\Lambda V^*$. Let $dv_V(Z)$ be the volume form on $V$. Let $S$ be the set of the square integrable sections of $\Lambda V^*$ over $V$. For $s_1, s_2 \in S$, set
\begin{equation}
\langle s_1, s_2 \rangle = \int_V \langle s_1, s_2 \rangle_{\Lambda V} \cdot dv_V(Z).
\end{equation}

Let $d$ be the exterior differential operator acting on the smooth section of $\Lambda V^*$, and let $\delta$ be the formal adjoint of $d$ with respect to the Euclidean product (2.2).

Let $C(V)$ be the Clifford algebra of $V$, i.e., the algebra generated over $\mathbb{R}$ by $e \in V$ and the commutation relations $ee' + e'e = -2\langle e, e' \rangle$ for $e, e' \in V$. Let $c(e), \tilde{c}(e)$ be the Clifford operators acting on $\Lambda V^*$ defined by
\begin{equation}
c(e) = e^* \wedge -i_e, \quad \tilde{c}(e) = e^* \wedge +i_e,
\end{equation}
where $e^* \wedge$ and $i_e$ are the standard notation for exterior and interior multiplication and $e^*$ denotes the dual of $e$ with respect to the Euclidean scalar product on $V$. Then $\Lambda V^*$ is a Clifford module. If $X, Y \in V$, one has
\begin{align}
c(X)c(Y) + c(Y)c(X) &= -2\langle X, Y \rangle, \\
\tilde{c}(X)\tilde{c}(Y) + \tilde{c}(Y)\tilde{c}(X) &= 2\langle X, Y \rangle, \\
c(X)\tilde{c}(Y) + \tilde{c}(Y)c(X) &= 0.
\end{align}

If we denote by $v$ the gradient of $f$ with respect to the given Euclidean scalar product, then
\begin{equation}
v(Z) = -\sum_{\alpha=1}^{n^-} Z_\alpha e_\alpha + \sum_{\alpha=n^-+1}^l Z_\alpha e_\alpha.
\end{equation}

Let $\Delta$ be the standard Laplacian on $V$, i.e.,
\begin{equation}
\Delta = -\sum_{\alpha=1}^l \left( \frac{\partial}{\partial Z_\alpha} \right)^2.
\end{equation}

Set
\begin{equation}
d_T = e^{-Tf}d \cdot e^{Tf}, \quad \delta_T = e^{Tf}\delta \cdot e^{-Tf}.
\end{equation}

The deformed de Rham operator on the Euclidean space is defined by
\begin{equation}
D_{T,v} = d_T + \delta_T = d + \delta + T\tilde{c}(v).
\end{equation}

Let $e_1, \ldots, e_l$ be the dual basis of $e_1, \ldots, e_l$. Then we have the following result [15, 17, Prop. 4.9].

**Proposition 2.1.** The kernel of $D_{T,v}^2$ is one-dimensional and is spanned by
\begin{equation}
\beta = \exp\left( -\frac{T|Z|^2}{2} \right) e_1 \wedge \ldots \wedge e^{n^-}.
\end{equation}
Moreover, all nonzero eigenvalues of $D_{T,v}^2$ are $\geq 2T$. 
Proof. We recall the proof for the reader’s convenience. For \( e \in V \), let \( \nabla_e \) be the differential operator along the vector \( e \). It is easy to calculate the square of \( D_{T,v} \),

\[
D_{T,v}^2 = \Delta + T^2 |Z|^2 + T \sum_{\alpha=1}^l c(e_\alpha) \hat{c}(\nabla_{e_\alpha} v)
\]

(2.10) \[= (\Delta + T^2 |Z|^2 - Tl) + T \sum_{\alpha=1}^n \left[ 1 - c(e_\alpha) \hat{c}(e_\alpha) \right] + T \sum_{\alpha=n+1}^l \left[ 1 + c(e_\alpha) \hat{c}(e_\alpha) \right] \]

\[= (\Delta + T^2 |Z|^2 - Tl) + 2T \left( \sum_{\alpha=1}^{n^-} i_{e_\alpha} e^\alpha \wedge + \sum_{\alpha=n+1}^l e^\alpha \wedge i_{e_\alpha} \right). \]

The operator

(2.11) \[L_T = \Delta + T^2 |Z|^2 - Tl\]

is the harmonic oscillator operator on \( V \). By [9 Th.1.5.1], [11 Appendix E], we know that \( L_T \) is a positive elliptic operator with one-dimensional kernel generated by \( \exp(-\frac{T|Z|^2}{2}) \). Moreover, the nonzero eigenvalues of \( L_T \) are all greater than \( 2T \). It is also easy to verify that the linear operator

(2.12) \[\sum_{\alpha=1}^{n^-} i_{e_\alpha} e^\alpha \wedge + \sum_{\alpha=n+1}^l e^\alpha \wedge i_{e_\alpha}\]

is positive and has one-dimensional kernel generated by \( e^1 \wedge \ldots \wedge e^{n^-} \). The proof of Proposition 2.1 is complete. \( \square \)

2.2. Local analysis near critical manifolds. Let \( B \) be an equivariant critical submanifold of the Morse-Bott function \( f \). By equivariant Morse’s Lemma [14 Lemma 4.1], we know that \( B \) possesses a \( G \)-invariant tubular neighborhood \((h, N)\) such that:

1. \( N \) is a \( G \)-vector bundle over \( B \), which is endowed with \( G \)-invariant scalar product \( g^N \). Moreover \( N \), which has rank \( m - n \), splits into two orthogonal \( G \)-subbundles \( N = N^- \oplus N^+ \), where the rank of \( N^- \) is \( n^- \).
2. \( h \) equivariantly embeds \( N \) into \( M \). Moreover there is an open \( G \)-invariant neighborhood \( \mathcal{B} \) of \( B \) in \( N \) such that if \( Z = (Z^-, Z^+) \in \mathcal{B} \), then

(2.13) \[f(h(Z)) = c - \frac{|Z^-|^2}{2} + \frac{|Z^+|^2}{2}, \]

where \( c \) denotes the value of the constant function \( f|_B \).

In the sequel, we will identify \( N \) and \( h(N) \). Let \( \pi \) be the projection \( N \to B \). We denote by \( o(N^-) \) the orientation bundle of \( N^- \).

Let \( g^{TB} \) be a \( G \)-invariant Riemannian metric on \( TB \) and \( \nabla^{TB} \) be the Levi-Civita connection on \( TB \) which is then \( G \)-invariant. As Euclidean \( G \)-bundles, \((N^-, g^{N^-})\) (resp. \((N^+, g^{N^+})\)) can be endowed with \( G \)-invariant Euclidean connections \( \nabla^{N^-} \) (resp. \( \nabla^{N^+} \)), where \( g^{N^-} \) (resp. \( g^{N^+} \)) denotes the restriction of the scalar product \( g^N \) to the subbundle \( N^- \) (resp. \( N^+ \)). We then have a natural Euclidean connection \( \nabla^N \) on \( N \), i.e.,

(2.14) \[\nabla^N = \nabla^{N^-} \oplus \nabla^{N^+}.\]
The Euclidean connection $\nabla^N$ on $N$ induce a splitting $TN = T^HN \oplus T^VN$ of the tangent space of the total space $N$ [1, Prop. 1.20], where $T^HN$ is the horizontal part of $TN$ with respect to the Euclidean connection $\nabla^N$. If $X \in TB$, let $X^H$ denote the horizontal lift of $X$ in $T^HN$ such that $X^H \in T^HN, \pi_*X^H = X$.

If $y \in N$, then $\pi_*$ identifies $T^H_N$ with $T_{\pi(y)}B$. Moreover, $T^VN$ and $N$ can be naturally identified. In this way, $T^H_N$ and $T^V_N$ are both endowed with a scalar product. We can assume as well that they are orthogonal for the metric $g^{TN}$ which splits into $g^{TN} = \pi^*(g^B) \oplus g^N$. Let $\nabla^{TN}$ be the Levi-Civita connection on $N$ associated to the Riemannian metric $g^{TN}$.

Let $TN|_B$ be the restriction of the tangent bundle $TN$ to $B$. Recall that $N$ is identified with the bundle orthogonal to $TB$ in $TN|_B$, i.e., $TN|_B = TB \oplus N$. Let $\nabla^{TN|_B}$ be the restriction of $\nabla^{TN}$ to $TN|_B$.

**Lemma 2.2.** The following identity holds:

$$\nabla^{TN|_B} = \nabla^{TB} \oplus \nabla^N.$$  

*Proof.* The proof is straightforward and is left to the reader. \hfill \Box

2.3. **The deformed de Rham operator.** Let $g^{TM}$ be a $G$-invariant Riemannian metric on $M$ which coincides with $g^{TN}$ in a neighborhood of $B$ via the embedding $h$ (this is always possible by a partition of unity argument).

Let $o(TM)$ be the orientation line bundle on $M$ and let $dv_M$ be the density (or Riemannian volume form) on $M$. Note that we do not assume that $M$ is oriented; thus $dv_M \in C^\infty(M, \Lambda^m(T^*M) \otimes o(TM))$ (see [1, p. 29], [3, p. 88]). Let $E$ be the set of smooth sections of $\Lambda(T^*M)$ on $M$. For $s_1, s_2 \in E$, set

$$\langle s_1, s_2 \rangle = \int_M \langle s_1, s_2 \rangle(x) dv_M(x).$$  

Let $D^M$ be the classical Dirac operator on $M$, i.e., $D^M = d + \delta$, where $d$ is the exterior differential operator and $\delta$ is the adjoint of $d$ with respect to the metric (2.16).

Set

$$d_T = e^{-TF}d \cdot e^{TF}, \quad \delta_T = e^{TF} \delta \cdot e^{-TF}.$$  

The deformed de Rham operator $D_T$ is defined by

$$D_T = d_T + \delta_T = D^M + T\delta(\nabla f),$$  

where $\nabla f$ is the gradient vector field of $f$ with respect to the Riemannian metric $g^{TM}$ of $M$. We denote by $\Omega^j(M)$ the smooth sections of $j$-forms of $M$. The next result describes the lower part of the spectrum of $D_T^2$ for large $T$. It will be proved in Section 2.6.

**Proposition 2.3.** There exist $C_0 > 0, T_0 > 0$ such that for $T > T_0$, the number of eigenvalues of $D_T^2|_{\Omega^j(M)}$ in $[0, C_0]$ equals $q_j$. Moreover, the direct sum of eigenspaces of $D_T^j$ with eigenvalues in $[0, C_0]$ is a $G$-vector space.
2.4. Local expansion of the operator $D_T$ near the critical submanifold $B$. We first introduce a coordinate system on $M$ near $B$. If $y \in B, Z \in N_y$, let $y_t = \exp_y(tZ), t \in \mathbb{R}$ be the geodesic in $M$ with $y_0 = y, y_0 = Z$, where $\dot{y}_0$ denotes $dy_t/dt$ evaluated at $t = 0$. For $\varepsilon > 0$, set $B_\varepsilon = \{(y, Z) \in N; y \in B, |Z| < \varepsilon \}$. In the following we denote $|Z|_{g^0_y}$ simply by $|Z|$. Since $B$ and $M$ are compact, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, the map $(y, Z) \in N \rightarrow \exp_y Z \in M$ is a diffeomorphism from $B_\varepsilon$ onto a tubular neighborhood $U_\varepsilon$ of $B$ in $M$. From now on, we identify $B_\varepsilon$ with $U_\varepsilon$ and use the notation $x = (y, Z)$ instead of $x = \exp_y Z$. Finally, we identify $y \in B$ with $(y, 0) \in N$.

The symbols $dv_B$ and $dv_N$ are understood in the same manner as $dv_M$. Let $\{f_1, \ldots, f_n, e_1, \ldots, e_l\}$ be a local orthonormal frame of $TN|_B$ with $\{f_1, \ldots, f_n\}$ being an orthonormal frame for $TB$ and $\{e_1, \ldots, e_l\}$ an orthonormal frame for $N$. By the definition of $g^TN$, we know that $e_1, \ldots, e_l$ are also orthonormal basis at the points $(y, Z)$ on the total space $N$. It is clear that

$$dv_N(y, Z) = dv_B(y)dv_{N_y}(Z).$$

Take $\alpha > 0$. Let $E$ (resp. $E_\alpha$) be the set of smooth sections of $\pi^*(\Lambda^*(T^*M)|_B)$ on the total space of $N$ (resp. of $\pi^*(\Lambda(T^*M)|_B)$ over $B_\alpha$).

For $s_1, s_2 \in E$ have compact support, set

$$\langle s_1, s_2 \rangle = \int_B \left( \int_{N_y} \langle s_1, s_2 \rangle(y, Z)dv_{N_y}(Z) \right)dv_B(y).$$

If $s \in E$ has compact support in $B_\alpha$, we will identify $s$ with an element of $E$ which has compact support in $U_\alpha$. This identification is unitary with respect to the Euclidean product (2.16) and (2.20).

The Levi-Civita connection $\nabla^TM$ on $TM$ induces a connection on $\Lambda(T^*M)$, which we denote by $\nabla^\Lambda(T^*M)|_B$. Let $\nabla^\Lambda(T^*M)|_B$ be the restriction of $\nabla^\Lambda(T^*M)$ to $\Lambda(T^*M)|_B$. The connection $\nabla^\Lambda(T^*M)|_B$ on $\Lambda(T^*M)|_B$ can be lift to a connection on the bundle $\pi^*(\Lambda(T^*M)|_B)$, which we denote by $\pi^*(\nabla^\Lambda(T^*M)|_B)$.

**Definition 2.4.** Let $D^H, D^N$ be the following operators acting on $E$:

$$D^H = \sum_{j=1}^n c(f_j)\pi^*(\nabla^\Lambda(T^*M)|_B)f_j^H,$$

$$D^N = \sum_{\alpha=1}^l c(e_{\alpha})\pi^*(\nabla^\Lambda(T^*M)|_B)e_{\alpha}.\tag{2.21}$$

One verifies directly that $D^H, D^N$ is self-adjoint with respect to metric (2.20). Indeed, $D^N$ is formally self-adjoint along the fibres of $N$, i.e., for $s_1, s_2 \in E$ with compact supports, $y \in B$,

$$\int_{N_y} \langle D^Ns_1, s_2 \rangle(y, Z)dv_{N_y}(Z) = \int_{N_y} \langle s_1, D^Ns_2 \rangle(y, Z)dv_{N_y}(Z).\tag{2.22}$$

Using the identification $(\Lambda(T^*M))_{(y, Z)}$ with $(\Lambda(T^*M))_y$ by parallel transport along the geodesic $t \rightarrow (y, tZ), t \in [0, 1]$ with respect to the connection $\nabla^\Lambda(T^*M)$, we can now consider the connection $\nabla^\Lambda(T^*M)$ as a Euclidean connection on $\pi^*(\Lambda(T^*M)|_B)$ over $B_\varepsilon$. 

Recall that the vector field $v$ is defined as in (2.5). Set
\begin{equation}
D^N_T = D_N + T \tilde{c}(v).
\end{equation}
Then we have the following analogue of [8, Lemma 2.4], [3, Th. 8.18].

**Theorem 2.5.** The following asymptotic formula holds on $E_{e_0}$ as $T \to +\infty$,
\begin{equation}
D_T = D^N_T + D^H + O(|Z|^2 \partial^N + |Z|^2 \partial^H + |Z| + T|Z|),
\end{equation}
where $\partial^H$ and $\partial^N$ represent horizontal and vertical differential operators, respectively.

**Proof.** We adapt the proof from [3, Th. 8.18], [8, Lemma 2.3] and show how the proof simplifies in our case, due to the fact that $B$ is now a totally geodesic submanifold of the total manifold $N$. For $(y, Z) \in B_{\alpha}, X \in T_y N$, let $\tilde{X}$ be the parallel transport of $X$ with respect to the connection $\nabla^{TM}$ along the geodesic $t \to (y, tZ), t \in [0, 1]$, i.e.,
\begin{equation}
(\nabla^Z_{\tilde{X}})(y, Z) = 0.
\end{equation}
Then $\tilde{e}_\alpha(y, Z) = e_{\alpha}(y)$ and
\begin{equation}
D^M = \sum_{j=1}^n c(f_j) \nabla^{\Lambda(T^*M)}_f + \sum_{\alpha=1}^l c(e_{\alpha}) \nabla^{\Lambda(T^*M)}_{e_{\alpha}}.
\end{equation}
For $1 \leq j \leq n$, set
\begin{equation}
\tilde{f}_j(y, Z) = f_j(y) + \sum_{k=1}^n \sum_{\alpha=1}^l c^\alpha_{kj}(y) Z_{\alpha} f_k + \sum_{\beta=1}^l \sum_{\alpha=1}^l c^\alpha_{\beta j}(y) Z_{\alpha} e_{\beta} + O(|Z|^2),
\end{equation}
where $c^\alpha_{kj}(y), c^\alpha_{\beta j}(y)$ are smooth functions of $y$. Using (2.25) and Lemma 2.2 we find that
\begin{equation}
\tilde{f}_j(y, Z) = f^H_j(y, Z) + O(|Z|^2).
\end{equation}
Set
\begin{equation}
\Gamma = \nabla^{\Lambda(T^*M)} - \pi^* (\nabla^{\Lambda(T^*M)}|B).
\end{equation}
By Lemma 2.2 $\Gamma_y = 0$. Combining (2.26), (2.28) and (2.29), we get
\begin{equation}
D^M = D^H + D^N + O(|Z|^2 \partial^H + |Z|^2 \partial^N + |Z|).
\end{equation}
Set
\begin{equation}
\nabla f(y, Z) = \sum_{j=1}^n v_j(y, Z) \tilde{f}_j + \sum_{\alpha=1}^l v_{\alpha}(y, Z) e_{\alpha},
\end{equation}
where
\begin{equation}
v_j(y, Z) = (\tilde{f}_j f)(y, Z), \quad v_{\alpha}(y, Z) = (e_{\alpha} f)(y, Z).
\end{equation}
Using (2.13), we find that
\begin{equation}
v_{\alpha}(y, Z) = -Z_{\alpha}, \text{ if } 1 \leq \alpha \leq n^-; \quad v_{\alpha}(y, Z) = -Z_{\alpha}, \text{ if } n^- + 1 \leq \alpha \leq l.
\end{equation}
From (2.13) and (2.28), we have
\begin{equation}
v_j(y, Z) = O(|Z|^4).
\end{equation}
Substituting (2.33) and (2.34) into (2.31), we get
\[
(2.35) \quad \nabla f(y, Z) = v + O(|Z|^4).
\]
Now (2.24) follow immediately from (2.30) and (2.35). \(\square\)

Note that \(D_T^2\) is actually an elliptic operator acting fibrewise on \(\pi^*(\Lambda N^*)\). We now formalize Witten’s description of the spectrum of \(D_T^2\) (\[15\] pp. 674-675) for the equivariant case by using the argument of \[17\] Prop. 4.9.

**Theorem 2.6.** For any \(y \in B\), the restriction of \((D_T^N)^2\) to \(C^\infty(N_y, \Lambda N_y^*)\) is a positive operator with kernel generated by
\[
(2.36) \quad \beta_y = \exp\left(-\frac{|Z|^2}{2}\right) \theta_y,
\]
where \(\theta_y\) is the volume form of \(N_y^*\). Moreover, all the nonzero eigenvalues of \((D_T^N)^2\) on \(C^\infty(N_y, \Lambda N_y^*)\) are \(\geq 2T\).

**Proof.** Let \(\Delta^N\) be the positive Laplacian along the fibres of \(N\). From (2.23), it is clear that on \(\pi^*(\Lambda(T^*M)|_B) = \pi^*(\Lambda T^*B) \otimes \Lambda N^*\),
\[
(2.37) \quad (D_T^N)^2 = -\sum_{\alpha=1}^l (\pi^*\nabla^{TM|B})^2 + T^2 |v|^2 + T \sum_{\alpha=1}^l c(e_\alpha) \bar{c}(\pi^*\nabla^{TM|B}v).
\]
By (2.5), we obtain that
\[
(2.38) \quad (D_T^N)^2 = \Delta^N + T^2 |Z|^2 - T \sum_{\alpha=1}^{n^*} c(e_\alpha) \bar{c}(e_\alpha) + T \sum_{\alpha=n^*+1}^l c(e_\alpha) \bar{c}(e_\alpha).
\]
Hence Theorem 2.6 follows from Proposition 2.4. \(\square\)

### 2.5. Estimates of the components of \(D_T\) as \(T \to +\infty\)

In this section, we will give a decomposition of \(D_T = \sum_{j=1}^4 D_{T,j}\) (see (2.39)) and establish estimates of \(D_{T,j}\) as \(T \to +\infty\) by using Bismut-Lebeau analytic localization techniques \[3\].

We denote by \(\det(N^-)^*\) the determinant line bundle of \((N^-)^*\). The connection \(\nabla^{N^-}\) on \(N^-\) induces naturally an Euclidean connection \(\nabla^{\det(N^-)^*}\) on \(\det(N^-)^*\). Let \(\Phi : \det(N^-)^* \to o(N^-)\) denote the canonical isomorphism over \(B\). Let \(\nabla^{o(N^-)}\) be the Euclidean connection on \(o(N^-)\) induced by \(\nabla^{\det(N^-)^*}\) via canonical isomorphism \(\Phi : \det(N^-)^* \to o(N^-)\).

For any \(\mu > 0\), let \(E^\mu\) (resp. \(E^\mu\), resp. \(F^\mu\)) be the set of sections of \(\Lambda(T^*M)|_M\) (resp. of \(\pi^*(\Lambda(T^*M)|_B)\) on the total space \(N\), resp. of \(\Lambda(T^*B) \otimes o(N^-)\) on \(B\)) which lies in the \(\mu\)-th Sobolev spaces. Let \(\|\|_{E^\mu}\) (resp. \(\|\|_{E^\mu}\), resp. \(\|\|_{F^\mu}\)) be the Sobolev norm on \(E^\mu\) (resp. \(E^\mu\), resp. \(F^\mu\)). We will always assume that the norm \(\|\|_{F^0}\) (resp. \(\|\|_{E^0}\)) is the norm associated with the Euclidean product (2.16) (resp. (2.20)). The norm \(\|\|_{F^\mu}\) defined on the sections of \(\Lambda(T^*B) \otimes o(N^-)\) is associated with a Euclidean product similarly to (2.16).

Take \(\varepsilon \in (0, \frac{\mu_0}{2})\). Let \(\varphi\) be a smooth function on \(\mathbb{R}\) with values in \([0, 1]\) such that
\[
(2.39) \quad \varphi(a) = \begin{cases} 1 & \text{if } a \leq \frac{1}{2}, \\ 0 & \text{if } a \geq 1. \end{cases}
\]
For $y \in B, Z \in N_y$, set
\begin{equation}
(2.40) \quad \rho(Z) = \varphi(\frac{|Z|}{\varepsilon}).
\end{equation}
For $T > 0$, set
\begin{equation}
(2.41) \quad \alpha_T(y) = \int_{N_y} \exp(-T|Z|^2)\rho^2(Z)dv_{N_y}(Z).
\end{equation}
Clearly, $y \mapsto \alpha_T(y)$ is a constant function on $B$. Since for $|Z| \leq \varepsilon/2$, $\rho(Z) = 1$, there exist $c > 0, C > 0$ such that for $T \geq 1$,
\begin{equation}
(2.42) \quad \frac{c}{T^{n/2}} \leq \alpha_T \leq \frac{C}{T^{n/2}}.
\end{equation}
Here $l = m - n$ denotes the rank of $N$.

**Definition 2.7.** For $\mu \geq 0, T > 0$, define $J_T : F^\mu \to E^\mu$ by
\begin{equation}
(2.43) \quad J_T s(y, Z) = \frac{1}{\sqrt{\alpha_T}} \rho(Z) \exp\left(-\frac{T|Z|^2}{2}\right) s(y) \wedge \theta_y \in E^\mu, \ s \in F^\mu,
\end{equation}
where the smooth section $\theta$ of $\Lambda^n(N^{-})^* \otimes o(N^{-})$ is given by
\begin{equation}
(2.44) \quad u^1 \wedge \ldots \wedge u^n \otimes \Phi(u^1 \wedge \ldots \wedge u^n)
\end{equation}
for any orthonormal basis $\{u^j\}_{j=1}^n$ of $N_y^{-}$.

It is easy to see that $J_T$ is an isometry from $F^0$ onto its image.

For $\mu \geq 0, T > 0$, let $E^\mu_T$ be the image of $F^\mu$ in $E^\mu$ by $J_T$. Let $E^0_T, E^{0, \perp}_T$ be the orthogonal space to $E^0_T$ in $E^0$, and let $p_T, p_T^\perp$ be the orthogonal projection operators from $E^0$ on $E^0_T, E^{0, \perp}_T$, respectively.

Recall that $\Lambda(T^*M)$ is identified with $\pi^*(\Lambda(T^*M)|_B)$ on $B_{\varepsilon_0} \simeq U_{\varepsilon_0}$. Therefore if $s \in F^\mu$, we can also consider $J_T s$ as an element of $E^\mu$. Let $E^\mu_T$ be the image of $F^\mu$ in $E^\mu$ by $J_T$. In particular, $E^0_T$ may be identified isometrically with $E^0_T$. Let $E^{0, \perp}_T$ be the orthogonal space to $E^0_T$ in $E^0$. Then $E^0$ splits orthogonally into
\begin{equation}
(2.45) \quad E^0 = E^0_T \oplus E^{0, \perp}_T.
\end{equation}

Let $p_T, p_T^\perp$ be the orthogonal projection operators from $E^0$ on $E^0_T, E^{0, \perp}_T$, respectively. Since $E^0_T$ may be identified isometrically with $E^0_T$, we find that
\begin{equation}
(2.46) \quad p_T s = p_T J_T s, \text{ for any } s \in E^0, \text{ supp}(s) \subset B_{\varepsilon_0}.
\end{equation}
In particular,
\begin{equation}
(2.47) \quad p_T J_T s = p_T J_T s, \text{ for any } s \in F^0.
\end{equation}

According to the decomposition (2.45) we set:
\begin{equation}
(2.48) \quad D_{T,1} = p_T D_T p_T, \quad D_{T,2} = p_T D_T p_T^\perp, \quad D_{T,3} = p_T^\perp D_T p_T, \quad D_{T,4} = p_T^\perp D_T p_T^\perp.
\end{equation}
Then
\begin{equation}
(2.49) \quad D_T = D_{T,1} + D_{T,2} + D_{T,3} + D_{T,4}.
\end{equation}
We will now establish various estimates for the $D_{T,j}$’s as $T \to +\infty$. We define a twisted de Rham operator

\begin{equation}
D^B = \sum_{j=1}^{n} c(f_j) \nabla^B_{f_j} : \Omega(B, o(N^-)) \to \Omega(B, o(N^-)),
\end{equation}

where $\nabla^B = \nabla^TB \otimes 1 + 1 \otimes \nabla^o(N^-)$. The following Lemma is similar to [3, Th. 9.8] and [8, Lemma 3.1]

**Proposition 2.8.** As $T \to +\infty$, the following formula holds

\begin{equation}
J_T^{-1}D_{T,1}J_T = D^B + O\left(\frac{1}{\sqrt{T}}\right),
\end{equation}

where $O\left(\frac{1}{\sqrt{T}}\right)$ is a first order differential operator with smooth coefficients dominated by $C/\sqrt{T}$.

**Proof.** We can proceed as in [3, Th. 9.8], [8, Lemma 3.1]. The proof becomes easier because of the simpler local formula (2.35) of the gradient of $f$. By (2.24),

\begin{equation}
D_{T,1} = p_T D_T p_T = p_T (D^H + D^N_T + R_T) p_T,
\end{equation}

where

\begin{equation}
R_T = O\left(|Z|^2 \partial^H + |Z|^2 \partial^N + |Z| + T|Z|^4\right).
\end{equation}

From (2.46), (2.47) and (2.52), we find that

\begin{equation}
J_T^{-1}D_{T,1}J_T = J_T^{-1} p_T (D^H + D^N_T + R_T) p_T J_T.
\end{equation}

We may write out the projection $p_T$ explicitly. From (2.43), one verifies directly that for $s \in F^0$,

\begin{equation}
p_T s(y, Z) = \frac{1}{\alpha_T(y)} \rho(Z) \exp\left(-\frac{T|Z|^2}{2}\right)
\end{equation}

\begin{equation}
\int_{N_y} \langle s(y, Z'), \theta_y \rangle \rho(Z') \exp\left(-\frac{T|Z'|^2}{2}\right) dv_{N_y}(Z') \wedge \theta_y.
\end{equation}

From (2.14) and [1, Prop. 1.20], we find

\begin{equation}
\nabla^N_{f_j} Z = 0, \ \nabla^N \theta_y = 0.
\end{equation}

For $s \in F^1$, (2.56) yields

\begin{equation}
D^H J_T s(y, Z) = \sum_{j=1}^{n} c(f_j) \pi^* \nabla^{A(T+M)}_{f_j} \left[ \frac{1}{\sqrt{\alpha_T}} \rho(Z) \exp\left(-\frac{T|Z|^2}{2}\right) s(y) \wedge \theta_y \right]
\end{equation}

\begin{equation}
= \frac{1}{\sqrt{\alpha_T}} \rho(Z) \exp\left(-\frac{T|Z|^2}{2}\right) \sum_{j=1}^{n} c(f_j) \pi^* \left( \nabla^B_{f_j} s(y) \right) \wedge \theta_y
\end{equation}

\begin{equation}
= J_T D^B s(y).
\end{equation}

For $s \in F^0$

\begin{equation}
D^N_{T} J_T s = \frac{(-1)^{|s|}}{\sqrt{\alpha_T}} \exp\left(-\frac{T|Z|^2}{2}\right) s(y) \wedge c(\nabla \rho(Z)) \theta_y,
\end{equation}
where \(|s|\) denotes the degree of \(s\) and \(\nabla \rho(Z)\) is calculated in the fiber direction, i.e.,

\[
\nabla \rho(Z) = \sum_{\alpha=1}^{l} (e_{\alpha} \rho)(Z)e_{\alpha}.
\]

From (2.55), (2.58) and (2.59), we get that

\[
p_{T}D_{T}^{N}p_{T}J_{T}s = 0.
\]

For the term containing \(R_{T}\), one verifies directly that when \(T \geq 1, \gamma \in \mathbb{R}, s \in E^{0}\),

\[
\|p_{T}|Z|^{\gamma}s\|_{E^{0}} \leq \frac{C}{T^{\frac{1}{2}}} \|s\|_{E^{0}}.
\]

Using (2.53) and (2.61), we find

\[
J_{T}^{-1}p_{T}R_{T}p_{T}J_{T} = O\left(\frac{1}{\sqrt{T}}\right), \quad T \to \infty.
\]

Finally (2.57), (2.60) and (2.62) imply the conclusion of Proposition 2.8.

Set

\[
E^{\mu,\perp}_{T} = E^{\mu} \cap E^{0,\perp}_{T}.
\]

Similarly to the proof of Theorems 9.10, 9.11 and 9.14 from [3, §9], we also have the following results.

**Lemma 2.9.** There exists \(T_{0} > 0, C_{1} > 0, C_{2} > 0\) such that for any \(T \geq T_{0}, s \in E^{1,\perp}_{T}, s_{1} \in E^{1}_{T}\), we have

\[
\|D_{T,2}s\|_{E^{0}} \leq \frac{C_{1}}{\sqrt{T}} \|s\|_{E^{1}};
\]

\[
\|D_{T,3}s_{1}\|_{E^{0}} \leq \frac{C_{1}}{\sqrt{T}} \|s_{1}\|_{E^{1}},
\]

\[
\|D_{T,4}s\|_{E^{0}} \geq C_{2}(\|s\|_{E^{1}} + \sqrt{T} \|s\|_{E^{0}}).
\]

2.6. **Proof of (1.9) and (1.10).** In the first part of this section, we prove Proposition 2.3 and then (1.9) and (1.10). Let \(C_{0} \in (0, 1]\) be a constant such that

\[
\text{Spec } (D^{B}) \cap [-2\sqrt{C_{0}}, 2\sqrt{C_{0}}] \subset \{0\},
\]

where Spec \((D^{B})\) denotes the spectrum of the operator \(D^{B}\).

Let \(\mathcal{L}(E^{0})\) denote the space of all bounded linear operators from \(E^{0}\) into itself. For \(A \in \mathcal{L}(E^{0})\) and \(T \geq 1\), we write \(A\) as a matrix with respect to the splitting \(E^{0} = E^{0}_{T} \oplus E^{0,\perp}_{T}\) in the form

\[
A = \begin{pmatrix} A_{1} & A_{2} \\ A_{3} & A_{4} \end{pmatrix}.
\]

**Definition 2.10.** For \(A \in \mathcal{L}(E^{0}), P \in \mathcal{L}(F^{0})\), set

\[
d(A, P) = \sum_{j=2}^{4} \|A_{j}\|_{1} + \|J_{T}^{-1}A_{1}J_{T} - P\|_{1},
\]

where the operator norm \(\|\cdot\|_{1}\) is given by \(\|A\|_{1} = \text{Tr}[(A^{*}A)^{\frac{1}{2}}].\)
Let $F^C_0$ be the direct sum of eigenspaces of $D^2_T$ associated to the eigenvalues lying in $[0, C_0)$. Let $P^C_0$ be the orthogonal projection operator from $E^0$ on $F^C_0$. Let $Q$ be the orthogonal projection from $F^0$ to $K = \text{Ker } D^B$. Similar to [3, (9.155)], we also have the following:

**Proposition 2.11.** For $T$ large enough, we have

\[
(2.67) \quad d(P^C_0, Q) \leq C \sqrt{T}.
\]

**Proof of Proposition 2.3.** From (2.67), we see that for $T$ large enough,

\[
(2.68) \quad \dim F^C_0 = \dim K.
\]

Let $P_j$ denote the orthogonal projection operator from $E^0$ onto the $L^2$-completion space of $\Omega^j(M)$ with respect to the metric (2.16). We need to show that when $T$ is large enough,

\[
(2.69) \quad \dim P_j(F^C_0) = q_j.
\]

By (2.68), we find that

\[
(2.70) \quad \sum_{j=0}^m \dim P_j(F^C_0) \leq \dim F^C_0 = \sum_{j=0}^m q_j.
\]

Also, we find that for any $s_j \in F_j, \|s_j\|_{E^0} = 1,

\[
(2.71) \quad \|P_jP^C_0 J Ts_j - J Ts_j\|_{E^0} \leq d(P^C_0, Q).
\]

Thus from (2.67) and (2.71), we have for $s \in K,$

\[
(2.72) \quad \|P_jP^C_0 J Ts - J Ts\|_{E^0} \leq \frac{C}{\sqrt{T}} \|s\|_{E^0}.
\]

From (2.72), one deduces that for sufficiently large $T$,

\[
(2.73) \quad \dim P_j(F^C_0) \geq q_j.
\]

From (2.70) and (2.73), we get (2.69). Since the action of $G$ commutes with the deformed de Rham operator $D_T$, the eigenspaces of $D^2_T$ with eigenvalues in $[0, C_0)$ are $G$-vector spaces. This completes the proof of Proposition 2.3. \hfill \Box

Let $F^C_{T,j}$ denotes the $q_j$-dimensional vector space generated by the eigenspaces of $D^2_T|_{\Omega^j(M)}$ associated with the eigenvalues lying in $[0, C_0), j = 0, 1, \ldots, m$. Then $G$ maps $F^C_{T,j}$ into itself.

Recall that the isometric map $J_T : F^0 \to E^0$ is defined by (2.43). We define the map $e_T : F^0 \to F^C_0$ by $e_T = P^C_0 J_T$. We will prove that $e_T$ is a $G$-isomorphism from $F_j$ onto its image when $T$ is large enough.

**Lemma 2.12.** There exists $C > 0$ such that for any $s \in F_j$,

\[
(2.74) \quad \|(e_T - J_T)s\|_{E^0} = O\left(\frac{C}{\sqrt{T}}\right) \|s\|_{E^0} \quad \text{as } T \to +\infty.
\]

In particular, $e_T$ is an $G$-isomorphism from $F_j$ onto $F^C_{T,j}$. 
Proof. It is clear that $e_T$ maps $F_j$ into $F^C_{T,j}$ and
\[(2.75)\quad (e_T - J_T)s = p_T P^C_{T,j} J_T s - J_T s + p_T P^C_{T,j} J_T s.\]
By Proposition 2.11 for any $s \in F_j$,
\[(2.76)\quad \| (e_T - J_T)s \|_{E^0} \leq \| (P^C_{T,j})_1 J_T s - J_T s \|_{E^0} + \| (P^C_{T,j})_3 J_T s \|_{E^0} \leq \frac{C}{\sqrt{T}} \| s \|_{E^0}.
\]
Therefore, $e_T|_{F_j}$ is injective for $T$ large enough. Moreover,
\[(2.77)\quad \dim F_J = \dim F^C_{T,j} = q_j.
\]
Thus $e_T$ is an isomorphism from $F_j$ onto $F^C_{T,j}$. Since $g^N$ is $G$-invariant,
\[(2.78)\quad |g^{-1} \cdot Z|_{g^{-1} \cdot y} = |Z|_{y}, \quad g \cdot \theta = \theta.
\]
From (2.41) we have $\alpha_T(y) = \alpha_T(g^{-1} \cdot y)$. From (2.43) and (2.78), we find that for any $s \in \Gamma^0$,
\[(g \cdot J_T s)(y, Z) = g \cdot (J_T s)(g^{-1} \cdot y, g^{-1} \cdot Z)
\]
\[= \frac{1}{\sqrt{\alpha_T(g^{-1} \cdot y)}} \rho \left( |g^{-1} \cdot Z|_{g^{-1} \cdot y} \right) \exp \left( - \frac{T |g^{-1} \cdot Z|_{g^{-1} \cdot y}^2}{2} \right) \times g \cdot s(g^{-1} \cdot y) \wedge g \cdot \theta_{g^{-1} \cdot y}
\]
\[(2.79)\quad = \frac{1}{\sqrt{\alpha_T(y)}} \rho \left( |Z|_{y} \right) \exp \left( - \frac{T |Z|_{y}^2}{2} \right) \left( g \cdot s \right)(y) \wedge (g \cdot \theta)(y)
\]
\[= J_T(g \cdot s)(y, Z).
\]
This shows that $g$ commutes with $J_T$. Since $g$ commutes with $D_T$, $g$ commutes with $P^C_{T,j}$. Therefore, $e_T$ is a $G$-map, i.e., it commutes with the action of $G$. The proof of Lemma 2.12 is complete. \hfill \square

Proof of (1.9) and (1.10). As $G$-representation space, $F^C_{T,j}$ can be decomposed as:
\[(2.80)\quad F^C_{T,j} = \sum_{\alpha=1}^{l_0} \text{Hom}_G(V^\alpha, F^C_{T,j}) \otimes V^\alpha.
\]
Then $(\text{Hom}_G(V^\alpha, F^C_{T,j}), d_T)$ is a $G$-subcomplex of the complex $(F^C_{T,j}, d_T)$. From (1.8) and Lemma 2.12 we see that
\[(2.81)\quad \dim \text{Hom}_G(V^\alpha, F^C_{T,j}) = d^*_j.
\]
From the Hodge theorem for complexes of finite-dimensional vector spaces, we know that the $j$-th cohomology group of the complex $(F^C_{T,j}, d_T)$ is isomorphic to $\text{Ker} D^2_{j,\Omega(M)}$. Thus the dimension of the $j$-th cohomology group associated to the complex $(\text{Hom}_G(V^\alpha, F^C_{T,j}), d_T)$ is $b^j_\alpha$. Then the inequalities (1.9) and (1.10) hold by standard algebraic techniques \cite{11} Lemma 3.2.12. This completes the proof of Theorem 1.1. \hfill \square
3. An application of equivariant Morse inequalities

In this Section, we apply the equivariant Morse inequalities to prove Theorem 1.2 which is a generalization of [16] Th. 1. We first prove a crucial Lemma.

**Lemma 3.1.** The following inequalities holds for \( k = 0, 1, \ldots, m \),

\[
(3.1) \quad \sum_{j=0}^{k} (-1)^{k-j} \beta_j(M, N_+) \leq \sum_{j=0}^{k} (-1)^{k} q_j.
\]

The equality holds in \((3.1)\) for \( k = m \).

**Proof.** Set

\[
M_1 = M \cup_{N_+} (-M), \quad M_2 = M \cup_{N_+} (-M_1),
\]

where \((-M), (-M_1)\) are copies of \( M \) and \( M_1 \), respectively, and \( N_+ \) is the boundary of \( M_1 \), i.e., \( N'_+ = N_- \cup (-N_-) \).

Denote by \( \mathbb{R}^+, \mathbb{R}^- \) the trivial and the nontrivial one-dimensional real \( \mathbb{Z}_2 \)-representation, respectively. It is well-known that, as \( \mathbb{Z}_2 \)-representation spaces, \( H^j(M_2) \) and \( H^j(M_1) \) have the following decompositions:

\[
H^j(M_2) = H^j(M_1) \cdot \mathbb{R}^+ \oplus H^j(M_1, N'_+) \cdot \mathbb{R}^-,
\]

\[
H^j(M_1) = H^j(M) \cdot \mathbb{R}^+ \oplus H^j(M, N_+) \cdot \mathbb{R}^-.
\]

Let \( \tau_1 \) and \( \tau_2 \) be the flip maps of \( M_1 \) and \( M_2 \), respectively. Let \( e \) and \( g \) be the trivial and nontrivial element, respectively, in \( \mathbb{Z}_2 \), which can be viewed as a multiplication group, i.e., \( g^2 = e = e^2 \). Then \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) acts naturally on \( M_2 \) by

\[
(e, g) \cdot x = \tau_1(x), \quad (g, e) \cdot x = \tau_2(x), \quad \forall \ x \in M_2.
\]

Let \( \{W^a\}_{a=1}^4 \) be the set of non-isomorphic irreducible representations of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \). As vector space, \( W^j = \mathbb{R} \) but \( (e, g) \) acts as \( \text{Id} \) on \( W^1, W^2 \) and acts as \( -\text{Id} \) on \( W^3, W^4 \); besides \( (g, e) \) acts as \( \text{Id} \) on \( W^1, W^3 \) and acts as \( -\text{Id} \) on \( W^2, W^4 \).

Recall that \( b^u_j \) and \( d^a_j \) are defined in \((1.5)\). Using \((3.3)\) and the Poincaré duality theorem for manifolds with boundary [13 Chap. 5, Prop. 9.12], we calculate directly,

\[
(3.5) \quad b^1_j = \beta_j(M), \quad b^2_j = \beta_{m-j}(M), \quad b^3_j = \beta_j(M, N_+), \quad b^4_j = \beta_{m-j}(M, N_+).
\]

To clarify our statement, we replace the vector space \( F_j, F_{a+j} \) and \( F_{r-j} \) by \( F_j(f), F_{a+j}(f) \) and \( F_{r-j}(f) \), respectively. The Poincaré duality theorem yields

\[
(3.6) \quad F_j(-f) \simeq F_{m-j}(f).
\]

By considering the Morse-Bott function \(-f\) instead of \( f \), we obtain that

\[
(3.7) \quad d^1_j = q_{m-j} + \dim \mathbb{R} F_{r-j-1}(-f) + \dim \mathbb{R} F_{a+j}(-f), \quad d^2_j = q_{m-j} + \dim \mathbb{R} F_{r-j-1}(-f); \quad d^3_j = q_{m-j} + \dim \mathbb{R} F_{a+j}(-f), \quad d^4_j = q_{m-j}.
\]

Now we prove \((3.7)\) as follows. For \( w \in F_j(-f) \), let \( \{w\} \) be the real line generated by \( w \). Set

\[
(3.8) \quad W = \{w\} \oplus \{\tau_1(w)\} \oplus \{\tau_2(w)\} \oplus \{\tau_1 \tau_2(w)\}.
\]
Then $W$ is a 4-dimensional vector space spanned by \{w\}, \{\tau_1(w)\}, \{\tau_2(w)\} and \{\tau_1\tau_2(w)\}. Note that $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts naturally on $W$, which can be rewritten as

$$W = \{w + \tau_1(w) + \tau_2(w) + \tau_2\tau_1(w)\} \oplus \{w + \tau_1(w) - \tau_2(w) - \tau_2\tau_1(w)\}$$

Moreover, the 1-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$-space \{w + \tau_1(w) + \tau_2(w) + \tau_2\tau_1(w)\} (resp. \{w + \tau_1(w) - \tau_2(w) - \tau_2\tau_1(w)\}, resp. \{w - \tau_1(w) + \tau_2(w) - \tau_2\tau_1(w)\}, resp. \{w - \tau_1(w) - \tau_2(w) + \tau_2\tau_1(w)\}) is isomorphic to $W^1$ (resp $W^2$, resp. $W^3$, resp. $W^4$) as $\mathbb{Z}_2 \times \mathbb{Z}_2$ representation space. Thus as a $\mathbb{Z}_2 \times \mathbb{Z}_2$ representation space,

$$W \simeq W^1 \oplus W^2 \oplus W^3 \oplus W^4.$$ For nondegenerate critical manifolds on the boundary, we have for $w \in F_{r-j-1}(-f)$, $w' \in F_{a+j}(-f)$,

$$\begin{align*}
\{w\} \oplus \{\tau_2(w)\} &= \{w + \tau_2(w)\} \oplus \{w - \tau_2(w)\}; \\
\{w'\} \oplus \{\tau_1(w')\} &= \{w' + \tau_1(w')\} \oplus \{w' - \tau_1(w')\}.
\end{align*}$$

From (3.6), (3.10) and (3.11), one get (3.7) immediately.

Applying the equivariant Morse inequalities (1.9) to $\alpha = 4$, we deduce that

$$\sum_{j=0}^{k} (-1)^{k-j} \beta_{m-j}(M, N_+) \leq \sum_{j=0}^{k} (-1)^{k-j} q_{m-j}.$$ From (3.12), one verifies directly that

$$\sum_{j=0}^{k} (-1)^{k-j} \beta_{j}(M, N_+) \leq \sum_{j=0}^{k} (-1)^{k-j} q_{j}.$$ One verifies easily that the equality in (3.13) holds when $k = m$. The proof of Lemma 3.1 is complete.

**Proof of (1.13).** We now consider the Mayer-Vietories sequence [12 pp. 185] associated with the triad $(M, N_+, N_{r+})$:

$$\ldots \rightarrow H^{j-1}(N_{a+}) \rightarrow H^{j}(M, N_+) \rightarrow H^{j}(M, N_{r+}) \rightarrow H^{j}(N_{a+}) \rightarrow \ldots$$

From (3.14) and [11] Lemma 3.2.12, we get

$$\sum_{j=0}^{k} (-1)^{k-j}[\beta_{j}(N_{a+}) - \beta_{j}(M, N_{r+}) + \beta_{j}(M, N_+)] = \text{dim } \text{Im } \delta_{k}^{1},$$

where $\delta_{k}^{1}$ denotes the connecting morphism $H^{k}(N_{a+}) \rightarrow H^{k+1}(M, N_{+})$ in the long exact sequence (3.14).

Next we consider the triad $(M, N_r, N_{r+})$:

$$\ldots \rightarrow H^{j-1}(N_{r-}) \rightarrow H^{j}(M, N_r) \rightarrow H^{j}(M, N_{r+}) \rightarrow H^{j}(N_{r-}) \rightarrow \ldots$$
From \((3.16)\) and \([11, \text{Lemma 3.2.12}]\), we find that
\[
\sum_{j=0}^{k} (-1)^{k-j} [\beta_j(M, N_{r+}) - \beta_j(M, N_r) + \beta_{j-1}(N_{r-})] = \dim \text{Im } \delta_k^2,
\]
where \(\delta_k^2\) denotes the morphism \(H^k(M, N_{r+}) \to H^k(N_{r-})\) in the long exact sequence \((3.16)\) induced by the inclusion \((N_r, N_{r+}) \hookrightarrow (M, N_{r+})\).

From \((3.13)\), \((3.15)\) and \((3.17)\), we get that for \(k = 0, 1, \ldots, m\),
\[
\sum_{j=0}^{k} (-1)^{k-j} \beta_j(M, N_r) \leq \sum_{j=0}^{k} (-1)^{k-j} \nu_j,
\]
where
\[
\nu_j = q_j + \beta_j(N_{a+}) + \beta_{j-1}(N_{r-}).
\]
The equality holds in \((3.18)\) for \(k = m\).

We now directly apply the degenerate Morse inequalities \([2, (2.101)]\) to the closed submanifolds \(N_{a+}\) and \(N_{r-}\) respectively: for \(0 \leq k \leq m - 1\),
\[
\sum_{j=0}^{k} (-1)^{k-j} \beta_j(N_{a+}) \leq \sum_{j=0}^{k} (-1)^{k-j} q_{a+j}, \quad \sum_{j=0}^{k} (-1)^{k-j} \beta_j(N_{r-}) \leq \sum_{j=0}^{k} (-1)^{k-j} q_{r-j},
\]
with equality for \(k = m - 1\). Note that
\[
\beta_m(N_{a+}) = 0 = q_{a+m}, \quad \beta_m(N_{r-}) = 0 = q_{r-m}.
\]
Due to \((3.20)\) and \((3.21)\), the equality in \((3.20)\) holds also for \(k = m\).

Now \((1.13)\) follows from \((3.18)\) and \((3.20)\). One verifies easily that the equality in \((1.13)\) holds when \(k = m\). This finishes the proof of Theorem 1.2. \(\square\)

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