Tableaux for First Order Logic of Proofs

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Abstract. In this paper we present a tableau proof system for first order logic of proofs FOLP. We show that the tableau system is sound and complete with respect to Mkrtychev models of FOLP.

1 Introduction
Artemov in [1, 2] introduced the first propositional justification logic LP, the Logic of Proofs (for more information about justification logics see [3, 4]). Later Artemov and Yavorskaya (Sidon) introduced in [5] the first order logic of proofs FOLP. The language of FOLP extends the language of first order logic by justification terms and expressions of the form $t : X \ A$, where $X$ is a set of individual variables. The intended meaning of $t : X \ A$ is “$t$ justifies $A$ in which the variables in $X$ can be substituted for and cannot be quantified.” Fitting in [8, 9] proposed possible world semantics and Mkrtychev semantics for FOLP.

Various tableau proof systems have been developed for the logic of proofs (see [6, 7, 10, 12, 13]). The aim of this paper is to present a tableau proof system for FOLP. Our tableau rules are extensions of Renne’s tableau rules [12] for LP. We show that our tableau proof system is sound and complete with respect to Mkrtychev models of FOLP.

2 The logic FOLP
The language of FOLP is an extension of the language of first order logic by expressions of the form $t : X \ A$, where $A$ is a formula, $t$ is a justification term and $X$ is a set of individual variables. Following [5] we consider a first order language in which there are no constant symbols, function symbols, and identity, but of course a countable set of individual variables $Var$ (denoted by $x, y, z, \ldots$).

Justification terms are built up from a countable set of justification variables $JVar$ and a countable set of justification constants $JCons$ by the following grammar:

$$t ::= p \mid c \mid t + t \mid t \cdot t \mid !t \mid gen_x(t),$$

where $p \in JVar$, $c \in JCons$, and $x \in Var$. FOLP formulas are constructed from a countable set of predicate symbols of any arity by the following grammar:

$$A ::= Q(x_1, \ldots, x_n) \mid \neg A \mid A \to A \mid \forall x A \mid \exists x A \mid t : X \ A,$$

where $Q$ is an $n$-place predicate symbol, $t$ is a justification term, and $X \subseteq Var$. 
Free individual variable occurrences in formulas are defined as in the first order logic, with the following addition: the free individual variable occurrences in $t : X A$ are the free individual variable occurrences in $A$, provided the variables also occur in $X$, together with all variable occurrences in $X$ itself. The set of all free individual variables of the formula $A$ is denoted by $FVar(A)$. Thus $FVar(t : X A) = X$. The universal closure of a formula $A$ will be denoted by $\forall A$. The notion of substitution of an individual variable for another individual variable is defined as in the first order logic.

If $y$ is an individual variable, then $Xy$ is short for $X \cup \{y\}$, and in addition it means $y \not\in X$.

**Definition 2.1.** Axioms schemes and rules of FOLP are:

- **FOL.** Axiom schemes of first order logic,
- **Ctr.** $t : X y A \rightarrow t : X A$, provided $y \notin FVar(A)$.
- **Exp.** $t : X A \rightarrow t : X y A$.
- **Sum.** $s : X A \rightarrow (s + t) : X A$, $s : X A \rightarrow (t + s) : X A$.
- **jK.** $s : X (A \rightarrow B) \rightarrow (t : X A \rightarrow (s \cdot t) : X B)$.
- **jT.** $t : X A \rightarrow A$.
- **j4.** $t : X A \rightarrow t : X A$.
- **Gen.** $t : X A \rightarrow gen_x(t) : X \forall x A$, provided $x \notin X$.
- **MP.** From $\vdash A$ and $\vdash A \rightarrow B$ infer $\vdash B$.
- **UG.** From $\vdash A$ infer $\vdash \forall x A$.
- **AN.** $\vdash c : A$, where $A$ is an axiom instance and $c$ is an arbitrary justification constant.

**Definition 2.2.**

1. A constant specification $CS$ for FOLP is a set of formulas of the form $c : A$, where $c$ is a justification constant and $A$ is an axiom instance of FOLP.
2. A constant specification $CS$ is axiomatically appropriate if for every axiom instance $A$ there is a justification constant $c$ such that $c : A \in CS$.
3. Two formulas are variable variants if each can be turned into the other by a renaming of free and bound individual variables.
4. A constant specification $CS$ is variant closed provided that whenever $A$ and $B$ are variable variants, $c : A \in CS$ if and only if $c : B \in CS$.

Let $\text{FOLP}_{CS}$ be the fragment of FOLP where the Axiom Necessitation rule only produces formulas from the given $CS$.

In the remaining of this section, we recall the definition of Mkrtchyan models for FOLP from [9] (Mkrtchyan models was first introduced for LP in [11]). First we need the following auxiliary definition.

**Definition 2.3.** Let $K$ be a non-empty set.

1. A $K$-formula is the result of substituting some free individual variables in an FOLP formula with members of $K$.

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1 Ctr and Exp are abbreviations for Contraction and Expansion respectively. The rule AN is called Axiom Necessitation.
2. A $K$-formula is closed if it contains no free occurrences of individual variables.
3. For a $K$-formula $A$, let $K(A)$ be the set of all members of $K$ that occur in $A$.
4. For a formula $F(\bar{x})$ and $\bar{a} \in K$, by $F(\bar{a})$ we mean that all free occurrences of the individual variables in $\bar{x}$ have been replaced with corresponding occurrences in $\bar{a}$. This is sometimes denoted by $F(\bar{x}/\bar{a})$.

**Definition 2.4.** A Mkrtychev model $M = (D, I, E)$ for FOLP$_{CS}$ (or FOLP$_{CS}$-model, for short) consists of:

- A non-empty set $D$, called the domain of the model. The definitions of (closed) $D$-formulas and $D(A)$, for a $D$-formula $A$, are similar to Definition 2.3, where $K$ is replaced by $D$.
- The interpretation $I$ assigns to each $n$-place predicate symbol some $n$-ary relation on $D$.
- The admissible evidence function $E$ assigns to each justification term a set of $D$-formulas meeting the following conditions:
  
  $E$1. $c : A \in CS$ implies $A \in E(c)$.
  $E$2. $A \rightarrow B \in E(s)$ and $A \in E(t)$ implies $B \in E(s \cdot t)$.
  $E$3. $E(s) \cup E(t) \subseteq E(s + t)$.
  $E$4. $A \in E(t)$ implies $t :_X A \in E(!t)$, where $D(A) \subseteq X \subseteq D$.
  $E$5. $A \in E(t)$ implies $\forall x A \in E(gen_x(t))$.
  $E$6. $a \in D$ and $A(x) \in E(t)$ implies $A(a) \in E(t)$.

Condition $E$6 is called the Instantiation Condition in [9].

**Definition 2.5.** For an FOLP$_{CS}$-model $M = (D, I, E)$ and a closed $D$-formula we define when the formula is true in $M$ as follows:

1. $M \vdash Q(\bar{a})$ iff $\bar{a} \in I(Q)$, for $n$-place predicate symbol $Q$ and $\bar{a} \in D$.
2. $M \vdash \neg A$ iff $M \not\vdash A$.
3. $M \vdash A \rightarrow B$ iff $M \not\vdash A$ or $M \vdash B$.
4. $M \vdash \forall x A(x)$ iff $M \vdash A(a)$ for every $a \in D$.
5. $M \vdash \exists x A(x)$ iff $M \vdash A(a)$ for some $a \in D$.
6. $M \vdash t :_X A$ iff $A \in E(t)$ and $M \vdash \forall A$.

If $M \vdash F$ then it is said that $F$ is true in $M$ or $M$ satisfies $F$.

A sentence $F$ is FOLP$_{CS}$-valid if it is true in every FOLP$_{CS}$-model. For a set $S$ of sentences, $M \vdash S$ provided that $M \vdash F$ for all formulas $F$ in $S$. Note that given a constant specification $CS$ for FOLP, and a model $M$ of FOLP$_{CS}$ we have $M \vdash CS$ (in this case it is said that $M$ respects $CS$).

The proof of soundness and completeness theorems of FOLP are given in [9].

**Theorem 2.1.** Let $CS$ be an axiomatically appropriate and variant closed constant specification for FOLP. Then a sentence $F$ is provable in FOLP$_{CS}$ iff $F$ is FOLP$_{CS}$-valid.
Tableaux

Tableau proof systems for the logic of proofs are given in [7, 12, 13]. In this section we extend them and present tableaux for FOLP.

Let Par be a denumerable set of new individual variables, i.e. Par ∩ Var = ∅. The members of Par are called parameters, with typical members denoted u, v, w. Parameters are never quantified. The definitions of (closed) Par-formulas, Par-instance of a formula, and Par(A), for a Par-formula A, are similar to Definition 2.3, where K is replaced by Par. Notice that closed Par-formulas may contain free parameters but do not contain free individual variables.

Tableau proofs will be of sentences of FOLP but will use closed Par-formulas. An FOLP CS-tableau for a sentence is a binary tree labeled by closed Par-formulas with the negation of that sentence at the root constructed by applying FOLP tableau rules from Table 1. An FOLP CS-tableau branch closes if one of the following holds:

1. Both A and ¬A occurs in the branch, for some closed Par-formula A.
2. ¬c : A occurs in the branch, where c : A ∈ CS.

A tableau closes if all branches of the tableau close. An FOLP CS-tableau proof for a sentence F is a closed tableau beginning with ¬F (the root of the tableau) using only FOLP tableau rules. An FOLP CS-tableau for a finite set S of closed Par-formulas begins with a single branch whose nodes consist of the formulas of S as roots.

Example 3.1. We give an FOLP CS-tableau proof of the sentence

\[ p \colon \forall x A(x) \rightarrow \forall x (c \cdot p) :_{\{x\}} A(x), \]

where p ∈ JVar and CS contains c : (\forall x A(x) \rightarrow A(x)). An axiomatic proof of this sentence is given in [5]. This sentence is an explicit counterpart of the Converse Barcan Formula □\forall x A(x) → \forall x □ A(x).

\[ 1. \neg(p : \forall x A(x) \rightarrow \forall x (c \cdot p) :_{\{x\}} A(x)) \]

\[ 2. p : \forall x A(x) \]

\[ 3. \neg\forall x (c \cdot p) :_{\{x\}} A(x) \]

\[ 4. \neg(c \cdot p) :_{\{u\}} A(u) \]

\[ 5. \neg c :_{\{u\}} (\forall x A(x) \rightarrow A(u)) \]

\[ 6. \neg p :_{\{u\}} \forall x A(x) \]

\[ 7. \neg c :_{\{u\}} (\forall x A(x) \rightarrow A(x)) \]

\[ 8. \neg p : \forall x A(x) \]

\[ 9. \neg c : (\forall x A(x) \rightarrow A(x)) \]

Formulas 2 and 3 are from 1 by rule (F →), 4 is from 3 by rule (F∀), where u is a new parameter, 5 and 6 are from 4 by rule (F), 7 is from 5 by rule (Ins), and 8 and 9 are from 6 and 7, respectively, by rule (Exp).
Let us now show the soundness of FOLP tableau system.

**Definition 3.1.** A Par-formula $A(u_1, \ldots, u_n)$, where $u_1, \ldots, u_n$ are all parameters of $A$, is satisfiable in a FOLP$_{CS}$-model $M = (D, I, E)$, denoted by $M \models A(u_1, \ldots, u_n)$, if $M \models A(a_1, \ldots, a_n)$ for some $a_1, \ldots, a_n \in D$. A tableau branch is satisfiable in a model $M$ if every formula of the branch is satisfiable.

**Lemma 3.1.** Let $\pi$ be any branch of an FOLP$_{CS}$-tableau and $M$ be an FOLP$_{CS}$-model that satisfies all the formulas occur in $\pi$. If an FOLP tableau rule is applied to $\pi$, then it produces at least one extension $\pi'$ such that $M$ satisfies all the formulas occur in $\pi'$.

**Proof.** Suppose that a tableau branch $\pi$ is satisfiable in the model $M = (D, I, E)$, and $\pi'$ is obtained by applying an FOLP tableau rule to $\pi$. To prove the lemma, we
consider each rule in turn. The cases for the propositional logic rules are standard.
Hence, we need consider only the rules for quantifiers and FOLP rules.

Suppose that the rule (T∀) is applied

\[
\frac{\forall x.A(x, \vec{w})}{A(u, \vec{w})} \quad (T∀)
\]

where \(u, \vec{w} \in Par\). Since \(\mathcal{M} \models \forall x.A(x, \vec{w})\), we have \(\mathcal{M} \models \forall x.A(x, \vec{b})\) for some \(\vec{b} \in D\). Thus, \(\mathcal{M} \models A(a, \vec{b})\) for every \(a \in D\). Then, obviously \(\mathcal{M} \models A(u, \vec{w})\). Hence \(\mathcal{M} \models \pi'\) as desired. The case of the rule (F∃) is similar.

Suppose that the rule (T∃) is applied

\[
\frac{\exists x.A(x, \vec{w})}{A(u, \vec{w})} \quad (T∃)
\]

where \(\vec{w} \in Par\) and \(u\) is a new parameter in \(\pi\). Since \(\mathcal{M} \models \exists x.A(x, \vec{w})\), we have \(\mathcal{M} \models \exists x.A(x, \vec{b})\) for some \(\vec{b} \in D\). Thus, \(\mathcal{M} \models A(a, \vec{b})\) for some \(a \in D\). Then, obviously \(\mathcal{M} \models A(u, \vec{w})\). Hence \(\mathcal{M}' \models \pi'\) as desired. The case of the rule (T:) is similar.

Suppose that the rule (T:) is applied

\[
\frac{t : (\vec{w}, \vec{x}) A(\vec{w}, \vec{x})}{\forall \vec{x} A(\vec{w}, \vec{x})} \quad (T:)
\]

Since \(\mathcal{M} \models t : (\vec{w}, \vec{x}) A(\vec{w}, \vec{x})\), we have \(\mathcal{M} \models t : (\vec{a}, \vec{x}) A(\vec{a}, \vec{x})\) for some \(\vec{a}, \vec{b} \in \mathcal{D}\). Thus \(\mathcal{M} \models \forall \vec{x} A(\vec{w}, \vec{x})\). Then, obviously \(\mathcal{M} \models \forall \vec{x} A(\vec{w}, \vec{x})\). Hence \(\mathcal{M} \models \pi'\) as desired.

Suppose that the rule (F+) is applied

\[
\frac{-t + s : (\vec{w}, \vec{x}) A(\vec{w}, \vec{x})}{-t : (\vec{w}, \vec{x}) A(\vec{w}, \vec{x})} \quad (F+)
\]

Since \(\mathcal{M} \models -t + s : (\vec{w}, \vec{x}) A(\vec{w}, \vec{x})\), we have \(\mathcal{M} \models -t + s : (\vec{a}, \vec{x}) A(\vec{a}, \vec{x})\) for some \(\vec{a}, \vec{b} \in \mathcal{D}\). Thus either \(A(\vec{a}, \vec{x}) \notin \mathcal{E}(t + s)\) or \(\mathcal{M} \not\models \forall \vec{x} A(\vec{a}, \vec{x})\). In the former case we have \(A(\vec{a}, \vec{x}) \notin \mathcal{E}(t) \cup \mathcal{E}(s)\), and hence \(\mathcal{M} \not\models t : (\vec{a}, \vec{x}) A(\vec{a}, \vec{x})\) and \(\mathcal{M} \not\models s : (\vec{a}, \vec{x}) A(\vec{a}, \vec{x})\).

We get the same results in the latter case. In either case \(\mathcal{M} \models -t : (\vec{w}, \vec{x}) A(\vec{w}, \vec{x})\) and \(\mathcal{M} \models -s : (\vec{w}, \vec{x}) A(\vec{w}, \vec{x})\). Hence \(\mathcal{M} \models \pi'\) as desired.

Suppose that the rule (F−) is applied

\[
\frac{-s \cdot t : (\vec{w}, \vec{x}) B(\vec{w}, \vec{x})}{-s : (\vec{w}, \vec{x}) B(\vec{w}, \vec{x})} \quad (F−)
\]

where \(\{\vec{w}'\} \subseteq \{\vec{w}\}\) and \(\{\vec{y}'\} \subseteq \{\vec{y}\}\). Since \(\mathcal{M} \models -s \cdot t : (\vec{w}, \vec{x}) B(\vec{w}, \vec{x})\), we have \(\mathcal{M} \models -s \cdot t : (\vec{a}, \vec{x}) B(\vec{a}, \vec{x})\) for some \(\vec{a}, \vec{b} \in \mathcal{D}\). Thus either \(B(\vec{a}, \vec{x}) \notin \mathcal{E}(s \cdot t)\) or \(\mathcal{M} \not\models \forall \vec{x} B(\vec{a}, \vec{x})\). In the former case we have either \(A(\vec{a}', \vec{b}', \vec{y}) \rightarrow B(\vec{a}, \vec{x}) \notin \mathcal{E}(s)\) or \(A(\vec{a}', \vec{b}', \vec{y}) \notin \mathcal{E}(t)\), where \(A(\vec{a}', \vec{b}', \vec{y}) = A(\vec{a}', \vec{b}', \vec{y})\{\vec{w}/\vec{a}', \vec{y}/\vec{b}'\}\), and hence either \(\mathcal{M} \not\models s : (\vec{a}, \vec{x}) A(\vec{a}, \vec{x})\) or \(\mathcal{M} \not\models t : (\vec{a}, \vec{x}) A(\vec{a}, \vec{x})\). In the latter case,
either $\mathcal{M} \not\models \forall A$ and hence $\mathcal{M} \not\models \forall t : \langle \vec{a}, \vec{b} \rangle \ A(\vec{a}', \vec{b}', \vec{y})$, or $\mathcal{M} \models \forall A$ and hence $\mathcal{M} \not\models s : \langle \vec{a}, \vec{b} \rangle \ (A(\vec{a}', \vec{b}', \vec{y}) \rightarrow B(\vec{a}, \vec{x}))$, since $\mathcal{M} \not\models \forall (A \rightarrow B)$. Thus, in both cases we have either $\mathcal{M} \models \neg s : \langle \vec{a}, \vec{b} \rangle \ A(\vec{a}', \vec{b}', \vec{y})$ or $\mathcal{M} \models \neg t : \langle \vec{a}, \vec{b} \rangle \ A(\vec{a}', \vec{b}', \vec{y})$. Therefore either $\mathcal{M} \models \neg t : \langle \vec{v}, \vec{w}, \vec{y} \rangle \ (A(\vec{v}', \vec{w}', \vec{y}) \rightarrow B(\vec{w}, \vec{x}))$ or $\mathcal{M} \models \neg t : \langle \vec{v}, \vec{w} \rangle \ A(\vec{w}', \vec{y}, \vec{y})$.

Suppose that the rule (F!) is applied

$$\frac{\neg \mathcal{M} \models \exists t : \langle \vec{v}, \vec{w} \rangle \ A(\vec{w}, \vec{x})}{\neg \mathcal{M} \models \exists t : \langle \vec{v}, \vec{w} \rangle \ A(\vec{w}, \vec{x})} \quad (F!)
$$

Since $\mathcal{M} \models \neg \mathcal{M} \models \exists t : \langle \vec{v}, \vec{w} \rangle \ A(\vec{w}, \vec{x})$, we have $\mathcal{M} \models \neg t : \langle \vec{v}, \vec{w} \rangle \ A(\vec{w}, \vec{x})$ for some $\vec{a}, \vec{b} \in D$. Thus either $t : \langle \vec{v}, \vec{w} \rangle \ A(\vec{a}, \vec{x}) \notin \mathcal{F}(t)$ or $\mathcal{M} \models \forall t : \langle \vec{v}, \vec{w} \rangle \ A(\vec{a}, \vec{x})$. In the former case we have $A(\vec{a}, \vec{x}) \notin \mathcal{E}(t)$, and hence $\mathcal{M} \models \neg t : \langle \vec{v}, \vec{w} \rangle \ A(\vec{a}, \vec{x})$. In either case $\mathcal{M} \models \neg t : \langle \vec{v}, \vec{w} \rangle \ A(\vec{w}, \vec{x})$. Hence $\mathcal{M} \models \pi'$ as desired.

Suppose that the rule (Ctr) is applied

$$\frac{\neg t : \langle \vec{v}, \vec{w} \rangle \ A(\vec{w}, \vec{x})}{\neg t : \langle \vec{v}, \vec{w}, \vec{u} \rangle \ A(\vec{w}, \vec{x})} \quad (Ctr)
$$

Since $\mathcal{M} \models \neg t : \langle \vec{v}, \vec{w} \rangle \ A(\vec{w}, \vec{x})$, we have $\mathcal{M} \models \neg t : \langle \vec{v}, \vec{w} \rangle \ A(\vec{a}, \vec{x})$ for some $\vec{a}, \vec{b} \in D$. Thus either $A(\vec{a}, \vec{x}) \notin \mathcal{E}(t)$ or $\mathcal{M} \models \forall t : \langle \vec{v}, \vec{w} \rangle \ A(\vec{a}, \vec{x})$ for an arbitrary $d \in D$. Therefore $\mathcal{M} \models \neg t : \langle \vec{v}, \vec{w}, \vec{u} \rangle \ A(\vec{a}, \vec{x})$. Hence $\mathcal{M} \models \pi'$ as desired.

Suppose that the rule (Exp) is applied

$$\frac{\neg t : \langle \vec{v}, \vec{w}, \vec{u} \rangle \ A(\vec{w}, \vec{x})}{\neg t : \langle \vec{v}, \vec{w} \rangle \ A(\vec{w}, \vec{x})} \quad (Exp)
$$

where $u \notin \text{Par}(A)$. Since $\mathcal{M} \models \neg t : \langle \vec{v}, \vec{w}, \vec{u} \rangle \ A(\vec{w}, \vec{x})$, we have $\mathcal{M} \models \neg t : \langle \vec{v}, \vec{w}, \vec{u} \rangle \ A(\vec{a}, \vec{x})$ for some $\vec{a}, \vec{b}, d \in D$. Thus either $A(\vec{a}, \vec{x}) \notin \mathcal{E}(t)$ or $\mathcal{M} \models \forall t : \langle \vec{v}, \vec{w} \rangle \ A(\vec{a}, \vec{x})$. From this it follows that $\mathcal{M} \models \neg t : \langle \vec{v}, \vec{w} \rangle \ A(\vec{a}, \vec{x})$. Therefore $\mathcal{M} \models \neg t : \langle \vec{v}, \vec{w} \rangle \ A(\vec{w}, \vec{x})$. Hence $\mathcal{M} \models \pi'$ as desired.

Suppose that the rule (Ins) is applied

$$\frac{\neg t : \langle \vec{v}, \vec{w}, \vec{u} \rangle \ A(\vec{w}, \vec{y}, u)}{\neg t : \langle \vec{v}, \vec{w}, \vec{x} \rangle \ A(\vec{w}, \vec{y}, x)} \quad (Ins)
$$

Since $\mathcal{M} \models \neg t : \langle \vec{v}, \vec{w}, \vec{u} \rangle \ A(\vec{w}, \vec{y}, u)$, we have $\mathcal{M} \models \neg t : \langle \vec{v}, \vec{w}, \vec{u} \rangle \ A(\vec{a}, \vec{y}, d)$ for some $\vec{a}, \vec{b}, d \in D$. Thus either $A(\vec{a}, \vec{y}, d) \notin \mathcal{E}(t)$ or $\mathcal{M} \models \forall \vec{y} A(\vec{a}, \vec{y}, d)$. By the In-stantiation Condition (E6), either $A(\vec{a}, \vec{y}, x) \notin \mathcal{E}(t)$ or $\mathcal{M} \models \forall \vec{x} \vec{y} A(\vec{a}, \vec{y}, x)$. Thus $\mathcal{M} \models \neg t : \langle \vec{v}, \vec{w}, \vec{u} \rangle \ A(\vec{a}, \vec{y}, x)$. Therefore $\mathcal{M} \models \neg t : \langle \vec{v}, \vec{w}, \vec{u} \rangle \ A(\vec{w}, \vec{y}, x)$. Hence $\mathcal{M} \models \pi'$ as desired.

Suppose that the rule (gen$_x$) is applied

$$\frac{\neg \text{gen}_x(t) : \langle \vec{v}, \vec{w} \rangle \ \forall x A(\vec{w}, \vec{y}, x)}{\neg t : \langle \vec{v}, \vec{w} \rangle \ A(\vec{w}, \vec{y}, x)} \quad (gen_x)
$$

We consider the case where $A = A(\vec{w}, \vec{y}, x)$, i.e. $x \in \text{FVar}(A)$. The case that $x$ is not free in $A$ is treated similarly. Since $\mathcal{M} \models \neg \text{gen}_x(t) : \langle \vec{v}, \vec{w} \rangle \ \forall x A(\vec{w}, \vec{y}, x)$, we have
Theorem 3.1 (Soundness). Let $A$ be a sentence of FOLP. If $A$ has an FOLP\textsubscript{CS}-
tableau proof, then it is FOLP\textsubscript{CS}-valid.

Proof. If the sentence $A$ is not FOLP\textsubscript{CS}-valid, then there is an FOLP\textsubscript{CS}-
model $M$ such that $M \models \neg A$. Then, by Lemma 3.1, there is no closed FOLP\textsubscript{CS}-tableau beginning
with $\neg A$. Therefore, $A$ does not have an FOLP\textsubscript{CS}-tableau proof.

Next we will prove the completeness theorem by making use of maximal consistent sets.

Definition 3.2. Suppose $\Gamma$ is a set of closed Par-formulas.

1. $\Gamma$ is tableau FOLP\textsubscript{CS}-consistent if there is no closed FOLP\textsubscript{CS}-tableau beginning
   with any finite subset of $\Gamma$.
2. $\Gamma$ is maximal if it has no proper tableau consistent extension (w.r.t. closed Par
   formulas).
3. $\Gamma$ is E-complete (with members of Par as witnesses) if
   - $\exists x A(x) \in \Gamma$ implies $A(u) \in \Gamma$ for some $u \in Par$.
   - $\forall x A(x) \in \Gamma$ implies $\neg A(u) \in \Gamma$ for some $u \in Par$.

By making use of the Henkin construction it is not hard to show the following result.

Lemma 3.2. Every tableau FOLP\textsubscript{CS}-consistent set of sentences of FOLP can be
extended to a tableau FOLP\textsubscript{CS}-consistent, maximal and E-complete set of closed Par-formulas.

It is easy to show that E-complete maximally tableau FOLP\textsubscript{CS}-consistent sets are closed under FOLP\textsubscript{CS}-tableau rules. For a non-branching rule like

\[
\frac{\alpha}{\alpha_1} \frac{\alpha_2}
\]

this means that if $\alpha$ is in a E-complete maximally tableau FOLP\textsubscript{CS}-consistent set $\Gamma$, then both $\alpha_1 \in \Gamma$ and $\alpha_2 \in \Gamma$. For a branching rule like

\[
\frac{\beta}{\beta_1 \beta_2}
\]

this means that if $\beta$ is in a E-complete maximally tableau FOLP\textsubscript{CS}-consistent set $\Gamma$, then $\beta_1 \in \Gamma$ or $\beta_2 \in \Gamma$. For the rule $(F\cdot)$ this means that if $\neg s \cdot t : X B \in \Gamma$, then for every formula $A$ such that $Par(A) \subseteq X$ either $\neg s : X (A \to B) \in \Gamma$ or $\neg t : X A \in \Gamma$.

Lemma 3.3. Suppose $\Gamma$ is an E-complete maximally tableau FOLP\textsubscript{CS}-consistent set of closed Par-formulas. Then $\Gamma$ is closed under FOLP\textsubscript{CS}-tableau rules.
Proof. The proof for rules \((F\neg)\), \((F\rightarrow)\), and \((T\rightarrow)\), are standard. We detail the proof for other tableau rules.

\((TV)\) Suppose \(\forall x A \in \Gamma\) and \(u\) is an arbitrary parameter. We want to show that \(A(u) \in \Gamma\). If this is not the case, since \(\Gamma\) is maximal, then \(\Gamma \cup \{A(u)\}\) is not tableau FOLPC\(_{CS}\)-consistent. Hence there is a closed FOLPC\(_{CS}\)-tableau for a finite subset, say \(\Gamma_0 \cup \{A(u)\}\). But \(\Gamma_0 \cup \{\forall x A\}\) is a finite subset of \(\Gamma\) and, using rule \((TV)\), there is a closed FOLPC\(_{CS}\)-tableau for it, contra the tableau consistency of \(\Gamma\). The case of \((F\exists)\) is similar.

\((T\exists)\) Suppose \(\exists x A \in \Gamma\). Since \(\Gamma\) is \(E\)-complete, \(A(u) \in \Gamma\) for some parameter \(u\). The case of \((F\forall)\) is similar.

\((F-\) Suppose \(\neg s : t : X B \in \Gamma\). Suppose towards a contradiction that for some formula \(A\) such that \(Par(A) \subseteq X\) we have \(\neg s : t : (A \rightarrow B) \not\in \Gamma\) and \(\neg t : X A \not\in \Gamma\). Since \(\Gamma\) is maximal, \(\Gamma \cup \{\neg s : X (A \rightarrow B)\}\) and \(\Gamma \cup \{\neg t : X A\}\) are not tableau consistent. Thus there are closed tableaux for finite subsets, say \(\Gamma_1 \cup \{\neg s : X (A \rightarrow B)\}\) and \(\Gamma_2 \cup \{\neg t : X A\}\). But \(\Gamma_1 \cup \Gamma_2 \cup \{\neg s : X B\}\) is a finite subset of \(\Gamma\) and, using rule \((F-)\), there is a closed FOLPC\(_{CS}\)-tableau for it, contra the tableau consistency of \(\Gamma\). The cases of rules \((T:)\) and \((F+)\) are similar.

\((Ctr)\) Suppose \(\neg t : X A \in \Gamma\). We want to show that \(\neg t : X A \in \Gamma\). If it is not the case, since \(\Gamma\) is maximal, then \(\Gamma \cup \{\neg t : X A\}\) is not tableau FOLPC\(_{CS}\)-consistent. Hence there is a closed FOLPC\(_{CS}\)-tableau for a finite subset, say \(\Gamma_0 \cup \{\neg t : X A\}\). But \(\Gamma_0 \cup \{\neg t : X A\}\) is a finite subset of \(\Gamma\) and, using rule \((Ctr)\), there is a closed FOLPC\(_{CS}\)-tableau for it, contra the tableau consistency of \(\Gamma\).

\((Exp)\) Suppose \(\neg t : X u A \in \Gamma\) and \(u \not\in Par(A)\). We want to show that \(\neg t : X A \in \Gamma\). If it is not the case, since \(\Gamma\) is maximal, then \(\Gamma \cup \{\neg t : X A\}\) is not tableau FOLPC\(_{CS}\)-consistent. Hence there is a closed FOLPC\(_{CS}\)-tableau for a finite subset, say \(\Gamma_0 \cup \{\neg t : X A\}\). But \(\Gamma_0 \cup \{\neg t : X u A\}\) is a finite subset of \(\Gamma\) and, using rule \((Exp)\), there is a closed FOLPC\(_{CS}\)-tableau for it, contra the tableau consistency of \(\Gamma\).

\((Ins)\) Suppose \(\neg t : X A(u) \in \Gamma\). We want to show that \(\neg t : X A(x) \in \Gamma\). If it is not the case, since \(\Gamma\) is maximal, then \(\Gamma \cup \{\neg t : X A(x)\}\) is not tableau FOLPC\(_{CS}\)-consistent. Hence there is a closed FOLPC\(_{CS}\)-tableau for a finite subset, say \(\Gamma_0 \cup \{\neg t : X A(x)\}\). But \(\Gamma_0 \cup \{\neg t : X A(u)\}\) is a finite subset of \(\Gamma\) and, using rule \((Ins)\), there is a closed FOLPC\(_{CS}\)-tableau for it, contra the tableau consistency of \(\Gamma\).

\((genx)\) Suppose \(\neg gen_x(t) : X \forall x A \in \Gamma\). We want to show that \(\neg t : X A \in \Gamma\). If it is not the case, since \(\Gamma\) is maximal, then \(\Gamma \cup \{\neg t : X A\}\) is not tableau FOLPC\(_{CS}\)-consistent. Hence there is a closed FOLPC\(_{CS}\)-tableau for a finite subset, say \(\Gamma_0 \cup \{\neg t : X A\}\). But \(\Gamma_0 \cup \{\neg gen_x(t) : X \forall x A\}\) is a finite subset of \(\Gamma\) and, using rule \((genx)\), there is a closed FOLPC\(_{CS}\)-tableau for it, contra the tableau consistency of \(\Gamma\). \(\square\)
Definition 3.3. Given an E-complete maximally tableau FOLP\(_{CS}\)-consistent set \( \Gamma \) of closed Par-formulas, the canonical model \( \mathcal{M} = (\mathcal{D}, \mathcal{I}, \mathcal{E}) \) with respect to \( \Gamma \) and \( \mathcal{CS} \) is defined as follows:

- \( \mathcal{D} = \text{Par} \),
- \( \mathcal{I}(Q) = \{ (u_1, \ldots, u_n) \in \mathcal{D} \mid Q(u_1, \ldots, u_n) \in \Gamma \} \), for any \( n \)-place relation symbol \( Q \),
- \( \mathcal{E}(t) = \{ A \mid \neg t ::_{\text{Par}(A)} A \not\in \Gamma \} \).

Lemma 3.4. Given an E-complete maximally tableau FOLP\(_{CS}\)-consistent set \( \Gamma \) of closed Par-formulas, the canonical model \( \mathcal{M} = (\mathcal{D}, \mathcal{I}, \mathcal{E}) \) with respect to \( \Gamma \) and \( \mathcal{CS} \) is an FOLP\(_{CS}\)-model.

Proof. Suppose \( \Gamma \) is an E-complete maximally tableau FOLP\(_{CS}\)-consistent set, and \( \mathcal{M} = (\mathcal{D}, \mathcal{I}, \mathcal{E}) \) is the canonical model with respect to \( \Gamma \) and \( \mathcal{CS} \). We will show that the admissible evidence function \( \mathcal{E} \) satisfies \( \mathcal{E}1-\mathcal{E}6 \) from Definition 2.4.

(\( \mathcal{E}1 \)) Suppose that \( c : A \in \mathcal{CS} \). Then \( \text{Par}(A) = \emptyset \). We have to show that \( A \in \mathcal{E}(c) \). Since \( \Gamma \) is tableau FOLP\(_{CS}\)-consistent, \( \neg c : A \not\in \Gamma \) and hence \( \neg c ::_{\text{Par}(A)} A \not\in \Gamma \). Thus \( A \in \mathcal{E}(c) \).

(\( \mathcal{E}2 \)) Suppose that \( A \in \mathcal{E}(t) \) and \( A \not\rightarrow B \in \mathcal{E}(s) \). We have to show that \( B \in \mathcal{E}(s \cdot t) \). Let \( \mathcal{X} = \text{Par}(A \rightarrow B) = \text{Par}(A) \cup \text{Par}(B) \). By the definition of \( \mathcal{E} \), \( \neg t ::_{\text{Par}(A)} A \not\in \Gamma \) and \( \neg s ::_{\mathcal{X}} (A \rightarrow B) \not\in \Gamma \). Since \( \Gamma \) is closed under rule (\( \text{Exp} \)), \( \neg t ::_{\mathcal{X}} A \not\in \Gamma \). Therefore, \( \Gamma \) is closed under rule (\( \text{Gen} \)), \( \neg s ::_{\mathcal{X}} B \not\in \Gamma \). Since \( \Gamma \) is closed under rule (\( \text{Gen} \)), \( \neg s ::_{\mathcal{X}} B \not\in \Gamma \). Hence, by the definition of \( \mathcal{E} \), \( B \in \mathcal{E}(s \cdot t) \).

(\( \mathcal{E}3 \)) Suppose that \( A \in \mathcal{E}(s) \cup \mathcal{E}(t) \). We have to show that \( A \in \mathcal{E}(s + t) \). If \( A \in \mathcal{E}(s) \), then \( \neg s ::_{\text{Par}(A)} A \not\in \Gamma \). Since \( \Gamma \) is closed under rule (\( \text{Gen} \)), \( \neg s ::_{\text{Par}(A)} A \not\in \Gamma \). Therefore, \( A \in \mathcal{E}(s + t) \). The case that \( A \in \mathcal{E}(t) \) is similar.

(\( \mathcal{E}4 \)) Suppose that \( A \in \mathcal{E}(t) \) and \( \mathcal{D}(A) = \text{Par}(A) \subseteq \mathcal{X} \). First consider the case that \( X \neq \text{Par}(A) \). We have to show that \( t ::_{\mathcal{X}} A \in \mathcal{E}(t) \). By the definition of \( \mathcal{E} \), \( \neg t ::_{\text{Par}(A)} A \not\in \Gamma \). Since \( \Gamma \) is closed under rule (\( \text{Exp} \)), \( \neg t ::_{\mathcal{X}} A \not\in \Gamma \). Since \( \Gamma \) is closed under rule (\( \text{Gen} \)), \( \neg t ::_{\mathcal{X}} A \not\in \Gamma \). Therefore, \( t ::_{\mathcal{X}} A \in \mathcal{E}(t) \). The case that \( X = \text{Par}(A) \) is similar.

(\( \mathcal{E}5 \)) Suppose that \( A \in \mathcal{E}(t) \). We have to show that \( \forall x A \in \mathcal{E}(\text{gen}_x(t)) \). By the definition of \( \mathcal{E} \), \( \neg t ::_{\text{Par}(A)} A \not\in \Gamma \). Since \( \Gamma \) is closed under rule (\( \text{Gen} \)), \( \neg \text{gen}_x(t) ::_{\text{Par}(A)} A \not\in \Gamma \). Therefore, \( \forall x A \in \mathcal{E}(\text{gen}_x(t)) \).

(\( \mathcal{E}6 \)) Suppose that \( A(x) \in \mathcal{E}(t) \) and \( u \in \mathcal{D} = \text{Par} \). We have to show that \( A(u) \in \mathcal{E}(t) \). Let \( X = \text{Par}(A(x)) \). By the definition of \( \mathcal{E} \), \( \neg t ::_{\mathcal{X}} A(x) \not\in \Gamma \). We distinguish two cases. (1) Suppose \( u \not\in X \). Since \( \Gamma \) is closed under rule (\( \text{Exp} \)), \( \neg t ::_{X \setminus u} A(x) \not\in \Gamma \). Since \( \Gamma \) is closed under rule (\( \text{Ins} \)), \( \neg t ::_{X \setminus u} A(u) \not\in \Gamma \). Therefore, \( A(u) \in \mathcal{E}(t) \). (2) Suppose \( u \in X \). Since \( \Gamma \) is closed under rule (\( \text{Ins} \)), \( \neg t ::_{X \setminus u} A(u) \not\in \Gamma \). Therefore, \( A(u) \in \mathcal{E}(t) \).

\( \square \)

Lemma 3.5 (Truth Lemma). Suppose \( \Gamma \) is an E-complete maximally tableau FOLP\(_{CS}\)-consistent set of closed Par-formulas and \( \mathcal{M} = (\mathcal{D}, \mathcal{I}, \mathcal{E}) \) is the canonical model with respect to \( \Gamma \) and \( \mathcal{CS} \). Then for every closed Par-formula \( F \):

1. \( F \in \Gamma \) implies \( \mathcal{M} \models F \).
2. \( \neg F \in \Gamma \) implies \( M \not\models F \).

Proof. By induction on the complexity of \( F \). The base case and the propositional and quantified inductive cases are standard. The proof for the case that \( F = t : X A \) is as follows. Note that \( Par(A) \subseteq X \).

Assume \( t : X A \in \Gamma \). Since \( \Gamma \) is \( \text{FOLP}_{\text{CS}} \)-consistent, \( \neg t : X A \not\in \Gamma \). If \( X = Par(A) \), then \( A \in E(t) \). If \( X \neq Par(A) \), then since \( \Gamma \) is closed under rule \((\text{Ctr})\), \( \neg t : Par(A) A \not\in \Gamma \). Thus \( A \in E(t) \). On the other hand, since \( t : X A \in \Gamma \) and \( \Gamma \) is closed under rule \((\text{T}_r)\), \( \forall A \in \Gamma \). Let \( FV\text{ar}(A) = \{ \overline{x} \} \). Thus \( \forall \overline{x} A(\overline{x}) \in \Gamma \). Since \( \Gamma \) is closed under rule \((\text{T}_\forall)\), \( A(\overline{u}) \in \Gamma \) for any \( \overline{u} \in Par \). Hence, by the induction hypothesis, \( M \models A(\overline{u}) \) for any \( \overline{u} \in Par = D \), and thus \( M \models \forall A \). Therefore, \( M \models t : X A \).

Assume \( \neg t : X A \in \Gamma \). If \( X = Par(A) \), then \( A \not\in E(t) \). On the other hand, if \( X \neq Par(A) \), then since \( \Gamma \) is closed under rule \((\text{Exp})\), \( \neg t : Par(A) A \in \Gamma \), and hence \( A \not\in E(t) \). In either cases \( M \not\models t : X A \). \( \Box \)

Theorem 3.2 (Completeness). Let \( A \) be a sentence of \( \text{FOLP} \). If \( A \) is \( \text{FOLP}_{\text{CS}} \)-valid, then it has a \( \text{FOLP}_{\text{CS}} \)-tableau proof.

Proof. If \( A \) does not have a \( \text{FOLP}_{\text{CS}} \)-tableau proof, then \( \{ \neg A \} \) is a \( \text{FOLP}_{\text{CS}} \)-consistent set and can be extended to a tableau \( \text{FOLP}_{\text{CS}} \)-consistent, maximal and \( E \)-complete set \( \Gamma \) of closed \( Par \)-formulas. Since \( \neg A \in \Gamma \), by the Truth Lemma, \( M \not\models A \), where \( M \) is the canonical model of \( \text{FOLP}_{\text{CS}} \) with respect to \( \Gamma \) and \( \text{CS} \). Therefore \( A \) is not \( \text{FOLP}_{\text{CS}} \)-valid. \( \Box \)

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