Capacity of Noisy Permutation Channels

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Abstract

We establish the capacity of a class of communication channels introduced in [1]. The \( n \)-letter input from a finite alphabet is passed through a discrete memoryless channel \( P_{Z|X} \) and then the output \( n \)-letter sequence is uniformly permuted. We show that the maximal communication rate (normalized by \( \log n \)) equals \( \frac{1}{2}(\text{rank}(P_{Z|X}) - 1) \) whenever \( P_{Z|X} \) is strictly positive. This is done by establishing a converse bound matching the achievability of [1]. The two main ingredients of our proof are (1) a sharp bound on the entropy of a uniformly sampled vector from a type class and observed through a DMC; and (2) the covering \( \varepsilon \)-net of a probability simplex with Kullback-Leibler divergence as a metric. In addition to strictly positive DMC we also find the noisy permutation capacity for \( q \)-ary erasure channels, the Z-channel and others.

I. Problem Statement and Main Results

The noisy permutation channel, as formally introduced in [1], is a communication model in which an \( n \)-letter input undergoes a concatenation of a discrete memoryless channel (DMC) and a uniform permutation of the \( n \) letters. Since the receiver observes a uniformly permuted output, the order of symbols conveys no information. See Section I-B for a motivation of this model. More formally, the channel \( P_{Y^n|X^n} \) can be described by the following Markov chain:

\[
X^n \rightarrow Z^n \rightarrow Y^n.
\]

Here the channel input \( X^n \) is a length \( n \) sequence where each position takes a value in \( \mathcal{X} = [q] \) (where \( [q] = \{1, 2, \ldots, q\} \)). The sequence \( X^n \) goes through the DMC which operates independently and identically on each symbol. This results in a sequence \( Z^n \) where each position takes a value in \( \mathcal{Y} = [k] \). The DMC transition probabilities can be represented as a \( q \times k \) matrix \( P_{Z|X} \). After the DMC, the sequence \( Z^n \) goes through the permutation part of the channel and results in sequence \( Y^n \) which is a uniformly random permutation of symbols on \( Z^n \).

Let \( f_n \) and \( g_n \) be the channel encoder and decoder respectively. For each message \( W \in [M] \), the input to the channel is \( X^n = f_n(W) \). The output is \( Y^n \), which the decoder decodes as \( \hat{W} = g_n(Y^n) \). The probability of error is given by \( P_{\text{error}}^n \triangleq \mathbb{P}[W \neq \hat{W}] \). The rate \(^1\) for the encoder-decoder pair \((f_n, g_n)\) is defined as

\[
R \triangleq \frac{\log M}{\log n}.
\]

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\(^1\) Notice that rate \( R \) for the noisy permutation channel is not the typical definition of information theory rate where \( R = \frac{\log M}{n} \). The noisy permutation channel would have rate 0 under this typical definition.
A rate $R$ is achievable if there is a sequence of encoder-decoder pairs $(f_n, g_n)$ with rate $R$ such that $\lim_{n \to \infty} P_{\text{error}}^{(n)} = 0$. The capacity for the noisy permutation channel with DMC $P_{Z|X}$ is $C_{\text{perm}}(P_{Z|X}) \triangleq \sup\{ R \geq 0 : R \text{ is achievable} \}$.

In [1], the author determined that the noisy permutation channel capacity is, for DMC $P_{Z|X}$, bounded by

$$C_{\text{perm}}(P_{Z|X}) \geq \frac{\text{rank}(P_{Z|X}) - 1}{2}. \tag{3}$$

For strictly positive matrices $P_{Z|X}$ (meaning all the transition probabilities are greater than 0), the author shows a converse bound

$$C_{\text{perm}}(P_{Z|X}) \leq \frac{|Y| - 1}{2}. \tag{4}$$

The author also gives a second converse bound: $C_{\text{perm}}(P_{Z|X}) \leq (\mathbf{ext}(P_{Z|X}) - 1)/2$, where $\mathbf{ext}(P)$ is the number of extreme points of the convex hull of the rows of $P$. For the case of strictly positive DMC $P_{Z|X}$, these upper and lower bounds do not necessarily match if the rank of matrix $P_{Z|X}$ does not equal to $|Y|$ or $\mathbf{ext}(P_{Z|X})$.

\section*{A. Main Results}

Our main result is establishing tightness of the lower bound \textsuperscript{3}, resolving Conjecture 1 of [1].

\begin{theorem} \textbf{(Strictly Positive DMC).} For strictly positive $P_{Z|X}$,

$$C_{\text{perm}}(P_{Z|X}) = \frac{\text{rank}(P_{Z|X}) - 1}{2}. \tag{5}$$

\end{theorem}

Our proof uses the idea of covering the space of distributions via an $\varepsilon$-net under the Kullback-Leibler (KL) divergence distance, following upon our investigations of a similar question in [2]. In order to reduce to the covering question, we first need another result that is, perhaps, of separate interest as well.

We let $\mathcal{P}_n$ be the set of $n$-types (probabilities which can be written with denominator $n$). For $P \in \mathcal{P}_n$, let $T_n(P)$ be the set of sequences of length $n$ in the type class of $P$. For instance\textsuperscript{2} when working with 3 symbols, the sequence 1233 is in $T_4(P)$ where $P = (1/4, 1/4, 1/4, 1/2)$ and $P$ is a 4-type. The notation $Q_Y^n$ means a distribution on random variable $Y$. For any distribution $Q_Y^n$, we will use $Q_Y^n$ to mean the product distribution $Q_Y^n(y^n) = \prod_{t=1}^n Q_Y(y_t)$. For any distribution $U$ on length $n$ sequences, the distribution $P_{Y|X}^o U$ can be understood as the distribution on random sequences derived by first randomly selecting a sequence according to $U$, then passing each symbol in this sequence through the transition probabilities $P_{Y|X}$ independently. (See Section I-C for more discussion.)

\begin{theorem} Fix channel $P_{Y|X}$ which is strictly positive. Then there exists a constant $c = c(P_{Y|X})$ such that the following holds: For any $n$-type $P$, let $U$ be uniform on $T_n(P)$. For all $Q_Y$ we have

$$nD(P_Y\|Q_Y) \leq D(P_{Y|X}^o U \| Q_Y^n) \leq nD(P_Y\|Q_Y) + c \tag{6}$$

where $P_Y$ is the marginal distribution of $Y$ under $(P \times P_{Y|X})$.

\end{theorem}\textsuperscript{2}\textsuperscript{3}

While it might seem that the noisy permutation channel capacity should be a continuous function of the values in $P_{Z|X}$, note that this is not the case due to how capacity is defined. Changing values in $P_{Z|X}$ by a small $\delta$ could change the rank of $P_{Z|X}$ by 1, but no matter how small $\delta$ is, there exists an $n$ large enough so its effects can make a difference.

\textsuperscript{3}See Section 11.1 of [3] for background on types.
\[ \hat{X}^n \sim U \sim \text{unif}(T_n(P)) \quad \text{such that} \quad P_{Y|X} \quad \rightarrow \quad \hat{Y}^n \]
\[ X^n \overset{iid}{\sim} P \quad \rightarrow \quad Y^n \]

Fig. 1. This diagram illustrates the special case of Theorem 2 where \( Q_Y = P_Y \). It shows how \( \hat{Y}^n \) relates to \( Y^n \).

**Remark 1.** It can be shown that the constant \( c \) in Theorem 2 is
\[ c \leq \frac{q-1}{2} \log \frac{2\pi \alpha^2}{c_*} + \frac{q}{12n} \leq \frac{q-1}{2} \log \frac{2\pi \alpha^2}{c_*} + \frac{q}{12} \]  
(7)
where \( \alpha \) is a universal constant defined in Theorem 3 (see Section III-C) and if \( p_{bj} \) denote the values in matrix \( P_{Y|X} \).

\[ c_* = \min_b \min_j p_{bj} \max_j p_{bj} . \]

(8)

Theorem 2 deals with the following scenario: Select some \( P \in \mathcal{P}_n \) and suppose we have two sequences, \( X^n \) and \( \hat{X}^n \). The sequence \( X^n \) is generated iid using the probability \( P \). On the other hand, \( \hat{X}^n \) has uniform probability over all sequences in the type \( T_n(P) \). Both sequences \( X^n \) and \( \hat{X}^n \) undergo the transition \( P_{Y|X} \) applied independently on each symbol and respectively results in \( Y^n \) and \( \hat{Y}^n \). How different are the distributions of \( Y^n \) and \( \hat{Y}^n \) under KL divergence? See Figure 1 for a diagram. Another interpretation of this scenario is if there are \( n \) balls of \( q \) colors in an urn. The sequence \( X^n \) are \( n \) draws from the urn with replacement and \( \hat{X}^n \) are \( n \) draws without replacement (all the balls are drawn). These observations then both go through the same noisy process to produce \( Y^n \) and \( \hat{Y}^n \).

It turns out that if \( P_{Y|X} \) is strictly positive, then regardless of the sequence length \( n \),
\[ D(P_{Y^n} \| P_{Y^n}) \leq c \]
(9)
where \( c \) is a constant that only depends on \( P_{Y|X} \). Theorem 2 actually shows something more general. The sequence \( X^n \) can be generated iid with another distribution \( Q \), and the KL divergence can still be bounded by constant \( c \) plus another term which is the KL divergence of the marginals on \( Y \) generated by \( P \) and \( Q \). In other words, the divergence of (a complicated distribution) \( P_{Y^n} \) to any iid distribution \( Q_{Y^n} \) can be approximated with \( nD(P_Y \| Q_Y) \) and this approximation will only be off by an additive constant.

**Remark 2.** Note that \( D(P_{X^n} \| P_{X^n}^m) \) describes the difference between sampling \( m \) balls from an \( n \)-urn with and without replacement. This is a classical question studied in [4]. Our setting studies this question for the particular case when \( n = m \) and when the observations are noisy. Bounds for the noiseless case \( D(P_{X^n} \| P_{X^n}^m) \) can still be an upper bound for the noisy case if we apply the data processing inequality. This shows that \( D(P_{Y^n|X} \circ U \| P_{Y^n}^m) \leq D(P_{X^n} \| P_{X^n}^m) \leq \frac{k-1}{2} (\log n + c) \), where the second inequality is shown using Stirling’s approximation. Our result removes the \( \log n \) term in this bound, but only under the assumption of a strictly positive \( P_{Y|X} \). See Appendix A for more details on comparing our bound to that of [4] when \( m < n \). We also note that results of [4] as shown in [5] imply the finitary case of de Finetti’s theorem.
Other contributions of this work use similar techniques to get converse results in other settings which do not have strictly positive DMC matrices.

**Theorem 3.** *Other channel results:*

1) Suppose $P_{Z|X}$ can be written as a block diagonal matrix with $\beta$ blocks where each block is strictly positive. Then,

$$C_{\text{perm}}(P_{Z|X}) = \frac{\text{rank}(P_{Z|X}) + \beta - 2}{2}.$$  \hspace{1cm} (10)

2) For DMC $P_{Z|X}$ which is a $q$-ary erasure channel for $q \geq 2$ (assuming non-zero transition probabilities), then

$$C_{\text{perm}}(P_{Z|X}) = \frac{q - 1}{2}.$$  \hspace{1cm} (11)

3) For DMC $P_{Z|X}$ which is a Z-channel (assuming non-zero probabilities on the edges), then

$$C_{\text{perm}}(P_{Z|X}) = \frac{1}{2}.$$  \hspace{1cm} (12)

The first result in Theorem 3 applies to DMC $P_{Z|X}$ which are block diagonal matrices where each block is strictly positive. As eq. (10) of Theorem 3 implies, we are able to show both the achievability and converse results for block diagonal DMC matrices. We will prove both these results in Appendix B.

The second result is for binary erasure channels and $q$-ary erasure channels. The work in [1] determines the capacity for when the DMC matrix is the binary symmetric channel, but leaves the binary and $q$-ary erasure channels as open problems. Item 2 of Theorem 3 resolves Conjecture 2 presented in [1] regarding the capacity of binary erasure channels and the conjecture regarding $q$-ary erasure channels. These results use (3) as the tight achievability.

The third result in Theorem 3 deals with DMC which is the Z-channel. While this is tight, for a $q$-ary Z-channel (if this exists) or what we call a “zigzag” channel, we are not able to find tight results with our current covering technique. The erasure channels, Z-channels, and a brief analysis on the zigzag channel are discussed in Appendix D.

All of these results also use the method of covering. A *covering* is a set of points in a space (we will call them centers) for which all other points in the space are within a certain distance $\varepsilon$ to (see Definition 1). Using covering as a technique to determine the capacity for the noisy permutation channel is reasonable because the centers which are far apart can intuitively be equated with messages that are distinguishable. When the messages correspond to two distributions $Q_1$ and $Q_2$ which are far in KL divergence, it is unlikely that noisy versions of $Q_1$ will be close to noisy versions of $Q_2$. If two distributions are close in KL divergence, their noisy versions are likely to be confused. If the messages in our communication are centers of a covering, then we know that if we add another center (or message), it will be close to one of the existing covering centers and thus cause error in determining which of the

\footnote{Note that while binary erasure channels and $q$-ary erasure channels usually have the same probabilities for each symbol getting erased, item 2 of Theorem 3 is still true even if these probabilities are different. The only requirement is that the probability of getting erased is not 0 or 1. The capacity when the transition probabilities are 0 or 1 is discussed in [1].}
centers (or messages) was sent. This gives us a limit on the total number of messages which can be sent, creating a converse bound\(^5\).

In order to use this intuition mathematically, we need to overcome the obstacle of computing the KL divergence over the noisy output distributions of the messages. This is difficult to do because these output distributions are not iid. This is where Theorem\(^2\) is useful, as it allows us to use KL divergences over iid distributions in place of the KL divergence over this output distribution (since we can replace a hypergeometric distribution which undergoes noise with a multinomial distribution). Other obstacles include determining the covering number under KL divergence (see Section II-B).

\(a\) Paper Organization: We continue this section with the motivation and the notation. In Section III we discuss how covering is used to determine the capacity of the noisy permutation channel along with some basics in covering. We begin the proof of Theorem\(^2\) in Section III. At the end of Section III we prove Theorem\(^1\). We prove all the parts of Theorem\(^3\) in the appendix.

B. Motivation

The motivation for studying the permutation channel is that it captures a setting where codewords get reordered. This occurs in applications such as communication networks and biological storage systems. We briefly describe some of these applications. More details on these applications and other relevant work can be found in [1].

\(a\) Communication Networks: Suppose we have a point-to-point communication network where the information is transmitted through a multipath routed network. Different packets are transmitted through different routes in the network, and each route has its own amount of latency, causing packets traveling on different routes to arrive at different times. The order in which the sender transmits packets is no longer preserved at the receiver end. Such a scenario is studied in [6] where the authors are primarily concerned with reducing delay in their channel. Unlike our work, they do not consider noisy symbols. Another line of work on packet-switched networks deals with the permutation channel along with errors such as insertions, deletions, and substitutions of symbols [7], [8]. Their work primarily focuses on building minimum distance codes and perfect codes for the permutation channel.

\(b\) DNA Storage Systems: DNA-based storage systems are an attractive option for data storage due to its ability to withstand time and encode a very high-density of information [9], [10]. The state-of-the-art technology for storing information on DNA uses nucleotides with relatively small lengths (few hundreds) [11]. Each of these DNA molecules are stored in a pool without any regard to order. The different molecule types can be treated as symbols in the setting of the permutation channel. Noise in this channel models any error that can occur, whether it is in synthesizing the DNA molecules or in reading the molecules. DNA storage is also the motivation for studying the permutation channel in [12], [13].

As typical in information theory, a question of fundamental interest is to determine the capacity of channels. We determine the capacity of the noisy permutation channel in the strictly positive case, settling the problem introduced

\(^5\)A similar notion to covering is packing, which is a set of centers in a space where all the centers in the set are at least distance \(2\varepsilon\) from another. Intuitively, covering corresponds to a converse bound while packing corresponds to an achievability bound.
This setting differs from some of the models studied in the works described in the motivations, as it looks at the problem from a purely information theoretic standpoint and does not include assumptions which might be specific to the application.

Among the works relevant to the motivations described, those that have some information theoretical flavors include [12], which deals with asymptotic bounds on rate, but for a fixed number of errors rather than probabilistic errors. The work in [11] finds the capacity when the symbols are sampled randomly then read, something relevant to DNA models, but not to general permutation channels. The results in [13] are specifically for when the permuted objects are a string of symbols and the noisy process is applied to symbols on a string; the set of strings are permuted but symbols in each string are not.

Our results for when the DMC is the erasure channel are particularly interesting to DNA storage applications since the erased symbol can model deletion errors. Permutation channels with deletions are central to the work in [12]. In our work, we find the probabilistic analogue of their bounds.

C. Notation

The set of all probability distributions on $q$ symbols is defined as the probability simplex

$$
\Delta_{q-1} \triangleq \left\{ (\pi_1, ..., \pi_q) : \sum_{i=1}^{q} \pi_i = 1, 0 \leq \pi_i \leq 1 \right\}.
$$

(13)

For a $q \times k$ DMC matrix $P_{Z|X}$, we can express the individual transitions as

$$
P_{Z|X} = \begin{bmatrix}
p_{11} & p_{12} & \cdots & p_{1d} \\
p_{21} & p_{22} & \cdots & p_{2d} \\
\vdots & \vdots & \ddots & \vdots \\
p_{q1} & p_{q2} & \cdots & p_{qd}
\end{bmatrix}.
$$

(14)

The values in each row of the matrix sums up to 1 (i.e, the matrix is stochastic). Symbol $b \in \mathcal{X}$ has probability $p_{bj}$ of becoming symbol $j \in \mathcal{Y}$. We can also write this probability as $P_{Z|X}(j|b)$. We say that the DMC matrix (or a submatrix) is strictly positive if $p_{bj} > 0$ for all $b$ and $j$ in the matrix (or submatrix).

For example, we can write the DMC matrix for the binary symmetric channel (BSC) with crossover probability $\delta$ as

$$
P_{Z|X} = \begin{bmatrix}
1 - \delta & \delta \\
\delta & 1 - \delta
\end{bmatrix}.
$$

(15)

If $0 < \delta < 1$, then this DMC matrix is strictly positive.

Though different from how we described it in the introduction, it is convenient to describe the Markov chain of the noisy permutation channel as

$$
\pi \rightarrow X^n \rightarrow Z^n \rightarrow Y^n.
$$

(16)

Each $\pi = (\pi_1,..,\pi_q) \in \Delta_{q-1}$ corresponds to a possible channel input. For each $n$, we will restrict $\pi$ to be in $\mathcal{P}_n$. The value of $\pi_b$ represents the proportion of positions in sequence $X^n$ which have symbol $b$. We can also understand $\pi_b$ to be the proportion of the $n$ symbols which undergo the transition given in row $b$ of $P_{Z|X}$. 

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The order of the symbols in $Y^n$ does not matter for decoding the messages. The only relevant statistic the receiver would use from $Y^n$ is which type class $Y^n$ belongs to. Note that it is entirely equivalent to perform the permutation on the sequence $X^n$ first and then apply the DMC. In this case, we no longer need the random variable $Z^n$. Because of this, we will also use $P_{Y|X}$ to specify the transition matrix, where $P_{Y|X}$ and $P_{Z|X}$ are the same and interchangeable.

We will specify a way to parameterize the distributions on $Y$. We use the notation $Q_{Y|\mu}$ for $\mu = (\mu_1, \ldots, \mu_k) \in \Delta_{k-1}$ to mean a distribution on symbols $Y$ where the probability of symbol $j \in Y$ is

$$Q_{Y|\mu}(j) = \mu_j.$$  

The distribution $Q_{Y|\mu}$ is the multinomial distribution with parameters $\mu$ and number of independent trials $n$. These distributions do not (directly) relate the permutation channel; we define them since they are important for our analysis.

On the other hand, the distribution $P_{Y^n|\pi}$ refers the the distribution on sequences $Y^n$ when $\pi \in \mathcal{P}_n$ is the input to the noisy permutation channel on $n$ letters as described in (16). Note that in general $P_{Y^n|\pi}$ is not a multinomial distribution. As seen in Theorem 2

$$P_{Y^n|\pi} = P_{Y|X} \circ U$$

where $U$ is a uniform distribution on $T_n(\pi)$. Both represent the distribution on the output of the noisy permutation channel. Permuting the input symbols gives a sequence in the support of $U$, and then each permuted symbol goes through the transition probabilities $P_{Y|X}$ independently.

When it is clear what $\pi$ is, we use $P_Y$ to mean the marginal distribution for each $Y_t$ in the sequence $Y^n \sim P_{Y^n|\pi} \circ U$. This distribution does not depend on the index $t$ since $U$ is uniform on all permutations.

II. COVERING CONVERSE

Our core method for finding our new results is to use divergence covering of the probability simplex. We first show how covering can be applied to the noisy permutation channel and then give the necessary covering results.

A. Covering Basics

The main concept of our proof uses covering ideas similar to [14, Theorem 1] in order to upper bound the mutual information $I(\pi; Y^n)$. In summary, we need to find a set of covering centers which are close in Kullback-Leibler (KL) divergence to all the possible distributions on $Y^n$ that can occur as outputs of the noisy permutation channel. Our set of centers need not be a possible distribution over $Y^n$ generated by the channel. We will opt for using multinomial distributions as our set of covering centers.

Let $\mathcal{N}_n$ be a discrete set in $\Delta_{k-1}$ which we will specify (later) for each $n$ (these will be the covering centers). Mutual information has the property that

$$I(\pi; Y^n) \leq \max_{\pi} D(P_{Y^n|\pi}\|\tilde{Q}_{Y^n}).$$

(19)
The above holds for any $\tilde{Q}_{Y^n}$, thus we can choose
\[
\tilde{Q}_{Y^n}(y^n) = \frac{1}{|\mathcal{N}_n|} \sum_{\mu \in \mathcal{N}_n} Q^n_{Y|\mu}(y^n) = \frac{1}{|\mathcal{N}_n|} \sum_{\mu \in \mathcal{N}_n} \prod_{t=1}^n Q_{Y|\mu}(y_t).
\] (20)

The following proposition is the main work-horse of all our converse results.

**Proposition 1** (Covering for Noisy Permutation Channels). Suppose that for the noisy permutation channel with DMC $P_{Y|X}$, we have that for any $\pi \in \mathcal{P}_n$,
\[
D(P^n_{Y|X} \circ U \| Q^n_Y) \leq nD(P_Y\|Q_Y) + f(n)
\] (21)
where $U$ is uniform on the type $T_n(\pi)$, $P_Y$ is the marginal distribution of $P^n_{Y|X} \circ U$ and $f$ is only a function of $n$ and $P_{Y|X}$. Then
\[
C_{\text{perm}}(P_{Y|X}) \leq \frac{\text{rank}(P_{Y|X}) - 1}{2} + \lim_{n \to \infty} \frac{f(n)}{\log n}.
\] (22)

In Proposition 1, when the DMC is strictly positive, the $f(n)$ term is constant in $n$ (which is shown via Theorem 2 and gives the proof for Theorem 1). However, when the DMC is not strictly positive, $f(n)$ is not necessarily constant in $n$. Non-constant values of $f(n)$ are used in deriving some of the results in Theorem 3.

For the proof, we need to define for any $\pi \in \Delta_{q-1}$
\[
\mu^M(\pi) \triangleq \left( \sum_i \pi_i p_{i1}, \ldots, \sum_i \pi_i p_{ik} \right).
\] (23)

The vector $\mu^M(\pi)$ is the mean of the distribution $P^n_{Y|\pi} = P^n_{Y|X} \circ U$. Note that $P_Y(j) = \sum_i \pi_i p_{ij}$. Also if $\mu = \mu^M(\pi)$, we can write $P_Y = Q_{Y|\mu}$.

**Proof.** Following techniques used in the proof of Theorem 1 from [14], we can upper bound the mutual information given in (19) by
\[
I(\pi; Y^n) \leq \log |\mathcal{N}_n| + \max_{\pi \in \mathcal{P}_n} \min_{\mu \in \mathcal{N}_n} D(P^n_{Y|\pi}\|Q_Y^n|\mu).
\] (24)

To specify $\mathcal{N}_n$, first define
\[
\mathcal{L}(P_{Y|X}) = \bigcup_{\pi \in \Delta_{q-1}} \mu^M(\pi).
\] (25)

This is the space of all possible marginals $P_Y$.

Let $\mathcal{N}_n$ be a covering of $\mathcal{L}(P_{Y|X})$ under KL divergence with covering radius $1/n$. In other words, $\mathcal{N}_n = \{\bar{\mu}^{(1)}, \ldots, \bar{\mu}^{(m)}\}$ so that
\[
\max_{\mu \in \mathcal{L}(P_{Y|X})} \min_{\bar{\mu} \in \mathcal{N}_n} D(Q_Y|\mu\|Q_Y|\bar{\mu}) \leq \frac{1}{n}.
\] (26)

Let $\ell$ be the dimension of $\mathcal{L}(P_{Z|X})$. Using divergence covering results and the result specifically about covering an $\ell$-dimensional subspace (see next part Section II-B),
\[
|\mathcal{N}_n| \leq C(q, \ell) \left( \frac{\ell}{1/n} \right)^{\frac{q}{2}}
\] (27)
where $C(q, \ell)$ depends on $q$ and $\ell$ but not $n$.
Using assumption (21) and putting this into (24), gives

\[ I(\pi; Y^n) \leq \log |N_n| + \max_{\pi \in \mathcal{P}_n} \min_{\mu \in \mathcal{N}_n} D(P_{Y^n|\pi} \| Q_{Y|\bar{\mu}}^n) \]

\[ \leq \log \left( C(q, \ell) \left( \frac{\ell}{1/n} \right)^{\frac{1}{2}} \right) + f(n) + \max_{\pi \in \mathcal{P}_n} \min_{\mu \in \mathcal{N}_n} nD(P_{Y} \| Q_{Y|\bar{\mu}}) \]

\[ \leq \frac{\ell}{2} \log n + \log C(q, \ell) + \frac{\ell}{2} \log \ell + f(n) + n \frac{1}{n} \]

\[ \leq \frac{\ell}{2} \log n + c' + f(n) \]

where \( c' \) is a constant which does not depend on \( n \).

For the noisy permutation channel, recall that the rate is defined as (2). Since asymptotically \( \log M \leq I(\pi, Y^n) \leq \frac{\ell}{2} \log n + c' + f(n) \), we have

\[ R \leq \frac{\ell}{2} \log n + c' + f(n) \]

\[ \quad \rightarrow \frac{\ell}{2} + \lim_{n \to \infty} f(n) \log n \] \hspace{1cm} \text{(32)}

Since \( \ell = \text{rank}(P_{Z|X}) - 1 \), we have an upper bound for the capacity of the noisy permutation channel.

(To see that \( \ell = \text{rank}(P_{Z|X}) - 1 \), let \( r = \text{rank}(P_{Z|X}) \). When the domain is any vector, the image space of \( P_{Z|X} \) is \( r \) dimensional. But since we are restricting the domain and image to probability vectors, this adds an additional constraint to the image space and reduces the dimension by 1.)

\[ \Box \]

B. Covering Definition and Results

In order to show (27), we have the following definitions and results.

**Definition 1** (Divergence Covering Number).

\[ M(k, \varepsilon) = \inf \{ m : \exists \{Q_1, ..., Q_m\} \text{ s.t.} \max_{P \in \Delta_{k-1}} \min_{Q_i} D(P || Q_i) \leq \varepsilon \} \] \hspace{1cm} \text{(33)}

Let \( M(k, \varepsilon, B) \) be defined like \( M(k, \varepsilon) \) except that \( P \in B \) for a subspace \( B \subset \Delta_{k-1} \).

Here, \( \varepsilon \) is the covering radius.

**Theorem 4** (Upper Bound on Divergence Covering). For \( 0 < \varepsilon \leq 1 \),

\[ M(k, \varepsilon) \leq c^{k-1} \left( \frac{k - 1}{\varepsilon} \right)^{\frac{k-1}{2}} \] \hspace{1cm} \text{(34)}

for some constant \( c \).

The above result is sufficient for showing our theorems. However, stronger bounds do exist (see [15]).

**Proposition 2**. For \( B \subset \Delta_{k-1} \), suppose there is a stochastic matrix \( F \) which maps \( \Delta_{q-1} \) onto \( B \). Suppose that \( B \) is a space of dimension \( \ell - 1 \) (or likewise, \( F \) has rank \( \ell \)). Then,

\[ M(k, \varepsilon, B) \leq \left( \frac{q}{\ell} \right) M(\ell, \varepsilon) \] \hspace{1cm} \text{(35)}

The proofs are in Appendix A.
III. DIVERGENCE UNDER FIXED TYPES

For computing our converse bounds, we need to determine the expression (21) for our DMC matrices. This is where we need Theorem 2 which gives the divergence between transition on a fixed type compared to an iid distribution.

We will prove Theorem 2 by showing some relevant intermediate results. The techniques in these intermediate results are also useful for when $P_{Z|X}$ is not strictly positive and therefore relevant for getting results in Theorem 3. Before doing so, we will briefly discuss the tightness of Theorem 2.

A. Tightness of Theorem 2

Here we show a proof sketch that the constant $c$ in Theorem 2 is sharp (cannot be improved to $o(1)$). One tool we will need is the following theorem by Marton [16] shown below.

**Lemma 1** (Marton’s Transportation Inequality). Let $X^n \sim \prod_{t=1}^{n} P_{X_t}$ and $\hat{X}^n \sim P_{\hat{X}^n}$. Then there exists a joint probability measure $P_{X^n, \hat{X}^n}$ with these given marginals such that

$$\frac{1}{n} \mathbb{E}[d(X^n, \hat{X}^n)] = \frac{1}{n} \sum_{t=1}^{n} \mathbb{P}[X_t \neq \hat{X}_t] \leq \left( \frac{1}{n} D(P_{\hat{X}^n} \| \prod_{t=1}^{n} P_{X_t}) \right)^{1/2}$$

where $d(X,Y)$ is the Hamming distance.

Suppose that we are only working with 2 symbols, $\{0,1\}$, for the space of $X$ and $Y$. Using the notation in Theorem 2 let $Y^n \sim P^n_{Y|X} \circ U$ and $\hat{Y}^n \sim Q^n_{Y}$. Choose $P = (n/2, n/2)$ and $P_{Y|X}$ to be that of a binary symmetric channel with crossover probability $\epsilon$. Choose $Q_Y = P_Y$. Distribution $Q_Y$ will be uniform on the two symbols.

We can choose $\epsilon$ to be non-zero but also small enough (for instance $\epsilon = 1/n$) so that all sequences likely to occur under $P^n_{Y|X} \circ U$ have a type differing by only a constant number of changes from the type $(n/2, n/2)$. On the other hand, under $Q^n_Y$, with constant probability, we expect sequences to have types that differ by $\sqrt{n}$ changes from the type $(n/2, n/2)$. Hence no matter the coupling chosen, the expected Hamming distance is

$$\mathbb{E}[d(Y^n, \hat{Y}^n)] \approx \sqrt{n}.$$ (38)

We use Lemma 1 to get the lower bound,

$$\frac{1}{\sqrt{n}} \approx \frac{1}{n} \mathbb{E}[d(Y^n, \hat{Y}^n)] \leq \left( \frac{1}{n} D(P^n_{Y|X} \circ U \| Q^n_Y) \right)^{1/2}$$

$$\leq \left( \frac{1}{n} (nD(P_Y \| Q_Y) + c) \right)^{1/2}$$

$$= \sqrt{\frac{c}{\sqrt{n}}}.$$ (41)

Improving $c$ to be $o(1)$ in $n$ would violate this lower bound.
Intuitively, think of what happens when we let $P_{Y|X}$ be the identity matrix where $Y = X$. In such a case, Theorem 2 is not true (in order to get a true statement, the constant $c$ should be replaced with a value that grows logarithmically with $n$, see Section III-B1). This is the same setting as the example given above but with $\varepsilon = 0$. It is clear here that $Y^n$ will have $\sqrt{n}$ deviations in the number of 0's from the mean whereas any sequence $\tilde{Y}^n$ will have exactly the type $(n/2, n/2)$. This creates an expected Hamming distance of $\sqrt{n}$. Slightly increasing $\varepsilon$ above zero will not change the Hamming distance by much but make $P_{Z|X}$ strictly positive.

B. Expression for Divergence Under Fixed Types

We will use the following to show Theorem 2. This proposition can be used for any $P_{Y|X}$, not just those which are strictly positive.

Proposition 3. Let $U$ be uniform on the type $T_n(P)$ and $(X, Y)^n$ be iid from $(P \times P_{Y|X})$. Let $P_Y$ be the marginal distribution of $Y$ under $(P \times P_{Y|X})$.

Then for all $Q_Y$,

$$D(P^n_{Y|X} \circ U \| Q^n_Y) = nD(P_Y || Q_Y) + \sum_{y^n \in Y^n} \mathbb{P}[Y^n = y^n | A = 1] \log \frac{\mathbb{P}[A = 1 | Y^n = y^n]}{\mathbb{P}[A = 1]}$$

(43)

where $A = \mathbb{I}\{X^n \in T_n(P)\}$ (and we use $\mathbb{P}$ to mean under the probability where $(X, Y)^n$ is iid from $(P \times P_{Y|X})$).

The second term on the right-hand side of (43) can be written as an expected value:

$$\sum_{y^n \in Y^n} \mathbb{P}[Y^n = y^n | A = 1] \log \frac{\mathbb{P}[A = 1 | Y^n = y^n]}{\mathbb{P}[A = 1]}$$

$$= \mathbb{E}_{(X, Y)^n \sim (P \times P_{Y|X})} \left[ \log \frac{\mathbb{P}(X, Y)^n \sim (P \times P_{Y|X})[\tilde{X}^n \in T(P)]\tilde{Y}^n = Y^n)}{\mathbb{P}(\tilde{X}, \tilde{Y})^n \sim (P \times P_{Y|X})[X^n \in T(P)]} \right] (X, Y)^n \text{ where } X^n \in T(P)$$

(44)

For ease of notation, we choose to express the term above as

$$\mathbb{E} \left[ \log \frac{\mathbb{P}[A = 1 | \tilde{Y}^n = Y^n]}{\mathbb{P}[A = 1]} \right] | A = 1$$

(45)

where the $\tilde{}$ notation emphasizes that the variables are associated with an independent copy $(\tilde{X}, \tilde{Y})^n$ drawn from the same distribution $(P \times P_{Y|X})$ as $(X, Y)^n$ and where $\tilde{A} = \mathbb{I}\{\tilde{X}^n \in T_n(P)\}$.

Proof. Note that $(P_{Y|X}^n \circ U)(y^n) = \mathbb{P}[Y^n = y^n | A = 1]$.

$$D(P^n_{Y|X} \circ U \| Q^n_Y) = \sum_{y^n} \mathbb{P}[Y^n = y^n | A = 1] \log \frac{\mathbb{P}[Y^n = y^n | A = 1]}{Q^n_Y(y^n)}$$

(46)

$$= \sum_{y^n} \mathbb{P}[Y^n = y^n | A = 1] \log \frac{\mathbb{P}[A = 1 | Y^n = y^n] \mathbb{P}[Y^n = y^n]}{\mathbb{P}[A = 1] Q^n_Y(y^n)}$$

(47)

$$= \mathbb{E} \left[ \log \frac{\mathbb{P}[Y^n]}{Q^n_Y(Y^n)} \right] | A = 1 + \sum_{y^n} \mathbb{P}[Y^n = y^n | A = 1] \log \frac{\mathbb{P}[A = 1 | Y^n = y^n]}{\mathbb{P}[A = 1]}$$

(48)

$$= \mathbb{E} \left[ \log \frac{\mathbb{P}[Y^n]}{Q^n_Y(Y^n)} \right] | A = 1 + \mathbb{E} \left[ \log \frac{\mathbb{P}[\tilde{A} = 1 | \tilde{Y}^n = Y^n]}{\mathbb{P}[\tilde{A} = 1]} \right] | A = 1$$

(49)
The marginal distribution $P_Y(a)$ is also the probability that any position $t$ in sequence $Y^n$ takes the value $a$, i.e. $P_Y(a) = \Pr[Y_t = a | A = 1]$. This occurs since $U$ is uniform on all permutations of type $T_n(P)$. We get for the first term in the sum,

$$
\mathbb{E} \left[ \log \frac{P[Y^n]}{Q_Y(Y^n)} \middle| A = 1 \right] = \sum_{y^n} \Pr[Y^n = y^n | A = 1] \log \frac{P^n_Y(y^n)}{Q^n_Y(y^n)}
$$

$$
= \sum_{y^n} \Pr[Y^n = y^n | A = 1] \sum_{t} \frac{\{t : y_t = a\}}{n} \log \frac{P_Y(a)}{Q_Y(a)}
$$

$$
= n \sum_{a} P_Y(a) \log \frac{P_Y(a)}{Q_Y(a)}
$$

$$
= n D(P_Y || Q_Y).
$$

This give the result (53).

**Lemma 2.** Let $P = (p_1, \ldots, p_q) \in \mathcal{P}_n$ and let $A = \{X^n \in T_n(P)\}$. Then if $(X, Y)^n$ is drawn iid from $(P \times P_{Y|X})$, $\log \frac{1}{\Pr[A = 1]} \leq -\frac{1}{2} \log n + \frac{1}{2} \log p_i n + \frac{q - 1}{2} \log 2 \pi + \frac{1}{12n}.$

For this proof, we will use a Stirling approximation type bound from [17]: For positive integers $n$,

$$
\sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{\pi/n} \leq n! \leq 2\sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{\pi/n}.
$$

**Proof.** We will assume that all $p_i > 0$ since we can always reduce $P$ to a shorter vector and decrease $q$.

The probability a specific type occurs is given by the multinomial distribution.

$$
- \log \Pr[A = 1] = - \log \left( \frac{n!}{\prod_{i=1}^{q} (p_i n)!} \prod_{i=1}^{q} p_i^{n_i} \right)
$$

$$
= - \log \left( \frac{n!}{n^n} \right) - \log \left( \frac{q!}{\prod_{i=1}^{q} (p_i n)!} \right)
$$

$$
\leq n - \frac{1}{2} \log n - \frac{1}{2} \log 2 \pi + \sum_{i=1}^{q} \left( -p_i n + \frac{1}{2} \log p_i n + \frac{1}{2} \log 2 \pi + \frac{q}{12n} \right).
$$

We used (55) last inequality (we can do this since each $p_i n$ is an integer greater than 0). Combining terms gives the result.

**1) Theorem 2 Without Strictly Positive:** We briefly mentioned above that if we remove the strictly positive requirement for $P_{Y|X}$ in Theorem 2 in the worst case, the constant $c$ would need to be replaced with a logarithmic term. To be exact, if $P_{Y|X}$ is not strictly positive, $c$ needs to be replaced with a poly-logarithmic term in $n$. Using Proposition 3 and Lemma 2 we can get an upperbound on $c$ with

$$
c = \mathbb{E} \left[ \log \frac{\Pr[\tilde{A} = 1| Y^n = Y^n]}{\Pr[\tilde{A} = 1]} \middle| A = 1 \right]
$$

$$
\leq \frac{q - 1}{2} \log n + c' + \mathbb{E} \left[ \log \Pr[\tilde{A} = 1| \tilde{Y}^n = Y^n] \middle| A = 1 \right]
$$

$$
\leq \frac{q - 1}{2} \log n + c'.
$$
We used the fact that the largest value $P[A = 1|Y^n]$ can take is 1 since it is a probability. The inequality is tight when $P_{Y|X}$ is the identity matrix (when $Y = X$). We also discuss this in later sections where we look at DMC matrices that are not strictly positive.

C. Concentration of Sums of Independent Variables

The concentration function $Q(Z; \lambda)$ of random variable $Z$ is defined by

$$Q(Z; \lambda) = \sup_z P[z \leq Z \leq z + \lambda]$$

for every $\lambda \geq 0$ \cite{18}. Let $S_n = \sum_{i=1}^{n} W_i$ where the $W_i$ are independent.

**Theorem 5** (Petrov \cite{18}). Let the numbers $a_i$ and $b_i$ be such that

$$\mathbb{P} \left[ W_i - a_i \leq -\frac{\lambda_i}{2} \right] \geq b_i$$

$$\mathbb{P} \left[ W_i - a_i \geq \frac{\lambda_i}{2} \right] \geq b_i$$

for $i = 1, ..., n$. Then there exists a universal constant $\alpha$ so that

$$Q(S_n; \lambda) \leq \alpha \left( \sum_{i=1}^{n} \lambda_i^2 b_i \right)$$

for every positive $\lambda_1, ..., \lambda_n$ none of which exceeds $\lambda$.

To apply Theorem 5 to our problem, each $W_i$ is a Bernoulli random variable where the probability $W_i = 1$ is $p_i$. Let $b_i = \min\{p_i, 1 - p_i\}$. We can fix $a_i = 1/2$. We can also fix $\lambda_i = 1/2$ and $\lambda_i = 1/2$, though this exact value does not matter so long as $\lambda < 1 - \varepsilon$ for a small $\varepsilon > 0$.

This gives that for any integer $z$

$$\mathbb{P}[S_n = z] \leq Q(S_n; 1/2)$$

$$\leq \frac{\alpha (1/2)}{\sqrt{\sum_{i=1}^{n} (1/2)^2 \min\{p_i, 1 - p_i\}}}$$

$$\leq \frac{\alpha}{\sqrt{\sum_{i=1}^{n} \min\{p_i, 1 - p_i\}}}.$$  

We will use this in the next lemma which is key to computing the second term in \eqref{43}.

**Lemma 3.** There are $n$ balls thrown in $q$ bins independently, so that for the $i$-th ball, the probability of landing in bin $b$ is $p_{i,b}$.

Let $N_b$ be the ball count of the $b$-th bin. Then if $\pi_b > 0$ for all $b$ and $\sum_b \pi_b = 1$, we have

$$\mathbb{P}[N_1 = n\pi_1, \ldots, N_q = n\pi_q] \leq \frac{\alpha^{q-1}}{n^{(q-1)/2} \sqrt{B}}$$  

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where

\[ B = c_\pi^{q-1} \prod_{b} \frac{\pi_b}{\pi_{\text{max}}} \]  

(70)

\[ c_a = \min_i \frac{c_-(i)}{c_+(i)} \]  

(71)

\[ c_-(i) = \min_b \frac{P_{i,b}}{\pi_b} \]  

(72)

\[ c_+(i) = \max_b \frac{P_{i,b}}{\pi_b} \]  

(73)

\[ \pi_{\text{max}} = \max_b \pi_b \]  

(74)

and \( \alpha \) is the universal constant used in Theorem 5.

Proof. For notation, let \( W_{i,b} \) be the indicator variable of whether ball \( i \) was thrown into bin \( b \). We can express \( N_b = \sum_{i=1}^n W_{i,b} \). Arrange the indices so that \( \pi_1 \leq \pi_2 \cdots \leq \pi_q \).

First observe that

\[
P[N_1 = n\pi_1, \ldots, N_q = n\pi_q] = \prod_{b=1}^q P[N_b = n\pi_b|N_1 = n\pi_1, \ldots, N_{b-1} = n\pi_{b-1}].
\]  

(75)

For \( b = q \),

\[
P[N_b = n\pi_b|N_1 = n\pi_1, \ldots, N_{b-1} = n\pi_{b-1}] = 1.
\]  

(76)

For \( b < q \), we can compute for any \( i \) that

\[
\min \left\{ \frac{P_{i,b}}{\sum_{a=b}^q P_{i,a}}, 1 - \frac{P_{i,b}}{\sum_{a=b}^q P_{i,a}} \right\} = \min \left\{ \frac{P_{i,b}}{\sum_{a=b}^q P_{i,a}}, \frac{\sum_{a=b}^q \pi_{a} P_{i,a}}{\sum_{a=b}^q \pi_{a}} \right\}
\]  

(77)

\[
= \min \left\{ \frac{P_{i,b}}{\sum_{a=b}^q \pi_{a}}, \frac{\sum_{a=b}^q \pi_{a} P_{i,a}}{\sum_{a=b}^q \pi_{a}} \right\}
\]  

(78)

\[
\geq \min \frac{P_{i,b}}{\pi_a} \min \left\{ \frac{\pi_b}{\sum_{a=b}^q \pi_{a}}, \frac{\sum_{a=b}^q \pi_{a}}{\pi_a} \right\}
\]  

(79)

\[
\geq \min \frac{c_-(i)}{c_+(i)} \frac{1}{\sum_{a=b}^q \pi_{a}} \min \left\{ \pi_b, \sum_{a=b}^q \pi_{a} \right\}
\]  

(80)

\[
= c_a \pi_b \sum_{a=b}^q \pi_{a}.
\]  

(81)

We get the last equality because we have arranged \( \pi_b \) in increasing order. Hence by (68)

\[
P[N_b = n\pi_b|N_1 = n\pi_1, \ldots, N_{b-1} = n\pi_{b-1}] \leq \frac{\alpha}{\sqrt{n - \sum_{a=1}^{b-1} n\pi_a} c_a \pi_b \sum_{a=b}^q \pi_{a}}
\]  

(82)

\[
= \frac{\alpha}{\sqrt{n - \sum_{a=1}^{b-1} n\pi_a} n c_a \pi_b \sum_{a=b}^q \pi_{a}}
\]  

(83)

\[
= \frac{\alpha}{n^{1/2} c_a \pi_b}
\]  

(84)
where we used that $n - \sum_{a=1}^{b-1} n \pi_a = n \sum_{a=b}^{q} \pi_a$ to get the last inequality. Taking a product of all terms in (75), gives

$$\mathbb{P}[N_1 = n\pi_1, \ldots, N_q = n\pi_q] \leq \prod_{b=1}^{q-1} \frac{\alpha}{\sqrt{c_s \pi_b}}$$

(85)

$$= \frac{\alpha^{q-1}}{n^{(q-1)/2} \sqrt{c_s^{q-1} \prod_{b=1}^{q} \pi_b}}$$

(86)

$$= \frac{\alpha^{q-1}}{n^{(q-1)/2} \sqrt{c_s^{q-1} \pi_a}}$$

(87)

$$= \frac{\alpha^{q-1}}{n^{(q-1)/2} \sqrt{B}} .$$

(88)

\[ \square \]

D. Completing Proof of Theorem 2 and Determining Capacity.

Proof of Theorem 2 First we show the easier lower bound. Using Proposition 3 we need only to show that

$$\mathbb{E} \left[ \log \frac{\mathbb{P}[A = 1 | \hat{Y}^n = Y^n]}{\mathbb{P}[A = 1]} \right] \geq 0 .$$

(89)

We do this by,

$$\mathbb{E} \left[ \log \frac{\mathbb{P}[A = 1 | \hat{Y}^n = Y^n]}{\mathbb{P}[A = 1]} \right] = \sum_{y^n} \mathbb{P}[Y^n = y^n | A = 1] \log \frac{\mathbb{P}[A = 1 | Y^n = y^n]}{\mathbb{P}[A = 1]}$$

(90)

$$= \sum_{y^n} \mathbb{P}[Y^n = y^n | A = 1] \log \frac{\mathbb{P}[Y^n = y^n | A = 1] \mathbb{P}[A = 1]}{\mathbb{P}[Y^n = y^n | A = 1]}$$

(91)

$$= \sum_{y^n} \mathbb{P}[Y^n = y^n | A = 1] \log \frac{\mathbb{P}[Y^n = y^n | A = 1]}{\mathbb{P}[Y^n = y^n]}$$

(92)

$$= D(\mathbb{P}[Y^n | A = 1] || \mathbb{P}[Y^n])$$

(93)

$$\geq 0$$

(94)

since divergences are always positive.

To get the upper bound, we will express

$$\mathbb{E} \left[ \log \frac{\mathbb{P}[A = 1 | \hat{Y}^n = Y^n]}{\mathbb{P}[A = 1]} | A = 1 \right] = \mathbb{E} \left[ \log \frac{\mathbb{P}[\hat{A} = 1 | \hat{Y}^n = Y^n]}{\mathbb{P}[\hat{A} = 1]} \right] - \log \mathbb{P}[A = 1]$$

(95)

and use Lemma 3 for the first term in the sum and Lemma 2 for the second term in sum.

To see why Lemma 3 applies to the first term, note that the first term is trying to calculate given some $Y^n$, what the probability that the type of $X^n$ is equal to $T_n(P)$. This under a distribution where $(X, Y)^n \sim (P_{Y|X} \times P)$.

We will express the type $T_n(P)$ with $P = (\pi_1, \ldots, \pi_q)$ where $P \in \mathcal{P}_n$. This implies that $\pi_b = \mathbb{P}[X = b]$. Let the balls described in Lemma 2 be each of the elements of $Y^n$. If $Y_i = y_i$, then let $p_{i,b} = \mathbb{P}[X_i = b | Y_i = y_i] = \mathbb{P}[X = b | Y = y_i]$ (because the symbols are iid). This way $p_{i,b}$ is appropriately the probability that the $i$th symbol lands in bin $b$. As in Lemma 3 $N_b$ is the number of balls in bin $b$. Then the probability that $X^n \in T_n(P)$ is equivalent
to $\mathbb{P}[N_1 = n\pi_1, \ldots, N_q = n\pi_q]$. This is computed for a specific value of $Y^n$, but notice that the expression we derived for $\mathbb{P}[N_1 = n\pi_1, \ldots, N_q = n\pi_q]$ in Lemma 3 does not depend on $Y^n$.

Before computing the rest of the expression, we need to pay particular attention to the case when there exists a $b$ such that $\pi_b = 0$. If $\pi_b = 0$, then $\mathbb{P}[X = b] = 0$, which would also imply that $p_{i,b} = 0$ for all $i$. In this case, we can remove the symbol $b$ (or bin $b$ in the interpretation of Lemma 3) from consideration and apply Lemma 3 to just the symbols with non-zero probability. We can always reorder the symbols, so that the first $b > q$ are such that $\pi_b = 0$. Like in Lemma 3 we can define

$$B = c_*^{q'-1} \prod_{b=1}^{q'} \frac{\pi_b}{\pi_{max}}$$

(96)

$$c_* = \min_i \frac{c_-(i)}{c_+(i)}$$

(97)

$$c_-(i) = \min_{b : b \leq q'} \frac{p_{b,i}}{\pi_b}$$

(98)

$$c_+(i) = \max_{b : b \leq q'} \frac{p_{b,i}}{\pi_b}$$

(99)

$$E \left[ \log \mathbb{P}[\hat{A} = 1 | \hat{Y}^n = Y^n] \bigg| A = 1 \right] = \log \mathbb{P}[N_1 = n\pi_1, \ldots, N_q = n\pi_q]$$

(100)

$$= \log \frac{\alpha^{q'-1}}{n^{(q'-1)/2} \sqrt{B}}$$

(101)

$$= \log \left( \frac{\alpha^{q'-1}}{n^{(q'-1)/2} c_*} \left( \frac{\pi_{max}}{\prod_b \pi_b} \right)^{1/2} \right)$$

(102)

$$= \log \left( \frac{\alpha^{q'-1}}{c_*} \left( \frac{n\pi_{max}}{\prod_b \pi_b} \right)^{1/2} \right)$$

(103)

$$= \frac{1}{2} \log n \pi_{max} - \sum_{b : \pi_b > 0} \frac{1}{2} \log n\pi_b + (q'-1) \log \left( \frac{\alpha}{\sqrt{c_*}} \right)$$

(104)

$$\leq \frac{1}{2} \log n - \sum_{b : \pi_b > 0} \frac{1}{2} \log n\pi_b + c' .$$

(105)

where $c'$ is constant that does not depend on $n$. Importantly, the value of $c'$ also does not depend on $\pi_b$ for any $b$.

The quantity $c_*$ which we can compute with:

$$\frac{p_{b,i}}{\pi_b} = \frac{\mathbb{P}[X = b | Y = y_i]}{\mathbb{P}[Y = y_i]} \mathbb{P}[X = b] = \frac{\mathbb{P}[Y = y_i | X = b \mathbb{P}[X = b]}{\mathbb{P}[Y = y_i]} = \frac{\mathbb{P}[Y = y_i | X = b]}{\mathbb{P}[Y = y_i]} = \frac{P_{Y|X}(y_i | b)}{\mathbb{P}[Y = y_i]}$$

(106)

and

$$c_* = \min_i \frac{\min_b \frac{p_{i,b}}{\pi_b}}{\max_b \frac{p_{i,b}}{\pi_b}}$$

(107)

$$= \min_i \frac{\min_b P_{Y|X}(y_i | b)}{\max_b P_{Y|X}(y_i | b)}$$

(108)

$$= \min_y \frac{\min_b P_{Y|X}(y | b)}{\max_b P_{Y|X}(y | b)}$$

(109)

So $c_*$ only depends on $P_{Y|X}$. 

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Combining these terms with those from Lemma 2 gives that the expression in (95) is a constant when $P_{Y|X}$ is strictly positive. This constant depends on $q$ and $P_{Y|X}$ but not on $n$ or $\pi_b$ for any $b$.

Proof of Theorem 1 Using Theorem 2 with Proposition 1 completes the proof for strictly positive DMC.

APPENDIX

Here we give some details on how our result relates to that of Stam in [4]. First, Stam’s setting has noiseless observations whereas our setting has noise. However, using data processing inequality, we can always use Stam’s result as an upperbound for the noisy case. Second, Stam’s result generalizes to $m$ observations, where $m$ can be less than $n$. Our result Theorem 2 only applies to exactly $n$ observations. However, we can also use a version of Han’s inequality for divergence [19, Proposition 5.5] (applies when the second probability argument is independent over the entries of the vector $Y^n$), to get the following corollary:

Corollary 1. Let $Y^n \sim P_{Y^n}^n \circ U$, so that $P_{Y^n}$ is the distribution as in Theorem 2. Then for every $m \leq n$ we have:

$$D(P_{Y^m} \| Q_{Y^m}^m) \leq mD(P_Y \| Q_Y) + \frac{m}{n}c$$  \hspace{1cm} (110)

where $c$ is the same constant as in Theorem 2

Here, $Y^m$ are the first $m$ entries of vector $Y^n$. In Stam’s setting we have $Q_Y = P_Y$, so

$$D(P_{Y^m} \| P_Y^m) \leq \frac{m}{n}c.$$  \hspace{1cm} (111)

Using Stam’s result with data processing gives

$$D(P_{Y^m} \| P_Y^m) \leq D(P_{X^m} \| P_X^m) \leq \frac{(q-1)}{2} \frac{m(m-1)}{(n-1)(n-m+1)}.$$  \hspace{1cm} (112)

The above equation, which is presented as Stam’s final result, is actually not the tightest when $m$ is close to $n$. For instance, when $m = n$, (112) gives $\frac{q-1}{2}n$ which is far from $\frac{q-1}{2}(\log n + c')$, the actual divergence when computed directly. Work in [20] is able to make for an improvement on Stam’s bound when $m$ is close to $n$. We show an
easier improvement, using an intermediate result in the proof of (112). We can derive for larger \( m \) that

\[
D(P_{Y^m} || P_{X^m}) \leq D(P_{X^m} || P_{X})
\]

(113)

\[
\leq \frac{q - 1}{n - 1} \sum_{t=1}^{m-1} \frac{t}{n - t}
\]

(114)

\[
= \frac{q - 1}{n - 1} \sum_{j=n-m+1}^{n-1} \frac{n - j}{j}
\]

(115)

\[
= \frac{q - 1}{n - 1} \left( n \left( \sum_{j=n-m+1}^{n-1} \frac{1}{j} \right) - (m - 1) \right)
\]

(116)

\[
= \frac{q - 1}{n - 1} \left( n \log(n - 1) - \log(n - m) + c'' \right) - (m - 1)
\]

(117)

\[
= \frac{q - 1}{n - 1} \left( n \log \frac{n - 1}{n - m} + nc'' - (m - 1) \right)
\]

(118)

\[
= (q - 1) \log \frac{n - 1}{n - m} + O(q)
\]

(119)

for \( m < n \) and \( c'' \) is a constant leftover from approximating the harmonic sum by a log.

When \( m \ll n \), (112) is the better bound. The bound in (111) will be a tighter bound than both (112) and (119) for the case when \( m \) is very close to \( n \), such as when \( n - m = o(n) \). When \( m \) is linear in \( n \), then it becomes important to compare the constant factors. Let \( \gamma = m/n \). To get an estimate on when our bound is tighter, we will first assume \( n \) is large and ignore the lower order constants which appear in the bounds. Using Remark 1, the bound in (111) can be tighter than (112) and (119) for large \( n \) if

\[
\frac{1}{2} \log \frac{2\pi \alpha^2}{c_*} \leq \min \left\{ \frac{\gamma}{1 - \gamma}, \frac{1}{\gamma} \log \frac{1}{1 - \gamma} \right\}.
\]

(120)

This can occur for certain values of \( \gamma \) depending on the size of \( c_* \), which is a function of \( P_{Y|X} \).

In this section we give the covering results necessary for proving Proposition 1. Again, the bound below is sufficient but not the best possible covering bound. Other bounds are explored in [15].

A. Divergence Covering Upperbound

To define our covering centers for the simplex, we will start with a set of scalars. Let

\[
\Lambda(\varepsilon) \triangleq \left\{ \varepsilon i^2 : \text{ for } i \in \mathbb{Z}_{>0}, \varepsilon i^2 < \frac{1}{2} \right\} \cup \left\{ 1 - \varepsilon i^2 : \text{ for } i \in \mathbb{Z}_{>0}, \varepsilon i^2 < \frac{1}{2} \right\} \cup \left\{ \frac{1}{2} \right\}
\]

(121)

Define

\[
\Lambda_2(\varepsilon) = \{ (\lambda, 1 - \lambda) : \lambda \in \Lambda(\varepsilon) \}
\]

(122)

For each \( k \), let \( u_k \in \Delta_{k-1} \) be \( u_k = (0, \ldots, 0, 1) \). For each \( q \in \Delta_{k-2} \), let \( \hat{q} \) be the corresponding \( \hat{q} \in \Delta_{k-1} \) where \( \hat{q} = (q_1, \ldots, q_{k-1}, 0) \).

For each \( q \in \Delta_{k-2} \), define \( q^{(\lambda)} \) such that

\[
q^{(\lambda)} = \lambda u_k + (1 - \lambda)\hat{q}.
\]

(123)
For $k > 2$, recursively define
\[
\Lambda_k(\varepsilon) \triangleq \bigcup_{\lambda \in \Lambda(\varepsilon)} \left\{ q^{(\lambda)} : q \in \Lambda_{k-1}\left(\frac{k-1}{k} \varepsilon\right) \right\}
\] (124)

Lemma 4. For any $p \in \Delta_{k-1}$,
\[
\min_{q \in \Lambda_k(\varepsilon)} D(p||q) \leq \gamma \varepsilon
\] (125)

where $\gamma$ is a constant.

Proof. We will show this by using induction. First, for any $p \in \Delta_1$, we want to show that
\[
\min_{q \in \Lambda_2(\varepsilon)} D(p||q) \leq \gamma \varepsilon
\] (126)

We will use the following fact from [21]. For probabilities $P_1$ and $P_2$ on $k$ symbols, we have
\[
D(P_1||P_2) \leq \sum_{a=1}^{k} \frac{(P_1(a) - P_2(a))^2}{P_2(a)}.
\] (127)

This implies that for any $p, q \in \Delta_1$, where $p = (p_1, p_2)$ and $q = (q_1, q_2)$
\[
D(p||q) \leq \frac{(p_1 - q_1)^2}{q_1} + \frac{(p_2 - q_2)^2}{1 - q_1} = \frac{(p_1 - q_1)^2}{q_1(1 - q_1)}.
\] (128)

Suppose that $p \in \Delta_1$ and $p_1 < 1/2$. Then $\varepsilon(i - 1)^2 < p_1 \leq \varepsilon i^2$ for some positive integer $i$. Assume for now that $\varepsilon i^2 < 1/2$. Choose $q = (\varepsilon i^2, 1 - \varepsilon i^2) \in \Lambda_2(\varepsilon)$. Note that we must have $1 - \varepsilon i^2 > 1/2$.
\[
D(p||q) \leq \frac{(p_1 - q_1)^2}{q_1(1 - q_1)}
\] (129)
\[
= \frac{(p_1 - \varepsilon i^2)^2}{(\varepsilon i^2)(1 - \varepsilon i^2)}
\] (130)
\[
\leq \frac{(\varepsilon(i - 1)^2 - \varepsilon i^2)^2}{(\varepsilon i^2)(1 - \varepsilon i^2)}
\] (131)
\[
\leq \varepsilon \frac{(-2i + 1)^2}{i^2(1/2)}
\] (132)
\[
\leq \varepsilon \frac{4i^2 - 4i + 1}{i^2/2}
\] (133)
\[
\leq 4\varepsilon
\] (134)

If $\varepsilon i^2 > 1/2$, we can choose $q = (1/2, 1/2)$. For this case, we can also assume $i > 1$, otherwise one center point, $q = (1/2, 1/2)$ is sufficient for covering the whole simplex. Then

\footnote{This fact is that KL divergence is upper-bounded by $\chi^2$-divergence.}
\[ D(p\|q) \leq \frac{(p_1 - 1/2)^2}{(1/2)(1/2)} \]
\[ \leq \frac{(\varepsilon(i - 1)^2 - \varepsilon i^2)^2}{1/4} \]
\[ \leq 4\varepsilon^2(-2i + 1)^2 \]
\[ \leq \frac{4}{2} \frac{4i^2 - 4i + 1}{(i - 1)^2} \]
\[ \leq 18\varepsilon \]

where we used that \( \varepsilon < 1/(2(i - 1)^2) \). This shows that we can set \( \gamma = 18 \). By symmetry, \( \min_{q \in \Lambda_2(\varepsilon)} D(p\|q) \leq \gamma \varepsilon \) holds for \( p_1 > 1/2 \) as well.

Suppose in dimension \( k - 1 \), that we have for any \( p \in \Delta_{k-2} \),
\[ \min_{q \in \Lambda_{k-1}(\varepsilon)} D(p\|q) \leq \gamma \varepsilon \]
(140)

For each \( p = (p_1, \ldots, p_k) \in \Delta_{k-1} \), we will specify a scalar quantity \( \lambda_p \in [0, 1] \). If \( p_k < 1/2 \), like above, we can find a positive integer \( i \) where
\[ \frac{\varepsilon}{k}(i - 1)^2 \leq p_k \leq \frac{\varepsilon}{k} i^2 \]
(141)

and set
\[ \lambda_p = \min \left\{ \frac{\varepsilon}{k} i^2, \frac{1}{2} \right\} \in \Lambda \left( \frac{\varepsilon}{k} \right) . \]
(142)

If \( p_k > 1/2 \), find \( i \) such that
\[ 1 - \frac{\varepsilon}{k} i^2 < p_k \leq 1 - \frac{\varepsilon}{k} (i - 1)^2 \]
(143)

and set
\[ \lambda_p = \max \left\{ 1 - \frac{\varepsilon}{k} i^2, \frac{1}{2} \right\} \in \Lambda \left( \frac{\varepsilon}{k} \right) . \]
(144)

Define \( p'_{k} = (p_k, 1 - p_k) \) and \( \lambda'_{p} = (\lambda_p, 1 - \lambda_p) \), then similar to above
\[ D(p'_{k}\|\lambda'_{p}) \leq \gamma \frac{\varepsilon}{k} . \]
(145)
\[
\min_{q \in \Lambda_k(\varepsilon)} D(p\|q) \leq \min_{q \in \Lambda_k(\varepsilon): q_k = \lambda_p} p_k \log \frac{p_k}{q_k} + \sum_{i=1}^{k-1} p_i \log \frac{p_i}{q_i} \tag{146}
\]
\[
\leq p_k \log \frac{p_k}{\lambda_p} + \min_{q \in \Lambda_k(\varepsilon): q_k = \lambda_p} \sum_{i=1}^{k-1} p_i \log \frac{p_i}{q_i} \tag{147}
\]
\[
\leq p_k \log \frac{p_k}{\lambda_p} + \min_{q \in \Lambda_k(\varepsilon): q_k = \lambda_p} (1 - p_k) \log \frac{1 - p_k}{1 - \lambda_p} + (1 - p_k) \sum_{i=1}^{k-1} \frac{p_i}{1 - p_k} \log \frac{p_i}{(1 - p_k) q_i/(1 - \lambda_p)} \tag{148}
\]
\[
\leq D(p_k'\|\lambda_p') + (1 - p_k) \min_{q' \in \Lambda_{k-1}(\varepsilon/\gamma)} \sum_{i=1}^{k-1} \frac{p_i}{1 - p_k} \log \frac{p_i/(1 - p_k)}{q_i'} \tag{149}
\]
\[
\leq \gamma \varepsilon + (1 - p_k) \frac{k - 1}{k} \varepsilon \tag{150}
\]
\[
\leq \gamma \varepsilon. \tag{151}
\]

\[\square\]

**Proof of Theorem 4** We will use \(Q_k(\varepsilon)\) to denote the set of centers we need to cover \(\Delta_{k-1}\) with radius \(\varepsilon\).

Let
\[
Q_k(\varepsilon) = \Lambda_k \left( \frac{\varepsilon}{\gamma} \right) \tag{152}
\]
where \(\gamma\) is the constant in Lemma 4. Then using Lemma 4 for \(p \in \Delta_{k-1},\)
\[
\min_{q \in Q_k(\varepsilon)} D(p\|q) \leq \varepsilon. \tag{153}
\]
Since, \(M(k, \varepsilon) \leq |Q_k(\varepsilon)|\), it remains to count the size of each \(Q_k(\varepsilon)\). We will show its size by induction. First, we have that
\[
\left| \Lambda \left( \frac{\varepsilon}{\gamma} \right) \right| \leq 2 \sqrt{\frac{\gamma}{2\varepsilon}} + 1 = \sqrt{\frac{2\gamma}{\varepsilon}} + 1 \leq \sqrt{\frac{2\gamma + 1}{\varepsilon}} \tag{154}
\]
where the last inequality holds if \(\varepsilon \leq 1\). Therefore we have for some constant \(c\) (we can show \(c \leq 7\),
\[
|Q_2(\varepsilon)| \leq c \frac{1}{\sqrt{\varepsilon}}. \tag{155}
\]
For the inductive case, given alphabet size \(k\) and any \(\varepsilon \leq 1\), we have \(|\Lambda_k \left( \frac{\varepsilon}{\gamma} \right)| \leq c^{k-1} \left( \frac{k - 1}{\varepsilon} \right)^{\frac{k-1}{2}}\).

Now consider the case of alphabet size \(k + 1\). The set \(\Lambda_{k+1} \left( \frac{\varepsilon}{\gamma} \right)\) is defined as a set of points which is a product of sets \(\Lambda \left( \frac{\varepsilon}{k \gamma} \right)\) and \(\Lambda_{k+1} \left( \frac{k - 1}{\varepsilon} \frac{\varepsilon}{k} \right)\). This gives
\[
|Q_{k+1}(\varepsilon)| = \left| \Lambda_{k+1} \left( \frac{\varepsilon}{\gamma} \right) \right| = \left| \Lambda \left( \frac{1 \varepsilon}{k \gamma} \right) \right| \left| \Lambda_k \left( \frac{k - 1 \varepsilon}{k \gamma} \right) \right| \leq \left( c \frac{1}{\sqrt{k}} \right) \left( c^{k-1} \left( \frac{k - 1}{\varepsilon} \right)^{\frac{k-1}{2}} \right) \tag{156}
\]
\[
= c \frac{\sqrt{k}}{\sqrt{\varepsilon}} c^{k-1} \left( \frac{k - 1}{\varepsilon} \right)^{\frac{k-1}{2}} \tag{157}
\]
\[
= c^{k} \left( \frac{k}{\varepsilon} \right)^{\frac{k}{2}} \tag{158}
\]
as the number of centers. \[\square\]
B. Subspace Covering

To use our covering result for noisy permutation channels, we actually need to cover a lower dimensional subspace of a \((k-1)\)-dimensional simplex.

**Lemma 5.** For \(B \subset \Delta_{k-1}\), suppose there is a stochastic matrix \(F\) which maps \(\Delta_{\ell-1}\) onto \(B\). Then,

\[
M(k, \varepsilon, B) \leq M(\ell, \varepsilon) .
\]

**Proof.** Let \(N_c(\ell, \varepsilon)\) be the set of points which are centers for a divergence covering of \(\Delta_{\ell-1}\) with covering radius \(\varepsilon\). For each \(b \in B\), there exists a \(p \in \Delta_{\ell-1}\) such that \(pF = b\). For this \(p\), let \(r \in N_c(\ell, \varepsilon)\) be such that \(D(p||r) \leq \varepsilon\). Let \(b^* = rF\). By data processing inequality [22, Theorem 2.2],

\[
D(b||b^*) \leq D(p||r) \leq \varepsilon .
\]

Hence the image of the set of centers in \(N_c(\ell, \varepsilon)\) mapped using \(F\), becomes the set of centers for a divergence covering on \(B\) with radius \(\varepsilon\).

**Proof of Proposition 2**

The key to this proof is to divide the space \(B\) into simplices of dimension \(\ell - 1\).

We will upper bound the number of simplices needed for a partition of \(B\). The image of \(F\) is a convex hull of at most \(q\) points (recall \(q\) is the size of the input symbols). We will call these corner points. Consider all possible choices of \(\ell\) of these \(q\) corner points. Let this set of all combinations be \(S\), where

\[
|S| = \binom{q}{\ell} .
\]

For each \(s \in S\), let \(B_s\) be the simplex which is the convex hull of the \(\ell\) corner points in set \(s\).

For each point \(x\) in the image of \(F\), since \(F\) has rank \(\ell\), there exists some linear combination of \(\ell\) corner points which results in \(x\). If \(s\) is this set of \(\ell\) points, then \(x \in B_s\). Thus for all \(x \in B\), there exists some \(s \in S\), so that \(x \in B_s\).

Label each of these simplices as \(B_1, \ldots, B_{|S|}\). There exists a stochastic matrix \(F_i\) which maps from space \(\Delta_{\ell-1}\) onto the space \(B_i\). In particular, we can find this map \(F_i\) by mapping each of the \(\ell\) corners of \(\Delta_{\ell-1}\) into one of the \(\ell\) corner points of \(B_i\). This map will cover all of \(B_i\) by linearity.

Hence using Lemma 5, we can find a divergence covering of size \(M(\ell, \varepsilon)\) for each \(B_i\). Combining these covering centers together for all \(i\), we get a covering of size

\[
\binom{q}{\ell} M(\ell, \varepsilon) .
\]

We are most assuredly over counting the number of simplices \(B\) has to be divided up into. However, this number does not depend on \(\varepsilon\), which is sufficient for our application to noisy permutation channels.

With a small modification to the proof of Theorem 1, we can show a converse bound for block diagonal matrices where each block is strictly positive. The modification uses that each block is independent from all the other blocks,

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so we can apply the bound for strictly positive matrices separately to each block. We will need to show a separate
achievability result to match this converse bound.

As a sanity check, the block diagonal case captures the situation where \( P_{Z|X} \) is the identity matrix. In which
case, it is possible to use all possible permutations of symbols as messages. No errors are allowed so decoding is
straight-forward. Using an identity matrix of size \( q \times q \) for the DMC, for each \( n \),

\[
R = \frac{\log M}{\log n} = \frac{\log \left( q^{-1} \right)}{\log n} \approx \frac{\log(c^{q-1}n^{q-1}/(q-1)^{q-1})}{\log n} = q - 1 + \frac{c'}{\log n}
\]

which goes to \( q - 1 \) asymptotically. This matches our block diagonal converse result.

C. Converse

**Proposition 4** (Block Diagonal Converse). Suppose \( P_{Z|X} \) can be written as a block diagonal matrix with \( \beta \) blocks, so that each block is strictly positive. Then,

\[
C_{\text{perm}}(P_{Z|X}) \leq \text{rank}(P_{Z|X}) + \beta - 2.
\]

**Proof.** We will use Proposition 1 but we need to show a version of the upper bound in Theorem 2 which applies
to block diagonal matrices instead of strictly positive matrices.

Fix \( \pi \in P_n \). Arrange the matrix \( P_{Z|X} \) in block diagonal form and let \( X_b \) be the set of symbols in \( X \) which are
in the \( b \)th block. Let \( (X,Y)^n \) be generated iid from \( (\pi \times P_{Y|X}) \). Let \( W_i \) be the number of \( X \) which equals \( i \), i.e.

\[
W_i = |\{t : X_t = i\}|.
\]

Define

\[
A_b = \left\{ \bigcap_{i \in X_b} W_i = \pi_i n \right\}.
\]

This is the event that all symbols \( i \) associated with block \( b \) occur with the count \( \pi_i n \). Each block has its own
separate set of output symbols in \( Y \). The probability of \( W_i \) is independent of what happens in other blocks. Let \( Y^n(b) \) (and \( y^n(b) \)) be notation for the counts restricted to just the output symbols associated with the \( b \)th block.

Using the definition in Proposition 3 notice that \( \mathbb{I}[A = 1] = \mathbb{I} \left[ \bigcap_{b=1}^\beta A_b \right] \). (Recall the notation \( \tilde{A} = \{ \tilde{X}^n \in T(P) \} \)
where \( (\tilde{X}, \tilde{Y})^n \) is independent copy under the same distribution \((P \times P_{Y|X})\) as \((X,Y)^n\). Then using 43 with

\[
D(P^\circ_{Y|X} \circ U \| Q_Y) = nD(P_Y \| Q_Y) + \mathbb{E} \left[ \log \frac{\mathbb{P}[\tilde{A} = 1|\tilde{Y}^n = Y^n]}{\mathbb{P}[A = 1]} \right] \bigg| A = 1
\]

\[
\leq nD(P_Y \| Q_Y) - \frac{1}{2} \log n + \sum_{i: \pi_i > 0} \frac{1}{2} \log \pi_i n + c + \mathbb{E} \left[ \log \mathbb{P}[\tilde{A} = 1|\tilde{Y}^n = Y^n] \right] | A = 1
\]

For any \( Y^n \),

\[
\mathbb{P}[A = 1|Y^n] = \prod_{b=1}^\beta \mathbb{P}[A_b = 1|Y^n(b) = y^n(b)].
\]
Each block is a strictly positive matrix. From Lemma 3 and following the same calculations that results in (105), we know that

\[
\log \mathbb{P}[A_b = 1 | Y^n(b) = y^n(b)] \leq \frac{1}{2} \log n - \sum_{i \in X_b: \pi_i > 0} \frac{1}{2} \log n \pi_i + c'
\]

and thus

\[
\log \mathbb{P}[A = 1 | Y^n] \leq \sum_{b=1}^{\beta} \left( \frac{1}{2} \log n - \sum_{i \in X_b: \pi_i > 0} \frac{1}{2} \log n \pi_i + c' \right)
\]

\[
= \frac{\beta}{2} \log n - \sum_{i: \pi_i > 0} \frac{1}{2} \log n \pi_i + \beta c'.
\]

This holds for all \( Y^n \) so it automatically gives the expected value. Putting all these terms together, for any \( \pi \), we get

\[
D(P^n_{Y|X} \circ U || Q^n_Y) = nD(Y||Q_Y) + \frac{\beta - 1}{2} \log n + c''
\]

where \( c'' \) combines all the constants. Using Proposition 1 gives

\[
C_{\text{perm}}(P_{Z|X}) \leq \frac{\text{rank}(P_{Z|X}) - 1}{2} + \lim_{n \to \infty} \frac{\beta - 1}{2} \log n + c''
\]

\[
= \frac{\text{rank}(P_{Z|X}) - 1}{2} + \frac{\beta - 1}{2}
\]

\[
= \frac{\text{rank}(P_{Z|X}) + \beta - 2}{2}.
\]

D. Achievability

**Proposition 5** (Block Diagonal Achievability). Suppose \( P_{Z|X} \) can be written as a block diagonal matrix with \( \beta \) blocks, so that each block is strictly positive. Then,

\[
C_{\text{perm}}(P_{Z|X}) \geq \frac{\text{rank}(P_{Z|X}) + \beta - 2}{2}.
\]

**Proof.** The achievability proof encodes using two steps. The first step is a zero-error code based on which block in the block diagonal matrix the symbols are associated with. Let \( M_1 \) denote the total possible messages (or rather message stems) for the first step. The second step operates only on each block independently, and uses the achievability given by (3). Let \( M_2 \) denote the total messages (or message tails) possible here.

Label the \( \beta \) blocks in \( P_{Z|X} \) as \( B_1, ..., B_\beta \). Define the sets of input symbols \( X_1, ..., X_\beta \) and output symbols \( Y_1, ..., Y_\beta \), so that \( \mathcal{A}_b \) and \( \mathcal{Y}_b \) are the input and output symbols respectively associated with block \( B_b \). (In other words, if \( p_{ij} > 0 \) and \( p_{ij} \) falls into block \( B_b \), then \( i \in \mathcal{X}_b \) and \( j \in \mathcal{Y}_b \).) These sets \( \mathcal{X}_1, ..., \mathcal{X}_\beta \) and \( \mathcal{Y}_1, ..., \mathcal{Y}_\beta \) will both be disjoint.

Let \( L = \text{rank}(P_{Z|X}) \) and let \( L_b = \text{rank}(B_b) \). Because of the block diagonal structure, \( L = \sum_{b=1}^{\beta} L_b \).

For fixed \( n \), set aside the first \( n/2 \) input symbol positions so that exactly \( n/(2\beta) \) are from set \( \mathcal{X}_b \) for each \( b \). These will not be used for the first step of the two-step code and is used to make the analysis of the second step.
easier. The remaining $n/2$ positions can be encoded using symbols from any set and this is used to make the first step of the code. There are
\[
\binom{n/2}{\beta - 1} \geq \left( \frac{n/2}{\beta - 1} \right)^{\beta - 1}
\] (179)
possible combinations of symbols chosen from $\beta$ blocks, disregarding order. The DMC will map the symbols in set $\mathcal{X}_b$ to symbols in set $\mathcal{Y}_b$ without any error. Hence, (179) is the number of messages $M_1$ the first step can encode without any error.

Once it is determined how many symbols of each set will be used, we can determine which symbol in the set will be used for the second step. Suppose there are $n_b$ positions which are designated for symbols in set $\mathcal{X}_b$. This includes the $n/(2\beta)$ we set aside in the beginning and however many were chosen to make the first step of the code. Using (3), we know there exists an encoder-decoder pair $(f_{n_b}, g_{n_b})$ so that the decoding error is vanishingly small as $n_b \to \infty$. Just by choosing which symbol in $\mathcal{X}_b$ to send, for some $\varepsilon_{n_b} > 0$ where $\varepsilon_{n_b} \to 0$, we can encode a set of messages with size $M_b$ satisfying
\[
\log M_b \geq \left( \frac{L_b - 1}{2} - \varepsilon_{n_b} \right) \log n_b.
\] (180)

The set of messages possible for all the $\beta$ different sets is
\[
\log M_2 = \log \prod_{b=1}^{\beta} M_b \geq \sum_{b=1}^{\beta} \left( \frac{L_b - 1}{2} - \varepsilon_{n_b} \right) \log n_b \geq \sum_{b=1}^{\beta} \left( \frac{L_b - 1}{2} - \varepsilon_{n_b} \right) \log \frac{n}{2\beta} \geq \left( \frac{L - \beta}{2} - \sum_{b=1}^{\beta} \varepsilon_{n_b} \right) \log \frac{n}{2\beta}. \] (181)
(182)
(183)
(184)
The total number of messages is the product of those available at the first and second steps.
\[
\log M = \log M_1 + \log M_2 \geq \log \left( \frac{n/2}{\beta - 1} \right)^{\beta - 1} + \left( \frac{L - \beta}{2} - \sum_{b=1}^{\beta} \varepsilon_{n_b} \right) \log \frac{n}{2\beta} \geq (\beta - 1) \log \frac{n}{2\beta} + \left( \frac{L - \beta}{2} - \sum_{b=1}^{\beta} \varepsilon_{n_b} \right) \log \frac{n}{2\beta} \geq \left( \frac{2\beta - 2}{2} + \frac{L - \beta}{2} - \sum_{b=1}^{\beta} \varepsilon_{n_b} \right) \log \frac{n}{2\beta} \geq \left( \frac{L + \beta - 2}{2} - \sum_{b=1}^{\beta} \varepsilon_{n_b} \right) \log \frac{n}{2\beta}. \] (185)
(186)
(187)
(188)
(189)
Since each $n_b \geq \frac{n}{2\beta} \to \infty$ as $n \to \infty$, asymptotically the term $\sum_{b=1}^{\beta} \varepsilon_{n_b}$ disappears.
The achievable rate is given by

\[ R = \frac{\log M}{\log n} \geq \left( \frac{L - \beta - 2}{2} + \sum_{b=1}^{\beta} \epsilon_{n_b} \right) \frac{\log n - \log 2 \beta}{\log n} \rightarrow \frac{L - \beta - 2}{2}. \]  

(190)

Combining Proposition 4 and Proposition 5 gives the first result in Theorem 3.

E. Preliminaries

The following lemma will be useful for computing the probability of event \( A \) (see Proposition 3) when the DMC matrix is not strictly positive. It is a straightforward concentration bound which is a direct application of Bernstein’s inequality. We choose to write the proof anyways for completeness.

**Lemma 6.** Suppose that \( Z \) is a sum of \( n \) independent Bernoulli random variables. Let \( \mathbb{E}[Z] \) be the expected value of \( Z \).

Fix \( \gamma \). If \( \mathbb{E}[Z] > 2\gamma \log n \), then with probability at least \( 1 - 2/n^{\gamma/4} \), we have that

\[ \mathbb{E}[Z] > \mathbb{E}[Z] - \sqrt{\mathbb{E}[Z] \gamma \log n} \geq \frac{1}{5} \mathbb{E}[Z]. \]  

(191)

**Proof.** Let \( Z = \sum_{i=1}^{n} W_i \) where \( W_i \) is the \( i \)th Bernoulli random variable. Let \( 0 < p_i < 1 \) be the probability of \( W_i = 1 \).

\[ \sum_{i=1}^{n} \mathbb{E}[(W_i - \mathbb{E}[W_i])^2] = \sum_{i=1}^{n} p_i(1 - p_i) \leq \sum_{i=1}^{n} p_i = \mathbb{E}[Z]. \]  

(192)

We will use Bernstein’s inequality for bounded variables \([23, \text{Theorem 2.8.4}]\).

\[ \mathbb{P} \left[ Z - \mathbb{E}[Z] \leq -\sqrt{\mathbb{E}[Z] \gamma \log n} \right] \leq 2 \exp \left( \frac{-1}{2} \frac{\mathbb{E}[Z] \gamma \log n}{\sum_{i=1}^{n} \mathbb{E}[(W_i - \mathbb{E}[W_i])^2] + \frac{1}{3} \sqrt{\mathbb{E}[Z] \gamma \log n}} \right)  
\leq 2 \exp \left( \frac{-1}{2} \frac{\mathbb{E}[Z] \gamma \log n}{\mathbb{E}[Z] + \frac{1}{3} \sqrt{\mathbb{E}[Z] \gamma \log n}} \right) 
\leq 2 \exp \left( \frac{-1}{2} \frac{\gamma \log n}{1 + \frac{1}{3} \sqrt{\gamma \log n}} \right). \]  

(193)

Using that \( \mathbb{E}[Z] > 2\gamma \log n \), we have \( 1 + \frac{1}{3} \sqrt{\gamma \log n} \leq 1 + \frac{1}{3} \sqrt{2\gamma \log n} \leq 2. \)

\[ \mathbb{P} \left[ Z - \mathbb{E}[Z] \leq -\sqrt{\mathbb{E}[Z] \gamma \log n} \right] \leq 2 \exp \left( \frac{-1}{4} \gamma \log n \right) \leq \frac{2}{n^{\gamma/4}}. \]  

(194)

Hence, with probability \( 1 - 2/n^{\gamma/4} \),

\[ \mathbb{E}[Z] \geq \mathbb{E}[Z] - \sqrt{\mathbb{E}[Z] \gamma \log n} \geq \mathbb{E}[Z] - \sqrt{\mathbb{E}[Z] \frac{1}{2} \mathbb{E}[Z]} \geq \left( 1 - \frac{1}{\sqrt{2}} \right) \mathbb{E}[Z] \geq \frac{1}{5} \mathbb{E}[Z]. \]  

(195)
F. The q-ary Erasure Channel

We now can prove the converse bound for q-ary erasure channels, where q is the number of input symbols. Let k = q + 1 represent the erased symbol.

The matrix $P_{Z|X}$ for a q-ary erasure channel has the following structure:

$$P_{Z|X} = \begin{bmatrix} p_{11} & 0 & \cdots & 0 & p_{1k} \\ 0 & p_{22} & \cdots & 0 & p_{2k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & p_{qq} & p_{qk} \end{bmatrix}.$$  \hspace{1cm} (198)

We will assume that each $p_{ik} > 0$.

**Proof of Item 2 of Theorem 3.** Fix $\pi = (\pi_1, \ldots, \pi_q)$ where $\pi \in P_n$. (We will assume each $\pi_i > 0$, otherwise we can remove it.) Reorder the symbols in $\{1, \ldots, q\}$ so that $\pi_1 \leq \pi_2 \leq \ldots \leq \pi_q$. (We will use $P_{Y|X} = P_{Z|X}$.)

Following Proposition 3 let $(X,Y)^n$ be generated iid according to $(\pi \times P_{Y|X})$. To use Proposition 3 we need to determine $P[\log P[A=1|Y^n = Y^n]|A=1]$.

Unlike the case of strictly positive $P_{Y|X}$, the value of $P[A=1|Y^n]$ depends on $Y^n$. For instance, it is easy to see that when the erasure symbol k does not appear, then $P[A=1|Y^n] = 1$. While $Y^n$ like this can occur under the event $A=1$, we want to show that these events are rare, this way the expected value of $P[A=1|Y^n]$ given that $A=1$ will be much smaller than 1 and close to the value which will give our result. We will first show a concentration result on $Y^n$ given that $A=1$.

Let $U_b$ be the random variable which gives the count of the number of times the symbol $b$ is erased, i.e

$$U_b = \sum_{i=1}^n \{(X_i,Y_i) = (b,k)\}.$$  \hspace{1cm} (199)

Let $v_b(y^n) = \{U_b|A = 1, Y^n = y^n\}$. Note that $v_b(y^n)$ is deterministic. If $A=1$ and $Y^n$ is known, we can determine exactly what $U_b$ is.

Define $S_b = \sum_{a \geq b} U_a$. Given $Y^n$, $S_1$ is deterministic. Given $Y^n$ and $U_1, \ldots, U_{b-1}$, $S_b$ is deterministic. Using Lemma 6 since $E[S_b] = n \sum_{a \geq b} \pi_a p_{ak} \geq n \pi_q p_{qk} > 2\gamma \log n$ for some $\gamma$ (chosen later) and all $b$ for large enough $n$, we have

$$P \left[ S_b > \frac{1}{5} n \sum_{a \geq b} \pi_a p_{ak} \right] \geq 1 - 2/n^{\gamma/4}. \hspace{1cm} (200)$$

Using the union bound,

$$P \left[ \bigcap_{b=1}^{q} \left\{ S_b > \frac{1}{5} n \sum_{a \geq b} \pi_a p_{ak} \right\} \right] \geq 1 - 2q/n^{\gamma/4}. \hspace{1cm} (201)$$

Next, for any $y^n$ which has positive probability given $A=1$,

$$P[A=1|Y^n = y^n] = P \left[ \bigcap_{b=1}^{q} U_b = v_b(y^n) \bigg| Y^n = y^n \right] = \prod_{b=1}^{q-1} P \left[ U_b = v_b(y^n) \bigg| \bigcap_{a=1}^{b-1} U_a = v_a(y^n), Y^n = y^n \right]. \hspace{1cm} (202)$$
We compute the following which is like the proof Lemma 3 of but with appropriate adjustments. Using (68),

\[
P \left[ U_b = v_b(y^n) \right] \leq \frac{\alpha}{\sqrt{\sum_{b=1}^{S_b} \min \{ \sum_{a \geq b} \pi_a p_{ab}, \sum_{a \geq b} \pi_a p_{ab} \}}}.
\]

Like in Lemma 4 define \( c_- = \min_i p_{ik} \).

\[
\min \left\{ \frac{\pi_b p_{ab}}{\sum_{a \geq b} \pi_a p_{ab}}, \frac{\sum_{a > b} \pi_a p_{ab}}{\sum_{a \geq b} \pi_a p_{ab}} \right\} = (\min_i p_{ik}) \min \left\{ \frac{\pi_b}{\sum_{a \geq b} \pi_a p_{ab}}, \frac{\sum_{a > b} \pi_a}{\sum_{a \geq b} \pi_a p_{ab}} \right\}
\]

\[
= \frac{c_- \pi_b}{\sum_{a \geq b} \pi_a p_{ab}}.
\]

We get the last equality since \( \pi_i \) is in increasing order. Hence

\[
P \left[ U_b = v_b(y^n) \right] \leq \frac{\alpha}{\sqrt{S_b} \sum_{a \geq b} \pi_a p_{ab}}.
\]

We can now compute

\[
\mathbb{E} \log \mathbb{P}[A = 1 | Y^n = y^n] = \sum_{y^n} \mathbb{P}[Y^n = y^n | A = 1] \log \mathbb{P}[A = 1 | Y^n = y^n]
\]

\[
\leq \log \sum_{y^n} \mathbb{P}[Y^n = y^n | A = 1] \mathbb{P}[A = 1 | Y^n = y^n]
\]

\[
\leq \log \sum_{y^n} \mathbb{P}[Y^n = y^n | A = 1] \alpha^{q-1} \prod_{b=1}^{\alpha} \frac{1}{\pi_{b}^{c_-}} \left( \frac{1}{\pi_{b}^{c_-}} \right) \sum_{a \geq b} \pi_a p_{ab}
\]

\[
\leq \log \left( 2q/n^{\gamma/4} + (1 - 2q/n^{\gamma/4}) \right) \prod_{b=1}^{\alpha} \frac{1}{\pi_{b}^{c_-}} \left( \frac{1}{\pi_{b}^{c_-}} \right) \sum_{a \geq b} \pi_a p_{ab}
\]

\[
\leq \log \left( 2q/n^{\gamma/4} + \frac{\alpha^q}{\left( \frac{c_-}{\pi_{\max}} \right)^{\frac{q-1}{2}}} \sqrt{\frac{\prod_{i=1}^{q-1} \pi_i}{\pi_{\max}}} \right)
\]

where in (211) we used (201). We can pick \( \gamma \) large enough\(^7\) so that the first term in the logarithm is negligible compared to the second term for large \( n \).

This gives

\[
P[\log \mathbb{P}[A = 1 | Y^n = y^n] | A = 1] \leq \log \left( \frac{2\alpha^q}{\left( \frac{c_-}{\pi_{\max}} \right)^{\frac{q-1}{2}}} \right)
\]

\[
\leq \frac{1}{2} \log n - \frac{1}{2} \sum_{b=1}^{q} \log \pi_b n + c'.
\]

The value \( c' \) collects all the constants. Combining with Lemma 2 we get that for the \( q \)-ary erasure channel and sufficiently large \( n \) that

\[
D(P_{Y|X} \circ U || Q_Y^n) \leq n D(P_Y || Q_Y) + c
\]

where \( c \) does not depend on \( n \) or \( \pi \). Using Proposition 1 completes the proof.

---

\(^7\)For instance, we can pick \( \gamma = 40q \). For large enough \( n \), we still get that \( \mathbb{E}[S_b] = n \sum_{a \geq b} \pi_a p_{ab} > 2\gamma \log n \) is true for all \( b \).
G. Z-Channel

The matrix for the Z-channel [3 p 225] is

\[
\begin{bmatrix}
p_{11} & p_{12} \\
0 & 1
\end{bmatrix}
\]

(216)

where we require that \( p_{ij} > 0 \). (Typically, \( p_{11} = p_{12} = 1/2 \), but we will consider a more general case.)

We can actually get the capacity of the noisy permutation channel with the Z-channel without altering the proof for the \( q \)-ary erasure channels. The transition matrix for the Z-channel can be written as

\[
\begin{bmatrix}
p_{11} & p_{12} & 0 \\
p_{21} & 0 & p_{23}
\end{bmatrix}
\]

(217)

setting \( p_{23} = 0 \). This does not change the rank of the matrix or the analysis in the proof.

**Corollary 2.** Let \( P_{Z|X} \) be a stochastic matrix for the Z-channel, then

\[
C_{\text{perm}}(P_{Z|X}) = \frac{1}{2}.
\]

(218)

This is item 3 of Theorem 3.

H. “Zigzag” Channel

In this section, we explore the limits of our approach. We have a particular DMC matrix which is similar to the \( q \)-ary erasure channel, but our method is not known to give a tight converse. We will use a matrix which could be considered a \( q \)-ary Z-channel. We will call it a “zigzag” channel since its edges in a transition diagram form a zigzag.

The matrix has the form:

\[
\begin{bmatrix}
p_{11} & p_{12} & 0 & \cdots & 0 & 0 \\
0 & p_{22} & p_{23} & \cdots & 0 & 0 \\
0 & 0 & p_{33} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & p_{q-1,q-1} & p_{q-1,q} \\
0 & 0 & 0 & \cdots & 0 & p_{qq}
\end{bmatrix}
\]

(219)

where each \( p_{ij} > 0 \). This matrix has rank \( q \).
Suppose that \( q \) is odd and that \( \pi \) is such that \( \pi_i = 0 \) for all even values of \( i \). Following the notation and method in Proposition 3, \( \mathbb{P}[A = 1 | Y^n] = 1 \), since any output symbol can be decoded to exactly one input symbol. For any \( \pi \) of this choice,

\[
D(P_{Y|X} || Q_Y^n) = -n D(P_Y || Q_Y) - \frac{1}{2} \log n + \sum_{i: \pi_i > 0} \frac{1}{2} \log \pi_i n + c + \mathbb{E}[\log \mathbb{P}[A = 1 | Y^n]|A = 1] = -n D(P_Y || Q_Y) + \frac{q - 1}{2} \frac{1}{2} \log n + c.
\]

(220)

If \( \pi \) of this form is the worst case \( \pi \) to use, meaning it gives the largest possible value of \( D(P_{Y|X} \circ U || Q_{Y^n}) \) for any \( Q_Y \), then we get that

\[
C_{\text{perm}}(P_{Z|X}) \leq q - \frac{1}{2} + \frac{q - 1}{4} = \frac{3(q - 1)}{4}.
\]

(224)

If there is another \( \pi \) which is the worst, then our upper bound on the capacity will be larger than the value on the right side of (224). In either case, there is a gap between our upper bound for the capacity and the lower bound of \((q - 1)/2\) given by (3). Exploring this gap is an opportunity for future work.

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