PAPER

Special exceptional point acting as Dirac point in one dimensional $\mathcal{PT}$-symmetric photonic crystal

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Abstract

We reveal that a special exceptional point (EP) can act as a Dirac point in a one-dimensional $\mathcal{PT}$-symmetric photonic crystal and prove it in detail using our extended first-principle theory. This theory was developed by applying biorthogonal bases of the non-Hermitian Hamiltonian to the $\mathbf{k} \cdot \mathbf{p}$ method to study the dispersion relations of non-Hermitian systems. By using this theory, we demonstrate that two linear dispersions can cross at a critical touching point, which is a special EP, and the corresponding effective Hamiltonian can be cast into a massless Dirac Hamiltonian under the biorthogonal bases of the non-Hermitian Hamiltonian; therefore, this point can be called the Dirac point, and the linear slope can also be predicted. In contrast to the Dirac point in a Hermitian system, which is induced by the degeneracy of two different eigenstates, the Dirac point here is induced by the degeneracy of parallel eigenstates in a non-Hermitian system. In addition, after increasing the non-Hermiticity, the Dirac point evolves into a pair of EPs, and their positions in the band structure can be predicted well by our theory. Our findings and theory are important for further understanding the physics of the Dirac point in non-Hermitian wave systems.

1. Introduction

In recent decades, there has been significant interest in the Dirac cones of classical wave systems. Studies show that many interesting phenomena in these classical wave systems as a consequence of Dirac cones can be supported, such as the unconventional quantum Hall effect [1–4], Klein tunneling [5–7], Zitterbewegung [8–11], and antilocalization [12–14], which are motivated by the study of Dirac particles in graphene [15, 16]. Researchers have explained the linear cross-dispersion in graphene by applying a tight-binding model to electronic wavefunctions in a honeycomb lattice [15, 16]. Studies of classical wave analogs have shown that pseudospin-1/2 Dirac cones can be realized in hexagonal photonic or phononic analogs at the corners of the Brillouin zone [17, 18], which are spanned in the basis functions of two degenerate eigenstates at the Dirac point. In addition, pseudospin-1 Dirac-like cones, which comprise two linear bands and one additional flat band, can be realized in photonic or phononic crystals at the centers of Brillouin zones [19–22]. While these Dirac cones are spanned on the basis functions of accidental triple degenerate eigenstates at the Dirac point [22], they can also be understood from the perspective of an effective medium at the Dirac point, where both the effective permittivity and effective permeability are equal to zero simultaneously [19, 23]. In addition to the findings of Dirac cones in classical wave analogs, interesting phenomena and many important properties, such as non-trivial Berry phases supported by Dirac cones in electronic systems, can exist in classical wave analogs [24].

Many studies have investigated the combination of a Dirac point and an exceptional point (EP) [25–29]. EPs are degenerate points in non-Hermitian systems where two or more eigenvalues coalesce into one and eigenfunctions become parallel [30–34], which support abundant novel phenomena and are of many applications, such as single-mode lasers [35, 36], unidirectional light propagation [37], coherent absorption [38], enhanced optical sensing [39, 40], laser-absorbers [41–45], topological light steering [46] and others.
[47–49]. Recently, researchers have shown that Dirac points can occur in non-Hermitian systems using symmetry constraints to enforce eigenstate orthogonality, which are diabolic points, not EPs [29]. Literature [25–27] has revealed that when certain non-Hermitian perturbations are introduced into a Hermitian system that supports a Dirac cone, it can spawn a ring of EPs. Furthermore, in [50], researchers proposed a non-Hermitian effective medium theory to interpret the spawning rings of EPs out of Dirac cones in the band structures of photonic crystals with loss/gain.

In this study, we focus on the Dirac point and EP in a one-dimensional $\mathcal{PT}$-symmetric photonic crystal (1DPTSPC), which is a special non-Hermitian system in which loss is balanced with gain $n(x) = n^*(-x)$, which is obtained from the band structure of photonic crystals [51–54]. We theoretically reveal and prove that in the 1DPTSPC, a special EP at the center of the Brillouin zone can also act as a Dirac point. This theory was developed by applying the biorthogonal bases of the non-Hermitian Hamiltonian [55, 56] to $\vec{k} \cdot \vec{p}$ method [57, 58]. By tuning the non-Hermiticity in 1DPTSPC, two linear dispersions can cross at a critical EP, and the corresponding effective Hamiltonian can be cast into a massless Dirac Hamiltonian under biorthogonal bases of the non-Hermitian Hamiltonian; therefore, this point can be called the Dirac point. The remainder of this paper is organized as follows: in section 2, we explicate our extended $\vec{k} \cdot \vec{p}$ theory for 1DPTSPC, and then, by using this theory, we analyze the dispersion relations and $\mathcal{PT}$-exact/broken phases. In section 3, we compare our theoretical predictions with the corresponding simulated results obtained using the transfer matrix method. Section 4 presents our conclusions.

2. Model and methodology

Consider a 1DPTSPC as shown in figure 1, where the unit cell comprises four layers with a balanced loss and gain. The relative permittivities of these layers are represented by $\varepsilon'$ and $\varepsilon''$ and their corresponding thicknesses are denoted by $d_0, d_3, d_2$ and $d_1$. In this study, we choose $\varepsilon''_0 = \varepsilon''_3 = \varepsilon''$ for simplicity. We set $\varepsilon''_1 = 1.1, \varepsilon''_d = 4.0, d_0 = 0.35\Lambda$ and $d_3 = 0.15\Lambda$, where $\Lambda$ is the thickness of a unit cell.

Using the transfer matrix method, we depicted in figure 2 that the band structures of the 1DPTSPC are represented by black lines at various values of the non-Hermiticity $\varepsilon''$. When $\varepsilon''$ is slightly increased from zero, all bands remain real and are in $\mathcal{PT}$-exact phase, as shown in figures 2(a) and (b). As $\varepsilon''$ is increased to a critical value $\varepsilon''_C$, the second band gap starts to vanish, and the 2nd and 3rd bands coalesce; simultaneously, an EP emerges at the center of the Brillouin zone, as shown in figures 2(c) and (d). As $\varepsilon''$ is increased further, this EP starts to be split into two, and these two EPs are located on either side of the center of the Brillouin center, as shown in figures 2(e) and (f). At the critical configuration $\varepsilon'' = \varepsilon''_C$, does the two bands cross linearly in the vicinity of the center of the Brillouin zone and can the special touching EP be treated as a Dirac point? We now solve this problem by applying the plane wave expansion method for periodic photonic structures [59–61] and biorthogonal bases for non-Hermitian systems to $\vec{k} \cdot \vec{p}$ method [57, 58].

Before the description of our theory, it should be pointed out that the simplest 1DPTSPC with two layers in a unit cell does not support the simultaneous occurrence of Dirac point and EP, this can be understood from our previous work on the band structures of the simplest 1DPTSPC [62]. In the figure 2 of that previous work, when there is no non-Hermiticity, the model is uniform medium, its bands are real and touch at the centers and boundaries of the Brillouin zone. When the non-Hermiticity is increased slightly, two bands touching each other at the Brillouin center are opened and forbidden gap forms. The touching points at the Brillouin boundaries evolve into EPs in the vicinity of Brillouin boundaries. As the non-Hermiticity is increased continuously, the forbidden gaps broaden gradually, and EP pairs move away from the Brillouin boundaries to the Brillouin center. At some critical values, these EP pairs coalesce at the Brillouin center. Subsequently, the corresponding conduction bands completely evolve into complex bands. Therefore, in the light of this evolution of the band structure as non-Hermiticity is increased, the simultaneous occurrence of Dirac point and EP cannot be produced in the simplest 1DPTSPC.

The Bloch wave function of the magnetic field $H_{ik}(x)$ in a one-dimensional periodic structure for normal incidence can be expanded as plane waves as follows:

$$H_{ik}(x) = \sum_G H'_{ik}(G)e^{i(k+G)x}, \quad (1)$$

Here, the direction of the magnetic field is along the $z$ axis, $n$ and $k$ are the band index and Bloch wave vector in the Brillouin zone, respectively, $G$ is the reciprocal lattice, which is an integer multiple of the corresponding unit vector, $G \equiv g2\pi/\Lambda$ is associated with the sorted integer $g = -N, \ldots -2, -1, 0, 1, 2 \ldots N$, where truncation is taken, and $H'_{ik}(G)$ is the expansion coefficient. The
magnetic field fulfills the master equation derived from Maxwell’s equations:

\[- \frac{d}{dx} \left( \frac{1}{\varepsilon(x)} \frac{d}{dx} \right) H_{nk}(x) = \frac{\omega_n^2}{c^2} H_{nk}(x). \tag{2}\]

This equation can be regarded as the eigenequation of the Hamiltonian in the coordinate space. Subsequently, substituting the plane-wave expansion in equation (1) of the magnetic field into this equation, we obtain

\[ \sum_{G'} \chi(G - G')(k + G')H'_{nk}(G') = \frac{\omega_n^2}{c^2} H_{nk}(G), \tag{3}\]

where \( \chi(G - G') \) is the expansion coefficient of the reciprocal of the dielectric function; that is, \( 1/\varepsilon(x) = \sum G \chi(G)e^{iGx} \). The matrix form of equation (3) can be regarded as the eigenequation of the Hamiltonian in the wave vector space with eigenvalue \( \omega_n^2/c^2 = \kappa_{nk} \). For zero non-Hermiticity, this Hamiltonian is Hermitian out of \( \chi^*(G - G') = \chi(G) \). For general non-Hermiticity, taking the Hermitian conjugate on both sides of equation (3), we obtain the eigen-equation of the Hermitian conjugate of the Hamiltonians

\[ \sum_{G'} \chi^*(G' - G)(k + G')\tilde{H}'_{nk}(G') = \frac{\omega_n^2}{c^2} \tilde{H}'_{nk}(G). \tag{4}\]

\( \tilde{H}'_{nk}(G) \) and \( H'_{nk}(G) \) are the left and right eigenvectors of the Hamiltonian, respectively, and satisfy the biorthogonal relation:

\[ \sum_{G} \tilde{H}'_{in}(G)H'_{in}(G) = \eta_{nk}\delta_{ln}. \tag{5}\]

Let us now study the two bands that can form a touching EP, e.g., the 2nd \((n = 1)\) and 3rd \((n = 2)\) bands in figure 2. When \( \varepsilon'' \) is close to \( \varepsilon''_0 \), we assume that the eigenvalues \( \varepsilon_{n0} \) and eigenfields \( H_{n0}(x) \) at the center of the Brillouin zone \( k = 0 \) are known. Then, according to \( \vec{k} \cdot \vec{p} \) method, we can write the unknown magnetic field \( H_{nk}(x) \) at wave vector \( k \) near the center of the Brillouin zone as follows:

\[ H_{nk}(x) = \sum_n A_{mn}(k)e^{ikx}H_{n0}(x), \tag{6}\]

where \( A_{mn}(k) \) is the coefficient of linear combination. By substituting this equation into the master equation (2) and using the plane-wave expansion in equation (1), we have that

\[ \sum_{G'} \chi(G - G')\left[GG' + k^2 + (G + G')k\right]\sum_n A_{mn}(k)\tilde{H}'_{n0}(G') = \frac{\omega_n^2}{c^2} \sum_n A_{mn}(k)H'_{n0}(G). \tag{7}\]

Multiplying both sides of this equation by \( 1/\eta_{0n}\tilde{H}'_{n0}(G) \) and utilizing the biorthogonal relationship in equation (5), we can derive

\[ \sum_n \left[ \tilde{H}'_{ln}(k) - (\kappa_{nk} - \kappa_{l0})\delta_{ln} \right] A_{mn}(k) = 0, \tag{8}\]

where \( \tilde{H}'_{ln}(k) \) is the effective Hamiltonian

\[ \tilde{H}'_{ln}(k) = (\kappa_{n0} - \kappa_{l0})\delta_{ln} + p_{ln}k + q_{ln}k^2, \tag{9}\]

associated with

\[ p_{ln} = 1/\eta_{0n}\sum_{G,G'} \tilde{H}'_{l0}(G)\chi(G - G')(G + G')H'_{n0}(G'), \tag{10}\]
Figure 2. Complex band structures of 1DPTSPC for $\varepsilon'' = 0.05$ (a) and (b), $\varepsilon'' = 0.17123$ (c) and (d), $\varepsilon'' = 0.4$ (e) and (f). Black curves are obtained by plane wave expansion method, and red circles show the results predicted by equation (15).

and

$$q_{kn} = 1/\eta_0 \sum_{G,G'} \tilde{H}^*_n(G) \chi(G - G') H^*_{m0}(G').$$

(11)

In addition, we present some important relations that are valid for the 1DPTSPC in the following. Components $\chi(G)$ are real because of the balanced loss and gain $\varepsilon'(-x) = \varepsilon(x)$. The left and right eigenfields of the Hamiltonian in the coordinate space and wave vector space are related by:

$$\H_{nk}(x) = H^*_n(-k)(x),$$

(12a)

$$\H_{nk}(G) = H^*_n(-k)(-G),$$

(12b)

A detailed derivation is provided in appendix A. Moreover, by utilizing these equations in appendix B we derive the relations for the coefficients $p_{ln}$ and $q_{ln}$

$$p_{11} = p_{22} = 0, \quad p_{21} = -\eta_{10}/\eta_{20} p_{12},$$

(13a)

$$q_{21} = \eta_{10}/\eta_{20} q_{12}.$$  

(13b)

On this basis, we next analyze the eigenvalues and eigenfields of the effective Hamiltonian $H^{eff}(k)$.

### 2.1. Second order term is omitted

When $k$ is sufficiently close to the center of the Brillouin zone, the second-order term $q_{00} k^2$ in the effective Hamiltonian in equation (9) can be approximately omitted; then, this Hamiltonian can be written as

$$H^{eff}(k) = \begin{pmatrix} 0 & p_{12} k \\ p_{21} k & \kappa_{20} - \kappa_{10} \end{pmatrix}.$$  

(14)
Here, we emphasize that this matrix is expanded under the biorthogonal bases $H_{10,20}(G)$ and $\vec{H}_{10,20}(G)$, and not under the conventional orthogonal bases that are used in Hermitian systems. Two eigenvalues of this effective Hamiltonian can be calculated as

$$\kappa_{1k,2k} = \frac{1}{2} \left[ \kappa_{10} + \kappa_{20} \pm \sqrt{(\kappa_{20} - \kappa_{10})^2 + 4p_{12}p_{21}k^2} \right]. \quad (15)$$

The ratios of the two components of the corresponding eigenvectors can also be calculated:

$$z_{1k,2k} = \frac{\kappa_{20} - \kappa_{10} \pm \sqrt{(\kappa_{10} - \kappa_{20})^2 + 4p_{12}p_{21}k^2}}{2p_{12}k}. \quad (16)$$

Moreover, the group velocity can be derived from the dispersion relationship in equation (15)

$$\frac{d\omega_{1k,2k}}{dk} = \pm \frac{c^2}{\omega_{1k,2k}} \frac{p_{12}p_{21}k}{\sqrt{(\kappa_{10} - \kappa_{20})^2 + 4p_{12}p_{21}k^2}}. \quad (17)$$

When the system at the center of the Brillouin zone is in $PT$-broken phase, the known eigenvalues $\kappa_{10,20}$ are taken in complex conjugate pair, hence assume $\kappa_{10} = \kappa_r - i\kappa_i$ and $\kappa_{20} = \kappa_r + i\kappa_i$ ($\kappa_i > 0$), and the corresponding known eigenfields satisfy $PTH_{10}(x) = \xi H_{20}(x)$; here, $\xi$ is a complex number, and it is also a gauge induced by the freedom that the parallel fields to $H_{20}(x)$ are also the eigenfields in equation (2). Specifically, $c_n H_{10}(x)$ are also eigenfields; then, the gauge $\xi$ is transformed into $\xi \epsilon_j / \epsilon'_j$, and thus the value of this gauge depends on our choice. Under this prerequisite, from equation (15), the eigenvalues in this case can be expressed as

$$\kappa_{1k,2k} = \kappa_r \pm \sqrt{-\kappa_i^2 + p_{12}p_{21}k^2}, \quad (18)$$

where $p_{12}p_{21} = |\xi p_{12}|^2$ is positive (appendix C). It can be observed that near the center of the Brillouin zone, these eigenvalues are complex conjugates with respect to each other, and thus $PT$-broken region forms. As $k$ deviates from 0, their real parts remain the same and the modulus of their imaginary parts decreases. When $k$ deviates further across the transition points, these eigenvalues become real, and these regions correspond to the $PT$-exact phase. The pair of EPs is located at the transition points $k = \pm \kappa_i / \sqrt{p_{12}p_{21}}$. As the EPs approach, the group velocities approach infinity, which can be obtained from equation (17) directly. Consequently, in the band diagrams, the tangent lines at the EPs are vertical.

The other situation is that the system at the center of Brillouin zone is in the $PT$-exact phase, the known eigenvalues $\kappa_{10,20}$ are real, and the corresponding known eigenfields fulfill $PTH_{10}(x) = \zeta_0 H_{10}(x)$, where $\zeta_0$ is a phase, and also a gauge that is induced by the same reason as the gauge $\xi$; if the eigenfields $H_{10}(x)$ are changed into $c_n H_{10}(x)$, the gauge $\zeta_0$ is transformed into $c_n \zeta_0 / \zeta'_0$ correspondingly, and thus the value of the gauge also depends on our choice. Thus, it can be derived that $p_{12}p_{21} = -\frac{\zeta_0}{2 \zeta'_0} |\xi p_{12}|^2$, which is a real number (refer to appendix D). There are two possible situations. If $p_{12}p_{21} > 0$, both eigenvalues $\kappa_{1k}$ and $\kappa_{2k}$ are real, and $PT$-exact phase forms in the entire Brillouin zone, which is discussed in this study. If $p_{12}p_{21} < 0$, near the center of the Brillouin zone, $PT$, the exact region forms. When $k$ deviates from 0 across the EPs, the real parts of the eigenvalues $\kappa_{1k}$ and $\kappa_{2k}$ remain the same and the modulus of their imaginary parts increases from zero, which corresponds to $PT$-broken phase. This case will be explained in a future study.

As the non-Hermiticity $\epsilon''$ approaches the critical value $\epsilon''_C$ from $PT$-exact phase or $PT$-broken phase, a special touching EP at the center of the Brillouin zone arises; simultaneously, $\kappa_{10}$ and $\kappa_{20}$ become real and equal to each other $\kappa_{10} = \kappa_{20} = \kappa_C$; in this critical case, the dispersion relation can be written as follows from equations (15) and (18),

$$\kappa_{1k,2k} = \kappa_C \pm s|p_{12}|k, \quad (19)$$

where $s = |\xi| = \sqrt{-\frac{\kappa_{10}}{\kappa_{20}}}$ is positive, at $\epsilon'' = \epsilon''_C$. If we choose a special gauge $\xi = -\frac{\kappa_{10}}{\kappa_{20}}$, then $p_{21} = p_{12}^*$ (refer to appendix C), which has the following form derived from equation (14).

$$H = \begin{pmatrix} 0 & p_{12}k \\ p_{12}^*k & 0 \end{pmatrix} = d_x \sigma_x + d_y \sigma_y, \quad (20)$$

where $\sigma_x$ and $\sigma_y$ are the first and second Pauli matrices in the 1/2 pseudo-spin space associated with coefficients $d_x = k \text{ Re } p_{12}$ and $d_y = -k \text{ Im } p_{12}$, respectively. Equation (20) can be mapped into the massless Dirac Hamiltonian in quantum mechanics; thus, the special touching EP satisfying the linear dispersion in equation (19) and Dirac Hamiltonian in equation (20) is called the Dirac point.
Figure 3. Complex band structures of 1DPTSPC for $\varepsilon'' = 0.05$ (a) and (b), $\varepsilon'' = 0.171$ (c) and (d), $\varepsilon'' = 0.4$ (e) and (f). Black curves are obtained by using the plane wave expansion method, and red circles show the results predicted by equation (23).

Furthermore, the linear slope at the Dirac point can be predicted using equation (17) or linear dispersion equation (19)

$$\frac{d\omega_{1k,2k}}{dk} = \pm s|p_{12}| \frac{q_{12}^2}{2\omega_C},$$

where $\omega_C$ denotes the eigenfrequency at the Dirac point. When the system near the center of the Brillouin zone transitions from $\mathcal{PT}$-exact phase to $\mathcal{PT}$-broken phase across the Dirac point, the group velocity at the center of Brillouin undergoes a sudden change from zero to nonzero and then to infinity.

2.2. Second order term is not omitted

When the second-order term $q_{ln}k^2$ in the effective Hamiltonian is considered, equation (9), the Hamiltonian is given by the matrix

$$H^\text{eff}(k) = \begin{pmatrix} q_{11}k^2 & p_{12}k + q_{12}k^2 \\ p_{21}k + q_{21}k^2 & \kappa_{20} - \kappa_{10} + q_{22}k^2 \end{pmatrix},$$

and is suitable for wave vectors $k$ in a larger range. Then, its eigenvalues and eigenvectors can be solved as

$$\kappa_{1k,2k} = \frac{1}{2} \left[ \kappa_{10} + \kappa_{20} + (q_{11} + q_{22})k^2 \pm \sqrt{\Delta} \right],$$

$$z_{1k,2k} = \frac{\kappa_{20} - \kappa_{10} + (q_{22} - q_{11})k^2 \pm \sqrt{\Delta}}{2(p_{12}k + q_{12}k^2)},$$

where we assume $\Delta = [\kappa_{10} - \kappa_{20} + (q_{11} - q_{22})k^2]^2 + 4(p_{12}k + q_{12}k^2)(p_{21}k + q_{21}k^2)$. When the system at the center of the Brillouin zone is in $\mathcal{PT}$-broken phase or $\mathcal{PT}$-exact phase, we can obtain similar results to those in the above subsection by using a similar method. In this process, the following relations are used: $q_{22} = q_{11}, q_{21} = |\xi|^2 q_{12}$ (refer to appendix C) and $q_{ln} = \zeta_l \zeta_0 d_{ln}$ (refer to appendix D). In particular, when
Figure 4. Real part (a) and imaginary part (b) of the reduced complex eigenfrequency $\omega \Lambda / 2\pi c$ as functions of the reduced Bloch wave vector $k \Lambda / 2\pi$ and non-Hermiticity $\epsilon''$. The real part (c) and imaginary part (d) of reduced complex eigenfrequency $\omega \Lambda / 2\pi c$ as functions of non-Hermiticity $\epsilon''$ at the center of Brillouin zone $k = 0$. These results are obtained by using the plane wave expansion method. The blue and red regions represent the second and third bands, respectively, and the purple region is the overlap between these two bands.

$\mathcal{PT}$ broken phase emerges, the eigenvalues are given by

$$\kappa_{1,2k} = \kappa_0 + q_{11r} k^2 \pm \sqrt{\left| \kappa_i - q_{11i} k^2 \right|^2 + \left| \xi \right|^2 \left| p_{12} k + q_{12r} k^2 \right|^2},$$

where $q_{11r}$ and $q_{11i}$ are the real and imaginary parts of coefficient $q_{11}$. Therefore, a pair of EPs are located at $|\xi (p_{12} k + q_{12r} k^2)| = |\kappa_i - q_{11i} k^2|$. In addition, when the second-order is considered, the effective Hamiltonian in equation (22) cannot be written in the form of a massless Dirac Hamiltonian in quantum mechanics because of the quadratic terms of the Bloch wave vector $k$ in this equation, which does not exist in the massless Dirac Hamiltonian in quantum mechanics; however, the approximation that we only consider the first-order term is sufficiently good near the center of the Brillouin zone.

3. Example and analysis

We now compare the theoretical predictions with the corresponding simulated results in figures 2 and 3, where the band structures of the 1DPTSPC are illustrated for different values of non-Hermiticity, the black curves represent the simulated dispersion relations obtained by the transfer matrix method, and the red circles are the theoretical predictions depicted by our extended $\vec{k} \cdot \vec{p}$ method. In figure 2 where the second-order term in our theory is omitted if the system at the center of the Brillouin zone is in $\mathcal{PT}$ exact phase (figures 2(a) and (b)), we can observe that our theoretical predictions agree well with the simulated results only in a narrow region near the center of the Brillouin zone. As the non-Hermiticity $\epsilon''$ approaches the critical value $\epsilon''_c$ (figures 2(c) and (d)), a touching EP emerges, which coincides with the simulated result near the center of the Brillouin zone; therefore, we prove that the linearly intersecting dispersion relation in equation (19) is valid, and the effective Hamiltonian can be mapped into the massless Dirac Hamiltonian, as shown in equation (20); hence, this special touching EP in figures 2(c) and (d) show the Dirac point. If the system at the center of the Brillouin zone is in $\mathcal{PT}$-broken phase (figures 2(e) and (f)), our theoretical predictions were consistent with the simulated results for the $\mathcal{PT}$-broken region and near the pair of EPs.

The slope at the Dirac point calculated using equation (21) is

$$\frac{d(\omega \Lambda / 2\pi c)}{d(k \Lambda / 2\pi)} = 0.7174,$$

which fits the corresponding simulated results. In figure 3 where the second order term is considered, it can be observed that the theoretical predictions coincide well with the simulated results in almost the whole Brillouin zone.
except the regions near edges. In general, the comparison in figures 2 and 3 confirm the validity of our theory.

To observe the effect of non-Hermiticity $\epsilon''$ on the complex band structure, as shown in figure 4, we plot the complex bands as the Bloch wave vector $k$ and non-Hermiticity $\epsilon''$. As $\epsilon''$ is increased from zero, the gap between the second and third dispersions, which refers to the eigenfrequency $\omega$ as a function of the variable Bloch wave vector $k$ at fixed non-Hermiticity $\epsilon''$, is reduced, becomes zero, and begins to coalesce at the critical point, which is a Dirac point and an EP simultaneously. As $\epsilon''$ is increased continuously, a complex eigenfrequency arises, and the corresponding interval in the Bloch wave vector axis is gradually enlarged.

Figure 5 presents the distributions of the eigenfields $\text{real}(H_{n0}(x))$ at the center of the Brillouin zone for different values of non-Hermiticity. In figure 5(a), the system is in the $P\bar{T}$-exact phase, and the eigenfields are symmetric because $H_{n0}(-x) = H_{n0}(x)$, where we choose a special gauge $\xi = 1$. In figure 5(c), the system is in $P\bar{T}$-broken phase, which agrees with the theoretical relation $H_{n0}(-x) = H_{2n}(x)$ for the specially selected gauge $\xi = 1$. When the Dirac point approaches from both phases, the field distributions at the two bands coincide gradually with each other, as shown in figure 5(b): consequently, compared with the Dirac point in the Hermitian system, which is induced by the degeneracy of two orthogonal eigenstates, such as a two-dimensional photonic crystal without loss and gain, the Dirac point here is a special EP in a non-Hermitian system, where two eigenstates are parallel to each other.

Moreover, when the system approaches the Dirac point, two eigenvalues become closer and simultaneously two right eigenfields become parallel; thus, from the biorthogonal relation in equation (5), we can deduce that the two corresponding left eigenfields also become parallel, and the right eigenfields
become orthogonal to their left eigenfields in the same band, that is, \( \eta_{\mathbf{k}} \) is zero at the EPs. In figures 6(a) and (b), we plot the normalized coefficients \( \eta_{\mathbf{k}0} \) and \( \eta_{\mathbf{k}2} \) as functions of the non-Hermiticity \( \varepsilon'' \). It can be observed that the coefficients \( \eta_{\mathbf{k}0} \) and \( \eta_{\mathbf{k}2} \) approach zero at the critical value \( \varepsilon''\text{C} \) corresponding to the Dirac point. Figures 6(c) and (d) show \( \eta_{\mathbf{k}1} \) and \( \eta_{\mathbf{k}3} \) as functions of the wave vector \( \mathbf{k} \); here, the non-Hermiticity is fixed at \( \varepsilon'' = 0.4 \); it can be also observed that the coefficients \( \eta_{\mathbf{k}1} \) and \( \eta_{\mathbf{k}3} \) approach zero at the pair of EPs; in addition, we observe that \( \eta_{\mathbf{k}2} \) approaches infinity in the \( PT \)-exact phase near these EPs, and thus the new axis \( 4 \arctan(\eta_{\mathbf{k}2})/\pi \) is chosen to observe the variation. In this figure, we also used the same gauge as that in figure 5. In addition, we calculated the inner products of the left and right eigenstates at different bands; they are zero and consistent with the biorthogonal relation in equation (3), which is widely known but not shown in this figure.

For conventional photonic crystals without loss and gain, the Dirac point can be supported by a two-dimensional photonic crystal rather than a one-dimensional photonic crystal. However, in our study, we have demonstrated that after introducing loss and gain, the Dirac point can emerge in 1DPTSPC, while it is induced by parallel degenerate eigenstates, not two orthogonal eigenstates. This study opens up for other relevant research. For instance, exploring other non-Hermitian optical systems that can support this type of Dirac point; in addition, out of the similarities and differences between the Dirac points in non-Hermitian systems and Hermitian systems, it is worth determining whether the interesting phenomena supported by the conventional Dirac point can emerge in non-Hermitian optical systems. These will be addressed in our next study.

4. Conclusion

By combining the biorthogonal bases of the non-Hermitian Hamiltonian and \( \vec{k} \cdot \vec{p} \) method, we propose a first-principles theory to study the dispersion relations of 1DPTSPCs. When the second-order term is omitted, this theory is valid near the center of the Brillouin zone if the non-Hermiticity \( \varepsilon'' \) is less than the critical value \( \varepsilon''\text{C} \) and it is valid between and near a pair of EPs if the non-Hermiticity \( \varepsilon'' \) is greater than the critical value \( \varepsilon''\text{C} \). If the critical value is reached, two linear dispersions cross at a special touching EP, and the corresponding effective Hamiltonian can be cast into a massless Dirac Hamiltonian under the biorthogonal bases of the non-Hermitian Hamiltonian, which can be called the Dirac point. This Dirac point in a non-Hermitian system, where the eigenstates are parallel to each other, is distinct from the conventional Dirac point in a Hermitian system, where the eigenstates are orthogonal to each other. When we consider the second-order term, our theoretical predictions agree well with the simulated results in almost the entire Brillouin zone, except in the vicinity of the edges.

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Appendix A. Relations between left and right eigenfields of Hamiltonian in 1DPTSPC

By performing a time-reversal transformation on both sides of equation (2) and taking the opposite of the wave vector \( \mathbf{k} \), we have

\[
\frac{d}{dx} \left( \frac{1}{\varepsilon'(x)} \frac{d}{dx} \right) H_{n(-k)}(x) = \frac{\omega_{n(k)}}{c} H_{n(-k)}(x),
\]

(A1)

where we used the relation \( \omega_{n(-k)} = \omega_{n(k)}. \) By substituting the plane-wave expansion in equation (1) of the eigenfield into this equation, we obtain

\[
\sum_{G'} \chi(G' - G)(k + G)H_{n(k)}^{*}(G')H_{n(-k)}(G') = \frac{\omega_{n(k)}}{c} H_{n(-k)}^{*}(-G).
\]

(A2)

Comparing equations (4) and (A2), we can deduce an important relationship between the left and right eigenvectors:

\[
\tilde{H}'_{nk}(G) = H_{n(-k)}^{*}(-G).
\]

(A3)

This relationship can be expressed in the following matrix form.
\[
\tilde{H}_{nk} = \sigma H_n^{*}(-k),
\]
(A4)

Here, the components \(\tilde{H}_{nk}^{*}(G)\) and \(H_n^{*}(-k)(G)\) of columns \(\tilde{H}_{nk}\) and \(H_n^{*}(-k)\) are sorted in the order \(g = -N, \ldots, -2, -1, 0, 1, 2 \ldots N,\) and \(\sigma\) denotes the generalized first Pauli matrix in the wave vector space

\[
\sigma = \begin{pmatrix}
0 & \ldots & 1 \\
\vdots & \ddots & \vdots \\
1 & \ldots & 0
\end{pmatrix},
\]
(A5)

where its components are zero except that the secondary diagonal components are equal to one. Using the plane wave expansion of the eigenfield, we can derive the relation between the corresponding left and right eigenfields in the coordinate space from equation (A3)

\[
\tilde{H}_{nk}(x) = H_n^{*}(-k)(x).
\]
(A6)

In addition, by utilizing the relationship between the left and right eigenvectors in equation (A3), the normalized coefficients \(\eta_{n0}\) in the biorthogonal relationship in equation (5) can be expressed in terms of eigenfields as:

\[
\eta_{n0} = \sum_{G} H_{n0}^{*}(-G) H_{l0}^{*}(G)
\]
(A7)

### Appendix B. Relations between coefficients in effective Hamiltonian

By utilizing the relationship between the left and right eigenfields, equation (A4), the coefficients \(p_{nl}\) defined in equation (10) can be expressed in the following matrix form:

\[
p_{nl} = 1/\eta_{nl} H_{nt}^{T}\sigma N H_{ln}^{*},
\]
(B1)

where \(N(G, G') = \chi(G - G')(G + G')\), this matrix \(N\) fulfills

\[
\sigma N \sigma = -N^T.
\]
(B2)

By transposing both sides of equation (B1), we obtain

\[
p_{nl} = 1/\eta_{nl} H_{nt}^{T}N^{T}\sigma H_{ln}^{*}.
\]
(B3)

Using the relationship in equations (B2) and (B1) can be expressed as follows:

\[
p_{nl} = -1/\eta_{nl} H_{nt}^{T}N^{T}\sigma H_{ln}^{*}.
\]
(B4)

Then, by comparing expressions (B3) and (B4), we find that the coefficients \(p_{nl}\) satisfy

\[
p_{nl} = -\eta_{l0} / \eta_{n0} p_{ln}.
\]
(B5)

Similarly, the coefficients \(q_{ln}\) can also be written in matrix form:

\[
q_{ln} = 1/\eta_{ln} H_{lt}^{T}\sigma L H_{ln}^{*},
\]
(B6)

associated with \(L(G, G') = \chi(G - G')\), satisfying

\[
\sigma L \sigma = L^T.
\]
(B7)

Then, in a manner similar to that of coefficients \(p_{nl}\), we obtain

\[
q_{ln} = \eta_{n0} / \eta_{l0} q_{ln}.
\]
(B8)

Thus far, we have derived the relations for the coefficients \(p_{nl}\) and \(q_{ln}\) shown in equations (B5) and (B8).

### Appendix C. System at the center of Brillouin zone in \(\mathcal{PT}\)-broken phase

When the system at the center of the Brillouin zone is in \(\mathcal{PT}\)-broken phase, the eigenvalues of the Hamiltonian always exist in the conjugate pair and the corresponding eigenfields fulfill \(\mathcal{P}\mathcal{T}H_{10}(x) = \xi H_{20}(x)\). Then, by using the plane wave expansion of eigenfields, we obtain
\[ H'_{\text{so}}(G) = \xi H''_{\text{so}}(G). \] (C1)

On this basis, the normalized coefficients \( \eta_{n0} \) at the two bands that can be written in the form of equation (A7) are related by

\[ \eta_{n0} = \xi^2 \eta_{l0}. \] (C2)

Substituting these relations into the definition of coefficients \( p_{l_{\text{n}}} \) in equation (10), we can derive the relationship between \( p_{12} \) and \( p_{21} \)

\[ p_{21} = |\xi|^2 p_{12}'. \] (C3)

Then, we prove \( p_{12} p_{21} = |\xi p_{12}'|^2 \) shown in the main text. Similarly, the relationship between \( q_{12} \) and \( q_{21} \) can also be calculated as

\[ q_{21} = |\xi|^2 q_{12}'^* q_{21} = q_{11}'. \] (C4)

Appendix D. System at the center of Brillouin zone in \( \mathcal{PT} \)-exact phase

When the system at the center of the Brillouin zone is in \( \mathcal{PT} \)-exact phase, the two eigenvalues are real numbers, and the corresponding eigenfields satisfy \( \mathcal{PT} H_{\text{so}}(x) = \zeta_n H_{\text{so}}(x) \), from which we can derive that

\[ H''_{\text{so}}(G) = \zeta_n H'_{\text{so}}(G). \] (D1)

Therefore, it can be observed that the modulus of \( \zeta_n \) is 1, and the phases of all the elements \( H''_{\text{so}}(G) \) are the same, except that the signs of these elements may be different. Substituting this relation (D1) into the general expression (A7) of the coefficients \( \eta_{n0} \), we find that \( \eta_{n0} \zeta_n \) can also be expressed in terms of eigenfields as:

\[ \eta_{n0} \zeta_n = \sum_G H''_{\text{so}}(-G) H'_{\text{so}}(G). \] (D2)

Thus, \( \eta_{n0} \zeta_n \) is a real number. Similarly, by substituting equation (D1) into the definition of coefficients \( p_{l_{\text{n}}} \) in equation (10), we can observe that the coefficients satisfy

\[ p_{l_{\text{n}}} = \zeta_l' \zeta_n p_{l_{\text{n}}}. \] (D3)

Based on this equation, and the general relationship in equation (B5) in appendix B, we can prove that

\[ p_{12} p_{21} = -\frac{\eta_{n0} \zeta_n}{|\eta_{l0}|^2} |p_{12}'|^2 \] and thus it is real. In addition, by substituting equation (D1) into the definition of the coefficients \( q_{l_{\text{n}}} \) equation (11), we have that

\[ q_{l_{\text{n}}} = \zeta_l' \zeta_n q_{l_{\text{n}}}. \] (D4)

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