Isotrivial unfoldings and structural theorems for foliations on Projective spaces.

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Abstract. Following T. Suwa, we study unfoldings of algebraic foliations and their relationship with families of foliations, making focus on those unfoldings related to trivial families. The results obtained in the study of unfoldings are then applied to obtain information on the structure of foliations on projective spaces, related to questions posed by Cerveau and Lins Neto.

1. Introduction

In [1] Cerveau and Lins Neto pose a question regarding the structure of codimension 1 foliations on projective spaces. Specifically they ask under the name of Main Conjecture whether the following alternative holds:

A codimension 1 foliation on \( \mathbb{P}^n \) is birationally equivalent to the pull-back of a foliation on \( \mathbb{P}^2 \) or it admits a transverse projective structure with poles on some hypersurface. They provide a plausibility arguments for low degree foliations to observe such an alternative.

In this work, in Corollary 5.3 we prove a weak version of the conjecture in [1]. Lossely speaking we prove that for codimension 1 foliations on \( \mathbb{P}^n \) the following alternative holds:

A codimension 1 foliation on \( \mathbb{P}^n \) is birationally equivalent to the pull-back of a foliation on an algebraic surface or it admits up to a meromorphic discrete covering an integrating factor.

The result of Corollary 5.3 is obtained here by considering unfoldings of foliations on \( \mathbb{P}^2 \).

The concept of unfoldings of foliations and its relations with deformation theory of foliations were first introduced and studied by Suwa in a series of papers (see [6, 7, 9] and references therein).

An unfolding of a foliation \( \mathcal{F}_0 \) on a variety \( X \) is a "bigger" foliation \( \mathcal{F} \) on a variety \( X \times S \) whose leaves contains those of \( \mathcal{F}_0 \) (see below for a precise definition). So we can always think of a foliation of codimension \( q \) on \( \mathbb{P}^n \) as being birationally equivalent to an unfolding of a foliation by curves on \( \mathbb{P}^{q+1} \). This, in general, does not provide us with much information, but under certain hypotheses we can control what kind of unfolding we may consider.

The special kind of unfolding that will give us structural information on the foliation will be Isotrivial unfoldings. Unfoldings may be thought of as special kinds of families of foliations, a family on which not only the differential equations defining the foliation vary continuously but also the solutions i.e.: the leaves vary continuously as well (this is, of course, very vague, again, see below for precise definitions). Isotrivial unfoldings are those that are related to trivial families of foliations, with this vague intuition, they are
families in which the equations stays still while the solutions vary. They were studied by Suwa in the infinitesimal case, see [9].

Here we generalize Suwa’s results on isotrivial unfolding to be able to deal with unfoldings and families parametrized by arbitrary schemes. It turns out that isotrivial unfoldings have a rather rigid structure and so being able to identify a foliation on $\mathbb{P}^n$ with an isotrivial unfolding says something about the structure of the foliation.

In Section 2 we give the principal definitions and state the theorems on unfoldings that will serve us as tools to conclude statements about foliations on $\mathbb{P}^n$. The main result is Theorem 2.12 which can be viewed as a first step in determining the representation of the functor that to every scheme $S$ associates the unfoldings of a given foliation parametrized by $S$, although such a line of investigation will not be pursued in this work.

In Section 3 we explain how foliations on $\mathbb{P}^n$ may be considered as giving rise to unfoldings of foliations by curves, and when this unfoldings may be taken isotrivial. The main result here is Theorem 3.3.

In Section 4 we deal with the technical issue of transversality, which is a condition we need to have to be able to apply the results of Section 2.

Finally, in Section 5 we apply the results of the previous sections to conclude some structural statements on foliations on $\mathbb{P}^n(\mathbb{C})$. In particular, Theorem 5.2 follows as a particular case of a statement (Proposition 5.1) valid for arbitrary dimensional foliations, although restrictions on the degree are always required.

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2. Isotrivial unfoldings

In order to treat infinitesimal unfoldings and its relations with deformations of foliations we will need to consider non-reduced schemes, and generalize the notion of a foliation to this setting. Luckily, a straightforward generalization will serve our purposes just fine.

Definition 2.1. Let $\mathcal{X}$ be a (separated) scheme of finite type over an algebraically closed field $k$. An involutive distribution over $\mathcal{X}$ is a subsheaf $\mathcal{T}\mathcal{F} \subseteq T\mathcal{X}$ that is closed under the Lie bracket, i.e.: for every open subscheme $U \subset \mathcal{X}$ we have

$$[\mathcal{T}\mathcal{F}(U),\mathcal{T}\mathcal{F}(U)] \subseteq \mathcal{T}\mathcal{F}(U).$$

The annihilator of an involutive distribution $I(\mathcal{F}) := \text{ann}(\mathcal{T}\mathcal{F}) \subseteq \Omega^1_{\mathcal{X}}$ is an integrable Pfaff system. An integrable pfaff system $I$ verifies the equation

$$d(I(\mathcal{F})) \wedge \bigwedge^r I(\mathcal{F}) = 0 \subset \Omega^{r+2}_{\mathcal{X}};$$
Definition 2.3 (Unfolding). Let \( X \) be any scheme (of finite type over the base field of \( X \)) and \( s \in S \) a closed point. We denote by \( \pi_1 \) and \( \pi_2 \) the projections of \( X \times S \) to \( X \) and \( S \) respectively. We denote by \( D\pi_2 \) the differential map 
\[
D\pi_2|_{(x,s)}: T(X \times S) \otimes k((x,s)) \to TS \otimes k(s).
\]
An unfolding of \( \mathcal{F}_0 \) parametrized by \((S, s)\) is a foliation \( \mathcal{F} \) on \( X \times S \) such that

1. The restriction of \( \mathcal{F} \) to \( X \times s \) is \( \mathcal{F}_0 \) i.e.: if we take the pull-back foliation \( \iota^* (\mathcal{F}) \) as in the above definition we get \( \iota^* (\mathcal{F}) = \mathcal{F}_0 \), here \( \iota : X \times s \to X \times S \) is the inclusion.
2. Dimensions of \( \mathcal{F} \) and \( \mathcal{F}_0 \) are related as thus: \( \dim \mathcal{F} = \dim \mathcal{F}_0 + \dim S \).

In the case where \( X \) and \( S \) are non-singular varieties over \( \mathbb{C} \), the leaves of the foliation \( \mathcal{F}_0 \) are contained in the larger dimensional leaves of the unfolding \( \mathcal{F} \).

Now we remind the definition of the relative tangent sheaf. Given a morphism \( f : \mathcal{X} \to \mathcal{Y} \) the relative tangent sheaf \( T_f \mathcal{X} \) is the dual of \( \Omega^1_{\mathcal{X}|\mathcal{Y}} \). Remember that \( T_f \mathcal{F} \) is naturally a sub-sheaf of the tangent sheaf \( T\mathcal{X} \), its sections are the vector fields on \( T\mathcal{X} \) that are tangent to the fibers of \( f \). In the case where \( f = \pi_2 : X \times S \to S \) is the projection we note \( T\pi_2 |_{(x,s)} = TS \otimes k(s) \).

Remark 2.4. In the case of the product of \( X \) and \( S \) we have the decomposition of sheaves.
\[
T(X \times S) \cong TS(X \times S) \oplus T_X(X \times S).
\]
Moreover we have
\[
T\pi_1(X \times S) \cong \pi_1^*(TX), \quad T\pi_2(X \times S) \cong \pi_2^*(TS),
\]
where \( \pi_1 \) and \( \pi_2 \) are the projections.

Remark 2.5. Let \( \mathcal{F}_0 \) be a foliation over a variety \( X \) and \( \mathcal{F} \) an unfolding of \( \mathcal{F}_0 \) parametrized by \((S, s_0)\). Then \( \mathcal{F} \) induces a family of foliations over \( X \) parametrized by \( S \) (see [5]) in the following way:

Let \( I(\mathcal{F}) \) be the Pfaff system associated with \( \mathcal{F} \). Let \( p : \Omega^1_{X \times S} \to \Omega^1_{X \times S|S} \) be the projection from the sheaf of differential to the sheaf of relative differentials. We set
\[ I_{S}(F) := p(I(F)) \subseteq \Omega^{1}_{X \times S|S}. \] Note that \( I_{S}(F) \) is a family of integrable Pfaff systems in the sense of [5] such that its restriction to \( s_{0} \) is \( I(F_{0}). \)

We can calculate the annihilator \( T_{S}F \) of \( I_{S}(F) \) which will be, of course, a family of involutive distributions. Indeed we obtain \( T_{S}(F) \) as the intersection of the subsheaves \( TF \) and \( T_{S}(X \times S) \) of \( T(X \times S) \).

In general, given a family of foliations, it is not possible to glue the leaves of the different foliations in the family to higher dimensional leaves. So not every family of foliation comes from an unfolding, as a matter of fact, that is almost never the case.

**Definition 2.6** (Isotriviality). We say the unfolding \( F \) of \( F_{0} \) is isotrivial if it induces a trivial family of Pfaff systems (equivalently of distributions), i.e.: if \( I_{S}(F) = \pi_{1}^{*}I(F_{0}) \) as subsheaves of \( \Omega^{1}_{X \times S|S}, \) where \( \pi_{1} : X \times S \to X \) is the projection and the morphism \( \pi_{1}^{*} : \Omega^{1}_{X} \otimes \mathcal{O}_{X \times S} \to \Omega^{1}_{X \times S} \) is the pull-back of forms.

**Definition 2.7** (Transversality). Given a foliation \( F_{0} \) and an unfolding \( F \) of \( F_{0} \) parametrized by \((S, s_{0})\) we have exact sequences.

\[
\begin{array}{ccc}
0 & \longrightarrow & T_{S}F \\
\downarrow & & \downarrow \\
0 & \longrightarrow & T(F) \\
\downarrow & & \downarrow \\
& N_{S}F & \longrightarrow \ N \longrightarrow 0.
\end{array}
\]

Where \( T_{S}F \) is defined as in Remark 2.5. We say \( F \) is transversal to \( S \) if \( NF/N_{S}F = 0. \)

**Example 2.8.** Let \( X = \mathbb{A}^{n} = \text{Spec}(k[x_{1}, \ldots, x_{n}]) \) and \( S = \mathbb{A}^{1} = \text{Spec}(k[y]) \) be affine spaces. Let \( F_{0} \) be a foliation on \( \mathbb{A}^{n} \) with tangent sheaf \( TF_{0} = k[x_{1}, \ldots, x_{n}][X_{1}, \ldots, X_{r}], \) the \( X_{i} \)’s being vector fields on \( \mathbb{A}^{n}. \)

Lets see what an isotrivial and transversal unfolding \( F \) of \( F_{0} \) parametrized by \( \mathbb{A}^{1} \) should look like.

In the first place, being isotrivial, the subsheaf \( T_{\mathbb{A}^{1}}F \subseteq TF \) will be generated by the vector fields \( X_{1}, \ldots, X_{r}, \) viewed as vector fields on \( \mathbb{A}^{n} \times \mathbb{A}^{1}. \) So we can write \( TF \) as generated by vector fields \( X_{1}, \ldots, X_{r}, Y_{1}, \ldots, Y_{s}. \) Now as \( F \) is an unfolding of \( F_{0}, \) then \( Y_{1}, \ldots, Y_{r} \) must span a space of dimension \( \leq 1 \) on the tangent space to each point. So we have

\[ TF = k[x_{1}, \ldots, x_{n}, y] \cdot (X_{1}, \ldots, X_{r}, Y). \]

Moreover, as \( F \) is transversal over \( \mathbb{A}^{1} \), we can write \( Y = \tilde{Y} + \frac{\partial}{\partial y}, \) where

\[ \tilde{Y} = f_{1}(x, y) \frac{\partial}{\partial x_{1}} + \cdots + f_{n}(x, y) \frac{\partial}{\partial x_{n}}. \]

Then the involutivity of \( TF \) implies that, for each fixed \( y_{0} \in \mathbb{A}^{1} \) we have a vector field \( \tilde{Y}(\cdot, y_{0}) \) on \( \mathbb{A}^{n} \) that verifies

\[ [TF_{0}, \tilde{Y}(\cdot, y_{0})] \subseteq TF_{0}. \]

Note that every family \( \tilde{Y} \) of vector fields on \( \mathbb{A}^{n} \) satisfying the above condition give rise to an isotrivial transversal unfolding of \( F_{0}. \)
Essentially, the same is true for any isotrivial transversal unfolding of a foliation on a non-singular variety $X$, although the role of the family of vector fields $\tilde{Y}$ will be taken by a section of a certain sheaf on the parameter space $S$. The rest of the section is devoted to the generalization of the above example.

**Definition 2.9.** Remember that, given an involutive distribution $TF$, the vector field bracket defines a map $[\cdot,\cdot] : TF \otimes NF \to NF$ known as Bott connection. Using the Bott connection we will define a subsheaf of $N^1F$, which we call $u(F)$, as the subsheaf of $N^1F$ whose local sections on an open set $V \subset X \times S$ are the following:

$$u(F)(V) := \text{Def } \{ s \in N^1S\mathcal{F}(V) \text{ s.t.: } [T_S\mathcal{F}, s] = 0 \}.$$ 

Note that $u(F)$ is not a coherent subsheaf of $N^1S\mathcal{F}$ but only a subsheaf of $k$-vector spaces.

**Notation 1.** We denote $Y(F_0) := \text{Def } \Gamma(X, u(F_0))$.

**Remark 2.10.** The sheaf $u(F)$ inherits from $T_S(X \times S)$ the structure of a sheaf of Lie algebras. Indeed, if $W \subseteq T_S(X \times S)$ is the preimage of $u(F)$ by the projection $T_S(X \times S) \to N^1_S(F)$, then $T_S\mathcal{F}$ is an ideal inside of $W$. In particular $Y(F)$ is a Lie algebra over the field of definition $k$.

The sheaf $u(F)$ will be useful to study the relation between infinitesimal unfoldings (i.e.: those parametrized by the Spec of artinian algebras) and unfoldings parametrized by varieties such as $\mathbb{P}^1$. Let us begin by considering an unfolding $\mathcal{F}$ of a foliation $\mathcal{F}_0$ parametrized by $S$, that is a foliation over $X \times S$. By Remark 2.4 we have a projection $T(X \times S) \to T_S(X \times S)$, from this we get a projection

$$T(X \times S)/T_S\mathcal{F} \to T_S(X \times S)/T_S\mathcal{F} = N^1_S\mathcal{F}.$$ 

We will focus now on the properties of the map $v_F$ one gets by composing

$$TF/T_S\mathcal{F} \to T(X \times S)/T_S\mathcal{F} \to N^1_S\mathcal{F}.$$ 

Note that, when the unfolding is transversal, we have an isomorphism $\pi^1_2TS \cong TF/T_S\mathcal{F}$, so we can consider $v_F$ to be a morphism

$$v_F : \pi^1_2TS \to N^1_S\mathcal{F}.$$ 

**Proposition 2.11.** If $\mathcal{F}$ is transversal to $S$ then $v_F(\pi^{-1}_2TS)$ is a subsheaf of $u(F) \subseteq N^1_S\mathcal{F}$.

**Proof.** Note that the statement is making reference to $\pi^{-1}_2TS \subset \pi^1_2TS$, that is the sheaf of vector fields that are constant along $X$.

So, given a local section $s \in v_F(\pi^{-1}_2TS)$ we need to compute $[T_S\mathcal{F}, s]$. To do this we take a pre-image of $s$, say $\tilde{s} \in TF$. As $p_2 : T(X \times S) \to \pi^1_2TS$ induces an isomorphism between $TF/T_S\mathcal{F}$ and $\pi^1_2TS$ and $s \in v_F(\pi^{-1}_2TS)$, we can take $\tilde{s}$ of the form $Y + Z$ with $Y \in \pi^1_1TX$ and $Z \in \pi^{-1}_1TS$. Given $W \in T_S\mathcal{F}$ we compute

$$[W, \tilde{s}] = [W, Y + Z] = [W, Y] - Z(W),$$

as $W(Z) = 0$, being $Z \in \pi^{-1}_1TS$. Then $[W, \tilde{s}]$ is in $\pi^1_1TX$, and also in $TF$, so it is in $T_S\mathcal{F}$. Therefore $[T_S\mathcal{F}, s] = 0$. \qed
THEOREM 2.12. Let $X$ be a non-singular variety and $\mathcal{F}_0$ a foliation on $X$. There is, for each scheme $S$, a $1$ to $1$ correspondence:

\[
\begin{aligned}
\left\{ \text{isotrivial transversal unfoldings} \quad \text{of } \mathcal{F} \text{ parametrized by } S \right\} & \leftrightarrow \left\{ \text{sections } v \in H^0(S, \Omega^1_S) \otimes \Upsilon(\mathcal{F}_0) \quad \text{s.t.} \quad dv + \frac{1}{2}[v,v] = 0 \right\}.
\end{aligned}
\]

Proof.

Form associated to an unfolding: Given an isotrivial transversal unfolding $\mathcal{F}$ of $\mathcal{F}_0$ we associate to it the map

\[
T\mathcal{F}/T_S\mathcal{F} \to N_S\mathcal{F},
\]

as in eq. (1). As $\mathcal{F}$ is transversal we have $T\mathcal{F}/T_S\mathcal{F} \cong T_X(X \times S) \cong \pi^*_2(TS)$.

So we have a map

\[
v_F : \pi^*_2(TS) \to N_S\mathcal{F}.
\]

By Proposition 2.11, the image of $\pi^*_2^{-1}(TS)$ under $v_F$ is a subsheaf of $\mathcal{U}(\mathcal{F})$. Now, $\mathcal{F}$ being isotrivial implies that $N_S\mathcal{F} \cong \pi^*_1(N\mathcal{F}_0)$ and that $\mathcal{U}(\mathcal{F}) \cong \pi^*_1(\mathcal{U}(\mathcal{F}_0))$.

Then we have a global section $v_F \in H^0(\pi^*_2\Omega^1_S \otimes \pi^*_1(N\mathcal{F}_0))$, by K"unneth isomorphism this is a section in $H^0(\Omega^1_S) \otimes H^0(N\mathcal{F}_0)$ and by Proposition 2.11 this section actually belongs to $H^0(\Omega^1_S) \otimes H^0(\mathcal{U}(\mathcal{F}_0))$. So we get a global form

\[
v_F \in H^0(S, \Omega^1_S) \otimes \Upsilon(\mathcal{F}_0)
\]

associated with an isotrivial transversal unfolding $\mathcal{F}$ parametrized by $S$.

Now to establish the first part of the correspondence we need the following:

Lemma 2.13. Let $\mathcal{F}_0$ be a foliation on a nonsingular variety $X$, and let $\mathcal{F}$ be an isotrivial and transversal unfolding of $\mathcal{F}_0$ parametrized by a variety $S$. The global $1$-form $v_F$ verifies the Maurer-Cartan equation:

\[
dv_F + \frac{1}{2}[v_F, v_F] = 0.
\]

Proof of Lemma 2.13. Set $v = v_F$, now given two vector fields $Y$ and $Z$ defined on $S$ we use Cartan’s formula for the differential

\[
dv(Y,Z) = Y(v(Z)) - Z(v(Y)) - v([Y,Z]).
\]

Now we want to calculate $v([Y,Z])$, for this we look at the definition of $v_F$. Given a local section $Y$ of $T_S \mathcal{F}$ we look at it as a section of $\pi^*(TS)$. As was noted above, transversality of $\mathcal{F}$ gives us an isomorphism $T\mathcal{F}/T_S\mathcal{F} \cong \pi^*_2(TS)$, so we can associate to $Y$ a corresponding global section of $T\mathcal{F}/T_S\mathcal{F}$.

We then apply the restriction of the morphism $p_1 : T(X \times S) \to \pi^*_1(TX)$ of Remark 2.4 to the above section obtaining by Proposition 2.11 a section of $\mathcal{U}(\mathcal{F})$.

So $Y$ viewed as a section of $T(X \times S)$ is of the form $Y + Y'$, with $Y'$ in $\pi^*_1(\mathcal{U}(\mathcal{F}))$. Then we have $Y' = v(Y)$.

To calculate $v([Y,Z])$ we first observe that

\[
[Y,Z] = [\tilde{Y}, \tilde{Z}].
\]
Indeed, the isomorphism between $T\mathcal{F}/T\mathcal{S}$ and $\pi_2^\ast(TS)$ that we use to define $\tilde{Y}$ comes from the inclusion of $T\mathcal{F}$ in $T(X \times S)$, so it respects Lie brackets. So now $\nu([Y,Z])$ is simply $p_1([\tilde{Y},\tilde{Z}])$.

We can write $\tilde{Y} = Y + Y'$ and similarly with $Z$ and, noting that $p_1([Y,Z]) = 0$, compute

$$
p_1([\tilde{Y},\tilde{Z}]) = p_1([Y + Y', Z + Z'])
$$

$$
= [Y', Z'] + p_1([Y, Z']) - p_1([Z, Y']).
$$

Now, as $p_1([Y, Z']) = Y(Z')$ we have

$$
\nu([Y, Z]) = [\nu(Y), \nu(Z)] + Y(\nu(Z)) - Z(\nu(Y)),
$$

from which the Maurer-Cartan equation, and therefore the lemma, follows. □

**Unfolding associated with a form:** Conversely, given $\nu \in H^0(S, \Omega_S^1) \otimes \Upsilon(\mathcal{F}_0)$ we have an associated morphism

$$
\nu : \pi_2^\ast(TS) \to \pi_1^\ast(N\mathcal{F}_0).
$$

So we consider the morphism

$$
\nu \oplus \text{id} : \pi_2^\ast(TS) \to \pi_1^\ast(N\mathcal{F}_0) \oplus \pi_2^\ast(TS)
$$

given by $\nu \oplus \text{id}(s) = (\nu(s), s)$. We also have the projection $\phi : T(X \times S) \to \pi_1^\ast(N\mathcal{F}_0) \oplus \pi_2^\ast(TS)$. So we consider the diagram

$$
\pi_2^\ast(TS) \xrightarrow{\nu \oplus \text{id}} \pi_1^\ast(N\mathcal{F}_0) \oplus \pi_2^\ast(TS) \xleftarrow{\phi} T(X \times S).
$$

We then take $T\mathcal{F}_\nu \subseteq T(X \times S)$ to be the sub-sheaf generated by $\phi^{-1}(\nu(\pi_2^\ast(TS)))$.

The fact that the image of $\nu$ is within $\Upsilon(\mathcal{F}_0)$ and that $\nu$ satisfies the Maurer-Cartan equation implies that $T\mathcal{F}_\nu$ is involutive.

Indeed, let $\tilde{Y}$ and $\tilde{Z}$ be local sections of $T\mathcal{F}_\nu$, we need to check that $[\tilde{Y}, \tilde{Z}]$ is also a section of $T\mathcal{F}_\nu$. So we take have section $Y$ and $Z$ in $\pi_2^\ast(TS)$ such that

$$
\phi(\tilde{Y}) = (\nu(Y), Y)
$$

$$
\phi(\tilde{Z}) = (\nu(Z), Z)).
$$

Moreover, we may assume that $Z$ and $Y$ are local sections of $\pi_2^{-1}(TS)$, as this latter sheaf generates $\pi_2^\ast(TS)$ and the general result will follow from the fact that the Lie bracket is a derivation on both of its inputs. So, given that $Y, Z \in \pi_2^{-1}(TS)$ we have

$$
\phi([\tilde{Y}, \tilde{Z}]) = \phi(\tilde{Y}), \phi(\tilde{Z})) =
$$

$$
= [Y, Z] + Y(\nu(Z)) - Z(\nu(Y)) + [\nu(Y), \nu(Z)] =
$$

$$
= [Y, Z] + \nu([Y, Z]).
$$

So, $T\mathcal{F}_\nu$ is involutive.

Also, by construction, $\mathcal{F}_\nu$ induces a trivial family. Hence $\mathcal{F}_\nu$ is an isotrivial, transversal unfolding of $\mathcal{F}_0$, associated to an $\nu \in H^0(S, \Omega_S^1) \otimes \Upsilon(\mathcal{F}_0)$.

It follows routinely that the constructions of $\nu_F$ and of $\mathcal{F}_\nu$ are inverse to each other. □
We thus see that the space $\Upsilon(\mathcal{F})$ will be an important ingredient in studying isotrivial unfoldings of a foliation $\mathcal{F}$. This space is acted upon by the group of automorphism of the foliation, that is the group 

$$\text{Aut}(\mathcal{F}) := \{g \in \text{Aut}(X) \text{ s.t.: } g_*(T\mathcal{F}) = T\mathcal{F}\}.$$ 

The Lie algebra of this group may be naturally identified with the global sections of the sheaf $\text{aut}(\mathcal{F})$ whose local sections are 

$$\text{aut}(\mathcal{F})(V) := \{\theta \in TX(V) \text{ s.t.: } [T\mathcal{F}, \theta] \subseteq T\mathcal{F}\},$$

so $\text{Lie}(\text{Aut}(\mathcal{F})) = H^0(X, \text{aut}(\mathcal{F}))$.

**Remark 2.14.** Note also that we have a short exact sequence of sheaves 

$$0 \to T\mathcal{F} \to \text{aut}(\mathcal{F}) \to u(\mathcal{F}) \to 0.$$ 

In the particular case when $H^0(X, T\mathcal{F}) = H^1(X, T\mathcal{F}) = 0$, which will be important to us later, we have the equality 

$$\text{Lie}(\text{Aut}(\mathcal{F})) = H^0(X, \text{aut}(\mathcal{F})) = H^0(X, u(\mathcal{F})) = \Upsilon(\mathcal{F}).$$

In particular, in the case where $X$ a complex variety and $H^0(X, T\mathcal{F}) = H^1(X, T\mathcal{F}) = 0$, there is an unfolding associated to the Maurer-Cartan form $\mathcal{F}$, which in this case take values in $\Upsilon(\mathcal{F})$. So there is, by Theorem 2.12, an unfolding associated to the Maurer-Cartan form.

In the situation where $X$ and $S$ be varieties over $\mathbb{C}$, $\mathcal{F}_0$ a foliation on $X$ such that $H^0(X, T\mathcal{F}_0) = H^1(X, T\mathcal{F}_0) = 0$ and $\mathcal{F}$ a transversal isotrivial unfolding of $\mathcal{F}_0$ parametrized by $S$. Denote $\pi: \tilde{S} \to S$ the universal covering of $S$, $\upsilon_{MC}$ the Maurer-Cartan form of $\text{Aut}(\mathcal{F}_0)$. Let $\mathcal{F}_{MC}$ the unfolding associated to the Maurer-Cartan form (cf.: Remark 2.14), and $\upsilon \in \Omega^1_{\tilde{S}} \otimes \Upsilon(\mathcal{F}_0)$ the pull-back of $\upsilon_{\mathcal{F}}$ by the universal covering map. A direct application of Darboux’s existence theorem gives us:

**Corollary 2.15.** With hypotheses as in the above paragraph, there is a morphism $f: \tilde{S} \to \text{Aut}(\mathcal{F}_0)$ such that $\upsilon$ is the pull-back of $\upsilon_{MC}$. Equivalently, $f^*(\mathcal{F}_{MC}) = \pi^*(\mathcal{F})$ as unfoldings of $\mathcal{F}_0$.

**Example 2.16.** Let $\mathcal{F}_0$ be a foliation by curves on $\mathbb{P}^2(\mathbb{C})$ such that $H^0(X, \text{aut}(\mathcal{F})) \neq 0$. Then $T\mathcal{F}_0$ is a line bundle on $\mathbb{P}^2$, so $H^0(X, T\mathcal{F}_0) = H^1(X, T\mathcal{F}_0) = 0$. Also, being a foliation by curves in $\mathbb{P}^2$ implies that $\dim \text{Aut}(\mathcal{F}_0) \leq 1$. So there is an infinitesimal symmetry $Y$ such that $H^0(X, \text{aut}(\mathcal{F})) = (Y)$ and 

$$\text{Aut}(\mathcal{F}_0)^0 = \exp(tY) \cong \mathbb{C}^*,$$

where $\text{Aut}(\mathcal{F}_0)^0$ is the connected component of the identity. In particular, we have that the universal covering $A$ of $\text{Aut}(\mathcal{F}_0)^0$ is isomorphic as a complex variety to $\mathbb{C}$ On $A$ we have the Maurer-Cartan form 

$$\upsilon_{MC} = dz \otimes Y,$$

here we are taking $z$ to be a coordinate of $A \cong \mathbb{C}$ and we are identifying $\text{Lie}(A) = \text{Lie}(\text{Aut}(\mathcal{F}_0)) = (Y)$. 


Considering that $\text{Lie}(\text{Aut}(\mathcal{F}_0)) = \Upsilon(\mathcal{F}_0)$ and applying Theorem 2.12, this gives us the unfolding $\mathcal{F}$ such that:

$$T\mathcal{F} = (\pi_1^* T\mathcal{F}_0 \oplus \left( \frac{\partial}{\partial z} + Y \right))$$

on $\mathbb{P}^2 \times \mathbb{C}$. If $\omega$ is the rational 1-form on $\mathbb{P}^2$ annihilating $T\mathcal{F}_0$, then 1-form annihilating $T\mathcal{F}$ is

\[ \varpi = \omega + \omega(Y) dz \in \Omega^1_{\mathbb{P}^2 \times \mathbb{C}}. \]

so $\mathcal{F}$ posses the integrating factor $\omega(Y)$ (considered as a rational function on $\mathbb{P}^2 \times \mathbb{C}$).

Note that $\omega(Y)$ considered as a rational function on $\mathbb{P}^2 \times \mathbb{C}$ is an integrating factor for $\mathcal{F}_0$.

Remark 2.17. In a sense, the previous results generalize those of Suwa in [9]. There is proven, in the context of codimension 1 foliations on $\mathbb{P}^n$, a correspondence between infinitesimal isotrivial unfoldings (i.e.: isotrivial unfoldings parametrized by $k[x]/(x^2)$) and rational integrating factors of the foliation. In our context this is understood the following way:

An infinitesimal isotrivial unfolding of a foliation $\mathcal{F}$ will be transversal on some open set $\iota : U \hookrightarrow \mathbb{P}^n$. As we have seen, a transversal isotrivial unfolding of $\iota^* \mathcal{F}$ give rise to a global section $s \in \Upsilon(\iota^* \mathcal{F})$. Restricting the open set $U$ further if needed we can take a section $Y$ in $\text{Lie}(\text{Aut}(\iota^* \mathcal{F}))$ representing $s$ modulo $T\iota^* \mathcal{F}$. In other words $Y$ is a rational symmetry of $\mathcal{F}$. It is well known, see [4], that to every rational symmetry of a codimension 1 foliation in a complete variety corresponds a rational integrating factor. Thus we can recover Suwa’s theorem from [9].

3. Foliations on $\mathbb{P}^n$ viewed as unfoldings

In this section we will be interested in foliations on $\mathbb{P}^n(\mathbb{C})$ up to birational equivalence. Moreover we will study the relations of foliations of arbitrary dimension on $\mathbb{P}^n(\mathbb{C})$ with foliations by curves on projective spaces of lower dimension. First we recall an important definition.

Definition 3.1. A codimension $q$ foliation $\mathcal{F}$ on $\mathbb{P}^n$ is said to be of degree $d$ if and only if the associated integrable Pfaff system $I(\mathcal{F})$ have the property that

$$\wedge^q I(\mathcal{F}) \cong \mathcal{O}_{\mathbb{P}^n}(-d - q - 1).$$

Given a foliation $\mathcal{F}$ on $\mathbb{P}^n(\mathbb{C})$ of codimension $q$, we fix a (rational, linear) projection $p : \mathbb{P}^n \dasharrow \mathbb{P}^{q+1}$. Now, let $\text{Gr}^n_{q+1}$ be the Grassmannian of $q + 1$-dimensional linear spaces on $\mathbb{P}^n$, define the open set

$$U = \{ P \in \text{Gr}^n_{q+1} \text{ s.t.: } p|_P : P \to \mathbb{P}^{q+1} \text{ is a (regular) isomorphism} \} \subset \text{Gr}^n_{q+1}. $$

Then we have for every $P \in U$ a foliation in $\mathbb{P}^{q+1}$ given by first restricting $\mathcal{F}$ to $P$ and then applying the isomorphism $p$. Note that if $\mathcal{F}$ is of degree $d$ so is every foliation on $\mathbb{P}^{q+1}$ obtained this way.

In other words, what we are doing here is considering the incidence correspondence

$$Z = \{ (x, P) \text{ s.t.: } x \in P \} \subset \mathbb{P}^n \times \text{Gr}^n_{q+1},$$

intersecting with $\mathbb{P}^n \times U$ gives us

$$Z \cap (\mathbb{P}^n \times U) \cong P \times U$$
and hence we have a diagram

\[
\begin{array}{ccc}
P \times U & \xrightarrow{\pi_1} & \mathbb{P}^n \\
& \searrow \pi_2 & \\
& & U.
\end{array}
\]

So taking the pull-back of \( F \) we have \( \pi_1^* F \) as a foliation on \( P \times U \). Now, we can take the family of foliations over \( U \) induced by \( \pi_1^* F \), as in Section 2. Restricting \( U \) if necessary, we may assume that \( \pi_1^* F \) induces a flat family of involutive distributions over \( \mathbb{P}^{q+1} \), parametrized by \( U \). By the results of [5] such a family defines a morphism between \( U \) and the moduli space of involutive distributions over \( \mathbb{P}^{q+1} \) of codimension \( q \) and degree \( d \). This later space is isomorphic to \( \mathbb{P}(H^0(\mathbb{P}^{q+1}, T\mathbb{P}^{q+1}(d-1))) \).

Then we have a morphism

\( \phi_F : U \rightarrow \mathbb{P}(H^0(\mathbb{P}^{q+1}, T\mathbb{P}^{q+1}(d-1))) \),

the later being a projective space of dimension \( (q+2)(d+q+1) - (d+q) \). In particular if the dimension of \( U \) (which is that of \( \text{Gr}_{q+1}^n \)) is greater than that of the target space, the morphism will have fibers of positive dimension. Moreover if

\[
\text{(3)} \quad \dim U = (q+2)(n-q-1) > (q+2) \left( \binom{d+q+1}{d} - \binom{d+q}{d-1} \right) + n - q - 1,
\]

then all non empty fibers will be of dimension \( \geq n - q - 1 \). Observe that, fixed \( q \) and \( d \), the inequality is satisfied for \( n \) large enough.

Now, suppose \( n, q \) and \( d \) are such that the inequality (3) is satisfied, and let \( S = \phi_F^{-1}(x) \subseteq U \) be a fiber. It is, in particular, a locally closed set of \( \text{Gr}_{q+1}^n \) of dimension \( \geq n - q - 1 \). We can then apply the following classical lemma:

**Lemma 3.2.** Let \( S \subseteq \text{Gr}_{q+1}^n \) be a locally closed set of dimension \( \geq n - q - 1 \). Then \( S \) spans a Zariski dense set in \( \mathbb{P}^n \), i.e.:

\[
V = \bigcup_{P \in S} P \subseteq \mathbb{P}^n
\]

contains an open subset.

In other words, if we set \( Z_S = \{(x, P) \text{ s.t.: } P \in S, x \in P \} \subseteq \mathbb{P}^n \times \text{Gr}_{q+1}^n \) the morphism

\[
Z_S \quad \xrightarrow{\quad} \quad \mathbb{P}^{q+1}
\]

given by the projection to the first factor, is dominant. Moreover, as \( S \subseteq U \), we have an isomorphism

\[
Z_S \quad \rightarrow \quad \mathbb{P}^{q+1} \times S \\
(x, P) \quad \mapsto \quad (p(x), P).
\]
So, summing up, we have a dominant rational morphism
\[ \psi : \mathbb{P}^{q+1} \times S \to \mathbb{P}^n. \]

**Theorem 3.3.** Given a foliation \( F \) of codimension \( q \) and degree \( d \) on \( \mathbb{P}^n \), if the condition
\[
(q + 1)n > (q + 2)\left(\frac{d + q + 1}{d} - \frac{d + q}{d - 1}\right) + (q + 1)^2
\]
is satisfied then there is a locally closed subset \( S \) of \( \text{Gr}_{q+1}^n \) and a dominant rational map \( \psi : \mathbb{P}^{q+1} \times S \to \mathbb{P}^n \) such that the foliation \( \psi^*(F) \) is an isotrivial unfolding of a foliation of dimension 1 on \( \mathbb{P}^{q+1} \). Moreover we can take \( S \) in such a way that \( \psi \) have generically finite fibers.

**Proof.** The above condition is equivalent to eq. (3), and so if it is satisfied we can construct \( \psi \) as above. Moreover, we can take \( S \) to be a subset of a fiber of \( \phi_F \) having dimension exactly \( n - q - 1 \), so \( \psi \) will have generically finite fibres. The isotriviality of \( \psi^*(F) \) follows from the fact that \( X \) is a subset of a fiber of \( \phi_F \), this being a morphism to the moduli space of involutive distributions on \( \mathbb{P}^{q+1} \). \( \square \)

**Corollary 3.4.** Even if condition (3) is not satisfied, we still have a rational map \( \psi : \mathbb{P}^{q+1} \times S \to \mathbb{P}^n \) such that \( \psi^*(F) \) is an isotrivial unfolding of a foliation of dimension 1 on \( \mathbb{P}^{q+1} \), although \( \psi \) may not have generically finite fibers in this case.

**Proof.** Taking a rational linear map \( \gamma : \mathbb{P}^N \to \mathbb{P}^n \) with \( N > n \) and taking the pull-back foliation \( \gamma^*F \) does not change the codimension or the degree, so we can take \( N \gg n \) large enough so eq. (3) is valid for \( (N,q,d) \). Then we apply the above theorem to \( \gamma^*F \) to obtain a dominant map \( \varphi : \mathbb{P}^{q+1} \times S \to \mathbb{P}^N \). The map \( \psi = \gamma \circ \varphi : \mathbb{P}^{q+1} \times S \to \mathbb{P}^n \) is such that \( \psi^*(F) \) is an isotrivial unfolding of a foliation of dimension 1 on \( \mathbb{P}^{q+1} \). \( \square \)

### 4. Generic transversality

By Theorem 3.3 we have a condition under which a foliation on \( \mathbb{P}^n \) admits a generically finite morphism \( \psi : \mathbb{P}^{q+1} \times S \to \mathbb{P}^n \) such that the pull-back is an isotrivial unfolding. Now we investigate conditions under which such an unfolding turns out to be not only isotrivial but also transversal, so we can apply to it the theory of Section 2.

**Definition 4.1.** Let \( F \) be a foliation on \( X \times S \) viewed as an unfolding of foliations on \( X \). We define \( \mathcal{Z}_F \) to be the schematic support of the sheaf \( N_{F}/N_{S,F} \).

The subscheme \( \mathcal{Z}_F \) will be of interest as it is the locus of points where transversality fails.

Recall that \( \text{sing}(F) \), the singular locus of a foliation \( F \) on a scheme \( X \), is defined to be the schematic support of the sheaf \( \text{Ext}^1_X(NF, O_X) \) (local Ext). Similarly, if we have a family of involutive distributions parametrized by a scheme \( S \) its singular locus is the scheme theoretic support of the sheaf \( \text{Ext}^1_X(N_{S,F}, O_{X \times S}) \).

**Lemma 4.2.** Let \( X \) and \( S \) be non-singular varieties. Let \( F \) be an unfolding of foliations on \( X \) parametrized by \( S \). Suppose that \( F \) is an isotrivial unfolding of a foliation \( F_0 \) on \( X \), and that \( \mathcal{Z}_F \cap \text{sing}(F) = \emptyset \). Then \( \mathcal{Z}_F \subseteq \text{sing}(F_0) \times S \).
Proof. As $\mathcal{F}$ is an isotrivial unfolding $\text{sing}(\mathcal{F}_0) \times S$ is the singular locus of the (trivial) family of involutive distributions induced by $\mathcal{F}$. So let $p \notin \text{sing}(\mathcal{F}_0) \times S$, then the localization $(N_S\mathcal{F})_p$ is a free $\mathcal{O}_{X \times S}$-module and the short sequence
\[ 0 \rightarrow T_S\mathcal{F} \otimes k(p) \rightarrow T_S(X \times S) \otimes k(p) \rightarrow N_S\mathcal{F} \otimes k(p) \rightarrow 0 \]
is exact. If $p \notin \text{sing}(\mathcal{F})$ then the sequence
\[ 0 \rightarrow T\mathcal{F} \otimes k(p) \rightarrow T(X \times S) \otimes k(p) \rightarrow N\mathcal{F} \otimes k(p) \rightarrow 0 \]
is exact. On the other hand we always have an immersion
\[ T_S(X \times S) \otimes k(p) \rightarrow T(X \times S) \otimes k(p). \]
Then if $p$ is a point neither in $\text{sing}(\mathcal{F}_0) \times S$ nor in $\text{sing}(\mathcal{F})$ we have an immersion
\[ T_S\mathcal{F} \otimes k(p) \rightarrow T\mathcal{F} \otimes k(p). \]
So we have another immersion
\[ T\mathcal{F}/T_S\mathcal{F} \otimes k(p) \rightarrow T_X(X \times S). \]
As $\dim T\mathcal{F} = \dim T\mathcal{F}_0 + \dim S$ the dimension of the above vector spaces are equal to $\dim S$, so
\[ N\mathcal{F}/N_S\mathcal{F} \otimes k(p) = 0. \]
Then, if $p \notin \text{sing}(\mathcal{F}_0) \times S$ and $p \notin \text{sing}(\mathcal{F})$, the point $p$ is not in $\mathcal{F}_0$. □

Lemma 4.3. Let $X$ and $S$ be non-singular varieties over $\mathbb{C}$, denote $\pi_1, \pi_2$ the projections of $X \times S$ to the first and second factor, respectively. Let $\mathcal{F}$ be an isotrivial unfolding of a foliation $\mathcal{F}_0$ on $X$ parametrized by $S$. Suppose that $\mathcal{F}_0$ is non-singular and that $\dim(\text{sing}(\mathcal{F})) \leq \dim(\mathcal{F}) - 1$. Then $\pi_2|_{\text{sing}(\mathcal{F})} : \text{sing}(\mathcal{F}) \rightarrow S$ is not dominant.

Proof. This follows from Theorem 2.7 of [8], actually we will use the following weaker version of the theorem in [8]:

If we take the reduced structure $\text{sing}(\mathcal{F})^{\text{red}} \subseteq \text{sing}(\mathcal{F})$ then, at a regular point $p$ of $\text{sing}(\mathcal{F})^{\text{red}}$, the image of the map
\[ T\mathcal{F} \otimes k(p) \rightarrow T(X \times S) \otimes k(p) \]
falls within the tangent space to $\text{sing}(\mathcal{F})^{\text{red}}$ at $p$.

To prove our assertion suppose that $\text{sing}(\mathcal{F})$ is dominant over $S$. Take $p$ to be a regular point of $\text{sing}(\mathcal{F})^{\text{red}}$. Then $\text{sing}(\mathcal{F})^{\text{red}}$ is dominant over $S$ as well, so it has dimension at least that of $S$. On the other hand by Theorem 2.7 of [8] the image of
\[ (T\mathcal{F}_0) \otimes k(\pi_1(p)) \cong \pi_1^*(T\mathcal{F}_0) \otimes k(p) \cong T_S\mathcal{F} \otimes k(p) \rightarrow T(X \times S) \otimes k(p) \]
falls within the tangent space to $\text{sing}(\mathcal{F})^{\text{red}}$ at $p$. And, as $\mathcal{F}_0$ is a non-singular foliation, we have that the dimension of that image is that of the leaves of $T\mathcal{F}_0$. As $T_S\mathcal{F}$ is tangent to the fibers of $\pi_2$ the differential of the projection $D\pi_2$ satisfy
\[ D\pi_2(T_S\mathcal{F}) = 0. \]

Then, comparing dimensions of tangent spaces we get
\[ \dim(\text{sing}(\mathcal{F})) \geq \dim \mathcal{F}_0 + \dim S = \dim \mathcal{F}, \]
obtaining a contradiction, so $\text{sing}(\mathcal{F})$ cannot be dominant over $S$. □
Putting together the last two lemmas we obtain the following.

**Proposition 4.4.** Let $\mathcal{F}$ be an isotrivial unfolding of a foliation $\mathcal{F}_0$ on a non-singular variety $X$ parametrized by a non-singular variety $S$, such that $\dim(\text{sing}(\mathcal{F})) \leq \dim(\mathcal{F}) - 1$. Set $Y = X \setminus \text{sing}(\mathcal{F}_0)$. Then there is an open set $U \subset S$ such that the restriction of $\mathcal{F}$ to $Y \times U$ is a transversal isotrivial unfolding.

If we apply the above proposition to the situation of Theorem 3.3 we get:

**Corollary 4.5.** Let $\mathcal{F}$ be a foliation of codimension $q$ and degree $d$ on $\mathbb{P}^n$. Suppose $\dim(\text{sing}(\mathcal{F})) \leq \dim(\mathcal{F}) - 1$ and the condition 3 is satisfied.

Then there is a locally closed subset $U$ of $\text{Gr}_{q+1}$, an open set $Y \subseteq \mathbb{P}^{q+1}$ and a dominant rational map $\psi : Y \times U \to \mathbb{P}^n$ such that the foliation $\psi^*(\mathcal{F})$ is an isotrivial and transversal unfolding of a foliation of dimension 1 on $Y$.

Moreover we can take $U$ in such a way that $\psi$ have generically finite fibers and $Y$ such that $\mathbb{P}^{q+1} \setminus Y$ is of codimension at least 2.

**Proof.** Under this conditions we can apply Theorem 3.3 and Proposition 4.4. So we can restrict the foliation $\mathcal{F}_0$ on $\mathbb{P}^{q+1}$, that we obtain by Theorem 3.3 to the open set $Y \subseteq \mathbb{P}^{q+1}$, where $Y = \mathbb{P}^{q+1} \setminus \text{sing}(\mathcal{F}_0)$. Observe that, with the hypotheses on the singularities of $\mathcal{F}$, we can take $\mathcal{F}_0$ so that $\mathbb{P}^{q+1} \setminus Y = \text{sing}(\mathcal{F}_0)$ is of codimension at least 2.

We obtain in such a manner an isotrivial (not necessarily transversal) unfolding of a foliation on $Y$ parametrized by a variety $X$. Then we can apply Proposition 4.4 to the unfolding in $Y \times X$ and restrict it to an unfolding parametrized by an open set $U \subseteq X$ such that the restriction will be isotrivial and transversal. □

5. Unfoldings of Foliations by curves

Now we are in conditions of applying the results of Section 2 to the situation in Corollary 4.5. Indeed, we have in Corollary 4.5 a transversal isotrivial unfolding of a foliation $\mathcal{F}_0$ of dimension 1 on a variety $Y$. The variety $Y$ is an open subset of $\mathbb{P}^{q+1}$ such that $\dim(\mathbb{P}^{q+1} \setminus Y) \leq q - 1$, and $\mathcal{F}_0$ is the restriction to $Y$ of a foliation $\mathcal{F}_0$ on $\mathbb{P}^{q+1}$.

Then we have

**Proposition 5.1.** Let $\mathcal{F}$ be a foliation of codimension $q$ and degree $d$ on $\mathbb{P}^n$. Suppose $\dim(\text{sing}(\mathcal{F})) \leq \dim(\mathcal{F}) - 1$ and the condition 3 is satisfied. Then we have the following alternative:

(1) Either there exist a map $\psi : S \times \mathbb{P}^{q+1} \to \mathbb{P}^n$ with finite generic fiber such that $\psi^*\mathcal{F}$ is pull-back of a foliation by curves on $\mathbb{P}^{q+1}$, or

(2) For every linear subspace $P \subseteq \mathbb{P}^n$ of dimension $q + 1$, the restriction $\mathcal{F}|_P$ is a foliation with rational infinitesimal symmetries.

**Proof.** As condition 3 is satisfied, for a generic linear subspace $P \subseteq \mathbb{P}^n$ of dimension $q + 1$ we have a rational morphism $\psi : S \times \mathbb{P}^{q+1} \to \mathbb{P}^n$ with generically finite fibers such that $\psi^*\mathcal{F}$ is an isotrivial unfolding. Let write $\mathcal{F}|_P$ for the pull-back of $\mathcal{F}$ with respect to the inclusion $P \subseteq \mathbb{P}^n$. 

If $\mathcal{F}|_P$ have no rational symmetries, then the sheaf $\text{aut}(\mathcal{F}|_P)$ is trivial. Then, because of the short exact sequence
\[
0 \to T\mathcal{F}|_P \to \text{aut}(\mathcal{F}|_P) \to u(\mathcal{F}|_P) \to 0,
\]
also $u(\mathcal{F}|_P) = 0$. As was said at the beginning of this section, by Corollary 4.3 we can restrict the unfolding $\psi^*\mathcal{F}$ to $Y \times S$ in such a way that the unfolding is now transversal and isotrivial on $Y \times S$. Note that $u(\mathcal{F}|_Y) = u(\mathcal{F}|_P)|_Y = 0$. So the unfolding is trivial and thus $\psi^*\mathcal{F}$ is pull-back of a foliation by curves on $\mathbb{P}^{q+1}$.

Otherwise, for every $P \subseteq \mathbb{P}^n$, we have that $\mathcal{F}|_P$ possess a rational symmetry.

In particular, if $q = 1$, we have the following:

**Theorem 5.2.** Let $\mathcal{F}$ be a foliation of codimension 1 and degree $d$ on $\mathbb{P}^n(\mathbb{C})$. Suppose $\dim(\text{sing}(\mathcal{F})) \leq n - 2$ and the condition
\[
2(n - 2) > 3 \left(\frac{d + 2}{d}\right) - \left(\frac{d + 1}{d - 1}\right)
\]
is satisfied. Then we have the following alternative:

1. Either there exist a map $\psi : S \times \mathbb{P}^2 \to \mathbb{P}^n$ with finite generic fiber such that $\psi^*\mathcal{F}$ is pull-back of a foliation by curves on $\mathbb{P}^2$, or

2. There are holomorphic varieties $\tilde{S}$ and $\tilde{Y}$ and a meromorphic map with discrete (not necessarily finite) generic fiber, $\phi : \tilde{S} \times \mathbb{P}^2 \to \mathbb{P}^n$, such that $\phi^*\mathcal{F}$ has a meromorphic first integral.

**Proof.** Indeed, for codimension $q = 1$ the inequality in the hypotheses is equivalent to eq. [3] and so we can apply Proposition 5.1. Then, either we have the map $\psi$, or we have that for every 2-dimensional linear subspace $P \subset \mathbb{P}^n$ the restriction $\mathcal{F}|_P$ is a foliation with rational symmetries.

In the second case $\mathcal{F}|_P$, has a rational integrating factor $f$. Then, by [2], if we take $p : \tilde{Y} \to P \setminus \text{Div}(f)$ to be the universal covering map, $p^*\mathcal{F}|_P$ has a first integral.

Then $(p \times \text{id})^*\phi^*\mathcal{F}$ is a transversal isotrivial unfolding of a foliation over $\tilde{Y}$ parametrized by $S$. It is an unfolding of a foliation with a first integral. As $(p \times \text{id})^*\phi^*\mathcal{F}$ is the unfolding of a foliation with a first integral defined on a simply connected space, then by [9, 5.3], $(p \times \text{id})^*\phi^*\mathcal{F}$ has itself a first integral (Suwa’s original result implies the existence of a local first integral, which we can extend globally on account of $Y$ being simply connected).

The following important corollary of this theorem, and its proof, were communicated to the author by Jorge Vitória Pereira:

**Corollary 5.3.** Let $\mathcal{F}$ be a codimension 1 foliation on $\mathbb{P}^n(\mathbb{C})$. Then either $\mathcal{F}$ is the pull-back of a foliation in a surface or there are holomorphic varieties $\tilde{S}$ and $\tilde{Y}$ and a meromorphic map with discrete (not necessarily finite) generic fiber, $\phi : \tilde{S} \times \mathbb{P}^2 \to \mathbb{P}^n$, such that $\phi^*\mathcal{F}$ has a meromorphic first integral.
Proof. Given $\mathcal{F}$, say of degree $d$, we can take the pull-back of $\mathcal{F}$ by a linear projection $f : \mathbb{P}^N \to \mathbb{P}^n$ with $N \gg n$ big enough for the inequality in Theorem 5.2 to be satisfied. Then we can apply Theorem 5.2 to $\mathcal{G} := f^* \mathcal{F}$.

If the first alternative of the theorem holds, then through a generic point of $\mathbb{P}^N$ there is a codimension 2 algebraic variety tangent to $\mathcal{G}$. This implies, by [3] Lemma 2.4], that $\mathcal{G}$ is the pull-back of a foliation on a surface.

If the second alternative holds, we have the map $\phi$ as in the theorem, such that $\phi^* \mathcal{G}$ has a meromorphic first integral.

In any case, taking a generic $\mathbb{P}^n \subset \mathbb{P}^N$ we recover the original foliation $\mathcal{F}$, obtaining thus the corollary. □

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