A Galileon Primer

Thomas Curtright§, David Fairlie⊠, and Hassan Alshal§,△

§ Department of Physics, University of Miami
  Coral Gables, Florida 33124-8046, USA
⊠ Department of Mathematical Sciences, University of Durham
  Durham, DH1 3LE, UK
△ Department of Physics, Faculty of Science, Cairo University
  Giza, 12613, Egypt

curtright@physics.miami.edu  dma6dbf@durham.ac.uk  halshal@sci.cu.edu.eg

Abstract

Elementary features of galileon models are discussed at an introductory level. Following a simple example, a general formalism leading to a hierarchy of field equations and Lagrangians is developed for flat spacetimes. Legendre duality is discussed. Implicit and explicit solutions are then constructed and analyzed in some detail. Galileon shock fronts are conjectured to exist. Finally, some interesting general relativistic effects are studied for galileons coupled minimally to gravity. Spherically symmetric galileon and metric solutions with naked curvature singularities are obtained and are shown to be separated from solutions which exhibit event horizons by a critical curve in the space of boundary data.
Contents

1 Introduction 3

2 A simple example 3

3 General formalism 5
  3.1 Determinant and trace identities 5
  3.2 Field equations 5
  3.3 Lagrangians 5
  3.4 Universal field equations 5
  3.5 More on determinant identities 9
  3.6 Legendre transformations 11
  3.7 Legendre self-dual models 12
  3.8 Legendre transformations of the action 12
  3.9 Hidden symmetry 14
  3.10 Summary of results for Euclidean metrics 15

4 Classical solutions 16
  4.1 Implicit solutions 16
  4.2 Envelope method 17
  4.3 Power law solutions in other dimensions and Legendre equivalences 18
  4.4 Self-Dual Solutions 19

5 Mixtures 20
  5.1 Symmetric spacetime solutions 20
  5.2 Static, spherically symmetric solutions 21
  5.3 Energy considerations 23
  5.4 Perturbative scattering 24

6 Effects of $\phi\Theta[\phi]$ self-couplings 25
  6.1 A non-vanishing trace 25
  6.2 A model with additional quartic self-coupling 26
  6.3 Static solutions 26
  6.4 Energy considerations again 28
  6.5 Scalar geons and a shock-front conjecture 30
  6.6 Comparison to the self-dual model 30

7 General relativistic effects 33
  7.1 Minimal coupling to gravity 33
  7.2 Static spherical solutions 33
  7.3 Numerical results 35
  7.4 Censored and naked phases 40

8 Conclusions 41
1 Introduction

Galileon theories are a class of models for hypothetical scalar fields whose Lagrangians involve multilinearss of first and second derivatives, but whose nonlinear field equations are still only second order. They may be important for the description of large-scale features in astrophysics as well as for elementary particle theory [11, 15]. Hierarchies of galileon Lagrangians were discussed mathematically for flat spacetime in [18, 19, 20], independently of an earlier systematic survey of second-order scalar-tensor field equations in curved 4D spacetime [25]. The simplest example involves a single scalar field, $\phi$. This galileon field may be coupled “universally” to the trace of the energy-momentum tensor, $\Theta$, and upon so doing, it is gravitation-like by virtue of the similarity between this universal coupling and that of the metric $g_{\mu\nu}$ to $\Theta_{\mu\nu}$ in general relativity. As might be expected from this similarity and the ubiquitous generation of scalar fields by the process of dimensional reduction, it is possible to obtain some galileon models from limits of higher dimensional gravitation theories. Indeed, galileon models were discovered yet again by this process [13].

2 A simple example

Although higher derivative actions usually lead to higher derivative field equations, as is well-known, nonetheless it is possible to accommodate higher derivatives in the action while retaining second-order field equations if the Lagrangian is not quadratic in the fields. The price to be paid is that the second-order field equations are always nonlinear. This is the basic ingredient that underlies all galileon models.

This point has already been well-appreciated in the literature, of course, but for purposes of illustration, consider the “simplest” cubic, fourth-order Lagrangian density,

$$L = \phi^2 \phi_{\alpha\alpha\beta\beta}, \quad (1)$$

where $\phi_{\alpha} = \partial \phi / \partial x^\alpha$, etc., and repeated indices are summed using either Lorentzian or Euclidean signatures.

The corresponding action $A = \int L$ has local variation,

$$\frac{\delta A}{\delta \phi} = 2 \phi \phi_{\alpha\alpha\beta\beta} + (\phi^2)_{\alpha\alpha\beta\beta}. \quad (2)$$

Surface terms, irrelevant for the field equations in the bulk, have been discarded. That is to say,

$$\frac{\delta A}{\delta \phi} = 4 \phi \phi_{\alpha\alpha\beta\beta} + 8 \phi_{\alpha} \phi_{\alpha\beta\beta} + 2 \phi_{\alpha\alpha} \phi_{\beta\beta} + 4 \phi_{\alpha\beta} \phi_{\alpha\beta}, \quad (3)$$

and the field equation, $\delta A / \delta \phi = 0$, is both nonlinear and fourth-order.

However, the order of the field equation may be reduced by adding to the Lagrangian a judicious amount of the “next-to-simplest” cubic term that involves first, second, or third derivatives. So far as the field equations are concerned, there is actually only one other term that can be added, namely, $\phi \phi_{\alpha\alpha} \phi_{\beta\beta}$. Superficially different terms, e.g. $\phi \phi_{\alpha} \phi_{\alpha\beta\beta}$, $\phi \phi_{\alpha\alpha\beta} \phi_{\beta\beta}$, $\phi_{\alpha\alpha\beta} \phi_{\alpha\beta\beta}$, and $\phi_{\alpha\beta} \phi_{\alpha\beta}$, do not give independent contributions to the local variation of the action in the bulk, although they differ in their surface contributions. In particular,

$$2 \phi \phi_{\alpha\alpha} \phi_{\beta\beta} + 4 \phi \phi_{\alpha\beta} \phi_{\alpha\beta} - 3 \phi^2 \phi_{\alpha\alpha\beta\beta} = (4 \phi \phi_{\alpha} \phi_{\alpha\beta} + 2 \phi \phi_{\beta} \phi_{\alpha\alpha} - 3 \phi^2 \phi_{\alpha\alpha\beta\beta} - 2 \phi_{\alpha} \phi_{\alpha} \phi_{\beta \beta}), \quad (4)$$

i.e. a total divergence. Thus it is sufficient to include in the action any two of the three terms on the LHS, with an arbitrary relative coefficient.

So, rather than $\phi^2$, consider instead the Lagrangian density

$$L = \phi^2 \phi_{\alpha\alpha\beta\beta} - \lambda \phi \phi_{\alpha\alpha} \phi_{\beta\beta}, \quad (5)$$

with constant $\lambda$. The variation of the action obtained from $\delta L$ is

$$\frac{\delta A}{\delta \phi} = 2 \phi \phi_{\alpha\alpha\beta\beta} + (\phi^2)_{\alpha\alpha\beta\beta} - \lambda \left( \phi_{\alpha\alpha} \phi_{\beta\beta} + 2 \phi \phi_{\alpha\alpha} \phi_{\beta\beta} \right) = (4 - 2\lambda) \left( \phi \phi_{\alpha\alpha\beta\beta} + 2 \phi_{\alpha} \phi_{\alpha\beta\beta} \right) + (2 - 3\lambda) \phi_{\alpha\alpha} \phi_{\beta\beta} + 4 \phi_{\alpha\beta} \phi_{\alpha\beta}. \quad (6)$$

Thus $\lambda = 2$ uniquely eliminates from the variation all derivatives higher than the second, leaving just

$$\left. \frac{\delta A}{\delta \phi} \right|_{\lambda=2} = -4 \left( \phi_{\alpha\alpha} \phi_{\beta\beta} - \phi_{\alpha\beta} \phi_{\alpha\beta} \right). \quad (7)$$
While this equation is still nonlinear, it is now only second-order.

Moreover, the action for the $\lambda = 2$ model can be rewritten in various ways upon integrating by parts. Perhaps the most compact and memorable of these is

$$A_2 = \int \phi_\alpha \phi_\alpha \phi_\beta d^n x .$$

(8)

This differs from the previous $A|_{\lambda=2}$ by a factor of 2 and a boundary term,

$$A_2 = \frac{1}{2} \int (\phi^2 \phi_\alpha \phi_\beta - 2 \phi \phi_\alpha \phi_\beta) d^n x - \frac{1}{2} \int \partial_\alpha B_\alpha d^n x ,$$

involving the current

$$B_\alpha = \phi^2 \overrightarrow{\partial_\alpha} \phi_\beta = \phi^2 \phi_\alpha \phi_\beta - 2 \phi \phi_\alpha \phi_\beta .$$

(9)

Indeed, most discussions of this model are developed around $A_2$, after defining the system’s Lagrangian to be

$$L_2 = \phi_\alpha \phi_\alpha \phi_\beta .$$

(10)

To complete our discussion of this elementary case, consider the energy-momentum density arising from $L_2$. The canonical result is straightforwardly obtained, even though the Lagrangian involves higher derivatives, but the resulting density is not a symmetric tensor. However, minimal coupling to gravity is guaranteed to yield a symmetric tensor, so we take that route. Generally covariant forms of (10) and (8) are obtained through the replacements $\phi_\alpha \phi_\alpha d^n x \rightarrow g^{\alpha\beta} \phi_\alpha \phi_\beta \sqrt{-g} d^n x$ and $\phi_\alpha \phi_\beta \rightarrow \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \phi_\nu)$. Thus an invariant action is

$$A_2|_{\text{curved space}} = \int g^{\alpha\beta} \phi_\alpha \phi_\beta \partial_\mu (\sqrt{-g} g^{\mu\nu} \phi_\nu) d^n x .$$

(11)

Varying the metric gives $\Theta_{\alpha\beta}$. In the flat-space limit, the result is

$$\Theta_{\mu\nu} = \phi_\mu \phi_\nu \phi_\alpha \phi_\alpha - \phi_\alpha \phi_\alpha \phi_\mu \phi_\mu - \phi_\alpha \phi_\alpha \phi_\mu \phi_\nu + \delta_{\mu\nu} \phi_\alpha \phi_\beta \phi_\alpha \phi_\beta .$$

(12)

This is seen to be conserved

$$\partial_\mu \Theta_{\mu\nu} = \phi_\nu \mathcal{E}_2 ,$$

(13)

upon using the field equation, $\mathcal{E}_2 = 0$, where

$$\mathcal{E}_2 \equiv \phi_\alpha \phi_\beta \phi_\alpha \phi_\beta .$$

(14)

The justification for the name “galileon” is now apparent. Any shift of the field by a constant, or by a term linear in $x$, as is reminiscent of a galilean transformation in classical mechanics, will leave the action (8) invariant, up to surface terms, and therefore not falsify a solution of the field equation.

An interesting wrinkle now appears: $\Theta_{\mu\nu}$ is not traceless on-shell. Consequently, the usual form of the scale current, $x_\mu \Theta_{\alpha\mu}$, is not conserved. On the other hand, the action (8) is homogeneous in $\phi$ and its derivatives, and is clearly invariant under the scale transformations $x \rightarrow sx$ and $\phi (x) \rightarrow s^{(4-n)/3} \phi (sx)$. Hence the corresponding Noether current must be conserved. This current is easily found, at least in four dimensions, since the trace is obviously a total divergence in that case:

$$\Theta_{\mu\nu}|_{n=4} = \phi_\alpha \phi_\beta \phi_\alpha \beta + 2 \phi_\beta \phi_\beta \phi_\alpha = \partial_\alpha (\phi_\alpha \phi_\beta \phi_\beta) .$$

(15)

That is to say, for $n = 4$ the virial is the trilinear $V_\alpha = \phi_\alpha \phi_\beta \phi_\beta$. So a conserved scale current is given by the combination,

$$S_{\mu}|_{n=4} = x_\alpha \Theta_{\alpha\mu} - \phi_\alpha \phi_\alpha \phi_\mu .$$

(16)

However, the virial here is not a divergence modulo a conserved current. So the theory is not conformally invariant despite being scale invariant [27].

Some additional algebra is needed for $n \neq 4$, but eventually one finds:

$$S_\mu = x_\alpha \Theta_{\alpha\mu} - V_\mu ,$$

(17)

$$V_\mu = \frac{n - 1}{3} \phi_\alpha \phi_\alpha \phi_\mu + \frac{4 - n}{3} \phi J_\mu ,$$

(18)

$$\partial_\mu S_\mu = \left[ x_\alpha \phi_\alpha + \frac{n - 4}{3} \phi \right] \mathcal{E}_2 .$$

(19)
The last term in the virial $V_{\mu}$ for $n \neq 4$ involves a bilinear current which is conserved merely as a restatement of the field equation:

$$J_\mu = \phi_\mu \overrightarrow{\partial_\nu} \phi_\alpha , \quad \mathcal{E}_2 = \partial_\mu J_\mu .$$

(20)

In fact, this current is itself a total divergence,

$$J_\mu = \partial_\nu (\delta_{\mu \nu} \phi \phi_\alpha - \phi \phi_{\mu \nu}) ,$$

(21)

so the field equation for the model is a double divergence for any $n$. But once again, although the model is scale invariant in any number of dimensions, it is not conformally invariant for $n > 2$.

Since (10) has the form of the conventional free field Lagrangian density $\times$ the Klein-Gordon equation, for a massless scalar field, it immediately suggests a generalization to a hierarchy of such systems, where the Lagrangian density for the $k$th system is just a product of the free field Lagrangian density and the equation of motion for the $(k-1)$st system. In fact, this simple generalization is easy to formulate in explicit detail. A systematic theory for the hierarchy is elegantly expressed using determinants.

### 3 General formalism

This section may be skipped by anyone with a phobia for determinants, and definitely should be passed over by anyone under a doctor’s orders to cut back on tensor index shuffling. Later sections of the paper rarely invoke results obtained in this section. However, the material presented here may be helpful for applications beyond those considered in this primer. Accordingly, the last part (§3.9) provides an encapsulation of the results in terms of determinants and standard Kronecker symbols.

#### 3.1 Determinant and trace identities

For any $n \times n$ matrix $M$, consider the expansion

$$\det (1 + \lambda M) = \sum_{k=0}^{n} \frac{\lambda^k}{k!} \mathcal{E}_k (M) ,$$

(22)

where $\mathcal{E}_0 \equiv 1$. Elementary cases are $\mathcal{E}_1 = \text{Tr} (M)$ and $\mathcal{E}_n = n! \det (M)$. Other cases may not be so familiar. However, from the identity

$$\det (1 + \lambda M) = \exp (\text{Tr} \ln (1 + \lambda M)) ,$$

(23)

it follows that the $\mathcal{E}_k$ obey a recursion relation for any $M$,\n
$$\frac{1}{k!} \mathcal{E}_k = \sum_{\ell=0}^{k-1} \frac{(-1)^{k-1-\ell}}{\ell!} T_{k-\ell} \mathcal{E}_\ell , \quad \text{for } k \leq n , \quad \text{where} \quad T_m \equiv \text{Tr} (M^m) .$$

(24)

The solution of this recursion for all $k \leq n$ can be expressed in terms of another set of determinants,

$$\mathcal{E}_k = \det (T_k) ,$$

(25)

where $T_k$ is an auxiliary $k \times k$ matrix containing the various traces:

$$T_k = \begin{pmatrix}
T_1 & k-1 & 0 & \cdots & 0 & 0 & 0 \\
T_2 & T_1 & k-2 & \cdots & 0 & 0 & 0 \\
T_3 & T_2 & T_1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
T_{k-2} & T_{k-3} & T_{k-4} & \cdots & T_1 & 2 & 0 \\
T_{k-1} & T_{k-2} & T_{k-3} & \cdots & T_2 & T_1 & 1 \\
T_k & T_{k-1} & T_{k-2} & \cdots & T_3 & T_2 & T_1
\end{pmatrix} .$$

(26)

The recursion relation (24) is recovered by expanding $\det (T_k)$ in the minors of the first column. In addition to (25) we also note the identity

$$k \mathcal{E}_{k-1} = \text{Tr} (\text{adj} T_k) .$$

(27)
In this last trace relation, we use the adjugate (a.k.a. the classical adjoint) matrix notation, \( \text{adj}(T) = (\det(T))^{-1} \).

For example,

\[
\mathcal{E}_1 = \det T_1 = T_1,
\]

\[
\mathcal{E}_2 = \det T_2 = \det \begin{pmatrix}
T_1 & 1 \\
T_2 & T_1
\end{pmatrix} = T_1^2 - T_2,
\]

\[
\mathcal{E}_3 = \det T_3 = \det \begin{pmatrix}
T_1 & 2 & 0 \\
T_2 & T_1 & 1 \\
T_3 & T_2 & T_1
\end{pmatrix} = T_1^3 - 3T_1T_2 + 2T_3,
\]

\[
\mathcal{E}_4 = \det T_4 = \det \begin{pmatrix}
T_1 & 3 & 0 & 0 \\
T_2 & T_1 & 2 & 0 \\
T_3 & T_2 & T_1 & 1 \\
T_4 & T_3 & T_2 & T_1
\end{pmatrix} = T_1^4 - 6T_1^2T_2 + 8T_1T_3 + 3T_2^2 - 6T_4,
\]

etc. Actually, for an \( n \times n \) matrix \( M \), explicit computation of the traces inserted into the expression \( (32) \) gives a null determinant for \( k > n \). That is, \( \mathcal{E}_{k>n} = 0 \), as would be expected from the expansion of \( \det(1 + \lambda M) \).

Moreover, a slight modification of the auxiliary matrix in \( (26) \) gives directly the characteristic polynomial for any \( n \times n \) matrix \( M \),

\[
det(M - \lambda \mathbf{1}) = \frac{1}{n!} \det \begin{pmatrix}
1 & n & 0 & \cdots & 0 & 0 & 0 \\
\lambda & T_1 & n - 1 & \cdots & 0 & 0 & 0 \\
\lambda^2 & T_2 & T_1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\lambda^{n-2} & T_{n-2} & T_{n-3} & \cdots & T_1 & 2 & 0 \\
\lambda^{n-1} & T_{n-1} & T_{n-2} & \cdots & T_2 & T_1 & 1 \\
\lambda^n & T_n & T_{n-1} & \cdots & T_3 & T_2 & T_1
\end{pmatrix}.
\]

This follows immediately from expanding in the minors of the first column, using \( (22) \) and \( (24) \).

From the elementary identity \( \det(AB) = (\det A)(\det B) \) we have

\[
\det(1 + \lambda M) = \lambda^n (\det M) \det(1 + \lambda^{-1}M^{-1}) ,
\]

for an \( n \times n \) nonsingular \( M \). It follows for \( 0 \leq k \leq n \) that

\[
\frac{1}{k!} \mathcal{E}_k(M) = (\det(M)) \frac{1}{(n-k)!} \mathcal{E}_{n-k}(M^{-1}) .
\]

For example,

\[
\mathcal{E}_n(M) = n! (\det M) ,
\]

\[
\mathcal{E}_{n-1}(M) = (n-1)! (\det M) \text{ Tr}(M^{-1}) = (n-1)! \mathcal{E}_1(\text{adj} M) ,
\]

etc. Or, to rewrite \( (34) \) more symmetrically, for \( n \times n \) nonsingular \( M \),

\[
\frac{1}{\sqrt{\det(M)}} \frac{1}{k!} \mathcal{E}_k(M) = \frac{1}{\sqrt{\det(M^{-1})}} \frac{1}{(n-k)!} \mathcal{E}_{n-k}(M^{-1}) .
\]

### 3.2 Field equations

Take \( M \) to be \( \mathcal{H} = \partial \partial \phi \), the Hessian matrix of second partial derivatives of \( \phi(x_1, \cdots, x_n) \), then

\[
det(1 + \lambda \partial \partial \phi) = \sum_{k=0}^{n} \frac{\lambda^k}{k!} \mathcal{E}_k(\partial \partial \phi) .
\]

For \( k \geq 1 \) also define “the equation of motion at level \( k \)” as \( \mathcal{E}_k(\partial \partial \phi) = 0 \). For example, with \( \phi_{\alpha\beta} = \partial_\alpha \partial_\beta \phi \),

\[
\mathcal{E}_1(\partial \partial \phi) = \text{Tr}(\partial \partial \phi) = \phi_{\alpha\alpha} ,
\]

\[
\mathcal{E}_2(\partial \partial \phi) = (\text{Tr}(\partial \partial \phi))^2 - \text{Tr}((\partial \partial \phi)^2) = \phi_{\alpha\alpha} \phi_{\beta\beta} - \phi_{\alpha\beta} \phi_{\alpha\beta} ,
\]

etc.
etc., while at the highest levels, in $n$ dimensions,

$$\mathcal{E}_n (\partial \partial \phi) = n! \det (\partial \partial \phi) ,$$  \hspace{1cm} (41)

$$\mathcal{E}_{n-1} (\partial \partial \phi) = (n-1)! \det (\partial \partial \phi) \text{ Tr} \left( (\partial \partial \phi)^{-1} \right) = (n-1)! \mathcal{E}_1 (\text{adj} (\partial \partial \phi)) ,$$  \hspace{1cm} (42)

etc. We shall refer to the $\mathcal{E}_n (\partial \partial \phi) = 0$ case as the “maximal” galileon field equations.

### 3.3 Lagrangians

These may be defined recursively and yield the above equations of motion after varying $\phi$ and integrating by parts:

$$\mathcal{L}_k = \phi_\alpha \phi_\alpha \mathcal{E}_{k-1} (\partial \partial \phi) , \quad \delta \int \mathcal{L}_k \, d^n x = -2 \int \mathcal{E}_k (\partial \partial \phi) \, \delta \phi \, d^n x .$$  \hspace{1cm} (43)

Thus, stationarity of the action for $\mathcal{L}_k$ implies the equation of motion:

$$0 = \mathcal{E}_k (\partial \partial \phi) .$$  \hspace{1cm} (44)

A systematic method to obtain this recursion is to work out the variation of

$$A = \int \phi_\alpha \phi_\alpha \det (1 + \lambda \partial \partial \phi) \, d^n x ,$$  \hspace{1cm} (45)

and then use (22) to single out the action for $\mathcal{L}_k$. More generally, with

$$A [F] = \int F (\phi_\alpha \phi_\alpha) \det (1 + \lambda \partial \partial \phi) \, d^n x ,$$  \hspace{1cm} (46)

we find

$$\delta A [F] = -2 \int \mathcal{E} [F] \, \delta \phi \, d^n x ,$$  \hspace{1cm} (47)

where

$$\mathcal{E} [F] = \det (1 + \lambda \partial \partial \phi) \left\{ \left( \phi_\alpha - \lambda (1 + \lambda \partial \partial \phi)_{\mu \nu}^{-1} \phi_\mu \phi_\nu \right) F' + \left( \phi_\mu (1 + \lambda \partial \partial \phi)_{\mu \nu}^{-1} \partial_\nu (\phi_\alpha \phi_\alpha) \right) F'' \right\} .$$  \hspace{1cm} (48)

Setting $F (\phi_\alpha \phi_\alpha) = \phi_\alpha \phi_\alpha$, i.e. $F' = 1$ and $F'' = 0$, and expanding the RHS of (48) in powers of $\lambda$ leads to

### 3.4 Universal field equations

Given in $n$ dimensions an arbitrary Lagrangian dependent only upon first derivatives of the field, $\phi$, and homogeneous of weight one, there is an iterative procedure for calculating a sequence of equations of motion which always terminates with the same final equation, $\mathcal{U}_n = 0$, independent of the starting Lagrangian. This final equation has therefore been called a “universal field equation” (UFE). It involves only first and second derivatives of $\phi$. Here we describe the relation between $\mathcal{U}_n$ and the galileon Lagrangian $\mathcal{L}_n$ in $n$ dimensions.

The functional form appearing in the UFE in $n$ dimensions can be expressed as a “bordered determinant”

$$\mathcal{U}_n [\phi] = \det \left( \begin{array}{cc} 0 & \partial \phi \\ \partial \phi & \partial \partial \phi \end{array} \right) ,$$  \hspace{1cm} (49)

where the entries in the top row, and in the left column, are 0 and $\phi_\alpha$ for $\alpha = 1, \cdots, n$, and the Hessian matrix occupies the $n \times n$ block on the lower right. However, unlike $\mathcal{E}_k (\partial \partial \phi)$ that appears in the galileon field equations, $\mathcal{U}_n$ is not identical to a total divergence, so the integral $\int \mathcal{U}_n [\phi] \, d^n x$ can serve to specify nontrivial dynamics in the bulk.

---

1In two dimensions, the UFE is just the Bateman equation, $\phi_{xx} \phi_t^2 - 2 \phi_{xt} \phi_x \phi_t + \phi_{tt} \phi_t^2 = 0$.\[1\]
For example, in $n = 2$ Euclidean dimensions,

$$U_2 = \det \begin{pmatrix} 0 & \phi_1 & \phi_2 \\ \phi_1 & \phi_{11} & \phi_{12} \\ \phi_2 & \phi_{21} & \phi_{22} \end{pmatrix}$$

$$= -\phi_1 (\phi_1 \phi_{22} - \phi_2 \phi_{12}) + \phi_2 (\phi_{21} \phi_1 - \phi_2 \phi_{11})$$

$$= -\phi_1 n_{\alpha \beta} \phi_{\alpha \beta} + \phi_2 n_{\alpha \beta} \phi_{\alpha \beta} .$$

(50)

But then $\phi_1 n_{\alpha \beta} \phi_{\alpha \beta} = \partial_\beta (\phi_1 n_{\alpha \beta} \phi_{\alpha \beta}) - \phi_1 n_{\alpha \beta} \partial_\beta \phi_{\alpha \beta} - \phi_\alpha n_{\alpha \beta} \phi_{\alpha \beta} \phi_{\beta \gamma}$ so $\phi_1 n_{\alpha \beta} \phi_{\alpha \beta} = \frac{1}{2} \partial_\beta (\phi_1 n_{\alpha \beta} \phi_{\alpha \beta}) - \frac{1}{2} \phi_\alpha n_{\alpha \beta} \phi_{\beta \gamma}$. Thus

$$U_2 = -\frac{3}{2} \phi_1 n_{\alpha \beta} \phi_{\alpha \beta} + \frac{1}{2} \partial_\beta (\phi_1 n_{\alpha \beta} \phi_{\alpha \beta}) .$$

That is to say,

$$U_2 = -\frac{3}{2} L_2 + \frac{1}{2} \partial_\beta (\phi_1 n_{\alpha \beta} \phi_{\alpha \beta}) .$$

(51)

The same result applies in spaces with Lorentz signature, when repeated indices are summed with the Lorentz metric.

Similarly, in $n = 3$ dimensions, $U_3$ differs from a constant multiple of $L_3$ just by a divergence,

$$U_3 = -L_3 + \frac{1}{2} n_{\alpha \beta} (\phi_{1,\alpha} n_{\beta \gamma} \phi_{\gamma \gamma} - \phi_{\alpha \beta} n_{\gamma \gamma} \phi_{\gamma \gamma}) .$$

(52)

Indeed, it turns out that in any $n$ dimensions, $U_n$ is always proportional to the maximal galileon Lagrangian $L_n$ modulo a divergence, or boundary term, $B_n$.

$$(n-1)! U_n = -\frac{1}{2} (n+1) L_n + \frac{1}{2} (n-1) B_n .$$

(53)

The relative coefficient between $U_n$ and $L_n$ is worked out explicitly in the next subsection, where an explicit form for $B_n$ is also given. The upshot is that the action for maximal galileon fields that vanish on the spacetime boundary is obtained just by integrating the functional form appearing in the UFE, $\int U_n [\phi] d^n x$.

As someone well-schooled in determinants might guess, especially in light of the discussion following (22), there is another way to express the UFE in $n$ dimensions, in terms of traces. This is given by

$$V_n = \det \begin{pmatrix} S_0 & n-1 & 0 & \cdots & 0 & 0 & 0 \\ S_1 & T_1 & n-2 & \cdots & 0 & 0 & 0 \\ S_2 & T_2 & T_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ S_{n-3} & T_{n-3} & T_{n-4} & \cdots & T_1 & 2 & 0 \\ S_{n-2} & T_{n-2} & T_{n-3} & \cdots & T_2 & T_1 & 1 \\ S_{n-1} & T_{n-1} & T_{n-2} & \cdots & T_3 & T_2 & T_1 \end{pmatrix} = -(n-1)! U_n ,$$

(54)

where we have defined

$$T_k = \text{Tr} \left[ (\partial \partial \phi)^k \right], \quad S_k = \text{Tr} \left[ (\partial \phi \partial \phi) (\partial \partial \phi)^k \right] .$$

(55)

The special results in (41) and (42) may be confirmed from (51).

Perhaps the most elegant proof that the determinants in (49) and (54) are proportional is to make use of an orthogonal transformation at each point $x$ to find local frames such that the symmetric Hessian matrix is diagonal,

$$(\partial \partial \phi) = \text{diag} (\lambda_1, \cdots, \lambda_n) .$$

(56)

(We do not diagonalize the full $(n+1) \times (n+1)$ matrix appearing in (49) because we wish to keep track of the first derivatives, $\partial \phi$.) In such frames it is straightforward to show, from either (49) or (54), that

$$U_n = -\sum_{\alpha=1}^{n} \phi_\alpha^2 \left( \prod_{\beta=1}^{n} \lambda_\beta \right) .$$

(57)

\(^2\)Note this would still be true if the zero in the upper left corner of $U_n$ were replaced by any constant $c$, for then $\det \begin{pmatrix} c & \partial \phi \\ \partial \phi & \partial \partial \phi \end{pmatrix} = U_n + \frac{c}{2!} E_n$ and the last term is again a total divergence.
We have defined more on determinant identities

\[ \det \begin{pmatrix} S_0 & k-1 & 0 & \cdots & 0 & 0 \\ S_1 & T_1 & k-2 & \cdots & 0 & 0 \\ S_2 & T_2 & T_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ S_{k-3} & T_{k-3} & T_{k-4} & \cdots & T_1 & 2 \\ S_{k-2} & T_{k-2} & T_{k-3} & \cdots & T_2 & T_1 \\ S_{k-1} & T_{k-1} & T_{k-2} & \cdots & T_3 & T_2 & T_1 \end{pmatrix} \] \hspace{1cm} (58)

Expand this determinant in the minors of the first column to obtain

\[ \det S_0 \mathcal{E}_{k-1} - (k-1) \mathcal{S}_1 \mathcal{E}_{k-2} + (k-1)(k-2) \mathcal{S}_2 \mathcal{E}_{k-3} - \cdots + (-1)^{k-1}(k-1)! S_{k-1} \mathcal{E}_0 . \] \hspace{1cm} (59)

This suggests that we consider more generally, for \( 0 \leq k \leq n \), the determinant

\[ \det (1 + \lambda M) \]

where the generalized Kronecker symbols are defined by

\[ \delta_{\alpha_1 \cdots \alpha_n}^{\beta_1 \cdots \beta_n} = \frac{1}{(n-k)!} \delta_{\beta_1 \cdots \beta_{k+1} \cdots \beta_n}^{\alpha_1 \cdots \alpha_{k+1} \cdots \alpha_n} \]

\[ \frac{1}{(n-k)!} \varepsilon_{\alpha_1 \cdots \alpha_{k+1} \cdots \alpha_n} \varepsilon_{\beta_1 \cdots \beta_{k+1} \cdots \beta_n} \]

\[ = \delta_{\beta_1 \cdots \beta_{k+1} \cdots \beta_n}^{\alpha_1 \cdots \alpha_{k+1} \cdots \alpha_n} \]

\[ \pm \text{permutations of } \alpha \text{ or } \beta \text{, but not both.} \]

For emphasis, on the RHS we have underlined the repeated indices that are implicitly summed. Thus the \( \mathcal{E}_k (M) \) can be expressed in terms of generalized Kronecker symbols:

\[ \mathcal{E}_k (M) = \delta_{\beta_1 \beta_2 \cdots \beta_k}^{\alpha_1 \alpha_2 \cdots \alpha_k} \times M_{\alpha_1 \beta_1} \cdots M_{\alpha_k \beta_k} . \] \hspace{1cm} (62)

Note the last term in the expansion \( (60) \) is the familiar

\[ \mathcal{E}_n (M) = \delta_{\beta_1 \beta_2 \cdots \beta_n}^{\alpha_1 \alpha_2 \cdots \alpha_n} \times M_{\alpha_1 \beta_1} \cdots M_{\alpha_n \beta_n} = n! \det M . \] \hspace{1cm} (63)

3There is a difference here between Euclidean and Minkowski metrics. For Euclidean space, \( \varepsilon_{\alpha_1 \cdots \alpha_n}^{\epsilon_{\alpha_1 \cdots \alpha_n}} = \delta_{\alpha_1 \cdots \alpha_n}^{\alpha_1 \cdots \alpha_n} \)

is true for any \( n \), but the corresponding identity in Minkowski space is

\[ \varepsilon_{\alpha_1 \cdots \alpha_n}^{\epsilon_{\alpha_1 \cdots \alpha_n}} = (-1)^{n-1} \delta_{\alpha_1 \cdots \alpha_n}^{\alpha_1 \cdots \alpha_n} . \]

The remaining discussion in this subsection will be given for the Euclidean case.
Applying generalized Kronecker symbol methods to bordered determinants, and making use of (62), leads to relations that may be usefully applied to the UFE. Consider
\[
\mathcal{M} = \begin{pmatrix} 0 & \bar{v} \\ v & M \end{pmatrix},
\]
where \( M \) is any symmetric \( n \times n \) matrix, \( v \) is an arbitrary \( n \times 1 \) column matrix, and its transpose \( \bar{v} \) is a \( 1 \times n \) row matrix. Clearly \( \det \mathcal{M} \) is bilinear in the components of \( v \). For convenience, we index the rows and columns of \( \mathcal{M} \) from 0 to \( n \). Since \( \mathcal{M}_{00} = 0 \) it follows that
\[
\det \mathcal{M} = (n+1)! \prod_{i,j=0}^{n} \delta_{\beta_i \beta_j} \mathcal{M}_{\alpha_i \beta_i} \mathcal{M}_{\alpha_j \beta_j},
\]
where other 0 subscripts can not appear because of the antisymmetry of the Kronecker delta. So, substituting \( v \) and \( M \) for the components of \( \mathcal{M} \),
\[
\det \mathcal{M} = \frac{-1}{(n+1)!} \prod_{i,j=0}^{n} \delta_{\beta_i \beta_j} \mathcal{M}_{\alpha_i \beta_i} \mathcal{M}_{\alpha_j \beta_j}
\]
and columns of \( M \) to diagonalize the Hessian matrix.

Next, we consider the local variation of \( \int \mathcal{U}_n [\phi] \, d^n x \) using (63). Thus
\[
\int \mathcal{U}_n [\phi] \, d^n x = \frac{-1}{(n-1)!} \int \delta_{\beta_1 \beta_2 \ldots \beta_{n-1} \beta} \phi_\beta \phi_\alpha \phi_{\alpha_1 \beta_1} \ldots \phi_{\alpha_{n-1} \beta_{n-1}} \, d^n x,
\]
and dropping surface terms in the variation,
\[
\delta \int \mathcal{U}_n [\phi] \, d^n x = \mathcal{U}_n [\phi] \, d^n x.
\]

\[
\begin{align*}
&= \frac{-1}{(n+1)!} \int \delta_{\beta_1 \beta_2 \ldots \beta_{n-1} \beta} \phi_\beta \phi_\alpha \phi_{\alpha_1 \beta_1} \ldots \phi_{\alpha_{n-1} \beta_{n-1}} \, d^n x \\
&= \frac{-1}{(n+1)!} \int \delta_{\beta_1 \beta_2 \ldots \beta_{n-1} \beta} \phi_\beta \phi_\alpha \phi_{\alpha_1 \beta_1} \ldots \phi_{\alpha_{n-1} \beta_{n-1}} \, d^n x \\
&= \frac{-1}{(n+1)!} \int \delta_{\beta_1 \beta_2 \ldots \beta_{n-1} \beta} \phi_\beta \phi_\alpha \phi_{\alpha_1 \beta_1} \ldots \phi_{\alpha_{n-1} \beta_{n-1}} \, d^n x \\
&= \frac{-1}{(n+1)!} \int \delta_{\beta_1 \beta_2 \ldots \beta_{n-1} \beta} \phi_\beta \phi_\alpha \phi_{\alpha_1 \beta_1} \ldots \phi_{\alpha_{n-1} \beta_{n-1}} \, d^n x \\
&= \frac{-1}{(n+1)!} \int \delta_{\beta_1 \beta_2 \ldots \beta_{n-1} \beta} \phi_\beta \phi_\alpha \phi_{\alpha_1 \beta_1} \ldots \phi_{\alpha_{n-1} \beta_{n-1}} \, d^n x \\
&= \frac{-1}{(n+1)!} \int \delta_{\beta_1 \beta_2 \ldots \beta_{n-1} \beta} \phi_\beta \phi_\alpha \phi_{\alpha_1 \beta_1} \ldots \phi_{\alpha_{n-1} \beta_{n-1}} \, d^n x \\
\end{align*}
\]
Now use $\mathcal{E}_n (\partial \phi)$ as given by (62) to obtain the result,
\[
\delta \int \mathcal{U}_n [\phi] \, d^n x = \frac{n+1}{(n-1)!} \int \mathcal{E}_n (\partial \phi) \, \delta \phi \, d^n x .
\]  
(71)

Compare this to $\delta \int \mathcal{L}_n \, d^n x = -2 \int \mathcal{E}_n (\partial \phi) \, \delta \phi \, d^n x$, as given in (63) for $k = n$, to conclude
\[
\delta \int \mathcal{U}_n [\phi] \, d^n x = \delta \int \left( -\frac{n+1}{2(n-1)!} \mathcal{L}_n [\phi] \right) \, d^n x ,
\]  
(72)

where spacetime boundary terms have been dropped. Thus the unvaried integrands can only differ by a divergence. This establishes (63). An explicit form for the divergence $\mathcal{B}_n$, as normalized in (63), can be found by keeping track of the discarded boundary terms produced by integrating by parts in (63) and in (70). This is left as an exercise. The result is
\[
\mathcal{B}_n = \partial_\sigma \left( \delta^\alpha_\beta_1 \alpha_2 \cdots \alpha_{n-2} \times \phi_{\alpha_1 \beta_1} \cdots \phi_{\alpha_{n-2} \beta_{n-2}} \times \phi_{\mu \nu} \phi_{\mu \nu} \right) .
\]  
(73)

For example, this reduces to the divergence terms in (51) and (52), for $n = 2$ and $n = 3$, respectively.

To complete our discussion of the UFE, we consider the effects of a quadratic constraint on $\phi_n$ for a field in $n+1$ dimensions. The constraint will effectively reduce the number of dimensions to be $n$. For convenience, we let indices $\alpha, \beta = 0, 1, \cdots, n$, while we let $\lambda, \mu, \nu = 1, \cdots, n$. Then the constraint of interest to us is
\[
\phi_\alpha \phi_\alpha = 0 .
\]  
(74)

Nontrivial solutions would of course require complex fields in the Euclidean case, but for the time being, let us not be deterred by this. Solve for $\phi_0^2$ and differentiate to obtain
\[
\phi_0^2 = -\phi_\mu \phi_\mu , \quad \phi_{0\mu} = -\phi_{\mu \nu} \phi_{\mu \nu} , \quad \phi_{00} = \frac{\phi_{\mu \nu} \phi_{\nu \mu}}{\phi_0^2} .
\]  
(75)

Now compute $\mathcal{E}_k (\partial \phi)$ subject to the constraint, specifically displaying the occurrences of $\phi_{0\mu}$ and $\phi_{00}$. Thus
\[
\mathcal{E}_k (\partial \phi) |_{n+1 \text{ dimensions with } \phi_n, \phi_{n=0}} = \delta^\alpha_\beta_1 \alpha_2 \cdots \alpha_k \times \phi_{\alpha_1 \beta_1} \cdots \phi_{\alpha_k \beta_k}
\]
\[
= k \delta^\mu_\nu_1 \mu_2 \cdots \mu_{k-1} \times \phi_{\mu_1 \nu_1} \cdots \phi_{\mu_{k-1} \nu_{k-1}} \times \phi_{00} - k (k-1) \delta^\mu_\nu_1 \mu_2 \cdots \mu_{k-2} \times \phi_{\mu_1 \nu_1} \cdots \phi_{\mu_{k-2} \nu_{k-2}} \times \phi_{0\alpha} \phi_{0\lambda} + \delta^\mu_\nu_1 \mu_2 \cdots \mu_k \times \phi_{\mu_1 \nu_1} \cdots \phi_{\mu_k \nu_k}
\]
\[
= k \delta^\mu_\nu_1 \mu_2 \cdots \mu_{k-1} \times \phi_{\mu_1 \nu_1} \cdots \phi_{\mu_{k-1} \nu_{k-1}} \times \phi_\nu \phi_{\nu \mu} \phi_{\mu \nu} / \phi_0^2 - k (k-1) \delta^\mu_\nu_1 \mu_2 \cdots \mu_{k-2} \times \phi_{\mu_1 \nu_1} \cdots \phi_{\mu_{k-2} \nu_{k-2}} \times \phi_\nu \phi_{\nu \mu} \phi_{\mu \lambda} / \phi_0^2 + \delta^\mu_\nu_1 \mu_2 \cdots \mu_k \times \phi_{\mu_1 \nu_1} \cdots \phi_{\mu_k \nu_k}
\]  
(76)

Factoring out $\phi_0^2 = -\phi_\lambda \phi_\lambda$ we arrive at
\[
\mathcal{E}_k (\partial \phi) |_{n+1 \text{ dimensions with } \phi_n, \phi_{n=0}} = \frac{1}{\phi_\lambda \phi_\lambda} \delta^\mu_\nu_1 \mu_2 \cdots \mu_{k+1} \times \phi_{\mu_1 \nu_1} \cdots \phi_{\mu_{k+1} \nu_{k+1}} |_{n \text{ dimensions}} .
\]  
(77)

That is to say
\[
\mathcal{E}_k (\partial \phi) |_{n+1 \text{ dimensions with } \phi_n, \phi_{n=0}} = \frac{1}{\phi_\lambda \phi_\lambda} \mathcal{V}_{k+1} |_{n \text{ dimensions with varying } \phi_\lambda \phi_\lambda} .
\]  
(78)

### 3.6 Legendre transformations

The standard form for a Legendre transformation $\phi, x \leftrightarrow \Phi, X$ is given by
\[
\phi (x) + \Phi (X) = \sum_{\alpha=1}^n x_\alpha X_\alpha , \quad (79)
\]
\[
X_\alpha (x) = \frac{\partial \phi (x)}{\partial x_\alpha} = \partial_\alpha \phi , \quad x_\alpha (X) = \frac{\partial \Phi (X)}{\partial X_\alpha} = \nabla_\alpha \Phi .
\]  
(80)
It follows that the Hessian matrices for $\phi$ and $\Phi$ are related by
\[
(\partial\partial\phi)^{-1} = (\nabla\nabla\Phi) .
\] (81)

From this and the previous matrix identity it follows in $n$ dimensions that
\[
\frac{1}{\sqrt{\det(\partial\partial\phi)}} \frac{1}{k!} \mathcal{E}_k(\partial\partial\phi) = \frac{1}{\sqrt{\det(\nabla\nabla\Phi)}} \frac{1}{(n-k)!} \mathcal{E}_{n-k}(\nabla\nabla\Phi) .
\] (82)

That is to say, field equations for $\phi$ and $\Phi$ are related by the Legendre transform, and so are their solutions. The transformation gives a one-to-one local map between solutions of the nonlinear equations $\mathcal{E}_k(\partial\partial\phi) = 0$ and $\mathcal{E}_{n-k}(\nabla\nabla\Phi) = 0$, valid for all $x$ or $X$ such that the corresponding Hessian matrices are nonsingular, i.e. for all $x$ or $X$ such that $\det(\partial\partial\phi) \neq 0 \neq \det(\nabla\nabla\Phi)$.

This then is a general, implicit procedure for the construction of solutions to the equation $\mathcal{E}_k = 0$ given solutions to $\mathcal{E}_{n-k} = 0$. In practice it is challenging to find tractable examples where the procedure can be fully realized. We will say more about solutions in §4.

### 3.7 Legendre self-dual models

The basic self-(anti)dual action consists of a pair of terms,
\[
A_\pm = \frac{1}{k!} A_k \pm \frac{1}{(n-k)!} A_{n-k} = \int \phi_\alpha \phi_\alpha \left( \frac{1}{k!} \mathcal{E}_{k-1}(\partial\partial\phi) \pm \frac{1}{(n-k)!} \mathcal{E}_{n-k-1}(\partial\partial\phi) \right) .
\] (83)

Thus from the first variation is
\[
\delta A_\pm = -2 \int \delta\phi \left( \frac{1}{k!} \mathcal{E}_k(\partial\partial\phi) \pm \frac{1}{(n-k)!} \mathcal{E}_{n-k}(\partial\partial\phi) \right) .
\] (84)

This exhibits a classical self-(anti)duality for the resulting field equations, and their solutions, under the Legendre transformation. Again in $n$ dimensions,
\[
\frac{1}{\sqrt{\det(\partial\partial\phi)}} \left( \frac{1}{k!} \mathcal{E}_k(\partial\partial\phi) \pm \frac{1}{(n-k)!} \mathcal{E}_{n-k}(\partial\partial\phi) \right)
= \frac{\pm 1}{\sqrt{\det(\nabla\nabla\Phi)}} \left( \frac{1}{k!} \mathcal{E}_k(\nabla\nabla\Phi) \pm \frac{1}{(n-k)!} \mathcal{E}_{n-k}(\nabla\nabla\Phi) \right) .
\] (85)

In particular, for $k = 1$ this becomes
\[
\frac{1}{\sqrt{\det(\partial\partial\phi)}} (\mathcal{E}_1(\partial\partial\phi) \pm \mathcal{E}_1(\adj(\partial\partial\phi)))
= \frac{\pm 1}{\sqrt{\det(\nabla\nabla\Phi)}} (\mathcal{E}_1(\nabla\nabla\Phi) \pm \mathcal{E}_1(\adj(\nabla\nabla\Phi))) ,
\] (86)

where we have made use of (12).

The consequences of this duality for quantized systems requires consideration of how the Legendre transformation directly affects the actions, $A_\pm$, and not just the field equations. We consider this next.

### 3.8 Legendre transformations of the action

When $M$ is taken to be the Hessian matrix, say, $M = \partial\partial\phi$, then every $\mathcal{E}_k(\partial\partial\phi)$ is actually a double divergence,
\[
\mathcal{E}_k(\partial\partial\phi) = \delta_{\alpha_1\alpha_2\cdots\alpha_k} \times \phi_\alpha \partial_\beta_1 \phi_\alpha \partial_\beta_2 \cdots \phi_\alpha \partial_\beta_k = \partial_\alpha_1 \partial_\beta_1 \left( \delta_{\alpha_1\alpha_2\cdots\alpha_k} \times \phi_\alpha \partial_\beta_2 \cdots \phi_\alpha \partial_\beta_k \right) .
\] (87)

Recall the Legendre transformation result
\[
\mathcal{E}_k(\partial\partial\phi) = \frac{k!}{(n-k)! \det(\nabla\nabla\Phi)} \mathcal{E}_{n-k}(\nabla\nabla\Phi) .
\] (88)
Thus, using \( \det \left( \frac{\partial k}{\partial X} \right) = \det (\nabla \nabla \Phi) \), we also have

\[
\int \phi_\mu (x) \phi_\mu (x) \, \mathcal{E}_k (\partial \partial \Phi) \, d^n x = \int X_\mu X_\mu \frac{k!}{(n-k)! \det (\nabla \nabla \Phi)} \epsilon_{n-k} (\nabla \nabla \Phi) \det \left( \frac{\partial X}{\partial X} \right) \, d^n X
\]

\[
= \frac{k!}{(n-k)!} \int X_\mu X_\mu \, \mathcal{E}_{n-k} (\nabla \nabla \Phi) \, d^n X
\]

\[
= \frac{k!}{(n-k)!} \int X_\mu X_\mu \nabla_\alpha \nabla_\beta \left( \delta^{\alpha_2 \alpha_3 \ldots \alpha_{n-k}}_{\beta_2 \beta_3 \ldots \beta_{n-k}} \right) \, d^n X
\]

\[
= \frac{k!}{(n-k)!} 2 \int \delta^{\alpha_2 \alpha_3 \ldots \alpha_{n-k}}_{\beta_2 \beta_3 \ldots \beta_{n-k}} \, \Phi \, \Phi_{\alpha_2} \beta_2 \ldots \Phi_{\alpha_{n-k}} \beta_{n-k} \, d^n X
\]

\[
= \frac{k!}{(n-k)!} 2 (k+1) \int \delta^{\alpha_2 \alpha_3 \ldots \alpha_{n-k}}_{\beta_2 \beta_3 \ldots \beta_{n-k}} \, \Phi \, \Phi_{\alpha_2} \beta_2 \ldots \Phi_{\alpha_{n-k}} \beta_{n-k} \, d^n X
\]

where in the last step we used

\[
\delta^{\mu_1 \mu_2 \ldots \mu_m}_{\lambda_1 \lambda_2 \ldots \lambda_m} = (n-m) \delta^{\nu_1 \nu_2 \ldots \nu_m}_{\lambda_1 \lambda_2 \ldots \lambda_m}
\]

The effect of the Legendre transformation is therefore

\[
A_{k+1} = \int \phi_\mu (x) \phi_\mu (x) \, \mathcal{E}_k (\partial \partial \Phi) \, d^n x
\]

\[
= \frac{(k+1)!}{(n-k)!} 2 \int \delta^{\alpha_2 \alpha_3 \ldots \alpha_{n-k}}_{\beta_2 \beta_3 \ldots \beta_{n-k}} \, \Phi \, \Phi_{\alpha_2} \beta_2 \ldots \Phi_{\alpha_{n-k}} \beta_{n-k} \, d^n X
\]

\[
= \frac{(k+1)!}{(n-k)!} 2 \int \mathcal{E}_{n-k-1} (\nabla \nabla \Phi) \, d^n X
\]

(89)

After the usual integrations by parts, the latter Lagrangian has variation

\[
\delta \int \delta^{\alpha_2 \alpha_3 \ldots \alpha_{n-k}}_{\beta_2 \beta_3 \ldots \beta_{n-k}} \, \Phi \, \Phi_{\alpha_2} \beta_2 \ldots \Phi_{\alpha_{n-k}} \beta_{n-k} \, d^n X = (n-k) \int (\delta \Phi) \times \delta^{\alpha_2 \alpha_3 \ldots \alpha_{n-k}}_{\beta_2 \beta_3 \ldots \beta_{n-k}} \, \Phi \, \Phi_{\alpha_2} \beta_2 \ldots \Phi_{\alpha_{n-k}} \beta_{n-k} \, d^n X
\]

\[
giving the expected equation of motion
\]

\[
0 = \mathcal{E}_{n-k-1} (\nabla \nabla \Phi) = \delta^{\alpha_2 \alpha_3 \ldots \alpha_{n-k}}_{\beta_2 \beta_3 \ldots \beta_{n-k}} \, \Phi \, \Phi_{\alpha_2} \beta_2 \ldots \Phi_{\alpha_{n-k}} \beta_{n-k}
\]

(92)

It should be possible, therefore, to express the transformed Lagrangian in the standard form, upon integrating by parts:

\[
\int \delta^{\nu_1 \nu_2 \ldots \nu_m}_{\lambda_1 \ldots \lambda_m} \times \Phi \, \Phi_{\nu_1} \lambda_1 \ldots \Phi_{\nu_m} \lambda_m \, d^n X
\]

\[
= - \int \delta^{\nu_1 \nu_2 \ldots \nu_m}_{\lambda_1 \ldots \lambda_m} \times \Phi_{\lambda_1} \Phi_{\nu_1} \Phi_{\nu_2} \lambda_2 \ldots \Phi_{\nu_m} \lambda_m \, d^n X \quad (\lambda_1 \text{ integrated by parts})
\]

\[
= - \int \left( \delta^{\nu_1 \nu_2 \ldots \nu_m}_{\lambda_1 \lambda_2 \ldots \lambda_m} - (m-1) \delta^{\nu_1 \nu_2 \ldots \nu_m}_{\lambda_2 \lambda_3 \ldots \lambda_m} \right) \times \left( \Phi_{\lambda_1} \Phi_{\nu_1} \Phi_{\nu_2} \lambda_2 \right) \Phi_{\nu_3} \lambda_3 \ldots \Phi_{\nu_m} \lambda_m \, d^n X
\]

\[
= - \int \left( \Phi_\lambda \Phi_\lambda \right) \left( \delta^{\nu_1 \nu_2 \ldots \nu_m}_{\lambda_2 \lambda_3 \ldots \lambda_m} \times \Phi_{\nu_2} \lambda_2 \Phi_{\nu_3} \lambda_3 \ldots \Phi_{\nu_m} \lambda_m \right) \, d^n X
\]

\[
+ \frac{1}{2} (m-1) \int \delta^{\nu_2 \nu_3 \ldots \nu_m}_{\lambda_1 \lambda_2 \ldots \lambda_m} \times \left( \Phi_{\lambda_1} \partial_{\nu_2} \left( \Phi_\lambda \Phi_\lambda \right) \right) \Phi_{\nu_3} \lambda_3 \ldots \Phi_{\nu_m} \lambda_m \, d^n X
\]

\[
= - \int \left( \Phi_\lambda \Phi_\lambda \right) \left( \delta^{\nu_2 \nu_3 \ldots \nu_m}_{\lambda_1 \lambda_2 \ldots \lambda_m} \times \Phi_{\nu_2} \lambda_2 \Phi_{\nu_3} \lambda_3 \ldots \Phi_{\nu_m} \lambda_m \right) \, d^n X
\]

\[
- \frac{1}{2} (m-1) \int \delta^{\nu_2 \nu_3 \ldots \nu_m}_{\lambda_1 \lambda_2 \lambda_3 \lambda_4 \ldots \lambda_m} \times \left( \Phi_\lambda \Phi_\lambda \right) \Phi_{\nu_2} \lambda_2 \Phi_{\nu_3} \lambda_3 \ldots \Phi_{\nu_m} \lambda_m \, d^n X \quad (\nu_2 \text{ integrated by parts})
\]

So the re-expressed Lagrangian is simply

\[
\int \delta^{\nu_1 \nu_2 \ldots \nu_m}_{\lambda_1 \ldots \lambda_m} \times \Phi \, \Phi_{\nu_1} \lambda_1 \ldots \Phi_{\nu_m} \lambda_m \, d^n X = \int \Phi \, \mathcal{E}_m (\nabla \nabla \Phi) \, d^n X = - \frac{1}{2} (m+1) \int \left( \Phi_\lambda \Phi_\lambda \right) \mathcal{E}_{m-1} (\nabla \nabla \Phi) \, d^n X
\]

(94)
OK then, we have
\[ A_{k+1} \left[ \phi \right] = \int \phi_\mu(x) \phi_\mu(x) \ E_k(\partial \phi) \ d^n x = \frac{(k+1)!}{(n-k)!} \left( \frac{1}{2} \right) \int (\Phi_\Lambda \Phi_\Lambda) E_{n-k-1-1}(\nabla \nabla \Phi) \ d^n X \]
\[ = -\frac{(k+1)!}{(n-k-1)!} \int (\Phi_\Lambda \Phi_\Lambda) E_{n-k-1-1}(\nabla \nabla \Phi) \ d^n X \]
\[ = -\frac{(k+1)!}{(n-k-1)!} A_{n-k-1} \left[ \Phi \right] . \]  

(95)

If we shift the index, this may be written more symmetrically.

Thus we have established that the Legendre transform \([79]\) gives directly a relation between the actions for the two theories:
\[ \frac{1}{k!} A_k \left[ \phi \right] = \frac{(-1)^n}{(n-k)!} A_{n-k} \left[ \Phi \right] \quad \text{Euclidean,} \]
\[ \frac{1}{k!} A_k \left[ \phi \right] = \frac{(-1)^n}{(n-k)!} A_{n-k} \left[ \Phi \right] \quad \text{Lorentzian.} \]

(96)

provided boundary terms from integrating by parts may be discarded, where
\[ A_k \left[ \phi \right] = \int \phi_\mu(x) \phi_\mu(x) \ E_k(\partial \phi) \ d^n x = \frac{-2}{k+1} \int \phi(x) \ E_k(\partial \phi) \ d^n x , \]
\[ A_{n-k} \left[ \Phi \right] = \int \Phi_\mu(x) \Phi_\mu(x) \ E_{n-k-1}(\nabla \nabla \Phi) \ d^n X = \frac{-2}{n-k+1} \int \Phi(x) \ E_{n-k}(\nabla \nabla \Phi) \ d^n X . \]

(97)

(98)

But alas, the sign in \([96]\) disagrees with the result from the explicit calculation of the \(n = 2, k = 1\) case with Lorentz signature! Namely,
\[ \int \partial_\alpha \Phi \partial^\alpha \Phi \ d^2 x = \int \nabla_\alpha \Phi \nabla^\alpha \Phi \ d^2 X . \]

(99)

However, this discrepancy is due to the difference between the Euclidean and Lorentzian space identities for the product of two Levi-Civita symbols, namely, in \(n\) dimensions with Lorentz metric sign conventions \((+,-,-,-,\cdots)\),
\[ \varepsilon_{\alpha_1 \cdots \alpha_n} \varepsilon^{\beta_1 \cdots \beta_n} = \delta^{\alpha_1 \cdots \alpha_n}_{\beta_1 \cdots \beta_n} \quad \text{Euclidean,} \]
\[ \varepsilon_{\alpha_1 \cdots \alpha_n} \varepsilon^{\beta_1 \cdots \beta_n} = (-1)^{n-1} \delta^{\alpha_1 \cdots \alpha_n}_{\beta_1 \cdots \beta_n} \quad \text{Lorentzian.} \]

(100)

(101)

In Lorentz space then, the appropriate identity in \(n\) dimensions is
\[ \frac{1}{k!} A_k \left[ \phi \right] = \frac{(-1)^n}{(n-k)!} A_{n-k} \left[ \Phi \right] \quad \text{Lorentzian.} \]

(102)

3.9 Hidden symmetry

Hinterbichler and Joyce (HJ) \([24]\) have pointed out that Legendre self-(anti)dua substitutions in four dimensions
\[ A_2 = A_1 \left[ \phi \right] \pm A_3 \left[ \phi \right] \]

(103)

realize (nonlinearly) a surprising amount of symmetry, namely, the semidirect sum of the Heisenberg and special linear algebras: \(h(4) \oplus sl(4)\), although HJ do not identify the algebra by these standard names. Moreover, HJ show that additional symmetries are also present if particular linear combinations of Legendre dual galileon Lagrangians are considered in \(D\) spacetime dimensions.

The most succinct verbal description is just to say the HJ galileon symmetry algebra is isomorphic to a semidirect sum of the Heisenberg algebra \(h(D)\) and \(sl(D)\). Thus,
\[ h(D) \oplus sl(D) \]

(104)

Recall that \(h(D)\) is realized by \(\{x_a, p_b, C\}\) where \(C\) is the central charge appearing in \([x_a, p_b] = \delta_{ab}C\), and \(sl(D)\) is realized by \(\{x_a p_b - \frac{1}{D} \delta_{ab} \} \).
3.10 Summary of results for Euclidean metrics

Local relations:

\[ \mathcal{L}_k = \phi_\alpha \phi_\alpha \mathcal{E}_{k-1} \quad \mathcal{E}_{k-1} = \delta^{\alpha_1 \alpha_2 \cdots \alpha_{k-1}}_{\beta_1 \beta_2 \cdots \beta_{k-1}} \times \phi_{\alpha_1} \phi_{\alpha_2} \cdots \phi_{\alpha_{k-1}} \beta_{k-1} \]

\[ \mathcal{E}_k = \partial_{\rho} \partial_{\sigma} \left( \delta^{\alpha_1 \alpha_2 \cdots \alpha_k}_{\sigma \beta_2 \cdots \beta_k} \times \phi_{\alpha_2} \cdots \phi_{\alpha_k} \phi_{\beta_2} \cdots \phi_{\beta_k} \times \phi \right) \]

\[ \mathcal{D}_k = \partial_{\rho} \partial_{\sigma} \left( \delta^{\alpha_1 \alpha_2 \cdots \alpha_k}_{\sigma \beta_2 \cdots \beta_k} \times \phi_{\alpha_2} \cdots \phi_{\alpha_k} \phi_{\beta_2} \cdots \phi_{\beta_k} \times \frac{1}{2} \phi' \right)^2 \]

\[ \mathcal{B}_k = \partial_{\rho} \left( \delta^{\alpha_1 \alpha_2 \cdots \alpha_k}_{\rho \beta_2 \cdots \beta_k} \times \phi_{\alpha_1} \phi_{\alpha_2} \cdots \phi_{\alpha_k} \beta_{k-2} \cdots \beta_{k-2} \times \phi \phi_\tau \rho \phi_\tau \right) \]

\[ \mathcal{V}_k = \delta^{\alpha_1 \alpha_2 \cdots \alpha_k}_{\sigma \beta_2 \cdots \beta_k} \times \phi_{\alpha_1} \phi_{\alpha_2} \cdots \phi_{\alpha_k} \beta_{k-1} \times \phi \phi_\beta_\beta \times \frac{1}{2} \phi' \]

Note that \( \mathcal{E}_k \) and \( \mathcal{D}_k \) are double divergences.

Integrated relations:

\[ \mathcal{A}_k = \frac{2}{k+1} \int \mathcal{V}_k \, d^n x \quad \text{upon setting} \quad \int \mathcal{B}_k \, d^n x = 0 \]

\[ \int \mathcal{V}_k \, d^n x = - \int \phi \mathcal{E}_k \, d^n x \quad \text{upon setting} \quad \int \mathcal{D}_k \, d^n x = 0 \]

\[ \mathcal{A}_k = \frac{2}{k+1} \int \phi \mathcal{E}_k \, d^n x \quad \text{upon discarding both of the above boundary terms} \]

Constraint relation:

\[ \mathcal{E}_k \left( \partial \partial \phi \right)_{n+1} \, \text{dimensions with} \, \phi_\alpha \phi_\alpha = 0 = \frac{1}{\phi_\alpha \phi_\alpha} \mathcal{V}_{n+1} \mid \, n \, \text{dimensions with varying} \, \phi_\alpha \phi_\alpha \]

Legendre relations:

\[ \mathcal{E}_k \left( \partial \partial \phi \right) = \frac{1}{\sqrt{\det(\partial \partial \phi)}} \left[ \frac{1}{k!} \mathcal{E}_k \left( \partial \partial \phi \right) \right] \]

\[ \frac{1}{\det(\partial \partial \phi)} \frac{1}{k!} \mathcal{E}_k \left( \partial \partial \phi \right) = \frac{1}{\sqrt{\det(\nabla \nabla \phi)}} \frac{1}{(n-k)!} \mathcal{E}_{n-k} \left( \nabla \nabla \phi \right) \]

\[ \frac{1}{k!} \mathcal{E}_k \left( \partial \partial \phi \right) = - \frac{1}{(n-k)!} \mathcal{A}_{n-k} \left[ \Phi \right] \]

Trace relations:

\[ \mathcal{T}_k = \text{Tr} \left[ \left( \partial \partial \phi \right)^k \right] \quad \mathcal{S}_k = \text{Tr} \left[ \left( \partial \partial \phi \right) \left( \partial \partial \phi \right)^{k-1} \right] \]

Lorentz metric results depend on sign conventions, i.e. \((+,-,--,\ldots)\) or \((-+,+,+,\ldots)\), and are left to the reader to determine.
4 Classical solutions

In this section we consider classical solutions of the field equations stemming from individual $A_k$. We present and illustrate a variety of methods to find solutions of $\mathcal{E}_k(\partial \partial \phi) = 0$.

As a warm-up, consider the first example beyond the standard free massless field, namely, $\mathcal{E}_2(\partial \partial \phi) = 0$. For an extremely simple case, a “spherically symmetric” solution is valid almost everywhere in $n$ dimensions:

$$\phi(x) = \begin{cases} \sqrt{x_1 x_2 \cdots x_n} & \text{if } n \neq 4 \\ \ln(x_{\alpha} x_\alpha) & \text{if } n = 4 \end{cases}$$  \hspace{1cm} (105)

where $\alpha = 1, \ldots, n$ is summed. However, in $n$-dimensional Minkowski space (except for $n = 8, 12, \ldots$) this solution obviously has an imaginary part outside the light-cone. So, in general, such functions with branch points give real solutions only on subspaces. We may think of the branch points as defining boundaries for the subspaces, with some particular boundary conditions.

At the opposite extreme, another solution is

$$\phi(x) = \sqrt{x_1 x_2 \cdots x_n}.$$  \hspace{1cm} (106)

It is straightforward to check the equation of motion $\mathcal{E}_2 = 0$ is satisfied. This solution has branch points for both Euclidean and Minkowski space.

Next consider $\mathcal{E}_k = 0$ for general $k$. Although $\mathcal{E}_{k>1}$ is nonlinear in $\phi$, it is nevertheless still true that some plane waves are exact solutions. For “light-ray” plane waves,

$$\mathcal{E}_k[A \exp (i q_\alpha x_\alpha)] = 0$$  \hspace{1cm} (107)

for constant $A$ and $q_\alpha$, if $q_\alpha q_\alpha = 0$ with $A$ arbitrary. In this case, each of the terms in $\mathcal{E}_k$ vanish separately. In fact, light-ray plane waves are only one among many possible solutions for which both $\phi_{\alpha\alpha} = 0$ and $\phi_{\alpha\beta} = 0$. That is to say, the general galileon equation $\mathcal{E}_k = 0$ possesses a class of solutions given by the simultaneous solution of

$$\frac{\partial^2 \phi}{\partial x^\alpha \partial x^\alpha} = 0 \quad \text{and} \quad \left( \frac{\partial \phi}{\partial x^\alpha} \right) \left( \frac{\partial \phi}{\partial x^\alpha} \right) = 0.$$  \hspace{1cm} (108)

The proof of this statement is elementary for $\mathcal{E}_2 = 0$, while for higher $k$, the hierarchical construction in and the nature of the variation procedure guarantees that $\mathcal{E}_k = 0$ will hold if both $\phi_{\alpha\alpha} = 0$ and $\mathcal{E}_{k-1} = 0$, thereby establishing the general result.

In three dimensions it is only necessary to take the single constraint $\phi_{\alpha\alpha} = 0$ since a consequence of taking additional derivatives of this one constraint is $\phi_{\alpha\alpha} = 0$. This is not true in higher dimensions, but it does suggest another method for $n = 3$.

4.1 Implicit solutions

In three dimensions, choose four arbitrary functions $f(u, v), g(u, v), h(u, v), k(u, v)$ constrained by the three relations

$$xf(u, v) + yg(u, v) + zh(u, v) + k(u, v) = \phi(x, y, z),$$  \hspace{1cm} (109)

$$xf(u, v)u + yg(u, v) + zh(u, v) + k(u, v) = 0,$$  \hspace{1cm} (110)

$$xf(u, v) + yg(u, v) + zh(u, v) + k(u, v) = 0.$$  \hspace{1cm} (111)

Here subscripts denote partial differentiation with respect to $u, v$. Then the implicit solution of these equations for $\phi(x, y, z)$ is a solution to the Monge-Ampere equation $\det(\phi_{\mu\nu}) = 0$ in 3 dimensions. Here

$$\phi_x = f(u, v), \phi_y = g(u, v), \phi_z = h(u, v),$$  \hspace{1cm} (112)

so the solution implies that there exists a functional relationship amongst $(\phi_x, \phi_y, \phi_z)$. This remark is enough to guarantee that the elimination of $(u, v)$ will give a solution to Monge Ampere. To solve the galileon equation some further constraints are necessary. Suppose

$$\phi_z = Q(\phi_x, \phi_y) = Q(\alpha, \beta)$$  \hspace{1cm} (113)

It is interesting that this class of solutions is not given by the method of Legendre transformations, described above in §3.6 and to be discussed further below. The Legendre method fails for this class because the second equation in (108) implies the existence of a functional relation among the Legendre transformed variables so that $\det(\partial X_{\alpha}/\partial x_{\beta}) = 0$. 

16
The first of these equations possesses a tantalizing similarity to the Legendre transform. But here the functions depend only upon one function:

\[ \phi_{xx} = Q_\alpha \phi_{x} + Q_\beta \phi_{xy} , \]  
\[ \phi_{x} = Q_\alpha \phi_{x} + Q_\beta \phi_{yy} , \]  
\[ \phi_{zz} = Q_\alpha \phi_{z} + Q_\beta \phi_{yz} , \]  
\[ = (Q_\alpha)^2 \phi_{xx} + 2Q_\alpha Q_\beta \phi_{xy} + (Q_\beta)^2 . \]  

Combining these relations we obtain

\[ (\phi_{xx} \phi_{yy} - (\phi_{xy})^2 ) (1 + Q_\alpha^2 + Q_\beta^2) , \]  
so either the constraint is

\[ 1 + Q_\alpha^2 + Q_\beta^2 = 0 \]  
\[ \phi_{xx} \phi_{yy} - (\phi_{xy})^2 = 0 . \]  

In the case \[ (1.16) \] this places a constraint upon the functional dependence of the derivatives. For example, if \( Q = i \sqrt{\phi_x^2 + \phi_y^2} \) then the constraint is satisfied automatically. Then \( \phi_x^2 + \phi_y^2 + \phi_z^2 = 0 \), a constraint encountered earlier. In turn, this constrains the functions \( (f(u, v), g(u, v), h(u, v)) \) to satisfy \( f^2 + g^2 + h^2 = 0 \). Another possibility is \( Q = i \sin(\phi_x) + i \cos(\phi_y) \).

In the other case where \[ (1.17) \] holds it is not sufficient to impose just this to solve the galileon equations, but we also require that \( \phi_x = Q(\phi_x, \phi_y) \), or else demand that all leading subdeterminants vanish, i.e.

\[ \phi_{xx} \phi_{yy} - (\phi_{xy})^2 = 0, \quad \phi_{yy} \phi_{zz} - (\phi_{yz})^2 = 0, \quad \phi_{zz} \phi_{xx} - (\phi_{xy})^2 = 0 . \]  

A solution of this type is given by \( \phi(x, y, z) = \sqrt{xyz} \).

### 4.2 Envelope method

The envelope method also gives solutions. To be explicit, in 3D, the method is to take

\[ xf(u) + yg(u) + zh(u) = \phi(x, y, z) , \]  
\[ xf'(u) + yg'(u) + zh'(u) = 0 , \]

and then choose various functions \( f, g, \) and \( h \) to determine \( u(x, y, z) \). For example, inserting \( f(u) = u, g(u) = \frac{1}{2} u^2, \) and \( h(u) = \frac{1}{3} u^3 \) into the second of these two equations gives \( x + yu + zu^2 = 0 \), whose solutions are

\[ u(x, y, z) = \frac{1}{2z} \left( y \pm \sqrt{y^2 - 4xz} \right) . \]

Therefore the corresponding solutions for \( \phi \) are

\[ \phi(x, y, z) = -\frac{1}{12xz} \left( y^3 + 6xyz \pm (y^2 - 4xz) \sqrt{y^2 - 4xz} \right) . \]

Again, these functions have a branch point, as well as a pole, so they are not real solutions unless \( y^2 \geq 4xz \) and \( z \neq 0 \). For these examples, it is again straightforward to check the equation of motion is satisfied.

This procedure can be extended if all three derivatives are functionally related, and the auxiliary functions depend only upon one function:

\[ xf(u) + yg(u) + zh(u) + \omega(u) = \phi(x, y, z) , \]
\[ xf'_u(u) + yg'_u(u) + zh_u(u) + \omega_u(u) = 0 , \]

The first of these equations possesses a tantalizing similarity to the Legendre transform. But here the procedure is to solve the second equation for \( u \) in terms of \( x, y, z \) which results in a solution of \( E_2 = 0 \) when
Thus the power

Taking a limit, a nontrivial solution for \( n \) may be obtained by replacing \( f(u), g(u), h(u), \) and \( \omega(u) \) by

\[
F(u) = q(u)f(u) - \int q'(u)f(u)du ,
\]

\[
G(u) = q(u)g(u) - \int q'(u)g(u)du ,
\]

\[
H(u) = q(u)h(u) - \int q'(u)h(u)du ,
\]

\[
\Omega(u) = q(u)\omega(u) - \int q'(u)\omega(u)du .
\]

The equation to determine \( u \) is still the same as before, so a new solution is generated. This may be checked on the specific example, taking \( q(u) = u \).

**Another example**

Take \( f(u) = u, \ g(u) = u^2, \ h(u) = u^3 \). Solving the second of these equations for \( u \),

\[
u = 1/3 \frac{-y + \sqrt{y^2 - 3zx}}{z},
\]

and

\[
\phi = 1/27 \left( -y + \sqrt{y^2 - 3zx} \right) \left( 6zx - y^2 + y\sqrt{y^2 - 3zx} \right). 
\]

This is a solution to the \( n = 3 \) case. Note this is of weight one, so it is also a solution of the Monge-Ampere equation.

### 4.3 Power law solutions in other dimensions and Legendre equivalences

In general, the equation \( \mathcal{E}_k (\partial \partial \phi) = 0 \) is homogeneous in \( \phi \), of degree \( k \), and therefore the overall normalization of any solution is not determined. Consider again spherically symmetric solutions as given by a power ansatz:

\[
\phi(x) = (x_\alpha x_\alpha)^p .
\]

In \( n \) dimensions, for this ansatz, the products and traces of \( \partial \partial \phi \) are:

\[
(\partial \partial \phi)^k_{\mu\nu} = (2p)^k \left( x_\alpha x_\alpha \delta_{\mu\nu} + \left( (2p - 1) - 1 \right) x_\mu x_\nu \right) (x_\alpha x_\alpha)^{k-p-1} ,
\]

\[
(\partial \partial \phi)^k_{\mu\mu} = (2p)^k \left( n - 1 + (2p - 1) \right) (x_\alpha x_\alpha)^{k-p} .
\]

Inserting these traces into (26) and evaluating (25) we find, for example,

\[
\mathcal{E}_1 = 2p \left( n - 2 + 2p \right) (x_\alpha x_\alpha)^{p-1} = 0 \Rightarrow p = 1 - n/2 ,
\]

\[
\mathcal{E}_2 = 4p^2 \left( n - 1 \right) \left( n + 4p - 4 \right) (x_\alpha x_\alpha)^{2p-2} = 0 \Rightarrow p = 1 - n/4 ,
\]

\[
\mathcal{E}_3 = 8p^3 \left( n - 1 \right) \left( n - 2 \right) \left( n + 6p - 6 \right) (x_\alpha x_\alpha)^{3p-3} = 0 \Rightarrow p = 1 - n/6 ,
\]

Thus the power \( p \) required for a solution is determined, as indicated.

For other levels in the hierarchy,

\[
\mathcal{E}_k (\partial \partial \phi)_{\phi(x)=(x_\alpha x_\alpha)^p} = (x_\alpha x_\alpha)^{k-p} (n + 2kp - 2k) (2p)^k \frac{(n+1-k)!}{(n-2)!} .
\]

The condition on \( p \) for the ansatz (131) to be a solution of \( \mathcal{E}_k = 0 \) is therefore

\[
p = 1 - \frac{n}{2k} .
\]

By taking a limit, a nontrivial solution for \( n = 2k \) is easily found to be \( \ln(x_\alpha x_\alpha) \).
So, in \( n \) dimensions the \( k \)th equation of motion of the hierarchy is solved by

\[
\phi ( x ) = \begin{cases} 
(x_\alpha x_\alpha)^{1 - \frac{2}{k}} & \text{if } n \neq 2k \\
\ln (x_\alpha x_\alpha) & \text{if } n = 2k 
\end{cases}
\] (138)

Moreover, under the Legendre transformation, the ansatz solution for level \( k \) is mapped into the ansatz solution for level \( n - k \).

In particular, when \( n = k + 1 \) the ansatz is mapped into an harmonic function by the Legendre transformation.\(^5\) In \( n \) dimensions the spherically symmetric harmonic solution is given by \( 0 = \nabla_\beta \nabla_\beta (X_\alpha X_\alpha)^{\frac{2-n}{n}} \), so this last statement is equivalent to

\[
(x_\alpha x_\alpha)^{\frac{k-1}{k}} = (x_\alpha x_\alpha)^{\frac{k-1}{k}} \quad \text{Legendre for } n = k + 1
\]

(141)

Note the effect of the transformation is just to replace \( k \mapsto \frac{1}{k} \) in the exponent, thereby changing the scaling properties of the solution:

\[
(X_\beta \nabla_\beta - 1) (X_\alpha X_\alpha)^{\frac{2-n}{n}} = (1 - n) (X_\alpha X_\alpha)^{\frac{2-n}{n}},
\] (142)

\[
(x_\beta \partial_\beta - 1) (x_\alpha x_\alpha)^{\frac{k-2}{k}} = \left( \frac{1}{1 - n} \right) (x_\alpha x_\alpha)^{\frac{k-2}{k}}.
\] (143)

Let’s go through the details for the \( n = k + 1 \) case. Under the Legendre transformation:

\[
x_\beta X_\beta = \phi ( x ) + \Phi ( X ) ,
\] (144a)

\[
X_\beta = \frac{\partial \phi ( x )}{\partial x_\beta} = \frac{\partial}{\partial x_\beta} (x_\alpha x_\alpha)^{\frac{k-1}{k}} = \frac{k-1}{k} x_\beta (x_\alpha x_\alpha)^{\frac{k-1}{k}} ,
\] (144b)

\[
x_\beta X_\beta = \frac{k-1}{k} (x_\alpha x_\alpha)^{\frac{k-1}{k}} = \frac{k-1}{k} \phi ( x ) ,
\] (144c)

\[
\Phi ( X ) = - \frac{1}{k} \phi ( x ) ,
\] (144d)

\[
X_\beta X_\beta = \left( \frac{k-1}{k} \right)^2 (x_\alpha x_\alpha)^{\frac{k-1}{k}}, 
\]

\[
x_\alpha x_\alpha = \left( \frac{k-1}{k} \right)^{2k} (X_\beta X_\beta)^{-k},
\] (144e)

\[
(x_\alpha x_\alpha)^{\frac{k-1}{k}} = \left( \frac{k-1}{k} \right)^{k-1} (X_\beta X_\beta)^{\frac{k-1}{k}}.
\] (144f)

So then, an harmonic function of \( X \) is indeed the result of transforming \( (x_\alpha x_\alpha)^{\frac{k-1}{k}} \), for dimension \( n = k + 1 \). Including a convenient normalization,

\[
\phi ( x ) = k^k (x_\alpha x_\alpha)^{\frac{k-1}{k}} \quad \text{Legendre for } n = k + 1
\]

\[
\Phi ( X ) = -(k - 1)^{k-1} (X_\beta X_\beta)^{\frac{k-1}{k}}.
\] (145)

Note that this procedure could be reversed, starting from the harmonic solution. Thus there is a local one-to-one map between harmonic functions \( \Phi ( X ) \) and solutions of the nonlinear equation \( \mathcal{E}_{n-1} (\partial \phi) = 0 \), in \( n \) dimensions, valid so long as \( 0 < |\det (\nabla \Phi)| < \infty \).

### 4.4 Self-Dual Solutions

Another approach to the solution provides a class of self dual solutions; i.e. solutions both to the original equation, and to the Legendre transformed equation (in the same variables). Suppose we impose the ansatz

\[
x_\mu V_\mu (\phi) = 1.
\] (146)

\(^5\)Two such examples for solutions of \( \mathcal{E}_2 (\partial \phi) = 0 \) and \( \mathcal{E}_1 (\nabla \nabla \Phi) = 0 \) in three dimensions are

\[
\Phi = XYZ, \quad \phi = \sqrt{XYZ},
\] (139)

and

\[
\Phi = \frac{1}{\sqrt{X^2 + Y^2 + Z^2}}, \quad \phi = (x^2 + y^2 + z^2)^{1/4}.
\] (140)

For another, take \( \Phi = Z (Z^2 + Y^2 - 4X^2) \), etc. We leave it as an exercise for the reader to find the corresponding \( \phi \).
Make a spherically symmetric spacetime ansatz: 

\[ \frac{\partial \phi}{\partial x_\mu} = -\frac{V_\mu}{\sum x_\mu V_\mu'}, \]

\[ \frac{\partial^2 \phi}{\partial x_\mu \partial x_\nu} = -\frac{(V_\mu V_\nu' + V_\nu V_\mu')}{(\sum x_\alpha V_\alpha')^2} + \frac{V_\mu V_\nu(\sum x_\beta V_\beta'')}{(\sum x_\alpha V_\alpha')^3}. \]  

Then

\[ \sum_{\text{dimension}, \text{as every term contains at least one factor of the form} \phi'}. \]

Therefore, solutions of this type will inevitably be complex in Euclidean space, but may be real in Minkowski space. This is clearly zero if the constraint \( \sum V_\mu V_\mu' = 0 \) is imposed. Moreover, it is easy to see that this solution also solves \( \nabla^2 \phi = 0 \), the Legendre transform of \( E_2 \) in 3 dimensions. Another remarkable property (10) of this solution is that if you replace \( \phi \) by any function of \( \phi \), it remains a solution of these equations! Indeed, this class of solutions is universal, as it solves all \( E_k \) in the appropriate dimension, as every term contains at least one factor of the form \( \sum V_\mu V_\mu' \) or \( \sum V_\mu V_\lambda' \), which both vanish by the constraint. Furthermore this extends to covariant equations which also include first, as well as second derivatives; thus the Lagrangians themselves vanish on this class of solutions.

**Example**

As a simple example consider the equation

\[ x\phi + iy\sqrt{1 + \phi'^2} + z = 1 \]  

whose coefficients satisfy the constraint. Solving for \( \phi \) we obtain

\[ \phi = \frac{\pm (x - z x) + \sqrt{-y^4 - y^2 z^2 + 2 y^2 z - y^2 x^2}}{y^2 + x^2}. \]

These solutions may be verified to satisfy \( E_2 = 0, E_3 = 0, \) and \( E_3 = 0 \).

**5 Mixtures**

*In this section we consider solutions of the field equations for systems governed by linear combinations of the \( A_k \) for different \( k \).*

**5.1 Symmetric spacetime solutions**

Consider the modified field equation that follows from \( A_1 - \kappa A_2 \):

\[ \phi_{\lambda\lambda} = \kappa (\phi_{\nu\nu} \phi_{\lambda\lambda} - \phi_{\nu\lambda} \phi_{\nu\lambda}) \]  

Make a spherically symmetric spacetime ansatz:

\[ \phi = f(\sigma), \quad \sigma \equiv x_\lambda x_\lambda. \]

Then

\[ \phi_{\lambda\lambda} = \partial_\lambda (2x_\lambda f') = 2n f' + 4x_\lambda x_\lambda f'', \quad \phi_{\nu\lambda} = \partial_\nu (2x_\lambda f') = 2\delta_{\nu\lambda} f' + 4x_\nu x_\lambda f'' \]

\[ \phi_{\lambda\lambda} \phi_{\nu\nu} - \phi_{\nu\lambda} \phi_{\nu\lambda} = (2n f' + 4x_\lambda x_\lambda f'')^2 - (2\delta_{\nu\lambda} f' + 4x_\nu x_\lambda f'') (2\delta_{\nu\lambda} f' + 4x_\nu x_\lambda f'') \]

\[ = 4n (n - 1) (f')^2 + 16 (n - 1) x_\lambda x_\lambda f' f'' \]

\[ = 4 (n - 1) (f') (n f' + 4\sigma f'') \]

and the field equation (152) becomes

\[ 2n f' + 4\sigma f'' = 4\kappa (n - 1) (f') (n f' + 4\sigma f'') \]
This is again a first order differential equation for \( g = f' \):

\[
\frac{2}{n} \sigma g' = -g + \frac{2\kappa (n-1)}{(4\kappa (n-1) g - 1)} g^2.
\]

Therefore

\[
\ln \left( g \left( g + \frac{1}{2\kappa (1-n)} \right) \right) = -\frac{n}{2} \ln \sigma ,
\]

and we have

\[
g \left( g + \frac{1}{2\kappa (1-n)} \right) = C \sigma^{-n/2} \quad \text{where} \quad C = g_1 \left( g_1 + \frac{1}{2\kappa (1-n)} \right) , \quad g_1 \equiv g (\sigma = 1) .
\]

That is to say,

\[
g (\sigma) = \frac{1}{4\kappa (n-1)} \begin{cases} 1 - \sqrt{1 + 4\kappa^2 (n-1)^2 C \sigma^{-n/2}} & \text{if} \quad 4\kappa (n-1) g_1 < 1 \\ 1 + \sqrt{1 + 4\kappa^2 (n-1)^2 C \sigma^{-n/2}} & \text{if} \quad 4\kappa (n-1) g_1 > 1 . \end{cases}
\]

One more integration gives \( f = \int g \):

\[
f (\sigma) = f (\sigma_0) + \int_{\sigma_0}^\sigma g (\rho) d\rho .
\]

For convenience, let us take \( \sigma_0 = 1 \). The result of the integral is then

\[
\int \sqrt{1 + 16\kappa^2 (n-1)^2 C \sigma^{-n/2}} d\sigma = \sigma \cdot _2F_1 \left( -\frac{1}{2}, -\frac{2}{n}; 1 - \frac{2}{n}; -16\kappa^2 (n-1)^2 C \sigma^{-\frac{4}{n}} \right) ,
\]

where \(_2F_1\) is the usual Gauss hypergeometric function,

\[
_2F_1 (a, b; c; z) = 1 + \frac{abz}{c} + \frac{a(1+a)b(1+b)z^2}{c(1+c)2!} + \frac{a(1+a)(2+a)b(1+b)(2+b)z^3}{c(1+c)(2+c)3!} + \cdots .
\]

The final result for the spherically symmetric spacetime solution (recall \( \sigma \equiv x_\lambda x_\lambda \)) is then

\[
f (\sigma) = f (1) + \frac{1}{4\kappa (n-1)} (\sigma - 1)
\]

\[
+ \frac{\pm 1}{4\kappa (n-1)} \begin{cases} \phantom{-} _2F_1 \left( -\frac{1}{2}, -\frac{2}{n}; 1 - \frac{2}{n}; -16\kappa^2 (n-1)^2 C \right) & \text{if} \quad 4\kappa (n-1) g_1 \leq 1 \quad \text{and} \quad 0 \leq z < 1/\sqrt{2} \\ -_2F_1 \left( -\frac{1}{2}, -\frac{2}{n}; 1 - \frac{2}{n}; -16\kappa^2 (n-1)^2 C \sigma^{-\frac{4}{n}} \right) & \text{if} \quad 4\kappa (n-1) g_1 > 1 \quad \text{and} \quad 0 < z < 1/\sqrt{2} \end{cases}
\]

where the \( \pm 1 \) choice is made depending on whether \( 4\kappa (n-1) g_1 \leq 1 \), thereby giving various values for

\[-16\kappa^2 (n-1)^2 C = (2 - 4\kappa (n-1) g_1) \times 4\kappa (n-1) g_1 . \]

At the critical value \( 4\kappa (n-1) g_1 = 1 \) we have \(-16\kappa^2 (n-1)^2 C = 1 \). \( \text{(Note: The series expansion is not valid for the hypergeometric function if} \quad 16\kappa^2 (n-1)^2 C \sigma^{-\frac{4}{n}} > 1 . \)

With \( z = 4\kappa (n-1) g_1 \) we have \(-16\kappa^2 (n-1)^2 C = z (2 - z) \), and we note for \( z (2 - z) = -1 \), the solutions are: \( z = 1 \pm \sqrt{2} \).

### 5.2 Static, spherically symmetric solutions

For example, for the free field in 4 spacetime dimensions, \( E_1 = 0 \), the static, spherically symmetric solutions of \( \nabla^2 \phi (r) = 0 \) in 3 space dimensions are of course

\[
\phi_1 (r) = C_0 + \frac{C_1}{r} .
\]

For the next step up in the hierarchy, \( E_2 = 0 \), the static, spherically symmetric solutions in 3 space dimensions are

\[
\phi_2 (r) = C_0 + C_1 \sqrt{r} .
\]

These two solutions are Legendre duals in 3D space (but not in 1+3 spacetime). So, what happens if we mix them up?
For example, take
\[ E_1 = \lambda E_2. \] (168)
The static, spherically symmetric solutions in this case satisfy
\[ -\frac{1}{r^2} \partial_r (r^2 \partial_r \phi) = \frac{2\lambda}{r^2} \partial_r \left( r (\partial_r \phi)^2 \right), \] (169)
which has an immediate first integral, and solution,
\[ C_1 = r^2 \partial_r \phi + 2\lambda r (\partial_r \phi)^2, \] (170)
\[ \partial_r \phi = \frac{r}{4\lambda} \left( -1 \pm \sqrt{1 + \frac{8\lambda C_1}{r^3}} \right). \] (171)
Integrating this gives
\[ \phi(r) = \phi(0) + \frac{R^2}{8\lambda} \left\{ -\frac{r^2}{R^2} \pm \frac{r}{R} \sqrt{1 + \frac{r^3}{R^3} + 3 \, _2F_1 \left( \frac{1}{2}, \frac{1}{6}, \frac{7}{6}; -\frac{r^3}{R^3} \right)} \right\}, \] (172)
where the length scale is related to the previous first integral by
\[ R = (8\lambda C_1)^{1/3}. \] (173)
For small \( r \) the solution (172) behaves like \( \phi_2(r) \) in (107), and therefore it is not singular at the origin. On the other hand, for large \( r \), upon taking the upper + sign in (172), the solution behaves like \( \phi_1(r) \) in (106), while the lower − sign choice in (172) gives a solution that grows like \( r^2 \) for large \( r \).
Taking the upper sign in (172),
\[ \phi(r) \biggr|_{r \ll R} = \phi(0) + \frac{R^2}{8\lambda} \left\{ 4 \sqrt{\frac{r}{R}} - \frac{r^2}{R^2} + O \left( \left( \frac{r}{R} \right)^{7/2} \right) \right\}, \] (174)
\[ \phi(r) \biggr|_{r \gg R} = \phi(0) + \frac{R^2}{8\lambda} \left\{ \frac{3\Gamma(1/3)\Gamma(7/6)}{\sqrt{\pi}} - \frac{R}{r} + O \left( \left( \frac{R}{r} \right)^4 \right) \right\}, \] (175)
where \( \frac{3\Gamma(1/3)\Gamma(7/6)}{\sqrt{\pi}} = 4.2065 \cdots \). Taking the upper sign, the graph of \( 8\lambda (\phi(r) - \phi(0)) / R^2 \) follows.

Another branch, obtained by taking the lower sign in (172), is not so well-behaved for large \( r \).
Two branches of the mixed solution, $\frac{8}{7\pi} (\phi(r) - \phi(0))$, versus $r/R$, for $R > 0$.

### 5.3 Energy considerations

A symmetric energy-momentum tensor for the mixed $\mathcal{E}_1 - \mathcal{E}_2$ model is

\[
\Theta_{\mu\nu} = \phi_{\mu} \phi_{\nu} - \frac{1}{2} \delta_{\mu\nu} \phi_{\alpha} \phi_{\alpha} - \lambda \left( \phi_{\mu} \phi_{\nu} \phi_{\alpha} - \phi_{\alpha} \phi_{\mu} \phi_{\nu} - \phi_{\alpha} \phi_{\mu} \phi_{\nu} + \delta_{\mu\nu} \phi_{\alpha} \phi_{\beta} \phi_{\alpha} \beta \right),
\]

\[
\partial_{\mu} \Theta_{\mu\nu} = \mathcal{E} [\phi] \phi_{\nu},
\]

\[
\mathcal{E} [\phi] = \phi_{\alpha\alpha} - \lambda \left( \phi_{\alpha\alpha} \phi_{\beta\beta} - \phi_{\alpha\beta} \phi_{\alpha\beta} \right),
\]

where the equation of motion is $\mathcal{E} [\phi] = 0$. For static, spherically symmetric $\phi$, the energy density is

\[
\Theta_{00} = \frac{1}{2} \left( \partial_r \phi(r) \right)^2 - \frac{\lambda}{3} \partial_r \left( \partial_r \phi(r) \right)^3,
\]

and the total energy is

\[
E = 4\pi \int_0^\infty \Theta_{00} r^2 dr.
\]

This is finite for the bounded static solution that goes like $1/r$ for large $r$, but it is not finite for the solution that goes like $r^2$. For the finite case,

\[
\partial_r \phi = \frac{r}{4\lambda} \left( -1 + \sqrt{1 + \frac{R^3}{r^3}} \right),
\]

where $R^3 = 8\lambda C_1$ gives a length scale set by the first integral of the static equation. After some playing around, we find

\[
E = \frac{\pi R^5}{6\lambda^2} \int_0^\infty s^4 \left( -1 + \sqrt{1 + \frac{1}{s^3}} \right)^2 ds = \frac{1}{90} \frac{2^{2/3} \pi^3}{\Gamma \left( \frac{4}{3} \right)^2} \frac{R^5}{\lambda^2} = 0.22025 \frac{R^5}{\lambda^2}.
\]

This is true for $R \geq 0$, but actually, it is also of interest to consider cases where $R < 0$.

The bounded solution for $R < 0$ is real for $r \geq |R|$, as obtained by integrating

\[
\partial_r \phi = \frac{r}{4\lambda} \left( -1 + \sqrt{1 - \frac{|R|^3}{r^3}} \right).
\]
Two branches of the mixed solution, $\frac{8\lambda}{R^2} (\phi(r) - \phi(|R|))$, versus $r/|R|$, for $R < 0$.

The energy density has the same form as before, but now the total energy outside the singularity at $r = |R|$ is

$$E_{r \geq |R|} = \frac{\pi |R|^5}{6\lambda^2} \int_1^{\infty} s^4 \left( -1 + \sqrt{1 - \frac{1}{s^8}} \right)^2 ds + \frac{4\pi\lambda}{3} r^2 (\partial_r \phi)^3 \bigg|_{r = |R|}$$

$$= \frac{1}{180} \left( \frac{2\pi^3}{\Gamma \left( \frac{3}{2} \right)^2} - \frac{3}{4} \pi \right) \frac{|R|^5}{\lambda^2} = 0.097 \, 037 \, |R|^5 \lambda^2. \quad (184)$$

5.4 Perturbative scattering

Consider $p + q \rightarrow p' + q'$ for the $L_1 + \lambda L_2$ model, perturbatively on-shell, i.e. $p^2 = q^2 = p'^2 = q'^2 = 0$. The lowest-order scattering amplitude is

$$M = \frac{1}{4} \lambda^2 \left( s^3 + t^3 + u^3 \right)_{|u = -s - t} = \frac{3}{4} \lambda^2 stu_{|u = -s - t} = -\frac{3}{4} \lambda^2 s t (s + t). \quad (185)$$

In the CM frame, in terms of the incident energy and scattering angle, $p = (E, \vec{p})$ and $p' = E^2 \cos \theta$, we have $s = 4E^2$, $t = -2E^2 (1 - \cos \theta)$, and so

$$M_{CM} = 12\lambda^2 E^6 \sin^2 \theta = 12\lambda^2 E^6 \left( 1 - \cos^2 \theta \right) = 8\lambda^2 E^6 \left( P_0 (\cos \theta) - P_2 (\cos \theta) \right).$$

By the usual rules for the differential cross section in 4D, we then have

$$\frac{d\sigma}{d\Omega}_{CM} = \frac{1}{64\pi^2} \frac{1}{2!} \left( \frac{|M_{CM}|^2}{4E^2} \right) = \frac{9\lambda^4 E^{10} \sin^4 \theta}{32\pi^2}, \quad (186)$$

and total cross section

$$\sigma_{CM} = \frac{3\lambda^4 E^{10}}{5\pi}. \quad (187)$$

Note that $[\lambda] = 1/m^3$ in 4D. This approximation for $\sigma$ obviously exceeds the Froissart bound ($\propto \ln^2 E$) as the energy increases.
6 Effects of $\phi \Theta[\phi]$ self-couplings

In this section, we consider galileon theories with an additional self-coupling of the fields to the trace of their own energy-momentum tensor. We explore the classical features of one such model, in flat 4D spacetime, with emphasis on solutions that are scalar analogues of gravitational geons. We discuss the stability of these scalar geons, and some of their possible signatures, including shock fronts.

For the simplest example, the galileon field is usually coupled to all other matter through the trace of the energy-momentum tensor, $\Theta^{(\text{matter})}$. But surely, in a self-consistent theory the galileon should also be coupled to its own energy-momentum trace, even in the flat spacetime limit. Some consequences of this additional self-coupling are considered in this section, based on work published in \[8\].

Recall the action for the lowest non-trivial member of the galileon hierarchy,

$$A_2 = \frac{1}{2} \int \phi_{\alpha} \phi_{\beta} \phi_{\alpha \beta} \, d^4 x , \quad (188)$$

where $\phi$ is the scalar galileon field, $\phi_{\alpha} = \partial_{\alpha}(x) / \partial x^\alpha$, etc., and where repeated indices are summed using the Lorentz metric $\delta_{\mu \nu} = \text{diag}(1, -1, -1, \cdots)$.

### 6.1 A non-vanishing trace

As discussed above, including in $A_2$ a minimal coupling to a background spacetime metric yields a symmetric energy-momentum tensor, which becomes in the flat-space limit:

$$\Theta^{(2)}_{\mu \nu} = \phi_{\mu} \phi_{\nu} \phi_{\alpha \alpha} - \phi_{\mu} \phi_{\alpha \nu} \phi_{\beta \alpha} - \phi_{\alpha \mu} \phi_{\alpha \nu} + \delta_{\mu \nu} \phi_{\alpha \beta} \phi_{\alpha \beta} . \quad (189)$$

This is seen to be conserved,

$$\partial_{\mu} \Theta^{(2)}_{\mu \nu} = \phi_{\nu} \mathcal{E}_2[\phi] , \quad (190)$$

upon using the field equation that follows from locally extremizing $A_2$, $0 = \delta A_2 / \delta \phi = -\mathcal{E}_2[\phi]$, where

$$\mathcal{E}_2[\phi] \equiv \phi_{\alpha \beta} \phi_{\beta \alpha} - \phi_{\alpha \beta} \phi_{\alpha \beta} . \quad (191)$$

But, as previously noted, this $\Theta^{(2)}_{\mu \nu}$ is not traceless. Consequently, the usual form of the scale current, $x_n \Theta^{(2)}_{\alpha \mu}$, is not conserved \[27\]. On the other hand, the action \[153\] is homogeneous in $\phi$ and its derivatives, and is clearly invariant under the scale transformations $x \rightarrow s x$ and $\phi(x) \rightarrow s^{(4-n)/3} \phi(s x)$. Hence the corresponding Noether current must be conserved. This current is easily found, especially for $n = 4$, so let us restrict our attention to four spacetime dimensions in the following.

In that case the trace is obviously a total divergence:

$$\Theta^{(2)} = \delta_{\mu \nu} \Theta^{(2)}_{\mu \nu} = \partial_{\alpha} \left( \phi_{\alpha \beta} \phi_{\beta} \right) . \quad (192)$$

That is to say, for $n = 4$ the virial is the trilinear $V_{\alpha} = \phi_{\alpha \beta} \phi_{\beta}$. So a conserved scale current is given by the combination,

$$S_\mu = x_n \Theta^{(2)}_{\alpha \mu} - \phi_{\alpha \beta} \phi_{\alpha \beta} . \quad (193)$$

Interestingly, this virial is not a divergence modulo a conserved current, so this model is not conformally invariant despite being scale invariant. Be that as it may, it is not our principal concern here.

Our interest here is that the nonzero trace suggests an additional interaction where $\phi$ couples directly to its own $\Theta^{(2)}$. This is similar to coupling a conventional massive scalar to the trace of its own energy-momentum tensor \[21\]. In that previously considered example, however, the consistent coupling of the field to its trace required an iteration to all orders in the coupling. Upon summing the iteration and making a field redefinition, the Nambu-Goldstone model emerged. But, for the simplest galileon model in four spacetime dimensions, \[153\], a consistent coupling of field and trace is much easier to implement. No iteration is required. The first-order coupling alone is consistent, after integrating by parts and ignoring boundary contributions, so that\[6\]

$$- \frac{1}{3} \int \phi \partial_{\alpha} \left( \phi_{\alpha \beta} \phi_{\beta} \right) \, d^4 x = \frac{1}{3} \int \phi_{\alpha} \phi_{\alpha \beta} \phi_{\beta} \, d^4 x . \quad (194)$$

\[6\] Also note that $A_2$ follows from coupling $\phi$ to the trace of the manifestly chargeless tensor $(\partial_{\mu} \partial_{\nu} - \delta_{\mu \nu} \partial_{\alpha} \partial_{\alpha}) \phi_{\alpha \beta} \phi_{\beta}$.
(Similar quadrilinear terms have appeared previously in [10, 9, only multiplied there by scalar curvature $R$ so that they would drop out in the flat spacetime limit that we consider.) Consistency follows because [14] gives an additional contribution to the energy-momentum tensor which is traceless, in 4D spacetime:

$$\Theta^{(3)}_{\mu\nu} = \phi_\mu \phi_\nu - \frac{1}{4} \delta_{\mu\nu} \phi_\alpha \phi_\alpha \phi_\beta \phi_\beta, \quad \Theta^{(3)} = 0.$$  

(195)

Of course, coupling $\phi$ to its own trace may impact the Vainstein mechanism [37] by changing the effective coupling of $\Theta_{\text{matter}}$ to both backgrounds and fluctuations in $\phi$. We leave this as an exercise for the reader.

### 6.2 A model with additional quartic self-coupling

Based on these elementary observations, we consider a model with action

$$A = \int \left( \frac{1}{2} \phi_\alpha \phi_\alpha - \frac{1}{2} \lambda \phi_\alpha \phi_\beta \phi_\alpha \phi_\beta - \frac{1}{4} \kappa \phi_\alpha \phi_\beta \phi_\beta \phi_\beta \right) d^4x,$$

(196)

where for the Lagrangian $L$ we take a mixture of three terms: the standard bilinear, the trilinear galileon, and its corresponding quadrilinear trace-coupling. The quadrilinear is reminiscent of the Skyrme term in nonlinear $\sigma$ models [34] although here the topology would appear to be always trivial.

The second and third terms in $A$ are logically connected, as we have indicated. But why include in $A$ the standard bilinear term? The reasons for including this term are to soften the behavior of solutions at large distances, as will be evident below, and also to satisfy Derrick’s criterion for classical stability under the standard bilinear term? The reasons for including this term are to soften the behavior of solutions at large distances, as will be evident below, and also to satisfy Derrick’s criterion for classical stability under the standard bilinear term.

Based on these elementary observations, we consider a model with action

$$A = \int \left( \frac{1}{2} \phi_\alpha \phi_\alpha - \frac{1}{2} \lambda \phi_\alpha \phi_\beta \phi_\alpha \phi_\beta - \frac{1}{4} \kappa \phi_\alpha \phi_\beta \phi_\beta \phi_\beta \right) d^4x,$$

(196)

where for the Lagrangian $L$ we take a mixture of three terms: the standard bilinear, the trilinear galileon, and its corresponding quadrilinear trace-coupling. The quadrilinear is reminiscent of the Skyrme term in nonlinear $\sigma$ models [34] although here the topology would appear to be always trivial.

The second and third terms in $A$ are logically connected, as we have indicated. But why include in $A$ the standard bilinear term? The reasons for including this term are to soften the behavior of solutions at large distances, as will be evident below, and also to satisfy Derrick’s criterion for classical stability under the standard bilinear term.

Similarly, for positive $\kappa$, the last term in $A$ ensures the energy density of static solutions is always bounded below under a rescaling of the field $\phi$, a feature that would not be true if $\kappa = 0$ but $\lambda \neq 0$. So, we only consider $\kappa > 0$ in the following. But before discussing the complete $\Theta_{\mu\nu}$ for the model, we note that we did not include in $A$ a term coupling $\phi$ to the trace of the energy-momentum due to the standard bilinear term, namely, $\int \phi \Theta^{(1)} d^4x$, where

$$\Theta^{(1)}_{\mu\nu} = \phi_\mu \phi_\nu - \frac{1}{2} \delta_{\mu\nu} \phi_\alpha \phi_\alpha, \quad \Theta^{(1)} = -\phi_\alpha \phi_\alpha.$$  

(197)

We have omitted such an additional term in $A$ solely as a matter of taste, thereby ensuring that $L$ is invariant under constant shifts of the field. Among other things, this greatly simplifies the task of finding solutions to the equations of motion.

The field equation of motion for the model is $0 = \delta A / \delta \phi = -\mathcal{E} [\phi]$, where

$$\mathcal{E} [\phi] \equiv \phi_\alpha \phi_\alpha - \lambda (\phi_\alpha \phi_\beta - \phi_\beta \phi_\alpha) - \kappa (\phi_\alpha \phi_\beta).$$

(198)

As expected, this field equation is second-order, albeit nonlinear. Also note, under a rescaling of both $x$ and $\phi$, nonzero parameters $\lambda$ and $\kappa$ can be scaled out of the equation. Define

$$\phi (x) = \frac{\lambda}{\kappa} \psi \left( \sqrt{\frac{\kappa}{\lambda}} x \right).$$

(199)

Then the field equation for $\psi (z)$ becomes

$$\psi_\alpha - \left( \psi_\alpha \psi_\beta - \psi_\beta \psi_\alpha \right) = 0,$$

(200)

where $\psi_\alpha = \partial \psi (z) / \partial z^\alpha$, etc. In effect then, if both $\lambda$ and $\kappa$ do not vanish, it is only necessary to solve the model’s field equation for $\lambda = \kappa = 1$.

### 6.3 Static solutions

For static, spherically symmetric solutions, $\phi = \phi (r)$, the field equation of motion becomes

$$0 = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \left( \phi' + \lambda \frac{2}{r} (\phi')^2 + \kappa (\phi')^3 \right) \right).$$

(201)

where $\phi' = d\phi / dr$. This is immediately integrated once to obtain a cubic equation,

$$r^2 \phi' + 2\lambda r \phi'^2 + \kappa r^2 (\phi')^3 = C,$$

(202)
where $C$ is the constant of integration. Now, without loss of generality (cf. (199) and (200)) we may choose $\lambda > 0$. Then, if $C = 0$, either $\phi'$ vanishes, or else there are two solutions that are real only within a finite sphere of radius $r = \sqrt{\lambda^2 / \kappa}$. These two “interior” solutions are given exactly by

$$
\phi'_{\pm} = -\frac{1}{rk} \left( \lambda \pm \sqrt{\lambda^2 - r^2 \kappa} \right).
$$

Note that these solutions always have $\phi' < 0$ within the finite sphere.

Otherwise, if $C \neq 0$, then examination of the cubic equation for small and large $|\phi'|$ determines the asymptotic behavior of $\phi'$ for large and small $r$. In particular, there is only one type of asymptotic behavior for large $r$:

$$
\phi' \sim \frac{C}{r^2} \text{ for either sign of } C.
$$

However, there are two types of behavior for large $|\phi'|$, corresponding to small $r$. Either

$$
r = -\frac{2\lambda}{\phi' \kappa} \left( 1 + O \left( \frac{1}{\phi'} \right) \right)
$$

provided $\phi' < 0$, but with either sign of $C$; or else

$$
r = \frac{1}{\phi'^2} \left( \frac{C}{2\lambda} + O \left( \frac{1}{\phi'} \right) \right)
$$

provided $C > 0$, but with either sign of $\phi'$. The corresponding real solutions behave as

$$
\phi' \sim \frac{-2\lambda}{\kappa r} \text{ for either sign of } C, \text{ or }
$$

$$
\phi' \sim \pm \sqrt{\frac{C}{2\lambda r}} \text{ provided } C > 0.
$$

Comparison of the small $r$ behavior to the large $r$ asymptotics shows that in half these cases the solutions would require zeroes to be real and continuous for all $r$. But such zeroes do not occur. Instead, half of the cases provide real solutions only over a finite interval of $r$, somewhat similar to the $C = 0$ solutions in (203), but not so easily expressed, analytically.

The solutions which are real for all $r > 0$ boil down to two cases, with small and large $r$ behavior given by either

$$
\phi' \sim \sqrt{\frac{C}{2\lambda r}} \text{ and } \phi' \sim \frac{C}{r^2} \text{ for } C > 0,
$$

or else

$$
\phi' \sim \frac{-2\lambda}{\kappa r} \text{ and } \phi' \sim \frac{C}{r^2} \text{ for } C < 0.
$$

From further inspection of the cubic equation to determine the behavior of $\phi'$ for intermediate values of $r$, when $C > 0$ it turns out that $\phi'$ is a single-valued, positive function for all $r > 0$, joining smoothly with the asymptotic behaviors given in (209). However, it also turns out there is an additional complication when $C < 0$. In this case there is a critical value $(\kappa^3/2 / \lambda^2)_{\text{critical}} = -4\sqrt{3} / 27 \approx -0.2566$ such that, if $C \leq (\kappa^3/2 / \lambda^2)_{\text{critical}}$ then $\phi'$ is a single-valued, negative function for all $r > 0$, while if $C > (\kappa^3/2 / \lambda^2)_{\text{critical}}$ then $\phi'$ is triple-valued for an open interval in $r > 0$. It is not completely clear to us what physics underlies this multivalued-ness for some negative $C$. But in any case, when $C < 0$ it is also true that $\phi'$ joins smoothly with the asymptotic behaviors given in (210). All this is illustrated in the two Figures to follow, for $\lambda = \kappa = 1$.

A test particle coupled by $\phi \Theta_{\text{matter}}$ to any of these galileon field configurations would see an effective potential which is not $1/r$, for intermediate and small $r$. Therefore its orbit would show deviations from the usual Kepler laws, including precession that is possibly at variance with the predictions of conventional general relativity. It would be interesting to search for such effects, say, by considering stars orbiting around the galactic center [29 28].
\[ \psi' (r) \text{ for } C = +1/4^N, \text{ with } N = 0, 1, 2, 3 \text{ for top to bottom curves, respectively.} \]

\[ \psi' (r) \text{ for } C = -1/2^N, \text{ with } N = 6, 5, 4, 3, 2, 1, 0 \text{ from left to right, respectively. The thin black curve is a union of the two } C = 0 \text{ solutions in (203).} \]

For the solutions described by (209) and (210), the total energy outside any large radius is obviously finite for both \( C > 0 \) and \( C < 0 \). And if \( C > 0 \), the total energy within a small sphere surrounding the origin is also manifestly finite. But if \( C < 0 \) the energy within that same small sphere could be infinite unless there is a cancellation between the galileon term and the trace interaction term. Remarkably, this cancellation does occur\(^7\). So both \( C > 0 \) and \( C < 0 \) static solutions for the model have finite total energy.

6.4 Energy considerations again

Complete information about the distribution of energy is provided by the model’s energy-momentum tensor,

\[ \Theta_{\mu \nu} = \Theta^{(1)}_{\mu \nu} - \lambda \Theta^{(2)}_{\mu \nu} - \kappa \Theta^{(3)}_{\mu \nu}. \quad (211) \]

As expected, this is conserved, given the field equation \( E [\phi] = 0 \), since

\[ \partial_\mu \Theta_{\mu \nu} = \phi_\nu E [\phi]. \quad (212) \]

\(^7\)For \( C < 0 \), to see cancellation between the individually divergent galileon and trace interaction energies for small \( r \) requires leading and next-to-leading terms in the expansion: \( \phi' \sim \frac{2 \lambda}{4 \pi r} + \frac{O}{4 \pi r^2} + O (r) \).
The energy density for static solutions differs from the canonical energy density for such solutions (namely, \(-L\)) by a total spatial divergence that arises from the galileon term:

\[ \Theta_{00} = -L|_{\text{static}} - \frac{1}{2} \lambda \vec{\nabla} \cdot \left( (\nabla \phi)^2 \vec{\nabla} \phi \right). \] (213)

This divergence will not contribute to the total energy for fields such that \(\lim_{r \to \infty} (\phi/\ln r)\) exists. Assuming that is the case, Derrick’s scaling argument for static, finite energy solutions of the equations of motion [12] shows the energy is just twice that due to the bilinear \(\Theta^{(1)}_{00}\). Thus,

\[ E = \int \Theta_{00} \, d^3r = \int \left( \vec{\nabla} \phi \right)^2 \, d^3r. \] (214)

For the spherically symmetric static solutions of (202), this becomes an expression of the energy as a function of the parameters and the constant of integration \(C\):

\[ E[\lambda, \kappa, C] = 4\pi \int_0^\infty (\phi')^2 r^2 \, dr. \] (215)

Again without loss of generality, consider \(\lambda = \kappa = 1\). Then for either \(C > 0\) or for \(C < C_{\text{critical}} < 0\), change integration variables from \(r\) to \(s \equiv \phi'\) to find:

\[ E(C \gtrless 0) = I(|C|) \mp (|C| + \frac{1}{2} \pi), \] (216)

\[ I(C > 0) = \frac{1}{2} \int_0^\infty \frac{P(s, C) \, ds}{(s^2 + 1)^4 \sqrt{s^4 + s(s^2 + 1)C}}, \] (217)

where the numerator of the integrand is an eighth-order polynomial in \(s\), namely,

\[ P(s, C) = 8s^8 + 12Cs^7 + (3C^2 - 8)s^6 + 8Cs^5 + 7C^2s^4 - 4Cs^3 + 5C^2s^2 + C^2. \] (218)

Thus, \(I(C)\) is an elliptic integral. But rather than express the final result in terms of standard functions, it suffices here just to plot \(E(C)\), in the Figure below. Note that \(E\) increases monotonically with \(|C|\).

For other values of \(\lambda\) and \(\kappa\) with the constant of integration \(C\) specified as in (202), the energy of the solution is given in terms of the function defined by (216, 217):

\[ E[\lambda, \kappa, C] = \left( \frac{\lambda^3}{\kappa^{5/2}} \right) E \left( \frac{\kappa^{3/2} C}{\lambda^2} \right). \] (219)

The energy curves indicate double degeneracy in \(E\), for different values of \(|C|\), when \(E[\lambda, \kappa, C] > \pi \lambda^3 / \kappa^{5/2}\). Also, for a given \(|C|\) the negative \(C\) solutions are higher in energy, with \(E[\lambda, \kappa, -|C|] - E[\lambda, \kappa, |C|] = \pi \lambda^3 / \kappa^{5/2} + 2|C|\lambda^3 / \kappa\). Or at least this is true for all \(|C| \geq |C_{\text{critical}}|\) in which case \(E[\lambda, \kappa, C] \geq \frac{\lambda^3}{\kappa^{5/2}} E \left( \frac{\kappa^{3/2} C_{\text{critical}}}{\lambda^2} \right) \approx 3.7396 \lambda^3 / \kappa^{5/2}\).
**6.5 Scalar geons and a shock-front conjecture**

Finite energy classical solutions of gravity-like theories bring to mind the “geons” proposed long ago by Wheeler [40]. These were envisioned in their purest form as distributions of only gravitational energy held together solely by gravitational interaction. Combinations of electromagnetic energy and gravity were also considered, as were systems containing neutrinos. Wheeler argued that such configurations would be relatively stable, if they existed, but would eventually dissipate due to a variety of both classical and quantum effects, including light-light scattering, as well as production and absorption of quanta. While plausible distributions were sketched, and decay rates were estimated, exact classical solutions were not found.

The same mechanisms would seem to apply to any hypothetical classical galileon distributions such as those discussed here, the main difference being that analytic spherically symmetric solutions might still be obtainable even if conventional gravitational effects were included. Perhaps these gravitational effects would not alter the qualitative features of the static pure $\phi$ configurations given above. Should they really exist, presumably these galileon geons could also be dissipated by various classical and quantum effects. All this is far beyond our current abilities and the scope of this paper, of course, but the general ideas suggest some interesting possibilities.

Whatever the cause, if the configuration’s energy loss were gradual, as a first step it might suffice to model the time-dependent system quasi-statically, as a continuous flow from one static solution to another. That is to say, perhaps a good approximation would be to take $C (t)$, with $|C|$ and $E (C)$ decreasing monotonically with time. For the positive $C$ case, this would be more or less uneventful as the whole configuration would just slowly disappear without any abrupt changes. But for the negative $C$ case, as $t$ increased $C_{\text{critical}}$ would be reached, beyond which the solution would begin to fold over, exhibiting the multivalued features shown in the Figure. But this is just the usual picture for the formation of a shock front. These particular galileon shocks would implode, converging towards the origin, as shown here. We believe this is a plausible scenario and a reasonable physical interpretation of the model’s multivalued solutions. Moreover, it would seem to provide a signature for their existence.

As is clear from the Figure, the shock front would form when $d\phi / dr = \infty$. For the $C < 0$ static solutions of (202) it is not difficult to determine the locus of such singular points. It is given by the intersection of the solutions, for various $C$, and the curve $(1 + 3\kappa \phi^2) r = 4\lambda \phi'$. As usual for singular points in the development of a shock, almost certainly there is some physics missing from the equations. Since $\phi''$ is large, the obvious modification would be to include higher derivative terms in the action, which is tantamount to attempting an ultraviolet completion of the model. This is an open question. Perhaps higher terms in the galileon hierarchy would be natural candidates to be included.

**6.6 Comparison to the self-dual model**

To get a handle on such terms, and for purposes of comparison to the model in [196], consider briefly another model somewhat similar in form, but whose Lagrangian consists only of terms taken from the galileon hierarchy, without any coupling to $\Theta$. After rescaling the field and coordinates to achieve a standard form, this alternate model may be defined by

$$A_{\text{self-dual}} [\psi] = \int \left( \frac{1}{4} \psi_{\alpha} \psi_{\alpha} - \frac{1}{4} \psi_{\alpha} \psi_{\beta\beta} + \frac{1}{12} \psi_{\alpha} \psi_{\beta\beta} \left( \psi_{\beta\gamma} \psi_{\beta\gamma} - \psi_{\beta\gamma} \psi_{\beta\gamma} \right) \right) d^4x .$$  (220)

The difference with (196) lies in the last term, which is quadrilinear in the field, as before, but now has two fields with second derivatives.

As the name suggests, this model is self-dual, in the following sense: The action retains its form under a Legendre transformation [19] (also see [22]) to a new field $\Psi$ and new coordinates $X$, as defined by:

$$\psi (x) + \Psi (X) = x_\alpha X_\alpha .$$  (221)

Thus $A_{\text{self-dual}} [\psi] = A_{\text{self-dual}} [\Psi]$, provided integrations by parts give no surface contributions. This identity suggests that there are interesting properties for the quantized model, such as its ultraviolet behavior, but that is outside the scope of the present discussion.
Here it suffices to compare the classical physics following from (220) with that following from (196). Upon integrating once the classical equations of motion for static, spherically symmetric solutions of the field equations for (220), the result is again a cubic equation,

$$r^2 \psi' + r (\psi')^2 + \frac{1}{3} (\psi')^3 = C,$$

(222)

but the $(\psi')^3$ term is no longer weighted by $r^2$ as it was in (202). Thus the small and large $r$ behaviors are now given by

$$\psi' \sim \frac{3C}{r^{1/3}} \quad \text{and} \quad \psi' \sim \frac{C}{r^{2}},$$

(223)

for either sign of the constant of integration, $C$. These static solutions have finite total energy for either sign of $C$, as before, only now $\psi'$ is always bounded. Moreover, upon inspection of the behavior of $\psi'$ for intermediate $r$, and various $C$, unlike the previous model the solutions are now always single-valued for either $C > 0$ or $C < 0$. Thus there are no multivalued solutions like those shown in the previous Figure for various $C < 0$. However, each of the $C < 0$ static solutions now has a single point for which $d\psi'/dr = \infty$, namely, $r = (3|C|/2)^{1/3}$. So there is still a reason to expect the existence of shock fronts for quasi-static time-dependent fields in this alternate model. Finally, again for $C < 0$, to have $\phi'$ real for all $r > 0$, it is necessary to join together “interior” and “exterior” solutions at $r = (3|C|/2)^{1/3}$. These features are illustrated in the following Figure.
Static solutions of the self-dual model for $C > 0$ upper half-plane, and for $C < 0$, lower half plane. The solutions $r_{\pm} = \frac{\phi'}{2} \left( -1 \pm \sqrt{\frac{4C}{(\phi')^2} - \frac{1}{3}} \right)$ are shown in orange/green.

In these graphs, the solutions of (222) are shown for both physical $r > 0$ and unphysical $r < 0$ to display some symmetry relations between the $C > 0$ cases and the interior and exterior solutions for $C < 0$. The straight lines, in gray, are the loci of points where the solutions have zero and infinite slopes, for different values of $C$.

It remains to investigate the stability of these spherically symmetric solutions under perturbations, especially to check for the existence of superluminal modes, along the lines of [23]. Evidently, superluminal modes are a possible feature for models of this type.
7 General relativistic effects

In this section, the simple trace-coupled Galileon model of the previous section is coupled minimally to gravitation (GR) and shown to admit spherically symmetric static solutions with naked spacetime curvature singularities.

In the previous section, based on [8], the effects of coupling a Galileon to its own energy-momentum trace were considered in the flat spacetime limit. Here, general relativistic effects are taken into consideration and additional features of this same model are explored in curved spacetime [9, 10]. Such features have been explored in the literature (see [7], and for a related class of models, [2]). The main point to be emphasized here is that there can be solutions with naked singularities when the energy in the scalar field is finite and not too large, and for which the effective mass of the system is positive. Thus for the simple model at hand there is an open set of physically acceptable scalar field data for which curvature singularities are not hidden inside event horizons [31] [32]. This would seem to have important implications for the cosmic censorship conjecture [30, 31, 33]. It is worthwhile to note that, in general, naked singularities have observable consequences that differ from those due to black holes [38].

7.1 Minimal coupling to gravity

The scalar field part of the action in curved space is

\[
A = \frac{1}{2} \int g^{\alpha\beta} \phi_\alpha \phi_\beta \left(1 - \frac{1}{\sqrt{-g}} g^\rho_\mu \phi_\rho \right) \left(1 - \frac{1}{2} g^{\mu\nu} \phi_\mu \phi_\nu \right) \sqrt{-g} \, d^4 x . \tag{224}
\]

This gives a symmetric energy-momentum tensor \( \Theta_{\alpha\beta} \) for \( \phi \) upon variation of the metric.

\[
\delta A = \frac{1}{2} \int \sqrt{-g} \Theta_{\alpha\beta} \delta g^{\alpha\beta} \, d^4 x . \tag{225}
\]

\[
\Theta_{\alpha\beta} = \phi_\alpha \phi_\beta \left(1 - g^{\mu\nu} \phi_\mu \phi_\nu \right) - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} \phi_\mu \phi_\nu \left(1 - \frac{1}{2} g^{\rho\sigma} \phi_\rho \phi_\sigma \right)
- \phi_\alpha \phi_\beta \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \phi_\nu \right) + \frac{1}{2} \partial_\alpha \left( g^{\mu\nu} \phi_\mu \phi_\nu \right) \phi_\beta + \frac{1}{2} \partial_\beta \left( g^{\mu\nu} \phi_\mu \phi_\nu \right) \phi_\alpha - \frac{1}{2} g_{\alpha\beta} \partial_\mu \left( g^{\mu\nu} \phi_\mu \phi_\nu \right) g^{\rho\sigma} \phi_\rho \phi_\sigma . \tag{226}
\]

It also gives the field equation for \( \phi \) upon variation of the scalar field, \( E [\phi] = 0 \), where

\[
\delta A = - \int \sqrt{-g} E [\phi] \, \delta \phi \, d^4 x , \tag{227}
\]

\[
E [\phi] = \partial_\alpha \left[ g^{\alpha\beta} \phi_\beta \sqrt{-g} - g^{\alpha\beta} \phi_\beta g^{\mu\nu} \phi_\mu \phi_\nu \sqrt{-g} - g^{\alpha\beta} \phi_\beta \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \phi_\nu \right) + \frac{1}{2} \sqrt{-g} g^{\alpha\beta} \partial_\beta \left( g^{\mu\nu} \phi_\mu \phi_\nu \right) \right] . \tag{228}
\]

Since \( E [\phi] \) is a total divergence, it easily admits a first integral for static, spherically symmetric configurations. Consider only those situations in the following.

7.2 Static spherical solutions

For such configurations the metric in generalized Schwarzschild coordinates is [36]

\[
(ds)^2 = e^{N(r)} \left( \frac{dr}{e^{L(r)}} \right)^2 - \frac{1}{2} \frac{c_1}{r^2} \sin^2 \theta \left( \frac{dr}{e^{L(r)}} \right)^2 - \frac{1}{2} \frac{c_2}{r^2} \sin^2 \theta \left( \frac{d\phi}{e^{L(r)}} \right)^2 . \tag{229}
\]

Thus for static, spherically symmetric \( \phi \), with covariantly conserved energy-momentum tensor (226), Einstein’s equations reduce to just a pair of coupled 1st-order nonlinear equations:

\[
r^2 \Theta^t_\text{t} = e^{-L} \left( r L' - 1 \right) + 1 , \tag{230}
\]

\[
r^2 \Theta^r_\text{r} = e^{-L} \left( -r N' - 1 \right) + 1 . \tag{231}
\]

These are to be combined with the first integral of the \( \phi \) field equation in this situation. Defining

\[
\eta (r) \equiv e^{-L(r)/2} , \quad \varpi (r) \equiv \eta (r) \phi' (r) , \tag{232}
\]

that first integral becomes

\[
\frac{C e^{-N/2}}{r^2} = \varpi \left( 1 + \varpi^2 \right) + \frac{1}{2} \left( N' + \frac{4}{r} \right) \eta \varpi^2 , \tag{233}
\]

33
where for asymptotically flat spacetime the constant \( C \) is given by \( \lim_{r \to \infty} r^2 \phi' (r) = C \). Then upon using

\[
\begin{align*}
\Theta^t_t &= \Theta^{\phi}_\phi = \Theta^{\chi}_\chi = \frac{1}{2} \mathcal{R} (1 + \frac{1}{2} \mathcal{R}^2) - \frac{1}{2} \phi \phi', \\
\Theta^{\eta}_\eta &= -\frac{1}{2} \mathcal{R}^2 (1 + \frac{3}{2} \mathcal{R}^2) - \frac{1}{2} \eta \mathcal{R} (N' + \frac{4}{7} C),
\end{align*}
\]

(234) \hspace{1cm} (235)

the remaining steps to follow are clear.

First, for \( C \neq 0 \), one can eliminate \( N' \) from (231) and (233) to obtain an exact expression for \( N \) in terms of \( \eta, \varpi, \) and \( C \):

\[
e^{N/2} = \frac{8C}{r \varpi (4\varpi - 2r^2 \mathcal{R}^3 - r^2 \mathcal{R}^5 + 8r \eta + 12r \varpi \eta^2 + 8r \varpi^2 \eta)}.
\]

(236)

If the numerator of this last expression vanishes there is an event horizon, otherwise not. When \( \eta = \frac{1}{2} r \varpi^3 \) the denominator of (236) is positive definite.

Next, in addition to (230) one can now eliminate \( N \) from either (231) or (233) to obtain two coupled first-order nonlinear equations for \( \eta \) and \( \varpi \). These can be integrated, at least numerically. Or they can be used to determine analytically the large and small \( r \) behaviors, hence to see if the energy and curvature are finite. For example, again for asymptotically flat spacetime, it follows that

\[
e^{L/2} \sim 1 + \frac{M}{r} + \frac{1}{4} \left( 6M^2 - C^2 \right) \frac{1}{r^2} + \frac{1}{2} \left( 5M^2 - 2C^2 \right) \frac{1}{r^3} + O \left( \frac{1}{r^4} \right),
\]

(237)

\[
e^{N/2} \sim 1 - \frac{M}{r} - \frac{1}{2} M \frac{2}{r^2} + \frac{1}{12} M^2 \frac{1}{r^3} + O \left( \frac{1}{r^4} \right),
\]

(238)

\[
\varpi \sim C \frac{r^2}{r} \left( 1 + \frac{M}{r} + \frac{3}{2} \frac{M^2}{r^2} \right) + O \left( \frac{1}{r^3} \right),
\]

(239)

for constant \( C \) and \( M \).

As of this writing the details of the two remaining first-order ordinary differential equations are not pretty, but the equations are numerically tractable. In terms of the variables defined in (232), in light of (230), Einstein’s equation (231) becomes

\[
F (r, \varpi, \eta) \frac{d}{dr} \varpi + G (r, \varpi, \eta) r \frac{d}{dr} \eta = H (r, \varpi, \eta),
\]

(240)

\[
F (r, \varpi, \eta) = -4 \eta \left[ 2r^2 \varpi^6 + 3r^3 \varpi^8 + 16 \varpi \eta + 4r \varpi^4 \\
+ 16 \eta^2 + 48 \varpi \eta^3 + 48r \varpi^2 \eta^2 + 12r \varpi^4 \eta^2 - 12r^2 \varpi^5 \eta \right],
\]

(241)

\[
G (r, \varpi, \eta) = 8 \eta \varpi \left[ 2r^2 \varpi^2 + 3r^2 \varpi^4 - 12 \eta^2 + 12 \varpi \eta^2 + 4 \right],
\]

(242)

\[
H (r, \varpi, \eta) = \varpi \left[ 8 \eta \varpi \left( 4r \varpi^3 - 4 \eta + 2r \varpi^2 \eta + 3r^2 \varpi^4 \eta + 12r \varpi^3 \eta^2 - 12 \eta^3 \right) \\
+ \left( 4 + 3r^2 \varpi^4 + 2r^2 \varpi^2 + 12 \eta^2 \right) \left( 4 \varpi - r^2 \varpi^5 - 2r^2 \varpi^3 + 8r \varpi^2 \eta + 8 \eta + 12 \varpi \eta^2 \right) \right],
\]

(243)

while Einstein’s equation (230) becomes

\[
I (r, \varpi, \eta) \frac{d}{dr} \varpi + J (r, \varpi, \eta) r \frac{d}{dr} \eta = K (r, \varpi, \eta),
\]

(244)

\[
I (r, \varpi, \eta) = r \eta \varpi^2, \quad J (r, \varpi, \eta) = -2 \eta,
\]

(245)

\[
K (r, \varpi, \eta) = \frac{1}{2} r \varpi^2 (1 + \frac{3}{2} \varpi^2) + \eta^2 - 1.
\]

(246)

### 7.3 Numerical results

As a representative example with \( \varpi > 0 \), (244) and (240) were integrated numerically to obtain the results shown in the Figure, for data initialized as \( \varpi |_{r=1} = 0.5 \) and \( \eta |_{r=1} = 1 \). Evidently it is true that \( \eta (r) \neq \frac{1}{2} r \varpi^3 (r) \) for this case, so \( e^{N(r)} \) does not vanish for any \( r > 0 \) and there is no event horizon.
For initial values $\varpi(s)|_{s=0} = 0.5$ and $\eta(s)|_{s=0} = 1.0$, $d\phi/dr = \varpi/\eta$ is shown in red, $e^L = 1/\eta^2$ in green, and $e^N$ in blue, where $r = e^s$. For comparison, Schwarzschild $e^L$ and $e^N$ are also shown as resp. green and blue dashed curves for the same $M \approx 0.21$.

“... an exotic type of matter with which human science is entirely unfamiliar is required for such a geometry to exist.” — B K Tippett [35]
However, there is a geometric singularity at \( r = 0 \) with divergent scalar curvature: \( \lim_{r \to 0} r^{3/2}R = \text{const.} \) Since \( R = -\Theta_{\mu}^{\mu} \), and \( \lim_{r \to 0} \varpi \) is finite, this divergence in \( R \) comes from the last term in \( \Box \), which in turn comes from the second term in \( A \), i.e. the covariant \( \partial \phi \partial \hat{\phi} \partial^2 \phi \) in \( \Box \). In fact, it is not difficult to establish analytically for a class of solutions of the model, for which the example in the Figure is representative, the following limiting behavior holds.

\[
\lim_{r \to 0} \left( e^{L/2} / \sqrt{r} \right) = \ell , \\
\lim_{r \to 0} \left( \sqrt{r} e^{N/2} \right) = n , \\
\lim_{r \to 0} \varpi = p , \\
\lim_{r \to 0} \left( \phi' / \sqrt{r} \right) = p\ell ,
\]

(247)

where \( \ell, n, \) and \( p \) are constants related to the constant \( C \) in \( \Box \):

\[
2C = 3np^2 / \ell .
\]

(248)

It follows that for solutions in this class,

\[
\lim_{r \to 0} r^{3/2}R = pC/n .
\]

(249)

For the example shown in the Figure: \( \ell \approx 1.5, n \approx 0.086, \, p \approx 3.3, \, C \approx 0.94, \) and \( pC/n \approx 36 \).

For the same \( \eta \big|_{r=1} = 1 \), further numerical results show there are also curvature singularities without horizons for smaller \( \varpi \big|_{r=1} > 0 \), but event horizons are present for larger scalar fields (roughly when \( \varpi \big|_{r=1} > 2/3 \)). A more precise and complete characterization of the data set \( \{ \varpi \big|_{r=1}, \eta \big|_{r=1} \} \) for which there are naked singularities is in progress, but it is already evident from the preceding remarks that the set has nonzero measure.

The energy contained in only the scalar field in the curved spacetime is given by

\[
E_{\text{Galileon}} = \int_0^\infty \mathcal{H}(r) \, dr = \int_{-\infty}^\infty e^s \mathcal{H}(e^s) \, ds ,
\]

(250)

\[
\mathcal{H}(r) \equiv 4\pi r^2 e^{L/2} e^{N/2} \Theta_{\epsilon}^{\ell} = 2\pi e^{2s} e^{L/2} e^{N/2} \varpi^2 (s) \left( 1 + \frac{1}{2} \varpi^2 (s) \right) - 4\pi e^{N/2} \varpi^2 (s) \frac{d}{ds} \varpi (s) .
\]

(251)

For the above numerical example, the integrand \( e^s \mathcal{H}(e^s) \) is shown in the Figure. Evidently, \( E_{\text{Galileon}} \) is finite in this case. It is also clear from the Figures that the Galileon field has significant effects on the geometry in the vicinity of the peak of its radial energy density. There the metric coefficients are greatly distorted from the familiar Schwarzschild values, and as a consequence, the horizon is eliminated.

36
Other numerical examples

Here are additional plots for $\varpi(s)|_{s=0} = 1.0$ and various initial values $\varpi(s)|_{s=0}$. As before, $d\phi/dr = \varpi/\eta$ is shown in red, $e^L = 1/\eta^2$ in green, and $e^N$ in blue, where $r = e^s$. For comparison, Schwarzschild $e^L$ and $e^N$ are also shown as resp. green and blue dashed curves for the same $M$, as given in the Figure labels.

Initial values $\varpi(s)|_{s=0} = 0.100$ and $\eta(s)|_{s=0} = 1.00$ corresponding to $M = 0.00358$ and $C = 0.121$

Initial values $\varpi(s)|_{s=0} = 0.200$ and $\eta(s)|_{s=0} = 1.00$ corresponding to $M = 0.0191$ and $C = 0.283$

Initial values $\varpi(s)|_{s=0} = 0.300$ and $\eta(s)|_{s=0} = 1.00$ corresponding to $M = 0.0546$ and $C = 0.484$

Initial values $\varpi(s)|_{s=0} = 0.400$ and $\eta(s)|_{s=0} = 1.00$ corresponding to $M = 0.117$ and $C = 0.710
Initial values $\varpi(s)|_{s=0} = 0.500$ and $\eta(s)|_{s=0} = 1.00$ corresponding to $M = 0.209$ and $C = 0.936$

Initial values $\varpi(s)|_{s=0} = 0.600$ and $\eta(s)|_{s=0} = 1.00$ corresponding to $M = 0.326$ and $C = 1.13$

Initial values $\varpi(s)|_{s=0} = 0.700$ and $\eta(s)|_{s=0} = 1.00$ corresponding to $M = 0.453$ and $C = 1.26$

Initial values $\varpi(s)|_{s=0} = 0.800$ and $\eta(s)|_{s=0} = 1.00$ corresponding to $M = 0.573$ and $C = 1.32$

For each of the last two plots, the numerical integration of the coupled galileon-GR equations has encountered a mathematical (as opposed to physical) singularity and terminated, resp. at $r \approx e^{-2.5} = 0.082$ and $r \approx e^{-1.3} = 0.27$, as is indicative of an horizon for which $e^N = 0$. This feature persists for initial data with larger values of $\varpi(s)|_{s=0}$, when $\eta(s)|_{s=0} = 1$. 
Here are two more cases, just below and just above the point where horizons are formed. Again, for the second of these plots, the numerical integration of the coupled galileon-GR equations has encountered a mathematical singularity, and terminated at the point where $e^{N(r)}$ (blue curve) vanishes.

Initial values $\varpi(s)|_{s=0} = 0.645$ and $\eta(s)|_{s=0} = 1.00$ corresponding to $M = 0.383$ and $C = 1.199$

Initial values $\varpi(s)|_{s=0} = 0.652$ and $\eta(s)|_{s=0} = 1.00$ corresponding to $M = 0.392$ and $C = 1.209$

A useful test for an horizon is provided by the numerator of $e^N$ in (236). Define the discriminant

$$disc(r) = 1 - \frac{r \, \varpi(r)^3}{2 \, \eta(r)}.$$  \hspace{1cm} (252)

Should this vanish at some radius for which $\eta(r)$ is finite, then at that radius $e^{N(r)} = 0$, thereby indicating an horizon at that radius.

The critical case, separating solutions with naked singularities from those with event horizons, has the small $r$ limiting behavior $\eta(r) \sim r^{-3} \varpi^3(r)$, such that the discriminant $\frac{\text{disc} \, r}{r \rightarrow 0} \sim \frac{1}{2}$ as illustrated here for specific data.
The discriminant \( \text{disc} = 1 - \frac{1}{2} r \varpi^3 / \eta \) versus \( s = \ln r \) for various \( \varpi \rvert_{r=1} \) (namely, 0.4, 0.5, 0.6, 0.64, critical, 0.66, and 0.7) with \( \eta \rvert_{r=1} = 1 \). The critical initial value for the separatrix, for which \( \text{disc} \sim 1/2 \), is \( \varpi \rvert_{r=1} = 0.65002917 \cdots \).

For initial data giving rise to naked singularities, \( \text{disc} > 1/2 \) (cf. the upper curves in the Figure above), while for data leading to horizons, \( e^N \) vanishes at the horizon radius, and therefore at that radius \( \text{disc} = 0 \) (cf. the lower two curves in the Figure). When the limiting critical behavior \( \eta(r) \sim r^{\varpi^3(r)} \) is inserted into the differential equations (241) and (240) we find the power law behavior:

\[
\eta_{\text{critical}}(r) \sim r^{c_3 r^{-4/5}}, \quad \varpi_{\text{critical}}(r) \sim c r^{-3/5}, \quad \phi'_{\text{critical}}(r) \sim \frac{r^{1/5}}{c^2}.
\]

Moreover, critical cases are easily determined numerically for various initial data, \( \{ \varpi(s) \rvert_{s=0}, \eta(s) \rvert_{s=0} \} \), thereby allowing determination of a curve that separates the open set of initial data that exhibits naked singularities from the set that exhibits event horizons.

### 7.4 Censored and naked phases

The situation for a portion of the initial data plane is as follows.

\( (\varpi \rvert_{r=1}, \eta \rvert_{r=1}) \) boundary separating initial data that exhibit naked singularities from data that exhibit horizons. The curve is a fourth-order polynomial fit to the numerically computed critical points (dots), namely, \( \eta_{\text{fit}}(\varpi) = 1 + 0.0255538 \varpi - 1.34405 \varpi^2 + 2.20589 \varpi^3 - 0.304933 \varpi^4 \).

This shows naked singularities for the model exist for an initial data set of non-zero measure, and are actually encountered for a significant portion of the initial data plane.
A similar demarcation between naked/censored solutions can be presented in terms of asymptotic \( r \to \infty \) data instead of initial \( r = 1 \) data. With \( M \) and \( C \) defined as in (237), (238), and (239), we find the following curve separating the two types of solutions. Solutions for points above the red curve have naked singularities, while solutions for points below that curve have event horizons.

![Computed points (red circles) and an interpolating curve (solid red) separating the \( r \to \infty \) asymptotic data for solutions with naked singularities from that for solutions with event horizons.](image)

By imposing the same \( \eta(s)|_{s=0} \) initial condition for various values of \( \varpi(s)|_{s=0} \), the numerical data also shows that the corresponding \( C(M) \) has a local maximum, and hence \( M(C) \) becomes double-valued near that point. For example, when \( \eta(s)|_{s=0} = 1 \) the local maximum for \( C(M) \) is near \( \varpi(s)|_{s=0} \approx 0.78 \). By examining larger \( \varpi(s)|_{s=0} \) for the same \( \eta(s)|_{s=0} \), it is apparent that \( C(M) \) can also be double-valued. All this is evident in a parametric plot of the corresponding \((C,M)\) points on the data plane. For example, for \( \eta(s)|_{s=0} = 1 \) and various \( \varpi(s)|_{s=0} \in [0.1, 1.25992 = \sqrt{2}] \), we find the naked (green circle) and censored (black circle) data as included in the last Figure, with a fitted interpolating curve (orange dashes) connecting the computed points. In this numerical analysis, care should be taken not to have \( \varpi(s)|_{s=0} \) larger than \( \sqrt{2} \eta(s)|_{s=0} \) because otherwise this would place data initialized at \( r = 1 \) within the horizon. The horizon is exactly at the radius \( r = 1 \) when \( \varpi(s)|_{s=0} = \sqrt{2} \eta(s)|_{s=0} \). The gray curve in the last Figure is the image of \( \varpi(s)|_{s=0} = \sqrt{2} \eta(s)|_{s=0} \) on the \((M,C)\) plane. Points below this gray curve can be investigated numerically using Schwarzshild coordinates but only if the initial data is specified for \( r > 1 \), i.e. outside the horizon.

(Also note the portion of the initial data plane shown in the previous Figure lies entirely above the curve \( \eta(s)|_{s=0} = \frac{1}{\pi} \varpi^3(s)|_{s=0} \), so all initial data points in that Figure lie outside of any horizons.)

8 Conclusions

In conclusion, as previously emphasized by many authors it would be interesting to search for evidence of galileons at all distance scales, including galactic and sub-galactic, as well as cosmological. Perhaps a combination of trace couplings and various galileon terms, such as those in (196) and (220) extended to included GR effects, will ultimately lead to a realistic physical model. In particular, it is important to investigate the stability of galileon solutions and to consider the dynamical evolution of generic galileon and other matter field initial data, along the lines of [4, 5], to determine under what physical conditions naked singularities are actually formed.

Acknowledgements: We thank S Deser and C Zachos for constructive comments. We also thank S K Rama, N Rinaldi, K S Virbhadra, and R Wald for discussions of naked singularities. This research was supported by a University of Miami Cooper Fellowship, and by NSF Awards PHY-0855386 and PHY-1214521.
References

[1] S. A. Appleby and E. V. Linder, “Trial of Galileon gravity by cosmological expansion and growth observations” JCAP 1208 (2012) 026. [arXiv:1204.4314 [astro-ph.CO]] doi:10.1088/1475-7516/2012/08/026

[2] C. Charmousis, B. Gouteraux, and E. Kiritsis, “Higher-derivative scalar-vector-tensor theories: black holes, Galileons, singularity cloaking and holography” JHEP 1209 (2012) 011. [arXiv:1206.1499 [hep-th]] doi:10.1007/JHEP09(2012)011

[3] T. Chaundy, The Differential Calculus, The Clarendon Press, Oxford (1935).

[4] M. W. Choptuik, “Universality and scaling in gravitational collapse of a massless scalar field” Phys. Rev. Lett. 70 (1993) 9-12. doi:10.1103/PhysRevLett.70.9

[5] M. W. Choptuik, E. W. Hirschmann, S. L. Liebling, and F. Pretorius, “Critical collapse of the massless scalar field in axisymmetry” Phys. Rev. D68 (2003) 044007. [arXiv:gr-qc/0305003]

[6] P. Creminelli, K. Hinterbichler, J. Khoury, A. Nicolis, and E. Trincherini, “Subluminal Galilean Genesis” JHEP 1302 (2013) 006. [arXiv:1209.3768 [hep-th]] doi:10.1007/JHEP02(2013)006

[7] T. Curtright, “Galileons and Naked Singularities” Phys. Lett. B716 (2012) 366-369. [arXiv:1208.1205 [hep-th]] doi:10.1016/j.physletb.2012.08.047

[8] T. Curtright and D. Fairlie, “Geons of Galileons” Phys. Lett. B716 (2012) 356-360. [arXiv:1206.3616 [hep-th]] doi:10.1016/j.physletb.2012.08.040

[9] C. Deffayet, S. Deser, and G. Esposito-Farese, “Generalized Galileons: All scalar models whose curved background extensions maintain second-order field equations and stress tensors” Phys. Rev. D80 (2009) 064015. [arXiv:0906.1967 [gr-qc]] doi:10.1103/PhysRevD.80.064015

[10] C. Deffayet, G. Esposito-Farese, and A. Vikman, “Covariant Galileon” Phys. Rev. D79 (2009) 084003. [arXiv:0901.1314 [hep-th]] doi:10.1103/PhysRevD.79.084003

[11] C. de Rham, “Galileons in the Sky” Comptes Rendus Physique 13 (2012) 666 [arXiv:1204.5492 [astro-ph.CO]] doi:10.1016/j.crhy.2012.04.006

[12] G. H. Derrick, “Comments on Nonlinear Wave Equations as Models for Elementary Particles” J. Math. Phys. 5 (1964) 1252-1254. doi:10.1063/1.1704233

[13] G. R. Dvali, G. Gabadadze, and M. Porrati, “4-D gravity on a brane in 5-D Minkowski space” Phys. Lett. B 485 (2000) 208 [arXiv:hep-th/0005010] doi:10.1016/S0370-2693(00)00669-9; M. A. Luty, M. Porrati, and R. Rattazzi, “Strong interactions and stability in the DGP model” JHEP 0309 (2003) 029 [arXiv:hep-th/0303116] doi:10.1088/1126-6708/2003/09/029; A. Nicolis and R. Rattazzi, “Classical and quantum consistency of the DGP model” JHEP 0406 (2004) 059 [arXiv:hep-th/0404159] doi:10.1088/1126-6708/2004/06/059

[14] S. Endlich, K. Hinterbichler, L. Hui, A. Nicolis, and J. Wang, “Derrick’s theorem beyond a potential” JHEP 05 (2011) 073. [arXiv:1002.4873 [hep-th]] doi:10.1007/JHEP05(2011)073; A. Padilla, P.M. Safin, and S.-Y. Zhou, “Multi-galileons, solitons and Derrick’s theorem” Phys. Rev. D83 (2011) 045009. [arXiv:1008.0745 [hep-th]] doi:10.1103/PhysRevD.83.045009

[15] D. B. Fairlie, “Comments on Galileons” J. Phys. A44 (2011) 305201. [arXiv:1102.1594 [hep-th]] doi:10.1088/1751-8113/44/30/305201

[16] D. B. Fairlie, “Implicit solutions to some Lorentz invariant non-linear equations revisited” J. Nonlinear Math. Phys. 12 (2005) 449-456 [arXiv:math-ph/0412005]

[17] D. B. Fairlie and J. Govaerts, “Universal field equations with reparametrisation invariance” Phys. Lett. B 281 (1992) 49-53. [arXiv:hep-th/9202056] doi:10.1016/0370-2693(92)90273-7

[18] D. B. Fairlie and J. Govaerts, “Euler hierarchies and universal equations” J. Math. Phys. 33 (1992) 3543-3566. [arXiv:hep-th/9204074] doi:10.1063/1.529904
[19] D. B. Fairlie and J. Govaerts, “Linearisation of Universal Field Equations” J. Phys. A26 (1993) 3339-3347. [arXiv:hep-th/9212005] doi:10.1088/0305-4470/26/13/037

[20] D. B. Fairlie, J. Govaerts, and A. Morozov “Universal field equations with covariant solutions” Nucl. Phys. B373 (1992) 214-232. [arXiv:hep-th/9110022] doi:10.1016/0550-3213(92)90455-K

[21] P. G. O. Freund and Y. Nambu, “Scalar field coupled to the trace of the energy-momentum tensor” Phys. Rev. 174 (1968) 1741-1743. doi:10.1103/PhysRev.174.1741; R. H. Kraichnan, “Special-Relativistic Derivation of Generally Covariant Gravitaton Theory” Phys. Rev. 98 (1955) 1118-1122, especially Appendix II. doi:10.1103/PhysRev.98.1118; S. Deser and L. Halpern “Self-coupled scalar gravitation” Gen. Rel. Grav. 1 (1970) 131-136. doi:10.1007/BF00756892; K. Hinterbichler, “Theoretical Aspects of Massive Gravity” Rev. Mod. Phys. 84 (2012) 671-710. [arXiv:1105.3735 [hep-th]] doi:10.1103/RevModPhys.84.671

[22] G. Gabadadze, K. Hinterbichler, and D. Pirtskhalava, “Classical Duals of Derivatively Self-Coupled Theories” Phys. Rev. D85 (2012) 125007. [arXiv:1202.6364 [hep-th]] doi:10.1103/PhysRevD.85.125007

[23] G. L. Goon, K. Hinterbichler, M. Trodden, “Stability and superluminality of spherical DBI galileon solutions” Phys. Rev. D83 (2011) 085015. [arXiv:1008.4580 [hep-th]] doi:10.1103/PhysRevD.83.085015

[24] K. Hinterbichler and A. Joyce, “Hidden Symmetry of the Galileon” Phys. Rev. D92 (2015) 023503. [arXiv:1501.07660 [hep-th]] doi:10.1103/PhysRevD.92.023503

[25] G. W. Horndeski, “Second-order scalar-tensor field equations in a four-dimensional space” Int. J. Theor. Physics 10 (1974) 363-384. doi:10.1007/BF01807638; C. Deffayet, Xian Gao, D. A. Steer, and G. Zahariade, “From k-essence to generalised Galileons” Phys. Rev. D84 (2011) 064039. [arXiv:1103.3260 [hep-th]] doi:10.1103/PhysRevD.84.064039

[26] L. Iorio, “Constraints on Galileon-induced precessions from solar system orbital motions” JCAP 07 (2012) 001. doi:10.1088/1475-7516/2012/07/001

[27] R. Jackiw and S.-Y. Pi, “Tutorial on Scale and Conformal Symmetries in Diverse Dimensions” J. Phys. A44 (2011) 223001. [arXiv:1101.4886 [math-ph]]. doi:10.1088/1751-8113/44/22/223001

[28] L. Meyer, A. M. Ghez, R. Schödel, S. Yelda, A. Boehle, J. R. Lu, T. Do, M. R. Morris, E. E. Becklin, and K. Matthews, “The Shortest-Known–Period Star Orbiting Our Galaxy’s Supermassive Black Hole” Science 338 (2012) 84-87. doi:10.1126/science.1225506

[29] M. Morris, L. Meyer, and A. Ghez, “Galactic Center Research: Manifestations of the Central Black Hole” Res. Astron. Astrophys. 12 (2012) 995. [arXiv:1207.6755 [astro-ph.GA]] doi:10.1088/1674-4527/12/8/007

[30] R. Penrose, “The Question of Cosmic Censorship” J. Astrophys. Astr. 20 (1999) 233–248. doi:10.1007/BF02702355

[31] Horizon-less solutions with naked singularities have also appeared in recent studies on solutions to higher dimensional Einstein equations in vacuum, suitable for describing intersecting brane solutions in string/M theory. In particular, see S. K. Rama, “Static brane–like vacuum solutions in $D \geq 5$ dimensional spacetime with positive ADM mass but no horizon” [arXiv:1111.1897 [hep-th]]

[32] In the context of a different Galileon model, numerical evidence of naked singularities is also mentioned in passing by M. Rinaldi, “Black holes with non-minimal derivative coupling” Phys. Rev. D 86 (2012) 084048. [arXiv:1208.0103 [gr-qc]] doi:10.1103/PhysRevD.86.084048

[33] T. P. Singh, “Gravitational Collapse, Black Holes and Naked Singularities” J. Astrophys. Astr. 20 (1999) 221–232. [arXiv:gr-qc/9805066] doi:10.1007/BF02702354

[34] T. H. R. Skyrme, “A unified field theory of mesons and baryons” Nucl. Phys. 31 (1962) 556-569. doi:10.1016/0029-5582(62)90775-7

[35] B. K. Tippett, “Possible Bubbles of Spacetime Curvature in the South Pacific” [arXiv:1210.8144 [physics.pop-ph]]
[36] R. C. Tolman, *Relativity, Thermodynamics, and Cosmology*, Dover Publications (1987)

[37] A. I. Vainshtein, “To the problem of nonvanishing gravitation mass” Phys. Lett. B 39 (1972) 393-394. doi:10.1016/0370-2693(72)90147-5

[38] K. S. Virbhadra, D. Narasimha, and S. M. Chitre, “Role of the scalar field in gravitational lensing” Astron. Astrophys. 337 (1998) 1-8. [arXiv:astro-ph/9801174]; K. S. Virbhadra and G. F. R. Ellis, “Gravitational lensing by naked singularities” Phys. Rev. D 65 (2002) 103004. doi:10.1103/PhysRevD.65.103004; K. S. Virbhadra and C. R. Keeton, “Time delay and magnification centroid due to gravitational lensing by black holes and naked singularities” Phys. Rev. D 77 (2008) 124014. [arXiv:0710.2333] doi:10.1103/PhysRevD.77.124014

[39] R. M. Wald, “Gravitational collapse and cosmic censorship” pp 69-85 in Iyer, B.R. (ed.) et al.: Black holes, gravitational radiation and the universe. [gr-qc/9710068] doi:10.1007/978-94-017-0934-7.5

[40] J. A. Wheeler, “Geons” Phys. Rev. 97 (1955) 511-536. doi:10.1103/PhysRev.97.511; P. R. Anderson and D. R. Brill, “Gravitational Geons Revisited” Phys. Rev. D56 (1997) 4824-4833. [arXiv:gr-qc/9610074] doi:10.1103/PhysRevD.56.4824