HEGY test under seasonal heterogeneity

Nan Zou* and Dimitris Politis†

Department of Mathematics, University of California-San Diego,
La Jolla, CA 92093

Abstract

Both seasonal unit roots and seasonal heterogeneity are common in seasonal data. When testing seasonal unit roots under seasonal heterogeneity, it is unclear if we can apply tests designed for seasonal homogeneous settings, i.e. the HEGY test (Hylleberg, Engle, Granger, and Yoo, 1990). In this paper, the validity of both augmented HEGY test and unaugmented HEGY test is analyzed. The asymptotic null distributions of the statistics testing the single roots at 1 or \(-1\) turn out to be standard and pivotal, but the asymptotic null distributions of the statistics testing any coexistence of roots at 1, \(-1\), i, or \(-i\) are non-standard, non-pivotal, and not directly pivotable. Therefore, the HEGY tests are not directly applicable to the joint tests for the concurrence of the roots. As a remedy, we bootstrap augmented HEGY with seasonal independent and identically distributed (iid) bootstrap, and unaugmented HEGY with seasonal block bootstrap. The consistency of both bootstrap procedures is established. Simulations indicate that for roots at 1 and \(-1\) seasonal iid bootstrap augmented HEGY test prevails, but for roots at \(\pm i\) seasonal block bootstrap unaugmented HEGY test enjoys better performance.

Keywords: Seasonality, Unit root, AR sieve bootstrap, Block bootstrap, Functional central limit theorem.

1 Introduction

Both seasonal unit roots and seasonal heterogeneity are common in seasonal data. Hence, it is important to design seasonal unit root tests that allow for seasonal heterogeneity. In particular, consider quarterly data \(\{Y_{4t+s} : t = 1, \ldots, T, s = -3, \ldots, 0\}\) generated by

\[
\alpha_s(L)Y_{4t+s} = V_{4t+s},
\]

where \(\alpha_s(L)\) are seasonally varying AutoRegressive (AR) filters, and \(\{V_{4t+s}\}\) have seasonally varying autocovariances. For more information on seasonal time series, see Ghysels and Osborn (2001), and Franses and Paap (2004).

Now suppose \(V = (V_{4t-3}, \ldots, V_{4t})'\) is a weakly stationary vector-valued process, and for all \(s = -3, \ldots, 0\), the roots of \(\alpha_s(L)\) are on or outside the unit circle. If for all \(s\), \(\alpha_s(L)\) have roots at 1, \(-1\), or \(\pm i\), then respectively \(\{V_{4t+s}\}\) has stochastic trends with period \(+\infty\), 2, or 4. To remove these stochastic trends, we need to test the roots at 1, \(-1\), or \(\pm i\). To address this task, Franses (1994) and Boswijk, Franses, and Haldrup (1997) limit their scope to finite order seasonal AR data and apply Johansen’s method (1988). However, their approaches cannot directly test the existence

*Email address: nzou@ucsd.edu.
†Email address: dpolitis@ucsd.edu.
of a certain root without first checking the number of seasonal unit roots. As a remedy, Ghysels, Hall, and Lee (1996) designs a Wald test that directly tests whether a certain root exists. However, in their own simulation, the Wald test turn out less powerful than the augmented HEGY test.

Does HEGY test work in the seasonally heterogeneous setting (1.1)? To the best of our knowledge, no literature has offered a satisfactory answer. Burridge and Taylor (2001a) analyze the behavior of augmented HEGY test when only seasonal heteroscedasticity exists; del Barrio Castro and Osborn (2008) put augmented HEGY test in the periodic integrated model, a model related but different from model (1.1). No literature has ever touched the behavior of unaugmented HEGY test proposed by Breitung and Franses (1998), the important semi-parametric version of HEGY test. Since unaugmented HEGY test does not assume the noise having an AR structure, it may suit our non-parametric noise in (1.1) better.

To check the legitimacy of HEGY tests in the seasonally heterogeneous setting (1.1), this paper derives the asymptotic null distributions of the unaugmented HEGY test and the augmented HEGY test whose order of lags goes to infinity. It turns out that, the asymptotic null distributions of the statistics testing single roots at 1 or \(-1\) are standard. More specifically, for each single root at 1 or \(-1\), the asymptotic null distributions of the augmented HEGY statistics are identical to that of Augmented Dickey-Fuller (ADF) test (Dickey and Fuller, 1979), and the asymptotic null distributions of the unaugmented HEGY statistics are identical to those of Phillips-Perron test (Phillips and Perron, 1988). However, the asymptotic null distributions of the statistics testing any combination of roots at 1, \(-1\), \(i\), or \(-i\) depend on the seasonal heterogeneity parameters, and are non-standard, non-pivotal, and not directly pivotable. Therefore, when seasonal heterogeneity exists, both augmented HEGY and unaugmented HEGY tests can be straightforwardly applied to single roots at 1 or \(-1\), but cannot be directly applied to the coexistence of any roots.

As a remedy, this paper proposes the application of bootstrap. In general, bootstrap’s advantages are two fold. Firstly, bootstrap helps when the asymptotic distributions of the statistics of interest cannot be found or simulated. Secondly, even when the asymptotic distributions can be found and simulated, bootstrap method may enjoy second order efficiency. For the aforementioned problem, bootstrap therefore serves as an appealing solution. Firstly, it is hard to estimate the seasonal heterogeneity parameters in the asymptotic null distribution, and to simulate the asymptotic null distribution. Secondly, it can be conjectured that bootstrap seasonal unit root test inherits second order efficiency from bootstrap non-seasonal unit root test (Park, 2003). The only methodological literature we find on bootstrapping HEGY test is Burridge and Taylor (2004). Their paper centers on seasonal heteroscedasticity, designs a bootstrap-aided augmented HEGY test, reports its simulation result, but does not give theoretical justification for their test. It will be shown (Remark 3.8) that their bootstrap approach is inconsistent under the general seasonal heterogeneous setting (1.1).

To cater to the general heterogeneous setting (1.1), this paper designs new bootstrap tests, namely 1) seasonal iid bootstrap augmented HEGY test, and 2) seasonal block bootstrap unaugmented HEGY test. To generate bootstrap replicates, the first test get residuals from season-by-season augmented HEGY regressions, and then applies seasonal iid bootstrap to the whitened regression errors. On the other hand, the second test starts with season-by-season unaugmented HEGY regressions, and then handles the correlated errors with seasonal block bootstrap proposed by Dudek, Lekow, Paparoditis, and Politis (2014). Our paper establishes the Functional Central Limit Theorem (FCLT) for both bootstrap tests. Based on the FCLT, the consistency for both bootstrap approaches is proven. To the best of our knowledge, this result gives the first justification for bootstrapping HEGY tests under (1.1).

This paper proceeds as follows. Section 2 formalizes the settings, presents the assumptions, and states the hypotheses. Section 3 gives the asymptotic null distributions of the augmented HEGY test statistics, details the algorithm of seasonal iid bootstrap augmented HEGY test, and establishes the
consistency of the bootstrap. Section 4 presents the asymptotic null distributions of the unaugmented HEGY test statistics, specifies the algorithm of seasonal block bootstrap unaugmented HEGY test, and proves the consistency of the bootstrap. Section 5 compares the simulation performance of the two aforementioned tests. Appendix includes all technical proofs.

2 Seasonal heterogeneous time series

Recall the quarterly data \( \{ Y_{4t+s} : t = 1, ..., T, s = -3, ..., 0 \} \) generated by the seasonal AR model,

\[
\alpha_s(L)Y_{4t+s} = V_{4t+s},
\]

where \( LY_{4t+s} = Y_{4t+s-1} \), \( \alpha_s(L) = 1 - \sum_{j=1}^{4} \alpha_j L^j \). If for all \( s \), \( \alpha_s(L) \) has roots on the unit circle, we suppose that all \( \alpha_s(L) \) share the same set of roots on the unit circle, this set of roots on the unit circle is a subset of \( \{ 1, -1, \pm i \} \), and \( Y_{-3} = Y_{-2} = Y_{-1} = Y_0 = 0 \); otherwise, suppose our data is a stretch of the process \( \{ Y_{4t+s}, t = ..., -1, 0, 1, ..., s = -3, ..., 0 \} \). Let \( V_{4t+s} \) and \( \alpha_j L^j \) be the regression errors and regression coefficients of \( \{ 2.1 \} \), respectively. More specifically, \( V_{4t+s} \) is the distance between \( Y_{4t+s} \) and the vector space generated by \( Y_{4t+s-j}, j = 1, ..., 4 \), and \( \alpha_j L^j \) is the coefficient of the projection of \( V_{4t+s} \) on the aforementioned vector space. Let \( \epsilon_t = (\epsilon_{4t-3}, ..., \epsilon_t) \), \( \Re \epsilon_t = \epsilon_{t-1} \). Denote by \( \text{AR}(p) \) an AutoRegressive process with order \( p \), by \( \text{VMA}(\infty) \) a Vector Moving Average process with infinite moving average order, and by \( \text{VARMA}(p, q) \) a Vector AutoRegressive Moving Average process with autoregressive order \( p \) and moving average order \( q \). Let \( \Re(z) \) be the real part of complex number \( z \). Let \( \lfloor x \rfloor \) be the largest integer smaller or equal to real number \( x \), and \( \lceil x \rceil \) be the smallest integer larger or equal to \( x \).

**Assumption 1.A.** Assume

\[
V_t = \Theta(B) \epsilon_t
\]

where \( B = L^4 \), \( \Theta(B) = \sum_{i=0}^{\infty} \Theta_i B_i \); the \( (j, k) \) entry of \( \Theta_i \), denoted by \( \Theta_i^{(j, k)} \), satisfies \( \sum_{i=1}^{\infty} |i| \Theta_i^{(j, k)} \) for all \( j \) and \( k \); the determinant of \( \Theta(z) \) has all roots outside the unit circle; \( \Theta_0 \) is a lower diagonal matrix whose diagonal entries equal 1; \( \epsilon_t \) is a vector-valued white noise process with mean zero and covariance matrix \( \Omega \); and \( \Omega \) is diagonal.

**Assumption 1.A** assumes that \( \{ V_t \} \) is \( \text{VMA}(\infty) \) with respect to white noise innovation. This is equivalent to the assumption that \( \{ V_t \} \) is a weakly stationary process with no deterministic part in the multivariate Wold decomposition. The assumptions on \( \Theta_0 \) and the determinant of \( \Theta(z) \) ensure the causality and the invertibility of \( \{ V_t \} \) and the identifiability of \( \Omega \).

**Assumption 1.B.** Assume

\[
V_t = \Psi(B)^{-1} \Lambda(B) \equiv \Theta(B) \epsilon_t
\]

where \( \Psi(B) = \sum_{i=0}^{p} \Psi_i B_i \); \( \Lambda(B) = \sum_{i=0}^{\infty} \Lambda_i B_i \); determinants of \( \Psi(z) \) and \( \Lambda(z) \) have all roots outside the unit circle; \( \Psi_0 \) is the identity matrix; \( \Lambda_0 \) is a lower diagonal matrix whose diagonal entries equal 1; \( \epsilon_t \) is a vector-valued white noise process with mean zero and covariance matrix \( \Omega \); and \( \Omega \) is diagonal.

**Assumption 1.B** restricts \( \{ V_t \} \) to be \( \text{VARMA}(p, q) \) with respect to white noise innovation. Compared to the \( \text{VMA}(\infty) \) model in Assumption 1.A, \( \text{VARMA}(p, q) \) ’s main restraint is its exponentially decaying autocovariance. Again, the assumptions on \( \Psi_0, \Lambda_0 \) and the determinant of \( \Psi(z) \) and \( \Lambda(z) \) in Assumption 1.B ensure the causality and the invertibility of \( \{ V_t \} \) and the identifiability of \( \Omega \).

At this stage \( \{ \epsilon_t \} \) is only assumed to be a white noise sequence of random vectors. In fact, \( \{ \epsilon_t \} \) needs to be weakly dependent as well.
Assumption 2.A. (i) \{ɛ_i\} is a fourth-order stationary martingale difference sequence with finite 4 + δ moment for some δ > 0. (ii) \( \exists K > 0, \forall i, j, k, \) and \( l, \sum_{h=-\infty}^{\infty} |\text{Cov}(\varepsilon_t, \varepsilon_k)\varepsilon_{t-h}| < K. \)

Assumption 2.B. (i) \{ɛ_i\} is a strictly stationary strong mixing sequence with finite 4 + δ moment for some δ > 0. (ii) \{ɛ_i\}’s strong mixing coefficient \( a(k) \) satisfies \( \sum_{k=1}^{\infty} k(a(k))^{\delta/(4+\delta)} < \infty. \)

Notice the higher moment \( \{ɛ_i\} \) has, the weaker assumption we require on the strong mixing coefficient of \( \{ɛ_i\} \) in Assumption 2.B. The strong mixing condition in Assumption 2.B actually guarantees (ii) of Assumption 2.A (see Lemma 4).

Hypotheses. We tackle the following set of null hypotheses. The alternative hypotheses are the complement of the null hypotheses.

\[ H_0^1: \quad \alpha_s(1) = 0, \forall s = -3, ..., 0. \]
\[ H_0^2: \quad \alpha_s(-1) = 0, \forall s = -3, ..., 0. \]
\[ H_0^{1.2}: \quad \alpha_s(1) = \alpha_s(-1) = 0, \forall s = -3, ..., 0. \]
\[ H_0^{1.4}: \quad \alpha_s(i) = \alpha_s(-i) = 0, \forall s = -3, ..., 0. \]
\[ H_0^{1.3.4}: \quad \alpha_s(1) = \alpha_s(i) = \alpha_s(-i) = 0, \forall s = -3, ..., 0. \]
\[ H_0^{2.3.4}: \quad \alpha_s(-1) = \alpha_s(i) = \alpha_s(-i) = 0, \forall s = -3, ..., 0. \]
\[ H_0^{1.2.3.4}: \quad \alpha_s(1) = \alpha_s(-1) = \alpha_s(i) = \alpha_s(-i) = 0, \forall s = -3, ..., 0. \]

Indeed, the alternative hypotheses can be written as one-sided. Recall we suppose that for all \( s = -3, ..., 0 \), the roots of \( \alpha_s(L) \) are either on or outside the unit circle. Since \( \alpha_s(0) = 1 \), by the intermediate value theorem, \( \alpha_s(1) \neq 0 \) implies \( \alpha_s(1) > 0 \), \( \alpha_s(-1) \neq 0 \) implies \( \alpha_s(-1) > 0 \), and \( \alpha_s(i) \neq 0 \) implies \( \text{Re}(\alpha_s(i)) > 0 \). To further analyze the roots of \( \alpha_s(L) \), HEGY (Hylleberg, Engle, Granger, and Yoo, 1990) propose the partial fraction decomposition

\[
\alpha_s(L) = \frac{\alpha_0}{1 - L^4} + \frac{\lambda_1}{1 - L} + \frac{\lambda_2}{1 + L} + \frac{\lambda_3}{1 + L^2} + \frac{\lambda_4}{1 - L^2};
\]

thus

\[
\alpha_s(L) = \frac{\alpha_0}{1 - L^4} + \frac{\lambda_1}{1 + L^2} + \frac{\lambda_2}{2} + \frac{\lambda_3}{1 - L} + \frac{\lambda_4}{1 - L^2} + \frac{\lambda_5}{1 - L} + \frac{\lambda_6}{1 + L^2} + \frac{\lambda_7}{1 + L} + \frac{\lambda_8}{1 - L};
\]

Substituting (2.2) into (2.1), we get

\[
(1 - L^4)Y_{4t+s} = \sum_{j=1}^{4} \pi_{j,s}Y_{j,4t+s-1} + V_{4t+s},
\]

where

\[
Y_{1,4t+s} = (1 + L)(1 + L^2)Y_{4t+s}, \quad Y_{2,4t+s} = -(1 - L)(1 + L^2)Y_{4t+s},
Y_{3,4t+s} = -(1 - L)(1 - L^2)Y_{4t+s}, \quad Y_{4,4t+s} = -(1 - L^2)Y_{4t+s},
\]

\[
\pi_{1,s} = -\lambda_1, \quad \pi_{2,s} = -\lambda_2, \quad \pi_{3,s} = -\lambda_3, \quad \pi_{4,s} = \lambda_4.
\]

Indeed, \( \pi_{j,s} \) relates to the root of \( \alpha_s(z) \), i.e., \( \alpha_s(1) = 4\lambda_1 \); hence the proposition below.
Proposition 2.1 (HEGY, 1990).

\[ \alpha_2(1) = 0 \iff \pi_{1,s} = 0, \quad \alpha_2(1) \neq 0 \iff \pi_{1,s} < 0, \]
\[ \alpha_3(-1) = 0 \iff \pi_{2,s} = 0, \quad \alpha_3(-1) \neq 0 \iff \pi_{2,s} < 0, \]
\[ \alpha_3(i) = 0 \iff \alpha_3(-i) = 0 \iff \pi_{3,s} = \pi_{4,s} = 0, \quad \alpha_3(i) \neq 0 \iff \alpha_3(-i) \neq 0 \iff \pi_{3,s} < 0. \]

By Proposition 2.1, the test for the null hypotheses can be carried on by checking the corresponding \(\pi_{j,s}\). Further, \(\pi_{j,s}\) can be estimated by Ordinary Least Squares (OLS). Unfortunately, OLS cannot be readily applied to \((2.3)\) season by season, because \(Y_{j,t+s-1}, j = 1, \ldots, 4\) in \((2.3)\) are not asymptotically orthogonal for any fixed \(s\). (See also Ghysels and Osborn, 2001, p. 158.) On the other hand, \(Y_{j,t-1}, j = 1, \ldots, 4\) in non-periodic regression equations \((3.1)\) and \((4.1)\) are asymptotically orthogonal (see Lemma 1). So we wonder if the OLS estimators based on \((3.1)\) and \((4.1)\) can be used to test the null hypotheses.

When we regress \(\{Y_{4t+s}\}\) with non-periodic regression equations \((5.1)\) and \((6.1)\), the seasonally heterogeneous sequence \(\{V_{4t+s}\}\) is fitted in seasonal homogeneous AR models. Consider, as an example, fitting \(\{V_{4t+s}\}\) in a misspecified AR(1) model \(V_t = \phi V_{t-1} + e_t\). Then \(\hat{\phi} = \frac{\tilde{\gamma}(1)}{\tilde{\gamma}(0)} + o_p(1)\), where

\[ \tilde{\gamma}(h) = \frac{1}{4} \sum_{s=-3}^{0} E[V_{4t+s}V_{4t+s-k}]. \]

Since \(\tilde{\gamma}(\cdot)\) is positive semi-definite, we can find a weakly stationary sequence \(\{\tilde{V}_t\}\) with mean zero and autocovariance function \(\tilde{\gamma}\). We call \(\{\tilde{V}_t\}\) a misspecified constant parameter representation (see also Osborn, 1991) of \(\{V_{4t+s}\}\), and will refer to this concept in later sections.

### 3 Seasonal iid bootstrap augmented HEGY Test

#### 3.1 Augmented HEGY test

In seasonally homogeneous setting

\[ \alpha(L)Y_t = V_t, \quad t = 1 + k, \ldots, 4T, \]

where \(\alpha(L) = \sum_{i=0}^{4} \alpha_i L^i\), the augmented HEGY test detailed below copes with the roots of \(\alpha(L)\) at 1, -1, and \(\pm i\). By calculations similar to \((2.2)\), HEGY (1990) get

\[ (1 - L^4)Y_t = \sum_{j=1}^{4} \pi_j Y_{j,t-1} + \sum_{i=1}^{k} \phi_i (1 - L^4)Y_{t-i} + e_t, \quad (3.1) \]

where augmentations \((1 - L^4)Y_{t-i}, i = 1, 2, \ldots, k\), pre-whiten the time series \((1 - L^4)Y_t\) up to an order of \(k\). As the sample size \(T \to \infty\), let \(k \to \infty\), so that the residual \(\{e_t\}\) is asymptotically uncorrelated. Let \(\hat{\gamma}\) be the OLS estimator of \(\pi_i\), \(t_i\) be the t-statistics corresponding to \(\hat{\gamma}_i\), and \(F_{3,4}\) be the F-statistic corresponding to \(\hat{\gamma}_3\) and \(\hat{\gamma}_4\). Other F-statistics \(F_{1,2}, F_{1,3,4}, F_{2,3,4}\), and \(F_{1,2,3,4}\) can be defined similarly. In seasonally homogeneous configuration, HEGY (1990) proposes to reject \(H_0^3\) if \(\hat{\gamma}_3\) is too small, reject \(H_0^4\) if \(\hat{\gamma}_4\) is too small, reject \(H_0^{3,4}\) if \(F_{3,4}\) is too large, and reject other composite hypotheses if their corresponding F-statistics are too large.
3.2 Augmented HEGY test under model misspecification

Now we apply the augmented HEGY test to seasonally heterogeneous processes. Namely, we run regression equation (3.1) with \{Y_{4t+s}\} generated by (2.1). Our results show that when testing roots at 1 or \(-1\) individually, the t-statistics \(t_1, t_2\), and the F-statistics have standard and pivotal asymptotic distributions. On the other hand, when testing joint roots at 1 and \(-1\), and when testing hypotheses that involve roots at \(\pm i\), the asymptotic distributions of the t-statistics and the F-statistics are non-standard, non-pivotal, and not directly pivotal.

**Theorem 3.1.** Assume that Assumption 1.B and one of Assumption 2.A or 2.B hold. Further, assume \(T \to \infty, k = k_T \to \infty, \) \(k = o(T^{1/3}),\) and \(ck > T^{1/\alpha}\) for some \(c > 0, \alpha > 0.\) Then under \(H_{1,2,3,4}^0,\) the asymptotic distributions of \(t_1, i = 1, 2,\) and F-statistics are given by

\[
t_j = \frac{I^{1}_0 W_{ij}(r) dW_{ij}(r)}{\sqrt{I^{1}_0 W_{ij}^2(r) dr}} = \xi_j, \ j = 1, 2,
\]

\[
F_{1,2} = \frac{1}{2}(\xi_1^2 + \xi_2^2), \quad F_{3,4} = \frac{1}{2}(\xi_3^2 + \xi_4^2),
\]

\[
F_{1,3,4} = \frac{1}{3}(\xi_1^2 + \xi_2^2 + \xi_3^2), \quad F_{2,3,4} = \frac{1}{3}(\xi_1^2 + \xi_2^2 + \xi_4^2),
\]

\[
F_{1,2,3,4} = \frac{1}{4}(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2), \quad \text{with}
\]

\[
\xi_3 = \frac{\lambda_3^4 I^{1}_0 W_3(r) dW_3(r) + \lambda_4^4 I^{1}_0 W_4(r) dW_4(r)}{\sqrt{(\lambda_3^4 + \lambda_4^4)(\frac{1}{2}\lambda_3^4 I^{1}_0 W_3^2(r) dr + \frac{1}{2}\lambda_4^4 I^{1}_0 W_4^2(r) dr)}},
\]

\[
\xi_4 = \frac{\lambda_3^4 \lambda_4^4 I^{1}_0 W_3(r) dW_4(r) - I^{1}_0 W_3(r) dW_3(r)}{\sqrt{(\lambda_3^4 + \lambda_4^4)(\frac{1}{2}\lambda_3^4 I^{1}_0 W_3^2(r) dr + \frac{1}{2}\lambda_4^4 I^{1}_0 W_4^2(r) dr)}},
\]

where \(c_1 = (1, 1, 1, 1)\), \(c_2 = (1, -1, 1, -1)\), \(c_3 = (0, -1, 0, 1)\), and \(c_4 = (-1, 0, 1, 0)\), \(\lambda_i = \sqrt{c_i^2 \Theta(1) \Omega(1)} c_{i/4}\), \(W_t = c_i \Theta(1) \Omega^{1/2} W(2\lambda_i)\), \(W(t) = (W_{i1}(t), W_{i2}(t), W_{i3}(t), W_{i4}(t))\) is a four-dimensional standard Brownian motion.

**Remark 3.1.** The asymptotic distributions presented in Theorem 3.1 degenerate to the distributions in Burridge and Taylor (2001b) and del Barrio Castro, Osborn and Taylor (2012) when \{\(Y_{4t+s}\)\} is a seasonally homogeneous sequence with homoscedastic noise, and to the distributions in Burridge and Taylor (2001a) when \{\(Y_{4t+s}\)\} is a seasonally homogeneous finite-order AR sequence with heteroscedastic noise.

**Remark 3.2.** Notice \(W_t\)'s are standard Brownian motions. When \{\(Y_{4t+s}\)\} is seasonally homogeneous (Burridge and Taylor, 2001b, del Barrio Castro et al., 2012), \(W_t\)'s are independent, so are the asymptotic distributions of \(t_1\) and \(t_2\). On the other hand, when \{\(Y_{4t+s}\)\} has seasonally heterogeneity, \(W_t\)'s are in general independent, so \(t_1\) and \(t_2\) are in general dependent, even asymptotically. Hence, when testing \(H_{1,2}^0\), it is problematic to test \(H_1^0\) and \(H_2^0\) separately and calculate the level of the test with the independence of \(t_1\) and \(t_2\) in mind. Instead, the test of \(H_{1,2}^0\) should be handled with \(F_{1,2}\). Further, because of the dependence of \(t_1\) and \(t_2\), the asymptotic distribution of \(F_{1,2}\) under heterogeneity is different from its counterpart when \{\(Y_{4t+s}\)\} is seasonally homogeneous. Hence, the augmented HEGY test cannot be directly applied to test \(H_{1,2}^0\).

**Remark 3.3.** When \{\(Y_{4t+s}\)\} is only seasonally heteroscedastic (Burridge and Taylor, 2001a), \(\Theta(1)\) does not occur in the asymptotic distributions of the F-statistics. On the other hand, when \{\(Y_{4t+s}\)\}
has generic seasonal heterogeneity, \( \Theta(1) \) impacts firstly the correlation between Brownian motions \( W_3 \) and \( W_4 \), and secondly the weights \( \lambda_3 \) and \( \lambda_4 \).

Remark 3.4. As Burridge and Taylor (2001a) point out, the dependence of the asymptotic distributions on weights \( \lambda_3 \) and \( \lambda_4 \) can be expected. Indeed, \( Y_{3,4t+s} = Y_{4,4t+s-1} \) is the partial sum of \( \{-V_{4t+s-1}, V_{4t+s-3}, \ldots\} \), while \( Y_{3,4t+s+1} = Y_{4,4t+s} \) is the partial sum of \( \{-V_{4t+s}, V_{4t+s-2}, \ldots\} \). Since these two partial sums differ in their variances, both \( \sum_{t=1}^{s_i} Y_{3,4t+s} \) and \( \sum_{s,t} Y_{4,4t+s} \) involve two different weights \( \lambda_3 \) and \( \lambda_4 \).

Remark 3.5. Theorem 3.1 presents the asymptotics when \( \{Y_t\} \) has all roots at 1, \(-1\), and \( \pm i \). When \( \{Y_{4t+s}\} \) has some but not all roots at 1, \(-1\), and \( \pm i \), we let \( U_t = (1 - L^4)Y_t \), \( U = (U_{4t-3}, U_{4t-2}, U_{4t-1}, U_{4t})' \), and calculate \( H(z) \) such that \( U_t = H(z)E_t \). The asymptotic distributions can be expressed with respect to \( H(z) \) and end up having the same form with those given in Theorem 3.1 where \( \{Y_{4t+s}\} \) has all roots.

Remark 3.6. The preceding results give the asymptotic behaviors of the testing statistics under the null hypotheses. Under the alternative hypotheses, we conjecture the powers of the augmented HEGY tests tend to one, as the sample size goes to infinity. To see this, we can without loss of generality assume that \( \{Y_{4t+s}\} \) has root at none of 1, \(-1\), or \( \pm i \). Then \( \{Y_{4t+s}\} \) is stationary, and thus for \( j = 1, 2, 3 \), the \( \pi_j \) corresponding to (the misspecified constant parameter representation of) \( \{Y_{4t+s}\} \) is negative, due to Proposition 2.1. We conjecture that for \( j = 1, 2, 3 \), the OLS estimators \( \hat{\pi}_j \) in (3.11) converge in probability to \( \pi_j \) and as a result the powers of the tests converge to one. See also Theorem 2.2 of Paparoditis and Politis (2016).

3.3 Seasonal iid bootstrap algorithm

To accommodate the non-standard, non-pivotal asymptotic null distributions of the augmented HEGY test statistics, we propose the application of bootstrap. In particular, the bootstrap replications are created as follows. Firstly, we pre-whiten the data season by season to obtain uncorrelated noises. Although these noises are uncorrelated, they are not white due to seasonally heteroscedasticity. Hence secondly we resample season by season in order to generate bootstrapped noise, as in Burridge and Taylor (2001b). Finally, we post-color the bootstrapped noise. The detailed algorithm of this seasonal iid bootstrap augmented HEGY test is given below.

**Algorithm 3.1.** Step 1: calculate the \( t \)-statistics \( t_1, t_2 \), and the \( F \)-statistics \( F_A \), \( A = \{1, 2\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\} \), and \( \{1, 2, 3, 4\} \), from the augmented HEGY test regression

\[
(1 - L^4)Y_t = \sum_{j=1}^{4} \hat{\pi}_j Y_{j,t-1} + \sum_{i=1}^{k} \hat{\phi}_i (1 - L^4)Y_{t-i} + \epsilon_t;
\]

Step 2: record OLS estimators \( \hat{\pi}_{j,s}, \hat{\phi}_{i,s} \) and residuals \( \hat{\epsilon}_{4t+s} \) from the season-by-season regression

\[
(1 - L^4)Y_{4t+s} = \sum_{j=1}^{4} \hat{\pi}_{j,s} Y_{j,4t+s-1} + \sum_{i=1}^{k} \hat{\phi}_{i,s} (1 - L^4)Y_{4t+s-i} + \hat{\epsilon}_{4t+s};
\]

Step 3: let \( \hat{\epsilon}_{4t+s} = \hat{\epsilon}_{4t+s} - \frac{1}{T} \sum_{t=[4k/4]+1}^{T} \hat{\epsilon}_{4t+s} \). Store demeaned residuals \( \{\hat{\epsilon}_{4t+s}\} \) of the four seasons separately, then independently draw four iid samples from each of their empirical distributions, and then combine these four samples into the vector \( \{\epsilon_{4t+s}\} \), with their seasonal orders preserved;
Proposition 3.1. Suppose the assumptions in Theorem 3.1 hold. Let \( Y_{4t+s}^* \) be generated by
\[
(1 - L^4) Y_{4t+s}^* = \sum_{j=1}^{4} \hat{\pi}_{j,s} Y_{4t+s-1}^* + \sum_{i=1}^{k} \hat{\phi}_{i,s}(1 - L^4) Y_{4t+s-i}^* + \epsilon_{4t+s}^*;
\]

Step 5: get t-statistics \( t_1^*, t_2^* \), and the F-statistics \( F_A^* \) from the regression
\[
(1 - L^4) Y_t^* = \sum_{j=1}^{4} \hat{\pi}_{j,t} Y_{t-1}^* + \sum_{i=1}^{k} \hat{\phi}_{i,t}(1 - L^4) Y_{t-i}^* + \epsilon_t^*;
\]

Step 6: run step 3, 4, and 5 for \( B \) times to get \( B \) sets of statistics \( t_1^*, t_2^* \), and the bootstrapped F-statistics \( F_A^* \). Count separately the numbers of \( t^* \)'s and \( F^* \)'s that are more extreme. If these numbers are higher than \( B(1 - \text{level}) \), then we consider \( t_1, t_2, \) and the F-statistics \( F_A \) more extreme, and reject the corresponding hypotheses.

Remark 3.7. It seems also reasonable to keep steps 1, 2, 3, 5, and 6 of the Algorithm 3.1 but change the generation of \( \{Y_{4t+s}^*\} \) in step 4 to
\[
(1 - L^4) Y_{4t+s}^* = \sum_{i=1}^{k} \hat{\phi}_{i,s}(1 - L^4) Y_{4t+s-i}^* + \epsilon_{4t+s}^*.
\]

This new algorithm is in fact theoretically invalid for the tests of any coexistence of roots (see Remark 3.2, 3.3, and 3.5), but it is valid for individual tests of roots at 1 or \(-1\), due to the pivotal asymptotic distributions of \( t_1 \) and \( t_2 \) in Theorem 3.1.

Remark 3.8. If we keep steps 1, 3, 5, and 6 of Algorithm 3.1 but run regression equations with seasonally homogeneous coefficients \( \hat{\pi}_j \) and \( \hat{\phi}_i \) in steps 2 and 4, then this algorithm is identical with Burridge and Taylor (2004). However, this algorithm cannot in step 2 fully pre-whiten the time series, and it leaves the regression error \( \{\epsilon_t\} \) serially correlated. When \( \{\epsilon_t\} \) is bootstrapped by seasonal iid bootstrap, this serial correlation structure is ruined. As a result, \( (1 - L^4) Y_t^* \) differs from \( (1 - L^4) Y_t \) in its correlation structure, in particular \( \Theta(1) \), and the conditional distributions of the bootstrapped F-statistics \( F_A^* \) differ from the distributions of the original F-statistics \( F_A \) (see Remark 3.2 and 3.3).

3.4 Consistency of seasonal iid bootstrap

Now we justify the seasonal iid bootstrap augmented HEGY test (Algorithm 3.1). Since the derivation of the real-world asymptotic distributions in Theorem 3.1 calls on FCLT (see Lemma 1), the justification of bootstrap approach also requires FCLT in the bootstrap world. From now on, let \( P^o \), \( E^o \), \( \text{Var}^o \), \( \text{Std}^o \), \( \text{Cov}^o \) be the probability, expectation, variance, standard deviation, and covariance, respectively, conditional on our data \( \{Y_{4t+s}\} \).

Proposition 3.1. Suppose the assumptions in Theorem 3.1 hold. Let \( S_t^u(u_1, u_2, u_3, u_4) \)
\[
= \frac{1}{\sqrt{4T}} \left( \sum_{t=1}^{[4T u_1]} \epsilon_t^* / \sigma_t^*, \sum_{t=1}^{[4T u_2]} (-1)^t \epsilon_t^* / \sigma_t^*, \sum_{t=1}^{[4T u_3]} \sqrt{2} \sin \left( \frac{\pi t}{2} \right) \epsilon_t^* / \sigma_t^*, \sum_{t=1}^{[4T u_4]} \sqrt{2} \cos \left( \frac{\pi t}{2} \right) \epsilon_t^* / \sigma_t^* \right)^t;
\]
where,

\[
\sigma_1^* = \text{Std}^c[\frac{1}{\sqrt{4T}} \sum_{t=1}^{4T} \epsilon_i^*], \\
\sigma_2^* = \text{Std}^c[\frac{1}{\sqrt{4T}} \sum_{t=1}^{4T} (-1)^i \epsilon_i^*], \\
\sigma_3^* = \text{Std}^c[\frac{1}{\sqrt{4T}} \sum_{t=1}^{4T} \sqrt{2} \sin(\frac{\pi t}{2}) \epsilon_i^*], \\
\sigma_4^* = \text{Std}^c[\frac{1}{\sqrt{4T}} \sum_{t=1}^{4T} \sqrt{2} \cos(\frac{\pi t}{2}) \epsilon_i^*].
\]

Then, no matter which hypothesis is true, \( S_T^2 \Rightarrow W \) in probability as \( T \to \infty \), where \( W(t) \) is a four-dimensional standard Brownian motion.

By the FCLT given by Proposition 3.1 and the proof of Theorem 3.1 in probability the conditional distributions of \( t_i^* \), \( i = 1, 2 \), and \( F_A^j \) converge to the limiting distributions of \( t_i \), \( i = 1, 2 \), and \( F_A \), respectively. Since conditional on \( \{Y_{4t+s}\} \), \( \{Y_{4t+s}\} \) is a finite-order seasonal AR process, the derivation of the conditional distributions of \( t_i^* \), \( i = 1, 2 \), and \( F_A^j \) turns out easier than that of Theorem 3.1 and in particular does not involve the fourth moments of \( \{Y_{4t+s}\} \). Hence the consistency of the bootstrap.

**Theorem 3.2.** Suppose the assumptions in Theorem 3.1 hold. Let \( P^A \) be the probability measure corresponding to the null hypothesis \( H_0^A \). Then,

\[
\sup_x |P^\circ(t_i^* \leq x) - P^i(t_i \leq x)| \xrightarrow{P} 0, \ i = 1, 2,
\]

\[
\sup_x |P^\circ(F_A^j \leq x) - P^A(F_A \leq x)| \xrightarrow{P} 0, \ A = \{1, 2\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}.
\]

### 4 Seasonal block bootstrap unaugmented HEGY test

#### 4.1 Unaugmented HEGY test

In the proceeding section our analysis focuses on the augmented HEGY test, an extension of the ADF test to the seasonal unit root setting. An important alternative of the ADF test is the Phillips-Perron test (Phillips and Perron, 1988). While the ADF test assumes an AR structure over the noise and thus becomes parametric, its semi-parametric counterpart, Phillips-Perron test, allows a wide class of weakly dependent noises. Unaugmented HEGY test (Breitung and Franses, 1998), as the extension of Phillips-Perron test to the seasonal unit root, inherits the semi-parametric nature and does not assume the noise to be AR. Given seasonal heterogeneity, it will be shown in Theorem 4.1 that the unaugmented HEGY test estimates seasonal unit root consistently under the very general VMA(\( \infty \)) class of noise (Assumption 1.A), instead of the more restrictive VARMA(p, q) class of noise (Assumption 1.B), which is needed for the augmented HEGY test.

Now we specify the unaugmented HEGY test. Consider regression

\[
(1 - L^4)Y_t = \sum_{j=1}^{4} \hat{\pi}_j Y_{j,t-1} + V_t.
\]

Let \( \hat{\pi}_j \) be the OLS estimator of \( \pi_j \), \( t_j \) be the t-statistic corresponding to \( \hat{\pi}_j \), and \( F_{4,4} \) be the F-statistic corresponding to \( \hat{\pi}_3 \) and \( \hat{\pi}_4 \). Other F-statistics \( F_{1,2}, F_{1,3,4}, F_{2,3,4}, \) and \( F_{1,2,3,4} \) can be defined analogously. Similar to the Phillips-Perron test (Phillips and Perron, 1988), the unaugmented HEGY test can use both \( \hat{\pi}_j \) and \( t_j \) when testing roots at 1 or -1. As in the augmented HEGY test, we reject \( H_0^j \) if \( \hat{\pi}_1 \) (or \( t_1 \)) is too small, reject \( H_0^2 \) if \( \hat{\pi}_2 \) (or \( t_2 \)) is too small, and reject the joint hypotheses if the corresponding F-statistics are too large. The following results give the asymptotic null distributions of \( \hat{\pi}_i, t_j, j = 1, 2 \), and the F-statistics.
4.2 Unaugmented HEGY test under model misspecification

Theorem 4.1. Assume that Assumption 1.A and one of Assumption 2.A or Assumption 2.B hold. Then under $H_{0,2,3,4}^t$, as $T \to \infty$,

$$ (4T) \pi_i \equiv \frac{\lambda_i^2 \int_0^t W_i(r) dW_i(r) + \Gamma^{(i)}}{\lambda_i^2 \int_0^t W_i^2(r) dr}, \text{ for } i = 1, 2, $$

$$ (4T) \pi_3 \equiv \frac{\lambda_3^2 \int_0^t W_3(r) dW_3(r) + \lambda_3^2 \int_0^t W_4(r) dW_4(r) + \Gamma^{(3)}}{\frac{1}{2} (\lambda_3^2 \int_0^t W_3^2(r) dr + \lambda_3^2 \int_0^t W_4^2(r) dr)}, $$

$$ (4T) \pi_4 \equiv \frac{\lambda_3 \lambda_4 (\int_0^t W_3(r) dW_3(r) - \int_0^t W_4(r) dW_4(r)) + \Gamma^{(4)}}{\frac{1}{2} (\lambda_3^2 \int_0^t W_3^2(r) dr + \lambda_3^2 \int_0^t W_4^2(r) dr)}, $$

$$ t_i \equiv \frac{\lambda_i^2 \int_0^t W_i(r) dW_i(r) + \Gamma^{(i)}}{\sqrt{\gamma(0) \lambda_i^2 \int_0^t W_i^2(r) dr}} \equiv \mathcal{D}_i, \text{ for } i = 1, 2, $$

$$ t_3 \equiv \frac{\lambda_3^2 \int_0^t W_3(r) dW_3(r) + \lambda_3^2 \int_0^t W_4(r) dW_4(r) + \Gamma^{(3)}}{\sqrt{\gamma(0) \frac{1}{2} (\lambda_3^2 \int_0^t W_3^2(r) dr + \lambda_3^2 \int_0^t W_4^2(r) dr)}} \equiv \mathcal{D}_3, $$

$$ t_4 \equiv \frac{\lambda_3 \lambda_4 (\int_0^t W_3(r) dW_3(r) - \int_0^t W_4(r) dW_4(r)) + \Gamma^{(4)}}{\sqrt{\gamma(0) \frac{1}{2} (\lambda_3^2 \int_0^t W_3^2(r) dr + \lambda_3^2 \int_0^t W_4^2(r) dr)}}, \quad F_{1,2} \Rightarrow \frac{1}{2} (\mathcal{D}_2 + \mathcal{D}_3^2), \quad F_{3,4} \Rightarrow \frac{1}{2} (\mathcal{D}_4^2 + \mathcal{D}_4^2), $$

$$ F_{1,3,4} \Rightarrow \frac{1}{3} (\mathcal{D}_2 + \mathcal{D}_3 + \mathcal{D}_4^2), \quad F_{2,3,4} \Rightarrow \frac{1}{3} (\mathcal{D}_2 + \mathcal{D}_3 + \mathcal{D}_4^2), \quad F_{1,2,3,4} \Rightarrow \frac{1}{4} (\mathcal{D}_2 + \mathcal{D}_3 + \mathcal{D}_4 + \mathcal{D}_4^2), $$

where $c_1 = (1,1,1,1)^t$, $c_2 = (1,-1,1,-1)^t$, $c_3 = (0,-1,0,1)^t$, $c_4 = (-1,0,1,0)^t$, $\lambda_1 = \sqrt{c_1^t \Theta(1) \Omega(1) \Theta(1)} c_1^t/4$, $W_i = c_i^t \Theta(1) \Omega^{1/2} W/2 \lambda_i$, $W(t) = (W_{(1)}(t), W_{(2)}(t), W_{(3)}(t), W_{(4)}(t))^t$ is a four-dimensional standard Brownian motion, $\tilde{\gamma}(j)$ are defined in (2.5), $\Gamma^{(1)} = \sum_{j=1}^{\infty} \tilde{\gamma}(j)$, $\Gamma^{(2)} = \sum_{j=1}^{\infty} (-1)^j \tilde{\gamma}(j)$, $\Gamma^{(3)} = \sum_{j=1}^{\infty} \sin(\pi j/2) \tilde{\gamma}(j)$, and $\Gamma^{(4)} = -\sum_{j=1}^{\infty} \sin(\pi j/2) \tilde{\gamma}(j)$.

Remark 4.1. The results in Theorem 4.1 degenerate to the asymptotics in Burridge and Taylor (2001ab) when $\{V_{4t+s}\}$ is uncorrelated, and degenerate to the asymptotics in Breitung and Franses (1998) when $\{V_{4t+s}\}$ is seasonally homogeneous.

Remark 4.2. When $\{V_{4t+s}\}$ is seasonally homogeneous (Breitung and Franses, 1998), the asymptotic distributions of $(\hat{\pi}_1, \hat{t}_1)$ and $(\hat{\pi}_2, \hat{t}_2)$ are independent. On the other hand, when $\{V_{4t+s}\}$ has seasonal heterogeneity, $(\hat{\pi}_1, \hat{t}_1)$ and $(\hat{\pi}_2, \hat{t}_2)$ are dependent, as we have seen for augmented HEGY test (Remark 3.2). Hence, when testing $H_{0,2}^1$, it is problematic to test $H_{0,2}^1$ and $H_{0,2}^2$ separately. Instead, the test of $H_{0,2}^1$ should be handled with $F_{1,2}$.

Remark 4.3. The parameters $\lambda_i$ have the same definition as in Theorem 3.1. Since $\lambda_1^2 = \sum_{j=-\infty}^{\infty} \tilde{\gamma}(j)$, and $\lambda_2^2 = \sum_{j=-\infty}^{\infty} (-1)^j \tilde{\gamma}(j)$, the asymptotic distributions of $\hat{\pi}_i$ and $\hat{t}_i$, $i = 1, 2$, only depend on the autocorrelation function of $\{V_t\}$, the misspecified constant parameter representation of $\{V_{4t+s}\}$. Since $\{V_t\}$ can be considered as a seasonally homogeneous version of $\{V_{4t+s}\}$, we can conclude that
the asymptotic behaviors of the tests for single roots at 1 or \(-1\) are not affected by the seasonal heterogeneity in \(\{V_{4t+s}\}\). On the other side, the asymptotic distributions of the F-statistics do not solely depend on \(\{V_t\}\). Hence, the test for the concurrence of roots at 1 and \(-1\) and the tests involving roots at \(\pm i\) are affected by the seasonal heterogeneity.

**Remark 4.4.** To remove the nuisance parameters in the asymptotic distributions, we notice that the asymptotic behaviors of \(\hat{\pi}_i\) and \(t_i\), \(i = 1, 2\), have identical forms as in Phillips and Perron (1988). In light of their approach, we can construct pivotal versions of \(\hat{\pi}_i\) and \(t_i\), \(i = 1, 2\), that converge in distribution to standard Dickey-Fuller distributions (Dickey and Fuller, 1979). More specifically, for \(i = 1, 2\), we can substitute any consistent estimator for \(\lambda_i^2\) and \(\tilde{\gamma}(0)\) below:

\[
(4T)\hat{\pi}_i - \frac{\frac{1}{2}(\lambda_i^2 - \tilde{\gamma}(0))}{(4T)^{-2} \sum_t Y_{i,t-1}^2} \Rightarrow \int_0^1 W_t(r) dW_i(r),
\]

\[
\frac{\sqrt{\gamma(0)}}{\lambda_i} t_i - \frac{\frac{1}{2}(\lambda_i^2 - \tilde{\gamma}(0))}{\lambda_i^2 (4T)^{-2} \sum_t Y_{i,t-1}^2} \Rightarrow \int_0^1 W_t(r) dW_i(r).
\]

**Remark 4.5.** However, there is no easy way to construct pivotal statistics for \(\hat{\pi}_3\), \(t_3\), \(\hat{\pi}_4\), \(t_4\), and F-statistics such as \(F_{3,4}\). The difficulties are two-fold. Firstly the denominators of the asymptotic distributions of these statistics contain weighted sums with unknown weights \(\lambda_3^2\) and \(\lambda_4^2\); secondly \(W_3\) and \(W_4\) are in general correlated standard Brownian motions as in Theorem 3.1.

**Remark 4.6.** The result in Theorem 4.1 can be generalized. Suppose \(\{Y_{4t+s}\}\) is not generated by \(H_{0}^{1,2,3,4}\), and only has some of the seasonal unit roots. Let \(U_t = (1 - L^4)Y_t\), and \(U_t = \langle U_{4t-3}, U_{4t-2}, U_{4t-1}, U_{4t} \rangle\). Then we can find \(H(z)\) such that \(U_t = H(B)\epsilon_t\). The asymptotic distributions of \(\hat{\pi}_i\), \(t_i\), \(i = 1, 2\), and the F-statistics have the same forms as those in Theorem 4.1 with \(\Theta(1)\) substituted by \(H(1)\), and \(\tilde{\gamma}\) based on \(\{U_t\}\).

**Remark 4.7.** As for the asymptotic results under the alternative hypothesis, we conjecture that the powers of the unaugmented HEGY tests converge to one as sample size goes to infinity. As in Remark 3.6 we can assume without loss of generality that \(\{Y_{4t+s}\}\) has no root at 1, \(-1\), or \(\pm i\). Then for \(j = 1, 2, 3\), the coefficient \(\pi_j\) corresponding to (the misspecified constant parameter representation of) \(\{Y_{4t+s}\}\) are negative, according to Proposition 2.1. We conjecture that for \(j = 1, 2, 3\), the OLS estimators \(\hat{\pi}_j\) in (4.1) converge to \(\pi_j\), and as a result the power of the tests tend to one.

### 4.3 Seasonal block bootstrap algorithm

Since many of the asymptotic distributions delivered in Theorem 4.1 are non-standard, non-pivital, and not directly pivotable, we propose the application of bootstrap. Since the regression error \(\{V_{4t+s}\}\) of (4.1) is seasonally stationary, we in particular apply the seasonal block bootstrap of Dudek et al. (2014). The algorithm of seasonal block bootstrap unaugmented HEGY test is illustrated below.

**Algorithm 4.1.** Step 1: get the OLS estimators \(\hat{\pi}_1\), \(\hat{\pi}_2\), \(t_1\), \(t_2\), and the F-statistics \(F_A\), \(A = \{1, 2\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\), and \(\{1, 2, 3, 4\}\), from the regression of the unaugmented HEGY test

\[
(1 - L^4)Y_t = \sum_{j=1}^{4} \tilde{\pi}_j Y_{j,t-1} + \epsilon_t, \quad t = 1, ..., 4T;
\]
Proposition 4.1. Let $\pi_{j,s} = 1$ for all $s$ when testing roots at $\pm \pi$. For example, set $\pi_{3,s} = \pi_{4,s} = 0$ for all $s$ when testing roots at $\pm \pi$. Generate $\{Y^*_{4t+s}\}$ by

$$(1 - L^4)Y^*_{4t+s} = \sum_{j=1}^{4} \tilde{\pi}_{j,s}Y_{4j,4t+s-1} + V^*_{4t+s};$$

Step 2: record residual $\hat{V}_t$ from regression

$$(1 - L^4)Y_{4t+s} = \sum_{j=1}^{4} \tilde{\pi}_{j,s}Y_{j,4t+s-1} + \hat{V}_{4t+s};$$

Step 3: let $\hat{V}_{4t+s} = \hat{V}_{4t+s} - \frac{1}{b} \sum_{t=1}^{T} \hat{V}_{4t+s}$, choose a integer block size $b$, and let $l = \lfloor 4T/b \rfloor$. For $t = 1, b + 1, \ldots, (l - 1)b + 1$, let

$$(V^*_t, \ldots, V^*_t + b - 1) = (\hat{V}_t, \ldots, \hat{V}_{t + b - 1}),$$

where $\{I_t\}$ is a sequence of iid uniform random variables taking values in $\{t - 4R_1, t - 4, t, t + 4, \ldots, t + 4R_2\}$ with $R_1 = \lfloor (t - 1)/4 \rfloor$ and $R_2 = \lfloor (n - t + 1)/4 \rfloor$;

Step 4: set the $\tilde{\pi}_{j,s}$ corresponding to the null hypothesis to be zero. For example, set $\pi_{3,s} = \pi_{4,s} = 0$ for all $s$ when testing roots at $\pm \pi$. Generate $\{Y^*_{4t+s}\}$ by

$$(1 - L^4)Y^*_{4t+s} = \sum_{j=1}^{4} \tilde{\pi}_{j,s}Y^*_{j,4t+s-1} + V^*_{4t+s};$$

Step 5: get OLS estimates $\hat{\pi}^*_1, \hat{\pi}^*_2, t^*_1, t^*_2$, and F-statistics $F^*_A$ from regression

$$(1 - L^4)Y^*_t = \sum_{j=1}^{4} \tilde{\pi}^*_j Y^*_{j,t-1} + e^*_t, \quad t = 1, \ldots, 4T;$$

Step 6: run step 3, 4, and 5 for $B$ times to get $B$ sets of statistics $\hat{\pi}^*_1, \hat{\pi}^*_2, t^*_1, t^*_2$, and $F^*_A$. Count separately the numbers of $\hat{\pi}^*_1, \hat{\pi}^*_2, t^*_1, t^*_2$, and $F^*_A$ than which $\hat{\pi}_1, \hat{\pi}_2, t_1, t_2$, and $F_A$ are more extreme. If these numbers are higher than $B(1 - \alpha)$, then consider $\hat{\pi}_1, \hat{\pi}_2, t_1, t_2$, and $F_A$ extreme, and reject the corresponding hypotheses.

4.4 Consistency of seasonal block bootstrap

Proposition 4.1. Let $S^*_T(u_1, u_2, u_3, u_4)$

$$= \frac{1}{\sqrt{4T}} \left( \sum_{t=1}^{4T} V^*_t / \sigma^*_1, \sum_{t=1}^{4T} (-1)^t V^*_t / \sigma^*_2, \sum_{t=1}^{4T} \sqrt{2} \sin(\pi t/2) V^*_t / \sigma^*_3, \sum_{t=1}^{4T} \sqrt{2} \cos(\pi t/2) V^*_t / \sigma^*_4 \right),$$

where

$$\sigma^*_1 = \text{Std}^*[\frac{1}{\sqrt{4T}} \sum_{t=1}^{4T} V^*_t], \quad \sigma^*_2 = \text{Std}^*[\frac{1}{\sqrt{4T}} \sum_{t=1}^{4T} (-1)^t V^*_t],$$

$$\sigma^*_3 = \text{Std}^*[\frac{1}{\sqrt{4T}} \sum_{t=1}^{4T} \sqrt{2} \sin(\pi t/2) V^*_t], \quad \sigma^*_4 = \text{Std}^*[\frac{1}{\sqrt{4T}} \sum_{t=1}^{4T} \sqrt{2} \cos(\pi t/2) V^*_t].$$

If $b \to \infty, T \to \infty, b/\sqrt{T} \to 0$, then no matter which hypothesis is true, $S^*_T \Rightarrow W$ in probability, where $W(t)$ is a four-dimensional standard Brownian motion.
By the FCLT given by Proposition 4.1 and the convergence of the bootstrap standard deviation $\sigma^*_i$ (Dudek et al., 2014), we have that the conditional distribution of $t^*_i$, $\hat{\pi}^*_i$, $i = 1, 2,$ and $F^*_A$ in probability converges to the limiting distribution of $\hat{\pi}_i, t_i, i = 1, 2,$ and $F_A$, respectively. Hence the consistence of the bootstrap.

**Theorem 4.2.** Suppose the assumptions in Theorem 4.1 hold. Let $P^A$ be the probability measure corresponding to the null hypothesis $H^A_1$. If $b \to \infty$, $T \to \infty$, $b/\sqrt{T} \to 0$, then

$$
sup_x |P^\circ(\pi^*_i \leq x) - P^i(\pi_i \leq x)| \overset{p}{\to} 0, \quad i = 1, 2,$$

$$sup_x |P^\circ(t^*_i \leq x) - P^i(t_i \leq x)| \overset{p}{\to} 0, \quad i = 1, 2,$$

$$sup_x |P^\circ(F^*_A \leq x) - P^A(F_A \leq x)| \overset{p}{\to} 0, \quad A = \{1, 2, 3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}.$$

5 Simulation

5.1 Data generating process

We focus on the hypotheses test for root at 1 ($H^1_0$ against $H^1_1$), root at $-1$ ($H^2_0$ against $H^2_1$), and root at $\pm i$ ($H^{3.4}_0$ against $H^{3.4}_1$). In each hypothesis test, we equip one sequence with all nuisance unit roots at 1, $-1$, and $\pm i$, and the other with none of the nuisance unit roots. The detailed data generation processes are listed in Table 1. To produce power curves, we let parameter $\rho = 0, 0.004, 0.008, 0.012, 0.016,$ and 0.020. Notice that $\rho$ is set to be seasonally homogeneous for the sake of simplicity. Further, we generate six types of innovations $\{V_{4t+s}\}$ according to Table 2, where $\epsilon_t \sim iid N(0, 1)$. The values of $\phi_s$ are assigned so that the misspecified constant parameter representation (see Section 2) of the “period” sequence has almost the same AR structure as the “ar” sequence.

Table 1: Data generation processes

| Data Generating Processes | No                     | Yes                      |
|---------------------------|------------------------|--------------------------|
| Root                      | $1$                    | $(1 - (1 - \rho)L)Y_t = V_t$ |
|                           | $-1$                   | $(1 + (1 - \rho)L)Y_t = V_t$ |
|                           | $\pm i$                | $(1 + (1 - \rho)L^2)Y_t = V_t$ |

Table 2: Types of noises

| Noise Type | $V_t = \epsilon_t$ | $V_{4t+s} = \sigma_4 \epsilon_{4t+s}$, $\sigma_1 = 10, \sigma_2 = \sigma_3 = \sigma_4 = 1$ |
|------------|-------------------|----------------------------------|
| heter      |                   | $V_t = \epsilon_t + 0.5\epsilon_{t-1}$ |
| mapos      |                   | $V_t = \epsilon_t - 0.5\epsilon_{t-1}$ |
| maseg      |                   | $V_t = \epsilon_t + 0.5V_{t-1}$ |
| ar         |                   | $V_{4t+s} = \epsilon_{4t+s} + \phi_s V_{4t+s-1}$, $\phi_1 = 0.2, \phi_2 = 0.45, \phi_3 = 0.65, \phi_4 = 0.8$ |
5.2 Testing procedure

Here we give additional implemental details for the algorithms of the seasonal iid bootstrap augmented HEGY test (Algorithm 3.1) and the seasonal block bootstrap unaugmented HEGY test (Algorithm 4.1).

5.2.1 Seasonal iid bootstrap augmented HEGY test

To improve the empirical performance of seasonal iid bootstrap algorithm (Algorithm 3.1), we select stepwise, truncate the coefficient estimators, and use (3.2) when testing roots at 1 or $-1$. Firstly, a stepwise selection procedure is applied to the regression in step 2 of Algorithm 3.1. To begin with, we choose a maximum lag $k_{\text{max}}$. $k_{\text{max}}$ may be chosen by AIC, BIC, or modified information criterion (for further discussions, see del Barrio Castro, Osborn, and Taylor, 2016). In our simulation we fix $k_{\text{max}} = 4$ for simplicity. Afterward, we apply a stepwise selection with Variance Inflating Factor (VIF) criterion to solve the multicollinearity between the regressors. In this selection, we locate the regressor with the largest VIF, remove this regressor from the regression if its VIF is larger than 10, and rerun the regression. Then we implement another stepwise selection on lags $(1 - L^4)Y_{4t+s-i}$, $i = 1, 2, ..., k$, by iteratively removing lags of which the absolute values of the t-statistics are smaller than 1.65 (see also Burridge and Taylor, 2004). Then the estimated coefficients of the deleted regressors are set to be zero, while the estimated coefficients of the remaining regressors are recorded and used in step 2 and 4. The stepwise selection of the lags based on their t-statistics is also applied to step 1 and 5.

Secondly, notice that in step 2, the true parameters $\pi_{j,s}$, $j = 1, 2, 3$, are smaller or equal to zero under both null and alternative hypotheses. However, the OLS estimators $\hat{\pi}_{j,s}$, $j = 1, 2, 3$, are often positive, especially when $\pi_{j,s} = 0$. This positivity not only renders the estimation of $\pi_{j,s}$ inaccurate, but also makes the equation in step 4 of Algorithm 3.1 non-causal, and the bootstrapped sequence $\{Y_{4t+s}^{\star}\}$ explosive. The solution of this problem is to truncate the OLS estimator. Let $\tilde{\pi}_{j,s} = \min(0, \hat{\pi}_{j,s})$, $j = 1, 2, 3$. Immediately we get $|\tilde{\pi}_{j,s} - \pi_{j,s}| \leq |\hat{\pi}_{j,s} - \pi_{j,s}|$. After we substitute $\tilde{\pi}_{j,s}$ for $\hat{\pi}_{j,s}$ in step 4, the empirical performance of seasonal iid bootstrap improves significantly.

Thirdly, we use the original step 4 of Algorithm 3.1 when testing roots at $\pm i$, but apply the alternative step (3.2) to the test of root at 1 or $-1$. (When apply the alternative step (3.2), we similarly select stepwise the lags and truncate the coefficients.) Unpublished simulation result shows an advantage of (3.2) when testing root at 1 or $-1$. This advantage occurs especially when all nuisance roots occur, or equivalently when all of the true $\pi_{j,s}$’s are zero, since in this case the inclusion of $Y_{j,4t+s}^{\star}$ in the original step 4 becomes redundant.

5.2.2 Seasonal block bootstrap unaugmented HEGY test

To improve the empirical performance of seasonal block bootstrap algorithm (Algorithm 4.1), we truncate the coefficient estimators, taper the blocks, and optimize the block size. Firstly, as in the seasonal iid bootstrap algorithm, we let $\tilde{\pi}_{j,s} = \min(0, \hat{\pi}_{j,s})$, $j = 1, 2, 3$, and substitute $\tilde{\pi}_{j,s}$ for $\hat{\pi}_{j,s}$ in step 4.

Secondly, it is known that the bootstrapped data around the edges of the bootstrap blocks are not good imitations of the original data. To reduce this “edge effect”, we apply tapered seasonal block bootstrap proposed by Dudek, Paparoditis, and Politis (2016), which put less weight on the bootstrapped data around the edges. In our simulation the weight function is set identical to the function suggested by Dudek et al. (2016).

Thirdly, both test statistics $\hat{t}_j$ and $t_j$ can be employed to run seasonal block bootstrap unaugmented HEGY test. So do various block sizes. In the following preliminary simulation we check the
impact of test statistics and block sizes on empirical sizes (for a thorough discussion on optimal block size, see Paparoditis and Politis, 2003). Let \( \hat{\pi}^{(i)} \) indicates the bootstrap test based on coefficient estimator \( \hat{\pi} \) with block size \( i \), and \( t^{(i)} \) indicates the bootstrap test based on t-statistics \( t \) with block size \( i \). Set the sample size \( T = 120 \); in each test \( B = 250 \) bootstrap replicates are created; the nominal size \( \alpha = 0.05 \); the empirical sizes are calculated using \( N = 300 \) iterations. The results on the empirical sizes of the tests are included in Table 3, 4, and 5.

From Table 3, 4, and 5 we can see that the choice of statistics and block sizes does not affect the empirical sizes of the tests very much. (Indeed, unpublished simulations show that empirical powers are not much affected either.) We also find that the distortion of empirical size becomes the worst when testing root at \(-1\) with nuisance roots and \( ma_{pos} \) noise. Noticing \( t^{(4)} \) gives the best result in the worst scenario, we base the test on the t-statistics and let the block size be four in the succeeding simulations.

Table 3: Empirical sizes of tests for unit root at 1

| Nuisance Root | Noise Type | Tests |
|---------------|------------|-------|
|               | \( \hat{\pi}^{(4)} \) | \( \hat{\pi}^{(8)} \) | \( \hat{\pi}^{(12)} \) | \( t^{(4)} \) | \( t^{(8)} \) | \( t^{(12)} \) |
| False         | iid        | 0.067 | 0.047 | 0.043 | 0.067 | 0.050 | 0.040 |
|               | heter      | 0.057 | 0.067 | 0.050 | 0.053 | 0.063 | 0.040 |
|               | \( ma_{pos} \) | 0.090 | 0.050 | 0.030 | 0.087 | 0.050 | 0.023 |
|               | \( ma_{neg} \) | 0.080 | 0.073 | 0.093 | 0.080 | 0.060 | 0.093 |
|               | ar         | 0.043 | 0.047 | 0.063 | 0.047 | 0.053 | 0.060 |
|               | period     | 0.043 | 0.043 | 0.047 | 0.047 | 0.043 | 0.047 |
| True          | iid        | 0.137 | 0.123 | 0.110 | 0.117 | 0.110 | 0.110 |
|               | heter      | 0.160 | 0.160 | 0.193 | 0.160 | 0.150 | 0.190 |
|               | \( ma_{pos} \) | 0.063 | 0.053 | 0.073 | 0.053 | 0.043 | 0.057 |
|               | \( ma_{neg} \) | 0.517 | 0.500 | 0.570 | 0.527 | 0.500 | 0.567 |
|               | ar         | 0.010 | 0.023 | 0.033 | 0.010 | 0.020 | 0.030 |
|               | period     | 0.017 | 0.003 | 0.023 | 0.017 | 0.007 | 0.023 |

Table 4: Empirical sizes of tests for unit root at \(-1\)

| Nuisance Root | Noise Type | Tests |
|---------------|------------|-------|
|               | \( \hat{\pi}^{(4)} \) | \( \hat{\pi}^{(8)} \) | \( \hat{\pi}^{(12)} \) | \( t^{(4)} \) | \( t^{(8)} \) | \( t^{(12)} \) |
| False         | iid        | 0.040 | 0.043 | 0.053 | 0.040 | 0.047 | 0.050 |
|               | heter      | 0.040 | 0.073 | 0.040 | 0.047 | 0.060 | 0.033 |
|               | \( ma_{pos} \) | 0.080 | 0.080 | 0.073 | 0.073 | 0.080 | 0.073 |
|               | \( ma_{neg} \) | 0.060 | 0.063 | 0.043 | 0.063 | 0.067 | 0.043 |
|               | ar         | 0.040 | 0.047 | 0.050 | 0.047 | 0.047 | 0.053 |
|               | period     | 0.030 | 0.037 | 0.050 | 0.037 | 0.033 | 0.063 |
| True          | iid        | 0.143 | 0.127 | 0.127 | 0.143 | 0.120 | 0.130 |
|               | heter      | 0.123 | 0.147 | 0.177 | 0.120 | 0.140 | 0.173 |
|               | \( ma_{pos} \) | 0.483 | 0.543 | 0.533 | 0.463 | 0.550 | 0.523 |
|               | \( ma_{neg} \) | 0.070 | 0.083 | 0.077 | 0.070 | 0.070 | 0.077 |
|               | ar         | 0.240 | 0.313 | 0.343 | 0.233 | 0.313 | 0.333 |
|               | period     | 0.247 | 0.327 | 0.310 | 0.243 | 0.310 | 0.303 |
Table 5: Empirical sizes of tests for unit roots at ±i

| Nuisance Root | Noise Type | Tests |
|---------------|------------|-------|
|               |            | $F^{(4)}$ | $F^{(8)}$ | $F^{(12)}$ |
| False         | iid        | 0.053    | 0.050    | 0.047    |
|               | heter      | 0.067    | 0.090    | 0.073    |
|               | ma<sub>pos</sub> | 0.061    | 0.060    | 0.047    |
|               | ma<sub>neg</sub> | 0.073    | 0.040    | 0.083    |
|               | ar         | 0.047    | 0.030    | 0.030    |
|               | period     | 0.053    | 0.040    | 0.027    |
| True          | iid        | 0.017    | 0.020    | 0.017    |
|               | heter      | 0.013    | 0.020    | 0.010    |
|               | ma<sub>pos</sub> | 0.087    | 0.063    | 0.097    |
|               | ma<sub>neg</sub> | 0.060    | 0.067    | 0.123    |
|               | ar         | 0.113    | 0.147    | 0.120    |
|               | period     | 0.093    | 0.100    | 0.090    |

5.3 Results

Now we present in Figure 1, 2, and 3 the main simulation result of the seasonal iid bootstrap augmented HEGY test and the seasonal block bootstrap unaugmented HEGY test. This simulation includes two cases of nuisance roots (see Table 1) and six types of noises (see Table 2), and sets sample size $T = 120$, number of bootstrap replicates $B = 500$, number of iterations $N = 600$, and nominal size $\alpha = 0.05$.

5.3.1 Root at 1

When our data have a potential root at 1, but no other nuisance roots at $-1$ or $\pm i$, the power curves of the both bootstrap tests almost overlap, according to (a)-(f) in Figure 1. Further, both power curves start at the correct size, $\alpha = 0.05$, and tend to one when $\rho$ departs from zero. Hence both tests work well when no nuisance root occurs.

When data have a potential root at 1 and all nuisance roots at $-1$ and $\pm i$, the sizes of seasonal block bootstrap unaugmented HEGY test are distorted in (g), (h), (j), and (l) in Figure 1. These distortions may result from the errors in estimating $\pi_{j,s}$ and the need to recover $\{Y_{4t+s}\}$ with the estimated $\pi_{j,s}$. The size distortion in (j) is particularly serious, since the unit root filter $(1 - L)$ is partially cancelled by the Moving Average (MA) filter $(1 - 0.5L)$, and this cancellation cannot be handled well by block bootstrap (Paparoditis and Politis, 2003). In contrast, in (l) the filter $(1 - L)$ is enhanced by the AR filters $(1 - \phi_s L)$, thus the size is distorted toward zero.

On the other hand, seasonal iid bootstrap augmented HEGY test is free of the size distortions when data have nuisance roots. This is in part because the test recovers $\{Y_{4t+s}\}$ using the true values of $\pi_{j,s}$, namely zero, instead of using the estimated values. Moreover, when both HEGY tests have almost the correct sizes as in (i) and (k), seasonal iid bootstrap augmented HEGY test attains equal or higher powers. Therefore, when testing the root at 1, seasonal iid bootstrap augmented HEGY test is recommended.
Figure 1: Powers as a function of $\rho$ when testing roots at 1
(a)-(f) have no nuisance roots; (g)-(l) have all nuisance roots;
blue dotted curve is for seasonal iid bootstrap; red solid curve is for seasonal block bootstrap.

5.3.2 Root at -1

Now we come to the tests for root at $-1$. When none of the nuisance root at 1 or $\pm i$ exists, the power curves of the two tests are very close to each other, as (a)-(f) in Figure 2 indicate. This patterns of curves have been seen in (a)-(f) in Figure 1 and indicate the nice performance of both tests.

When nuisance roots are present, sizes of seasonal block bootstrap unaugmented HEGY test are distorted in nearly all scenarios in (g)-(l) in Figure 2. In particular, the size distortion in (i) is the worst, because of the partial cancellation of the seasonal unit root filter $(1 + L)$ and the MA filter $(1 + 0.5L)$. However, the power curves of seasonal iid bootstrap augmented HEGY test start around the nominal size 0.05 in all of (g)-(l). Further, these curves tend to 1, as $\rho$ grows larger. Therefore, we recommend seasonal iid bootstrap test for testing root at $-1$. 
5.3.3 Root at $\pm i$

Finally we discuss the tests for roots at $\pm i$. With none of the nuisance root at 1 or $-1$, (a)-(f) in Figure 3 illustrate that both tests achieve sizes that are close to the nominal size, and powers that tend to one. When all of nuisance roots show up, both tests suffer from some size distortions. The empirical sizes of seasonal iid bootstrap augmented HEGY test are biased toward zero in (g)-(l); the sizes of seasonal block bootstrap unaugmented HEGY test are biased toward zero in (g) and (h), but are biased toward one in (j)-(l). On the other hand, seasonal block bootstrap unaugmented HEGY test’s empirical powers prevail throughout (g)-(l), and therefore shall be recommended for testing roots at $\pm i$. 

Figure 2: Powers as a function of $\rho$ when testing roots at $-1$
(a)-(f) have no nuisance roots; (g)-(l) have all nuisance roots;
blue dotted curve is for seasonal iid bootstrap; red solid curve is for seasonal block bootstrap.
6 Conclusion

In this paper we analyze the augmented and unaugmented HEGY tests in the seasonal heterogeneous setting. Given root at 1 or $-1$, the asymptotic distributions of the test statistics are standard. However, given concurrent roots at 1 and $-1$, or roots at $\pm i$, the asymptotic distributions are neither standard, pivotal, nor directly pivotable. Therefore, when seasonal heterogeneity exists, HEGY tests can be used to test the single roots at 1 or $-1$, but cannot be directly applied to any combinations of roots.

Bootstrap proves to be an effective remedy for HEGY tests in the seasonal heterogeneous setting. The two bootstrap approaches, namely 1) seasonal iid bootstrap augmented HEGY test and 2) seasonal block bootstrap unaugmented HEGY test, turn out both theoretically solid. In the comparative simulation study, seasonal iid bootstrap augmented HEGY test has better performance when testing roots at 1 or $-1$, but seasonal block bootstrap unaugmented HEGY test outperforms when testing roots at $\pm i$.

Therefore, when testing seasonal unit roots under seasonal heterogeneity, the aforementioned
Lemma 1. Suppose Assumption 2.A or Assumption 2.B hold. Then under 7.1 Proof of Theorem 4.1.

Let $Y_t = (Y_{t-3}, Y_{t-2}, Y_{t-1}, Y_t)'$, $V_t = (V_{t-3}, V_{t-2}, V_{t-1}, V_t)'$, $\Gamma_j = E[V_t V_{t-j}']$, $W(t) = (W(1)(t), W(2)(t), W(3)(t), W(4)(t))'$ be a four-dimensional standard Brownian motion, $\int W_dW'$ denotes $\int_0^T W(r)dW(r)'$, and $\int WW'$ denotes $\int_0^T W(r)W(r)'dr$.

7 Appendix

The appendix includes the proof of the theorems in this paper. We first present the proof for the asymptotics of the unaugmented HEGY test, then the asymptotics of the augmented HEGY test. Thoughout the appendix, let $Y_t = (Y_{t-3}, Y_{t-2}, Y_{t-1}, Y_t)'$, $V_t = (V_{t-3}, V_{t-2}, V_{t-1}, V_t)'$, $\Gamma_j = E[V_t V_{t-j}']$, $W(t) = (W(1)(t), W(2)(t), W(3)(t), W(4)(t))'$ be a four-dimensional standard Brownian motion, $\int W_dW'$ denotes $\int_0^T W(r)dW(r)'$, and $\int WW'$ denotes $\int_0^T W(r)W(r)'dr$.

7.1 Proof of Theorem 4.1

Lemma 1. Suppose Assumption 2.A or Assumption 2.B hold. Then under $H_1^{1,2,3,4},$

$$T^{-1} \sum_{t=1}^T Y_{t-1} V_t' \Rightarrow \Theta(1)\Omega^{1/2}\{ \int W_dW' \} \Omega^{1/2}\Theta(1)' + \sum_{j=1}^{\infty} \Gamma_j' \equiv Q_1,$$

$$T^{-2} \sum_{t=1}^T Y_{t-1} Y_t' \Rightarrow \Theta(1)\Omega^{1/2}\{ \int WW' \} \Omega^{1/2}\Theta(1) \equiv Q_2,$$

$$T^{-1} \sum_{t=1}^T V_t V_{t-j}' \Rightarrow \Gamma_j.$$

Proof. See Hamilton (1994, Proposition 18.1, pp. 547-548) for the proof with iid innovations, Chan and Wei (1988) for the proof under Assumption 2.A, and De Jong and Davidson (2000) for the proof under Assumption 2.B.

Lemma 2. Let $X_{U,j} = (Y_{j,0}, ..., Y_{j,4T-1})'$, and $X_U = (X_{U,1}, X_{U,2}, X_{U,3}, X_{U,4})$, where $U$ stands for unaugmented HEGY, and $\{Y_{j,4T+s}\}$ is defined in (2.4). Let $V = (V_1, ..., V_{4T})'$, $\Upsilon$ be the matrix generated by assigning zero to all entries of $\Gamma_0$ but those above the main diagonal. Then, under $H_1^{1,2,3,4},$

$$(a) (AT)^{-2} (X_U'X_U)_{11} \Rightarrow \frac{1}{4} c_1^2 Q_2 e_1 \equiv \eta_1,$$

$$(AT)^{-2} (X_U'X_U)_{22} \Rightarrow \frac{1}{4} c_1^2 Q_2 e_2 \equiv \eta_2,$$

$$(AT)^{-2} (X_U'X_U)_{33} \Rightarrow \frac{1}{8} (c_1^2 Q_2 e_3 + c_2^2 Q_2 e_4) \equiv \eta_3,$$

$$(AT)^{-2} (X_U'X_U)_{44} \Rightarrow \frac{1}{8} (c_1^2 Q_2 e_3 + c_2^2 Q_2 e_4) \equiv \eta_3,$$

$$(AT)^{-1} (X_U'X_U)_{ij} \Rightarrow 0, \text{ for } i \neq j.$$
The proof follows the lines of Said and Dickey (1984) and contains two parts. Firstly, we show when \( T \to \infty \) the statistic of interest tends to a limit free of \( k \), and then we prove this limit tends to a certain distribution as \( T \to \infty \). For part (b), we only present the proof of the first statement. Other statements are proven in similar ways. By Lemma [1],

\[
(4T)^{-1} X'_{t,1} V = (4T)^{-1} \sum_{t=1}^{T} \sum_{s=-3}^{0} Y_{t,4t+s-1} V_{4t+s}
\]

Further, the asymptotic distributions of F-statistics are identical with the asymptotic distributions of the averages of the squares of the corresponding t-statistics, i.e., \( F_{3,4} \to \frac{1}{\frac{1}{2}(t_{3}^{2} + t_{4}^{2})} \to 0 \), due to the asymptotic orthogonality indicated by Lemma [2] (a).

### 7.2 Proof of Theorem 3.1

The proof follows the lines of Said and Dickey (1984) and contains two parts. Firstly, we show when \( T \to \infty \) and \( k = k_{T} \to \infty \) simultaneously, the statistic of interest tends to a limit free of \( k \), and then we prove this limit tends to a certain distribution as \( T \to \infty \).

To begin with, notice that when \( k \to \infty \), the error term of regression (3.1) tends to a limit. Surprisingly, this limit is in general not \( \epsilon_{i} \), because the regression (3.1) falsely assumes seasonally
homogeneous coefficients and thus in general cannot find the correct residuals $\epsilon_t$. To find the limit, recall that $\{\tilde{V}_t\}$ is defined as a misspecified constant parameter representation of $\{V_{4t+s}\}$. Under Assumption 1.B, the spectral densities of $\{\tilde{V}_t\}$ are finite and positive everywhere, so $\{\tilde{V}_t\}$ has AR$(\infty)$ and MA$(\infty)$ expressions

$$\tilde{V}(L)\tilde{V}_t = \tilde{\zeta}_t$$

(7.1)

where $\tilde{V}(z) = 1 - \sum_{i=1}^{\infty} \tilde{\psi}_i z^i$, $\tilde{\psi}(z) = 1 + \sum_{i=1}^{\infty} \tilde{\psi}_i z^i$. Let $\zeta_t^{(k)} = V_t - \sum_{i=1}^{k} \tilde{\psi}_i V_{t-i}$, and $\zeta_t = V_t - \sum_{i=1}^{\infty} \tilde{\psi}_i V_{t-i}$, where $\{\tilde{\psi}_i\}$ are the AR coefficients defined in (7.1). Since a misspecified constant parameter representation of $\zeta_t$ is $\tilde{V}_t - \sum_{i=1}^{\infty} \tilde{\psi}_i \tilde{V}_t$, which is exactly $\tilde{\zeta}_t$ defined in (7.1), no ambiguity arises. The following lemma provides two properties of $\{\zeta_t\}$, whose proof is left to the readers.

**Lemma 3.**

\[
(a) \frac{1}{4} \sum_{s=-3}^{0} \text{Cov}(\zeta_{4t+s-j}, \zeta_{4t+s}) = 0, \ \forall j = 1, 2, \ldots ,
\]

\[
(b) \frac{1}{4} \sum_{s=-3}^{0} \text{Cov}(V_{4t+s-j}, \zeta_{4t+s}) = 0, \ \forall j = 1, 2, \ldots ,
\]

Now we show when $T \to \infty$ and $k \to \infty$ simultaneously, the statistics of interest tend to certain limits. Let $X$ be the design matrix of regression equation (3.1), $\hat{\beta} = (\hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3, \hat{\pi}_4, \hat{\phi}_1, \ldots, \hat{\phi}_k)'$ be the estimated coefficient vector of regression equation (3.1), $\beta = (0, 0, 0, 0, \hat{\psi}_1, \ldots, \hat{\psi}_k)'$, $\zeta_t^{(k)} = (\zeta_{1+k}, \ldots, \zeta_{4T})'$, and $\zeta = (\zeta_{1+k}, \ldots, \zeta_{4T})'$. Define the $(4+k) \times (4+k)$ dimensional scaling matrix $D_T = \text{diag}((4 - k)^{-1}, (4T - k)^{-1}, (4T - k)^{-1}, (4T - k)^{-1/2}, \ldots, (4T - k)^{-1/2})$. Then $D_T^{-1}(\hat{\beta} - \beta) = (D_T X' XD_T)^{-1} D_T X' \zeta^{(k)}$.

Let $\| \cdot \|$ be the $L_2$ induced norm of matrices. Now we want to define a diagonal matrix $R$ such that $\| D_T X' XD_T - R \|$ converges to 0 in probability. By the multivariate Beveridge-Nelson Decomposition (see Hamilton, 1994, pp. 545-546), since $(4T - k)^{-1} \sum (1 - L^4) Y_{t-i} (1 - L^4) Y_{t-j}$ converges in probability to the seasonal average of autocovariance of $V_t$ of lag $|i-j|$, we let $R = \text{diag}(R_1, R_2, R_3, R_4, \tilde{\Gamma})$, where

\[
R_1 = \frac{c_1' \Theta(1) \sum S_i S_i' \Theta(1)' c_1}{(4T - k)^2}
\]

\[
R_2 = \frac{c_2' \Theta(1) \sum S_i S_i' \Theta(1)' c_2}{(4T - k)^2}
\]

\[
R_3 = \frac{c_3' \Theta(1) \sum S_i S_i' \Theta(1)' c_3 + c_4' \Theta(1) \sum S_i S_i' \Theta(1)' c_4}{2(4T - k)^2}
\]

\[
R_4 = R_3, \quad S_t = \sum_{i=1}^{t} \epsilon_i, \quad \tilde{\Gamma}_{i,j} = \tilde{\gamma}(|i-j|).
\]
Following the definition of $R$, we make the following decomposition:

$$
D_T^{-1}(\hat{\beta} - \beta) = (D_T'XD_T)^{-1}D_T'X'\zeta^{(k)} \\
= [(D_T'XD_T)^{-1} - R^{-1}]D_T'X'\zeta^{(k)} + R^{-1}D_T'X' (\zeta^{(k)} - \zeta) + R^{-1}D_T'X' \zeta.
$$

(7.2)

Notice the last term in the right hand side summation, $R^{-1}D_T'X' \zeta$, is free of $k$. Later we will find out its asymptotic distribution as $T \to \infty$. But now we need to prove the first two terms in the right hand side of (7.2) converge to zero as $T \to \infty$ and $k \to \infty$. Indeed,

$$
|| (D_T'XD_T)^{-1} - R^{-1}|| = o_p(k^{-1/2}),
$$

(7.3)

$$
|| D_T'X' (\zeta^{(k)} - \zeta) || = o_p(1),
$$

(7.4)

$$
|| D_T'X' \zeta || = O_p(k^{1/2}),
$$

(7.5)

$$
|| R^{-1} || = O_p(1).
$$

(7.6)

Equation (7.3) can be proven straightforwardly (see Said and Dickey, 1984). For (7.4), notice

$$
E || D_T'X' (\zeta^{(k)} - \zeta) ||^2 = E[(4T - k)^{-2} \sum_{j=1}^{k} \sum_{i=0}^{k} (\zeta^{(k)} - \zeta_i)^2 + (4T - k)^{-1} \sum_{i=1}^{k} (\zeta^{(k)} - \zeta_i)^2] = o(1).
$$

Hence, $\tilde{\psi}(L)\tilde{\vartheta}(L)$ has exponentially decaying coefficient $\tilde{\psi}_i$. It follows straightforwardly that $E || D_T'X' (\zeta^{(k)} - \zeta) ||^2 \to 0$. For (7.5), notice that

$$
E || D_T'X' \zeta ||^2 = E[(4T - k)^{-2} \sum_{j=1}^{k} \sum_{i=0}^{k} (\zeta^{(k)} - \zeta_i)^2 + (4T - k)^{-1} \sum_{i=1}^{k} (\zeta^{(k)} - \zeta_i)^2] = o(1).
$$

By Lemma 3 and the stationarity of $\{\epsilon_t\}$,

$$
E[(4T - k)^{-1/2} \sum_{t=0}^{4T} V_{t-i} (\zeta_t)^2] = \frac{1}{4} \sum_{s=-3}^{0} \sum_{h=-\infty}^{\infty} \text{Cov}(V_{4t+s-i} \zeta_{4t+s}, V_{4t+s-h-i} \zeta_{4t+s-h}) + o(1)
$$

$$
= \frac{1}{4} \sum_{s=-3}^{0} \sum_{h=-\infty}^{\infty} \text{Cov}(V_s \zeta_s, V_{s-h-i} \zeta_{s-h}) + o(1).
$$

Without loss of generality we can focus on $i = 1$ and $s = 0$. By writing $V_t$ and $\zeta_t$ as linear combinations of $\epsilon_t$,

$$
\sum_{h=-\infty}^{\infty} \text{Cov}(V_{t-i} \zeta_t, V_{t-h-i} \zeta_{t-h}) \leq \text{const. sup}_{i,j_1,j_2} \sum_{h=-\infty}^{\infty} |\text{Cov}(\epsilon_{i_1} \epsilon_{j_1}, \epsilon_{i_2} \epsilon_{j_2} \epsilon_{j_2-h})|.
$$
The right hand side of this inequality is assumed to be bounded under Assumption 2.A. On the other hand, the right hand side is also bounded under Assumption 2.B, by the lemma below.

**Lemma 4.** Suppose (i) \{z_t\}_{t=1}^\infty is a strictly stationary strong mixing time series with mean zero and finite \(4 + \delta\) moment for some \(\delta > 0\), and (ii) \{z_t\}'s strong mixing coefficient \(\alpha(h)\) satisfies \(\sum_{h=1}^\infty h\alpha^{3/(4+\delta)}(h) < \infty\). Then \(\exists K > 0\) such that for all \(i_1, i_2, j_1,\) and \(j_2\),

\[
\sum_{h=-\infty}^{\infty} \text{Cov}(z_{i_1} z_{j_1}, z_{i_2} z_{j_2}) < K.
\]

**Proof.** Let \(h_1 = h + i_1 - i_2, h_2 = h + j_1 - j_2, h_3 = h + i_1 - j_2, h_4 = h + j_1 - i_2\). By Lemma A.0.1 of Politis, Romano, and Wolf (1999),

\[
\sum_{h=-\infty}^{\infty} \text{Cov}(z_{i_1} z_{j_1}, z_{i_2} z_{j_2}) 
\leq \text{const.} \sum_{h=-\infty}^{\infty} \left(\alpha(\min(|h_1|, |h_2|, |h_3|, |h_4|))\right)^{\delta/4} 
\leq \text{const.} \sum_{h=-\infty}^{\infty} \left(\alpha(|h_1|) + \alpha(|h_2|) + \alpha(|h_3|) + \alpha(|h_4|)\right)^{\delta/4} 
\leq \text{const.} \sum_{h=-\infty}^{\infty} \alpha(|h|) \tag{\text{Q.E.D.}}
\]

We have proven that \(E[|(4T - k)^{-1/2} \sum_{t=k+1}^{4T} V_{t-1} \zeta(t)|^2] = O(1)\). Similarly, it can be shown that \(E[|(4T - k)^{-1} \sum_{t=k+1}^{4T} Y_{j,t-1} \zeta_t|^2] = O(1)\). Hence, (7.5) follows. To justify (7.6), notice

\[
\frac{c_4 \Theta(1) \Omega^{1/2} \sum_{i=0}^{\infty} \sum_{s} S_i S_i' \Omega^{1/2} \Theta(1)' c_i}{(4T - k)^2} \Rightarrow c_4 \Theta(1) \Omega^{1/2} \int W W' \Omega^{1/2} \Theta(1)' c_i,
\]

where \(W\) indicates standard four-dimensional Brownian Motion. Since

\[
P(c_4 \Theta(1) \Omega^{1/2} \int W W' \Omega^{1/2} \Theta(1)' c_i = 0) = 0,
\]

\(\forall \epsilon > 0, \exists M_\epsilon > 0\), such that \(P(c_4 \Theta(1) \Omega^{1/2} \int W W' \Omega^{1/2} \Theta(1)' c_i < M_\epsilon) < \epsilon\). (7.6) follows from the definition of \(O_p(1)\).

Combining equations (7.3), (7.4), (7.5), and (7.6), we have

\[
\|((D_T X' XD_T)^{-1} - R^{-1} D_T X' \zeta(k)) \| = o_p(1) \\
\|R^{-1} D_T X' (\zeta(k) - \zeta)) \| = o_p(1) \\
\|R^{-1} D_T X' \zeta) \| = O_p(k^{1/2}).
\]

From these results, we can immediately show the consistency of \(\hat{\beta}\). Notice \(D_T^{-1}(\hat{\beta} - \beta) = O_p(k^{1/2})\) by (7.2). The consistency follows from \(\|D_T\| = O((4T - k)^{-1})\) and \(k = o(T^{1/3})\). Further, the asymptotic distribution of \(\hat{\beta}\) can be derived with the asymptotic equivalence of \(D_T^{-1}(\hat{\beta} - \beta)\) and \(R^{-1} D_T X' \zeta\). Notice \(R^{-1} D_T X' \zeta\) is free of \(k\). As \(T \to \infty\), \(R^{-1}\) converges in distribution to a functional of Brownian motion, and the asymptotics of \(D_T X' \zeta\) can be found with the following lemma.

24
Lemma 5.

\[
\frac{1}{4T} \sum_{t=1}^{4T} Y_{1,t-1} \xi_t = Var(\tilde{\xi}) \tilde{\theta}(1) \int_0^1 W_1(r) dW_1(r),
\]

\[
\frac{1}{4T} \sum_{t=1}^{4T} Y_{2,t-1} \xi_t = Var(\tilde{\xi}) \tilde{\theta}(-1) \int_0^1 W_2(r) dW_2(r),
\]

\[
\left( \frac{1}{4T} \sum_{t=1}^{4T} Y_{3, t-1} \xi_t \right)^2 + \left( \frac{1}{4T} \sum_{t=1}^{4T} Y_{4, t-1} \xi_t \right)^2
\]

\[
= Var(\tilde{\xi}) \left[ \frac{1}{4} c_4^T (\Omega(1) \theta(1))' c_4 \int W_4(r) dW_4(r) + \frac{1}{4} c_4^T (\Omega(1) \theta(1))' c_4 \int W_3(r) dW_3(r) \right]^2
\]

\[
\frac{1}{4} (c_4^T (\Omega(1) \theta(1))' c_4 + c_4^T (\Omega(1) \theta(1))' c_3)
\]

\[
\Rightarrow Var(\tilde{\xi}) \left[ \frac{1}{2} c_4^T (\Omega(1) \theta(1))' c_4 + \frac{1}{2} c_4^T (\Omega(1) \theta(1))' c_3 (\int_0^1 W_3(r) dW_3(r) - \int_3 W_4(r) dW_4(r)) \right]^2
\]

\[
\frac{1}{4} (c_4^T \Omega(1) \theta(1)) c_4 + c_4^T (\Omega(1) \theta(1)) c_3.
\]

Proof of Lemma 5. Firstly we focus on the convergence of \( \frac{1}{4T} \sum_{t=1}^{4T} Y_{1,t-1} \xi_t \). The convergence of \( \frac{1}{4T} \sum_{t=2}^{4T} Y_{2,t-1} \xi_t \) can be proven analogously. Define \( \Psi(z) \) such that \( \xi_t = \Psi(B) V_t \). Let \( \xi_t = \tilde{\psi}(L) Y_t \), \( \xi_{1, t} = \tilde{\psi}(L) Y_{1, t}, \xi_t = (\xi_{4, t-3}, \xi_{4, t-2}, \xi_{4, t-1}, \xi_t)' \), \( \xi_t = (\xi_{4, t-3}, \xi_{4, t-2}, \xi_{4, t-1}, \xi_t)' \). Then \( \mathbb{B} \xi_t = \xi_t \), and

\[
\frac{1}{4T} \sum_{t=1}^{4T} Y_{1, t-1} \xi_t
\]

\[
= \tilde{\theta}(1) \frac{1}{4T} \sum_{t=1}^{4T} \sum_{s=-3}^{0} \xi_{1, 4t+s-1} \xi_{4t+s} \quad \text{(by Beveridge-Nielson Decomposition, up to } o_p(1))
\]

\[
= \tilde{\theta}(1) \frac{1}{4T} \sum_{t=1}^{T} \sum_{s=-3}^{0} \sum_{k=-3}^{s} \xi_{1, 4t+s-1} \xi_{4t+s}
\]

\[
\Rightarrow \frac{1}{4} \tilde{\theta}(1) c_4^T \Psi(1) (\Omega(1))^{1/2} \int W dW' \Omega(1)^{1/2} \Psi(1)' c_4 \quad \text{(by Lemma 5 and FCLT)}
\]

\[
= \frac{1}{4} \tilde{\theta}(1) (\int_0^1 W_1(r) dW_1(r)
\]

\[
\text{(by Osborn (1991, p. 378)), } \frac{1}{4} c_4^T \Omega(1) \theta(1)' c_4 = Var(\xi_t) \tilde{\theta}(1)^2.
\]

Secondly we show the convergence of \( \left( \frac{1}{4T} \sum_{t=1}^{4T} Y_{3, t-1} \xi_t \right)^2 + \left( \frac{1}{4T} \sum_{t=1}^{4T} Y_{4, t-1} \xi_t \right)^2 \). Let \( \xi_{3, t} = \tilde{\psi}(L) Y_{3, t}, \xi_{4, t} = \tilde{\psi}(L) Y_{4, t}, \tilde{\psi}_a = (\tilde{\psi}(i) + \tilde{\psi}(-i))/2, \tilde{\psi}_b = (\tilde{\psi}(i) - \tilde{\psi}(-i))/2i, \tilde{\theta}_a = (\tilde{\theta}(i) + \tilde{\theta}(-i))/2, \tilde{\theta}_b = (\tilde{\theta}(i) - \tilde{\theta}(-i))/2i. \)
Then
\[
\frac{1}{4T} \sum_{t=1}^{4T} Y_{3,t-1} \zeta_t
\]
\[
= \frac{1}{4T} \sum_{t=1}^{4T} (\bar{\theta}_a \xi_{3,t-1} - \bar{\theta}_b \xi_{4,t-1}) \zeta_t
\]
(by Beveridge-Nielson Decomposition, up to \(o_p(1)\))
\[
= \frac{1}{4T} \sum_{t=1}^{T} \left[ \bar{\theta}_a [c_3' \xi_{t-1} \zeta c_3 + c_4' \xi_{t-1} \zeta c_4] - \sum_{s=-3}^{-2} \zeta_{4t+s} \zeta_{4t+s+2} \right]
\]
\[
- \frac{1}{4T} \sum_{t=1}^{T} \left[ \bar{\theta}_b [c_3' \xi_{t-1} \zeta c_3 - c_4' \xi_{t-1} \zeta c_4] - \sum_{s=-3}^{-1} \zeta_{4t+s} \zeta_{4t+s+1} + \zeta_{4t-3} \zeta_{4t} \right]
\]
\[\Rightarrow \frac{1}{4} \bar{\theta}_a [c_3' \bar{\Psi}(1) \Theta(1) \Omega^{1/2} \int W dW' \Omega^{1/2} \Theta(1)' \bar{\Psi}(1)' c_3
\]
\[+ c_4' \bar{\Psi}(1) \Theta(1) \Omega^{1/2} \int W dW' \Omega^{1/2} \Theta(1)' \bar{\Psi}(1)' c_4]
\]
\[- \frac{1}{4} \bar{\theta}_b [c_3' \bar{\Psi}(1) \Theta(1) \Omega^{1/2} \int W dW' \Omega^{1/2} \Theta(1)' \bar{\Psi}(1)' c_4
\]
\[- c_4' \bar{\Psi}(1) \Theta(1) \Omega^{1/2} \int W dW' \Omega^{1/2} \Theta(1)' \bar{\Psi}(1)' c_3] \]
(by Lemma 3 and FCLT, the covariances of \(\zeta_t\) cancel out since \(\{\zeta_t\}\) is white noise)
\[
= \frac{1}{4} \bar{\theta}_a [\bar{\psi}(i)^2] [c_3' \Theta(1) \Omega^{1/2} \int W dW' \Omega^{1/2} \Theta(1)' c_3 + c_4' \Theta(1) \Omega^{1/2} \int W dW' \Omega^{1/2} \Theta(1)' c_4]
\]
\[- \frac{1}{4} \bar{\theta}_b [\bar{\psi}(i)^2] [c_3' \Theta(1) \Omega^{1/2} \int W dW' \Omega^{1/2} \Theta(1)' c_4 - c_4' \Theta(1) \Omega^{1/2} \int W dW' \Omega^{1/2} \Theta(1)' c_3]
\]
(since \(c_3' \bar{\Psi}(1) = \bar{\psi}_a c_3 + \bar{\psi}_b c_3', \ c_4' \bar{\Psi}(1) = \bar{\psi}_a c_4 - \bar{\psi}_b c_4', \) and \(\bar{\psi}_a^2 + \bar{\psi}_b^2 = |\bar{\psi}(i)|^2).\)

Similarly,
\[
\frac{1}{4T} \sum_{t=1}^{4T} Y_{4,t-1} \zeta_t
\]
\[
= \frac{1}{4} \bar{\theta}_b [\bar{\psi}(i)^2] [c_3' \Theta(1) \Omega^{1/2} \int W dW' \Omega^{1/2} \Theta(1)' c_3 + c_4' \Theta(1) \Omega^{1/2} \int W dW' \Omega^{1/2} \Theta(1)' c_4]
\]
\[+ \frac{1}{4} \bar{\theta}_a [\bar{\psi}(i)^2] [c_3' \Theta(1) \Omega^{1/2} \int W dW' \Omega^{1/2} \Theta(1)' c_4 - c_4' \Theta(1) \Omega^{1/2} \int W dW' \Omega^{1/2} \Theta(1)' c_3]
\]
The lemma follows from \(|\bar{\psi}(i)|^2 = |\bar{\theta}(i)|^{-2}\) and (Osborn, 1991)
\[
Var(\zeta_t) |\bar{\theta}(i)|^2 = \frac{1}{4} (c_3' \Theta(1) \Omega \Theta(1)' c_3 + c_4' \Theta(1) \Omega \Theta(1)' c_3).
\]

Now we come to the asymptotic distribution of the t-statistics and the F-statistics. Notice,
\[
t_i = \sigma^{-1} [(X'X)^{-1}]_{ii}^{-1/2} [(X'X)^{-1} X' \zeta^{(k)}],
\]
26
where independent but not identical random variables, while

\[ \text{Define} \]

\[ 7.3 \text{ Proof of Theorem 3.1.} \]

\[ 3.1 \text{ is complete.} \]

\[ \text{are identical with the asymptotic distributions of the averages of the squares of the corresponding } \]

\[ \hat{\beta} \text{, we have } \hat{\sigma}^2 \overset{p}{=} \text{Var}(\hat{\zeta}). \]

Further, the asymptotic distributions of the F-statistics

\[ \text{By the consistency of } \hat{\beta} \text{, } \hat{\sigma}^2 \overset{p}{=} \text{Var}(\hat{\zeta}). \]

\[ \text{further, the asymptotic distributions of the F-statistics are identical with the asymptotic distributions of the averages of the squares of the corresponding t-statistics because of the asymptotic orthogonality of the regression. Hence, the proof of Theorem 3.1 is complete.} \]

\[ \text{7.3 Proof of Theorem 3.1.} \]

Define \( \{i_t\} \) and \( \{I_t\} \) such that \( e_t = \hat{\epsilon}_i \) and \( e_{4t+s} = \hat{\epsilon}_{4I_t+s} \). By Algorithm 3.1, \( \{i_t\} \) is a sequence of independent but not identical random variables, while \( \{I_t\} \) is a sequence of iid random variables. Recall

\[ (1 - L^4)Y_{4t+s} = \sum_{j=1}^{4} \pi_{j,s}Y_{j,4t+s-1} + \sum_{i=1}^{k} \phi_{i,s}(1 - L^4)Y_{4t+s-i} + e_{4t+s}, \]

where \( \{e_{4t+s}\} \) is the regression error. Let

\[ v_{T,t}^{(1)} = (e_{it} - E^o e_{it})/\text{Std}^o(e_{it}) \]

\[ v_{T,t}^{(2)} = (-1)^t(e_{it} - E^o e_{it})/\text{Std}^o((-1)^t e_{it}) \]

\[ v_{T,t}^{(3)} = \sqrt{2} \sin(\frac{\pi t}{2}) (e_{it} - E^o e_{it})/\text{Std}^o(\sqrt{2} \sin(\frac{\pi t}{2}) e_{it}) \]

\[ v_{T,t}^{(4)} = \sqrt{2} \cos(\frac{\pi t}{2}) (e_{it} - E^o e_{it})/\text{Std}^o(\sqrt{2} \cos(\frac{\pi t}{2}) e_{it}) \]

Let \( R_T^t \) be the partial sum of \( v_{T,t} \) above. Formally,

\[ R_T^t(u_1, u_2, u_3, u_4) = \left( \frac{1}{\sqrt{AT}} \sum_{t=1}^{[4Tu_1]} v_{T,t}^{(1)} \right) \left( \frac{1}{\sqrt{AT}} \sum_{t=1}^{[4Tu_2]} v_{T,t}^{(2)} \right) \left( \frac{1}{\sqrt{AT}} \sum_{t=1}^{[4Tu_3]} v_{T,t}^{(3)} \right) \left( \frac{1}{\sqrt{AT}} \sum_{t=1}^{[4Tu_4]} v_{T,t}^{(4)} \right). \]

To justify theorem 3.1, it suffices to show

\[ \|S_T^o - R_T^o\| \overset{p}{\to} 0 \text{ uniformly in } u_1, u_2, u_3 \text{ and } u_4, \quad (7.8) \]

and \( R_T^o = W \) in probability,

\[ (7.9) \]

because the unconditional convergence in (7.8) implies that in probability the conditional distribution of \( \|S_T^o - R_T^o\| \) given \( \{Y_{4t+s}\} \) converges to zero. To prove (7.8), we can without loss of generality focus on the uniform convergence of the first coordinate, that is, uniformly in \( u_1 \),

\[ \left| \frac{1}{\sqrt{AT}} \sum_{t=1}^{[4Tu_1]} \epsilon_t^*/\sigma_t^* - \frac{1}{\sqrt{AT}} \sum_{t=1}^{[4Tu_1]} v_{T,t}^{(1)} \right| \overset{p}{\to} 0. \quad (7.10) \]
Notice that uniformly in $u_1$,
\[
\frac{1}{\sqrt{T}} \sum_{t=1}^{[4T u_1]} \epsilon_t^r
\]
\[
= \frac{1}{\sqrt{T}} \sum_{s=-3}^{[T u_1]} \sum_{t=[k/4]+1}^T \epsilon_{4t+s} + o_p(1)
\]
\[
= \frac{1}{\sqrt{T}} \sum_{s=-3}^{[T u_1]} \sum_{t=[k/4]+1}^T \left( \hat{e}_{4t+s} - \frac{1}{T} \sum_{t=[k/4]+1}^T \epsilon_{4t+s} \right) + o_p(1)
\]
\[
= \frac{1}{\sqrt{T}} \sum_{s=-3}^{[T u_1]} \sum_{t=[k/4]+1}^T \left( \epsilon_{4t+s} - \frac{1}{T} \sum_{t=[k/4]+1}^T \epsilon_{4t+s} \right)
\]
\[
= \frac{1}{\sqrt{T}} \sum_{s=-3}^{[T u_1]} \sum_{t=[k/4]+1}^T \left( \epsilon_{4t+s} - \frac{1}{T} \sum_{t=[k/4]+1}^T \epsilon_{4t+s} \right) - B_T(u_1) - C_T(u_1) + o_p(1)
\]
\[
= \frac{1}{\sqrt{T}} \sum_{t=1}^{[4T u_1]} \left( e_{i_t} - E^o e_{i_t} \right) - B_T(u_1) - C_T(u_1) + o_p(1),
\]
where $B_T(u_1)$ and $C_T(u_1)$ have obvious definitions.

Now we show $B_T(u_1) \overset{p}{\to} 0$, and $C_T(u_1) \overset{p}{\to} 0$, uniformly in $u_1$. For $B_T(u_1)$, notice if $\pi_{j,s} \neq 0$, then $\{Y_{j,4t+s}\}$ is weakly stationary, so $\hat{\pi}_{j,s} - \pi_{j,s}$ is $O_p(T^{-1/2})$ (see Berk, 1974), and it follows straightforwardly that $B_T(u_1) \overset{p}{\to} 0$ uniformly in $u_1$. On the other hand, if $\pi_{j,s} = 0$, then by Theorem 3.1 $\hat{\pi}_{j,s} - \pi_{j,s} = O_p(T^{-1})$. Let
\[
Q_T(u_1) = \frac{1}{\sqrt{T}} \sum_{t=[k/4]+1}^T \left( Y_{j,4t+s} - \frac{1}{T} \sum_{t=[k/4]+1}^T Y_{j,4t+s} \right).
\]
It suffices to show that $\sup_{u_1 \leq u_1 \leq 1} Q_T(u_1) = o_p(T)$. By continuous mapping theorem, it suffices to prove $(4T)^{-1} Q_T(\cdot) \Rightarrow 0(\cdot)$, where $0(\cdot) \equiv 0$. It is straightforward to show the weak convergence of the finite dimensional distributions of $(4T)^{-1} Q_T(\cdot)$. Furthermore, $(4T)^{-1} Q_T(\cdot)$ is tight, since (see Billingsley, 1999, pp. 146-147) $\forall r_1 \leq r \leq r_2$,
\[
E\left[ \left( \frac{Q_T(r_2)}{T} - \frac{Q_T(r)}{T} \right)^2 \left( \frac{Q_T(r_2)}{T} - \frac{Q_T(r_1)}{T} \right)^2 \right]
\]
\[
= E\left[ \text{Var} \left[ \frac{Q_T(r_2)}{T} - \frac{Q_T(r)}{T} \right] \text{Var} \left[ \frac{Q_T(r)}{T} - \frac{Q_T(r_1)}{T} \right] \right] \to 0.
\]
Hence \((4T)^{-1}Q_T(\cdot) \Rightarrow 0(\cdot)\), and \(B_T(u_1) \overset{p}{\rightarrow} 0\) uniformly in \(u_1\) follows. For \(C_T(u_1)\), in light of the derivation of Theorem 3.1, it can be shown that \(\phi_{i,s} - \phi_{i,s} = O_p(T^{-1/2})\) holds not only under alternative hypotheses but also under the null. Hence, it follows that uniformly in \(u_1\), \(C_T(u_1) \overset{p}{\rightarrow} 0\). Therefore, recalling (7.11), we have
\[
\frac{1}{\sqrt{4T}} \sum_{t=1}^{[4Tu_1]} e_t^* - \frac{1}{\sqrt{4T}} \sum_{t=1}^{[4Tu_1]} (\epsilon_{i_t} - E^o\epsilon_{i_t}) \overset{p}{\rightarrow} 0.
\]
Further, it is straightforward to show \(E[B_T^2(1)] \overset{p}{\rightarrow} 0\), and \(E[C_T^2(1)] \overset{p}{\rightarrow} 0\). Using the same decomposition as in (7.11), \(\sigma_t^2 = \text{Std}^2(\epsilon_{i_t}) \overset{p}{\rightarrow} 0\). Hence we have proven (7.3).

Secondly we prove (7.9). Notice that the standard deviations in the definition of \(\{v_{T,t}^{(j)}\}\) are bounded in probability. For example,
\[
\text{Std}^2(\epsilon_{i_t}) = \text{Std}^2(\epsilon_{4l+m+s}) = \text{Std}(\epsilon_{4l+m+s}) + o_p(1) = \text{Std}(\epsilon_{4l+m+s}) + o_p(1),
\]
Further, given \(\{Y_{4l+m+s}\}\), for fixed \(j = 1, \ldots, 4\), \(v_{T,t}^{(j)}, v_{T,t}^{(2)}, \ldots, v_{T,t}^{(4)}\) are conditionally iid random variables. Finally, for all \(u \geq 0\),
\[
\text{Var}^o\left(\frac{1}{\sqrt{4T}} \sum_{m=1}^{[4Tu]} v_{T,m}^{(j)} \right) \overset{p}{\rightarrow} u,
\]
\[
\text{Cov}^o\left(\frac{1}{\sqrt{4T}} \sum_{m=1}^{[4Tu]} v_{T,m}^{(j)}, \frac{1}{\sqrt{4T}} \sum_{m=1}^{[4Tu]} v_{T,m}^{(i)} \right) \overset{p}{\rightarrow} 0 \quad \text{for } i \neq j.
\]
The convergence \(R_T^*\) of to \(W\) follows by generalizing (see Kreiss and Paparoditis, 2015) the real world result of Helland (1982, Theorem 3.3) to the bootstrap world.

7.4 Proof of Theorem 4.1

Proof. Without loss of generality, assume block size \(b\) is a multiple of four. Let \(i_m = I(m-1)b+1\). Then the \(m\)th block of \(\{V_t^*\}\) starts from \(V_{i_m}\). Let \(v_{i,m}^{(j)}\) be the rescaled aggregation of the \(m\)th block, defined by
\[
\begin{align*}
v_{i,m}^{(1)} &= \frac{1}{\sqrt{b}} \sum_{h=1}^{b} (V_{i_m+h-1} - E^o V_{i_m+h-1}) / \text{Std}^o\left(\frac{1}{\sqrt{b}} \sum_{h=1}^{b} V_{i_m+h-1}\right) \\
v_{i,m}^{(2)} &= \frac{1}{\sqrt{b}} \sum_{h=1}^{b} (-1)^h (V_{i_m+h-1} - E^o V_{i_m+h-1}) / \text{Std}^o\left(\frac{1}{\sqrt{b}} \sum_{h=1}^{b} (-1)^h V_{i_m+h-1}\right) \\
v_{i,m}^{(3)} &= \frac{1}{\sqrt{b}} \sum_{h=1}^{b} \sqrt{2} \sin\left(\frac{\pi h}{2}\right) (V_{i_m+h-1} - E^o V_{i_m+h-1}) / \text{Std}^o\left(\frac{1}{\sqrt{b}} \sum_{h=1}^{b} \sqrt{2} \sin\left(\frac{\pi h}{2}\right) V_{i_m+h-1}\right) \\
v_{i,m}^{(4)} &= \frac{1}{\sqrt{b}} \sum_{h=1}^{b} \sqrt{2} \cos\left(\frac{\pi h}{2}\right) (V_{i_m+h-1} - E^o V_{i_m+h-1}) / \text{Std}^o\left(\frac{1}{\sqrt{b}} \sum_{h=1}^{b} \sqrt{2} \cos\left(\frac{\pi h}{2}\right) V_{i_m+h-1}\right)
\end{align*}
\]
Let \(R_T^*\) be the partial sum of the block aggregations above. Formally,
\[
R_T^*(u_1, u_2, u_3, u_4) = \left(\frac{1}{\sqrt{l}} \sum_{m=1}^{[lu_1]} v_{i,m}^{(1)}, \frac{1}{\sqrt{l}} \sum_{m=1}^{[lu_2]} v_{i,m}^{(2)}, \frac{1}{\sqrt{l}} \sum_{m=1}^{[lu_3]} v_{i,m}^{(3)}, \frac{1}{\sqrt{l}} \sum_{m=1}^{[lu_4]} v_{i,m}^{(4)}\right).
\]

To prove theorem 4.1, it suffices to show

$$\|S_T - R_T^\star\|_{p} \to 0$$ uniformly in \( u_1, u_2, u_3 \) and \( u_4 \),

and \( R_T^\star \Rightarrow W \) in probability,

where \( \| \cdot \| \) denotes the \( L_2 \) norm. To show \( (7.13) \), without loss of generality we focus on the uniform convergence of the first coordinate, that is, uniformly in \( u_1 \),

$$\frac{1}{\sqrt{4T}} \sum_{t=1}^{4Tu_1} V_t^* / \sigma_t^* \to \frac{1}{\sqrt{4T}} \sum_{m=1}^{[4Tu_1]} \sum_{h=1}^{(m-1)b} \tilde{V}_{im+h-1}$$

(7.15)

where \( M(u_1) = \lceil [4Tu_1]/b \rceil \) denotes the total number of the blocks, and \( B_m = \min(b, [4Tu_1] - (m-1)b) \) is the length of the \( m \)th block. It suffices to only consider the first term in \( (7.15) \), since

$$\sup_{0 \leq u_1 \leq 1} \frac{1}{\sqrt{4T}} \sum_{h=B_M(u_1)+1}^{b} \tilde{V}_{iM(u_1)+h} = O_p(\frac{1}{\sqrt{t}} \ln l).$$

By the definition of \( \tilde{V}_t \),

$$\frac{1}{\sqrt{4T}} \sum_{m=1}^{M(u_1)} \sum_{h=1}^{b} \tilde{V}_{im+h-1}$$

$$= \frac{1}{\sqrt{4T}} \sum_{m=1}^{M(u_1)} \sum_{s=-3}^{0} \sum_{t=1}^{4T} (V_{im+4t+s-1} - \frac{1}{T} \sum_{t=1}^{T} V_{4t+s})$$

$$- \frac{4}{T} \sum_{j=1}^{4} \sum_{s=-3}^{0} (\hat{\pi}_{j,s} - \pi_{j,s}) \frac{1}{\sqrt{4T}} \sum_{m=1}^{M(u_1)} \sum_{t=1}^{b/4} (Y_{j,im+4t+s-1} - \frac{1}{T} \sum_{t=1}^{T} Y_{4t+s}).$$

Now we show the second term on the right hand side of the equation above converges uniformly in \( u_1 \) to 0 in probability. Here we only present the result for \( j=1, s=0 \). Notice if \( \hat{\pi}_{1,0} \neq 0 \) for some \( s \), then \( (\hat{\pi}_{1,0} - \pi_{1,0}) = o_p(1) \). Hence, the result follows the weakly stationarity of the vector sequence \( \{Y_t \} \). On the other hand, if \( \pi_{1,s} = 0 \) for all \( s \), then \( (\hat{\pi}_{1,0} - \pi_{1,0}) = O_p(T^{-1}) \). Hence, we only need to show that

$$Q_T(u) \overset{def}{=} \frac{1}{\sqrt{4T}} \sum_{m=1}^{M(u_1)} \sum_{t=1}^{b/4} (Y_{1,im+4t+s-1} - \frac{1}{T} \sum_{t=1}^{T} Y_{4t+s})$$

has \( Q_T(u) \Rightarrow 0(.) \), where \( 0(u_1) \equiv 0 \). The convergence of finite dimensional distribution of \( Q_T(.) \) can be proven by the line of Politis and Paparaditis (2003, p. 841). Furthermore, it can be shown that \( Q_T(.) \) is tight using \( (7.12) \). Hence \( Q_T(.) \Rightarrow 0(.) \). Therefore,

$$\left| \frac{1}{\sqrt{4T}} \sum_{m=1}^{M(u_1)} \sum_{h=1}^{b} \tilde{V}_{im+h-1} - \frac{1}{\sqrt{4T}} \sum_{m=1}^{M(u_1)} \sum_{s=-3}^{0} \sum_{t=1}^{4T} (V_{im+4t+s-1} - \frac{1}{T} \sum_{t=1}^{T} Y_{4t+s}) \right|_{p} \to 0$$

30
uniformly in \( u \). Since it is straightforward to show

\[
\left| \frac{1}{\sqrt{4T}} \sum_{t=1}^{4T} V_t - \frac{1}{\sqrt{T}} \sum_{t=1}^{T} V_t \right| - \mathcal{E} 0
\]

uniformly in \( u \), and

\[
\left| \frac{1}{\sqrt{4T}} \sum_{m=1}^{M(u_1)} \sum_{h=1}^{b} (V_m + h - E[V_m + h]) - \frac{1}{\sqrt{4T}} \sum_{m=1}^{M(u_1)} \sum_{h=1}^{b} (V_m + h) \right| - \mathcal{E} 0
\]

uniformly in \( u_1 \), we have obtained that

\[
\left| \frac{1}{\sqrt{4T}} \sum_{t=1}^{4T} V_t^* - \frac{1}{\sqrt{4T}} \sum_{t=1}^{4T} V_t \right| - \mathcal{E} 0
\]

uniformly in \( u_1 \). Now we show that \( \text{Var}^\circ \left[ \frac{1}{\sqrt{4T}} \sum_{t=1}^{4T} V_t^* \right] = \text{Var}^\circ \left[ \frac{1}{\sqrt{4T}} \sum_{h=1}^{b} V_{m+h-1} \right] - \mathcal{E} 0. \) Notice,

\[
\frac{1}{\sqrt{4T}} \sum_{t=1}^{4T} V_t^* = \frac{1}{\sqrt{4T}} \sum_{m=1}^{b} \sum_{h=1}^{h} \hat{V}_{m+h-1}
\]

\[
= \frac{1}{\sqrt{4T}} \sum_{m=1}^{b} \sum_{s=-3}^{3} (V_{m+s} - E[V_{m+s}]) + \frac{1}{T} \sum_{t=1}^{T} Y_{4t+s}
\]

\[
- \sum_{j=1}^{4} \sum_{t=1}^{T} (\pi_{j,s} - \pi_{j,s}) \frac{1}{\sqrt{4T}} \sum_{m=1}^{b} \sum_{s=-3}^{3} (Y_{j,m+s} - E[Y_{j,m+s}])
\]

\[
= \frac{1}{\sqrt{4T}} \sum_{m=1}^{b} (V_{m+h-1} - E[V_{m+h-1}]) + \frac{1}{T} \sum_{t=1}^{T} Y_{4t+s}
\]

\[
- \sum_{j=1}^{4} \sum_{t=1}^{T} (\pi_{j,s} - \pi_{j,s}) \frac{1}{\sqrt{4T}} \sum_{m=1}^{b} \sum_{s=-3}^{3} (Y_{j,m+s} - E[Y_{j,m+s}])
\]

\[
= A_T + B_T + \sum_{j=1}^{4} C_{T,j}
\]

where \( A_T, B_T \) and \( C_{T,j}, j = 1, ..., 4 \) have obvious definitions. It is straightforward to show \( E[V_2^2] - \mathcal{E} 0, E[C_{T,j}^2] - \mathcal{E} 0 \) for \( j = 1, ..., 4 \), and \( \text{Var}^\circ [A_T] = \text{Var}^\circ [\frac{1}{\sqrt{b}} \sum_{h=1}^{b} V_{m+h-1}] \). Hence, we have

\[
\text{Var}^\circ \left[ \frac{1}{\sqrt{4T}} \sum_{t=1}^{4T} V_t^* \right] = \text{Var}^\circ \left[ \frac{1}{\sqrt{b}} \sum_{h=1}^{b} V_{m+h-1} \right] - \mathcal{E} 0.
\]

By (7.16) and (7.17), we have shown

\[
\left| \frac{1}{\sqrt{4T}} \sum_{t=1}^{4T} V_t^*/\sigma^*_1 - \sum_{m=1}^{M(u_1)} V_{t,m}^{(1)} \right| - \mathcal{E} 0
\]
uniformly in \( u_1 \), and thus \( \| S_T^* - R_T^* \| \overset{p}{\to} 0 \) uniformly in \( u_1, u_2, u_3 \) and \( u_4 \).

Secondly we prove (7.14). Given assumption B.1, it is sufficient to show that the following three properties hold:

\[
\begin{align*}
\left\{ \sum_{m=1}^{[lu]} E^\circ[v^{(i)}_{l,m}] \right\} & \overset{p}{\to} u, \forall u \geq 0, \text{ and } \forall i = 1, \ldots, 4, \\
\left\{ \sum_{m=1}^{[lu]} E^\circ[v^{(i)}_{l,m}] 1(|v_{l,m}| > \epsilon) \right\} & \overset{p}{\to} 0, \forall u \geq 0, \forall i = 1, \ldots, 4, \\
\left\{ \sum_{m=1}^{[lu]} E^\circ[v^{(i)}_{l,m} 1_{i,j}] \right\} & \overset{p}{\to} 0, \forall u \geq 0, \forall i, j \in \{1, 2, 3, 4\}, i \neq j.
\end{align*}
\] (7.18) (7.19) (7.20)

Helland (1982) shows that if \( \{v_{l,m}\} \) is a martingale difference array and the above three properties hold in real world, then \( \sum_{m=1}^{[lu]} v_{l,m} \Rightarrow W(u) \). By Beveridge-Neilson Decomposition (Hamilton, 1994, Proposition 17.2, p. 504), Helland’s result can be generalized to the case when \( \{v_{l,m}\} \) is a convolution of a constant array and a martingale difference array. Further, Helland’s result can be generalized to the bootstrap world (see Kreiss and Paparoditis, 2015). Hence the sufficiency of the three properties above.

To verify (7.18) and (7.19), notice that

\[
\sum_{m=1}^{[lt]} E^\circ[v^{(i)}_{l,m}] = [H]/l \to t,
\]

and, by the dominated convergence theorem,

\[
\sum_{m=1}^{[lt]} E^\circ[v^{(i)}_{l,m} 1(|v_{l,m}| > \epsilon)] \overset{p}{\to} 0.
\]

Hence, it remains to verify the (7.20), which indicates asymptotic independence between coordinates of \( R_T^* \). Note that the third property need to be proved for all \( i, j \in \{1, 2, 3, 4\}, i \neq j \). Here we cite as an example the case \( i = 1 \) and \( j = 3 \). The rest of cases can be shown by similar calculations. Notice,

\[
\sum_{m=1}^{[lt]} E^\circ[v^{(1)}_{l,m} v^{(3)}_{l,m}] = E^\circ[\frac{1}{\sqrt{b}} \sum_{h=1}^{b} V_{i+h-1} 1 - \frac{1}{\sqrt{b}} \sum_{r=1}^{b} \sqrt{2} \sin(\pi r/2) V_{i+r-1}] \overbrace{\text{Std}^\circ[\frac{1}{\sqrt{b}} \sum_{h=1}^{b} V_{i+h-1}]\text{Std}^\circ[\frac{1}{\sqrt{b}} \sum_{r=1}^{b} \sqrt{2} \sin(\pi r/2) V_{i+r-1}]} - E^\circ[\frac{1}{\sqrt{b}} \sum_{h=1}^{b} V_{i+h-1}]E^\circ[\frac{1}{\sqrt{b}} \sum_{r=1}^{b} \sqrt{2} \sin(\pi r/2) V_{i+r-1}] \overbrace{\text{Std}^\circ[\frac{1}{\sqrt{b}} \sum_{h=1}^{b} V_{i+h-1}]\text{Std}^\circ[\frac{1}{\sqrt{b}} \sum_{r=1}^{b} \sqrt{2} \sin(\pi r/2) V_{i+r-1}]].
\]

Since

\[
E^\circ[\frac{1}{\sqrt{b}} \sum_{h=1}^{b} V_{i+h-1}] \overset{p}{\to} 0, \quad E^\circ[\frac{1}{\sqrt{b}} \sum_{r=1}^{b} \sqrt{2} \sin(\pi r/2) V_{i+r-1}] \overset{p}{\to} 0,
\]

and both \( \text{Std}^\circ[\frac{1}{\sqrt{b}} \sum_{h=1}^{b} V_{i+h-1}] \) and \( \text{Std}^\circ[\frac{1}{\sqrt{b}} \sum_{r=1}^{b} \sqrt{2} \sin(\pi r/2) V_{i+r-1}] \) converge in probability to constants (Dudek et al., 2014), we only need to show that

\[
E^\circ[\frac{1}{\sqrt{b}} \sum_{h=1}^{b} V_{i+h-1} 1 - \frac{1}{\sqrt{b}} \sum_{r=1}^{b} \sqrt{2} \sin(\pi r/2) V_{i+r-1}] \overset{p}{\to} 0.
\]
Notice,
\[ E^c[\frac{1}{\sqrt{b}} \sum_{h=1}^{b} V_{i+h-1} \frac{1}{\sqrt{b}} \sum_{r=1}^{b} \sqrt{2} \sin(\pi r/2) V_{i+r-1}] = \frac{\sqrt{2}}{b(T - b/4)} \sum_{h=1}^{b} \sum_{j=1}^{T-b/4} V_{4j-3} V_{4j-4h-6}, \]
where
\[ A = \frac{\sqrt{2}}{b(T - b/4)} \sum_{h=1}^{b/4} \sum_{j=1}^{T-b/4} V_{4j-3}, \]
\[ B = \frac{\sqrt{2}}{b(T - b/4)} \sum_{h=1}^{b/4} \sum_{j=1}^{T-b/4} V_{4j-3} V_{4j-4h-4}. \]

The proof under Assumption 1.B is complete after showing
\[ A \xrightarrow{P} 0, B \xrightarrow{P} 0. \quad (7.21) \]

by Lemma 6 below. Now consider Assumption 2.B. Let \( v_{i,m} = (v_{i,m}^{(1)}, v_{i,m}^{(2)}, v_{i,m}^{(3)}, v_{i,m}^{(4)})' \). Let \( (\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4})' \) be the eigenvalues of \( \text{Var} \sum_{i} v_{i,m} \). It is sufficient (Wooldridge and White, 1988, Corollary 4.2) to show that the following two properties hold:
\[ E^c(\frac{1}{\sqrt{l}} \sum_{i=1}^{[l]} v_{i,m}^{(1)})(\frac{1}{\sqrt{l}} \sum_{j=1}^{[l]} v_{j,m}^{(j)}) \xrightarrow{P} t \{ i = j \}, \forall l \geq 0, \forall i, j, \quad (7.22) \]
\[ (\lambda_{1}^{-1}, \lambda_{2}^{-1}, \lambda_{3}^{-1}, \lambda_{4}^{-1}) = O(l^{-1}). \quad (7.23) \]

Notice, to show (7.22), it suffices to show (7.21), which is ensured by Lemma 4 and Lemma 6. Equation (7.23) follows from the continuity of the eigenvalue function. Hence we have completed the proof when block size \( b \) is a multiple of four.

When \( b \) is not a multiple of four, it is straightforward to show (7.13). For (7.14), let
\[ R_{T,s}^* = \frac{1}{\sqrt{l/4}} \sum_{k=1}^{[l/4]} v_{i,m}^{(1)}(k), \frac{1}{\sqrt{l/4}} \sum_{k=1}^{[l/4]} v_{i,m}^{(2)}(k), \frac{1}{\sqrt{l/4}} \sum_{k=1}^{[l/4]} v_{i,m}^{(3)}(k), \frac{1}{\sqrt{l/4}} \sum_{k=1}^{[l/4]} v_{i,m}^{(4)}(k)' . \]

Since \( \{ R_{T,s}^*, s = -3, ..., 0 \} \) are mutually independent with respect to \( P^c \), and \( R_{T,s}^* \Rightarrow W \) in probability for all \( s = -3, ..., 0 \), we have \( R_{T,s}^* = \frac{1}{n} \sum_{s=-3}^{0} R_{T,s}^* + o_p(1) \Rightarrow W \) in probability. \( \square \)

Lemma 6. Suppose (i) \( \{ z_t \}_{t=1}^n \) is a fourth-order stationary time series with finite \( 4+\delta \) moment for some \( \delta > 0 \). (ii) \( \exists K > 0, \forall i, j, k, s, \text{ and } l, \sum_{h=-\infty}^{\infty} |\text{Cov}(z_i z_j, z_k z_l-h)| < K. \) Suppose \( b \to \infty \) and \( n \to \infty \). Then,
\[ \text{Var} \left[ \frac{1}{bn} \sum_{t=1}^{b} \sum_{j=1}^{n} z_{i} z_{i+j} \right] \to 0. \]
Proof.

\[
\text{Var}\left[ \frac{1}{bn} \sum_{t=1}^{n} \sum_{j=1}^{b} z_{t}z_{t-j} \right]
= \frac{1}{b^2n^2} \sum_{t_1=1}^{n} \sum_{t_2=1}^{n} \sum_{j_1=1}^{b} \sum_{j_2=1}^{b} \text{cov}[z_{0}z_{-j_{1}},z_{t_2-t_{1}}z_{t_1-t_{1}-j_{1}}]
= \frac{1}{b^2n^2} \sum_{h=1-n}^{n-1} (n - |h|) \sum_{j_1=1}^{b} \sum_{j_2=1}^{b} \text{cov}[z_{0}z_{-j_{1}},z_{h}z_{h-j_{1}}]
< \frac{K}{n} \to 0.
\]

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