OPTIMAL $H_2$ MOMENT MATCHING-BASED MODEL REDUCTION FOR LINEAR SYSTEMS
BY (NON)CONVEX OPTIMIZATION

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Abstract. In this paper we compute families of reduced order models that match a prescribed set of $\nu$ moments of a highly dimensional linear time-invariant system. First, we fully parametrize the models in the interpolation points and in the free parameters, and then we fix the set of interpolation points and parametrize the models only in the free parameters. Based on these two parametrizations and using as objective function the $H_2$-norm of the error approximation we derive non-convex optimization problems, i.e., we search for the optimal free parameters and even the interpolation points to determine the approximation model yielding the minimal $H_2$-norm error. Further, we provide the necessary first-order optimality conditions for these optimization problems given explicitly in terms of the controllability and the observability Gramians of a minimal realization of the error system. Using the optimality conditions, we propose gradient type methods for solving the corresponding optimization problems, with mathematical guarantees on their convergence. We also derive convex SDP relaxations for these problems and analyze when the convex relaxations are exact. We illustrate numerically the efficiency of our results on several test examples.

Key words. Model order reduction, moment matching, optimal $H_2$-norm, (non)convex optimization, gradient method.

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1. Introduction. Today we are living in a complex and interconnected world. Mathematical tools yield complex and highly dimensional dynamical models, e.g., from partial-differential equations or networks of interconnected subsystems. Hence, for purposes such as simulation and control design, scientists and engineers need tweaking of such models rendering them simpler and useful. To this end, model reduction is called for. The main idea of model order reduction is to find a low-order mathematical model that approximates the given highly dimensional dynamical system. The approximation is accurate if the approximation error is small and if the most important physical properties/structure, such as the stability of the given system are preserved. A large number of methods have been developed for linear systems, split in two major categories. The first category consists of the so-called SVD-based methods, such as balanced truncation and Hankel norm approximation, described, e.g., in [31]. The second category contains moment matching-based methods as, e.g., in [4, 19, 36]. For a survey on model reduction of linear systems see the monograph [2].

State-of-the-art. Balancing is a tool using an energy measure of the states of the system to determine whether that state can be neglected in the dynamics or not, introduced by Moore in [31] for stable linear systems. It measures the controllability and the observability of a given state through the Hankel singular values, input-output invariants of the system. From a model reduction point of view, one may choose to truncate the states which are badly controllable and badly observable, corresponding to the smaller Hankel singular values, yielding a lower-order dynamical system. The balanced truncation model preserves the stability of the given system. Another important property of this approach is the analytic upper error bound found of the infinity norm of the error-system, see, e.g., the work of Glover [16]. However, the balanced truncation-based approximation does not minimize any norm [2] associated to the error system. A refinement of the balanced truncation leads to an approximation method, with respect to the 2-induced norm of the Hankel operator associated to the system, known as optimal approximation in the Hankel norm.

The second category of model reduction techniques is based on moment matching. Model reduction moment matching techniques represent an efficient tool for reducing the dimension of the system, see, e.g., [9, 3] and [2] for an overview for linear systems. In such techniques the reduced order model is obtained by
constructing a lower degree rational function that approximates the given higher degree transfer function. The low degree rational function matches the original transfer function and its derivatives at various points in the complex plane. The notion of moments has been given in [2], through the series expansion of the transfer function of the linear system, see also [17, 36, 12]. Hence, one can write equivalent definitions of moments. A first equivalent definition is in terms of \( \nu \) right Krylov projections and a second equivalent definition is the dual of the previous one, in terms of \( \nu \) left Krylov projections. The reduced order models obtained through Krylov projections match a prescribed number of moments, say \( \nu \). Alternatively, in [15], the Krylov projections are obtained solving Sylvester equations. To improve the accuracy of the reduced order models that achieve moment matching, two-sided projections have been employed, see, e.g., [2, 9, 18]. The simultaneous application of the left and the right projections yields a reduced order model that matches \( 2\nu \) moments at two sets of \( \nu \) interpolation points, respectively. Recently, in [21, 1], using two-sided projection-based interpolatory methods, the model that minimizes the \( H_2 \)-norm of the approximation error is computed. Here, a unifying framework for the optimal \( H_2 \) approximant has been obtained using best approximation properties in the underlying Hilbert space. A set of local optimality conditions taking the form of a structured orthogonality condition have been developed. Based on the interpolation framework, [21] has provided an iteratively corrected rational Krylov algorithm (IRKA) for \( H_2 \) model reduction. The resulting model interpolates the transfer function of the given system and its derivatives at the mirror images of the poles of the approximant, see also [30]. Further, in [29], a new framework has been proposed for the solution to the realization problem. More precisely, the moment matching problem has been recast in terms of the Loewner matrix and the solutions of Sylvester equations, with matrices constructed from tangential interpolation data. The result is a reduced order model that achieves moment matching and is minimal, while the corresponding Loewner/Krylov matrix-based algorithm is highly efficient numerically, involving only matrix-vector multiplications.

**Related work.** Most recently, in [5, 6, 24] new definitions of moments in a time-domain framework have been given. Algebraically, the moments of a linear system are defined in terms of the solution of a Sylvester equation. From a systemic viewpoint, moments are in (a one-to-one) relation with the well-defined steady-state response of the system driven by a signal generator (a novel interpretation of the results in [15]). An approximation achieves moment matching if the steady-state of its response to a signal generator matches the steady-state response of the original system at the same signal generator. Imposing such moment matching conditions yields a family of reduced order parametrized approximations. The degrees of freedom are used such that properties, like stability, are preserved. Based on a dual Sylvester equation, a new definition of moments dual to the previous one results, related to the well-defined steady-state response of a generalized signal generator driven by the system. The resulting (family of) reduced order models that achieve moment matching are also parametric. Employing both time-domain notions of moments, two-sided moment matching can be also achieved [22]. The resulting model that matches \( 2\nu \) moments is computed by a specific selection of the free parameters. Moreover, the reduced order model matching the moments of both the transfer function of the system and its derivative is determined by another specific choice of the free parameters. The results therein follow the necessary first-order optimality conditions of optimal \( H_2 \) model reduction. Experimentally, these models exhibit low \( H_2/H_\infty \)-norm of the error following the arguments in [21] that a reduced order locally minimizing the \( H_2 \)-norm of the approximation error is achieving moment matching of both the transfer function of the given system and its derivative.

**Motivation.** However, in the time-domain moment matching framework finding a model in the family of \( \nu \) order models that matches a set of \( \nu \) prescribed moments and approximates a system with the minimal \( H_2 \)-norm of the error is an open question. Even the relaxed version of this general optimal \( H_2 \) model reduction problem, where we seek only the free parameters that yield a model from a family of \( \nu \) order models that match \( \nu \) prescribed moments at a set of \( \nu \) fixed interpolation points and minimizing the \( H_2 \)-norm of the approximation error has not been addressed yet. Some initial progress has been made in [22], see related work paragraph, by matching the derivatives of the given system. However, the reduced model does not yield the optimal \( H_2 \) approximation since there is no degree of freedom left in the parameters to ensure a minimal norm of the error. Hence, in the time-domain moment matching framework, the problem of finding the free parameters and eventually also the interpolation points yielding a \( \nu \) order model that matches \( \nu \)
moments and minimizes the $H_2$-norm of the approximation error has not been addressed or fully understood. Furthermore, the algorithms developed so far for the problem of optimal $H_2$-norm model reduction do not guarantee preservation of stability or other physical properties. Thus, the problem of finding the stable reduced order model achieving the minimal possible approximation for $H_2$-norm motivates our work here.

Contributions. In this paper we write families of reduced order models that match a prescribed set of $\nu$ moments of a highly dimensional linear time-invariant system. First, we fully parametrize the models in the interpolation points and in the free parameters, and then we fix the set of interpolation points and parametrize the models only in the free parameters. Based on the parametrizations and using as objective function the $H_2$-norm of the error approximation we derive non-convex optimization problems, i.e., we search for the optimal free parameters and eventually the interpolation points to determine the approximation model yielding the minimal $H_2$-norm error. For all the optimization problems we compute the necessary first-order optimality conditions given explicitly in terms of the controllability and the observability Gramians of a minimal realization of the error system. Furthermore, using the optimality conditions, we propose several gradient-type methods for solving the corresponding optimization problems, with mathematical guarantees on their convergence. We also provide convex SDP relaxations for these non-convex optimization problems and analyze when these relaxations are exact. Our contributions are summarized as follows:

(i) We first formulate a general model reduction problem with reduced models from the family of models matching $\nu$ moments parameterized in the interpolation points and in the free parameters. A corresponding optimization formulation is derived, where the objective function is the $H_2$-norm of the approximation error, written explicitly in terms of the controllability and observability Gramians of a minimal realization of the error system. We also write the necessary first-order optimality conditions (KKT system) of this optimization problem, in terms of these Gramians.

(ii) For this general model reduction problem we propose several numerical optimization algorithms. The first method is using a gradient update for solving the KKT system, leading to a simple iteration involving only matrix multiplications. However, with this update the stability of the approximation is achieved asymptotically. The second solution is based on a partial minimization approach. We show that for the evaluation of the gradient of the objective function we need to solve two Lyapunov equations associated to the Gramians, but the gradient is Lipschitz continuous. Therefore, a gradient-based algorithm is developed, ensuring convergence due to the smoothness of the objective function. Although the gradient evaluation is expensive, each iteration provides a stable reduced order model, whereas the first method yields a stable reduced order model only asymptotically. Finally, we propose a convex SDP relaxation of the original optimization problem and derive sufficient conditions when this relaxation is exact. Note that the interpolation points obtained are the spectrum of a squared matrix computed by each of these algorithms.

(iii) We also consider a relaxed version of the general model reduction problem, searching only for the free parameters that yield the optimal reduced order model from the family of models matching $\nu$ moments at fixed interpolation points. Optimization formulations for this particular problem are also proposed and subsequently the previous numerical optimization algorithms can be also applied to solve this simpler problem, obtaining similar convergence guarantees. Finally, we illustrate the efficiency of our results numerically using several test problems.

2. Preliminaries. In this section we briefly review the main results for Sylvester equation-based time-domain moment matching model reduction framework for linear time-invariant systems. In Section 3, we formulate two optimal $H_2$-norm model reduction problems, recast them as optimization problems with a Gramian-based cost function, and derive the corresponding first-order optimality conditions. We also analyze several numerical optimization methods for solving these problems. Finally, in Section 4 we illustrate the efficiency of our theory on several test examples.
2.1. Linear systems. Consider a linear time invariant (LTI), minimal, square, dynamical system:

\begin{equation}
\Sigma : \quad \dot{x} = Ax + Bu, \quad y = Cx,
\end{equation}

with the state $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}^m$ and the output $y \in \mathbb{R}^p$. The transfer function of (2.1) is:

\begin{equation}
K(s) = C(sI - A)^{-1}B, \quad K : \mathbb{C} \to \mathbb{C}^{p \times m}.
\end{equation}

Throughout the rest of the paper we assume that the system (2.1) is stable, i.e., $\sigma(A) \subset \mathbb{C}^-.

2.2. Sylvester equation-based moment matching. Assume that (2.1) is a minimal realization of the transfer function $K(s)$. The moments of (2.2) are defined as follows:

**Definition 2.1.** [2, 6] The $k$-moment of (2.1) at $s_1$, along direction $\ell \in \mathbb{C}^m$ is:

$$\eta_k(s_1) = \frac{(-1)^k}{k!} \frac{d^k K(s)}{ds^k} \ell, \quad k \geq 0.$$ 

The $k$-moment of system (2.1) at $s_1$, along direction $r \in \mathbb{C}^{1 \times p}$ is:

$$\eta_k(s_1) = r \frac{(-1)^k}{k!} \frac{d^k K(s)}{ds^k} r^T \in \mathbb{C}^{1 \times p}, \quad k \geq 0.$$ 

Consider the linear system (2.1) and let the matrices $S \in \mathbb{R}^{\nu \times \nu}$, $L = [\ell_1 \ell_2 \ldots \ell_\nu] \in \mathbb{C}^{m \times \nu}$, where $\ell_1 \in \mathbb{C}^m$, and $Q \in \mathbb{R}^{\nu \times \nu}$, $R = [r_1^* \ldots r_\nu^*] \in \mathbb{C}^{\nu \times p}$, where $r_i \in \mathbb{C}^{1 \times p}$, be such that the pair $(L, S)$ is observable and $(Q, R)$ is controllable, respectively. Let $\Pi \in \mathbb{R}^{n \times \nu}$ and $\Upsilon \in \mathbb{R}^{p \times \nu}$ be the solutions of Sylvester equations:

\begin{align}
(2.3a) \quad & A\Pi + BL = PS, \\
(2.3b) \quad & \Upsilon A + RC = Q\Upsilon,
\end{align}

respectively. Furthermore, since the system is minimal, assuming that $\sigma(A) \cap \sigma(S) = \emptyset$, then $\Pi$ is the unique solution of the equation (2.3a) and rank $\Pi = \nu$. Assuming that $\sigma(A) \cap \sigma(Q) = \emptyset$, then $\Upsilon$ is the unique solution of the equation (2.3b) and rank $\Upsilon = \nu$, see, e.g., [8]. Then, the moments of system (2.1) are characterized as follows:

**Proposition 2.2.** [6, 7]

1. The moments of system (2.1) at the interpolation points $\{s_1, s_2, \ldots, s_\nu\} = \sigma(S)$ are determined by the elements of the matrix $\Pi$.

2. The moments of system (2.1) at the interpolation points $\{s_1, s_2, \ldots, s_\nu\} = \sigma(Q)$ are determined by the elements of the matrix $\Upsilon B$.

The next result gives necessary and sufficient conditions for a low order system to achieve moment matching:

**Proposition 2.3.** [6, 7] Consider the reduced order system:

\begin{equation}
\dot{\xi} = F\xi + Gu, \quad \psi = H\xi,
\end{equation}

with $F \in \mathbb{R}^{\nu \times \nu}$, $G \in \mathbb{R}^{\nu \times m}$, $H \in \mathbb{R}^{p \times \nu}$ and the corresponding transfer function:

\begin{equation}
\tilde{K}(s) = H(sI - F)^{-1}G.
\end{equation}

Let $S \in \mathbb{C}^{\nu \times \nu}$ and $L \in \mathbb{C}^{m \times \nu}$ be such that the pair $(L, S)$ is observable, and let $Q \in \mathbb{C}^{\nu \times \nu}$ and $R \in \mathbb{C}^{\nu \times p}$ be such that the pair $(Q, R)$ is controllable. Moreover, assume that $\sigma(S) \cap \sigma(A) = \emptyset$ and $\sigma(Q) \cap \sigma(A) = \emptyset$.

Then, the following statements hold:

1. Assume that $\sigma(F) \cap \sigma(S) = \emptyset$. Then, the system (2.4) matches the moments of the original system (2.1) at $\sigma(S)$ if and only if:

\begin{equation}
HP = CPI,
\end{equation}

where the invertible matrix $P \in \mathbb{C}^{\nu \times \nu}$ is the unique solution of the Sylvester equation

$$FP + GL = PS.$$
2. Assume that $\sigma(F) \cap \sigma(Q) = \emptyset$. Then, the system (2.4) matches the moments of the original system (2.1) at $\sigma(Q)$ if and only if:

$$(2.7) \quad \Upsilon B = PG,$$

where the invertible matrix $P \in \mathbb{C}^{\nu \times \nu}$ is the unique solution of the Sylvester equation

$$QP = PF + RH.$$ 

We are now ready to present families of $\nu$ order models that match $\nu$ moments of the given system (2.1):

(I) The approximation, parameterized in the interpolations points given by the spectrum of $S$ and the free parameters given by $G$ and $L$

$$(2.8) \quad \hat{\Sigma}_{(S,G,L)} : \quad \dot{x} = (S - GL)x + Gu, \quad \psi = CPIx,$$

with the transfer function

$$(2.9) \quad \hat{K}(s) = CPI(sI - S + GL)^{-1}G,$$

describes a family of $\nu$ order models that achieve moment matching at $\sigma(S)$ satisfying the following properties and constraints:

(a) $\hat{\Sigma}_{(S,G,L)}$ is parameterized in the triplet $(S, G, L)$, with $S \in \mathbb{C}^{\nu \times \nu}$, $G \in \mathbb{C}^{\nu \times m}$ and $L \in \mathbb{C}^{m \times \nu}$ such that the pair $(L, S)$ is observable

(b) $\sigma(S) \cap \sigma(A) = \emptyset$

(c) $\sigma(S - GL) \cap \sigma(S) = \emptyset$.

If the pair of observable matrices $(L, S)$ is a priori fixed, and consequently $\nu$ interpolation points in $\sigma(S)$ are fixed, then the system $\hat{\Sigma}_{G}$ from (2.8) defines a family of $\nu$ order models that match $\nu$ moments along directions $\xi$, of the original system (2.1) at $\sigma(S)$ and satisfies the following properties and constraints:

(a) $\hat{\Sigma}_{G}$ is parametrized in free parameters $G$

(b) $\sigma(S - GL) \cap \sigma(S) = \emptyset$.

(II) Similarly, the approximation, parameterized in the interpolations points given by the spectrum of $Q$ and the free parameters given by $H$ and $R$

$$(2.10) \quad \hat{\Sigma}_{(Q,H,R)} : \quad \dot{x} = (Q - RH)x + YBu, \quad \psi = Hx,$$

with the transfer function

$$(2.11) \quad \hat{K}(s) = H(sI - S + GL)\Upsilon B,$$

describes a family of $\nu$ order models that achieve moment matching at $\sigma(Q)$ satisfying the following properties and constraints:

(a) $\hat{\Sigma}_{(Q,R,H)}$ is parameterized in the triplet $(Q, R, H)$, with $Q \in \mathbb{C}^{\nu \times \nu}$, $HT \in \mathbb{C}^{p \times \nu}$ and $R \in \mathbb{C}^{\nu \times p}$ such that the pair $(Q, R)$ is controllable

(b) $\sigma(Q) \cap \sigma(A) = \emptyset$

(c) $\sigma(Q - RH) \cap \sigma(Q) = \emptyset$.

If the pair of controllable matrices $(Q, R)$ is a priori fixed, then $\hat{\Sigma}_{H}$ yielded by (2.10) defines a family of $\nu$ order models that match $\nu$ moments along prescribed directions $\rho$, of (2.1) at $\sigma(Q)$ fixed, satisfying the following properties and constraints:

(a) $\hat{\Sigma}_{H}$ is parametrized in $H$

(b) $\sigma(Q - RH) \cap \sigma(Q) = \emptyset$.

2.3. Computation of moments. In practice, the moments $CPI$ and $\Upsilon B$ are not computed solving the Sylvester equation (2.3), but using Krylov projections. In this section we recall two different notions of moments based on Krylov projections. This definition allows for development of efficient numerical
algorithms for the computation of reduced order models, i.e., the Lanczos procedures, see, e.g., [9, 10, 13, 19, 25, 20] and references therein. These algorithms achieve moment matching through iterative procedures. As presented in [24], given a set of points in the complex plane, not among the poles of the given system, Krylov projections may be constructed. In particular, let \( s_1, s_2, \ldots, s_\nu, s_{\nu+1}, s_{\nu+2}, \ldots, s_{2\nu} \in \mathbb{C} \setminus \sigma(A) \), \( s_i \neq s_j, \ i \neq j \) and let \( V \in \mathbb{C}^{n \times \nu} \) and \( W \in \mathbb{C}^{n \times \nu} \) be, respectively:

\[
(2.12a) \quad V = \begin{bmatrix}
(s_1 I - A)^{-1} B & (s_2 I - A)^{-1} B & \cdots & (s_\nu I - A)^{-1} B 
\end{bmatrix},
\]

\[
(2.12b) \quad W = \begin{bmatrix}
(s_{\nu+1} I - A^*)^{-1} C^* & (s_{\nu+2} I - A^*)^{-1} C^* & \cdots & (s_{2\nu} I - A^*)^{-1} C^*
\end{bmatrix}.
\]

The next result follows from Definition 2.1, writing the moments at each point \( s_i \) in matrix form:

**Proposition 2.4.** [24] The moments of system (2.1) at \( s_1, s_2, \ldots, s_\nu \notin \sigma(A) \) are the elements of the matrix \( CV \). We call \( V \) the right Krylov projection matrix. Furthermore, the moments of system (2.1) at \( s_{\nu+1}, s_{\nu+2}, \ldots, s_{2\nu} \notin \sigma(A) \) are the elements of the matrix \( WB \). We call \( W \) the left Krylov projection.

In the sequel, we briefly overview the equivalent relation between the moments described in Proposition 2.2 and the moments described by Proposition 2.4. In [24, 7] relations between the projections \( V \) and \( W \) and the solutions of the Sylvester equations \( \Pi = CVT \) and \( \Upsilon = TW \) were established:

**Lemma 2.5.** [7]

1. Let \( \Pi = \Pi(\nu) \) be the solution of the Sylvester equation (2.3a) and let the projector \( V = \Pi(\nu) \) be as in (2.12a). Then, there exists a square, non-singular matrix \( T \in \mathbb{C}^{n \times \nu} \) such that \( \Pi = VT \). For \( T = I_\nu \), \( V \) from (2.12a) is the unique solution of equation (2.3a) for \( S = \text{diag}(s_1, s_2, \ldots, s_\nu) \) and \( L = [\ell_1 \ell_2 \cdots \ell_\nu] \in \mathbb{R}^{\nu \times \nu} \).

2. Let \( \Upsilon = \Upsilon(\nu) \) be the solution of the Sylvester equation (2.3b) and let the projector \( W = \Upsilon(\nu) \) be as in (2.12b). Then, there exists a square, non-singular matrix \( T \in \mathbb{C}^{n \times \nu} \) such that \( \Upsilon = TW \). For \( T = I_\nu \), \( W \) from (2.12b) is the unique solution of equation (2.3b) for \( Q = \text{diag}(s_{\nu+1}, s_{\nu+2}, \ldots, s_{2\nu}) \) and \( R = [r_1^* \cdots r_\nu^*]^* \in \mathbb{R}^{\nu \times \nu} \).

Hence, the moments of system (2.1) at \( \sigma(S) \) and/or \( \sigma(Q) \) as in Proposition 2.2, are computed as follows:

**Corollary 1.** Consider system (2.1). Let \( (L, S) \) be a pair of observable matrices of appropriate dimensions and let \( (Q, R) \) be another pair of controllable matrices of appropriate dimension, respectively, such that \( \sigma(S) \cap \sigma(Q) \). Then:

i) the moments of system (2.1) at \( \sigma(S) \) are given by \( C \Pi = CVT \), where \( \Pi = \Pi(\nu) \) is the unique solution of the Sylvester equation (2.3a) and \( V \) is given by (2.12a).

ii) the moments of system (2.1) at \( \sigma(Q) \) are given by \( \Upsilon B = TWB \), where \( \Upsilon = \Upsilon(\nu) \) is the unique solution of the Sylvester equation (2.3b) and \( W \) is given by (2.12b).

The results of Proposition 2.4, Lemma 2.5, and Corollary 1 also hold for higher order moments at a set of interpolation points \( s_1, \ldots, s_l \in \mathbb{C} \) which are not poles of the given transfer function \( K \). Let \( s_i, i = 0, \ldots, l \) and \( l \geq 0 \). To this end, take \( j_i \geq 0 \) such that:

\[
(2.13) \quad \sum_{i=0}^{l} (j_i - 1) = \nu.
\]

For each \( i \), let \( \eta_0(s_i), \ldots, \eta_{j_i}(s_i) \) denote the first \( j_i + 1 \) moments of the system defined by (2.1) at the given points \( s_i \). Then, these moments are characterized by the matrix \( CV \), with:

\[
V = \begin{bmatrix}
V_0 & \ldots & V_l
\end{bmatrix} \in \mathbb{C}^{n \times \nu},
\]

\[
V_i = \begin{bmatrix}
(s_1 I - A)^{-1} B & (s_2 I - A)^{-1} B & \cdots & (s_\nu I - A)^{-1} B
\end{bmatrix} \in \mathbb{C}^{n \times j_i}.
\]

Furthermore, \( V \) is the solution of the Sylvester equation (2.3a), for \( S = \text{diag}(\Sigma_0, \ldots, \Sigma_l) \), with \( \Sigma_i \in \mathbb{C}^{j_i \times j_i} \) a Jordan block matrix of the eigenvalue \( s_i \) with multiplicity \( j_i \) and \( L = [\ell_0 \cdots \ell_l] \in \mathbb{R}^{\nu \times \nu} \), with \( \ell_i = [1 \ 0 \ \cdots \ 0] \in \mathbb{R}^{1 \times j_i} \). The results follow directly from the arguments used in [6, 24]. Note that the moment matching conditions (2.6) and (2.7) are equivalent [14], up to a constant coordinate transformation.
to the right tangential interpolation conditions:

\[ K(s_i) \ell_j = \hat{K}(s_i) \ell_j, \quad K'(s_i) \ell_j = \hat{K}'(s_i) \ell_j, \]
\[ \cdots \]
\[ \frac{d^{i_j}}{ds^{i_j+1}} K(s) \ell_j = \frac{d^{i_j}}{ds^{i_j+1}} \hat{K}(s) \ell_j, \quad i = 0 : l \]

and the left tangential interpolation conditions:

\[ r_j, K(s_i) = r_j, \hat{K}(s_i), \quad r_j, K'(s_i) = r_j, \hat{K}'(s_i), \]
\[ \cdots \]
\[ r_j, \frac{d^{i_j}}{ds^{i_j+1}} K(s) = r_j, \frac{d^{i_j}}{ds^{i_j+1}} \hat{K}(s), \quad i = 0 : l. \]

Note that these reduced order models are parameterized in \( L \) and \( R \), respectively. Their choice is important for computing subfamilies of models that preserve specific properties, for establishing appropriate directions for interpolation and for finding the most accurate approximants. Note that we can also do moment matching with prescribed poles. The poles of the reduced order model may be placed, for example, in the open left half plane, by properly selecting \( G \) or \( H \) respectively, yielding the subfamily of stable reduced order models that match the moments of (2.1) at \( \sigma(S) \) or \( \sigma(Q) \), respectively.

**Proposition 2.6.** [6] Consider an LTI system (2.1). Furthermore, consider the families of reduced order models \( \Sigma \) and \( \hat{\Sigma} \) that match the moments of (2.1) at \( \sigma(S) \) and \( \sigma(Q) \), respectively. Let \( \lambda_i \in \mathbb{C}, i = 1, \ldots, \nu \), be such that \( \lambda_i \notin \sigma(S) \) or \( \lambda_i \notin \sigma(Q) \). Then:

1. There exists a subfamily of models of the form \( \Sigma \), with the property that the spectrum of each model contains \( \lambda_1, \ldots, \lambda_\nu \), i.e., there exists \( G \) such that \( \{\lambda_1, \ldots, \lambda_\nu\} = \sigma(S - GL) \).
2. There exists a subfamily of models of the form \( \hat{\Sigma} \), with the property that the spectrum of each model contains \( \lambda_1, \ldots, \lambda_\nu \), i.e., there exists \( H \) such that \( \{\lambda_1, \ldots, \lambda_\nu\} = \sigma(Q - RH) \).

Based on the previous discussion in the following we make the following working assumption: Matrices \( \Pi \) and \( \Upsilon \), unique solutions of (2.3) are formed using Krylov projections \( V \) and \( W \) in (2.12a), respectively, by Lemma 2.5. Furthermore, the moments of system (2.1) at \( \sigma(S) \) and at \( \sigma(Q) \) are computed efficiently based on Corollary 1, respectively. It is not required to explicitly solve equations (2.3).

**2.4. \( H_2 \)-norm based on the Gramians of linear systems.** Let us also briefly recall the definition of the \( H_2 \)-norm of an LTI system and its computation based on the controllability and the observability Gramians, respectively. Given the LTI system (2.1), the controllability Gramian \( W \) and the observability Gramian \( M \) are the solutions of the following Lyapunov equations [2]:

\[ \begin{align*}
AW + WA^T + BB^T &= 0, \\
A^T M + MA + C^T C &= 0.
\end{align*} \]

Let \( H_2 \) denote the Hilbert space of complex functions analytic in the open right-half plane and square integrable. Note that the transfer functions \( K \) and \( \hat{K} \) are elements of \( H_2 \). By [21] \( H_2 \)-norm is defined as:

\[ \| K \|_{H_2}^2 = \sqrt{\int_{-\infty}^{\infty} |K(j\omega)|^2 \, d\omega}. \]

The following result provides a computation formula for \( H_2 \)-norm of a rational transfer function \( K \).

**Lemma 2.7.** [21] Consider the LTI system (2.1) with the transfer function (2.2). Then:

\[ \| K \|_{H_2}^2 = C^T WC = B^T MB, \]

where \( W \) and \( M \) the solutions of equations (2.14).
3. $H_2$ model reduction by moment matching and optimization. This section presents the main theoretical contribution of our paper. Based on the previous parametrizations of the reduced models and using the $H_2$-norm of the approximation error as objective function, we write several optimization problems to optimally determine the approximation yielding the minimal $H_2$-norm error. We also derive the KKT (optimality) conditions for this optimization problems in terms of the controllability and the observability Gramians of the error system. Finally, based on these optimality conditions we propose several gradient-based methods for solving the corresponding optimization problems, with mathematical guarantees on its convergence. Due to the symmetry of families $\hat{\Sigma}$ and $\hat{\Sigma}$, in the sequel we focus on the parametrization of reduced system $\hat{\Sigma}$ in $(S, G, L)$. The general optimal $H_2$ model reduction problem by moment matching is formulated as follows:

**Problem 1.** Given an LTI system (2.1) with the transfer function $K$ given in (2.2), find a reduced order LTI system $\hat{\Sigma}_{(S,G,L)}$ in the family (2.8) with the transfer function $\hat{K}$ defined in (2.9), given in terms of the interpolations points $\sigma(S)$ and free parameters $L$ and $G$, that match $\nu$ fixed moments of (2.1) at $\sigma(S)$ and the following conditions are satisfied:

(i) the $H_2$ norm of the error system $\|K - \hat{K}\|_2$ is minimal

(ii) the reduced model $\hat{K}$ is stable, i.e. $\sigma(S - GL) \subset \mathbb{C}^-$

(iii) the pair $(S, L)$ is observable, $\sigma(S) \cap \sigma(A) = \emptyset$, and $\sigma(S) \cap \sigma(S - GL) = \emptyset$. 

We can also consider a relaxed formulation of this problem. Assuming that the pair $(L, S)$ is fixed a priori for the reduced order model (2.8), such that $(L, S)$ is observable and $\sigma(S) \cap \sigma(A) = \emptyset$, we search for a reduced order model $\hat{\Sigma}_{G}$ in the family (2.8) parameterized only in $G$, that yields the minimal $H_2-$norm of the approximation error. Hence, we compute the best $\nu$ order model from the family $\hat{\Sigma}_{G}$, matching fixed $\nu$ moments of (2.1). Thus, we may formulate the following particular instance of Problem 1, separately:

**Problem 2.** Fix $S \in \mathbb{C}^{\nu \times \nu}$ and $L \in \mathbb{C}^{m \times \nu}$ two matrices such that the pair $(L, S)$ is observable and $\sigma(S) \cap \sigma(A) = \emptyset$. Given the LTI system (2.1) with the transfer function (2.2) and the family of reduced order models $\hat{\Sigma}_{G}$ as in (2.8) that match the $\nu$ fixed moments of (2.1) at $\sigma(S)$, find the free parameters defined in terms of matrix $G$ such that the following conditions are satisfied:

(i) the $H_2$ norm of the error system $\|K - \hat{K}\|_2$ is minimal;

(ii) the reduced model $\hat{K}$ is stable, i.e., $\sigma(S - GL) \subset \mathbb{C}^-$;

(iii) $\sigma(S) \cap \sigma(S - GL) = \emptyset$.

Problems 1 and 2 can be recast in terms of the computation of the $H_2$ norm of the Gramians of the realization of the error system:

$$K_e = K - \hat{K},$$

with $\hat{K}$ from (2.9), parameterized in $(S, G, L)$ or $G$, respectively. Let $(A_c, B_c, C_c)$ be a state-space realization of the error transfer function $K_e$:

$$K_e(s) = C_c(sI - A_c)^{-1}B_c,$$

where

\begin{equation}
A_c = \begin{bmatrix} A & 0 \\ 0 & S - GL \end{bmatrix}, \quad B_c = \begin{bmatrix} B \\ G \end{bmatrix}, \quad C_c = C \begin{bmatrix} I & -\Pi \end{bmatrix}.
\end{equation}

Denote the controllability and the observability Gramians of (3.1) by $\mathcal{W}$ and $\mathcal{M}$, respectively. They are solutions of the following Lyapunov equations:

\begin{align}
(3.2a) \quad & A_c \mathcal{W} + \mathcal{W} A_c^T + B_c B_c^T = 0, \\
(3.2b) \quad & A_c^T \mathcal{M} + \mathcal{M} A_c + C_c^T C_c = 0.
\end{align}

Let us also recall a standard result for Lyapunov equations:

**Lemma 3.1.** Let $A_c$ be given stable matrix, i.e., $\sigma(A_c) \subset \mathbb{C}^-$. Then, there exist unique solutions $\mathcal{W}$ and $\mathcal{M}$ positive semidefinite of (3.2a) and (3.2b), respectively.
Below, we partition \( \mathcal{W} \) and \( \mathcal{M} \) following the block structure of matrix \( \mathcal{A}_e \):

\[
\mathcal{W} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21}^T & W_{22} \end{bmatrix}, \quad \mathcal{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21}^T & M_{22} \end{bmatrix}.
\]

### 3.1. Optimization formulation of Problem 1

In this section we propose an optimization formulation for the general Problem 1, where recall that the parametrisation of the reduced order model is done through matrices \( S, G \) and \( L \). Let us define the feasible set for the reduced model:

\[
\mathcal{R} = \{ (S, G, L) : (S, L) \text{ obs., } \sigma(S - GL) \subset \mathbb{C}^-, \sigma(S) \cap \sigma(A) = \emptyset, \text{ and } \sigma(S) \cap \sigma(S - GL) = \emptyset \}.
\]

By (2.15), the Problem 1 becomes:

\[
\min_{(S,G,L) \in \mathcal{R}} \left\| \mathcal{K}_e \right\|_2^2 = \min_{(S,G,L) \in \mathcal{R}, \mathcal{W} \text{ s.t. (3.2a)}} C \begin{bmatrix} I & -\Pi \end{bmatrix} \begin{bmatrix} W_{11} \\ W_{12} \\ W_{21}^T \\ W_{22} \end{bmatrix} \begin{bmatrix} I \\ -\Pi^T \end{bmatrix} C^T = \min_{(S,G,L) \in \mathcal{R}, \mathcal{M} \text{ s.t. (3.2b)}} \begin{bmatrix} B^T & G^T \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21}^T & M_{22} \end{bmatrix} \begin{bmatrix} B \\ G \end{bmatrix}.
\]

We now consider the problem formulation in terms of the observability Gramian \( \mathcal{M} \), written explicitly in matrix form as:

\[
\min_{(S,G,L,M)} \text{Trace}(\mathcal{B}_e^T \mathcal{M} \mathcal{B}_e) \\
\text{s.t.: (S, L) observable, } \sigma(S) \cap \sigma(A) = \emptyset, \text{ and } \sigma(S) \cap \sigma(S - GL) = \emptyset,
\]

\[A\Pi + BL = \Pi S, \quad \sigma(S - GL) \subset \mathbb{C}^-, \quad \mathcal{A}_e^T \mathcal{M} + \mathcal{M} \mathcal{A}_e + C_e^T C_e = 0.
\]

Note that, since \( \sigma(S) \cap \sigma(A) = \emptyset \), the Sylvester equation (2.3a) has a unique solution \( \Pi = \Pi(S, L) \). However, by our working assumptions, (2.3a) need not be solved since, by Lemma 2.5, we can take \( \Pi = VT \), with \( V \) from (2.12a) and \( T \) some non-singular matrix. Therefore, Problem 1 can be reformulated equivalently as:

\[
\min_{(S,G,L,M)} \text{Trace}(\mathcal{B}_e^T \mathcal{M} \mathcal{B}_e) \\
\text{s.t.: (S, L) observable, } \sigma(S) \cap \sigma(A) = \emptyset, \text{ and } \sigma(S) \cap \sigma(S - GL) = \emptyset,
\]

\[\sigma(S - GL) \subset \mathbb{C}^-, \quad \mathcal{A}_e^T \mathcal{M} + \mathcal{M} \mathcal{A}_e + C_e^T C_e = 0,
\]

with \( \Pi = VT \), for \( V \) as in (2.12a) and \( T \) some fixed non-singular matrix. However, it is difficult to deal with the restrictions (iii) in Problem 1, i.e., the constraints \( (S, L) \) observable, \( \sigma(S) \cap \sigma(A) = \emptyset \), and \( \sigma(S) \cap \sigma(S - GL) = \emptyset \). Hence, we can consider a triplet \( (S, L, G) \) such that the pair \( (S, L) \) is observable. Then, the unknowns are the diagonal matrix \( S \) and the vector \( G \), while \( L \) is fixed and \( \Pi = VT \). For example, without loss of generality, we can consider a canonical form for the triplet \( (S, L, G) \) as:

\[
S = \text{diag}(s_1, s_2, \ldots, s_\nu), \quad L = [\ell_1 \ell_2 \ldots \ell_\nu] \in \mathbb{R}^{m \times \nu} \quad \text{and} \quad G \in \mathbb{R}^{\nu \times m},
\]

such that the pair \( (S, L) \) is automatically observable, provided that we choose \( \ell_i \neq 0 \), regardless of the values of \( s_i \) for all \( i \). In this case we take \( \Pi = V \). Moreover, the constraints \( \sigma(S) \cap \sigma(A) = \emptyset \) and \( \sigma(S) \cap \sigma(S - GL) = \emptyset \) are not imposed in the numerical algorithms and are usually checked at the solution of the problem. Therefore, in the next sections we provide several numerical methods for solving the general non-convex optimization problem (3.5), with unknowns \( \mathcal{M}, S \) and \( G \), while \( L = [\ell_1 \ell_2 \ldots \ell_\nu] \) is fixed a priori. Moreover, following our discussion above, the constraints \( (S, L) \) observable, \( \sigma(S) \cap \sigma(A) = \emptyset \) and \( \sigma(S) \cap \sigma(S - GL) = \emptyset \) are also removed. In this case from (3.5) we get the simplified non-convex optimization formulation for Problem 1 analyzed in the sequel:

\[
\min_{(S,G,\mathcal{M})} \text{Trace}(\mathcal{B}_e^T \mathcal{M} \mathcal{B}_e) \\
\text{s.t.: } \sigma(S - GL) \subset \mathbb{C}^-, \quad \mathcal{A}_e^T \mathcal{M} + \mathcal{M} \mathcal{A}_e + C_e^T C_e = 0.
\]
Note that any (local) solution of optimization problem (3.6) satisfying the constraints \((S,L)\) observable, \(\sigma(S) \cap \sigma(A) = \emptyset\) and \(\sigma(S) \cap \sigma(S-GL) = \emptyset\) is also a local/global solution of Problem 1. How to efficiently numerically tackle the aforementioned constraints (i.e., non-convex problem (3.5)) so that to easily include them in an optimization algorithm remains an open question that will be investigated in the future. Let:

\[
\mathcal{X} = [S \quad G] \in \mathbb{R}^{\nu \times (\nu+m)}, \quad \mathcal{L} = \begin{bmatrix} I_{\nu} \\ -L \end{bmatrix} \in \mathbb{R}^{(\nu+m) \times \nu} \quad \text{and} \quad \mathcal{E} = \begin{bmatrix} 0_{\nu \times m} \\ I_m \end{bmatrix} \in \mathbb{R}^{(\nu+m) \times m},
\]

yielding

\[
G = \mathcal{X}\mathcal{E} \quad \text{and} \quad S - GL = \mathcal{X}\mathcal{L}.
\]

In order to clearly see the dependence on \(\mathcal{X}\), let us also define:

\[
A(\mathcal{X}) = A_e = \begin{bmatrix} A & 0 \\ 0 & \mathcal{X}\mathcal{L} \end{bmatrix}, \quad B(\mathcal{X}) = B_eB_e^T = \begin{bmatrix} B \\ \mathcal{X}\mathcal{E} \end{bmatrix}^T, \quad C = C_e^TC_e.
\]

In the next sections we present several (equivalent) reformulations of the nonconvex problem (3.6), accompanied by their first-order optimality conditions.

### 3.1.1. KKT approach.

Recall that our goal is to find a (local) minimum point of the non-convex problem (3.6). However, for a non-convex problem a minimum point is among the KKT points, i.e. it satisfies the KKT system. Using \(\text{Trace}(MN) = \text{Trace}(NM)\) for any matrices \(M, N\) of compatible sizes, in the sequel we derive the KKT system for the non-convex problem (3.6), which in compact form, in terms of \(\mathcal{X}\), can be written as follows:

\[
\begin{aligned}
\min_{(\mathcal{M}, \mathcal{X})} & \quad \text{Trace}(\mathcal{M}B(\mathcal{X})) \\
\text{s.t.:} & \quad \mathcal{X} \in \mathcal{D}_L, \quad A^T(\mathcal{X})\mathcal{M} + \mathcal{M}A(\mathcal{X}) + C = 0,
\end{aligned}
\]

where the open set \(\mathcal{D}_L = \{\mathcal{X}: \sigma(\mathcal{X}\mathcal{L}) \subset C^-\}\) and recall that \(L\) is fixed a priori. The Lagrangian function associated to problem (3.6), or equivalently (3.9), is given by:

\[
\Gamma(\mathcal{W}, \mathcal{M}, \mathcal{X}) = \text{Trace}(\mathcal{M}B(\mathcal{X})) + \text{Trace}(\mathcal{W}(A^T(\mathcal{X})\mathcal{M} + \mathcal{M}A(\mathcal{X}) + C)),
\]

where the multiplier \(\mathcal{W}\) is associated to the equality constraint in (3.9). Then, we write the optimization problem (3.9) into the max-min form:

\[
\max_{\mathcal{W}} \min_{\mathcal{M}, \mathcal{X} \in \mathcal{D}_L} \Gamma(\mathcal{W}, \mathcal{M}, \mathcal{X}).
\]

From standard optimization arguments we know that for any solution (also called KKT or saddle point) \((\mathcal{W}, \mathcal{M}, \mathcal{X})\) of problem (3.11), we have that \((\mathcal{M}, \mathcal{X})\) is a (possibly local) minimum point of the original problem (3.9) [32]. Moreover, if \((\mathcal{W}, \mathcal{M}, \mathcal{X})\) is a solution of problem (3.11) with \(\mathcal{X} \in \mathcal{D}_L\), then it must satisfy the KKT system:

\[
\nabla \Gamma(\mathcal{W}, \mathcal{M}, \mathcal{X}) = 0 \iff \begin{cases} 
\nabla_\mathcal{W} \Gamma(\mathcal{W}, \mathcal{M}, \mathcal{X}) = 0 \\
\nabla_\mathcal{M} \Gamma(\mathcal{W}, \mathcal{M}, \mathcal{X}) = 0.
\end{cases}
\]

The next theorem provides the explicit form of the KKT system:

**Theorem 3.2.** The KKT system of optimization problem (3.9) is given by:

\[
\nabla \Gamma(\mathcal{W}, \mathcal{M}, \mathcal{X}) = 0 \iff \begin{cases} 
A^T(\mathcal{X})\mathcal{M} + \mathcal{M}A(\mathcal{X}) + C = 0 \\
A(\mathcal{X})\mathcal{W} + \mathcal{W}A^T(\mathcal{X}) + B(\mathcal{X}) = 0 \\
M_{12}^TBE^T + M_{22}\mathcal{X}\mathcal{E}\mathcal{E}^T + M_{12}^TW_{12}\mathcal{L}^T + M_{22}W_{22}\mathcal{L}^T = 0.
\end{cases}
\]
Proof. Note that the KKT system has the form:
\[
\nabla \Gamma (W, M, \lambda) = \begin{bmatrix}
\nabla_W \Gamma (W, M, \lambda) \\
\nabla_M \Gamma (W, M, \lambda) \\
\nabla_{\lambda} \Gamma (W, M, \lambda)
\end{bmatrix} = \begin{bmatrix}
\mathcal{A}^T(\lambda)M + MA(\lambda) + C \\
\mathcal{A}(\lambda)W + WA^T(\lambda) + B(\lambda)
\end{bmatrix}.
\]

It remains to explicitly compute \( \nabla_{\lambda} \Gamma (W, M, \lambda) \). However, to write the gradient expression, we introduce a gradient of \( \Gamma \) w.r.t. \( \lambda \) using the trace of a matrix:
\[
\Gamma'(W, M, \lambda) \, d\lambda = \text{Trace}(\nabla^T_{\lambda} \Gamma (W, M, \lambda) \, d\lambda), \quad \text{with} \quad d\lambda \in \mathbb{R}^{p \times (p+m)}.
\]

Then, we have:
\[
\text{Trace}(\nabla^T_{\lambda} \Gamma (W, M, \lambda) \, d\lambda) = \text{Trace}(\mathcal{M}B'(\lambda) + W((A'(\lambda))^T\mathcal{M} + MA'(\lambda)))
\]
\[
= \text{Trace} \left( \mathcal{M} \left[ \begin{array}{cc}
0 & B \mathcal{E}^T \lambda T \\
\mathcal{E} \lambda T & \mathcal{E} \lambda T^T + \lambda \mathcal{E} \mathcal{E}^T \lambda T^T
\end{array} \right] + W \left[ \begin{array}{cc}
0 & 0 \\
0 & \lambda \mathcal{E} \lambda^T
\end{array} \right]^T \lambda + \mathcal{M} \left[ \begin{array}{cc}
0 & 0 \\
0 & \lambda \mathcal{E} \lambda^T
\end{array} \right] W \right)
\]
\[
= 2 \text{Trace} \left( \mathcal{E} \lambda T^T \mathcal{M} \lambda T + \mathcal{E} \mathcal{E}^T \lambda T^T \mathcal{M} \lambda T + \lambda \mathcal{E} \mathcal{E}^T \lambda T^T \mathcal{M} \lambda T \right) + \lambda \mathcal{E} \mathcal{E}^T \lambda T^T \mathcal{M} \lambda T + \lambda \mathcal{E} \mathcal{E}^T \lambda T^T \mathcal{M} \lambda T + \lambda \mathcal{E} \mathcal{E}^T \lambda T^T \mathcal{M} \lambda T) \right),
\]

where in the last equality we used the block structure of \( \mathcal{W} \) and \( \mathcal{M} \). Hence, we have:
\[
\nabla_{\lambda} \Gamma (W, M, \lambda) = 2 \left( \mathcal{M}^T \mathcal{E} \lambda T^T \mathcal{M} + \lambda \mathcal{E} \mathcal{E}^T \lambda T^T \mathcal{M} + \lambda \mathcal{E} \mathcal{E}^T \lambda T^T \mathcal{M} \right).
\]

Finally, we get the KKT system from (3.12).

The result of previous theorem also yields the necessary optimality condition for the optimization problem (3.9) of the general model reduction Problem 1:

**Lemma 3.3.** If \( \mathcal{M} \) and \( \lambda \) ∈ \( D_L \), where \( \lambda = [S \ G] \), solves the optimization problem (3.9) corresponding to the model reduction Problem 1, then there exists \( \mathcal{W} \) such that the triplet \( (W, M, \lambda) \) solves the KKT system (3.12).

### 3.1.2. Partial minimization approach

Consider the non-convex optimization problem (3.6), where \( \mathcal{L} \) is fixed a priori, \( \Pi = VT \), and \( \mathcal{A}_e = \mathcal{A}_e(S, G) \). Then, the following partial minimization holds for (3.6):
\[
\text{min}_{(S,G): \sigma(S-GL) \subset \mathcal{C}^{-}} \left( \text{min}_{\mathcal{M}: A^T_e \mathcal{M} + MA_e + C^T_e \mathcal{C}_e = 0} \text{Trace}(B^T_e \mathcal{M} B_e)) \right).
\]

However, if \( S-GL \) and \( A \) are stable, it follows from Lemma 3.1 that there exists unique \( \mathcal{M} = \mathcal{M}(S, G) \geq 0 \) solution of the Lyapunov equation:
\[
A^T_e \mathcal{M} + MA_e + C^T_e \mathcal{C}_e = 0.
\]

Hence, for any pair \( (S, G) \) stable, the partial minimization in \( \mathcal{M} \) leads to an optimal value \( f(S, G) = \min_{\mathcal{M}: A^T_e \mathcal{M} + MA_e + C^T_e \mathcal{C}_e = 0} \text{Trace}(B^T_e \mathcal{M} B_e)) \) which can be written explicitly as:
\[
f(S, G) = \text{Trace} \left( \begin{bmatrix} B^T_e \\ G \end{bmatrix} \mathcal{M}(S, G) \begin{bmatrix} B \\ G \end{bmatrix} \right),
\]

where \( \mathcal{M}(S, G) \) is the unique solution of the Lyapunov equation:
\[
(3.13) \quad \begin{bmatrix} A & 0 \\ S-GL \end{bmatrix}^T \mathcal{M} + \mathcal{M} \begin{bmatrix} A & 0 \\ S-GL \end{bmatrix} + \begin{bmatrix} C^T \mathcal{C} & -C^T \mathcal{C} \\ -C^T \mathcal{C} & C^T \mathcal{C} \end{bmatrix} = 0,
\]
with \( C_V = C \Pi = CVT \). Therefore, we get the following equivalent reformulation for (3.6):

\[
\min_{(S,G)} \text{Trace} \left( \begin{bmatrix} B^T & M(S,G) \end{bmatrix} \begin{bmatrix} B \\ G \end{bmatrix} \right)
\]
\[
\text{s.t.: } \sigma(S - GL) \subset C^- \quad \text{and} \quad (3.13).
\]

Using the notation in (3.7), the non-convex problem (3.14) becomes:

\[
\min_{\mathcal{X}} \text{Trace} \left( \begin{bmatrix} B^T & M(\mathcal{X}) \end{bmatrix} \begin{bmatrix} B \\ \mathcal{X}E \end{bmatrix} \right)
\]
\[
\text{s.t.: } \sigma(\mathcal{X}L) \subset C^- \quad \text{and} \quad (3.16),
\]

where \( M(\mathcal{X}) \) is the unique positive semidefinite solution of the Lyapunov equation:

\[
A \begin{bmatrix} 0 & \mathcal{X}L \end{bmatrix}^T M(\mathcal{X}) + M(\mathcal{X}) \begin{bmatrix} A & 0 \\ \mathcal{X}L \end{bmatrix} + \begin{bmatrix} C^T C & -C^T C_V \\ -C^T C_V & C_V^2 C_V \end{bmatrix} = 0.
\]

For solving the equivalent non-convex problem (3.15) we can apply any first- or second-order optimization method. For this type of optimization scheme we need to compute the gradient and even the Hessian of the objective function. In the sequel, we show that we can compute the gradient of the objective function of (3.15) solving two Lyapunov equations. Indeed, by \( \text{Trace}(MN) = \text{Trace}(NM) \) for any matrices \( M, N \) of compatible sizes, the non-convex objective function of (3.15) becomes in terms of the notation (3.7):

\[
f(\mathcal{X}) = \text{Trace} \left( \begin{bmatrix} B^T & M(\mathcal{X}) \end{bmatrix} \begin{bmatrix} B \\ \mathcal{X}E \end{bmatrix} \right) = \text{Trace} (M(\mathcal{X})B(\mathcal{X})).
\]

**Theorem 3.4.** The objective function \( f \) of (3.15) is differentiable on the set of stable matrices \( \mathcal{D}_L \) and the gradient of \( f \) at \( \mathcal{X} \in \mathcal{D}_L \) is given by:

\[
\nabla f(\mathcal{X}) = 2 \left[ M_{12}^T(\mathcal{X})W_{12}(\mathcal{X})L^T + M_{22}(\mathcal{X})W_{22}(\mathcal{X})\mathcal{L}^T + M_{12}^T(\mathcal{X})B\mathcal{E}^T + M_{22}(\mathcal{X})\mathcal{X}\mathcal{E}^T \right],
\]

where \( M(\mathcal{X}) \) solves the Lyapunov equation (3.16) and \( W(\mathcal{X}) \) solves the Lyapunov equation:

\[
\begin{bmatrix} A & 0 \\ \mathcal{X}L \end{bmatrix} W(\mathcal{X}) + W(\mathcal{X}) \begin{bmatrix} A & 0 \\ \mathcal{X}L \end{bmatrix}^T + B(\mathcal{X}) = 0.
\]

**Proof.** To compute the gradient \( \nabla f(\mathcal{X}) \), we write the derivative \( f'(\mathcal{X}) \, d\mathcal{X} \) for some \( d\mathcal{X} \in \mathbb{R}^{(\nu + m) \times \nu} \) in gradient form using the trace. We introduce the gradient as:

\[
f'(\mathcal{X}) \, d\mathcal{X} = \text{Trace} \left( \nabla f(\mathcal{X})^T \, d\mathcal{X} \right).
\]

Then, we have:

\[
f'(\mathcal{X}) \, d\mathcal{X} = \text{Trace} (M'(\mathcal{X})B(\mathcal{X}) + M(\mathcal{X})B'(\mathcal{X})).
\]

We compute separately the two terms in the above expression. Let

\[
\Phi(\mathcal{X}, M) = \begin{bmatrix} A & 0 \\ 0 & \mathcal{X}L \end{bmatrix}^T M + M \begin{bmatrix} A & 0 \\ 0 & \mathcal{X}L \end{bmatrix}.
\]

Since \( \mathcal{X} \in \mathcal{D}_L \) and \( \mathcal{D}_L \) is an open set, then by Lemma 3.1 we have that \( \Phi_M(\mathcal{X}, M) \, dM \) given by:

\[
\Phi_M(\mathcal{X}, M) \, dM = \begin{bmatrix} A & 0 \\ 0 & \mathcal{X}L \end{bmatrix}^T dM + dM \begin{bmatrix} A & 0 \\ 0 & \mathcal{X}L \end{bmatrix}.
\]
is surjective and also we have:

$$\Phi_{\mathcal{X}}(\mathcal{X}, \mathcal{M}) \, d\mathcal{X} = \begin{bmatrix} 0 & 0 \\ 0 & d\mathcal{X} \end{bmatrix}^T \mathcal{M} + \mathcal{M} \begin{bmatrix} 0 & 0 \\ 0 & d\mathcal{X} \end{bmatrix}.$$ 

Since $\Phi(\mathcal{X}, \mathcal{M}) + C = 0$, the Implicit Function Theorem yields the differentiability of $\mathcal{M}(\mathcal{X})$ and the following relation:

$$\tag{3.19} \begin{bmatrix} A & 0 \\ 0 & \mathcal{X} \end{bmatrix}^T \mathcal{M}'(\mathcal{X}) + \mathcal{M}'(\mathcal{X}) \begin{bmatrix} A & 0 \\ 0 & \mathcal{X} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d\mathcal{X} \end{bmatrix}^T \mathcal{M}(\mathcal{X}) + \mathcal{M}(\mathcal{X}) \begin{bmatrix} 0 & 0 \\ 0 & d\mathcal{X} \end{bmatrix} = 0.$$ 

Moreover, by (3.2a) the Gramian $\mathcal{W}(\mathcal{X})$ is the unique solution of the Lyapunov equation (3.18). Subtracting (3.19) multiplied by $\mathcal{W}(\mathcal{X})$ to the left from (3.18) multiplied by $\mathcal{M}'(\mathcal{X})$ to the right, taking the trace, and reducing the appropriate terms, we get:

$$\text{Trace} (\mathcal{M}'(\mathcal{X}) \mathcal{B}(\mathcal{X})) = \text{Trace} \left( \mathcal{W}(\mathcal{X}) \begin{bmatrix} 0 & 0 \\ 0 & d\mathcal{X} \end{bmatrix}^T \mathcal{M}(\mathcal{X}) + \mathcal{M}(\mathcal{X}) \begin{bmatrix} 0 & 0 \\ 0 & d\mathcal{X} \end{bmatrix} \mathcal{W}(\mathcal{X}) \right)$$

$$= 2 \text{Trace} \left( \mathcal{L}W_{12}^T(\mathcal{X}) M_{12}(\mathcal{X}) \, d\mathcal{X} + \mathcal{L}W_{22}(\mathcal{X}) M_{22}(\mathcal{X}) \, d\mathcal{X} \right),$$

where for the second equality we used the block structure of $\mathcal{W}$ and $\mathcal{M}$ and the definition of trace. Similarly, for the second term using the block structure of $\mathcal{M}$ and the definition of trace, we get:

$$\text{Trace} (\mathcal{M}(\mathcal{X}) \mathcal{B}'(\mathcal{X})) = \text{Trace} \left( \mathcal{M}(\mathcal{X}) \begin{bmatrix} 0 & \mathcal{B}^T \\ d\mathcal{X} \mathcal{E} \mathcal{B}^T & \mathcal{E} \mathcal{B} \mathcal{X}^T + \mathcal{X} \mathcal{E} \mathcal{B}^T \end{bmatrix} \right)$$

$$= 2 \text{Trace} \left( \mathcal{E} \mathcal{B} \mathcal{X} M_{12}(\mathcal{X}) \, d\mathcal{X} + \mathcal{E} \mathcal{B} \mathcal{X} \mathcal{E} \mathcal{B} \mathcal{X}^T M_{22}(\mathcal{X}) \, d\mathcal{X} \right).$$

Hence, from (3.20) and (3.21) we get the closed form expression for the gradient (3.17).

Note that the expression of the gradient $\nabla f$ from (3.17) is the same as the partial gradient of the Lagrangian $\nabla X \Gamma$ from (3.12). The result of previous theorem also yields the necessary optimality condition for the model reduction Problem 1 expressed in terms of the optimization problem (3.15):

**Lemma 3.5.** If $\mathcal{X} \in D_L$, where $\mathcal{X} = [S \ G]$, solves the optimization problem (3.15) corresponding to the model reduction Problem 1, then

$$M_{12}^T(\mathcal{X}) W_{12}(\mathcal{X}) \mathcal{L}^T + M_{22}(\mathcal{X}) W_{22}(\mathcal{X}) \mathcal{L}^T + M_{12}^T(\mathcal{X}) \mathcal{B} \mathcal{E}^T + M_{22}(\mathcal{X}) \mathcal{X} \mathcal{E} \mathcal{B}^T = 0,$$

where $\mathcal{M}(\mathcal{X})$ solves the Lyapunov equation (3.16) and $\mathcal{W}(\mathcal{X})$ solves the Lyapunov equation (3.18).

We can replace the open set $D_L$ with any sublevel set:

$$\mathcal{N}_L^{X_0} = \{ \mathcal{X} \in D_L : f(\mathcal{X}) \leq f(X_0) \},$$

where $X_0 \in D_L$ is any initial stable reduced order system matrix. Arguments as in [35] we can show that $\mathcal{N}_L^{X_0}$ is a compact set. Then, the theorem of Weierstrass implies that for any given matrix $X_0 \in D_L$, the model reduction Problem 1 given by optimization formulation (3.15) has a global minimum in the sublevel set $\mathcal{N}_L^{X_0}$. We can also show that the gradient $\nabla f(\mathcal{X})$ is Lipschitz continuous on the compact sublevel set $\mathcal{N}_L^{X_0}$. Let us briefly sketch the proof of this statement. First we observe that $\mathcal{M}(\mathcal{X})$ and $\mathcal{W}(\mathcal{X})$ are continuous functions and moreover there exists finite $\ell_M > 0$ such that:

$$\| \mathcal{M}(\mathcal{X}) - \mathcal{M}(\mathcal{Y}) \| \leq \ell_M \| \mathcal{X} - \mathcal{Y} \| \quad \forall \mathcal{X}, \mathcal{Y} \in \mathcal{N}_L^{X_0}.$$ 

Then, using the expression of $\nabla f(\mathcal{X})$, compactness of $\mathcal{N}_L^{X_0}$, continuity of $\mathcal{M}(\mathcal{X})$ and $\mathcal{W}(\mathcal{X})$, and the previous relation we conclude that there exists $\ell_f > 0$ such that:

$$\| \nabla f(\mathcal{X}) - \nabla f(\mathcal{Y}) \| \leq \ell_f \| \mathcal{X} - \mathcal{Y} \| \quad \forall \mathcal{X}, \mathcal{Y} \in \mathcal{N}_L^{X_0}.$$ 

This property of the gradient is useful when analyzing the convergence behavior of the first-order algorithm we propose for solving (3.15).
3.1.3. SDP approach. Alternatively, the non-convex problem (3.6) can be written equivalently in terms of matrix inequalities (semidefinite programming):

\[
\begin{align*}
\min_{(S,G,M)} & \quad \text{Trace} \left( B^T G + M \begin{bmatrix} B \\ G \end{bmatrix} \right) \\
\text{s.t.} & \quad \begin{bmatrix} A \\ S - GL \end{bmatrix}^T M + M \begin{bmatrix} A \\ S - GL \end{bmatrix} + \begin{bmatrix} C^T C & -C^T (C T) \\ -(C T)^T C & (C T)^T (C T) \end{bmatrix} \preceq 0,
\end{align*}
\]

(3.22)

where recall that \( L \) is fixed a priori. Clearly, SDP problem (3.22) is not convex since it contains bilinear matrix inequalities (BMIs). However, next theorem proves that we can obtain a suboptimal solution through convex relaxation:

**Theorem 3.6.** If the following convex SDP relaxation:

\[
\begin{align*}
\min & \quad \text{Trace} \left( B^T M_{11} B + X_{22} \right) \\
\text{s.t.} & \quad \Theta^T - L^T Z_{22}^T + \Theta - Z_{22} L + C_V^T C_V \preceq Y_{22} \\
& \quad \begin{bmatrix} X_{22} & Z_{22} \\ Z_{22}^T & M_{22} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} A^T M_{11} + M_{11} A^T + C^T C & A^T M_{12} + M_{12} (S - GL) - C^T C_V \\ M_{11} A + (S - GL)^T M_{12} - C_V^T C & (S - GL)^T M_{22} + M_{22} (S - GL) + C_V^T C_V \end{bmatrix} \succeq 0
\end{align*}
\]

(3.23)

has a solution, then we can recover a suboptimal solution of the model reduction Problem 1 expressed in terms of the SDP problem (3.22) through the relations:

\[
G = M_{22}^{-1} Z_{22}, \quad S = M_{22}^{-1} \Theta_{22} \quad \text{and} \quad M = \text{diag}(M_{11}, M_{22}).
\]

**Proof.** Using the block form of \( M \) and the notation \( C_V = C T = CVT \), (3.22) yields the equivalent SDP problem:

\[
\begin{align*}
\min_{(S,G,M)} & \quad \text{Trace} \left( B^T M_{11} B + B^T M_{12} G + G^T M_{12}^T B + G^T M_{22} G \right) \\
\text{s.t.} & \quad \begin{bmatrix} A^T M_{11} + M_{11} A^T + C^T C \\ M_{12} A + (S - GL)^T M_{12} - C_V^T C \end{bmatrix} \succeq 0, \quad (S - GL)^T M_{22} + M_{22} (S - GL) + C_V^T C_V \succeq 0.
\end{align*}
\]

(3.24)

Note that problem (3.24) is not convex since it contains bilinear matrix terms and we are not aware of any change of variables that might lead to a convex reformulation (note that if we assume \( M_{12} \neq 0 \), then we cannot convexify the previous BMIs since we need to define \( M_{12} G = Z_{12} \) and \( M_{22} G = Z_{22} \) and require \( M \succeq 0 \). However, if we assume the block \( M_{12} = 0 \), then problem (3.24) can be recast as a convex SDP. More precisely, if we introduce additional variables, then we have:

\[
\begin{align*}
\min & \quad \text{Trace} \left( B^T M_{11} B + X_{22} \right) \\
\text{s.t.} & \quad X_{22} \succeq G^T M_{22} G, \quad (S - GL)^T M_{22} + M_{22} (S - GL) + C_V^T C_V \succeq Y_{22} \\
& \quad \begin{bmatrix} A^T M_{11} + M_{11} A^T + C^T C & -C^T C_V \\ -C_V^T C & \end{bmatrix} \succeq Y_{22}
\end{align*}
\]

(3.25)

Denoting \( Z_{22} = M_{22} G, \Theta_{22} = M_{22} S \) and using the Schur complement, problem (3.25) becomes the convex SDP (3.23). Moreover, we can recover a suboptimal solution of the original problem through the relations:

\[
G = M_{22}^{-1} Z_{22}, \quad S = M_{22}^{-1} \Theta_{22} \quad \text{and} \quad M = \text{diag}(M_{11}, M_{22}).
\]

Clearly, this is a suboptimal solution of the original SDP problem (3.22) since we restrict the matrix \( M \) to have the block \( M_{12} = 0 \). Hence, (3.23) is a convex SDP relaxation of the original problem (3.22).

**3.2. Numerical optimization algorithms for Problem 1.** In this section we present several optimization algorithms for solving the model reduction Problem 1. For solving the associated KKT system (3.12) of the non-convex problem (3.6) or the non-convex (partial) optimization problem (3.15) we propose first-order
methods since they are adequate for large-scale optimization problems, i.e. the dimension $n$ is very large. Of course, we can also apply second-order methods to solve these optimization problems, but they require more expensive computations at each iteration (e.g., evaluation of Hessians and finding solutions of linear system), making them intractable when dimension $n$ of the original linear system (2.1) is large.

3.2.1. Gradient type method for KKT system. One optimization algorithm that can be used for solving the KKT system (3.12) is the gradient method. Starting from an initial triplet $(W_0,M_0,X_0)$ update:

$$
W_{k+1} = W_k + \alpha_k \nabla_W (W_k,M_k,X_k)
$$

$$
M_{k+1} = M_k - \alpha_k \nabla_M (W_k,M_k,X_k),
$$

$$
X_{k+1} = X_k - \alpha_k \nabla_{X_k} (W_k,M_k,X_k),
$$

where $\alpha_k$ is a stepsize selected to minimize an appropriate merit function in the search direction at each step. Under some mild assumptions it is possible to show that the iterative process (3.26) converges locally to a KKT point, see, e.g., [28] (Chapter 14). Moreover, if we start sufficiently close to a KKT point we can even choose $\alpha_k$ constant and the sequence will converge linearly to a KKT point, with a speed of convergence depending on the starting point.

If the convex SDP relaxation (3.23) admits a solution, then we can consider as a starting point the suboptimal solution provided by this relaxation, i.e., $X_0 = [S_0 G_0]$ with $G_0 = M_0^{-1}Z_{22}, S_0 = M_0^{-1}\Theta_{22}$ and $M_0 = \text{diag}(M_{11},M_{22})$. Moreover, we can take $W_0$ as the solution of the Lyapunov equation (3.2a) with $S = S_0$ and $G = G_0$ given before. Otherwise, we can fix $S_0$ and $L$ such that the pair $(L,S_0)$ is observable, and select a set $\{\lambda_1,\ldots,\lambda_\nu\} \subset \mathbb{C}^-$. Then, from control theory it is known that there exists (stabilizing) $G_0$, computed by standard control algorithms, such that the spectrum $\sigma(S_0 - G_0 L) = \{\lambda_1,\ldots,\lambda_\nu\}$. Based on the explicit form of the KKT system (3.12) we get the following simple iterative process:

$$
W_{k+1} = W_k + \alpha_k (A^T(X_k) M_k + M_k A(X_k) + C)
$$

$$
M_{k+1} = M_k - \alpha_k (A(X_k) W_k + W_k A^T(X_k) + B(X_k))
$$

$$
X_{k+1} = X_k - \alpha_k (M_{22,k} B E^T + M_{22,k} X_k E E^T + M_{22,k} W_{12,k} E^T + M_{22,k} W_{22,k} E^T).
$$

This algorithm has a cheap iteration since it requires only matrix multiplications. The update in (3.27) has the disadvantage however that only the asymptotic $X_k = [S_k G_k]$ leads to a reduced order stable system while the intermediate iterates can lead to unstable systems.

3.2.2. Gradient method for partial minimization problem. We have proved that the non-convex optimization problem (3.15) has differentiable objective function and its gradient is given in (3.17). Moreover, the gradient is Lipschitz continuous on any compact set. Then, we can apply gradient method for solving (3.15). Starting from the initial stable matrix $X_0 \in \mathcal{D}_L$ we consider the following update:

$$
X_{k+1} = X_k - \alpha_k \nabla f(X_k),
$$

where the stepsize $\alpha_k$ can be chosen by a backtracking procedure or constant in the interval $(0,2/\ell_f)$ (where $\ell_f$ denotes the Lipschitz constant of the gradient). With these choices for the stepsize and using the Lipschitz gradient property for the objective function the sequence of value functions $f(X_k)$ is nonincreasing [32]:

$$
f(X_{k+1}) \leq f(X_k) - \Delta \cdot \|\nabla f(X_k)\|^2 \quad \forall k \geq 0,
$$

for some constant $\Delta > 0$. Therefore all the iterates remain in the compact sublevel set $\mathcal{X}_0^{X_0}$. Moreover, since $f$ is bounded from below by zero, then for any positive integer $K$ it is straightforward to prove from the previous descent inequality the following global convergence rate:

$$
\min_{i=0:k} \|\nabla f(X_k)\|^2 \leq \frac{f(X_0) - f^*}{\Delta \cdot K} \quad \forall k \geq 0,
$$

where $f^*$ is the optimal value of problem (3.15). Under some mild assumptions, such as the Hessian of $f$ at a local minimum is positive definite and bounded, then starting sufficiently close to this local optimum the
gradient iteration converges linearly to this solution [32]. Therefore, the speed of convergence of this iterative process depends on the starting point. For choices of the starting point we can consider the procedures described in the previous section.

Note that the gradient iteration has the explicit form:

\[
\mathcal{X}_{k+1} = \mathcal{X}_k - \alpha_k (\mathcal{M}_{12,k} B E^T + M_{22,k} \mathcal{X}_k E E^T + M_{12,k} W_{12,k} \mathcal{L}^T + M_{22,k} W_{22,k} \mathcal{L}^T),
\]

where \( \mathcal{M}_k \) and \( \mathcal{W}_k \) are the unique positive semidefinite solutions of the Lyapunov equations in \( \mathcal{X}_k \) (3.16) and (3.18), respectively. Therefore, this iterative process has expensive iterations since it requires solving two Lyapunov equations, which can be prohibitive when dimension \( n \) of the original system is large. On the other hand the update in (3.28) has the advantage that any iterate \( \mathcal{X}_k = [S_k \ G_k] \) leads to a stable reduced order model, while for the iteration (3.27) only the asymptotic \( \mathcal{X}_k \) leads to a stable system.

### 3.2.3. Convex SDP relaxation.

There are several methods available for solving SDP problems with convex objective function and constraints of type BMIs, see, e.g., [26]. However, there are more efficient solvers for convex SDPs (as problem (3.23)) that can scale to large instances such as first order methods or interior point methods [32]. Note that in the general case, i.e., for general matrices \( A \), the convex SDP relaxation (3.23) is not exact, since imposing the block \( M_{12} = 0 \), its solution is suboptimal for the original SDP problem (3.22). If the convex SDP relaxation (3.23) admits a solution, then we can initialize the gradient-based methods from previous two sections with the suboptimal solution provided by this relaxation.

On the other hand, for certain particular systems the convex SDP relaxation (3.23) is exact. Indeed, this is the case, e.g., for positive systems. Let us briefly introduce the notion of positive systems and their main properties, see, e.g., [33] for a detailed exposition. A matrix is said to be Metzler if all offdiagonal elements are non-negative. Further, the LTI system (2.1) is said to be a positive system if \( A \) is Metzler and \( B, C \geq 0 \). Then, one basic result for positive systems states that they admit diagonal Lyapunov matrices:

\[
A \text{ stable } \iff \exists P > 0 \text{ diagonal s.t. } A^T P + PA < 0.
\]

Recently, a high interest in positive systems has been shown in the literature. Positive systems occur in modelling of applications with special structures from, e.g., biomedicine, economics, data networks, etc., [33]. Naturally, these systems are generally highly dimensional and need to be approximated with the help of model order reduction techniques. Unfortunately, conventional model reduction techniques do not preserve the positivity. However, working with an approximation violating basic physical constraints it always leaves the question of how conclusive results on this basis are. Recently, balanced truncation-based methods that preserve positivity have been proposed in, e.g., [27, 34]. Note that in all our optimization formulations we proposed we can easily impose additional convex constraints for preserving positivity: offdiagonal(\( S - GL \)) \( \geq 0 \) and \( G \geq 0 \), where \( L \) is fixed a priori. Then, we can apply, e.g., a projected gradient type algorithm for solving the corresponding first two problems. Moreover, for positive systems the convex SDP relaxation (3.23) is exact since there exists diagonal Gramian \( \mathcal{M} \) satisfying the Lyapunov equation (3.2b) (see, e.g., [27]), and consequently requiring the block \( M_{12} = 0 \) is not restricting the feasible set of the original SDP problem (3.22). Furthermore, in the convex SDP problem (3.23) positivity can be imposed through new additional convex constraints:

\[
\text{offdiagonal}(\Theta_{22} - Z_{22} L) \geq 0, \ Z_{22} \geq 0.
\]

It is clear that the reduced order model is also a positive system, i.e., offdiagonal(\( S - GL \)) \( \geq 0 \) and \( G \geq 0 \), provided that \( G = M_{22}^{-1} Z_{22}, S = M_{22}^{-1} \Theta_{22}, \mathcal{M} \) diagonal, and \( \Theta_{22}, Z_{22} \) satisfy the new constraints from above. Hence, our model reduction techniques are flexible, allowing to incorporate easily constraints for preserving positivity and/or stability.

### 3.3. Optimization formulation of Problem 2.

Using similar arguments as for the general Problem 1 we can derive optimization formulations for the particular Problem 2, where now the parametrisation of the reduced order model is done only through matrix \( G \). Note that in Problem 2 the pair \( (S, L) \) is fixed a priori such that it is observable. Further, we can find CTI based on Corollary 1. Moreover, if we choose \( S \) unstable,
that is $\sigma(S) \subseteq \mathbb{C}^+$, then the optimal solution of Problem 2 automatically satisfies $\sigma(S) \cap \sigma(S - GL) = \emptyset$. Then, from (2.15) it follows that Problem 2 can be written as:

$$\min_{G \text{ s.t. } \sigma(S - GL) \subseteq \mathbb{C}^-} ||K_G||^2 = \min_{(G,W) \text{ s.t. } \sigma(S - GL) \subseteq \mathbb{C}^-} C \begin{bmatrix} I & -I \end{bmatrix} ^T \begin{bmatrix} W_{11} & W_{12} \\ W_{12}^T & W_{22} \end{bmatrix} \begin{bmatrix} I \\ -I \end{bmatrix} ^T G^T$$

Below we consider again only the formulation in terms of the observability Gramian $M$:

$$\min_{(G,M)} \text{Trace}(B_e^T M B_e)$$

s.t. $\sigma(S - GL) \subseteq \mathbb{C}^-$ and $A_c^T M + MA_e + C_c^T C_e = 0$,

with $(S, L)$ fixed and $\Pi = VT$, where $V$ as in (2.12a) and $T$ some fixed non-singular matrix. We clearly observe that in this case the reduced order model is parametrized only in the matrix $G$. Let us denote:

$$A(G) = \begin{bmatrix} A & 0 \\ 0 & S - GL \end{bmatrix}, \quad B(G) = \begin{bmatrix} B \\ G \end{bmatrix}, \quad C = C_c^T C_e = \begin{bmatrix} I \\ -T^T V^T \end{bmatrix} \begin{bmatrix} I \\ -T^T V^T \end{bmatrix}^T$$

In the next sections we present several (equivalent) reformulations of the nonconvex problem (3.29), accompanied by their first-order optimality conditions.

### 3.3.1. KKT approach.

We determine the corresponding KKT system for optimization problem (3.29). We first define the open set $D_{(SL)} = \{ G : \sigma(S - GL) \subseteq \mathbb{C}^- \}$ where the pair $(S, L)$ is fixed a priori. Using again that Trace$(MN) =$ Trace$(NM)$, Lagrangian function associated to problem (3.29) is given by:

$$\Gamma(W, M, G) = \text{Trace}(MB_e(G)) + \text{Trace}(W(A^T(G)M + MA_e(G) + C))$$

where the multiplier $W$ is associated to the equality constraint in (3.29). Then, we write (3.30) into the max-min form:

$$\max_{W, M, G \in D_{(SL)}} \min_{W, M, G} \Gamma(W, M, G).$$

From standard optimization arguments we know that any solution $(W, M, G)$ of problem (3.31) implies that $(M, G)$ is a (possibly local) minimum point of the original problem (3.29) and needs to satisfy the KKT system:

$$\nabla \Gamma(W, M, G) = 0 \iff \begin{cases} \nabla W \Gamma(W, M, G) = 0 \\ \nabla_{(M,G)} \Gamma(W, M, G) = 0. \end{cases}$$

Next theorem derives explicitly the corresponding KKT system:

**Theorem 3.7.** The KKT system of optimization problem (3.29) is given by:

\[ \nabla \Gamma(W, M, G) = 0 \iff \begin{cases} A^T(G)M + MA_e(G) + C = 0 \\ A(G)W + WA_e^T(G) + B(G) = 0 \\ M_{12}^T B + M_{22} G - M_{12}^T W_{12} L^T - M_{22} W_{22} L^T = 0. \end{cases} \]

**Proof.** Note that the KKT system has the following explicit form:

$$\nabla \Gamma(W, M, G) = \begin{bmatrix} \nabla_W \Gamma(W, M, G) \\ \nabla_M \Gamma(W, M, G) \\ \nabla_G \Gamma(W, M, G) \end{bmatrix} = \begin{bmatrix} A^T(G)M + MA_e(G) + C \\ A(G)W + WA_e^T(G) + B(G) \\ \nabla_G \Gamma(W, M, G) \end{bmatrix}.$$
It remains to explicitly compute $\nabla G \Gamma(W, M, G)$. However, to write the gradient expression, we introduce a gradient of $\Gamma$ w.r.t. $G$ using the trace of a matrix:

$$\Gamma'(W, M, G) dG = \text{Trace}(\nabla^T G \Gamma(W, M, G) dG), \quad \text{with } dG \in \mathbb{R}^{n \times m}.$$ 

Then:

$$\text{Trace}(\nabla^T G \Gamma(W, M, G) dG)$$

$$= \text{Trace}(MB'(G)) + \text{Trace}(W((A'(G))^T M + MA'(G)))$$

$$= \text{Trace}
\begin{pmatrix}
0 & B dG^T \\
dG B^T & dG G^T + G dG^T
\end{pmatrix}
+ W
\begin{bmatrix}
0 & 0 \\
0 & -dGL
\end{bmatrix}
\begin{pmatrix}
M + M
0 & 0
\end{pmatrix}
W$$

$$= 2 \text{Trace}
\begin{pmatrix}
B^T M_{12} dG + G^T M_{22} dG - LW^T_{12} M_{12} dG - LW_{22} M_{22} dG
\end{pmatrix},$$

where in the last equality we used the block structure of $W$ and $M$. Then:

$$\nabla G \Gamma(W, M, G) = 2 \left(M^T_{12} B + M_{22} G - M^T_{12} W_{12} L^T - M_{22} W_{22} L^T\right).$$

Hence, we get the KKT system from (3.32).

The result of previous theorem also yields the necessary optimality condition for the optimization problem (3.29) of the model reduction Problem 2:

**Lemma 3.8.** If $M$ and $G \in D_{(SL)}$ solves the optimization problem (3.29) corresponding to the model reduction Problem 2, then there exists $W$ such that the triplet $(W, M, G)$ solves the KKT system (3.32).

**3.3.2. Partial minimization approach.** Consider now the optimization problem (3.29) where recall that the pair $(S, L)$ is fixed a priori, $\Pi = VT$, and $A$ depends on $G$, i.e., $A = A(G)$. Then, the partial minimization holds for (3.29):

$$(3.29) = \min_{G : \sigma(S - GL) \subset \mathbb{C}^-} \min_{M : A^T \Pi M + MA_e + C_e^T C_e = 0} \text{Trace}(B^T_e MB_e).$$

However, if $S - GL$ and $A$ are stable, it follows from Lemma 3.1 that there exists unique $M = \mathcal{M}(G) \succeq 0$ solution of the Lyapunov equation:

$$A^T M + MA_e + C_e^T C_e = 0.$$ 

Hence, for any stabilizable $G$, the partial minimization in $M$ leads to an optimal value:

$$f(G) = \min_{M : A^T \Pi M + MA_e + C_e^T C_e = 0} \text{Trace}(B^T_e MB_e) = \text{Trace}
\begin{pmatrix}
B^T \\
G
\end{pmatrix}
\mathcal{M}(G)
\begin{pmatrix}
B \\
G
\end{pmatrix},$$

where $\mathcal{M}(G)$ is the unique solution of the Lyapunov equation:

$$(3.33) \begin{pmatrix}
A & 0 \\
0 & S - GL
\end{pmatrix} \mathcal{M} + \mathcal{M}
\begin{pmatrix}
A & 0 \\
0 & S - GL
\end{pmatrix}
+ \begin{pmatrix}
C^T C & -C^T C_V \\
-C^T C_V & C^T C_V
\end{pmatrix} = 0,$$

with $C_V = C \Pi = CVT$. Explicitly, in terms of $G$, we have:

$$(3.34) \min_G f(G) = \text{Trace}
\begin{pmatrix}
B^T \\
G
\end{pmatrix}
\mathcal{M}(G)
\begin{pmatrix}
B \\
G
\end{pmatrix}
\text{s.t. : } \sigma(S - GL) \subset \mathbb{C}^- \text{ and } (3.35),$$
where \( M(G) \) is the solution of the Lyapunov equation:

\[
\begin{bmatrix}
A & 0 \\
0 & S - GL
\end{bmatrix}^T M(G) + M(G) \begin{bmatrix}
A & 0 \\
0 & S - GL
\end{bmatrix} + \begin{bmatrix}
C^T C & -C^T C_V \\
-C_V^T C & C_V^T C_V
\end{bmatrix} = 0.
\]

We now compute the expression of the gradient of the objective function of (3.34). Using again that \( \text{Trace}(MN) = \text{Trace}(NM) \), the non-convex objective function of (3.34) becomes:

\[
f(G) = \text{Trace} \left( \begin{bmatrix} B \\ G \end{bmatrix}^T M(G) \begin{bmatrix} B \\ G \end{bmatrix} \right) = \text{Trace} \left( M(G) \begin{bmatrix} B \\ G \end{bmatrix} \begin{bmatrix} B & [B]^T \\ G & [G]^T \end{bmatrix} \right) = \text{Trace} (M(G)B(G)).
\]

**Theorem 3.9.** The objective function \( f \) of (3.34) is differentiable on the set of stable matrices \( D_{(SL)} \) and the gradient of \( f \) at \( G \in D_{(SL)} \) is given by:

\[
\nabla f(G) = 2 \left[ -M_{12}^T(G)W_{12}(G)L^T - M_{22}(G)W_{22}(G)L^T + M_{12}^T(B) + M_{22}(G) \right],
\]

where \( M(G) \) solves the Lyapunov equation (3.35) and \( W(G) \) solves the Lyapunov equation:

\[
\begin{bmatrix}
A & 0 \\
0 & S - GL
\end{bmatrix} W(G) + W(G) \begin{bmatrix}
A & 0 \\
0 & S - GL
\end{bmatrix}^T + B(G) = 0.
\]

**Proof.** To computing the expression of the gradient \( \nabla f(G) \in \mathbb{R}^{n \times m} \) we write the derivative \( f'(G) \) \( dG \) for some \( dG \in \mathbb{R}^{n \times m} \) in gradient form using the trace. We introduce the gradient as:

\[
f'(G) \, dG = \text{Trace} \left( \nabla f(G)^T \, dG \right).
\]

From the expression of \( f(G) \) we have:

\[
f'(G) \, dG = \text{Trace} (M'(G)B(G) + M(G)B'(G)).
\]

We compute separately the two terms in the above expression. Let

\[
\Phi(G, M) = \begin{bmatrix}
A & 0 \\
0 & S - GL
\end{bmatrix}^T M + M \begin{bmatrix}
A & 0 \\
0 & S - GL
\end{bmatrix}.
\]

Since \( G \in D_{(SL)} \) and \( D_{(SL)} \) is an open set, then by Lemma 3.1 we have that \( \Phi_M(G, M) \, dM \) given by:

\[
\Phi_M(G, M) \, dM = \begin{bmatrix}
A & 0 \\
0 & S - GL
\end{bmatrix}^T \, dM + dM \begin{bmatrix}
A & 0 \\
0 & S - GL
\end{bmatrix}
\]

is surjective and also we have:

\[
\Phi_G(G, M) \, dG = \begin{bmatrix}
0 & 0 \\
0 & -dGL
\end{bmatrix}^T M + M \begin{bmatrix}
0 & 0 \\
0 & -dGL
\end{bmatrix}.
\]

Since \( \Phi(G, M) + C = 0 \), the Implicit Function Theorem yields the differentiability of \( M(G) \) and the following relation:

\[
\begin{bmatrix}
A & 0 \\
0 & S - GL
\end{bmatrix}^T M'(G) + M'(G) \begin{bmatrix}
A & 0 \\
0 & S - GL
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
0 & -dGL
\end{bmatrix}^T M(G) + M(G) \begin{bmatrix}
0 & 0 \\
0 & -dGL
\end{bmatrix} = 0.
\]

(3.38)
Consider the Lyapunov equation (3.2a) with the unique solution $W(G)$ provided $G$ is stabilizable:

$$
(3.39) \begin{bmatrix} A & 0 \\ 0 & S - GL \end{bmatrix} W(G) + W(G) \begin{bmatrix} A & 0 \\ 0 & S - GL \end{bmatrix}^T + B(G) = 0.
$$

Subtracting (3.38) multiplied by $W(G)$ to the left from (3.39) multiplied by $M'(G)$ to the right, taking the trace, and reducing the appropriate terms, yields:

$$
\text{Trace} (M'(G)B(G)) = \text{Trace} \left( W(G) \begin{bmatrix} 0 & 0 \\ 0 & dGL \end{bmatrix}^T M(G) + M(G) \begin{bmatrix} 0 & 0 \\ 0 & -dGL \end{bmatrix} W(G) \right)
$$

(3.40)

$$
= 2\text{Trace} (-LW_{12}^T(G)M_{12}(G)dG - LW_{22}(G)M_{22}(G)dG).
$$

Similarly, for the second term using the block structure of $M$ and the definition of trace, we get:

$$
\text{Trace} (M(G)B'(G)) = \text{Trace} \left( M(G) \begin{bmatrix} 0 & B \text{d}G^T \\ dGB^T & dGG^T + G \text{d}G^T \end{bmatrix} \right)
$$

(3.41)

$$
= 2\text{Trace} (B^T M_{12}(G) dG + G^T M_{22}(G) dG).
$$

Hence, from (3.40) and (3.41) we get the closed form expression for the gradient (3.36).

Note that the expression of the gradient $\nabla f$ from (3.36) is the same as the partial gradient of the Lagrangian $\nabla \mathcal{L}$ from (3.32). The previous result also yields the necessary optimality condition for the model reduction Problem 2 expressed in terms of the optimization problem (3.34):

**Lemma 3.10.** If $G \in \mathcal{D}_{(SL)}$ solves the optimization problem (3.34) corresponding to the model reduction Problem 2, then

$$
M_{12}(G)W_{12}(G)L^T + M_{22}(G)W_{22}(G)L^T = M_{12}(G)B + M_{22}(G)G
$$

where $\mathcal{M}(G)$ solves the Lyapunov equation (3.35) and $W(G)$ solves the Lyapunov equation (3.37).

We can replace the open set $\mathcal{D}_{(SL)}$ with any sublevel set:

$$
\mathcal{N}^{G_0}_{(SL)} = \{ G \in \mathcal{D}_{(SL)} : f(G) \leq f(G_0) \},
$$

where $G_0 \in \mathcal{D}_{(SL)}$ is any initial stable reduced order system matrix. Using similar arguments as in [35] we can show that $\mathcal{N}^{G_0}_{(SL)}$ is a compact set. Then, the theorem of Weierstrass implies that for any given matrix $G_0 \in \mathcal{D}_{(SL)}$, the model reduction Problem 2 given by optimization formulation (3.29) has a global minimum in the sublevel set $\mathcal{N}^{G_0}_{(SL)}$. We can also show that the gradient $\nabla f(G)$ is Lipschitz continuous on the compact sublevel set $\mathcal{N}^{G_0}_{(SL)}$. Let us briefly sketch the proof of this statement. First we observe that $\mathcal{M}(G)$ and $W(G)$ are continuous functions and moreover there exists finite $\ell_M > 0$ such that:

$$
\| \mathcal{M}(G) - \mathcal{M}(\bar{G}) \| \leq \ell_M \| G - \bar{G} \| \quad \forall G, \bar{G} \in \mathcal{N}^{G_0}_{(SL)}.
$$

Then, using the expression of $\nabla f(G)$, compactness of $\mathcal{N}^{G_0}_{(SL)}$, continuity of $\mathcal{M}(G)$ and $W(G)$, and the previous relation we conclude that there exists $\ell_f > 0$ such that:

$$
\| \nabla f(G) - \nabla f(\bar{G}) \| \leq \ell_f \| G - \bar{G} \| \quad \forall G, \bar{G} \in \mathcal{N}^{G_0}_{(SL)}.
$$

This property of the gradient is useful when analyzing the convergence behavior of the algorithm we propose for solving the optimization problem (3.34).
3.3.3. SDP approach. Alternatively, problem (3.29) can be written equivalently in terms of matrix inequalities (semidefinite programming):

\[
\min_{(G,M)} \text{Trace} \left( \begin{bmatrix} B \\ G \end{bmatrix}^T \mathcal{M} \begin{bmatrix} B \\ G \end{bmatrix} \right)
\]

s.t. : \( \mathcal{M} \succeq 0 \) and \( \begin{bmatrix} A \\ 0 \\ S - GL \end{bmatrix}^T \mathcal{M} + \mathcal{M} \begin{bmatrix} A \\ 0 \\ S - GL \end{bmatrix} + \begin{bmatrix} C \mathcal{C}^T \\ \mathcal{C}^T \mathcal{C} \end{bmatrix} \preceq 0 \),

where recall that the pair \((S, L)\) is fixed a priori. Clearly, SDP problem (3.42) is not convex since it contains bilinear matrix inequalities. However, next theorem proves that we can obtain a suboptimal solution through convex relaxation.

**Theorem 3.11.** If the following convex SDP relaxation:

\[
\min_{(G,X_{22},Y_{22},Z_{22})} \text{Trace} \left( B^T M_{11} B + X_{22} \right)
\]

s.t. : \( S^T M_{22} - L^T Z_{22}^T + M_{22} S - Z_{22} L + C Y_{22} \preceq Y_{22} \)

\[
\begin{bmatrix} X_{22} \\ Z_{22} \\ M_{22} \end{bmatrix} \succeq 0, \begin{bmatrix} A^T M_{11} + M_{11} A^T + C^T C \\ B^T M_{12} + M_{12} (S - GL) - C^T C \end{bmatrix} \succeq 0
\]

has a solution, then we can recover a suboptimal solution of the model reduction Problem 2 expressed in terms of the SDP problem (3.29) through the relations:

\( G = M_{22}^{-1} Z_{22} \) and \( \mathcal{M} = \text{diag}(M_{11}, M_{22}) \).

**Proof.** Using the block form of \( \mathcal{M} \) we get the equivalent problem:

\[
\min_{G,M} \text{Trace} \left( B^T M_{11} B + B^T M_{12} G + G^T M_{12}^T B + G^T M_{22} G \right)
\]

s.t. : \( A^T M_{11} + M_{11} A^T + C^T C \)

\[
\begin{bmatrix} A^T M_{12} + M_{12} (S - GL) - C^T C \\ (S - GL)^T M_{12} + M_{12} A - C^T Y_{22} \end{bmatrix} \succeq 0.
\]

If we introduce additional variables we can reformulate the previous problem as an SDP subject to bilinear matrix inequalities. Indeed, we have the equivalent formulation:

\[
\min_{(G,X_{22},Y_{22})} \text{Trace} \left( B^T M_{11} B + X_{22} \right)
\]

s.t. : \( X_{22} \succeq B^T M_{12} G + G^T M_{12}^T B + G^T M_{22} G \)

\[
\begin{bmatrix} A^T M_{11} + M_{11} A^T + C^T C \\ (S - GL)^T M_{12} + M_{12} A - C^T Y_{22} \end{bmatrix} \succeq 0,
\]

and using now Schur complement we arrive at an SDP with convex objective function but with non-convex constraints of type BMIs:

\[
\min_{(G,X_{22},Y_{22})} \text{Trace} \left( B^T M_{11} B + X_{22} \right)
\]

s.t. : \( (S - GL)^T M_{22} + M_{22} (S - GL) + C^T C \succeq Y_{22} \)

\[
\begin{bmatrix} X_{22} - B^T M_{12} G - G^T M_{12} B - G^T M_{22} G \\ M_{22} G \end{bmatrix} \succeq 0
\]

\[
\begin{bmatrix} A^T M_{11} + M_{11} A^T + C^T C \\ (S - GL)^T M_{12} + M_{12} A - C^T C \end{bmatrix} \succeq 0
\]
Problem (3.46) is not convex since it contains bilinear matrix terms. However, if we assume that the block $M_{12} = 0$, then problem (3.46) can be recast as a convex SDP. That is, for $M_{12} = 0$ from (3.46) we get:

\begin{equation}
\begin{aligned}
\min_{(G,X_{22},Y_{22}), M_{11} \succeq 0, M_{22} \succeq 0} & \quad \text{Trace} \left( B^T M_{11} B + X_{22} \right) \\
\text{s.t.} : & \quad X_{22} \succeq G^T M_{22} G, \quad (S - GL)^T M_{22} + M_{22}(S - GL) + C_v^T C_v \preceq Y_{22} \\
& \quad \begin{bmatrix} A^T M_{11} + M_{11} A^T + C^T C & -C^T C_v \\ -C_v^T C & Y_{22} \end{bmatrix} \preceq 0.
\end{aligned}
\end{equation}

Denoting by $Z_{22} = M_{22} G$ and using the Schur complement, problem (3.47) becomes the convex SDP (3.43). Moreover, if (3.43) has a solution, then we can recover $G = M_{22}^{-1} Z_{22}$ and $M = \text{diag}(M_{11}, M_{22})$. Note also that the solution $(G, M)$ of this convex SDP problem is a suboptimal solution of the original problem (3.29) since we restrict $M_{12} = 0$.

All the numerical optimization algorithms presented in Section 3.2 can be applied to also solve the three optimization problems from this section corresponding to the relaxed Problem 2. Thus we omit these details here and refer to Section 3.2.

4. Illustrative examples. In this section, we illustrate the efficiency of our results numerically on examples such as a double-pendulum [11] or a CD player [21]. In particular, we compute and compare reduced order models for these test systems achieving (possibly) the minimum $H_2$-norm.

4.1. Cart with a double pendulum controller. Consider the following cart system with a double-pendulum controller, depicted in Figure 4.1, see also [11, 23].

![Cart system with a double-pendulum controller](image)

Defining the state as $x = [q_1, \dot{q}_1, q_2, \dot{q}_2, q_3, \dot{q}_3]^T \in \mathbb{R}^6$ and selecting the output as $y = x_1$, we obtain a 6th order system described by equations of the form (2.1), with:

\begin{equation}
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-1 & -1 & 98/5 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & -196/5 & -2 & 49/5 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 98/5 & 1 & -98/5 & -2
\end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\end{equation}

Fix the observable pair $(S, L)$:

$S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $L = [1 \ 1]$.

A family of second order models that match the first two moments at zero of (4.1) is described by $\Sigma_G$ as in (2.8) with

\begin{equation}
F = \begin{bmatrix} -g_1 & 1 \\ -g_2 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}, \quad H = [1 \ -1],
\end{equation}
where \( g_1, g_2 \in \mathbb{C} \) are the free parameters. Selecting e.g. \( g_1 = 1 \) and \( g_2 = 0.5 \) yields a model (4.2) with

\[
F = \begin{bmatrix}
-1 & 1 \\
-0.5 & 0
\end{bmatrix}, \quad G = \begin{bmatrix}
1 \\
0.5
\end{bmatrix}, \quad H = [1 \ -1],
\]

which is stable, but not a very accurate approximation error as revealed by the first row in Table I. By [22], the unique second order model that matches the first two moments of \( K(s) \) and \( K'(s) \) is

\[
F = \begin{bmatrix}
-1 & 0.999 \\
0.999 & 2
\end{bmatrix}, \quad G = \begin{bmatrix}
0.333 \\
0.333
\end{bmatrix}, \quad H = [1 \ -1].
\]

This model is stable, although not apriorically guaranteed. Furthermore, this model exhibits a significant decrease in the \( H_2 \)-norm of the error (see the second row of Table 4.1), since matching the derivative of the transfer function is a necessary first-order optimality condition. However, since the interpolation points are fixed, there is no optimality here, whatsoever. On the other hand, using the gradient-based solution for Problem 2 for solving either the KKT system or the partial minimization problem, we compute the optimal model (4.2) in the family (4.2), with

\[
F = \begin{bmatrix}
-0.2505 & 1.0000 \\
-0.1500 & 0
\end{bmatrix}, \quad G = \begin{bmatrix}
0.2505 \\
0.1500
\end{bmatrix}, \quad H = [1 \ -1],
\]

that matches the first two moments at zero and yields a minimal \( H_2 \)-norm of the approximation error in (4.2), see the third row in Table 4.1. Although it is difficult to prove (due to non-convexity of the optimization problem), in principle the “optimal” reduced model is the unique model that minimizes the approximation error in the family (4.2).

Finally, employing the gradient-based solution for Problem 1 for solving either the KKT system or the partial minimization problem corresponding to it, yields the optimal second order moment matching-based stable reduced order model. The matrix \( S \) is not fixed, but the algorithm has been initialized with a diagonal matrix having subunitary positive eigenvalues generated at random. The optimal model is obtained interpolating at \( 0.0109 \pm 0.0946 j \), which are the eigenvalues of the optimal matrix

\[
S = \begin{bmatrix}
0.0113 & 0.9953 \\
-0.0090 & 0.0105
\end{bmatrix}
\]

given by the gradient algorithm. The optimal approximation is:

\[
F = \begin{bmatrix}
-0.3354 & 0.6486 \\
-0.3250 & -0.3060
\end{bmatrix}, \quad G = \begin{bmatrix}
0.3467 \\
0.3160
\end{bmatrix}, \quad H = [1.0049 \ -1.0832].
\]

The reduced order model is stable with \( \sigma(F) = -0.1624 \pm 0.5422 \). Note that the matrix \( S \) is unstable and since \( A \) is stable, \( \sigma(S) \cap \sigma(A) = \emptyset \) is satisfied. Furthermore, the resulting \( G \) is such that \( \sigma(S) \cap \sigma(S-GL) = \emptyset \) is also verified.

| Second order model \( \Sigma_G \), \( G = [g_1 \ g_2]^T \) | \( H_2 \)-norm of approx. error |
|----------------------------------------------------------|-----------------------------|
| \( \Sigma_G, g_1 = 1, \ g_2 = 0.5 \), matching 2 mom. at 0 | \( 13.91 \cdot 10^{-1} \) |
| \( \Sigma_G, g_1 = 0.333, \ g_2 = 0.333 \) matching 2 mom. at 0 of \( K \) and \( K' \) | \( 1.86 \cdot 10^{-1} \) |
| \( \Sigma_G \), by Problem 2, \( g_1 = 0.2507, \ g_2 = 0.15 \) matching 2 mom. at 0 | \( 1.474 \cdot 10^{-1} \) |
| \( \Sigma_G \) by Problem 1, \( g_1 = 0.3467, \ g_2 = 0.3160 \) matching 2 mom. at 0 \( 0.0109 \pm 0.0946 j \) yielding minimal \( H_2 \)-norm | \( 0.6232 \cdot 10^{-4} \) |

**Table 4.1.** \( H_2 \)-norms of the approximation errors for different scenarios.
4.2. CD player. In the second test we perform model reduction on the CD player system, which has 120 states, i.e., $n = 120$, with a single input and a single output, see, e.g., [2, 21]. We obtain the optimal model through the solution to Problem 2 based on the gradient iteration at orders $\nu = 1 : 10$. In Figure 4.2, we plot the $H_2$-norm of the approximation error versus the reduced order index. We compute the solution to Problem 2 yielding the $H_2$-norm of the approximation errors in families of reduced order models that achieve moment matching at a set of $\nu$ moments at $\nu$ fixed interpolation points. The interpolation points have been chosen at low frequencies and dense, e.g., 0, 0.2, 0.4, 0.6, ..., as well as rare, e.g., 0, 2, 4, 6, ... Note that interpolating at zero ensures preservation of DC-gain of the step response of the system. Figure 4.2 also shows that a rare choice of the interpolation points yields better optimal approximations.

![Image](image-url)

**Fig. 4.2.** $H_2$-norm of the error versus $\nu$ for CD player.

5. Conclusions. In this paper we have formulated several optimization problems with respect to $H_2$-norm minimal error approximation in a family of reduced order models that match a prescribed set of fixed $\nu$ moments. For these optimization problems we have derived first-order optimality conditions and numerical solutions has been proposed in terms of the gradient method or SDP. Using test examples from model reduction literature, such as a cart controlled by a double-pendulum or a CD player, we have also verified the efficiency of our results numerically.

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