Enumerable Distributions, Randomness, Dependence

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Abstract

Mutual information $I$ in infinite sequences (and in their finite prefixes) is essential in theoretical analysis of many situations. Yet its right definition has been elusive for a long time. I address it by generalizing Kolmogorov Complexity theory from measures to semimeasures i.e., infimums of sets of measures. Being concave rather than linear functionals, semimeasures are quite delicate to handle. Yet, they adequately grasp various theoretical and practical scenario.

A simple lower bound $i(\alpha : \beta) = \sup_{x \in \mathbb{N}}(K(x) - K(x|\alpha) - K(x|\beta))$ of information turns out tight for Martin-Löf random $\alpha, \beta \in \{0, 1\}^\mathbb{N}$. For all sequences $I(\alpha : \beta)$ is characterized by the minimum of $i(\alpha' : \beta')$ over random $\alpha', \beta'$ with $U(\alpha')=\alpha, U(\beta')=\beta$.

1 Introduction

Kolmogorov Information theory applies to individual objects, in contrast to Shannon theories that apply to the models of processes that generated such objects. It thus has a much wider domain since many objects (e.g., Shakespeare plays) have no realistic generation models. For completed objects, such as integers, the concept is simple and robust: $I(x : y) = K(x) + K(y) - K(x, y)$.

Yet, the concept is also needed for emerging objects, such as, e.g., prefixes of infinite sequences. Encoding prefixes as integers distorts the information by specifying their (arbitrary) cut-off point. This cut-off information is not a part of the original sequence and can be smaller in a longer prefix. In fact, this distortion can overwhelm the actual mutual information between the sequences.

This issue complicates many studies forcing one to use (as, e.g., in [L 13]) concepts of information that are merely lower bounds, differ between applications, and known not to be tight.

For the related concept of rarity (randomness deficiency) Per Martin-Löf proposed an extension that works well for infinite sequences under computable distributions. Yet, computability of distributions requires a running time limit for the processes generating them. Such limits then must be accounted for in all formulas, obscuring the simplicity of purely informational values, at a great cost to elegance and transparency. Without such limit many important distributions are only lower-enumerable (r.e.). For instance, universal probability $M$ is the largest within a constant factor r.e. distribution. It is extraordinarily flat: all sequences are random with respect to it.

Yet $M$ is instrumental in defining other interesting distributions. In particular, Mutual Information in two sequences is their dependence, i.e., rarity with respect to the distribution $M \otimes M$ generating them independently with universal probability each. R.e. distributions are of necessity semimeasures: concave rather than linear functionals. Semimeasures also are relevant in more mundane and widespread situations where the specific probability distribution is not fully known (e.g., due to interaction with a party that cannot be modeled). They require much more delicate handling than measures. This article considers many subtleties that arise in such generalization of complexity theory. The concept of rarity for such distribution considered here respects randomness conservation inequalities and is the strongest (i.e., largest) possible such definition. The definition of mutual information arising from this concept is shown to allow rather simple descriptions.

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1Measureless in this world of measures. – Marina Tsvetaeva
2 Conventions and Background

Let \( \mathbb{R}, \mathbb{Q}, \mathbb{N}, \mathbb{S} = \{0, 1\}^* \) be, respectively, the sets of reals, rationals, integers, finite, and infinite binary sequences; \( x_{[n]} \) is the \( n \)-bit prefix and \(|x|\) is the bit-length of \( x \in \mathbb{S} \); for \( a \in \mathbb{R}^+ \), \(|a| \equiv \lceil \log a \rceil - 1 \). A function \( f \) and its values are \textit{enumerable} or \textit{r.e.} (\( \neg f \) is \textit{co-r.e.}) if its subgraph \( \{(x, t) : t < f(x)\} \) is r.e., i.e. a union of an r.e. set of open balls. \( X^+ \) means \( X \cap \{x \geq 0\} \). \textbf{Elementary} (\( f \in \mathcal{E} \)) are functions \( f : \Omega \rightarrow \mathbb{Q} \) depending on a finite number of digits; \( 1 \in \mathcal{E} \) is their unity: \( 1(\alpha) = 1 \). \( \overline{\mathcal{E}} \) is the set of all supremaums of subsets of \( E \). \( f^+ \) for \( f : \Omega \rightarrow \mathbb{R} \), denotes \( \sup\{g : f > g \in \mathcal{E}\} \).

\textbf{Majorant} is an r.e. function largest, up to constant factors, among r.e. functions in its class. \(<f, >f, \times f, \text{and} \lesssim f, \gtrsim f, \sim f \) denote \( f + O(1) \), \( f - O(1) \), \( f \pm O(1) \), and \( \lesssim f + O(\|f + 1\|) \), respectively. \([A] \equiv 1 \) if statement \( A \) holds, else \([A] \equiv 0 \).

When unambiguous, I identify objects in clear correspondence: e.g., prefixes with their codes or their sets of extensions, sets with their characteristic functions, etc.

2.1 Integers: Complexity, Randomness, Rarity

Let us define Kolmogorov complexity \( K(x) \) as \( \|m(x)\| \) where \( m : \mathbb{N} \rightarrow \mathbb{R} \) is the universal distribution, i.e., a majorant r.e. function with \( \sum_x m(x) \leq 1 \). It was introduced in \([ZL 70]\), and noted in \([Gacs 74]\) to be a modification of the least length of binary programs for \( x \) defined in \([Kolmogorov 65]\). The modification restricts the domain \( D \) of the universal algorithm \( u \) to be prefixless. While technically different, \( m \) relies on intuition similar to that of \([Solomonoff 64]\). The proof of the existence of a majorant function was a direct modification of \([Solomonoff 64]\) \([Kolmogorov 65]\) proofs which have been a keystone of the informational complexity theory.

For \( x \in \mathbb{N}, y \in \mathbb{N} \) or \( y \in \Omega \), similarly, \( m(\cdot) \) is a majorant r.e. real function with \( \sum_x m(x|y) \leq 1 \); \( K(x|y) \equiv \|m(x|y)\| (= \text{the least length of prefixless programs transforming } y \text{ into } x) \).

\([Kolmogorov 65]\) considers \textbf{rarity} \( d(x) \equiv \|x| - K(x) \) of uniformly distributed \( x \in \{0, 1\}^n \). Our modified \( K \) allows extending this to other measures \( \mu \) on \( \mathbb{N} \). A \( \mu \)-test is \( f : \mathbb{N} \rightarrow \mathbb{R} \) with mean \( \mu(f) \leq 1 \) (and, thus, small values \( f(x) \) on randomly chosen \( x \)). For computable \( \mu \), a majorant r.e. test is \( t(x) \equiv m(x)/\mu(x) \). This suggests defining \( d_\mu(x) \) as \( \|t(x)\| \times \|\mu(x)\| - K(x) \).

2.2 Integers: Information

In particular, \( x = (a, b) \) distributed with \( \mu = m \otimes m \), is a pair of two independent, but otherwise completely generic, finite objects. Then, \( I(a : b) \equiv d_m \otimes m((a, b)) \times K(a) + K(b) - K(a, b) \) measures their \textit{dependence} or \textit{mutual information}. It was shown (see \([ZL 70]\)) by Kolmogorov and Levin to be close (within \( \pm O(\log K(a, b)) \)) to the expression \( K(a) - K(a|b) \) of \([Kolmogorov 65]\). Unlike this earlier expression (see \([Gacs 74]\)), our \( I \) is symmetric and monotone: \( I(a : b) \leq I((a, x) : b) \) (which will allow extending \( I \) to \( \Omega \)); it equals \( \times K(a) - K(a|b) \), where by \( b \) we will denote \( (b, K(b)) \). (The \( I_\mu \) variation of \( I \) with all algorithms accessing oracle \( z \), works similarly.)

\( I \) satisfies the following Independence Conservation Inequalities \([L 74]; L 84] \):

For any computable transformation \( A \) and measure \( \mu \), and some family \( t_{a, b} \) of \( \mu \)-tests

\[
(1) \quad I(A(a : b) \leq I(a : b);
(2) \quad I((a, w) : b) \leq I(a : b) + \log t_{a, b}(w).
\]

(The \( O(1) \) error terms reflect the constant complexities of \( A, \mu \).) So, independence of \( a \) from \( b \) is preserved in random processes, in deterministic computations, their combinations, etc. These inequalities are not obvious (and false for the original 1965 expression \( I(a : b) = K(a) - K(a|b) \)) even with \( A \), say, simply cutting off half of \( a \). An unexpected aspect of \( I \) is that \( x \) contains all information about \( k = K(x) \), \( I(x : k) \asymp K(k) \), despite \( K(k|x) \) being \( \sim \|k\| \), or \( \sim \log \|x\| \) in the worst case \([Gacs 74]\). One can view this as an "Occam Razor" effect: with no initial information about it, \( x \) is as hard to obtain as its simplest (\( k \)-bit) description.
2.3 Reals: Measures and Rarity

A measure on $\Omega$ is a function $\mu(x) = \mu(x_0) + \mu(x_1)$, for $x \in S$. Its mean $\mu(f)$ is a functional on $\mathcal{E}$, linear: $\mu(cf + g) = c\mu(f) + \mu(g)$ and normal: $\mu(\pm 1) = \pm 1, \mu(|f|) \geq 0$. It extends to other functions, as usual. An example is $\lambda(x \Omega) \overset{2}{=} 2^{-|x|}$ (or $\lambda(x)$ for short). I use $\mu(\alpha)(A)$ to treat the expression $A$ as a function of $\alpha$, taking other variables as parameters.

$\mu$-tests are functions $f \in \bar{\mathcal{E}}$, $\mu(f) \leq 1$; computable $\mu$ have universal (i.e., majorant r.e.) tests $T^\mu_\alpha = \sum_i m(\alpha[i])/\mu(\alpha[i]),$ called Martin-Löf tests. Indeed, let $t$ be an r.e. $\mu$-test, and $S_k$ be an r.e. family of prefixless subsets of $S$ such that $\cup_{x \in S_k} x \Omega = \{\alpha : t(\alpha) > 2^{k+1}\}$. Then $t(\alpha) = \Theta(\sum_{k,x \in S_k}(2^k[\alpha \in x\Omega])) = \Theta(\sup_{k,x \in S_k}(2^k[\alpha \in x\Omega])).$ Now, $\sum_{k,x \in S_k}(2^k \mu(x)) < \mu(t) \leq 1$, so $2^k \mu(x) = O(\mu(x))$ for $x \in S_k$ and $t(\alpha) = O(\sup_{k,x \in S_k}([\alpha \in x\Omega]\mu(x)) = O(\sup_{k}(\mu(\alpha[i]) / \mu(\alpha[i])))$.

Martin-Löf random are $\alpha$ with finite rarity $d_\mu(\alpha) = \inf \|T^\mu_\alpha\| > \inf \|\mu(\alpha[i])\| - K(\alpha[i])$ and we also use $d_\mu(\alpha|x) \overset{2}{=} \sup_i(\|\mu(\alpha[i])\| - K(\alpha[i]|x))$.

Continuous transformations $A : \Omega \to \Omega$ induce normal linear operators $A^* : f \to g$ over $\mathcal{E}$, where $g(\omega) = f(A(\omega))$. So obtained, $A^*$ are deterministic: $A^*(\min\{f, f'\}) = \min\{A^*(f), A^*(f')\}$. Operators that are not, correspond to probabilistic transformations (their inclusion is the benefit of the dual representation), and $g(\omega)$ is then the expected value of $f(A(\omega))$. Such $A$ also induce $A^*$ transforming input distributions $\mu$ to output distributions $\varphi = A^*(\mu) : \varphi(f) = \mu(A^*(f))$. I treat $A, A^*, A^*$ as one function $A$ acting as $A^*$, or $A^*$ on the respective (disjoint) domains. Same for partial transformations below and their concave duals. I also identify $\omega \in \Omega$ with measures $f \to f(\omega)$.

3 Partial Operators, Semimeasures, Complexity of Prefixes

Not all algorithms are total: narrowing down the output to a single sequence may go slowly and fail (due to divergence or missing information in the input), leaving a compact set of eligible results:

**Definition 1.** 1. Partial continuous transformations (PCT) are compact subsets $A \subset \Omega \times \Omega$ with $A(\alpha) = \{\beta : (\alpha, \beta) \in A\} \neq \emptyset$. When not confusing I identify singletons $\{\beta\}$ with $\beta \in \Omega$.

Computable PCT are r.e., i.e., enumerate the open complement of $A$;

2. a PCT $A$ is clopen if co-images $A^{-1}(s) = \{\alpha : A(\alpha) \subset s\}$ of all clopen $s \subset \Omega$ are clopen.

$A$ is t-clopen if $A^{-1}(x\Omega)$ depend only on $\alpha[t(x\Omega)]$ for some $i$.

3. Dual of PCT $A$ is the operator $A^* : \mathcal{E} \to \bar{\mathcal{E}}$, where $A^*(f) = g : \alpha \to \min_{\beta \in A(\alpha)} f(\beta)$.

An important example is a universal algorithm $U$. It enumerates all algorithms $A_i$ with a prefixless set $P$ of indexes $i$ and sets $(i\alpha, \beta) \in U$ iff $(\alpha, \beta) \in A_i, i \in P$.

**Remark 1.** Composing PCT with linear operators produces normal concave operators, all of them by Hahn–Banach theorem. Indeed, each such $C(f)$ is a composition $A(R(B(f)))$: Here a PCT $A(\alpha)$ relates each $\alpha$ to the binary encodings $\{\mu\}$ of measures $\mu \geq C(\alpha)$; $R$ transforms $\{\mu\}$ into a distribution $\{\mu\} \otimes \lambda$; and $B(\{\mu\}, \beta)$ relates $\lambda$-distributed $\beta$ to $\mu$-distributed $\gamma$ with $\mu[0, \gamma] \leq \beta \leq \mu[0, \gamma]$.

Normal concave operators transform measures into semimeasures:

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The condition $\mu(T_\mu) \leq 1$, slightly stronger (in log scale) than the original one of Martin-Löf 66, was required in [L 76] in order to satisfy conservation of randomness. Both types of tests diverge simultaneously. [Schnorr 73] (for divergence of $T_\chi$), [L 73], [Gács 80] characterized the tests in complexity terms.
Definition 2. 1. A semimeasure $\mu : \mathcal{E} \to \mathbb{R}$ is a normal ($\mu(\pm 1) = 1$, $\mu(f) > 0$) functional that is concave: $\mu(cf + g) \geq c\mu(f) + \mu(g)$, $c \in \mathbb{R}^+$, e.g., $\mu(x) \geq \mu(x_0) + \mu(x_1)$, for $x \in S$. $\mu$ extends to $f \in \mathcal{E}$ as $\inf\{\mu(g) : f \leq g \in \mathcal{E}\}$, and to other functions as $\sup\{\mu(g) : f \geq g \in \mathcal{E}\}$, as usual for inner measures. $\mu$ is deterministic if $\mu(\min\{f, g\}) = \min\{\mu(f), \mu(g)\}$.

2. Normal $(A(\pm 1) = 1, A(|f|) > 0)$ concave operators $A : \mathcal{E} \to \mathcal{E}$ transform input points $\alpha$ and distributions $\varphi$ (measures or semimeasures) into their output distributions $A(\varphi) : f \mapsto \varphi(A(f))$. Operators $A$ are deterministic if semimeasures $A(\alpha)$ are.

Regular are semimeasures $A(\lambda)$ for deterministic r.e. $A$; t-regular for a t-clopen $A$.

Proposition 1. 1. Each deterministic $\mu$ is $\mu(f) = \min_{\omega \in S} f(\omega)$ for some compact $S \subset \Omega$.

2. Dual of PCT are those and only those operators that are normal, concave, and deterministic.

3. Each $f \in \mathcal{E}$ has a unique form $f = \sum r_i f_i$ with distinct boolean $f_i \geq f_{i+1}$, $f_0 = 0$, $r_i > 0$ for $i > 0$.

Then $\mu(f) = \sum r_i \mu(f_i)$, $\mu = \mu$ if $\mu$ is regular. All r.e. measures are regular.

4. Each r.e. semimeasure $\mu$ has a regular r.e. $\mu' \leq \mu$ with $\mu'(x) = \mu(x)$ for all $x \in S$.

$\mu'$ is t-regular for a computable $t$ if $\mu(x)$ have $< t(x)$ bits.

Proof. Note, $p(\beta) \equiv \inf_{g, \mu(g) > 1} |g(\beta)| \in \{0, 1\}$. Indeed, if $\mu(f) - f(\beta) = t > 0$ and $g = (f - f(\beta))t$ then $g(\beta) = 0$, $\mu(g) > 1$. Then $S \equiv \{\beta : p(\beta) = 1\}$. $\mu = A(\alpha)$ is deterministic, so $\mu(f) = \min_{\beta \in S} f(\beta)$. $\mu$ is since regular $\mu$ are averages of deterministic ones. $\mu$ is by Theorem 3.2 of [ZL 70].

3.1: Complexity: General Case

Proposition 2. There exists a universal, i.e., majorant (on $\mathcal{E}^+$) r.e. semimeasure $M$. The values $M(x)$ can have $K(x)$ bits. (Thus t-clopen PCT can generate $M$ for any computable $t(x) > K(x)$).

Proof. For an r.e. family $\mu_i$ of all r.e. semimeasures, take $M(x) = \sum_i \mu_i(x)/2^i$. $M(x)$ can be rounded-up to $K(x)$ bits after adding $\sum_{y \not= \{\}} m(xy)$ (to keep $M(x) \geq M(x_0) + M(x_1)$).

As in [ZL 70], $K(x) \equiv ||M(x)||$. Same for $M_\alpha$, r.e. w.r.t. $\alpha$ and $M(\alpha) \equiv ||M_\alpha(x)||$.

$K(x|y)$, $K(x)$ are examples of the many types of complexity measures on $S$.

L.6.3 gives the general construction of Kolmogorov-like complexities $K_v$. I summarize it here.

$K_v$ are associated with classes $v$ of functions $m: \mathcal{S} \to (0, 1]$, in linear scale, and their logarithmic scale projections $\bar{v} = \{K = ||m|| : m \in v\}$. Thus, $K(x|y)$ is $K_v$ for $v = \{m : \sup_y \sum_x m(x|y) \leq 1\}$.

These $v$ are closed-down, weakly compact, and decidable on tables with finite support. $\bar{v}$ will have a minimal, up to $\propto$, co-r.e. function $K_v$. This justifies the logarithmic scale where the values of $K_v$ are well defined up to $O(1)$ adjacent integers. (Though linear scale is often clearer, analytically.)

$K_v$ minimality requires $\min\{K', K''\} + O(1) \in \bar{v}$ for any $K', K''$ in $\bar{v}$. In the linear scale of $m$ this comes to $(m' + m'')/c \in v$ for some $c = O(1)$. I tightened this to convexity with $c = 2$; this changes $K$ in $\bar{v}$ by just $\Theta(1)$ factors: a matter of choosing bits as units of complexity.

Similarly to Proposition 2 this condition suffices for $\bar{v}$ to have a minimal, up to $\propto$, co-r.e. $K_v$. Each such $K_v < \infty$ has a computable lower bound $B_v(x) = \min_{K \in \bar{v}} K(x)$, largest up to $\infty$, among r.e. bounds. And $K_v - B_v$, too, is such a $K_v$: I call $v'$ normal as $B_v = 0$. Let $E_1 = \mathcal{E}^+ \cap \{f : \max_\alpha f(\alpha) = 1\}$. $\bar{v}$ is a normal complexity measure and all others are its special cases.

Proposition 3. For each normal $v$ a computable representation $t_x \in E_1$ for $x \in S$ exists such that $K_v(x) \asymp K_M(t_x) \asymp K(x)$.

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4For $t(x) \sim ||x||$ shown in [L 71], Th.13; also mentioned in [ZL 70], Prp.3.2.
Proof. $K_w(x)<K(x)$ follows from normality ($B_u<0$) and convexity of $v$. Thus $m_v(x)$ needs $<K(x)<2\|x\|$ bits. Let $m'$ be $m\in v$ so rounded-down. For $m\in v$, let $m_x$ be a prefix code of $(x,m'(x))$, and $m[m]'=m_1m_2\ldots m_x$. Then $t_x(\alpha)\equiv m'(x)$ if $\alpha=m_x\beta, m\in v$; otherwise $t_x(\alpha)\equiv 0$.

The measure concentrated in a single $\alpha$ has some $m\in v$ for which it maps each $t_x$ to $m'(x)$. Other measures $\mu$ also have $\tau_\mu : x\mapsto \mu(t_x)$ in $v$ by convexity of $v$. As $v$ is closed down, $\tau_M \in v$, too, and so, $\tau_M = O(\tau_v)$. Conversely, some measure $\alpha$ has $\tau_\alpha = m_v$. As $m_v$ is r.e., the minimal semimeasure $\mu$ with $\tau_\mu \geq m_v$ is r.e., too, and so, $m_v \leq \tau_\mu = O(\tau_M)$. □

4 Complete Sequences

L.76a calls complete sequences $\alpha$ that are $\mu$-random for a computable $\mu$. This class is closed under all total recursive operators. Here I use this term complete also for $\alpha'$ Turing-equivalent to such $\alpha$. This is identical to $\alpha'$ being either recursive or Turing-equivalent to a $\lambda$-random sequence. By Kucera 85 [Gács 86, Barmpalias, Lewis-Pye 18], each $\alpha,\beta \in \Omega$ is w.t.t. -reducible to a $\lambda$-random $\omega$. Indeed, for $P(x,\alpha) = x\alpha$, let measure $\rho$ be $\lambda$-integral of $T_\lambda : \rho = P(m\otimes \lambda)$. Let $R = \{\alpha : T_\lambda(\alpha) \leq c\}$ for a convenient constant $c$. When $A(\lambda)$ generates $M(x)$, the co-images of all prefixes intersect $R$. (Otherwise $A(\rho)$ would exceed $M = A(\lambda)$.) But for clopen $A$ (see Prp. 2), co-image of any $\alpha \in \Omega$ is the intersection of (non-empty in $R$) clopen co-images of its prefixes $\alpha_n$, so intersects $R$, too.

Yet partial algorithms can generate incomplete sequences with positive probability: [V'yugin 82]. I extend $K(\beta|\alpha)$ to $\alpha, \beta \in \Omega$ using a universal PCT $U(p,\alpha)$ that runs on $\alpha$ a program $p$ given on a separate tape; $\alpha_p$ combines bits of $p, \alpha$ in order read by $U$. $p$ must be prefixless: $U$ diverges and $\alpha_p$ is undefined unless $U$ detects the end of $p$ and does not try to move beyond its end of tape.

Definition 3. Here $\alpha, \beta \in \Omega$. $K(\beta|\alpha) \equiv \min_p\{|p| : U(p,\alpha) = \beta\}$.

The codeset $R_\alpha$ for $\alpha$ is $\{\beta : U(\beta)=\alpha, d_\lambda(\beta)<c\}$ where $c$ is a constant such that the incompleteness $\lambda(\alpha) \equiv \min_{\beta \in R_\alpha} K(\beta|\alpha)$ of any $\alpha$ is $\lesssim \|d_\lambda(\alpha)\|$.

Tight complexity $K(x|\alpha)$ is $\|\hat{m}(x|\alpha)\|$ where $x \in \mathbb{N}$, $\hat{m}_x(\alpha) \equiv \min_{\beta \in R_\alpha} m(x|\beta)$, $\hat{m}(x|\alpha) \equiv \lim_{\beta \uparrow \alpha} (\hat{m}_x(\alpha)|x,y)$. These concepts satisfy many properties similar to those given (for integers) in Gács 74, L.74—

Proposition 4. 1. $K(\beta|\alpha) \sim K_M(\beta|\alpha)$.
2. $d_\lambda(\beta_q) \geq d_\lambda(\beta) + \|q\| - K(q|\beta, d_\lambda(\beta))$.
3. $\lambda(\alpha) \simeq \min_{\beta} \{K(\alpha|\beta) + K(\beta|\alpha) + d_\lambda(\beta)\}$.
4. $\hat{K}(x|\alpha) \equiv \hat{K}(\bar{\alpha})$. (Recall: $\bar{\alpha}$ is $(x, K(x))$.)
5. $\hat{I}(\alpha : x) \equiv K(x) - \hat{K}(x) \leq \hat{K}(\alpha : x, y)$.
6. $\hat{I}(\alpha : x) \approx (\min_{\beta \in R_\alpha} d_\lambda(\beta|\bar{\alpha})) \uparrow \approx (\min_{\beta \in U^{-1}(\alpha)} d_\lambda(\beta|\bar{\alpha})) \uparrow$.

Proof. 1. Let $k = K_M(\beta|\alpha)$, $s_{k,\alpha} \equiv \{x, 1 : K_M(x|0\alpha) < k, K_M(x|1\alpha) < k\}$, so, $|s_{k,\alpha}| < 2^k$. Let $x$ be the longest prefix of $\beta$ in $s_{k,\alpha}$. Then $K(x|\alpha) < k$, and $\beta$ can be computed from $x, k, \alpha$.

2. "$d_\lambda(\beta_q) \simeq"$ is by $t_{\beta_q} \equiv T_\lambda(\beta)^{2\|q\|} m_q(\beta, d_\lambda(\beta))$ being r.e. with $\lambda(\beta_q)(t_{\beta_q}) \leq 1$. For "$<"$ take a distribution $\mu_{\beta,d}(q) \equiv T_\lambda(\beta_q/2\|q\|+d$ enumerated for each $\beta, d$ only while $\delta_q \equiv \|\sum_q 2^d \mu_{\beta,d}(q)\| \leq d$; so enumeration of $\mu_{\beta,d}(q)$ is not stopped. Now, $\delta_q \simeq d_\lambda(\beta)$ since $\lambda(\beta)(2^\delta_q) \leq 1$. Also, $\sum_q \mu_{\beta,d}(q) = O(1)$, so $\mu_{\beta,d}(q) = O(m_q(\beta|d))$. Thus, $d_\lambda(\beta) + \|q\| - d_\lambda(\beta_q) \geq \|\mu_{\beta,d}(q)\| \geq K(q\beta, d_\lambda(\beta))$.

4 For some applications of $\chi$ its lower bound $\|M_{\alpha}(R_\alpha)\|$ may suffice.
5 By finding $p$ to replace a prefix $q = U(p)$ where $\|q\| - \|p\|$ is the rarity.
Take \( p, q, \beta = U(p, \alpha) \) with \( U(q, \beta) = \alpha \), \( \chi(\alpha) \approx ||p|| + ||q|| + d_1(\beta) \).

Then \( d_1(\beta) \approx 0 \), \( K(q, \beta) \approx ||q|| \), else \( \beta \) or \( q \) could be shrunk decreasing \( \chi(\alpha) \).

Then \( d_1(\beta) \approx 0 \) by \( \sum \) and the claim follows by appending \( q \) to map \( \alpha \mapsto (q, \beta) \mapsto \beta \).

Let \( \beta = v \omega \), \( d_1(\beta) \approx 0 \), \( ||p|| = K(p, \beta) \) (and so, \( K(p, \beta) \)), and \( U(p, \beta) = x \) reads only \( p, v \), so, \( K(p, v) < ||pv|| \).

This, finding \( i, j \) with \( K(x) < i, K((p, v) x, i) < j, i + j < ||pv|| \) computes \( K(x) \approx i \) from \( p, v \).

By \( \sum \) and \( K(\mathcal{O}(x, y)) \approx 0 \), we can replace \( x \) with \( \mathcal{O} \). Let \( d_1(\beta) \approx 0 \).

Then \( K(\mathcal{O}) - K(\mathcal{O}(\beta)) - K(\mathcal{O}(\beta)) \approx K((\mathcal{O}(\beta), \mathcal{O}(\beta)) - K(\mathcal{O}(\beta)) < 0 \).

Indeed, the r.e. \( \sum x \in A \) \( d_1(\mathcal{O}(x, y)) \) is \( O(\mathcal{O}(\beta)) \approx O(1) \) since \( \lambda(\beta) \approx \sum x \in A \) \( d_1(\mathcal{O}(x, y)) \approx \sum x \in A \) \( d_1(\mathcal{O}(x, y)) \) is \( O(\mathcal{O}(\beta)) \approx O(1) \). So and \( K(x) - K(x, \beta) < d_1(\beta) \).

And any \( \beta \in U^{-1}(\alpha) \) can be compressed\(^5\) to \( \beta' \in R_\alpha \) with \( d_1(\beta') < d_1(\beta) \).

\[ \square \]

5 Rarity

5.1 Non-algorithmic Distributions

\[ \text{L 73} \] considered a definition of rarity \( T_\mu(\alpha) \) for arbitrary measures \( \mu \) where \( T_\mu \) is r.e. only relative to \( \mu \) used as an oracle. This concept gives interesting results on testing for co-r.e. \( \mu \).

Yet, for individual \( \mu \) it is peculiar in its strong dependence on insignificant digits of \( \mu \) that have little effect on probabilities. \[ \text{L 76, Gács 80} \] confronted this aspect by restrictions making \( 1/T_\mu(\alpha) \) monotone, homogeneous, and concave in \( \mu \).

\[ \text{L 84} \] used another construction for \( T_\mu(\alpha) \). It generates \( \mu \)-tests by randomized algorithms and averages their values on \( \alpha \). For computable \( \mu \) the tests’ \( \leq \)-mean can be forced by the generating algorithm, so the definition agrees with the standard one. But for other \( \mu \) the \( \leq \)-mean needs to be imposed externally. \[ \text{L 84} \] does this by just replacing the tests of higher mean with \( 1 \) (thus tarnishing the purity of the algorithmic generation aspect). That definition respects the conservation inequalities, so for r.e. semimeasures it gives a lower bound for our \( d_1(\beta) \) below (by Prop\[\square\]).

5.2 R.E. Semimeasures

Coarse Graining. I use \( \lambda \) as a typical continuous computable measure on \( \Omega \), though any of them can be equivalently used instead. Also, any recursive tree of clopen subsets can serve in place of \( S \).

Restricting inputs \( \omega \) of a PCT \( A \) to those with converging outputs (i.e., a singletons \( A(\omega) \subseteq \Omega \)) truncate the output semimeasure to a smaller linear functional: a maximal measure \( \mu^E \leq \mu = A(\lambda) \).

Yet, much information is lost this way: e.g., \( \|\mu^E(x)\|, x \in \mathcal{O} \) has no recursive \( \|\mu^E(x)\| \) upper bound. To keep information about generated prefixes, I will require linearity of \( \mu^E \) only on a subspace \( E \subseteq \mathcal{F} \).

\( E \) will play a role of space of \( \mu^E \)-measurable functions. E.g., relaxing \( A(\omega) \) restriction from singletons to sets of radius \( \leq 2^{-n} \), produces a semimeasure linear on the subspace of \( f \) with \( f(\alpha) \) dependent only on \( \alpha_{[n]} \).

Subspaces \( E \subseteq \mathcal{F} \) used below are generated by subtrees \( S \subseteq \mathcal{S} \), i.e., are spaces of linear combinations of functions in \( S \). By \( E \)-measures I call semimeasures linear on such \( E \).

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\(^5\)The Definition in \[ \text{L 76} \] has a typo: "\( Q(f) \)" meant to be "\( Q(g) \)". Also, in English version "concave relative to \( P \)" would be clearer as "for any measure \( Q \) concave over \( P \)". So, our \( T_\mu(\alpha) \) is \( \sup_{f \in \mathcal{F}} \|f(x)\|/\mu(\alpha) \).

Restrictions on \( t \) (e.g., \( t \subseteq S \times \mathcal{F} \), \( T_\mu(\alpha) \approx \sup_{f(\alpha) \in \mathcal{F}} \|f(\alpha)/\mu(\alpha)\| \) can reduce redundancy with no loss of generality.

\(^7\)If a non-binary tree is used instead of \( S \) then any \( x \in S \) must have either all its children in \( S \) or none.
Proposition 5. Each semimeasure $\mu$, for each $E$, has the largest (on $\mathcal{E}^+$) $E$-measure $\mu^E \leq \mu$.

Proof. Let $X$ be the set of all measures $\varphi$ which, for some $F \subset \mathcal{E}^+$ with $\sum_{f \in F} f > 0$ and all $g \in \mathcal{E}^+$, $g \leq f \in F$, have $\varphi(g) \geq \mu(g)$. Then $\mu^E(f) = \inf_{\varphi \in X} \varphi(f)$.

Now, I will extend the concept of rarity $T_\mu$, $d \overset{df}{=} || T ||$ from computable measures $\mu$ to r.e. semimeasures. The idea is for $d_\mu(\alpha)$ to be bounded by $d_\lambda(\omega)$ if $\alpha = A(\omega)$, $\mu \geq A(\lambda)$. Coarse graining on a space rougher than the whole $\mathcal{E}$, allows to define rarity not only for $\alpha \in \Omega$ but also for its prefixes. For semimeasures, rarity of extensions does not determine the rarity of a prefix.

$T_\mu$ for a computable measure $\mu$ is a single r.e. function $\Omega \to \mathbb{R}^+$ with $\leq 1$ mean. It is obtained by averaging the r.e. family of all such functions. This fails if $\mu$ is a semimeasure: its mean of sum can exceed the sum of means. So, our extended $T_\mu$ will be refined with a subspace $E \subset \mathcal{E}$ parameter.

Definition 4. For an $E \subset \mathcal{E}$ and a PCT $A$, $t_A^E$ is sup\{f $\in E$ : $A(f) \leq T_\lambda$\}.

Proposition 6. Each r.e. $\mu$, among all r.e. PCT $A$ with $A(\lambda) \leq \mu$, has a universal one $U_\mu$, i.e., such that $t_{U_\mu}^E = O(t_\lambda^E)$ for each $E$ and all $E$. $\mu(f) \leq \lambda(2U_\mu(f))$ if $f \in S$ or $\mu$ is regular.

Proof. $U(i\omega) \overset{df}{=} A_i(\omega)$ for a prefixless enumeration $A_i$ of all such $A$.

Definition 5. $T_\mu^E(\varphi)$ for semimeasures $\varphi$, r.e. $\mu$ is the mean: $\varphi^E(2U_\mu)$ for $U_\mu$ defined in Prop. 6.

Lemma 1. (1) $d_\mu^E \times d_\mu$ for computable measures $\mu$ (So, if $E = \mathcal{E}$, we omit $E$ in $d_\mu^E \overset{df}{=} || T_\mu^E ||$.)
(2) $d_\mu(\mu) = 0$.
(3) $d_M \propto 0$ for the universal semimeasure $M$.

Proof. (1) follows from [ZL 70] Th. 3.1 and enumerability of $T_\mu$.
(2) Let $A = U_\mu$. By Prop. 6 $\mu^E(f)/2 \leq \lambda(A(f))$ for $f \in S$, and thus for $f \in \mathcal{E}^+$. Also any $f << t_{U_\mu}^E$ is $< \sum f_i$ where $f_i \in \mathcal{E}^+$, $f_i f_j \neq 0$, and $A(f_i) \leq T_\lambda$. Now, $T_\mu^E(\mu) = sup_{f \in \mathcal{E}^+, f << t_{U_\mu}^E} \mu^E(f)/2$, and $\mu^E(f)/2 \leq \sum_i \mu^E(f_i)/2 \leq \lambda(\sum_i A(f_i)) = \mu(\sum_i A(f_i)) \leq \lambda(T_\mu) \leq 1$.
(3) By [Gács 86; Kucera 83], an r.e. PCT $A$ exists such that any $\alpha = A(\omega)$ with $d_\lambda(\omega) = 0$. Then $g = A(f) \leq T_\lambda$ means $g(\omega) = f(A(\omega)) = f(\alpha) \leq T_\lambda(\omega) \leq 2$. For a universal $M$, $d_M \propto d_A(\lambda) \propto 0$.

Let the semimeasure $\nu = \mu \otimes \varphi$ on $\Omega^2$ be the minimum of $\mu' \otimes \varphi'$ over all measures $\mu' \geq \mu$, $\varphi' \geq \varphi$. Then $\nu(h) = \mu(f) \varphi(g)$ for $h(\alpha, \beta) = f(\alpha) g(\beta)$, and for all $h$, if $\varphi$ is a measure, $\nu(h) = \mu(\varphi(\beta) h(\alpha, \beta))$. Let $E \otimes \mathcal{E}$ be the space generated by \{f(\varphi,g), g \in E, f \in \mathcal{E}\}. Adding coin-flips preserves randomness:

Lemma 2. $d^E_{\mu \otimes \varphi}$ for all r.e. $\mu$, space $E \subset \mathcal{E}$.

Proof. Let $\phi = \varphi \otimes \lambda$, $\nu = \mu \otimes \lambda$, $E' = \mathcal{E} \otimes \mathcal{E}$, $A(\alpha, \beta) = (U_\mu(\alpha), \beta)$, $\nu = T^E(\phi) = \phi^E(t_{U_\mu}^E)$. Then for some $c \in \mathbb{Q}^+$, $t/c < \phi^E(t_{U_\mu}^E) = \phi^E(sup H)$ where $H = \{h \in E' : A(h) \leq T_\lambda\}$. So $t/c < \phi^E(sup G)$ for a finite set $G = \{f_i(\alpha) g_i(\beta) : H \text{ with } \lambda(g_i) = 1 \}$ and $f_i f_j g_i = 0$, thus sup $G = \sum G$. Now, $U_\mu(f_i) g_i < T_\lambda$, thus $U_\mu(f_i) < \lambda(\beta) \langle T_\lambda(\alpha, \beta) \rangle = O(\lambda(\alpha))$. Then, $t/c < \phi^E(\sum_i f_i g_i) = \phi^E(\sum_i f_i) = \varphi^E(\sum_i f_i) = \varphi^E(sup f_i) = O(\varphi^E(t_{U_\mu}^E(\varphi)))$.

Let $A(E)$ be \{f $\in \mathcal{E}$ : $A(f) \in \mathcal{E}\}$. Deterministic processing preserves randomness, too:

Lemma 3. $d^{A(E)}_{A(\mu)}(\varphi) \propto d^E_\mu(\varphi)$ for each r.e. PCT $A$, all r.e. $\varphi$, r.e. $\mu$, space $E \subset \mathcal{E}$. 

Proof. Let \( E' \equiv A(E), \phi \equiv A(\varphi)^{E'} \leq A(\varphi^E), A_\mu(f) \equiv U_\mu(A(f)) \). So, \( t \equiv T^E_{A(\mu)}(A(\varphi)) = \phi(t^{E}_{U_{A(\mu)}}) < c \phi(t^E_{A(\mu)}) < c \phi(\text{sup } F) \) for \( F \equiv \{ f \in E^+ : U_{A(\mu)}(A(f)) \leq T_\lambda \} \) and some \( c \in \mathbb{Q}^+ \).

Then \( t < \phi(\text{sup } G) \) for a finite set \( G \subset F \) that can be made disjoint, i.e., \( gg' = 0 \) for \( g \neq g' \) in \( G \) (and thus \( A(g)A(g') = 0 \) as \( A \) is deterministic), so \( \text{sup } G = \sum G \).

Now, \( U_{A(\mu)}(h) \leq T_\lambda \) for \( h \equiv \text{sup } \{ A(f) : f \in E \} \in E^+ \), so \( h \leq t^{E}_{U_{A(\mu)}} \). Then \( t/c < \phi(\text{sup } G) = \phi(\sum G) = \sum_{g \in G} \phi(g) \leq \sum_{\varphi \in G} \varphi^{E}(A(g)) = \varphi^{E}(\sum_{g \in G} A(g)) = \varphi^{E}(\text{sup}_{g \in G} A(g)) \leq \varphi^{E}(h) \leq 2T^E_{A(\mu)}(\varphi). \)

By the remark \([\mathbb{I}]\), Lemmas \([\mathbb{II}] \) and \([\mathbb{III}] \) imply the following theorem:

**Theorem 1** (Randomness Conservation). The test \( d \) satisfies \( d^E_{A(\mu)}(A(\varphi)) < d^E_{\mu}(\varphi) \) for each normal concave r.e. operator \( A \), all \( \varphi \), r.e. \( \mu \), space \( E \subseteq \mathcal{E} \).

These tests \( d^E_{\mu} \) are the strongest (largest) extensions of Martin-Löf tests for computable \( \mu \):

**Proposition 7.** \( T^E_{\mu}(\omega) \) is majorant among extensions \( \tau_\mu \in E^+ \) of Martin-Löf test \( T_\lambda = \tau_\lambda \) that are non-increasing on \( \mu \) and obey Lemma \([\mathbb{III}] \) for \( \| [\tau] \| \) with \( \tau^E_{A(\mu)}(\varphi) \equiv \varphi^{E}(\tau_\mu) \).

Proof. With \( A \equiv U^*, A(\tau_\mu) \leq A(\tau_{A(\lambda)}) \) and Lemma \([\mathbb{III}] \) for \( \| [\tau] \| \) gives \( A(\tau_{A(\lambda)})(\omega) = \tau_{A(\lambda)}(A^*(\omega)) \leq c \tau_\lambda(\omega) = cT_\lambda(\omega) \) for some \( c \in \mathbb{Q}^+ \). If \( \tau_\mu > 2c \) then \( 2c A(f) < A(\tau_\mu) \leq cT_\lambda, \) so \( T^E_{\mu} \geq f \) as defined. \( \square \)

# 6 Information and its Bounds

Now, like for the integer case, mutual information \( I(\alpha : \beta) \) can be defined as the deficiency of independence, i.e., rarity for the distribution where \( \alpha, \beta \) are assumed each universally distributed (a vacuous assumption, see e.g., Lemma \([\mathbb{I}] \) (3)) but independent of each other:

\[ I(\alpha : \beta) \equiv d_{M \otimes M}(\alpha, \beta). \]

Its conservation inequalities are just special cases of Theorem \([\mathbb{I}] \) and supply \( I(\alpha : \beta) \) with lower bounds \( I(A(\alpha) : B(\beta)) \) for various operators \( A, B \). In particular transforming \( \alpha, \beta \) into distributions \( m(\cdot | \alpha), m(\cdot | \beta) \), gives \( I(\alpha : \beta) \geq \chi(\alpha, \beta) \equiv \| \sum_{x,y \in N} m(x|\alpha)m(y|\beta)\|_2^I \).\footnote{Same for \( \hat{I}(\alpha : \beta) \equiv \| \sum_{x,y \in N} \hat{m}(x|\alpha)\hat{m}(y|\beta)\|_2 \geq I(\alpha : \beta) \). These bounds also satisfy the conservation inequalities, and agree with \( I(\alpha : \beta) \) for \( \alpha, \beta \in \mathbb{N} \). While \( \chi \) is the largest such extension from \( N \), \( \chi \) is the smallest one. Interestingly, not only for integers, but also for all complete sequences this simple bound \( \chi \) is tight, as is an even simpler one \( \hat{I}(\alpha : \beta) \geq \| \sum_{x,y \in N} (K(x) - \hat{K}(x|\alpha) - \hat{K}(x|\beta)) \| \geq I(\alpha : \beta) \).

**Proposition 8.** For \( \alpha, \beta \in \Omega, \ b \in \mathbb{N} : \ (1) \ I(\alpha : b) \geq K(b) - \hat{K}(b|\alpha) \) (follows from Prop \([\mathbb{IV}] \) (6));

(2) \( I(\alpha : \beta) < \chi \equiv \min_{\alpha' \in R_\alpha, \beta' \in R_\beta} \hat{I}(\alpha' : \beta') \) (following Prop \([\mathbb{IV}] \) (7)).

In particular, this can be used for \( \alpha \) being the Halting Problem sequence (which is complete, being Turing-equivalent to any random r.e. real, such as, e.g., one constructed in sec. 4.4 of \([\mathbb{ZL} 70] \)).

Proof. We can replace \( \alpha, \beta \) with \( \alpha' \in R_\alpha, \beta' \in R_\beta \). Let \( h_n \equiv (\alpha_n, \beta_n) \).

\[ \chi^2 \equiv \chi \otimes \lambda = O(M^2), \] so \( I(\alpha : \beta) \leq d_{A_2}((\alpha, \beta)) \leq \| \| \sum_{n \in N} 4^n \hat{m}(h_n) \| \| \leq \sum_{n \in N} K(h_n) - 2(K(h_n) - n) \).

Also \( t \equiv \sum_{n \in N} 2^n \hat{m}(\alpha_n, \nu) = \Theta(T_\lambda(\alpha)), \) so \( 2^n \hat{m}(\alpha_n, \nu)/t = O(\mu(n, \nu)|\alpha, \|\|t\|\|)), \) and

\[ K(h_n|\alpha) - K(h_n) \leq \|\|t\|\| \times 0. \] Thus \( K(h_n|\alpha) \leq K(h_n) - n \) and \( K(h_n|\beta) \leq K(h_n) - n. \)

Then \( I(\alpha : \beta) \leq \sum_{n \in N} (K(h_n) - 2(K(h_n) - n)) < \sum_{n \in N} (K(h_n) - K(h_n|\alpha) - K(h_n|\beta)) < \hat{I}(\alpha : \beta). \) \( \square \)

\( ^8 \)This \( \hat{I} \) was used as the definition of information in \([\mathbb{T}, \mathbb{Y}] \).
Proposition 9. Let \( A \subset \Omega \). Then \( M^\mathbb{P}(A) = 0 \) iff \( \exists \alpha \forall \beta \in A I(\beta : \alpha) = \infty \).

Proof. "If" is by Theorem 1. Now, any \( A \) with \( M^\mathbb{P}(A) = 0 \) has a sequence \( \alpha \) of clopen sets \( \alpha_i \subset \Omega \) with shrinking \( M(\alpha_i) \), i.e., \( \lambda\{\gamma : \exists x U(\gamma) \subset x \Omega \subset \alpha_i\} < 2^{-i} \), and s.t. each \( \beta \in A \) is in infinitely many \( \alpha_i \). Then, by Prop. 4.6, \( \hat{I}(\beta : (i, \alpha_i)) \succ (\min_{\gamma \in U^{-1}(\beta)} d_\lambda(\gamma | i, \alpha_i)) \uparrow > i \) and so \( I(\beta : \alpha) \succ \hat{I}(\beta : \alpha) = \infty \). \( \square \)
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