A FAMILY OF 4-BRANCH-POINT COVERS WITH MONODROMY GROUP $\text{PSL}_6(2)$

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ABSTRACT. We describe the explicit computation of a family of 4-branch-point rational functions of degree 63 with monodromy group $\text{PSL}_6(2)$. This, in particular, negatively answers a question by J. König whether there exists a such a function with rational coefficients. The computed family also gives rise to non-regular degree-126 realizations of $\text{Aut}(\text{PSL}_6(2))$ over $\mathbb{Q}(t)$.

1. Introduction

Let $C := (C_1, C_2, C_3)$ be the genus-0 class vector of $\text{PSL}_6(2)$ in its natural 2-transitive action on the 63 non-zero elements of $\mathbb{F}_2^6$, where $C_1$, $C_2$ and $C_3$ are the unique conjugacy classes of cycle structure $2^{28}.1^7$, $2^{16}.1^{31}$ and $3^{20}.1^3$, respectively. Then, using the theory of Hurwitz spaces J. König [9, p. 109] established the theoretical existence of a hyperelliptic genus-3 curve $H$ defined over $\mathbb{Q}$ and polynomials $p, q \in \mathbb{Q}(H)[X]$ satisfying the following:

- The family $\mathcal{F}$ of normalized covers with ramification locus $(0, \infty, 1+\sqrt{\lambda}, 1-\sqrt{\lambda})$ where $\lambda \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and ramification structure $C$ can be parameterized by a rational function $F = \frac{p}{q} \in \mathbb{Q}(H)(X)$.
- $\text{Gal}(p - tq | \mathbb{Q}(H)(t)) \cong \text{PSL}_6(2)$.

In order to decide whether $\text{PSL}_6(2)$ occurs regularly as a Galois group over $\mathbb{Q}(t)$ with ramification structure $C$, one has to check the existence of $\mathbb{Q}$-rational points on $H$ that lead to Galois group preserving specializations. König also mentions that without explicit computation of $H$ there seems to be no way of finding an answer to this question.

Note that $\text{PSL}_6(2)$ and $\text{PSp}_6(2)$ of degree 63 are expected to be the largest (with respect to the permutation degree) almost simple primitive groups having a generating genus-0 tuple of length at least 4 with the socle being a simple group of Lie type. While multi-parameter families of polynomials with Galois group $\text{PSp}_6(2)$ of degree 28 and 36 were calculated in [2], the case $\text{PSL}_6(2)$ remained open. With the recent development in computing multi-branch-point covers in [2] we are able to give explicit defining equations for $H$ and $F$. Alternative techniques
for such calculations are described by Couveignes [4], Hallouin [6], König [8, 9], Malle [10] and Müller [12].

This paper is structured as follows: Section 2 depicts the computation of $\mathcal{H}$ and $F$. The computed results are verified in section 3 and we will show that $\mathcal{H}$ does not have $\mathbb{Q}$-rational points that lead to Galois group preserving specializations. As a consequence we deduce that $\text{PSL}_6(2)$ does not occur as the monodromy group of a rational function with rational coefficients ramified over at least 4 points. Furthermore we obtain explicit polynomials of degree 126 over $\mathbb{Q}(t)$ with Galois groups isomorphic to $\text{Aut}(\text{PSL}_6(2))$ which are presented in section 4.

2. Computation

Let $C = (C_1, C_2, C_3, C_3)$ be the genus-0 class vector from the introduction and $F$ the family of all $\text{PSL}_6(2)$-covers $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree 63 such that:

(i) $f$ is a 4-branch-point cover ramified over $0, \infty, 1, \pm \sqrt{\lambda}$ for some $\lambda \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ with ramification structure:

| branch point | 0 | $\infty$ | $1 + \sqrt{\lambda}$ | $1 - \sqrt{\lambda}$ |
|--------------|---|--------|----------------|----------------|
| inertia class| $C_1$ | $C_2$ | $C_3$ | $C_3$ |

(ii) $f$ is normalized in the following sense: The sum of all simple roots of $f$ is 0 and the sum of all double poles is 1. Furthermore, $\infty$ is the unique simple pole of $f$ fixed under the action of the normalizer of the inertia group at $\infty$. Note that for any $g \in C_2$ exactly one length-1-cycle of $g$ is fixed under $N_{\text{PSL}_6(2)}(\langle g \rangle)$.

2.1. Properties of $F$. The straight inner Nielsen class $\text{SNi}^\text{in}(C)$ of $C$ is the set of quadruples $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in C_1 \times C_2 \times C_3 \times C_3$ up to simultaneous conjugation satisfying both $\sigma_1 \sigma_2 \sigma_3 \sigma_4 = 1$ and $\langle \sigma_1, \sigma_2, \sigma_3, \sigma_4 \rangle = \text{PSL}_6(2)$. A computer computation with Magma [3] yields $|\text{SNi}^\text{in}(C)| = 48$.

Since $F$ carries the structure of an algebraic variety, various properties of $F$ can be studied via the branch-point reference map:

$$\Psi : \begin{cases} F \to \mathbb{P}^1_{\lambda} \\
\text{cover with ramification locus } \{0, \infty, 1, \pm \sqrt{\lambda_0}\} \mapsto \lambda
\end{cases}$$

By Riemann’s existence theorem for each $\lambda_0 \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and $\sigma \in \text{SNi}^\text{in}(C)$ there is a unique cover (up to inner Möbius transformation) with ramification locus $(0, \infty, 1, \pm \sqrt{\lambda_0})$ and ramification $\sigma$. The normalization conditions stated in (ii) guarantee that $F$ contains exactly one such cover. As a consequence $F$ is a curve and $\Psi$ turns out to be a Belyi map of degree $|\text{SNi}^\text{in}(C)| = 48$ with ramification locus $(0, 1, \infty)$. The ramification of $\Psi$, denoted by $(x, y, z) \in \text{Sym}(\text{SNi}^\text{in}(C))^3$, is also well
studied and can be calculated explicitly using the formula in [11] Theorem III.7.8\] which arises from the action of the braid group on $\text{SNi}^\text{in}(C)$. This triple generates a transitive group and consists of cycle structures $(6^5, 4^4, 2^1, 7^4, 4^3, 3^2, 2^1, 9^{24})$. From this we can deduce that $\mathcal{F}$ is connected of genus 3 (by the Riemann-Hurwitz formula). Furthermore note that $\mathcal{F}$ can be defined over $\mathbb{Q}$ since all classes of $C$ are rational.

In the following the function field of $\mathcal{F}$ will be denoted by $\mathbb{Q}(\mathcal{F})$. The family $\mathcal{F}$ can be parameterized by a rational function

$$ F = \frac{p}{q} \in \mathbb{Q}(\mathcal{F})(X) $$

with $p, q \in \mathbb{Q}(\mathcal{F})[X]$ such that any element of $\mathcal{F}$ is obtained via specializing $F$ at some point in $\mathcal{F}$.

2.2. Defining equations for elements in $\mathcal{F}$. Fix $f_{\lambda_0} \in \mathcal{F}$ with $\Psi(f_{\lambda_0}) = \lambda_0$ for some $\lambda_0 \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$. According to (i) and (ii) there exist a scalar $c_0$ and separable, monic and mutually coprime polynomials $p_7, p_{28}, q_{16}, q_{30}, r_3, r_{20}, s_3, s_{20}$ of respective degree denoted in the index such that

$$ f_{\lambda_0} = \frac{c_0 \cdot p_7 \cdot p_{28}^2}{q_{30} \cdot q_{16}^2} = 1 + \sqrt{\lambda_0} + \frac{c_0 \cdot r_3 \cdot r_{20}^3}{q_{30} \cdot q_{16}^2} = 1 - \sqrt{\lambda_0} + \frac{c_3 \cdot s_3 \cdot s_{20}^3}{q_{30} \cdot q_{16}^2} $$

where the traces of $p_7$ and $q_{16}$ are 0 and 1, respectively.

By comparing coefficients (3) can be considered as a system of polynomial equations where $c_0$ and the coefficients of $p_7, p_{28}, \ldots, s_{20}$ are considered to be the unknowns. This system consists of 126 unknowns and 126 equations, hence it is expected to have at most finitely many solutions with $f_{\lambda_0}$ being one of them.

2.3. Walking on $\mathcal{F}$. Assume we are given an explicit approximative equation for $f_{\lambda_0}$, then we are able to compute another approximative equation of a cover $\lambda_{\lambda_0 + \delta} \in \mathcal{F}$ with $\Psi(f_{\lambda_0 + \delta}) = \lambda_0 + \delta$ for some sufficiently small $\delta \in \mathbb{C}$. This can be achieved via Newton iteration by assembling the corresponding polynomial equations similar to (3) and using $f_{\lambda_0}$ as the initial value.

Starting from an approximative equation of a cover $f_{\text{start}} \in \mathcal{F}$ we can find an approximative equation for another cover $f_{\text{end}} \in \mathcal{F}$ with prescribed $\lambda_{\text{end}} := \Psi(f_{\text{end}}) \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and prescribed ramification $\sigma_{\text{end}} \in \text{SNi}^\text{in}(C)$.

Let $\lambda_{\text{start}} := \Psi(f_{\text{start}})$ and $\gamma_1$ be a path in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ connecting $\lambda_{\text{start}}$ to $\lambda_{\text{end}}$. Lift $\gamma_1$ via $\Psi$ to $\mathcal{F}$ from a path starting in $f_{\text{start}}$ and ending in some element denoted by $f_{\text{end}} \in \mathcal{F}$, then $\Psi(f_{\text{end}}) = \lambda_{\text{end}}$. The ramification of $f_{\text{end}}$ will be denoted by $\sigma_{\text{end}}$. According to the ramification of $\Psi$ we can give a closed path $\gamma_2$ in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ starting in $\lambda_{\text{end}}$ with the property: The lifted path of $\gamma_2$ in $\mathcal{F}$ via $\Psi$ connects $f_{\text{end}}$ to another element $f_{\text{end}}$ with $\Psi(f_{\text{end}}) = \lambda_{\text{end}}$ and ramification $\sigma_{\text{end}}$. Using Newton
iteration as explained before we can slightly deform $f_{\text{start}}$ at its ramification locus along $\gamma_2 \circ \gamma_1$ to obtain an approximate equation for $f_{\text{end}}$ having the prescribed ramification data.

2.4. Splitting behaviour of $\Psi$. The monodromy group of $\Psi$, generated by $x, y, z$, turns out to be imprimitive acting on 24 blocks, each of size 2. The induced action of $(x, y, z)$ on the set $B$ of these blocks, denoted by $(x', y', z') \in \text{Sym}(B)^3$, consists of cycle structures $(4^2.3^5.11, 7^2.4^1.3^1.2^1.1^1, 2^{12})$. Since $(x', y', z')$ describes a genus-0 triple the cover $\Psi$ splits as follows:

\[
\Psi : \mathcal{F} \xrightarrow{\Psi_2} \mathbb{P}^1_{\mu} \xrightarrow{\Psi_{24}} \mathbb{P}^1_{\lambda}
\]

with a degree-2 subcover $\Psi_2$ and a degree-24 subcover $\Psi_{24}$ with ramification $(x', y', z')$ over $(0, 1, \infty)$. The latter cover can be computed explicitly (using for example the method explained in [2]):

\[
\lambda = \Psi_{24}(\mu) = \frac{p_{24}}{q_{24}} = 1 - \frac{r_{24}}{q_{24}}
\]

where

\[
p_{24} := \left(\mu - \frac{1}{4}\right) \left(\mu^2 - \frac{11}{16}\mu + \frac{1}{8}\right)^4 \left(\mu^5 - \frac{137}{4}\mu^4 + \frac{178}{3}\mu^3 - 34\mu^2 + 8\mu - \frac{2}{3}\right)^3,
\]

\[
r_{24} := 243 \left(\mu - \frac{1}{2}\right)^3 \left(\mu - \frac{1}{3}\right)^4 \left(\mu - \frac{5}{16}\right)^2 \left(\mu^2 + \frac{1}{3}\mu - \frac{1}{6}\right)^7,
\]

\[
q_{24} := p_{24} + r_{24}.
\]

Recall that the cycle structures of $(x, y, z)$ and $(x', y', z')$ are given by

$(6^5.4^4.2^1, 7^4.4^3.3^2.2^1, z^{24})$ and $(4^2.3^5.1^1, 7^2.4^1.3^1.2^1.1^1, 2^{12})$.

It is now easy to see, that these cycle structures in combination with $p_{24}$, $q_{24}$ and $r_{24}$ uniquely determine the ramification locus $R_{\Psi_2} \subseteq \mathbb{P}^1_{\mu}$ of the degree-2 subcover $\Psi_2$. We find $R_{\Psi_2} = R_0 \cup R_1 \cup R_\infty$ with

\[
R_0 := \Psi_2^{-1}(0) \cap R_{\Psi_2} = \left\{\frac{1}{4}\right\} \cup \left\{\text{roots of } \mu^5 - \frac{137}{4}\mu^4 + \frac{178}{3}\mu^3 - 34\mu^2 + 8\mu - \frac{2}{3}\right\},
\]

\[
R_1 := \Psi_2^{-1}(1) \cap R_{\Psi_2} = \left\{\frac{5}{16}, \infty\right\},
\]

\[
R_\infty := \Psi_2^{-1}(\infty) \cap R_{\Psi_2} = \emptyset.
\]
A model for \( \mathcal{F} \). Since \( z' \) has a unique fixed point and \( \mathcal{F} \) is defined over \( \mathbb{Q} \) the function field analogue of (4) can be stated as

\[
Q(\mathcal{F}) \geq Q(\mu) \geq Q(\lambda).
\]

where \( \mu \) is a root of \( p_{24} - \lambda q_{24} \in \mathbb{Q}(\lambda)[X] \) and \( Q(\mathcal{F}) \) being the degree-2 extension of \( \mathbb{Q}(\mu) \) corresponding to \( \Psi_2 \). The computation of \( R_{\Psi_2} \) guarantees the existence of a primitive element \( y \in \mathbb{Q}(F) \), i.e., \( \mathbb{Q}(F) = \mathbb{Q}(\mu, y) \), with defining equation

\[
y^2 = cP(\mu) := c\left(\mu^5 - \frac{137}{4}\mu^4 + \frac{178}{3}\mu^3 - 34\mu^2 + 8\mu - \frac{2}{3}\right)\left(\mu - \frac{1}{4}\right)\left(\mu - \frac{5}{16}\right)
\]

for some square-free \( c \in \mathbb{Q} \) which will be determined in 2.7. For this reason a hyperelliptic \( \mathbb{Q} \)-model for \( \mathcal{F} \) can be chosen to be

\[
\mathcal{H} := \{ (\mu, y) : y^2 = cP(\mu) \}.
\]

Using this particular model \( \Psi_2 \) is then given by \( \Psi_2(\mu, y) = \mu \) for all \( (\mu, y) \in \mathcal{H} \).

Field of definition for elements in \( \mathcal{F} \). Since \( \mathcal{H} \) is a model for \( \mathcal{F} \), elements of \( \mathcal{F} \) are obtained via specializing \( F \) at points in \( \mathcal{H} \). The coefficients of a cover \( f_0 \in \mathcal{F} \) are therefore contained in

\[
\mathbb{Q} \left( \mu_0, \sqrt{cP(\mu_0)} \right)
\]

where \( \mu_0 := \Psi_2(f_0) \).

Obtaining elements in \( \mathcal{F} \). By Riemann’s existence theorem there exists a \( \text{PSL}_6(2) \)-cover \( h : \mathbb{P}^1 \to \mathbb{P}^1 \) ramified over \( (0, \infty, -1, 1) \) with ramification structure \((C_3, C_3, C_1, C_2)\). Then \( h^2 \) turns out to be a Belyi map with ramification locus \((0, \infty, 1)\) and monodromy group contained in \( \text{PSL}_6(2) \cap C_2 \leq S_{126} \). Its ramification consists of cycle structures \((6^{20}, 2^3, 6^{20}, 2^3, 2^{14}, 1^{38})\). Using the method described in [2], this Belyi map of degree 126 can be computed explicitly. Clearly, this yields a defining (approximative) equation for \( h \).

After applying suitable Möbius transformations and slightly moving the ramification points of \( h \) using Newton iteration we obtain a complex approximation of a cover \( f_{\text{start}} \in \mathcal{F} \) with \( \Psi(f_{\text{start}}) = \lambda_0 := \Psi_{24}(\frac{1}{6}) \). The approach described
in 2.3 allows the computation of a complex approximation of a cover \( f_{\text{end}} \in \mathcal{F} \) with \( \Psi(f_{\text{end}}) = \lambda_0 \) and ramification contained in \( B \in \mathcal{B} \) such that \( \chi(B) = \frac{1}{6} \). In combination with (7) this implies \( \Psi_2(f_{\text{end}}) = \frac{1}{6} \in \mathbb{Q} \). Due to (8) the coefficients of \( f_{\text{end}} \) can be recognized in the quadratic number field \( \mathbb{Q}(\sqrt{cP(\Psi_2(f_{\text{end}}))}) = \mathbb{Q}(\sqrt{-c \cdot 3 \cdot 7 \cdot 457}) \). With the help of Magma we find \( c = 3 \). Note that \( \mathcal{H} \) from (7) is finally computed.

2.8. Computing the universal cover \( \mathcal{F} \). Any coefficient of \( \mathcal{F} \in \mathbb{Q}(\mathcal{F})(X) = \mathbb{Q}(\mu, y)(X) \) from (2) can be expressed as

\[
H_1(\mu) + yH_2(\mu)
\]

where \( H_1, H_2 \in \mathbb{Q}(\mu) \). By slightly moving the ramification points of \( f_{\text{end}} \) via Newton iteration as described in 2.3 we obtain many defining equations of covers \( f \in \mathcal{F} \) such that \( \Psi_2(f) \) is a rational number close to \( \Psi_2(f_{\text{end}}) \). Considering (8) the coefficients of \( f \) are then contained in \( \mathbb{Q}(\sqrt{3P(\Psi_2(f))}) \), allowing us to read off \( H_1(\Psi_2(f)) \) and \( H_2(\Psi_2(f)) \). Therefore, both \( H_1 \) and \( H_2 \) can be computed by interpolation. The resulting universal cover \( \mathcal{F} = \frac{p}{q} \) is presented in file 3.3A.

Remark. The standard approach of computing a hyperelliptic model \( \mathcal{H} \) for \( \mathcal{F} \) consists of finding a polynomial relation between \( \lambda \) and a fixed coefficient of \( \mathcal{F} \) which are usually expected to generate the entire function field \( \mathbb{Q}(\mathcal{F}) \) with \( [\mathbb{Q}(\mathcal{F}) : \mathbb{Q}(\lambda)] = 48 \). This is achieved by interpolation via computing several elements \( f \in \mathcal{F} \) such that \( \Psi(f) \in \mathbb{Q} \) and recognizing the previously fixed coefficient as algebraic degree-48 numbers. A Riemann-Roch space computation then leads to the hyperelliptic model \( \mathcal{H} \).

Our approach takes advantage that the monodromy group of \( \Psi \) is imprimitive with an explicitly computable genus-0 subcover \( \Psi_{24} \). As explained in 2.5 and 2.7 this yields a defining equation for \( \mathcal{H} \) after recognizing only a degree-2 number.

3. Verification and Consequences

An essential tool for the upcoming verification process is the following criterion that guarantees the existence of subgroups of a Galois group having specific properties. Similar techniques have already been applied for example by Malle, see [10].

**Lemma 3.1.** Let \( K \) be an arbitrary field and \( f(t, X) \in K(t)[X] \) a separable and irreducible polynomial. Furthermore, let \( p, q \in K[X] \) be coprime polynomials such that \( p - tq \in K(t)[X] \) is separable and \( f(p(t), X) \in K(t)[X] \) splits nontrivially into irreducible factors of degree \( d_1, \ldots, d_r \). Then the following holds:

(a) The Galois group \( \text{Gal}(f \mid K(t)) \) has a subgroup of index dividing \( \text{deg}(p - tq) \) with orbit lengths \( d_1, \ldots, d_r \).
(b) If \( \text{Gal}(p - tq \mid K(t)) \) is primitive and both \( \text{Gal}(p - tq \mid K(t)) \) and \( \text{Gal}(f \mid K(t)) \) have the same order, then the splitting fields of \( p - tq \) and \( f \) over \( K(t) \) coincide.

**Proof.** (a) Let \( \Omega_f \) (resp. \( \Omega_{p-tq} \)) be the splitting field of \( f \) (resp. \( p - tq \)) over \( K(t) \) and \( s \) a root of the irreducible polynomial \( p - tq \in K(t)[X] \). Then \( t = \frac{p(s)}{q(s)} \) and, according to the assumption, \( f(t, X) = f\left(\frac{p(s)}{q(s)}, X\right) \) splits over \( K(s) \) into irreducible factors of degree \( d_1, \ldots, d_r \). This also holds if we factorize \( f \) over \( \Omega_f \cap K(s) \). Therefore, \( \text{Gal}(\Omega_f \mid \Omega_f \cap K(s)) \leq \text{Gal}(\Omega_f \mid K(t)) \) is of index dividing \([K(s) : K(t)] = \deg(p - tq)\) with orbit lengths \( d_1, \ldots, d_r \).

(b) Recall that \( f \) splits nontrivially over \( K(s) \), thus \( K(s) \cap \Omega_f \neq K(t) \). Since \( \text{Gal}(p - tq \mid K(t)) \) is primitive, the latter yields \( K(s) \cap \Omega_f = K(s) \), therefore \( K(s) \leq \Omega_f \). Of course, the normal closure of \( K(s) \) over \( K(t) \) is also contained in \( \Omega_f \), thus \( \Omega_{p-tq} \leq \Omega_f \). Due to \(|\text{Gal}(p - tq \mid K(t))| = |\text{Gal}(f \mid K(t))|\) we find \( \Omega_{p-tq} = \Omega_f \).

**Lemma 3.2.** Let \( G \) be a 2-transitive subgroup of \( S_{63} \) that contains a subgroup of index dividing 63 with orbit lengths 31 and 32. Then, \( G \) is isomorphic to \( \text{PSL}_6(2) \).

**Proof.** This follows immediately from the classification of finite 2-transitive groups which relies on the classification of finite simple groups. We indeed do not require such a strong result:

As explained by Dembowski [5, 2.4.3 and 2.4.5] we have \( G \leq \text{Aut} (\mathcal{D}) \) where \( \mathcal{D} \) is a symmetric 2-(63, 31, \( \lambda \))-design \( \mathcal{D} \) for some \( \lambda \in \mathbb{N} \). An easy combinatorial consideration yields \( \lambda = 15 \). Thus, by a result of Kantor [7], \( \mathcal{D} \) must be isomorphic to the projective space \( \text{PG}(5, 2) \). Since \( \text{Aut}(\text{PG}(5, 2)) = \text{PSL}_6(2) \) does not contain any proper 2-transitive subgroups, we conclude \( G \cong \text{PSL}_6(2) \).

**Theorem 3.3.** Let \( \mathcal{H} \) be the curve computed in [2.3 and 2.7] with defining equation

\[
y^2 = 3 \left( \mu^5 - \frac{137}{4} \mu^4 + \frac{178}{3} \mu^3 - 34 \mu^2 + 8 \mu - \frac{2}{3} \right) \left( \mu - \frac{1}{4} \right) \left( \mu - \frac{5}{16} \right).
\]

Furthermore, let

\[
F := \frac{p}{q} \in \mathbb{Q}(\mathcal{H})(X) = \mathbb{Q}(\mu, y)(X)
\]

be the rational function computed in [2.8] see ancillary file 3.3A, and \( \Psi_{24} = \frac{p_{24}}{q_{24}} \) the map from [5]. Then the following holds:

(a) The polynomial \( p - tq \) defines a regular \( \text{PSL}_6(2) \)-extension of \( \mathbb{Q}(\mu, y, t) \). The ramification locus with respect to \( t \) is given by \( \mathcal{R} := (0, \infty, 1 + \sqrt[5]{\Psi_{24}(\mu)}, 1 - \sqrt[2]{\Psi_{24}(\mu)}) \) with ramification structure \( (2^{28}.1^7, 2^{16}.1^{31}, 3^{20}.1^3, 3^{20}.1^3) \).

(b) Every cover in \( \mathcal{F} \) is obtained in a unique way via specialization of \( F \) at some point in \( \mathcal{H} \).
Proof. (a) We firstly verify that \( f := p - tq \) is ramified over \( R \) with ramification structure \( C \) from the introduction. This can be done by studying the inseparability behaviour of \( f \) at the places \( t \mapsto t_0 \) for \( t_0 \in R \). The corresponding factorizations are given in the file 3.3B. In particular, the behaviour above \( 1 \pm \sqrt{\Psi_{24}(\mu)} \) was obtained by interpolating the factorizations of several specialized polynomials. The ramification locus of \( f \) cannot be larger than \( R \), otherwise it would contradict the Riemann-Hurwitz formula.

Let \( \Omega \) be the splitting field of \( p - tq \) over \( \mathbb{Q}(\mu, y, t) \). Then, the geometric monodromy group \( G := \text{Gal}(\Omega | (\Omega \cap \mathbb{Q}(\mu, y))(t)) \) is normal in \( A := \text{Gal}(\Omega | \mathbb{Q}(\mu, y, t)) \). We now consider the specialization of \( f = p - tq \) at the point \( (0, \frac{1}{8}\sqrt{-10}) \in H \), denoted by

\[
(10) \quad f_0 = p_0 - tq_0 \in \mathbb{Q}(\sqrt{-10}, t)[X].
\]

Note that \( f_0 \) is still ramified over 4 points. Write \( \Omega_0 \) for the splitting field of \( f_0 \) over \( \mathbb{Q}(\sqrt{-10}, t) \). Then, by [11] Theorem III.6.4 and its proof, we find \( G \cong G_0 := \text{Gal}(\Omega_0 | (\Omega_0 \cap \mathbb{Q}(\mu))(t)) \). Using the fact that \( f_0(\mathbb{Q}(\mu)(t), X) \) and \( f_0(\mathbb{Q}(\mu)(t), X) \) split over \( \mathbb{Q}(\sqrt{-10}, t) \) into irreducible factors of degree 1, 62 and 31, 32, see file 3.3C, Lemma 3.1(a) implies that \( A_0 := \text{Gal}(\Omega_0 | \mathbb{Q}(\sqrt{-10}, t)) \) must be a 2-transitive group that contains a subgroup of index dividing 63 with orbit lengths 31 and 32. According Lemma 3.2 the group \( A_0 \) turns out to be \( \text{PSL}_6(2) \). Since \( A_0 \) is simple and \( G_0 \) is normal in \( A \) we find \( G \cong G_0 \cong \text{PSL}_6(2) \). As \( \text{PSL}_6(2) \) is also self-normalizing in \( S_{63} \), we end up with \( A \cong \text{PSL}_6(2) \).

(b) We will use the following notation: For a rational function \( P \) over a field of characteristic 0 we denote by \( Q_P \) the field extension of \( Q \) generated by the coefficients of \( P \).

The normalized discriminant \( \delta \) of \( f = p - tq \) is a polynomial in \( Q_F[t] \). Since the roots of \( \delta \) are given by the ramification locus of \( f \) its factorization in \( Q_F[t] \) is either of the form \( \delta = t^k(t - (1 + \sqrt{\lambda}))^\ell(t - (1 - \sqrt{\lambda}))^h \) or \( \delta = t^k(t^2 - 2t + 1 - \lambda)^\ell \) for some \( k, \ell, h \in \mathbb{N} \) where \( \lambda := \Psi_{24}(\mu) \). Both cases yield \( \lambda \in Q_F \), therefore \( Q(\lambda) \subseteq Q_F \subseteq Q(\mu, y) \) with \( [Q(\mu, y) : Q(\lambda)] = 48 \). Fix \((\mu_0, y_0) \in H \) such that \( \Psi_{24}(\mu_0) = \frac{1}{4} \). Then, for the specialization of \( F \) at \((\mu_0, y_0) \), denoted by \( F_{(\mu_0, y_0)} \), we compute \([Q_{F_{(\mu_0, y_0)}} : Q] = 48 \) using Magma. We end up with \( Q_F = Q(\mu, y) \). From the latter we see that \( \mu \) and \( y \) are rational functions in the coefficients of \( F \). Recall that for any \( \lambda_0 \in \mathbb{P}^1 \{0, 1, \infty\} \) we find distinct points \((\mu_1, y_1), \ldots, (\mu_{48}, y_{48}) \in H \) such that \( \Psi_{24}(\mu_k) = \lambda_0 \) for \( k = 1, \ldots, 48 \). If we specialize \( F \) at these points we obtain 48 distinct \( \text{PSL}_6(2) \)-covers \( F_{(\mu_1, y_1)}, \ldots, F_{(\mu_{48}, y_{48})} \) with ramification locus \((0, \infty, 1 \pm \sqrt{\lambda_0}) \) and ramification structure \( C \), which are all normalized with respect to inner Möbius transformations (in the sense of (ii)). Therefore, all covers \( F_{(\mu_1, y_1)}, \ldots, F_{(\mu_{48}, y_{48})} \) lie in \( F \) and correspond to all distinct 48 quadruples in \( \text{SNI}^0(C) \). As a consequence, each element in \( F \) can be obtained uniquely via specialization. \( \square \)
Remark. By looking at Theorem 3.3(b) and its proof we do not get any information about the specialization behaviour of $F$ at a point $(\mu_0, y_0) \in \mathcal{H}$ with $\Psi_{24}(\mu_0) \in \{0, 1, \infty\}$. Assume, the specialization of $F$ at $(\mu_0, y_0)$, denoted by $F(\mu_0, y_0)$, is a degree-63 cover, then one of the following cases occurs:

| $\Psi_{24}(\mu_0)$ | ramification locus of $F(\mu_0, y_0)$ | ramification structure of $F(\mu_0, y_0)$ |
|---------------------|---------------------------------------|----------------------------------------|
| 0                   | $\{0, 1, \infty\}$                    | contains $C_1, C_2$                    |
| 1                   | $\{0, 2, \infty\}$                    | contains $C_2, C_3$                    |
| $\infty$            | $\{0, \infty\}$                       | only contains $C_1$                    |

In none of these cases $\text{PSL}_6(2)$ is the monodromy group of $F(\mu_0, y_0)$. Using Magma we see that $\text{PSL}_6(2)$ does not contain generating tuples of length at most 3 satisfying the product 1 condition that correspond to the respective conjugacy classes.

With a little more effort we can deduce from Theorem 3.3 that $\text{PSL}_6(2)$ does not occur as the monodromy group of a rational function in $\mathbb{Q}(X)$ ramified over at least 4 points.

In order to achieve this result, we still have to study $\text{PSL}_6(2)$-covers with ramification structure $C$ and ramification locus of type $(0, \infty, \pm \sqrt{c})$. These covers can be calculated explicitly by deforming the ramification locus of covers contained in $\mathcal{F}$ via Newton iteration by assembling the defining equations explained in subsection 2.2.

**Theorem 3.4.** Let $K$ be the degree-24 number field, $c \in K$ the non-square and $p, q \in K[X]$ the monic polynomials given in the ancillary file 3.4A. Then the Galois group $A := \text{Gal}(\mathbb{Q} - t q \mid K(t))$ is isomorphic to $\text{PSL}_6(2)$ in its natural 2-transitive action on 63 elements. The ramification structure is given by $(2^{28}.1^7, 2^{16}.1^{31}, 3^{20}.1^3, 3^{20}.1^3)$ and the ramification locus with respect to $t$ is given by $(0, \infty, \sqrt{c}, -\sqrt{c})$.

**Proof.** In the same fashion as in the proof of Theorem 3.3(a) it can be calculated easily that the ramification locus of $p - tq$ is indeed given by $(0, \infty, \sqrt{c}, -\sqrt{c})$ with ramification structure $(2^{28}.1^7, 2^{16}.1^{31}, 3^{20}.1^3, 3^{20}.1^3)$, see file 3.4B.

Let $\Omega$ be the splitting field of $p - tq$ over $K(t)$. Recall that the geometric monodromy group $G := \text{Gal}(\Omega \mid (\Omega \cap K)(t))$ is normal in $A$. Let $p = (67, a + 7)$ and $q = (67, a + 42)$ be the unique prime ideals of norm 67 in the ring of integers $\mathcal{O}_K$ of $K$ where $a$ denotes the primitive element of $K$ used in file 3.4A. Write $p_\mathbb{Q}$ and $q_\mathbb{Q}$ for the reduction of $p$ and $q$ modulo $\mathbb{p}$. Accordingly, we define $A_p := \text{Gal}(\Omega_p \mid (\mathcal{O}_K/p)(t))$ and $G_p := \text{Gal}(\Omega_p \mid (\Omega_p \cap \mathcal{O}_K/p)(t))$ with $\Omega_p$ being the splitting field of $p_\mathbb{Q} - tq_\mathbb{Q}$ over $(\mathcal{O}_K/p)(t)$. Again, $G_p$ is normal in $A_p$. Of course, we will use the same notation for the reduction modulo $q$. 

Note that $p_q - p_q(t)q_p$ and $p_q - 16q(t)q_p$ split into irreducible factors of 1, 62 and 31, 32 over $(O_{\mathbb K}/p)(t) \cong \mathbb F_{67}(t)$. Therefore, by Lemma 3.1 and 3.2 the group $A_p$ must be isomorphic to $\mathrm{PSL}_6(2)$. As $A_p$ is simple, we see that $A_p$ and $G_p$ coincide. Since $p$ is a prime of good reduction for $p - tq$, we have $G_p \cong G$ by a theorem of Beckmann, see [11] Proposition I.10.9. Due to the fact that $\mathrm{PSL}_6(2)$ is self-normalizing in $S_{63}$ we end up with $A = \mathrm{PSL}_6(2)$.

\[\square\]

**Corollary 3.5.** The group $\mathrm{PSL}_6(2)$ does not occur as the monodromy group of a rational function in $\mathbb Q(X)$ ramified over at least 4 points.

**Proof.** Suppose, there exists a rational function $f$ defined over $\mathbb Q$ ramified over at least 4 points and monodromy group $\mathrm{PSL}_6(2)$. As $\mathrm{PSL}_6(2)$ is simple any non-trivial decomposition $f = g \circ h$ implies $\mathrm{Mon}(h) \cong \mathrm{PSL}_6(2)$, therefore we may assume that $f$ is indecomposable with primitive monodromy group. A Magma computation shows that $C$ is the only genus-0 class vector of length at least 4 containing generating tuples for $\mathrm{PSL}_6(2)$ in a primitive permutation action, thus $f$ has degree 63 with ramification structure $C = (C_1, C_2, C_3, C_4)$. The branch cycle lemma, see [14] Lemma 2.8, asserts that the ramification locus of $f$ is of the form $(a_1, a_2, a_3, a_4)$ where $a_1, a_2 \in \mathbb P^1(\mathbb Q)$ and $a_3, a_4$ fulfil a degree-2 relation over $\mathbb Q$. Hence, after applying a suitable outer Möbius transformation we may assume — without altering the field of definition — that $f$ either has ramification locus $(0, \infty, 1 \pm \sqrt{\lambda_0})$ or $(0, \infty, \pm \sqrt{\lambda_0})$ for some $\lambda_0 \in \mathbb P^1(\mathbb Q) \setminus \{0, 1, \infty\}$. We will now study both cases:

1. case $(0, \infty, 1 \pm \sqrt{\lambda_0})$: Using the notation and result from Theorem 3.3(b) there exist 48 specialized covers $F(\mu_k, y_{k_1}), \ldots, F(\mu_k, y_{k_4}) \in \mathcal F$ with $\Psi(F(\mu_k, y_k)) = \lambda_0$ for $k \in \{1, \ldots, 48\}$. Up to inner Möbius transformations $f$ has to coincide with $F(\mu_k, y_k)$ for some $k \in \{1, \ldots, 48\}$, therefore $F(\mu_k, y_k)$ also has to be defined over $\mathbb Q$, in particular $(\mu_k, y_k)$ must be a $\mathbb Q$-rational point on $\mathcal H$ with $\lambda_0 = \Psi_{24}(\mu_k) \notin \{0, 1, \infty\}$.

Since $\mathcal H$ is given by a hyperelliptic genus-3 model and its Jacobian is of Mordell-Weil rank 1, Chabauty’s algorithm (with the implementation in Sage [13] presented in [11]) gives us the complete list of $\mathbb Q$-rational points of $\mathcal H$. We find $\mu_k \in \{1, \frac{1}{3}, \frac{2}{3}, \frac{5}{9}, \infty\}$ and for all these values we see $\Psi_{24}(\mu_k) \in \{0, 1\}$, a contradiction.

2. case $(0, \infty, \pm \sqrt{\lambda_0})$: After a suitable scaling process Theorem 3.4 gives us 48 different $\mathrm{PSL}_6(2)$-covers $f_1, \ldots, f_{48}$ that satisfy condition (ii) with ramification locus $(0, \infty, \pm 1)$. Each cover is defined over a degree-48 number field.

Since $\frac{f_1}{\sqrt{\lambda_0}}$ has ramification locus $(0, \infty, \pm 1)$ the cover $\frac{f_2}{\sqrt{\lambda_0}}$ defined over a quadratic number field has to coincide with $f_k$ for some $k \in \{1, \ldots, 48\}$ up to inner Möbius transformations, a contradiction.
This shows that $\text{PSL}_6(2)$ cannot be the monodromy group of $f$. \hfill \square

4. Non-regular extensions of $\mathbb{Q}(t)$ with Galois group $\text{Aut}(\text{PSL}_6(2))$

Although our approach does not yield a $\text{PSL}_6(2)$-polynomial over $\mathbb{Q}(t)$ we at least get an explicit non-regular realization of $\text{Aut}(\text{PSL}_6(2))$ over $\mathbb{Q}(t)$.

**Theorem 4.1.** Let $f_0 \in \mathbb{Q}(\sqrt{-10}, t)[X]$ be the polynomial from (110), then the Galois group of $f_0f_0$ over $\mathbb{Q}(t)$ is isomorphic to $\text{Aut}(\text{PSL}_6(2))$ in its imprimitive action on 126 points.

**Proof.** The Galois groups of $f_0 = p_0 - tq_0$ and $\overline{f}_0 = \overline{p}_0 - t\overline{q}_0$ over $\mathbb{Q}(\sqrt{-10}, t)$ are isomorphic to the primitive group $\text{PSL}_6(2)$. According to Lemma 3.1(b) both $f_0$ and $\overline{f}_0$ have the same splitting field $\Omega$ over $\mathbb{Q}(\sqrt{-10}, t)$ since $f_0(\overline{p}(\overline{t}))X$ is reducible over $\mathbb{Q}(\sqrt{-10}, t)$. Let $\Omega'$ be the splitting field of $f_0\overline{f}_0$ over $\mathbb{Q}(t)$ and $G := \text{Gal}(\Omega' / \mathbb{Q}(t))$. Clearly, $\Omega' \leq \Omega$. Since $f_0\overline{f}_0$ is irreducible over $\mathbb{Q}(t)$ but obviously reducible over $\mathbb{Q}(\sqrt{-10}, t)$ we find $\sqrt{-10} \in \Omega'$, therefore $\Omega' = \Omega$ and $H := \text{Gal}(\Omega / \mathbb{Q}(\sqrt{-10}, t))$ is a subgroup of $G$ with index $[G : H] = [\mathbb{Q}(\sqrt{-10}, t) : \mathbb{Q}(t)] = 2$.

Let $\varphi : G \to \text{Aut}(H)$ be the conjugation action of $G$ on the normal subgroup $H$, $x$ a root of $f_0$ and $y$ a root of $\overline{f}_0$. The point stabilizers $G_x$ and $G_y$ are conjugate in $G$ but not in $H$, because $x$ and $y$ have the same minimal polynomial over $\mathbb{Q}(t)$ but not over $\mathbb{Q}(\sqrt{-10}, t)$, therefore $\varphi(H) < \varphi(G)$ and $\text{Inn}(H) = \varphi(H) < \varphi(G) \leq \text{Aut}(H)$. Since $|\text{Out}(\text{PSL}_6(2))| = 2$ and $H \cong \text{PSL}_6(2)$ this implies $\varphi(G) = \text{Aut}(H)$. In combination with $|G| = 2 \cdot |H| = |\text{Aut}(H)|$ we see that $\varphi$ is an isomorphism. \hfill \square

Coincidentally, $\text{PSL}_6(2)$ also happens to contain a rigid, $\mathbb{Q}(\sqrt{-10})$-rational genus-0 generating triple sharing similar properties, in particular leading to (another) non-regular $\text{Aut}(\text{PSL}_6(2))$-extension of $\mathbb{Q}(t)$. For the explicit realization we again apply the method explained in [4].

**Theorem 4.2.** Let $p, q \in \mathbb{Q}(\sqrt{-10}, t)[X]$ be the polynomials of degree 63 from the ancillary file 4.2A.

(a) The polynomial $p - tq$ has Galois group $\text{PSL}_6(2) \leq S_{63}$ over $\mathbb{Q}(\sqrt{-10}) / \mathbb{Q}(t)$ with ramification locus $(0, 1, \infty)$ and ramification structure $(21^3, 4^8, 2^{12}, 1^7, 2^{28}, 1^7)$.

(b) The product $(p - tq)(\overline{p} - \overline{tq})$ has Galois group $\text{Aut}(\text{PSL}_6(2)) \leq S_{126}$ over $\mathbb{Q}(t)$.

**Proof.** The ramification can be checked by inspecting the inseparability behaviour of $p$, $q$ and $p - q$. A computation with Magma yields that $p - \frac{p(t)}{\overline{q}(t)}q$ and $p - \frac{p(t)}{\overline{q}(t)}q$ split in $\mathbb{Q}(\sqrt{-10}, t)[X]$ into irreducible factors of degree 1, 62 and 31, 32, see file 4.2B. By repeating the arguments from the previous proofs both assertions follow. \hfill \square
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