ON THE CONSTANT FACTOR IN SEVERAL RELATED ASYMPTOTIC ESTIMATES

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Abstract. We establish formulas for the constant factor in several asymptotic estimates related to the distribution of integer and polynomial divisors. The formulas are then used to approximate these factors numerically.

1. Introduction

A number of asymptotic estimates [7, 19, 20], related to the distribution of divisors of integers and of polynomials, contain a constant factor that is as yet undetermined. In this note, we give an explicit formula for this constant as the sum of an infinite series. As a result, we are able to approximate this factor numerically in several instances and improve some of the error terms in [19, 20]. For more extensive background information, we refer the reader to [7, 19, 20] and the references therein.

We begin by recalling the general setup from [19]. Let \( \theta \) be a real-valued arithmetic function. Let \( B = B_\theta \) be the set of positive integers containing \( n = 1 \) and all those \( n \geq 2 \) with prime factorization \( n = p_1^{a_1} \cdots p_k^{a_k} \), \( p_1 < p_2 < \ldots < p_k \), which satisfy

\[
p_i \leq \theta(p_1^{a_1} \cdots p_i^{a_i-1}) \quad (1 \leq i \leq k).
\]

We write \( B(x) \) to denote the number of integers \( n \leq x \) in \( B \). Theorem 1.2 of [19] states that, if

\[
\theta(1) \geq 2, \quad n \leq \theta(n) \leq An (\log 2n)^a (\log \log 3n)^b \quad (n \geq 2) \tag{1}
\]

for suitable constants \( A \geq 1, a < 1, b \), then

\[
B(x) = \frac{c_\theta x}{\log x} \left( 1 + O_\theta ((\log x)^{a-1}(\log \log x)^b) \right), \tag{2}
\]

for some positive constant \( c_\theta \). This result still holds if \( a = 1 \) and \( b < -1 \), provided \( b \) is replaced by \( b + 1 \) in the error term of (2).

Theorem 1. Assume \( \theta \) satisfies (1). The constant \( c_\theta \) in (2) is given by

\[
c_\theta = \frac{1}{1 - e^{-\gamma}} \sum_{n \in \mathcal{B}} \frac{1}{n} \left( \sum_{p \leq \theta(n)} \frac{\log p}{p-1} - \log n \right) \prod_{p \leq \theta(n)} \left( 1 - \frac{1}{p} \right),
\]

where \( \gamma \) is Euler’s constant and \( p \) runs over primes.
1.1. **Practical numbers.** A well known example is the set $\mathcal{P}$ of practical numbers [14], i.e. integers $n$ with the property that every natural number $m \leq n$ can be expressed as a sum of distinct positive divisors of $n$. Stewart [15] and Sierpinski [13] found that $\mathcal{P} = \mathcal{B}_\theta$ if $\theta(n) = \sigma(n) + 1$, where $\sigma(n)$ denotes the sum of the positive divisors on $n$. Since $n + 1 \leq \sigma(n) + 1 \ll n \log \log 3n$, (2) shows that the number of practical numbers up to $x$ satisfies

$$P(x) = \frac{cx}{\log x} \left\{ 1 + O\left( \frac{\log \log x}{\log x} \right) \right\},$$

(3)

for some $c > 0$. Theorem 1 states that

$$c = \frac{1}{1 - e^{-\gamma}} \sum_{n \in \mathcal{P}} \frac{1}{n} \left( \sum_{p \leq \sigma(n)+1} \frac{\log p}{p-1} - \log n \right) \prod_{p \leq \sigma(n)+1} \left( 1 - \frac{1}{p} \right),$$

(4)

from which we will derive the following bounds.

**Corollary 1.** The constant $c$ in (3) satisfies $1.311 < c < 1.693$.

Corollary 1 is consistent with the empirical estimate $c \approx 1.341$ given by Margenstern [4]. The lack of precision in Corollary 1, when compared with Corollary 2, is due to the fact that $\theta(n)/n$ is not bounded when $\theta(n) = \sigma(n) + 1$, which makes it more difficult to estimate the tail of the series (4).

1.2. **The distribution of divisors.** Another example are integers with $t$-dense divisors [11, 16], i.e. integers $n$ whose divisors $1 = d_1 < d_2 < \ldots < d_{\tau(n)} = n$ satisfy $d_{i+1} \leq td_i$ for all $1 \leq i < \tau(n)$. Tenenbaum [16, Lemma 2.2] showed that these integers are exactly the members of $\mathcal{B}_\theta$ if $\theta(n) = tn$. When $t \geq 2$ is fixed, (2) implies that the number of such integers up to $x$ satisfies

$$D(x, t) = \frac{c_t x}{\log x} \left\{ 1 + O_t \left( \frac{1}{\log x} \right) \right\}.$$

(5)

We can now give numerical approximations for $c_t$ based on Theorem 1. The details behind these calculations will be described in Section 5.

**Corollary 2.** Table 1 shows values of the factor $c_t$ appearing in (5).

| $t$ | $c_t$     | $t$ | $c_t$     | $t$ | $c_t$     | $t$ | $c_t$     |
|-----|-----------|-----|-----------|-----|-----------|-----|-----------|
| 2   | 1.2248... | 6   | 3.247...  | 20  | 5.742...  | 10^4| 14.449... |
| $e$ | 1.5242... | 7   | 3.644...  | 40  | 7.210...  | 10^4| 19.689... |
| 3   | 2.0554... | 8   | 3.850...  | 60  | 8.113...  | 10^5| 24.937... |
| 4   | 2.4496... | 9   | 4.041...  | 80  | 8.761...  | 10^6| 30.187... |
| 5   | 2.9541... | 10  | 4.227...  | 100 | 9.248...  | 10^7| 35.435... |

**Table 1.** Truncated values of $c_t$ derived from Theorem 1.
For example, the number of integers \( n \leq x \), which have a divisor in the interval \((y, 2y]\) for every \( y \in [1, n) \), is

\[
D(x, 2) = 1.2248\ldots \frac{x}{\log x} \left\{ 1 + O\left( \frac{1}{\log x} \right) \right\},
\]

so that these integers are about 22.5\% more numerous than the primes.

Corollary 1.1 of [19] gives an estimate for \( D(x, t) \) which holds uniformly in \( t \). It states that, uniformly for \( x \geq t \geq 2 \),

\[
D(x, t) = \frac{c_t x}{\log(tx)} \left\{ 1 + O\left( \frac{1}{\log x} + \frac{\log^2 t}{\log^2 x} \right) \right\}, \tag{6}
\]

where \( c_t = (1 - e^{-\gamma})^{-1} \log t + O(1) \). We can improve the estimate for \( c_t \) with the help of Theorem 1.

**Corollary 3.** Let \( c_t \) be the factor in (5) and (6). Define \( \delta_t \) implicitly by

\[
c_t = \frac{\log(te^{-\gamma}) + \delta_t}{1 - e^{-\gamma}}. \tag{7}
\]

We have \( \delta_t \ll \exp(\sqrt{\log t}) \) and

\[
|\delta_t| \leq \frac{0.084}{\log^2 t} \quad (t \geq 2^{25}). \tag{8}
\]

Assuming the Riemann hypothesis, we have \( |\delta_t| \leq \frac{\log^2 t}{\sqrt{t}} \) for \( t \geq 55 \).

| Table 2. Truncated values of \( c_t \) derived from (7) and (8). |
| --- |
| \( t \) | \( c_t \) | \( t \) | \( c_t \) | \( t \) | \( c_t \) |
| \( 10^8 \) | 40.68... | \( 10^{20} \) | 103.69... | \( 10^{30} \) | 313.7176... |
| \( 10^9 \) | 45.93... | \( 10^{40} \) | 156.200... | \( 10^{40} \) | 418.7289... |
| \( 10^{10} \) | 51.189... | \( 10^{60} \) | 208.7063... | \( 10^{100} \) | 523.7401... |

Assuming the Riemann hypothesis, the last entry in Table 2 is

\[
c_{10^{100}} = 523.74019053615422813260729554671054989578274943\ldots
\]

Combining the estimate (6) with Corollary 3, we obtain the following improvement of [19, Corollary 1.2].

**Corollary 4.** Uniformly for \( x \geq t \geq 2 \), we have

\[
D(x, t) = \frac{x \log(te^{-\gamma})}{(1 - e^{-\gamma}) \log(tx)} \left\{ 1 + O\left( \frac{1}{\log x} + \frac{\log^2 t}{\log^2 x} + \frac{1}{\exp(\sqrt{\log t})} \right) \right\}.
\]

The error term \( O(\exp(\sqrt{\log t})) \) can be replaced by \( O((\log t)/\sqrt{t}) \) if the Riemann hypothesis holds.
1.3. \(\varphi\)-practical numbers. An integer \(n\) is called \(\varphi\)-practical [18] if \(X^n - 1\) has divisors in \(\mathbb{Z}[X]\) of every degree up to \(n\). The name comes from the fact that \(X^n - 1\) has this property if and only if each natural number \(m \leq n\) is a subsum of the multiset \(\{\varphi(d) : d|n\}\), where \(\varphi\) is Euler’s function. These numbers were first studied by Thompson [18], who showed that their counting function \(P_\varphi(x)\) has order of magnitude \(x/\log x\). Pomerance, Thompson and the author [7] established the asymptotic result

\[
P_\varphi(x) = \frac{C x}{\log x} \left( 1 + O \left( \frac{1}{\log x} \right) \right),
\]

for some positive constant \(C\).

Although the set of \(\varphi\)-practical numbers, say \(\mathcal{A}\), is not exactly an example of a set \(\mathcal{B}_\varphi\) as described earlier, Thompson [18] showed that \(\mathcal{B}_{\theta_1} \subset \mathcal{A} \subset \mathcal{B}_{\theta_2}\), where \(\theta_1(n) = n + 1\) and \(\theta_2(n) = n + 2\). \(\mathcal{B}_{\theta_1}\) is the set of even \(\varphi\)-practical numbers, while the integers in \(\mathcal{B}_{\theta_2}\) are called \(\text{weakly }\varphi\)-practical in [18]. We can use Theorem 1 to estimate the constants \(c_{\theta_1}\) and \(c_{\theta_2}\).

**Corollary 5.** If \(\theta(n) = n + 1\), \(c_\varphi = 0.8622\ldots\). If \(\theta(n) = n + 2\), \(c_\varphi = 1.079\ldots\)

It follows that the constant \(C\) in (9) satisfies \(0.8622 < C < 1.080\). Our goal is to give a formula for the exact value of \(C\). As the proof of (9) in [7] is more general and applies to other similar sequences, so does Theorem 2 below. For simplicity, we assume \(\max(2, n) \leq \theta(n) \ll n\), as in [7]. Let \(P^+(n)\) denote the largest prime factor of \(n\) and put \(P^+(1) = 1\). For a given integer \(m\), which we call a \textit{starter}, let \(\mathcal{A}_m\) be the set of all integers of the form \(mp_1p_2\ldots p_k\), \(P^+(m) < p_1 < \ldots < p_k\), which satisfy \(p_i \leq \theta(mp_1\ldots p_i-1)\) for all \(1 \leq i \leq k\). Theorem 3.1 of [7] states that the counting function of \(\mathcal{A}_m\) satisfies

\[
A_m(x) = \frac{c_m x}{\log x} + O \left( \frac{\log^6(2m)x}{m \log^2 x} \right) \quad (m \geq 1, x \geq 2),
\]

for some constant \(c_m\).

Let \(\mathcal{S}\) be a set of natural numbers (starters) with the property that \(\mathcal{A}_m \cap \mathcal{A}_{m_2} = \emptyset\) for all \(m_1 \neq m_2 \in \mathcal{S}\), and \(\sum_{m \in \mathcal{S}} m^{-1} \log^6 m \ll 1\). Let \(\mathcal{A} = \bigcup_{m \in \mathcal{S}} A_m\) and assume that its counting function satisfies \(A(x) \ll x/\log x\). As in [7], summing (10) over \(m \in \mathcal{S}\) yields

\[
A(x) = \frac{C x}{\log x} + O \left( \frac{1}{\log^2 x} \right),
\]

where \(C = \sum_{m \in \mathcal{S}} c_m\). For \(n \in \mathcal{A}_m\), define \(s(n) = m\), the starter for \(n\).

**Theorem 2.** The constant \(C\) in (11) is given by

\[
C = \frac{6/\pi^2}{1 - e^{-\gamma}} \sum_{n \in \mathcal{A}} \frac{1}{n} \left( \sum_{p^+ (s(n)) < p^+ \leq \theta(n)} \frac{\log p}{p + 1} - \log \left( \frac{n}{s(n)} \right) \right) \prod_{p \leq \theta(n)} \left( 1 + \frac{1}{p} \right)^{-1},
\]

where \(s(n)\) is the starter for \(n\).
For the set of ϕ-practical numbers, θ(n) = n + 2 and the set of starters S will be described in Section 6.

**Corollary 6.** The constant C in (9) satisfies 0.945 < C < 0.967.

Corollary 6 is consistent with the empirical estimate C ≈ 0.96 given in [7, Section 6], which is based on values of Pϕ(2k) for k ≤ 34 and nonlinear regression.

### 1.4. Squarefree analogues

Let D∗ denote the set of squarefree integers with t-dense divisors and let D∗(x, t) be its counting function. With θ(n) = tn and m = 1, (10) shows that

\[ D^*(x, t) = c^*_t x \left\{1 + O_t \left(\frac{1}{\log x}\right)\right\}, \tag{12} \]

while Theorem 2 with S = \{1\}, s(n) = 1 and A = A1 = D∗ yields

\[ c^*_t = \frac{6/\pi^2}{1 - e^{-\gamma}} \sum_{n \in D^*_t} \frac{1}{n} \left(\sum_{p \leq tn} \frac{\log p}{p+1} - \log n\right) \prod_{p \leq tn} \left(1 + \frac{1}{p}\right)^{-1}. \tag{13} \]

**Corollary 7.** Table 3 shows values of the factor c∗ t appearing in (12).

| t  | c∗ t  |
|----|------|
| 2  | 0.06864.. |
| e  | 0.1495..  |
| 3  | 0.2618..  |
| 4  | 0.4001..  |
| 5  | 0.5898..  |
| 6  | 0.7142..  |
| 7  | 0.923...  |
| 8  | 1.0065... |
| 9  | 1.0978... |
| 10 | 1.1868... |
| 20 | 2.017...  |
| 40 | 2.854...  |
| 60 | 3.389...  |
| 80 | 3.778...  |
| 100| 4.066...  |
| 103| 7.208...  |
| 104| 10.390... |
| 105| 13.580... |
| 106| 16.771... |
| 107| 19.963... |

**Table 3.** Truncated values of c∗ t derived from (13).

The squarefree analogue of Corollary 3 is as follows.

**Corollary 8.** Let c∗ t be the factor in (12). Define δ∗ t implicitly by

\[ c^*_t = \frac{\log t - \gamma - h + \delta^*_t}{(1 - e^{-\gamma})\pi^2/6}, \tag{14} \]

where

\[ h = \sum_{p \geq 2} \frac{2\log p}{p^2 - 1} = 1.139921... \tag{15} \]

We have δ∗ t ≪ \exp(-\sqrt{\log t}) and

\[ |\delta^*_t| \leq \frac{0.084}{\log^2 t} \quad (t \geq 2^{25}). \tag{16} \]

Assuming the Riemann hypothesis, we have |δ∗ t| ≤ \frac{\log^2 t}{t\sqrt{t}} for t ≥ 55.
We briefly mention two other squarefree analogues. The estimate (10) and Theorem 2, with $\theta(n) = n + 2$ and $s(n) \equiv 1$, give the asymptotic estimate and the constant factor for the count of squarefree $\varphi$-practical numbers. For the count of squarefree practical numbers, one would first derive (10) under the condition $\theta(n) \ll n \log \log n$, which introduces an extra factor of $\log \log x$ in the error term. Theorem 2 then gives the constant factor with $\theta(n) = \sigma(n) + 1$ and $s(n) \equiv 1$.

1.5. Polynomial divisors over finite fields. Let $F_q$ be the finite field with $q$ elements. Let $f_q(n, m)$ be the proportion of polynomials $F$ of degree $n$ over $F_q$, with the property that the set of degrees of divisors of $F$ has no gaps of size greater than $m$. For example, $f_q(n, 1)$ is the proportion of polynomials of degree $n$ over $F_q$ which have a divisor of every degree up to $n$. Corollary 1 of [20] states that, uniformly for $q \geq 2$, $n \geq m \geq 1$, we have

$$f_q(n, m) = \frac{c_q(m)m}{n + m} \left\{ 1 + O \left( \frac{1}{n} + \frac{m^2}{n^2} \right) \right\},$$

where $0 < c_q(m) = (1 - e^{-\gamma})^{-1} + O \left( m^{-1} q^{-(m+1)\tau} \right)$ and $\tau = 0.205466 \ldots$. The estimate (17) can be viewed as the polynomial analogue of (6). By adapting the proof of Theorem 1 to polynomials over finite fields, we obtain an expression for the factor $c_q(m)$.

**Theorem 3.** The factor $c_q(m)$ in (17) is given by

$$c_q(m) = \frac{1/m}{1 - e^{-\gamma}} \sum_{n \geq 0} f_q(n, m) \left( \sum_{k=1}^{n+m} \frac{k I_k}{q^k - 1} - n \right) \prod_{k=1}^{n+m} \left( 1 - \frac{1}{q^k} \right)^{I_k},$$

where $I_k$ is the number of monic irreducible polynomials of degree $k$ over $F_q$.

**Corollary 9.** Table 5 shows values of the factor $c_q(m)$ appearing in (17).

For example, the proportion of polynomials of degree $n$ over $F_2$, which have a divisor of every degree up to $n$, is given by

$$3.400335 \ldots \left\{ 1 + O \left( \frac{1}{n} \right) \right\}. $$

Theorem 3 leads to an improvement of the asymptotic estimate for $c_q(m)$ mentioned below (17).

| $t$ | $c_t^*$ |
|-----|--------|
| $10^8$ | 23.15... |
| $10^9$ | 26.34... |
| $10^{10}$ | 29.53... |
| $10^{11}$ | 32.72... |

| $t$ | $c_t^*$ |
|-----|--------|
| $10^{12}$ | 35.91... |
| $10^{13}$ | 39.10... |
| $10^{14}$ | 42.30... |
| $10^{15}$ | 45.50... |

| $t$ | $c_t^*$ |
|-----|--------|
| $10^{16}$ | 48.70... |
| $10^{17}$ | 51.90... |
| $10^{18}$ | 55.11... |
| $10^{19}$ | 58.31... |

**Table 4.** Truncated values of $c_t^*$ derived from (14) and (16).
Corollary 10. Uniformly for $q \geq 2$, $m \geq 1$, we have
\[ c_q(m) = \frac{1}{1 - e^{-\gamma}} + O\left(\frac{1}{m^2 q^{(m+1)/2}}\right). \]

Combining Corollary 10 with (17), we obtain the following improvement of [20, Corollary 2]. Corollary 11 is the polynomial analogue of Corollary 4.

Corollary 11. Uniformly for $q \geq 2$, $n \geq m \geq 1$, we have
\[ f_q(n, m) = \frac{m}{(1 - e^{-\gamma})(n + m)} \left\{ 1 + O\left(\frac{1}{n} + \frac{m^2}{n^2} + \frac{1}{m^2 q^{(m+1)/2}}\right) \right\}. \]

2. Proof of Theorem 1

Let $\chi(n)$ be the characteristic function of the set $B_\theta$. Theorem 1 of [21] shows that
\[ 1 = \sum_{n \geq 1} \frac{\chi(n)}{n} \prod_{p \leq \theta(n)} \left(1 - \frac{1}{p}\right) \quad (18) \]
if and only if $B(x) = o(x)$. Lemma 1 extends this to an identity involving Dirichlet series for $\text{Re}(s) > 1$, valid without any conditions on $\theta$ or $B(x)$.

Lemma 1. For $\text{Re}(s) > 1$ we have
\[ 1 = \sum_{n \geq 1} \frac{\chi(n)}{n^s} \prod_{p \leq \theta(n)} \left(1 - \frac{1}{p^s}\right). \quad (19) \]

Proof. Let $P^-(n)$ denote the smallest prime factor of $n$ and put $P^-(1) = \infty$. Each natural number $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $p_1 < p_2 < \ldots < p_k$, factors uniquely as $m = nr$, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_j^{\alpha_j} \in B_\theta$ and $P^-(r) = p_{j+1} > \theta(n)$. It follows that, for $\text{Re}(s) > 1$,
\[ \zeta(s) = \sum_{m \geq 1} \frac{1}{m^s} = \sum_{n \geq 1} \frac{\chi(n)}{n^s} \prod_{p > \theta(n)} \left(1 - \frac{1}{p^s}\right)^{-1}. \]
Dividing by $\zeta(s) = \prod_{p \geq 2} (1 - p^{-s})^{-1}$ yields the result. \qed
Lemma 2. For \( \Re(s) > 1 \) we have

\[
0 = \sum_{n \geq 1} \frac{\chi(n)}{n^s} \left( \sum_{p \leq \theta(n)} \frac{\log p}{p^s - 1} - \log n \right) \prod_{p \leq \theta(n)} \left( 1 - \frac{1}{p^s} \right).
\]

(20)

Proof. Differentiate (19) with respect to \( s \).

While (18) shows that (19) remains valid at \( s = 1 \) if \( B(x) = o(x) \), (20) does not hold at \( s = 1 \). To see this, note that each term on the right-hand side of (20) is non-negative if \( s = 1 \) and \( \theta(n) = tn \), where \( t \) is a sufficiently large constant. Define

\[
\alpha = \sum_{n \geq 1} \frac{\chi(n)}{n^s} \left( \sum_{p \leq \theta(n)} \frac{\log p}{p^s - 1} - \log n \right) \prod_{p \leq \theta(n)} \left( 1 - \frac{1}{p^s} \right),
\]

\[
F_N(s) = \sum_{1 \leq n \leq N} \frac{\chi(n)}{n^s} \left( \sum_{p \leq \theta(n)} \frac{\log p}{p^s - 1} - \log n \right) \prod_{p \leq \theta(n)} \left( 1 - \frac{1}{p^s} \right),
\]

\[
G_N(s) = \sum_{n > N} \frac{\chi(n)}{n^s} \left( \log n - \sum_{p \leq \theta(n)} \frac{\log p}{p^s - 1} \right) \prod_{p \leq \theta(n)} \left( 1 - \frac{1}{p^s} \right),
\]

and let \( s_N = 1 + 1/\log^2 N \) for \( N \geq 2 \). We have \( F_N(s_N) = G_N(s_N) \) by Lemma 2, \( \lim_{N \to \infty} F_N(s_N) = \alpha \) by Lemma 3, and \( \lim_{N \to \infty} G_N(s_N) = (1 - e^{-\gamma})c_\theta \) by Lemma 4. Thus \( \alpha = (1 - e^{-\gamma})c_\theta \), which establishes Theorem 1. It remains to prove Lemmas 3 and 4, where we will assume

\[
n \leq \theta(n) \ll n \log 2n(\log \log 3n)^b
\]

(21)

for some constant \( b < -1 \).

Lemma 3. If \( \theta \) satisfies (21), \( \lim_{N \to \infty} F_N \left( 1 + 1/\log^2 N \right) = \alpha \).

Proof. Let \( s = s_N = 1 + 1/\log^2 N \) and write

\[
|F_N(s) - \alpha| \leq |F_N(s) - F_N(1)| + |F_N(1) - \alpha| = E_1 + E_2,
\]

say. Since \( B(x) \ll x/\log x \) and \( \log n \leq \log \theta(n) \leq \log n + O(\log \log n) \),

\[
E_2 \ll \sum_{n > N} \frac{\chi(n)}{n \log \theta(n)} |\log \theta(n) + O(1) - \log n| \ll \sum_{n > N} \frac{\log \log n}{n \log^2 n} \ll \frac{\log \log N}{\log N}.
\]

To estimate \( E_1 \), note that for \( n \leq N \),

\[
n^{-s} = n^{-1}(1 + O((s - 1) \log n)) = n^{-1}(1 + O(1/\log N)).
\]

Similarly, \( p^s - 1 = (p - 1)(1 + O((s - 1) \log p)) \), so that

\[
\sum_{p \leq \theta(n)} \frac{\log p}{p^s - 1} = O((s - 1) \log^2 n) + \sum_{p \leq \theta(n)} \frac{\log p}{p - 1}.
\]
By the mean value theorem, there is an $\tilde{s}$ with $1 < \tilde{s} < s$ such that

$$0 < \prod_{p \leq \theta(n)} \left(1 - \frac{1}{p^s}\right) - \prod_{p \leq \theta(n)} \left(1 - \frac{1}{p}\right) = (s-1) \sum_{p \leq \theta(n)} \frac{\log p}{p^s - 1} \prod_{p \leq \theta(n)} \left(1 - \frac{1}{p^s}\right)$$

$$\ll (s-1) \log(\theta(n)) \prod_{p \leq \theta(n)} \left(1 - \frac{1}{p^s}\right) \ll (1/\log N) \prod_{p \leq \theta(n)} \left(1 - \frac{1}{p^s}\right),$$

for $n \leq N$. These estimates show that

$$F_N(s) = \sum_{1 \leq n \leq N} \frac{\chi(n)}{n} \left(1 + O\left(\frac{1}{\log N}\right)\right) \cdot \left(O\left(\frac{\log^2 n}{\log^2 N}\right) + \sum_{p \leq \theta(n)} \frac{\log p}{p-1} - \log n\right) \prod_{p \leq \theta(n)} \left(1 - \frac{1}{p}\right).$$

The contribution to the last sum from each of the two error terms is $\ll 1/\log N$. Hence $E_1 \ll 1/\log N$ and the proof of Lemma 3 is complete. □

**Lemma 4.** If $\theta$ satisfies (21), \( \lim_{N \to \infty} G_N \frac{(1 + 1/\log^2 N)}{1 - e^{-\gamma}c_\theta} = 1 \).

**Proof.** Let $s = s_N = 1 + 1/\log^2 N$ and write $I(y) = \int_0^y (1 - e^{-t}) \frac{dt}{t}$. Lemma 9.1 of [17] shows that

$$\prod_{p \leq \theta(n)} \left(1 - \frac{1}{p^s}\right) = \exp\left\{-\gamma + I((s-1) \log \theta(n))\right\} \left(1 + O\left(\frac{1}{\log \theta(n)}\right)\right)$$

$$= \frac{\exp\left\{-\gamma + I((s-1) \log n)\right\}}{\log n} \left(1 + O\left(\frac{\log \log n}{\log n}\right)\right),$$

by (21). By the prime number theorem,

$$\sum_{p \leq \theta(n)} \frac{\log p}{p^s - 1} = \sum_{p \leq n} \frac{\log p}{p^s - 1} + \sum_{n < p \leq \theta(n)} \frac{\log p}{p^s - 1}$$

$$= O(1) + \frac{1 - n^{1-s}}{s-1} + O(1/\log(\theta(n)/n)), $$

for $n > N$. The details behind the estimate for the sum over $p \leq n$ are explained in [17, Ex.1 of Sec.III.5]. For the sum over $n < p \leq \theta(n)$, note that the terms are $\ll \log(p)/p$. With these two estimates we have

$$G_N(s) = \sum_{n > N} \frac{\chi(n)}{n^s} \left(\log n - \frac{1 - n^{1-s}}{s-1} + O(\log \log n)\right) \times \frac{\exp\left\{-\gamma + I((s-1) \log n)\right\}}{\log n} \left(1 + O\left(\frac{\log \log n}{\log n}\right)\right).$$

Since $e^{I(y)} \ll 1 + y$ and $B(x) \ll x/\log x$, the contribution to the last sum from each of the two error terms is $\ll \log \log N/\log N$. Abel summation
and the asymptotic estimate (2) show that

\[ G_N(s) = o(1) \]
\[ + \int_N^\infty \frac{c_0}{y^s \log y} \left( \log y - \frac{1 - y^1 - s}{s - 1} \right) \exp \left\{ -\gamma + I((s - 1) \log y) \right\} \, dy, \]

as \( N \to \infty \). With the change of variables \( u = (s - 1) \log y \), this simplifies to

\[ G_N(s) = o(1) + e^{-\gamma} c_0 \int_{1/\log N}^\infty \frac{u - 1 + e^{-u}}{u^2 e^u} \exp(I(u)) \, du. \]

Note that the integrand is equal to \((I'(u))^2 + I''(u)) \exp(I(u))\), so that an antiderivative is \(I'(u) \exp(I(u))\). Thus the last integral equals

\[ \lim_{u \to \infty} I'(u) \exp(I(u)) - I'(1/\log N) \exp(I(1/\log N)) = e^\gamma - 1 + o(1), \]

as \( N \to \infty \), since \( I(u) = \gamma + \log u + \int_u^\infty e^{-t} \, dt \) by [5, Ex.1 of Sec.7.2.1]. □

3. PROOF OF THEOREM 2

The proof of Theorem 2 closely follows that of Theorem 1.

**Lemma 5.** For \( m \geq 1 \) and Re\((s) > 1\) we have

\[ \frac{1}{m^s} \prod_{p \leq P^+(m)} \left( 1 + \frac{1}{p^s} \right)^{-1} = \sum_{n \in A_m} \frac{1}{n^s} \prod_{p \leq \theta(n)} \left( 1 + \frac{1}{p^s} \right)^{-1}. \quad (22) \]

**Proof.** Each natural number of the form \( mp_1 \cdots p_k, P^+(m) < p_1 < \cdots < p_k \), factors uniquely as \( nr \), where \( n = mp_1 p_2 \cdots p_j \in A_m \) and \( P^-(r) = p_{j+1} > \theta(n) \). Thus, for Re\((s) > 1\),

\[ \frac{1}{m^s} \prod_{p > P^+(m)} \left( 1 + \frac{1}{p^s} \right) = \sum_{n \in A_m} \frac{1}{n^s} \prod_{p > \theta(n)} \left( 1 + \frac{1}{p^s} \right). \]

The result follows from dividing by \( \prod_{p \geq 2} (1 + 1/p^s) \). □

**Lemma 6.** For \( m \geq 1 \) and Re\((s) > 1\) we have

\[ 0 = \sum_{n \in A_m} \frac{1}{n^s} \left( \sum_{p^+(m) < p \leq \theta(n)} \frac{\log p}{p^s} - \log \left( \frac{n}{m} \right) \right) \prod_{p \leq \theta(n)} \left( 1 + \frac{1}{p^s} \right)^{-1}. \quad (23) \]

**Proof.** Differentiating (22) with respect to \( s \) shows that

\[ \frac{1}{m^s} \left( \sum_{p \leq P^+(m)} \frac{\log p}{p^s + 1} - \log m \right) \prod_{p \leq P^+(m)} \left( 1 + \frac{1}{p^s} \right)^{-1} \]
\[ = \sum_{n \in A_m} \frac{1}{n^s} \left( \sum_{p \leq \theta(n)} \frac{\log p}{p^s + 1} - \log n \right) \prod_{p \leq \theta(n)} \left( 1 + \frac{1}{p^s} \right)^{-1}. \]

The result now follows from Lemma 5. □
Define
\[ \alpha_m = \sum_{n \in A_m} \frac{1}{n} \left( \sum_{p^{+}(m) < p \leq \theta(n)} \frac{\log p}{p + 1} \right) \prod_{p\leq\theta(n)} \left( 1 + \frac{1}{p} \right)^{-1}, \]
\[ F_{m,N}(s) = \sum_{n \in A_m \atop n \leq N} \frac{1}{n^s} \left( \sum_{p^{+}(m) < p \leq \theta(n)} \frac{\log p}{p^s + 1} \right) \prod_{p\leq\theta(n)} \left( 1 + \frac{1}{p^s} \right)^{-1}, \]
\[ G_{m,N}(s) = \sum_{n \in A_m \atop n > N} \frac{1}{n^s} \left( \log \left( \frac{n}{m} \right) - \sum_{p^{+}(m) < p \leq \theta(n)} \frac{\log p}{p^s + 1} \right) \prod_{p\leq\theta(n)} \left( 1 + \frac{1}{p^s} \right)^{-1}, \]
and let \( s_N = 1 + 1/\log^2 N \) for \( N \geq 2 \). We have \( F_{m,N}(s_N) = G_{m,N}(s_N) \) by Lemma 6, \( \lim_{N \to \infty} F_{m,N}(s_N) = \alpha_m \) by Lemma 7, and \( \lim_{N \to \infty} G_{m,N}(s_N) = \zeta(2)(1 - e^{-\gamma})c_m \) by Lemma 8, where \( c_m \) is the constant in (10). Thus \( \alpha_m = \zeta(2)(1 - e^{-\gamma})c_m \). Theorem 2 now follows from summing over \( m \in S \). The proofs of Lemmas 7 and 8 are almost identical to those of Lemmas 3 and 4.

**Lemma 7.** Let \( m \geq 1 \) be fixed and assume \( \theta \) satisfies (21). Then
\[ \lim_{N \to \infty} F_{m,N} \left( 1 + 1/\log^2 N \right) = \alpha_m. \]

**Lemma 8.** Let \( m \geq 1 \) be fixed and assume \( \theta \) satisfies (21). Then
\[ \lim_{N \to \infty} G_{m,N} \left( 1 + 1/\log^2 N \right) = \zeta(2)(1 - e^{-\gamma})c_m. \]

4. **Proof of Theorem 3**

The proof of Theorem 3 is analogous to that of Theorem 1, with power series replacing Dirichlet series.

**Lemma 9.** For \( m \geq 1 \) and \( |z| < 1 \) we have
\[ 1 = \sum_{n \geq 0} f_q(n, m) z^n \prod_{k=1}^{n+m} \left( 1 - \frac{z^k}{q^k} \right)^{I_k}. \]  
(24)

**Proof.** Lemma 5 of [20] implies that
\[ z^j = \sum_{n=0}^{j} f_q(n, m) z^n r_q(j - n, n + m) z^{j-n} \quad (j \geq 0, m \geq 0), \]
where \( r_q(n, m) \) denotes the proportion of polynomials of degree \( n \) over \( \mathbb{F}_q \), all of whose non-constant divisors have degree \( > m \). Summing over \( j \geq 0 \) yields
\[ \frac{1}{1 - z} = \sum_{n \geq 0} f_q(n, m) z^n \sum_{j \geq n} r_q(j - n, n + m) z^{j-n}, \]
for $|z| < 1$. The inner sum equals
\[
\sum_{j \geq 0} r_q(j, n + m) z^j = \prod_{k > n + m} \left(1 - \frac{z^k}{q^k}\right)^{-I_k} = \frac{1}{1 - z} \prod_{k=1}^{n+m} \left(1 - \frac{z^k}{q^k}\right) .
\]

The result now follows from multiplying by $(1 - z)$. □

**Lemma 10.** For $m \geq 1$ and $|z| < 1$ we have
\[
0 = \sum_{n \geq 0} f_q(n, m) z^{n-1} \left(n - \sum_{k=1}^{n+m} \frac{kI_k}{(q/z)^k - 1}\right)\prod_{k=1}^{n+m} \left(1 - \frac{z^k}{q^k}\right) .
\]

**Proof.** Differentiate (24) with respect to $z$. □

Define
\[
\alpha_{q,m} = \sum_{n \geq 0} f_q(n, m) \left(\sum_{k=1}^{n+m} \frac{kI_k}{q^k - 1} - n\right)\prod_{k=1}^{n+m} \left(1 - \frac{1}{q^k}\right) I_k ,
\]
\[
F_{q,m,N}(z) = \sum_{n=0}^{N} f_q(n, m) z^{n-1} \left(\sum_{k=1}^{n+m} \frac{kI_k}{(q/z)^k - 1} - n\right)\prod_{k=1}^{n+m} \left(1 - \frac{z^k}{q^k}\right) I_k ,
\]
\[
G_{q,m,N}(z) = \sum_{n>N} f_q(n, m) z^{n-1} \left(n - \sum_{k=1}^{n+m} \frac{kI_k}{(q/z)^k - 1}\right)\prod_{k=1}^{n+m} \left(1 - \frac{z^k}{q^k}\right) I_k ,
\]

and let $z_N = \exp(-1/N^2)$ for $N \geq 1$. We have $F_{q,m,N}(z_N) = G_{q,m,N}(z_N)$ by Lemma 10, $\lim_{N \to \infty} F_{q,m,N}(z_N) = \alpha_{q,m}$ by Lemma 11, and $\lim_{N \to \infty} G_{q,m,N}(z_N) = (1 - e^{-\gamma})mc_q(m)$ by Lemma 12, where $c_q(m)$ is the constant in (17). Thus $\alpha_{q,m} = (1 - e^{-\gamma})mc_q(m)$, which is what we need to show. The proofs of Lemmas 11 and 12 are analogous to those of Lemmas 3 and 4, with (37) playing the role of the prime number theorem.

**Lemma 11.** For $m \geq 1$ we have
\[
\lim_{N \to \infty} F_{q,m,N}(\exp(-1/N^2)) = \alpha_{q,m} .
\]

**Lemma 12.** For $m \geq 1$ we have
\[
\lim_{N \to \infty} G_{q,m,N}(\exp(-1/N^2)) = (1 - e^{-\gamma})mc_q(m) .
\]

5. **Proofs of corollaries to Theorem 1**

We need to estimate $\alpha = (1 - e^{-\gamma})c_\theta = \lim_{N \to \infty} \alpha_N$, where
\[
\alpha_N = \sum_{n \leq N} \frac{\chi(n)}{n} \Delta(n) \prod_{p \leq \theta(n)} \left(1 - \frac{1}{p}\right) \quad (26)
\]

and
\[
\Delta(n) = \sum_{p \leq \theta(n)} \frac{\log p}{p - 1} - \log n .
\]
Assume that there are real numbers $L_N$ and $R_N$ such that
\[ L_N < \Delta(n) < R_N \quad (n > N), \tag{27} \]
and let
\[ \varepsilon_N = \sum_{n > N} \frac{\chi(n)}{n} \prod_{p \leq \theta(n)} \left( 1 - \frac{1}{p} \right) = 1 - \sum_{n \leq N} \frac{\chi(n)}{n} \prod_{p \leq \theta(n)} \left( 1 - \frac{1}{p} \right), \]
by (18). The last equation allows us to calculate $\varepsilon_N$ on a computer based on values of $\chi(n)$ and $\theta(n)$ for $n \leq N$. We have
\[ \alpha_N + L_N \varepsilon_N < \alpha < \alpha_N + R_N \varepsilon_N. \tag{28} \]

To determine values for $L_N$ and $R_N$ which satisfy (27), we need an effective estimate for the sum over primes in the definition of $\Delta(n)$.

**Lemma 13.** Let
\[ \eta(x) = \sum_{p \leq x} \frac{\log p}{p - 1} - \log x + \gamma. \tag{29} \]
We have $\eta(x) \ll \exp(-\sqrt{\log x})$ and
\[ |\eta(x)| \leq E(x) := \frac{0.084}{\log^2 x} \quad (x \geq 2^{25}). \]
Assuming the Riemann hypothesis, we have $|\eta(x)| \leq \frac{\log^2 x}{\sqrt{x}}$ for $x \geq 25$.

**Proof.** Rosser and Schoenfeld [10, Eq. 4.21] give the relation
\[ \tilde{\eta}(x) := \eta(x) + \sum_{p > x} \frac{\log p}{p(p - 1)} = \vartheta(x) - x - \int_x^\infty \frac{\vartheta(y) - y}{y^2} \, dy, \]
where $\vartheta(x) = \sum_{p \leq x} \log p$. The estimate $\eta(x) \ll \exp(-\sqrt{\log x})$ now follows from the prime number theorem.

Axler [1, Prop. 5.2] shows that for $x \geq 30972320 = 2^{24.88...}$,
\[ |\tilde{\eta}(x)| \leq \frac{3}{40 \log^2 x} \left( 1 + \frac{2}{\log x} \right), \]
which implies our estimate $|\eta(x)| \leq E(x)$ for $x \geq 2^{25}$, since
\[ 0 < \sum_{p > x} \frac{\log p}{p(p - 1)} < \int_x^\infty \frac{\log y}{y^2} \, dy = \frac{1 + \log x}{x}. \tag{30} \]
Assuming the Riemann hypothesis, Schoenfeld [12, Cor. 2] gives a bound for $|\tilde{\eta}(x)|$, which together with (30) yields our bound for $|\eta(x)|$ if $x \geq 160000$. For $25 \leq x \leq 160000$, we verify the result with a computer. □
5.1. **Proof of Corollaries 2 and 5.** For Corollary 2 we have $\theta(n) = tn$ and

$$
\Delta(n) = \sum_{p \leq tn} \frac{\log p}{p - 1} - \log n = \eta(tn) + \log t - \gamma.
$$

(31)

Lemma 13 shows that condition (27) is satisfied with $R_N = \log t - \gamma + E(tN)$ and $L_N = \log t - \gamma - E(tN)$, if $tN \geq 2^{25}$. For $N = 2^{25}$ and $t = 2$, we calculate $\alpha_N$ and $\varepsilon_N$ with a computer and find that (28) yields $1.224806 < c_2 < 1.224852$, hence $c_2 = 1.2248$. All the other estimates in Corollaries 2 and 5 are derived similarly. To obtain the decimal places as shown, $tN \leq 2^{30}$ suffices in all cases.

5.2. **Proof of Corollary 3.** Theorem 1 and (31) yield

$$(1 - e^{-\gamma})c_t = \sum_{n \in B_{\log t}} \frac{1}{n} \left( \log t + \eta(tn) \right) \prod_{p \leq nt} \left( 1 - \frac{1}{p} \right).$$

Together with (18) we obtain

$$
\inf_{n \geq 1} \eta(nt) \leq \delta_t \leq \sup_{n \geq 1} \eta(nt).
$$

(32)

The other estimates for $\delta_t$ follow from (32) and Lemma 13, since $E(x)$ is decreasing for $x \geq 2$ and $\frac{\log^2 x}{\sqrt{x}}$ is decreasing for $x \geq 55$.

5.3. **Proof of Corollary 1.** We use the fact that $\theta(n) = \sigma(n) + 1 \geq 2n$ whenever $n$ is practical [4, Lemma 2]. We have

$$
\Delta(n) \geq \sum_{p \leq 2n} \frac{\log p}{p - 1} - \log n = \eta(2n) + \log 2 - \gamma > \log 2 - \gamma - E(2n),
$$

for $2n \geq 2^{25}$, and hence

$$
\alpha \geq \alpha_N + (\log 2 - \gamma - E(2N))\varepsilon_N,
$$

for $2N \geq 2^{25}$. The lower bound in Corollary 1 now follows from calculating $\alpha_N$ and $\varepsilon_N$ for $N = 2^{26}$.

For the upper bound in Corollary 1, we have, for $n \geq 2^{25}$,

$$
\Delta(n) = \sum_{p \leq \sigma(n) + 1} \frac{\log p}{p - 1} - \log n
= \eta(\sigma(n) + 1) + \log(\sigma(n) + 1) - \gamma - \log n
\leq E(2n) + \log((\sigma(n) + 1)/n) - \gamma
\leq \log_3(n) + E(2n) + 1/n + \frac{0.6483}{e^\gamma (\log_2 n)^2},
=: \log_3(n) + \beta_n,
$$

(33)
say, where \( \log_k \) denotes the \( k \)-fold logarithm. For the last inequality of (33) we used Robin’s [8, Theorem 2] unconditional upper bound

\[
\frac{\sigma(n)}{n} \leq e^\gamma \log_2 n \left(1 + \frac{0.6483}{e^\gamma \log^2(n)}\right) \quad (n \geq 3)
\]

and the inequality \( \log(1 + x) \leq x \). If \( n \geq 2 \) is practical,

\[
\prod_{p \leq \sigma(n)+1} \left(1 - \frac{1}{p}\right) \leq \prod_{p \leq 2n} \left(1 - \frac{1}{p}\right) \leq \frac{e^{-\gamma}}{\log 2n} \left(1 + \frac{1}{2 \log^2(2n)}\right) < \frac{e^{-\gamma}}{\log n},
\]

by [10, Theorem 7]. For \( M > N \geq 2^{25} \), we get

\[
\alpha - \alpha_N \leq \varepsilon N \beta_N + \varepsilon N \log_3 n \sum_{n > N} \frac{\chi(n)}{n} \log_3 n \prod_{p \leq \sigma(n)+1} \left(1 - \frac{1}{p}\right)
\]

\[
\leq \varepsilon N \beta_N + \varepsilon N \log_3 M + \sum_{n > M} \frac{\chi(n)}{n} (\log_3 n - \log_3 M) \prod_{p \leq \sigma(n)+1} \left(1 - \frac{1}{p}\right)
\]

\[
\leq \varepsilon N \beta_N + \varepsilon N \log_3 M + e^{-\gamma} \sum_{n > M} \frac{\chi(n)}{n \log n} (\log_3 n - \log_3 M)
\]

\[
\leq \varepsilon N \beta_N + \varepsilon N \log_3 M + e^{-\gamma} \frac{a(1 + 1/\log M)}{b^2(\log M)^b \log_2 M},
\]

for \( a = 1.185 \) and \( b = \mu - \nu \), by Lemmas 16 and 17. For \( N = 2^{26} \), the last expression is minimized when \( \log_3 M = 4.15 \), which results in the upper bound \( c = \alpha/(1 - e^{-\gamma}) < 1.693 \). If one could use \( a = 2 \) and \( b = 1 \), i.e. improve Lemma 16 to \( P(x) \leq 2x/\log x \) for \( x \geq 2^{26} \), which is likely true based on empirical evidence, the same method would yield \( c < 1.441 \).

6. Proofs of Corollaries to Theorem 2

Following [7], the set of \( \phi \)-practical numbers arises as described in Section 1.3 with \( \theta(n) = n + 2 \) and a set of starters \( S \) defined as follows. Let \( P^+(n) \) (resp. \( P^-(n) \)) denote the largest (resp. smallest) prime factor of \( n \). We call \( d \) an initial divisor of \( n \) if \( d|n \) and \( P^+(d) < P^-(n/d) \). A starter is a \( \phi \)-practical number \( m \) such that either \( m/P^+(m) \) is not \( \phi \)-practical or \( P^+(m)^2|m \). A \( \phi \)-practical number \( n \) is said to have starter \( m \) if \( m \) is a starter, \( m \) is an initial divisor of \( n \), and \( n/m \) is squarefree. Each \( \phi \)-practical number \( n \) has a unique starter, which we will denote by \( s(n) \).

6.1. The lower bound in Corollary 6. Let \( h \) be as in (15) and write

\[
H(x) = \sum_{p \leq x} \frac{\log p}{p + 1}.
\]

We have

\[
H(x) + h + \gamma - \log x = \eta(x) + \sum_{p > x} \frac{2\log p}{p^2 - 1} < \eta(x) + h.
\]
Lemma 13 implies
\[ -E(x) < H(x) + h + \gamma - \log x < 2, \]  
(34)
where \( x \geq 2^{25} \) in the first inequality, and \( x \geq 1 \) in the second. We need a lower bound for \( C\zeta(2)(1 - e^{-\gamma}) \), which by Theorem 2 equals
\[
\sum_{m \in \mathcal{S}} \frac{1}{n} \left\{ \log m - H(P^+(m)) - \lambda \right\} \prod_{p \leq P^+(m)} \left(1 + \frac{1}{p}\right)^{-1} + \sum_{n \in \mathcal{A}} \frac{1}{n} \left\{ H(n + 2) - \log n + \lambda \right\} \prod_{p \leq n + 2} \left(1 + \frac{1}{p}\right)^{-1} = U(\lambda) + V(\lambda),
\]
say. Let \( U_N(\lambda) \) and \( V_N(\lambda) \) denote the corresponding partial sums. For a given \( N \), we pick \( \lambda \) such that the terms of both series are positive for \( m, n > N \). Then \( C\zeta(2)(1 - e^{-\gamma}) = U(\lambda) + V(\lambda) > U_N(\lambda) + V_N(\lambda) \), which yields a lower bound for \( C \) after dividing by \( \zeta(2)(1 - e^{-\gamma}) \). For \( n > N = 2^{30} \), (34) implies
\[ H(n + 2) - \log n \geq H(n) - \log n \geq -h - \gamma - E(n) > -1.7174. \]
For the series \( U(\lambda) \), note that \( m \in \mathcal{S} \) implies \( m \) is \( \varphi \)-practical. Thus \( P^+(m) \leq 2 + m/P^+(m) \), which yields \( P^+(m) \leq 2 + \sqrt{m} \). We have
\[ \log m - H(P^+(m)) \geq \log m - H(2 + \sqrt{m}) \geq \log(m/(\sqrt{m} + 2)) - 1 \geq 2, \]
by (34), for \( m \geq 500 \). Thus \( \lambda = 1.7174 \) ensures that the terms in both series are positive for \( m, n > N \). With \( N = 2^{30} \), we get \( C > (U_N(\lambda) + V_N(\lambda))/\zeta(2)(1 - e^{-\gamma}) > 0.945 \).

6.2. The upper bound in Corollary 6. Using a similar strategy as for the lower bound would require an explicit upper bound for the counting function of starters, since \( \log m - H(P^+(m)) \) grows unbounded. Instead, we will define a function \( \theta(n) \) such that \( \mathcal{A} \subset \mathcal{B}_\theta \) and hence \( C \leq c_\theta \). We then estimate \( c_\theta \) as in Section 5.

Let
\[
\theta(n) = \begin{cases} 
  n + 2 & \text{if } n \in \mathcal{A}; \\
  mp + 2 & \text{if } n = mp^a, m \in \mathcal{A}, p = m + 2, a \geq 2; \\
  n + 2 - m\varphi(n/m) & \text{else},
\end{cases}
\]
where \( m \) denotes the largest initial divisor of \( n \) with \( m \in \mathcal{A} \).

To show that \( \mathcal{A} \subset \mathcal{B}_\theta \), assume that \( n \not\in \mathcal{B}_\theta \). Then \( n \) has an initial divisor \( \tilde{n} \) such that \( q := P^-(n/\tilde{n}) \) satisfies \( q > \theta(\tilde{n}) \). First, if \( \tilde{n} \in \mathcal{A} \), then
q > \tilde{n} + 2 and n \notin A by [18, Lemma 3.3]. Second, if \tilde{n} = mp^a, m \in A, p = m + 2, a \geq 2, then q > mp + 2. Since \varphi(p^2) = p(p - 1) > mp + 1 and \varphi(q) = q - 1 > mp + 1, the number mp + 1 cannot be written as a subsum of \sum d|n \varphi(d), so n \notin A. Third, if m < \tilde{n} is the largest initial divisor of \tilde{n} with m \in A, then q > \tilde{n} + 2 - m\varphi(\tilde{n}/m), hence \varphi(q) > \tilde{n} + 1 - m\varphi(\tilde{n}/m).

Lemmas 5.2 and 5.3 of [7] show that the number \tilde{n} + 1 - m\varphi(\tilde{n}/m) cannot be written as a subsum of \sum d|n \varphi(d), so n \notin A.

To estimate \( c_\theta \), we use Theorem 1 and proceed as in Section 5. Since \( \theta(n) \leq n + 2 \), we have

\[
\Delta(n) \leq \sum_{p \leq n} \frac{\log p}{p - 1} - \log n = \eta(n + 2) - \gamma + \log(\log n) - \log n \\
\leq \frac{0.225}{\log^2(n + 2)} - \gamma + \frac{2}{n} =: R_n,
\]

for \( n \geq 2^{2^1} \), where the estimate for \( \eta(n + 2) \) follows from Lemma 13 for \( n \geq 2^{2^5} \), and for \( 2^{2^1} \leq n < 2^{2^5} \) we verify it by computation. For \( N = 2^{2^1} \), we get \( c_\theta < (\alpha_N + \varepsilon_N R_N)/(1 - e^{-\gamma}) < 0.967. \)

6.3. Proof of Corollaries 7 and 8. The calculations for Corollary 7 are analogous to those for Corollary 2, with (31) replaced by

\[
\Delta^*(n) = \sum_{p \leq tn} \frac{\log p}{p + 1} - \log n = \log t - \gamma - h + \eta^*(tn),
\]

where \( h \) is given by (15),

\[
\eta^*(x) = \eta(x) + \sum_{p > x} \frac{2\log p}{p^2 - 1}, \tag{35}
\]

and \( \eta(x) \) is as in (29). Lemma 3.5 of [7] shows that (22) is valid for \( s = 1 \), that is

\[
1 = \sum_{n \in \mathcal{D}_t^*} \frac{1}{n} \prod_{p \leq nt} \left( 1 + \frac{1}{p} \right)^{-1},
\]

since \( m = 1 \) and \( \theta(n) = nt \). From (13) we have

\[
c_t^*(1 - e^{-\gamma})\pi^2/6 = \sum_{n \in \mathcal{D}_t^*} \frac{1}{n} \left( \log t - \gamma - h + \eta^*(tn) \right) \prod_{p \leq nt} \left( 1 + \frac{1}{p} \right)^{-1},
\]

which yields

\[
\inf_{n \geq 1} \eta^*(nt) \leq \delta_t^* \leq \sup_{n \geq 1} \eta^*(nt). \tag{36}
\]

The other assertions follow from (36), because Lemma 13 remains valid when \( \eta(x) \) is replaced by \( \eta^*(x) \), if we replace the range \( x \geq 25 \) by \( x \geq 33 \) for the bound that assumes the Riemann hypothesis. Indeed, we have

\[
0 < \eta^*(x) - \tilde{\eta}(x) = \sum_{p > x} \frac{2\log p}{p^2 - 1} - \frac{\log p}{p(p - 1)} < \sum_{p > x} \frac{\log p}{p^2 - 1} < \frac{1 + \log x}{x},
\]
the same upper bound as we used for \( \tilde{\eta}(x) - \eta(x) \) in the proof of Lemma 13.

7. Proofs of corollaries to Theorem 3

Theorem 3 says that 
\[
c_q(m) = \frac{\alpha/m}{1 - e^{-\gamma}},
\]
where \( \alpha = \lim_{N \to \infty} \alpha_N \),
\[
\alpha_N = \sum_{0 \leq n \leq N} f_q(n, m) \Delta_m(n) \lambda(n + m),
\]
\[
\lambda(n) = \prod_{k=1}^{n} \left( 1 - \frac{1}{q^k} \right)^{I_k}
\]
and
\[
\Delta_m(n) = \sum_{k=1}^{n+m} \frac{kI_k}{q^k - 1} - n.
\]

We have
\[
\Delta_m(n) = m + \sum_{k=1}^{n+m} \frac{kI_k - (q^k - 1)}{q^k - 1} = m + \sum_{k>n+m} \frac{q^k - 1 - kI_k}{q^k - 1},
\]
by Lemma 14. With the bounds [3, p. 142, Ex. 3.26 and Ex. 3.27]
\[
\frac{q^k}{k} - \frac{2q^{k/2}}{k} < I_k \leq \frac{q^k}{k} \quad (k \geq 1),
\]
we obtain
\[
-L_{n+m} := - \sum_{k>n+m} \frac{1}{q^k - 1} \leq \Delta_m(n) - m \leq \sum_{k>n+m} \frac{2q^{k/2} - 1}{q^k - 1} =: R_{n+m}.
\]
(38)

Lemma 7 of [20] shows that
\[
1 = \sum_{n \geq 0} f_q(n, m) \lambda(n + m),
\]
so that
\[
\varepsilon_N := \sum_{n>N} f_q(n, m) \lambda(n + m) = 1 - \sum_{0 \leq n \leq N} f_q(n, m) \lambda(n + m),
\]
which we can calculate on a computer. We have
\[
\alpha_N + (m - L_{n+m}) \varepsilon_N \leq \alpha \leq \alpha_N + (m + R_{n+m}) \varepsilon_N,
\]
which yields bounds for \( c_q(m) \) upon dividing by \( m(1 - e^{-\gamma}) \). To obtain the accuracy as shown in Table 2, \( N = 50 \) or less suffices in all cases.
From (38) and (39) we get
\[ \alpha = \sum_{n \geq 0} f_q(n, m) \Delta_m(n) \lambda(n + m) \]
\[ = \sum_{n \geq 0} f_q(n, m) \left\{ m + O \left( q^{-(n+m+1)/2} \right) \right\} \lambda(n + m) \]
\[ = m + O \left( \frac{1}{mq(m+1)/2} \right) , \]
since \( f_q(n, m) \ll m/(n + m) \) and \( \lambda(n + m) \ll 1/(n + m) \). Dividing by \( m(1 - e^{-\gamma}) \) yields Corollary 10.

**Lemma 14.** We have
\[ \sum_{k=1}^{\infty} kI_k - \frac{(q^k - 1)}{q^k - 1} = 0. \]

**Proof.** For \(|z| < 1\) we have [9, p. 13]
\[ \frac{1}{1-z} = \prod_{k \geq 1} \left( 1 - \frac{z^k}{q^k} \right)^{-I_k} . \] (40)
Taking logarithms and differentiating yields
\[ \frac{1}{1-z} = \frac{1}{z} \sum_{k \geq 1} \frac{kI_k}{q^k z^k - 1} . \]
Now write the left-hand side as \( z^{-1} \sum_{k \geq 1} z^k \) and subtract to get
\[ 0 = \sum_{k \geq 1} \frac{(kI_k - q^k + z^k)z^k}{q^k - z^k} . \]
If \(|z| < \sqrt{q}\), the numerators in the last sum are \( \ll q^{k/2}z^k \) by (37), while the denominators are \( \gg q^k \). Thus the last series converges uniformly on the disk \(|z| < (1 + \sqrt{q})/2\) and is therefore continuous at \( z = 1 \), which is all we need. \( \square \)

8. An explicit upper bound for \( P(x) \)

We first need an explicit upper bound for sums of powers of \( \tau(n) \), the number of divisors of \( n \). Let
\[ \mu = - \log(\log 2)/\log 2 = 0.528766... \]
and let
\[ \nu = 2^\mu - 1 = 1/\log 2 - 1 = 0.442695... \]
This choice of \( \mu \) maximizes \( \mu - \nu = \mu - 2^\mu + 1 \).
Lemma 15. For \( x \geq 2 \),
\[
\sum_{n \leq x} (\tau(n))^{\mu} \leq 1.315 x (\log x)^{\nu} \left(1 + \frac{0.2}{(\log x)^2}\right)^{\nu}.
\]

Proof. Lemma 2.5 of Norton [6] implies
\[
\sum_{n \leq x} (\tau(n))^{\mu} \leq x \prod_{p \leq x} \left(1 + \frac{1}{p} \sum_{k \geq 1} \frac{(k+1)^{\mu} - k^{\mu}}{p^k}\right)
\]
\[
= x \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-\nu} \prod_{p \leq x} \left\{ \left(1 - \frac{1}{p}\right)^{\nu} \left(1 + \frac{1}{p} \sum_{k \geq 1} \frac{(k+1)^{\mu} - k^{\mu}}{p^k}\right) \right\}.
\]
The second product clearly converges. With the help of a computer we find that it is less than 1.0181 for all \( x \geq 1 \). To estimate the first product, we use the following result by Dusart [2, Theorem 6.12]: For \( x \geq 2973 \),
\[
\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{\mu} < e^{\gamma} \log x \left(1 + \frac{0.2}{\log^2 x}\right).
\]
Since \( 1.0181 e^{\gamma \nu} < 1.315 \), the result follows for \( x \geq 2973 \). For \( x < 2973 \), we verify the lemma with a computer. \( \Box \)

Lemma 16. For \( x \geq 2 \), \( P(x) \leq \frac{1.185 x}{(\log x)^{\mu - \nu}} \).

Proof. If \( n \) is practical, then \( 2^{\tau(n)} \geq n \), since every natural number \( m \leq n \) can be expressed as a subsum of \( \sum_{d \mid n} d \), and the number of subsums is \( 2^{\tau(n)} \).

Thus
\[
P(x) \leq 1 + \sum_{2 \leq n \leq x} \left( \tau(n) \frac{\log 2}{\log n} \right)^{\mu}.
\]
Partial summation and Lemma 15 yield
\[
P(x) \leq 1.315 (\log 2)^{\mu} \frac{x}{(\log x)^{\mu - \nu}} f(x),
\]
where
\[
f(x) = \left(1 + \frac{0.2}{(\log x)^2}\right)^{\nu} + \mu(\log x)^{\mu - \nu} \frac{x}{x} \int_2^x \frac{\left(1 + \frac{0.2}{(\log t)^2}\right)^{\nu}}{t^{\mu - \nu + 1}} dt.
\]
Since \( f(x) \) is decreasing for \( x \geq 6 \) and \( 1.315 (\log 2)^{\mu} f(1320) < 1.185 \), the result follows for \( x \geq 1320 \). For \( x < 1320 \), the trivial bound \( P(x) \leq x \) is sufficient. \( \Box \)

Lemma 17. If \( P(x) \leq ax(\log x)^{-b} \) for all \( x \geq N \) and some constants \( a, b > 0 \), then
\[
\sum_{n \geq N} \frac{\chi(n)}{n \log n} \left(\log n - \log_3 N\right) \leq \frac{a(1 + 1/\log N)}{b^2 (\log N)^b \log \log N}.
\]
Proof. This is a standard exercise using partial summation and integration by parts.

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