Modified Gauss-Bonnet theory as gravitational alternative for dark energy

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We suggest the modified gravity where some arbitrary function of Gauss-Bonnet (GB) term is added to Einstein action as gravitational dark energy. It is shown that such theory may pass solar system tests. It is demonstrated that modified GB gravity may describe the most interesting features of late-time cosmology: the transition from deceleration to acceleration, crossing the phantom divide, current acceleration with effective (cosmological constant, quintessence or phantom) equation of state of the universe.

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1. The explanation of the current acceleration of the universe (dark energy problem) remains to be a challenge for theoretical physics. Among the number of the approaches to dark energy, the very interesting one is related with the modifications of gravity at large distances. For instance, adding \(1/R\) term to Einstein action leads to gravitational alternative for dark energy where late-time acceleration is caused by the universe expansion. Unfortunately, such \(1/R\) gravity contains the instabilities of gravitationally bound objects. These instabilities may disappear with the account of higher derivative terms leading to consistent modified gravity. Another proposals for modified gravity suggest \(lnR\) or \(Tr1/R\) terms, account of inverse powers of Riemann invariants or some other modifications. The one-loop quantization of general \(f(R)\) in de Sitter space is also done. In addition to the stability condition which significantly restricts the possible form of \(f(R)\) gravity, another restriction comes from the study of its newtonian limit. Passing these two solar system tests leads to necessity of fine-tuning of the form and coefficients in \(f(R)\) action, like in consistent modified gravity. That is why it has been even suggested to consider such alternative gravities in Palatini formulation (for recent discussion and list of references, see [11]).

In the present paper we suggest new class of modified gravity, where Einstein action is modified by the function \(f(G)\), \(G\) being Gauss-Bonnet (GB) invariant. It is known that \(G\) is topological invariant in four dimensions while it may lead to number of interesting cosmological effects in higher dimensional brane-world approach (for review, see [12]). It naturally appears in the low energy effective action from string/M-theory (for recent discussion of late-time cosmology in stringy gravity with GB term, see [13]). As we demonstrate below, modified \(f(G)\) gravity passes solar system tests for reasonable choice of the function \(f\). Moreover, it is shown that such modified GB gravity may describe late-time (effective quintessence, phantom or cosmological constant) acceleration of the universe. For quite large class of functions \(f\) it is possible to describe the transition from deceleration to acceleration or from non-phantom phase to phantom phase in the late universe within such theory. Thus, modified GB gravity represents quite interesting gravitational alternative for dark energy with more freedom if compare with \(f(R)\) gravity.

2. Let us start from the following action:

\[
S = \int d^4 x \sqrt{-g} \left( \frac{1}{2\kappa^2} R + f(G) \right) . \tag{1}
\]

Here \(G\) is the GB invariant: \(G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma}\). By introducing two auxiliary fields \(A\) and \(B\), one may rewrite the action [11] as

\[
S = \int d^4 x \sqrt{-g} \left( \frac{1}{2\kappa^2} R + B (G - A) + f(A) \right) . \tag{2}
\]

Varying over \(B\), it follows \(A = G\). Using it in [2], the action [11] is recovered. On the other hand, by the

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variation over $A$ in (2), one gets $B = f'(A)$. Hence,

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{2\kappa^2} R + f'(A)G - Af'(A) + f(A) \right)$$

(3)

The scalar is not dynamical, it has no kinetic term and is introduced for simplicity. Varying over $A$, the relation $A = G$ is obtained again.

The spatially-flat FRW universe metric is chosen as

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1}^{3} (dx^i)^2.$$  

(4)

The first FRW equation has the following form:

$$0 = -\frac{3}{\kappa^2} H^2 + Af'(A) - f(A) - 2A f''(A) H^3.$$  

(5)

Here the Hubble rate $H$ is defined by $H \equiv \dot{a}/a$. For (1), GB invariant $G$ ( $A$) has the following form:

$$G = A = 24 \left( \dot{H} H^2 + H^4 \right).$$  

(6)

In general, Eq. (6) has deSitter universe solution where $H$ and therefore $A = G$ are constants. If $H = H_0$ with constant $H_0$, Eq. (5) looks as:

$$0 = -\frac{3}{\kappa^2} H_0^2 + 24 H_0^4 f'(24 H_0^3) - f'(24 H_0^3).$$  

(7)

For large number of choices of the function $f$, Eq. (7) has a non-trivial ($H_0 \neq 0$) real solution for $H_0$ (deSitter universe). Hence, such deSitter solution may be applied for description of the early-time inflationary as well as late-time accelerating universe.

Let us check how modified GB gravity passes the solar system tests. The GB correction to the Newton law may be found from the coupling matter to the action (1). Varying over $g_{\mu\nu}$, we obtain

$$0 = \frac{1}{2\kappa^2} \left( -R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R \right) + T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} f(G)$$

$$-2 f'(G) R_{\mu\rho} R_{\nu\sigma} + 4 f'(G) R_{\mu\rho} R_{\nu\sigma}$$

$$-2 f'(G) R_{\mu\rho\sigma\tau} R_{\nu\rho\tau} - 4 f'(G) R_{\mu\rho\sigma\tau} R_{\nu\rho\tau}$$

$$+2 \left( \nabla_{\mu} \nabla_{\nu} f'(G) \right) R - 2 g_{\mu\nu} \left( \nabla^2 f'(G) \right)$$

$$-4 \left( \nabla_{\mu} \nabla_{\nu} f'(G) \right) R_{\rho\sigma} - 4 \left( \nabla_{\rho} \nabla_{\nu} f'(G) \right) R_{\mu\rho\sigma}$$

$$+4 \left( \nabla^2 f'(G) \right) R_{\mu\rho\sigma} + 4 g_{\mu\nu} \left( \nabla_{\rho} \nabla_{\sigma} f'(G) \right) R_{\rho\sigma}$$

$$-4 \left( \nabla_{\rho} \nabla_{\sigma} f'(G) \right) R_{\mu\rho\sigma\tau}.$$  

(8)

Here $T_{\mu\nu}$ is matter EMT. In the expression (8), the third derivative $f'''(G)$ is included as $\nabla^2 f'(G) = f'''(G) \nabla^2 G + f''(G) \nabla^\mu G \nabla_{\mu} G$, for example. In the equation (3), corresponding to the first FRW equation, however, the terms including $f'''$ do not appear. When $T_{\mu\nu} = 0$, the $(t, t)$-component of Eq. (3) reproduces Eq. (5) by identifying $G$ with $A$. The perturbation around the deSitter background which is a solution of (7) may be easily constructed. We now write the deSitter space metric as $g_{(0)\mu\nu}$, which gives the following Riemann tensor:

$$R_{(0)\mu\nu\rho\sigma} = H_0^2 \left( g_{(0)\mu\rho} g_{(0)\nu\sigma} - g_{(0)\mu\sigma} g_{(0)\nu\rho} \right).$$  

(9)

The flat background corresponds to the limit of $H_0 \to 0$. Represent $g_{\mu\nu} = g(0)_{\mu\nu} + h_{\mu\nu}$. For simplicity, the following gauge condition is chosen:

$$g_{(0)\mu\nu} h_{\mu\nu} = \nabla^\mu h_{\mu\nu} = 0.$$  

(10)

The GB term contribution does not appear except the length parameter $1/H_0$ of the deSitter space which is determined with the account of GB term. Eq. (10) proves that there is no correction to Newton law in deSitter and even in the flat background corresponding to $H_0 \to 0$ whatever is the form of $f$. We should note that the expression (10) could be valid only in the deSitter background. In more general FRW universe, there could appear the corrections coming from $f(G)$ term. We should also note that in deriving (10), we have used a gauge condition $g_{(0)\mu\nu} h_{\mu\nu} = 0$ but if we include the mode corresponding to $g_{(0)\mu\nu} h_{\mu\nu}$, there might appear corrections from $f(G)$ term. Eq. (10) only shows that for the mode corresponding to the usual graviton, any correction coming from $f(G)$ does not appear in the deSitter background.

In case of $f(R)$-gravity (2), for most interesting choices of the function $f$ an instability was observed in (3). The instability is generated since the FRW equation, contains derivatives of the fourth order and therefore the scalar curvature propagates. This may cause the appearance of growing force between the galaxies. Only special forms of $f(R)$ may be free of such instability (3).

In $f(G)$-gravity case (1), the scalar field denoted by $A$ in (3) has no kinetic term. Then the scalar field or the curvature itself does not propagate and therefore there is no such instability as in (3), which is clear from the (no) correction to the Newton law in (3). Thus, modified GB gravity may pass the solar system tests (at least, for some functions $f(G)$).

Having in mind the fundamental property of the current accelerating universe, it is interesting to
study the possible transition between deceleration and acceleration of the universe for the action (1). First, Eq. (10) could be rewritten as $G = A = 24H^2\ddot{a}/a$. Then at the transition point $\ddot{a} = 0$, the Gauss-Bonnet term vanishes: $G = A = 0$. Let us assume the transition occurs at $t = t_0$. The Hubble rate $H$ may be expanded as

$$H = H_0 + H_1 (t - t_0) + H_2 (t - t_0)^2 + H_3 (t - t_0)^3 + \mathcal{O}((t - t_0)^4).$$  

(11)

At the transition point $t = t_0$, one finds

$$G = 48H_0^2(H_0^2 + H_1).$$  

(12)

Since $G = 0$ there, it follows

$$H_1 = -H_0^2.$$  

(13)

Hence, $G (A)$ is:

$$G = 96H_0^2(H_2 - H_3^3)(t - t_0) + 48H_0^2(5H_3^4 - 2H_0H_2 + 3H_3)(t - t_0)^2 + \mathcal{O}((t - t_0)^3).$$  

(14)

We now also assume that $f(A)$ could be expanded as

$$f(A) = f_0 + f_1A + f_2A^2 + f_3A^3 + \mathcal{O}(A^4).$$  

(15)

Then substituting (11), (14), and (15) into (5), we find

$$H_2 = H_0^3 - \frac{1}{96H_0^2f_2} \left( \frac{1}{16\kappa^2} + \frac{f_0}{48H_0^2} \right),$$

$$H_3 = -H_0^4 - \frac{1}{96 \times 16\kappa^2 f_2 H_0^2}$$

$$- \frac{f_3}{96 f_2^2 H_0^2} \left( \frac{1}{16\kappa^2} + \frac{f_0}{48H_0^2} \right)^2.$$  

(16)

Combining (13) and (16), one can show that the Hubble rate can be determined consistently, which suggests the existence of the transition between deceleration and acceleration of the universe. We should note that $H_0$ could be determined by a proper initial condition. Then the transition condition could be $f_2 \neq 0$, that is, $f(A)$ contains the quadratic term on $A$.

Let us consider now the possible transition between non-phantom phase and phantom phase of the universe (if current universe is phantom one). We now assume that the transition occurs at $t = t_1$.

Since $\dot{H} = 0$ at the transition point, it is natural to assume the Hubble rate behaves as

$$H = \dot{H}_0 + \dot{H}_1 (t - t_1)^2 + \mathcal{O}((t - t_1)^3).$$  

(17)

Hence, $A (G)$ behaves as

$$G = A = 24H_0^4 + 48H_0^2H_1(t - t_1) + \mathcal{O}((t - t_1)^2).$$  

(18)

Eq. (5) shows

$$\dot{H}_1 = \frac{-\frac{3}{\kappa^2} \dot{H}_0^3 + 24\dot{H}_0 f' (24\dot{H}_0^4) - f (24\dot{H}_0^4)}{1152\dot{H}_0^2 f'' (24\dot{H}_0^4)},$$  

(19)

if $f'' (24\dot{H}_0^4) \neq 0$. We also note that $\dot{H}_0$ and $\dot{H}_1$ should be positive. Eq. (15) suggests the existence of the consistent solution. Then the crossing of phantom divide could occur for large class of functions $f(G)$. The condition could be $f'' \neq 0$ and $\dot{H}_1 > 0$ when $H = \dot{H}_0 > 0$.

The transition between non-phantom phase and phantom phase could be regarded as a perturbation from the deSitter solution in (7). The following perturbation may be suggested

$$H = H_0 + \delta H,$$  

(20)

where $H_0$ satisfies (7). Eq. (6) gives

$$0 = \delta \ddot{H} + 3H_0 \delta H + \left( -4H_0^2 + \frac{1}{96\kappa^2 H_0^2 f'' (24H_0^4)} \right) \delta H.$$  

(21)

Here, it is supposed that $f'' (24H_0^4) \neq 0$. Let $\lambda$ satisfies

$$0 = \lambda^2 + 3H_0 \lambda - 4H_0^2 + \frac{1}{96\kappa^2 H_0^2 f'' (24H_0^4)}.$$  

(22)

Then $\delta H$ behaves as $\delta H \sim e^{\lambda t}$. If $\lambda$ is real, $\delta H$ is monotonically increasing or decreasing, $\dot{H}$ does not vanish and therefore there is no transition. If

$$600\kappa^2 H_0^4 f'' (24H_0^4) < 1,$$  

(23)

$\lambda$ becomes imaginary and $\delta H$ oscillates. Therefore the transition between the phantom phase, where $\dot{H} = \delta \dot{H} > 0$, and the non-phantom phase, where $\dot{H} = \delta \dot{H} < 0$ could be repeated in oscillation regime. The amplitude of the oscillation of $\delta H$ decreases as $|\delta H| \sim e^{-3H_0 t/2}$ and the universe asymptotically goes to deSitter space.
3. To show that modified GB gravity may lead to quite rich and realistic cosmological dynamics we consider some explicit examples of the function \( f(G) \). The following solvable model, which does not belong to the above class, may be discussed

\[
f(A) = f_0 |A|^{1/2} .
\]

Eq. (24) gives

\[
0 = -\frac{3}{\kappa^2} h_0^2 - \frac{f_0 (1 + h_0)}{2 (h_0 - 1)} |24 h_0 (h_0 - 1)|^{1/2} .
\] (26)

The solution differs for \( h_0 > 1 \) and \( h_0 < 0 \) cases. The simple analysis shows that when \( f_0^2 > 3/2\kappa^4 \), there are two negative solutions, which describe effective phantom universe with \( w < -1 \). When \( f_0^2 < 3/2\kappa^4 \), we have one \( h_0 > 1 \) solution, which describes the effective quintessence \(-1 < w < -1/3 \) and one \( h_0 < 0 \) solution describing effective phantom. If \( 16 f_0^2/3\kappa^4 < 1/\kappa^8 \), there are two real solutions which satisfy \( 0 < h_0 < 1 \).

For the second model let \( f(A) \) behaves as

\[
f(A) \sim f_0 |A|^{1/2} .
\] (27)

in a proper limit with constants \( f_0 \) and \( \beta \). We now assume in a limit \( t \to \infty \), \( H \) behaves as

\[
H \sim h_0^\alpha t^\alpha ,
\] (28)

with constants \( h_0 \) and \( \alpha \). If \( \alpha > -1 \), \( A \) (\( G \)) behaves as

\[
A = G \sim 24 t_0^4 t^{4\alpha} .
\] (29)

Then \( \beta = 4\alpha \) which is consistent with \( \alpha < -1 \). The assumption \( \alpha < -1 \) and \( \beta = 2\alpha/(3\alpha - 1) \) restricts \( 1/2 < \beta < 2/3 \). Eq. (29) also gives

\[
0 = -\frac{3}{\kappa^2} h_0^2 + (\beta - 1) f_0 |\alpha h_0^\beta| ,
\] (31)

which leads to non-trivial solution for \( h_0 \) if \( f_0 < 0 \).

Let us consider \( t \to 0 \) or \( t_0 - t \to 0 \) limit. If \( \alpha > -1 \), we obtain Eq. (30). If one requires \( 2\alpha = (3\alpha - 1)\beta = \alpha (3\beta + 1) \), then \( \beta = -\alpha \) and \( \alpha = 0, -1/3 \). In case \( \beta = -\alpha = 1/3 \), Eq. (30) tells

\[
0 = -\frac{3}{\kappa^2} h_0^2 + \frac{94}{3} f_0 |h_0^3|^{1/3} ,
\] (32)

which requires \( f_0 > 0 \) and gives a non-trivial solution for \( h_0 \). If one restricts \( 2\alpha = \alpha (3\beta + 1) < (3\alpha - 1)\beta \), it follows \( \beta = -1/3 \). In this case Eq. (30) gives

\[
0 = -\frac{3}{\kappa^2} h_0^2 + \frac{32 (3\alpha - 1) f_0}{3\alpha} |\alpha h_0^3|^{1/3} .
\] (33)

where \( \alpha \) can be arbitrary as long as \( \alpha > -1 \).

If \( \alpha < -1 \) (see Eq. (29)), Eq. (30) gives

\[
\beta = \frac{1}{2} + \frac{1}{4\alpha} ,
\]

\[
0 = -\frac{3}{\kappa^2} h_0^2 + 24 (4\beta - 5)\beta (\beta - 1) |\frac{h_0^3}{2 (2\beta - 1)}|^{\beta} .
\] (34)

The first equation shows \( 1/2 > \beta > 1/4 \). The second equation in (34) gives a non-trivial solution for \( h_0 \) if \( f_0 > 0 \).

The above results show that the asymptotic solution behaving as (29) with \( \alpha \neq 0 \) can be obtained only for the cases where \( \beta = 1/2 \) or \( 1/2 < \beta < 2/3 \) when \( t \to \infty \) or \( \beta = -1/3 \) or \( 1/4 < \beta < 1/2 \) when \( t_0 - t \to 0 \). In other situations, the asymptotic solution corresponds to the deSitter space (7). The following model may be taken as next example:

\[
f(A) = -\frac{\alpha}{A} + \beta A^2 .
\] (35)

Then Eq. (17) gives

\[
0 = -\frac{3}{\kappa^2} H_0^2 + \frac{\alpha}{12 H_0^2} + 576 \beta H_0^2 .
\] (36)

It could be difficult to solve the above equation exactly, but if the curvature and therefore \( H_0 \) is small, one gets

\[
H_0^6 \sim H_s^6 \equiv \frac{\alpha \kappa^2}{3} .
\] (37)
On the other hand if the curvature is large, we obtain
\[
H_0 \sim H_l^6 \equiv \frac{1}{192\beta \kappa^2}.
\] (38)

For the consistency \( H_l \gg H_s \), hence \( \alpha \beta \kappa^4 \ll 1 \). Then the large curvature solution \( (38) \) might correspond to the inflation in the early universe and the small curvature one in \( (37) \) to the late time acceleration. Similarly, one can construct many more explicit examples of modified GB gravity which complies with solar system tests (at least, for some backgrounds) and leads to accelerating (effective cosmological constant, phantom or quintessence) late-time cosmology.

4. In summary, it is shown that modified GB gravity passes two fundamental solar system tests in the same sense as General Relativity. It easily describes the late-time acceleration of the universe whatever is the effective equation of state (effective cosmological constant, quintessence or phantom). Moreover, we demonstrated that such modified GB gravity may describe the transition from deceleration to acceleration as well as crossing of phantom divide. It remains to understand whether any limitations to the function \( f(G) \) from solar system gravitational physics may be found. More serious limitations to its form may be searched fitting the theory against the observational data.

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