Symbolic extensions and dominated splittings for generic $C^1$-diffeomorphisms

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Abstract Let Diff$^1(M)$ be the set of all $C^1$-diffeomorphisms $f : M \to M$, where $M$ is a compact boundaryless $d$-dimensional manifold, $d \geq 2$. We prove that there is a residual subset $\mathcal{R}$ of Diff$^1(M)$ such that if $f \in \mathcal{R}$ and if $H(p)$ is the homoclinic class associated with a hyperbolic periodic point $p$, then either $H(p)$ admits a dominated splitting of the form $E \oplus F_1 \oplus \cdots \oplus F_k \oplus G$, where $F_i$ is not hyperbolic and one-dimensional, or $f|_{H(p)}$ has no symbolic extensions.

1 Introduction

Expansiveness is an important notion in the theory of dynamical systems. Let $M$ be a compact manifold and $f : M \to M$ be a homeomorphism. Roughly, it says that if the orbits of different points must separate in finite time. More precisely, there exists $\varepsilon > 0$ such that for any point $x$, the $\varepsilon$-set of $x$, given by the points $y$ such that $d(f^n(x), f^n(y)) < \varepsilon$ for every integer $n$, reduces to the point $x$. This notion is somewhat related with the well known notion of sensitivity to initial conditions, commonly known as chaos, which means that for any point, there exists a point such that the future orbit of these two points separated. Moreover, expansiveness naturally appears in hyperbolic sets, and together with the shadowing property, play a central role to prove their stability.
However, it is important to look for weaker forms of expansiveness. Clearly, expansiveness implies \( h\)-\textit{expansiveness}, i.e. for some \( \varepsilon > 0 \) the entropy of the \( \varepsilon \)-set of any point \( x \) is zero. This notion implies uppersemicontinuity of the entropy function defined on measures, henceforth leading to the existence of equilibrium states, which is a well know problem in ergodic theory. We remark that \( h\)-expansiveness do not imply expansiveness. Indeed, any system with zero topological entropy is \( h\)-expansive. This weaker property holds for partially hyperbolic diffeomorphisms such that their central subbundle admits a dominated splitting by one-dimensional subbundles, see [17]. More generally it holds for diffeomorphisms away from tangencies see [22].

It turns out that \( h\)-expansiveness implies the existence of \textit{symbolic extensions}, see [7]. This means that the system is a factor of a subshift over a finite alphabet. Actually, we can ask if the entropy of invariant measures is preserved, in this case we say that the extension is \textit{principal}. The existence of a principal extension is equivalent to asymptotic \( h\)-expansiveness, which requires that the entropy of the \( \varepsilon \)-sets goes to zero if \( \varepsilon \) goes to zero. However, in general the existence of symbolic extensions does not imply any kind of expansiveness. In particular, the \textit{non existence} of symbolic extensions implies that a positive amount of entropy, far from zero, can be found in arbitrarily small sets, given some complexity of the dynamics, see [6].

In the other hand, symbolic extensions are used in the theory of data transmission, see [18]. It is worthing to remark that any \( C^\infty \) diffeomorphism is asymptotically \( h\)-expansive, see [8].

For hyperbolic dynamical systems it’s possible to find principal extensions, in fact there exists finite to one symbolic extension which are of finite type by using Markov partitions.

In the other hand, the existence of symbolic extensions is somewhat rare in non-hyperbolic dynamics in \( C^1 \) topology. Indeed, it was proved by Downarowicz and Newhouse [19] that \( C^1\)-generic non-Anosov symplectic diffeomorphisms in surfaces do not have symbolic extensions. This result was extended to higher dimensions by Catalan and Tahzibi [13]. By generic, we mean that this holds for systems in a residual subset of such diffeomorphisms.

A natural question in dynamical systems is to know whether the presence of a dynamical property in a \( C^1\)-robust way implies some hyperbolicity. For instance, [5] shows that robust transitivity implies the existence of a dominated splitting. Naturally, some authors asked this question using expansiveness. Indeed, Mañé [25] shows that any robustly expansive diffeomorphisms is Axiom A. The same question can be asked in a semi-local way. More precisely, we can ask if a homoclinic class has some expansiveness in a robust way then it is hyperbolic. By homoclinic class we mean the closure of the transversal homoclinic intersections of a periodic orbit. The series of papers [27,28,31] and [32], essentially proves that robustly expansive homoclinic classes are hyperbolic, see the articles for more details. In [29], it was proved that any robustly \( h\)-expansive homoclinic class has a dominated splitting of the form \( E \oplus F_1 \oplus \cdots \oplus F_k \oplus G \), where \( F_i \) is not hyperbolic and one-dimensional. A related result was proved by Li in the context of \( R\)-robustly \( h\)-expansive homoclinic classes \( H(p,f) \), i.e. if for any \( g \) in a local residual subset around \( f \) the \( \varepsilon \)-set of any point of the homoclinic class of the continuation \( p_g \) has zero topological entropy, see [24] for more details.

Another related question is the existence of a residual subset where the presence of a dynamical property implies hyperbolicity in the global and semi-local case. For instance, in [3] it is proved that any generic expansive diffeomorphism is Axiom A. In [12], it was proved that generic volume preserving diffeomorphisms have symbolic extensions if, and only if, they are partially hyperbolic. In the semi-local case, [34] proved that for a generic diffeomorphisms, any expansive homoclinic class is hyperbolic.

In this article we study these questions for generic diffeomorphisms in the semi-local case but using symbolic extensions, that as we saw before, is much weaker than expansiveness. Another results dealing with the non-existence of symbolic extensions are: Diaz and Fisher.
[16] constructed a locally residual subset of $C^1$-partially hyperbolic diffeomorphisms without symbolic extensions, Asaoka [1] also constructed other examples, for smoother systems Diaz and Fisher [19] conjectured that $C^r$-diffeomorphisms have symbolic extensions if $r > 1$, Burguet [10] proved this conjecture for surfaces diffeomorphisms, Burguet [11] extended this result for higher dimensions with 2-dimensional center subbundle. Any $C^r$-one-dimensional transformation, with $r > 1$, has symbolic extensions, this was proved by Downarowicz and Maass [20].

Now, we give precise definitions and state our main results.

We consider a compact boundaryless $d$-dimensional Riemmanian manifold $M$, $d \geq 2$, and denote by $\text{Diff}(M) = \text{Diff}^1(M)$ the set of $C^1$ diffeomorphisms on $M$ endowed with the $C^1$ topology.

**Definition 1** A dynamical system $f : M \to M$ has a symbolic extension if there exists a subshift $\sigma : N \to N$ over a finite alphabet and a continuous surjective map $\pi : N \to M$ such that $\pi \circ \sigma = f \circ \pi$. By a subshift over a finite alphabet $\Sigma$ we mean a $\sigma$-invariant subset in $\Sigma^\mathbb{Z}$.

In this case the system $\sigma : N \to N$ is called a symbolic extension of $f : M \to M$ and $f$ is called a factor of $\sigma$. If $h_{\pi^*\mu}(f) = h_\mu(\sigma)$ for every invariant measure $\mu$ of $\sigma$ then the extension is called principal.

We say that $f : M \to M$ has a good decomposition over an invariant and compact subset $\Lambda$ if there exists a dominated splitting $T_\Lambda M = E_1 \oplus \cdots \oplus E_k$ such that $\dim(E_1) = s$, $\dim(E_k) = n - u$ and for every $1 < j < k$ we have $\dim(E_j) = 1$. Here, $s$ (resp. $u$) denotes the smallest (resp. greatest) index of a hyperbolic periodic point in $\Lambda$. Recall that the index of a hyperbolic periodic point $p$ is the dimension of its stable manifold.

For a hyperbolic periodic point $p$, we set the homoclinic class $H(p, f)$ of $p$ as the closure of the set of transversal intersections between the stable manifold and unstable manifold of the orbit of $p$. These sets are always transitive, invariant and compact sets. We also say that $H(p, f)$ is isolated if there exists a neighborhood $U$ of the class such that if the orbit of a point $x$ is contained in $U$ then $x$ belongs to the class.

**Theorem 2** There is a residual subset $\mathcal{R}$ of $\text{Diff}^1(M)$ such that if $f \in \mathcal{R}$, then for every homoclinic class $H(p, f)$,

(a) either $H(p, f)$ has a good decomposition,
(b) or $f|_{H(p, f)}$ has no symbolic extensions.

To prove this theorem we will use a dichotomy between good decompositions and the existence of a homoclinic tangency, see [4] and [23]. With this homoclinic tangency we want to use criterions to the non existence of symbolic extensions developed by Downarowicz and Newhouse [19], constructing several nice Horseshoes. However, since we do not have robustness, first we need to perform several perturbations to construct more tangencies and more Horseshoes. Then we will apply a generic lemma to prove that those criterions can be carried from the Horseshoes of those perturbations for the initial system.

Even so, once that one obtain a good decomposition, is somewhat folklore to obtain partial hyperbolicity when the class is isolated. In particular, we obtain the following theorem and prove it just for sake of completeness.

**Theorem 3** There is a residual subset $\mathcal{R}$ of $\text{Diff}^1(M)$ such that if $f \in \mathcal{R}$, then for every isolated homoclinic class $H(p, f)$

(a) either $H(p, f)$ is partially hyperbolic,
(b) or $f|_{H(p, f)}$ has no symbolic extensions.
However, this theorem together with the result of Diaz et al. [17] has an interesting directly consequence.

**Corollary 4** There is a residual subset $\mathcal{R}$ of $\text{Diff}^1(M)$ such that if $f \in \mathcal{R}$, any isolated homoclinic class of $f$ has a symbolic extension if, and only if, it has a principal symbolic extension.

Finally, as a byproduct of the techniques used in the proof of the main theorem we also get the following interesting consequence, which is somewhat related to the previous result by Pacifico and Vieitez [29] (see also [30]) and Gang et al. [22] mentioned before.

**Proposition 5** Let $HT \subset \text{Diff}^1(M)$ be the set of diffeomorphisms exhibiting a homoclinic tangency, $NHE \subset \text{Diff}^1(M)$ the set of diffeomorphisms that are no $h$-expansive and $NAHE \subset \text{Diff}^1(M)$ be the set of diffeomorphisms that are not asymptotically $h$-expansive. Then $HT = NHE = NAHE$.

As a consequence, if a diffeomorphism is stably asymptotically $h$-expansive then it has a dominated splitting in the pre-periodic set, using a result of Wen, see [35]. Moreover, if the diffeomorphism is generic then it is partially hyperbolic due to [15]. Also, we would like to point out that as a consequence of the proof of Proposition 5 we could obtain that if a diffeomorphism belongs to $HT$, then it is also approximated by diffeomorphisms for which the measure theoretical entropy map is not uppersemicontinuous. See Remark 16.

This article is organized as follows: In Sect. 2, we define precisely the notions and objects used in this paper, in Sect. 3 we define and study the $S_{n,p}$ property, which is our tool to find diffeomorphisms that has no symbolic extensions, in Sect. 4 we prove a local version of the Theorem 2, and thus we give a proof for Theorem 2 and for the isolated case. Finally, in Sect. 5 we prove Proposition 5.

2 Definitions

In this section we define precisely the notions and objects used in the introduction.

We say that $p$ is a **periodic point** if $f^n(p) = p$ for some $n \geq 1$, the minimal such natural is called the **period** of $p$ and it is denoted by $\tau(p, f)$, or simply by $\tau(p)$ if the diffeomorphisms $f$ is fixed. The periodic point is **hyperbolic** if the eigenvalues of $Df^{\tau(p)}(p)$ do not belong to $S^1$. Let $\text{Per}_h^n(f)$ be the collection of hyperbolic periodic points of $f$ of period less than or equal to $n$, and let $\text{Per}_h(f) = \bigcup_{n \geq 1} \text{Per}_h^n(f)$.

If $p$ is a hyperbolic periodic point then its **homoclinic class** $H(p, f)$ is the closure of the transversal intersections of the stable manifold and unstable manifold of the orbit of $p$:

$$H(p, f) = \overline{W^s(p) \cap W^u(p)}.$$  

It is well known that a homoclinic class is transitive. Moreover, we say that a hyperbolic periodic point $q$ is **related** to $p$ if $W^s(p) \cap W^u(q) \neq \emptyset$ and $W^u(p) \cap W^s(q) \neq \emptyset$, it can be proved that the homoclinic class of $p$ is also the closure of the hyperbolic periodic points related to $p$.

2.1 Domination

We say that a compact $f$-invariant set $\Lambda \subset M$ admits a **dominated splitting** if the tangent bundle $T_\Lambda M$ has a continuous $Df$-invariant splitting $E_1 \oplus \cdots \oplus E_k$ and there exist constants
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C > 0, 0 < λ < 1, such that

||Df^n|E_i(x)|| · ||Df^{-n}|E_j(f^n(x))|| ≤ C\lambda^n, \forall x \in \Lambda, n ≥ 0, for every i < j.

We say that T_\Lambda M = E_1 \oplus \cdots \oplus E_k is the finest dominated splitting if there is no dominated splitting of E_i for every 1 < l < k.

2.2 Hyperbolicity

If \Lambda is a compact invariant set of a diffeomorphism f then \Lambda is said to be a hyperbolic set if we have a Df -invariant continuous splitting T_\Lambda M = E^s \oplus E^u and constants C > 0 and \kappa < 1 such that

||Df^{-n}(x)|E^s|| \leq C\kappa^n \quad \text{and} \quad ||Df^n(x)|E^u|| \leq C\kappa^n,

for every x \in \Lambda and n \in \mathbb{N}. We define s = dim E^s the index of an hyperbolic set.

Let E \oplus F_1 \oplus \cdots \oplus F_k \oplus G be a dominated splitting over \Lambda. If E contracts and G expands, like E^s and E^u, respectively, in the previous paragraph then we say that \Lambda is partially hyperbolic.

Let \Lambda be a hyperbolic set for f. We call \Lambda a hyperbolic basic set if

- it is isolated, i.e. there is a neighborhood U of \Lambda such that
  \[
  \bigcap_{n \in \mathbb{Z}} f^n(U) = \Lambda \quad \text{and}
  \]
  - f has a dense orbit in \Lambda.

2.3 Genericity

We say that a subset \mathcal{R} \subset Diff^1(M) is a residual subset if contains a countable intersection of open and dense sets.

The countable intersection of residual subsets is also a residual subset. Since Diff^1(M) is a Baire space when endowed with the C^1-topology, any residual subset of Diff^1(M) is dense.

We will say that a property (P) holds generically if there exists a residual subset \mathcal{R} such that any f \in \mathcal{R} has the property (P).

2.4 Measures and exponents

A measure \mu is f -invariant if \mu(f^{-1}(B)) = \mu(B) for every measurable set B. An invariant measure is ergodic if the measure of any invariant set is zero or one. Let \mathcal{M}(f) be the space of f -invariant probability measures on M, and let \mathcal{M}_e(f) denote the ergodic elements of \mathcal{M}(f).

For a hyperbolic periodic point p of f with period \tau(p), we let \mu_p denote the hyperbolic periodic measure given by

\[
\mu_p = \frac{1}{\tau(p)} \sum_{x \in O(p)} \delta_x
\]

where O(p) denotes the orbit of p and \delta_x is the Dirac measure at x.

Let C(M, \mathbb{R}) be the set of all continuous functions h : M \to \mathbb{R}. If h \in C(M, \mathbb{R}) then

\[
\mu(h) = \int_M h d\mu.
\]

Let us denote by \rho the metric on \mathcal{M}(f) which defines the weak-* topology.

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as follows. Let $\phi_1, \phi_2, \ldots$ be a countable dense subset of the unit ball in $C(M, \mathbb{R})$ and set
\[
\rho(\mu, \nu) = \sum_{i \geq 1} \frac{1}{2^i} |\mu(\phi_i) - \nu(\phi_i)|.
\]

Given a periodic ergodic measure $\mu_p \in \mathcal{M}_e(f)$, we denote by $\chi^+(p, f)$ and $\chi^-(p, f)$ the smallest positive Lyapunov exponent and the biggest negative Lyapunov exponent of $\mu_p$, respectively. Then we define $\chi(p, f) = \min\{\chi^+(p, f), -\chi^-(p, f)\}$.

### 3 The property $S_{n, p}$

In this section, we define and study the $S_{n, p}$ property. This property is in the spirit of Downarowicz and Newhouse [19], in order to find diffeomorphisms that has no symbolic extensions.

**Definition 6** Given a positive integer $n$, we say that a diffeomorphism $f$ satisfies property $S_{n, p}$ if $p$ is a hyperbolic periodic point of $f$, and for any $\tilde{p} \in Per^n_h(f)$ related to $p$ there is a zero dimensional periodic hyperbolic basic set $\Lambda(\tilde{p}, n) \subset H(p, f)$ for $f$ with the same index that $p$, such that the following happens:

(a) there is $\nu \in \mathcal{M}_e(\Lambda(\tilde{p}, n))$ such that
\[
h_{\nu}(f) > \chi(\tilde{p}, f) - \frac{1}{n},
\]

(b) for every $\mu \in \mathcal{M}_e(\Lambda(\tilde{p}, n))$, we have
\[
\rho(\mu, \mu_{\tilde{p}}) < \frac{1}{n}.
\]

(c) for every hyperbolic periodic point $q \in \Lambda(\tilde{p}, n)$, we have
\[
\chi(q, f) > \chi(\tilde{p}, f) - \frac{1}{n}.
\]

The following result concern about the abundance of diffeomorphisms satisfying property $S_{n, p}$ near diffeomorphism with homoclinic classes admitting no dominated splittings.

**Proposition 7** Let $f$ be a generic diffeomorphism with a hyperbolic periodic point $p$ of index $i$. If $H(p, f)$ is a non-trivial homoclinic class admitting no $i$-dominated splitting, then for any neighborhood $U$ of $f$ and any positive integer $n$, there exists an open subset $V \subset U$ such that every $g \in V$ satisfies property $S_{n, p(g)}$.

The idea to prove this Proposition is to produce many nice horseshoes, as done by Downarowicz and Newhouse [19]. However, in their context, there is an abundance of homoclinic tangencies to produce such horseshoes. In our context we will use Lemma 8, which is a key and technical lemma, to overcome the lack of such abundance in general.

**Proof** First of all, we can suppose that $f$ is Kupka–Smale, since we are assuming that $f$ is generic. We recall that $f$ is Kupka–Smale if all of its periodic points are hyperbolic and the intersections between the invariant manifolds of any two periodic points is transversal.

We can suppose that every periodic orbit of $Per^n_h(f)$ and the orbit of $p$ has an analytic continuation on $U$. Moreover, by semicontinuity arguments, there exists $k$ such that for every $g \in U$ we have
\[
\#\{q \in Per^n_h(g); \text{ homoclinically related with } p\} = k.
\]
We denote the elements of this set for \( f \) by \( \{p_1, \ldots, p_k\} \).

Since \( H(p, f) = H(p_1, f) \) admits no \( i \)-dominated splitting, Gourmelon’s result [23] implies that, after some perturbation, we can suppose that \( f \) exhibits a homoclinic tangency for \( p_1 \), i.e., there exists a non transversal intersection between \( W^s(O(p_1), f) \) and \( W^u(O(p_1), f) \).

Now we state a technical lemma.

**Lemma 8** Let \( f \) be a diffeomorphism exhibiting a homoclinic tangency for a hyperbolic periodic point \( p_1 \), if \( p_2 \) is a hyperbolic periodic point homoclinically related with \( p_1 \) and \( n \) is large enough, then for any neighborhood \( \mathcal{U} \) of \( f \) there exist a diffeomorphism \( g \in \mathcal{U} \) and a small neighborhood \( \mathcal{V} \subset \mathcal{U} \) of \( g \) such that for every \( h \in \mathcal{V} \) the items of property \( S_{n, p(h)} \) holds for \( p_1(h) \), and moreover \( g \) exhibits a homoclinic tangency for \( p_2(g) \).

We postpone the proof of this lemma and finish the proof of the Proposition.

Given \( n \) a large positive integer, provided \( f \) exhibits a homoclinic tangency for \( p_1 \), and \( p_2 \) is homoclinically related with \( p_1 \), by Lemma 8 there exists a diffeomorphism \( g_1 \) and a neighborhood \( \mathcal{V}_1 \subset \mathcal{U} \) of \( g_1 \), such that \( g_1 \) exhibits a homoclinic tangency for \( p_2 \) and for every diffeomorphism \( h \in \mathcal{V}_1 \) the items of property \( S_{n, p(h)} \) holds for \( p_1(h) \). Now, since \( g_1 \in \mathcal{U} \), we have \( p_3(g_1) \) is still homoclinically related to \( p_2(g_1) \) and thus we can use again Lemma 8 to obtain a diffeomorphism \( g_2 \) and a neighborhood \( \mathcal{V}_2 \subset \mathcal{V}_1 \) of \( g_2 \), such that \( g_2 \) exhibits a homoclinic tangency to \( p_3(g_2) \), and now for every diffeomorphism \( h \in \mathcal{V}_2 \) the items of property \( S_{n, p(h)} \) also holds for \( p_3(h) \).

Now, repeating the above process finitely many times, we obtain a diffeomorphism \( g = g_k \) and a neighborhood \( \mathcal{V} = \mathcal{V}_k \subset \mathcal{V}_{k-1} \cdots \subset \mathcal{V}_1 \subset \mathcal{U} \) of \( g \) such that the items of property \( S_{n, p(h)} \) holds for any \( p_i(h), i = 1, \ldots, k \), and any \( h \in \mathcal{V} \). Then, by choice of \( \mathcal{U} \), every diffeomorphism \( h \in \mathcal{V} \) satisfy property \( S_{n, p(h)} \).

\[ \square \]

### 3.1 Proof of Lemma 8

First of all, we observe that many times in this proof we use expressions like “by some perturbation”, or “we can perturb \( f \)”, to say we can take a diffeomorphism arbitrary close to \( f \). Sometimes, in order to not complicate the notation we use the same letter to denote the new diffeomorphism. Also, when we say “by a local perturbation” we mean that we can perform a perturbation of \( f \) keeping the new diffeomorphism equal to \( f \) outside some small open set.

Let \( q \) be a point of homoclinic tangency of \( p_1 \), and \( V \) be a small neighborhood of \( O(p_1) \) such that \( f^{-1}(q) \) is not in \( V \). Shrinking \( V \), if necessary, we can suppose \( f^{\tau(p_1, f)} = Df^{\tau(p_1, f)} \) (in local coordinates on \( V \)) after a perturbation (see Franks’ lemma [21]). We remark that after this perturbation the homoclinic tangency could disappear. Nevertheless, since \( f^{-1}(q) \) is not in \( V \), using the continuity of compact parts of unstable and stable manifolds of \( p_1 \), by a local perturbation in some neighborhood of \( f^{-1}(q) \) we can recover the homoclinic tangency.

Up to take another point of the orbit of \( q \), we can suppose that \( q \in V \) and \( f^{-1}(q) \notin V \). Now, if \( j \) is a positive integer such that \( f^{-j-1}(q) \in W^u_{loc}(p_1) \), then we can take a neighborhood \( U \) of \( q \) such that \( f^{-1}(U) \cap V = \emptyset \), and moreover such that \( U \cap (\bigcup_{i=1}^{j} f^i(W^u_{loc}(p_1))) = \emptyset \). We denote by \( D \) the connected component of \( W^u(p_1, f) \cap U \) that contains \( q \). Note, by choice of \( U \) that \( (\bigcup_{i=1}^{j+1} f^i(W^u_{loc}(p_1))) \cap U = D \).

Now, we look to \( U \) in some local coordinates with the splitting \( T_qD \oplus T_qD^\perp \), and such that \( q = 0 \) in these coordinates. Since \( D \subset W^u(p_1, f) \) we have that \( D \) is a graph of a \( C^1 \) map \( r : T_qD \rightarrow T_qD^\perp \), i.e. \( D = (x, r(x)) \). Moreover, \( Dr(q) \) zero, and thus \( Dr \) is close to zero.
on its domain. Hence, the diffeomorphism $\phi(x, y) = (x, y - r(x))$ is $C^1$ close to identity in a small neighborhood of $q$. In particular, there exists a diffeomorphism $h$, $C^1$-close to identity, such that $h = \phi$ in some small neighborhood of $q$, and $h = Id$ for points far away from $q$. Thus, $f_1 := h \circ f$ is a $C^1$ local perturbation of $f$ such that $T_q D \cap U \subset W^u(p_1, f_1)$.

Since $f^{-1}(U) \cap V = \emptyset$, we have that $f_1 = f$ in $V$, as a consequence $f_1|V$ is still linear, and $W_{loc}^s(p_1, f_1)$ remains unchanged in $U$. Since $q$ is a non transversal homoclinic point we have that $T_q D \cap E^s(p_1, f)$ is a non trivial subspace. Actually, we can assume that $T_q D \cap E^s(p_1, f)$ is an one-dimensional subspace, after some local perturbation if necessary. Thus, $f_1$ exhibits an interval of homoclinic tangencies containing $q$.

Let $I$ be this interval of homoclinic tangencies. Replacing the local coordinates in $U$, if necessary, we can suppose that $\{(x_1, 0, \ldots, 0), -3a \leq x_1 \leq 3a\} \subset I$, for some $a > 0$ small enough.

Let $N$ be a large positive integer. Taking $I$ smaller, if necessary, we can construct a diffeomorphism $\Theta : M \to M$, such that $\Theta = Id$ in $B(0, 2a)^c$ and

$$\Theta(x, y) = \left(x_1, \ldots, x_s, y_1 + A \cos \frac{\pi x_1 N}{2a}, y_2, \ldots, y_u\right), \quad \text{for} \ (x, y) \in B(0, a) \subset U,$$

for $A = \frac{2Ka\delta}{\pi N}$, where $K$ is a constant which depends only on the local coordinates over $U$ and $\delta > 0$ is so small as we want. See Fig. 1.

Hence, taking $g = \Theta \circ f_1$, we have that $g$ is $\delta - C^1$ close to $f_1$ and moreover $g = f_1$ in the complement of $f_1^{-1}(B(q, 2a))$. Note that $g$ depends on $N$ but to not complicate the notation we denote this diffeomorphism by $g$, independent of $N$.

Remark 9 The most important properties of this new diffeomorphism is that $g$ has $N$ transversal homoclinic points for $p_1$ inside $U$, but $g$ still has an interval of homoclinic tangency inside $U$, in fact there are two intervals of homoclinic tangency in $U$: one inside $\{(x_1, 0, \ldots, 0), -3a \leq x_1 \leq -2a\}$ and other inside $\{(x_1, 0, \ldots, 0), 2a \leq x_1 \leq 3a\}$, in local coordinates.

To simplify notation we assume $p_1$ is a fixed point, being similar the general case.
We remark that $g|V$ is still linear in local coordinates, since $f$ is equal $g$ in $V$. Let $D_t = D^s \times D_t^u$ be a small rectangle, with $D^s = W^s_{loc}(p_1, g) \cap U$, and $D_t^u$ a small disk in $\{(0, \ldots, 0, y_1, \ldots, y_n), \ y_i \in \mathbb{R}^+ \text{ and } |y_i| < A/4, \text{ such that } t \text{ is the smallest positive integer} \text{ such that } g'(D_t) \text{ is a disk } A/4 - C^1 \text{ close to the connected component of } W^u(p_1, g) \cap U \text{ containing the } N \text{ transversal homoclinic points built before. We remark that } t \text{ depends on } N, \text{ and } t \to \infty \text{ when } N \to \infty.$

Observe that $A$ is small if $N$ is large, and by choice of $D_t$, we have that $g(D_t) \cap D_t$ has $N$ disjoint connected components. Moreover, note that for $N$ large enough the approximated angle that $g(D_t)$ intersects $D_t$ is near $AN$ which is close to $\epsilon$, independently of $N$. Hence, taking $N$ larger (which implies $A$ smaller), if necessary, we have that the maximal invariant set in $D_t$ for $g^t$

$$\tilde{\Lambda}(p_1, N) = \bigcap_{j \in \mathbb{Z}} g^j(D_t)$$

is a hyperbolic set inside $H(p_1, g)$.

Let $\Lambda(p_1, N) = \bigcup_{0 \leq j \leq t} g^j(\tilde{\Lambda}(p_1, N))$ be the hyperbolic periodic set of $g$ induced by $\tilde{\Lambda}(p_1, N)$. Since $g'|\tilde{\Lambda}(p_1, N)$ is conjugated with the full shift of $N$ symbols, we have that $h(g|\Lambda(p_1, N)) = \frac{1}{t} \log N$.

We recall that $g|V$ is linear. So, if $m$ is the largest positive integer such that $g^j(x) \in V$ for $0 \leq j \leq m$, there exist constants $K_1$ and $K_2$ depending on the local coordinate on $V$ such that

$$K_1\|Dg(p_1)^m|E^u\|^{-1} \leq d(x, W^s_{loc}(p_1, g)) \leq K_2\|Dg(p_1)^{-m}|E^u\|, \quad (1)$$

for $x \in V$. Analogously, if $m$ is the largest positive integer such that $g^{-j}(x) \in V$ for $0 \leq j \leq m$, then there exist constants $K_3$ and $K_4$ such that

$$K_3\|Dg(p_1)^{-m}|E^s\|^{-1} \leq d(x, W^u_{loc}(p_1, g)) \leq K_4\|Dg(p_1)^m|E^s\|. \quad (2)$$

Now, as a consequence of inequalities (1) and (2) and the choice of $t$ we have the following result, which also appears in [13]. It’s worth to point out that this result and Lemma 11 is an adaptation for higher dimensions of some results obtained in [19].

**Lemma 10** (Lemma 4.2 of [13]) For $A$ and $t$ defined as before, there exists a positive integer $K_5$, which is independent of $A$, such that

$$A < K_5 \max\{\|Dg(p_1)^{-t}|E^u\|, \|Dg(p_1)^t|E^s\|\}.$$ 

Let $n$ be a large positive integer. Since $A = \frac{2K_a \delta}{\pi N}$, using Lemma 10 and recalling that $N \to \infty$ implies $t \to \infty$, we can select a large positive integer $N$, such that

$$\frac{1}{t} \log N > \min \left\{ \frac{1}{t} \log \|Dg(p_1)^{-t}|E^u\|^{-1}, \frac{1}{t} \log \|Dg(p_1)^t|E^s\|^{-1} \right\} - \frac{1}{2n}.$$ 

But, when $t$ goes to infinity the above minimum converges to $\chi(p_1, g)$, by definition. Therefore, there exists a large positive integer $N_1$ such that

$$\frac{1}{t} \log N_1 > \chi(p_1, g) - \frac{1}{n}.$$ 

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So, it is possible to find a $C^1$-perturbation $g$ of $f$ such that

$$h(g|\Lambda(p_1, N_1)) > \chi(p_1, g) - \frac{1}{n}.$$ 

Now, by the variational principle there exists an ergodic measure $\mu_N \in \mathcal{M}(\Lambda(p_1, N))$ such that

$$h_{\mu_N}(g) > \chi(p_1, g) - \frac{1}{n}, \text{ for } N \geq N_1. \quad (3)$$

Observe that the orbit of points in the hyperbolic set $\Lambda(g, N)$, when $N$ is large enough, stay almost all the time inside the neighborhood $V$ of $p_1$, which one could be assumed so small as we wanted. Hence, there exists a positive integer $N_2$ such that if $\mu \in \mathcal{M}(f|\Lambda(g, N))$ is ergodic then $\rho(\mu, \mu_{p_1}) < 1/n$, for every $N \geq N_2$.

Finally, we find $N_3$ in order to obtain property (c) of $S_{n, p(g)}$ for $\Lambda(p_1, N)$ with $N \geq N_3$. We define

$$V_k^u = V \cap g(V) \cap \cdots \cap g^k(V), \text{ and }$$

$$V_k^s = V \cap g^{-1}(V) \cap \cdots \cap g^{-k}(V).$$

Given vectors $v, w \in \mathbb{R}^{2n}$ and subspaces $E, F \subset \mathbb{R}^{2n}$ we define

$$\text{ang}(v, w) := \left| \tan \left[ \arccos \left( \frac{<v, w>}{|v||w|} \right) \right] \right|,$$

$$\text{ang}(v, E) = \min_{w \in E, |w|=1} \text{ang}(v, w) \text{ and } \text{ang}(E, F) = \min_{w \in E, |w|=1} \text{ang}(w, F).$$

The following lemma, is also a straightforward consequence of inequalities (1) and (2), as in Lemma 4.4 in [13].

**Lemma 11** With above definitions, there exists positive constants $K_6$ and $K_7$, such that

1. if $z \in V_k^u, v \in \mathbb{R}^{2n} \setminus E_{p_1}^u$ and $\text{ang}(g^{-k}(v), E_{p_1}^u) > 1$, then

$$K_6 \| Dg_{p_1}^k |E_{p_1}^s||v| \min\{|\text{ang}(v, E_{p_1}^u), 1\} < |Dg_{p_1}^{-k}(z)(v)| < K_7 \| Dg_{p_1}^{-k} |E_{p_1}^s||v|$$

2. if $z \in V_k^s, v \in \mathbb{R}^{2n} \setminus E_{p_1}^s$ and $\text{ang}(g^k(v), E_{p_1}^s) > 1$, then

$$K_6 \| Dg_{p_1}^{-k} |E_{p_1}^u|^{-1}|v| \min\{|\text{ang}(v, E_{p_1}^s), 1\} < |Dg_{p_1}^k(z)(v)| < K_7 \| Dg_{p_1}^{k} |E_{p_1}^u||v|$$

Now, since $\Lambda(p_1, N) = \bigcup_{i=0}^{k-1} g^i(\tilde{\Lambda}(p_1, N))$ with $\tilde{\Lambda}(p_1, N) \subset V$, then we can take positive integers $k$ and $T$ such that $t = k + T$, and $g^t(\tilde{\Lambda}(p_1, N)) \subset V$ for $0 \leq i \leq k$. Moreover, by construction of $\tilde{\Lambda}(p_1, N)$ this $T$ can be taken independent of $N$. Hence, provided $t$ goes to infinity when $N$ goes to infinity, we have that $k$ also goes to infinity. Now, we know that the hyperbolic decomposition $T_{\tilde{\Lambda}(p_1, N)}M = \tilde{E}^s \oplus \tilde{E}^u$ of the hyperbolic set $\tilde{\Lambda}(p_1, N)$ is such that $\tilde{E}^s(g^{-k}(z_1))$ and $\tilde{E}^u(g^k(z_2))$ are close to $E_{p_1}^s$ and $E_{p_1}^u$, respectively, for every $z_1 \in g^k(\tilde{\Lambda}(p_1, N))$ and $z_2 \in \tilde{\Lambda}(p_1, N)$. In particular,

$$\text{ang}(Dg_{p_1}^{-k}(z_1)(v), E_{p_1}^u) > 1 \text{ for } v \in \tilde{E}^s(z_1) \text{ and }$$

$$\text{ang}(Dg_{p_1}^k(z_2)(v), E_{p_1}^u) > 1 \text{ for } v \in \tilde{E}^u(z_2).$$
Moreover, and the most important argument in this case, is that although \( \text{ang}(v, E^u_{p_1}) \) for \( v \in \tilde{\mathcal{E}}^s(z_1) \), and \( \text{ang}(v, E^s_{p_1}) \) for \( v \in \tilde{\mathcal{E}}^u(z_2) \) are a very small constant, independent of \( N \), we have ensured that

\[
\text{ang}(Dg^{-k}(z_1)(v), E^u_{p_1}) > 1 \quad \text{and} \quad \text{ang}(Dg^k(z_2)(v), E^s_{p_1}) > 1.
\]

So, using these informations and Lemma 11 we can find constants \( K_6 \) and \( K_7 \), such that for every \( z \in \tilde{\Lambda}(p_1, N) \), \( r = I(k + T) \) and for every \( l \in \mathbb{N} \):

1. if \( v \in \tilde{\mathcal{E}}^s(z) \) then
   \[
   |Dg^{-r}(z)(v)| \geq (C_1 K_6)^l \|Dg^k_p\|E^s\|^{-l}|v|
   \]
2. if \( v \in \tilde{\mathcal{E}}^u(z) \) then
   \[
   |Dg^r(z)(v)| \geq (C_1 K_7)^l \|Dg^{-k}_p\|E^u\|^{-l}|v|,
   \]
   where

   \[
   C_1 = \inf_{z \in V \setminus g^{-1}(V), |v| = 1} \|Dg^T(z)(v)\|.
   \]

Therefore, for \( N \) large enough, all points in \( \tilde{\Lambda}(p_1, N) \) have Lyapunov exponents with absolute values larger than \( \chi(p, g) - 1/n \). In particular, we can choose \( N_3 \), in order to get \( k \gg T \), such that for any periodic point \( \tilde{g} \in \Lambda(p_1, N) \), with \( N > N_3 \), we have

\[
\chi(\tilde{g}) > \chi(p_1, g) - \frac{1}{n}.
\]

Hence, if we take \( \Lambda(p_1, n) = \Lambda(p_1, N) \) for \( N = \max\{N_1, N_2, N_3\} \), the items of property \( S_{n, p(g)} \) are satisfied for the perturbation \( g \) of \( f \) and the hyperbolic periodic point \( p_1 \) of \( g \).

**Remark 12** Provided \( \Lambda(p_1, n) \) is a basic hyperbolic set, item (a) and (b) in the property \( S_{n, p} \) are also satisfied for continuations of this hyperbolic set, since we have also continuity for the the Lyapunov exponents of a periodic point. Now, although item (c) could be not an open property, note that in the above proof this property is obtained a consequence of Lemma 11, which one is an open property. Therefore, here we can obtain an open set close to \( g \), such that for every continuation of \( \Lambda(p_1, n) \) for a diffeomorphism \( \tilde{g} \) in this open set all the items of property \( S_{n, p(\tilde{g})} \) is still true for \( p_1(\tilde{g}) \).

Hence, by the previous remark there exists a neighborhood \( V \subset \mathcal{U} \) of \( g \), such that every diffeomorphism \( h \in V \) satisfies the items of property \( S_{n, p(h)} \) for \( p_1(h) \).

Now, since the diffeomorphism \( g \) belongs to \( \mathcal{U} \), we know that the hyperbolic periodic point \( p_2(g) \) still is homoclinic related with \( p_1(g) \). Also, by Remark 9, \( g \) still exhibits a homoclinic tangency for \( p_1(g) \). Now, by a perturbation using Franks Lemma, we can find a transversal homoclinic point to \( p_1(g) \), such that the angle between \( W^s(p_1(g), g) \) and \( W^u(p_1(g), g) \) is so small as we want. Hence, since \( p_1(g) \) and \( p_2(g) \) are related, there exists a transversal homoclinic point for \( p_2(g) \) such that the angle between \( W^s(p_2(g), g) \) and \( W^u(p_2(g), g) \) is so small, too. Finally, using Franks Lemma once more, we can perturb \( g \) such that this transversal homoclinic point become a homoclinic tangency. Since this perturbation can be find in \( V \), we finish the proof.
4 Non existence of symbolic extensions versus good decomposition

First we recall some knowns residual subsets. We denote by \( \mathcal{R}_1 \subset \text{Diff}^1(M) \) the residual subset given by Carballo et al. [14], such that for every diffeomorphism \( g \in \mathcal{R}_1 \) two homoclinic classes are either disjoint or coincide. By \( \mathcal{R}_2 \subset \text{Diff}^1(M) \) the residual subset given by Abdenur et al. [4], such that for every diffeomorphism \( g \in \mathcal{R}_2 \), every homoclinic class having a hyperbolic periodic point with index \( i \) and a hyperbolic periodic point with index \( j \), with \( i < j \), has a dense set of hyperbolic periodic points with index \( k \) for every \( i \leq k \leq j \). And by \( KS \) the residual subset of Kupka–Smale diffeomorphisms. Hence, we define \( \mathcal{R}_4 = \mathcal{R}_1 \cap \mathcal{R}_2 \cap KS \).

The following proposition is a local version of the main Theorem 2.

**Proposition 13** Let \( f \in \mathcal{R}_3 \), \( p \) be a hyperbolic periodic point of \( f \). If \( U(f) \subset \text{Diff}^1(M) \) is a small enough neighborhood of \( f \), there is a residual subset \( \mathcal{R} \subset U(f) \) such that every \( g \in \mathcal{R} \) satisfies only one of the following statements:

(i) \( H(p_g, g) \) has a good decomposition;

(ii) \( g|_{H(p_g, g)} \) has no symbolic extensions.

**Proof** Since \( f \in \mathcal{R}_3 \) if \( U(f) \) is small enough then there exist \( i \) and \( j \), and hyperbolic periodic points \( p_i, p_{i+1}, \ldots, p_j \) of \( f \) with \( \text{ind} \ p_k = k \), for \( i \leq k \leq j \), such that:

- \( H(p, f) = H(p_i, f) = H(p_{i+1}, f) = \cdots = H(p_j, f) \),
- for every hyperbolic periodic point \( q \in H(p, f) \) we have \( i \leq \text{ind} \ q \leq j \).

By [4, Lemma 4.2, pg.20], there exists an open and dense subset of \( U(f) \) over \( \mathcal{R}_3 \) such that for every \( g \in \mathcal{R} \), for every such \( g \), we still have that:

- \( H(p(g), g) = H(p_i(g), g) = H(p_{i+1}(g), g) = \cdots = H(p_j(g), g) \),
- for every hyperbolic periodic point \( q \in H(p, g) \) we have \( i \leq \text{ind} \ q \leq j \).

Now, for any positive integer \( n \) and any \( i \leq k \leq j \), we define \( B_{n, p_k} \subset U(f) \) as the subset of diffeomorphisms that robustly satisfies property \( S_{n, p_k} \), i.e., \( g \in B_{n, p_k} \) if there is a small neighborhood of \( g \) where every diffeomorphism \( h \) satisfy property \( S_{n, p_k(h)} \).

**Lemma 14** There is a residual subset of \( \mathcal{R}_4 \subset \mathcal{U}(f) \), such that for any positive integer \( n \) and any \( i \leq k \leq j \), if \( g \in \mathcal{R}_4 \) and there is a sequence of diffeomorphisms \( \{g_m\} \subset B_{n, p_k} \) that converges to \( g \), then \( g \) satisfies property \( S_{n, p_k(g)} \).

**Proof** Let us define \( \mathcal{V}_{n, k} = B_{n, p_k} \cup \overline{B_{n, p_k}}^c \) be an open and dense subset in \( \mathcal{U}(f) \), for every positive integer \( n \) and every \( i \leq k \leq j \). Then, \( \mathcal{R}_4 = \bigcap_{n \geq 0} \bigcap_{i \leq k \leq j} \mathcal{V}_{n, k} \) is a residual subset in \( \mathcal{U}(f) \). To finish the proof, let \( g \in \mathcal{R}_4 \). Given a positive integer \( n \) and \( i \leq k \leq j \), if there exists diffeomorphisms \( g_m \in B_{n, p_k} \) converging to \( g \), then \( g \notin \overline{B_{n, p_k}}^c \). Therefore, since \( g \in \mathcal{V}_{n, k} \) we have that \( g \in B_{n, p_k} \) and then satisfies property \( S_{n, p_k(g)} \).

Using Lemma 14, we define \( \mathcal{R} = \mathcal{R}_3 \cap \mathcal{R}_4 \), which is a residual subset in \( \mathcal{U}(f) \). Now, we will verify that a diffeomorphism in this residual subset satisfies one of the two properties claimed in the proposition which finishes the proof.

For this we will use the following result of Burguet.

**Proposition 15** (Corollary 1 of [9]) Let \( f : M \to M \) be a dynamical system admitting a symbolic extension. Then the entropy function \( h : M(f) \to \mathbb{R} \) is a difference of nonnegative upper semicontinuous functions. In particular the entropy function \( h \) restict to any compact set of measures has a large set of continuity points.
Let $g \in R \cap U(f)$, by choice of $R$, $i$ and $j$ are the two extreme indices in $H(p(g), g)$ and $H(p(g), g) = H(p_i(g), g) = H(p_{i+1}(g), g) = \cdots = H(p_j(g), g)$.

Suppose $H(p(g), g)$ admits no good decomposition. Hence, there is some $i \leq k \leq j$ such that $H(p(g), g) = H(p_k(g), g)$ admits no $k$-dominated splitting.

By Proposition 7, for every $n > 0$, we can find a sequence of diffeomorphisms $\{g_{n,m}\}_{m\in\mathbb{N}}$ converging to $g$, such that each $g_{n,m} \in B_{n,p_k}$. Therefore, by Lemma 14, $g$ satisfies property $S_{n,p_k(g)}$ for every $n > 0$, since $g \in R_4$.

We define $\rho_0 = \max\{\chi(\tilde{p}, g); \tilde{p} \in Per_h(g)\}$ and related to $p_k(g)$, and

$$\xi_1(g) = \left\{ \mu_{\tilde{p}} : \tilde{p} \in Per_h(g), \text{ related to } p_k(g) \text{ and } \chi(\tilde{p}, g) > \rho_0 \right\}$$

which is a non empty subset in $M(f)$. Then, we consider the compact subset $\xi(g) = \overline{\xi_1(g)}$ in $M(g)$.

Now, let $\mu_{\tilde{p}} \in \xi_1$ and $t$ be a positive integer. Since $g$ satisfies property $S_{n,p_k(g)}$ for every positive integer $n$, there exist ergodic measures $\nu_m \rightarrow \mu_{\tilde{p}}$ such that $h_{\nu_m}(g) > \rho_0/2$, for every $m$. Moreover, since these measures are supported on hyperbolic sets with the same index that $p_k(g)$, by Sigmund [33], they are approximated by hyperbolic periodic measures also supported in these hyperbolic sets, and by item (c) of property $S_{n,p_k(g)}$, they belong to $\xi_1(g)$ for large $n$. Hence, $\nu_m \in \xi(g)$ for every $m$, and then

$$\limsup_{\nu_m \rightarrow \mu_{\tilde{p}}, \nu_m \in \xi(g)} h_{\nu_m}(g) > \frac{\rho_0}{2}.$$  

Therefore, since $p$ is arbitrary and $\xi_1(g)$ has dense periodic measures, there is no continuity point for the entropy function $h$. Thus, by Proposition 15, this implies that $f$ has no symbolic extensions.

Finally, we will use Proposition 13 and the generic machinery to prove Theorem 2.

**Proof of Theorem 2.** Since $Diff^1(M)$ is separable, there is a countable and dense subset $A \subset Diff^1(M)$. Moreover, we can assume that $A \subset R_3$, the residual subset of $Diff^1(M)$ in the hypothesis of Proposition 13.

Now, for any $f \in A$ and a small enough neighborhood $U(f)$ of $f$, we consider the residual subset $R_f$ in $U(f)$ given by Proposition 13. Thus, we define

$$R_f = R_f \cup (U(f))^c,$$

which is a residual subset in $Diff^1(M)$, indeed. Also, since $A$ is a dense subset

$$U = \bigcup_{f \in A} U(f),$$

is an open and dense subset of $Diff^1(M)$.

Finally, we define the following residual subset

$$R = \bigcap_{f \in A} R_f \cap U.$$

Now, let $g \in R$ and $H(p, g)$ be a homoclinic class of $g$. Since $g \in U$, there exists $f \in A$ such that $g \in U(f)$, and then provided $g$ also belongs to $R_f$, $g$ should belongs to $R_f$. Therefore, by Proposition 13 we have that either $H(p, g)$ has a good decomposition, or $f \mid H(p, g)$ has no symbolic extensions. This completes the proof.

To finish this section we prove the isolated case.
Proof of Theorem 3  It is enough to prove that for \( f \in \mathcal{R} \) of Theorem 2, if \( H(p, f) \) has a good decomposition and it is isolated then it is partially hyperbolic. Let \( U \) be a neighborhood of \( H(p, f) \) such that \( H(p, f) = \bigcap_{n \in \mathbb{Z}} f^n(U) \). Also, let \( E \oplus E_1 \oplus \cdots \oplus E_l \oplus F \) be the good decomposition. We will prove that \( E \) is contracting, a similar argument will prove that \( F \) is expanding.

We recall that \( \dim(E) \) is the smallest index of a periodic point in the class. By Abdenur et al. [4], there is another residual subset where we know that there exists a neighborhood \( \mathcal{U} \) of \( f \) such that \( \dim(E) \) is still the smallest index of a periodic point in the class \( H(p, f) \), for any \( g \in \mathcal{U} \), where \( p_g \) is the analytic continuations of \( p \). The intersection of this two residual subsets is the one claimed in the statement of the theorem.

Now, if \( E \) does not contract, using the Ergodic Closing Lemma, as Mañé did in [26], then it is possible to find \( g \in \mathcal{U} \) with a periodic orbit \( O(q) \subset U \) with index smaller than \( \dim(E) \).

However, we can consider the result of Abdenur proved in [2] to relative homoclinic class. Here, a relative homoclinic class of \( p \) for \( U \) is the subset of \( H(p, f) \) of points that have the whole orbit inside \( U \), which we denote by \( H_U(p, f) \). Therefore, since \( H_U(p, f) = H(p, f) \) isolated homoclinic class, for any \( h \in \mathcal{R} \) close enough to \( f \), \( H_U(p(h), h) = \bigcap_{n \in \mathbb{Z}} h^n(U) \) and then \( q \in H_U(p(h), h) \subset H(p(h), h) \), which is a contradiction.

\( \square \)

5 Proof of Proposition 5

First, by definition \( NAHE \subset NHE \). Moreover, one inclusion is a directly consequence of the result of Gang et al. [22]. More precisely, they have proved that far away from homoclinic tangencies every diffeomorphism is \( h \)-expansive. So \( NHE \subset HT \). For the other inclusion we will use Lemma 8.

Let \( f \in HT \), that is, \( f \) exhibits a homoclinic tangency, say \( q \), for a hyperbolic periodic point \( p \). Given \( \varepsilon > 0 \) small, let us consider a small neighborhood \( \mathcal{U} \) of \( f \), with \( \text{diam}(\mathcal{U}) < \varepsilon \). Then by Lemma 8 there is a perturbation \( f_1 \in \mathcal{U} \) of \( f \) such that \( f_1 \) has a periodic hyperbolic basic set \( \Lambda_1 \) satisfying

\[
h(f_1|\Lambda_1) > \chi(p(f_1), f_1) - \frac{1}{n_0 + 1},
\]

for a big positive integer \( n_0 \) fixed, and moreover \( f_1 \) still exhibits a homoclinic tangency for \( p(f_1) \). As we can see in the proof of the Lemma 8, \( \Lambda_1 \) can be found such that the base set \( \overline{\Lambda}_1 \), i.e., \( \Lambda_1 = \bigcup f_1^n(\overline{\Lambda}_1) \), is contained in a ball of radius so small, in particular, we can assume it is in a ball with radius \( \frac{1}{n_0 + 1} \).

In the sequence, we consider a small neighborhood \( \mathcal{U}_1 \) of \( f_1 \), such that for all diffeomorphisms in \( \mathcal{U}_1 \) there is a continuation for \( \Lambda_1 \), and moreover \( \text{diam}(\mathcal{U}_1) < \frac{\varepsilon}{n_0 + 1} \). Now, using again Lemma 8 we can find a diffeomorphism \( f_2 \in \mathcal{U}_1 \), and a periodic hyperbolic basic set \( \Lambda_2 \), with base set contained in a ball with radius \( \frac{1}{n_0 + 2} \), such that

\[
h(f_2|\Lambda_2) > \chi(p(f_2), f_2) - \frac{1}{n_0 + 2},
\]

and \( f_2 \) still exhibits a homoclinic tangency for \( p(f_2) \).

Following this process inductively we can find a sequence of diffeomorphism \( f_n \in \mathcal{U}_{n-1} \), with \( \text{diam}(\mathcal{U}_n) < \frac{1}{n_0 + n} \), \( \mathcal{U} \supset \mathcal{U}_1 \supset \cdots \supset \mathcal{U}_n \supset \cdots \), and moreover, by construction, \( f_n \) is such that there exists periodic hyperbolic sets \( \Lambda_1, \ldots, \Lambda_n \) with \( \text{diam}(\Lambda_i) < \frac{1}{n_0 + i} \), for every
Since this sequence of diffeomorphism is a Cauchy sequence, it converges to a diffeomorphism $g$, that is $\epsilon$-close to $f$. Now, by choice of the open sets $U_n$, $g$ has periodic hyperbolic basic sets with diameter so small as we want with topological entropy away from zero, since $\chi(p(f), f)$ varies continuously with the diffeomorphism $f$. Therefore, $g$ can not be asymptotically $h$-expansive. And then, we have proved that $HT \subset NAHE$.

**Remark 16** In the previous proof we can suppose that the basic hyperbolic sets $\Lambda_i$ have the following property: all ergodic measures with support contained in it are close enough to the hyperbolic periodic measure $\mu_p$. Hence, the measure theoretical entropy map of the diffeomorphism $g$ found in the proof of Proposition 5 is not upper semicontinuous.

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