ON A TWISTED JACQUET MODULE OF GL(2n) OVER A FINITE FIELD

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Abstract. Let $F$ be a finite field and $G = \text{GL}(2n, F)$. In this paper, we explicitly describe a certain twisted Jacquet module of an irreducible cuspidal representation of $G$.

1. Introduction

Let $F$ be a finite field and $G = \text{GL}(2n, F)$. Let $P$ be a parabolic subgroup of $G$ with Levi decomposition $P = MN$. Let $\pi$ be any irreducible finite dimensional complex representation of $G$ and $\psi$ be an irreducible representation of $N$. Let $\pi_{N, \psi}$ be the sum of all irreducible representations of $N$ inside $\pi$, on which $\pi$ acts via the character $\psi$. It is easy to see that $\pi_{N, \psi}$ is a representation of the subgroup $M_\psi$ of $M$, consisting of those elements in $M$ which leave the isomorphism class of $\psi$ invariant under the inner conjugation action of $M$ on $N$. The space $\pi_{N, \psi}$ is called the twisted Jacquet module of the representation $\pi$. It is an interesting question to understand for which irreducible representations $\pi$, the twisted Jacquet module $\pi_{N, \psi}$ is non-zero and to understand its structure as a module for $M_\psi$.

In [2],[1], inspired by the work of Prasad in [6], we studied the structure of a certain twisted Jacquet module of a cuspidal representation of $GL(4,F)$ and $GL(6,F)$. Based on our calculations, we had conjectured the structure of the module for $GL(2n,F)$ (see Section 1 in [1]). For a more detailed introduction and the motivation to study the problem, we refer the reader to Section 1 in [2].

Before we state our result, we set up some notation. Let $F$ be a finite field and $F_n$ be the unique field extension of $F$ of degree $n$. Let $G = \text{GL}(2n, F)$ and $P = MN$ be the standard maximal parabolic subgroup of $G$ corresponding to the partition $(n,n)$. We have, $M \simeq \text{GL}(n, F) \times \text{GL}(n, F)$ and $N \simeq M(n, F)$. We let $\pi = \pi_\theta$ to be an irreducible cuspidal representation of $G$ associated to the regular character $\theta$. Let $\psi$ be any character of $N \simeq M(n, F)$ and $\psi_0$ be a fixed non-trivial character of $F$. We let

$$A_i = \begin{bmatrix} I_i & 0 \\ 0 & 0 \end{bmatrix} \in M(n, F).$$

Let $\psi_A : N \to \mathbb{C}^\times$ be the character given by

$$\psi_A \left( \begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \right) = \psi_0(\text{Tr}(AX)).$$

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Let $H_A = M_1 \times M_2$ where $M_1$ is the Mirabolic subgroup of $GL(n, F)$ and $M_2 = w_0 M_1^T w_0^{-1}$ where

$$
w_0 = \begin{bmatrix}
0 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 0
\end{bmatrix}
$$

Let $U$ be the subgroup of unipotent matrices in $GL(2n, F)$ and $U_A = U \cap H_A$. Then, we get $U_A \simeq U_1 \times U_2$ where $U_1$ and $U_2$ are the upper triangular unipotent subgroups of $GL(n, F)$. For $k = 1, 2$, let $\mu_k : U_k \rightarrow \mathbb{C}^\times$ be the non-degenerate character of $U_k$ given by

$$
\mu_k\left(\begin{bmatrix}
1 & x_{12} & x_{13} & \ldots & x_{1,n} \\
1 & x_{23} & \ldots & x_{2,n} \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots \\
& & & x_{(n-1),n} \\
1 & & & & 1
\end{bmatrix}\right) = \psi_{0}(x_{12} + x_{23} + \cdots + x_{(n-1),n}).
$$

Let $\mu : U_A \rightarrow \mathbb{C}^\times$ be the character of $U_A$ given by

$$
\mu(u) = \mu_1(u_1) \mu_2(u_2)
$$

where $u = \begin{bmatrix} u_1 & 0 \\
0 & u_2 \end{bmatrix} \in U_A$.

**Theorem 1.1.** Let $\theta$ be a regular character of $F_\infty^\times$ and $\pi = \pi_\theta$ be an irreducible cuspidal representation of $G$. Then

$$
\pi_{N, \psi_A} \simeq \theta|_{F_\infty^\times} \otimes \text{ind}^{H_A}_{U_A} \mu
$$

as $M_{\psi_A}$ modules.

2. Preliminaries

In this section, we mention some preliminary results that we need in our paper.

2.1. Character of a Cuspidal Representation. Let $F$ be the finite field of order $q$ and $G = GL(m, F)$. Let $F_m$ be the unique field extension of $F$ of degree $m$. A character $\theta$ of $F_m^\times$ is called a “regular” character, if under the action of the Galois group of $F_m$ over $F$, $\theta$ gives rise to $m$ distinct characters of $F_m^\times$. It is a well known fact that the cuspidal representations of $GL(m, F)$ are parametrized by the regular characters of $F_m^\times$. To avoid introducing more notation, we mention below only the relevant statements on computing the character values that we have used. We refer the reader to Section 6 in [4] for more precise statements on computing character values.

**Theorem 2.1.** Let $\theta$ be a regular character of $F_m^\times$. Let $\pi = \pi_\theta$ be an irreducible cuspidal representation of $GL(m, F)$ associated to $\theta$. Let $\Theta_\theta$ be its character. If $g \in GL(m, F)$ is such that the characteristic polynomial of $g$ is not a power of a polynomial irreducible over $F$. Then, we have

$$
\Theta_\theta(g) = 0.
$$

**Theorem 2.2.** Let $\theta$ be a regular character of $F_m^\times$. Let $\pi = \pi_\theta$ be an irreducible cuspidal representation of $GL(m, F)$ associated to $\theta$. Let $\Theta_\theta$ be its character. Suppose that $g = s.u$ is the Jordan decomposition of an element $g$ in $GL(m, F)$. If $\Theta_\theta(g) \neq 0$, then the semisimple element $s$ must come from $F_\infty^\times$. Suppose that $s$
comes from $F_m^n$. Let $z$ be an eigenvalue of $s$ in $F_m$ and let $t$ be the dimension of the kernel of $g-z$ over $F_m$. Then

$$\Theta_s(g) = (-1)^{m-1} \left[ \sum_{\alpha=0}^{d-1} \theta(z^\alpha) \right] (1 - q^d)(1 - (q^d)^2) \cdots (1 - (q^d)^{t-1}).$$

where $q^d$ is the cardinality of the field generated by $z$ over $F$, and the summation is over the distinct Galois conjugates of $z$.

See Theorem 2 in [6] for this version.

2.2. Kirillov Representation. Let $F$ be a finite field with $q$ elements and $G = \text{GL}(n, F)$. Let $P_n$ be the Mirabolic subgroup of $G$ and let $U$ be the subgroup of unipotent matrices of $G$. In this section, we recall the Kirillov representation of the Mirabolic subgroup $P_n$ of $G$. Let $\psi_0$ be a non-trivial character of $F$ and let $\psi : U \to \mathbb{C}^\times$ be the non-degenerate character of $U$ given by

$$\psi = \begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1,n} \\ 1 & x_{23} & & & x_{2,n} \\ & \ddots & \ddots & \ddots & \vdots \\ & & 1 & \cdots & x_{(n-1),n} \\ & & & 1 & x_{(n-1),n} \end{pmatrix} = \psi_0(x_{12} + x_{23} + \cdots + x_{(n-1),n}).$$

Then, $K = \text{ind}_{U}^{P_n} \psi$ is called the Kirillov representation of $P_n$.

**Theorem 2.3.** $K = \text{ind}_{U}^{P_n} \psi$ is an irreducible representation of $P_n$.

We refer the reader to Theorem 5.1 in [3] for a proof.

2.3. Multiplicity one Theorem for $\text{GL}(n, F)$ over a finite field $F$. We continue with the notation of section 2.2.

**Theorem 2.4.** Let $G = \text{ind}_{U}^{G} (\psi)$. The representation $G$ of $G$ is multiplicity free.

We refer to Theorem 6.1 in [3] for a proof.

2.4. Twisted Jacquet Module. In this section, we recall the character and the dimension formula of the twisted Jacquet module of a representation $\pi$.

Let $G = \text{GL}(k, F)$ and $P = MN$ be a parabolic subgroup of $G$. Let $\psi$ be a character of $N$. For $m \in M$, let $\psi^m$ be the character of $N$ defined by $\psi^m(n) = \psi(mnm^{-1})$. Let

$$V(N, \psi) = \text{Span}_C \{ \pi(n)v - \psi(n)v \mid n \in N, v \in V \}$$

and

$$M_\psi = \{ m \in M \mid \psi^m(n) = \psi(n), \forall n \in N \}.$$

Clearly, $M_\psi$ is a subgroup of $M$ and it is easy to see that $V(N, \psi)$ is an $M_\psi$-invariant subspace of $V$. Hence, we get a representation $(\pi_N, V/V(N, \psi))$ of $M_\psi$. We call $(\pi_N, V/V(N, \psi))$ the twisted Jacquet module of $\pi$ with respect to $\psi$. We write $\Theta_{N, \psi}$ for the character of $\pi_N, \psi$.

**Proposition 2.5.** Let $(\pi, V)$ be a representation of $\text{GL}(k, F)$ and $\Theta_\pi$ be the character of $\pi$. We have

$$\Theta_{N, \psi}(m) = \frac{1}{|N|} \sum_{n \in N} \Theta_\pi(mn) \overline{\psi(n)}.$$

We refer the reader to Proposition 2.3 in [2] for a proof.
Remark 2.6. Taking \( m = 1 \), we get the dimension of \( \pi_{N,\psi} \). To be precise, we have

\[
\dim_{\mathbb{C}}(\pi_{N,\psi}) = \frac{1}{|N|} \sum_{n \in \mathbb{N}} \Theta_{\pi}(n) \overline{\psi}(n).
\]

2.5. \( q \)-Hypergeometric Identity. In this section, we record a certain \( q \)-identity from [5] which we use in calculating the dimension of the twisted Jacquet module. Before we state it, we set up some notation. Let \( M(n,m,r,q) \) be the set of all \( n \times m \) matrices of rank \( r \) over the finite field \( F \) of order \( q \) and \( (a;q)_n \) be the \( q \)-Pochhammer symbol defined by

\[
(a;q)_n = \prod_{i=0}^{n-1} (1 - aq^i).
\]

Proposition 2.7. Let \( a \) be an integer greater than or equal to \( 2n \). Then

\[
\sum_{r \geq 0} M(n,n,r,q)(q;q)_{a-r} = q^n \frac{(q;q)^2_{a-n}}{(q;q)_{a-2n}}.
\]

We refer the reader to Lemma 2.1 in [5] for a proof of the above proposition in a more general set up.

3. Dimension of the Twisted Jacquet Module

Let \( \pi = \pi_\theta \) be an irreducible cuspidal representation of \( G \) corresponding to the regular character \( \theta \) of \( F_2^\times \) and \( \Theta_\theta \) be its character. In this section, we calculate the dimension of \( \pi_{N,\psi,A} \), where

\[
A = \begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0
\end{bmatrix}.
\]

Throughout, we write \( M(n,m,r,q) \) denote the set of \( n \times m \) matrices of rank \( r \) over the finite field \( F \) of cardinality \( q \). For \( \alpha \in F \) and \( 0 \leq r \leq n \), consider the subset \( Y^\alpha_{n,r} \) of \( M(n,F) \) given by

\[
Y^\alpha_{n,r} = \{ X \in M(n,F) \mid \text{Rank}(X) = r, \text{Tr}(AX) = \alpha \}.
\]

Lemma 3.1. We have

\[
| M(n,n,r,q) | = q^n | M(n,n-1,r,q) | + (q^n - q^{r-1}) | M(n,n-1,r-1,q) |.
\]

Proof. Let \( S = q^n | M(n,n-1,r,q) | + (q^n - q^{r-1}) | M(n,n-1,r-1,q) | \). It is well known that

\[
| M(n,m,r,q) | = \prod_{j=0}^{r-1} \frac{(q^n - q^j)(q^m - q^j)}{(q^j - q^j)}.
\]
Thus, we have

\[
S = q^r \prod_{j=0}^{r-1} \frac{(q^n - q^j)(q^{n-1} - q^j)}{(q^r - q^j)} + (q^n - q^{r-1}) \prod_{j=0}^{r-2} \frac{(q^n - q^j)(q^{n-1} - q^j)}{(q^r - q^j)}
\]

\[
= \prod_{j=0}^{r-1} \frac{(q^n - q^j)(q^{n-j+1})}{(q^r - q^j)} + (q^n - q^{r-1}) \prod_{j=0}^{r-2} \frac{(q^n - q^j)(q^{n-j+1})}{(q^r - q^{j+1})}
\]

\[
= \frac{q^n - q^r}{q^n-1} |M(n,n,r,q)| + \frac{(q^n - q^{r-1})(q^r - 1)}{(q^n - q^{r-1})(q^n - 1)} |M(n,n,r,q)|
\]

\[
= |M(n,n,r,q)|. \quad \square
\]

**Lemma 3.2.** Let \( r \in \{1,2,3,\ldots,n\} \) and \( \alpha, \beta \in F^\times \). Then we have

\[ |Y_{n,r}^\alpha| = |Y_{n,r}^\beta|. \]

**Proof.** Consider the map \( \phi : Y_{n,r}^\alpha \to Y_{n,r}^\beta \) given by

\[ \phi(X) = \alpha^{-1} \beta X. \]

Suppose that \( \phi(X) = \phi(Y) \). Since \( \alpha^{-1} \beta \neq 0 \), it follows that \( \phi \) is injective. For \( Y \in Y_{n,r}^\beta \), let \( X = \alpha \beta^{-1} Y \). Clearly, we have \( \text{Tr}(AX) = \alpha \) and \( \text{Rank}(X) = \text{Rank}(Y) = r \). Thus \( \phi \) is surjective and hence the result. \( \square \)

**Lemma 3.3.**

\[ |Y_{n,r}^0| = q^{-1} |M(n,n,r,q)| + (q^r - q^{r-1}) |M(n-1,n-1,r,q)| + (q^{r-2} - q^{r-1}) |M(n-1,n-1,r-1,q)|. \]

**Proof.** Let \( \mathcal{B} = \{e_1,e_2,\ldots,e_n\} \) be a basis of \( F^n \) over \( F \) and \( X \in Y_{n,r}^0 \). Then,

\[
[X]_{\mathcal{B}} = \begin{bmatrix} 0 & w \\ v & Y \end{bmatrix}
\]

where \( w \) is an \( 1 \times (n-1) \) row vector, \( v \) is an \( (n-1) \times 1 \) column vector and \( Y \) is an \( (n-1) \times (n-1) \) block matrix. We also write

\[
\begin{bmatrix} w \\ Y \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_{n-1} \end{bmatrix}
\]

where \( v_i \) is an \( n \times 1 \) column vector for \( 1 \leq i \leq n - 1 \).

Let \( V \) be the \( n-1 \) dimensional hyperplane spanned by the vectors \( \{e_2,e_3,\ldots,e_n\} \).

It is easy to see that \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in V \). We let \( W \) be the space spanned by the vectors \( \{v_1,v_2,\ldots,v_{n-1}\} \). Since \( X \in Y_{n,r}^0 \), the rank of the \( n \times (n-1) \) matrix

\[
\begin{bmatrix} w & v_1 & v_2 & \cdots & v_{n-1} \end{bmatrix}
\]

has only two possibilities, either \( r \) or \( r - 1 \). We consider both these cases separately.

**Case 1)** Suppose that

\[ \text{Rank} \left( \begin{bmatrix} w & v_1 & v_2 & \cdots & v_{n-1} \end{bmatrix} \right) = r. \]
Then \( \dim W = r \). It follows that, \( \begin{bmatrix} 0 \\ v \end{bmatrix} \in W \) and hence \( \begin{bmatrix} 0 \\ v \end{bmatrix} \in V \cap W \). Therefore, the number of choices for \( [v_1 \ v_2 \ \cdots \ \ v_{n-1}] \) is \( |M(n,n-1,r,q)| \).

a) If \( w = 0 \), then
\[
W \subseteq V.
\]
Hence, \( V \cap W = W \) and \( \dim(V \cap W) = \dim W = r \). Since \( \begin{bmatrix} 0 \\ v \end{bmatrix} \in V \cap W \), the number of possibilities of \( \begin{bmatrix} 0 \\ v \end{bmatrix} \) will be \( q^r \). Also, the total number of matrices \( \begin{bmatrix} w \\ Y \end{bmatrix} \) with rank \( r \) and \( w = 0 \) is \( |M(n-1,n-1,r,q)| \).

b) If \( w \neq 0 \), we have \( W \not\subseteq V \). Therefore,
\[
\dim(W \cap V) = \dim V + \dim W - \dim(V + W)
= n - 1 + r - n = r - 1.
\]
Since \( \begin{bmatrix} 0 \\ v \end{bmatrix} \in V \cap W \), the number of possibilities of \( \begin{bmatrix} 0 \\ v \end{bmatrix} \) will be \( q^{r-1} \). The number of matrices \( \begin{bmatrix} w \\ Y \end{bmatrix} \) with rank \( r \) and \( w \neq 0 \), is
\[
|M(n,n-1,r,q)| - |M(n-1,n-1,r,q)|.
\]

Case 2) Suppose that
\[
\text{Rank} \left( \begin{bmatrix} w \\ Y \end{bmatrix} \right) = \text{Rank} \left( [v_1 \ v_2 \ \cdots \ \ v_{n-1}] \right) = r - 1.
\]
Then \( \dim W = r - 1 \). Therefore, \( v \not\in W \) and hence \( \begin{bmatrix} 0 \\ v \end{bmatrix} \in V \setminus W \). Also, we have that the total number of matrices \( \begin{bmatrix} w \\ Y \end{bmatrix} \) with rank \( r - 1 \) is \( |M(n,n-1,r-1,q)| \).

a) If \( w = 0 \), then \( W \subseteq V \). Therefore, \( V \cap W = W \) and \( \dim(V \cap W) = \dim W = r - 1 \). Since \( \begin{bmatrix} 0 \\ v \end{bmatrix} \in V \setminus W \), the number of possibilities of \( \begin{bmatrix} 0 \\ v \end{bmatrix} \) will be \( q^{n-1} - q^{r-1} \). Furthermore, The total number of matrices \( \begin{bmatrix} w \\ Y \end{bmatrix} \) with rank \( r - 1 \) and \( w = 0 \) is \( |M(n-1,n-1,r-1,q)| \).

b) If \( w \neq 0 \), then \( W \not\subseteq V \). Therefore,
\[
\dim(W \cap V) = \dim V + \dim W - \dim(V + W)
= n - 1 + r - 1 - n = r - 2.
\]
Since \( v \in V \setminus W \), the number of possibilities of \( \begin{bmatrix} 0 \\ v \end{bmatrix} \) will be \( q^{n-1} - q^{r-2} \). The total number of matrices in this case will be \( |M(n,n-1,r-1,q)| - |M(n-1,n-1,r-1,q)| \).

Using Lemma 3.1 and the above computations, we have
\[ |Y_{n,r}^1| = q^{-1} |M(n-1, n-1, r, q)| - M(n-1, n-1, r, q) | - q^{-1} |M(n-1, n-1, r, q)| + q^{-2} |M(n-1, n-1, r, q)| \]
\[ + (q^{-1} - q^{-2}) |M(n-1, n-1, r, q)| + (q^{-1} - q^{-2}) |M(n-1, n-1, r, q)| \]
\[ + (q^{-1} - q^{-2}) |M(n-1, n-1, r, q)| \]
\[ = q^{-1} |M(n-1, n-1, r, q)| + q^{-1} |M(n-1, n-1, r, q)| + q^{-1} |M(n-1, n-1, r, q)| \]
\[ = q^{-1} |M(n-1, n-1, r, q)| + q^{-1} |M(n-1, n-1, r, q)| + q^{-1} |M(n-1, n-1, r, q)| \]
\[ = q^{-1} |M(n-1, n-1, r, q)| + q^{-1} |M(n-1, n-1, r, q)| + q^{-1} |M(n-1, n-1, r, q)| \]
\[ = q^{-1} |M(n-1, n-1, r, q)| + q^{-1} |M(n-1, n-1, r, q)| + q^{-1} |M(n-1, n-1, r, q)| \]

\[ \square \]

**Lemma 3.4.** We have
\[ |Y_{n,r}^1| = q^{-1} |M(n, n, r, q)| - q^{-1} |M(n-1, n-1, r, q)| + q^{-2} |M(n-1, n-1, r, q)|. \]

**Proof.** Using Lemma 3.2, we have
\[ |Y_{n,r}^0| + (q-1)|Y_{n,r}^1| = |M(n, n, r, q)|. \]

Thus we get,
\[ |Y_{n,r}^1| = \frac{M(n, n, r, q)| - |Y_{n,r}^0|}{q-1} \]
\[ = \frac{M(n, n, r, q)| - q^{-1} |M(n, n, r, q)|}{q-1} \]
\[ = \frac{(q^{-1} - q^{-1}) |M(n-1, n-1, r, q)| + (q^{-2} - q^{-1}) |M(n-1, n-1, r, q)|}{q-1} \]
\[ = q^{-1} |M(n-1, n-1, r, q)| - q^{-1} |M(n-1, n-1, r, q)| + q^{-2} |M(n-1, n-1, r, q)|. \]
\[ \square \]

**Lemma 3.5.** We have
\[ |Y_{n,r}^0| - |Y_{n,r}^1| = q^{-1} |M(n-1, n-1, r, q)| - q^{-1} |M(n-1, n-1, r, q)|. \]

**Proof.** Follows from Lemma 3.4 and Lemma 3.3. \[ \square \]

**Lemma 3.6.** Let \( r \in \{0, 1, 2, \ldots, n\} \) and \( X \in M(n, n, r, q) \). We have
\[ \Theta_{\theta} \left( \begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \right) = \begin{cases} (-1)(q; q)_{2n-1}, & \text{if } r = 0 \\ (-1)(q; q)_{2n-2}, & \text{if } r = 1 \\ \vdots & \\ (-1)(q; q)_{n-1}, & \text{if } r = n \end{cases} \]

**Proof.** The proof follows from Theorem 2.2 above and rewriting the character values using the \( q \)-Pochhammer symbol. \[ \square \]
\textbf{Theorem 3.7.} Let \( \theta \) be a regular character of \( F_{2n}^\times \) and \( \pi = \pi_\theta \) be an irreducible cuspidal representation of \( \text{GL}(2n,F) \). We have
\[
\dim_{C}(\pi_{N,\psi_A}) = (q-1)^2(q^2-1)^2 \cdots (q^{n-1}-1)^2 = (q;q)_{n-1}^2.
\]

\textit{Proof.} It is easy to see that the dimension of \( \pi_{N,\psi_A} \) is given by
\[
\dim_{C}(\pi_{N,\psi_A}) = \frac{1}{q^n} \sum_{X \in M(n,F)} \Theta_\theta \left( \begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \right) \psi_0(\text{Tr}(AX)).
\]
Clearly, we have \( M(n,F) = \bigcup_{r=0}^{n} \bigcup_{\alpha \in F} Y^\alpha_{n,r} \). Using this, we see that
\[
\dim_{C}(\pi_{N,\psi_A}) = \frac{1}{q^n} \sum_{r=0}^{n} \sum_{\alpha \in F} \Theta_\theta \left( \begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \right) \psi_0(\alpha)
\]
\[
= \frac{1}{q^n} \sum_{r=0}^{n} \sum_{\alpha \in F} (-1)(q;q)_{2n-1}^{-r} (|Y^0_{n,r}| - |Y^1_{n,r}|)
\]
\[
= -\frac{1}{q^n} \sum_{r=0}^{n} \sum_{\alpha \in F} (-1)(q;q)_{2n-1}^{-r} |M(n-1, n-1, 0, q)|
\]
\[
= \frac{1}{q^n} \sum_{r=0}^{n} \sum_{\alpha \in F} (-1)(q;q)_{2n-1}^{-r} |M(n-1, n-1, r, q)| - q^{-r} |M(n-1, n-1, r-1, q)|)
\]
\[
= \frac{1}{q^n} \sum_{r=0}^{n} \sum_{\alpha \in F} (-1)(q;q)_{2n-1}^{-r} |M(n-1, n-1, r, q)| - (q;q)_{2n-2}^{-r}
\]
\[
= \frac{1}{q^n} \sum_{r=0}^{n} \sum_{\alpha \in F} (-1)(q;q)_{2n-1}^{-r} |M(n-1, n-1, r, q)| - (q;q)_{2n-2}^{-r}
\]
\[
= (q;q)_{n-1}^2.
\]

\[\square\]

\textit{Remark 3.8.} Suppose that \( B = Aw_0 \). It is easy to see that \( \Theta_{N,\psi_A} \left( \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \right) = \Theta_{N,\psi_B} \left( \begin{bmatrix} w_0m_1w_0 & 0 \\ 0 & m_2 \end{bmatrix} \right) \). Thus we have that \( \dim_{C}(\pi_{N,\psi_A}) = \dim_{C}(\pi_{N,\psi_B}) \).

4. Main Theorem

In this section, we prove the main result of this paper. Before we continue, we set up some notation and record a few preliminary results that we need. Let \( G = \text{GL}(2n,F) \) and \( P \) be the maximal parabolic subgroup of \( G \) with Levi decomposition \( P = MN \), where \( M \simeq \text{GL}(n,F) \times \text{GL}(n,F) \) and \( N \simeq M(n,F) \). We write \( F_n \) for the unique field extension of \( F \) of degree \( n \). Let \( \psi_0 \) be a fixed non-trivial additive character of \( F \). Let
\[
A = \begin{bmatrix}
0 & \ldots & 0 & 1 \\
0 & \ldots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & 0
\end{bmatrix}
\]
Let \( \psi_A : N \rightarrow \mathbb{C}^\times \) be the character of \( N \) given by
\[
\psi_A \left( \begin{array}{cc} 1 & X \\ 0 & 1 \end{array} \right) = \psi_0(\text{Tr}(AX)).
\]

Let \( H_A = M_1 \times M_2 \) where \( M_1 \) is the Mirabolic subgroup of \( \text{GL}(n,F) \) and \( M_2 = w_0M_1w_0^{-1} \). Let \( U \) be the subgroup of unipotent matrices in \( \text{GL}(2n,F) \). Let \( U_A = H_A \cap U \). Clearly, we have \( U_A \simeq U_1 \times U_2 \) where \( U_1 \) and \( U_2 \) are the upper triangular unipotent subgroups of \( \text{GL}(n,F) \). For \( k = 1, 2 \), let \( \mu_k : U_k \rightarrow \mathbb{C}^\times \) be the non-degenerate character of \( U_k \) given by
\[
\mu_k \left( \begin{array}{ccc} 1 & x_{12} & x_{13} & \cdots & x_{1n} \\ 0 & 1 & x_{23} & \cdots & x_{2n} \\ & 1 & & \cdots & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{array} \right) = \psi_0(x_{12} + x_{23} + \cdots + x_{n-1,n}).
\]

Let \( \mu : U_A \rightarrow \mathbb{C}^\times \) be the character of \( U_A \) given by
\[
\mu(u) = \mu_1(u_1)\mu_2(u_2)
\]
where \( u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \).

**Lemma 4.1.** Let \( M_{\psi_A} = \{ m \in M \mid \psi_A^m(n) = \psi_A(n), \forall n \in N \} \). Then we have
\[
M_{\psi_A} = \left\{ \begin{bmatrix} C & x \\ 0 & a \\ C & y \\ 0 & D \end{bmatrix} \mid a \in F^\times, C, D \in \text{GL}(n-1,F), x, y \in F^{n-1} \right\}.
\]

**Proof.** Let \( g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \in M \). Then \( g \in M_{\psi_A} \) if and only if \( Ag_1 = g_2A \). It follows that \( g \in M_{\psi_A} \) if and only if \( g_1 = \begin{bmatrix} C & x \\ 0 & a \end{bmatrix} \) and \( g_2 = \begin{bmatrix} a & y \\ 0 & D \end{bmatrix} \). \( \square \)

**Lemma 4.2.** Let \( Z \) be the center of \( G = \text{GL}(2n,F) \). Let \( H_A \) be a subgroup of \( G \) as above. Then,
\[
M_{\psi_A} \simeq Z \times H_A.
\]

**Proof.** Trivial. \( \square \)

**Lemma 4.3.** Let \( \rho_1 = \text{ind}_{U_1}^{M_1} \mu_1 \) and \( \rho_2 = \text{ind}_{U_2}^{M_2} \rho_2 \). Consider the representation \((\rho,V)\) of \( M_{\psi_A} \) given by
\[
\rho = \theta |_{F^\times} \otimes \text{ind}_{U_A}^{H_A} \mu = \theta |_{F^\times} \otimes (\rho_1 \otimes \rho_2).
\]
Then \((\rho,V)\) is an irreducible representation of \( M_{\psi_A} \).

**Proof.** Since \( \rho_1 \) is the Kirillov representation of the Mirabolic subgroup \( M_1 \) of \( \text{GL}(n,F) \), we have that \( \rho_1 \) is irreducible (see Theorem 2.3). In a similar way, we can see that \( \rho_2 \) is also irreducible. Hence the result. \( \square \)

**Lemma 4.4.** Let \( P_{\psi_A} = M_{\psi_A}N \). Consider the map \( \tilde{\rho} : P_{\psi_A} \rightarrow \text{GL}(V) \) given by
\[
\tilde{\rho}(p) = \tilde{\rho}(mn) = \psi_A(mnm^{-1})\rho(m),
\]
where \( m \in M_{\psi_A}, n \in N \). Then \((\tilde{\rho},V)\) is a representation of \( P_{\psi_A} \).
Proof. Let $p_1 = m_1 n_1, p_2 = m_2 n_2 \in P_{\psi A}$. Then, we have
\[
\tilde{\rho}(p_1 p_2) = \tilde{\rho}(m_1 n_1 m_2 n_2) \\
= \tilde{\rho}(m_1 m_2 (m_2^{-1} n_1 m_2) n_2) \\
= \psi_A(m_1 m_2 (m_2^{-1} n_1 m_2) m_2^{-1} m_1^{-1}) \rho(m_1 m_2) \\
= \psi_A(m_1 m_2) \psi_A(m_2^{-1} m_1^{-1}) \rho(m_1) \rho(m_2) \\
= \psi_A(m_1) \psi_A(m_2^{-1}) \rho(m_1) \rho(m_2) \\
= \psi_A(m_1) \psi_A(m_2^{-1}) \rho(m_1) \rho(m_2) \\
= \tilde{\rho}(p_1) \tilde{\rho}(p_2).
\]

\[\square\]

Lemma 4.5. Let $\tilde{(\rho, V)}$ be the representation of $P_{\psi A}$ given by
\[
\tilde{\rho}(p) = \tilde{\rho}(n) = \psi_A(m n m^{-1}) \rho(m),
\]
where $m \in M_{\psi A}, n \in N$. Then, $\tilde{(\rho, V)}$ is irreducible.

Proof. Let $W$ be a non-trivial $P_{\psi A}$-invariant subspace of $V$. For $w \in W, p \in P_{\psi A}$, we have
\[
\tilde{\rho}(p) w = \psi_A(m n m^{-1}) \rho(m) w \in W.
\]
Therefore $\rho(m) w \in W$, for all $m \in M_{\psi A}, w \in W$. Since $\rho$ is irreducible (see Lemma 4.3), the result follows. \[\square\]

Lemma 4.6. Consider the representation $\tilde{\rho}$ of $P_{\psi A}$. We have
\[
\tilde{\rho}|_U = \psi_A \otimes \rho|_{U_A}.
\]

Proof. Clearly we have $U = U_A N$. Hence for $u = x n \in U$, we have
\[
\tilde{\rho}(u) = \psi_A(x n x^{-1}) \rho(x) = \psi_A(n) \rho(x).
\]

\[\square\]

Lemma 4.7. Let $\rho = \theta|_{F^x} \otimes \text{ind}^{H_A}_{U_A} \mu$ be the representation of $M_{\psi A}$ and $\tilde{\rho}$ be the corresponding representation of $P_{\psi A}$. For any $z \in Z$, we have
\[
\omega_{\tilde{\rho}}(z) = \omega_{\rho}(z) = \theta(z).
\]

Proof. For $z \in Z$, we have
\[
\chi_{\tilde{\rho}}(z) = \text{Tr}(\tilde{\rho}(z)) \\
= \omega_{\tilde{\rho}}(z) \deg(\rho) \\
= \text{Tr}(\rho(z)) \\
= \omega_{\rho}(z) \deg(\rho) \\
= \text{Tr}(\theta|_{F^x}(z) \otimes \text{ind}^{H_A}_{U_A} \mu(1)) \\
= \theta(z) \deg(\rho)
\]

It follows that $\omega_{\tilde{\rho}}(z) = \omega_{\rho}(z) = \theta(z).$ \[\square\]

Lemma 4.8. Let $\chi : F^x \rightarrow \mathbb{C}^x$ be a character of $F^x$. Consider the representation $\tilde{(\rho, V)}$ of $P_{\psi A}$ defined above. Let $\sigma_\chi : P_{\psi A} \rightarrow \text{GL}(V)$ be the map
\[
\sigma_\chi(p) = \sigma_\chi(z h n) = \chi(z) \tilde{\rho}(h n),
\]
where $z \in Z, h \in H_A, n \in N$. Then $\sigma_\chi$ is an irreducible representation of $P_{\psi A}$.
Proof. It is easy to see that $\sigma_\chi$ is a representation of $P_{\psi_A}$. Let $W$ be a non-trivial subspace of $V$ invariant under $P_{\psi_A}$ and let $w \neq 0 \in W$. We have

$$\sigma_\chi(zhn)w = \chi(z)\tilde{\rho}(hn)w \in W.$$ 

Therefore,

$$\tilde{\rho}(zhn)w = \tilde{\rho}(z)\tilde{\rho}(hn)w = \omega_\rho(z)\tilde{\rho}(hn)w \in W.$$ 

Since $\tilde{\rho}$ is irreducible, it follows that $V = W$ and hence the result.

**Lemma 4.9.** Let $\chi_1, \chi_2 \in \hat{F}^\times$ such that $\chi_1 \neq \chi_2$. Then,

$$\sigma_{\chi_1} \not\simeq \sigma_{\chi_2}.$$ 

**Proof.** Let $z_0 \in Z$ such that $\chi_1(z_0) \neq \chi_2(z_0)$. Let $\chi_{\sigma_{\chi_1}}, \chi_{\sigma_{\chi_2}}$ be the characters of $\sigma_{\chi_1}$ and $\sigma_{\chi_2}$. Suppose that $\sigma_{\chi_1} \simeq \sigma_{\chi_2}$. We have

$$\chi_{\sigma_{\chi_1}}(z_0) = \text{Tr}(\sigma_{\chi_1})(z_0)$$

$$= \chi_1(z_0) \deg(\rho)$$

$$= \chi_{\sigma_{\chi_2}}(z_0)$$

$$= \text{Tr}(\sigma_{\chi_2})(z_0)$$

$$= \chi_2(z_0) \deg(\rho).$$

The result follows.

**Lemma 4.10.** For $\chi \in \hat{F}^\times$, we have

$$\text{Hom}_{P_{\psi_A}}(\sigma_\chi, \text{ind}_{P_{\psi_A}} U_{\psi_A}) \neq 0.$$ 

**Proof.** Using Fröbenius Reciprocity, we have

$$\text{Hom}_{P_{\psi_A}}(\sigma_\chi, \text{ind}_{U_{\psi_A}} ^{P_{\psi_A}} \psi) = \text{Hom}_U(\sigma_\chi |_U, \psi).$$

Thus it is enough to show that $\text{Hom}_U(\sigma_\chi |_U, \psi) \neq 0$. For $u \in U$, we have

$$\sigma_\chi |_U(u) = \sigma_\chi(u) = \chi(1)\tilde{\rho}(u) = \tilde{\rho}(u)_U.$$ 

Therefore,

$$\text{Hom}_U(\sigma_\chi |_U, \psi) = \text{Hom}_U(\tilde{\rho}|_U, \psi)$$

$$= \text{Hom}_U(\tilde{\rho}|_U, \psi)$$

$$= \text{Hom}_U \left( \psi_A \otimes \bigoplus_{s \in U_A \setminus H_A / U_A} \text{ind}_{s^{-1} U_A s \cap U_A} ^{U_A} \mu_s, \psi \right)$$

$$= \text{Hom}_U(\psi_A, \mu, \psi) \oplus \bigoplus_{1 \neq s \in U_A \setminus H_A / U_A} \text{Hom}_U \left( \psi_A \otimes \text{ind}_{s^{-1} U_A s \cap U_A} ^{U_A} \mu_s, \psi \right)$$

$$= \text{Hom}_U(\psi_A, \mu, \psi) \oplus \bigoplus_{1 \neq s \in U_A \setminus H_A / U_A} \text{Hom}_U \left( \psi_A \otimes \text{ind}_{s^{-1} U_A s \cap U_A} ^{U_A} \mu_s, \psi \right)$$

$$\neq 0.$$
Lemma 4.11. Let $\chi \in \hat{F}^\times$ and $\sigma_\chi$ be the irreducible representation of $P_{\psi_A}$. Then

$$\text{ind}_{U_{\psi_A}}^{P_{\psi_A}} \psi = \bigoplus_{\chi \in \hat{F}^\times} \sigma_\chi.$$

Proof. The result clearly follows from a simple application of Lemma 4.9 and Lemma 4.10, and computing the degree of $\text{ind}_{U_{\psi_A}}^{P_{\psi_A}} (\psi)$. To be precise, suppose that

$$\text{ind}_{U_{\psi_A}}^{P_{\psi_A}} (\psi) = (\bigoplus_{x \in \hat{F}^\times} d_x \sigma_\chi) \oplus d\sigma$$

where $d_x \geq 1$, $d \geq 0$ and $\sigma$ is some representation of $P_{\psi_A}$. By degree comparison, we have that

$$\deg (\bigoplus_{x \in \hat{F}^\times} d_x \sigma_\chi) = \sum_{x \in \hat{F}^\times} d_x \deg (\sigma_\chi) = \sum_{x \in \hat{F}^\times} d_x \deg (\rho)$$

Clearly

$$\sum_{x \in \hat{F}^\times} d_x \deg (\rho) \geq (q - 1) \deg (\rho) = \deg (\text{ind}_{U_{\psi_A}}^{P_{\psi_A}} (\psi)),$$

On the other hand, we have

$$\deg (\bigoplus_{x \in \hat{F}^\times} d_x \sigma_\chi) + d \deg (\sigma) = \deg (\text{ind}_{U_{\psi_A}}^{P_{\psi_A}} (\psi)).$$

It follows that

$$d = 0, d_x = 1, \forall x \in \hat{F}^\times.$$

Hence the result. \qed

Lemma 4.12. Let $m = ah \in M_{\psi_A}$, where $a \in Z$ and $h \in H_A$. Then,

$$\Theta_{N,\psi_A}(m) = \theta(a) \Theta_{N,\psi_A}(h).$$

Proof. We have

$$\Theta_{N,\psi_A}(m) = \Theta_{N,\psi_A}(ah)$$

$$= \frac{1}{|N|} \sum_{n \in N} \Theta_\theta(ahn)\psi_A(n)$$

$$= \frac{1}{|N|} \sum_{n \in N} \text{Tr}(\pi(ahn)\psi_A(n))$$

$$= \frac{1}{|N|} \sum_{n \in N} \text{Tr}(\pi(a)\pi(hn)\psi_A(n))$$

$$= \omega_\pi(a) \frac{1}{|N|} \sum_{n \in N} \text{Tr}(\pi(hn)\psi_A(n))$$

$$= \omega_\pi(a) \Theta_{N,\psi_A}(h)$$

where $\omega_\pi$ is the central character of $\pi$. Explicitly, we have

$$\Theta_\theta(a) = \text{Tr}(\pi(a)) = \text{Tr}(\omega_\pi(a)) = \omega_\pi(a) \dim(\pi).$$

Using Theorem 2.2, it is easy to see that

$$\Theta_\theta(a) = \theta(a) \dim(\pi).$$

Thus, we have $\omega_\pi(a) = \theta(a)$ and the result follows. \qed

Lemma 4.13. Let $\chi \neq \theta \in \hat{F}^\times$. Then

$$\text{Hom}_{P_{\psi_A}} (\pi|_{P_{\psi_A}}, \sigma_\chi) = 0.$$
Proof. It is enough to show that \[ \dim \mathbb{C} \text{Hom}_{P_{\psi A}}(\pi|_{P_{\psi A}}, \sigma_{\chi}) = 0. \] Clearly, we have

\[
\dim \mathbb{C} \text{Hom}_{P_{\psi A}}(\pi|_{P_{\psi A}}, \sigma_{\chi}) = \langle \chi_{\pi|_{P_{\psi A}}}, \chi_{\sigma_{\chi}} \rangle = \sum_{zhn \in P_{\psi A}} \chi_{\pi}(zhn)\overline{\chi_{\sigma_{\chi}}(zhn)} = \\
= \sum_{hn \in H_{A}N} \sum_{z \in Z} \omega_{\pi}(z)\chi_{\chi}(hn)\overline{\chi_{\sigma}(hn)} = \sum_{zn \in H_{A}N} \sum_{z \in Z} \theta(z)\overline{\chi(z)}\chi_{\chi}(hn)\overline{\chi_{\sigma}(hn)} = \\
= \langle \theta, \chi \rangle \sum_{hn \in H_{A}N} \chi_{\pi}(hn)\overline{\chi_{\sigma}(hn)} = \langle \theta, \chi \rangle \\
= 0
\]

It follows that \[ \text{Hom}_{P_{\psi A}}(\pi|_{P_{\psi A}}, \sigma_{\chi}) = 0, \forall \chi \in \widehat{F}, \chi \neq \theta. \]

\[ \square \]

Lemma 4.14. Consider the restriction \( \theta|_{F^{\times}} \) of the regular character \( \theta \). Then

\[ \sigma_{\theta} = \tilde{\rho} \]

as \( P_{\psi A} \) representations.

Proof. Using Lemma 4.7, we have \( \omega_{\tilde{\rho}}(z) = \theta(z) \). Thus for \( p = zhn \in P_{\psi A} \), we have

\[
\sigma_{\theta}(zhn) = \theta(z)\rho(hn) = \omega_{\tilde{\rho}}(z)\tilde{\rho}(hn) = \tilde{\rho}(zhn).
\]

\[ \square \]

4.1. Proof of the Main Theorem. For the sake of completeness, we recall the statement below.

Theorem 4.15. Let \( \theta \) be a regular character of \( F_{2m}^{\times} \) and \( \pi = \pi_{\theta} \) be an irreducible cuspidal representation of \( G \). Then

\[ \pi_{N,\psi A} \simeq \theta|_{F^{\times}} \otimes \text{ind}_{U_{A}}^{H_{A}} \mu \]

as \( M_{\psi A} \) modules.
Proof. Using transitivity of induction and Lemma 4.11 we have that
\[ \text{Hom}_G(\pi, \text{ind}_U^G \psi) = \text{Hom}_G(\pi, \text{ind}_{P_{\psi A}}^G (\text{ind}_{U_{\psi}}^G \psi)) \]
\[ = \text{Hom}_G(\pi, \text{ind}_{P_{\psi A}}^G (\bigoplus_{\chi \in F^\times} \sigma_{\chi})) \]
\[ = \bigoplus_{\chi \in F^\times} \text{Hom}_G(\pi, \text{ind}_{P_{\psi A}}^G \sigma_{\chi}) \]
\[ = \bigoplus_{\chi \in F^\times} \text{Hom}_{P_{\psi A}}(\pi|_{P_{\psi A}}, \sigma_{\chi}) \]
\[ = \text{Hom}_{P_{\psi A}}(\pi|_{P_{\psi A}}, \sigma_{\theta}) \oplus \bigoplus_{\theta \neq \chi \in F^\times} \text{Hom}_{P_{\psi A}}(\pi|_{P_{\psi A}}, \sigma_{\chi}) \]
\[ = \text{Hom}_{P_{\psi A}}(\pi|_{P_{\psi A}}, \tilde{\rho}) \]
Hence,
\[ \text{Hom}_G(\pi, \text{ind}_U^G \psi) = \text{Hom}_{P_{\psi A}}(\pi|_{P_{\psi A}}, \tilde{\rho}) \simeq \text{Hom}_G(\pi, \text{ind}_{P_{\psi A}}^G \tilde{\rho}) \simeq \text{Hom}_{M_{\psi A}}(\pi_{N,\psi A}, \rho). \]
Using the multiplicity one theorem for GL(n) (give precise statement in preliminaries), we conclude that
\[ \text{dim}_C \text{Hom}_{M_{\psi A}}(\pi_{N,\psi A}, \rho) = 1 \]
and it follows that
\[ \pi_{N,\psi A} \simeq \rho \]
as $M_{\psi A}$ representations.

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