Predictive SUSY $\text{SO}(10) \times \Delta(48) \times \text{U}(1)$

Model for CP Violation, Neutrino Oscillation, 

Fermion Masses and Mixings 

with Small $\tan \beta$

K.C. Chou$^\dagger$ and Y.L. Wu$^\ddagger$

$^\dagger$Chinese Academy of Sciences, Beijing 100864, China

$^\ddagger$Institute of Theoretical Physics, Chinese Academy of Sciences

P.O. Box 2735, Beijing, 100080, P.R. China
Predictive SUSY SO(10)×Δ(48)× U(1) Model for CP Violation,
Neutrino Oscillation, Fermion Masses and Mixings

with Small $\tan \beta$

K.C. Chou† and Y.L. Wu‡∗

†Chinese Academy of Sciences, Beijing 100864, China
‡Institute of Theoretical Physics, Chinese Academy of Sciences
P.O. Box 2735, Beijing, 100080, P.R. China
(May, 1997)

Abstract

CP violation, fermion masses and mixing angles including that of neutrinos are studied in an SUSY SO(10)×Δ(48)× U(1) model with small $\tan \beta$. The family symmetry $\Delta(48)$ associated with a simple scheme of U(1) charge assignment on various fields concerned in superpotential leads to unique Yukawa coupling matrices with zero textures. Thirteen parameters involving masses and mixing angles in the quark and charged lepton sector are successfully predicted by only four parameters. The masses and mixing angles for the neutrino sector can also be predicted without involving new parameters. It is found that the atmospheric neutrino deficit, the mass limit put by hot dark matter and the LSND $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ events can be naturally explained. Solar neutrino puzzle can be solved only by introducing a sterile neutrino. An additional parameter is added to obtain the mass and mixing of the sterile neutrino. The hadronic parameters $B_K$ and $f_{B\sqrt{B}}$ are extracted from the ob-

*supported in part by Outstanding Young Scientist Found of China
served $K^0-\bar{K}^0$ and $B^0-\bar{B}^0$ mixings respectively. The direct CP violation ($\varepsilon'/\varepsilon$) in kaon decays and the three angles $\alpha$, $\beta$ and $\gamma$ of the unitarity triangle in the CKM matrix are also presented. More precise measurements of $\alpha_s(M_Z)$, $|V_{cb}|$, $|V_{ub}/V_{cb}|$, $m_t$, as well as various CP violation and neutrino oscillation experiments will provide an important test for the present model and guide us to a more fundamental theory.

**Keywords:** Fermion masses, mixing angles, CP violation, neutrino oscillations

PACS numbers: 12.15.Ff, 14.60.Pq, 11.30.Er, 12.10.Dm, 12.60.Jv
I. INTRODUCTION

The standard model (SM) is a great success. Eighteen phenomenological parameters in the SM, which are introduced to describe all the low energy data in the quark and charged lepton sector, have been extracted from various experiments although they are not yet equally well known. Some of them have an accuracy of better than 1%, but some others less than 10%. To improve the accuracy of these parameters and understand them is a big challenge for particle physics. The mass spectrum and the mixing angles observed remind us that we are in a stage similar to that of atomic spectroscopy before Balmer. Much effort has been made along this direction. It was first observed by Gatto et al, Cabbibo and Maiani \[1\] that the Cabbibo angle is close to $\sqrt{m_d/m_s}$. This observation initiated the investigation of the texture structure with zero elements \[2\] in the fermion Yukawa coupling matrices. The well-known examples are the Fritzsch ansatz \[3\] and Georgi-Jarlskog texture \[4\], which has been extensively studied and improved substantially in the literature \[3\]. Ramond, Robert and Ross \[6\] presented recently a general analysis on five symmetric texture structures with zeros in the quark Yukawa coupling matrices. A general analysis and review of the previous studies on the texture structure was given by Raby in \[7\]. Recently, Babu and Barr \[8\], Babu and Mohapatra \[9\], Babu and Shafi \[10\], Hall and Raby \[11\], Berezhiani \[12\], Kaplan and Schmaltz \[13\], Kusenko and Shrock \[14\] constructed some interesting models with texture zeros based on supersymmetric (SUSY) SO(10). Anderson, Dimopoulos, Hall, Raby and Starkman \[15\] presented a general operator analysis for the quark and charged lepton Yukawa coupling matrices with two zero textures ‘11’ and ‘13’. Though the texture ‘22’ and ‘32’ are not unique they could fit successfully the 13 observables in the quark and charged lepton sector with only six parameters. Recently, we have shown \[16\] that the same 13 parameters as well as 10 parameters concerning the neutrino sector (though not unique for this sector) can be successfully described in an SUSY SO(10) × $\Delta(48)$ × U(1) model with large tan $\beta$, where the universality of Yukawa coupling of superpotential was assumed. The resulting texture of mass matrices in the low energy region is quite unique and depends
only on a single coupling constant and some vacuum expectation values (VEVs) caused by necessary symmetry breaking. The 23 parameters were predicted by only five parameters with three of them determined by the symmetry breaking scales of U(1), SO(10), SU(5) and SU(2)\textsubscript{L}. In that model, the ratio of the VEVs of two light Higgs \(\tan \beta \equiv v_2/v_1\) has large value \(\tan \beta \sim m_t/m_b\). In general, there exists another interesting solution with small value of \(\tan \beta \sim 1\). Such a class of model could also give a consistent prediction on top quark mass and other low energy parameters. Furthermore, models with small value of \(\tan \beta \sim 1\) are of phenomenological interest in testing Higgs sector in the minimum supersymmetric standard model (MSSM) at the Colliders \cite{17}. Most of the existing models with small values of \(\tan \beta\) in the literature have more parameters than those with large values of \(\tan \beta \sim m_t/m_b\). This is because the third family unification condition \(\lambda_t^G = \lambda_b^G = \lambda_\tau^G\) has been changed to \(\lambda_t^G \neq \lambda_b^G = \lambda_\tau^G\). Besides, some relations between the up-type and down-type quark (or charged lepton) mass matrices have also been lost in the small \(\tan \beta\) case when two light Higgs doublets needed for SU(2)\textsubscript{L} symmetry breaking belong to different 10s of SO(10). Although models with large \(\tan \beta\) have less parameters, large radiative corrections \cite{18} to the bottom quark mass and Cabibbo-Kobayashi-Maskawa (CKM) mixing angles might arise depending on an unknown spectrum of supersymmetric particles.

In a recent Rapid Communication \cite{19}, we have presented an alternative model with small value of \(\tan \beta \sim 1\) based on the same symmetry group SUSY SO(10)\(\times\Delta(48)\times U(1)\) as the model \cite{16} with large value of \(\tan \beta\). It is amazing to find out that the model with small \(\tan \beta \sim 1\) in \cite{19} has more predictive power on fermion masses and mixings. For convenience, we refer the model in \cite{16} as Model I (with large \(\tan \beta \sim m_t/m_b\)) and the model in \cite{16,20} as Model II (with small \(\tan \beta \sim 1\)).

In this paper, we will present in much greater detail an analysis for the model II. Our main considerations can be summarized as follows:
1) SO(10) is chosen as the unification group\(^1\) so that the quarks and leptons in each family are unified into a 16-dimensional spinor representation of SO(10).

2) The non-abelian dihedral group $\Delta(48)$, a subgroup of SU(3) ($\Delta(3n^2)$ with $n = 4$), is taken as the family group. Thus, the three families can be unified into a triplet 16-dimensional spinor representation of $\text{SO}(10) \times \Delta(48)$. $U(1)$ is family-independent and is introduced to distinguish various fields which belong to the same representations of $\text{SO}(10) \times \Delta(48)$. The irreducible representations of $\Delta(48)$ consisting of five triplets and three singlets are found to be sufficient to build interesting texture structures for fermion mass matrices. The symmetry $\Delta(48) \times U(1)$ naturally ensures the texture structure with zeros for fermion Yukawa coupling matrices. Furthermore, the non-abelian flavor symmetries provides a super-GIM mechanism to supress flavor changing neutral currents induced by supersymmetric particles [23,24,13,25].

3) The universality of Yukawa coupling of the superpotential before symmetry breaking is simply assumed to reduce possible free parameters, i.e., all the coupling coefficients in the renormalizable superpotentials are assumed to be equal and have the same origins from perhaps a more fundamental theory. We know in general that universality of charges occurs only in the gauge interactions due to charge conservation like the electric charge of different particles. In the absence of strong interactions, family symmetry could keep the universality of weak interactions in a good approximation after breaking. In the present theory, there are very rich structures above the grand unification theory (GUT) scale with many heavy fermions and scalars and their interactions are taken to be universal before symmetry breaking. All heavy fields must have some reasons to exist and interact which we do not understand at this moment. So that it can only be an ansatz at the present moment since we do not know the answer governing the behavior of nature above the GUT scale. As the

\(^1\)Recently, a three-family SO(10) grand unification theory was found in the string theories from orbifold approach [21]. Other possible theories can be found from the free fermionic approach [22].
Yukawa coupling matrices of the quarks and leptons in the present model are generated at the GUT scale, so that the initial conditions of the renormalization group evaluation for them will be set at the GUT scale. As the resulting Yukawa couplings only rely on the ratios of the coupling constants of the renormalizable superpotentials at the GUT scale, the predictions on the low energy observables will not be affected by the renormalization group (RG) effects running from the Planck scale to the GUT scale as long as the relative value of the ratios for the ‘22’ and ‘32’ textures is unchanged. For this aim, the ‘22’ and ‘32’ textures are constructed in such a way that they have a similar superpotential structure and the fields concerned belong to the same representations of the symmetry group. As we know that the renormalization group evaluation does not change the representations of a symmetry group, thus the ratios of the coupling constants for the ‘22’ and ‘32’ textures should remain equal at the GUT scale. As we will see below, even if we abandon the general assumption of an universal coupling for all the Yukawa terms in the superpotential, the above feature can still be ensured by imposing a permutation symmetry among the fields concerning the ‘22’ and ‘32’ textures after family symmetry breaking. As the numerical predictions on the low energy parameters so found are very encouraging and interesting, we believe that there must be a deeper reason that has to be found in the future.

4) The two light Higgs doublets are assumed to belong to an unique 10 representation Higgs of SO(10).

5) Both the symmetry breaking direction of SO(10) down to SU(5) and the two symmetry breaking directions of SU(5) down to SU(3)c × SU(2)L × U(1) are carefully chosen to ensure the needed Clebsch coefficients for quark and lepton mass matrices. The mass splitting

---

\(^2\)For models in which the third family Yukawa interaction is considered to be a renormalizable one starting from the Planck scale and the other two family Yukawa interactions are effectively generated at the GUT scale, one then needs to consider the renormalization effect the third family Yukawa coupling from the Planck scale down to the GUT scale.
between the up-type quark and down-type quark (or charged lepton) Yukawa couplings is attributed to the Clebsch factors caused by the SO(10) symmetry breaking direction. Thus the third family four-Yukawa coupling relation at the GUT scale will be given by

\[
\lambda_b^{G} = \lambda_r^{G} = \frac{1}{3^n} \lambda_t^{G} = 5^{n+1} \lambda_{\nu_{\tau}}^{G}
\]  

(1)

where the factors $1/3^n$ and $5^{n+1}$ with $n$ being an integer are the Clebsch factors. A factor $1/3^n$ will also multiply the down-type quark and charged lepton Yukawa coupling matrices.

6) CP symmetry is broken spontaneously in the model, a maximal CP violation is assumed to further diminish free parameters.

With the above considerations, the resulting model has found to provide a successful prediction on 13 parameters in the quark and charged lepton sector as well as an interesting prediction on 10 parameters in the neutrino sector with only four parameters. One is the universal coupling constant and the other three are determined by the ratios of vacuum expectation values (VEVs) of the symmetry breaking scales and the RG effects above the GUT scale. One additional parameter resulting from the VEV of a singlet scalar is introduced to obtain the mass and mixing angle of a sterile neutrino. Our paper is organized as follows: In section 2, we will present the results of the Yukawa coupling matrices. The resulting masses and CKM quark mixings are presented in section 3. In section 4 neutrino masses and CKM-type mixings in the lepton sector are presented. All existing neutrino experiments are discussed and found to be understandable in the present model. In section 5, the representations of the dihedral group $\Delta(48)$ and their tensor products are explicitly presented. In section 6, the model with superfields and superpotential is constructed in detail. Conclusions and remarks are presented in the last section.

II. YUKAWA COUPLING MATRICES

With the above considerations, a model based on the symmetry group SUSY $\text{SO}(10) \times \Delta(48) \times \text{U}(1)$ with a single coupling constant and small value of $\tan \beta$ is constructed.
Yukawa coupling matrices which determine the masses and mixings of all quarks and leptons are obtained by carefully choosing the structure of the physical vacuum and integrating out the heavy fermions at the GUT scale. We find

$$\Gamma^G_u = \frac{2}{3} \lambda_H \begin{pmatrix} 0 & \frac{3}{2} z_u \epsilon_P^2 & 0 \\ \frac{3}{2} z_u \epsilon_P^2 & -3 y_u \epsilon_G^2 e^{i \phi} & -\frac{\sqrt{3}}{2} x_u \epsilon_G^2 \\ 0 & -\frac{\sqrt{3}}{2} x_u \epsilon_G^2 & w_u \end{pmatrix}$$

(2)

and

$$\Gamma^G_f = \frac{2}{3} \lambda_H \frac{(-1)^{n+1}}{3^n} \begin{pmatrix} 0 & -\frac{3}{2} z_f \epsilon_P^2 & 0 \\ -\frac{3}{2} z_f \epsilon_P^2 & 3 y_f \epsilon_G^2 e^{i \phi} & -\frac{1}{2} x_f \epsilon_G^2 \\ 0 & -\frac{1}{2} x_f \epsilon_G^2 & w_f \end{pmatrix}$$

(3)

for $f = d, e$, and

$$\Gamma^G_\nu = \frac{2}{3} \lambda_H \frac{(-1)^{n+1}}{3^n} \frac{1}{5^n+1} \begin{pmatrix} 0 & -\frac{3}{2} z_\nu \epsilon_P^2 & 0 \\ -\frac{3}{2} z_\nu \epsilon_P^2 & 15 y_\nu \epsilon_G^2 e^{i \phi} & -\frac{1}{2} x_\nu \epsilon_G^2 \\ 0 & -\frac{1}{2} x_\nu \epsilon_G^2 & w_\nu \end{pmatrix}$$

(4)

for Dirac-type neutrino coupling. We will choose $n = 4$ in the following considerations. $\lambda_H = \lambda^0_H r_3$, $\epsilon_G \equiv \left(\frac{\lambda^0_H}{M_P}\right) \sqrt{\frac{r_1}{r_3}}$ and $\epsilon_P \equiv \left(\frac{\lambda^0_H}{M_P}\right) \sqrt{\frac{r_2}{r_3}}$ are three parameters. Where $\lambda^0_H$ is a universal coupling constant expected to be of order one, $r_1, r_2$ and $r_3$ denote the ratios of the coupling constants of the superpotential at the GUT scale for the textures ‘12’, ‘22’ (‘32’) and ‘33’ respectively. They represent the possible renormalization group (RG) effects running from the scale $\tilde{M}_P$ to the GUT scale. Note that the RG effects for the textures ‘22’ and ‘32’ are considered to be the same since they are generated from a similar superpotential structure after integrating out the heavy fermions and the fields concerned belong to the same representations of the symmetry group. This can be explicitly seen from their effective operators $W_{22}$ and $W_{32}$ given in eq. (6). $\tilde{M}_P$, $v_{10}$ and $v_5$ are the VEVs for $U(1) \times \Delta(48)$, $SO(10)$ and $SU(5)$ symmetry breaking respectively. $\phi$ is the physical CP phase arising from

\[3\] We have rotated away other possible phases by a phase redefinition of the fermion fields.
the VEVs. The assumption of maximum CP violation implies that $\phi = \pi/2$. $x_f$, $y_f$, $z_f$, and $w_f$ ($f = u, d, e, \nu$) are the Clebsch factors of SO(10) determined by the directions of symmetry breaking of the adjoints $45$'s. The following three directions have been chosen for symmetry breaking, namely:

\[
\begin{align*}
< A_X > &= 2v_{10} \text{ diag.}(1, 1, 1, 1, 1) \otimes \tau_2, \\
< A_z > &= 2v_5 \text{ diag.}( -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -1, -1) \otimes \tau_2, \\
< A_u > &= \frac{1}{\sqrt{3}}v_5 \text{ diag.}(2, 2, 2, 1, 1) \otimes \tau_2
\end{align*}
\]

Their corresponding U(1) hypercharges are given in Table I.

The Clebsch factors associated with the symmetry breaking directions can be easily read off from the U(1) hypercharges of the above table. The related effective operators obtained after the heavy fermion pairs integrated out are:

\[
W_{33} = \lambda_H^0 r_3 16_3 \eta_X \eta_A 10_1 \eta_A 16_3 \\
W_{32} = \lambda_H^0 r_2 16_3 \eta_X \eta_A \left( \frac{A_z}{A_X} \right) 10_1 \left( \frac{A_z}{A_X} \right) \eta_A 16_2 \\
W_{22} = \lambda_H^0 r_2 16_2 \eta_A \left( \frac{A_u}{A_X} \right) 10_1 \left( \frac{A_u}{A_X} \right) \eta_A 16_2 e^{i\phi} \\
W_{12} = \lambda_H^0 r_1 16_1 \left[ \left( \frac{v_5}{M_P} \right)^2 \eta_A 10_1 \eta_A' \right. \\
\left. + \left( \frac{v_{10}}{M_P} \right)^2 \eta_A \left( \frac{A_u}{A_X} \right) 10_1 \left( \frac{A_u}{A_X} \right) \eta_A \right] 16_2
\]

with $n = 4$ and $\phi = \pi/2$. $\eta_A = (v_{10}/A_X)^{n+1}$ and $\eta_A' = (v_{10}/A_X)^{n-3}$. The factor $\eta_X = 1/\sqrt{1 + 2\eta_A^2}$ in eq. (6) arises from mixing, and provides a factor of $1/\sqrt{3}$ for the up-type quark. It remains almost unity for the down-type quark and charged lepton as well as neutrino due to the suppression of large Clebsch factors in the second term of the square.

\[\text{Note that } W_{22} \text{ is slightly modified in comparison with the one in } [19] \text{ since we have renormalized the VEV } < A_u >. \text{ As a consequence, only the Clebsch factor } y_\nu \text{ is modified, which does not affect all the numerical predictions.}\]
root. The relative phase (or sign) between the two terms in the operator \( W_{12} \) has been fixed. The resulting Clebsch factors are

\[
\begin{align*}
    w_u &= w_d = w_e = w_\nu = 1, \\
    x_u &= 5/9, \quad x_d = 7/27, \quad x_e = -1/3, \quad x_\nu = 1/5, \\
    y_u &= 0, \quad y_d = y_e/3 = 2/27, \quad y_\nu = 4/225, \\
    z_u &= 1, \quad z_d = z_e = -27, \quad z_\nu = -15^3 = -3375, \\
    z'_u &= 1 - 5/9 = 4/9, \quad z'_d = z_d + 7/729 \simeq z_d, \\
    z'_e &= z_e - 1/81 \simeq z_e, \quad z'_\nu = z_\nu + 1/15^3 \simeq z_\nu.
\end{align*}
\]  

(7)

In obtaining the \( \Gamma^G_f \) matrices, some small terms arising from mixings between the chiral fermion 16; and the heavy fermion pairs \( \psi_j (\bar{\psi}_j) \) are neglected. They are expected to change the numerical results no more than a few percent for the up-type quark mass matrix and are negligible for the down-type quark and lepton mass matrices due to the strong suppression of the Clebsch factors. This set of effective operators which lead to the above given Yukawa coupling matrices \( \Gamma^G_f \) is quite unique for a successful prediction on fermion masses and mixings. A general superpotential leading to the above effective operators will be given in section 6. We would like to point out that unlike many other models in which \( W_{33} \) is assumed to be a renormalizable interaction before symmetry breaking, the Yukawa couplings of all the quarks and leptons (both heavy and light) in both Model II and Model I are generated at the GUT scale after the breakdown of the family group and SO(10). Therefore, initial conditions for renormalization group (RG) evolution will be set at the GUT scale for all the quark and lepton Yukawa couplings. The hierarchy among the three families is described by the two ratios \( \epsilon_G \) and \( \epsilon_P \). The mass splittings between the quarks and leptons as well as between the up and down quarks are determined by the Clebsch factors of SO(10). From the GUT scale down to low energies, Renormalization Group (RG) evolution has been taken into account. The top-bottom splitting in the present model is mainly attributed to the Clebsch factor \( 1/3^n \) with \( n = 4 \) rather than the large value of \( \tan \beta \) caused by the hierarchy
of the VEVs $v_1$ and $v_2$ of the two light Higgs doublets.

An adjoint 45 $A_X$ and a 16-dimensional representation Higgs field $\Phi$ ($\bar{\Phi}$) are needed for breaking $SO(10)$ down to $SU(5)$. Adjoint 45 $A_z$ and $A_u$ are needed to break $SU(5)$ further down to the standard model $SU(3)_c \times SU_L(2) \times U(1)_Y$.

### III. PREDICTIONS

From the Yukawa coupling matrices given above with $n = 4$ and $\phi = \pi/2$, the 13 parameters in the SM can be determined by only four parameters: a universal coupling constant $\lambda_H$ and three ratios: $\epsilon_G$, $\epsilon_P$ and $\tan \beta = v_2/v_1$. In obtaining physical masses and mixings, renormalization group (RG) effects below the GUT scale has been further taken into consideration. The result at the low energy obtained by scaling down from the GUT scale will depend on the strong coupling constant $\alpha_s$. From low-energy measurements and lattice calculations, $\alpha_s$ at the scale $M_Z$, has value around $\alpha_s(M_Z) = 0.113$, which was also found to be consistent with a recent global fit to the LEP data. This value might be reached in nonminimal SUSY GUT models through large threshold effects. As our focus here is on the fermion masses and mixings, we shall not discuss it in this paper. In the present consideration, we take $\alpha_s(M_Z) \simeq 0.113$. The prediction on fermion masses and mixings thus obtained is found to be remarkable. Our numerical predictions are given in Tables II and III with four input parameters, three of them are the well measured charged lepton masses and another is the bottom quark mass.

The predictions on the quark masses and mixings as well as CP-violating effects presented in Table IIb agree remarkably with those extracted from various experimental data. Especially, there are four predictions on $|V_{us}|$, $|V_{ub}/V_{cb}|$, $|V_{td}/V_{ts}|$ and $m_d/m_s$ which are independent of the RG scaling (see eqs. (41)-(44) below).

Let us now analyze in detail the above predictions. To a good approximation, the up-type and down-type quark Yukawa coupling matrices can be diagonalized in the form
The CKM matrix at the GUT scale is then given by $V_{CKM} = V_u V_d^\dagger$. Where $s_i \equiv \sin(\theta_i)$ and $c_i \equiv \cos(\theta_i)$ ($i = 1, 2, u, d$). For $\phi = \pi/2$, the angles $\theta_i$ at the GUT scale are given by

$$
\tan(\theta_1) \simeq -\frac{z_d \epsilon_d^p}{2y_d \epsilon_G^s}, \quad \tan(\theta_2) \simeq \frac{2w_u z_u \epsilon_d^p}{x_u^2 \epsilon_G^s},
$$

$$
\tan(\theta_d) \simeq \frac{x_d}{2w_d \epsilon_G^s}, \quad \tan(\theta_u) \simeq \frac{\sqrt{3} x_u}{2 w_u \epsilon_G^s},
$$

and the Yukawa eigenvalues at the GUT scale are found to be

$$
\frac{\lambda_u^G}{\lambda_e^G} = \frac{4w_u^2 z_u' \epsilon_d^p}{x_u^4 \epsilon_G^s}, \quad \frac{\lambda_e^G}{\lambda_u^G} = \frac{3 x_u^2 \epsilon_d^p}{4 w_u^2 \epsilon_G^s}, \quad \lambda_{\mu}^G = \frac{3 y_d}{w_d} \epsilon_G^s, \quad \lambda_{\tau}^G = \frac{3 y_e}{w_e} \epsilon_G^s.
$$

Using the eigenvalues and angles of these Yukawa matrices, one can easily find the following ten relations among fermion masses and CKM matrix elements at the GUT scale

$$
\frac{m_b}{m_\tau}_G = 1,
$$

$$
\frac{m_s}{m_\mu}_G = \frac{1}{3}, \quad \text{or} \quad \frac{m_s}{m_b}_G = \frac{1}{3} \left( \frac{m_\mu}{m_\tau}_G \right)
$$

$$
\left( \frac{m_d}{m_s}_G \right) \left( 1 - \left( \frac{m_d}{m_s}_G \right)^{-2} \right) = 9 \left( \frac{m_e}{m_\mu}_G \right) \left( 1 - \left( \frac{m_e}{m_\mu}_G \right)^{-2} \right),
$$

$$
\frac{m_t}{m_\tau}_G = 81 \tan \beta,
$$

$$
\frac{m_c}{m_t}_G = \frac{25}{48} \left( \frac{m_\mu}{m_\tau}_G \right)^2.
$$
\[
\left( \frac{m_u}{m_c} \right)_G = \frac{4}{9} \left( \frac{14}{15} \right)^4 \left( \frac{m_e m^2 \tau}{m^3 \mu} \right)_G^4, \tag{20}
\]

\[
|V_{ub}|_G = \tan(\theta_2) = \left( \frac{4}{15} \right)^2 \left( \frac{m_e}{m_\mu} \right)_G \sqrt{\left( \frac{m_e}{m_\mu} \right)_G}, \tag{21}
\]

\[
|V_{cb}|_G = \tan(\theta_1) = 3 \sqrt{\left( \frac{m_e}{m_\mu} \right)_G}, \tag{22}
\]

\[
|V_{us}|_G = c_1 c_2 \sqrt{\tan^2(\theta_1) + \tan^2(\theta_2)} = 3 \sqrt{\left( \frac{m_e}{m_\mu} \right)_G \left( \frac{1 + \left( \frac{16}{675} \left( \frac{m_e}{m_\tau} \right)_G \right)^2}{1 + \left( \frac{1}{675} \left( \frac{m_e}{m_\mu} \right)_G \right)^2} \right)^{1/2}}, \tag{23}
\]

\[
|V_{cb}|_G = c_2 c_d c_u (\tan(\theta_u) - \tan(\theta_d)) = \frac{15\sqrt{3} - 7}{15\sqrt{3}} \frac{5}{4\sqrt{3}} \left( \frac{m_\mu}{m_\tau} \right)_G. \tag{24}
\]

The Clebsch factors in eq. (7) appeared as those miraculous numbers in the above relations. The index ‘G’ refers throughout to quantities at the GUT scale. The first two relations are well-known in the Georgi-Jarlskog texture. The physical fermion masses and mixing angles are related to the above Yukawa eigenvalues and angles through the renormalization group (RG) equations [32]. As most Yukawa couplings in the present model are much smaller than the top quark Yukawa coupling \( \lambda_t^G \sim 1 \). In a good approximation, we will only keep top quark Yukawa coupling terms in the RG equations and neglect all other Yukawa coupling terms in the RG equations. The RG evolution will be described by three kinds of scaling factors. Two of them (\( \eta_F \) and R_t ) arise from running the Yukawa parameters from the GUT scale down to the SUSY breaking scale \( M_S \) which is chosen to be close to the top quark mass, i.e., \( M_S \simeq m_t \simeq 170 \) GeV, and are defined as

\[
m_t(M_S) = \eta_U(M_S) \lambda_t^G R_t^{-6} \frac{v}{\sqrt{2}} \sin \beta, \tag{25}
\]

\[
m_b(M_S) = \eta_D(M_S) \lambda_b^G R_t^{-1} \frac{v}{\sqrt{2}} \cos \beta, \tag{26}
\]

\[
m_i(M_S) = \eta_U(M_S) \lambda_i^G R_t^{-3} \frac{v}{\sqrt{2}} \sin \beta, \quad i = u, c, \tag{27}
\]

\[
m_i(M_S) = \eta_D(M_S) \lambda_i^G \frac{v}{\sqrt{2}} \cos \beta, \quad i = d, s, \tag{28}
\]

\[
m_i(M_S) = \eta_E(M_S) \lambda_i^G \frac{v}{\sqrt{2}} \cos \beta, \quad i = e, \mu, \tau, \tag{29}
\]

\[
\lambda_i(M_S) = \eta_N(M_S) \lambda_i^G R_t^{-3}, \quad i = \nu_e, \nu_\mu, \nu_\tau. \tag{30}
\]

with \( v = 246 \) GeV. \( \eta_F(M_S) \) and \( R_t \) are given by
\[ \eta_F(M_S) = \prod_{i=1}^{3} \left( \frac{\alpha_i(M_G)}{\alpha_i(M_S)} \right)^{c_i^F / 2b_i}, \quad F = U, D, E, N \]  

(31)

\[ R_t^{-1} = e^{-\int_{\ln M_S}^{\ln M_G} \frac{(\alpha_i^G)^2}{4\pi^2} dt} = (1 + (\lambda_t^G)^2 K_t)^{-1/12} = \left( 1 - \frac{\lambda_t^2(M_S)}{\lambda_f^2} \right)^{1/12} \]  

(32)

with \( c_i^U = \left( \frac{13}{15}, 3, \frac{16}{3} \right) \), \( c_i^D = \left( \frac{7}{15}, 3, \frac{16}{3} \right) \), \( c_i^F = \left( \frac{27}{15}, 3, 0 \right) \), \( c_i^N = \left( \frac{9}{2}, 3, 0 \right) \), and \( b_i = \left( \frac{33}{5}, 1, -3 \right) \), where \( \lambda_f \) is the fixed point value of \( \lambda_t \) and is given by

\[ \lambda_f = \frac{2\pi \eta_U^2}{\sqrt{3I(M_S)}}, \quad I(M_S) = \int_{\ln M_S}^{\ln M_G} \eta_U^2(t) dt \]  

(33)

The factor \( K_t \) is related to the fixed point value via \( K_t = \frac{\eta_U^2}{\lambda_f^2} = \frac{3I(M_S)}{4\pi^2} \). The numerical value for \( I \) taken from Ref. [33] is 113.8 for \( M_S \simeq m_t = 170 \text{GeV} \). \( \lambda_f \) cannot be equal to \( \lambda_t(M_S) \) exactly, since that would correspond to infinite \( \lambda_t^G \), and lead to the so called Landau pole problem at the GUT scale. Other RG scaling factors are derived by running Yukawa couplings below \( M_S \)

\[ m_i(m_i) = \eta_i m_i(M_S), \quad i = c, b, \]  

(34)

\[ m_i(1\text{GeV}) = \eta_i m_i(M_S), \quad i = u, d, s \]  

(35)

where \( \eta_i \) are the renormalization factors. The physical top quark mass is given by

\[ M_t = m_t(m_t) \left( 1 + \frac{4 \alpha_s(m_t)}{3 \pi} \right) \]  

(36)

In numerical calculations, we take \( \alpha^{-1}(M_Z) = 127.9, \ s^2(M_Z) = 0.2319, \ M_Z = 91.187 \text{ GeV} \) and use the gauge couplings at \( M_G \sim 2 \times 10^{16} \text{ GeV} \) at GUT scale and that of \( \alpha_1 \) and \( \alpha_2 \) at \( M_S \simeq m_t \simeq 170 \text{ GeV} \)

\[ \alpha_1^{-1}(m_t) = \alpha_1^{-1}(M_Z) + \frac{53}{30\pi} \ln \frac{M_Z}{m_t} = 58.59, \]  

(37)

\[ \alpha_2^{-1}(m_t) = \alpha_2^{-1}(M_Z) - \frac{11}{6\pi} \ln \frac{M_Z}{m_t} = 30.02, \]  

(38)

\[ \alpha_1^{-1}(M_G) = \alpha_2^{-1}(M_G) = \alpha_3^{-1}(M_G) \simeq 24 \]  

(39)

we keep \( \alpha_3(M_Z) \) as a free parameter in this note. The precise prediction on \( \alpha_3(M_Z) \) concerns GUT and SUSY threshold corrections. We shall not discuss it here since our focus in this
note is the fermion masses and mixings. Including the three-loop QCD and one-loop QED contributions, the values of $\eta_i$ in Table IV will be used in numerical calculations.

It is interesting to note that the mass ratios of the charged leptons are almost independent of the RG scaling factors since $\eta_e = \eta_\mu = \eta_\tau$ (up to an accuracy $O(10^{-3})$), namely

$$\frac{m_e}{m_\mu} = \left(\frac{m_e}{m_\mu}\right)_G, \quad \frac{m_\mu}{m_\tau} = \left(\frac{m_\mu}{m_\tau}\right)_G$$

(40)

which is different from the models with large $\tan\beta$. In the present model the $\tau$ lepton Yukawa coupling is small. It is easily seen that four relations represented by eqs. (21)-(23) and (17) hold at low energies. Using the known lepton masses $m_e = 0.511$ MeV, $m_\mu = 105.66$ MeV, and $m_\tau = 1.777$ GeV, we obtain four important RG scaling-independent predictions:

$$|V_{us}| = |V_{us}|_G = \lambda \simeq 3 \sqrt{\frac{m_e}{m_\mu}} \left(1 + \frac{16}{675} \frac{m_e}{m_\mu} \right)^2 \frac{1}{1 + \frac{m_e}{m_\mu}} = 0.22,$$

(41)

$$|V_{cb}| = |V_{cb}|_G = \lambda \sqrt{\rho^2 + \eta^2} \simeq \left(\frac{4}{15}\right)^2 \frac{m_\tau}{m_\mu} \sqrt{\frac{m_e}{m_\mu}} = 0.083,$$

(42)

$$|V_{ts}| = |V_{ts}|_G = \lambda \sqrt{(1 - \rho)^2 + \eta^2} \simeq 3 \sqrt{\frac{m_e}{m_\mu}} = 0.209,$$

(43)

$$\frac{m_d}{m_s} \left(1 - \frac{m_d}{m_s}\right)^{-2} = 9 \frac{m_e}{m_\mu} \left(1 - \frac{m_e}{m_\mu}\right)^{-2}, \quad \text{i.e.}, \quad \frac{m_d}{m_s} = 0.040$$

(44)

and six RG scaling-dependent predictions:

$$|V_{cb}| = |V_{cb}|_G R_t = A \lambda^2 = \frac{15\sqrt{3} - 7}{15\sqrt{3}} \frac{5}{4\sqrt{3}} \frac{m_\mu}{m_\tau} R_t = 0.0391 \left(\frac{0.80}{R_t^{-1}}\right),$$

(45)

$$m_s(1\text{GeV}) = \frac{1}{3} m_\mu \frac{\eta_s}{\eta_D/E} = 159.53 \left(\frac{\eta_s}{2.2}\right) \left(\frac{\eta_D/E}{2.1}\right) \text{MeV},$$

(46)

$$m_b(m_b) = m_\tau \frac{\eta_b}{\eta_D/E} R_t^{-1} = 4.25 \left(\frac{\eta_b}{1.49}\right) \left(\frac{\eta_D/E}{2.04}\right) \left(\frac{R_t^{-1}}{0.80}\right) \text{GeV},$$

(47)

$$m_u(1\text{GeV}) = \frac{5}{3} \frac{4}{45} \frac{m_e}{m_\mu} \eta_u R_t^3 m_t = 4.23 \left(\frac{\eta_u}{2.2}\right) \left(\frac{0.80}{R_t^{-1}}\right) \left(\frac{m_t(m_t)}{174\text{GeV}}\right) \text{MeV},$$

(48)

$$m_c(m_c) = \frac{25}{48} \frac{m_\mu^2}{m_\tau} \eta_c R_t^3 m_t = 1.25 \left(\frac{\eta_c}{2.0}\right) \left(\frac{0.80}{R_t^{-1}}\right) \left(\frac{m_t(m_t)}{174\text{GeV}}\right) \text{GeV},$$

(49)

$$m_t(m_t) = \frac{\eta_t}{\sqrt{K_t}} \sqrt{1 - R_t^{-12}} \frac{v}{\sqrt{2}} \sin\beta = 174.9 \left(\frac{\sin\beta}{0.92}\right) \left(\frac{\eta_U}{3.33}\right) \left(\frac{8.65}{K_t}\right) \left(\frac{1 - R_t^{-12}}{0.965}\right) \text{GeV}$$

(50)

We have used the fixed point property for the top quark mass. These predictions depend on two parameters $R_t$ and $\sin\beta$ (or $\lambda^G_t$ and $\tan\beta$). In general, the present model contains
four parameters: $\epsilon_G$, $\epsilon_P$, $\tan \beta = v_2/v_1$, and $\lambda^G_t = 81 \lambda^G_b = 81 \lambda^G_\tau = \frac{2}{3} \lambda_H$. It is not difficult to notice that $\epsilon_G$ and $\epsilon_P$ are determined solely by the Clebsch factors and mass ratios of the charged leptons

\[
\epsilon_G = \left( \frac{v_5}{v_{10}} \right) \sqrt{\frac{r_2}{r_3}} = \sqrt{\frac{m_\mu \eta_{\tau} w_e}{m_\tau \eta_\mu 3y_e}} = 2.987 \times 10^{-1},
\]

\[
\epsilon_P = \left( \frac{v_5}{M_P} \right) \sqrt{\frac{r_1}{r_3}} = \left( \frac{4 m_e m_\mu \eta^2_\tau w^2_e}{9 m^2_\tau \eta_\mu \eta_\mu z^2_e} \right)^{1/4} = 1.011 \times 10^{-2}.
\]

The coupling $\lambda^G_t$ (or $R_t$) can be determined by the mass ratio of the bottom quark and $\tau$ lepton

\[
\lambda^G_t = \frac{1}{\sqrt{K_t}} \sqrt{\frac{1 - R_t^{-12}}{R_t^{-6}}} = 1.25 \zeta_t,
\]

\[
\zeta_t = \left( \frac{8.65}{K_t} \right)^6 \left( \frac{0.80}{R_t^{-1}} \right) \left( \frac{1 - R_t^{-12}}{0.965} \right),
\]

\[
R_t^{-1} = \frac{m_b \eta_{\tau}}{m_\tau \eta_b \eta_{D/E}} = 0.80 \left( \frac{m_b (m_b)}{4.25 GeV} \right) \left( \frac{1.49}{\eta_b} \right) \left( \frac{2.04}{\eta_{D/E}} \right).
\]

$\tan \beta$ is fixed by the $\tau$ lepton mass

\[
\cos \beta = \frac{m_\tau \sqrt{2}}{\eta_E \eta_{\tau} \lambda^G_\tau} = \left( \frac{0.41}{\zeta_t} \right) \left( \frac{3^a}{81} \right),
\]

\[
\sin \beta = \sqrt{1 - \left( \frac{0.41}{\zeta_t} \right)^2 3^a} = 0.912 \left( \sqrt{\frac{1 - (0.41/\zeta_t)^2 3^a}{0.912}} \right),
\]

\[
\tan \beta = 2.225 \left( \frac{81}{3^a} \right) \left( \frac{\sqrt{\zeta^2_t - (0.41)^2 (3^a/81)^2}}{0.912} \right).
\]

With these considerations, the top quark mass is given by

\[
m_t(m_t) = 173.4 \left( \frac{\eta_U}{3.33} \right) \left( \frac{8.65}{K_t} \right) \left( \frac{1 - R_t^{-12}}{0.965} \right) \left( \frac{1 - (0.41/\zeta_t)^2}{0.912} \right) GeV
\]

Given $\epsilon_G$ and $\epsilon_P$ as well $\lambda^G_t$, the Yukawa coupling matrices of the fermions at the GUT scale are then known. It is of interest to expand the above fermion Yukawa coupling matrices $\Gamma^G_f$ in terms of the parameter $\lambda = 0.22$ (the Cabbibo angle), as Wolfenstein [34] did for the CKM mixing matrix.
\[
\Gamma_u^G = 1.25\zeta_t \begin{pmatrix}
0 & 0.60\lambda^6 & 0 \\
1.35\lambda^6 & 0 & -0.89\lambda^2 \\
0 & -0.89\lambda^2 & 1
\end{pmatrix},
\]

(58)

\[
\Gamma_d^G = -\frac{1.25\zeta_t}{81} \begin{pmatrix}
0 & 1.77\lambda^4 & 0 \\
1.77\lambda^4 & 0.41\lambda^2 e^{i\frac{\pi}{2}} & -1.09\lambda^3 \\
0 & -1.09\lambda^3 & 1
\end{pmatrix},
\]

(59)

\[
\Gamma_e^G = -\frac{1.25\zeta_t}{81} \begin{pmatrix}
0 & 1.77\lambda^4 & 0 \\
1.77\lambda^4 & 1.23\lambda^2 e^{i\frac{\pi}{2}} & 1.40\lambda^2 \\
0 & 1.40\lambda^2 & 1
\end{pmatrix},
\]

(60)

\[
\Gamma_\nu^G = -\left(\frac{1.25\zeta_t}{81}\right) \left(\frac{2.581}{5^5}\right) \begin{pmatrix}
0 & 1 & 0 \\
1 & 0.86\lambda^3 e^{i\frac{\pi}{2}} & 1.472\lambda^4 \\
0 & 1.472\lambda^4 & 1.757\lambda
\end{pmatrix}
\]

(61)

Using the CKM parameters and quark masses predicted in the present model, the bag parameter \(B_K\) can be extracted from the indirect CP-violating parameter \(|\varepsilon_K| = 2.6 \times 10^{-3}\) in \(K^0-\bar{K}^0\) system via

\[
B_K = 0.90 \left(\frac{0.57}{\eta_2}\right) \left(\frac{|\varepsilon_K|}{2.6 \times 10^{-3}}\right) \left(\frac{0.138 y_t^{1.55}}{A^4(1 - \rho)\eta}\right) \left(\frac{1.41}{1 + \frac{0.246 y_t^{0.34}}{A^2(1 - \rho)}}\right)
\]

(62)

The B-meson decay constant can also be obtained from fitting the \(B^0-\bar{B}^0\) mixing

\[
f_B\sqrt{B} = 207 \left(\sqrt{\frac{0.55}{\eta_B}}\right) \left(\frac{\Delta M_{B_d}(ps^{-1})}{0.465}\right) \left(\frac{0.77 y_t^{0.76}}{A\sqrt{(1 - \rho)^2 + \eta^2}}\right) \text{MeV}
\]

(63)

with \(y_t = 175 GeV/m_t(m_t)\) and \(\eta_2\) and \(\eta_B\) being the QCD corrections [35]. Note that we did not consider the possible contributions to \(\varepsilon_K\) and \(\Delta M_{B_d}\) from box diagrams through exchanges of superparticles. To have a complete analysis, these contributions should be included in a more detailed consideration in the future. The parameter \(B_K\) was estimated ranging from 1/3 to 1 based on various approaches. Recent analysis using the lattice methods [36,37] gives \(B_K = 0.82 \pm 0.1\). There are also various calculations on the parameter \(f_{B_d}\). From the recent lattice analyses [36,38], \(f_{B_d} = (200 \pm 40)\) MeV, \(B_{B_d} = 1.0 \pm 0.2\). QCD sum

17
rule calculations \[39\] also gave a compatible result. An interesting upper bound \[40\] \( f_B \sqrt{B} < 213\text{MeV} \) for \( m_c = 1.4\text{GeV} \) and \( m_b = 4.6 \text{ GeV} \) or \( f_B \sqrt{B} < 263\text{MeV} \) for \( m_c = 1.5\text{GeV} \) and \( m_b = 5.0 \text{ GeV} \) has been obtained by relating the hadronic mixing matrix element, \( \Gamma_{12} \), to the decay rate of the bottom quark.

The direct CP-violating parameter \( \text{Re}(\varepsilon'/\varepsilon) \) in the K-system has been estimated by the standard method. The uncertainties mainly arise from the hadronic matrix elements \[41\]. We have included the next-to-leading order contributions from the chiral-loop \[42–44\] and the next-to-leading order perturbative contributions \[45,46\] to the Wilson coefficients together with a consistent analysis of the \( \Delta I = 1/2 \) rule. Experimental results on \( \text{Re}(\varepsilon'/\varepsilon) \) is inconclusive. The NA31 collaboration at CERN reported a value \( \text{Re}(\varepsilon'/\varepsilon) = (2.3 \pm 0.7) \cdot 10^{-3} \) \[47\] which clearly indicates direct CP violation, while the value given by E731 at Fermilab, \( \text{Re}(\varepsilon'/\varepsilon) = (0.74 \pm 0.59) \cdot 10^{-3} \) \[48\] is compatible with superweak theories \[49\] in which \( \varepsilon'/\varepsilon = 0 \). The average value quoted in \[28\] is \( \text{Re}(\varepsilon'/\varepsilon) = (1.5 \pm 0.8) \cdot 10^{-3} \).

For predicting physical observables, it is better to use \( J_{CP} \), the rephase-invariant CP-violating quantity, together with \( \alpha, \beta \) and \( \gamma \), the three angles of the unitarity triangle of a three-family CKM matrix

\[
\alpha = \text{arg.} \left( -\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*} \right), \quad \beta = \text{arg.} \left( -\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*} \right), \quad \gamma = \text{arg.} \left( -\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} \right)
\]

(64)

where \( \sin 2\alpha, \sin 2\beta \) and \( \sin 2\gamma \) can in principle be measured in \( B^0/\bar{B}^0 \to \pi^+\pi^- \) \[50\], \( J/\psi K_S \) \[51\] and \( B^- \to K^- D \) \[52\], respectively. \( |V_{us}| \) has been extracted with good accuracy from \( K \to \pi e\nu \) and hyperon decays \[28\]. \( |V_{cb}| \) can be determined from both exclusive and inclusive semileptonic \( B \) decays with values given by

\[
|V_{cb}| = \begin{cases} 
0.039 \pm 0.001 \text{ (exp.)} \pm 0.005 \text{ (theor.)}; & \text{measurements at } \Upsilon(4s), \\
0.042 \pm 0.002 \text{ (exp.)} \pm 0.005 \text{ (theor.)}; & \text{measurements at } Z^0
\end{cases}
\]

(65)

from inclusive semileptonic \( B \) decays \[53\] and

\[
|V_{cb}| = \begin{cases} 
0.0407 \pm 0.0027 \text{ (exp.)} \pm 0.0016 \text{ (theor.)}; & \text{[54]} \\
0.0388 \pm 0.0019 \text{ (exp.)} \pm 0.0017 \text{ (theor.)} & \text{[55]}
\end{cases}
\]

(66)
from the exclusive semileptonic B decays. The data from the exclusive channels is taken from the results by CLEO, ALEPH, ARGUS and DELPHI

\[
|V_{cb}|F(1) = \begin{cases} 
0.0351 \pm 0.0019 \pm 0.0020; & \text{[CLEO]} \\
0.0314 \pm 0.0023 \pm 0.0025; & \text{[ALEPH]} \\
0.0388 \pm 0.0043 \pm 0.0025; & \text{[ARGUS]} \\
0.0374 \pm 0.0021 \pm 0.0034; & \text{[DELPHI]} \\
0.0370 \pm 0.0025; \text{WEIGHTED AVERAGE in} [54] \\
0.0353 \pm 0.0018; \text{WEIGHTED AVERAGE in} [55]
\end{cases}
\]

with the related Isgur-Wise function \( F(1) \) taking the value \( 0.91 \pm 0.04 \)

The above values are also in good agreement with the value \( |V_{cb}| = 0.037^{+0.003}_{-0.002} \) obtained from the exclusive decay \( B \to D^* \ell \nu_\ell \) by using a dispersion relation approach [57].

Another CKM parameter \( |V_{ub}/V_{cb}| \) is extracted from a study of the semileptonic B decays near the end point region of the lepton spectrum. The present experimental measurements are compatible with

\[
\left| \frac{V_{ub}}{V_{cb}} \right| = 0.08 \pm 0.01 \text{ (exp.)} \pm 0.02 \text{ (theor.)} \quad (67)
\]

The CKM parameter \( |V_{td}/V_{ts}| \) is constrained [55] by the indirect CP-violating parameter \( |\varepsilon| \) in kaon decays and \( B^0 - \bar{B}^0 \) mixing \( x_d \). Large uncertainties of \( |V_{td}/V_{ts}| \) are caused by the bag parameter \( B_K \) and the leptonic B decay constant \( f_B \).

A detail analysis of neutrino masses and mixings will be presented in the next section. Before proceeding further, we would like to address the following points: Firstly, given \( \alpha_s(M_Z) \) and \( m_b(m_b) \), the value of tan \( \beta \) depends, as one sees from eq.(56), on the choice of the integer ‘n’ in an over all factor \( 1/3^n \), so do the masses of all the up-type quarks (see eqs. (48)-(50)). For \( n > 4 \), the value of tan \( \beta \) becomes too small, as a consequence, the resulting top quark mass will be below the present experimental lower bound, so do the masses of the up and charm quarks. In contrast, for \( 1 < n < 4 \), the values of tan \( \beta \) will become larger,
the resulting charm quark mass will be above the present upper bound and the top quark mass is very close to the present upper bound. Secondly, given $m_b(m_b)$ and integer ‘n’, all other quark masses increase with $\alpha_s(M_Z)$. This is because the RG scaling factors $\eta_i$ and $R_t$ increase with $\alpha_s(M_Z)$. When $\alpha_s(M_Z)$ is larger than 0.117 and n=4, either charm quark mass or bottom quark mass will be above the present upper bound. Finally, the symmetry breaking direction of the adjoint $45 A_z$ or the Clebsch factor $x_u$ is strongly restricted by both $|V_{ub}|/|V_{cb}|$ and charm quark mass $m_c(m_c)$. From these considerations, we conclude that the best choice of n will be 4 for small tan $\beta$ and the value of $\alpha_s$ should around $\alpha_s(M_Z) \simeq 0.113$, which can be seen from table 2b.

**IV. NEUTRINO MASSES AND MIXINGS**

Neutrino masses and mixings, if they exist, are very important in astrophysics and crucial for model building. Many unification theories predict a see-saw type mass $m_{\nu_i} \sim m_{u_i}^2/M_N$ with $u_i = u, c, t$ being up-type quarks. For $M_N \simeq (10^{-3} \sim 10^{-4}) M_{GUT} \simeq 10^{12} - 10^{13}$ GeV, one has

$$m_{\nu_e} < 10^{-7} eV, \quad m_{\nu_\mu} \sim 10^{-3} eV, \quad m_{\nu_\tau} \sim (3 - 21) eV$$

In this case solar neutrino anomalous could be explained by $\nu_e \rightarrow \nu_\mu$ oscillation, and the mass of $\nu_\tau$ is in the range relevant to hot dark matter. However, LSND events and atmospheric neutrino deficit can not be explained in this scenario.

By choosing Majorana type Yukawa coupling matrix differently, one can construct many models of neutrino mass matrix. As we have shown in the Model I that by choosing an appropriate texture structure with some diagonal zero elements in the right-handed Majorana mass matrix, one can explain the recent LSND events, atmospheric neutrino deficit and hot dark matter, however, the solar neutrino anomalous can only be explained by introducing a sterile neutrino. A similar consideration can be applied to the present model. The following texture structure with zeros is found to be interesting for the present model.
The corresponding effective operators are given by

\[
M_N^G = M_R \begin{pmatrix}
0 & 0 & \frac{1}{2} z_N \epsilon_P^2 e^{i(\delta_\nu + \phi_3)} \\
0 & y_N e^{2i\phi_2} & 0 \\
\frac{1}{2} z_N \epsilon_P^2 e^{i(\delta_\nu + \phi_3)} & 0 & w_N \epsilon_P^4 e^{2i\phi_3}
\end{pmatrix}
\]

(69)

The light neutrino mass matrix is then given via see-saw mechanism as follows

\[
M_\nu = \Gamma_\nu^G (M_N^G)^{-1} (\Gamma_\nu^G) \cdot \frac{v^2}{2 R_t^{-6} \eta_N^2}
\]

where \( \delta_\nu, \phi_2 \) and \( \phi_3 \) are three phases. For convenience, we first redefine the phases of the three right-handed neutrinos \( \nu_{R1} \rightarrow e^{i\delta_\nu} \nu_{R1}, \nu_{R2} \rightarrow e^{i\phi_2} \nu_{R2}, \) and \( \nu_{R3} \rightarrow e^{i\phi_3} \nu_{R3} \), so that the matrix \( M_N^G \) becomes real.

The light neutrino mass matrix is then given via see-saw mechanism as follows

\[
M_\nu = \Gamma_\nu^G (M_N^G)^{-1} (\Gamma_\nu^G) \cdot \frac{v^2}{2 R_t^{-6} \eta_N^2}
\]

where \( \delta_\nu, \phi_2 \) and \( \phi_3 \) are three phases. For convenience, \( \phi_2 \) and \( \phi_3 \) are three phases. For convenience, we first redefine the phases of the three right-handed neutrinos \( \nu_{R1} \rightarrow e^{i\delta_\nu} \nu_{R1}, \nu_{R2} \rightarrow e^{i\phi_2} \nu_{R2}, \) and \( \nu_{R3} \rightarrow e^{i\phi_3} \nu_{R3} \), so that the matrix \( M_N^G \) becomes real.

\[
M_\nu = \Gamma_\nu^G (M_N^G)^{-1} (\Gamma_\nu^G) \cdot \frac{v^2}{2 R_t^{-6} \eta_N^2}
\]

(70)

\[
M_0 = \left( \frac{2}{15^5} \right)^2 \left( \frac{15}{\epsilon_P^2} \right) \left( -\frac{w v z}{y N N} \right) \left( \frac{v^2}{2 v_5} \right) \left( \frac{R_t^{-6} \eta_N^2 \lambda_H}{\zeta_t} \right) e V
\]

(72)
It is seen that only one phase, $\delta_{\nu}$, is physical. We shall assume again maximum CP violation with $\delta_{\nu} = \pi/2$. Neglecting the small terms of order above $O(\lambda^7)$, the neutrino mass matrix can be simply diagonalized by

$$V_{\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{\nu} & -s_{\nu} \\ 0 & s_{\nu} & c_{\nu} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta_{\nu}} \end{pmatrix}$$  \hspace{1cm} (73)$$

and the charged lepton mass matrix by

$$V_e = \begin{pmatrix} \bar{c}_1 & -\bar{s}_1 & 0 \\ \bar{s}_1 & \bar{c}_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} i & 0 & 0 \\ 0 & c_e & -s_e \\ 0 & s_e & c_e \end{pmatrix}$$  \hspace{1cm} (74)$$

The CKM-type lepton mixing matrix is then given by

$$V_{\text{LEP}} = V_{\nu} V_e^\dagger = \begin{pmatrix} V_{\nu e} & V_{\nu \mu} & V_{\nu \tau} \\ V_{\mu e} & V_{\mu \mu} & V_{\mu \tau} \\ V_{\tau e} & V_{\tau \mu} & V_{\tau \tau} \end{pmatrix}$$

$$= \begin{pmatrix} \bar{c}_1 & -\bar{s}_1 & 0 \\ \bar{s}_1(c_{\nu}c_{e} + s_{\nu}s_{e}e^{i\delta_{\nu}}) & \bar{c}_1(c_{\nu}c_{e} + s_{\nu}s_{e}e^{i\delta_{\nu}}) - (s_{\nu}c_{e} - c_{\nu}s_{e}e^{i\delta_{\nu}}) \\ -\bar{s}_1(s_{\nu}c_{e} - c_{\nu}s_{e}e^{i\delta_{\nu}}) & \bar{c}_1(s_{\nu}c_{e} - c_{\nu}s_{e}e^{i\delta_{\nu}}) - (c_{\nu}c_{e} + s_{\nu}s_{e}e^{i\delta_{\nu}}) \end{pmatrix}$$  \hspace{1cm} (75)$$

where the angles are found to be

$$\tan \bar{\theta}_1 = \sqrt{\frac{m_e}{m_{\mu}}} = 0.0695$$  \hspace{1cm} (76)$$

$$\tan \theta_e = -\frac{x_e}{2w_e} \epsilon_G = -\frac{m_{\mu}}{m_{\tau}} \frac{x_e}{6y_e} = 0.0149$$  \hspace{1cm} (77)$$

$$\tan \theta_{\nu} = 1$$  \hspace{1cm} (78)$$

It is of interest to note that these predictions are solely determined without involving any new parameters.

For masses of light Majorana neutrinos we have
\[ m_{\nu_e} = \frac{1}{4} \frac{z_\nu z_N}{w_\nu} M_0 = 1.27 \times 10^{-3} \text{ eV}, \quad (79) \]
\[ m_{\nu_\mu} = \left( 1 + \frac{15}{2} \frac{z_\nu w_N}{w_\nu z_N} \epsilon_p^4 \right) M_0 \simeq 2.449 \text{ eV} \quad (80) \]
\[ m_{\nu_\tau} = \left( 1 - \frac{15}{2} \frac{z_\nu w_N}{w_\nu z_N} \epsilon_p^4 \right) M_0 \simeq 2.452 \text{ eV} \quad (81) \]

The three heavy Majorana neutrinos have masses

\[ M_{N_1} \simeq M_{N_3} \simeq \frac{1}{2} y_N z_N \epsilon_p^7 v_5 \lambda_H \simeq 333 \left( \frac{v_5}{2.36 \times 10^{16} \text{ GeV}} \right) \text{ GeV} \quad (82) \]
\[ M_{N_2} = y_N \epsilon_p^5 v_5 \lambda_H = 1.63 \times 10^6 \left( \frac{v_5}{2.36 \times 10^{16} \text{ GeV}} \right) \text{ GeV} \quad (83) \]

The RG effects above the GUT scale may be absorbed into the mass \( M_0 \). The three heavy Majorana neutrinos in the present model have their masses much below the GUT scale, unlike many other GUT models with corresponding masses near the GUT scale. In fact, two of them have masses in the range comparable with the electroweak scale.

As the masses of the three light neutrinos are very small, a direct measurement for their masses would be too difficult. An efficient detection on light neutrino masses can be achieved through their oscillations. The probability that an initial \( \nu_\alpha \) of energy \( E \) (in unit MeV) gets converted to a \( \nu_\beta \) after travelling a distance \( L \) (in unit \( m \)) is

\[ P_{\nu_\alpha \nu_\beta} = \delta_{\alpha\beta} - 4 \sum_{j>i} V_{\alpha i} V_{\beta i} V_{\beta j} V_{\alpha j}^* \sin^2 \left( \frac{1.27 L \Delta m_{ij}^2}{E} \right) \quad (84) \]

with \( \Delta m_{ij}^2 = m_j^2 - m_i^2 \) (in unit \( eV^2 \)). From the above results, we observe the following

1. a \( \nu_\mu (\bar{\nu}_\mu) \rightarrow \nu_e (\bar{\nu}_e) \) short wave-length oscillation with

\[ \Delta m_{e\mu}^2 = m_{\nu_e}^2 - m_{\nu_\mu}^2 \simeq 6 \text{ eV}^2, \quad \sin^2 2\theta_{e\mu} \simeq 1.0 \times 10^{-2}, \quad (85) \]

which is consistent with the LSND experiment \[\text{[60]}\]

\[ \Delta m_{e\mu}^2 = m_{\nu_e}^2 - m_{\nu_\mu}^2 \simeq (4 - 6) eV^2, \quad \sin^2 2\theta_{e\mu} \simeq 1.8 \times 10^{-2} \sim 3 \times 10^{-3}; \quad (86) \]

2. a \( \nu_\mu (\bar{\nu}_\mu) \rightarrow \nu_\tau (\bar{\nu}_\tau) \) long-wave length oscillation with

\[ \Delta m_{\mu\tau}^2 = m_{\nu_\tau}^2 - m_{\nu_\mu}^2 \simeq 1.5 \times 10^{-2} eV^2, \quad \sin^2 2\theta_{\mu\tau} \simeq 0.987, \quad (87) \]
which could explain the atmospheric neutrino deficit \[61\]:

$$\Delta m_{\mu\tau}^2 = m_{\nu_\tau}^2 - m_{\nu_\mu}^2 \simeq (0.5 - 2.4) \times 10^{-2} eV^2 , \quad \sin^2 2\theta_{\mu\tau} \simeq 0.6 - 1.0 , \quad (88)$$

with the best fit \[61\]

$$\Delta m_{\mu\tau}^2 = m_{\nu_\tau}^2 - m_{\nu_\mu}^2 \simeq 1.6 \times 10^{-2} eV^2 , \quad \sin^2 2\theta_{\mu\tau} \simeq 1.0 ; \quad (89)$$

3. Two massive neutrinos $\nu_\mu$ and $\nu_\tau$ with

$$m_{\nu_\mu} \simeq m_{\nu_\tau} \simeq 2.45 \, eV , \quad (90)$$

fall in the range required by possible hot dark matter \[62\].

4. ($\nu_\mu - \nu_\tau$) oscillation will be beyond the reach of CHORUS/NOMAD and E803. However, ($\nu_e - \nu_\tau$) oscillation may become interesting as a short wavelength oscillation with

$$\Delta m_{e\tau}^2 = m_{\nu_\tau}^2 - m_{\nu_e}^2 \simeq 6 \, eV^2 , \quad \sin^2 2\theta_{e\tau} \simeq 1.0 \times 10^{-2} , \quad (91)$$

which should provide an independent test on the pattern of the present Majorana neutrino mass matrix.

5. Majorana neutrino allows neutrinoless double beta decay ($\beta\beta_0\nu$) \[63\]. Its decay amplitude is known to depend on the masses of Majorana neutrinos $m_{\nu_i}$ and the lepton mixing matrix elements $V_{ei}$. The present model is compatible with the present experimental upper bound on neutrinoless double beta decay

$$\bar{m}_{\nu_e} = \sum_{i=1}^{3} [V_{ei}^2 m_{\nu_i} \zeta_i] \simeq 1.18 \times 10^{-2} \, eV < \bar{m}_{\nu}^{upper} \simeq 0.7 \, eV \quad (92)$$

The decay rate is found to be

$$\Gamma_{\beta\beta} \simeq \frac{Q^5 G_F^4 \bar{m}_{\nu_e}^2 p_F^2}{60 \pi^3} \simeq 1.0 \times 10^{-61} GeV \quad (93)$$

with the two electron energy $Q \simeq 2 \, MeV$ and $p_F \simeq 50 \, MeV$.

6. In this case, solar neutrino deficit has to be explained by oscillation between $\nu_e$ and a sterile neutrino $\nu_s$ \[64,10,13\]. Since strong bounds on the number of neutrino species both
from the invisible $Z^0$-width and from primordial nucleosynthesis \[53,60\] require the additional neutrino to be sterile (singlet of SU(2) \(\times\) U(1), or singlet of SO(10) in the GUT SO(10) model). Masses and mixings of the triplet sterile neutrinos can be chosen by introducing an additional singlet scalar with VEV $v_s \simeq 336$ GeV. We find
\[
m_{\nu_s} = \lambda H v_s^2 / v_{10} \simeq 2.8 \times 10^{-3} eV
\]
\[
\sin \theta_{es} \simeq \frac{m_{\nu_L \nu_s}}{m_{\nu_s}} = \frac{v_2 \epsilon_P}{2v_s \epsilon^2 t} \simeq 3.8 \times 10^{-2}
\]
with the mixing angle consistent with the requirement necessary for primordial nucleosynthesis \[67\] given in \[65\]. The resulting parameters
\[
\Delta m^2_{es} = m^2_{\nu_s} - m^2_{\nu_e} \simeq 6.2 \times 10^{-6} eV^2, \quad \sin^2 2\theta_{es} \simeq 5.8 \times 10^{-3}
\]
are consistent with the values \[64\] obtained from fitting the experimental data:
\[
\Delta m^2_{es} = m^2_{\nu_s} - m^2_{\nu_e} \simeq (4 - 9) \times 10^{-6} eV^2, \quad \sin^2 2\theta_{es} \simeq (1.6 - 14) \times 10^{-3}
\]

This scenario can be tested by the next generation solar neutrino experiments in Sudbury Neutrino Observatory (SNO) and Super-kamiokanda (Super-K), both planning to start operation in 1996. From measuring neutral current events, one could identify $\nu_e \rightarrow \nu_s$ or $\nu_e \rightarrow \nu_\mu (\nu_\tau)$ since the sterile neutrinos have no weak gauge interactions. From measuring seasonal variation, one can further distinguish the small-angle MSW \[68\] oscillation from vacuum mixing oscillation.

V. DIHEDRAL GROUP $\Delta(48)$

For completeness, we present in this section some features of the non-Abelian discrete dihedral group $\Delta(3n^2)$, a subgroup of SU(3). The generators of the $\Delta(3n^2)$ group consist of the matrices
\[
E(0, 0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}
\]
and

\[
A_n(p, q) = \begin{pmatrix}
e^{\frac{2\pi i}{n}p} & 0 & 0 \\
0 & e^{\frac{2\pi i}{n}q} & 0 \\
0 & 0 & e^{-\frac{2\pi i}{n}(p+q)}
\end{pmatrix}
\] (98)

It is clear that there are \(n^2\) different elements \(A_n(p, q)\) since if \(p\) is fixed, \(q\) can take on \(n\) different values. There are three different types of elements \(A_n(p, q)\), \(E_n(p, q) = A_n(p, q)E(0, 0)\), \(C_n(p, q) = A_n(p, q)E^2(0, 0)\)
in the \(\Delta(3n^2)\) group, therefore the order of the \(\Delta(3n^2)\) group is \(3n^2\). The irreducible representations of the \(\Delta(3n^2)\) groups consist of i) \((n^2 - 1)/3\) triplets and three singlets when \(n/3\) is not an integer and ii) \((n^2 - 3)/3\) triplets and nine singlets when \(n/3\) is an integer.

The characters of the triplet representations can be expressed as

\[
\Delta^m_{m_1 m_2}(A_n(p, q)) = e^{\frac{2\pi i}{n}m_1p + m_2q} + e^{\frac{2\pi i}{n}(m_1q - m_2(p+q))} + e^{\frac{2\pi i}{n}(-m_1(p+q) + m_2p)}
\] (99)

\[
\Delta^m_{m_1 m_2}(E_n(p, q)) = \Delta^m_{m_1 m_2}(C_n(p, q)) = 0
\]

with \(m_1, m_2 = 0, 1, \ldots, n-1\). Note that \((-m_1 + m_2, -m_1)\) and \((-m_2, m_1 - m_2)\) are equivalent to \((m_1, m_2)\).

One will see that \(\Delta(48)\) is the smallest of the dihedral group \(\Delta(3n^2)\) with sufficient triplets for constructing interesting texture structures of the Yukawa coupling matrices.

The irreducible triplet representations of \(\Delta(48)\) consist of two complex triplets \(T_1 = (x, y, z), \bar{T}_1 = (\bar{x}, \bar{y}, \bar{z})\) and \(T_3 = (\alpha, \beta, \gamma), \bar{T}_3 = (\bar{\alpha}, \bar{\beta}, \bar{\gamma})\), one real triplet \(T_2 = \bar{T}_2 = (a, b, c)\) as well as three singlet representations. For a similar consideration as in Ref. [13] for \(\Delta(75)\), the basis of the triplet representations of \(\Delta(48)\) is chosen as

\[
T_1 \otimes T_1 |_{T_2} = \begin{bmatrix} x^2 \\ y^2 \\ z^2 \end{bmatrix}
\] (100)
\[ T_1 \otimes \bar{T}_1 |_{T_3} = \begin{bmatrix} y \bar{z} \\ z \bar{x} \\ x \bar{y} \end{bmatrix} \]  

(101)

Thus the generator \( \hat{E}(0,0) \) has the same representation matrix \( D_R(\hat{E}(0,0)) \) for all of the triplet representations \( R \):

\[
D_R(\hat{E}(0,0)) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad R = \{T_1, \bar{T}_1, T_2, T_3, \bar{T}_3\} \quad (102)
\]

The representation matrices corresponding to the generator \( \hat{A}_4(1,0) \) are given by

\[
D_1(\hat{A}_4(1,0)) = A_4(1,0) = \begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -i \end{pmatrix},
\]

\[
D_1(\bar{A}_4(1,0)) = \bar{A}_4(1,0) = \begin{pmatrix} -i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix},
\]

\[
D_2(\hat{A}_4(1,0)) = A_4(2,0) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (103)
\]

\[
D_3(\hat{A}_4(1,0)) = A_4(1,2) = \begin{pmatrix} i & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{pmatrix},
\]

\[
D_3(\bar{A}_4(1,0)) = \bar{A}_4(1,2) = \begin{pmatrix} -i & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -i \end{pmatrix},
\]

where \( D_i \) is the representation matrix for the triplet \( T_i \) and \( D_i \) for \( \bar{T}_i \). The \( A_4(p,q) \) matrices are defined in eq.(99). With the above basis and representations, one can explicitly construct
the invariant tensors.

It is seen from Table IV that each $T_i \otimes \bar{T}_i$ ($i=1,2,3$) contains all three singlet representations

$$T_i \otimes \bar{T}_i \mid_{A_1} = x \bar{x} + y \bar{y} + z \bar{z},$$

$$T_i \otimes \bar{T}_i \mid_{A_2} = x \bar{x} + \omega y \bar{y} + \omega^2 z \bar{z},$$

$$T_i \otimes \bar{T}_i \mid_{\bar{A}_2} = x \bar{x} + \omega^2 y \bar{y} + \omega z \bar{z},$$

with $\omega = e^{i2\pi/3}$. From Table V, one can obtain easily the structure of all three-triplet invariants. Following a similar consideration as in [13] for $\Delta(75)$, the three triplet invariant $(ABC)$ can be specified by three numbers $\{ijk\}$ due to the property of the matrix representation under cyclic permutation in eq.(103), i.e., $(ABC) = A_i B_j C_k + c.p. = \{ijk\}$, where c.p. represent cyclic permutation of each representation’s index. As an example, $\{112\} = (ABC) = (A_1 B_1 C_2 + A_2 B_2 C_3 + A_3 B_3 C_1)$. The product of three same triplets always contains two invariants

$$(T_i T_i T_i) = \{123\} + \{213\}$$

(105)

The remaining five independent invariants with three triplets are

$$\{111\} : \{112\}; \quad \{112\} : \{132\};$$

$$\{113\} : \{132\}; \quad \{123\} : \{311\}, \{133\}.$$  

(106)

With the above structure, if one wants to find, for example, the invariant of the product $T_1 \otimes T_3 \otimes T_2$, one notes that $(T_1 T_2 T_3)$ is an invariant of the $\{113\}$ type, thus $T_1 \otimes T_3 \otimes T_2 \mid_{A_1} = x \alpha c + y \beta a + z \gamma b$. Similarly, to find the $T_3$ contained in $\bar{T}_1 \otimes \bar{T}_2$, one yields from the same $\{113\}$ that

$$\bar{T}_2 \otimes \bar{T}_1 \mid_{T_3} = \begin{bmatrix} \gamma \bar{x} \\ \bar{\alpha} \bar{y} \\ \bar{\beta} \bar{z} \end{bmatrix}$$

(107)
VI. SUPERPOTENTIAL FOR FERMION YUKAWA INTERACTIONS

From the above properties of the dihedral group $\Delta(48)$, we can now construct the model in details. All three families with $3 \times 16 = 48$ chiral fermions are unified into a triplet 16-dimensional spinor representation of $\text{SO}(10) \times \Delta(48)$. Without losing generality, one can assign the three chiral families into one triplet representation $T_1$, which may be simply denoted as $\hat{16} = (16, T_1)$. All the fermions are assumed to obtain their masses through a single $10_1$ of $\text{SO}(10)$ into which the needed two Higgs doublets are unified. It is possible to have a triplet sterile neutrino which has small mixings with the ordinary neutrinos. A singlet scalar near the electroweak scale is necessary to generate small masses for the sterile neutrinos.

The superpotentials which lead to the above texture structures (eqs. (2)-(4) and (69)) with zeros and effective operators (eqs. (6) and (70)) are found to be

\[
W_Y = \sum_{a=0}^{4} \psi_{a1} 10_1 \psi_{a2} + \bar{\psi}_{11} \chi_1 \psi_{n+1} + \bar{\psi}_{12} \chi_2 \psi_{n+1} + \bar{\psi}_{21} \chi_2 \psi_{13} + \bar{\psi}_{22} \chi \psi_{13} \\
+ \bar{\psi}_{13} A_z \psi_{n+1} + \bar{\psi}_{31} \chi_3 \psi_{23} + \bar{\psi}_{32} \chi_2 \psi_{23} + \bar{\psi}_{23} A_u \psi_{n+1} + \bar{\psi}_{01} \chi_0 \psi_{33} \\
+ \bar{\psi}_{02} \chi_3 \psi_{33} + \bar{\psi}_{33} S_G \psi_{n-3} + \bar{\psi}_{41} \chi_0 \psi_{03} + \bar{\psi}_{42} \chi_3 \psi_{43} + \bar{\psi}_{43} S_I \psi_{23} \\
+ \bar{\psi}_{03} S_I \psi_{13} + \bar{\psi}_{0} S_G \hat{16} + \sum_{j=1}^{n+1} \bar{\psi}_{j} S_I \psi_{j-1} + \sum_{a=0}^{4} \sum_{i=0}^{2} S_G \bar{\psi}_{ai} \psi_{ai} \\
+ \sum_{i=1}^{2} \bar{\psi}_{i} A_X \psi_{i3} + \sum_{j=1}^{n+1} \bar{\psi}_{j} A_X \psi_{j} + S_P(\bar{\psi}_{0} \psi_{0} + \sum_{i=0,3,4} \bar{\psi}_{i3} \psi_{i3})
\] (108)

for the fermion Yukawa coupling matrices,

\[
W_R = \sum_{i=1}^{3} (\bar{\psi}_{i1}^T \Phi N_i + \bar{\tilde{N}}_i^T \tilde{\Phi}^T \psi_{i2} + \bar{\tilde{\psi}}_{i1}^T \chi_i \psi_{i3}) + \sum_{i=1}^{2} \bar{\psi}_{i2} \chi_i \psi_{i2} + \bar{\psi}_{32} \chi \psi_{33} \\
+ \bar{\psi}_{13} A_z \psi_{2} + \bar{\psi}_{23} S_G \psi_{33} + \bar{\psi}_{33} A_z \psi_{0} + \bar{\psi}_{3} S_G \psi_{2} + \bar{\psi}_{4} S_G \psi_{1} \\
+ \bar{\psi}_{2} A_u \psi_{1} + \bar{\tilde{\psi}}_{1} A_u \psi_{0} + \bar{\psi}_{0} S_G \hat{16} + \sum_{i=1}^{2} \bar{\psi}_{3i} A_X \psi_{3i} \\
+ \sum_{i=1}^{2} \sum_{j=1}^{2} S_I \bar{\psi}_{ij} \psi_{ij} + S_P(3 \bar{\tilde{N}}_i^T N_i + \sum_{i=1}^{3} \bar{\tilde{\psi}}_{i3} \psi_{i3} + \sum_{a=0}^{4} \bar{\tilde{\psi}}_{a} \psi_{a})
\] (109)

for the right-handed Majorana neutrinos, and
\[ W_S = \psi_1'' \psi_2' + \bar{\psi}_1'' \Phi \nu_s + \bar{\psi}_2'' \phi_s + (\bar{\nu}_s \phi_s N_s + h.c.) \]
\[ + S_I \bar{N}_s N_s + S_G \bar{\psi}_2'' \psi_2' + S_P \bar{\psi}_1'' \psi_1' \]

(110)

for the sterile neutrino masses and their mixings with the ordinary neutrinos.

In the above superpotentials, all \( \psi \) fields are triplet 16-dimensional spinor heavy fermions. Where the fields \( \psi_{a3} \{ \bar{\psi}_{a3} \} \), \( (a = 0, 1, 2, 3, 4) \), \( \psi_{i3} \{ \bar{\psi}_{i3} \} \), \( (i = 1, 2, 3) \), \( \psi_i \{ \bar{\psi}_i \} \) \( (i = 0, 1, \cdots n+1) \), \( \psi'_a \{ \bar{\psi}'_a \} \), \( (a = 0, 1, 2, 3, 4) \), \( \psi''_a \{ \bar{\psi}''_a \} \), and \( \bar{\psi}'_{i2} \{ \psi'_i \} \) \( (i = 0, 2, 3, 4) \) belong to \((16, T_1)\{(16, T_1)\})\) representations of \( SO(10) \times \Delta(48) \); \( \psi_{11} \{ \bar{\psi}_{11} \} \) and \( \psi_{12} \{ \bar{\psi}_{12} \} \) belong to \((16, T_2)\{(16, T_2)\})\); \( \psi_{i1} \{ \bar{\psi}_{i1} \} \) and \( \bar{\psi}_{i1} \{ \psi'_i \} \) \( (i = 1, 2) \) belong to \((16, T_3)\{(16, T_3)\})\); \( \psi_{31} \{ \bar{\psi}_{31} \} \) and \( \psi_{32} \{ \bar{\psi}_{32} \} \) belong to \((16, T_2)\{(16, T_2)\})\); \( \bar{N}_i \{ N_i \} \) \( (i = 1, 2) \) belong to \((1, \bar{T}_3)\{(1, T_3)\})\); \( \bar{N}_3 \{ N_3 \} \) belong to \((1, T_2)\{(1, T_2)\})\); \( \Phi \) belong to \((16, 1)\); \( S_G \), \( S_I \), \( S_P \) and \( \phi_s \) are singlet scalars of \( SO(10) \times \Delta(48) \). \( \nu_s \) and \( N_s \) are \( SO(10) \) singlet and \( \Delta(48) \) triplet fermions. The \( SO(10) \) singlets \( N_i \) and \( \bar{N}_i \) \((i = 1, 2, 3)\) are \( \Delta(48) \) triplets heavy neutral fermions above the GUT scale. They are introduced to generate the right-handed Majorana neutrino masses and mixings in the \( SO(10) \) grand unified models if the \( 126 \)-dimensional representation Higgs fields do not allow to exist in a fundamental theory. Recently, it was shown in ref. [70] that for fermionic compactification schemes the \( 126 \)-dimensional representations appear unlikely to emerge from the compactification of heterotic string models. All \( SO(10) \) singlet \( \chi \) fields are triplets of \( \Delta(48) \). Where \( (\chi_1, \chi_2, \chi_3, \chi_0, \chi) \) belong to triplet representations \((\bar{T}_3, T_3, \bar{T}_1, T_1, \bar{T}_3)\) respectively; \( (\chi'_1, \chi'_2, \chi'_3, \chi') \) belong to triplet representations \((\bar{T}_1, T_2, T_3, \bar{T}_3)\) respectively. With the above assignment for various fields, one can check that once the triplet field \( \chi \) develops VEV only along the third direction, i.e., \( < \chi^{(3)}> \neq 0 \), and \( \chi' \) develops VEV only along the second direction, i.e., \( < \chi'^{(2)}>< \neq 0 \), the resulting fermion Yukawa coupling matrices at the GUT scale will automatically have, due to the special features of \( \Delta(48) \), the interesting texture structure with four non-zero textures ‘33’, ‘32’, ‘22’ and ‘12’ characterized by \( \chi_1 \), \( \chi_2 \), \( \chi_3 \), and \( \chi_0 \) respectively, and the resulting right-handed Majorana neutrino mass matrix has three non-zero textures ‘33’, ‘13’ and ‘22’ characterized by \( \chi'_1 \), \( \chi'_2 \), and \( \chi'_3 \) respectively. It is seen that five triplets are needed. Where one triplet

30
is necessary for unification of the three family fermions, and four triplets are required for obtaining the needed minimal non-zero textures. In figures 1 and 2, we have illustrated the non-zero textures needed for the Dirac fermion Yukawa coupling matrices and Majorana mass matrix, respectively. To obtain the realistic fermion Yukawa coupling matrices and Majorana mass matrix, one uses the Froggatt-Nielsen mechanism [71] to understand the small mass ratios and applies an effective operator analysis to yield the appropriate Clebsch-Gordan coefficients.

Before proceeding, we would like to address the following points: Firstly, in the above superpotentials, each term is ensured by the U(1) symmetry. An appropriate assignment of U(1) charges for the various fields is implied. As explained in the introduction, the U(1) symmetry in this model is family-independent and introduced to distinguish various fields which belong to the same representations of \( \text{SO}(10) \times \Delta(48) \) so that it is manifest to make the U(1) charge assignment for the various fields. Let the spinor fields \( \psi_I \) have the U(1) charges \( \alpha_I \) and \( \tilde{\psi}_I \) have the U(1) charges \( \bar{\alpha}_I \), the scalars (45’s, singlets, triplets) have the U(1) charges \( \beta_J \), \( (I,J,=1,2, \cdots) \), i.e., all the fields are assigned to have different U(1) charges. What we need to do is to keep the U(1) charge conservation and ensure the uniqueness for each interaction term in the superpotentials. For instance, for an interaction term \( \bar{\psi}_J A_K \psi_I \), the U(1) charge conservation requires \( \alpha_I + \bar{\alpha}_J + \beta_K = 0 \). The uniqueness of each interaction term can be arrived by using the following procedure for the construction of the superpotentials: firstly, writing down a needed interaction term by assigning appropriate U(1) charges to the relevant fields, one then figures out all the forbidden U(1) charges for assigning new fields in order to ensure the uniqueness of the given interaction term. The next step is to construct a new interaction term by assigning the allowed U(1) charges beyond the forbidden U(1) charges to the relevant new fields and to figure out the additional forbidden U(1) charges for keeping the uniqueness of the existing interaction terms. Applying this rule step by step to all the interaction terms, one finally completes the assignment of the U(1) charges for all the fields in the superpotentials. In such a way, all the interaction terms and the resulting texture patterns will be uniquely determined. Thus, the high order correction terms that
contribute to the given textures at the tree level are absent. Due to the special properties of the \( \Delta(48) \), one can easily make the U(1) charge assignment to ensure the needed zero texture structure. This is because one only needs to keep the uniqueness of the interaction terms which concern the heavy fermion pairs \( \psi_{ai}(\bar{\psi}_{ai}) \) with \( a = 0, 1, 2, 3, 4; i = 1, 2 \). This may distinguish from other models in which the U(1) symmetries are family dependent. As the present model concerns so many fields, a careful U(1) charge assignment in applying for the above rule is still needed. Note also that the absolute values of the U(1) charges for the various fields cannot be uniquely fixed by only the condition of the U(1) charge conservation. Here, we have presented a general assignment of the U(1) charges for constructing unique superpotentials. An explicit assignment of U(1) charges for various fields in the above specific superpotentials \( W_Y, W_R \) and \( W_S \) will be presented below. It is of interest to see that one only needs to appropriately assign the U(1) charges of the 10-representation 10\(_1\) that is necessary for obtaining the masses of the quarks and leptons, and the U(1) charges of the \( \Delta(48) \) triplets \( \chi_k \) (\( k = 0, 1, 2, 3 \)), \( \chi, \chi'_i \) (\( i = 1, 2, 3 \)) and \( \chi' \) that characterize the structures of the mass matrices with zero textures, as well as the U(1) charge of the singlet scalar \( \phi_s \) that is introduced to obtain the sterile neutrino mass and mixing, then the U(1) charges of the other fields will be completely determined by the U(1) charge conservations.

Secondly, unlike many other models in which the third family Yukawa coupling is assumed to be renormalizable, the third family Yukawa coupling in the present model, similar to the first and the second families, is an effective one\(^5\) generated at the GUT scale so that the three families are treated on the same footing. As the top quark is heavy, the third family Yukawa coupling must be large. This requires a large mixing among the super heavy fermions and the third family. Thus the effective Yukawa coupling of the third family shall be obtained by diagonalizing the mixing mass matrices. The perturbation expansion is no longer valid

\(^5\)In ref. \cite{9}, the third family Yukawa coupling was also considered to be an effective one, but it is considered to be generated above the GUT scale
for yielding the third family Yukawa coupling. For illustration, let us first consider a toy model with the following superpotential

\[ W_{\text{TOY}} = \lambda H (1_{10} \psi_2 + \bar{\psi}_1 S_1 16 + \bar{\psi}_2 S_2 16 + \bar{\psi}_1 A_x \psi_1 + \bar{\psi}_2 A_x \psi_2) \]

The U(1) charges of the various fields \((\psi_1, \psi_2, \bar{\psi}_1, \bar{\psi}_2, 16, 10_1, S_1, S_2, A_x)\) are corresponding to \((1/2, 3/2, -3/2, -5/2, -1/2, -2, 2, 3, 1)\).

After symmetry breaking, i.e., the fields \(S_1, S_2\) and \(A_x\) get the VEVs \(< S_1 >, < S_2 >\) and \(< A_x >\), the mixing mass matrix \(M\) defined from \(\bar{\Psi} M \Psi\) with \(\bar{\Psi} = (\bar{16}, \bar{\psi}_1, \bar{\psi}_2)\) (here \(16\) is introduced as an auxiliary field for convenience of discussions) and \(\Psi = (16, \psi_1, \psi_2)\), is easily found to be (for simplicity, we will omit the bracket \(< >\) for the VEVs)

\[
M = \begin{pmatrix}
0 & 0 & 0 \\
S_1 & A_x & 0 \\
S_2 & 0 & A_x
\end{pmatrix}; \quad M^\dagger M = \begin{pmatrix}
S_1^2 + S_2^2 & S_1 A_x & S_2 A_x \\
S_1 A_x & A_x^2 & 0 \\
S_2 A_x & 0 & A_x^2
\end{pmatrix}
\]

Diagonalizing the hermitian mass matrix square \(M^\dagger M\) (note that \(16\) remains massless), one easily obtains the mixing angles among the heavy fermions \(\psi_1, \psi_2\) and the massless fermion \(16\). Explicitly, one has,

\[
\psi_1 = \frac{\left(\frac{S_1}{A_x}\right)}{\sqrt{1 + \left(\frac{S_1}{A_x}\right)^2 + \left(\frac{S_2}{A_x}\right)^2}} 16 + \cdots
\]

\[
\psi_2 = \frac{\left(\frac{S_2}{A_x}\right)}{\sqrt{1 + \left(\frac{S_1}{A_x}\right)^2 + \left(\frac{S_2}{A_x}\right)^2}} 16 + \cdots
\]

Thus the effective operator for the Yukawa interaction is given by

\[
O = \lambda_H 16 \frac{\left(\frac{S_1}{A_x}\right)}{\sqrt{1 + \left(\frac{S_1}{A_x}\right)^2 + \left(\frac{S_2}{A_x}\right)^2}} 10_1 \frac{\left(\frac{S_2}{A_x}\right)}{\sqrt{1 + \left(\frac{S_1}{A_x}\right)^2 + \left(\frac{S_2}{A_x}\right)^2}} 16
\]

when the ratios \((S_1/A_x)^2 \ll 1\) and \((S_2/A_x)^2 \ll 1\), one can apply the perturbation expansion to the above effective operator. In a good approximation, one has

\[
O \simeq \lambda_H 16 \left(\frac{S_1}{A_x}\right) 10_1 \left(\frac{S_2}{A_x}\right) 16
\]
If $S_1/A_x = S_2/A_x = 1$, the mixing becomes maximal, one then obtains

$$O = \lambda^G \mathbf{1610_116}$$

with the effective Yukawa coupling at the GUT scale $\lambda^G = \lambda_H/3$, here the factor $1/3 = (1/\sqrt{3})^2$ is due to the maximal mixing.

A similar analysis has been used to obtain the realistic Yukawa coupling matrices (eqs. (2-4)), where one needs to diagonalize a high rank mass matrix. Note that in writing down eqs. (2-4), the small terms arising from the mixings have been neglected. We shall not explicitly present the calculations here for all the operators in eqs. (6) as it is straightforward but tedious. For illustration, we will explicitly show how to obtain the operator $W_{33}$ below.

Finally, the initial conditions of the renormalization group evaluation for the Yukawa coupling matrices of all the quarks and leptons must be set at the GUT scale since all the Yukawa couplings of the quarks and leptons in this model are generated at the GUT scale. Above the GUT scale, the three family quarks and leptons which belong to the three $\mathbf{16}$'s all couple to the heavy fermions. Note that all the resulting Yukawa couplings of the quarks and leptons in this model only rely on the ratios of the coupling constants appearing in the renormalizable superpotential at the GUT scale. Though the coupling constants of the superpotential at the GUT scale may become different due to the renormalization group effect running from the scale $\bar{M}_P$ to the GUT scale, the relative values of their ratios for the textures can remain the same when the textures are generated from a similar superpotential structure and concern the fields which belong to the same representations of the symmetry group. This is because the renormalization group evaluation does not change the representations of the symmetry group. This is the case for the ‘22’ and ‘32’ textures in this model. Therefore our predictions for the quark and charged lepton sector will not be affected by the renormalization group effects when running from the Planck scale to the GUT scale. It is very interesting to note that this may also be arrived by using a permutation symmetry among the fields concerning the ‘22’ and ‘32’ textures instead of assuming the universality for all the terms in the superpotentials. Explicitly, the permutation symmetry operates on
the fields in the superpotential $W_Y$ in eq.(108) as $\psi_{2i} \leftrightarrow \psi_{3i}$ ($i = 1, 2, 3$) and $\chi_2 \leftrightarrow \chi_3$.

Let us now demonstrate in details how to assign the U(1) charges for various fields in the superpotentials $W_Y$, $W_R$ and $W_S$. Let $(\alpha, \alpha_j, \bar{\alpha}_j, \alpha_{ai}, \bar{\alpha}_{ai})$ and $(\beta_{10}, \beta, \beta_k, \beta_X, \beta_G, \beta_z, \beta_a, \beta_I, \beta_P)$ with $j = 0, 1, \ldots, n + 1$, $a = 0, 1, 2, 3, 4$, $i = 1, 2, 3$, $k = 0, 1, 2, 3$ be the U(1) charges of the corresponding fields (16, $\psi_j, \bar{\psi}_j, \psi_{ai}, \bar{\psi}_{ai}$) and (101, $\chi, \chi_k, A_X, S_G, A_2, A_u, S_I, S_P$) which appear in the superpotential $W_Y$ (some of which also appear in the superpotentials $W_R$ and $W_S$), and let $(\alpha'_a, \bar{\alpha}'_a, \alpha'_{ij}, \bar{\alpha}'_{ij}, \gamma_i, \bar{\gamma}_i)$ and $(\beta'_i, \beta'_j, \beta_\phi)$ with $(a = 0, 1, 2, 3, 4)$, $i, j = 1, 2, 3$ be the U(1) charges of the corresponding fields $(\psi'_i, \bar{\psi}'_{a_i}, \psi'_{ij}, \bar{\psi}'_{ij}, N_i, \bar{N}_i)$ and $(\chi', \chi'_i, \bar{\Phi})$ appearing in the superpotential $W_R$, as well as let $(\alpha''_i, \bar{\alpha}'_{ii}, \alpha_{ij}, \bar{\alpha}_{ij}, \alpha_{N}, \bar{\alpha}_N)$ and $\beta_\phi$ with $i = 1, 2$ be the U(1) charges of the corresponding fields $(\psi''_i, \bar{\psi}''_{i}, \nu_s, \bar{\nu}_s, N_s, \bar{N}_s)$ and $\phi_s$ appearing in the superpotential $W_S$. We will show that once the U(1) charges $(\beta_{10}, \beta, \beta_k, \beta'_i, \beta'_j, \beta_\phi)$ corresponding to the fields (101, $\chi, \chi_k, \chi', \chi'_i, \phi_s$) are appropriately chosen, then the U(1) charges of the other fields in the superpotentials $W_Y$, $W_R$ and $W_S$ will be completely determined by the U(1) charge conservations.

Firstly, we will see that by using the U(1) charge conservations the U(1) charges of various fields appearing in the superpotential $W_Y$ can be expressed in terms of the U(1) charges $(\beta_{10}, \beta, \beta_k, \beta_G, \beta_{10}, \beta_X)$ corresponding to the fields (101, $\chi, \chi_k, S_G, A_X$). One can always set $\beta_{10} = 0$, then the U(1) charge conservation of the terms $\sum_{a=0}^4 \psi_{1a}101\psi_{a2}$ leads to,

$$\alpha_{a1} = -\alpha_{a2}, \quad a = 0, 1, 2, 3, 4$$

(111)

The charge conservation of the terms $\sum_{a=0}^4 \sum_{i=1}^2 S_G(\bar{\psi}_{ai}\psi_{ai})$ leads to

$$\bar{\alpha}_{a1} + \alpha_{a1} + \beta_G = 0, \quad \bar{\alpha}_{a1} + \alpha_{a1} + \beta_G = 0$$

(112)

From the charge conservation of the terms $\bar{\psi}_{11}\chi_{1}\psi_{n+1}$, $\bar{\psi}_{12}\chi_{1}\psi_{n+1}$, $\bar{\psi}_{a1}\chi_{a}\psi_{(a+1)3}$ and $\bar{\psi}_{a2}\chi_{a}\psi_{(a+1)3}$ with $a = 0, 2, 3$, we have

$$\bar{\alpha}_{11} + \beta_1 + \alpha_{a+1} = 0, \quad \bar{\alpha}_{12} + \beta + \alpha_{a+1} = 0, \quad \bar{\alpha}_{a1} + \beta_a + \alpha_{(a+1)3} = 0, \quad \bar{\alpha}_{a2} + \beta + \alpha_{(a+1)3} = 0$$

(113)

(114)
Combining the above equations, we obtain

\[ \alpha_{a1} = -\alpha_{a2} = \frac{1}{2}(\beta_a - \beta), \quad a = 0, 1, 2, 3, \]  
(115)

\[ \bar{\alpha}_{a1} = -\frac{1}{2}(\beta_a - \beta) - \beta_G, \quad \bar{\alpha}_{a2} = \frac{1}{2}(\beta_a - \beta) - \beta_G, \]  
(116)

\[ \alpha_{n+1} = -\frac{1}{2}(\beta_1 + \beta) + \beta_G, \quad \alpha_{13} = -\frac{1}{2}(\beta_2 + \beta) + \beta_G, \]  
\[ \alpha_{23} = -\frac{1}{2}(\beta_3 + \beta) + \beta_G, \quad \alpha_{33} = -\frac{1}{2}(\beta_0 + \beta) + \beta_G \]  
(117)

From \( \sum_{j=1}^{n+1} \bar{\psi}_j A_X \psi_j \) and \( \sum_{j=1}^{n+1} \bar{\psi}_j S_I \psi_{j-1} \), we read off

\[ \bar{\alpha}_j + \alpha_{j-1} + \beta_I = 0, \quad \bar{\alpha}_j + \alpha_j + \beta_X = 0, \quad (j \neq 0) \]  
(118)

Substituting the above result for \( \alpha_{n+1} \), we yield

\[ \alpha_j = -\frac{1}{2}(\beta_1 + \beta) - (n - j + 1)(\beta_I - \beta_X) + \beta_G, \quad j = 0, \cdots n, \]  
(119)

\[ \bar{\alpha}_j = \frac{1}{2}(\beta_1 + \beta) + (n - j + 1)(\beta_I - \beta_X) - \beta_G, \quad j = 1, \cdots n, \]  
(120)

From the terms \( S_P \bar{\psi}_0 \psi_0 \) and \( \bar{\psi}_0 S_G \hat{16} \), we have

\[ \bar{\alpha}_0 = -\alpha_0 - \beta_P = \frac{1}{2}(\beta_1 + \beta) + (n + 1)(\beta_I - \beta_X) - \beta_G - \beta_P, \]  
(121)

\[ \alpha = -\bar{\alpha}_0 - \beta_G = -\frac{1}{2}(\beta_1 + \beta) - (n + 1)(\beta_I - \beta_X) + \beta_P \]  
(122)

The terms \( \sum_{a=1}^{2} \bar{\psi}_{a3} A_X \psi_{a3} \) and \( S_P \bar{\psi}_{33} \psi_{33} \) determine the U(1) charges

\[ \bar{\alpha}_{a3} = -\alpha_{a3} - \beta_X = \frac{1}{2}(\beta_{a+1} + \beta) - \beta_G - \beta_X, \quad a = 1, 2 \]  
(123)

\[ \bar{\alpha}_{33} = -\alpha_{33} - \beta_P = \frac{1}{2}(\beta_0 + \beta) - \beta_G - \beta_P, \]  
(124)

and the terms \( \bar{\psi}_{43} S_I \psi_{23} \) and \( \bar{\psi}_{03} S_I \psi_{13} \) determine the U(1) charges

\[ \bar{\alpha}_{43} = -\alpha_{23} - \beta_I = \frac{1}{2}(\beta_3 + \beta) - \beta_G - \beta_I, \]  
(125)

\[ \bar{\alpha}_{03} = -\alpha_{03} - \beta_I = \frac{1}{2}(\beta_2 + \beta) - \beta_G - \beta_I, \]  
(126)

From \( S_P(\bar{\psi}_{43} \psi_{43} + \bar{\psi}_{03} \psi_{03}) \), we have
\[
\alpha_{43} = -\bar{\alpha}_{43} - \beta_P = -\frac{1}{2} (\beta_3 + \beta) + \beta_G + \beta_I - \beta_P,
\]
\[
\alpha_{03} = -\bar{\alpha}_{03} - \beta_P = -\frac{1}{2} (\beta_2 + \beta) + \beta_G + \beta_I - \beta_P,
\]
(127)
\[
\alpha_{03} = -\bar{\alpha}_{03} - \beta_P = -\frac{1}{2} (\beta_2 + \beta) + \beta_G + \beta_I - \beta_P,
\]
(128)

The terms \(\bar{\psi}_{41} \chi_0 \psi_{03}, \bar{\psi}_{42} \chi \psi_{43}\) and \(\sum_{i=1}^{2} S_G \bar{\psi}_{4i} \psi_{4i}\) then lead to
\[
\bar{\alpha}_{42} = -\alpha_{43} - \beta = \frac{1}{2} (\beta_3 - \beta) - \beta_G - \beta_I + \beta_P,
\]
(129)
\[
\bar{\alpha}_{41} = -\alpha_{03} - \beta_0 = \frac{1}{2} (\beta_2 + \beta) - \beta_0 - \beta_G - \beta_I + \beta_P,
\]
(130)

and
\[
\alpha_{42} = -\bar{\alpha}_{42} - \beta_G = -\frac{1}{2} (\beta_3 - \beta) + \beta_I - \beta_P,
\]
(131)
\[
\alpha_{41} = -\bar{\alpha}_{41} - \beta_G = -\frac{1}{2} (\beta_2 + \beta) + \beta_0 + \beta_I - \beta_P,
\]
(132)

The term \(\bar{\psi}_{33} S_G \psi_{n-3}\) provides a relation for \(\beta_P\) as
\[
\beta_P = \beta_G - 4(\beta_I - \beta_X) - \frac{1}{2}(\beta_1 - \beta_0)
\]
(133)

Noticing the relation \(\alpha_{41} = -\alpha_{42}\) and using the above result for \(\beta_P\), we find that \(\beta_I\) and \(\beta_P\) are given by
\[
\beta_I = \frac{1}{5} (\beta_G + 4\beta_X) + \frac{1}{10} (\beta_2 + \beta_3 - \beta_0 - \beta_1),
\]
(134)
\[
\beta_P = \frac{1}{5} (\beta_G + 4\beta_X) - \frac{2}{5} (\beta_2 + \beta_3 - \beta_0 - \beta_1) - \frac{1}{5} (\beta_1 - \beta_0)
\]
(135)

Finally, \(\beta_u\) and \(\beta_z\) are determined by the terms \(\bar{\psi}_{13} A_{2} \psi_{n+1}\) and \(\bar{\psi}_{23} A_{u} \psi_{n+1}\),
\[
\beta_u = -\bar{\alpha}_{23} - \alpha_{n+1} = \frac{1}{2} (\beta_1 - \beta_3),
\]
(136)
\[
\beta_z = -\bar{\alpha}_{13} - \alpha_{n+1} = \frac{1}{2} (\beta_1 - \beta_2)
\]
(137)

It is seen that by using the U(1) charge conservations in the superpotential \(W_Y\) the U(1) charges of the various fields in the superpotential \(W_Y\) can be simply expressed in terms of the U(1) charges \(\beta_{10}, \beta, \beta_k, \beta_X\) and \(\beta_G\) corresponding to the fields \(10_1, \chi, \chi_k\) (k=0,1,2,3), \(A_X\) and \(S_G\). We now further apply the U(1) charge conservations to the superpotential \(W_R\) and show that the U(1) charges of all new fields in addition to the fields appearing in the
superpotential $W_Y$ will be determined by the U(1) charges $\beta'_i (i=1,2,3)$ and $\beta'$. Furthermore, the U(1) charges $\beta_G$ and $A_X$ will also be fixed by the U(1) charges $\beta'_i (i=1,2,3)$ and $\beta'$. We start from the term $\bar{\psi}'_i A_u \psi_0$ in which the U(1) charges $\beta_u$ and $\alpha_0$ of the fields $A_u$ and $\psi_0$ have been given above in terms of the U(1) charges $\beta$, $\beta_i (i = 0, 1, 2, 3)$, $\beta_G$ and $\beta_X$, the U(1) charge conservation leads to

$$\bar{\alpha}'_1 = -\beta_u - \alpha_0 = -\beta_X + \frac{1}{2} (\beta_3 + \sum_{i=0}^{3} \beta_i) + \frac{n-4}{10} \left( \sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X) \right)$$

(138)

From U(1) charge conservation of the terms $S_P \bar{\psi}'_i \psi'_i (i = 1, 2, 3)$, $\bar{\psi}'_2 A_u \psi_0$, $\bar{\psi}'_3 S_G \psi'_2$, $\bar{\psi}'_{12} \chi'_3$, $S_I \bar{\psi}'_{11} (i = 1, 2)$, $N_{11} \bar{\Phi} T \psi'_1$, $S_P \bar{N}_{11} T N_1$, $\psi'_1 \bar{\Phi} N_1$, $\bar{\psi}'_{11} \chi'_1$ and $\psi'_3 A_u \psi'_2$, we have

$$\alpha'_1 = -\bar{\alpha}'_1 - \beta_P = \beta_X - \frac{1}{2} (\beta_3 + \sum_{i=0}^{3} \beta_i) - \beta_P$$

(139)

$$\alpha'_2 = -\alpha'_2 - \beta_P = \beta_X - \frac{1}{2} (2\beta_3 - \beta_1 + \sum_{i=0}^{3} \beta_i) + \beta_P$$

(140)

$$\alpha'_2 = -\bar{\alpha}'_2 - \beta_G = -\beta_X + \frac{1}{2} (2\beta_3 - \beta_1 + \sum_{i=0}^{3} \beta_i) - 2\beta_P$$

(141)

$$\alpha'_3 = -\alpha'_3 - \beta_G = -\beta_X + \frac{1}{2} (2\beta_3 - \beta_1 + \sum_{i=0}^{3} \beta_i) + 2\beta_P - \beta_G$$

(142)

$$\alpha'_3 = -\bar{\alpha}'_3 - \beta_P = \beta_X - \frac{1}{2} (2\beta_3 - \beta_1 + \sum_{i=0}^{3} \beta_i) - 3\beta_P + \beta_G$$

(143)

$$\bar{\alpha}'_{12} = -\alpha'_3 - \beta' = -\beta_X + \frac{1}{2} (2\beta_3 - \beta_1 + \sum_{i=0}^{3} \beta_i) + 3\beta_P - \beta_G$$

(144)
\[
\alpha'_{12} = -\bar{\alpha}'_{12} - \beta_I = \beta_X - \frac{1}{2}(2\beta_3 - \beta_1 + \sum_{i=0}^{3} \beta_i) - 3\beta_P + \beta_G \\
+\beta' - \beta_I - \frac{n-4}{10}(\sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X))
\]

\[
\bar{\alpha}'_1 = -\bar{\alpha}'_{12} - \beta_\Phi = -\beta_X + \frac{1}{2}(2\beta_3 - \beta_1 + \sum_{i=0}^{3} \beta_i) + 3\beta_P - \beta_G
\]

\[
-\beta' + \beta_I - \beta_\Phi + \frac{n-4}{10}(\sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X))
\]

\[
\gamma_1 = -\bar{\gamma}_1 - \beta_P = \beta_X - \frac{1}{2}(2\beta_3 - \beta_1 + \sum_{i=0}^{3} \beta_i) - 4\beta_P + \beta_G
\]

\[
+\beta' - \beta_I + \beta_\Phi - \frac{n-4}{10}(\sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X))
\]

\[
\bar{\alpha}'_{13} = -\bar{\alpha}'_{12} - \beta_z = -\beta_X + \frac{1}{2}(2\beta_3 - 2\beta_1 + \beta_2 + \sum_{i=0}^{3} \beta_i) + 2\beta_P
\]

\[
+\frac{n-4}{10}(\sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X))
\]

\[
\bar{\alpha}'_{13} = -\bar{\alpha}'_{13} - \beta_P = \beta_X - \frac{1}{2}(2\beta_3 - 2\beta_1 + \beta_2 + \sum_{i=0}^{3} \beta_i) - 3\beta_P
\]

\[
-\frac{n-4}{10}(\sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X))
\]

\[
\bar{\alpha}'_{11} = -\bar{\alpha}'_{13} - \beta_1 = -\beta_X + \frac{1}{2}(2\beta_3 - 2\beta_1 + \beta_2 + \sum_{i=0}^{3} \beta_i) + 3\beta_P
\]

\[
-\beta_1 + \frac{n-4}{10}(\sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X))
\]

\[
\bar{\alpha}'_{11} = -\bar{\alpha}'_{11} - \beta_I = \beta_X - \frac{1}{2}(2\beta_3 - 2\beta_1 + \beta_2 + \sum_{i=0}^{3} \beta_i) - 3\beta_P
\]

\[
+\beta_1' - \beta_I - \frac{n-4}{10}(\sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X))
\]

\[
\gamma_1 = -\bar{\alpha}'_{11} - \beta_\Phi = -\beta_X + \frac{1}{2}(2\beta_3 - 2\beta_1 + \beta_2 + \sum_{i=0}^{3} \beta_i) + 3\beta_P
\]

\[
-\beta_1' + \beta_I - \beta_\Phi + \frac{n-4}{10}(\sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X))
\]

The U(1) charge conservation of the terms $\bar{\psi}'_4 S_G \psi'_1$, $\bar{\psi}'_2 \chi' \psi'_4$, $\bar{\psi}'_4 \Phi \psi'_4$, $S_P \bar{N}_2 \bar{T}_2 \bar{N}_2$, $\bar{\psi}'_{23} A_2 \psi_0$, $\bar{\psi}'_{23} S_G \psi_{33}$, $\bar{\psi}'_{21} \chi' \psi'_{23}$ and $\psi'_{21} \Phi \bar{N}_2$ leads to

\[
\bar{\alpha}'_4 = -\alpha'_4 - \beta_G = -\beta_X + \frac{1}{2}(3 + \sum_{i=0}^{3} \beta_i) + \beta_P - \beta_G
\]
\[ + \frac{n - 4}{10} \left( \sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X) \right) \]

\[ \alpha'_4 = -\alpha'_4 - \beta_P = +\beta_X - \frac{1}{2}(\beta_3 + \sum_{i=0}^{3} \beta_i) - 2\beta_P + \beta_G \quad \text{(154)} \]

\[ - \frac{n - 4}{10} \left( \sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X) \right) \]

\[ \alpha'_{22} = -\alpha'_4 - \beta' = -\beta_X + \frac{1}{2}(\beta_3 + \sum_{i=0}^{3} \beta_i) + 2\beta_P - \beta_G \quad \text{(155)} \]

\[ -\beta' + \frac{n - 4}{10} \left( \sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X) \right) \]

\[ \alpha'_2 = -\alpha'_{22} - \beta_I = \beta_X - \frac{1}{2}(\beta_3 + \sum_{i=0}^{3} \beta_i) - 2\beta_P + \beta_G \quad \text{(156)} \]

\[ + \beta' - \beta_I - \frac{n - 4}{10} \left( \sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X) \right) \]

\[ \gamma_2 = -\alpha'_{22} - \beta_\Phi = -\beta_X + \frac{1}{2}(\beta_3 + \sum_{i=0}^{3} \beta_i) + 2\beta_P - \beta_G \quad \text{(157)} \]

\[ -\beta' + \beta_I - \beta_\Phi + \frac{n - 4}{10} \left( \sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X) \right) \]

\[ \gamma_2 = -\gamma_2 - \beta_P = \beta_X - \frac{1}{2}(\beta_3 + \sum_{i=0}^{3} \beta_i) - 3\beta_P + \beta_G \quad \text{(158)} \]

\[ + \beta' - \beta_I + \beta_\Phi - \frac{n - 4}{10} \left( \sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X) \right) \]

\[ \bar{\alpha}'_{33} = -\beta_z - \alpha_0 = -\beta_X + \frac{1}{2}(\beta_2 + \sum_{i=0}^{3} \beta_i) \quad \text{(159)} \]

\[ + \frac{n - 4}{10} \left( \sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X) \right) \]

\[ \alpha'_{33} = -\bar{\alpha}_{33} - \beta_P = \beta_X - \frac{1}{2}(\beta_2 + \sum_{i=0}^{3} \beta_i) - \beta_P \quad \text{(160)} \]

\[ - \frac{n - 4}{10} \left( \sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X) \right) \]

\[ \bar{\alpha}'_{23} = -\alpha_{33} - \beta_G = -\beta_X + \frac{1}{2}(\beta_2 + \sum_{i=0}^{3} \beta_i) + \beta_P - \beta_G \quad \text{(161)} \]

\[ + \frac{n - 4}{10} \left( \sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X) \right) \]

\[ \alpha'_{23} = -\bar{\alpha}_{23} - \beta_P = \beta_X - \frac{1}{2}(\beta_2 + \sum_{i=0}^{3} \beta_i) - 2\beta_P + \beta_G \quad \text{(162)} \]
\[ -\frac{n - 4}{10} \left( \sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X) \right) \]

\[ \alpha'_{21} = -\alpha'_{23} - \beta'_{2} = -\beta_X + \frac{1}{2}(\beta_2 + \sum_{i=0}^{3} \beta_i) + 2\beta_P - \beta_G \quad (163) \]

\[ -\beta'_{2} + \frac{n - 4}{10} \left( \sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X) \right) \]

\[ \alpha'_{21} = -\alpha'_{21} - \beta_I = +\beta_X - \frac{1}{2}(\beta_2 + \sum_{i=0}^{3} \beta_i) - 2\beta_P + \beta_G \quad (164) \]

\[ +\beta'_{2} - \beta_I - \frac{n - 4}{10} \left( \sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X) \right) \]

\[ \gamma_{2} = -\alpha'_{21} - \beta_X = -\beta_X + \frac{1}{2}(\beta_2 + \sum_{i=0}^{3} \beta_i) + 2\beta_P - \beta_G \quad (165) \]

\[ -\beta'_{2} + \beta_I - \beta_X + \frac{n - 4}{10} \left( \sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X) \right) \]

From U(1) charge conservation of the remaining terms \( \bar{\psi}'_{32} \chi'_{33}, \bar{\psi}'_{32} A_X \psi'_{32}, \bar{N}'_{3} \tilde{\Phi}^T \psi'_{32}, \)

\( S_P \bar{N}'_{3} N_3, \bar{\psi}'_{31} \chi'_{33}, \bar{\psi}'_{31} A_X \psi'_{31} \) and \( \bar{\psi}'_{31} \tilde{\Phi} N_3, \) we yield

\[ \bar{\alpha}'_{32} = -\alpha'_{33} - \beta' = -\beta_X + \frac{1}{2}(\beta_2 + \sum_{i=0}^{3} \beta_i) + \beta_P \quad (166) \]

\[ -\beta' + \frac{n - 4}{10} \left( \sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X) \right) \]

\[ \alpha'_{32} = -\alpha'_{32} - \beta_X = -\frac{1}{2}(\beta_2 + \sum_{i=0}^{3} \beta_i) - \beta_P \quad (167) \]

\[ +\beta' - \frac{n - 4}{10} \left( \sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X) \right) \]

\[ \bar{\gamma}_{3} = -\alpha'_{32} - \beta_\Phi = +\frac{1}{2}(\beta_2 + \sum_{i=0}^{3} \beta_i) + \beta_P \quad (168) \]

\[ -\beta' - \beta_\Phi + \frac{n - 4}{10} \left( \sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X) \right) \]

\[ \gamma_{3} = -\bar{\gamma}_{3} - \beta_P = -\frac{1}{2}(\beta_2 + \sum_{i=0}^{3} \beta_i) - 2\beta_P \quad (169) \]

\[ +\beta' + \beta_\Phi - \frac{n - 4}{10} \left( \sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X) \right) \]

\[ \alpha'_{31} = -\alpha'_{33} - \beta_3 = -\beta_X + \frac{1}{2}(\beta_2 + \sum_{i=0}^{3} \beta_i) + \beta_P \quad (170) \]

\[ -\beta'_{3} + \frac{n - 4}{10} \left( \sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X) \right) \]
Finally, applying the U(1) charge conservation to the terms in the superpotential $W_S$: \[ \bar{\psi}\phi_i 16, S_G \bar{\psi}\phi_i 16, \bar{\psi}_1 10_2 \psi_i 1, S_P \bar{\psi}_1 \phi_i \nu_s, \bar{\psi}_1 \Phi \nu_s, 5_N_3 N_s \nu_s, \bar{\nu}_s \phi_i N_s \], we obtain

\[
\begin{align*}
\alpha'_{3i} &= -\alpha'_{3i} - \beta_X = -\frac{1}{2}(\beta_2 + \sum_{i=0}^{3} \beta_i) - \beta_P \\
+ \beta_3' &= \frac{n - 4}{10} \left( \sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X) \right) \\
\gamma_3 &= -\alpha'_{31} - \beta_\Phi = \frac{1}{2}(\beta_2 + \sum_{i=0}^{3} \beta_i) + \beta_P \\
- \beta_3' &= -\beta_\Phi + \frac{n - 4}{10} \left( \sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X) \right)
\end{align*}
\]

(171)

(172)

(173)

(174)

(175)

(176)

(177)

(178)

(179)

(180)

Notice that in the above relations each U(1) charge $\gamma_i \ (i = 1, 2, 3)$ is given by two relations. Equal of the two relations leads to

\[
3 \beta_P - 2 \beta_\Phi = -\beta_2 - \sum_{i=0}^{3} \beta_i + \beta' + \beta'_3
\]

- \frac{n - 4}{10} \left( \sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X) \right)

\[
5 \beta_P - 2 \beta_\Phi - 2 \beta_X - 2 \beta_G + 2 \beta_I = -\frac{1}{2}(\beta_2 + \beta_3) - \sum_{i=0}^{3} \beta_i + \beta' + \beta'_2
\]

42
\[- \frac{n-4}{10}(\sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X)) \]

\[7\beta_P - 2\beta_\Phi - 2\beta_X - 2\beta_G + 2\beta_I = -2\beta_3 + \beta_1 - \sum_{i=0}^{3} \beta_i + \beta' + \beta'_1 \quad (183)\]

\[- \frac{n-4}{10}(\sum_{i=0}^{3} \beta_i + 2(\beta_G - \beta_X)) \]

From the above three relations together with the two relations for $\beta_P$ and $\beta_I$ obtained from the superpotential $W_Y$, we find, for the realistic case $n = 4$, that

\[
\beta_\Phi = -\frac{1}{3} \beta_3 + \frac{3}{5} \beta_2 + \frac{22}{15} \beta_1 - \frac{23}{30} \beta_0 - \frac{1}{6} \beta'_3 - \frac{1}{3} \beta'_2 + \frac{1}{6} \beta'_1 - \frac{1}{3} \beta', \quad (184)\]

\[
\beta_X = -\frac{5}{36} \beta_3 + \frac{5}{12} \beta_2 + \frac{37}{90} \beta_1 - \frac{29}{18} \beta_0 - \frac{1}{18} \beta'_3 - \frac{1}{18} \beta'_2 - \frac{1}{9} \beta'_1 + \frac{1}{9} \beta', \quad (185)\]

\[
\beta_G = \frac{2}{9} \beta_3 - \frac{4}{3} \beta_2 + \frac{26}{45} \beta_1 - \frac{1}{9} \beta_0 - \frac{8}{9} \beta'_3 - \frac{8}{9} \beta'_2 + \frac{1}{9} \beta'_1 + \frac{13}{45} \beta', \quad (186)\]

\[
\beta_I = -\frac{1}{10} \beta_3 + \frac{1}{10} \beta_2 + \frac{43}{90} \beta_1 - \frac{139}{90} \beta_0 - \frac{1}{9} \beta'_3 - \frac{2}{9} \beta'_2 + \frac{1}{9} \beta'_1 + \frac{1}{9} \beta', \quad (187)\]

\[
\beta_P = -\frac{4}{5} \beta_3 - \frac{1}{3} \beta_2 + \frac{7}{9} \beta_1 - \frac{2}{5} \beta_0 - \frac{2}{9} \beta'_3 - \frac{2}{9} \beta'_2 + \frac{1}{9} \beta'_1 + \frac{1}{9} \beta'. \quad (188)\]

Substituting these five relations to all of the relations given above for the U(1) charges of the various fields, we come to our conclusions that once the U(1) charges $\beta_k$ ($k = 0, 1, 2, 3$), $\beta$, $\beta'_i$ ($i = 1, 2, 3$), $\beta'$, $\beta_{10}$ and $\beta_{\Phi}$ are appropriately chosen, the U(1) charges of all other fields will be completely determined. For instance, the U(1) charge of the 16-representation quarks and leptons is given by

\[
\alpha = -\frac{44}{45} \beta_3 + \frac{14}{15} \beta_2 + \frac{7}{90} \beta_1 - \frac{2}{3} \beta_0 - \frac{1}{2} \beta \quad (189)\]

\[-\frac{4}{9} \beta'_3 + \frac{4}{9} \beta'_2 + \frac{1}{9} \beta'_1 + \frac{1}{9} \beta'. \]

This is not difficult to understand, as we have discussed before and also explicitly shown in the figures that the texture structures of mass matrices in the present model are characterized by the $\Delta(48)$ triplets $\chi_k$ ($k=0,1,2,3$), $\chi$, $\chi'_i$ ($i = 1, 2, 3$) and $\chi'$.

To describe the real world, the following symmetry breaking scenario and the structure of the physical vacuum are considered

\[
SO(10) \times \Delta(48) \times U(1) \xrightarrow{\text{symmetry breaking}} SO(10) \xrightarrow{\text{vacuum}} SU(5) \]

\[
\xrightarrow{\text{vacuum}} SU(3)_c \times SU(2)_L \times U(1)_Y \xrightarrow{\text{symmetry breaking}} SU(3)_c \times U(1)_{em} \quad (190)\]
and: \[< S_P >= \bar{M}_P, < S_I > = v_{10}, < \Phi^{(16)}> = < \bar{\Phi}^{(16)} >= v_{10}/\sqrt{2}, < S_G >= v_5,\]
\[< \chi^{(3)} >= < \chi_a^{(i)} >= M_P, < \chi^{(2)} >= < \chi_j^{(i)} >= v_5 \text{ with } (i = 1, 2, 3; a = 0, 1, 2, 3; j = 1, 2, 3),\]
\[< \chi^{(1)} >= < \chi^{(1)} >= < \chi^{(3)} >= 0, < \phi_s >= v_s \simeq 336 \text{ GeV}, < H_2 >= v_2 = v \sin \beta\]
and \[< H_1 > = v_1 = v \cos \beta \text{ with } v = \sqrt{v_1^2 + v_2^2} = 246 \text{ GeV}.\]

We now explicitly derive the operator \(W_{33}\). It concerns a \((7 + n) \times (7 + n)\) mass matrix. For the realistic case \(n=4\), we need to treat an \(11 \times 11\) mass matrix. Let us denote \(\Psi_3^T = (16_3, \psi_0^{(3)}, \psi_1^{(3)}, \psi_2^{(3)}, \psi_3^{(3)}, \psi_4^{(3)}, \psi_5^{(3)}, \psi_{11}^{(1)}, \psi_{12}^{(1)}, \psi_{13}^{(3)}, \psi_{21}^{(1)})^T\) and \(\bar{\Psi}_3^T = (\bar{16}_3, \bar{\psi}_0^{(3)}, \bar{\psi}_1^{(3)}, \bar{\psi}_2^{(3)}, \bar{\psi}_3^{(3)}, \bar{\psi}_4^{(3)}, \bar{\psi}_5^{(3)}, \bar{\psi}_{11}^{(1)}, \bar{\psi}_{12}^{(1)}, \bar{\psi}_{13}^{(3)}, \bar{\psi}_{21}^{(1)})^T\). Here the upper indeces \(i\) \((i = 1, 2, 3)\) label the components of the \(\Delta(48)\) triplets. \(16\) is introduced as an auxiliary field for convenience of discussions. The mass matrix \(M_{33}\) defined by \(\bar{\Psi}_3 M_{33} \Psi_3\) is found to be

\[
M_{33} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_5 & \bar{M}_P & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & v_{10} & A_X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & v_{10} & \bar{A}_X & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & v_{10} & \bar{A}_X & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & v_{10} & A_X & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & v_{10} & A_X & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & v_{10} & A_X & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \chi_1^{(3)} & v_5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \chi_1^{(3)} & v_5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \chi_2^{(2)} & v_5 & 0 & 0 \\
\end{pmatrix}
\]  
(191)

Here, we have taken \(< S_P > = \bar{M}_P, < S_I > = v_{10} \text{ and } < S_G > = v_5\). For simplicity, we have omitted the bracket \(<>\) for the VEVs of other non-singlet fields. To work in the mass eigenstates, we make unitary transformations \(\Psi_3 \rightarrow U \Psi_3\) and \(\bar{\Psi}_3 \rightarrow \bar{V}^* \bar{\Psi}_3\), so that \(U^T M_{33} U\) becomes a diagonal mass matrix. Diagonalizing the hermitian mass matrix squires \(M_{33}^d M_{33}^t\) and \(M_{33} M_{33}^d\), one yields the unitary matrices \(U\) and \(V\) respectively. To obtain Yukawa coupling matrices of the quarks and leptons, the interesting mixings are between the fields

44
ψ_{a1} (a = 0, 1, 2, 3, 4) and 16_3 as well as between the fields ψ_{a2} (a = 0, 1, 2, 3, 4) and 16_3. For that, we only need to find the normalized eigenvector corresponding to the mass eigenvalue of the 16-representation quarks and leptons 16 which remains massless. By solving simple $n+7$ linear equations, it is not difficult to find the normalized eigenvector corresponding to the zero eigenvalue of $M_{33}^\dagger M_{33}$, i.e., $U_{i1}$ ($i = 1, \ldots, n+7$) with $\sum_{i=1}^{n+7} |U_{i1}|^2 = 1$, as follows

$$U_{i1} = \eta_X \left\{ \left( \frac{v_5}{M_P} \right), \left( \frac{v_5}{M_P} \right), \left( \frac{v_{10}}{M_P} \right), \left( \frac{v_{10}}{A_X} \right)^2, \ldots, \left( \frac{v_5}{M_P} \right), \left( \frac{v_{10}}{A_X} \right)^{n+1} \right\},$$

$$\left( \frac{v_5}{M_P} \right), \left( \frac{v_{10}}{A_X} \right)^{n+1} \left( \frac{\chi^{(3)}}{v_5} \right), \left( \frac{v_5}{M_P} \right), \left( \frac{v_{10}}{A_X} \right)^{n+1} \left( \frac{\chi^{(3)}}{v_5} \right),$$

$$\left( \frac{v_5}{M_P} \right), \left( \frac{v_{10}}{A_X} \right)^{n+1} \left( \frac{A_z}{A_X} \right), - \left( \frac{v_5}{M_P} \right), \left( \frac{v_{10}}{A_X} \right)^{n+1} \left( \frac{A_z}{A_X} \right) \left( \frac{\chi^{(2)}}{v_5} \right) \right\} \right\} \right\} \right\} \right\} \right\}$$

with

$$\eta_X^{-2} = 1 + \left| \left( \frac{v_5}{M_P} \right) \right|^2 + \left| \left( \frac{v_5}{M_P} \right) \right|^2 + \cdots + \left| \left( \frac{v_5}{M_P} \right) \right|^2 \left| \left( \frac{v_{10}}{A_X} \right)^{n+1} \right|^2$$

$$+ \left| \left( \frac{v_5}{M_P} \right) \right|^2 + \left| \left( \frac{v_5}{M_P} \right) \right|^2 \left| \left( \frac{\chi^{(3)}}{v_5} \right) \right|^2 \right\} \right\} \right\} \right\} \right\} \right\}$$

Noticing the symmetry breaking scenario considered above, i.e., $< \chi^{(3)} > = < \chi_a^{(i)} > = \tilde{M}_P$, and the smallness of the ratios $v_5/\tilde{M}_P \ll 1$ and $v_5/v_{10} < 1$, we yield

$$\eta_X \simeq \frac{1}{\sqrt{1 + 2\eta_A}}, \quad \eta_A = \left( \frac{v_{10}}{A_X} \right)^{n+1}$$

and

$$\psi_{11}^{(1)} = \eta_X \eta_A 16_3 + \cdots,$$

$$\psi_{12}^{(1)} = \eta_X \eta_A 16_3 + \cdots,$$

$$\psi_{21}^{(1)} = \eta_X \eta_A \left( \frac{A_z}{A_X} \right) 16_3 + \cdots,$$
VII. CONCLUSIONS AND REMARKS

Based on the symmetry group SUSY $SO(10) \times \Delta(48) \times U(1)$, we have presented in much greater detail an alternative interesting model with small $\tan \beta$. It is amazing that nature has allowed us to make predictions in terms of a single Yukawa coupling constant and three ratios of the VEVs determined by the structure of the physical vacuum and understand the low energy physics from the GUT scale physics. It has also suggested that nature favors maximal spontaneous CP violation. In comparison with the model with large $\tan \beta \sim m_t/m_b$, i.e., Model I, the model analyzed here with low $\tan \beta$, i.e., Model II has provided a consistent picture on the 23 parameters with better accuracy. Besides, ten relations involving fermion masses and CKM matrix elements are obtained with four of them independent of the RG scaling effects. Five relations in the light neutrino sector are also found to be independent of the RG scaling effects. These relations are our main results which contain only low energy observables. As an analogy to the Balmer series formula, these relations may remain to be considered as empirical at the present moment. They have been tested by the existing experimental data to a good approximation and can be tested further directly by more precise experiments in the future. The two types of the models corresponding to the large $\tan \beta$ (Model I) and low $\tan \beta$ (Model II) might be distinguished in testing the MSSM Higgs sector at Colliders as well as by precisely measuring the ratio $|V_{ub}/V_{cb}|$ since this ratio does not receive radiative corrections in both models. The neutrino sector is of special interest for further study. Though the recent LSND experiment, atmospheric neutrino deficit, and hot dark matter could be simultaneously explained in the present model, solar neutrino puzzle can be understood only by introducing an $SO(10)$ singlet sterile neutrino. The scenario for the neutrino sector can be further tested through ($\nu_e - \nu_\tau$) and ($\nu_\mu - \nu_\tau$) oscillations since the present scenario has predicted a short wave ($\nu_e - \nu_\tau$) oscillation. However, the ($\nu_\mu - \nu_\tau$) oscillation is beyond the reach of CHORUS/NOMAD and E803. As we have also shown that one may abandon the assumption of universality for all terms in the superpotential and use a permutation symmetry among the fields concerning the ‘22’ and ‘32’ textures, consequently,
the resulting predictions in the quark and charged lepton sector are unchanged and remain involving four parameters. It is also interesting to note that even if without imposing the permutation symmetry and universality of the coupling constants, the resulting Yukawa coupling matrices of the quarks and leptons only add one additional parameter which can be determined by the charm quark mass. For $m_c (m_c) = 1.27 \pm 0.05 \text{GeV}$, the resulting predictions remain the same as those in Tables II and III for the quark and charged lepton sector. For the neutrino sector, three additional parameters corresponding to the three nonzero textures are involved. Nevertheless, it remains amazing that nature allows us to make predictions on 23 observables by only using nine parameters for this general case. It is expected that more precise measurements from CP violation, neutrino oscillation and various low energy experiments in the near future could provide an important test on the present model and guide us to a more fundamental theory.

**ACKNOWLEDGEMENTS:** YLW has greatly benefited from valuable conversation and discussions with H. Goldberg, G. Kane, P. Nath, W. Palmer, S. Raby and G. Steigman. He would also like to thank the Ohio State University where part of this work was done and supported in part by the US Department of Energy Grant DOE/ER/01545-675.
REFERENCES

[1] R. Gatto, G. Sartori and M. Tonin, “Weak self-masses, Cabbibo Angle, and Broken SU(2)× SU(2)”, Phys. Lett. 28B, 128 (1968);
N. Cabbibo and L. Maiani, “Dynamical Interrelation of weak, electromagnetic and strong interactions and the value of θ”, Phys. Lett. 28B, 131 (1968);
R.J. Oakes, “SU(2)× SU(2) breaking and the Cabbibo angle”, Phys. Lett. 29B, 683 (1969).

[2] S. Weinberg, I.I. Rabi, Festschrift (1977);
F. Wilczek and A. Zee, “Discrete flavor symmetries and a formula for the Cabbibo angle”, Phys. Lett. 70B, 418 (1977).

[3] H. Fritzsch, “Calculating the Cabbibo Angle”, Phys. Lett. 70B, 436 (1977).

[4] H. Georgi and C. Jarlskog, “A new lepton-quark mass relation in a unified theory”,
Phys. Lett. 86B, 297 (1979).

[5] J. Harvey, P. Ramond and D. Reiss, “CP violation and mass relations in SO(10)”, Phys.
Lett. 92B, 309 (1980); “Mass relations and neutrino oscillations in an SO(10) model”,
Nucl. Phys. B199. 223 (1982);
X.G. He and W.S. Hou, “Relating the long B lifetime to a very heavy top quark”, Phys.
Rev. D 41, 1517 (1990);
S. Dimopoulos, L.J. Hall and S. Raby, “Predictive framework for fermion masses in supersymmetric theories”, Phys. Rev. Lett. 68, 752 (1992); ibid., “Predictive Ansatz for fermion mass matrices in supersymmetric grand unified theories”, Phys. Rev. D 45, 4192 (1992); ibid., “Predictions for neutral K and B meson physics”, ibid. 46, R4793 (1992);
L.J. Hall and A. Rasin, “On the generality of certain predictions for quark mixing”,
Phys. Lett. 315B, 164 (1993);
H. Arason, D. Castano, E.J. Piard and P. Ramond, “Mass and Mixing angle patterns
in the standard model and its minimal supersymmetric extension”, Phys. Rev. D 47, 232 (1993);
V. Barger, M.S. Berger, T. Han, and M. Zralek, “Test of the Dimopoulos-Hall-Raby Ansatz for fermion mass matrix”, Phys. Rev. Lett. 68, 3394 (1992).

[6] P. Ramond, R.G. Roberts and G.G. Ross, “Stitching the Yukawa quilt”, Nucl. Phys. B406, 19 (1993).

[7] For a recent review see, S. Raby, “Introduction to theories of fermion masses”, Ohio State University Report No. OHSTPY-HEP-T-95-024, 1995.

[8] K.S. Babu and S.B. Barr, “An SO(10) solution to the puzzle of quark and lepton masses”, Phys. Rev. Lett. 75, 2088 (1995).

[9] K.S. Babu and R.N. Mohapatra, “Mass matrix textures from superstring inspired SO(10) models”, Phys. Rev. Lett. 74, 2418 (1995).

[10] K.S. Babu and Q. Shafi, “A predictive SO(10) scheme for fermion masses and mixings”, Phys. Lett. 357B, 365 (1995).

[11] L.J. Hall and S. Raby, “A complete supersymmetric SO(10) model”, Phys. Rev. D 51, 6524 (1995).

[12] Z.G. Berezhiani, “Predictive SUSY SO(10) model with very low tan β”, Phys. Lett. 355B, 178, (1995).

[13] D. Kaplan and M. Schmaltz, “Flavor unification and discrete non-abelian symmetries”, Phys. Rev. D 49, 3741 (1994); M. Schmaltz, “Neutrino oscillations from discrete non-abelian family symmetries”, .ibid. 52, 1643 (1995), hep-ph/9411383.

[14] A. Kusenko and R. Shrock, Phys. Rev. D 49, 4962 (1994).

[15] G. Anderson, S. Raby, S. Dimopoulos, L.J. Hall, and G.D. Starkman, “Systematic SO(10) operator analysis for fermion masses”, Phys. Rev. D 49, 3660 (1994).
[16] K.C. Chou (Zhou Guangzhao) and Y.L. Wu (Wu Yueliang), “Low energy phenomena in a model with symmetry group SUSY SO(10)×Δ(48)× U(1)”, Science in China (Sci. Sin.), 39A, 65 (1996); hep-ph/9508402.

[17] see for example, J. Ellis, “Supersymmetry and grand unified theories”, presented at the 17th International Symposium on Lepton-photon Interactions, Beijing, China, 1995.

[18] L. Hall, R. Rattazzi, and U. Sarid, “The top quark mass in supersymmetric SO(10) unification”, Phys. Rev. D 50, 7048 (1994);
M. Carena, M. Olechowski, S. Pokorski and C. Wagner, “Electroweak symmetry breaking and bottom-top Yukawa unification”, Nucl. Phys. B 426, 269 (1994);
R. Rattazzi, U. Sarid and L.J. Hall, “Yukawa unification: The good, the bad and the ugly”, Stanford University Report No. SU-ITP-94-15, hep-ph/9405313;
T. Blažek, S. Raby and S. Pokorski, “Finite supersymmetric threshold corrections to CKM matrix elements in the large tan β regime”, Phys. Rev. D 52, 4151 (1995), and references therein;
H. Murayama, M. Olechowski and S. Pokorski, “Viable t − b − τ Yukawa unification in SUSY SO(10)”, Report No. MPI-PhT/95-100, DFTT 63/95, LBL-37884, UCB-PTH-95/34, hep-ph/9510327, 1995.

[19] K.C. Chou and Y.L. Wu, “CP violation, Fermion masses and Mixings in a Predictive SUSY SO(10)×Δ(48)× U(1) Model with small tan β”, Phys. Rev. D 53, 3492 (1996) (Rapid Communication); hep-ph/9511327.

[20] K.C. Chou and Y.L. Wu, “A Solution to the Puzzles of CP Violation, Neutrino Oscillation, Fermion Masses and Mixings in an SUSY GUT Model With Small tan β”, Nucl. Phys. Proc. Suppl. 53A (1997) 159-163.

[21] Z. Kakushadze and S.-H. Tye, “Three-Family SO(10) Grand Unification in String Theory”, Cornell University Rep. No., CLNS 96/1412, 1996.
[22] J. Cleaver, private communication.

[23] Y. Nir and N. Seiberg, Phys. Lett. 309B, 337 (1993); M. Leurer, Y. Nir and N. Seiberg, Nucl. Phys. B420, 468 (1994).

[24] M. Dine, R. Leigh and A. Kagan, Phys. Rev. D 48, 4269 (1993).

[25] P.H. Frampton and O.C.W. Kong, Phys. Rev. Lett. 75, 781 (1995).

[26] CDF Collaboration, F. Abe et al., “Evidence for top quark production in $\bar{p}p$ collisions at $\sqrt{s} = 1.8$ TeV”, Phys. Rev. D 50, 2966 (1994); Phys. Rev. Lett. 73, 225 (1994); “Observation of Top quark production in $\bar{p}p$ collisions with the collider detector at Fermilab”, Phys. Rev. Lett. 74, 2626 (1995); D0 Collaboration, S. Abachi et al., “Observation of the Top quark”, Phys. Rev. Lett. 74, 2632 (1995).

[27] J. Gasser and H. Leutwyler, “Quark masses”, Phys. Rep. 87, 77 (1982); H. Leutwyler, “$m_u$ is not equal to zero”, Nucl. Phys. B337, 108 (1990).

[28] Particle Data Group, L. Montanet et al., Phys. Rev. D 50, 1173, (1994).

[29] M. Shifman, “Determining $\alpha_s$ from measurements at $Z$: how nature prompts us about new physics”, Mod. Phys. Lett. A10, 605 (1995); M.B. Voloshin, “Precision determination of $\alpha_s$ and $m_b$ from QCD sum rules for $b\bar{b}$”, Int. J. Mod. Phys. A10, 2865 (1995).

[30] For recent review, see for example, C. Michael, “Hadronic Physics from the Lattice”, Liverpool Preprint, LTH 351, hep-lat/9509091, and references therein.

[31] G.L. Kane, R.G. Stuart and J.D. Wells, “A Global fit of LEP/SLC data with light superpartners”, Phys. Lett. 354B, 350 (1995).

[32] T.P. Cheng, E. Eichten, and L.F. Li, “Higgs phenomena in asymptotically free gauge theories”, Phys. Rev. D 9, 2259 (1974);
M.S. Chanowitz, J. Ellis, and M.K. Gaillard, Nucl. Phys. B128, 506 (1977);
A. Buras, J. Ellis, M.K. Gaillard, and D. Nanopoulos, “Aspects of the grand unification of strong, weak and electromagnetic interactions”, Nucl. Phys. B135, 66 (1978);
C.T. Hill, C.N. Leung and S. Rao, “Renormalization group fixed points and the Higgs boson spectrum”, Nuc. Phys. B262, 517 (1985);
K. Inoue, A. Kakuto, H. Komatsu, and S. Takeshita, “Low-energy parameters and particle masses in a supersymmetric grand unified model”, Prog. Theor. Phys. 67, 1889 (1982);
L. Ibanez and C. Lopez, “N=1 supergravity, the weak scale and the low-energy particle spectrum”, Nucl. Phys. B233, 511 (1984);
E. Ma and S. Pakvasa, “Cabbibo angle and quark masses in the Weinberg-Salam model”, Phys. Lett. B66B, 43 (1979); “Variation of mixing angles and masses with $Q^2$ in the standard six quark model”, Phys. Rev. D 20, 2899 (1979);
B. Ananthanarayan, G. Lazarides and Q. Shafi, “Top-quark-mass prediction from supersymmetric grand unified theories”, Phys. Rev. D 44, 1613 (1991);
H. Arason et al., “Top-quark and Higgs-boson mass bounds from a numerical study of supersymmetric grand unified theories”, Phys. Rev. Lett. 67, 2933 (1991);
S. Kelley, J. Lopez and D. Nanopoulos, “Yukawa Unification”, Phys. Lett. 274B, 387 (1992);
V. Barger, M.S. Berger, and P. Ohmann, “Supersymmetric grand unified theories: Two-loop evolution of gauge and Yukawa couplings”, Phys. Rev. D 47, 1093 (1993);
S. G. Naculich, “Third-generation effects on fermion mass predictions in supersymmetric grand unified theories”, Phys. Rev. D 48, 5293 (1993);
P. Langacker and N. Polonsky, “Bottom quark mass prediction in supersymmetric grand unification, Uncertainties and constraints”, Phys. Rev. D 49, 1454 (1994);
J. Bagger, K. Matchev and D. Pierce, “Precision corrections to supersymmetric unification”, Phys. Lett. 348B, 443 (1995).
[33] V. Barger, M.S. Berger, T. Han, and M. Zralek, in Ref. [5]; S.G. Na culich, in Ref. [32].

[34] L. Wolfenstein, “Parametrization of the Kobayashi-Maskawa matrix”, Phys. Rev. Lett. **51**, 1945 (1983).

[35] A. Datta, E.A. Paschos, J.-M. Schwartz and M.N. Sinha Roy, “ QCD corrections for the $K^0$-$\bar{K}^0$ and $B^0$-$\bar{B}^0$ system”, Dortmund University Report No. DO-TH-95-12, [hep-ph/9509420], 1995;
S. Herrlich and U. Nierste, “Indirect CP violation in the neutral K system beyond leading logarithms”, Report No. TUM-T31-81/95, [hep-ph/9507262], 1995;
A.J. Buras, M. Jamin, and P.H. Weisz, “Leading and next-to-leading QCD corrections to epsilon parameter and $B^0 - \bar{B}^0$ mixing in the presence of a heavy top quark”, Nucl. Phys. **B370**, 69 (1992).

[36] J. Shigemitsu, in: *Proceedings of the XXVII International Conference on High Energy Physics*, Glasgow, Scotland, 1994, eds. P.J. Bussey and I.G. Knowles (IOP, Bristol, 1995).

[37] C. Bernard and A. Soni, “Update on $B_K$ with Wilson fermions”, in *Lattice ’94*: Proceedings of the International Symposium, Bielefeld, Germany, edited by F. Karsch et al. [Nucl. Phys. B (Proc. Suppl.) 42 (1995)];
S.R. Sharpe, “$B_K$ using staggered fermions: An Update”, in *Lattice ’93*: Proceedings of the International Symposium, Dallas, Texas, edited by T. Draper et al. [ *ibid* **34**, 403 (1994)].

[38] C. Allton, “Leptonic Decays of Heavy-Light systems”, [hep-lat/9509084], 1995, and references therein.

[39] S. Narison, “Precise determination of $F_{P_\pi}/F_P$ and measurement of the perturbative pole mass from $F_P$”, Phys. Lett. **322B**, 247 (1994); S. Narison and A. Pivovarov, “QSSR estimate of the $B_B$ parameter at next-to-leading order”, Phys. Lett. **327B**, 341 (1994);
A. Bagan, P. Ball, V.M. Brau and H.G. Dosch, “QCD sum rules in the effective heavy quark theory”, Phys. Lett. 278B, 457 (1992);
M. Neubert, “Heavy meson form-factors from QCD sum rules”, Phys. Rev. D 45, 2451 (1992);
K. Schilcher and Y.L. Wu, “Unified QCD determination of all pseudoscalar leptonic decay constants”, Z. Phys. C54, 163 (1992);
N.F. Nasrallah, K. Schilcher and Y.L. Wu, “1/M Correction to $f_B$”, Phys. Lett. 261B, 131 (1991).

[40] Y.L. Wu, “Upper bounds for heavy meson matrix elements”, Nucl. Phys. B324, 296 (1989).

[41] for a recent summary, see for example, E.A. Paschos, Dortmund University Report No. DO-TH-96-01, presented at the 17th International Symposium on Lepton Photon Interactions, Beijing, China, 1995 and at Hellenic School on Elementary Particle Physics, Corfu, Greece, 1995.

[42] W.A. Bardeen, A.J. Buras and J.-M. Gerard, “A consistent analysis of the $\Delta I = 1/2$ rule for K decays ”, Phys. Lett. 192B, 138 (1987).

[43] J. Heinrich, E.A. Paschos, J.-M. Schwartz and Y.L. Wu, “Accuracy of the predictions for Direct CP violation”, Phys. Lett. 279B, 140 (1992).

[44] Y.L. Wu, “A simultaneous analysis for the $\Delta I = 1/2$ rule, direct CP violation, $K_0$-$\bar{K}^0$ and $B^0$-$\bar{B}^0$ mixings”, Intern. J. of Mod. Phys. A7, 2863 (1992).

[45] A.J. Buras, M. Jamin, M.E. Lautenbacher, and P.H. Weisz, “Two loop anomalous dimension matrix for $\Delta S = 1$ weak nonleptonic decays. 1. $O(\alpha_s^2)$ ”, Nucl. Phys. B400, 37 (1993); A.J. Buras, M. Jamin, and M.E. Lautenbacher, “Two loop anomalous dimension matrix for $\Delta S = 1$ weak nonleptonic decays. 2. $O(\alpha_s^2)$ ”, Nucl. Phys. B400, 75 (1993).

[46] M. Ciuchini, E. Franco, G. Martinelli, L. Reina and L. Silvestrini, “ An upgraded
analysis of $\varepsilon'/\varepsilon$ at the next-to-leading order”, CERN Report No. CERN-TH. 7514/94, ROME preprint 94/1024, Jan. 1995.

[47] G.D. Barr et al., “A new measurement of direct CP violation in the neutral kaon system”, Phys. Lett. 317B, 233 (1993).

[48] L.K. Gibbons et al., “Measurement of the CP violation parameter $Re(\varepsilon'/\varepsilon)$”, Phys. Rev. Lett. 70, 1203 (1993).

[49] L. Wolfenstein, Phys. Rev. Lett. 13, 562 (1964); A superweak model with $\varepsilon'/\varepsilon < O(10^{-4})$ could be realized in a general two-Higgs doublet model with flavor changing processes mediated by the exchange of neutral scalar boson (FCNE), see for example, J. Liu and L. Wolfenstein, Nuc. Phys. B289, 1 (1987);
Y.L. Wu and L. Wolfenstein, “Sources of CP violation in the two-Higgs-doublet model”, Phys. Rev. Lett. 73, 1762 (1994), and references therein;
G.C. Branco, W. Grimus, L. Lavoura, “Relating the scalar flavor changing neutral coupling to the CKM matrix”, FISIST-1-96-CFIF, 1996.

[50] G. Kramer, W.F. Palmer and Y.L. Wu, “Extraction of $\alpha$ from the CP asymmetry in $B^0/\bar{B}^0 \rightarrow \pi^+\pi^-$ Decays”, DESY Report No. DESY 95-246, hep-ph/9512341 and references therein.

[51] I.I. Bigi and S.I. Sanda, Nuc. Phys. B193, 85 (1981); B281, 41 (1987).

[52] M. Gronau and D. Wyler, Phys. Lett. 256B, 172 (1991); I. Dunietz, Phys. Lett. B270, 75 (1991); R. Aleksan, I. Dunietz and B. Kayser, Z. Phys. C54, 653 (1992).

[53] R. Patterson, in Proceedings of the XXVII International Conference on High Energy Physics, Glasgow, Scotland, 1994, eds. P.J. Bussey and I.G. Knowles (IOP, Bristol, 1995), Vol. 1, p. 149.

[54] M. Neubert, “Uncertainties in the determination of $|V_{cb}|$”, CERN Report No. CERN-TH/95-107, hep-ph/9505238, talk given at 30th Rencontres de Moriond: Electroweak
Interactions and Unified Theories, Meribel les Allues, France, 11-18 Mar 1995. 1995.

[55] A. Ali and D. London, “CP violation and Flavour mixing in the standard model”, DESY Report No. DESY 95-148, UdeM-GPP-TH-95-32, hep-ph/9508272. Presented at 6th International Symposium on Heavy Flavor Physics, Pisa, Italy, 6-9 Jun 1995.

[56] M. Shifman, “Recent progress in the Heavy Quark Theory”, TPI-MINN-95/15-T, UMN-TH-1345-95, hep-ph/9505289. 1995, and references therein.

[57] C.G., Boyd, B. Grinstein and R.F. Lebed, “Model independent extraction of $|V_{cb}|$ using dispersion relations”, Phys. Lett. 353B, 306 (1995).

[58] M. Gell-Mann, P. Ramond and R. Slansky, in Supergravity, ed. F. van Nieuwenhuizen and D. Freedman (Nort Holland, Amsterdam, 1979) p. 315; T. Yanagida, Proc. of the Workshop on Unified Theory and Baryon Number of Universe, KEK, Japan, 1979; S. Bludman, D. Kennedy, and P. Langacker, “Seesaw model predictions for the $\tau$-neutrino mass”, Phys. Rev. D 45, 1810 (1992).

[59] S. Dimopoulos and F. Wilczek, Proceedings Erice Summer School, Ed. A. Zichichi (1981).

[60] C. Athanassopoulos et al., “Candidate events in a search for $\bar{\nu}_\mu - \bar{\nu}_e$ oscillations”, Phys. Rev. Lett., nucl-ex/9504002 (1995).

[61] Y. Fukuda et al., “Atmospheric $\nu_\mu/\nu_e$ ratio in multi-GeV energy range”, Phys. Lett. 335B, 237 (1994).

[62] J. Primack, J. Holtzman, A. Klypin and D. O. Caldwell, “Cold + Hot dark matter cosmology with $m(\nu_\mu) \simeq m(\nu_e) = 2.4$ eV ” Phys. Rev. Lett. 74, 2160 (1995); K.S. Babu, R.K. Schaefer, and Q. Shafi, “Cold plus hot dark matter cosmology in the light of solar and atmospheric neutrino oscillations”, Phys. Rev. D 53, 606 (1996), and references therein.
[63] For recent review see, R.N. Mohapatra, “Neutrinoless Double Beta decay and physics beyond the standard model”, Maryland University Report No. UMD-PP-95-147, hep-ph/9507234, talk given at the International Workshop on Neutrinoless Double Beta Decay and Related Topics, Trento, Italy, 24 Apr - 5 May 1995.

[64] D.O. Caldwell and R.N. Mohapatra, “Neutrino mass explanations of solar and atmospheric neutrino deficits and hot dark matter”, Phys. Rev. D 48, 3259 (1993);
J. Peltoniemi, D. Tommasini, and J.W.F. Valle, “Reconciling dark matter and solar neutrinos”, Phys. Lett. 298B, 383 (1993).

[65] T. Walker, G. Steigman, D.N. Schramm, K. Olive, and H. Kang, “Primordial Nucleosynthesis redux” Astrophys. J. 376, 51 (1991);
P. Kernan and L.M. Krauss, “Refined Big Bang nucleosynthesis constraints on $\Omega_B$ and $N_\nu$”, Phys. Rev. Lett. 72, 3309 (1994).

[66] Recently, a much more stringent constraint with $N_\nu < 2.6$ for $\Delta Y = 0.005$ was reported by N. Hata, R. J. Scherrer, G. Steigman, D. Thomas, T.P. Walker, S. Bludman, and P. Langacker, “Big bang nucleosynthesis in crisis?”, Phys. Rev. Lett. 75, 3981 (1995);
While the analyses by C. J. Copi, D.N. Schramm and M.S. Turner, ”Assessing Big-Bang Nucleosynthesis”, Phys. Rev. Lett. 75, 3977 (1995), provided an upper bound $N_\nu < 3.9$ with a suggestion that the $^4He$ abundance may have been systematically underestimated.

[67] R. Barbieri and A. Dolgov, “Neutrino oscillations in the early universe”, Nucl. Phys. B349, 743 (1991);
K. Enqvist et al., “Stringent cosmological bounds on inert neutrino mixing”, Nucl. Phys. B373, 498 (1992);
X. Shi, D. N. Schramm, and B.D. Fields, “Constraints on neutrino oscillations from big bang nucleosynthesis”, Phys. Rev. D 48, 2563 (1993).

[68] L. Wolfenstein, “Neutrino Oscillation in matter”, Phys. Rev. D 17, 2369 (1978);
S.P. Mikheyev and A. Yu. Smirnov, Sov. J. Nucl. Phys. 42, 913 (1986).

[69] W.M. Fairbairn, T. Fultz, and W.H. Klink, “Finite and disconnected subgroups of SU(3) and their application to the elementary-particle spectrum”, Journal of Math. Phys. 5, 1038 (1964).

[70] S. Chaudhuri, S. Chung and J. Lykken, FERMILAB-PUB-94/137-T (1994); G. Cleaver, Ohio State University Report No. OHSTPY-HEP-T-94-007 (1994); A. Aldazabal, A. Font, L.E. Ibanez and A.M. Uranga, hep-th/9410206 (1994).

[71] C.D. Froggatt and H.B. Nielsen, Nucl. Phys. B147, 277 (1979).
FIGURE CAPTIONS

**FIG. 1.** four non-zero textures resulting from the family symmetry $\Delta(48)$ and $U(1)$ symmetry, are needed for constructing fermion Yukawa coupling matrices.

**FIG. 2.** three non-zero textures resulting from the family symmetry $\Delta(48)$ and $U(1)$ symmetry, are needed for constructing right-handed Majorana neutrino mass matrix.
TABLE I. U(1) Hypercharge Quantum Number

|   | 'X' | 'u' | 'z' | B-L | \(T_{3R}\) |
|---|-----|-----|-----|-----|------------|
| q | 1   | \(\frac{1}{3}\) | \(\frac{1}{3}\) | \(\frac{1}{3}\) | 0          |
| \(u^c\) | 1   | 0   | \(\frac{5}{3}\) | \(-\frac{1}{3}\) | \(\frac{1}{2}\) |
| \(d^c\) | -3  | -\(\frac{2}{3}\) | \(\frac{7}{3}\) | \(-\frac{1}{3}\) | \(\frac{1}{2}\) |
| l  | -3  | -1  | -1  | -1  | 0          |
| \(e^c\) | 1   | \(\frac{2}{3}\) | -1  | 1   | \(\frac{1}{2}\) |
| \(\nu^c\) | 5   | \(\frac{1}{3}\) | 3   | 1   | \(\frac{1}{2}\) |
**TABLE II.** Output observables and model parameters and their predicted values with input parameters $m_e = 0.511$ MeV, $m_\mu = 105.66$ MeV, $m_\tau = 1.777$ GeV and $m_b(m_b) = 4.25$ GeV for $\alpha_s(M_Z) = 0.113$.

| Output parameters | Output values | Data \[26–28\] | Output para. | Output values |
|-------------------|---------------|----------------|--------------|---------------|
| $M_t$ [GeV]       | 182           | 175 ± 6        | $J_{CP} = A^2 \lambda^6 \eta$ | $2.68 \times 10^{-5}$ |
| $m_e(m_e)$ [GeV]  | 1.27          | 1.27 ± 0.05    | $\alpha$     | 86.28°        |
| $m_\mu$ [1GeV]    | 4.31          | 4.75 ± 1.65    | $\beta$      | 22.11°        |
| $m_\tau$ [1GeV]   | 156.5         | 165 ± 65       | $\gamma$     | 71.61°        |
| $m_d$ [1GeV]      | 6.26          | 8.5 ± 3.0      | $m_{\nu_e}$ [eV] | 2.4515       |
| $|V_{us}| = \lambda$ | 0.22        | 0.221 ± 0.003  | $m_{\nu_\mu}$ [eV] | 2.4485       |
| $\frac{|V_{ub}|}{|V_{cb}|} = \lambda \sqrt{\rho^2 + \eta^2}$ | 0.083 | 0.08 ± 0.03   | $m_{\nu_\tau}$ [eV] | $1.27 \times 10^{-3}$ |
| $\frac{|V_{cd}|}{|V_{cs}|} = \lambda \sqrt{(1 - \rho)^2 + \eta^2}$ | 0.209 | 0.24 ± 0.11   | $m_{\nu_s}$ [eV] | $2.8 \times 10^{-3}$ |
| $|V_{cb}| = A \lambda^2$ | 0.0393 | 0.039 ± 0.005 | $|V_{\nu_{\mu e}}|$ | -0.049       |
| $\lambda G$       | 1.30          | -              | $|V_{\nu_\tau}|$ | 0.000         |
| $\tan \beta = v_2/v_1$ | 2.33          | -              | $|V_{\nu_{e}}|$ | -0.049       |
| $\epsilon G$      | 0.2987        | -              | $|V_{\nu_{\tau}}|$ | -0.707       |
| $\epsilon P$      | 0.0101        | -              | $|V_{\nu_{s}}|$ | 0.038         |
| $B_K$              | 0.90          | 0.82 ± 0.10    | $M_{N_1}$ [GeV] | $\sim 333$   |
| $f_B \sqrt{B}$ [MeV] | 207          | 200 ± 70       | $M_{N_2}$ [GeV] | $1.63 \times 10^6$ |
| Re($\epsilon'/\epsilon$)/10^{-3} | 1.4 ± 1.0 | 1.5 ± 0.8 | $M_{N_3}$ [GeV] | 333         |
TABLE III. Output observables and model parameters and their predicted values with input parameters $m_e = 0.511$ MeV, $m_\mu = 105.66$ MeV, $m_\tau = 1.777$ GeV and $m_b(m_b) = 4.32$ GeV for $\alpha_s(M_Z) = 0.113$

| Output parameters   | Output values | Data [26–28] | Output para. | Output values |
|---------------------|---------------|--------------|--------------|---------------|
| $M_t$ [GeV]         | 179           | 175 ± 6      | $J_{CP} = A^2 \lambda^6 \eta$ | $2.62 \times 10^{-5}$ |
| $m_e(m_e)$ [GeV]    | 1.21          | 1.27 ± 0.05  | $\alpha$     | 86.28°        |
| $m_\mu(1\text{GeV})$ [MeV] | 4.11           | 4.75 ± 1.65  | $\beta$      | 22.11°        |
| $m_\tau(1\text{GeV})$ [MeV] | 156.5        | 165 ± 65     | $\gamma$     | 71.61°        |
| $m_\nu_e$ [eV]     | 2.4515        |              | $m_\nu_\mu$  [eV] | 2.4485        |
| $|V_{us}| = \lambda$ | 0.22          | 0.221 ± 0.003 | $m_\nu_\tau$ [eV] | 1.27 $\times 10^{-3}$ |
| $\frac{|V_{ub}|}{|V_{cb}|} = \lambda \sqrt{\rho^2 + \eta^2}$ | 0.083         | 0.08 ± 0.03  | $m_\nu_s$ [eV] | 2.8 $\times 10^{-3}$ |
| $|V_{cb}| = A \lambda^2$ | 0.0389        | 0.039 ± 0.005 | $|V_{he}|$    | -0.049        |
| $\chi^G_\ell$      | 1.20          | -            | $|V_{he}|$    | -0.049        |
| $\tan \beta = v_2/v_1$ | 2.12          | -            | $|V_{he}|$    | -0.707        |
| $\epsilon_G$       | 0.2987        | -            | $|V_{he}|$    | 0.038         |
| $\epsilon_P$       | 0.0101        | -            | $|V_{he}|$    | -0.049        |
| $B_K$               | 0.96          | 0.82 ± 0.10  | $M_{N_1}$ [GeV] | $\sim 361$    |
| $f_B\sqrt{B}$ [MeV] | 212           | 200 ± 70     | $M_{N_2}$ [GeV] | $1.77 \times 10^6$ |
| $\text{Re}(\varepsilon'/\varepsilon)/10^{-3}$ | 1.4 ± 1.0     | 1.5 ± 0.8    | $M_{N_3}$ [GeV] | 361           |
TABLE IV. Values of $\eta_i$ and $\eta_F$ as a function of the strong coupling $\alpha_s(M_Z)$

| $\alpha_s(M_Z)$ | 0.110 | 0.113 | 0.115 | 0.117 | 0.120 |
|-----------------|-------|-------|-------|-------|-------|
| $\eta_{u,d,s}$  | 2.08  | 2.20  | 2.26  | 2.36  | 2.50  |
| $\eta_c$        | 1.90  | 2.00  | 2.05  | 2.12  | 2.25  |
| $\eta_b$        | 1.46  | 1.49  | 1.50  | 1.52  | 1.55  |
| $\eta_{e,\mu,\tau}$ | 1.02  | 1.02  | 1.02  | 1.02  | 1.02  |
| $\eta_U$        | 3.26  | 3.33  | 3.38  | 3.44  | 3.50  |
| $\eta_D/\eta_E \equiv \eta_{D/E}$ | 2.01  | 2.06  | 2.09  | 2.12  | 2.16  |
| $\eta_E$        | 1.58  | 1.58  | 1.58  | 1.58  | 1.58  |
| $\eta_N$        | 1.41  | 1.41  | 1.41  | 1.41  | 1.41  |

TABLE V. Decomposition of the product of two triplets, $T_i \otimes T_j$ and $T_i \otimes \bar{T}_j$ in $\Delta(48)$. Triplets $T_i$ and $\bar{T}_i$ are simply denoted by $i$ and $\bar{i}$ respectively. For example $T_1 \otimes \bar{T}_1 = A \oplus T_3 \oplus \bar{T}_3 \equiv A3\bar{3}$, here $A$ represents singlets.

| $\Delta(48)$ | 1   | 1   | 2   | 3   | 3   |
|--------------|-----|-----|-----|-----|-----|
| 1            | I1  | A3  | I3  | 12  | 12  |
| 2            | I3  | I3  | A2  | I1  | I1 |
| 3            | 12  | I2  | I3  | 23  | A1  |

63
FIG. 1. four non-zero textures resulting from the family symmetry $\Delta(48)$ and $U(1)$ symmetry, are needed for constructing fermion Yukawa coupling matrices.
FIG. 2. three non-zero textures resulting from the family symmetry $\Delta(48)$ and $U(1)$ symmetry, are needed for constructing right-handed Majorana neutrino mass matrix.