DIMENSION IS POLYNOMIAL IN HEIGHT FOR POSETS WITH PLANAR COVER GRAPHS

JAKUB KOZIK, PIOTR MICEK, AND WILLIAM T. TROTTER

Abstract. We show that height \( h \) posets that have planar cover graphs have dimension \( O(h^6) \). Previously, the best upper bound was \( 2^{O(h^3)} \). Planarity plays a key role in our arguments, since there are posets such that (1) dimension is exponential in height and (2) the cover graph excludes \( K_5 \) as a minor.

1. Introduction

In this paper, we study finite partially ordered sets, posets for short, and we assume that readers are familiar with the basics of the subject, including chains and antichains; minimal and maximal elements; height and width; order diagrams (also called Hasse diagrams); and linear extensions. For readers who are new to combinatorics on posets, several of the recent research papers cited in our bibliography include extensive background information.

Following the traditions of the subject, elements of a poset are also called points. Recall that when \( P \) is a poset, an element \( x \) is covered by an element \( y \) in \( P \) when \( x < y \) in \( P \) and there is no element \( z \) of \( P \) with \( x < z < y \) in \( P \). We associate with \( P \) an ordinary graph \( G \), called the cover graph of \( P \), defined as follows. The vertex set of \( G \) is the ground set of \( P \), and distinct elements (now also called vertices) \( x \) and \( y \) are adjacent in \( G \) when either \( x \) is covered by \( y \) in \( P \) or \( y \) is covered by \( x \) in \( P \). We consider the edges of \( G \) oriented by the order relation of \( P \), i.e. an edge \( xy \) is oriented from \( x \) to \( y \) when \( x \) is covered by \( y \) in \( P \).

Dushnik and Miller [1] defined the dimension of a poset \( P \), denoted \( \dim(P) \), as the least positive integer \( d \) such that there are \( d \) linear orders \( L_1, \ldots, L_d \) on the ground set of

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such that \( x \leq y \) in \( P \) if and only if \( x \leq y \) in \( L_i \) for each \( i \in \{1, \ldots, d\} \). In general, there are many posets that have the same cover graph, and among them, there may be posets which have markedly different values of height, width and dimension. Indeed, it is somewhat surprising that we are able to bound any combinatorial property of a finite poset in terms of graph theoretic properties of its cover graph.

However, Streib and Trotter [11] proved that dimension is bounded in terms of height for posets that have a planar cover graph. This stands in sharp contrast with a number of well-known families of posets that have height 2 but unbounded dimension (e.g. the standard examples discussed below). The result from [11] prompted researchers to investigate in greater depth connections between dimension and graph theoretic properties of cover graphs. Subsequently, it has been shown that dimension is bounded in terms of height for posets whose cover graphs:

- Have bounded treewidth, bounded genus, or more generally exclude an apex-graph as minor [4];
- Exclude a fixed graph as a (topological) minor [17, 10];
- Belong to a fixed class with bounded expansion [7].

Moreover, the existence of bounds for dimension of posets with cover graphs in a fixed class can say something about the sparsity of the class. Joret, Micek, Ossona de Mendez, and Wiechert [3] proved that a monotone class of graphs is nowhere dense if and only if for every \( h \geq 1 \) and every \( \varepsilon > 0 \), posets of height \( h \) with \( n \) elements whose cover graphs are in the class have dimension \( O(n^\varepsilon) \).

The best upper bound to date on dimension in terms of height for posets that have planar cover graphs is \( 2^{O(h^3)} \). This result can be extracted from [3] via connections between dimension for posets and weak-coloring numbers of their cover graphs. We will give additional details on this work in the next section.

Our main theorem improves this exponential bound to one which is polynomial in \( h \).

**Theorem 1.** If \( P \) is a poset of height \( h \) and the cover graph of \( P \) is planar, then \( \dim(P) = O(h^6) \).

Planarity plays a crucial role in the existence of a polynomial bound. In [6], Joret, Micek and Wiechert show that for each even integer \( h \geq 2 \), there is a height \( h \) poset \( P \) with dimension at least \( 2^{h/2} \) such that the cover graph of \( P \) excludes \( K_5 \) as a minor.

To discuss lower bounds, we pause to give the following construction which first appears in [1]. For each \( n \geq 2 \), let \( S_n \) be the height 2 poset with \( \{a_1, a_2, \ldots, a_n\} \) the set of minimal elements, \( \{b_1, b_2, \ldots, b_n\} \) the set of maximal elements, and \( a_i < b_j \) in \( S_n \) if and only if \( i \neq j \). Posets in the family \( \{S_n : n \geq 2\} \) are now called standard examples, as \( \dim(S_n) = n \) for every \( n \geq 2 \).

To date, the best lower bound for the maximum dimension of a height \( h \) poset with a planar cover graph is \( 2h - 2 \), and this bound comes from the “double wheel” construction
Figure 1. We illustrate the double wheel construction when \( h = 5 \). Note that the elements \( a_1, \ldots, a_{10} \) and \( b_1, \ldots, b_{10} \) induce a standard example, so the dimension of the depicted poset is at least 10. On the other hand, the height of \( P \) is 5.

given in [6], and illustrated here in Figure 1. To avoid clutter, we do not show arrowheads in our figures. Instead, we indicate directions using color and accompanying narrative. In this figure, the black edges are oriented in each individual wheel from outside to inside. The elements of \( \{a_1, \ldots, a_n\} \) are minimal elements so the red edges are oriented “left-to-right” and the blue edges are oriented “right-to-left.”

Requiring that the diagram of a poset \( P \) is planar is a stronger restriction than requiring that the cover graph of \( P \) is planar. Accordingly among posets that have planar cover graphs, some but not all also have planar order diagrams. Among the class of posets with planar diagrams, Joret, Micek and Wiechert [6] showed that \( \dim(P) \leq 192h + 96 \) when \( P \) has height \( h \).

The remainder of this paper is organized as follows. In the next section, we prove three reductions to simpler problems, and we give essential background material. The proof of Theorem 1 is given in the following two sections, and we close with brief comments on challenging open problems that remain.

2. Preliminary Reductions and Background Material

We list below some elementary, and well known, properties of dimension.

(i) Dimension is monotonic, i.e., if \( Q \) is an induced subposet of \( P \), then \( \dim(Q) \leq \dim(P) \).

(ii) The dual of a poset \( P \) is the poset \( P' \) on the same ground set of \( P \) with \( x < y \) in \( P' \) if and only \( x > y \) in \( P \). Then \( \dim(P) = \dim(P') \).

For the balance of this preliminary section, we fix a poset \( P \). We let \( \text{Min}(P) \) and \( \text{Max}(P) \) denote, respectively, the set of minimal elements and the set of maximal elements of \( P \).
When \( x \) is in \( P \), we let \( U_P(x) \) consist of all elements \( u \) such that \( x < u \) in \( P \). Dually, \( D_P(x) \) consist of all elements \( u \) such that \( x > u \) in \( P \).

We write \( x \parallel_P y \) (also \( x \parallel y \) in \( P \)) when \( x \) and \( y \) are incomparable. Also, we let \( \text{Inc}(P) \) denote the set of all ordered pairs \((x, y)\) with \( x \parallel_P y \). We will assume \( \text{Inc}(P) \neq \emptyset \); otherwise \( P \) is a chain and \( \text{dim}(P) = 1 \). When \((x, y) \in \text{Inc}(P) \) and \( L \) is a linear extension of \( P \), we say that \( L \) reverses \((x, y)\) when \( y < x \) in \( L \). A set \( I \subseteq \text{Inc}(P) \) is reversible if there is a linear extension \( L \) of \( P \) which reverses every pair in \( I \). Vacuously, the empty set is reversible. We then define \( \text{dim}_P(I) \) as the least \( d \geq 1 \) such that \( I \) can be covered by \( d \) reversible sets. It is easily seen that \( \text{dim}(P) \) is equal to \( \text{dim}_P(\text{Inc}(P)) \).

Given sets \( A, B \subseteq P \), we let \( \text{Inc}_P(A, B) \) be the set of pairs \((a, b) \in \text{Inc}(P) \) with \( a \in A \) and \( b \in B \). We use the abbreviation \( \text{dim}_P(A, B) \) for \( \text{dim}_P(\text{Inc}(A, B)) \). Again, \( \text{dim}_P(A, B) = 1 \) when \( \text{Inc}_P(A, B) = \emptyset \). When the meaning of the poset \( P \) remains fixed in the discussion, we will drop the subscript in the notation \( \text{dim}_P(I) \) and \( \text{dim}_P(A, B) \).

A sequence \(((x_1, y_1), \ldots, (x_k, y_k))\) of pairs from \( \text{Inc}(P) \) with \( k \geq 2 \) is an alternating cycle of size \( k \) if \( x_i \leq_P y_{i+1} \) for all \( i \in \{1, \ldots, k\} \), cyclically (so \( x_k \leq_P y_1 \) is required). Observe that if \(((x_1, y_1), \ldots, (x_k, y_k))\) is an alternating cycle in \( P \), then any subset \( I \subseteq \text{Inc}(P) \) containing all the pairs on this cycle is not reversible.

An alternating cycle \(((x_1, y_1), \ldots, (x_k, y_k))\) is strict if we have \( x_i \leq_P y_j \) if and only if \( j = i + 1 \) (cyclically). Note that in this case, \( \{x_1, \ldots, x_k\} \) and \( \{y_1, \ldots, y_k\} \) are \( k \)-element antichains. Note also that in alternating cycles, we allow that \( x_i = y_{i+1} \) for some or even all values of \( i \). Trotter and Moore [14] observed the following: A subset \( I \subseteq \text{Inc}(P) \) is reversible if and only if \( I \) contains no strict alternating cycle.

When \( x <_P y \), a sequence \( W = (u_0, u_1, \ldots, u_t) \) is called a witnessing path (from \( x \) to \( y \)) when \( u_0 = x \), \( u_t = y \) and \( u_i \) is covered by \( u_{i+1} \) in \( P \) for each \( i \in \{0, 1, \ldots, t-1\} \).

The following elementary lemma allows us to concentrate our attention on incomparable pairs from \( \text{Inc}(\text{Min}(P), \text{Max}(P)) \). See for instance [5, Observation 3] for a proof.

**Lemma 2** (Reduction to min-max). For every poset \( P \), there is a poset \( Q \) containing \( P \) as an induced subposet such that

(i) The height of \( P \) is the same as the height of \( Q \);

(ii) The cover graph of \( Q \) is obtained from the cover graph of \( P \) by adding some degree 1 vertices; and

\[
\text{dim}(P) \leq \text{dim}_Q(\text{Min}(Q), \text{Max}(Q)).
\]

2.1. **Constrained Subsets and Weak-Coloring Numbers.** Let \( P \) be a poset. We say that a subset \( I \subseteq \text{Inc}(P) \) is singly constrained in \( P \) if there is an element \( x_0 \in P \) such that \( x_0 < b \) in \( P \) for every \((a, b) \in I \). To identify the element \( x_0 \), we will also say \( I \) is singly constrained by \( x_0 \).
The following lemma was used first in [11] for posets with planar cover graphs and in a more complex form in [5]. The underlying principle is the concept of unfolding, which is an analogue of breadth first search for posets.

**Lemma 3 (Reduction to singly constrained).** For every non-empty poset $P$, there exists a poset $Q$ such that

(i) The height of $Q$ is at most the height of $P$.
(ii) The cover graph of $Q$ is a minor of the cover graph of $P$.
(iii) There is a minimal element $x_0$ in $Q$ such that $x_0 \leq q$ in $Q$ for all $q \in \text{Max}(Q)$, and
\[
\dim_P(\text{Min}(P), \text{Max}(P)) \leq 2 \dim_Q(\text{Min}(Q), \text{Max}(Q)).
\]

In particular, the set $\text{Inc}(\text{Min}(Q), \text{Max}(Q))$ is singly constrained by $x_0$ in $Q$.

We say that a subset $I$ of $\text{Inc}(P)$ is **doubly constrained** in $P$ when there is a pair of elements $(x_0, y_0)$ in $P$ such that

(i) $x_0 < y_0$ in $P$,
(ii) $x_0 < b$ in $P$ for every $(a, b) \in I$, and
(iii) $a < y_0$ in $P$ for every $(a, b) \in I$.

As before, we will also say that $I$ is doubly constrained by $(x_0, y_0)$.

We would very much like to reduce to the case where we are bounding $\dim(I)$ when $I \subseteq \text{Inc}(P)$ is doubly constrained. Unfortunately, Lemma 3 will not be of assistance. Instead, we will use a different reduction, one that will cost us an $O(h^3)$-factor in the final bound.

The length of a path in a graph is the number of its edges. For two vertices $u$ and $v$ in a graph $G$, an $u$–$v$ path is a path in $G$ with ends in $u$ and $v$. Let $G$ be a graph, and let $\sigma$ be an ordering of the vertices of $G$. For $r \in \{0, 1, 2, \ldots \} \cup \{\infty\}$ and two vertices $u$ and $v$ of $G$, we say that $u$ is weakly $r$-reachable from $v$ in $\sigma$, if there exists an $u$–$v$ path of length at most $r$ such that for every vertex $w$ on the path, $u \leq_{\sigma} w$. The set of vertices that are weakly $r$-reachable from a vertex $v$ in $\sigma$ is denoted by $\text{WReach}_r[G, \sigma, v]$. The weak $r$-coloring number $\text{wcol}_r(G)$ of $G$ is defined as
\[
\text{wcol}_r(G) := \min_{\sigma} \max_{v \in V(G)} |\text{WReach}_r[G, \sigma, v]|.
\]

where $\sigma$ ranges over the set of all vertex orderings of $G$. We call $\text{wcol}_r(G)$ the $r$-th weak coloring number of $G$.

Weak coloring numbers were originally introduced by Kierstead and Yang [9] as a generalization of the degeneracy of a graph (also known as the coloring number). Since then, they have been applied in several novel situations (see Zhu [18] and Van den Heuvel et al. [16], for examples). We also have good bounds on weak coloring numbers. For planar graphs, van den Heuvel et al. [15] have shown that the $r$-th weak coloring number
is at most $(r^2 + 2) \cdot (2r + 1) = O(r^3)$. See also a recent paper [2] with a lower bound in $\Omega(r^2 \log r)$.

Here is a lemma on weak coloring numbers from [3] that will play an important role in the reduction to the doubly constrained case.

**Lemma 4.** Let $P$ be a height $h$ poset, let $G$ be the cover graph of $P$, and let $c := \text{wcol}_{4h-4}(G)$. If $I \subseteq \text{Inc}(P)$, then there is an element $z_0 \in P$ such that the set $J = \{(a,b) \in I : a < z_0 \text{ in } P\}$ satisfies

$$\dim(J) \geq \frac{\dim(I)}{c} - 2.$$

We then have the following immediate corollary.

**Corollary 5.** Let $P$ be a poset with a planar cover graph, and let $x_0$ be an element of $P$ such that $x_0 < b$ in $P$ for every $b \in \text{Max}(P)$. Let $I$ be a subset of $\text{Inc}(P)$ that is singly constrained by $x_0$. Then there is a set $J \subseteq I$ and an element $y_0$ of $P$ such that $J$ is doubly constrained by $(x_0,y_0)$ in $P$ and

$$\dim(I) = O(h^3) \cdot \dim(J).$$

**Proof.** Let $G$ be the cover graph of $P$. Apply Lemma 4 with $c = \text{wcol}_{4h-4}(G) = O(h^3)$ to obtain the element $z_0$ and the set $J \subseteq I$ such that $J = \{(a,b) \in I : a < z_0 \text{ in } P\}$ and $\dim(J) \geq \dim(I)/c - 2$. Let $y_0$ be any maximal element with $z_0 \leq y_0$ in $P$. Since $y_0 \in \text{Max}(P)$ we have $x_0 < y_0$ in $P$. Evidently, $J$ is doubly constrained by the pair $(x_0,y_0)$. The inequality from Lemma 4 becomes $\dim(I) \leq c \cdot (2 + \dim(J))$, and with this observation, the proof of the corollary is complete. \hfill $\square$

2.2. A Reduction to the Doubly Exposed Case. Let $P$ be a poset and let $I \subseteq \text{Inc}(P)$ that is doubly constrained by $(x_0,y_0)$ in $P$. For convenience, we will say that a sequence $(P,x_0,y_0,I)$ is **doubly constrained**.

Let $(P,x_0,y_0,I)$ be doubly constrained. Furthermore, suppose that the cover graph $G$ of $P$ is planar. Fix a plane drawing $D$ of $G$ with $x_0$ on the exterior face. Also, we append an imaginary edge $e_{-\infty}$ attached to $x_0$ in the exterior face.

Let $z$ be an element of $P$, and let $e_0$ be an edge of $G$ incident to $z$. With $(z,e_0)$ fixed, we consider all the edges incident to $z$ ordered by the clockwise traversal around $z$ starting at $e_0$—this constitutes the **left-to-right** $(z,e_0)$-**ordering** of edges around $z$. Thus, $e$ is left of $e'$ in this ordering if the clockwise traversal around $z$ starting at $e_0$ visits $e$ before $e'$, see Figure 2.

Let $b$ and $b'$ be distinct elements of $P$. Also, let $W$ and $W'$ be paths in $G$ from $x_0$ to $b$ and $x_0$ to $b'$, respectively. We say that $W$ and $W'$ are $x_0$-**consistent** if there is an element $z \not\in \{b,b'\}$ common to $W$ and $W'$ such that (1) $x_0Wz = x_0W'z$; and (2) $z$ is the only vertex common to $zWb$ and $zW'b$. If $z \neq x_0$, let $e_0$ be the last edge of $x_0Wz$. If $z = x_0$, let $e_0$ be the imaginary edge $e_{-\infty}$. Let $e$ and $e'$ be the first edge of $zWb$ and
Figure 2. Left: The edges incident to \( z \) are enumerated with respect to the \((z, e_0)\)-ordering. Right: The paths \( W \) and \( W' \) are \( x_0 \)-consistent, and \( W \) is \( x_0 \)-left of \( W' \).

For the remainder of the paper, we say that a sequence \((P, G, x_0, y_0, I, \mathcal{D})\) is **doubly exposed** if

(i) \( P \) is a poset and \( G \) is the cover graph of \( P \);
(ii) \( x_0 \) and \( y_0 \) are elements in \( P \) with \( x_0 < y_0 \);
(iii) \( I \) is doubly constrained by \((x_0, y_0)\) in \( P \);
(iv) \( G \) is planar and \( \mathcal{D} \) is a plane drawing of \( G \) with \( x_0 \) and \( y_0 \) on the exterior face.

**Lemma 6.** Let \((P, x_0, y_0, I)\) be doubly constrained, and assume that \( P \) has a planar cover graph and height at most \( h \). Then there exists a sequence \((Q, G, u_0, v_0, J, \mathcal{D})\) that is doubly exposed and

\[
\dim_P(I) \leq 2(h - 1) \dim_Q(J) + 1.
\]

**Proof.** Fix a plane drawing of the cover graph of \( P \) with \( x_0 \) on the exterior face. Fix also a witnessing path \( W^* \) from \( x_0 \) to \( y_0 \) and refer to this chain as the spine. Label the points on the spine as \( \{u_0, u_1, \ldots, u_t\} \) with \( x_0 = u_0, y_0 = u_t \) and \( u_i \) covered by \( u_{i+1} \) in \( P \) for each \( i \in \{0, \ldots, t - 1\} \). Note that \( t \leq h - 1 \).

Recall that \( I \subseteq \text{Inc}(D_P(y_0), U_P(x_0)) \). Note that the set \( S = \{(a, b) \in I \mid b \in W^*\} \) is reversible (as you cannot build a strict alternating cycle with pairs within the set). Let \( J = I - S \). Thus, \( \dim(I) \leq \dim(J) + 1 \).

For each \( b \in U_P(x_0) \), let \( \tau(b) \) be the largest integer \( i \) so that \( u_i <_P b \). Note that \( 0 \leq \tau(b) \leq t - 1 \). Let \( W_b \) be a witnessing path from \( x_0 \) to \( b \) such that \( W_b \) shares the initial segment \((u_0, u_1, \ldots, u_{\tau(b)})\) with the spine. Note that \( W^* \) and \( W_b \) are \( x_0 \)-consistent for each \( b \in U_P(x_0) \) as long as \( b \not\in W^* \).
We partition $U_P(x_0) - W^*$ into $B_{\text{left}}$ and $B_{\text{right}}$ in such a way that $b$ is assigned to the set $B_{\text{left}}$ if $W_b$ is $x_0$-left of $W^*$. Dually, we assign $b$ to $B_{\text{right}}$ if $W_b$ is $x_0$-right of $W^*$.

For each $a \in D_P(y_0)$, let $\tau(a)$ be the least integer $i$ so that $a < u_i$ in $P$. Now we have $1 \leq \tau(a) \leq t$. We partition the set $D_P(y_0)$ into $A_1 \cup A_2 \cup \cdots \cup A_t$ by assigning $a$ to $A_i$ when $\tau(a) = i$. Clearly,

$$\dim(I) \leq 1 + \dim(J) \leq 1 + \sum_{s \in \{1, \ldots, t\}} \sum_{\text{dir} \in \{\text{left, right}\}} \dim(A_s, B_{\text{dir}}).$$

It follows that there is some $s \in \{1, \ldots, t\}$ and $\text{dir} \in \{\text{left, right}\}$ so that

$$\dim(I) \leq 1 + 2(h-1) \dim(A_s, B_{\text{dir}}).$$

We assume that $\text{dir} = \text{right}$. From the details of the argument, it will be clear that the proof is symmetric in the other case.

We say that an edge $e = u_i v$ in the cover graph of $P$ is bad if $0 \leq i < s$, $v$ is not on the spine, and $e$ is left of $u_i u_{i+1}$ in the $(u_i, u_{i-1} u_i)$-ordering. (Note that $e = u_i v$ in the cover graph means $u_i < v$ in $P$ or $v < u_i$ in $P$.) We then define a poset $Q$ having the same ground set as $P$ with $x \leq y$ in $Q$ if and only if there is a witnessing path in $P$ from $x$ to $y$ avoiding bad edges.

We claim that for every $a \in A_s$ and every $b \in B_{\text{right}}$, we have $a \leq b$ in $Q$ if and only if $a \leq b$ in $P$. The forward implication is obvious. To see the backward one, let $a \in A_s$ and $b \in B_{\text{right}}$ with $a < b$ in $P$. Then let $W$ be a witnessing path from $a$ to $b$ in $P$. This path cannot use a bad edge as this would make $a < u_i$ in $P$ for some $i \in \{1, \ldots, s-1\}$ contradicting $a \in A_s$. Therefore, the claim holds and also $\dim(A_s, B_{\text{right}})$ in $Q$ is the same as $\dim(A_s, B_{\text{right}})$ in $P$.

Note that the diagram and the cover graph of $Q$ are obtained simply by removing the bad edges from the diagram and cover graph, respectively, of $P$. It follows that the cover graph of $Q$ is planar. Furthermore, $x_0$ and $u_s$ are on the same face, and the set $\text{Inc}(A_s, B_{\text{right}})$ is doubly exposed by the pair $(x_0, u_s)$. With this observation, the proof of the lemma is complete. \hfill \square

Summarizing, we can combine Lemma 3, Corollary 5, and Lemma 6 to obtain:

**Corollary 7.** Let $P$ be a height $h$ poset with a planar cover graph. Then there is a sequence $(Q, G, x_0, y_0, I, D)$ that is doubly exposed such that $Q$ has height at most $h$ and

$$\dim(P) = O(h^4) \cdot \dim_Q(I).$$

The reader may note that the argument for the reduction actually proves that we may assume that the pairs in $I$ are min-max pairs. In order that our results can be applied in a more general setting, we elect to proceed with only the assumption that $I \subseteq \text{Inc}(Q)$ is doubly exposed by $(x_0, y_0)$.

We are now ready to begin the proof of our main theorem.
3. Large Standard Examples in the Doubly Exposed Case

We pause here to make the following important comment: Height plays no role in the arguments given in this section.

Throughout this section, we discuss a sequence \((P,G,x_0,y_0,I,D)\) that is doubly exposed.

We add to the drawing \(D\) an imaginary edge \(e_{-\infty}\) in the exterior face to \(x_0\), and we add an imaginary edge \(e_{+\infty}\) in the exterior face to \(y_0\).

Let \(B\) be an antichain in \(P\) with \(B \subseteq U_P(x_0)\). Let \(W_b\) be a witnessing path from \(x_0\) to \(b\), for each \(b \in B\). We say that the family \(\{W_b | b \in B\}\) is \(x_0\)-consistent if \(W_b\) and \(W_{b'}\) are \(x_0\)-consistent whenever \(b\) and \(b'\) are distinct elements of \(B\). In this case, the edges of the paths in this family form a tree. We call this tree a witnessing tree for \(B\). It is natural to use a single symbol, such as \(T\), to denote this tree. Now for each \(b \in B\), the path \(W_b\) becomes \(x_0Tb\). When \(T\) is a witnessing tree for \(B\), we write \(b <_T b'\) when \(x_0Tb\) is \(x_0\)-left of \(x_0Tb'\). Note that \(<_T\) is a linear order on elements of \(B\).

We observe that if \(B \subsetneq B'\) are both antichains in \(U_P(x_0)\), and \(T\) is a witnessing tree for \(B\), then there is a witnessing tree \(T'\) for \(B'\) such that \(x_0T'b = x_0Tb\) for all \(b \in B\). In this case, \(b <_T b'\) if and only if \(b <_{T'} b'\) for all \(b, b' \in B\).

Let \(b\) and \(b'\) be incomparable elements of \(U_P(x_0)\). We say that \(b\) is \(x_0\)-left of \(b'\) if \(W\) is \(x_0\)-left of \(W'\) whenever \(W\) and \(W'\) are \(x_0\)-consistent witnessing paths from \(x_0\) to \(b\) and \(b'\), respectively. When \(b\) is \(x_0\)-left of \(b'\), we also say \(b'\) is \(x_0\)-right of \(b\).

**Proposition 8.** Let \(b, b', b''\) be elements of \(U_P(x_0)\). If \((b,b'),(b',b'') \in \text{Inc}(P)\), \(b\) is \(x_0\)-left of \(b'\) and \(b'\) is \(x_0\)-left of \(b''\), then \((b,b'') \in \text{Inc}(P)\) and \(b\) is \(x_0\)-left of \(b''\).

**Proof.** We first show that \((b,b'') \in \text{Inc}(P)\). Let \(W'\) be an arbitrary witnessing path from \(x_0\) to \(b'\). Then, let \(W\) and \(W''\) be witnessing paths from \(x_0\) to \(b\) and \(b''\), respectively, such that \(\{W,W'\}\) is \(x_0\)-consistent and \(\{W',W''\}\) is \(x_0\)-consistent. Let \(w\) be the least element of \(W\) which is not on \(W'\). Also, let \(w''\) be the least element of \(W''\) which is not on \(W'\). We claim that \(ww'b\) and \(ww''b''\) are disjoint. To the contrary suppose that \(z\) is a common element. Now, \(x_0WzW''b''\) and \(W'\) are \(x_0\)-consistent and \(x_0WzW''b''\) is \(x_0\)-left of \(W'\), contradicting the assumption that \(b'\) is \(x_0\)-left of \(b''\).

Now, suppose \(b < b''\) in \(P\) and let \(U\) be a witnessing path from \(b\) to \(b''\). Then \(x_0WbUb''\) and \(W'\) are \(x_0\)-consistent, and \(x_0WbUb''\) is \(x_0\)-left of \(W''\), contradicting that \(b'\) is \(x_0\)-left of \(b''\). A symmetric shows that \(b'' \not< b'\) in \(P\). Thus, \((b,b'') \in \text{Inc}(P)\).

Now we will argue that \(b\) is \(x_0\)-left of \(b''\). To the contrary, let \(W\) and \(W''\) be witnessing paths from \(x_0\) to \(b\) and \(b''\), respectively, such that \(W\) and \(W''\) are \(x_0\)-consistent, and \(W''\) is \(x_0\)-left of \(W\). Choose a witnessing \(W'\) from \(x_0\) to \(b'\) such that the union of \(W, W', W''\) form a witnessing tree. Then \(W''\) is \(x_0\)-left of \(W''\) and \(x_0\)-right of \(W\). Clearly, this is impossible. ∎
In the doubly exposed setting $\mathcal{F} = (G, P, x_0, y_0, I, \mathcal{D})$, we have two distinguished elements on the exterior face, namely $x_0$ and $y_0$. We proceed with a compact description of necessary definitions and notations involving $y_0$ that are dual to those introduced for $x_0$.

Let $a$ and $a'$ be distinct elements of $P$. Also, let $W$ and $W'$ be paths in $G$ from $a$ to $y_0$ and $a'$ to $y_0$, respectively. We say that $W$ and $W'$ are $y_0$-consistent if there is an element $z \notin \{a, a'\}$ common to $W$ and $W'$ such that (1) $y_0Wz = y_0W'z$ and (2) $z$ is the only vertex common to $zWa$ and $zW'a'$. If $z \neq y_0$, let $e_0$ be the last edge of $y_0Wz$. If $z = y_0$, let $e_0$ be the imaginary edge $e_{+\infty}$. Let $e$ and $e'$ be the first edge of $zWa$ and $zW'a'$, respectively. Now, we say that $W$ is $y_0$-left ($y_0$-right) of $W'$ if $e$ is left (right) of $e'$ in the $(z, e_0)$-ordering. Note that either $W$ is $y_0$-left of $W'$ or $W$ is $y_0$-right of $W'$.

Let $A$ be an antichain in $P$ with $A \subseteq D_P(y_0)$. Let $W_a$ be a witnessing path from $a$ to $y_0$, for each $a \in A$. We say that the family $\{W_a \mid a \in A\}$ is $y_0$-consistent if $W_a$ and $W_a'$ are $y_0$-consistent whenever $a$ and $a'$ are distinct elements of $A$. In this case, the edges of the paths in this family form a tree. Let $S$ be the obtained tree, we call this tree a witnessing tree for $A$. Now for each $a \in A$, the path $W_a$ becomes $aS_{y_0}$. When $S$ is a witnessing tree for $A$, we write $a <_S a'$ when $aS_{y_0}$ is $y_0$-left of $a'S_{y_0}$.

Let $a$ and $a'$ be incomparable elements of $D_P(y_0)$. We say that $a$ is $y_0$-left of $a'$ if $W$ is $y_0$-left of $W'$ whenever $W$ and $W'$ are $y_0$-consistent witnessing paths from $a$ to $y_0$ and from $a'$ to $y_0$, respectively. When $a$ is $y_0$-left of $a'$, we also say $a'$ is $y_0$-right of $a$. We state for emphasis a dual statement to Proposition 8.

**Proposition 9.** Let $a, a', a''$ be elements of $D_P(y_0)$. If $(a, a'), (a', a'') \in \text{Inc}(P)$, $a$ is $y_0$-left of $a'$ and $a'$ is $y_0$-left of $a''$, then $(a, a'') \in \text{Inc}(P)$ and $a$ is $y_0$-left of $a''$.

When $C$ is a simple closed curve in the plane, it splits the points of the plane not on $C$ into those that are in the interior of the region bounded by $C$ and those in the exterior of this region. In the discussion to follow, we will abuse terminology slightly and say that a point not on $C$ is either in the interior of $C$ or it is in the exterior of $C$, dropping the reference to the region bounded by $C$. We then fix a simple closed curve $C$ such that $x_0$ and $y_0$ are on $C$ while all other vertices and edges of $G$ are in the interior of $C$.

Let $N$ be a path from $x_0$ to $y_0$ in $G$. Then the clockwise portion of $C$ beginning at $x_0$ and ending at $y_0$ together with $N$ traversed backwards is a simple closed curve. Elements of $G$ that are in the interior of this curve are said to be left of $N$. Analogously, elements of $G$ that are not in $N$ and not left of $N$ are said to be right of $N$. These conventions are illustrated in Figure 3.

The following self-evident proposition is stated for emphasis.

**Proposition 10.** Let $N$ be a path from $x_0$ to $y_0$ in $G$ and let $u$ be a vertex on $N$. If $u = x_0$, set $e_0$ to be the imaginary edge $e_{-\infty}$. Otherwise, set $e_0$ to be the last edge of $x_0Nu$. If $u = y_0$, set $e_1$ to be the imaginary edge $e_{+\infty}$. Otherwise, set $e_1$ to be the first edge of $uNy_0$. Let $N'$ be a non-trivial path starting at $u$ and let $e$ be the first edge of $N'$.

(i) If $e$ left (right) of $e_1$ in the $(u, e_0)$-ordering, then $e$ is left (right) of $N$. 
(ii) All edges and vertices of $N'$, except $u$, are on the same side of $N$ unless there is a vertex $u' \neq u$ that is common to $N$ and $N'$.

Let $N$ be a path from $x_0$ to $y_0$ in $G$. We call $N$ a separating path if there exist (not necessarily distinct) elements $u$, $v$ of $N$ such that (1) $v \leq u$ in $P$ and $v$ does not occur before $u$ in traversing $N$ from $x_0$ to $y_0$; (2) $x_0Nu$ and $vNy_0$ are witnessing paths; and (3) $uNv$ is a witnessing path from $v$ to $u$ traversed backwards. To identify the elements $u$ and $v$ in this definition, we will write $N$ as $N(u,v)$. When $N = N(u,v)$ is a separating path, we refer to $x_0Nu$ as the blue part of $N$. Analogously, $vNy_0$ is the red part of $N$, while $uNv$ is the black part of $N$. Note that if $z$ is on the path $N$, and $z$ is either on the red part or the black part, then $v \leq_P z$. Symmetrically, if $z$ is on the blue part or the black part, then $z \leq_P u$.

A separating path $N(u,v)$ is associated with a comparability $a \leq b$ in $P$ when $a \leq v$, and $u \leq b$ in $P$. The next proposition states that for all $a$, $b$ with $a \leq b$ in $P$, and all $W$, $W'$ witnessing paths from $x_0$ to $b$ and $b'$, respectively, we can find a separating path associated with $a \leq b$ that aligns well with $W$ and $W'$. See Figure 3.

**Proposition 11.** Let $(a, b) \in D_P(y_0) \times U_P(x_0)$ with $a \leq b$ in $P$. Let $W$ be a witnessing path from $a$ to $y_0$, and let $W'$ be a witnessing path from $x_0$ to $b$. Then there exists a separating path $N = N(u,v)$ associated with $a \leq b$ in $P$ such that (1) $u \in W'$ and the blue part of $N$ is $x_0W'u$; and (2) $v \in W$ and the red part of $N$ is $vWy_0$.

**Proof.** Suppose first that $W$ and $W'$ intersect and let $u$ be a common point. Then $N = x_0W'uWy_0$ satisfies the statement. Otherwise, let $u$ be the least point in $P$ on $W$ such that $a < u$ in $P$. Then take $v$ to be the greatest point in $P$ on $W'$ such that $v < u$ in $P$. Set $N_2$ to be an arbitrary witnessing path from $v$ to $u$ traversed backwards. Then $x_0W'uN_2vWy_0$ is the desired separating path. \[\square\]
The following elementary proposition has four symmetric statements: two for $D_P(y_0)$ and two for $U_P(x_0)$.

**Proposition 12.** Suppose $a_1, a_2 \in D_P(y_0), b \in U_P(x_0),$ and $a_2 < b$ in $P$. For $i \in \{1, 2\}$, let $W_i$ be a witnessing path from $a_i$ to $y_0$, and suppose that $W_1$ and $W_2$ are $y_0$-consistent. Suppose further that $W_1$ is $y_0$-left of $W_2$. Let $N = N(u, v)$ be a separating path associated with $a_2 < b$ in $P$ with $v$ on $W_2$ and $vNy_0 = vW_2y_0$.

(i) If $a_1 \parallel b$ in $P$, then $a_1$ is right of $N$.
(ii) If $a_1$ is not right of $N$, then $a_1 \leq b$ in $P$, and $W_1$ contains a point from $x_0Nv$.

**Proof.** Let $v'$ be the least point of $P$ common to $vNy_0$ and $W_1$. If $v' = v$, then $a_1 \leq v' = v \leq u \leq b$ in $P$. Therefore, $a_1 \leq b$ in $P$ and $W_1$ contains $v$ which belongs to $x_0Nv$, as desired. If $v' \neq v$, then $v < v'$ in $P$. Let $e' = (w, v')$ be the edge of $W_1$ that occurs immediately before $v'$. Then since $W_1$ is $y_0$-left of $W_2$, Proposition 10 implies that $e'$ is right of $N$. If $a_1$ is not right of $N$, there exists a vertex $z$ of $a_1W_1w$ which is in $N$. By the choice of $v'$, $z$ must belong to the $x_0Nv$. Therefore, $a_1 \leq z \leq u \leq b$ in $P$, so $a_1 \leq b$ in $P$, as desired.

**Proposition 13.** Let $N$ be a separating path associated with $a \leq b$ in $P$. If $w \prec_P z$, $w$ is on one side of $N$ and $z$ is on the other, then either $w \prec_P b$ or $a \prec_P z$.

**Proof.** Let $W$ be a witnessing path from $w$ to $z$. Then $W$ and $N$ must intersect. Let $q$ be a common point. If $q$ is on the blue part of $N$, then $w \prec_P q \leq_P b$. If $q$ is on the red part of $N$, then $a \leq_P q < z$. If $q$ is on the black part of $N$, then both $w \prec_P b$ and $a \prec_P z$ hold.

Let $z$ be an element of $P$ and $X$ be a subset of elements of $P$. We say that $z$ is **enclosed by $X$** if there is a cycle $D$ in $G$ such that (1) all vertices of $D$ are in $X$; and (2) $z$ is in the interior of $D$.

**Proposition 14.** Let $((a_1, b_1), \ldots, (a_k, b_k))$ be a strict alternating cycle of incomparable pairs from $D_P(y_0) \times U_P(x_0)$. Also, let $i, j$ be distinct integers from $[k]$. Then the following statements hold:

(i) \(a_i\) is not enclosed by $U_P(a_j)$;
(ii) \(b_i\) is not enclosed by $D_P(b_j)$;
(iii) either $a_i$ is $y_0$-left of $a_j$ or $a_i$ is $y_0$-right $a_j$;
(iv) either $b_i$ is $x_0$-left of $b_j$ or $b_i$ is $x_0$-right of $b_j$;
(v) $a_i$ is $y_0$-left of $a_j$ if and only if $b_{i+1}$ is $x_0$-right of $b_{j+1}$ (cyclically).

The statement of Propositions 14 and 15 are illustrated at Figure 4.

**Proof.** Suppose first that $a_i$ is enclosed by $U_P(a_j)$ as evidenced by cycle $D \subseteq U_P(a_j)$ in $G$ with $a_i$ in the interior of $D$. Note also that $x_0$ is not in the interior of $D$ as it is on the exterior face. Consider a witnessing path $W$ from $x_0$ to $b_{i+1}$. If $W$ intersects $D,$
We continue with an argument for (v). Note that the arguments given for (iii) and (iv) show that if \( b_{i+1} \) is \( x_0 \)-left of \( b_{j+1} \), then \( a_i \) is \( y_0 \)-right of \( a_j \). Also if \( b_{i+1} \) is \( x_0 \)-right of \( b_{j+1} \), then \( a_i \) is \( y_0 \)-left of \( a_j \), as desired. 

The following special case of the preceding proposition is stated for emphasis.

**Proposition 15.** If \( ((a_1, b_1), (a_2, b_2)) \) is a strict alternating cycle of incomparable pairs from \( D_P(y_0) \times U_P(x_0) \), then \( a_1 \) is \( y_0 \)-left of \( a_2 \) if and only if \( b_1 \) is \( x_0 \)-left of \( b_2 \).
We define an auxiliary digraph $H$ whose vertex set is $Inc(D_P(y_0), U_P(x_0))$. When $(a, b)$ and $(a', b')$ are vertices in $H$, we have a directed edge from $(a, b)$ to $(a', b')$ in $H$ when $((a, b), (a', b'))$ is an alternating cycle, and $a$ is $y_0$-left of $a'$ (therefore, $b$ is $x_0$-left of $b'$ by Proposition 15).

When $n \geq 1$, a sequence $((a_1, b_1), \ldots, (a_n, b_n))$ of vertices from $H$ is a directed path of length $n$ in $H$, if there is a directed edge in $H$ from $(a_i, b_i)$ to $(a_{i+1}, b_{i+1})$ for all $i$ such that $1 \leq i \leq n - 1$. Note that when $(a, b)$ is a vertex in $H$, we consider $((a, b))$ as a directed path of length one. Note further that $H$ is acyclic.

The next proposition implies a notion of transitivity for directed paths in $H$, and this concept will prove to be fundamentally important.

**Proposition 16.** Let $n \geq 3$ and let $((a_1, b_1), \ldots, (a_n, b_n))$ be a directed path in $H$. Then $((a_i, b_i), (a_j, b_j))$ is an edge in $H$ for all $i, j$ with $1 \leq i < j \leq n$. In particular, these pairs form a copy of the standard example $S_n$.

**Proof.** Using induction, it is clear that the lemma holds in general if it holds when $n = 3$. Now there are two statements that need to be proved: (1) there is an edge in $H$ from $(a_1, b_1)$ to $(a_3, b_3)$; (2) the sets $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are disjoint. This will force a standard example of size 3 on these six elements.

Now to prove statement (1), we observe first that $a_1$ is $y_0$-left of $a_2$, and $a_2$ is $y_0$-left of $a_3$. Proposition 9 implies $a_1 \parallel a_3$ in $P$, and $a_1$ is $y_0$-left of $a_3$. It suffices to show that $a_1 \leq_P b_3$ and $a_3 \leq_P b_1$. We first show that $a_1 \leq_P b_3$.

Let $S$ and $T$ be witnessing trees for $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, respectively. Let $N = N(u, v)$ be a separating path associated with $a_2 \leq b_3$ in $P$ such that (1) $u$ is on $x_0Tb_3$ and $x_0Nu = x_0Tu$; and (2) $v$ is on $a_2Sy_0$ and $vNy_0 = vSy_0$. Since $b_2 \parallel a_2$ and $b_2$ is left of $b_3$, Proposition 12 implies $b_2$ is left of $N$. If $a_1$ is not right of $N$ (see the left part of Figure 5), then Proposition 12 implies $a_1 \leq b_3$ in $P$. Accordingly, we may assume $a_1$ is right of $N$, see the right part of Figure 5. Since $a_1 \leq b_2$ and $a_1$ and $b_2$ are on opposite sides of $N$, Proposition 13 forces either $a_1 < b_3$ or $a_2 < b_2$ in $P$. The second option is false, so we conclude that $a_1 < b_3$ in $P$. The argument for $a_3 \leq_P b_1$ is symmetric, and this completes the proof of statement (1).

Now suppose there are integers $i, j \in [3]$ such that $a_i = b_j$. Let $k$ be the other integer in $[3]$. Then $a_k \leq b_j = a_i$ so $a_k$ is comparable to $a_i$, which is false. This proves statement (2). \qed

Let $\mathbb{F} = (P, G, x_0, y_0, I, D)$ be our fixed doubly exposed sequence. Then let $\rho(\mathbb{F})$ be the maximum size (number of vertices) of a directed path in the auxiliary digraph $H$. The proof of the following lemma is (essentially) the same as the argument given for Lemma 5.9 in [11], although we are working here in a more general setting.

**Lemma 17.** If $\mathbb{F} = (P, G, x_0, y_0, I, D)$ is doubly exposed, then

$$\dim(I) \leq \rho(\mathbb{F})^2.$$
Figure 5. Left: $a_1$ is left of $N$ and therefore $a_1 \leq b_3$ in $P$. Right: $a_1$ is right of $N$ and $b_2$ is left of $N$. Thus, a witnessing path from $a_1$ to $b_2$ intersects $N$ and witnesses $a_1 \leq b_3$ in $P$ as well.

In particular, for $d \geq 3$, if $d$ is the largest size of a standard example in $P$, then $\dim(I) \leq d^2$.

Proof. We show $\dim(I) \leq \rho(\mathbb{F})^2$ by exhibiting a partition of $I$ into $\rho(\mathbb{F})^2$ reversible sets. These sets will have the form $I(m,n)$ where $1 \leq m, n \leq \rho(\mathbb{F})$. A pair $(a, b) \in I$ belongs to $I(m,n)$ if

(i) the longest directed path in $H$ starting from $(a, b)$ has size $m$, and
(ii) the longest directed path in $H$ ending at $(a, b)$ has size $n$.

To complete the proof, it suffices to show that each $I(m,n)$ is reversible. We argue by contradiction.

Suppose that for some pair $(m,n)$, the set $I(m,n)$ is not reversible. Therefore there is a strict alternating cycle $((a_1, b_1), \ldots, (a_k, b_k))$ of size $k \geq 2$ with all pairs from $I(m,n)$. Fix $S$ and $T$ to be witnessing trees for $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$, respectively. Without loss of generality, $a_1 <_S a_i$ for each $i \in \{2, \ldots, k\}$.

If $k = 2$, then there is a directed edge from $(a_1, b_1)$ to $(a_2, b_2)$ in $H$. It follows that any directed path in $H$ starting at $(a_2, b_2)$ can be extended by prepending $(a_1, b_1)$. Thus $(a_1, b_1), (a_2, b_2)$ cannot both belong to $I(m,n)$. We conclude that $k \geq 3$.

The balance of the proof divides into two cases:

$$a_1 <_S a_k <_S a_2 \quad \text{or} \quad a_1 <_S a_2 <_S a_k.$$
We have already noted that \((a_1, b_1)\). In the second case, we will show that there is a directed path in \(H\) of size \(n + 1\) ending at \((a_2, b_2)\). Both implications are contradictions. We will give details of the proof for the first case. It will be clear that the argument for the second case is symmetric.

Therefore we assume \(a_1 <_S a_k <_S a_2\). Since the pairs \((a_1, b_1)\), \((a_k, b_2)\) form an alternating cycle of size 2 and \(a_1 <_S a_k\), we have an edge in \(H\) from \((a_1, b_1)\) to \((a_k, b_2)\). Since \((a_1, b_1)\) is the first vertex on this edge, we know \(m \geq 2\).

Fix a directed path \(((w_1, z_1), (w_2, z_2), \ldots, (w_m, z_m))\) in \(H\) with \((w_1, z_1) = (a_2, b_2)\). (Recall that \(m \geq 2\).) Now consider the sequence

\[\((a_1, b_1), (a_k, b_2), (w_2, z_2), \ldots, (w_m, z_m))\].

We claim that this sequence is a directed path in \(H\). Since it has size \(m + 1\) and it starts at \((a_1, b_1)\), this will be a contradiction.

We have already noted that \(((a_1, b_1), (a_k, b_2))\) is an edge in \(H\). Note that for all \(i\) with \(2 \leq i < m\), \(((w_i, z_i), (w_{i+1}, z_{i+1}))\) is an edge in \(H\) as well. It remains only to show that there is an edge from \((a_k, b_2)\) to \((w_2, z_2)\) in \(H\). Since \(a_k\) is \(y_0\)-left of \(a_2\) and \(a_2 = w_1\) is \(y_0\)-left of \(w_2\), Proposition 9 we know that \(a_k\) is \(y_0\)-left of \(w_2\). It remains only to show that \(a_k \leq z_2\) and \(w_2 \leq b_2\) in \(P\). Note that \(w_2 \leq_P z_1 = b_2\). Therefore, we only need to show that \(a_k \leq_P z_2\).

Let \(N = N(u, v)\) be a separating path for \(a_k \leq b_1\) in \(P\) such that (1) \(u\) is on \(x_0Tb_1\) and \(x_0Nu = x_0Tu\); and (2) \(v\) is on \(a_kSy_0\) and \(vNy_0 = vSy_0\). See Figure 6. Note that Proposition 14, \(b_1 <_T b_2 = z_1 <_T z_2\). If \(z_2\) is not right of \(N\), then Proposition 12 implies \(a_k \leq z_2\) in \(P\), as desired. Therefore, we may assume that \(z_2\) is right of \(N\). Since \(a_k <_S a_2\) and \(a_2 \parallel_P b_1\), Proposition 12 implies \(a_2\) is left of \(N\). Since \(a_2 = w_1 \leq z_2\) in \(P\),

**Figure 6.** The argument shows that \(a_2\) is left of \(N\) and \(z_2\) is right of \(N\). Therefore, a witnessing path associated with \(a_2 \leq_P z_2\) in \(P\) has to cross \(N\), and this forces \(a_k \leq_P z_2\) in \(P\).
Proposition 13 implies either $a_2 < b_1$ or $a_k < z_2$ in $P$. Since the first option is false, we must have $a_k < z_2$ in $P$, as desired. □

When $I$ is doubly exposed, we now have $\dim(I)$ bounded in terms of $\rho(F)$, independent of the height $h$ of $P$. Now we turn our attention to bounding $\rho(F)$ in terms of $h$.

4. Restrictions Resulting from Bounded Height

This section is devoted to proving the following lemma.

**Lemma 18.** If the sequence $F = (P, G, x_0, y_0, I, D)$ is doubly exposed, and $P$ is of height at most $h$, then

$$\rho(F) \leq 58h + 11.$$ 

Once this lemma has been proven, the proof of our main theorem will be complete. To see this, recall that using Corollary 7, we paid a price of $O(h^4)$ to reduce to the case where we need to bound $\dim(I)$ for $I$ doubly exposed in $P$. Lemma 17 asserts that $\dim(I) \leq \rho(F)^2$. Combining this with Lemma 18, we obtain the bound $O(h^6)$.

Our final bound on $\rho(F)$ will emerge from a series of preliminary results all working within the following context. We fix a sequence $F = (P, G, x_0, y_0, I, D)$ which is doubly exposed and let $h$ be the height of $P$. We may assume that $h \geq 2$. We also let $H$ be the auxiliary digraph of $F$.

The following elementary proposition will play a key role in subsequent arguments. There are actually two versions, one for $\{a_1, \ldots, a_n\} \subseteq D_P(y_0)$ and one for $\{b_1, \ldots, b_n\} \subseteq U_P(x_0)$. The impact of the proposition is illustrated in Figure 7.

**Proposition 19.** Let $((a_1, b_1), \ldots, (a_n, b_n))$ be a directed path in $H$, with $n \geq 3$, and let $S$ be a witnessing tree for $\{a_1, \ldots, a_n\}$. If $1 \leq i < j < k \leq n$ and $W$ is a witnessing path intersecting both $a_iS_{y_0}$ and $a_kS_{y_0}$, then $W$ intersects $a_jS_{y_0}$.

*Proof.* Since $S$ is a tree, if there is an element $w$ of $W$ common to $a_iS_{y_0}$ and $a_kS_{y_0}$, then $w$ is in $a_jS_{y_0}$. Accordingly, we may assume that (1) $W$ is a non-trivial path intersecting both $a_iS_{y_0}$ and $a_kS_{y_0}$ at distinct points $s_i$ and $s_k$, respectively; and (2) no proper subpath of $W$ intersects both $a_iS_{y_0}$ and $a_kS_{y_0}$.

In this case, the paths $W$ and $s_iS_k$ form a cycle $D$ in $G$. If $W$ does not intersect $a_jS_{y_0}$, then by planarity, $a_j$ is in the interior of $D$. Since $W$ is a witnessing path, we have either $s_i < s_k$ or $s_k < s_i$ in $P$. In the first case, $a_i \leq s_i \leq d$ for all $d \in D$. This implies that $a_j$ is enclosed by $U_P(a_i)$, which is false by Proposition 14. A symmetric argument shows that if $s_k < s_i$ in $P$, then $a_j$ is enclosed by $U_P(a_k)$. The contradiction completes the proof. □
Let $N$ be a separating path associated with a comparability $a \leq b$ in $P$. Let $A$ and $B$ be two subsets of elements of $P$. We will say that $N$ separates $A$ from $B$ if all points of $A$ are on one side of $N$ and all points of $B$ are on the other side.

We present the first of three key results bounding $\rho(F)$ in terms of the height of $P$.

**Proposition 20.** Let $((a_1, b_1), \ldots, (a_n, b_n))$ be a directed path in $H$. Let $\alpha, \beta$ be distinct integers from $[n]$ and let $N$ be a separating path associated with $a_\alpha < b_\beta$ in $P$. Let $X \subseteq [n] - \{\alpha, \beta\}$ such that either $i < \min(\alpha, \beta)$ for all $i \in X$, or $i > \max(\alpha, \beta)$ for all $i \in X$. If $N$ separates $A(X)$ from $B(X)$, then $|X| \leq 2h - 1$.

**Proof.** We give the argument when $i < \min(\alpha, \beta)$ for all $i \in X$. The argument for the other case is symmetric.

Let $N = N(u, v)$. Fix a witnessing tree $S$ for $\{a_1, \ldots, a_n\}$ such that $vNy_0$ is a terminal portion of $a_\alpha Sy_0$. Also, fix a witnessing tree $T$ for $\{b_1, \ldots, b_n\}$ such that $x_0Nu$ is an initial portion of $x_0Tb_\beta$.

Consider the red portion of $N$, a chain on at most $h$ elements. For each $a \in A(X)$, let $\sigma(a)$ be the lowest element of this chain such that $a <_P \sigma(a)$. Consider also the blue portion of $N$. For each $b \in B(X)$, let $\tau(b)$ be the highest element of this chain such that $\tau(b) <_P b$.

We claim that when $i, j \in X$ and $i < j$, then $\sigma(a_j) \leq \sigma(a_i)$ in $P$. Consider a witnessing path $W$ from $a_i$ to $\sigma(a_i)$. The path $W$ intersects $a_i Sy_0$ and $a_\alpha Sy_0$. Proposition 19 implies
that $W$ also intersects $a_jS_y$ (recall that $i < j < \alpha$ by our assumption). Therefore, $a_j \leq \sigma(a_i)$ in $P$ which implies $\sigma(a_j) \leq \sigma(a_i)$ in $P$. A dual argument shows that when $i, j \in X$ and $i < j$, we have $\tau(b_j) \geq \tau(b_i)$ in $P$.

When $i < j$, we claim that at least one of the two inequalities $\sigma(a_j) \leq \sigma(a_i)$, $\tau(b_j) \geq \tau(b_i)$ must be strict. To see this, assume that $\sigma(a_i) = \sigma(a_j)$ and $\tau(b_j) = \tau(b_i)$. Consider a witnessing path $W_{j,i}$ from $a_j$ to $b_i$. Since $a_j$ and $b_i$ are on opposite sides of $N$, we know that $W_{j,i}$ intersects $N$. Let $z$ be a common point of $W_{j,i}$ and $N$. If $z$ is on the red portion of $N$, then $b_i > z \geq \sigma(a_j) = \sigma(a_i) \geq a_i$ in $P$ which is a contradiction. If $z$ is on the blue portion of $N$, then $a_j < z \leq \tau(b_i) = \tau(b_j) \leq b_j$ in $P$ which is a contradiction. Thus $z$ is in the black part of $N$. Similarly, a witnessing path $W_{i,j}$ from $a_i$ to $b_j$ must intersect $N$ at a point $z'$ which is also in the black part of $N$. Since the black part is a chain, $z$ and $z'$ are comparable in $P$. If $z \leq_P z'$, then $a_i \leq z \leq z' \leq b_i$ in $P$. If $z' \geq_P z$, then $a_i \leq z' \leq z \leq b_i$ in $P$. Both statements are false. This observation confirms our claim.

Consider the following two sets \{\{\sigma(a) \mid a \in A(X)\}, \{\tau(b) \mid b \in B(X)\}\}. Each of these can be considered as a sequence sorted by the linear order on $X$ as a set of integers. The first sequence is non-increasing on the red chain in $N$. The second sequence is non-decreasing on the blue chain in $N$. Now moving along elements in $X$ in their natural ordering, there are $|X| - 1$ consecutive pairs. For each such pair, at least one of the two sequences changes. Therefore,

$$|X| - 1 \leq |\{\sigma(a) \mid a \in A(X)\}| - 1 + |\{\tau(b) \mid b \in B(X)\}| - 1 \leq 2(h - 1),$$

so $|X| \leq 2h - 1$. With this observation, the proof is complete. \hfill \Box

**Proposition 21.** Let $((a_1,b_1),\ldots,(a_n,b_n))$ be a directed path in $H$. If $S$ and $T$ are witnessing trees for $\{a_1,\ldots,a_n\}$ and $\{b_1,\ldots,b_n\}$, respectively, and $S \cap T$ is empty, then $n \leq 6h + 1$.

**Proof.** We assume that $n \geq 6h + 2$ and argue to a contradiction. Let $N = N(u,v)$ be a separating path associated with the comparability $a_{4h} < b_{4h+1}$ such that the red part of $N$ is a terminal portion of $a_{4h}S_y$, and the blue portion of $N$ is an initial portion of $x_0Tb_{4h+1}$. Let $W$ be the black portion of $N$. We split the elements of the pairs into $A_1 = \{a_1,a_2,\ldots,a_{4h-1}\}$, $A_2 = \{a_{4h+2},a_{4h+3},\ldots,a_{6h+2}\}$, $B_1 = \{b_1,b_2,\ldots,b_{4h-1}\}$ and $B_2 = \{b_{4h+2},b_{4h+3},\ldots,b_{6h+2}\}$.

Since $a_{4h+1} \parallel b_{4h+1}$ in $P$, the path $W$ does not intersect $a_{4h+1}S_y$. Furthermore, by Proposition 12, $a_{4h+1}$ is left of $N$. Proposition 19 implies that if $a \in A_2$, then $W$ does not intersect $aS_y$. Since $S$ and $T$ are disjoint, Proposition 12 now implies $a$ is left of $N$ as well. Dually, $W$ does not intersect $x_0Tb_{4h}$ and $b_{4h}$ is left of $N$. Now, Proposition 19 implies that if $b \in B_1$, then $W$ does not intersect $x_0Tb$, and $b$ is left of $N$ as well. On the other hand, elements of $A_1 \cup B_2$ may be on either side of $N$.

We partition the set $\{1,2,\ldots,4h-1\}$ as $X_1 \cup X_2$, where $i \in X_1$ if and only if $a_i$ is left of $N$. Since $N$ separates $A(X_2)$ from $B(X_2) \subseteq B_1$, it follows from Proposition 20 that $|X_2| \leq 2h - 1$. Therefore $|X_1| \geq 4h - 1 - (2h - 1) = 2h$. Similarly, we partition
\{4h + 2, 4h + 3, \ldots, 6h + 2\} as \(Y_1 \cup Y_2\), where \(i \in Y_1\) if and only if \(b_i\) is left of \(N\). Now Proposition 20 implies that \(|Y_2| \leq 2h - 1\) and therefore \(|Y_1| \geq 2h + 1 - (2h - 1) = 2\). Set \(m = 2h\). Now we are going to discard excess elements and relabel those that remain. Let \(A'\) be a subset of \(A(X_1)\) of size \(m\) with elements relabeled as \(\{w_1, \ldots, w_m\}\) so that \(w_1 <_S \cdots <_S w_m\). Let \(B'\) be the corresponding subset of elements of \(B(Y_1)\) with elements relabeled correspondingly as \(\{z_1, \ldots, z_m\}\). Let \(\{z_{m+1}, z_{m+2}\}\) be a subset of \(B(Y_1)\) of size 2 so that \(z_{m+1} \prec_T z_{m+2}\). Let \(\{w_{m+1}, w_{m+2}\}\) be the corresponding subset of elements of \(A(Y_1)\). Note that we have

\[
\begin{align*}
w_1 &<_S \cdots <_S w_m <_S a_{4h} <_S a_{4h+1} <_S w_{m+1} <_S w_{m+2}, \\
z_1 &<_T \cdots <_T z_m <_T b_{4h} <_T b_{4h+1} <_T z_{m+1} <_T z_{m+2}.
\end{align*}
\]

Let \(N' = N(u', v')\) be a separating path associated with \(w_{m+1} < z_{m+2}\) such that the red part of \(N'\) is a terminal portion of \(w_{m+1} S y_0\) and the blue part of \(N'\) is an initial portion of \(x_0 T z_{m+2}\). Also, let \(W'\) denote the black part of \(N'\). Since \(z_{m+1} \parallel w_{m+1}\), \(W\) does not intersect \(x_0 T z_{m+1}\). Now Proposition 12 implies that \(z_{m+1}\) is left of \(N'\). Using Proposition 19, it follows that if \(b \in B'\), then \(W'\) does not intersect \(x_0 T b\), and \(b\) is left of \(N'\).

**Claim.** All elements of \(A'\) are right of \(N'\).

**Proof.** Consider an element \(a \in A'\). Since \(a\) is left of \(N\), \(a S y_0\) contains a point from the union of the black and blue parts of \(N\). Since \(S \cap T\) is empty, \(W\) intersects \(a S y_0\). Let \(p\) be the largest point of \(W\) that is also on \(a S y_0\). Dually, since \(z_{m+2}\) is left of \(N\), we know that \(W\) intersects \(x_0 T z_{m+2}\). Let \(q\) be the least element of \(W\) that is also on \(x_0 T z_{m+2}\). Since \(S\) and \(T\) are disjoint and by planarity, we know \(v \leq p < q \leq u\) in \(P\). Clearly, \(q\) is the first point of \(x_0 T z_{m+2}\) that lies in \(W\), and \(p\) is the last point of \(a S y_0\) that lies in \(W\).

Proposition 19 implies that there is a point \(r\) common to \(q W u\) and \(x_0 T z_{m+1}\). In particular, we have \(q \leq r \leq z_{m+1}\) in \(P\).

Recall that \(W'\) is a witnessing path from \(v'\) to \(u'\), and \(u'\) is in \(x_0 T z_{m+2}\). If \(u'\) is on \(x_0 T q\) then

\[w_{m+1} < u' \leq q < r \leq z_{m+1}\] in \(P\),

which is a contradiction. We conclude that \(u'\) is an element of \(x_0 T z_{m+2}\) that occurs after \(q\). Let \(e\) be the first edge of \(q T z_{m+2}\) and let \(e_0\) be the last edge of \(x_0 T q\). Now let \(e'\) be the first edge of \(q W p\) (it does exist as \(q \neq p\)). See Figure 8. We assert that \(e\) is left of \(e'\) in the \((q, e_0)\)-ordering. To verify this assertion, consider the path \(N'' = x_0 T q W v S y_0\). Since \(z_{m+2}\) is left of \(N\), it is left of \(N''\) as well. Now suppose to the contrary that \(e'\) is left of \(e\) in the \((q, e_0)\)-ordering. Then \(e\) is right of \(N''\). Now Proposition 10 implies that there is a vertex \(q'\), with \(q' \neq q\) such that \(q'\) is common to \(N''\) and \(q T z_{m+2}\). Since \(T\) is a tree, \(q'\) cannot be on \(x_0 N'' q = x_0 T q\). Since \(S\) and \(T\) are disjoint, \(q'\) cannot be on \(v N'' y_0 = v S y_0\). Therefore, \(q'\) is on \(q N'' v = q W v\). This contradicts the choice of \(q\) as the least element common to \(W\) and \(x_0 T z_{m+2}\). This observation completes the proof of the assertion.
Note that by the definition of $p$ and $q$, $M = qWpSa$ is a path. Recall that $e$ in on $N'$ and $e'$ is on $M$. Since $e$ is left of $e'$ in the $(q, e_0)$-ordering, Proposition 10 implies that $e'$ is right of $N'$.

To complete the proof, we observe that if $a$ is not right of $N'$, then Proposition 10 implies that there is a vertex $q'$, with $q' \neq q$, common to $M$ and $N'$. If $q'$ is in the black or red part of $N'$, i.e. $u'N'y_0$, then $v' \leq q'$ in $P$ and

$$w_{m+1} \leq v' \leq q' \leq q \leq r \leq z_{m+1}$$

in $P$, a contradiction. If $q'$ is in the blue part of $N'$, i.e. $x_0 Tu'$, then $q'$ cannot belong to $aMp = aSp$ as $S$ and $T$ are disjoint. Finally, if $q'$ is in the blue part of $N'$ and $q'$ is in $pMq = pWq$, then we contradict the choice of $q$.

We have now reached a contradiction since we have shown that $N'$ separates $A'$ and $B'$ with $|A'| = |B'| = m = 2h$, contradicting Proposition 20. This completes the proof of Proposition 21.

4.1. Separating witnessing trees. In Figure 9, we illustrate some of the challenges we face in finding disjoint witnessing trees. Now we begin the material to address this challenge. Let $Z$ be the non-empty subposet of $P$ consisting of all elements of $P$ that belong to a witnessing path from $x_0$ to $y_0$. If we restrict our drawing of $G$ to the induced subgraph determined by the elements of $Z$, we obtain a drawing without edge crossings of the cover graph of $Z$. Furthermore, $x_0$ is the unique minimal element of $Z$, and $y_0$ is the unique maximal element of $Z$, and
Figure 9. Left: Two copies of $S_3$ are stacked in a vertical manner. Clearly, this construction can be expanded for two copies of an arbitrarily large standard example. Right: Three incomparable pairs are stacked vertically to form a copy of $S_3$.

the induced subgraph of $G$ determined by the elements of $Z$ is the cover graph of the subposet $Z$.

Our fixed drawing of the cover graph of $Z$ splits the plane into regions: some number of bounded regions and one unbounded. We call such a bounded region a $Z$-face. Each element of $P$ that is not in $Z$ is in the interior of one of the regions. After adding two dummy elements $z', z''$ into $P$ (and $Z$) such that (1) $x_0 < z' < y_0$, $x_0 < z'' < y_0$ in $P$ and all these relations are covers; (2) $x_0z'$ is the leftmost edge in the $(x_0, e_{-\infty})$-ordering; (3) $x_0z''$ is the rightmost edge in the $(x_0, e_{-\infty})$-ordering; we can assume that any element of $P$ that is not in $Z$ is in the interior of one of the (bounded) $Z$-faces.

Let $S($left, not-left$)$ consist of all pairs $(a, b) \in \text{Inc}(D_P(y_0), U_P(x_0))$ for which there exists a witnessing path from $x_0$ to $y_0$ such that $a$ is left of $W$ and $b$ is not left of $W$. Similarly, let $S($not-left, left$)$ consist of all pairs $(a, b) \in \text{Inc}(D_P(y_0), U_P(x_0))$ for which there exists a witnessing path from $x_0$ to $y_0$ such that $a$ is not left of $W$ and $b$ is left of $W$. The other two sets $S($right, not-right$)$ and $S($not-right, right$)$ are defined in a symmetric manner.

**Proposition 22.** Each of the sets $S($left, not-left$)$, $S($right, not-right$)$, $S($not-left, left$)$, $S($not-right, right$)$ is reversible.

**Proof.** Using an argument by contradiction, we show that $S($left, not-left$)$ is reversible. The argument for the other three sets is symmetric. Let $((a_1, b_1), \ldots, (a_k, b_k))$ be a strict alternating cycle in $S($left, not-left$)$. Let $i \in [k]$. Fix a witnessing $W$ from $x_0$ to $y_0$ such that $a_i$ is left of $W$ and $b_i$ is not left of $W$. If $b_i$ is on $W$ then let $W_i = x_0 W b_i$. Otherwise, let $W_i$ be a witnessing path from $x_0$ to $b_i$ that is $x_0$-consistent with $W$, so we have that $W_i$ is $x_0$-right of $W$. 


If $b_{i+1}$ is left of $W$, then let $W_{i+1}$ be a witnessing path from $x_0$ to $b_{i+1}$ that is $x_0$-consistent with $W$. Clearly, $W_{i+1}$ is $x_0$-left of $W$. Since $b_{i+1} \parallel b_i$ in $P$ we conclude that $W_{i+1}$ and $W_i$ are $x_0$-consistent and $W_{i+1}$ is $x_0$-left of $W_i$. Thus in this case, Proposition 14 implies that $b_{i+1}$ is $x_0$-left of $b_i$.

Now consider the case when $b_{i+1}$ is not left of $W$, and let $W'$ be a witnessing path from $a_i$ to $b_{i+1}$. Since $a_i$ is left of $W$ and $b_{i+1}$ is not left $W$, we conclude that $W'$ and $W$ must intersect, say at element $z$. Note that $z$ is not in $W_i$, as this would imply $a_i < z < b_i$ in $P$, which is false. Let $W_{i+1} = x_0 W z W' b_{i+1}$. Then $W_{i+1}$ and $W_i$ are $x_0$-consistent and $W_{i+1}$ is $x_0$-left of $W_i$. Again, we conclude by Proposition 14 that $b_{i+1}$ is $x_0$-left of $b_i$.

We have now shown that $b_{i+1}$ is $x_0$-left of $b_i$. Clearly, this statement cannot hold for all $i \in [k]$. The contradiction completes the proof.

Let $J$ be the subset of pairs in $\text{Inc}(D_P(y_0), U_P(x_0))$ that do not belong to any of $S(\text{left, not-left}), S(\text{not-left, left}), S(\text{right, not-right}), S(\text{not-right, right})$. Note that when $(a, b) \in J$ then neither $a$ nor $b$ belongs to $Z$. Proposition 22 implies

$$\dim(I) \leq \dim(\text{Inc}(D_P(y_0), U_P(x_0))) \leq \dim(J) + 4.$$ 

Each $Z$-face $F$ is bounded by two distinct witnessing paths that have only their starting and ending points in common. We let $x_F$ denote the common starting point, and we let $y_F$ denote the common ending point of these two witnessing paths. When we start at $x_F$ and traverse the boundary of $F$ in a clockwise manner, we follow the left side of $F$ until we reach $y_F$. Then we traverse the right side of $F$ backwards until we arrive back at $x_F$. An element on the left (right) side of $F$ that is not in $\{x_F, y_F\}$ is said to be strictly on the left (right) side of $F$. Note that if $u$ is strictly on the left side of $F$, and $v$ is strictly on the right side of $F$, then $u \parallel v$ in $P$. Note that $u$ is $x_0$-left of $v$ and $y_0$-right of $v$. Note also that there is always at least one point strictly on the left (right) side of $F$.

When $F$ is a $Z$-face, no element $u$ of $P$ that is in the interior of $F$ satisfies $x_F <_P u <_P y_F$; otherwise this region would be split into smaller $Z$-faces. Also, a $Z$-face has no chords.

When $u \in P$ and $u$ is not in $Z$, there is a unique $Z$-face $F_u$ containing $u$ in its interior. We let $y_u = y_{F_u}$ and $x_u = x_{F_u}$. Let $(a, b) \in J$. A witnessing path from $a$ to $y_0$ has to leave the interior of $F_a$, and this implies $a < y_a$ in $P$. Dually, a witnessing path from $b$ to $x_0$ (going backward) has to leave the interior of $F_b$, and this implies $x_b < b$ in $P$.

A pair $(a, b) \in J$ is called a same-face pair if $F_a = F_b$.

**Proposition 23.** All pairs in $J$ are same-face pairs.

**Proof.** To the contrary, suppose $(a, b) \in J$ and $F_a \neq F_b$. Let $W_0$ be a witnessing path from $x_0$ to $x_b$, and let $W_1$ be a witnessing path from $y_b$ to $y_0$. Now let $W$ be the witnessing path from $x_0$ to $y_0$ formed by concatenating $W_0$, the left side of $F_b$, and $W_1$. Also let $W'$ be the witnessing path from $x_0$ to $y_0$ formed by concatenating $W_0$, the right side of $F_b$, and $W_1$. See Figure 10.
Figure 10. An incomparable pair \((a, b)\) with \(a\) and \(b\) in different \(Z\)-faces. Left: \(a\) is left of \(W\) and \(b\) is right of \(W\). Right: \(a\) is right of \(W'\) and \(b\) is left of \(W'\).

The elements in the interior of \(F_b\) are the only points in the plane that are right of \(W\) and left of \(W'\). It follows that either (1) \(a\) is left of \(W\) and \(b\) is right of \(W\); or (2) \(a\) is right of \(W'\) and \(b\) is left of \(W'\). If (1) holds then \((a, b) \in S(\text{left, not-left})\), and if (2) holds then \((a, b) \in S(\text{right, not-right})\).

Here is another self-evident proposition stated for emphasis.

**Proposition 24.** If \(W\) is a witnessing path from \(x_0\) to \(y_0\) in \(Z\) and \(F\) is a \(Z\)-face, then there do not exist points \(u, v\) on the boundary of \(F\) such that \(u\) is left of \(W\) and \(v\) is right of \(W\).

**Proposition 25.** Let \(u, v\) be incomparable elements of \(Z\).

(i) Then either \(u\) is \(x_0\)-left of \(v\), or \(u\) is \(x_0\)-right of \(v\).
(ii) If \(W\) is a witnessing path from \(x_0\) to \(y_0\) with \(v \in W\). Then, \(u\) is \(x_0\)-left of \(v\) if and only if \(u\) is left of \(W\).

**Proof.** Note that \(((u, v), (v, u))\) is a strict alternating cycle of elements in \(Z\). Proposition 14 implies that either \(u\) is \(x_0\)-left of \(v\) or \(u\) is \(x_0\)-right of \(v\).

Let \(W\) be a witnessing path from \(x_0\) to \(y_0\) with \(v \in W\). Suppose \(u\) is \(x_0\)-left of \(v\) and let \(W_u\) be a witnessing path from \(x_0\) to \(u\) such that the pair \((W_u, x_0Wv)\) is \(x_0\)-consistent. Proposition 10 implies that \(u\) is left of \(W\).

Conversely, suppose that \(u\) is left of \(W\), and let \(z\) be the largest element of \(W\) such that \(z < u\) in \(P\). Let \(U\) be a witnessing path from \(z\) to \(u\). Then \(x_0WzUu\) and \(x_0Wv\) are \(x_0\)-consistent. Again by Proposition 10, the first path is \(x_0\)-left of the second. This shows \(u\) is \(x_0\)-left of \(v\).
Claim 1. There do not exist distinct integers \(i, j \in [k]\) such that the pairs \((a_i, b_i)\) and \((a_j, b_j)\) are in the same \(Z\)-face.

Proof. Suppose that for some \(i \neq j\) all four elements involved in \((a_i, b_i), (a_j, b_j)\) lie in the same \(Z\)-face. Since our alternating cycle is a counterexample, we do not have all the pairs lying in the same \(Z\)-face, so we know that \(k \geq 3\). After a relabeling, we may assume that \(j = k\) and \(2 \leq i \leq k - 1\). However, this implies that

\[
((a_1, b_1), \ldots, (a_{i-1}, b_{i-1}), (a_k, b_k))
\]

is an alternating cycle of same-face pairs from \(J\). This is a contradiction unless all the pairs on this cycle belong the same \(Z\)-face. In this case, we consider the strict alternating cycle

\[
((a_i, b_i), (a_{i+1}, b_{i+1}), \ldots, (a_k, b_k)).
\]

Again, we have a strict alternating cycle of pairs from \(J\). However, now it is clear that not all the pairs on this cycle belong to the same \(Z\)-face. Furthermore, the length of this cycle is less than \(k\). The contradiction completes the proof of the claim. \(\square\)

For each \(i \in [k]\), let \(\mathcal{F}_i\) be the common \(Z\)-face \(\mathcal{F}_{a_i} = \mathcal{F}_{b_i}\), let \(x_i = x_{b_i}\), and let \(y_i = y_{a_i}\). Let \(W_i\) be a witnessing path from \(a_i\) to \(b_{i+1}\). Then let \(u_i\) be the lowest point of \(W_i\) that is on the boundary of \(\mathcal{F}_i\), and let \(v_{i+1}\) be the highest point of \(W_i\) that is on the boundary of \(\mathcal{F}_{i+1}\). We note that \(a_i <_{P} u_i <_{P} v_{i+1} <_{P} b_{i+1}\).

Claim 2. For all \(i, j \in [k]\), \(u_i \leq_P v_j\) if and only if \(j = i + 1\) (cyclically).

Proof. We already know that \(u_i \leq_P v_{i+1}\) for all \(i \in [k]\). Now suppose \(j \neq i + 1\) and \(u_i \leq v_j\). Then \(a_i <_{P} u_i <_{P} v_j <_{P} b_j\). This implies \(a_i < b_j\). Now we have contradicted the assumption that our original cycle is strict. With this observation, the proof of the claim is complete. \(\square\)

Claim 2 implies that \((u_1, v_1), \ldots, (u_k, v_k)\) is a strict alternating cycle of incomparable pairs in \(Z\). Let \(i \in [k]\). Since \(u_i \parallel_P v_i\), and both \(u_i\) and \(v_i\) are on the boundary of \(\mathcal{F}_i\), it implies that they are on opposite sides of \(\mathcal{F}_i\). Also, \(\{u_i, v_i\} \cap \{x_i, y_i\} = \emptyset\).

Claim 3. For each \(i \in [k]\), the following statements hold.

(i) If \(u_i\) is \(x_0\)-left of \(v_i\), then \(u_{i+1}\) is \(x_0\)-left of \(v_{i+1}\) and \(u_{i+1}\) is \(x_0\)-left of \(u_i\).

(ii) If \(u_i\) is \(x_0\)-right of \(v_i\), then \(u_{i+1}\) is \(x_0\)-right of \(v_{i+1}\) and \(u_{i+1}\) is \(x_0\)-right of \(u_i\).
As illustrated on the left side of Figure 11, we first consider the case that $z < v_{i+1}$ in $P$, so $v_{i+1}$ is left of $U$.

**Proof.** We prove the first statement. The proof of the second is symmetric. Let $i \in [k]$. Suppose that $u_i$ is $x_0$-left of $v_i$. This implies that $u_i$ is strictly on the left side of $\mathcal{F}_i$ and $v_i$ is strictly on the right side of $\mathcal{F}_i$.

Let $W_0$ be a witnessing path from $x_0$ to $x_i$ and let $W_1$ be a witnessing path from $y_i$ to $y_0$. Let $U$ be the concatenation of $W_0$, the left side of $\mathcal{F}_i$, and $W_1$.

Recall that $u_i \leq v_{i+1}$ in $P$. Let $z$ be the largest element on $U$ such that $z \leq v_{i+1}$ in $P$. Since $v_{i+1} \parallel v_i$ in $P$, we must have $z \in u_iUy_i$ but excluding $y_i$, i.e., $z$ is strictly on the left side of $\mathcal{F}_i$. We split the proof into two cases: $z = v_{i+1}$ and $z < v_{i+1}$ in $P$.

As illustrated on the left side of Figure 11, we first consider the case that $z = v_{i+1}$. Since $\mathcal{F}_i \neq \mathcal{F}_{i+1}$, $v_{i+1}$ must be strictly on the right side of $\mathcal{F}_{i+1}$. Therefore, $u_{i+1}$ must be strictly on the left side of $\mathcal{F}_{i+1}$, $u_{i+1}$ is $x_0$-left of $v_{i+1}$, and $u_{i+1}$ is left of $U$. Proposition 25 implies that $u_{i+1}$ is $x_0$-left of $u_i$, as desired.

Now we consider the case that $z < v_{i+1}$ in $P$, as illustrated on the right side of Figure 11. Let $W$ be a witnessing path from $z$ to $v_{i+1}$. By the definition of $z$, $W$ is disjoint from $U$ except the element $z$. Note also that the first edge of $W$ is left of $U$, as no edge of a witnessing path between two elements in $Z$ is in the interior of a $Z$-face. Proposition 10 implies that all edges and vertices of $W$ except $z$ are left of $U$. Let $z'$ be the least element on $U$ such that $v_{i+1} < z'$ in $P$. Let $W'$ be a witnessing path from $v_{i+1}$ to $z'$.

Now let $U' = x_0UzWv_{i+1}W'z'Uy_0$. We claim that $u_{i+1}$ is left of $U'$. Clearly, since $u_{i+1} \parallel v_{i+1}$ in $P$, $u_{i+1}$ is not on $U'$. Suppose to the contrary that $u_{i+1}$ is right of $U'$. Recall that $v_{i+1}$ is left of $U$. Proposition 24 implies that $u_{i+1}$ is not right of $U$. Since $u_{i+1} \parallel u_i$ in $P$, we conclude $u_{i+1}$ is left of $U$. Note that all points in the plane that are right of $U'$ and left of $U$ are in the interior of the cycle $zWv_{i+1}W'z'Uz$. However, all
elements of the cycle are in \( U_P(u_i) \). This means that \( u_{i+1} \) is enclosed by the \( U_P(u_i) \), a contradiction. This proves that \( u_{i+1} \) is left \( U' \). Now, Proposition 25 implies that \( u_{i+1} \) is \( x_0 \)-left of \( u_{i+1} \), and \( u_{i+1} \) is \( x_0 \)-left of \( u_i \), as desired. \( \square \)

To complete the proof of the lemma, we simply note that the statement of the claim cannot hold for all \( i \in [k] \) cyclically. \( \square \)

Let \( F \) be a Z-face. We define \( P_F \) as the subposet of \( P \) containing all elements inside or on the boundary of \( F \). Note that the cover graph \( G_F \) of \( P_F \) is the induced subgraph of \( G \) on elements of \( P_F \). Let \( J_F \) consist of those pairs \((a, b)\) \( \in J \) such that \( F = F_a = F_b \). Finally, let \( D_F \) be the restriction of the plane drawing \( D \) to vertices and edges \( G_F \). Observe that \( F_F = (P_F, G_F, x_F, y_F, J_F, D_F) \) is doubly exposed. Then, Propositions 22, 23, 26 imply

\[
\rho(F) \leq 4 + \max_F \rho(F). \]

**Proposition 27.** Let \( F = (P, G, x_0, y_0, I, D) \) be doubly exposed, where \( P \) is a poset of height \( h \). Suppose further that:

(i) The boundary of the exterior face of \( G \) in \( D \) is the boundary of a Z-face \( F \).
(ii) If \((a, b)\) \( \in I \), then both \( a \) and \( b \) are in the interior of \( F \).

Then

\[
\rho(F) \leq 58h + 7. \]

**Proof.** Suppose to the contrary that \((a_1, b_1), \ldots, (a_n, b_n)\) is a directed path in the auxiliary graph \( H \) of \( F \) with \( n \geq 58h + 8 \). Let \( S \) and \( T \) be witnessing trees for \( \{a_1, \ldots, a_n\} \) and \( \{b_1, \ldots, b_n\} \), respectively. Suppose that \((\alpha, \beta)\) is a pair of distinct integers in \([n]\) such that \( x_0 T \beta \) intersects \( y_0 S \alpha \). Let \( c = c(a_\alpha, b_\beta) \) and \( d = d(a_\alpha, b_\beta) \) be, respectively, the least and the greatest element of \( P \) common to \( x_0 T \beta \) and \( y_0 S \alpha \). Then \( x_0 < c < d < y_0 \) in \( P \). Since the only witnessing paths from \( x_0 \) to \( y_0 \) are the two sides of \( F \), it follows that \( W = x_0 T c T d S y_0 = x_0 T c S d S y_0 \) is one of the sides of \( F \).

If \( \alpha < \beta \), we assert (as illustrated on the left side of Figure 12) that the following three statements hold:

(i) \( W \) is the right side of \( F \).
(ii) If \( 1 \leq i < \alpha \), then \( c S y_0 \) is a terminal portion of \( a_i S y_0 \). Furthermore, all edges and vertices of \( a_i S c \), except \( c \), are in the interior of \( F \).
(iii) If \( \beta < j \leq n \), then \( x_0 T d \) is an initial portion of \( x_0 T b_j \). Furthermore, all edges and vertices of \( d T b_j \), except \( d \), are in the interior of \( F \).

To verify this assertion, suppose first that \( \alpha < \beta \) and \( W \) is the left side of \( F \). Recall that \( a_\alpha \) is \( y_0 \)-left of \( a_\beta \). Since \( c \in a_\alpha S y_0 \) and \( c \) is on the left side of \( F \), it follows that \( c \in a_\beta S y_0 \). This forces \( a_\beta < c < b_\beta \) in \( P \), which is false. The contradiction shows that \( W \) is on the right side of \( F \).
If $1 \leq i < \alpha$, then $a_i$ is $y_0$-left of $a_\alpha$. Since $c \in a_\alpha S y_0$ and $c$ is on the right side of $\mathcal{F}$, it follows that $c \in a_i S y_0$. Therefore $c S y_0$ is a terminal portion of $a_i S y_0$. The definition of $c$ implies that the first edge of $c S a_i$ is in the interior of $\mathcal{F}$. Since $\mathcal{F}$ is a $Z$-face, all edges and vertices of $c S a_i$, except $c$, are in the interior of $\mathcal{F}$. A symmetric argument shows that if $\beta < j \leq n$, then $x_0 T d$ is an initial portion of $x_0 T b_j$, and all edges and vertices of $d T b_j$, except $d$, are in the interior of $\mathcal{F}$.

Symmetrically, if $\alpha > \beta$, the following three statements hold:

(i) $W$ is the left side of $\mathcal{F}$.
(ii) If $\alpha < i \leq n$, then $c S y_0$ is a terminal portion of $a_i S y_0$. Furthermore, all edges and vertices of $a_i S c$, except $c$, are in the interior of $\mathcal{F}$.
(iii) If $1 \leq j < \beta$, then $x_0 T d$ is an initial portion of $x_0 T b_j$. Furthermore, all edges and vertices of $d T b_j$, except $d$, are in the interior of $\mathcal{F}$.

Let $s = 6h + 1$. Note that $n \geq 9s + 4h - 1$. If $X$ is any subset of $[n]$ with $|X| = s + 1$, then Proposition 21 implies that there are distinct integers $\alpha, \beta \in X$ such that $x_0 T b_\beta$ intersects $a_\alpha S y_0$. We apply this observation to the set $X = [s + 1]$. We give the balance of the argument under the assumption that $\alpha < \beta$. From the details of the argument (illustrated on the right side of Figure 12), it will be clear that the proof when $\alpha > \beta$ is symmetric.

For all $i \in [n]$, let $c_i$ be the least element of $P$ common to $a_i S y_0$ and the right side of $\mathcal{F}$. Dually, for all $j \in [n]$, let $d_j$ be the greatest element of $P$ common to $x_0 T b_j$ and the
right side of $\mathcal{F}$. Since $a_1 < a_2 < \cdots < a_n$, we have $$c_1 \leq c_2 \leq \cdots \leq c_n \text{ in } P.$$ Symmetrically, since $b_1 < b_2 < \cdots < b_n$, we have $$d_1 \leq d_2 \leq \cdots \leq d_n \text{ in } P.$$ Note also that since $a_i \parallel b_i$ in $P$, we must have $$d_i < c_i \text{ in } P, \text{ for all } i \in [n].$$ (1)

From our statements above and our assumption that $\alpha < \beta$, we conclude $x_0 < c_\alpha \leq d_\beta < y_0$ in $P$. Also, both $c_\alpha$ and $d_\beta$ are strictly on the right side of $\mathcal{F}$. Again from the statements above we have $d_j$ strictly on the right side of $\mathcal{F}$ for all $j \geq \beta$. Therefore, the only intersection of $x_0Tb_j$ with the left side of $\mathcal{F}$ is at $x_0$, for all $j \geq \beta$.

Claim 1. If $i \in \{s + 1, \ldots, n - 2s\}$, then $c_i < d_{i+2s}$ in $P$.

Proof. Consider $X' = \{i, \ldots, i+s\}$. Since $|X'| = s + 1$, Proposition 21 implies that there are distinct $\alpha', \beta' \in X'$ such that $a_{\alpha'}Sy_0$ and $x_0Tb_{\beta'}$ intersect. Since $\beta' \geq s + 1 \geq \beta$, we conclude that $x_0Tb_{\beta'}$ is disjoint from the left side of $\mathcal{F}$ (except $x_0$). Therefore, the intersection of $a_{\alpha'}Sy_0$ and $x_0Tb_{\beta'}$ is on the right side of $\mathcal{F}$, so $\alpha' < \beta'$, and $c_{\alpha'} \leq d_{\beta'}$ in $P$. Consider also $X'' = \{i + s, \ldots, i + 2s\}$. Again $|X''| = s + 1$ and therefore as before, we conclude that there exist $\alpha'' \in X''$ such that $c_{\alpha''} \leq d_{\beta''}$ in $P$.

Now we have $$c_i \leq c_{\alpha'} \leq d_{\beta'} < c_{\beta'} \leq c_{i+s} \leq c_{\alpha''} \leq d_{\beta''} \leq d_{i+2s} \text{ in } P.$$ This proves Claim 1. \hfill \square

Now consider the following chain of nine inequalities.

\[
d_{s+1} < c_{s+1} < d_{3s+1} < c_{3s+2} < d_{5s+2} < c_{5s+2} < d_{7s+2} < c_{7s+4h-1} < d_{9s+4h-1} \leq c_{9s+4h-1} \text{ in } P.
\]

The first, third, fifth, seventh, and ninth follow by (1). The second, fourth, sixth, and eighth inequalities follow by Claim 1.

Let $X_1 = \{3s + 1, \ldots, 3s + 2h\}$ and $X_2 = \{7s + 2h, \ldots, 7s + 4h - 1\}$. Note that $|X_1| = |X_2| = 2h$.

Let $N = N(u, v)$ be a separating path associated with $a_{5s+2h} < b_{s+1}$ in $P$ such that $x_0Tu = x_0Nu$ and $vSy_0 = vNy_0$. Let $W = vNu$.

Now consider an element $b_j$ with $j \in X_2$. Recall that $x_0Tb_j$ leaves the right side of $\mathcal{F}$ at element $d_j$. Since $c_{5s+2h} < d_{7s+2h} \leq d_j$ in $P$, the first edge of $x_0Tb$ that is not on the right side of $\mathcal{F}$ is left of $N$. Proposition 10 implies that either all edges and vertices of $d_jTb_j$ (except $d_j$) are left of $N$ or $d_jTb_j$ intersects $N$ at a point distinct from $d_j$. We
want to rule out the second possibility. Suppose to the contrary that \( w \neq d_j \) is an element common to \( d_j Tb_j \) and \( N \). Then \( w \) is in the interior of \( F \). The only section of \( N \) in the interior of \( F \) is \( d_{s+1}Nc_{5s+2h} \), except its endpoints. Note that \( w \) cannot be in \( vNC_{5s+2h} \) as this would imply \( v < c_{5s+2h} < d_j < v \) in \( P \). Clearly that is impossible. Next, \( w \) is not on \( d_{s+1}Nu \) as \( T \) is a tree. Finally, if \( w \) is on \( W = uNu \) then \( W \) intersects \( x_0Tb_j \) and \( x_0Tb_{s+1} \), so Proposition 19 implies \( W \) intersects \( x_0Tb_{5s+2h} \), which implies \( a_{5s+2h} \leq u \leq a_{5s+2h} \) in \( P \), a clear contradiction. Thus, indeed \( b_j \) is left of \( N \), for all \( j \in X_2 \).

**Claim 2.** \( a_{9s+4h-1}Sy_0 \) does not intersect \( W \).

**Proof.** Suppose the conclusion does not hold and take \( w \) to be the greatest element in \( P \) common to \( W \) and \( a_{9s+4h-1}Sy_0 \). Note that \( D = wSca_{9s+4h-1}Wy \) is a cycle in \( G \) with \( D \subseteq Uv(a_{5s+2h}) \). Proposition 14 implies that all elements \( a_i \) with \( i \neq 5s+2h \) are in the exterior of \( D \).

Consider an element \( a_i \) with \( i \in X_2 \). Since \( c_{5s+2h} < c_{7s+2h} \leq c_i \leq c_{7s+4h-1} < c_{9s+4h-1} \) in \( P \), we conclude that the first edge of \( c_iSa_i \) is in the interior of \( D \). Since \( a_i \) is in the exterior of \( D \), the path \( c_iSa_i \) has to intersect \( D \) in a vertex distinct from \( c_i \). Let \( z \) be the least element common to \( a_iSc_i \) and \( D \). Since all of \( D \), except \( uWw \), is contained in the tree \( S \), we must have \( z \) in the interior of \( uWw \). It follows that all edges and vertices of \( a_iSz \), except \( z \) are in the exterior of \( D \).

Let \( e_0 \) and \( e_1 \) be edges incident to \( z \) traversing \( D \) counterclockwise. Let \( e \) be the first edge of \( zSa_i \). Now since \( e \) is in the exterior of \( D \), we have \( e \) right of \( e_1 \) in the \((z,e_0)\)-ordering. Since \( e_0 \) and \( e_1 \) are consecutive edges in \( N \), we conclude that \( e \) is right of \( N \).

We assert that \( a_i \) is right of \( N \). If this statement fails to hold then there is an element \( z' \) distinct from \( z \) common to \( N \) and \( zSa_i \). Note that \( z' \) is not in \( vNC_{5s+2h} = vSc_{5s+2h} \) as \( S \) is a tree. Note also that \( z' \) is not in \( d_{s+1}Nu = d_{s+1}Tu \) as \( F \) is a \( Z \)-face. Also \( z' \) is not in \( vNZ = vWz \) by the choice of \( z \). Finally, \( z' \) is not in \( zNu = zWu \) as this way \( zWz' \) implies \( z < z' \) in \( P \), and \( z'Sz \) implies \( z' < z \) in \( P \), which is a contradiction. This completes the proof that \( a_i \) is right of \( N \).

Thus, all elements of \( A(X_2) \) are right of \( N \). However, this is impossible, by Proposition 20 as \( N \) now separates \( A(X_2) \) from \( B(X_2) \), and \( |X_2| > 2h - 1 \). This completes the proof of Claim 2. \( \square \)

Let \( N' = N'(u',v') \) be a separating path associated with \( a_{9s+4h-1} < b_{5s+2h} \) such that \( x_0Nu' = x_0Tu' \) and \( v'Ny_0 = v'Sy_0 \). Let \( W' = v'Nu' \). The proof of the following claim is dual to the proof of Claim 2.

**Claim 3.** \( x_0Tb_{s+1} \) does not intersect \( W' \).

Note that \( u' \parallel a_{5s+2h} \) in \( P \). Indeed, if \( a_{5s+2h} \leq u' \) in \( P \) then \( a_{5s+2h} \leq u' \leq b_{5s+2h} \) in \( P \) which is false. Also, if \( u' \leq a_{5s+2h} \) in \( P \), then \( x_0 < u' \leq a_{5s+2h} < y_0 \) in \( P \), so we have a
chain from $x_0$ to $y_0$ that goes through $u'$ which is in the interior of $\mathcal{F}$. This contradicts the fact that $\mathcal{F}$ is a $Z$-face. Now Proposition 12 implies that $u'$ is right of $N$.

Now we argue that $v'$ is left of $N$. Proposition 12 implies that either $v'$ is left of $N$ or $v'Sy_0$ intersects $x_0Nv$. However, Claim 2 forbids intersections of $v'Sy_0$ and $uNv = uWv$. Also, $v'Sy_0$ does not intersect $x_0Nv = x_0Tu$ as the first branches of the right side at $c_{9s+4h-1}$ and the latter branches of the right side at $d_{s+1}$. Thus, $v'$ is left of $N$.

Finally, recall that $W'$ is a witnessing path from $v'$ to $u'$. Since $v'$ is left of $N$ and $u'$ is right of $N$, $W'$ must intersect $N$, say at element $w$. Since all elements on $W'$ are interior in $\mathcal{F}$, $w$ must be in the section $d_{s+1}Nc_{5s+2h}$. However, Claim 3 says that $W'$ is disjoint from $d_{s+1}Nu = d_{s+1}Tu$. So $w$ must be in $uNc_{5s+2h}$. But now we have

$$a_{5s+2h} \leq c_{5s+2h} \leq w \leq u' \leq b_{5s+2h}$$

in $P$, which is the final contradiction. □

And as noted previously, this completes the proof of Lemma 18, as well as the principal theorem of the paper.

5. Closing Comments

Since we have not been able to disprove that $\dim(P) = \mathcal{O}(h)$ we comment that our proof for $\mathcal{O}(h^6)$ has three steps where improvements might be possible. Do we really need the $\mathcal{O}(h^3)$ factor in the transition from singly constrained to doubly constrained set of incomparable pairs? When $I$ is a set of doubly constrained pairs, did we need another factor of $h$ to transition to the doubly exposed case? Could $\dim(I)$ be linear in $\rho(\mathcal{F})$ when $\mathcal{F} = (P, G, x_0, y_0, I, D)$ is doubly exposed?

Although we believe the establishment of a polynomial bound for dimension in terms of height for posets with planar cover graphs is intrinsically interesting, we find the results of Section 3, where height plays no role, particularly intriguing. Indeed, we hope that insights from this line of research may help to resolve the following long-standing conjecture.

The standard example number $\text{se}(P)$ of a poset $P$ is 1, if $P$ does not contain a copy of the standard example $S_2$; otherwise it is the largest integer $d \geq 2$ such that $P$ contains a copy of $S_d$. Clearly, $\text{se}(P) \leq \dim(P)$, for all posets $P$. A class $\mathcal{C}$ of posets is dim-bounded if there is a function $f$ such that $\dim(P) \leq f(\text{se}(P))$, for all $P$ in $\mathcal{C}$.

**Conjecture 28.** The class of posets with planar cover graphs is dim-bounded.

Apparently, the first reference in print to Conjecture 28 is in an informal comment on page 119 of [13], published in 1991. However, the problem goes back at least 10 years earlier. In 1978, Trotter [12] showed that there are posets that have large dimension and have planar cover graphs. In 1981, Kelly [8] showed that there are posets that have large dimension and have planar order diagrams. In both of these constructions, the
fact that the posets have large dimension is evidenced by large standard examples that they contain. The belief that large standard examples are necessary for large dimension among posets with planar cover graphs grew naturally from these observations.

To attack Conjecture 28, it is tempting to believe that we can achieve a transition from a singly constrained poset to a doubly exposed poset, independent of height, by allowing a considerable reduction in the dimension $d$.

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