Lie symmetries of the energy-momentum tensor for plane symmetric static spacetimes

K. Saifullah

Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan
Electronic address: saifullah@qau.edu.pk

Abstract: Matter collineations (MCs) are the vector fields along which the energy-momentum tensor remains invariant under the Lie transport. Invariance of the metric, the Ricci and the Riemann tensors have been studied extensively and the vectors along which these tensors remain invariant are called Killing vectors (KVs), Ricci collineations (RCs) and curvature collineations (CCs), respectively. In this paper plane symmetric static spacetimes have been studied for their MCs. Explicit form of MCs together with the Lie algebra admitted by them has been presented. Examples of spacetimes have been constructed for which MCs have been compared with their RCs and KVs. The comparison shows that neither of the sets of RCs and MCs contains the other, in general.
1. Introduction

One of the principal applications of Lie derivatives in theoretical physics is to express
the notion that a tensor field is invariant under some transformation. We say that a
tensor field $T$ is invariant under a vector field $V$ if

$$\mathcal{L}_V T = 0,$$

(1.1)

where $\mathcal{L}_V$ denotes Lie differentiation with respect to the vector $V$. If $T$ has physical
importance then those special vector fields under which $T$ is invariant will also be
important. The manifolds of interest in theoretical physics have metrics, and it is
therefore of considerable interest whenever the metric is invariant with respect to
some vector field. These vector fields are called the Killing vectors (KVs) or isome-
tries. After the spacetime metric, the curvature, the Ricci and the energy-momentum
tensors are other important candidates which play a significant role in understand-
ing the geometric structure and physical properties of spacetimes in relativity. While
the isometries, provide information of the symmetries inherent in the spacetime, the
symmetries of the matter-energy field are provided by MCs, vector fields along which
the energy momentum tensor is invariant under the Lie transport. These symmetry
properties are described by continuous groups of motions or collineations and they
lead to conservation laws. For an introduction to spacetime symmetries and their
significance the reader may see References [1, 2, 3].

Formally, the KV is defined as follows. A manifold $M$ is said to admit a KV (or motion) $\xi^a$ if the Lie derivative of the metric $g_{ab}$ with respect to $\xi^a$ is conserved, i.e.

$$\mathcal{L}_\xi g_{ab} = 0.$$

(1.2)

The vector $\xi^a$ is a Ricci collineation (RC) if the Lie derivative of the Ricci tensor,
$R_{ab}$, with respect to $\xi^a$ is conserved, i.e.

$$\mathcal{L}_\xi R_{ab} = 0.$$

(1.3)

Since the Ricci tensor is built from the metric tensor, it must inherit its symmetries.
Thus if the Lie derivative of $g_{ab}$ vanishes, it must vanish for $R_{ab}$ also. Hence every
KV is an RC but the converse may not be true. The RCs which are not KVs are
called proper RCs [4]. For plane symmetric spacetimes the RCs are finite if the Ricci
tensor is non-degenerate; for degenerate case the RCs may be finite as well as infinite
dimensional [4, 5]. If $R_{ab}$ in Eq. (1.3) is replaced by the energy-momentum tensor,
$T_{ab}$, then the vector $\xi$ is called an MC. In component form this can be expressed as

$$\xi^c T_{ab,c} + T_{ac} \xi^c_b + T_{bc} \xi^c_a = 0.$$

(1.4)
Clearly, every KV is an MC also but the converse is not true. Recently, MCs for different spacetimes have been discussed in the literature [3, 6]. However, very little is known on this important subject, particularly, the relationship between RCs and MCs and there is a need for more research [7]. In this paper we study the Lie symmetries of the energy–momentum tensor, called the matter collineations (MCs), for plane symmetric static spacetimes, and compare them with their KVs and RCs [4, 5, 8].

The plan of the paper is as follows. In Sections 2 and 3 we construct and solve the MC equations. Section 5 contains the algebra of the MCs obtained in the previous section. Examples of the metrics are given in Section 6, where MCs are compared with their RCs and KVs. Concluding remarks are given at the end.

2. The matter collineation equations

We take \((x^0, x^1, x^2, x^3) = (t, x, y, z)\), so that, the most general plane symmetric static line element can be written as [9]

\[
ds^2 = e^{\nu(x)} dt^2 - dx^2 - e^{\mu(x)} (dy^2 + dz^2), \tag{2.1}
\]

where \(\nu\) and \(\mu\) are functions of \(x\) only. For this metric the non-vanishing components of \(T_{ab}\) are

\[
T_{00} = -\frac{e^{\nu(x)}}{4} (4\nu'' + 3\mu^2),
T_{11} = \frac{1}{4} (2\nu'\mu' + \mu^2),
T_{22} = \frac{e^{\nu(x)}}{4} (2\nu'' + 2\mu'' + \nu^2 + \mu^2 + \nu'\mu') = T_{33},
\tag{2.2}
\]

and those of the Ricci tensor are

\[
R_{00} = \frac{e^{\nu}}{4} (2\nu'' + \nu^2 + 2\nu'\mu'),
R_{11} = -\left(\frac{\nu''}{2} + \mu'' + \frac{\nu^2}{4} + \frac{\mu^2}{2}\right),
R_{22} = -\frac{e^{\nu}}{4} (2\mu'' + \nu'\mu' + 2\mu^2) = R_{33}. \tag{2.3}
\]

Here \(\cdot\) denotes differentiation with respect to \(x\). The Ricci scalar is given by

\[
R = \nu'' + 2\mu'' + \frac{1}{2}\left(\nu^2 + 3\mu^2 + 2\nu'\mu'\right). \tag{2.4}
\]

Writing \(T_{ii} = T_i\), for \(i = 0, 1, 2, 3\), the MC equations Eq. (1.4) for the MC vector \(\xi\) take the form

\[
defn{\text{- 3 -}}
\begin{align*}
T'_0 \xi^1 + 2T_0 \xi^0_0 &= 0, \quad (2.5) \\
T_0 \xi^0_1 + T_1 \xi^1_0 &= 0, \quad (2.6) \\
T_0 \xi^0_2 + T_2 \xi^2_0 &= 0, \quad (2.7) \\
T_0 \xi^0_3 + T_3 \xi^3_0 &= 0, \quad (2.8) \\
T'_1 \xi^1_1 + 2T_1 \xi^1_1 &= 0, \quad (2.9) \\
T_1 \xi^1_2 + T_2 \xi^2_1 &= 0, \quad (2.10) \\
T_1 \xi^1_3 + T_2 \xi^3_1 &= 0, \quad (2.11) \\
T'_2 \xi^1_2 + 2T_2 \xi^2_2 &= 0, \quad (2.12) \\
T_2 (\xi^3_3 + \xi^2_2) &= 0, \quad (2.13) \\
T'_2 \xi^1_3 + 2T_2 \xi^3_3 &= 0. \quad (2.14)
\end{align*}

These are ten non-linear coupled partial differential equations for \( \xi^0, \xi^1, \xi^2, \xi^3 \), and \( T_0, T_1, T_2, T_3 \). The \( \xi^i \) depend on \( t, x, y \) and \( z \); and the \( T_i \) on \( x \) only.

### 3. Solution of the matter collineation equations

Solving Eqs. (2.5)- (2.14) simultaneously we obtain the components of the MC vector, \( \xi \). Since the procedure for the solution of similar systems of partial differential equations has appeared in the literature [4, 10], we do not give the calculations and only present the results. The procedure, roughly speaking, is as follows. We first integrate any of these equations to obtain the components of \( \xi \) in terms of arbitrary functions of the coordinates. Using this form of \( \xi \) in other MC equations will give conditions on these arbitrary functions. Going back and forth in this way and checking consistency with the MC equations at every step until these functions are determined explicitly yields the final form of \( \xi \) involving arbitrary constants. In the course of finding these solutions we get constraints on the components of \( T \). Thus we will arrive at various cases of MCs corresponding to these constraints. Solving these constraints, which are often differential in nature, gives the spacetimes. Now, we list these cases.

#### Case 1

**Constraints on \( T_{ab} \):**

\[ T'_0 \neq 0, \left( \frac{T'_2}{T_2 \sqrt{T_1}} \right) \neq 0, \left( \frac{T_0}{T_2} \right)' = 0. \]
MCs: Let $\frac{T_0}{T_2} = k$.

\[
\xi^0 = -\frac{1}{k} (c_6 z + c_5 y) + c_1, \\
\xi^1 = 0, \\
\xi^2 = c_4 z + k c_5 t + c_2, \\
\xi^3 = -y c_4 + k c_6 t + c_3.
\]  
(3.1)

Case 2
Constraints on $T_{ab}$:

\[
T'_0 \neq 0, \left( \frac{T'_2}{T_2 \sqrt{T_1}} \right)' \neq 0, \left( \frac{T'_0}{T_2} \right)' \neq 0.
\]

MCs:

\[
\xi^0 = c_1, \\
\xi^1 = 0, \\
\xi^2 = c_4 z + c_2, \\
\xi^3 = -c_4 y + c_3.
\]  
(3.2)

Case 3
Constraints on $T_{ab}$:

\[
T'_0 \neq 0, \left( \frac{T'_2}{T_2 \sqrt{T_1}} \right)' = 0, T'_2 \neq 0, \left( \frac{T'_2}{T_0} \right)' \neq 0, \left( \frac{T'_0}{T_0 \sqrt{T_1}} \right)' \neq 0.
\]

MCs:

\[
\xi^0 = c_1, \\
\xi^1 = 0, \\
\xi^2 = c_4 z + c_2, \\
\xi^3 = -c_4 y + c_3.
\]  
(3.3)

Case 4
Constraints on $T_{ab}$:

\[
T'_0 \neq 0, \left( \frac{T'_2}{T_2 \sqrt{T_1}} \right)' = 0, T'_2 \neq 0, \left( \frac{T'_2}{T_0} \right)' \neq 0, \left( \frac{T'_0}{T_0 \sqrt{T_1}} \right)' = 0.
\]

MCs: Let $\frac{R'_0}{R_2 \sqrt{R_1}} = \alpha$, $\frac{T'_0}{T_0 \sqrt{T_1}} = \beta \neq \alpha$, where $\alpha$ and $\beta$ are constants.

\[
\xi^0 = \frac{\beta}{\alpha} c_5 t + c_1, \\
\xi^1 = -c_5 \frac{2}{\alpha \sqrt{T_1}}, \\
\xi^2 = c_4 z + c_5 y + c_2, \\
\xi^3 = -c_4 y + c_5 z + c_3.
\]  
(3.4)
Case 5

Constraints on $T_{ab}$:

$$T_0' 
eq 0, \left( \frac{T_2'}{T_2 \sqrt{T_1}} \right)' = 0, T_2' \neq 0, \left( \frac{T_2'}{T_0} \right)' = 0.$$  

MCs: Here we write $\frac{R_2'}{R_2 \sqrt{R_1}} = \alpha$, $T_2 = -\delta T_0$.

$$\begin{align*}
\xi^0 &= \frac{c}{2} \left( t^2 - \frac{4}{\alpha t_0} + \delta y^2 + \delta z^2 \right) + c_6 \delta z - c_8 yt + c_9 tz + c_5 \delta y + c_{10} t + c_1, \\
\xi^1 &= -\frac{2}{\sqrt{\alpha t_1}} (c_7 t - c_8 y + c_9 z + c_{10}), \\
\xi^2 &= c_7 yt + \frac{c}{\sqrt{2}} \left( -t^2 + \frac{4}{\alpha t_2} - y^2 + z^2 \right) + c_5 t + c_9 yz + c_{10} y + c_4 z + c_2, \\
\xi^3 &= c_7 zt - c_8 yz - \frac{c}{\sqrt{2}} \left( -t^2 + \frac{4}{\alpha t_2} + y^2 - z^2 \right) + c_6 t + c_{10} z - c_4 y + c_3.
\end{align*}$$  \hspace{1cm} (3.5)

Case 6

Constraints on $T_{ab}$:

$$T_0' 
eq 0, T_2' = 0, \left( \frac{\sqrt{T_0}'}{\sqrt{T_1}} \right)' = 0.$$  

MCs: Put $\frac{\sqrt{T_0}'}{\sqrt{T_1}} = \gamma$ a constant.

$$\begin{align*}
\xi^0 &= \frac{1}{\sqrt{T_0}} [z (c_7 \sin \gamma t - c_8 \cos \gamma t) + y (c_5 \sin \gamma t - c_6 \cos \gamma t) \\
&\quad - (c_9 \sin \gamma t - c_{10} \cos \gamma t)] + c_1, \\
\xi^1 &= -\frac{1}{\sqrt{T_1}} [z (c_7 \cos \gamma t + c_8 \sin \gamma t) + y (c_5 \cos \gamma t + c_6 \sin \gamma t) \\
&\quad - (c_9 \cos \gamma t + c_{10} \sin \gamma t)], \\
\xi^2 &= \frac{\sqrt{\gamma t_0}}{\gamma t_2} (c_5 \cos \gamma t + c_6 \sin \gamma t) + c_4 z + c_2, \\
\xi^3 &= \frac{\sqrt{T_0}}{\gamma t_2} (c_7 \cos \gamma t + c_8 \sin \gamma t) - c_4 y + c_3.
\end{align*}$$  \hspace{1cm} (3.6)

Case 7

Constraints on $T_{ab}$:

$$T_0' 
eq 0, T_2' = 0, \left( \frac{\sqrt{T_0}'}{\sqrt{T_1}} \right)' \neq 0, \left[ \frac{T_0}{2 \sqrt{T_1}} \left( \frac{T_0'}{T_0} \right) \right]' = 0.$$  

MCs: This implies that $\frac{T_0}{T_0 \sqrt{T_1}}$ = constant = $\lambda \neq 0$.

$$\begin{align*}
\xi^0 &= c_5 \left( \frac{1}{\sqrt{T_0}} - \frac{\lambda}{4} t^2 \right) - c_6 \frac{\lambda}{2} t + c_1, \\
\xi^1 &= \frac{1}{\sqrt{T_1}} (c_5 t + c_6), \\
\xi^2 &= c_4 z + c_2, \\
\xi^3 &= -c_4 y + c_3.
\end{align*}$$  \hspace{1cm} (3.7)
Case 8

Constraints on $T_{ab}$:

\[ T_0' \neq 0, \quad T_2' = 0, \quad \left( \frac{(\sqrt{T_0}')'}{\sqrt{T_1}} \right) \neq 0, \quad \left[ \frac{T_0}{\sqrt{T_1}} \left( \frac{T_0'}{T_0' \sqrt{T_1}} \right) \right]' \neq 0. \]

MCs:

\[
\begin{align*}
\xi^0 &= c_1, \\
\xi^1 &= 0, \\
\xi^2 &= c_4z + c_2, \\
\xi^3 &= -c_4y + c_3.
\end{align*}
\]

(3.8)

Case 9

Constraints on $T_{ab}$:

\[ T_0' = 0, \quad T_2' = 0. \]

MCs: We put $T_0 = -\alpha$ and $T_2 = \beta$.

\[
\begin{align*}
\xi^0 &= -\frac{\alpha}{\beta} \int \sqrt{T_1} \, dx + c_7y + c_8z + c_1, \\
\xi^1 &= \frac{1}{\sqrt{T_1}} \left( c_6t + c_9y + c_{10}z + c_5 \right), \\
\xi^2 &= \frac{\alpha}{\beta} c_7 t - \frac{1}{\beta} c_9 \int \sqrt{T_1} \, dx + c_4z + c_2, \\
\xi^3 &= \frac{\alpha}{\beta} c_8 t - \frac{1}{\beta} c_{10} \int \sqrt{T_1} \, dx - \beta c_4y + c_3.
\end{align*}
\]

(3.9)

Case 10

Constraints on $T_{ab}$:

\[ T_2' \neq 0, \quad \left[ \frac{T_2}{\sqrt{T_1}} \left( \frac{T_2'}{T_2 \sqrt{T_1}} \right) \right]' \neq 0. \]

MCs:

\[
\begin{align*}
\xi^0 &= c_1, \\
\xi^1 &= 0, \\
\xi^2 &= c_4z + c_2, \\
\xi^3 &= -c_4y + c_3.
\end{align*}
\]

(3.10)

Case 11

Constraints on $T_{ab}$:

\[ T_2' \neq 0, \quad \left[ \frac{T_2}{\sqrt{T_1}} \left( \frac{T_2'}{2T_2 \sqrt{T_1}} \right) \right]' = 0, \quad \left( \frac{T_2'}{T_2 \sqrt{T_1}} \right)' \neq 0. \]
MCs:
\[
\begin{align*}
\xi^0 &= c_1, \\
\xi^1 &= 0, \\
\xi^2 &= c_4 z + c_2, \\
\xi^3 &= -c_4 y + c_3.
\end{align*}
\tag{3.11}
\]

Case 12
Constraints on \( T_{ab} \):
\[
T'_0 = 0, T'_2 \neq 0, \left( \frac{T'_2}{T_2 \sqrt{T_1}} \right)' = 0.
\]

MCs: Put \( \frac{T'_2}{T_2 \sqrt{T_1}} = k_1 \)
\[
\begin{align*}
\xi^0 &= c_1, \\
\xi^1 &= \frac{1}{\sqrt{T_1}} (c_6 y + c_7 z + c_5), \\
\xi^2 &= -c_6 \left( \int \frac{\sqrt{T_1}}{T_2} dx + k_1 y^2 - k_1 \frac{z^2}{2} \right) - k_1 c_7 y z - k_1 c_5 y - c_4 z + c_2, \\
\xi^3 &= -k_1 c_6 y z - c_7 \left( \int \frac{\sqrt{T_1}}{T_2} dx - k_1 \frac{z^2}{2} y^2 + k_1 \frac{z^2}{2} \right) - k_1 c_5 z + c_4 y + c_3.
\end{align*}
\tag{3.12}
\]

For the degenerate \( T_{ab} \), i.e. when \( \text{det}(T_{ab}) = 0 \), we get MCs admitting infinite dimensional Lie algebras, except in one case when \( T_1 = 0, T_i \neq 0 \) for \( i = 0, 2, 3 \). Cases 13 and 14 have degenerate energy-momentum tensor but admit finite MCs.

Case 13
Constraints on \( T_{ab} \):
\[
T_1 = 0, T_0 \neq 0, T_2 \neq 0, \left( \frac{T_0 T_2}{T_0 T_2} \right)' = 0.
\]

MCs: Put \( \frac{T'_1 T_2}{T'_0 T_2} = k_2 \)
\[
\begin{align*}
\xi^0 &= c_5 t + c_1, \\
\xi^1 &= -c_5 \frac{2T_1}{T_1}, \\
\xi^2 &= c_4 z + \frac{1}{k_2} c_5 y + c_2, \\
\xi^3 &= -c_4 y + \frac{1}{k_2} c_5 z + c_3.
\end{align*}
\tag{3.13}
\]

Case 14
Constraints on \( T_{ab} \):
\[
T_1 = 0, T_0 \neq 0, T_2 \neq 0, T'_0 \neq 0, T'_2 \neq 0, \left( \frac{T_0}{T_2} \right)' = 0.
\]
MCs: Here we write $T_0 = -\alpha T_2$.

$$\xi^0 = c_7 \left( \frac{y^2}{2} + \frac{z^2}{2} + \alpha t^2 \right) + c_8 t y + c_5 y + c_9 t z + c_6 z + c_{10} t + c_1,$$

$$\xi^1 = -\frac{2T_0}{T_0^2} (\alpha c_7 t + c_8 y + c_9 z + c_{10}),$$

$$\xi^2 = \alpha c_7 t y + c_8 \left( \frac{y^2}{2} - \frac{z^2}{2} \right) + \alpha c_5 t + c_9 y z + c_{10} y - c_4 z + c_2,$$

$$\xi^3 = \alpha c_7 t z + c_8 y z + c_9 \left( \frac{y^2}{2} - \frac{z^2}{2} \right) + \alpha c_6 t + c_{10} z + c_4 y + c_3.$$  \hfill (3.14)

Case 15

If either $T_0$ or $T_2$ (or both) are zero $T_{ab}$ becomes degenerate and MCs admit infinite dimensional Lie algebra.

4. Lie algebras of matter collineations

If a set of vector fields on a manifold under the operation of Lie bracket (defined by the Lie derivative on a manifold) satisfies the conditions of anti-commutativity and Jacobi’s identity, one gets a Lie algebra. Here we provide the Lie algebraic structure for the MC vector fields obtained in the last section and identify their nature. We also classify them into solvable and semisimple algebras and identify some of their sub-algebras.

Case 1

Generators:

$$X_1 = \partial_t,$$

$$X_2 = \partial_y,$$

$$X_3 = \partial_z,$$

$$X_4 = z \partial_y - y \partial_z,$$

$$X_5 = y \partial_t + k^2 t \partial_y,$$

$$X_6 = z \partial_t + k^2 t \partial_z.$$

Algebra:

$$[X_1, X_5] = k^2 X_2,$$

$$[X_1, X_6] = k^2 X_3,$$

$$[X_2, X_4] = -X_3,$$

$$[X_2, X_5] = X_1,$$

$$[X_3, X_4] = X_2,$$

$$[X_3, X_6] = X_1,$$

$$[X_4, X_5] = X_6,$$

$$[X_4, X_6] = -X_5,$$

$$[X_5, X_6] = -k^2 X_4,$$

$$[X_i, X_j] = 0,$$ otherwise.

This is $SO(1, 2) \times [SO(2) \otimes \mathbb{R}^2]$ where ‘×’ represents the semi-direct and ‘⊗’ the direct product. Here $X_5$ and $X_6$ are the Lorentz boosts in $y$ and $z$ directions.
$X_4$ is a rotation in $y$ and $z$. This is a semisimple algebra having $\langle X_4, X_5, X_6 \rangle$ as a subalgebra.

**Case 2**

Generators:

\[
X_1 = \partial_t, \\
X_2 = \partial_y, \\
X_3 = \partial_z, \\
X_4 = z\partial_y - y\partial_z.
\]

Algebra:

\[
[X_2, X_4] = -X_3, \quad [X_3, X_4] = X_2, \quad [X_i, X_j] = 0, \text{ otherwise.}
\]

This can be written as $\{SO(2) \times [\mathbb{R} \otimes SO(2)]\} \otimes \mathbb{R}$ and is solvable.

**Case 3**

Generators:

\[
X_1 = \partial_t, \\
X_2 = \partial_y, \\
X_3 = \partial_z, \\
X_4 = z\partial_y - y\partial_z.
\]

Algebra:

\[
[X_2, X_4] = -X_3, \quad [X_3, X_4] = X_2, \quad [X_i, X_j] = 0, \text{ otherwise.}
\]

This is the same as the previous case.

**Case 4**

Generators:

\[
X_1 = \partial_t, \\
X_2 = \partial_y, \\
X_3 = \partial_z, \\
X_4 = z\partial_y - y\partial_z, \\
X_5 = \frac{\beta}{\alpha}t\partial_t - \frac{2}{\alpha\sqrt{T_1}}\partial_x + y\partial_y + z\partial_z.
\]

Algebra:

\[
[X_1, X_5] = \frac{\beta}{\alpha}X_1, \quad [X_2, X_4] = -X_3, \quad [X_2, X_5] = X_2, \\
[X_3, X_4] = X_2, \quad [X_3, X_5] = X_3, \quad [X_i, X_j] = 0, \text{ otherwise.}
\]

This is a solvable algebra which can be written as $G_5 = \langle G_4, X_5 \rangle$, where
\( G_4 = \{ \text{SO}(2) \times [\mathbb{R} \otimes \text{SO}(2)] \} \otimes \mathbb{R} \).

Case 5

Generators:
\[
\begin{align*}
X_1 &= \partial_t , \\
X_2 &= \partial_y , \\
X_3 &= \partial_z , \\
X_4 &= z\partial_y - y\partial_z , \\
X_5 &= \delta y\partial_t + t\partial_y , \\
X_6 &= \delta z\partial_t + t\partial_z , \\
X_7 &= \frac{1}{2} \left( t^2 - \frac{4}{\sqrt{4} T_0} + \delta y^2 + \delta z^2 \right) \partial_t - \frac{2}{\alpha T_0} t\partial_x + y\partial_y + z t\partial_z , \\
X_8 &= -yt\partial_t + \frac{2}{\alpha T_0} y\partial_x + \frac{1}{2} \left( -t^2 + \frac{4}{\alpha T_0} - y^2 + z^2 \right) \partial_y - yz\partial_z , \\
X_9 &= tz\partial_t - \frac{2}{\alpha T_0} z\partial_x + yz\partial_y - \frac{1}{2} \left( -t^2 + \frac{4}{\alpha T_0} + y^2 - z^2 \right) \partial_z , \\
X_{10} &= t\partial_t - \frac{2}{\alpha T_0} \partial_x + y\partial_y + z\partial_z .
\end{align*}
\]

Algebra:
\[
\begin{align*}
[X_1, X_5] &= X_2 , & [X_1, X_6] &= X_3 , & [X_1, X_7] &= X_{10} , \\
[X_1, X_8] &= \frac{1}{2} X_5 , & [X_1, X_9] &= \frac{1}{2} X_6 , & [X_1, X_{10}] &= X_1 , \\
[X_2, X_4] &= -X_3 , & [X_2, X_5] &= \delta X_1 , & [X_2, X_7] &= X_5 , \\
[X_2, X_8] &= -X_{10} & [X_2, X_9] &= X_4 , & [X_2, X_{10}] &= X_2 , \\
[X_3, X_4] &= X_2 , & [X_3, X_6] &= \delta X_1 , & [X_3, X_7] &= X_6 , \\
[X_3, X_8] &= X_4 , & [X_3, X_9] &= X_{10} , & [X_3, X_{10}] &= X_3 , \\
[X_4, X_5] &= X_6 , & [X_4, X_6] &= -X_5 , & [X_4, X_8] &= -X_9 , \\
[X_4, X_9] &= X_8 , & [X_5, X_6] &= -\delta X_1 , & [X_5, X_7] &= -\delta X_8 , \\
[X_5, X_8] &= -X_7 , & [X_6, X_7] &= \delta X_9 , & [X_6, X_9] &= X_7 , \\
[X_7, X_{10}] &= -X_7 , & [X_8, X_{10}] &= -X_8 , & [X_9, X_{10}] &= -X_9 , \\
[X_i, X_j] &= 0 , \text{ otherwise}.
\end{align*}
\]

This SO(1, 4) or SO(2, 3) is the maximal semisimple anti-de Sitter algebra. It has 3 dimensional subalgebras \( \{ X_4, X_5, X_6 \} \) of rotations and \( \{ X_8, X_9, X_{10} \} \); 4 dimensional subalgebras \( \{ X_1, X_2, X_3, X_4 \} \) and \( \{ X_7, X_8, X_9, X_{10} \} \); and 6 dimensional subalgebras \( \{ X_1, X_2, X_3, X_4, X_5, X_6 \} \) in it.
Case 6

Generators:
\[ X_1 = \partial_t, \]
\[ X_2 = \partial_y, \]
\[ X_3 = \partial_z, \]
\[ X_4 = z\partial_y - y\partial_z, \]
\[ X_5 = \frac{1}{\sqrt{\gamma}} y \sin \gamma t \partial_t - \frac{1}{\sqrt{\gamma}} y \cos \gamma t \partial_x + \frac{\sqrt{\gamma}}{\gamma} \cos \gamma t \partial_y, \]
\[ X_6 = -\frac{1}{\sqrt{\gamma}} y \cos \gamma t \partial_t - \frac{1}{\sqrt{\gamma}} y \sin \gamma t \partial_x + \frac{\sqrt{\gamma}}{\gamma} \sin \gamma t \partial_y, \]
\[ X_7 = \frac{1}{\sqrt{\gamma}} z \sin \gamma t \partial_t - \frac{1}{\sqrt{\gamma}} z \cos \gamma t \partial_x + \frac{\sqrt{\gamma}}{\gamma} \cos \gamma t \partial_z, \]
\[ X_8 = -\frac{1}{\sqrt{\gamma}} z \cos \gamma t \partial_t - \frac{1}{\sqrt{\gamma}} z \sin \gamma t \partial_x + \frac{\sqrt{\gamma}}{\gamma} \sin \gamma t \partial_z, \]
\[ X_9 = -\frac{1}{\sqrt{\gamma}} \sin \gamma t \partial_t + \frac{1}{\sqrt{\gamma}} \cos \gamma t \partial_x, \]
\[ X_{10} = \frac{1}{\sqrt{\gamma}} \cos \gamma t \partial_t + \frac{1}{\sqrt{\gamma}} \sin \gamma t \partial_z. \]

Algebra:
\[
[X_1, X_5] = -X_6, \quad [X_1, X_6] = X_5, \quad [X_1, X_7] = -X_8, \]
\[
[X_1, X_8] = X_7, \quad [X_1, X_9] = -X_{10}, \quad [X_1, X_{10}] = X_9, \]
\[
[X_2, X_4] = -X_3, \quad [X_2, X_5] = -X_9, \quad [X_2, X_6] = -X_{10}, \]
\[
[X_3, X_4] = X_2, \quad [X_3, X_7] = -X_9, \quad [X_3, X_8] = -X_{10}, \]
\[
[X_4, X_5] = X_2, \quad [X_4, X_6] = X_8, \quad [X_4, X_7] = -X_5, \]
\[
[X_4, X_8] = -X_6, \quad [X_5, X_6] = -\frac{1}{\sqrt{\gamma}} X_1, \quad [X_5, X_7] = \frac{1}{\gamma} X_4, \]
\[
[X_5, X_9] = -\frac{1}{\gamma^2} X_2, \quad [X_6, X_8] = \frac{1}{\gamma^2} X_4, \quad [X_6, X_{10}] = -\frac{1}{\gamma^2} X_2, \]
\[
[X_7, X_8] = -\frac{1}{\gamma^2} X_1, \quad [X_7, X_9] = -\frac{1}{\gamma^2} X_3, \quad [X_8, X_{10}] = -\frac{1}{\gamma^2} X_3, \]
\[
[X_i, X_j] = 0, \quad \text{otherwise.} \]

This is again a 10 dimensional semisimple algebra and has \( \{X_1, X_2, X_3, X_4\} \) as a subalgebra.

Case 7

Generators:
\[ X_1 = \partial_t, \]
\[ X_2 = \partial_y, \]
\[ X_3 = \partial_z, \]
\[ X_4 = z\partial_y - y\partial_z, \]
\[ X_5 = \left( \frac{1}{\lambda_0} - \frac{\lambda t^2}{4} \right) \partial_x + \frac{1}{\sqrt{\lambda_1}} t \partial_x, \]
\[ X_6 = -\frac{\lambda}{2} t \partial_t + \frac{1}{\sqrt{\lambda_1}} t \partial_x. \]
Algebra:

\[
\begin{align*}
[X_1, X_5] &= X_6, & [X_1, X_6] &= -\frac{1}{2}X_1, & [X_2, X_4] &= -X_3, \\
[X_3, X_4] &= X_2, & [X_5, X_6] &= \frac{1}{2}X_5, & [X_i, X_j] &= 0, \text{ otherwise.}
\end{align*}
\]

This is a semisimple algebra having \{X_1, X_2, X_3, X_4\} and \{X_1, X_5, X_6\} as sub-algebras.

Case 8

Generators:

\[
\begin{align*}
X_1 &= \partial_t, \\
X_2 &= \partial_y, \\
X_3 &= \partial_z, \\
X_4 &= z\partial_y - y\partial_z.
\end{align*}
\]

Algebra:

\[
\begin{align*}
[X_2, X_4] &= -X_3, & [X_3, X_4] &= X_2, & [X_i, X_j] &= 0, \text{ otherwise.}
\end{align*}
\]

This is the same as in Case 2.

Case 9

Generators:

\[
\begin{align*}
X_1 &= \partial_t, \\
X_2 &= \partial_y, \\
X_3 &= \partial_z, \\
X_4 &= z\partial_y - \frac{\beta}{\gamma} y\partial_z, \\
X_5 &= \frac{1}{\sqrt{T_1}} \partial_x, \\
X_6 &= \frac{1}{\alpha} \int \sqrt{T_1} dx \partial_t + \frac{1}{\sqrt{T_1}} t \partial_x, \\
X_7 &= y\partial_t + \frac{\alpha}{\beta} t \partial_y, \\
X_8 &= z\partial_t + \frac{\alpha}{\gamma} t \partial_z, \\
X_9 &= \frac{1}{\sqrt{T_1}} \partial_x - \frac{1}{\beta} \int \sqrt{T_1} dx \partial_y, \\
X_{10} &= \frac{1}{\sqrt{T_1}} \partial_x - \frac{1}{\gamma} \int \sqrt{T_1} dx \partial_z.
\end{align*}
\]
Algebra:
\[
[X_1, X_6] = X_5, \\
[X_2, X_4] = -\frac{\gamma}{\beta} X_3, \\
[X_3, X_4] = X_2, \\
[X_4, X_7] = X_8, \\
[X_4, X_{10}] = -\frac{\beta}{\alpha} X_9, \\
[X_5, X_{10}] = -\frac{\gamma}{\alpha} X_6, \\
[X_7, X_9] = \frac{\alpha}{\beta} X_6, \\
[X_i, X_j] = 0, \text{ otherwise.}
\]

This semisimple algebra has a 3 dimensional subalgebra \{X_4, X_9, X_{10}\} of rotations, 4 dimensional subalgebra \{X_1, X_2, X_3, X_4\} and 7 dimensional subalgebra \{X_1, X_2, X_3, X_4, X_5, X_9, X_{10}\} in it.

Case 10

Generators:
\[
X_1 = \partial_t, \\
X_2 = \partial_y, \\
X_3 = \partial_z, \\
X_4 = z\partial_y - ky\partial_z.
\]

Algebra:
\[
[X_2, X_4] = -kX_3, \\
[X_3, X_4] = X_2, \\
[X_i, X_j] = 0, \text{ otherwise.}
\]

Case 11

Generators:
\[
X_1 = \partial_t, \\
X_2 = \partial_y, \\
X_3 = \partial_z, \\
X_4 = z\partial_y - y\partial_z.
\]

Algebra:
\[
[X_2, X_4] = -X_3, \\
[X_3, X_4] = X_2, \\
[X_i, X_j] = 0, \text{ otherwise.}
\]

Its structure is similar to that of Case 2.
Case 12

Generators:
\[ X_1 = \partial_t, \quad X_2 = \partial_y, \quad X_3 = \partial_z, \quad X_4 = -z \partial_y + y \partial_z, \quad X_5 = \frac{1}{\sqrt{T_1}} \partial_x - k_1 y \partial_y - k_1 z \partial_z, \quad X_6 = \frac{y}{\sqrt{T_1}} \partial_x - \left( \int \frac{\sqrt{T_1}}{T_2} dx + k_1 \frac{y^2}{2} - k_1 \frac{z^2}{2} \right) \partial_y - k_1 y z \partial_z, \quad X_7 = \frac{-z}{\sqrt{T_1}} \partial_x - k_1 y z \partial_y - \left( \int \frac{\sqrt{T_1}}{T_2} dx - k_1 \frac{y^2}{2} + k_1 \frac{z^2}{2} \right) \partial_z. \]

Algebra:
\[
\begin{align*}
[X_2, X_4] &= X_3, & [X_2, X_5] &= -k_1 X_2, & [X_2, X_6] &= X_5, \\
[X_2, X_7] &= k_1 X_4, & [X_3, X_4] &= -X_2, & [X_3, X_5] &= -k_1 X_3, \\
[X_3, X_6] &= -k_1 X_4, & [X_3, X_7] &= X_5, & [X_4, X_6] &= -X_7, \\
[X_4, X_7] &= -X_5, & [X_5, X_6] &= -X_6, & [X_5, X_7] &= -X_7, \\
[X_6, X_7] &= K X_4, & [X_i, X_j] &= 0 \text{, otherwise.}
\end{align*}
\]

where \( K = -2k_1 \left( \int \frac{\sqrt{T_1}}{T_2} dx + \frac{1}{2k_1 T_2} \right) \) is a constant. This is a semisimple algebra having \( \{X_1, X_2, X_3, X_4\} \) and \( \{X_4, X_5, X_6, X_7\} \) as 4 dimensional subalgebras and a 6 dimensional subalgebra \( \{X_2, X_3, X_4, X_5, X_6, X_7\} \) in it. We write this as \( G_7 = \langle G_4, X_5, X_6, X_7 \rangle \) where \( G_4 = \{SO(2) \times [\mathbb{R} \otimes SO(2)]\} \otimes \mathbb{R}. \)

Case 13

Generators:
\[ X_1 = \partial_t, \quad X_2 = \partial_y, \quad X_3 = \partial_z, \quad X_4 = z \partial_y - y \partial_z, \quad X_5 = t \partial_t - \frac{2T_0}{T_0} \partial_x + \frac{1}{k_2} y \partial_y + \frac{1}{k_2} z \partial_z. \]

Algebra:
\[
\begin{align*}
[X_1, X_5] &= X_1, & [X_2, X_4] &= -X_3, & [X_2, X_5] &= \frac{1}{k_2} X_2, \\
[X_3, X_4] &= X_2, & [X_3, X_5] &= \frac{1}{k_2} X_3, & [X_i, X_j] &= 0 \text{, otherwise.}
\end{align*}
\]

This is a solvable algebra which can be written as \( G = \langle G_4, X_5 \rangle \), where \( G_4 = \{SO(2) \times [\mathbb{R} \otimes SO(2)]\} \otimes \mathbb{R}. \)
Case 14

Generators:
\[
\begin{align*}
X_1 &= \partial_t, \\
X_2 &= \partial_y, \\
X_3 &= \partial_z, \\
X_4 &= -z\partial_y + y\partial_z, \\
X_5 &= y\partial_t + \alpha t\partial_y, \\
X_6 &= z\partial_t + \alpha t\partial_z, \\
X_7 &= \left(\frac{\alpha^2}{2} + \frac{\alpha^4}{2} + \alpha^2 \frac{\alpha^2}{2} \right) \partial_t - 2\frac{\alpha^3}{t_0} t\partial_x + \alpha y\partial_y + \alpha z\partial_z, \\
X_8 &= ty\partial_t - \frac{2\alpha}{t_0} y\partial_x + \left(\frac{\alpha^2}{2} + \frac{\alpha^2}{2} - \frac{\alpha^2}{2} \right) \partial_y + yz\partial_z, \\
X_9 &= tz\partial_t - \frac{2\alpha}{t_0} z\partial_x + yz\partial_y + \left(\frac{\alpha^2}{2} - \frac{\alpha^2}{2} + \frac{\alpha^2}{2} \right) \partial_z, \\
X_{10} &= t\partial_t - \frac{2\alpha}{t_0} \partial_x + y\partial_y + z\partial_z.
\end{align*}
\]

Algebra:
\[
\begin{align*}
[X_1, X_5] &= \alpha X_2, & [X_1, X_6] &= \alpha X_3, & [X_1, X_7] &= \alpha X_{10}, \\
[X_1, X_8] &= X_5, & [X_1, X_9] &= X_6, & [X_1, X_{10}] &= X_1, \\
[X_2, X_4] &= X_3, & [X_2, X_5] &= X_1, & [X_2, X_7] &= X_5, \\
[X_2, X_8] &= X_{10}, & [X_2, X_9] &= -X_4, & [X_2, X_{10}] &= X_2, \\
[X_3, X_4] &= -X_2, & [X_3, X_6] &= X_1, & [X_3, X_7] &= X_6, \\
[X_3, X_8] &= X_4, & [X_3, X_9] &= X_{10}, & [X_3, X_{10}] &= X_3, \\
[X_4, X_5] &= -X_6, & [X_4, X_6] &= X_5, & [X_4, X_8] &= -X_9, \\
[X_4, X_9] &= X_8, & [X_5, X_6] &= \alpha X_4, & [X_5, X_7] &= \alpha X_8, \\
[X_5, X_8] &= X_7, & [X_6, X_7] &= -X_9, & [X_6, X_9] &= X_7, \\
[X_7, X_{10}] &= -X_7, & [X_8, X_{10}] &= -X_8, & [X_9, X_{10}] &= -X_9, \\
[X_i, X_j] &= 0, \text{ otherwise.}
\end{align*}
\]

It has 3 dimensional subalgebras \{X_4, X_5, X_6\} of rotations and \{X_8, X_9, X_{10}\}; 4 dimensional subalgebras \{X_1, X_2, X_3, X_4\} and \{X_7, X_8, X_9, X_{10}\}; and 6 dimensional subalgebras \{X_1, X_2, X_3, X_4, X_5, X_6\} in it. This SO(1,4) or SO(2,3) anti-de Sitter Lie algebra is the maximal semisimple algebra of the degenerate case.

5. Examples of metrics

Here we give a few examples of spacetimes comparing MCs with their RCs [4, 5] and KV [8].

---

\[ -16 - \]
1. 
\[ ds^2 = e^\nu (dt^2 - dy^2 - dz^2) - dx^2, (\nu'' \neq 0). \]

For this metric the non-vanishing components of \( T_{ab} \) are
\[ T_{00} = -\frac{e^\nu}{4} \left( 4\nu'' + 3\nu'^2 \right), \]
\[ T_{11} = \frac{1}{4} \left( 3\mu'' \right), \]
\[ T_{22} = \frac{e^\nu}{4} \left( 4\nu'' + 3\nu'^2 \right) = T_{33}, \]
and those of the Ricci tensor are
\[ R_{00} = \frac{e^\nu}{4} \left( 2\nu'' + 3\nu'^2 \right), \]
\[ R_{11} = -3 \left( \frac{\nu''}{4} + \frac{\nu'^2}{4} \right), \]
\[ R_{22} = -\frac{e^\nu}{4} \left( 2\mu'' + 3\nu'^2 \right) = R_{33}. \]

It has 6 KVs, 6 RCs and 6 MCs given in Case 1.

2. 
\[ ds^2 = e^\nu dt^2 - dx^2 - e^\mu (dy^2 + dz^2), (\nu'' \neq 0, \mu'' \neq 0). \]

It admits 4 KVs, 4 RCs and 4 MCs given by Case 2.

3. 
\[ ds^2 = e^{Ax} (dt^2 - dy^2 - dz^2) - dx^2. \]

This is an anti-de Sitter metric admitting 10 KVs, 10 RCs and 10 MCs given by Case 5.

4. 
\[ ds^2 = x^a dt^2 - dx^2 - x^b (dy^2 + dz^2). \]

The \( T_{ab} \) in this case are
\[ T_{00} = (a - \frac{3}{4}a^2) x^{c-2}, \]
\[ T_{11} = \left( \frac{ca}{4} + \frac{a^2}{4} \right) x^2, \]
\[ T_{22} = \frac{1}{4} \left( a^2 + c^2 - 2c - 2a + ac \right) x^{a-2} = T_{33}. \]

This is an example of Case 4 with 4 KVs, 5 RCs and 5 MCs.
5.

\[ ds^2 = (x/x_0)^{2a} \, dt^2 - dx^2 - (x/x_0)^2 \left( dy^2 + dz^2 \right) , \]

\( a \) and \( x_0 \) are constants and \( a \neq 0, 1, -1 \). For this metric \( R_{ab} \) are given by

\[
R_{00} = a (1 + a) x^{2a-2} / x_0^{2a}, \\
R_{11} = -(-a + a^2) / x^2, \\
R_{22} = -(1 + a) / x_0^2 = R_{33}.
\] (5.4)

The energy-momentum tensor for this metric can be written as with

\[
T_{00} = -x^{2a-2} / x_0^{2a}, \\
T_{11} = (2a + 1) / x^2, \\
T_{22} = a^2 / x_0^2 = T_{33}.
\] (5.5)

It has 4 KVs, 6 RCs and 6 MCs (Case 7). The energy density is negative and cannot be made positive by introducing a cosmological constant, therefore, it is unphysical.

6.

\[ ds^2 = dt^2 - dx^2 - dy^2 - dz^2 . \]

For this metric \( T_{ab} = R_{ab} = 0 \), for all \( a, b \), which means that every direction is an MC and similarly RCs are also arbitrary functions of the coordinates \( t, x, y \) and \( z \) giving an infinite dimensional Lie algebra. However, it has 10 KVs. This is a wrapped Minkowski spacetime.

7.

\[ ds^2 = dt^2 - dx^2 - e^{Ax} \left( dy^2 - dz^2 \right) , \]

\( A \) is a non-zero constant. For this metric the non-vanishing components of \( T_{ab} \) are

\[
T_{00} = -\frac{3A^2}{4}, \\
T_{11} = \frac{A^2}{4}, \\
T_{22} = \frac{A^2 e^{Ax}}{4} = T_{33},
\] (5.6)

and those of the Ricci tensor are

\[
R_{00} = 0, \\
R_{11} = -\frac{A^2}{4}, \\
R_{22} = -\frac{A^2 e^{Ax}}{2} = R_{33}.
\] (5.7)

It admits 7 KVs, infinitely many RCs and 7 MCs, given in Case 12.
\[ ds^2 = e^{Ax} dt^2 - dx^2 - e^{Bx} \left( dy^2 + dz^2 \right), \]

For this metric the non-vanishing components of \( T_{ab} \) are
\[
T_{00} = -\frac{3B^2 e^{Ax}}{4}, \\
T_{11} = \frac{1}{4} (2AB + B^2), \\
T_{22} = \frac{e^{Bx}}{4} (A^2 + B^2 + AB) = T_{33},
\]
and those of the Ricci tensor are
\[
R_{00} = \frac{e^{Ax}}{4} (A^2 + 2AB), \\
R_{11} = - \left( \frac{A^2}{4} + \frac{B^2}{2} \right), \\
R_{22} = -\frac{e^{Bx}}{4} (AB + 2B^2) = R_{33}.
\]

Now, if \( A \neq B \) it admits 5 KVs, 5 RCs and 5 MCs, otherwise it has 10 RCs and 10 MCs.

\[ ds^2 = \left( x/x_0 \right)^{2a} dt^2 - dx^2 - \left( x/x_0 \right)^{4/3} \left( dy^2 + dz^2 \right), \]

\( a \) and \( x_0 \) are constants and \( a \neq 0, 1, -1 \). The energy-momentum tensor for this metric can be written as with
\[
T_{00} = 0, \\
T_{11} = 16/9x^2, \\
T_{22} = - \left( a^2 - \frac{a}{3} - \frac{2}{9} \right) / x^{2/3} x_0^{4/3} = T_{33}.
\]

For this metric \( R_{ab} \) are given by
\[
R_{00} = a \left( a + \frac{1}{3} \right) x^{2a-2} / x_0^{2a}, \\
R_{11} = \left( a - a^2 - \frac{4}{3} \right) / x^2, \\
R_{22} = -\frac{2}{3} \left( a + 3 \right) / x^{2/3} x_0^{4/3} = R_{33}.
\]

This space admits 4 KVs, 5 RCs and infinitely many MCs (Case 15).
6. Conclusion

The plane symmetric static spacetimes have been studied for their MCs. The MC equations have been solved giving rise to various cases characterized by the constraints on the components of the energy-momentum tensor. This includes cases of the non-degenerate as well the degenerate tensor. Their Lie algebra structure has also been given. Particular examples of metrics have been provided for which MCs have been compared with their KVs and RCs. In some spaces the RCs are greater than the MCs, while in others the MCs are more than the RCs, which shows that neither of the sets contains the other, in general.

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