Abstract

In this work we present a useful way to introduce the octonionic projective and hyperbolic plane $OP^2$ through the use of Veronese vectors. Then we focus on their relation with the exceptional Jordan algebra $J_3^0$ and show that the Veronese vectors are the rank-1 elements of the algebra. We then study groups of motions over the octonionic plane recovering all real forms of $G_2$, $F_4$ and $E_6$ groups and finally give a classification of all octonionic and split-octonionic planes as symmetric spaces.

MSC: 17C36, 17C60, 17C90, 22E15, 32M15
Keywords: Exceptional Lie Groups, Jordan Algebra, Octonionic Projective Plane, Real Forms, Veronese embedding.

1 Introduction

The study of the exceptional Jordan algebra and its complexification has been of interest in recent papers of theoretical physics. Todorov, Dubois-Violette[17] and Krasnov[11] characterized the Standard Model gauge group $G_{SM}$ as a subgroup of automorphisms of the exceptional Jordan algebra $J_3^0(0)$ while Boyle[4, 5] pointed to its complexification $J_3^C(0)$. An equivalent well known view of the exceptional Jordan algebra is the one of projective geometry, in which the automorphism group of $J_3^0(0)$ is the group of motions of the octonionic projective plane[10]. Making use of Veronese coordinates we will explore these relations and show how all real forms of $F_4$ and $E_6$ can be recovered as group of motions of projective or hyperbolic planes defined over division Octonions $\mathbb{O}$ or split Octonions $\mathbb{O}_s$.

In sec. 2 we introduce the octonionic projective plane through the use of Veronese coordinates, we define the projective lines and relate the construction with the octonionic affine plane. In sec. 3 we show the correspondance with the exceptional Jordan algebra $J_3^0(0)$ while in sec. 4 we show how real forms of the exceptional Lie Groups $E_6$, $F_4$ and $G_2$, arise as specific groups of collineations of the octonionic projective plane. In the last section we proceed defining in a systematic way all possible octonionic planes as symmetric spaces.
2 The Octonionic Projective Plane

Octonions \( \mathbb{O} \) are, along with Real numbers \( \mathbb{R} \), Complex numbers \( \mathbb{C} \) and Quaternions \( \mathbb{H} \), one of the four Hurwitz algebras, more specifically are the only unital non-associative normed division algebra. A practical way to work with them is to consider their \( \mathbb{R}^8 \) decomposition, i.e.,

\[
x = \sum_{k=0}^{7} x_k i_k
\]  

(1)

where \( \{i_0, i_1, \ldots, i_7\} \) is a basis of \( \mathbb{R}^8 \) and the multiplication rules are mnemonically encoded in the Fano plane (Fig. 1) along with \( i_k^2 = -1 \) for \( k = 1, \ldots, 7 \).

We then define the octonionic conjugate of \( x \) as

\[
x := x_0 i_0 - \sum_{k=1}^{7} x_k i_k
\]  

(2)

with the usual norm

\[
\|x\|^2 = \overline{x} x = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2
\]  

(3)

and the inner product given by the polarisation of the norm, i.e.,

\[
\langle x, y \rangle = \|x + y\|^2 - \|x\|^2 - \|y\|^2 = \overline{y} x + y \overline{x}.
\]  

(4)

In respect to this norm the Octonions are a composition algebra, i.e. \( \|xy\| = \|x\| \|y\| \), which will be of paramount importance in the following sections. Finally, we denote with \( \mathbb{O}_s \) the split-octonionic algebra, whose definition can be found in [13, 14].
The Projective Plane  It is a common practice defining a projective plane 
over an associative division algebra starting from a vector space over the given 
algebra, e.g. \( \mathbb{R}^{n+1} \), and then define the projective space as the quotient 
\[ \mathbb{R}P^n = \mathbb{R}^{n+1} / \sim \] 

where \( x \sim y \) if \( x \) and \( y \) are multiple through the scalar field, i.e. \( \lambda x = y, \lambda \in \mathbb{R}, \lambda \neq 0, x, y \in \mathbb{R}^{n+1} \). But, since the algebra of Octonions is not associative, we 
have that \( x(\lambda \mu) \neq (x\lambda)\mu \) when \( \lambda, \mu \in \mathbb{O} \). If we then try to define the equivalence 
relation as above, we then might have \( x \sim y = x\lambda, \) and \( x \sim z = x(\lambda \mu), \) but \( z \) not related to \( y \) since 
\[ z = x(\lambda \mu) \neq (x\lambda)\mu = y\mu. \] 

Therefore the previous is not an equivalence relation and the quotient cannot 
be defined. A method for overcoming such an issue is based on determining 
an equivalent algebraic definition of the rank-one idempotent of the exceptional 
Jordan algebra in order define points in the projective plane, but here we want to 
use a direct and less known way to proceed making use of the Veronese vectors.

Veronese coordinates  Let \( V \cong \mathbb{O}^3 \times \mathbb{R}^3 \) be a real vector space, with elements of the form 
\[ (x_\nu; \lambda_\nu)_\nu = (x_1, x_2, x_3; \lambda_1, \lambda_2, \lambda_3) \] 
where \( x_\nu \in \mathbb{O}, \lambda_\nu \in \mathbb{R} \) and \( \nu = 1, 2, 3 \). A vector \( w \in V \) is called Veronese if 
\[ \lambda_1 x_1 = x_2 x_3, \lambda_2 x_2 = x_3 x_1, \lambda_3 x_3 = x_1 x_2 \] 
\[ \|x_1\|^2 = \lambda_2 \lambda_3, \|x_2\|^2 = \lambda_3 \lambda_1, \|x_3\|^2 = \lambda_1 \lambda_2. \] 

Let \( H \subset V \) be the subset of Veronese vectors. If \( w = (x_\nu; \lambda_\nu)_\nu \) is a Veronese 
vector then also \( \mu w = \mu (x_\nu; \lambda_\nu)_\nu \) is a Veronese vector, that is \( \mathbb{R}w \subset H \). We 
define the Octonionic plane \( \mathbb{O}P^2 \) as the geometry having this one-dimensional 
subspaces \( \mathbb{R}w \) as points, i.e. 
\[ \mathbb{O}P^2 = \{ \mathbb{R}w : w \in H \setminus \{0\} \}. \] 

Remark 2.1. The point in the projective plane is defined as the equivalence class 
\( \mathbb{R}w \) of the Veronese vector \( w \), but, in order to determine an explicit relation 
between points in the projective plane and rank-one idempotent elements of the 
Jordan algebra \( \mathfrak{J}_0^3 \), we will choose when as representative of the class the vector 
\( v = (y_\nu; \xi_\nu)_\nu \in \mathbb{R}w \) such that \( \xi_1 + \xi_2 + \xi_3 = 1 \). Then \( (y_\nu; \xi_\nu)_\nu \) are called Veronese 
coordinates of the projective point.

Projective lines  We then define projective lines of \( \mathbb{O}P^2 \) as the vectors orthogonal 
to the points \( \mathbb{R}w \). Let \( \beta \) be the bilinear form over \( \mathbb{O}^3 \times \mathbb{R}^3 \) defined as
\[
\beta(w_1, w_2) = \sum_{\nu=1}^{3} (\langle x_1^\nu, x_2^\nu \rangle + \lambda_1^\nu \lambda_2^\nu)
\]  
(10)

where \( w_1 = (x_1^\nu; \lambda_1^\nu) \), \( w_2 = (x_2^\nu; \lambda_2^\nu) \) \( \in \mathbb{O}^3 \times \mathbb{R}^3 \). Then, for every Veronese vector \( w \), corresponding to the point \( \mathbb{R}w \) in \( \mathbb{O}P^2 \), we define a line \( \ell \) in \( \mathbb{O}P^2 \) as the orthogonal space

\[
\ell := w^\perp = \{ z \in \mathbb{O}^3 \times \mathbb{R}^3 : \beta(z, w) = 0 \}.
\]  
(11)

The bilinear form \( \beta \) also defines the elliptic polarity, i.e. the map \( \pi^+ \) that corresponds points to lines and lines to points, i.e.

\[
\pi^+(w) = w^\perp, \pi^+(w^\perp) = w
\]  
(12)

where the orthogonal space to a vector is defined by the bilinear form \( \beta(\cdot, \cdot) \), so that

\[
\pi^+: w \rightarrow \{ \beta(\cdot, w) = 0 \}
\]  
(13)

\[
\ell \rightarrow w
\]  
(14)

when \( \ell \) is given by \( \{ \beta(\cdot, w) = 0 \} \). Explicitly, \( \beta(w_1, w_2) = 0 \) when

\[
2\pi_1^1x_1^2 + 2\pi_2^1x_2^2 + 2\pi_3^1x_3^2 + \lambda_1^1\lambda_1^2 + \lambda_2^1\lambda_2^2 + \lambda_3^1\lambda_3^2 = 0.
\]  
(15)

In addition to the elliptic polarity defined above, we then define the hyperbolic polarity \( \pi^- \), which still has

\[
\pi^-(w) = w^\perp, \pi^-(w^\perp) = w
\]  
(16)

but through the use of the bilinear form \( \beta_- \) that has a change of sign in the last coordinate, i.e. \( \beta_-(w_1, w_2) = 0 \) is given by

\[
2\pi_1^1x_1^2 + 2\pi_2^1x_2^2 - 2\pi_3^1x_3^2 + \lambda_1^1\lambda_1^2 + \lambda_2^1\lambda_2^2 - \lambda_3^1\lambda_3^2 = 0.
\]  
(17)

A projective plane equipped with the hyperbolic polarity will be called hyperbolic plane and denoted as \( \mathbb{O}H^2 \).

**The Affine Plane** The octonionic projective plane is also the completion of the octonionic affine plane. The embedding of the affine plane can be explicated through the use of Veronese coordinates defining the map that sends a point \((x, y)\) of the affine plane to the projective point \( \mathbb{R} \left( x, y; \|y\|^2, \|x\|^2, 1 \right) \), i.e.

\[
(x, y) \mapsto \mathbb{R} \left( x, y; \|y\|^2, \|x\|^2, 1 \right)
\]  
(18)

which is an homeomorphism. To complete the affine plane, we then have to extend the map to another set of coordinates, i.e.

\[
(x) \mapsto \mathbb{R} \left( 0, 0, x; \|x\|^2, 1, 0 \right)
\]  
(19)

\[
(\infty) \mapsto \mathbb{R} \left( 0, 0, 0; 1, 0, 0 \right).
\]  
(20)
Figure 2: Representation of the affine plane: (0, 0) represents the origin, (0) the point at the infinity on the x-axis, (s) is the point at infinity of the line [s, t] of slope s while (∞) is the point at the infinity on the y-axis and of vertical lines [c].

Remark 2.2. To show that the above is a Veronese vector and therefore that the map is well defined, we made essential use of alternativity of the Octonions and fact that Octonions are a composition algebra. In case of non-composition algebra, though the definition of the projective and hyperbolic planes would still be valid using Veronese coordinates, the geometry of these planes will not satisfy the basic axioms of projective and affine geometry and therefore they would have to be considered as "generalised" projective or hyperbolic planes.

Moreover, let [s, t] be a line in the affine plane OA² of the form

\[ [s, t] = \{ (x, sx + t) : x \in \mathbb{O} \} \]  \hspace{1cm} (21)

where s is the slope of the line. Then [s, t] is mapped into the projective line orthogonal to the vector \( \left( s\bar{t}, -\bar{t}, -s; 1, \|s\|^2, \|t\|^2 \right) \), i.e.

\[ [s, t] \mapsto \mathbb{P} \left( s\bar{t}, -\bar{t}, -s; 1, \|s\|^2, \|t\|^2 \right)^\perp. \]  \hspace{1cm} (22)

Vertical lines [c] that are of the form \( \{ c \} \times (\mathbb{O}) \), are mapped into lines of \( \mathbb{O}P^2 \) given by

\[ [c] \mapsto \mathbb{P} \left( -c, 0, 0; 0, 1, \|c\|^2 \right)^\perp. \]  \hspace{1cm} (23)

Finally the line at infinity [∞] is mapped to the orthogonal space of the vector

\[ [\infty] \mapsto \mathbb{P} (0, 0, 0; 0, 1)^\perp. \]  \hspace{1cm} (24)
3 The Exceptional Jordan Algebra $J_3 (O)$

The exceptional Jordan algebra $J_3 (O)$ is the algebra of Hermitian three by three octonionic matrices with the Jordan product

$$X \circ Y = \frac{1}{2} (XY + YX).$$  \hspace{1cm} (25)

It is easy to see that $J_3 (O)$ is commutative, i.e. $X \circ Y = Y \circ X$ and satisfies the Jordan identity

$$ (X^2 \circ Y) \circ X = X^2 \circ (Y \circ X).$$  \hspace{1cm} (26)

We then define the bilinear form

$$ (X, Y) = \frac{1}{2} \text{tr}(X \circ Y) $$  \hspace{1cm} (27)

the quadratic form whose the previous bilinear form is a polarisation

$$ Q(X) = \frac{1}{2} \text{tr}(X^2) $$  \hspace{1cm} (28)

and the Freudenthal product, i.e.

$$ X \ast Y = X \circ Y - \frac{1}{4} (X \text{tr}(Y) + Y \text{tr}(X)) + \frac{1}{4} (\text{tr}(X) \text{tr}(Y) - \text{tr}(X \circ Y)) I_3 $$  \hspace{1cm} (29)

where $I_3 = \text{diag}(+, +, +)$, along with the symmetric trilinear form

$$ (X, Y, Z) = \frac{1}{3} (X, Y \ast Z) $$  \hspace{1cm} (30)

and the determinant

$$ \det (X) = \frac{1}{3} (X, X, X). $$  \hspace{1cm} (31)

Now, let $\mathbb{R} w$ be a point in the projective plane $O P^2$, related to a vector in $O^3 \times \mathbb{R}^3$ with Veronese coordinates $w = (x_\nu; \lambda_\nu)$, and consider the map from $V \cong O^3 \times \mathbb{R}^3$ into the space of three by three Hermitian matrices with octonionic coefficients, defined as

$$(x_\nu; \lambda_\nu) \mapsto \left( \begin{array}{ccc} \lambda_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \lambda_2 & x_1 \\ \overline{x_2} & \overline{x_1} & \lambda_3 \end{array} \right).$$  \hspace{1cm} (32)

We then have that

$$ \det (X) = \lambda_1 \lambda_2 \lambda_3 - \lambda_1 \| x_1 \|^2 - \lambda_2 \| x_2 \|^2 - \lambda_3 \| x_3 \|^2 + 2 \text{Re} ((x_1 x_2) x_3) $$  \hspace{1cm} (33)

that, imposing the Veronese conditions translates to $\det (X) = 0$. 

Moreover, let $X^\sharp$ be the image of a non-zero element $X$ under the adjoint $(\cdot)$-map of $\mathfrak{j}_3^6$, which is given by (cf. Example 5 of [12])

$$X^\sharp := \begin{pmatrix}
\lambda_2 \lambda_3 - \|x_1\|^2 & x_1 x_3 - \lambda_2 x_2 & x_3 x_1 - \lambda_2 x_2 \\
x_1 x_2 - \lambda_3 x_3 & \lambda_1 \lambda_3 - \|x_2\|^2 & x_3 x_2 - \lambda_1 x_1 \\
x_1 x_3 - \lambda_2 x_2 & x_2 x_3 - \lambda_1 x_1 & \lambda_1 \lambda_2 - \|x_3\|^2
\end{pmatrix}.$$  

(34)

From this explicit expression, it is immediate to realize that the Veronese conditions are equivalent to the vanishing of $X^\sharp$. Then, by the $\text{Str}(\mathfrak{j}_3(\mathbb{O}))$-invariant definition of the rank of an element of $\mathfrak{j}_3(\mathbb{O})$ [9], one obtains that the Veronese conditions are equivalent to the rank-1 condition for an element of $\mathfrak{j}_3(\mathbb{O})$.

Thus, from the knowledge of the orbit stratification of $\mathfrak{j}_3^6$ under the non-transitive action of its reduced structure group $\text{Str}_0(\mathfrak{j}_3(\mathbb{O})) \simeq E_{6(-26)}$, it follows that the Veronese conditions for a non-zero element of $\mathfrak{j}_3(\mathbb{O})$ are equivalent to imposing that such an element belongs to the (unique) rank-1 orbit of $E_{6(-26)}$ in $\mathfrak{j}_3(\mathbb{O})$ (cf. [17], and Refs. therein).

Now we want to show that in the rank-1 (unique) orbit of $\mathfrak{j}_3(\mathbb{O})$, idempotency is equivalent to the condition of unitary trace. In order to do that, let us consider the element $X^\sharp$ and let us impose the condition of $X$ being of rank $= 1$. Since this condition is equivalent to the Veronese conditions, one obtains

$$X^\sharp = \begin{pmatrix}
\lambda_1^2 + \lambda_2 \lambda_3 & (\lambda_1 + \lambda_2 + \lambda_3) x_3 & (\lambda_1 + \lambda_2 + \lambda_3) x_1 \\
(\lambda_1 + \lambda_2 + \lambda_3) x_3 & \lambda_2^2 + \lambda_2 \lambda_3 & (\lambda_1 + \lambda_2 + \lambda_3) x_2 \\
(\lambda_1 + \lambda_2 + \lambda_3) x_2 & (\lambda_1 + \lambda_2 + \lambda_3) x_1 & \lambda_3^2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3
\end{pmatrix}.$$  

(35)

from which it follows that $X^\sharp = X$ if and only if

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$  

(36)

i.e. if and only if $\text{tr}(X) = 1$. Thus, the idempotency condition for rank-1 elements of $\mathfrak{j}_3^6$ is equivalent to the condition of unitary trace.

4 Lie Groups of Type $G_2$, $F_4$ and $E_6$ as Groups of Collineations

We are now interested in the motions and symmetries of the octonionic projective plane. More specifically we are interested in collineations that are transformations of the projective plane that send lines into lines. If the collineation preserves the elliptic polarity or the hyperbolic polarity is then called elliptic or hyperbolic motion. Elliptic and hyperbolic motions are an equivalent characterization of the isometries of the projective or hyperbolic plane respectively, thus the elliptic motion group of the projective plane will be indicated as $\text{Iso}(OP^2)$ and the hyperbolic motion group as $\text{Iso}(OH^2)$.

Collineations of the Octonionic Projective Plane A collineation is a bijection $\varphi$ of the set of points of the plane onto itself, mapping lines onto lines.
It is straightforward to see that the identity map is a collineation, as the inverse \( \varphi^{-1} \) and the composition \( \varphi \circ \varphi' \) are if \( \varphi, \varphi' \) are both collineation. Therefore the set \( \text{Coll}(\mathbb{O}P^2) \) of collineations is a group under composition of maps. It also has a proper subgroup of order three generated by the triality collineation that permutes three special points of the affine/projective octonionic plane, i.e. the origin of coordinate \((0,0)\), the point at the origin of the line at infinity which has coordinate \((0)\) and the point at infinity of the line at the infinity which has affine coordinate \((\infty)\). In Veronese coordinates these three points are images of the following vectors

\[
(0,0) \rightarrow \mathbb{R}(0,0;0,0,1) \\
(0) \rightarrow \mathbb{R}(0,0;0,1,0) \\
(\infty) \rightarrow \mathbb{R}(0,0;1,0,0)
\]  

and the triality collineation \( \tau \) is given by

\[
(x_1, x_2, x_3; \lambda_1, \lambda_2, \lambda_3) \mapsto (x_2, x_3, x_1; \lambda_2, \lambda_3, \lambda_1)
\]  

that is a cyclic permutation of order three that leaves invariant the Veronese vectors. This means that it induces a bijection \( \tau \) on \( \mathbb{O}P^2 \) that is unseen by the bilinear form \( \beta \) and therefore maps lines into lines, since lines are constructed as the orthogonal space of a vector through the bilinear form \( \beta \).

Let us now consider the transformations \( T_{a,b} \) of \( \mathbb{O}^3 \times \mathbb{R}^3 \) into itself defined on the Veronese coordinates as

\[
\begin{align*}
 x_1 &\rightarrow x_1 + \lambda_3 a \\
 x_2 &\rightarrow x_2 + \lambda_3 b \\
 x_3 &\rightarrow x_3 + \langle x_3, a \rangle + \lambda_3 \|a\|^2 \\
 \lambda_1 &\rightarrow \lambda_1 + \langle x_2, a \rangle + \lambda_3 \|b\|^2 \\
 \lambda_2 &\rightarrow \lambda_2 + \langle x_1, a \rangle + \lambda_3 \|a\|^2 \\
 \lambda_3 &\rightarrow \lambda_3 
\end{align*}
\]  

Those are in fact translations on the affine plane corresponding to the transformation \((x_1, x_2) \rightarrow (x_1 + a, x_2 + b)\) and they all induce collineations \( T_{a,b} \) on \( \mathbb{O}P^2 \).

It can be shown that all collineations are generated by the interplay between a translation and the conjugation of a power of the triality collineation, i.e. are of the form

\[
T_{a,b}, \quad \tau T_{a,b} \tau^{-1}, \quad \tau^2 T_{a,b} \tau^{-2}.
\]  

From another perspective, collineations transform lines of \( \mathbb{O}P^2 \) in lines of \( \mathbb{O}P^2 \). This is equivalent to find all the linear transformations \( A \) of \( V \) in itself such that the image of Veronese vectors is still a Veronese vector \( A(H) \subset H \). If this condition is fulfilled, the linear transformation \( A \) in \( \text{End}(V) \) will induce
a collineation $\varphi$ on $O^2_P$, i.e.

\[
\begin{array}{ccc}
\mathbb{O}^3 \times \mathbb{R}^3 & \xrightarrow{A} & \mathbb{O}^3 \times \mathbb{R}^3 \\
\uparrow & & \uparrow \\
O^2_P & \xrightarrow{A} & O^2_P \\
\end{array}
\] (43)

Since all linear multiple of the transformation $A$ will produce the same collineation $\varphi$, to have a bijection between linear transformations and collineations we have to impose also $\det(A) = 1$. That is that the group of collineation $\text{Coll}(O^2_P)$ is

\[
\text{SL}(V, H) := \{ A \in \text{End}(V) : A(H) \subseteq H; \det(A) = 1 \}.
\] (44)

If we also impose the preservation of the elliptic polarity, i.e. of the bilinear form $\beta$, we will then have the group of elliptic motion $\text{Iso}(O^2_P)$ that is

\[
\text{SU}(V, H) = \{ A \in \text{End}(V) : A(H) \subseteq H; \det(A) = 1; \text{tr}(A) = 1 \}.
\] (45)

Those two groups are in fact two exceptional Lie Groups, i.e.

\[
\text{Coll}(O^2_P) \cong \text{SL}(V, H) \cong E_6
\] (46)

\[
\text{Iso}(O^2_P) \cong \text{SU}(V, H) \cong F_4.
\] (47)

The identification of this two group is done through a direct determination of the generators as in [13]; instead, we will here follow Rosenfeld in [4] focusing on the Lie algebra of the group of collineations on $O^2_P$, i.e. $\text{coll}(O^2_P)$, which is given by

\[
\text{coll}(O^2_P) = g_2 \oplus a_3 (0)
\] (48)

where $g_2 \cong \text{der}(0)$ and $a_3$ are the three by three matrices on 0 with null trace, i.e. $\text{tr}(A) = 0$. The dimension count on the possible generators of this algebra, since the only condition you have is to have null trace, i.e. $\text{tr}(A) = 0$, gives as only condition on

\[
A = \begin{pmatrix}
    a_1^1 & a_1^2 & a_1^3 \\
    a_2^1 & a_2^2 & a_2^3 \\
    a_3^1 & a_3^2 & a_3^3
\end{pmatrix}
\] (49)

the condition on the trace, i.e. $a_3^3 = -(a_1^1 + a_2^2)$, and therefore we have 8 entries of dimension 8 and $\dim_{\mathbb{R}} a_3 = 64$. We therefore have

\[
\dim_{\mathbb{R}} \text{coll}(O^2_P) \cong 78 = 64 + 14.
\] (50)

Since $\text{coll}(O^2_P)$ is a Lie group, simple and of dimension 78, then it must be of $E_6$ type.

**Isometries of the Plane** Again, following Rosenfeld [4] we look at the elliptic motions of $O^2_P$, which are the collineations that preserve also the polarity $\pi^+$ or equivalently the form $\beta$; they are given by

\[
\text{iso}(O^2_P) = g_2 \oplus sa(3)
\] (51)
where we notated $sa(3)$ the skew-Hermitian matrices with null trace. Here the elements of $sa(3)$ are of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & -\bar{a}_{13} \\ -\bar{a}_{21} & a_{22} & a_{23} \\ a_{31} & -a_{32} & a_{33} \end{pmatrix}$$ (52)

with $a_{ij} = \bar{a}_{ji}$, $a_{33} = -(a_{11} + a_{22})$ and $\text{Re} \ (a_{11}) = \text{Re} \ (a_{22}) = 0$. We therefore have 3 coefficient of dimension 8, 2 entries of dimension 7 and therefore $\dim_{\mathbb{R}} sa(3) = 38$ so that

$$\dim_{\mathbb{R}} \text{iso}(OP^2) \cong 52 = 38 + 14$$ (53)

and the group of elliptic motion $\text{Iso}(OP^2)$, being simple and of dimension 52, is of the $F_4$ type. Moreover we can proceed as in [4] to find the collineations that preserve the hyperbolic polarity $\pi^-$ or equivalently the form $\beta_-$ we previously defined. Here the element of the Lie algebra are of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \bar{a}_{13} \\ \bar{a}_{21} & a_{22} & a_{23} \\ a_{31} & -a_{32} & a_{33} \end{pmatrix}$$ (54)

with $a_{11}^2 = (a_{22} + a_{33})$ and $\text{Re} \ (a_{11}) = \text{Re} \ (a_{22}) = 0$, therefore leading to the same count of the dimension of

$$\dim_{\mathbb{R}} \text{iso}(OH^2) \cong 52 = 38 + 14$$ (55)

deducing that $\text{Iso}(OH^2)$ is again an $F_4$ type group.

**Collineations with a Fixed Triangle or Quadrangle** We are now interested in studying the collineations $\varphi$ on the affine plane that fix every point of $\triangle$, i.e. $\varphi((0,0)) = (0,0)$, $\varphi((0)) = (0)$ and $\varphi((\infty)) = (\infty)$.

**Proposition 1.** The group $\Gamma(\triangle, 0)$ of collineations that fix every point of $\triangle$ are transformations of this form

$$(x, y) \mapsto (A(x), B(y)) \quad (s) \mapsto (C(s)) \quad (\infty) \mapsto (\infty)$$ (56, 57, 58)

where $A$, $B$ and $C$ are automorphisms with respect to the sum over $O$ and that satisfy

$$B(sx) = C(s)A(x).$$ (59)

**Proof.** A collineation $\varphi$ that fixes $(0,0)$, $(0)$ and $(\infty)$, also fixes the $x$-axis and $y$-axis and all lines that are parallel to them. This means that the first coordinate is the image of a function that does not depend on $y$ and the second coordinate is image of a function that does not depend of $x$, i.e. $(x, y) \mapsto (A(x), B(y))$ and
$(s) \mapsto (C(s))$. Now consider the image of a point on the line $[s,t]$. The point is of the form $(x, sx + t)$ and its image goes to

$$(x, sx + t) \mapsto (A(x), B(sx + t)).$$

(60)

If we want this to be a collineation, the points of the line $[s,t]$ must all belong to the same line which can be easily identified setting $x = 0$, i.e. the image of $[s,t]$ is the line that joins the points $(0, B(t))$ and $(C(s))$. We now have that the condition for $(A(x), B(sx + t))$ to be in the image of $[s,t]$ is

$$B(sx + t) = C(s)A(x) + B(t).$$

(61)

Now, if $B$ is an automorphism with respect to the sum over $O$, we then have the condition $B(sx) = C(s)A(x)$. Conversely if $B(sx) = C(s)A(x)$ is true then $B(sx + t) = B(sx) + B(t)$, and $B$ is an automorphism with respect to the sum.

Let us consider the quadrangle $\square$ given by the points $(0,0), (1,1), (0)$ and $(\infty)$, that is $\square = \triangle \cup \{(1,1)\}$, and consider the collineations that fix the $\square$. Since in addition to the previous case we also have to impose

$$(1,1) \mapsto (A(1), B(1)) = (1,1)$$

(62)

then $C(1) = 1$ and, therefore $A = B = C$ and therefore $A$ is an automorphism of $O$. We then have the following

**Proposition 2.** The collineations that fix every point of $\square$ are transformations of the type

$$(x,y) \mapsto (A(x), A(y))$$

(63)

$$(s) \mapsto (A(s))$$

(64)

$$(\infty) \mapsto (\infty)$$

(65)

where $A$ is an automorphism of $O$.

Moreover, since $\text{Aut}(O) = G_{2(-14)}$ and $\text{Aut}(O_+)$ is $G_{2(2)}$, we have the following

**Corollary 1.** The group of collineations $\Gamma(\square,0)$ that fix $(0,0), (1,1), (0)$ and $(\infty)$ is isomorphic to $\text{Aut}(O)$. Therefore $\Gamma(\square,0)$ is isomorphic to $G_{2(-14)}$, while in the case of split octonions $O_+$ is isomorphic to $G_{2(2)}$.

It can be shown that the group of collineations $\Gamma(\triangle,0)$ is in fact the double cover of $SO_8(R)$, i.e. $\text{Spin}_8(R)$, that we define here as

$$\text{Spin}(O) = \left\{(A,B,C) \in O^+(O)^3 : A(xy) = B(x)C(y) \ \forall x,y \in O\right\}$$

(66)

where $O^+$ is the connected component of the orthogonal group with the identity.
Proposition 3. The Lie algebra $\text{Lie} (\Gamma (\triangle, 0))$ of the group of collineation that fixes $(0,0),(0)$ and $(\infty)$

$$\text{tri} (0) = \left\{ (T_1, T_2, T_3) \in \mathfrak{so} (0)^3 : T_1 (xy) = T_2 (x) y + x T_3 (y) \right\} \quad (67)$$

while the Lie algebra $\text{Lie} (\Gamma (\square, 0))$ of the group of collineation that fixes $(0,0),(1,1),(0)$ and $(\infty)$ is

$$\text{der} (0) = \left\{ T \in \mathfrak{so} (0) : T (xy) = T (x) y + x T (y) \right\} . \quad (68)$$

Proof. $\Gamma (\triangle, 0)$ is a Lie group since it is a closed subgroup of the Lie group of collineations. We will find directly its Lie algebra considering the elements $A,B,C \in \Gamma (\triangle, 0)$ in a neighbourhood of the identity and writing them as

$$(A, B, C) \rightarrow (\text{Id} + \epsilon T_1, \text{Id} + \epsilon T_2, \text{Id} + \epsilon T_3)$$

where $T_1, T_2, T_3 \in \mathfrak{so} (0)$. Imposing the condition $A (xy) = B (x) C (y)$ and then we obtain

$$(\text{Id} + \epsilon T_1) (xy) = (\text{Id} + \epsilon T_2) (x) (\text{Id} + \epsilon T_3) (x) \quad (69)$$

which, considering $\epsilon^2 = 0$, yields to

$$T_1 (xy) = T_2 (x) y + x T_3 (y) . \quad (70)$$

The second part of the proposition is obtained imposing $T_1 = T_2 = T_3 = T$. $\Box$

We then have the following

$$\Gamma (\triangle, 0) \cong \text{Spin} (0) \cong \text{Spin}_8 (\mathbb{R}) \quad (71)$$
$$\Gamma (\square, 0) \cong \text{Aut} (0) \cong G_{2(-14)} \quad (72)$$

and, passing to Lie algebras, we obtain

$$\text{Lie} (\Gamma (\triangle, 0)) \cong \text{tri} (0) \cong \mathfrak{so} (0) \quad (73)$$
$$\text{Lie} (\Gamma (\square, 0)) \cong \text{der} (0) \cong \mathfrak{g}_{2(-14)}. \quad (74)$$

By considering the split Octonions $\mathcal{O}_s$, previous formulas yield to

$$\Gamma (\triangle, \mathcal{O}_s) \cong \text{Spin} (\mathcal{O}_s) \cong \text{Spin}_{(4,4)} (\mathbb{R}) \quad (75)$$
$$\Gamma (\square, \mathcal{O}_s) \cong \text{Aut} (0) \cong G_{2(2)} \quad (76)$$

and, passing to Lie algebras, we obtain

$$\text{Lie} (\Gamma (\triangle, \mathcal{O}_s)) \cong \text{tri} (\mathcal{O}_s) \cong \mathfrak{so} (\mathcal{O}_s) \cong \mathfrak{so}_{4,4} \quad (77)$$
$$\text{Lie} (\Gamma (\square, \mathcal{O}_s)) \cong \text{der} (\mathcal{O}_s) \cong \mathfrak{g}_{2(2)}. \quad (78)$$

Resuming all the findings, following Yokota [18, p.105] in the definition of real forms of $E_6$, we then have the following motion groups arising from the octonionic and split-octonionic projective and hyperbolic plane, i.e.
5 Classification of the Octonionic Projective Planes

Thus, the space of rank-1 idempotent elements of $J_3(\mathbb{O})$ enjoys the following expression as an homogeneous space

$$F_{4(-52)} \mathbb{S}^{pin_9},$$

which is a compact Riemannian symmetric space, of (geodesic) rank $= 1$ and of real dimension

$$\dim_{\mathbb{R}} \left( \frac{F_{4(-52)}}{Spin_9} \right) = \dim_{\mathbb{R}} \left( F_{4(-52)} \right) - \dim_{\mathbb{R}} \left( Spin_9 \right) = 52 - 36 = 16,$$

as expected from the number of degrees of freedom characterizing rank-1 idempotents of $J_3(\mathbb{O})$ itself. Since the unitary trace condition is imposed on top of Veronese conditions, the coset (79) is a (proper) submanifold of $O_{\text{rank} \ 1} \left( J_3(\mathbb{O}) \right)$, i.e.

$$\frac{F_{4(-52)}}{Spin_9} \subset \frac{E_{6(-26)}}{Spin_{9,1} \ltimes \mathbb{R}^{16}}.$$  

(81)

The space (79) of rank-1 idempotent (or, equivalently, trace-1) elements of $J_3(\mathbb{O})$ can be identified with the (compact real form of the) octonionic projective plane $\mathbb{O}P^2$, which is the largest octonionic (projective) geometry; this can also be hinted from the fact that the tangent space to the coset $F_{4(-52)}/Spin_9$ transforms under the isotropy group $Spin_9$ as its spinor irreducible representation $16$, which can indeed be realized as a pair of octonions $16$:

$$f_{4(-52)} = so_9 \oplus 16 \Rightarrow T \left( \frac{F_{4(-52)}}{Spin_9} \right) \simeq 16 \ (\text{of} \ Spin_9) \simeq 0 \oplus 0.$$

Thus, one obtains that

$$O_{\text{rank} \ 1} \left( J_3(\mathbb{O}) \right) \cong \frac{E_{6(-26)}}{Spin_{9,1} \ltimes \mathbb{R}^{16}}$$

(83)

$$\cup \cup \cong \frac{F_{4(-52)}}{Spin_9}$$

which gives an alternative definition of the octonionic projective plane. From the table in the previous section, we can then classify all possible octonionic planes. We start from the complexification of the Cayley plane

$$\mathbb{O}P^2(\mathbb{C}) \simeq \frac{F_4(\mathbb{C})}{Spin_9(\mathbb{C})}$$

(84)
Figure 3: Satake diagrams of the real forms of $F_4$, their character $\chi$ and the corresponding octonionic plane whose they are the isometry group.

and define three different real forms of the plane: a totally compact real coset, that identifies with $\mathbb{O}P^2$; a totally non compact which is $\mathbb{O}H^2$; a pseudo-Riemannian real coset that we will define as $\mathbb{O}\tilde{H}^2$. Those octonionic planes will be defined taking as isometry group $F_4(-52)$ and $F_4(-20)$, while the last real form $F_4(4)$ will yield to projective planes on the split-octonionic algebra $\mathbb{O}_s$, i.e.

$$\mathbb{O}P^2 \cong \frac{F_4(-52)}{Spin_9}$$

$$\mathbb{O}H^2 \cong \frac{F_4(-20)}{Spin_9}$$

$$\mathbb{O}\tilde{H}^2 \cong \frac{F_4(-20)}{Spin_{8,1}}$$

$$\mathbb{O}_s\tilde{H}^2 \cong \mathbb{O}_sP^2 \cong \mathbb{O}_sH^2 \cong \frac{F_4(4)}{Spin_{5,4}}.$$ 

Moreover, if we consider the type of the plane, i.e. the cardinality of non-compact and compact generators ($\#_{nc}, \#_c$), and the character $\chi$, i.e. the difference between the two, $\chi = \#_{nc} - \#_c$, we then note that: the totally compact plane, i.e. the classical Cayley plane or the octonionic projective plane $\mathbb{O}P^2$, is of type $(0, 16)$ and character $\chi = 16$; the totally non-compact one, i.e. hyperbolic octonionic plane $\mathbb{O}H^2$, is of type $(16, 0)$ and character $\chi = -16$; while the other two planes named $\mathbb{O}\tilde{H}^2$ and $\mathbb{O}_s\tilde{H}^2$ are of type $(8, 8)$ and character $\chi = 0$.

6 Conclusions

We have presented an explicit construction of the octonionic projective and hyperbolic planes and showed how Lie groups of type $G_2$, $F_4$ and $E_6$ arise naturally as groups of motion of such planes. The fact that all different real forms of $E_6$,
F₄ and G₂ can be recovered from similar constructions is of the uttermost physical importance since different physical theories require different real forms of Lie groups. Compact and non-compact forms of G₂ are notoriously known to be isomorphic to automorphisms of Octonions and split Octonions. In this paper we show how they can be thought as the subgroup of collineations that fix a quadrangle of the Projective plane over Octonions and Split-Octonions. Different compact and non compact real forms of E₆ and F₄ are related to different but analogue geometric frameworks such as projective planes or hyperbolic planes over the algebra of Octonions and split Octonions. Indeed while we recover E₆(−26) and F₄(−52) as collineation and isometry group of the octonionic projective plane OP², we have E₆(6) and F₄(4) for the split case OsP². The hyperbolic plane over Octonions and split-Octonions lead to E₆(−14) and F₄(−20) in the octonionic case OH² or to E₆(2) and F₄(4) in the split case OsH². The only real form left out is the compact E₆(−78) which is obtained as isometry group of the complex Cayley plane or projective Rosenfeld plane over Bioctonions (C ⊗ O) P².

7 Acknowledgments

The work of D.Corradietti is supported by a grant of the Quantum Gravity Research Institute. The work of AM is supported by a “Maria Zambrano" distinguished researcher fellowship, financed by the European Union within the NextGenerationEU program.

References

[1] Baez J., The Octonions, Bull. Amer. Math. Soc. 39 (2002) 145-205.
[2] Borsten L., Marrani A., A Kind of Magic, Class. Quant. Grav. 34 (2017) 23.
[3] Borsten L., Duff M.J., Ferrara S., Marrani A., Rubens W., Small Orbits, Phys. Rev. D85 (2012) 086002.
[4] Boyle L., The Standard Model, The Exceptional Jordan Algebra, and Triality arXiv:2006.16265 (2020).
[5] Corradetti D., Complexification of the Exceptional Jordan algebra and its relation with particle Physics, J. Geom. Symmetry Phys.61 (2021) 1-16.
[6] Corradetti D., Marrani A., Chester D. and Aschheim R., Conjugation Matters. Bioctonionic Veronese Vectors and Cayley-Rosenfeld Planes, ArXiv (2022) 2202.02050.
[7] Freudenthal H., Beziehungen der E7 und E8 zur Oktavenebene. I-XI, Indag. Math.16 (1954) 218-230.
[8] Hurwitz A., Über die Composition der quadratischen Formen von beliebig vielen Variablen, Nachr. Ges. Wiss. Göttingen, 1898.

[9] Jacobson N., Some groups of transformations defined by Jordan algebras. III, J. Reine Angew. Math. 207 (1961) 61–85.

[10] Jordan P., von Neumann J. and Wigner E., On an Algebraic Generalization of the Quantum Mechanical Formalism, Ann. Math. 35 (1934) 29-64.

[11] Krasnov, K., SO(9) characterisation of the Standard Model gauge group, Journal of Mathematical Physics 62 (2021) 021703.

[12] Krutelevich S., Jordan algebras, exceptional groups, and higher composition laws, J. Algebra 314 (2007) 924.

[13] Manogue C.A. and Dray T., The Geometry of Octonions, World Scientific 2015.

[14] Rosenfeld B., Geometry of Lie Groups, Kluwer 1997.

[15] Rosenfeld B., Geometry of Planes over Nonassociative Algebras, Acta Appl. Math. 50 (1998) 103-110.

[16] Salzmann H., Betten D., Grundhöfer T., Hähl H., Löwen R. and Stroppel M., Compact Projective Planes: With an Introduction to Octonion Geometry, New York: De Gruyter, 2011.

[17] Sudbery A., Division algebras,(pseudo) orthogonal groups and spinors, J. Phys. A17 5 (1984) 939.

[18] Todorov I. and Dubois-Violette M., Deducing the Symmetry of the Standard Model from the Automorphism and Structure Groups of the Exceptional Jordan Algebra, Int. J. Mod. Phys. A 33 (2018) 1850118.

[19] Yokota I., Exceptional Lie Groups, arXiv:0902.0431 (2009).

1Departamento de Matemática
Universidade do Algarve
Campus de Gambelas
8005-139 Faro, Portugal
email: a55499@ualg.pt

2Instituto de Física Teórica, Dep.to de Física,
Universidad de Murcia, Campus de Espinardo, E-30100, Spain
email:jazzphyzz@gmail.com

3 Quantum Gravity Research,
Los Angeles, California, CA 90290, USA
davidC@quantumgravityresearch.com
raymond@quantumgravityresearch.com