Bayesian doubly robust causal inference via loss functions

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Abstract

Doubly robust causal inference has a well-established basis in frequentist semi-parametric theory, with estimation of causal parameters typically conducted via outcome regression and propensity score adjustment. A Bayesian counterpart, however, is not obvious as doubly robust estimation involves a semi-parametric formulation in the absence of a fully specified likelihood function. In this paper, we propose a Bayesian approach for doubly robust causal inference via two general Bayesian updating approaches based on loss functions. First, we specify a loss function for a doubly robust propensity score augmented outcome regression model and apply the traditional Bayesian updating mechanism which uses a prior belief distribution to calculate the posterior. Secondly, we draw inference for the posterior from a Bayesian predictive distribution via a Dirichlet process model, extending the Bayesian bootstrap. We show that these updating procedures yield valid posterior distributions of parameters which exhibit double robustness. Simulation studies show that the proposed methods can recover the true causal effect efficiently and achieve frequentist coverage even when the sample size is small or if the propensity score distribution is highly skewed. Finally, we apply our methods to evaluate the causal impact of speed cameras on traffic collisions in England.

Key words: Causal inference; General Bayesian updating; Gibbs posteriors; Loss functions; Double robustness; Bayesian predictive inference

1 Introduction

Causal inference based on the potential outcomes framework has been extensively developed in the past few decades. The inference target is a causal estimand which measures the change in the expected outcome of unit under study of different interventions. However, data arising from observational studies often exhibit confounding of the effect of intervention due to dependence between characteristics of units and outcomes, and this can result in inconsistent estimators of the causal effect if the dependence is ignored. Propensity score (PS) adjustment and outcome regression (OR) models are commonly used to remove the effect of confounding. PS-based adjustment can take several forms, but typically involves two steps of implementation: first, model the treatment assignment mechanism to obtain the PS which is defined as the conditional probability of the treatment assignment given confounders; second, adjust the treatment-outcome relationship via estimated PSs in either form of weighting, matching, stratification, or covariate adjustment. This adjustment corrects for confounding bias under certain assumptions (Rosenbaum and Rubin [1983]) by inducing balance, that is, by rendering confounders and treatment independent upon conditioning. Outcome regression involves postulating a conditional mean of the outcome

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given treatment and confounding covariates. In order to obtain a consistent estimator, we normally need to correctly specify the outcome model, or in the doubly robust version, at least one of the OR and PS models (Scharfstein et al., 1999; Bang and Robins, 2005; Cao et al., 2009).

Bayesian approaches to causal inference focus on the posterior distribution of causal estimands. The Bayesian framework offers calibrated uncertainty quantification and prediction, which can render a more thorough understanding of the causal effect from the practical perspective. There is growing interest in the application of Bayesian methodology to causal problems, but most proposed methods appeal to the principles of two-stage PS-adjustment or flexible outcome modeling; see the survey in Stephens et al. (2022). In the two-stage PS approach, a natural Bayesian solution is to fit a joint model for the treatment and outcome (McCandless et al., 2010); however, the propensity score function estimated by this joint approach no longer possesses the required balancing property, and therefore this joint approach is not a true PS adjustment method. As a development, McCandless et al. (2010) suggested using the posterior distribution for the PS as a covariate in the OR model as a technique for cutting feedback. In addition, an alternative approach assumes a complete separation between the PS and OR models, and conducts a two-step procedure (Kaplan and Chen, 2012), which has been shown to provide a robust estimation procedure for mis-specification. All of these methods adopt standard Bayesian calculation methods, including making parametric assumptions. Several semi-parametric methods have also been proposed (for example, Graham et al., 2016; Saarela et al., 2016); these methods typically exploit computational approaches, in particular the Bayesian bootstrap (Rubin, 1981) to perform inference.

It has been contended that exact, fully Bayesian doubly robust inference is not possible. First, it is argued that the semi-parametric efficient adjustment via the fitted propensity score (see for example Robins et al., 1992) or augmented inverse probability weighting (Bang and Robins, 2005) necessarily involves a plug-in strategy, and as such cannot be regarded as fully Bayesian. Secondly, it is claimed that computational approaches based on the Bayesian bootstrap only provide approximate inference. However, there are strong counterarguments against both of these points: whereas it is true that conventional parametric Bayesian inference computed using the product of a likelihood and a prior cannot legitimately deploy plug-in approaches, this issue can be overcome using Bayesian non-parametric calculations and an alternative view of Bayesian estimation that adopts a decision theoretic standpoint; furthermore, the Bayesian bootstrap, and extensions, can again be justified as a (Monte Carlo-based) exact Bayesian inference procedure. See Stephens et al. (2022) for more extensive discussion.

In this paper, we aim to build a comprehensive Bayesian updating framework for doubly robust causal inference via loss functions. In the usual Bayesian setting, a full probabilistic model is required. Instead we partially specify an augmented outcome regression (AOR) through a mean model. We specifically conduct Bayesian inference, first via the usual prior-to-posterior procedure, and then via a predictive-to-posterior procedure. For prior-to-posterior updating, we construct the inference procedure using recent advances in learning theory for Bayesian loss-type inference (Zhang, 2006; Jiang and Tanner, 2008). This formulates Bayesian inference without the concept of likelihood, instead relying entirely on a loss-type (or utility) specification to update a priori. For predictive inference, we generalize the previous Bayesian bootstrap approach, and specify a Dirichlet process model for the predictive distribution in light of the data. Under the DP representation, the prior distribution can be incorporated into the model through the base measure, which, unlike the Bayesian bootstrap, does not rely on sampling-importance resampling or Markov chain Monte Carlo (MCMC) via the change of measure (Bornn et al., 2019).

The rest of the paper is organized as follows. We connect formulations of causal inference via loss functions and two Bayesian general updating mechanisms in Section 2, specifically via prior-to-posterior and predictive-to-posterior updates. Section 3 outlines Bayesian doubly robust causal inference via an AOR, followed by the asymptotic justification in Section 4. Section 5 demonstrates the proposed method with simulation studies. We apply this method in a real example from England to study the causal effect of the installation of speed cameras on the number of personal injury collisions in Section 6. Finally, Section 7 presents some concluding remarks and future research directions.
2 Bayesian inference via loss functions

We consider a data generating model encapsulated in distribution $F_0(\cdot)$, and assume that realizations, $z$, of random variable $Z$ are available to form inference. In some settings, a parametric specification may be regarded as appropriate; in this case $F_0(z) \equiv F_0(z|\xi^*)$, where $\xi^*$ is the data generating parameter, and $\xi$ denotes a generic element of the parameter space $\Xi$. However, more generally $F_0(\cdot)$ cannot be represented parametrically.

Conventional Bayesian inference concerning the true value of the model parameter, $\xi^*$, in a parametric model focuses on the posterior distribution, $\pi(\xi|z) \propto \mathcal{L}(\xi) \times \pi_0(\xi)$, where $\mathcal{L}(\xi)$ is the likelihood and $\pi_0(\xi)$ is the prior density for $\xi$; for independent and identically distributed random variables $Z_i, i = 1, \ldots, n$, the likelihood is defined in the usual way as $\mathcal{L}(\xi) = \prod_{i=1}^n f(z_i|\xi)$. In the conventional Bayesian setting, a full probabilistic model is required, but this is challenging when the model is only partially specified, say when the regression model is specified via a conditional mean but without specification of the residual error distribution model.

We first present two fundamental approaches to general Bayesian updating. The upper part of Figure in the Supplement shows the standard Bayesian prior-to-posterior update, in which the first step requires the specification of a prior distribution. The posterior distribution is obtained by updating this prior distribution in light of data $z_1, \ldots, z_n$. The lower part represents another perspective of Bayesian inference, starting with a predictive distribution, $p(z|z_{1:n})$, the posterior distribution of $\xi$ can be derived from the relationship between $Z$ and $\xi$ via repeatedly sampling from $p(z|z_{1:n})$.

2.1 Inference via loss minimization

To allow for partial specification of the inference model, we consider inference about parameter $\xi$ and model specification via the loss function, $\ell(z, \xi)$, and minimization of the expected loss taken with respect to the data generating model $F_0(\cdot)$. The ‘true’ value of the parameter, $\xi_0$, in the inference model can be defined as that value which minimizes the expected loss

$$\xi_0 = \arg \min_{\xi \in \Xi} \mathbb{E}[\ell(Z, \xi)] = \arg \min_{\xi \in \Xi} \int \ell(z, \xi) dF_0(z)$$  \hspace{1cm} (1)

where the integral is presumed finite for at least one $\xi \in \Xi$.

In the parametric case, if $F_0(z|\xi^*)$ admits a density $f_0(z|\xi^*)$ with respect to Lebesgue measure, and if $\ell(z, \xi) = -\log f(z|\xi)$ for some other density $f$, the expectation becomes the Kullback-Leibler (KL) divergence between the true model $f_0(z|\xi^*)$ and $f(z|\xi)$. If $f$ matches $f_0$, so that the inference model is correctly specified, and identifiable, we have that $\xi_0 = \xi^*$. If $f$ is mis-specified, this definition of the ‘true’ parameter is in line with standard frequentist arguments. If we further assume $\ell(z, \xi)$ is differentiable with respect to $\xi \in \Xi$ for all $z$, then $\xi_0$ is the solution of the unbiased estimating equation

$$\int \frac{\partial \ell(z, \xi)}{\partial \xi} dF_0(z) = \mathbb{E} \left[ \frac{\partial \ell(Z, \xi)}{\partial \xi} \right] = 0.$$  

If $F_0(z)$ is represented using a non-parametric specification, we can retain most of the parametric calculations but with $\xi_0$ a functional of $F_0$.

From a prior-to-posterior perspective, the decision task is to construct a similar minimization problem with a new objective function involving a measure on $\xi$. For the predictive-to-posterior approach, because the relationship between $z$ and $\xi$ is encapsulated in a loss in (1), we develop a predictive distribution as our best estimate for $F_0(z|\xi^*)$. The following sections elaborate steps to generate the posterior distribution of $\xi$ via the two frameworks.
2.2 Bayesian inference via loss functions

A valid Bayesian inference mechanism can be derived using a loss function connecting \( Z \) and \( \xi \) in terms of a probability measure of \( \xi \), which consequently defines the posterior distribution \( \pi (\xi \mid z) \) given a prior \( \pi_0 (\xi) \) and data \( z \). This formulation (Zhang, 2006; Jiang and Tanner, 2008; Walker, 2010; Bissiri et al., 2016) adopts the decision-making approach taken in (1), and formulates Bayesian inference without the concept of likelihood, instead relying entirely on a loss-type (or utility) specification, and the identification of some function, \( \varphi \), such that \( \pi(\xi \mid z) = \varphi \{ \ell (z, \xi), \pi_0 (\xi) \} \). Specifically, Zhang (2006) defined the objective function for Bayesian loss-based inference with a probability measure \( \mu \) on \( \Xi \) as

\[
\arg \inf_{\mu \in \mathcal{M}_{\pi_0}} \left\{ \int_{\Theta} \ell (z, \xi) \mu (d\xi) + K (\mu, \pi_0) \right\}
= \arg \inf_{\mu \in \mathcal{M}_{\pi_0}} \int \log \left[ \frac{\mu (\xi)}{\exp \left( -\ell (z, \xi) \right) \pi_0 (\xi)} \right] \mu (d\xi)
\tag{2}
\]

where \( \mathcal{M}_v \) is the space which is absolutely continuous with respect to \( v \), \( \ell \) is some measurable function, such that \( \ell(\cdot, z) : \Xi \to \mathbb{R} \) is measurable with respect to \( \mu \) for every \( z \) in the support, and \( K (\mu, \pi_0) \) is the KL divergence between two probability measures \( \mu \) and \( \pi_0 \). The KL divergence is chosen for coherent inference (Bissiri and Walker, 2012), so that whenever a new observation, \( z_{n+1} \), is available, it satisfies

\[
\varphi \{ \ell (z_{n+1}, \xi), \varphi [\ell (z_{1:n}, \xi), \pi_0 (\xi)] \} = \varphi \{ \ell (z_{1:n}, \xi) + \ell (z_{n+1}, \xi), \pi_0 (\xi) \}.
\]

The solution for this optimization problem in (2) is the so-called Gibbs posterior

\[
\pi (\xi \mid z_{1:n}) = \frac{\exp (-\ell (z_{1:n}, \xi)) \times \pi_0 (\xi)}{\int_{\Xi} \exp (-\ell (z_{1:n}, \varphi)) \times \pi_0 (d\varphi)}
\tag{3}
\]

defined if and only if the denominator is finite. The posterior distribution in (3) gives a formal Bayesian procedure to update prior beliefs on \( \xi \) to posterior beliefs based on the loss function and decision-theoretic arguments. The term \( \exp (-\ell (z_{1:n}, \xi)) \) replaces the ‘likelihood’ in conventional Bayesian updating. Further, if

\[
c(\xi) = \int \exp (-\ell (z, \xi)) \, dz
\]

does not depend on \( \xi \), as is the case in location problems, then the form in (3) is equivalent to a de Finetti representation for exchangeable random variables based on this likelihood, with marginal law for the observables given by

\[
p(z_{1:n}) = \int_{\Xi} \exp (-\ell (z_{1:n}, \xi)) \pi_0 (d\xi) \, d\xi.
\]

and the loss-based approach matches the conventional (de Finetti) approach, and the loss-based method is always equivalent to conventional Bayesian inference. For example, if \( \ell (z, \xi) = (z - \xi)^2 \), then the loss-based approach is identical to a conventional Bayesian analysis based on a Gaussian likelihood. If \( c(\xi) \) does depend on \( \xi \), there may still be a a de Finetti representation that matches the loss-based form.

2.3 Prior-to-posterior updating via Bayesian non-parametric modelling

In a decision-theoretic formulation, and in light of posterior distribution \( \pi (\xi \mid z_{1:n}) \) formed in a conventional calculation based on correct specification via model \( f_0 (\cdot | \xi) \), the Bayes estimator of a target parameter minimizes the posterior expected loss, given by

\[
\hat{\xi} = \arg \min_{\xi \in \Xi} \int_{\Xi} u (\xi', \xi) \pi (\xi \mid z_{1:n}) \, d\xi
\tag{4}
\]
where \( u \) is a real-valued function quantifying the loss between the \( \xi \) and \( \xi' \). The minimizer in (4) is typically a function of \( z_{1:n} \).

The formulation allows consideration of the case when \( \xi' \) and the associated loss function relate to an alternative inference model, with this alternative model linked to the presumed data generating model via a utility function. For example, we may specify \( u (\xi', \xi) = E_{Z|\xi} \left[ f (Z, \xi') \right] \) for the observable \( Z \). The conventional Bayesian calculation renders the posterior and posterior predictive deterministic functions as the solution to (4) is a single point. An extension that makes the posterior distribution itself a random quantity is necessary, and this leads to the consideration of Bayesian non-parametric formulations. From a non-parametric Bayesian perspective, the parameter \( \xi^* \) in the presumed data-generating model is infinite dimensional, and effectively corresponds to the distribution, henceforth denoted \( F \). In this case, \( \pi (F | z_{1:n}) \) is a probability distribution on the space of distribution functions, and a draw from this posterior is a random distribution which can be transformed via (4) into a sampled variate \( \hat{\xi} \), which may be replicated to reproduce the posterior for the minimizing quantity from (4).

A simple implementation of a Bayesian non-parametric theory via Monte Carlo is given by the Bayesian bootstrap (Rubin, 1981). The classical formulation of the Bayesian bootstrap assumes that the data points are realizations from a multinomial model on the finite set \( \{z_1, \ldots, z_n\} \) with unknown probability \( \omega = (\omega_1, \ldots, \omega_n) \), and assuming a priori that \( \omega \sim \text{Dirichlet} (\alpha, \ldots, \alpha) \), then, a posteriori \( \omega \sim \text{Dirichlet} (\alpha + 1, \ldots, \alpha + 1) \). Conditional on a draw \( \omega \) from the posterior distribution, samples from the posterior predictive can be made by drawing independently from \( \{z_1, \ldots, z_n\} \) with associated probabilities \( \{\omega_1, \ldots, \omega_n\} \). The Bayesian bootstrap is obtained under the improper specification \( \alpha = 0 \). Referring to (4), the parameter \( \xi \) can then be derived via

\[
\hat{\xi} (\omega) = \arg \min_{\xi} \sum_{k=1}^{n} \omega_k \ell (z_k, \xi)
\]

that is, via a deterministic transformation of \( \omega \). A sample from the posterior distribution for \( \hat{\xi} \) can be obtained by repeatedly drawing \( \omega \sim \text{Dirichlet} (1, \ldots, 1) \) and obtaining the solutions to (5) (Chamberlain and Imbens, 2003; Graham et al., 2016).

If direct minimization in (5) is not feasible, an equivalent Gibbs posterior embedded in the Bayesian bootstrap implementation strategy would be possible. For a posterior distribution defined in (5), we first draw the weights, \( \omega \sim \text{Dirichlet} (1, \ldots, 1) \) and then generate the MCMC samples from

\[
\pi (\xi | z_{1:n}) \propto \prod_{i=1}^{n} \exp \left( -\ell (z_{1:n}, \xi) \right)^{n\omega_i} \pi_0 (\xi).
\]

We can obtain a posterior estimate by taking the mode of the MCMC samples. We repeat this procedure \( N \) times to get a sample of the posterior distribution. Under a non-informative prior, this posterior sample will be similar to the sample obtained via (5).

The Bayesian bootstrap is a consequence of a Dirichlet process (DP) specification that can be implemented in a more general form. If \( F \) parameterizes the true but unknown distribution, a priori\( F \sim \text{DP} (\alpha, G_0) \) where \( \alpha > 0 \) is the concentration parameter and \( G_0 \) is the base measure. In light of data \( (z_1, \ldots, z_n) \), the resulting posterior distribution of \( F \) is \( \text{DP} (\alpha_n, G_n) \), where \( \alpha_n = \alpha + n \) and \( G_n (\cdot) = \alpha G_0 (\cdot)/(\alpha + n) + \sum_{k=1}^{n} \delta_{z_k} (\cdot)/(\alpha + n) \).

### 2.4 Predictive-to-posterior updating via Bayesian non-parametric modelling

If the function \( u \) in (4) is taken to be the Kullback-Leibler divergence,

\[
u (\xi', \xi) = \int \log \left( \frac{f_0 (z | \xi)}{f_0 (z | \xi')} \right) f_0 (z | \xi) \, dz
\]
then the minimizing value of $\xi'$ is that for which
\[
\int \log f_0(z \mid \xi') \left\{ \int f_0(z \mid \xi) \pi(\xi \mid z_{1:n}) \, d\xi \right\} \, dz = \int \log f_0(z \mid \xi') p_n(z) \, dz
\]
is maximized, where
\[
p_n(z) = \int f_0(z \mid \xi) \pi(\xi \mid z_{1:n}) \, d\xi
\]
is the usual Bayesian posterior predictive distribution given data $z_{1:n}$.

This calculation stresses the importance of the predictive distribution in inference. It can be adapted to consider the estimation of parameters in models that are different from the proposed data generating model that leads to $\pi(\xi \mid z_{1:n})$. Consider a new set of exchangeable data, $z_1^*, \ldots, z_N^*$, and take $u(\xi', \xi)$ as
\[
\mathbb{E}_{Z_{1:N}^* \mid \xi} [\ell (Z_{1:N}^*, \xi')] = \sum_{i=1}^N \mathbb{E}_{Z_i^* \mid \xi} [\ell (Z_i^*, \xi')] = \sum_{i=1}^N \int \ell(z_i^*, \xi') f_0(z_i^* \mid \xi) \, d\xi,
\]
that is, the expected loss under the ‘correct specification’ that presumes $\xi^* = \xi$. Then
\[
\arg \min_{\xi' \in \Xi} \int \sum_{i=1}^N \mathbb{E}_{Z_i^* \mid \xi} [\ell (Z_i^*, \xi')] \pi(\xi \mid z_{1:n}) \, d\xi
\]
\[
= \arg \min_{\xi' \in \Xi} \int \sum_{i=1}^N \ell(z_i^*, \xi') f_0(z_1^*, \ldots, z_N^* \mid \xi) \, dz^* \pi(\xi \mid z_{1:n}) \, d\xi \quad \text{(8)}
\]
\[
= \arg \min_{\xi' \in \Xi} \int \sum_{i=1}^N \ell(z_i^*, \xi') p(z_1^*, \ldots, z_N^* \mid z_{1:n}) \, dz^*
\]
where $p(z_1^*, \ldots, z_N^* \mid z_{1:n})$ is the $N$-fold posterior predictive distribution. Therefore, the solution to the minimization problem (8) is the Bayesian estimator that mimics the calculation in (1), with the posterior predictive distribution replacing $F_0(\cdot \mid \xi^*)$.

The integral in (8) may not be analytically tractable, but typically may be approximated using Monte Carlo methods. If $z^* = (z_1^*, \ldots, z_N^*)$ are drawn from $p(z_{1:N}^* \mid z_{1:n})$, then the finite sample approximation to (8) is
\[
\hat{\xi}^*(z^*) = \arg \min_{\xi} \sum_{i=1}^N \ell(z_i^*, \xi).
\]
As $N \to \infty$, under mild regularity conditions, and under correct specification of the model leading to $\pi(\xi \mid z_{1:n})$, the minimizer from (8) converges to $\xi_0$ defined in (1). Now, with $\hat{\xi}^*(z_{1:n})$ in a neighbourhood of $\xi^*$, following Bernardo (1979), we have
\[
p(z_1^*, \ldots, z_N^* \mid z_{1:n}) = f_0(z_N^* \mid \xi^*) \cdots f_0(z_1^* \mid \xi^*) + o(1)
\]
and therefore, a draw from the predictive $p(z_1^*, \ldots, z_N^* \mid z_{1:n})$ suitably simulates a collection of $N$ sample points from the true data generating model $F_0(z \mid \xi^*)$ as $n \to \infty$. The minimizer of (8) will be become degenerate at $\xi_0$ as both $N \to \infty$ and $n \to \infty$.

In the Dirichlet process formulation, the posterior predictive distribution $p(z_{1:N}^* \mid z_{1:n})$ is also a random distribution. Draws from it can be generated directly via Pólya urn schemes (Blackwell and MacQueen 1973) that integrate out the posterior distribution, and which allow direct draws of variates from the predictive distribution in a dependent fashion. We can simulate $S$ datasets of size $N$, with each dataset $z^* = \{z_1^*, \ldots, z_N^*\}, s = 1, \ldots, S$, where $z^*$ is generated in a sequential fashion: $z_1^* \sim G_n$, and then for $j = 2, \ldots, N$,
\[
\begin{align*}
z_j^* \mid z_1^*, \ldots, z_{j-1}^* & \sim \frac{\alpha + n}{\alpha + n + j - 1} G_n(.) + \frac{1}{\alpha + n + j - 1} \sum_{k=1}^{j-1} \delta_{z_k^*}() \equiv G_{n+j-1}.
\end{align*}
\]
Each of the $S$ data sets generates a sampled variate from the posterior distribution by solving (5) or (8) to yield $(\hat{\xi}^1, \ldots, \hat{\xi}^S)$, which, in the limit as $N \to \infty$, is an exact sample from the distribution for $\xi$. Algorithm 1 describes the predictive-to-posterior update from a DP model to obtain posterior samples of $\xi$.

**Algorithm 1:** Predictive-to-posterior inference based on a DP model.

Data: $z_1, \ldots, z_n$

for $s$ to 1 : $S$

for $j$ to 1 : $N$

Sample $z_j^s \sim G_{n+j-1}$;

Update $G_{n+j} \leftarrow \{z_j^s, G_{n+j-1}\}$;

Obtain a set $z^s = \{z_1^s, \ldots, z_N^s\}$;

Compute $\hat{\xi}^s$ by solving the minimization problem in (5);

return $(\hat{\xi}^1, \ldots, \hat{\xi}^S)$.

3 Doubly robust causal inference via propensity score regression

We now focus on the motivating example, which is a Bayesian representation for the regression approach to causal estimation. In a causal inference setting, for the $i$th unit of observation, $Y_i$ denotes a response, $d_i$ the treatment (or exposure) received, and $x_i$ a vector of pre-treatment covariates or confounder variables. Suppose the data generating structural model is

$$Y_i = \theta d_i + h_0(x_i) + \epsilon_i, \quad \forall i = 1, 2, \ldots, n$$

(10)

where $E(\epsilon_i | x_i, d_i) = 0$ and $\text{Var}(\epsilon_i | x_i, d_i) = \sigma^2 < \infty$, with $\epsilon_1, \ldots, \epsilon_n$ independent, and where $h_0(x_i)$ is an unknown real-valued function of the vector $x_i$. In this setting, $\theta^*$ is the average treatment effect (ATE) and the parameter of interest.

3.1 Frequentist inference in propensity score regression

A typical approach to causal adjustment uses the propensity score. With the PS estimated either via maximum likelihood or a fully Bayesian procedure summarized by the posterior mean, the outcome is modelled by adding a likelihood or a fully Bayesian procedure summarized by the posterior mean, the outcome is modelled by adding

$$A = \theta d_i + h_1(x_i) + \phi \epsilon(x_i; \hat{\gamma}) + \epsilon_i, \quad \forall i = 1, 2, \ldots, n.$$  (11)

In frequentist estimation, this model leads to doubly robust inference. If $h_1(x) = h_0(x)$, so that (11) matches (10) and the model correctly specified, then the estimator of the true ATE will be consistent irrespective of whether the PS model is correctly specified because $\phi \to 0$; if the PS is correctly modelled, conditioning on it will block the confounding path from $D$ to $Y$ via $X$ so that $X \perp D \mid e(X)$ so that (11) will still yield a consistent estimator of $\theta$, even if $h_1(x)$ is incorrectly specified. We will proceed by assuming that the functional forms of $h_0(x_i, \beta_0)$ and $h_1(x_i, \beta)$ are parametric with associated parameter vectors $\beta_0$ and $\beta$. For example, linear regression assumes that $h_0(x_i, \beta_0) = x_i^\top \beta_0$ and $h_1(x_i, \beta) = x_i^\top \beta$.

Let $z_i = (y_i, d_i, x_i)$, $i = 1, \ldots, n$, be the observed data, and $Z_i = (Y_i, d_i, x_i)$, $i = 1, \ldots, n$ be the random variable representing the random component in the conditional model. Estimation of $\xi = (\theta, \beta, \phi)$ in the conditional mean model (11) can proceed by defining a loss function which is the sum of squares of the residual error, i.e.,

$$\ell ((y_1:n, x_1:n, d_1:n), \xi) = \frac{1}{2} \sum_{i=1}^n \left[ y_i - (\theta d_i + h_1(x_i, \beta) + \phi \epsilon(x_i; \hat{\gamma})) \right]^2.$$  (12)
Such frequentist semi-parametric inference does not make any additional distributional assumption about \( \epsilon_i \). A key aspect of this frequentist approach is the use of plug-in estimation for parameter \( \gamma \); it can be demonstrated that this approach is locally optimal, in that it provides efficient estimation of \( \theta \) under the assumption that the propensity score model is correctly specified.

### 3.2 Bayesian inference in propensity score regression

The plug-in approach can be justified in a fully Bayesian framework under the loss-based approach to inference as described in section 2. In the Supplement, we establish the following theorem for consistency with a mis-specified PS model and a correctly specified OR model.

**Theorem 1** (Consistency-mis-specified PS). Under Assumptions 1-5, and if \( U = \{ \xi : \| \xi - \xi_0 \| < \epsilon \} \) is an open
neighbourhood of \(\xi_0\), then,
\[
\Pi^n(U) = \frac{\int_U \exp \left[ - \sum_{i=1}^{n} \{\ell(z_i, \xi) - \ell(z_i, \xi_0)\} \right] \pi_0(d\xi)}{\int_{\Xi} \exp \left[ - \sum_{i=1}^{n} \{\ell(z_i, \xi) - \ell(z_i, \xi_0)\} \right] \pi_0(d\xi)} \rightarrow 1 \text{ almost surely.}
\]

For this prior-to-posterior update, we can also consider the limiting behaviour of the estimator in terms of the probability law, specifically, that it exhibits posterior asymptotic normality. In the empirical measure, the estimating equations becomes \(\sum_{i=1}^{n} U_i(\xi_n) = 0\) and with some regularity conditions, we have that the solution \(\xi_n\) has the property that
\[
\sqrt{n}(\xi_n - \xi_0) \overset{d}{\rightarrow} \text{Normal}_p(0, V)
\]
where \(V = J^{-1}IJJ^{-\top}\) with
\[
I = E[U(\xi_0)U(\xi_0)^\top], \quad J = -E[\dot{U}(\xi_0)]
\]
both \((p \times p)\) matrices, and \(\dot{U}(\xi_0) = \partial U(\xi)/\partial \xi|_{\xi=\xi_0}\). The Bayesian analogy is the Bernstein-von Mises theorem, which establishes the limiting behaviour of the posterior distribution \(\pi(\sqrt{n}(\xi - \xi_0)|z_1, \ldots, z_n)\). With some additional assumptions, we have the following Bernstein-von Mises theorem under a mis-specified PS model.

**Theorem 2.** For arbitrary \(\delta > 0\), suppose that there exists an \(\epsilon > 0\) so that
\[
\lim_{n \to \infty} P(\sup_{||\xi - \xi_0|| > \delta} \frac{1}{n} \sum_{i=1}^{n} (\ell(z_i, \xi) - \ell(z_i, \xi_0)) \leq \epsilon) = 1.
\]

Then under Assumptions 1-6, the posterior for the model, \(\pi(\sqrt{n}(\xi - \xi_0)|z_1, \ldots, z_n)\), assigning to an arbitrary set \(\Xi \subseteq \Xi\) converges to the mass given by a Normal measure. Specifically, if \(Z \sim \text{Normal}_p(0, V)\) is an arbitrary random variable independent from all other random variables, then
\[
\sup_{A} \left| \pi(\sqrt{n}(\xi - \xi_0) \in A | z_1, \ldots, z_n) - P(Z \in A) \right| \overset{P}{\to} 0.
\]

This theorem essentially shows the limiting posterior distribution of \(\xi\) from the prior-to-posterior update concentrates on a \(\sqrt{n}\)-ball centered at \(\xi_0\) with the sandwich covariance matrix.

**Mis-specified outcome model:** When the outcome model is mis-specified but the PS model is correctly specified, \(X \perp D | e(x; \gamma_0)\) and \(\hat{\gamma} \to \gamma_0\). Therefore, \(e(x; \hat{\gamma})\) is an asymptotic balancing score. Suppose we specify the mean model as in (11), and assume that the dependence of the mean model on \(D\) is correctly specified, and the effect of \(D\) is captured via \(\theta D\). Under the assumption of no unmeasured confounding, we can find the pseudo-true value, \(\xi_0^* = (\theta_0^*, \beta_0^*, \phi_0^*)\), such that \(\xi_0^* = \arg \min_\xi \ell(Y, x, d, \xi)dF_0(y)\), where \(\ell(Y, x, d, \xi)\) is the loss function corresponding to all mis-specified OR models defined in (11). If \(\ell(z, \xi) = -\log f(y | x, d, \xi)\), \(\xi_0^*\) is the minimizer of the KL divergence between the mis-specified OR model and the true data generating model. This is again in line with the standard frequentist approach for mis-specified models. As the PS is correctly modelled and \(X \perp D | e(x; \gamma_0)\), then \(\theta_0^* = \theta_0\). With Assumptions 7 and 8 on the identification of the pseudo-true value, we can construct the same asymptotic results (consistency and asymptotic normality) as the mis-specified PS case by replacing \(\xi_0\) to \(\xi_0^*\).

### 4.2 Predictive-to-posterior via predictive sampling

Fong et al. (2021) describe methods of sampling \(\xi\) through predictive imputation. Suppose unobserved data, \(Z_{n+1}, Z_{n+2} \ldots\) are sampled from a sequence of conditional (predictive) models, \(p_{n+1}, p_{n+2}, \ldots\) as described in (9).
until we have complete information, i.e., $Z_{1:∞}$. Given this random infinite dataset, we then compute the limiting point estimate, $ξ_{∞}$, and the uncertainty in $ξ_{∞}$ is equivalent to prior uncertainty in $Ξ$. [Fong et al. (2021)] argued that $ξ_{∞}$ offers a deterministic mapping from the data to $Ξ$, yielding a full recovery of the uncertainty of $ξ$ when $Z_{1:∞}$ is available. Therefore, in order to obtain a valid posterior from the predictive distribution, the sequence of predictive distributions has to converge to a random distribution. [Fong et al. (2021)] stated two conditions which the predictive distribution has to satisfy.

**Condition 1.** The sequence of predictive distributions, $p_{n+1}(y|d,x), p_{n+2}(y|d,x), \ldots$, converges almost surely to a random probability distribution $p_{∞}(y|d,x)$, for all $y \in \mathbb{R}$.

**Condition 2.** The posterior expectation of the random $p_{∞}(y|d,x)$ satisfies $E[p_{∞}(y|d,x) | Z_{1:n}] = p_{n}(y|d,x)$ almost surely for all $y \in \mathbb{R}$.

Assuming these two conditions, $p_{∞}(y|d,x)$ is considered as the best estimate of the unknown true data generating mechanism under the specified model sequence, and gives a mechanism for generating posterior uncertainty of $ξ$ without applying Bayes rule. [Berti et al. (2006)] showed that the conditional distribution, $p_{n+N}(y|d,x)$, converges weakly to a random probability measure almost surely for each pair of $(d,x)$ if these two conditions are satisfied.

In the predictive resampling approach derived from the Dirichlet process and indicated in (9), the sequence $\{G_j\}_{j=1}^N$ are precisely predictive models that align with the theory of [Berti et al. (2006)], and therefore we have the following theorem.

**Theorem 3.** There exists a random probability measure $G_{∞}$ such that $G_{n+1}$ converges weakly to $G_{∞}$.

This theorem confirms that predictive resampling via the Dirichlet process is a valid Bayesian update and gives the same uncertainty quantification as the prior-to-posterior update. From (7), we may deduce that the value obtained from solving the minimization problem in (8) is a sample from the posterior distribution of the target parameter. To show the consistency, we need to consider the limiting case as $n \rightarrow \infty$. This requires $ξ$ in (4) to be degenerate at $ξ_0$ if all the information is available.

**Theorem 4.** Suppose the prior has full Hellinger support, and $ℓ(z,ξ)$ is continuous $∀ ξ \in Ξ$ and $\int \log [1 + |ℓ(z,ξ)|] dG_0(z) < +∞$, then

$$\sum_{k=1}^{n} \omega_k ℓ \left( z_k, \bar{ξ}(\omega) \right) \rightarrow \min_{ξ \in Ξ} \int ℓ(z,ξ) dF_0(z) \quad n \rightarrow \infty$$

and

$$\int ℓ(z,\bar{ξ}(z^*)) dG_{n+N-1}(z) \rightarrow \min_{ξ \in Ξ} \int ℓ(z,ξ) dF_0(z) \quad n \rightarrow \infty, N \rightarrow \infty$$

The proof of this theorem is sketched in Section 2.4 and given in full in [Lijoi et al. (2004)] (Theorem 1). This theorem confirms the consistency results for the Bayesian bootstrap and predictive-to-posterior inference via predictive sampling, which both agree with the standard Bayesian prior-to-posterior consistency result.

### 5 Simulation

We examine the performance of the Bayesian computational methods via loss functions described in Section 2 with the two updating frameworks. For each example, we consider the following methods:

- Method I: Gibbs posterior from (3) via MCMC;
• Method II: Gibbs posterior from (6) via the Bayesian bootstrap and MCMC;
• Method III: Estimation via the Bayesian bootstrap from (5);
• Method IV: Predictive-to-posterior inference via the Dirichlet process from Algorithm 1.

The prior-to-posterior update via the Gibbs posterior is implemented using MCMC and the Bayesian bootstrap approaches from Sections 2.2 and 2.3, and non-informative priors are placed for all the parameters with 10,000 MCMC samples and 1,000 burn-in iterations. For the predictive-to-posterior update, we generate $S = 1,000$ sets, each with $N = 10,000$ new data points and with $\alpha = 1$.

5.1 Example 1: Double robustness

In this example, we consider the simulation study constructed by Saarela et al. (2016). The data are simulated as follows:

$$X_1, X_2, X_3, X_4 \sim N(0, 1), \quad U_1 = \frac{|X_1|}{\sqrt{1 - 2/\pi}}$$

$$D \mid U_1, X_2, X_3 \sim \text{Bernoulli}(\expit(0.4U_1 + 0.4X_2 + 0.8X_3))$$

$$Y \mid D, U_1, X_2, X_4 \sim N(D - U_1 - X_2 - X_4, 1)$$

In this case, three scenarios are considered:

• Scenario A: Mis-specify the OR model using covariates $(x_1, x_2, x_4)$ and correctly specify a treatment assignment model using covariates $(u_1, x_2, x_3)$.
• Scenario B: Correctly specify the OR model using covariates $(u_1, x_2, x_4)$ and mis-specify a treatment assignment model using covariates $(x_1, x_2, x_3)$.
• Scenario C: Mis-specify the OR model using covariates $(x_1, x_2, x_4)$ and mis-specify a treatment assignment model using covariates $(x_1, x_2, x_3)$. This is not originally considered in Saarela et al. (2016).

For $n = 20$, we also place an informative normal prior with mean at the true value and standard deviation 2 for the Gibbs posterior using MCMC. Table 1 shows the results of 1,000 Monte Carlo replicates of the averages of the posterior means, variances and coverage rates for $\theta$ with different sample sizes. The coverage rates are computed by constructing a 95% credible interval for $\theta$ from the the 2.5% and 97.5% posterior sample quantiles. When the sample size is small, the Bayesian bootstrap (Method III) and predictive inference models (Method IV) exhibit poor coverage while the Gibbs posterior (Methods I and II) returns coverage rates at the nominal level; however, the Bayesian bootstrap approach presents rather larger variances. The difference in variances diminishes as the sample size increases, or when the informative prior is considered (demonstrated in the bracket for $n = 20$). As expected, the two updating approaches yield unbiased estimates in both scenarios and show agreements in the variance and the coverage rate when the sample size is over 100. The Bayesian bootstrap and DP-based predictive inference generate similar results, as the prior information in predictive inference does not carry much weight when $\alpha/N$ is small. When both models are mis-specified, all cases yield significantly biased estimates unless the informative prior is used.

5.2 Example 2: PS distribution

In this example, we examine the performance of the proposed updating approaches under some extreme PS
Table 1: Simulation results of the marginal causal contrast, with true value equal to 1, on 1000 simulation runs on generated datasets of size $n$.

| $n$   | Scenario A | Scenario B | Scenario C |
|-------|------------|------------|------------|
|       | 20         | 50         | 100        | 500        | 20         | 50         | 100        | 500        |
| Mean  |            |            |            |            |            |            |            |            |
| Method I | 0.980      | 0.980      | 1.005      | 1.001      | 0.623      | 0.77       | 0.621      | 0.613      |
|        | (1.049)    |            |            |            | (0.793)    |            |            |            |
| Method II | 0.914      | 1.001      | 1.008      | 1.001      | 0.677      | 0.641      | 0.642      | 0.626      |
|        |            | (1.025)    |            |            | (0.793)    |            |            |            |
| Method III | 0.925      | 0.990      | 1.003      | 0.999      | 0.621      | 0.77       | 0.623      | 0.625      |
|        |            |            | (1.049)    |            |            | (0.793)    |            |            |
| Method IV | 0.945      | 1.005      | 0.992      | 1.003      | 0.597      | 0.628      | 0.620      | 0.623      |
|        |            | (1.049)    |            |            | (0.793)    |            |            |            |
| Variance |            |            |            |            |            |            |            |            |
| Method I | 1.115      | 0.129      | 0.058      | 0.015      | 0.523      | 0.203      | 0.092      | 0.018      |
|        | (3.48)     |            |            |            | (2.21)     |            |            |            |
| Method II | 2.431      | 0.119      | 0.056      | 0.010      | 0.758      | 0.205      | 0.094      | 0.020      |
|        |            | (3.20)     |            |            | (2.21)     |            |            |            |
| Method III | 8.396      | 0.118      | 0.050      | 0.010      | 69.893     | 0.230      | 0.094      | 0.020      |
|        |            | (3.20)     |            |            | (2.21)     |            |            |            |
| Method IV | 0.461      | 0.125      | 0.054      | 0.011      | 0.676      | 0.194      | 0.089      | 0.020      |
|        | (3.48)     |            |            |            | (2.21)     |            |            |            |
| Coverage, % |           |            |            |            |            |            |            |            |
| Method I | 96.5       | 95.5       | 95.1       | 94.9       | 93.2       | 91.2       | 85.3       | 25.6       |
|        | (97.3)     |            |            |            | (95.6)     |            |            |            |
| Method II | 94.3       | 95.2       | 94.6       | 95.3       | 91.7       | 88.4       | 80.6       | 22.3       |
|        |            | (95.9)     |            |            | (95.6)     |            |            |            |
| Method III | 89.3       | 92.2       | 93.5       | 94.2       | 77.4       | 77.2       | 74.2       | 20.9       |
|        |            | (95.9)     |            |            | (95.6)     |            |            |            |
| Method IV | 92.2       | 95.0       | 94.9       | 93.8       | 77.5       | 81.4       | 76.8       | 35.4       |
|        | (95.9)     |            |            |            | (95.6)     |            |            |            |

In the analyses, the PS model is assumed to be correctly specified. For the outcome model, we fit the model with the treatment indicator and estimated PSs only as covariates. In order to investigate how the PS distribution affects the estimation of the treatment effect, different PS distributions are considered using three scenarios with different values of γ:

- **Scenario A**: γ = (0.00, 0.30, 0.80, 0.30, 0.80), generating a nearly uniform distribution of propensity scores, i.e., no skewness.
- **Scenario B**: γ = (0.50, 0.50, 0.75, 1.00, 1.00), having a greater density of lower scores, i.e., mildly skewed.
- **Scenario C**: γ = (0.00, 0.45, 0.90, 1.35, 1.80), having very few high scores, i.e., highly skewed.

In this example, we also fit Bayesian regression on the correctly specified OR. Table 2 summarizes the estimates of θ over 1,000 Monte Carlo replicates for three different scenarios described above. For a correctly specified OR, the coverage rate is around the nominal level. For the proposed methods, the results suggest that all approaches yield unbiased estimates across all scenarios (see the Supplement). As in Example 1, under a uniform distribution of PSs, these approaches indicate nearly the same performance, and the results agree in terms of posterior mean and variance. However, when the PS distribution is slightly skewed (Scenario B), BB exhibits a slightly higher bias and greater variance, notably when $n$ is small but these differences diminish as $n$ increases. The bias and greater variance in BB become more obvious when the PS distribution is highly skewed (Scenario C). Also in Scenario C, the Gibbs posterior implemented via MCMC has consistently the smallest variance. In general, Pred-P-DP and the prior-to-posterior updating using the Bayesian bootstrap have very similar performance in those scenarios.
Table 2: Simulation results of the marginal causal contrast, with true value equal to 0, on 1000 simulation runs on generated datasets of size $n$.

| $n$   | Scenario A |         | Scenario B |         | Scenario C |         |
|-------|------------|---------|------------|---------|------------|---------|
|       | 100        | 200     | 500        | 1000    | 100        | 200     | 500     | 1000    | 100        | 200     | 500     | 1000    |
| Var   | Method I   | 0.148   | 0.067      | 0.028    | 0.013     | 0.154   | 0.079    | 0.031    | 0.013    | 0.144   | 0.069   | 0.043   | 0.014   |
|       | Method II  | 0.156   | 0.076      | 0.027    | 0.017     | 0.165   | 0.077    | 0.033    | 0.018    | 0.219   | 0.107   | 0.046   | 0.022   |
|       | Method III | 0.155   | 0.071      | 0.030    | 0.017     | 0.163   | 0.080    | 0.032    | 0.018    | 0.233   | 0.112   | 0.043   | 0.023   |
|       | Method IV  | 0.145   | 0.074      | 0.029    | 0.014     | 0.139   | 0.080    | 0.032    | 0.015    | 0.234   | 0.109   | 0.044   | 0.021   |
|       | Bayes-OR   | 0.055   | 0.026      | 0.010    | 0.006     | 0.066   | 0.031    | 0.012    | 0.006    | 0.087   | 0.040   | 0.017   | 0.008   |
| Cov   | Method I   | 94.4    | 97.2       | 96.8     | 97.4      | 96.8    | 96.0     | 95.2     | 97.0     | 97.1    | 96.0    | 95.9    | 95.7    |
|       | Method II  | 93.9    | 93.4       | 95.2     | 96.2      | 92.0    | 92.2     | 93.7     | 95.2     | 91.7    | 92.8    | 95.4    | 95.2    |
|       | Method III | 91.6    | 93.1       | 94.4     | 94.7      | 91.5    | 93.0     | 94.8     | 95.1     | 89.2    | 92.8    | 94.7    | 94.3    |
|       | Method IV  | 93.6    | 95.9       | 95.5     | 95.8      | 95.1    | 94.9     | 95.9     | 95.5     | 91.0    | 94.6    | 94.5    | 95.1    |
|       | Bayes-OR   | 95.2    | 96.0       | 95.9     | 94.4      | 94.9    | 94.3     | 95.2     | 94.0     | 94.2    | 96.0    | 94.2    | 95.0    |

Bayes-OR represents standard Bayesian inference for the correctly specified OR with non-informative priors.

6 Application

6.1 UK Speed Camera Data

Our real example consists of data on the location of fixed speed cameras for 771 camera sites in the eight English administrative districts, including Cheshire, Dorset, Greater Manchester, Lancashire, Leicester, Merseyside, Sussex and the West Midlands. These data form the treated group. For the untreated group, we randomly select a sample of 4,787 points on the network within our eight administrative districts. Details of these data can be found in Graham et al. (2019). The objective of this study is to evaluate the causal effect of the installation of speed cameras on the number of personal injury collisions.

The outcome of interest is the number of personal injury collisions per kilometre as recorded from the location with or without speed cameras. The data are taken from police reports collated and processed by the Department for Transport in the UK in the ‘STATS 19’ data set. The location of each personal injury collision is recorded using the British National Grid coordinate system and can be located on a map using Geographical Information System software. Data are collected from 1999 to 2007 to ensure the availability of collision data for the years before and after the camera installation for every camera site as speed cameras were introduced varying from 2002 to 2004. There is a formal set of location selection guidelines for speed cameras in the UK (Gains et al., 2004). These guidelines inform the selection of covariates which represent the characteristics of units that simultaneously determine the treatment assignment (camera location) and outcome (number of accidents). Primary guidelines for site section include the site length, the number of fatal and serious collisions and the number of personal injury collisions in a preceding time period. In addition, drivers might try to avoid the routes with speed cameras, and the reduction in collisions may come from a reduced traffic flow. Therefore, we include the annual average daily flow (AADF) as a confounder to control the effect due to the traffic flow. We also include factors that would have safety impacts on speed cameras, such as road types, speed limit, and the number of minor junctions within site length (Christie et al., 2003).
6.2 Results

We apply our proposed Bayesian methods to the speed camera data with the loss defined in (12). Graham et al. (2019) estimated the PS with a generalized additive model by including smooth functions on the AADF and the number of minor junctions and achieved balance and overlap. We adopt this model to estimate PSs. For the augmented outcome model, we include all the confounders described previously and the estimated PS. We place non-informative priors for all the parameters in prior and predictive inference based on the general DP representation with $\alpha = 100$ and $N = 10,000$. Table 3 shows summary statistics of the ATE based on 20,000 posterior samples. All three updating methods indicate that the installation of the speed camera can reduce road traffic collisions by 1.4 incidents per site on average, demonstrating agreement in the estimated ATE across these three methods; however, we notice that the Bayesian bootstrap based approaches yield a slightly higher variation. The Gibbs posterior using MCMC has the smallest variance; we calibrated the Gibbs posterior using a scaling based on the estimated residual variance (18.473). Graham et al. (2019) reported the percentage reduction in an average change in road traffic collisions with the presence of speed cameras, and therefore we also report the posterior predictive distribution of this quantity in Table 3. Compared to inverse probability weighting (IPW) estimation, our models demonstrate that there is about an 18% reduction in road traffic collisions in locations where a speed camera is installed, indicating a stronger causal relationship. Compared to the IPW analysis, which only has the inverse weighting adjustment, the AOR has an additional treatment-free component, and therefore offers an additional degree of robustness if one of the component models is correctly specified. We also obtain narrower 95% credible intervals and smaller standard deviations because the Bayesian bootstrap strategy reduces the influence of extreme PSs.

| ATE          | Posterior Mean | Standard Deviation | 95% Credible Interval |
|--------------|----------------|--------------------|-----------------------|
| Method I     | -1.419         | 0.173              | (-1.751, -1.078)      |
| Method II    | -1.411         | 0.183              | (-1.772, -1.049)      |
| Method III   | -1.411         | 0.184              | (-1.772, -1.048)      |
| Method IV    | -1.411         | 0.181              | (-1.767, -1.055)      |
| IPW-BB       | -1.089         | 0.203              | (-1.486, -0.679)      |
| IPW-BB (plug-in) | -1.088    | 0.209              | (-1.484, -0.663)      |

| Percentage Change of the ATE | Posterior Mean | Standard Deviation | 95% Credible Interval |
|------------------------------|----------------|--------------------|-----------------------|
| Method I                     | -18.727        | 2.233              | (-23.062, -14.308)    |
| Method II                    | -18.736        | 2.349              | (-23.231, -14.027)    |
| Method III                   | -18.603        | 2.352              | (-23.224, -13.951)    |
| Method IV                    | -18.627        | 2.320              | (-23.164, -14.032)    |
| IPW-BB                       | -14.338        | 2.622              | (-19.419, -9.036)     |
| IPW-BB (plug-in)             | -14.625        | 2.807              | (-19.978, -8.877)     |

IPW-BB represents results using the update two-step Bayesian bootstrap approach based on inverse probability weighting estimation, while IPW-BB (plug-in) represents the plug-in approach using the two-step Bayesian bootstrap.

7 Discussion

In this paper, we have formulated the causal inference problem in a formal Bayesian setting without relying on conventional prior-likelihood calculations. Specifically, we consider two distinct methods of sampling the posterior distribution of the target parameter, the average treatment effect. First, from the traditional Bayesian updating
approach, we obtain the posterior distribution in light of the data from a loss-based decision theoretic perspective. Subsequently, by sequentially imputing the unobserved data, we then compute the point estimate by minimizing a specified loss function, which yields quantification of uncertainty in the ATE. This method is equivalent to the usual prior-to-posterior update. Secondly, in predictive inference, we focus on the DP construction and argue that the Bayesian bootstrap method is a limiting case of the DP construction. The asymptotic results show the double robustness of the two general updating mechanisms. Simulation examples demonstrated that two proposed approaches have good Bayesian and frequentist properties. Finally, we have applied both methods to study road safety outcomes and quantified the causal effect of speed cameras on road traffic accidents, concluding that the presence of speed cameras can reduce the number of personal injury collisions. Such inference could aid transportation authorities to propose a more effective installation plan of speed cameras to improve road safety.

Bayesian methods are generally applicable in causal inference for real applications. They can yield interpretable variability estimates in finite samples and also allow the statistician to impose structures onto the inference problem. The usual prior updating framework provides a means of informed and coherent decision making in the presence of uncertainty. Predictive inference sheds lights on how to quantify Bayesian uncertainty, where the model is specified via a sequence of predictive distributions without a prior-likelihood construction. When the sample size is small, the standard prior updating framework offers more stable results as informative priors are easy to incorporate. As the sample size increases, predictive inference yields a computationally more efficient calculation because it relies purely on optimization instead of MCMC, which provides a ‘shortcut’ to fully Bayesian causal inference.

One issue in Gibbs posterior is to calibrate the scaling parameter, which we fixed it to be the reciprocal of the estimated residual variance in our analysis and have obtained similar results with other methods. However, if the variance is greater than 1, the Gibbs posterior will have a low coverage rate (Syring and Martin, 2019) while other proposed approaches will be unaffected as they do not rely on scaling. The principles presented in this paper can also be applied in much more general settings. In addition, the proposed methodology can be widely applied in other causal settings when the traditional Bayesian set-up requires over-specifying the model condition, clashing with the partial specified restriction. In our outcome model, we focus on the treatment effect as a single parameter, which can be extended to include further interaction terms with confounders. Moreover, our model can be straightforwardly extended to the marginal structural model (MSM) for longitudinal settings, where the weighted pseudo-likelihood derived from the MSM can be used as the utility function.

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Supplementary material for “Bayesian doubly robust causal inference via loss functions”

A Bayesian general updating mechanisms

\[
\begin{array}{ccc}
\text{Prior} & \text{Data} & \text{Posterior} \\
\pi_0(\xi) & z_1, \ldots, z_n & \pi(\xi | z_{1:n})
\end{array}
\]

\[
\begin{array}{ccc}
\text{Data} & \text{Predictive} & \text{Posterior} \\
z_1, \ldots, z_n & p(z|z_{1:n}) & \pi(\xi | z_{1:n})
\end{array}
\]

Figure 1: Bayesian general updating mechanisms. The upper diagram represents the Bayesian update from prior to posterior. The lower diagram represents the Bayesian update from predictive to posterior.

B Predictive-to-posterior inference via KL divergence

The Bayes estimator of \(\xi\) is that which minimizes the posterior expected loss, given by

\[
\arg\min_{\xi' \in \Xi} \int u(\xi', \xi) \pi(\xi | z_{1:n}) d\xi.
\]

If \(u\) is taken to be the KL divergence, i.e., \(u(\xi', \xi) = \int \log \frac{f(z | \xi')}{f(z | \xi)} f(z | \xi) dz\), then the objective function becomes

\[
\arg\min_{\xi' \in \Xi} \int \int \log \frac{f(z | \xi)}{f(z | \xi')} f(z | \xi) dz \pi(\xi | z_{1:n}) d\xi = \arg\min_{\xi' \in \Xi} \int \log \frac{f(z | \xi)}{f(z | \xi')} p(z|z_{1:n}) dz.
\]

(13)

Under standard regularity conditions, exchanging differentiation and integration, we can deduce that (13) is the solution to

\[
\int \frac{\partial}{\partial \xi'} \log f(z | \xi') p(z|z_{1:n}) dz = \int S(z, \xi') p(z|z_{1:n}) dz
\]

where \(S(z, \xi')\) is the score function. The minimization in (13) does not involve prior opinion concerning \(\xi'\), but (13) can be modified to

\[
\arg\min_{\xi' \in \Xi} \left\{ \int \log \frac{f(z | \xi)}{f(z | \xi')} p(z|z_{1:n}) dz - \log \pi_0(\xi') \right\} = \arg\min_{\xi' \in \Xi} \left\{ \int \log \frac{f(z | \xi)}{f(z | \xi') \pi_0(\xi')} dP(z|z_{1:n}) \right\}
\]

with the modified score function \(S^*(z, \xi') = S(z, \xi') + \frac{\partial}{\partial \xi'} \log \pi_0(\xi')\). We specified a non-parametric model for \(P(z|z_{1:n})\). A distribution sampled from the posterior on \(P(z|z_{1:n})\) can be converted to a sampled value of \(\xi'\) in the same fashion as discussed in the main paper, which yield a fully Bayesian procedure with the solution of the usual likelihood-based posterior distribution.
C Asymptotics

C.1 Definition of Bayesian Consistency

**Definition 1.** [Walker and Hjort (2001)] For realizations $z_1, z_2, \ldots, z_n$ drawn independently from some unknown underlying distribution parameterized by $\xi$, with true data generating value $\xi^*$ in the interior of the parameter space $\Xi$, the posterior mass assigning to a subset $A \subseteq \Xi$ is given by

$$
\Pi^n (A) = \pi (\xi \in A | z_1, \ldots, z_n) = \frac{\int_A R_n (\xi) \pi_0 (d\xi)}{\int \pi_0 (d\xi)}
$$

where

$$
R_n (\xi) = \prod_{i=1}^{n} \exp \{- \ell (z_i, \xi) - \ell (z_i, \xi^*)\}
$$

and $\pi_0 (\xi)$ is the prior density for $\xi$. If $A_\epsilon = \{\xi : d(\xi, \xi^*) > \epsilon\}$ where $d(\xi, \xi^*)$ is some distance measure, the posterior distribution is consistent in the Bayesian sense if $\Pi^n (A_\epsilon) \to 0$ almost surely.

C.2 Assumptions

**Assumption 1.** Assume that the ‘true’ parameter $\xi_0$ defined in (1) lies in support of the prior distribution, so that $\pi_0 (\xi \in U) > 0$, for every neighborhood $U$ of $\xi_0$. The prior is on a space of probability densities, and puts positive mass on all KL divergences of $F_0$.

**Assumption 2.** $Y_1, \ldots, Y_n$, take values in $Y \subseteq \mathbb{R}$.

**Assumption 3.** $\ell (z, \xi)$ is continuous $\forall \xi \in \Xi$.

**Assumption 4.** $\xi_0 = (\theta_0, \beta_0, \phi_0) \in \Xi$ is the unique solution to $\mathbb{E}_{Y|x,d}[U(\xi)] = 0$.

**Assumption 5.** $\mathbb{E} \left[ \sup_{\xi \in \Xi} \|U(\xi)\|_F^\gamma \right] < \infty$ for $\gamma > 2$. Suppose there exists a neighborhood, $\tilde{\Xi}$ of $\xi_0$ within which $U(\xi)$ is continuously differentiable.

$$
\mathbb{E} \left[ \sup_{\xi \in \tilde{\Xi}} \| \hat{U}(\xi) \|_F \right] < \infty,
$$

with $\| \cdot \|_F$ denoting the Frobenius norm.

**Assumption 6.** $I$ non-singular and $J$ is full rank, i.e., $\text{rank}(J) = p$.

**Assumption 7.** Assume that the pseudo-true value $\xi_0^*$ belongs to the support of the prior distribution, i.e. $\pi_0 (\xi_0^* \in U) > 0$, for every neighborhood $U$ of $\xi_0^*$.

**Assumption 8.** For a fixed $\xi \in \Xi$, there exists $Q \in \{f_1 : \text{all misspecified loss models in (11)}\}$ such that $Q$ is mutually absolutely continuous with respect to the loss function corresponding to (10).

Assumption 8 essentially implies that the set of all misspecified models is in a nonempty set [Kleijn and van der Vaart, 2012].
C.3 Proof of Theorem 1

Proof. By Jensen’s inequality, for $\xi \in \Xi$, we have

$$d (\xi, \xi_0) = \mathbb{E}_{F_0} [\ell (Y, x, d, \xi) - \ell (Y, x, d, \xi_0)]$$

$$= \int_Y [\ell (Y, x, d, \xi) - \ell (Y, x, d, \xi_0)] dF_0 \geq 0$$

The last line follows as $\xi_0$ is defined as the minimizer. By the strong law of large numbers, one can obtain

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (\ell (Y_i, x_i, d_i, \xi) - \ell (Y_i, x_i, d_i, \xi_0)) \to d (\xi, \xi_0) \quad \text{almost surely for } \forall \xi \in \Xi. \quad (14)$$

Assume that there is some positive real number $M > 0$ and a set $S_M = \{ \xi : \| \xi - \xi_0 \| \leq M \}$ such that $\pi_0 (\xi \in S_M) \geq 0$. Then by Fatou’s lemma, there exists a neighborhood $U$ of $\xi_0$ such that $\pi_0 (\xi \in S_M \cap U) \geq 0$.

$$\liminf_{n \to \infty} e^{2nM} \int_U \exp \left[ - \sum_{i=1}^{n} (\ell (z_i, \xi) - \ell (z_i, \xi_0)) \right] \pi_0 (d\xi)$$

$$\geq \int_U \liminf_{n \to \infty} \exp \left[ 2nM - \sum_{i=1}^{n} (\ell (z_i, \xi) - \ell (z_i, \xi_0)) \right] \pi_0 (d\xi) \quad (15)$$

$$\geq \int_U \limsup_{n \to \infty} \exp \left[ n \left( 2M - \frac{1}{n} \sum_{i=1}^{n} (\ell (z_i, \xi) - \ell (z_i, \xi_0)) \right) \right] \pi_0 (d\xi)$$

$$= \int_{U \cap S_M} \limn \exp [n (2M - d (\xi, \xi_0))] \pi_0 (d\xi) = \infty.$$ 

In fact since $M$ is arbitrary, there is a neighborhood $U \subset S_M$ of $\xi_0$ which is small enough so that $\pi_0 (\xi \in U) < \pi_0 (\xi \in S_M)$. Therefore,

$$\int_U \exp \left[ - \sum_{i=1}^{n} (\ell (z_i, \xi) - \ell (z_i, \xi_0)) \right] \pi_0 (d\xi) \geq e^{-2nM}.$$ 

Similarly, by reverse Fatou’s lemma,

$$\limsup_{n \to \infty} \int_{U^c} \exp \left[ - \sum_{i=1}^{n} (\ell (z_i, \xi) - \ell (z_i, \xi_0)) \right] \pi_0 (d\xi)$$

$$\leq \limsup_{n \to \infty} \int_{\{x \in U^c : d (x, \xi_0) = 0\}} \exp (-nd (\xi, \xi_0)) d\pi_0 (\xi) \quad (16)$$

$$+ \int_{\{x \in U^c : d (x, \xi_0) > 0\}} \limsup_{n \to \infty} \exp (-nd (\xi, \xi_0)) \pi_0 (d\xi)$$

$$\leq \pi_0 (\xi \in U^c : \xi = \xi_0) + 0 = 0.$$ 

Therefore, $\limn \int_{U^c} \exp [- \sum_{i=1}^{n} (\ell (z_i, \xi) - \ell (z_i, \xi_0))] \pi_0 (d\xi) = 0$. Combine (15) and (16) to obtain

$$\frac{\int_{U^c} \exp [- \sum_{i=1}^{n} (\ell (z_i, \xi) - \ell (z_i, \xi_0))] \pi_0 (d\xi)}{\int_U \exp [- \sum_{i=1}^{n} (\ell (z_i, \xi) - \ell (z_i, \xi_0))] \pi_0 (d\xi)} \to 0 \quad \text{almost surely.}$$

This implies that $\Pi^n (U) \to 1$ almost surely.
C.4 Proof of Theorem 2

The proof of this theorem is based on [Chib et al. 2018; Ghosh and Ramamoorthi 2003; van der Vaart 2000].

Proof. Denote \( w = \sqrt{n}(\xi - \xi_0) \) and define the posterior distribution

\[
\pi_n (w) := \pi \left( \sqrt{n}(\xi - \xi_0) | z_1, \ldots, z_n \right).
\]

Then by the change of variable

\[
\pi_n (w) = C_n^{-1} \pi_0 (\xi_0 + \frac{w}{\sqrt{n}}) \exp \left[ - \sum_{i=1}^{n} \left( \ell \left( z_i, \xi_0 + \frac{w}{\sqrt{n}} \right) - \ell (z_i, \xi_0) \right) \right]
\]

where \( C_n = \int \pi_0 (\xi_0 + \frac{w}{\sqrt{n}}) \exp \left[ - \sum_{i=1}^{n} \left( \ell \left( z_i, \xi_0 + \frac{w}{\sqrt{n}} \right) - \ell (z_i, \xi_0) \right) \right] dw. \) Then in order to show that

\[
\sup_A | \pi_n (w \in A) - \text{Normal}_p(0, V) \in A | \overset{p}{\to} 0,
\]

we need to show

\[
\int_A \left| C_n^{-1} \pi_0 (\xi_0 + \frac{w}{\sqrt{n}}) \exp \left[ - \sum_{i=1}^{n} \left( \ell \left( z_i, \xi_0 + \frac{w}{\sqrt{n}} \right) - \ell (z_i, \xi_0) \right) \right] - (2\pi)^{-\frac{n}{2}} | V |^{-\frac{1}{2}} e^{-\frac{1}{2}(w^T V^{-1} w)} \right| dw \overset{p}{\to} 0.
\]

Note that (17) can be written as

\[
\int_A \left| C_n^{-1} \pi_0 (\xi_0 + \frac{w}{\sqrt{n}}) \exp \left[ - \sum_{i=1}^{n} \left( \ell \left( z_i, \xi_0 + \frac{w}{\sqrt{n}} \right) - \ell (z_i, \xi_0) \right) \right] - C_n^{-1} \pi_0 (\xi_0) e^{-\frac{1}{2}(w^T V^{-1} w)} \right| dw \\
+ C_n^{-1} \pi_0 (\xi_0) e^{-\frac{1}{2}(w^T V^{-1} w)} - (2\pi)^{-\frac{n}{2}} | V |^{-\frac{1}{2}} e^{-\frac{1}{2}(w^T V^{-1} w)} \right| dw \\
\leq C_n^{-1} \int_A \pi_0 (\xi_0 + \frac{w}{\sqrt{n}}) \exp \left[ - \sum_{i=1}^{n} \left( \ell \left( z_i, \xi_0 + \frac{w}{\sqrt{n}} \right) - \ell (z_i, \xi_0) \right) \right] - \pi_0 (\xi_0) e^{-\frac{1}{2}(w^T V^{-1} w)} \right| dw \\
+ \int_A \left| C_n^{-1} \pi_0 (\xi_0) e^{-\frac{1}{2}(w^T V^{-1} w)} - (2\pi)^{-\frac{n}{2}} | V |^{-\frac{1}{2}} e^{-\frac{1}{2}(w^T V^{-1} w)} \right| dw.
\]

For the first term in (18), it follows by Lemma C.1 in [Chib et al. 2018] with Assumption 1-7 and assuming that

\[
\lim_{n \to \infty} \mathbb{P} \left( \sup_{\|\xi - \xi_0\| > \delta} \frac{1}{n} \sum_{i=1}^{n} (\ell(z_i, \xi) - \ell(z_i, \xi_0)) \leq \epsilon \right) = 1.
\]

It can be shown that

\[
\int_A \pi_0 (\xi_0 + \frac{w}{\sqrt{n}}) \exp \left[ - \sum_{i=1}^{n} \left( \ell \left( z_i, \xi_0 + \frac{w}{\sqrt{n}} \right) - \ell (z_i, \xi_0) \right) \right] - \pi_0 (\xi_0) e^{-\frac{1}{2}(w^T V^{-1} w)} \right| dw \overset{p}{\to} 0
\]

by breaking the domain of integration A into different regions, i.e., \( A_1 = \{ w : \|w\| < c \log \sqrt{n} \} \), \( A_2 = \{ w : c \log \sqrt{n} \leq \|w\| \leq \delta \sqrt{n} \} \) and \( A_3 = \{ w : \|w\| > \delta \sqrt{n} \} \) for any given \( c \) and \( \delta \). Then therefore

\[
C_n^{-1} \int_A \pi_0 (\xi_0 + \frac{w}{\sqrt{n}}) \exp \left[ - \sum_{i=1}^{n} \left( \ell \left( z_i, \xi_0 + \frac{w}{\sqrt{n}} \right) - \ell (z_i, \xi_0) \right) \right] - \pi_0 (\xi_0) e^{-\frac{1}{2}(w^T V^{-1} w)} \right| dw \overset{p}{\to} 0.
\]
Secondly, by Lemma C.2 in [Chib et al., 2018] and Assumption 1-7, it can be shown that
\[
\ell \left( z_i, \xi_0 + \frac{w}{\sqrt{n}} \right) - \ell \left( z_i, \xi_0 \right) \to -\frac{1}{2} w^\top V^{-1} w.
\]

Therefore,
\[
C_n = \int \pi_0 \left( \xi_0 + \frac{w}{\sqrt{n}} \right) \exp \left[ -\sum_{i=1}^n \left( \ell \left( z_i, \xi_0 + \frac{w}{\sqrt{n}} \right) - \ell \left( z_i, \xi_0 \right) \right) \right] dw \xrightarrow{P} \pi_0 \left( \xi_0 \right) (2\pi)^{\frac{p}{2}} |V|^\frac{1}{2}.
\]

Then the second term in (18) becomes
\[
\int_A C_n^{-1} \pi_0 \left( \xi_0 \right) e^{-\frac{1}{2}(w^\top V^{-1} w - (2\pi)^{-\frac{p}{2}} |V|^{-\frac{1}{2}} e^{-\frac{1}{2}(w^\top V^{-1} w)})} dw \xrightarrow{P} 0 \tag{20}
\]

Combining the results in (19) and (20), we can obtain the result of the theorem. i.e.,
\[
\sup_A \left| \pi \left( \sqrt{n}(\xi - \xi_0) \in A | z_1, \ldots, z_n \right) - P(Z \in A) \right| \xrightarrow{P} 0.
\]

where \( Z \sim \text{Normal}_p(0, V) \) is an arbitrary random variable independent from all other random variables. \(\blacksquare\)

### C.5 Proof of Theorem 3

**Proof.** For the sequence of random probability measures based on the DP construction \( \{G_N, G_{N+1}, \ldots\} \) defined on the probability space \((\Omega, \mathcal{A}, P)\), take values in the measurable space \((\mathcal{Y}, \mathcal{Y})\), we define
\[
G_N (f | x, d) = \int f(y) dG_N (y | x, d) \quad \text{all bounded measurable } f : \mathcal{Y} \to \mathbb{R}.
\]

This integral is finite if \( \int \log(1 + |f(y)|) dG_0 (y | x, d) < +\infty \) [Feigin and Tweedie, 1989]. We denote a filtration, \( \mathcal{F}_t = \sigma(Z_1, \ldots, Z_t) \). Taking the conditional expectation, from Fubini’s theorem, we have
\[
\mathbb{E} \left[ G_{N+1} (f | x, d) | \mathcal{F}_N \right] = \int f(y) \mathbb{E} \left[ dG_{N+1} (y | x, d) | \mathcal{F}_N \right] = G_N (f | x, d)
\]

because \( G_N (y | x, d) \) is a martingale with respect to \( \mathcal{F}_N \) regardless of the draw for the pair of \( x, d \). As \( f \) is bounded, then \( \mathbb{E} \left[ |G_N (f | x, d)| \right] \) is also bounded. Therefore, \( G_N (f | x, d) \) is also a martingale with respect to \( \mathcal{F}_N \). By Theorem 2.2 in [Berti et al., 2006], so there exists a random probability measure \( G_\infty \) defined on \((\Omega, \mathcal{A}, P)\) such that \( G_N \to G_\infty \) weakly almost surely. \(\blacksquare\)

### D Additional simulation results

Table 4 summarizes the mean estimates of \( \theta \) over 1,000 Monte Carlo replicates for three different scenarios described above.
Table 4: Example 2: Simulation results of the marginal causal contrast, with true value equal to 0, on 1000 simulation runs on generated datasets of size \( n \). The rows correspond to mean the point estimates.

| \( n \)  | Scenario A |         |         |         | Scenario B |         |         |         | Scenario C |         |         |         |
|---------|------------|---------|---------|---------|------------|---------|---------|---------|------------|---------|---------|---------|
|         | 100        | 200     | 500     | 1000    | 100        | 200     | 500     | 1000    | 100        | 200     | 500     | 1000    |
| Mean    | -0.004     | -0.005  | -0.010  | 0.002   | 0.001      | -0.007  | 0.005   | 0.003   | -0.007     | -0.006  | 0.004   | 0.000   |
| Method I| -0.001     | 0.009   | 0.003   | 0.004   | -0.012     | 0.000   | -0.013  | 0.002   | 0.007      | -0.003  | -0.006  | 0.009   |
| Method II| -0.010    | -0.001  | 0.003   | -0.002  | -0.001     | 0.007   | -0.002  | -0.002  | 0.038      | 0.024   | 0.000   | 0.002   |
| Method III| -0.003   | -0.004  | 0.001   | 0.004   | -0.007     | 0.006   | 0.000   | -0.008  | 0.054      | -0.008  | 0.012   | 0.002   |
| Bayes-OR | 0.000     | -0.001  | 0.002   | -0.004  | 0.008      | 0.001   | -0.004  | -0.007  | -0.005     | 0.002   | 0.000   | 0.002   |