A common Misconception about the Categorical Arithmetic

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Abstract

Although the categorical arithmetic is not effectively axiomatizable, the belief that the incompleteness Theorems can apply to it is fairly common. Furthermore, the so-called essential (or inherent) semantic incompleteness of the second-order Logic that can be deduced by these same Theorems does not imply the standard semantic incompleteness that can be derived using the Löwenheim-Skolem or the compactness Theorem. This state of affairs has its origins in an incorrect and misinterpreted Gödel's comment at the Königsberg congress of 1930 and has consolidated due to different circumstances. This paper aims to clear up these questions and proposes an alternative interpretation for the Gödel’s statement.

Keywords: arithmetic, categoricity, semantic completeness, syntactic incompleteness, second-order languages.

1 The Categorical Arithmetic

The categorical arithmetic (AR) is a theory where the induction principle is introduced as a second-order axiom and interpreted according to the standard semantics. This interpretation, briefly called full (since the predicates range over the entire power set of the universe of discourse), is necessary for the categoricity [1].

AR is not a formal theory: it is impossible to dispense with the meaning of every its formula. This conclusion can be stated as follow. Since the standard model of AR is infinite and unique under isomorphism, according to the Löwenheim-Skolem Theorem, this theory cannot be expressed in a semantically complete language\(^1\). Then, it cannot be formal, because every formal theory has a semantically complete language [3].

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1This also can be concluded by noting that, due to the categoricity, the compactness Theorem (which applies to every theory with a semantically complete language), cannot be applicable to AR. Here, with a semantically complete language/axiomatic theory we understand a system always interpretable if consistent (that is equivalent to affirm that all the valid formulas are deducible). That is the usual way, but notice that occasionally others take a quite different meaning for the same expression (see, e.g. [2]).

A fortiori, AR is not effectively axiomatizable\textsuperscript{2}, because every effectively axiomatizable theory is formal.

Since AR is finitely axiomatizable, the last conclusion may surprise. Indeed, the diffuse opinion that every finite set of exhibited elements always is effectively enumerable, certainly is true when pure symbols suffice to characterize each one of the elements. But if we have even one with a meaning, a machine could identify it only if such meaning is entirely reproducible by mechanical operations \textit{(i.e. eliminable)} \textsuperscript{[4-5]}. It is not difficult to realize that in the case of the full second-order induction axiom this does not occur. Trying to clarify its meaning by a more precise language, we could resort to that set-theoretical one, representing AR within the Set Theory. Here, introduced a set for the axioms of AR, it turns out that the full second-order induction principle is no longer representable by a single axiom but it is equivalent to an axiomatic scheme capable of generating an uncountable number of axioms\textsuperscript{3}.

Therefore, against the widespread view, the incompleteness Theorems cannot be applied to AR. This misconception has dragged on for too long and has its origins in an incorrect, although absolutely excusable, Gödel’s comment at the Königsberg congress of 1930.

2 The Gödel’s Statement

According to the editors, the document \textsuperscript{*1930c} in the third volume of the \textit{Kurt Gödel’s collected works} (1995), is in all probability the text presented by Gödel at the Königsberg congress on September 6, 1930 \textsuperscript{[7]}. In the first part of the document, Gödel presents his semantic completeness Theorem. After he adds \textsuperscript{[8]}:

\[\text{[...]} \text{If the completeness theorem could also be proved for the higher parts of logic (the extended functional calculus), then it could be shown in complete generality that syntactical completeness follows from monomorphic [categoricity]; and since we know, for example, that the Peano axiom system is monomorphic [categorical], from that the solvability of every problem of arithmetic and analysis expressible in Principia mathematica would follow. Such an extension of the completeness theorem is, however, impossible, as I have recently proved [...]. This fact can also be expressed thus: The Peano axiom system, with the logic of Principia mathematica added as superstructure, is not syntactically complete.}\]

In summary, Gödel affirms that is impossible to generalize the semantic completeness Theorem to the “extended functional calculus”. In fact, in this case also the Peano axiomatic system, structured with the logic of the \textit{Principia}
Mathematica (PM), would be semantically complete. But since this theory is categorical, it would follow that it is also syntactically complete. But just this last thing is false, as he — surprise — announces to have proved.

Now, regardless of what Gödel meant by “extended functional calculus”, this affirmation contains a mistake. We have in fact two cases:

a) If Gödel understands by “Peano axiomatic system structured with the logic of the PM” an any type of formal arithmetic theory, the error is precisely to regard it as categorical.

b) If rather he alludes to the unique categorical arithmetic, that is AR, then Gödel errs applying to it his first incompleteness Theorem.

Of the two, just the second belief has been consolidating but without reporting the error. Rather, exalting the merit of having detected for the first time the semantic incompleteness of the (full) second-order Logic.

Probably, to forming this opinion has been important the influence of the following sentence contained in the second edition (1938) of Grundzüge theoretischen der Logik by Hilbert and Ackermann [9]:

Let us remark at once that a complete axiom system for the universally valid formulas of the predicate calculus of second order does not exist. Rather, as K. Gödel has shown [K. Gödel, Über formal unentscheidbare Sätze der Principia Mathematica und verwandter systeme, Mh. Math. Physik Vol. 38 (1931)], for any system of primitive formulas and rules of inference we can find universally valid formulas which cannot be deduced.

The echo of the Gödel’s incorrect words at the Congress pushes the authors (in particular Ackermann, given the age of Hilbert) to attest that the first incompleteness Theorem concludes directly the semantic incompleteness of the second-order Logic! False. Furthermore, as we will try to show, this (true) conclusion really does not appear to follow by the incompleteness Theorems.

Even in the introductory note of the aforementioned document, Goldfarb writes [10]:

Finally, Gödel considers categoricity and syntactic completeness in the setting of higher-order logics. [...] Noting then that Peano Arithmetic is categorical — where by Peano Arithmetic he means the second-order formulation — Gödel infers that if higher-order logic is [semantically] complete, then there will be a syntactically complete axiom system for Peano Arithmetic. At this point, he announces his incompleteness theorem: “The Peano axiom system, with the logic of Principia mathematica added as superstructure, is not syntactically complete”. He uses the result to conclude that there is no (semantically) complete axiom system for higher-order logic.

So interpreting, without the slightest doubt, that Gödel refers to the second-order categorical arithmetic.
indeed, today the belief that the incompleteness theorems can also apply to ar and, above all, that they have as a corollary the semantic incompleteness of the second-order logic is widespread. nevertheless, it is very rare that someone infers the semantic incompleteness of the (full) second-order logic in the easy and direct way that — according to the b. interpretation — gödel would follow, i.e. passing by the (alleged) syntactic incompleteness of ar. almost all the authors follow the alternative to prove that the valid formulas of ar cannot be effectively enumerable (see e.g. [11] and [12]): by contradiction, also the true (in the standard model) sentences of the formal (first-order) arithmetic would be effectively enumerable, against the first incompleteness theorem. that is not only more complex but also quite different: the genuine semantic completeness of a system, simply requires that all the valid formulas are theorems, not necessarily effectively enumerable theorems. actually, the intrinsic non-formality of ar entails that it really makes use of a non-effective deductive method. so, these proofs really do not conclude the (genuine) semantic incompleteness of the second-order logic (as, on the contrary, the use of the löwenheim-skolem theorem or compactness theorem can do).

the only explanation of this approach is that the authors are not sure about the direct applicability of the first incompleteness theorem to ar. that is not at all surprising in view of the evidences shown in the previous section; but nothing more is said.

too respect for the stature of gödel may have affected this state of affairs, but the main reasons of this misunderstanding are due probably to ambiguities of the used terminology, both ancient and modern.

3 clearing up the terms

the expression “extended predicate calculus” is for the first time used by hilbert in the first edition (1928) of the aforementioned grundzüge der theoretischen logik where, with no doubt, indicates the full second-order logic, which was considered for the first time in the principia mathematica. the belief that gödel, in the aforementioned phrase, refers to the ar theory (explanation b.), implies that he, with “extended functional calculus”, intends the same thing. but in which work he has shown or at least suggested that the incompleteness theorems can apply to the full second-order logic? in none.

in his proof of 1931, gödel refers to a formal system with a language that, in addition to the first-order classical logic, allows the use of non-bound functional variables [14]. then he proves that this is not a real extension of the language, able, in particular, to hinder the applicability of the semantic completeness theorem.

in the 1932b publication, gödel declares the validity of the incompleteness theorems for a formal system (z), based on first-order logic, with the axioms

4 this type of semantic incompleteness is called sometimes essential [13] or inherent [12], but these adjectives are not very appropriate because it does not imply the (standard) semantic incompleteness.
of Peano and an induction principle defined by a recursive function. Certainly not a full induction. He adds [15]:

If we imagine that the system $Z$ is successively enlarged by the introduction of variables for classes of numbers, classes of classes of numbers, and so forth, together with the corresponding comprehension axioms, we obtain a sequence (continuable into the transfinite) of formal systems that satisfy the assumptions mentioned above [...] Speaking explicitly of *comprehension axioms*, able to limit to the countable the number of the sentences, and formal systems.

Finally, in the publication of 1934, which contains the last and definitive proof of the first incompleteness Theorem, Gödel, having the aim both to generalize and to simplify the proof, allows the quantification either on the functional or propositional variables: a declared type of second-order. However, appropriate comprehension axioms limit again to infinite countable the number of sentences [16]. Gödel never misses an opportunity to point out carefully that always is referring to a formal system and that the formulas are enumerable [17]:

Different formal systems are determined according to how many of these types of variables are used. We shall restrict ourselves to the first two types; that is, we shall use variables of the three sorts $p$, $q$, $r$, ... [propositional variables]; $x$, $y$, $z$, ... [natural numbers variables]; $f$, $g$, $h$, ... [functional variables]. We assume that a denumerably infinite number of each are included among the undefined terms (as may be secured, for example, by the use of letters with numerical sub- scripts). [...] For undefined terms (hence the formulas and proofs) are countable, and hence a representation of the system by a system of positive integers can be constructed, as we shall now do.

Therefore, certainly we are not in the *full* second-order. Nevertheless, in the introduction to the same paper, Kleene, in summarizing the work of Gödel, does not avoid commenting ambiguously [18]:

Quantified propositional variables are eliminable in favor of function quantifiers. Thus the whole system is a form of full second-order arithmetic (now frequently called the system of “analysis”).

But he could only mean that the whole system is a *formal* version (perhaps as large as possible) of the full second-order arithmetic. Maybe is exactly this one the “extended functional calculus” to which Gödel was referring in the examined words at the Congress? We will discuss it in the next section.

Another source of mistake is probably related to use of the term metamathematics. Although Gödel intends it in the modern broad sense that includes any kind of argument beyond to the coded formal language of Mathematics, in his theorems always he employs this term limiting it to a formalizable (though often not yet formalized) use deductive (and, indeed, even decidable): therefore, only with purpose of brevity.
In the short paper that anticipates his incompleteness Theorems, for example, Gödel invokes a metamathematics able to decide whether a formula is an axiom or not [19]:

[...] IV. Theorem I [first incompleteness Theorem] still holds for all $\omega$-consistent extensions of the system $S$ that are obtained by the addition of infinitely many axioms, provided the added class of axioms is decidable, that is, provided for every formula it is metamathematically decidable whether it is an axiom or not (here again we suppose that in metamathematics we have at our disposal the logical devices of PM). Theorems I, III [as the IV, but the added axioms are finite], and IV can be extended also to other formal systems, for example, to the Zermelo-Fraenkel axiom system of set theory, provided the systems in question are $\omega$-consistent.

But in both the subsequent rigorous proofs, he will formalize this process, which now is called metamathematical, using the recursive functions, so revealing that, in the words just quoted, he refers to the usual “mechanical” decidability. By the same token, even in the theorem that concludes the consistency of the axiom of choice and of the continuum hypothesis with the other axioms of the formal Set Theory, he does the same: he uses the metamathematics only as a simplification, stating explicitly that all “the proofs could be formalized” and that “the general metamathematical considerations could be left out entirely” [20].

4 An Alternative Explanation

As noted, Gödel has never put in writing that his proofs of incompleteness may be applied to the uncountable full second-order arithmetic and it looks absolutely not reasonable to believe that he deems it. In this section, therefore, we will examine the other possibility, namely the a. of the second section. It pretends that Gödel in 1930 believed, mistakenly, categorical a kind of formal arithmetic and, in consequence of his incompleteness Theorems, semantically incomplete its language. Is this reasonable (or more reasonable than the previous case)?

Certainly not for the system considered by Gödel in his first proof of 1931: in fact, the semantic completeness Theorem applies to it, as Gödel himself remarks in note n. 55 of the publication [23]. Indeed, this is the first time in which the existence of non-standard models for a formal arithmetic is proved: why Gödel does not report it? The topic deserves a brief analysis.

More generally than the use of the incompleteness Theorems, the existence of non-standard models for any formal arithmetic theory can be proved using the compactness Theorem, the upward Löwenheim-Skolem one or a theorem proved

\[\text{I myself have changed my opinion reported in [21] and [22] after a deeper analysis of the Kurt Gödel’s collected works.}\]
by Skolem in 1933. The compactness Theorem is due precisely to Gödel (1930) and derives from his semantic completeness Theorem; but in none of his works Gödel ever uses it. Moreover, despite its fundamental importance for the model theory, nobody — except Maltsev in 1936 and 1941 — uses it before 1945.

Not much more fortunate is the story of the Löwenheim-Skolem Theorem. The first proof, by Löwenheim (1915), will be simplified by Skolem in 1920. In both cases, these theorems are downward versions, able to conclude the non-categoricity of the formal theory of the real numbers and of the formal Set Theory, but not of the formal Peano arithmetic. However, Skolem and Von Neumann suspect a much more general validity of the result. It seems that also Tarski was interested to this argument at that time, probably getting the upward version of the Theorem in a seminar of 1928. In any case, the argument continues to have low popularity, at least until the generalization of Maltsev in 1936, which, including for the first time the upward version, will allow the general conclusion that all the theories equipped with an infinite model and a semantically complete language are not categorical.

In this context of disinterest for the topic, Gödel not only is no exception, but his notorious Platonist inclination pushes him to distrust and/or despise any interpretation that refers to objects foreign to those that he believes existing independently of the considered theory; which, in all plausibility, also believes unique. As a matter of fact, in the introduction of his first paper on the semantic completeness, he shows to believe categorical even the first-order formal theory of the real numbers.

On this basis, one can surmise the following alternative for the option a. When he discovers the non-categoricity of the formal arithmetical system where his original incompleteness Theorems are applied, Gödel is not so glad and immediately looks for an extension that, though formal, is able to ensure the categoricity. Probably he believes to have identified it in a formal version of the full second order arithmetic: just that one that will be considered in his generalized proof of the first incompleteness Theorem of 1934, where quantification on the functional and propositional variables are allowed, while the formality is respected. This hypothesis is consistent with the fact that Gödel could admit the possibility that this theory uses a semantically incomplete language, because in both the versions of his semantic completeness Theorem, he does not allow the use of quantifiers on functional variables. Just the planning of this generalization (literally extended to the functional calculus) pushes him, in the meantime, to communicate the result without mentioning the discovery of the non-standard models. For example, just after his famous announcement at the

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6 That resolves that the system of the formal (first-order) sentences that are true in the standard model is non-categorical [24].
7 This is confirmed by Feferman [25].
8 This is attested both by Vaught and Fenstad in [26] and [27].
9 The respective works, relative to the first-order classical Logic (instead of the semantic completeness), are in [28].
10 This information is due to Maltsev: in [30] he claims to have known it by Skolem.
11 It is significant, for example, that Hilbert did not mention this theme in the seminar of Hamburg in 1927 [31].
Congress of Königsberg on September 6, 1930, Gödel declares [35]:

(Assuming the consistency of classical mathematics) one can even give examples of propositions (and in fact of those of the type of Goldbach or Fermat) that, while contentually true, are unprovable in the formal system of classical mathematics. Therefore, if one adjoining the negation of such a proposition to the axioms of classical mathematics, one obtains a consistent system in which a contentually false proposition is provable.

Indubitably here, admitted the soundness, he has in mind a formal theory that, being syntactically incomplete and categorical, has a semantically incomplete language.

The first surprise comes to him with the Skolem’s proof of 1933: neither the system of the formal (first-order) sentences that are true in the standard model is categorical [24]. A first disturbing clue that the non-categoricity covers all the formal (with at least an infinite model) systems, regardless of the syntactic completeness or incompleteness. Gödel, in reviewing the Skolem’s paper, laconically observes — finally! — that a consequence of this result, that is the non-categoricity of the formal Peano arithmetic, was already derivable from his incompleteness Theorems [36]. Later, in any work (not only in the cited generalization of 1934), he always will ignore the issue of the categoricity, nor ever will return to state that by his incompleteness Theorems can be derived the semantic incompleteness of some language or theory.

Ultimately, the Henkin’s Theorem of 1950 [3] will prove that in every formal system (and so, anywhere the incompleteness Theorems could be applied) there is semantic completeness of the language and therefore, if at least an infinite model exists, there cannot be categoricity.

5 Conclusions

Since the categorical arithmetic is not effectively axiomatizable and any type of formal arithmetic is not categorical, the text of the Gödel’s communication at the conference in Königsberg on September 6, 1930 (never published by him) contains a mistake. In the common understanding this error is not reported and thus it is wrongly believed that: a) the incompleteness Theorems also can be applied to the categorical arithmetic; b) the semantic incompleteness of the second-order Logic is a consequence of the incompleteness Theorems.

The previous interpretation is untenable, nor supported by the Gödel’s publications. As a matter of fact, the semantic incompleteness of the second-order Logic is due to the fact that this language allows to get the categoricity of theories (not only $AR$) equipped with at least an infinite model. And this is independent of the syntactic incompleteness of the formal arithmetic. By the incompleteness Theorems it is only possible to derive the so-called essential (or inherent) semantic incompleteness of the second-order Logic, which however does not imply the standard semantic incompleteness.
As an alternative interpretation of the manuscript in question, it is very plausible that Gödel was referring to a formal arithmetic (later specified in his proof of 1934) in which the quantification on the functional and propositional variables is allowed. If so, in 1930 he believed that this theory was categorical and, as a consequence of its syntactic incompleteness, equipped with a semantically incomplete language. This explanation is consistent with the fact that both the version of his semantic completeness Theorem cannot be applied to this system, due to the said quantification.

We wish to emphasize that this alternative in no way shades the luster of Gödel, because it makes no sense to pretend that he, in 1930, could know that every formal system, equipped with at least one infinite model, is not categorical. Conversely, it absolves him from a blunder and also explains why, becoming more and more evident, as time passes, the difficulty for the condition of categoricity, he never will repeat alike affirmations. On the other hand, Gödel never corrected the phrase presumably because he was not worrying about rectifying an unpublished text.

Finally, about the possible syntactic (and, by the categoricity, also semantic) completeness of the categorical arithmetic, we just observe that it would not be incompatible with the fact that the language of this theory is semantically incomplete. In fact, although an axiomatic system that uses a semantically complete language always is semantically complete, the reverse is not always true [37].

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On the basis of that label, *1930c ought to be the text of Gödel’s presentation to Menger’s colloquium on 14 May 1930 - the only occasion, aside from the meeting in Königsberg, on which Gödel is known to have lectured on his dissertation results [...]. Internal evidence, however, especially the reference on the last page to the incompleteness discovery, suggests that the text must be that of the later talk. Since no other lecture text on this topic has been found, it may well be that Gödel used the same basic text on both occasions, with a few later additions”.

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